Extended state observer–based sliding mode learning control for mechanical system

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Abstract
A novel sliding mode learning controller is proposed for uncertain mechanical system in this paper. The model of uncertain mechanical system is listed first, and then extended state observer is designed for the estimation of the uncertainty. Then, an extended state observer–based sliding surface is constructed. The sliding surface parameters are solved by Lyapunov function approach. Then, a sliding mode learning controller is proposed for uncertain mechanical system to overcome the inherent chattering. Finally, a numerical simulation is given to show the effectiveness of the proposed sliding mode learning controller.

Keywords
Uncertain mechanical system, extended state observer, sliding mode learning control

Introduction
Uncertainty is inevitable in actual production process and will certainly bring bad influence to the control performance of a real system. How to estimate and deal with the uncertainty of the closed control loop is very important.¹–³ Generally speaking, there are mainly two effective ways to handle uncertainty: (1) estimating the uncertainty online and compensating it in the designed process, and (2) designing a robust controller to reduce its influence. In the estimating and compensating process, neural network and fuzzy modeling technology is utilized frequently, but the amount of computation is really big. In the robust controller designing process, some information of the uncertainty, such as its upper bound, is needed. If the uncertainty can be estimated online, and a robust controller can compensate it in real time, the control performance will be much better and the amount of computation will greatly reduce.

Extended state observer (ESO) is an effective way for the estimation of unknown uncertainty and has been utilized for the observation of uncertainty and external disturbance.⁴–⁶ In ESO, the uncertainty and perturbation are all viewed as extended state, and an observer is constructed to estimate them. Based on the estimating result, a controller is designed to compensate it, and then the control performance is improved. In view of the advantages of ESO, it is utilized in this paper to estimate the uncertainty and external disturbance real time in this paper.

Since an observer is decided, a robust controller is then needed. Sliding mode control (SMC) is an effective robust controller.⁷,⁸ For SMC controller design, a reduced order sliding surface is constructed first, in which the reduced dynamics are stable. Then, a discontinuous controller is constructed to force the model dynamics to arrive and stay in the sliding surface. In SMC, the system dynamics are unaffected by the disturbance,⁹ and so the controller is robust. SMC has been applied to the robust control of uncertain system, such as adaptive control of hypersonic flight vehicle,¹⁰ active suspension vehicle systems,⁷ adaptive control of fuzzy system,¹¹,¹² and SMC of master–slave time-delay systems.¹³ But for the application of SMC, there are also shortcomings.¹⁴,¹⁵ When utilizing SMC, some information of uncertainty and disturbance, such as the upper bound of them, must be known. For a real system, the interference and uncertainty are difficult to be modeled or measured, so the special upper bound is usually unknown. An effective way for this question is...
choosing a big enough upper bound for the interference and uncertainty, but this will certainly cause chattering in the control input.\textsuperscript{16–18} In this case, novel SMC strategy is needed.

Recently, sliding mode learning control (SMLC) strategy is proposed for the controller design of uncertain system.\textsuperscript{19} Similar with traditional SMC, a sliding surface is first constructed, and its stability is guaranteed by correctly selecting the parameter of the sliding surface. Then, a learning controller is introduced. The learning controller can greatly reduce chattering, so it is more practical in real application. By the way, SMLC need not prior information of uncertainty and disturbance, no matter whether they are matched or mismatched, it has been widely studied.\textsuperscript{20} But the traditional SMLC is certainly conservative since it ignores the information of interference and uncertainty. If the interference and uncertainty can be estimated online, and a corresponding improved strategy is made to SMLC, the conservatism of traditional SMLC will be decreased and the control performance will be much better. Considering the advantage of it, ESO is adopted to estimate the uncertainty real time, and then an SMLC is constructed to deal with the influence of the uncertainty.

A lot of industrial system can be represented by a mechanical system, such as active suspension of vehicle,\textsuperscript{21} serial robot arm,\textsuperscript{22} planar three-link mechanical system,\textsuperscript{23} and rotational mechanical system.\textsuperscript{24} For the successful application of industrial system, the control of mechanical system is a hot issue in recent years.\textsuperscript{25} However, because of the change of working environment and the existence of various external disturbances, uncertainty and disturbance are inevitable in mechanical system. In view of the representative significance of mechanical system, it is adopted in this paper as the research object. Motivated by the above discussion, an ESO-based improved SMLC is presented for uncertain mechanical system. An uncertain model of mechanical system is proposed first, and then the model is transformed into a standard one, more specifically, a linear uncertain system with disturbances. ESO is utilized here to estimate the uncertainty and disturbance, and then an ESO-based sliding surface is constructed and its stability is guaranteed by selecting parameters appropriately. After getting the sliding surface, a learning controller is proposed for the uncertain mechanical system model. Finally, the proposed ESO-based SMLC is confirmed by a numerical example.

