2-Dimensional Groups with Action: The Category of Crossed Module of Groups with Action

A. Odabaş and E. Soylu Yılmaz

Department of Mathematics and Computer Science, Osmangazi University, Eskisehir, Turkey

Abstract

In this paper, we define the notion of crossed modules of groups with action and investigate related structures. Functions for computing of these structures have been written using the GAP computational discrete algebra programming language.

Key Words: Group with action; category; Gap computational discrete algebra programming language.

Classification: 18-08, 18G50.

1 Introduction

Crossed modules can be considered as two-dimensional of algebraic structures were first mentioned by Whitehead in 1941 [21]. Later Whitehead named crossed modules in [22] as an additional note of his early work. These concepts aroused in the paper entitled ‘Combinatorial homotopy II’ [23] paper which also introduced the substantial concept of a free crossed module. Many generalizations of crossed module are given in the paper of Janelidze [12].

The term 2-group (two-dimensional group) can be considered as a cluster of well-matched categories of crossed modules and also the cat-1 groups which are the pair of groups. Both crossed modules and cat-1 groups can be viewed as a Moore complex of a simplicial group.

The notion of the group with action first appeared in Datuashvili’s paper [9]. Datuashvili demonstrated this structure by one of the three problems of Loday mentioned in [15, 16]. The problem is associated with the Leibniz algebras to define the algebraic correspondence called ‘coquecigrue’ object as Lie group structure. Leibniz algebras are closely related with the lower central series of a group constructed and examined in [24] by Witt.

Leibniz algebras was firstly identified with the research of Loday in 1989 [14]. A Leibniz algebra is a non-associative equivalent of a Lie algebra. To explain this association, the functor

\[ F : \text{Gr}_{\text{Lie}} \rightarrow \text{Lie} \]

was constructed by Witt [24] in 1937 where \( \text{Gr}_{\text{Lie}} \) is the category of Lie groups and \( \text{Lie} \) is the category of Lie algebras. A group with action arises from the analogous version of the above functor taking the Leibniz algebras instead of Lie algebras. Since Leibniz algebras are non-associative congruence of Lie algebras, a group with action takes place of Lie groups. Therefore, Datuashvili defined the functor

\[ \text{Gr}^* \rightarrow \text{Leibniz} \]

from the category, \( \text{Gr}^* \), of group with action to the category, \( \text{Leibniz} \), of Leibniz algebras (see for details [9, 10]).
Likewise obtaining a group version of Lie algebras, the category of Leibniz algebras are equivalent to the category of group with action.

A shared package XMod [1], for the GAP [11], computational discrete algebra system was described by Wensley et al. which contains functions for computing crossed modules of groups and cat\(^1\)-groups and their morphisms. Thereafter, the algebraic version of a GAP package XModAlg [5] was given by Arvasi and Odabas (see [4]). In this paper, we describe a package XModGwA for GAP which constructs crossed modules of groups with action (see [18]).

In this paper, we investigate the simplicial group with action corresponding to a crossed module of group with action. We also give a natural equivalence for these structures.

Shortly, we can summarize the purpose of this paper as:

- To construct the two-dimensional group with action as crossed module of group with action,
- To determine the action conditions between two group with actions,
- To compose a group with action crossed module by an action,
- To give the categorical equivalences between other structures.

2 The Main Text

The concept of crossed module, generalizing the notion of a G-module, was introduced by Whitehead [23] in the course of his studies on the algebraic structure on the second relative homotopy group. We first recall the crossed modules of groups from [19, 3]. A crossed module of groups is a homomorphism \( \partial : S \to R \) where \( S \) and \( R \) are groups and there exist an action of \( R \) on \( S \) denoted by \((r,s) \mapsto r \cdot s\). These data must satisfy the following two conditions,

\[
\text{CM1) \ } \partial \text{ is } R\text{-equivariant, so } \partial(r \cdot s) = r \partial(s)r^{-1} \\
\text{CM2) Peiffer rule; } \partial(s) \cdot s_1 = ss_1s^{-1}
\]

where \( r \in R \) and \( s, s_1 \in S \).

A crossed module is written by \( \mathcal{X} = (\partial : S \to R) \) notation. The groups \( S,R \) and the group homomorphism \( \partial \) are called the source, range and boundary of \( \mathcal{X} \) respectively. When only the first of these conditions is satisfied, the resulting structure is a pre-crossed module. Given a pre-crossed module \( \mathcal{X} = (\partial : S \to R) \), one can form an internal directed graph in the category of groups simply by forming the semidirect product \( S \rtimes R \) and taking the source and target to send an element \((s,r)\) to \( r \) and \( \partial(s)r \) respectively.

Example 1.

