ON THE HAUSDORFF DIMENSION OF THE SPECTRUM
OF THUE-MORSE HAMILTONIAN

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ABSTRACT. We show that the Hausdorff dimension of the spectrum of
the Thue-Morse Hamiltonian has a common positive lower bound for all
coupling.

1. INTRODUCTION

Given a bounded real sequence \( v = \{v(n)\}_{n \in \mathbb{Z}} \) and a real number \( \lambda \in \mathbb{R} \),
we can define the so-called discrete Schrödinger operator \( H_{\lambda,v} \) acting on
\( \mathcal{L}^2(\mathbb{Z}) \) as

\[
(H_{\lambda,v}\psi)(n) = \psi(n + 1) + \psi(n - 1) + \lambda v(n) \psi(n), \quad \forall n \in \mathbb{Z}.
\]

Here \( \lambda \) is called the coupling constant and \( \lambda v \) is called the potential. It is
well known that \( H_{\lambda,v} \) is a self-adjoint operator and the spectrum of \( H_{\lambda,v} \) is
a compact subset of \( \mathbb{R} \), which we denote by \( \sigma(H_{\lambda,v}) \) (see for example [8]).
We concern about the size of \( \sigma(H_{\lambda,v}) \). For example whether it has positive
Lebesgue measure? If not, what is the Hausdorff dimension of it? Note
that zero Lebesgue measure spectrum implies absence of absolutely continu-
ous spectrum, and dimension of spectrum has some relation with quantum
dynamics ([21, 14]).

When \( v \) is periodic, the spectral property of \( H_{\lambda,v} \) is well understood, it is
known that \( \sigma(H_{\lambda,v}) \) is a union of finite intervals and has positive Lebesgue
measure (see for example [8]).

When \( v \) is less ordered, the situation is more complicated. Several classes
of quasi-periodic potentials are extensively studied during the previous three
decades. One famous class is the so-called almost Mathieu potential, where
\( v^{am}(n) = \cos(n\alpha + \theta) \) with \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \) and \( \theta \in \mathbb{R} \). It is known that in this case
\( \sigma(H_{\lambda,v^{am}}) = [4 - 2\lambda]([20, 17, 1]) \). Another class is the potential generated
by primitive substitution. It is shown in [6, 23, 22] that if \( v \) is generated by
a primitive substitution, then the spectrum \( \sigma(H_{\lambda,v}) \) has Lebesgue measure
0.
Among substitution class, the most famous one is the so-called Fibonacci potential, which is defined as
\[ v^F(n) = \chi(1-\alpha,1)(n\alpha + \theta), \]
where \( \alpha = (\sqrt{5} - 1)/2 \) is the Golden number and \( \theta \in \mathbb{R} \). We note that when \( \theta = 0 \), \( v^F \) can also be defined through the Famous Fibonacci substitution \( \tau \):
\[ \tau(a) = ab \text{ and } \tau(b) = a \text{ (see for example [16] Section 5.4)}. \]
The operator \( H_{\lambda,v^F} \) is called Fibonacci Hamiltonian. Since the pioneer works [19, 27], Fibonacci Hamiltonian is always the central model in quasicrystal and is extensively studied, see for example the survey [9] and the most recent progress [13].

The dimensional properties of \( \sigma(H_{\lambda,v^F}) \) has been well understood until now, see [28, 18, 24, 10, 7, 11, 12]. In particular the following property is shown in [10]:
\[ \lim_{|\lambda| \to \infty} |\lambda| \dim_H \sigma(H_{\lambda,v^F}) \ln |\lambda| = \ln(1 + \sqrt{2}). \]
This implies that \( \dim_H \sigma(H_{\lambda,v^F}) \to 0 \) with the speed \( 1/\ln |\lambda| \) when \( |\lambda| \to \infty \).

Another famous potential is the so-called Thue-Morse potential \( w \), which is defined as follows: Let \( \sigma \) be the Thue-Morse substitution such that \( \sigma(a) = ab \) and \( \sigma(b) = ba \), let \( u = u_1u_2 \cdots = \sigma^n(a) \). For \( n \geq 1 \), let \( w(n) = 1 \) if \( u_n = a \); let \( w(n) = -1 \) if \( u_n = b \); let \( w(1-n) = w(n) \) for \( n \geq 1 \). The operator \( H_{\lambda,w} \) is called Thue-Morse Hamiltonian. Thue-Morse Hamiltonian is also studied by many authors, see [2, 26, 29, 3, 4, 25, 14] and so on. However compared with Fibonacci case, almost noting is known about the dimensional properties of \( \sigma(H_{\lambda,w}) \), except some numerical result about the box dimension given in [4].

In this paper, we will show the following

**Theorem 1.1.** For Thue-Morse potential \( w \) and any \( \lambda \in \mathbb{R} \),
\[ \dim_H \sigma(H_{\lambda,w}) \geq \frac{\ln 2}{140 \ln 2.1}. \]

**Remark 1.2.**
1. Compare with the Fibonacci case, our result is a bit surprising since in Fibonacci case \( \dim_H \sigma(H_{\lambda,v^F}) \) will tends to 0 with speed \( \ln |\lambda| \) when \( |\lambda| \to \infty \), however in Thue-Morse case there exists an absolute positive lower bound for \( \dim_H \sigma(H_{\lambda,w}) \) for all \( \lambda \in \mathbb{R} \).
2. The lower bound is not optimal. It can be improved through a finer estimation. We do not pursue this since it does not give the exact dimension.

In the following we say a few words about the idea of the proof. Our method bases on the analysis of the behavior of the trace polynomials related to Thue-Morse Hamiltonian. Some convergence behavior hidden in
these polynomials enable us to construct a Cantor subset of the spectrum, meanwhile the Hausdorff dimension of the Cantor set can be estimated, which in turn offer a lower bound of the spectrum.

Let us recall the definition of trace polynomials of Thue-Morse Hamiltonian, see \[4, 5\] for more details and motivations about trace polynomials. Recall that \(\sigma\) is the Thue-Morse substitution such that \(\sigma(a) = ab\) and \(\sigma(b) = ba\). Denote the free group generated by \(a, b\) as \(FG(a, b)\). Given \(\lambda, x \in \mathbb{R}\), define a homomorphism \(\tau : FG(a, b) \to SL(2, \mathbb{R})\) as

\[
\tau(a) = \begin{bmatrix} x - \lambda & -1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \tau(b) = \begin{bmatrix} x + \lambda & -1 \\ 1 & 0 \end{bmatrix}
\]

and \(\tau(a_1 \cdots a_n) = \tau(a_n) \cdots \tau(a_1)\). Define \(h_n(x) := \text{tr}(\tau(\sigma^n(a)))\) (where \(\text{tr}(A)\) denotes the trace of the matrix \(A\)), then \((4, 5)\)

\[
h_1(x) = x^2 - \lambda^2 - 2, \quad h_2(x) = (x^2 - \lambda^2)^2 - 4x^2 + 2, \quad (1)
\]

and for \(n \geq 2\),

\[
h_{n+1}(x) = h_{n-1}^2(x)(h_n(x) - 2) + 2. \quad (2)
\]

\(\{h_n : n \geq 1\}\) is called the family of trace polynomials related to Thue-Morse Hamiltonian.

Define the zero set of the trace polynomials as

\[
\Sigma = \{x \in \mathbb{R} : \exists n \geq 1, \ s.t. \ h_n(x) = 0\}. \quad (3)
\]

It is shown in \([4, 5]\) that \(\Sigma \subset \sigma(H_{\lambda, w})\). Since \(\sigma(H_{\lambda, w})\) is closed, the closure of \(\Sigma\) is also a subset of the spectrum (indeed the closure of \(\Sigma\) is exactly the spectrum). The set \(\Sigma\) play a crucial role in our proof. By the recurrence relation \((2)\) it is direct to check that, for any \(x \in \Sigma\), if \(h_n(x) = 0\), then \(h_k(x) = 2\) for any \(k \geq n + 2\). Moreover, \(x\) is a local maximal point of \(h_k\) and the graph of \(h_k\) is tangent to the horizontal line \(y = 2\) at point \((x, 2)\).

Figure 1 shows the typical configuration when we plot the graphs of \(\{h_n(x)\}\) around a point \(x \in \Sigma\).

At first we describe a naive way of obtaining a lower bound for dimension of spectrum. In Figure 2 \(a\) is such that \(h_1(a) = 0\), then \(h_4(a) = 2\). \(h_4(x)\) is decreasing on the interval \([a, b]\) with \(h_4(b) = 0\), \(h_6(x)\) is decreasing on \([a, c]\) with \(h_6(c) = 0\) and increasing on \([d, b]\) with \(h_6(d) = 0\). Consequently \(a, b, c, d \in \Sigma\). Similarly when restricting to \([a, c]\) and drawing the graphs of \(h_6\) and \(h_8\), we can find two subintervals of \([a, c]\) such that the endpoints of these intervals are all in \(\Sigma\). For \([d, b]\) the situation is the same. If we continue this process, we can obtain a covering structure which determines a Cantor
set $E$ as its limit set. Moreover by the construction all the endpoints of those intervals are in $\Sigma$. Since $E$ is the closure of all these endpoints, we conclude that $E$ is contained in the closure of $\Sigma$, and hence contained in the spectrum. The Hausdorff dimension of $E$ offers a lower bound of Hausdorff dimension of the spectrum.

