Vacua and RG flows in $N = 9$ three dimensional gauged supergravity

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Abstract: We study some vacua of $N = 9$ three dimensional gauged supergravity. The theory contains sixteen scalar fields parametrizing the exceptional coset space $F_4(-20)/(SO(9))$. Various supersymmetric and some non-supersymmetric AdS$_3$ vacua are found in both compact and non-compact gaugings with gauge groups $SO(p) \times SO(9 - p)$ for $p = 0, 1, 2, 3, 4$, $G_2(-14) \times SL(2)$ and $Sp(1, 2) \times SU(2)$. We also study many RG flow solutions, both analytic and numerical, interpolating between supersymmetric AdS$_3$ critical points in this theory. All the flows considered here are driven by a relevant operator of dimension $\Delta = \frac{3}{2}$. This operator breaks conformal symmetry as well as supersymmetry and drives the CFT in the UV to another CFT in the IR with lower supersymmetries.

Keywords: AdS-CFT Correspondence, Gauge-gravity correspondence, Supergravity Models.
1. Introduction

Three dimensional Chern-Simons gauged supergravity has a very rich structure. The theory admits gauge groups of various types namely compact, noncompact, non semisimple and complex gauge groups \[1, 2, 3\]. This stems from the fact that there is no restriction on the number of gauge fields. The gauge fields are introduced to the gauged theory by a Chern-Simons kinetic term which results in their non propagating nature in the theory. This peculiar feature comes from the duality between vectors and scalars in three dimensions. All the bosonic propagating degrees of freedom are carried by the scalars because pure supergravity in three dimensions is also topological.

Maximal gauged supergravity in three dimensions has been constructed in \[1, 2, 3\]. The construction of the \(N = 8\) theory can be found in \[4\]. All extended three dimensional gauged supergravities with \(N \leq 16\) have been given in a unique formulation in \[5\]. This is a gauged version of the ungauged theory constructed in \[6\]. Vacua of these theories have been studied in some details e.g. see \[7, 8\] for \(N = 16\) theory and \[9, 10\] for \(N = 4\) and \(N = 8\) theories.

In three dimensional supergravities with \(N > 4\), the scalar target space manifold is a symmetric space and can be written as a coset space \(G/H\), where \(G\) is a global symmetry group, and \(H\) is its maximal compact subgroup. For the theories with \(N > 8\), the target space is unique because there is only one supermultiplet in these cases \[6\]. In this paper, we study \(N = 9\) gauged theory in which the scalar manifold is given by the exceptional coset \(F_{4(-20)}/SO(9)\). We will study some vacua of this theory with gauge groups \(SO(p) \times SO(9 - p)\) for \(p = 0, 1, 2, 3, 4, G_{2(-14)} \times SL(2)\) and \(Sp(1, 2) \times SU(2)\). All these gauge groups have been shown to be consistent gaugings in \[5\]. We will study some vacua of these gaugings and give relevant superconformal groups for maximally supersymmetric vacua with all scalars being zero.

The possibility to study holographic renormalization group flows is one of the interesting consequences of the AdS/CFT correspondence \[11\]. We will also study some supersymmetric RG flow solutions interpolating between supersymmetric AdS\(_3\) vacua. In gauged supergravity, these solutions are domain walls interpolating between critical points of the scalar potential. They have an interpretation in the dual field theory as an RG flow driving the UV CFT to the IR fixed point corresponding to the CFT in the IR. In this paper, we will find flow solutions in three dimensional gauged supergravity. The solutions describe the RG flows in two dimensional field theories. Some supersymmetric flow solutions in three dimensional gauged supergravity have been studied in \[11, 9\] for \(N = 8\) and \(N = 4\) theories, respectively. Some flow solutions of \(N = 2\) models have been studied in \[12\]. In this paper, we give the analogous analysis in the \(N = 9\) theory. The is the largest amount of supersymmetry ever studied so far in the
context of RG flows in three dimensional gauged supergravities.

The paper is organized as follows. In section 2, we review some useful ingredients to construct the $N = 9$ gauged theory. We also give the explicit construction of the gauged theory with symmetric scalar manifold $F_4(-20)$ in detail. The procedure can be applied to other theories with different values of $N$ as well. Various vacua are found in section 3. We then find some flow solutions in section 4. Finally, we give some conclusions and comments in section 5.

2. $N = 9$ three dimensional gauged supergravity

In this section, we construct $N = 9$ three dimensional gauged supergravity using the formulation given in [5]. The $N = 9$, $SO(9)$ gauged three dimensional supergravity has also been constructed in [13], but we will follow the construction of [5] because this formulation can be easily extended to other gauge groups. We start by reviewing some formulae and all the ingredients needed in this paper.

In symmetric spaces $G/H$, we have the following decompositions of the $G$ generators $t^M$ into $\{X^{IJ}, X^\alpha, Y^A\}$. The maximal compact subgroup $H$ of $G$, consists of the $SO(N)$ R-symmetry and an additional factor $H'$ such that $H = SO(N) \times H'$. The scalar fields parametrizing the target space are encoded in the coset representative $L$. This transforms under global $G$ and local $H$ symmetries by multiplications from the left and right, respectively. The latter can be used to eliminate the spurious degrees of freedom such that $L$ is parametrized by $\text{dim}(G/H)$ physical scalars. In our case, the maximal compact subgroup of $G = F_4(-20)$ is $SO(9)$, so there is no factor $H'$. Generators $X^{IJ}, I, J = 1, 2, \ldots N$ generate $SO(N)$, and $Y^A$, transforming in a spinor representation of $SO(N)$, are non-compact generators of $G$. The target space has a metric $g_{ij}, i, j = 1, 2, \ldots d = \text{dim}(G/H)$ given by

$$g_{ij} = \epsilon^A_i \epsilon^B_j \delta_{AB}. \tag{2.1}$$

Extended supersymmetries are described by $N - 1$ almost complex structures $f^{P_i}_j$, $P = 2, \ldots, N$. We can construct $SO(N)$ generators from these $f^{P_i}_j$'s by forming tensors $f^{IJ}_{ij}$ via [5]

$$f^{PQ} = f^{[P} f^{Q]}, \quad f^{1P} = -f^{P1} = f^P. \tag{2.2}$$

The tensors $f^{IJ}_{ij}$, being generators of $SO(N)$ in the spinor representation, are given in terms of $SO(N)$ gamma matrices by

$$f^{IJ}_{ij} = -\Gamma^{IJ}_{AB} \epsilon^A_i \epsilon^B_j. \tag{2.3}$$
The indices $A$ and $B$ are tangent space or “flat” indices on the scalar target space. The vielbein of the target space is encoded in the expansion
\[ L^{-1} \partial_i L = \frac{1}{2} Q_i^{IJ} X^{IJ} + Q_\alpha^a X^\alpha + e_i^A Y^A. \]  
(2.4)

$Q_i^{IJ}$ and $Q_\alpha^a$ are composite connections for $SO(N)$ and $H'$, respectively.

The gaugings are described by the gauge invariant embedding tensor $\Theta_{MN}$. From $\Theta_{MN}$, we can compute the $A_1$ and $A_2$ tensors as well as the scalar potential via the so-called $T$-tensor using
\[ A_1^{IJ} = -\frac{4}{N-2} T^{I M, J M} + \frac{2}{(N-1)(N-2)} \delta^{I J} T^{M N, M N}, \]
\[ A_2^{IJ} = \frac{2}{N} T^{I J, j} + \frac{4}{N(N-2)} f^{M (I m, T) M}_j \frac{1}{m} + \frac{2}{N(N-1)(N-2)} \delta^{I J} f^{K L, m} T^{K L} m, \]
\[ V = -\frac{4}{N} g^2 (A_1^{IJ} A_1^{IJ} - \frac{1}{2} N g^{i j} A_2^{IJ} A_2^{IJ}). \]  
(2.5)

The $T$-tensors are defined by
\[ T_{AB} = \mathcal{V}_A^M \Theta_{MN} \mathcal{V}_B^N. \]  
(2.6)

All the $\mathcal{V}$'s are given by
\[ L^{-1} t^M L = \frac{1}{2} \mathcal{V}^M I J X^{IJ} + \mathcal{V}_\alpha^a X^\alpha + \mathcal{V}^A Y^A. \]  
(2.7)

