SEMI-ADDITIVITY AND ACYCLICITY

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ABSTRACT. We generalize the notion of length to an ordinal-valued invariant defined on the class of finitely generated modules over a Noetherian ring. A key property of this invariant is its semi-additivity on short exact sequences. As an application, we prove some general acyclicity theorems.

1. INTRODUCTION

The length \( \text{len}(M) \) of an Artinian, finitely generated module \( M \) is defined as the longest chain of submodules in \( M \). Since we have the descending chain condition, such a chain is finite, and hence can be viewed as a finite ordinal. Hence we can immediately generalize this by transfinite induction to arbitrary Artinian modules, getting an ordinal-valued length function. To remain in the more familiar category of finitely generated modules, observe that at least over a complete Noetherian local ring, the latter is anti-equivalent with the class of Artinian modules via Matlis duality. We could have used this perspective (which we will discuss in [?]), but a moment’s reflection directs us to a simpler solution: just reverse the order. Indeed, if we view the class of all submodules \(^1\) of \( M \), the Grassmanian \( \text{Grass}_R(M) \), as a partially ordered set by reverse inclusion, then \( \text{Grass}_R(M) \) admits the descending chain condition, and hence any subchain is well-ordered, that is to say an ordinal. We then simply define \( \text{len}(M) \) as the supremum of all such chains/ordinals in \( \text{Grass}_R(M) \). Viewed as a module over itself, this yields the length \( \text{len}(R) \) of a Noetherian ring \( R \).

The key property of ordinary length is its additivity on short exact sequences. An example like \( 0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 0 \) immediately shows this can no longer hold in the general transfinite case. Moreover, even the formulation of additivity becomes problematic since ordinal sum is no longer commutative. There does exist a different, commutative sum, called in this paper the shuffle sum \( \oplus \) (see Appendix 7), which, as our main result shows, also plays a role:

Theorem (Semi-additivity, Theorem 3.1). If \( 0 \rightarrow M \rightarrow N \rightarrow Q \rightarrow 0 \) is exact, then
\[
\text{len}(Q) + \text{len}(N) \leq \text{len}(M) \leq \text{len}(Q) \oplus \text{len}(N).
\]

To appreciate the power of this result, notice that we instantaneously recover Vasconcelos’ observation that a surjective endomorphism is also injective (see Corollary 4.1 below). Extending this observation, we formulate in Theorem 5.2 a general acyclicity criterion in terms of a certain “ordinal Euler characteristic” (since we do not have a well-behaved subtraction, this is in fact a pair of two ordinals). As an application, we see that \( f : M \rightarrow N \) is injective whenever \( \text{len}(N) = \text{len}(M) \oplus \text{len}(\text{coker } f) \). We also get a new proof and a generalization of a result by Miyata in [6]: any exact sequence of the form \( M \rightarrow M \oplus N \rightarrow N \rightarrow 0 \) must be split exact. The length of a module encodes quite

\(^1\)Throughout the paper, \( R \) is a finite dimensional Noetherian ring and \( M, N, \ldots \) are finitely generated \( R \)-modules.
some information of the module. For instance, its degree, that is to say, the highest power \( \omega^d \) occurring in \( \text{len}(M) \), is precisely the dimension of \( M \) (Theorem 3.5). It follows that a Noetherian ring has length \( \omega^d \) if and only if it is a \( d \)-dimensional domain. In [7], we give a formula expressing length via (zero-th) local cohomology, but in this paper, we content ourself with proving the following special case: all associated primes of \( M \) have the same dimension—that is to say, \( M \) is unmixed—if and only if \( \text{len}(M) \) is a monomial of the form \( a\omega^d \); moreover, \( a \) is then the generic length \( \ell_{\text{gen}}(M) \) and \( d \) its dimension (Theorem 3.8).

Since ordinals and ordered structures are not the usual protagonists in commutative algebra, the paper starts with a section on this, and in an appendix, I explain shuffle sums. In §3, I prove semi-additivity, and the next sections contain some applications.

2. Notation and generalities on ordered sets

An ordered set \( P \) (also called a partially ordered set or poset), is a set together with a reflexive, antisymmetric and transitive binary relation \( \leq \), called the ordering of \( P \), and almost always written as \( \leq \), without a subscript. A partial order is total if for any two elements \( a, b \in P \) either \( a \leq b \) or \( b \leq a \). A subset \( C \subseteq P \) is called a chain, if its induced order is total. If \( a \leq b \), then we may express this by saying that \( a \) is below \( b \); if \( a < b \) (meaning that \( a \leq b \) and \( a \neq b \)), we also say that \( a \) is strictly below \( b \). More generally, for subsets \( A, B \subseteq P \), we say \( A \) is below \( B \), and write \( A \leq B \), to mean that \( a \leq b \) for all \( a \in A \) and all \( b \in B \).

The initial closed interval determined by \( a \in P \) is by definition the set of \( b \in P \) with \( b \leq a \) and will be denoted \([P, a]\). Dually, the terminal closed interval of \( a \), denoted \([a, P]\), is the collection of all \( b \in P \) with \( a \leq b \).

2.1. Ordinals. A partial ordering is called a partial well-order if it has the descending chain condition, that is to say, any descending chain must eventually be constant. A total order is a well-order if every non-empty subset has a minimal element.

Recall that an ordinal is an equivalence class, up to an order-preserving isomorphism, of a total well-order. The class of all ordinals is denoted \( \text{Ord} \); any bounded subset of \( \text{Ord} \) has then an infimum and a supremum. For generalities on ordinals, see any elementary textbook on set-theory (but see also Appendix 7 for the notion of shuffle sum). Let me remind the reader of the fact that ordinal sum (see §2.5 below) is not commutative: \( 1 + \omega \neq \omega + 1 \) since the former is just \( \omega \). We will adopt the usual notations except for one, where we will reverse the order. Frankly, the common notation for multiplication goes against any (modern) sense of logic, aesthetics or analogy. Therefore, in this paper, \( \alpha \beta \) will simply mean \( \alpha \) copies of \( \beta \), that is to say, \( \alpha \beta \) is equal to the lexicographic ordering on \( \alpha \times \beta \). After all, \( 2\omega \) should mean ‘two omega’, that is to say, \( \omega + \omega \).

All ordinals considered will be less than \( \omega^\omega \), and hence can be written uniquely in Cantor normal form

\[
\alpha = \sum_{i=0}^{d} a_i \omega^i
\]

with \( a_i \in \mathbb{N} \). We call the least \( i \) (respectively, the largest \( i \)) such that \( a_i \neq 0 \) the order \( \text{ord}(\alpha) \) (respectively, the degree \( \deg(\alpha) \)) of \( \alpha \); the sum of all \( a_i \) is called its valence \( \text{val}(\alpha) \).

An ordinal \( \alpha \) is called a successor ordinal if it has an immediate predecessor, denoted simply \( \alpha - 1 \). This is equivalent with \( \text{ord}(\alpha) = 0 \). Given any \( e \geq 0 \), we will write

\[
\alpha_e^+ := \sum_{i=e}^{d} a_i \omega^i \quad \text{and} \quad \alpha_e^- := \sum_{i=0}^{e-1} a_i \omega^i,
\]
where the $a_i$ are given by (1). In particular, $\alpha = \alpha^+_e + \alpha^-_e$. We will write $\alpha \preceq_e \beta$ (respectively, $\alpha =_e \beta$) if $\alpha^+_e \leq \beta^+_e$ (respectively, $\alpha^+_e = \beta^+_e$). To make statements uniform in all $e$, we assign degree $-1$ to 0, and dimension $-1$ to a zero module. We will need:

2.2. Lemma. Let $\alpha$ and $\beta$ be ordinals and let $e \in \mathbb{N}$ be such that $\text{ord}(\alpha) \geq e$ and $\alpha =_e \beta$. If for some ordinal $\lambda$ we have $\alpha + \lambda \leq \beta$, then $\lambda$ has degree at most $e - 1$.

Proof. We use the notation from (2). By assumption, $\alpha = \alpha^+_e = \beta^+_e$. Let $d$ be larger than the degree of any of the ordinals involved. For $e \leq q \leq d$, we prove by downward induction on $q$ that $\text{deg} \lambda \leq q$. The case $q = d$ is clear, so assume $\text{deg} \lambda \leq q$. Write $\alpha = \sum_{i \geq 0} a_i \omega^i$ and $\lambda = \sum_{i=0}^q l_i \omega^i$. The coefficient of $\omega^q$ in $\alpha + \lambda$ is equal to $a_q + l_q$, and since $\alpha^+_e = \beta^+_e$, this is at most $a_q$, proving that $l_q = 0$.

2.3. The length of a partial well-order. Let $P$ be a partial well-order. We define the height rank $\text{lP}(\cdot)$ on $P$ by transfinite induction as follows: at successor stages, we say that $\text{lP}(a) \geq \alpha + 1$, if there exists $b \leq a$ with $\text{lP}(b) \geq \alpha$, and at limit stages, that $\text{lP}(a) \geq \lambda$, if there exists for each $\alpha < \lambda$ some $b_\alpha \leq a$ with $\text{lP}(b_\alpha) \geq \alpha$. We then say that $\text{lP}(a) = \alpha$ if $\text{lP}(a) \geq \alpha$ but not $\text{lP}(a) \geq \alpha + 1$. In particular, $\text{lP}(a) = 0$ if and only if $a$ is a minimal element of $P$. For a subset $A \subseteq P$, we set $\text{lP}(A)$ equal to the supremum of all $\text{lP}(a)$ with $a \in A$. Finally, we define the (ordinal) length of $P$ as $\text{len}(P) := \text{lP}(P)$. In particular, if $P$ has a maximum $\top$, then $\text{len}(P) = \text{lP}(\top)$.