The novelties and main contributions of the paper can be summarized and listed as follows:

1. A robust controller is designed for uncertain mechanical system with both parameter uncertainty and disturbance; what’s more, the priori information of parameter uncertainty and disturbance are assumed to be unknown.

2. An ESO is constructed to estimate the unknown parameter uncertainty and disturbance, and then an ESO-based sliding surface is proposed for uncertain mechanical system;

3. A novel SMLC is proposed for mechanical system. The proposed controller can deal with both parameter uncertainty and unmodeled dynamic without any information of them and can also greatly reduce the chattering of traditional SMC.

This paper is organized as follows. The uncertain mechanical system is listed in the “Problem formulation” section, and the main results are listed in the “Main result” section. Numerical simulation results are proposed in the “Numerical simulation” section. The paper is summarized in the “Conclusion” section.

Problem formulation

Model of mechanical system

The mechanical system is a 2-degree-of-freedom and the sketch of it is given in Figure 1. The equation is listed as

$$
\begin{align*}
M' \ddot{r}(t) + (G' + \Delta G') \dot{r}(t) + \left(K' + \Delta K'\right) r(t)
& = B' \dot{u}(t) + E \ddot{d}(t) \\
& = B' \dot{u}(t) + E \dot{d}(t)
\end{align*}
$$

(1)

where $r(t) = [r_1(t) \, r_2(t)]^T$ is the position vector of the mechanical system, correspondingly, and $\dot{r}(t)$ and $\ddot{r}(t)$ represent the velocity vector and the acceleration vector, respectively. $M'$ represents the mass of the system, $G'$ represents the gyroscopic/dissipation characteristics, $K'$ represents the stiffness characteristics, and $\Delta G'$ and $\Delta K'$ are unknown uncertainties in gyroscopic/dissipation characteristics and stiffness characteristics, respectively.

$$
M' = \begin{bmatrix}
m_{11} & 0 \\
0 & m_{22}
\end{bmatrix}, 
G' = \begin{bmatrix}
c_1 & -c_2 \\
-c_2 & c_3
\end{bmatrix}, 
K' = \begin{bmatrix}
k_s & -k_s \\
-k_s & k_s
\end{bmatrix}
$$

$$
\Delta G' = \begin{bmatrix}
\Delta c_1 & -\Delta c_2 \\
-\Delta c_2 & \Delta c_3
\end{bmatrix}, \Delta K' = \begin{bmatrix}
\Delta k_s & -\Delta k_s \\
-\Delta k_s & \Delta k_s
\end{bmatrix}
$$

$$
B' = \begin{bmatrix}
1 \\
0
\end{bmatrix}, E' = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
$$

According to the physical meaning of mechanical system, equation (1) can be rewritten as
\hat{r}(t) = -M^{-1}(G + \Delta G')\hat{r}(t) - M^{-1}(K' + \Delta K')r(t) \\
+ M^{-1}B'u(t) + M^{-1}E'd(t) \\
= -M^{-1}G\hat{r}(t) - M^{-1}K'r(t) + M^{-1}B'u(t) \\
- M^{-1}\Delta G\hat{r}(t) - M^{-1}\Delta K'r(t) + M^{-1}E'd(t) \\
(2)

For mechanical system model (equation (2)), choosing the state as

\[ x(t) = \begin{bmatrix} \hat{r}_1(t) \\ \hat{r}_2(t) \\ r_1(t) \\ r_2(t) \end{bmatrix} \]