Using these, we can now construct the $N = 9$ theory. We give the procedure in detail but leave some formulae to the appendix. We begin with the $F_4(-20)$ coset. The 52 generators of the compact $F_4$ have been explicitly constructed by realizing $F_4$ as an automorphism group of the Jordan algebra $J_3$ in [14]. There are 16 non-compact and 36 compact generators in $F_4(-20)$. Under $SO(9)$, the 52 generators decompose as
\[ 52 \rightarrow 36 + 16 \]
where $36$ and $16$ are adjoint and spinor representations of $SO(9)$, respectively. The non-compact $F_4(-20)$ can be obtained from the compact $F_4$ by using “Weyl unitarity trick”, see [14] for an example with $G_2$. This is achieved by introducing a factor of $i$ to each generator corresponding to the non-compact generators. From [14], the compact subgroup $SO(9)$ is generated by, in the notation of [14], $c_1, \ldots, c_{21}, c_{30}, \ldots, c_{36}, c_{45}, \ldots, c_{52}$. We have chosen the same $SO(9)$ subgroup as in [14] among the three possibilities, see [14] for a discussion. The remaining 16 generators are our non-compact ones which we will define by
\[ Y^A = \begin{cases} 
i c_{A+21} & \text{for } A = 1, \ldots, 8, \\ i c_{A+28} & \text{for } A = 9, \ldots, 16. \end{cases} \]  
(2.8)
Note that the $SO(9)$ generators $c_i$ in [14], are labeled by the $F_4$ adjoint index. In order to apply the $SO(9)$ covariant formulation of $N = 9$ theory, we need to relabel them by using the $SO(9)$ antisymmetric tensor indices i.e. $X^{IJ}$. To do this, we first note the relevant algebra from [5]

$$
[t^{IJ}, t^{KL}] = -4\delta^{[I[K} t^{L]J]}, \quad [t^{IJ}, t^{A}] = \frac{1}{2} f^{IJAB} t^{B}, \quad [t^{A}, t^{B}] = \frac{1}{4} f^{AB}_{IJ} t^{IJ} \tag{2.9}
$$

where we have used the flat target space indices in $f^{IJ}_{AB}$ and the non-compact generators, $t^A$. Using the first commutator in (2.9), we can map all $c_i$’s forming $SO(9)$ to the desired form $X^{IJ}$. The detail of this is given in the appendix. The next step is to find the $f^{IJ}$. In order to be compatible with the $F_4$ algebra given in [14], we need to use the second and the third commutators in (2.9) to extract the component of $f^{IJ}_{AB}$ rather than putting the explicit forms of gamma matrices from another basis. There are eight independent $f^{IJ}$ from which all other components follow from (2.2). We will not give all of the $f^{IJ}$ here due to their complicated form.

We now come to various gaugings characterized by the embedding tensors $\Theta$. The embedding tensors for the compact gaugings with gauge groups $SO(p) \times SO(9 - p)$, $p = 0, \ldots, 4$ are given by [5]

$$
\Theta_{IJ,KL} = \theta \delta_{IJ}^{KL} + \delta_{[I[K} \Xi_{L]J]} \tag{2.10}
$$

where

$$
\Xi_{IJ} = \begin{cases} 
2 \left(1 - \frac{p}{9}\right) \delta_{IJ} & \text{for } I \leq p, \\
-2\frac{9}{p} \delta_{IJ} & \text{for } I > p,
\end{cases} \quad \theta = \frac{2p - 9}{9}. \tag{2.11}
$$

There is only one independent coupling constant, $g$. The gauge generators can be easily obtained from $SO(9)$ generators $X^{IJ}$ by choosing appropriate values for the indices $I, J$. For example, in the case of $SO(2) \times SO(7)$ gauging, we have the following gauge generators

$$
SO(7) : T^{ab}_1 = X^{ab}, \quad a, b = 1, \ldots, 7, \\
SO(2) : T_2 = X^{89}. \tag{2.12}
$$

We then move to non-compact gaugings with gauge groups $G_{2(-14)} \times SL(2)$ and $Sp(1,2) \times SU(2)$. We find the following embedding tensors

$$
G_{2(-14)} \times SL(2) : \Theta_{MN} = \eta_{MN}^{G_{2(-14)}} - \frac{1}{6} \eta_{MN}^{SL(2)}, \tag{2.13}
$$

$$
Sp(1,2) \times SU(2) : \Theta_{MN} = \eta_{MN}^{Sp(1,2)} - 12 \eta_{MN}^{SU(2)} \tag{2.14}
$$

where $\eta_{G_0}$ is the Cartan Killing form of the gauge group $G_0$. The gauge generators in these two gaugings are given in the appendix.
Using these embedding tensors and equation (2.7), we can find all the \( V \)'s and T-tensors. With the help of the computer algebra system Mathematica [16], it is then straightforward to compute \( A_1 \) and \( A_2 \) tensors and finally the scalar potential for each gauge group. In the next section, we will give all of these potentials but refer the readers to the appendix for \( V \)'s and T-tensors. For completeness, we also give here the condition for finding stationary points of the potential. We are most interested in supersymmetric AdS\(_3\) vacua, so we mainly work with the condition for supersymmetric stationary points. As given in [5], see also [2] for \( N = 16 \), the supersymmetric stationary points satisfy the two equivalent conditions

\[
A_{2i}^{II} \epsilon^I = 0 \quad \text{and} \quad A_1^{IK} A_1^{KJ} \epsilon^J = -\frac{V_0}{4g^2} \epsilon^I = \frac{1}{N} (A_1^{IJ} A_1^{IJ} - 1) N g^{ij} A_2^{IJ} A_2^{IJ} \epsilon^I, \tag{2.15}
\]

where \( V_0 \) is the value of the potential at the critical point i.e. the cosmological constant. \( \epsilon^I \) are the Killing spinors corresponding to the residual supersymmetries at the stationary point. The second condition simply says that \( \epsilon^I \) is an eigenvector of \( A_1 \) with an eigenvalue \( \sqrt{-\frac{V_0}{4g^2}} \) or \( -\sqrt{-\frac{V_0}{4g^2}} \). In addition, these two conditions are indeed equivalent as shown in [5].

The condition for any stationary points, not necessarily supersymmetric, is \[ 3 A_1^{IK} A_2^{KJ} + N g^{kl} A_2^{IK} A_3^{KL} = 0 \tag{2.16} \]

where \( A_3^{KL} \) is defined by

\[
A_3^{ij} = \frac{1}{N^2} \left[ -2D_i D_j A_1^{IJ} + g_{ij} A_1^{IJ} + A_1^{K[I} f^{J]K} + 2 T_{ij} \delta^{IJ} - 4 D_i T^{IJ} - 2 T_{kj} f^{IJ} \right]. \tag{2.17}
\]

For supersymmetric critical points, we will mostly work with the two equivalent conditions given by (2.15). However, for non-supersymmetric points, the condition (2.16) is necessary to ensure that all the points are indeed stationary points.

3. Vacua of \( N = 9 \) gauged supergravity

In this section, we give some vacua of the \( N = 9 \) gauged theory with the gaugings mentioned in the previous section. We will discuss the isometry groups of the background with maximal supersymmetries at \( L = \mathbf{I} \). This is a supersymmetric extension of the \( SO(2,2) \sim SO(1,2) \times SO(1,2) \) isometry group of AdS\(_3\). The superconformal group can be identified by finding its bosonic subgroup and representations of supercharges under this group. A similar study has been done in [8] for models with \( N = 16 \) supersymmetry. The full list of superconformal groups in two dimensions can be found in [17]. We first start with compact gaugings.
3.1 Vacua of compact gaugings

It has been shown in [18] that the critical points obtained from the potential restricted on a scalar manifold which is invariant under some subgroups of the gauge group are critical points of the full potential. This invariant manifold is parametrized by all scalars which are singlets under the chosen symmetry. To make things more manageable, we will not study the scalar potential with more than four scalars. We choose to parametrize the scalars by using the coset representative

\[ L = e^{a_1 Y_1} e^{a_2 Y_2} e^{a_3 Y_3} e^{a_4 Y_4}. \] (3.1)

For any invariant manifold with the certain residual symmetry, our choice for \( L \) in (3.1) certainly does not cover the whole invariant manifold. Therefore, the critical points on this submanifold may not be critical points of the potential on the whole scalar manifold. Nevertheless, we can use the argument of [18] as a guideline to find critical points. After identifying the critical points, we then use the stationarity condition (2.16) to check whether our critical points are truly critical points of the scalar potential.

Let us identify some residual symmetries of (3.1). In \( SO(9) \) gauging, with only \( a_1 \neq 0 \), \( L \) has \( SO(7) \) symmetry. For \( a_1, a_2 \neq 0 \), \( L \) preserves \( SO(6) \) symmetry. With \( a_1, a_2, a_3 \neq 0 \) and \( a_1, a_2, a_3, a_4 \neq 0 \), \( L \) preserves \( SU(3) \) and \( SU(2) \), respectively. In other gauge groups, \( L \) will have different residual symmetry. We will discuss the residual gauge symmetry of each critical point, separately. We find that in all cases, non trivial supersymmetric critical points arise with at most two non zero scalars. With all four scalar fields turned on, the conditions \( A^I_J \epsilon^J = 0 \) are satisfied if and only if two of the scalars vanish. So, we give below only potentials with two scalars.

In (3.1), we have used the basis elements of \( Y \)'s to parametrize each scalar field. We also find that, in this parametrization, all the sixteen scalars are on equal footing in the sense that any four of the \( Y \)’s among sixteen of them give the same structure of the potential. As a consequence, any two non zero scalars in (3.1) give rise to the same critical points with the same location and cosmological constant. Notice that this is not the case if we use different parametrization of \( L \). For example, by using linear combinations of \( Y_i \)'s as basis for the four scalars in (3.1), different choices of \( Y_i \)'s in each basis may give rise to different structures of the scalar potential.

