Abstract. The separating time for two probability measures on a filtered space is an extended stopping time which captures the phase transition between equivalence and singularity. More specifically, two probability measures are equivalent before their separating time and singular afterwards. In this paper, we investigate the separating time for two laws of general one-dimensional regular continuous strong Markov processes, so-called general diffusions, which are parameterized via scale functions and speed measures. Our main result is a representation of the corresponding separating time as (loosely speaking) a hitting time of a deterministic set which is characterized via speed and scale. As hitting times are fairly easy to understand, our result gives access to explicit and easy-to-check sufficient and necessary conditions for two laws of general diffusions to be (locally) absolutely continuous and/or singular. Most of the related literature treats the case of stochastic differential equations. In our setting we encounter several novel features, which are due to general speed and scale on the one hand, and to the fact that we do not exclude (instantaneous or sticky) reflection on the other hand. These new features are discussed in a variety of examples. As an application of our main theorem, we investigate the no arbitrage concept no free lunch with vanishing risk (NFLVR) for a single asset financial market whose (discounted) asset is modeled as a general diffusion which is bounded from below (e.g., non-negative). More precisely, we derive deterministic criteria for NFLVR and we identify the (unique) equivalent local martingale measure as the law of a certain general diffusion on natural scale.

1. Introduction

The purpose of this paper is to give explicit and easy-to-check sufficient and necessary conditions for two laws of general one-dimensional regular continuous strong Markov processes (so-called general diffusions) to be (locally) absolutely continuous and/or singular. In contrast to the (strict) subclass of diffusions that can be characterized via stochastic differential equations (SDEs), and which are called Itô diffusions in this paper, general diffusions are, in general, not semimartingales. The law of a general diffusion can be characterized via two deterministic objects, namely its scale function and its speed measure.

Our interest in general diffusions is motivated by applications to mathematical finance. It is well-known that Itô diffusion models with non-vanishing volatility coefficients cannot reflect occupation time effects such as stickiness of the price curve around some price. However, such effects can, for instance, be observed when a company got a takeover offer. To give an explicit example, we draw the reader’s attention to Figure 1, which shows the share price of Hansen Technologies Limited. On June 7, 2021, BGH Capital made an offer to acquire Hansen Technologies Limited at a price of AUD 6.5 in cash per share. The acquisition was cancelled by BGH Capital on September 6, 2021. Figure 1 shows a sticky behavior of the price curve while the takeover offer was active. Contrary to Itô diffusions with non-vanishing volatility, general diffusions are able to capture such occupation time effects.

Further, general diffusions can be used to model non-negative volatility processes which are allowed to vanish and reflect off the origin. For instance, there is empirical evidence (see, e.g., [11, Section 6.3.1]) that for the classical Heston model the Feller condition is often violated, which means its volatility process can
reach the origin and reflect from it. In general, it might not always be possible to capture such properties with Itô diffusions or semimartingales. The class of general diffusions is more flexible when it comes to modeling fine local time effects.

Let us now turn to our main results. Studying (local) absolute continuity and singularity of probability measures on a path space has a very long tradition, see [25, pp. 631–634] and [32, pp. 314–315] for bibliographic notes. A classical approach to tackle such questions is by studying certain candidate density processes. In this paper we use a somehow different idea based on the concept of separating times. For two given probability measures on a filtered space there exists ([8]) a so-called separating time which captures the phase transition between equivalence and singularity. More precisely, the two probabilities are equivalent before their separating time and singular afterwards. It is possible to describe all standard notions such as (local) absolute continuity, equivalence and singularity via the separating time. Broadly speaking, in case the separating time is known in a tractable form, one can answer any type of absolute continuity or singularity question. In particular, the separating time even encodes the information when absolute continuity is lost.

Our main result is a representation of the separating time for two laws of general diffusions as “something like” the hitting time of a deterministic set which is described in terms of the scale functions and the speed measures. As easy corollaries, we obtain necessary and sufficient conditions for two laws of general diffusions to be (locally) absolutely continuous and/or singular. In contrast to such results for Itô diffusions, in our setting we encounter several novel features. Some of them are due to general scale functions and speed measures. Other ones are due to the fact that we allow boundary points of the state space to be accessible and, in this case, allow both absorption and (instantaneous or sticky) reflection. These new features are illustrated by a variety of examples. As hitting times are fairly easy to understand, our result appears to be very useful for applications.

To illustrate such an application, we consider a single asset financial market where the discounted asset price is modeled as a general diffusion bounded from below (e.g., non-negative). We establish the existence of a unique candidate for an equivalent local martingale measure (ELMM), which is a certain general diffusion on natural scale, and, in terms of scale and speed, we give precise deterministic criteria for the candidate to be an ELMM. In financial terms, our result provides a precise deterministic description of the no arbitrage concept no free lunch with vanishing risk (NFLVR) as introduced in [15].

We comment now on related literature. The separating time for two solutions to SDEs under the Engelbert–Schmidt conditions was computed in [8] (see also [35]). A characterization of local absolute continuity of two general diffusions with open state space was proved in [37]. Both of these results follow directly from our main theorem.
Local absolute continuity is closely related to the martingale property of a certain local martingale, which can be viewed as a candidate density process. In this connection, we mention the paper [29], which provides necessary and sufficient conditions for the martingale property of diffusions on natural scale. These conditions are expressed solely in terms of the speed measure of the diffusion. For the same general framework as considered in this paper, a characterization of the martingale property of certain non-negative continuous local martingales was recently proved in [17]. In Remark 2.27 we comment in more detail on the relation to this work.

Among other things, the existence and absence of NFLVR for one-dimensional Itô diffusion markets has been studied in [34]. Our work extends those from [34] (for NFLVR) to a general diffusion framework. In particular, when compared to [34], we make an interesting structural observation. Namely, in the Itô diffusion framework from [34] the scale functions and the speed measures are assumed to be absolutely continuous w.r.t. the Lebesgue measure. We prove that the absolute continuity of the scale functions (but not of the speed measures) is a necessary condition for NFLVR. At first glance, this appears to be quite surprising.

In general, (local) absolute continuity and singularity have also been studied for frameworks which are multidimensional, which include jumps and which are even non-Markovian. For instance, Itô diffusions with and without jumps were studied in [7, 13], (non-Markovian) continuous Itô processes were studied in [12, 42], and more general semimartingales have been considered in [14]. Of course, this short list is quite incomplete and we refer to the references of these papers for more literature. We contribute to this list, as our paper seems to be the first to cover the full class of general diffusions.

This article is structured as follows. In Section 2.1 we recall the concept of separating times. An introduction to our canonical diffusion framework is given in Section 2.2. Our main result is presented in Section 2.3 and examples are discussed in Section A. The financial application is given in Section 3. Finally, the proof of our main result is presented in Section 4. We also added three appendices to make our paper as self-contained as possible. In Appendix B we discuss the martingale problem for general diffusions, which we also think to be of independent interest. Further technical facts about general diffusions, semimartingales and differentiation of measures are collected in Appendicies C and D.

2. Separating Times for General Diffusions

This section is divided into four parts. In the first two parts we recall the concept of separating times and we introduce our canonical diffusion setting. In the third part we present our main results and finally we discuss some corollaries and examples.

2.1. Separating Times: Encoding Absolute Continuity and Singularity. We take a filtered space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,\infty)})\) with a right-continuous filtration. Recall that, for a stopping time \(\tau\), the \(\sigma\)-field \(\mathcal{F}_\tau\) is defined by the formula

\[
\mathcal{F}_\tau \equiv \{ B \in \mathcal{F} : B \cap \{ \tau \leq t \} \in \mathcal{F}_t \ \text{for all} \ t \in [0, \infty) \},
\]

in particular, \(\mathcal{F}_\infty = \mathcal{F}\). Let \(\delta\) be a point outside \([0, \infty]\) and endow \([0, \infty] \cup \{\delta\}\) with the ordering \(\delta > t\) for all \(t \in [0, \infty]\).

Definition 2.1. A map \(S : \Omega \to [0, \infty] \cup \{\delta\}\) is called an extended stopping time if \(\{S \leq t\} \in \mathcal{F}_t\) for all \(t \in [0, \infty]\).

Let \(P\) and \(\hat{P}\) be two probability measures on \((\Omega, \mathcal{F})\). The following result is a restatement of [8, Theorem 2.1].

Theorem 2.2. There exists a \(P, \hat{P}\)-a.s. unique extended stopping time \(S\) such that for any stopping time \(\tau\) we have

\[
P \sim \hat{P} \text{ on } \mathcal{F}_\tau \cap \{\tau < S\}, \quad P \perp \hat{P} \text{ on } \mathcal{F}_\tau \cap \{\tau \geq S\}.
\]

This extended stopping time is called the separating time for \(P\) and \(\hat{P}\).
We have already mentioned in the Introduction that separating times encode (local) absolute continuity and singularity properties. This claim is verified by the following proposition, which is a (slightly extended) restatement of [8, Lemma 2.1].

**Proposition 2.3.** Let $S$ be the separating time for $\mathbb{P}$ and $\tilde{\mathbb{P}}$ and let $t \in [0, \infty]$. Then, the following hold:

(i) $\mathbb{P} \ll \tilde{\mathbb{P}}$ if and only if $\mathbb{P}$-a.s. $S = \delta$.

(ii) $\tilde{\mathbb{P}} \ll \mathbb{P}$ if and only if $\tilde{\mathbb{P}}$-a.s. $S = \delta$.

(iii) $\mathbb{P} \ll_{\text{loc}} \tilde{\mathbb{P}}$ if and only if $\mathbb{P}$-a.s. $S \geq \infty$.

(iv) $\tilde{\mathbb{P}} \ll_{\text{loc}} \mathbb{P}$ if and only if $\tilde{\mathbb{P}}$-a.s. $S \geq \infty$.

(v) $\mathbb{P} \ll \mathbb{P}$ on $\mathcal{F}_t$ if and only if $\mathbb{P}$-a.s. $S > t$.

(vi) $\tilde{\mathbb{P}} \ll \mathbb{P}$ on $\mathcal{F}_t$ if and only if $\tilde{\mathbb{P}}$-a.s. $S > t$.

(vii) $\mathbb{P} \perp \tilde{\mathbb{P}}$ on $\mathcal{F}_t$ if and only if $\mathbb{P}$, $\tilde{\mathbb{P}}$-a.s. $S \leq t$ if and only if $\mathbb{P}$-a.s. $S \leq t$.

The following generalization of Proposition 2.3 is sometimes also useful. Again, its proof is straightforward.

**Proposition 2.4.** Let $S$ be the separating time for $\mathbb{P}$ and $\tilde{\mathbb{P}}$, let $\tau$ be a stopping time and $B \in \mathcal{F}_\tau$. Then, the following hold:

(i) $\mathbb{P} \ll \tilde{\mathbb{P}}$ on $\mathcal{F}_\tau \cap B$ if and only if $S > \tau$ $\mathbb{P}$-a.s. on $B$.

(ii) $\tilde{\mathbb{P}} \ll \mathbb{P}$ on $\mathcal{F}_\tau \cap B$ if and only if $S > \tau$ $\tilde{\mathbb{P}}$-a.s. on $B$.

(iii) The following are equivalent:

(a) $\mathbb{P} \perp \tilde{\mathbb{P}}$ on $\mathcal{F}_\tau \cap B$.

(b) $S \leq \tau$ $\mathbb{P}$, $\tilde{\mathbb{P}}$-a.s. on $B$.

(c) $S \leq \tau$ $\mathbb{P}$-a.s. on $B$.

For several classes of probability measures the separating times are explicitly known. For instance, this is the case if $\mathbb{P}$ and $\tilde{\mathbb{P}}$ are laws of one-dimensional Itô diffusions under the Engelbert–Schmidt conditions ([8, 35]) or Levy processes ([8]). In Section 2.3 we prove an explicit characterization of the separating times for general one-dimensional regular continuous strong Markov processes. Our result generalizes those from [8, 35] for Itô diffusions in several directions. For instance, we allow sticky points in the interior and also (both instantaneous and slow) reflection at the boundaries.

### 2.2. A Recap on the Canonical Diffusion Framework.

The purpose of this section is to introduce our general setting and to fix some notation related to the theory of one-dimensional regular continuous strong Markov processes (general diffusions). Before we start this program, let us mention some of the most classical references. The definitive account of the theory is given in the seminal monograph [23] by Itô and McKean. More introductory treatments can be found in [5, Chapter 16], [20, Chapter 2], [26, Chapter 33], [39, Section VII.3] and [40, Section V.7]. For an introduction with many useful examples, we also refer to [28, Chapter 15]. An overview on formulae is given in Chapter II of the “Handbook” [4] by Borodin and Salminen.

Let $J \subset [-\infty, \infty]$ be a bounded or unbounded, closed, open or half-open interval. We denote the closure of $J$ in the extended reals $[-\infty, \infty]$ by $\text{cl}(J)$, its interior by $J^\circ$ and its boundary by $\partial J$ ($= \text{cl}(J) \setminus J^\circ$). Further, we define $\Omega \triangleq C(\mathbb{R}_+; [-\infty, \infty])$, the space of continuous functions $\mathbb{R}_+ \to [-\infty, \infty]$. The coordinate process on $\Omega$ is denoted by $X$, i.e., $X_t(\omega) = \omega(t)$ for $t \in \mathbb{R}_+$ and $\omega \in \Omega$. We also set $\mathcal{F} \triangleq \sigma(X_s, s \geq 0)$ and $\mathcal{F}_t \triangleq \bigcap_{s \geq t} \sigma(X_r, r \leq s)$ for $t \in \mathbb{R}_+$. Except in Section 3, where we discuss an application to mathematical finance, $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0})$ will be the underlying filtered space.

In accordance with [40, Definition V.45.1], we call a map $(J \ni x \mapsto P_x)$, from $J$ into the set of probability measures on $(\Omega, \mathcal{F})$, a **general diffusion** (with state space $J$) if the following hold:

(i) $P_x(X_0 = x) = 1$ and $P_x(C(\mathbb{R}_+; J)) = 1$ for all $x \in J$;

(ii) the map $x \mapsto P_x(B)$ is measurable for all $B \in \mathcal{F}$;

(iii) the **strong Markov property** holds, i.e., for any stopping time $\tau$ and any $x \in J$, the kernel $P_{X_\tau}$ is the regular conditional $P_x$-distribution of $(X_{t+\tau})_{t \geq 0}$ given $\mathcal{F}_\tau$ on $\{\tau < \infty\}$, i.e., $P_x$-a.s. on $\{\tau < \infty\}$

$$P_x(X_{\tau+} \in \text{d}\omega|\mathcal{F}_\tau) = P_{X_\tau}(\text{d}\omega).$$
For abbreviation, we call “general diffusions” simply “diffusions” in this paper. A diffusion \((x \mapsto P_x)\) is called regular if, for all \(x \in J^0\) and \(y \in J\),

\[
P_x(T_y < \infty) > 0,
\]

where\(^2\)

\[
T_y \triangleq \inf(s \geq 0 : X_s = y)
\]

with the usual convention \(\inf(\emptyset) \triangleq \infty\). In this article we will only work with regular diffusions.

Next, we recall the important concepts of scale and speed. There exists a strictly increasing continuous function \(s : J \to \mathbb{R}\), which is unique up to increasing affine transformations, such that, for any interval \(I = (a, b)\) with \([a, b] \subset J\), we have

\[
P_x(T_b < T_a) = 1 - P_x(T_a < T_b) = \frac{s(x) - s(a)}{s(b) - s(a)}, \quad x \in I,
\]

see [5, Theorem 16.27]. Any such function \(s\) is called a scale function. In case \(s(x) = x\), we say that the diffusion is on natural scale. For an interval \(I = (a, b)\) with \([a, b] \subset J\), we set

\[
G_I(x, y) \triangleq \begin{cases} 
\frac{2(s(x \land y) - s(a))(s(b) - s(x \lor y))}{s(b) - s(a)}, & x, y \in I, \\
0, & \text{otherwise},
\end{cases}
\]

which is often called Green’s function. There exists a unique locally finite measure \(m\) on \((J^0, \mathcal{B}(J^0))\) such that, for any interval \(I = (a, b)\) with \([a, b] \subset J\),

\[
E_x[T_a \land T_b] = \int G_I(x, y)m(dy), \quad x \in I,
\]

see [39, Theorem VII.3.6] (notice that \(m\) is determined uniquely given the version of \(s\); if we replace \(s\) with \(ks + l\), \(k > 0\), \(l \in \mathbb{R}\), then \(m\) is replaced with \(m/k\)). The measure \(m\) is called the speed measure of \((x \mapsto P_x)\).\(^3\) Scale and speed determine the potential operator of the diffusion killed when exiting an interval. We take a reference point \(c \in J^0\) and define, for \(x \in J^0\),

\[
u(x) \triangleq \begin{cases} 
\int_{(c, x]} m((c, z])s(dz), & \text{for } x \geq c, \\
\int_{(x, c]} m((z, c])s(dz), & \text{for } x \leq c,
\end{cases}
\]

\[
u(x) \triangleq \begin{cases} 
\int_{(c, x]} s((c, y])m(dy), & \text{for } x \geq c, \\
\int_{(x, c]} s((y, c])m(dy), & \text{for } x \leq c,
\end{cases}
\]

where we identify \(s\) with the locally finite measure on \((J^0, \mathcal{B}(J^0))\) defined via \(s((x, y]) \triangleq s(y) - s(x)\), for \(x < y\) in \(J^0\). For \(b \in \partial J\), we write \(u(b) \triangleq \lim_{x \searrow x \mapsto b} u(x)\) and \(v(b) \triangleq \lim_{x \searrow x \mapsto b} v(x)\). A boundary point \(b \in \partial J\) is called

- regular if \(u(b) < \infty\) and \(v(b) < \infty\),
- exit if \(u(b) < \infty\) and \(v(b) = \infty\),
- entrance if \(u(b) = \infty\) and \(v(b) < \infty\),
- natural if \(u(b) = \infty\) and \(v(b) = \infty\).

\(^2\)Later we use the notation (2.3) for arbitrary \(y \in [-\infty, \infty]\) regardless of whether \(y\) belongs to \(J\).

\(^3\)We remark that our speed measure is one half of the speed measure in [4, 20, 39]. Our normalization is consistent with [5, 28, 40]. For instance, our speed measure of a Brownian motion is the Lebesgue measure (when we take the scale function \(s(x) = x\)).
These definitions are independent of the choice of the reference point \( c \in J^o \). Regular and exit boundaries are called accessible, and entrance and natural boundaries are called inaccessible. As already indicated by their names, inaccessible boundaries are not in the state space \( J \), while accessible ones are, see [5, Proposition 16.43].

For \( b \in \partial J \), we set \( s(b) \equiv \lim_{x \to b} s(x) \, (\in [-\infty, \infty]) \), and, for a reference point \( c \in J^o \), we also use the notation

\[
\langle b, c \rangle \equiv \begin{cases} 
(b, c), & b < c, \\
(c, b), & c < b. 
\end{cases}
\]

Straightforward calculations show that, for \( b \in \partial J \),

\[
(2.7) \quad u(b) < \infty \implies |s(b)| < \infty,
\]

\[
(2.8) \quad v(b) < \infty \implies m(\langle b, c \rangle) < \infty,
\]

\[
(2.9) \quad b \text{ is regular} \iff |s(b)| < \infty \text{ and } m(\langle b, c \rangle) < \infty,
\]

where \( c \in J^o \) is arbitrary (as \( m \) is a locally finite measure on \( (J^o, \mathcal{B}(J^o)) \), the finiteness of \( m(\langle b, c \rangle) \) does not depend on \( c \in J^o \)). For the sake of comparing the above statements with each other, we emphasize that the converse implications in (2.8) and (2.9) are false, whereas (2.10) is indeed an equivalence. Sometimes the following characterization for \( b \in \partial J \) to be accessible (i.e., for \( u(b) < \infty \) is convenient):

\[
(2.11) \quad b \text{ is accessible} \iff |s(b)| < \infty \text{ and } \int_{\langle b, c \rangle} |s(z) - s(b)| \, m(dz) < \infty
\]

for all (equivalently, for some) \( c \in J^o \). The characterization (2.11) follows from straightforward calculations.

The behavior of the diffusion at exit, entrance and natural boundaries is fully specified by \( s \) and \( m \). Regular boundaries are different in this regard. To see this, consider Brownian motion with state space \([0, \infty)\) and absorption or reflection in the origin ([5, Section 16.3]). In both cases the speed measure coincides with the Lebesgue measure on \([0, \infty)\) (when we take the scale function \( s(x) = x \)) and the origin is regular. Hence, knowing the speed measure on \( J^o = (0, \infty) \) does not suffice to decide whether the origin is absorbing or reflecting. To fix this issue, it is convenient to extend the speed measure \( m \) to a measure on \((J, \mathcal{B}(J))\). In the following we explain how this can be done for the case \( J = [0, \infty) \) and \( s(0) = 0 \). Define \( s^* : \mathbb{R} \to \mathbb{R} \) by setting \( s^*(x) \equiv s(x) \) and \( s^*(-x) \equiv -s(x) \) for \( x \in \mathbb{R}_+ \). For \( I = [0, a) \) with \( a > 0 \), define \( G_I^I(x, y) \) as \( G_{(-a, a)} \) from (2.5) with \( s \) replaced by \( s^* \), and set

\[
G_I^I(x, y) \equiv G_I^I(x, y) + G_I^I(x, -y), \quad x, y \in J = \mathbb{R}_+.
\]

By [39, Proposition VII.3.10], it is possible to define \( m(\{0\}) \in [0, \infty) \) such that, for any interval \( I = [0, a) \) with \( a > 0 \) and any Borel function \( f : \mathbb{R}_+ = J \to \mathbb{R}_+ \) with \( f(0) > 0 \),

\[
(2.12) \quad \mathbb{E}_x \left[ \int_0^{T_a} f(X_s) \, ds \right] = \int_I G_I^I(x, y) f(y) \, m(dy), \quad x \in I.
\]

Let us convince ourselves that \( m(\{0\}) \) distinguishes absorption and reflection. Taking \( f \equiv 1_{\{0\}} \) in (2.12) yields

\[
\mathbb{E}_0 \left[ \int_0^{T_a} 1_{\{X_s = 0\}} \, ds \right] = 2s(a) m(\{0\}), \quad a > 0.
\]

This formula motivates the following definitions: a regular boundary point \( b \) is called absorbing if \( m(\{b\}) = \infty \), slowly reflecting if \( 0 < m(\{b\}) < \infty \), and instantaneously reflecting if \( m(\{b\}) = 0 \). In what follows, we call a boundary point simply “reflecting” if it is “either instantaneously or slowly reflecting”.

We stress at this point that exit boundaries are also absorbing in the sense that such a boundary point cannot be left by the diffusion. However, in contrast to the regular case, the behavior of an exit boundary
is fully characterized by \( m \) on \((J^0, B(J^0))\). To guarantee that (2.12) holds we use the convention that \( m(\{b\}) = \infty \) in case \( b \) is an exit boundary.

The scale function and the extended speed measure determine the law of the diffusion uniquely, see [5, Corollary 16.73]. Therefore, we call the pair \((m, s)\) the characteristics of the diffusion. To avoid confusions, we stress that here \( m \) is the extended speed measure, i.e., it is defined on \((J,B(J))\).

**Remark 2.5.** We will also need a kind of converse to the discussion above: given a function and a measure with properties to be specified precisely in this remark they are the scale function and the speed measure of some diffusion. To this end, we first recall from above that the restriction \( m|_{J^0} \) of the speed measure \( m \) to \( J^0 \) is necessarily a locally finite measure on \((J^0, B(J^0))\). Moreover, by virtue of (2.6), the speed measure also satisfies

\[
m([a,b]) > 0 \quad \text{for all } a < b \text{ in } J^0.
\]

Conversely, given an open interval \( I \subset \mathbb{R} \), a continuous and strictly increasing function \( s^0 : I \to \mathbb{R} \) and a locally finite measure \( m^0 \) on \((I, B(I))\) satisfying (2.13) with \( I \) in place of \( J^0 \), there exists a diffusion with state space \( J \), scale function \( s \) and speed measure \( m \) such that \( J^0 = I \), \( s|_{J^0} = s^0 \) and \( m|_{J^0} = m^0 \) (see [26, Theorem 33.9]).

**2.3. Main Results.** We take two state spaces \( J \) and \( \tilde{J} \) such that \( J^0 = \tilde{J}^0 \). Let \((J \ni x \mapsto \mathbb{P}_x)\) and \((\tilde{J} \ni x \mapsto \mathbb{P}_x)\) be two regular diffusions with characteristics \((m, s)\) and \((\tilde{m}, \tilde{s})\), respectively. Our goal is to compute the separating time for \( \mathbb{P}_{x_0} \) and \( \tilde{\mathbb{P}}_{x_0} \) and an arbitrary initial value \( x_0 \in J \cap \tilde{J} \).

To give some guidance on the structure of this section, in the first part, only in terms of the diffusion characteristics \((m, s)\) and \((\tilde{m}, \tilde{s})\), we introduce the concept of separating points for \( \mathbb{P}_{x_0} \) and \( \tilde{\mathbb{P}}_{x_0} \) (these are certain points from \( \text{cl}(J) \)). Our main Theorem 2.18 then shows that the separating time of \( \mathbb{P}_{x_0} \) and \( \tilde{\mathbb{P}}_{x_0} \) is “something like” the hitting time of the set \( J_{\text{sep}} \) of separating points, where we need to carefully distinguish between the values \( \infty \) and \( \delta \). A precise mathematical formulation is given below.

**Definition 2.6 (Non-separating or good interior point).** We say that a point \( x \in J^0 (= \tilde{J}^0) \) is non-separating (or good) if there exists an open neighborhood \( U(x) \subset J^0 \) of \( x \) such that

(i) the differential quotient

\[
\left( \frac{d^+ s}{d \tilde{s}} \right)(z) \triangleq \lim_{h \downarrow 0} \frac{s(z + h) - s(z)}{\tilde{s}(z + h) - \tilde{s}(z)}
\]

exists for all \( z \in U(x) \) and is strictly positive and finite, i.e., \( d^+ s/d \tilde{s} \) is a function \( U(x) \to (0, \infty) \);

(ii) there exists a Borel function \( \beta : U(x) \to \mathbb{R} \) such that

\[
\int_{U(x)} \left( \beta(z) \right)^2 \tilde{s}(dz) < \infty
\]

and, for all \( y \in U(x) \),

\[
\left( \frac{d^+ \tilde{s}}{d \tilde{s}} \right)(y) - \left( \frac{d^+ s}{d \tilde{s}} \right)(x) = \int_x^y \beta(z) \tilde{s}(dz);
\]

(iii) the differential quotient

\[
\left( \frac{dm}{dm} \right)(z) \triangleq \lim_{h \downarrow 0} \frac{m((z - h, z + h))}{m((z - h, z + h))}
\]

\[\text{The law of such a diffusion can happen to be non-unique, as we can have different boundary behavior (instantaneous or slow reflection, absorption) at regular boundary points. More precisely, as we have seen above, } s^0 \text{ and } m^0 \text{ alone determine which boundary points of } J^0 \text{ are regular, exit or inaccessible. Accessible boundary points need to be in } J, \text{ so } J \text{ is uniquely determined by } J^0, s^0 \text{ and } m^0. \text{ For an exit boundary point } b \in \partial J, \text{ the only possible boundary behavior is absorption, and we need to set } m(\{b\}) = \infty \text{ according to the convention above. For a regular boundary point } b \in \partial J, \text{ however, we can choose any } m(\{b\}) \in [0, \infty], \text{ i.e., we can decide about the boundary behavior at a regular boundary in addition to the information carried in } s^0 \text{ and } m^0.\]
exists for all $z \in U(x)$ and

\begin{equation}
\frac{d\mu}{d\nu} \frac{d^+ s}{d\nu} = 1
\end{equation}
on $U(x)$.

We stress that the integration in (2.16) is indeed meant w.r.t. $s$. One could alternatively write the right-hand side of (2.16) as $f'_x \beta (z) \tilde{s}(dz)$, where $\beta = \beta (d^+ s/d\tilde{s})$, but the function $\beta$ (rather than $\tilde{\beta}$) is explicitly used many times below, which is the reason for the chosen form of (2.16).

**Remark 2.7.** Replacing the right differential quotient $d^+ s/d\tilde{s}$ with the left one $d^- s/d\tilde{s}$ everywhere in Definition 2.6 results in an equivalent definition. Indeed, if $x \in J^0$ is non-separating according to Definition 2.6 (i.e., with $d^+ s/d\tilde{s}$), then, due to (2.16), $d^+ s/d\tilde{s}$ is continuous on $U(x)$ and hence, thanks to [43, p. 204] applied to the function $\tilde{s} \circ \tilde{s}^{-1}$ on $\tilde{s}(U(x))$, $d^- s(z)/d\tilde{s}$ exists for all $z \in U(x)$ and coincides with $d^+ s/d\tilde{s}$. A similar argument also applies to the converse.

**Intuition 2.8.** Let us outline the main ideas that explain that (i)–(iii) from Definition 2.6 must hold in case $\mathbb{P}_{x_0}$ and $\tilde{\mathbb{P}}_{x_0}$ are equivalent up to the exit time of a neighborhood of $x_0$, emphasizing that the following discussion is by no means rigorous nor complete. We refer to Section 4 for all the fizzy details. We restrict our attention to the case $\tilde{s} = \text{Id}$. In fact this simplification is without loss of generality, as the general case then can be obtained by Lemma C.4. Take a point $x_0 \in J^0$, an open neighborhood $V(x_0) \subset J^0$ of $x_0$ and set $\tau \equiv \inf(t \geq 0; X_t \notin V(x_0))$. We assume that $\mathbb{P}_{x_0} \sim \tilde{\mathbb{P}}_{x_0}$ on $\mathcal{F}_\tau$.

Let us try to understand (i) from Definition 2.6. Since the process $s(X_{\cdot \wedge \tau})$ is a $\mathbb{P}_x$-martingale (see Lemma C.3), $\mathbb{P}_{x_0} \sim \tilde{\mathbb{P}}_{x_0}$ on $\mathcal{F}_\tau$ yields that $s(X_{\cdot \wedge \tau})$ is a $\tilde{\mathbb{P}}_x$-semimartingale. But, under $\tilde{\mathbb{P}}_{x_0}$, $X_{\cdot \wedge \tau}$ is a time-change of Brownian motion (because $\tilde{s} = \text{Id}$) and, as the semimartingale property is invariant under changes of time (see Lemma C.29), also $s(W_{\cdot \wedge \tau})$ is a semimartingale, where $W$ is a Brownian motion starting at $x_0$ and $\tau' \equiv \inf(t \geq 0; W_t \notin V(x_0))$. Thanks to the seminal work [10], for a Borel function $f: \mathbb{R} \to \mathbb{R}$, we know that the process $f(W)$ is a continuous semimartingale if and only if $f$ is the difference of two convex functions (a so-called dc function). In Theorem C.31, we prove a local version of this deep result, which allows us to conclude that $s$ is a dc function in a neighborhood of $x_0$. For now, we assume that $s$ is a dc function on $V(x_0)$ (possibly making $V(x_0)$ a bit smaller), which, by [10, Proposition 5.1], means that it is continuous with a right-continuous right-hand derivative whose second derivative measure has locally finite variation. In particular, part (i) follows (except the strict positivity, which we explain below).

To understand (ii), let $\{L^x_t(X); (t, x) \in \mathbb{R}_+ \times J^0\}$ be a jointly continuous modification of the semimartingale local time processes of $X$ under $\mathbb{P}_{x_0}$ (it exists for diffusions on natural scale, see Lemma C.15 and Remark C.16 (a)). Similarly, let $\{\tilde{L}^x_t(s(X)); (t, x) \in \mathbb{R}_+ \times \tilde{s}(J^0)\}$ be a jointly continuous modification of the semimartingale local time processes of $s(X)$ under $\mathbb{P}_{x_0}$ (which is a diffusion on natural scale, see Lemma C.4). Then, by Lemma C.26, we must have $\mathbb{P}_{x_0} \sim \tilde{\mathbb{P}}_{x_0}$-a.s.

\begin{equation}
L^x_t(s(X)) = \left(\frac{d^+ s}{dx}\right)(x)L^x_t(X), \quad x \in V(x_0), \ t \leq \tau.
\end{equation}

As both local times are continuous in the space variable, $y \mapsto d^+ s(y)/dx$ needs to be continuous on $V(x_0)$. Furthermore, inherited from the respective property of the Brownian local time, one can prove that $\mathbb{P}_{x_0}, \tilde{\mathbb{P}}_{x_0}$-a.s. $L^x_t(s(X)), L^x_t(X) > 0$ on $V(x_0)$ (see Lemma C.18). Hence, $y \mapsto d^+ s(y)/dx$ is a strictly positive continuous function on $V(x_0)$. In fact, the above regularity of $y \mapsto d^+ s(y)/dx$ can be improved to absolute continuity. To understand this, using the Itô–Tanaka formula (see Lemma C.26), one gets that $\tilde{\mathbb{P}}_{x_0}$-a.s., for all $t < \tau$, 

\begin{equation}
ds_t(X_t) = \left(\frac{d^+ s}{dx}\right)(X_t) \, dX_t + \frac{1}{2} \int L^x_t(X) \, s''(dx).
\end{equation}
From now on, suppose that \( X \) is a Brownian motion till \( \tau \) under \( \tilde{\mathbb{P}}_{x_0} \), which can be made rigorous by a change of time. Then, Girsanov’s theorem (see Lemma C.30) yields that \( \tilde{\mathbb{P}}_{x_0} \)-a.s.

\[
d \int L^\tau_t(X) s''(dx) \ll dt \quad \text{on } [0, \tau].
\]

But, as \( \tilde{\mathbb{P}}_{x_0} \)-a.s., for all \( t \in [0, \tau] \), \( t = \int L^\tau_t(X) dx \), by the occupation time formula (see Lemma C.26), one might find it convincing that \( \tilde{\mathbb{P}}_{x_0} \)-a.s.

\[
L^\tau_t(X) s''(dx) \ll L^\tau_t(X) dx, \quad t \in [0, \tau],
\]

which yields \( s''(dx) \ll dx \), as \( \tilde{\mathbb{P}}_{x_0} \)-a.s. \( \tilde{\mathcal{L}}_x^\tau(X) > 0 \) for all \( x \in V(x_0) \) (see Lemma C.18). As a consequence, up to increasing affine transformations, we have

\[
s(x) = \int^x \exp \left( \int^y \beta(z)dz \right) dy, \quad x \in V(x_0),
\]

which means that on \( V(x_0) \) the scale function has the form of a scale function of an Itô diffusion (which is given through an SDE). In particular, we get (2.16) with (a priori) some integrable \( \beta \). The final part of (ii) deals with the square-integrability of \( \beta \), i.e., \( \int_{V(x_0)} \beta(z)^2dz < \infty \), where we recall that \( \tilde{s} = \text{Id} \). To understand this condition, notice that (2.20) reformulates to

\[
ds(X_t) = s'(X_t) dX_t + \frac{1}{2} d \int L^\tau_t(X) s'(x) \beta(x) dx
\]

where we use the occupation time formula (see Lemma C.26) and \( s' \) denotes the derivative of \( s \). Now, Girsanov’s theorem (see Lemma C.30) yields that \( \tilde{\mathbb{P}}_{x_0} \)-a.s. \( \int^\tau_0 \beta(X_s)^2 d\langle X, X \rangle_s < \infty \). Because \( \tilde{\mathbb{P}}_{x_0} \)-a.s.

\[
\int^\tau_0 \beta(X_s)^2 d\langle X, X \rangle_s = \int_{V(x_0)} \beta(x)^2 L^\tau_x(X) dx, \quad L^\tau_x(X) > 0, \quad x \in V(x_0),
\]

we conclude that \( \int_{V(x_0)} \beta(z)^2dz < \infty \) (possibly making \( V(x_0) \) a bit smaller), as needed.

Finally, we comment on part (iii). Thanks to the occupation time formula for diffusions (see Lemma C.15), for \( z \in J^0 \), we have \( \tilde{\mathbb{P}}_{x_0} \)-a.s.

\[
\int_{V(x_0)} \mathbb{1}_{\{z-h < x < z+h\}} ds \rightarrow L^\tau_{z}(s(X)), \quad h \searrow 0,
\]

and \( \tilde{\mathbb{P}}_{x_0} \)-a.s.

\[
\int_{V(x_0)} \mathbb{1}_{\{z-h < x < z+h\}} ds \rightarrow L^\tau_{z}(X), \quad h \searrow 0.
\]

Hence, as \( \tilde{\mathbb{P}}_{x_0} \)-a.s. \( L^z_{\tau}(s(X)) > 0 \) on \( V(x_0) \), using also \( \tilde{\mathbb{P}}_{x_0} \sim \tilde{\mathbb{P}}_{x_0} \) on \( \mathcal{F}_\tau \) and (2.19), we get that \( \tilde{\mathbb{P}}_{x_0} \)-a.s.

\[
dm/d\tilde{m}(x) = L^\tau_x(X)/L^z_{\tau}(s(X)) = 1/(d^zs(x)/dx), \quad x \in V(x_0),
\]

which yields (iii).

The next result provides a measure-theoretical view on Definition 2.6 and a measure-theoretical meaning of the quantities appearing there. We recall that \( s \) and \( \tilde{s} \) are identified with locally finite measures on \( (J^0, \mathcal{B}(J^0)) \) defined via \( s((x, y]) \triangleq s(y) - s(x) \) and \( \tilde{s}((x, y]) \triangleq \tilde{s}(y) - \tilde{s}(x) \), where \( x < y \) are in \( J_0^\circ \).

**Lemma 2.9** (Equivalent definition of a non-separating interior point).

(i) A point \( x \in J_0^\circ (= J^0) \) is non-separating if and only if there exists an open neighborhood \( U(x) \subset J_0^\circ \) of \( x \) such that

\[
(a) \ s \ll \tilde{s} \text{ on } \mathcal{B}(U(x)) \text{ and } m \ll \tilde{m} \text{ on } \mathcal{B}(U(x));
\]
(b) there exists a version of the Radon–Nikodym derivative \( \partial s / \partial \tilde{s} \) and a Borel function \( \beta : U(x) \to \mathbb{R} \) such that

\[
\left( \frac{\partial s}{\partial \tilde{s}} \right) (y) > 0 \quad \text{for all } y \in U(x),
\]

\[
\left( \frac{\partial s}{\partial \tilde{s}} \right) (y) - \left( \frac{\partial s}{\partial \tilde{s}} \right) (x) = \int_x^y \beta(z) \tilde{s}(dz) \quad \text{for all } y \in U(x),
\]

\[
\int_{U(z)} (\beta(z))^2 \tilde{s}(dz) < \infty;
\]

(c) for the version of \( \partial s / \partial \tilde{s} \) described in (b) we have

\[
\frac{\partial m}{\partial \tilde{m}} = 1 \quad \text{\( \tilde{m} \)-a.e. on } U(x),
\]

where \( \partial m / \partial \tilde{m} \) denotes (any version of) the Radon–Nikodym derivative of \( m \) w.r.t. \( \tilde{m} \).

(ii) Furthermore, let \( x \in J^o (= J^o) \) be a non-separating point. Then, the differential quotient \( d^+ s / d \tilde{s} \) from Definition 2.6 equals the version \( \partial s / \partial \tilde{s} \) of the Radon–Nikodym derivative described in (b) and the differential quotient \( d m / d \tilde{m} \) from Definition 2.6 is a version of the Radon–Nikodym derivative \( \partial m / \partial \tilde{m} \). Finally, the function \( \beta \) from part (ii) of Definition 2.6 satisfies

\[
\beta(z) = \lim_{h \to 0} \frac{d^+ s(z + h) / d \tilde{s} - d^+ s(z - h) / d \tilde{s}}{s(z + h) - s(z - h)}
\]

for \( s, \tilde{s} \)-a.a. \( z \in U(x) \).

**Remark 2.10.** As an immediate consequence of Lemma 2.9, we make the following note. In a neighborhood of a non-separating point we must have \( s \sim \tilde{s} \) and \( m \sim \tilde{m} \) with continuous (!) Radon–Nikodym derivatives.

**Proof of Lemma 2.9.** The facts that (a)–(c) of Lemma 2.9 imply (i)–(iii) of Definition 2.6 and that the differential quotients from Definition 2.6 are versions of the Radon-Nikodym derivatives as described in Lemma 2.9 (ii) are straightforward due to the continuity of the version \( \partial s / \partial \tilde{s} \) of the Radon–Nikodym derivative described in (b) and the continuity of the version \( d m / d \tilde{m} \) w.r.t. \( \tilde{m} \). We now prove that the formula (2.21) holds for \( s \)-a.a. \( z \in U(x) \). Let \( \nu \) be the signed measure defined by

\[
\nu((a, b]) \triangleq d^+ s / d \tilde{s}(b) - d^+ s / d \tilde{s}(a), \quad a < b, a, b \in U(x).
\]

Then, we get from (2.16) and a monotone class argument that

\[
\nu(B) = \int_B \beta(y) \tilde{s}(dy), \quad B \in B(U(x)).
\]

As a consequence, \( \nu \ll s \) on \( B(U(x)) \) with Radon–Nikodym density \( \partial \nu / \partial s = \beta \). Finally, Theorem D.7 yields that (2.21) holds for \( s \)-a.a. \( z \in U(x) \). Finally, as \( s \sim \tilde{s} \) on \( B(U(x)) \), (2.21) also holds \( \tilde{s} \)-a.e. on \( U(x) \).

