Split dimensional regularization for the Coulomb gauge at two loops

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Abstract

We evaluate the coefficients of the leading poles of the complete two-loop quark self-energy $\Sigma(p)$ in the Coulomb gauge. Working in the framework of split dimensional regularization, with complex regulating parameters $\sigma$ and $n/2 - \sigma$ for the energy and space components of the loop momentum, respectively, we find that split dimensional regularization leads to well-defined two-loop integrals, and that the overall coefficient of the leading pole term for $\Sigma(p)$ is strictly local. Extensive tables showing the pole parts of one- and two-loop Coulomb integrals are given. We also comment on some general implications of split dimensional regularization, discussing in particular the limit $\sigma \to 1/2$ and the subleading terms in the $\epsilon$-expansion of noncovariant integrals.
1 Introduction

Despite serious efforts during the past twenty years to place the Coulomb gauge on the same rigorous footing as covariant gauges, we still have no consistent rules for renormalizing non-Abelian theories in this gauge. There can be little doubt, however, that the Coulomb gauge is superior to other gauges in at least two respects, namely in the treatment of bound states and in the study of confinement in QCD. Since these topics play a crucial role in our understanding of the strong interactions, there is clearly a need to put this gauge on a sound theoretical basis. To gain a better understanding of the various advantages, as well as disadvantages, of the Coulomb gauge, we refer the reader to the vast literature on the subject. (For some recent references see, for example, refs. [1]–[9].)

Unfortunately, progress in the Coulomb gauge continues to be hampered by the operator ordering problem in the quantum Hamiltonian, as noted by Schwinger in 1962 [10]. The ordering problem was later re-examined by Christ and Lee who demonstrated that the quantum Hamiltonian differed from the classical Hamiltonian by special Coulomb interaction terms, labelled \( (V_1 + V_2) \) [11]. A few years later, Cheng and Tsai [12] pointed out that the \( (V_1 + V_2) \) -terms are equivalent to a distinct class of integrals, called energy integrals, which lead to two different types of divergences [12, 13]:

(a) ordinary UV divergences associated with the structure of space-time, and
(b) divergences characteristic of the Coulomb gauge, arising from the integration over the energy variable \( q_0 \) in integrals of the form

\[
\int_{-\infty}^{\infty} dq_0 \frac{q_0^2}{q_0^2 - \vec{q}^2 + i\varepsilon}.
\]

It is the divergences from such energy integrals that give rise to ambiguities in the Coulomb gauge. These ambiguities come as no surprise, since the Coulomb-gauge condition

\[
\vec{\nabla} \cdot \vec{A}(x) = 0
\]

(1)

does not fix the gauge completely, but leaves a residual gauge freedom for gauge transformations \( g(t) \) that do not depend on the space coordinates \( \vec{x} \). In 1987, Doust and Taylor [14, 15] came to the conclusion that standard dimensional regularization is incapable of regulating both types of divergences
Renormalization in the Coulomb gauge has recently been examined by Baulieu and Zwanziger [4], who treated the Coulomb gauge as the singular limit of the Landau-Coulomb interpolating gauge. Zwanziger also exploited the Coulomb gauge in the study of confinement [5]. Employing a diagrammatic representation, he showed that the problematic energy integrals cancel, at least to one-loop order. Based on this observation, he assigned to these integrals the value zero. However, it is not obvious that this cancellation procedure can also be carried out explicitly beyond one-loop order [13].

Accordingly, it seems desirable to regulate the Coulomb-gauge integrals on an individual basis. To this effect, a novel technique called split dimensional regularization was introduced by one of the authors [16]. The idea is to replace the measure \( d^n q \) by

\[
d^{2(\sigma+\omega)} q = d^{2\sigma} q_0 d^{2\omega} \vec{q},
\]

where \( \sigma \) and \( \omega \) are understood as parameters in the complex plane satisfying \( \sigma + \omega = n/2 \), and the limits \( \sigma \to 1/2 \) and \( \omega \to 3/2 \) are taken only at the end of integration. One may think of split dimensional regularization as a special form of dimensional regularization, the special feature being that the dimension of the energy component is explicitly specified to be non-integer. Whereas in [5] the ambiguous energy integrals were defined to be zero, in the context of split dimensional regularization these integrals turn out to be zero in a natural way.

To date, split dimensional regularization has been tested at one loop for the gluon self-energy [16], and the quark self-energy and quark-gluon three-point function [17]. All Coulomb-gauge integrals appearing in these calculations are free of ambiguities and respect the appropriate Ward identities.

We note in passing that split dimensional regularization has also been applied in the context of non-relativistic QCD [3].

However, as already alluded to in ref. [17], the real challenge comes from energy integrals at two loops and beyond [15]. In order to investigate the associated ambiguities, we have evaluated the contributions of the leading \( 1/\epsilon^2 \) poles of the complete two-loop quark self-energy. Working in the framework of split dimensional regularization, we find that all integrals can be

\footnote{These authors assume that the energy component \( q_0 \) is one-dimensional, while \( \vec{q} \) is \((n-1)\)-dimensional.}
calculated consistently. But, whereas the coefficients of the leading poles can be evaluated explicitly, the coefficients of the subleading poles are generally more difficult to compute, since many of the parameter integrals can no longer be expressed in closed form. For this reason, the subleading poles have not been evaluated in their entirety in this paper.

It is also worth noting at this stage that the subleading poles exhibit the following interesting feature: the regulators \( \sigma \) and \( \omega \), characterizing the time and space components, respectively, appear \textit{independently} in the various \( \Gamma \)-functions, a clear indication of the special role played by the time component in the Coulomb gauge. (See Section 2.3 for more details.) By comparison, in the \( \Gamma \)-functions for the leading poles, the regulators \( \sigma \) and \( \omega \) always appear in the combination \( \sigma + \omega = n/2 \).

The paper is organized as follows. In Section 2, we first discuss a specific integral for which standard dimensional regularization fails to regulate the divergence from the energy component, if the latter component is assumed to be one-dimensional, and then show how split dimensional regularization cures the ambiguity. In the second part of Section 2, we concentrate on two-loop integrals, analyzing in particular the properties of the subleading poles. Then we comment on some general properties of split dimensional regularization. We demonstrate that the limit \( \sigma \to 1/2 \) \textit{after} integration is always well defined and discuss the implications of split dimensional regularization for the subleading terms in the \( \epsilon \)-expansion of noncovariant integrals. In Section 3, we outline the calculation of the leading divergence of the two-loop quark self-energy and present our results. The highlights of this paper are summarized in Section 4. Appendix A contains results for the pole parts of one-loop integrals (for both integer and non-integer powers of propagators), while Appendix B shows the results for the leading poles of several two-loop integrals. In Appendix C and D we give general formulas for Coulomb integrals at one and two loops in Feynman parameter space.

2 Split dimensional regularization
2.1 Problems with standard dimensional regularization

The gluon propagator in the Coulomb gauge, given by [16]

\[
G_{\mu\nu}^{ab}(q) = \frac{-i\delta^{ab}}{q^2 + i\epsilon} d_{\mu\nu}^{\text{con}}(q),
\]

\[
d_{\mu\nu}^{\text{con}}(q) = g_{\mu\nu} + \frac{n^2}{q^2} q_\mu q_\nu - \frac{q n}{q^2} (q_\mu n_\nu + n_\mu q_\nu),
\]

\[n_\mu = (1, 0, 0, 0),\]

is seen to contain the factor $1/\vec{q}^2$. While most Coulomb-gauge integrals can be computed consistently with standard dimensional regularization, it is no secret that the appearance of the noncovariant factor $1/\vec{q}^2$ in the integrand may, under certain circumstances, lead to spurious singularities that require special attention. The presence of two or more of these factors, in combination with powers of $q_0$ in the numerator, is particularly troublesome and can lead to the ambiguous energy integrals mentioned in Section 1. To illustrate this point, consider the integral

\[
I_{\mu\nu} = \int d^n q \frac{q_\mu q_\nu}{q^2 \vec{q}^2 (\vec{p} - \vec{q})^2}.
\]

We shall first show that standard dimensional regularization fails to regulate the singularity in the energy component of $I_{\mu\nu}$ (i.e. in $I_{00}$), if the energy component $q_0$ of a vector $q_\mu$ is assumed to be one-dimensional. After Feynman parametrization, we obtain

\[
I_{\mu\nu} = \Gamma(3) \int_0^1 dx \int_0^1 dy \ y \int d^n q \frac{q_\mu q_\nu}{[qA^{-1}q - 2qP + M^2]^3},
\]

where

\[
A^{-1} = \begin{pmatrix} xy & 0 \\ 0 & -1 \end{pmatrix}; \quad P = \begin{pmatrix} 0 \\ (1-y)p \end{pmatrix}; \quad M^2 = (1-y)p^2.
\]

\[^2\text{Note that in Abelian gauge theories two noncovariant factors containing the same loop momentum cannot occur if the number of external legs is } \leq 3, \text{ i.e. in self-energy and vertex diagrams.}\]
Integration over \( n = (4 - 2\epsilon) \)-dimensional momentum space leads to

\[
I_{\mu\nu} = i\pi^{\frac{3}{2}} \int_0^1 dx \int_0^1 dy |\text{Det} A|^\frac{1}{2} \left\{ \Gamma(3 - \frac{n}{2}) (\mathcal{P} A)_{\mu} (\mathcal{P} A)_{\nu} [M^2 - \mathcal{P} a \mathcal{P}]^\frac{n}{2} - 3 \\
+ \frac{1}{2} \Gamma(2 - \frac{n}{2}) A_{\mu\nu} [M^2 - \mathcal{P} a \mathcal{P}]^\frac{n}{2} - 2 \right\}.
\] (7)

