Domains of analyticity for response solutions in strongly dissipative forced systems

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Abstract

We study the ordinary differential equation $\varepsilon \ddot{x} + \dot{x} + \varepsilon g(x) = \varepsilon f(\omega t)$, where $g$ and $f$ are real-analytic functions, with $f$ quasi-periodic in $t$ with frequency vector $\omega$. If $c_0 \in \mathbb{R}$ is such that $g(c_0)$ equals the average of $f$ and $g'(c_0) \neq 0$, under very mild assumptions on $\omega$ there exists a quasi-periodic solution close to $c_0$ with frequency vector $\omega$. We show that such a solution depends analytically on $\varepsilon$ in a domain of the complex plane tangent more than quadratically to the imaginary axis at the origin.

1 Introduction

Consider the ordinary differential equation in $\mathbb{R}$

$$\varepsilon \ddot{x} + \dot{x} + \varepsilon g(x) = \varepsilon f(\omega t), \quad (1.1)$$

where $\varepsilon \in \mathbb{R}$ is small and $\omega \in \mathbb{R}^d$, with $d \in \mathbb{N}$, is assumed (without loss of generality) to have rationally independent components, i.e. $\omega \cdot \nu \neq 0 \forall \nu \in \mathbb{Z}^d := \mathbb{Z}^d \setminus \{0\}$. For $\varepsilon > 0$ the equation describes a one-dimensional system with mechanical force $g$, subject to a quasi-periodic forcing $f$ with frequency vector $\omega$ and in the presence of strong dissipation. We refer to [3] for some physical background. A quasi-periodic solution to (1.1) with the same frequency vector $\omega$ as the forcing will be called a response solution.

Hypothesis 1. The functions $g: \mathbb{R} \to \mathbb{R}$ and $f: \mathbb{T}^d \to \mathbb{R}$ are real-analytic. There is $c_0 \in \mathbb{R}$ such that $g(c_0) = f_0$, where $f_0$ is the average of $f$ on $\mathbb{T}^d$, and $a := g'(c_0) \neq 0$.

In other words we assume that $c_0$ is a simple zero of the function $g(x) - f_0$. Denote by $\Sigma_\xi := \{\psi = (\psi_1, \ldots, \psi_d) \in (\mathbb{C}/2\pi\mathbb{Z})^d : |\text{Im} \psi_k| \leq \xi \text{ for } k = 1, \ldots, d\}$, with $\xi > 0$, the strip where $f$ is analytic. By the analyticity assumptions one can write

$$f(\psi) = \sum_{\nu \in \mathbb{Z}^d} e^{i \nu \cdot \psi} f_\nu, \quad g(x) = \sum_{p=0}^\infty a_p (x - c_0)^p,$$

where

$$|f_\nu| \leq \Phi e^{-\xi |\nu|}, \quad a_p := \frac{1}{p!} \frac{d^p g}{dx^p}(c_0), \quad |a_p| \leq \Gamma p^p,$$
for suitable constants $\Phi$, $\Gamma$ and $\rho$. Set $N(f) = N$ if $f$ is a trigonometric polynomial of degree $N$ and $N(f) = \infty$ otherwise, and define
\[
\beta_n(\omega) := \min \left\{ |\omega \cdot \nu| : 0 < |\nu| \leq 2^n, |\nu| \leq N(f) \right\}, \quad \epsilon_n(\omega) := \frac{1}{2^n} \log \frac{1}{\beta_n(\omega)},
\]
\[
\alpha_n(\omega) := \min \left\{ |\omega \cdot \nu| : 0 < |\nu| \leq 2^n \right\}, \quad \mathfrak{B}(\omega) := \sum_{n=0}^{\infty} \frac{1}{2^n} \log \frac{1}{\alpha_n(\omega)}.
\]

**Hypothesis 2.** $\lim_{n \to \infty} \epsilon_n(\omega) = 0$.

In particular no assumption at all is required on $\omega$ if $f$ is a trigonometric polynomial, since $\beta_n(\omega)$ is definitely constant in that case.

