Spin-orbit-phonon interaction as an origin of helical-symmetry breaking spin-triplet superconducting state

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The excess increase of the local field distribution width \( \Delta H \equiv \sigma/\gamma_\mu \) (with \( \gamma_\mu = 135.5 \text{ MHz/T} \) being the gyromagnetic ratio of \( \mu^+ \)), and/or the spontaneous magnetic field \( H_{\text{sp}} \), have been reported in the superconducting state of a series of compounds more than 10 by \( \mu\text{SR} \) (muon spin rotation or relaxation) experiment. Sizes of these quantities are of the order of 1G on the whole. We propose that the increase of the local field distribution \( \Delta H \) in those compounds, including ions with strong spin-orbit interaction of 5d electrons, such as La in LaNiGa\(_2\) or Re in Re\(_6\)Zr, is possible through the cooperation of the spin-orbit coupling between the quasiparticles and ionic vibrations (phonons) and the conventional quasiparticles-phonon coupling, inducing the spin-triplet p-wave pairing with helical symmetry breaking superconducting state with chiral super spin current around stopped \( \mu^+ \) site. The size of \( \Delta H \) is estimated as of the order of 1G, in consistent with experimental observations.

1. Introduction

In a past decade or so, the extra increase of the local field distribution (LFD) in the superconducting state has been observed by the muon spin rotation (\( \mu\text{SR} \)) measurements in a series of compounds [1]. In these measurements, the time dependence of \( \mu\text{SR} \) spectrum was analyzed by fitting to the sinusoidal oscillating function with Gaussian relaxation \( \propto \exp(-\sigma^2t^2/2) \), from which the LFD is estimated as \( \Delta H \equiv \sigma/\gamma_\mu \) with the muon gyromagnetic ratio \( \gamma_\mu = 2\pi \times 135.5 \text{ MHz T}^{-1} \) [2]. The sizes of the LFD so determined are all of the order of 1G on the whole, suggesting that there exists a common physical basis for the increase of LFD in the superconducting states.

On the other hand, the spontaneous magnetic field \( H_{\text{sp}} \), of the order of 1G has been observed by \( \mu\text{SR} \) in some superconductors, e.g., \( \text{Sr}_2\text{RuO}_4 \) [3] which is believed to be in the unitary spin-triplet p-wave chiral superconducting state \( \Delta_k = \Delta (\sin k_x a + i \sin k_y a) \) with \( a \) being the lattice constant in the \( ab \)-plane [4, 5]. This phenomenon was shown to be understood as a pair-breaking effect of \( \mu^+ \) which attracts electron on the Ru site and breaks locally the chiral Cooper pairs around \( \mu^+ \), causing the circulating super current around \( \mu^+ \) resulting in \( H_{\text{sp}} \) on \( \mu^+ \) in turn [6]. Namely, it is crucial to realize that the muon is not only the probe measuring properties of the system but changes also the local physical property of the system.

The purpose of this paper is to propose a new mechanism for the extra LFD \( \Delta H \) on the basis of a helical-symmetry breaking spin-triplet p-wave superconducting state which can give rise to the spontaneous super spin current around stopped \( \mu^+ \) leading to the increase in the LFD \( \Delta H \) in the superconducting state through the spin-flipping of \( \mu^+ \) caused by the dipole interaction between the spin of the Cooper pairs circulating around the muon. Such a superconducting state is shown to be possible by a cooperation of strong spin-orbit-phonon coupling and electron-phonon interaction.
2. Pairing Interaction Induced by Spin-Orbit-Phonon and Electron-Phonon Interactions

2.1 Spin-orbit-phonon interaction

The scattering vertex $\Gamma_{k,\sigma;k',\sigma'}^{\text{so}}(\mathbf{R}_n)$ of conduction electrons from $(\mathbf{k}, \sigma)$ to $(\mathbf{k}', \sigma')$ by spin-orbit interaction from the atom located at $\mathbf{R}_n$ is given as

$$
\Gamma_{k,\sigma;k',\sigma'}^{\text{so}}(\mathbf{R}_n) = -ig \sum_{\sigma'=\pm} \sum_{k,k'} [s_{\sigma\sigma'} \cdot (\mathbf{k} \times \mathbf{k}')] U_{k-k'}(\mathbf{R}_n) c_{k\sigma}^\dagger c_{k'\sigma'},
$$

