The stein characterization of $M$-Wright distributions

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**ABSTRACT**

In this paper, we use the Stein method to characterize the $M$-Wright distribution $M_{1/3}$ and its symmetrization. The Stein operator is associated with the general Airy equation and the corresponding Stein equation is nothing but a general inhomogeneous Airy equation.

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1. Introduction

Stein’s method is a powerful technic used for studying approximations of probability distributions, and is best known for its ability to establish convergence rates. It was initially conceived by Charles Stein in the seminal paper [14] to provide errors in the approximation by the normal distribution of the distribution of the sum of dependent random variables of a certain structure. However, the ideas presented are sufficiently powerful to be able to work well beyond that intended purpose, applying to approximation of more general random variables by distributions other than the normal (such as the Poisson, exponential, Gamma, etc). We refer to [2,3,13,15], and references therein for more details on this method.

The $M$-Wright probability density function $M_\beta$, $0 < \beta < 1$ defined on $\mathbb{R}_+$, was introduced by F. Mainardi in order to study time-fractional diffusion-wave equation, see [6]. The function $M_\beta$ is a special case of the Wright function, see [7] for more details. The distribution $\nu_\beta$ on $\mathbb{R}_+$ with density $M_\beta$ with respect to the Lebesgue measure, i.e. $d\nu_\beta(x) = M_\beta(x) \, dx$, $x \geq 0$, we call $M$-Wright distribution. Its Laplace transform is given by

$$
\int_0^\infty e^{-xt} M_\beta(x) \, dx = E_\beta(-t), \quad \forall t \geq 0,
$$

where $E_\beta$ is the Mittag-Leffler function, cf. [5] for details, defined by

$$
E_\beta(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\beta n + 1)}, \quad z \in \mathbb{C}.
$$
The density $M_{1/q}$, $q = 2, 3, \ldots$ satisfies the ODE of order $q-1$, cf. [8]

$$\frac{d^{q-1}}{dx^{q-1}}M_{1/q}(x) + \frac{(-1)^q}{q} x M_{1/q}(x) = 0, \quad x \geq 0. \quad (1)$$

For the special cases $q = 2$ and $q = 3$, we have

$$M_{1/2}(x) = \frac{1}{\sqrt{\pi}} \exp \left( -\frac{x^2}{4} \right),$$

$$M_{1/3}(x) = 3^{2/3} \text{Ai} \left( \frac{x}{3^{1/3}} \right), \quad (2)$$

where $\text{Ai}$ is the Airy function, see Appendix. The density $M_{1/2}$ was characterized in the pioneering work of Stein [14], see also [15] for more details.

In this paper, we investigate the Stein method for the distribution $M_{1/3}$ as well as its symmetrization $\hat{M}_{1/3}$ and characterize them via the Stein equation, see Theorems 2.1 and 3.1 below. The distribution $\hat{M}_{1/3}$ is squeezed between the Gaussian and the symmetric Laplacian distributions, see Figure 1, which were characterized in [15] and [12], respectively. In Section 2, we characterize the distribution $M_{1/3}$ and in Section 3 we do the same for the symmetric case $\hat{M}_{1/3}$.

It follows from Equation (1), with $q = 3$, that $M_{1/3}$ solves the Airy differential equation

$$y''(x) - \frac{1}{3} xy(x) = 0, \quad x \geq 0. \quad (3)$$

It is well known that the two independent solutions of Equation (3) are the Airy functions of the first and second kind, $\text{Ai}$ and $\text{Bi}$, respectively, see Appendix for more details on these functions. The first step in extending the Stein method for the distribution $M_{1/3}$ is
to find a proper Stein equation from (3). This leads us to consider as Stein’s equation the inhomogeneous Airy differential equation

$$y''(x) - \frac{1}{3}xy(x) = h(x) - \mathbb{E}(h(Y)), \quad x \geq 0,$$

(4)

