An axiomatic characterization of generalized entropies under analyticity condition

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Abstract

We present the characterization of the Nath, Rényi and Havrda-Charvát-Tsallis entropies under the assumption that they are analytic function with respect to the distribution dimension, unlike the the previous characterizations, which supposes that they are expandable maximized for uniform distribution.

Keywords: Axiomatic characterization, Information measures, Shannon entropy, Nath entropy, Rényi entropy, Havrda-Charvát-Tsallis entropy.

1. Introduction

Nath [1], Rényi [2] and Havrda-Charvát-Tsallis [3], [4] entropies are well known generalizations of the Shannon entropy. All of them have more general strong additivity property in comparison to the Shannon entropy. By the strong additivity property, the entropy of joint distribution can be represented as the sum of the the entropy of the first one and the conditional entropy of the second one with respect to the first one. The conditional entropy is defined as the $P$ expected value of the entropy of the conditional distribution $Q$ conditioned on $P$. For Nath entropy, and its normalized instance, the Rényi entropy, the definition of the expectation is generalized from linear to the quasi-linear mean, while in the Havrda-Charvát-Tsallis case linear expectation is used but the additivity is generalized to the $\gamma$-additivity [5].

Previous axiomatic systems [6], [7], [8], [9] take additivity condition as an axiom, and another three axioms - continuity, expandability (which means that adding the zero probability event in the sample space does not affect the entropy distribution) and maximality (which means that the entropy is maximized for uniform distribution). In this letter we give alternative axiomatic systems, which replace the expandability and maximality axioms with the axiom that states the uniform distribution entropy can be analytical continued if it is taken as the function of the distribution dimension, the property that has an important role in asymptotic analysis of entropy [10] (see [11] for alternative approach). The presented results generalizes discussion in [12], where the Shannon entropy is considered.

The letter is organized as follows. In section 2 we review the Shannon entropy uniqueness theorem given by Nambiar et. al [12]. The theorem is generalized to the Nath and Rényi entropies in section 3 and to the Havrda-Charvát-Tsallis entropy in section 4.

2. Shannon entropy

Let the set of all $n$-dimensional distributions be denoted with

$$\Delta_n \equiv \left\{ (p_1, \ldots, p_n) | p_i \geq 0, \sum_{i=1}^{n} p_i = 1 \right\}, \quad n > 1$$

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and the \( n \)-dimensional uniform distribution be denoted with

\[
U_n = \left( \frac{1}{n}, \ldots, \frac{1}{n} \right) \in \Delta_n.
\]  

(2)

Shannon entropy [13] of \( n \)-dimensional distribution is a function \( \mathcal{H}_n : \Delta_n \to \mathbb{R}_{>0} \) given with the family parameterized by \( \tau \in \mathbb{R} \):

\[
S_n(P) = \tau \cdot \sum_{k=1}^n p_k \log p_k; \quad \tau < 0,
\]

where \( \log \) stands for the logarithm to the base 2. The following theorem characterizes the Shannon entropy.

**Theorem 2.1.** Let the function \( \mathcal{H}_n : \Delta_n \to \mathbb{R}_{>0} \) satisfies the following axioms, for all \( n \in \mathbb{N}, n > 1 \):

**SA1:** \( \mathcal{H}_n(U_n) = s(1/n), \) where \( s : \mathbb{C} \to \mathbb{C} \) is an analytic function.

**SA2:** Let \( P = (p_1, \ldots, p_n) \in \Delta_n, \) \( PQ = (r_{11}, r_{12}, \ldots, r_{nm}) \in \Delta_{nm}, n, m \in \mathbb{N} \) such that \( p_i = \sum_{j=1}^n r_{ij} \) and \( Q_{ik} = (q_{1k}, \ldots, q_{mk}) \in \Delta_m, \) where \( q_{ik} = r_{ik}/p_k. \) Then,

\[
\mathcal{H}_{nm}(PQ) = \mathcal{H}_n(P) + \mathcal{H}_m(Q|P), \quad \text{where} \quad \mathcal{H}_m(Q|P) = \sum_k p_k \mathcal{H}_m(Q_k)
\]

Then, \( \mathcal{H}_n \) is the Shannon entropy.

The previous theorem slightly differs from the one presented in [12], but it can be proven by straightforward repetition of the steps from [12]. First, we do not assume the normalization condition \( s(1/2) = 1 \), which fixes the constant to \( \tau = -1 \). Second, the statement of the theorem from [12] assumes that the entropy is complex analytic function with respect to the distribution. We assume the equivalent statement that the entropy is continuous real function (note that the assumption about the analyticity with respect to the distribution dimension is kept).

