Supplementary Information to “Intrinsic curvature determines the crinkled edges of “crenellated disks”.”

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A. Total free energy

For a single surface of revolution terminated by two space curves, in terms of the angles \( \alpha, \beta_i, \) and \( \phi_i, \) and the differential geometric quantities defined in the main text, the total free energy is given by

\[
F = F_H + F_{edge_1} + F_{edge_2} + F_{\lambda_1} + F_{\lambda_2}, \tag{1}
\]

where

\[
F_H = \int_\epsilon^R dr (\pi + \phi_1 + \phi_2) \left[ \frac{sr}{\cos \alpha} + \frac{k}{2r} \sin \alpha \tan \alpha \right. \\
+ \frac{k}{2r} \cos \alpha \left( \frac{d\alpha}{dr} \right)^2 \\
+ (k + \vec{k}) \int_\epsilon^R \frac{dr}{r} (\tan \beta_1 + \tan \beta_2) \\
- (k + \vec{k}) \cos \alpha (R) (\pi + \phi_1 (R) + \phi_2 (R)) \\
+ (k + \vec{k}) \cos \alpha (\epsilon) (\pi + \phi_1 (\epsilon) + \phi_2 (\epsilon)) \\
\]

\[
F_{edge_i} = \int_\epsilon^R dr \left[ \frac{\gamma_i}{\cos \alpha \cos \beta_i} + k_b \kappa_{g,1}^2 + \kappa_{n,1}^2 \right. \\
- h_i \sin \frac{\beta_i}{\cos \alpha} (\kappa_2 - \kappa_1) \left( \kappa_2 - \kappa_1 \right), \tag{3}
\]

\[
F_{\lambda_i} = \int_\epsilon^R dr \lambda_i \left[ r \cos \alpha \frac{d\phi_i}{dr} - \tan \beta_i \right]. \tag{4}
\]

The index of the edges is \( i = \{1, 2\}, \) \( \epsilon \) is the cut-off length, which is given by \( \epsilon \equiv -R \sin \alpha (R) \) for the catenoid.

B. Euler-Lagrange equations

The variations \( \delta F \), where \( g_j = \{\alpha, \beta_1, \beta_2, \phi_1, \phi_2\}, \) are equivalent to the Euler-Lagrange (EL) equations. For one independent and several dependent variables, the EL equations are given by [1]

\[
\frac{\partial f}{\partial g_j} = \frac{d}{dx} \frac{\partial f}{\partial g'_j}, \tag{5}
\]

where \( f = f(r, g_j, g'_j) \) is the free energy density given by \( F = \int dr f(r, g_j, g'_j) \). Primes denote derivatives \( \frac{dg_j}{dr} \). The set of EL Equations in Eq. (5) contains ten coupled nonlinear first-order differential equations. Evaluating Eq. (5), these equations are calculated as

\[
u_1 \equiv \frac{\partial f}{\partial \alpha'} = \pi kr \cos \alpha \frac{d\alpha}{dr} + 2k_b \kappa_{g,1} \cos \beta_1 + 2k_b \kappa_{n,2} \cos \beta_2 \\
+ h_1 \cot \beta_1 + h_2 \sin \beta_2, \tag{6}
\]

\[
u_2 \equiv \frac{\partial f}{\partial \beta_1'} = 2k_b \kappa_{g,1}, \quad \nu_3 \equiv \frac{\partial f}{\partial \beta_2'} = 2k_b \kappa_{g,2}, \tag{7}
\]

\[
u_4 \equiv \frac{\partial f}{\partial \phi_1'} = \lambda_1 r \cos \alpha, \quad \nu_5 \equiv \frac{\partial f}{\partial \phi_2'} = \lambda_2 r \cos \alpha, \tag{8}
\]

\[
u_1' \cos^2 \alpha = \pi r \sin \alpha + k \frac{r}{2} \sin \alpha \frac{d\alpha}{dr} \left[ \frac{2}{r} - \frac{\sin^2 \alpha}{r} - r \cos^2 \alpha \right. \\
+ \sin \alpha \sec \beta_1 \left( \gamma_1 + k_b \kappa_{g,1}^2 - k_b \kappa_{g,1}^2 \right) \\
+ \sin \alpha \sec \beta_2 \left( \gamma_2 + k_b \kappa_{n,2}^2 - k_b \kappa_{n,2}^2 \right) \\
+ 2k_b \kappa_{g,1} \sec \beta_1 \left( \frac{\cos^2 \alpha \sin^2 \beta_1}{r} - \frac{2 \cos \alpha}{dr} \cos \beta_1 \right) \\
+ 2k_b \kappa_{n,2} \sec \beta_2 \left( \frac{\cos^2 \alpha \sin^2 \beta_2}{r} - \frac{2 \cos \alpha}{dr} \cos \beta_2 \right) \\
- \frac{1}{r} \left( h_1 \sin \beta_1 + h_2 \sin \beta_2 \right) \\
- \frac{\sin 2 \alpha}{r} \left( \lambda_1 \tan \beta_1 + \lambda_2 \tan \beta_2 \right), \tag{9}
\]

