Integrable orbit equivalence rigidity for free groups

Lewis Bowen
University of Texas at Austin
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Abstract

It is shown that every accessible group which is integrable orbit equivalent to a free group is virtually free. Moreover, we also show that any integrable orbit-equivalence between finitely generated groups extends to their end compactifications.

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1 Introduction

Measure equivalence (ME) is an equivalence relation on groups, introduced by M. Gromov [Gro93] as a measure-theoretic counterpart to quasi-isometry. Most of the research in this area has focussed on rigidity phenomena. For example, Furman proved [Fur99a, Fur99b] that any group ME to a lattice in a higher rank simple Lie group has a finite normal subgroup whose quotient is commensurable to a lattice in the same Lie group. See [Fur11] for a survey of further results.

Here we consider the class of free groups. This class is far from rigid: there is a large variety of groups measure-equivalent to a free group [Gab05] and we do not even have a conjectural classification of such groups. So it makes sense to consider the more restrictive
notion of measure-equivalence known as integrable orbit equivalence (IOE) or $L^1$-orbit-equivalence ($L^1$-OE) which takes into account the metric in addition to measure-theoretic structure. This notion is stricter than integrable measure equivalence (IME) (also called $L^1$-measurable equivalence) that first appeared in [BFS13] where it was shown that any group that is $L^1$-ME to a lattice in $SO(n,1)$ ($n \geq 3$) has a finite normal subgroup whose quotient is a lattice in $SO(n,1)$. Another milestone is T. Austin’s work proving $L^1$-ME rigidity for nilpotent groups [Aus13] (see also M. Cantrell’s recent strengthening of Austin’s results [Can15]).

The main result of this paper is that any finitely generated accessible group that admits a strict integrable embedding into a free group is virtually free. In particular, any finitely generated accessible group that is IOE to a free group is virtually free. These terms are defined next.

1.1 Accessible groups

According to Stallings’ Ends Theorem [Sta68], if a finitely generated group $\Gamma$ has more than one end then it splits as either a nontrivial free product with amalgamation or as an HNN extension over a finite subgroup. If such splittings cannot occur indefinitely, then the group is called accessible. C.T.C. Wall conjectured [Wal71] that all finitely generated groups are accessible. A counterexample was obtained by Dunwoody [Dun93]. However, Dunwoody showed that all finitely presented groups are accessible [Dun85].

1.2 Strict integrable embeddings

Let $\Gamma, G$ be finitely generated groups. Intuitively, a strict integrable embedding of $\Gamma$ into $G$ is a random map from $\Gamma$ into $G$ that is ‘Lipschitz on average’ and has a bounded number of preimages. To be precise, fix a finite symmetric generating set $S_G$ of $G$ and define the word length of any $g \in G$ by $|g|_G := n$ where $n$ is the smallest natural number such that there exist $s_1, \ldots, s_n \in S_G$ with $g = s_1 \cdots s_n$.

Given an action of $\Gamma$ on a set $X$, a cocycle into $G$ is a map $\alpha : \Gamma \times X \to G$ such that

$$\alpha(g_1g_2, x) = \alpha(g_1, g_2x)\alpha(g_2, x) \quad \forall g_1, g_2 \in \Gamma, x \in X.$$ 

In the case of concern, $X$ is endowed with a probability measure $\mu$, the action $\Gamma \acts (X, \mu)$ is measure-preserving and $\alpha$ is measurable. Then we say that $\alpha$ is integrable if

$$\int |\alpha(g, x)|_G \, d\mu(x) < \infty$$

for every $g \in \Gamma$. While the precise value of $\int |\alpha(g, x)|_G \, d\mu(x)$ depends on the generating set $S_G$, its finiteness does not and therefore integrability of $\alpha$ does not depend on $S_G$.

We say that $\alpha$ is a strict integrable embedding if in addition to being integrable there is a constant $C > 0$ such that for every $h \in G$,

$$\#\{g \in \Gamma : \alpha(g, x) = h\} \leq C$$

for every $g \in \Gamma$. While the precise value of $\int |\alpha(g, x)|_G \, d\mu(x)$ depends on the generating set $S_G$, its finiteness does not and therefore integrability of $\alpha$ does not depend on $S_G$. However, Dunwoody showed that all finitely presented groups are accessible [Dun85].
for a.e. \( x \). This notion is more restrictive than the notion of integrable embedding defined in the appendix of [Aus13].

