Amina Cherifi Hadjiat · Azzeddine Lansari

Surjectivity of certain adjoint operators and applications

1 Introduction

The ideals of finite codimension in Lie algebras of vector fields have recently received a lot of attention. Some authors such as Pursel and Shanks [9], by studying the invertibility of the Lie bracket \([X, Y] = \text{ad}_X(Y)\) which is an infinitesimal generator of an one-parameter group \(t, \gamma_t = (\exp tX)^*\), in Lie algebras containing a germ of vector fields \(X\) do not vanish at the origin \(O\), have treated the finite-codimensional ideals of these algebras. This result has been prolonged in the Banach-Lie algebras of vector fields infinitely flat at 0 containing germs which vanish at the origin of the form \(X_0 = X^+_0 + X^-_0 + Z_0\), where \(X^-\) (respectively, \(X^+\)) a symmetric matrix having eigenvalues \(\lambda < 0\) (respectively, \(\lambda > 0\)) and \(Z_0\) are germs infinitely flat at the origin. This sub-algebra admits a hyperbolic structure for the diffeomorphism \(\psi_{t\ast} = (\exp \cdot tY_0)\). In a second step, we will show that the infinitesimal generator \(\text{ad}_{-X}\) is an epimorphism of this admissible Lie sub-algebra \(U\). We then deduce, by our fundamental lemma, that \(U = E\).

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• The properties of the injectivity of the exponential function of a vector field have given rise to the existence of the Fourier series [11].

• The properties of the surjectivity of the directional derivative of the exponential function have given rise to the existence of the inversibility of the exponential function through the Nash–Moser theorem where positive results were obtained first in a Fréchet space and then in a hyperbolic type Fréchet space by integrating the diffeomorphisms in the smooth flows [12].

• Boris Kolev in [7] studied the particular case of a Lie–Poisson canonical structure.
• M. BENALILI in [1] has studied suitable spectral properties of the adjoint operators induced by appropriate perturbations of some hyperbolic linear vector fields of the form $Y_0 = X^+ + X^- + Z_0$, where $Z_0$ is $k$-flat in the unit ball.

This paper is an extension on the basis of these works where we will first study the sub-algebra $U$ of the Lie–Fréchet space $E$, containing vector fields of the form $Y_0 = X^+ + X^- + Z_0$, such as $X_0(x, y) = A(x, y) = \left( A^-(x), A^+(y) \right)$, with $A^-$ (respectively, $A^+$) a symmetric matrix having eigenvalues $\lambda < 0$ (respectively, $\lambda > 0$) and $Z_0$ are germs infinitely flat at the origin. This sub-algebra admits a hyperbolic structure for the diffeomorphism $\psi_{Y_0} = \exp(tY_0)$. In a second step, we will show that the infinitesimal generator $ad_{-X}$ is an epimorphism of this admissible Lie sub-algebra $U$. We then deduce, by our fundamental lemma, that $U = E$.

**Part I: Admissible hyperbolic-type algebra**

### 2 Definitions

#### 2.1 Fréchet space

Let $\mathbb{R}^a$ be the Euclidean space provided with the scalar product $(.,.)$ and $\| . \|$ the norm induced by this scalar product.

**(a)** Let $E$ be the space of vector fields $X$ of class $C^\infty$ on $\mathbb{R}^a$, satisfying:

$$\forall r \in \mathbb{N}, \exists M_r > 0 \text{ such that:}$$

$$\forall x \in \mathbb{R}^a, \exists M_r > 0 \text{ such that:}$$

$$\forall k \in \mathbb{N}$$

we have $\| D^\alpha X(x) \| (1 + \| x \|^2)^{k/2} \leq M_r$.

We define on $E$ a graduation of seminorms:

$$\| X \|_r = \sup_{x \in \mathbb{R}^a} \max_{k + |\alpha| \leq r} \| D^\alpha X(x) \| (1 + \| x \|^2)^{k/2}$$

so $(E, \| \cdot \|_r)$ is called a Fréchet space [6].

**(b)** Let $G$ the Schwartz space, which is the vector space of class $C^\infty$ functions on $\mathbb{R}^a$ satisfying:

$$G = \{ f \in C^\infty(\mathbb{R}^a)/Np \geq 0, \forall x \in \mathbb{R}^a, \exists C_p > 0 \text{ satisfying } \| D^\alpha f(x) \| (1 + \| x \|^2)^p \leq C_p \}.$$  

$G$ is a space where the Fourier transform exists as well as its inverse.

We define on $G$ a graduation of seminorms as follows:

$$\| f \|_r = \sup_{x \in \mathbb{R}^a} \max_{k + |\alpha| \leq r} \| D^\alpha f(x) \| (1 + \| x \|^2)^{k/2}$$

so $(G, \| \cdot \|_r)$ is called a Fréchet space [6].