Then, mechanical system (equation (2)) can be rewritten as

\[ \dot{x}(t) = (A + \Delta A)x(t) + Bu(t) + Ed(t) \]

where

\[ A = \begin{bmatrix} -\frac{c_1}{m_1} & \frac{c_1}{m_1} & -\frac{k_1}{m_1} & \frac{k_1}{m_1} \\ -\frac{c_2}{m_2} & \frac{c_2}{m_2} & -\frac{k_2}{m_2} & \frac{k_2}{m_2} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \]

\[ \Delta A = \begin{bmatrix} \frac{\Delta c_1}{m_1} & \frac{\Delta c_2}{m_2} & \frac{\Delta k_1}{m_1} & \frac{\Delta k_2}{m_2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \]

\[ B = \begin{bmatrix} \frac{1}{m_2} \\ 0 \\ 0 \end{bmatrix}, \]

\[ E = \begin{bmatrix} \frac{1}{m_1} & 0 \\ \frac{1}{m_1} & 0 \\ 0 & \frac{1}{m_1} \end{bmatrix} \]

**Main result**

**ESO design**

Equation (3) can be simplified as

\[ \dot{x}(t) = (A + \Delta A)x(t) + Bu(t) + Ed(t) \]

\[ = Ax(t) + Bu(t) + \Delta A\dot{x}(t) + Ed(t) \]

\[ = Ax(t) + Bu(t) + F\dot{x}(t) \]

where

\[ F = \begin{bmatrix} \frac{1}{m_1} \\ 0 \\ \frac{1}{m_2} \\ 0 \end{bmatrix} \]

\[ \chi(t) = \begin{bmatrix} -\Delta c_1\hat{r}_1(t) - \Delta c_2\hat{r}_2(t) - \Delta k_1r_1(t) + \Delta k_2r_2(t) + d_1(t) \\ -\Delta c_1\hat{r}_1(t) - \Delta c_2\hat{r}_2(t) + \Delta k_1r_1(t) - \Delta k_2r_2(t) + d_2(t) \end{bmatrix} \]

**Assumption 1.** \[ \chi(t) \] is unknown, but \[ \chi(t) \] and its derivative are bounded, which means that

\[ \lim_{0 \leq t < \infty} \|\chi(t)\| \leq N_\chi \]

\[ \lim_{0 \leq t < \infty} \|\dot{\chi}(t)\| \leq \dot{N}_\chi \]

where \[ N_\chi \] and \[ \dot{N}_\chi \] are unknown constant vectors.

**Remark 1.** From equation (4), we can see that \[ \chi(t) \] can be viewed as the sum of \[ \Delta A\dot{x}(t) \] and \[ d(t) \]. For the perturbation \[ d(t) \], it is bounded and the derivative can also be viewed as bounded. Even when a perturbation occurring suddenly, the derivative of it can also be viewed as bounded for long time scales. For \[ \Delta A\dot{x}(t) \], we assume that if the proposed controller \[ u(t) \] can guarantee the stability of the closed-loop system, then the state \[ x(t) \] of equation (4) will converge to a stable value. Since \[ \Delta A \] is bounded, \[ \Delta A\dot{x}(t) \] is also bounded. Then, \[ \chi(t) \] and its derivative can be considered as bounded.

For system (equation (4)), designing a sliding surface directly is difficult because of the existence of unknown \[ \chi(t) \], especially when they do not satisfy the matched condition. Since the boundary information of uncertainty and perturbation are unknown, the traditional robust sliding surface cannot be utilized here, so an improved sliding surface design method is needed. For the controller design of equation (4), if \[ \chi(t) \] can be estimated online, it will bring great benefit for the sliding surface design. ESO is an efficient way for the estimation of unknown dynamics, so it is utilized here.

For system (equation (4)), the extended state is chosen as \[ \xi(t) = [x(t), \chi(t)] \], and

\[ \dot{\xi}(t) = A\xi(t) + B'u(t) + Dh(t) \]

where

\[ A = \begin{bmatrix} A & F_m \\ 0_{2 \times 4} & 0_{2 \times 2} \end{bmatrix}, \]

\[ B = \begin{bmatrix} B \\ 0_{2 \times 1} \end{bmatrix} \]
\[ D = \begin{bmatrix} 0 & \nu \v 2 \\ I \end{bmatrix}, h(t) = \frac{d \chi(t)}{dt} \]