where $Y$ is a random variable with density $M_{1/3}$ and $h$ is a suitable function, see Lemma 2.2 below for details. The solution of Equation (4) is obtained from the two linearly independent solutions $A_i$ and $B_i$ of Equation (3) using the method of variation of parameters. We show that for any real-valued continuous bounded function $h$ on $\mathbb{R}_+$ the solution of Equation (4) belongs to $C^2_b(\mathbb{R}_+)$ (the space of bounded twice continuously differentiable functions on $\mathbb{R}_+$ with bounded derivatives), see Lemma 3.1. Once this is done, we characterize $M_{1/3}$ via the second order Stein’s operator

$$(A_{1/3}f)(x) := f''(x) - \frac{1}{3}xf(x), \quad x \geq 0,$$

(5)

for a sufficiently smooth function $f$, see Section 2 for details. This is the contents of Section 2.

In Section 3, we apply the above scheme to the symmetric density $\hat{M}_{1/3}$ on $\mathbb{R}$. It turns out that $\hat{M}_{1/3}$ solves the ODE

$$y''(x) - \frac{1}{3}|x|y(x) = 0, \quad x \in \mathbb{R},$$

and as Stein’s equation, we consider

$$y''(x) - \frac{1}{3}|x|y(x) = \hat{h}(x),$$

(6)

where $\hat{h}$ is defined

$$\hat{h}(x) := \left[ h(x) - \mathbb{E}(h(Y)) \right]1_{[0,\infty)}(x) + \left[ h(x) - \mathbb{E}(h(-Y)) \right]1_{(-\infty,0)}(x).$$

Notice that in general $\hat{h}$ is not continuous at $x=0$ which implies less regularity of the solution of Equation (6). More precisely, in Lemma 3.2 we show that the solution $f$ of Equation (6) is such $f \in C^1_b(\mathbb{R})$ and $f'' \in C_b(\mathbb{R}^*)$, see also Remark 3.3 for more details. Here and below $C^k_b(X)$ denotes the Banach space of bounded $k$th continuous differentiable functions $f : X \longrightarrow \mathbb{R}$ endowed with the supremum norm $\| \cdot \|_\infty$.

2. The stein characterization of $M_{1/3}$ distribution

In this section, we use the Stein method to characterize the distribution $M_{1/3}$. To this end, at first we introduce the functional framework. Define the space $\mathcal{D}_{1/3}$ of functions in $C^2_b(\mathbb{R}_+)$ such that

$$\frac{f'(0)}{\Gamma\left(\frac{2}{3}\right)} - \frac{f(0)}{\Gamma\left(\frac{1}{3}\right)} = 0,$$

(7)

and consider the operator $(A_{1/3}, \mathcal{D}_{1/3})$.

The following theorem states the main result of this section.
Theorem 2.1 (Characterization of $M_{1/3}$): Let $X$ be a positive random variable. Then $X$ follows the $M_{1/3}$ distribution if and only if

$$
\mathbb{E}((A_{1/3}f)(X)) = 0, \ \forall f \in \mathcal{D}_{1/3}.
$$

Before proving Theorem 2.1, we need a technical lemma.

Lemma 2.2: Let $h : \mathbb{R}_+ \rightarrow \mathbb{R}$ be a bounded continuous function and denote by $\tilde{h}(x) := h(x) - \mathbb{E}(h(Y))$ where $Y$ has $M_{1/3}$ distribution. Then the function

$$
f_h(x) = -3^{1/3}\pi \left[ \text{Ai} \left( \frac{x}{3^{1/3}} \right) \int_0^x \text{Bi} \left( \frac{t}{3^{1/3}} \right) \tilde{h}(t) \, dt + \text{Bi} \left( \frac{x}{3^{1/3}} \right) \int_x^\infty \text{Ai} \left( \frac{t}{3^{1/3}} \right) \tilde{h}(t) \, dt \right],
$$

solves the Stein Equation (4). In addition, there exists non-negative constants $\tilde{C}_1, \tilde{C}_2$ and $\tilde{C}_3$ such that $\|f_h\|_\infty \leq \tilde{C}_1 \|\tilde{h}\|_\infty$, $\|f'_h\|_\infty \leq \tilde{C}_2 \|\tilde{h}\|_\infty$, $\|f''_h\|_\infty \leq \tilde{C}_3 \|\tilde{h}\|_\infty$ and $f_h$ belongs to $\mathcal{D}_{1/3}$.