Before stating our results we need some more notations. We define the sets $C_R := \{ \varepsilon \in \mathbb{C} : |\Re \varepsilon^{-1}| > (2R)^{-1} \}$ and $\Omega_{R,B} := \{ \varepsilon \in \mathbb{C} : |\Re \varepsilon| \geq B (\Im \varepsilon)^2 \text{ and } 0 < |\varepsilon| < 2R \}$. $C_R$ consists of two disks with radius $R$ and centers $(R,0)$ and $(-R,0)$, while $\Omega_{R,B}$ is the intersection of the disk of center $(0,0)$ and radius $2R$ with two parabolas with vertex at the origin: all such sets are tangent at the origin to the imaginary axis. Note that the smaller $B$, the more flattened are the parabolas. If $2RB < 1$ one has $C_R \subset \Omega_{R,B}$.

The following result has been proved in [1].

**Theorem 1.1.** Assume Hypotheses [1] and [2] for the system (1.1) and denote by $\Sigma_\xi$ the strip of analyticity of $f$. Then there exist $\varepsilon_0 > 0$ and $B_0 > 0$ such that for all $B > B_0$ there is a response solution $x(t) = c_0 + u(\omega t, \varepsilon)$ to (1.1), with $u(\psi, \varepsilon) = O(\varepsilon)$ analytic in $\psi \in \Sigma_{\xi'}$ and $\varepsilon \in \Omega_{\varepsilon_0,B}$, for some $\xi' < \xi$.

In the theorem above $\varepsilon_0$ has to be small, while $B_0$ must be large enough. However, for $B$ as close as wished to $B_0$ one can take $\overline{\tau} < \varepsilon_0$ small enough for the condition $\overline{\tau} B < 1$ to be satisfied, so as to obtain that $C_{\tau/2}$ is contained inside the analyticity domain. In this respect Theorem [1,1] extends previous results in the literature [3, 4], where analyticity in a pair of disks was obtained under stronger conditions on $\omega$, such as the standard Diophantine condition
\[
|\omega \cdot \nu| > \frac{\gamma}{|\nu|^\tau}, \quad \forall \nu \in \mathbb{Z}^d,
\]
(1.2)
or the Bryuno condition $B(\omega) < \infty$ If either $d = 1$ or $d = 2$ and $\omega$ satisfies the standard Diophantine condition (1.2) with $\tau = 1$, the response solution is Borel-summable.

In the present letter we remove in Theorem [1,1] the condition for $B$ to be large, by proving the following result.

**Theorem 1.2.** Assume Hypotheses [1] and [2] for the system (1.1) and denote by $\Sigma_\xi$ the strip of analyticity of $f$. Then for all $B > 0$ there exists $\varepsilon_0 > 0$ such that there is a response solution $x(t) = c_0 + u(\omega t, \varepsilon)$ to (1.1), with $u(\psi, \varepsilon) = O(\varepsilon)$ analytic in $\psi \in \Sigma_{\xi'}$ and $\varepsilon \in \Omega_{\varepsilon_0,B}$, for some $\xi' < \xi$. The dependence of $\varepsilon_0$ on $B$ is of the form $\varepsilon_0 = \varepsilon_1 B^\alpha$, for some $\alpha > 0$ and $\varepsilon_1$ independent of $B$.

The proof of the theorem given in Section [3] yields the value $\alpha = 8$: such a value is non-optimal and could be improved by a more careful analysis. Thanks to Theorem [1,2] we can estimate the domain of analyticity by the union of the domains $\Omega_{\varepsilon_0,B}$, with $\varepsilon_0 = B^\alpha \varepsilon_1$, by letting $B$ varying in $(0,1]$. This provides a domain that near the origin has boundary of the form $|\Re \varepsilon| \approx \varepsilon_1^{-\beta} |\Im \varepsilon|^{2+\beta}$, where $\beta = 1/\alpha$. 