Defining the measurable output of equation (5) as \( y(t) \)

\[ y(t) = C \xi(t) \]  
(6)

with \( C \) is the output matrix, and the pair of \((A^T, C)\) is observable, and then considering the following observer for equation (5)

\[ \dot{\hat{\xi}}(t) = A^T \hat{\xi}(t) + B^T u(t) + L(\hat{y}(t) - y(t)) \]

\[ \hat{y}(t) = C \hat{\xi}(t) \]  
(7)

where \( L \) is the observer gain which will be designed later. Defining the estimation error as \( e_\xi(t) = \hat{\xi}(t) - \xi(t) \), then we can get the estimation error dynamic

\[ \dot{e}_\xi(t) = (A^T + LC)e_\xi(t) - Dh(t) \]

\[ = [A, e_\xi(t) - Dh(t)] \]  
(8)

where \( A_e = A^T + LC \). Because of the existence of \( h(t) \), the designed observer gain \( L \) should guarantee the following \( H_\infty \) performance for equation (8)

\[ \int_0^T e_{\xi(t)}^T e_{\xi(t)} dt \leq \rho^2 \int_0^T h(t) h(t) dt \]  
(9)

where \( \rho \) is a prescribed attenuation level. Then, Theorem 1 can be constructed.

**Theorem 1.** For observer error system (equation (8)), if there exists a matrix \( P_e \geq 0 \) and matrix \( L \) with appropriate dimension, satisfying

\[ P_e A_e + A_e^T P_e + \rho^{-2} P_e D D^T P_e < 0 \]  
(10)

Then, equation (8) is asymptotically stable and the \( H_\infty \) performance (equation (9)) is satisfied.

**Proof.** Defining Lyapunov function for equation (8) as

\[ V_1(t) = e_{\xi(t)}^T P_e e_{\xi(t)} \]

and taking time derivative of \( V_1(t) \), we have

\[ \dot{V}_1(t) = e_{\xi(t)}^T \left\{ P_e [A_e e_{\xi(t)} - Dh(t)] + [A_e e_{\xi(t)} - Dh(t)]^T P_e \right\} e_{\xi(t)} \]

\[ = e_{\xi(t)}^T (P_e A_e + A_e^T P_e) e_{\xi(t)} - 2 e_{\xi(t)}^T P_e D h(t) \]

\[ \leq e_{\xi(t)}^T (P_e A_e + A_e^T P_e) e_{\xi(t)} + \rho^{-2} e_{\xi(t)}^T P_e D D^T P_e e_{\xi(t)} \]

\[ + \rho^{-2} h(t) h(t) \]

System (equation (8)) is \( H_\infty \) stable with \( \rho \) if the following inequality holds

\[ \dot{V}_1(t) - \rho^{-2} h(t) h(t) < 0 \]  
(11)

From equation (10), we can get that equation (11) leads to

\[ P_e A_e + A_e^T P_e + \rho^{-2} P_e D D^T P_e < 0 \]

If equation (10) is held, equation (8) is stable and \( H_\infty \) performance (equation (9)) is satisfied. The proof is completed. □

From Theorem 1, the observer error system is stable, then the designed observer (equation (7)) can estimate the unknown uncertainty \( \chi(t) \) exactly. The linear matrix inequality (LMI) (equation (10)) proposed in Theorem 1 can be easily solved by LMI toolbox in MATLAB, so the computational complexity of the observer gain \( L \) is really low.

**ESO-based sliding surface design**

Considering the construction of \( B \)

\[ B = \begin{bmatrix} \frac{1}{m_0} \\ 0 \end{bmatrix} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \]

Then, the linear system can accordingly be rewritten as

\[ \dot{x}(t) = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ F_2 \end{bmatrix} \xi(t) \]

\[ + \begin{bmatrix} F_1 \end{bmatrix} \tilde{r}(t) \]

where

\[ A_{11} = \begin{bmatrix} -e_1 \\ \frac{c_e}{m_3} \end{bmatrix} \]

\[ A_{12} = \begin{bmatrix} \frac{c_e}{m_3} & \frac{k_e}{m_3} & \frac{k_s}{m_3} \\ \frac{c_e}{m_2} & 1 & 0 \end{bmatrix} \]

\[ A_{21} = \begin{bmatrix} -\frac{c_e}{m_2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]

\[ A_{22} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]

\[ B_1 = \begin{bmatrix} 1 \\ \frac{1}{m_0} \end{bmatrix} \]

\[ F_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \]

\[ F_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]