#### 2.2 Smooth flow

**(a) Adjoint diffeomorphisms**

**Definition 2.1** For all $X, Y \in E$ such that $Y = \sum_{i=1}^n u_i(x) \frac{\partial}{\partial x_i}$, where $u_i \in C^\infty(\mathbb{R}^a)$, we define the adjoint diffeomorphisms $\phi_i^*$ and $(\phi_i)_*$ by

$$\begin{cases}
(\phi_i)^*Y = (\exp tX)^* \cdot Y = ((D\phi_i)^{-1} \cdot Y) \circ \phi_i = \sum_{j=1}^n e^{-tX} \cdot u_j(e^{tX} \cdot x) \frac{\partial}{\partial x_j} \\
(\phi_i)_*Y = (\exp tX)_* \cdot Y = (D\phi_i) \cdot Y \circ (\phi_i)^{-1} = \sum_{j=1}^n e^{tX} \cdot u_j(e^{-tX} \cdot x) \frac{\partial}{\partial x_j}
\end{cases}$$
**Properties.** For all $X \in E$, we associate the $X$–flow $\phi_t = \text{exp} X$, so (i) $ad_X$ (respectively, $ad_{-X}$) is an infinitesimal generator of the one-parameter group $\phi_t^*$ (respectively, $(\phi_t)_*$) on $E$; that is, $\phi_t^*$ is a solution of the following dynamic system:

\[
\begin{align*}
\frac{d}{dt} (\phi_t^*) \ Y = (\phi_t^*) \cdot ad_X (Y) = \phi_t^* \ [X, Y] \\
(\phi_0^*) Y = Y
\end{align*}
\]

and, respectively, $(\phi_t)_*$ is the solution of

\[
\begin{align*}
\frac{d}{dt} (\phi_t)_* \ Y = (\phi_t)_* \cdot ad_{-X} (Y) = (\phi_t)_* \ [Y, X] \\
(\phi_0)_* Y = Y
\end{align*}
\]

(ii) $(\phi_t)_* = \phi_{-t}^*$; $\phi_t^* = (\phi_{-t})_*$, $\forall t > 0$.

**Proof** (i) In fact,

\[
\frac{d}{dt} ((\phi_t)_* \ Y) (x) = \frac{d}{dt} D \left[ \phi_t (\phi_{-t} (x)) \right] \cdot Y (\phi_{-t} (x)) + D \phi_t (\phi_{-t} (x)) \cdot \frac{d}{dt} Y (\phi_{-t} (x))
\]

\[
= D_x X (\phi_t \circ \phi_{-t} (x)) \cdot D \phi_t (\phi_{-t} (x)) \cdot Y (\phi_{-t} (x))
\]

\[
- D \phi_t (\phi_{-t} (x)) \cdot D_y (Y (\phi_{-t} (x))) \cdot X \circ \phi_{-t} (x) \quad \text{where } Y = \phi_{-t} (x)
\]

\[
= D_x X (x) \left( D \phi_t \cdot Y \right) (\phi_{-t} (x)) - (D \phi_t \cdot X) (\phi_{-t} (x)) \cdot D_y Y (y)
\]

\[
= D_x X (x) \left( (\phi_t)_* \cdot Y \right) (x) - \left( (\phi_t)_* \cdot X \right) (x) \cdot D_y Y (y)
\]

\[
= (\phi_t)_* [Y, X] (x)
\]

\[
= (\phi_t)_* \ ad_{-X} (Y) (x).
\]

That is to say:

\[
\frac{d}{dt} (\phi_t)_* \ Y = (\phi_t)_* \ [Y, X];
\]

so,

\[
\frac{d}{dt} ((\phi_t)_* Y) (x) \mid_{t=0^+} = [Y, X] (x) = ad_{-X} (Y) (x),
\]

i.e. that $ad_{-X}$ is an infinitesimal generator of the one-parameter group $(\phi_t)_*$ on $E$, and by the same reasoning, we will have

\[
\frac{d}{dt} (\phi_t^*) \ Y = (\phi_t^*)^* \ [X, Y].
\]

(ii) Let us show that $(\phi_t)_* = \phi_{-t}^*$; we put $y = \phi_{-t} (x)$:

\[
\frac{d}{dt} ((\phi_t)_* \ Y) (x) = D_x X (\phi_t \circ \phi_{-t} (x)) \cdot D \phi_t (\phi_{-t} (x)) \cdot Y (\phi_{-t} (x))
\]

\[
- D \phi_t (\phi_{-t} (x)) \cdot D (Y (\phi_{-t} (x))) \cdot X \circ \phi_{-t} (x)
\]

\[
= D_x X (x) \left( (D \phi_{-t})^{-1} \cdot Y \right) (\phi_{-t} (x)) - \left( (D \phi_{-t})^{-1} \cdot X \right) (\phi_{-t} (x)) \cdot D_y Y (y)
\]

\[
= D_x X (x) \left( (\phi_{-t})^* \cdot Y \right) (x) - \left( (\phi_{-t})^* \cdot X \right) (x) \cdot D_y Y (y)
\]

\[
= (\phi_{-t})^* \ [Y, X] (x)
\]

\[
= (\phi_{-t})^* \ ad_{-X} (Y) (x).
\]

Now, according to Property (i):

\[
\frac{d}{dt} ((\phi_t)_* \ Y) (x) = (\phi_t)_* \ ad_{-X} (Y) (x),
\]

then $(\phi_t)_* = (\phi_{-t})^*$.