\[
\cos \alpha \cos^2 \beta_1 \nu_2' = \left( k + \vec{k} \right) \frac{\cos \alpha}{r} + \gamma_1 \sin \beta_1 - \lambda_1 \cos \alpha \\
+ k_b \sin \beta_1 \left( \kappa_{g,1}^2 \kappa_{g,1}^2 - \kappa_{g,1}^2 \kappa_{g,1}^2 \right) + 2k_b \kappa_{g,1} \cos \alpha \\
+ (4k_b \kappa_{n,1} \cos \beta_1 - h_1 \cos \beta_1 \cot \beta_1) \tau_{g,1}, \tag{10}
\]
\[
\cos \alpha \cos^2 \beta_2 u_3' = (k + \bar{k}) \frac{\cos \alpha}{r} + \gamma_2 \sin \beta_2 - \lambda_2 \cos \alpha \\
+ k_0 \sin \beta_2 (\kappa_{n,2}^2 - \kappa_{g,2}^2) + \frac{2k_0}{r} \kappa_{g,2} \cos \alpha \\
+ (4k_0 \kappa_{n,2} \cos \beta_2 - h_2 \cos \beta_2 \cot \beta_2) \tau_{g,2}.
\]

(11)

\[
u_4' = u'_5 = \frac{\sigma r}{\cos \alpha} + \frac{k}{2r} \sin \alpha \tan \alpha + \frac{k}{2r} \cos \alpha \left(\frac{d\alpha}{dr}\right)^2.
\]

(12)

In Eq. (9), the two constraints \(r \cos \alpha \frac{d\phi_1}{dr} = \tan \beta_1\) are used. Based on Eqs. (8) and (12), we conclude that \(\lambda \equiv \lambda_1 = \lambda_2\). Then, the nine unknowns to be solved are \(\alpha, \alpha', \beta_1, \beta_1', \beta_2, \beta_2', \phi_1, \phi_2, \lambda\).

Note that we neglect the contribution of the variations \(\delta f/\delta \phi_i\) to the surface and edge deformations, as explained in the main text. Therefore, the structure in our approximation is governed solely by \(\alpha, \alpha', \beta_1, \beta_1', \beta_2, \beta_2', \phi_1, \phi_2, \lambda\).

C. Boundary conditions

The torque-free boundary conditions are given by Eqs. (6), (7), and (8) being equal to zero \([1]\). Sufficiently away from the rotation axis of the surface, in 2D membrane limit, \(\alpha = 0, \alpha' = 0, \) and \(\alpha'' = 0\). Then, \(\kappa_{n,i} = 0\). Plugging in these results to Eq. (9), we find that \(\sin \beta_i = A_i/r\). This rather expected result applies to a straight line in polar coordinates, where the shortest distance between the origin and the straight line is \(A_i\) (or, the in-plane protrusion amplitude). When \(h_i = 0\) as in the achiral limit, then the same relation still holds, and this time the torque-free condition \(\frac{\partial f}{\partial \phi_i} = 0\) is satisfied. The geodesic curvature \(\kappa_{g,i}\) of a straight line vanishes in 2D, hence \(\frac{\partial f}{\partial \phi_i} = 0\). In the 2D membrane limit \(\lambda\) must be zero. This is best seen when a curve with the free energy given in Eq. (3) is minimized on a plane. The resulting EL equations are independent of \(\phi_i\), then \(\lambda = 0\). Hence \(\frac{\partial f}{\partial \phi_i} = 0\), away from the rotation axis of the nonplanar surface. Plugging in these results into Eqs. (10) and (11), we find that \(k + \bar{k} = \gamma_1 A_1 = \gamma_2 A_2\). For a minimal surface, \(k\) does not contribute to this equation.

At the cusp, ideally, the boundary conditions are \(\alpha = -\pi/2\) and \(\beta_i = 0\) (again, we ignore \(\delta f/\delta \phi_i\) dependence of the structure). In the case of the minimal surface, since \(\epsilon \approx 10^{-2} R\), we take \(\alpha = 0.95\) which nearly corresponds to a 70° slope. This slope is steep enough as evidenced in Fig. 3(a) in the main text.

[1] Arfken G., Weber H. Mathematical Methods for Physicists, Sixth Edition (Elsevier Inc., 2005).