A Borsuk-Ulam theorem for $(\mathbb{Z}_p)^k$-actions on products of (mod $p$) homology spheres

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Abstract

It is proved that for a product action of $(\mathbb{Z}_p)^k$ on a product of (mod $p$) homology spheres $N^{n_1} \times \ldots \times N^{n_k}$, where all $n_i$'s are assumed to be odd if $p$ is odd, and any continuous map $f: N^{n_1} \times \ldots \times N^{n_k} \to \mathbb{R}^m$ the set $A(f) = \{x \in N^{n_1} \times \ldots \times N^{n_k} | f(x) = f(gx) \forall g \in (\mathbb{Z}_p)^k\}$ has dimension at least $n_1 + \ldots + n_k - m(p^k - 1)$, provided $n_i \geq mp^{k-1}(p-1)$ for all $i (1 \leq i \leq k)$. Moreover, if $n_i \geq mp^{k-1}(p-1)$ for all $i (1 \leq i \leq k)$ then the free action $\mu$ can be assumed arbitrary.

Keywords: vector bundle, Euler class, Chern classes, Stiefel-Whitney classes

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1 Introduction

The famous Borsuk-Ulam theorem has been generalized by a number of authors. One of the first and memorable generalizations is due to C.T. Yang and D.G. Bourgin:

Theorem 1.1. Let $T$ be a fixed point free involution on a sphere $S^n$ and let $f: S^n \to \mathbb{R}^m$ be a continuous map into Euclidean space. Then the dimension of the coincidence set $A(f) = \{x \in S^n | f(x) = f(Tx)\}$ is at least $n - m$.

The theorem above was generalized in works of several authors. First, P.E. Conner and E.E. Floyd proved the following generalization:

Theorem 1.2. Let $T$ be a differentiable involution on a sphere $S^n$ and let $f: S^n \to M^m$ be a continuous map into a differentiable manifold $M^m$ of dimension $m$. Assume that $f_\ast: H_n(S^n; \mathbb{Z}_2) \to H_n(M; \mathbb{Z}_2)$ is trivial. Then the dimension of the coincidence set $A(f) = \{x \in S^n | f(x) = f(Tx)\}$ is at least $n - m$. 
In their consequent works H. Munkholm [7] and M. Nakaoka [10] showed that the differentiability condition on the involution $T$ can be dropped provided the target topological manifold $M^m$ is assumed to be compact. Moreover, they generalized the previous theorem to the case of free actions of a cyclic group $\mathbb{Z}_p$ on (mod $p$) homology spheres. They proved the following:

**Theorem 1.3.** Let a cyclic group $\mathbb{Z}_p$ of a prime order act freely on a (mod $p$) homology $n$-sphere $N^n$, and let $f:N^n \to M^m$ be a continuous map into a compact topological manifold $M^m$ of dimension $m$. If $p$ is odd also assume that $M$ is orientable. Suppose that $f_*:H_n(N;\mathbb{Z}_p) \to H_n(M;\mathbb{Z}_p)$ is trivial. Then the dimension of the coincidence set $A(f) = \{x \in N| f(x) = f(gx) \\forall g \in \mathbb{Z}_p\}$ is at least $n - m(p - 1)$.

One of the corollaries of the theorem above is the famous theorem by J. Milnor [6] which asserts that every element of order two in a group which acts freely on a sphere must be central (see [10] for details).

The purpose of this paper is to suggest another generalization of the Borsuk-Ulam theorem, namely, to prove the next theorem. Further and until the rest of the paper $p$ is always assumed to be a prime number.

**Theorem 1.4.** Let $M := N^{n_1} \times \ldots \times N^{n_k}$ be a product of (mod $p$) homology $n_i$-spheres and let $\mu: (\mathbb{Z}_p)^k \wr M$ be a product of free actions $\mu_i: \mathbb{Z}_p \wr N^{n_i}$ ($1 \leq i \leq k$). If $p$ is odd also assume that all $n_i$'s are odd. For a map $f: M \to \mathbb{R}^m$ define a coincidence set $A(f) := \{x \in M| f(x) = f(gx) \\forall g \in \mathbb{Z}_p^k\}$. Then

$$\dim A(f) \geq \dim M - m(p^k - 1)$$

provided $n_i \geq mp^{i-1}(p - 1)$ for all $i(1 \leq i \leq k)$. Moreover, if one assumes $n_i \geq mp^{k-1}(p - 1)$ for all $i(1 \leq i \leq k)$ then the free action $\mu$ can be assumed arbitrary.

