Estimation of ordinal pattern probabilities in fractional Brownian motion

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Abstract

For equidistant discretizations of fractional Brownian motion (fBm), the probabilities of ordinal patterns of order $d = 2$ are monotonically related to the Hurst parameter $H$. By plugging the sample relative frequency of those patterns indicating changes between up and down into the monotonic relation to $H$, one obtains the Zero Crossing (ZC) estimator of the Hurst parameter which has found considerable attention in mathematical and applied research.

In this paper, we generally discuss the estimation of ordinal pattern probabilities in fBm. As it turns out, according to the sufficiency principle, for ordinal patterns of order $d = 2$ any reasonable estimator is an affine functional of the sample relative frequency of changes. We establish strong consistency of the estimators and show them to be asymptotically normal for $H < \frac{3}{4}$. Further, we derive confidence intervals for the Hurst parameter. Simulation studies show that the ZC estimator has larger variance but less bias than the HEAF estimator of the Hurst parameter.

Keywords: Ordinal pattern, fractional Brownian motion, estimation, Hurst parameter.

1 Introduction

Probabilities of ordinal patterns in equidistant discretizations of fractional Brownian motion (fBm) have first been analyzed by Bandt and Shiha [3]. Ordinal patterns represent the rank order of successive equidistant values of a discrete time series. For example, for three successive values there are six different possible outcomes of the rank order, which we call ordinal patterns of order $d = 2$.

As Bandt and Shiha have shown, for equidistant discretizations of fBm the distribution of ordinal patterns is stationary and does not depend on the particular sampling interval length. Further, the probabilities of ordinal patterns of order $d = 2$ are all monotonically related to the Hurst parameter, with the probability of ordinal patterns indicating changes from up to down and from down to up, respectively, being strictly monotonically decreasing in $H$.

The estimator of the Hurst parameter obtained by plugging the sample relative frequency of changes between up and down into the monotonic functional relation to $H$ has been
known for some time, running under the label 'Zero Crossing' (ZC) estimator since changes between up and down correspond to zero crossings of the first order differences.

Note that more generally, but not focussing on fBm, Kedem [11] has considered the estimation of parts of the autocorrelation structure of a stationary Gaussian process by counting (higher order) zero crossings, that is, zero crossings of the first (or higher) order differences.

The ZC estimator is asymptotically normally distributed for $H < \frac{3}{4}$, a result essentially due to Ho and Sun [10], who proved that the sample relative frequency of changes is asymptotically normally distributed in this case. Coeurjolly [6] has resumed properties of the ZC estimator including strong consistency. In the sequel, the applicability of the ZC estimator has been examined by Marković and Koch [17] and Shi et al. [20], including simulation studies as well as the application to hydrological and meteorological data.

The distribution of ordinal patterns is the base of ordinal time series analysis, being a new fast, robust and flexible approach to the investigation of large and complex time series (see Bandt [2] and Keller et al. [13, 12]). From the viewpoint of ordinal time series analysis, the estimation of ordinal pattern probabilities in stochastic processes is of special interest.

This paper is structured as follows: Sec. 2 is devoted to a general discussion of the estimation of ordinal pattern probabilities in fBm where we consider estimators based on counting the occurrence of ordinal patterns in realizations. Obviously, the sample relative frequency of ordinal patterns is an unbiased estimate of the probability of the patterns. As Theorem 4 shows, averaging the sample relative frequencies of ordinal patterns and of their ‘time’ and ‘spatial’ reversals, one obtains estimates which are strictly more concentrated in convex order. Hence, we restrict our following considerations to such ‘reasonable’ estimators. Notice that Theorem 4 has important consequences for the estimation of functionals of ordinal pattern distributions such as the permutation entropy (see [2], [12]).

For ordinal patterns of order $d = 2$ one obtains two different estimators which can both be expressed as affine functionals of the sample relative frequency of changes. By Lemma 5 we establish strong consistency and asymptotically unbiasedness for the estimators of bounded continuous functionals of ordinal pattern probabilities. Theorem 8 states that estimators of ordinal pattern probabilities are asymptotically normal for $H < \frac{3}{4}$.

In Sec. 3, we consider the estimation of the probability of a change by the sample relative frequency of changes in some detail. We give a formula for the precise numerical evaluation of the variance of the sample relative frequency of changes as well as asymptotically equivalent expressions. Based on these results, confidence intervals for the Hurst parameter are provided in Sec. 4.

In Sec. 5 we compare the previous results to findings for simulations of fBm. As it turns out, the confidence intervals obtained from the ZC estimator in Sec. 3 cover the true unknown value of the Hurst parameter for about 95 per cent of the cases, even for small sample sizes and for the values of the Hurst parameter larger than $\frac{3}{4}$, where asymptotic normality of the ZC estimator is not necessarily expected to hold. Compared to the HEAF estimator which estimates the Hurst parameter by plugging the sample autocovariance into a monotonic functional relation to $H$, the ZC estimator has larger variance but much less bias, in particular for small sample sizes.

Notice that the results given in this work for the increments of fBm similarly apply to
FARIMA\((0,d,0)\) processes. Furthermore, the statistical properties of the estimates of ordinal pattern probabilities do not only apply to fBm, but also to monotonic transformations of fBm. In particular, the estimates of ordinal pattern probabilities are invariant with respect to (unknown) non-linear monotonic transformations of processes.

2 Estimating ordinal pattern probabilities

2.1 General aspects

Let \((\Omega, \mathcal{A})\) be a measurable space equipped with a family \((\mathbb{P}_\vartheta)_{\vartheta \in \Theta}\) of probability measures with \(\Theta\) non-empty. The subscript \(\vartheta\) indicates that a corresponding quantity (\(\mathbb{E}_\vartheta, \text{Var}_\vartheta\), etc.) is taken with respect to \(\mathbb{P}_\vartheta\). For the special case of fractional Brownian motion where \(\Theta = [0, 1]\), we write \(H\) instead of \(\vartheta\).

Let \((X_k)_{k \in \mathbb{N}_0}\) be a given real-valued stochastic process defined on \((\Omega, \mathcal{A})\). Define the process of increments \((Y_k)_{k \in \mathbb{N}}\) by \(Y_k := X_k - X_{k-1}\) for \(k \in \mathbb{N}\). For \(d \in \mathbb{N}\) let \(S_d\) denote the set of permutations of \(\{0, 1, \ldots, d\}\).

**Definition 1.** For \(d \in \mathbb{N}\) let the mapping \(\pi : \mathbb{R}^{d+1} \to S_d\) be defined by
\[
\pi((x_0, x_1, \ldots, x_d)) = \left(\begin{array}{cccc}
0 & 1 & 2 & \ldots & d \\
r_0 & r_1 & r_2 & \ldots & r_d
\end{array}\right)
\]
for \((x_0, x_1, \ldots, x_d) \in \mathbb{R}^{d+1}\) with the permutation \((r_0, r_1, \ldots, r_d)\) of \(\{0, 1, \ldots, d\}\) satisfying
\[
x_{d-r_0} \geq x_{d-r_1} \geq \ldots \geq x_{d-r_d}, \text{ and } r_l-1 > r_l \text{ if } x_{d-r_l-1} = x_{d-r_l} \text{ for } l = 1, 2, \ldots, d.
\]
For \(k \in \mathbb{N}_0\) we call \(\Pi_d(k) := \pi((X_k, X_{k+1}, \ldots, X_{k+d}))\) the (random) ordinal pattern of order \(d\) at time \(k\).

The permutation \(\pi((x_0, x_1, \ldots, x_d))\) describes the rank order of the values \(x_0, x_1, \ldots, x_d\), where in case \(x_{d-i} = x_{d-j}\) for \(0 \leq i < j \leq d\) the ‘earlier’ \(x_{d-j}\) is ranked higher than \(x_{d-i}\).

Notice that if \((X_k)_{k \in \mathbb{N}_0}\) is pairwise distinct, that is, \(X_i \neq X_j \, \text{\(\mathbb{P}_\vartheta\)-a.s.}\) for all \(i, j \in \mathbb{N}_0\) with \(i \neq j\) and \(\vartheta \in \Theta\), then, apart from sets with probability zero for all \(\vartheta \in \Theta\), \(\Pi_d(k)\) generates the same \(\sigma\)-algebra as the rank vector \((R_0, R_1, \ldots, R_d)\) of \((X_k, X_{k+1}, \ldots, X_{k+d})\) given by
\[
R_j = \sum_{i=0}^{d} 1_{\{x_{k+i} \geq x_{k+j}\}} \text{ for } j = 0, 1, \ldots, d \text{ (see [15], p. 286).}
\]

**Stationarity.** For \(\vartheta \in \Theta\) let \(\overset{\vartheta}{=}\) denote equality in distribution with respect to \(\mathbb{P}_\vartheta\). A stochastic process \((Z_k)_{k \in T}\) defined on \((\Omega, \mathcal{A})\) for \(T = \mathbb{N}\) or \(T = \mathbb{N}_0 := \mathbb{N} \cup \{0\}\) is called **stationary** iff with respect to each \(\vartheta \in \Theta\)
\[
(Z_{k_1}, Z_{k_2}, \ldots, Z_{k_n}) \overset{\mathbb{P}_\vartheta}{=} (Z_{k_1+l}, Z_{k_2+l}, \ldots, Z_{k_n+l})
\]
for all \(k_1, k_2, \ldots, k_n \in T\) with \(n \in \mathbb{N}\) and for all \(l \in \mathbb{N}\).

In fact, \(\Pi_d(k)\) only depends on \((Y_{k+1}, Y_{k+2}, \ldots, Y_{k+d})\) for all \(d \in \mathbb{N}\) and \(k \in \mathbb{N}_0\). In particular, let
\[
\tilde{\pi}((y_1, y_2, \ldots, y_d)) = (r_0, r_1, \ldots, r_d)
\]
be the unique permutation of \( \{0, 1, \ldots, d\} \) for \((y_1, y_2, \ldots, y_d) \in \mathbb{R}^d\) such that

\[
\sum_{j=1}^{d-r_0} y_j \geq \sum_{j=1}^{d-r_1} y_j \geq \ldots \geq \sum_{j=1}^{d-r_{d-1}} y_j \geq \sum_{j=1}^{d-r_d} y_j,
\]

and \( r_{l-1} > r_l \) if \( \sum_{j=1}^{d-r_{l-1}} y_j = \sum_{j=1}^{d-r_l} y_j \) for \( l = 1, 2, \ldots, d \). Obviously,

\[
\pi((x_0, x_1, \ldots, x_d)) = \tilde{\pi}((x_1 - x_0, x_2 - x_1, \ldots, x_d - x_{d-1}))
\]

for all \((x_0, x_1, \ldots, x_d) \in \mathbb{R}^{d+1}\), and hence \( \Pi_d(k) = \tilde{\pi}(Y_{k+1}, Y_{k+2}, \ldots, Y_{k+d}) \) for all \( k \in \mathbb{N}_0 \). This immediately yields the following statement.