It remains to prove that (i)–(iii) of Definition 2.6 imply (a)–(c) of Lemma 2.9. Let \( x \in J^o \) and suppose that there exists an open neighborhood \( U(x) \subset J^o \) of \( x \) such that (i)–(iii) of Definition 2.6 hold true. Remark 2.7 implies that

\[
\left( \frac{d^+ s}{d \tilde{s}} \right) (z) = \lim_{h \to 0} \frac{s(z + h) - s(z - h)}{\tilde{s}(z + h) - \tilde{s}(z - h)}
\]

for all \( z \in U(x) \). Using the notation from Section D.3, this means that \( d^+ s / d \tilde{s} = D^y m(z) / s \) on \( U(x) \).

By Theorem D.7, \( d^+ s / d \tilde{s} \) is a version of the generalized Radon–Nikodym derivative \( \partial s / \partial \tilde{s} \) (the notion is defined in Section D.1). As \( d^+ s / d \tilde{s} \) is \((0, \infty)\)-valued, Lemma D.1 implies that \( s \sim \tilde{s} \) on \( B(U(x)) \) (alternatively, this follows from Corollaries D.2 and D.3); in particular, the generalized Radon–Nikodym derivative \( \partial s / \partial \tilde{s} \) is the standard one. In a similar way, (iii) of Definition 2.6 and Theorem D.7 yield that \( m \sim \tilde{m} \) on \( B(U(x)) \) and that \( d m / d \tilde{m} \) is a version of the Radon–Nikodym derivative \( \partial m / \partial \tilde{m} \). The remaining claims are now obvious. \( \square \)
**Definition 2.11 (Half-good boundary point).** Take \( b \in \partial J = \partial \tilde{J} \). We say that \( b \) is **half-good** if there exists a non-empty open interval \( B \subseteq J^\circ(= \tilde{J}^\circ) \) with \( b \) as endpoint such that

(i) all points in \( B \) are good in the sense of Definition 2.6;
(ii) \( \tilde{s}(b) \triangleq \lim_{z \to b} \tilde{s}(x) \in \mathbb{R} \) and

\[
(2.22) \quad \int_B |\tilde{s}(z) - \tilde{s}(b)|^2 \tilde{s}(dz) < \infty,
\]

where \( \beta \) is as in Definition 2.6, which can be defined unambiguously by the formula (2.21).

**Intuition 2.12.** Part (ii) from Definition 2.11 encodes that, for every \( x \in B, \mathbb{P}_x, \tilde{\mathbb{P}}_x \)-a.s.

\[
\lim_{t \to T_b} \frac{d\tilde{\mathbb{P}}_x}{d\mathbb{P}_x} \geq 0
\]
on the trajectories such that \( X_t \to b, t \not\in T_b \), and \( X \in B \) on \([0, T_b)\). It is known that this property is related to the non-explosion of a non-negative additive functional (cf., e.g., \([13, 14, 17, 35]\)). By similar ideas as outlined in Intuition 2.8, again considering the case \( \tilde{s} = \text{Id} \), this question can be reduced to the problem whether

\[
\int_0^{T_b(W)} \beta(W_s)^2 ds, \quad W = \text{Brownian motion},
\]
is almost surely finite. Such questions were for example investigated in \([33]\), leading to the integral condition from (ii) in Definition 2.11. Of course, this provides only a rough idea where (2.22) comes from, while a rigorous proof needs a careful investigation.

**Lemma 2.13.** Assume that \( b \in \partial J(= \partial \tilde{J}) \) is half-good in the sense of Definition 2.11. Then, one of the following alternatives holds:

(a) \( b \in J \) and \( b \in \tilde{J} \), i.e., \( b \) is an accessible boundary point both for \((x \mapsto \mathbb{P}_x)\) and for \((x \mapsto \tilde{\mathbb{P}}_x)\);
(b) \( b \notin J \) and \( b \notin \tilde{J} \), i.e., \( b \) is an inaccessible boundary point both for \((x \mapsto \mathbb{P}_x)\) and for \((x \mapsto \tilde{\mathbb{P}}_x)\).

This lemma is proved in Section 4.1.1. At this point, Lemma 2.13 justifies the equivalence between \( b \in \tilde{J} \) and \( b \in J \) in the following definition. For \( b \in \partial J(= \partial \tilde{J}) \) and an arbitrary reference point \( c \in J^\circ(= \tilde{J}^\circ) \), we set

\[
[b, c] \triangleq \begin{cases} [b, c], & b < c, \\ (c, b], & c < b. \end{cases}
\]

**Definition 2.14 (Non-separating or good boundary point).** Take \( b \in \partial J = \partial \tilde{J} \). We say that \( b \) is **non-separating** (or **good**) if

(i) \( b \) is half-good in the sense of Definition 2.11;
(ii) if \( b \in \tilde{J} \) (equivalently, \( b \in J \)), then either it holds that \( m(\{b\}) = \tilde{m}(\{b\}) = \infty \) or it holds that \( m(\{b\}) < \infty, \tilde{m}(\{b\}) < \infty \);
(iii) if \( b \in \tilde{J} \) (equivalently, \( b \in J \)) and \( m(\{b\}) < \infty, \tilde{m}(\{b\}) < \infty \), then also the following conditions hold:

(a) the differential quotient

\[
\left( \frac{d^{+} \tilde{s}}{d \tilde{s}} \right)(b) \triangleq \lim_{z \to c \in b} \frac{\tilde{s}(b) - \tilde{s}(c)}{\tilde{s}(b) - \tilde{s}(c)},
\]

exists as a strictly positive and finite number;

(b) with \( B \) as in Definition 2.11, there exists a Borel function \( \beta: B \to \mathbb{R} \) such that

\[
\int_B (\beta(z))^2 \tilde{s}(dz) < \infty,
\]

and, for all \( y \in B \),

\[
\left( \frac{d^{+} \tilde{s}}{d \tilde{s}} \right)(b \vee y) - \left( \frac{d^{+} \tilde{s}}{d \tilde{s}} \right)(b \wedge y) = \int_{b \wedge y}^{b \vee y} \beta(z) \tilde{s}(dz);
\]
(c) the differential quotient

\[
\left(\frac{d\mu}{dm}\right)(b) \triangleq \lim_{J \ni \xi \to b} \frac{\mu([b, c])}{\mu([b, \xi])}
\]

exists as a strictly positive and finite number and

\[
\left(\frac{d\mu}{dm}\right)(b) \left(\frac{d^+s}{ds}\right)(b) = 1.
\]

We notice that, although the superscript "+" in the notation \(d^+s/ds\) does not really make sense when \(b\) is a right boundary point in (2.24), we keep it for conformity with Definition 2.6 (compare (2.18) with (2.28)).

**Discussion 2.15.** What is encoded in Definition 2.14 is worth a discussion. Let \(b \in \partial J\) be a non-separating boundary point. Then, the following statements hold true, where the phrase “for both diffusions" always means “both for \((x \mapsto P_x)\) and for \((x \mapsto \tilde{P}_x)\)."

(i) As already mentioned, \(b\) is either accessible for both diffusions or inaccessible for both (Lemma 2.13).

(ii) Let \(b\) be accessible for both diffusions. Then, by (2.8), \(|s(b)| < \infty\) and \(|\tilde{s}(b)| < \infty\). Furthermore, \(b\) is either absorbing for both diffusions (\(m(\{b\}) = \tilde{m}(\{b\}) = \infty\)) or reflecting for both ones (\(m(\{b\}) < \infty\) and \(\tilde{m}(\{b\}) < \infty\)).

(iii) Let \(b\) be reflecting for both diffusions. Then, by (2.9) or (2.10), \(m([b, c]) < \infty\) and \(\tilde{m}([b, c]) < \infty\) for all \(c \in J^0\) (recall the convention in the second paragraph before Remark 2.5 that \(m(\{b\}) = \tilde{m}(\{b\})\) whenever \(b\) is an exit boundary, hence \(m(\{b\}) < \infty\) necessarily means that \(b\) is a regular boundary). Together with \(|s(b)| < \infty\) and \(|\tilde{s}(b)| < \infty\) discussed above, this means that the quotients \((s(b) - s(c))/s(b) - \tilde{s}(c))\) and \(m([b, c])/\tilde{m}([b, c])\) on the right-hand sides of (2.24) and (2.27) are well-defined.

(iv) Let \(b\) be slowly reflecting for both diffusions. Then, the limit in (2.27) is nothing else but the quotient \(m(\{b\})/\tilde{m}(\{b\})\).

**Intuition 2.16.** In addition to “half-goodness" from Definition 2.11, Definition 2.14 imposes additional constraints if \(b \in \partial J\) is reflecting at least for one of the diffusions (and, for \(b\) to be non-separating, necessarily, for both diffusions). The structure of these additional parts appear to be very similar to the definition of an interior non-separating point (see Definition 2.6). This is no coincidence, because reflecting boundary points are related to interior points of a symmetrized diffusion with a larger state space. To see the idea, for a standard Brownian motion \(W\) with \(W_0 = 0\), the formula \(|W|\) defines a Brownian motion with reflection at the origin. So to say, standard Brownian motion is the symmetrization of a Brownian motion with reflection at the origin. We refer to Lemmata C.12 and C.13 for the general picture of the symmetrization procedure. Finally, the additional constraints in Definition 2.14 are nothing else but the requirement that \(b\) is a non-separating interior point for the symmetrized diffusion. Let us emphasize that it is by no means obvious that symmetrization is useful for the computation of the separating time (to get an idea of the problem, notice that, in the above example with \(|W|\) and \(W\), the filtration of \(|W|\) is strictly smaller than that of \(W\)). This requires a careful investigation.

**Definition 2.17** (Separating or bad point). We say that a point \(x \in \text{cl}(J)(= \text{cl}(\tilde{J}))\) is separating (or bad) if it is not non-separating. We denote the set of all separating points in \(\text{cl}(J)\) by \(J_{\text{sep}}\). Notice that \(J_{\text{sep}}\) is closed in \(\text{cl}(J)\).
We set
\[ l \triangleq \inf J \quad \text{and} \quad r \triangleq \sup J. \]
Let \( \Delta \) be a point outside \([-\infty, \infty] \). We denote by \( \alpha \) the separating point which is closest to \( x_0 \) from the left in the following sense:
\[
\alpha \triangleq \begin{cases} 
\sup ([l, x_0] \cap J_{\text{sep}}), & [l, x_0] \cap J_{\text{sep}} \neq \emptyset, \\
\Delta, & [l, x_0] \cap J_{\text{sep}} = \emptyset.
\end{cases}
\]
Next, we define an extended stopping time \( U \), which is “a variant of the hitting time” of the point \( \alpha \).
- If \( \alpha = \Delta \), then we set \( U \triangleq \delta \).
- If \( \alpha = x_0 \), then we set \( U \triangleq 0 \).
- Otherwise (i.e., in case \( \text{cl}(J) \ni \alpha < x_0 \)) we set
\[
U \triangleq \begin{cases} 
T_\alpha, & \liminf_{t \nearrow T_\alpha} X_t = \alpha, \\
\delta, & \liminf_{t \nearrow T_\alpha} X_t > \alpha.
\end{cases}
\]
The structure of \( U \) in (2.29) is worth a discussion. On the event \( \{T_\alpha < \infty\} \), the right-hand side of (2.29) equals \( T_\alpha \). However, on \( \{T_\alpha = \infty\} \), the right-hand side of (2.29) is either \( \infty \) or \( \delta \) depending on whether the point \( \alpha \) is hit asymptotically or not.

In a symmetric way, we define \( V \) to be “a variant of the hitting time” of the separating point which is closest to \( x_0 \) from the right. To avoid confusion, we give a precise definition. We set
\[
\gamma \triangleq \begin{cases} 
\inf ([x_0, r] \cap J_{\text{sep}}), & [x_0, r] \cap J_{\text{sep}} \neq \emptyset, \\
\Delta, & [x_0, r] \cap J_{\text{sep}} = \emptyset.
\end{cases}
\]
and proceed as above:
- If \( \gamma = \Delta \), we set \( V \triangleq \delta \).
- If \( \gamma = x_0 \), we set \( V \triangleq 0 \).
- Otherwise (i.e., in case \( \text{cl}(J) \ni \gamma > x_0 \)), we set
\[
V \triangleq \begin{cases} 
T_\gamma, & \limsup_{t \nearrow T_\gamma} X_t = \gamma, \\
\delta, & \limsup_{t \nearrow T_\gamma} X_t < \gamma.
\end{cases}
\]
Of course, the structure of \( V \) in (2.30) can be discussed in the same way as the structure of \( U \) in (2.29) is discussed above.

We also introduce the deterministic time

\[
R \triangleq \begin{cases} 
\infty, & \alpha = \gamma = \Delta, \ l \text{ and } r \text{ are reflecting for one (equivalently, for both) of the diffusions,} \\
\delta, & \text{otherwise.}
\end{cases}
\]
Finally, we are in a position to present our main result. Intuitively speaking, it shows that the separating time of two non-identical diffusions is given by “something like” the first time the coordinate process hits a separating point (with a careful distinction between \( \infty \) and \( \delta \) on the trajectories that do not hit separating points in finite time).

**Theorem 2.18.** Let \( x_0 \in J \cap \bar{J} \) and let \( S \) be the separating time for \( P_{x_0} \) and \( \bar{P}_{x_0} \).

(i) If \( P_{x_0} = \bar{P}_{x_0} \), then \( P_{x_0}, \bar{P}_{x_0} \)-a.s. \( S = \delta \).
(ii) If \( P_{x_0} \neq \bar{P}_{x_0} \), then \( P_{x_0}, \bar{P}_{x_0} \)-a.s. \( S = U \wedge V \wedge R \).

**Remark 2.19.** (a) We notice that the (trivial) case \( P_{x_0} = \bar{P}_{x_0} \) needs to be treated separately in Theorem 2.18. Technically, the reason is that a boundary point \( b \) can be separating also in the case \( P_{x_0} = \bar{P}_{x_0} \). For example, in case both \( P_{x_0} \) and \( \bar{P}_{x_0} \) are the standard Wiener measure, trivially \( P_{x_0}, \bar{P}_{x_0} \)-a.s. \( S = \delta \), but \( P_{x_0}, \bar{P}_{x_0} \)-a.s. \( U \wedge V \wedge R = \infty \), because \( \pm \infty \) are separating points and \( P_{x_0}, \bar{P}_{x_0} \)-a.s. \( \limsup_{t \to \infty} X_t = -\liminf_{t \to \infty} X_t = \infty \) by standard properties of Brownian motion.

(b) It is possible to formulate part (ii) from Theorem 2.18 without the deterministic time \( R \):
(ii.1) If all points of \(\text{cl}(J)\) are non-separating and the boundary points \(l\) and \(r\) are reflecting for one (equivalently, for both) of the diffusions, then \(P_{x_0}, P_{x_0}\)-a.s. \(S = \infty\).

(ii.2) Otherwise, \(P_{x_0}, P_{x_0}\)-a.s. \(S = U \wedge V\).

The situation from (ii.1) needs to be treated separately, because diffusions with two reflecting boundary points are necessarily recurrent, which entails that they cannot be equivalent on the infinite time horizon. This is the only case where \(P_{x_0} \neq \tilde{P}_{x_0}\) but the separating time still cannot be captured via the set \(J_{\text{sep}}\) of separating points alone. In Example A.13 and Discussion A.14 we discuss this issue in more detail.

**Discussion 2.20.** In order to apply Theorem 2.18 in specific situations, we need to compute the set \(J_{\text{sep}}\) of separating points in \(\text{cl}(J)(= \text{cl}(\tilde{J}))\). Given the above definitions this might be a very computational task. Fortunately, there are many interesting interdependencies between the notions, which allow to reduce the computations in many specific situations considerably. In the following, we collect several helpful observations.

(i) If a boundary point \(b \in \partial J(= \partial \tilde{J})\) is accessible for one of the diffusions but inaccessible for the other one, then \(b \in J_{\text{sep}}\) (that is, in such a case we do not need to verify (2.22)). This follows directly from Lemma 2.13.

(ii) The roles of the diffusions \((x \mapsto P_x)\) and \((x \mapsto \tilde{P}_x)\) in each of the three building blocks in the definition of a non-separating point, namely, Definitions 2.6, 2.11 and 2.14, are symmetric. That is, in any of these definitions we can interchange the roles between \((m, \bar{s})\) and \((\bar{m}, \bar{s})\). To avoid confusions, we sketch this in more detail.

- Instead of working with \(d^+s/d\tilde{s}\) and \(dm/d\tilde{m}\) one may use \(d^+\tilde{s}/ds\) and \(d\tilde{m}/dm\) to define a good (non-separating) interior point. Instead of \(\beta\) we then get another function \(\tilde{\beta}\). The relation between \(\beta\) and \(\tilde{\beta}\) is

\[
\tilde{\beta} = -\beta \frac{d^+\tilde{s}}{ds}.
\]

- To define a half-good and a good (non-separating) boundary point exchange the roles of \((m, \bar{s})\) and \((\bar{m}, \bar{s})\) everywhere and do not forget to replace \(\beta\) with \(\tilde{\beta}\) in (2.22), (2.25) and (2.26).

- It is possible to exchange the roles in *some* but not in *all* of the three definitions. For instance, one might inspect the goodness of the interior points exactly as in Definition 2.6 (which also provides the function \(\beta\) on the way) and inspect the goodness of the boundary points using the definitions for the interchanged diffusions (in which case use \(\tilde{\beta}\) from (2.31)).

The possibility to interchange the roles of \((m, \bar{s})\) and \((\bar{m}, \bar{s})\) follows from Theorem 2.18 together with the symmetry in the notion of the separating time: the \((P_{x_0}, P_{x_0})\)-a.s. unique separating time for \(P_{x_0}\) and \(\tilde{P}_{x_0}\) is the same as the separating time for \(P_{x_0}\) and \(P_{x_0}\). The formula (2.31) is a straightforward calculation.

(iii) A boundary point \(b\) with either \(|\tilde{s}(b)| = \infty\) or \(|\bar{s}(b)| = \infty\) is automatically separating, i.e., \(b \in J_{\text{sep}}\). In the case \(|\tilde{s}(b)| = \infty\) this is seen directly from the definitions. In the case \(|\bar{s}(b)| = \infty\) this follows from the previous point in this discussion, i.e., the symmetric roles of \((x \mapsto P_x)\) and \((x \mapsto \tilde{P}_x)\).

By virtue of Proposition 2.3, Theorem 2.18 yields a variety of corollaries concerning absolute continuity and singularity of \(P_{x_0}\) and \(\tilde{P}_{x_0}\).

**Corollary 2.21.** Let \(P_{x_0} \neq \tilde{P}_{x_0}\) and \(x_0 \in J \cap \tilde{J}\). Then, the following are equivalent:

(i) \(P_{x_0} \ll \tilde{P}_{x_0}\) on \(F_t\) for some \(t > 0\).

(ii) \(P_{x_0} \ll \tilde{P}_{x_0}\) on \(F_t\) for all \(t > 0\), i.e., \(P_{x_0} \ll_{\text{loc}} \tilde{P}_{x_0}\).

(iii) All points in \(J\) are non-separating (in other words, all points in \(J(= \tilde{J})\) are non-separating and boundary points that are accessible for \((x \mapsto P_x)\) are non-separating).

**Remark 2.22.** In case \(x_0\) is not an absorbing boundary point at least for one of the diffusions, the assumption \(P_{x_0} \neq \tilde{P}_{x_0}\) can be removed from Corollary 2.21. To see this, consider the case where \(x_0\) is not absorbing and \(P_{x_0} = \tilde{P}_{x_0}\). Then, all conditions (i), (ii) and (iii) in Corollary 2.21 are satisfied. While this is clear for (i) and (ii), for (iii) this follows from Lemma C.1. As a consequence, the assumption \(P_{x_0} \neq \tilde{P}_{x_0}\) in Corollary 2.21 can be dropped whenever \(x_0 \in J(= \tilde{J})\).
Corollary 2.23. Let $\mathbb{P}_{x_0} \neq \tilde{\mathbb{P}}_{x_0}$ and $x_0 \in J \cap \tilde{J}$. Then, $\mathbb{P}_{x_0} \ll \tilde{\mathbb{P}}_{x_0}$ if and only if

(a) all points in $J^o$ are non-separating,
(b) each boundary point $b$ of $J$ satisfies one of the following:
   (b.i) $b$ is non-separating,
   (b.ii) $|s(b)| = \infty$ and the other boundary point $b^*$ is non-separating,
(c) and, in case both boundary points are non-separating, at least one of the boundary points is not reflecting (for one, equivalently for both, of the diffusions).

It is worth noting that, for a boundary point $b$, (b.i) and (b.ii) above exclude each other because $|s(b)| = \infty$ implies that $b$ is separating (recall Discussion 2.20 (iii)).

Remark 2.24. Contrary to Corollary 2.21 and Remark 2.22, in Corollary 2.23 the assumption $\mathbb{P}_{x_0} \neq \tilde{\mathbb{P}}_{x_0}$ cannot be dropped even in the case $x_0 \in J^o$. The reason is discussed in Remark 2.19 (a). For instance, if $\mathbb{P}_{x_0} = \mathbb{P}_{x_0}$ is the standard Wiener measure, then both $\infty$ and $-\infty$ are separating boundary points, and hence (b) in Corollary 2.23 is not satisfied.

Proof of Corollaries 2.21 and 2.23. The (trivial) case that $x_0$ is an absorbing boundary for both diffusions is excluded by the assumption $\mathbb{P}_{x_0} \neq \tilde{\mathbb{P}}_{x_0}$. Now, the claims of both corollaries follow from Theorem 2.18, Proposition 2.3 and Lemmata C.5 and C.8.

Corollary 2.25. Let $x_0 \in J \cap \tilde{J}$. We have $\mathbb{P}_{x_0} \perp \tilde{\mathbb{P}}_{x_0}$ on $\mathcal{F}_0$ if and only if $x_0 \in J_{sep}$.

Corollary 2.26. Take $x_0 \in J \cap \tilde{J}$ and suppose that either $s = \tilde{s}$ on $J^o$ or $m = \tilde{m}$ on $\mathcal{B}(J^o)$. The following are equivalent:

(i) $\mathbb{P}_{x_0} \ll \tilde{\mathbb{P}}_{x_0}$ on $\mathcal{F}_t$ for some $t > 0$.
(ii) $\mathbb{P}_{x_0} = \tilde{\mathbb{P}}_{x_0}$.

Proof. Of course, (ii) $\Rightarrow$ (i) is trivial. Assume that (i) holds and, for contradiction, further assume that $\mathbb{P}_{x_0} \neq \tilde{\mathbb{P}}_{x_0}$. By Corollary 2.21, all points in $J$ are non-separating. In particular, all points in $J^o(= \tilde{J}^o)$ are non-separating. Thus, if $s = \tilde{s}$ on $J^o$ then $dm/d\tilde{m} = 1$ on $J^o$, and if $m = \tilde{m}$ on $\mathcal{B}(J^o)$ then $d^+s/d\tilde{s} = 1$ on $J^o$. Consequently, irrespective of our hypothesis, both $s = \tilde{s}$ on $J^o$ and $m = \tilde{m}$ on $\mathcal{B}(J^o)$ hold. As accessibility is characterized by scale and speed on the interior of the state space, we have $J = \tilde{J}$ and, by continuity, $s = \tilde{s}$ on $J$. Finally, let $b \in J \setminus J^o$ be accessible for $(x \mapsto \mathbb{P}_x)$. As noted before, $b$ is non-separating and we get $m\{\{\}\} = \tilde{m}\{\{\}\}$ from Definition 2.14. Thus, $m = \tilde{m}$ on $\mathcal{B}(J)$ and we conclude $\mathbb{P}_{x_0} = \tilde{\mathbb{P}}_{x_0}$ from the fact that scale and speed characterize a diffusion uniquely. This is a contradiction and the proof is complete.

Remark 2.27. The main result from [17] provides necessary and sufficient conditions for the martingale property of certain non-negative local martingales. Equivalently, these are necessary and sufficient conditions for $\mathbb{P}_{x_0} \ll_{loc} \mathbb{P}_{x_0}$ in case

$$d\tilde{s} = \varphi^2 ds, \quad d\tilde{m} = \varphi^2 dm,$$

where $\varphi$ is a positive function in the domain of the extended generator of the diffusion $(x \mapsto \mathbb{P}_x)$. Corollary 2.21 gives a complete characterization of local absolute continuity for two general diffusions without predetermined structural assumptions. As a consequence, our result shows that the above structure of $(\tilde{s}, \tilde{m})$ is necessary, where, in general, $\varphi$ does not need to be in the domain of the extended generator, although $\varphi^2$ always needs to be continuous (recall Remark 2.10). This observation confirms a conjecture of Chris Rogers (personal communication) about the structure of $\tilde{s}$ and $\tilde{m}$. For diffusions with open state space, the conjecture also follows from the main result in [37].

We stress that our approach is quite different from the one in [17], as we do not work with a candidate density process. In fact, because in our general framework there is no obvious candidate for a density process, a different approach seems to be necessary.

In the next section we discuss an application of our main result to mathematical finance.
3. Deterministic Conditions for NFLVR in General Diffusion Models

In this section we discuss an application of our main result to mathematical finance. Namely, we derive deterministic conditions for the existence and absence of arbitrage in the sense of the notion no free lunch with vanishing risk (NFLVR) as introduced in [15]. We do this in the most general single-asset regular diffusion model whose price process is bounded from below.

Let us start with an introduction to our financial framework. Take \( l \in \mathbb{R} \) and let \( J \) be either \([l, \infty)\) or \((l, \infty)\). Moreover, let \( s : J \to \mathbb{R} \) be a scale function, let \( m \) be a speed measure on \((J, \mathcal{B}(J))\) and let \((J \ni x \mapsto \mathbb{P}_x)\) be a regular filtration with characteristics \((s, m)\).

In the following we fix an initial value \( x_0 \in J^0 \) and a time horizon \( T \in (0, \infty] \). In case \( T = \infty \) we understand the interval \([0, T]\) as \( \mathbb{R}_+ \) and read expressions like “\( s \leq T \)” as “\( s < \infty \)”. Define \( \mathcal{G}_t = \mathcal{F}_t \) for \( t < T \) and \( \mathcal{G}_T = \sigma(X_s, s \leq T) \). In the following, we consider \((\Omega, \mathcal{G}_T, (\mathcal{G}_t)_{t \leq T}, \mathbb{P}_{x_0})\) as our underlying filtered probability space. Moreover, we use the convention that all processes are time indexed over \([0, T]\).

The process \( X \) will represent an asset price process. The interest rate is assumed to be zero.

We now recall the notion NFLVR. To introduce necessary terminology, suppose for a moment that \( X \) is a semimartingale. A one-dimensional predictable process \( H = (H_t)_{t \leq T} \) is called a trading strategy if the stochastic integral \( \int_0^T H_s dX_s \) is well-defined. Further, a trading strategy \( H \) is called admissible if there exists a constant \( c \geq 0 \) such that a.s. \( \int_0^T H_s dX_s \geq -c \) for all \( t \leq T \). The convex cone of all contingent claims attainable from zero initial capital is given by

\[
K \triangleq \left\{ \int_0^T H_s dX_s : H \text{ admissible, and if } T = \infty, \text{ then } \lim_{t \to \infty} \int_0^t H_s dX_s \text{ exists a.s.} \right\}.
\]

Let \( C \) be the set of essentially bounded random variables that are dominated by claims in \( K \), i.e.,

\[
C \triangleq \{ g \in L^\infty : \exists f \in K \text{ such that } g \leq f \text{ a.s.} \}.
\]

**Definition 3.1.** We say that NFLVR holds in our market if \( X \) is a semimartingale and \( \overline{C} \cap L^\infty_+ = \{0\} \), where \( \overline{C} \) denotes the closure of \( C \) in \( L^\infty_+ \) w.r.t. the norm topology and \( L^\infty_+ \) denotes the cone of non-negative random variables in \( L^\infty_+ \).

According to the celebrated fundamental theorem of asset pricing ([15]), NFLVR is equivalent to the existence of an equivalent local martingale measure (ELMM), i.e., a probability measure \( Q \) on the filtered space \((\Omega, \mathcal{G}_T, (\mathcal{G}_t)_{t \leq T})\) such that \( \mathbb{P}_{x_0} \sim Q \) on \( \mathcal{G}_T \) and \( X \) is a local \( Q \)-martingale. We emphasize that, thanks to Girsanov’s theorem (Lemma C.30), the existence of an ELMM immediately implies that \( X \) is a \( \mathbb{P}_{x_0} \)-semimartingale. In the following, we describe NFLVR in a deterministic manner for our single asset diffusion market.

**Condition 3.2.** There exists a function \( \beta : J^0 \to \mathbb{R} \) such that

\[
\beta^2 \in L^1_{\text{loc}}(J^0), \quad (3.1)
\]

and, up to increasing affine transformations,

\[
s(x) = \int_x^\infty \exp \left( \int_y^\infty \beta(z)dz \right)dy, \quad x \in J^0. \quad (3.2)
\]

Moreover, if \( J = [l, \infty) \), then \( m(\{l\}) = \infty \), i.e., the boundary point \( l \) is absorbing for the diffusion \((x \mapsto \mathbb{P}_x)\) whenever it is accessible.

**Condition 3.3.** Condition 3.2 holds and, for \( \beta \) as in Condition 3.2,

\[
\int_{l^+} (x - l) \beta(x)^2 dx < \infty. \quad (3.3)
\]

**Condition 3.4.** Condition 3.2 holds and

\[
either \quad s(l) = -\infty \quad or \quad s(l) > -\infty \text{ and } \int_{l^+} |s(x) - s(l)| m(dx) = \infty, \quad (3.4)
\]


(3.5) \[ \int_{t^+} (x-t) s'(x) m(dx) = \infty, \]
where \( s' \) denotes the derivative of \( s \), see (3.2).

**Theorem 3.5.** If \( T < \infty \), then NFLVR holds if and only if at least one of Conditions 3.3 and 3.4 holds. Moreover, if NFLVR holds, the unique ELMM is given by \( \tilde{P}_x \), where \((x \mapsto \tilde{P}_x) \) is the diffusion with the interior of the state space \( J^\circ \), characteristics \((|dL|, s' \omega \, dm)\) on \( J^\circ \) and the boundary point \( l \) being absorbing whenever it is accessible.

It is worth noting that Conditions 3.3 and 3.4 do not exclude each other.

**Remark 3.6.** Assume that NFLVR holds for some \( T < \infty \). Then either the boundary point \( l \) is inaccessible for both diffusions \((x \mapsto P_x)\) and \((x \mapsto \tilde{P}_x)\) or \( l \) is accessible (and absorbing) for both diffusions. This follows from Theorem 3.5 and Lemma 2.13 together with the facts that

- Condition 3.3 means that \( l \) is half-good for \( P_x \) and \( \tilde{P}_x \) (compare, in particular, (2.22) with (3.3));
- (3.4) (resp., (3.5)) means that \( l \) is inaccessible under \( P_x \) (resp., under \( \tilde{P}_x \)).

Furthermore, \( \infty \) is inaccessible both under \( P_x \) and under \( \tilde{P}_x \). The former is contained in our setting \((\infty \notin J)\), while the latter follows from \( \tilde{s}(\infty) = \infty \) (with \( \tilde{s} = \text{Id} \)).

**Proof of Theorem 3.5.** We start with the necessity of the conditions. Suppose that NFLVR holds, which means, by the fundamental theorem of asset pricing ([15, Corollary 1.2]), that an ELMM \( Q \) exists. As non-negative local martingales (which are supermartingales) cannot resurrect from zero ([39, Proposition II.3.4]), in case \( J = [l, \infty) \), using \( Q \sim P_x \) we get that the left boundary \( l \) has to be an absorbing state for \((x \mapsto P_x)\), i.e., \( m(\{l\}) = \infty \).

Now we investigate properties of the scale function \( s \). Take an arbitrary point \( y_0 \in J^\circ \) and define \( Y_t \triangleq X_{(t+T_{y_0})\wedge T} \) and \( A_t \triangleq G_{(t+T_{y_0})\wedge T} \) for \( t \in [0,T] \). Recall from Lemma C.5 that \( P_x(T_{y_0} < T) > 0 \). In particular, as \( P_x \sim Q \) on \( G_T \), we have \( Q(T_{y_0} < T) > 0 \), which means that we can define a probability measure \( K \) on \((\Omega, G_T)\) by the formula

\[ K(\omega | {T_{y_0} < T}) = \frac{Q(\omega \cap \{T_{y_0} < T\})}{Q(T_{y_0} < T)}, \]

By the definition of an ELMM and Lemma C.29, \( Y \) is a local \( Q-(A_t)_{t \leq T}\)-martingale. Moreover, \( \{T_{y_0} < T\} \in A_0 \), \( Y \) is a local \( K-(A_t)_{t \leq T}\)-martingale. We define

\[ L(t) \triangleq \inf(s \in [0,T] : (Y,Y)_s > t) \wedge T, \quad t \in \mathbb{R}_+, \]

where \( (Y,Y) \) denotes the \( K-(A_t)_{t \leq T}\)-quadratic variation process of \( Y \). As, in case \( J = [l, \infty) \), the left boundary point \( l \) is absorbing for \((x \mapsto P_x)\), we get from Lemmata C.3 and C.29 that \( s(Y) \) is a local \( P_x\)-\( (A_t)_{t \leq T}\)-martingale. Hence, by Girsanov’s theorem (Lemma C.30), as \( K \ll Q \sim P_x \) on \( A_T = G_T \), the process \( s(Y) \) is a \( K-(A_t)_{t \leq T}\)-semimartingale. Using once again Lemma C.29, this implies that \( s(Y_L) \) is an \( (A_{L(t)})_{t \geq 0}\)-Brownian motion stopped at \((Y,Y)_T\), which is an \( (A_{L(t)})_{t \geq 0}\)-stopping time by [24, Lemma 10.5]. It follows from the fact ([39, Proposition IV.1.13]) that continuous local martingales and their quadratic variation processes have the same intervals of constancy, and Lemma C.6, that \( K\)-a.s. \((Y,Y)_T > 0 \) and \( L(0) = 0 \). Consequently,

\[ K(Y_{L(0)} = y_0) = \frac{Q(X_{T_{y_0} \wedge T} = y_0, T_{y_0} < T)}{Q(T_{y_0} < T)} = 1, \]

and we deduce from Theorem C.31 that, in an open neighborhood of \( y_0 \), the scale function \( s \) is the difference of two convex functions. Recall that being the difference of two convex functions on an open (or closed) convex subset of a finite-dimensional Euclidean space is a local property (see [21, (I) on p. 707]). Hence, as \( y_0 \) was arbitrary, we conclude that the scale function \( s \) is the difference of two convex functions on \( J^\circ \). In particular, \( s \) has a right-continuous right-hand derivative (and a left-continuous left-hand one) and they are locally bounded on \( J^\circ \). With a little abuse of notation, we denote the right-hand
derivative of \( s \) by \( s' \). This notation is motivated by the fact that \( s \) is differentiable, as we prove below, cf. (3.2).

Next, let us understand the structure of \( Q \) in a more precise manner. We discuss the case \( J = [l, \infty) \) where \( l \) is accessible (and absorbing) under \( \mathbb{P}_{x_0} \). The inaccessible case \( J = (l, \infty) \) follows the same way. Notice that \( s'dm \) is a valid speed measure (recall Remark 2.5). Indeed, its local finiteness on \( J^o \) follows from the local boundedness of \( s' \) on \( J^o \), so we only need to verify (2.13). As \( s \) is strictly increasing, \( s' \) cannot have intervals of zeros. Therefore, for any \( a < b \) in \( J^o \), there exists an \( x \in [a, b] \) with \( s'(x) > 0 \), hence \( s' > 0 \) on some \( (x, x + \varepsilon) \subset [a, b] \) due to the right continuity of \( s' \), and this implies (2.13). Now take \( f \in C_b([l, \infty); \mathbb{R}) \) such that the following holds true: the restriction \( f|_{(l, \infty)} \) is a difference of two convex functions \( (l, \infty) \to \mathbb{R} \) and, denoting the right-hand derivative of \( f \) on \((l, \infty) \) by \( f'_+ \) (which necessarily has locally finite variation on \((l, \infty)) \), there exists a function \( g \in C_b([l, \infty); \mathbb{R}) \) with \( g(l) = 0 \) such that \( df'_+ = 2gs'dm \) on \((l, \infty) \) in the sense explained in (B.3). Let \( \{L^T_t(X): (t, x) \in \mathbb{R}_+ \times \mathbb{R} \} \) be the (continuous in \( t \) and right-continuous in \( x \)) semimartingale local time of the coordinate process \( X \) under \( \mathbb{P}_{x_0} \). Under \( \mathbb{P}_{x_0} \), we obtain, for all \( t < T_1 \),

\[
\begin{align*}
f(X_t) &= f(x_0) + \int_0^t \left( \frac{d-f}{dx} \right)(X_s) dX_s + \frac{1}{2} \int_0^t L_t^2(X) 2g(x) s'(x) m(dx) \\
&= f(x_0) + \int_0^t \left( \frac{d-f}{dx} \right)(X_s) dX_s + \int_0^t L_t^2(s(X)) g(x) m(dx) \\
&= f(x_0) + \int_0^t \left( \frac{d-f}{dx} \right)(X_s) dX_s + \int_{s(J^c)} L_t^2(s(X)) g(s^{-1}(x)) m \circ s^{-1}(dx) \\
&= f(x_0) + \int_0^t \left( \frac{d-f}{dx} \right)(X_s) dX_s + \int_0^t g(s^{-1}(X_s)) ds, \\
&= f(x_0) + \int_0^t \left( \frac{d-f}{dx} \right)(X_s) dX_s + \int_0^t g(X_s) ds,
\end{align*}
\]

where we use Lemma C.26 in the first and second line and Lemmata C.4 and C.15 (more precisely, formula (C.7)) in the fourth line. Now, the coordinate process \( X \) is a local \( Q \)-martingale and hence, by the above formula, the process

\[
f(X) - f(x_0) - \int_0^t g(X_s) ds
\]

is a local \( Q \)-martingale on \([0, T_1)\). Using [24, Proposition 5.9] and the fact that the process in (3.6) is bounded on any finite time interval, we obtain that it is a global \( Q \)-martingale. Consequently, we deduce \( Q = \tilde{\mathbb{P}}_{x_0} \) on \( \mathcal{U}_T \) from Lemmata B.4 and B.5.

In summary, if NFLVR holds, then \( \mathbb{P}_{x_0} \sim \tilde{\mathbb{P}}_{x_0} \) on \( \mathcal{U}_T \) and the unique ELMM is given by \( \tilde{\mathbb{P}}_{x_0} \). By virtue of Corollary 2.21 and Remark 2.22, \( \mathbb{P}_{x_0} \sim \tilde{\mathbb{P}}_{x_0} \) on \( \mathcal{U}_T \) if and only if all interior points are non-separating (which implies Condition 3.2) and in addition either \( l \) is non-separating (which means that Condition 3.3 holds) or \( \mathbb{P}_{x_0}, \tilde{\mathbb{P}}_{x_0} \) do not reach \( l \) (which means that Condition 3.4 holds). This proves the necessity of the conditions.

It remains to discuss the sufficiency. As explained above, if either Condition 3.3 or Condition 3.4 holds, then \( \mathbb{P}_{x_0} \sim \tilde{\mathbb{P}}_{x_0} \) on \( \mathcal{U}_T \). As \( X \) is a local \( \tilde{\mathbb{P}}_{x_0} \)-martingale (Lemma C.3), we can conclude that \( \mathbb{P}_{x_0} \) is an ELMM and hence, NFLVR holds by the fundamental theorem of asset pricing. The proof is complete.

**Remark 3.7.** Under the NFLVR condition, in case the asset represented by \( X \) gets bankrupt in the sense \( X \) reaches the boundary point \( l \), it remains there. Of course, this observation holds beyond our diffusion framework, as it is a consequence of the fact that non-negative local martingales are non-negative supermartingales, which cannot resurrect from zero.

**Remark 3.8.** In the Itô diffusion setting (see [34]) conditions (3.1)–(3.5) have the following financial interpretations: condition (3.1) means that the market price of risk is \( \mathbb{P}_{x_0}, \tilde{\mathbb{P}}_{x_0} \)-a.s. square integrable on
Theorem 3.9. If $T = \infty$, then NFLVR holds if and only if Condition 3.3 and $s(\infty) = \infty$ hold. Moreover, in case NFLVR holds, the unique ELMM is given by $\tilde{P}_{x_0}$, where $(x \mapsto \tilde{P}_x)$ is the diffusion with the interior of the state space $J^0$, characteristics $(\text{Id}, s'd\mu)$ on $J^0$ and the boundary point $l$ being absorbing whenever it is accessible.

For completeness, we notice that, as NFLVR with $T = \infty$ implies NFLVR with any $T < \infty$, the messages of Remark 3.6 apply also under NFLVR with $T = \infty$.