Our main result is:

**Theorem 1.1.** Let \( \Gamma \) be a finitely generated accessible group. If \( \Gamma \) admits a strict integrable embedding into a free group then \( \Gamma \) is virtually free.

**Definition 1.** Two groups \( \Gamma, \Lambda \) are **integrably orbit equivalent** (IOE) if there exist probability measure-preserving essentially free ergodic actions \( \Gamma \curvearrowright (X, \mu) \) and \( \Lambda \curvearrowright (X, \mu) \) such that for a.e. \( x \in X \), \( \Gamma x = \Lambda x \) and the orbit cycles, defined by

\[
\alpha : \Gamma \times X \to \Lambda, \quad \alpha(g, x)x = gx,
\]

\[
\beta : \Lambda \times X \to \Gamma, \quad \beta(h, x)x = hx
\]

are integrable.

Clearly an IOE cocycle is a strict integrable embedding. So Theorem 1.1 implies that any finitely generated accessible group IOE to a free group is virtually free.

We do not know whether accessibility is a necessary condition nor whether ‘strict integrable embedding’ can be weakened to ‘integrable embedding’ or IOE weakened to IME.

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## 2 Preliminaries

**Definition 2.** If \( X \) is a connected locally connected \( \sigma \)-compact topological space then let

\[
\text{End}(X) := \lim_{\leftarrow K} \pi_0(X \setminus K).
\]

To be precise, the inverse limit is over all compact subsets \( K \subset X \) and \( \pi_0(X \setminus K) \) denotes the set of noncompact connected components of \( X \setminus K \). We give \( \pi_0(X \setminus K) \) the discrete topology. If \( K \subset L \) are compact subsets of \( X \) then there is a natural map from \( \pi_0(X \setminus L) \) to \( \pi_0(X \setminus K) \) and \( \text{End}(X) \) is the inverse limit of this system (where the collection of compacts of \( X \) is ordered by inclusion). In particular, for every compact \( K \), there is a natural map \( \pi_K : \text{End}(X) \to \pi_0(X \setminus K) \).

The **end compactification** of \( X \), denoted \( \overline{X} \) is the space \( \overline{X} := X \cup \text{End}(X) \) with the following topology: every open subset of \( X \) is \( \overline{X} \). Also, for every compact \( K \subset X \) and \( C \in \pi_0(X \setminus K) \), the set

\[
C \cup \{ \xi \in \text{End}(X) : \pi_K(\xi) \in C \} \subset \overline{X}
\]

is open. These sets form a basis for the topology of \( \overline{X} \).
Let $\Gamma$ be a group with a finite generating set $S$. The \textbf{Cayley graph of} $(\Gamma, S)$, denoted $\text{Cay}(\Gamma, S)$, has vertex set $\Gamma$ and edge set $\{(g, gs) : g \in G, s \in S\}$. We usually let $\bar{\Gamma}$ denote the end compactification $\text{Cay}(\Gamma, S)$, leaving the generating set implicit.

It is well-known that any finitely generated group quasi-isometric to a free group is itself virtually free. This is usually attributed to Gromov via Stallings’ Ends Theorem. Alternatively, it follows from Thomassen-Woess [TW93] that accessibility is a quasi-isometric-invariant and from Papasoglu-Whyte [PW02] that any accessible group quasi-isometric to a free group must be virtually free. Indeed, this implies more:

**Theorem 2.1.** If $\Gamma$ is the fundamental group of a finite graph of groups in which all vertex and edge groups are finite then $\Gamma$ is virtually free.

**Proof.** This follows from [PW02] although it may have been known earlier. \hfill \Box

We also note:

**Lemma 2.2.** If $\Gamma$ is the fundamental group of a finite graph of groups in which all edge groups are finite and $\Gamma_v \leq \Gamma$ is a vertex subgroup then $\Gamma_v$ is quasi-isometrically embedded in $\Gamma$.

