Kardar-Parisi-Zhang Equation from
Non-Simple Variations on Open-ASEP

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ABSTRACT. The current article concerns two primary interests. The first of these goals is the universality of the KPZ stochastic PDE among fluctuations of dynamical interfaces associated to interacting particle systems in the presence of a boundary; more precisely, we consider variations on the open-ASEP from [6] and [22] but admitting non-simple interactions both at the boundary and within the bulk of the particle system. These variations on open-ASEP are not integrable models, similar to those long-variations on ASEP considered in [9] and [23]. We establish the KPZ equation with the appropriate Robin boundary conditions as the continuum limits for the fluctuations of height functions associated to these non-integrable models, providing further evidence for the aforementioned universality of the KPZ equation. We specialize to compact domains; we study the non-compact regime in a second article [24].

The procedure we apply to establish the aforementioned theorem is the second primary goal for the current article. The invariant measures in the presence of boundary interactions generally lack reasonable descriptions, and global analysis through the invariant measure including the theory of energy solutions in [11], [12], and [13] is immediately obstructed. To circumvent this obstruction, we entirely localize the analysis in [23].

Lastly, we provide the first application of the Nash theory towards heat kernel estimates within the literature concerning SPDE models for interacting particle systems, to the author’s knowledge.

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The Kardar-Parisi-Zhang equation, abbreviated to be the KPZ equation, on a one-dimensional domain \( \mathcal{D} \) is the following nonlinear stochastic partial differential equation:

\[
\partial_T \mathcal{H}_{T,X} = \frac{\alpha^2}{2} \partial_X^2 \mathcal{H}_{T,X} - \frac{\alpha'}{2} \left| \partial_X \mathcal{H}_{T,X} \right|^2 + a \frac{\partial}{\partial T} \mathcal{H}_{T,X}; \quad (T,X) \in \mathbb{R}_{>0} \times \mathcal{D}.
\]

(1.1)

Above, \( \alpha \in \mathbb{R}_{>0} \) is the diffusivity parameter, while \( \alpha' \in \mathbb{R} \) provides the effective drift for the field \( \mathcal{H} \). Moreover, the field \( \mathcal{W} \) is the Gaussian space-time white noise. Lastly, it is implicitly understood that if \( \mathcal{D} \subseteq \mathbb{R} \) has some one-sided or two-sided boundary, the Laplacian is equipped with Robin boundary parameters \( A_x \subseteq \mathbb{R} \); precisely, if we have \( \mathcal{D} = [a, b] \), then we stipulate the boundary conditions \( \partial_X \mathcal{H} |_{X=a} = A_- \) and \( \partial_X \mathcal{H} |_{X=b} = A_+ \), where we certainly allow \( a = -\infty \) or \( b = \infty \), in which case the corresponding boundary condition is removed.

Historically, the KPZ equation (1.1) was obtained through physical reasoning via the renormalization group to provide a universal model for the fluctuations of dynamical random interfaces; this was performed within [19]. From the perspective of physics and applied sciences, several examples of random interfaces of primary interest include paper-wetting, burning fronts, crack formation, and epidemics; precise examples include the ballistic deposition model, and the Eden model, and we refer to [5] for a more detailed discussion.

Though the renormalization group methods employed within [19] are non-rigorous, within the work of Bertini-Giacomin in [3], the authors rigorously prove that the fluctuations of the dynamical interface known as the solid-on-solid model exhibit the KPZ equation for its continuum limit. Inspired via this result with some duality of this interface with the interacting particle system known as the asymmetric simple exclusion process, abbreviated into ASEP for convenience, the KPZ equation has been rigorously established as the continuum model for fluctuations of interfaces associated to a handful of other interacting particle systems, which we discuss in more detail below.

First, let us somewhat precisely introduce the aforementioned interacting particle system of ASEP as a continuous-time stochastic process.

- Consider any configuration of particles on the full-lattice \( \mathbb{Z} \) subject to the constraint that each site admits at most one particle.
- The collection of particles in the configuration performs jointly independent simple random walks with the "weak" asymmetry; here, the adjective "weak" refers to an asymmetry that vanishes as the system becomes larger, namely with more particles and the speed of the simple random walks increasing.
- The simple random walk dynamics in the previous bullet point are subject to another exclusion principle, namely that particles attempting to jump to already-occupied sites are prohibited from executing the jump.
- The associated interface is defined in Definition 1.5.

Since the work of Bertini-Giacomin in [3], a resulting open problem became to establish the same result but for variations on ASEP though replacing the simple random walk for the individual particles with random walks whose step distributions admitted long-range jumps; for example, we cite the list of "Big Picture Questions" from the workshop for the KPZ equation from the American Institute of Mathematics in 2013. Assuming the restrictive assumption of beginning the particle system at an invariant measure, this was achieved in [12]. Concerning generic initial data, such a universality result was ultimately established in [23] by localizing certain aspects of the analysis within [12]; for the maximal jump-length less than or equal to 3, however, it was achieved through much simpler means in [9].

Meanwhile, within the work of Corwin-Shen in [6], the authors analyze the interface fluctuations for open-ASEP, which is a variation of ASEP although on the half-space with additional boundary interactions. In particular, within [6] the authors establish the continuum limit of the KPZ equation with boundary conditions determined by the boundary interactions from the interacting particle system. This was furthered in [22] for a wider family of allowable boundary interactions. For some more detailed discussion of open-ASEP, we refer the reader to both of these aforementioned articles [6] and [22]; because our focus is towards universality of this continuum model as discussed in the sequel, we do not provide these details.

Again, let us somewhat precisely introduce open-ASEP as another continuous-time stochastic process.
• First, we introduce open-ASEP on the half-space \( \mathbb{Z}_{\geq 1} \).
  - Again, consider any particle configuration on \( \mathbb{Z}_{\geq 1} \) subject to the constraint that each site hosts at most one particle. The system of particles performs jointly independent simple random walks on \( \mathbb{Z}_{\geq 1} \) with the "weak" right-wards asymmetry subject to preventing any attempt jumps onto already-occupied sites.
  - Additionally consider annihilation-creation interactions at the site \( 1 \in \mathbb{Z}_{\geq 1} \). More precisely, with some "weak" asymmetry biased towards either one of the annihilation or creation dynamic, the site \( 1 \in \mathbb{Z}_{\geq 1} \) interacts with a reservoir that either annihilates an existing particle at this site or creates a particle when the site is vacant.

• Second, we introduce open-ASEP on the interval \([1,N]\) \( \cap \mathbb{Z} \), where \( N \in \mathbb{Z}_{\geq 0} \) is the underlying scaling parameter from which we obtain the continuum limit. This process is defined in exactly the same manner upon replacing the half-space with the interval; we also introduce analogous boundary interactions at the boundary \( N \in [1,N] \cap \mathbb{Z} \).

Moreover, we now define precisely what it exactly means to solve this KPZ equation in (1.1); similar to [3], [6], [9], [22], and [23], the following Cole-Hopf solution theory employs another auxiliary linear SPDE which is well-poised in the space of space-time continuous functions with probability 1.

• First, consider any pair of constants \( \lambda \in \mathbb{R} \) providing the Robin boundary parameter.
• Second, define \( \mathcal{Z}_+ \) as the mild solution to the following SPDE, which we define as SHE\( _{\lambda} \):

\[
\partial_t \mathcal{Z}_{T,X} = \alpha \frac{\lambda}{2} \partial_x^2 \mathcal{Z}_{T,X} + \lambda \alpha \frac{x}{2} \mathcal{Z}_{T,X} + \nu(x) \mathcal{Z}_{T,X}, \quad (T,X) \in [0,\infty) \times \mathcal{D} \tag{1.2}
\]

above, the Laplacian in (1.2) is additionally equipped with the same Robin boundary conditions.

• Third, courtesy of the comparison result in Proposition 2.7 from [6], which is identical to the comparison theorem in [21] at the level of proof, positive initial data remains positive with probability 1; define \( \mathcal{H} = -\frac{1}{2} \log \mathcal{Z} \).

We now combine the two respective stories from the previous two discussions. More precisely, we consider variations on open-ASEP upon replacing the nearest-neighbor structure of the random walks with random walks whose step distribution allow for jumps of arbitrary length, and we establish the identical KPZ equation with boundary to be the continuum model for the associated interface fluctuations. The difficulties and interesting aspects of such a result are explained as follows.

• Proceeding in reverse order, the open-ASEP system considered in both [6] and [22] exhibits a self-duality property which reveals its integrability property as an interacting particle system. The minute we adapt the step distribution for the underlying random walks for open-ASEP to admit jumps of arbitrary length, this algebraic property is lost. In particular, like [9], [12], and [23], within the current article we take a step towards the universality of the KPZ equation with boundary.

• Although the current discussion presents [23] as a possible robust method towards establishing the universality for the KPZ equation among some family of interface fluctuations associated to interacting particle systems, there is a significant deficiency – we require an explicit understanding of the invariant measures for ASEP and its variations with long-range interactions on the full-line \( \mathbb{Z} \) or the discrete-torus.

Such understanding is attainable through completely elementary means quite easily as fortunate coincidence. However, for open-ASEP on a domain with boundary, this requires a sophisticated approach known as the matrix product ansatz whose analysis is facilitated through another algebraic duality with a rather complicated stochastic process. Moreover, for long-range variations on open-ASEP, the matrix product ansatz likely fails.

The solution discussed in great detail within this article to the deficiencies in existing technology as discussed above is the following strategy – we adopt the framework within [23] to analyze fluctuations of the interfaces associated to non-simple variations on open-ASEP. However, as alluded to in the second bullet point, the analysis in [23] cannot be applied directly.

The key observation is that the aforementioned analysis within [23] is entirely local, and away from the boundary, which constitutes almost all of the domain at hand, the local dynamics are exactly that of non-simple variations on ASEP without boundary interactions. A primary objective for this article is to fully exploit and develop the local nature of the techniques introduced within [23] towards analyzing the fluctuations of interfaces within the KPZ-class corresponding to some family of interesting particle systems given by long-range variations on ASEP as well as many other interacting particle systems.
Indeed, there is current work-in-progress by the author to implement the local analysis developed for this article towards establishing KPZ-type fluctuations for the interfaces associated to variations on ASEP with slow bonds.

We conclude the current introductory discussion with the following alternative approaches towards universality of the KPZ equation among interfaces associated to interacting particle systems.

- First, we consider the energy solution theory within [11], [12], and [13], for example. This probabilistic approach, which was substantially furthered in [14], was designed as some approach for the KPZ equation in (1.1) through a martingale problem for its weak derivative. Provided this structure as a martingale problem, this solution theory was engineered to establish convergence in fashion almost identical to our primary interest though with hopes to successfully treat a large class of interacting particle systems.

However, its methodology towards defining the problematic nonlinearity in the KPZ equation is entirely reliant on analyzing exclusively interacting particle systems both admitting and starting at an invariant measure satisfying strong mixing and ergodicity assumptions. In particular, any attempt to approach via energy solutions for general Robin boundary parameters $\mathcal{A}_k \in \mathbb{R}$ is currently immediately obstructed by the lack of any reasonable description for the invariant measure of the non-simple variations on open-ASEP considered in the current article; even in the nearest-neighbor open-ASEP system, the invariant measure is quite delicate to analyze because it depends heavily on the matrix product ansatz; see [6] for a further discussion.

- Second, we consider the approach via the theory of regularity structures initiated in [16] and developed in greater sophistication within [17]. Currently, to the author's knowledge there has not yet been any literature for analyzing the interface fluctuations associated to interacting particle systems through the technology of regularity structures. Moreover, the theory of regularity structures itself is inherently limited in the following departments.
  
  - The underlying spatial geometry $\mathcal{D}$ of the KPZ equation (1.1) is necessarily assumed as compact. Meanwhile, many interacting particle systems whose associated interface fluctuations are conjectured to exhibit the KPZ equation as a continuum model are of primary interest on non-compact domains; ASEP on $\mathbb{Z}$ is a prototypical example. Although the primary setting we assume in the current article specializes to compact domains, we demonstrate the methods in the current article are sufficiently robust to employ for the non-compact setting in the second article [24]. Moreover, even the methods within the current article apply for models on domains which are "between" the compact and non-compact regimes exactly as written; we remark on this later.
  
  - Within [16], the initial data required for the KPZ equation (1.1) is assumed to exhibit sufficient regularity. In particular, the narrow-wedge initial data, which morally amounts to having initial data equal to the logarithm of some Dirac point mass, certainly falls outside this class of the allowable initial data, although the narrow-wedge initial data is of utmost importance provided its integrable structure and connection to random matrix theory as established in [1]. This disadvantage with the approach via regularity structures towards studying the KPZ equation is also a disadvantage with the approach via energy solution theory discussed above.
  
  - The approach with regularity structures relies heavily on some precise regularity estimates for the heat kernel corresponding to the parabolic component within the KPZ equation. However, in this article, the dynamical height function at the microscopic level of the interacting particle system evolves according to some stochastic evolution equation in which the analogous parabolic component does not clearly admit heat kernel estimates of the necessary precision.

1.1. Definitions. We proceed to precisely introduce the relevant interacting particle systems. For efforts towards precision and also convenience later on in this article, we define the non-simple variation on open-ASEP as a Markov process.

- We consider the interacting particles evolving on the asymptotically compact space $\mathcal{F}_{N,0} = [0,N\kappa]$; although any universal constant $\kappa \in \mathbb{R}_{>0}$ is allowable, we specialize to $\kappa = 1$ for convenience. Lastly, define $\mathcal{F}_{\infty} = [0,1]$.

  We define the state space of our Markov process to be $\Omega = \{\pm 1\}^{\mathcal{F}_{N,0}}$. Moreover, provided any sub-lattice $\Lambda \subseteq \mathbb{Z}$, let us define the space $\Omega_\Lambda = \{\pm 1\}^\Lambda$. Equivalently, we define the space $\Omega_\Lambda$ will denote the set of admissible particle configurations, or equivalently spin configurations, on the sub-lattice $\Lambda \subseteq \mathbb{Z}$. Elements in $\Omega_\Lambda$ are denoted by $\eta$. 


Lastly, provided any pair of sub-lattices satisfying the containment relation \( \Lambda \subseteq \Lambda' \subseteq \mathbb{Z} \), we obtain the induced canonical "contravariant" projection operator \( \Pi_{\Lambda \to \Lambda'} : \Omega_{\Lambda'} \to \Omega_\Lambda \) defined by the formula
\[
\Pi_{\Lambda \to \Lambda'} : \Omega_\Lambda \to \Omega_{\Lambda'}, \quad (\eta_x)_{x \in \Lambda} \mapsto (\eta_x)_{x \in \Lambda'}.
\] (1.3)

In particular, this justifies the lack of explicitly labeling to which space \( \Omega_\Lambda \) the particle configuration \( \eta \) belongs to; thus, for any \( \eta \in \Omega_\Lambda \), we obtain a canonical particle configuration \( \eta \in \Omega_\Lambda \) for any sub-lattice \( \Lambda \subseteq \mathcal{F}_{x,0} \).

- We proceed to define the Markov process on \( \Omega \) which encodes the non-simple variations on open-ASEP of primary interest in this article.

First, consider any possibly \( N \)-dependent positive integer \( m_N \in \mathbb{Z}_{>0} \) along with three generic sets:
\[
A_N := \left\{ \alpha^N_k \in \mathbb{R}_{>0} \right\}_{k=1}^{m_N}, \quad B_N := \left\{ \beta^N_{k,\pm} \in \mathbb{R}_{>0} \right\}_{k=1}^{m_N}, \quad I_N := \left\{ \gamma^N_k \in \mathbb{R}_{>0} \right\}_{k=1}^{m_N}.
\] (1.4a, 1.4b, 1.4c)

Provided an index \( k \in \mathbb{Z}_{>0} \), there exist four possible choices of signs for the coefficients \( \beta^N_{k,\pm} \). Moreover, we remark that the non-positivity constraint for the set \( I_N \) may be removed upon possibly redefining the narrow-wedge initial data in Definition 1.7 below.

Second, provided any pair of sites \( x, y \in \mathbb{Z} \), let us write \( \Xi_{x,y} \) to be the generator corresponding to the symmetric simple exclusion process on the bond \( \{x, y\} \) with rate 1. Moreover, provided any site \( x \in \mathbb{Z} \), let us denote by \( \Xi_{x,+} \) to be the generator corresponding to the totally asymmetric Glauber dynamic for this site, which updates the spin from \( - \) to \( + \) with rate 1. Similarly, we define \( \Xi_{x,-} \) to be the generator for the totally asymmetric Glauber dynamic which updates the spin from \( + \) to \( - \) with rate 1.

- Define \( \mathcal{L}^{N,\dagger} : L^2(\Omega) \to L^2(\Omega) \) through the following action on any test function \( \varphi \in L^2(\Omega) \):
\[
\mathcal{L}^{N,\dagger} \varphi : \eta \mapsto \frac{1}{2} N^2 \sum_{k=1}^{m_N} \alpha^N_k \sum_{x \in \mathcal{F}_{x,0}} \left[ \left( 1 + \frac{\gamma^N_k}{\sqrt{N}} \right) \frac{1 - \eta_k}{2} + \left( 1 - \frac{\gamma^N_k}{\sqrt{N}} \right) \frac{1 + \eta_k}{2} \right] \Xi_{x,x+k} \varphi + \frac{1}{2} \eta_k \Xi_{x,+} \varphi + \frac{1}{2} \eta_k \Xi_{x,-} \varphi.
\] (1.5)

Provided any \( T \in \mathbb{R}_{>0} \), let us denote by \( \eta^N_T \) as the particle configuration obtained at this time after evolution under the appropriate dynamic.

Moreover, we define \( \mathcal{F}_\bullet \) as the canonical filtration associated to these processes, again omitting the regime.

### 1.2. Main Results.

Before introducing the associated interface fluctuations dual to the aforementioned interacting particle system, a few preliminary assumptions concerning the structural aspects of the interacting particle system itself.

The first assumption is a combination of the separate assumptions within [23].

**Assumption 1.1.** First, we assume the maximal jump-length is uniformly bounded; equivalently, we assume \( m_N \leq 1 \) with a universal implied constant. Moreover, there exist universal constants \( \alpha, \lambda \) such that
\[
\sum_{k=1}^{m_N} k^2 \alpha^N_k \to_{N \to \infty} \alpha_k; \quad \sum_{k=1}^{m_N} k \alpha^N_k \gamma^N_k \to_{N \to \infty} \lambda_k.
\] (1.6)

Additionally, we assume \( \alpha^N_k \gtrsim 1 \) for some universal implied constant; equivalently, the coefficient corresponding to the nearest-neighbor symmetric component of the random walk is uniformly elliptic. Second, we define the specialized asymmetry:
\[
\alpha^N_k \gamma^N_k := 2 \lambda_N \sum_{\ell = k+1}^{m_N} \frac{\ell - k}{k} \alpha^N_k + \lambda_N \alpha^N_k; \quad \lambda_N := \sum_{k=1}^{m_N} \alpha^N_k \gamma^N_k.
\] (1.7)
We assume the following a priori bound for some sufficiently small though universal positive constant $\beta_c \in \mathbb{R}_{>0}$:
\[
\sum_{k=1}^{m_u} k^{\alpha_k} N \left| \gamma_k^N - \gamma_k^{N,-} \right| \lesssim N^{-\frac{1}{2} + \beta_c}.
\] (1.8)

**Remark 1.2.** We address a few subtleties in Assumption 1.1 above.

- We have assumed a universal upper bound on the maximal jump-length for the random walk. This should certainly be removable upon appropriate modifications of our analysis, but this point seems to be just an elementary exercise in the theory of random walks and the associated elliptic and parabolic problems.
- Though the second a priori estimate in Assumption 1.1 above could be replaced by a supremum over jump-lengths or any topology, if one were to extend our analysis to unbounded jump-length then the topology used above would be the appropriate topology to introduce the assumption; see [23] for details.

Roughly speaking, Assumption 1.1 presents structural assumptions concerning the "Kawasaki" component of the particle system, or precisely spin-exchanges. The second and final assumption we require concerns the "Glauber" component within the particle system, or more precisely the boundary interactions which create and annihilate particles when appropriate.

**Assumption 1.3.** Consider any pair of universal constants $\alpha_x \in \mathbb{R}$; moreover, within the non-compact regime, all quantities to follow with a subscript of superscript containing $+$ are neglected.

We will say Assumption 1.3 holds with parameters $\alpha_x \in \mathbb{R}$ if the collection of coefficients \{\(\beta_{j,\pm}^{N,\pm}\)\} \(j=1\) satisfies
\[
\beta_{j,+,N}^{N,\pm} + \beta_{j,-,N}^{N,\pm} = \sum_{k=j}^{m_u} \alpha_k^N + O(N^{-\frac{1}{2}});
\] (1.9a)
\[
\beta_{j,+,N}^{N,\pm} - \beta_{j,-,N}^{N,\pm} = I_+ + II_\pm + III_\pm + IV_\pm;
\] (1.9b)

above, we have introduced the quantities
\[
I_\pm \equiv -\sum_{\ell=j+1}^{m_u} \left( \beta_{\ell,+,N}^{N,\pm} - \beta_{\ell,-,N}^{N,\pm} \right);
\] (1.10a)
\[
II_\pm \equiv \frac{1}{2} \lambda N N^{-\frac{1}{2}} \left[ \sum_{k=1}^{m_u} k \alpha_k^N + \sum_{k=1}^{j-1} \sum_{k=j}^{m_u} k \alpha_k^N + (j-1) \sum_{k=j}^{m_u} \alpha_k^N \right];
\] (1.10b)
\[
III_\pm \equiv \frac{1}{2} \lambda N N^{-\frac{1}{2}} \sum_{k=1}^{m_u} k \alpha_k^N + \sum_{k=1}^{j-1} \sum_{k=j}^{m_u} k \alpha_k^N + (j-1) \sum_{k=j}^{m_u} \alpha_k^N;
\] (1.10c)
\[
IV_\pm \equiv -2 \lambda N^{-\frac{1}{2}} \alpha_x^2 N^{-\frac{1}{2}} k_{N,j-1,\pm};
\] (1.10d)

where in what follows, provided \(x \in \mathbb{Z}_{\geq 0}\), we have \(k_x^- = k \wedge x\) and \(k_x^+ = k \vee (N - x + 1)\):
\[
\kappa_{N,x}^\pm \equiv \frac{1}{4} \sum_{k=1}^{m_u} \left( \alpha_k^N + \frac{\alpha_k^N \gamma_k^N}{\sqrt{N}} \right) \times k_x^\pm \left( e^{-2 \lambda N^{-\frac{1}{2}}} - 1 \right) + \frac{1}{4} \sum_{k=1}^{m_u} \left( \alpha_k^N - \frac{\alpha_k^N \gamma_k^N}{\sqrt{N}} \right) \times k_x^\pm \left( e^{2 \lambda N^{-\frac{1}{2}}} - 1 \right) + O(N^{-\frac{1}{2}}).
\] (1.11)

**Remark 1.4.** Assumption 1.3, though technically involved, is a non-simple variation of "Liggett's condition" for open-ASEP; see Definition 2.8 and Remark 2.9 in [6]. In particular, the role behind Assumption 1.3 is exactly that of the aforementioned Liggett's condition in [6] and [22]; it identifies which boundary condition for the KPZ equation appears in the continuum limit, which is manifest in its facilitation of an "approximate" duality for the particle system of interest in this article.

We proceed to introduce the interface fluctuations associated to the interacting particle system; the following definition is taken directly from Definition 2.3 in [6], for example.
\textbf{Definition 1.5.} First, provided any $T \in \mathbb{R}_{>0}$, we first define $h_{T,0}^N$ to be the net flux of particles at the site $1 \in \mathbb{Z}_{>0}$, with the convention that annihilation of particles contributes a positive flux of $+1$ and creation of particles contributes a negative flux of $-1$, afterwards providing an additional factor of $2$. Observe $h_{0,0}^N = 0$.

Moreover, provided any $x \in \mathcal{J}_{N,0}$, let us now define the associated \textit{height function} and \textit{microscopic Cole-Hopf transform}:

\begin{align}
\mathbf{b}_{T,x}^N & \triangleq h_{T,0}^N + N^{\frac{1}{2}} \sum_{y=1}^{x} \eta_{T,y}^N; \\
\mathbf{3}_{T,x}^N & \triangleq e^{-\lambda u_{T,x} + v_T},
\end{align}

where

\begin{equation}
v_N \triangleq \frac{1}{4} N \sum_{k=1}^{\infty} k \left[ y_k^N u(N^{-1} - \alpha_k^N v(N^{-1}) \right] + \lambda N^2 \sum_{k=1}^{\infty} \alpha_k^N \left( k + \frac{k^2}{12N} - 6k \right); \tag{1.13}
\end{equation}

above, we defined $u(x) = x^{-\frac{1}{4}} \sin(2\lambda_N x)$ and $v(x) = x^{-1}[\cosh(2\lambda_N x) - 1]$, where we have introduced special parameters \{\alpha_k^N\}_{k=1}^{\infty}$ satisfying $|\alpha_k^N - \alpha_k^N| \leq k^{-1}$ for a universal implied constant; we refer to Notation 2.2 for a precise definition.

The final assumptions we introduce before presenting the primary results of this article concern some "analytic" shape of stability for the height function $h^N$ defined above. These assumptions both take the form of definitions, with the first of these encoding spatial regularity of the height function initially.

\textbf{Definition 1.6.} We define any \textit{near-stationary} probability measure $\mu_0^N$ on $\Omega$ to be any probability measure which satisfies the following moment estimates provided any $p \in \mathbb{R}_{>1}$ along with any $u \in (0, \frac{1}{2})$:

\begin{align}
\left\| 3_{0,x}^N \right\|_{L^2^p} & \leq_p 1; \\
\left\| 3_{0,x}^N - 3_{0,y}^N \right\|_{L^2^p} & \leq_p |X - Y|^u;
\end{align}

moreover, we require some continuous function $\mathcal{Z}_{0,\bullet}$ such that $3_{0,N,\bullet}^{N} \rightarrow \mathcal{Z}_{0,\bullet}$ in the uniform topology on $\mathcal{C}(\mathcal{J}_\infty)$.

The second category of allowable initial data is highly-singular and in particular far from continuous at the level of the height function initially; its interest and connection to random matrix theory is deeply discussed in \cite{[1]}, while analytically, the microscopic Cole-Hopf transform $3_{0,\bullet}^{N}$ approximates the fundamental solution of the SHE under appropriate scaling.

\textbf{Definition 1.7.} The \textit{narrow-wedge} initial data is the probability measure on $\Omega$ that is supported on the unique configuration with no particles; namely, under the narrow-wedge initial data, with probability $1$ we have $\eta_x = -1$ for all $x \in \mathcal{J}_{N,0}$.

We now introduce our main results in the current article; the following Theorem 1.8 depends heavily on the technology of the Skorokhod space $\mathcal{D}_\infty \triangleq \mathcal{D}(\mathbb{R}_{\geq 0}, \mathcal{C}(\mathcal{J}_\infty))$; we refer to \cite{[4]} for details. It will additionally serve convenient to define the Skorokhod space $\mathcal{D}_0^\bullet \triangleq \mathcal{D}(\mathbb{R}_{\geq 0}, \mathcal{C}(\mathcal{J}_\infty))$.

\textbf{Theorem 1.8.} Suppose Assumption 1.3 holds for a pair of universal parameters $\mathcal{A}_\bullet \in \mathbb{R}$.

- Assume near-stationary initial data. The space-time field $3_{0,N,\bullet}^{N}$ converges to the solution of the SHE$_{\mathcal{A}_\bullet}$ with parameters $\alpha, \lambda \in \mathbb{R}$ defined in Assumption 1.1 in the Skorokhod space $\mathcal{D}_\infty$.
- Assume narrow-wedge initial data and further consider the rescaled field $3_{N,\bullet}^{N} \equiv \frac{1}{2} \lambda^{-1} N^{\frac{1}{2}} \mathcal{Z}_{N,\bullet}^{N}$. The space-time field $3_{N,\bullet}^{N}$ converges to the solution of the SHE$_{\mathcal{A}_\bullet}$ with the parameters $\alpha, \lambda \in \mathbb{R}$ defined in Assumption 1.1, and initial data the Dirac point mass $\delta_0$ supported at the origin in the Skorokhod space $\mathcal{D}_\infty$.

We conclude the current section with commentary concerning Theorem 1.8.

- For the proof of Theorem 1.8, we will conveniently assume that $\beta_c = 0$, where we recall $\beta_c \in \mathbb{R}_{\geq 0}$ is the sufficiently small constant from Assumption 1.1. The interested reader is certainly invited to find the optimal value of $\beta_c \in \mathbb{R}_{>0}$ allowable under our proof of Theorem 1.8.
Moreover, we present only the proof of Theorem 1.8 under the assumption of near-stationary initial data. Again, this is for our convenience; however, the extension of our analysis for near-stationary initial data may be adapted in exactly the same fashion as in the extension to narrow-wedge initial data in [23] with minor and straightforward additional gymnastics.

- Replace all time-intervals \( \mathbb{R}_{\geq 0} \) defining the Skorokhod spaces within Theorem 1.8 by the family of compact time-intervals \( [0, T_f] \subseteq \mathbb{R}_{\geq 0} \), where the time-parameter \( T_f \in \mathbb{R}_{\geq 0} \) is given by a generic \( N \)-independent terminal time. Indeed, convergence in \( \mathcal{D}_\infty \) and \( \mathcal{D}_\infty^c \) is equivalent to convergence up to any arbitrary though finite terminal time.

- Alternatively, to flirt with the non-compact regime, we proceed to consider the identical particle system although where the domain \( \mathcal{S}_{N,0} \subseteq \mathbb{Z}_{\geq 0} \) is now replaced by the non-compact domain \( \mathcal{S}'_{N,0} \subseteq [0, N^{1+\varepsilon}] \subseteq \mathbb{Z}_{\geq 0} \) with \( \varepsilon \in \mathbb{R}_{>0} \).

Provided that this parameter \( \varepsilon \in \mathbb{R}_{>0} \) is sufficiently small, the methods in this article also provide the scaling limit for the associated height function, which is the solution to SHE_{\alpha} on the non-compact domain \( \mathbb{R}_{\geq 0} \); in particular, the Robin boundary parameter \( \alpha \in \mathbb{R} \) becomes irrelevant within this large-\( N \) limit. We do not explicitly provide any analysis for this, however, because the forthcoming subsequent article [24] addresses a more general problem of extension to the genuinely non-compact setting which requires significantly more analysis.

### 1.3. Outline of Proof

We provide an outline of the proof behind Theorem 1.8, or more specifically those components of the proof detailed in the list of bullet points following the statement of Theorem 1.8, in another list of bullet points below.

- Following [3], [6], [9], [22], and [23], for both the non-compact and compact regimes, the proof of Theorem 1.8 genuinely begins with identifying a microscopic approximation to the SHE_{\alpha} for the stochastic evolution equation of the microscopic Cole-Hopf transform \( \mathfrak{Z}^N \). Unlike those nearest-neighbor interacting particle systems within [3], [6], and [22], and like the particle systems from [9] and [23], the non-simple variations on open-ASEP considered for this article do not admit any integrable structure, duality, or exact microscopic analog of the SHE; an attempt at establishing such duality, as performed in [9], results in error quantities.

Concerning the non-simple variations on open-ASEP, the stochastic evolution equation of the microscopic Cole-Hopf transform \( \mathfrak{Z}^N \) at any spatial coordinates for which particles cannot instantly interact with boundary is exactly that which is computed in Section 2 of [9]; this is consequence of observation that at such coordinates, the particle system "looks like" the non-simple variations on ASEP on the full-lattice \( \mathbb{Z} \). Meanwhile, although we cannot directly "inherit" those expansions provided from Section 2 of [9], the number of such "problematic boundary coordinates" is uniformly bounded. Applying similar expansions, the error quantities near this boundary ultimately assume the form of functionals which vanish in expectation respect to the product Bernoulli measure with site-wise mean 0.

This whole bullet point is the content of Section 2.

- As discussed in this previous bullet point, Section 2 is dedicated towards approximating the stochastic evolution equation of the microscopic Cole-Hopf transform \( \mathfrak{Z}^N \), and we are left with several highly nontrivial error quantities to analyze. Before discussing these error quantities, the approximation obtained within Section 2 towards SHE_{\alpha} approximates the Laplacian with appropriate Robin boundary parameters with a discrete differential-type operator which lacks any explicit analysis via Fourier analysis or the method-of-images; this is in stark contrast to all of the aforementioned articles in [3], [6], [9], [22], and [23], in which heat kernel estimates, which are crucial to these articles, are obtained through applying such exact formulas. Moreover, the differential operator we arrive upon in Section 2 to approximate the Laplacian with Robin boundary conditions, which is a self-adjoint in an appropriate space, is itself not self-adjoint with respect to any reasonable space and is thus only asymptotically self-adjoint.

The strategy we employ to establish the necessary delicate heat kernel estimates begins with its most important ingredient which is the suitable Nash-type inequality, which in turns depends heavily on the heat kernel estimates for the nearest-neighbor version of our approximating operator to the Laplacian with appropriate Robin boundary conditions analyzed within both of [6] and [22], itself admitting an exact formula through the method-of-images. We proceed to deduce the optimal "on-diagonal" pointwise heat kernel upper bound through the strategy via Nash inequality and afterwards upgrade this to an off-diagonal heat kernel estimates through probabilistic random walk.
considerations. We emphasize that such an approach towards heat kernel estimates through the Nash inequalities has not yet been employed or even yet considered before this article within the literature concerning SPDE models for interacting particle systems to the author’s awareness or knowledge, although it promises to serve substantially more robust and analytically-inspired than the method-of-images used crucially in [6] and [22].

The previous paragraph concerns heat kernel estimates which would be sufficient if we had exact approximation to SHE$\mathbb{R}$ without the aforementioned error quantities; we reemphasize that such is actually true for open-ASEP. To analyze these error quantities, the procedure we will outline shortly reduces their analysis to somewhat precise heat kernel regularity estimates. Our strategy towards this goal begins with considering first exactly the Neumann boundary parameters $\mathcal{A}_z = 0$. Through some Duhamel-type formula, we then approximate the method-of-images expansions within both of [6] and [22] for the heat kernel through the auxiliary heat kernel corresponding to the non-simple symmetric random walk on the entire lattice $\mathbb{Z}$, for which we establish heat kernel regularity estimates via Proposition A.1 in [9]. This Duhamel-type formula then allows us to transfer the regularity estimates.

The extension from the Neumann boundary parameters $\mathcal{A}_z = 0$ to arbitrary Robin boundary parameters begins another perturbative Duhamel-type formula which ultimately recovers the method-of-images expansion in [6] and [22]. Similar to the previous paragraph, this allows us to transfer heat kernel estimates with Neumann boundary condition to those with arbitrary Robin boundary conditions.

The aforementioned procedure provides one robust, analytic, and more general strategy to heat kernel analysis compared to both [6] and [22].

- Having equipped ourselves with precise heat kernel regularity estimates, we proceed to analyze the error quantities in our approximation of SHE$\mathbb{R}$ in the stochastic evolution equation for $\mathbb{R}^N$.
  - The error quantities arising away from the boundary, or equivalently where the evolution of the microscopic Cole-Hopf transform $\mathbb{R}^N$ "looks like" that for the interacting particle system on the entire lattice, are identically those of primary interest in [23], namely the pseudo-gradient fields defined therein.
  - The primary obstruction towards simply applying the results in [23] is the observation that, unless we choose specific values of the Robin boundary parameters $\mathcal{A}_z \in \mathbb{R}$, the "grand-canonical ensembles" and "canonical ensembles" are absolutely not invariant measures for the interacting particle system; in fact, actually outside the regime of exclusively nearest-neighbor interactions, there exists no reasonable explicit description of the relevant invariant measures. This simple though important obstruction, for example, obstructs a possibility of adopting the approach via energy solutions, discussed towards the beginning of this article, for this problem.
  - Fortunately, away from the boundary, the "local" dynamics are unaware of any boundary interactions, so thus the "local" invariant measures away from the boundary are determined by those "local" canonical ensembles. Moreover, on mesoscopic time-scales barely above the microscopic time-scale, the interacting particle system near the boundary is roughly exactly the interacting particle system obtained after eliminating all asymmetric dynamics in both the Glauber and Kawasaki dynamics whose unique invariant measure is equal to the product Bernoulli measure whose parameter is the global particle density. Roughly speaking, these two observations are certainly sufficient to adopt the strategy in [23] because this strategy is entirely based on space-time local analysis. In particular, localizing certain aspects of the energy solution theory in [23] makes it substantially more robust.

Concerning the error quantities which are supported entirely at the boundary, another application of the dynamical-averaging principle introduced in [23] suffices because the time-scale at which we ultimately employ this principle is barely above the microscopic time-scale, and these error quantities vanish with respect to the "sufficiently close" invariant measure given by the product Bernoulli measure of parameter equal to the global particle density.

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2. Derivation of Microscopic Stochastic Heat Equation

In the current section, we first derive a microscopic stochastic heat equation for the dynamic interface associated to the particle system on the compact domain \( \mathcal{J}_{N,0} \subseteq \mathbb{Z}_{\geq 0} \).

For the reader’s convenience, we recall the following notation for the underlying Poisson clocks in our particle system.

**Notation 2.1.** Consider any \( T \in \mathbb{R}_{\geq 0}, \) any \( x \in \mathcal{J}_{N,0} \) and any \( k \in [1, m_N] \).

We define \( \xi_{T,x}^{N,k}(\pm) \) to be a Poisson process of dynamical time-inhomogeneous rate

\[
(e^{-2\lambda_0 N^{-\frac{1}{2}}} - 1) \left( \alpha_k^N \pm \frac{\alpha_k^N \gamma_k^N}{\sqrt{N}} \right) \frac{1 \mp \eta_{T,x}^N}{2} \frac{1 \pm \eta_{T,x+k}^N}{2}.
\]

(2.1)

In particular, \( \xi_{T,x}^{N,k}(+) \) admits the interpretation of the Poisson clock that determines the total number of times a particle jumps from \( x \rightsquigarrow x + k \) in the time interval \([0, T] \subseteq \mathbb{R}_{\geq 0}\) and \( \xi_{T,x}^{N,k}(-) \) admits the interpretation of the Poisson clock that determines the total number of times a particle jumps from \( x + k \rightsquigarrow x \) in the time interval \([0, T] \subseteq \mathbb{R}_{\geq 0}\), up to \( N \)-dependent factors.

We proceed to define the auxiliary Poisson clocks associated to the boundary interactions. Provided \( T \in \mathbb{R}_{\geq 0} \) and \( k \in [1, m_N] \), we define \( \xi_{T,-}^{N,k}(\pm) \) and \( \xi_{T,+}^{N,k}(\pm) \) to be Poisson processes of rate \( r_{T,k,-}^{N,\pm}, r_{T,k,+}^{N,\pm} \in \mathbb{R}_{\geq 0} \), respectively, where

\[
r_{T,k,-}^{N,\pm} = \left( e^{\pm 2\lambda_0 N^{-\frac{1}{2}}} - 1 \right) \beta_{k,-}^N \frac{1 \mp \eta_{T,k}^N}{2}, \quad r_{T,k,+}^{N,\pm} = \left( e^{\pm 2\lambda_0 N^{-\frac{1}{2}}} - 1 \right) \beta_{k,+}^N \frac{1 \pm \eta_{T,N-k+1}^N}{2}.
\]

(2.2)

Lastly, denote by \( \tilde{\xi}^N \) the associated Poisson-martingale to \( \xi^N \); we make this more precise in the proof of Proposition 2.3.

We additionally recall the auxiliary coefficients which are suitable approximations of the coefficients \( A^N = \{ a_k^N \}_{k=1}^{m_N} \).

**Notation 2.2.** Provided any \( k \in [1, m_N] \), let us define

\[
\tilde{\alpha}_k^N = \alpha_k^N + N^{-1} \lambda^2 \frac{k-2}{2} \alpha_k^N - N^{-1} \frac{k^2}{2} \sum_{\ell=k+1}^{m_N} (2\ell-k) \alpha_{k}^N.
\]

(2.3)

2.1. Stochastic Differential Equation. Let us establish the SDE evolution governing the microscopic Cole-Hopf transform for the current subsection. Provided the expansion via SDEs performed within Section 2 from [9] for the exclusion dynamic on the full lattice \( \mathbb{Z} \), for the interactions are local it will often serve convenient to cite the expansion therein when boundary interactions are absent. Indeed, the purpose of this subsection is to address the contribution from boundary interactions.

**Proposition 2.3.** Recall the maximal jump-length \( m_N \in \mathbb{Z}_{\geq 0} \).

- Consider any \( x \in [0, m_N-1] \), for some deterministic coefficient \( \kappa_{N,x}^N \in \mathbb{R}_{\geq 0} \) and deterministic \( x \)-dependent coefficients \( \kappa_{j,l}^N \in \mathbb{R}_{\geq 0} \), we have

\[
d\tilde{\xi}_{T,x}^N = \frac{1}{4} \sum_{\ell=0}^{x-1} \eta_{T,x-\ell}^N \left[ -\left( e^{-2\lambda_0 N^{-\frac{1}{2}}} - 1 \right) \sum_{k=0}^{m_N} \left( \alpha_k^N - \frac{\alpha_k^N \gamma_k^N}{\sqrt{N}} \right) \right] \tilde{\xi}_{T,x}^N dT
\]

\[
+ \frac{1}{4} \sum_{\ell=1}^{x} \eta_{T,x+\ell}^N \left[ \left( e^{-2\lambda_0 N^{-\frac{1}{2}}} - 1 \right) \sum_{k=0}^{m_N} \left( \alpha_k^N - \frac{\alpha_k^N \gamma_k^N}{\sqrt{N}} \right) - \left( e^{2\lambda_0 N^{-\frac{1}{2}}} - 1 \right) \sum_{k=0}^{m_N} \left( \alpha_k^N + \frac{\alpha_k^N \gamma_k^N}{\sqrt{N}} \right) \right] \tilde{\xi}_{T,x}^N dT
\]

\[
+ \frac{1}{2} \sum_{\ell=1}^{x} \eta_{T,x+\ell}^N \left[ -\frac{1}{2} \beta_{x+l,+}^N \left( e^{2\lambda_0 N^{-\frac{1}{2}}} - 1 \right) + \frac{1}{2} \beta_{x+l,-}^N \left( e^{-2\lambda_0 N^{-\frac{1}{2}}} - 1 \right) \right] \tilde{\xi}_{T,x}^N dT
\]

\[
+ \kappa_{N,x}^N \tilde{\xi}_{T,x}^N dT + \sum_{j,l \in \mathbb{Z}_{\geq 0}; j+l \leq m_N} \kappa_{j,l}^N \eta_{T,x-j}^N \tilde{\xi}_{T,x+l}^N dT + \tilde{\xi}_{T,x}^N d\tilde{\xi}_{T,x}^N,
\]

the coefficients \( \kappa_{j,l}^N \) satisfy the bounds \( \kappa_{j,l}^N \ll |j-l|^{-p} \) for all \( p \in \mathbb{R}_{> 1} \), and for \( \kappa_{x}^N = k \wedge x \), we have

\[
\kappa_{N,x}^N = \frac{1}{4} \sum_{k=1}^{m_N} \left( \alpha_k^N + \frac{\alpha_k^N \gamma_k^N}{\sqrt{N}} \right) x_k^{-1} \left( e^{-2\lambda_0 N^{-\frac{1}{2}}} - 1 \right) + \frac{1}{4} \sum_{k=1}^{m_N} \left( \alpha_k^N - \frac{\alpha_k^N \gamma_k^N}{\sqrt{N}} \right) x_k \left( e^{2\lambda_0 N^{-\frac{1}{2}}} - 1 \right) + \mathcal{O}(N^{-\frac{1}{2}})
\]

(2.5)

where the implied constant is universal. In particular, we have \( \kappa_{N,x}^N = \mathcal{O}(N^{-1}) \) with universal implied constant.
Consider any point \( x \in [N - m_N + 1, N] \), for some deterministic coefficient \( \kappa^+_{N,x} \in \mathbb{R}_{>0} \) and deterministic \( x \)-dependent coefficients \( \{ \kappa_{j,l}^+ \}_{j,l} \in \mathbb{R}_{>0} \), we have

\[
\begin{align*}
\text{d} \mathcal{N}^{N}_{T,x} &= \frac{1}{4} \sum_{l=1}^{N-x} \eta^{N}_{T,x+l} \left[ \left( -e^{2\lambda_N N^{-\frac{1}{2}} - 1} \right) \sum_{k=1}^{m_x} \left( a_k^N - \frac{\alpha_k^N Y_k}{\sqrt{N}} \right) + \left( e^{2\lambda_N N^{-\frac{1}{2}} - 1} \right) \sum_{k=1}^{m_x} \left( a_k^N + \frac{\alpha_k^N Y_k}{\sqrt{N}} \right) \right] \mathcal{N}^{N}_{T,x} \text{d}T \\
&\quad + \frac{1}{4} \sum_{l=0}^{m_x-1} \eta^{N}_{T,x-l} \left[ \left( -e^{2\lambda_N N^{-\frac{1}{2}} - 1} \right) \sum_{k=1}^{N-x} \left( a_k^N - \frac{\alpha_k^N Y_k}{\sqrt{N}} \right) + \left( e^{2\lambda_N N^{-\frac{1}{2}} - 1} \right) \sum_{k=1}^{N-x} \left( a_k^N + \frac{\alpha_k^N Y_k}{\sqrt{N}} \right) \right] \mathcal{N}^{N}_{T,x} \text{d}T \\
&\quad + \frac{1}{2} \sum_{l=0}^{m_x-N+x-1} \eta^{N}_{T,x-l} \left[ -\frac{1}{2} \beta^{N,+}_{N-x+1+l,0} \left( e^{2\lambda_N N^{-\frac{1}{2}} - 1} \right) + \frac{1}{2} \beta^{N,+}_{N-x+1+l,0} \left( e^{-2\lambda_N N^{-\frac{1}{2}} - 1} \right) \right] \mathcal{N}^{N}_{T,x} \text{d}T \\
&\quad + \kappa^+_{N,x} \mathcal{N}^{N}_{T,x} \text{d}T + \sum_{j,l \in \mathbb{Z}_{>0}} \kappa^+_{j,l} \mathcal{N}^{N}_{T,x-j} \mathcal{N}^{N}_{T,x+l} \mathcal{N}^{N}_{T,x} \text{d}T + \mathcal{N}^{N}_{T,x} \text{d}\xi^N_{T,x}.
\end{align*}
\]

for all \( p \in \mathbb{R}_{>0} \), and for \( k^*_x = k \wedge (N - x + 1) \),

\[
\kappa^+_{N,x} = \frac{1}{4} \sum_{k=1}^{m_x} \left( a_k^N + \frac{\alpha_k^N Y_k}{\sqrt{N}} \right) \times k^*_x \left( e^{2\lambda_N N^{-\frac{1}{2}} - 1} \right) + \frac{1}{4} \sum_{k=1}^{m_x} \left( a_k^N - \frac{\alpha_k^N Y_k}{\sqrt{N}} \right) \times k^*_x \left( e^{2\lambda_N N^{-\frac{1}{2}} - 1} \right) + \mathcal{O}(N^{-\frac{1}{2}}).
\]

Proof. We provide the proof of the dynamics for \( x \in [0, m_N - 1] \); for sites \( x \in [N - m_N + 1, N] \), the same argument applies upon reflection.