3. Nath entropy

For a distribution \( P \in \Delta_n \) we define the \( \alpha \)-escort distribution \( P^{(\alpha)} = (p_1^{(\alpha)}, \ldots, p_n^{(\alpha)}) \), where

\[
p_k^{(\alpha)} = \frac{p_k^\alpha}{\sum_{i=1}^n p_i^\alpha}; \quad k = 1, \ldots, n.
\]

(5)

If \( P = (p_1, \ldots, p_n) \in \Delta_n \) and \( Q = (q_1, \ldots, q_m) \in \Delta_m \) the direct product, \( P \star Q \in \Delta_{nm} \), is defined as

\[
P \star Q = (p_1q_1, p_1q_2, \ldots, p_nq_m).
\]

(6)

The proof of the following theorem can be found in [7], [14].

**Theorem 3.1.** Let \( g : \mathbb{R} \to \mathbb{R} \) be a continuous invertible function and \( \mathcal{H}_n : \Delta_n \to \mathbb{R}_{>0} \) a continuous function

\[
\mathcal{H}_n(P) = g^{-1} \left( \sum_{\alpha=1}^n p_k^{(\alpha)}(\alpha)g(\tau \cdot \log p_k) \right); \quad \tau < 0,
\]

(7)

for all \( P = (p_1, \ldots, p_n) \in \Delta_n \), and let \( \mathcal{H}_n \) be additive for all \( n \in \mathbb{N} \), i.e. \( \mathcal{H}_{nm}(P \star Q) = \mathcal{H}_n(P) + \mathcal{H}_m(Q) \) for all \( P = (p_1, \ldots, p_n) \in \Delta_n, Q = (q_1, \ldots, q_m) \in \Delta_m, n, m \in \mathbb{N} \). Then, \( g \) is the function from the class parameterized by \( \lambda \in \mathbb{R} \):

\[
g(x) = \begin{cases} 
-\frac{c}{\alpha} x, & \text{for } \lambda = 0 \\
2^{-\lambda x} - 1, & \text{for } \lambda \neq 0
\end{cases}
\]

\[
g^{-1}(x) = \begin{cases} 
\frac{\lambda}{c} x, & \text{for } \lambda = 0 \\
\lambda \log(\gamma x + 1), & \text{for } \lambda \neq 0
\end{cases}
\]

(8)
where \( c, \gamma \neq 0 \), and the entropy is uniquely determined with

\[
\mathcal{H}_n(P) = \begin{cases} 
\tau \cdot \sum_{k=1}^{n} p_k^{(0)} \log p_k & \text{for } \lambda = 0 \\
- \frac{1}{\lambda} \log \left( \frac{\sum_{k=1}^{n} p_k^{(\tau - \lambda)}}{\sum_{k=1}^{n} p_k^2} \right) & \text{for } \lambda \neq 0
\end{cases}
\]

(9)

where \( \tau < 0 \) and \( \alpha - \tau \lambda > 0 \).

Nath entropy of \( n \)-dimensional distribution is a function \( \mathcal{H}_n : \Delta_n \to \mathbb{R}_{>0} \) from the family parameterized by \( \alpha, \lambda, \tau \in \mathbb{R}:

\[
\mathcal{H}_n(P) = \begin{cases} 
\tau \cdot \sum_{k=1}^{n} p_k \log p_k ; & \tau < 0 \quad \text{for } \alpha = 1 \\
- \frac{1}{\lambda} \log \left( \sum_{k=1}^{n} p_k^\alpha \right) ; & \frac{1 - \alpha}{\lambda} > 0 , \lambda \neq 0 \quad \text{for } \alpha \neq 1.
\end{cases}
\]

(10)

Rényi entropy is a function \( \mathcal{R}_\alpha : \Delta_n \to \mathbb{R}_{>0} \) from the family (10) with \( \lambda = 1 - \alpha \) and \( \tau = -1 \). The following theorem characterizes the Nath and Rényi entropies.

**Theorem 3.2.** Let the function \( \mathcal{H}_n : \Delta_n \to \mathbb{R}_{>0} \) satisfies the following axioms, for all \( n \in \mathbb{N}, n > 1 \):

**NA1:** \( \mathcal{H}_n(\mathcal{U}_n) = \nu(1/n) \), where \( \nu : \mathbb{C} \to \mathbb{C} \) is an analytic function.