We use the same notation as in [9] namely \( V_0 \) is the cosmological constant, and \((n_-, n_+)\) refers to the number of supersymmetries in the dual two dimensional field theory. On the other hand, the \( n_+ (n_-) \) corresponds to the number of positive (negative) eigenvalues of \( A^I_J \). For definiteness, we will keep \( a_1 \) and \( a_2 \) non zero. Furthermore, we give the values of scalar fields up to a trivial sign change.
• **$SO(9)$ gauging:**

The scalar potential is

\[
V = \frac{1}{32} g^2 (-1390 - 232 \cosh(2a_1) + 6 \cosh(4a_1) + 4 \cosh[2(a_1 - 2a_2)] \\
+ 4 \cosh(4a_1 - 2a_2) - 112 \cosh[2(a_1 - a_2)] + \cosh[4(a_1 - a_2)] \\
- 232 \cosh(2a_2) + 6 \cosh(4a_2) - 112 \cosh[2(a_1 + a_2)] \\
+ \cosh[4(a_1 + a_2)] + 4 \cosh[2(2a_1 + a_2)] + 4 \cosh[2(a_1 + 2a_2)]). \quad (3.2)
\]

This is the case in which the full R-symmetry group $SO(9)$ is gauged. There is no nontrivial critical point with two scalars. For $a_2 = 0$, there are two critical points, but only the $L = I$ solution has any supersymmetry.

| Critical points | $a_1$ | $V_0$ | Preserved supersymmetry |
|----------------|-------|-------|-------------------------|
| 1              | 0     | $-64g^2$ | (9,0)                   |
| 2              | $\cosh^{-1}2$ | $-100g^2$ | -                      |

The corresponding $A_1$ tensor at the supersymmetric point is

\[
A_1^{(1)} = \text{diag}(-4, -4, -4, -4, -4, -4, -4, -4, -4). \quad (3.3)
\]

The notation $A_1^{(1)}$ means that this is the value of the $A_1$ tensor evaluated at the critical point number 1 in the table. For $L = I$, the background isometry is given by $Osp(9|2, \mathbb{R}) \times SO(1,2)$. The non-supersymmetric critical point has unbroken $SO(7)$ gauge symmetry. This point is closely related to the non-supersymmetric $SO(7) \times SO(7)$ point found in $N = 16$ $SO(8) \times SO(8)$ gauged supergravity [8]. Both the location and the value of the cosmological constants compared to the $L = I$ point are very similar to that in [8].

• **$SO(8)$ gauging:**

The potential is

\[
V = -\frac{1}{16} g^2 [(26 + 2 \cosh(2a_1) + \cosh[2(a_1 - a_2)]) + 2 \cosh(2a_2) \\
+ \cosh[2(a_1 + a_2)])^2 - 32(\cosh^2 a_2 \sinh^2(2a_1) \\
+ \cosh^4 a_1 \sinh^2(2a_2))]. \quad (3.4)
\]

This case is very similar to the $SO(9)$ gauging. There are two critical points with a single scalar.
The $A_1$ tensor is
\[
A_1^{(1)} = \text{diag}(-4, -4, -4, -4, -4, -4, -4, -4, -4, -4, -4, -4).
\tag{3.5}
\]

For $L = 1$, the background isometry is given by $Osp(8|2, \mathbb{R}) \times Osp(1|2, \mathbb{R})$. The critical point 2 is invariant under $G_2$ subgroup of $SO(8)$. Apart from the splitting of supercharges and residual gauge symmetry, the critical points in this gauging are the same as the $SO(9)$ gauging.

- $SO(7) \times SO(2)$ gauging:

  In this gauging, the potential is
  \[
  V = -\frac{1}{36864}g^2[9(342 + 40 \cosh a_1 + 18 \cosh(2a_1) - 4 \cosh(a_1 - 2a_2) \\
  + 16 \cosh(a_1 - a_2) + 3 \cosh[2(a_1 - a_2)] + 12 \cosh(2a_1 - a_2) \\
  + 8 \cosh a_2 + 50 \cosh(2a_2) + 16 \cosh(a_1 + a_2) + 3 \cosh[2(a_1 + a_2)] \\
  + 12 \cosh(2a_1 + a_2) - 4 \cosh(a_1 + 2a_2))^2 + 8(-576 \cosh^2 a_2(-3 \\
  + \cosh a_2 - 3 \cosh a_1(1 + \cosh a_2))^2 \sinh^2 a_1 - 9(-1 \\
  - 8 \cosh a_1(-1 + \cosh a_2) + 47 \cosh a_2 + 3 \cosh(2a_1)(1 + \cosh a_2) \\
  + 6 \cosh^2 a_1(1 + \cosh a_2)^2 \sinh^2 a_2)]].
  \tag{3.6}
  \]

  We find one supersymmetric critical point with
  \[
  V_0 = -144g^2, \quad a_1 = \cosh^{-1}\frac{5}{3}, \quad a_2 = \cosh^{-1}2
  \tag{3.7}
  \]
  with the value of the $A_1$ tensor
  \[
  A_1 = \begin{pmatrix}
  -10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & -10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8\sqrt{2} & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8\sqrt{2} & 0 & 0 & 0 \\
  0 & 0 & 0 & -10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & -10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & -10 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & -10 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & -10 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 \\
  0 & 0 & -8\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 14 & 3 \\
  \end{pmatrix}.
  \tag{3.8}
  \]
After diagonalization, we find

\[ A_1 = \text{diag}(-10, -10, -6, -10, -10, -10, 6, 6). \]  \hspace{1cm} (3.9)

This is a (1,2) point with SU(2) symmetry. With \( a_2 = 0 \), we find the following critical points

| Critical points | \( a_1 \) | \( V_0 \) | Preserved supersymmetry |
|-----------------|----------|-----------|------------------------|
| 1               | 0        | \(-64g^2\) | (7,2)                  |
| 2               | \( \cosh^{-1} \frac{7}{3} \) | \(-\frac{1024}{9}g^2\) | (0,1)                  |

The corresponding values of the \( A_1 \) tensor are

\[ A_1^{(1)} = \text{diag}(-4, -4, -4, -4, -4, -4, -4, 16, 8) \]

and \( A_1^{(2)} = \text{diag}(-4, -4, -4, -4, -4, -4, -4, \frac{16}{3}, 8) \).

(3.10)

For \( L = \mathbf{I} \), the background isometry is given by \( Osp(7|2, \mathbb{R}) \times Osp(2|2, \mathbb{R}) \). The critical point 2 preserves \( SU(3) \) symmetry. The location and value of the cosmological constant relative to the \( L = \mathbf{I} \) point are similar to the \( G_2 \times G_2 \) point in \( SO(8) \times SO(8) \) gauged \( N = 16 \) supergravity. In our result, the residual gauge symmetry is the \( SU(3) \) subgroup of \( G_2 \) which is in turn a subgroup of \( SO(7) \).

- \( SO(6) \times SO(3) \) gauging:

We find the potential

\[
V = \frac{1}{128}g^2(-3886 - 424 \cosh(2a_1) + 6 \cosh(4a_1) + 4 \cosh[2(a_1 - 2a_2)] \\
+ 4 \cosh(4a_1 - 2a_2) - 1536 \cosh(a_1 - a_2) - 208 \cosh[2(a_1 - a_2)] \\
+ \cosh[4(a_1 - a_2)] - 424 \cosh(2a_2) + 6 \cosh(4a_2) \\
- 1536 \cosh(a_1 + a_2) - 208 \cosh[2(a_1 + a_2)] + \cosh[4(a_1 + a_2)] \\
+ 4 \cosh[2(2a_1 + a_2)] + 4 \cosh[2(a_1 + 2a_2)]).
\]  \hspace{1cm} (3.11)

One supersymmetric critical point is

\[ V_0 = -256g^2, \quad a_1 = \cosh^{-1} 2, \quad a_2 = \cosh^{-1} 3. \]  \hspace{1cm} (3.12)
with the value of the $A_1$ tensor

$$A_1 = \begin{pmatrix} -16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & 0 & 0 & 0 & 0 & -4\sqrt{3} \\ 0 & 0 & 0 & -16 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -16 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -16 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 0 \\ 0 & 0 & -4\sqrt{3} & 0 & 0 & 0 & 0 & 0 & 4 \end{pmatrix}. \quad (3.13)$$

This can be diagonalized to

$$A_1 = \text{diag} \left( -16, -16, -8, -16, -16, -16, 8, 8, 8 \right). \quad (3.14)$$

This is a $(1,3)$ point and has $SO(3) \subset SO(6)$ symmetry. With $a_2 = 0$, we find the following critical points

| Critical points | $a_1$ | $V_0$ | Preserved supersymmetry |
|-----------------|-------|-------|-------------------------|
| 1               | 0     | $-64g^2$ | $(6,3)$ |
| 2               | $\cosh^{-1} 3$ | $-144g^2$ | $(0,2)$ |

The corresponding values of the $A_1$ tensor are

$$A_1^{(1)} = \text{diag} \left( -4, -4, -4, -4, -4, 4, 4, 4 \right)$$

and

$$A_1^{(2)} = \text{diag} \left( -10, -10, -10, -10, -10, -10, 6, 6, 10 \right). \quad (3.15)$$

For $L = I$, the background isometry is given by $Osp(6|2, \mathbb{R}) \times Osp(3|2, \mathbb{R})$. The critical point 2 is also invariant under $SO(3)$ subgroup of $SO(6)$.