Finally, we introduce a notion of $L^1$-embedding:

**Definition 3.** Let $\Gamma \curvearrowright (X, \mu)$ a probability measure-preserving action and $\alpha : \Gamma \times X \to G$ a measurable cocycle. We say $\alpha$ is an \textit{L}^1-embedding if

- $\alpha$ is $L^1$: for every $g \in \Gamma$, $\int |\alpha(g, x)|_G \, d\mu(x) < \infty$ where $|\cdot|_G$ denotes length with respect to a fixed word metric on $G$;

- there is a constant $C > 0$ such that for any $h \in G$,
  $$\# \{g \in \Gamma : \alpha(g, x) = h\} \leq C$$

for a.e. $x$.

**Remark 1.** It is straightforward to check that a composition of $L^1$-embeddings is an $L^1$-embedding and that any cocycle arising from an $L^1$-OE is an $L^1$-embedding.

In order to prove Theorem 1.1 it now suffices to show the following. Let $\Gamma \curvearrowright (X, \mu)$ be a probability measure-preserving action and $\alpha : \Gamma \times X \to G$ an $L^1$-embedding. Then $\Gamma$ is not 1-ended. We will show this in the next section.

3 \quad \textbf{The Space of Ends}

**Theorem 3.1.** Let $\Gamma, G$ be finitely generated groups, $\Gamma \curvearrowright (X, \mu)$ a probability measure-preserving action, $\alpha : \Gamma \times X \to G$ an $L^1$-embedding. Define $\alpha' : \Gamma \times X \to G$ by

$$\alpha'(g, x) = \alpha(g^{-1}, x)^{-1}.$$ 

Let $\bar{\Gamma}, \bar{G}$ denote the end-compactifications of $\Gamma, G$ respectively with respect to fixed finite generating subsets. Then $\alpha'$ extends to a map, also denoted by $\alpha'$ from $\bar{\Gamma} \times X \to \bar{G}$ such that
• \( \alpha'_x(g \xi) = \alpha(g, x) \alpha'_x(\xi) \) (for \( g \in G, \xi \in \hat{\Gamma} \) and a.e. \( x \in X \));

• \( \alpha'_x : \hat{\Gamma} \to \hat{G} \) is continuous for a.e. \( x \).

Here, \( \alpha'_x(g) := \alpha'(g, x) \).

\textbf{Remark 2.} The Theorem above implies that every finitely generated group is 1-taut relative to its space of ends, in the terminology of [BFS13]. We will not need this fact.

Theorem 3.1 follows immediately from the next two lemmas.

\textbf{Lemma 3.2.} Let \( X, Y \) be connected locally connected \( \sigma \)-compact topological spaces. Let \( \alpha : X \to Y \) be a continuous map. Assume that for every compact \( K \subset Y \) there exists a compact \( F \subset X \) such that \( \alpha(X \setminus F) \subset Y \setminus K \) and \( \alpha \) descends to a well defined map \( \pi_0(X \setminus F) \to \pi_0(Y \setminus K) \). Then \( \alpha \) extends continuously to \( \tilde{\alpha} : \overline{X} \to \overline{Y} \).

\textbf{Proof.} For every \( K \subset Y \) we have a map

\[ \text{End}(X) \to \pi_0(X \setminus D) \to \pi_0(Y \setminus K), \]

so the lemma follows by the definition of the inverse limit \( \varprojlim_{D} \pi_0(Y \setminus K) \). \hfill \Box

\textbf{Lemma 3.3.} For a.e. \( x \in X \) and every finite set \( K \subset G \) there exists a finite set \( F \subset \Gamma \) (depending on \( x \) and \( K \)) such that \( \alpha'_x(\Gamma \setminus F) \subset G \setminus K \) and \( \alpha'_x \) descends to a map \( \pi_0(\Gamma \setminus F) \to \pi_0(G \setminus K) \).

\textbf{Proof.} Let \( S_\Gamma, S_G \) be finite generating sets for \( \Gamma, G \) respectively. Let \( | \cdot |_\Gamma, | \cdot |_G \) denote word length on \( \Gamma, G \) respectively.