The important term in Eq. (7) is \( |\text{Det} A|^\frac{1}{2} \). If we assume that the energy component is one-dimensional, while the identity matrix contained in \( A \) is defined in \( (n - 1) \) dimensions, we obtain \( |\text{Det} A|^\frac{1}{2} = (xy)^{-\frac{1}{2}} \), so that

\[
I_{\mu\nu} = i\pi^{\frac{3}{2}} (p^2)^{-\epsilon} \int_0^1 dx x^{-\frac{3}{2}} \int_0^1 dy y^\frac{3}{2} \left\{ \Gamma(1 + \epsilon)(p_{\mu} - p_0 n_{\mu})(p_{\nu} - p_0 n_{\nu}) y^{1-\epsilon}(1 - y)^{1-\epsilon} \\
+ \frac{1}{2} \Gamma(\epsilon) \left[ \frac{1}{xy} n_{\mu} n_{\nu} + (g_{\mu\nu} - n_{\mu} n_{\nu}) \right] y^{-\epsilon}(1 - y)^{-\epsilon} \right\}.
\] (8)

Thus, we see that the “energy component” \( I_{00} \) is ill defined, since it leads to the parameter integral

\[
\int_0^1 dx x^{-\frac{3}{2}}.
\]

The example above demonstrates that the application of standard dimensional regularization in the noncovariant Coulomb gauge leads to unavoidable difficulties, as long as the energy component \( q_0 \) is assumed to be one-dimensional. This conclusion raises another question, namely, how is the dimensionality of \( q_0 \) defined in standard dimensional regularization? The answer to this question is that in general, the dimension of \( q_0 \) need not to be specified. In covariant integrals, we could distribute arbitrary fractions of \( \epsilon \) to the individual components of a Lorentz vector, as, for example, in

\[
\int d^n q \rightarrow \int d^{\sigma_0} q_0 d^{\sigma_1} q_1 d^{\sigma_2} q_2 d^{n-\sigma_0-\sigma_1-\sigma_2} q_3.
\]

Due to covariance, the result will invariably depend only on \( n \). However, in noncovariant gauges, the situation is quite different. In our example, we could as well have assumed that the identity matrix contained in \( A \) is three-dimensional, thereby leading to an energy component which is \( (n - 3) \)-dimensional, so that

\[
|\text{Det} A|^\frac{1}{2} = (xy)^{-\frac{n-3}{2}} = (xy)^{-\frac{1}{2} + \epsilon}.
\] (9)
In that case, the $x$-integral in $I_{00}$ is given by $\int_0^1 dx \, x^{-3/2 + \epsilon}$, which is well defined by invoking analytic continuation in the context of dimensional regularization. This is exactly the idea of split dimensional regularization. It asserts that in the noncovariant Coulomb gauge, the energy component $q_0$ of a Lorentz vector (as a loop momentum) cannot be treated as one-dimensional. Instead, a nonzero “fraction” of the regulator $\epsilon$ in $n = 4 - 2 \epsilon$ has to be assigned to the energy component, for instance by defining

$$\sigma = \frac{1}{2} (1 - \epsilon), \quad \epsilon = c_{\sigma} \cdot \epsilon, \quad (10)$$

$$\omega = \frac{3}{2} - \epsilon (1 - \frac{c_{\sigma}}{2}),$$

such that $\sigma + \omega = n/2$. The parameter $c_{\sigma}$ is arbitrary\(^3\), as long as we do not set it to zero before all parameter integrals have been carried out. After integration, we can define the integral at $\sigma = 1/2$ by analytic continuation, invoking the same arguments as in standard dimensional regularization. Thus, the technique of split dimensional regularization imposes the condition that the energy component $q_0$ must be of non-integer dimension.

Using split dimensional regularization, we find for the integral $I_{\mu\nu}$:

$$I_{\mu\nu} = i \pi \frac{2}{\epsilon} (p^2)^{-\epsilon} n_\mu n_\nu \Gamma(\epsilon) \cdot \sigma \int_0^1 dx \, x^{-1-\sigma} \int_0^1 dy \, y^{-\sigma-\epsilon}(1 - y)^{-\epsilon} \quad (11)$$

$$+ p_i p_j - \text{ and } \delta_{ij} - \text{ terms } ; \quad i, j \in \{1, 2, 3\},$$

where $(\sigma + \omega)$ has been replaced by $n/2 = 2 - \epsilon$ whenever $\sigma, \omega$ occur in this particular combination. Contraction of the result (11) with $n^\mu n^\nu$ yields for the pole part of $I_{00} = \int d^n q \, q^2_0 / \{ q^2 \vec{q}^2 (\vec{p} - \vec{q})^2 \}$

$$\text{div } I_{00} = \text{div} \left\{ -i \pi \frac{2}{\epsilon} (p^2)^{-\epsilon} \Gamma(\epsilon) \text{Beta}(1 - \sigma - \epsilon, 1 - \epsilon) \right\} \quad (12)$$

$$\sigma \rightarrow \frac{1}{2}, \quad i \pi \frac{1}{\epsilon} (-2).$$

As a crosscheck, we may write

$$\text{div } I_{00} = \text{div} \int d^n q \, \frac{q^2 + \vec{q}^2}{q^2 \vec{q}^2 (\vec{p} - \vec{q})^2} = \text{div} \int d^n q \, \frac{1}{q^2 (\vec{p} - \vec{q})} = i \pi \frac{1}{\epsilon} (-2).$$

\(^3\)Note that in [18], a special version of split dimensional regularization with $c_{\sigma} = 1/2$ has been used.
Note that \[ \int d^2q_0 \int \frac{d^n-2\sigma}{\vec{q}^2(\vec{p} - \vec{q})^2} = 0 \]
in the context of split dimensional regularization, by the same arguments as \( \int d^nq = 0 \) in standard dimensional regularization. The breaking of covariance is manifest in Eq. (12) from the function \( \text{Beta}(1-\sigma-\epsilon, 1-\epsilon) \), which contains \( \sigma \), rather than \( \sigma + \omega \), in the energy component of \( I_{\mu\nu} \). It is evident from this example that split dimensional regularization leads to well-defined parameter integrals.

### 2.2 Subleading poles

In order to extract interesting quantities such as the two-loop renormalization constant for the quark self-energy in the Coulomb gauge, it is of course necessary to determine not only the leading \( (1/\epsilon^2) \) poles, but also the subleading \( (1/\epsilon) \) poles of the two-loop integrals. Since the expressions for some of the Coulomb integrals occurring in the two-loop quark self-energy are quite involved (as already noted above), the subleading poles of these integrals can no longer be extracted in the form of analytic functions. Instead, they have to be expressed as combinations of infinite series and hypergeometric functions. We shall illustrate the extraction of the subleading poles by means of two examples. The first example consists of an integral \( I_1 \) where all integrations can be done in closed form:

\[ I_1 = \frac{1}{i^2\pi^n} \int \frac{d^n k_1 \, d^n k_2}{k_1^2 (k_1 - k_2)^2 \vec{k}_2^2 (\vec{p} - \vec{k}_1)^2} \]

\[ = -\frac{1}{i\pi^{n/2}} \int \frac{d^n k_1}{k_1^2 (\vec{p} - \vec{k}_1)^2 (\vec{k}_1^2 \epsilon)} \Gamma(\epsilon) \text{Beta}(\omega - 1, 1 - \epsilon) \]

\[ = (\vec{p}^2)^{-2\epsilon} \Gamma(\epsilon) \Gamma(2\epsilon) \cdot G(\epsilon, \sigma, \omega), \quad (13) \]

\[ G(\epsilon, \sigma, \omega) = \frac{\Gamma(1-\sigma)\Gamma^2(\omega-1)\Gamma(1-\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1-\sigma+\epsilon)\Gamma(\omega-\epsilon)\Gamma(\omega-2\epsilon)}, \]

where \( \sigma + \omega = 2 - \epsilon \) has been used in Eq. (13). But note that \( \sigma \) and \( \omega \) also occur separately in \( G(\epsilon, \sigma, \omega) \), not only in the combination \( \sigma + \omega \). In order to extract the leading and subleading poles in \( \epsilon \), we use Eq. (10) and expand

\[^4\text{A factor } \Gamma^2(1+\epsilon) \text{ has been extracted in order to avoid the Euler } \gamma_E \text{ in the expanded result.}\]
in $\epsilon$. This yields

$$I_1 = (\tilde{p}^2)^{-2\epsilon} \Gamma^2(1 + \epsilon) \left\{ \frac{2}{\epsilon^2} + \frac{4}{\epsilon} \left[ 5 - c_\sigma - 2 \log(2) \right] + \text{finite} \right\}. \quad (14)$$

We will comment on the appearance of the parameter $c_\sigma$, in the subleading pole term, in Section 2.3. Note that the breaking of covariance is also manifest in the factor $(\tilde{p}^2)^{-2\epsilon}$.