- $SO(5) \times SO(4)$ gauging:

The potential for this gauging is

$$V = \frac{1}{32} g^2 \left( 3 + \cosh a_1 \cosh a_2 \right)^2 \left( -86 + 2 \cosh(2a_1) - 24 \cosh(a_1 - a_2) \right)$$

$$+ \cosh[2(a_1 - a_2)] + 2 \cosh(2a_2) - 24 \cosh(a_1 + a_2)$$

$$+ \cosh[2(a_1 + a_2)]). \quad (3.16)$$

There is no critical point with two non zero scalars. With $a_2 = 0$, we find the following critical points:
Critical points | $a_1$ | $V_0$ | Preserved supersymmetry
--- | --- | --- | ---
1 | 0 | $-64g^2$ | (5,4)
2 | $\cosh^{-1} 5$ | $-256g^2$ | (0,3)

The corresponding values of the $A_1$ tensor are

\[
A_1^{(1)} = \text{diag} (-4, -4, -4, -4, 4, 4, 4, 4),
\]
\[
\text{and } A_1^{(2)} = \text{diag} (-16, -16, -16, -16, 8, 8, 8, 16).
\] (3.17)

For $L = \mathbf{I}$, the background isometry is given by $Osp(5|2, \mathbb{R}) \times Osp(4|2, \mathbb{R})$. The critical point 2 preserves $SO(4)_{\text{diag}}$ symmetry which is the diagonal subgroup of $SO(4) \times SO(4)$ with the first $SO(4)$ being a subgroup of $SO(5)$.

### 3.2 Vacua of non-compact gaugings

We now give some critical points of the non-compact gaugings. The isometry group of the background with $L = \mathbf{I}$ consists of the maximal compact subgroup of the gauge group and $SO(2,2)$ as the bosonic subgroup. Using the generators given in the appendix, we can compute the scalar potentials for these two gaugings. Notice that in the non-compact gaugings, all sixteen scalars are not equivalent. At the maximally symmetric vacua, the gauge group is broken down to its maximal compact subgroup, and some of the scalars become Goldstone bosons making some of the vector fields massive. This “Higgs-mechanism” results in the propagating $n_{ng}$ massive vector fields where $n_{ng}$ denotes the number of non compact generators which are broken at the critical point. The total number of degrees of freedom remains the same because of the disappearance of the $n_{ng}$ scalars, Goldstone bosons. For further detail, see [8] in the context of $N = 16$ models.

- $G_{2(-14)} \times SL(2)$ gauging:
  The coset representative is chosen to be

\[
L = e^{a_1 Y_3} e^{a_2 Y_5}. \tag{3.18}
\]

This parametrization has residual gauge symmetry $SU(2)$ which is a subgroup of $G_{2(-14)}$. With one of the scalars vanishing, $L$ has $SU(3)$ symmetry. The potential
with two scalars is given by

\[ V = \frac{1}{4608} g^2 \left[ -23406 - 2520 \cosh(2a_1) + 70 \cosh(4a_1) + 8 \cosh(4a_1 - 3a_2) \
+ 28 \cosh[2(a_1 - 2a_2)] + 28 \cosh(4a_1 - 2a_2) - 560 \cosh[2(a_1 - a_2)] \
+ \cosh[4(a_1 - a_2)] - 1792 \cosh(2a_1 - a_2) + 56 \cosh(4a_1 - a_2) \
+ 3472 \cosh(a_2) - 6104 \cosh(2a_2) - 16 \cosh(3a_2) + 198 \cosh(4a_2) \
- 560 \cosh[2(a_1 + a_2)] + \cosh[4(a_1 + a_2)] - 1792 \cosh(2a_1 + a_2) \
+ 28 \cosh[2(2a_1 + a_2)] + 56 \cosh(4a_1 + a_2) + 28 \cosh[2(a_1 + 2a_2)] \
+ 8 \cosh(4a_1 + 3a_2) \right]. \] (3.19)

We find the following critical points:

| critical point | \( a_1 \) | \( a_2 \) | \( V_0 \) | preserved supersymmetries |
|----------------|---------|---------|---------|-----------------------------|
| 1              | 0       | 0       | \(-\frac{64}{7} g^2\) | (7,2)                        |
| 2              | 0       | \(\cosh^{-1} \frac{1}{2} \sqrt{\frac{11+\sqrt{57}}{2}}\) | \(-\frac{100}{7} g^2\) | -                            |
| 3              | \(\cosh^{-1} \frac{1}{2}\) | 0       | \(-\frac{1024}{81} g^2\) | (0,1)                        |
| 4              | \(\cosh^{-1} \frac{3}{2}\) | \(\cosh^{-1} \frac{2}{\sqrt{3}}\) | \(-\frac{1024}{81} g^2\) | (1,2)                        |

The corresponding values of the \( A_1 \) tensor are

\[ A^{(1)}_1 = \text{diag} \left( -\frac{4}{3}, -\frac{4}{3}, -\frac{4}{3}, -\frac{4}{3}, -\frac{4}{3}, -\frac{4}{3}, -\frac{4}{3}, -\frac{4}{3}, -\frac{4}{3} \right), \]

\[ A^{(3)}_1 = \begin{pmatrix}
-\frac{7}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -3 & 0 & 0 & 0 & 0 & \frac{2}{3} & 0 & 0 \\
0 & 0 & -\frac{7}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{7}{3} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{7}{3} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{7}{3} & 0 & 0 & 0 \\
0 & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 & -3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{3} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{3} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{7}{3}
\end{pmatrix} \] (3.20)
and

\[
A_1^{(4)} = \begin{pmatrix}
-\frac{28}{9} & 0 & 0 & 0 & 0 & -\frac{4}{9} & 0 & 0 & 0 \\
0 & -\frac{28}{9} & 0 & 0 & 0 & \frac{4}{9} & 0 & 0 & 0 \\
0 & 0 & -\frac{8}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{14}{9} & 0 & 0 & 0 & 0 & -\frac{2}{3}\sqrt{\frac{5}{3}} \\
0 & 0 & 0 & 0 & -\frac{8}{3} & 0 & 0 & 0 & 0 \\
-\frac{4}{9} & 0 & 0 & 0 & 0 & -\frac{28}{9} & 0 & 0 & 0 \\
0 & \frac{4}{9} & 0 & 0 & 0 & 0 & \frac{16}{9} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{2}{3}\sqrt{\frac{5}{3}} & 0 \\
0 & 0 & 0 & -\frac{2}{3}\sqrt{\frac{5}{3}} & 0 & 0 & 0 & 0 & \frac{14}{9}
\end{pmatrix}.
\]  (3.21)

\[
A_1^{(3)} \text{ and } A_1^{(4)} \text{ can be diagonalized to}
\]

\[
A_1^{(3)} = \text{diag} \left( -\frac{7}{3}, -\frac{11}{3}, -\frac{7}{3}, -\frac{7}{3}, -\frac{7}{3}, -\frac{7}{3}, -\frac{5}{3}, -\frac{7}{3} \right),
\]

\[
A_1^{(4)} = \text{diag} \left( -\frac{32}{9}, -\frac{32}{9}, -\frac{8}{3}, -\frac{16}{9}, -\frac{8}{3}, -\frac{8}{3}, -\frac{16}{9}, -\frac{16}{9} \right). \]  (3.22)

For \( L = I \), the gauge group is broken down to its compact subgroup \( G_2 \times SO(2) \).

The background isometry is given by \( G(3) \times Osp(2|2, \mathbb{R}) \). There are two \( SU(3) \)
points with completely broken supersymmetry (point 2) and (0,1) supersymmetry (point 3). Point 4 has \( SU(2) \) symmetry.