**Corollary 2.** If \((Y_k)_{k \in \mathbb{N}}\) is stationary then \((\Pi_d(k))_{k \in \mathbb{N}_0}\) is stationary.

**Space and time symmetry.** For \( d \in \mathbb{N} \) let the mappings \( \alpha, \beta \) from \( S_d \) onto \( S_d \) be defined by

\[
\alpha(r) := (r_d, r_{d-1}, \ldots, r_0), \quad \beta(r) := (d - r_0, d - r_1, \ldots, d - r_d)
\]

for \( r = (r_0, r_1, \ldots, r_d) \in S_d \). Geometrically, \( \alpha(r) \) and \( \beta(r) \) can be seen as the spatial and time reversal of \( r \), respectively (see Figure 1). Let the set \( \tau \) be defined by

\[
\tau := \{ r, \alpha(r), \beta(r), \beta \circ \alpha(r) \}
\]

with \( \circ \) denoting the usual composition of functions. As \( \alpha \circ \beta = \beta \circ \alpha \) and \( \alpha \circ \alpha = \beta \circ \beta = r \), one has \( \alpha(\tau) = \beta(\tau) = \tau \). Clearly, if \( s \in \tau \) for \( r, s \in S_d \), then \( s = \tau \). This provides a division of each \( S_d \) into classes, consisting of 2 or 4 elements. For \( d = 1 \) the only class is \( S_d = \{(0, 1), (1, 0)\} \), for \( d = 2 \) there are the two classes \( \{(0, 1, 2), (2, 1, 0)\} \) and \( \{(0, 2, 1), (2, 0, 1), (1, 2, 0), (1, 0, 2)\} \), and for \( d = 3 \) there are 8 classes. Note that for \( d \geq 3 \) classes of both 2 and 4 elements are possible.

![Figure 1: The ordinal pattern](image)

\( r = (0, 2, 1) \) \( \quad \alpha(r) = (1, 2, 0) \quad \beta(r) = (2, 0, 1) \quad \beta \circ \alpha(r) = (1, 0, 2) \)

Figure 1: The ordinal pattern \( r = (0, 2, 1) \), its spatial reversal \( \alpha(r) \), its time reversal \( \beta(r) \), and its spatial and time reversal \( \beta \circ \alpha(r) \).

For \( d \in \mathbb{N} \) and \( n \in \mathbb{N} \) let \((S_d)^n\) denote the \( n \)-fold Cartesian product of \( S_d \). Define the mappings \( A, B \) from \((S_d)^n\) onto \((S_d)^n\) by

\[
A((r^{(1)}, r^{(2)}, \ldots, r^{(n)})) := (\alpha(r^{(1)}), \alpha(r^{(2)}), \ldots, \alpha(r^{(n)})),
\]

\[
B((r^{(1)}, r^{(2)}, \ldots, r^{(n)})) := (\beta(r^{(n)}), \beta(r^{(n-1)}), \ldots, \beta(r^{(1)}))
\]

for \((r^{(1)}, r^{(2)}, \ldots, r^{(n)}) \in (S_d)^n \). Further, let \( \text{id} \) denote the identity map on \((S_d)^n \). Note that \( \{\text{id}, A, B, B \circ A\} \) together with \( \circ \) forms an Abelian group.
We say that \((Y_k)_{k \in \mathbb{N}}\) is symmetric in space and time iff with respect to each \(\vartheta \in \Theta\)

\[
(Y_{k_1}, Y_{k_2}, \ldots, Y_{k_n}) \overset{\text{d}}{=} (-Y_{k_1}, -Y_{k_2}, \ldots, -Y_{k_n})
\]

and \((Y_{k_1}, Y_{k_2}, \ldots, Y_{k_n}) \overset{\text{d}}{=} (Y_{k_n}, Y_{k_{n-1}}, \ldots, Y_{k_1})\)

for all \(k_1, k_2, \ldots, k_n \in \mathbb{N}\) with \(n \in \mathbb{N}\).

**Lemma 3.** If \((X_k)_{k \in \mathbb{N}_0}\) is pairwise distinct and \((Y_k)_{k \in \mathbb{N}}\) is symmetric in space and time, then the distribution of \(\Pi = (\Pi_d(0), \Pi_d(1), \ldots, \Pi_d(n-1))\) is invariant under the group of transformations \(\{\text{id}, A, B, B \circ A\}\) on \((S_d)^n\) with respect to \(\mathbb{P}_\vartheta\) for each \(\vartheta \in \Theta\), i.e.

\[
\Pi \overset{\text{d}}{=} A(\Pi) \overset{\text{d}}{=} B(\Pi) \overset{\text{d}}{=} B \circ A(\Pi).
\]

**Proof.** If \((X_k)_{k \in \mathbb{N}_0}\) is pairwise distinct then, according to the definition of \(\bar{\pi}\),

\[
\mathbb{P}_\vartheta\text{-a.s. with respect to each } \vartheta \in \Theta, \text{ see } (1).
\]

Further, symmetry in space and time of \((Y_k)_{k \in \mathbb{N}}\) implies that \((Y_1, Y_2, \ldots, Y_{d+n-1}), (-Y_1, -Y_2, \ldots, -Y_{d+n-1}), (Y_{d+n-1}, \ldots, Y_2, Y_1)\) and \((-Y_{d+n-1}, \ldots, -Y_2, -Y_1)\) have the same distribution with respect to \(\mathbb{P}_\vartheta\) for each \(\vartheta \in \Theta\), and hence the statement follows.

**A Rao-Blackwellization.** For fixed \(r \in S_d\) with \(d \in \mathbb{N}\) consider the functional \(p_r(\cdot) := \mathbb{P}_\vartheta(\Pi_d(0) = r)\) for \(\vartheta \in \Theta\). Obviously, if \((Y_k)_{k \in \mathbb{N}}\) is stationary then the statistic

\[
\hat{\rho}_{r,n} = \hat{\rho}_{r,n}(\Pi) := \frac{1}{n} \sum_{k=0}^{n-1} 1_{\{\Pi_d(k) = r\}}
\]

of \(\Pi = (\Pi_d(0), \Pi_d(1), \ldots, \Pi_d(n-1))\) is an unbiased estimate of \(p_r(\cdot)\), that is \(\mathbb{E}_\vartheta(\hat{\rho}_{r,n}) = p_r(\vartheta)\) for all \(\vartheta \in \Theta\) and \(n \in \mathbb{N}\). If, additionally, \((X_k)_{k \in \mathbb{N}_0}\) is pairwise distinct and \((Y_k)_{k \in \mathbb{N}}\) is symmetric in space and time then, according to Lemma 3, the statistic

\[
\hat{\rho}_{r,n} = \hat{\rho}_{r,n}(\Pi) := \frac{1}{d} \left( \hat{\rho}_{r,n}(\Pi) + \hat{\rho}_{r,n}(A(\Pi)) + \hat{\rho}_{r,n}(B(\Pi)) + \hat{\rho}_{r,n}(B \circ A(\Pi)) \right)
\]

of \(\Pi\) is a Rao-Blackwellization of \(\hat{\rho}_{r,n}\) (see Theorem 3.2.1 in [13]). This proves the following Theorem.

**Theorem 4.** Let \(r \in \bigcup_{d \in \mathbb{N}} S_d\). If \((X_k)_{k \in \mathbb{N}_0}\) is pairwise distinct and \((Y_k)_{k \in \mathbb{N}}\) is stationary and symmetric in space and time then the estimate \(\hat{\rho}_{r,n}\) of \(p_r(\cdot)\) is unbiased and more concentrated in convex order than \(\hat{\rho}_{r,n}\), that is

\[
\mathbb{E}_\vartheta(\varphi(\hat{\rho}_{r,n}, p_r(\vartheta))) \leq \mathbb{E}_\vartheta(\varphi(\hat{\rho}_{r,n}, p_r(\vartheta)))
\]

for all \(\vartheta \in \Theta\) with respect to each function \(\varphi: [0, 1] \times [0, 1] \rightarrow [0, \infty]\) with \(\varphi(p, p) = 0\) and \(\varphi(\cdot, p)\) being convex for every \(p \in [0, 1]\).
According to the strictness of the Jensen inequality for strictly convex functions, if \((X_k)_{k \in \mathbb{N}}\) is pairwise distinct and \((Y_k)_{k \in \mathbb{N}}\) is stationary and symmetric in space and time, one has strict inequality in (i) whenever \(\varphi(\cdot, p)\) is strictly convex for every \(p \in [0, 1]\) and, additionally, \(\mathbb{P}_\varphi(\widehat{\varphi}_{r,n} \neq \widetilde{\varphi}_{r,n}) > 0\) (see [18], Theorem 3.2.1). Since \((\cdot - p)^2\) is strictly convex for every \(p \in [0, 1]\), in particular

\[
\text{Var}_\varphi(\widehat{\varphi}_{r,n}) < \text{Var}_\varphi(\widetilde{\varphi}_{r,n})
\]

in this case. Note that for \(r \in S_d\) with \(d = 1\), the estimator \(\widehat{\varphi}_{r,n}\) simply estimates the constant functional \(p_r(\cdot) = \frac{1}{2}\).

**Ergodicity.** Consider the measurable space \((\Omega', \mathcal{A}') := (\mathbb{R}^N, \mathcal{B}(\mathbb{R}^N))\) of infinite sequences of real numbers and let the mapping \(T : \Omega' \to \Omega'\) be defined by \(T(\omega') = (\omega_2', \omega_3', \ldots)\) for \(\omega' = (\omega_1', \omega_2', \ldots) \in \Omega'\). The process \((Y_k)_{k \in \mathbb{N}}\) is called *ergodic* iff \((Y_k)_{k \in \mathbb{N}}\) is stationary and, additionally, for every \(A \in \mathcal{A}'\) such that \(T^{-1}(A) = A\) one has \(\mathbb{P}_\varphi((Y_k)_{k \in \mathbb{N}} \in A) = 0\) or \(\mathbb{P}_\varphi((Y_k)_{k \in \mathbb{N}} \in A) = 1\) for each \(\varphi \in \Theta\).

As the next Lemma shows, if \((Y_k)_{k \in \mathbb{N}}\) is ergodic then the estimators of continuous and bounded functionals of ordinal pattern probabilities are strongly consistent and asymptotically unbiased.

**Lemma 5.** Let \(r \in \bigcup_{d \in \mathbb{N}} S_d\). If \((Y_k)_{k \in \mathbb{N}}\) is ergodic and \(h : [0, 1] \to \mathbb{R}\) is continuous then

\[
\lim_{n \to \infty} h(\widehat{\varphi}_{r,n}) = h(p_r(\varphi))
\]

\(\mathbb{P}_\varphi\)-a.s. for all \(\varphi \in \Theta\). If \(h\) is continuous and bounded, then with respect to each \(\varphi \in \Theta\)

\[
\lim_{n \to \infty} \mathbb{E}_\varphi\left(h(\widehat{\varphi}_{r,n})\right) = h(p_r(\varphi)).
\]

**Proof.** For fixed \(\varphi \in \Theta\) let the probability measure \(\mu\) on \((\Omega', \mathcal{A}') = (\mathbb{R}^N, \mathcal{B}(\mathbb{R}^N))\) be defined by \(\mu(A) := \mathbb{P}_\varphi((Y_k)_{k \in \mathbb{N}} \in A)\) for \(A \in \mathcal{A}'\). Further, let the mapping \(T : \Omega' \to \Omega'\) be given as above and define \(f : \Omega' \to \mathbb{R}\) by

\[
f(\omega') := \begin{cases} 
1 & \text{for } \widehat{\pi}(\omega_1', \omega_2', \ldots, \omega_n') \in \overline{\varphi} \\
0 & \text{else}
\end{cases}
\]

for \(\omega' = (\omega_1', \omega_2', \ldots) \in \Omega'\). Obviously, \(f\) is Borel-measurable and \(\int_{\Omega'} |f| d\mu < \infty\), hence, according to Birkhoff’s Theorem, if \((Y_k)_{k \in \mathbb{N}}\) is ergodic then

\[
\lim_{n \to \infty} \widehat{\varphi}_{r,n} = \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j((Y_k)_{k \in \mathbb{N}})) = \mathbb{E}_\varphi(f((Y_k)_{k \in \mathbb{N}})) = p_r(\varphi)
\]

\(\mathbb{P}_\varphi\)-a.s. (see [7], Theorem 1.2.1) and the first statement follows since \(h\) is continuous. If additionally \(h\) is bounded then

\[
\lim_{n \to \infty} \mathbb{E}_\varphi\left(h(\widehat{\varphi}_{r,n})\right) = \mathbb{E}_\varphi\left(\lim_{n \to \infty} h(\widehat{\varphi}_{r,n})\right) = h(p_r(\varphi))
\]

according to the dominated convergence Theorem. \(\square\)
2.2 Specialization to fBm

In the following we specialize the considerations of Subsection 2.1 to equidistant discretizations of fractional Brownian motion (fBm). We start with the definition of fBm.