- Let \( G \) be a group and \( N \) be a normal subgroup of \( G \). \( \mathcal{X} = (\text{inc} : N \hookrightarrow G) \) is a crossed module. Where \( G \) acts on \( N \) by conjugation.

- Let \( M \) be a \( G \)-module. \( \mathcal{X} \) is a crossed module with trivial morphism \( 0_M : M \to G, m \mapsto 1_G \).

- For a group \( H, \alpha : H \to \text{Aut}(H) \) homomorphism represents the action so that \( \mathcal{X} = (\alpha : H \to \text{Aut}(H)) \) is a crossed module.

- A central extension crossed module has boundary as surjection \( \partial : S \to R \) with the central kernel, where \( r \in R \) acts on \( S \) by conjugation with \( \partial^{-1}r \).

- The direct product of \( \mathcal{X} = (\partial : S \to R) \) and \( \mathcal{X}' = (\partial' : S' \to R') \) is \( \mathcal{X} \times \mathcal{X}' = (\partial \times \partial' : S \times S' \to R \times R') \) with \( R, R' \) acting trivially on \( S, S' \) respectively.
Let $X = (\partial : S \to R)$ and $X' = (\partial' : S' \to R')$ be two crossed modules. The diagram

\[
\begin{array}{ccc}
S & \xrightarrow{\theta} & S' \\
\downarrow \partial & & \downarrow \partial' \\
R & \xrightarrow{\psi} & R'
\end{array}
\]

is commutative. That is for $r \in R$ and $s \in S$

\[\partial'(\theta(s)) = \psi(\partial(s))\]

and $\theta$ preserves the action of $R$ on $S$.

\[\theta(r \cdot s) = \psi(r) \cdot \theta(s)\]

then $\theta$ is called the morphism of crossed modules. This states a category of crossed modules, $\textbf{XMod}$.

A simplicial group $G$ consists of a family of groups $\{G_n\}$ together with face and degeneracy maps $d^i_n : G_n \to G_{n-1}$, $0 \leq i \leq n$, $(n \neq 0)$ and $s^i_n : G_n \to G_{n+1}$, $0 \leq i \leq n$, $(n \neq 0)$ satisfying the usual simplicial identities given in [17, 8]. The category of simplicial groups is denoted by $\textbf{SimpGrp}$.

The Moore complex $NG$ of a simplicial group $G$ is defined to be the normal chain complex $(NG, \partial)$ with

\[NG_n = \bigcap_{i=0}^{n-1} \ker d_i\]

and with $\partial_n : NG_n \to NG_{n-1}$ induced from $d_n$ by restriction. The $n^{th}$ homotopy group $\pi_n(G)$ of $G$ is the $n^{th}$ homology of the Moore complex of $G$, i.e.

\[\pi_n(G) \cong H_n(NG, \partial) = \bigcap_{i=0}^{n} \ker d_i^n / d_{i+1}^{n+1}(\bigcap_{i=0}^{n} \ker d_{i+1}^{n+1}).\]

We say that the Moore complex $NG$ of a simplicial group $G$ is of length $k$ if $NG_n = 1$ for all $n \geq k + 1$. We denote the category of simplicial groups with Moore complex of length $k$ by $\textbf{SimpGrp}_{\leq k}$.

**Theorem 1.** The category of crossed modules is equivalent to the category of simplicial groups with Moore complex of length 1. (see [13])

### 3 Group with Action

In this section, we recall the definition of group with action given by Datuashvili in [9].

Let $G$ be a group. A map $\varepsilon : G \times G \to G$ represents the right action on itself. For $g, g', g'' \in G$,

\[
\varepsilon(g, g' + g'') = \varepsilon(\varepsilon(g, g'), g'') \\
\varepsilon(g, 0) = g \\
\varepsilon(g' + g'', g) = \varepsilon(g', g) + \varepsilon(g'', g) \\
\varepsilon(0, g) = 0
\]

with the above conditions, $G^*$ is called the group with action. The action is denoted by $\varepsilon(g, h) = g^h$, for $g, h \in G$. Here the group operation is addition.
Let \((G, \epsilon)\) and \((G', \epsilon')\) be group with actions. A morphism between group with actions is denoted with the following diagram.

\[
\begin{array}{c}
G \times G \\
\downarrow \phi \\
G'
\end{array} \quad \begin{array}{c}
G' \times G' \\
\downarrow \phi' \\
G''
\end{array}
\]

with \(\phi : G \to G'\) map. Furthermore for \(g, h \in G\)

\((G, \epsilon) \to (G', \epsilon')\)