- \( Sp(1, 2) \times SU(2) \) gauging:

We choose the coset representative

\[
L = e^{a_1(Y_1 - Y_{10})} e^{a_2(Y_2 + Y_0)}.
\]  (3.23)

This has symmetry \( SO(3) \times SO(3) \) if any one of the scalars vanishes. This is the case in which our critical points lie. This symmetry is a subgroup of the \( SO(5) \times SO(3) \) compact subgroup of \( Sp(1, 2) \) with the first \( SO(3) \) being a subgroup of \( SO(5) \). We find the potential

\[
V = \frac{1}{32} g^2 \left[ -1390 - 232 \cosh(2\sqrt{2}a_1) + 6 \cosh(4\sqrt{2}a_1) \\
+ 4 \cosh(2\sqrt{2}(a_1 - 2a_2)) - 112 \cosh(2\sqrt{2}(a_1 - a_2)) \\
+ \cosh(4\sqrt{2}(a_1 - a_2)) + 4 \cosh(2\sqrt{2}(2a_1 - a_2)) - 232 \cosh(2\sqrt{2}a_2) \\
+ 6 \cosh(4\sqrt{2}a_2) - 112 \cosh(2\sqrt{2}(a_1 + a_2)) + \cosh(4\sqrt{2}(a_1 + a_2)) \\
+ 4 \cosh(2\sqrt{2}(a_1 + a_2)) + 4 \cosh(2\sqrt{2}(a_1 + 2a_2)) \right].
\]  (3.24)
Some of the critical points are given by

| critical point | $a_1$ | $a_2$ | $V_0$ | preserved supersymmetries |
|---------------|-------|-------|-------|---------------------------|
| 1             | 0     | 0     | $-64g^2$ | (5,4)                     |
| 2             | 0     | $\frac{\cosh^{-1}2}{\sqrt{2}}$ | $-100g^2$ | -                         |
| 3             | $\frac{\ln(2-\sqrt{3})}{\sqrt{2}}$ | 0     | $-100g^2$ | -                         |
| 4             | $\frac{\ln(2+\sqrt{3})}{\sqrt{2}}$ | 0     | $-100g^2$ | -                         |

with the corresponding $A_1$ tensor

$$A_1^{(1)} = \text{diag}(-4, -4, -4, -4, 4, 4, 4, 4)$$

for the critical point 1. For $L = \mathbf{1}$, the gauge group is broken down to its compact subgroup $Sp(1) \times Sp(2) \times SU(2) \sim SU(2) \times SO(5) \times SU(2)$. The two $SU(2)$’s factors combine to $SO(4)$ under which the right handed supercharges transform as 4. So, the background isometry is given by $Osp(5|2, \mathbb{R}) \times Osp(4|2, \mathbb{R})$. Point 2, 3, and 4 are $SO(3) \times SO(3)$ points with completely broken supersymmetry.

We have checked that all critical points given above are truely critical points of the corresponding potential. In the next section, we will find RG flow solutions interpolating between some of these vacua.

### 4. RG flow solutions

In this section, we study RG flow solutions in the $N = 9$ theory whose vacua are obtained in the previous section. We start by giving the general formulae we will use in various gaugings. The strategy to find supersymmetric flow solutions is to find the solutions to the BPS equations coming from the supersymmetry transformations of fermions which in this case, are $\delta \chi^{iI}$ and $\delta \psi^{I}$.

We start by giving an ansatz for the metric

$$ds^2 = e^{2A(r)} dx_{1,1}^2 + dr^2.$$ (4.1)

The relevant spin connection is

$$\omega^{\hat{\nu}}_{\hat{\mu}} = A' \delta^\nu_\mu$$ (4.2)

where hatted indices denote the tangent space indices, $\hat{\mu}, \hat{\nu} = 0, 1$. We use the notation $' \equiv \frac{d}{dr}$ from now on. We then recall the supersymmetry transformations from [5]

$$\delta \psi^{I} = D_{\mu} \epsilon^{I} + gA^{IJ}_1 \gamma_{\mu} \epsilon^{J} ,
\delta \chi^{iI} = \frac{1}{2} (\delta^{I} J - f^{IJ})_{i} \mathcal{D}_{\bar{J}} \bar{\epsilon}^{J} - g N A_{2}^{J} \bar{\epsilon}^{J}.$$ (4.3)
We will not repeat the meaning of all the notations here but refer the readers to [5] for the detailed explanation. Using (2.4), we find, in our normalization,

\[
\frac{d\phi^A}{dr} = \frac{1}{6} \text{Tr}(Y^AL^{-1}L').
\] (4.4)

With this information, we are now in a position to set up the BPS equations which are our flow equations. The \(\delta\chi^I = 0\) equation gives flow equations for the scalars while the \(\delta\psi^I = 0\) is used to determine \(A(r)\) in the metric. In order to obtain the equation for \(A(r)\), we impose \(\gamma^I\epsilon^I = \epsilon^I\), so the solution preserves half of the original supersymmetries. We now apply this result to various gaugings. In the gauging that admits a supersymmetric flow solution, there must exist at least two AdS supersymmetric critical points with different cosmological constants. The latter is related to the central charge of the dual CFT as

\[
c \sim \frac{1}{\sqrt{-V_0}}.
\] (4.5)

According to the c-theorem, the c-function interpolating between the central charges in the UV and IR fixed points is a monotonically decreasing function along the flow from the UV to the IR. From the previous section, there is no flow solution in the \(SO(9), SO(8)\) and \(Sp(1,2) \times SU(2)\) gaugings because there is only one supersymmetric critical point.

### 4.1 RG flows in compact gaugings

We start by finding flow solutions in the compact gaugings.

#### 4.1.1 \(SO(7) \times SO(2)\) gauging

With a single scalar, the flow equation is given by

\[
\frac{da_1}{dr} = g \sinh a_1(3 \cosh a_1 - 7).
\] (4.6)

Changing the variable to \(b = \cosh a_1\), we find the solution

\[
r = \frac{1}{20g} \ln(1 + b) - \frac{1}{8g} \ln(b - 1) + \frac{3}{40g} \ln(3b - 7).
\] (4.7)

The supersymmetry transformation of the gravitino gives

\[
\frac{dA}{dr} = -\frac{1}{2} g(b - 5)(1 + 3b).
\] (4.8)

We can solve this equation to obtain \(A\) as a function of \(b\) using the equation for \(\frac{db}{dr}\). The solution is

\[
A = -\ln(b - 1) - \frac{3}{10} \ln(1 + b) + \frac{4}{5} \ln(3b - 7).
\] (4.9)
In all these solutions, we have neglected all the additive constants to \( A \) and \( r \) because we can always shift \( A \) and \( r \) to absorb them. From (4.7), we see that as \( a_1 = 0 \), \( r \to \infty \) and \( r \to -\infty \) when \( a_1 = \cosh^{-1} \frac{7}{3} \). The UV point corresponds to \( a_1 = 0 \), and the IR point is at \( a_1 = \cosh^{-1} \frac{7}{3} \). The ratio of the central charges is given by

\[
\frac{c_{\text{UV}}}{c_{\text{IR}}} = \sqrt{\frac{V_{0\text{IR}}}{V_{0\text{UV}}}} = \frac{4}{3}.
\]

(4.10)

At the UV point, the AdS\(_3\) radius is \( L = \frac{1}{8g} \). Near this point, the scalar fluctuation behaves as

\[
\delta a_1 \sim e^{-4gr} = e^{-\frac{r}{L}}.
\]

(4.11)

Using the argument in [19, 20], we find that the flow is driven by a relevant operator of dimension \( \Delta = \frac{3}{2} \). In the IR, we find

\[
\delta a_1 \sim e^{\frac{r}{L}}, \quad L = \frac{3}{32g}.
\]

(4.12)

The corresponding operator is irrelevant with dimension \( \Delta = \frac{13}{4} \). The UV and IR points have supersymmetries (7,2) and (0,1), respectively. Our scalars are canonically normalized as can be easily checked by looking at the scalar kinetic terms, so we can directly read off the value of \( m^2 \) from the potential. Near the UV point, we find

\[
V = -64g^2 - 24g^2a_1^2.
\]

(4.13)

The mass squared in unit of \( \frac{1}{L^2} \) is \( m^2L^2 = -\frac{3}{4} \). The mass-dimension formula \( \Delta(\Delta - 2) = m^2L^2 \) gives \( \Delta = \frac{3}{2} \) in agreement with what we have found from the behavior of the scalar near the critical point. At the IR point, we find

\[
V = -\frac{1024}{9}g^2 + \frac{2080}{9}g^2a_1^2.
\]

(4.14)

The mass squared is \( m^2L^2 = \frac{65}{16} \) which gives precisely \( \Delta = \frac{13}{4} \).

We now consider a flow solution with two non zero scalars. Unfortunately, we are not able to find an analytic solution in this case. We do find a numerical solution interpolating between maximal supersymmetric point at \( L = 1 \) and the non trivial critical point with two scalars given in the previous section. We start by giving flow equations

\[
\frac{da_1}{dr} = g \frac{e^{\frac{a_2}{2}} \cosh \frac{a_2}{2} \sinh a_1}{1 + e^{a_2}} [3 \cosh a_1 (1 + \cosh a_2) - \cosh a_2 - 13],
\]

(4.15)

\[
\frac{da_2}{dr} = \frac{g}{16} (-65 + 9 \cosh^2 a_1 (1 + \cosh a_2) - 8 \cosh a_1 (7 + \cosh a_2) + 3 \sinh^2 a_1 + \cosh a_2 (47 + 3 \sinh^2 a_1)) \sinh a_2.
\]

(4.16)
Changing the variables to
\[ a_1 = \cosh^{-1} b_1, \quad a_2 = \cosh^{-1} b_2, \] (4.17)
we can rewrite (4.15) and (4.16) as
\[ b'_1 = \frac{g}{2} (b_2^2 - 1)[3b_1(1 + b_2) - b_2 - 13], \] (4.18)
\[ b'_2 = \frac{g}{4} (b_2^2 - 1)[11b_2 - 17 + 3b_1^2(1 + b_2) - 2b_1(7 + b_2)]. \] (4.19)