**Definition 6.** For \( H \in [0, 1] \) let \( (B(t))_{t \in [0, \infty[} \) be a mean-zero Gaussian process on a probability space \((\Omega, \mathcal{A}, \mathbb{P})\) with the covariance function

\[
\text{Cov}(B(t), B(s)) = \frac{1}{2} (t^{2H} + s^{2H} - |t - s|^{2H}) \text{Var}(B(1))
\]

for \( s, t \in [0, \infty[. \) Then \((B(t))_{t \in [0, \infty[}\) is called fractional Brownian motion (fBm) with the Hurst parameter \( H \).

It is well-known that each fBm possesses a modification with \( \mathbb{P}\)-a.s. continuous paths (see [9]). We are only interested in the distribution of fBm, so let for the rest of the paper \((\mathbb{P}_H)_{t \in [0, 1]}\) be a family of probability measures on a measurable space \((\Omega, \mathcal{A})\), and \((B(t))_{t \in [0, \infty[}\) be a family of real-valued random variables on \((\Omega, \mathcal{A})\) such that \((B(t))_{t \in [0, \infty[}\) is fBm with the Hurst parameter \( H \). E.g., \((B(t))_{t \in [0, \infty[}\) can be defined as the identity on the set of continuous functions on \([0, \infty[\).

For the **sampling interval length** \( \delta > 0 \) consider the equidistant discretization

\[
(X^\delta_k)_{k \in \mathbb{N}_0} := (B(k\delta))_{k \in \mathbb{N}_0}
\]

of fBm. Further, let \( Y^\delta_k := X^\delta_k - X^\delta_{k-1} \) for \( k \in \mathbb{N} \). Note that fBm is \( H\)-self-similar (see [2]), that is, for every \( H \in ]0, 1[ \) and for all \( \alpha \in ]0, \infty[ \) it holds

\[
(B(at))_{t \in [0, \infty[} \overset{\mathbb{P}_H}{=} (a^H B(t))_{t \in [0, \infty[}
\]

with \( \overset{\mathbb{P}_H}{=} \) denoting equality of all finite-dimensional distributions with respect to \( \mathbb{P}_H \) here. Since it holds

\[
\overline{\pi}((y_1, y_2, \ldots, y_d)) = \overline{\pi}((a^H y_1, a^H y_2, \ldots, a^H y_d)) \text{ for all } a > 0, H \in ]0, 1[ \text{ and } (y_1, y_2, \ldots, y_d) \in \mathbb{R}^d,
\]

one has

\[
\overline{\pi}((Y^\delta_{k+1}, \ldots, Y^\delta_{k+d})) \overset{\mathbb{P}_H}{=} \overline{\pi}((Y^1_{k+1}, \ldots, Y^1_{k+d}))
\]

with respect to each \( H \in ]0, 1[ \) for all \( \delta > 0 \). This means that the distribution of ordinal patterns for equidistant discretizations of fBm does not depend on the particular sampling interval length. Therefore, in the following we only consider the discretization \((X^\delta_k)_{k \in \mathbb{N}_0} = (X^\delta_k)_{k \in \mathbb{N}_0} = (X^1_k - X^1_{k-1})_{k \in \mathbb{N}}\) with the increment process \((Y^\delta_k)_{k \in \mathbb{N}_0} = (X^1_k - X^1_{k-1})_{k \in \mathbb{N}}\).

Note that, similarly to the sampling interval length, one can show that the particular scaling \( \text{Var}_H(B(1)) \) of fBm has no effect on the distribution of ordinal patterns for equidistant discretizations. Hence, we always assume the case of standard fBm where \( \text{Var}_H(B(1)) = 1 \) for all \( H \in ]0, 1[ \).

According to Definition [6], \( \text{Var}_H(X_i - X_j) > 0 \) for all \( i, j \in \mathbb{N}_0 \) with \( i \neq j \) and \( H \in ]0, 1[ \), and since \( X_i - X_j \) is Gaussian with respect to \( \mathbb{P}_H \) for all \( H \in ]0, 1[ \), the stochastic process \((X_k)_{k \in \mathbb{N}_0} \) is pairwise distinct. Further, obviously \((Y^\delta_k)_{k \in \mathbb{N}_0} \) is mean-zero Gaussian with respect to \( \mathbb{P}_H \) for each \( H \in ]0, 1[ \), and by Definition [5] one obtains

\[
\rho_H(k) := \text{Cov}_H(Y_1, Y_{1+k}) = \frac{1}{2} (|k + 1|^{2H} - 2|k|^{2H} + |k - 1|^{2H})
\]

(7)
for all $H \in [0,1]$ and $k \in \mathbb{N}_0$, that is, the stochastic process $(Y_k)_{k \in \mathbb{N}}$ is stationary. Furthermore, for all $n \in \mathbb{N}$ and $k_1, k_2, \ldots, k_n \in \mathbb{N}$ the random vectors $(Y_{k_1}, Y_{k_2}, \ldots, Y_{k_n})$, $(-Y_{k_1}, -Y_{k_2}, \ldots, -Y_{k_n})$ and $(Y_{k_1}, \ldots, Y_{k_2}, Y_{k_1})$ have the same covariance structure, hence $(Y_k)_{k \in \mathbb{N}}$ is space and time symmetric. Consequently, the conclusion of Theorem 5 applies to the estimation of ordinal pattern probabilities for equidistant discretizations of fBm.

As the following Lemma shows, one has strict inequality in (3) whenever $\varphi(\cdot, p)$ is strictly convex for every $p \in [0,1]$ and $H \in [0,1]$ (compare to the remark after Theorem 4).

**Lemma 7.** If $H \in [0,1]$ then $\mathbb{P}_H(\hat{p}_{r,n} \neq \tilde{p}_{r,n}) > 0$ for all $r \in \bigcup_{d \in \mathbb{N}} S_d$ and $n \in \mathbb{N}$.

**Proof.** Let $r \in S_d$ with $d \in \mathbb{N}$ and $n \in \mathbb{N}$ be fixed. It is easy to construct a sequence of permutations $(r^{(1)}, r^{(2)}, \ldots, r^{(n)}) \in (S_d)^n$ such that the $(n+d-1)$-dimensional Lebesgue measure of $\cap_{k=1}^{n} \{(y_1, y_2, \ldots, y_{n+d-1}) \in \mathbb{R}^{n+d-1} \mid \tilde{\pi}((y_k, y_{k+1}, \ldots, y_{k+d-1}) = r^{(k)})\}$ is strictly positive and

$$\# \tau \cdot \# \{k \in \{0,1,\ldots,n-1\} \mid r^{(k)} = r\} \neq \# \{k \in \{0,1,\ldots,n-1\} \mid r^{(k)} \in \tau\}.$$ 

According to (7), if $H \in [0,1]$ then $(Y_1, Y_2, \ldots, Y_{n+d-1})$ is non-degenerate Gaussian with respect to $\mathbb{P}_H$ and, consequently, $\mathbb{P}_H((Y_1, Y_2, \ldots, Y_{n+d-1}) \in A) > 0$ for each Borel-set $A \subset \mathbb{R}^{d+n-1}$ with strictly positive $(n+d-1)$-dimensional Lebesgue measure. Since $\tilde{p}_{r,n} = \frac{1}{|\tau|} \sum_{\tau \in \tilde{\pi}} \hat{p}_{r,n}$ (see (1) and (5)) the statement follows. \hfill $\square$

Note that if $H = 1$ then $\mathbb{P}_H(\hat{p}_{r,n} \neq \tilde{p}_{r,n}) > 0$ if and only if $r \in \{(0,1,\ldots,d), (d,\ldots,1,0)\}$. It is well-known that for each $H \in [0,1]$ the spectral distribution function of $(Y_k)_{k \in \mathbb{N}}$ is absolutely continuous (see [4]) which is a sufficient condition for $(Y_k)_{k \in \mathbb{N}}$ to be ergodic with respect to $\mathbb{P}_H$ (see [7], Theorem 14.2.1). In the case $H = 1$ one has $X_k = kX_1$ $\mathbb{P}_H$-a.s. for every $k \in \mathbb{N}_0$ (see [4]), hence

$$\hat{p}_{r,n} = p_r(H) = \begin{cases} 0 & \text{if } r \in \bigcup_{d \in \mathbb{N}} \left(S_d \setminus \{(0,1,\ldots,d), (d,\ldots,1,0)\}\right) \\ \frac{1}{2} & \text{if } r \in \bigcup_{d \in \mathbb{N}} \{(0,1,\ldots,d), (d,\ldots,1,0)\} \end{cases}$$

$\mathbb{P}_H$-a.s. for all $n \in \mathbb{N}$. Consequently, the conclusions of Lemma 5 apply to the estimation of ordinal pattern probabilities for equidistant discretizations of fBm.

**Asymptotic normality.** Next, we discuss asymptotic normality of the estimators of ordinal pattern probabilities for equidistant discretizations of fBm. We restrict our consideration to the ‘reasonable’ estimators as provided by Theorem 4. Let $\overline{p}_H$ denote convergence in distribution with respect to $\mathbb{P}_H$, and $\mathcal{N}(\mu, \sigma^2)$ be the normal distribution with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 > 0$. Given a mean-zero Gaussian vector $Z = (Z_1, Z_2, \ldots, Z_d)$ on a probability space $(\Omega', \mathcal{A}', \mathbb{P})$, and a Borel-measurable function $f : \mathbb{R}^d \to \mathbb{R}$ such that $\text{Var}(f(Z)) < \infty$, define the rank of $f$ with respect to $Z$ by

$$\text{rank}(f) := \min \{\kappa \in \mathbb{N} : \text{There exists a real polynomial } q : \mathbb{R}^d \to \mathbb{R} \text{ of degree } \kappa \text{ with } \mathbb{E}([f(Z) - \mathbb{E}(f(Z))]q(Z)) \neq 0\};$$

where the minimum of the empty set is infinity. We write $f(k) \sim g(k)$ for mappings $f, g$ from $\mathbb{N}_0$ onto $\mathbb{R}$ and say that $f$ is asymptotically equivalent to $g$ iff $\lim_{k \to \infty} f(k)/g(k) = 1$ where $\frac{0}{0} := 1$. Note that for $\rho_H(k)$ as defined in (7), one has the asymptotic equivalence

$$\rho_H(k) \sim H(2H - 1)k^{2H-2}$$

(9)
for each $H \in ]0, 1]$ (see [9]). Further, we write $f(k) = O(g(k))$ iff $\sup_{k \in \mathbb{N}_0} f(k)/g(k) < \infty$, and $f(k) = o(g(k))$ iff $\lim_{k \to \infty} f(k)/g(k) = 0$.

