Signatures of pairing and spin-orbit coupling in correlation functions of Fermi gases

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We derive expressions for spin and density correlation functions in the (greatly enhanced) pseudogap phase of spin-orbit coupled Fermi superfluids. Density-density correlation functions are found to be relatively insensitive to the presence of these Rashba effects. To arrive at spin-spin correlation functions we derive new $f$-sum rules, valid even in the absence of a spin conservation law. Our spin-spin correlation functions are shown to be fully consistent with these $f$-sum rules. Importantly, they provide a clear signature of the Rashba band-structure and separately help to establish the presence of a pseudogap.

**Introduction.**—Spin-orbit coupling (SOC) in superconductors and superfluids is a topic of much current interest [1–3]. This is in large part because there is some hope that (particularly in the presence of a magnetic field) they may relate to the much sought after spinless $p_x + ip_y$ superfluid [4]. Two communities have united around these issues: those working on cold Fermi superfluids with intrinsic Rashba SOC [5, 6] and those studying superconductivity that is proximity induced in a spin-orbit coupled material [7]. To achieve this ultimate goal it is important to establish that a given candidate for the $p_x + ip_y$ superfluid simultaneously exhibits signatures of both pairing and spin orbit coupling. This would provide minimal evidence for a properly engineered ultracold atomic gas. One therefore needs experimental signatures of these simultaneous effects and this provides a central goal for the present paper.

Here we address the signatures of this anomalous spin-orbit coupled superfluid as reflected in spin-spin and density-density correlation functions. Our work builds on the observation [8–13] that in the presence of Rashba SOC, pairing (in the form of pseudogap effects [14, 15]) is significantly enhanced. For this reason (and because the correlation functions are free of the complications of collective mode effects) we focus here on the normal phase. We show how even without condensation, the frequency dependent spin response exhibits features which relate to the Rashba ring band-structure, as well as to the presence of a pairing gap. In contrast, the density response is relatively unaffected by SOC. Previous work has focused on identifying SOC without [16, 17] or with pairing [18] via the one body spectral function. As compared with Ref. [18], we find less subtle features in the two particle response. At the very least the spin correlation functions provide complementary and accessible (via neutrons or two photon Bragg scattering [19]) information.

Validating any theory of correlation functions requires satisfying important constraints [20]. Indeed, the absence of conservation laws complicates all spin transport in spin-orbit coupled materials. Thus, it is extremely important to find underlying principles for establishing self consistency. To address this issue, here we use the Heisenberg equations of motion to derive $f$-sum rules for the spin-spin correlation functions, which also provides important constraints on our numerical calculations. While a magnetic field is necessary for arriving at topological order, we begin by ignoring this additional complication. There is a substantial literature investigating correlation functions in the superfluid phase (without pseudogap effects) which we note here [21–24].

In this Rapid Communication we present two main results. The first is a consistent derivation of spin-spin correlation functions in spin-orbit coupled Fermi gases. The second establishes qualitative experimental signatures reflecting separately the presence of a pairing gap and of SOC. Readers interested in the experimental signatures need only a cursory exploration of the mathematical derivation that precedes it.

**Background Theory.**—We consider a gas of fermions whose single particle Hamiltonian is $H^0(k) = \hbar^2/2m - \mu + \lambda \sigma \cdot \mathbf{k}_\perp/m$ for a particle of mass $m$, spin-orbit coupling momentum $\lambda$, momentum $\mathbf{k} = (k_x, k_y, k_z)$, in-plane
momentum $k_{\perp} = (k_x, k_y, 0)$, and vector of Pauli matrices $\sigma = (\sigma_x, \sigma_y, \sigma_z)$. Throughout this paper we set $\hbar = k_B = 1$. To describe our spin-orbit coupled Fermi gas with pairing, we use a $4 \times 4$ inverse Nambu Green’s function
\[
G^{-1}(K) = \left( \frac{G_0^{-1}(K)}{\Delta}, \frac{\Delta}{G_0^{-1}(K)} \right),
\]
that acts on the spinor $\Psi_k^T = \left( c_{k}, c_{\perp k}, -c_{-k}, c_{-\perp k}^\dagger \right)$ for a fermion annihilation (creation) operator $c_{k}\sigma_k^{\dagger}\sigma_k$ of spin $s = \uparrow, \downarrow$ and momentum $k$. In the Green’s function $\Delta$ is a pairing gap, the 4-vector $G_M$ Matsubara frequency of spin $a$ that is obtained from Eq. (2) leads to a modified phase [27] must be modified to include a well developed pairing gap (pg) effects, in a fashion fully consistent with both the ground state and the mean field equations that have been extensively studied in previous work [8–13]. The latter is the temperature at which the pairing gap, $T^∗$, in the absence (a) and presence (b) of Rashba SOC. The latter is in reasonable agreement with the results of Ref. [26]. We restrict these plots to the weak pairing side of resonance. Here we observe a greatly enhanced pseudogap regime denoted by an enhancement of $T^*$ without significant enhancement of $T_c$. The behavior of $T^*$ has been attributed to the enhancement of the pairing attraction [8–13], due to an increased density of states near the minimum of the Rashba ring. Since $T_c$ is obtained in the presence of a gap at $T_c$, stronger pairing (reflected in $T^*$) is offset by an increasingly gapped density of states. This leads to a relatively constant $T_c$ as a function of interaction strength.