- The same reasoning can be applied in the case: $\phi_t^* = (\phi_{-t})_*$
(b) Smooth flow Let $X \in E$ and the $X$-flow $\phi_t = \exp tX$. We say that an adjoint flow $\phi_t^*$ decays tamely on $E$ of degree $r$ and base $b$ if

(i) $\phi_t^*$ preserves $E \forall t \geq 0$,
(ii) for any integer $k \geq b$ there is an integer $l_k = k + r$ and a strictly positive, continuous and decreasing function $C_k(t)$ defined on $[0, \infty)$ satisfying:

$$\|\phi_t^* Z\|_k \leq C_k(t) \|Z\|_k \quad (\forall t \geq 0; \forall Z \in E),$$

and the improper integral:

$$\int_0^\infty C_k(t) dt \quad \text{converges } \forall k \geq b.$$

Alternately, $\phi_t^*$ can be replaced by $(\phi_t)_s = \phi_{s-t}^*$ according to the asymptotic behaviour of $\phi_t$ [12].

2.3 Fréchet space with hyperbolic structure

(a) Admissible algebra

Definition 2.2 (Admissible algebra) Let $U$ be a Lie–Fréchet sub-algebra of $E$. $U$ is said to be admissible with respect to the vector field $Y_0 = X_0 + Z_0$ if and only if it satisfies the following conditions:

(i) Let $X_0$ be a vector field on $E$ such that

$$X_0 - A \begin{pmatrix} x \\ y \end{pmatrix} = A - A^+ \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A^- y \\ A^+ y \end{pmatrix}$$

$$\forall (x, y) \in K \subset \mathbb{R}^k \times \mathbb{R}^l = \mathbb{R}^n$$

$A^-$ (respectively, $A^+$) is a real symmetric matrix of type $(k \times k)$ (respectively, $l \times l$) with $k + l = n$, having strictly negative eigenvalues (respectively, strictly positive).

The matrix $A^-$ satisfies: $\forall m \geq 2, \forall i = 1, k : \exists a_R, a_L > 0, \rho_1 > 1$, such that

$$\begin{cases} -a_L \leq \lambda_i \leq -a_R < 0 \\ m \cdot a_R - a_L \geq \rho_1 > 1 \end{cases} \quad (I).$$

And, respectively, for $A^+$:

$\forall m \geq 2, \forall i = 1, l : \exists b_R, b_L > 0, \rho_2 > 1$ such that

$$\begin{cases} 0 < b_L \leq \lambda_i \leq b_R \\ m \cdot b_L - b_R \geq \rho_2 > 1 \end{cases} \quad (II).$$

We adopt the following notation for the rest of this paper:

$$X_0 = X_0^- + X_0^+; \quad X_0(x, y) = (X_0^-(x), X_0^+(y)) \in \mathbb{R}^k \times \mathbb{R}^l,$$

from which we have

$$\phi_t(x, y) = (\exp tX_0)(x, y) = ((\exp tX_0^-)(x), (\exp tX_0^+)(y)); \forall (x, y) \in \mathbb{R}^k \times \mathbb{R}^l.$$
\(Z_0^-(\text{respectively}, Z_0^+)\) is an infinitely flat germ at the origin, i.e. satisfying the following estimate:

\[
\forall x \in \mathbb{R}^n, \left\{ \forall \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n \right\} \quad \left| \alpha \right| = \sum_{i=1}^n \alpha_i
\]

There exists \(M_\alpha > 0\) such that

\[
\|D^\alpha Z_0^-(x)\| \leq M_\alpha \cdot \|x\|^k_1, \quad \forall k_1 > 1 \quad \forall x \in \mathbb{R}^k,
\]

(resp)

\[
\|D^\alpha Z_0^+(y)\| \leq M_\alpha \cdot \|y\|^k_2, \quad \forall k_2 > 1 \quad \forall y \in \mathbb{R}^l.
\]

(iii) \(\forall Y \in U; \exists Y^i \in U_i (i = 1, 2)\) such that \(Y = Y^1 + Y^2\) and \(\forall(x, y) \in \mathbb{R}^n = \Omega_1 \cup \Omega_2\), we have

\[
Y(x, y) = (Y^1(x), Y^2(y)) \in \Omega_1 \cup \Omega_2;
\]

\[
(expt Y_0) \ast Y(x, y) = (expt Y_0^1) \ast Y^1(x), (expt Y_0^2) \ast Y^2(y) \in \Omega_1 \cup \Omega_2,
\]

where

\[
\begin{align*}
\Omega_1 &= \{(x, y) \in \mathbb{R}^k \times \mathbb{R}^l / Y(x, y) = (Y^1(x), 0)\} \\
\Omega_2 &= \{(x, y) \in \mathbb{R}^k \times \mathbb{R}^l / Y(x, y) = (0, Y^2(y))\}.
\end{align*}
\]

(4i) There exists \(\delta > 0\) such that \(U_i^\delta = \{Y^i \in U_i / supp Y^i \subset \Omega_i^\delta\}\) with \(\Omega_i^\delta = \Omega_i + B_\delta\) and \(B_\delta \subset \Omega_j, i \neq j\).