**Theorem 8.** If $H < \frac{3}{4}$ then

\[
(\text{Var}_H(\hat{p}_{\tau,n}))^{-\frac{1}{2}} (\hat{p}_{\tau,n} - p_r(H)) \xrightarrow{\mathbb{P}_H} N(0, 1)
\]

for all $r \in S_d$ with $d \in \mathbb{N} \setminus \{1\}$.

**Proof.** Let $r \in S_d$ with $d \in \mathbb{N} \setminus \{1\}$ be fixed and define $f : \mathbb{R}^d \to \mathbb{R}$ by

\[
f((y_1, y_2, \ldots, y_d)) := \begin{cases} 1 & \text{for } \hat{\pi}((y_1, y_2, \ldots, y_d)) \in \mathcal{T} \\ 0 & \text{else} \end{cases}
\]

for $(y_1, y_2, \ldots, y_d) \in \mathbb{R}^d$. Write $Y = (Y_1, Y_2, \ldots, Y_d)$. We first show that $\text{rank}(f) > 1$ with respect to $Y$ and $\mathbb{P}_H$ for all $H \in ]0, 1]$. Obviously, $f$ is Borel-measurable, and $\text{Var}_H(f(Y)) < \infty$ for all $H \in ]0, 1]$. Now, let $i \in \{1, 2, \ldots, d\}$ be fixed. Because $(Y_k)_{k \in \mathbb{N}}$ is symmetric in space and time, the random vectors $(Y_1, Y_2, \ldots, Y_d)$ and $(-Y_1, -Y_2, \ldots, -Y_d)$ have the same distribution with respect to $\mathbb{P}_H$ for all $H \in ]0, 1]$, and hence, according to the definition of $\alpha$ (see [2])

\[
E_H(1_{\{\hat{\pi}(Y) = s\}} Y_i) + E_H(1_{\{\hat{\pi}(Y) = \alpha(s)\}} Y_i) = E_H(1_{\{\hat{\pi}(Y) = s\}} Y_i) - E_H(1_{\{\hat{\pi}(Y) = s\}} (-Y_i)) = 0 \quad (10)
\]

for each $s \in S_d$. Since $Y_i$ is mean-zero Gaussian with respect to $\mathbb{P}_H$ for all $H \in ]0, 1]$, one has $E_H(f(Y)) E_H(Y_i) = 0$, and this together with (10) yields in case $\sharp \mathcal{T} = 2$

\[
E_H([f(Y) - E_H(f(Y))] Y_i) = E_H(1_{\{\hat{\pi}(Y) = r\}} Y_i) + E_H(1_{\{\hat{\pi}(Y) = \alpha(r)\}} Y_i) = 0
\]

with respect to each $H \in ]0, 1]$, and in case $\sharp \mathcal{T} = 4$

\[
E_H([f(Y) - E_H(f(Y))] Y_i) = E_H(1_{\{\hat{\pi}(Y) = r\}} Y_i) + E_H(1_{\{\hat{\pi}(Y) = \alpha(r)\}} Y_i)
+ E_H(1_{\{\hat{\pi}(Y) = \beta(r)\}} Y_i) + E_H(1_{\{\hat{\pi}(Y) = \beta\alpha(r)\}} Y_i) = 0
\]

with respect to each $H \in ]0, 1]$. Consequently, $\text{rank}(f) > 1$ with respect to $Y$ and $\mathbb{P}_H$ for all $H \in ]0, 1]$. According to [2], $|\rho_H(k)|^{\text{rank}(f)} = O(k^{4H-4})$ for each $H \in ]0, 1]$, thus

\[
\sum_{k=0}^{\infty} |\rho_H(k)|^{\text{rank}(f)} < \infty
\]

for $H < \frac{3}{4}$. Since, by definition one has $\frac{1}{n} \sum_{k=0}^{n-1} f((Y_{k+1}, Y_{k+2}, \ldots, Y_{k+d})) = \hat{p}_{\tau,n}$ for every $n \in \mathbb{N}$, the statement follows from Theorem 4 of Arcones [1].

Note that Theorem 8 can also be proven by the Central Limit Theorem for non-instantaneous filters of a stationary Gaussian process given by Ho and Sun [10].

We leave it as an open question whether $H < \frac{3}{4}$ is also a necessary condition for $\hat{p}_{\tau,n}$ to be asymptotically normally distributed. Indeed, at least for $r \in S_d$ with $d = 2$ simulations suggest that $\hat{p}_{\tau,n}$ is not asymptotically normally distributed with respect to $\mathbb{P}_H$ if $H \geq \frac{3}{4}$ (see Figure 3 below).
3 Estimating the probability of a change

3.1 Ordinal patterns of order $d=2$

The results of this Section are mainly based on the analysis of normal orthant probabilities as given by the following definition.

**Definition 9.** Let $n \in \mathbb{N}$ be fixed. For a non-singular strictly positive definite and symmetric matrix $\Sigma \in \mathbb{R}^{n \times n}$ let $\phi(\Sigma, \cdot)$ denote the Lebesgue density of the $n$-dimensional normal distribution with zero means and the covariance matrix $\Sigma$, that is

$$
\phi(\Sigma, x) = \left(\frac{1}{2\pi}\right)^\frac{n}{2} |\Sigma|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2} x^T \Sigma^{-1} x\right\}
$$

for $x \in \mathbb{R}^n$. We call

$$
\Phi(\Sigma) := \int_{[0, \infty]^n} \phi(\Sigma, x) \, dx,
$$

(11)

the $n$-dimensional normal orthant probability with respect to $\Sigma$. □

Clearly, if $(Z_1, Z_2, \ldots, Z_n)$ is a non-degenerate mean-zero Gaussian random vector on a probability space $(\Omega', \mathcal{A}', \mathbb{P})$, then

$$
\Phi\left(\left(\text{Cov}(Z_i, Z_j)\right)_{i,j=1}^n\right) = \mathbb{P}(Z_1 > 0, Z_2 > 0, \ldots, Z_n > 0).
$$

The following result is well-known (see [19]).

**Lemma 10.** If $(Z_1, Z_2, \ldots, Z_n)$ is a non-degenerate mean-zero Gaussian random vector on a probability space $(\Omega', \mathcal{A}', \mathbb{P})$ such that $\text{Var}(Z_k) > 0$ for all $k \in \mathbb{N}$, then

$$
\mathbb{P}(Z_1 > 0, Z_2 > 0) = \frac{1}{4} + \frac{1}{2\pi} \arcsin \rho_{12},
$$

$$
\mathbb{P}(Z_1 > 0, Z_2 > 0, Z_3 > 0) = \frac{1}{8} + \frac{1}{4\pi} \arcsin \rho_{12} + \frac{1}{4\pi} \arcsin \rho_{13} + \frac{1}{4\pi} \arcsin \rho_{23}
$$

where $\rho_{ij} = \text{Corr}(Z_i, Z_j)$ for $i, j \in \{1, 2, 3\}$.

Note that, in general, no closed-form expressions are available for normal orthant probabilities of dimension $n \geq 4$.

Same as in Subsection 2.2 let $(X_k)_{k \in \mathbb{N}_0}$ be an equidistant discretization of fBm with the increment process $(Y_k)_{k \in \mathbb{N}} = (X_k - X_{k-1})_{k \in \mathbb{N}}$. The following statement is due to Bandt and Shiha [3]. We refer to parts of the proof below, and thus include it here.

**Corollary 11.** For $H \in [0, 1]$ one has

$$
p_r(H) = \begin{cases} 
\frac{1}{4} - \frac{1}{2\pi} \arcsin 2^H - 1 & \text{if } r \in \{(0, 1, 2), (2, 1, 0)\} \\
\frac{1}{4} - \frac{1}{2\pi} \arcsin 2^H - 1 & \text{if } r \in \{(1, 0, 2), (1, 2, 0), (0, 2, 1), (2, 0, 1)\}
\end{cases}.
$$
**Proof.** We first show the statement for \( r = (0, 1, 2) \). According to (8), if \( H = 1 \) then \( p_{(0,1,2)}(H) = \frac{1}{2} \). By formula (7) one has \( \text{Var}_H(Y_k) = 1 \) for each \( H \in [0, 1] \) and \( k \in \mathbb{N}_0 \), and hence hence \( \text{Cov}_H(Y_i, Y_j) = \text{Corr}_H(Y_i, Y_j) \) for each \( H \in [0, 1] \) and \( i, j \in \mathbb{N} \). Consequently, by Lemma 10 one obtains

\[
p_{(0,1,2)}(H) = \mathbb{P}_H(X_2 > X_1 > X_0) = \mathbb{P}_H(Y_1 > 0, Y_2 > 0) = \frac{1}{4} + \frac{1}{2\pi} \arcsin \rho_H(1)
\]

for each \( H \in [0, 1] \) with \( \rho_H(1) \) as given in (7). Since \( \arcsin \rho = 2 \arcsin \sqrt{(1 + \rho)/2} - \frac{\pi}{2} \) for each \( \rho \in [-1, 1] \), one gets \( p_{(0,1,2)}(H) = \frac{1}{\pi} \arcsin 2^{H-1} \). Now, by Lemma 3 it holds

\[
p_{(0,1,2)}(H) = p_{(2,1,0)}(H) \quad \text{and} \quad p_{(1,0,2)}(H) = p_{(1,2,0)}(H) = p_{(0,2,1)}(H) = p_{(2,0,1)}(H)
\]

for each \( H \in [0, 1] \), and hence the statement follows.

According to Corollary 11, the probabilities of ordinal patterns of order \( d = 2 \) are all monotonically related to the Hurst parameter. In particular, consider the *indicator for a change between up to down* at time \( k \) defined by

\[
C_k := 1_{\{x_k \geq x_{k+1} < x_{k+2}\}} + 1_{\{x_k < x_{k+1} \geq x_{k+2}\}}
\]

for \( k \in \mathbb{N}_0 \) and the *probability of a change* \( c(\cdot) \) given by \( c(H) := \mathbb{P}_H(C_0 = 1) \) for \( H \in [0, 1] \). Because \( C_k = 1_{\{\Pi_2(k) = (1,0,2)\}} + 1_{\{\Pi_2(k) = (1,2,0)\}} + 1_{\{\Pi_2(k) = (0,2,1)\}} + 1_{\{\Pi_2(k) = (2,0,1)\}} \) for each \( k \in \mathbb{N}_0 \), one has

\[
c(H) = p_{(1,0,2)}(H) + p_{(1,2,0)}(H) + p_{(0,2,1)}(H) + p_{(2,0,1)}(H) = 1 - \frac{2}{\pi} \arcsin 2^{H-1}
\]

for \( H \in [0, 1] \). Hence, the probability of a change and the Hurst parameter \( H \) are monotonically related: the larger \( H \), the smaller \( c(H) \). In particular, \( c(H) \) tends to \( \frac{2}{\pi} \) as \( H \) tends to 0, and \( c(H) = 0 \) for \( H = 1 \).