Density/current and f-sum rules.— To characterize the anomalous, normal, and superfluid phases in more detail,
we investigate both the density/current and spin correlation functions, considering the former first. For systems with a $U(1)$ symmetry, the Ward-Takahashi identity (WTI) provides an important constraint on the full vertex $\Gamma^\mu(K, K)$ which enters into the correlation functions. Given a mean field like self energy, it is possible to analytically solve the WTI, and obtain the full vertex function along with the full correlation function [20, 27].

We define the generalized correlation function

$$P^{\mu\nu}(Q) = \sum_K \text{Tr} \left[ G(K) \Gamma^\mu(K, K) G(K) \gamma^\nu(K, K) \right], \quad (6)$$

where $K = K + Q$ and $\gamma^\nu(K, K)$ is a bare vertex. From this we have the density-density $\chi_{\rho\rho}(Q) = P^{00}(Q)$ and current-current correlation functions $\chi_{\sigma j j}(Q) = P^{\sigma j j}(Q), i, j \in \{1, 2, 3\}$. The bare and full vertices satisfy respectively

$$q_\rho \gamma^\rho(K, K) = G_0^{-1}(K) - G_0^{-1}(K), \quad (7)$$

$$q_\mu \Gamma^\mu(K, K) = G^{-1}(K) - G^{-1}(K), \quad (8)$$

with the latter a consequence of the WTI. We now specialize to systems with the self energy as in Eq. (3). Using the WTI above $T_c$, we have

$$\Gamma^\mu(K, K) = \gamma^\mu(K, K) + \Delta^2 G_0(K) \bar{\gamma}^\mu(K, K) G_0(K), \quad (9)$$

where $\bar{\gamma}^\mu(K, K) = \sigma_\rho \gamma^\rho(-K, -K)^T \sigma_\rho$ is a time-reversed vertex. Inserting the full vertex into Eq. (6) then gives the correlation functions above $T_c$.

One can incorporate superconducting (or equivalently superfluid) terms within this formalism building on Eq. (4) and, for example, address the superfluid density [29], as outlined in the supplement. One considers the transverse response $P_T^{\mu\nu}(Q)$ which contains no collective modes:

$$P_T^{\mu\nu}(Q) = \sum_K \text{Tr} \left[ G(K) \gamma^\mu(K, K) G(K) \right. $$

$$+ F_{\rho\rho}(K) \bar{\gamma}^\rho(K, K) \bar{F}_{\rho\rho}(K) $$

$$- F_{\mu\nu}(K) \gamma^\nu(K, K) \bar{F}_{\mu\nu}(K) \right] \gamma^\nu(K, K), \quad (10)$$

where $F_m(K) = \Delta_m G_0(K) G(K) = \bar{F}_m(K)$ for $m \in \{\rho, \rho\}$. Note that $F_{\rho\rho}$ does not represent an anomalous Green’s function, but rather reflects a vertex correction to the correlation functions [20]. There is a disagreement in the literature [18-21] as to the importance of collective modes in the spin response below $T_c$. We agree with the results of Ref. [21], where collective modes were included.

As shown in the supplementary material, when one integrates over the entire frequency range, a consequence of the WTI is that the f-sum rule is satisfied:

$$\int \frac{d\omega}{\pi} (-\omega \chi''_{\rho\rho}(\omega, \bm{q})) = \frac{nq^2}{m}, \quad (11)$$

where $\chi''_{\rho\rho}$ is the imaginary part of the density response function. This f-sum rule depends on the total particle number $n$ and the bare mass $m$. Since $\lambda$ does not enter, the presence of spin-orbit coupling does not modify the weight of the f-sum rule.

**Spin response and f-sum rules.** In the spin channel, where there is no $U(1)$ symmetry to justify the use of the WTI. Nevertheless, we are able to provide an a posteriori check on any proposed correlation function via a sum rule which we now derive. We define $\chi_{S, S'}(\omega, \bm{q}) = \sum_{k s} \epsilon_{ks} c_{ks'}^\dagger (\sigma_s)_{ss'} c_{k+\bm{q}s'}$ is the many-body spin density operator. Using the Heisenberg
equations of motion and the properties of Fourier transforms, the sum rule for the spin-spin correlation function $\chi''_{S,S}$ can be shown to be

$$\int \frac{d\omega}{\pi} \left(-\omega \chi''_{S,S}(\omega, q)\right) = \left\langle \left[ H_0, S_{\mathbf{q}} \right], S_{-\mathbf{q}} \right\rangle, \quad (12)$$

where $H_0 = \sum_{s,s'} \eta_{s,s'}^\dagger \eta_{s,s'}(\mathbf{k}) \eta_{s,s'}^\dagger \eta_{s,s'}$, and $\chi''_{S,S}$ is the singular part of $\chi_{S,S}$ found by analytically continuing $\omega \rightarrow \omega + i\delta$ and then taking the $\delta \rightarrow 0$ limit.