Examples

- Let \(G\) be a group. Consider \(G^\ast\) as a group with action with the (right) action by conjugation.
- Consider the group klein four \(Kl_4 = \{e, a, b, ab\}\). We have ten groups with action obtained from \(Kl_4\). Three of them are denoted by \((Kl_4, \epsilon_i), i = 1, 2, 3\). The tables of the actions \(\epsilon_i\) are as follows:

| \(\epsilon_1\) | e | a | b | ab |
|------------|---|---|---|----|
| e          | e | a | b | ab |
| a          | e | a | b | ab |
| b          | e | a | ab| b  |
| ab         | e | a | ab| b  |

| \(\epsilon_2\) | e | a | b | ab |
|------------|---|---|---|----|
| e          | e | a | b | ab |
| a          | e | ab| b | a  |
| b          | e | a | ab| b  |
| ab         | e | ab| b | a  |

| \(\epsilon_3\) | e | a | b | ab |
|------------|---|---|---|----|
| e          | e | a | b | ab |
| a          | e | a | b | ab |
| b          | e | a | ab| b  |
| ab         | e | a | b | ab |

where the \(ij\)th element shows the right action of the \(i\)th element on the \(j\)th element for \(i, j \in \{1, 2, 3, 4\}\)

Definition 1. Let \(G^\ast\) be a group with action and \(A\) be a nonempty subset of \(G\). If the conditions

i) \(A\) is a normal subgroup of \(G\) as a group,

ii) \(a^g \in A\), for \(a \in A\) and \(g \in G\),

iii) \(-g + g^a \in A\), for \(a \in A\) and \(g \in G\),

satisfied, then \(A^\ast\) is called an ideal of \(G^\ast\) [9].

Condition 1: For each \(x, y, z \in G\),

\[
x - x^{(z^*)} + x^{y+z^*} - x + x^z - x^{z+y^z} = 0.
\]

In [9], category of Abelian groups with action satisfying this condition and category of Lie-Leibniz algebras were defined. Then it was proved that the analogue of Witt’s construction defined a functor from the category of groups with action to category of Lie-Leibniz algebras which gave rise to Leibniz algebras (introduced in [15]) over the ring of integers.

Example 2. Each group with the trivial action satisfies Condition 1.
4 Crossed Modules of Groups with Action

In this section, we will define an action between two groups with actions and a new category \( \text{XMod}^* \) called the category of crossed module of groups with action. Let \( S^* = (S, \varepsilon_S) \) and \( R^* = (R, \varepsilon_R) \) be groups with action. We can use the exact sequence to define the action. For \( s \in S, r \in R \) the action \((r, s) \mapsto r \cdot s\) of \( R \) on \( S \) can be represented with the following sequence:

\[
0 \longrightarrow S \xrightarrow{i} K \xrightarrow{j} R \longrightarrow 0
\]

Existence of the function \( t \) with \( jt = 1_R \) is the main property for the described exact sequence and the equation \( r \cdot s = -t(r) + i(s) + t(r) \) denotes the \( R \)-action on \( S \). For \( s, s_1 \in S \) and \( r, r_1 \in R \)

i) \((r + r_1) \cdot s = r \cdot (r_1 \cdot s)\)

ii) \( r \cdot (s + s_1) = r \cdot s + r \cdot s_1 \)

iii) \( 0_R \cdot s = s, r \cdot 0_S = 0_s \)

In this category, there must be two derived actions of \( R \) on \( S \) corresponding to the group operations. The first one is defined above. We define the second action as \( r \cdot s = s^{t(r)} \).

The exact sequences with left group actions on itself can be represented by the following diagram.

\[
\begin{array}{ccc}
S & \xrightarrow{i} & K & \xrightarrow{j} & R \\
\downarrow{i_S} & & \downarrow{j} & & \downarrow{j_R} \\
S & \xrightarrow{i} & K & \xrightarrow{j} & R \\
\downarrow{j} & & \downarrow{j} & & \downarrow{j} \\
\end{array}
\]

We obtained the following equalities via the diagram.

iv) \((r + r_1) \cdot s = r \cdot (r_1 \cdot s)\)

v) \( r \cdot (s + s_1) = r \cdot s + r \cdot s_1 \)

vi) \( 0_R \cdot s = s, r \cdot 0_S = 0_s \)

vii) \( r \cdot (r_1 \cdot s) = (r_1^r) \cdot (r \cdot s) \)

viii) \((r \cdot s)^{(r, s_1)} = r \cdot s^{s_1}\)