Proof: First consider $w_1$ and $w_2$ the two independent solutions of the corresponding homogeneous Stein Equation (4). They are given in terms of the Airy functions

$$
w_1(x) = M_{1/3}(x) = 3^{2/3} \text{Ai} \left( \frac{x}{3^{1/3}} \right), \quad x \geq 0,
$$

$$
w_2(x) = 3^{2/3} \text{Bi} \left( \frac{x}{3^{1/3}} \right), \quad x \geq 0.
$$

The solution of the inhomogeneous Stein equation is given using the method of the variation of the parameters, see for example [4], in explicit

$$
f_h(x) = -w_1(x) \int_0^x w_2(t) \tilde{h}(t) \, dt - w_2(x) \int_x^\infty \frac{w_1(t) \tilde{h}(t)}{W(t)} \, dt,
$$

where $W$ is the Wronskian of the solutions $w_1, w_2$ equal to $3/\pi$. Hence, the solution (8) results from (11) using (9)–(11). As $h$ is a continuous function, then it follows that $f_h$ is twice continuously differentiable. Moreover, for any $x \geq 0$, we have

$$
|f_h(x)| \leq 3^{1/3}\pi \|\tilde{h}\|_\infty \left( \text{Ai} \left( \frac{x}{3^{1/3}} \right) \int_0^x \text{Bi} \left( \frac{t}{3^{1/3}} \right) \, dt + \text{Bi} \left( \frac{x}{3^{1/3}} \right) \int_x^\infty \text{Ai} \left( \frac{t}{3^{1/3}} \right) \, dt \right)
$$

$$
= 3^{2/3}\pi \|\tilde{h}\|_\infty \left( \text{Ai} \left( \frac{x}{3^{1/3}} \right) \int_0^{x/3^{1/3}} \text{Bi}(t) \, dt + \text{Bi} \left( \frac{x}{3^{1/3}} \right) \int_{x/3^{1/3}}^\infty \text{Ai}(t) \, dt \right)
$$

$$
\leq 3^{2/3}\pi \|G_i\|_\infty \|\tilde{h}\|_\infty,
$$

where $G_i$ is the Scorer function, see Appendix, Equation (A3).
Now differentiating (8), we obtain
\[ f'_h(x) = -\pi \left[ Ai' \left( \frac{x}{3^{1/3}} \right) \int_0^x \text{Bi} \left( \frac{t}{3^{1/3}} \right) \tilde{h}(t) \, dt + \text{Bi}' \left( \frac{x}{3^{1/3}} \right) \int_x^\infty \text{Ai} \left( \frac{t}{3^{1/3}} \right) \tilde{h}(t) \, dt \right] \]
and estimating as above implies, using Equation (A5), yields
\[ |f'_h(x)| \leq 3^{1/3} \pi \| \text{Gi}' \|_\infty \| \tilde{h} \|_\infty. \]
Since \( f'_h \) is the solution of the Stein Equation (4), then
\[ |f''_h(x)| \leq \frac{1}{3} |xf_h(x)| + \| \tilde{h} \|_\infty \leq 3^{-(2/3)} \pi |x \text{Gi}(x)| \| \tilde{h} \|_\infty + \| \tilde{h} \|_\infty, \quad \forall x \in \mathbb{R}_+. \]
As the function \( \mathbb{R}_+ \ni x \mapsto x \text{Gi}(x) \in \mathbb{R} \) is bounded, cf. Appendix Equation (A4), it follows that \( f''_h \) is bounded and we have
\[ |f''_h(x)| \leq \left( 3^{-(2/3)} \pi \sup_{x \in \mathbb{R}_+} |x \text{Gi}(x)| + 1 \right) \| \tilde{h} \|_\infty. \]
Finally, to see that \( f'_h \) satisfies Equation (7) we notice that \( \text{Bi}(0) = 1/(3^{1/6} \Gamma(2/3)) \) and \( \text{Bi}'(0) = 3^{1/6}/\Gamma(1/3) \).