As a second example, consider the integral

$$I_2 = \frac{1}{i^2\pi^n} \int \frac{d^n k_1 \, d^n k_2}{k_1^2 k_2^2 (p - k_2)^2 (k_1 - k_2)^2 k_2^2} \left\{ -\frac{1}{\epsilon^2} \Gamma(1 - \epsilon, 1 - \epsilon) \frac{1}{i\pi^2} \int \frac{d^n k_2}{(k_2^2)^{1+\epsilon} (p - k_2)^2 k_2^2} ight\} \cdot \int_0^1 du \frac{u^{\omega-2}}{2} F_1(1 + 2\epsilon, -2\epsilon; 1 - \epsilon; B(u)) \quad B(u) = \frac{\tilde{p}_0^2 - up^2}{p^2},$$

$$= (-p^2)^{-1-2\epsilon} \Gamma(1 + \epsilon) \left\{ \frac{1}{\omega - 1} \left\{ -\frac{1}{2\epsilon^2} \right\} - \frac{1}{\epsilon} \left[ 1 - \sum_{m=1}^{\infty} \frac{1}{m} \left( \frac{\tilde{p}_0^2}{p^2} \right)^m 2F_1(-m, \omega - 1; \omega; \tilde{p}_0^2 / p_0^2) \right] + \text{finite} \right\}. \quad (15)$$

Using Eq. (11), we obtain

$$I_2 = (-p^2)^{-1-2\epsilon} \Gamma(1 + \epsilon) \left\{ -\frac{1}{\epsilon^2} - \frac{1}{\epsilon} \left[ 3 - c_\sigma - \sum_{m=1}^{\infty} \frac{1}{m} \left( \frac{\tilde{p}_0^2}{p^2} \right)^m 2F_1(-m, \frac{1}{2}; \frac{3}{2}; \tilde{p}_0^2 / p_0^2) \right] + \text{finite} \right\}. \quad (16)$$

Again, we see that the breaking of Lorentz covariance manifests itself in terms such as $\tilde{p}^2 / p_0^2$ and in the fact that $c_\sigma$ appears in the subleading pole term.

It should be clear from the result for the subleading poles of $I_2$ in Eq. (16) that a complete determination of the subleading poles in the two-loop quark self-energy is beyond the scope of this paper. Afterall, $I_2$ is still a relatively simple Coulomb integral since it has only one noncovariant denominator.
2.3 Comment on noncovariance in the context of dimensional regularization

In the context of split dimensional regularization, as already explained in Section 2.1, the dimension $d(q_0)$ of the energy component of an $n$-dimensional vector $q_\mu$ is given by $d(q_0) = 2\sigma$, where (cf. Eq. (14))

$$\sigma = \frac{1}{2}(1 - \epsilon_\sigma) \quad , \quad \epsilon_\sigma \equiv c_\sigma \cdot \epsilon \ , \quad (17)$$

and $\epsilon_\sigma$ serves to regulate the spurious divergences in the energy component, inherent in the Coulomb gauge. The important point is that the limit $\epsilon_\sigma \to 0$ in the energy integrals after integration always exists. This means that the “singularities” in the energy integrals, which would be unregulated for $\epsilon_\sigma = 0$, are really spurious. They do not show up as poles in $\epsilon_\sigma$. Only the “usual” UV and IR poles of Coulomb-gauge integrals appear as poles in $\epsilon = 2 - (\sigma + \omega)$, whereas the potential singularities in the energy component never show up as $\Gamma(\pm k (1/2 - \sigma))$, but rather as $\Gamma(1 - k - \sigma)$, where $k$ is a positive integer. Hence, the limit $\epsilon_\sigma \to 0$ with $\epsilon$ fixed, i.e. the limit $c_\sigma \to 0$, indeed exists. This observation follows from the expressions in Feynman parameter space for general Coulomb-gauge integrals given in Appendix C and D. First, consider a scalar one-loop integral with $r \ (r > 0)$ covariant and $m$ noncovariant denominators in $n = 2(\sigma + \omega)$ dimensions (cf. Eq. (41)):

$$I_{r+m}(\alpha_j, \beta_l) = \int \frac{d^n q}{i \pi^n} \prod_{j=1}^{r} \prod_{l=1}^{m} \frac{1}{(q - p_j)^{2\alpha_j} (q - \bar{s_l})^{2\beta_l}}$$

$$= (-1)^{\sum_j \alpha_j} \frac{\Gamma(\lambda(\alpha, \beta) - \sigma - \omega)}{\prod_{j=1}^{r} \Gamma(\alpha_j) \prod_{l=1}^{m} \Gamma(\beta_l)} \int_{0}^{1} \prod_{j=1}^{r} dx_j \ x_j^{\alpha_j - 1} \left( \sum_{j=1}^{r} x_j \right)^{-\sigma}$$

$$\prod_{l=1}^{m} dy_l \ y_l^{\beta_l - 1} \delta(1 - r \sum_{j=1}^{r} x_j - m \sum_{l=1}^{m} y_l)$$

$$\left[ - \sum_{j=1}^{r} x_j p_{0,j} + \left( \frac{\sum_{j=1}^{r} x_j p_{0,j}}{\sum_{j=1}^{r} x_j} \right)^2 + \sum_{j=1}^{r} x_j \bar{p}_j^2 + \sum_{l=1}^{m} y_l \bar{s_l}^2 - \left( \sum_{j=1}^{r} x_j \bar{p}_j + \sum_{l=1}^{m} y_l \bar{s_l} \right)^2 \right]^{\sigma + \omega - \lambda(\alpha, \beta)} , \quad (18)$$

$$\lambda(\alpha, \beta) = \sum_{j=1}^{r} \alpha_j + m \beta_l .$$
Concerning the appearance of the parameters \( \sigma \) and \( \omega \), we see that the overall \( \Gamma \)-function (which indicates the UV behaviour), as well as the term in square brackets containing the momentum dependence, always depend on the sum \( \sigma + \omega = n/2 \). Only the integration over the Feynman parameters \( x_j \) (which are associated with the covariant denominators) leads to the isolated parameter \( \sigma \), the reason being that the Feynman parameters \( y_l \) never multiply a \( q_0 \) component.

Referring to Eqs. (37) – (39) in Appendix C, we observe that the most dangerous case for a potential singularity at \( \sigma = 1/2 \) occurs when \( r = 1, l \geq 2 \), and with factors of \( q_0^b \) in the numerator, that is, in the following type of integral (for simplicity, we assume \( \beta_l = 1 \) for all \( l \), since more general powers \( \beta_l \) do not spoil the argument):

\[
I^{(b)}(\alpha) = \int \frac{d^nq}{i\pi^\frac{n}{2}} \prod_{l=1}^m \frac{q_0^b}{(q-p)^{2\alpha} (q-\bar{s}_l)^2} = (-1)^\alpha b^\alpha \frac{\Gamma(\alpha)}{\Gamma(\alpha + b)} \int_0^1 dx x^{\alpha - 1 - \sigma} \int \prod_{l=1}^m dy_l \delta(1 - x -\sum_{l=1}^m y_l)
\]
\[
\int_0^\infty dz z^{\alpha + m - \frac{n}{2} - 1} \exp\{-z [x\bar{p}^2 + \sum_{l=1}^m y_l \bar{s}_l^2 - (x\bar{p} + \sum_{l=1}^m y_l \bar{s}_l)^2]\}
\cdot \left(\frac{-\partial}{\partial a_4}\right)^b \exp\{-\frac{1}{x} (a_4 p_4 - \frac{a_4^2}{4x})\} \bigg|_{a_4=0}.
\]  

Carrying out the derivative with respect to \( a_4 \), we see that the resulting \( x \)-integral will be of the form

\[
\int_0^1 dx x^{\alpha - 1 - \sigma - \gamma} f(x, y_l), \quad \text{where} \quad \gamma = \begin{cases} b/2 & \text{for } b \text{ even} \\ (b-1)/2 & \text{for } b \text{ odd} \end{cases}
\]  

such that \( \gamma \) is always an integer. The function \( f(x, y_l) \) corresponds to the term in square brackets in Eq. (19) and will, after integration over \( z \), always contain the sum \( \sigma + \omega \) in the exponent. Since \( f(x, y_l) \) in general does not contribute to the leading behaviour for \( x \to 0 \), it is irrelevant for our considerations\(^5\).