**Remark.** For $p = 2$ and $m = 1$ the theorem above was implicitly proved by A.N. Dranishnikov in [3]. In the case $n_i \geq m(p^k - 1)$ for all $i(1 \leq i \leq k)$ the theorem above was proved by V.V. Volovikov in [12].

Let $G$ be a group and let $R$ be a commutative ring with one. Then by $I_R(G)$ we denote the augmentation ideal of the group ring $R[G]$, i.e. the kernel of the augmentation homomorphism $R[G] \to R$.

The key ingredient in the proofs of the most Borsuk-Ulam type theorems for maps into Euclidean spaces is the following basic observation:
Lemma 1.1. Let $G \acts M$ be a free action of a finite group $G$ on a topological manifold $M$. For a continuous map $f: M \to \mathbb{R}^m$ define a coincidence set $A(f) := \{x \in M | f(x) = f(gx) \ \forall g \in G\}$. Then $A(f) \neq \emptyset$ if and only if the vector bundle $\xi: M \times_G \mathbb{R}^m(G) \to M/G$ does not have a non-vanishing section.

Proof. First, note that every continuous map $f: M \to \mathbb{R}^m$ gives rise to a continuous section $\hat{s}(f): M/G \to M \times_G \mathbb{R}^m(G)$ of a vector bundle $\hat{\xi}: M \times_G \mathbb{R}^m(G) \to M/G$ defined by a formula:

$$\hat{s}(f)(xG) = (x, \sum_{g \in G} f(xg^{-1})g)G.$$

Observe that $\hat{\xi} = \xi \oplus \mathbb{R}^m$ where $\mathbb{R}^m$ is a trivial $m$-dimensional real vector bundle. Therefore a projection $\pi: M \times_G \mathbb{R}^m(G) \to M \times_G \mathbb{R}^m(G)$ is well defined. Now define a continuous section $s(f): M/G \to M \times_G \mathbb{R}^m(G)$ of $\xi$ by a formula $s(f) := \pi \circ \hat{s}(f)$. It is easy to see that $s(f)(xG) = 0$ if and only if the orbit of $x \in M$ is mapped by $f$ to a point.

Conversely, given a continuous section $s$ of $\xi$, it defines a $G$-equivariant map $\bar{s}: M \to M \times \mathbb{R}^m(G)$ which is due to its equivariance must be of the form $\bar{s}(x) = (x, \sum_{g \in G} f(xg^{-1})g)$ for some $f: M \to \mathbb{R}^m$, and the lemma follows. \qed

Usually, to prove a Borsuk-Ulam type theorem for maps into Euclidean spaces one shows that the Euler class of the vector bundle $\xi: M \times_G \mathbb{R}^m(G) \to M/G$ in a suitable cohomology theory is non-trivial. Then the dimension restrictions on the coincidence set $A(f)$ follow (see the proof of Theorem 1.4). For instance, the theorems from \cite{7, 8, 11} were proved in this way. Unfortunately, when one uses ordinary cohomology theory, Euler class of $\xi$ very often turns out to be trivial (see \cite{11}). This in fact is an explanation of why all available results in the area are restricted to the actions of so few groups. In this setting the results of H. Munkholm from \cite{8} are especially interesting. In that paper he proves a Borsuk-Ulam type theorem for $\mathbb{Z}_p$-actions, $p$ is odd, on odd dimensional spheres using a $\tilde{KU}$-theory Euler class.