Here we give the explicit result, for two example cases of interest and present further details in the supplementary material:

$$\int \frac{d\omega}{\pi} \left(-\omega \chi''_{S,S}(\omega, q)\right) = \frac{nq^2}{m} - \frac{4\lambda}{m} \sum_{\mathbf{k}_{\alpha}} f_{\mathbf{k}_{\alpha}} n_{\mathbf{k}_{\alpha}}, \quad (13)$$

where $i \in \{x, z\}$, $f_{zz} = k_\perp$, $f_{xx} = k_\perp^2/k_\perp$, and $n_{\mathbf{k}_{\alpha}} = T \sum_n G_{K}^\alpha(K_x, K_y, K_z)$ with $G_{K}^\alpha(K)$ a helicity Green’s functions to be defined in the next section.

**Correlation functions in the helicity basis.** In the absence of a magnetic field, helicity is a good quantum number, and the correlation functions are most easily expressed in terms of the helicity Green’s functions [20]:

$$G_{H}^\alpha(K_x, K_y, K_z) = \frac{u_{k_{\alpha}}^2}{\omega - E_{\mathbf{k}_{\alpha}}} + \frac{v_{k_{\alpha}}^2}{\omega + E_{\mathbf{k}_{\alpha}}}, \quad (14)$$

$$F_{H}^\alpha(K_x, K_y, K_z) = \frac{u_{k_{\alpha}} v_{k_{\alpha}}}{\omega + E_{\mathbf{k}_{\alpha}}} - \frac{1}{\omega - E_{\mathbf{k}_{\alpha}}}, \quad (15)$$

where $E_{\mathbf{k}_{\alpha}} = \sqrt{k_{\alpha}^2 + \Delta^2}$, $\xi_{\mathbf{k}_{\alpha}} = k^2/2m + \mu + \alpha \lambda k_\perp/m$ is an eigenvalue of $H_{H}(\mathbf{k})$, and $F_{H}^\alpha(K)$ represents the pseudogap, or equivalently vertex contribution. Here $\alpha = \pm$ denotes the helicity index and the coherence factors satisfy $u_{k_{\alpha}}^2 = \frac{1}{2}(1 + \xi_{\mathbf{k}_{\alpha}}/E_{\mathbf{k}_{\alpha}})$, $u_{k_{\alpha}}^2 + v_{k_{\alpha}}^2 = 1$.

It follows from the vertex function in Eq. [9] that the explicit form for the $f$-sum rule [20] consistent density-density correlation function is

$$\chi_{\mu\nu}(\omega, q) = \frac{1}{2} \sum_{\mathbf{k}, \alpha, \alpha'} \left(1 + \alpha \alpha' \cos(\phi_{\mathbf{k}_{\alpha} + \mathbf{q}} - \phi_{\mathbf{k}})\right) \times \left[ G_{H}^\alpha(K_x, K_y, K_z) \right. \left. + F_{H}^\alpha(K_x, K_y, K_z) \right]. \quad (16)$$

The angle $\exp(i\phi_{\mathbf{k}}) = (k_x + ik_y)/k_\perp$, so that $\exp(i\phi_{-\mathbf{k}}) = -\exp(i\phi_{\mathbf{k}})$.

The spin-spin correlation functions are constructed using their form below $T_c$ (deduced using the path integral [21]) with appropriate sign changes in the pseudogap relative to the condensate gap. These sign changes, which appear in the $T < T_c$ Ward-Takahashi identity [27] are essential for satisfying sum rules.

As can be shown, in the normal phase the following expression for the spin-spin correlation functions are fully compatible with the spin $f$-sum rules given in Eq. [13]:

$$\chi_{S,S}(\omega, q) = \frac{1}{2} \sum_{\mathbf{k}, \alpha, \alpha'} \left(1 \pm \alpha \alpha' \cos(\phi_{\mathbf{k}_{\alpha} + \mathbf{q}} \pm \phi_{\mathbf{k}})\right) \times \left[ G_{H}^\alpha(K_x, K_y, K_z) \right. \left. + F_{H}^\alpha(K_x, K_y, K_z) \right]. \quad (17)$$

where the $+, -$ signs are for $\chi_{S,S}$, $\chi_{S,S}$ respectively.

**Numerical Results.** We now look for qualitative new physics in the spin-spin response functions. We numerically calculate the response function $\chi''_{S,S}(q, \omega)$ at fixed $q = (0.5, 0, 0)k_F$ as a function of $\omega$ [31], and for definiteness consider $T = 0.9T_F > T_c$ and unitary scattering, $1/k_F a = 0$. We plot the results in Fig. [2]. In order to illustrate the physics, in Fig. [2(a) and Fig. [2(b)], Rashba SOC or pseudogap effects were set to zero respectively, while Fig. [2(c)] shows their combined effects. The $f$-sum rules derived above are important for constraining numerical results of the spin-spin and density-density correlation functions. Comparison between our numerical calculations and the exact $f$-sum rules agreed to within a few percent.

In Fig. [2(a)] we set $\lambda = 0$. In this case, the spin and density correlations are equal ($\chi''_{S,S} = \chi''_{\mu\nu}$) and this function is plotted in the figure. Two low energy peaks are observable, as found in our earlier work [32]. The lower frequency peak reflects contributions from thermally excited fermions, while the higher frequency peak is associated with the contribution from broken pairs which appears at a threshold associated with the pseudogap.

In Fig. [2(b)] we set $\Delta_{pg} = 0$ and plot $\chi''_{S,S}$ for a pure SOC system with $\lambda = 1.2k_F$. (We do not show $\chi''_{\mu\nu}$ since there is still no qualitative signature of $\lambda \neq 0$.) The response $\chi''_{S,S}$ shows two peaks, but one is at a considerably higher energy compared to Fig. [2(a)]. The lower frequency peak reflects intra-helicity band contributions while the larger frequency peak is due to inter-helicity effects.