Figure 1. Graph of $h_k(x)$, $k = 2, \cdots, 9$ ($\lambda = 3$).

To estimate the dimension of $E$, we need to estimate ratios such as $(c - a)/(b - a)$ and $(b - d)/(b - a)$. However Figure 2 already suggests that the ratios may be out of control. We need more information about trace polynomials and more delicate construction of subset.

A key observation is the following: fix any point $x_0$ in $\Sigma$ and assume $h_n(x_0) = 0$, then there exists a scaling factor $\rho$ such that the rescaled sequence $h_{n+3+k}(\frac{x}{2\rho} + x_0)$ will be more and more close to $2\cos x$ on any fixed interval $[-c, c]$ as $k$ tends to infinity (see again Figure 1 for this phenomenon). This closeness enable us to construct a Cantor subset $E$ of $\sigma(H_{\lambda,w})$ in

Figure 2. The graph of $h_4(x), h_6(x), h_8(x)$ for $x \in (3.3166, 3.3175)$.
a controllable way such that the lower bound of \( \dim_H E \) can be estimated explicitly.

More precisely we will start with a polynomial pair \((P_{-1}, P_0)\), which have the following expansions at \(x_0\):

\[
\begin{align*}
P_{-1}(x) &= 2 - \rho^2 (x - x_0)^2 + O((x - x_0)^3); \\
P_0(x) &= 2 - \rho^2 (x - x_0)^2 + O((x - x_0)^3).
\end{align*}
\]

We can define two kinds of closeness of \((P_{-1}, P_0)\) to \(2 \cos x\) at \(x_0\): weak one and strong one (see Remark \[2.3\]). Then we define a polynomial sequence \(\{P_n : n \geq -1\}\) according to \(\[2\]\).

A crucial step is to establish the following inductive lemma (Lemma \[2.4\]):

There exists an absolute constant \(K = 140\) such that the following holds. Assume \((P_{-1}, P_0)\) is strongly close to \(2 \cos x\) at \(x_0\). Let \(y_0\) be the minimal \(y > x_0\) such that \(P_0(y_0) = 0\), then \((P_{K-1}, P_K)\) is strongly close to \(2 \cos x\) at both \(x_0\) and \(y_0\). Moreover the following estimates hold:

\[
\frac{y_K - x_0}{y_0 - x_0} \geq 2.1^{-K} \quad \text{and} \quad \frac{y_0 - x_K}{y_0 - x_0} \geq 2.1^{-K},
\]

where \(y_K\) is the minimal \(y > x_0\) such that \(P_K(y) = 0\), and \(x_K\) is the maximal \(y < y_0\) such that \(P_K(y) = 0\).

![Figure 3: Illustration of our process](image)

Note that this is the right-side version of Lemma \[2.4\]. If we define \(y_0\) to be the maximal \(y < x_0\) such that \(P_0(y_0) = 0\), then we can state the left-side version similarly.

Now for the pair \((P_{K-1}, P_K)\), since it is strongly close to \(2 \cos x\) at both \(x_0\) and \(y_0\), we can continue the process. Then inductively we can construct a Cantor set, for which we can estimate the dimension.

Then what left is to find a trace polynomial pair such that it is indeed strongly close to \(2 \cos x\). We achieve this by two steps: at first we show that
if \((P_{-1}, P_0)\) is weakly close to \(2 \cos x\) at \(x_0\), then \((P_{K-1}, P_K)\) is strongly close to \(2 \cos x\) at \(x_0\) (see Lemma 2.2); next we show that \((h_4, h_5)\) is weakly close to \(2 \cos x\) at \(a_0\), where \(a_0\) is a zero of \(h_1\) (see Lemma 2.1). This will finish the proof of the main result.

The rest of the paper is organized as follows. In Section 2 we introduce the basic notations, state the main lemmas which are needed for the proof of main theorem. Then we finish the proof of Theorem 1.1. In Section 3 we prove Lemma 2.1 and Lemma 2.2. In Section 4 we prove Lemma 2.4.

2. Germ, closeness, proof of Theorem 1.1

In this section, at first we will introduce the notation of germ, which is the typical configuration of the trace polynomial pairs \((h_{n-1}, h_n)\) around a zero point. Next we will define the regularity of germ, which measure the closeness between the rescaled polynomial pair and \(2 \cos x\). Then we will state several lemmas which describe the properties of regular germs. Based on these lemmas we will prove the main theorem.

2.1. Regular germ.

Given a polynomial pair \((P_{-1}, P_0)\). Assume at \(x_0 \in \mathbb{R}\), there exists \(\rho > 0\) such that

\[
\begin{align*}
P_{-1}(x) &= 2 - \frac{x^2}{4} (x - x_0)^2 + O((x - x_0)^3); \\
P_0(x) &= 2 - \rho^2 (x - x_0)^2 + O((x - x_0)^3)
\end{align*}
\]

Then we say \((P_{-1}, P_0)\) has a \(\rho\)-germ at \(x_0\).

We want to rescale \(P_{-1}\) and \(P_0\) and compare them with \(2 \cos x\). For this purpose, for \(k = -1\) and 0 define

\[Q_k(x) = P_k\left(\frac{x}{2k\rho} + x_0\right)\]

It is ready to see that \(Q_k(x) = 2 - x^2 + O(x^3)\). Since \(2 \cos x = 2 - x^2 + O(x^3)\), we have \(Q_k(x) = 2 \cos x + O(x^3)\). Write \(\Delta_k(x) = Q_k(x) - 2 \cos x\), then

\[\Delta_k(x) = Q_k(x) - 2 \cos x = \sum_{k \geq 3} \Delta_{k,n} x^n.
\]

We want to define a kind of smallness for \(\Delta_k\) through its coefficients. Let us do some preparation. Given two formal series with real coefficients

\[f(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad g(x) = \sum_{n=0}^{\infty} b_n x^n,
\]
define the following partial order:

\[ f \preceq g \iff a_n \leq b_n \quad (\forall n \geq 0). \]

We further define \(|f(x)|^* := \sum_{n=0}^{\infty} |a_n|x^n\). Then it is easy to check that

\[ |fg|^* \leq |f|^*|g|^* \quad \text{and} \quad |f + g|^* \leq |f|^* + |g|^*. \]

Moreover if \(|f|^* \preceq \tilde{f}\) and \(|g|^* \preceq \tilde{g}\), then it is seen that \(|fg|^* \preceq \tilde{f}\tilde{g}\). Later we will use these properties repeatedly to estimate the coefficients of certain series.

Let us go back to \((P_{-1}, P_0)\). If moreover there exist \(\delta > 0\) and \(\beta \geq 1\) such that

\[ |\Delta_{-1}|^*, |\Delta_0|^* \leq \delta \sum_{n=3}^{\infty} \frac{x^n}{\beta^n} \]

Then we say that \((P_{-1}, P_0)\) has a \((\delta, \beta)\)-regular \(\rho\)-germ at \(x_0\). We also say that \((P_{-1}, P_0)\) is \((\delta, \beta)\)-regular at \(x_0\) with scaling factor \(\rho\), or simply \((\delta, \beta)\)-regular at \(x_0\). An immediate observation is that if \(\delta \leq \delta'\) and \(\beta \geq \beta'\) and \((P_{-1}, P_0)\) is \((\delta, \beta)\)-regular at \(x_0\), then \((P_{-1}, P_0)\) is also \((\delta', \beta')\)-regular at \(x_0\).

Recall that \(\{h_n(x) : n \geq 1\}\) is the family of trace polynomials of Thue-Morse Hamiltonian satisfy (1) and (2). Let

\[ a_0 = \sqrt{2 + \lambda^2}, \quad (4) \]

then \(h_1(a_0) = 0\). The following lemma is the starting point of our whole proof.

**Lemma 2.1.** \((h_4, h_5)\) is \((1, 1)\)-regular at \(a_0\).

The following lemma shows that the germ can keep when we iterating the pair. Moreover the regularity will become better and better.

**Lemma 2.2.** Assume \((P_{-1}, P_0)\) has a \(\rho\)-germ at \(x_0\). For \(k \geq 1\), define

\[ P_k = P_{k-2}^2(P_{k-1} - 2) + 2. \quad (5) \]

Then

1) \((P_{k-1}, P_k)\) has a \(2^k\rho\)-germ at \(x_0\) for any \(k \geq 1\).

2) If \((P_{-1}, P_0)\) is \((1, 1)\)-regular at \(x_0\), then \((P_{k-1}, P_k)\) is \((3200 \cdot 2^{-k/2}, 4)\)-regular at \(x_0\) for any \(k \geq 4\).
2.2. Closeness and consequences.