\textbf{Proof of Proposition 2.1: Necessity.} Let \( Y \) be a random variable with \( M_{1/3} \) distribution and \( f \in \mathcal{D}_{1/3} \). Then, it is clear that
\[ \mathbb{E}(\langle Af \rangle(Y)) = \int_0^\infty \left( f''(x) - \frac{1}{3} xf(x) \right) M_{1/3}(x) \, dx \]
and an integration by parts yields
\[ \mathbb{E}(\langle Af \rangle(Y)) = \int_0^\infty \left( M''_{1/3}(x) - \frac{1}{3} x M_{1/3}(x) \right) f(x) \, dx \]
\[ + f'(x) M_{1/3}(x) \bigg|_0^\infty - f(x) M'_{1/3}(x) \bigg|_0^\infty. \]
Taking into account that \( M_{1/3} \) satisfies (3), \( M_{1/3}(0) = 1/ \Gamma(2/3), M'_{1/3}(0) = -(1/ \Gamma(1/3)) \) and the asymptotic behaviour for \( M_{1/3}, M'_{1/3} \) in terms of the Airy function, see Equation (2) and Appendix, Equation (A1), we obtain \( \mathbb{E}(\langle Af \rangle(Y)) = 0 \).

\textbf{Sufficiency.} Let \( h \) be a continuous bounded function. By Lemma 2.2, the function \( f_h \in \mathcal{D}_{1/3} \) and satisfies the Stein Equation (4):
\[ \langle Af_h \rangle(x) = f''_h(x) - \frac{1}{3} xf_h(x) = \tilde{h}(x). \]
Taking expectation in both sides, yields
\[ \mathbb{E}(\langle Af_h \rangle(X)) = \mathbb{E}(\tilde{h}(X)) = \mathbb{E}(h(X)) - \mathbb{E}(h(Y)). \]
Since \( \mathbb{E}(\langle Af_h \rangle(X)) = 0 \), then
\[ \mathbb{E}(h(X)) - \mathbb{E}(h(Y)) = 0. \]
Consequently, \( X \) and \( Y \) have the same law.
3. Stein’s characterization of the symmetric $\hat{M}_{1/3}$ distribution

The extended and normalized $M$-Wright function $M_{\beta}$, over the negative real axis as an even function, becomes a probability density in $\mathbb{R}$. We denote this extension by $\hat{M}_{\beta}$, it is defined by

$$\hat{M}_{1/3}(x) := \frac{1}{2}M_{1/3}(|x|) := \begin{cases} 
\frac{1}{2}M_{1/3}(x), & x \geq 0, \\
\frac{1}{2}M_{1/3}(-x), & x < 0.
\end{cases}$$

In this section, we use the Stein method to characterize $\hat{M}_{1/3}$. To this end, we define the space $\hat{D}_{1/3}$ by

$$\hat{D}_{1/3} := \{ f \in C^1_b(\mathbb{R}), f'' \in C_b(\mathbb{R}^+) \mid f(0) = 0 \}.$$

The Stein operator $\hat{A}_{1/3}$ on $C^2_b(\mathbb{R})$ is defined by

$$(\hat{A}_{1/3}f)(x) := f''(x) + \frac{1}{3}|x|f(x), \quad x \in \mathbb{R},$$

and as Stein’s equation associated to $\hat{A}_{1/3}$

$$y''(x) - \frac{1}{3}|x|y(x) = \hat{h}(x), \quad (12)$$

where

$$\hat{h}(x) := \left[ h(x) - \mathbb{E}(h(Y)) \right] \mathbb{1}_{[0,\infty)}(x) + \left[ h(x) - \mathbb{E}(h(-Y)) \right] \mathbb{1}_{(-\infty,0)}(x),$$

$h$ is a real-valued function and $Y$ has $M_{1/3}$ distribution. Now, we are ready to state the main result of this section.