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2 Tree representation

We can rewrite (1.1) as

\[ \varepsilon \ddot{x} + \dot{x} + \varepsilon a (x - c_0) + \mu \varepsilon \sum_{p=2}^{\infty} a_p (x - c_0)^p = \mu \varepsilon \sum_{v \in \mathbb{Z}^d} e^{i \nu \cdot \psi} f_{\nu}, \] (2.1)

where \( a := a_1 \) and \( \mu = 1 \). However, we can consider \( \mu \) as a free parameter and study (2.1) for \( \varepsilon \in \mathbb{C} \) and \( \mu \in \mathbb{R} \). Then we look for a quasi-periodic solution to (2.1) of the form

\[ x(t, \varepsilon, \mu) = c_0 + u(\omega t, \varepsilon, \mu), \quad u(\psi_0, \varepsilon, \mu) = \sum_{k=1}^{\infty} \sum_{\nu \in \mathbb{Z}^d} \mu_k e^{i \nu \cdot \psi} u_{\nu}^{(k)}(\varepsilon). \] (2.2)

By inserting (2.2) into (2.1) we obtain a recursive definition for the coefficients \( u_{\nu}^{(k)}(\varepsilon) \), which admits a natural graphical representation in terms of trees.

A rooted tree \( \theta \) is a graph with no cycle, such that all the lines are oriented toward a unique point (root) which has only one incident line (root line). All the points in \( \theta \) except the root are called nodes. The orientation of the lines in \( \theta \) induces a partial ordering relation (\( \preceq \)) between the nodes. Given two nodes \( v \) and \( w \), we shall write \( w \prec v \) every time \( v \) is along the path (of lines) which connects \( w \) to the root. We shall write \( w \prec \ell \) if \( w \preceq v \) and \( v \) is the node which \( \ell \) exits. For any node \( v \) denote by \( p_v \) the number of lines entering \( v: v \) is called and end node if \( p_v = 0 \) and an internal node if \( p_v > 0 \). We denote by \( N(\theta) \) the set of nodes, by \( E(\theta) \) the set of end nodes, by \( V(\theta) \) the set of internal nodes and by \( L(\theta) \) the set of lines; one has \( N(\theta) = E(\theta) \cup V(\theta) \).

We associate with each end node \( v \in E(\theta) \) a mode label \( \nu_v \in \mathbb{Z}_+^d \) and with each internal node an degree label \( d_v \in \{0, 1\} \). With each line \( \ell \in L(\theta) \) we associate a momentum \( \nu_{\ell} \in \mathbb{Z}^d \). We impose the following constraints on the labels:

1. \( \nu_{\ell} = \sum_{w \in E(\theta)} \nu_w \), where \( E(\theta) := \{ w \in V(\theta) : w \prec \ell \} \);
2. \( p_v \geq 2 \ \forall v \in V(\theta) \);
3. if \( d_v = 0 \) then the line \( \ell \) exiting \( v \) has \( \nu_{\ell} = 0 \).

We shall write \( V(\theta) = V_0(\theta) \cup V_1(\theta) \), where \( V_0(\theta) := \{ v \in V(\theta) : d_v = 0 \} \). For any discrete set \( A \) we denote by \( |A| \) its cardinality. Define the degree and the order of \( \theta \) as \( d(\theta) := |E(\theta)| + |V_1(\theta)| \) and \( k(\theta) := |N(\theta)| \), respectively.

We call equivalent two labelled rooted trees which can be transformed into each other by continuously deforming the lines in such a way that they do not cross each other. In the following we shall consider only inequivalent labelled rooted trees, and we shall call them trees tout court, for simplicity.