(1)

where $g \approx \hbar e^2/4nm^*e^2 > 0$ with $m^*$ being the effective mass of quasiparticles, and $s$ is the spin operator in the unit of $\hbar$, and $U_{k-k'}(\mathbf{R}_n)$ is defined in terms of the atomic potential located at $\mathbf{R}_n$ as

$$
U_{k-k'}(\mathbf{R}_n) \equiv \int d\mathbf{r} e^{-i(k-k')\mathbf{r}} U(\mathbf{r} - \mathbf{R}_n) = e^{-i(k-k')\mathbf{R}_n} U_{k-k'},
$$

(2)

where $U_q \equiv \int d\mathbf{r} e^{-iq\mathbf{r}} U(\mathbf{r})$. Hereafter we assume that the conduction electrons are described essentially by the free dispersion and the spin degrees of freedom. It is crucial to note that such a strong spin-orbit interaction can be induced through the hybridization between conduction electrons and the electrons in the atomic orbitals which are subject to the strong spin-orbit interaction from the positive nuclear charge in heavy ions such as Re, while the direct screened Coulomb interaction from the nuclear charge is far less important.

Since $\mathbf{R}_n$ oscillates by the influence of the phonon vibrations, the position of the atom is expressed as $\mathbf{R}_n = \bar{\mathbf{R}}_n + \mathbf{u}_n$ where $\bar{\mathbf{R}}_n$ and $\mathbf{u}_n$ are the equilibrium position of $n$-the atom and the deviation from it, respectively. Then, by taking the summation with respect to $\bar{\mathbf{R}}_n$, the interaction $\Gamma^{\text{so-ph}}$ between the spin-orbit interaction and phonon vibrations is given as

$$
\Gamma^{\text{so-ph}} = -ig \sum_{\sigma,\sigma'} \sum_{k,k'} [s_{\sigma\sigma'} \cdot (\mathbf{k} \times \mathbf{k}')] (-i)(\mathbf{k} - \mathbf{k}') \cdot \mathbf{u}_{k-k'} U_{k-k'} c_{k\sigma}^\dagger c_{k'\sigma'},
$$

(3)

where $\mathbf{u}_n \equiv \sum_q \mathbf{q} e^{-iq\bar{\mathbf{R}}_n}$. Since $i\mathbf{q} \cdot \mathbf{u}_n$ is described by the phonon creation and annihilation operators as $A_q(b_q + b_q^\dagger)$ with $\mathbf{q}$ dependent coefficient $A_q$ [7], the spin-orbit coupling and phonon interaction is given by a simple form as

$$
\Gamma^{\text{so-ph}} = ig \sum_{\sigma,\sigma'} \sum_{k,k'} [s_{\sigma\sigma'} \cdot (\mathbf{k} \times \mathbf{k}')] U_{k-k'} A_{k-k'} \left( b_{k-k'} + b_{k+k'}^\dagger \right) c_{k\sigma}^\dagger c_{k'\sigma'}.
$$

(4)

2.2 Quasiparticle-phonon interaction

The quasiparticle-phonon interaction $\Gamma^{\text{el-ph}}$ is represented as

$$
\Gamma^{\text{el-ph}} = \sum_{\sigma} \sum_{k,k'} W_{k,k'} A_{k-k'} \left( b_{k-k'} + b_{k+k'}^\dagger \right) c_{k\sigma}^\dagger c_{k'\sigma'},
$$

(5)

where the structure factor $W_{k,k'}$ depends on the origin of the electron-phonon interaction. Namely, in the free electron picture where the coupling arises from the ionic charge accumulation $(-e)[-\text{div}\mathbf{u}(\mathbf{r})]$ at $\mathbf{r} = \mathbf{R}_n$ which influences the electrons through the screened Coulomb potential [7], while in the tight-binding picture, variation of ions does not break charge neutrality associated with motion of ions so that the electron-phonon coupling arises through the variation of the transfer integral corresponding to the variation of distance among ions due to the lattice vibrations [8].
2.3 General expression of pairing interaction triggered by spin-orbit-phonon and electron-phonon interactions