(5i) \(U\) is locally closed, i.e. for any sequence \((X^i_m)_{m \geq 0}\) in \(U_i\) such that \(supp X^i_m \subset \Omega_i\) converges to \(X^i\) in \(\Omega_i\).

(b) Hyperbolic structure

**Definition 2.3** Let \(E\) be a Fréchet space of vector fields, \(X \in E\) and the \(X\)--flow \(\psi_t = exptX. E\) has a tame hyperbolic structure for \((\psi_t)_t\) if and only if \(\exists \delta > 0 /\)

- (i) \(\psi_{t\alpha}\) is invariant on \(E; \forall t \in \mathbb{R},\)
- (ii) \((\psi_t)_t\) decays on \(E_1^\delta; \forall t \geq 0,\)
- (iii) \((\psi_{-t})_t\) decays on \(E_2^\delta; \forall t \geq 0,\)

where \(E_1^\delta = \{Z \in E / supp Z \subset \Omega_1^\delta\}\) with \(\Omega_1^\delta = \Omega_1 + B_\delta\), such that \(B_\delta \subset \Omega_j, i \neq j\) and \(E = E_1 \oplus E_2 = E_1^\delta + E_2^\delta\) \([12]\).

### 3 Estimations

To show that \(U\) has a hyperbolic structure, we need some estimates.

3.1 Estimation of \(expt Y_0^+\) and \(expt Y_0^-\)

(a) **Estimation of \(expt X_0^+\) and \(expt X_0^-\)**

**Lemma 3.1** Let \(X_0^+ \in U_2\) (respectively, \(X_0^- \in U_1\)) Then \(X_0^+\)--flow has for estimate, for all \(t \geq 0:\)

\[
\begin{align*}
\| y \| e^{bLt} &\leq \| (expt X_0^+) (y) \| \leq \| y \| e^{bLt} \\
\| y \| e^{-bLt} &\leq \| (expt -tX_0^+) (y) \| \leq \| y \| e^{-bLt} \quad (\forall y \in \mathbb{R}^l),
\end{align*}
\]

respectively, for the \(X_0^-\)--flow:

\[
\begin{align*}
\| x \| e^{-aLt} &\leq \| (expt X_0^-) (x) \| \leq \| x \| e^{-aLt} \\
\| x \| e^{aLt} &\leq \| (expt -tX_0^-) (x) \| \leq \| x \| e^{aLt} \quad (\forall x \in \mathbb{R}^k).
\end{align*}
\]
Proof We put $\phi_i^+(y) = (\exp tX_0^+)(y)$, from which

$$\frac{1}{2}\frac{d}{dt} \| \phi_i^+(y) \|^2 = \langle \phi_i^+(y), X_0^+ \circ \phi_i^+(y) \rangle.$$ 

According to Wintner theorem [10], $\langle y, A^+ y \rangle \leq d \| y \|^2$, $y \in \mathbb{R}^l$, where the constant $d$ is the largest eigenvalue of the symmetric matrix $A^+$. We get

$$\inf_{i=1,\ldots,d} \lambda_i \cdot \| \phi_i^+(y) \|^2 \leq \frac{1}{2} \frac{d}{dt} \| \phi_i^+(y) \|^2 \leq \sup_{i=1,\ldots,d} \lambda_i \cdot \| \phi_i^+(y) \|^2.$$ 

We finally have

$$\| y \| e^{bLt} \leq \| \phi_i^+(y) \| \leq \| y \| e^{bLt}.$$ 

The same should be applicable to $\phi_i^t$ by replacing $t$ by $(-t)$:

$$\| y \| e^{-bLt} \leq \| \phi_i^t(y) \| \leq \| y \| e^{-bLt}.$$ 

On the other hand, applying the same reasoning to $X_0^-$ shows that

$$\begin{cases} 
\| x \| e^{-aLt} \leq \| \phi_i^t(x) \| \leq \| x \| e^{-aLt} \\
\| x \| e^{aLt} \leq \| \phi_i^t(x) \| \leq \| x \| e^{aLt}
\end{cases}$$

\(\Box\)

(b) Estimation of $\exp tY_0^-$ and $\exp tY_0^+$

We put $Y_0^- = X_0^- + Z_0^-$ (respectively, $Y_0^+ = X_0^+ + Z_0^+$). Then the vector form of $Y_0^- - flow$ (respectively, $Y_0^+ - flow$) will be

$$\psi_i^-(x) = \exp t(X_0^- + Z_0^-)(x) = \phi_i^-(x) + \int_0^t \phi_i^{-s} \circ Z_0^-(\psi_i^-(x))ds$$

(respectively,

$$\psi_i^+(y) = \exp t(X_0^+ + Z_0^+)(y) = \phi_i^+(y) + \int_0^t \phi_i^{s+} \circ Z_0^+(\psi_i^+(y))ds.$$ 

Solution of the following dynamic system:

$$\begin{cases} 
\frac{d}{dt} \psi_i^-(x) = X_0^- \circ \psi_i^-(x) + Z_0^- \circ \psi_i^-(x) \\
\psi_0^-(x) = x.
\end{cases}$$