Importantly, this figure shows how the physics of the Rashba ring band-structure can be directly probed by the spin-spin response function. To illustrate this, in the inset we plot the dispersion relation of two helicity bands. The horizontal line denotes the self-consistently determined chemical potential, chosen so that occupied fermions mostly reside in the Rashba ring. The onset of the inter-helicity band transition energy is given by the energy difference between two bands positioned on the inner circle of the ring, while the endpoint frequency for this peak is determined by the outer circle. These energy differences roughly match the width observed in the high frequency peak in the main plot. (The smearing of the width is because we have a non-zero momentum $\mathbf{q}$ and $T \neq 0$.)

Finally, in Fig. [2(c)] we plot $\chi''_{S,S}$ for the case where both pseudogap and Rashba SOC are present. Here we observe three distinct peaks. The first is associated with thermally excited fermions within the lowest helicity band, the second with the breaking of the preformed (pg) pairs and the third mainly with the inter-helicity transitions discussed in the previous panel. We also observe some inter-play between pairing and the high frequency SOC peak, as this inter-helicity band peak is pushed toward slightly higher energies.
Conclusion.— A major finding of this paper is that spin-spin correlations provide a clear signature of the simultaneous presence of Rashba modified band-structure and of a pairing gap. Signatures of both are a necessary (but clearly not sufficient) condition for ultimately obtaining a topological superfluid. This should complement observations which are based on the single particle response functions in different experiments either in cold gases [15][18] or in condensed matter. Our spin correlation functions are consistent with sum rules which we derive in this paper. These provide important constraints on the spin response which is complicated by the fact that spin conservation laws are unavailable for spin-orbit coupled systems.

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Supplementary Material: Signatures of pairing and spin-orbit coupling in correlation functions of Fermi gases

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I. GREEN’S FUNCTIONS AND SELF ENERGY IN THE HELICITY BASIS

In evaluating the correlation functions, it is convenient to have the Green’s functions written in the helicity basis. Here we present a brief discussion of the simple expressions used in the main text:

\[ G_{H}^\alpha(K) = \sum_\alpha G_{\alpha} H(K) P_{\alpha}, \quad F_{H}^\alpha(K) = \sum_\alpha F_{\alpha} H(K) P_{\alpha}. \]

where the Bogoliubov spectrum \( E_{K\alpha} = \sqrt{\varepsilon_{K\alpha}^2 + \Delta^2} \) with corresponding coherence factors which satisfy \( u_{K\alpha}^2 = \frac{1}{2}(1 + \xi_{K\alpha}/E_{K\alpha}), \) \( u_{K\alpha}^2 + v_{K\alpha}^2 = 1 \). Note that the helicity Green’s functions have the standard form, aside from the index \( \alpha \).

We define the Nambu spinor \( \Psi_k^+ = (\alpha_{k\uparrow}, \alpha_{k\downarrow}, -c_{k\downarrow}^\dagger, c_{k\uparrow}^\dagger) \) of annihilation and creation operators. The many-body Nambu Green’s functions are then defined as

\[ G(K) = -\int_0^\beta d\tau e^{i\omega \tau} \langle T_\tau \Psi_k^\dagger(\tau) \Psi_k^0(0) \rangle = \left( \begin{array}{cc} G(1) & F(1) \\ F(\bar{K}) & G(\bar{K}) \end{array} \right), \]

where \( \beta = 1/T \) is the inverse temperature. The Green’s function \( G(1) \), time-reversed Green’s function \( \tilde{G}(1) = i\sigma_y[G(1)]^T i\sigma_y \), and the anomalous Green’s functions \( F(1) \) and \( \tilde{F}(1) \) are now in general \( 2 \times 2 \) matrices.

Using the equations of motion for the Nambu Green’s function in the pairing approximation we have

\[ G(0) = \left( \begin{array}{cc} G_0^{-1}(1) & \Delta \\ \Delta^\ast & G_0^{-1}(1) \end{array} \right), \]

(4)

For the remainder of this supplement, as well as in the main text, we chose \( \Delta \) to be real. The non-interacting Green’s function of the particle sector is \( G_0^{-1}(1) = i\omega - H^0(k) \), and the hole Green’s function is defined through \( \tilde{G}_0^{-1}(1) = i\sigma_y[G_0^{-1}(1)]^T i\sigma_y = i\omega + H^0(k) \). This non-interacting Greens function depends on the single particle Hamiltonian through \( H^0(k) = \hbar^2/2m - \mu + k\sigma \cdot \mathbf{\sigma}/m \) where \( \sigma = (\sigma_x, \sigma_y, \sigma_z) \) is a vector of Pauli matrices in spin space. We consider a 2D Rashba SOC, so that \( \sigma \) is coupled only to the in plane momentum \( k_\perp = (k_x, k_y, 0) \). Taking the block matrix inverse gives the following \( 2 \times 2 \) matrix Green’s functions:

\[ G(1) = \left( G_0^{-1}(1) + \Delta^2 \tilde{G}_0(1) \right)^{-1}, \]
\[ F(1) = \Delta G_0(1) \tilde{G}(1). \]

(5)

(6)

In the pairing approximation, the structure of the block matrix inverse forces \( \tilde{F}(1) = \Delta^\ast / \Delta F(1) = F(1) \).