Now, consider the *sample relative frequency of changes* \( \hat{c}_n \) given by \( \hat{c}_n := \frac{1}{n} \sum_{k=0}^{n-1} C_k \) for \( n \in \mathbb{N} \). Since obviously

\[
\hat{\rho}_{r,n} = \left\{ \begin{array}{ll}
\frac{1}{2} (1 - \hat{c}_n) & \text{if } r \in \{(0, 1, 2), (2, 1, 0)\} \\
\frac{1}{4} \hat{c}_n & \text{if } r \in \{(1, 0, 2), (1, 2, 0), (0, 2, 1), (2, 0, 1)\}
\end{array} \right.
\]

in the case \( d = 2 \) any reasonable estimator of ordinal pattern probabilities as provided by Theorem 4 is an affine functional of \( \hat{c}_n \) and hence essentially has the same statistical properties as \( \hat{c}_n \). In particular, according to Lemma 5 and Theorem 8 respectively, \( \hat{c}_n \) is a strongly consistent estimator of \( c(\cdot) \), and asymptotically normally distributed with respect to \( \mathbb{P}_H \) for \( H < \frac{3}{4} \).

In the rest of this Section, we investigate the variance of \( \hat{c}_n \). In Sec. 4 we consider the estimator of \( H \) obtained by plugging the estimate \( \hat{c}_n \) of \( c(\cdot) \) into the monotonic functional relation (13).

Notice that, Lemma 10 yields closed-form expressions also for probabilities of ordinal patterns of order \( d = 3 \) (see [3]). According to Theorem 4 one obtains eight different reasonable estimators of ordinal pattern probabilities in this case where not all probabilities seem to be monotonically related to the Hurst parameter.
3.2 Variance of the sample relative frequency of changes

Let \( \gamma_H \) denote the autocovariance function of the stochastic process \((C_k)_{k \in \mathbb{N}_0}\) with respect to \(P_H\) for \(H \in [0,1]\), that is, \(\gamma_H(k) = \text{Cov}_H(C_0, C_k)\) for \(H \in [0,1]\) and \(k \in \mathbb{N}_0\). Then \(\text{Var}_H(\hat{c}_n)\) can be written as

\[
\text{Var}_H(\hat{c}_n) = \frac{1}{n^2} (n \gamma_H(0) + 2 \sum_{k=1}^{n-1} (n - k) \gamma_H(k)).
\]

(15)

for \(n \in \mathbb{N}\). Clearly, \(\gamma_H(0) = c(H)(1 - c(H))\), and, according to Lemma 10 one has

\[
\gamma_H(1) = 2 P_H(Y_1 > 0, -Y_2 > 0, Y_3 > 0) - c(H)^2
\]

\[
= \frac{1}{2\pi} \arcsin \rho_H(2) - \frac{1}{\pi^2} \left( \arcsin \rho_H(1) \right)^2
\]

(16)

for \(H \in [0,1]\) with \(\rho_H(k)\) as given in (7). For \(k \in \mathbb{N} \setminus \{1\}\) one obtains

\[
\gamma_H(k) = \text{Cov}_H(1 - C_0, 1 - C_k)
\]

\[
= P_H(C_0 = 0, C_k = 0) - P_H(C_0 = 0)^2
\]

\[
= 2 P_H(Y_1 > 0, Y_2 > 0, Y_{k+1} > 0, Y_{k+2} > 0)
\]

\[
+ 2 P_H(Y_1 > 0, Y_2 > 0, -Y_{k+1} > 0, -Y_{k+2} > 0)
\]

\[- 4 P_H(Y_1 > 0, Y_2 > 0)^2
\]

(17)

for \(H \in [0,1]\) where, in general, no closed-form expressions are known for the probabilities \(P_H(Y_1 > 0, Y_2 > 0, Y_{k+1} > 0, Y_{k+2} > 0)\) and \(P_H(Y_1 > 0, Y_2 > 0, -Y_{k+1} > 0, -Y_{k+2} > 0)\), respectively.

Obviously, if \(H = 1\) then \(\hat{c}_n = 0\) \(P_H\)-a.s. and hence \(\text{Var}_H(\hat{c}_n) = 0\) for all \(n \in \mathbb{N}\) (compare to (14) and (8)). Note that, indeed, \(\gamma_H(k) = 0\) for each \(k \in \mathbb{N}_0\) in this case which can be seen from the fact that \(Y_i = Y_j\) \(P_H\)-a.s. for all \(i, j \in \mathbb{N}\) if \(H = 1\).

In the case \(H = \frac{1}{2}\) one obtains \(\gamma_H(0) = \frac{1}{4}\) and \(\gamma_H(k) = 0\) for all \(k \in \mathbb{N}\), and hence, according to formula (15), \(\text{Var}_H(\hat{c}_n) = \frac{1}{4n}\) for \(n \in \mathbb{N}\). In particular, \(\rho_H(k) = 0\) for all \(k \in \mathbb{N}\) if \(H = \frac{1}{2}\) (see (7)), hence the \(Y_1, Y_2, Y_{k+1}, Y_{k+2}\) in (17) are stochastically independent and thus

\[
P_H(Y_1 > 0, Y_2 > 0, Y_{k+1} > 0, Y_{k+2} > 0)
\]

\[= P_H(Y_1 > 0, Y_2 > 0, -Y_{k+1} > 0, -Y_{k+2} > 0) = P_H(Y_1 > 0, Y_2 > 0)^2 = \frac{1}{16}.
\]

Numerical evaluation of the covariances. Next, we provide a way for the numerical evaluation of \(\gamma_H(k)\) in the non-trivial case \(H \notin \{\frac{1}{2}, 1\}\) and \(k \in \mathbb{N} \setminus \{1\}\). Let \(\mathcal{R}\) denote the set of \(r = (r_1, r_2, r_3, r_4) \in [-1,1]^4\) such that the matrix

\[
\Sigma(r) := \begin{pmatrix}
1 & r_1 & r_2 & r_3 \\
r_1 & 1 & r_4 & r_2 \\
r_2 & r_4 & 1 & r_1 \\
r_3 & r_2 & r_1 & 1
\end{pmatrix}
\]

(18)

is strictly positive definite. Note that, if \(r, s \in \mathcal{R}\) then \(x^T \Sigma(h \cdot r + (1 - h) \cdot s) x = h x^T \Sigma(r) x + (1 - h) x^T \Sigma(s) x > 0\) for each \(x \in \mathbb{R}^4 \setminus \{0\}\) and \(h \in [0,1]\), hence \(\mathcal{R}\) is convex.

Define

\[
\rho((k, H, h)) := (\rho_H(1), h \cdot \rho_H(k), h \cdot \rho_H(k+1), h \cdot \rho_H(k-1))
\]

(19)
for $k \in \mathbb{N} \setminus \{1\}$, $H \in ]0,1[$ and $h \in [-1,1]$. Obviously, $\Sigma(\rho((k,H,1)))$ is the covariance matrix of $(Y_1, Y_2, Y_{k+1}, Y_{k+2})$, and $\Sigma(\rho((k,H,-1)))$ is the covariance matrix of $(Y_1, Y_2, -Y_{k+1}, -Y_{k+2})$, both with respect to $\mathbb{P}_H$, and since $(Y_1, Y_2, Y_{k+1}, Y_{k+2})$ and $(Y_1, Y_2, -Y_{k+1}, -Y_{k+2})$ are non-degenerate Gaussian with respect to $\mathbb{P}_H$ if $H \in ]0,1[$, one has $\rho((k,H,1)) \in \mathcal{R}$ and $\rho((k,H,-1)) \in \mathcal{R}$ for each $k \in \mathbb{N} \setminus \{1\}$ and $H \in ]0,1[$. From the convexity of $\mathcal{R}$ it follows that $\rho((k,H,h)) \in \mathcal{R}$ for each $k \in \mathbb{N} \setminus \{1\}$, $H \in ]0,1[$ and $h \in [-1,1]$, consequently,

$$v((k,H,h)) := (\Phi \circ \Sigma)(\rho((k,H,h)))$$

is well-defined for all $k \in \mathbb{N} \setminus \{1\}$, $H \in ]0,1[$ and $h \in [-1,1]$ (compare to Definition 9). Note that

$$v((k,H,1)) = \mathbb{P}_H(Y_1 > 0, Y_2 > 0, Y_{k+1} > 0, Y_{k+2} > 0),$$
$$v((k,H,-1)) = \mathbb{P}_H(Y_1 > 0, Y_2 > 0, -Y_{k+1} > 0, -Y_{k+2} > 0),$$
$$v((k,H,0)) = \mathbb{P}_H(Y_1 > 0, Y_2 > 0)^2$$

for all $k \in \mathbb{N} \setminus \{1\}$ and $H \in ]0,1[$. Hence, by inserting the expressions on the left side into (17) one obtains

$$\gamma_H(k) = 2 \int_0^1 \left( \frac{\partial v}{\partial h}((k,H,x)) - \frac{\partial v}{\partial h}((k,H,-x)) \right) dx$$

(20)

for $k \in \mathbb{N} \setminus \{1\}$ and $H \in ]0,1[$. The partial derivative of $v$ with respect to $h$ is given by

$$\frac{\partial v}{\partial h}((k,H,x)) = \rho_H(k) \frac{\partial (\Phi \circ \Sigma)}{\partial r_2}(\rho((k,H,x))) + \rho_H(k+1) \frac{\partial (\Phi \circ \Sigma)}{\partial r_3}(\rho((k,H,x))) + \rho_H(k-1) \frac{\partial (\Phi \circ \Sigma)}{\partial r_4}(\rho((k,H,x)))$$

for $k \in \mathbb{N} \setminus \{1\}$, $H \in ]0,1[$ and $x \in [-1,1]$. By the reduction formula for normal orthant probabilities given by Plackett (see (19)), one gets the first partial derivatives of $(\Phi \circ \Sigma)$ with respect to $r_2, r_3, r_4$, namely

$$\frac{\partial (\Phi \circ \Sigma)}{\partial r_2}(s) = \frac{1}{\pi (1-s_2^2)^{\frac{3}{2}}} \left( \frac{1}{4} - \frac{1}{2\pi} \arcsin \frac{\Sigma(s)_{13}}{(\Sigma(s)_{11})(\Sigma(s)_{22})^{\frac{1}{2}}} \right),$$
$$\frac{\partial (\Phi \circ \Sigma)}{\partial r_3}(s) = \frac{1}{2\pi (1-s_3^3)^{\frac{3}{2}}} \left( \frac{1}{4} + \frac{1}{2\pi} \arcsin \frac{\Sigma(s)_{23}}{(\Sigma(s)_{22})^{\frac{1}{2}}} \right),$$
$$\frac{\partial (\Phi \circ \Sigma)}{\partial r_4}(s) = \frac{1}{2\pi (1-s_4^4)^{\frac{3}{2}}} \left( \frac{1}{4} + \frac{1}{2\pi} \arcsin \frac{\Sigma(s)_{14}}{(\Sigma(s)_{11})^{\frac{1}{2}}} \right)$$

(21)

for $s = (s_1, s_2, s_3, s_4) \in \mathcal{R}$, where $\Sigma(s)_{ij}$ denotes the matrix obtained from $\Sigma(s)$ by deleting the $i$-th row and $j$-th column of $\Sigma(s)$.