2.4. Lemma. Let $P$ be a partial well-order and let $A, B \subseteq P$ be subsets. If $A \subseteq B$, then

$$\text{lA}(A) + \text{lB}(B) \leq \text{lB}(B)$$

(3)

Proof. Let $\alpha := \text{lA}(A) = \text{len}(A)$. Since $\text{lB}(B)$ is the supremum of all $\text{lP}(b)$ with $b \in B$, it suffices to show that

$$\alpha + \text{lB}(b) \leq \text{lP}(b).$$

(4)

We will prove (4) by induction on $\beta := \text{lP}(b)$. Assume first that $\beta = 0$. Let $\theta := \text{lP}(A)$. Since $\alpha$ is the supremum of all $\text{lA}(a)$ for $a \in A$, and since $\text{lA}(a) \leq \text{lP}(a)$, we get $\alpha \leq \theta$. Since $A \subseteq b$, we have $\theta \leq \text{lP}(b)$, and hence we are done in this case.

Next, assume $\beta$ is a successor ordinal, and denote its predecessor by $\beta - 1$. By definition, there exists $b' \in B$ below $b$ such that $\text{lB}(b') \geq \beta - 1$. By induction, we get $\text{lP}(b') \geq \alpha + \beta - 1$. This in turn shows that $\text{lP}(b)$ is at least $\alpha + \beta$. Finally, assume $\beta$ is a limit ordinal. Hence for each $\gamma < \beta$, there exists $b_\gamma \in B$ below $b$ such that $\text{lB}(b_\gamma) = \gamma$. By induction, $\text{lP}(b_\gamma) \geq \alpha + \gamma \leq \alpha + \beta$ and hence also $\text{lP}(b) \geq \alpha + \beta$.

2.5. Sum Orders. By the sum $P + Q$ of two partially ordered sets $P$ and $Q$, we mean the partial order induced on their disjoint union $P \sqcup Q$ by declaring any element in $P$ to lie below any element in $Q$. In fact, if $\alpha$ and $\beta$ are ordinals, then their sum is just $\alpha \sqcup \beta$, customarily denoted $\alpha + \beta$. We may represent elements in the disjoint union $P \sqcup Q$ as pairs $(i, a)$ with $i = 0$ if $a \in P$ and $i = 1$ if $a \in Q$. The ordering $P + Q$ is then the lexicographical ordering on such pairs, that is to say, $(i, a) \leq (j, b)$ if $i < j$ or if $i = j$ and $a \leq b$.

2.6. Proposition. If $P$ and $Q$ are partial well-orders, then so is $P + Q$. If $P$ has moreover a maximum, then $\text{len}(P + Q) = \text{len}(P) + \text{len}(Q)$.

Proof. We leave it as an exercise to show that $P + Q$ is a partial well-order. Let $\pi := \text{lP}(\top P) = \text{len}(P)$. For a pair $(i, a)$ in $P + Q$, let $\nu(i, a)$ be equal to $\text{lP}(a)$ if $i = 0$ and to $\pi + \text{lQ}(a)$ if $i = 1$. The assertion will follow once we showed that $\nu(i, a) = \text{l}(i, a)$, for
all \((i,a) \in P + Q\), where we wrote \(l(i,a)\) for \(l_{P+Q}(i,a)\). We use transfinite induction. If \(i = 0\), that is to say, if \(a \in P\), then the claim is easy to check, since no element from \(Q\) lies below \(a\). So we may assume \(i = 1\) and \(a \in Q\). Let \(\alpha := l_Q(a)\) and suppose first that \(\alpha = 0\). Since any element of \(P\) lies below \(a\), in any case \(\pi \leq l(i,a)\). If this were strict, then there would be an element \((j,b)\) below \((i,a)\) of height rank \(\pi\). Lest \(l(0,\top_P)\) would be bigger than \(\pi\), we must have \(j = 1\) whence \(b \in Q\). Since \(b \leq_Q a\), we get \(l_Q(a) \geq 1\), contradiction. This concludes the case \(\alpha = 0\), so assume \(\alpha > 0\). We leave the limit case to the reader and assume moreover that \(\alpha\) is a successor ordinal. Hence there exists some \(b \in Q\) below \(a\) with \(\alpha_Q(b) = \alpha - 1\). By induction, \(l(1,b) = \nu(1,b) = \pi + \alpha - 1\), and hence \(l(1,a) \geq \pi + \alpha\). By a similar argument as above, one then easily shows that this must in fact be an equality, as we wanted to show. 

\[\square\]

2.7. Product Orders. The product of two partially ordered sets \(P\) and \(Q\) is defined to be the Cartesian product \(P \times Q\) ordered by the rule \((a,b) \leq (a',b')\) if and only if \(a \leq a'\) and \(b \leq b'\). Note that this is not the lexicographical ordering, even if \(P\) and \(Q\) are total orders. The map \((a,b) \mapsto (b,a)\) is an order-preserving bijection between \(P \times Q\) and \(Q \times P\). It is easy to check that if both \(P\) and \(Q\) are partial well-orders, then so is \(P \times Q\).

For the next result, we make use of the ordinal sum \(\oplus\) defined in Appendix 7 below.

2.8. Theorem (Product Formula). Given partial well-orders \(P\) and \(Q\), we have an equality \(\text{len}(P \times Q) = \text{len}(P) \oplus \text{len}(Q)\).

Proof. We prove the more general fact that

\[(5)\]

\[l(a,b) = l(a) \oplus l(b)\]

for all \(a \in P\) and \(b \in Q\), from which the assertion follows by taking suprema over all elements in \(P\) and \(Q\). Note that we have not written superscripts to denote on which or-

dered set the height rank is calculated since this is clear from the context. To prove (5), we may assume by transfinite induction that it holds for all pairs \((a',b')\) strictly below \((a,b)\). Put \(\alpha := l(a),\ \beta := l(b)\) and \(\gamma := l(a,b)\). Since \(l(a,b) = l(b,a)\), via the isomorphism \(P \times Q \cong Q \times P\), we may assume that \(\text{ord}(\alpha) \leq \text{ord}(\beta)\) whenever this assumption is required (namely, when dealing with limit ordinals). Here \(\text{ord}(\alpha)\) denotes the order of \(\alpha\), that is to say, the lowest exponent in the Cantor normal form; see §7 for details.

We start with proving the inequality \(\alpha \oplus \beta \leq \gamma\). If \(\alpha = \beta = 0\) then \(a\) and \(b\) are minimal elements in respectively \(P\) and \(Q\), whence so is \((a,b)\) in \(P \times Q\), that is to say, \(\gamma = 0\). So we may assume, after perhaps exchanging \(P\) with \(Q\) that \(\alpha > 0\). Suppose \(\alpha\) is a successor ordinal. Hence there exists \(a' < a\) in \(P\) with \(l(a') = \alpha - 1\). By induction, \(l(a',b) = (\alpha - 1) \oplus \beta\) and hence \(\gamma = l(a,b)\) is at least \((\alpha - 1) \oplus \beta + 1 = \alpha \oplus \beta\), where the last equality follows from Theorem 7.1. If \(\alpha\) is a limit ordinal, then there exists for each \(\delta < \alpha\) an element \(a_\delta < a\) of height rank \(\delta\). By induction \(l(a_\delta,b) = \delta \oplus \beta\) and hence \(l(a,b)\) is at least \(\alpha \oplus \beta\) by Theorem 7.1. This concludes the proof that \(\alpha \oplus \beta \leq \gamma\).

For the converse inequality, assume first that \(\gamma\) is a successor ordinal. By definition, there exists \((a',b') < (a,b)\) in \(P \times Q\) of height rank \(\gamma - 1\). By induction, \(\gamma - 1 = l(a',b') = l(a') \oplus l(b')\), from which we get \(\gamma \leq \alpha \oplus \beta\). A similar argument can be used to treat the limit case and the details are left to the reader. 

\[\square\]

Increasing functions. We conclude this section with the behavior of height rank under an increasing function. Let \(f : P \to Q\) be an increasing (=order-preserving) map between ordered sets. We say that \(f\) is strictly increasing, if \(a < b\) then \(f(a) < f(b)\). For instance, an increasing, injective map is strictly increasing.
2.9. Theorem. Let \( f : P \to Q \) be a strictly increasing map between partial well-orders. If \( P \) has a minimum \( \bot_P \), then
\[
\ell_Q(f(\bot_P)) + \ell_P(a) \leq \ell_Q(f(a)).
\]
for all \( a \in P \).

Proof. From the context, it will be clear in which ordered set we calculate the rank and hence we will drop the superscripts. Let \( \gamma := \ell(f(\bot)) \). We induct on \( \alpha := \ell(a) \), where the case \( \alpha = 0 \) holds trivially. We leave the limit case to the reader and assume that \( \alpha \) is a successor ordinal. By definition, there exists \( b < a \) with \( \ell(b) = \alpha - 1 \). By induction, the height rank of \( f(b) \) is at least \( \gamma + \alpha - 1 \). By assumption, \( f(b) < f(a) \), showing that \( f(a) \) has height rank at least \( \gamma + \alpha \). \qed

Even in the absence of a minimum, the inequality still holds, upon replacing the first ordinal in the formula by the minimum of the ranks of all \( f(a) \) for \( a \in P \). In particular, height rank always increases.