For pure SOC (i.e., no magnetic field), the system is time-reversal invariant and helicity is a good quantum number. Therefore, the hole Green’s function and the particle Green’s function can be diagonalized simultaneously with the same unitary transformation: \( U_k = \exp \left[ -i\phi_k \pi/4 \left( \hat{k}_\perp \times \hat{z} \right) \cdot \mathbf{\sigma} \right] \) with \( \hat{k}_\perp = k_\perp/k_\perp \) and \( e^{2i\phi_k} = (k_x + ik_y)/k_\perp \). This unitary transformation will then diagonalize the non-interacting Green’s function \( G_0(K) \), and leads to the identity:

\[ G_0(K) = \sum_\alpha \frac{1}{i\omega - \xi_{K\alpha}} P_{K\alpha}, \]

(7)

where \( P_{K\alpha} = U_k(1 + \alpha \sigma_z)U_k^\dagger/2 = (1 + \alpha k_\perp \sigma_\perp /k_\perp /2) \) is a projector into the helicity state \( \alpha = \pm 1 \). We can now express the \( G(K) \) and \( F(K) \) in terms of the projector \( P_{K\alpha} \) and the helicity Green’s functions as Eq. (1) and Eq. (2) through:

\[ G(K) = \sum_\alpha G_{H}^\alpha(K) P_{K\alpha}, \]
\[ F(K) = \sum_\alpha F_{H}^\alpha(K) P_{K\alpha}. \]

(8)

(9)
II. THE WARD-Takahashi Identity

A. Deriving the full vertex

The Ward-Takahashi identity (WTI) in quantum field theory is a diagrammatic identity that imposes a symmetry between response functions. The particular symmetry we are interested in is the \( U(1) \) abelian gauge symmetry that is present in the density/current channel. Since the pseudogap phase self energy is mean field, the WTI provides a powerful tool in deriving the full vertex that appears in correlation functions. This is because the WTI can be explicitly used to solve for the full vertex function, and thereby obtain the exact correlation function.

We confine our attention here to the normal phase, as the density-density correlation function has collective mode effects which are complicated below \( T_c \).

The WTI for the full vertex \( \Gamma^\mu(\vec{K}, K) \) is

\[
q_\mu \Gamma^\mu(\vec{K}, K) = G^{-1}(\vec{K}) - G^{-1}(K). 
\]

Similarly the WTI for the bare vertex \( \gamma^\mu(\vec{K}, K) \) is \( q_\mu \gamma^\mu(\vec{K}, K) = G^{-1}_0(\vec{K}) - G^{-1}_0(K) \) and we also define \( q_\mu \tilde{\gamma}^\mu(\vec{K}, K) = \tilde{G}^{-1}_0(\vec{K}) - \tilde{G}^{-1}_0(K) \). Here \( \vec{K} \equiv K + Q \). In the pseudogap phase the mean field like self energy is \( \Sigma(K) = \Delta^2 \tilde{G}_0(K) \). Using this self energy, the WTI becomes

\[
q_\mu \Gamma^\mu(\vec{K}, K) = G^{-1}_0(\vec{K}) - G^{-1}_0(K) - \Sigma(\vec{K}) + \Sigma(K) \\
= G^{-1}_0(\vec{K}) - G^{-1}_0(K) - \Delta^2 \left( \tilde{G}_0(\vec{K}) - \tilde{G}_0(K) \right) \\
= q_\mu \gamma^\mu(\vec{K}, K) - \Delta^2 \tilde{G}_0(\vec{K}) \left( \tilde{G}^{-1}_0(\vec{K}) - \tilde{G}^{-1}_0(K) \right) \tilde{G}_0(K). \\
= q_\mu \gamma^\mu(\vec{K}, K) - \Delta^2 \tilde{G}_0(\vec{K}) \left( -q_\mu \tilde{\gamma}^\mu(\vec{K}, K) \right) \tilde{G}_0(K). 
\]

It follows that the full vertex is then

\[
\Gamma^\mu(\vec{K}, K) = \gamma^\mu(\vec{K}, K) + \Delta^2 \tilde{G}_0(\vec{K}) \tilde{\gamma}^\mu(\vec{K}, K) \tilde{G}_0(K). 
\]

It is important to note that, this expression is exact, and is not a perturbation expansion in terms of bare vertices.

B. \( f \) and longitudinal sum rules

We now show that, given the exact correlation function obtained by using the WTI, the \( f \)-sum rule is explicitly satisfied. This ensures our correlation functions are consistent with charge conservation.

The correlation function is given by

\[
P^{\mu\nu}(Q) = \sum_K \text{Tr} \left[ G(\vec{K}) \Gamma^\mu(\vec{K}, K) G(K) \gamma^\nu(K, \vec{K}) \right]. 
\]

Inserting the full vertex from Eq. (11) gives the correlation function above \( T_c \) as:

\[
P^{\mu\nu}(Q) = \sum_K \text{Tr} \left\{ [G(\vec{K}) \gamma^\mu(\vec{K}, K) G(K) + F_{pg}(\vec{K}) \tilde{\gamma}^\mu(\vec{K}, K) \tilde{F}_{pg}(K)] \gamma^\nu(K, \vec{K}) \right\}. 
\]

where \( F_{pg}(K) = \Delta_{pg} G_0(K) \tilde{G}(K), \tilde{F}_{pg}(K) = \Delta_{pg} \tilde{G}_0(K) G(K) = F_{pg}(K) \).