Fix \( \delta_0 = 10^{-2}, \delta_1 = 10^{-4}, \delta_2 = 10^{-16} \) and \( K = 140 \). Then
\[
20 \delta_1 \leq \delta_0; \quad 400^4 \times 4 \times 3 \delta_2 \leq \delta_1; \quad 3200 \cdot 2^{-(K-4)/2} < \delta_2.
\] (6)

Remark 2.3. Later in Section 4 we will see that if \((P_{-1}, P_0)\) is \((\delta, \beta)\)-regular with \( \delta \) small and \( \beta \) big, then the rescaled polynomials \( Q_{-1}, Q_0 \) will be close to \( 2 \cos x \) in a bounded neighborhood of 0. In this sense \((\delta, \beta)\) give a measurement of closeness between \((P_{-1}, P_0)\) and \( 2 \cos x \) around \( x_0 \). We will use the following three levels of closeness: Assume \((P_{-1}, P_0)\) has a germ at \( x_0 \). If \((P_{-1}, P_0)\) is \((1, 1)\)-regular (or \((\delta_1, 2)\)-regular, or \((\delta_2, 4)\)-regular) at \( x_0 \), we say that \((P_{-1}, P_0)\) is weakly close ( close, or strongly close ) to \( 2 \cos x \).

Our key technique lemma is the following:

Lemma 2.4. Let \((P_k)_{k \geq -1}\) satisfy the recurrence relation (5). Assume \((P_{-1}, P_0)\) is \((\delta_2, 4)\)-regular at \( x_0 \). For \( k = 0, K \), let \( y_k^+ \) (resp. \( y_k^- \)) be the minimal \( t > x_0 \) (resp. maximal \( t < x_0 \)) such that \( P_k(t) = 0 \). Let \( x_K^+ \) (resp. \( x_K^- \)) be the maximal \( t < y_0^+ \) (minimal \( t > y_0^- \)) such that \( P_K(t) = 0 \). Then
\[
\left| \frac{y_k^+ - x_0}{y_0^+ - x_0} \right| \geq 2.1^{-K}, \quad \left| \frac{y_0^- - x_K^+}{y_0^- - x_0} \right| \geq 2.1^{-K},
\]
and \((P_{K-1}, P_K)\) is \((\delta_2, 4)\)-regular at \( x_0 \) and \( y_0^\pm \) respectively.

See Figure 3 for an illustration of the lemma. The crucial point is that, the strong closeness can pass to smaller scale, which enable us to iterate the process.

2.3. Proof of Theorem 1.1

Now we will construct the desired Cantor set. To simplify the notation we write \( \tilde{P}_k(x) := h_{k+5}(x) \) for \( k \geq -1 \). By Lemma 2.1 we know that \((\tilde{P}_{-1}, \tilde{P}_0)\) is \((1, 1)\)-regular at \( a_\emptyset \).

We have fixed \( K = 140 \). Assume \( b_\emptyset > a_\emptyset \) is the first zero of \( \tilde{P}_K \) to the right of \( a_\emptyset \). Define \( I_0 := [a_\emptyset, b_\emptyset] \). Assume \( b_0 \) is the smallest zero of \( \tilde{P}_{2K} \) in \( I_0 \) and \( a_1 \) is the biggest zero of \( \tilde{P}_{2K} \) in \( I_0 \). Define
\[
I_0 := [a_\emptyset, b_\emptyset] = [a_0, b_0] \quad \text{and} \quad I_1 := [a_1, b_\emptyset] = [a_1, b_1].
\]
Take any \( w \in \{0, 1\}^k \), suppose \( I_w := [a_w, b_w] \) is defined. Assume \( b_{w_0} \) is the smallest zero of \( \tilde{P}_{(k+2)K} \) in \( I_w \) and assume \( a_{w_1} \) is the biggest zero of \( \tilde{P}_{(k+2)K} \) in \( I_w \). Write \( a_{w_0} = a_w, b_{w_1} = b_w \) and define
\[
I_{w_0} = [a_w, b_{w_0}] = [a_{w_0}, b_{w_0}] \quad \text{and} \quad I_{w_1} = [a_{w_1}, b_w] = [a_{w_1}, b_{w_1}].
\]
Define a Cantor set as
\[ C := \bigcap_{n \geq 0} \bigcup_{|w| = n} I_w \]

**Proposition 2.5.**
\[ \dim_H C \geq \frac{\ln 2}{K \ln 2}. \]

**Proof.** We claim that
\[ \frac{|I_{w0}|}{|I_w|} \geq 2.1^{-K}, \quad \frac{|I_{w1}|}{|I_w|} \geq 2.1^{-K} \]
and \( (\tilde{P}_{(k+2)K-1}, \tilde{P}_{(k+2)K}) \) is \((\delta_2, 4)\)-regular at \( a_w, b_w \) for any \( w \in \{0, 1\}^* \). We show it by induction.

At first take \( w = \emptyset \). Recall that \( (\tilde{P}_{-1}, \tilde{P}_0) \) is \((1, 1)\)-regular at \( a_\emptyset \). By Lemma 2.2 and [6], \( (\tilde{P}_{K-1}, \tilde{P}_K) \) is \((\delta_2, 4)\)-regular at \( a_\emptyset \). By applying Lemma 2.4 to the pair \( (\tilde{P}_{K-1}, \tilde{P}_K) \), we conclude that
\[ \frac{|I_0|}{|I_\emptyset|} \geq 2.1^{-K}, \quad \frac{|I_1|}{|I_\emptyset|} \geq 2.1^{-K} \]
and \( (\tilde{P}_{2K-1}, \tilde{P}_{2K}) \) is \((\delta_2, 4)\)-regular at both \( a_\emptyset \) and \( b_\emptyset \).

Next take \( k > 0 \) and assume the result holds for any \( w \in \{0, 1\}^{k-1} \). Then fix \( w \in \{0, 1\}^k \) and write \( w = \bar{w}j \) with \( j = 0 \) or \( 1 \). If \( w = \bar{w}0 \), then by the construction of \( C \), \( b_w = b_{\bar{w}0} \) is the smallest zero of \( \tilde{P}_{(k+1)K} \) in \( I_{\bar{w}} \). Moreover by induction assumption, \( (\tilde{P}_{(k+1)K-1}, \tilde{P}_{(k+1)K}) \) is \((\delta_2, 4)\)-regular at \( a_{\bar{w}} = a_w \). By applying Lemma 2.4 to pair \( (\tilde{P}_{(k+1)K-1}, \tilde{P}_{(k+1)K}) \) we conclude that
\[ \frac{|I_{w0}|}{|I_w|} \geq 2.1^{-K}, \quad \frac{|I_{w1}|}{|I_w|} \geq 2.1^{-K} \]
and \( (\tilde{P}_{(k+2)K-1}, \tilde{P}_{(k+2)K}) \) is \((\delta_2, 4)\)-regular at both \( a_w \) and \( b_w \). If \( w = \bar{w}1 \), then by the construction of \( C \), \( a_w \) is the biggest zero of \( \tilde{P}_{(k+1)K} \) in \( I_{\bar{w}} \).
Moreover by induction assumption, \((\tilde{P}_{(k+1)K-1}, \tilde{P}_{(k+1)K})\) is \((\delta_2, 4)\)-regular at \(b_w = b_w\). Again by applying Lemma 2.4, the desired result holds.

By induction, the claim is proven.

Now it is well known that (see for example [15])
\[
\dim_H C \geq \frac{\ln 2}{-\ln 2.1} = \frac{\ln 2}{K\ln 2.1}.
\]

\[\square\]

Proof of Theorem 1.1. Recall that \(\Sigma\) is defined by (3). By definition, for any \(w \in \{0, 1\}^*, a_w, b_w \in \Sigma \subset \sigma(H_{\lambda,w})\). Since \(C = \{a_w, b_w : w \in \{0, 1\}^*\}\), we conclude that \(C \subset \sigma(H_{\lambda,w})\). Consequently
\[
\dim_H \sigma(H_{\lambda,w}) \geq \dim_H C \geq \frac{\ln 2}{K\ln 2.1}.
\]

Recall that \(K = 140\) is an absolute positive constant, the result follows. \[\square\]

3. Proof of Lemma 2.1 and 2.2

In this section, at first we will show how a germ can appear. Next we show that \((h, h)\) has a \((1, 1)\)-regular germ at \(a\) (Lemma 2.1). Then we will show that when iterate the polynomial pairs, the regularity will become better and better (Lemma 2.2).

3.1. Generating a germ.

At first we present a sufficient condition on how we can produce a germ when a family of polynomials satisfies the recurrence relation (2).

Given polynomial pair \((f_0, f_1)\). Define \(f_{n+1} = f_n^2(f_n - 2) + 2\) for \(n \geq 1\).

Lemma 3.1. Assume \(f_0(x_0) = 0\). If \(f_1(x_0) < 2\) and \(f_0'(x_0) f_1(x_0) \neq 0\), then \((f_{k-1}, f_k)\) has a germ at \(x_0\) for \(k \geq 4\).