3. Semi-additivity

Let \( R \) be a ring and \( M \) a Noetherian \( R \)-module. The Grassmanian of \( M \) (over \( R \)) is by definition the collection \( \text{Grass}_R(M) \) of all submodules of \( M \), ordered by reverse inclusion. Note that this is a natural generalization of the notion of a Grassmanian of a vector space \( V \) over a field \( K \). The height rank of \( \text{Grass}(M) \) will be called the length \( \text{len}_R(M) \) of \( M \) as an \( R \)-module. This is well-defined, since \( \text{Grass}_R(M) \) is a well-partial order. Thus, for \( N \subseteq M \), we have \( \ell(N) \geq \alpha + 1 \), if there exists \( N' \) containing \( N \) with \( \ell(N') \geq \alpha \). Since the initial closed interval \( [\text{Grass}(M), N] \) is isomorphic to \( \text{Grass}_R(M/N) \), we get
\[
\ell(N) = \ell_{\text{Grass}_R(M)}(N) = \ell_{\text{Grass}_R(M/N)}(0_{M/N}) = \text{len}_R(M/N)
\]
and hence in particular \( \ell(0_M) = \text{len}_R(M) \), where \( 0_M \) denotes the zero module of \( M \). Similarly, \( [\text{Grass}_R(M), N] \) consists of all submodules of \( M \) contained in \( N \), whence is equal to \( \text{Grass}_R(N) \). Note that if \( I \) is an ideal in the annihilator of \( M \), then \( \text{Grass}_R(M) = \text{Grass}_R(M/I) \), so that in order to calculate the length or the order dimension of \( M \), it makes no difference whether we view it as an \( R \)-module or as an \( R/I \)-module. We call the length of \( R \), denoted \( \text{len}(R) \), its length when viewed as a module over itself. Hence, the length of \( R/I \) as an \( R \)-module is the same as that of \( R/I \) viewed as a ring. We define the order, \( \text{ord}_R(M) \), and valence, \( \text{val}_R(M) \), as the respective orders and valence of \( \text{len}_R(M) \).

3.1. Theorem (Semi-additivity). If \( 0 \to N \to M \to Q \to 0 \) is an exact sequence of Noetherian \( R \)-modules, then
\[
\text{len}_R(Q) + \text{len}_R(N) \leq \text{len}_R(M) \leq \text{len}_R(Q) + \text{len}_R(N)
\]
Moreover, if the sequence is split, then the last inequality is an equality.

Proof. The last assertion follows from the first, Theorem 2.8, and the fact that then
\[
\text{Grass}_R(N) \times \text{Grass}_R(Q) \subseteq \text{Grass}_R(M).
\]
To prove the lower estimate, let \( A \) be the initial closed interval \( [\text{Grass}_R(M), N] \) and let \( B \) be the terminal closed interval \( [N, \text{Grass}_R(M)] \). By our discussion above, \( A = \text{Grass}_R(M/N) = \text{Grass}(Q) \), since \( M/N \cong Q \), with maximum, viewed in \( \text{Grass}_R(Q) \), equal to \( 0_Q \). By the same discussion, \( B = \text{Grass}_R(N) \) with maximum \( 0_N \). Since \( A \leq B \), we may apply Lemma 2.4 to get an inequality
\[
\ell_{\text{Grass}_R(Q)}(0_Q) + \ell_{\text{Grass}_R(N)}(0_N) \leq \ell_{\text{Grass}_R(M)}(0_M).
\]
from which the assertion follows.

To prove the upper bound, let \( f : \text{Grass}_R(M) \to \text{Grass}_R(N) \times \text{Grass}_R(Q) \) be the map sending a submodule \( H \subseteq M \) to the pair \((H \cap N, \pi(H))\), where \( \pi \) denotes the morphism \( M \to Q \). It is not hard to see that this is an increasing function. Although it is in general not injective, I claim that \( f \) is strictly increasing, so that we can apply Theorem 2.9. Together with Theorem 2.8, this gives us the desired inequality. So remains to verify the claim: suppose \( H < H' \) but \( f(H) = f(H') \). Hence \( H' \not\subset H \), but \( H \cap N = H' \cap N \) and \( \pi(H) = \pi(H') \). Applying the last equality to an element \( h \in H \setminus H' \), we get \( \pi(h) \in \pi(H') \), whence \( h \in H' + N \). Hence, there exists \( h' \in H' \) such that \( h - h' \) lies in \( H \cap N \) whence in \( H' \cap N \). This in turn would mean \( h \in H' \), contradicting our assumption on \( h \).

Using that \( \alpha + n = \alpha \oplus n \), when \( n \) is finite, we immediately get:

3.2. Corollary. If \( N \) is a submodule of \( M \) of finite length, then
\[
\text{len}_R(M) = \text{len}_R(M/N) + \text{len}_R(N).
\]

3.3. Corollary. Let \( R \) be a Noetherian ring. If \( x \) is an \( R \)-regular element and \( I \subseteq R \) an arbitrary ideal, then
\[
\text{len}(R/xR) + \text{len}(R/I) \leq \text{len}(R/xI).
\]

Proof. Apply Theorem 3.1 to the exact sequence
\[
0 \to R/I \xrightarrow{x} R/xI \to R/xR \to 0.
\]

\[\square\]

Applying Corollary 3.3 to the zero ideal and observing that \( \alpha + \beta = \beta \) if and only if \( \text{deg} \alpha < \text{deg} \beta \), we get:

3.4. Corollary. If \( x \) is an \( R \)-regular element, then the degree of \( \text{len}(R/xR) \) is strictly less than the degree of \( \text{len}(R) \).

\[\square\]

3.5. Theorem. Let \( M \) be a finitely generated module over a Noetherian ring \( R \). Then the degree of \( \text{len}_R(M) \) is equal to the dimension of \( M \). In particular, \( R \) is a \( d \)-dimensional domain if and only if \( \text{len}(R) = \omega^d \).

Proof. Let \( \mu \) be the length of \( M \) and \( d \) its dimension. We start with proving the inequality
\[
\omega^d \leq \mu.
\]

We will do this first for \( M = R \), by induction on \( d \), where the case \( d = 1 \) is clear, since \( R \) does not have finite length. Hence we may assume \( d > 1 \). Taking the residue modulo a \( d \)-dimensional prime ideal (which only can lower length), we may assume that \( R \) is a domain. Let \( p \) be a \((d - 1)\)-dimensional prime ideal and let \( x \) be a non-zero element in \( p \). By Corollary 3.4, the degree of \( \text{len}(R/xR) \) is at most \( d - 1 \). By induction,
\[
\omega^{d-1} \leq \text{len}(R/p) \leq \text{len}(R/xR),
\]
whence \( d - 1 \leq \text{deg} \mu - 1 \), proving (8).

For \( M \) an arbitrary \( R \)-module, let \( p \) be a \( d \)-dimensional associated prime of \( M \), so that we can find an exact sequence
\[
0 \to R/p \to M \to \bar{M} \to 0.
\]

With \( \bar{\mu} := \text{len}(\bar{M}) \), Theorem 3.1 yields
\[
\bar{\mu} + \text{len}(R/p) \leq \mu \leq \bar{\mu} \oplus \text{len}(R/p).
\]

In particular, \( \omega^d \leq \text{len}(R/p) \leq \mu \), proving (8).
Next we show that
\[ \mu < \omega^{d+1}, \]
again by induction on \( d \). Assume first that \( M = R \) is a domain. Since \( R/I \) has then dimension at most \( d - 1 \) for any non-zero ideal \( I \), we get \( \text{len}(R/I) < \omega^d \) by our induction hypothesis. By (6), this means that any non-zero ideal has height rank less than \( \omega^d \), and hence \( R \) itself has length at most \( \omega^d \). Together with (8), this already proves one direction in the second assertion. For the general case, we do a second induction, this time on \( \mu \). With \( p \) as above, a \( d \)-dimensional associated prime of \( M \), we get \( \mu \leq \bar{\mu} \oplus \text{len}(R/p) \) by (10). By what we just proved, \( \text{len}(R/p) = \omega^d \), and hence by induction \( \mu \leq \bar{\mu} \oplus \omega^d < \omega^{d+1} \). The first assertion is now immediate from (8) and (11).