We first claim the following preliminary calculation in which \( k_x = k \wedge x \) for any \( k \in \mathbb{Z}_{>0} \):

\[
\begin{align*}
\text{d} \mathcal{N}^{N}_{T,x} &= \sum_{k=1}^{m_x} \left( a_k^N + \frac{\alpha_k^N Y_k}{\sqrt{N}} \right) \sum_{l=0}^{k-1} \left( 1 - \frac{\eta^{N}_{T,x-l} + \eta^{N}_{T,x-l+k}}{2} \right) \left( e^{2\lambda_N N^{-\frac{1}{2}} - 1} \right) \mathcal{N}^{N}_{T,x} \text{d}T \\
&\quad + \sum_{k=1}^{m_x} \left( a_k^N - \frac{\alpha_k^N Y_k}{\sqrt{N}} \right) \sum_{l=0}^{k-1} \left( 1 - \frac{\eta^{N}_{T,x-l} + \eta^{N}_{T,x-l+k}}{2} \right) \left( e^{2\lambda_N N^{-\frac{1}{2}} - 1} \right) \mathcal{N}^{N}_{T,x} \text{d}T \\
&\quad + \sum_{k=1}^{m_x} \beta^{N,+}_{k,x} \left( 1 - \frac{\eta^{N}_{T,x} + \eta^{N}_{T,x+k}}{2} \right) \mathcal{N}^{N}_{T,x} \text{d}T + \sum_{k=1}^{m_x} \beta^{N,-}_{k,x} \left( 1 + \frac{\eta^{N}_{T,x} + \eta^{N}_{T,x+k}}{2} \right) \mathcal{N}^{N}_{T,x} \text{d}T \\
&\quad + \mathcal{N}^{N}_{T,x} \text{d}T + \mathcal{N}^{N}_{T,x} \text{d}\xi^N_{T,x}.
\end{align*}
\]

Indeed, the first-step (2.8) is a direct consequence of the Ito formula which we summarize in words through the following observations; as a guide, we refer to the SDE expansion within Section 2 of [9]:

- The constant drift \( \gamma_N \in \mathbb{R}_{\neq 0} \) arises straightforwardly from the constant growth in the exponential defining \( \mathcal{N}^{N}_{T,x} \).
- The martingale integrator \( \mathcal{N}^{N}_{T,x} \) is a sum of the underlying Poisson clock processes associated to bonds and boundary interactions crossing \( x \mathcal{S}_{N,0} \), afterwards compensated by their respective drifts. More precisely, we have

\[
\begin{align*}
\text{d} \mathcal{N}^{N}_{T,x} &= \sum_{k=1}^{m_x} \sum_{l=0}^{k-1} \mathcal{N}^{N}_{T,x-l} \left( - \left( e^{2\lambda_N N^{-\frac{1}{2}} - 1} \right) \sum_{k=1}^{m_x} \left( a_k^N + \frac{\alpha_k^N Y_k}{\sqrt{N}} \right) \sum_{l=0}^{k-1} \left( 1 - \frac{\eta^{N}_{T,x-l} + \eta^{N}_{T,x-l+k}}{2} \right) \right) \mathcal{N}^{N}_{T,x} \text{d}T \\
&\quad + \sum_{k=1}^{m_x} \sum_{l=0}^{k-1} \mathcal{N}^{N}_{T,x-l} \left( + \left( e^{2\lambda_N N^{-\frac{1}{2}} - 1} \right) \sum_{k=1}^{m_x} \left( a_k^N - \frac{\alpha_k^N Y_k}{\sqrt{N}} \right) \sum_{l=0}^{k-1} \left( 1 - \frac{\eta^{N}_{T,x-l} + \eta^{N}_{T,x-l+k}}{2} \right) \right) \mathcal{N}^{N}_{T,x} \text{d}T \\
&\quad + \sum_{k=1}^{m_x} \sum_{l=0}^{k-1} \mathcal{N}^{N}_{T,x-l} \left( + \sum_{k=1}^{m_x} \left( a_k^N + \frac{\alpha_k^N Y_k}{\sqrt{N}} \right) \sum_{l=0}^{k-1} \left( 1 - \frac{\eta^{N}_{T,x-l} + \eta^{N}_{T,x-l+k}}{2} \right) \right) \mathcal{N}^{N}_{T,x} \text{d}T \\
&\quad + \sum_{k=1}^{m_x} \sum_{l=0}^{k-1} \mathcal{N}^{N}_{T,x-l} \left( + \sum_{k=1}^{m_x} \left( a_k^N - \frac{\alpha_k^N Y_k}{\sqrt{N}} \right) \sum_{l=0}^{k-1} \left( 1 - \frac{\eta^{N}_{T,x-l} + \eta^{N}_{T,x-l+k}}{2} \right) \right) \mathcal{N}^{N}_{T,x} \text{d}T.
\end{align*}
\]

- The aforementioned drifts of the Poisson processes contribute the first three lines within the RHS of (2.8), almost exactly as in the expansion performed in Section 2 of [9]. The only difference between the particle system for this article and the particle system without boundary interactions in [9] amount to the following two bullet points.
- The prevalent interactions within the current particle system with boundary involve points to the left of \( x \in \mathcal{S}_{N,0} \), of which there exists strictly less than the maximal jump-length \( m_N \in \mathbb{Z}_{>0} \) in number. This provides summations over \( l \in [0, k_x - 1] \), which would be replaced by a summation over \( l \in [0, k - 1] \) if without the boundary condition.
Second, concerning those quantities within the RHS of (2.8) involving $\beta^{N,-}_{k,\pm}$-coefficients, it suffices to observe that adding a particle to a site $k \in [x + 1, m_N]$ occurs with rate $\beta^{N,-}_k \in \mathbb{R}_{>0}$ if the given site is empty. Moreover, the total change before and after this interaction to the flux of the system $\eta^{N}_T$ is $2\lambda_N N^{-\frac{3}{2}}$; it also provides the only change in the height fluctuation $h^{N}_T$, because the occupation variables in $[1, x]$ are all unchanged. A similar observation concerning the action of removing a particle yields the quantity involving $\beta^{N,-}_k$-coefficients.

Provided (2.8), however, the desired equation now follows from organizing quantities within the RHS of (2.8) in terms of dependence on the particle system; such reorganization consists of a complicated but elementary calculation that matches that of Section 2 in [9] almost identically, so we omit it.

Remark 2.4. Technically, the martingale quantity $\mathcal{N} \, d\xi^{N}$ appearing in the stochastic evolution equation within Proposition 2.3 is not the product of the microscopic Cole-Hopf transform $\mathcal{N}$ with a Poisson-martingale $\xi^{N}$. Actually, it is a martingale $m^{N}$ whose jumps at the space-time coordinates $(T, x) \in \mathbb{R}_{>0} \times \mathbb{Z}_{\geq 0}$ are given by those of $\xi^{N}$ although scaled by the stochastic though adapted factor $\mathcal{N}^{N}_{T,x}$.

In general, let us declare that any martingale quantity of the form $\Phi \, d\xi^{N}$ provided $\Phi$ a space-time adapted random field is the Poisson martingale whose jumps at space-time coordinates $(T, x) \in \mathbb{R}_{>0} \times \mathbb{Z}_{\geq 0}$ are exactly those of $\xi^{N}$ scaled via the additional factor of $\Phi_{T,x}$.

2.2. Differential Operator Expansion. We proceed to identify the SDE-type expansion in Proposition 2.3 with the action of some discrete-type Laplacian on the microscopic Cole-Hopf transform. In particular, the expression which we ultimately match with the evolution equations in Proposition 2.3 are provided in the following result; first we introduce some notation that we employ throughout the remainder of this article.

Notation 2.5. Consider the operator $\mathcal{L}_{\text{Lap}}^{N}$ defined by the following action on functions $\varphi : \mathcal{N}_{N,0} \rightarrow \mathbb{R}$:

$$
\mathcal{L}_{\text{Lap}}^{N} \varphi_x = 1_{x \in [m_N, N-m_N]} \left\{ \frac{1}{2} \sum_{k=1}^{m_N} \alpha^N_k \varphi_x + 1_{x \in [0, m_N-1]} \left\{ \frac{1}{2} \sum_{k=1}^{m_N} \alpha^N_k \nabla^T_k \varphi_x + 1_{x \in [0, m_N-1]} \cdot \frac{1}{2} \sum_{k=1}^{m_N} \alpha^N_k \nabla^T_k \varphi_x \right\} + 1_{x \in [N-m_N+1, N]} \left\{ \frac{1}{2} \sum_{k=1}^{m_N} \alpha^N_k \varphi_x \right\}
\right.
$$

(2.10)

Proposition 2.6. Recall the maximal jump-length $m_N \in \mathbb{Z}_{>0}$.

- Consider any point $x \in [1, m_N-1]$; for deterministic though $x$-dependent coefficients $\{ \tilde{K}^N_{j,l} \}_{j,l}$ and for some functional $\tilde{\eta}^{N,+}_{T,x}$ of the particle system,

$$
\mathcal{L}_{\text{Lap}}^{N} \tilde{\eta}^{N}_{T,x} = \sum_{\ell=1}^{m_N} \tilde{\eta}^{N}_{T,x+\ell} \left[ \frac{1}{2} \lambda^N \nabla^T \cdot \sum_{k=1}^{m_N} \alpha^N_k \tilde{\eta}^{N}_{T,x-\ell} + \sum_{\ell=0}^{x-1} \nabla^T_{\tilde{\xi}^{N}_{T,x-\ell}} \left[ \frac{1}{2} \lambda^N \nabla^T \cdot \sum_{k=1}^{m_N} \alpha^N_k \right] \tilde{\eta}^{N}_{T,x-\ell} \right]
$$

(2.11)

moreover, these $\tilde{K}^N_{j,l}$-coefficients and the functional $\tilde{\eta}^{N,+}_{T,x}$ are uniformly bounded.

- For deterministic coefficients $\{ \tilde{K}^N_{j,l} \}_{j,l}$ and for some functional $\tilde{\eta}^{N,-}_{T,0}$ of the particle system, we have

$$
\mathcal{L}_{\text{Lap}}^{N} \tilde{\eta}^{N}_{T,0} = \sum_{\ell=1}^{m_N} \tilde{\eta}^{N}_{T,x+\ell} \left[ \frac{1}{2} \lambda^N \nabla^T \cdot \sum_{k=1}^{m_N} \alpha^N_k \tilde{\eta}^{N}_{T,0} + \frac{1}{2} \lambda^N \nabla^T \sum_{k=1}^{m_N} k \alpha^N_k \tilde{\eta}^{N}_{T,0} - \mathcal{L}^{N} \tilde{\eta}^{N}_{T,0} \right]
$$

(2.12)

moreover, these $\tilde{K}^N_{j,l}$-coefficients and the functional $\tilde{\eta}^{N,-}_{T,0}$ are uniformly bounded.
• Consider any $x \in [N - m_N, 1, N - 1]$; we have

$$\mathcal{L}^{N}_{\text{Lap}} \mathcal{N}^{N}_{T,x} = \sum_{\ell=0}^{m_N-1} \eta_{T,x-\ell} \times \left[ -\frac{1}{2} \lambda N N^{-\frac{1}{2}} \sum_{k=1}^{m_N} \tilde{a}_k \right] \mathcal{N}^{N}_{T,x} + \sum_{\ell=1}^{N-x} \eta_{T,x+\ell} \times \left[ \frac{1}{2} \lambda N N^{-\frac{1}{2}} \sum_{k=1}^{m_N-1} \tilde{a}_k \right]$$

$$+ \frac{1}{4} \lambda^2 N N^{-1} \left[ \sum_{k=1}^{m_N} k \tilde{a}_k + (N - x) \sum_{k=N-N+1}^{m_N} \tilde{a}_k \right] \mathcal{N}^{N}_{T,x} + N^{-\frac{3}{2}} \mathcal{N}^{N}_{T,x} \mathcal{N}^{N}_{T,x}$$

$$+ \lambda^2 N N^{-1} \sum_{j \neq \ell \in \mathbb{Z}^2 \cap [0, m_N]} \left[ \tilde{\kappa}^{N}_{j, \ell} \eta_{T,x-j} \eta_{T,x+\ell} \mathcal{N}^{N}_{T,x} \right];$$

moreover, these $\tilde{\kappa}^{N}_{j, \ell}$-coefficients and the functional $\mathcal{N}^{N}_{T,0}$ are uniformly bounded.

**Proof.** We first assume $x \in [0, m_N - 1]$; for those points $x \in [N - m_N + 1, N]$, an identical argument holds upon reflection just as in the proof of Proposition 2.3. Moreover, we will first consider $x \neq 0$.

Provided $k \in [1, m_N]$, we employ Taylor expansion to compute the following forward-difference operator:

$$3^{N}_{T,x+k} - 3^{N}_{T,x} = -\lambda N N^{-\frac{1}{2}} \sum_{\ell=1}^{k} \eta_{T,x+\ell} \mathcal{N}^{N}_{T,x} + \frac{1}{2} \lambda^2 N N^{-1} \left( \sum_{\ell=1}^{k} \eta_{T,x+\ell} \right) \mathcal{N}^{N}_{T,x} + N^{-\frac{3}{2}} \mathcal{N}^{N}_{T,x} \mathcal{N}^{N}_{T,x},$$

where $|k|^{-3} \mathcal{N}^{N}_{T,x}$ is a uniformly bounded functional of the particle system.

We proceed to analyze the backward-difference operator. To this end, consider $k \in [1, x]$; Taylor expansion gives us

$$3^{N}_{T,x-k} - 3^{N}_{T,x} = \lambda N N^{-\frac{1}{2}} \sum_{\ell=0}^{k-1} \eta_{T,x-\ell} \mathcal{N}^{N}_{T,x} + \frac{1}{2} \lambda^2 N N^{-1} \left( \sum_{\ell=0}^{k-1} \eta_{T,x-\ell} \right) \mathcal{N}^{N}_{T,x} + N^{-\frac{3}{2}} \mathcal{N}^{N}_{T,x} \mathcal{N}^{N}_{T,x},$$

where again $|k|^{-3} \mathcal{N}^{N}_{T,x}$ is another uniformly bounded functional of the particle system. Combining these Taylor expansions (2.15) and (2.16), we obtain

$$\mathcal{L}^{N}_{\text{Lap}} \mathcal{N}^{N}_{T,x} = \frac{1}{2} \sum_{k=1}^{m_N} \left( 3^{N}_{T,x+k} - 3^{N}_{T,x} \right) + \frac{1}{2} \sum_{k=1}^{N} \tilde{a}_k \left( 3^{N}_{T,x-k} - 3^{N}_{T,x} \right) + \frac{1}{2} \sum_{k=x+1}^{m_N} \tilde{a}_k \left( 3^{N}_{T,0} - 3^{N}_{T,x} \right)$$

$$= \frac{1}{2} \sum_{k=1}^{m_N} \tilde{a}_k \left[ -\lambda N N^{-\frac{1}{2}} \sum_{\ell=1}^{k} \eta_{T,x+\ell} + \frac{1}{2} \lambda^2 N N^{-1} \left( \sum_{\ell=1}^{k} \eta_{T,x+\ell} \right) \right] \mathcal{N}^{N}_{T,x}$$

$$+ \frac{1}{2} \sum_{k=1}^{N} \tilde{a}_k \left[ \lambda N N^{-\frac{1}{2}} \sum_{\ell=0}^{k-1} \eta_{T,x-\ell} + \frac{1}{2} \lambda^2 N N^{-1} \left( \sum_{\ell=0}^{k-1} \eta_{T,x-\ell} \right) \right] \mathcal{N}^{N}_{T,x}$$

$$+ \frac{1}{2} \sum_{k=x+1}^{m_N} \tilde{a}_k \left[ \lambda N N^{-\frac{1}{2}} \sum_{\ell=0}^{x-1} \eta_{T,x-\ell} + \frac{1}{2} \lambda^2 N N^{-1} \left( \sum_{\ell=0}^{x-1} \eta_{T,x-\ell} \right) \right] \mathcal{N}^{N}_{T,x}$$

$$+ \frac{1}{2} N^{-\frac{3}{2}} \sum_{k=1}^{m_N} \tilde{a}_k \mathcal{N}^{N}_{T,x} \mathcal{N}^{N}_{T,x} + \frac{1}{2} N^{-\frac{3}{2}} \sum_{k=1}^{x} \tilde{a}_k \mathcal{N}^{N}_{T,x} \mathcal{N}^{N}_{T,x} + \frac{1}{2} N^{-\frac{3}{2}} \sum_{k=x+1}^{m_N} \tilde{a}_k \mathcal{N}^{N}_{T,x} \mathcal{N}^{N}_{T,x}.$$
We now expand the squares appearing on the RHS of (2.18) within the following calculation which crucially relies on the observation the spin at each site is valued in \( \{ \pm 1 \} \):

\[
\left( \sum_{\ell=1}^{k} \eta_{T,x+\ell}^{N} \right)^{2} = k + \sum_{\ell_{1},\ell_{2}=1}^{k} \mathbf{1}_{\ell_{1}\neq\ell_{2}} \eta_{T,x+\ell_{1}}^{N} \eta_{T,x+\ell_{2}}^{N}; \tag{2.19a}
\]

\[
\left( \sum_{\ell=0}^{k-1} \eta_{T,x-\ell}^{N} \right)^{2} = k + \sum_{\ell_{1},\ell_{2}=0}^{k-1} \mathbf{1}_{\ell_{1}\neq\ell_{2}} \eta_{T,x-\ell_{1}}^{N} \eta_{T,x-\ell_{2}}^{N}; \tag{2.19b}
\]

\[
\left( \sum_{\ell=0}^{x-1} \eta_{T,x+\ell}^{N} \right)^{2} = x + \sum_{\ell_{1},\ell_{2}=0}^{x-1} \mathbf{1}_{\ell_{1}\neq\ell_{2}} \eta_{T,x-\ell_{1}}^{N} \eta_{T,x-\ell_{2}}^{N}. \tag{2.19c}
\]

Rearranging quantities as in the proof of Proposition 2.3 completes the proof; once again, although this calculation is quite involved, it is elementary and a simpler version of the proof of Proposition 2.2 in [9], so we omit it.

We proceed to compute the evolution equation for \( x = 0 \). To this end, observe that it suffices to compute only forward-derivatives. Proceeding with the previous calculation and using the Robin boundary condition, we have

\[
\mathcal{L}_{\text{Lap}}^{N} \bar{z}^{N}_{T,0} = \frac{1}{2} \sum_{k=1}^{m_{c}} \alpha_{k} \left( 3_{T,x+k}^{N} - 3_{T,0}^{N} \right) + \left( 3_{T,-1}^{N} - 3_{T,0}^{N} \right) \tag{2.20}
\]

\[
= \frac{1}{2} \sum_{k=1}^{m_{c}} \alpha_{k} \times \left[ -\lambda_{N} N^{-\frac{1}{2}} \sum_{k=1}^{k} \eta_{T,x+k}^{N} + \frac{1}{2} \lambda_{N}^{2} N^{-1} \left( \sum_{k=1}^{k} \eta_{T,x+k}^{N} \right) \right] 3_{T,0}^{N} - \mathcal{L}^{N} 3_{T,0}^{N}, \tag{2.21}
\]

from which the evolution equation follows via another elementary calculation as before. This completes the proof. \( \square \)

2.3. Matching Expressions. The purpose of the final subsection is to present the matching between the expressions within Proposition 2.3 and Proposition 2.6, respectively. However, the details behind the proof for the main result in Proposition 2.11 is the Taylor expansion procedure performed in detail throughout Section 2 within [9] and Proposition 2.7 from [23], and so, provided their elementary nature, we omit them from the present article.

Before we may state the primary result, we introduce a few important definitions from Section 2 of [23], one of which is an important modification of "weakly-vanishing term" from Section 2 of [9]. These are essentially stated verbatim although with minor modifications taking into account the domain \( \mathcal{H}_{N,0} \); for the philosophies behind these definitions, see Section 2 of [23].

**Definition 2.7.** A space-time random field \( w_{T,X}(\eta) : \mathbb{R}_{\geq 0} \times \mathcal{H}_{N,0} \times \Omega \rightarrow \mathbb{R} \) is a weakly vanishing random field if the following conditions are satisfied:

- For all \( (T,X,\eta) \in \mathbb{R}_{\geq 0} \times \mathcal{H}_{N,0} \times \Omega \), we have \( w_{T,X}(\eta) = \tau_{T,X} w_{0,0}(\eta) \).
- We have \( E^{w_{0,0}} w_{0,0}(\eta) = 0 \), where \( \mu_{N}^{w_{0,0}} \) is the grand-canonical ensemble on \( \mathcal{H} \) of parameter \( \frac{1}{2} \).
- For some universal constant \( \kappa' \in \mathbb{R}_{> 0} \), we have \( \| w_{0,0}(\bullet) \mathcal{L}^{N} \|_{\mathcal{L}_{\infty}} \lesssim 1 \) uniformly in \( N \in \mathbb{Z}_{> 0} \).

**Remark 2.8.** Per Lemma 2.5 in [9], degree-\( n \) polynomials of the following form are weakly vanishing quantities:

\[
\Phi(\eta) = \prod_{j=1}^{n} \eta_{x_{j}}, \quad x_{1}, \ldots, x_{n} \in \mathcal{H}_{N,0}. \tag{2.22}
\]

**Definition 2.9.** A space-time random field \( \bar{v}_{T,X}(\eta) : \mathbb{R}_{\geq 0} \times \mathcal{H}_{N,0} \times \Omega \rightarrow \mathbb{R} \) is said to be a pseudo-gradient field if the following conditions are satisfied:

- For all \( (T,X,\eta) \in \mathbb{R}_{\geq 0} \times \mathcal{H}_{N,0} \times \Omega \), we have \( \bar{v}_{T,X}(\eta) = \tau_{T,X} \bar{v}_{0,0}(\eta) \).
- For any canonical-ensemble parameter \( \varrho \in [0,1] \), we have \( E^{w_{0,0}} \bar{v}_{0,0}(\eta) = 0 \).
- We have the universal bound \( \sup_{N \in \mathbb{Z}_{> 0}} \| \bar{v}_{0,0}(\bullet) \|_{\mathcal{L}_{\infty}} \lesssim 1 \).
- The support of \( \bar{v}_{T,X,\eta} \) has its size bounded above by \( N^{\varrho_{G}} \in \mathbb{R}_{> 0} \) for some arbitrarily small though universal constant \( \varrho_{G} \in \mathbb{R}_{> 0} \).
Definition 2.10. A given space-time random field $\tilde{g}_{T,X}(\eta) : \mathbb{R}_{\geq 0} \times \mathcal{S}_{\mathcal{N},0} \times \Omega \to \mathbb{R}$ is said to admit some pseudo-gradient factor if it is uniformly bounded and

$$\tilde{g}_{T,X}(\eta) = \tilde{g}_{T,X}(\eta) \cdot f_{T,X}(\eta), \quad (2.23)$$

where the following constraints are satisfied:

- We have $f_{T,X}(\eta) = \tau_{T,X} f_{0,0}(\eta)$ for all $(T,X,\eta) \in \mathbb{R}_{\geq 0} \times \mathcal{S}_{\mathcal{N},0} \times \Omega$, and $\| f_{0,0}(\cdot) \|_{\mathcal{S}_{\mathcal{N},0}} < 1$.
- The factor $\tilde{g}_{T,X}(\eta)$ is a pseudo-gradient field.
- The $\eta$-wise supports of $\tilde{g}_{T,X}(\eta)$ and $f_{T,X}(\eta)$, which are subsets of $\mathcal{S}_{\mathcal{N},0}$, are disjoint.

Proposition 2.11. Recall the maximal jump-length $m_\mathcal{N} \in \mathbb{Z}_{\geq 0}$ and both Assumption 1.1 and Assumption 1.3. Moreover, define $\beta_x = \frac{1}{4} + \varepsilon_x$ with $\varepsilon_x \in \mathbb{R}_{\geq 0}$ arbitrarily small but universal.

- Suppose $x \in [m_\mathcal{N}, N - m_\mathcal{N}]$; then
  
  $$\delta X_T^\mathcal{N} = \mathcal{L}_{\text{Lap}} X_T^\mathcal{N} dT + \mathcal{X}_{\mathcal{N}} X_T^\mathcal{N} dT + N^\frac{1}{2} \sum_{w=1}^{N^I} \tau_w X_T^\mathcal{N} X_T^\mathcal{N} dT + N^{\beta_x} \mathcal{g}_{T,X} X_T^\mathcal{N} X_T^\mathcal{N} dT$$

  above, we have introduce the following data:
  - The field $\mathcal{g}_{T,X}^\mathcal{N}$ is a pseudo-gradient field.
  - The field $\tilde{g}_{T,X}^\mathcal{N}$ admits a pseudo-gradient factor whose support is contained in a sub-lattice whose size is bounded above by $N^{\varepsilon_x} \in \mathbb{R}_{\geq 0}$; if $\mathcal{g}_{T,X}^\mathcal{N}$ denotes this pseudo-gradient factor, then $[\mathcal{g}_{T,X}^\mathcal{N}]^{-1} \mathcal{g}_{T,X}^\mathcal{N}$ is the average of some monomial functionals, in space, in the occupation variables.
  - The fields $\mathcal{g}_{T,X}^\mathcal{N}, \mathcal{g}_{T,X}^{\mathcal{N}_T}, \ldots, \mathcal{g}_{T,X}^{\mathcal{N}_{\mathcal{M}_T}}$ weakly vanishing quantities.
  - The coefficients $\{c_k\}_{k=1}^{\infty}$ are deterministic and admit all moments as measures on $\mathbb{R}_{\geq 0}$.

- Suppose $x \in [0, m_\mathcal{N}]$; then
  
  $$\delta X_T^{m_\mathcal{N}} = \mathcal{L}_{\text{Lap}} X_T^{m_\mathcal{N}} dT + \mathcal{X}_{\mathcal{N}} X_T^{m_\mathcal{N}} dT + N \mathcal{w}_T^{m_\mathcal{N}} X_T^{m_\mathcal{N}} dT,$$

  where $\mathcal{w}_T^{m_\mathcal{N}}$ is a weakly vanishing quantity supported on $[0, 2m_\mathcal{N}]$.

- Suppose $x \in [N - m_\mathcal{N} + 1, N]$; then
  
  $$\delta X_T^{N-m_\mathcal{N}} = \mathcal{L}_{\text{Lap}} X_T^{N-m_\mathcal{N}} dT + \mathcal{X}_{\mathcal{N}} X_T^{N-m_\mathcal{N}} dT + N \mathcal{w}_T^{N-m_\mathcal{N}} X_T^{N-m_\mathcal{N}} dT,$$

  where $\mathcal{w}_T^{N-m_\mathcal{N}}$ is a weakly vanishing quantity supported on $[N - 2m_\mathcal{N}, N]$.

Proof. For $x \in [m_\mathcal{N}, N - m_\mathcal{N}]$, observe that the infinitesimal evolution of $\delta X_T^{N-m_\mathcal{N}}$ does not interact with the boundary, and thus the evolution equation follows from Proposition 2.7 of [23].

For points $x \in [0, m_\mathcal{N}]$, the evolution equation holds upon combining both Proposition 2.3 and Proposition 2.6 and a further Taylor expansion of the exponential quantities within the former through an identical procedure similar to Section 2 of [9]. This same calculation, upon the reflection change-of-variables $x \mapsto N - x$, yields the claimed evolution equation for points $x \in [N - m_\mathcal{N} + 1, N]$.

For both of these latter two boundary-type domains, the weakly vanishing quantities $\mathcal{w}_T^{N-m_\mathcal{N}}$ are given by the error terms obtained by trying to match the equations from Proposition 2.3 and Proposition 2.6 exactly and observing that these error terms are equipped with coefficients of the form $\Phi(\eta)$ explicitly addressed in Remark 2.8.

Remark 2.12. Combining the three bullet points within Proposition 2.11 provides the observation that $\delta X_T^{N-m_\mathcal{N}}$ evolves according to the microscopic SHE propagated by the heat kernel corresponding to the parabolic operator $\tilde{\partial}_T - \mathcal{L}_{\text{Lap}}$, with contribution of the various functionals of the particle system varying between both the bulk and the edge. Analysis of the equations in Proposition 2.11 thus naturally separates into two components – analysis of bulk functionals and of the edge functionals.
3. Heat Kernel Estimates I – Preliminaries

In view of Proposition 2.11, within the current subsection we are primarily concerned with obtaining a priori pointwise and regularity estimates for the following heat kernel in which \( \mu_{\mathcal{A}} = 1 - N^{-1} \mathcal{A} \) for \( \mathcal{A} \in \mathbb{R} \):

\[
\begin{align*}
\partial_t u^N_{S,t,x,y} &= \mathcal{L}^N u^N_{S,t,x,y}; \\
u^N_{S,x,y} &= 1_{x=y}; \\
u^N_{S,t,y} &= \mu_{\mathcal{A}} u^N_{S,t,0,y}; \\
u^N_{S,t,N+1,y} &= \mu_{\mathcal{A}} u^N_{S,t,N,y}.
\end{align*}
\]

Above, the understanding is that

- the times \( S, T \in \mathbb{R}_{\geq 0} \) satisfy \( S \leq T \);
- the spatial coordinates satisfy \( x, y \in \mathcal{I}_{N,0} \);
- the operator \( \mathcal{L}^{N,0} \) acts on the backwards spatial variable.

Towards our analysis for the heat kernel \( u^N \), it will provide immensely helpful to consider the following auxiliary nearest-neighbor specialization \( \tilde{u}^N \):

\[
\begin{align*}
\partial_t \tilde{u}^N_{S,t,x,y} &= \frac{1}{2} \left( \sum_{k=1}^{m_0} \tilde{a}_k | \kappa |^2 \right) \Delta^N \tilde{u}^N_{S,t,x,y}; \\
\tilde{u}^N_{S,x,y} &= 1_{x=y}; \\
\tilde{u}^N_{S,t,y} &= \mu_{\mathcal{A}} \tilde{u}^N_{S,t,0,y}; \\
\tilde{u}^N_{S,t,N+1,y} &= \mu_{\mathcal{A}} \tilde{u}^N_{S,t,N,y}.
\end{align*}
\]

Above, the understanding is that

- the times \( S, T \in \mathbb{R}_{\geq 0} \) satisfy \( S \leq T \);
- the spatial coordinates satisfy \( x, y \in \mathcal{I}_{N,0} \);
- the operator \( \Delta^N \) acts on the backwards spatial variable.

Moreover, by \( \tilde{u}^{N,0} \) we refer to the heat kernel with Neumann boundary conditions, or equivalently \( \mathcal{A}_\mu = 0 \).

As suggested within the title of the current section, we redirect our attention towards various heat kernel estimates for \( u^N \); this discussion is focused towards the necessary preliminary ingredients, which we afterwards combine via a suitable fashion in a subsequent section. Purely for organizational clarity, we present these ingredients in the following list.

- The first ingredient consists of establishing an elliptic-type estimate for the invariant measure associated to \( \tilde{u}^{N,0} \), or equivalently the kernel of the operator \( \left( \mathcal{L}^{N,0} \right)^* \) with respect to the uniform "Lebesgue measure" on \( \mathcal{I}_{N,0} \subseteq \mathbb{Z}_{\geq 0} \).

  We perform this step via a discrete-type elliptic maximum principle with additional direct analysis at the boundary.

- Provided the elliptic-type estimate mentioned within the previous bullet point, we proceed to employ the auxiliary nearest-neighbor specialization \( \tilde{u}^{N,0} \) to establish a suitable Nash-Sobolev inequality with respect to the invariant measure associated to \( \tilde{u}^{N,0} \), which is actually the uniform "Lebesgue measure". To this end, we require an a priori estimate for \( \tilde{u}^{N,0} \) achieved through the method-of-images calculations in [6], for example.

- Lastly, we require a set of Duhamel-type perturbative schemes; these will serve convenient both to establish sub-optimal regularity estimates for the heat kernel \( u^{N,0} \) which are robust at the mesoscopic scales and to extend our heat kernel estimates for \( u^{N,0} \) to arbitrary Robin boundary parameters.

3.1. Elliptic Estimates. The primary result for the current subsection is presented precisely as follows; roughly speaking, it is a elliptic-type stability estimate.

**Lemma 3.1.** Consider the linear space

\[
\Pi^N = \ker \left( \mathcal{L}^{N,0} \right)^*,
\]

\begin{align*}
\partial_t u^N_{S,t,x,y} &= \mathcal{L}^N u^N_{S,t,x,y}; \\
u^N_{S,x,y} &= 1_{x=y}; \\
u^N_{S,t,y} &= \mu_{\mathcal{A}} u^N_{S,t,0,y}; \\
u^N_{S,t,N+1,y} &= \mu_{\mathcal{A}} u^N_{S,t,N,y}.
\end{align*}

\[
\tilde{u}^N_{S,t,x,y} \quad \text{and} \quad \tilde{u}^N_{S,x,y}.
\]

where this adjoint above is taken with respect to the uniform measure on \( \mathcal{S}_{N,0} \subseteq \mathbb{Z}_{\geq 0} \). Provided any positive measure \( \pi \in \Pi^N \), we deduce the following upper bound with implied constants depending only on the maximal jump-length \( m_N \in \mathbb{Z}_{\geq 0} \):

\[
\sup_{x \in \mathcal{S}_{N,0}} \pi_x \lesssim_{m_N} \inf_{x \in \mathcal{S}_{N,0}} \pi_x.
\]

(3.4)

In particular, there exists an invariant measure, which we denote by \( \pi \) throughout the remainder of this section, that satisfies

\[
1 \lesssim_{m_N} \inf_{x \in \mathcal{S}_{N,0}} \pi_x \leq \sup_{x \in \mathcal{S}_{N,0}} \pi_x \lesssim_{m_N} 1.
\]

(3.5)

Before proving Lemma 3.1, it will serve convenient beyond the current section to compute the adjoint appearing in the statement of Lemma 3.1. Roughly speaking, such an adjoint is effectively equal to the operator \( \mathcal{L}^N_{\text{Lap}} \) itself within the bulk of \( \mathcal{S}_{N,0} \subseteq \mathbb{Z}_{\geq 0} \) with additional quantities appearing at the edge.

**Lemma 3.2.** Provided any function \( \varphi : \mathcal{S}_{N,0} \to \mathbb{R} \), we have, for all \( x \in \mathbb{Z}_{\geq 0} \),

\[
\left( \mathcal{L}^N_{\text{Lap}, \infty} \right)^* \varphi_x = \frac{1}{2} \sum_{k=1}^{m_N} \alpha^N_k \left[ 1_{x \geq k} \varphi^N_k \varphi_x + 1_{x \in \{0,k-1\}} \nabla^N_k \varphi_x + N^2 1_{x=0} \nabla^N_1 \varphi_x \right]
\]

\[
- \frac{1}{2} \sum_{k=1}^{m_N} \alpha^N_k \left[ N^2 1_{x \in \{0,k-1\}} \varphi_x + N^2 1_{x=0} \left( \sum_{j \in \{0,k-1\}} \varphi_j \right) \right]
\]

\[
- \frac{1}{2} \sum_{k=1}^{m_N} \alpha^N_k \left[ N^2 1_{x \in \{N-k+1,N\}} \varphi_x + N^2 1_{x=0} \left( \sum_{j \in \{N-k+1,N\}} \varphi_j \right) \right].
\]

(3.6)

**Proof.** Towards our convenience, consider any functions \( \varphi, \psi : \mathcal{S}_{N,0} \to \mathbb{R} \) such that \( \varphi \) has support contained in the left-half \( \frac{1}{2} \mathcal{S}_{N,0} \subseteq \mathcal{S}_{N,0} \); we compute as follows:

\[
\sum_{x \in \mathcal{S}_{N,0}} \varphi_x \mathcal{L}^N_{\text{Lap}} \psi_x = \frac{1}{2} \sum_{k=1}^{m_N} \alpha^N_k \sum_{x=0}^{\infty} \varphi_x (\psi_{x+k} - \psi_x) + \frac{1}{2} \sum_{k=1}^{m_N} \alpha^N_k \sum_{x=0}^{\infty} \varphi_x (\psi_{x-k} - \psi_x) + \frac{1}{2} \sum_{k=1}^{m_N} \alpha^N_k \sum_{x=0}^{k-1} \varphi_x (\psi_0 - \psi_x)
\]

(3.7)

\[
= \frac{1}{2} \sum_{k=1}^{m_N} \alpha^N_k \left[ \sum_{x=k}^{\infty} \varphi_{x-k} \psi_x - \sum_{x=0}^{\infty} \varphi_x \psi_x \right] + \frac{1}{2} \sum_{k=1}^{m_N} \alpha^N_k \left[ \sum_{x=0}^{\infty} \varphi_x \psi_x - \sum_{x=k}^{\infty} \varphi_{x+k} \psi_x \right]
\]

(3.8)

\[
+ \left[ \frac{1}{2} \sum_{k=1}^{m_N} \alpha^N_k \left( \sum_{x=0}^{k-1} \varphi_x \right) \right] \psi_0 - \frac{1}{2} \sum_{k=1}^{m_N} \alpha^N_k \sum_{x=0}^{k-1} \varphi_x \psi_x
\]

(3.9)

To deduce the desired relation, it suffices to replicate this argument on the right-half \( \mathcal{S}_{N,0} \setminus \frac{1}{2} \mathcal{S}_{N,0} \subseteq \mathcal{S}_{N,0} \), which ultimately results in an identical calculation upon changing coordinates. This completes the proof.

The proof of Lemma 3.1 begins with the following consequence of the maximum principle; roughly speaking, it localizes the proof of Lemma 3.1 to the boundary.

**Lemma 3.3.** Suppose \( \pi \in \Pi^N \); then

\[
\sup_{x \in \mathcal{S}_{N,0}} \pi_x = \max \left\{ \max_{x \in \{0,m_N-1\}} \pi_x, \max_{x \in \{N-m_N+1,N\}} \pi_x \right\};
\]

(3.10a)

\[
\inf_{x \in \mathcal{S}_{N,0}} \pi_x = \min \left\{ \min_{x \in \{0,m_N-1\}} \pi_x, \min_{x \in \{N-m_N+1,N\}} \pi_x \right\}.
\]

(3.10b)
Proof. Observe it suffices to exactly one of these identities, because the other is obtained by the former applied to \( \pi \rightarrow -\pi \).

Throughout the proof of Lemma 3.3 only, we define the following pseudo-norms on functions \( \varphi : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{R} \):

\[
[\varphi]_\infty^+ = \sup_{x \in \mathcal{J}_{\alpha,\beta}} \varphi_x, \quad [\varphi]_{\infty}^- = \inf_{x \in \mathcal{J}_{\alpha,\beta}} \varphi_x.
\]  

(3.11)

We consider the following scenario.

- Suppose \( x \in [m_N, N - m_N] \) satisfies \( [\pi]_\infty^+ = \pi_x \); courtesy of Lemma 3.2 we obtain the difference equation

\[
\frac{1}{2} \sum_{k=1}^{m_N} \alpha_k^N (\pi_{x+k} + \pi_{x-k} - 2\pi_x) = 0.
\]  

(3.12)

By definition, we observe \( \nabla_{\pm k} \pi_x \leq 0 \), and thereby provided the above difference equation, we deduce \( \nabla_{\pm k} \pi_x = 0 \).

Because \( \alpha_1^N \in \mathbb{R}_{>0} \) is strictly positive, we deduce \( \pi_{x+1} = \pi_x = [\pi]_\infty^+ \).

Continuing inductively until we arrive upon the boundary completes the proof. \( \square \)

 Courtesy of Lemma 3.3, to estimate any invariant measure globally from above or below, it suffices to provide estimates at the boundary. This is exactly the content of this second result; we emphasize the importance of the assumption \( m_N \lesssim 1 \) for this estimate, which is the only part of our analysis that requires such a priori \( O(1) \) bound on the maximal jump-length.

**Lemma 3.4.** Consider any \( \pi \in \Pi_N \) satisfying \( \pi_x \geq 0 \) for all \( x \in \mathcal{J}_{\alpha,\beta} \). Then for some universal constant \( \kappa \in \mathbb{R}_{>0} \), we have

\[
\max_{x \in [0,m_N-1]} \pi_x \leq \pi_0 \leq \kappa^m \min_{x \in [0,m_N-1]} \pi_x; \quad \min_{x \in [N-m_N+1,N]} \pi_x \leq \pi_N \leq \kappa^m \min_{x \in [N-m_N+1,N]} \pi_x.
\]  

(3.13a)

(3.13b)

Moreover, we have \( \pi_0 = \pi_N \).

Proof. We provide a proof of the former estimate; the latter follows from identical considerations upon reflection of coordinates. We first claim the following relation:

\[
\max_{x \in [0,m_N-1]} \pi_x = \pi_0.
\]  

(3.14)

To this end, suppose there exists some \( j_0 \in [1,m_N-1] \) such that \( \pi_{j_0} = \max_{j \in [0,m_N-1]} \pi_j \). Courtesy of Lemma 3.2, we have

\[
\left( \mathcal{L}^{\mathcal{N}}_{\text{Lap}} \right)^* \pi_{j_0} = \frac{1}{2} \sum_{k=1}^{j_0} \alpha_k^N \Delta_k \pi_{j_0} + \frac{1}{2} \sum_{k=j_0+1}^{m_N} \alpha_k^N \nabla_{k+} \pi_{j_0} - \left[ \frac{1}{2} \sum_{k=j_0+1}^{m_N} \alpha_k^N \right] \pi_{j_0} = 0.
\]  

(3.15)

(3.16)

As with the proof of Lemma 3.3, the first two summations in the expansion above are non-positive. However, the positivity constraint shows that this final quantity is strictly negative as well, from which we obtain a contradiction to this previous adjoint relation. Thus, we obtain the stated relation (3.14).

It remains to prove the stated upper bound. To this end, we again appeal to the exact calculation in Lemma 3.2 – given \( j \in [1,m_N-1] \), we have the following by continuing the previous relevant calculation:

\[
\left( \mathcal{L}^{\mathcal{N}}_{\text{Lap}} \right)^* \pi_j = \frac{1}{2} \sum_{k=1}^{j} \alpha_k^N \pi_{j-k} + \frac{1}{2} \sum_{k=1}^{m_N} \alpha_k^N \pi_{j+k} - \left[ \frac{1}{2} \sum_{k=1}^{m_N} \alpha_k^N + \frac{1}{2} \sum_{k=j+1}^{m_N} \alpha_k^N \right] \pi_j = 0.
\]  

(3.17)

Provided the positivity assumption, we deduce

\[
\frac{1}{2} \alpha_1^N \pi_{j-1} \leq \left[ \frac{1}{2} \sum_{k=1}^{m_N} \alpha_k^N + \frac{1}{2} \sum_{k=j+1}^{m_N} \alpha_k^N \right] \pi_j \lesssim \pi_j
\]  

(3.18)
with a universal implied constant. As \( \bar{\alpha}^N_1 \geq 1 \) with another universal implied constant, we establish \( \pi_{j-1} \lesssim \pi_j \). Iterating inductively provides the desired upper bound.

Finally, it suffices to justify \( \pi_0 = \pi_N \). To this end, we simply observe that the operator \( \mathcal{L}^N_{\text{Lap}} \) is invariant under reflection about the midpoint of \( \mathcal{S}_{N,0} \), this completes the proof.

3.2. Nash Inequalities. We redirect our attention towards the second ingredient necessary to establish the relevant heat kernel estimates for \( \mathcal{H}_N \), which ultimately take the form of suitable Nash inequalities. Although Nash inequalities are generally employed in the non-compact regime, we adapt them to the compact regime in this paper; the important ingredient for our approach begins with heat kernel estimates \( \bar{\mathcal{H}}_N^{N,0} \).

Concerning the relevant heat kernel estimates for \( \bar{\mathcal{H}}_N^{N,0} \), one important distinction between the asymptotically compact geometry and the non-compact geometry is the lack of any similar long-time estimates. Indeed, in the long-time limit with respect to the parabolic space-time scaling, the heat kernel \( \bar{\mathcal{H}}_N^{N,0} \) approximates the flat measure on \( \mathcal{S}_{N,0} \). We account for this non-vanishing long-time behavior in the following estimates which actually categorizes the universal implied constant as well which implies the estimate

\[
0 \leq \sup_{x,y \in \mathcal{S}_{N,0}} \bar{\mathcal{H}}_N^{N,0}_{S,T,x,y} \lesssim \bar{\alpha}_1^N \frac{N}{S^2} + N^{-\frac{3}{2}} \mathcal{G}_{S,T}.
\]

**Proof.** The desired lower bound is a straightforward consequence of a discrete-type parabolic maximum principle for \( \bar{\mathcal{H}}_N^{N,0} \), or alternatively its interpretation as a probability. Moreover, it suffices to assume \( \mathcal{G}_{S,T} \geq N^{-2} \), because the aforementioned parabolic maximum principle also provides a uniform upper bound for the heat kernel provided its initial data.

To provide the desired upper bound, we first recall the following explicit representation of the heat kernel \( \bar{\mathcal{H}}_N^{N,0} \) from Section 3.2 of [22].

- First, let us define \( \mathcal{G}_N^{N,0} \) to be the heat kernel associated to the following parabolic problem with spatial coordinate in the boundary-less lattice \( \mathbb{Z} \):

\[
\partial_t \mathcal{G}_N^{N,0}_{S,T,x,y} = \frac{1}{2} \left( \sum_{k=1}^m \bar{\alpha}_k^N |k|^2 \right) \Delta \mathcal{G}_N^{N,0}_{S,T,x,y};
\]

\[
\mathcal{G}_N^{N,0}_{S,T,x,y} = 1_{x=y}. \tag{3.20b}
\]

- Courtesy of Section 3.2 of [22], the method-of-images principle provides the following formula:

\[
\bar{\mathcal{H}}_N^{N,0}_{S,T,x,y} = \sum_{k \in \mathbb{Z}} \mathcal{G}_N^{N,0}_{S,T,x,i_y,k}. \tag{3.21}
\]

above, we have introduced the notation \( i_y,k \in \mathbb{Z} \) from Section 3.2 of [22].

Employing both the heat kernel estimates within Proposition A.1 in [9] and an elementary calculation like within the proof of Proposition 3.15 of [22], we have the following upper bound per any universal constants \( \kappa, \kappa' \in \mathbb{R}_{>0} \):

\[
\bar{\mathcal{H}}_N^{N,0}_{S,T,x,y} \lesssim \kappa N^{-\frac{3}{2}} \mathcal{G}_{S,T}^{-\frac{1}{2}} \sum_{k \in \mathbb{Z}} \exp \left( -\kappa' \frac{|x - i_y,k|}{\mathcal{G}_{S,T}^{1/2}} \right). \tag{3.22}
\]

\[
\lesssim_{\kappa, \kappa'} N^{-\frac{3}{2}} \mathcal{G}_{S,T}^{-\frac{1}{2}} \sum_{k \in \mathbb{Z}} \exp \left( -\kappa' \frac{|k|}{\mathcal{G}_{S,T}^{1/2}} \right) \tag{3.23}
\]

\[
\lesssim N^{-\frac{3}{2}} \mathcal{G}_{S,T}^{-\frac{1}{2}} \left[ 1 - \exp \left( -\kappa' \mathcal{G}_{S,T}^{-\frac{1}{2}} \right) \right]^{-1}. \tag{3.24}
\]

We now observe that if \( \mathcal{G}_{S,T} \lesssim 1 \) with some sufficiently large though universal implied constant, then \( \mathcal{G}_{S,T}^{-1} \gtrsim 1 \) with some universal implied constant as well which implies the estimate

\[
N^{-\frac{3}{2}} \mathcal{G}_{S,T}^{-\frac{1}{2}} \left[ 1 - \exp \left( -\kappa' \mathcal{G}_{S,T}^{-\frac{1}{2}} \right) \right]^{-1} \lesssim_{\kappa'} N^{-\frac{3}{2}} \mathcal{G}_{S,T}^{-\frac{1}{2}}. \tag{3.25}
\]
On the other hand, in the complementary scenario for which \( \rho_{S,T} \geq 1 \) with an identical arbitrarily large though universal implied constant, Taylor expansion provides

\[
N^{-1} \rho_{S,T}^{-\frac{1}{2}} \left[ 1 - \exp\left( -\kappa' \rho_{S,T}^{-\frac{1}{2}} \right) \right]^{-1} \lesssim x^{-N^{-1}}.
\]

(3.26)

Combining the estimates (3.24), (3.25), and (3.26) completes the proof. \( \square \)

Provided this a priori pointwise upper bound for the heat kernel \( \mathcal{U}_{S,T}^{N,0} \) corresponding to the nearest-neighbor Laplacian, we obtain a general functional inequality reminiscent of the classical Nash-Sobolev inequality although with an additional quantity arising from the optimal long-time behavior of the upper bound in Lemma 3.5.

**Lemma 3.6.** Define the operator \( \mathcal{D} = -\Delta; \) this operator is self-adjoint with respect to the uniform "discrete Lebesgue" measure when equipped with Neumann boundary conditions \( \mathcal{A}_x = 0; \) moreover, it is positive semidefinite.

Moreover, provided any function \( \varphi : \mathcal{S}_{N,0} \rightarrow \mathbb{R}, \) we have

\[
\sum_{x \in \mathcal{S}_{N,0}} \varphi_x^2 \leq \mathcal{D} \left[ \sum_{x \in \mathcal{S}_{N,0}} |\varphi_x| \right]^2 + \left[ \sum_{x \in \mathcal{S}_{N,0}} |\mathcal{D}^{1/2} \varphi_x| \right]^2 \]  

(3.27)

**Proof.** The self-adjoint property and the positive semidefinite property of \( \mathcal{D} \) follows from some straightforward summation-by-parts calculation.