Spin-orbit coupling and phonon interaction and electron phonon interaction induces the pairing interaction \( V_{k-k'}^{\text{so-ph}} \) by the Feynman diagram shown in Fig. 1. Explicit form of \( V_{k-k'}^{\text{so-ph}} \) in the static limit is given as follows:

\[
V_{k-k',\sigma\sigma'}^{\text{so-ph}} = ig[\mathbf{s}_{\sigma\sigma'} \cdot (\mathbf{k} \times \mathbf{k}')]U_{k-k'}A_{k-k'}^2W_{k-k'}D_{\text{ph}}(k-k',0) + \text{ig}[\mathbf{s}_{\sigma\sigma'} \cdot (\mathbf{k} \times \mathbf{k}')]U_{k'-k}A_{k'-k}^2W_{k'-k}D_{\text{ph}}(k-k,0),
\]

where \( D_{\text{ph}}(k-k',i\epsilon_n-i\epsilon_{n'}) \) is the Matsubara Green function of phonons with \( i\epsilon_n \)'s being the fermionic Matsubara frequencies. Since the wave vector dependence of \( U_{\pm k-k'}A_{\pm k-k'}^2W_{\pm k-k'}D_{\text{ph}}(\pm k-k,0) \) is expected to be weak and negative, according to the fact that the conventional phonon mediated Cooper pair is the s-wave which is essentially wave vector independent. Then, denoting it by \(-\Lambda\), the pairing interaction [Eq. (6)] is reduced to a simple form as

\[
V_{k-k',\sigma\sigma'}^{\text{so-ph}} = -2ig\Lambda[\mathbf{s}_{\sigma\sigma'} \cdot (\mathbf{k} \times \mathbf{k}')].
\]

\[
\text{Fig. 1. Feynman diagram for the pairing interaction induced by the spin-orbit-phonon and electron-phonon interactions. The open square and filled circle represent the spin-orbit-phonon coupling and the conventional electron-phonon coupling, respectively.}
\]

3. Helical-Symmetry Breaking Cooper Pairs

The pairing interaction [Eq. (7)] has different wave vector dependence depending on the existence or non-existence of spin flipping process. Namely, in the case without spin flip \((\sigma' = \sigma = \pm)\),

\[
V_{k-k',\sigma\sigma'}^{\text{so-ph}} = -ig\Lambda\sigma(k_xk'_y - k_yk'_x),
\]

while in the case with spin flip \((\sigma' = -\sigma)\),

\[
V_{k-k',\sigma\sigma'}^{\text{so-ph}} = -g\Lambda \left[ -(k_x - ik_y)k'_x + k_y(k'_x - ik_y) \right],
\]

or

\[
V_{k-k',\sigma\sigma'}^{\text{so-ph}} = -g\Lambda \left[ (k_x + ik_y)k'_x - k_y(k'_x + ik_y) \right].
\]

Note that paring interactions [Eqs. (8)-(10)] induce triplet p-wave pairings in one form or another, and are independent of \(\sigma''\), the spin component of electrons scattered by phonons.

Hereafter, we choose the \(z\)-direction as a special and favorable one for the Cooper pair formation on the basis of the assumption that the effective mass of the quasiparticles in the \(xy\)-plane is considerably larger than that in the \(z\)-direction, which makes the spin-orbit interaction for \(S_z\) dominant compared to those for \(S_x\) and \(S_y\).
3.1 Gap equations at transition temperature

Gap symmetry at the transition temperature $T_c$ is determined by the gap equation with the pairing interactions [Eqs. (8)-(10)]. Considering these interactions induce the spin-triplet p-wave pairing, let us introduce the four gap functions $\Delta_{\sigma\sigma'}(k)$ with $(\sigma, \sigma' = +$ or $-).$ Then, the gap equations at $T = T_c$ are given by two coupled equations as follows:

$$\Delta_{++}(k) = -\sum_{k'} \left[ V_{k-k',+}^{so-ph} + V_{k-k',-}^{so-ph} \right] \Phi_{k'}$$  \hspace{1cm} (11)

$$\Delta_{--}(k) = -\sum_{k'} \left[ V_{k-k',-}^{so-ph} - V_{k-k',+}^{so-ph} \right] \Phi_{k'}$$  \hspace{1cm} (12)

and

$$\Delta_{--}(k) = -\sum_{k'} \left[ V_{k-k',+}^{so-ph} - V_{k-k',-}^{so-ph} \right] \Phi_{k'}$$  \hspace{1cm} (13)

$$\Delta_{++}(k) = -\sum_{k'} \left[ V_{k-k',-}^{so-ph} + V_{k-k',+}^{so-ph} \right] \Phi_{k'}.$$  \hspace{1cm} (14)

where $\Phi_{k'} \equiv \tanh(\xi_{k'}/2T_c)/(2\xi_{k'})$ with the dispersion $\xi_{k'}$ of the quasiparticles.