Conversely, if \( R \) has length \( \omega^d \), then for any non-zero ideal \( I \), the length of \( R/I \) is strictly less than \( \omega^d \), whence its dimension is strictly less than \( d \) by what we just proved. This shows that \( R \) must be a domain. \( \square \)

Let \( \ell^\text{gen}_R(M) \) be the generic length of a Noetherian module \( M \), defined as the sum
\[ \ell^\text{gen}_R(M) = \sum_{\dim p = \dim(M)} \text{len}(M_p). \]
Note that we only have a non-zero contribution in this sum if \( p \) is a minimal prime of \( M \), and the corresponding localization \( M_p \) then has finite length, so that \( \ell^\text{gen}(M) \) is well-defined. If \( M \) has finite length, \( \ell^\text{gen}(M) = \text{len}(M) \). With the notation from (2), we get

3.6. Proposition. For \( d = \dim(M) \), we have \( \text{len}_R(M)_\omega^{d+1} = \ell^\text{gen}_R(M)\omega^d \).

Proof. We induct on \( \mu := \text{len}(M) \), where the case for finite \( \mu \) is clear. By Theorem 3.5 we can write \( \mu = a\omega^d + \mu^- \). Let \( p \) be a \( d \)-dimensional associated prime of \( M \), so that we have an exact sequence (9) and let \( \bar{\mu} := \text{len}(M) \). By induction, we have \( \bar{\mu} = \ell^\text{gen}(M)\omega^d + \bar{\mu}^- \). Applying Theorem 3.1 to (9) yields inequalities \( \bar{\mu} + \omega^d \leq \mu \leq \bar{\mu} \oplus \omega^d \). Looking at the coefficients of the degree \( d \) terms, this implies \( a = \ell^\text{gen}(M) + 1 \). On the other hand, localizing (9) at \( p \) and taking lengths gives \( \text{len}(M_p) = \text{len}(M_p) + 1 \), whereas localizing at any other \( d \)-dimensional prime ideal \( q \) gives \( M_q = M_q \), showing that \( \ell^\text{gen}(M) = \ell^\text{gen}(M) + 1 = a \). \( \square \)

3.7. Remark. In [7], we will extend this formula by calculating all coefficients in \( \text{len}_R(M) \).

As a corollary, we will obtain that the order of \( M \) is the minimal dimension of an associated prime of \( M \). In this paper, we only prove the following consequence of this characterization, where we call a module \( M \) unmixed, if all its associated primes have the same dimension (equivalently, if all non-zero modules have the same dimension).

3.8. Theorem. A module is unmixed if and only if its length is a monomial, that is to say, of the form \( a\omega^d \). Moreover, \( a \) is then the generic length of the module and \( d \) its dimension.

Proof. The second assertion is just Proposition 3.6. Let \( d \) be the dimension of \( M \). Suppose \( M \) is unmixed and let \( \mu \) be its length. By Theorem 3.5, we can write \( \mu = a\omega^d + \mu^- \), with \( a \) a positive integer and \( \mu^- := \mu^{\mu^- - 1} \) as in (2). We need to show that \( \mu^- = 0 \). Suppose it is not, so that there must exist a non-zero submodule \( N \) with \( \text{len}(N) = a\omega^d \). By (6), we have \( \text{len}(M/N) = a\omega^d \). By Theorem 3.1, with \( \nu := \text{len}(N) \), we get \( a\omega^d + \nu \leq \mu = a\omega^d + \mu^- \). Hence \( \nu \leq \mu^- \), and so \( \nu \) has degree at most \( d - 1 \), which means that \( \dim(N) \) is less than \( d \) by Theorem 3.5. Since \( M \) is unmixed, we must have \( N = 0 \), contradiction.

Conversely, suppose \( \mu = a\omega^d \). Suppose there exists a non-zero sub-module \( N \subseteq M \) of dimension at most \( d - 1 \). Let \( \nu \) and \( \gamma \) be the respective lengths of \( N \) and \( M/N \). By
semi-additivity, we have
\[ \gamma + \nu \leq a\omega^d \leq \gamma \oplus \nu. \]
By Theorem 3.5, the degree of \( \nu \) is at most \( d - 1 \). Hence for the first equality to hold, we must have \( \gamma < a\omega^d \). However, \( \gamma \oplus \nu \) is then also strictly less than \( a\omega^d \), contradicting the second inequality. \( \square \)

Immediately from Theorem 3.5 and Theorem 3.8, since \( \omega^d + \nu = \omega^d \oplus \nu \), for any ordinal \( \nu \) of degree at most \( d \), we get

**3.9. Corollary.** If \( 0 \to N \to M \to Q \to 0 \) is exact and \( Q \) is unmixed with \( \dim Q = \dim M \), then \( \text{len}(M) = \text{len}(Q) \oplus \text{len}(N) \).

There is another measure for how long a partial well-order \( P \) is, namely the maximal length of a chain in \( P \). More precisely, given a chain \( C \) in \( P \), let \( \sigma(C) \) be the ordinal giving the order type of \( C \). Assume \( P \) has a maximum \( \top \). Define the **chain length of \( P \)** as the supremum of all \( \sigma(C) \), where \( C \) runs over all chains in \( P \) not containing \( \top \). We have to omit \( \top \) here to conform with the notion of length for finite chains as one less than their cardinality. The chain length can be smaller than the length: for each \( \alpha \), let \( C_\alpha \) be a chain of length \( \alpha \), let \( P \) be obtained from the disjoint union of all \( C_\alpha \) by adding two more elements \( a < b \) above all these. In particular, as each chain in \( P \) is finite, but of arbitrarily large size, its chain length is \( \omega \). However, the height rank of \( a \) is \( \omega \), and hence \( \text{len}(P) = \text{I}_P(b) = \text{I}_P(a) + 1 = \omega + 1 \). This phenomenon cannot occur in Grassmanians, and, in fact, the chain length is even a maximum.

**3.10. Theorem.** The length of \( M \) is equal to the chain length of \( \text{Grass}_R(M) \). In fact, \( \text{len}_R(M) \) is the maximum of all \( \sigma(C) \), where \( C \) runs over all chains of non-zero submodules in \( \text{Grass}_R(M) \).

**Proof.** Let \( C \) be a chain of non-zero submodules and let \( \rho := \sigma(C) \). Let \( M_\alpha \) for \( \alpha < \rho \) be the \( \alpha \)-th element in this chain. In particular, \( 0_M \) is the \( \rho \)-th element in the chain \( C \cup \{0_M\} \).

By definition, \( I(M_\alpha) \geq \alpha \). Hence \( \text{len}(M) = I(0_M) \geq \sigma(C) \). So the result will follows once we prove the existence of a chain of non-zero modules of length \( \mu := \text{len}(M) \), and we do this by induction on \( \mu \). For \( \mu < \omega \), this is just the classical Jordan-Holder theorem for modules of finite length. If \( \mu < \omega \), we can write it as \( \mu \oplus e \), where \( e \) is the order of \( \mu \). Let \( N \) be a submodule of height rank \( \mu \), so that \( M/N \) has height \( \mu \) by (6).

If \( \nu := \text{len}(N) \), then \( \nu \leq \mu \). Hence, by semi-additivity, we get \( \mu + \nu \leq \mu \oplus e \leq \mu \). The latter implies that \( \omega^e \leq \nu \), and by the former inequality, it cannot be bigger either. By induction we can find a chain of non-zero submodules in \( \text{Grass}(M/N) \) of length \( \mu \), and hence, upon lifting these to \( M \), we get a chain in \( \text{Grass}(M) \) of submodules strictly containing \( N \). Induction also gives a chain of non-zero submodules of \( N \) of length \( \omega^e \), and putting these two chains together, we get a chain of length \( \mu \).

So remains the case that \( \mu = \omega^d \). We prove this case independently by induction on \( d \), where the case \( d = 1 \) is classical: if there were no infinite chains, then \( M \) is both Artinian and Noetherian, whence of finite length ([1, Proposition 6.8]). Put \( M_0 := M \) and choose a submodule \( M_1 \) of height rank \( \omega^{d-1} \). Applying the induction hypothesis to \( M/M_1 \), we can find, as above, a chain \( C_1 \) of submodules strictly containing \( M_1 \), of length \( \omega^{d-1} \). Since \( M_1 \) is non-zero, it has dimension \( d \), whence its length must be at least \( \omega^d \) by Theorem 3.5, and therefore, by semi-additivity, equal to it. Choose a submodule \( M_2 \) of \( M_1 \) of height rank \( \omega^{d-1} \) in \( \text{Grass}(M_1) \), and as before, find a \( \omega^{d-1} \)-chain \( C_2 \) in this Grassmanian of submodules strictly containing \( M_2 \). Continuing in this manner, we get a descending chain.
$M_0 \supseteq M_1 \supseteq M_2 \supseteq \ldots$ and $\omega^{d-1}$-chains $C_n$ from $M_n$ down to $M_{n+1}$. The union of all these chains is therefore a chain of length $\omega^d$, as we needed to construct. \hfill \Box

4. ACYCLICITY

For the remainder of this paper $R$ is a Noetherian ring, $d$ its dimension, and $\rho$ its length. Furthermore, $M$, $N$, $\ldots$ are finitely generated modules over $R$, of length $\mu$, $\nu$, etc. We start with reproving the observation of Vasconcelos [8] that a surjective endomorphism on a Noetherian module must be an isomorphism (the usual proof uses the determinant trick; see for instance [5, Theorem 2.4]).

4.1. Corollary. Any surjective endomorphism is an isomorphism.

Proof. Let $M \to M$ be a surjective endomorphism with kernel $N$, so that we have an exact sequence $0 \to N \to M \to M \to 0$, and therefore, by Theorem 3.1, an inequality $\text{len}(M) + \text{len}(N) \leq \text{len}(M)$. By simple ordinal arithmetic, this implies $\text{len}(N) = 0$, whence $N = 0$. \hfill \Box

4.2. Remark. Our argument in fact proves that any surjection between modules of the same length must be an isomorphism, or more generally, if $f : M \to N$ is an epimorphism and $\text{len}(M) \leq \text{len}(N)$, then $f$ is an isomorphism and $\text{len}(M) = \text{len}(N)$.