Then, we can design a sliding surface for system (equation (12)). Designing the following sliding surface

\[ \sigma(t) = K \tilde{\xi}(t) = K \xi(t) + K e_{\xi(t)} = \begin{bmatrix} I & -K_2 & K_d \xi(t) + K e_{\xi(t)} \end{bmatrix} = x_1(t) - K_2 x_2(t) + K_d \xi(t) + K e_{\xi(t)} \]  
(13)

\( K_2 \) and \( K_d \) are gain matrices needed to be designed later. According to the SMC theory, \( \sigma(t) = 0 \), then
\[ x_1(t) = K_2 x_2(t) - K_d x(t) - K e(t) \]

From equation (12)
\[ \dot{x}_2(t) = A_{21} x_1(t) + A_{22} x_2(t) + F_2 x(t) \]

So the reduced order dynamic is
\[ \dot{x}_2(t) = (A_{21} K_2 + A_{22}) x_2(t) + (F_2 - A_{21} K_d) \chi(t) - A_{21} K e(t) \]

Then, the sliding surface control design problem is transformed into the design of main matrix \( K_2 \) and \( K_d \) for the stability of equation (14). From equation (14), we should find a \( K_d \) to minimize the norm of \( F_2 - A_{21} K_d \). Then, the design of sliding surface (equation (13)) can then be divided into the following two parts:

**Step 1.** Finding a matrix \( K_d \), which can minimize \( \|F_2 - A_{21} K_d\| = \|H\| \);

**Step 2.** Constructing a gain matrix \( K_2 \), which can guarantee the stability of the following equation:

\[ \dot{x}_2(t) = (A_{21} K_2 + A_{22}) x_2(t) + H \chi(t) - A_{21} K e(t) \]

(15)

For Step 1, the minimizing problem can be transformed into the following optimal problem:

\[ \min_{K_d} \text{Trace} \left( (F_2 - A_{21} K_d)(F_2 - A_{21} K_d)^T \right) \]

(16)

Through the optimal problem (equation (16)), \( K_d \) can be determined and then the value of \( H \) is determined. Considering equation (8), the reduced closed-loop system can be rewritten as

\[ \dot{e}(t) = \begin{pmatrix} A_{21} K_2 + A_{22} & -A_{21} K_d \\ 0 & A_d \end{pmatrix} e(t) + \begin{pmatrix} H \\ -D \end{pmatrix} \Omega(t) \]

(17)

where

\[ e(t) = \begin{pmatrix} x_2(t) \\ e(t) \end{pmatrix}, \quad \Omega(t) = \begin{pmatrix} \chi(t) \\ h(t) \end{pmatrix} \]

For the above system, if the designed \( K_2 \) can guarantee the following \( H_\infty \) performance, then the designed sliding surface (equation (13)) is stable for both matched and unmatched uncertainty

\[ \int_0^\infty e(t) e^T(t) dt \leq \int_0^\infty \Omega(t) \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} \Omega(t) dt \]

(18)

with \( p_1 \) and \( p_2 \) are prescribed attenuation levels and will be discussed later. Then, for Step 2, we have the following theorem.

**Theorem 2.** For system (equation (17)), if there exists matrices \( P_1 > 0 \), \( P_2 > 0 \), \( K_2 \), and a scalar \( \varepsilon > 0 \), satisfying

\[
\begin{align*}
&\begin{bmatrix} P_1 (A_{21} K_2 + A_{22}) + (A_{21} K_2 + A_{22})^T P_1 + \rho_1^{-2} P_1 DD^T P_1 + \varepsilon P_1 A_{21} A_{21}^T P_1^T \end{bmatrix} \leq 0 \\
&\begin{bmatrix} (P_2 A_d + A_d^T P_2 + \rho_2^{-2} P_2 DD^T P_2 + \varepsilon^{-1} K^T K) \leq 0 \end{bmatrix}
\end{align*}
\]

(19)

(20)

Then, the reduced closed-loop system (equation (17)) is stable, and \( H_\infty \) tracking performance (equation (18)) is satisfied.