Consider first the following semigroup action on \( \varphi : \mathcal{S}_{N,0} \rightarrow \mathbb{R}: \)

\[
\varphi_{T,x} = \sum_{y \in \mathcal{S}_{N,0}} \mathcal{D}^{\frac{N,0}{0},T,x,y} \varphi_y.
\]

(3.28)

Recall via either a standard convexity argument or Riesz-Thorin interpolation combined with the maximum principle that this semigroup action is contractive with respect to every \( \ell^2 \)-norm. Upon an elementary differentiation with respect to the time-coordinate combined with the heat kernel estimates in Lemma 3.5, we have

\[
\sum_{x \in \mathcal{S}_{N,0}} \varphi_x^2 = \sum_{x \in \mathcal{S}_{N,0}} \varphi_{T,x} \varphi_x + \frac{1}{2} \sum_{x \in \mathcal{S}_{N,0}} \int_0^T \mathcal{D} \varphi_x \, dS
\]

(3.29)

\[
= \sum_{x \in \mathcal{S}_{N,0}} \varphi_{T,x} \varphi_x + \frac{1}{2} \sum_{x \in \mathcal{S}_{N,0}} \left( \mathcal{D}^{\frac{1}{2}} \varphi \right)_{S,x} \mathcal{D}^{\frac{1}{2}} \varphi_x \, dS
\]

(3.30)

\[
\lesssim \mathcal{D} \left[ \sum_{x \in \mathcal{S}_{N,0}} |\varphi_x| \right]^2 + \mathcal{D} \left[ \sum_{x \in \mathcal{S}_{N,0}} |\varphi_x| \right]^2 + \frac{1}{2} \sum_{x \in \mathcal{S}_{N,0}} \left( \mathcal{D}^{\frac{1}{2}} \varphi_x \right)^2.
\]

(3.31)

We complete the proof by optimizing the contribution from the latter two quantities on the RHS with respect to \( T \in \mathbb{R}_{\geq 0} \) to obtain the desired estimate. \( \square \)

3.3. **Perturbative Analysis I.** As far as preliminary estimates go, we have completed those of the non-perturbative variety. The first preliminary estimate of the perturbative variety that we introduce is a Duhamel-type expansion for the heat kernel \( \mathcal{U}_{S,T}^{N,0}, \) realizing this kernel as perturbation of another explicit and analytically tractable function of the full-line heat kernel addressed in [9], for example. Moreover, we explicitly emphasize that the current subsection is focused exclusively towards the heat kernel with Neumann boundary conditions \( \mathcal{A}_x = 0. \)

**Remark 3.7.** In principle, via straightforward adaptations of our analysis we may apply the exact same procedure towards the heat kernel \( \mathcal{U}_{S,T}^{N,0} \) corresponding to any Robin boundary parameter \( \mathcal{A} \in \mathbb{R}. \) The only additional complexity arises from bookkeeping the evolution of this heat kernel and the auxiliary field \( \mathcal{F}_{S,T}^{N,0} \) defined below near the boundary. It will suffice for our purposes to dodge these additional complexities and focus only on the Neumann parameters \( \mathcal{A}_x = 0, \) because we later introduce another second perturbative mechanism to transfer regularity estimates for \( \mathcal{A}_x = 0 \) to those given arbitrary Robin boundary parameters. However, we make this remark in case of possible interests beyond what we provide to obtain Theorem 1.8.
To begin, we first define the object around which we would like to perturb to obtain $\mathcal{U}^{N,0}_{S,T}$.  

**Notation 3.8.** First, consider $\mathcal{G}^{N,0}$ to be the heat kernel associated to the following problem on the lattice $\mathbb{Z}$:

$$\partial_T \mathcal{G}^{N,0}_{S,T,x,y} = \frac{1}{2} \sum_{k=1}^{m_N} \mathcal{G}_k^N \Delta_k^{\|} \mathcal{G}^{N,0}_{S,T,x,y};$$

$$\mathcal{G}^{N,0}_{S,T,x,y} = 1_{x=y}.$$  \hspace{1cm} (3.32a)  

Moreover, we define the following auxiliary space-time kernel $\mathcal{F}^{N,0}_{S,T,x,y}$ for $S, T \in \mathbb{R}_{\geq 0}$ satisfying $S \leq T$ and for $x, y \in \mathbb{Z}$:

$$\mathcal{F}^{N,0}_{S,T,x,y} = \sum_{k \in \mathbb{Z}} \mathcal{G}^{N,0}_{S,T,x,y}.$$ \hspace{1cm} (3.33)  

Second, we introduce a relevant “perturbation” at the level of differential operators which even appears in the classical Duhamel formula as well.

**Notation 3.9.** Consider the following operator $\mathcal{D}^N_\partial$:

$$\mathcal{D}^N_\partial \cdot = \mathcal{L}^{\text{Lap}} \partial - \frac{1}{2} \sum_{k=1}^{m_N} \mathcal{G}_k^N \Delta_k^{\|}. \hspace{1cm} (3.34)$$

The primary result within the current subsection is the following Duhamel-type perturbative scheme.

**Lemma 3.10.** Provided any $S, T \in \mathbb{R}_{\geq 0}$ satisfying $S \leq T$ and any pair of points $x, y \in \mathcal{S}_{N,0}$, we have

$$\mathcal{U}^{N,0}_{S,T,x,y} = \mathcal{F}^{N,0}_{S,T,x,y} + \int_S^T \sum_{w=0}^{m_N-1} \mathcal{U}^{N,0}_{R,T,x,w} \cdot \mathcal{D}^N_\partial \mathcal{F}^{N,0}_{R,T,x,y} \cdot \mathcal{D}^N_\partial \mathcal{F}^{N,0}_{S,R,w,y} \cdot dR + \int_S^T \sum_{w=N-m_N+1}^{N} \mathcal{U}^{N,0}_{R,T,x,w} \cdot \mathcal{D}^N_\partial \mathcal{F}^{N,0}_{S,R,w,y} \cdot dR, \hspace{1cm} (3.35)$$

where the operator $\mathcal{D}^N_\partial$ acts on the backwards spatial coordinate $w \in \mathbb{Z}$.

**Proof.** Lemma 3.10 is an honest consequence of the Duhamel formula combined with the observation that the heat kernel $\mathcal{U}^{N,0}$ is supported, at least in the spatial directions, on $\mathcal{S}_{N,0} \times \mathcal{S}_{N,0} \subseteq \mathbb{Z}^2$.

To be more precise, we first observe that this kernel $\mathcal{F}^{N,0}$ solves the exact same PDE as $\mathcal{G}^{N,0}$ with identical initial data upon the restriction of spatial coordinates in $\mathcal{S}_{N,0} \times \mathcal{S}_{N,0} \subseteq \mathbb{Z}^2$. We proceed to consider the quantity

$$\Phi^{N,0}_{R,T,x,w} = \sum_{w \in \mathcal{S}_{N,0}} \mathcal{U}^{N,0}_{R,T,x,w} \cdot \mathcal{F}^{N,0}_{S,R,w,y}. \hspace{1cm} (3.36)$$

Applying the fundamental theorem of calculus with respect to the time coordinate $R \in [S, T]$ using the respective relevant PDEs, we have

$$\mathcal{F}^{N,0}_{S,T,x,y} = \mathcal{U}^{N,0}_{S,T,x,y} + \int_S^T \partial_R \Phi^{N,0}_{R,T,x,y} \cdot dR \hspace{1cm} (3.37)$$

with

$$\int_S^T \partial_R \Phi^{N,0}_{R,T,x,y} \cdot dR = -\int_S^T \sum_{w \in \mathcal{S}_{N,0}} \mathcal{U}^{N,0}_{R,T,x,w} \cdot \mathcal{L}^{\text{Lap}} \Phi^{N,0}_{R,T,x,y} \cdot \mathcal{F}^{N,0}_{S,R,w,y} \cdot dR + \int_S^T \sum_{w \in \mathcal{S}_{N,0}} \mathcal{U}^{N,0}_{R,T,x,w} \cdot \partial_R \Phi^{N,0}_{R,T,x,y} \cdot \mathcal{F}^{N,0}_{S,R,w,y} \cdot dR; \hspace{1cm} (3.38)$$

to be totally explicit, we remark that the differentiation performed in the previous time-integral is done so while recalling all the relevant heat kernels are time-homogeneous, so differentiation in the backwards time-coordinate is differentiation in the forwards time-coordinate with an additional sign. Upon elementary rearrangement, this completes the proof.  

The utility of this particular perturbative representation for $\mathcal{U}^{N,0}_{S,T}$ obtained in Lemma 3.10 is precisely the local nature of the perturbative time-integral supported near the boundary. In particular, the perturbative integral from within Lemma 3.10 has spatial support contained microscopically towards the boundary, so we are not responsible for keeping any record of global perturbative effects on macroscopic blocks, for example.

Moreover, given the support of these perturbative effects is concentrated specifically near the boundary we can exploit the Neumann boundary behavior of $\mathcal{F}^{N,0}_{S,T}$ to establish "upgraded" regularity estimates for $\mathcal{F}^{N,0}_{S,T}$ within Lemma B.2 whose
proof relies on the higher-order extension of the gradient estimates from Proposition A.1 from [9], all of which we establish in the appendix of this article.

3.4. Perturbative Analysis II. As promised both in the outline of the current section and at the beginning of the previous subsection, we provide a mechanism to transfer both pointwise estimates and regularity estimates for the heat kernel \( \mathcal{U}_{S,T}^{N,0} \) of specific Robin boundary parameters \( \mathcal{A}_{\pm} = 0 \) to the heat kernel \( \mathcal{U}_{S,T}^{N} \) of arbitrary Robin boundary parameter.

We emphasize the following disclaimers.

- Concerning pointwise estimates, this is only necessary for negative Robin boundary parameters \( \mathcal{A}_{\pm} \in \mathbb{R}_{<0} \).
- Concerning regularity estimates, this is necessary for no values of the Robin boundary parameters \( \mathcal{A}_{\pm} \in \mathbb{R} \) assuming a priori pointwise upper bounds and upon analyzing additional complexities arising in the proof of Lemma 3.10 that we avoided by assuming \( \mathcal{A}_{\pm} = 0 \); otherwise, if avoiding these additional complexities, then this procedure is necessary for all nonzero parameters \( \mathcal{A}_{\pm} \in \mathbb{R}_{\neq 0} \).
- Because this perturbative procedure is still necessary in our proof of Theorem 1.8, we may just as well employ it further and avoid the aforementioned complexities.

We now present the second perturbative strategy we employ to transfer estimates from \( \mathcal{U}_{S,T}^{N,0} \) to arbitrary Robin boundary parameters.

Lemma 3.11. Provided any Robin boundary parameters \( \mathcal{A}_{\pm} \in \mathbb{R} \), the following identity holds for all \( S, T \in \mathbb{R}_{\geq 0} \) satisfying \( S \leq T \) and for all \( x, y \in \mathcal{I}_{N,0} \):

\[
\mathcal{U}_{S,T,x,y}^{N} = \mathcal{U}_{S,T,x,y}^{N,0} - \int_{S}^{T} \mathcal{U}_{R,R,T,x,0}^{N,0} \left[ N \mathcal{A}_{\pm} \mathcal{U}_{S,R,0,y}^{N,0} \right] \, dR - \int_{S}^{T} \mathcal{U}_{R,R,T,x,N}^{N,0} \left[ N \mathcal{A}_{\pm} \mathcal{U}_{S,R,N,y}^{N} \right] \, dR. \tag{3.39}
\]

Proof. Identical to the proof of Lemma 3.10, the proof of Lemma 3.11 follows via interpolation with the semigroup associated to the heat kernel \( \mathcal{U}_{S,T}^{N,0} \) combined with the following observation:

\[
\partial_{R} \mathcal{U}_{S,R,x,y}^{N} - \mathcal{L}_{\text{Lap}} \mathcal{U}_{S,R,x,y}^{N} = 1_{x=0} N \mathcal{A}_{\pm} \mathcal{U}_{S,R,0,y}^{N} + 1_{x=N} N \mathcal{A}_{\pm} \mathcal{U}_{S,R,x,y}^{N}. \tag{3.40}
\]

This completes the proof. \( \square \)

4. Heat Kernel Estimates II – Nash-Type Estimates

Let us retain the framework, notation, and definitions from the previous Section 3 and proceed to use the results therein to establish the desired heat kernel estimates for \( \mathcal{U}_{S,T}^{N,0} \) and \( \mathcal{U}_{S,T}^{N} \). We emphasize that the heat kernel estimates established in the current section are of on-diagonal-type because the domain is compact with respect to the "heat kernel scale", making the hypothetical off-diagonal factors irrelevant. However, we moreover establish off-diagonal estimates as well which will provide utility for short-times; this is done at the end of the section.

To provide some organizational clarity, we provide the following outline for the current section.

- We first specialize our analysis to the specific heat kernel \( \mathcal{U}_{S,T}^{N,0} \) with Neumann boundary conditions, or equivalently specializing to the case \( \mathcal{A}_{\pm} = 0 \); ultimately, we obtain Nash-type upper bounds for this heat kernel.
- Second, we extend our analysis to any arbitrary Robin boundary parameters with the perturbative schemes from Section 3. Although these resulting heat kernel estimates are not stable in the long-time regime, they still resemble Nash-type upper bounds for uniformly bounded times.

4.1. Nash-Type Estimates for \( \mathcal{U}_{S,T}^{N,0} \). As mentioned prior, the first step in our analysis of relevant heat kernels specializes to Neumann boundary parameters \( \mathcal{A}_{\pm} = 0 \); the upshot to this specialization is the applicability of the Nash inequality in Lemma 3.6, which provides the following main result for the current subsection.

Lemma 4.1. Provided times \( S, T \in \mathbb{R}_{\geq 0} \) satisfying \( S \leq T \leq T_{f} \), we have the following pointwise upper bound with an implied constant depending only on the maximal jump-length \( m_{N} \in \mathbb{Z}_{>0} \) and also \( \mathcal{A}_{1}, T_{f} \in \mathbb{R}_{\geq 0} \):

\[
0 \leq \sup_{x,y \in \mathcal{I}_{N,0}} \mathcal{U}_{S,T,x,y}^{N,0} \lesssim m_{N, \mathcal{A}_{1}, T_{f}} \left[ N^{-1} + N^{-1} \mathcal{A}_{1}^{-
frac{1}{2}} \right] \wedge 1. \tag{4.1}
\]
Towards the proof of Lemma 4.1, we establish the following convenient notation.

**Notation 4.2.** We fix a strictly positive invariant measure \( \pi \in \Pi^N \). With respect to such invariant measure, define the adjoint \((\mathcal{L}_{\text{Lap}}^N)^! \); we now define the symmetric operator

\[
(\mathcal{L}_{\text{Lap}}^N)^\text{Sym} = \frac{1}{2} \left[ (\mathcal{L}_{\text{Lap}}^N)^! + (\mathcal{L}_{\text{Lap}}^N)^! \right].
\]

Provided any function \( \varphi : \mathcal{A}_{N,0} \to \mathbb{C} \), we define the following pair of semigroup evolutions:

\[
\varphi_{T,x} = \left[ e^{T \mathcal{L}_{\text{Lap}}^N} \varphi \right]_x = \sum_{y \in \mathcal{A}_{N,0}} \mathcal{W}_{0,T,x,y}^N \varphi_y,
\]

\[
\varphi_{T,x}^\dagger = \left[ e^{T (\mathcal{L}_{\text{Lap}}^N)^!} \varphi \right]_x = \sum_{y \in \mathcal{A}_{N,0}} \tilde{\pi}_x^\dagger \mathcal{W}_{0,T,x,y}^N \varphi_y.
\]

**Proof.** Because the heat kernel is time-homogeneous, it suffices to assume \( S = 0 \).

Observe first that the proposed lower bound follows from either the parabolic maximum principle or the interpretation of \( \mathcal{W}_{0,T,x,y}^N \) as a probability; the same is true for the trivial upper bound.

To provide the precise upper bound, consider any function \( \varphi : \mathcal{A}_{N,0} \to \mathbb{C} \) satisfying \( \sum_{x \in \mathcal{A}_{N,0}} |\varphi_x| = 1 \). We have

\[
\partial_T \sum_{x \in \mathcal{A}_{N,0}} |\varphi_{T,x}|^2 \tilde{\pi}_x = -2 \sum_{x \in \mathcal{A}_{N,0}} \varphi_{T,x} \left[ -(\mathcal{L}_{\text{Lap}}^N)^\text{Sym} \varphi_{T,x} \right] \tilde{\pi}_x.
\]

The remaining quantity on the RHS of (4.5) is exactly the Dirichlet form corresponding to the operator \((\mathcal{L}_{\text{Lap}}^N)^\text{Sym}\) with reversible measure \( \tilde{\pi}_x \), evaluated at \( \varphi_{T,x} \). In particular, we establish the following lower bounds, with the first failing to be an equality only due to possibly missing constant prefactors and the second failing to be an equality only due to forgetting interactions beyond length 1 – see Proposition 10.1 in Appendix 1 of [20]:

\[
\sum_{x \in \mathcal{A}_{N,0}} \varphi_{T,x} \left[ -(\mathcal{L}_{\text{Lap}}^N)^\text{Sym} \varphi_{T,x} \right] \tilde{\pi}_x \gtrsim N^2 \sum_{x,w \in \mathcal{A}_{N,0} : |x-w| \leq m} \tilde{a}_x^N \varphi_{T,x} - \varphi_{T,w}^2 \tilde{\pi}_x
\]

\[
\gtrsim \tilde{a}_x^N \sum_{x,w \in \mathcal{A}_{N,0} : |x-w| \leq 1} \varphi_{T,x} - \varphi_{T,w}^2 \tilde{\pi}_x.
\]

Similarly for the nearest-neighbor case, recalling the definition of the operator \( \mathcal{D} = \Delta_1 \), as the flat measure on \( \mathbb{Z}_{\geq 0} \) is a reversible measure for the operator \( \mathcal{D} \), for \( \mathcal{A} = 0 \) we have

\[
\sum_{x \in \mathcal{A}_{N,0}} \left| \mathcal{D}_x^{\frac{1}{2}} \varphi_{T,x} \right|^2 \lesssim \sum_{x,w \in \mathcal{A}_{N,0} : |x-w| \leq 1} \left| \varphi_{T,x} - \varphi_{T,w} \right|^2.
\]

Combining this bound of (4.8) with the preceding inequalities (4.5) and (4.7) along with Lemma 3.1 and the assumption \( \tilde{a}_x^N \gtrsim 1 \) with universal implied constant, we deduce

\[
\partial_T \sum_{x \in \mathcal{A}_{N,0}} |\varphi_{T,x}|^2 \tilde{\pi}_x \lesssim -N^2 \sum_{x,w \in \mathcal{A}_{N,0} : |x-w| \leq 1} \left| \varphi_{T,x} - \varphi_{T,w} \right|^2.
\]

Before we proceed, we remark that the analysis presented until this point is rather standard in the theory of Nash inequalities; precisely, we have not yet seen the influence or role of the underlying compact geometry in the proof of Lemma 4.1 at this point. Indeed, this influence is manifest in the Nash inequality of Lemma 3.6, which we discuss as follows.
We employ the Nash inequality of Lemma 3.6 along with the elliptic estimate Lemma 3.1; this provides the lower bounds in which the implied constant is universal even in its dependence on \( \tilde{\alpha}^N_1 \in \mathbb{R}_{>0} \) and \( m_N \in \mathbb{Z}_{>0} \):

\[
N^2 \sum_{x,w \in \mathcal{F}_{N,0}} |\varphi_{T,x} - \varphi_{T,w}|^2 \gtrsim -N^{-1} \left[ \sum_{x \in \mathcal{F}_{N,0}} |\varphi_{T,x}|^6 \right] + N^2 \left[ \sum_{x \in \mathcal{F}_{N,0}} |\varphi_{T,x}|^2 \pi_x \right]^3
\]

\[
= -N^{-1} + N^2 \left[ \sum_{x \in \mathcal{F}_{N,0}} |\varphi_{T,x}|^2 \pi_x \right]^3
\]

in the final estimate above, we recall our normalization of the function \( \varphi \), which is then preserved because of the contractive estimates of the semigroup corresponding to \( \mathcal{W}^{N,0} \) with respect to the \( \ell^1 \)-norm. Combining this above estimate with (4.9) then provides

\[
\partial_T \sum_{x \in \mathbb{Z}_{>0}} |\varphi_{T,x}|^2 \pi_x \gtrsim N^{-1} - N^2 \left[ \sum_{x \in \mathbb{Z}_{>0}} |\varphi_{T,x}|^2 \pi_x \right]^3
\]

from which classical ODE theory gives the following upper bound:

\[
\sum_{x \in \mathbb{Z}_{>0}} |\varphi_{T,x}|^2 \pi_x \lesssim N^{-1} \mathcal{E}_{0,T} + N^{-1} \mathcal{E}_{0,T}^{-\frac{1}{2}}.
\]

In particular, recalling the normalization for this test function \( \varphi \) provides the operator norm estimate, in which \( \ell^p \)-norms are taken not with respect to the uniform "discrete Lebesgue" measure, but rather with the invariant measure \( \pi \in \Pi^N \):

\[
\left\| e^{T \mathcal{L}_{\text{lap}}^{N,n}} \right\|^{2}_{1 \rightarrow 2} \lesssim N^{-1} \mathcal{E}_{0,T} + N^{-1} \mathcal{E}_{0,T}^{-\frac{1}{2}}.
\]

Repeating the entire preceding calculation with the adjoint heat flow \( \varphi^\dagger \), we deduce an identical operator-norm estimate for the adjoint flow, which by duality gives us the estimate

\[
\left\| e^{T \mathcal{L}_{\text{lap}}^{N,n}} \right\|^2_{2 \rightarrow \infty} \lesssim N^{-1} \mathcal{E}_{0,T} + N^{-1} \mathcal{E}_{0,T}^{-\frac{1}{2}}.
\]

The Chapman-Kolmogorov equation then provides the estimate

\[
\left\| e^{T \mathcal{L}_{\text{lap}}^{N,n}} \right\|_{1 \rightarrow \infty} \lesssim \left\| e^{T \mathcal{L}_{\text{lap}}^{N,n}} \right\|_{1 \rightarrow 2} \left\| e^{T \mathcal{L}_{\text{lap}}^{N,n}} \right\|_{2 \rightarrow \infty}
\]

\[
\lesssim N^{-1} \mathcal{E}_{0,T} + N^{-1} \mathcal{E}_{0,T}^{-\frac{1}{2}}.
\]

because \( \mathcal{W}^{N,0} \) is the kernel associated to the above exponential operators, and because all \( \ell^p \)-norms are equivalent to the corresponding norms with respect to uniform "discrete Lebesgue" measure, this completes the proof. \( \square \)

**Remark 4.3.** The quantity \( N^{-1} \mathcal{E}_{S,T} \) appearing in the heat kernel upper bound from Lemma 4.1 is probably sub-optimal in time-dependence, although because we are concerned only with compact time-domains, this estimate is certainly sufficient for our purposes. We provide this remark in case of any possible future interest.

### 4.2. Nash-Type Estimates for \( \mathcal{W}^{N}_{S,T} \)

Through a perturbative scheme, we achieve similar Nash-type estimates for the heat kernel \( \mathcal{W}^{N}_{S,T} \) satisfying any arbitrary Robin boundary conditions. As alluded to earlier throughout both the current section and the previous, the mechanism we employ for this task is the perturbative Duhamel-type formula in Lemma 3.11.

**Lemma 4.4.** Provided any Robin boundary parameters \( \mathcal{A} \in \mathbb{R} \) along with any \( S, T \in \mathbb{R}_{>0} \) satisfying \( S \leq T \leq T_f \), we have the following estimates:

\[
\sup_{x,y \in \mathcal{F}_{N,0}} \left| \mathcal{W}^{N}_{S,T,x,y} \right| \lesssim m_N \tilde{\alpha}^N_1 T_f, \mathcal{A} \left[ N^{-1} + N^{-1} \mathcal{E}_{S,T}^{-\frac{1}{2}} \right] \wedge 1;
\]

\[
\sup_{x \in \mathcal{F}_{N,0}} \sum_{y \in \mathcal{F}_{N,0}} \left| \mathcal{W}^{N}_{S,T,x,y} \right| \lesssim m_N \tilde{\alpha}^N_1 T_f, \mathcal{A} 1.
\]

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Proof. Purely for notational convenience, let us denote by $\Phi^N_{S,T}$ the supremum within the statement of Lemma 4.4. Courtesy of Lemma 3.11 and the heat kernel estimate in Lemma 4.1, we have the following upper bound

$$\Phi^N_{S,T} \leq m_w \Phi^N_{S,T} \sup_{x,y \in \mathcal{C}_{S,T}} \mathcal{U}^N_{S,T,x,y} + \int_S^T \mathcal{U}^N_{S,T} \Phi^N_{S,T} \, dR,$$

(4.19)

Thus, courtesy of Lemma 4.1 again, we have

$$\mathcal{U}^N_{S,T} \Phi^N_{S,T} \leq m_w \Phi^N_{S,T} N^{-1} \sup_{x,y \in \mathcal{C}_{S,T}} \mathcal{U}^N_{S,T,x,y} + \int_S^T \mathcal{U}^N_{S,T,q} \Phi^N_{S,T,q} \, dR,$$

(4.20)

from which we obtain the following estimate courtesy of the singular Gronwall inequality combined with Lemma C.1:

$$\Phi^N_{S,T} \leq m_w \Phi^N_{S,T} N^{-1} \sup_{x,y \in \mathcal{C}_{S,T}} \mathcal{U}^N_{S,T,x,y}.$$  

(4.21)

On the other hand, if we employ (4.19) with the trivial upper bound $\mathcal{U}^N_{S,T,x,y} \leq 1$, we establish the alternative upper bound for $\Phi^N_{S,T}$; this completes the proof of the pointwise estimate. The proof behind the spatially-averaged estimate follows from identical considerations.

4.3. Feller Continuity. Observe the a priori estimates in Lemma 4.1 and Lemma 4.4 provide optimal a priori heat kernel estimates. However, towards the proof of Theorem 1.8, it will serve important to show that the associated heat semigroups admit the Feller continuity property; otherwise, the convergence result of Theorem 1.8 holds only in either the Skorokhod space $\mathcal{D}^\infty_0$ or a weighted version of the Skorokhod space $\mathcal{D}^\infty_0$.

Our method for establishing the aforementioned Feller property is some off-diagonal improvement of Lemma 4.1 and Lemma 4.4; this is precise the following sub-optimal though certainly sufficient and convenient sub-exponential estimate.

Proposition 4.5. Provided any $\kappa \in \mathbb{R}_{>0}$ arbitrarily large but universal, the estimates of Lemma 4.1 and Lemma 4.4 remain valid upon the replacement

$$\mathcal{U}^N_{S,T,x,y} \to \mathcal{U}^N_{S,T,x,y} \exp \left( \kappa \frac{|x-y|}{N^{\frac{1}{2}} \mathcal{H}_{S,T} + 1} \right),$$

(4.22a)

$$\mathcal{U}^N_{S,T,x,y} \to \mathcal{U}^N_{S,T,x,y} \exp \left( \kappa \frac{|x-y|}{N^{\frac{1}{2}} \mathcal{H}_{S,T} + 1} \right),$$

(4.22b)

above, we have introduced the five-parameter family of quantities

$$\mathcal{E}^N_{S,T,x,y} \equiv \exp \left( \kappa \frac{|x-y|}{N^{\frac{1}{2}} \mathcal{H}_{S,T} + 1} \right).$$

(4.23)

Courtesy of the perturbative mechanism from Lemma 3.11, it will suffice to establish the estimate from Proposition 4.5 for the specialization $\mathcal{U}^N_{S,T,x,y}$ satisfying Neumann boundary conditions; we make this precise later. For Neumann boundary conditions, we use the probabilistic interpretation of $\mathcal{U}^N_{S,T,x,y}$ as transition probabilities for to a random walk on $\mathcal{J}_{S,T} \subseteq \mathbb{Z}_{>0}$.

Notation 4.6. Define $T \mapsto \mathcal{X}_{T,x}^{Z,N}$ as the random walk on $\mathbb{Z}$ with initial condition $\mathcal{X}_{0,x}^{Z,N} = x$ and time-homogeneous transition probabilities given by

$$\mathbb{P}[\mathcal{X}_{T,x}^{Z,N} = y] = \mathcal{U}^N_{0,T,x,y}.$$  

(4.24)

Similarly, we define $T \mapsto \mathcal{X}_{T,x}^{N}$ to be the random walk on $\mathcal{J}_{S,T}$ with elastic reflection at the boundaries, with initial condition $\mathcal{X}_{0,x}^{N} = x$, and with time-homogeneous transition probabilities given by $\mathcal{U}^N_{0,T,x,y}$ in similar fashion.

We proceed to present the probabilistic hitting-time estimates for both the random walk $\mathcal{X}_{T,x}^{Z,N}$ and its adjoint walk $\mathcal{X}_{T,x}^{N}$, in what follows, the numbers of 77 and 17 can be replaced by any sufficiently large but universal prefactor.

Lemma 4.7. Consider any $S, T \in \mathbb{R}_{>0}$ satisfying $S \leq T$, and moreover consider any $x \in \mathbb{Z}_{>0}$. We have the following upper bounds uniformly in $\mathcal{L} \in \mathbb{R}_{>0}$ satisfying $\mathcal{L} \geq 77 m_N$ with a universal implied constant:

$$\sum_{w \in \mathcal{J}_{S,T,x}, |w-x| \geq \mathcal{L}} \mathcal{U}^N_{S,T,x,w} \sup_{y \in \mathcal{J}_{S,T,x,w}} \mathcal{U}^N_{S,T,y,w} \leq \mathbb{P} \left[ \sup_{y \in \mathcal{J}_{S,T,x}, |w-x| \geq \mathcal{L}} \left| \mathcal{X}_{R,x}^{Z,N} - \mathcal{X}_{0,x}^{Z,N} \right| \geq \frac{1}{17} \mathcal{L} \right].$$

(4.25)
Proof. Certainly it suffices to prove the upper bound on the RHS of (4.25) for both quantities on the LHS individually. For this, we begin with the second summation on the LHS, because the bound for the first summation on the LHS follows from the exact same calculation with the additional input from Lemma 3.1 which we explain more precisely later.

First, provided that all relevant heat kernels are time-homogeneous, it suffices to assume $S = 0$. Proving the first desired estimate then amounts to proving the following probabilistic inequality with some universal implied constant:

$$\mathbb{P}\left[ \left| Y_{T_x} - x_{0,x} \right| \geq \mathcal{L} \right] \leq \mathbb{P}\left[ \sup_{R \in (0,T)} \left| Y_{Rx} - x_{0,x} \right| \geq \frac{1}{17} \mathcal{L} \right].$$

(4.26)

We establish the previous hitting-time upper bound applying the following argument via pathwise-coupling; it will serve presentationally convenient to define the following path-space events:

$$\mathcal{E}_{T,x}^{\mathcal{L}} = \left\{ \left| Y_{T_x} - x_{0,x} \right| \geq \mathcal{L} \right\}, \quad \mathcal{E}_{R_x}^{\mathcal{L}} = \left\{ \sup_{R \in (0,T)} \left| Y_{Rx} - x_{0,x} \right| \geq \frac{1}{17} \mathcal{L} \right\}.$$  

(4.27)

• Suppose first that $x \geq \frac{2}{3} \mathcal{L} + m_N$; in particular, the random walk begins a distance at least $\frac{2}{3} \mathcal{L}$ away from the subset of the lattice $I \times \mathbb{N}$ on which the generator $\mathcal{L}_{\text{Lap}}^{N,0}$ of the random walk $X_{\cdot, x}$ does not agree with that of the $\mathbb{Z}$-valued random walk $Y_{\cdot, x}$.

- For such initial conditions, we construct any coupled random walk $Y_{\cdot, x}$ with the same initial condition $Y_{0,x}$ but undergoing dynamics corresponding to the heat kernel $\mathcal{G}_{K,T}^{N,0}$. More precisely, define the stopping time

$$\tau_{m_N} = \inf \left\{ R \in [0, T) : Y_{Rx} \notin [0, m_N] \right\} \wedge T.$$  

(4.28)

For times in the random interval $[0, \tau_{m_N}] \subseteq [0, T]$, we couple the walks $Y_{\cdot, x}$ and $X_{\cdot, x}$ by coupling the Poisson clocks for each walk corresponding to the same jump-length and direction; indeed, as noted at the beginning of this case of $x \geq \frac{2}{3} \mathcal{L} + m_N$, this is a well-defined coupling between the two random walk laws.

- Consider a trajectory $Y_{T_x}$ of the $\mathcal{F}_{N,0}$-valued random walk with times in $[0, T] \subseteq \mathbb{R}_{\geq 0}$ belonging to the event $\mathcal{E}_{T,x}^{\mathcal{L}}$. In particular, because we have assumed $x \geq \frac{2}{3} \mathcal{L} + m_N$, we deduce the coupled $\mathbb{Z}$-valued random walk from the previous bullet point $Y_{\cdot, x}$ belongs in the event $\mathcal{E}_{T,x}^{\mathcal{L}}$. This completes the proof for initial conditions satisfying $x \geq \frac{2}{3} \mathcal{L} + m_N$.

Reflecting this argument completes the proof for initial conditions satisfying $N - x \geq \frac{2}{3} \mathcal{L} + m_N$ as well.

• Suppose now that $x \leq \frac{2}{3} \mathcal{L} + m_N$; which by the assumption $\mathcal{L} \geq 7m_N$ yields the constraint $x \leq \frac{6}{\mathcal{L}}$, for example. Suppose further that we have sampled any generic trajectory $Y_{\cdot, x}$ with times in $[0, T] \subseteq \mathbb{R}_{\geq 0}$ belonging to $\mathcal{E}_{T,x}^{\mathcal{L}}$.

- Observe that for a stopping time $\tau \in \mathbb{R}_{\geq 0}$ bounded by $T \in \mathbb{R}_{\geq 0}$ on the event $\mathcal{E}_{T,x}^{\mathcal{L}}$ with probability 1, we have $Y_{\tau_x} \geq \frac{2}{3} \mathcal{L} + m_N$. Indeed, if this is not the case, then the entire path of $Y_{\cdot, x}$ on the time interval $[0, T] \subseteq \mathbb{R}_{\geq 0}$ is contained in $[0, \frac{2}{3} \mathcal{L} + m_N] \subseteq \mathbb{Z}_{\geq 0}$, which clearly contradicts the constraint defining $\mathcal{E}_{T,x}^{\mathcal{L}}$.

- Continuing with the previous bullet point, we define the stopping time

$$\tau = \inf \left\{ R \in [0, T) : Y_{Rx} \geq \frac{2}{3} \mathcal{L} + m_N \right\} \wedge T.$$  

(4.29)

We now construct a $\mathbb{Z}_{\geq 0}$-valued random walk $\tilde{Y}_{\cdot, x}$ as follows.

* Provided $\tau \in [0, T]$, we first define $\tilde{Y}_{\tau,x} = x$ for all $R \in (0, \tau)$.

* We further define $\tilde{Y}_{\tau,x}$ to be equal to $Y_{\cdot, x}$, so that $\tilde{Y}_{\cdot, x}$ exhibits a jump not encoded in the generator of $\mathcal{G}_{S,T}^{N,0}$.

* We proceed to modify the previous stopping time on which our coupling between $\mathcal{G}_{S,T}^{N,0}$-dynamics and $\mathcal{G}_{S,T}^{N,0}$-dynamics is defined as follows:

$$\tau_{m_N} = \inf \left\{ R \in [\tau, T) : Y_{Rx} \notin [0, m_N] \right\} \wedge T.$$  

(4.30)

Observe that, by the strong Markov property for our random walk, the law of $\tilde{Y}_{\cdot, x}$ on the time-interval $[\tau, T]$ is equal to the law of the random walk with transition probabilities given by $\mathcal{G}_{S,T}^{N,0}$ on an identical
time interval with the same initial condition, which we recall is exactly the initial condition of $X_{t,x}^N$ on this time-interval. Thus, we employ the coupling of the previous situation on this time-interval $[0, \tau]$.

- Observe that any trajectory $X_{t,x}^N$, belonging to $\Omega_{t,x}^N$, automatically satisfies, for example, the lower bound
\begin{align}
|X_{t,x}^N - X_{t,0,x}^N| \geq |X_{t,x}^N - X_{t,0,x}^N| - |X_{t,\tau,x}^N - X_{t,0,x}^N| \geq \mathcal{L} - \frac{6}{7}\mathcal{L} \geq \frac{1}{7}\mathcal{L}.
\end{align}

Thus, appealing to the argument in the previous case of $X_{0,x}^N \geq \frac{2}{3}\mathcal{L} + m_N$, we see the coupled trajectory $\bar{X}_{t,x}^{Z,N}$ must satisfy the following lower bound, for example:
\begin{align}
\sup_{R \in [\tau, T]} |\bar{X}_{R,x}^{Z,N} - \bar{X}_{t,x}^{Z,N}| \geq \frac{1}{17}\mathcal{L}.
\end{align}

Again, because the law of $\bar{X}_{t,x}^{Z,N}$ within the time-interval $[\tau, T] \subseteq \mathbb{R}_{\geq 0}$ is exactly that of the random walk with time-homogeneous transition probabilities given by $\mathcal{P}^{Z,N}_{S,T}$ and initial condition $\bar{X}_{t,x}^{Z,N}$, we have the desired estimate in this case $x \leq \frac{2}{3}\mathcal{L} + m_N$ as well; note the simple but important observation that $\mathcal{P}_{\tau,T} \subseteq T$.

Again, reflection of the argument completes the proof for initial conditions satisfying $N - x \leq \frac{2}{3}\mathcal{L} + m_N$.

To prove the upper bound for the first quantity on the LHS of (4.25), we observe that the same pathwise-coupling argument applies to the adjoint random walk $X_{t,x}^N$ given by the adjoint calculation from Lemma 5.2. Therefore, we may establish the upper bound for the first summation on the LHS of (4.25) if we somehow had an additional factor of $\pi_{x,w}^{-1} \in \mathbb{R}_{\geq 0}$ within this summation, where $\pi \in \Pi^N$ is any strictly positive invariant measure. However, observe that the elliptic estimate from Lemma 3.1 allows us to insert such a factor at the cost of an $m_N$-dependent constant, so we are done.

Combining Lemma 4.7 with a standard martingale maximal inequality gives the following large-deviations-type bound for the LHS of (4.25).

**Corollary 4.8.** Retain the setting from Lemma 4.7; for some constant $\kappa \in \mathbb{R}_{>0}$ depending only on $m_N \in \mathbb{Z}_{>0}$ and for some universal implied constant, we have
\begin{align}
\sum_{w \in \mathcal{F}_{S,T,w}} \psi_{S,T,w}^{N,0} &+ \sum_{w \in \mathcal{F}_{S,T,w}} \psi_{S,T,w}^{N,0} \leq 1_{|\mathcal{L}| \leq N_{1/2}^{1/2}} \exp \left[ -\kappa \frac{\mathcal{L}^2}{N^2 \mathcal{P}_{S,T}} \right] + 1_{|\mathcal{L}| \leq N_{1/2}^{1/2}} \exp \left[ -\kappa \mathcal{L} \right];
\end{align}
above, the implied constants within the indicator functions are both universal outside their dependence on $m_N \in \mathbb{Z}_{>0}$; moreover, these implied constants between the two indicator functions are equal.

**Proof.** Observe that $X_{t,x}^{Z,N} - X_{0,x}^{Z,N}$ is a continuous-time random-walk martingale with uniformly bounded jumps with initial condition equal to 0 with probability 1. Employing the Doob maximal inequality, we have the following inequality for any $\beta \in \mathbb{R}_{>0}$:
\begin{align}
P \left[ \sup_{R \in [0, T]} \left| X_{R,x}^{Z,N} - X_{0,x}^{Z,N} \right| \geq \frac{1}{17}\mathcal{L} \right] &= P \left[ \sup_{R \in [0, T]} \exp \left[ \beta \left| X_{R,x}^{Z,N} - X_{0,x}^{Z,N} \right| \right] \geq \exp \left[ \frac{1}{17}\mathcal{L} \right] \right] \leq \exp \left[ -\beta \frac{1}{17}\mathcal{L} \right] \cdot \mathbb{E} \left[ \beta \left| X_{T,x}^{Z,N} - X_{0,x}^{Z,N} \right| \right],
\end{align}
which we may estimate by conditioning on the total number of jumps and then analyze the induced discrete-time random walk via the Azuma-Hoeffding inequality because the jumps in this martingale are uniformly bounded above by $m_N \in \mathbb{Z}_{>0}$. This procedure is rather standard; for example, such a calculation is performed in the proof of Theorem 5.17 in [2]. Thus, we omit this calculation and thus complete the proof.
Proof of Proposition 4.5. The proof of the estimate corresponding to the specialization $Ψ^{N,0}$ satisfying Neumann boundary conditions follows immediately from Corollary 4.8 and elementary calculations with the Chapman-Kolmogorov equation.

To establish the sub-exponential estimate for general heat kernels $Ψ^N$, we proceed to employ Lemma 3.11 to establish the elementary inequality

$$\mathcal{H}^N_{S,T,x,y} \cdot ϵ^N_{S,T,x,y} \lesssim_{\text{a.e.}} \mathcal{H}^N_{S,T,x,y} \cdot ϵ^N_{S,T,x,y} + \mathcal{H}^N_{S,T,x,y} + \mathcal{Y}^N_{S,T,x,y},$$

where

$$\mathcal{H}^N_{S,T,x,y} = \int_T^S \mathcal{H}^N_{S,R,x,y} \cdot ϵ^N_{S,R,x,y} \times \left[ N \mathcal{H}^N_{S,R,y} \cdot ϵ^N_{S,R,y} \right] dR;$$
$$\mathcal{Y}^N_{S,T,x,y} = \int_T^S \mathcal{H}^N_{S,R,x,y} \cdot ϵ^N_{S,R,x,y} \times \left[ N \mathcal{H}^N_{S,R,y} \cdot ϵ^N_{S,R,y} \right] dR.$$  

We proceed to establish notation for a topology with respect to which we may employ a fixed-point-type argument; define

$$\mathcal{Y}^N_{S,T,x,y} = \mathcal{H}^{1/2} \mathcal{H}^N_{S,T,x,y} \cdot ϵ^N_{S,T,x,y}.$$  

(4.40)

Courtesy of the optimal off-diagonal estimate established for $Ψ^{N,0}$ via Corollary 4.8 as mentioned above, we first have

$$\mathcal{H}^N_{S,T,x,y} \cdot ϵ^N_{S,T,x,y} \lesssim_{\text{a.e.}} N^{-1} + N^{-1} \mathcal{Y}^N_{S,T}.$$  

(4.41)

The result follows from an identical argument via singular Gronwall inequality as in the proof of Lemma 4.4; this completes the proof.  

□

4.4. Additional Remarks on Nash Inequalities. The current subsection is purely for possible future interest; in particular, it will serve no impact on our analysis, though we include this discussion because of its potential applicability for stochastic PDE limits associated to some interacting particle systems where the associated heat kernels are elliptic though with rough coefficients.

Observe that the evolution equation satisfied by these heat kernels $Ψ^N$ are uniformly elliptic if away from the boundary; let $Ψ_t \subseteq Ψ_t,0$ provide any example of some sub-lattice whose distance from the boundary is roughly $εN \in \mathbb{R}_{>0}$. Moreover, the associated elliptic differential operator on $Ψ_t \subseteq Ψ_t,0$ is of purely second-order; thus, there exist no boundary dynamics on this bulk sub-lattice by assumption. In particular, any appropriate discrete analog of the robust De Giorgi-Nash-Moser parabolic Harnack inequalities provide $ε$-dependent space-time Holder regularity estimates for the heat kernels $Ψ^N$. Upon re-working through again the estimates of Bertini-Giacomin within [3], for example, choosing the cutoff parameter $ε \in \mathbb{R}_{>0}$ appropriately, this space-time Holder regularity is all one requires to prove tightness of the microscopic Cole-Hopf transform $Z^N$ if the model at hand is integrable; in particular, the delicate spectral estimates established in [3] are significantly more than sufficient towards the proof of tightness of $Z^N$.

To identify subsequential limit points as the solutions to the appropriate SHE, in [3] the authors rely crucially on delicate heat kernel estimates established through this aforementioned spectral theory. As noted in both [9] and [23], it turns out that this approach may be avoided with a hydrodynamic-type analysis even with sub-optimal entropy production estimates. One ultimate conclusion of this discussion is that the theory of De Giorgi-Nash-Moser provides a significantly more robust alternative to the approach via spectral theory employed in previous relevant articles; in particular, the perspective taken in this article is that refined heat kernel estimates are employed only for implementing the dynamical-averaging strategy at the heart of the proof of Theorem 1.8.

5. Heat Kernel Estimate III – Regularity Estimates

The current and final section of this article that is exclusively interested in heat kernel estimates aims to obtain regularity estimates with respect to both mesoscopic and macroscopic spatial scales provided the Nash-type estimates established in Section 4; we reemphasize the importance of the respective roles of Lemma 4.1 and Lemma 4.4 in providing a priori heat kernel estimates which the current section would fail without.

Again, to provide some organizational clarity, we begin with a table-of-contents for the current section.
• The first ingredient we establish consists of regularity estimates in space-time of the heat kernel $\mathcal{U}^{N,0}$; the primary tools in this direction are the perturbative mechanism in Lemma 3.10 combined with the Chapman-Kolmogorov equation for $\mathcal{U}^{N,0}$; the latter ingredient will be important to provide some "smoothing" effect for the heat kernel towards the boundary of the lattice where the perturbative scheme in Lemma 3.10 is actually ineffective.

• The second ingredient we establish consists of comparison between the heat kernel $\mathcal{U}^{N,0}$ and its nearest-neighbor specialization; although this does not resemble any version of a regularity estimate, our comparison will allow us to transfer some regularity estimates from the nearest-neighbor specialization to $\mathcal{U}^{N,0}$ itself; we remark here that the nearest-neighbor specialization is amenable to exact formulas through the method-of-images, from which the appropriate regularity estimates are established in [6] and [22].

• Finally, we transfer those results within the aforementioned two bullet points to heat kernels with arbitrary Robin parameters, once again through the perturbative scheme of Lemma 3.11.

Before we proceed, throughout the entirety of this article, it will serve convenient to introduce the following notation.

**Notation 5.1.** Provided any $\beta \in \mathbb{R}_{>0}$, let us define the sub-domain $\mathcal{I}_{N,\beta} \subseteq \mathcal{I}_{N,0}$ via the prescription

$$\mathcal{I}_{N,\beta} := (\mathcal{I}_{N,0} \setminus [0,N^\beta]) \cup [N-N^\beta,N].$$

(5.1)

Roughly speaking, the sub-lattice $\mathcal{I}_{N,\beta} \subseteq \mathcal{I}_{N,0} \subseteq \mathbb{Z}_{\geq 0}$ provides a cutoff away from the boundary.

Moreover, it will serve convenient as well to introduce the following discrete-time-differential operator.

**Notation 5.2.** Suppose $\mathcal{F}_{S,T} : \mathbb{R}^2_{\geq 0} \to \mathbb{R}$ is any function supported on times $S,T \in \mathbb{R}_{\geq 0}$ satisfying $S \leq T$. For times $S,T \in \mathbb{R}_{\geq 0}$ and any time-scale $\tau \in \mathbb{R}_{\geq 0}$ satisfying $\tau \leq \mathcal{O}_{S,T}$, we define

$$\mathcal{D}_\tau \mathcal{F}_{S,T} := \mathcal{F}_{S+\tau,T} - \mathcal{F}_{S,T}.$$  

(5.2)

5.1. **Regularity of $\mathcal{U}^{N,0}$.** The first primary result of the current subsection is precisely stated as follows; roughly speaking, it provides spatial regularity of the heat kernel $\mathcal{U}^{N,0}$ away from the boundary.

**Lemma 5.3.** Consider times $S,T \in \mathbb{R}_{\geq 0}$ satisfying $S \leq T$, along with any $x \in \mathcal{I}_{N,0}$ and any $\beta_2 \in \mathbb{R}_{>0}$ arbitrarily small but universal.

- There exists $\epsilon \in \mathbb{R}_{>0}$ satisfying $\epsilon \lesssim \beta_2$ with some universal implied constant such that provided any uniformly bounded $k \in \mathbb{Z}$, we have

$$\sum_{y \in \mathcal{F}_{S,T}} \left| \nabla^I_{k,y} \mathcal{U}^{N,0}_{S,T,x,y} - \nabla^I_{k,y} \mathcal{F}^{N,0}_{S,T,x,y} \right| \lesssim_{\epsilon} N^{-1} \mathcal{O}_{S,T} + e^{-\log^{100} N};$$

(5.3a)

$$\sum_{y \in \mathcal{F}_{S,T}} \left| \nabla^I_{k,y} \mathcal{F}^{N,0}_{S,T,x,y} \right| \lesssim_{\epsilon} \mathcal{O}_{S,T} + e^{-\log^{100} N}.  