We associate with each node \( v \in N(\theta) \) a node factor \( F_v \), and with each line \( \ell \in L(\theta) \) a propagator \( G_{\ell} \), such that

\[ F_v := \begin{cases} 1/D(\varepsilon, \omega \cdot \nu_{\ell}), & v \in V(\theta), \\ 1/a, & v \in E(\theta), \end{cases} \]

\[ G_{\ell} := \begin{cases} 1/\varepsilon \nu_{\ell}, & \nu_{\ell} \neq 0, \\ 1/a, & \nu_{\ell} = 0, \end{cases} \]
where \( D(\epsilon, s) := -\epsilon s^2 + is + \epsilon a \). Then, by defining

\[
\mathcal{V}(\theta, \epsilon) := \left( \prod_{v \in N(\theta)} F_v \right) \left( \prod_{\ell \in L(\theta)} G_\ell \right) \tag{2.3}
\]

one has

\[
u^{(k)}(\epsilon) = \sum_{\theta \in T_k} \mathcal{V}(\theta, \epsilon), \quad \nu \in \mathbb{Z}^d \tag{2.4}
\]

where \( T_{k, \nu} \) is the set of trees of order \( k \) and momentum \( \nu \) associated with the root line. Note that \( \nu^{(1)}_0 = 0 \) and \( \nu^{(2)}_0 = 0 \) for all \( \nu \in \mathbb{Z}^d \).

3 Proof of Theorem 1.2

We shall prove Theorem 1.2 in the case in which \( N(f) = \infty \). The case of trigonometric polynomials is in fact easier and can be dealt with as shown in [2].

**Lemma 3.1.** Set \( c_0 = \min\{1/8, B/18, B/8|a|, |a|/8, |a|B/4, \sqrt{|a|}/2\} \). There exists \( \epsilon_1 > 0 \) such that one has \( |D(\epsilon, s)| \geq c_0 \max\{1, s^2\} |\epsilon| \) for all \( s \in \mathbb{R} \) and all \( \epsilon \in \Omega_{B, \epsilon_1} \).

**Proof.** Write \( \epsilon = x + iy \), with \( |x| \geq By^2 \) and \( x \) small enough. By symmetry it is enough to study \( y \geq 0 \). One has \( |D(\epsilon, s)|^2 = (s + ya - ys^2)^2 + x^2(a - s^2)^2 \). If \( y = 0 \) the bound is straightforward. If \( y > 0 \) denote by \( s_1 \) and \( s_2 \) the two roots of \( s + ya - ys^2 = 0 \): one has \( s_1 = -ay + O(y^2) \) and \( s_2 = 1/y + ay + O(y^2) \). Let \( \epsilon_1 \) be so small that \( |s_1 + ay| \leq |a|y/2, |s_2 - 1/y| \leq 1/6y \) and \( 18|a|y^2 \leq 1 \) for \( \epsilon \leq \epsilon_1 \). The following inequalities are easily checked:

1. (1) if \( |s| < 2|a|y \), then \( |x||a - s^2| \geq |ax|/2 \geq |a|By^2/2 \geq Bs^2/8|a| \); (2) if \( |s - s_2| < 1/2y \), then \( |x||a - s^2| \geq |x|s^2/2 \geq |x|18y^2 \geq B/18 \); (3) if \( |s| \geq 2|a|y \) and \( |s - s_2| \geq 1/2y \), then \( (3.1) |s + ya - ys^2| \geq |s - s_1| |s - s_2| \geq |a|y/4, (3.2) |s + ya - ys^2| \geq |s - s_1|/2 \geq |s|/8, (3.3) \) if either \( a < 0 \) or \( a > 0 \) and \( |a - s^2| > |a|/2 \) one has \( |x||a - s^2| > |ax|/2 \), while if \( a > 0 \) and \( |a - s^2| \leq |a|/2 \) one has \( |s + ya - ys^2| \geq |s| - |y||a - s^2| \geq \sqrt{a}/2 \). By collecting together all the bounds the assertion follows. 

**Lemma 3.2.** For any tree \( \theta \) one has \( |E(\theta)| \geq |V(\theta)| + 1 \) and hence \( 2|E(\theta)| \geq k(\theta) + 1 \).