Proof of Theorem 3.9. Using the arguments explained in the proof of Theorem 3.5, it suffices to understand when $P_{x_0} \sim \tilde{P}_{x_0}$ on $\mathcal{F} (= \mathcal{G}_T$ as $T = \infty)$. By Corollary 2.23, in the case when $P_{x_0} \neq \tilde{P}_{x_0}$ (cf. Remark 2.24), we have $P_{x_0} \sim \tilde{P}_{x_0}$ on $\mathcal{F}$ if and only if all points in $J^0$ are non-separating (which means that Condition 3.2 holds), $l$ is non-separating (which upgrades Condition 3.2 to Condition 3.3) and $s(\infty) = \infty$. In the other case $P_{x_0} = \tilde{P}_{x_0}$, clearly, $P_{x_0} \sim \tilde{P}_{x_0}$ on $\mathcal{F}$, as well as both Condition 3.3 and $s(\infty) = \infty$ hold. This concludes the proof. \qed

Remark 3.10. (a) Theorem 3.9 shows that if NFLVR holds for $T = \infty$, then bankruptcy is certain on the long run, i.e., $P_{x_0}$-a.s. $\lim_{t \to \infty} X_t = l$. Indeed, Condition 3.3 means that $l$ is a non-separating boundary point for the diffusions $(x \mapsto P_x)$ and $(x \mapsto \tilde{P}_x)$, hence, by Discussion 2.20, we have $s(l) > -\infty$. Together with $s(\infty) = \infty$ this yields the statement via Lemma C.8.

(b) As pointed out by the referee, it is interesting to notice that, under NFLVR in the case $T = \infty$, the price process cannot be a uniformly integrable martingale under the ELMM $\tilde{P}_{x_0}$. Let us mention two ways to see this. First, by (a) and the equivalence $P_{x_0} \sim P_{x_0}$, we have $P_{x_0}$-a.s. $\lim_{t \to \infty} X_t = l$. Now, if $X$ were a uniformly integrable martingale under $\tilde{P}_{x_0}$, then we would obtain the contradiction $x_0 = E^{P_{x_0}}[X_t] \to l$ as $t \to \infty$. Alternatively, the fact can be deduced from [22, Theorem 1.1] because, as explained in [22, Example 3.1], Condition (A) from [22] is violated in diffusion settings with state space $[l, \infty)$.

Remark 3.11. The state spaces $[l, \infty)$ or $(l, \infty)$ seem to be economically interesting choices. Nevertheless, using identical arguments as in the proofs of Theorems 3.5 and 3.9, Theorem 2.18 also yields deterministic characterizations for NFLVR in case $J$ is any other interval in $\mathbb{R}$. The precise statements are left to the reader. We emphasize that for financial applications we need to consider $J \subset \mathbb{R}$ (as opposed to $J \subset [-\infty, \infty]$ in the general setting from Section 2.2), as this is necessary to define NFLVR (if $\infty \in J$ or $-\infty \in J$, then the coordinate process cannot be a semimartingale).

Remark 3.12. Theorems 3.5 and 3.9 include their counterparts [34, Theorems 3.1, 3.5] for the Itô diffusion setting. In the framework from [34] the scale function $s$ is assumed to be continuously differentiable with absolutely continuous derivative. Our results show that this assumption is necessary, which seems to be surprising at first glance. Although for NFLVR to hold the scale function has to have the same structure as in [34], our results apply for arbitrary speed measures, while in [34] these have to be absolutely continuous w.r.t. the Lebesgue measure. In particular, in contrast to the results in [34], ours cover diffusions with sticky points, which are interesting models in the presence of takeover offers, as explained the Introduction.

Example 3.13. (a) The famous Black–Scholes model with drift $\mu \in \mathbb{R}$ and volatility $\sigma \neq 0$ can be rephrased in our diffusion language by taking $J = (0, \infty)$ and

$$s(x) = \begin{cases} -\frac{x^{-2\nu}}{2\nu}, & \nu \neq 0, \\ \log(x), & \nu = 0, \end{cases} \quad m(dx) = \frac{\nu^{2\nu-1}}{\sigma^2} dx,$$
where \( \nu = \mu/\sigma^2 - 1/2 \). In this case Condition 3.2 holds with \( \beta(x) = -2\mu/(\sigma^2 x) \). Moreover, it is straightforward to check that Condition 3.4 is satisfied, while Condition 3.3 is violated whenever \( \mu \neq 0 \) (but satisfied in the case \( \mu = 0 \)). As a consequence, the Theorems 3.5 and 3.9 show that NFLVR holds for \( T < \infty \) but fails for \( T = \infty \) in the case \( \mu \neq 0 \) (and holds for \( T = \infty \) in the trivial case \( \mu = 0 \)). Of course, this recovers the very classical results from the literature.

(b) A classical counterexample in arbitrage theory is the three-dimensional Bessel process (see, e.g., [16]). In our language, this corresponds to the case \( J = (0, \infty) \), \( s(x) = -1/x \) and \( m(dx) = x^2 dx \). Condition 3.2 holds with \( \beta(x) = -2/x \), but, as the reader easily checks, both Condition 3.3 and Condition 3.4 are violated. As a consequence, Theorem 3.5 yields that NFLVR fails for any finite time horizon. Again, we recover the known results from the literature.

4. Proof of the Main Theorem

As homeomorphic space transformations do not affect the question of equivalence and singularity, we can assume that one of the diffusions of interest is on natural scale. More precisely, we assume that \( (x \mapsto \tilde{P}_x) \) is on natural scale, i.e., \( \tilde{s} = \text{Id} \). The general result then follows from Lemma C.4.

4.1. Some Preparations. In this subsection we collect some auxiliary results which are needed in the proof of Theorem 2.18. One of our main tools is a time-change argument which we learned from [37], where it was used to prove local equivalence for diffusions with open state space. Excluding, for a moment, the possibility of (instantaneously or slowly) reflecting boundaries, the idea is roughly speaking as follows: via a change of time we reduce certain questions related to equivalence of two diffusions to the same question for Brownian motions with and without (generalized) drift and possibly absorbing boundaries. This reduction brings us into the position to apply one of the main results from [8], i.e., that Theorem 2.18 holds in Itô diffusion settings with possibly absorbing boundaries. To treat (instantaneously and slowly) reflecting boundaries, we combine the time change argument with a symmetrization trick, which, roughly speaking, states that reflecting boundaries can be considered as interior points of a symmetrized diffusion. As equivalence of laws of diffusions is typically not preserved by symmetrization, this part of the proof requires some additional technical considerations.

4.1.1. Proof of Lemma 2.13. Suppose that \( b \in \partial J \) is half-good. As \( \tilde{s} = \text{Id} \), this implies that \( b \in \mathbb{R} \). By definition, there exists a non-empty open interval \( B \subseteq J^o \) with \( b \) as endpoint such that all points in \( B \) and the other endpoint of \( B \) are non-separating (good). By virtue of (i) and (ii) in Definition 2.6 and recalling our standing assumption \( \tilde{s} = \text{Id} \), it follows that the scale function \( s \) is differentiable on \( B \) (see [43, p. 204]) and that its derivative \( s' \) satisfies the equation

\[
 ds'(x) = \beta(x)s'(x)dx.
\]

This means that, up to increasing affine transformations,

\[
 s(x) = \int^x \exp \left( \int^y \beta(z)dz \right)dy, \quad x \in B.
\]

Take \( x_0 \in B \), let \( \tilde{P}_{x_0}^o \) be the law of a Brownian motion which is absorbed in the boundaries of \( B \) and let \( P_{x_0}^o \) be the law of a diffusion started at \( x_0 \) absorbed in accessible boundaries of \( B \) with scale function \( s \) and speed measure \( m^o(dx) = dx/s'(x) \) on \( \mathcal{B}(B) \). We deduce from Lemma C.25 that \( P_{x_0}^o \sim \tilde{P}_{x_0}^o \). In particular, this means that \( P_{x_0} \) and \( \tilde{P}_{x_0} \) have the same state space \( \text{cl}(B) \). Next, we transfer this equivalence to the (stopped) diffusions \( P_{x_0} \circ X_{\lambda}^1 \) and \( \tilde{P}_{x_0} \circ X_{\lambda}^1 \), where \( \lambda \triangleq \inf(t \geq 0: X_t \notin B) \). Define

\[
 \tilde{v}^*(t, x) \triangleq \begin{cases} \limsup_{h \downarrow 0} \frac{\int_0^h 1(x-h<x<s+h)ds}{2h} & \text{if } x \in B, \\ 0 & \text{if } x \in \partial B, \end{cases}
\]

and set

\[
 g(t) \triangleq \begin{cases} \int_B \tilde{v}^*(t, x)m(dx), & t < T, \\ \infty, & t \geq T. \end{cases}
\]
Further, let \( g^{-1} \) be the right-inverse of \( g \), i.e., \( g^{-1}(t) \triangleq \inf(s \geq 0: g(s+) > t), t \in \mathbb{R}_+ \). Then, according to the chain rule for diffusions (Theorem C.24), we get that

\[
\hat{P}^o_{x_0} \circ X_{g^{-1}(\cdot)} = \hat{P}^o_{x_0} \circ X_{\Lambda T}.
\]

Set

\[
\hat{\ell}^*(t, x) \triangleq \begin{cases} 
\limsup_{h \to 0} \frac{\int_0^h \mathbb{1}_{\{x-h<X_s<x+h\}} ds}{m'(x-h,x+h)} & \text{if } x \in B, \\
0 & \text{if } x \in \partial B.
\end{cases}
\]

Now, for \( t < T \), using part (iii) of Definition 2.6, together with Lemma 2.9, we obtain

\[
G(t) = \int_B \hat{\ell}^*(t,x) \hat{m}(dx) = \int_B \ell^*(t,x) \left( \frac{dm^o}{dx} \right)(x) \hat{m}(dx) = \int_B \ell^*(t,x) \hat{m}(dx).
\]

Hence, again by the chain rule, we have

\[
\hat{P}^o_{x_0} \circ X_{g^{-1}(\cdot)} = \hat{P}^o_{x_0} \circ X_{\Lambda T}.
\]

Take \( G \in \mathcal{F} \) such that \( \hat{P}^o_{x_0}(X_{\Lambda T} \in G) = 0 \). Then, by (4.4), also \( P^o_{x_0}(X_{g^{-1}(\cdot)} \in G) = 0 \), which yields that \( \hat{P}^o_{x_0}(X_{g^{-1}(\cdot)} \in G) = 0 \), by the equivalence of \( P^o_{x_0} \) and \( \hat{P}^o_{x_0} \) on \( \mathcal{F} \), and finally, using (4.2), we get \( \hat{P}^o_{x_0}(X_{\Lambda T} \in G) = 0 \). Conversely, if \( G \in \mathcal{F} \) is such that \( \hat{P}^o_{x_0}(X_{\Lambda T} \in G) = 0 \), then we get \( \hat{P}^o_{x_0}(X_{\Lambda T} \in G) = 0 \) by a similar reasoning. Thus, we conclude that \( \hat{P}^o_{x_0} \circ X_{\Lambda T} \sim \hat{P}^o_{x_0} \circ X_{\Lambda T} \). As an equivalent change of measure does not change the state space, the claim of Lemma 2.13 follows. \( \square \)

4.1.2. Criteria for Equivalence. In the following, we investigate equivalence up to a hitting time. Recall that \( J_{\text{sep}} \) denotes the set of separating points.

**Lemma 4.1.** Suppose that \( x_0 \in \mathcal{J} \) and that \( a, c \in \mathcal{J} \) are such that \( a < x_0 < c \) and \( [a,c] \subset \mathcal{J} \setminus J_{\text{sep}} \). Then, \( \hat{P}^o_{x_0} \sim \hat{P}^o_{x_0} \) on \( \mathcal{F}_{T_a \wedge T_c} \).

**Proof.** Take \( a' < a \) and \( c' > c \) such that \( [a',c'] \subset \mathcal{J} \setminus J_{\text{sep}} \). We stress that \( a' \) and \( c' \) exist as \( \mathcal{J} \setminus J_{\text{sep}} \) is open. To simplify our notation, we set \( \Delta_x \) as in the proof of Lemma 2.13, \( \delta' \). Further, as in the proof of Lemma 2.13, \( \mathcal{F} \) has the representation (4.1) on \( [a',c'] \), while, due to (2.18), \( m^o(dx) = dx/s'(x) \) on \( [a',c'] \). Therefore, Lemma 2.25 applies and yields \( \hat{P}^o_{x_0} \sim \hat{P}^o_{x_0} \).

We now deduce the claim of the lemma from this equivalence. We use a refined version of the time-change argument from the proof of Lemma 2.13. Define

\[
\hat{\ell}^*(t, x) \triangleq \begin{cases} 
\limsup_{h \to 0} \frac{\int_0^h \mathbb{1}_{\{x-h<X_s<x+h\}} ds}{2h} & \text{if } x \in (a',c'), \\
0 & \text{if } x \in [a',c'),
\end{cases}
\]

and set

\[
G(t) \triangleq \int_0^c \hat{\ell}^*(t,x) \hat{m}(dx), \quad t < T',
\]

\[
\epsilon, \quad t \geq T'.
\]
Further, \( g^{-1}(t) = \inf(s \geq 0: g(s+) > t) \) denotes the right-inverse of \( g \). According to the chain rule for diffusions (Theorem C.24), we get that \( \tilde{P}^0_x \circ X^{-1}_g(t) = P_x \circ X^{-1}_{g^{-1}(t)} \). Set

\[
\ell^*(t, x) \triangleq \begin{cases} 
\limsup_{h \searrow 0} \frac{\int_{h}^{0} 1_{[x-h, x+h]} dx}{m([x-h, x+h])} & \text{if } x \in (a', c'), \\
0 & \text{if } x \in \{a', c'\}.
\end{cases}
\]

Now, for \( t < T' \), as in (4.3), we obtain

\[
g(t) = \int_{a'}^{c'} \ell^*(t, x) m(dx).
\]

Hence, again by the chain rule, we have \( P_x \circ X^{-1}_{g^{-1}(t)} = P_x \circ X^{-1}_{g^{-1}(t)} \). Finally, take \( G \in \mathcal{F}_{T_a \wedge T_c} \) such that \( P_{x_0}(G) = 0 \). Using Galmarino’s test in the form [23, 10 c), p. 87] (or adjusting [39, Exercise I.4.21] for the right-continuous filtration \( (\mathcal{F}_t)_{t \geq 0} \)), we obtain

\[
0 = P_{x_0}(G) = P_{x_0}(X_{T'} \in G) = P^0_{x_0}(X_{g^{-1}(t)} \in G).
\]

Since \( P^0_{x_0} \sim \tilde{P}^0_{x_0} \), we obtain

\[
0 = \tilde{P}^0_{x_0}(X_{g^{-1}(t)} \in G) = \tilde{P}_{x_0}(X_{T'} \in G) = \tilde{P}_{x_0}(G),
\]

which proves \( \tilde{P}_{x_0} \ll P_{x_0} \) on \( \mathcal{F}_{T_a \wedge T_c} \). The reverse absolute continuity follows by a similar reasoning. This completes the proof. \( \square \)

A minor variation of argument for Lemma 4.1 also shows the next lemma. We leave the details to the reader. Recall that \( l = \inf J(= \inf \tilde{J}) \) and \( r = \sup J(= \sup \tilde{J}) \).

**Lemma 4.2.** Suppose that the left boundary \( l \) is non-separating and that it is either inaccessible or absorbing for one (equivalently, for both) of the diffusions. Further, assume that \( c \in (x_0, r) \) is such that all points in \( [l, c] \) are non-separating. Then, \( P_{x_0} \sim \tilde{P}_{x_0} \) on \( \mathcal{F}_{T_c} \).

Clearly, there is also a version of Lemma 4.2 for the right boundary point.

### 4.1.3. Singularity under time transformations.

In the following, we give a result, which shows that singularity propagates through certain changes of time.

**Lemma 4.3.** Let \( P^0_{x_0} \) and \( \tilde{P}^0_{x_0} \) be as in the first step of the proof of Lemma 4.1 with some \( a', c' \in J^0 \), \( a' < x_0 < c' \). If \( P^0_{x_0} \perp \tilde{P}^0_{x_0} \) on \( \mathcal{F}_0 \), then \( P_{x_0} \perp \tilde{P}_{x_0} \) on \( \mathcal{F}_0 \).

**Proof.** For \( t \in \mathbb{R}_+ \), define

\[
\tilde{\ell}(t, x) \triangleq \begin{cases} 
\limsup_{h \searrow 0} \frac{\int_{h}^{0} 1_{[x-h, x+h]} dx}{m([x-h, x+h])} & \text{if } x \in (a', c'), \\
0 & \text{if } x \in \{a', c'\},
\end{cases}
\]

\[
f(t) \triangleq \begin{cases} 
\int_{a'}^{c'} \tilde{\ell}(t, x) dx, & t < T' = T_{a'} \wedge T_{c'}, \\
\infty, & t \geq T = T_{a'} \wedge T_{c'},
\end{cases}
\]

and \( f^{-1}(t) \triangleq \inf(s \geq 0: f(s+) > t) \). The chain rule for diffusions as given by Theorem C.24 yields that

\[
P^0_{x_0} = P_{x_0} \circ X^{-1}_{f^{-1}(\cdot)}, \quad \tilde{P}^0_{x_0} = \tilde{P}_{x_0} \circ X^{-1}_{f^{-1}(\cdot)}.
\]

Take \( G \in \mathcal{F}_0 \). By definition, for every \( t > 0 \), we have \( G \in \sigma(X_s, s \leq t) \) and hence,

\[
\{X_{f^{-1}(\cdot)} \in G\} \in \sigma(X_{f^{-1}(\cdot)}, s \leq t) \subset \mathcal{F}_{f^{-1}(t)}.
\]

Consequently, we get

\[
\{X_{f^{-1}(\cdot)} \in G\} \in \bigcap_{t > 0} \mathcal{F}_{f^{-1}(t)} = \mathcal{F}_{\inf_{t > 0} f^{-1}(t)} = \mathcal{F}_{f^{-1}(0)},
\]

and hence

\[
P^0_{x_0} \perp \tilde{P}^0_{x_0} \text{ on } \mathcal{F}_0.
\]
see [26, Lemma 9.3] for the first equality. Thus, \( \{X_{t-1}(\cdot) \in G, f^{-1}(0) = 0 \} \in \mathcal{F}_0 \). Now, assume that \( \mathbb{P}^\circ \perp \tilde{\mathbb{P}}^\circ \) on \( \mathcal{F}_0 \). Then, there exists a set \( G \in \mathcal{F}_0 \) such that \( \mathbb{P}^\circ_{x_0}(G) = 1 - \tilde{\mathbb{P}}^\circ_{x_0}(G) = 0 \). Hence, using (4.6), we get
\[
\mathbb{P}_{x_0}(X_{t-1}(\cdot) \in G, f^{-1}(0) = 0) \leq \mathbb{P}_{x_0}(X_{t-1}(\cdot) \in G) = \mathbb{P}^\circ_{x_0}(G) = 0,
\]
and
\[
\tilde{\mathbb{P}}_{x_0}(X_{t-1}(\cdot) \notin G \text{ or } f^{-1}(0) \neq 0) = \tilde{\mathbb{P}}_{x_0}(X_{t-1}(\cdot) \notin G) = 1 - \tilde{\mathbb{P}}^\circ_{x_0}(G) = 0.
\]
Here, we also used that \( \tilde{\mathbb{P}}_{x_0}(f^{-1}(0) = 0) = 1 \) by Lemma C.18. We conclude that \( \mathbb{P}_{x_0} \perp \tilde{\mathbb{P}}_{x_0} \) and the proof is complete.

4.1.4. Criteria for Singularity. In Section 4.1.2 we studied criteria for equivalence up to hitting times and in Section 4.1.3 we provided some preliminary observations in the direction of singularity. In the following, we take a more complete look at singularity. The following lemma is a preparatory result.

**Lemma 4.4.** Let \( S \) be the separating time for \( \mathbb{P}_{x_0} \) and \( \tilde{\mathbb{P}}_{x_0} \) with \( x_0 \in J \cap \tilde{J} \). If \( \mathbb{P}_{x_0}(S > 0) > 0 \), then there exists a stopping time \( \xi \) such that \( \mathbb{P}_{x_0}, \tilde{\mathbb{P}}_{x_0} \)-a.s. \( 0 < \xi < S \land \infty \).

**Proof.** First of all, since \( \{S > 0\} \in \mathcal{F}_0 \), Blumenthal’s zero-one law (Lemma C.2) yields that the event \( \{S > 0\} \) has \( \mathbb{P}_{x_0} \)-probability zero or one and \( \mathbb{P}_{x_0} \)-probability zero or one. Consequently, by Proposition 2.3, \( \mathbb{P}_{x_0}(S > 0) = \tilde{\mathbb{P}}_{x_0}(S > 0) = 1 \). Let \( S' \triangleq S \land \infty \). Clearly, as \( S \) is an extended stopping time, \( S' \) is a stopping time. By Lemma C.23, i.e., Meyer’s theorem on predictability, any stopping time coincides \( \mathbb{P}_{x_0} \)-a.s. with a predictable time. Let \( S_1', S_2', \ldots \) be an announcing sequence for \( S' \) under \( \mathbb{P}_{x_0} \), i.e., \( S_1', S_2', \ldots \) is an increasing sequence of stopping times such that \( \mathbb{P}_{x_0} \)-a.s. \( S_n' \to S' \) and \( S_n' < S' \) for all \( n \in \mathbb{N} \). There exists an \( N \in \mathbb{N} \) such that \( \mathbb{P}_{x_0}(S_N' > 0) > 0 \), because \( \mathbb{P}_{x_0} \)-a.s. \( S_n' \to S' > 0 \). Using again Blumenthal’s zero-one law, we get that \( \mathbb{P}_{x_0}(S_N' > 0) = 1 \). Now, \( \xi_1 \triangleq S_N' \) satisfies \( \mathbb{P}_{x_0} \)-a.s. \( 0 < \xi_1 < S' \). In the same manner, we obtain the existence of a stopping time \( \xi_2 \) such that \( \mathbb{P}_{x_0} \)-a.s. \( 0 < \xi_2 < S' \). Finally, set \( \xi \triangleq \xi_1 \land \xi_2 \). We clearly have \( \mathbb{P}_{x_0}, \tilde{\mathbb{P}}_{x_0} \)-a.s. \( \xi < S' \). It remains to check that also \( \mathbb{P}_{x_0}, \tilde{\mathbb{P}}_{x_0} \)-a.s. \( \xi > 0 \). Since \( \mathbb{P}_{x_0}, \tilde{\mathbb{P}}_{x_0} \)-a.s. \( S > 0 \), we get from Proposition 2.3 that \( \mathbb{P}_{x_0} \sim \tilde{\mathbb{P}}_{x_0} \) on \( \mathcal{F}_0 \). Thus, since \( \{\xi_1 > 0\} \in \mathcal{F}_0 \), \( \mathbb{P}_{x_0}(\xi_1 > 0) = 1 \) implies \( \tilde{\mathbb{P}}_{x_0}(\xi_1 > 0) = 1 \), and similarly, \( \mathbb{P}_{x_0}(\xi_2 > 0) = 1 \) follows from \( \tilde{\mathbb{P}}_{x_0}(\xi_2 > 0) = 1 \). In summary, we have \( \mathbb{P}_{x_0}, \tilde{\mathbb{P}}_{x_0} \)-a.s. \( \xi > 0 \) and the proof is complete.

**Lemma 4.5.** Suppose that \( x_0 \in J^c \) and consider the following three conditions:

(a) for every open neighborhood \( U(x_0) \subset J^c \) of \( x_0 \) either there exists a point \( z \in U(x_0) \) such that the differential quotient \( dm(z)/d\tilde{m} \) does not exist, or \( dm/d\tilde{m} \) exists on \( U(x_0) \) but not as a strictly positive continuous function;
(b) for every open neighborhood \( U(x_0) \subset J^c \) of \( x_0 \) either there exists a point \( z \in U(x_0) \) such that the differential quotient \( d^+s(z)/dx \) does not exist, or \( d^+s/dx \) exists on \( U(x_0) \) but not as a strictly positive absolutely continuous function such that \( ds'(z) = s'(z)\beta(z)dz \) with \( \beta \in L^2(U(x_0)) \);
(c) there exists an open neighborhood \( U(x_0) \subset J^c \) of \( x_0 \) such that the differential quotients \( dm/d\tilde{m} \) and \( d^+s/dx \) exist on \( U(x_0) \) and for all sub-neighborhoods \( V(x_0) \subset U(x_0) \) with \( x_0 \in V(x_0) \) there exists a point \( z \in V(x_0) \) such that \( dm(z)/d\tilde{m} \cdot d^+s(z)/dx \neq 1 \).

Then, the following equivalent statements hold:

(i) If at least one of the conditions (a)–(c) above holds, then \( \mathbb{P}_{x_0} \perp \tilde{\mathbb{P}}_{x_0} \) on \( \mathcal{F}_0 \).
(ii) If \( \mathbb{P}_{x_0} \sim \tilde{\mathbb{P}}_{x_0} \) on \( \mathcal{F}_0 \), then all three conditions (a)–(c) above are violated.

**Proof.** First, notice that (i) and (ii) are equivalent, because, as a consequence of Blumenthal’s zero-one law (Lemma C.2), either \( \mathbb{P}_{x_0} \perp \tilde{\mathbb{P}}_{x_0} \) on \( \mathcal{F}_0 \) or \( \mathbb{P}_{x_0} \sim \tilde{\mathbb{P}}_{x_0} \) on \( \mathcal{F}_0 \). In the following, we will prove (ii), i.e., we assume that \( \mathbb{P}_{x_0} \sim \tilde{\mathbb{P}}_{x_0} \) on \( \mathcal{F}_0 \) and prove that (a)–(c) are violated.

In view of Proposition 2.3, this means that \( \mathbb{P}_{x_0}, \tilde{\mathbb{P}}_{x_0} \)-a.s. \( S > 0 \), where \( S \) denotes the separating time for \( \mathbb{P}_{x_0} \) and \( \tilde{\mathbb{P}}_{x_0} \). Thanks to Lemma 4.4, there exists a stopping time \( \xi \) such that \( \mathbb{P}_{x_0}, \tilde{\mathbb{P}}_{x_0} \)-a.s. \( 0 < \xi < S \land \infty \) and, by the definition of separating time, \( \mathbb{P}_{x_0} \sim \tilde{\mathbb{P}}_{x_0} \) on \( \mathcal{F}_\xi \).
Let \( \{ L^f_t(X) : (t, x) \in \mathbb{R}_+ \times \mathbb{R} \} \) be the on \( \mathbb{R}_+ \times J^o \) jointly continuous \( \mathbb{P}_{x_0} \)-modification of the semimartingale local time of \( X \) and let \( \{ L^f_t(s(X)) : (t, x) \in \mathbb{R}_+ \times \mathbb{R} \} \) be the on \( \mathbb{R}_+ \times s(J^o) \) jointly continuous \( \mathbb{P}_{x_0} \) -modification of the semimartingale local time of \( s(X) \), see Lemmata C.4 and C.15 and Remark C.16 (a). We define \( G = G_1 \cap G_2 \cap G_3 \), where
\[
G_1 \triangleq \left\{ 0 < \xi < \infty, L^z_\xi(X)(s(X)) > 0 \right\},
\]
\[
G_2 \triangleq \left\{ 0 < \xi < \infty, \int_0^\xi f(X_s)ds = \int f(x)L^z_\xi(X)\tilde{m}(dx) \forall f \in \mathcal{B}_0^+ \right\},
\]
\[
G_3 \triangleq \left\{ 0 < \xi < \infty, \int_0^\xi f(X_s)ds = \int f(x)L^z_\xi(s(X))\tilde{m}(dx) \forall f \in \mathcal{B}_0^+ \right\}
\]
and \( \mathcal{B}_0^+ \triangleq \{ f : \text{cl}(J) \to \mathbb{R}_+, \text{Borel}, f|_{\partial J} = 0 \} \). By Lemmata C.15 and C.18 and Remark C.16 (b), the equivalence \( \mathbb{P}_{x_0} \sim \mathbb{P}_{x_0} \) on \( J^o \) and the fact that \( \mathbb{P}_{x_0} \), \( \mathbb{P}_{x_0} \)-a.s. \( 0 < \xi < \infty, \) we have \( \mathbb{P}_{x_0}(G) = \mathbb{P}_{x_0}(G) = 1 \), which in particular implies that \( G \neq \emptyset \). We fix some \( \omega \in G \). As the functions
\[
x \mapsto L^z_\xi(X)(\omega) \quad \text{and} \quad x \mapsto L^z_\xi(s(X))(\omega)
\]
are continuous on \( J^o \), using the definition of \( G_1 \), we can find an open neighborhood \( V(x_0) \) of \( x_0 \) such that
\[
L^z_\xi(X)(\omega), L^z_\xi(s(X))(\omega) > 0 \quad \forall x \in V(x_0).
\]
Using the definition of \( G_2 \) and \( G_3 \), for every \( z \in V(x_0) \), we get, as \( h \downarrow 0 \),
\[
\int_0^\xi \mathbb{I}\{ z - h < X_s(\omega) < z + h \}ds \frac{m((z-h, z+h))}{\tilde{m}((z-h, z+h))} \to L^z_\xi(X)(\omega) > 0,
\]
and
\[
\int_0^\xi \mathbb{I}\{ z - h < X_s(\omega) < z + h \}ds \frac{m((z-h, z+h))}{\tilde{m}((z-h, z+h))} \to L^z_\xi(s(X))(\omega) > 0,
\]
which shows that
\[
(4.7) \quad \left( \frac{dm}{d\tilde{m}} \right)(z) = \frac{L^z_\xi(X)(\omega)}{L^z_\xi(s(X))(\omega)} > 0.
\]
Thanks to the continuity of the function
\[
V(x_0) \ni z \mapsto \frac{L^z_\xi(X)(\omega)}{L^z_\xi(s(X))(\omega)}
\]
we conclude that \( dm/d\tilde{m} \) exists as a strictly positive continuous function on \( V(x_0) \), i.e., (a) is violated.

Next, we show that (b) is violated, too. Let \( \mathbb{P}^o_{x_0} \) and \( \mathbb{P}^o_{x_0} \) be as in the first step of the proof of Lemma 4.1
with \( (a', c') = V(x_0) \). By virtue of Lemma 4.3 and Blumenthal’s zero-one law, \( \mathbb{P}_{x_0} \sim \mathbb{P}_{x_0} \) on \( \mathcal{F}_0 \) implies
that \( \mathbb{P}^o_{x_0} \sim \mathbb{P}^o_{x_0} \) on \( \mathcal{F}_0 \). Therefore, to simplify our notation, we can and will w.l.o.g. assume that \( \mathbb{P}_{x_0} \) is the law of a Brownian motion stopped at the boundaries of an open neighborhood \( V(x_0) \) of \( x_0 \) and that \( \mathbb{P}_{x_0} \) is the law of a diffusion with scale function \( s \) and speed measure
\[
\frac{m^o(dx)}{dx} \triangleq \left( \frac{dm}{d\tilde{m}} \right)(x)
\]
also stopped at the boundaries of \( V(x_0) \). These assumptions are in force for the remainder of this proof.

By Lemma C.3, \( Y \triangleq s(X_\xi) \) is a \( \mathbb{P}_{x_0} \)-semimartingale (in fact, \( Y \) is even a local \( \mathbb{P}_{x_0} \)-martingale). Consequently, as \( \mathbb{P}_{x_0} \sim \mathbb{P}_{x_0} \) on \( \mathcal{F}_\xi \), Girsanov’s theorem (Lemma C.30) yields that \( Y \) is also a \( \mathbb{P}_{x_0} \)-semimartingale. As \( \mathbb{P}_{x_0} \) is the law of a Brownian motion with absorbing boundaries, thanks to Theorem C.31, possibly making \( V(x_0) \) a bit smaller, we get that \( s \) is the difference of two convex functions on \( V(x_0) \). In particular, \( d^o \bar{s}/dx \equiv \bar{s}' \) exists as a right-continuous function of (locally) finite variation \([10, \text{Proposition 5.1}]\). Let \( a < c \) denote the boundary points of \( V(x_0) \). Take \( a < a' < x_0 < c' < c \) and set
$T \triangleq T_{a'} \wedge T_{c'}$. Since $\mathbb{P}_{x_0} \sim \tilde{\mathbb{P}}_{x_0}$ on $\mathcal{F}_\xi$ and because $\tilde{\mathbb{P}}_{x_0}$ is the law of a Brownian motion stopped at the boundaries of $V(x_0)$, Girsanov's theorem (Lemma C.30) shows that under $\mathbb{P}_{x_0}$ the stopped process $X_{\wedge \xi \wedge T}$ is a Brownian motion stopped at $\xi \wedge T$ with additional drift $\int_0^{\xi \wedge T} \tilde{\beta}_s \, ds$ for a predictable process $\tilde{\beta}$ such that $\mathbb{P}_{x_0}, \tilde{\mathbb{P}}_{x_0}$-a.s.

$$\int_0^{\xi \wedge T} \tilde{\beta}_s^2 \, ds < \infty$$

for all $t \in \mathbb{R}_+$. In other words, under $\mathbb{P}_{x_0}$, we have, informally,

$$X_{\wedge \xi \wedge T} = x_0 + \int_0^{\wedge \xi \wedge T} \tilde{\beta}_s \, ds + W_{\wedge \xi \wedge T},$$

where $W$ is a standard Brownian motion. Let $\{L^x_\tau(X): (t,x) \in \mathbb{R}_+ \times \mathbb{R}\}$ be a jointly continuous modification\(^6\) of the semimartingale local time of $X$ under $\mathbb{P}_{x_0}$. Then, as $\mathcal{F}$ is the difference of two convex functions on $[a', c'] \subset (a, c)$, the generalized Itô formula (Lemma C.26 and Remark C.27) yields that $\mathbb{P}_{x_0}$-a.s.

$$\mathbb{F}(X_{\wedge \xi \wedge T}) = \mathbb{F}(x_0) + \int_0^{\wedge \xi \wedge T} \mathbb{F}'(X_s) \, ds + \frac{1}{2} \int_0^{\wedge \xi \wedge T} \mathbb{F}'(X_s) \tilde{\beta}_s \, ds$$

where $\mathbb{F}''(dx)$ denotes the (signed) second derivative measure of $\mathcal{F}$ defined on $(a', c']$ by $\mathbb{F}''((x, y]) = \mathbb{F}'(y) - \mathbb{F}'(x)$ for all $a' \leq x \leq y \leq c'$. Of course, by the equivalence of $\mathbb{P}_{x_0}$ and $\tilde{\mathbb{P}}_{x_0}$ on $\mathcal{F}_\xi$, this equality also holds under $\mathbb{P}_{x_0}$. Thus, under $\mathbb{P}_{x_0}$, we get

$$\mathbb{F}(X_{\wedge \xi \wedge T}) - \int_0^{\wedge \xi \wedge T} \mathbb{F}'(X_s) \tilde{\beta}_s \, ds = \mathbb{F}(x_0) + \int_0^{\wedge \xi \wedge T} \mathbb{F}'(X_s) \, ds - \int_0^{\wedge \xi \wedge T} \mathbb{F}''(X_s) \, ds = \text{local } \mathbb{P}_{x_0}$-martingale.$$

By Lemma C.3, the stopped process $\mathbb{F}(X_{\wedge \xi \wedge T})$ is a local $\mathbb{P}_{x_0}$-martingale, too. Hence, using the fact that continuous local martingales of (locally) finite variation are constant, we obtain that $\mathbb{P}_{x_0}$-a.s., and hence also $\tilde{\mathbb{P}}_{x_0}$-a.s.,

(4.8) \[ \int_0^{\wedge \xi \wedge T} L^x_\tau(X) \mathbb{F}''(dx) = - \int_0^{\wedge \xi \wedge T} 2\mathbb{F}'(X_s) \tilde{\beta}_s \, ds. \]

Computing the variation of both sides (see [36, pp. 915–916] for a detailed computation of the variation of the first integral) yields that $\tilde{\mathbb{P}}_{x_0}$-a.s.

$$\int_0^{\wedge \xi \wedge T} L^x_\tau(X) |\mathbb{F}''|(dx) = \int_0^{\wedge \xi \wedge T} 2|\mathbb{F}'(X_s)\tilde{\beta}_s| \, ds,$$

where $|\mathbb{F}''|$ denotes the variation measure of the measure $\mathbb{F}''$. Let $\mathcal{N}$ be the collection of all Lebesgue null sets in $B((a', c')]$. The occupation time formula (Lemma C.26) yields that $\tilde{\mathbb{P}}_{x_0}$-a.s., for all $G \in \mathcal{N}$,

$$\int_0^{\wedge \xi \wedge T} \mathbb{1}_G(X_s) \, ds = 0 \quad \text{and} \quad \int_0^{\wedge \xi \wedge T} \mathbb{1}_G(X_s) \mathbb{1}_{\{X_s = x\}} \, ds \mathbb{E}_W(L^x_\tau(X)) |\mathbb{F}''|(dx) = 0.$$

Moreover, we compute that $\tilde{\mathbb{P}}_{x_0}$-a.s. for all $G \in \mathcal{N}$

$$\int_0^{\wedge \xi \wedge T} \mathbb{1}_G(X_s) \mathbb{E}_W(L^x_\tau(X)) |\mathbb{F}''|(dx) = \int_0^{\wedge \xi \wedge T} \mathbb{1}_G(X_s) \mathbb{E}_W(L^x_\tau(X)) |\mathbb{F}''|(dx).$$

\(^6\)Notice that $X$ is a continuous $\tilde{\mathbb{P}}_{x_0}$-martingale, as it is a stopped Brownian motion under $\tilde{\mathbb{P}}_{x_0}$, and recall [39, Theorem VI.1.7].

\(^7\)Recall that here $\mathbb{F}'$ denotes the right-hand derivative, so we need the left-continuous (in the space variable) local time process in the generalized Itô formula. But in the present context the local time process is jointly continuous.
where we use Lemma C.26 for the last two lines. Together, we get \( \bar{P}_{x_0} \)-a.s. \( \int_G L_{\xi(T)}(X)|\omega''|(dx) = 0 \) for all \( G \in \mathcal{N} \). By the above observations, Lemma C.18 and the fact that \( \bar{P}_{x_0} \)-a.s. \( \xi \in (0, \infty) \), for \( \bar{P}_{x_0} \)-a.s. \( \omega \in \Omega \), we have

\begin{equation}
\begin{cases}
0 < \xi(\omega) < \infty, \\
L_{\xi(T)}(X)(\omega) > 0, \\
\int_G L_{\xi(T)}(X)(\omega)|\omega''|(dx) = 0 \quad \forall \ G \in \mathcal{N}.
\end{cases}
\end{equation}

Take an \( \omega \in \Omega \) such that (4.9) holds. As \( x \mapsto L_{\xi(T)}(X)(\omega) \) is continuous, there exists an open neighborhood \( V^o(x_0) \subset (a', c') \) of \( x_0 \) such that \( L_{\xi(T)}(X)(\omega) > 0 \) for all \( x \in V^o(x_0) \). Together with the third part of (4.9), we conclude that \( \omega''(dx) \ll dx \) on \( B(V^o(x_0)) \). Thus, \( \omega'' \) is an absolutely continuous function on \( V^o(x_0) \). Let \( \{ L_t^z(s(X)) : (t, x) \in \mathbb{R}_+ \times \mathbb{R} \} \) be a jointly continuous modification\(^8\) of the semimartingale local time of \( s(X) \) under \( \bar{P}_{x_0} \). Then, \( \bar{P}_{x_0} \)-a.s. for all \( (t, z) \in \mathbb{R}_+ \times J \) we have

\begin{equation}
L_{\xi(T)}(s(X)) = L_t^z(s(X)_{\Lambda \Delta T}).
\end{equation}

As the local time process is preserved under an absolutely continuous measure change (which is a consequence of [39, Corollary VI.1.9]) and \( \bar{P}_{x_0} \sim \bar{P}_{x_0} \) on \( \mathcal{F}_\xi \), it follows that (4.10) holds also \( \bar{P}_{x_0} \)-a.s. for all \( (t, z) \in \mathbb{R}_+ \times J \). Further, \( \bar{P}_{x_0} \)-a.s. for all \( (t, z) \in \mathbb{R}_+ \times (a', c') \) we have

\begin{equation}
L_t^z(s(X)_{\Lambda \Delta T}) = \omega''(z)L_t^z(s(X)_{\Lambda \Delta T}) = \omega''(z)L_t^z(s(X)_{\Lambda \Delta T}),
\end{equation}

where in the first equality we use part (ii) of Lemma C.26 (recall that \( s \) is the difference of two convex functions on \( V(x_0) = (a, c) \), that \( a < a' < x_0 < c' < c \) and that \( T = T_{a'} \wedge T_c \)). In particular, (4.10) and (4.11) imply that \( \bar{P}_{x_0} \)-a.s. for all \( z \in (a', c') \)

\begin{equation}
L_{\xi(T)}(s(X)) = \omega''(z)L_{\xi(T)}(X).
\end{equation}

By Lemma C.18, the equivalence \( \bar{P}_{x_0} \sim \bar{P}_{x_0} \) on \( \mathcal{F}_\xi \) and the continuity of the local times in the space variable, we can take an \( \omega \in \Omega \) such that (4.12) holds for all \( z \in (a', c') \) and the functions

\[ z \mapsto L_{\xi(T)}(X)(\omega) \quad \text{and} \quad z \mapsto L_t^z(s(X)_{\Lambda \Delta T})(\omega) \]

are strictly positive in a neighborhood of \( x_0 \). It follows that \( \omega'' > 0 \) in a sufficiently small open neighborhood \( V^s(x_0) \subset (a', c') \) of \( x_0 \). We now define the open neighborhood \( U(x_0) \equiv V^s(x_0) \cap V^*(x_0) \) of \( x_0 \) and observe that \( \omega'' \) is a strictly positive absolutely continuous function on \( U(x_0) \). Let \( \zeta : U(x_0) \to \mathbb{R} \) be a Borel function such that \( \omega''(dx) = \zeta(x)dx \) on \( B(U(x_0)) \) (in other words, \( \zeta \) equals the second derivative \( \omega'' \) almost everywhere on \( B(U(x_0)) \) w.r.t. the Lebesgue measure). Set

\begin{equation}
H \equiv \inf(t \geq 0 : X_t \not\in U(x_0)).
\end{equation}

Recalling (4.8) and using the occupation time formula (Lemma C.26), we get \( \bar{P}_{x_0} \)-a.s.

\[ \int_0^{\wedge \xi \wedge H} \zeta(X_s)ds = \int L_{\xi(T)}(X)(\xi(x)dx) = \int L_{\xi(T)}(X)(\omega''(dx) = -\int_0^{\wedge \xi \wedge H} 2\zeta(X_s)\beta(s)ds, \]

which implies that \( \bar{P}_{x_0} \)-a.s. \( \beta_s = -\zeta(X_s)/(2\zeta(X_s)) \equiv -\beta(X_s)/2 \) for \( \mu_L \)-a.a. \( s \leq \xi \wedge H \) with \( \beta = \zeta/\omega'' \) (cf. the formulation of part (b)). Thanks to the (mentioned above) square integrability of \( \beta \), which follows from Girsanov’s theorem, we obtain

\[ \bar{P}_{x_0} \left( \int_0^{\wedge \xi \wedge H} (\beta(X_s))^2ds < \infty \right) = 1, \quad t \in \mathbb{R}_+. \]

\(^8\)Notice that \( s(X) \) is a continuous local \( \bar{P}_{x_0} \)-martingale, as it is stopped at the boundaries of \( V(x_0) \) (Lemma C.3), and recall [39, Theorem VI.1.7].
Now, Lemma C.19 implies that the function $\beta$ is square integrable in a neighborhood of $x_0$. All in all, we proved that (b) does not hold.