**Proof.** For system (equation (17)), considering the following Lyapunov function candidate

\[ V_2(t) = \begin{pmatrix} x_2(t) \\ e(t) \end{pmatrix}^T \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} \begin{pmatrix} x_2(t) \\ e(t) \end{pmatrix} \]

Taking time derivative of \( V_2(t) \), we have

\[
\dot{V}_2(t) = \begin{pmatrix} 2 P_1 (A_{21} K_2 + A_{22}) + 2 A_{21} e(t) e^T(t) \\ 2 e(t) e^T(t) \end{pmatrix} P_2 \dot{x}_2(t) + \begin{pmatrix} 2 P_1 (A_{21} K_2 + A_{22}) + 2 A_{21} e(t) e^T(t) \\ 2 e(t) e^T(t) \end{pmatrix} P_2 \dot{x}_2(t)
\]

\[ + e^T(t) \begin{pmatrix} 2 P_2 A_d + A_d^T P_2 + \rho_2^{-2} P_2 DD^T P_2 \end{pmatrix} e(t) - 2 e(t) e^T(t) P_2 D h(t) \]

(21)

Since

\[ 2 e^T(t) P_1 H \chi(t) \leq \rho_1^{-2} e^T(t) P_1 H H^T P x_2(t) + \rho_2^{-1} \chi(t) \chi(t) \]

\[ - 2 e^T(t) P_1 A_{21} K e(t) \chi(t) \]

and

\[ - 2 e^T(t) P_2 D h(t) \leq \rho_2^{-2} e^T(t) P_2 D D^T P_2 e(t) + \rho_2^{-2} h^T(t) h(t) \]

equation (21) can be rewritten as

\[
\dot{V}_2(t) \leq \begin{pmatrix} 2 P_1 (A_{21} K_2 + A_{22}) + 2 A_{21} e(t) e^T(t) \\ 2 e(t) e^T(t) \end{pmatrix} P_2 \dot{x}_2(t) + \begin{pmatrix} 2 P_1 (A_{21} K_2 + A_{22}) + 2 A_{21} e(t) e^T(t) \\ 2 e(t) e^T(t) \end{pmatrix} P_2 \dot{x}_2(t)
\]

\[ + e^T(t) \begin{pmatrix} 2 P_2 A_d + A_d^T P_2 + \rho_2^{-2} P_2 DD^T P_2 \end{pmatrix} e(t) - 2 e^T(t) P_2 D h(t) \]

\[ + e^T(t) \begin{pmatrix} 2 P_2 A_d + A_d^T P_2 + \rho_2^{-2} P_2 DD^T P_2 \end{pmatrix} e(t) - 2 e^T(t) P_2 D h(t) \]

(22)

From equations (19) and (20)

\[ \begin{pmatrix} (P_1 (A_{21} K_2 + A_{22}) + (A_{21} K_2 + A_{22})^T P_1 + \rho_1^{-2} P_1 DD^T P_1 + \varepsilon P_1 A_{21} A_{21}^T P_1^T \end{pmatrix} \leq 0 \\
\begin{pmatrix} (P_2 A_d + A_d^T P_2 + \rho_2^{-2} P_2 DD^T P_2 + \varepsilon^{-1} K^T K) \leq 0 \end{pmatrix}
\]

(23)

(24)
Then, equation (22) is guaranteed, and the stability and the reduced sliding dynamics (equation (15)) are stable. The proof is completed.

Learning controller design

By choosing \(K_s\) and \(K_d\), the reduced dynamics on the surface can be guaranteed, then a discontinuous sliding mode controller should be designed to guarantee the stability of the closed-loop system. Unfortunately, traditional SMC will cause chattering in control input, while chattering has extremely bad impact on the actual system. In this case, a novel controller is needed.

For avoiding or at least reducing chattering in the control input, a sliding mode–based learning controller is proposed in this section. Quite different from traditional sliding mode discontinuous controller, the proposed learning controller has the form of

\[
u(t) = u(t) - \tau + \Delta u(t)
\]

with \(\tau\) as a time delay and \(\Delta u(t)\) as an adaptation term needed to be designed. \(\Delta u(t)\) has the form of

\[
\Delta u(t) = \begin{cases} 
-B^{-1}_s(a\dot{V}_3(t) - \beta V_3(t)) & \text{for } \sigma(t) \neq 0 \\
0 & \text{for } \sigma(t) = 0 
\end{cases}
\]

where \(V_3(t)\) is a Lyapunov function which will be given later, \(a\) and \(\beta\) are designed parameters, and \(a > 0\), \(\beta > 0\). \(\dot{V}_3(t - \tau)\) is the numerical solution of \(\dot{V}_3(t - \tau)\), and the approximation accuracy of \(\dot{V}(t - \tau)\) is high enough, such that

\[
\text{sign}(\dot{V}(t - \tau)) = \text{sign}(\dot{V}(t - \tau))
\]

\[
|\dot{V}(t - \tau) - \dot{V}(t - \tau)| < \gamma |\dot{V}(t - \tau)|
\]

for \(\dot{V}(t - \tau) \neq 0\), \(\dot{V}(t - \tau) \neq 0\), and \(0 < \gamma \ll 1\).