With the use of the expressions [Eqs. (8)-(10)] for the pairing interactions, Eqs. (11) and (12) are given explicitly as

$$\Delta_{++}(k) = ig\Lambda \sum_{k'} (k_x k'_y - k_y k'_x) \Delta_{++}(k') \Phi_{k'}$$  \hspace{1cm} (15)

$$\Delta_{--}(k) = -ig\Lambda \sum_{k'} (k_x k'_y - k_y k'_x) \Delta_{--}(k') \Phi_{k'}$$  \hspace{1cm} (16)

Similarly, Eqs. (13) and (14) are given explicitly as

$$\Delta_{++}(k) = -ig\Lambda \sum_{k'} (k_x k'_y - k_y k'_x) \Delta_{--}(k') \Phi_{k'}$$  \hspace{1cm} (17)

$$\Delta_{--}(k) = ig\Lambda \sum_{k'} (k_x k'_y - k_y k'_x) \Delta_{++}(k') \Phi_{k'}.$$  \hspace{1cm} (18)

In order to find possible gap symmetries satisfying Eqs. (15)-(18), we postulate the $k$ dependence of gap functions as follows:

$$\Delta_{++}(k) = \tilde{\Delta}[a_k(k_x - i k_y) + b_k(k_x + i k_y) + c_k],$$  \hspace{1cm} (19)

$$\Delta_{--}(k) = \tilde{\Delta}[a_{kd}(k_x - i k_y) + b_{kd}(k_x + i k_y) + c_{kd}],$$  \hspace{1cm} (20)

$$\Delta_{--}(k) = \tilde{\Delta}[a_k(k_x + i k_y) + b_k(k_x - i k_y) + c_k],$$  \hspace{1cm} (21)

$$\Delta_{++}(k) = \tilde{\Delta}[a_{kd}(k_x + i k_y) + b_{kd}(k_x + i k_y) + c_{kd}].$$  \hspace{1cm} (22)
Note that the two off-diagonal gaps should be the same, i.e., $\Delta_{-}(k) = \Delta_{-}(k)$, by the symmetry requirement in the manifold of spin-triplet odd-parity pairings.

Hereafter, to grasp the essence of the paring mechanism triggered by the spin-orbit interaction, we assume the dispersion of the quasiparticles is even function of $k_z$, or the inversion symmetry is preserved in the $z$-direction. Then, Eqs. (15) and (16) are reduced to

$$a_{+}(k_x - ik_y) + b_{+}(k_x + ik_y) + c_{+}k_z =$$
$$g\Lambda [a_{+}(k_x - ik_y)F_{+} + a_{+}(k_x + ik_y)F_{-} - b_{+}(k_x + ik_y)F_{+} - b_{+}(k_x - ik_y)F_{-} + 2a_{od}k_z(F_{+} - iF_{xy}) + 2b_{od}k_zF_{-} - c_{od}(k_x - ik_y)F_{z}],$$  \(23\)

and

$$a_{od}(k_x - ik_y) + b_{od}(k_x + ik_y) + c_{od}k_z =$$
$$g\Lambda [-a_{od}(k_x - ik_y)F_{+} - a_{od}(k_x + ik_y)F_{-} + b_{od}(k_x + ik_y)F_{+} + b_{od}(k_x - ik_y)F_{-} - 2a_{od}k_zF_{+} - 2b_{od}k_zF_{-} + c_{od}(k_x + ik_y)F_{z}],$$  \(24\)

respectively. In deriving these equations, we have used the relations

$$\sum_{k'}(k_{x' \gamma} - k_{y' \gamma})(k'_{x' \gamma} \pm ik'_{y' \gamma})\Phi_{k'} = \mp i[(k_{x' \gamma} \mp ik'_{y' \gamma})F_{+} + (k_{x' \gamma} \mp ik'_{y' \gamma})F_{-}],$$  \(25\)

where $F_{\pm} = (1/2) \sum_{k}[k_{x}^{2} \pm k_{y}^{2}]\Phi_{k}$, $F_{z} = \sum_{k}k_{z}^{2}\Phi_{k}$, and $F_{xy} = \sum_{k}k_{x}k_{y}\Phi_{k}$.