Finally, for the last part, we notice that formulas (4.7) and (4.12) entail that (c) is violated (more precisely, use (4.7) with $\xi$ replaced by $\xi \wedge T$). The proof is complete. \hfill $\square$

For the following lemma we use the notation $[b, c]$, which is defined similar to $(b, c)$ and $[b, c)$ from (2.7) and (2.23).

**Lemma 4.6.** Suppose that $b \in \partial J$ is an accessible boundary for both $(x \mapsto \mathbb{P}_x)$ and $(x \mapsto \tilde{\mathbb{P}}_x)$, and that there exists a non-empty open interval $B \subseteq J^o$ with $b$ as endpoint such that all points in $B$ are non-separating. Furthermore, assume $\mathbb{m}(\{b\}), \tilde{\mathbb{m}}(\{b\}) < \infty$.

Consider the following three conditions:

(a) for every $c \in B$ the differential quotient $d\mathbb{m}/d\tilde{\mathbb{m}}$ does not exist as a continuous function from $[b, c]$ into $(0, \infty)$;

(b) for every $c \in B$ the differential quotient $d^+s/dx$ ($\equiv s'$) does not exist as an absolutely continuous function from $[b, c]$ into $(0, \infty)$ such that $ds'(z) = s'(z)\beta(z)dz$ with $\beta \in L^2([b, c])$;

(c) the differential quotients $d\mathbb{m}(b)/d\tilde{\mathbb{m}}$ and $d^+s(b)/dx$ exist but

$$\left(\frac{d\mathbb{m}}{d\tilde{\mathbb{m}}}(b)\right) \left(\frac{d^+s}{dx}(b)\right) \neq 1.$$ 

Then, the following equivalent statements hold:

(i) If at least one of the conditions (a)–(c) above holds, then $\mathbb{P}_b \perp \tilde{\mathbb{P}}_b$ on $\mathcal{F}_0$.

(ii) If $\mathbb{P}_b \sim \tilde{\mathbb{P}}_b$ on $\mathcal{F}_0$, then all three conditions (a)–(c) above are violated.

Related to the assumption of Lemma 4.6, we remark the following: as $\tilde{s} = \mathrm{Id}$, the accessibility of $b$ implies that $b \in \mathbb{R}$ (see (2.8)).

**Proof.** As in the proof of Lemma 4.5, (i) and (ii) are equivalent by Blumenthal’s zero-one law (Lemma C.2). In the following, we prove (ii), i.e., we assume that $\mathbb{P}_b \sim \tilde{\mathbb{P}}_b$ on $\mathcal{F}_0$. Equivalently, this means that $\mathbb{P}_b, \tilde{\mathbb{P}}_b$-a.s. $S > 0$, where $S$ is the separating time of $\mathbb{P}_b$ and $\tilde{\mathbb{P}}_b$.

Thanks to Lemma 4.4, there exists a stopping time $\xi$ such that $\mathbb{P}_b, \tilde{\mathbb{P}}_b$-a.s. $0 < \xi < S \wedge \infty$. By the definition of separating time, $\mathbb{P}_b \sim \tilde{\mathbb{P}}_b$ on $\mathcal{F}_\xi$. Take $c \in B$. Let $L(X) = \{L^\xi_t(X) : (t, x) \in \mathbb{R}_+ \times \mathbb{R}\}$ (resp., $L(s(X)) = \{L^\xi_t(s(X)) : (t, x) \in \mathbb{R}_+ \times \mathbb{R}\}$) be the semimartingale local time of $X$ under $\tilde{\mathbb{P}}_b$ (resp., of $s(X)$ under $\mathbb{P}_b$), see Lemma C.4 and Lemma C.15 (i). Notice that, if $b$ is the left (resp., right) boundary point of $B$, then, $\mathbb{P}_b$-a.s., $L(X)$ (resp., $L^-(X) \triangleq \{L^-_t(X) : (t, x) \in \mathbb{R}_+ \times \mathbb{R}\}$) is jointly continuous for $(t, x) \in \mathbb{R}_+ \times [b, c]$, see Lemma C.15 (ii) and Remark C.16 (a). Similarly, $\tilde{\mathbb{P}}_b$-a.s., $L(s(X))$ or $L^-(s(X)) \triangleq \{L^-_t(s(X)) : (t, x) \in \mathbb{R}_+ \times \mathbb{R}\}$ is jointly continuous on $\mathbb{R}_+ \times [b, c]$ depending on whether $b < c$ or $b > c$. For notational convenience only, below we assume w.l.o.g. that $b = 0$, $c \in (0, \infty)$ and $s(0) = 0$.\footnote{This is the case $b < c$. In the opposite case $b > c$ we need to work with $L^-(X)$ in place of $L(X)$ and $L^-(s(X))$ in place of $L(s(X))$.} We define $G \triangleq G_1 \cap G_2 \cap G_3$ by

$$G_1 \triangleq \left\{0 < \xi < \infty, \ L^0_\xi(X), L^0_\xi(s(X)) > 0\right\},$$

$$G_2 \triangleq \left\{0 < \xi < \infty, \int_0^\xi f(X_s)ds = \int f(x)L^\xi_\xi(X)m(dx) \ \forall f \in \mathcal{B}_0^+\right\},$$

$$G_3 \triangleq \left\{0 < \xi < \infty, \int_0^\xi f(X_s)ds = \int f(x)L^\xi_\xi(s(X))m(dx) \ \forall f \in \mathcal{B}_0^+\right\},$$

with $\mathcal{B}_0^+ = \{f : \mathbb{R}_+ \rightarrow \mathbb{R}_+, \text{Borel}, f|_{[0, \infty)} = 0\}$. By Lemmata C.15 and C.18 and Remark C.16 (b), the equivalence $\mathbb{P}_0 \sim \tilde{\mathbb{P}}_0$ on $\mathcal{F}_\xi$ and the fact that $\mathbb{P}_0, \tilde{\mathbb{P}}_0$-a.s. $\xi \in (0, \infty)$, we have $\mathbb{P}_0(G) = \tilde{\mathbb{P}}_0(G) = 1$. Take $\omega \in G$. By the space continuity of the local time processes and the definition of the set $G_1$, there exists
a number \( c^o = c^o(\omega) \in (0, c) \) such that \( L^z_{\tilde{\xi}(\omega)}(X)(\omega), L^z_{\tilde{\xi}(\omega)}(\mathfrak{s}(X))(\omega) > 0 \) for all \( z \in [0, c^o] \). Using the definition of \( G_2 \) and \( G_3 \), for every \( z \in [0, c^o] \), we get

\[
\lim_{h \to 0} \int_0^z \mathbb{I}\{z \leq X_s(\omega) < z + h\} \, ds = L^z_{\tilde{\xi}(\omega)}(X)(\omega) > 0,
\]
and

\[
\lim_{h \to 0} \int_0^z \mathbb{I}\{z \leq X_s(\omega) < z + h\} \, ds = L^z_{\tilde{\xi}(\omega)}(\mathfrak{s}(X))(\omega) > 0,
\]

which shows that

\[
(4.14) \quad \left( \frac{dm}{d\tilde{m}} \right)(z) = \frac{L^z_{\tilde{\xi}(\omega)}(X)(\omega)}{L^z_{\tilde{\xi}(\omega)}(\mathfrak{s}(X))(\omega)} > 0.
\]

Thanks to the continuity of the function

\[
[0, c^o] \ni z \mapsto \frac{L^z_{\tilde{\xi}(\omega)}(X)(\omega)}{L^z_{\tilde{\xi}(\omega)}(\mathfrak{s}(X))(\omega)},
\]
we conclude that \( dm/d\tilde{m} \) exists as a continuous function from \([0, c^o]\) into \((0, \infty)\). This means that (a) has to be violated.

Next, we show that (b) is violated, too. Let \([0, c] \ni x \mapsto \tilde{P}^o_x\) be a diffusion on natural scale with speed measure

\[
\tilde{m}^o(dx) \equiv dx \text{ on } (0, c), \quad \tilde{m}^o(\{0\}) \equiv 0, \quad \tilde{m}^o(\{c\}) \equiv \infty,
\]
i.e., \(([0, c] \ni x \mapsto \tilde{P}^o_x)\) is a Brownian motion which is instantaneously reflected from 0 and absorbed in \( c \). Moreover, let \((0, c] \ni x \mapsto P^o_x)\) be a diffusion with characteristics \((\mathfrak{s}, \tilde{m}^o)\), where

\[
\tilde{m}^o(dx) \equiv \left( \frac{dm}{d\tilde{m}} \right)(x) \text{ on } (0, c), \quad m^o(\{0\}) \equiv 0, \quad m^o(\{c\}) \equiv \infty.
\]

**Lemma 4.7.** If \( P^o_0 \perp \tilde{P}^o_0 \) on \( \mathcal{F}_0 \), then \( P_0 \perp \tilde{P}_0 \) on \( \mathcal{F}_0 \). Equivalently, if \( P_0 \sim \tilde{P}_0 \) on \( \mathcal{F}_0 \), then \( P^o_0 \sim \tilde{P}^o_0 \) on \( \mathcal{F}_0 \).

**Proof.** The equivalence between the two claims follows from Blumenthal’s zero-one law (Lemma C.2). We prove the first claim. For \( t \in \mathbb{R}_+ \), define

\[
\tilde{\varepsilon}(t, x) \equiv \lim_{h \to 0} \sup \int_0^t \mathbb{I}\{x - h < X_s < x + h\} \, ds, \quad x \in (0, c),
\]

\[
f(t) \equiv \begin{cases} \int_{(0,c)} \tilde{\varepsilon}(t, x) \, dx, & t < T_c, \\ \infty, & t \geq T_c, \end{cases}
\]

and \( f^{-1}(t) \equiv \inf(s \geq 0: f(s+) > t) \). The chain rule for diffusions as given by Theorem C.24 yields that

\[
P^o_0 = P_0 \circ X_t^{-1}_{f^{-1}(\cdot)}, \quad \tilde{P}^o_0 = \tilde{P}_0 \circ X_t^{-1}_{f^{-1}(\cdot)},
\]

From this point on, we can argue verbatim as in the proof of Lemma 4.3 to obtain the first claim of the lemma. We omit the details. \( \square \)

Thanks to Lemma 4.7, we can and will assume (in the part of the proof, where we show that (b) is violated) that \( \tilde{P}_0 = \tilde{P}^o_0 \) and that \( P_0 = P^o_0 \). These assumptions are in force for the remainder of this proof. Define

\[
\mathfrak{g}^{**}(x) \equiv sgn(x)s(|x|), \quad x \in [-c, c],
\]
and
\[ m^+(A) = \begin{cases} m(A), & A \in B((0,c)), \\ m(-A), & A \in B((-c,0)), \\ 0, & A = \{0\}, \\ +\infty, & A \subset \{-c,c\}, A \neq \varnothing. \end{cases} \]

Let \([-c,c] \ni x \mapsto \mathbb{P}_x^+\) be the diffusion with characteristics \((\sigma^+,m^+)\). Furthermore, let \([-c,c] \ni x \mapsto \tilde{\mathbb{P}}_x^+\) be a Brownian motion with absorbing boundaries \(-c\) and \(c\). By a version of Lemma C.12 (cf. [1, Section 6]), we have
\[ \mathbb{P}_0 = \mathbb{P}_0^\ast \circ Y^{-1} = \tilde{\mathbb{P}}_0^\ast \circ Y^{-1}, \quad Y \triangleq |X|. \]

Notice that \(\mathbb{P}_0 \sim \tilde{\mathbb{P}}_0\) on \(\mathcal{F}_t\) implies that \(\mathbb{P}_0^\ast \sim \tilde{\mathbb{P}}_0^\ast\) on \(Y^{-1}(\mathcal{F}_t) = \mathcal{F}_t^Y\) with \(\rho = \xi \circ Y\), see Lemma C.34 for the identity of the \(\sigma\)-fields and the fact that \(\rho\) is an \((\mathcal{F}_t^Y)_{t \geq 0}\)-stopping time. Let \((\mathbb{R} \ni x \mapsto W_x)\) be the Wiener measure, take \(0 < c' < c\) and set
\[ T \triangleq T_{-c'} \wedge T_c = T_c \circ Y, \quad T^* \triangleq T_{-c'}^* \wedge T_{c'}^* = T_{c'} \circ Y. \]

Then, \(W_0 \circ X^{-1}_\lambda = \tilde{\mathbb{P}}_0^\ast\) and, for every \(A \in \mathcal{F}_\rho^\ast\), Galmirano’s test in the form [23, 10 c), p. 87] (alternatively, one can adjust [39, Exercise I.4.21]) to the right-continuous filtration \((\mathcal{F}_t)_{t \geq 0}\) yields that
\[ W_0(A) = W_0(X_{\wedge T} \in A) = \tilde{\mathbb{P}}_0^\ast(A). \]

Hence, \(P_0^\ast \sim W_0\) on \(\mathcal{F}_\rho^\ast\). We deduce from Lemmata C.3 and C.26 that the process \(s(Y) = |s^+(X)|\) is a \(P_0^\ast\)-\((\mathcal{F}_t)_{t \geq 0}\)-semimartingale (since it is the absolute value of a semimartingale). Thus, by Stricker’s theorem (see, e.g., [24, Theorem 9.19]), the process \(s(Y)\) is also a \(P_0^\ast\)-\((\mathcal{F}_t^Y)_{t \geq 0}\)-semimartingale and, by GirSANov’s theorem (Lemma C.30), as \(P_0^\ast \sim W_0\) on \(\mathcal{F}_\rho^\ast\), the stopped process \(s(Y_{\wedge \rho \wedge T^*})\) is a \(W_0\)-\((\mathcal{F}_t^Y)_{t \geq 0}\)-semimartingale.

**Lemma 4.8.** \(s(Y_{\wedge \rho \wedge T^*}) = s(|X_{\wedge \rho \wedge T^*}|)\) is also a \(W_0\)-\((\mathcal{F}_t)_{t \geq 0}\)-semimartingale.

**Proof.** Essentially, the claim follows from [10, (3.24.1)] and [30, Exercise 6.23]. We provide the details. Thanks to [30, Exercise 6.23], \(Y\) is an \((\mathcal{F}_t)_{t \geq 0}\)-Markov process under \(W_0\). Hence, we get for all \(s_1 < s_2 < \cdots < s_m \leq t < t_1 < t_2 < \cdots < t_m\) and every bounded Borel functions \(f: \mathbb{R}^m \to \mathbb{R}\) and \(g: \mathbb{R}^n \to \mathbb{R}\), that \(W_0\)-a.s.

\[ \mathbb{E}^W_0[f(Y_{s_1}, \ldots, Y_{s_n})g(Y_{t_1}, \ldots, Y_{t_m})|\mathcal{F}_t] = f(Y_{s_1}, \ldots, Y_{s_n})\mathbb{E}^W_0[g(Y_{t_1}, \ldots, Y_{t_m})|\mathcal{F}_t]. \]

By the tower property, this yields that \(W_0\)-a.s.

\[ \mathbb{E}^W_0[f(Y_{s_1}, \ldots, Y_{s_n})g(Y_{t_1}, \ldots, Y_{t_m})|\mathcal{F}_t^Y] = \mathbb{E}^W_0[\mathbb{E}^W_0[f(Y_{s_1}, \ldots, Y_{s_n})g(Y_{t_1}, \ldots, Y_{t_m})|\mathcal{F}_t]|\mathcal{F}_t^Y] = \mathbb{E}^W_0[f(Y_{s_1}, \ldots, Y_{s_n})g(Y_{t_1}, \ldots, Y_{t_m})|\mathcal{F}_t]. \]

Therefore, by a monotone class argument, for every \(W_0\)-integrable \(\mathcal{F}_\infty^Y\)-measurable random variable \(Z\) and any \(t \in \mathbb{R}_+\), we have \(W_0\)-a.s.

\[ \mathbb{E}^W_0[Z|\mathcal{F}_t] = \mathbb{E}^W_0[Z|\mathcal{F}_t^Y]. \]

This implies that any \(W_0\)-\((\mathcal{F}_t^Y)_{t \geq 0}\)-martingale is also a \(W_0\)-\((\mathcal{F}_t)_{t \geq 0}\)-martingale. Finally, by [24, Proposition 9.28], this implies that the same implication also holds for semimartingales and hence, the claim follows.

Since \(P_0^\ast \sim W_0\) on \(\mathcal{F}_\rho^\ast\) and \(P_0^\ast(\rho \wedge T^* > 0) = P_0(\xi \wedge T_{-c'} \wedge T_{c^*} > 0) = 1\), we have \(W_0(\rho \wedge T^* > 0) = 1\). Now, we deduce from Lemma 4.8 and Theorem C.31 that \(s(\cdot|\cdot)\) is the difference of two convex functions on a closed interval \([-c',c]\) with \(0 < c' < c < c\). Set \(T' \triangleq T_{-c'} \wedge T_{c'}\) and, for \(t \in \mathbb{R}_+\), let \(G_{t}^W\) be the \(W_0\)-completion of \(\mathcal{F}_{t \wedge \rho \wedge T^*}\) (with subsets of \(W_0\)-null sets from \(\mathcal{F}\)) and \(G_t \triangleq G_t^W \cap \mathcal{F}_{\rho \wedge T^*}^Y\). Since
$P_0^{++} \sim W_0$ on $F_{\rho \land T}$, we also have $G_t = G_t^{P_0^{++}} \cap F_{\rho \land T}$, and $P_0^{++} \sim W_0$ on $G_{\rho \land T}$. Notice that the integral process

$$\int_0^{\wedge \rho \land T} \text{sgn}(X_s) dX_s = |X_{\wedge \rho \land T}|-L_0^{\wedge \rho \land T}(X)$$

is $(G_t)_{t \geq 0}$-adapted by virtue of [39, Corollary VI.1.9]. Hence, we conclude, by the tower property, that it is a $W_0(\mathcal{G}_t)_{t \geq 0}$-martingale (notice that $\text{sgn}(X)$ is bounded, which yields the true martingale property). By Girsanov’s theorem, there exists a $(G_t)_{t \geq 0}$-predictable process $\hat{\beta}$ such that $P_0^{++}$-a.s.

$$(4.15) \quad \int_0^{\wedge \rho \land T} \text{sgn}(X_s) dX_s = \text{local } P_0^{++}(G_t)_{t \geq 0}-\text{martingale} + \int_0^{\wedge \rho \land T} \hat{\beta}_s ds.$$ 

As above, notice that the integral process

$$\int_0^{\wedge \rho \land T} \text{sgn}(s^+(X_s)) ds^+(X_s) = s(|X_{\wedge \rho \land T}|) - L_0^{\wedge \rho \land T}(s^+(X))$$

is $(G_t)_{t \geq 0}$-adapted by [39, Corollary VI.1.9]. We conclude, by the tower property, that it is a $P_0^{++}(G_t)_{t \geq 0}$-martingale. Next, by the generalized Itô formula (Lemma C.26), we get that $P_0^{++}$-a.s.

$$(4.17) \quad \int_0^{\wedge \rho \land T} \text{sgn}(s^+(X_s)) ds^+(X_s) = \int_0^{\wedge \rho \land T} \text{sgn}(s^+(X_s)) \cdot |s^+|'(X_s) dX_s$$

$$+ \frac{1}{2} \int_0^{\wedge \rho \land T} \text{sgn}(s^+(X_s)) d \left[ \int L^s_+(X)|s^+|''(dx) \right].$$

Notice that $\text{sgn}(s^+(X)) = \text{sgn}(X)$. Hence, by virtue of (4.15), we obtain that $P_0^{++}$-a.s.

$$(4.18) \quad \int_0^{\wedge \rho \land T} \text{sgn}(s^+(X_s)) \cdot |s^+|'(X_s) dX_s = \text{local } P_0^{++}(G_t)_{t \geq 0}-\text{martingale} + \int_0^{\wedge \rho \land T} |s^+|'(X_s) \hat{\beta}_s ds.$$ 

Therefore, as the process in (4.16) is a (local) $P_0^{++}(G_t)_{t \geq 0}$-martingale, (4.17) and (4.18), together with the fact that continuous local martingales of (locally) finite variation are constant, imply that $P_0^{++}$-a.s.

$$\int_0^{\wedge \rho \land T} \text{sgn}(s^+(X_s)) d \left[ \int L^s_+(X)|s^+|''(dx) \right] = - \int_0^{\wedge \rho \land T} 2|s^+|'(X_s) \hat{\beta}_s ds.$$ 

Integrating $\text{sgn}(s^+(X)) = \text{sgn}(X)$ against both sides yields that $P_0^{++}$-a.s.

$$\int L^s_+(X)|s^+|''(dx) = - \int 2\text{sgn}(X_s)|s^+|'(X_s) \hat{\beta}_s ds.$$ 

From this point on we can argue almost verbatim as in the proof of Lemma 4.5 to obtain the existence of a Borel map $\tilde{\beta} : [-c', c'] \to \mathbb{R}$, where $0 < c' < c$ might have been replaced by a smaller value, such that $|s^+|''(dx) = \tilde{\beta}(x)|s^+|'(x) dx$ and $\tilde{\beta} \in L^2([-c', c'])$. We omit the details. In summary, condition (b) is violated.

Finally, we prove that (c) is violated. We are no longer assuming that $\tilde{P}_0 = \tilde{P}_0^0$ and that $P_0 = P_0^0$. To achieve the aim, it suffices only to observe that the formula (4.14) at the point $z = 0 (= b)$ and part (iii) of Lemma C.26 (see also (4.12)) provide a contradiction to part (c) (more precisely, we need to use (4.14) with $\xi$ replaced by $\xi \land T$ because $s$ is the difference of two convex functions only on $[0, c']$, whereas such a structure of $s$ is needed to apply part (iii) of Lemma C.26). The proof is complete.

4.2. Proof of Theorem 2.18. First of all, part (i) is trivial. Let us note that if $x_0$ is an absorbing boundary point for both diffusions ($x \mapsto P_x$ and $x \mapsto \tilde{P}_x$), then $P_{x_0} = \tilde{P}_{x_0}$, which is covered by (i). We now prove part (ii). That is, below we always assume that $P_{x_0} \neq \tilde{P}_{x_0}$.

Next, we discuss the case where $x_0$ is a boundary point which is absorbing for one of the diffusions, but not for the other. In this case, $x_0$ is separating (recall Definition 2.14). Consequently, $U \lor V \land R = 0$. We now show that $P_{x_0} \perp \tilde{P}_{x_0}$ on $\mathcal{F}_0$, which then yields that $P_{x_0}, \tilde{P}_{x_0}$-a.s. $S = 0 = U \lor V \land R$ and hence the claim. Set $C = \inf(t > 0 : X_t \neq x_0)$. In case $x_0$ is an absorbing boundary point for ($x \mapsto P_x$) and a reflecting boundary point for ($x \mapsto \tilde{P}_x$), we have $P_{x_0}(C > 0) = 1$ and $\tilde{P}_{x_0}(C > 0) = 0$. Thus,
as \( \{C > 0\} \in \mathcal{F}_0 \), it follows that \( \mathbb{P}_{x_0} \perp \mathbb{P}_{\tilde{x}_0} \) on \( \mathcal{F}_0 \). The case where \( x_0 \) is absorbing for \( (x \mapsto \tilde{\mathbb{P}}_x) \) and reflecting for \( (x \mapsto \mathbb{P}_x) \) follows by symmetry.

We now discuss all remaining cases, where we can assume that, in case \( x_0 \) is a boundary point, it is reflecting for both diffusions. The proof is split into two parts. First, we show that \( \mathbb{P}_{x_0} \perp \mathbb{P}_{\tilde{x}_0} \)-a.s. \( U \wedge V \wedge R \leq S \) and, second, we prove that \( \mathbb{P}_{x_0}, \tilde{\mathbb{P}}_{x_0} \)-a.s. \( S \leq U \wedge V \wedge R \). Recall that \( J_{\text{sep}} \) denotes the set of separating points.

**Proof of \( U \wedge V \wedge R \leq S \):** If \( x_0 \in J_{\text{sep}} \), then \( U \wedge V \wedge R = 0 \) and the claim is trivial. Thus, we can and will assume that \( x_0 \notin J_{\text{sep}} \). In the following, we will consider the case \( x_0 \in J^0 \). The case where \( x_0 \) is a boundary point (which is reflecting for both diffusions) can be handled the same way. Define

\[
\alpha' \triangleq \begin{cases} l, & \alpha = \Delta, \\ \alpha, & \alpha \neq \Delta, \end{cases} \quad \gamma' \triangleq \begin{cases} r, & \gamma = \Delta, \\ \gamma, & \gamma \neq \Delta, \end{cases}
\]

and take two sequences \( (a_n)_{n \in \mathbb{N}} \) and \( (c_n)_{n \in \mathbb{N}} \) such that \( a_1 < x_0 < c_1, a_{n+1} < a_n, c_{n+1} > c_n, \lim_{n \to \infty} a_n = \alpha' \) and \( \lim_{n \to \infty} c_n = \gamma' \). By Lemma 4.1 and Proposition 2.4, we have \( \mathbb{P}_{x_0}, \tilde{\mathbb{P}}_{x_0} \)-a.s. \( T_{a_n} \wedge T_{c_n} < S \) for all \( n \in \mathbb{N} \).

**Case 1 (\( \alpha \neq \Delta \) and \( \gamma \neq \Delta \)):** In this case, we have \( R = \delta \) and, by virtue of Lemma C.8, \( \mathbb{P}_{x_0}, \tilde{\mathbb{P}}_{x_0} \)-a.s. \( T_{a_n} \wedge T_{c_n} \nearrow U \wedge V \). Hence, \( \mathbb{P}_{x_0}, \tilde{\mathbb{P}}_{x_0} \)-a.s. \( U \wedge V \wedge R \leq S \) follows.

**Case 2 (\( \alpha = \Delta, \gamma \neq \Delta \) and \( l \) is inaccessible or absorbing for one, equivalently for both, of the diffusions):** In this case, \( U = R = \delta \). By Lemma 4.2 and Proposition 2.4, it holds that \( \mathbb{P}_{x_0} \sim \tilde{\mathbb{P}}_{x_0} \) on \( \mathcal{F}_{T_{a_n}} \) for all \( n \in \mathbb{N} \). Hence, \( \mathbb{P}_{x_0}, \tilde{\mathbb{P}}_{x_0} \)-a.s. \( T_{c_n} < S \). Notice that \( \{T_{c_n} = \infty\} \nearrow \{V = \delta\} \). Therefore, \( \mathbb{P}_{x_0}, \tilde{\mathbb{P}}_{x_0} \)-a.s. we have \( S = \delta \) on \( \{V = \delta\} \). Together with the fact that \( \mathbb{P}_{x_0}, \tilde{\mathbb{P}}_{x_0} \)-a.s. \( T_{c_n} \nearrow V \wedge \infty \) we conclude that \( \mathbb{P}_{x_0}, \tilde{\mathbb{P}}_{x_0} \)-a.s. \( U \wedge V \wedge R = S \).

**Case 3 (\( \alpha = \Delta, \gamma \neq \Delta \) and \( l \) is reflecting for one, equivalently for both, of the diffusions):** Here, we again have \( U = R = \delta \). To simplify our notation, we assume that \( J = \mathbb{R}_+ \), in particular, \( l = 0 \). Let \( (\mathbb{R} \ni x \mapsto \mathbb{P}_{x}^{++}) \) and \( (\mathbb{R} \ni x \mapsto \tilde{\mathbb{P}}_{x}^{++}) \) be diffusions constructed from \( (\mathbb{R}_+ \ni x \mapsto \mathbb{P}_x) \) and \( (\mathbb{R}_+ \ni x \mapsto \tilde{\mathbb{P}}_x) \) as in Lemma C.12. In particular, we have

\[
\mathbb{P}_{x_0} = \mathbb{P}_{x_0}^{++} \circ |X|^{-1} \quad \text{and} \quad \tilde{\mathbb{P}}_{x_0} = \tilde{\mathbb{P}}_{x_0}^{++} \circ |X|^{-1}.
\]

For a moment fix \( n \in \mathbb{N} \). Since all points in \( \{c_n, -c_n\} \) are non-separating for the symmetrized setting (see Lemma C.12 for the structure of scale and speed of the symmetrized diffusions), Lemma 4.1 yields that \( \mathbb{P}_{x_0}^{++} \sim \tilde{\mathbb{P}}_{x_0}^{++} \) on \( \mathcal{F}_{T_{a_n} \wedge T_{c_n}} \) and, consequently, \( \mathbb{P}_{x_0} \sim \tilde{\mathbb{P}}_{x_0} \) on \( \mathcal{F}_{T_{a_n}} \). Therefore, \( \mathbb{P}_{x_0}, \tilde{\mathbb{P}}_{x_0} \)-a.s. \( T_{c_n} < S \). By Lemma C.7, \( \mathbb{P}_{x_0}, \tilde{\mathbb{P}}_{x_0} \)-a.s. \( T_{c_n} \nearrow V \). Thus, \( \mathbb{P}_{x_0}, \tilde{\mathbb{P}}_{x_0} \)-a.s. \( U \wedge V \wedge R = S \).

**Case 4 (\( \alpha = \Delta, \gamma = \Delta \) and both \( l \) and \( r \) are inaccessible or absorbing):** First of all, notice that in this case we have \( x_0 \in J^0 \) and that \( l \) and \( r \) are finite (because they are half-good and \( \tilde{s} = \text{Id} \)). Let \( ([l, r] \ni x \mapsto \mathbb{P}_x^\circ) \) be a Brownian motion which is absorbed in \( l \) and \( r \). Further, let \( ([l, r] \ni x \mapsto \tilde{\mathbb{P}}_x^\circ) \) be a diffusion with characteristics \( (s, m^\circ) \), where the measure \( m^\circ \) on \( \mathcal{B}([l, r]) \) is given by

\[
\frac{m^\circ(dx)}{dx} \triangleq \left( \frac{dm}{dm} \right)(x) \text{ on } (l, r), \quad \frac{m^\circ([l])}{m^\circ([r])} \triangleq \infty.
\]

That \( m^\circ \) is a valid speed measure and the state space of the latter diffusion is indeed \( [l, r] \) (i.e., both endpoints are accessible) follows as in the proof of Lemma 4.1. Moreover, again as in the proof of Lemma 4.1, we argue that Lemma C.25 applies to the diffusions \( ([l, r] \ni x \mapsto \mathbb{P}_x^\circ) \) and \( ([l, r] \ni x \mapsto \tilde{\mathbb{P}}_x^\circ) \) and consequently, \( \mathbb{P}_{x_0}^\circ \sim \tilde{\mathbb{P}}_{x_0}^\circ \).

We now use the time-change argument from Lemma 4.1 to conclude that \( \mathbb{P}_{x_0} \sim \tilde{\mathbb{P}}_{x_0} \). Define

\[
\tilde{\mathbb{P}}^*(t, x) \triangleq \begin{cases} \limsup_{h \downarrow 0} \int_0^h 1_{\{x-h<\xi_s<s+x+h\}}ds \quad & \text{if } x \in (l, r), \\ 0 & \text{if } x \in \{l, r\}, \end{cases}
\]
and set

\[ g(t) \triangleq \begin{cases} \int_0^t \hat{g}^*(t, x) \mathfrak{m}(dx), & t < T_l \land T_r, \\ \infty, & t \geq T_l \land T_r, \end{cases} \]

Let \( g^{-1} \) be the right-inverse of \( g \). According to the chain rule for diffusions (Theorem C.24), we get that

\[ P_{x_0}^\circ \circ X^{-1}_g(\cdot) = P_{x_0}^\circ \quad \text{and} \quad P_{x_0}^\circ \circ \tilde{X}^{-1}_g(\cdot) = \tilde{P}_{x_0} \]

(see the proof of Lemma 4.1 for more details). In particular, at this stage we used that \( l \) and \( r \) are either inaccessible or absorbing for both diffusions \((x \mapsto P_x)\) and \((x \mapsto \tilde{P}_x)\). Thanks to these equalities, \( P_{x_0}^\circ \sim \tilde{P}_{x_0} \) implies \( P_{x_0} \sim \tilde{P}_{x_0} \) and, consequently, \( P_{x_0}, \tilde{P}_{x_0}\text{-a.s.} \quad S = \delta \).

**Case 5** \((\alpha = \Delta, \gamma = \Delta, l \text{ is reflecting and } r \text{ is inaccessible or absorbing})\): This case is reduced to the previous one via Lemma C.12 (cf. Cases 2 and 3).

**Case 6** \((\alpha = \Delta, \gamma = \Delta \text{ and both } l \text{ and } r \text{ are reflecting})\): Here \( U = V = \delta \) and \( R = \infty \). We need to prove that \( P_{x_0}, \tilde{P}_{x_0}\text{-a.s.} \quad S \geq \infty \). First, we notice that, in this case, \( l \) and \( r \) are finite (because they are half-good and \( \bar{s} = 1d \)). To simplify our notation, we assume that \( J = [0,1] \), in particular, \( l = 0 \) and \( r = 1 \). Let \((R \ni x \mapsto Q_x)\) (resp., \((R \ni x \mapsto \tilde{Q}_x)\)) be the diffusion constructed from \(([0,1] \ni x \mapsto P_x)\) (resp., \(([0,1] \ni x \mapsto \tilde{P}_x)\)) as in Lemma C.13. Let \( \mathcal{S} \) be the separating time for \( Q_{x_0} \) and \( \tilde{Q}_{x_0} \). All points in \( \mathcal{S} \) are non-separating for \((x \mapsto Q_x)\) and \((x \mapsto \tilde{Q}_x)\). As the boundaries \( \pm \infty \) are inaccessible for both these diffusions, Case 1 above yields \( Q_{x_0}, \tilde{Q}_{x_0}\text{-a.s.} \quad \mathcal{S} \geq \infty \), hence, by Proposition 2.3, \( Q_{x_0} \sim_{\operatorname{loc}} \tilde{Q}_{x_0} \). As \( P_{x_0} = Q_{x_0} \circ f(X)^{-1} \) and \( \tilde{P}_{x_0} = \tilde{Q}_{x_0} \circ (f(X)^{-1}) \) for a function \( f \) that is described in Lemma C.13, it follows that \( P_{x_0} \sim_{\operatorname{loc}} \tilde{P}_{x_0} \). Applying Proposition 2.3 once again, we obtain \( P_{x_0}, \tilde{P}_{x_0}\text{-a.s.} \quad S \geq \infty \), as required.

Up to symmetry we considered all possible cases.

**Proof of** \( S \leq U \land V \land R \): \( S \leq U \land V \land R \): We will distinguish several cases.

**Case 1** \((x_0 \in J_{\text{sep}} \cap J^c)\): Lemma 4.5 yields \( P_{x_0} \perp \tilde{P}_{x_0} \text{-a.s.} \quad S = 0 = U \land V \land R \).

**Case 2** \((x_0 \in J_{\text{sep}} \cap \partial J)\): Recall that we suppose that, in case \( x_0 \) is a boundary point, it is reflecting for both diffusions \((x \mapsto P_x)\) and \((x \mapsto \tilde{P}_x)\).

First, suppose that there is a point \( z \in J^c \) such that all points in \((x_0, z)\) (recall this notation from (2.7)) are non-separating. Then, Lemma 4.6 yields \( P_{x_0}, \tilde{P}_{x_0}\text{-a.s.} \quad S = 0 = U \land V \land R \).

Second, suppose that we can find a monotone sequence \( z_1, z_2, \ldots \in J^c \) of separating points such that \( z_n \to x_0 \). As \( x_0 \) is a reflecting boundary point, Lemma C.7 yields \( P_{x_0}, \tilde{P}_{x_0}\text{-a.s.} \quad T_{z_n} < \infty \) for all \( n \in \mathbb{N} \). Now, the strong Markov property and Lemma 4.5 yield that \( P_{x_0} \perp \tilde{P}_{x_0} \text{-a.s.} \quad S \leq T_{z_n} \), for all \( n \in \mathbb{N} \). Define \( T_{z_0} = \lim_{n \to \infty} T_{z_n} \). By Lemma C.6, \( P_{x_0}, \tilde{P}_{x_0}\text{-a.s.} \quad T_{z_0} = 0 \). Thus, \( P_{x_0}, \tilde{P}_{x_0}\text{-a.s.} \quad S = 0 = U \land V \land R \).

**Case 3** \((l < \alpha < x_0 < \gamma < r)\): In this case, we have \( P_{x_0}, \tilde{P}_{x_0}\text{-a.s.} \quad U \land V \land R = T_\alpha \land T_\gamma < \infty \) by (2.4). Using the strong Markov property and the result from Case 1, we obtain that \( P_{x_0} \perp \tilde{P}_{x_0} \text{-a.s.} \quad S \leq T_\alpha \land T_\gamma \}. \)

**Proposition 2.4** yields \( P_{x_0}, \tilde{P}_{x_0}\text{-a.s.} \quad S \leq T_\alpha \land T_\gamma \}. \) Similarly, we get that \( P_{x_0}, \tilde{P}_{x_0}\text{-a.s.} \quad S \leq T_\alpha \land T_\gamma \}. \) In summary, \( P_{x_0}, \tilde{P}_{x_0}\text{-a.s.} \quad S \leq T_\alpha \land T_\gamma \}. \)

**Case 4** \((l < \alpha < x_0, \gamma = \Delta)\): It suffices to prove that \( P_{x_0}, \tilde{P}_{x_0}\text{-a.s.} \quad S \leq T_\alpha \}. \) This follows from the strong Markov property, Lemma 4.5 and Proposition 2.4.

**Case 5** \((l < \alpha < x_0 < \gamma = r)\): Recall from Lemma C.8 that, restricted to the set \( \{T_\alpha = T_r \} \), a.a. paths of a diffusion with state space \( J \) travel to \( r \) in the sense that \( X_t \to r \) as \( t \to \infty \). Consequently, as \( R = \delta \), we have \( P_{x_0}, \tilde{P}_{x_0}\text{-a.s.} \quad U \land V \land R = T_\alpha \land T_r \}. \) It suffices to prove that \( P_{x_0}, \tilde{P}_{x_0}\text{-a.s.} \quad S \leq T_\alpha \}. \)

The first part, i.e., \( P_{x_0}, \tilde{P}_{x_0}\text{-a.s.} \quad S \leq T_\alpha \}. \) follows from the strong Markov property, Lemma 4.5 and Proposition 2.4.

In the following, we prove that \( P_{x_0}, \tilde{P}_{x_0}\text{-a.s.} \quad S \leq T_r \} \). We distinguish several cases. First, suppose that \( r \) has different boundary classifications for the diffusions \( (x \mapsto P_x) \) and \( (x \mapsto \tilde{P}_x) \). In case \( r \) is accessible for one of the diffusions and inaccessible for the other, the set \( \{T_r < \infty \} \in \mathcal{F}_T \) shows that
$\mathbb{P}_{x_0} \perp \bar{\mathbb{P}}_{x_0}$ on $\mathcal{F}_T \cap \{ T_\alpha \geq T_r \}$ (apply Lemma C.9). In case $r$ is reflecting for one of the diffusions and absorbing for the other, the set $\{ \tau = 0, T_r < \infty \} \in \mathcal{F}_T$, with $\tau \triangleq \inf \{ t \geq 0 : X_{t+T_r} \neq r \}$, shows that $\mathbb{P}_{x_0} \perp \bar{\mathbb{P}}_{x_0}$ on $\mathcal{F}_T \cap \{ T_\alpha \geq T_r \}$ (here we use that, in case of reflection, $\{ \tau > 0 \}$ happens with probability 0 by Lemma C.6). By Proposition 2.4, in both cases, $\mathbb{P}_{x_0}, \bar{\mathbb{P}}_{x_0}$-a.s. $S \leq T_r$ on $\{ T_\alpha \geq T_r \}$.