**NA2:** Let \( P = (p_1, \ldots, p_n) \in \Delta_n \), \( PQ = (r_{11}, r_{12}, \ldots, r_{nm}) \in \Delta_{mn} \), \( n, m \in \mathbb{N} \) such that \( p_i = \sum_{j=1}^{m} r_{ij} \), and \( Q_{ik} = (q_{ik1}, \ldots, q_{ikm}) \in \Delta_{n}, k = 1, \ldots, m \) where \( q_{ik} = r_{ik}/p_k \) and \( \alpha \in (0, \infty) \) is some fixed parameter. Then,

\[
\mathcal{H}_{nm}(PQ) = \mathcal{H}_n(P) + \mathcal{H}_m(Q|P), \quad \text{where } \mathcal{H}_m(Q|P) = f^{-1}\left( \frac{1}{m} \sum_k p_k^{(\alpha)} f(\mathcal{H}_m(Q_{ik})) \right),
\]

(11)

where \( f \) is invertible continuous function.

Then, \( \mathcal{H}_n \) is the Nath entropy. If, in addition, normalization axiom \( \nu(1/2) = 1 \) is satisfied, \( \mathcal{H}_n \) is the Rényi entropy.

**Proof.** Let \( A = (1/2, 1/2) \). By successive use of formula (11) we get

\[
\nu\left( \frac{1}{2^n} \right) = \mathcal{H}_n(A \ast A \ast \ldots \ast A) = \sum_{k=1}^{n} \mathcal{H}_2(A) = n \cdot \mathcal{H}_2(A) = - \log \frac{1}{2^n} \cdot \mathcal{H}_2\left( \frac{1}{2} \right) = \tau \cdot \log \frac{1}{2^n}
\]

(12)

where \( \tau = -\mathcal{H}_2(1/2, 1/2) \) is a negative real value, since \( \mathcal{H}_2(1/2, 1/2) \) is positive by assumption of the theorem. Accordingly, the values of functions \( \nu(z) \) and \( \tau \cdot \log z \) coincide at an infinite number of points converging to zero, \( |z| = 1/2^n \), \( n \in \mathbb{N} \). Since both \( f(z) \) and \( \tau \cdot \log z \) are analytic functions, they must be the same,

\[
\nu(z) = \tau \cdot \log z; \quad \tau < 0.
\]

(13)

Let us determine the entropy form for the distribution \( P = (p_1, \ldots, p_n) \in \mathbb{C}^n \) when \( p_i \) are rational numbers and the case for irrational numbers follows from the continuity of entropy. Let \( P = (p_1, \ldots, p_n) \in \mathbb{C}^n \), \( Q_{ik} = (q_{ik1}, \ldots, q_{im_i}) \in \mathbb{C}^{m_i} ; k = 1, \ldots, n \) and \( PQ = (r_{11}, r_{12}, \ldots, r_{nm}) \in \mathbb{C}^{mn} \), for \( n, m \in \mathbb{N} \), and \( p_j = m_j/m \); \( r_{ij} = 1/m, q_{ij} = 1/m_i \), where \( m = \sum_{i=1}^{m} m_i \) and \( m_i \in \mathbb{N} \) for any \( i = 1, \ldots, n \) and \( j = 1, \ldots, m_i \). Then we
have $\mathcal{H}_n(PQ) = \mathcal{H}_m(U_m) = v(1/m) = -\tau \cdot \log m$ and $\mathcal{H}_m(Q_k) = \mathcal{H}_m(U_m) = v(1/m_k) = -\tau \cdot \log m_k$. Since $p_i = \sum_{j=1}^n r_{ij}$ and $q_{jk} = r_{jk}/p_k$, we can apply the axiom [NA2] and we get

$$\mathcal{H}_n(P) = -\tau \cdot \log m - f^{-1} \left( \sum_{k=1}^n p_k^{(a)} \left( -\tau \cdot \log m_k \right) \right) = -\tau \cdot \log m - f^{-1} \left( \sum_{k=1}^n p_k^{(a)} \left( -\tau \cdot \log p_k - \tau \cdot \log m \right) \right).$$ (14)

Let us define $f_y(x) = f(-x - y)$, and $f_y^{-1}(z) = -y - f^{-1}(z)$. If we set $y = \tau \cdot \log m$ the equality (14) becomes

$$\mathcal{H}(P) = f^{-1}_y \left( \sum_{k=1}^n p_k^{(a)} f_y \left( \tau \cdot \log p_k \right) \right); \quad \tau < 0.$$ (15)