Similarly, Eqs. (17) and (18) are reduced to

$$a_{-}(k_x - ik_y) + b_{-}(k_x + ik_y) + c_{-}k_z =$$
$$g\Lambda [-a_{-}(k_x - ik_y)F_{+} - a_{-}(k_x + ik_y)F_{-} + b_{-}(k_x + ik_y)F_{+} + b_{-}(k_x - ik_y)F_{-} - 2a_{od}k_zF_{+} - 2b_{od}k_zF_{-} + c_{od}(k_x + ik_y)F_{z}],$$  \(26\)

and

$$a_{od}(k_x - ik_y) + b_{od}(k_x + ik_y) + c_{od}k_z =$$
$$g\Lambda [a_{od}(k_x - ik_y)F_{+} + a_{od}(k_x + ik_y)F_{-} - b_{od}(k_x + ik_y)F_{+} - b_{od}(k_x - ik_y)F_{-} + 2a_{-}k_z(F_{+} - iF_{xy}) + 2b_{-}k_zF_{-} - c_{-}(k_x - ik_y)F_{z}].$$  \(27\)

### 3.2 Helical-symmetry broken pairing

Since coupled equations Eqs. (23), (24), (26) and (27) are still complicated, we make further simplification by assuming the dispersion of the quasiparticles satisfies the mirror symmetry concerning $xz$, $yz$ and $(x-y)z$ planes, resulting in $F_{xy} = F_{-} = 0$. Then, we obtain the relations among coefficients in a compact form. Indeed, from a set of equations [Eqs. (23) and (24)], we obtain

$$a_{+} = g\Lambda (a_{+}F_{+} - c_{od}F_{z}),$$  \(28\)
$$b_{+} = -g\Lambda F_{+}b_{+},$$  \(29\)
$$c_{+} = 2g\Lambda (a_{od} + b_{od})F_{+},$$  \(30\)
$$a_{od} = -g\Lambda F_{+}a_{od},$$  \(31\)
$$b_{od} = g\Lambda (b_{od}F_{+} + c_{+}F_{z}),$$  \(32\)
$$c_{od} = -2g\Lambda (a_{+} + b_{+})F_{+}.$$  \(33\)

Similarly, from a set of equations [Eq. (26) and Eq. (27)], we obtain

$$a_{-} = -g\Lambda F_{+}a_{-}.$$  \(34\)
From Eqs. (29), (31), (34) and (38), it is obvious that \( b_+ = a_{od} = a_- = b_{od} = 0 \), resulting in \( c_+ = c_- = 0 \) through the relations [Eqs. (30) and (36)]. Therefore, \( \Delta_{++} \) [Eq. (19)], \( \Delta_{--} \) [Eq. (21)], and \( \Delta_{+-} \) [Eq. (20)], \( \Delta_{+--} \) [Eq. 22)] are given as

\[
\Delta_{++}(k) = \tilde{\Delta} a_+(k_x - ik_y), \\
\Delta_{--}(k) = 0,
\]
and

\[
\Delta_{+-}(k) = \tilde{\Delta} b_-(k_x + ik_y), \\
\Delta_{+--}(k) = 0,
\]
respectively. This superconducting gap matrix \( \tilde{\Delta}(k) \) represents a sort of helical-symmetry broken state in the sense that the direction of the circulating spin current carried by the Cooper pairs is in the xy-plane and fixed in the clockwise when looking from positive side of the z-axis. This state is similar to the BW state in the sense that the Cooper pairs with \( \uparrow \uparrow \) and \( \downarrow \downarrow \) components have opposite circulations resulting in the spin current near the system boundary (wall of the vessel in the case of superfluid \( ^3\text{He} \)) in principle. Indeed, the gap \( \Delta_{\uparrow \uparrow} \) and \( \Delta_{\downarrow \downarrow} \) in the BW state are reproduced by taking \( a_+ = -b_- \) in Eqs. (40) and (42). Of course we need to take into account the effect of higher order terms in \( \Delta \)'s in the Landau-Ginzburg free energy expansion, which is out of scope of the present note.