Next, suppose that $r$ is reflecting for both diffusions. Then, we can again use the strong Markov property and Lemma 4.6 to conclude that $\mathbb{P}_{x_0}, \bar{\mathbb{P}}_{x_0}$-a.s. $S \leq T_r$ on $\{ T_\alpha \geq T_r \}$.

Now, assume that $r$ is either absorbing for both diffusions or inaccessible for both diffusions. In particular, that means that $r$ cannot be half-good. First, suppose that $r = \infty$. Then, by Lemma C.8 and the fact that $\tilde{s} = \text{Id}$, $\mathbb{P}_{x_0}(T_\alpha \geq T_r) = 0$ and hence, $\mathbb{P}_{x_0} \perp \bar{\mathbb{P}}_{x_0}$ on $\mathcal{F}_T \cap \{ T_\alpha \geq T_r \}$. This yields that $\mathbb{P}_{x_0}, \bar{\mathbb{P}}_{x_0}$-a.s. $S \leq T_r$ on $\{ T_\alpha \geq T_r \}$. From now on, suppose that $r < \infty$. We fix a number $c \in (\alpha, x_0)$. Recall that all points $[c, r)$ are non-separating and let $\beta : [c, r) \to \mathbb{R}$ be the function as in Definition 2.6 (see also (2.21)). Notice that $\beta \in L^2_{\text{loc}}([c, r))$, as all points in $[c, r)$ are non-separating and $\tilde{s} = \text{Id}$. Furthermore, again using that $\tilde{s} = \text{Id}$, $X$ is a $\bar{\mathbb{P}}_{x_0}$-semimartingale (see Lemma C.15) and the stopped process $X_{\wedge T_r \wedge T_c} = X_{\wedge T_c}$ is a continuous local $\mathbb{P}_{x_0}$-martingale. The semimartingale occupation time formula (see Lemma C.26) yields that, $\mathbb{P}_{x_0}$-a.s. for all $t \in \mathbb{R}_+$,

$$\int_0^t [\beta(X_s)]^2 d\langle X, X \rangle_s = \int_{-\infty}^\infty [\beta(x)]^2 L_r^x(X)dx.$$  

As $\beta \in L^2_{\text{loc}}([c, r))$, this identity shows that, $\mathbb{P}_{x_0}$-a.s. for all $t < T_r$,

$$\int_0^{t\wedge T_r} [\beta(X_s)]^2 d\langle X, X \rangle_s < \infty.$$  

Hence, we can define the process

$$Z_t \triangleq \begin{cases} 
\exp \left( -\frac{1}{2} \int_0^{t\wedge T_r} \beta(X_s) dX_s - \frac{1}{8} \int_0^{t\wedge T_r} |\beta(X_s)|^2 d\langle X, X \rangle_s \right), & t < T_r, \\
\exp \left( -\frac{1}{2} \int_0^{T_r} \beta(X_s) dX_s - \frac{1}{8} \int_0^{T_r} |\beta(X_s)|^2 d\langle X, X \rangle_s \right), & t \geq T_r, T_r < T_c, \\
0, & t \geq T_r, T_c \geq T_r.
\end{cases}$$

Recall that $r$ is finite but not half-good. Thus, it holds that $\int_0^t (r - x)^2 dx = \infty$, and Lemma C.21 shows that $\mathbb{P}_{x_0}$-a.s.

$$\int_0^{T_r} [\beta(X_s)]^2 d\langle X, X \rangle_s = \infty \text{ on } \left\{ \lim_{t \uparrow T \wedge T_r} X_t = r \right\}.$$  

Notice that $\mathbb{P}_{x_0}$-a.s. $\{ T_c \geq T_r \} \subset \{ \lim_{t \uparrow T \wedge T_r} X_t = r \}$ by Lemma C.8, recalling $\tilde{s}(r) = r < \infty$. Thus, $\mathbb{P}_{x_0}$-a.s. $Z_{T_r} = 0$ on $\{ T_c \geq T_r \}$, which proves that $Z$ is a continuous process. Further, thanks to [24, Lemma 12.43], $Z$ is even a continuous local $\bar{\mathbb{P}}_{x_0}$-martingale. Take a sequence $x_0 < x_1 < x_2 < \cdots$ such that $r_n \to r$. For $m, n \in \mathbb{N}$, define

$$\sigma_m^n \triangleq \inf \{ t \geq 0 : \int_0^{t\wedge T_r} [\beta(X_s)]^2 d\langle X, X \rangle_s \geq m \}.$$  

The process

$$t \mapsto \int_0^{t\wedge T_r} [\beta(X_s)]^2 d\langle X, X \rangle_s \in [0, \infty]$$

is left-continuous (by the monotone convergence theorem) but, in general, it might jump to infinity. Therefore, $\sigma_m^n$ is an $(\mathcal{F}_t)_{t \geq 0}$-stopping time but it might fail to be $(\mathcal{F}_t)_{t \geq 0}$-predictable (cf. [25, p. 193] for a more detailed discussion). We now pass to a suitable predictable version. Notice that $T_c \wedge T_{r_n} = (T_c \wedge T_{r_n})(X_{\wedge T_c \wedge T_{r_n}})$ by Galmarino’s test (cf. [25, Lemma III.2.43]). Hence, we obtain

$$\sigma_m^n(X_{\wedge T_c \wedge T_{r_n}}) = \inf \{ t \geq 0 : \int_0^{t\wedge T_r} [\beta(X_s^\wedge T_c \wedge T_{r_n})]^2 d\langle X_{\wedge T_c \wedge T_{r_n}}, X_{\wedge T_c \wedge T_{r_n}} \rangle_s \}$$

$$= \inf \{ t \geq 0 : \int_0^{t\wedge T_r} [\beta(X_s^\wedge T_c \wedge T_{r_n})]^2 d\langle X_{\wedge T_c \wedge T_{r_n}}, X_{\wedge T_c \wedge T_{r_n}} \rangle_s \} = \sigma_m^n.$$
It follows from Lemma C.23 applied to the diffusion \( \tilde{P}_{x_0} \circ X_{\cap T_r \wedge T_n}^{-1} \) that there exists an \((\mathcal{F}_t)_{t \geq 0}\)-predictable time \( \sigma_m^n \) such that \( \tilde{P}_{x_0} \circ X_{\cap T_r \wedge T_n}^{-1} \cdot \sigma_m^n = \tilde{\sigma}_m^n \). As \( \tilde{P}_{x_0}\)-a.s. \( \sigma_m^n > 0 \), we may take \( \tilde{\sigma}_m^n > 0 \) identically. Hence, by [25, III.2.36], the time \( \tilde{\sigma}_m^n \) is an \((\mathcal{F}_t^n)_{t \geq 0}\)-stopping time, where \( \mathcal{F}_t^n = \sigma(X, s \leq t) \). We also observe that \( \tilde{\sigma}_m^n \triangleq \tilde{\sigma}_m^n(X_{\cap T_r \wedge T_n}) \) is an \((\mathcal{F}_t^n_{1\cap T_r \wedge T_n})_{t \geq 0}\)-stopping time (see [24, Proposition 10.35]). In particular, it is an \((\mathcal{F}_t^n)_{t \geq 0}\)-stopping time. Below we use this observation to apply Lemma B.5. Since \( \tilde{P}_{x_0} \circ X_{\cap T_r \wedge T_n}^{-1} \cdot \sigma_m^n = \tilde{\sigma}_m^n \), we have \( \tilde{P}_{x_0}\)-a.s. \( \sigma_m^n = \tilde{\sigma}_m^n \). Hence, \( \tilde{\sigma}_m^n(X_{\cap T_r \wedge T_n}) = \tilde{\sigma}_m^n \). By Lemma C.7, \( \tilde{P}_{x_0}\)-a.s. \( T_c \wedge T_n < \infty \), hence \( T_c \wedge T_n < T_r \) and therefore, by (4.19), \( \tilde{P}_{x_0}\)-a.s.

\[
\int_0^{T_c \wedge T_n} [\beta(X_s)]^2 \, d\langle X, X \rangle_s < \infty.
\]

Thus, by Novikov’s condition, \( Z_{\cap T_r \wedge T_n} \cdot \tilde{\sigma}_m^n \) is a uniformly integrable \( \tilde{P}_{x_0}\)-martingale that starts in 1. Notice that \( \tilde{P}_{x_0}\)-a.s. \( Z_{T_r \wedge \tilde{\sigma}_m^n} > 0 \). We define a probability measure \( K^{n,m} \) by the formula

\[
K^{n,m}(G) \triangleq \tilde{P}_{x_0} \left[ Z_{T_r \wedge \tilde{\sigma}_m^n} \mathbf{1}_G \right], \quad G \in \mathcal{F}.
\]

Below we will prove that

\[
(4.21) \quad K^{n,m} = P_{x_0} \text{ on } \mathcal{F}_{T_r \wedge \tilde{\sigma}_m^n} \wedge \tilde{\sigma}_m^n, \quad n, m \in \mathbb{N}.
\]

Suppose for a moment that this identity is established. Given a \( \sigma \)-field \( G \) on \( \Omega \) such that \( G \subset \mathcal{F} \), let \( (P_{x_0} | G)^{ac} \) denote the absolutely continuous part of the restriction \( P_{x_0} | G \) with respect to the restriction \( P_{x_0} | G \). By Jessen’s theorem (see Corollary D.5), \( \tilde{P}_{x_0}\)-a.s.

\[
(4.22) \quad Z_{T_r \wedge \tilde{\sigma}_m^n} = \frac{d(P_{x_0} | \mathcal{F}_{T_r \wedge \tilde{\sigma}_m^n} \wedge \tilde{\sigma}_m^n)}{d(P_{x_0} | \mathcal{F}_{T_r \wedge \tilde{\sigma}_m^n} \wedge \tilde{\sigma}_m^n)} \rightarrow \frac{d(P_{x_0} | \bigvee_{k \in \mathbb{N}} \mathcal{F}_{T_r \wedge \tilde{\sigma}_m^n} \wedge \tilde{\sigma}_m^n)}{d(P_{x_0} | \bigvee_{k \in \mathbb{N}} \mathcal{F}_{T_r \wedge \tilde{\sigma}_m^n} \wedge \tilde{\sigma}_m^n)}, \quad m \rightarrow \infty.
\]

Since \( P_{x_0} \sim \tilde{P}_{x_0} \) on \( \mathcal{F}_{T_r \wedge \tilde{\sigma}_m^n} \) by Lemma 4.1, (4.20) holds \( P_{x_0} - \tilde{P}_{x_0} \)-a.s. This yields that \( P_{x_0} - \tilde{P}_{x_0} \)-a.s. \( \lim_{m \rightarrow \infty} \sigma_m^n = \infty \). Moreover, recalling that \( \sigma_m^n = \sigma_m^n(X_{\cap T_r \wedge T_n}) \) and that \( \tilde{\sigma}_m^n \) is an \((\mathcal{F}_t^n_{1\cap T_r \wedge T_n})_{t \geq 0}\)-stopping time (hence, both \( \sigma_m^n \) and \( \tilde{\sigma}_m^n \) are \( \mathcal{F}_{T_r \wedge T_n} \)-measurable), \( P_{x_0} \sim \tilde{P}_{x_0} \) on \( \mathcal{F}_{T_r \wedge T_n} \) yields that \( \sigma_m^n = \tilde{\sigma}_m^n \) holds not only \( \tilde{P}_{x_0}\)-a.s. but also \( P_{x_0}\)-a.s. Consequently, \( P_{x_0}, \tilde{P}_{x_0}\)-a.s. \( \lim_{m \rightarrow \infty} \tilde{\sigma}_m^n = \infty \), hence, \( P_{x_0}, \tilde{P}_{x_0}\)-a.s.

\[
(4.23) \quad \bigvee_{k \in \mathbb{N}} \mathcal{F}_{T_r \wedge \tilde{\sigma}_m^n} \wedge \tilde{\sigma}_m^n = \mathcal{F}_{T_r \wedge T_n},
\]

therefore, \( \tilde{P}_{x_0}\)-a.s.

\[
(4.24) \quad \frac{d(P_{x_0} | \bigvee_{k \in \mathbb{N}} \mathcal{F}_{T_r \wedge \tilde{\sigma}_m^n} \wedge \tilde{\sigma}_m^n)^{ac}}{d(P_{x_0} | \bigvee_{k \in \mathbb{N}} \mathcal{F}_{T_r \wedge \tilde{\sigma}_m^n} \wedge \tilde{\sigma}_m^n)} \rightarrow \frac{d(P_{x_0} | \mathcal{F}_{T_r \wedge T_n})^{ac}}{d(P_{x_0} | \mathcal{F}_{T_r \wedge T_n})}, \quad n \rightarrow \infty.
\]

(Notice that we need (4.23) to hold \( P_{x_0}\)-a.s. and \( \tilde{P}_{x_0}\)-a.s. to conclude that (4.24) holds \( P_{x_0}\)-a.s. Indeed, if the \( \sigma \)-fields \( \bigvee_{k \in \mathbb{N}} \mathcal{F}_{T_r \wedge \tilde{\sigma}_m^n} \wedge \tilde{\sigma}_m^n \) and \( \mathcal{F}_{T_r \wedge T_n} \) were essentially different under \( P_{x_0} \), then the restrictions of the measures to these \( \sigma \)-fields could have essentially different absolutely continuous parts.) Now, (4.22) and (4.24) yield that \( \tilde{P}_{x_0}\)-a.s.

\[
Z_{T_r} = \frac{d(P_{x_0} | \mathcal{F}_{T_r \wedge T_n})^{ac}}{d(P_{x_0} | \mathcal{F}_{T_r \wedge T_n})}.
\]

Using Jessen’s theorem together with the fact that \( \bigvee_{n \in \mathbb{N}} \mathcal{F}_{T_r \wedge T_n} = \mathcal{F}_{T_r \wedge T_r} \), we get that \( \tilde{P}_{x_0}\)-a.s.

\[
Z_{T_r} = \frac{d(P_{x_0} | \mathcal{F}_{T_r \wedge T_n})^{ac}}{d(P_{x_0} | \mathcal{F}_{T_r \wedge T_n})}, \quad n \rightarrow \infty.
\]

This yields that \( \tilde{P}_{x_0}\)-a.s.

\[
Z_{T_r} = \frac{d(P_{x_0} | \mathcal{F}_{T_r \wedge T_n})^{ac}}{d(P_{x_0} | \mathcal{F}_{T_r \wedge T_n})}.
\]
Using that $\tilde{P}_{x_0}$-a.s. $Z_{r_n} = 0$ on $\{T_c \geq T_r\}$, we conclude that $\tilde{P}_{x_0} \perp \tilde{P}_{x_0}$ on $\mathcal{F}_{r_n} \cap \{T_c \geq T_r\}$. Further, as $\mathcal{F}_{T_r} \subset \mathcal{F}_{r_n}$, we obtain that $\tilde{P}_{x_0}, \tilde{P}_{x_0}$-a.s. $S \leq T_r$ on $\{T_c \geq T_r\}$. It follows from Lemma C.8 that $\tilde{P}_{x_0}, \tilde{P}_{x_0}$-a.s. $\{T_c \geq T_r\} \not\supset \{T_c \geq T_r\}$ as $c \rightarrow \alpha$. Hence $\tilde{P}_{x_0}, \tilde{P}_{x_0}$-a.s. $S \leq T_r$ on $\{T_c \geq T_r\}$.

It remains to prove (4.21). Let $f \in C_b(\tilde{\mathcal{S}}(c), \tilde{\mathcal{S}}(r_n))$ be such that $f|_{\tilde{\mathcal{S}}(c), \tilde{\mathcal{S}}(r_n)}$ is the difference of two convex functions on $(\tilde{\mathcal{S}}(c), \tilde{\mathcal{S}}(r_n))$ and $df^p = 2gd\tilde{\alpha}^{-1}$ on $(\tilde{\mathcal{S}}(c), \tilde{\mathcal{S}}(r_n))$ for some $g \in C_b(\tilde{\mathcal{S}}(c), \tilde{\mathcal{S}}(r_n))$ and $\tilde{\alpha}$. Let $\tilde{x}$ be such that $\tilde{x}$ is a local $\tilde{\alpha}$-skeleton at $\tilde{x}$, which proves that the process is local $\tilde{\alpha}$-skeleton at $\tilde{x}$.

Furthermore, applying part (iii) of Lemma C.26 together with the fact that $\tilde{d} \tilde{m} = \tilde{s}' \tilde{d}m$ for all $t < T_c \wedge T_r$.

As all points in $[c, r_n]$ are non-separating, $s$ is a $C^1$-function with absolutely continuous derivative on $[c, r_n]$ (recall Definition 2.6). Hence, $\tilde{P}_{x_0}$-a.s. for $t < T_c \wedge T_r$,

$$\frac{1}{2} \int L_t^f(s(X_t)) df'_t(y) = \int L_t^f(s(X_t))g(y) m \circ \tilde{s}^{-1}(dy) = \int L_t^\tilde{s} g(s(x))(s(x)) m(dx)$$

(4.27)

where the last equality is the occupation time formula for diffusions (more precisely, see (C.7)). Substituting (4.26) and (4.27) into (4.25) yields that, $\tilde{P}_{x_0}$-a.s. for $t < T_c \wedge T_r$,

$$df(s(X_t)) = f'_t(s(X_t)) ds(X_t) + \frac{1}{2} f''_t(s(X_t)) ds(X_t) + g(s(X_t)) dt.$$ 

Recall that, $\tilde{P}_{x_0}$-a.s. $dZ_t = -\frac{1}{2} \tilde{Z}_t \tilde{\alpha}(X_t) dX_t$ for $t < T_c \wedge T_r$. Hence, $\tilde{P}_{x_0}$-a.s. for $t < T_c \wedge T_r$,

$$d(f(s(X_t)) Z_t) = Z_t df(s(X_t)) + f(s(X_t))dZ_t + d(f(s(X_t)), Z_t)$$

$$= dM_t + Z_t g(s(X_t)) dt - \frac{1}{2} \tilde{Z}_t f''(s(X_t)) ds(X_t)$$

$$= dM_t + Z_t g(s(X_t)) dt$$

with some local $\tilde{P}_{x_0}$-martingale $M$, where the integrals with respect to $(X, X_t)$ are cancelled in the last equality due to $s'' = \tilde{s}'$ on $[c, r_n]$ (recall Definition 2.6). Hence, $\tilde{P}_{x_0}$-a.s. for all $t < T_c \wedge T_r$,

$$d \left[ (f(s(X_t)) - \int_0^t g(s(X_s)) ds) Z_t \right] = dM_t - \left( \int_0^t g(s(X_s)) ds \right) dZ_t,$$

which proves that the process

$$\left( f(s(X_t)) - \int_0^{T_c \wedge T_r} g(s(X_s)) ds \right) Z_{T_c \wedge T_r}$$

is a local $\tilde{P}_{x_0}$-martingale. Consequently, by [25, Proposition III.3.8], the stopped process

$$f(s(X_{T_c \wedge T_r \wedge \sigma_m})) - \int_0^{T_c \wedge T_r \wedge \sigma_m} g(s(X_s)) ds$$

is a local $K^{n,m}$-martingale. By the Lemmata B.4 and B.5, this proves (4.21).

Case 6 ($\alpha = \gamma = \Delta$): In this case we have $U \wedge V \wedge R = R$. Thus, if $R = \delta$, i.e., the diffusions have a non-reflecting boundary point, the inequality $S \leq \delta = U \wedge V \wedge R = \delta$ is trivial. We now discuss the case where $R = \infty$, i.e., we assume that the boundaries of $(x \mapsto P_x)$ and $(x \mapsto \tilde{P}_x)$ are reflecting. Notice that in this case $J = \bar{J}$ is compact and, by (2.10), that $\tilde{m}$ and $\tilde{m}$ are finite measures. Further, thanks to Lemma C.10, both diffusions are recurrent. Let $C$ be a countable set of bounded continuous
functions \( f : J \to \mathbb{R} \) that is probability measure determining. It follows from the ratio ergodic theorem (see Lemma C.11) that \( \mathbb{P}_{x_0} \)-a.s.

\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t f(X_s)ds = \frac{1}{m(J)} \int f(x)m(dx), \quad \text{for all } f \in C,
\]

and \( \mathbb{P}_{x_0} \)-a.s.

\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t f(X_s)ds = \frac{1}{\tilde{m}(J)} \int f(x)\tilde{m}(dx), \quad \text{for all } f \in C.
\]

For contradiction, assume that \( \mathbb{P}_{x_0}(S = \delta) > 0 \). Then, by the definition of the separating time,

\[
\hat{P}_{x_0} \left( \lim_{t \to \infty} \frac{1}{t} \int_0^t f(X_s)ds = \frac{1}{\tilde{m}(J)} \int f(x)\tilde{m}(dx), \quad \text{for all } f \in C, \quad S = \delta \right) > 0,
\]

and consequently,

\[
\hat{P}_{x_0} \left( \frac{1}{\tilde{m}(J)} \int f(x)\tilde{m}(dx) = \lim_{t \to \infty} \frac{1}{t} \int_0^t f(X_s)ds = \frac{1}{\tilde{m}(J)} \int f(x)\tilde{m}(dx), \quad \text{for all } f \in C \right) > 0.
\]

By the uniqueness theorem for probability measures, and the assumption that \( C \) is probability measure determining, we conclude that \( \hat{m}/\tilde{m}(J) = \hat{m}/\tilde{m}(J) \), or, equivalently, \( \hat{m} = c\tilde{m} \), where \( c \) is a constant (necessarily, \( c = \hat{m}(J)/\tilde{m}(J) \)). As all points in \( J \) are non-separating and \( \delta = \text{Id} \), we observe that \( s \) is continuously differentiable and \( s' = 1/c \) on \( J \). Recall that a speed measure \( \hat{m} \) is determined uniquely given the scale function \( s \) in the sense that if we replace \( s \) with \( ks + l \), \( k > 0 \), \( l \in \mathbb{R} \), then \( \hat{m} \) is replaced with \( k\tilde{m} \). Thanks to this observation, and the fact that speed and scale determine a diffusion uniquely, we conclude that \( \mathbb{P}_{x_0} = \hat{P}_{x_0} \). This, however, is a contradiction to our general assumption that \( \mathbb{P}_{x_0} \neq \hat{P}_{x_0} \). Hence, \( \mathbb{P}_{x_0} \)-a.s. \( S \leq \infty \) and, by Proposition 2.3, \( \mathbb{P}_{x_0}, \mathbb{P}_{x_0} \)-a.s. \( S \leq \infty = U \vee V \wedge R \).

The cases \((\alpha = \Delta, x_0 < \gamma = r)\) and \((\alpha = x_0 < \gamma = r)\) can be treated with the techniques developed in Cases 5 and 6. The only point where additional arguments are needed is the situation where \( \alpha = \Delta \), \( l \) is a reflecting boundary for (necessarily both) diffusions and \( r \) is either inaccessible for both diffusions or absorbing for both diffusions. The reason for this is that \( X \) is no local \( \mathbb{P}_{x_0} \)-martingale anymore. On the contrary, in the proof of Case 5, the local martingale property of \( X \) was used twice, namely in the definition of the candidate density \( Z \) and in the proof of (4.21). We now discuss the necessary changes.

Suppose that \( \alpha = \Delta \), \( x_0 < \gamma = r \), that \( l \) is reflecting for both diffusions and that \( r \) is inaccessible for both diffusions.\(^{10}\) We then infer from Lemma C.7 that \( \mathbb{P}_{x_0}, \mathbb{P}_{x_0} \)-a.s. \( U \vee V \wedge R = T_r = \infty \), that is, we need only to prove that \( \mathbb{P}_{x_0} \perp \hat{P}_{x_0} \). To simplify our notation, we also assume that \( l = 0 \).

Recall from Lemma C.15 (i) that \( X \) is a \( \hat{P}_{x_0} \)-semimartingale and denote its continuous local \( \hat{P}_{x_0} \)-martingale part by \( X^c \). Quite similar as in Case 5 above, we define

\[
Z \triangleq \exp \left( -\frac{1}{2} \int_0^t \beta(X_s)dX^c_s - \frac{1}{8} \int_0^t [\beta(X_s)]^2 d(X,X)_s \right).
\]

Using that \( \beta \in L^2_{\text{loc}}([0,r]) \), it follows as in Case 5 that \( Z \) is well-defined as a continuous local \( \hat{P}_{x_0} \)-martingale. We, further, observe that

\[
\hat{P}_{x_0} \text{-a.s.} \quad Z_{\infty-} (= Z_{T_{r-}}) = 0.
\]

Indeed, if \( r < \infty \), then (4.28) follows as in Case 5 from Lemma C.21 due to the fact that \( r \) necessarily fails to be half-good. In the opposite case \( r = \infty \), the diffusion \( (x \mapsto \hat{P}_x) \) is recurrent by Lemma C.10, hence (4.28) follows from Lemma C.22, once we establish that \( \beta \) is non-vanishing. Now, assuming that \( \beta \) vanishes (a.e. with respect to the Lebesgue measure) and noting that all points in \([0,r)\) \((= J = \hat{J})\) are non-separating yields that \( s' \) equals some positive constant \( c \) on \([0,r)\) and, hence, \( \hat{m} = \frac{1}{c} \tilde{m} \) (recall Definitions 2.6 and 2.14). As discussed in Case 6, this would imply \( \mathbb{P}_{x_0} = \hat{P}_{x_0} \), which contradicts to our

\(^{10}\)The remaining case where \( r \) is absorbing for both diffusions is handled via a sequence \( x_0 < r_1 < r_2 < \cdots \) such that \( r_n \to r \) in exactly the same way as in Case 5.
general assumption $\mathbb{P}_{x_0} \neq \tilde{\mathbb{P}}_{x_0}$ and concludes the proof of (4.28). Next, let $\sigma$ be an $(\mathcal{F}_t^\circ)_{t \geq 0}$-stopping time such that $\mathbb{E}_{x_0}^\circ[Z_x] = 1$. Then, we can define a probability measure $\mathbb{K}$ by the formula

$$
\mathbb{K}(G) \triangleq \mathbb{E}_{x_0}^\circ[Z_x1_G], \quad G \in \mathcal{F}.
$$

Below, we prove that $\mathbb{K} = \mathbb{P}_{x_0}$ on $\mathcal{F}_\sigma^\circ$. Once this identity is established, we can reuse arguments from Case 5 and deduce $\tilde{\mathbb{P}}_{x_0} \perp \mathbb{P}_{x_0}$ from the fact that $\tilde{\mathbb{P}}_{x_0}$-a.s. $Z_{x_0} = 0$ and Jessen’s theorem. We omit this part of the proof and concentrate on the proof for $\mathbb{K} = \mathbb{P}_{x_0}$ on $\mathcal{F}_\sigma^\circ$. Again, as in Case 5, we use a martingale problem argument. Take $f \in C_b([s(0), s(r)]; \mathbb{R})$ such that $f^*_t$ exists on $[s(0), s(r))$ as a right-continuous function of locally finite variation, $d\tilde{f}^*_t = 2gd\circ s^{-1}$ on $(s(0), s(r))$ and $f^*_t(s(0)) = 2g(s(0))\mathbb{m}(\{0\})$ for some $g \in C_b([s(0), s(r)]; \mathbb{R})$. We deduce from Lemma C.17, that $X^e = X - \frac{1}{2}L^0(X)$, where $L(X) = \{L^0_t(X); (t, y) \in \mathbb{R}_+ \times J\}$ denotes the semimartingale local time of $X$ under $\tilde{\mathbb{P}}_{x_0}$. For what follows, we notice that $L(X)$ is jointly continuous on $\mathbb{R}_+ \times J (= \mathbb{R}_+ \times [0, r])$. This follows from (C.4) and the joint continuity of the diffusion local time $\tilde{L}(X)$ in Lemma C.15. As all points in $[0, r)$ are non-separating, $s$ is a $C^1$-function with absolutely continuous derivative on $[0, r)$ (recall the Definitions 2.6 and 2.14 (iii)). Thus, we can apply Lemma C.28 with $\alpha$ and obtain that $\tilde{\mathbb{P}}_{x_0}$-a.s.

$$
ds(X_t) = \sigma'(X_t)dX_t + \frac{1}{2}\sigma''(X_t)d\langle X, X \rangle_t
$$

Using that $df^*_t = 2gd\circ s^{-1}$ on $s(J^c)$, $d\mathbb{m} = \sigma'\mathbb{m}$ on $J^c$ (see Definition 2.6) and Lemma C.26 (iii), we obtain $\tilde{\mathbb{P}}_{x_0}$-a.s.

$$
\frac{1}{2} \int_{(s(0), s(\tau))} L^\circ_t(s(X))f'_+ (dx) = \int_{(0, r)} L^\circ_t(s(X))g(s(x))\mathbb{m}(dx) = \int_{(0, r)} L^\circ_t(X)g(s(x))\tilde{\mathbb{m}}(dx).
$$

Using Lemma C.28 with $f$, (4.29), (4.30), $\frac{1}{2}f'_+ (s(0))\sigma'(0) = g(s(0))\tilde{\mathbb{m}}(\{0\})$, which uses the fact that 0 is non-separating (see Definition 2.14 (iii)), and the occupation time formula for diffusions (see Lemma C.15, in particular, formulas (C.4) and (C.6)), we get $\tilde{\mathbb{P}}_{x_0}$-a.s.

$$
df(s(X_t)) = f'_+ (s(X_t))d\sigma(X_t) + \frac{1}{2} \int_{(s(0), s(\tau))} L^\circ_t(s(X))f'_+ (dx)
$$

where $d\sigma_t = f'_+ (s(X_t))\sigma'(X_t)dX_t$. Then $\tilde{\mathbb{P}}_{x_0}$-a.s. it holds

$$
d\langle Z, f(s(X)) \rangle_t = -\frac{1}{2}Z_t\beta(X_t) f'_+ (X_t)\sigma'(X_t)d\langle X, X \rangle_t = -\frac{1}{2}Z_t f'_+ (s(X_t))\sigma''(X_t)d\langle X, X \rangle_t,
$$

where the second equality follows from the fact that $\mu_L$-a.e. $\sigma'' = \beta \sigma'$ (see Definition 2.6) and the semimartingale occupation time formula (Lemma C.26). By integration by parts, (3.41) and (3.42), we obtain that $\tilde{\mathbb{P}}_{x_0}$-a.s.

$$
df(s(X_t))Z_t = f(s(X_t))dZ_t + Z_tdf(X_t) + d\langle Z, f(X) \rangle_t = f(s(X_t))dZ_t + Z_t d\sigma_t + Z_t g(s(X_t))dt.
$$

Hence, as in Case 5, using [25, Proposition III.3.8], we get that $f(s(X_{t\wedge \sigma})) - \int_0^{t\wedge \sigma} g(s(X_u))du$ is a local $\mathbb{K}$-martingale and, by Lemmata B.4 and B.5, $\tilde{\mathbb{P}}_{x_0}$ on $\mathcal{F}_\sigma^\circ$. This finishes our discussion of this case.

Up to symmetry we considered all possible cases.

\[\square\]
Appendix A. Examples for Separating Times

In the first example we relate our definition of non-separating (good) points to those from \[8, 35\] for the Itô diffusion setting. Not surprisingly, we will see that the definitions coincide in this case.

**Example A.1** (Itô diffusion setting). Let \( J^o = (l, r) \) and take four Borel measurable functions \( b, \tilde{b}, \sigma, \tilde{\sigma} : J^o \to \mathbb{R} \) such that the Engelbert–Schmidt conditions hold, i.e.,
\[
\sigma^2, \tilde{\sigma}^2 > 0 \text{ everywhere on } J^o, \quad \frac{1 + |b|}{\sigma^2}, \quad \frac{1 + |\tilde{b}|}{\tilde{\sigma}^2} \in L_{\text{loc}}^1(J^o).
\]

Suppose that \( x_0 \in J^o \) and define
\[
s(x) \triangleq \int_x^r \exp \left( -2 \int \frac{b(y)dy}{\sigma^2(y)} \right) dz, \quad \tilde{s}(x) \triangleq \int_x^r \exp \left( -2 \int \frac{\tilde{b}(y)dy}{\tilde{\sigma}^2(y)} \right) dz,
\]
and
\[
(A.1) \quad m(dx) \triangleq \frac{dx}{s'(x)\sigma^2(x)}, \quad \tilde{m}(dx) \triangleq \frac{dx}{\tilde{s}'(x)\tilde{\sigma}^2(x)}.
\]

Moreover, we suppose that \( m(\{l\}) \equiv m(\{r\}) \equiv \tilde{m}(\{l\}) \equiv \tilde{m}(\{r\}) \equiv \infty \) in case the points are accessible. In other words, we stipulate that the diffusions are absorbed in the boundaries of their state spaces in case they can reach them in finite time. Providing some intuition, in this case \( \mathbb{P}_{x_0} \) is the law of an Itô diffusion \( X \) with dynamics
\[
dX_t = b(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x_0,
\]
up to the hitting time of the boundaries, and \( \tilde{\mathbb{P}}_{x_0} \) is the law of an Itô diffusion \( Y \) with dynamics
\[
dY_t = \tilde{b}(Y_t)dt + \tilde{\sigma}(Y_t)dB_t, \quad Y_0 = x_0,
\]
up to the hitting time of the boundaries, where \( W \) and \( B \) are standard Brownian motions. We refer to \[27, \text{Section 5.5}\] for precise definitions.

Let us now understand Definitions 2.6, 2.11 and 2.14 in this specific setting. First, the differential quotient \( d^+s/d\tilde{s} \) clearly exists everywhere on \( J^o \) and it equals
\[
\frac{d^+s}{d\tilde{s}} = \frac{s'}{\tilde{s}'} = \exp \left( 2 \int \frac{b(y)dy}{\sigma^2(y)} - 2 \int \frac{\tilde{b}(y)dy}{\tilde{\sigma}^2(y)} \right).
\]

In particular, \( d^+s/d\tilde{s} \) is an absolutely continuous function. This yields that (2.16) is satisfied on \( J^o \) with
\[
\beta = \frac{d^2s}{d\tilde{s}d\tilde{s}} = \frac{s'}{\tilde{s}'} \frac{1}{\tilde{s}'} = 2 \frac{\tilde{b}}{\tilde{s}'} \left( \frac{1}{\tilde{\sigma}^2} - \frac{b}{\sigma^2} \right).
\]

Let \( \mu_L \) denote the Lebesgue measure. We deduce from (A.1) that
\[
\frac{dm}{d\tilde{m}} = \frac{\tilde{s}'\tilde{\sigma}^2}{s'\sigma^2} \mu_L\text{-a.e. on } J^o
\]

(more precisely, this holds for any Lebesgue point, see [41, Chapter 7]). As in Definition 2.6, we now consider a point \( x \in J^o \) and an open neighborhood \( U(x) \subset J^o \) of \( x \). We see that
\[
\frac{dm}{d\tilde{m}} \frac{d^+s}{d\tilde{s}} = 1 \text{ on } U(x) \quad \implies \quad \tilde{\sigma}^2 = \sigma^2 \text{ } \mu_L\text{-a.e. on } U(x).
\]

Conversely, if \( \tilde{\sigma}^2 = \sigma^2 \text{ } \mu_L\text{-a.e. on } U(x) \), then we get everywhere on \( U(x) \)
\[
\frac{dm}{d\tilde{m}} = \frac{\tilde{s}'}{s'}
\]

by the continuity of \( \tilde{s}' \) and \( s' \) and the mean value theorem for Riemann–Stieltjes integrals ([46, p. 197]). In summary, we have
\[
\frac{dm}{d\tilde{m}} \frac{d^+s}{d\tilde{s}} = 1 \text{ on } U(x) \iff \tilde{\sigma}^2 = \sigma^2 \text{ } \mu_L\text{-a.e. on } U(x).
\]
Now, suppose that \( \mu_L \text{-a.e. on } U(x) \) we have \( \tilde{s}^2 = \sigma^2 \). Hence, \( \mu_L \text{-a.e. on } U(x) \)
\[
\beta = \frac{2}{\tilde{\gamma}} \left( \frac{\tilde{b} - b}{\sigma^2} \right)
\]
and
\[
(A.2) \quad (\beta(z))^2 \tilde{s}(dz) = \frac{4}{\tilde{\gamma}^2(z)} \frac{\tilde{b}(z) - b(z))^2}{\sigma^4(z)} dz.
\]
As \( \tilde{s}' \) is positive and continuous on \( J^0 \), provided that \( \partial(U(x)) \subset J^0 \), we have
\[
\int_{U(x)} (\beta(z))^2 \tilde{s}(dz) < \infty \iff \int_{U(x)} \frac{(\tilde{b}(z) - b(z))^2}{\sigma^4(z)} dz < \infty.
\]
We conclude that a point \( x \in J^0 \) is non-separating in the sense of Definition 2.6 if and only if there exists an open neighborhood \( U(x) \subset J^0 \) of \( x \) such that \( s^2 = \sigma^2 \mu_L \text{-a.e. on } U(x) \) and \( (\tilde{b} - b)^2/\sigma^4 \in L^1(U(x)) \). This is precisely the definition of a non-separating (good) interior point from [8, 35]. Recalling (2.22) and (A.2), we also see that, for the boundary points \( l \) and \( r \), Definition 2.11 of half-goodness coincides with the definition of a non-separating (good) boundary point from [8, 35]. It remains to notice that in this case, where accessible boundaries are forced to be absorbing, a boundary point is non-separating (good) in the sense of Definition 2.14 if and only if it is half-good in the sense of Definition 2.11.\(^{11}\)

In summary, Theorem 2.18 includes [8, Theorem 5.1] and [35, Theorem 5.5].

Example A.1 gives the impression that \( dm/d\tilde{m} \cdot d^2s/d\tilde{s} = 1 \) means that the diffusion coefficients coincide, which is well-known to be necessary for (local) absolute continuity, recall Girsanov’s theorem (Lemma C.30). However, the condition encodes much more as the following example illustrates.

**Example A.2 (Sticky Brownian motions).** Suppose that \( J = \tilde{J} = \mathbb{R} \) and that \( (x \mapsto P_x) \) and \( (x \mapsto \tilde{P}_x) \) are Brownian motions sticky at the origin. More precisely, we assume that both \( (x \mapsto P_x) \) and \( (x \mapsto \tilde{P}_x) \) are on natural scale and that
\[
\begin{align*}
\tilde{m}(dx) &= dx + \gamma \delta_0(dx), \\
m(dx) &= dx + \tilde{\gamma} \delta_0(dx),
\end{align*}
\]
where \( \gamma, \tilde{\gamma} \in (0, \infty) \). It is well-known ([2, 18]) that \( P_{x_0} \) is the law of a solution process to
\[
(A.3) \quad dX_t = 1_{\{X_t \neq 0\}} dW_t, \quad 1_{\{X_t = 0\}} dt = \gamma dL_t^0(X), \quad X_0 = x_0,
\]
and \( \tilde{P}_{x_0} \) is the law of a solution process to
\[
(A.4) \quad dY_t = 1_{\{Y_t \neq 0\}} dB_t, \quad 1_{\{Y_t = 0\}} dt = \tilde{\gamma} dL_t^0(Y), \quad Y_0 = x_0,
\]
where \( L^0(X) \) and \( L^0(Y) \) denote the semimartingale local times of \( X \) and \( Y \) in zero, and \( W \) and \( B \) are standard Brownian motions. In this setting we clearly have \( d^2s/d\tilde{s} = 1 \) and
\[
\left( \frac{dm}{d\tilde{m}} \right)(z) = \begin{cases} 1, & z \neq 0; \\
\frac{\gamma}{\tilde{\gamma}}, & z = 0.
\end{cases}
\]
Consequently, the origin is separating if and only if \( \gamma \neq \tilde{\gamma} \). All other points in \( \mathbb{R} \) are non-separating. Finally, since \( \tilde{s}(\pm \infty) = \tilde{\gamma}(\pm \infty) = \pm \infty \), the boundary points \( \pm \infty \) are separating. In summary,
\[
J_{\text{sep}} = \begin{cases} (-\infty, \infty), & \gamma = \tilde{\gamma}; \\
(-\infty, 0, \infty), & \gamma \neq \tilde{\gamma}.
\end{cases}
\]
With this observation at hand, we can deduce the following result from our main theorem.