Since $f$ is continuous, both $f_y$ and $f_y^{-1}$ are continuous, as well as the entropy, and we may extend the result (15) from rational $p_i$'s to any real valued $p_i$'s defined in $[0,1]$. Now, if the axiom [NA2] is used with $PQ = P \ast Q$, the conditions from the Theorem 3.1 are satisfied so that $f_y(x) = -cx$, for $\lambda = 0$ and $f_y(x) = (2^{-\lambda} - 1)/\gamma$, for $\lambda \neq 0$. Accordingly, the entropy is uniquely determined by the class $\mathfrak{C}$. The relationship between the parameters $a$, $\tau$ and $\lambda$ is determined by use of the axiom [NA2].

For $\lambda = 0$, since $f_y(x) = f(z)$, where $z = -x - y$, we get $f(z) = f_y(x) = f_y(-z - y) = cz + cy$. If the equality (9) is substituted in [NA2] with $f(z) = cz + cy$, we get

$$\sum_{k=1}^n \sum_{j=1}^m p_k^{(a)} \log (r_{jk}) = \sum_{k=1}^n p_k^{(a)} \log p_k + \sum_{k=1}^n \sum_{j=1}^m q_{ij}^{(a)} \log (q_{jk}) = \sum_{k=1}^n \sum_{j=1}^m r_{jk} \log (r_{ij}),$$

which can be transformed to

$$\sum_{k=1}^n \sum_{j=1}^m p_k^{(a)} q_{ij}^{(a)} = \sum_{k=1}^n p_k^{(a)} \sum_{j=1}^m q_{ij}^{(a)} \log (r_{ij}) \quad \text{and} \quad \tau_k = \tau.$$ (17)

The equality (17) holds for all distributions and we may consider the case $n = m = 2$, and the distributions $P = (1/2, 1/2)$, $Q_{11} = (1, 0)$, $Q_{12} = (1/2, 1/2)$. If we set $x_2 = \sum_{i=1}^2 q_{i1}^{(a)} = 1$, $x_2 = \sum_{i=1}^2 q_{i2}^{(a)} = 2^{1-\alpha}$ and $u_k = p_k^{(a)}$, $v_k = q_{k2}^{(a)}$, the equality (17) can be transformed as follows:

$$u_1 x_1 + u_2 x_2 = v_1 x_1 + v_2 x_2 \iff u_1 x_1 + (1 - u_1) x_2 = v_1 x_1 + (1 - v_1) x_2 \iff (u_1 - v_1) x_1 = (u_1 - v_1) x_2.$$ (18)

By using of $u_1 = u_2 \neq v_1 = v_2$, we get $x_1 = x_2$, i.e. $1 = 2^{1-\alpha}$, which implies $\alpha = 1$. Accordingly, the case $\lambda = 0$ from (9) reduces to the case $\alpha = 1$ from (10). Positivity of entropy implies that $\tau < 0$.

If $\lambda \neq 0$, and the equality (9) is substituted in [NA2], we get

$$\lambda = \log \left( \frac{\sum_{k=1}^n \sum_{i=1}^m q_{ij}^{(a) \tau}}{\sum_{k=1}^n \sum_{i=1}^m p_k^{(a) \tau}} \right) = \log \left( \frac{\sum_{k=1}^n p_k^{(a) \tau}}{\sum_{k=1}^n p_k^{(a) \tau}} \right) + f^{-1} \left( \sum_{k=1}^n p_k^{(a)} f \left( \frac{1}{\lambda} \log \left( \sum_{k=1}^n q_{ij}^{(a) \tau} \right) \right) \right).$$ (19)

where $f(z) = (2^{\lambda(z+y)} - 1)/\gamma$ (since for $z = -x - y$, we have $f(z) = f_y(-z - y) = (2^{\lambda(z+y)} - 1)/\gamma$ or, equivalently,

$$\sum_{k=1}^n \sum_{i=1}^m f_k^{(a) \tau} = \frac{\sum_{k=1}^n p_k^{(a) \tau}}{\sum_{k=1}^n q_{ij}^{(a) \tau} \sum_{k=1}^n p_k^{(a) \tau}} \iff \sum_{k=1}^n \sum_{i=1}^m p_k^{(a) \tau} \cdot \sum_{k=1}^n q_{ij}^{(a) \tau} = \frac{1}{\sum_{j=1}^m q_{j2}^{(a) \gamma}}.$$