Proof. Write
\[
f_0(x) = f'(x_0)(x - x_0) + O((x - x_0)^2) \quad \text{and} \quad f_1(x) = f_1(x_0) + O((x - x_0)).
\]

By the recurrence relation we have
\[
f_2(x) = 2 - (2 - f_1(x_0)) f_0^2(x_0)(x - x_0)^2 + O((x - x_0)^3)
\]
\[
f_k(x) = 2 - 4^{k-3}(2 - f_1(x_0))(f_0'(x_0)f_1(x_0))^2 (x - x_0)^2
\]
\[
+ O((x - x_0)^3) \quad (k \geq 3).
\]

If \(f_1(x_0) < 2\) and \(f_0'(x_0), f_1(x_0) \neq 0\), then
\[
\rho := \sqrt{2 - f_1(x_0)|f_0'(x_0)f_1(x_0)|} > 0
\]
and for \( k \geq 3 \)
\[
f_k(x) = 2 - (2^{k-3} \rho)^2(x - x_0)^2 + O((x - x_0)^3). \tag{7}
\]
Then by the definition, \((f_{k-1}, f_k)\) has a \(2^{k-3}\rho\)-germ at \( x_0 \) for all \( k \geq 4 \). \( \Box \)

Now we prove Lemma 2.1.

**Proof of Lemma 2.1** By (1), (2) and (4),
\[
h_1(a_0) = 0, \quad h_2(a_0) = -2 - 4\lambda^2 < 2, \quad h_1'(a_0) = 2a_0 \neq 0, \quad h_2(a_0) \neq 0.
\]
Write \( \tau := 2^3(1 + 2\lambda^2) \sqrt{(1 + \lambda^2)(2 + \lambda^2)} \). By Lemma 3.1, \((h_4, h_5)\) has a \(2\tau\)-germ at \( a_0 \).

In the following we show that this germ is \((1, 1)\)-regular. Write \( t := 2a_0 \). Then \( t \geq 2\sqrt{2} \). Define
\[
g_n(x) = h_n(x + a_0).
\]
Then by direct computation,
\[
g_1(x) = tx + x^2 \quad \text{and} \quad g_2(x) = (6 - t^2) + t^2x^2 + 2tx^3 + x^4.
\]
Then we can compute that for \( n \geq 4 \),
\[
g_n(x) = 2 - 4^{n-4}t^2(t^2 - 6)^2(t^2 - 4)x^2 + O(x^3).
\]
By computation, \( \tau = t(t^2 - 6)\sqrt{t^2 - 4} \). Define \( f_n(x) = g_n(x/\tau) \). Then for \( n \geq 4 \) we have
\[
f_n(x) = 2 - 4^{n-4}x^2 + O(x^3).
\]
We also have
\[
f_1(x) = t\tau^{-1}x + \tau^{-2}x^2
\]
\[
f_2(x) = (6 - t^2) + (t/\tau)^2x^2 + (2t/\tau^3)x^3 + x^4/\tau^4
\]
\[
f_3(x) = 2 + O(x^2).
\]
By the fact that \( t \geq 2\sqrt{2} \) and \( \tau = t(t^2 - 6)\sqrt{t^2 - 4} \), it is direct to verify that
\[
|f_1(x)|^* \leq \frac{t}{\tau}x e^{x^2/32}, \quad |f_2(x)|^* \leq (t^2 - 6)e^{x^4/4}, \quad |f_2(x) - 2|^* \leq (t^2 - 4)e^{x^4/4}.
\]
Then, by \( f_n = 2 + f_{n-2}^2(f_{n-1} - 2) \), we have
\[
|f_3(x)|^* \leq 2 + (|f_1(x)|^*)^2 \cdot |f_2(x) - 2|^* \leq 2 + \frac{1}{(t^2 - 6)^2}x^2 e^{5x/16}. \tag{8}
\]
Since \( f_3(x) = 2 + O(x^2) \), by (8) we have \( |f_3(x) - 2|^* \leq (t^2 - 6)^{-2}x^2 e^{5x/16} \).
Consequently
\[
|f_4(x)|^* \leq 2 + (|f_2(x)|^*)^2 \cdot |f_3(x) - 2|^* \leq 2 + x^2 e^{13x/16}. \tag{9}
\]
Since \( f_4(x) = 2 - x^2 + O(x^3) \), by (9) we have
\[
|f_4(x) - 2 + x^2|^* \leq \sum_{n \geq 3} \frac{x^n}{(n-2)!}.
\]

We also have
\[
|2 \cos x - 2 + x^2|^* \leq \sum_{n \geq 4} \frac{x^n}{n!}.
\]

Thus we conclude that
\[
|f_4(x) - 2 \cos x|^* \leq \sum_{n \geq 3} x^n. \tag{10}
\]

Similarly since \( f_4(x) = 2 + O(x^2) \), by (9) we have \( |f_4(x) - 2|^* \leq x^2 e^{13x/16} \).

Consequently
\[
|f_5(x)|^* \leq 2 + (|f_3(x)|^*)^2 \cdot |f_4(x) - 2|^* \\
\leq 2 + 4x^2 e^{13x/16} + 4x^4 e^{18x/16} + \frac{x^6 e^{23x/16}}{(t^2 - 6)^2}.
\]

Then we have
\[
|f_5(x/2)|^* \leq 2 + x^2 e^{x/2} + \frac{x^4 e^x}{16} + \frac{x^6 e^x}{210}.
\]

Since \( f_5(x/2) = 2 - x^2 + O(x^3) \) and \( 2 \cos x = 2 - x^2 + O(x^4) \), by similar argument as (10), we can show that
\[
|f_5(x/2) - 2 \cos x|^* \leq \sum_{n \geq 3} x^n. \tag{11}
\]

(10) and (11) implies that \((h_4, h_5)\) is \((1, 1)\)-regular at \(a_0\) with scaling factor \(2\tau\). \(\square\)

3.2. Iteration of regular polynomial pairs.

In this subsection we will prove a strengthen version of Lemma 2.2 which will be needed in the proof of Lemma 2.4.

Let us recall the setting in subsection 2.1. Assume \((P_{-1}, P_0)\) has a \(\rho\)-germ at \(x_0\). For \(k \geq 1\), define \(P_k\) by the recurrence relation (5). Then it is direct to check that
\[
P_k(x) = 2 - 4^k \rho^2 (x - x_0)^2 + O((x - x_0)^3), \quad (\forall k \geq 1). \tag{12}
\]

For \(k \geq -1\) define
\[
Q_k(x) = P_k \left( \frac{x}{2^k \rho} + x_0 \right). \tag{13}
\]
It is ready to show that $Q_k(x) = 2 - x^2 + O(x^3)$. Since $2 \cos x = 2 - x^2 + O(x^3)$, we conclude that $Q_k(x) = 2 \cos x + O(x^3)$. Write $\Delta_k(x) = Q_k(x) - 2 \cos x$, then

$$\Delta_k(x) = Q_k(x) - 2 \cos x = \sum_{k \geq 3} \Delta_{k,n} x^n.$$  \hfill (14)

By the recurrence relation (5), we have for $k \geq 1$

$$Q_k(x) = Q_{k-2}^2(x/4)(Q_{k-1}(x/2) - 2) + 2 = \left(2 \cos x/4 + \Delta_{k-2}(x/4)\right)^2 \left(2 \cos x/2 - 2 + \Delta_{k-1}(x/2)\right) + 2 = 2 \cos x + (2 + 2 \cos x^2/2) \cdot \Delta_{k-1}(x^2/2) +$$

$$\Delta_{k-2}(x^2/4) \left(4 \cos x^2/4 + \Delta_{k-2}(x^2/4)\right) \left(2 \cos x^2/2 - 2 + \Delta_{k-1}(x^2/2)\right).$$

Thus we conclude that for $k \geq 1$

$$\Delta_k(x) = (2 + 2 \cos x^2/2) \cdot \Delta_{k-1}(x^2/2) +$$

$$\Delta_{k-2}(x^2/4) \left(4 \cos x^2/4 + \Delta_{k-2}(x^2/4)\right) \left(2 \cos x^2/2 - 2 + \Delta_{k-1}(x^2/2)\right).$$  \hfill (15)

The following proposition shows that how the coefficients evolves.