Applying the WTI to \( P^{\mu\nu}(Q) \), and using the cyclic property of the trace, then gives

\[
\Omega P^{\mu\nu}(Q) - q \cdot P^{\mu\nu}(Q) = \sum_K \text{Tr} \left\{ G(K) [\gamma^\nu(K, K + Q) - \gamma^\nu(K, K - Q)] \right\}. 
\]

For the SOC system the bare vertex is \( \gamma^\mu(\vec{K}, K) = \left( 1, \frac{k_+ \gamma}{m} + \frac{\Delta}{m} \sigma_\perp \right) \). Using this yields

\[
\Omega P^{\mu\nu}(Q) - q \cdot P^{\mu\nu}(Q) = \frac{q^\nu}{m} (1 - \delta_{0, \nu}) \sum_K \text{Tr} \{ G(K) \} = \frac{m}{m} (1 - \delta_{0, \nu}). 
\]
Here we have used $n = \sum_K \text{Tr} \{ G(K) \}$. In components, this equation becomes
\begin{align}
\omega P^{00}(\omega, q) - q \cdot P^{i0}(\omega, q) &= 0, \\
\omega P^{ij}(\omega, q) - q \cdot \tilde{P}^{ij}(\omega, q) &= \frac{nq^2}{m}.
\end{align}
Setting $\omega = 0$ in the second equation and then taking the dot product with $q$ gives
\[ q \cdot \tilde{P}^{ij}(0, q) \cdot q = -\frac{nq^2}{m}. \tag{18} \]
Now use the identity $\text{Im} P^{0i}(\omega, q) = -\text{Im} P^{0i}(\omega, -q)$ and solve for $\text{Im} P^{00}$ in terms of $\text{Im} \tilde{P}^{ij}$. Applying the Kramers-Kronig relations and Eq. (18) then gives
\[ \int \frac{d\omega}{\pi} \left( -\omega \text{Im} P^{00}(\omega, q) \right) = \int \frac{d\omega}{\pi} \left( - \frac{q \cdot \text{Im} \tilde{P}^{ij}(\omega, q) \cdot q}{\omega} \right) = -q \cdot \text{Re} \tilde{P}^{ij}(0, q) \cdot q = \frac{nq^2}{m}. \tag{19} \]
Defining $\chi_{\rho\rho}(Q) \equiv P^{00}(Q), \tilde{\chi}_{JJ}(Q) \equiv P^{ij}(Q), i, j \in \{1, 2, 3\}$, the $f$-sum rule, which holds for all $q$, is thus
\[ \int \frac{d\omega}{\pi} \left( -\omega \chi''_{\rho\rho}(\omega, q) \right) = \frac{nq^2}{m}. \tag{20} \]
Here the singular part of the correlation function, $\chi''_{\rho\rho}(\omega, q)$, is equal to $\text{Im} \chi_{\rho\rho}(\omega, q)$, which is true only in the density/current channel. The next section on the spin-spin correlation functions elaborates further on this point. Similarly the longitudinal sum rule, which holds for all $q$ in this continuum case, and above $T_c$, is thus
\[ \int \frac{d\omega}{\pi} \left( - \frac{q \cdot \tilde{\chi}_{JJ}(\omega, q) \cdot q}{\omega} \right) = \frac{nq^2}{m}. \tag{21} \]

### III. SPIN-SPIN CORRELATION FUNCTIONS AND SUM RULES

#### A. General sum rules for spin density-spin density correlation functions

Here we investigate both the spin-spin correlation functions and sum rules. In the presence of SOC, spin is no longer conserved. The WTI is therefore inapplicable to deduce the form of the spin-spin correlation functions. However, the Heisenberg equations of motion can be utilized to derive a sum rule for spin density-spin density correlation functions.

We start with the definition for the spin-spin correlation function in the main text, $\chi_{S_iS_j}(i\omega, q) = \langle \langle S_{qi}(t), S_{-qj}(0) \rangle \rangle$, for a many-body spin operator $S_{qi} = \sum_{s\sigma} c_{ks}^\dagger (\sigma_i)_{s\sigma}, c_{k+q\sigma'}$. We will consider a general many-body Hamiltonian $H = H_0 + H_I$, where the contribution $H_0 = \sum_{ss'kk} c_{ks}^\dagger H_{ss'}^{0}(k)c_{kc'}$ is constructed from an arbitrary single particle Hamiltonian $H_{ss'}^{0}(k)$. We also assume that the interaction Hamiltonian commutes with the spin-operator, $[H_I, S_{qi}] = 0$.