**Theorem 3.1:** Let $X$ be a real-valued random variable such that

$$P(X \geq 0) = P(X < 0) = \frac{1}{2}. \quad (13)$$

Then $X$ follows the $\hat{M}_{1/3}$ distribution if and only if

$$\mathbb{E}\left( (\hat{A}_{1/3}f)(X) \right) = 0, \quad \forall f \in \hat{D}_{1/3}.$$

Before proving the above theorem, we first show that the solution of the Stein Equation (12) belongs to $\hat{D}_{1/3}$. This the contents of the following lemma.

**Lemma 3.2:** Let $h : \mathbb{R} \rightarrow \mathbb{R}$ be a bounded continuous function. Then the function

$$f_h(x) = -3^{1/3}\pi \left\{ \left[ \text{Ai} \left( \frac{x}{3^{1/3}} \right) \int_0^x \text{Bi} \left( \frac{t}{3^{1/3}} \right) \hat{h}(t) \, dt \right. \right.$$

$$\left. + \text{Bi} \left( \frac{x}{3^{1/3}} \right) \int_x^\infty \text{Ai} \left( \frac{t}{3^{1/3}} \right) \hat{h}(t) \, dt \right\} \mathbb{1}_{[0,\infty)}(x)$$

$\hat{h}$ is a real-valued function and $Y$ has $M_{1/3}$ distribution. Now, we are ready to state the main result of this section.
we have such that
\[ \|u\|_{\infty} \leq \hat{C}_1 \|\hat{h}\|_{\infty}, \|f'_{\hat{h}}\|_{\infty} \leq \hat{C}_2 \|\hat{h}\|_{\infty}, \|f''_{\hat{h}}\|_{\infty} \leq \hat{C}_3 \|\hat{h}\|_{\infty} \text{ and } f_{\hat{h}} \text{ belongs to } \hat{D}_{1/3}. \]

**Proof:** If follows from Lemma 2.2 that the solution \( f_{\hat{h}} \) of Equation (12) for \( x \geq 0 \) is given by
\[
f_{\hat{h}}(x) = -3^{1/3} \pi \left[ \text{Ai} \left( \frac{x}{3^{1/3}} \right) \int_{0}^{x} \text{Bi} \left( \frac{t}{3^{1/3}} \right) \hat{h}(t) \, dt + \text{Bi} \left( \frac{x}{3^{1/3}} \right) \int_{0}^{\infty} \text{Ai} \left( \frac{t}{3^{1/3}} \right) \hat{h}(t) \, dt \right].
\]

On the other hand, for \( x < 0 \) the Equation (12) turns into
\[
y''(x) + \frac{1}{3} xy(x) = \hat{h}(x).
\]

With a change of variables \( u = -x \) the solution \( f_{\hat{h}} \) of the above differential equation is given by
\[
f_{\hat{h}}(x) = -3^{1/3} \pi \left[ \text{Ai} \left( \frac{x}{3^{1/3}} \right) \int_{0}^{x} \text{Bi} \left( \frac{t}{3^{1/3}} \right) \hat{h}(t) \, dt + \text{Bi} \left( \frac{x}{3^{1/3}} \right) \int_{0}^{\infty} \text{Ai} \left( \frac{t}{3^{1/3}} \right) \hat{h}(t) \, dt \right].
\]

Then the solution (14) follows putting together both Equations (15) and (16). It remains to show that \( f_{\hat{h}} \in \hat{D}_{1/3} \). As \( \hat{h} \) is a continuous function on \( \mathbb{R}^* \), then \( f_{\hat{h}} \in C^2(\mathbb{R}^*) \). At \( x = 0 \), we have
\[
f_{\hat{h}}(0) = -3^{-(1/3)} \pi \text{Bi}(0) \int_{0}^{\infty} M_{1/3}(t) \left[ h(t) - \mathbb{E}(h(Y)) \right] \, dt = 0.
\]
and
\[
f_{\hat{h}}(0^-) = -3^{-(1/3)} \pi \text{Bi}(0) \int_{-\infty}^{0} M_{1/3}(-t) \left[ h(t) - \mathbb{E}(h(-Y)) \right] \, dt = 0,
\]
then \( f_{\hat{h}} \) is continuous at \( x = 0 \) and \( f_{\hat{h}}(0) = 0 \). On the other hand,
\[
f'_{\hat{h}}(0) = -3^{-(1/3)} \pi \text{Bi}'(0) \int_{0}^{\infty} M_{1/3}(t) \left[ h(t) - \mathbb{E}(h(Y)) \right] \, dt = 0
\]
and
\[
f'_{\hat{h}}(0^-) = -3^{1/3} \pi \text{Bi}'(0) \int_{-\infty}^{0} M_{1/3}(-t) \left[ h(t) - \mathbb{E}(h(-Y)) \right] \, dt = 0.
Therefore, \( f'_h \) is also continuous at \( x = 0 \) and consequently \( f_h \in C^1(\mathbb{R}) \). It is easy to see that \( f''_h \) is continuous in \( \mathbb{R}^+ \) and