4.3. Corollary. If $N$ is a homomorphic image of $M$ which contains a submodule isomorphic to $M$, then $M \cong N$.

Proof. Since $M \to N$, semi-additivity yields $\text{len}(M) \leq \text{len}(N)$. By Remark 4.2, the epimorphism $M \to N$ must then be an isomorphism. \hfill \Box

The following result generalizes Miyata’s result [6] as we do not need to assume that the given sequence is left exact.

4.4. Theorem. An exact sequence $M \to N \to C \to 0$ is split exact if and only if $N \cong M \oplus C$.

Proof. One direction is just the definition of split exact. Let $\overline{M}$ be the image of $M$ and apply Theorem 3.1 to $0 \to \overline{M} \to N \to C \to 0$ to get $\text{len}(N) \leq \text{len}(C) \oplus \text{len}(\overline{M})$. On the other hand, $N \cong M \oplus C$ yields $\text{len}(N) = \text{len}(M) \oplus \text{len}(C)$, whence $\text{len}(M) \leq \text{len}(\overline{M})$. Since $\overline{M}$ is a homomorphic image of $M$, they must be isomorphic by Remark 4.2. Hence, we showed $M \to N$ is injective. At this point we could invoke [6], but we can as easily give a direct proof of splitness as follows. Given a finitely generated $R$-module $H$, since $M \otimes H \cong (N \otimes H) \oplus (C \otimes H)$, the same argument applied to the tensored exact sequence

$$M \otimes H \to N \otimes H \to C \otimes H \to 0,$$

gives the injectivity of the first arrow. We therefore showed that $M \to N$ is pure, whence split by [5, Theorem 7.14]. \hfill \Box

4.5. Theorem. Let $X$ be a non-singular variety over an algebraically closed field $k$. Then a closed subscheme $Y \subseteq X$ with ideal of definition $I$ is non-singular if and only if $\Omega_{X/k} \otimes \mathcal{O}_Y$ is locally isomorphic to $I/I^2 \oplus \Omega_{Y/k}$.

Proof. Since $X$ is non-singular, its module of differentials $\Omega_{X/k}$ is locally free ([4, Theorem 8.15]), whence so is $\Omega_{X/k} \otimes \mathcal{O}_Y$, and therefore so is its direct summand $\Omega_{Y/k}$. Moreover, by Theorem 4.4, the conormal sequence

$$I/I^2 \to \Omega_{X/k} \otimes \mathcal{O}_Y \to \Omega_{Y/k} \to 0$$

is then split exact, and the result now follows from [4, Theorem 8.17]. \hfill \Box
4.6. **Theorem.** Let \( A \) be a finitely generated \( R \)-algebra, \( I \subseteq A \) an ideal, and \( \bar{A} := A/I \). The closed immersion \( \text{Spec} \bar{A} \subseteq \text{Spec}(A/I^2) \) is a retract over \( R \) if and only if we have an isomorphism of \( A \)-modules

\[
\Omega_{A/R}/I\Omega_{A/R} \cong \Omega_{\bar{A}/R} + I/I^2.
\]

**Proof.** One direction is easy, and if (12) holds, then the conormal sequence

\[
I/I^2 \rightarrow \Omega_{A/R}/I\Omega_{A/R} \rightarrow \Omega_{\bar{A}/R} \rightarrow 0
\]

is split exact by Theorem 4.4, so that the result follows from [3, Proposition 16.12]. \( \square \)

4.7. **Remark.** We can also formulate a necessary and sufficient condition for the cotangent sequence to be split, although I do not know of any consequences of this fact: given homomorphisms \( R \rightarrow S \rightarrow T \), the sequence

\[
T \otimes_S \Omega_{S/R} \rightarrow \Omega_{T/R} \rightarrow \Omega_{S/R} \rightarrow 0
\]

is split exact if and only if \( \Omega_{T/R} \cong (T \otimes_S \Omega_{S/R}) \oplus \Omega_{S/R} \).

4.8. **Proposition.** Let \((R,m)\) be a \( d \)-dimensional local Cohen-Macaulay ring with canonical module \( \omega_R \). Assume there exist exact sequences \( 0 \rightarrow N \rightarrow M \rightarrow X \rightarrow 0 \) and \( 0 \rightarrow M \rightarrow N \rightarrow Y \rightarrow 0 \). For all \( e \) such that \( \dim X < e \leq \depth Y \), we have

\[
\Ext_R^{d-e}(M,\omega_R) \cong \Ext_R^{d-e}(N,\omega_R).
\]

**Proof.** By faithfully flat descent, we may pass to the completion of \( R \), and therefore assume from the start that \( R \) is complete. By Grothendieck vanishing, the local cohomology groups \( \hat{H}_m^e(X) \) and \( \hat{H}_m^{e-1}(Y) \) vanish. Taking local cohomology of the two respective sequences therefore yields

\[
\hat{H}_m^e(N) \rightarrow \hat{H}_m^e(M) \rightarrow 0 = \hat{H}_m^e(X)
\]

\[
\hat{H}_m^{e-1}(Y) = 0 \rightarrow \hat{H}_m^e(M) \rightarrow \hat{H}_m^e(N)
\]

Taking Matlis duals and using Grothendieck duality ([2, Theorem 3.5.8]), we get exact sequences

\[
0 \rightarrow \Ext_R^{d-e}(M,\omega_R) \rightarrow \Ext_R^{d-e}(N,\omega_R)
\]

\[
\Ext_R^{d-e}(N,\omega_R) \rightarrow \Ext_R^{d-e}(M,\omega_R) \rightarrow 0
\]

and the result now follows from Corollary 4.3. \( \square \)

Inspired by the previous results, we introduce the following measures for two modules \( M \) and \( N \) to be non-isomorphic: let \( \kappa(M,N) \) (respectively, \( \gamma(M,N) \)) be the infimum of all \( \text{len}(\ker f) \) (respectively, \( \text{len}(\coker f) \)) for \( f \in \text{Hom}_R(M,N) \). We may rephrase Corollary 4.3 as

\[
M \cong N \quad \text{if and only if} \quad \gamma(M,N) + \kappa(M,N) = 0.
\]

In fact, as the previous examples suggest, \( \kappa \) is often bounded by \( \gamma \):

4.9. **Lemma.** If \( \text{len}_R(M) = e \) \( \text{len}_R(N) \) and \( \deg \gamma(M,N) \leq e - 1 \), for some \( e \), then \( \deg \kappa(M,N) \leq e - 1 \).

**Proof.** By assumption, we can find a morphism \( M \rightarrow N \) whose cokernel \( C \) has length \( \gamma = \gamma(M,N) \). We use the notation from (2) but drop the subscript \( e \) as this will not change throughout the proof. Let \( K \) and \( M \) the respective kernel and image, and let \( \kappa, \bar{\mu}, \mu, \nu \) be
the lengths of $K, \tilde{M}, M, N$. By assumption, $\mu^+ = \nu^+$ and $\gamma = \gamma^-$. By Theorem 3.1, the exact sequence $0 \rightarrow \tilde{M} \rightarrow N \rightarrow C \rightarrow 0$ gives
\begin{equation}
\gamma^- + \nu^+ + \mu^- \leq \tilde{\mu}^+ + \tilde{\mu}^- \leq \nu^+ + \gamma^- + \nu^-,
\end{equation}
from which it follows that $\nu^+ = \tilde{\mu}^+$. Applying semi-additivity instead to $0 \rightarrow K \rightarrow M \rightarrow \tilde{M} \rightarrow 0$ gives $\tilde{\mu} + \kappa \leq \mu$. Since $\tilde{\mu}^+ = \nu^+ = \mu^+$, we get $\mu = \kappa = \mu$, and hence $\deg \kappa < e$ by Lemma 2.2. Since $\kappa(M, N) \leq \kappa$, our claim follows.

Let us say that $M$ and $N$ are isomorphic at level $e$, denoted $M \cong_e N$, if there exists a morphism $M \rightarrow N$, called an isomorphism at level $e$, whose kernel and cokernel both have dimension strictly less than $e$. We similarly define an epimorphism at level $e$ as one whose cokernel has dimension strictly less than $e$. Of course $\cong_0$ just means isomorphic, whereas $\cong_{\dim R}$ gives the notion of being generically isomorphic.

4.10. Proposition. For $e$ equal to the degree of $\kappa(M, N) + \gamma(M, N)$, we have an isomorphism $M \cong_{e+1} N$ at level $e + 1$.

Proof. By assumption, there exist $f : M \rightarrow N$ and $g : M \rightarrow N$ such that $K := \ker(f)$ and $C := \text{coker}(g)$ have respective lengths $\kappa(M, N)$ and $\gamma(M, N)$. By Theorem 3.5, this means that $K$ and $C$ have dimension at most $e$. Let $p$ be an arbitrary prime ideal of dimension strictly bigger than $e$. Since $K_p$ and $C_p$ are then both zero, $\kappa(M_p, N_p) = \gamma(M_p, N_p) = 0$, and hence $M_p \cong_{e+1} N_p$ by (13). Since they have therefore the same length, the epimorphism $g_p : M_p \rightarrow N_p$ must be an isomorphism by Remark 4.2. Let $H$ be the kernel of $g$. We showed that $H_p = 0$. Since this holds for all $p$ of dimension $> e$, we must have $\dim(H) \leq e$, showing that $g$ is an isomorphism at level $e + 1$.