It can be easily checked that \( b_1 = \frac{2}{3}, b_2 = 2 \) is a fixed point of these equations. In order to find a numerical solution, we set \( g = 1 \) and \( b_2 = z \). Taking \( b_1 \) as a function of \( z \), we can write the two equations as a single equation
\[ \frac{db_1}{dz} = \frac{2(-13 - z + 3(1 + z)b_1)(-1 + b_1^2)}{(-1 + z^2)(-17 + 11z - 2(7 + z)b_1 + 3(1 + z)b_1^2)}. \] (4.20)

The numerical solution to this equation is shown in Figure 1. The gravitino variation gives
\[ \frac{dA}{dr} = -\frac{1}{8} g[3 - 34z + 11z^2 - 2(13 + 14z + z^2)b_1 + 3(1 + z)^2b_1^2] \] (4.21)
or
\[ \frac{dA}{dz} = \frac{3 - 34z + 11z^2 - 2(13 + 14z + z^2)b_1 + 3(1 + z)^2b_1^2}{2(-1 + z^2)(-17 + 11z - 2(7 + z)b_1 + 3(1 + z)b_1^2)}. \] (4.22)

The numerical solution for \( A \) is shown in Figure 2. The UV point is at \( r \to \infty \) and has (7,2) supersymmetries. The IR point has (1,2) supersymmetries and corresponds to \( r \to -\infty \). The ratio of the central charges is
\[ \frac{c_{UV}}{c_{IR}} = \frac{3}{2}. \] (4.23)

The behavior of the fluctuations of \( a_1 \) and \( a_2 \) near the fixed point can be found by linearizing (4.15) and (4.16). We find
\[ \delta a_1 \sim e^{-\frac{r}{2}}, \quad \delta a_2 \sim e^{-\frac{r}{2}}, \] (4.24)
near the UV point with \( L = \frac{1}{8g} \). We see that the flow is driven by a relevant operator of dimension \( \frac{3}{2} \). Near the IR point with \( L = \frac{1}{12g} \), we find
\[ \delta a_1 \sim \delta a_2 \sim e^{4gr} = e^{\frac{r}{3}}. \] (4.25)

So, the operator becomes irrelevant at the IR and has dimension \( \Delta = \frac{7}{3} \). We can also check this by computing the scalar masses from the potential although it is more
complicated in this case because we will need to diagonalize the mass matrix. We only give the analysis at the UV point. The potential is fortunately diagonal and given by

\[ V = -64g^2 - 24g^2(a_1^2 + a_2^2). \] (4.26)

We find \( m^2 L^2 = -\frac{3}{4} \) which gives \( \Delta = \frac{3}{2} \).

In all other gaugings studied here, the same pattern appears, and the analysis is the same. So, we will quickly go through these cases and give only the main results without giving all the details. In particular, we will not give the scalar masses. These can be worked out as above.

\[ \begin{align*}
\text{Figure 1:} & \quad \text{Solution for } b_1(z) \text{ in } SO(7) \times SO(2) \text{ gauging.} \\
\text{Figure 2:} & \quad \text{Solution for } A(z) \text{ in } SO(7) \times SO(2) \text{ gauging.}
\end{align*} \]
4.1.2 $SO(6) \times SO(3)$ gauging

We begin with a flow with one scalar. The flow equation is

$$a'_1 = g(2 \cosh a_1 - 6) \sinh a_1. \quad (4.27)$$

With $a_1 = \cosh^{-1} b$, we find

$$b' = 2g(3 - b - 3b^2 + b^3). \quad (4.28)$$

This can be solved directly and gives

$$r = \frac{1}{16} \ln(9 + 6b - 3b^2) - \frac{1}{8} \ln(b - 1). \quad (4.29)$$

The gravitino variation gives

$$A'_1 = -\frac{g}{2}(\cosh(2a_1) - 12 \cosh a_1 - 5). \quad (4.30)$$

The solution is given by

$$A = \frac{3}{4} \ln(b - 3) - \ln(b - 1) - \frac{1}{4} \ln(1 + b). \quad (4.31)$$

Near the UV point with (6,3) supersymmetries, the fluctuation behaves as

$$\delta a_1 \sim e^{-\frac{ir}{L}}, \quad L = \frac{1}{8g^2}. \quad (4.32)$$

The flow is driven by a relevant operator of dimension $\frac{3}{2}$. At the IR (0,2) point, we find

$$\delta a_1 \sim e^{\frac{ir}{L}}, \quad L = \frac{1}{12g}. \quad (4.33)$$

The operator becomes irrelevant with dimension $\Delta = \frac{10}{3}$. The ratio of the central charges is

$$\frac{c_{UV}}{c_{IR}} = \frac{3}{2}. \quad (4.34)$$

We then move to a flow with two scalars. With

$$a_1 = \cosh^{-1} b_1, \quad a_2 = \cosh^{-1} b_2, \quad (4.35)$$

the flow equations are given by

$$b'_1 = g(b_1^2 - 1)(b_1 - 5 - b_2 + b_1 b_2), \quad (4.36)$$

$$b'_2 = \frac{g}{2}(b_2^2 - 1)[b_2^2(1 + b_2) - 7 + 5b_2 - 2b_1(3 + b_2)]. \quad (4.37)$$
Taking $b_1$ as a function of $z = b_2$, we find

$$\frac{db_1}{dz} = \frac{2(-5 - z + (1 + z)b_1)(-1 + b_1^2)}{(-1 + z^2)(-7 + 5z - 2(3 + z)b_1 + (1 + z)b_1^2)}. \quad (4.38)$$

The numerical solution is given in Figure 3. The metric function $A$ can be determined by using the equation

$$\frac{dA}{dz} = \frac{-3 - 14z + 5z^2 - 2(5 + 6z + z^2)b_1 + (1 + z)^2b_1^2}{2(-1 + z^2)(-7 + 5z - 2(3 + z)b_1 + (1 + z)b_1^2)}. \quad (4.39)$$

The numerical solution is given in Figure 4. The linearized equations give

$$\delta a_1 \sim \delta a_2 \sim e^{-\frac{3}{2}r}, \quad L = \frac{1}{8g} \quad (4.40)$$

near the UV point. The flow is driven by a relevant operator of dimension $\frac{3}{2}$ and interpolates between (6,3) and (1,3) critical points. Near the IR, we find

$$\delta a_1 \sim \delta a_2 \sim e^{\frac{5}{2}r}, \quad L = \frac{1}{16g}. \quad (4.41)$$

So, in the IR the operator has dimension $\frac{5}{2}$. The ratio of the central charges is

$$\frac{c_{\text{UV}}}{c_{\text{IR}}} = 2. \quad (4.42)$$

![Figure 3: Solution for $b_1(z)$ in $SO(6) \times SO(3)$ gauging.](image)
4.1.3 $SO(5) \times SO(4)$ gauging

In this gauging, there is no critical point with two non zero scalars, so there is no flow with two scalars. The flow equation with one scalar is

$$a'_1 = g \sinh a_1 (\cosh a_1 - 5).$$  \hfill (4.43)

The solution for $r$ as a function of $b = \cosh a_1$ is

$$r = \frac{1}{24g} \ln(b - 5) - \frac{1}{8g} \ln(b - 1) + \frac{1}{12g} \ln(1 + b).$$  \hfill (4.44)

The gravitino variation gives

$$A' = -\frac{g}{4} (\cosh(2a_2) - 20 \cosh a_1 - 13).$$  \hfill (4.45)

The solution for $A$ as a function of $b$ is

$$A = \frac{2}{3} \ln(b - 5) - \ln(b - 1) - \frac{1}{6} \ln[20(1 + b)].$$  \hfill (4.46)

We find the scalar fluctuations near the UV and IR point as

$$\text{UV} : \; \delta a_1 \sim e^{-\frac{\pi}{4}}, \quad L_{UV} = \frac{1}{8g},$$  \hfill (4.47)

$$\text{IR} : \; \delta a_1 \sim e^{\frac{\pi}{4}}, \quad L_{IR} = \frac{1}{16g}.  \hfill (4.48)$$

From these, we find that the flow is driven by a relevant operator of dimension $\frac{3}{2}$. The operator has dimension $\frac{7}{2}$ in the IR. The ratio of the central charges is

$$\frac{c_{UV}}{c_{IR}} = 2.$$  \hfill (4.49)
4.2 RG flows in non-compact gaugings

4.2.1 $G_{2(-14)} \times SL(2)$ gauging

Remarkably, there exist flow solutions in this non-compact exceptional gauging. We start with a single scalar giving rise to the flow equation

$$a'_1 = -\frac{4g}{3} \sinh a_1 (\cosh a_1 - 2).$$

(4.50)

The solution to this equation is

$$r = \frac{3}{4g} \left[ \frac{1}{3} \ln(b - 2) - \frac{1}{2} \ln(b - 1) + \frac{1}{6} \ln(1 + b) \right]$$

(4.51)

where as usual $b = \cosh a_1$. The equation for $A$ and its solution are given by

$$A' = \frac{8g}{3} \cosh a_1 - \frac{2g}{3} \sinh^2 a_1$$

(4.52)

and

$$A = \frac{5 \ln(b - 2) - 6 \ln(b - 1) - 2 \ln(1 + b)}{6}.$$  

(4.53)

The solution interpolates between (7,2) and (0,1) critical points with the ratio of the central charges

$$\frac{c_{UV}}{c_{IR}} = \frac{5}{4}.$$  

(4.54)

The linearized equation gives

UV : $\delta a_1 \sim e^{-\frac{r}{L_{UV}}}$, \quad $L_{UV} = \frac{3}{8g}$,

(4.55)

IR : $\delta a_1 \sim e^{\frac{r}{L_{IR}}}$, \quad $L_{IR} = \frac{3}{10g}.$

(4.56)

The flow is driven by a relevant operator of dimension $\frac{3}{2}$. In the IR, the operator has dimension $\frac{88}{5}$.