**Corollary A.3.** Let \( S \) be the separating time for \( P_{x_0} \) and \( \tilde{P}_{x_0} \).

(i) If \( \gamma = \tilde{\gamma} \), then \( P_{x_0}, \tilde{P}_{x_0} \text{-a.s. } S = \delta. \)

(ii) If \( \gamma \neq \tilde{\gamma} \), then \( P_{x_0}, \tilde{P}_{x_0} \text{-a.s. } S = T_0 \text{ and, in particular, } P_0 \perp \tilde{P}_0 \text{ on } \mathcal{F}_0. \)

\(^{11}\)Definition 2.14 starts to be essential as long as we include instantaneously or slowly reflecting boundaries into consideration.
are skew Brownian motions. More precisely, we assume that 

Example A.4 (Skew Brownian motions). Suppose that $J = \mathbb{R}$ and that $(x \mapsto \mathbb{P}_x)$ and $(x \mapsto \mathbb{P}^-_x)$ are skew Brownian motions. More precisely, we assume that

$$s(x) = \begin{cases} (1 - \alpha)x, & x \geq 0, \\ \alpha x, & x < 0, \end{cases}$$

and

$$\hat{s}(x) = \begin{cases} (1 - \hat{\alpha})x, & x \geq 0, \\ \hat{\alpha} x, & x < 0, \end{cases}$$

and

$$m(dx) = \left((1 - \alpha)^{-1}1_{\{x > 0\}} + \alpha^{-1}1_{\{x < 0\}}\right)dx,$$

$$\hat{m}(dx) = \left((1 - \hat{\alpha})^{-1}1_{\{x > 0\}} + \hat{\alpha}^{-1}1_{\{x < 0\}}\right)dx,$$

where $\alpha, \hat{\alpha} \in (0, 1)$. We easily see that

$$\left(\frac{d^+s}{d\hat{s}}\right)(x) = \begin{cases} (1 - \alpha)/(1 - \hat{\alpha}), & x \geq 0, \\ \alpha/\hat{\alpha}, & x < 0. \end{cases}$$

Hence, in case $\alpha \neq \hat{\alpha}$ the origin is separating. Moreover, as

$$\left(\frac{dm}{d\hat{m}}\right)(x) = \begin{cases} (1 - \alpha)/(1 - \hat{\alpha}), & x > 0, \\ [1 - (1 - \alpha)\hat{\alpha}]/[(1 - \alpha)\alpha], & x = 0, \\ \hat{\alpha}/\alpha, & x < 0, \end{cases}$$

we note that all interior points except the origin are non-separating regardless of $\alpha$ and $\hat{\alpha}$. Let us stress that (i) as well as (iii) from Definition 2.6 fail for the origin. Finally, as $s(\pm\infty) = \hat{s}(\pm\infty) = \pm\infty$, the boundary points $\pm\infty$ are separating. Now, as in the proof of Corollary A.3, we get the following:

Corollary A.5. Let $S$ be the separating time for $\mathbb{P}_{x_0}$ and $\mathbb{P}^-_{x_0}$.

(i) If $\alpha = \hat{\alpha}$, then $\mathbb{P}_{x_0}, \mathbb{P}^-_{x_0}$-a.s. $S = \delta$.

(ii) If $\alpha \neq \hat{\alpha}$, then $\mathbb{P}_{x_0}, \mathbb{P}^-_{x_0}$-a.s. $S = T_0$ and, in particular, $\mathbb{P}_{x_0} \perp \mathbb{P}^-_{x_0}$ on $\mathcal{F}_0$.

In [8, Theorem 4.1] it was shown that the first hitting time of the origin is the separating time for two Bessel processes with different dimensions. In the next example we deduce this result from Theorem 2.18.

Example A.6 (Generalized Bessel processes). For $\gamma > 0$ and $x_0 \in \mathbb{R}_+$, a solution $Y$ to

$$dY_t = \gamma dt + 2\sqrt{Y_t}dW_t, \quad Y_0 = x_0, \quad W = \text{Brownian motion},$$

is called a square Bessel process of dimension $\gamma$ started at $x_0$. The number $\nu \triangleq \gamma/2 - 1$ is called its index. The square root $Z \triangleq \sqrt{Y}$ is called a Bessel process of dimension $\gamma$. A detailed discussion of (square) Bessel processes can be found in [39, Chapter XI].

In the following we discuss the separating time for generalized Bessel processes (in the sense that we allow for arbitrary boundary behavior in the origin whenever the origin is accessible). More precisely, we

Proof. Of course, in case $\gamma = \hat{\gamma}$ we have $\mathbb{P}_{x_0} = \mathbb{P}^-_{x_0}$ and hence, (i) is trivial. We now discuss part (ii) and therefore assume that $\gamma \neq \hat{\gamma}$. Clearly, we have $R = \delta$. If $x_0 = 0$, then $U = V = T_0 = 0$ and the claim follows. Suppose now that $x_0 > 0$ (resp. $x_0 < 0$). By virtue of Lemma C.8, we get $\mathbb{P}_{x_0}, \mathbb{P}^-_{x_0}$-a.s. $U = T_0$ (resp. $V = T_0$) and $\mathbb{P}_{x_0}, \mathbb{P}^-_{x_0}$-a.s. $V = T_\infty = \infty$ (resp. $U = T_{-\infty} = \infty$). We conclude from Theorem 2.18 that $\mathbb{P}_{x_0}, \mathbb{P}^-_{x_0}$-a.s. $S = T_0$. □
take \( J^o = (0, \infty) \) and \( (x \mapsto \mathbb{P}^x_\gamma) \) to be a regular diffusion with scale function, for \( x > 0 \),
\[
g^\gamma(x) \triangleq \begin{cases} -\text{sgn}(\nu)x^{-2\nu}, & \nu \neq 0, \\ 2\log(x), & \nu = 0, \end{cases}
\]
and speed measure, on \( \mathcal{B}((0, \infty)) \),
\[
m^\gamma(dx) \triangleq \begin{cases} x^{2\nu+1}dx/2|\nu|, & \nu \neq 0, \\ \frac{1}{2}x dx, & \nu = 0. \end{cases}
\]
The above scale and speed coincide with those of the Bessel process \( Z \). For all \( \gamma > 0 \) we have that \( \infty \) is inaccessible. In case \( 0 < \gamma < 2 \) the origin is regular (in particular, \( J = [0, \infty) \)) and in case \( \gamma \geq 2 \) the origin is inaccessible (so, \( J = (0, \infty) \)). For the square Bessel process as defined in (A.5) with \( 0 < \gamma < 2 \) the origin is instantaneously reflecting, which corresponds to \( m^\gamma(\{0\}) = 0 \) for the Bessel process. For our generalized Bessel process, we allow for all values \( m^\gamma(\{0\}) \in [0, \infty] \) in case \( 0 < \gamma < 2 \). Notice that stopping at the origin is included as particular case \( m^\gamma(\{0\}) = \infty \).

From now on, fix \( x_0, \gamma, \bar{\gamma} > 0 \) such that \( \gamma \neq \bar{\gamma} \).

**Lemma A.7.** Recalling that \( J_{\text{sep}} \subset [0, \infty) \) is the set of separating points for \( \mathbb{P}^x_\gamma \) and \( \mathbb{P}^x_{\bar{\gamma}} \), \( J_{\text{sep}} = \{0, \infty\} \).

**Proof.** We write \( \bar{\nu} \triangleq \bar{\gamma}/2 - 1 \). In the following we investigate which points in \([0, \infty]\) are separating. If \( \gamma, \bar{\gamma} \neq 2 \), we get, for \( x > 0 \),
\[
\left( \frac{d^+ g^\gamma}{ds^\gamma} \right)(x) = \frac{(s^\gamma)'(x)}{(s^\gamma)'(x)} = \frac{\nu}{|\nu|} x^{2(\bar{\nu}-\nu)}, \quad \left( \frac{dm^\gamma}{dm^\bar{\gamma}} \right)(x) = \left| \frac{\bar{\nu}}{\nu} \right| x^{2(\nu-\bar{\nu})},
\]
and
\[
\beta(x) \equiv \frac{(d^+ g^\gamma/ds^\bar{\gamma})'(x)}{(s^\gamma)'(x)} = \frac{\bar{\nu} - \nu}{|\nu|} x^{2\bar{\nu}}.
\]
Further, if \( \gamma = 2 \) and \( \bar{\gamma} \neq 2 \), we obtain, for \( x > 0 \),
\[
\left( \frac{d^+ g^\gamma}{ds^\gamma} \right)(x) = \frac{x^{2\bar{\nu}}}{|\nu|}, \quad \left( \frac{dm^\gamma}{dm^\bar{\gamma}} \right)(x) = \frac{|\bar{\nu}|}{x^{2\bar{\nu}}}, \quad \beta(x) = 2 \text{sgn}(\bar{\nu})x^{2\bar{\nu}}.
\]
The case where \( \gamma \neq 2 \) and \( \bar{\gamma} = 2 \) looks similar. Thus, all points in \((0, \infty)\) are non-separating.

Next, we discuss the boundary points. Notice the following:
\[
\begin{array}{c|ccc}
\bar{\gamma} \\
A \infty \\
\gamma(0) & < 2 & = 2 & > 2 \\
\gamma(\infty) & \infty & \infty & 0 \\
\end{array}
\]
Thus, \( \infty \) is separating if \( \bar{\gamma} \leq 2 \) and \( \gamma(0) \) is separating if \( \bar{\gamma} \geq 2 \). For \( \bar{\gamma} < 2 \) the origin is separating, because
\[
\int_{0+} |g^\gamma(x) - g^\gamma(0)| (\beta(x))^2 g^\gamma(dx) \overset{\varepsilon}{=} \int_{0+} x^{-2\bar{\nu}} x^{4\bar{\nu}} x^{-2\bar{\nu}-1} dx = \int_{0+} \frac{dx}{x} = \infty,
\]
where \( \overset{\varepsilon}{=} \) denotes an equality up to a positive multiplicative constant, and for \( \bar{\gamma} > 2 \) the boundary \( \infty \) is separating, as
\[
\int_{0}^\infty |g^\gamma(x) - g^\gamma(\infty)| (\beta(x))^2 g^\gamma(dx) \overset{\varepsilon}{=} \int_{0}^{\infty} \frac{dx}{x} = \infty.
\]
We conclude that \( J_{\text{sep}} = \{0, \infty\} \).

Thanks to this lemma, we obtain the following representation of the separating time for \( \mathbb{P}^x_\gamma \) and \( \mathbb{P}^x_{\bar{\gamma}} \).

**Corollary A.8.** Let \( S \) be the separating time for \( \mathbb{P}^x_\gamma \) and \( \mathbb{P}^x_{\bar{\gamma}} \). Then, \( \mathbb{P}^x_\gamma, \mathbb{P}^x_{\bar{\gamma}} \)-a.s. \( S = T_0 \).

**Proof.** Clearly, \( R = \delta \). Notice the following:
Thus, by virtue of Lemmata A.7 and C.8, we obtain $P_{x_0}^\gamma$-a.s.

$$U = \begin{cases} T_0, & \gamma \leq 2, \\ \delta, & \gamma > 2, \end{cases}, \quad V = \begin{cases} T_\infty(= \infty), & \gamma < 2, \ m^\gamma(\{0\}) < \infty, \\ \delta, & \gamma < 2, \ m^\gamma(\{0\}) = \infty, \\ T_\infty(= T_0), & \gamma \geq 2 \end{cases}$$

(notice that, in the case $\gamma < 2$ and $m^\gamma(\{0\}) < \infty$, Lemma C.7 yields $P_{x_0}^\gamma$-a.s. $\limsup_t \nu_{T_\infty} X_t = \infty$), therefore, $P_{x_0}^\gamma$-a.s. $U \land V \land R = T_0$. As this computation is independent of $\gamma$, the corollary follows from Theorem 2.18. 

□

Remark A.9. It is interesting to observe that separability of the origin is independent of the boundary behavior, i.e., in particular of its attainability or the values $m^\gamma(\{0\})$ and $\tilde{m}(\{0\})$ in the attainable case. This fact shows that equivalence of $P_{x_0}^\gamma$ and $P_{x_0}^\tilde{m}$ is already lost at the time the origin is hit and that the separating time is not affected by stopping at the origin. This is different for sticky and skew Brownian motions (see Examples A.2 and A.4). Indeed, if these processes are stopped in the origin, they coincide, which means they are trivially equivalent. In the following, we present a non-trivial example of two processes whose equivalence is lost right after the time a boundary point is hit (but not at this time).

Example A.10. We take $J = \tilde{J} = \mathbb{R}_+$, and, for $x \in \mathbb{R}_+$,

$$s(x) = \int^x \exp \left( - \int^y \frac{dz}{\sqrt{2}} \right) dy = \text{const} \int^x e^{-2\sqrt{y}} dy \quad \left( = \text{const} - \text{const} e^{-2\sqrt{x} (\sqrt{x} + \frac{1}{2})} \right),$$

$$\tilde{s}(x) = x,$$

and, on $B((0, \infty))$,

$$m(dx) = \frac{dx}{\tilde{s}(x)}, \quad \tilde{m}(dx) = dx.$$

Notice that

$$\int^\infty (s(\infty) - s(x)) m(dx) = \int^\infty \left( \sqrt{x} + \frac{1}{2} \right) dx = \infty.$$ 

It follows from (2.11) that $\infty$ is not accessible for the diffusion with characteristics $(s, m)$. Clearly, the same is true for the diffusion with characteristics $(\tilde{s}, \tilde{m})$. Furthermore, (2.10) yields that the origin is regular for both diffusions. Therefore, the values $m(\{0\}), \tilde{m}(\{0\})$ can be arbitrarily chosen from $[0, \infty]$. We thus take an arbitrary $m(\{0\}) \in [0, \infty]$ and an arbitrary $\tilde{m}(\{0\}) \in [0, \infty]$.

Fix $x_0 > 0$. Providing an intuition, under $P_{x_0}$, we have

$$dX_t = \frac{dt}{2\sqrt{x_0}} + dW_t, \quad t < T_0,$$

and, under $\tilde{P}_{x_0}$, we have $d\tilde{X}_t = dB_t$ for $t < T_0$, where $W$ and $B$ are standard Brownian motions. The behavior after $T_0$ under $P_{x_0}$ (resp., $\tilde{P}_{x_0}$) depends on the choice of $m(\{0\})$ (resp., $\tilde{m}(\{0\})$).

Lemma A.11. Recalling that $J_{\text{sep}} \subset [0, \infty]$ is the set of separating points for $P_{x_0}$ and $\tilde{P}_{x_0}$,

$$J_{\text{sep}} = \begin{cases} \{\infty\}, & m(\{0\}) = \tilde{m}(\{0\}) = \infty, \\ \{0, \infty\}, & m(\{0\}) \land \tilde{m}(\{0\}) < \infty. \end{cases}$$

Proof. We compute

$$\left( \frac{d^+ s}{d\tilde{s}} \right)(x) = s'(x) = e^{-2\sqrt{x}}, \quad \beta(x) = \frac{d^+ s}{d\tilde{s}} \left( \frac{d^+ s}{d\tilde{s}} \right)'(x) = \frac{(d^+ s/d\tilde{s})'(x)}{s'(x)} = - \frac{1}{\sqrt{x}}.$$
SEPARATING TIMES FOR DIFFUSIONS

It is easy to see that all points in \((0, \infty)\) are non-separating. As \(\tilde{s}(\infty) = \infty\), \(\infty\) is separating. Notice that
\[
\int_{0^+} |\tilde{s}(x) - \tilde{s}(0)| (\beta(x))^2 \tilde{s}(dx) = \int_{0^+} dx < \infty,
\]
that is, the origin is in any case half-good, but
\[
\int_{0^+} (\beta(x))^2 \tilde{s}(dx) = \int_{0^+} \frac{dx}{x} = \infty.
\]
Hence, the origin is non-separating if and only if \(m(\{0\}) = \tilde{m}(\{0\}) = \infty\). \(\square\)

With this representation of \(J_{\text{sep}}\) at hand, one can deduce from Theorem 2.18 the following result on the separating time.

**Corollary A.12.** Let \(S\) be the separating time for \(P_{x_0}\) and \(\tilde{P}_{x_0}\).

(i) Let \(m(\{0\}) = \tilde{m}(\{0\}) = \infty\). Then, the following hold:
- \(P_{x_0}\)-a.s. \(S = \delta\), while \(P_{x_0}\)-a.s. \(S \geq \infty\).
- \(P_{x_0}(S = \infty) > 0\) and \(P_{x_0}(S = \delta) > 0\).
- We have the following mutual arrangement between \(P_{x_0}\) and \(\tilde{P}_{x_0}\) from the viewpoint of their (local) absolute continuity and singularity:
  \[ P_{x_0} \sim_{\text{loc}} \tilde{P}_{x_0}, \quad \tilde{P}_{x_0} \ll P_{x_0}, \quad P_{x_0} \not\ll \tilde{P}_{x_0}, \quad P_{x_0} \not\perp \tilde{P}_{x_0}. \]

(ii) Let \(m(\{0\}) \wedge \tilde{m}(\{0\}) < \infty\). Then, the following hold:
- \(P_{x_0}, \tilde{P}_{x_0}\)-a.s. \(S = T_0\).
- \(P_{x_0}(T_0 < \infty) = 1, \ P_{x_0}(T_0 < \infty) > 0, \ P_{x_0}(T_0 = \infty) > 0\).
- We have the following mutual arrangement between \(P_{x_0}\) and \(\tilde{P}_{x_0}\) from the viewpoint of their (local) absolute continuity and singularity:
  \[ P_{x_0} \not\ll_{\text{loc}} \tilde{P}_{x_0}, \quad \tilde{P}_{x_0} \not\ll_{\text{loc}} P_{x_0}, \quad P_{x_0} \perp \tilde{P}_{x_0}. \]

**Proof.** Let us first notice the following:

\[
\begin{array}{cccc}
s(0) & s(\infty) & \tilde{s}(0) & \tilde{s}(\infty) \\
> -\infty & < \infty & = 0 & = \infty
\end{array}
\]

Moreover, as discussed above, \(\infty\) is inaccessible and 0 is regular for both diffusions. The claims, therefore, follow from Theorem 2.18, Proposition 2.3 and Lemmata A.11 and C.8. \(\square\)

For \(m(\{0\}) \neq \tilde{m}(\{0\})\) it is intuitive that equivalence might get lost right after the origin is hit, because the diffusions have different boundary behavior. However, Corollary A.12 shows that even in the case \(m(\{0\}) = \tilde{m}(\{0\}) < \infty\) the equivalence of \(P_{x_0}\) and \(\tilde{P}_{x_0}\) is lost right after the origin is hit. On the other hand, stopping the processes in the origin transfers us to the case \(m(\{0\}) = \tilde{m}(\{0\}) = \infty\), where hitting the origin does not break the equivalence any longer.

In our final example we discuss the importance of taking the (deterministic) time \(R\) into consideration. We thank Paul Jenkins for bringing the issue and the example to our attention.

**Example A.13.** Suppose that \((0, 1] \ni x \mapsto P_x\) is a Brownian motion and that \((0, 1] \ni x \mapsto \tilde{P}_x\) is a Brownian motion with a constant drift \(\mu \neq 0\), both with instantaneous reflection in their boundaries 0 and 1. To be more precise, the corresponding scale functions on \([0, 1]\) are given by
\[
\tilde{s}(x) = x, \quad s(x) = -\frac{e^{-2\mu x}}{2\mu}
\]
and the speed measures on \(B([0, 1])\) are given by
\[
\tilde{m}(dx) = dx, \quad m(dx) = e^{2\mu^2} dx.
\]
We take an arbitrary \(x_0 \in [0, 1]\) and discuss the separating time \(S\) for \(P_{x_0}\) and \(\tilde{P}_{x_0}\). It is not hard to see that all points in \([0, 1]\) are non-separating, which means that \(U = V = \delta\). However, the separating time for
\( \mathbb{P}_{x_0} \) and \( \tilde{\mathbb{P}}_{x_0} \) cannot be \( \delta \). The reason for this stems from the fact that a diffusion whose boundaries are both accessible and reflecting is necessarily recurrent (see Lemma C.10). More precisely, using this fact and the ratio ergodic theorem restated as Lemma C.11, we obtain that, for all bounded Borel functions \( f : [0, 1] \to \mathbb{R}_+ \), it holds \( \mathbb{P}_{x_0} \)-a.s.

\[
\frac{1}{t} \int_0^t f(X_s)ds \to \frac{1}{m([0, 1])} \int_0^1 f(x)m(dx), \quad t \to \infty,
\]

and \( \tilde{\mathbb{P}}_{x_0} \)-a.s.

\[
\frac{1}{t} \int_0^t f(X_s)ds \to \int_0^1 f(x)\tilde{m}(dx), \quad t \to \infty.
\]

Hence, in the case \( \mathbb{P}_{x_0}(S = \delta) \vee \tilde{\mathbb{P}}_{x_0}(S = \delta) > 0 \), we get

\[
\frac{1}{m([0, 1])} \int_0^1 f(x)m(dx) = \int_0^1 f(x)\tilde{m}(dx)
\]

for any countable collection of test functions \( f \). By the uniqueness theorem for probability measures, this implies that \( m/m([0, 1]) = \tilde{m} \). The latter is, however, false, as \( \mu \neq 0 \). Therefore, we must have \( \mathbb{P}_{x_0}, \tilde{\mathbb{P}}_{x_0} \)-a.s. \( S \leq \infty \). And, indeed, as \( R = \infty \), Theorem 2.18 yields \( \mathbb{P}_{x_0}, \tilde{\mathbb{P}}_{x_0} \)-a.s. \( S = U \wedge V \wedge R = \infty \).

**Discussion A.14.** More generally, in the case where both diffusions \((x \mapsto \mathbb{P}_x)\) and \((x \mapsto \tilde{\mathbb{P}}_x)\) are recurrent, an argument based on the ratio ergodic theorem similar to those from Example A.13 shows that \( \mathbb{P}_{x_0}, \tilde{\mathbb{P}}_{x_0} \)-a.s. \( S \leq \infty \). The question arises how this fact is encoded in Theorem 2.18 for the cases when \( R = \delta \). If there is a separating point in \( J^o \), then, by recurrence, \( \mathbb{P}_{x_0}, \tilde{\mathbb{P}}_{x_0} \)-a.s. \( U \wedge V \leq \infty \). Now, assume that all points in \( J^o \) are non-separating, that both diffusions are recurrent, and that at least one boundary point (say, \( l \)) is not reflecting for at least one of the diffusions (say, for \((x \mapsto \mathbb{P}_x))\). By Lemma C.10, in this case we have \( s(l) = -\infty \). As for the other boundary point, using that \((x \mapsto \mathbb{P}_x)\) and \((x \mapsto \tilde{\mathbb{P}}_x)\) are recurrent, we deduce from Lemma C.10 that

- either \( s(r) = \infty \) or \( r \) is reflecting for \((x \mapsto \mathbb{P}_x)\) and
- either \( \tilde{s}(r) = \infty \) or \( r \) is reflecting for \((x \mapsto \tilde{\mathbb{P}}_x)\).

Then, it follows that

- \( \mathbb{P}_{x_0} \)-a.s. \( U \leq \infty \) (by Lemma C.8 in the case \( s(r) = \infty \); by Lemma C.7 (iii) in the case \( r \) is reflecting for \((x \mapsto \mathbb{P}_x)\)).
- \( \tilde{\mathbb{P}}_{x_0} \)-a.s. \( U \leq \infty \) (by the same argument as for the other diffusion).

In summary, we see that \( \mathbb{P}_{x_0}, \tilde{\mathbb{P}}_{x_0} \)-a.s. \( U \wedge V \leq \infty \) in all cases when both \((x \mapsto \mathbb{P}_x)\) and \((x \mapsto \tilde{\mathbb{P}}_x)\) are recurrent except the only case, when all points in \( \text{cl}(J) \) are non-separating and both boundaries are reflecting for one, equivalently for both, of the diffusions. To account for this case, we need the deterministic time \( R \).

**Appendix B. Martingale Problem for Diffusions**

The martingale problem method is one of the key techniques to analyze Markov processes. The first martingale problem was introduced by Stroock and Varadhan for multidimensional diffusions which can be described via SDEs (see [27, Section 5.4] for an overview). It seems to us that the literature contains no martingale problem for general one-dimensional regular diffusions, although all required tools can be found in the monographs [20, 31]. In this paper, we need such a martingale problem for the proof of our main Theorem 2.18 as well as Theorem 3.5. In general, we also think that such a martingale problem is of independent interest.

Let \((J \ni x \mapsto \mathbb{P}_x)\) be a regular diffusion and set

\[
(G_{\alpha}f)(x) \triangleq \int_0^\infty e^{-\alpha t}\mathbb{E}_x[f(X_t)]dt, \quad f \in C_0(J; \mathbb{R}), \alpha > 0,
\]
where $C_b(J; \mathbb{R})$ denotes the space of bounded continuous functions $J \to \mathbb{R}$. Notice that $G_\alpha$ is a bounded linear operator from $C_b(J; \mathbb{R})$ into $C_b(J; \mathbb{R})$ (see [20, Lemma 30, p. 116]). Let us collect some useful facts (see [20, Corollaries 43, 44, p. 119]):

**Lemma B.1.**

(i) $G_\alpha f \equiv 0$ if and only if $f \equiv 0$.

(ii) The range of $G_\alpha$ does not depend on $\alpha$.

We define
\[
\Delta \triangleq \text{rng}(G_1) = \text{rng}(G_\alpha), \quad \Gamma \triangleq \text{Id} - G_1^{-1} \text{ on } \Delta.
\]
Notice that $\Delta \subset C_b(J; \mathbb{R})$. The operator $(\Gamma, \Delta)$ is called the (extended) generator of the diffusion $(x \mapsto \mathbb{P}_x)$. Here, we use the term “extended” to emphasize that we work on the space $C_b(J; \mathbb{R})$, which is less conventional than the space $C_0(J; \mathbb{R})$ of continuous functions vanishing at infinity.\textsuperscript{12}

The following result is given in [20, Lemma 46, p. 119].

**Lemma B.2.** $G_\alpha^{-1} = \alpha \text{ Id} - \Gamma$ on $\Delta$.

The next result can be proved (with minor changes) as [31, Theorem 3.33]. For reader’s convenience we present a proof, which is a slight modification of the one from [31]. The theorem provides the martingale problem for $(x \mapsto \mathbb{P}_x)$.

**Theorem B.3.** If $\mathbb{P}$ is a probability measure on $(\Omega, \mathcal{F})$ such that $\mathbb{P}(C(\mathbb{R}_+; J)) = 1$, $\mathbb{P}(X_0 = x) = 1$ and
\[
M^f \triangleq f(X) - f(x) - \int_0^t (\Gamma f)(X_s)ds
\]
is a local\textsuperscript{13} $\mathbb{P}$-martingale for all $f \in \Delta$, then $\mathbb{P} = \mathbb{P}_x$.

To understand the condition $\mathbb{P}(C(\mathbb{R}_+; J)) = 1$, recall that $\Omega = C([\mathbb{R}_+; [-\infty, \infty]]$ (see Section 2.2), while $f \in \Delta \subset C_b(J; \mathbb{R})$ is defined only on $J$. That is, the assumption $\mathbb{P}(C(\mathbb{R}_+; J)) = 1$ guarantees that $M^f$ is well-defined under $\mathbb{P}$.

**Proof.** Take $g \in C_b(J; \mathbb{R})$ and $\alpha > 0$. Set $f \triangleq G_\alpha g$ and note that $\alpha f - \Gamma f = G_\alpha^{-1} f = g$ by Lemma B.2. Further, we emphasize that $f(X)$ is a $\mathbb{P}$-semimartingale by hypothesis (because $f \in \Delta$). Using integration by parts yields that (under $\mathbb{P}$)
\[
d e^{-\alpha t} f(X_t) = f(X_t) d e^{-\alpha t} + e^{-\alpha t} df(X_t)
\]
\[
= -\alpha e^{-\alpha t} f(X_t) dt + e^{-\alpha t} dM_t^{f} + e^{-\alpha t} (\Gamma f)(X_t) dt
\]
\[
= -e^{-\alpha t} (\alpha f(X_t) - (\Gamma f)(X_t)) dt + e^{-\alpha t} dM_t^{f}
\]
\[
= -e^{-\alpha t} g(X_t) dt + e^{-\alpha t} dM_t^{f}.
\]
Consequently, the process
\[
e^{-\alpha t} f(X_t) + \int_0^t e^{-\alpha s} g(X_s) ds, \quad t \in \mathbb{R}_+,
\]
is a $\mathbb{P}$-martingale. Fix $s \in \mathbb{R}_+$. Then, for every $t > s$, we have $\mathbb{P}$-a.s.
\[
\mathbb{E}^\mathbb{P}[e^{-\alpha t} f(X_t) + \int_s^t e^{-\alpha r} g(X_r) dr | \mathcal{F}_s] = e^{-\alpha s} f(X_s).
\]

\textsuperscript{12}The conventional approach is to work with Feller processes, which correspond to Feller semigroups, i.e., strongly continuous contraction semigroups on $C_0(J; \mathbb{R})$. This, however, excludes diffusions with entrance boundaries, as $C_0(J; \mathbb{R})$ is not invariant under the respective semigroup. To include all diffusions, in particular, those with entrance boundaries, we work with $C_b(J; \mathbb{R})$. In this way, we gain generality but face the problem that, on the space $C_b(J; \mathbb{R})$, many diffusion semigroups (e.g., the Brownian semigroup on $C_b(\mathbb{R}; \mathbb{R})$) are not strongly continuous. Therefore, standard results for Feller processes should be applied with care. In particular, this is the reason for presenting a full proof of Theorem B.3.

\textsuperscript{13}Hence, necessarily, a $\mathbb{P}$-martingale, as the process $M^f$ is bounded on finite time intervals due to $\Delta \subset C_b(J; \mathbb{R})$. 

Letting $t \to \infty$ and using the dominated convergence theorem yields $\mathbb{P}$-a.s.

$$\mathbb{E}^\mathbb{P} \left[ \int_s^\infty e^{-ar} g(X_r) dr \bigg| \mathcal{F}_s \right] = e^{-as} f(X_s).$$

Thus, by the definition of the conditional expectation, we have for all $A \in \mathcal{F}_s$

$$\mathbb{E}^\mathbb{P} \left[ \int_0^\infty e^{-a(r+s)} g(X_{r+s}) dr \mathbb{1}_A \right] = \mathbb{E}^\mathbb{P} \left[ e^{-as} f(X_s) \mathbb{1}_A \right],$$

which means

(B.1) $$\mathbb{E}^\mathbb{P} \left[ \int_0^\infty e^{-ar} g(X_{r+s}) dr \mathbb{1}_A \right] = \mathbb{E}^\mathbb{P} \left[ f(X_s) \mathbb{1}_A \right].$$

By virtue of Dynkin’s formula ([20, Lemma 48, p. 119]), also the measure $\mathbb{P}_x$ satisfies the hypothesis of the theorem. Thus, the above equation also holds for $\mathbb{P}$ replaced by $\mathbb{P}_x$, i.e.,

(B.2) $$\mathbb{E}^{\mathbb{P}_x} \left[ \int_0^\infty e^{-ar} g(X_{r+s}) dr \mathbb{1}_A \right] = \mathbb{E}^{\mathbb{P}_x} \left[ f(X_s) \mathbb{1}_A \right].$$

Taking $s = 0$ and $A = \Omega$ in (B.1) and (B.2) yields that

$$\int_0^\infty e^{-ar} \mathbb{E}^\mathbb{P} \left[ g(X_t) \right] dt = f(x) = \int_0^\infty e^{-\alpha t} \mathbb{E}^{\mathbb{P}_x} \left[ g(X_t) \right] dt.$$

Hence, the uniqueness theorem for Laplace transforms (the maps $t \mapsto \mathbb{E}^\mathbb{P} \left[ g(X_t) \right], \mathbb{E}^{\mathbb{P}_x} \left[ g(X_t) \right]$)

are continuous by the continuity of $t \mapsto X_t$ and the dominated convergence theorem) yields that $\mathbb{E}^\mathbb{P} [g(X_t)] = \mathbb{E}^{\mathbb{P}_x} [g(X_t)]$ for all $t \in \mathbb{R}_+$. As $g \in C_0(J; \mathbb{R})$ was arbitrary, this implies that $\mathbb{P}$ and $\mathbb{P}_x$ have the same one-dimensional distributions. This result can be extended to the finite-dimensional distributions by induction. Assume that the $n$-dimensional distributions coincide and take $0 \leq t_1 \leq \cdots \leq t_n < \infty$. Further, let $A = \{X_{t_1} \in F_1, \ldots, X_{t_n} \in F_n\}$ for $F_1, \ldots, F_n \in \mathcal{B}(J)$. Applying (B.1) and (B.2) with $s = t_n$ and using the induction hypothesis (i.e., that $\mathbb{E}^\mathbb{P} [f(X_{t_n}) \mathbb{1}_A] = \mathbb{E}^{\mathbb{P}_x} [f(X_{t_n}) \mathbb{1}_A]$) yields

$$\mathbb{E}^\mathbb{P} \left[ \int_0^\infty e^{-ar} g(X_{r+t_n}) dr \mathbb{1}_A \right] = \mathbb{E}^{\mathbb{P}_x} \left[ \int_0^\infty e^{-\alpha r} g(X_{r+t_n}) dr \mathbb{1}_A \right].$$

Using again the uniqueness theorem for Laplace transforms, we get that the distribution of $(X_{t_1}, \ldots, X_{t_n}, X_{t_n+t})$ coincide under $\mathbb{P}$ and $\mathbb{P}_x$ for all $t \in \mathbb{R}_+$. In summary, $\mathbb{P}$ and $\mathbb{P}_x$ have the same finite-dimensional distributions and the usual monotone class argument yields that $\mathbb{P} = \mathbb{P}_x$. The proof is complete. \hfill \Box

The generator $(\Gamma, \Delta)$ is known explicitly (see [20] or [23]). For reader’s convenience and for later reference we collect the most important cases for diffusions on natural scale. To this end, we first prepare a couple of notations. Consider an open interval $I \subset \mathbb{R}$ and recall ([10, Proposition 5.1]) that the following are equivalent:

(a) $f: I \to \mathbb{R}$ is a difference of two convex functions;

(b) $f: I \to \mathbb{R}$ is a continuous function such that its right-hand derivative $f'_+ ( = d^+ f/dx)$ exists everywhere on $I$ and $f'_+$ is a right-continuous function with locally finite variation on $I$.

(There is, of course, a symmetric equivalent condition that involves the left-hand derivative.) We stress that, if $I$ is not open, then the equivalence between (a) and (b) breaks: for instance, on $[0, \infty)$ the root $\sqrt{\cdot}$ satisfies (a) but not (b). In fact, for the interval $[0, \infty)$, we easily conclude from above that the following are equivalent:

(a') $f: [0, \infty) \to \mathbb{R}$ has a representation $f = h_1 - h_2$ such that $h_i: [0, \infty) \to \mathbb{R}$ are convex functions with $\|h_i'\_+(0)\|_\infty < \infty$, $i = 1, 2$;

(a'') $f: [0, \infty) \to \mathbb{R}$ has a representation $f = h_1 - h_2$ such that $h_i: [0, \infty) \to \mathbb{R}$ are convex functions that can be extended beyond zero to convex functions $\mathbb{R} \to \mathbb{R}$;
(b') \( f : [0, \infty) \to \mathbb{R} \) is a continuous function such that \( f'_+ \) exists everywhere on \([0, \infty)\) and is a right-continuous function with locally finite variation on \([0, \infty)\).

Further, given an open interval \( I \subset \mathbb{R} \), a function \( f : I \to \mathbb{R} \) satisfying (a) (equivalently, (b)) and a locally finite measure \( \mu \) on \((I, \mathcal{B}(I))\) the notation \( \text{“} df'_+ = d\mu \text{“} \) on \( I \) is a shorthand for

\[
(B.3) \quad f'_+(b) - f'_+(a) = \mu([a, b)) \quad \text{for all } a < b \text{ in } I.
\]

**Lemma B.4.** Assume that \((J \ni x \mapsto \mathbb{P}_x)\) has characteristics \((\text{Id}, \mathbb{m}).\)

(i) Suppose that \( J = \mathbb{R} \). Let \( \Delta^1 \) be the set of all \( f \in C_b(\mathbb{R}; \mathbb{R}) \) such that \( f \) is a difference of two convex functions and \( df'_+ = 2g dm \) on \( \mathbb{R} \) for some \( g \in C_b(\mathbb{R}; \mathbb{R}) \). Further, set \( \Gamma^1 f = g \). Then, \( (\Gamma^1, \Delta^1) = (\Gamma, \Delta) \).

(ii) Suppose that \( J = (0, \infty) \). Let \( \Delta^2 \) be the set of all \( f \in C_b((0, \infty); \mathbb{R}) \) such that \( f \) is a difference of two convex functions \((0, \infty) \to \mathbb{R} \) and \( df'_+ = 2g dm \) on \((0, \infty)\) for some \( g \in C_b((0, \infty); \mathbb{R}) \). Further, set \( \Gamma^2 f = g \). Then, \( (\Gamma^2, \Delta^2) = (\Gamma, \Delta) \).

(iii) Suppose that \( J = [0, \infty) \) and that \( \mathbb{m}(\{0\}) = \infty \). Let \( \Delta^3 \) be the set of all \( f \in C_b([0, \infty); \mathbb{R}) \) such that, restricted to \((0, \infty)\), \( f \) is a difference of two convex functions \((0, \infty) \to \mathbb{R} \) and \( df'_+ = 2g dm \) on \((0, \infty)\) for some \( g \in C_b((0, \infty); \mathbb{R}) \) with \( g(0) = 0 \). Further, set \( \Gamma^3 f = g \). Then, \( (\Gamma^3, \Delta^3) = (\Gamma, \Delta) \).

(iv) Suppose that \( J = [0, \infty) \) and that \( \mathbb{m}(\{0\}) < \infty \). Let \( \Delta^4 \) be the set of all \( f \in C_b([0, \infty); \mathbb{R}) \) such that \( f \) satisfies the equivalent conditions \((a'), (a''), (b')\) above, \( df'_+ = 2g dm \) on \((0, \infty)\) and \( f'_+(0) = 2g(0)\mathbb{m}(\{0\}) \) for some \( g \in C_b([0, \infty); \mathbb{R}) \). Further, set \( \Gamma^4 f = g \). Then, \( (\Gamma^4, \Delta^4) = (\Gamma, \Delta) \).

Alternative possible cases (e.g., \( J = [0, 1] \), \( \mathbb{m}(\{0\}) < \infty \), \( \mathbb{m}(\{1\}) < \infty \)) are similar to the ones above. Alternatively, with somewhat more effort, descriptions of such type for general diffusion generators can be read off from [23, Section 4.7, p. 135].

**Proof.** For (i) see [20, Theorem 75, p. 131], for (ii) see [20, Theorem 79, p. 133], for (iii) see [20, Theorem 81, p. 135]. We now prove (iv). [20, Theorem 89, p. 137] yields a description similar to that in (iv) with the only difference that the requirement that \( f \) satisfies (b') is replaced by a seemingly weaker requirement that

\[
(b'') \quad f : [0, \infty) \to \mathbb{R} \text{ is a continuous function such that } f'_+ \text{ exists, is finite and right-continuous on } [0, \infty) \text{, and } f'_+ \text{ has locally finite variation on the open interval } (0, \infty).
\]

It remains to establish that, if \( f \) satisfies (b'') together with other things listed in (iv), then \( f \) satisfies (b'). For \( a \in (0, 1) \) we have

\[
\int_{(a,1]} |df'_+| = \int_{(a,1]} 2|g| dm.
\]

As the boundary 0 is regular in (iv), by (2.10), \( \mathbb{m}(\{0,1\}) < \infty \). Letting \( a \downarrow 0 \), we obtain

\[
(\text{variation of } f'_+ \text{ on } [0, 1]) = \int_{(0,1]} |df'_+| = \int_{(0,1]} 2|g| dm < \infty.
\]

Together with (b''), this yields (b') and concludes the proof. \( \square \)

Finally, we also need a local uniqueness property of martingale problems. We define \( \mathcal{F}_t^\omega \triangleq \sigma(X_s, s \leq t) \) for \( t \in \mathbb{R}_+ \). The following result is, in fact, a generalization of Theorem B.3, and it can be proved similarly to [12, Lemma 9.1]. We emphasize that Theorem B.3 is needed for its proof.

**Lemma B.5.** Let \( \tau \) be a stopping time for the filtration \((\mathcal{F}_t^\omega)_{t \geq 0}\) and let \( \mathbb{P} \) be a probability measure on \((\Omega, \mathcal{F})\) such that \( \mathbb{P}(C(\mathbb{R}_+; J)) = 1 \), \( \mathbb{P}(X_0 = x) = 1 \) and

\[
M^f \triangleq f(X_{\cdot \wedge \tau}) - f(x) - \int_0^{\wedge \tau} (\Gamma f)(X_s) ds
\]

is a local \(^{14}\) \( \mathbb{P} \)-martingale for all \( f \in \Delta \). Then, \( \mathbb{P} = \mathbb{P}_x \) on \( \mathcal{F}_\tau^\omega \).

\(^{14}\)Hence, necessarily, a \( \mathbb{P} \)-martingale, as \( M^f \) is bounded on finite time intervals due to \( \Delta \subset C_b(J; \mathbb{R}) \).
Appendix C. Technical Facts on Diffusions and Semimartingales

To make our paper as self-contained as possible we collect technical facts about diffusions and semimartingales which are used in our proofs.

C.1. Facts on Diffusions. In this section, \((J \ni x \mapsto P_x)\) is a regular diffusion with scale function \(s\) and speed measure \(m\). Recall from (2.3) that \(T_y = \inf(s \geq 0: X_s = y)\). We denote \(l \triangleq \inf J\) and \(r \triangleq \sup J\).

Lemma C.1. In addition to the regular diffusion \((J \ni x \mapsto P_x)\) consider a state space \(\tilde{J}\) with \(J^0 = \tilde{J}^0\) and a regular diffusion \((\tilde{J} \ni x \mapsto \tilde{P}_x)\). Let \(x_0 \in \tilde{J} \cap J\) be such that either \(x_0 \in J^0\) or \(x_0 \in \partial J\) is not an absorbing boundary for at least one of the diffusions. Assume that \(P_{x_0} = \tilde{P}_{x_0}\). Then \(J = \tilde{J}\) and \(P_y = \tilde{P}_y\) for all \(y \in J\) (that is, the diffusions coincide).

It is worth noting that the assumption on \(x_0\) cannot be dropped: if \(x_0\) is an absorbing boundary point for both diffusions, then \(P_{x_0} \neq \tilde{P}_{x_0}\), but, clearly, the diffusions can be different.