Formula (20) together with formula (15) allows to compute numerical values of $\text{Var}_H(\zeta_n)$ to any desired precision. See [3] for details on the evaluation of the integral in (20). The following Lemma will be needed below.

**Lemma 12.** For each $k \in \mathbb{N}_0$ the mapping $H \mapsto \gamma_H(k)$ is continuous on $]0,1[$.
Proof. For \( k = 0 \) the statement is valid since \( H \mapsto c(H)(1 - c(H)) \) is continuous on \([0, 1]\). Note that \( H \mapsto \rho_H(k) \) is continuous on \([0, 1]\) for each \( k \in \mathbb{N} \) (see (11)). Consequently, the statement is valid for \( k = 1 \) (compare to (10)) and, furthermore, the mapping \( H \mapsto v((k, H, h)) \) is continuous on \([0, 1]\) for each \( k \in \mathbb{N} \setminus \{1\} \) and \( h \in [-1, 1] \). Since \( \gamma_H(k) = 2v((k, H, 1)) + 2v((k, H, -1)) - 4v((k, H, 0)) \), the statement follows. 

\[ \Box \]

### 3.3 Limit behaviour

The next Theorem establishes an asymptotically equivalent expression for \( \gamma_H(k) \).

**Theorem 13.** For each \( H \in [0, 1] \) one has

\[
\gamma_H(k) \sim \frac{2(1 - \rho_H(1))}{\pi^2(1 + \rho_H(1))} H^2(2H - 1)^2 k^{4H - 4}
\]

as \( k \) tends to \( \infty \).

**Proof.** First note, for \( H = \frac{1}{2} \) and \( H = 1 \) the statement is true since both sides evaluate to 0 for each \( k \in \mathbb{N} \) (in particular \( \rho_H(1) = 1 \) for \( H = 1 \)). Let the mapping \((\Phi \circ \Sigma) : \mathcal{R} \to [0, 1]\) be given as in (11) and (13), respectively. For each \( s = (s_1, s_2, s_3, s_4) \in \mathcal{R} \) Taylor’s Theorem asserts the existence of some \( h \in [0, 1] \) such that

\[
(\Phi \circ \Sigma)(s) = (\Phi \circ \Sigma)((s_1, 0, 0, 0)) + \sum_{i=2}^{4} s_i \frac{\partial (\Phi \circ \Sigma)}{\partial r_i}((s_1, 0, 0, 0))
\]

\[
+ \frac{1}{2} \sum_{i,j=2}^{4} s_i s_j \frac{\partial^2 (\Phi \circ \Sigma)}{\partial r_i \partial r_j}((s_1, 0, 0, 0))
\]

\[
+ \frac{1}{6} \sum_{i,j,l=2}^{4} s_i s_j s_l \frac{\partial^3 (\Phi \circ \Sigma)}{\partial r_i \partial r_j \partial r_l}((s_1, h \cdot s_2, h \cdot s_3, h \cdot s_4)).
\]

Consequently, for each \( (s_1, s_2, s_3, s_4) \in \mathcal{R} \) with \( (s_1, -s_2, -s_3, -s_4) \in \mathcal{R} \) there exist \( h_1 \in [0, 1] \) and \( h_2 \in [-1, 0] \) such that

\[
(\Phi \circ \Sigma)((s_1, s_2, s_3, s_4)) + (\Phi \circ \Sigma)((s_1, -s_2, -s_3, -s_4)) =
\]

\[
2(\Phi \circ \Sigma)((s_1, 0, 0, 0)) + \sum_{i,j=2}^{4} s_i s_j \frac{\partial^2 (\Phi \circ \Sigma)}{\partial r_i \partial r_j}((s_1, 0, 0, 0))
\]

\[
+ \frac{1}{3} \sum_{i,j,l=2}^{4} s_i s_j s_l \frac{\partial^3 (\Phi \circ \Sigma)}{\partial r_i \partial r_j \partial r_l}((s_1, h_1 \cdot s_2, h_1 \cdot s_3, h_1 \cdot s_4))
\]

\[
- \frac{1}{3} \sum_{i,j,l=2}^{4} s_i s_j s_l \frac{\partial^3 (\Phi \circ \Sigma)}{\partial r_i \partial r_j \partial r_l}((s_1, h_2 \cdot s_2, h_2 \cdot s_3, h_2 \cdot s_4)).
\]  

(22)

Now, according to equation (17), for \( H \in [0, 1] \) and \( k \in \mathbb{N} \setminus \{1\} \) one has

\[
\gamma_H(k) = 2(\Phi \circ \Sigma)((\rho_H(1), \rho_H(k), \rho_H(k + 1), \rho_H(k - 1)))
\]

\[
+ 2(\Phi \circ \Sigma)((\rho_H(1), -\rho_H(k), -\rho_H(k + 1), -\rho_H(k - 1)))
\]

\[
- 4(\Phi \circ \Sigma)((\rho_H(1), 0, 0, 0)).
\]  

(23)
Let $\tilde{\rho}_H(k) := H(2H-1)k^{2H-2}$ for $H \in [0,1]$ and $k \in \mathbb{N} \setminus \{1\}$. By formula (9), $\rho_H(k)$ is asymptotically equivalent to $\tilde{\rho}_H(k)$ for each $H \in [0,1]$, hence for all $i,j,l \in \mathbb{N}$ one has the asymptotic equivalence

$$(\rho_H(k))^i (\rho_H(k+1))^j (\rho_H(k-1))^l \sim (\tilde{\rho}_H(k))^i (\tilde{\rho}_H(k))^j (\tilde{\rho}_H(k))^l$$

for each $H \in [0,1]$. Consequently, according to formula (22) and (23), respectively, for each $H \in [0,1]$ and $k \in \mathbb{N} \setminus \{1\}$ there exist $h_1 \in [0,1]$ and $h_2 \in [-1,0]$ such that

$$\gamma_H(k) \sim 2 (\tilde{\rho}_H(k))^2 \sum_{i,j=2}^4 \frac{\partial^2 (\Phi \circ \Sigma)}{\partial r_i \partial r_j} ((\rho_H(1),0,0,0)) + 2 (\tilde{\rho}_H(k))^3 R_H(k) \quad (24)$$

where

$$R_H(k) = \frac{1}{3} \sum_{i,j,l=2}^4 \frac{\partial^3 (\Phi \circ \Sigma)}{\partial r_i \partial r_j \partial r_l} (\rho((k,H,h_1))) - \frac{1}{3} \sum_{i,j,l=2}^4 \frac{\partial^3 (\Phi \circ \Sigma)}{\partial r_i \partial r_j \partial r_l} (\rho((k,H,h_2)))$$

with $\rho((k,H,h))$ as defined in (19). Now, by the formulas for the first partial derivatives of $(\Phi \circ \Sigma)$ given in (21) one obtains

$$\frac{\partial^2 (\Phi \circ \Sigma)}{\partial^2 r_2}(s) = \frac{1 + s^2_1}{2\pi^2(1 - s^2_1)}, \quad \frac{\partial^2 (\Phi \circ \Sigma)}{\partial r_2 \partial r_3}(s) = \frac{s_1}{2\pi^2(1 - s^2_1)}, \quad \frac{\partial^2 (\Phi \circ \Sigma)}{\partial^2 r_3}(s) = \frac{s_1}{4\pi^2(1 - s^2_1)}, \quad \frac{\partial^2 (\Phi \circ \Sigma)}{\partial r_3 \partial r_4}(s) = \frac{1}{4\pi^2(1 - s^2_1)}$$

for $s = (s_1,0,0,0) \in \mathcal{R}$. In particular, one has $|\Sigma(s)_{13}| = |\Sigma(s)_{23}| = |\Sigma(s)_{14}| = 0$ for $s = (s_1,0,0,0) \in \mathcal{R}$. Putting these terms with $(s_1,0,0,0) = (\rho_H(1),0,0,0)$ into (24), one gets

$$\gamma_H(k) \sim 2 (\tilde{\rho}_H(k))^2 \frac{4 - 8\rho_H(1) + 4\rho_H(1)^2}{4\pi^2(1 - \rho_H(1)^2)} + 2 (\tilde{\rho}_H(k))^3 R_H(k) = \frac{2(1 - \rho_H(1))}{\pi^2(1 + \rho_H(1))} H^2(2H-1)^2 k^{4H-4} + O(k^{6H-6} R_H(k))$$

for each $H \in [0,1]$. The final claim is $k^{6H-6} R_H(k) = o(k^{4H-4})$ for each $H \in [0,1]$. Let $H \in [0,1]$ be fixed. Since $\rho((k,H,1))$ and $\rho((k,H,-1))$ both converge to $(\rho_H(1),0,0,0)$ as $k$ tends to $\infty$, the convex hull of the subset

$$\mathcal{R}_H := \bigcup_{k \in \mathbb{N} \setminus \{2\}} \{\rho((k,H,1)), \rho((k,H,-1))\} \cup \{(\rho_H(1),0,0,0)\}$$

of $\mathcal{R}$ is closed in $[-1,1]^4$. Note that $\bigcup_{k \geq 2} \{\rho((k,H,x)) : x \in [-1,1]\}$ is a subset of the convex hull of $\mathcal{R}_H$. Consequently, since $\frac{\partial^3 (\Phi \circ \Sigma)}{\partial r_i \partial r_j \partial r_l}$ is continuous on $\mathcal{R}$ for all $i,j,l \in \{2,3,4\}$ (see [19]), one has

$$\sup_{k \in \mathbb{N} \setminus \{2\}} R_H(k) < \infty$$

which shows the claim. \qed
In the case $H \geq \frac{3}{4}$, equation (13) together with Theorem 13 yields simple closed-form asymptotically equivalent expressions for $\text{Var}_H(\hat{c}_n)$, namely

$$
\text{Var}_H(\hat{c}_n) \sim \begin{cases} 
\frac{d_H \ln(n)}{n} & \text{if } H = \frac{3}{4} \\
\frac{d_H n^{4H-4}}{4H-3} & \text{if } H > \frac{3}{4}
\end{cases}
$$

(25)

where $d_H = \frac{4(1-H)}{\pi^2(1+H)(1-H)} H^2 (2H - 1)^2$. As one can see, $\text{Var}_H(\hat{c}_n)$ tends to 0 with a rate slower than $n^{-1}$ in this case. Note that, from the practical viewpoint, the expressions in (25) are useful approximations of $\text{Var}_H(\hat{c}_n)$ only if $n$ is excessively large. We provide a way for getting more accurate practical approximations at the end of Sec. 4.

If $H < \frac{3}{4}$ then, by equation (15) and Theorem 13 one has $\text{Var}_H(\hat{c}_n) = O(n^{-1})$. Let a family of mappings $(f_n)_{n \in \mathbb{N}}$ from $]0, \frac{3}{4}[$ onto $\mathbb{R}$ be defined by

$$
f_n(H) := n \cdot \text{Var}_H(\hat{c}_n)
$$

(26)

for $H \in ]0, \frac{3}{4}[$ and $n \in \mathbb{N}$. Further, define $f_\infty(H) := \lim_{n \to \infty} f_n(H)$ for $H \in ]0, \frac{3}{4}[$. The following Lemma will be needed in the next section.