(5.3b)

Observe that Lemma 5.3 is certainly sub-optimal; not only is the stated result an averaged estimate, but it also imposes some artificial constraint on the forward spatial variable. An application of the Chapman-Kolmogorov equation is certainly a legitimate and successful method to remove this constraint up to an additional error of the same order as the RHS of the estimate in Lemma 5.3 for times $S,T \in \mathbb{R}_{\geq 0}$ satisfying $\mathcal{O}_{S,T} \geq N^{-2}$, for example. However, for just the purposes of proving Theorem 1.8 this will not be necessary for spatial regularity, so we will not include it now. Such an upgrade will turn out to be necessary, though, for forthcoming time regularity of $\mathcal{U}^{N,0}_{S,T}$, so we will illustrate this method when relevant.

**Proof.** Appealing directly to Lemma 3.10, we have

$$\nabla^I_{k,y} \mathcal{U}^{N,0}_{S,T,x,y} = \nabla^I_{k,y} \mathcal{F}^{N,0}_{S,T,x,y} + \int_S^T \sum_{m=0}^{m-1} \mathcal{F}^{N,0}_{R,T,x,w} \cdot \mathcal{D}^N_{x,y} \nabla^I_{k,y} \mathcal{F}^{N,0}_{R,T,R_w,w} dR + \int_S^T \sum_{w=0}^{N-N_{\mathcal{O}_{S,T}}} \mathcal{F}^{N,0}_{R,T,x,w} \cdot \mathcal{D}^N_{x,y} \nabla^I_{k,y} \mathcal{F}^{N,0}_{R,T,R_w,w} dR;$$

(5.4)

indeed, as the operator acting on $\mathcal{F}_{S,T}^{Z_x}$ within the integral on the RHS of the formula in Lemma 3.10 act on the backwards spatial variable, such operator certainly commutes with $\nabla^I_{k,y}$ Directly applying the regularity estimates for $\mathcal{F}_{S,T}^{N,0}$ obtained
in Lemma B.2 gives the following upper bound with universal implied constant:

$$\sum_{y \in F_{N,0}} \left| \nabla_{k,y}^1 \mathcal{G}_{S,T,x,y}^{N,0} \right| \lesssim \Theta_{S,T}^{-\frac{3}{4}}. \tag{5.5}$$

It remains to estimate both of these integrals within the RHS of (5.4); we analyze explicitly only the first of these integrals, which we denote by $\Xi_{S,T}$, for convenience, because the second integral is analyzed in analogous fashion.

We decompose this integral $\Xi_{S,T} = \Xi_{S,T}^{N,1} + \Xi_{S,T}^{N,2}$ into two separate time-intervals as follows for $\varepsilon \in \mathbb{R}_{>0}$ arbitrarily small although universal and determined shortly:

$$\Xi_{S,T}^{N,1} = \int_{S}^{S+N-2^+} \sum_{w=0}^{m_N-1} \sum_{y \in F_{N,0}} \left| D_y \nabla_{k,y}^1 \mathcal{G}_{S,T,x,y}^{N,0} \mathcal{G}_{S,R,w,y}^{N,0} \right| \, dR; \tag{5.6a}$$

$$\Xi_{S,T}^{N,2} = \int_{S+N-2^+}^{T} \sum_{w=0}^{m_N-1} \sum_{y \in F_{N,0}} \left| D_y \nabla_{k,y}^1 \mathcal{G}_{S,T,x,y}^{N,0} \right| \, dR. \tag{5.6b}$$

Let us analyze the second object $\Xi_{S,T}^{N,2}$ first. We first observe that within both of the integrals $\Xi_{S,T}^{N,1}$ and $\Xi_{S,T}^{N,2}$, the composition of the operators acting on the $\mathcal{G}_{S,R}^{N,0}$ factor is actually a third-order differential operator acting on $\mathcal{G}_{S,R}^{N,0}$; this follows from the definition of $\mathcal{G}_{S,R}^{N,0}$. Thus, Lemma B.2 combined with the on-diagonal estimate in Lemma 4.1 provides the estimate

$$\sum_{y \in F_{N,0}} \left| \Xi_{S,T}^{N,1} \right| \lesssim \int_{S}^{T} \sum_{w=0}^{m_N-1} \left| D_y \nabla_{k,y}^1 \mathcal{G}_{S,T,x,y}^{N,0} \mathcal{G}_{S,R,w,y}^{N,0} \right| \, dR \tag{5.7}$$

$$\leq \int_{S}^{T} \sum_{w=0}^{m_N-1} \Theta_{S,R}^{-\frac{3}{4}} \left| D_y \nabla_{k,y}^1 \mathcal{G}_{S,T,x,y}^{N,0} \right| \, dR \tag{5.8}$$

$$\lesssim m_N^{-1} \int_{S+N-2^+}^{T} \Theta_{S,R}^{-\frac{3}{4}} \, dR \tag{5.9}$$

$$\lesssim N^{-\frac{3}{4}} \Theta_{S,T}^{-\frac{3}{4}}, \tag{5.10}$$

where the last estimate follows from Lemma C.2. Thus, we are left with analyzing the first integral $\Xi_{S,T}^{N,1}$. Before we begin, observe that none of our analysis so far requires the assumption $y \gtrsim N^{\beta_0}$. Indeed, such an assumption is important simply for the following straightforward and brutal calculation:

$$\sum_{y \in F_{N,0}} \left| \Xi_{S,T}^{N,2} \right| \lesssim \int_{S}^{T} \sum_{w=0}^{m_N-1} \left| D_y \nabla_{k,y}^1 \mathcal{G}_{S,T,x,y}^{N,0} \mathcal{G}_{S,R,w,y}^{N,0} \right| \, dR \tag{5.11}$$

$$\lesssim m_N \int_{S}^{T} \sup_{|w| \leq m_N} \left| \mathcal{G}_{S,R,w,y}^{N,0} \right| \, dR. \tag{5.12}$$

Suppose we now choose $\varepsilon \lesssim \beta_0$ sufficiently small although universal outside dependence of $\beta_0 \in \mathbb{R}_{>0}$. In this case, observe the next calculation for $\kappa \in \mathbb{R}_{>0}$ a constant depending only on $m_N \in \mathbb{Z}_{>0}$; this follows from the definition of $\mathcal{G}_{S,R}^{N,0}$ and an elementary change-of-variables-type calculation, in which $\kappa \in \mathbb{R}_{>0}$ is universal:

$$N^3 \int_{S}^{T} \sup_{|w| \leq m_N} \left| \mathcal{G}_{S,R,w,y}^{N,0} \right| \, dR \lesssim N^3 \int_{S}^{T} \sup_{|w| \leq m_N} \left| \mathcal{G}_{S,R,w,y}^{N,0} \right| \, dR \tag{5.13}$$

$$\lesssim \kappa N^{3 - \varepsilon N^{\beta_0} - \varepsilon}, \tag{5.14}$$

the latter estimate as certainly sufficient and following from Lemma B.1. In particular, combining our estimates for $\Xi_{S,T}^{N,1}$ and $\Xi_{S,T}^{N,2}$ along with the analogous parallel estimates for the second integral on the RHS of (5.4), we deduce

$$\sum_{y \in F_{N,0}} \left| \nabla_{k,y}^1 \mathcal{G}_{S,T,x,y}^{N,0} \mathcal{G}_{S,R,w,y}^{N,0} \right| \lesssim N^{-\frac{3}{4}} \Theta_{S,T}^{-\frac{3}{4}} + e^{-\kappa N^{3 - \varepsilon N^{\beta_0} - \varepsilon}}, \tag{5.15}$$
which gives the first desired estimate. The remaining desired estimate is consequence of this first estimate combined with (5.5); this completes the proof.

We proceed to establish estimates for the time-regularity of \( \mathcal{U}^{N,0} \). Similarly to Lemma 5.3 above, we establish a spatially-averaged estimate, although distinct from Lemma 5.3 we additionally employ the Chapman-Kolmogorov equation after to upgrade this spatially-averaged estimate to a pointwise estimate.

We emphasize that the constant 7 in the following result is certainly unimportant.

**Lemma 5.4.** Recall the setting from Lemma 5.3. Provided any time-scale \( \tau \in \mathbb{R} \geq 0 \) satisfying \( \tau \leq 7\mathcal{E}_{S,T} \), we have the following estimate for any \( \epsilon \in \mathbb{R}_{>0} \):

\[
\sup_{x,y \in \mathcal{F}_{S,T}} \left| D_{x,y} \mathcal{U}^{N,0}_{S,T} \right| \lesssim m_{\epsilon} \mathcal{E}_{S,T}^{\mathcal{E}_{S,T}^{N^2+\epsilon}} + N^{-2+2\epsilon} \mathcal{E}_{S,T}^{-2} + N^{-1+\epsilon} \mathcal{E}_{S,T}^{-2}. \tag{5.16}
\]

**Proof.** Once again appealing to Lemma 3.10, we have

\[
D_{x,y} \mathcal{U}^{N,0}_{S,T} = \mathcal{E}_{x,y}^{N,0}_{S,T} + \mathcal{E}_{x,y}^{N,0}_{S,T} + \mathcal{E}_{x,y}^{N,0}_{S,T}, \tag{5.17}
\]

where we have introduced the quantities

\[
\mathcal{E}_{x,y}^{N,-}_{S,T} = \sum_{m=0}^{S+\tau} \sum_{w=0}^{m-1} \mathcal{U}^{N,0}_{R,T,x,w} \cdot D_{r,s}^{N,0}_{S,R,w,y} \mathcal{E}_{S,T}, \tag{5.18a}
\]

\[
\mathcal{E}_{x,y}^{N,+}_{S,T} = \sum_{m=0}^{S+\tau} \sum_{w=0}^{N-m+1} \mathcal{U}^{N,0}_{R,T,x,w} \cdot D_{r,s}^{N,0}_{S,R,w,y} \mathcal{E}_{S,T}, \tag{5.18b}
\]

\[
\mathcal{E}_{x,y}^{N,-}_{S,T} = \sum_{m=0}^{S+\tau} \sum_{w=0}^{N-m+1} \mathcal{U}^{N,0}_{R,T,x,w} \cdot D_{r,s}^{N,0}_{S,R,w,y} \mathcal{E}_{S,T}, \tag{5.18c}
\]

\[
\mathcal{E}_{x,y}^{N,+}_{S,T} = \sum_{m=0}^{S+\tau} \sum_{w=0}^{N-m+1} \mathcal{U}^{N,0}_{R,T,x,w} \cdot D_{r,s}^{N,0}_{S,R,w,y} \mathcal{E}_{S,T}. \tag{5.18d}
\]

To be completely transparent, as alluded to before the statement of Lemma 3.10, we first establish some spatially averaged estimate. Moreover, our analysis for the quantities \( \mathcal{E}_{x,y}^{N,-}_{S,T} \) and \( \mathcal{E}_{x,y}^{N,+}_{S,T} \) is identical to analysis of the pair \( \mathcal{E}_{x,y}^{N,-}_{S,T} \) and \( \mathcal{E}_{x,y}^{N,+}_{S,T} \), so we detail the analysis only for the latter pair.

Concerning the first quantity from the RHS of (5.17), we simply employ the temporal regularity of the heat kernel \( \mathcal{E}_{S,T}^{N,0} \) in Proposition A.1 from [9] and obtain

\[
\sum_{y \in \mathcal{F}_{S,T}} \left| D_{x,y} \mathcal{U}^{N,0}_{S,T} \right| \lesssim \tau \mathcal{E}_{S,T}^{-1} \wedge 1. \tag{5.19}
\]

We proceed to analyze the first integral \( \mathcal{E}_{S,T}^{N,-} \) on the RHS of (5.17); for this, we appeal to the regularity estimate in Lemma B.2 to deduce the following upper bounds valid for any \( \epsilon \in \mathbb{R}_{>0} \) arbitrarily small but universal:

\[
\sum_{y \in \mathcal{F}_{S,T}} \left| \mathcal{E}_{x,y}^{N,-}_{S,T} \right| \lesssim \sum_{m=0}^{S+\tau} \sum_{w=0}^{m-1} \mathcal{U}^{N,0}_{R,T,x,w} \cdot D_{r,s}^{N,0}_{S,R,w,y} \mathcal{E}_{S,T} \lesssim N^{2\epsilon} \int_{S}^{S+\tau} \mathcal{E}_{S,T}^{-1+2\epsilon} \sum_{w=0}^{m-1} \mathcal{U}^{N,0}_{R,T,x,w} dR \lesssim \sum_{m,\epsilon} N^{-1+2\epsilon} \mathcal{E}_{S,T}. \tag{5.20}
\]

We deduce an identical upper bound for the second integral \( \mathcal{E}_{S,T}^{N,-} \) on the RHS of (5.17) as well upon brutally bounding the spatial derivatives of the time-gradient \( D_{x,y} \mathcal{U}^{N,0}_{S,T} \) by the spatial derivatives of the original kernel \( \mathcal{U}^{N,0}_{S,T} \) up to some universal
constant and following the proof of Lemma 5.3 provided we perform the summation over only the sub-lattice $\mathcal{I}_{N, \beta} \subseteq \mathcal{I}_{N, 0}$ with $\beta \in \mathbb{R}_{> 0}$ arbitrarily small but universal. Ultimately, we deduce
\begin{equation}
\sum_{y \in \mathcal{I}_{N, \beta}} \left| \nabla \mathcal{Y}_{N, 0} \right|_{x, y} \lesssim N^{-1+\epsilon} \mathcal{Y}_{T, \mathcal{S_T}}^{-1} + N^{-2+2\epsilon} \mathcal{Y}_{T, \mathcal{S_T}}^{-1} + N^{-\epsilon} \mathcal{Y}_{T, \mathcal{S_T}}^{-1}.
\end{equation}

To establish the ultimate pointwise estimate, we will employ the Chapman-Kolmogorov equation; this provides the upper bound
\begin{equation}
\left| \nabla \mathcal{Y}_{N, 0} \right|_{x, y} \lesssim N^{-1+\epsilon} \mathcal{Y}_{T, \mathcal{S_T}}^{-1} + N^{-2+2\epsilon} \mathcal{Y}_{T, \mathcal{S_T}}^{-1} + N^{-\epsilon} \mathcal{Y}_{T, \mathcal{S_T}}^{-1}.
\end{equation}

To estimate the first quantity from within the RHS of (5.25), we employ the heat kernel estimate from Lemma 4.1 combined with the spatially-averaged estimate (5.23); for the latter quantity within the RHS from (5.25), we employ the on-diagonal estimate in Lemma 4.1 with an additional observation that the second summation is over $N\beta$-many sites, up to a universal constant. In particular, we obtain
\begin{equation}
\left| \nabla \mathcal{Y}_{N, 0} \right|_{x, y} \lesssim N^{-1+\epsilon} \mathcal{Y}_{T, \mathcal{S_T}}^{-1} + N^{-2+2\epsilon} \mathcal{Y}_{T, \mathcal{S_T}}^{-1} + N^{-\epsilon} \mathcal{Y}_{T, \mathcal{S_T}}^{-1},
\end{equation}

this completes the proof.

We complete the current discussion concerning regularity estimates for $\mathcal{Y}_{N, 0}$ with a spatial-temporal gradient estimate. The only motivation we provide for this estimate is that the dynamical averaging scheme which is at the heart of the proof of Theorem 1.8 requires delicate heat kernel regularity estimates, and this is simply one of those estimates.

**Lemma 5.5.** Consider times $S, T \in \mathbb{R}_{> 0}$ satisfying $S \leq T$ along with any spatial coordinate $x \in \mathcal{I}_{N, 0}$. Provided any time-scale $\tau \in \mathbb{R}_{> 0}$ satisfying $\tau \leq \mathcal{Y}_{T, \mathcal{S_T}}$ along with any $k \in \mathbb{Z}$ uniformly bounded, we have the following estimate for any $\beta \in \mathbb{R}_{> 0}$ and $\epsilon \in \mathbb{R}_{> 0}$ arbitrarily small but universal:
\begin{equation}
\nabla_{k, y} \left| \nabla \mathcal{Y}_{N, 0} \right|_{x, y} \lesssim \mathcal{Y}_{T, \mathcal{S_T}}^{-1+\epsilon} + N^{-2+2\epsilon} \mathcal{Y}_{T, \mathcal{S_T}}^{-1} + N^{-\epsilon} \mathcal{Y}_{T, \mathcal{S_T}}^{-1}. \tag{5.27}
\end{equation}

**Proof.** We assume $S = 0$ due to time-homogeneity of the heat kernel once again.

We employ the Chapman-Kolmogorov equation to write
\begin{equation}
\nabla_{k, y} \left| \nabla \mathcal{Y}_{N, 0} \right|_{x, y} = \sum_{w \in \mathcal{I}_{N, \beta}} \mathcal{Y}_{N, 0} \left| \nabla \mathcal{Y}_{N, 0} \right|_{x, w, y} + \sum_{w \in \mathcal{I}_{N, \beta}} \mathcal{Y}_{N, 0} \left| \nabla \mathcal{Y}_{N, 0} \right|_{x, w, y}. \tag{5.28}
\end{equation}

In particular, courtesy of the gradient estimate from Lemma 5.3 and the spatially-averaged time-regularity estimate (5.23) from the proof of Lemma 5.4, we obtain a suitable estimate for the first summation within the RHS of the above Chapman-Kolmogorov equation after performing a summation over the forward spatial variable over $\mathcal{I}_{N, \beta} \subseteq \mathcal{I}_{N, 0}$. To estimate the second quantity, we again employ Lemma 5.3 along with the estimate in Lemma 5.4. \qed

5.2. **Comparison with $\mathcal{Y}_{N, 0}$.** At this point, we admit that the previous regularity estimates are sub-optimal with respect to the macroscopic scale, either in $N$-dependent scaling or in their spatially-averaged nature. Indeed, this previous subsection of heat kernel regularity estimates will be employed exclusively with respect to the mesoscopic scale to perform the local dynamical strategy at the heart of the proof of Theorem 1.8.

Meanwhile, the current subsection will be exclusively dedicated towards establishing the required space-time regularity of the heat kernel $\mathcal{Y}_{N, 0}$ at the macroscopic scale which serves key to the proof behind Theorem 1.8, namely for establishing tightness. Towards this goal, the strategy we employ is a comparison between $\mathcal{Y}_{N, 0}$ and its nearest-neighbor specialization $\mathcal{Y}_{N, 0}$. In particular, if we could actually replace the relevant heat kernel $\mathcal{Y}_{N, 0}$ within the context of the evolution equation for $\mathcal{Y}_{N}$ by its nearest-neighbor specialization, we may simply inherit regularity estimates and tightness from the article [22], for example. Indeed, this is the strategy we employ in the proof of Theorem 1.8.
Remark 5.6. Although we held a previous discussion about proving tightness of the microscopic Cole-Hopf transform $\mathcal{Z}^N$ via regularity estimates for $\mathcal{W}^{N,0}$ which may be established via the De Giorgi-Nash-Moser theory, the strategy we outlined directly above will make the problem of identifying subsequential limits significantly simpler and will avoid the necessity of introducing additional technology.

Remark 5.7. We have already introduced a result of "comparison-type" in Lemma 5.3; namely, comparison with the nearest-neighbor specialization $\mathcal{J}^{N,0}$ in a discrete-type $\mathcal{W}^{1,1}$-Sobolev topology.

In view of the previous discussion, it will serve convenient to adopt the following notation.

Notation 5.8. Provided any $S, T \in \mathbb{R}_{\geq 0}$ satisfying $S \leq T$ along with any pair $x, y \in \mathcal{I}_{N,0}$ we define

$$\mathcal{D}_{S,T,x,y}^{N,0} = \mathcal{W}_{S,T,x,y}^{N,0} - \mathcal{W}_{S,T,x,y}^{N,0}.$$ \hspace{1cm} (5.29)

We additionally define the following two quantities:

$$\mathcal{D}_{S,T,x,y}^{N,0,1} = \mathcal{W}_{S,T,x,y}^{N,0} - \mathcal{W}_{S,T,x,y}^{N,0},$$ \hspace{1cm} (5.30a)

$$\mathcal{D}_{S,T,x,y}^{N,0,2} = \mathcal{W}_{S,T,x,y}^{N,0} - \mathcal{W}_{S,T,x,y}^{N,0}.$$ \hspace{1cm} (5.30b)

In particular, we have $\mathcal{D}^{N,0} = \mathcal{D}^{N,0,1} + \mathcal{D}^{N,0,2}.$

The primary goal of the current subsection is to establish the following result.

Proposition 5.9. Provided times $S, T \in \mathbb{R}_{\geq 0}$ satisfying $S \leq T$ along with any $x, y \in \mathcal{I}_{N,0}$ we have the following estimate in which $\epsilon \in \mathbb{R}_{>0}$ is universal:

$$1_{y \in \mathcal{I}_{S,T,0}} \left| \mathcal{D}_{S,T,x,y}^{N,0} \right| \lesssim N^{-1-\epsilon} \mathcal{B}_{S,T}^{-1} + N^{-2} \mathcal{B}_{S,T}^{-1} + \epsilon^{-\log 100 N}. \hspace{1cm} (5.31)$$

Moreover, provided $\beta_0 \in \mathbb{R}_{>0}$ is arbitrarily small though universal, we have the following spatially-averaged estimates uniformly in uniformly bounded indices $k \in \mathbb{Z}$:

$$\sum_{y \in \mathcal{I}_{S,T,0}} \left| \mathcal{D}_{k,y}^{N,0} \right| \lesssim N^{-\epsilon} \mathcal{B}_{S,T}^{-1} + N^{-1+\beta_0} \mathcal{B}_{S,T}^{-1} + N^{-1} \mathcal{B}_{S,T}^{-1} + \epsilon^{-\log 100 N}, \hspace{1cm} (5.32a)$$

$$\sum_{y \in \mathcal{I}_{S,T,0}} \left| \nabla_{k,y} \mathcal{D}_{S,T,x,y}^{N,0} \right| \lesssim N^{-\epsilon} \mathcal{B}_{S,T}^{-1} + N^{-1} \mathcal{B}_{S,T}^{-1}. \hspace{1cm} (5.32b)$$

As suggested with the preceding notation, we first estimate $\mathcal{D}^{N,0,1}$ and then afterwards estimate $\mathcal{D}^{N,0,2}.$ Beginning with the former quantity, we present the next estimate.

Lemma 5.10. Consider times $S, T \in \mathbb{R}_{\geq 0}$ satisfying $S \leq T$ along with any spatial coordinates $x, y \in \mathcal{I}_{N,0}$. Moreover, consider any parameter $\beta_2 \in \mathbb{R}_{>0}$ arbitrarily small but universal.

There exists a universal constant $\epsilon \in \mathbb{R}_{>0}$ satisfying $\epsilon \lesssim \beta_2$ such that for any $y \in \mathcal{I}_{N,\beta_2}$ we have

$$\left| \mathcal{D}_{S,T,x,y}^{N,0,1} \right| \lesssim m_{N,\epsilon,\beta_2} \left[ N^{-1-\epsilon} \mathcal{B}_{S,T}^{-1} + \epsilon^{-\log 100 N} \right] \wedge 1. \hspace{1cm} (5.33)$$

Moreover, we also have

$$\sum_{y \in \mathcal{I}_{S,T,0}} \left| \mathcal{D}_{S,T,x,y}^{N,0,1} \right| \lesssim m_{N,\epsilon,\beta_2} \left[ N^{-\epsilon} \mathcal{B}_{S,T}^{-1} + \epsilon^{-\log 100 N} \right] \wedge 1 + N^{-1+\beta_2} \mathcal{B}_{S,T}^{-1}. \hspace{1cm} (5.34)$$

Proof. Observe that we may assume the a priori lower bound $\mathcal{B}_{S,T} \gtrsim N^{-2+\epsilon}$ with arbitrarily large though universal implied constant; indeed, in this case, the RHS in the stated bound is simply 1.

We now appeal to Lemma 3.10 combined with the regularity estimates in Lemma B.2 and the off-diagonal estimate in Lemma B.1 to establish the following estimate for $\epsilon \in \mathbb{R}_{>0}$ satisfying $\epsilon \lesssim \beta_2$ with a sufficiently small but universal implied
constant:

\[
1_{y \in I_{\kappa, \beta}} \left| \mathcal{G}^{N,0,1}_{S,T,x,y} \right| \lesssim \int_S^T \sum_{w=0}^{m-1} \left| \mathcal{G}^{N,0}_{R,T,x,w} \right| \left| \mathcal{D}^N_Z \mathcal{G}^{N,0}_{S,R,w,y} \right| \, dR \tag{5.35}
\]

\[
\lesssim_{m_0, \kappa} N^{-2} \int_S^T \epsilon_{S,T}^{-\frac{1}{2}} dR + e^{-\log^{100} N} \tag{5.36}
\]

\[
\lesssim N^{-1+\beta} \epsilon_{S,T}^{-\frac{1}{2}} + e^{-\log^{100} N}. \tag{5.37}
\]

Indeed, above we have crucially used the assumption \( y \in I_{\kappa, \beta} \) like for the proofs of Lemma 5.3 and Lemma 5.4 to obtain the second inequality, and this final inequality is consequence of Lemma C.2. It remains to observe \( |\mathcal{G}^{N,0}_{S,T,x,y}| \lesssim 1 \) courtesy of those Nash-type heat kernel estimates within both of Lemma 4.1 and Lemma 3.5, the latter of which applying equally well to the kernel \( \mathcal{G}^{N,0} \) provided the heat kernel estimates in Lemma B.1. This provides the pointwise estimate.

To obtain the spatially-averaged estimate, we simply employ those heat kernel estimates within Lemma 4.1 and Lemma 3.5 once again as follows:

\[
\sum_{y \in I_{\kappa}} \left| \mathcal{G}^{N,0,1}_{S,T,x,y} \right| \lesssim \sum_{y \in I_{\kappa}} \left| \mathcal{G}^{N,0,1}_{S,T,x,y} \right| + \sum_{y \notin I_{\kappa}} \left| \mathcal{G}^{N,0,1}_{S,T,x,y} \right| \tag{5.38}
\]

\[
\lesssim \sum_{y \in I_{\kappa}} \left| \mathcal{G}^{N,0,1}_{S,T,x,y} \right| + N^{-1+\beta} \epsilon_{S,T}^{-\frac{1}{2}}. \tag{5.39}
\]

To estimate the remaining summation, we proceed exactly as with the pointwise estimate; this completes the proof. \( \square \)

We proceed to estimate the quantity \( \mathcal{G}^{N,0,2} \); to this end, it will serve convenient to introduce the following operator.

**Notation 5.11.** We define the operator

\[
\mathcal{D}^N_Z = \frac{1}{2} \sum_{k=1}^{m} \alpha_k \mathbb{D}^N_k - \left( \sum_{k=1}^{m} k^2 \alpha_k \right) \mathbb{D}_1^N. \tag{5.40}
\]

Moreover, provided \( S, T \in \mathbb{R}_{>0} \) satisfying \( S \leq T \) along with any \( x, y \in \mathbb{Z} \), we define

\[
\mathcal{G}^{N,0,2}_{S,T,x,y} = \mathcal{G}^{N,0}_{S,T,x,y} - \mathcal{G}^{N,0}_{S,T,x,y} \tag{5.41}
\]

**Lemma 5.12.** Consider times \( S, T \in \mathbb{R}_{>0} \) satisfying \( S \leq T \) along with any spatial coordinates \( x, y \in \mathbb{Z} \). Given any \( \kappa \in \mathbb{R}_{>0} \) and any uniformly bounded \( k \in \mathbb{Z} \), we have

\[
\left| \mathcal{G}^{N,0,2}_{S,T,x,y} \right| \lesssim \kappa N^{-2} \epsilon_{S,T}^{-\frac{1}{2}} e^{\kappa N, \kappa}. \tag{5.42a}
\]

\[
\left| \nabla_{k,y}^{\text{T}} \mathcal{G}^{N,0,2}_{S,T,x,y} \right| \lesssim \kappa N^{-2} \epsilon_{S,T}^{-\frac{1}{2}} e^{\kappa N, \kappa}. \tag{5.42b}
\]

Moreover, we have the spatially-averaged estimates

\[
\sum_{y \in \mathbb{Z}} \left| \mathcal{G}^{N,0,2}_{S,T,x,y} \right| \epsilon_{S,T}^{N, \kappa} \lesssim \kappa N^{-1} \epsilon_{S,T}^{-\frac{1}{2}}. \tag{5.43a}
\]

\[
\sum_{y \in \mathbb{Z}} \left| \nabla_{k,y}^{\text{T}} \mathcal{G}^{N,0,2}_{S,T,x,y} \right| \epsilon_{S,T}^{N, \kappa} \lesssim \kappa N^{-1} \epsilon_{S,T}^{-1}. \tag{5.43b}
\]

**Proof.** We first observe that the spatially-averaged estimates follow from the pointwise estimates combined with elementary calculations. Moreover, we will assume \( \kappa = 0 \) purely for convenience; the subsequent analysis applies equally well for any \( \kappa \in \mathbb{R}_{>0} \) upon inserting additional \( \kappa \)-dependent constants. Courtesy of the Duhamel principle, we have

\[
\mathcal{G}^{N,0,2}_{S,T,x,y} = \int_S^T \sum_{w \in \mathbb{Z}} \mathcal{G}^{N,0}_{R,T,x,w} \mathcal{D}^N_Z \mathcal{G}^{N,0}_{S,R,w,y} \, dR. \tag{5.44}
\]
We decompose the integral within the RHS into an integral on \([S, \frac{T+S}{2}]\) and another on \([\frac{T+S}{2}, T]\). On the latter of the two, we simply apply Lemma B.4 and obtain

\[
\int_{\frac{T+S}{2}}^{T} \sum_{w \in \mathbb{Z}} \left| \mathcal{D}_{Z} \mathcal{G}_{R,T,x,w}^{N,0} \right| \mathcal{D}_{Z} \mathcal{G}_{S,R,w,y}^{N,0} \, dR \lesssim \int_{\frac{T+S}{2}}^{T} N^{-2} \mathcal{C}_{S}^{-2} \sum_{w \in \mathbb{Z}} \mathcal{G}_{R,T,x,w}^{N,0} \, dR \lesssim N^{-2} \mathcal{C}_{S,T}^{-1}.
\]  

(5.45)

(5.46)

On the former domain \([S, \frac{T+S}{2}]\), we cannot hope to apply Lemma B.4 in the same fashion and proceed similarly; indeed, this integral contains the non-integrable singularity of the resulting upper bound. We remedy this by exploiting the geometry \(\mathcal{Z}\) and the resulting self-adjoint property of all relevant second-order differential operators to move the differential operators onto the first heat kernel \(\mathcal{G}_{R,T}^{N,0}\) via summation-by-parts. We ultimately obtain the upper bound on this integral of

\[
\int_{S}^{\frac{T+S}{2}} \sum_{w \in \mathbb{Z}} \left| \mathcal{D}_{Z} \mathcal{G}_{R,T,x,w}^{N,0} \right| \mathcal{D}_{Z} \mathcal{G}_{S,R,w,y}^{N,0} \, dR \lesssim \int_{S}^{\frac{T+S}{2}} N^{-2} \mathcal{C}_{S}^{-2} \sum_{w \in \mathbb{Z}} \mathcal{G}_{R,T,x,w}^{N,0} \, dR \lesssim N^{-2} \mathcal{C}_{S,T}^{-1}.
\]  

(5.47)

(5.48)

to obtain the first inequality above, we have applied Lemma B.4 for the nearest-neighbor heat kernel \(\mathcal{G}_{R,T}^{N,0}\). Thus, combining all estimates thus far, we deduce the following bound with universal implied constant:

\[
\sup_{x,y \in \mathcal{Z}} \left| \mathcal{G}_{S,T,x,y}^{N,0} \right| \lesssim N^{-2} \mathcal{C}_{S,T}^{-1}.
\]  

(5.49)

The proof for the gradient estimate follows from almost identical considerations; this completes the proof. \(\square\)

As an immediate consequence of Lemma 5.12, we deduce the following consequence for the kernels \(\mathcal{U}^{N,0}\) and \(\mathcal{U}^{N,0}\).

\textbf{Corollary 5.13.} \textit{Retain the setting of Lemma 5.12; we have}

\[
\left| \mathcal{G}_{S,T,x,y}^{N,0} \right| \lesssim N^{-2} \mathcal{C}_{S,T}^{-1};
\]  

(5.50a)

\[
\left| \nabla_{k,y}^{l} \mathcal{G}_{S,T,x,y}^{N,0} \right| \lesssim N^{-2} \mathcal{C}_{S,T}^{-1};
\]  

(5.50b)

Moreover, we have the spatially-averaged estimates

\[
\sum_{y \in \mathcal{F}_{N,0}} \left| \mathcal{G}_{S,T,x,y}^{N,0} \right| \lesssim N^{-1} \mathcal{C}_{S,T}^{-1};
\]  

(5.51a)

\[
\sum_{y \in \mathcal{F}_{N,0}} \left| \nabla_{k,y}^{l} \mathcal{G}_{S,T,x,y}^{N,0} \right| \lesssim N^{-1} \mathcal{C}_{S,T}^{-1}.
\]  

(5.51b)

\textbf{Proof of Proposition 5.9.} The only necessary remark we have not yet made is to employ the precise gradient estimate from Lemma 5.3; otherwise, this follows from combining the aforementioned precise gradient estimate in Lemma 5.10, Lemma 5.12, and Corollary 5.13. \(\square\)

\textbf{5.3. Estimates for \(\mathcal{U}^{N}\).} We conclude with this last subsection on heat kernel estimates with extensions to the heat kernels satisfying arbitrary Robin boundary conditions; throughout the entirety of this subsection, the Robin boundary parameters \(\alpha, \beta \in \mathbb{R}\) will be implicit, and all estimates will depend on these parameters.

We begin by extending the mesoscopic regularity estimates for \(\mathcal{U}^{N,0}\) to the heat kernel \(\mathcal{U}^{N}\) in the following result.

\textbf{Lemma 5.14.} \textit{Consider Robin boundary parameters \(\alpha, \beta \in \mathbb{R}\). Moreover, consider \(S, T \in \mathbb{R}_{>0}\) satisfying \(S < T \leq T_{f}\) along with any \(x \in \mathcal{F}_{N,0}\). Moreover throughout the following list of bullet points, we choose an arbitrarily small although universal parameter \(\beta_{\delta} \in \mathbb{R}_{>0}\):}

\begin{itemize}
  \item \textit{First, provided any \(k \in \mathbb{Z}\) uniformly bounded, we have}
  \[
  \sum_{y \in \mathcal{F}_{N,0}} \left| \nabla_{k,y}^{l} \mathcal{U}^{N}_{S,T,x,y} \right| \lesssim_{m_{0}, \alpha, \beta_{\delta}, T_{f}, \mathcal{T}_{x,k}} \mathcal{C}_{S,T}^{-1} + e^{-\log^{10} N}.
  \]  
  \end{itemize}
• Second, provided any time-scale $\tau \in \mathbb{R}_{>0}$ satisfying $\tau \leq T_{S,T}$ along with any $\varepsilon \in \mathbb{R}_{>0}$ arbitrarily small but universal, we have
\[
\sum_{y \in \mathcal{F}_h \beta_j} \left| \mathcal{D}_T \mathcal{W}^N_{S,T,x,y} \right| \lesssim m_{S,T,x,y} \tau^{1-\varepsilon} \theta_{S,T}^{-1+\varepsilon} + N^{-1+2\varepsilon} \theta_{S,T}^{-1+\tau} + \tau. \tag{5.53}
\]

This can be upgraded to the pointwise estimate
\[
\left| \mathcal{D}_T \mathcal{W}^N_{S,T,x,y} \right| \lesssim m_{S,T,x,y} \tau^{1-\varepsilon} \theta_{S,T}^{-1+\varepsilon} + N^{-2+2\varepsilon} \theta_{S,T}^{-1} + N^{-1} \theta_{S,T}^{-1+\tau}. \tag{5.54}
\]

• Third, retaining the setting of the previous bullet point, we have
\[
\sum_{y \in \mathcal{F}_h \beta_j} \left| \nabla^1_{k,y} \mathcal{D}_T \mathcal{W}^N_{S,T,x,y} \right| \lesssim m_{S,T,x,y} \theta_{S,T}^{-1+\varepsilon} + N^{1-2\varepsilon} \theta_{S,T}^{-1} + N^{1-2\varepsilon} \theta_{S,T}^{-1+\tau}. \tag{5.55}
\]

\textbf{Proof.} For total clarity, we decompose the proof of Lemma 5.14 into the following list of bullet points written in the order the estimates above are presented. Moreover, the quantities $\Xi^{N,0}$ and $\Upsilon^{N,0}$ will be updated within each new bullet point.

• We first prove the spatially-averaged spatial-regularity estimate. To this end, we employ the perturbative Duhamel-type formula in Lemma 3.11; along with Lemma 5.3 this provides
\[
\sum_{y \in \mathcal{F}_h \beta_j} \left| \nabla^1_{k,y} \mathcal{W}^N_{S,T,x,y} \right| \lesssim m_{S,T,x,y} \sum_{y \in \mathcal{F}_h \beta_j} \left| \nabla^1_{k,y} \mathcal{W}^{N,0}_{S,T,x,y} \right| + \sum_{y \in \mathcal{F}_h \beta_j} \left| \Xi^{N,0}_{S,T,x,y} \right| + \sum_{y \in \mathcal{F}_h \beta_j} \left| \Upsilon^{N,0}_{S,T,x,y} \right| \tag{5.56}
\]
\[
\lesssim k \theta_{S,T}^{-1} + e^{-\log_{10} N} \sum_{y \in \mathcal{F}_h \beta_j} \left| \Xi^{N,0}_{S,T,x,y} \right| + \sum_{y \in \mathcal{F}_h \beta_j} \left| \Upsilon^{N,0}_{S,T,x,y} \right|, \tag{5.57}
\]

in which we have introduced the integrals
\[
\Xi^{N,0}_{S,T,x,y} \cdot \int_S^T \mathcal{W}^{N,0}_{R,T,x,y} \cdot N \left| \nabla^1_{k,y} \mathcal{W}^{N}_{S,R,0,y} \right| \, dR; \tag{5.58a}
\]
\[
\Upsilon^{N,0}_{S,T,x,y} \cdot \int_S^T \mathcal{W}^{N,0}_{R,T,x,y} \cdot N \left| \nabla^1_{k,y} \mathcal{W}^{N}_{S,R,N,y} \right| \, dR. \tag{5.58b}
\]

In particular, we deduce from this previous calculation combined with the Nash-type heat kernel estimate within Lemma 4.1 that
\[
\sum_{y \in \mathcal{F}_h \beta_j} \left| \nabla^1_{k,y} \mathcal{W}^N_{S,T,x,y} \right| \lesssim m_{S,T,x,y} \theta_{S,T}^{-1} + \int_S^T \theta_{R,T}^{-1} \sum_{y \in \mathcal{F}_h \beta_j} \left| \nabla^1_{k,y} \mathcal{W}^{N}_{S,R,x,y} \right| \, dR. \tag{5.59}
\]

The spatially-averaged spatial-regularity estimate is now consequence of the singular Gronwall inequality.

• We proceed with establishing the spatially-averaged time-regularity estimate. We once again employ Lemma 3.11:
\[
\sum_{y \in \mathcal{F}_h \beta_j} \left| \mathcal{D}_T \mathcal{W}^N_{S,T,x,y} \right| \lesssim m_{S,T,x,y} \sum_{y \in \mathcal{F}_h \beta_j} \left| \mathcal{D}_T \mathcal{W}^{N,0}_{S,T,x,y} \right| + \sum_{y \in \mathcal{F}_h \beta_j} \left| \Xi^{N,0}_{S,T,x,y} \right| + \sum_{y \in \mathcal{F}_h \beta_j} \left| \Upsilon^{N,0}_{S,T,x,y} \right| \tag{5.60}
\]
\[
\lesssim \theta_{S,T}^{-1+\varepsilon} + N^{-1+2\varepsilon} \theta_{S,T}^{-1} + N^{-\varepsilon} \tau + \sum_{y \in \mathcal{F}_h \beta_j} \left| \Xi^{N,0}_{S,T,x,y} \right| + \sum_{y \in \mathcal{F}_h \beta_j} \left| \Upsilon^{N,0}_{S,T,x,y} \right|, \tag{5.61}
\]

where the final estimate is consequence of (5.23), and we have introduced the integrals
\[
\Xi^{N,0}_{S,T,x,y} \cdot \int_S^T \mathcal{W}^{N,0}_{R,T,x,y} \cdot N \left| \mathcal{D}_T \mathcal{W}^{N}_{S,R,0,y} \right| \, dR; \tag{5.62}
\]
\[
\Upsilon^{N,0}_{S,T,x,y} \cdot \int_S^T \mathcal{W}^{N,0}_{R,T,x,y} \cdot N \left| \mathcal{D}_T \mathcal{W}^{N}_{S,R,N,y} \right| \, dR. \tag{5.63}
\]

In particular, proceeding exactly as in the previous bullet point completes the proof.

• To prove the pointwise time-regularity estimate, we proceed exactly as in the proof of Lemma 5.4 via the Chapman-Kolmogorov equation combined with the Nash-type heat kernel estimate in Lemma 4.4.
Finally, to prove the spatial-temporal regularity estimate, we proceed exactly as in the proof of Lemma 5.5 via the Chapman-Kolmogorov equation together with the spatially-averaged spatial-regularity estimate and the spatially-averaged time-regularity estimate of this result.

This completes the proof.

We conclude the current section with an analog of the comparison estimate in Proposition 5.9 for the heat kernel \( \mathcal{U}^N \) with any generic Robin boundary conditions. To this end, we proceed to introduce the following heat kernel \( \bar{\mathcal{U}}^N \) associated to the specialized nearest-neighbor parabolic problem on \( \mathbb{R}_{\geq 0} \times \mathcal{A}_{N,0} \).

**Notation 5.15.** Provided any \( (S, T) \in \mathbb{R}_{\geq 0} \) and \( x, y \in \mathcal{A}_{N,0} \), define the heat kernel \( \bar{\mathcal{U}}^N \) via the following integral equation:

\[
\bar{\mathcal{U}}^N_{S,T,x,y} = \mathcal{U}^N_{S,T,x,y} - \int_S^T \mathcal{U}^N_{R,T,x,y} \left[ N \mathcal{A}_+ \mathcal{U}^N_{R,T,0,y} \right] dR - \int_S^T \mathcal{U}^N_{R,T,x,N} \left[ N \mathcal{A}_+ \mathcal{U}^N_{R,T,0,y} \right] dR. \quad (5.64)
\]

Alternatively, \( \bar{\mathcal{U}}^N \) is the heat kernel associated to the parabolic problem:

\[
\begin{align*}
\partial_t \bar{\mathcal{U}}^N_{S,T,x,y} &= \mathcal{L}^N_{\text{lap}} \bar{\mathcal{U}}^N_{S,T,x,y}; \\
\bar{\mathcal{U}}^N_{S,x,y} &= 1_x \delta_y; \\
\bar{\mathcal{U}}^N_{S,T,-1,y} &= \mu \mathcal{A}_+ \bar{\mathcal{U}}^N_{S,T,-1,y}; \\
\bar{\mathcal{U}}^N_{S,T,N+1,y} &= \mu \mathcal{A}_+ \bar{\mathcal{U}}^N_{S,T,N+y}.
\end{align*}
\]

Moreover, we define

\[
\mathcal{D}^N_{S,T,x,y} = \mathcal{U}^N_{S,T,x,y} - \bar{\mathcal{U}}^N_{S,T,x,y}. \quad (5.66)
\]

**Remark 5.16.** We briefly note that the kernel \( \bar{\mathcal{U}}^N \) benefits from the heat kernel estimates within Lemma 4.4 upon realizing this heat kernel as the nearest-neighbor specialization of \( \mathcal{U}^N \) up to uniformly elliptic and uniformly bounded time-change.

The proposed analog of Proposition 5.9 for any generic Robin boundary parameters are the following estimates, which admit a perspective as a perturbative deduction of Proposition 5.9 itself.

**Proposition 5.17.** The estimates within Proposition 5.9 remain valid upon replacing \( \mathcal{D}^N_{0} \) with \( \mathcal{D}^N \).

**Proof.** Towards organizational clarity, we split the proof of Proposition 5.17 into a list of bullet points, one for each proposed estimate. The starting point for each estimate is the following dynamical integral equation for \( \mathcal{D}^N \):

\[
\begin{align*}
\mathcal{D}^N_{S,T,x,y} &= \mathcal{D}^N_{S,T,x,y} - \int_S^T \mathcal{D}^N_{R,T,x,y} \left[ N \mathcal{A}_+ \mathcal{D}^N_{R,T,0,y} \right] dR - \int_S^T \mathcal{D}^N_{R,T,x,N} \left[ N \mathcal{A}_+ \mathcal{D}^N_{R,T,0,y} \right] dR \\
&\quad - \int_S^T \mathcal{D}^N_{R,T,x,N} \left[ N \mathcal{A}_+ \mathcal{D}^N_{R,T,0,y} \right] dR - \int_S^T \mathcal{D}^N_{R,T,x,N} \left[ N \mathcal{A}_+ \mathcal{D}^N_{R,T,0,y} \right] dR. \quad (5.67)
\end{align*}
\]

- To establish a space-time pointwise estimate for \( \mathcal{D}^N \), we simply view (5.67) as a perturbation of (5.67). In particular, upon implementing the heat kernel estimates in both of Lemma 4.1 and Lemma 4.4 combined with the comparison estimate in Proposition 5.9, we have the following estimate:

\[
\begin{align*}
\mathcal{V}^N_{S,T} &\leq \mathcal{D}^N_{S,T} + \int_S^T \mathcal{D}^N_{S,R} dR + \int_S^T e^{-\frac{1}{2} \mathcal{D}^N_{R,T} \mathcal{D}^N_{S,R}} dR. \quad (5.68)
\end{align*}
\]

in which

\[
\begin{align*}
\mathcal{V}^N_{S,T} &\equiv \sup_{x,y \in \mathcal{A}_{N,0}} \left| \mathcal{D}^N_{S,T,x,y} \right|; \\
\mathcal{D}^N_{S,T} &\equiv \left[ N^{-1-r} \mathcal{D}^N_{S,T} + N^{-2} \mathcal{D}^N_{S,T} + e^{-\log^{10} N} \right] \wedge 1. \quad (5.69a)
\end{align*}
\]
Because $\Phi^N$ admits the cutoff near the potential short-time singularity, this $\Phi^N$ quantity is integrable with integral satisfying the estimate
\[
\int_S^T \Phi^N_{S,R} \, dR \lesssim \varepsilon \, N^{-1-\varepsilon}. \tag{5.70}
\]
In particular, the desired estimate on $\mathcal{T}^N_{S,T}$ follows from the singular Gronwall inequality; see Lemma C.1.

- To obtain the spatially-averaged estimate, we simply follow the previous bullet point upon the replacements
  \[
  \mathcal{T}^N_{S,T} = \sup_{x \in \mathcal{F}_{N,R}, y \in \mathcal{F}_{N,0}} \sum_{x \times y} \left| \mathcal{G}^N_{S,T,x,y} \right|; \tag{5.71a}
  \]
  \[
  \Phi^N_{S,T} = N^{-\varepsilon} \mathcal{G}^N_{S,T} + N^{-1} \mathcal{G}^N_{S,T} + e^{-\log^{100} N}. \tag{5.71b}
  \]
- To obtain the spatially-averaged gradient estimate, we again follow the procedure detailed in the first bullet point upon the replacements
  \[
  \mathcal{T}^N_{S,T} = \sup_{x \in \mathcal{F}_{N,R}, y \in \mathcal{F}_{N,0}} \sum_{x \times y} \left| \nabla_k \mathcal{G}^N_{S,T,x,y} \right|; \tag{5.72a}
  \]
  \[
  \Phi^N_{S,T} = \left[ N^{-\varepsilon} \mathcal{G}^N_{S,T} + N^{-1} \mathcal{G}^N_{S,T} \right] \wedge N. \tag{5.72b}
  \]
This completes the proof. \[\square\]

To illustrate the utility of this auxiliary heat kernel $\mathcal{G}^N$, we include one more regularity estimate.