**Proposition 3.2.** Assume $\Phi_0, \Phi_1, \Phi_2$ are real analytic functions with Taylor expansion $\Phi_k(x) = \sum_{n \geq 3} \Phi_{k,n} x^n$, $k = 0, 1, 2$ and satisfy the following relation:

$$\begin{align*}
\Phi_2(x) & = (2 + 2 \cos x^2/2) \cdot \Phi_{1}(x^2/2) + \\
& \quad \Phi_0(x^2/4) \left(4 \cos x^2/4 + \Phi_0(x^2/4)\right) \left(2 \cos x^2/2 - 2 + \Phi_1(x^2/2)\right).
\end{align*}$$

If there exist $0 < \delta \leq 1$ and $\beta \geq 1$ such that

$$|\Phi_{0,n}|, |\Phi_{1,n}| \leq \delta \beta^{-n}, \quad \forall n \geq 3,$$

then when $\beta = 1$,

$$|\Phi_2(x)|^* \leq \delta \left(4 \frac{x^3}{2^3} + 4 \frac{x^4}{2^4} + 9 \sum_{n \geq 5} \frac{x^n}{2^n}\right),$$  \hfill (16)

when $\beta = 2$,

$$|\Phi_2(x)|^* \leq \delta \left(4 \frac{x^3}{4^3} + 4 \frac{x^4}{4^4} + 24 \frac{x^5}{4^5} + 43 \sum_{n \geq 6} \frac{x^n}{4^n}\right).$$  \hfill (17)
Proof. Write

\[
\begin{align*}
(1) &= (2 + 2 \cos \frac{x}{2}) \cdot \Phi_1(\frac{x}{2}), \\
(II) &= 4 \cos \frac{x}{4} + \Phi_0(\frac{x}{2}), \\
(III) &= 2 \cos \frac{x}{2} - 2 + \Phi_1(\frac{x}{2}).
\end{align*}
\]

Then

\[
\Phi_2(x) = (I) + \Phi_0(\frac{x}{4})(II)(III),
\]

(18)

We will frequently use the following facts:

\[
|\cos x|^n \leq 1 + \sum_{n \geq 2} \frac{x^n}{n!}, \quad \sum_{k=2}^{n} \frac{\beta^k}{k!} < e^\beta - 1 - \beta, \quad \sum_{k=3}^{n} 2^{-k} < \frac{1}{4}.
\]

(19)

For example by the last two inequalities of (19) we have

\[
\begin{align*}
\sum_{n \geq 2} \frac{x^n}{n!} \sum_{n \geq 2} \frac{x^n}{(2 \beta)^n} &= \sum_{n \geq 5} \frac{x^n}{(2 \beta)^n} \sum_{k=2}^{n-3} \frac{\beta^k}{k!} < (e^\beta - 1 - \beta) \sum_{n \geq 5} \frac{x^n}{(2 \beta)^n} \\
\sum_{n \geq 3} \frac{x^n}{(2 \beta)^n} \sum_{n \geq 3} \frac{x^n}{(2 \beta)^n} &= \sum_{n \geq 6} \frac{x^n}{(2 \beta)^n} \sum_{k=3}^{n-6} 2^{-k} < \frac{1}{4} \sum_{n \geq 6} \frac{x^n}{(2 \beta)^n}.
\end{align*}
\]

(20)

By the assumption, (19) and (20), we have

\[
|I|^n \leq \left(4 + 2 \sum_{n \geq 2} \frac{x^n}{n!} \right) \sum_{n \geq 3} \frac{\delta x^n}{(2 \beta)^n} \\
\leq 4 \delta \sum_{n \geq 3} \frac{x^n}{(2 \beta)^n} + 2(e^\beta - 1 - \beta) \delta \sum_{n \geq 5} \frac{x^n}{(2 \beta)^n} \\
\leq 4 \delta \sum_{n=3}^{4} \frac{x^n}{(2 \beta)^n} + 2(e^\beta - 1 - \beta) \delta \sum_{n \geq 5} \frac{x^n}{(2 \beta)^n}.
\]

(21)

By \(|II|^n \leq 4 + 4 \sum_{n \geq 2} \frac{x^n}{n!} + |\Phi_0(\frac{x}{4})|^n\) and \(|\Phi_0(\frac{x}{4})|^n \leq \delta \sum_{n \geq 3} \frac{x^n}{(2 \beta)^n}\) we get

\[
|\Phi_0(\frac{x}{4}) \times (II)|^n \leq 4 \delta \sum_{n \geq 3} \frac{x^n}{(2 \beta)^n} + 4 \delta \sum_{n \geq 3} \frac{x^n}{(4 \beta)^n} \sum_{k=2}^{n-3} \frac{\beta^k}{k!} + (|\Phi_0(\frac{x}{4})|^n)^2 \\
\leq 4 \delta \sum_{n \geq 3} \frac{x^n}{(2 \beta)^n} + 4(e^\beta - 1 - \beta) \delta \sum_{n \geq 5} \frac{x^n}{(4 \beta)^n} + (|\Phi_0(\frac{x}{4})|^n)^2 \\
\leq 4(e^\beta - \beta) \delta \sum_{n \geq 3} \frac{x^n}{(2 \beta)^n} + (|\Phi_0(\frac{x}{4})|^n)^2.
\]

Since \(|III|^n \leq 2 \sum_{n \geq 2} \frac{x^n}{2^n n!} + \delta \sum_{n \geq 3} \frac{x^n}{(2 \beta)^n}\), by (20) we have

\[
|\Phi_0(\frac{x}{4}) \times (II) \times (III)|^n \\
\leq 8(e^\beta - \beta) \delta \sum_{n \geq 2} \frac{x^n}{(2 \beta)^n} \sum_{n \geq 2} \frac{x^n}{14 \cdot 2^n n!}
\]
\[
+4(e^{\beta} - \beta)\delta^2 \left( \sum_{n \geq 3} \frac{x^n}{(4\beta)^n} \right) \left( \sum_{n \geq 3} \frac{x^n}{(2\beta)^n} \right) \\
+2\delta^2 \left( \sum_{n \geq 3} \frac{x^n}{(4\beta)^n} \right) \left( \sum_{n \geq 3} \frac{x^n}{(2\beta)^n} \right) \left( \sum_{n \geq 2} \frac{x^n}{2^n n!} \right) \\
+\delta^3 \left( \sum_{n \geq 3} \frac{x^n}{(4\beta)^n} \right) \left( \sum_{n \geq 3} \frac{x^n}{(2\beta)^n} \right) \left( \sum_{n \geq 3} \frac{x^n}{(2\beta)^n} \right) \\
\leq \left( e^{\beta} - \beta \right)\delta \left( \sum_{n \geq 3} \frac{x^n}{(2\beta)^n} \right) \left( \sum_{n \geq 2} \frac{x^n}{2^n n!} \right) \\
+4(e^{\beta} - \beta)\delta^2 \left( \sum_{n \geq 3} \frac{x^n}{(4\beta)^n} \right) \left( \sum_{n \geq 3} \frac{x^n}{(2\beta)^n} \right) \\
+2^{-2}\delta^2 \left( \sum_{n \geq 3} \frac{x^n}{(4\beta)^n} \right) \left( \sum_{n \geq 3} \frac{x^n}{(2\beta)^n} \right) \left( \sum_{n \geq 2} \frac{x^n}{2^n n!} \right) \\
+\delta^3 \left( \sum_{n \geq 3} \frac{x^n}{(4\beta)^n} \right) \left( \sum_{n \geq 3} \frac{x^n}{(2\beta)^n} \right) \left( \sum_{n \geq 3} \frac{x^n}{(2\beta)^n} \right) \\
\leq \left( e^{\beta} - \beta \right)\delta \sum_{n \geq 5} \frac{x^n}{(2\beta)^n} \sum_{k=2}^{n-3} \frac{\beta^k}{k!} \\
+(e^{\beta} - \beta)\delta^2 \sum_{n \geq 6} \frac{x^n}{(2\beta)^n} \\
+2^{-2}(e^{\beta} - 1 - \beta)\delta^2 \left( \sum_{n \geq 3} \frac{x^n}{(4\beta)^n} \right) \left( \sum_{n \geq 3} \frac{x^n}{(2\beta)^n} \right) \\
+\delta^3 \left( \sum_{n \geq 3} \frac{x^n}{(4\beta)^n} \right) \left( \frac{1}{4} \sum_{n \geq 6} \frac{x^n}{(2\beta)^n} \right) \\
\leq \frac{\beta^2(e^{\beta} - \beta)}{2} \delta \frac{x^5}{(2\beta)^5} + (e^{\beta} - \beta)(e^{\beta} - 1 - \beta)\delta \sum_{n \geq 6} \frac{x^n}{(2\beta)^n} + \\
(e^{\beta} - \beta)\delta^2 \sum_{n \geq 6} \frac{x^n}{(2\beta)^n} + \frac{e^\beta - 1 - \beta}{16} \delta^2 \sum_{n \geq 8} \frac{x^n}{(2\beta)^n} + \frac{1}{16} \delta^3 \sum_{n \geq 9} \frac{x^n}{(2\beta)^n},
\]
where, to get the first term of the last inequality, we notice that $\sum_{k=2}^{n-3} \frac{\beta^k}{k!} = \beta^2/2$ for $n = 5$. Together with (21) and (18) and induction, for any $\beta$ we notice that $\sum_{n=3}^{4} x^n$ implies that $(24)$ and is $2$ for any $0 < \delta \leq 1$, we get

$$|\Phi_2(x)|^* \leq 4\delta \sum_{n=3}^{4} \frac{x^n}{(2\beta)^n} + \left[(e^\beta - \beta)(2 + \frac{\beta^2}{2}) + 2\right] \delta \frac{x^5}{(2\beta)^n} + \left[(e^\beta - \beta)^2 + \frac{33}{16}(e^\beta - \beta) + 2\right] \delta \sum_{n\geq 6} \frac{x^n}{(2\beta)^n}.$$

By taking $\beta = 1$ and $\beta = 2$ respectively, we prove the proposition. □

Now we show a strengthen version of Lemma 2.2.