**Lemma 14.** $f_n$ uniformly converges to $f_\infty$ on every compact subset $I \subset ]0, \frac{3}{4}[$. Moreover, $f_\infty$ is continuous and strictly positive on $]0, \frac{3}{4}[$.

**Proof.** First, by Theorem 13 one has $\sum_{k=1}^\infty \gamma_H(k) < \infty$ for each $H \in ]0, \frac{3}{4}[$. Hence, according to equation (13) and the dominated convergence Theorem one obtains

$$
f_\infty(H) - \gamma_H(0) = \lim_{n \to \infty} 2 \sum_{k=1}^{n-1} \frac{n-k}{n} \gamma_H(k) = \lim_{n \to \infty} 2 \sum_{k=1}^\infty \max \{0, \frac{n-k}{n}\} \gamma_H(k)
$$

$$
= 2 \sum_{k=1}^\infty \lim_{n \to \infty} \max \{0, \frac{n-k}{n}\} \gamma_H(k) = 2 \sum_{k=1}^\infty \gamma_H(k)
$$

(27)

and, consequently,

$$
f_\infty(H) - f_n(H) = 2 \left( \sum_{k=1}^\infty \gamma_H(k) - \sum_{k=1}^{n-1} \frac{n-k}{n} \gamma_H(k) \right)
$$

$$
= 2 \sum_{k=1}^\infty \min \{k/n, 1\} \gamma_H(k)
$$

(28)

for each $H \in ]0, \frac{3}{4}[$. Now, let $I \subset ]0, \frac{3}{4}[$ be compact. By Lemma 12, the mapping $H \mapsto \gamma_H(k)$ is continuous on $I$ for every $k \in \mathbb{N}$, hence one can define the family $(H_k)_{k \in \mathbb{N}}$ of numbers in $I$ to be such that $\gamma_{H_k}(k) = \max \{\gamma_H(k) : H \in I\}$ for every $k \in \mathbb{N}$. Then one has

$$
|f_\infty(H) - f_n(H)| \leq 2 \sum_{k=1}^\infty \min \{k/n, 1\} \gamma_{H_k}(k)
$$

(29)

uniformly for all $H \in I$. To prove the uniform convergence of $f_n$ to $f_\infty$ on $I$, it is sufficient to show that the right side of (29) tends to 0 as $n$ tends to $\infty$. Let $\delta \in ]0, \frac{1}{4}[$ and define $H_* := \max \left( I \cup \{\frac{1}{2} + \delta\} \right)$. By Theorem 13 one has $|\gamma_{H_k}(k)| = O(|\gamma_{H_*}(k)|)$, and hence

$$
\sum_{k=1}^\infty \min \{k/n, 1\} |\gamma_{H_k}(k)| = O\left( \sum_{k=1}^\infty \min \{k/n, 1\} |\gamma_{H_*}(k)| \right).
$$
With the same arguments as in (27) one obtains
\[
\lim_{n \to \infty} \sum_{k=1}^{\infty} \min \{k/n, 1\} |\gamma_{H_n}(k)| = \sum_{k=1}^{\infty} \lim_{n \to \infty} \min \{k/n, 1\} |\gamma_{H_n}(k)| = 0
\]
showing that the right side of (29) tends to 0 as \(n\) tends to \(\infty\). Now, the continuity of \(f_\infty\) follows since \(f_n\) is continuous on \([0, \frac{3}{4}]\) for every \(n \in \mathbb{N}\) which is a direct consequence of formula (15) together with Lemma 12. To establish the strict positiveness of \(\gamma\) and set \(\tilde{g}\), according to equation (13), the probability of a change \(c(H)\) and the Hurst parameter \(H\) are monotonically related. Here we consider the estimator \(\hat{g}_n\) of \(c(\cdot)\) into the monotonic functional relation (13). In order to get finite non-negative estimates, we define
\[
g(x) := \begin{cases} 
\log_2 \sin(\pi(1-x)/2) + 1 & \text{for } x \in \left[0, \frac{2}{3}\right] \\
0 & \text{for } x \in \left[\frac{2}{3}, 1\right]
\end{cases}
\]
(30)
and set \(\hat{H}_n := g(\hat{c}_n)\). Obviously, the first derivative \(g'\) of \(g\) exists and is non-zero on \([0, \frac{2}{3}]\), namely one has
\[
g'(x) = -\frac{\pi}{2 \ln 2} \frac{\cos(\pi(1-x)/2)}{\sin(\pi(1-x)/2)} < 0
\]
(31)
for \(x \in \left[0, \frac{2}{3}\right]\). Hence, by Theorem 8 together with Theorem 2.5.2 in [16] one obtains
\[
(g'(c(H)))^{-1} \left( \text{Var}_H(c_n) \right)^{-\frac{1}{2}} \left( \hat{H}_n - H \right) \xrightarrow{\text{P}_H} N(0, 1)
\]
for each \(H \in \left[0, \frac{2}{3}\right]\). Note that \(g\) is continuous on \([0, 1]\), and thus \(\lim_{n \to \infty} \hat{H}_n = H \) \(\text{P}_H\)-a.s. for each \(H \in [0, 1]\) (see Lemma 5). Let \(f_n(H)\) be defined as in (28). According to Lemma 14 and because the mapping \(H \mapsto g'(c(H))\) is continuous on \([0, 1]\), one obtains
\[
\lim_{n \to \infty} \left( \frac{g'(c(\hat{H}_n)))}{g'(c(H))} \right)^2 f_n(\hat{H}_n) = \left( \frac{g'(c(H)))}{g'(c(H))} \right)^2 f_\infty(H) = 1
\]
\(\text{P}_H\)-a.s. for each \(H \in \left[0, \frac{2}{3}\right]\). By Theorem 2.3.3 in [16] it follows that
\[
(g'(c(\hat{H}_n)))^{-1} \left( \left( f_n(\hat{H}_n) \right)^\frac{1}{2} \sqrt{n} \left( \hat{H}_n - H \right) \right) \xrightarrow{\text{P}_H} N(0, 1)
\]
(32)
for each \(H \in \left[0, \frac{2}{3}\right]\). Now, let the family of mappings \((s_n)_{n \in \mathbb{N}}\) from \([0, 1]\) on \(\mathbb{R}\) be defined by \(s_n(1) := 0, s_n(H) := n^{-1} (g'(c(H)))^2 \text{Var}_H(c_n)\)
for $H \in ]0, 1[$, and $s_n(0) := \lim_{H \to 0} s_n(H)$. Note that $s_n(0)$ is well-defined for each $n \in \mathbb{N}$ since, according to Lemma 12 and equation (31), the mappings $H \mapsto \text{Var}_H(\widehat{c}_n)$ and $H \mapsto g'(c(H))$, respectively, are continuous on $]0, 1[$. For $n \in \mathbb{N}$ consider the random (confidence) interval

$$K_n := \left[ \widehat{H}_n - 1.96 \left( s_n(\widehat{H}_n) \right)^{1/2}, \widehat{H}_n + 1.96 \left( s_n(\widehat{H}_n) \right)^{1/2} \right] \cap [0, 1]. \quad (33)$$

Let $\Phi$ denote the cumulative distribution function of $N(0, 1)$ here. According to (32) one obtains

$$\mathbb{P}_H(H \in K_n) \longrightarrow 2 \Phi(1.96) - 1 = 0.9500... \quad (34)$$

as $n$ tends to $\infty$. We compare this to simulations of $\mathbb{P}_H(H \in K_n)$ below (see Table 2). The computations of $s_n(\widehat{H}_n)$ were carried out using formula (37) given below. See Figure 2 (a) for a visualization of the confidence intervals.

**Asymptotic expectation and variance.** Notice that, simulations suggest that the actual expectation $\mathbb{E}_H(\widehat{H}_n)$ and the actual variance $\text{Var}_H(\widehat{H}_n)$ of $\widehat{H}_n$ asymptotically behave like the expectation and the variance of the asymptotic distribution of $\widehat{H}_n$ obtained by the Delta method (see [16]), namely,

$$\mathbb{E}_H(\widehat{H}_n) \sim H - \frac{1}{2} g''(c(H)) \text{Var}_H(\widehat{c}_n) \quad (35)$$

as well as

$$\text{Var}_H(\widehat{H}_n) \sim g'(c(H))^2 \text{Var}_H(\widehat{c}_n) \quad (36)$$

for each $H \in ]0, 1[$ (see Table 2 below). In order to avoid confusion with the actual expectation and the actual variance, we call the expressions on the right side of (35) and (36) the asymptotic expectation and the asymptotic variance, respectively, of $\widehat{H}_n$. See Figure 2 (b), (c) for plots of the asymptotic bias, which is the difference between the asymptotic expectation and $H$, and of the variance. The computation of $\text{Var}_H(\widehat{c}_n)$ was carried out using formula (37) given below.

![Figure 2](image_url)

**Figure 2:** (a) Confidence intervals for the Hurst parameter obtained from $\widehat{H}_n$, for $n = 128$ (black), $n = 1024$ (dark grey) and $n = 8192$ (light grey). For the value $x$ on the abscissa, the ordinate shows the confidence interval $K_n$ obtained if $\widehat{H}_n(\omega) = x$. (b) Asymptotic bias of $\widehat{H}_n$, plotted against $H$. (c) Asymptotic variance of $\widehat{H}_n$, plotted against $H$. 


Practical approximations. For the practical purpose of computing approximate numerical values of $\text{Var}_H(\hat{c}_n)$ for given $H \in ]0, 1[$ and $n \in \mathbb{N}$, first note that the approximation of $\gamma_H(k)$ by the asymptotically equivalent expression

$$\tilde{\gamma}_H(k) := \frac{2(1-\rho_H(1))}{\pi^2(1+\rho_H(1))} H^2(2H-1)^2 k^{4H-4}$$

obtained in Theorem 13 can be improved in the following way. For $(r_1, r_2) \in \mathbb{R}^2$ such that $(r_1, r_2, r_2, r_2) \in \mathcal{R}$ (with $\mathcal{R}$ as given in the preceding section), let

$$F((r_1, r_2)) := \Phi \circ \Sigma((r_1, r_2, r_2, r_2)).$$

Further, define

$$\tilde{\gamma}_H^{(m)}(k) := 4 \sum_{l=1}^{m} \frac{1}{(2l)!} \frac{\partial^{2l} F}{\partial^2 r_2}((\rho_H(1), 0))(H(2H-1)k^{2H-2})^{2l}$$

for $H \in ]0, 1[$, $m \in \mathbb{N}$ and $k \geq 2$. Note that $\tilde{\gamma}_H^{(1)}(k) = \tilde{\gamma}_H(k)$ for all $H \in ]0, 1[$ and $k \geq 2$. For $m = 3$, $\epsilon = 0.01, 0.001$ and several values of $H$, Table 1 provides the number

$$k_H^{(m)}(\epsilon) := \min \{k \geq 2 : \frac{|\tilde{\gamma}_H^{(m)}(k) - \gamma_H(k)|}{\gamma_H(k)} < \epsilon\},$$

that is the least $k \geq 2$ such that the relative error caused by approximating $\gamma_H(k)$ by $\tilde{\gamma}_H^{(m)}(k)$ is less than $\epsilon$. Because $\gamma_H(k)$ and $\tilde{\gamma}_H^{(m)}(k)$ are asymptotically equivalent for all $m \in \mathbb{N}$, the relative error tends to 0 as $k$ tends to $\infty$ (see [16], Lemma 1.1.1), that is, $k_H^{(m)}(\epsilon) < \infty$ for every $\epsilon > 0$. Numerical experiments suggest that $\tilde{\gamma}_H^{(m)}(k) \leq \gamma_H(k)$ for all $m \in \mathbb{N}$, $H \in ]0, 1[$ and $k \geq 2$. As one can see in Table 1, $\tilde{\gamma}_H^{(3)}(k)$ already yields good approximations of $\gamma_H(k)$ for small $k$ if $H$ is not too close to 1.

| $H$     | 0.05 | 0.15 | 0.25 | 0.35 | 0.45 | 0.55 | 0.65 | 0.75 | 0.85 | 0.95 |
|---------|------|------|------|------|------|------|------|------|------|------|
| $k_H^{(3)}(0.01)$ | 18   | 16   | 14   | 12   | 11   | 9    | 7    | 5    | 5    | 226  |
| $k_H^{(3)}(0.001)$ | 55   | 50   | 44   | 38   | 32   | 26   | 21   | 15   | 13   | 10040 |

Table 1: The numbers $k_H^{(3)}(\epsilon)$ for $\epsilon = 0.01, 0.001$ and several values of the Hurst parameter.