Definition 2. Let \( S^* = (S, \varepsilon_S) \) and \( R^* = (R, \varepsilon_R) \) be groups with action. Denote \( \varepsilon_S(s, s_1) = s_1^s \) and \( \varepsilon_R(r, r_1) = r_1^r \) for \( s, s_1 \in S \) and \( r, r_1 \in R \). Let

\[
\cdot : R \times S \longrightarrow S \quad \text{and} \quad * : R \times S \longrightarrow S
\]

be actions of \( R \) on \( S \). We denote the group operation additively, nevertheless the group is not abelian. Moreover, let \( \varphi_S \) and \( \varphi_R \) be conjugate actions on \( S \) and \( R \) respectively. Using commutativity of the diagrams,
Definition 3. Let \( \alpha \) be a crossed module of groups with action. A crossed module of groups with action is written by \( \alpha \in \text{XMod}^* \). \( \alpha \) is a crossed module in the category of groups with action by following conditions

\[
\begin{align*}
\text{CM1)} & \quad \partial(r \cdot s) = \varphi_R(r, \partial(s)) = -r + \partial(s) + r \\
\text{CM2)} & \quad \partial(s) \cdot s_1 = 1_S(\varphi_S(s, s_1)) = \varphi_S(s, s_1) = -s + s_1 + s \\
\text{CM3)} & \quad \partial(r \ast s) = \varepsilon_R(r, \partial(s)) = \partial(s)^r \\
\text{CM4)} & \quad \partial(s) \ast s_1 = 1_S(\varepsilon_S(s, s_1)) = \varepsilon_S(s, s_1) = s_1^s
\end{align*}
\]

A crossed module of groups with action is written by \( \lambda^* = (\partial : S^* \to R^*) \) notation. When only \( \text{CM1} \) and \( \text{CM3} \) conditions are satisfied, the resulting structure is a pre-crossed module.

Examples

- Let \( R^* \) be a group with action and \( S^* \) be an ideal of \( R^* \). \( \lambda^* = (\text{inc} : S^* \hookrightarrow R^*) \) is a crossed module of groups with action. Where \( R \) acts on \( S \) by \( r \cdot s = -r + s + r \) and \( r \ast s = \varepsilon_R(r, s) = s^r \).

- Let \( G^* \) be a group with action and \( M \) be any \( G \)-module. Using \( \varepsilon_M(m, m_1) = m_1^m = m_1 \) trivial action \( M^* \) is a group with action. Then \( \lambda^* = (0 : M^* \to G^*) \) is a crossed module of groups with action. Where the boundary of \( \lambda^* \) is \( (m, g) \mapsto e_G \) zero morphism and \( G \) acts on \( M \) by \( (g, m) \mapsto g \cdot m \) and \( (g, m) \mapsto g \ast m \) any two actions.

- The direct product of \( \lambda_1^* = (\partial_1 : S_1^* \to R_1^*) \) and \( \lambda_2^* = (\partial_2 : S_2^* \to R_2^*) \) is \( \lambda_1^* \times \lambda_2^* = (\partial_1 \times \partial_2 : S_1^* \times S_2^* \to R_1^* \times R_2^*) \) a crossed module of groups with action with \( R_1, R_2 \) acting trivially on \( S_1, S_2 \) respectively.

Definition 3. Let \( \lambda_1^* = (\partial_1 : S_1^* \to R_1^*) \) and \( \lambda_2^* = (\partial_2 : S_2^* \to R_2^*) \) be crossed modules of groups with action. A crossed module of groups with action morphism

\[
(\alpha, \beta) : (\partial_1 : S_1^* \to R_1^*) \longrightarrow (\partial_2 : S_2^* \to R_2^*)
\]

is a pair of homomorphisms \( \alpha : S_1^* \to S_2^*, \beta : R_1^* \to R_2^* \) such that

\[
\begin{align*}
i) & \quad \beta \partial_1(s_1) = \partial_2 \alpha(s_1) \quad \text{for all} \ s_1 \in S_1 \\
ii) & \quad \alpha(r_1 \cdot s_1) = \beta(r_1) \cdot \alpha(s_1) \quad \text{for all} \ s_1 \in S_1, r_1 \in R_1 \\
iii) & \quad \alpha(r_1 \ast s_1) = \beta(r_1) \ast \alpha(s_1)
\end{align*}
\]

So we get the category of crossed module of groups with action. It is denoted by \( \text{XMod}^* \). If \( (\alpha, \beta) : (\partial_1 : S_1^* \to R_1^*) \longrightarrow (\partial_2 : S_2^* \to R_2^*) \) is a crossed module of groups with action morphism such that \( \alpha \) and \( \beta \) both
isomorphisms then \((\alpha, \beta)\) is called an isomorphism. The kernel of \((\alpha, \beta), \ker(\alpha, \beta)\) is the crossed module of groups with action \((\partial : \ker \alpha \to \ker \beta)\).

5 Simplicial Group with Action

Definition 4. Let \([n]\) be an ordered set. For \([n]\) and \([m]\) ordered sets, \(f : [n] \to [m]\) monotone function is called the operator. \(\Delta [n]\) category have \([n]\) objects and \(f\) operator morphism. \(\Delta^{\text{op}}[n]\) is the dual of the \(\Delta [n]\) category.