Proof. Notice first that if \(x_0\) is a boundary point, then it is clearly reflecting for both diffusions. Now, fix some \(y \in J\) and observe that

\[
P_{x_0}(T_y < \infty) > 0. \tag{C.1}
\]

Indeed, if \(x_0 \in J^0\), then (C.1) follows from the regularity of the diffusion \((x \mapsto P_x)\) (recall (2.2)); if \(x_0 \in \partial J\), then (C.1) follows from Lemma C.7 below. Take any \(A \in \mathcal{F}\). By the strong Markov property,

\[
P_{x_0}(T_y < \infty, X_{+T_y} \in A) = E_{x_0}[1_{\{T_y < \infty\}}P_{x_0}(X_{+T_y} \in A|\mathcal{F}_{T_y})] = P_y(A)P_{x_0}(T_y < \infty).
\]

Thus,

\[
P_y(A) = \frac{P_{x_0}(T_y < \infty, X_{+T_y} \in A)}{P_{x_0}(T_y < \infty)}. \tag{C.2}
\]

As \(y \in J\) and \(A \in \mathcal{F}\) are arbitrary, it follows that the measure \(P_{x_0}\) uniquely determines the whole diffusion \((J \ni x \mapsto P_x)\). Similarly, the measure \(P_{x_0}\) uniquely determines the whole diffusion \((\tilde{J} \ni x \mapsto \tilde{P}_x)\).

Moreover, the determining formula (C.2), its analogue for the other diffusion and the fact that \(P_{x_0} = \tilde{P}_{x_0}\) yield that \(P_y = \tilde{P}_y\) for all \(y \in J \cap \tilde{J}\) (in particular, for all \(y \in J^0\)).

It remains only to prove that \(J = \tilde{J}\). To this end, take an interior point \(c \in J^0(= \tilde{J}^0)\) and a boundary point \(b \in \partial J(= \partial \tilde{J})\). Recall that \(b \in J\) (resp., \(b \in \tilde{J}\)) if and only if \(P_c(T_b < \infty) > 0\) (resp., \(\tilde{P}_c(T_b < \infty) > 0\)). As we already established that \(P_c = \tilde{P}_c\), we obtain that \(b \in J\) is equivalent to \(b \in \tilde{J}\). It follows that \(J = \tilde{J}\). This concludes the proof.

The following lemma is Blumenthal’s zero-one law ([20, Lemma 4, p. 106]).

Lemma C.2. For any \(x \in J\) the \(\sigma\)-field \(\mathcal{F}_x\) is \(P_x\)-trivial, i.e., \(P_x(A) \in \{0, 1\}\) for all \(A \in \mathcal{F}_0\).

The following lemma is a restatement of [40, Corollary V.46.15].

Lemma C.3. The process \(s(X_{\wedge T_x \land T_c})\) is a continuous local \(P_x\)-martingale for all \(x \in J\).

The next lemma explains how the characteristics are changed via an homeomorphic change of space. It is a restatement of [39, Exercise VII.3.18].

Lemma C.4. Let \(\phi\) be a homeomorphism from \(J\) onto an interval \(I\). Then, \((I \ni x \mapsto P\phi^{-1}(x) \circ \phi(X)^{-1})\) is also a regular diffusion with speed measure \(m \circ \phi^{-1}\) and scale function \(\pm s \circ \phi^{-1}\) (the sign is “+” if \(\phi\) is increasing and “−” of \(\phi\) is decreasing).

The next lemma is a restatement of [6, Theorem 1.1]. It says that diffusions hit points arbitrarily fast with positive probability, which can be viewed as an irreducibility property.

Lemma C.5. Take \(x, z \in J\) such that \(x \neq z\) and \(P_x(T_z < \infty) > 0\). Then,

\[P_x(T_z < \varepsilon) > 0\]

for all \(\varepsilon > 0\).
The following lemma explains that diffusions exit non-absorbing points immediately. It follows, for instance, from [20, Corollary 5, Fact 6 on p. 107, Lemma 12 on p. 109].

**Lemma C.6.** For every $x \in J$ with $m\{x\} < \infty$\footnote{In other words, we exclude only the case where $x$ is an absorbing boundary point.}, $\mathbb{P}_x(S_x = 0) = 1$, where $S_x \triangleq \inf\{t \geq 0 : X_t \neq x\}$.

Part (i) of the following lemma is a restatement of [20, Corollary 19, p. 112], parts (ii) and (iii) are restatements of [20, Lemmata 20, 21, p. 112].

**Lemma C.7.**

(i) If $a, x, c \in J$ are such that $a \leq x \leq c$, then $\mathbb{E}_x[T_a \wedge T_c] < \infty$.

(ii) If $l$ is reflecting and $x, c \in J$ are such that $x \leq c$, then $\mathbb{E}_x[T_c] < \infty$.

(iii) If $r$ is reflecting and $a, x \in J$ are such that $a \leq x$, then $\mathbb{E}_x[T_a] < \infty$.

Let us also recall some general path properties of diffusions. The following lemma is a consequence of (2.4) (see, e.g., [27, Proposition 5.5.22] for a detailed proof).

**Lemma C.8.** Take $x \in J^o$.

(i) If $s(l) = -\infty, s(r) = \infty$, then

$$
\mathbb{P}_x\left(T_l \wedge T_r = \infty\right) = \mathbb{P}_x\left(\limsup_{t \to \infty} X_t = r\right) = \mathbb{P}_x\left(\liminf_{t \to \infty} X_t = l\right) = 1.
$$

(ii) If $s(l) > -\infty, s(r) = \infty$, then

$$
\mathbb{P}_x\left(\sup_{t < T_l \wedge T_r} X_t < r\right) = \mathbb{P}_x\left(\lim_{t \to T_l \wedge T_r} X_t = l\right) = 1.
$$

(iii) If $s(l) = -\infty, s(r) < \infty$, then

$$
\mathbb{P}_x\left(\lim_{t \to T_l \wedge T_r} X_t = r\right) = \mathbb{P}_x\left(\inf_{t < T_l \wedge T_r} X_t > l\right) = 1.
$$

(iv) If $s(l) > -\infty, s(r) < \infty$, then

$$
\mathbb{P}_x\left(\lim_{t \to T_l \wedge T_r} X_t = l\right) = 1 - \mathbb{P}_x\left(\lim_{t \to T_l \wedge T_r} X_t = r\right) = \frac{s(r) - s(l)}{s(r) - s(l)}.
$$

The next result implies that there is a dichotomy: in the case when with positive probability the trajectories tend to a boundary point $b \in \{l, r\}$, as $t \nearrow T_b$\footnote{From Lemma C.8 we know that this is exactly the case $|s(b)| < \infty$.} either they do this in infinite time only (when $r$ is inaccessible) or they do this in finite time only (when $r$ is accessible). It is the latter “only” that needs to be formally stated. We do this in the next lemma, which follows from [26, Theorem 33.15] (in fact, [26, Theorem 33.15] contains much more information).

**Lemma C.9.** Let $b \in \{l, r\}$ be an accessible boundary and $x \in J^o$. Then

$$
\mathbb{P}_x\left(T_b = \infty, \lim_{t \to T_b} X_t = b\right) = 0.
$$

We recall that a diffusion $(J \ni x \mapsto \mathbb{P}_x)$ is called **recurrent** if $\mathbb{P}_x(T_y < \infty) = 1$ for all $x, y \in J$. The following characterization of recurrence is a consequence of Lemmata C.7 and C.8.

**Lemma C.10.** Diffusion $(J \ni x \mapsto \mathbb{P}_x)$ is recurrent if and only if each boundary point $b \in \{l, r\}$ satisfies one of the following:

(i) $|s(b)| = \infty$,

(ii) $b$ is reflecting.

The following version of the ratio ergodic theorem follows from [26, Theorem 33.14] and Lemma C.4.

**Lemma C.11.** Let $(J \ni x \mapsto \mathbb{P}_x)$ be recurrent. Then, for any Borel functions $f, g : J \to \mathbb{R}_+$ with $\int f \, d\mathbb{m} < \infty$ and $\int g \, d\mathbb{m} > 0$, we have

$$
\lim_{t \to \infty} \frac{\int_0^t f(X_s) \, ds}{\int_0^t g(X_s) \, ds} = \frac{\int f \, d\mathbb{m}}{\int g \, d\mathbb{m}} \mathbb{P}_x\text{-a.s. for all } x \in J.
$$
The law of a reflecting Brownian motion coincides with the law of the absolute value of a standard Brownian motion. More generally, for any diffusion with an (instantaneously or slowly) reflecting boundary point one can find a diffusion on an extended state space with inaccessible or absorbing boundaries and a Lipschitz function $f$ such that in law the original one is obtained via space transformation by $f$ from the extended one. In the following we explain this fact for the cases $J = [0,\infty)$ (Lemma C.12) and $J = [0,1]$ where both 0 and 1 are reflecting boundaries (Lemma C.13). Lemmata C.12 and C.13 follow from the proof of [39, Proposition VII.3.10]. We also refer to [1, Section 6] for a similar discussion for diffusions on natural scale.

**Lemma C.12.** Suppose that $J = \mathbb{R}_+$, $s(0) = 0$ and $m(\{0\}) < \infty$. Define scale function $s^+: \mathbb{R} \to \mathbb{R}$ by

$$s^+(x) = \begin{cases} s(x), & x \geq 0, \\ -s(-x), & x < 0, \end{cases}$$

and speed measure $m^+$ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ by

$$m^+(A) = \begin{cases} m(A), & A \in \mathcal{B}((0,\infty)), \\ m(-A), & A \in \mathcal{B}((-,0)) \quad (-A \triangleq \{x \in (0,\infty) : -x \in A\}), \\ 2m(\{0\}), & A = \{0\}. \end{cases}$$

(C.3)

Let $(\mathbb{R} \ni x \mapsto P_x^+)$ be regular diffusion with characteristics $(s^+, m^+)$. Then,

$$P_x = P_x^+ \circ |X|^{-1}, \quad x \in \mathbb{R}_+ = J.$$

We emphasize that in the situation of Lemma C.12 formula (C.3) always defines a valid speed measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ (recall Remark 2.5). Indeed, only the fact that $m^+$ is locally finite on $\mathbb{R}$ requires an explanation, and this is a direct consequence of (2.9) or (2.10) (notice that boundary point 0 is regular due to the assumption $m(\{0\}) < \infty$; to this end, recall the convention in the second paragraph before Remark 2.5 that $m(\{b\}) = \infty$ whenever $b$ is an exit boundary).

**Lemma C.13.** Suppose that $J = [0,1]$, $s(0) = 0$, $m(\{0\}) < \infty$ and $m(\{1\}) < \infty$. Define scale function $\overline{s}: \mathbb{R} \to \mathbb{R}$ by

- $\overline{s}(x) \triangleq s(x)$, $x \in [0,1]$,
- $\overline{s}(x) \triangleq -s(-x)$, $x \in [-1,0]$,
- $\overline{s}(x + 2k) \triangleq \overline{s}(x) + 2k s(1)$, $k \in \mathbb{Z} \setminus \{0\}$, $x \in [-1,1]$,

and speed measure $\overline{m}$ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ by

- $\overline{m}(A) = m(-A) \triangleq m(-A)$, $A \in \mathcal{B}((0,1))$,
- $\overline{m}(A + 2k) \triangleq \overline{m}(A)$, $k \in \mathbb{Z} \setminus \{0\}$, $A \in \mathcal{B}((-1,0) \cup (0,1))$,
- $\overline{m}(\{2k\}) = 2m(\{0\})$, $k \in \mathbb{Z}$,
- $\overline{m}(\{2k + 1\}) = 2m(\{1\})$, $k \in \mathbb{Z}$.

Let $(\mathbb{R} \ni x \mapsto Q_x)$ be regular diffusion with characteristics $(\overline{s}, \overline{m})$. Then,

$$P_x = Q_x \circ f(X)^{-1}, \quad x \in [0,1] = J,$$

where $f: \mathbb{R} \to [0,1]$ is the periodic function with period 2 satisfying $f(x) = |x|$, $x \in [-1,1]$.

It is necessary to remark that in the situation of Lemma C.13 both $\infty$ and $\infty$ are inaccessible boundaries for $(x \mapsto Q_x)$ (that is, the state space of that diffusion is indeed $\mathbb{R}$), as $|\overline{s}(\pm \infty)| = \infty$. Furthermore, as both 0 and 1 are regular boundaries for $(x \mapsto P_x)$, the measure $\overline{m}$ is locally finite on $\mathbb{R}$ due to (2.9) or (2.10), which is necessary for $\overline{m}$ to be a valid speed measure (cf. with the discussion after Lemma C.12).

The next lemma explains the general structure of diffusions on natural scale. It is a restatement of [26, Theorem 33.9].

**Lemma C.14.** Suppose that $s = Id$ and take $x \in J$. 

(i) Possibly on an extension of the underlying filtered probability space, there exists a Brownian motion \( W \) starting in \( x \) such that \( \mathbb{P}_x \)-a.s. \( X = W_\gamma \), where

\[
\gamma_t \triangleq \inf(s \geq 0: A_s > t), \quad A_t \triangleq \int_0^t L_t^y(W) \mathbb{d}m(dy), \quad t \in \mathbb{R}_+,
\]

and \( \{L_t^y(W): (t, y) \in \mathbb{R}_+ \times \mathbb{R}\} \) is the local time process of \( W \).

(ii) The law of any time-changed Brownian motion \( W_\gamma \) as in (i) above coincides with \( \mathbb{P}_x \).

Notice that, as a.s. \( L_\infty^y(W) = \infty \) for all \( y \in \mathbb{R} \) and, hence, \( A_\infty = \infty \), it follows that the time-change \( \gamma \) in Lemma C.14 is always finite. Further notice that \( A \) can have intervals of constancy, i.e., the time-change \( \gamma \) might have jumps. For instance, think about \( J = [0, \infty) \) with \( m([0]) < \infty \) (in particular, \( 0 \) is a regular boundary, i.e., also \( m((0, 1]) < \infty \), see (2.10)). Then, \( A \) is constant (and finite) during the negative excursions of \( W \), i.e., \( \gamma \) overjumps the negative excursions of \( W \).

By virtue of Lemma C.4 any diffusion can be brought to natural scale (i.e., \( s = \text{Id} \)) by a homeomorphic space transformation via its scale function. Thus, Lemma C.14 shows that any regular diffusion is a space and time-changed Brownian motion. In fact, any diffusion on natural scale is a semimartingale.\(^{17}\) The next lemma formalizes this fact, providing the occupation time formula for diffusions. In particular, it emphasizes the difference between the semimartingale local time and the diffusion local time.\(^{18}\)

**Lemma C.15.** Suppose that \( s = \text{Id} \) and take \( x \in J \).

(i) The process \( X \) is a continuous \( \mathbb{P}_x \)-semimartingale. In particular, it has continuous in time and càdlàg in space semimartingale local time process \( L(X) = \{L_t^y(X): (t, y) \in \mathbb{R}_+ \times \mathbb{R}\} \) (see Lemma C.26 below for a recap).

(ii) For the diffusion local time process \( \tilde{L}(X) = \{\tilde{L}_t^y(X): (t, y) \in \mathbb{R}_+ \times J\} \) defined as

\[
\tilde{L}_t^y(X) \triangleq \begin{cases} 
L_t^y(X), & (t, y) \in \mathbb{R}_+ \times (J \setminus \{r\}), \\
L_t^y(X), & (t, y) \in \mathbb{R}_+ \times \{r\} \quad \text{(in case } r \in J) 
\end{cases}
\]

(recall that \( r = \sup J \)), \( \mathbb{P}_x \)-a.s. for all \( (t, y) \in \mathbb{R}_+ \times J \),

\[
\tilde{L}_t^y(X) = L_t^y(W),
\]

where we use the notation from Lemma C.14. Moreover, \( \mathbb{P}_x \)-a.s. the diffusion local time \( \tilde{L}(X) \) is jointly continuous in time and space on \( \mathbb{R}_+ \times J \), and \( \mathbb{P}_x \)-a.s. we have

\[
\int_0^t f(X_s)ds = \int_0^t f(y)\tilde{L}_t^y(X)\mathbb{d}m(dy)
\]

simultaneously for all \( t \in \mathbb{R}_+ \) and all Borel functions \( f: J \to [0, \infty] \) with \( f(b) = 0 \) for all boundary points \( b \in J \setminus J^\circ \) with \( m(\{b\}) = \infty \).

**Remark C.16.** (a) We emphasize that the semimartingale local time is defined for semimartingales and for \( (t, y) \in \mathbb{R}_+ \times \mathbb{R} \). On the other hand, the diffusion local time is defined for diffusions on natural scale and for \( (t, y) \in \mathbb{R}_+ \times J \). Comparing them for levels \( y \in J \), we see that, by (C.4), \( L(X) \) and \( \tilde{L}(X) \) can only differ at the level \( y = r \). To see the difference, the right boundary point \( r \) needs to be in \( J \) and therefore, as \( X \) is on natural scale, \( r \) is necessarily finite. In that case, as \( X \) is \( J \)-valued and càdlàg in the space variable, we always have \( L_r^r(X) = 0 \), but \( L_r^r(X) > 0 \) is also possible. Thus, the only correction at the level \( y = r \) performed in (C.4) is needed to obtain the joint continuity of \( \tilde{L}(X) \) in time and space (but of course on \( \mathbb{R}_+ \times J \) only; not on \( \mathbb{R}_+ \times \mathbb{R} ! \)). To ease several references, we also note the following consequence of the joint continuity of \( \tilde{L}(X) \) on \( \mathbb{R}_+ \times J \): \( \mathbb{P}_x \)-a.s. the semimartingale local time \( L(X) \) is jointly continuous in time and space on \( \mathbb{R}_+ \times J^\circ \).

\(^{17}\)We stress that the semimartingale property might be lost via the space transformation by the scale function. For instance, the square root of reflecting Brownian motion (which is a semimartingale and a diffusion on natural scale) is not a semimartingale. This example is known under the name Yor’s example.

\(^{18}\)In this connection, we thank Zhesheng Liu and Mihail Zervos for bringing formula (C.4) to our attention.
(b) Finally, we stress that, in the above generality, it is essential to have the diffusion local time \( \tilde{L}(X) \) in (C.6) (with \( L(X) \) in place of \( \tilde{L}(X) \) the claim would be incorrect). On the other hand, (C.6) obviously implies that \( \mathbb{P}_x \)-a.s. we have

\[
(C.7) \quad \int_0^t f(X_s)ds = \int_f f(y)\tilde{L}_y(X)m(dy)
\]
simultaneously for all \( t \in \mathbb{R}_+ \) and all Borel functions \( f : J \to [0, \infty] \) with \( f(b) = 0 \) for all boundary points \( b \in J \setminus J^o \) (cf. [40, Theorem V.49.1]).

**Proof of Lemma C.15.** The semimartingale property follows from Lemma C.14 and Lemma C.29 below.\(^\text{19}\) The representation (C.5) of the diffusion local time and the occupation time formula (C.6) are restated in (C.6) (with \( 52 \text{ D. CRIENS AND M. URUSOV} \)). By Lemma C.15, using also its notation, we stress that, contrary to the semimartingale property, the (local) martingale property is in general not preserved by the change of time. Indeed, a reflected Brownian motion is a time-changed Brownian motion by Lemma C.14 but it is not a (local) martingale.

**Lemma C.17.** Take \( J = [l, r] \), \( s = \text{Id} \) and assume that \( l \) is a reflecting boundary point. Fix \( x \in J \) and let \( \{ \tilde{L}_s(x) : (t, y) \in \mathbb{R}_+ \times J \} \) be the diffusion local time of \( X \) under \( \mathbb{P}_x \) (see Lemma C.15). Then, \( X - \frac{1}{2} \tilde{L}(X) \) is a continuous local \( \mathbb{P}_x \)-martingale.

**Lemma C.18.** Suppose that \( s = \text{Id} \) and take \( x \in J \) with \( m(\{x\}) < \infty \). Let \( \{ \tilde{L}_s(x) : (t, y) \in \mathbb{R}_+ \times J \} \) be the diffusion local time of \( X \) under \( \mathbb{P}_x \) (see Lemma C.15). Then, \( \mathbb{P}_x \)-a.s. \( \tilde{L}_s(X) > 0 \) for all \( t > 0 \).

**Proof.** By Lemma C.15, using also its notation, \( \mathbb{P}_x \)-a.s. \( \tilde{L}(X) = L_s(W) \), where we recall that \( W \) is a Brownian motion starting in \( x \). Furthermore, as \( x \) is non-absorbing (which is the meaning of the assumption \( m(\{x\}) < \infty \)), \( \mathbb{P}_x \)-a.s. \( \gamma_t > 0 \) for all \( t > 0 \) see Lemma C.14 (i) or [20, Section 2.9]. Finally, since a.s. \( L_t^z(W) > 0 \) for all \( t > 0 \) by [20, Lemma 106, p. 146], the claim follows.

**Lemma C.19.** Take \( x \in J \) with \( m(\{x\}) < \infty \). Furthermore, take a Borel function \( f : J \to [0, \infty] \) and set

\[
A \triangleq \int_0^t f(X_s)ds.
\]

The following are equivalent:

(i) There exists an open in \( J \) neighborhood \( I \) of \( x \) such that \( \int_I f(y)m(dy) < \infty \).

(ii) There exists a random time \( \rho \) such that \( \mathbb{P}_x \)-a.s. \( \rho > 0 \) and \( \mathbb{P}_x(A_\rho < \infty) > 0 \).

\(^{19}\)We stress that, contrary to the semimartingale property, the (local) martingale property is in general not preserved by the change of time. Indeed, a reflected Brownian motion is a time-changed Brownian motion by Lemma C.14 but it is not a (local) martingale.
(iii) There exists a stopping time $\tau$ such that $P_x$-a.s. $\tau > 0$ and $A_\tau < \infty$.

In relation with Lemma C.19 we emphasize that, in (i), the neighborhood $I$ of $x$ should be open in $J$. In particular, if $x$ is left (resp., right) boundary point of $J$, then we search for $I$ of the form $[x, c)$ (resp., $(c, x]$) for some $c \in J^\circ$.

**Remark C.20.** In an equivalent and condensed form, the claim in Lemma C.19 can be restated as follows:

- If (i) holds, then (iii) holds.
- If (i) does not hold, then $P_x$-a.s. we have $A_t = \infty$ for all $t > 0$.

**Proof of Lemma C.19.** By Lemmata C.4 and C.15, $P_x$-a.s. it holds for all $t \in \mathbb{R}_+$

$$A_t = \int_0^t f(X_u) \, du = \int_0^t f \circ s^{-1}(s(X_u)) \, du$$

(C.8)

$$= \int_{s(J)} f \circ s^{-1}(z) \tilde{L}_t^{z(s)}(s(X)) \, m \circ s^{-1}(dz)$$

$$= \int_J f(y) \tilde{L}_t^{s(y)} (s(X)) \, m(dy).$$

Assume that (i) holds and let $I$ be an open in $J$ neighborhood of $x$ such that $\int_I f(y) m(dy) < \infty$. Via shrinking, we can w.l.o.g. assume that $I$ is relatively compact in $J$. Define the stopping time

$$\tau \triangleq \inf(t \geq 0: X_t \notin I)$$

and notice that for $P_x$-a.a. $\omega \in \Omega$, $J \ni y \mapsto \tilde{L}_t^{s(y)}(s(X))(\omega)$ is continuous with compact support, because it vanishes outside the relatively compact set $I$. Now, (C.8) implies that (iii) holds with the stopping time $\tau$. Statement (iii), in turn, implies (ii).

We finally show that (ii) implies (i) or, equivalently, that the negation of (i) implies the negation of (ii). So we assume that for any open in $J$ neighborhood $I$ of $x$ we have $\int_I f(y) m(dy) = \infty$ and take any random time $\rho$ such that $P_x$-a.s. $\rho > 0$. Thanks to Lemma C.18, we have $P_x$-a.s. $\tilde{L}_\rho^{s(x)}(s(X)) > 0$. Further, by continuity (Lemma C.15), for $P_x$-a.a. $\omega \in \Omega$, the function $J \ni y \mapsto \tilde{L}_\rho^{s(y)}(s(X))(\omega)$ is bounded away from zero in a small neighborhood of $x$ in $J$. Then, (C.8) implies that $P_x$-a.s. $A_\rho = \infty$. This concludes the proof.

The next lemma can be viewed as an extension of [33, Theorem 2.11] beyond the class of Itô diffusions. For its statement, recall that diffusions on natural scale are semimartingales (Lemma C.15 (i)). Of course, Lemma C.21 also has an analogue for the left boundary point.

**Lemma C.21.** Assume that $s = \text{Id}$ and $r < \infty$ (recall the notations $r = \sup J$ and $l = \inf J$). Let $f: J \to \mathbb{R}_+$ be a Borel function such that $f \in L^1_{\text{loc}}(J^\circ)$ and set

$$\zeta \triangleq \inf(t \geq 0: X_t \notin J^\circ) \quad \text{and} \quad D \triangleq \left\{ \lim_{t \uparrow \zeta} X_t = r \right\}.$$

With $(X, X)$ denoting the quadratic variation process of $X$, we have:

(C.9) \quad $\int_{r-}^{T_r} (r-y) f(y) \, dy < \infty \implies \forall x \in J^\circ: \int_0^{T_r} f(X_s) d(X, X)_s < \infty \quad P_x$-a.s. on $D$,

(C.10) \quad $\int_{r-}^{T_r} (r-y) f(y) \, dy = \infty \implies \forall x \in J^\circ: \int_0^{T_r} f(X_s) d(X, X)_s = \infty \quad P_x$-a.s. on $D$.

Furthermore, if $l$ is reflecting (necessarily, $l > -\infty$) and $f \in L^1_{\text{loc}}([l, r])$, then (C.9) and (C.10) also hold for all $x \in [l, r)$ and with $D$ replaced by $\Omega$. 
Proof. In case \( l \) is reflecting and \( f \in L^1_\text{loc}([l, r]) \) take some \( x \in [l, r) \). Otherwise take some \( x \in J^o \). By Lemma C.14, we have \( P_x \)-a.s. \( X = W^r \), where, on an extension of the underlying space, \( W \) is a Brownian motion starting in \( x \) and the time-change \( \gamma \) is defined as in Lemma C.14. We extend the function \( f : J \to \mathbb{R} \) to a function \( \mathbb{R} \to \mathbb{R}_+ \) by setting \( f(x) \triangleq 0 \) for \( x \in \mathbb{R} \setminus J \). By the semimartingale occupation time formula (see Lemma C.26 below) together with (C.4) and (C.5) in Lemma C.15, we have a.s. for all \( t \in \mathbb{R}_+ \)

\[
(C.11) \quad \int_0^t f(x_s)d(X,X)_s = \int_J f(y)L^y_t(X)dy = \int_{-\infty}^\infty f(y)L^y_t(W)dy = \int_0^\gamma f(W_s)ds.
\]

Here the following comments are in order:

- The third and the fourth expressions in (C.11) require the extended function \( f \).
- The seemingly indirect way of proving (C.11) via the occupation time formula is due to the fact that the time-change \( \gamma \) can have jumps.

Notice that a.s. \( \gamma_{T_x} = T_x(W) \triangleq \inf(t \geq 0 : W_t = r) \) on \( \{\lim_{t \uparrow T_x} X_t = r\} \), while this event equals a.s.

- \( \Omega \) if \( l \) is reflecting,
- \( D \) otherwise.

By [33, Lemma 4.1], for any nonnegative Borel function \( g \in L^1_\text{loc}((-\infty, r)) \),
\[
\int_{r^-}^r (r - y)g(y)dy < \infty \quad \implies \quad \text{a.s.} \quad \int_0^{T_x(W)} g(W_s)ds < \infty,
\]
\[
\int_{r^-}^r (r - y)g(y)dy = \infty \quad \implies \quad \text{a.s.} \quad \int_0^{T_x(W)} g(W_s)ds = \infty.
\]

Therefore, the claims follow. \( \square \)

We also need an analogue of the previous lemma for the recurrent case.

**Lemma C.22.** Assume that \( \mathcal{s} = \text{Id} \) and that \( (x \mapsto P_x) \) is recurrent. Let \( f : J \to \mathbb{R}_+ \) be a non-vanishing Borel function in the sense that \( \int_J f(y)dy > 0 \). Then
\[
\forall x \in J : \int_0^\infty f(X_s)d(X,X)_s = \infty \quad P_x\text{-a.s.}
\]

**Proof.** Take some \( x \in J \). Using the notation from the beginning of the proof of Lemma C.21 and extending the function \( f \) to the whole \( \mathbb{R} \) by zero, we obtain a.s. for \( t \in \mathbb{R}_+ \)
\[
\int_0^t f(x_s)d(X,X)_s = \int_{-\infty}^\infty f(y)L^y_t(W)dy.
\]

Letting \( t \to \infty \) and noting that

- a.s. \( \gamma_\infty = \infty \) (due to the recurrence of \( X \)) and
- a.s., for all \( y \in \mathbb{R} \), it holds \( L^y_{\gamma_\infty}(W) = \infty \),

we deduce the claim. \( \square \)

We end this subsection with a version of Meyer’s theorem on predictability, see [9, Proposition 4] and [25, Lemma I.2.17].

**Lemma C.23.** For every \( x \in J \), any stopping time coincides \( P_x\text{-a.s. with a predictable time.} \)

C.2. **Chain Rule for Diffusions.** In this subsection we put the chain rule for diffusions from [23, Section 5.5] to a general scale.

Let \( (J \ni x \mapsto P_x) \) and \( (I \ni x \mapsto \hat{P}_x) \) be regular diffusions with characteristics \( (\mathcal{s},m) \) and \( (\hat{\mathcal{s}},\hat{m}) \), respectively. Suppose the following:

(a) \( I \subset J \).
(b) \( \mathcal{s} = \hat{\mathcal{s}} \) on \( I \).
Proof. Under (P, Theorem C.24. The following theorem provides the chain rule for diffusions:

\[ \limsup_{t \to 0} \frac{1}{t} \int_0^t \mathbf{1}_{\{x_h < t\}} \mathbf{1}_{\{x < x_h + h\}} \frac{ds}{m((x_h, x + h))} \]

which is a measurable function with values in \([0, \infty]\). Let \(G \subseteq \{t, r^*\}\) be the set of absorbing boundaries of \((x \mapsto \hat{P}_x)\) and set

\[ \epsilon \triangleq \inf\{t \geq 0 : X_t \in G\} \]

and

\[ \ell(t, x) \triangleq \begin{cases} \limsup_{h \to 0} \frac{1}{h} \int_0^h \mathbf{1}_{\{x_h < t\}} \mathbf{1}_{\{x < x_h + h\}} \frac{ds}{m((x_h, x + h))} & \text{if } x \in I^o, \\
\limsup_{h \to 0} \frac{1}{h} \int_0^h \mathbf{1}_{\{x < x_h < t\}} \frac{ds}{m((x_h, x + h))} & \text{if } x = l^* \in I,
\limsup_{h \to 0} \frac{1}{h} \int_0^h \mathbf{1}_{\{x_h < x < t\}} \frac{ds}{m((x_h, x + h))} & \text{if } x = r^* \in I, \end{cases} \]

where

\[ \mathbf{g}(t) \triangleq \begin{cases} \int_t^\infty \ell(t, x) \mathbf{m}(dx), & t < \epsilon, \\
\lim_{t \to \epsilon^-} \int_t^\epsilon \ell(t, x) \mathbf{m}(dx) & t \geq \epsilon, \end{cases} \]

which is a measurable function with values in \([0, \infty]\). Let \(G^{-1}\) be the right-inverse of \(G\), i.e.,

\[ G^{-1}(t) \triangleq \inf\{s \geq 0 : G(s) > t\}, \quad t \in \mathbb{R}_+. \]

The following theorem provides the chain rule for diffusions:

**Theorem C.24.** \(P_x \circ \hat{X}^{-1}_{g^{-1}(t)} = \hat{P}_x \) for all \(x \in I\).

**Proof.** Under \((J \ni x \mapsto P_x)\), the process \(g(X)\) is a diffusion on natural scale (Lemma C.4). Lemma C.15 yields for the diffusion local time process \(\hat{L}(g(X))\) that, for all \(x \in I, P_x\)-a.s. we have, for all \(t \in \mathbb{R}_+\) and all \(K \in \mathcal{B}(I)\),

\[ \int_0^t \mathbf{1}_{\{X_s \in K\}} ds = \int_0^t \mathbf{1}_{\{g(X_s) \in g(K)\}} ds = \int_{g(K)} \hat{L}^g_t(g(X)) \mathbf{m} \circ g^{-1}(dy) = \int_K \hat{L}^g_t(g(X)) \mathbf{m}(dy) = \int_K \hat{L}^\tilde{g}_t(y)(g(X)) \mathbf{m}(dy), \]

where in the last equality we use that \(g = \tilde{g}\) on \(I\). The continuity of \(\hat{L}(g(X))\) in the space variable implies that the above defined \(\{\ell(t, x) : (t, x) \in \mathbb{R}_+ \times I\}\) provides a version of the processes \(\{\hat{L}^g_t(g(X)) : t \in \mathbb{R}_+\}\) simultaneously for all \(x \in I\). This means that, for all \(x \in I, P_x\)-a.s. we have, for all \(t \in \mathbb{R}_+\),

\[ \mathbf{g}(t) = \begin{cases} \int_{g(I)} \hat{L}^g_t(g(X)) \mathbf{m} \circ g^{-1}(dx), & t < \epsilon, \\
\lim_{t \to \epsilon^-} \int_t^\epsilon \ell(t, x) \mathbf{m}(dx) & t \geq \epsilon, \end{cases} \]

Finally, the diffusion on natural scale \(\tilde{g}(I) \ni x \mapsto \hat{P}_{\tilde{g}^{-1}(x)} \circ \hat{g}(X)^{-1}\) with speed measure \(\mathbf{m} \circ g^{-1}\) is obtained from the diffusion\(^20\) \(g(X)\) considered above in this proof via the subordination inverse to (C.12) ([23, p. 177]). This concludes the proof.

**C.3. Separating Times for Itô Diffusions.** The separating time for Itô diffusions was studied in [8, 35]. Thanks to some time-change and symmetrization arguments we can reduce certain steps in the proof of Theorem 2.18 to the Itô diffusion setting. The purpose of this short subsection is to recall the result from [8, 35] for later reference. We pose ourselves in the setup from Example A.1, i.e., we assume that \(J^o = (l, r)\) and

\[ g(x) \triangleq \int_t^x \exp\left(-\int_z^t \frac{2b(y)}{\sigma^2(y)} \, dy\right) \, dz, \quad \tilde{g}(x) \triangleq \int_t^x \exp\left(-\int_z^t \frac{2b(y)}{\sigma^2(y)} \, dy\right) \, dz, \]

\(^20\)Viewed under \((J \ni x \mapsto P_x)\).
Furthermore, we suppose that \( m \) which we use in this paper.

In part (ii) in the formula for \( f \) (C.14)

\[ f(Y_t) = f(Y_0) + \int_0^t \left( \frac{d^+ f}{dx} \right)(Y_s) dY_s + \frac{1}{2} \int_0^t L^y_t(Y) f''(dy), \quad t \in \mathbb{R}_+, \]

where \( f''(dx) \) denotes the second derivative measure of \( f \) defined by

\[ f''((x, y]) = \left( \frac{d^+ f}{dx} \right)(y) - \left( \frac{d^+ f}{dx} \right)(x), \quad x \leq y. \]

In particular, \( f(Y) \) is a continuous semimartingale.

(iii) Let \( f: \mathbb{R} \to \mathbb{R} \) be a strictly increasing function which is a difference of two convex functions. Almost surely, simultaneously for all \( (t, z) \in \mathbb{R}_+ \times \mathbb{R} \) it holds

\[ L^f_t(z)(f(Y)) = \left( \frac{d^+ f}{dx} \right)(z)L^y_t(Y). \]

(iv) Almost surely, simultaneously for all \( a \in \mathbb{R} \) it holds \( \text{supp}(d_i L^a_t(Y)) \subset \{ t \in \mathbb{R}_+: Y_t = a \} \).

**Remark C.27.** In part (ii) in the formula for \( f(Y_t) \) one can replace the left-hand derivative \( d^- f/dx \) by the right-hand derivative \( d^+ f/dx \) when one takes the left-continuous (in the space variable) local time process. Similarly, the analogue of part (iii) for the left-continuous local time contains the left-hand derivative \( d^- f/dx \).

There is a delicate point related to part (ii) of Lemma C.26. It is tempting to write (C.13) also in case when, say, \( Y \) is a \([0, \infty)\)-valued semimartingale and \( f: [0, \infty) \to \mathbb{R} \) is a difference of two convex continuous functions \([0, \infty) \to \mathbb{R} \). But this would no longer be true in general. A counterexample is given by \( Y = |W| \) with a Brownian motion \( W \) and \( f = \sqrt{\cdot} \) (indeed, \( \sqrt{|W|} \) is not a semimartingale). We, therefore, present also the generalized Itô formula for semimartingales taking values in half-open intervals.

---

21 Regarding the normalizations in the literature, we stress that the local time in [27] is half of the local time in [26, 30, 39, 40] which we use in this paper.
Lemma C.28. Let $I = [a, b)$ be a half-open (possibly unbounded) interval and let $Y$ be an $I$-valued continuous semimartingale. Furthermore, let $f : I \to \mathbb{R}$ be a continuous function such that $d^+\frac{df}{dx}$ exists everywhere on $[a, b)$ as a right-continuous function with locally finite variation (recall the equivalent conditions $(a')$, $(a'')$ and $(b')$ preceding Lemma B.4). Then, a.s. for all $t \in \mathbb{R}_+$,

$$f(Y_t) = f(Y_0) + \int_0^t \left( \frac{d^+ f}{dx} \right)(Y_s) dY_s + \frac{1}{2} \int_{(a,b)} L^x_{t} - (Y) f''(dx),$$

where $\{L^x_t(Y) : (t, x) \in \mathbb{R}_+ \times \mathbb{R}\}$ denotes the continuous in time and càdlàg in space local time process of the process $Y$, and $f''(dx)$ is the measure defined on the open interval $(a, b)$ by (C.14) with $a \leq x \leq y < b$. In particular, if $f \in C^1([a, b); \mathbb{R})$ with absolutely continuous derivative, then a.s. for all $t \in \mathbb{R}_+$

$$f(Y_t) = f(Y_0) + \int_0^t f'(Y_s) dY_s + \frac{1}{2} \int_{0}^t f''(Y_s) d\langle Y, Y \rangle_s,$$

where the function $f''$ is the second derivative of $f$ (which is well-defined almost everywhere with respect to the Lebesgue measure).

Proof. For notational simplicity, we prove the claim for $I = [0, \infty)$. By hypothesis, there exists a function $g : \mathbb{R} \to \mathbb{R}$ that is the difference of two convex functions such that $g = f$ on $[0, \infty)$. Further, as a.s. $L^0(Y) = 0$ for all $y \in (\infty, 0)$, we have a.s. $L^0 - (Y) = 0$ for all $y \in (-\infty, 0]$. Now, Lemma C.26 and Remark C.27 yield that, a.s. for all $t \in \mathbb{R}_+$,

$$f(Y_t) = g(Y_0) + \int_0^t \left( \frac{d^+ g}{dx} \right)(Y_s) dY_s + \frac{1}{2} \int_{-\infty}^{\infty} L^x_{t} - (Y) g''(dx)$$

$$= f(Y_0) + \int_0^t \left( \frac{d^+ f}{dx} \right)(Y_s) dY_s + \frac{1}{2} \int_{(0,\infty)} L^x_{t} - (Y) f''(dx),$$

which is (C.15). It is worth noting that, if we applied (C.13) directly (i.e., without Remark C.27), we would get a dependence on $d^- g(0)/dx$ in both integral terms, which looks puzzling at first glance, as $d^- g(0)/dx$ is not uniquely defined. However, using [39, Theorem VI.1.7] one can see that $d^- g(0)/dx$ cancels.

For the second claim, suppose that $f \in C^1([a, b); \mathbb{R})$ with absolutely continuous derivative. Then, by the occupation time formula (part (i) of Lemma C.26), a.s. for all $t \in \mathbb{R}_+$,

$$\int_{(a,b)} L^x_{t} - (Y) f''(dx) = \int_a^b L^x_{t} - (Y) f''(x) dx = \int_a^b L^x_{t} - (Y) f''(x) dx = \int_0^t f''(Y_s) d\langle Y, Y \rangle_s.$$

This observation establishes (C.16). 

The following lemma contains classical facts on stability of the local martingale and semimartingale property under time-changes. It is implied by [24, Corollary 10.12, Theorem 10.16].

Lemma C.29. Let $(\mathcal{G}_t)_{t \geq 0}$ be a right-continuous (complete) filtration and let $(L(t))_{t \geq 0}$ be a finite time-change, i.e., a family of (a.s.) finite $(\mathcal{G}_t)_{t \geq 0}$-stopping times such that $t \mapsto L(t)$ is (a.s.) increasing and right-continuous.

(i) If $Y$ is a $(\mathcal{G}_t)_{t \geq 0}$-semimartingale, then $Y_L$ is a $(\mathcal{G}_{L(t)})_{t \geq 0}$-semimartingale.

(ii) If $Y$ is a (continuous) local $(\mathcal{G}_t)_{t \geq 0}$-martingale such that a.s. $Y$ is constant on every interval $[L(t-), L(t)]$, then $Y_L$ is a (continuous) local $(\mathcal{G}_{L(t)})_{t \geq 0}$-martingale.