By the formulas for the first partial derivatives of $(\Phi \circ \Sigma)$ given in (21) one obtains

$$\frac{\partial F}{\partial r_2}(s) = \frac{2}{\pi(1-s_2)^2} \left(\frac{1}{4} - \frac{1}{2\pi} \arcsin \frac{s_2(s_1 - 1)}{1 + s_1 - 2s_2^2}\right)$$

for every $s = (s_1, s_2)$ in the domain of $F$. The partial derivatives with respect to $r_2$ of higher orders are obtained by easy calculations. Those of 4th and 6th order in $(\rho_H(1), 0)$ are given by

$$\frac{\partial^4 F}{\partial^4 r_2}((\rho_H(1), 0)) = \frac{4(1 - \rho_H(1))(2 + \rho_H(1))^2}{\pi^2(1 + \rho_H(1))^3},$$

$$\frac{\partial^6 F}{\partial^6 r_2}((\rho_H(1), 0)) = \frac{16(1 - \rho_H(1))(7 + 6\rho_H(1) + 2\rho_H(1)^2)^2}{\pi^2(1 + \rho_H(1))^5}.$$
For a practical approximation of \( V_H(\hat{c}_n) \) we propose to sum up the precise numerical values \( \gamma_H(k) \) obtained by (20) for \( k = 2, 3, \ldots \) up to some specified \( \tilde{n} \leq n - 1 \), and then to take the approximate values \( \tilde{\gamma}_H^{(3)}(k) \) for \( k = \tilde{n} + 1, \ldots, n - 1 \), that is

\[
\text{Var}_H(\hat{c}_n) \approx \frac{1}{n^2} \left( n\gamma_H(0) + 2 \sum_{k=1}^{\tilde{n}-1} (n-k) \gamma_H(k) + 2 \sum_{k=\tilde{n}}^{n-1} (n-k) \tilde{\gamma}_H^{(3)}(k) \right). \tag{37}
\]

For the calculations behind Figure 2 we have chosen \( \tilde{n} \) depending on \( H \), namely \( \tilde{n} = \min\{k_H^{(3)}(0.01), 250, n\} \).

5 Simulation studies

For the simulation of fBm we used the algorithm of Davies and Harte [8] and the random number generator of Mathematica 5.0.1.0. For each of the values \( H = 0.55, 0.65, 0.75, 0.85, 0.95 \) and the sample lengths \( n = 128, 1024, 8192 \), we performed 50 000 simulations of fBm. We only chose values \( H > \frac{1}{2} \), because the increments of fBm are particularly interesting for the modelling of long range dependence in this case (see [4]).

5.1 Performance of the ZC estimator

Table 2 shows simulations of \( \mathbb{E}_H(\hat{H}_n) \) and \( \text{Var}_H(\hat{H}_n) \), given by the sample mean and the sample variance, respectively, of the 50 000 estimates \( \hat{H}_n \) obtained for simulated samples of length \( n \) with the true Hurst parameter \( H \). Additionally, Table 2 provides the asymptotic expectation (abbreviated as exp.) and the asymptotic variance (abbr. as var.) of \( \hat{H}_n \) as given by the expressions on the right side of equations (35) and (36), respectively. As one can see, the results obtained by the simulations and by equations (35) and (36), respectively, agree very well. Only for \( H = 0.85, 0.95 \) the asymptotic expressions seem to yield slightly larger values for \( \text{Var}_H(\hat{H}_n) \) than the simulations.

Table 2 also provides simulations of \( \mathbb{P}_H(H \in K_n) \), which is the probability that the true value of \( H \) falls within the interval \( K_n \) obtained from formula (33). Theoretically, \( \mathbb{P}_H(H \in K_n) \) tends to 0.95 as \( n \) tends to \( \infty \) if \( H < \frac{3}{4} \) (see (31)). As one can see, for \( H = 0.55, 0.65, 0.75 \) the simulations of \( \mathbb{P}_H(H \in K_n) \) always attain values larger than 0.95. For \( H = 0.85 \) this is still true for \( n = 1024 \) and \( n = 8096 \), while the simulations of \( \mathbb{P}_H(H \in K_n) \) are all clearly below 0.95 for \( H = 0.95 \).

For \( n = 8192 \) and \( H = 0.55, 0.75, 0.95 \), Figure 3 shows the standardized simulated distribution of \( \hat{H}_n \) with respect to \( \mathbb{P}_H \) compared to the density of the standard normal distribution. In accordance with (32), for \( H = 0.55 \) the standardized distribution of \( \hat{H}_n \) is very close to the standard normal distribution. For \( H = 0.75 \) there is a slight difference between both curves, in particular, the standardized distribution of \( \hat{H}_n \) seems to be more concentrated around 0 than the standard normal distribution. For \( H = 0.95 \), Figure 3 shows a remarkable contrast between the standardized distribution of \( \hat{H}_n \) and the standard normal distribution, suggesting that \( \hat{H}_n \) is not asymptotically normally distributed in this case.
Table 2: Simulations of $\mathbb{E}_H(\hat{H}_n)$, $\text{Var}_H(\hat{H}_n)$ and $\mathbb{P}_H(H \in K_n)$, obtained each for 50 000 simulations of fBm. The rows ‘as. exp.’ and ‘as. var.’ show the asymptotic expectation and variance, respectively, given by the right side of equations (35) and (36), respectively. Note that, according to (34), if $H < \frac{3}{4}$ then $\mathbb{P}_H(H \in K_n)$ tends to 0.95 as $n$ tends to $\infty$.

### 5.2 Comparison to the HEAF estimator

Let $(Y_k)_{k \in \mathbb{N}}$ be the increment process of an equidistant discretization of fBm, and define $\overline{Y}_n := \frac{1}{n} \sum_{k=1}^{n} Y_k$ for $n \in \mathbb{N}$. The HEAF estimator (HEAF meaning 'Hurst Exponent from Autocorrelation Function') proposed by Kettani and Gubner [14] puts the estimate

$$\hat{\rho}_n = \frac{\sum_{k=1}^{n-1} (Y_k - \overline{Y}_n)(Y_{k+1} - \overline{Y}_n)}{\sum_{k=1}^{n} (Y_k - \overline{Y}_n)^2},$$

of $\rho_H(1)$ into the monotonic functional relation $\rho_H(1) = 2^{2H-1} - 1$ obtained from (7). In order to receive only finite non-negative estimates of $H$, let

$$\hat{H}_n^{\text{HEAF}} = \frac{1}{2} \left( 1 + \log_2 \left( 1 + \max\{-1/2, \hat{\rho}_n\} \right) \right)$$

for $n \in \mathbb{N}$. Table 3 shows simulations of $\mathbb{E}_H(\hat{H}_n^{\text{HEAF}})$ and $\text{Var}_H(\hat{H}_n^{\text{HEAF}})$, each obtained for 50 000 simulations of fBm.

As one can see, $\hat{H}_n^{\text{HEAF}}$ has smaller variance but larger bias than the ZC estimator. For $n = 128$ the variance of the ZC estimator is about 2 times larger than the variance of
Figure 3: Standardized simulated distribution of $\hat{H}_n$ for $n = 8192$ and (a) $H = 0.55$, (b) $H = 0.75$ and (c) $H = 0.95$, compared to the standard normal distribution (light grey).

$\hat{H}_n^{\text{HEAF}}$, for $n = 1024$ it is about 2 to 4 times larger, and for $n = 8192$ it is about 2 to 8 times larger. Notice that the highest ratios are obtained for $H = 0.95$.

The loss of efficiency of $\hat{H}_n$ relative to $\hat{H}_n^{\text{HEAF}}$ is not surprising. In particular, $\hat{H}_n$ can be seen as the estimator obtained by plugging the estimate $\sin(\pi(1/2 - \bar{c}_n))$ of $\rho_H(1)$ into the monotonic functional relation $\rho_H(1) = 2^{2H-1} - 1$ (compare to (12) and (30), respectively), and clearly $\sin(\pi(1/2 - \bar{c}_n))$ loses efficiency relative to the estimate $\hat{\rho}_n$ of $\rho_H(1)$. However, because of the slow convergence of the sample mean for increments of fBm with Hurst parameter $H > \frac{1}{2}$ (see [4]), $\hat{H}_n^{\text{HEAF}}$ has a much larger bias than the ZC estimator, in particular for large values of $H$.

In a concluding remark, the ZC estimator of the Hurst parameter performs comparatively well, although it does not regard the metric structure but only the order relations between values of realizations. Notice that generally, for fixed $r \in S_d$ with $d \in \mathbb{N}$, the computation of $\hat{p}_{\tau,n}$ can be carried out by asymptotically $4 (\# \mathbb{X}) d n$ computational steps (see [12]).

| $H$   | 0.55 | 0.65 | 0.75 | 0.85 | 0.95 |
|-------|------|------|------|------|------|
| $n = 128$ | $\mathbb{E}_H(\hat{H}_n^{\text{HEAF}})$ | 0.538 | 0.628 | 0.712 | 0.787 | 0.849 |
|       | $\text{Var}_H(\hat{H}_n^{\text{HEAF}})$ | 0.00377 | 0.00322 | 0.00267 | 0.00218 | 0.00167 |
| $n = 1024$ | $\mathbb{E}_H(\hat{H}_n^{\text{HEAF}})$ | 0.548 | 0.646 | 0.739 | 0.824 | 0.893 |
|       | $\text{Var}_H(\hat{H}_n^{\text{HEAF}})$ | 0.000468 | 0.000399 | 0.000378 | 0.000373 | 0.000328 |
| $n = 8192$ | $\mathbb{E}_H(\hat{H}_n^{\text{HEAF}})$ | 0.55 | 0.649 | 0.746 | 0.837 | 0.912 |
|       | $\text{Var}_H(\hat{H}_n^{\text{HEAF}})$ | 0.0000584 | 0.0000515 | 0.0000573 | 0.0000836 | 0.000101 |

Table 3: Simulations of the mean and of the variance of the HEAF estimator.

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