We now move to a flow solution with two scalars. The flow equations are

$$a'_1 = -\frac{2g}{3} [4 \sinh a_2 - \cosh^2 a_2 \frac{a_2}{3} \sinh(2a_1)],$$

(4.57)

$$a'_2 = -\frac{g}{12} [2(9 + 8 \cosh a_1 - \cosh(2a_1)) \sinh a_2$$

$$- (7 + \cosh(2a_1)) \sinh(2a_2)].$$

(4.58)

Using

$$a_1 = \cosh^{-1} a, \quad a_2 = \cosh^{-1} b,$$

(4.59)
we can combine the two equations into one with $a$ being independent variable

\[
\frac{db}{da} = \frac{(b^2 - 1)(b(3 + a^2) + a^2 - 4a - 5)}{2(a^2 - 1)(ab + a - 4)}.
\]

(4.60)

We give a numerical solution to this equation in Figure 5. The equation for $A$ is

\[
\frac{dA}{da} = -\frac{3 - 8a + a^2 + 2(a^2 - 4a - 5)b + (3 + a^2)b^2}{4(a^2 - 1)(ab + a - 4)}
\]

(4.61)

whose solution is shown in Figure 6. The scalar fluctuations are given by

UV : $\delta a_1 \sim \delta a_2 \sim e^{\pi r}$, \quad $L_{UV} = \frac{3}{8g}$.

(4.62)

IR : $\delta a_1 \sim \delta a_2 \sim e^{\pi r}$, \quad $L_{IR} = \frac{9}{32g}$.

(4.63)

The flow interpolates between the (7,2) and (1,2) points with the ratio of the central charges

\[
\frac{c_{UV}}{c_{IR}} = \frac{4}{3}.
\]

(4.64)

The flow is driven by a relevant operator of dimension $\frac{3}{2}$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure5.png}
\caption{Solution for $b(a)$ in $G_{2(-14)} \times SL(2)$ gauging.}
\end{figure}

5. Conclusions

We have studied $N = 9$ three dimensional gauged supergravity with compact and non-compact gaugings. We have found some supersymmetric AdS$_3$ vacua corresponding to
some two dimensional CFT’s. We have identified the superconformal groups from the isometry of the AdS$_3$ backgrounds with $L = I$. These backgrounds have dual conformal field theories at their boundaries. We then studied RG flow solutions describing a deformation of the CFT in the UV to the CFT in the IR. In the scalar sector studied here, only $SO(7) \times SO(2)$, $SO(6) \times SO(3)$, $SO(5) \times SO(4)$ and $G_{2(-14)} \times SL(2)$ gaugings admit supersymmetric flow solutions. This is because there is only one supersymmetric critical point in $SO(9)$, $SO(8)$ and $Sp(1,2) \times SU(2)$ gaugings. This is not unexpected, the bigger gauge groups give rise to a simpler structure of vacua in general. We have found analytic flow solutions with one active scalar and numerical solutions for the flows with two active scalars. All the flows are operator flows driven by a relevant operator of dimension $\frac{3}{2}$. It is interesting to identify the CFT’s dual to these gravity solutions. Because two dimensional field theories are more controllable and the gravity solutions correspond to strong coupling limits of the dual field theories, we hope to understand many aspects of the AdS/CFT correspondence in the case of AdS$_3$/CFT$_2$.

The higher dimensional origin of many three dimensional gauged supergravities is still mysterious. Only the case of non semisimple gaugings is known to be related to dimensional reductions of higher dimensional theories [23]. It is interesting to study the non semisimple gaugings in this $N = 9$ theory although there is another subtlety with the theories with odd $N$. This is because we cannot obtain these theories directly from dimensional reductions due to the mismatch in the number of supercharges. The reduced theory, always having even $N$ in three dimensions, needs to be truncated in order to give odd values of $N$. The models with compact and non-compact gauge groups studied in this paper and elsewhere are not obtainable from dimensional reductions, so it is very interesting to study whether there exist any higher dimensional origin for
these models. This will provide an interpretation of our flow solutions and that studied in \cite{11} in terms of higher dimensional geometries.

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A. Essential formulae

In this appendix, we give all necessary formulae in order to obtain the scalar potential and flow equations. We use the 52 generators of \( F_4 \) from \cite{14}. The generators are normalized by

\[
\text{Tr}(c_ic_j) = -6\delta_{ij}. \tag{A.1}
\]

With this normalization, we find that

\[
V^{\alpha IJ} = -\frac{1}{6}\text{Tr}(L^{-1}T^\alpha_G L^{IJ}) \tag{A.2}
\]

\[
V^{\alpha A} = \frac{1}{6}\text{Tr}(L^{-1}T^\alpha_G L^A) \tag{A.3}
\]

where we have introduced the symbol \( T^\alpha_G \) for gauge group generators. \( T^\alpha_G \) will be replaced by some appropriate generators of the gauge group being considered in each gauging.

The following mapping provides the relation between \( c_i \) and \( X^{IJ} \), generators of \( SO(9) \),

\[
X^{12} = c_1, \quad X^{13} = -c_2, \quad X^{23} = c_3, \quad X^{34} = c_6, \quad X^{14} = c_4, \quad X^{24} = -c_5, \quad X^{15} = c_7, \quad X^{25} = -c_8, \quad X^{35} = c_9, \quad X^{45} = -c_{10}, \quad X^{56} = -c_{15}, \quad X^{16} = c_{11}, \quad X^{26} = -c_{12}, \quad X^{46} = -c_{14}, \quad X^{36} = c_{13}, \quad X^{17} = c_{16}, \quad X^{27} = -c_{17}, \quad X^{47} = -c_{19}, \quad X^{37} = c_{18}, \quad X^{67} = -c_{21}, \quad X^{57} = -c_{20}, \quad X^{78} = -c_{36}, \quad X^{18} = c_{30}, \quad X^{28} = -c_{31}, \quad X^{48} = -c_{33}, \quad X^{38} = c_{32}, \quad X^{68} = -c_{35}, \quad X^{58} = -c_{34}, \quad X^{29} = -c_{46}, \quad X^{19} = c_{45}, \quad X^{49} = -c_{48}, \quad X^{39} = c_{47}, \quad X^{69} = -c_{50}, \quad X^{59} = -c_{49}, \quad X^{89} = -c_{52}, \quad X^{79} = -c_{51}. \tag{A.4}
\]

All the \( f^{IJ} \)’s components can be obtained from the structure constants of the \([X^{IJ}, Y^A]\) given in \cite{14}, but we will not repeat them here.

In the non-compact \( G_{2(-14)} \times SL(2) \) gauging, we use the following generators. The
generators of $G_{2(-14)}$ are obtained by using the embedding of $G_{2(-14)}$ in $SO(7)$ generated by $X^{IJ}$, $I, J = 1, \ldots, 7$. The adjoint representation of $SO(7)$ decomposes under $G_{2(-14)}$ as

$$21 \rightarrow 14 + 7. \quad (A.5)$$

The generators of $G_{2(-14)}$ can be explicitly found by combinations of $SO(7)$ generators from [22]

$$T_1 = \frac{1}{\sqrt{2}}(X^{36} + X^{41}) , \quad T_2 = \frac{1}{\sqrt{2}}(X^{31} - X^{46}) ,$$

$$T_3 = \frac{1}{\sqrt{2}}(X^{43} - X^{16}) , \quad T_4 = \frac{1}{\sqrt{2}}(X^{73} - X^{24}) ,$$

$$T_5 = -\frac{1}{\sqrt{2}}(X^{23} + X^{47}) , \quad T_6 = -\frac{1}{\sqrt{2}}(X^{26} + X^{71}) ,$$

$$T_7 = \frac{1}{\sqrt{2}}(X^{76} - X^{21}) , \quad T_8 = \frac{1}{\sqrt{6}}(X^{16} + X^{43} - 2X^{72}) ,$$

$$T_9 = -\frac{1}{\sqrt{6}}(X^{41} - X^{36} + 2X^{25}) , \quad T_{10} = -\frac{1}{\sqrt{6}}(X^{31} + X^{46} - 2X^{57}) ,$$

$$T_{11} = \frac{1}{\sqrt{6}}(X^{73} + X^{24} + 2X^{15}) , \quad T_{12} = -\frac{1}{\sqrt{6}}(X^{74} - X^{23} + 2X^{65}) ,$$

$$T_{13} = \frac{1}{\sqrt{6}}(X^{26} - X^{71} + 2X^{35}) , \quad T_{14} = \frac{1}{\sqrt{6}}(X^{21} + X^{76} - 2X^{45}). \quad (A.6)$$

We have verified that these generators satisfy $G_2$ algebra given in [21]. The $SL(2)$ generators are

$$J_1 = i\sqrt{2}(c_{22} + c_{27}), \quad J_2 = i\sqrt{2}(c_{37} + c_{42}), \quad J_3 = 2c_{52} \quad (A.7)$$

which can be easily checked that they commute with all $T$'s and form $SL(2)$ algebra.