Note that values of \( \alpha > 1 \) actually improve the behaviour for \( x \to 0 \) in

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\(^5\)In the special cases where \( f(x, y_l) \) does contribute to the leading behaviour for \( x \to 0 \) (for example, if \( l = 1, r = 1, p = 0 \)), it also provides a regulator \( \epsilon \) for the exponent of \( x \) and thus is harmless.
Eq. (20). By contrast, it is the more severe *ultraviolet* behaviour, induced by powers of $q_0^b (b > 0)$ in the numerator, that can render the integral over $x$ singular unless the parameter $\sigma$ is complex$^6$. Generalization of the above arguments to $r > 1$ is straightforward. If $r$ is the number of covariant propagators, $b$ the power of $q_0$ in the numerator and $l \geq 2$, then integration over one of the Feynman parameters $x_i$ requires a complex value of $\sigma$, provided $r - b/2 \leq 0$ is fulfilled (or, respectively, $\sum_{j=1}^r \alpha_j - b/2 \leq 0$ for general propagator powers).

Summarizing, we conclude from the analysis above that all parameter integrals are well defined if $\sigma$ contains the regulator $\epsilon$, as in $\sigma = (1 - c_\sigma \epsilon)/2$, and that the limit $c_\sigma \to 0$ after integration is non-singular, since the exponents $\alpha$ and $\gamma$ in Eq. (20) are never half-integers.

At two loops, the situation is similar, as may be seen from the expressions in Appendix D. If $c_1$ is the number of covariant denominators containing $q$, but not $k$, and $c_2$ is the number of covariant denominators containing $k$, but not $q$, and if, furthermore, $b_1$ and $b_2$ are the powers of $q_0$ and $k_0$ in the numerator, respectively, and $i_B$ is defined as in Eq. (48), then a non-integer dimension of the energy components $q_0$, $k_0$ is mandatory in the following cases ($i \in \{1, 2\}$):

$$
c_i - b_i/2 + i_B \leq 0 \quad \text{for } c_1 \text{ and } c_2 \neq 0,
$$

$$
c_i - b_i/2 \leq 0 \quad \text{for } i_B = 1 \text{ and } c_1 \text{ or } c_2 = 0.
$$

As an example, consider the case $i_B = 0, \lambda_1 = 1, \rho_1 = 1, \lambda_i = 0 \text{ for } i > 1, \rho_j = 0 \text{ for } j > 1$, and all $\alpha_i, \beta_u = 1$ in Eqs. (12)–(19). (More general propagator powers do not change the arguments.) In that case, we have

$$
\mathcal{M}_4 = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} ; \quad (\det \mathcal{M}_4)^{-\frac{1}{2}} = (x_1 x_2)^{-\sigma}.
$$

Analogous to the one-loop case, additional powers of $q_0$ or $k_0$ in the numerator lead to more negative powers of $x_1$ or $x_2$, as can be seen from Eqs. (17). For example, two powers of $q_0$ or $k_0$ in the numerator lead to terms such as

$$
(\det \mathcal{M}_4)^{-\frac{1}{2}} J_i[l_4^{(i)} l_4^{(j)}] \to (\det \mathcal{M}_4)^{-\frac{1}{2}} \left(\mathcal{M}_4^{-1}\right)^{(ij)} J_i[1]
$$

$$
\to \delta_{ij} \int_0^1 dx_i x_i^{-1-\sigma}, \quad i, j \in \{1, 2\}.
$$

$^6$By complex we mean that $\sigma$ has to be understood as a parameter in the complex plane in order to define the analytic continuation of the corresponding $\Gamma$-function.
Although higher powers of \( q_0 \) or \( k_0 \) in the numerator will lead to even more negative powers of \( x_i \), such as \( x_i^{-1-\sigma-\gamma} \), the parameter \( \gamma \) can never be half-integer. As a result, the parameter integral is well defined for complex values of \( \sigma \), and the limit \( \sigma \to 1/2 \) after integration exists due to the analytic properties of the \( \Gamma \)-function.

These arguments and the conclusion that \( \gamma \) will never be half-integer, can be generalized to \( L \) loops by using the same formalism as in Appendix D, with the objects \( l_4, N_4, \ldots \) and \( \vec{l}, \vec{N}, \ldots \) being defined in \( L \cdot 2\sigma \) and \( L \cdot 2\omega \) dimensional space, respectively. Hence, it follows that a troublesome \( \Gamma \)-function of the form \( \Gamma(\pm k (1/2 - \sigma)) \) never arises.

The arbitrariness in the choice of \( c_\sigma \) is not problematic, since it is just the noncovariant version of the arbitrariness in the definition of the subtraction scheme in the context of renormalization. To illustrate this point, consider a simple one-loop integral \( T(p) \) in split dimensional regularization:

\[
T(p) = \int \frac{d^n q}{(2\pi)^n} \frac{1}{(p - q)^2} \vec{q}^2
= \frac{1}{(2\pi)^n} \int d^{2\sigma} q_0 d^{n-2\sigma} \vec{q} \frac{1}{(p - q)^2} \vec{q}^2
= -\frac{i\pi^{\frac{n}{2}}}{(2\pi)^n} (\vec{p}^2)^{\frac{n}{2}-2} \Gamma(2 - \frac{n}{2}) \text{Beta}(\frac{n}{2} - 1, \frac{n}{2} - 1 - \sigma). \tag{23}
\]

Using \( n = 4 - 2c_n \epsilon \), together with Eq. (17), and expanding in \( \epsilon \), we arrive at

\[
T(p) = -2\frac{i}{(4\pi)^2} (\vec{p}^2)^{-c_n \epsilon} \left\{ \frac{1}{c_n \epsilon} + 4 - 2\ln(2) - \frac{c_\sigma}{c_n} + \ln(4\pi) - \gamma_E \right\}. \tag{24}
\]

The factor of \( 1/c_n \) in the pole part is irrelevant since it will be removed by renormalization. As the terms \( \ln(4\pi) \) and \( \gamma_E \), in the subleading part of the \( \epsilon \)-expansion, are always present as an artifact of standard dimensional regularization, the additional term \( c_\sigma/c_n \) is just another aspect of the arbitrariness inherent in renormalization, caused by the breaking of covariance. The freedom in the definition of the finite part merely reflects the fact that renormalization schemes need to be specified. The procedure of dealing with the parameter \( c_\sigma \) during renormalization (i.e., whether or not it is subtracted together with the terms \( \ln(4\pi) - \gamma_E \)) defines a specific "noncovariant renormalization scheme" in the context of split dimensional regularization.
Of course, the easiest option would be to set $c_\sigma$ to zero after having evaluated the integral, since, as explained above, the operation of setting $c_\sigma = 0$ after integration is well defined, and is the natural choice as it corresponds to $d(q_0) = 1$ and $d(\vec{q}) = n - 1$ in standard dimensional regularization. Nevertheless, it may be advantageous to keep the parameter $c_\sigma$ as a useful check when computing gauge invariant quantities, since $c_\sigma$ obviously has to cancel out in any physical quantity.

3 Leading divergence of the two-loop quark self-energy

There are two methods of determining the overall divergence of the two-loop quark self-energy, namely by

1. direct calculation of the two-loop integrals corresponding to the graphs shown in Fig. 1;

2. computing the counterterm diagrams shown in Fig. 2 by first extracting the UV-divergent parts of the corresponding one-loop subgraphs, and then calculating the divergence of the overall diagram containing these one-loop insertions.

We have used both of these methods in order to check the consistency of our results. Note that method (1) requires us to keep the full $\epsilon$-dependence from the first integration if the momentum integrations are performed sequentially (this procedure is sometimes called the “nested method” [19]). Since it may not always be possible to maintain, without expansion, the full $\epsilon$-dependence, the majority of our two-loop integrals has been computed by integrating over both loop momenta simultaneously (also referred to as the “matrix method” [19]), as outlined in Appendix D.