The proof the Theorem 1.4 also will be based on the non-triviality of the $(\text{mod } p)$ Euler class of a corresponding vector bundle. The next two sections will be devoted to the calculation of Euler classes of relevant vector bundles.
2 Calculation of $w_{2k-1}(\eta)$

In this section assume that $G = (\mathbb{Z}_2)^k$. As usual $BG$ stands for the classifying space of $G$ and $EG$ stands for the total space of the universal $G$-bundle. This section is devoted to the calculation of the (mod 2) Euler class of a vector bundle $\eta: EG \times_G I_{\mathbb{R}}(G) \to BG$, i.e. its Stiefel-Whitney class $w_{2k-1}(\eta)$. These calculations are then needed in the proof of Theorem 1.4 in case $p = 2$. Recall that $H^*(BG; \mathbb{Z}_2)$ is a polynomial algebra $\mathbb{Z}_2[x_1, ..., x_k]$ on 1-dimensional generators.

Lemma 2.1. $w_{2k-1}(\eta) = \prod_{q=1}^{k} \prod_{1 \leq i_1 < ... < i_q \leq k} (x_{i_1} + ... + x_{i_q})$

Proof. Let $\mathbb{Z}_2$ act on $\mathbb{R}$ by an obvious involution. This involution induces on $\mathbb{R}$ a structure of an $\mathbb{R}[\mathbb{Z}_2]$-module which we will denote by $V$. Denote by $pr_i: BG \to \mathbb{R}P^\infty$ a projection on the $i^{th}$ coordinate. Then by $\lambda_i$ we denote a 1-dimensional real vector bundle obtained from the following diagram:

$$
\begin{align*}
E(\lambda_i) & \longrightarrow S^\infty \times \mathbb{Z}_2 V \\
\lambda_i & \downarrow \\
BG & \xrightarrow{pr_i} \mathbb{R}P^\infty
\end{align*}
$$

Here $S^\infty$ stands for the infinite dimensional sphere. From the construction of $\lambda_i$ it follows that $w_1(\lambda_i) = x_i$.

Let $\eta_i$ be a vector bundle obtained from the following diagram:

$$
\begin{align*}
E(\eta_i) & \longrightarrow S^\infty \times \mathbb{Z}_2 \mathbb{R}[\mathbb{Z}_2] \\
\eta_i & \downarrow \\
BG & \xrightarrow{pr_i} \mathbb{R}P^\infty
\end{align*}
$$

From the equality $\mathbb{R}[\mathbb{Z}_2] = V \oplus (V \otimes_{\mathbb{R}[\mathbb{Z}_2]} V) = V \oplus V^2$ it follows that $\eta_i = \lambda_i \oplus \lambda_i^2$ where $\lambda_i^2 = \lambda_i \otimes \lambda_i$ is a trivial 1-dimensional bundle. Recall an isomorphism of $\mathbb{R}$-modules: $\mathbb{R}[G] \simeq \mathbb{R}[\mathbb{Z}_2 \oplus ... \oplus \mathbb{Z}_2] \simeq \mathbb{R}[\mathbb{Z}_2] \otimes_{\mathbb{R}} ... \otimes_{\mathbb{R}} \mathbb{R}[\mathbb{Z}_2]$. From this isomorphism it follows that $\eta \oplus \varepsilon_1 = \eta_1 \otimes ... \otimes \eta_k$. Therefore there exists the following chain of isomorphisms of vector bundles:

$$
\eta \oplus \varepsilon_1 \simeq \bigotimes_{i=1}^{k} (\lambda_i \oplus \lambda_i^2) \simeq \bigoplus_{(\alpha_1, ..., \alpha_k) \in H} (\lambda_{\alpha_1} \otimes ... \otimes \lambda_{\alpha_k}).
$$
It is well known that the first Stiefel-Whitney class of a tensor product of 1-dimensional real vector bundles equals to the sum of the first Stiefel-Whitney classes of the multiplies. Then by this fact and a formula of Whitney we get the following chain of equalities:

\[
 w_{2k-1}(\eta) = w_{2k-1}(\eta \oplus \varepsilon_1^1) = \prod_{(\alpha_1, \ldots, \alpha_k) \neq 0} (\alpha_1 x_1 + \ldots + \alpha_k x_k) = \prod_{q=1}^k \prod_{1 \leq i_1 < \ldots < i_q \leq k} (x_{i_1} + \ldots + x_{i_q}).
\]