**Lemma 3.3.** Let $(P_k)_{k \geq 1}$ satisfy recurrence relation (5) and $\delta \leq 1$. Assume $(P_{-1}, P_0)$ has a $\rho$-germ at $x_0$. Then

1) $(P_{-1}, P_k)$ has a $2^k \rho$-germ at $x_0$ for any $k \geq 1$.

2) If $(P_{-1}, P_0)$ is $(\delta, 1)$-regular at $x_0$, then $(P_{-1}, P_k)$ is $(2 \cdot 2^{-k/2}\delta, 1)$-regular at $x_0$ for any $k \geq 1$, is $(36 \cdot 2^{-k/2}\delta, 2)$-regular at $x_0$ for any $k \geq 2$ and is $(3200 \cdot 2^{-k/2}\delta, 4)$-regular at $x_0$ for any $k \geq 4$.

3) If $(P_{-1}, P_0)$ is $(\delta, 2)$-regular at $x_0$, then for any $k \geq 0$, $(P_{-1}, P_k)$ is $(\delta, 2)$-regular.

**Proof.** 1) follows from the definition of germ and (12).

2) Define $(Q_k)_{k \geq -1}$ and $(\Delta_k)_{k \geq -1}$ as in (13) and (14). By (15), $\Delta_{k-1}, \Delta_k$ and $\Delta_{k+1}$ satisfies the assumption of Proposition 3.2 for $k \geq 0$. By the assumption of this lemma we have

$$|\Delta_{-1}(x)|^* \leq \delta \sum_{n \geq 3} x^n, \quad |\Delta_0(x)|^* \leq \delta \sum_{n \geq 3} x^n.$$

Then by (16) and induction, for any $k \geq 1$,

$$|\Delta_{2k-1}(x)|^*, |\Delta_{2k}(x)|^* \leq 2^{-k}\delta \sum_{n \geq 3} x^n. \quad (22)$$

Consequently by the assumption, (22), (16) and induction, for any $k \geq 1$,

$$|\Delta_{2k-1}(x)|^*, |\Delta_{2k}(x)|^* \leq 18 \cdot 2^{-k}\delta \sum_{n \geq 3} \frac{x^n}{2^n}. \quad (23)$$

By (23), (17) and induction, for any $k \geq 2$,

$$|\Delta_{2k-1}(x)|^*, |\Delta_{2k}(x)|^* \leq 1600 \cdot 2^{-k}\delta \sum_{n \geq 3} \frac{x^n}{4^n}. \quad (24)$$

(22) implies that $(P_{-1}, P_k)$ is $(2 \cdot 2^{-k/2}\delta, 1)$-regular at $x_0$ for any $k \geq 1$.

(23) implies that $(P_{-1}, P_k)$ is $(36 \cdot 2^{-k/2}\delta, 2)$-regular at $x_0$ for any $k \geq 2$.

(24) implies that $(P_{-1}, P_k)$ is $(3200 \cdot 2^{-k/2}\delta, 4)$-regular at $x_0$ for any $k \geq 4$. 

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3) By the condition we have
\[ |\Delta_{-1}(x)|^* \leq \delta \sum_{n \geq 3} \frac{x^n}{2^n}, \quad |\Delta_0(x)|^* \leq \delta \sum_{n \geq 3} \frac{x^n}{2^n}. \]
Then by (17) and induction, for any \( k \geq 1 \),
\[ |\Delta_k(x)|^* \leq \delta \sum_{n \geq 3} \frac{x^n}{2^n}, \]
which implies the result.

4. PROOF OF LEMMA 2.4

Recall that \( \delta_0 = 10^{-2}, \delta_1 = 10^{-4} \) and \( \delta_2 = 10^{-16} \). We start with two simple geometric lemmas:

**Lemma 4.1.** Assume \( \varphi \) is a polynomial satisfying
\[ |\varphi(x) - 2 \cos x| \leq \delta_0, \quad \forall x \in [0, 1.9] \ (\text{resp.} \ [-1.9, 0]). \]
We further assume \( x_+ \) (resp. \( x_- \)) is the minimal \( x \in [0, \pi] \) (resp. the maximal \( x \in [-\pi, 0] \)) such that \( \varphi(x) = 0 \). Then
\[ |x_+ - \pi/2| \leq \delta_0, \quad (\text{resp.} |x_- - (-\pi/2)| \leq \delta_0). \]

**Proof.** By the assumption we have
\[ |2 \cos x_+| = |\varphi(x_+) - 2 \cos x_+| \leq \delta_0. \]
On the other hand since \( |\sin x| \geq 2|x|/\pi \) for \( |x| \leq \pi/2 \) and \( |x_+| \leq \pi/2 \)
\[ |2 \cos x_+| = 2|\sin(\frac{\pi}{2} - x_+)| \geq \frac{4}{\pi} |x_+ - \frac{\pi}{2}|. \]
This prove the lemma for \( x_+ \). The proof for \( x_- \) is analogous.

**Proposition 4.2.** Assume polynomial pair \( (P_{-1}, P_0) \) is \( (\delta_1, 2) \)-regular at \( x_0 \). Then there exist \( y_{-1} < y_0 < y_0^+ < y_{-1}^+ \) such that, for \( k = -1, 0, \)
\[ P_k(y_k^-) = P_k(y_k^+) = 0; \quad P_k(x) > 0 \ (x \in I_k^- \cup I_k^+), \]
where \( I_k^- = (y_k^-, x_0], I_k^+ = [x_0, y_k^+). \) Moreover
\[ \frac{|I^-_0|}{|I^+_0|}, \quad \frac{|I^-_{-1}|}{|I^+_{-1}|} > 2.1^{-1}. \]
Proof. Let \((P_{-1}, P_0)\) be \((\delta_1, 2)\)-regular at \(x_0\) with scaling factor \(a\).

For \(k = -1, 0\), define \(Q_k(x)\) as in (13). Then for \(|x| < 2\),
\[
|Q_{-1}(x) - 2 \cos x|, |Q_0(x) - 2 \cos x| \leq \delta_1 \sum_{n=3}^{\infty} \frac{|x|^n}{2^n} = \frac{|x|^3}{8 - 4|x|} \delta_1.
\]

Especially for \(|x| \leq 1.9\), we have
\[
|Q_{-1}(x) - 2 \cos x|, |Q_0(x) - 2 \cos x| \leq \frac{1.9^3 \delta_1}{8 - 4 \cdot 1.9} < 20 \delta_1 < \delta_0.
\]

By Lemma 4.1,
\[
\left| \frac{a(y_0^+ - x_0) - \pi}{2} \right| < \delta_0, \quad \left| \frac{a(y_{-1}^- - x_0)}{2} - \frac{\pi}{2} \right| < \delta_0.
\]

Then
\[
\frac{|I_{-1}^+|}{|I_0^+|} = \frac{|y_{-1}^- - x_0|}{|y_0^+ - x_0|} \leq \frac{2(\frac{\pi}{2} + 0.01)}{\frac{\pi}{2} - 0.01} < 2.1.
\]

The proof of the other part of the proposition is analogous. \(\square\)

The proof of Lemma 2.4 rely on the following technical proposition.

**Proposition 4.3.** Let \((P_k)_{k \geq -1}\) satisfy recurrence relation (5). Assume \((P_{-1}, P_0)\) is \((\delta_2, 4)\)-regular at \(x_0\) with scaling factor \(a\). Let \(y_0^+\) (resp. \(y_0^-\)) be the minimal \(t > x_0\) (resp. maximal \(t < x_0\)) such that \(P_0(t) = 0\). Let \(x_3^+\) (resp. \(x_3^-\)) be the maximal \(t < y_0^+\) (resp. minimal \(t > y_0^-\)) such that \(P_3(t) = 0\). Let \(I_0^+ = [x_0, y_0^+]\), \(I_0^- = [y_0^-, x_0]\), \(I_3^+ = [x_3^+, y_0^+]\) and \(I_3^- = [y_0^-, x_3^-]\). Then
\[
\frac{|I_3^+|}{|I_0^+|} \geq 2.1^{-3}, \quad \frac{|I_3^-|}{|I_0^-|} \geq 2.1^{-3}.
\]

Moreover, \((P_3, P_1)\) is \((\delta_1, 2)\)-regular at both \(y_0^+\) and \(y_0^-\).

**Proof of Lemma 2.4** We only consider the case in the interval \([x_0, y_0^+]\), another case is the same.
For $k = 0, \cdots, K$, let $y_k$ be the minimal $t > x_0$ such that $P_k(t) = 0$. Then $y_0 = y_0^+$ and $y_K = y_K^+$. For $k = 3, \cdots, K$, let $x_k$ be the maximal $t < y_k^+$ such that $P_k(t) = 0$. Then $x_K = x_K^+$.