It is worth emphasizing that the (spurious) infrared divergences are more severe in the Coulomb gauge than they are, for example, in the Feynman gauge, the severity being caused by the noncovariant factor $1/\vec{q}^2$ in the gluon propagator. A partial list of the noncovariant integrals used in our calculation is given in the Tables in Appendix A and B. The number of integrals per graph obtained by method (1) is typically of the order of a few hundred. The algebra has been performed by using the symbolic manipulation program FORM [20].
Figure 1: Contributions to the two-loop quark self-energy
Application of method (2) requires insertion of the following one-loop counterterms [16, 17] (cf. Fig. 2):

\[ \Sigma(p) = i C_F \frac{\alpha_s}{4\pi} \frac{1}{\epsilon} \not{p}, \]

\[ \Gamma^a_\mu = T^a_\mu (\Gamma^{(b)}_\mu + \Gamma^{(c)}_\mu), \]

\[ \Gamma^{(b)}_\mu = i (C_F - \frac{N_c}{2}) \frac{\alpha_s}{4\pi} \frac{1}{\epsilon} (\not{-\gamma_\mu}), \]

\[ \Gamma^{(c)}_\mu = i \frac{N_c}{2} \frac{\alpha_s}{4\pi} \frac{1}{\epsilon} (\not{-\gamma_\mu}), \]

\[ \Pi^{G,ab}_{\mu\nu}(q) = i N_c \delta^{ab} \frac{\alpha_s}{4\pi} \frac{1}{\epsilon} \left[ q^2 g_{\mu\nu} - q_\mu q_\nu \right. \]
\[ \left. - \frac{4}{3} \frac{q n}{n^2} (q_\mu n_\nu + q_\nu n_\mu) + \frac{8}{3} \frac{q^2}{n^2} n_\mu n_\nu \right], \]

\[ \Pi^{F,ab}_{\mu\nu}(q) = i T_f \delta^{ab} \frac{\alpha_s}{4\pi} \frac{1}{\epsilon} \left( -\frac{4}{3} \right) (q^2 g_{\mu\nu} - q_\mu q_\nu), \]

\[ C_F = \frac{(N_c^2 - 1)}{2N_c}; \quad T_f = T_R \cdot n_f = \frac{1}{2} n_f. \]

Note that \( \Sigma(p) \) and \( \Gamma_\mu \) satisfy the Ward identity

\[ (p' - p)^\mu (\Gamma^{(b)}_\mu + \Gamma^{(c)}_\mu) = \Sigma(p) - \Sigma(p'), \]

and that the divergent part of the ghost contributions, discussed in ref. [16], is zero.
Figure 2: Counterterms for the two-loop quark self-energy. The grey circles denote the UV divergent part of the corresponding one-loop insertions given below.
The two-loop corrections to the quark self-energy shown in Fig. 1 are given by the expressions in Eqs. (29) to (31), where we use the following short-hand notations:

\[ c_1 = k_1^2 + i \varepsilon \; ; \; c_3 = (p - k_1)^2 + i \varepsilon \; , \]
\[ c_2 = k_2^2 + i \varepsilon \; ; \; c_4 = (p - k_2)^2 + i \varepsilon \; , \]
\[ c_5 = (k_1 - k_2)^2 + i \varepsilon \; , \] (27)

and
\[ V_{3g}^{\mu_1 \mu_2 \mu_3} (q_1, q_2, q_3) = g^{\mu_1 \mu_2} (q_1^{\mu_3} - q_2^{\mu_3}) + g^{\mu_2 \mu_3} (q_2^{\mu_1} - q_3^{\mu_1}) + g^{\mu_3 \mu_1} (q_3^{\mu_2} - q_1^{\mu_2}) \; , \]
\[ d_{\mu \nu}^{\text{con}} (q) = g_{\mu \nu} + \frac{n^2}{q^2} q_\mu q_\nu - \frac{q_n}{q^2} (q_\mu n_\nu + n_\mu q_\nu) \; . \] (28)

The “rainbow graph”, diagram (a), is given by
\[ \Sigma^{(a)} (p) = \frac{ig^4}{(2\pi)^{2n}} C_F^2 \int d^n k_1 d^n k_2 \frac{\gamma^\alpha k_1 \gamma^\mu (k_1 - k_2) \gamma^\nu k_1 \gamma^\beta}{c_1^2 c_2 c_3 c_5} \cdot d_{\alpha \beta}^{\text{con}} (p - k_1) d_{\mu \nu}^{\text{con}} (k_2) \; , \] (29)

while the graphs with the Abelian and non-Abelian vertex subgraphs have the form
\[ \Sigma^{(b)} (p) = \frac{ig^4}{(2\pi)^{2n}} \left( C_F - \frac{C_F N_c}{2} \right) \int d^n k_1 d^n k_2 \frac{\gamma^\alpha k_1 \gamma^\mu (k_1 - k_2) \gamma^\nu (p - k_2) \gamma^\beta}{c_1 c_2 c_3 c_4 c_5} \cdot d_{\mu \nu}^{\text{con}} (p - k_1) d_{\alpha \beta}^{\text{con}} (k_2) \; , \]
\[ \Sigma^{(c)} (p) = \frac{ig^4}{(2\pi)^{2n}} \frac{C_F N_c}{2} \int d^n k_1 d^n k_2 \frac{\gamma^\mu (p - k_1) \gamma^\nu (p - k_2) \gamma^\beta}{c_1 c_2 c_3 c_4 c_5} \cdot d_{\mu \nu}^{\text{con}} (k_1) d_{\alpha \beta}^{\text{con}} (k_1 - k_2) d_{\alpha \beta}^{\text{con}} (k_2) V_{3g}^{\alpha \lambda} (k_1, -k_2, k_2 - k_1) \; . \]

The gluon self-energy insertion reads
\[ \Pi_{ab}^{\mu \nu, G} (q) = \frac{g^2}{(2\pi)^n} \frac{N_c}{2} \delta_{ab} \int d^n k \frac{d_{\alpha \beta}^{\text{con}} (k) d_{\lambda \rho}^{\text{con}} (k - q)}{k^2 (k - q)^2} \cdot V_{3g}^{\mu \lambda} (q, -k, k - q) V_{3g}^{\nu \beta} (-q, k, q - k) \; , \]

such that
\[ \Sigma^{(d)} (p) = \frac{ig^2}{(2\pi)^n} \frac{T^a T^b}{T^n} \int d^n k_1 \frac{\gamma^\alpha (p - k_1) \gamma^\beta}{c_1^2 c_3} \cdot d_{\alpha \beta}^{\text{con}} (k_1) \Pi_{ab}^{\mu \nu, G} (k_1) d_{\mu \nu}^{\text{con}} (k_1) \; . \] (30)
The fermion loop insertion in diagram (e) is trivial (it does not contain a gluon propagator in the inner loop), and is given by

\[
\Pi_{ab}^{\mu\nu,F}(q) = -i T_f \delta_{ab} \frac{\alpha_s}{4\pi} (4\pi)^\epsilon \Gamma(\epsilon) \frac{\Gamma^2(2-\epsilon)}{\Gamma(4-2\epsilon)} \cdot 8 (-q^2)^{-\epsilon} (q^2 g^{\mu\nu} - q^\mu q^\nu),
\]

\[
\Sigma^{(e)}(p) = \frac{ig^2}{(2\pi)^2} T^a T^b \int d^4k_1 \frac{\gamma^\alpha (\not{p} - \not{k}_1) \gamma^\beta}{c_1^2 c_3} \alpha^{\alpha\beta \mu}(k_1) \Pi_{ab}^{\mu\nu,F}(k_1) d^{\alpha\beta \mu}(k_1).
\]

(31)

We would like to point out that \(\Sigma^{(d)}\) and \(\Sigma^{(e)}\) contain the double noncovariant denominator \(1/(k_1^2)^2\) stemming from the two terms of \(d^{\alpha\beta \mu}(k_1)\). In diagram (e), those double noncovariant denominators are reduced to single ones by corresponding terms in the numerator, whereas no such cancellation occurs in diagram (d). The integrals for graph (d) containing a factor \(1/(k_1^2)^2\) are those given in Table 3 for \(\beta = 2\).

After inserting the appropriate integrals and performing various crosschecks, as outlined above, we obtain the following results for the leading poles:

\[
\Sigma^{(a)}(p) = i C_F^2 \left( \frac{\alpha_s}{2\pi} \right)^2 \frac{1}{\epsilon^2} \left( -\frac{\not{p}}{4} \right),
\]

\[
\Sigma^{(b)}(p) = i (C_F^2 - \frac{C_F N_c}{2}) \left( \frac{\alpha_s}{2\pi} \right)^2 \frac{1}{\epsilon^2} \frac{\not{p}}{4},
\]

\[
\Sigma^{(c)}(p) = i \frac{C_F N_c}{2} \left( \frac{\alpha_s}{2\pi} \right)^2 \frac{1}{\epsilon^2} \frac{\not{p}}{4},
\]

\[
\Sigma^{(d)}(p) = i C_F^2 \left( \frac{\alpha_s}{2\pi} \right)^2 \frac{1}{\epsilon^2} \left\{ -\frac{41}{36} \not{p} + \frac{8 pm}{9n^2} \not{H} \right\},
\]

\[
\Sigma^{(e)}(p) = i C_F T_f \left( \frac{\alpha_s}{2\pi} \right)^2 \frac{1}{\epsilon^2} \left( -\frac{\not{p}}{3} \right).
\]

(32)

Note that the Ward identity (26) manifests itself in the equation

\[
\Sigma^{(a)} + \Sigma^{(b)} + \Sigma^{(c)} = 0.
\]

(33)

We also recall that the one-loop gluon self-energy, \(\Pi_{\mu\nu}^{G,ab}(q)\), is not transverse in the Coulomb gauge, since it satisfies the Ward identity (16)

\[
q^\mu \Pi_{\mu\nu}^{G,ab}(q) + (q^2 g_{\mu\nu} - q_\mu q_\nu) H^{ab\mu} = 0,
\]

(34)
where $H^{ab\mu}$ is a non-vanishing ghost term. It should come as no surprise, therefore, that the final expression for $\Sigma^{(d)}$ in Eqs. (32) still contains the vector $n_\mu$.