In order to constrain the spin-spin correlation function, we investigate the analogous sum rule for $\chi''_{S_iS_j}(\omega, q)$ that is derived in the previous section for the density/current response. To do this, we first analytically continue $\chi_{S_iS_j}(i\omega, q)$ to real frequency by setting $i\omega = \omega + i\delta$, then letting $\delta \to 0$, and finally invoking the Sokhotski-Plemelj theorem. This is expressed mathematically as
\[ \chi_{S_iS_j}(\omega, q) \equiv \lim_{\delta \to 0} \chi_{S_iS_j}(i\omega + i\delta) = \chi''_{S_iS_j}(\omega, q), \tag{22} \]
Here $\mathcal{P}$ denotes the Cauchy principal value. Note that, for $i \neq j$, $\chi''_{S_iS_j}(\omega, q)$ need not necessarily be real. Returning to the time domain, the singular response function is $\chi''_{S_iS_j}(t, q) = \frac{1}{2} \langle \langle S_{qi}(t), S_{-qj}(0) \rangle \rangle$. 

We now can derive the sum rule. Using the properties of Fourier transforms, we have

\[
\int \frac{d\omega}{\pi} \left( -\omega \chi''_{S_i,S_j}(\omega, q) \right) = -2i \partial_t \chi''_{S_i,S_j}(t, q) \big|_{t=0},
\]

where we have used the Heisenberg equations of motion \(i \partial_t S_{q\alpha}(t) = -[\mathcal{H}, S_{q\alpha}(t)]\) and the commutators on the last line are taken at equal time. The problem thus reduces to calculating the equal time commutator \(\langle [\mathcal{H}, S_{q\alpha}], S_{-q\alpha} \rangle \rangle = [\mathcal{H}_0, S_{q\alpha}]\).

The commutator in Eq. (23) is tedious but straightforward to calculate. After much algebra we arrive at the \(f\)-sum rule, or analogue thereof, for spin density-spin density response:

\[
\int \frac{d\omega}{\pi} \left( -\omega \chi''_{S_i,S_j}(\omega, q) \right) = \frac{1}{2} \sum_{kss'} \left\{ \{\sigma_i, H^0(k) - H^0(k + q)\} \sigma_j - \sigma_j \{\sigma_i, H^0(k - q) - H^0(k)\} \right. \\
\left. + \left[ H^0(k) + H^0(k + q)\right] \sigma_i \right\} \sigma_j - \sigma_j \left[ H^0(k - q) + H^0(k)\right] \sigma_i \right\} \delta_{ss'} \left( \epsilon^+_{ks} \epsilon_{ks'} \right),
\]

This expression can be further simplified as follows. The Hamiltonian can be expressed as \(H^0(k) = h_{\mu}(k) \sigma^\mu\) where \(h_{\mu}(k) = \frac{1}{2} \text{Tr} \left[ \sigma_{\mu} H^0(k) \right]\) and \(\sigma^\mu = (1, \sigma)\). After using the commutation and anti-commutation rules for Pauli matrices, further algebra reduces the sum rule to the form

\[
\int \frac{d\omega}{\pi} \left( -\omega \chi''_{S_i,S_j}(\omega, q) \right) = \sum_k \left\{ \text{Tr} \left[ \left( H^0(k + q) + H^0(k - q) - 2 H^0(k) \right) g_k \right] \delta_{ij} \\
- 2 \left( (h(k + q) + h(k - q)) \cdot s_k \delta_{ij} + (h_i(k + q) + h_i(k - q)) s_{ij} \right) \\
- i \left( [h_k(k + q) - h_k(k - q)] n_k - (h_0(k + q) - h_0(k - q)) s_{kk} \right) \right\} \epsilon_{ijk},
\]

where we have defined \(g_k = T \sum_{k'J} G(i\omega, k), n_k = \text{Tr} [g_k]\) and \(s_{kk} = \text{Tr} [\sigma_i g_k]\). The first line, appearing only for \(i = j\), can be seen to be equal to the density \(f\)-sum rule contribution. The second line can give diagonal contributions, resulting in a deviation of the spin density response from the density response in the presence of SOC. The third line vanishes for \(i = j\). It is of note that this line is purely imaginary and can give an imaginary \(f\)-sum contribution. This is only possible for correlation functions of two distinct operators, as was noted in the literature\(^3\). This implies that \(\chi''_{S_i,S_j} \neq \text{Im} \chi_{S_i,S_j}\) for \(i \neq j\). Finally, the first and third lines of Eq. (25) are symmetric under interchange of \(i \leftrightarrow j\) and \(q \leftrightarrow -q\), while the second line is not. This can be explained by observing that Eq. (23) is not fully symmetric under the interchange of \(i \leftrightarrow j\) and \(q \leftrightarrow -q\).

B. Explicit spin-spin correlation functions and sum rules for Rashba SOC

We now present simplified expressions for the correlation functions and sum rules for the case of Rashba SOC. Due to symmetry in the \(x-y\) plane, there are only four unique correlation functions of interest. The remaining spin density correlation functions can be found through exchanging \(x \rightarrow y\) where relevant. The explicit spin-spin correlation functions are

\[
\chi_{S_z,S_z}(\omega, q) = \frac{1}{2} \sum_{K,k,\alpha,\alpha'} \left( 1 + \alpha \alpha' \cos (\phi_{k+q} + \phi_k) \right) \left[ G''_{\alpha\alpha'}(K) G''_{\alpha'\alpha}(\bar{K}) + F''_{\alpha\alpha'}(K) F''_{\alpha'\alpha}(\bar{K}) \right],
\]