4.11. Corollary. Given $e \geq 0$, we have an isomorphism $M \cong_{e+1} N$ at level $e + 1$ if and only if $\text{len}_R(M) =_{e+1} \text{len}_R(N)$ and $\gamma(M, N) \leq e$.

Proof. The non-trivial direction follows from Proposition 4.10 and Lemma 4.9.

5. LENGTH CRITERION FOR ACYCLICITY

Given a complex
\begin{equation}
\mathcal{M} : 0 \rightarrow M_t \rightarrow M_{t-1} \rightarrow \cdots \rightarrow M_1 \rightarrow M_0 \rightarrow 0
\end{equation}
let us define its lower length $\text{Lolen} (\mathcal{M})$ and its upper length $\text{Hilen} (\mathcal{M})$ as the ordinals
\begin{align*}
\text{Lolen} (\mathcal{M}) & := \sum_{i \equiv t+1 \text{ mod } 2} \text{len}_R (M_i) \\
\text{Hilen} (\mathcal{M}) & := \bigoplus_{i \equiv t \text{ mod } 2} \text{len}_R (M_i),
\end{align*}
where we use the ascending order of the index set in the first sum.

5.1. Lemma. If $\mathcal{M}$ is exact, then $\text{Lolen} (\mathcal{M}) \leq \text{Hilen} (\mathcal{M})$.

Proof. We can break the exact sequence $\mathcal{M}$ into short exact sequences
\begin{equation}
0 \rightarrow Z_t \rightarrow M_t \rightarrow Z_{t-1} \rightarrow 0
\end{equation}
where $Z_0 = M_0$ and $Z_{t-1} = M_t$. Let $\zeta_i$ and $\mu_i$ be the respective lengths of $Z_i$ and $M_i$. By Theorem 3.1, we have for each $i$ an inequality
\begin{equation}
\zeta_{i-1} + \zeta_i \leq \mu_i \leq \zeta_{i-1} + \zeta_i.
\end{equation}
If \( t \) is even, then \( \text{Lolen}(\mathcal{M}) = \mu_1 + \mu_3 + \cdots + \mu_{t-1} \). Using respectively the upperbounds in (17) for odd \( i \), ordinal arithmetic, and the lowerbounds in (17) for even \( i \), gives

\[
\text{Lolen}(\mathcal{M}) \leq (\zeta_0 \oplus \zeta_1) + (\zeta_2 \oplus \zeta_3) + \cdots + (\zeta_{t-2} \oplus \zeta_{t-1}) \\
\leq \mu_0 \oplus (\zeta_1 + \zeta_2) \oplus \cdots \oplus (\zeta_{t-3} + \zeta_{t-2}) \oplus \mu_t \\
\leq \mu_0 \oplus \mu_2 \oplus \cdots \oplus \mu_{t-2} \oplus \mu_t = \text{Hilen}(\mathcal{M}).
\]

The proof for \( t \) odd is similar and left to the reader. \( \square \)

5.2. Theorem. Let \( e \geq -1 \) and suppose that all homology groups \( H_i(\mathcal{M}) \) for \( i < t \) have dimension at most \( e \), where \( \mathcal{M} \) is the complex \( (15) \). If \( \text{Hilen}(\mathcal{M}) \leq_e \text{Lolen}(\mathcal{M}) \), then \( H_t(\mathcal{M}) \) too has dimension at most \( e \).

Proof. The homology \( H_i := H_i(\mathcal{M}) \) of \( \mathcal{M} \), is given, for each \( i < t \), by two short exact sequences

\[
\begin{align*}
(18) & \quad 0 \to B_i \to Z_i \to H_i \to 0 \\
(19) & \quad 0 \to Z_{i+1} \to M_{i+1} \to B_i \to 0
\end{align*}
\]

with \( Z_0 = M_0 \) and \( Z_t = H_t \). We use the notation \( (2) \) with the value \( e + 1 \), but dropping the subscript. Let \( \mu_i, \theta_i, \beta_i \), and \( \zeta_i \) be the respective lengths of \( M_i, H_i, B_i \), and \( Z_i \). By Theorem 3.5, we have \( \theta_i^+ = 0 \), for all \( i < t \), and we want to show the same for \( i = t \). By semi-additivity, (18) yields

\[
\theta_i + \beta_i \leq \zeta_i \leq \theta_i \oplus \beta_i.
\]

Since \( \deg \theta_i \leq e \), for all \( i < t \), we get \( \beta_i^+ = \zeta_i^+ \). Semi-additivity applied to (19) for \( i - 1 \) gives

\[
\beta_{i-1} + \zeta_i \leq \mu_{i-1} \leq \beta_{i-1} \oplus \zeta_i,
\]

and hence, for \( i < t \), we have

\[
\zeta_{i-1}^+ + \mu_i^+ \leq \beta_i^+ \leq \zeta_i^+ \oplus \zeta_i^+.
\]

and for \( i = t \), using that \( Z_t = H_t \), we get

\[
\zeta_{t-1}^+ + \theta_t^+ \leq \mu_t^+.
\]

For simplicity, let us assume \( t \) is even (the odd case is similar). Using respectively the upperbounds in (21) for odd \( i < t \), ordinal arithmetic, the lowerbounds in (21) for even \( i < t \), and then for \( i = t \), we get inequalities

\[
\text{Lolen}(\mathcal{M})^+ = \mu_1^+ + \mu_3^+ + \cdots + \mu_{t-1}^+ \\
\leq (\zeta_0^+ \oplus \zeta_1^+) + (\zeta_2^+ \oplus \zeta_3^+) + \cdots + (\zeta_{t-2}^+ \oplus \zeta_{t-1}^+) \\
\leq \zeta_0^+ \oplus (\zeta_1^+ \oplus \zeta_2^+) \oplus \cdots \oplus (\zeta_{t-3}^+ \oplus \zeta_{t-2}^+) \oplus \zeta_{t-1}^+ \\
\leq \mu_0^+ \oplus \mu_2^+ \oplus \cdots \oplus \mu_{t-2}^+ \oplus \mu_t^+ = \text{Hilen}(\mathcal{M})^+
\]

(note that \( \beta_0^+ = \zeta_0^+ = \mu_0^+ \)). By assumption, the lowerbound is bigger than or equal to the upperbound, so that we have equalities throughout. In particular, the last of these gives \( \zeta_{t-1}^+ = \mu_t^+ \). Applied to (22), we then get the desired \( \theta_t^+ = 0 \). \( \square \)

Some special cases of this result are worth mentioning separately:
5.3. Corollary. Let \( S \colon 0 \to M_2 \to M_1 \to M_0 \to 0 \) be a complex such that \( \text{len}(M_0) \oplus \text{len}(M_2) \leq \text{len}(M_1) \). If \( S \) is right exact (respectively, right exact at all but finitely many maximal ideals), then \( S \) is exact (respectively, exact at all but finitely many maximal ideals).

5.4. Corollary. Let \((R, \mathfrak{m})\) be local ring with residue field \( k \) and let \( d \) and \( \rho \) be its respective dimension and length. Let \( b_i := \dim_k(\text{Tor}_i^R(M, k)) \) be the Betti numbers of \( M \). If \( \text{len}_R(M) = (b_0 - b_1) \odot \rho \), then \( M \) has projective dimension one, and the converse is true if \( M \) is moreover unmixed of dimension \( d \).

Proof. By assumption, there exists an exact sequence \( S_1 : \; R^{b_1} \to R^{b_0} \to M \to 0 \) (respectively, \( S_2 : R^{b_2} \to R^{b_1} \to R^{b_0} \to M \to 0 \)). If \( \mu = \text{len}(M) \), then \( \text{Lolen}(S_1) = b_0 \odot \rho \) and \( \text{Hilen}(S_1) = \mu \oplus (b_1 \circ \rho) \), proving that \( M \) has projective dimension one, since \( S_1 \) is then left exact by Corollary 5.3.

Suppose next that \( M \) is unmixed of dimension \( d \), so that \( \mu = q_\omega^d \) by Theorem 3.8. Hence if \( S_1 \) is also exact on the left, then \( q_\omega^d + (b_1 \circ \rho) \leq b_0 \odot \rho \leq q_\omega^d \oplus (b_1 \circ \rho) \) by Theorem 3.1, and both bounds are equal by ordinal arithmetic.

To prove the second case, assume \( q_\omega^d = \mu = (b_0 + b_2 - b_1) \circ \rho \), from which it follows that \( \mu + (b_1 \circ \rho) = (b_0 + b_2) \circ \rho \). Since the latter two ordinals are respectively \( \text{Lolen}(S_2) \) and \( \text{Hilen}(S_2) \), the sequence \( S_2 \) is also exact on the left by Theorem 5.2.

5.5. Corollary. Let \( M \) and \( N \) be modules of the same dimension with \( N \) moreover unmixed. If the complex \( \mathcal{S} : 0 \to N \to M \to N \to 0 \) is exact at all spots except possibly at the left most one, then it is in fact exact.

Proof. Let \( \mu \) and \( \nu \) be the respective lengths of \( M \) and \( N \). By Theorem 3.8, we get \( \text{Lolen}(\mathcal{S}) = \nu + \mu = \nu \oplus \mu = \text{Hilen}(\mathcal{S}) \). The result now follows from Theorem 5.2.

Given a complex (15), let us define its generic Euler characteristic \( \chi^{gen}(M) \) as the alternating sum \( \sum_i (-1)^i \chi^{gen}(M_i) \).