3 Euler class of \( \eta_\mathbb{C} : EG \times_G I_\mathbb{C}(G) \to BG \)

Throughout this section assume that \( p \) is a fixed odd prime and that \( G = (\mathbb{Z}_p)^k \). In this section we will calculate the \((\text{mod } p)\) Euler class of a complex vector bundle \( \eta_\mathbb{C} : EG \times_G I_\mathbb{C}(G) \to BG \) which equals to its Chern class \( c_{p^k-1}(\eta_\mathbb{C}) \). These calculations are then needed in the proof of Theorem 1.4 in case of odd primes. Recall that:

\[
 H^*(BG; \mathbb{Z}_p) = \Lambda_{\mathbb{Z}_p}(y_1, \ldots, y_k) \otimes \mathbb{Z}_p[x_1, \ldots, x_k],
\]

where \( \Lambda_{\mathbb{Z}_p}(y_1, \ldots, y_k) \) is an exterior algebra on 1-dimensional generators and \( \mathbb{Z}_p[x_1, \ldots, x_k] \) is a polynomial algebra on 2-dimensional generators.

Chern classes of a regular representation of \( G \), i.e. Chern classes of the vector bundle \( \eta_\mathbb{C} \oplus \varepsilon_1^1 : EG \times_G \mathbb{C}[G] \to BG \), were first computed by B.M. Mann and R.J. Milgram in [5]. The lemma which is stated after the next definition is essentially borrowed from their paper.

**Definition 3.1.** \( L_k = \prod_{i=1}^k \prod_{\alpha_j \in \mathbb{Z}_p} (\alpha_1 x_1 + \ldots + \alpha_{i-1} x_{i-1} + x_i) \)

The polynomial defined above is called the \( k^{th} \) Dickson’s polynomial (see [5] for more details).

**Lemma 3.1.** \( e(\eta_\mathbb{C}) = (-1)^k L_k^{p-1} \)

**Proof.** The action of \( \mathbb{Z}_p \) on \( \mathbb{C} \) by rotations by \( \frac{2\pi}{p} \) induces on \( \mathbb{C} \) a structure of a \( \mathbb{C}[\mathbb{Z}_p] \)-module which we will denote by \( L \). Let \( pr : BG \to B\mathbb{Z}_p \) be a
projection on the \(i^{th}\) coordinate. Then let \(\lambda_i\) be a 1-dimensional complex vector bundle obtained from the following diagram:

\[
\begin{array}{ccc}
E(\lambda_i) & \longrightarrow & S^n \times_{\mathbb{Z}_p} L \\
\lambda_i \downarrow & & \downarrow \\
BG & \overset{pr_i}{\longrightarrow} & B\mathbb{Z}_p
\end{array}
\]

It is not very difficult to show that \(c_1(\lambda_i) = x_i\).

Let \(\eta_i\) be a vector bundle obtained from the following diagram:

\[
\begin{array}{ccc}
E(\eta_i) & \longrightarrow & S^n \times_{\mathbb{Z}_p} \mathbb{C}[\mathbb{Z}_p] \\
\eta_i \downarrow & & \downarrow \\
BG & \overset{pr_i}{\longrightarrow} & B\mathbb{Z}_p
\end{array}
\]

It follows from the equality \(\mathbb{C}[\mathbb{Z}_p] = L \oplus \cdots \oplus L^p\), where \(L^j = L \otimes_{\mathbb{C}[\mathbb{Z}_p]} \cdots \otimes_{\mathbb{C}[\mathbb{Z}_p]} L\), that \(\eta_i = \lambda_i \oplus \cdots \oplus \lambda_i^p\). Here \(\lambda_i^j = \lambda_i \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \lambda_i\). Also note that \(\lambda_i^p\) is a trivial 1-dimensional complex bundle. Recall an isomorphism of \(\mathbb{C}\)-modules: \(\mathbb{C}[G] \simeq \mathbb{C}[\mathbb{Z}_p \oplus \cdots \oplus \mathbb{Z}_p] \simeq \mathbb{C}[\mathbb{Z}_p] \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \mathbb{C}[\mathbb{Z}_p]\). From this isomorphism it follows that \(\eta_\mathbb{C} \oplus \varepsilon_\mathbb{C} = \eta_1 \otimes \cdots \otimes \eta_k\). Therefore there exists the following chain of isomorphisms of vector bundles:

\[
\eta_\mathbb{C} \oplus \varepsilon_\mathbb{C} \simeq \bigotimes_{i=1}^{k} (\lambda_i \oplus \cdots \oplus \lambda_i^p) \simeq \bigoplus_{(\alpha_1, \ldots, \alpha_k) \in G} (\lambda_i^{\alpha_1} \otimes \cdots \otimes \lambda_i^{\alpha_k}).
\]

From a formula by Whitney and the fact that the first Chern class of a tensor product of 1-dimensional complex bundles equals to the sum of the first Chern classes of the multiples, it follows that:

\[
c_{p^k-1}(\eta_\mathbb{C}) = c_{p^k-1}(\eta_\mathbb{C} \oplus \varepsilon_\mathbb{C}) = \prod_{(\alpha_1, \ldots, \alpha_k) \neq 0} (\alpha_1 x_1 + \cdots + \alpha_k x_k) = \\
= \prod_{i=1}^{k} [(p - 1)!]^{p^k-1} \prod_{(\alpha_1, \ldots, \alpha_i-1,1,0,\ldots,0)} (\alpha_1 x_1 + \cdots + \alpha_{i-1} x_{i-1} + x_i)^{p-1} = \\
= [(p - 1)!]^k L_k^{p-1} = (-1)^k L_k^{p-1}.
\]

The last equality follows from a theorem of Wilson which states that \((p-1)! \equiv (-1)(\text{mod } p)\). Thus \(e(\eta_\mathbb{C}) = c_{p^k-1}(\eta_\mathbb{C}) = (-1)^k L_k^{p-1}\). \(\square\)
4 Proof of Theorem 1.4

In this section we will prove the main result of the paper Theorem 1.4. Here assume that $p$ is any prime number and that $G = (\mathbb{Z}_p)^k$.

Proof of Theorem 1.4. Recall that $M = N^{n_1} \times \ldots \times N^{n_k}$ is a product of $(\mod p)$ homology $n_i$-spheres. We will begin the proof by showing that under assumptions of the theorem the $(\mod p)$ Euler class of $\xi_M: M \times \mu I \mathbb{R}^m(G) \to M/G$ is non-trivial.

By universality property there exists the following commutative diagram:

$$
\begin{array}{ccc}
M \times \mu I \mathbb{R}^m(G) & \longrightarrow & EG \times G I \mathbb{R}^m(G) \\
\xi_M \downarrow & & \downarrow \xi \\
M/G & \xrightarrow{\varphi} & BG.
\end{array}
$$

Case $p=2$. Let $\eta$ be a vector bundle from section 2. Then from an isomorphism $I \mathbb{R}^m = I \mathbb{R} \oplus \ldots \oplus I \mathbb{R} = mI \mathbb{R}$ it follows that $\xi = \eta \oplus \ldots \oplus \eta = m\eta$. Thus $e_2(\xi) = w_{2k-1}(\xi) = w_{2k-1}(\eta)^m$. By Lemma 2.1 we have

$$
w_{2k-1}(\eta) = \prod_{q=1}^{k} \prod_{1 \leq i_1 < \ldots < i_q \leq k} (x_{i_1} + \ldots + x_{i_q}) =
$$

$$
x_k^{2^{k-1}} \prod_{q=1}^{k-1} \prod_{1 \leq i_1 < \ldots < i_q \leq k} (x_{i_1} + \ldots + x_{i_q}) + R_k,
$$

where $R_k$ contains monomials in powers less than $2^{k-1}$. Therefore

$$
e_2(\xi) = x_1^m x_2^{2m} \cdot \ldots \cdot x_k^{2^{k-1}m} + Q_k,
$$

where $Q_k$ does not contain monomials of the form $x_1^m x_2^{2m} \cdot \ldots \cdot x_k^{2^{k-1}m}$. It is easy to verify that