Let \(\text{Gr}^*\) be the category of group with action. The functor

\[
G^* : \Delta^{\text{op}}[n] \to \text{Gr}^*
\]

is the simplicial group with action.

The operator of the simplicial category can be represented with two special operators, \(\delta^n_i\) and \(\sigma^n_j\):

\[
G^*([n]) = G^n_n, G^*(\delta^n_i) = d^n_i \text{ and } G^*(\sigma^n_j) = s^n_j
\]

The category of simplicial group with action can be defined and is denoted by \(\text{SimpGrp}^*\).

Definition 5. Let \(G^*\) be a simplicial group with action. Let

\[
NG^n_n = \bigcap_{i=0}^{n-1} \ker d^n_i
\]

and the restriction of function \(d^n_n\)

\[
\partial_n : NG^n_n \to NG^*_{n-1}
\]

is defined. So the chain complex

\[
(NG^*) : \ldots NG^*_2 \xrightarrow{\partial_2} NG^*_1 \xrightarrow{\partial_1} NG^*_0 = G^*_0
\]

is called the Moore chain complex of simplicial group with action. In the chain complex

\[
NG^*_0 = G^*_0, NG^*_1 = \ker d^*_0, NG^*_2 = \ker d^*_0 \cap \ker d^*_1,
\]

are denoted.

The following result of different versions (such as for Lie, group, algebra) can be found in [2, 19]

Theorem 2. The category of crossed module of groups with action \(\text{XMod}^*\) is equivalent to the category of simplicial group with action \(\text{SimpGrp}^*\) with Moore Complex of length 1.

Proof. Let \(G^*\) be a simplicial group with action with Moore complex of length 1. For the group with action crossed modules,

\[
NG^*_2 \cap D_2 = 0 \text{ and } \partial_2(NG^*_2 \cap D_2) = \partial_2(G^*_2 \cap D_2) = \{0\}
\]

is satisfied. \(D_2\) is a subgroup generated by \(s_j\) degenerated operator in \(G^*_2\).

\[
(NG^*) : \ldots NG^*_2 \xrightarrow{\partial_2} NG^*_1 \xrightarrow{\partial_1} NG^*_0
\]

chain complex has a sub-complex as

\[
NG^*_1 \to NG^*_0
\]
defined sub-complex is a group with action homomorphism. An action $NG_0^\bullet \times NG_1^\bullet \rightarrow NG_1^\bullet$

\[ (r, s) \mapsto s' = -(s_0r) + s + (s_0r) \]

is constructed. For the first condition of group with action of crossed modules,

\[ \partial_1(s') = \partial_1(-(s_0r) + s + (s_0r)) = -r + d_1s + r \]

is satisfied properly. On the other hand, let $a, b \in NG_1^\bullet$.

\[ b^{\partial_1a} = -s_0d_1a + b + s_0d_1a \]

is defined by the action of group with action. $NG_2^\bullet \cap D_2^\bullet = N_2^\bullet \cap D_2^\bullet$ equality is satisfied with $a, b \in NG_1^\bullet$ and

\[ F_{(0)(1)}(a, b) = [s_0a, s_1b][s_1b, s_1a] \in NG_2^\bullet \cap D_2^\bullet \]

in $NG_2^\bullet \cap D_2^\bullet$. Since the length of Moore complex is 1, we have

\[ \partial_2(N_2^\bullet \cap D_2^\bullet) = \{0\} \]

and the following equality

\[ \partial_2(F_{(0)(1)}(a, b)) = d_2([s_0a, s_1b][s_1b, s_1a]) \]

\[ = -s_0d_1a - b + s_0d_1a + b(-b - a + b + a) \in NG_2^\bullet \cap D_2^\bullet \]

so

\[ -s_0d_1a + b + s_0d_1a = -a + b + a \]

is the second condition of group with action crossed modules. For $r \in NG_0^\bullet, s \in NG_1^\bullet$

\[ \partial_1(s^{s_0(\partial_1r)}) = \partial_1(s)^{\partial_1(r)} = \partial_1(s') \]

Since $\partial_1$ is one-to-one morphism, crossed module of group with action CM4 condition holds.

CM3)

\[ \partial_1(r * s) = \partial_1(s^{s_0(r)}) = \partial_1(s)^{\partial_1(s_0r)} = \partial_1(s)^r = \partial_1(s) * r \]

condition is obtained.