\[
\chi_{S_x,S_x}(\omega, q) = \frac{i}{2} \sum_{K,k,\alpha,\alpha'} \left( \alpha \alpha' \sin (\phi_{k+q} - \sin(\phi_k)) \right) \left[ G''_{\alpha\alpha'}(K) G''_{\alpha'\alpha}(\bar{K}) + F''_{\alpha\alpha'}(K) F''_{\alpha'\alpha}(\bar{K}) \right],
\]

\[
\chi_{S_x,S_y}(\omega, q) = \frac{1}{2} \sum_{K,k,\alpha,\alpha'} \left( 1 - \alpha \alpha' \cos (\phi_{k+q} - \phi_k) \right) \left[ G''_{\alpha\alpha'}(K) G''_{\alpha'\alpha}(\bar{K}) + F''_{\alpha\alpha'}(K) F''_{\alpha'\alpha}(\bar{K}) \right],
\]

\[
\chi_{S_z,S_y}(\omega, q) = 0.
\]

Note that the response \(\chi_{S_z,S_z}\) vanishes identically, as it must due to in-plane rotational symmetry.
For each of the above correlation functions, we have explicitly integrated the left hand side of the sum rule, Eq. (25), and found:

\[
\int \frac{d\omega}{\pi} (-\omega \chi''_{S_x S_x} (\omega, \q)) = \frac{nq^2}{m} - \frac{4\lambda}{m} \sum_{k\alpha} \frac{k_x^2}{k_{\perp}} n_{k\alpha},
\]

(30)

\[
\int \frac{d\omega}{\pi} (-\omega \chi''_{S_y S_y} (\omega, \q)) = -iqy \frac{4\lambda}{m} \sum_{k\alpha} \left( \frac{k_y^2}{k_{\perp}} - \lambda \right) n_{k\alpha},
\]

(31)

\[
\int \frac{d\omega}{\pi} (-\omega \chi''_{S_z S_z} (\omega, \q)) = \frac{nq^2}{m} - \frac{4\lambda}{m} \sum_{k\alpha} \alpha k_{\perp} n_{k\alpha},
\]

(32)

where \( n_{k\alpha} = \text{Tr} [p_{\alpha} g_K] = T \sum_{\omega \epsilon} G_{\alpha}^\dagger (K) \). These results are consistent with the form using the right hand side of Eq. (25). Note further that the sum rule for \( \chi_{S_x S_x} \) is purely imaginary. This is not surprising, as mentioned above.

IV. SUPERFLUID DENSITY

The formalism underlying this paper is capable of addressing the superfluid phase. However, many response functions in this phase require the inclusion of collective mode effects. The superfluid density, which can be obtained using a transverse response, does not require collective modes. The superfluid density \( \left( \frac{\nabla}{m} \right) \) is defined by

\[
\left( \frac{\nabla}{m} \right) = \left( \frac{\nabla}{m} \right)_{\text{dia}} - \frac{\chi'_{\perp}}{\chi_{JJ}(0)},
\]

(33)

where \( \chi_{\perp} (Q) \) is the transverse part of the current-current correlation function.

This contains two contributions from: (i) quasi-particles which are reflected in the current-current correlation function and (ii) the diamagnetic current which above \( T_c \) must precisely cancel the quasi-particle term. Below \( T_c \) the current-current correlation function acquires an addition contribution which depends on the anomalous Green's function \( F_{\text{sc}} \). Note that the diamagnetic current is dependent on the total pairing gap. The current-current correlation function depends on the function \( F_{\text{pg}} \) as defined above, and arises from vertex corrections.

Using Eq. (10) from the main paper, the current-current correlation function and diamagnetic response tensors are

\[
\chi_{JJ}(Q) = \sum_{K} \text{Tr} \left[ J(\k) G(\k) J(\k) G(\k) + J(\k) F_{\text{pg}}(\k) J(\k) \tilde{F}_{\text{pg}}(\k) - J(\k) F_{\text{sc}}(\k) J(\k) \tilde{F}_{\text{sc}}(\k) \right],
\]

(34)

\[
\left( \frac{\nabla}{m} \right)_{\text{dia}} = \sum_{K} \text{Tr} \left[ J(\k,0) G(\k) J(\k,0) G(\k) + J(\k,0) F_{\text{pg}}(\k) J(\k,0) \tilde{F}_{\text{pg}}(\k) + J(\k,0) F_{\text{sc}}(\k) J(\k,0) \tilde{F}_{\text{sc}}(\k) \right],
\]

(35)

where \( J(\k, \q) = \frac{\left[ k + \frac{\q^2}{m} + \lambda \sigma_{\perp} \right]}{m} \) is the current operator. Notice that the relative sign between the \( F_{\text{sc}} \) and \( F_{\text{pg}} \) contributions is different in \( \chi_{\perp}(Q) \) and in the diamagnetic current.

The superfluid density tensor is thus

\[
\left( \frac{\nabla}{m} \right)_{ij} = 2 \sum_{K} \text{Tr} \left[ J_i(\k,0) F_{\text{sc}}(\k) J_j(\k,0) \tilde{F}_{\text{sc}}(\k) \right].
\]

(36)

Due to symmetry, the off-diagonal \( (i \neq j) \) elements vanish. Furthermore, the SOC leads to different in-plane \( (i = j = x, y) \) and out-of-plane \( (i = j = z) \) response. Despite a significantly different form, we can show this expression for the superfluid density is consistent with that derived by adding a phase twist to the order parameter\textsuperscript{4–6}.

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