For each \( h_1, h_2 \in G \), choose a geodesic segment \( \gamma[h_1, h_2] \) from \( h_1 \) to \( h_2 \). More precisely, for every integer \( 0 \leq n \leq |h_1^{-1}h_2|_G \), there is an element \( \gamma[h_1, h_2](n) \in G \) so that

\[ \gamma[h_1, h_2](n)^{-1}\gamma[h_1, h_2](n + 1) \in S_G \]

if \( n < |h_1^{-1}h_2|_G \) and \( \gamma[h_1, h_2](0) = h_1, \gamma[h_1, h_2](|h_1^{-1}h_2|_G) = h_2 \). Let us also require that this choice is left-invariant so that \( h\gamma[h_1, h_2] = \gamma[hh_1, hh_2] \) for any \( h, h_1, h_2 \in G \).

For each \( x \in X, g \in \Gamma, s \in S_\Gamma \), we imagine an airplane flying from \( \alpha'_x(g) \) to \( \alpha'_x(gs) \). The path of the flight is the geodesic \( \gamma[\alpha'_x(g), \alpha'_x(gs)] \). We call this an \textbf{s-flight}. For \( k \in G \), we let \( F_{s,k}(x) \) denote the set of elements \( g \in \Gamma \) such that the s-flight from \( \alpha'_x(g) \) to \( \alpha'_x(gs) \) contains \( k \). That is:

\[ F_{s,k}(x) := \{ g \in \Gamma : \ k \in \gamma[\alpha'_x(g), \alpha'_x(gs)] \} \]

\textbf{Claim 1.} \( F_{s,k}(x) \) is finite for a.e. \( x \). In fact,

\[ \int \#F_{s,k}(x) \, d\mu(x) \leq C \int |\alpha(s^{-1}, x)^{-1}|_G \, d\mu(x) < \infty \]

where \( C > 0 \) is the constant in the definition of \( L^1 \)-embedding.
Proof of Claim 1. It suffices to show that \( \int \#F_{s,k}(x) \, d\mu(x) < \infty \). In order to prove this, let
\[
L_{s,k} = \{(g, x) \in \Gamma \times X : g^{-1} \in F_{s,k}(x)\}.
\]
Let \( c_{\Gamma} \) denote the counting measure on \( \Gamma \). Then \( \int \#F_{s,k} \, d\mu = c_{\Gamma} \times \mu(L_{s,k}) \). Because the action \( \Gamma \curvearrowright (X, \mu) \) is invariant,
\[
c_{\Gamma} \times \mu(L_{s,k}) = c_{\Gamma} \times \mu(R_{s,k})
\]
where \( R_{s,k} = \{(g^{-1}, gx) : (g, x) \in L_{s,k}\} \). By definition
\[
c_{\Gamma} \times \mu(R_{s,k}) = \int \#\{g \in \Gamma : (g, x) \in R_{s,k}\} \, d\mu(x).
\]
However, \( (g, x) \in R_{s,k} \) if and only if \( (g^{-1}, gx) \in L_{s,k} \) if and only if \( g \in F_{s,k}(gx) \) if and only if \( k \in \gamma[\alpha'_{gx}(g), \alpha'_{gx}(gs)] \) if and only if
\[
\alpha'_{gx}(g)^{-1}k \in \gamma[e, \alpha'_{gx}(g)^{-1}\alpha'_{gx}(gs)].
\]
Let us now compute
\[
\alpha'_{gx}(g)^{-1}\alpha'_{gx}(gs) = \alpha(g^{-1}, gx)\alpha(s^{-1}g^{-1}, gx)^{-1} = \alpha(s^{-1}, x)^{-1}
\]
by the cocycle equation. So
\[
\int \#F_{s,k} \, d\mu = c_{\Gamma} \times \mu(R_{s,k}) = \int \#\{g \in \Gamma : \alpha'_{gx}(g)^{-1}k \in \gamma[e, \alpha(s^{-1}, x)^{-1}]\} \, d\mu(x)
\leq C \int |\alpha(s^{-1}, x)^{-1}|_G \, d\mu(x) < \infty.
\]