**Lemma 5.18.** Moreover, provided any Robin boundary parameters $\mathcal{A}_\pm \in \mathbb{R}$, we have
\[
\sup_{x \in \mathcal{F}_{N,R}, y \in \mathcal{F}_{N,0}} \sum_{x \times y} \left| \nabla_k \mathcal{G}^N_{S,T,x,y} \right| \lesssim \left( k, T, m, \mathcal{A}_\pm, \varepsilon \right) \, N^{-1+2\varepsilon} \, \mathcal{G}^{1+\varepsilon}_{S,T}. \tag{5.73}
\]

### 6. Stochastic Regularity Estimates

The current section is dedicated towards important regularity estimates for the microscopic Cole-Hopf transform. Unlike previous articles including [3], [6], [9], [22], for example, although similar to [23], these regularity estimates will actually not yield macroscopic regularity estimates for the continuum limit. Instead, these results will be precise estimates employed for analyzing the microscopic Cole-Hopf transform at mesoscopic scales as crucial ingredients towards our dynamical one-block strategy.

#### 6.1. Stochastic Fundamental Solution

The first component to our analysis in this section is to understand the solution to the following integral equation.

**Notation 6.1.** Consider any Robin boundary parameters $\mathcal{A}_\pm \in \mathbb{R}$. Given any arbitrarily small but universal constant $\beta_D \in \mathbb{R}_{>0}$, we define $\mathcal{Q}^N_{S,T}$ as the unique solution to
\[
\mathcal{Q}^N_{S,T,x,y} = \mathcal{Q}^N_{S,T,x,y}^{1} + \mathcal{Q}^N_{S,T,x,y}^{2} + \mathcal{Q}^N_{S,T,x,y}^{3} + \mathcal{Q}^N_{S,T,x,y}^{4}, \tag{6.1}
\]
where
\[
\mathcal{Q}^N_{S,T,x,y}^{1} = \int_S^T \sum_{x \in \mathcal{F}_{N,R}, y \in \mathcal{F}_{N,0}} \mathcal{Q}^N_{S,R,x,w} \left[ \mathcal{Q}^N_{S,R,y,w} \, d\mathcal{G}^N_{R,w} \right]; \tag{6.2a}
\]
\[
\mathcal{Q}^N_{S,T,x,y}^{2} = \int_S^T \sum_{x \in \mathcal{F}_{N,R}, y \in \mathcal{F}_{N,0}} \mathcal{Q}^N_{S,R,x,w} \left[ \mathcal{Q}^N_{S,R,y,w} \, d\mathcal{G}^N_{R,w} \right] \, dR; \tag{6.2b}
\]
\[
\mathcal{Q}^N_{S,T,x,y}^{3} = \sum_{|k| \leq m} c_k \int_S^T \sum_{x \in \mathcal{F}_{N,R}, y \in \mathcal{F}_{N,0}} \mathcal{Q}^N_{S,R,x,w} \left[ \mathcal{Q}^N_{S,R,y,w} \, d\mathcal{G}^N_{R,w} \right] \, dR; \tag{6.2c}
\]
\[
\mathcal{Q}^N_{S,T,x,y}^{4} = \int_S^T \sum_{x \in \mathcal{F}_{N,R}, y \in \mathcal{F}_{N,0}} \mathcal{Q}^N_{S,R,x,w} \left[ \mathcal{Q}^N_{S,R,y,w} \, d\mathcal{G}^N_{R,w} \right] \, dR; \tag{6.2d}
\]
to be completely precise, we have the defined following quantity with an extension from the boundary to all of $\mathcal{S}_{N,0}$ by 0:

$$w_{R,w}^{N,\pm} \equiv m_{R,w}^{N,+} + m_{R,w}^{N,-}.$$  \hspace{1cm} (6.3)

The allowed space-time coordinates are provided by $S, T \in \mathbb{R}_{>0}$ satisfying $S \leq T \leq T_f$, and spatial coordinates $x, y \in \mathcal{S}_{N,0}$.

The entire philosophy behind considering this stochastic fundamental solution $\mathcal{Q}_{S,T}^N$ is to rewrite the microscopic Cole-Hopf transform $Z_N$ as the solution to a stochastic integral equation with propagator $\mathcal{Q}_{S,T}^N$, indeed, this allows to separate our hydrodynamical analysis which is mainly concerned with those quantities $\mathcal{Q}_{S,T}^{N,1}, \ldots, \mathcal{Q}_{S,T}^{N,4}$ defined above and our dynamical one-block analysis which concerns with the remaining quantities from the RHS of the evolution equation in Proposition 2.11. In particular, the latter dynamical one-block strategy is considerably delicate to implement, and actually dealing with both the aforementioned hydrodynamical analysis as well as the SPDE theory of the component to the RHS of the evolution equation in Proposition 2.11 providing the limiting SHE is quite complicated.

Moreover, it turns out that the only requirements for the propagator to implement the dynamical one-block strategy is suitable mesoscopic regularity estimates of this propagator that are uniform in space-time with high-probability. The first of these estimates is in the following result; this result is written as simply a pointwise $L_{p}^{2}$-estimate on the kernel $\mathcal{Q}_{S,T}^{N}$, but we later upgrade this to a space-time uniform estimate with high-probability with only slightly worse $N$-dependence.

**Lemma 6.2.** Consider any Robin boundary parameters $\mathcal{Q}_x \in \mathbb{R}$. Provided times $S, T \in \mathbb{R}_{>0}$ satisfying $S \leq T \leq T_f$ along with any $x, y \in \mathcal{S}_{N,0}$, the following estimate holds provided any $p \in \mathbb{R}_{>0}$ and any $\epsilon \in \mathbb{R}_{>0}$:

$$\left\| \mathcal{Q}_{S,T,x,y}^{N} \right\|_{L_{p}^{2}} \lesssim m_{x,y,T,T_f,\beta_1,\mathcal{Q}_x} N^{-1+\epsilon} \mathcal{Q}_{S,T}^{N-\frac{1}{2}+\epsilon}.$$  \hspace{1cm} (6.4)

Moreover, we have the spatially-averaged estimate

$$\sum_{y \in \mathcal{S}_{N,0}} \left\| \mathcal{Q}_{S,T,x,y}^{N} \right\|_{L_{p}^{2}}^{2} \lesssim m_{x,p,\beta,\mathcal{Q}_x} N^{-1+\epsilon} \mathcal{Q}_{S,T}^{N-\frac{1}{2}+\epsilon}.$$  \hspace{1cm} (6.5)

**Proof.** We observe that the heat kernel $\mathcal{Q}_{S,T,x,y}^{N}$ is deterministic, and the desired pointwise estimate for this quantity follows directly from Lemma 4.4. It remains to analyze those $L_{p}^{2}$-norms of each of $\mathcal{Q}_{S,T}^{N,1}, \mathcal{Q}_{S,T}^{N,2}, \mathcal{Q}_{S,T}^{N,3}, \mathcal{Q}_{S,T}^{N,4}$. Concerning the first of these, we employ Lemma 3.1 in [9] to obtain

$$\left\| \mathcal{Q}_{S,T,x,y}^{N,1} \right\|_{L_{p}^{2}}^{2} \lesssim m_{x,y,T_f,\mathcal{Q}_x} \int_{S}^{T} \sum_{w \in \mathcal{S}_{N,0}} \mathcal{Q}_{R,T,w}^{N} \left\| \mathcal{Q}_{S,R,w,y}^{N} \right\|_{L_{p}^{2}}^{2} dR + N^{-1} \sup_{w \in \mathcal{S}_{N,0}} \left\| \mathcal{Q}_{S,T,w,y}^{N} \right\|_{L_{p}^{2}}^{2}$$  \hspace{1cm} (6.6)

$$\lesssim m_{x,y,T_f,\mathcal{Q}_x} \int_{S}^{T} \sum_{w \in \mathcal{S}_{N,0}} \mathcal{Q}_{R,T,w}^{N} \left\| \mathcal{Q}_{S,R,w,y}^{N} \right\|_{L_{p}^{2}}^{2} dR + N^{-1} \sup_{w \in \mathcal{S}_{N,0}} \left\| \mathcal{Q}_{S,T,w,y}^{N} \right\|_{L_{p}^{2}}^{2};$$  \hspace{1cm} (6.7)

this final estimate above is correct of the on-diagonal Nash-type heat kernel estimate within Lemma 4.4 combined with observation that $S, T \leq T_f$, so that in particular the only relevant factor within the upper bound of Lemma 4.4 is that with the short-time singularity.

We now estimate the second quantity $\mathcal{Q}_{S,T}^{N,2}$, the uniform upper bound for $m_{N}^{N}$ provides the straightforward inequality

$$\left\| \mathcal{Q}_{S,T,x,y}^{N,2} \right\|_{L_{p}^{2}}^{2} \lesssim m_{N}^{N} \left[ \int_{S}^{T} \sum_{w \in \mathcal{S}_{N,0}} \mathcal{Q}_{R,T,w}^{N} \left\| \mathcal{Q}_{S,R,w,y}^{N} \right\|_{L_{p}^{2}}^{2} dR \right]^{2}$$  \hspace{1cm} (6.8)

$$\lesssim m_{N}^{N} \left[ \int_{S}^{T} \sum_{w \in \mathcal{S}_{N,0}} \mathcal{Q}_{R,T,w}^{N} \left\| \mathcal{Q}_{S,R,w,y}^{N} \right\|_{L_{p}^{2}}^{2} dR \right];$$  \hspace{1cm} (6.9)

indeed, the latter upper bound is the consequence of the Cauchy-Schwarz inequality applied with respect to the space-time kernel $\mathcal{Q}_{S,T}^{N}$; to this end, observe that the total space-time mass of $\mathcal{Q}_{S,T}^{N}$ is bounded uniformly above by a universal constant depending only on $T_f \in \mathbb{R}_{>0}$ and $\mathcal{Q}_x \in \mathbb{R}$ courtesy of the heat kernel estimates within Lemma 4.4.
Concerning the third quantity of $Q_{S,T}^{N,3}$, we require several intermediate steps before ultimately establishing a suitable $L^2_{\omega}$-estimate. First, following the proof of Lemma 3.2, summation-by-parts provides the adjoint calculation

$$Q_{S,T}^{N,3} = \sum_{|k| \leq n_k} c_k \int_{S} \left( \sum_{w \in F_{N,\delta}} \nabla^{1}_{-k,w} N_{R,T,x,w} \cdot \left[ w R_{S,w}^{-1} \phi_{S}^{N} \right] \right) dR. \quad (6.10)$$

Indeed, the complications arising from the boundary are completely absent courtesy of the cutoff from $1_{w \in F_{N,\delta}}$ within the gradient operator. Thus, courtesy of the strategy via Cauchy-Schwarz inequality used to estimate $Q_{S,T}^{N,2}$ combined with the gradient estimate of Lemma 5.14, we obtain

$$\left\| Q_{S,T,x,y}^{N} \right\|^2_{L^2_{\omega}} \lesssim \sum_{|k| \leq n_k} |c_k| \left[ \int_{S} \left( \sum_{w \in F_{N,\delta}} \left\| \nabla^{1}_{-k,w} N_{R,T,x,w} \right\| dR \right) \right]^2 \quad (6.11)$$

$$\lesssim_{T,T,\beta_1} \sum_{|k| \leq n_k} |k| |c_k| \int_{S} \left( \sum_{w \in F_{N,\delta}} \left\| \nabla^{1}_{-k,w} N_{R,T,x,w} \right\| \right)^2 dR. \quad (6.12)$$

Lastly, we estimate the quantity $Q_{S,T}^{N,4}$. To this end, we proceed as with the quantity $Q_{S,T}^{N,2}$ via the Cauchy-Schwarz inequality in combination with the heat kernel estimate in Lemma 4.4 as follows:

$$\left\| Q_{S,T,x,y}^{N} \right\|^2_{L^2_{\omega}} \lesssim_{w,w_{\beta},m_{\beta}} \left[ N \int_{S} \left( \sup_{w \in F_{N,0}} \left\| Q_{S,R,T,x,y}^{N} \right\| \right)^2 dR \right]^2 \quad (6.13)$$

$$\lesssim_{T} \int_{S} \left( \sup_{w \in F_{N,0}} \left\| Q_{S,R,T,x,y}^{N} \right\| \right)^2 dR. \quad (6.14)$$

We combine the estimates (6.7), (6.9), (6.12), and (6.14) to obtain the following estimate for any $\varepsilon \in \mathbb{R}_{>0}$:

$$\left\| Q_{S,T,x,y}^{N} \right\|^2_{L^2_{\omega}} \lesssim_{p,T,T_1,\beta_1,\delta,\varepsilon_1} \left\| Q_{S,T,x,y}^{N} \right\|^2_{L^2_{\omega}} + \int_{S} \left( \sup_{w \in F_{N,0}} \left\| Q_{S,R,T,x,y}^{N} \right\| \right)^2 dR + N^{1+\varepsilon} \sup_{w \in F_{N,0}} \left\| Q_{S,R,T,x,y}^{N} \right\|^2_{L^2_{\omega}} \quad (6.15)$$

$$\lesssim_{T,T_1,\delta} N^{-2+2\varepsilon_{1}} Q_{S,T}^{-1+\varepsilon} + \int_{S} \left( \sup_{w \in F_{N,0}} \left\| Q_{S,R,T,x,y}^{N} \right\| \right)^2 dR + N^{1+\varepsilon} \sup_{w \in F_{N,0}} \left\| Q_{S,R,T,x,y}^{N} \right\|^2_{L^2_{\omega}} \quad (6.16)$$

indeed, the final estimate is the consequence of Lemma 4.4 upon additional interpolation between the two upper bounds therein. To complete the proof, it suffices to replace each $L^2_{\omega}$-norm in the estimate (6.16) with its spatial-maximum and apply both the gradient estimate in Lemma 5.14 again along with the singular Gronwall inequality.

The proof of the spatially-averaged $L^2_{\omega}$-norm follows from identical considerations. \[\square\]

Although Lemma 6.2 will be important in its own right towards our proof of Theorem 1.8, it also serves as an important a priori estimate for a time-regularity estimate for $Q_{S}^{N}$. First, we introduce convenient notation.

**Notation 6.3.** Provided any $S,T \in \mathbb{R}_{>0}$ and $\tau \in \mathbb{R}_{>0}$ satisfying $S \leq T \leq T_{\tau}$ and $\partial S \geq \tau$, along with any $\kappa \in \mathbb{R}_{>0}$:

$$\Psi_{S,T,\tau}^{N,\kappa} \triangleq \sup_{w \in F_{N,0}} \left\| Q_{S,T,x,y}^{N} \right\|^2_{L^2_{\omega}}. \quad (6.17)$$

**Lemma 6.4.** Consider times $S,T \in \mathbb{R}_{>0}$ satisfying $S \leq T \leq T_{\tau}$. Moreover, let us suppose $T_{\tau}, m_{\theta} \lesssim 1$ with a universal implied constant. Provided $\tau \in \mathbb{R}_{>0}$ satisfying $N^{-2} \lesssim \tau \lesssim \frac{1}{2} Q_{S,T}^{N}$, for any $\varepsilon_{1}, \varepsilon_{2}, \varepsilon_{3} \in \mathbb{R}_{>0}$ arbitrarily small but universal, we have

$$\Psi_{S,T,\tau}^{N,\kappa} \lesssim \varepsilon_{1}, \varepsilon_{2}, \varepsilon_{3} \tau^2 + N^{-2+\varepsilon_{1}} Q_{S,T}^{-1+\varepsilon_{1}} \quad (6.18)$$

$$\lesssim \varepsilon_{1}, \varepsilon_{2}, \varepsilon_{3} N^{-2+\varepsilon_{1}} Q_{S,T}^{-1+\varepsilon_{1}} \quad (6.19)$$
Proof. By definition, we compute \( \mathcal{D}_\tau \mathcal{Q}^N = \mathcal{D}_\tau \mathcal{Q}^N + \sum_{j=1}^A \mathcal{D}_\tau \mathcal{Q}^{N,j} \); decomposing further, for \( j \in [1, 3] \), we have another decomposition of

\[
\mathcal{D}_\tau \mathcal{Q}^{N,j} = \mathcal{Q}^{N,j,1,\tau} + \mathcal{Q}^{N,j,2,\tau},
\]

where the new space-time kernels are defined by

\[
\mathcal{Q}^{N,j,1,\tau}_{S,T,x,y} = \int_{T}^{S+\tau} \sum_{w \in \mathcal{F}_{N,0}} \mathcal{W}^N_{R,T,x,w} \left[ \mathcal{D}_\tau \mathcal{Q}^N_{S,R,w,y} \right] \; d\xi^N_{R,w};
\]

\[
\mathcal{Q}^{N,j,2,\tau}_{S,T,x,y} = \int_{S}^{S+\tau} \sum_{w \in \mathcal{F}_{N,0}} \mathcal{W}^N_{R,T,x,w} \left[ \mathcal{D}_\tau \mathcal{Q}^N_{S,R,w,y} \right] \; d\xi^N_{R,w};
\]

\[
\mathcal{Q}^{N,j,3,\tau}_{S,T,x,y} = \sum_{|k| \leq m_N} c_k \int_{S+\tau}^{T} \sum_{w \in \mathcal{F}_{N,0}} \mathcal{W}^N_{R,T,x,w} \left[ \nabla \left( \mathcal{1}_{w \in \mathcal{F}_{N,0}} w^N_{R,w} \mathcal{D}_\tau \mathcal{Q}^N_{S,R,w,y} \right) \right] \; d\xi^N_{R,w};
\]

\[
\mathcal{Q}^{N,j,4,\tau}_{S,T,x,y} = \int_{S+\tau}^{T} \sum_{w \in \mathcal{F}_{N,0}} \mathcal{W}^N_{R,T,x,w} \left[ \mathcal{D}_\tau \mathcal{Q}^N_{S,R,w,y} \right] \; d\xi^N_{R,w};
\]

\[
\mathcal{Q}^{N,j,5,\tau}_{S,T,x,y} = \int_{S}^{S+\tau} \sum_{w \in \mathcal{F}_{N,0}} \mathcal{W}^N_{R,T,x,w} \left[ N w^N_{R,w} \mathcal{D}_\tau \mathcal{Q}^N_{S,R,w,y} \right] \; d\xi^N_{R,w};
\]

Thus, the remainder of this argument amounts to estimating each of the above six quantities, combined with the following upper bound on the deterministic heat kernel, provided any \( \epsilon \in \mathbb{R}_{>0} \) sufficiently small though universal courtesy of Lemma 5.14:

\[
\sum_{y \in \mathcal{F}_{N,0}} \left| \mathcal{D}_\tau \mathcal{Q}^N_{S,T,x,y} \right|^2 \lesssim_{\epsilon} N^{-1+2\epsilon} \Theta_{S,T}^{2+2\epsilon} + N^{-2+4\epsilon} \Theta_{S,T}^{-1+\epsilon};
\]

indeed, even if \( \tau \geq 7\Theta_{S,T} \), the estimate is true courtesy of Lemma 4.4.

Second, courtesy of Lemma 3.1 in [9], which certainly remains valid even outside the full-lattice, we obtain the following stochastic estimate provided any \( p \in \mathbb{R}_{>1} \):

\[
\sum_{y \in \mathcal{F}_{N,0}} \left\| \mathcal{Q}^{N,1,\tau}_{S,T,x,y} \right\|_{\mathcal{L}^p_{\mathcal{H}}}^2 \lesssim_{p,\epsilon} \int_{T}^{S+\tau} N \sum_{w \in \mathcal{F}_{N,0}} \left\| \mathcal{W}^N_{R,T,x,w} \right\|_{\mathcal{L}^p_{\mathcal{H}}}^2 \sum_{y \in \mathcal{F}_{N,0}} \left\| \mathcal{D}_\tau \mathcal{Q}^N_{S,R,w,y} \right\|_{\mathcal{L}^{p,2}_{\mathcal{H}}}^2 \; d\xi^N_{R,w} + N^{-1+2\epsilon} \Theta_{S,T}^{2+2\epsilon};
\]

\[
\sum_{y \in \mathcal{F}_{N,0}} \left\| \mathcal{Q}^{N,2,\tau}_{S,T,x,y} \right\|_{\mathcal{L}^p_{\mathcal{H}}}^2 \lesssim_{p,\epsilon} \int_{S+\tau}^{T} \sum_{w \in \mathcal{F}_{N,0}} \left\| \mathcal{W}^N_{R,T,x,w} \right\|_{\mathcal{L}^p_{\mathcal{H}}}^2 \sum_{y \in \mathcal{F}_{N,0}} \left\| \mathcal{D}_\tau \mathcal{Q}^N_{S,R,w,y} \right\|_{\mathcal{L}^{p,2}_{\mathcal{H}}}^2 \; d\xi^N_{R,w} + N^{-1+2\epsilon} \Theta_{S,T}^{2+2\epsilon};
\]

indeed, the quantity \( N^{-1} \Theta_{S,T}^{2+2\epsilon} \) arising from counting the total number of jumps per site is defined using universal implied constant, and its validity as an upper bound relies on the bound \( \tau \geq N^{-2} \) and the a priori estimate in Lemma 6.2, giving

\[
\sum_{y \in \mathcal{F}_{N,0}} \left\| \mathcal{D}_\tau \mathcal{Q}^N_{S,R,w,y} \right\|_{\mathcal{L}^p_{\mathcal{H}}}^2 \lesssim_{p,\epsilon} N^{-1+2\epsilon} \Theta_{S,T}^{2+2\epsilon}.
\]

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Along a similar line with another application of Lemma 6.2 and the integration-lemma in Lemma C.1, we obtain
\[
\sum_{y \in \mathcal{F}_{N,0}} \left\| \mathcal{Q}^{N,1,2,\tau}_{S,T,x,y} \right\|_{\mathcal{L}^2_{\tau,x}}^2 \lesssim \int_{S}^{T} \frac{1}{R_{T}} \sum_{w \in \mathcal{F}_{N,0}} \sum_{y \in \mathcal{F}_{N,0}} \left\| \mathcal{Q}^{N}_{S,R,x,w} \right\|_{\mathcal{L}^2_{\tau,x}}^2 \, dR + N^{-1+2\epsilon} \frac{1}{\theta_{S,T}} \lesssim \tau \tag{6.26}
\]
\[
\lesssim \int_{S}^{T} \frac{1}{R_{T}} \sum_{w \in \mathcal{F}_{N,0}} \sum_{y \in \mathcal{F}_{N,0}} \left\| \mathcal{Q}^{N}_{S,R,x,w} \right\|_{\mathcal{L}^2_{\tau,x}}^2 \, dR + N^{-1+2\epsilon} \frac{1}{\theta_{S,T}} \tau^{-1-\epsilon} \tag{6.27}
\]
\[
\lesssim \tau_{T} N^{-1+2\epsilon} \frac{1}{\theta_{S,T}} \tau^{-1} + N^{-1+2\epsilon} \theta_{S,T}^{-1-\epsilon}. \tag{6.28}
\]
We proceed to analyze the quantities \{ \mathcal{Q}^{N,2,j,\tau}_{T} \}_{j=1,2}. To this end, we first observe that Proposition 4.5 allows us to employ the Cauchy-Schwarz inequality with respect to the space-time “integral” and deduce the upper bounds valid for any \( \epsilon \in \mathbb{R}_{>0} \) arbitrarily small but universal:
\[
\sum_{y \in \mathcal{F}_{N,0}} \left\| \mathcal{Q}^{N,2,1,\tau}_{S,T,x,y} \right\|_{\mathcal{L}^2_{\tau,x}}^2 \lesssim \int_{S}^{T} \frac{1}{R_{T}} \sum_{w \in \mathcal{F}_{N,0}} \sum_{y \in \mathcal{F}_{N,0}} \left\| \mathcal{Q}^{N}_{S,R,x,w} \right\|_{\mathcal{L}^2_{\tau,x}}^2 \, dR; \tag{6.29}
\]
\[
\sum_{y \in \mathcal{F}_{N,0}} \left\| \mathcal{Q}^{N,2,2,\tau}_{S,T,x,y} \right\|_{\mathcal{L}^2_{\tau,x}}^2 \lesssim \int_{S}^{T} \frac{1}{R_{T}} \sum_{w \in \mathcal{F}_{N,0}} \sum_{y \in \mathcal{F}_{N,0}} \left\| \mathcal{Q}^{N}_{S,R,x,w} \right\|_{\mathcal{L}^2_{\tau,x}}^2 \, dR \tag{6.30}
\]
\[
\lesssim \int_{S}^{T} \frac{1}{R_{T}} \sum_{w \in \mathcal{F}_{N,0}} \sum_{y \in \mathcal{F}_{N,0}} \left\| \mathcal{Q}^{N}_{S,R,x,w} \right\|_{\mathcal{L}^2_{\tau,x}}^2 \, dR \tag{6.31}
\]
\[
\lesssim N^{-1+4\epsilon} \tau^{1+2\epsilon}. \tag{6.32}
\]
Indeed, the estimate for \( \mathcal{Q}^{N,2,2,\tau}_{S,T,x,y} \) requires the off-diagonal estimate in Lemma 6.2. Analyzing the quantities \{ \mathcal{Q}^{N,3,j,\tau}_{T} \}_{j=1,2} requires the same procedure adopted above towards the quantities \{ \mathcal{Q}^{N,2,j,\tau}_{T} \}_{j=1,2} upon recalling the gradient estimate in Lemma 5.14 for the heat kernel \( \mathcal{W}^{N} \). In particular, we obtain the following for any \( \epsilon \in \mathbb{R}_{>0} \) arbitrarily small but universal:
\[
\sum_{y \in \mathcal{F}_{N,0}} \left\| \mathcal{Q}^{N,3,1,\tau}_{S,T,x,y} \right\|_{\mathcal{L}^2_{\tau,x}}^2 \lesssim \int_{S}^{T} \frac{1}{R_{T}} \sum_{w \in \mathcal{F}_{N,0}} \sum_{y \in \mathcal{F}_{N,0}} \left\| \mathcal{Q}^{N}_{S,R,x,w} \right\|_{\mathcal{L}^2_{\tau,x}}^2 \, dR; \tag{6.33}
\]
\[
\sum_{y \in \mathcal{F}_{N,0}} \left\| \mathcal{Q}^{N,3,2,\tau}_{S,T,x,y} \right\|_{\mathcal{L}^2_{\tau,x}}^2 \lesssim \int_{S}^{T} \frac{1}{R_{T}} \sum_{w \in \mathcal{F}_{N,0}} \sum_{y \in \mathcal{F}_{N,0}} \left\| \mathcal{Q}^{N}_{S,R,x,w} \right\|_{\mathcal{L}^2_{\tau,x}}^2 \, dR \tag{6.34}
\]
\[
\lesssim \int_{S}^{T} \frac{1}{R_{T}} \sum_{w \in \mathcal{F}_{N,0}} \sum_{y \in \mathcal{F}_{N,0}} \left\| \mathcal{Q}^{N}_{S,R,x,w} \right\|_{\mathcal{L}^2_{\tau,x}}^2 \, dR \tag{6.35}
\]
\[
\lesssim N^{-1+4\epsilon} \tau^{1+2\epsilon}. \tag{6.36}
\]
Finally, we analyze \{ \mathcal{Q}^{N,4,j,\tau}_{T} \}_{j=1,2} along a similar fashion as our analysis of \{ \mathcal{Q}^{N,2,j,\tau}_{T} \}_{j=1,2}; observe
\[
\sum_{y \in \mathcal{F}_{N,0}} \left\| \mathcal{Q}^{N,4,1,\tau}_{S,T,x,y} \right\|_{\mathcal{L}^2_{\tau,x}}^2 \lesssim \int_{S}^{T} \frac{1}{R_{T}} \sum_{w \in \mathcal{F}_{N,0}} \sum_{y \in \mathcal{F}_{N,0}} \mathcal{Q}^{N}_{S,R,x,w} \, N \left\| \mathcal{W}^{N}_{S,R} \right\|_{\mathcal{L}^2_{\tau,x}}^2 \tag{6.37}
\]
\[
\lesssim \int_{S}^{T} \frac{1}{R_{T}} \sum_{w \in \mathcal{F}_{N,0}} \mathcal{Q}^{N}_{S,R,x,w} \, N \left\| \mathcal{W}^{N}_{S,R} \right\|_{\mathcal{L}^2_{\tau,x}}^2 \, dR; \tag{6.38}
\]
\[
\sum_{y \in \mathcal{F}_{N,0}} \left\| \mathcal{Q}^{N,4,2,\tau}_{S,T,x,y} \right\|_{\mathcal{L}^2_{\tau,x}}^2 \lesssim \int_{S}^{T} \frac{1}{R_{T}} \sum_{w \in \mathcal{F}_{N,0}} \sum_{y \in \mathcal{F}_{N,0}} \mathcal{Q}^{N}_{S,R,x,w} \, N \left\| \mathcal{W}^{N}_{S,R} \right\|_{\mathcal{L}^2_{\tau,x}}^2 \tag{6.39}
\]
\[
\lesssim \int_{S}^{T} \frac{1}{R_{T}} \sum_{w \in \mathcal{F}_{N,0}} \sum_{y \in \mathcal{F}_{N,0}} \mathcal{Q}^{N}_{S,R,x,w} \, N \left\| \mathcal{W}^{N}_{S,R} \right\|_{\mathcal{L}^2_{\tau,x}}^2 \, dR \tag{6.40}
\]
\[
\lesssim N^{-1+2\epsilon} \tau^{1+2\epsilon}. \tag{6.41}
\]
\[
\lesssim N^{-1+2\epsilon} \tau^{1+2\epsilon}. \tag{6.42}
\]
Ultimately, we combine the estimates (6.22), (6.24), (6.28), (6.29), (6.32), (6.33), and (6.36) along with the heat kernel estimate in Lemma 4.4 to obtain

\[ \Psi_{S,T}^{N,0} \lesssim_{\varepsilon} \mathcal{T}_{Tj,m_0} K_{S,T}^{N,\varepsilon} + \int_{S+\tau}^{T} \Theta_{R,T}^{-\frac{1}{2}} \Psi_{S,R,\tau}^{N,0} \, dR, \]  

where we have introduced the quantity

\[ K_{S,T}^{N,\varepsilon} \equiv N^{-1+2\varepsilon} \Theta_{S,T}^{-1+\varepsilon/2} + N^{-2+4\varepsilon} \Theta_{S,T}^{-1+\varepsilon} + N^{-1+2\varepsilon} \Theta_{S,T}^{-1+\varepsilon} + N^{-1+2\varepsilon} \Theta_{S,T}^{-1+\varepsilon/2}. \]  

Before we insert the estimate (6.43) into the machine of the singular Gronwall inequality, we remark that the first quantity defining \( K_{S,T}^{N,\varepsilon} \) either outputs a sub-optimal estimate when compared to the stated estimate in the lemma or exhibits a non-integrable singularity at \( S = T \). To remedy this issue, we decompose the integration-domain on the RHS of (6.43) towards regularizing the aforementioned non-integrable singularity.

More precisely, let us first recall the lower bound \( \Theta_{S,T} \gtrsim 2\tau \). The first step consists of the obvious identity

\[ \int_{S+\tau}^{T} \Theta_{R,T}^{-\frac{1}{2}} \Psi_{S,R,\tau}^{N,0} \, dR = \int_{S+\tau}^{S+2\tau} \Theta_{R,T}^{-\frac{1}{2}} \Psi_{S,R,\tau}^{N,0} \, dR + \int_{S+2\tau}^{T} \Theta_{R,T}^{-\frac{1}{2}} \Psi_{S,R,\tau}^{N,0} \, dR. \]  

Concerning the first integral, courtesy of Lemma 6.2 and Lemma C.1, along with the a priori lower bound \( \Theta_{S,T} \gtrsim 2\tau \), we have

\[ \int_{S+\tau}^{S+2\tau} \Theta_{R,T}^{-\frac{1}{2}} \Psi_{S,R,\tau}^{N,0} \, dR \lesssim_{\varepsilon} N^{-1+4\varepsilon} \int_{S+\tau}^{S+2\tau} \Theta_{R,T}^{-\frac{1}{2}} \Theta_{S,R}^{1+2\varepsilon} \, dR \lesssim N^{-1+4\varepsilon} \Theta_{S,T}^{1+2\varepsilon}, \]  

indeed, at the cost of some acceptable upper bound, we replaced the action of the time-gradient operator on the stochastic fundamental solution \( \mathcal{P}^N \) within \( \Psi_{S,R,\tau}^{N,0} \) with the stochastic fundamental solution itself evaluated at two distinct backwards time-coordinates and afterwards employed, as mentioned above, Lemma 6.2 and Lemma C.2 to deduce such upper bound. Anyway, as consequence we obtain

\[ \Psi_{S,T}^{N,0} \lesssim_{\varepsilon} \mathcal{T}_{Tj,m_0} K_{S,T}^{N,\varepsilon} + \int_{S+\tau}^{T} \Theta_{R,T}^{-\frac{1}{2}} \Psi_{S,R,\tau}^{N,0} \, dR; \]  

from which we complete the proof after the smoke clears from applying the singular Gronwall inequality along with Lemma C.2.

The final preliminary ingredient we will require from the stochastic fundamental solution is upgrading of the arbitrarily high, although still universal, moment estimates obtained in both of Lemma 6.2 and Lemma 6.4 to high-probability space-time uniform estimates through stochastic continuity. Actually, the utility behind both of Lemma 6.2 and Lemma 6.4 within the current article is solely towards proving the following result; this idea was actually adopted in [23] in almost identical spirit for an almost identical purpose. Thus, for this reason, we omit the proof and instead refer the reader to [23], Section 4 or Section 6, for example.

Roughly speaking, despite the complicated facade behind this final result, all that is being stated is that you may remove the moments in Lemma 6.2 and Lemma 6.4 and the resulting random inequalities hold simultaneously in space-time with high-probability, in the sense to be made precise, as long as one inserts the additional factor of \( N^\delta \) to the RHS with \( \delta \in \mathbb{R}_{>0} \) arbitrarily small though universal.

**Lemma 6.5.** Consider any \( \varepsilon \in \mathbb{R}_{>0} \) arbitrarily small though universal; along with this data, now consider \( \bar{\varepsilon} = (\varepsilon_1, \varepsilon_2, \varepsilon_3) \in \mathbb{R}^3 \) with \( \varepsilon_1, \varepsilon_2, \varepsilon_3 \in \mathbb{R}_{>0} \) arbitrarily small though universal as well. Moreover, consider the space-time domains

\[ \Omega_{Tj,\tau} = \left\{ S, T \in \mathbb{R}_{>0} : S \leq T \leq T_j, \tau \leq \frac{1}{2} \Theta_{S,T} \right\}; \]  

\[ \Omega_{Tj,\tau,X} = \Omega_{Tj,\tau} \times \mathcal{F}_{N,0} \times \mathcal{F}_{N,0}. \]  

}\[43\]
Second, provided any $\delta \in \mathbb{R}_{>0}$, consider the events indexed by $(S, T, x, y) \in \Omega_{T_{I}, 0, x}$:

$$
\mathcal{E}_{S, T, x, y}^{N} = \{ Q_{S, T, x, y}^{N} \geq N^{-1+\varepsilon_{1}} e^{\varepsilon_{1}} \mathcal{E}_{S, T} \};
$$

$$
\mathcal{E}_{S, T, x, y}^{N, \delta} = \left\{ \sum_{y \in \mathcal{F}_{0, 0}} \left| \mathcal{D}_{x} Q_{S, T, x, y}^{N} \right|^{2} \geq K^{N, \delta}_{S, T} \right\},
$$

in which we have introduced the quantity

$$
K^{N, \delta}_{S, T} = N^{-1+\varepsilon_{1}+\varepsilon_{2}} \mathcal{E}_{S, T}^{-\varepsilon_{1}} e^{\varepsilon_{1}} + N^{-2+\varepsilon_{1}+\varepsilon_{2}} \mathcal{E}_{S, T}^{-\varepsilon_{2}} e^{\varepsilon_{2}} + N^{-1+\varepsilon_{1}+\varepsilon_{2}} \mathcal{E}_{S, T}^{-\varepsilon_{1}+\varepsilon_{2}} e^{\varepsilon_{1}+\varepsilon_{2}}.
$$

Provided any $D \in \mathbb{R}_{>0}$

$$
P \left[ \bigcup_{(S, T, x, y) \in \Omega^{(2)}_{T_{I}, 0, x}} \mathcal{E}_{S, T, x, y}^{N, \delta} \right] + P \left[ \bigcup_{(S, T, x, y) \in \Omega_{T_{I}, 0, x}} \mathcal{E}_{S, T, x, y}^{N, \delta, \tau} \right] \leq \alpha_{d, T_{I}, m_{0}, x, \delta, D} N^{-D}.
$$

6.2. Cole-Hopf Transform. To prove Theorem 1.8 through the dynamical one-block strategy, we require time-regularity estimates for the microscopic Cole-Hopf transform $\mathcal{Y}^{N}$ itself in addition to previous results from both of Lemma 6.5.

To this end, we will actually avoid the Duhamel representation of $\mathcal{Y}^{N}$ using stochastic fundamental solutions and instead appeal to the following Duhamel representation via the heat kernel $\mathcal{Y}^{N}$.

**Notation 6.6.** Retain the notation within Proposition 2.11. Provided times $S, T \in \mathbb{R}_{>0}$ satisfying $S \leq T$ along with $x \in \mathcal{F}_{N, 0}$, we consider the decomposition

$$
\mathcal{Y}^{N}_{T, x} = \mathcal{Y}^{N, 1}_{T, x} + \cdots + \mathcal{Y}^{N, 6}_{T, x},
$$

where

$$
\mathcal{Y}^{N, 1}_{T, x} = \sum_{y \in \mathcal{F}_{0, 0}} \mathcal{Y}^{N, y}_{0, T, x, y},
$$

$$
\mathcal{Y}^{N, 2}_{T, x} = \int_{0}^{T} \sum_{y \in \mathcal{F}_{0, 0}} \mathcal{Y}^{N}_{S, T, x, y} \left[ \mathcal{Y}^{N}_{S, y} \mathcal{D}_{x} \mathcal{Y}^{N, y}_{S, y} \right] dS;
$$

$$
\mathcal{Y}^{N, 3}_{T, x} = \int_{0}^{T} \sum_{y \in \mathcal{F}_{0, 0}} \mathcal{Y}^{N}_{S, T, x, y} \left[ N^{2} \sum_{w=1}^{N} \mathcal{Y}^{N}_{S, y} \mathcal{Y}^{N}_{S, y} \right] dS;
$$

$$
\mathcal{Y}^{N, 4}_{T, x} = \int_{0}^{T} \sum_{y \in \mathcal{F}_{0, 0}} \mathcal{Y}^{N}_{S, T, x, y} \left[ N^{2} \sum_{w=1}^{N} \mathcal{Y}^{N}_{S, y} \mathcal{Y}^{N}_{S, y} \right] dR;
$$

$$
\mathcal{Y}^{N, 5}_{T, x} = \int_{0}^{T} \sum_{y \in \mathcal{F}_{0, 0}} \mathcal{Y}^{N}_{S, T, x, y} \left[ N^{2} \sum_{w=1}^{N} \mathcal{Y}^{N}_{S, y} \mathcal{Y}^{N}_{S, y} \right] dR;
$$

$$
\mathcal{Y}^{N, 6}_{T, x} = \sum_{|k| \leq m_{0}} c_{k} \int_{0}^{T} \sum_{y \in \mathcal{F}_{0, 0}} \mathcal{Y}^{N}_{S, T, x, y} \left[ \nabla^{k} \left( m^{N}_{S, y} \mathcal{Y}^{N}_{S, y} \right) \right] dR.
$$

We further introduce, for the purposes of compact presentation, another time-differential operator acting on $\mathcal{Y}^{N}$.

**Notation 6.7.** Consider any function $\varphi : \mathbb{R}_{>0} \times \mathcal{F}_{N, 0} \rightarrow \mathbb{R}$.

Provided any $T \in \mathbb{R}_{>0}$ and $x \in \mathcal{F}_{N, 0}$ along with any time-scale $\tau \in \mathbb{R}_{>0}$, define

$$
\mathcal{D}_{T, \tau} \varphi_{T, x} = \varphi_{T+\tau, x} - \varphi_{T, x}.
$$

First, we will consider the situation of near-stationary initial data, so in particular $\mathcal{Y}^{N}_{0, x}$ admits suitable spatial regularity estimates with suitable sub-exponential growth. The following high-probability estimate is parallel to that from Proposition 6.2 in [23]; in particular, we will record only the necessary changes and additions to the proof of Proposition 6.2 in [23].
Lemma 6.8. Consider any \( \varepsilon \in \mathbb{R}_{>0} \) arbitrarily small though universal and any time-scale \( \tau \in \mathbb{R}_{>0} \) satisfying \( N^{-2} \lesssim \tau \lesssim N^{-1} \) with universal implied constant.

Provided any \( \varepsilon_1, \varepsilon_2, \varepsilon_3 \in \mathbb{R}_{>0} \) arbitrarily small but universal, define the event
\[
\mathcal{E}^N_{\tau, \varepsilon_1, \varepsilon_2, \varepsilon_3} \doteq \left\{ \left\| \mathcal{D}_\tau 3^N \right\|_{L^\infty_{T,X}} \gtrsim N^{e_1} \tau^\frac{1}{4} \varepsilon_2 + N^{e_1} \tau^\frac{1}{4} \varepsilon_3 \left\| 3^N \right\|_{L^\infty_{T,X}} \right\};
\] (6.56)
the implied constant defining this event is universal. Provided any \( D \in \mathbb{R}_{>0} \), we have
\[
P \left[ \mathcal{E}^N_{\tau, \varepsilon_1, \varepsilon_2, \varepsilon_3, D, T, m, m_0, \sigma_0} \right] \lesssim N^{-D}.
\] (6.57)

Proof. We outline the necessary adjustments and additions through the following list of bullet points; first, we emphasize that it suffices to compute sufficiently high moments in view of the proof of Lemma 6.5.

- The most annoying adjustment required is the observation that the time-regularity estimate within Lemma 5.14 contains some error of order \( N^{-1} \theta_{S,T}^{1/2} \) in comparison to the time-regularity of the continuum Gaussian heat kernel. In particular, we require the following modification of time-regularity estimates established towards the proof of Proposition 6.2 in [23]; in the following, the function \( \mathcal{J} \) is a uniformly bounded functional of the particle system:
\[
\sum_{y \in \mathcal{F}_{0,0}} \left| \mathcal{D}_\tau \mathcal{U}^N_{0, T, x, y} \right| 3^N_{S,y} \lesssim_{\tau, T, \varepsilon} N^{\varepsilon} \theta_{S,T}^{-1+\varepsilon} \tau^{1-\varepsilon} \left\| 3^N \right\|_{L^\infty_{T,X}} + N^{-1+2\varepsilon} \theta_{S,T}^{-\frac{1}{4}+\varepsilon} \varepsilon \left\| 3^N \right\|_{L^\infty_{T,X}}.
\] (6.58a)
\[
\int_0^T \sum_{y \in \mathcal{F}_{0,0}} \left| \mathcal{D}_\tau \mathcal{U}^N_{S,T,x,y} \right| \left[ \left.N^2 \mathcal{J}_{S,y} \right. \mathcal{J}_{S,y} \right] dS \lesssim_{\tau, T, \varepsilon} N^{\varepsilon} \theta_{S,T}^{-1+\varepsilon} \tau^{1-\varepsilon} \left\| 3^N \right\|_{L^\infty_{T,X}} + N^{-1+2\varepsilon} \theta_{S,T}^{-\frac{1}{4}+\varepsilon} \varepsilon \left\| 3^N \right\|_{L^\infty_{T,X}}.
\] (6.58b)

Indeed, the difference between the above pair of upper bounds and their parallels within the proof of Proposition 6.2 in [23] is the additional second quantities on the respective RHS above which arises from the estimate within Lemma 5.14; we emphasize the second of these estimates above provides estimates for the corresponding \( \mathcal{D}_\tau 3^N_{j, x} \) quantities for \( j = 3, 4, 5 \).

Concerning the stochastic-integral-type quantity for \( j = 2 \), we instead establish the following moment estimate, in which \( 3^N \) denotes a uniformly bounded functional of the particle dynamic or process; the ingredients behind this upper bound consist of the BDG-martingale inequality of Lemma 3.1 in [9], the heat kernel estimates in Lemma 5.14, and the integration-lemma in Lemma C.1:
\[
\left\| \int_0^T \sum_{y \in \mathcal{F}_{0,0}} \left[ \mathcal{D}_\tau \mathcal{U}^N_{S,T,x,y} \right] 3^N_{S,y} dS \right\|_{L^p_{T,X}}^2 \lesssim_p \int_0^T N \sum_{y \in \mathcal{F}_{0,0}} \left[ \mathcal{D}_\tau \mathcal{U}^N_{S,T,x,y} \right] dS + N^{-1}
\] (6.59)
\[
\lesssim_{\varepsilon} \int_0^T \theta_{S,T}^{-1+\varepsilon} \tau^{\frac{1}{2}+\varepsilon} dS + N^{-1+4\varepsilon} \int_0^T \theta_{S,T}^{-1+\varepsilon} dS + N^{-1}
\] (6.60)
\[
\lesssim_{\varepsilon, T, \varepsilon} \tau^{\frac{1}{2}+\varepsilon} + N^{-1+4\varepsilon}.
\] (6.61)

- The remaining ingredient is some appropriate analysis concerning the index \( j = 6 \). The first ingredient we require is the following decomposition reminiscent of the definition of the stochastic fundamental solution \( \mathcal{Q}^N \):
\[
3^N_{T,x} = 3^N_{T,x}, 1 + 3^N_{T,x}, 2
\] (6.62)
where
\[
3^N_{T,x}, 1 = \sum_{|k| \leq m} c_k \int_0^T \sum_{y \in \mathcal{F}_{0,0}} \mathcal{U}_{S,T,x,y}^N \left[ \nabla \left( 1_{y \in \mathcal{F}_{0,0}} N^k_{S,y} 3^N_{S,y} \right) \right] dR;
\] (6.63a)
\[
3^N_{T,x}, 2 = \sum_{|k| \leq m} c_k \int_0^T \sum_{y \in \mathcal{F}_{0,0}} \mathcal{U}_{S,T,x,y}^N \left[ \nabla \left( 1_{y \notin \mathcal{F}_{0,0}} N^k_{S,y} 3^N_{S,y} \right) \right] dR.
\] (6.63b)

Let us first focus on the second quantity \( 3^N_{T,x}, 2 \); for this quantity, we first write
\[
\mathcal{D}_\tau 3^N_{T,x}, 2 = 3^N_{T,x}, 2, 1 + 3^N_{T,x}, 2, 2
\] (6.64)
where

\[ 3^{N,6,2,1}_{T,\tau,x} = \sum_{|k| \leq m_n} c_k \int_0^T \sum_{y \in \mathcal{S}_N} \mathscr{D}_x \mathcal{W}_y^{N_{S,T,x,y}} \left[ \nabla_k' \left( 1_{y \notin \mathcal{S}_N} m_k^{N_{S,Y}} \mathcal{S}_y \right) \right] \, ds; \]  
\[ (6.65a) \]

\[ 3^{N,6,2,2}_{T,\tau,x} = \sum_{|k| \leq m_n} c_k \int_T^{T+\tau} \sum_{y \in \mathcal{S}_N} \mathcal{W}_y^{N_{S,T+\tau,x,y}} \left[ \nabla_k' \left( 1_{y \notin \mathcal{S}_N} m_k^{N_{S,Y}} \mathcal{S}_y \right) \right] \, ds. \]  
\[ (6.65b) \]

We first address \( 3^{N,6,2,1}_{T,\tau,x} \); because the functionals \( \{m_k^{N_{S,Y}}\}_{|k| \leq m_n} \) are uniformly bounded, we see

\[ \left| 3^{N,6,2,1}_{T,\tau,x} \right| \lesssim_{m_n} N^{1+\beta_\delta} \int_0^T \left\| \mathcal{D}_x \mathcal{W}_y^{N_{S,T,x,y}} \right\| \mathcal{L}_{x,y} \, ds \cdot \left\| 3^N \right\| \mathcal{L}_{x,y}. \]  
\[ (6.66) \]

We will now decompose the integral remaining on the RHS into integrals over different integration-domains; this is to address and regularize the non-integrable singularity arising in the regularity estimate in Lemma 5.14.