$$
H^*(M/G; \mathbb{Z}_2) = \mathbb{Z}_2[x_1, \ldots, x_k]/(x_1^{n_1+1}, \ldots, x_k^{n_k+1}),
$$

and $\varphi^*: H^*(BG; \mathbb{Z}_2) \to H^*(M/G; \mathbb{Z}_2)$ is an epimorphism with

$$
Ker \varphi^* = (x_1^{n_1+1}, \ldots, x_k^{n_k+1}).
$$

Thus from (1) and the assumption $n_i \geq m2^{i-1}$ for all $i (1 \leq i \leq k)$ it follows that $e_2(\xi_M) = \varphi^*(e_2(\xi)) \neq 0$. 

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Case $p > 2$. Let $\eta_C$ be a vector bundle from section 3. Then from an isomorphism $I_{C^m} = I_C \oplus \ldots \oplus I_C = mI_C$ it follows that $C\xi = \eta \oplus \ldots \oplus \eta = m\eta$, where $C\xi$ is a complexification of the vector bundle $\xi$. We have the following chain of equalities:

$$e_p(\xi)^2 = e_p(C\xi) = c_{2k-1}(C\xi) = c_{2k-1}(\eta_C)^m. \tag{2}$$

By Lemma 3.1 we have

$$e_p(\eta_C) = (-1)^k L_k^{p-1} = (-1)^k L_k^{p-1} \left[ \prod_{\alpha_j \in \mathbb{Z}_p} (\alpha_1 x_1 + \ldots + \alpha_{k-1} x_{k-1} + x_k) \right]^{p-1} = (-1)^k x_k^{k-1(p-1)} L_k^{p-1} + R_k,$$

where $R_k$ contains $x_k$ in powers less than $p^{k-1}(p-1)$. Thus

$$e_p(C\xi) = (-1)^k x_k^{mp^{k-1}(p-1)} L_k^{m(p-1)} + R_k,$$

where $R_k$ contains $x_k$ in powers less than $mp^{k-1}(p-1)$. Then by induction it follows that

$$e_p(C\xi) = (-1)^k x_1^{m(p-1)} x_2^{mp(p-1)} \cdots x_k^{mp^{k-1}(p-1)} + Q_k,$$

where $Q_k$ contains no monomials of the form

$$bx_1^{m(p-1)} x_2^{mp(p-1)} \cdots x_k^{mp^{k-1}(p-1)}, \ b \neq 0, \ b \in \mathbb{Z}_p.$$

Therefore from the previous and (2) it follows that

$$e_p(\xi) = ax_1^{m(p-1)/2} x_2^{mp(p-1)/2} \cdots x_k^{mp^{k-1}(p-1)/2} + \hat{Q}_k, \tag{3}$$

where $a^2 \equiv (-1)^q (mod p)$ for some $q \geq 0$ and $\hat{Q}_k$ contains no monomials of the form

$$bx_1^{m(p-1)/2} x_2^{mp(p-1)/2} \cdots x_k^{mp^{k-1}(p-1)/2}, \ b \neq 0, \ b \in \mathbb{Z}_p.$$

It is not very difficult to see that

$$H^*(M/G; \mathbb{Z}_p) = \Lambda_{\mathbb{Z}_p}(y_1, \ldots, y_k) \otimes_{\mathbb{Z}_p} \mathbb{Z}_2[x_1, \ldots, x_k]/(x_1^{n_1+1/2}, \ldots, x_k^{n_k+1/2}).$$
where \( \dim x_i = 2 \), and \( \varphi^*: H^*(BG; \mathbb{Z}_p) \to H^*(M/G; \mathbb{Z}_p) \) is an epimorphism with
\[
\text{Ker } \varphi^* = (x_1^{\frac{n_1+1}{2}}, \ldots, x_k^{\frac{n_k+1}{2}}).
\]
Then from (3) and the assumption \( n_i \geq mp_i^{-1}(p-1) \) for all \( i (1 \leq i \leq k) \) it follows that \( e_p(\xi_M) = \varphi^*(e_p(\xi)) \neq 0 \).