(respectively,

$$\begin{cases} 
\frac{d}{dt} \psi_i^+(y) = X_0^+ \circ \psi_i^+(y) + Z_0^+ \circ \psi_i^+(y) \\
\psi_0^+(y) = y.
\end{cases}$$

Lemma 3.2 Let $Y_0^- \in U_1$ (respectively, $Y_0^+ \in U_2$) Then the vector fields $Y_0^-$ (respectively, $Y_0^+$) is complete, and the flow $\psi_i^-$ (respectively, $\psi_i^+$) would satisfy the following estimates:

$$\begin{cases} 
\| x \| e^{-aLt} \leq \| \exp tY_0^- \| (x) \leq \| x \| e^{-aLt} ; \forall t \geq 0 \\
\| x \| e^{aLt} \leq \| \exp tY_0^- \| (x) \leq \| x \| e^{aLt} ; \forall t \geq 0
\end{cases}$$

(respectively,

$$\begin{cases} 
\| y \| e^{-bLt} \leq \| \exp tY_0^+ \| (y) \leq \| y \| e^{-bLt} ; \forall t \geq 0 \\
\| y \| e^{bLt} \leq \| \exp tY_0^+ \| (y) \leq \| y \| e^{bLt} ; \forall t \geq 0
\end{cases}$$

1 Springer
Proof Consider the equation below:

\[
\frac{1}{2} \frac{d}{dt} \| \psi_t^- (x) \|^2 = (\psi_t^- (x) , X_0^- (\psi_t^- (x)) + Z_0^- (\psi_t^- (x))).
\]

After taking \( \zeta = \| \psi_t^- (x) \| \), we have

\[
\begin{aligned}
- a_L \zeta^2 - M_0 \zeta^{k_1 + 1} &\leq \frac{1}{2} \frac{d}{dt} \zeta^2 \leq - a_R \zeta^2 + M_0 \zeta^{k_1 + 1} \\
\zeta (0) &= \| x \|.
\end{aligned}
\]

Then

\[
- a_L \zeta^{1-k_1} - M_0 \zeta^{k_1} \leq \zeta^{-k_1} \frac{d}{dt} \zeta \leq - a_R \zeta^{1-k_1} + M_0.
\]

If we put \( z = \zeta^{1-k_1} \), we will have \( dz = (1-k_1) \zeta^{-k_1} d\zeta \) and the system becomes

\[
- a_L z - M_0 \leq \frac{1}{1-k_1} \frac{d}{dt} z \leq - a_R z + M_0
\]

which has as solutions

\[
b_1 e^{-a_R (1-k_1)t} + \frac{M_0}{a_L} \leq z \leq b_2 e^{-a_R (1-k_1)t} - \frac{M_0}{a_R}
\]

such as \( b_1, b_2 \), two functions of \( x \) and as \( a_L, a_R > 0 \), then

\[
b_1 e^{-a_R (1-k_1)t} \leq z \leq b_2 e^{-a_R (1-k_1)t}.
\]

As \( 1-k_1 < 0 \), then

\[
b_3 e^{-a_L t} \leq \zeta \leq b_4 e^{-a_L t},
\]

where \( b_3, b_4 \) also two functions of \( x \), from which

\[
\| x \| \cdot e^{-a_L t} \leq \| \psi_t^- (x) \| \leq \| x \| \cdot e^{-a_R t}.
\]

Similarly, we will have

\[
\| x \| \cdot e^{a_L t} \leq \| \psi_t^- (x) \| \leq \| x \| \cdot e^{a_R t}.
\]

And by a similar reasoning, we come to

\[
\begin{aligned}
\| y \| \cdot e^{b_L t} \leq \| \psi_t^+ (y) \| \leq \| y \| \cdot e^{b_R t} \\
\| y \| \cdot e^{-b_L t} \leq \| \psi_t^- (y) \| \leq \| y \| \cdot e^{-b_R t}; \ \forall t \geq 0
\end{aligned}
\]

\( \square \)

(c) The final estimate of the \( l' - \)th derivative of \( Y_0 - \)flow

(i) Estimation of the first derivative of the \( Y_0 - \)flow:

We denote \( \eta^- (t, x, v_1) = D \psi_t^- (x) v_1 \) and \( \eta^+ (t, y, v_2) = D \psi_t^+ (y) v_2 \), where \( v_1, v_2 \in \mathbb{R}^k \). The first derivative with respect to \( x \) of \( Y_0^+ \)-flow (respectively, \( Y_0^- \)-flow) is a solution of the dynamic system:

\[
\begin{aligned}
\frac{d}{dt}\eta^- (t, x, v_1) &= (D_{\xi_1} X_0^- + D_{\xi_1} Z_0^-) \cdot \eta^- (t, x, v_1) \\
\eta^- (0, x, v_1) &= v_1
\end{aligned}
\]

with \( \xi_1 = \psi_t^- (x) \) and, respectively,

\[
\begin{aligned}
\frac{d}{dt}\eta^+ (t, y, v_2) &= (D_{\xi_2} X_0^+ + D_{\xi_2} Z_0^+) \cdot \eta^+ (t, y, v_2) \\
\eta^+ (0, y, v_2) &= v_2
\end{aligned}
\]

with \( \xi_2 = \psi_t^+ (y) \).
Lemma 3.3 The derivative of $Y_0^-$-flow (respectively, of $Y_0^+$-flow) has the following estimates for all $t \geq 0$:

\[
\begin{cases}
\| e^{-a_Lt} \| \leq \| (D \exp t Y_0^-) (x) \| \leq e^{-a_R t} \\
\| e^{a_R t} \| \leq \| (D \exp -t Y_0^-) (x) \| \leq e^{a_L^t}.
\end{cases}
\]