Let us also recall a useful formulation of Girsanov’s theorem, which is a version of [25, Theorem III.3.24] for one-dimensional (continuous) semimartingales.

Lemma C.30. We consider a filtered measurable space $(\Sigma, \mathcal{G}, (\mathcal{G}_t)_{t \geq 0})$ (with a right-continuous filtration) and two probability measures $\mathbb{P}$ and $\mathbb{P}'$ defined on it. Suppose that $\mathbb{P} \ll_{loc} \mathbb{P}$. If $Y$ is a (continuous) $\mathbb{P}$-semimartingale, then it is also a (continuous) $\mathbb{P}'$-semimartingale. Moreover, if $Y$ is a continuous local
\( \mathbb{P} \)-martingale with \((Y, Y) = \int_0^c c_s ds \) (where \( c \) is a non-negative predictable process and the integral is well-defined), then there exists a predictable process \( \beta \) such that \( \mathbb{P} \)-a.s.

\[
\int_0^t \beta_s^2 c_s ds < \infty, \quad t \in \mathbb{R}_+,
\]

is a continuous local \( \mathbb{P} \)-martingale with quadratic variation \( \int_0^c c_s ds \).

C.5. Semimartingale Functions in a Non-Markovian Setting. It is well-known ([10, Theorem 5.5]) that if \( W \) is a Brownian motion and \( f : \mathbb{R} \to \mathbb{R} \) is a Borel function, the process \( f(W) \) is a semimartingale\(^{22}\) if and only if \( f \) is the difference of two convex functions. The following theorem is a variation of this result in the sense that we consider Brownian motion up to a strictly positive stopping time and deduce properties of the function in a neighborhood of the origin. As a stopped Brownian motion is, in general, no longer Markovian, we cannot use the argument from [10], which essentially hinges on the Markov property.

Theorem C.31. Let \((\Sigma, \mathcal{G}, (\mathcal{G}_t)_{t \geq 0}, \mathbb{P}) \) be a filtered probability space which supports a standard \((\mathcal{G}_t)_{t \geq 0}\)-Brownian motion \( W \) (starting in the origin). Furthermore, let \( \tau \) be a \((\mathcal{G}_t)_{t \geq 0}\)-stopping time such that \( \mathbb{P}(\tau > 0) > 0 \) and let \( \mathfrak{f} : \mathbb{R} \to \mathbb{R} \) be a Borel function. Suppose that \( \mathfrak{f}(W_{\wedge \tau}) \) is a \((\mathcal{G}_t)_{t \geq 0}\)-semimartingale. Then, there exists a \( \delta > 0 \) such that the restriction \( \mathfrak{f}_{(-\delta, \delta)} \) is the difference of two convex functions from \((\delta, \delta)\) into \( \mathbb{R} \) (in particular, \( \mathfrak{f} \) is continuous on \((\delta, \delta)\)).

Before we prove this result, let us stress that we recover the classical global result from [10].

Corollary C.32. Let \( \mathfrak{f} : \mathbb{R} \to \mathbb{R} \) be a Borel function and let \( W \) be a standard Brownian motion. If \( \mathfrak{f}(W) \) is a semimartingale, then \( \mathfrak{f} \) is the difference of two convex functions from \( \mathbb{R} \) into \( \mathbb{R} \).

Proof. Take \( x \in \mathbb{R} \). By Lemma C.29, the process \( \mathfrak{f}(W_{\wedge T_x(W)}) \) is a semimartingale. As \( W_{\wedge T_x(W)} - x \) is a standard Brownian motion, Theorem C.31 yields the existence of a \( \delta > 0 \) such that \((\delta, \delta) \ni y \mapsto \mathfrak{f}(y + x)\) is the difference of two convex functions, i.e., \( \mathfrak{f} \) is the difference of two convex functions on \((x - \delta, x + \delta)\). Thus, as being the difference of two convex functions is a local property (use the equivalence between (a) and (b) before Lemma B.4, or, alternatively, apply [21, (I) on p. 707]), it follows that \( \mathfrak{f} : \mathbb{R} \to \mathbb{R} \) is the difference of two convex functions.

Theorem C.31 follows by an adjustment of the argument in [38, Theorem 2.1], which relies on [38, Proposition 6.3], which provides a useful criterion for a continuous function to be the difference of two convex ones. For convenience, let us restate [38, Proposition 6.3]:

Lemma C.33. Let \( \mathfrak{f} : \mathbb{R} \to \mathbb{R} \) be a continuous function and let \( I \) be an open interval. Then, \( \mathfrak{f} : I \to \mathbb{R} \) is the difference of two convex functions if and only if for any compact interval \( K \subset I \)

\[
\limsup_{\beta \searrow 0} \frac{1}{\beta} \sum_{x \in K \cap \beta\mathbb{Z}} |\mathfrak{f}(x + \beta) + \mathfrak{f}(x - \beta) - 2\mathfrak{f}(x)| < \infty.
\]

Proof of Theorem C.31. We first show that \( \mathfrak{f} \) is continuous in a sufficiently small neighborhood of zero. For contradiction, assume that there exists a sequence \((x_n)_{n \in \mathbb{N}}\) of points such that \( x_n \to 0 \) and such that \( \mathfrak{f} \) is not continuous at each \( x_n \). Since a.s. \( T_{x_n}(W) \to 0 \), we can take a sufficiently large \( N \in \mathbb{N} \) such that \( \mathbb{P}(T_{x_N}(W) < \tau) > 0 \). By the strong Markov property, \((W_{T_{x_N}(W) + s - x_N})_{s \geq 0}\) is a Brownian motion, hence has oscillating behavior as \( s \searrow 0 \). Therefore, a.s. on \( \{T_{x_N}(W) < \tau\} \) it holds

\[
\limsup_{t \searrow T_{x_N}(W)} \mathfrak{f}(W_{\wedge \tau}) = \limsup_{x \to x_N} \mathfrak{f}(x) > \liminf_{x \to x_N} \mathfrak{f}(x) = \liminf_{t \searrow T_{x_N}(W)} \mathfrak{f}(W_{\wedge \tau}).
\]

\(^{22}\)Semimartingales are always assumed to have càdlàg paths.
This contradicts the right-continuity and hence, the semimartingale property of \( f(W_{\cdot \tau}) \). Therefore, we can find an \( \varepsilon > 0 \) such that \( f \) is continuous on \((-2\varepsilon, 2\varepsilon)\) and \( \mathbb{P}(T_{-\varepsilon}(W) \wedge T_{\varepsilon}(W) < \tau) > 0 \). Now, replacing \( f \) with the continuous function \( \mathbb{I}_{(-\varepsilon, \varepsilon)} + \mathbb{I}_{[-\varepsilon, \varepsilon]} \) and \( \mathbb{I}_{(\varepsilon, \infty)} \) and \( \tau \) with \( \tau \wedge T_{-\varepsilon}(W) \wedge T_{\varepsilon}(W) \), we see that we can w.l.o.g. assume that \( f: \mathbb{R} \to \mathbb{R} \) is continuous.

Next, also notice that we can w.l.o.g. assume that \( \mathbb{P}(\tau > 0) = 1 \). Indeed, in case \( \mathbb{P}(\tau > 0) < 1 \), we can simply replace the probability measure \( \mathbb{P} \) by

\[
\hat{\mathbb{P}}(d\omega) \triangleq \mathbb{P}(d\omega|\tau > 0) = \frac{\mathbb{P}(d\omega \cap \{\tau > 0\})}{\mathbb{P}(\tau > 0)}.
\]

Clearly, as \( \{\tau > 0\} \in \mathcal{G}_0 \), \( W \) remains a \((\mathcal{G}_t)_{t \geq 0}\)-Brownian motion under \( \hat{\mathbb{P}} \) and, since \( \hat{\mathbb{P}} \ll \mathbb{P} \), Girsanov’s theorem yields that \( f(W_{\cdot \tau}) \) is also a \( \hat{\mathbb{P}}\)-\((\mathcal{G}_t)_{t \geq 0}\)-semimartingale. Thus, from now on, we assume that \( f: \mathbb{R} \to \mathbb{R} \) is continuous and \( \mathbb{P}(\tau > 0) = 1 \).

Fix \( \beta \in (0, 1) \) and define inductively

\[
\tau_0 \triangleq 0, \quad \tau_{n+1} \triangleq \inf(t > \tau_n - 1: |W_t - W_{\tau_n}| \geq \beta), \quad n \in \mathbb{N}.
\]

It is easy to see that \( (S_n \triangleq W_{\tau_n})_{n \in \mathbb{N}} \) is a simple random walk starting in the origin with step size \( \beta \). By hypothesis,

\[
f(W_{\cdot \tau}) = f(0) + M + A,
\]

where \( M \) is a continuous local martingale and \( A \) is a continuous process of (locally) finite variation both starting in zero. We denote the variation process of \( A \) by \( V \). Take some \( K > 1 \) and define

\[
\rho_K \triangleq \inf(t \geq 0: |M_t| \vee |V_t| \vee |W_t| \geq K) \wedge K \wedge \tau.
\]

Next, take \( I = (a, c) \) with \( a < 0 < c \) small enough such that

\[
\inf_{x \in I} \mathbb{E}[L_{p_K}^x(W)] > 0,
\]

where \( \{L_{p_K}^x(W): (t, x) \in \mathbb{R}_+ \times \mathbb{R}\} \) denotes the jointly continuous modification of the local time process of \( W \).

Let us prove that such an interval \( I \) exists. Thanks to the generalized Itô formula (Lemma C.26), we have

\[
\mathbb{E}[L_{p_K}^x(W)] = \mathbb{E}[|W_{p_K} - x|] - |x|,
\]

and hence, the map \( x \mapsto \mathbb{E}[L_{p_K}^x(W)] \) is continuous. Lemma C.18 implies that \( \mathbb{E}[L_{p_K}^0(W)] > 0 \). Now, for a continuous function which is strictly positive in zero, we can find a sufficiently small neighborhood \( I \) of the origin where this function is bounded away from zero.

We set

\[
l(x, t) \triangleq \sum_{m=0}^{\infty} \mathbb{I}\{S_m = x, \tau_m < t\}, \quad (x, t) \in \mathbb{R} \times \mathbb{R}_+.
\]

For \( n \in \mathbb{Z}_+ \), on the event \( \{\tau_n < \rho_K\} \) we get

\[
\frac{1}{2}(f(S_n + \beta) + f(S_n - \beta)) - f(S_n) = |\mathbb{E}[f(S_{n+1}) - f(S_n)|\mathcal{G}_{\tau_n}]|
\]

\[
\leq |\mathbb{E}[f(W_{\tau_{n+1}}) - f(W_{\tau_{n+1} \wedge \rho_K})|\mathcal{G}_{\tau_n}]| + \mathbb{E}[|A_{\tau_{n+1} \wedge \rho_K} - A_{\tau_n}|\mathcal{G}_{\tau_n}]|
\]

Consequently, we obtain

\[
\sum_{k: \tau_k \in I} \frac{1}{2}|f((k+1)\beta) + f((k-1)\beta) - 2f(k\beta)| \mathbb{E}[l(k\beta, \rho_K)]
\]

\[
= \mathbb{E}\left[ \sum_{m=0}^{\infty} \sum_{k: \tau_k \in I} \frac{1}{2}|f(S_m + \beta) + f(S_m - \beta) - 2f(S_m)| \mathbb{I}\{S_m = k\beta, \tau_m < \rho_K\} \right]
\]

\[
\leq \mathbb{E}\left[ \sum_{m=0}^{\infty} (|f(W_{\tau_{m+1}}) - f(W_{\tau_{m+1} \wedge \rho_K})|\mathcal{G}_{\tau_m}) + \mathbb{E}[|A_{\tau_{m+1} \wedge \rho_K} - A_{\tau_m}|\mathcal{G}_{\tau_m}] \mathbb{I}\{\tau_m < \rho_K\} \right]
\]
Consequently, taking the expectation of both sides, we obtain that

\[
\sum_{m=0}^{\infty} \mathbb{E}\left[ (f(W_{m+1}^\tau) - f(W_{m+1}^\tau)_{\rho_K}) | \mathcal{G}_m \right] + \mathbb{E}\left[ A_{m+1}^\tau - A_{m}^{\tau} | \mathcal{G}_m \right] I\{\tau_m < \rho_K\} = \sum_{m=0}^{\infty} \mathbb{E}\left[ (f(W_{m+1}^\tau) - f(W_{m+1}^\tau)_{\rho_K}) + |A_{m+1}^\tau - A_{m}^{\tau}| I\{\tau_m < \rho_K\}\right]
\]

\[
\leq \mathbb{E}\left[ \sum_{m=0}^{\infty} |f(W_{m+1}^\tau) - f(W_{m+1}^\tau)_{\rho_K})| I\{\tau_m < \rho_K < \tau_{m+1}\} + V_{\rho_K} \right] 
\]

\[
\leq \sup_{|x| \leq K+1} 2|f(x)| + K.
\]

We now estimate the expectation \(\mathbb{E}[l(k\beta, \rho_K)]\). Notice that, for all \(n \in \mathbb{Z}_+\), on the event \(\{\tau_n < \rho_K\}\)

\[
(C.17) \quad |W_{n+1}^\tau - k\beta| - |W_{n}^\tau - k\beta| = \begin{cases} W_{n+1}^\tau - W_n^\tau & \text{if } W_{\tau_n} \geq (k+1)\beta, \\ W_n^\tau - W_{n+1}^\tau & \text{if } W_{\tau_n} \leq (k-1)\beta, \\ |W_{n+1}^\tau - k\beta| & \text{if } S_n = W_{n} = k\beta. \end{cases}
\]

For every \(n \in \mathbb{N}\), the generalized Itô formula (Lemma C.26) yields that \(\mathbb{P}\)-a.s.

\[
(\|W_{n+1}^\tau - k\beta\| - |W_{n}^\tau - k\beta|) I\{\tau_n < \rho_K\} = \left( L_{k\beta}^{\tau_{n+1}^\tau} W_n^\tau - L_{k\beta}^{\tau_n^\tau} - \int_{\tau_n}^{\tau_{n+1}^\tau} \text{sgn}(W_s - k\beta) dW_s \right) I\{\tau_n < \rho_K\}.
\]

Computing the conditional expectation \(\mathbb{E}[\cdot | \mathcal{G}_{\tau_n}]\) of both sides and taking (C.17) into account, we obtain from the martingale property of Brownian motion and the stochastic integral that \(\mathbb{P}\)-a.s.

\[
\mathbb{E}[W_{n+1}^\tau - k\beta | S_n = k\beta, \tau_n < \rho_K] | \mathcal{G}_{\tau_n}] = \mathbb{E}[(L_{k\beta}^{\tau_{n+1}^\tau} - L_{k\beta}^{\tau_n^\tau}) I\{\tau_n < \rho_K\} | \mathcal{G}_{\tau_n}].
\]

Consequently, taking the expectation of both sides, we obtain that

\[
\mathbb{E}[(L_{k\beta}^{\tau_{n+1}^\tau} - L_{k\beta}^{\tau_n^\tau}) I\{\tau_n < \rho_K\}] \leq \beta \mathbb{P}(S_n = k\beta, \tau_n < \rho_K).
\]

Summing over \(n\) from 0 to \(\infty\) and using Fubini’s theorem yields that

\[
\mathbb{E}[L_{k\beta}^{\tau_{n+1}^\tau} W_n^\tau] \leq \beta \mathbb{E}[l(k\beta, \rho_K)].
\]

Putting all pieces together, we conclude that

\[
\sum_{k, \beta \in I} \frac{1}{\beta} |f((k+1)\beta) + f((k-1)\beta) - 2f(k\beta)| \leq \frac{\max_{|x| \leq K+1} 4|f(x)| + 2K}{\inf_{x \in I} \mathbb{E}[L_{\rho_K}^{\tau_{n+1}^\tau} W_n^\tau]} < \infty.
\]

Finally, we conclude from Lemma C.33 that the function \(f: I \to \mathbb{R}\) is the difference of two convex functions. \(\square\)

C.6. A useful identity for natural filtrations. For a process \(Y = (Y_t)_{t \geq 0}\) with paths in \(\Omega\), we define its natural filtration by

\[
\mathcal{F}_t^Y = \bigcap_{\varepsilon > 0} \sigma(Y_s, s \leq t + \varepsilon), \quad t \in \mathbb{R}_+.
\]

The following technical observation is useful in some of our arguments.

Lemma C.34. Let \(\tau\) be an \((\mathcal{F}_t)_{t \geq 0}\)-stopping time and let \(Y\) be a process with paths in \(\Omega\). Then, \(\rho \triangleq \tau \circ Y\) is an \((\mathcal{F}_t^Y)_{t \geq 0}\)-stopping time and

\[
Y^{-1}(\mathcal{F}_\tau) = \mathcal{F}_\rho^Y.
\]

Proof. The fact that \(\rho\) is an \((\mathcal{F}_t^Y)_{t \geq 0}\)-stopping time is part (a) of [24, Proposition 10.35]. We now prove identity (C.18). Recall from [23, Problem 1, p. 88] that

\[
\mathcal{F}_\tau = \bigcap_{\varepsilon > 0} \sigma(X_{t \wedge (\tau + \varepsilon)}, t \geq 0), \quad \mathcal{F}_\rho^Y = \bigcap_{\varepsilon > 0} \sigma(Y_{t \wedge (\rho + \varepsilon)}, t \geq 0).
\]
Thus, for every $\varepsilon > 0$, we have

$$Y^{-1}(F_\tau) \subset Y^{-1}(\sigma(X_{t+\tau}, t \geq 0)) = \sigma(Y_{t+\tau}, t \geq 0),$$

which implies that $Y^{-1}(F_\tau) \subset \mathcal{F}^Y_\tau$. To establish the converse inclusion, take $A \in \mathcal{F}^Y_\tau$. Then, for every $n \in \mathbb{N}$, $A \in \sigma(Y_{t+(\rho+1/n)}, t \geq 0) = Y^{-1}(\sigma(X_{t+(\rho+1/n)}, t \geq 0))$, which means there exists a set $B_n \in \sigma(X_{t+(\rho+1/n)}, t \geq 0)$ such that $A = Y^{-1}(B_n)$. Set $G \triangleq \bigcap_{n \in \mathbb{N}} \bigcup_{m \geq n} B_m$ and notice that $G \in \mathcal{F}_\tau$. Finally, as $Y^{-1}(G) = \bigcap_{n \in \mathbb{N}} \bigcup_{m \geq n} Y^{-1}(B_m) = A$, we conclude that $A \in Y^{-1}(F_\tau)$, which finishes the proof of (C.18).

\[\square\]

APPENDIX D. GENERALIZED DENSITY AND DIFFERENTIATION OF MEASURES

D.1. Generalized Density. Let $(\Omega, \mathcal{F})$ be a measurable space and let $\mu, \nu$ be $\sigma$-finite measures on $(\Omega, \mathcal{F})$. Generalized Radon–Nikodym derivative (or generalized density) $\partial \mu/\partial \nu$ is an $\mathcal{F}$-measurable mapping $\Omega \to [0, \infty]$ such that

$$\mu, \nu\text{-a.e. } \frac{\partial \mu}{\partial \nu} = \frac{d\mu/d\gamma}{d\nu/d\gamma},$$

where $\gamma$ is any $\sigma$-finite measure on $(\Omega, \mathcal{F})$ such that $\mu \ll \gamma$ and $\nu \ll \gamma$ ($d\mu/d\gamma$ and $d\nu/d\gamma$ denote the corresponding Radon–Nikodym derivatives). To define a version of $\partial \mu/\partial \nu$, we can simply take the right-hand side of (D.1) with any convention for how to understand $0/0$, e.g., $0/0 \triangleq 1$. We emphasize that this definition does not depend on the choice of the dominating measure $\gamma$ because it holds that $\mu + \nu \ll \gamma$,

$$\gamma\text{-a.e., hence, } \mu, \nu\text{-a.e. } \frac{d\mu}{d\gamma} = \frac{d\mu}{d(\mu + \nu)} \frac{d(\mu + \nu)}{d\gamma},$$

$$\gamma\text{-a.e., hence, } \mu, \nu\text{-a.e. } \frac{d\nu}{d\gamma} = \frac{d\nu}{d(\mu + \nu)} \frac{d(\mu + \nu)}{d\gamma},$$

and $(\mu + \nu)(d(\mu + \nu)/d\gamma) = 0 = 0$, thus,

$$\mu, \nu\text{-a.e. } \frac{d\mu/d\gamma}{d\nu/d\gamma} = \frac{d\mu/d(\mu + \nu)}{d\nu/d(\mu + \nu)},$$

where the latter expression does not depend on $\gamma$. We also notice that in the case $\mu \ll \nu$ the generalized Radon–Nikodym derivative is nothing else but the standard one (take $\gamma = \nu$ in (D.1)).

The generalized Radon–Nikodym derivative appears without a special name in [19, Theorem A.17] and in [24, Theorem 7.1], and it appears under the name Lebesgue derivative in formula (29) of [44, Section III.9]. This object is a convenient tool to concisely describe the mutual arrangement between $\mu$ and $\nu$ from the viewpoint of absolute continuity and singularity. We illustrate this by the following lemma. Its proof is straightforward and therefore omitted.

**Lemma D.1.** There exist pairwise disjoint sets $E, S_\mu, S_\nu \in \mathcal{F}$ such that

$$\Omega = E \cup S_\mu \cup S_\nu, \quad \mu \sim \nu \text{ on } \mathcal{F} \cap E, \quad \mu(S_\nu) = \nu(S_\mu) = 0$$

(in particular, $\mu \perp \nu$ on $\mathcal{F} \cap E^c$, where $E^c \triangleq \Omega \setminus E$). Such sets $E, S_\mu, S_\nu$ are $\mu, \nu$-a.e. unique. Moreover, $\mu, \nu$-a.e. it holds

$$E = \{ \frac{\partial \mu}{\partial \nu} \in (0, \infty) \}, \quad S_\mu = \{ \frac{\partial \mu}{\partial \nu} = \infty \}, \quad S_\nu = \{ \frac{\partial \mu}{\partial \nu} = 0 \}.$$

**Corollary D.2.**

(i) $\mu$-a.e. $\partial \mu/\partial \nu$ is $(0, \infty]$-valued.

(ii) $\mu \ll \nu$ if and only if $\mu$-a.e. $\partial \mu/\partial \nu < \infty$.

(iii) $\mu \perp \nu$ if and only if $\mu$-a.e. $\partial \mu/\partial \nu = \infty$.

\[\text{In fact, as } (\mu + \nu)(d\mu/d\gamma = d\nu/d\gamma) = 0, \text{ it does not matter which convention to choose for } 0/0.\]
Another convenient feature of the generalized density is its symmetry in $\mu$ and $\nu$ (regardless of the mutual arrangement between $\mu$ and $\nu$): $\mu, \nu$-a.e. we, clearly, have
\[
\frac{\partial \nu}{\partial \mu} = \frac{1}{\frac{\partial \mu}{\partial \nu}}.
\]
For an illustration, we now rewrite Corollary D.2, which views the generalized density $\partial \mu / \partial \nu$ only under $\mu$, into a similar result that views $\partial \mu / \partial \nu$ only under $\nu$.

**Corollary D.3.**
(i) $\nu$-a.e. $\partial \mu / \partial \nu$ is $[0, \infty)$-valued.
(ii) $\nu \ll \mu$ if and only if $\nu$-a.e. $\partial \mu / \partial \nu > 0$.
(iii) $\mu \perp \nu$ if and only if $\nu$-a.e. $\partial \mu / \partial \nu = 0$.

We, finally, remark that the unique (Lebesgue) decomposition $\mu = \mu^{ac} + \mu^s$, where $\mu^{ac}$ and $\mu^s$ are $\sigma$-finite measures on $(\Omega, \mathcal{F})$ such that $\mu^{ac} \ll \nu$ and $\mu^s \perp \nu$, can be easily expressed with the help of the generalized Radon–Nikodym derivative, namely

\[
\mu^{ac}(A) = \int_A \frac{\partial \mu}{\partial \nu} \, d\nu, \quad \mu^s(A) = \mu \left( A \cap \left\{ \frac{\partial \mu}{\partial \nu} = \infty \right\} \right), \quad A \in \mathcal{F},
\]

which easily follows from Lemma D.1 (alternatively, see [19, Theorem A.17] or [44, (29) in Section III.9]). In particular, viewed under $\nu$ (but not under $\mu$!), $\partial \mu / \partial \nu$ is a version of the Radon–Nikodym derivative $d\mu^{ac} / d\nu$ of the absolutely continuous part $\mu^{ac}$ of $\mu$ with respect to $\nu$.

### D.2. Generalized density process and Jessen’s theorem

With the concept of generalized density we can in a natural way generalize Jessen’s theorem (Corollary D.5 below) and provide a very simple proof.

We consider a filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n \in \mathbb{N}})$ with a discrete-time filtration such that $\mathcal{F} = \bigvee_{n \in \mathbb{N}} \mathcal{F}_n$ (i.e., $\bigcup_{n \in \mathbb{N}} \mathcal{F}_n$ generates $\mathcal{F}$). In this subsection, for any probability measure $P$ on $(\Omega, \mathcal{F})$, we use the notation

\[
P_n \doteq P|F_n
\]

for the restriction of $P$ to $\mathcal{F}_n$.

Let $P$ and $\tilde{P}$ be two arbitrary probability measures on $(\Omega, \mathcal{F})$. The process $(\partial P_n / \partial \tilde{P}_n)_{n \in \mathbb{N}}$ is called the **generalized density process**, where $\partial P_n / \partial \tilde{P}_n$ denote the corresponding generalized Radon-Nikodym derivatives.

**Theorem D.4.** $\tilde{P}$-a.s. it holds that

\[
\frac{\partial P_n}{\partial \tilde{P}_n} \to \frac{\partial P}{\partial \tilde{P}}, \quad n \to \infty.
\]

**Proof.** Set $Q \doteq \frac{P + \tilde{P}}{2}$, notice that $P \ll Q$ and consider the process $(Z_n)_{n \in \mathbb{N}}$, $Z_n \doteq dP_n / dQ_n$, where $dP_n / dQ_n$ denotes the (standard) Radon-Nikodym derivative. We have $Z_n = E^Q[dP / dQ|F_n]$, $n \in \mathbb{N}$, and consequently, the following:

- $(Z_n)_{n \in \mathbb{N}}$ is a uniformly integrable $Q$-martingale.
- $Q$-a.s. and in $L^1(Q)$, $Z_n \to Z_\infty$, $n \to \infty$, for some random variable $Z_\infty \in L^1(Q)$.
- For all $n_0 \in \mathbb{N}$ and $A \in \mathcal{F}_{n_0}$,

\[
\int_A Z_\infty \, dQ = \lim_{n_0 \leq n \to \infty} \int_A Z_n \, dQ = P(A) = \int_A \frac{dP}{dQ} \, dQ.
\]

- By a monotone class argument, for all $A \in \bigcup_{n \in \mathbb{N}} \mathcal{F}_n = \mathcal{F}$,

\[
\int_A Z_\infty \, dQ = \int_A \frac{dP}{dQ} \, dQ.
\]

This implies that $Q$-a.s. $Z_\infty = dP / dQ$. 

Concluding, Q-a.s. it holds
\[
\frac{d\mathbb{P}_n}{d\mathbb{Q}_n} \to \frac{d\mathbb{P}}{d\mathbb{Q}} \quad \text{and} \quad \frac{\hat{d}\mathbb{P}_n}{d\mathbb{Q}_n} \to \frac{\hat{d}\mathbb{P}}{d\mathbb{Q}}, \quad n \to \infty
\]
(the second claim holds by symmetry). Now the result follows from (D.1).

Let \( \mathbb{P}^ac \) denote the absolutely continuous part of \( \mathbb{P} \) with respect to \( \hat{\mathbb{P}} \). Similarly, let \( \mathbb{P}^ac_n \) be the absolutely continuous part of \( \mathbb{P}_n \) with respect to \( \hat{\mathbb{P}}_n \). Notice that, in general, \( \mathbb{P}^ac_n \) is not the restriction of \( \mathbb{P}^ac \) to \( \mathcal{F}_n \).

Recalling the discussion after Corollary D.3, we obtain Jessen’s theorem (see [45, Theorem 5.2.20]) as an immediate consequence of Theorem D.4.

**Corollary D.5 (Jessen’s theorem).** \( \hat{\mathbb{P}} \)-a.s. it holds that
\[
\frac{d\mathbb{P}^ac_n}{d\mathbb{P}_n} \to \frac{d\mathbb{P}^ac}{d\mathbb{P}}, \quad n \to \infty.
\]

### D.3. Differentiation of Measures

Let \( I \subset \mathbb{R} \) be an open interval and let \( \mu \) and \( \nu \) be locally finite measures on \((I, \mathcal{B}(I))\). For \( x \in \mathbb{R} \) and \( r > 0 \), we write \( B(x, r) \) for the open ball with center \( x \) and radius \( r \).

A sequence \((A_j)_{j=1}^\infty \subset \mathcal{B}(I)\) is said to converge \( \nu \)-measure-metrizably to \( x \in I \) if the following conditions hold:

- (a) For every \( j \in \mathbb{N} \), there exists an \( r_j > 0 \) such that \( A_j \subset B(x, r_j) \subset I \);
- (b) \( \lim_{j \to \infty} r_j = 0 \);
- (c) there exists an \( \alpha \in (0, 1] \) such that for every \( j \in \mathbb{N} \), \( \nu(A_j) \geq \alpha \nu(B(x, r_j)) \).

We now define two closely related derivatives of \( \mu \) w.r.t. \( \nu \) at a point \( x \in I \). The first one will be denoted by \( D^\nu_\nu(\mu)(x) \) and the second one by \( \overline{D}^\nu_\nu(\mu)(x) \). If there exists a number \( z \in [0, \infty] \) such that
\[
\lim_{j \to \infty} \mu(A_j) = z,
\]
for every sequence \((A_j)_{j=1}^\infty \) which converges \( \nu \)-measure-metrizably (resp., \( \mu \)-measure-metrizably and \( \nu \)-measure-metrizably) to \( x \), then we write \( D^\nu_\nu(\mu)(x) = z \) (resp., \( \overline{D}^\nu_\nu(\mu)(x) = z \)). If the limit
\[
\lim_{r \to 0} \frac{\mu(B(x, r))}{\nu(B(x, r))}
\]
exists in \([0, \infty] \), it is said to be the symmetric derivative of \( \mu \) w.r.t. \( \nu \) at \( x \) and we denote it by \( D^\nu_{sym}(\mu)(x) \).

We notice the obvious relations between the notions:

(D.2) \( D^\nu_\nu(\mu)(x) \) exists \( \iff \) \( \overline{D}^\nu_\nu(\mu)(x) \) exists and equals \( D^\nu_\nu(\mu)(x) \);

(D.3) \( \overline{D}^\nu_\nu(\mu)(x) \) exists \( \iff \) \( D^\nu_{sym}(\mu)(x) \) exists and equals \( \overline{D}^\nu_\nu(\mu)(x) \).

The following is a partial restatement of [3, Theorem 8.4.4, Corollary 8.4.5].

**Lemma D.6.** (i) \( D^\nu_\nu(\mu) \) exists \( \nu \)-a.e. and \( D^\nu_\nu(\mu) \in L^1_{loc}(I, \mathcal{B}(I), \nu) \).

(ii) \( \mu \ll \nu \) if and only if \( \mu(A) = \int_A D^\nu_\nu(\mu)d\nu \) for all \( A \in \mathcal{B}(I) \).

This observation allows us to infer that \( \overline{D}^\nu_\nu(\mu) \) is a version of the generalized Radon–Nikodym derivative \( \partial \mu/\partial \nu \).

**Theorem D.7.** For all locally finite measures \( \mu, \nu \) on \((I, \mathcal{B}(I))\), the derivative \( \overline{D}^\nu_\nu(\mu) \) exists \( \mu, \nu \)-a.e. on \( I \) and is a version of the generalized Radon–Nikodym derivative \( \partial \mu/\partial \nu \). In particular, the same claims hold also for the symmetric derivative \( D^\nu_{sym}(\mu) \).

**Proof.** By Lemma D.6, the derivatives \( D^\mu_\nu(\mu) \) and \( D^\nu_\nu(\mu) \) exist \( \mu, \nu \)-a.e. and are versions of the Radon–Nikodym derivatives \( d\mu/d(\mu + \nu) \) and \( d\nu/d(\mu + \nu) \). For a point \( x \in I \) such that \( D^\mu_\nu(\mu)(x) \) and \( D^\nu_\nu(\mu)(x) \) exist and
\[
\text{neither } D^\mu_\nu(\mu)(x) = D^\nu_\nu(\mu)(x) = 0 \text{ nor } D^\mu_\nu(\mu)(x) = D^\nu_\nu(\mu)(x) = \infty,
\]

...
we have $\mathcal{D}_\nu(\mu)(x) = D_{\mu+\nu}(\mu)(x)/D_{\mu+\nu}(\nu)(x)$ (notice here that, if a sequence $(A_j)_{j=1}^\infty$ converges $\mu$-measure-metrizably and $\nu$-measure-metrizably to $x$, then it converges $(\mu+\nu)$-measure-metrizably to $x$).

The claims for $\mathcal{D}_\nu(\mu)$ now follow from (D.1) with $\gamma = \mu + \nu$. The claims for $D_{\nu}^{\text{sym}}(\mu)$ are trivial consequences of (D.3). □

References

[1] S. Ankirchner, T. Kruse, and M. Urusov. A functional limit theorem for coin tossing Markov chains. *Ann. Inst. Henri Poincaré, Probab. Stat.*, 56(4):2996–3019, 2020.
[2] R. Bass. A stochastic differential equation with a sticky point. *Electron. J. Probab.*, 19:22, 2014. Id/No 32.
[3] J. J. Benedetto and W. Czaja. *Integration and modern analysis*. Birkhäuser Adv. Texts, Basler Lehrbüch. Basel: Birkhäuser, 2009.
[4] A. N. Borodin and P. Salminen. *Handbook of Brownian motion: Facts and formulae*. Probab. Appl. Basel: Birkhäuser, 2nd edition, 2002.
[5] L. Breiman. *Probability*, volume 7 of *Class. Appl. Math.* Society for Industrial and Applied Mathematics (SIAM), Philadelphia, 1992.
[6] C. Bruggeman and J. Ruf. A one-dimensional diffusion hits points fast. *Electron. Commun. Probab.*, 21:7, 2016. Id/No 22.
[7] P. Cheridito, D. Filipović, and M. Yor. Equivalent and absolutely continuous measure changes for jump-diffusion processes. *Ann. Appl. Probab.*, 15(3):1713–1732, 2005.
[8] A. Cherny and M. Urusov. On the absolute continuity and singularity of measures on filtered spaces: separating times. In *From stochastic calculus to mathematical finance*. The Shiryaev Festschrift., pages 125–168. Berlin: Springer, 2006.
[9] K. L. Chung and J. B. Walsh. Meyer’s theorem on predictability. *Z. Wahrscheinlichkeitstheor. Verw. Geb.*, 29:253–256, 1974.
[10] E. Cinlar, J. Jacod, P. Protter, and M. J. Sharpe. Semimartingales and Markov processes. *Z. Wahrscheinlichkeitstheor. Verw. Geb.*, 54:161–219, 1980.
[11] I. J. Clark. *Foreign exchange option pricing: A Practitioner’s Guide*. Wiley Finance Series. Wiley, 2011.
[12] D. Criens. No arbitrage in continuous financial markets. *Math. Financ. Econ.*, 14(3):461–506, 2020.
[13] D. Criens. On absolute continuity and singularity of multidimensional diffusions. *Electron. J. Probab.*, 26:26, 2021. Id/No 12.
[14] D. Criens and K. Glau. Absolute continuity of semimartingales. *Electron. J. Probab.*, 23:28 p, 2018. Id/No 125.
[15] F. Delbaen and W. Schachermayer. A general version of the fundamental theorem of asset pricing. *Math. Ann.*, 300(3):463–520, 1994.
[16] F. Delbaen and W. Schachermayer. Arbitrage possibilities in Bessel processes and their relations to local martingales. *Probab. Theory Relat. Fields*, 102(3):357–366, 1995.
[17] S. Desmettre, G. Leobacher, and L. C. G. Rogers. Change of drift in one-dimensional diffusions. *Finance Stoch.*, 25(2):359–381, 2021.
[18] H.-J. Engelbert and G. Peskir. Stochastic differential equations for sticky Brownian motion. *Stochastics*, 86(6):993–1021, 2014.
[19] H. Föllmer and A. Schied. *Stochastic finance. An introduction in discrete time*. De Gruyter Textb. Berlin: de Gruyter, 4th revised edition edition, 2016.
[20] D. Freedman. Brownian motion and diffusion. (Reprint of the 1971 orig., publ. by Holden-Day, Inc., San Francisco). New York - Heidelberg - Berlin: Springer-Verlag., 1983.
[21] P. Hartman. On functions representable as a difference of convex functions. *Pac. J. Math.*, 9:707–713, 1959.
[22] H. Hulley and J. Ruf. Weak tail conditions for local martingales. *Ann. Probab.*, 47(3):1811–1825, 2019.
[23] K. Itô and H. P. McKean jun. *Diffusion processes and their sample paths.* Class. Math. Berlin: Springer-Verlag, repr. of the 1974 edition, 1996.

[24] J. Jacod. *Calcul stochastique et problèmes de martingales,* volume 714 of *Lect. Notes Math.* Springer, Cham, 1979.

[25] J. Jacod and A. N. Shiryaev. *Limit theorems for stochastic processes,* volume 288 of *Grundlehren Math. Wiss.* Berlin: Springer, 2nd edition, 2003.

[26] O. Kallenberg. *Foundations of modern probability. In 2 volumes,* volume 99 of *Probab. Theory Stoch. Model.* Cham: Springer, 3rd revised and expanded edition, 2021.

[27] I. Karatzas and S. E. Shreve. *Brownian motion and stochastic calculus,* volume 113 of *Grad. Texts Math.* New York etc.: Springer-Verlag, 2nd edition, 1991.

[28] S. Karlin and H. M. Taylor. A second course in stochastic processes. New York etc.: Academic Press, A Subsidiary of Harcourt Brace Jovanovich, Publishers. XVIII, 542 p. $35.00 (1981), 1981.

[29] S. Kotani. On a condition that one-dimensional diffusion processes are martingales. In *In memoriam Paul-André Meyer. Séminaire de probabilités XXXIX.,* pages 149–156. Berlin: Springer, 2006.

[30] J.-F. Le Gall. *Brownian motion, martingales, and stochastic calculus,* volume 274 of *Grad. Texts Math.* Cham: Springer, 2016.

[31] T. M. Liggett. *Continuous time Markov processes. An introduction.,* volume 113 of *Grad. Stud. Math.* Providence, RI: American Mathematical Society (AMS), 2010.

[32] R. S. Liptser and A. N. Shiryaev. *Statistics of random processes. I. General theory. Translated by A. B. Aries,* volume 5 of *Appl. Math. (N. Y.).* Springer, New York, 1977.

[33] A. Mijatović and M. Urusov. Convergence of integral functionals of one-dimensional diffusions. *Electron. Commun. Probab.*, 17:13, 2012. Id/No 61.

[34] A. Mijatović and M. Urusov. Deterministic criteria for the absence of arbitrage in one-dimensional diffusion models. *Finance Stoch.*, 16(2):225–247, 2012.

[35] A. Mijatović and M. Urusov. On the martingale property of certain local martingales. *Probab. Theory Relat. Fields,* 152(1-2):1–30, 2012.

[36] A. Mijatović and M. Urusov. On the loss of the semimartingale property at the hitting time of a level. *J. Theor. Probab.*, 28(3):892–922, 2015.

[37] S. Orey. Conditions for the absolute continuity of two diffusions. *Trans. Am. Math. Soc.*, 193:413–426, 1974.

[38] V. Prokaj and L. Bondici. On the lack of semimartingale property. *Stochastic Processes Appl.*, 146:335–359, 2022.

[39] D. Revuz and M. Yor. *Continuous martingales and Brownian motion,* volume 293 of *Grundlehren Math. Wiss.* Berlin: Springer, 3rd. corrected printing, 3rd edition, 2005.

[40] L. C. G. Rogers and D. Williams. *Diffusions, Markov processes, and martingales. Volume 2: Itô calculus. Wiley Ser. Probab. Math. Stat. John Wiley & Sons, Hoboken, NJ, 1987.

[41] W. Rudin. *Real and complex analysis.* New York, NY: McGraw-Hill, 3rd edition, 1987.

[42] J. Ruf. The martingale property in the context of stochastic differential equations. *Electron. Commun. Probab.*, 20:10, 2015. Id/No 34.

[43] S. Saks. *Theory of the integral. 2. ed. English translation by L. C. Young. With two additional notes by S. Banach,* volume 7 of *Monogr. Mat., Warszawa.* PWN - Panstwowe Wydawnictwo Naukowe, Warszawa, 1937.

[44] A. N. Shiryaev. *Probability-1. Translated from the fourth Russian edition by R. P. Boas and D. M. Chibisov,* volume 95 of *Grad. Texts Math.* New York, NY: Springer, 3rd edition of the book previously published as a single-volume edition, 2016.

[45] D. W. Stroock. *Probability theory. An analytic view.* Cambridge: Cambridge University Press, 2nd edition, 2011.

[46] W. Walter. *Analysis 2.* Springer-Lehrb. Berlin: Springer, 5., extended edition, 2002.