First, provided any \( \varepsilon \in \mathbb{R}_{>0} \) arbitrarily small but universal, we have

\[ N^{1+\beta_\delta} \int_0^T \left\| \mathcal{D}_x \mathcal{W}_y^{N_{S,T,x,y}} \right\| \mathcal{L}_{x,y} \, ds \lesssim_{\varepsilon,\omega_z,T,\tau} N^{\beta_\delta+2\varepsilon} \int_0^{T-\varepsilon} T_{\omega_z,T} \, ds \cdot \tau^{-1-\varepsilon}. \]  
\[ (6.67) \]

Second, via the heat kernel estimate in Lemma 4.4, we have

\[ N^{1+\beta_\delta} \int_T^{T+\tau} \left\| \mathcal{D}_x \mathcal{W}_y^{N_{S,T+\tau,x,y}} \right\| \mathcal{L}_{x,y} \, ds \lesssim_{\varepsilon,\omega_z,T,\tau} N^{\beta_\delta} \int_T^{T+\tau} T_{\omega_z,T} \, ds \lesssim N^{\beta_\delta} \tau^{\frac{1}{2}}. \]  
\[ (6.69) \]

Ultimately, we obtain the estimate

\[ \left| 3^{N,6,2,1}_{T,\tau,x} \right| \lesssim_{\varepsilon,\omega_z,T,\tau} N^{\beta_\delta+2\varepsilon} \tau^{-\frac{1}{2}-\varepsilon}. \]  
\[ (6.71) \]

To address the quantity \( 3^{N,6,2,2}_{T,\tau,x} \), we proceed with a similar although simpler calculation relying on the heat kernel estimate in Lemma 4.4. This is done in the following:

\[ \left| 3^{N,6,2,2}_{T,\tau,x} \right| \lesssim_{m_n,\omega_z,T,\tau} N^{\beta_\delta} \int_T^{T+\tau} \left\| \mathcal{W}_y^{N_{S,T+\tau,x,y}} \right\| \mathcal{L}_{x,y} \, ds \lesssim_{\varepsilon,\omega_z,T} N^{\beta_\delta} \tau^{\frac{1}{2}}. \]  
\[ (6.73) \]

Thus, we have the first preliminary estimate

\[ \left| 3^{N,6,2}_{T,\tau,x} \right| \lesssim_{m_n,\omega_z,T,\tau} N^{\beta_\delta+2\varepsilon} \tau^{-\frac{1}{2}-\varepsilon}. \]  
\[ (6.75) \]

• For organizational purposes, we separate our analysis of \( 3^{N,6,1}_{T,\tau,x} \) into another bullet point; the approach is somewhat similar but the current bullet point requires an additional higher-order regularity estimate for the heat kernel \( \mathcal{W}_y^{N} \) in Lemma 5.14. More precisely, we again write

\[ \mathcal{D}_x 3^{N,6,1}_{T,\tau,x} = 3^{N,6,1,1}_{T,\tau,x} + 3^{N,6,1,2}_{T,\tau,x}, \]  
\[ (6.76) \]

where

\[ 3^{N,6,1,1}_{T,\tau,x} = \sum_{|k| \leq m_n} c_k \int_0^T \sum_{y \in \mathcal{S}_N} \mathcal{D}_x \mathcal{W}_y^{N_{S,T,x,y}} \left[ \nabla_k' \left( 1_{y \notin \mathcal{S}_N} m_k^{N_{S,Y}} \mathcal{S}_y \right) \right] \, ds; \]  
\[ (6.77a) \]

\[ 3^{N,6,1,2}_{T,\tau,x} = \sum_{|k| \leq m_n} c_k \int_T^{T+\tau} \sum_{y \in \mathcal{S}_N} \mathcal{W}_y^{N_{S,T+\tau,x,y}} \left[ \nabla_k' \left( 1_{y \notin \mathcal{S}_N} m_k^{N_{S,Y}} \mathcal{S}_y \right) \right] \, ds. \]  
\[ (6.77b) \]
To analyze the first of these two quantities, as with the proof of Lemma 6.2 and Lemma 6.4 along with the previous observation that the functionals \( \{ \nu^{N,k} \}_{|k| \leq m_N} \) are uniformly bounded, we observe

\[
\left| 3_{N,T,x}^{6,1,1} \right| \lesssim \sum_{|k| \leq m_N} \left| k \right| \sum_{y \in \mathcal{N}_{T,x}} \left| \nabla_{k,y}^{N} \mathcal{W}_{N,T,x,y} \right| \int_{0}^{T} \mathcal{W}_{N,T,x,y} \mathrm{d} \mathcal{S} \quad (6.78)
\]

\[
\lesssim \left\| 3_{N}^{6,1,1} \right\|_{L_{2}} \int_{0}^{T} \sum_{y \in \mathcal{N}_{T,x}} \left| \nabla_{k,y}^{N} \mathcal{W}_{N,T,x,y} \right| \mathrm{d} \mathcal{S}. \quad (6.79)
\]

In view of the non-integrable singularity appearing within the space-time regularity estimate of Lemma 5.14, we again decompose the integral into separate integration-domains. First, appealing to Lemma 5.14, we have

\[
\int_{0}^{T-\tau} \sum_{y \in \mathcal{N}_{T,x}} \left| \nabla_{k,y}^{N} \mathcal{W}_{N,T,x,y} \right| \mathrm{d} \mathcal{S} \lesssim_{\beta, \mathcal{A}, T, T, \tau} \tau \int_{0}^{T-\tau} \frac{1}{S_{N,T}^{1/2}} \mathrm{d} \mathcal{R} + N^{-1+2\epsilon} \int_{0}^{T-\tau} \frac{1}{S_{N,T}^{1/2}} \mathrm{d} \mathcal{S} \quad (6.80)
\]

\[
\lesssim \frac{\tau^{3}}{2} + N^{-1+3\epsilon}; \quad (6.81)
\]

above, we have used the estimate \( |\log \tau| \lesssim N^{\epsilon} \) provided that \( N^{-2} \lesssim \tau \lesssim N^{-1} \).

On the other hand, courtesy of the Nash-type heat kernel estimate within Lemma 4.4, we also have

\[
\int_{T-\tau}^{T} \sum_{y \in \mathcal{N}_{T,x}} \left| \nabla_{k,y}^{N} \mathcal{W}_{N,T,x,y} \right| \mathrm{d} \mathcal{S} \lesssim_{\beta, \mathcal{A}, T, T, \tau} \int_{T-\tau}^{T} \frac{1}{S_{N,T}^{1/2}} \mathrm{d} \mathcal{S} \quad (6.82)
\]

\[
\lesssim \frac{\tau^{3}}{2}. \quad (6.83)
\]

This completes the proof. \( \square \)

7. DYNAMICAL ONE-BLOCK ANALYSIS

For the current section, we provide the necessary ingredients towards the proof of Theorem 1.8 for developing the local dynamical one-block lemma first developed in [23], which we recall addresses the same problem for long-range variations on ASEP for the full-line \( \mathbb{Z} \). Similar to the previous article [23], the local dynamical procedure begins with the preliminary entropy production estimate; this is the content of Proposition 7.1 below. However, in contrast to [23], within the current article we are presented with the obstruction of having no reasonable and explicit representation of the invariant measure of the interacting particle system. In fact, the entropy production estimates that we establish for the local dynamical one-blocks strategy are not performed with respect to the invariant measure of the process but rather a specific grand-canonical ensemble, the point here being that the infinitesimal generator \( \mathcal{L}^{N,\text{II}} \) is sufficiently close to an infinitesimal generator that admits such a grand-canonical ensemble to be its unique invariant measure.

Though the resulting aforementioned entropy production estimates are sub-optimal due to its perturbative nature compared to those for the ASEP on the discrete torus, for example, they are ultimately beyond sufficient towards implementing the dynamical one-block strategy because of the latter’s local nature.

To provide a complete introduction for the current section, we introduce another much simpler variation of the dynamical one-block scheme to analyze fluctuations averaged with respect to time that are supported near the boundary where the particle system exhibits additional "Glauber-type" dynamics beyond the "Kawasaki-type" dynamics in the bulk.

7.1. Preliminaries. Before discussing the dynamical one-block analysis, we require some preliminaries concerning particle systems and their Lyapunov functionals in a significantly more general setting. First, we make the following declaration.

- We adopt the notation from Section 3 within [23] for both the canonical ensembles and grand-canonical ensembles parameterized by densities \( \varrho \in [-1, 1] \) and sub-lattices \( \Lambda \subseteq \mathcal{N}_{T} \). Moreover, if the sub-lattice is omitted from this notation, then it is assumed \( \Lambda = \mathcal{N}_{T} \).

- We also adopt notation from the same part of [23] for the relative entropy and Dirichlet form with respect to the above pair of probability measures. Additionally, we further assume all relevant entropy inequalities and spectral gap inequalities in Section 3.3, Section 3.4, and Section 3.5 of [23] without explicitly recording them or proving
them here. In particular, we will freely cite these aforementioned estimates therein unless they require some input of Proposition 3.5 in [23]; for this case, we re-present them below with modifications to Proposition 3.5 therein.

To be completely transparent, within the Dirichlet form that we have adopted notation for, we have accounted for the boundary "Glauber-type" dynamic as well.

The remaining preliminary ingredient is the following estimate for the entropy production with respect to a grand-canonical ensemble and the resulting Dirichlet form estimate.

**Proposition 7.1.** Consider a time $T \in \mathbb{R}_{\geq 0}$ satisfying $T \leq T_f$ and independent of $N \in \mathbb{Z}_{\geq 0}$. Provided any probability measure $\mu^N_0$ on $\Omega_{\mathcal{I}_N^0}$, we have

$$\int_0^T \mathbb{D}^{gc}_{0,\mathcal{I}_N^0} \left( \frac{d\mu^N_S}{d\mu^N_{0,\mathcal{I}_N^0}} \right) dS \lesssim T^{-1} N^{-1} + N^{-\frac{1}{2}};$$

(7.1)

above, we have defined the probability measure $\mu^N_T = e^{T \mathbb{D}^{gc}_{0,\mathcal{I}_N^0}} \mu^N_0$, and moreover the implied constant is universal.

**Remark 7.2.** The result of Proposition 7.1 might be striking or appear incorrect in view of the fact that the grand-canonical ensemble is not necessarily the invariant measure of the particle system. In particular, we strongly emphasize the Dirichlet form estimate within Proposition 7.1 does not imply that the law of the particle system equilibrates to this grand-canonical ensemble in the long-time limit. We address this in the following list of bullet points.

- In view of the aforementioned fact concerning the grand-canonical ensemble, the law of the particle system cannot equilibrate to the grand-canonical ensemble, because its invariant measure is actually somewhat sophisticated at the level of explicit formulas.
- Alternatively, the optimal LSI constant for the particle dynamic is diffusive, as indicated in both Lemma 7.5 below and by Theorem A within [25]. In particular, this implies that the time-averaged relative entropy of the measure $\mu^N_T$ on any sub-lattice $\Lambda \subseteq \mathcal{I}_N^0$ is of order $|\Lambda|^2$ above the time-averaged Dirichlet form. With $|\Lambda| \asymp N$, this suggests that the time-averaged relative entropy on macroscopic scales diverges in the large-$N$ limit.
- To provide an explicit example – for ASEP on the torus, any two product Bernoulli measures of constant parameter serve as invariant measures for this interacting particle system. Thus both of the entropy production and Dirichlet form of one of these invariant measures with respect to the other vanishes; however, these two invariant measures may be quite far apart.
- Philosophically, though with concrete illustration in a host of tractable examples as with the previous bullet point, the Dirichlet form instead effectively quantifies the rate or speed of "local equilibration", and not the distance from "global equilibrium"; for this reason, for example, the one-block and two-blocks strategies originally developed in [15] succeed because of the local nature of their respective fundamental mechanisms.

Continuing with the final bullet point above, the Dirichlet form estimate of Proposition 7.1 actually quantifies the following statement:

- The infinitesimal generator or dynamic is locally the distance of $N^{-\frac{1}{2}}$ of another infinitesimal generator or dynamic which does admit the grand-canonical ensemble as its unique invariant measure; this auxiliary dynamic is just the original particle dynamic but removing all asymmetry in the Kawasaki dynamic and all asymmetry between rates of creation/annihilation at the boundary.

The previous statement is further the reason why we consider the specific grand-canonical ensemble with density parameter $\rho = 0$ and not any other one – after removing the asymmetry in creation/annihilation at the boundary, for $\rho \not= 0$ it is false that the resulting "asymmetry-free" generator admits the $\rho$-canonical ensemble as its invariant measure.
Proof. It will serve convenient to define

\[
\begin{align*}
  f_T & \equiv \frac{\text{d}\mu_T}{\text{d}\mu_{0,N}}; \quad (7.2a) \\
  H_T & \equiv H_{0,N_0}^{\text{re}} (f_T); \quad (7.2b) \\
  D_T & \equiv D_{0,N_0}^{\text{re}} (f_T). \quad (7.2c)
\end{align*}
\]

Moreover, provided any \( k \in \mathbb{Z}_{>0} \), let us define the following for a reason to be made clear shortly, in which the randomness does not manifest in the indicator functions:

\[
\begin{align*}
  \beta_{k,\min}^{N,\pm} & \equiv \beta_{k,\min}^{N,\pm} \wedge \beta_{k,-}^{N,\pm}; \\
  \beta_{k,\text{rem}} (\eta) & \equiv 1_{\beta_{k,\min}^{N,\pm} = \beta_{k,+}^{N,\pm}} \left( \beta_{k,+}^{N,\pm} - \beta_{k,-}^{N,\pm} \right) \frac{1 + \eta_x}{2} + 1_{\beta_{k,\min}^{N,\pm} = \beta_{k,-}^{N,\pm}} \left( \beta_{k,+}^{N,\pm} - \beta_{k,-}^{N,\pm} \right) \frac{1 - \eta_x}{2}. \quad (7.3a, b)
\end{align*}
\]

We proceed to compute the time-derivative for the entropy functional \( H_T \), despite the fact the grand-canonical ensemble at hand is not an invariant measure of the particle system. Performing elementary standard calculations, we get the following in which all expectations are taken with respect to the grand-canonical ensemble at hand; see Appendix 1 in [20]:

\[
\begin{align*}
  \frac{\text{d}}{\text{d}T} H_T & = \frac{1}{2} N^2 E f_T \sum_{k=1}^{\infty} \sum_{x,k \in \mathbb{Z}_{\geq 0}} \left( \alpha_k^{N} + \frac{\alpha_k^{N} \gamma_k^{N}}{\sqrt{N}} \frac{1 + \eta_x}{2} \right) \left( \bar{\Sigma}_{x,k+} \log f_T - \log f_T \right) \\
  & \quad + \frac{1}{2} N^2 E f_T \sum_{x=1}^{m_0} \beta_{x,+}^{N,\pm} \frac{1 - \eta_x}{2} \left( \bar{\Sigma}_{x,+} \log f_T - \log f_T \right) \\
  & \quad + \frac{1}{2} N^2 E f_T \sum_{x=1}^{m_0} \beta_{x,-}^{N,\pm} \frac{1 + \eta_x}{2} \left( \bar{\Sigma}_{x,-} \log f_T - \log f_T \right) \\
  & \quad + \frac{1}{2} N^2 E f_T \sum_{x=N-m_0+1}^{N} \beta_{x,+}^{N,\pm} \frac{1 - \eta_x}{2} \left( \bar{\Sigma}_{x,+} \log f_T - \log f_T \right) \\
  & \quad + \frac{1}{2} N^2 E f_T \sum_{x=N-m_0+1}^{N} \beta_{x,-}^{N,\pm} \frac{1 + \eta_x}{2} \left( \bar{\Sigma}_{x,-} \log f_T - \log f_T \right) \\
  & \equiv I_T + II_T, \quad (7.5)
\end{align*}
\]

where

\[
\begin{align*}
  I_T & \equiv \frac{1}{2} N^2 E f_T \sum_{k=1}^{\infty} \sum_{x,k \in \mathbb{Z}_{\geq 0}} \alpha_k^{N} \left( \bar{\Sigma}_{x,k+} \log f_T - \log f_T \right) \quad (7.6a) \\
  & \quad + \frac{1}{2} N^2 E f_T \sum_{x=1}^{m_0} \beta_{x,\min}^{N,\pm} \frac{1 - \eta_x}{2} \left( \bar{\Sigma}_{x,+} \log f_T - \log f_T \right) + \frac{1}{2} N^2 E f_T \sum_{x=1}^{m_0} \beta_{x,\min}^{N,\pm} \frac{1 + \eta_x}{2} \left( \bar{\Sigma}_{x,-} \log f_T - \log f_T \right) \\
  & \quad + \frac{1}{2} N^2 E f_T \sum_{x=N-m_0+1}^{N} \beta_{x,\min}^{N,\pm} \frac{1 - \eta_x}{2} \left( \bar{\Sigma}_{x,+} \log f_T - \log f_T \right) + \frac{1}{2} N^2 E f_T \sum_{x=N-m_0+1}^{N} \beta_{x,\min}^{N,\pm} \frac{1 + \eta_x}{2} \left( \bar{\Sigma}_{x,-} \log f_T - \log f_T \right); \\
  II_T & \equiv \frac{1}{2} N^2 E f_T \sum_{k=1}^{\infty} \sum_{x,k \in \mathbb{Z}_{\geq 0}} \frac{\alpha_k^{N} \gamma_k^{N}}{\sqrt{N}} \frac{1 + \eta_x}{2} \left( \bar{\Sigma}_{x,k+} \log f_T - \log f_T \right) \quad (7.6b) \\
  & \quad + \frac{1}{2} N^2 E f_T \sum_{x=1}^{m_0} \beta_{x,\min}^{N,\pm} (\eta) \left( \bar{\Sigma}_{x} \log f_T - \log f_T \right) + \frac{1}{2} N^2 E f_T \sum_{x=N-m_0+1}^{N} \beta_{x,\min}^{N,\pm} (\eta) \left( \bar{\Sigma}_{x} \log f_T - \log f_T \right).
\end{align*}
\]

We proceed to analyze the quantity \( I_T \). This begins with the following two points.

- Observe the Kawasaki dynamics providing the first term defining \( I_T \) admit the grand-canonical ensemble at hand as an invariant measure.
Moreover, observe the Glauber dynamics at the boundary providing both the second and third summation defining \( I_T \) also admit the same grand-canonical ensemble at hand as an invariant measure as well, because the annihilation and creation coefficients are equal.

Thus, following standard procedure as in Theorem 9.2 in Appendix 1 of [20], we have

\[
I_{T,1} \lesssim c_1 - N^2 D_T; \quad (7.7)
\]

indeed, recall the Dirichlet form \( D_T \) includes the local Dirichlet form corresponding to the Glauber-type dynamics near the boundary per the remarks at the beginning of this subsection.

Proceeding to estimate the \( I_T \)-quantity, we first make the following declaration.

- The third quantity within the summation defining \( I_T \) is a mirror-image copy of the second quantity therein; as the precise values of the coefficients corresponding to the boundary "Glauber" interactions are irrelevant, analysis of this last third quantity follows exactly analysis of the second quantity in the summation. Simply for this reason, for notational convenience, we drop this third quantity in this summation defining \( I_T \) below with the understanding that our final estimate for \( I_T \) remains valid with or without this dropped quantity up to a universal constant.

We proceed to decompose this new version of \( I_T \) as follows:

\[
I_T = I_{T,1} + I_{T,2}, \quad (7.8)
\]

where

\[
I_{T,1} = \frac{1}{2} N^2 \mathbb{E} f_T \sum_{k=1}^{\infty} \sum_{x, x+k \in \mathcal{F}_{N,0}} \frac{\alpha_k N}{\sqrt{N}} \frac{1 + \eta_x - \eta_{x+k}}{2} \left( \mathcal{S}_{x,x+k} \log f_T - \log f_T \right); \quad (7.9a)
\]

\[
I_{T,2} = \frac{1}{2} N^2 \mathbb{E} f_T \sum_{x=1}^{m_N} \beta_{x, \text{rem}}^N (\eta) (\mathcal{T}_x \log f_T - \log f_T). \quad (7.9b)
\]

Second, because the logarithm is concave, we deduce the following upper bound courtesy of Taylor expansion, for example, combined with the invariance of the grand-canonical ensemble with respect to any local Kawasaki operator \( \mathcal{S}_{x,y} \):

\[
I_{T,1} \leq \frac{1}{2} N^2 \mathbb{E} f_T \sum_{k=1}^{\infty} \sum_{x, x+k \in \mathcal{F}_{N,0}} \frac{\alpha_k N}{\sqrt{N}} \frac{1 + \eta_x - \eta_{x+k}}{2} \left( \mathcal{S}_{x,x+k} f_T - f_T \right) \leq \frac{1}{8} N^2 \sum_{k=1}^{\infty} \alpha_k N \sum_{x, x+k \in \mathcal{F}_{N,0}} \mathbb{E} [f_T \cdot (\eta_{x+k} - \eta_x)] . \quad (7.10)
\]

Observe that provided any \( k \in \mathbb{Z}_{>0} \) fixed, the summation over pairs of points in \( \mathcal{F}_{N,0} \) gives a telescoping sum; in particular, because the occupation variables, or spins, at each site are uniformly bounded, we see

\[
I_{T,1} \lesssim N^{\frac{1}{2}} \sum_{k=1}^{\infty} \alpha_k N \cdot k \mathbb{E} f_T \lesssim N^{\frac{1}{2}}; \quad (7.12)
\]

since \( f_T \) is a probability density with respect to the grand-canonical ensemble and the moments of \( \{\alpha_k N \}_{k=1}^{\infty} \) as a measure on \( \mathbb{Z}_{>0} \) are uniformly bounded in \( N \in \mathbb{Z}_{>0} \).

To estimate \( I_{T,2} \), we rely on the following observations.

- By construction, observe the quantity \( \beta_{x, \text{rem}}^N (\eta) \) is non-negative with probability 1; this is exactly the utility of this coefficient introduced towards the beginning of this proof.

- Again employing the concavity of the logarithm, we have

\[
f_T (\mathcal{T}_x \log f_T - \log f_T) \lesssim \mathcal{T}_x f_T - f_T. \quad (7.14)
\]
Thus, we first obtain the following upper bound for which we again crucially rely on the invariance of the grand-canonical ensemble at hand under flipping the spin or occupation variable at any deterministic site:

\[ \mathbb{I}_{T,2} \leq \frac{1}{2} N^2 \sum_{x=1}^{m_0} \mathbb{E} \beta_{x,\text{rem}}^{N,-} (\eta) (\mathcal{S}_x f_T - f_T) \]

\[ \leq N^2 \sum_{x=1}^{m_0} \| \beta_{x,\text{rem}}^{N,-} \|_{L^\infty} \mathbb{E} f_T \]

\[ \lesssim N^\frac{2}{3}, \]

the implied constant in the final upper bound is universal, as by Assumption 1.3 it is bounded by moments of the coefficients \( \{a^N_k\}_{k=1}^\infty \) and \( \{a^N_{\nu,\nu'}\}_{k=1}^\infty \) themselves. Ultimately, we deduce

\[ \frac{d}{dt} H_T \lesssim -N^2 D_T + N^\frac{2}{3}, \]

from which the result follows via integration combined with the observation that \( H \lesssim N \).

7.2. Interior Entropy Inequalities. Provided the Dirichlet form estimate in Proposition 7.1, if we ignore the local Dirichlet form corresponding to the boundary Glauber-type dynamics, then we recover the following analog of Lemma 3.12 in [23].

Lemma 7.3. Consider a time \( T \in \mathbb{R}_{>0} \) and any sub-lattice \( \Lambda \subseteq \mathcal{I}_{N,0} \) satisfying the constraint \( \Lambda \cap \mathcal{I}_{N,\beta} = \emptyset \) for some universal constant \( \beta \in \mathbb{R}_{>0} \).

Provided any functional \( \varphi : \Omega_\Lambda \rightarrow \mathbb{R} \), for any \( \kappa \in \mathbb{R}_{>0} \) we have

\[ \mathbb{E}^{\mu_{\Lambda}} \left( \varphi \tilde{f}_{T,N}^\Lambda \right) \leq \kappa^{-1} T^{-1} N^{-2} |\Lambda|^3 + \kappa^{-1} N^{-\frac{2}{3}} |\Lambda|^3 + \kappa^{-1} \sup_{\nu \in [-1,1]} \log \mathbb{E}^{\mu_{\Lambda}} e^{\kappa \nu}. \]

Moreover, we further recall the probability density \( \tilde{f}_{T,N}^\Lambda \) within Notation 3.10 in [23] given by the Radon-Nikodym derivative with respect to the probability measure \( \mu_{0,\Lambda}^{\text{re}} \) corresponding to the average of space-time translates of \( \mu_{0,N}^{\text{re}} \) over the time-domain \([0,T]\) and over \( \mathcal{I}_{N,0} \), which we then afterwards take the conditional expectation of with respect to the sub-lattice \( \Lambda \subseteq \mathcal{I}_{N,0} \).

The proof of Lemma 7.3 is exactly that of Lemma 3.12 in [23], so we omit it entirely.

7.3. Boundary Entropy Inequalities. In contrast with Lemma 7.3 above, we now explicitly exploit the boundary Glauber-type dynamics and, in particular, the local Dirichlet form that it provides. The primary distinction between the consequential boundary entropy inequality and Lemma 7.3 is the creation and annihilation of particles accounted for by the boundary interactions forces uniqueness of the invariant measure "near the boundary"; meanwhile, Kawasaki dynamics in the interior as in Lemma 7.3 admits a one-parameter family of invariant measures.

Lemma 7.4. Consider any \( T \in \mathbb{R}_{>0} \), any \( \beta \in \mathbb{R}_{>0} \) arbitrarily small but universal, and any sub-lattice \( \Lambda \subseteq \mathcal{I}_{N,0} \setminus \mathcal{I}_{N,\beta} \).

Provided any functional \( \varphi : \Omega_\Lambda \rightarrow \mathbb{R} \), we have

\[ \mathbb{E}^{\mu_{\Lambda}} \left( \varphi \tilde{f}_{T,N}^\Lambda \right) \leq \kappa^{-1} T^{-1} N^{-1} |\Lambda|^3 + \kappa^{-1} N^{-\frac{2}{3}} |\Lambda|^3 + \kappa^{-1} \log \mathbb{E}^{\mu_{\Lambda}} e^{\kappa \nu}. \]

Moreover, we further recall the probability density \( \tilde{f}_{T,N}^\Lambda \) from Notation 3.4 in [23] given by the Radon-Nikodym derivative with respect to the ensemble \( \mu_{0,\Lambda}^{\text{re}} \) corresponding to the average of the time-translates of \( \mu_{0,N}^{\text{re}} \) through the time-domain of \([0,T]\), which we afterwards take the conditional expectation of with respect to \( \Lambda \subseteq \mathcal{I}_{N,0} \).

We remark that the difference in the \( N \)-dependent factors in Lemma 7.4 compared to the \( N \)-dependent factors in Lemma 7.3 is consequence of the lack of averaging with respect to the spatial coordinate.

Following the proof for Lemma 7.3, it turns out that the proof for Lemma 7.4 is missing exactly one ingredient which is the following log-Sobolev inequality for the grand-canonical ensemble \( \mu_{0,N}^{\text{re}} \) itself, as opposed to the one-parameter family of canonical ensembles that was proved in [25] and used crucially in [23].
Lemma 7.5. Consider any $\beta \in \mathbb{R}_{>0}$ arbitrarily small but universal along with any sub-lattice $\Lambda \subseteq \mathcal{I}_{N,0} \setminus \mathcal{I}_{N,\beta}$.

Provided any probability measure $f \, d\mu_{0,\Lambda}^G$ on $\Omega_\Lambda$, we have

$$H_{0,\Lambda}^G(f) \lesssim |\Lambda|^2 D_{0,\Lambda}^G(f),$$

(7.21)

where the implied constant is universal.

Proof. Before we begin, we now declare that all expectations are taken with respect to the grand-canonical ensemble $\mu_{0,\Lambda}^G$. Moreover, let us define $\bar{f} = \sqrt{f}$.

Lastly, let us assume that the sub-lattice $\Lambda \subseteq \mathcal{I}_{N,0} \setminus \mathcal{I}_{N,\beta}$ is completely contained in the left-half, so that $\sup \Lambda \subseteq N^\beta$. In general, for sub-lattices completely contained in the right-half of $\mathcal{I}_{N,0} \setminus \mathcal{I}_{N,\beta}$, a mirror-image of the following analysis succeeds. Moreover, provided any sub-lattice of $\mathcal{I}_{N,0} \setminus \mathcal{I}_{N,\beta}$, we decompose such a sub-lattice into one piece contained in the left-half and another disjoint piece contained in the right-half; courtesy of convexity of both the relative entropy and the Dirichlet form, it then suffices to combine the respective left-half and right-half estimates.

The first step towards establishing Lemma 7.5 is the following intermediate inequality with universal implied constant:

$$H_{0,\Lambda}^G(f) \lesssim \sum_{x \in \Lambda} \mathbb{E} \left[ \frac{1 + \eta_x}{2} (\mathbb{T}_{x,-f} - \bar{f})^2 \right] + \sum_{x \in \Lambda} \mathbb{E} \left[ \frac{1 - \eta_x}{2} (\mathbb{T}_{x,+f} - \bar{f})^2 \right];$$

(7.22)

indeed, this is the log-Sobolev inequality for the classical Glauber dynamic, which follows from diagonalizing the Glauber dynamic over the sub-lattice $\Lambda \subseteq \mathcal{I}_{N,0} \setminus \mathcal{I}_{N,\beta}$, computing the one-site LSI constant, and extension by tensorization.

Thus, it remains to prove

$$\sum_{x \in \Lambda} \mathbb{E} \left[ \frac{1 + \eta_x}{2} (\mathbb{T}_{x,-f} - \bar{f})^2 \right] + \sum_{x \in \Lambda} \mathbb{E} \left[ \frac{1 - \eta_x}{2} (\mathbb{T}_{x,+f} - \bar{f})^2 \right] \lesssim |\Lambda|^2 D_{0,\Lambda}^G(f).$$

(7.23)

Consider a generic site $x \in \Lambda$; we make the following observations.

- Provided $\eta_x = 1$ and $\eta_1 = 1$, then $\mathbb{T}_{x,-} = \mathbb{S}_{1,x} \mathbb{T}_{1,-}$. Provided $\eta_x = 1$ and $\eta_1 = -1$, then $\mathbb{T}_{x,-} = \mathbb{T}_{1,x} \mathbb{S}_{1,x}$.

Courtesy of the previous observations, we have

$$\frac{1 + \eta_x}{2} (\mathbb{T}_{x,-f} - \bar{f})^2 = \frac{1 + \eta_x}{2} (\mathbb{S}_{1,x} \mathbb{T}_{1,-f} - \bar{f})^2 + \frac{1 + \eta_x}{2} \eta_1 \frac{1 - \eta_1}{2} (\mathbb{T}_{x,-f} - \bar{f})^2$$

(7.24)

$$= \frac{1 + \eta_x}{2} \left( \mathbb{S}_{1,x} \mathbb{T}_{1,-f} - \mathbb{T}_{1,-f} \right)^2 + \frac{1 + \eta_x}{2} \frac{1 - \eta_1}{2} \left( \mathbb{T}_{x,-f} - \mathbb{T}_{1,-f} \right)^2$$

(7.25)

$$\lesssim \frac{1 + \eta_x}{2} \frac{1 + \eta_1}{2} \left( \mathbb{S}_{1,x} \mathbb{T}_{1,-f} - \mathbb{T}_{1,-f} \right)^2 + \frac{1 + \eta_x}{2} \frac{1 + \eta_1}{2} \left( \mathbb{T}_{x,-f} - \mathbb{T}_{1,-f} \right)^2$$

(7.26)

Taking expectations and crucially relying on the invariance of the grand-canonical ensemble $\mu_{0,\Lambda}^G$ with respect to flipping the occupation variable or spin at any deterministic site, as well as its invariance under swapping the occupation variables or spins at any pair of deterministic sites, we deduce

$$\mathbb{E} \left[ \frac{1 + \eta_x}{2} \frac{1 + \eta_1}{2} (\mathbb{S}_{1,x} \mathbb{T}_{1,-f} - \mathbb{T}_{1,-f})^2 \right] = \mathbb{E} \left[ \frac{1 + \eta_x}{2} \frac{1 - \eta_1}{2} (\mathbb{S}_{1,x} \mathbb{T}_{1,-f} - \mathbb{T}_{1,-f})^2 \right]$$

(7.27a)

$$\leq \mathbb{E} \left[ (\mathbb{S}_{1,x} \mathbb{T}_{1,-f})^2 \right];$$

(7.27b)

$$\mathbb{E} \left[ \frac{1 + \eta_x}{2} \frac{1 + \eta_1}{2} (\mathbb{T}_{1,-f} - \mathbb{T}_{1,-f})^2 \right] \leq \mathbb{E} \left[ \frac{1 + \eta_1}{2} (\mathbb{T}_{1,-f} - \mathbb{T}_{1,-f})^2 \right];$$

(7.27c)

$$\mathbb{E} \left[ \frac{1 + \eta_x}{2} \frac{1 - \eta_1}{2} (\mathbb{S}_{1,x} \mathbb{T}_{1,-f} - \mathbb{T}_{1,-f})^2 \right] = \mathbb{E} \left[ \frac{1 - \eta_x}{2} \frac{1 + \eta_1}{2} (\mathbb{S}_{1,x} \mathbb{T}_{1,-f} - \mathbb{T}_{1,-f})^2 \right]$$

(7.27d)

$$\leq \mathbb{E} \left[ \frac{1 + \eta_1}{2} (\mathbb{T}_{1,-f} - \mathbb{T}_{1,-f})^2 \right];$$

(7.27e)

$$\mathbb{E} \left[ \frac{1 + \eta_x}{2} \frac{1 - \eta_1}{2} (\mathbb{S}_{1,x} \mathbb{T}_{1,-f})^2 \right] \leq \mathbb{E} \left[ (\mathbb{S}_{1,x} \mathbb{T}_{1,-f})^2 \right].$$

(7.27f)
Moreover, by the classical moving-particle lemma as in the classical two-blocks scheme in [15], we have

\[ E[(S_{1,x}f - f)^2] \lesssim |x| \sum_{y=1}^{x-1} E[(S_{y,y+1}f - f)^2]. \] (7.28)

Now, observe that all of these upper bounds are local Dirichlet forms arising in $D^{ac}_{0,a}(f)$. In particular, provided the ellipticity bound $\alpha_1^N \geq 1$ and $\beta_1^N \geq 1$ both with universal implied constant, the latter courtesy of Assumption 1.3, we have

\[ \sum_{x \in \Lambda} E \left[ \frac{1 + \eta}{2} (\xi_{x,f} - f)^2 \right] \lesssim \sum_{x \in \Lambda} E \left[ (S_{1,x}f - f)^2 \right] + \sum_{x \in \Lambda} E \left[ \frac{1 + \eta}{2} (\xi_{1,f} - f)^2 \right] \] (7.29)

\[ \lesssim \sum_{x \in \Lambda} |x| \sum_{y=1}^{x-1} E[(S_{y,y+1}f - f)^2] + |\Lambda| E \left[ \frac{1 + \eta}{2} (\xi_{1,f} - f)^2 \right] \] (7.30)

\[ \lesssim a_1^\rho, \beta_1^\rho \sum_{x \in \Lambda} |x| \cdot D^{ac}_{0,a}(f) + |\Lambda| \cdot D^{ac}_{0,a}(f) \] (7.31)

\[ \lesssim |\Lambda|^2 D^{ac}_{0,a}(f). \] (7.32)

An entirely analogous calculation shows that

\[ \sum_{x \in \Lambda} E \left[ \frac{1 - \eta}{2} (\xi_{x,f} - f)^2 \right] \lesssim a_1^\rho, \beta_1^\rho |\Lambda|^2 D^{ac}_{0,a}(f). \] (7.33)

This completes the proof. \qed

7.4. Interior Dynamical Analysis. Having established the relevant preliminaries, the purposes for the current subsection are to introduce the local dynamical replacement lemma from [23] although localized to estimates for quantities supported uniformly away from the boundary. In particular, this first dynamical strategy concerns statistics of the particle system on domains where the particle dynamic is purely Kawasaki dynamics, hence the qualifier "interior".

Before we state the estimates, like Section 5 in [23] we introduce some notation for clean presentation.

Notation 7.6. Consider any universal constant $\beta \in \mathbb{R}_{>0}$ and any arbitrarily small but universal constant $\epsilon_x \in \mathbb{R}_{>0}$.

For any pseudo-gradient field $g^N$ and any field $\overline{g}^N$ admitting a pseudo-gradient factor, for $f^N = g^N$ and $\overline{f}^N = \overline{g}^N$, define

\[ \text{Av}_{t,x}^N = \sum_{y=1}^{N} \tau_{-\varphi}^N \tau_{t,x}^N \quad \text{and} \quad \text{Av}_{t,x}^N = \sum_{y=1}^{N} \tau_{-\varphi}^N \tau_{t,x}^N \] (7.34)

Moreover, consider any time-scale $\tau \in \mathbb{R}_{>0}$ and any universal constants $\beta_\pm \in \mathbb{R}_{>0}$; we define

\[ \Omega_{t,x,\tau}^N = \left| \int_0^\tau \tau \text{Av}_{t,x}^N \right| \] (7.35)

The following is the first primary estimate for this subsection; roughly speaking, it provides a priori multiscale estimates which are dynamical in nature.

Proposition 7.7. Throughout the entirety of this statement, we let $g^N$ denote a pseudo-gradient field and we recall $\beta = \frac{1}{4} + \epsilon$ for $\epsilon \in \mathbb{R}_{>0}$ arbitrarily small but universal.

We define two sequences $\{\beta_m\}_{m=0}^M$ and $\{\tau_N^{(m)}\}_{m=1}^M$ along with the terminal index $M \in \mathbb{Z}_{>0}$ via the following algorithm:

- Define $\beta_0 = \frac{1}{2} \beta - \epsilon$.
- Consider any sequence $\{\epsilon_m\}_{m=1}^M$ of arbitrarily small but universal constants depending only on $\epsilon \in \mathbb{R}_{>0}$.
- Provided any index $m \in \{0, M - 1\}$, define $\beta_{m+1} = \beta_m + \epsilon_m$.
- Moreover, provided any index $m \in \{1, M\}$, we define $\tau_N^{(m)} = N^{-\frac{1}{2} - \beta_{m+1} - \epsilon_{m+1}}$, recalling $\epsilon \in \mathbb{R}_{>0}$ is arbitrarily small but universal.
- Define $M \in \mathbb{Z}_{>0}$ as the smallest positive integer for which $\beta_M > \frac{1}{2}$.  


Provided that each constant in the sequence \( \{\varepsilon_m\}_{m=1}^M \) is sufficiently small but universal, we have \( M \lesssim \{\varepsilon_m\}_{m=1}^M \) 1, so, in particular, the sequences are uniformly bounded in length with respect to \( N \in \mathbb{Z}_{>0} \).

Moreover, choosing parameters \( \varepsilon, \{\varepsilon_m\}_{m=1}^M \) appropriately depending only on \( \varepsilon_N \in \mathbb{R}_{>0} \), we have \( \tau_{N}^{(m)} \lesssim N^{-\frac{1}{\beta}} \) with a universal implied constant uniformly in \( m \in [1, M] \). The sequence \( \{\tau_{N}^{(m)}\}_{m=1}^M \) is also strictly increasing.

Lastly, there exists a universal constant \( \beta_u \in \mathbb{R}_{>0} \) such that

\[
\sum_{m=1}^{M} \mathbb{E} \left[ \left\| \int_{0}^{T} \sum_{y \in \mathcal{F}_{N,1/2}} \phi_{S,T,x,y}^{N} \left[ N^{\frac{1}{2}} \mathcal{G}_{\varepsilon_{N} \phi_{S,T,x,y}^{N}}^{\beta_{\varepsilon_{N} \phi_{S,T,x,y}^{N}}} \right] \right\|_{L_{2}^{T}}^{2} \right] \lesssim \varepsilon, T_{1} N^{-\beta_{u}}. \tag{7.36}
\]

Consider another sequence \( \{\sigma_{N}^{(i)}\}_{i=0}^{L} \) of time-scales defined via the following algorithm:

- Define \( \sigma_{N}^{(0)} = N^{-2+2\beta_{u}} - 2x_{x} + \varepsilon' \).
- Provided any index \( \ell \in [0, L - 1] \), define \( \sigma_{N}^{(\ell+1)} = \sigma_{N}^{(\ell)} + \beta \), where \( \beta \) is an arbitrarily small but universal constant.
- Upon possibly adjusting the exponent \( \beta \) from the previous bullet point by a universal, uniformly positive factor, define the terminal index \( L \in \mathbb{Z}_{>0} \) to be the minimal positive integer for which \( \sigma_{N}^{(L)} \approx \varepsilon_{N} \).

As with the previous terminal index, we have \( L \lesssim \varepsilon_{N}^{1} \), so, in particular, this sequence \( \{\sigma_{N}^{(i)}\}_{i=0}^{L} \) is uniformly bounded in length with respect to \( N \in \mathbb{Z}_{>0} \). Moreover, this sequence of time-scales is strictly increasing and satisfies the estimate \( \sigma_{N}^{(i)} \lesssim N^{-\frac{1}{\beta}} \) for universal implied constant and uniformly over \( \ell \in [0, L] \).

Lastly, upon possibly adjusting \( \beta_u \in \mathbb{R}_{>0} \) by a universal and uniformly positive factor, we have

\[
\sum_{\ell=0}^{L-1} [\sigma_{N}^{(\ell+1)}]^{-\frac{1}{2}} \mathbb{E} \left[ \left\| \int_{0}^{T} \sum_{y \in \mathcal{F}_{N,1/2}} \phi_{S,T,x,y}^{N} \left[ N^{\frac{1}{2}} \mathcal{G}_{\varepsilon_{N} \phi_{S,T,x,y}^{N}}^{\beta_{S,T,x,y}^{N},\sigma_{N}^{(\ell)}} \right] \right\|_{L_{2}^{T}}^{2} \right] \lesssim \varepsilon, \beta \ N^{-\beta_{u}}. \tag{7.37}
\]

Proof. The proof of the first estimate (7.36) and preceding bounds on sequences prior to its statement within Proposition 7.7 follows exactly the proofs for Proposition 5.3 and Corollary 5.4 in [23] along with the following modifications.

For simplicity, define \( \tau = \tau_{N}^{(m)} \). Proceeding as in the proof of Proposition 5.3 in [23], we now write

\[
\int_{0}^{T} \sum_{y \in \mathcal{F}_{N,1/2}} \phi_{S,T,x,y}^{N} \left[ N^{\frac{1}{2}} \mathcal{G}_{\varepsilon_{N} \phi_{S,T,x,y}^{N}}^{\beta_{S,T,x,y}^{N},\tau} \right] dS = \int_{0}^{T_{1}} \sum_{y \in \mathcal{F}_{N,1/2}} \phi_{S,T,x,y}^{N} \left[ N^{\frac{1}{2}} \mathcal{G}_{\varepsilon_{N} \phi_{S,T,x,y}^{N}}^{\beta_{S,T,x,y}^{N},\tau} \right] dS + \int_{T_{1}}^{T} \sum_{y \in \mathcal{F}_{N,1/2}} \phi_{S,T,x,y}^{N} \left[ N^{\frac{1}{2}} \mathcal{G}_{\varepsilon_{N} \phi_{S,T,x,y}^{N}}^{\beta_{S,T,x,y}^{N},\tau} \right] dS, \tag{7.38}
\]

with \( T_{1} \in [0, T] \) to be determined. We apply the on-diagonal estimate in Lemma 6.5 for the heat kernel \( \mathcal{H}_{N} \) to obtain

\[
\int_{T_{1}}^{T} \sum_{y \in \mathcal{F}_{N,1/2}} \phi_{S,T,x,y}^{N} \left[ N^{\frac{1}{2}} \mathcal{G}_{\varepsilon_{N} \phi_{S,T,x,y}^{N}}^{\beta_{S,T,x,y}^{N},\tau} \right] dS \lesssim \varepsilon, \tau_{1} N^{-\frac{1}{2}} + 2\varepsilon \int_{T_{1}}^{T} \mathcal{G}_{\varepsilon_{N} \phi_{S,T,x,y}^{N}}^{\beta_{S,T,x,y}^{N},\tau} \sum_{y \in \mathcal{F}_{N,1/2}} \mathcal{G}_{\varepsilon_{N} \phi_{S,T,x,y}^{N}}^{\beta_{S,T,x,y}^{N},\tau} dS. \tag{7.39}
\]

At this point, we choose \( T_{1} \in [0, T] \) appropriately like in the proof of Proposition 5.3 in [23] and then follow the argument in the proof of this proposition as well to obtain (7.36); this additionally is at least expedited via the following observations.

- We emphasize the key remark that because \( \tau_{N}^{(m)} \lesssim N^{-\frac{1}{\beta}} \) uniformly in the index \( m \in [1, M] \) with a universal implied constant, the procedure of localization-via-coupling implemented for the proof of Proposition 5.3 in [23] remains equally valid in this case because the domains for the "localized" dynamics in this procedure are thus of size \( \ll N^{\frac{1}{\beta}} \), therefore avoiding the boundary of \( \mathcal{F}_{N,0} \) by definition of the block \( \mathcal{F}_{N,1/2} \subseteq \mathcal{F}_{N,0} \).

Concerning the estimate (7.37), we first mention the preceding bounds on the sequence \( \{\sigma_{N}^{(i)}\}_{i=0}^{L} \) follow exactly like in the proofs of both Proposition 5.10 and Corollary 5.11 in [23]; the estimate (7.37) also follows from the exact same arguments.
with the following preliminary adjustment of an estimate – like for our discussion for (7.36), we have

\[
\int_0^T \sum_{y \in \mathcal{F}_{N/2}} |\Omega_{S,T,x,y}^{N+1}| \cdot |N^+ \mathbf{q}_{u,y}^{(N+1)}| \, ds = \int_0^T \mathbf{q}_{S,T}^{1-2} \sum_{y \in \mathcal{F}_{N/2}} |\Omega_{S,T,x,y}^{N+1}| \cdot \left[ N^+ \mathbf{q}_{u,y}^{(N+1)} \right] \, ds
\]  

(7.40)

\[
\leq \left[ \int_0^T \mathbf{q}_{S,T}^{1-2} \, ds \right]^\frac{1}{2} \left[ \int_0^T \sum_{y \in \mathcal{F}_{N/2}} |\Omega_{S,T,x,y}^{N+1}| \cdot \left[ N^+ \mathbf{q}_{u,y}^{(N+1)} \right]^2 \, ds \right]^\frac{1}{2}
\]  

(7.41)

\[
\lesssim_{K,T,e} N^{(1+2\varepsilon} \left[ \int_0^T \sum_{y \in \mathcal{F}_{N/2}} \left| \mathbf{q}_{u,y}^{(N+1)} \right|^2 \, ds \right]^\frac{1}{2} + e^{-\log^{100} N}. 
\]  

(7.42)

At this point, we proceed exactly as in the proof of Proposition 5.10 to obtain (7.37) and this completes the proof. \(\square\)

In addition to the estimates within Proposition 7.7, we also require the following analog for the second set of multiscale estimates therein, in which the stochastic fundamental solution is replaced by its time-gradient.