5.6. Corollary. If \( M : M_1 \to M_{i-1} \to \cdots \to M_1 \to M_0 \to 0 \) is an exact sequence in which all modules are unmixed of dimension \( d \), then the generic length of \( \chi^{gen}(M) \) is equal to \( \chi^{gen}(M) \). In particular, \( M_i \to M_{i-1} \) is injective if and only if \( \chi^{gen}(M) = 0 \).

Proof. Since \( \chi^{gen}(M) \) is the kernel of \( M_i \to M_{i-1} \), it suffices to show the last assertion. By Theorem 3.8, ordinal sum and shuffle sum are the same here, so that \( \chi^{gen}(M) = 0 \) if and only if \( \text{Lolen}(M) = \text{Hilen}(M) \), and the result now follows from Theorem 5.2.

Recall that when \((R, \mathfrak{m})\) is local, any finitely generated \( R \)-module \( M \) has a uniquely defined syzygy, denoted \( \Omega M \), given by a minimal exact sequence \( 0 \to \Omega M \to R^n \to M \to 0 \), that is to say, such that \( \Omega M \subseteq mR^n \).

5.7. Theorem. Let \( 0 \to \Omega M \to X \to M \to 0 \) be an exact sequence. If \( M \) is unmixed of maximal dimension, then \( X \) is free if and only if \( \Omega M \subseteq \mathfrak{m}X \).

Proof. If \( X \) is free, then the result follows from the minimality and uniqueness of syzygies. Let \( N := \Omega M \), and let \( \mu, \nu, \) and \( \chi \) be the respective lengths of \( M, N, \) and \( X \). By applying Corollary 3.9 to the given exact sequence and to the minimal exact sequence \( 0 \to N \to R^n \to M \to 0 \) respectively, we get \( \chi = \mu + \nu = \text{len}(R^n) \). Since \( N \subseteq \mathfrak{m}X \), it follows from Nakayama’s lemma, that \( X \) and \( M \) have the same minimal number of generators, which by minimality is precisely \( n \). Hence there exists a surjective morphism \( R^n \to X \). As both have the same length, this must be an isomorphism by Remark 4.2.
5.8. Theorem. Let $\varphi: \tilde{M} \to M$ be a surjective morphism, $N \subseteq M$ a submodule, and $\tilde{N} = \varphi^{-1}(N)$ its pull-back inside $M$. Let $\psi: \tilde{N} \to M$ be an arbitrary morphism with cokernel $C$. If $M$ is unmixed of maximal dimension, and there exists an exact sequence $0 \to C \to M \to N \to 0$, then $\psi$ is injective.

Proof. By assumption, we have an exact sequence
\[
\mathbf{S}: \tilde{N} \xrightarrow{\psi} \tilde{M} \to M \to N \to 0.
\]
In particular, $\text{Hilens}(S) = \text{len}(N) + \text{len}(M)$ and $\text{Hilen}_R(S) = \text{len}(M) \oplus \text{len}(\tilde{N})$. Let $K$ be the kernel of $\varphi$, which is then also the kernel of the restriction of $\varphi$ to $\tilde{N}$. In other words, we have exact sequences $0 \to K \to \tilde{M} \to M \to 0$ and $0 \to K \to \tilde{N} \to N \to 0$. Since $M$ is unmixed, so is $N$, and hence Corollary 3.9 yields $\text{len}(M) = \text{len}(M) \oplus \text{len}(K)$ and $\text{len}(\tilde{N}) = \text{len}(N) \oplus \text{len}(K)$. Moreover, by unmixedness $\text{Hilen}(S)$ is equal to $\text{len}(N) \oplus \text{len}(M)$ whence equal to $\text{Hilen}(S)$. Therefore, $\psi$ is injective by Theorem 5.2.

6. Locally isomorphic modules

Given two modules $M$ and $N$, define $\mathfrak{t}(M, N)$ as the sum of all $\text{Ann}_R(\ker(f))$, for $f \in \text{Hom}_R(M, N)$, and similarly, define $\mathfrak{c}(M, N)$ as the sum of all $\text{Ann}_R(\text{coker}(f))$. In particular, if $N$ is a homomorphic image of $M$, then $\mathfrak{c}(M, N) = 1$, and if $M$ is (isomorphic to) a submodule of $N$, then $\mathfrak{t}(M, N) = 0$.

6.1. Lemma. $M$ and $N$ are locally isomorphic if and only if $\mathfrak{c}(M, N)\mathfrak{c}(N, M) = 1$.

Proof. For each prime ideal $p$, we can find surjective morphisms $M_{p} \to N_{p}$ and $N_{p} \to M_{p}$. The composition must then be an isomorphism by Corollary 4.1. The converse is also immediate.

Given an $R$-module $M$, let $\text{ass}_R(M)$ be the sum of all its associated primes.

6.2. Lemma. If $\mathfrak{c}(M, N)$ is not contained in $\text{ass}_R(N)$, then $\text{len}_R(N) \leq \text{len}_R(M)$. If $R$, moreover, is a complete local ring and $\mathfrak{t}(M, N)$ is not contained in $\text{ass}_R(M)$, then $\text{len}_R(M) \leq \text{len}_R(N)$.

Proof. Write $\mathfrak{c}(M, N)$ as a finite sum $\sum_i \text{Ann}_R(\text{coker}(f_i))$, for some $f_i: M \to N$. By assumption this is not contained in the sum of all associated primes of $N$. Therefore, there must be some $i$, such that $\text{Ann}_R(\text{coker}(f_i))$ contains an $N$-regular element $x$. Let $f := f_i$. Since then $xN \subseteq f(M)$, and since $x$ is $N$-regular, the morphism $N \to f(M)$ sending $a \in N$ to $xa$ is an injection. In particular, $\text{len}(N) \leq \text{len}(f(M))$ by semi-additivity. Since $f(M)$ is a homomorphic image of $M$, its length is at most $\text{len}(M)$.

To prove the last assertion, under the additional assumption that $R$ is complete and local, with residue field $k$, we will use Matlis duality, where we write $M^\dagger := \text{Hom}_R(M, E)$ for the Matlis dual of a module $M$, with $E$ the injective hull of $k$. By the same argument, there exists $g: M \to N$ and an $M$-regular element $x$ such that $xK = 0$, where $K$ is the kernel of $g$. Taking Matlis duals, we get an exact sequence
\[
N^\dagger \xrightarrow{g^\dagger} M^\dagger \to K^\dagger \to 0.
\]
Since $xK^\dagger = 0$, we get $xM^\dagger \subseteq W := g^\dagger(N^\dagger)$. On the other hand, since $0 \to M \xrightarrow{x} M$ is injective, the dual map $M^\dagger \xrightarrow{x^\dagger} M^\dagger \to 0$ is surjective, so that $xM^\dagger = M^\dagger$. Taking again duals then yields an epimorphism $W^\dagger \to M$, so that $\text{len}(M) \leq \text{len}(W^\dagger)$. Since we also have an epimorphism $N^\dagger \to W$, we get an embedding $W^\dagger \subseteq N$, so that $\text{len}(W^\dagger) \leq \text{len}(N)$.
6.3. **Theorem.** Let $M$ and $N$ be finitely generated $R$-modules. If

- (6.3.a) $c(M, N)$ is not contained in $\text{ass}_R(N)$;
- (6.3.b) $c(N, M)$ is not contained in $\text{ass}_R(M)$;
- (6.3.c) $c(M, N) + c(N, M) = 1$;

then $M$ and $N$ are locally isomorphic.

**Proof.** Assume first that (6.3.a)–(6.3.c) hold, and let $p \subseteq R$ be an arbitrary prime ideal. From (6.3.c), it follows that $p$ does not contain some annihilator of a cokernel of a morphism (in either direction) between $M$ and $N$. Since the conditions are symmetric in $M$ and $N$, we may assume without loss of generality that we have a morphism $f : M \to N$ such that $\text{Ann}_R(\text{coker}(f))$ does not contain $p$. This means that $f_p : M_p \to N_p$ is surjective. It is not hard to see that $c(N_p, M_p)$ contains $c(N, M)$ and $\text{ass}_R(M_p)$ is contained in $\text{ass}_R(M)R_p$. Hence by (6.3.b), we see that $c(N_p, M_p)$ is not contained in $\text{ass}_R(M_p)$, and so

\[(24) \quad \text{len}_{R_p}(M_p) \leq \text{len}_{R_p}(N_p)\]

by Lemma 6.2. It now follows from Remark 4.2 that $f_p$ is an isomorphism. \hfill $\Box$

6.4. **Remark.** We may replace in (6.3.a) and (6.3.b) the ideals $c$ by the ideals $\diamond$ defined on a pair of modules as the sum $\delta(M, N) := c(M, N) + \varepsilon(N, M)$. Indeed, since the problem is local, we may localize $R$ so that $p$ is its maximal ideal. If $\varepsilon(M, N)$ is not contained in $\text{ass}(M)$, then the second part of Lemma 6.2 gives (24), at least over the completion $\hat{R}$ of $R$. Hence $\hat{f}$ is an isomorphism, whence so is $f$ by faithfully flat descent.

6.5. **Corollary.** Let $M$ and $N$ be such that $c(M, N) + c(N, M) = 1$. If there exists some $H$ such that $M \oplus H$ and $N \oplus H$ are locally isomorphic, then $M$ and $N$ are already locally isomorphic.