Finally, we should like to draw the reader’s attention to the complete absence of nonlocal factors in any of the leading-pole expressions, Eqs. (32). The absence of terms such as $p_0^2/p^2, \vec{p}^2/p^2$, etc., which implies the cancellation of all spurious infrared divergences, is certainly remarkable in view of the ubiquitous appearance of nonlocal terms at intermediate stages of the calculation. Of course, the trend was already set at the one-loop level, where the divergent parts of both the gluon self-energy and the quark self-energy were shown to be local [16, 17].

4 Conclusion

In this paper we have tested the technique of split dimensional regularization by calculating the leading divergence of the two-loop quark self-energy $\Sigma(p)$ in the Coulomb gauge. We find that the application of split dimensional regularization enables us to compute all two-loop integrals consistently, and that the final expressions for the leading poles of each graph are local, despite the presence of nonlocal terms at intermediate steps of the calculation.

The leading pole parts of noncovariant integrals at one and two loops, used in the present calculation, can be found in Tables 1–5 (Appendix A, B). We have also derived general expressions, in Feynman parameter space, for both one- and two-loop Coulomb-gauge integrals (Appendix C, D).

The latter expressions are useful in analyzing some general properties of the technique of split dimensional regularization. First of all, these expressions allow us to identify the class of integrals for which a complex regulator of the form $\sigma = (1 - c_\sigma \epsilon)/2$ is mandatory in order that these integrals be at all well defined. Furthermore, it turned out that the leading pole of a typical Coulomb-gauge integral, evaluated with split dimensional regularization, depends only on the sum of the regulating parameters, i.e. on $\sigma + \omega = n/2$, whereas the subleading pole generally contains the parameter $c_\sigma$. The latter stems from the isolated appearance of the regulator $\sigma$ and is a direct consequence of breaking covariance. Finally, we have demonstrated that the limit $c_\sigma \to 0$ after integration always leads to non-singular expressions, thus establishing the consistency of split dimensional regularization in general.
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A Divergent parts of one-loop integrals

In the following tables we give the results in Minkowski space for the poles of the one-loop integrals entering in the calculation. All tables have to be read as follows: The result for the pole part of an integral

$$\frac{1}{i\pi^2} \int d^2(\sigma + \omega) q \frac{A}{B}$$

is listed such that the denominator $B$ is given in the first row and the corresponding numerators $A$ in the first column. All entries are implicitly multiplied by $\Gamma(\epsilon)$. Note that the poles which are of an infrared nature have a negative mass dimension (for example, a coefficient proportional to $1/(p^2)^a$, or $1/(p^2)\bar{a}$, where $a > 0$). The symbol “−” means that the corresponding integral does not occur in our calculation and hence has not been computed.

| $A$ | $B$ | $q^2(p - q)^2q^2(\vec{p} - \vec{q})^2$ | $q^2(p - q)^2(k - q)^2q^2$ | $(p - q)^2(k - q)^2q^2$ |
|-----|-----|----------------------------------|--------------------------|---------------------|
| 1   |      | $4/(p^2\bar{p}^2)$              | $2/(p^2k^2)$             | 0                   |
| $q_0$ |     | $2p_0/(p^2\bar{p}^2)$            | 0                        | 0                   |
| $q_i$ |     | $2p_i/(p^2\bar{p}^2)$            | 0                        | 0                   |
| $q_iq_0$ | | $2p_0p_0/(p^2\bar{p}^2)$         | 0                        | 0                   |
| $q_i^2$ |   | $2p_i^2/(p^2\bar{p}^2)$          | 0                        | $-1$                |
| $q_iq_j$ | | $2p_ip_j/(p^2\bar{p}^2)$         | 0                        | $1/3\delta_{ij}$    |

Table 1: Pole terms of one-loop integrals in Minkowski space.
\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
A & B & q^{2\alpha}(\vec{p} - \vec{q})^2 & 2 + \epsilon & q^{2\alpha}(p - q)^2 & 2 + \epsilon \\
\hline
\alpha & 1 & 1 & 1 + \epsilon & 1 & 1 + \epsilon \\
\hline
1 & -2 & -1 & 1/(2p^2) & 2/p^2 & 1/p^2 \\
q_0 & 0 & 0 & 0 & 0 & 0 \\
q_i & -4p_i/3 & -2p_i/3 & 0 & 0 & 0 \\
q_0q_0 & 0 & 0 & 0 & 0 & 0 \\
q_0^2 & -2\vec{p}^2/3 & -\vec{p}^2/3 & -1/2 & -1 & -1/2 \\
\vec{q}^2 & -2\vec{p}^2/3 & -\vec{p}^2/3 & 1/2 & 1 & 1/2 \\
q_0q_j & -16/15 p_ip_j + 2/15 \vec{p}^2 \delta_{ij} & -8/15 p_ip_j + 1/15\vec{p}^2 \delta_{ij} & 1/6 \delta_{ij} & 1/3 \delta_{ij} & 1/6 \delta_{ij} \\
\hline
\end{array}
\]

Table 2: Pole terms of one-loop integrals. All entries have to be multiplied by the overall factor \(\Gamma(\epsilon)\).
\[
B_A 
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
\alpha & (p - q)^{2\alpha}q^{2\beta} & q^{2\alpha}(p - q)^{2\alpha}q^{2\beta} & q^2(p - q)^{2\alpha}q^{2\beta} \\
\beta & \epsilon & 1 + \epsilon & \epsilon & 1 + \epsilon & \epsilon & 1 + \epsilon \\
\hline
1 & 0 & -1 & 0 & 0 & [4 + 16 \bar{p}^2/(3p^2)]/p^4 & 1 & -1 \\
\phi_0 & 0 & -p_0 & 0 & 0 & 2p_0/p^4 & 0 & 0 \\
\phi_i & 0 & -p_i/3 & 0 & 0 & - & 0 & -2p_i/3 \\
\phi_0^2 & - & p^2/3 - 4p_0^2/3 & -2 & - & - & 0 & -2p^2/15 \\
\phi_0^3 & - & - & -6p_0 & - & - & 0 & - \\
\phi_0^4 & - & - & 2p^2 - 14p_0^2 & - & - & - & - \\
\hline
\end{array}
\]

Table 3: Pole terms of one-loop integrals with non-integer denominator powers.
B  Leading poles of two-loop integrals

The integrals listed are of the form

\[ \frac{1}{\epsilon^2 \pi^4} \int d^{2(\sigma+\omega)} k_1 \, d^{2(\sigma+\omega)} k_2 \, \frac{A}{B}, \]

where the denominator \( B \) is given in the first row and the corresponding numerators \( A \) in the first column. For the denominators, the short-hand notations defined in Eq. (27), as well as \( k_3 = p - k_1 \), have been used. Note that \( k_0 = k^0 \), \( p_0 = p^0 \). All entries are implicitly multiplied by \( 1/\epsilon^2 \).

| \( A \) | \( B \) | \( c_1 c_2 c_3 c_4 k_2^2 k_3^2 \) | \( c_2 c_3 c_5 k_2^2 k_3^2 \) | \( c_1 c_3 c_5 k_2^2 k_3^2 \) | \( c_1 c_2 c_5 k_2^2 k_3^2 \) | \( c_1 c_2 c_4 k_2^2 k_3^2 \) |
|---|---|---|---|---|---|---|
| 1 \( k_2^0 k_3^0 \) | \( 4/p^2 \) | \( 4/p^2 \) | \( -4/p^2 \) | \( -1/p^2 \) | \( -4/p^2 \) | \( 0 \) |
| \( (p k_2) k_3^0 \) | \( -2p^0 / p^2 \) | \( 0 \) | \( -p^0 \) | \( - \) | \( p^0 \) | \( 2p^0 \) |
| \( (p k_2) k_2^0 k_3^0 \) | \( 0 \) | \( 0 \) | \( - \) | \( p_0^2 \) | \( 2p_0^2 \) | \( - \) |
| \( (k_2 k_3) k_2^0 k_3^0 \) | \( 1 \) | \( 0 \) | \( - \) | \( - \) | \( - \) | \( - \) |

| \( A \) | \( B \) | \( c_3 c_5 k_2^2 k_3^2 \) | \( c_1 c_4 k_2^2 k_3^2 \) | \( c_1 c_5 k_2^2 k_3^2 \) |
|---|---|---|---|---|
| 1 \( k_2^0 \) | \( -2 \) | \( 4 \) | \( 2 \) |
| \( k_3^0 \) | \( -2p^0 \) | \( 4p^0 \) | \( 0 \) |
| \( k_2^0 k_3^0 \) | \( 0 \) | \( 4p^0 \) | \( 2p^0 \) |
| \( p k_3 \) | \( 0 \) | \( 4p_0^2 \) | \( -2/3 \bar{p}^2 \) |
| \( p \bar{k}_3 \) | \( 0 \) | \( 4/3 (p^2 + 2p_0^2) \) | \( 2/3 (p^2 + 2p_0^2) \) |

Table 4: Leading pole terms of two-loop integrals with two noncovariant denominators.
Table 5: Leading pole terms of two-loop integrals with one noncovariant denominator. An overall factor $1/\epsilon^2$ is implicit.