Thus, there exists an obvious functor

\[ \triangle : XMod^\bullet \rightarrow SimpGrp^\bullet_{\leq 1} \]

\[ R^\bullet \mapsto \triangle(R^\bullet) = (S^\bullet \rightarrow R^\bullet) \]
Using the definition of semi-direct product the following equation holds
\[
\partial : S^* \rightarrow R^*
\]
is a crossed module of group with action. Since \( R^* \) acts on \( S^* \) we far from the semi-direct product \( R^* \rtimes S^* \). We define the semi-direct group with action \( R^* \rtimes S^* \) by
\[
(r, s)(r_1, s_1) = (r + r_1, s^r_1 + s_1)
\]
and
\[
(r, s)(r_1, s_1) = (r * r_1, r * s_1 + s * s_1 + s * r_1)
\]
an action on \( S^* \) by \( R^* \). Defining \( S^*_0 = R^* \) and \( S^*_1 = R^* \rtimes S^* \) we get
\[
d_1(r, s) = (\partial_1 s) + r
\]
\[
d_0(r, s) = r
\]
\[
s_0(r) = (0, r)
\]
These operators satisfy the following simplicial identities due to being group with action morphisms.
\[
d_1s_0(r) = d_1(r, 0) = (\partial_1 0) + r = r
\]
\[
d_0s_0(r) = d_0(r, 0) = r
\]
Thus, a 1-truncated simplicial group with action is \( \{S^*_1, S^*_0\} = S'^* \). From truncation we have the following equalities
\[
NS^*_0 = NS^*_0 = S^*_0 = N
\]
\[
NS^*_1 = NS^*_1 = (\ker d_0)NS^*_2 = \{0\}
\]
Equivalently, if
\[
\partial_2(NS^*_2 \cap D^*_2) = \partial_2(NS^*_2) = \{0\}
\]
is satisfied, \( S'^* \) is denoted a simplicial group with action with Moore complex of length 1. By the definition of \( F_{a,b} \) functions
\[
\partial_2(NS^*_2) = [\ker d_0, \ker d_1]
\]
is obtained. Also
\[
d_0 : R^* \rtimes S^* \rightarrow R^*
\]
\[
(r, s) \mapsto r
\]
with \( \ker d_0 = \{(0, s) : s \in S^*\} \cong S^* \) and
\[
d_1 : R^* \rtimes S^* \rightarrow R^*
\]
\[
(r, s) \mapsto (\partial_1 s) + r
\]
with \( \ker d_1 = \{(-\partial_1 s, s) : s \in S^*\} \) are conveniently determined. Since \((0, s_1) \in \ker d_0 \) and \((-\partial_1 s, s) \in \ker d_1 \), we have
\[
[(0, s_1), (-\partial_1 s, s)] = (0, 0).
\]
Using the definition of semi-direct product the following equation holds
\[
[(0, s_1), (-\partial_1 s, s)] = (0, s_1)(-\partial_1 s, s)(-\partial_1 s, s)(-\partial_1 s, s)(-\partial_1 s, s)
\]
\[
= (-\partial_1 s, (s_1)^{-\partial_1 s} + s)0, -s_1(\partial_1 s, (-s)^{\partial_1 s})
\]
\[
= (-\partial_1 s, (s_1)^{-\partial_1 s} + s)(\partial_1 s, (-s_1)^{\partial_1 s} + (-s)^{\partial_1 s})
\]
\[
= (0, ((s_1)^{-\partial_1 s} + s)^{\partial_1 s} + (-s_1)^{\partial_1 s} + (-s)^{\partial_1 s})
\]
\[
= (0, 0_S)
\]
where \((0_{R\ast}, 0_{S\ast})\) is the identity element.

Furthermore,
\[
(0, s_1)(-\partial_1 s, s) = (0 \ast (-\partial_1 s), 0 \ast s_1 \ast s + s_1 \ast (-\partial_1 s))
\]
\[
= (0, \partial_1 s_1 \ast s + \partial_1 s_1 \ast (-s))
\]
\[
= (0, s_1^\ast + (-s)^s_1)
\]
\[
= (0, s_1)
\]
and
\[
(-\partial_1 s, s)^{(0, s_1)} = (-\partial_1 s, -\partial_1 s \ast s_1 + s \ast s_1 + s \ast 0)
\]
\[
= (-\partial_1 s, (-s) \ast s_1 + s \ast s_1 + s)
\]
\[
= (-\partial_1 s, 0 \ast s_1 + s)
\]
\[
= (-\partial_1 s, s)
\]
equations are hold. So,
\[
[\ker d_0, \ker d_1] = (0_{R\ast}, 0_{S\ast})
\]
and
\[
\partial_2(N S_2^\ast) = \{0\}
\]
are obtained from \(S' = \{S_1^\ast, S_0^\ast\}\) with Moore complex of length 1.