Now let \( K \subset G \) be finite and define
\[
F_K(x) := \bigcup\{F_{s,k} : s \in S_\Gamma, k \in K\}.
\]
To finish the proof of the lemma, it suffices to show that if \( g_1, g_2 \in \Gamma \) are in the same connected component of \( \Gamma \setminus F_K(x) \) then \( \alpha'_x(g_1), \alpha'_x(g_2) \) are in the same connected component of \( G \setminus K \). Because \( S_\Gamma \) is a generating set, we may assume that \( g_2 = g_1s \) for some \( s \in S_\Gamma \). Because \( g_1 \notin F_K(x) \), it follows that
\[
K \cap \gamma[\alpha'_x(g_1), \alpha'_x(g_1s)] = \emptyset.
\]
So \( \alpha'_x(g_1), \alpha'_x(g_1s) \) are in the same connected component of \( G \setminus K \) as required.

**Definition 4.** Suppose \( H \) is a finitely generated group and \( S_H \subset H \) is a finite symmetric generating set. Let \( \text{Cay}(H, S_H) \) be the associated Cayley graph. Given a subset \( F \subset H \), let \( \partial F \) be the set of all edges \( e \) of \( \text{Cay}(H, S_H) \) with one endpoint in \( F \) and one endpoint in \( H \setminus F \).
Lemma 3.4. Suppose $H$ is a finitely generated group and $S_H \subset H$ is a finite symmetric generating set. Suppose there exists a constant $C > 0$ and finite subsets $F_n \subset H$ such that

- $|\partial F_n| \leq C$ for all $n \in \mathbb{N}$,
- $\lim_{n \to \infty} |F_n| = \infty$.

Then $H$ has at least 2 ends.

Proof. We identify each $F_n$ with its induced subgraph in $\text{Cay}(H, S_H)$. We may assume without loss of generality that every connected component of the complement $\text{Cay}(H, S_H) \setminus F_n$ is infinite. This is because we may add all finite components of $\text{Cay}(H, S_H) \setminus F_n$ to $F_n$ without increasing the size of its boundary.

Choose elements $g_n \in F_n$, $s_n \in S_H$ so that

- $(g_n, g_ns_n) \in \partial F_n$
- if $F_n^0 \subset F_n$ is the connected component of $F_n$ containing $g_n$ then $\lim_{n \to \infty} |F_n^0| = +\infty$
- there exists an infinite path $p_n \subset \text{Cay}(H, S_H) \setminus F_n$ starting from $g_ns_n$.

Let $F'_n = g_n^{-1}F_n$ and $p'_n = g_n^{-1}p_n$. After passing to a subsequence if necessary, we may assume that $F'_n$ converges to a limit $F'_\infty$ and $p'_n$ converges to a limit $p'_\infty$ (in the topology of uniform convergence on compact subsets). We observe that $F'_\infty$ is infinite, $p'_\infty \subset \text{Cay}(H, S_H) \setminus F'_\infty$ is an infinite path and $|\partial F'_\infty| \leq C$. Thus the compact set $K := \partial F'_\infty$ is such that there are at least two infinite components of $\text{Cay}(H, S_H) \setminus K$ (namely, the component containing $p'_\infty$ and the component containing $F'_\infty$). This proves that $H$ has at least two ends.

Proposition 3.5. Suppose $\Gamma$ is an infinite finitely generated group, $G = \mathbb{F}_r$ be a nonabelian free group, $\Gamma \acts (X, \mu)$ a probability measure-preserving action and $\alpha : \Gamma \times X \to G$ an $L^1$-embedding. Then $\Gamma$ has more than one end.

Proof. We fix a free generating set of $G$ from which we obtain a word metric and a Cayley graph (which is a regular tree since $G$ is a free group). We also fix a finite generating set $S_\Gamma$ for $\Gamma$.

To obtain a contradiction, we assume that $\text{End}(\Gamma) = \{\xi\}$ is a singleton. Define $\phi : X \to \text{End}(G)$ by $\phi(x) = \alpha'(\xi, x)$ where $\alpha'$ is as in Theorem 3.1. By Theorem 3.1, $\phi(hx) = \alpha(h, x)\phi(x)$.