**Theorem 3.** For system (equation (12)), if \(u(t)\) is adopted according to equations (24) and (25), and the parameters of \(a\) and \(\beta\) are chosen as

\[
\frac{1}{M} < a < 1 - \frac{1}{M} - \gamma, \beta > 0
\]

then, the closed-loop system (equation (12)) is asymptotically stable.

**Proof.** Choosing a Lyapunov function for equation (12)

\[
V_3(t) = \frac{1}{2} \sigma^T(t)\sigma(t)
\]

For sliding surface (equation (13))

\[
\dot{\sigma}(t) = \dot{x}_1(t) - K_2\dot{x}_2(t) + K_6\dot{x}_3(t) + K_6\dot{x}_4(t) + B_1u(t)
\]

\[
= A_{11}\chi_1(t) + A_{12}\chi_2(t) + F_{12}\chi(t) + B_1u(t)
\]

\[
- K_2A_{21}\chi_1(t) - K_2A_{22}\chi_2(t) - K_2E_{2}\chi(t)
\]

\[
+ K_2h(t) + K_{11}\chi_1(t) - D_1h(t)
\]

where \(B_1 = 1/m_t\),

\[
\Psi(t) = A_{11}\chi_1(t) + A_{12}\chi_2(t) + E_{1}\chi(t) - K_2A_{21}\chi_1(t) - K_2A_{22}\chi_2(t) + K_2E_{2}\chi(t) + K_2h(t) + K_{11}\chi_1(t) - D_1h(t).
\]

Then, for \(V_3(t)\)

\[
\dot{V}_3(t) = \sigma^T(t)\dot{\sigma}(t)
\]

\[
= \sigma^T(t)(\dot{\Psi}(t) + B_1u(t))
\]

Using the designed controller (equation (24)) we have

\[
\dot{V}_3(t) = \sigma^T(t)\dot{\Psi}(t) + \sigma^T(t)B_1u(t - \tau)
\]

\[
+ \sigma^T(t)B_1\Delta u(t)
\]

Considering the expression of \(\Delta u(t)\), we have

\[
\sigma^T(t)B_1\Delta u(t) = -\alpha\dot{V}(t - \tau) - \beta V(t)
\]

Then

\[
\dot{V}_3(t) = \sigma^T(t)\dot{\Psi}(t) + \sigma^T(t)B_1u(t - \tau)
\]

\[
- \alpha\dot{V}(t - \tau) - \beta V(t)
\]

The time delay \(\tau\) is reasonably small; then, if \(\dot{V}(t, t - \tau) \neq 0\), \(V(t - \tau) \neq 0\), and \(\dot{V}(t - \tau) \neq 0\)

\[
|\dot{V}(t, t - \tau) - \dot{V}(t - \tau)| < \frac{1}{M} \dot{V}(t - \tau)
\]

where \(M \gg 1\). Then

\[
\dot{V}_3(t) \leq \dot{V}(t, t - \tau) - \dot{V}(t - \tau) + \dot{V}(t - \tau)
\]

\[
- \alpha\dot{V}(t - \tau) - \beta V(t)
\]

\[
< \frac{1}{M} \dot{V}(t - \tau) + \dot{V}(t - \tau) - \alpha\dot{V}(t - \tau) - \beta V(t)
\]

(27)

From the proof of Theorem 2 in Hu et al.,\(^{20}\) whether \(\dot{V}_3(t - \tau) > 0\) or \(\dot{V}_3(t - \tau) < 0\), we can always get that \(\dot{V}(t)\) is always reducing and from equation (26), \(\frac{1}{M} + \gamma - 1 + \alpha < 0\), then

\[
\dot{V}_3(t) < - \frac{1}{M} + \gamma - 1 + \alpha |\dot{V}(t - \tau)| - \beta V_3(t)
\]

\[
< - BV_3(t)
\]

Since \(\beta > 0\), the closed-loop system (equation (12)) is stable. The proof is completed.