### C General formula for one-loop integrals in the Coulomb gauge

Here, we give the expression in Feynman parameter space for scalar one-loop integrals in the Coulomb gauge with an arbitrary number and arbitrary powers of propagators. We also outline how to obtain expressions in Feynman parameter space for the tensor integrals.

Consider a scalar integral with $r$ covariant and $m$ noncovariant denominators in $n$ dimensions:

$$ I_{r+m}(\alpha_j, \beta_l) = \int \frac{dnq}{i\pi^n} \prod_{j=1}^{r} \prod_{l=1}^{m} \frac{1}{D_j^{\alpha_j} C_l^{\beta_l}}, $$  

(35)
\[ B_j = (q - p_j)^2, \quad C_l = (\bar{q} - \bar{s}_l)^2. \]

Going to Euclidean space and introducing Feynman parameters in the usual way, we find that the integral \( I_{r+m}(\alpha_j, \beta_l) \) is given by

\[
I_{r+m}(\alpha_j, \beta_l) = (-1)^{\sum_j \alpha_j} \int \frac{d^n q_E}{\pi^{n+2}} \prod_{j=1}^r \prod_{l=1}^m \frac{1}{B_{E,j}^{\alpha_j} C_l^{\beta_l}}.
\]

\[
= (-1)^{\sum_j \alpha_j} \prod_{j=1}^r \Gamma(\alpha_j) \prod_{l=1}^m \Gamma(\beta_l) \int_0^1 \prod_{j=1}^r dx_j x_j^{\alpha_j-1} \prod_{l=1}^m dy_l y_l^{\beta_l-1} \delta(1 - \sum_{j=1}^r x_j - \sum_{l=1}^m y_l) \int \frac{d^n q_E}{\pi^{n+2}} \mathcal{D}^{-\lambda(\alpha, \beta)},
\]

where

\[
q_E^2 = q_4^2 + \vec{q}^2, \quad \lambda(\alpha, \beta) = \sum_{j=1}^r \alpha_j + \sum_{l=1}^m \beta_l,
\]

\[
\mathcal{D} = q^2 + \sum_{j=1}^r x_j q_4^2 - 2q_4 \sum_{j=1}^r x_j p_{4,j} - 2\vec{q} \left( \sum_{j=1}^r x_j \vec{p}_j + \sum_{l=1}^m y_l \vec{s}_l \right)
\]

\[
+ \sum_{j=1}^r x_j \vec{p}_j^2 + \sum_{l=1}^m y_l \vec{s}_l^2.
\]

In order to obtain a formula for general tensor integrals, we first define

\[
I_{r+m}(a, \alpha_j, \beta_l) = \int \frac{d^n q}{i \pi^{n+2}} \prod_{j=1}^r \prod_{l=1}^m \frac{e^{-a \cdot q}}{B_{E,j}^{\alpha_j} C_l^{\beta_l}},
\]

where \( a \) is an arbitrary Lorentz vector in Euclidean space, and then derive the tensor integrals by differentiation with respect to \( a \): 

\[
I_{r+m}^{\mu_1 \ldots \mu_s}(\alpha_j, \beta_l) = \int \frac{d^n q}{i \pi^{n+2}} \prod_{j=1}^r \prod_{l=1}^m \frac{q^{\mu_1} \ldots q^{\mu_s}}{B_{E,j}^{\alpha_j} C_l^{\beta_l}}
\]

\[
= (-1)^s \frac{\partial}{\partial a_{\mu_1}} \ldots \frac{\partial}{\partial a_{\mu_s}} \left. I_{r+m}(a, \alpha_j, \beta_l) \right|_{a=0}.
\]

Carrying out the momentum integration in \( I_{r+m}(a, \alpha_j, \beta_l) \), we obtain

\[
I_{r+m}(a, \alpha_j, \beta_l) = \frac{(-1)^{\sum_j \alpha_j}}{\prod_{j=1}^r \Gamma(\alpha_j) \prod_{l=1}^m \Gamma(\beta_l)} \int_0^1 \prod_{j=1}^r dx_j x_j^{\alpha_j-1} \left( \sum_{j=1}^r x_j \right)^{-\sigma}
\]

25
\begin{align}
\int_0^\infty dz \, z^{\lambda(\alpha, \beta) - \sigma - \omega - 1} \exp \{ - z | \sum_{j=1}^r x_j p^2_{4,j} - \left( \frac{\sum_{j=1}^r x_j p_{4,j}}{\sum_{j=1}^r x_j} \right)^2 + \sum_{j=1}^r x_j p_j^2 \}
+ \sum_{l=1}^m y_l s_l^2 - \left( \sum_{j=1}^r x_j \vec{p}_j + \sum_{l=1}^m y_l \vec{s}_l \right)^2 \} \right] \cdot \exp \{ - f(a_4) \} \cdot \exp \{ - g(\vec{a}) \},
\end{align}

\begin{align}
f(a_4) &= \frac{1}{\sum_{j=1}^r x_j} \left( a_4 \sum_{j=1}^r x_j p_{4,j} - \frac{1}{4} \frac{a_4^2}{z} \right), \quad (39) \\
g(\vec{a}) &= \vec{d} \left( \sum_{j=1}^r x_j \vec{p}_j + \sum_{l=1}^m y_l \vec{s}_l \right) - \frac{1}{4} \frac{\vec{a}^2}{z}. \quad (40)
\end{align}

Note that $f(a_4)$ and $g(\vec{a})$ depend on $z$, leading to more severe UV divergences for higher rank tensor integrals.

For a scalar integral, we can immediately set $a = 0$, carry out the $z-$integration and go back to Minkowski space to get

\begin{align}
I_{r+m}(\alpha_j, \beta_l) &= (-1)^{\sum_j \alpha_j} \frac{\Gamma(\lambda(\alpha, \beta) - \sigma - \omega)}{\prod_{j=1}^r \Gamma(\alpha_j) \prod_{l=1}^m \Gamma(\beta_l)} \int_0^1 dx_j x_j^{\alpha_j - 1} \left( \sum_{j=1}^r x_j \right)^{-\sigma} \prod_{l=1}^m dy_l y_l^{\beta_l - 1} \delta(1 - \sum_{j=1}^r x_j - \sum_{l=1}^m y_l) \\
&\left[ - \sum_{j=1}^r x_j p^2_{0,j} + \frac{\left( \sum_{j=1}^r x_j p_{0,j} \right)^2}{\sum_{j=1}^r x_j} + \sum_{j=1}^r x_j p_j^2 \right] \sigma + \omega - \lambda(\alpha, \beta). \quad (41)
\end{align}
D General formula for two-loop Coulomb gauge integrals

A general two-loop integral in the Coulomb gauge, with \( r + m + 1 \) covariant and \( a + b + 1 \) noncovariant denominators, is of the form

\[
J_{r,m,a,b} = \int \frac{d^n q d^n k}{i^2 \pi^n} \frac{\prod_{i=1}^{r} \prod_{j=r+1}^{r+m} \prod_{l=1}^{a} \prod_{u=a+1}^{a+b}}{(q - k)^2 (q - p_j)^2 (k - p_j)^2 \rho_j (q - s_l)^2 \alpha_l (k - s_u)^2 \beta_u}.
\]  

(42)

The general exponents \( \lambda_i, \rho_j, \alpha_l \) and \( \beta_u \) have been introduced in order to account for certain denominators such as \((q - p)^2\), occurring for example in the gluon self-energy correction.