$(P_{-1}, P_0)$ is $(\delta_2, 4)$-regular at $x_0$, then it is $(\delta_1, 2)$-regular and $(1, 1)$-regular at $x_0$ by the definition of regularity. By Lemma 3.3, $(P_{K-1}, P_K)$ is $(\delta_2, 4)$-regular at $x_0$.

By Lemma 3.3, $(P_{k-1}, P_k)$ is $(\delta_1, 2)$-regular at $x_0$ for any $0 < k \leq K$, thus by Proposition 4.2 for any $0 < k \leq K$,

$$\frac{|y_k - x_0|}{|y_k - x_0 - 1|} \geq 2.1^{-1}.$$  

And hence,

$$\frac{|y_K^+ - x_0|}{|y_0^+ - x_0|} = \frac{|y_K - x_0|}{|y_0 - x_0|} \geq 2.1^{-K}.$$  

By Proposition 4.3, $(P_3, P_4)$ is $(\delta_1, 2)$-regular at $y_0^+ = y_0$ and

$$\frac{y_0 - x_3}{y_0 - x_0} \geq 2.1^{-3}. \quad (25)$$

Same argument as above shows that $(P_{K-1}, P_K)$ is $(\delta_2, 4)$-regular at $y_0$ and $(P_{k-1}, P_k)$ is $(\delta_1, 2)$-regular at $y_0$ for any $3 < k \leq K$. By Proposition 4.2 for any $3 < k \leq K$,

$$\frac{|y_0 - x_k|}{|y_0 - x_{k-1}|} \geq 2.1^{-1}.$$  

Combining with (25) we conclude that

$$\frac{|y_K^+ - x_K^+|}{|y_0^+ - x_0|} = \frac{|y_0 - x_K|}{|y_0 - x_0|} \geq 2.1^{-K}.$$  

This finish the proof. \hfill \Box

To prove Proposition 4.3, we need the following variant of proposition 3.2.
Proposition 4.4. Assume \( \Psi_0, \Psi_1, \Psi_2 \) are real analytic functions with Taylor expansion \( \Psi_k(x) = \sum_{n \geq 0} \Psi_{k,n}x^n \), \( k = 0, 1, 2 \) and satisfy the following relation: for some \( \tau \in \mathbb{R} \)

\[
\Psi_2(x) = (2 + 2 \cos \frac{x + \tau}{2}) \cdot \Psi_1\left(\frac{x}{2}\right) + \Psi_0\left(\frac{x}{4}\right)\left(4 \cos \frac{x + \tau}{4} + \Psi_0\left(\frac{x}{4}\right)\right)\left(2 \cos \frac{x + \tau}{2} - 2 + \Psi_1\left(\frac{x}{2}\right)\right).
\]

If there exist \( 0 < \delta \leq 1 \) and \( 2 \leq \beta \leq 3 \) such that

\[
|\Psi_{0,n}|, |\Psi_{1,n}| \leq \delta \beta^{-n}, \quad \forall n \geq 0,
\]

then

\[
|\Psi_2(x)|^* \leq 400\delta \sum_{n \geq 0} \frac{x^n}{\beta^n}.
\]

Proof. Write

\[
(I) = \left(2 + 2 \cos\left(\frac{x}{2} + \frac{\tau}{2}\right)\right) \cdot \Psi_1\left(\frac{x}{2}\right),

(II) = 4 \cos\left(\frac{x}{4} + \frac{\tau}{4}\right) + \Psi_0\left(\frac{x}{4}\right),

(III) = 2 \cos\left(\frac{x}{2} + \frac{\tau}{2}\right) - 2 + \Psi_1\left(\frac{x}{2}\right).
\]

Then

\[
\Psi_2(x) = (I) + \Psi_0\left(\frac{x}{4}\right)(II)(III).
\]

Note that

\[
|\cos(x + x_0)|^* = \sum_{n \geq 0} \frac{\cos(n)x_0}{n!}x^n |^* \leq \sum_{n \geq 0} \frac{x^n}{n!}. \quad (28)
\]

We have

\[
|(I)|^* \leq \left(\sum_{n \geq 0} 4 \frac{x^n}{n!}2^n\right) \left(\sum_{n \geq 0} \frac{x^n}{2^n n!}\right) = 4\delta \sum_{n \geq 0} \frac{x^n}{(2\beta)^n} \sum_{k=0}^{n} \frac{\beta^k}{k!}
\]

\[
\leq 4\delta \sum_{n \geq 0} \frac{x^n}{\beta^n} \sum_{k=0}^{n} \left(\frac{\beta}{\delta}\right)^k \leq 4e\frac{\beta}{\delta} \delta \sum_{n \geq 0} \frac{x^n}{\beta^n}.
\]

By (28) and the assumption we also have

\[
|\Psi_0\left(\frac{x}{4}\right) \times (II)|^* \leq \left(\delta \sum_{n \geq 0} \frac{x^n}{(4\beta)^n}\right) \left(4 \sum_{n \geq 0} \frac{x^n}{4^n n!} + \delta \sum_{n \geq 0} \frac{x^n}{(4\beta)^n}\right)
\]

\[
\leq 4e\frac{\beta}{\delta} \delta \sum_{n \geq 0} \frac{x^n}{(2\beta)^n} + \delta^2 \sum_{n \geq 0} \frac{x^n}{(4\beta)^n} (n + 1)
\]

\[
\leq 4e\frac{\beta}{\delta} \delta \sum_{n \geq 0} \frac{x^n}{(2\beta)^n} + \delta^2 \sum_{n \geq 0} \frac{x^n}{(2\beta)^n}
\]

\[
\leq \left(4e\frac{\beta}{\delta} + \delta\right) \delta \sum_{n \geq 0} \frac{x^n}{(2\beta)^n}.
\]
Then we have
\[
|\Psi_0(\frac{x}{\pi}) \times (II) \times (III)|^* \
\leq (4e^{\frac{\beta}{2}} + \delta) \delta \left( \sum_{n \geq 0} \frac{x^n}{(2n)!} \right) \left( 4 \sum_{n \geq 0} \frac{x^n}{(2n)!} + \delta \sum_{n \geq 0} \frac{x^n}{(2n)!} \right) 
\]
\[
= (4e^{\frac{\beta}{2}} + \delta) \delta \left( 4 \sum_{n \geq 0} \frac{x^n}{(2n)!} \right) \left( 4 \sum_{n \geq 0} \frac{x^n}{(2n)!} + \delta \sum_{n \geq 0} \frac{x^n}{(2n)!} (n + 1) \right) 
\]
\[
\leq (4e^{\frac{\beta}{2}} + \delta)^2 \delta \sum_{n \geq 0} \frac{x^n}{(2n)!}. 
\]

Since $0 < \delta \leq 1$ and $2 \leq \beta \leq 3$, by (27) we have
\[
|\Psi_2(x)|^* \leq \left( 4e^{\frac{\beta}{2}} + (4e^{\frac{\beta}{2}} + \delta)^2 \right) \delta \sum_{n \geq 0} \frac{x^n}{(2n)!} \leq 400 \delta \sum_{n \geq 0} \frac{x^n}{(2n)!}. 
\]

This proves the proposition.\(\square\)

**Proof of Proposition 4.3.** Recall that we have defined
\[
Q_k(x) = P_k\left(\frac{x}{2^k} + x_0\right) = 2 \cos x + \Delta_k(x), \quad k = -1, 0. \tag{29}
\]

Since \((P_{-1}, P_0)\) is \((\delta_2, 4)\)-regular, we have
\[
|\Delta_{-1}(x)|^* \leq 2 \delta_2 \sum_{n=3}^{\infty} \frac{x^n}{4^n}, \quad |\Delta_0(x)|^* \leq 2 \delta_2 \sum_{n=3}^{\infty} \frac{x^n}{4^n}. 
\]

And consequently, for $x \in (0, 4)$ it is ready to show that for $k \geq 0$
\[
\begin{align*}
\left| \Delta_{-1}^{(k)}(x) \right| & \leq \delta_2 \left( \sum_{n=3}^{\infty} \frac{x^n}{4^n} \right)^{(k)} = \frac{\delta_2}{4^k} \left( \frac{x^3}{4-x} \right)^{(k)} \\
\left| \Delta_{-1}^{(k)}(x) \right| & = \begin{cases} 
\frac{\delta_2 x^3}{16(4-x)} & k = 0 \\
\frac{\delta_2 (x+2)}{8} + \frac{4 \delta_2}{(4-x)^3} & k = 1 \\
\frac{\delta_2}{8} + \frac{8 \delta_2}{(4-x)^5} & k = 2 \\
\frac{4k \delta_2}{(4-x)^{k+1}} & k \geq 3.
\end{cases} \tag{30}
\end{align*}
\]