The transition temperature \( T_c \) is given by solving the coupled equation [Eqs. (28) and (33); Eqs. (35) and (39)] as

\[
1 = g\Delta F_s(T_c) \left[ 1 + g\Delta F_e(T_c) \right].
\]

4. Extra Local Field Distribution \( \Delta H \) on \( \mu^+ \) by Cooper-Pair Spin Current through Magnetic Dipole Interaction among the Muon and the Cooper Pairs

The dipole-dipole interaction \( H_d \) between the muon spin \( \mu \) and electron spin \( S_i \) at \( r_i \) is given by

\[
H_d = \mu_B^2 \frac{\gamma_\mu \gamma_e (\mu \cdot S_i) r_i^2 - 3(\mu \cdot r_i)(S_i \cdot r_i)}{r_i^3},
\]

where \( \mu_B \) is the Bohr magneton, \( \gamma_\mu \) and \( \gamma_e \) are gyro-magnetic ratio of muon and electron, respectively, and \( r_i \) is the position vector of electron measured from the position of stopped muon in the crystal as shown in Fig. 2 which shows the clock-wise circular motion (along the dashed circle) of the Cooper pairs in the plane perpendicular to the z-axis with nearly constant angular frequency \( \omega_j \approx \hbar/2m^* r_{j,\perp} \) for a certain \( r_{j,\perp} \), with \( m^* \) being the effective mass of the quasiparticles. Note that such a spin current of Cooper-pairs is induced (by the pair-breaking effect of \( \mu^+ \) as discussed in Ref. 6 for the charge current of Cooper-pairs in the chiral superconducting state of \( \text{Sr}_2\text{RuO}_4 \)) around the stopped \( \mu^+ \) at \( r < \xi \), with \( \xi \) being the size of the Cooper pairs.

Hereafter, we retain only the terms including \( \tilde{\mu}_\pm \equiv \tilde{\mu}_x \pm i\tilde{\mu}_y \) in Eq. (45) because we are interested in the extra LFD \( \Delta H \) induced by the circular motions of the Cooper pairs. Note that the spins of
Cooper pairs are vanishing because \( \uparrow \uparrow \) and \( \downarrow \downarrow \) pairs gives no spin polarization while they induce the spin current in one-direction, breaking the helical-symmetry as shown in Fig. 2. Then, after straightforward calculations, we obtain

\[
H_d = \mu_B^2 g_\mu g_e \sum_i \left\{ \frac{3 \cos^2 \theta_i}{4r^3} \left[ \hat{\mu}_+ S_i e^{-2i\phi_i(t)} + \hat{\mu}_- S_i e^{2i\phi_i(t)} \right] + \frac{3 \sin(2\theta_i)}{4r^3} \left[ \hat{\mu}_+ e^{-i\phi_i(t)} + \hat{\mu}_- e^{i\phi_i(t)} \right] S_{iz} \right\},
\]  

(46)

where \( S_{i,\pm} \equiv S_{i,x} \pm S_{i,y}, \phi_i(t) \equiv \phi_i - \omega_it \), and only terms, including \( e^{\pm i\phi_i(t)} \) and \( e^{\pm 2i\phi_i(t)} \) have been retained as discussed above. Note, however, that the first term in Eq. (46) gives no contribution to \( \Delta H \) because \( S_{i,\pm} = 0 \) in the helical-symmetry broken superconducting state discussed in Sect. 3.

Since the spin current of the Cooper pairs flows at around \( r_i \sim \xi \), a fundamental magnetic field size \( \bar{H}_\mu \), which the \( \mu^+ \) feels from each \( S_{i,z} \) through the dipole-dipole interaction Eq. (46), is

\[
\bar{H}_\mu \sim \frac{\mu_B}{\xi^3} \langle S_{iz} \rangle,
\]

(47)

where \( \langle S_{iz} \rangle \) is the z-component of the spin of the Cooper pairs at \( r_i \) and is given roughly as

\[
\langle S_{iz} \rangle \sim \frac{\Delta}{E_F^*},
\]