**Proposition 7.8.** *Retain the setting of Proposition 7.7; we have*

\[
\sum_{l=0}^{L-1} E \left[ \left\| \int_0^T \sum_{y \in \mathcal{F}_{N/2}} |\Omega_{S,T,x,y}^{N} \cdot \left[ N^+ \mathbf{q}_{u,y}^{(N)} \right] | \, ds \right\|_{L^2_T} \right] \lesssim_{T,e} N^{-\beta_e}. 
\]  

(7.43)

**Proof.** As with the proof of Proposition 7.7, for \(e \in \mathbb{R}_{>0}\) arbitrarily small, we have

\[
\int_0^T \sum_{y \in \mathcal{F}_{N/2}} |\Omega_{S,T,x,y}^{N} \cdot \left[ N^+ \mathbf{q}_{u,y}^{(N)} \right] | \, ds = \int_0^T \mathbf{q}_{S,T}^{1-2\varepsilon} \sum_{y \in \mathcal{F}_{N/2}} \left| \Omega_{S,T,x,y}^{N} \cdot \left[ N^+ \mathbf{q}_{u,y}^{(N)} \right] \right| \, ds \]  

(7.44)

which is further bounded from above by the following quantity courtesy of applying the Cauchy-Schwarz inequality with respect to the time-integral up to a constant depending only on \(e, T_f \in \mathbb{R}_{>0}\):

\[
\Phi = \left[ \int_0^T \mathbf{q}_{S,T}^{1-2\varepsilon} \left( \sum_{y \in \mathcal{F}_{N/2}} \left| \Omega_{S,T,x,y}^{N} \cdot \left[ N^+ \mathbf{q}_{u,y}^{(N)} \right] \right|^2 \right) \, ds \right]^\frac{1}{2}.
\]  

(7.45)

Another application of the Cauchy-Schwarz inequality with respect to the spatial summation over \(\mathcal{F}_{N/2} \subseteq \mathcal{F}_{N,0}\) provides the following estimate courtesy of the estimates in Lemma 6.5 with probability at least \(1 - \kappa_D N^{-D}\) for any \(D \in \mathbb{R}_{>0}\) with \(\kappa_D \in \mathbb{R}_{>0}\) universal outside its dependence on \(D \in \mathbb{R}_{>0}\):

\[
\Phi \lesssim \int_0^T \mathbf{q}_{S,T}^{1-2\varepsilon} \left( \sum_{y \in \mathcal{F}_{N/2}} \left| \Omega_{S,T,x,y}^{N} \cdot \left[ N^+ \mathbf{q}_{u,y}^{(N)} \right] \right|^2 \right) \, ds
\]  

(7.46)

\[
\lesssim e^{N^{(1+2\varepsilon} \left[ \sigma_{N}^{(l)} \right]^{-1-\varepsilon} \int_0^T \sum_{y \in \mathcal{F}_{N/2}} \left[ N^+ \mathbf{q}_{u,y}^{(N)} \right]^2 \, ds,
\]  

(7.47)

from which we may estimate the expectation of the final quantity using the proof of (7.37) in Proposition 7.7. It remains to compute the desired expectation on the event for which the a priori estimates in Lemma 6.5 fail to hold; in this case, the result follows from the uniform \(\mathcal{L}_0^\infty\)-bound for the pseudo-gradient functionals and the high-moment bounds on \(\Omega^{N}\) from Lemma 6.2. This completes the proof. \(\square\)

Proposition 7.7 provides some interior-exclusive variation of the dynamical one-block strategy from [23] which succeeds even with the significantly less sharp and weaker estimate on entropy production from Proposition 7.1. Following Section 5 of [23], the following second primary set of estimates for the current subsection addresses the same problem for random fields with pseudo-gradient factors and weaker \(N\)-dependent prefactor but no a priori bounds coming from averaging with respect to the spatial coordinate.
Proposition 7.9. Throughout the entirety of this statement, we first let \( \widetilde{g}^{N} \) denote any space-time random field with a pseudo-gradient factor whose support is uniformly bounded in size with respect to \( N \in \mathbb{Z}_{>0} \). Moreover, we assume the support of \( \widetilde{g}^{N} \) is bounded above by \( N^{\beta_{X}+\epsilon} \) in size; recall \( \beta_{X} = \frac{1}{2} + \epsilon \) and \( \epsilon \in \mathbb{R}_{>0} \) is arbitrarily small but universal.

We define two sequences \( \{ \beta_{m}^{(m)} \}_{m=0}^{M} \) and \( \{ \tau_{N}^{(m)} \}_{m=1}^{M} \) along with the terminal index \( M \in \mathbb{Z}_{>0} \) via the following algorithm:

- Define \( \beta_{0} = 4\beta_{X} - \epsilon_{X} \).
- Consider any sequence \( \{ \epsilon_{m}^{(m)} \}_{m=1}^{M} \) of arbitrarily small but universal constants depending only on \( \epsilon \in \mathbb{R}_{>0} \).
- Provided any index \( m \in [0, M - 1] \), define \( \beta_{m+1}^{(m)} = \beta_{m}^{(m)} + \epsilon_{m}^{(m)} \).
- Moreover, provided any index \( m \in [1, M] \), define \( \tau_{N}^{(m)} = N^{-\frac{1}{2} \beta_{m}^{(m)} + \beta_{m}^{(m)} + \epsilon_{m}^{(m)}} \), recalling \( \epsilon \in \mathbb{R}_{>0} \) is arbitrarily small but universal.
- Define \( M \in \mathbb{Z}_{>0} \) as the smallest positive integer for which \( \beta_{M} > \beta_{X} + \epsilon \).

Provided that each constant in the sequence \( \{ \epsilon_{m}^{(m)} \}_{m=1}^{M} \) is sufficiently small but universal, we have \( M \lesssim (\epsilon_{m}^{(m)})_{m=1}^{M} \), so, in particular, the sequences are uniformly bounded in length with respect to \( N \in \mathbb{Z}_{>0} \).

Moreover, choosing parameters \( \epsilon, \{ \epsilon_{m}^{(m)} \}_{m=1}^{M} \) appropriately depending on \( \epsilon_{X} \in \mathbb{R}_{>0} \), we have \( \tau_{N}^{(m)} \lesssim N^{-\frac{1}{2} + \epsilon} \) with a universal implied constant uniformly in \( m \in [1, M] \). The sequence \( \{ \tau_{N}^{(m)} \}_{m=1}^{M} \) is also strictly increasing.

Lastly, there exists a universal constant \( \beta_{u} \in \mathbb{R}_{>0} \) such that

\[
\sum_{m=1}^{M} \mathbb{E} \left[ \left\| \int_{0}^{T} \sum_{y \in \mathcal{M}^{\infty}_{1/2}} \varphi_{S_{T},x,y}^{N} \cdot \left[ N_{\beta_{X} + \epsilon_{m}^{(m)} \beta_{m}^{(m)} + \beta_{m}^{(m)} + \epsilon_{m}^{(m)}} \right] \right\|_{L^{\infty}_{T}L^{\infty}_{x,y}} \right] \lesssim_{M,T_{f}} N^{-\beta_{u}}. \tag{7.48}
\]

Consider another sequence \( \{ \sigma_{N}^{(L)} \}_{L=0}^{L} \) of time-scales defined via the following algorithm:

- Define \( \sigma_{N}^{(0)} = N^{-2+\epsilon} \).
- Provided any index \( L \in [0, L - 1] \), define \( \sigma_{N}^{(L+1)} = \sigma_{N}^{(L)} N^{\epsilon_{L}} \), where \( \epsilon_{L} \in \mathbb{R}_{>0} \) is another arbitrarily small but universal constant.
- Upon possibly adjusting the exponent \( \epsilon_{L} \in \mathbb{R}_{>0} \) from the previous bullet point by a universal, uniformly positive factor, define the terminal index \( L \in \mathbb{Z}_{>0} \) to be the minimal positive integer for which \( \sigma_{N}^{(L)} = \tau_{N}^{(M)} \).

As with the previous terminal index, we have \( L \lesssim \epsilon_{L} \), so, in particular, the sequence \( \{ \sigma_{N}^{(L)} \}_{L=0}^{L} \) is uniformly bounded in length with respect to \( N \in \mathbb{Z}_{>0} \). Moreover, this sequence of time-scales is strictly increasing and satisfies the inequality \( \sigma_{N}^{(L)} \lesssim N^{-\frac{1}{2} + \epsilon} \) with universal implied constant and uniformly over \( L \in [0, L] \).

Lastly, upon possibly adjusting \( \beta_{u} \in \mathbb{R}_{>0} \) by a universal and uniformly positive factor, we have

\[
\sum_{L=0}^{L-1} \mathbb{E} \left[ \left\| \int_{0}^{T} \sum_{y \in \mathcal{M}^{\infty}_{1/2}} \varphi_{S_{T},x,y}^{N} \cdot \left[ N_{\beta_{X} + \epsilon_{L} \sigma_{N}^{(L)} + \beta_{L}^{(L)} + \epsilon_{L} \sigma_{N}^{(L)}} \right] \right\|_{L^{\infty}_{T}L^{\infty}_{x,y}} \right] \lesssim_{\epsilon_{L},\epsilon_{L}} N^{-\beta_{u}}; \tag{7.49a}
\]

\[
\sum_{L=0}^{L-1} \mathbb{E} \left[ \left\| \int_{0}^{T} \sum_{y \in \mathcal{M}^{\infty}_{1/2}} \varphi_{S_{T},x,y}^{N} \cdot \left[ N_{\beta_{X} + \epsilon_{L} \sigma_{N}^{(L)} + \beta_{L}^{(L)} + \epsilon_{L} \sigma_{N}^{(L)}} \right] \right\|_{L^{\infty}_{T}L^{\infty}_{x,y}} \right] \lesssim_{\epsilon_{L},\epsilon_{L}} N^{-\beta_{u}}. \tag{7.49b}
\]

The proof of Proposition 7.9 amounts to those exact same modifications for Proposition 5.8, Corollary 5.9, Proposition 5.12, and Corollary 5.13 in [23] that were introduced in the proof of Proposition 7.7 along with the proof of Proposition 7.8.

7.5. Boundary Dynamical Analysis. We proceed to develop a dynamical fluctuation analysis for functionals of the particle system supported near the boundary of the domain \( \mathcal{S}_{N,0} \subseteq \mathcal{S}_{>0} \) which serves as an analog of Proposition 7.7, Proposition 7.8, and Proposition 7.9 for the boundary interactions.

Proposition 7.10. Consider any \( \beta \in \mathbb{R}_{>0} \) and the corresponding sub-lattice \( \mathcal{S}_{N,\beta} \) along with any weakly-vanishing functional \( q^{N} \) supported in \( \mathcal{S}_{N,0} \setminus \mathcal{S}_{N,\beta} \). Now, define the time-scale \( \tau = N^{-2+\epsilon} \) with \( \epsilon \in \mathbb{R}_{>0} \) satisfying the inequality \( \epsilon \lesssim \beta \) with universal implied constant.
Provided \( \epsilon, \beta \in \mathbb{R}_{>0} \) are arbitrarily small although universal, there exists a constant \( \beta_u \in \mathbb{R}_{>0} \) depending only on \( \beta, \epsilon \in \mathbb{R}_{>0} \) such that

\[
\mathbb{E} \left[ \left\| \int_0^T \sum_{y \in \mathcal{J}_{N,0} \setminus \mathcal{J}_{N,\beta}} \frac{\partial^2}{\partial y^2} N^{q_{0}\tau} \cdot \left[ N^{q_{0}\tau} \right] dS \right\| \right] \approx_T N^{-\beta_u}. \tag{7.50}
\]

**Proof.** Courtesy of the estimate within Lemma 4.4 and the Holder inequality taken with respect to the space-time "integral", we have

\[
\int_0^T \sum_{y \in \mathcal{J}_{N,0} \setminus \mathcal{J}_{N,\beta}} \frac{\partial^2}{\partial y^2} N^{q_{0}\tau} \cdot \left[ N^{q_{0}\tau} \right] dS \approx_T N^{2\beta+\epsilon} \left[ \int_0^T \sum_{y \in \mathcal{J}_{N,0} \setminus \mathcal{J}_{N,\beta}} \left| \frac{\partial^2}{\partial y^2} N^{q_{0}\tau} \right|^3 dS \right] \tag{7.51}
\]

This final upper bound is uniform in space-time, so it remains to estimate its expectation like in the proofs of Proposition 7.7, Proposition 7.8, and Proposition 7.9. To this end, by the Holder inequality we first have

\[
\mathbb{E} \left[ \left( \int_0^T \sum_{y \in \mathcal{J}_{N,0} \setminus \mathcal{J}_{N,\beta}} |q_{0}\tau|^3 dS \right)^{\frac{1}{3}} \right] \leq \left( \mathbb{E} \left[ \int_0^T \sum_{y \in \mathcal{J}_{N,0} \setminus \mathcal{J}_{N,\beta}} |q_{0}\tau|^3 dS \right] \right)^{\frac{1}{3}} \tag{7.54}
\]

Again exactly as in the proofs of Proposition 7.7, Proposition 7.8, and Proposition 7.9, as detailed within Section 5 of [23], courtesy of the boundary entropy inequality of Lemma 7.4, we have the following for any \( \delta \in \mathbb{R}_{>0} \) arbitrarily small though universal, in which \( \Lambda_1 = \mathcal{J}_{N,0} \setminus \mathcal{J}_{N,\beta} \) and \( \Lambda_2 = \mathcal{J}_{N,0} \setminus \mathcal{J}_{N,2\beta} \):

\[
\mathbb{E} \left[ \int_0^T \sum_{y \in \mathcal{J}_{N,0} \setminus \mathcal{J}_{N,\beta}} |q_{0}\tau|^3 dS \right] \leq \delta N^{-\frac{1}{2}+\beta+\delta} + N^\beta \sup_{y \in \Lambda_1} \log \mathbb{E}^{\mu_{0,\Lambda}}_y \exp \left[ \left| q_{0}\tau \right| \right] \tag{7.55}
\]

above, the object \( \tilde{\mathcal{A}}^0 \) is the expectation over the particle dynamic, which we adjust to have the same boundary condition on \( \Lambda_2 \subseteq \mathcal{J}_{N,0} \) of \( \mathcal{A}^0 \) upon conditioning on the initial data which is the configuration sampled in the right-most expectation \( \mathbb{E}^{\mu_{0,\Lambda}}_y \).

For \( \beta \in \mathbb{R}_{>0} \) arbitrarily small but universal, we choose \( \delta \in \mathbb{R}_{>0} \) sufficiently small to estimate the first quantity. Concerning the remaining log-exponential moment, we proceed exactly like in the proof of either Proposition 7.7 or Proposition 5.3 in [23] and use elementary convexity inequalities; as \( q^N \) is uniformly bounded, this gives

\[
\sup_{y \in \Lambda_1} \log \mathbb{E}^{\mu_{0,\Lambda}}_y \exp \left[ \left| q_{0}\tau \right| \right] \leq \sup_{y \in \Lambda_1} \mathbb{E}^{\mu_{0,\Lambda}}_y \left| q_{0}\tau \right| \tag{7.56}
\]

Recall that \( \tilde{\mathcal{A}}^0 \) is the expectation with respect to the particle dynamic localized with the same boundary condition but on \( \Lambda_2 \subseteq \mathcal{J}_{N,0} \). In particular, we remark that the dynamic does not admit \( \mu_{0,\Lambda}^{\infty} \) as its invariant measure necessarily. However, because it is within \( N^{-\frac{1}{2}} \) of a dynamic which does, and because the time-scale of this path-space expectation is just beyond the microscopic scale, we may replace the particle dynamic defining \( \tilde{\mathcal{A}}^0 \) by another particle dynamic which admits \( \mu_{0,\Lambda}^{\infty} \) as an invariant measure satisfying the log Sobolev inequality in Lemma 7.5 up to suitable error; precisely, as in the proof of Proposition 5.3 in [23], if \( \epsilon \geq \beta \) with sufficiently large but universal implied constant, we have

\[
\mathbb{E}^{\mu_{0,\Lambda}}_y \left| q_{0}\tau \right| \leq N^{-2\beta_u} + N^{-\frac{1}{2}+\epsilon}, \tag{7.57}
\]

which completes the proof; indeed, we crucially rely on the relation \( \mathbb{E}^{\mu_{0,\Lambda}}_y q^N = 0 \) to make the Kipnis-Varadhan inequality applicable. \( \Box \)
7.6. Static Analysis. We continue the current section by replicating Section 5.5 in [23], which implements a precise and quantitative version of the static one-block scheme. This lets us replace the spatial average of any generic pseudo-gradient field with an additional a priori cutoff or upper bound courtesy of large deviations theory for the canonical ensembles.

**Proposition 7.11.** Define $\beta = \beta_x + \varepsilon$ for $\varepsilon \in \mathbb{R}_{>0}$ an arbitrarily small but universal constant, and consider $\varepsilon_x \in \mathbb{R}_{>0}$ another arbitrarily small but universal constant. Let $g^N$ denote a pseudo-gradient field, and define

$$\widehat{\mathbf{A}}_{\phi^N_x} = \mathbf{A}_\phi^N \mathbf{1} \left[ \mathbf{A}_\phi^N \geq N^{-\frac{1}{2}} \beta_x + \varepsilon_x \right].$$

(7.58)

For some universal constant $\beta_u \in \mathbb{R}_{>0}$ we have

$$\mathbb{E} \left[ \left\| \int_0^T \sum_{y \in \mathcal{N}, y \neq 0} \mathbf{Q}^N_{S,T,x,y} \cdot \left[ N^\frac{1}{2} \widehat{\mathbf{A}}_{\phi^N_x} \right] dS \right\|_{L^\infty \mathcal{H}_x} \right] \lesssim T^{-\beta_u} N^{-\beta_u}. \quad (7.59)$$

**Proof.** The result follows exactly from the proof of Proposition 5.15 within [23] with the entropy inequality in Lemma 7.3 in place of Lemma 3.12 from [23]; indeed, by the definition of $\beta \in \mathbb{R}_{>0}$, all the relevant functionals of the particle system are supported away from the boundary. \hfill \Box

7.7. Consequences of Dynamical Analysis. We conclude this section concerning the relevant dynamical strategy towards the proof for Theorem 1.8 with consequences of Proposition 7.7, Proposition 7.8, Proposition 7.9, Proposition 7.10, and Proposition 7.11 which are key to the proof of Theorem 1.8. Before we proceed, however, we emphasize that the analysis behind the respective proofs for the upcoming consequences are almost entirely detailed within Section 7 of [23]; for this reason, we address explicitly only those issues which are not addressed in detail in Section 7 from [23], and otherwise we refer the reader there.

The primary result for this subsection is the following high-probability estimate. First, we introduce additional notation for the remainder of this article which will make presentation more convenient.

**Notation 7.12.** Retaining the setting of Proposition 2.11, define the functional

$$\Omega^N_{T,x} = N^\frac{1}{2} \sum_{w=1}^{N^\beta_x} \tau_{\omega}^N \mathbf{g}_{T,x}^N + N^\beta_x \mathbf{g}_{T,x}^N; \quad (7.60)$$

in particular, we have

$$3^N_{T,x} = \sum_{y \in \mathcal{N}, y \neq 0} \mathbf{Q}^N_{0,T,x,y} \cdot \Omega^N_{S,T,x,y} + \int_0^T \sum_{y \in \mathcal{N}, y \neq 0} \mathbf{Q}^N_{S,T,x,y} \cdot \Omega^N_{S,Y_S,T,Y_S} dS. \quad (7.61)$$

Indeed, the integral equation (7.61) will be our starting point towards the proof of Theorem 1.8 in Section 8; precisely, our first goal within Section 8 will be to justify dropping the space-time “integral” on the RHS of (7.61), thus providing us with a problem similar to that in [9].

**Proposition 7.13.** Consider the following events parameterized by $\beta_u, \varepsilon \in \mathbb{R}_{>0}$, in which the implied constant is universal:

$$\mathcal{E}_{T,x,\beta_u,\varepsilon} = \left\{ \left\| \int_0^T \sum_{y \in \mathcal{N}, y \neq 0} \mathbf{Q}^N_{S,T,x,y} \cdot \Omega^N_{S,Y_S,T,Y_S} 3^N_{S,Y_S} dS \right\|_{L^\infty \mathcal{H}_x} \geq N^{-\beta_u} + N^{-\beta_u} \| 3^N \|_{L^1 \mathcal{H}_x} \right\}. \quad (7.62)$$

Provided $\beta_u, \varepsilon \in \mathbb{R}_{>0}$ is sufficiently small, we have the following estimate for $\varepsilon \in \mathbb{R}_{>0}$ universal and $\beta_u, \varepsilon \in \mathbb{R}_{>0}$ universal:

$$\mathbb{P} \left[ \mathcal{E}_{T,x,\beta_u,\varepsilon} \right] \lesssim T^{-\beta_u} N^{-\beta_u}. \quad (7.63)$$

**Proof.** Observe that it suffices to condition on the a priori estimates in Lemma 6.5 because of the high-probability estimate derived therein; the same is true for conditioning on the a priori estimates in Lemma 6.8.
With these a priori estimates, following the proof of Proposition 7.1 in [23] combined with the results from Proposition 7.7, Proposition 7.8, Proposition 7.9, and Proposition 7.11, we are left with establishing the following estimates, in which \( \tau_1 = \sigma_N^{(0)} = N^{−2+2β_x−2\epsilon+ε} \) and \( \tau_2 = N^{−2+ε} \):

\[
E \left[ \left\| \int_0^T \sum_{y \in \mathcal{N}_{1/2}} |\mathcal{D}_{\tau_1} \mathcal{D}_{S,T,x,y}^N| \cdot N^{\frac{1}{2}} \mathcal{A}_{N,y}^{\beta_x} |dS| \right\|_{L^2_{x,y}} \right] \lesssim N^{-\beta_x}; \tag{7.64a}
\]

\[
E \left[ \left\| \int_0^T \sum_{y \in \mathcal{N}_{1/2}} |\mathcal{D}_{\tau_2} \mathcal{D}_{S,T,x,y}^N| \cdot N^{\beta_x} |\mathcal{A}_{S,y}^{N}| |dS| \right\|_{L^2_{x,y}} \right] \lesssim N^{-\beta_x}. \tag{7.64b}
\]

As with the proof of Proposition 7.1 in [23], the respective estimates (7.64a) and (7.64b) follow from the time-regularity estimate in Lemma 6.5 in a fashion which we detail only for (7.64a):

\[
\int_0^T \sum_{y \in \mathcal{N}_{1/2}} |\mathcal{D}_{\tau_1} \mathcal{D}_{S,T,x,y}^N| \cdot N^{\frac{1}{2}} \mathcal{A}_{N,y}^{\beta_x} |dS| \lesssim N^{1−\frac{1}{2}\beta_x+\frac{ε}{2} + \epsilon} \int_0^T \left[ \sum_{y \in \mathcal{N}_{1/2}} |\mathcal{D}_{\tau_1} \mathcal{D}_{S,T,x,y}^N|^2 \right]^\frac{1}{2} |dS|. \tag{7.65}
\]

\[
\lesssim N^{1−\frac{1}{2}\beta_x+\frac{ε}{2} + \epsilon} \int_0^T \left( \kappa_{S,T,x}^{\epsilon,ε,\delta} \right)^{1/2} |dS|. \tag{7.66}
\]

\[
\lesssim_{\tau_1} N^{\frac{1}{2}−\frac{1}{2}\beta_x+\frac{ε}{2} + \epsilon} + N^{\epsilon−\frac{1}{2}\beta_x}; \tag{7.67}
\]

above, the quantity \( \kappa_{S,T,x}^{\epsilon,ε,\delta} \in \mathbb{R}_{≥0} \) is the quantity \( \kappa_{S,T,x}^{\epsilon,\delta} \in \mathbb{R}_{≥0} \) within the statement of Lemma 6.5, where \( \delta = (ε, ε, ε) \), and the parameter \( \epsilon' \in \mathbb{R}_{≥0} \) is arbitrarily small though universal. By construction of the time-scale \( \tau_1 = \sigma_N^{(0)} = N^{−2+2β_x−2\epsilon+ε} \), we obtain (7.64a); the estimate (7.64b) follows from identical considerations. This completes the proof. \( \square \)

8. PROOF OF THEOREM 1.8

We split up the proof of Theorem 1.8 into the following three components.

- First, we show that the second integral from the RHS of (7.61) is negligible in the large-N limit, so that it remains to analyze the first quantity on the RHS of (7.61), at least with high-probability. Courtesy of Proposition 7.13, the only required ingredient is a high-probability estimate on \( \|\mathcal{Z}^N\|_{L^2_{x,y}} \); we achieve this through a stochastic continuity argument similar to that within Section 9 of [23].

- Second, we unfold the stochastic fundamental solution \( \mathcal{D}_{S,T,x,y}^N \) and show that the quantities beyond whatever appears in the large-N limit within the SHE are negligible in the large-N limit. This is similar to the procedure adopted in [9] through Lemma 2.5 therein, though we perform this estimate in a quantitative fashion.

- Third, after the procedures outlined in the previous two bullet points, we are left with exactly the spatial action of the heat propagator corresponding to \( \mathcal{Z}_{S,T,x,y}^N \) on the initial data \( \mathcal{Z}_{0,x}^N \) and the discrete approximation to the space-time white noise \( \mathcal{F}_{0,x} \). In this step, we perform the replacement \( \mathcal{Z}^N \nrightarrow \mathcal{Z}^N \), in which case we can inherit tightness of \( \mathcal{Z}^N \) from [22]. To identify subsequential limit points as solutions to SHE, we follow again the hydrodynamical approach made precise in [9] that was made quantitative within the previous bullet point.

Combining these steps would complete the proof of Theorem 1.8.

Throughout this section, it will be convenient to establish the following notation.

**Notation 8.1.** Provided any \( T \in \mathbb{R}_{≥0} \) and \( x \in \mathcal{N}_{0,0} \) define

\[
\mathcal{D}_{S,T,x,y}^{N} \equiv \sum_{y \in \mathcal{N}_{0,0}} \mathcal{D}_{0,T,x,y}^{N} \mathcal{Z}_{0,y}^{N}. \tag{8.1}
\]
Equivalently, the field \( \mathcal{Y}^N_{T,x} \) is the unique solution to the following stochastic parabolic problem with \( \beta_\delta \in \mathbb{R}_{>0} \) arbitrarily small but universal:

\[
\mathcal{Y}^N_{T,x} = \sum_{y \in \mathcal{F}_{N,0}} \mathcal{W}^N_{0,T,x,y} \mathcal{Z}^N_{0,y} + \int_0^T \sum_{y \in \mathcal{F}_{N,0}} \mathcal{W}^N_{S,T,x,y} \left[ \mathcal{Y}^N_{S,y} \, d\xi^N_{S,y} \right] + \int_0^T \sum_{y \in \mathcal{F}_{N,0}} \mathcal{W}^N_{S,T,x,y} \left[ w^N_{S,y} \mathcal{Y}^N_{S,y} \right] \, dS
\]  

(8.2)

\[+ \sum_{|k| \leq m_0} c_k \int_0^T \sum_{y \in \mathcal{F}_{N,0}} \mathcal{W}^N_{S,T,x,y} \left[ \nabla^k \left( 1_{y \in \mathcal{F}_{N,0}} w^N_{S,y} \mathcal{Y}^N_{S,y} \right) \right] \, dS \]

\[+ \int_0^T \sum_{y \in \mathcal{F}_{N,0}} \mathcal{W}^N_{S,T,x,y} \left[ Nw^N_{S,y} \mathcal{Y}^N_{S,y} \right] \, dS. \]

Moreover, we define \( \mathcal{X}^N \) via the stochastic integral equation

\[
\mathcal{X}^N_{T,x} = \sum_{y \in \mathcal{F}_{N,0}} \mathcal{W}^N_{0,T,x,y} \mathcal{Z}^N_{0,y} + \int_0^T \sum_{y \in \mathcal{F}_{N,0}} \mathcal{W}^N_{S,T,x,y} \left[ \mathcal{X}^N_{S,x} \, d\xi^N_{S,y} \right].
\]  

(8.3)

Lastly, we define \( \tilde{\mathcal{X}}^N \) via the stochastic integral equation

\[
\tilde{\mathcal{X}}^N_{T,x} = \sum_{y \in \mathcal{F}_{N,0}} \mathcal{W}^N_{0,T,x,y} \mathcal{Z}^N_{0,y} + \int_0^T \sum_{y \in \mathcal{F}_{N,0}} \mathcal{W}^N_{S,T,x,y} \left[ \tilde{\mathcal{X}}^N_{S,x} \, d\xi^N_{S,y} \right].
\]  

(8.4)

We reiterate that our interpretation of the aforementioned fields is that \( \mathcal{Y}^N \) serves as a proxy for \( \mathcal{Z}^N \) courtesy of step 1 in the previous outline. Similarly, the field \( \mathcal{X}^N \) serves as a proxy for \( \mathcal{Y}^N \), and \( \tilde{\mathcal{X}}^N \) serves as a proxy for \( \mathcal{X}^N \). Lastly, the field \( \tilde{\mathcal{X}}^N \) is afterwards identified to converge to the solution to SHE_{\mathcal{F}_N}.

8.1. A Priori Estimates. Before we proceed with the three-step procedure outlined above, we collect a priori estimates which will provide useful throughout this section. The first of these is a high-probability estimate for \( \mathcal{Y}^N, \mathcal{X}^N, \) and \( \tilde{\mathcal{X}}^N \).

Lemma 8.2. Consider the following event parameterized by \( \varepsilon \in \mathbb{R}_{>0} \), in which the implied constant is universal:

\[
E^\varepsilon \doteq \left\{ \| \mathcal{Y}^N \|_{L^\infty_{T,x}} + \| \mathcal{X}^N \|_{L^\infty_{T,x}} + \| \tilde{\mathcal{X}}^N \|_{L^\infty_{T,x}} \geq N^\varepsilon \right\}.
\]  

(8.5)

Provided any \( D \in \mathbb{R}_{>0} \) and \( \varepsilon \in \mathbb{R}_{>0} \), we have

\[
\mathbb{P}[E^\varepsilon] \lesssim_{\varepsilon,D} N^{-D}.
\]  

(8.6)

Proof. Because the results hold for the initial data \( T = 0 \), the estimate for \( \mathcal{Y}^N \) follows immediately from the result for the initial data combined with Lemma 6.2. For \( \mathcal{X}^N \), an identical reasoning holds upon proving the analogous estimate for the stochastic fundamental solution corresponding to the stochastic problem defining \( \mathcal{X}^N \), although this follows from identical reasoning as the proof of Lemma 6.2. Estimating \( \tilde{\mathcal{X}}^N \) requires identical considerations. This completes the proof. \( \square \)

The second a priori estimate which will serve helpful is the following gradient estimate for \( \mathcal{Y}^N \).

Lemma 8.3. Provided any \( \varepsilon \in \mathbb{R}_{>0} \) arbitrarily small but universal, consider the following event in which \( t_N = N^{1-\varepsilon} \):

\[
E^\varepsilon \doteq \left\{ \left\| \nabla_{t_N} \mathcal{Y}^N_{t_N,x} \right\|_{L^\infty_{t_N,x}} \geq N^{-\frac{1}{2}} \right\}.
\]  

(8.7)

Then for some universal constant \( \delta \in \mathbb{R}_{>0} \), we have

\[
\mathbb{P}[E^\varepsilon] \lesssim_{\varepsilon} N^{-\delta}.
\]  

(8.8)

Proof. Similar to the proof for Lemma 6.5, it suffices to establish arbitrarily high moment estimates for the spatial gradient uniformly in space-time; this amounts to following the proof of Proposition 3.2 in [9] for \( \mathcal{Y}^N \) on the lattice \( \mathcal{F}_{N,0} \subseteq \mathbb{Z}_{>0} \). \( \square \)
8.2. Step I. As briefly described in the outline at the beginning of this section, the primary goal for the current subsection is the following high-probability estimate.

**Proposition 8.4.** Provided any \( \beta \in \mathbb{R}_{>0} \), consider the following event with the implied constant universal:

\[
\mathcal{E}_\beta := \left\{ \left\| \mathcal{X}^{N}_{T,R,x} - \mathcal{Y}^{N}_{T,R,x} \right\|_{\mathbb{L}^\infty_{T,R,x}} \gtrsim N^{-\beta} \right\}.
\]  

(8.9)

There exist universal constants \( \beta_{u,1}, \beta_{u,2} \in \mathbb{R}_{>0} \) such that

\[
P\left[ \mathcal{E}_{\beta_{u,1}} \right] \lesssim_{\epsilon, m_N, \beta, T} N^{-\beta_{u,2}}.
\]  

(8.10)

**Proof.** Courtesy of Proposition 7.13, it suffices to prove that for any \( \epsilon \in \mathbb{R}_{>0} \) sufficiently small, we establish the following estimate with high-probability on the event \( \mathcal{E}_{T,\beta,v} \) defined therein:

\[
\left\| \mathcal{X}^{N} \right\|_{\mathbb{L}^\infty_{T,R,x}} \lesssim \epsilon N^\epsilon.
\]  

(8.11)

To this end, we follow the pathwise analysis within Section 9 from [23] provided the a priori estimate in Lemma 8.2; this completes the proof. \( \square \)

8.3. Step II. As discussed prior, in this subsection we proceed to compare \( \mathcal{Y}^{N} \) and \( \mathcal{X}^{N} \); this is precisely stated as follows.

**Proposition 8.5.** Provided any \( \beta \in \mathbb{R}_{>0} \), define the following event with universal implied constant:

\[
\mathcal{E}_\beta := \left\{ \left\| \mathcal{X}^{N}_{T,R,x} - \mathcal{Y}^{N}_{T,R,x} \right\|_{\mathbb{L}^\infty_{T,R,x}} \gtrsim N^{-\beta} \right\}.
\]  

(8.12)

There exist universal constants \( \beta_{u,1}, \beta_{u,2} \in \mathbb{R}_{>0} \) such that

\[
P\left[ \mathcal{E}_{\beta_{u,1}} \right] \lesssim_{T,R,x} N^{-\beta_{u,2}}.
\]  

(8.13)

Before we provide the proof of Proposition 8.5, we require first the following probabilistic estimate which we establish through analytic means; observe that if the interacting particle system both exhibited the grand-canonical ensemble \( \mu_{\beta, \epsilon, R, x, 0} \) as its invariant measure and began at this invariant measure, the following estimate is effectively the classical CLT. Since this previous condition is certainly false in general, we follow [9] and establish the estimate instead through the regularity of the microscopic Cole-Hopf transform \( \mathcal{Z}^{N} \).

**Lemma 8.6.** Provided any \( \epsilon, \delta \in \mathbb{R}_{>0} \), consider the following event with universal implied constant:

\[
\mathcal{E}_{\epsilon, \delta} := \left\{ \left\| \sum_{w=1}^{\left\lfloor N^{-\beta} \right\rfloor} T_{w} \mathcal{Y}^{N}_{T,R,x} \right\|_{\mathbb{L}^\infty_{T,R,x}} \gtrsim N^{-\delta} \right\}.
\]  

(8.14)

For \( \epsilon, \delta \in \mathbb{R}_{>0} \) sufficiently small but universal, there exists a universal constant \( \delta' \in \mathbb{R}_{>0} \) such that

\[
P\left[ \mathcal{E}_{\epsilon, \delta} \right] \lesssim_{T,R,x, \epsilon, \delta, \delta'} N^{-\delta'}.
\]  

(8.15)

**Remark 8.7.** We emphasize the current estimate from Lemma 8.6 is sub-optimal if compared to the result with respect to the product Bernoulli measure of appropriate parameter; however, this will be more than sufficient.

**Proof.** All statements to be presented in this proof hold with high-probability of at least \( 1 - N^{-\beta_u} \) with \( \beta_u \in \mathbb{R}_{>0} \) universal. Because we require a uniformly bounded number of these statements, the final conclusion of this analysis will also occur with probability at least \( 1 - N^{-\beta_u'} \) for another universal constant \( \beta_u' \in \mathbb{R}_{>0} \).

First, let us introduce the following event for \( \beta \in \mathbb{R}_{>0} \) arbitrarily small but universal:

\[
\mathcal{E}_{T,R,x}^{N} := \left\{ \mathcal{Y}^{N}_{T,R,x} \leq N^{-\beta} \right\}.
\]  

(8.16)

Although quite unlikely, or equivalently although \( \mathcal{E}_{T,R,x}^{N} \) has small probability in all likelihood, we technically have no method of proving this; however, like in the proof of Lemma 2.5 from [9], this event is actually "good" towards proving the desired estimate.
Second, we introduce this cutoff as in the proof of Lemma 2.5 in [9]:

\[
\left| \sum_{w=1}^{N^{\frac{1}{1-\varepsilon}}} \tau_w \eta_{T,x}^{\mathbb{N}} \cdot \varphi_{T,x}^{\mathbb{N}} \right| = \left| \sum_{w=1}^{N^{\frac{1}{1-\varepsilon}}} \tau_w \eta_{T,x}^{\mathbb{N}} \cdot \varphi_{T,x}^{\mathbb{N}} \right| 1_{\delta \varepsilon} + \left| \sum_{w=1}^{N^{\frac{1}{1-\varepsilon}}} \tau_w \eta_{T,x}^{\mathbb{N}} \cdot \varphi_{T,x}^{\mathbb{N}} \right| 1_{\delta \varepsilon} \cdot \gamma_c. \tag{8.17}
\]

By construction of the aforementioned cutoff, the first quantity on the RHS is certainly bounded above by \(N^{-\beta}\). Concerning the second quantity, we first observe that Lemma 8.2 provides the following high-probability estimate given any \(\delta \in \mathbb{R}_{>0}\):

\[
\left\| \varphi_{T,x}^{\mathbb{N}} \right\|_{\mathcal{L}_2(0)} \lesssim_{T, m_{\delta}, \delta} N^{\delta}. \tag{8.18}
\]

Like in the proof of Lemma 2.5 in [9], by definition of the Gartner transform we may now write the following representation of the fluctuations of the particle density in terms of the regularity of the microscopic Cole-Hopf transform for \(\ell_N \equiv N^{\frac{1}{1-\varepsilon}}\):

\[
\sum_{w=1}^{N^{\frac{1}{1-\varepsilon}}} \tau_w \eta_{T,x}^{\mathbb{N}} = -\frac{1}{2\lambda_N} N^\varepsilon \log \left[ 1 + \frac{\nabla \ell_N \varphi_{T,x}^{\mathbb{N}}}{\varphi_{T,x}^{\mathbb{N}}} \right]. \tag{8.19}
\]

Outside the event \(\delta_{T,x}^{N}\) for \(\beta \in \mathbb{R}_{>0}\) sufficiently small, courtesy of Proposition 8.4 we also have \(\varphi_{T,x}^{\mathbb{N}} \lesssim N^{\frac{1}{1-\varepsilon}}\). Moreover, by Proposition 8.4 once again, we now establish the following estimate uniformly over \(T \in \mathbb{R}_{>0}\) and \(x \in \mathcal{S}_{N,0}\) with \(\beta_0 \in \mathbb{R}_{>0}\) universal with high-probability, in which the second inequality follows from an application of Lemma 8.3:

\[
\left| \nabla \ell_N \varphi_{T,x}^{\mathbb{N}} \right| \lesssim \left| \nabla \ell_N \varphi_{T,x}^{\mathbb{N}} \right| + N^{-\beta_0} \lesssim N^{-\frac{\beta_0}{2}} + N^{-\beta_0}. \tag{8.20}
\]

Thus, by standard inequalities concerning the logarithm function, we obtain the following estimate for the fluctuations of the particle density:

\[
\left| \sum_{w=1}^{N^{\frac{1}{1-\varepsilon}}} \tau_w \eta_{T,x}^{\mathbb{N}} \right| \lesssim \frac{1}{2\lambda_N} N^\varepsilon \left\| \nabla \ell_N \varphi_{T,x}^{\mathbb{N}} \right\|_1 \lesssim \frac{1}{2\lambda_N} N^\varepsilon \left| \nabla \ell_N \varphi_{T,x}^{\mathbb{N}} \right| \lesssim N^{\varepsilon + \frac{\beta}{2} - \frac{\beta_0}{2}} + N^{\varepsilon + \frac{\beta}{2} - \beta_0}. \tag{8.22}
\]

For \(\varepsilon, \beta \in \mathbb{R}_{>0}\) sufficiently small depending only on the universal constant \(\beta_0 \in \mathbb{R}_{>0}\), this completes the proof.

This next preliminary ingredient is another consequence of spatial regularity; roughly speaking, it permits us to replace the weakly vanishing quantities appearing the integral equation defining \(\varphi^{N}\) with appropriate spatial averages.

**Lemma 8.8.** Consider the following quantities for \(\varepsilon \in \mathbb{R}_{>0}\) arbitrarily small but universal:

\[
\Upsilon_N^{T, x} = \int_0^T \sum_{y \in \mathcal{S}_{N,0}} \bar{w}_{S,T,x,y}^{N} \left( \mathbb{N}_{S,Y} \varphi_{S,Y}^{N} \right) \text{d}S; \tag{8.25a}
\]

\[
\Phi_N^{T, x} = \int_0^T \sum_{y \in \mathcal{S}_{N,0}} \bar{w}_{S,T,x,y}^{N} \left( \mathbb{N}_{S,Y} \varphi_{S,Y}^{N} \right) \text{d}S; \tag{8.25b}
\]

\[
\Phi_N^{T, x} = \sum_{|k| \leq m_{\delta}} c_k \int_0^T \sum_{y \in \mathcal{S}_{N,0}} \nabla \bar{w}_{S,T,x,y}^{N} \left( \mathbb{N}_{S,Y} \varphi_{S,Y}^{N} \right) \text{d}S; \tag{8.25c}
\]

\[
\Phi_N^{T, x} = \sum_{|k| \leq m_{\delta}} c_k \int_0^T \sum_{y \in \mathcal{S}_{N,0}} \nabla \bar{w}_{S,T,x,y}^{N} \left( \mathbb{N}_{S,Y} \varphi_{S,Y}^{N} \right) \text{d}S. \tag{8.25d}
\]
Moreover, provided $\delta \in \mathbb{R}_{>0}$, define the following events with universal implied constant:

\[ E_{\delta,1}^* = \left\{ \left\| \mathcal{T}^N_{T,t} - \mathcal{T}_{T,t}^N \right\|_{L^\infty_{x,t}} \gtrsim N^{-\delta} \right\}; \quad (8.26a) \]

\[ E_{\delta,2}^* = \left\{ \left\| \Phi^N_{T,t} - \Phi_{T,t}^N \right\|_{L^\infty_{x,t}} \gtrsim N^{-\delta} \right\}. \quad (8.26b) \]

Provided $\delta \in \mathbb{R}_{>0}$ sufficiently small but universal, there exists a universal constant $\delta' \in \mathbb{R}_{>0}$ such that

\[ \mathbb{P}[E_{\delta,1}^*] \lesssim_T m_{u,t'}, \delta, \delta' N^{-\delta'}; \quad (8.27a) \]

\[ \mathbb{P}[E_{\delta,2}^*] \lesssim_T m_{u,t'}, \delta, \delta' N^{-\delta'}. \quad (8.27b) \]

**Proof.** We provide the same first paragraph of disclaimers from the proof of Lemma 8.6.

Courtesy of Lemma 5.14 and Proposition 5.17 combined with the a priori estimate for $\mathcal{Y}^N$ in Lemma 8.2, we have the following estimates with high-probability for some $\delta'' \in \mathbb{R}_{>0}$:

\[ \mathcal{T}^N_{T,t} = \int_0^T \sum_{y \in \mathcal{F}_{N,0}}^T \mathcal{W}^N_{S,T,x,y} \cdot \left[ w_{S,y}^N \mathcal{Y}^N_{S,y} \right] dS + O(N^{-\delta''}); \quad (8.28a) \]

\[ \mathcal{T}^N_{T,t} = \int_0^T \sum_{y \in \mathcal{F}_{N,0}}^T \mathcal{W}^N_{S,T,x,y} \cdot \left[ \mathcal{Y}^N_{S,y} \mathcal{Y}^N_{S,y} \right] dS + O(N^{-\delta''}); \quad (8.28b) \]

\[ \mathcal{Y}^N_{T,t} = \sum_{|k| \leq m_N} c_k \int_0^T \sum_{y \in \mathcal{F}_{N,0}}^T \nabla^k_{x,y} \mathcal{W}^N_{S,T,x,y} \cdot \left[ 1_{y \in \mathcal{F}_{N,0}} \mathcal{W}_{S,y}^N \mathcal{Y}_{S,y} \right] dS + O(N^{-\delta''}); \quad (8.28c) \]

\[ \mathcal{Y}^N_{T,t} = \sum_{|k| \leq m_N} c_k \int_0^T \sum_{y \in \mathcal{F}_{N,0}}^T \nabla^k_{x,y} \mathcal{W}^N_{S,T,x,y} \cdot \left[ 1_{y \in \mathcal{F}_{N,0}} \mathcal{Y}^N_{S,y} \mathcal{Y}^N_{S,y} \right] dS + O(N^{-\delta''}). \quad (8.28d) \]

Like in the proof of Lemma 2.5 in [9], provided the spatial regularity estimate for $\mathcal{Y}^N$ within Lemma 8.3 and the a priori estimate in Lemma 8.2, we establish the desired probability estimate for the event $E_{\delta,1}$; we emphasize the utility behind the regularity estimate for the kernel $\mathcal{W}^N$ from Corollary 3.3 in [22] as well, because the gradient estimate therein applies to either the backwards or forwards spatial coordinate as the nearest-neighbor Laplacian is self-adjoint with respect to the uniform measure and any Robin boundary parameter.

To establish the stated probability estimate for $E_{\delta,2}$, an identical argument we inherited via Lemma 2.5 in [9] with the regularity estimate within Corollary 3.3 in [22] to estimate the probability of $E_{\delta,1}$ applies equally well if we instead employ the regularity estimate in Lemma 5.18 in place of Corollary 3.3 in [22]. This completes the proof.

As an immediate consequence of the preceding few ingredients combined with the quantitative classical one-block and two-blocks estimates in Proposition D.1, we obtain our final preliminary estimate towards the proof of Proposition 8.5.

**Lemma 8.9.** Retain the setting from Lemma 8.8, and provided any $\delta \in \mathbb{R}_{>0}$ define the following events with universal implied constant:

\[ F_{\delta,1}^* = \left\{ \left\| \mathcal{T}^N_{T,t} \right\|_{L^\infty_{x,t}} \gtrsim N^{-\delta} \right\}; \quad (8.29a) \]

\[ F_{\delta,2}^* = \left\{ \left\| \Phi^N_{T,t} \right\|_{L^\infty_{x,t}} \gtrsim N^{-\delta} \right\}. \quad (8.29b) \]

Provided $\delta \in \mathbb{R}_{>0}$ sufficiently small but universal, there exists a universal constant $\delta' \in \mathbb{R}_{>0}$ such that

\[ \mathbb{P}[F_{\delta,1}^*] + \mathbb{P}[F_{\delta,2}^*] \lesssim_T m_{u,t'}, \delta, \delta' N^{-\delta'}. \quad (8.30) \]

**Proof.** This follows immediately from Lemma 8.2, Lemma 8.6, Lemma 8.8 and Proposition D.1. \qed
We conclude with one more ingredient similar to Lemma 8.8; in particular, we exploit the regularity of the heat kernel $\mathcal{H}^N$ and the field $\Psi^N$ to replace the error quantities $w_i^{+,\pm}$ supported at the boundary by time-averages; we emphasize that the aforementioned is the time-regularity.

**Lemma 8.10.** Consider the following family of events parameterized by $\tau, \delta \in \mathbb{R}_{>0}$, in which the implied constant is universal:

\[
\mathcal{E}_{T, \tau} = \left\{ \int_0^T \sum_{y \in \mathcal{N}_0} \mathcal{H}_{S,T,x,y}^N \left[ w_{S,y}^{N,\pm} - \bar{\Phi}_0^{S,y} \tau \right] \Phi_{S,y}^N \, dS \right\} \gtrsim N^{-\delta}. \tag{8.31}
\]

Provided $\tau = N^{-2+\epsilon}$ with $\epsilon \in \mathbb{R}_{>0}$ sufficiently small though still universal, there exist universal constants $\beta, \alpha, \beta_{u,1}, \beta_{u,2} \in \mathbb{R}_{>0}$ such that

\[
P[\mathcal{E}_{T, \tau, \beta_{u,1}}] \lesssim T, \mathcal{N}_0, N^{-\beta_{u,2}}. \tag{8.32}
\]

**Proof.** First, we observe

\[
\int_0^T \sum_{y \in \mathcal{N}_0} \mathcal{H}_{S,T,x,y}^N \left[ w_{S,y}^{N,\pm} - \bar{\Phi}_0^{S,y} \tau \right] \Phi_{S,y}^N \, dS \leq \Phi + \Psi \leq \tau \left\| \Psi_{T,x}^N \right\|_{\mathcal{L}_T}^N, \tag{8.33}
\]

in which we have defined the quantities

\[
\Phi \equiv \int_0^T \sum_{y \in \mathcal{N}_0} \left| \mathcal{H}_{S,T,x,y}^N \right| \cdot w_{S,y}^{N,\pm} \Phi_{S,y}^N \, dS; \tag{8.34a}
\]

\[
\Psi \equiv \int_0^T \sum_{y \in \mathcal{N}_0} \left( \mathcal{H}_{S,T,x,y}^N \right) \cdot \Phi_{S,y}^N \, dS. \tag{8.34b}
\]

We first analyze the quantity $\Phi$; to this end, we first decompose this integral $\Phi = \Phi_1 + \Phi_2$ with the following procedure, in which $T_N^\ast = T - N^{-\delta'}$ with $\delta' \in \mathbb{R}_{>0}$ arbitrarily small though universal:

\[
\Phi_1 \equiv \int_0^{T_N} \sum_{y \in \mathcal{N}_0} \left| \mathcal{H}_{S,T,x,y}^N \right| \cdot w_{S,y}^{N,\pm} \Phi_{S,y}^N \, dS; \tag{8.35a}
\]

\[
\Phi_2 \equiv \int_0^T \sum_{y \in \mathcal{N}_0} \left( \mathcal{H}_{S,T,x,y}^N \right) \cdot \Phi_{S,y}^N \, dS. \tag{8.35b}
\]

To analyze $\Phi_1$, we employ the pointwise time-regularity estimate from Lemma 5.14; this gives the following for $\epsilon \in \mathbb{R}_{>0}$ arbitrarily small but universal:

\[
\Phi_1 \lesssim m_0, a_1^N, T_f, \mathcal{N}_0, \epsilon, \left\| \Psi_{T,x}^N \right\|_{\mathcal{L}_T}^N, \int_0^{T_N} \kappa_{S,T} \, dS, \tag{8.36}
\]

in which

\[
\kappa_{S,T} \equiv \theta_{S,T}^{-1/2 + \epsilon} \tau^{-1} + N^{-1+2\epsilon} \theta_{S,T}^{-1} + \theta_{S,T}^{-1/2}. \tag{8.37}
\]

In particular, integration then yields the upper bound

\[
\Phi_1 \lesssim m_0, a_1^N, T_f, \mathcal{N}_0, \epsilon, N^{1/2}, \tau^{1-\epsilon} \left\| \Psi_{T,x}^N \right\|_{\mathcal{L}_T}^N + N^{-1+2\epsilon} \log N \cdot \left\| \Psi_{T,x}^N \right\|_{\mathcal{L}_T}^N + \tau \left\| \Psi_{T,x}^N \right\|_{\mathcal{L}_T}^N. \tag{8.38}
\]

To estimate the $\Phi_2$-quantity, we drop the time-gradient and replace it by the heat kernel at distinct times; this provides

\[
\Phi_2 \lesssim m_0, a_1^N, T_f, \mathcal{N}_0, \epsilon, \left\| \Psi_{T,x}^N \right\|_{\mathcal{L}_T}^N, \int_0^T \theta_{R,T}^{-1/2} \, dR \tag{8.39}
\]

\[
\lesssim N^{-1/2} \left\| \Psi_{T,x}^N \right\|_{\mathcal{L}_T}^N. \tag{8.40}
\]
Thus, combining (8.33), (8.38), and (8.40), we have the estimate
\[
\left\| \int_0^T \sum_{y \in \mathcal{Y}_N} \mathcal{W}_{S,T,x,y} \left( w_{S,y}^{N,-} - w_{S,y}^{N,+} - \mathcal{A}_{S,y}^0 \right) \mathcal{Q}_{S,y}^N \, ds \right\|_{L^\infty_{T,x}} \lesssim T \left\| \mathcal{Q}_{T,x}^N \right\|_{L^\infty_{T,x}} + N^{-1+2\epsilon} \log N \cdot \left\| \mathcal{Q}_{T,x}^N \right\|_{L^\infty_{T,x}}.
\]  
(8.41)

Observe this previous estimate is deterministic – to complete the proof of the high-probability estimate, we recall $\tau = N^{-2+\epsilon}$ with $\epsilon \in \mathbb{R}_{>0}$ arbitrarily small though universal and afterwards employ Lemma 6.8 and Lemma 8.2.