6 Computer Implementation

GAP (Groups, Algorithms, Programming [11]) is the leading symbolic computation system for solving computational discrete algebra problems. Symbolic computation has underpinned several key advances in Mathematics and Computer Science, for example, in number theory and coding theory (see [6]). GAP which is free, open source, and extensible system, can deal with different discrete mathematical problems, but it focuses on computational group theory. It is distributed under the GNU Public License. The system is delivered together with the source codes, which are written in two languages: the kernel of the system is written in C, and the library of functions and additional packages is in a special language, also called GAP. The GAP system and extension packages now comprise 360K lines of C and 90K lines of GAP code. The Small Groups library has been used in such landmark computations as the "Millennium Project" to classify all finite groups of order up to 2000 by Besche, Eick and O'Brien in [7].

The GwA package for GAP contains functions for groups with action and their morphisms and was first described in [20]. In this paper, we have developed new functions which construct (pre) crossed modules of groups with action and renamed the package to XModGwA.

The function GwA in the package may be used in two ways. GwA(G) returns the group with trivial action, while GwA(G, act) returns a group with action for this chosen group and an act : G \rightarrow \text{Aut}(G) action. Functions for groups with action include IsGwA, IsPerfectGwA, IsIdeal, AllIdealOnGwA, IsGwAC1, Commutator, LowerCentralSeriesOfGwA, IsNilpotent, and NilpotencyClassOfGwA. Attributes of a group with action constructed in this way include BaseGroup and BaseAction.

The function GwAMorphismObj defines morphisms of groups with action, and the function IsGwAMorphism which controls whether a map satisfies condition morphisms of groups with action or not. The function AllGwAMorphisms is used to find all morphisms between two groups with action.

The following GAP session illustrates the use of these functions.
The function `AllGwAOnGroup(G)` constructs a list of all groups with action over $G$. The function `AreIsomorphicGwA` is used for checking whether or not two groups with action are isomorphic, and `IsomorphicGwAFamily` returns a list of representatives of the isomorphism classes.

In the following GAP session, we compute all 736 groups with action on $C_2 \times C_2 \times C_2$; representatives of the 14 isomorphism classes; and the list of members of a family.

| Family | Number of Members | Representator | Number of Ideals | Nilpotency Class | Condition 1 |
|--------|------------------|---------------|-----------------|-----------------|-------------|
| 1      | 1                | 1/736         | 16              | 1               | ✓           |
| 2      | 84               | 2/736         | 7               | 0               | ✗           |

Six of the isomorphism families satisfy Condition 1 and there are six families with nilpotent and eight are not nilpotent. Other features obtained with these functions are given in the table below.
Function for the action between two groups with action include `IsGwAAction`. The function `IsGwAAction` is implemented for checking the structure of a group with action.

The group $R$ acts on $S$ by an action $\alpha, \beta : R \to \text{Aut}(S)$ and let $S^\ast$ and $R^\ast$ be groups with action on group $S$ and $R$, respectively. `IsGwAAction(S^\ast, R^\ast, \alpha, \beta)` is used to verify that the conditions of action between two groups with action in page 5 are satisfied. The function `AllXModGwAActions(S^\ast, R^\ast)` constructs a list of all actions of group with action $R^\ast$ on group with action $S^\ast$.

The following GAP session illustrates the use of these functions.

```gap
gap> G := Range(IsomorphismPermGroup(SmallGroup(8,2)));;
gap> allgwa_onG := AllGwAOnGroup(G);
gap> Length(allgwa_onG);
32
gap> SwA := allgwa_onG[2];
GroupWithAction [ Group( [ (1,2), (3,4), (5,6) ] ), * ]
gap> RwA := allgwa_onG[4];
GroupWithAction [ Group( [ (1,2), (3,4,5,6) ] ), * ]
gap> all_acts := AllXModGwAActions(SwA,RwA);
gap> Length(all_acts);
256
gap> act_pair := all_acts[13];
void
gap> IsGwAAction(SwA,RwA,act_pair[1],act_pair[2]);
true
```

Functions for crossed modules of groups with action include `PreXModGwAObj`, `IsPreXModGwA`, `IsXModGwA` and `XModGwAByIdeal`. Attributes of a group with action constructed in this way include `XModGwAAction`, `Range`, `Source` and `Boundary`.

A structure which has `IsXModGwAC1` is a pre-crossed module or a crossed module of groups with action whose source and range are both satisfies condition 1.

The following GAP session illustrates the use of these functions.

```gap
gap> all_bdys := AllGwAMorphisms(SwA,RwA);
gap> bdy := all_bdys[7];
GroupWithAction [ Group( [ (1,2), (3,4,5,6) ] ), * ] =>
GroupWithAction [ Group( [ (1,2), (3,4,5,6) ] ), * ] =>
gap> XM1 := PreXModGwAObj(bdy,act_pair[1], act_pair[2]);
GroupWithAction [ Group( [ (1,2), (3,4,5,6) ] ), * ] =>
```

12
The global function \texttt{AllXModsGwA} may be called in two ways: as \texttt{AllXModsGwA(S^*, R^*)} to compute all crossed modules with chosen source and range groups with action; as \texttt{AllXModsGwA(x, y, m, n)} to compute all crossed modules of groups with actions with given size and numbers of small groups. The function computes both all pre-crossed modules and crossed modules of groups with action.