Going to Euclidean space and introducing Feynman parameters \( x_j \) for the covariant denominators and \( y_l \) for the noncovariant denominators, where \( x_0 \) and \( y_0 \) are associated with \((q - k)^2\) and \((q - k)^2\), respectively, we find that

\[
J_{r,m,a,b} = (-1)^{\sum_i \lambda_i + \sum_j \rho_j + 1} \prod_{i=1}^{r} \prod_{j=r+1}^{r+m} \prod_{l=0}^{a} \prod_{u=a+1}^{a+b}
\]

\[
\int_0^1 dx_i x_i^{\lambda_i - 1} \prod_{j=r+1}^{r+m} dx_j x_j^{\rho_j - 1} \prod_{l=0}^{a} dy_l y_l^{\alpha_l - 1} \prod_{u=a+1}^{a+b} dy_u y_u^{\beta_u - 1}
\]

\[
\delta(1 - \sum_j x_j - \sum_{l=0}^{a+b} y_l) \int_0^\infty dz \, z^{(r,m,a,b)+1} \cdot J_l[1],
\]  

(43)

where

\[
J_l[1] = \int \frac{d^{2n} l}{i^2 \pi^n} \exp \left\{ -z[l_4^4 l_4 + \overline{l}^4 \overline{l} - 2N_4 l_4 - 2N_4 \overline{l} + R] \right\}
\]

\[
= \frac{\exp \left\{ -zR \right\}}{\pi^{2(\sigma + \omega)}} \cdot \int d^{4\sigma} l_4 \exp \left\{ -z[l_4^4 l_4 - 2N_4 l_4] \right\}
\]

\[
\cdot \int d^{4\omega} \overline{l} \exp \left\{ -z[\overline{l}^4 \overline{l} - 2N_4 \overline{l}] \right\},
\]  

(44)

\[
l_4 = \begin{pmatrix} q_4 \\ k_4 \end{pmatrix}, \quad \overline{l} = \begin{pmatrix} \overline{q} \\ \overline{k} \end{pmatrix},
\]

\[
M_4 = \begin{pmatrix} \sum_{j=0}^{r} x_j & -x_0 \\ -x_0 & x_0 + \sum_{j=r+1}^{r+m} x_j \end{pmatrix},
\]
\[ \mathcal{M} = \left( \begin{array}{c}
\sum_{j=0}^r x_j + \sum_{l=0}^a y_l - x_0 - y_0 \\
- x_0 - y_0
\end{array} \right)
\left( \begin{array}{c}
x_0 + y_0 + \sum_{j=r+1}^{r+m} x_j + \sum_{l=a+1}^{a+b} y_l
\end{array} \right), \]

\[ \mathcal{N}_4 = \left( \begin{array}{c}
\sum_{j=1}^r x_j p_{4j}
\end{array} \right), \]

\[ \mathcal{N} = \left( \begin{array}{c}
\sum_{j=1}^r x_j \vec{p}_j + \sum_{l=1}^a y_l \vec{s}_l
\end{array} \right), \]

\[ \mathcal{R} = \sum_{j=1}^{r+m} x_j \vec{p}_j^2 + \sum_{l=1}^{a+b} y_l \vec{s}_l^2, \]

\[ \xi(r, m, a, b) = \sum_{i=1}^r \lambda_i + \sum_{j=r+1}^{r+m} \rho_j + \sum_{l=1}^a \alpha_l + \sum_{u=a+1}^{a+b} \beta_u, \quad \lambda_0 = \alpha_0 = 1. \]

Observe that for the generic case \( \lambda_i = \rho_j = \alpha_l = \beta_u = 1 \), \( \xi(r, m, a, b) = r + m + a + b \). Integration over \( l \) in (44) leads to

\[ J_l[1] = z^{-2(\sigma+\omega)} (\text{Det} \mathcal{M}_4)^{-\frac{1}{2}} (\text{Det} \mathcal{M})^{-\frac{1}{2}} \]

\[ \cdot \exp \left\{ -z [\mathcal{R} - \mathcal{N}_4 \mathcal{M}_4^{-1} \mathcal{N}_4 - \mathcal{N} \mathcal{M}^{-1} \mathcal{N}] \right\}, \] (45)

where

\[ \text{Det} \mathcal{M}_4 = \left[ x_0 \cdot \sum_{j=1}^{r+m} x_j + \sum_{j=1}^r x_j \left( \sum_{j'=r+1}^{r+m} x_{j'} \right) \right]^{2\sigma}, \] (46)

\[ \text{Det} \mathcal{M} = \left[ \left( x_0 + y_0 \right) \left( \sum_{j=1}^{r+m} x_j + \sum_{l=1}^a y_l \right) + \left( \sum_{j=1}^r x_j + \sum_{l=1}^a y_l \right) \left( \sum_{j=r+1}^{r+m} x_j + \sum_{l=a+1}^{a+b} y_l \right) \right]^{2\omega}. \]

By differentiating the integral (44) repeatedly with respect to \( \mathcal{N}_4 \) and/or \( \mathcal{N} \), we may easily derive the appropriate expression for any tensor integral. Denoting the integrals with non-trivial numerators by \( J_l[l_4^{(i)}], J_l[l^{(i)}], J_l[l_4^{(i)}] l_4^{(j)}], \ldots \)

etc., where \( l_4^{(1)} = q_4, l_4^{(2)} = k_4, l^{(1)} = \vec{q}, l^{(2)} = \vec{k} \), and applying result (45), we get:

\[ \frac{\partial}{\partial N_4^{(i)}} J_l[1] = 2z l_4^{(i)} J_l[1] = 2z l[l_4^{(i)}] = 2z \left( N_4 \mathcal{M}_4^{-1} \right)^{(i)} J_l[1]. \]

The procedure for other integrals is similar, leading to expressions such as

\[ J_l[l_4^{(i)}] = \left( N_4 \mathcal{M}_4^{-1} \right)^{(i)} J_l[1], \]

\[ J_l[l^{(i)}] = \left( N \mathcal{M}^{-1} \right)^{(i)} J_l[1], \]
is absent. After integration over \( z \) (basic) integral (43) is then given by

\[
\begin{align*}
J_i[l_4^{(i)} \bar{I}^{(j)}] &= \left( N_4 M_{4}^{-1} \right)^{(i)} \left( N M^{-1} \right)^{(j)} J_i[1], \\
J_i[l_4^{(i)} l_4^{(j)}] &= \left\{ \left( N_4 M_{4}^{-1} \right)^{(i)} \left( N M_{4}^{-1} \right)^{(j)} + \frac{1}{2z} \left( M_{4}^{-1} \right)^{(ij)} \right\} J_i[1], \\
J_i[l_4^{(i)} \bar{I}^{(j)}] &= \left\{ \left( N M^{-1} \right)^{(i)} \left( N M^{-1} \right)^{(j)} + \frac{1}{2z} \left( M^{-1} \right)^{(ij)} \right\} J_i[1], \\
J_i[l_4^{(i)} l_4^{(j)} \bar{I}^{(k)}] &= \left\{ \left( N_4 M_{4}^{-1} \right)^{(i)} \left( N_4 M_{4}^{-1} \right)^{(j)} \left( N M^{-1} \right)^{(k)} + \frac{1}{2z} \left( M_{4}^{-1} \right)^{(ij)} \left( N M^{-1} \right)^{(k)} \right\} J_i[1],
\end{align*}
\]

To cope with integrals in which either the factor \((q - k)^2\), or \((\vec{q} - \vec{k})^2\), is absent in the basic integral \([12] \), we proceed by first introducing the indicator functions

\[
i_B = \begin{cases} 
1 & \text{if } (q - k)^2 \text{ is present} \\
0 & \text{if } (q - k)^2 \text{ is not present}
\end{cases}, \quad i_C = \begin{cases} 
1 & \text{if } (\vec{q} - \vec{k})^2 \text{ is present} \\
0 & \text{if } (\vec{q} - \vec{k})^2 \text{ is not present}
\end{cases}
\] (48)

and

\[
\text{ind}(i_B) = \begin{cases} 
\delta(x_0) & \text{for } i_B = 0 \\
1 & \text{for } i_B = 1
\end{cases}, \quad \text{ind}(i_C) = \begin{cases} 
\delta(y_0) & \text{for } i_C = 0 \\
1 & \text{for } i_C = 1
\end{cases}
\]

such that \( x_0 \) is set to zero if \((q - k)^2\) is absent, and \( y_0 \) is set to zero if \((\vec{q} - \vec{k})^2\) is absent. After integration over \( z \), the most general formula for the scalar (basic) integral \([13] \) is then given by

\[
J_{r,m,a,b} = (-1)^{\sigma + \sum_j \rho_j + b} \frac{\Gamma(\xi(r, m, a, b) + i_B + i_C - 2\sigma - 2\omega)}{\prod_{i=1}^{r+m} \prod_{j=r+1}^{j+1} \prod_{a=1}^{a} \prod_{b=1}^{b} \prod_{u=a+1}^{u} \Gamma(\lambda_i) \Gamma(\rho_j) \Gamma(\alpha_l) \Gamma(\beta_u)}
\]

\[
\int_0^1 \prod_{i=0}^{r} dx_i x_i^{\lambda_i-1} \prod_{j=r+1}^{j+1} dx_j x_j^{\rho_j-1} \prod_{l=0}^{a} dy_l y_l^{\alpha_l-1} \prod_{u=a+1}^{b} dy_u y_u^{\beta_u-1} \delta(1 - \sum_{j=0}^{r+m} x_j - \sum_{l=0}^{a+b} y_l) \cdot \text{ind}(i_B) \cdot \text{ind}(i_C) \cdot (\text{Det} M_4)^{-\frac{1}{2}} (\text{Det} M)^{-\frac{1}{2}}
\]

\[
\left[ R - N_4 M_{4}^{-1} N_4 - N M^{-1} N \right]^{2\sigma + 2\omega - \xi(r, m, a, b) - i_B - i_C}
\] (49)

The result in Minkowski space is obtained by making the replacements \( ip_4 \rightarrow p_0 \) and \( p_E^2 \rightarrow -p^2 \).
Although the expressions given above look somewhat complicated, since they are intended to represent all types of integrals that may occur in a two-loop calculation, they can be easily implemented into an algebraic manipulation program in order to yield the parameter representations of the various integrals.

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