**Proof.** By induction on the order \( k(\theta) \).

For \( v \in V_1(\theta) \) define \( E(\theta, v) := \{ w \in E(\theta) : \text{the line exiting } w \text{ enters } v \} \) and set \( r_v := |E(\theta, v)|, s_v := p_v - r_v, \mu_v := \sum_{w \in E(\theta, v)} \nu_w \) and \( \mu_v := |\mu_v| \). Define \( V_2(\theta) := \{ v \in V(\theta) : s_v = 0 \} \) and \( V_3(\theta) := \{ v \in V(\theta) : r_v = 1 \} \). For \( v \in V_2(\theta) \) call \( \ell_v \) the line exiting \( v \), and for \( v \in V_3(\theta) \) call \( \ell' \) the line entering \( v \) which does not exit an end node. Define \( \overline{V}_2(\theta) := \{ v \in V_2(\theta) : \nu_{\ell_v} \neq 0 \} \) and \( \overline{V}_3(\theta) := \{ v \in V_3(\theta) : \nu_{\ell_v} \neq 0 \} \), and set \( \overline{V}_1(\theta) = \overline{V}_2(\theta) \cup \overline{V}_3(\theta) \). By construction one has \( \overline{V}_1(\theta) \subset V_1(\theta) \).

**Lemma 3.3.** There exists \( C_0 > 0 \) such that \( C_0 |\omega \cdot \nu| \geq e^{-|\nu|/16} \forall \nu \in \mathbb{Z}^d \).

**Proof.** It follows from Hypothesis 2 by using that \( \beta_n(\omega) = \alpha_n(\omega) \) if \( N(f) = \infty \).
Lemma 3.4. One has $C_0|ω·ν_ℓ| ≥ e^{-ξ_ν/16}$ for $v ∈ V_2(θ)$ and $2C_0 \max\{|ω·ν_ℓ|, |ω·ν_ℓ′|\} ≥ e^{-ξ_ν/16}$ for $v ∈ V_3(θ)$.

Proof. For $v ∈ V_2(θ)$ one has $ν_ℓ = ν_ν$, so that one can use Lemma 3.3. For $v ∈ V_3(θ)$ one proceeds by contradiction. Suppose that the assertion is false: this would imply

$$e^{-ξ_ν/16} > C_0|ω·ν_ℓ| + C_0|ω·ν_ℓ′| ≥ C_0|ω·(ν_ℓ - ν_ℓ′)| = C_0|ω·ν_ν| ≥ e^{-ξ_ν/16},$$

where we have used that $E(θ, v)$ contains only one node $w$ and hence $μ_ν = ν_ν ≠ 0$.

Define $L_1(θ, v) := \{ℓ_ν\}$ for $v ∈ V_2(θ)$ and $L_1(θ, v) := \{ℓ ∈ \{ℓ_ν, ℓ′_ν\} : 2C_0|ω·ν_ℓ| ≥ e^{-ξ_ν/16}\}$ for $v ∈ V_3(θ)$. Lemma 3.4 yields $L_1(θ, v) ≠ \emptyset$ for all $v ∈ V_1(θ)$. Set also $L_1(θ) := \{ℓ ∈ L(θ) : \exists v ∈ V_1(θ)$ such that $ℓ ∈ L_1(θ, v)\}$, $L_{int}(θ) := \{ℓ ∈ L(θ) : ℓ$ exits a node $v ∈ V_1(θ)\}$ and $L_0(θ) := L_{int}(θ) \setminus L_1(θ)$.

Lemma 3.5. For any tree $θ$ one has $4|L_0(θ)| ≤ 3|E(θ)| - 4$.