(respectively,

\[
\begin{cases}
\| e^{b_L t} \| \leq \| (D \exp t Y_0^+) (y) \| \leq e^{b_R t} \\
\| e^{-b_R t} \| \leq \| (D \exp -t Y_0^+) (y) \| \leq e^{-b_L t}.
\end{cases}
\]

Proof Let $z_1 = \| \eta^- (t, x, v) \|$, and consider the following equation:

\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{1}{2} \frac{d}{dt} z_1^2 = \frac{1}{2} \frac{d}{dt} \| \eta^- (t, x, v_1) \|^2 \\
= (\eta^- (t, x, v_1), (D_{\xi_i} X_0^- + D_{\xi_i} Z_0^-) \cdot \eta^- (t, x, v_1)) \\
z_1(0) = \| v_1 \|.
\end{array} \right.
\end{align*}
\]

We will have

\[
\left\{ \begin{array}{l}
z_1^2 (-a_L - M_1 e^{-a_L t}) \leq \frac{1}{2} \frac{d}{dt} z_1^2 \leq z_1^2 (-a_R + M_1 e^{-a_R t}) \\
z_1(0) = \| v_1 \|.
\end{array} \right.
\]

Then

\[
(-a_L - M_1 e^{-a_L t}) \leq \frac{d z_1}{z_1} \leq (-a_R + M_1 e^{-a_R t}) d t.
\]

We will then have

\[
-a_L t + M_1 e^{-a_L t} k_1 + b_6 \leq ln z_1 \leq -a_R t - M_1 e^{-a_R t} k_1 + b_5,
\]

where $b_5$, $b_6$ are two functions depending on $x$.

As $-a_L t + b_6 \leq ln z_1 \leq -a_R t + b_5$, then $v_1 \cdot e^{-a_L t} \leq z_1 \leq \| v_1 \| \cdot e^{-a_R t}$. Taking $\| v_1 \| = 1$, we get

\[
e^{-a_L t} \leq \| D \psi^- (x) \| \leq e^{-a_R t} \quad \forall x \in \mathbb{R}^k, \forall t > 0.
\]

We can easily deduce

\[
e^{a_R t} \leq \| D \psi^- (x) \| \leq e^{a_L t}.
\]

And by a similar reasoning, we come to

\[
\left\{ \begin{array}{l}
\| e^{b_L t} \| \leq \| D \psi^+ (x) \| \leq e^{b_R t} \\
\| e^{-b_R t} \| \leq \| D \psi^+ (y) \| \leq e^{-b_L t}.
\end{array} \right.
\]

\[\square\]

(ii) The final estimate of the $l'$-th derivative of $Y_0^-$-flow

Lemma 3.4 The $l'$-th derivative of $Y_0^-$-flow (resp of $Y_0^+$-flow) has the following estimates for all $t \geq 0, \forall l' \in \mathbb{N}$ :

\[
\left\{ \begin{array}{l}
\| D^{l'} \psi^- (x) \| \leq M_1^{l'} e^{-a_R t} \\
\| D^{l'} \psi^- (x) \| \leq M_1^{l'} e^{a_L t} \quad \forall x \in \mathbb{R}^k
\end{array} \right.
\]

and

\[
\left\{ \begin{array}{l}
\| D^{l'} \psi^+ (y) \| \leq M_1^{l'} e^{b_R t} \\
\| D^{l'} \psi^+ (y) \| \leq M_1^{l'} e^{-b_L t} \quad \forall y \in \mathbb{R}^l.
\end{array} \right.
\]

Proof We can generalize the previous result using a recurrence reasoning as follows:

(1) For $j = 0, 1$, the result is verified.
(2) So, let us suppose the statement true until \((j - 1)\) order, i.e.

\[
\| D^{j'} \psi^+_t (x) \| \leq M'_j e^{-\alpha R t}; \quad \forall t' \leq j - 1.
\]

(3) Let us show that this last property is true at the order \(j\). Let the \(j^{th}\) derivative of \(Y_0^-\) flow

\[
\eta^+_j (t, x, v) = D^j \psi^+_t (x) v^j, \quad \forall v \in \mathbb{R}^n
\]

solution of the following dynamic system:

\[
\begin{aligned}
\frac{d}{dt} \eta^+_j (t, x, v) &= D_{x_1} Y_0^- \cdot \eta^+_j (t, x, v) + G^+_j (t, x, v) \\
\eta^+_j (0, x, v, \ldots, v) &= v
\end{aligned}
\]

with

\[
\begin{aligned}
G^+_j (t, x, v) &= \sum_{k'=2}^j D_{x_1}^k Z_0^- (\xi_1) \sum_{i_1 + i_2 + \cdots + i_{k'} = j} D^i \psi^+_t (x) \psi^+_0 (x) v^i \psi^+_0 (x) v^{i_k} \\
\xi_1 &= \psi^+_t (x).
\end{aligned}
\]