The generators of non-compact $Sp(1, 2)$ can be constructed by first finding its compact subgroup generators $Sp(1) \times Sp(2) \sim SO(3) \times SO(5)$. The latter can be obtained by taking $SO(8)$ with generators $X^{IJ}$, $I, J = 1, \ldots, 8$. We then identify the $SO(3)$ generators with $X^{IJ}$ for $I, J = 1, \ldots, 3$ and $SO(5)$ with $X^{IJ}$ for $I, J = 4, \ldots, 8$. The eight non-compact generators of $Sp(1, 2)$ can be obtained by taking combinations of $Y^A$'s which commute with the $SU(2)$ gauge group. The latter has three generators obtained by looking for the combinations of $SO(9)$ generators that commute with $SO(3) \times SO(5)$ mentioned above. We find the following gauge generators:
\begin{itemize}
    \item $\text{Sp}(1,2)$:
        \begin{align*}
            Q_1 &= \sqrt{2} c_1, \quad Q_2 = -\sqrt{2} c_2, \quad Q_3 = \sqrt{2} c_3, \quad Q_4 = \sqrt{2} c_4, \quad Q_5 = -\sqrt{2} c_5, \\
            Q_6 &= \sqrt{2} c_6, \quad Q_7 = \sqrt{2} c_7, \quad Q_8 = -\sqrt{2} c_8, \quad Q_9 = \sqrt{2} c_9, \quad Q_{10} = -\sqrt{2} c_{10}, \\
            Q_{11} &= -c_{21} - c_{52}, \quad Q_{12} = c_{51} - c_{35}, \quad Q_{13} = c_{50} + c_{36}, \\
            Q_{14} &= Y_1 + Y_{10}, \quad Q_{15} = Y_2 - Y_9, \quad Q_{16} = Y_3 + Y_{13}, \\
            Q_{17} &= Y_4 + Y_{16}, \quad Q_{18} = Y_5 - Y_{11}, \quad Q_{19} = Y_6 - Y_{15}, \\
            Q_{20} &= Y_7 + Y_{14}, \quad Q_{21} = Y_8 - Y_{12}.
        \end{align*}

    \item $\text{SU}(2)$:
        \begin{align*}
            K_1 &= \frac{1}{2} (c_{52} - c_{21}), \quad K_2 = -\frac{1}{2} (c_{35} + c_{51}), \quad K_3 = \frac{1}{2} (c_{36} - c_{50}).
        \end{align*}
\end{itemize}

With these generators and (A.3), we can compute the T-tensors

\begin{align*}
    T_{IJ,KL} &= V_{IJ,\alpha} V_{KL,\beta} \delta_{\alpha\beta}^{SO(p)} - V_{IJ,\alpha} V_{KL,\beta} \delta_{\alpha\beta}^{SO(9-p)}, \\
    T_{IJ,A} &= V_{IJ,\alpha} V_{A,\beta} \eta_{\alpha\beta}^{G_1} - K V_{IJ,\alpha} V_{A,\beta} \eta_{\alpha\beta}^{G_2},
\end{align*}

for compact gaugings and

\begin{align*}
    T_{IJ,KL} &= V_{IJ,\alpha} V_{KL,\beta} \eta_{\alpha\beta}^{G_1} - K V_{IJ,\alpha} V_{KL,\beta} \eta_{\alpha\beta}^{G_2}, \\
    T_{IJ,A} &= V_{IJ,\alpha} V_{A,\beta} \eta_{\alpha\beta}^{G_1} - K V_{IJ,\alpha} V_{A,\beta} \eta_{\alpha\beta}^{G_2},
\end{align*}

for non-compact gaugings with $K$ being $\frac{1}{6}$ and 12 for $G_1 \times G_2 = G_2(-14) \times SL(2)$ and $Sp(1,2) \times SU(2)$, respectively. We have used summation convention over gauge indices $\alpha$, $\beta$ with the notation $\delta^{G_0}$ and $\eta^{G_0}$ meaning that the summation is restricted to the $G_0$ generators.

References

[1] H. Nicolai and H. Samtleben, “Maximal gauged supergravity in three dimensions”, Phys. Rev. Lett. 86 (2001) 1686-1689, arXiv: hep-th/0010076.

[2] H. Nicolai and H. Samtleben, “Compact and noncompact gauged maximal supergravities in three dimensions”, JHEP 0104 (2001) 022, arXiv: hep-th/0103032.

[3] T. Fischbacher, H. Nicolai and H. Samtleben, “Non-semisimple and Complex Gaugings of $N = 16$ Supergravity”, Commun.Math.Phys. 249 (2004) 475-496, arXiv: hep-th/0306276.
[4] H. Nicolai and H. Samtleben, “$N = 8$ matter coupled $\text{AdS}_3$ supergravities”, Phys. Lett. B514 (2001) 165-172, arXiv: hep-th/0106153.

[5] Bernard de Wit, Ivan Herger and Henning Samtleben, “Gauged Locally Supersymmetric $D = 3$ Nonlinear Sigma Models”, Nucl. Phys. B671 (2003) 175-216, arXiv: hep-th/0307006.

[6] Bernard de Wit, A. K. Tollsten and H. Nicolai, “Locally supersymmetric $D = 3$ nonlinear sigma models”, Nucl. Phys. B392 (1993) 3-38, arXiv: hep-th/9208074.

[7] T. Fischbacher, “Some stationary points of gauged $N = 16$ $D = 3$ supergravity”, Nucl. Phys. B638 (2002) 207-219, arXiv: hep-th/0201030.

[8] T. Fischbacher, H. Nicolai and H. Samtleben, “Vacua of Maximal Gauged $D = 3$ Supergravities”, Class. Quant. Grav. 19 (2002) 5297-5334, arXiv: hep-th/0207206.

[9] Edi Gava, Parinya Karndumri and K. S. Narain, “$\text{AdS}_3$ Vacua and RG Flows in Three Dimensional Gauged Supergravities”, JHEP 04 (2010) 117, arXiv: 1002.3760.

[10] M. Berg and H. Samtleben, “An exact holographic RG Flow between 2d Conformal Field Theories”, JHEP 05 (2002) 006, arXiv: hep-th/0112154.

[11] J. M. Maldacena, “The large $N$ limit of superconformal field theories and supergravity”, Adv. Theor. Math. Phys. 2 (1998) 231-252, arXiv: hep-th/9711200.

[12] N. S. Deger, “Renormalization group flows from $D = 3$, $N = 2$ matter coupled gauged supergravities”, JHEP 0211 (2002) 025. arXiv: hep-th/0209188.

[13] Hitoshi Nishino and Subhash Rajpoot, “Topological Gauging of $N = 16$ Supergravity in Three-Dimensions”, Phys. Rev. D67 (2003) 025009, arXiv: hep-th/0209106.

[14] F. Bernardoni, S. L. Cacciatori, B. L. Cerchiai and A. Scotti, “Mapping the geometry of the $F_4$ group”, Adv. Theor. Math. Phys. Volume 12, Number 4 (2008), 889-994, arXiv: 07053978.

[15] Sergio L. Cacciatori and B. L. Cerchiai, “Exceptional groups, symmetric spaces and applications”, arXiv: 0906.0121 [math-ph].

[16] Stephen Wolfram, “The Mathematica Book”, 5th ed. Wolfram Media, 2003.

[17] E. S. Fradkin and Y. Ya. Linetsky, “Results of the classification of superconformal algebras in two dimensions”, Phys. Lett. B282 (1992) 352-356, arXiv: hep-th/9203045.

[18] N. P. Warner, “Some New Extrema of the Scalar Potential of Gauged $N = 8$ Supergravity”, Phys. Lett. B128 (1983) 169.
[19] Wolfgang M"uck, “Correlation functions in holographic renormalization group flows”, Nuclear Physics B 620 [FS] (2002) 477-500.

[20] Igor R. Klebanov and Edward Witten, “AdS/CFT Correspondence and Symmetry Breaking”, Nucl. Phys. B 556 (1999) 46-51. arXiv: hep-th/9905104.

[21] S. L. Cacciatori, B. L. Cerchiai, A. Della Vedova, G. Ortenzi and A. Scotti, “Euler angles for $G_2$”, J. Math. Phys. 46 (2005) 083512, arXiv: hep-th/0503106.

[22] Murat G"unaydin and Sergei V. Ketov, “Seven-sphere and the exceptional $N = 7$ and $N = 8$ superconformal algebras”, Nucl. Phys. B467 (1996) 215-246, arXiv: hep-th/9601072.

[23] H. Nicolai and H. Samtleben, “Chern-Simons vs Yang-Mills gaugings in three dimensions”, Nucl. Phys. B 638 (2002) 207-219 , arXiv: hep-th/0303213.