\((P_{-1}, P_0)\) is \((\delta_2, 4)\)-regular implies that it is also \((\delta_2, 2)\)-regular. Let $t_0$ be the minimal $t \in (0, 2)$ such that $Q_0(t) = 0$. Then as proof in Proposition 4.2 and Lemma 4.1
\[
P_0(t_0/a + x_0) = 0, \quad |t_0 - \frac{\pi}{2}| \leq 20 \delta_2. \tag{31}
\]
Consequently $y_0^+ = t_0/a + x_0$. We have
\[
P_{-1}(y_0^+) = Q_{-1}\left(\frac{t_0}{2}\right) = 2 \cos \frac{t_0}{2} + \Delta_{-1}\left(\frac{t_0}{2}\right). 
\]
By (30) and (31) we get $|P_{-1}(y_0^+) - \sqrt{2}| \leq 20\delta_2$. Since

$$P_1(y_0^+) = P_{-1}^2(y_0^+)(P_0(y_0^+) - 2) + 2,$$

we conclude that

$$|P_1(y_0^+) + 2| \leq 120\delta_2. \quad (32)$$

By (29), $P_0'(y_0^+) = aQ_0'(t_0) = a(-2 \sin t_0 + \Delta_0'(t_0))$. By (30) and (31),

$$\left| \frac{P_0'}{a} + 2 \right| \leq 2\delta_2. \quad (33)$$

By (7) and (32), (33), for $k = 3, 4$ we have

$$P_k(x) = 2 - (2^{k-3}\rho)^2(x - y_0^+)^2 + O((x - y_0^+)^3)$$

with $\rho = \sqrt{2 - P_1(y_0^+)|P_0'(y_0^+)P_1(y_0^+)|}$. Thus

$$\frac{\rho}{a} = \sqrt{4 + \varepsilon_1(2 + \varepsilon_2)(2 + \varepsilon_1)}$$

with $|\varepsilon_1| < 120\delta_2$ and $|\varepsilon_2| < 2\delta_2$. Consequently we have

$$\left| \frac{\rho}{8a} - 1 \right| < 100\delta_2, \quad \left| \frac{8a}{\rho} - 1 \right| < 100\delta_2.$$

For $k \geq 3$, if we define $\tilde{Q}_k(x) := P_k\left(\frac{x+y_0^+}{2^{k-3}\rho}\right)$, then

$$\tilde{Q}_k(x) = 2 - x^2 + O(x^3) = 2\cos x + O(x^3) =: 2\cos x + \tilde{\Delta}_k(x). \quad (34)$$

We need to show that $(P_3, P_4)$ is $(\delta_1, 2)$-regular, i.e.,

$$|\tilde{\Delta}_3(x)|^*, |\tilde{\Delta}_4(x)|^* \leq \delta_1 \sum_{n \geq 3} \frac{x^n}{2^n}.$$

To get this result, we study $\tilde{\Delta}_k(x) := \Delta_k(x + 2^k t_0)$ first. We have

$$P_k(\frac{x}{a} + y_0^+) = P_k\left(\frac{x+t_0}{a} + x_0\right) = Q_k(2^k(x + t_0))$$

$$= 2 \cos(2^k(x + t_0)) + \Delta_k(2^k(x + t_0)) \quad (35)$$

By the recurrence relation of $P_k$, it is ready to check that

$$\Psi_0 := \tilde{\Delta}_{k-2}, \Psi_1 := \tilde{\Delta}_{k-1}, \Psi_2 := \tilde{\Delta}_k$$

satisfies (26) with $\tau = 2^k t_0$ for $k \geq 1$.

On the other hand we have

$$\begin{cases}
\tilde{\Delta}_0(x) &= \Delta_0(x + t_0) = \sum_{n=0}^{\infty} \frac{\Delta_0^{(n)}(t_0)}{n!} x^n \\
\tilde{\Delta}_{-1}(x) &= \Delta_{-1}(x + t_0/2) = \sum_{n=0}^{\infty} \frac{\Delta_{-1}^{(n)}(t_0/2)}{n!} x^n.
\end{cases}$$
Write $\beta = 4 - \pi/2 - 0.01 = 2.419 \cdots$ By (30) and (31) we get

$$|\Delta_0(x)|^*, |\Delta_{-1}(x)|^* \leq 4\delta_2 \sum_{n=0}^{\infty} \frac{x^n}{\beta^n}.$$ 

Recall that by (6) we have $4 \cdot 400^4 \delta_2 \leq \delta_1/3$. By Proposition 4.4 and induction, for $k = 3, 4$ we get

$$|\Delta_k(x)|^* \leq 4 \cdot 400^k \delta_2 \sum_{n=0}^{\infty} \frac{x^n}{\beta^n} \leq \frac{\delta_1}{3} \sum_{n=0}^{\infty} \frac{x^n}{\beta^n}. \quad (36)$$

By (34) and (35), for $k = 3, 4$ we have

$$\Delta_k(x) = \tilde{Q}_k(x) - 2 \cos x = P_k(\frac{a}{\rho} + \frac{y_0}{\rho}) - 2 \cos x = \Delta_k(\frac{8a}{\rho}x) + 2 \cos(\frac{8a}{\rho}x) - 2 \cos x = \Delta_k(\frac{8a}{\rho}x) - 2 \sin 2k t_0 \sin(\frac{8a}{\rho}x) + 2 \cos 2k t_0 \cos(\frac{8a}{\rho}x) - 2 \cos x.$$ 

Notice that $|\frac{8a}{\rho} - 1| < 100 \delta_2 = 10^{-14}$, then by (36), for $k = 3, 4$ we have

$$|\Delta_k(\frac{8a}{\rho}x)|^* \leq \frac{\delta_1}{3} \sum_{n=0}^{\infty} \frac{x^n}{2^n}. \quad (37)$$

Moreover, $|t_0 - \frac{\pi}{2}| < 20 \delta_2$ implies for $k = 3, 4$,

$$|\sin 2k t_0| < 320 \delta_2, \quad |1 - \cos 2k t_0| < 320 \delta_2.$$ 

Consequently

$$|2 \sin 2k t_0 \sin(\frac{8a}{\rho}x)|^* \leq 640 \delta_2 \sum_{n=1}^{\infty} \frac{(\frac{8a}{\rho}x)^{2n-1}}{(2n-1)!} \leq \frac{\delta_1}{3} \sum_{n=0}^{\infty} \frac{x^n}{2^n}. \quad (38)$$

Finally we have

$$|2 \cos 2k t_0 \cos(\frac{8a}{\rho}x) - 2 \cos x|^* \leq 2 \sum_{n=0}^{\infty} \frac{|\cos 2k t_0(\frac{8a}{\rho})^{2n} - 1| x^{2n}}{(2n)!} \leq \frac{\delta_1}{3} \sum_{n=0}^{\infty} \frac{x^n}{2^n}, \quad (39)$$

where we have used the fact that for $n \leq 10$,

$$|\cos 2k t_0(\frac{8a}{\rho})^{2n} - 1| \leq |\cos 2k t_0 - 1| + |(\frac{8a}{\rho})^{2n} - 1| < \delta_1/16;$$

for $n > 10$,

$$\frac{2|\cos 2k t_0(\frac{8a}{\rho})^{2n} - 1|}{(2n)!} < \frac{2 + 2(1 + 100 \delta_2)^{2n}}{(2n)!} < \frac{\delta_1}{3 \times 2^{2n}}.$$
Combine (37), (38) and (39), we conclude that for \( k = 3, 4 \)

\[
|\tilde{\Delta}_k(x)|^* = |\tilde{Q}_k(x) - 2 \cos x| \leq \delta_1 \sum_{n \geq 3} \frac{x^n}{2^n}.
\]

This proves that \((P_3, P_4)\) is \((\delta_1, 2)\)-regular at \( y^+_0 \) with scaling factor \( 2\rho \).

Since \((P_3, P_4)\) is \((\delta_1, 2)\)-regular at \( y^+_0 \), analogous to the proof of Proposition 4.2, we have

\[
|\rho(y^+_0 - x^+_3) - \frac{\pi}{2}| < \delta_0.
\]

This implies \( |\rho| I^+_{3} | - \frac{\pi}{2} | < 0.01 \).

On the other hand since \((P_{-1}, P_0)\) is \((\delta_1, 2)\)-regular at \( x_0 \) with scaling factor \( a \), we have

\[
|a(y^+_0 - x_0) - \frac{\pi}{2}| < \delta_0.
\]

This implies \( |a| I^+_{0} | - \frac{\pi}{2} | < \delta_0 \). Hence

\[
\frac{|I^+_3|}{|I^+_0|} \geq \frac{18a \frac{\pi}{2} - 0.01}{8 \rho \frac{\pi}{2} + 0.01} \geq 2.1^{-3}.
\]

This prove the result for \( y^+_0 \). The proof for \( y^-_0 \) is the same. \( \square \)

**Acknowledgements.** The authors thank Professor Jean Bellissard for many valuable suggestions. Liu and Qu are supported by the National Natural Science Foundation of China, No. 11371055. Qu is supported by the National Natural Science Foundation of China, No. 11201256.

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