**Proof.** By assumption, any prime ideal $p$ does not contain either $c(M, N)$ or $c(N, M)$. Let us say the former holds. Localizing at $p$, we may assume $R$ is local and $N$ is a homomorphic image of $M$. Taking lengths, we get $\text{len}(M) \oplus \text{len}(H) = \text{len}(N) \oplus \text{len}(H)$ by semi-additivity, and hence $\text{len}(M) = \text{len}(N)$, whence $M \cong N$ by Remark 4.2. \hfill $\Box$

7. **Appendix: Shuffle Sums**

Recall that neither addition nor multiplication of ordinals is commutative. We will give three different but equivalent ways of defining a different, commutative addition operation on $\Omega$, which we temporarily will denote as $\oplus$, $\hat{\oplus}$ and $\hat{\hat{\oplus}}$. The sum $\oplus$ is also known as the natural (Hessenberg) sum and is often denoted $\#$. Recall our convention of writing multiplication from left-to-right (see §2.1). Every ordinal $\alpha$ can be written as a sum

\[(25) \quad \alpha = a_n\omega^{\nu_n} + \cdots + a_1\omega^{\nu_1}\]

where the $\nu_i$ (called the *exponents*) form a strictly ascending chain of ordinals, that is to say, $\nu_1 < \cdots < \nu_n$, and the $a_i$ (called the *coefficients*) are non-negative integers. This decomposition (in base $\omega$) is unique if we moreover require that all coefficients $a_i$ are non-zero, called the Cantor normal form (in base $\omega$) of $\alpha$. If (25) is in Cantor normal form, then we call the highest (respectively, lowest) occurring exponent, the degree (respectively, the order) of $\alpha$ and we denote these respectively by $\deg(\alpha) := \nu_n$ and $\ord(\alpha) := \nu_1$. Note that $\alpha$ is a successor ordinal if and only if $\ord(\alpha) = 0$. 


Given a second ordinal \( \beta \), we may assume that after possibly adding some more exponents, that it can also be written in the form (25), with coefficients \( b_i \geq 0 \) instead of the \( a_i \). We now define

\[
\alpha \oplus \beta := (a_n + b_n)\omega^{\nu_n} + \cdots + (a_1 + b_1)\omega^{\nu_1}.
\]

It follows that \( \alpha \oplus \beta \) is equal to \( \beta \oplus \alpha \) and is greater than or equal to both \( \alpha + \beta \) and \( \beta + \alpha \). For instance if \( \alpha = \omega + 1 \) then \( \alpha \oplus \alpha = 2\omega + 2 \) whereas \( \alpha + \alpha = 2\omega + 1 \). In case both ordinals are finite, \( \alpha \oplus \beta = \alpha + \beta \). It is easy to check that we have the following finite distributivity property:

(26) \[
(\alpha \oplus \beta) + 1 = (\alpha + 1) \oplus \beta = \alpha \oplus (\beta + 1).
\]

In fact, this follows from the more general property that \((\alpha \oplus \beta) + \theta = (\alpha + \theta) \oplus \beta = \alpha \oplus (\beta + \theta)\) for all \( \theta < \omega^{\alpha+1} \), where \( \omega \) is the minimum of \( \text{ord}(\alpha) \) and \( \text{ord}(\beta) \).

For the second definition, we use transfinite induction on the pairs \((\alpha, \beta)\) ordered lexicographically, that is to say, induction on the ordinal \( \alpha \beta \). Define \( \alpha \oplus 0 := \alpha \) and \( 0 \oplus \beta := \beta \) so that we may assume \( \alpha, \beta > 0 \). If \( \alpha \) is a successor ordinal (recall that its predecessor is then denoted \( \alpha - 1 \)), then we define \( \alpha \oplus \beta \) as \((\alpha - 1) \oplus \beta) + 1 \). Similarly, if \( \beta \) is a successor ordinal, then we define \( \alpha \oplus \beta \) as \((\alpha \oplus (\beta - 1)) + 1 \). Note that by transfinite induction, both definitions agree when both \( \alpha \) and \( \beta \) are successor ordinals, so that we have no ambiguity in defining this sum operation when at least one of the components is a successor ordinal. So remains the case that both are limit ordinals. If \( \text{ord}(\alpha) \leq \text{ord}(\beta) \), then we let \( \alpha \oplus \beta \) be equal to the supremum of the \( \delta \oplus \beta \) for all \( \delta < \alpha \). In the remaining case, when \( \text{ord}(\alpha) > \text{ord}(\beta) \), we let \( \alpha \oplus \beta \) be equal to the supremum of the \( \alpha \oplus \delta \) for all \( \delta < \beta \). This concludes the definition of \( \oplus \).

Finally, define \( \alpha \oplus \beta \) as the supremum of all sums \( \alpha_1 + \beta_1 + \cdots + \alpha_n + \beta_n \), where the supremum is taken over all \( n \) and all decompositions \( \alpha = \alpha_1 + \cdots + \alpha_n \) and \( \beta = \beta_1 + \cdots + \beta_n \). Loosely speaking, \( \alpha \oplus \beta \) is the largest possible ordering one can obtain by shuffling pieces of \( \alpha \) and \( \beta \). Since we may take \( \alpha_1 = 0 = \beta_n \), one checks that \( \alpha \oplus \beta = \beta \oplus \alpha \).

**7.1. Theorem.** For all ordinals \( \alpha, \beta \) we have \( \alpha \oplus \beta = \alpha \oplus \beta = \alpha \oplus \beta \).

**Proof.** Let \( \gamma := \alpha \oplus \beta, \tilde{\gamma} := \alpha \oplus \beta \) and \( \hat{\gamma} := \alpha \oplus \beta \). We first prove \( \gamma = \tilde{\gamma} \) by induction on \( \alpha \beta \). Since the case \( \alpha = 0 \) or \( \beta = 0 \) is trivial, we may take \( \alpha, \beta > 0 \). If \( \alpha \) is a successor ordinal, then

\[
\tilde{\gamma} = ((\alpha - 1) \oplus \beta) + 1 = ((\alpha - 1) \oplus \beta) + 1 = \alpha \oplus \beta = \gamma,
\]

where the first equality is by definition, the second by induction and the third by the finite distributivity property (26). Replacing the role of \( \alpha \) and \( \beta \), the same argument can be used to treat the case when \( \beta \) is a successor ordinal. So we may assume that both are limit ordinals. There are again two cases, namely \( \text{ord}(\alpha) \leq \text{ord}(\beta) \) and \( \text{ord}(\alpha) > \text{ord}(\beta) \). By symmetry, the argument for the second case is similar as for the first, so we will only give the details for the first case. Write \( \alpha \) as \( \alpha' + \omega^\alpha \) where \( \alpha := \text{ord}(\alpha) \). By definition, \( \tilde{\gamma} \) is the supremum of all \( \delta \oplus \beta \) with \( \delta < \alpha \). A cofinal subset of such \( \delta \) are the ones of the form \( \alpha' + \theta \) with \( 0 < \theta < \omega^\alpha \), so that \( \tilde{\gamma} \) is the supremum of all \( (\alpha' + \theta) \oplus \beta \) for \( 0 < \theta < \omega^\alpha \). By induction, \( \gamma \) is the supremum of all

(27)

\[
(\alpha' + \theta) \oplus \beta = (\alpha' \oplus \beta) + \theta,
\]

where the equality holds because \( \alpha \leq \text{ord}(\beta) \). Taking the supremum of the ordinals in (27) for \( \theta < \omega^\alpha \), we get that \( \gamma = (\alpha' \oplus \beta) + \omega^\alpha \). Using the remark following (26) one checks that this is just \( (\alpha' + \omega^\alpha) \oplus \beta = \alpha \oplus \beta = \gamma \).
The inequality $\gamma \leq \tilde{\gamma}$ is clear using the shuffle of the terms in the Cantor normal forms (25) for $\alpha$ and $\beta$. To finish the proof, we therefore need to show, by induction on $\alpha$, that

$$\alpha_1 + \beta_1 + \cdots + \alpha_n + \beta_n \leq \bar{\gamma},$$

for all decompositions $\alpha = \alpha_1 + \cdots + \alpha_n$ and $\beta = \beta_1 + \cdots + \beta_n$. Since $\oplus$ is commutative, we may assume $\text{ord}(\alpha) \leq \text{ord}(\beta)$ and, moreover, that $\alpha_n > 0$. Suppose first that $\alpha$ is a successor ordinal. In particular, $\alpha_n$ is also a successor ordinal. By definition, $\bar{\gamma} = ((\alpha - 1) \oplus \beta) + 1$. Using the decomposition $\alpha - 1 = \alpha_1 + \cdots + \alpha_{n-1} + (\alpha_n - 1)$ and induction, we get that $\alpha_1 + \beta_1 + \cdots + \beta_n + (\alpha_n - 1) \leq (\alpha - 1) \oplus \beta$. Taking successors of both ordinals then yields (28). Hence suppose $\alpha$ is a limit ordinal. Let $\theta < \alpha_n$ and apply the induction to each $\delta := \alpha_1 + \cdots + \alpha_{n-1} + \theta$, to get

$$\alpha_1 + \beta_1 + \cdots + \beta_{n-1} + \theta + \beta_n \leq \delta \oplus \beta.$$

Taking suprema of both sides then yields inequality (28).

We will denote this new sum simply by $\oplus$ and refer to it as the \textit{shuffle sum} of two ordinals, in view of its third equivalent form.

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