\[
\begin{align*}
\hat{f}''_h(0^+) &= \hat{h}(0^+) = h(0) - \mathbb{E}(h(Y)), \\
\hat{f}''_h(0^-) &= \hat{h}(0^-) = h(0) - \mathbb{E}(h(-Y)).
\end{align*}
\]

Finally, we show that \( f_h \) and \( f'_h \) are bounded. Moreover, as \( f_h \) satisfies the Equation (12) it will follows that \( f''_h \) will be also bounded. On one hand, for \( x \geq 0 \), it follows from Lemma 2.2 that \( f_h \) given by Equation (15) and its derivatives are bounded, namely

\[
\begin{align*}
|\hat{f}_h(x)| &\leq 3^{2/3} \pi \|G_\pi\|_\infty \|\hat{h}\|_\infty, \\
|\hat{f}'_h(x)| &\leq 3^{1/3} \pi \|G'_\pi\|_\infty \|\hat{h}\|_\infty, \\
|\hat{f}''_h(x)| &\leq (3^{-(2/3)} \pi \sup_{x \in \mathbb{R}_+} |xG_\pi(x)| + 1) \|\hat{h}\|_\infty.
\end{align*}
\]

In a similar way for \( x < 0 \), we estimate \( f_h \) given by Equation (16) and its derivatives in order to show that \( f_h \) and its derivatives are bounded. Thus, we conclude that \( f_h \in \hat{D}_{1/3} \). ■

**Proof of Proposition 3.1**: **Necessity.** Let \( \hat{Y} \) be a random variable with \( \hat{M}_{1/3} \) distribution and \( f \in \hat{D}_{1/3} \). Then, it is clear that

\[
\int_{\mathbb{R}} f''(x)\hat{M}_{1/3}(x) \, dx = \frac{1}{2} \left( \int_{-\infty}^{0} f''(x)\hat{M}_{1/3}(-x) \, dx + \int_{0}^{\infty} f''(x)\hat{M}_{1/3}(x) \, dx \right).
\]

Using the integration by parts and the asymptotic behaviour of \( M_{1/3}, \hat{M}'_{1/3} \), see Appendix, we have

\[
\int_{-\infty}^{0} f''(x)\hat{M}_{1/3}(-x) \, dx = f'(0)\hat{M}_{1/3}(0) + \int_{-\infty}^{0} f'(x)\hat{M}'_{1/3}(-x) \, dx
\]

\[
= f'(0)\hat{M}_{1/3}(0) + \int_{-\infty}^{0} f(x)\hat{M}''_{1/3}(-x) \, dx.
\]

But we know that

\[
\hat{M}''_{1/3}(-x) = -\frac{1}{3} x\hat{M}_{1/3}(-x),
\]

therefore we obtain

\[
\int_{-\infty}^{0} f''(x)\hat{M}_{1/3}(-x) \, dx = f'(0)\hat{M}_{1/3}(0) - \frac{1}{3} \int_{-\infty}^{0} xf(x)\hat{M}_{1/3}(-x) \, dx
\]

equivalently

\[
\int_{-\infty}^{0} \left( f''(x) + \frac{1}{3} xf(x) \right) \hat{M}_{1/3}(-x) \, dx = f'(0)\hat{M}_{1/3}(0).
\]
It follows from the proof of Theorem 2.1 that

\[
\int_0^\infty \left( f''(x) - \frac{1}{3} xf(x) \right) M_{1/3}(x) \, dx = -f'(0)M_{1/3}(0).
\]