**Proof of Proposition 8.5.** Courtesy of Lemma 8.9 and Lemma 8.10, we may proceed with pathwise analysis exactly like in the proof of Theorem 1.6 given in Section 8 of [25] and the proof of Proposition 8.4.

8.4. **Pathwise Analysis III.** Courtesy of Proposition 8.4 and Proposition 8.5, it remains to establish tightness for $\mathcal{X}^N$ in an appropriate function space and identify the subsequential limits.

**Remark 8.11.** Although we could certainly directly analyze $\mathcal{X}^N$ in the fashion similar to the combination of the respective analysis in [22] and [9] by establishing suitable heat kernel estimates for $\mathcal{Q}_{S,T,x}$, we find it certainly more convenient, though certainly extraneous, to compare $\mathcal{X}^N$ to $\mathcal{S}^N$, from which we can then directly inherit the analysis in [22] up to an additional ingredient for the martingale quantity.

In view of Remark 8.11, the primary result for the current subsection is the following comparison estimate.

**Proposition 8.12.** First, we define the following event with universal implied constant for any and $\delta \in \mathbb{R}_{>0}$.

\[
\mathcal{E}_\delta = \left\{ \left\| \mathcal{X}^N_{T,x} - \mathcal{S}^N_{T,x} \right\|_{L^\infty_{T,x}} \geq N^{-\delta} \right\},
\]  
(8.42)

If $\delta \in \mathbb{R}_{>0}$ is sufficiently small but universal, there exists a universal constant $\delta' \in \mathbb{R}_{>0}$ such that

\[
P[\mathcal{E}_\delta] \lesssim T^{\epsilon_0}, a, \delta, \delta' \cdot N^{-\delta'}.
\]  
(8.43)

**Proof.** For simplicity, it will be convenient to define

\[
\mathbb{T}^N_{T,x} = \mathcal{X}^N_{T,x} - \mathcal{S}^N_{T,x}.
\]  
(8.44)

We now consider two time-regimes, for which we fix an arbitrarily small but universal parameter $\epsilon \in \mathbb{R}_{>0}$.

- First, consider $T \in [0, N^{-\epsilon}]$. For this situation, we first note the following consequences of the a priori bounds for near-stationary initial condition and the off-diagonal heat kernel estimates for both $\mathcal{W}_{S,T,x}$ and $\mathcal{Q}_{S,T,x}$ from Proposition 4.5, in which $\epsilon' \in \mathbb{R}_{>0}$ is another universal parameter depending only on $\epsilon \in \mathbb{R}_{>0}$ and the near-stationary initial condition:

\[
\left\| \sum_{y \in \mathcal{Y}_N} \mathcal{W}_{0,T,x,y} \mathcal{S}^N_{0,y} - \mathcal{S}^N_{0,x} \right\|_{L^\infty_{T,x}} \lesssim N^{-\epsilon'},
\]  
(8.45)

and by a similar token, we also obtain the following estimate as well:

\[
\left\| \sum_{y \in \mathcal{Y}_N} \mathcal{W}_{0,T,x,y} \mathcal{S}^N_{0,y} - \mathcal{S}^N_{0,x} \right\|_{L^\infty_{T,x}} \lesssim N^{-\epsilon'}.
\]  
(8.46)

Moreover, again for $T \in [0, N^{-\epsilon}]$, courtesy of the martingale inequality from Lemma 3.1 in [9], we also have

\[
\left\| \int_0^T \sum_{y \in \mathcal{Y}_N} \mathcal{W}_{S,T,x,y} \left[ x_{S,y}^N \, dw_{S,y}^N \right] \right\|^2_{L^\infty_{T,x}} \lesssim N^{-1} \left\| x_{T,x}^N \right\|^2_{L^\infty_{T,x}} + \int_0^T \mathcal{W}_{S,T} \, ds \cdot \left\| x_{T,x}^N \right\|^2_{L^\infty_{T,x}}
\]  
(8.47)

\[
\lesssim N^{-1+2\epsilon} \left\| x_{T,x}^N \right\|^2_{L^\infty_{T,x}}.
\]  
(8.48)
while the same estimate holds upon replacing $\bar{\mathcal{N}}$ with $\mathcal{U}$ and $\mathcal{X}$ by $3\mathcal{N}$. In particular, combining all estimates in this bullet point for $T \in [0, N^{-\varepsilon}]$, we obtain the following for $x \in \mathcal{F}_{N,0}$ and $\varepsilon'' \in \mathbb{R}_{>0}$ a universal parameter:

$$
\left\| \mathcal{X}_{T,x}^N - 3_{0,x}^N \right\|_{L^p_{\omega}} + \left\| \mathcal{S}_{T,x}^N - 3_{0,x}^N \right\|_{L^p_{\omega}} \lesssim \delta' + N^{-\varepsilon''} + N^{-\varepsilon''} \left\| \mathcal{X}_{T,x}^N \right\|_{L^p_{\omega}} + N^{-\varepsilon''} \left\| \mathcal{S}_{T,x}^N \right\|_{L^p_{\omega}}
$$

(8.49)

where the final estimate is courtesy of Lemma 8.2. Upon yet another application of stochastic continuity as in the proof of Lemma 8.2, this completes the proof for $T \in [0, N^{-\varepsilon}]$.

\* Consider times $T \in [N^{-\varepsilon}, T_f]$ and recall $\mathcal{D} = \mathcal{N} - \mathcal{N}$. A straightforward calculation yields

$$
\mathcal{Y}_{T,x}^N = \sum_{y \in \mathcal{F}_{N,0}} \mathcal{D}_{0,T,y}^N \mathcal{X}_{0,y}^N + \int_0^T \sum_{y \in \mathcal{F}_{N,0}} \mathcal{D}_{S,T,x,y}^N \left[ \mathcal{X}_{S,y}^N \right] d \mathcal{S}_{S,y}^N + \int_0^T \sum_{y \in \mathcal{F}_{N,0}} \mathcal{D}_{S,T,x,y}^N \left[ \mathcal{Y}_{S,y}^N \right] d \mathcal{S}_{S,y}^N
$$

(8.51)

$$
\mathcal{Y}_{T,x}^N = \mathcal{Y}_{T,x}^{N,1} + \mathcal{Y}_{T,x}^{N,2} + \mathcal{Y}_{T,x}^{N,3}
$$

(8.52)

Recalling $T \geq N^{-\varepsilon}$, courtesy of Proposition 5.9, for $\varepsilon \in \mathbb{R}_{>0}$ sufficiently small, the first quantity yields the following estimate for $\beta \in \mathbb{R}_{>0}$ universal:

$$
\left\| \mathcal{Y}_{T,x}^{N,1} \right\|_{L^p_{\omega}} \lesssim \delta' \mathcal{X}_{0,x}^N \left\| \mathcal{X}_{T,x}^N \right\|_{L^p_{\omega}(0)}
$$

(8.53)

(8.54)

Concerning the second quantity, we proceed similarly using Lemma 3.1 in [9] and the a priori estimates in Lemma 8.2; this provides the following estimate for $\beta' \in \mathbb{R}_{>0}$ a universal constant and for $\delta'' \in \mathbb{R}_{>0}$ arbitrarily small:

$$
\left\| \mathcal{Y}_{T,x}^{N,2} \right\|_{L^p_{\omega}} \lesssim \mathcal{X}_{T,x}^N \left\| \mathcal{X}_{T,x}^N \right\|_{L^p_{\omega}(0)}
$$

(8.55)

$$
\left\| \mathcal{Y}_{T,x}^{N,2} \right\|_{L^p_{\omega}} \lesssim \delta' \left\| \mathcal{X}_{T,x}^N \right\|_{L^p_{\omega}(0)}
$$

(8.56)

$$
\left\| \mathcal{Y}_{T,x}^{N,2} \right\|_{L^p_{\omega}} \lesssim \delta' \left\| \mathcal{X}_{T,x}^N \right\|_{L^p_{\omega}(0)}
$$

(8.57)

to obtain the second inequality, we instead employ Lemma 4.4.

Finally, we analyze $\mathcal{Y}_{T,x}^{N,3}$ by first decomposing the time-interval $[0, T] = [0, N^{-\varepsilon}] \cup [N^{-\varepsilon}, T]$; proceeding again in similar fashion with the heat kernel estimates for $\mathcal{D}^N$, we have

$$
\left\| \mathcal{Y}_{T,x}^{N,3} \right\|_{L^p_{\omega}} \lesssim \delta' \int_0^T \mathcal{S}_{T}^N \sum_{y \in \mathcal{F}_{N,0}} \mathcal{D}_{S,T,x,y}^N \left\| \mathcal{Y}_{S,y}^N \right\|_{L^p_{\omega}}^2 d S
$$

(8.58)

$$
\left\| \mathcal{Y}_{T,x}^{N,3} \right\|_{L^p_{\omega}} \lesssim \mathcal{X}_{T,x}^N \left\| \mathcal{X}_{T,x}^N \right\|_{L^p_{\omega}(0)} + \int_0^T \sum_{y \in \mathcal{F}_{N,0}} \mathcal{D}_{S,T,x,y}^N \left\| \mathcal{Y}_{S,y}^N \right\|_{L^p_{\omega}}^2 d S
$$

(8.59)

$$
\left\| \mathcal{Y}_{T,x}^{N,3} \right\|_{L^p_{\omega}} \lesssim \mathcal{X}_{T,x}^N \left\| \mathcal{X}_{T,x}^N \right\|_{L^p_{\omega}(0)}
$$

(8.60)

Ultimately, we obtain

$$
\left\| \mathcal{Y}_{T,x}^{N,3} \right\|_{L^p_{\omega}} \lesssim \delta' \mathcal{X}_{T,x}^N \left\| \mathcal{X}_{T,x}^N \right\|_{L^p_{\omega}(0)} + \int_0^T \sum_{y \in \mathcal{F}_{N,0}} \mathcal{D}_{S,T,x,y}^N \left\| \mathcal{Y}_{S,y}^N \right\|_{L^p_{\omega}}^2 d S
$$

(8.61)

from which an application of the singular Gronwall inequality, as in the proof of Proposition 3.2 in [9], for example, yields the desired moment estimate for $T \in [N^{-\varepsilon}, T_f]$. This completes the proof. □
8.5. **Proof of Theorem 1.8.** Employing all of Proposition 8.4, Proposition 8.5, and Proposition 8.12, we first observe that tightness of the microscopic Cole-Hopf transform with respect to the Skorokhod topology on $\mathcal{D}(\mathbb{R}_{\geq 0}, C_{loc}(\mathbb{R}))$ is equivalent to tightness in the same space of the proxy $\tilde{Z}^N$; the latter result follows from the proof for Proposition 5.4 within [22] with the additional martingale inequality in Lemma 3.1 within [9] accounting for the non-simple nature of the particle random walks.

Thus, it remains to identify subsequential limits for $\tilde{Z}^N$ in the Skorokhod space $\mathcal{D}(\mathbb{R}_{\geq 0}, C_{loc}(\mathbb{R}))$. To this end, it suffices to follow the proof of Theorem 5.7 in [22]; the only missing ingredient is the following estimate.

**Lemma 8.13.** Consider any generic smooth test function $\varphi \in C^\infty(\mathcal{F}_\infty)$ with compact support along with any weakly vanishing quantity $w^N$. Provided any uniformly bounded time $T \in \mathbb{R}_{\geq 0}$, we have

$$
E\left[\left|\int_0^T \sum_{x \in \mathcal{S}_{N,0}} \varphi_{N^{-1},x} w^N_{S,x,3} \tilde{Z}^N_{S,x} \, ds\right|\right] \longrightarrow_{N \to \infty} 0. \tag{8.62}
$$

The proof of Lemma 8.13 follows from exactly the one-block and two-blocks argument in Proposition D.1, though it is actually simpler for this scenario because the test function $\varphi \in C^\infty(\mathcal{F}_\infty)$ admits no space-time singularities. Alternatively, we follow the proof of Lemma 2.5 in [9]. This completes the proof of Theorem 1.8.

**APPENDIX A. WELL-POSEDNESS OF THE BOUNDARY COEFFICIENTS**

The purpose of this appendix section is show the existence of a choice of boundary parameters $\beta^{\pm N}_{\ell,\pm}$ satisfying necessary constraints, for example those in Assumption 1.3.

The procedure we employ to solve the associated system of linear equations is some iterative approach which may be thought of as row reduction.

**Lemma A.1.** There exists a unique solution $\{\beta^{N,-}_{\ell,\pm}\}_{j=1}^{m_N}$ and $\{\beta^{N,\pm}_{\ell,\pm}\}_{j=1}^{m_N}$ to the respective systems of equations from Assumption 1.3.

**Proof.** We consider only $\beta^{N,-}_{\ell,\pm}$; for the coefficients $\beta^{N,\pm}_{\ell,\pm}$, the exact same calculation works upon swapping $\mathcal{A}_- \leftrightarrow \mathcal{A}_+$. We first observe the system of equations

$$
\beta^{N,-}_{m_N,\pm} + \beta^{N,-}_{m_N,\pm} = \bar{a}^N_{m_N}; \tag{A.1a}
$$

$$
\beta^{N,-}_{m_N,\pm} - \beta^{N,-}_{m_N,\pm} = \frac{1}{2} \lambda_N N^{-\frac{1}{2}} \left[ \sum_{k=1}^{m_N} k \bar{a}^N_k + \sum_{k=1}^{m_N-1} k \bar{a}^N_k + (m_N-1) \bar{a}^N_{m_N} \right] + \lambda_N^{-1} N^\frac{1}{2} \mathcal{A}_- - 2 \lambda_N N^\frac{1}{2} \kappa^{-N,-}_{N,m_N-1}. \tag{A.1b}
$$

In particular, as a function of the symmetric-jump coefficients we obtain exact formulas for $\beta^{N,-}_{m_N,\pm}$, to be more precise, first let us define $\beta^{N,-}_{m_N,\pm}$ to be the unique solution to this isolated couple of equations.

Suppose we now have unique and exact formulas for $\beta^{N,-}_{\ell,\pm}$ for $\ell \in [j+1, m_N]$ for some index $j \in [0, m_N - 1]$. Observe that the following induced system of equations provides us with some unique solution to $\beta^{N,-}_{\ell,\pm}$ as the isolated "sub-system" of equations:

$$
\beta^{N,-}_{\ell,\pm} + \beta^{N,-}_{\ell,\pm} = \bar{a}^N_{ \ell }; \tag{A.2a}
$$

$$
\beta^{N,-}_{\ell,\pm} + \beta^{N,-}_{\ell,\pm} = - \sum_{k=1}^{m_N} \left( \beta^{N,-}_{\ell,\pm} - \beta^{N,-}_{k,\pm} \right) + \frac{1}{2} \lambda_N N^{-\frac{1}{2}} \left[ \sum_{k=1}^{m_N} k \bar{a}^N_k + \sum_{k=1}^{m_N-1} k \bar{a}^N_k + (j-1) \sum_{k=j}^{m_N} \bar{a}^N_k \right] + \lambda_N^{-1} N^\frac{1}{2} \mathcal{A}_- + 2 \lambda_N N^\frac{1}{2} \kappa^{-N,-}_{N,j-1} \tag{A.2b}
$$

Indeed, the RHS of each equation is independent of $\{\beta^{N,-}_{\ell,\pm}\}_{\ell=1}^{m_N}$, so we obtain a unique exact formula for $\beta^{N,-}_{j,\pm}$ in addition these coefficients. Continuing inductively completes the proof of existence and uniqueness. \qed
The stated goal for this appendix section is to provide auxiliary estimates for the full-line heat kernel $\varrho_{S,T}^{N,0}$ not recorded in previous relevant papers, for example [6], [22], [9], or [23].

We record in this appendix section a precise extension of Proposition A.1 from [9] to higher-order derivatives; we recall that these ingredients are important towards the perturbative scheme begun in Lemma 3.10 we use to establish regularity estimates for the heat kernels $\varrho_{S,T}^{N}$ with arbitrary Robin boundary parameter $\mathcal{A} \in \mathbb{R}$.

**Lemma B.1.** Provided any $\ell \in \mathbb{Z}_{\geq 0}$ and any $\vec{n} = (n_1, \ldots, n_\ell) \in \mathbb{Z}^\ell$, we have the following estimate uniformly in $S, T \in \mathbb{R}_{\geq 0}$ in $x, y \in \mathbb{Z}$, and in $\kappa \in \mathbb{R}_{> 0}$ arbitrarily large but universal:

$$
\left| \nabla_{\vec{n}} \varrho_{S,T,y}^{N,0} \right| \lesssim \kappa, \| \vec{n} \|_{\infty} \cdot N^{-\ell-1} \varrho_{S,T}^{\frac{\ell}{2}-\frac{1}{2}} \sup_{w \in \mathcal{D}(x)} \exp \left( -\kappa \frac{|w-y|}{N \varrho_{S,T}^{\frac{1}{2}} \vee 1} \right) \tag{B.1}
$$

where $\mathcal{D}(x) = \{ w \in \mathbb{Z} : |w-x| \lesssim n_1 + \ldots + n_\ell \}$ with a universal implied constant.

**Proof.** First, by iteration, it suffices to assume that $|n_j| = 1$ for all $j \in [1, \ell]$. Following the proof of Proposition A.1 in [9], we obtain the following spectral representation for $\varrho_{S,T}^{N,0}$:

$$
\varrho_{0,T}^{N,0} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i\xi \cdot \vec{N} T} \Phi_{0,T}(\xi) \, d\xi, \tag{B.2}
$$

$$
\left| \Phi_{0,T}(\xi) \right| \lesssim e^{-\kappa N^2 T \xi^2}. \tag{B.3}
$$

Thus, for any vector $\vec{n} = (n_1, \ldots, n_\ell) \in \mathbb{Z}^\ell$, we have the following spectral formula for the gradients of $\varrho_{S,T}^{N,0}$:

$$
\nabla_{\vec{n}} \varrho_{S,T,y}^{N,0} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \prod_{j=1}^{\ell} \nabla_{\xi_j} e^{-i\xi \cdot \vec{n} \Phi_{0,T}(\xi)} \, d\xi, \tag{B.4a}
$$

$$
\nabla_{\vec{n}} \varrho_{S,T,y}^{N,0} \propto e^{-in_{\vec{n}} \cdot \vec{e} - 1}. \tag{B.4b}
$$

Proceeding to bound the $\nabla_{\vec{n}}$-quantities like in the proof of Proposition A.1 in [9], under our assumptions on the vector $\vec{n} \in \mathbb{Z}^\ell$ we obtain the following estimate:

$$
\left| \nabla_{\vec{n}} \varrho_{S,T,y}^{N,0} \right| \lesssim \prod_{j=1}^{\ell} |n_j| \int_{-\pi}^{\pi} \left( \varrho_{S,T}^{N} \right)^{\ell} e^{-i\xi \cdot \vec{n} \Phi_{0,T}(\xi)} \, d\xi, \tag{B.5}
$$

$$
\varrho_{S,T}^{N} \propto |\vec{e}| + N^{-1} \varrho_{0,T}^{\frac{1}{2}}. \tag{B.6}
$$

In particular, an elementary calculation implies

$$
\left| \nabla_{\vec{n}} \varrho_{S,T,y}^{N,0} \right| \lesssim \prod_{j=1}^{\ell} |n_j| \int_{-\pi}^{\pi} |\xi| \cdot e^{-\kappa N^2 T \xi^2} \, d\xi + N^{-\ell} \varrho_{0,T}^{\frac{1}{2}} \prod_{j=1}^{\ell} |n_j| \int_{-\pi}^{\pi} e^{-\kappa N^2 T \xi^2} \, d\xi \tag{B.7}
$$

$$
\lesssim \| \vec{n} \|_{\infty} \cdot N^{-\ell-1} \varrho_{S,T}^{\frac{\ell}{2}-\frac{1}{2}} \tag{B.8}
$$

where the latter upper bound follows from computing the integrals though on the integration domain $\mathbb{R}$.

To account for off-diagonal factor, we adapt the contour of integration as in the proof of Proposition A.1 in [9].  \( \square \)

Applying Lemma B.1, we obtain the following boundary regularity estimate.

**Lemma B.2.** Consider any $\ell \in \mathbb{Z}_{\geq 0}$ and $\vec{n} = (n_1, \ldots, n_\ell) \in \mathbb{Z}^\ell$; provided any pair of times $S, T \in \mathbb{R}_{\geq 0}$ such that $S \leq T$ along with any pair of points $x, y \in \mathbb{Z}$, we have

$$
\left| \nabla_{\vec{n}} \varrho_{S,T,y}^{N,0} \right| \lesssim \mathcal{J}_{x,\vec{n}} \cdot N^{-\ell-1} \varrho_{S,T}^{\frac{\ell}{2}-\frac{1}{2}} \wedge 1, \tag{B.9}
$$

$$
\sum_{y \in \mathbb{Z}} \left| \nabla_{\vec{n}} \varrho_{S,T,y}^{N,0} \right| \lesssim \mathcal{J}_{x,\vec{n}} \cdot N^{-\ell} \varrho_{S,T}^{\frac{\ell}{2}} \wedge 1, \tag{B.10}
$$

Applying Lemma B.2, we obtain the following boundary regularity estimate.
where \( \mathcal{S}_{x,R} = \mathbb{1}_{x} \prod_{j=1}^{\lfloor R/2 \rfloor} |n_j| + \prod_{j=1}^{\lfloor R/2 \rfloor^2} |n_j| \), where \( \mathbb{1}_{x} = |x| \land |N - x| \) is the distance from \( x \in \mathcal{I}_{N,0} \) to the boundary.

Remark B.3. Comparing Lemma B.2 above with the regularity estimates established within Proposition A.1 from [9], for example, we observe that near the boundary at a microscopic scale we have "gained" one derivative, at least from the PDE perspective. This is exactly the utility of the perturbative scheme in Lemma 3.10 mentioned just prior to the statement of Lemma B.2.

Proof. Suppose first that \( x = 0 \). Applying definition and afterwards rearranging terms, we have

\[
\nabla_n \mathcal{S}^{N,0}_{S,T,0,y} = \sum_{k \in \mathbb{Z}} \nabla_n \mathcal{S}^{N,0}_{S,T,y,k} = \sum_{k \in \mathbb{Z}} \left[ \nabla_n \mathcal{S}^{N,0}_{S,T,0,y,k} - \nabla_n \mathcal{S}^{N,0}_{S,T,0,y,k+1} \right] = \sum_{k \in \mathbb{Z}} \nabla_n \nabla_n \mathcal{S}^{N,0}_{S,T,0,y,k}.
\]

Employing Lemma B.1 provides the desired estimate for first-order gradients. Provided a general \( x \in \mathcal{I}_{N,0} \), we first write

\[
\nabla_k \mathcal{S}^{N,0}_{S,T,x,y} = \nabla_{x+k} \mathcal{S}^{N,0}_{S,T,0,y} - \nabla_x \mathcal{S}^{N,0}_{S,T,0,y},
\]

from which we deduce the result for first-order gradients. Concerning the higher-order gradients, we take gradients of the previous two sets of calculations and apply Lemma B.1 once again. \( \square \)

We moreover require another preparatory lemma which will serve as an important a priori estimate on the full-line heat kernel \( \mathcal{S}_{S,T}^{N,0} \) within the proof of Proposition 5.9.

Lemma B.4. Provided any \( S, T \in \mathbb{R}_{>0} \) satisfying \( S \leq T \) along with any \( x, y \in \mathbb{Z} \), we have

\[
\sup_{x,y \in \mathbb{Z}_{\geq 0}} \mathbb{D}_{x,y}^{N,0}_{S,T} \lesssim \left( \sum_{k=1}^{m} |k^2 \mathcal{S}_{k}^{N,0}| \right) N^{-2} \theta^{S,T};
\]

above, the implied constant is universal. Moreover, the same estimate holds upon replacing \( \mathcal{S}_{S,T}^{N,0} \) by the nearest-neighbor heat kernel \( \mathcal{S}_{S,T}^{N,0} \) on the full-line \( \mathbb{Z} \).

Proof. We provide a proof for \( \mathcal{S}_{S,T}^{N,0} \); the argument applies just as well, although, to the nearest-neighbor heat kernel \( \mathcal{S}_{S,T}^{N,0} \).

Moreover, it suffices to assume \( y = 0 \) since every relevant heat kernel is spatially-homogeneous; this is simply for notational convenience.

First, using the spectral integral representation of the heat kernel \( \mathcal{S}_{S,T}^{N,0} \) provided in equation (A.6) from [9], which we also use earlier in Lemma B.1, we have

\[
\frac{1}{2} \sum_{k=1}^{m} \frac{\mathcal{S}_{k}^{N,0}}{\mathcal{S}_{k}^{N,0}} = \frac{1}{2} \sum_{k=1}^{m} \mathcal{S}_{k}^{N,0} \frac{1}{\mathcal{S}_{k}^{N,0}} \mathcal{A}_{k}^{N} \Delta_{k}^{N} e^{i \theta - \theta + \theta} \exp \left[ -N^2 \mathcal{A}_{S,T}^{N} \sum_{\ell=1}^{\mathcal{A}_{S,T}^{N}} (1 - \cos(\ell \theta)) \right] \, d\theta
\]

\[
= \frac{1}{2} \sum_{k=1}^{m} \mathcal{S}_{k}^{N,0} \cdot N^2 e^{i \theta} \left[ e^{i \theta} - e^{-i \theta} - 2 \right] \exp \left[ -N^2 \mathcal{A}_{S,T}^{N} \sum_{\ell=1}^{\mathcal{A}_{S,T}^{N}} (1 - \cos(\ell \theta)) \right] \, d\theta.
\]

Meanwhile, by the same token we have

\[
\left( \sum_{k=1}^{m} k^2 \mathcal{S}_{k}^{N,0} \right) \Delta_{k}^{N} e^{i \theta - \theta + \theta} \exp \left[ -N^2 \mathcal{A}_{S,T}^{N} \sum_{\ell=1}^{\mathcal{A}_{S,T}^{N}} (1 - \cos(\ell \theta)) \right] \, d\theta.
\]

To proceed, we observe the relevant difference in the respective integrands is bounded via Taylor expansion as follows:

\[
\frac{1}{2} \sum_{k=1}^{m} \mathcal{S}_{k}^{N,0} \cdot N^2 e^{i \theta} \left[ e^{i \theta} - e^{-i \theta} - 2 \right] - \left( \sum_{k=1}^{m} k^2 \mathcal{S}_{k}^{N,0} \right) \cdot N^2 e^{i \theta} \left[ e^{i \theta} - e^{-i \theta} - 2 \right] \lesssim \left( \sum_{k=1}^{m} k^3 \mathcal{S}_{k}^{N,0} \right) \theta^3.
\]
Moreover, as noted in the two-sided bounds of equation (A.7) from [9], we also obtain the following upper bound on the other factor of the integrand for some $\kappa \in \mathbb{R}_{>0}$ universal outside its dependence on $\tilde{\alpha}_1^N \in \mathbb{R}_{>0}$:

$$\exp \left[ -N^2 \phi_{S,T} \sum_{\ell=1}^{m_N} \tilde{\alpha}_\ell^N (1 - \cos(\ell \theta)) \right] \leq \exp \left[ -\kappa N^2 \phi_{S,T} \theta^2 \right]. \quad (B.20)$$

Combining this with the straightforward bound $|e^{ix\theta}| \leq 1$, we have

$$\left| \mathcal{D}^{N,0}_{S,T} \right| \lesssim \left( \sum_{k=1}^{m_N} k^3 \tilde{\alpha}_k^N \right) N^2 \int_{-\pi}^{\pi} \theta^3 \exp \left[ -\kappa N^2 \phi_{S,T} \theta^2 \right] \, d\theta, \quad (B.21)$$

from which the desired estimate follows from a straightforward integral calculation with change-of-variables, for example. This completes the proof.

**Appendix C. An Elementary Integral Calculation**

Throughout our derivation of necessary a priori regularity estimates for relevant heat kernels and stochastic fundamental solutions, we appeal to two integral inequalities. The first of these concerns time-integrals of integrable singularities.

**Lemma C.1.** Provided any $S, T \in \mathbb{R}_{>0}$ satisfying $S \leq T$ and any pair $c_1, c_2 \in \mathbb{R}_{<1}$, we have

$$\int_{S}^{T} \rho_{r,T}^{-c_1} \rho_{S,r}^{-c_2} \, dr \lesssim_{c_1, c_2} \rho_{S,T}^{1-c_1-c_2}. \quad (C.1)$$

**Proof.** We first decompose the integral into halves; precisely, we first decompose $[S, T] = [S, \frac{T+S}{2}] \cup [\frac{T+S}{2}, T]$. This decomposition provides the upper bound

$$\int_{S}^{T} \rho_{r,T}^{-c_1} \rho_{S,r}^{-c_2} \, dr = \int_{S}^{\frac{T+S}{2}} \rho_{r,T}^{-c_1} \rho_{S,r}^{-c_2} \, dr + \int_{\frac{T+S}{2}}^{T} \rho_{r,T}^{-c_1} \rho_{S,r}^{-c_2} \, dr \quad (C.2)$$

$$\leq 2^{c_1} \rho_{S,T}^{-c_1} \int_{S}^{\frac{T+S}{2}} \rho_{S,r}^{-c_2} \, dr + 2^{c_2} \rho_{S,T}^{-c_2} \int_{\frac{T+S}{2}}^{T} \rho_{r,T}^{-c_1} \, dr \quad (C.3)$$

$$\leq \frac{1}{1-c_2} 2^{-1+c_1+c_2} \rho_{S,T}^{1-c_1-c_2} + \frac{1}{1-c_1} 2^{-1+c_2+c_1} \rho_{S,T}^{1-c_1-c_2}, \quad (C.4)$$

which completes the proof upon additional elementary bounds.

The second of these preliminary time-integral estimates concerns non-integrable singularities though with a cutoff away from said singularity. The point of the following elementary upper bound is to provide a precise estimate for the otherwise clear qualitative convergence of the integral.

**Lemma C.2.** Consider any $S, T \in \mathbb{R}_{>0}$ satisfying $S \leq T$ and any pair $c_1 \in \mathbb{R}_{<1}$ and $c_2 \in \mathbb{R}_{>1}$. Provided any $\varepsilon \in \mathbb{R}_{>0}$, we have

$$\int_{S+\varepsilon}^{T} \rho_{r,T}^{-c_1} \rho_{S,r}^{-c_2} \, dr \lesssim_{c_1, c_2} \rho_{S+\varepsilon,T}^{-c_1} \rho_{S,T}^{1-c_2+1}. \quad (C.5)$$

In particular, if $\rho_{S,T} \gtrsim \varepsilon$ with sufficiently large but universal implied constant, we have

$$\int_{S+\varepsilon}^{T} \rho_{r,T}^{-c_1} \rho_{S,r}^{-c_2} \, dr \lesssim_{c_1, c_2} \rho_{S,T}^{-c_1} \rho_{S,T}^{1-c_2+1}. \quad (C.6)$$

**Proof.** For notational convenience, let us consider $S = 0$; the proof for general $S \in \mathbb{R}_{>0}$ follows from a change-of-variables transformation provided by time-translation. Moreover, observe we may certainly assume, a priori, that $\varepsilon \leq T$. Otherwise the stated integral vanishes.
We again decompose the integral into halves as in the proof of Lemma C.1 as follows; this provides

\[
\int_\epsilon^T e_{\rho,\tau}^{-\xi} e_{\rho,\tau}^{-\xi} dR = \int_\epsilon^{T/2} e_{\rho,\tau}^{-\xi} e_{\rho,\tau}^{-\xi} dR + \int_{T/2}^T e_{\rho,\tau}^{-\xi} e_{\rho,\tau}^{-\xi} dR \tag{C.7}
\]

\[
\leq e_{\rho,\tau}^{-\xi_1} \int_\epsilon^{T/2} e_{\rho,\tau}^{-\xi} dR + e_{\rho,\tau}^{-\xi_2} \int_{T/2}^T e_{\rho,\tau}^{-\xi} dR \tag{C.8}
\]

\[
\lesssim \xi_1, \xi_2 e_{\rho,\tau}^{-\xi_1+1} + e_{\rho,\tau}^{-\xi_2} \leq e_{\rho,\tau}^{-\xi_1+1}; \tag{C.9}
\]

Observe that \( \frac{T}{2} \geq T \) clearly, and recall \( \epsilon \leq T \), which together imply that the second quantity on the upper bound above is bounded above by

\[
e_{\rho,\tau}^{-\xi_2} \leq e_{\rho,\tau}^{-\xi_1+1}; \tag{C.10}
\]

this completes the proof. □

**APPENDIX D. QUANTITATIVE CLASSICAL REPLACEMENT LEMMA**

The purpose of this appendix section is to make precise the classical one-block and two-blocks estimates of \([15]\), which is traditionally used in a topological framework. The only additional input is a precise equivalence of ensembles of estimates which we borrow from \([11]\) and the log-Sobolev inequality of Yau from \([23]\). We recall that the utility behind such a result is to address the weakly vanishing quantities arising in Proposition 2.11 in a quantitative variation of the fashion in \([9]\).

**Proposition D.1.** Consider any weakly-vanishing quantity \( w_0 \) and let \( \delta \in \mathbb{R}_{>0} \) denote an arbitrarily small though universal constant. There exists a universal constant \( \beta_u \) such that

\[
E \left[ \left\| \int_0^T \sum_{y \in \mathcal{F}_{N,y}} \mathcal{U}_{S,T,x,y}^N \cdot \left\| \frac{Av^\beta_{w_{x,y}}}{\partial y} - \left\langle \frac{Av^\beta_{w_{x,y}}}{\partial y} \right\rangle_\beta \right\| \, dS \right\|_{L^2_{T,x}} \right] \lesssim_{T, \beta} N^{-\beta_u}; \tag{D.1}
\]

\[
E \left[ \left\| \int_0^T \sum_{y \in \mathcal{F}_{N,y}} \mathcal{U}_{S,T,x,y}^N \cdot \left\| \frac{Av^\beta_{w_{x,y}}}{\partial y} - \left\langle \frac{Av^\beta_{w_{x,y}}}{\partial y} \right\rangle_\beta \right\| \, dS \right\|_{L^2_{T,x}} \right] \lesssim_{T, \beta} N^{-\beta_u}. \tag{D.2}
\]

above, provided any \( S \in \mathbb{R}_{>0} \) and \( y \in \mathcal{F}_{N,y} \) along with any \( \beta \in \mathbb{R}_{>0} \), we have defined the following local expectation

\[
\left\langle \frac{Av^\beta_{w_{x,y}}}{\partial y} \right\rangle_\beta \equiv E_{\beta_{x,y}}^{\mathcal{U}_{S,T,x,y}^N} w_0; \tag{D.3a}
\]

\[
\beta_{x,y} \equiv \frac{Av^\beta_{w_{x,y}}}{\partial y}. \tag{D.3b}
\]

Moreover, the same estimates hold upon replacing \( \mathcal{U}_{S,T,x,y}^N \) with \( \nabla_{k,y}^l \mathcal{U}_{S,T,x,y}^N \) for any \( k \in \mathbb{Z} \) uniformly bounded.

**Proof.** We first establish (D.1); like the respective proofs of Proposition 7.7, Proposition 7.8, and Proposition 7.9, provided any \( \epsilon \in \mathbb{R}_{>0} \), we have

\[
\int_0^T \sum_{y \in \mathcal{F}_{N,y}} \mathcal{U}_{S,T,x,y}^N \cdot \left\| \frac{Av^\beta_{w_{x,y}}}{\partial y} - \left\langle \frac{Av^\beta_{w_{x,y}}}{\partial y} \right\rangle_\beta \right\| \, dS \lesssim_{T, \epsilon} N^{2\epsilon} \left( \int_0^T \sum_{y \in \mathcal{F}_{N,y}} \left\| \frac{Av^\beta_{w_{x,y}}}{\partial y} - \left\langle \frac{Av^\beta_{w_{x,y}}}{\partial y} \right\rangle_\beta \right\|^2 \, dS \right)^{1/2}. \tag{D.4}
\]

Again, it remains to estimate the expectation for this final upper bound; moreover, with the Cauchy-Schwarz inequality it suffices to remove the square root if \( \epsilon \in \mathbb{R}_{>0} \) is chosen sufficiently small but still universal.

To this end, like in the classical one-block estimate or Proposition 5.3 in \([23]\) combined with the entropy inequality in Lemma 7.3, we have the following for \( \epsilon \in \mathbb{R}_{>0} \) arbitrarily small but universal, in which \( \Lambda = [-N^\beta, N^\beta] \subseteq \mathbb{Z} \):

\[
E \left[ \int_0^T \sum_{y \in \mathcal{F}_{N,y}} \left\| \frac{Av^\beta_{w_{x,y}}}{\partial y} - \left\langle \frac{Av^\beta_{w_{x,y}}}{\partial y} \right\rangle_\beta \right\|^2 \, dS \right] \lesssim_{\epsilon} N^{-\frac{1}{2}+\epsilon+3\beta} + \sup_{\epsilon \in [-1,1]} \log E_{\epsilon,\lambda}^{\mathcal{U}_{S,T,x,y}^N} \left[ \left\| \frac{Av^\beta_{w_{x,y}}}{\partial y} - \left\langle \frac{Av^\beta_{w_{x,y}}}{\partial y} \right\rangle_\beta \right\|^2 \right]. \tag{D.5}
\]
Because $\varepsilon, \beta \in \mathbb{R}_{>0}$ are arbitrarily small but universal, the first quantity is clearly okay. For the log-exponential moments, we observe the elementary inequality

$$\log \mathbb{E}^\mu_{\varepsilon, \Lambda} \exp \left( \frac{A^\beta \omega_{0,0} - \langle A^\beta \omega_{N,0} \rangle_\beta}{\beta} \right) \lesssim \|w_N\|_{\infty} \mathbb{E}^\mu_{\varepsilon, \Lambda} \left[ \left( \frac{A^\beta \omega_{0,0} - \langle A^\beta \omega_{N,0} \rangle_\beta}{\beta} \right)^2 \right].$$  \tag{D.6}

Courtesy of Proposition 3.6 in [11], we may replace the expectation with respect to $\mu_{\varepsilon, \Lambda}$ by the expectation with respect to $\mu_{\varepsilon, \Lambda}'$ at the cost of an allowable error. Moreover, the resulting expectation with respect to the grand-canonical ensemble $\mu_{\varepsilon, \Lambda}'$ admits an allowable estimate by standard probability theory as we then have independence of occupation variables. Now, for $\beta \in \mathbb{R}_{>0}$ sufficiently small and for $\varepsilon \in \mathbb{R}_{>0}$ sufficiently small depending only on $\beta \in \mathbb{R}_{>0}$, this completes the proof for (D.1).

We now proceed to prove the estimate (D.2); following the usual strategy for the two-blocks estimate in [15], for example illustrated in the proof of Proposition 4.4 in [9] combined with the preliminary calculations before taking expectation in our proof of (D.1), it suffices to estimate

$$\sup_{\ell \in [n^0, N^0 /2 - \delta]} \mathbb{E} \left[ \int_0^T \sum_{y \in \mathcal{F}_{N, \ell}} \left| g^\beta_{S,y} - g^\beta_{S,y+\ell} \right|^2 dS \right].$$  \tag{D.7}

Again, following standard practice with the two-blocks estimate in [15] as detailed in the proof of Proposition 4.4 in [9] and combining this with the entropy inequality from Lemma 7.3, we have the following estimate for any $\delta \in \mathbb{R}_{>0}$ arbitrarily small but universal:

$$\mathbb{E} \left[ \int_0^T \sum_{y \in \mathcal{F}_{N, \ell}} \left| g^\beta_{S,y} - g^\beta_{S,y+\ell} \right|^2 dS \right] \lesssim \delta N^{-\frac{1}{2} + \frac{3}{2}} + N^{-\beta_0};$$  \tag{D.8}

indeed, for $\Lambda' \subseteq \mathbb{Z}$ the union of two possibly disjoint sub-lattices of size $\lesssim N^\beta$, the classical moving-particle lemma provides the Dirichlet form bound on $\mathcal{F}_{N, \ell}$ of $N^{-\frac{1}{2} + \delta}$ up to a prefactor depending on $\delta \in \mathbb{R}_{>0}$ as detailed in the proof of Proposition 4.4 in [9], from which we obtain the above estimate by following the proof of Lemma 7.3 and the proof of Proposition 4.4 in [9]. In particular, choosing $\delta, \beta \in \mathbb{R}_{>0}$ arbitrarily small but universal, we obtain the desired estimate.

To obtain these same estimates but replacing $\mathcal{W}_{S,T, x, y}^N$ with $\nabla^l_{k,y} \mathcal{W}_{S,T, x, y}^N$ for any $k \in \mathbb{Z}$ uniformly bounded, we first choose $\delta \in \mathbb{R}_{>0}$ arbitrarily small but universal and write

$$\int_0^T \sum_{y \in \mathcal{F}_{N, \ell}} \nabla^l_{k,y} \mathcal{W}_{S,T, x, y}^N \cdot \left| A^\beta \omega_{y, x} - \langle A^\beta \omega_{y, x} \rangle_\beta \right| dS = I + II,$$  \tag{D.9}

where

$$I = \int_0^T \sum_{y \in \mathcal{F}_{N, \ell}} \nabla^l_{k,y} \mathcal{W}_{S,T, x, y}^N \cdot \left| A^\beta \omega_{y, x} - \langle A^\beta \omega_{y, x} \rangle_\beta \right| dS; \tag{D.10a}$$

$$II = \int_0^T \sum_{y \in \mathcal{F}_{N, \ell}} \nabla^l_{k,y} \mathcal{W}_{S,T, x, y}^N \cdot \left| A^\beta \omega_{y, x} - \langle A^\beta \omega_{y, x} \rangle_\beta \right| dS. \tag{D.10b}
$$

The quantity $II$ is analyzed directly via the regularity estimates in Lemma 5.14.

For the quantity $I$, first we observe that we may replace $\mathcal{W}_{S,T, x, y}^N$ with $\mathcal{W}_{S, y}^N$ like in the proof for Lemma 8.8. Employing the regularity estimate in Proposition 3.2 of [22], we have the following upper bound for $x \in \Lambda_{N} \subseteq \mathcal{F}_{N, 0}$:

$$|I| \lesssim_{T, m, \varepsilon, \delta} N^{25} \int_0^{T-N^\delta} \sum_{y \in \mathcal{F}_{N, \ell}} \left| A^\beta \omega_{y, x} - \langle A^\beta \omega_{y, x} \rangle_\beta \right| dS + e^{-\log N^\delta};$$  \tag{D.11}

indeed, by construction of the quantity $I$, the singularity in the gradient of the heat kernel $\mathcal{W}_{S, y}^N$ is cut-off up to an additional factor of $N^{\delta}$; as $\delta \in \mathbb{R}_{>0}$ is arbitrarily small but universal, we may proceed exactly as in the proof of the original estimate (D.1). Moreover, the analog of (D.2) but for the gradient $\nabla^l_{k,y} \mathcal{W}_{S, y}^N$ follows from an identical procedure beginning with the cutoff from the singularity of this gradient. This completes the proof. \hfill \square
E.1. Expectation Operators. First, provided any probability measure \( \mu \) on a generic probability space which is understood in context, we denote by \( \mathbb{E}^\mu \) the expectation with respect to this probability measure. Moreover, when additionally provided with a \( \sigma \)-algebra \( \mathcal{F} \), we denote by \( \mathbb{E}^\mu_{\mathcal{F}} \) the conditional expectation operator with respect to the probability measure \( \mu \) with respect to conditioning on this \( \sigma \)-algebra \( \mathcal{F} \).

E.2. Lattice Differential Operators and \( N \)-dependent Scaling. Provided any index \( k \in \mathbb{Z} \), we define the discrete differential operators \( \nabla_k, \Delta_k \) acting on any suitable space of functions \( \varphi : \mathbb{Z} \rightarrow \mathbb{R} \) through the following formula:

\[
\nabla_k \varphi_x \doteq \varphi_{x+k} - \varphi_x, \quad \Delta_k \varphi_x \doteq \varphi_{x+k} + \varphi_{x-k} - 2 \varphi_x. \tag{E.1}
\]

Moreover, we define the appropriately rescaled operators \( \nabla_k^N \doteq N \nabla_k \) and \( \Delta_k^N \doteq N^2 \Delta_k \), and these should be interpreted as approximations to their continuum differential counterparts. More generally, provided a generic bounded linear operator \( \mathcal{F} \) acting on any linear space, each additional \(!\) in the superscript denotes another scaling factor of \( N \in \mathbb{Z}_{>0} \). For example, we define \( \mathcal{F}^1 \doteq N \mathcal{F} \) and \( \mathcal{F}^2 \doteq N^2 \mathcal{F} \).

E.3. Landau Notation for Asymptotics. We will employ the Landau \( O \)-notation. We emphasize that provided any generic set \( \mathcal{F} \), the notation \( a \lesssim \mathcal{F} \) is equivalent to \( a = \mathcal{O}(b) \) for any real numbers \( a, b \in \mathbb{R} \), where the implied constant is allowed to depend on every element of \( \mathcal{F} \).

E.4. Miscellaneous Space-Time Objects. We conclude this notational index with a list of space-time objects we repeatedly use in this article.

- We define a space-time norm through the following maximal formula provided any \( \varphi_{T,x} : [0, T_f] \times \mathcal{F}_{N,0} \rightarrow \mathbb{R} 
\]

\[
\|\varphi_{T,x}\|_{\mathcal{F}_{N,0}} \doteq \sup_{T \in [0, T_f]} \sup_{x \in \mathcal{F}_{N,0}} |\varphi_{T,x}|. \tag{E.2}
\]

- Simply for notational convenience and compact presentation, provided any \( S, T \in \mathbb{R}_{\geq 0} \), we define \( \mathcal{G}_{S,T} \doteq |T - S| \).
- Provided coordinates \( (T, X) \in \mathbb{R}_{\geq 0} \times \mathbb{R} \), we define the associated space-time shift-operator \( \tau_{T,X} \) acting on possibly random fields as follows:

\[
\tau_{T,X} f(s, y, n_{T,x}) \doteq f(T + s, X + y, n_{T+s,x+y}). \tag{E.3}
\]

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