In the following GAP session, we get crossed modules using the function.

\begin{verbatim}
gap> all_XM3 := AllXModsGwA(SwA, RwA);;
gap> Length(all_XM3[1]); Length(all_XM3[2]);
66
16
all_XM4 := AllXModsGwA(4,1,4,2);
Length(all_XM4[1]); Length(all_XM4[2]);
416
184
gap> list := Filtered(all_XM4[2], XM -> IsXModGwAC1(XM));;
gap> Length(list);
88
\end{verbatim}

References

[1] M., Alp, A., Odabas, E.O, Uslu and C.D., Wensley, Crossed modules and cat$^1$-groups, (manual for the XMod package for GAP, version 2.77) (2019).

[2] Z., Arvasi and T., Porter, Higher Dimensional Peiffer Elements in Simplicial Commutative Algebras, Theory and Applications of Categories, \textbf{vol. 3, no. 1} (1997) 1–23.

[3] Z., Arvasi, T.S., Kuzpınarı and E. Ö., Uslu, Three Crossed Modules, Homology, Homotopy and Applications \textbf{11, no. 2} (2009) 161-187.

[4] Z., Arvasi and A., Odabas, Computing 2-dimensional algebras: Crossed modules and Cat$^1$-algebras, \textit{J. Algebra Appl.} \textbf{15} (2016) 165-185.
[5] Z., Arvasi and A., Odabas, Crossed Modules and cat\textsuperscript{1}-algebras. (manual for the textsf-XModAlg share package for GAP, version 1.17) (2018).

[6] R., Behrends, K., Hammond, V., Janjic, A., Konovalov, S., Linton, H-W., Loidl, P., Maier and P., Trinder, HPC-GAP: Engineering a 21st-century high-performance computer algebra system, Concurrency and Computation Practice and Experience 28 (2016) 3606-3636.

[7] B., Eick, H. U., Besche and E. A., O’Brien, A Millennium Project: Constructing Small Groups, International Journal of Algebra and Computation Vol. 12, No. 05 (2002) 623-644.

[8] E.B., Curtis, Simplicial Homotopy Theory, Adv. in Math. 6 (1971) 107-209.

[9] T., Datuashvili, Central Series For Groups With Action and Leibniz Algebras, Georgian Mathematical Journal Volume 9, Number 4 2002 671-682.

[10] T., Datuashvili, Witt’s theorem for groups with action and free Leibniz algebras, Georgian Mathematical Journal Volume 11, Number 4 (2004) 691-712.

[11] The GAP Group, GAP – Groups, Algorithms, and Programming, version 4.10.2 (https://www.gap-system.org) (2019).

[12] G., Janelidze, Internal crossed modules, Georgian Math. J. 10 (1) (2003) 99-114.

[13] J.-L., Loday, Spaces having finitely many non-trivial homotopy groups, Jour. Pure Appl. Algebra 24 (1982) 179-202.

[14] J.-L., Loday, Operations sur l’homologie cyclique des algebres commutatives, Invent. Math. 96(1989) 205-230.

[15] J.-L., Loday, Une version non commutative des algebres de Lie:les algebres de Leibniz, Enseign. Math. 39 (1993).

[16] J.-L., Loday, Algebraic K-theory and related operads, Lecture Notes in Math. Springer, Berlin (2001).

[17] J.P., May, Simplicial Objects in Algebraic Topology, Van Nostrand, Math. Studies 11 (1965).

[18] A., Odabas and E., Soylu, Crossed Modules of Groups with Action, (The XModGwA share package for GAP, version 1.12) (http://fef.ogu.edu.tr/aodabas/xmodgwa/) (2020).

[19] T., Porter, Homotopy Quantum Field Theories meets the Crossed Menagerie: an introduction to HQFTs and their relationship with things simplicial and with lots of crossed gadgetry, Lecture Notes (2011).

[20] E.O., Uslu, A.F., Aslan and A., Odabas, On groups with action on itself, Georgian Mathematical Journal 26(3) (2019) 459-470.

[21] J.H.C., Whitehead, On adding relations to homotopy groups, Annals of Mathematics 42 (2) (1941) 409-428.

[22] J.H.C., Whitehead, Note on a previous paper entitled “On adding relations to homotopy groups”, Annals of Mathematics 47 (2) (1946) 806-810.

[23] J.H.C., Whitehead, Combinatorial homotopy II, Bulletin of the American Mathematical Society 55 (3) (1949) 213-245.

[24] E., Witt, Treue Darstellung Liescher Ringe, J. Reine Angew. Math. 177 (1937) 152-160.