Putting together, yields

\[
\int \left( f''(x) + \frac{1}{3} |x| f(x) \right) M_{1/3}(x) \, dx = 0.
\]

Therefore, we have

\[
\mathbb{E}(\hat{A}_{1/3}f(\hat{Y})) = 0.
\]

**Sufficiency.** Let \( h \) be a bounded continuous function. By Lemma 3.2, the function \( f_h \in \hat{D}_{1/3} \) and satisfies the Stein equation:

\[
(\hat{A}_{1/3}f_h)(x) = f''_h(x) - \frac{1}{3} |x| f_h(x) = \hat{h}(x).
\]

Then for \( X \) a real-valued random variable satisfying the condition (13), we have

\[
\mathbb{E}(\hat{A}_{1/3}f_h(X)) = \mathbb{E}(\hat{h}(X)) = \mathbb{E}(h(X)) - \frac{1}{2} \left[ \mathbb{E}(h(Y)) + \mathbb{E}(h(-Y)) \right].
\]

Since \( \mathbb{E}(\hat{A}_{1/3}f_h(X)) = 0 \) and \( \mathbb{E}(h(\hat{Y})) = \frac{1}{2} \left[ \mathbb{E}(h(Y)) + \mathbb{E}(h(-Y)) \right] \), then

\[
\mathbb{E}(h(X)) - \mathbb{E}(h(\hat{Y})) = 0.
\]

Thus, \( X \) and \( \hat{Y} \) have the same law.

**Remark 3.3:** Assume that \( X \) does not satisfies (13) and take \( h \) an even continuous function on \( \mathbb{R} \), we have

1. \( \hat{h} \) is continuous and consequently the solution \( f_h \) of the Stein Equation (14) belongs to \( C^2_b(\mathbb{R}) \), i.e. the space of bounded twice continuously differentiable functions with bounded derivatives.
2. In addition, we obtain that the random variables \( |X| \) and \( |\hat{Y}| \) have the same law but it is not enough to conclude that \( X \) and \( \hat{Y} \) have the same law.

**4. Conclusion**

We have characterized the distribution with density \( M_{1/3} \) in \( \mathbb{R}_+ \) as well as its symmetrization \( \hat{M}_{1/3} \) through the Stein method. The Stein operator turns out to be the Airy equation, compare Equations (3) and (5). The Stein equation corresponds to the general inhomogeneous Airy Equation (5). As particular solution of the Stein equation, we take a generalization of the Scorer function \( G_i \), see Equation (A7).
The characterization of the class of distributions with density $M_{1/n}$, $n = 4, 5, \ldots$ leads us to consider the Stein operator of the following form

$$y^{(n-1)} - \frac{(-1)^n}{n} xy = 0. \quad (17)$$

This equation for $n \geq 4$ is akin to the hyper-Airy differential equation of order $n-1$, see [9]. In general for $q := (1/\beta) - 1$, $0 < \beta < 1$, the density $M_{\beta}$ satisfies the fractional differential equation

$$\frac{d^q}{dx^q} M_{\beta}(x) + \beta e^{\pm (i\pi/\beta)} x M_{\beta}(x) = 0. \quad (18)$$

In view of the above equation, the density $M_{\beta}$ is referred in [8] as a generalized hyper-Airy function. To find a particular solution for the above Equations (17) and (18) seems to be a non-trivial task and here new ideas are needed.

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Appendix. The Airy and Scorer Functions

In this appendix, we collect some properties of the Airy and Scorer functions which are used throughout this paper. We refer to the following books [1,10,11,16] for more details and properties.