**Remark 2.** Equations (24) and (25) give the expression of the designed SMLC, but how to applied it in practice is still not clear. From equation (24), we can see that the main computation of SMLC is \(\Delta u(t)\), while from equation (25), the difficult point of computing \(\Delta u(t)\) is
the value of $\dot{V}_3(t - \tau)$. Actually, with the wide application of digital computer, the calculation of differential and integral in control system is all approximated by numerical solutions. By choosing appropriate numerical approximation algorithm, the approximated error and the computation will be accepted as small. In this paper, we choose a three-point numerical differential formulas to compute $\dot{V}_3(t - \tau)$

$$\dot{V}_3(t - \tau) = \frac{3V_3(t) - 4V_3(t - \tau) + V_3(t - 2\tau)}{2\tau}$$

From the computing expression of $\dot{V}_3(t - \tau)$, we can see that the computational complexity is not so big.

**Remark 3.** Equations (13), (24), and (25) give the total expression of the designed SMLC, and Remark 2 analyzes the computational complexity of SMLC. From equations (13), (24), and (25), we can see that the designed SMLC is easily to be carried out in practice, so the application prospect of SMLC is broad.

**Numerical simulation**

For demonstrating the effectiveness of the proposed ESO-based sliding mode learning controller, a numerical simulation example is considered in this section. The parameters of equation (1) are chosen as follows

- $m_1 = 1 \text{ kg}$, $m_2 = 2 \text{ kg}$, $c_s = 3 \text{ Ns/m}$, $k_s = 5 \text{ N/m}$
- $\Delta c_s = 0.5 \cos(t)$, $\Delta k_s = 0.5 \sin(t)$
- $d(t) = \begin{bmatrix} 0.2 \exp(-0.01t) \sin(2t) \\ 0.2 \exp(-0.01t) \cos(2t) \end{bmatrix}$

Then, the matrices of equation (4) are

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -5 & 5 & -3 & 3 \\ 2.5 & -2.5 & 1.5 & -1.5 \end{bmatrix}$$

**Figure 2.** Position $r_1$.

**Figure 3.** Position $r_2$.

In the leaning controller, we choose $\alpha = 0.3$ and $\beta = 4$, and then the designed controller is applied on the 2-degree-of-freedom mechanical system. For testing the performance of the proposed SMLC, the improved SMLC is marked as $u_{usmlc}$ and is implemented on the 2-degree-of-freedom mechanical system, together with a traditional sliding model control $u_{usmc}$.

The initial states of the mechanical system is set to be $x(0) = [1 \ -0.8 \ 0 \ 0]^T$, and the simulation results are listed in Figures 2 and 3, and the dash–dot line represents the responses under controller $u_{usmlc}$ and the solid line is for traditional controller SMC $u_{usmc}$. Figures 4 and 5 are the input of $u_{usmc}$ and $u_{usmlc}$. From Figures 2 and 3, we can see that the system controlled by the proposed SMLC and SMC are all stable. But from equations (4) and (5), we can see that the input of $u_{usmlc}$ is smooth, while the input of $u_{usmc}$ is chattering. The
Control performance of $u_{\text{usmlc}}$ is obviously better than traditional controller $u_{\text{usmc}}$.

Conclusion

In this paper, an ESO-based SMLC has been proposed for mechanical system. An ESO is constructed to estimate the unknown uncertainty online first and then an ESO-based sliding mode surface is designed. The stability of the designed sliding surface is discussed. A learning controller is designed instead of the discontinuous controller to guarantee the stable of the mechanical system. Finally, a numerical simulation is proposed to show the good performance of SMLC.

Declaration of conflicting interests

The author(s) declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

Funding

The author(s) disclosed receipt of the following financial support for the research, authorship, and/or publication of this article: This work was partially supported by the Fundamental Research Funds for the Central Universities (3102019ZDHQD03, 3102019ZDHKY06), Natural Science Basic Research Plan in Shanxi Province of China (Program No. 2019JM-127), Aeronautical Science Foundation of China (201707U8003), and National Natural Science Foundation of China (61503392, 61304001, 61773386, 61673386).

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