Since \( A(f) \) is closed and \( G \)-invariant, the set \( M \setminus A(f) \) is also \( G \)-invariant, and therefore we can consider the following exact sequence of a pair:
\[
\ldots \to H^l(M/G, (M \setminus \varnothing)/G) \xrightarrow{\alpha} H^l(M/G) \xrightarrow{\beta} H^l((M \setminus A(f))/G) \to \ldots
\]
By Lemma 1.1 the vector bundle \( \xi_M \) has a non-vanishing section over \( M \setminus A(f) \). Thus \( \beta(e_p(\xi_M)) = 0 \). Therefore there exists a non-trivial element
\[
\mu \in H^{m(p^k-1)}(M/G, (M \setminus A(f))/G)
\]
such that \( \alpha(\mu) = e_p(\xi_M) \). Since we are working over coefficients in a field \( \mathbb{Z}_p \) there exists a corresponding non-trivial element \( \tilde{\mu} \in H_{m(p^k-1)}(M/G, (M \setminus A(f))/G) \). Then by Alexander duality we have
\[
H^{\dim M - m(p^k-1)}(A(f)/G; \mathbb{Z}_p) \neq 0,
\]
and thus \( \dim_{\mathbb{Z}_p} A(f)/G \geq \dim M - m(p^k - 1) \) (see [4]). Since the group \( G \) is finite it easily follows that
\[
\dim A(f) \geq \dim_{\mathbb{Z}_p} A(f) \geq \dim M - m(p^k - 1),
\]
and we are done. \( \square \)

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**References**

[1] P.E. Conner and E.E. Floyd, *Differentiable periodic maps*, Springer-Verlag, Berlin, 1964
[2] D.G. Bourgin, *Multiplicity of solutions in frame mappings*, Illinois J. Math., vol. 9 (1965), 169-177

[3] A.N. Dranishnikov, *On Q-fibrations without disjoint sections*, Funct. Anal. Appl., 22 (1988) no.2, 151-152

[4] A.N. Dranishnikov, *Cohomological dimension of compact metric spaces*, 6 issue 1 (2001), Topology Atlas Invited Contributions, [http://at.yorku.ca/topology/taic.html](http://at.yorku.ca/topology/taic.html) pp. 7-73

[5] B.M. Mann and R.J. Milgram *On the Chern classes of the regular representations of some finite groups*, Proc. Edinburgh Math. Soc (1982) 25, 259-268

[6] J. Milnor, *Groups which act on $S^n$ without fixed points*, Amer. J. Math., vol. 79, n. 3 (1957), pp. 623-630

[7] H.J. Munkholm *Borsuk-Ulam type theorems for proper $\mathbb{Z}_p$-actions on $(\mod p)$ homology n-spheres*, Math. Scand. 24 (1969) 167-185

[8] H.J. Munkholm *On the Borsuk-Ulam theorem for the $\mathbb{Z}_p$ actions on $S^{2n-1}$ and maps $S^{2n-1} \to \mathbb{R}^n$*, Osaka J. Math 7 (1970) 451-456

[9] H.J. Munkholm and M. Nakaoka, *The Borsuk-Ulam theorem and formal groups*, Osaka J. Math. 9(1972), 337-349

[10] M. Nakaoka *Generalizations of Borsuk-Ulam theorem*, Osaka J. Math. 7(1970), 423-441

[11] J.E. Roberts, *A stronger Borsuk-Ulam type theorem for proper $\mathbb{Z}_p$-actions on mod p homology n-spheres*, Proc. Amer. Math. Soc., vol. 72, n. 2 (1978), pp. 381-386

[12] V. Volovikov, *Bourgin-Yang theorem for $\mathbb{Z}_p^n$-actions*, Mat. Sb., vol. 183, n. 7 (1992), pp. 115-144

[13] C.T. Yang, *On maps from spheres to Euclidean spaces*, Amer. J. Math. vol. 79, no. 4(1957), 725-732

[14] C.Y. Yang, *On theorems of Borsuk-Ulam, Kakutani-Yamabe-Yajobo and Dyson, I*, Ann. Math. vol. 60, no. 2 (1954), 262-282