For $n \in \mathbb{N}, x \in X$, let $G(n, x)$ be the set of all $g \in G$ such that $(g|\phi(x))_e \leq n$ where $(\cdot|\cdot)_e$ is the Gromov product. To be precise $(g|\phi(x))_e = d(e, m)$ where, if $g \neq e$, $m \in G$ is the ‘midpoint’ of the geodesic triangle with vertices $\{g, \phi(x), e\}$. That is, $m$ is the unique element contained in all three geodesic sides of the triangle with vertices $\{g, \phi(x), e\}$. If $g = e$ then by definition $m = e$. Thus $G(n, x)$ is the set of all elements $g \in G$ such that the geodesic from $g$ to $\phi(x)$ contains a point of distance no more than $n$ from $e$. Let
\[ F(n, x) = \{ h \in \Gamma : \alpha'(h, x) \in G(n, x) \} \]

**Claim 1.** \( \int |\partial F(n, x)| \, d\mu(x) \leq C \sum_{s \in S_{x}} \int |\alpha(s, x)|_{G} \, d\mu(x) =: M \). Note \( M \) is independent of \( n \).

**Proof of Claim 1.** Let \( r_{n}(x) \in G \) be the unique element satisfying
\[ d(e, r_{n}(x)) = n = (r_{n}(x)|_{\phi(x)})_{e}. \]
In other words, \( n \mapsto r_{n}(x) \) is the geodesic from \( e \) to \( \phi(x) \). Observe that \( \partial G(n, x) \) is the unique edge from \( r_{n}(x) \) to \( r_{n+1}(x) \).

By definition \( \partial F(n, x) \) consists of all edges of the form \( (g, gs) \) such that \( g \in F(n, x) \) and \( gs \notin F(n, x) \) \((s \in S)\). Equivalently, \( \alpha'(g) \in G(n, x) \) and \( \alpha'(gs) \notin G(n, x) \). Equivalently, the \( s \)-flight from \( \alpha'(g) \) to \( \alpha'(gs) \) flies over \( r_{n}(x) \). The claim now follows as in the proof of Lemma \[ \S 3 \]

**Claim 2.** For every \( n \in \mathbb{N} \) and a.e. \( x \in X \), \( |F(n, x)| < \infty \).

**Proof.** The previous claim implies \( \partial F(n, x) \) is finite for a.e. \( x \). Because \( \Gamma \) is 1-ended, for a.e. \( x \in X \) either \( F(n, x) \) or \( \Gamma \setminus F(n, x) \) is finite. Because \( \alpha' : \Gamma \times X \to \tilde{G} \) is continuous and \( \alpha'(\xi, x) \notin G(n, x) = \alpha'(F(n, x), x) \), it follows that \( \Gamma \setminus F(n, x) \) must be infinite and therefore \( F(n, x) \) is finite.

Observe that \( G(n, x) \subset G(n+1, x) \) and \( \cup_{n \geq 0} G(n, x) = G \). Therefore \( F(n, x) \subset F(n+1, x) \) for all \( n \) and \( \cup_{n \geq 0} F(n, x) = \Gamma \) which in particular implies that \( \lim_{n \to \infty} |F(n, x)| = +\infty \).

Because
\[ \lim_{n \to \infty} \int |F(n, x)| \, d\mu(x) = +\infty, \quad \int |\partial F(n, x)| \, d\mu(x) \leq M, \]
we can choose \( x_{n} \in X \) so that \(|F(n, x_{n})| \to \infty \) while \(|\partial F(n, x_{n})| \) stays bounded. Lemma \[ \S 3 \] now implies that \( \Gamma \) has at least 2 ends, a contradiction.

**Proof of Theorem 1.1.** By assumption there exists an \( L^{1} \)-embedding \( \alpha : \Gamma \times X \to G \) and \( G \) is a free group. Since \( \Gamma \) is accessible, we may write it as the fundamental group of a finite graph of groups in which each edge group is finite and each vertex group has \( \leq 1 \) end. By Lemma \[ \S 2 \] each vertex group \( H \) quasi-isometrically embeds into \( \Gamma \). So if we restrict \( \alpha \) to \( H \times X \), it is still an \( L^{1} \)-embedding. So Proposition \[ \S 3 \] implies that \( H \) is not 1-ended. So every vertex group and edge group in the graph of groups decomposition of \( \Gamma \) is finite. This implies that \( \Gamma \) is virtually free by Theorem \[ \S 2 \]

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