(20)

where $\beta = \alpha - \tau \lambda$. Similarly to the case $\lambda = 0$ we note that the equality (17) holds for all distributions and we consider the case $n = m = 2$, and the distributions $P = (1/2, 1/2)$, $Q_{11} = (1, 0)$, $Q_{12} = (1/2, 1/2)$. If we set $x_1 = \sum_{i=1}^m q_{i1}^{(a) \gamma} = 1$, $x_2 = \sum_{i=1}^m q_{i2}^{(a) \gamma} = 2^{1-\alpha}$ and $u_k = \sum_{i=1}^m q_{ij}^{(a) \gamma}$, $v_k = \sum_{i=1}^m q_{ij}^{(a) \gamma}$, the equality (20) can be transformed into the form (18). By using of $u_1 = \frac{1}{1 + 2^{-\alpha}} \neq v_1 = \frac{1}{2^{1-\alpha}}$, we get $x_1 = x_2$, i.e. $1 = 2^{1-\alpha}$, which implies $\beta = 1$, i.e. $\alpha - \tau \lambda = 1$ with $\tau < 0$, and the case $\lambda \neq 0$ from (9) reduces to the case $\alpha \neq 1$ from (10). Positivity of entropy implies $(1 - a)/\lambda > 0$.

Finally, if $v(1/2) = \mathcal{H}_Q(1/2) = 1$, the equation (10) implies $\tau = -1$, $\lambda = 1 - a$, which proves the theorem.
4. Tsallis entropy

Havrda-Charvát-Tsallis entropy of $n$-dimensional distribution is a function from the family parameterized by $\tau, \lambda, \alpha \in \mathbb{R}$:

$$\mathcal{T}(P) = \begin{cases} \tau \cdot \sum_{k=1}^{n} p_k \log p_k; & \tau < 0, \quad \text{for } a = 0 \\ \frac{1}{\lambda} \left( \sum_{k} p_k^\alpha - 1 \right); & \alpha \neq 1, \quad \text{for } a \neq 0 \end{cases}$$ (21)

For $\lambda = 2^{1-a} - 1$ the entropy reduces to the Havrda-Charvát entropy [3], while in the case of $\lambda = 1 - \alpha$ it reduces to the Tsallis entropy [4]. Let us define $\oplus$-addition, defined as [5],

$$x \oplus_\lambda y = x + y + \lambda xy; \quad a \in \mathbb{R},$$ (22)

for all $x, y \in \mathbb{R}$. For the case $\lambda = 0$, $\oplus_\lambda$-addition reduces to ordinary addition. The pair $(\mathbb{R}, \oplus)$ forms commutative group where the inverse operation and the $\ominus_\lambda$-difference are defined as

$$\ominus_\lambda x = \frac{-x}{1 + \lambda}; \quad x \ominus_\lambda y = \frac{x - y}{1 + \lambda y}$$ (23)

It is easy to see that the structure $(\mathbb{R}, \oplus)$ is a topological group isomorphic to $(\mathbb{R}, +)$ with an isomorphism

$$h(x) = \begin{cases} x, & \text{for } \lambda = 0 \\ \frac{\lambda x - 1}{\lambda} & \text{for } \lambda \neq 0 \end{cases} \iff h^{-1}(y) = \begin{cases} y, & \text{for } \lambda = 0 \\ \frac{\log(\lambda \cdot y + 1)}{\lambda}, & \text{for } \lambda \neq 0 \end{cases}$$ (24)

so that,

$$h(x + y) = h(x) \oplus_\lambda h(y) \iff h^{-1}(x \ominus_\lambda y) = h^{-1}(x) + h^{-1}(y).$$ (25)

The following theorem characterizes the Havrda-Charvát-Tsallis entropy.

**Theorem 4.1.** Let the function $\mathcal{T}_n : \Delta_n \rightarrow \mathbb{R}_{>0}$ satisfies the following axioms, for all $n \in \mathbb{N}, n > 1$:

**TA1:** $\mathcal{T}_n(U_n) = h(1/n)$, where $t : \mathbb{C} \rightarrow \mathbb{C}$ is an analytic function.

**TA2:** Let $P = (p_1, \ldots, p_n) \in \Delta_n$, $PQ = (r_{11}, r_{12}, \ldots, r_{nm}) \in \Delta_{nm}$, $n, m \in \mathbb{N}$ such that $p_i = \sum_{j=1}^{x} r_{ij}$, and $Q_{jk} = (q_{1k}, \ldots, q_{mk}) \in \Delta_m$, where $q_{ik} = r_{ik}/p_k$. Then

$$\mathcal{T}_{nm}(PQ) = \mathcal{T}_n(P) \oplus_\lambda \mathcal{T}_m(Q|P), \text{ where } \mathcal{T}_m(Q|P) = \sum_k p_k^{(n)} \mathcal{T}_m(Q_{jk})$$ (26)

Then, $\mathcal{T}_n$ is the Havrda-Charvát-Tsallis entropy.