Proof. By induction on $V(θ)$. If $|V(θ)| = 1$ then either $V(θ) = V_0(θ)$ or $V(θ) = V_2(θ)$ and hence $|L_0(θ)| = 0$, so that the bound holds. If $|V(θ)| ≥ 2$ the root line $ℓ_0$ of $θ$ exits a node $v_0 ∈ V(θ)$ with $s_{v_0} + r_{v_0} ≥ 2$ and $s_{v_0} ≥ 1$. Call $θ_1, \ldots, θ_{s_{v_0}}$ the trees whose respective root lines $ℓ_1, \ldots, ℓ_{s_{v_0}}$ enter $v_0$. One has $|E(θ)| = |E(θ_1)| + \ldots + |E(θ_{s_{v_0}})| + r_{v_0}$. If $ℓ_0 ∉ L_0(θ)$ then $|L_0(θ)| = |L_0(θ)| + \ldots + |L_0(θ_{s_{v_0}})|$ and the bound follows from the inductive hypothesis.

If $ℓ_0 ∈ L_0(θ)$ then one has $|L_0(θ)| = 1 + |L_0(θ_1)| + \ldots + |L_0(θ_{s_{v_0}})|$, so that, again by the inductive hypothesis, $4|L_0(θ)| ≤ 3|E(θ)| - 3r_{v_0} - 4(s_{v_0} - 1)$. If either $r_{v_0} + s_{v_0} ≥ 3$ or $r_{v_0} + s_{v_0} = 2$ and $s_{v_0} = 2$, the bound follows. If $r_{v_0} + s_{v_0} = 2$ and $s_{v_0} = 1$, then $v_0 ∈ V_3(θ)$, so that either $ν_ℓ_1 = 0$ or $2C_0|ω·ν_ℓ_1| ≥ e^{-ξ_{v_0}/16}$, by Lemma 3.4, because $ℓ_0 ∈ L_0(θ)$ and hence $2C_0|ω·ν_ℓ_0| ≤ e^{-ξ_{v_0}/16}$. Therefore $ℓ_1 ∉ L_0(θ)$. If $v_1$ is the node which $ℓ_1$ exits, call $θ'_{d_1}, \ldots, θ'_{s_{v_1}}$ the trees whose root lines enter $v_1$. One has $|L_0(θ)| = 1 + |L_0(θ')| + \ldots + |L_0(θ'_{s_{v_1}})|$ and hence, by the inductive hypothesis, $4|L_0(θ)| ≤ 3|E(θ)| - 3r_{v_0} - 3r_{v_1} - 4(s_{v_1} - 1)$, where $3r_{v_0} + 3r_{v_1} + 4s_{v_1} - 4 ≥ 5$, so that the bound follows in this case too.

Lemma 3.6. For any $k ≥ 1$ and $ν ∈ \mathbb{Z}^d$ and any tree $θ ∈ Ξ_{k, ν}$ one has

$$|ψ(θ, ε)| ≤ A_0^k C_0^{-k} |ε|^{d(θ)} \prod_{v ∈ E(θ)} e^{-3|v|/4},$$

with $A_0$ a positive constant depending on $Φ, Γ$ and $ρ$, and $c_0$ as in Lemma 3.1.

Proof. One bounds (2.3) as

$$|ψ(θ, ε)| ≤ |ε|^{d(θ)} \left( \prod_{v ∈ V(θ)} \left| a_{ν_ν} \right| \right) \left( \prod_{v ∈ E(θ)} \left| f_{ν_ν} \right| \right) \left( \prod_{v ∈ L(θ)} \left| G_{ν_ν} \right| \right).$$

We deal with the propagators by using Lemma 3.1 as follows. If $ℓ$ exits a node $v ∈ V_2(θ)$, then we have

$$\left| G_{ν_ν} \right| \prod_{w ∈ E(θ,v)} \left| f_{ν_ν} \right| \left| G_{ν_ν} \right| ≤ \frac{1}{c_0|ω·ν_ν|^2} \prod_{w ∈ E(θ,v)} \left| f_{ν_ν} \right| ≤ c_0^{-1} C_0^2 (c_0^{-1} C_0^2 Φ)^{|E(θ,v)|} \prod_{w ∈ E(θ,v)} e^{-3|ν_ν|/4},$$

with $A_0$ as in Lemma 3.1.