The second order homogeneous differential equation, known as Airy equation

$$y'' - xy = 0,$$

has a pair of linear independent solutions $A_i$ and $B_i$, called the Airy function of the first and second kind, respectively. They are entire functions of $x$ with initial values

$$A_i(0) = \frac{1}{3^{2/3}\Gamma\left(\frac{2}{3}\right)}, \quad A_i'(0) = -\frac{1}{3^{1/3}\Gamma\left(\frac{1}{3}\right)},$$

$$B_i(0) = \frac{1}{3^{1/6}\Gamma\left(\frac{1}{3}\right)}, \quad B_i'(0) = \frac{3^{1/6}}{\Gamma\left(\frac{1}{3}\right)},$$

and their Wronskian is

$$W(A_i(x), B_i(x)) = \frac{1}{\pi}.$$

For the special case $x > 0$, the Airy functions can be written in terms of the Bessel functions

$$A_i(x) = \frac{1}{\pi} \sqrt{\frac{x}{3}} K_{1/3}(\zeta), \quad \zeta := \frac{2}{3} x^{3/2},$$

$$B_i(x) = \sqrt{\frac{x}{3}} \left[I_{1/3}(\zeta) + I_{-1/3}(\zeta)\right],$$

where $K_{1/3}$ and $I_{\pm 1/3}$ are the modified Bessel functions of the second and first second, respectively, see [1, Section 10.4]. The asymptotic behaviour of the Airy function $A_i$ and its derivative for $x \to \infty$

$$A_i(x) \sim \frac{1}{2} \pi^{-1/2} x^{-1/4} e^{-\zeta}, \quad A_i'(x) \sim -\frac{1}{2} \pi^{-1/2} x^{1/4} e^{-\zeta}.$$  \hfill (A1)
It is easy to see that the two linear independent solutions of the general Airy equation

\[ y'' - k^2 xy = 0 \]

are \( \text{Ai}(k^{2/3}x) \) and \( \text{Bi}(k^{2/3}x) \). The general solution of the inhomogeneous Airy equation

\[ y'' - xy = \frac{1}{\pi} \]  

(A2)

is

\[ y(x) = c_1 \text{Ai}(x) + c_2 \text{Bi}(x) + p(x), \]

where \( c_1, c_2 \) are arbitrary constants and \( p(x) \) is any particular solution of the Equation (A2). A standard particular solution may be found by the method of variation of parameters (see for example [4]), called Scorer’s function \( G_i(x) \) (also known as inhomogeneous Airy functions), given by

\[ G_i(x) := \text{Ai}(x) \int_0^x \text{Bi}(t) \, dt + \text{Bi}(x) \int_x^\infty \text{Ai}(t) \, dt. \]  

(A3)

The Scorer function \( G_i(x) \) is an entire bounded function of \( x \). Its asymptotic as \( x \to \infty \) is given by

\[ G_i(x) \sim \frac{1}{\pi x} \quad \Rightarrow \quad xG_i(x) \sim \frac{1}{\pi}. \]  

(A4)

The derivative of the Scorer function \( G_i(x) \) is given by

\[ G'_i(x) = \frac{1}{3} \text{Bi}'(x) + \int_0^x \left[ \text{Ai}'(x) \text{Bi}(t) - \text{Ai}(t) \text{Bi}'(t) \right] \, dt \]

and the following asymptotic as \( x \to \infty \) holds

\[ G'_i(x) \sim -\frac{1}{\pi x^2}, \quad x \in \mathbb{R}_+. \]  

(A5)

In Sections 2 and 3 we use a generalization of Equation (A2), namely

\[ y'' - k^2 xy = f(x), \]  

(A6)

with \( k = 3^{-(1/2)} \) and \( f \) a bounded measurable function. Using the above scheme, the general solution of Equation (A6) is given by

\[ y(x) = c_1 \text{Ai}(k^{2/3}x) + c_2 \text{Bi}(k^{2/3}x) + q(x), \]

where, as before \( c_1, c_2 \) are arbitrary constants and \( q(x) \) is any particular solution of the Equation (A6). One convenient choice of \( q(x) \) is

\[ q(x) = -k^{2/3} \pi \left( \text{Ai}(k^{-(2/3)}x) \int_0^x \text{Bi}(k^{-(2/3)}t)f(t) \, dt + \text{Bi}(k^{-(2/3)}x) \int_x^\infty \text{Ai}(k^{-(2/3)}t)f(t) \, dt \right). \]  

(A7)