(48)

where \( \Delta \) and \( E_F^* \) are the superconducting gap and the effective Fermi energy of the quasiparticles. Considering a typical case \( \xi \sim 10^{-7} [\text{m}] (=10^3 \text{ [\AA]} ) \) and using \( \mu_B \approx 9.3 \times 10^{-24} \text{ [JT]} \), \( \bar{H}_\mu/\langle S_{iz} \rangle \sim \mu_B/\xi^3 \) is estimated as

\[
\frac{\bar{H}_\mu}{\langle S_{iz} \rangle} \sim 10^{-2} \text{ [T]} = 10^2 \text{ [G]}. 
\]

(49)

The size of the extra local field distribution \( \Delta H \) is roughly estimated by summing up the magnetic field contribution on the \( \mu^+ \) [given by Eq. (46)] from the sites \( r_i \) extending within the distance \( \xi \) from a
\( \mu^+ \) site, and by obtaining the mean-square root over the circulating period \( T_{\text{period}} \sim 4\pi m^* \xi^2 / \hbar \) for \( r \sim \xi \), which is estimated to be far smaller than the life-time of the \( \mu^+ \) of the order of \( 10^{-6} \) [sec] because \( T_{\text{period}} \sim 10^{-8} \) [sec] for \( \xi = 10^{-7} \) [m] (= \( 10^3 \) [Å]) as above. Note that the typical angular frequency of \( \omega_i \) in \( \phi_i(t) \equiv \phi_i - \omega_i t \) is given by \( 2\pi / T_{\text{period}} \).

The quantity in the bracket of the second term in the brace of Eq. (46) is equal to \( 2(\mu_x + \mu_y) \sin \phi_i(t) \) so that this term represents the existence of an oscillating magnetic field (with the angular frequency \( \omega_i \)) on the muon spin. \( (\mu_x + \mu_y) \), from the Cooper pairs whose center is located at \( r_i \). At fixed time \( t \), the length \( r_i \equiv |r_i| \), and the polar angle \( \theta_i \), the summation with respect to \( \phi_i \) around the circular orbit gives the magnetic field of the order of \( \bar{H}_\mu \) [Eq. (47)]. This is because \( \phi_i \) is distributed discretely on the circular orbit due to the atomic structure of ions. Then, at fixed \( t \) and \( r_i \), the summation with respect to \( \theta_i \) gives a factor far smaller than 1 but non-vanishing in general, considering the fact that the distribution of the circular orbit is not symmetric (depending on the stopping position of \( \mu^+ \)) and discrete as in that of \( \phi_i \). Finally, the summation with respect to \( r_{i,z} \) gives a factor \( \xi / a \), with \( a \) being the unit-cell size. Therefore, with the use of Eqs. (46) - (48), the magnetic field \( H_\mu(t) \) acting on the \( \mu^+ \) spin component \( (\mu_x + \mu_y) \) is roughly estimated as follows:

\[
H_\mu(t) \sim -3 \langle \sin(2\theta_i) \rangle \frac{\bar{H}_\mu}{\langle S^z \rangle E^*_F} \frac{\Delta \xi}{a} \sin(\bar{\omega}t),
\]

(50)

where \( a \) is the lattice constant which is of the same order as the mean distance between quasiparticles, and \( \bar{\omega} (= 2\pi / T_{\text{period}}) \) is the average angular frequency of circular motion of the Cooper pairs.

The extra LFD \( \Delta H \) in the superconducting state is obtained by taking the mean-square root average of \( H_\mu(t) \) [Eq. (50)] with respect to the circular motion as

\[
\Delta H \sim \frac{3}{2} \langle \sin(2\theta_i) \rangle \frac{\bar{H}_\mu}{\langle S^z \rangle E^*_F} \frac{\Delta \xi}{a}.
\]

(51)

With the use of Eq. (49) and borrowing the relation of \( \Delta(T = 0) / E^*_F = (2/\pi^2)(a/\xi(T = 0)) \), valid in the s-wave pairing, the extra LFD \( \Delta H(T = 0) \) is estimated as

\[
\Delta H(T = 0) \sim \frac{3 \times 10^2}{\pi^2} \langle \sin(2\theta_i) \rangle [\text{G}].
\]

(52)

This value of \( \Delta H(T = 0) \) is consistent with that observed in a series of compounds by the \( \mu \)SR measurement if \( \langle \sin(2\theta_i) \rangle \) is \( O(10^{-1}) \), i.e., \( \Delta H(T = 0) \sim 1 \text{G} \).

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