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where $\ell_v$ denotes the line exiting $v$. For the other lines in $L_1(\theta)$ we distinguish three cases: given a node $v \in V_3(\theta)$ and denoting by $v'$ the node which the line $\ell'_v$ exits, (1) if either $\ell'_v \notin L_1(\theta, v)$ or $\ell'_v \in L_1(\theta, v')$, we proceed as for the nodes $v \in V_2(\theta)$ with $\ell = \ell_v$ and obtain the same bound; (2) if $L_1(\theta, v) = \{\ell'_v\}$ and $\ell'_v \notin L_1(\theta, v')$, we proceed as for the nodes $v \in V_2(\theta)$ with $\ell = \ell_v$ and we obtain the same bound once more; (3) if both lines $\ell_v, \ell'_v$ belong to $L_1(\theta, v)$ and $\ell'_v \notin L_1(\theta, w)$, we bound

$$|G_{\ell_v}G'_{\ell'_v}| \prod_{w \in E(\theta, v)} |f_{\nu_w}| |G_{\nu_w}| \leq C_0^{-2} C_0^4 \left| c_0^{-1} C_0^2 \Phi \right| |E(\theta, v)| \prod_{w \in E(\theta, v)} e^{-5|\nu_w|/8}.$$ 

For all the other propagators we bound (1) $|G_{\ell}| \leq 1/|a|$ if $\ell$ exits a node $v \in V_0(\theta)$, (2) $|G_{\ell}| \leq C_0^{-1}|\omega \cdot \nu|^{-2}$ if $\ell$ exits an end node and has not been already used in the bounds above for the lines $\ell \in L_1(\theta)$, and (3) $|G_{\ell}| \leq C_0^{-1}|c|^{-2}$ if $\ell \in L_0(\theta)$. Then we obtain

$$|\mathcal{V}(\theta, \epsilon)| \leq \left| c \right|^{d(\theta) - 2|L_0(\theta)|} \left| \Gamma(V(\theta)) \right| \rho^{N(\theta)} \left| (c_0^{-1} C_0^{-2})^{1/2} \right| \left| (c_0^{-1} C_0^{-2} \Phi)^{1/2} \right| |E(\theta)| \left| a \right|^{-|V_0(\theta)|} e^{-5|\nu|/8},$$

where we can bound, by using Lemma 3.2 and Lemma 3.3, $d(\theta) - 2|L_0(\theta)| = |E(\theta)| + |V_1(\theta)| - 2|L_0(\theta)| \geq |E(\theta)| - |L_0(\theta)| \geq 1 + |E(\theta)|/4 \geq 1 + (k(\theta) + 1)/4$, so that the assertion follows.

Lemma 3.7. For any $k \geq 1$ and $\nu \in \mathbb{Z}^d$ one has

$$|u^{(k)}_{\nu}(\epsilon)| \lesssim A_1^{k} c_0^{-k} e^{-\xi|\nu|/2} |\epsilon|^{1 + \frac{k+1}{8}},$$

with $A_1$ a positive constant $C$ depending on $\Phi, \Gamma, \xi$ and $\rho$, and $c_0$ as in Lemma 3.1.

Proof. The coefficients $u^{(k)}_{\nu}$ are given by (2.4). Each value $\mathcal{V}(\theta, \epsilon)$ is bounded through Lemma 3.6. The sum over the Fourier labels is performed by using a factor $e^{-\xi|\nu|/2}$ for each end node $v \in E(\theta)$. The sum over the other labels is easily bounded by a constant to the power $k$.

Lemma 3.7 implies that for $\epsilon$ small enough the series (2.2) converges uniformly to a function analytic in $\psi \in \Sigma^\prime$, with $\xi' < \xi/2$. Moreover such a function is analytic in $\epsilon \in \Omega_{c_0, B}$, provided $A_1^1 c_0^1 / c_0^3$ is small enough. This completes the proof of Theorem 1.2.

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