**Proof.** If $\lambda = 0$, the Theorem 4.1 reduces to the Theorem 2.1 so we prove the theorem for $\lambda \neq 0$. Similarly as in the proof of the Theorem 3.2 we set $A = (1/2, 1/2)$. By successive usage of formula (26), we get

$$h\left(\frac{1}{2^n}\right) = \mathcal{T}_{2^n}(A \ast A \ast \ldots \ast A) = \bigoplus_{k=1}^{n} \mathcal{T}_{2}(A) = h\left(\sum_{k=1}^{n} \mathcal{T}_{2}(A)\right) = h\left(n \cdot \mathcal{T}_{2}(A)\right) = h\left(\tau \cdot \log \frac{1}{2^n}\right),$$ (27)

where $\tau = -\mathcal{T}_2(1/2, 1/2)$ is a negative real value, since $\mathcal{T}_2(1/2, 1/2)$ is positive, by assumption of the theorem and $\text{sgn}(h(x)) = \text{sgn}(x)$. Accordingly, the values of functions $t(z)$ and $h(\tau \cdot \log z)$ coincide at an infinite
number of points converging to zero, \(z = 1/2^n\) for all \(n\). Since both \(t(z)\) and \(h(\tau \cdot \log z)\) are analytic functions, they must be the same,

\[
t(z) = h(\tau \cdot \log z) = \frac{1}{\lambda} \left( z^{\alpha} - 1 \right); \quad \tau < 0.
\]

Similarly as in the proof of the theorem (2.2), we determine the entropy form for the distribution \(P = (p_1, \ldots, p_n) \in \mathbb{C}^n\) when \(p_i\) are rational numbers and the case for irrational numbers follows from the continuity of the entropy. Let \(P = (p_1, \ldots, p_n) \in \mathbb{C}^n\), \(Q_k = (q_{1,k}, \ldots, q_{m,k}) \in \mathbb{C}^{m \times k}\); \(k = 1, \ldots, n\), and \(PQ = (r_{11}, r_{12}, \ldots, r_{mn}) \in \mathbb{C}^{mn}\), for \(n, m \in \mathbb{N}\), where \(p_i = m_i/m; r_{ij} = 1/m\) and \(q_{ij} = 1/m_i\); where \(m = \sum_{i=1}^m m_i\) and \(m_i \in \mathbb{N}\), for any \(i = 1, \ldots, n\) and \(j = 1, \ldots, m_i\). Then, \(p_i = \sum_{j=1}^k r_{ij}\) and \(q_{jk} = r_{ik}/p_k\) and we can apply the axiom [TA2], which gives

\[
T(P) = \frac{1}{h(\lambda)} \sum_{k=1}^n p_k^\alpha \log \left( \frac{1}{p_k} \right) = \frac{1}{1 + \lambda} \cdot \left( \sum_{k=1}^n p_k^\alpha \right) - 1.
\]

The relationship between \(\tau\) and \(\lambda\) is determined by use of the axiom [TA2]. If we apply the map \(h\) on both sides of the equality (25), using the equality (25), by which \(h^{-1}(x @_y z) = h^{-1}(x) + h^{-1}(y)\), we get

\[
h^{-1}(T(P)) = h^{-1}(T(P)) + h^{-1}\left( \sum_{k=1}^n p_k^\alpha \right)
\]

The function \(h(z) = (2^{1+z} - 1)/\lambda\) is the linear function of \(f(z) = (2^{1+z} - 1)/\gamma\). It is a well-known fact from the mean theory that, if \(h\) is a linear function of \(f\), they generate the same quasi-linear mean [15], and the function \(h\) in the \(\alpha\)-entropies can be substituted with \(f\). Accordingly, the equation (30) can be rewritten in the form of the equation (19). As shown in the proof of the Theorem 3.2, the equation (19) is satisfied iff \(\alpha - \tau = 1\), and the equation (29) reduces to the equation (21), which proves the theorem.

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