Using the so-called resolvent transform method discussed, for example, in [4], Y. Domar studied the closed ideals in some Banach algebras [5]), we deduce that

\[
\eta^+_j (t, x, v, \ldots, v) = D^j \psi^+_t (x) v + \int_0^t D \psi^+_s (\psi^- (x) \psi^- (x) (s, x, v) ds.
\]

The integral in the preceding expression is well defined at the point \(s = 0\) because

\[
\lim_{s \to 0^+} D \psi^+_s (\psi^- (x)) = D \psi^+_0 (x)
\]

and there are constants \(A_j > 0\) such that

\[
\lim_{s \to 0^+} G^+_j (s, x, v) = \sum_{k'=2}^m A_{k'} D_{x_1}^k Z_0^- (\xi_1) v^{i_k}.
\]

Since \(v\) is arbitrary, we can choose \(\| v \| = 1\), and put

\[
I_j = \int_0^t \| D \psi^+_s (\psi^- (x)) \| \cdot \| G^+_j (s, x, v) \| ds.
\]

Let \(K \subseteq \mathbb{R}^k\) a compact set, so \(I_j\) converges uniformly when \(t\) tends to \(+\infty\) for all \(x \in K\). As

\[
\| D^k Z^- (x) \| \leq M_k, \quad \forall k \geq 1, \quad \text{and} \quad \forall x \in \mathbb{R}^k
\]

and using the recurrence hypotheses, we arrive at

\[
I_j \leq \sum_{k'=2}^j M_{k'} \sum_{i_1 + i_2 + \cdots + i_{k'} = j} M_{i_1} \cdots M_{i_k} \int_0^t e^{-(t-s+k')a_R} ds \leq M'_j e^{-\alpha R t}.
\]

The integral \(I_j\) is then uniformly convergent with respect to \(x\) when \(t\) tends to \(+\infty\). Consequently, there are constants \(M''_j = \sup (1, M'_j) > 0\) such that

\[
\| \eta^+_j \| \leq \| D \psi^+_t (x) v \| + \int_0^{+\infty} \| D \psi^+_s (\psi^- (x)) \| \cdot \| G^+_j (s, x, v) \| ds \leq M''_j e^{-\alpha R t}.
\]
(4) Conclusion:

\[ \| D^l \psi^+_t(x) \| \leq M''_l e^{-a_{l+1}t}, \forall l' \in \mathbb{N}; \quad \forall x \in \mathbb{R}^k. \]

By similar reasoning, we shall have

\[ \| D^l \psi^-_t(x) \| \leq M''_l e^{a_{l+1}t} \]

and

\[
\begin{cases}
\| D^l \psi^+_{t}(y) \| \leq M''_l e^{b_{l+1}t} \\
\| D^l \psi^-_{t}(y) \| \leq M''_l e^{-b_{l+1}t} \quad \forall y \in \mathbb{R}^l.
\end{cases}
\]

\[ \boxdot \]

3.2 Estimation of \((\text{expt} Y^+_0)_s\) and \((\text{expt} Y^-_0)_s\)

As \(\psi_t = \phi_t \circ f_t\), the estimate of \(\psi_t\) is deduced according to the estimates of \(\phi_t\) and \(f_t\).

(a) The absorbents \(N^+_1\) and \(N^+_2\)

A closed subset \(N^+_i\) is called positive absorbent for the flow \(\phi_t\), if and only if for any compact \(K_i\) in \(\Omega_i\), there exists \(t_{K_i} > 0\), such that \(\phi_t(K_i) \subset N_i\).

Example: Let \(K_i\) be a compact of \(\Omega_i\), \((i = 1, 2)\) Then \(\exists \delta > 0\), such that \(\delta = \min_{x \in K_i} \|x\|\), we can then have the following absorbents:

\[
N^+_1 = \left\{ (x, y) \in \mathbb{R}^k \times \mathbb{R}^l / Y(x, y) = (Y^1(x), 0) \text{ and } \|x\| \geq \delta \right\} \subset \Omega_1
\]

\[
N^+_2 = \left\{ (x, y) \in \mathbb{R}^k \times \mathbb{R}^l / Y(x, y) = (0, Y^2(y)) \text{ and } \|y\| \geq \delta \right\} \subset \Omega_2
\]

(b) Estimation of \((\text{expt} X^+_0)_s\) and \((\text{expt} X^-_0)_s\)

Lemma 3.5 The diffeomorphism \((\phi^-_*)\) (respectively, \((\phi^+_*)\)) decays tamely on \(U_1\) (respectively, on \(U_2\)) of degree 0 and base \(m_1\) (respectively, \(m_2\)), in other words:

(i) For every arbitrary positive constant \(\rho'_1\), there is a constant \(m_1 = \left[ r, \frac{a_r}{\rho'_1 a_r} \right] + \rho'_1 > 0 \text{ and } C_1 > 0 \text{ such that}

\[ \|((\text{expt} X^-_0)_s \cdot Y^1)_{|\Omega^1_1} \| \leq C_1 \cdot e^{-r \cdot \rho'_1 a_r} \cdot \|Y^1\|_{\Omega^1_{r+m_1}} \quad \forall Y^1 \in U_1. \]

(ii) For every arbitrary positive constant \(\rho'_2\), there is a constant \(m_2 = \left[ r, \frac{b_r}{\rho'_2 b_r} \right] + \rho'_2 > 0 \text{ and } C_2 > 0 \text{ such that}

\[ \|((\text{expt} X^+_0)_s \cdot Y^2)_{|\Omega^2_1} \| \leq C_2 \cdot e^{-r \cdot \rho'_2 b_r} \cdot \|Y^2\|_{\Omega^2_{r+m_2}} \quad \forall Y^2 \in U_2. \]

Proof (i) Consider the following diffeomorphism:

\[ (\phi^-_*) \cdot Y^1 = (D\phi^-_t \cdot Y^1) \circ (\phi^-_t)^{-1} \]

\[ (\phi^-_*) \cdot Y^1 = \sum_{j=1}^{k} e^{tX_0} \left[ u_j \left( e^{-tX_0} \cdot x \right) \right] \frac{\partial}{\partial x_j} \]

with

\[ Y^1 = \sum_{j=1}^{k} u_j(x) \frac{\partial}{\partial x_j} \in U_1. \]

For any integer \(r \geq 0\) and any multi-index \(\beta = (\beta_1, \ldots, \beta_k) \in \mathbb{N}^k\) of module \(|\beta| = \beta_1 + \ldots + \beta_k\), we obtain for \(x \in \mathbb{R}^k:\n
\[ D^\beta \phi^-_t (x) \cdot Y^1(x) = (D^\beta e^{tX_0} \cdot u_1(e^{-tX_0} \cdot x), \ldots, D^\beta e^{tX_0} \cdot u_k(e^{-tX_0} \cdot x)). \]
Let \( \rho \) be an arbitrary positive constant. Then there is \( m_1 = \left[ r, \frac{a_r}{a} \right] + \rho'_1 > 0 \), we will have
\[
||(\phi_{-r})_+ \cdot Y^2||_{r} \leq e^{-t(r_0 + a_r)} ||Y^2||_{r+1}, \forall r \in \mathbb{N}.
\]

Let \( \rho'_1 \) be an arbitrary positive constant. Then there is \( m_1 = \left[ r, \frac{a_r}{a} \right] + \rho'_1 > 0 \), we will have
\[
||(\phi_{-r})_+ \cdot Y^1||_{r} \leq e^{-t(r_0 - ra)} ||Y^1||_{r+1}, \forall r \in \mathbb{N}.
\]

From Lemma 3.1, \( ||x|| e^{aRt} \leq || \phi_{-r}(x) || \), there is a constant \( C_1 > 0 \) such that
\[
||(\phi_{-r})_+ \cdot Y^1||_{r} \leq C_1 \cdot e^{-t \rho'_1 a_k}, ||Y^1||_{r+1}.
\]

And as the integral \( \int_0^\infty e^{-t r_0 a_k} dt \) is convergent, \( (\phi_{-r})_+ \) decays tamely on \( U_1 \).

(ii) We have, respectively,
\[
\text{For any vector field } Y^2 \in U_2, \text{ with } Y^2 = \sum_{i=1}^l v_j(y) \frac{\partial}{\partial y_j}, \text{ we consider}
\]
\[
(\phi_{-r})_+ \cdot Y^2 = \sum_{j=1}^l e^{-t x_0^+} \left[ v_j(e^{t x_0^+} \cdot y) \right] \frac{\partial}{\partial y_j}.
\]

For any integer \( r \geq 0 \) and any multi-index \( \beta' = (\beta'_1, \ldots, \beta'_l) \in \mathbb{N}^l \) of module \( ||\beta'|| = \beta'_1 + \ldots + \beta'_l \), we obtain
\[
D^\beta_y (\phi_{-r})_+ \cdot Y^1(y) = (D^\beta_y e^{-t x_0^+} \cdot v_1(e^{t x_0^+} \cdot y), \ldots, D^\beta_y e^{-t x_0^+} \cdot v_l(e^{t x_0^+} \cdot y)).
\]

We will have
\[
||(D^\beta_y (\phi_{-r})_+ \cdot Y^2(y))|| \leq e^{-b_1 t} \sup_{j=1,\ldots,l} ||D^\beta_y v_j(e^{t x_0^+} \cdot y)||.
\]

Since \( ||\phi_{-r}^+(y)|| \leq ||y|| e^{b_1 t} \), and posing \( z' = e^{t x_0^+} \cdot y \),
\[
||(D^\beta_y (\phi_{-r})_+ \cdot Y^2(y))|| \leq e^{-b_1 t} \cdot e^{t ||\beta'|| b_1} \sup_{j=1,\ldots,l} ||D^\beta_y v_j(z')||,
\]
from which we obtain
\[
||(\phi_{-r})_+ \cdot Y^2||_{r} \leq e^{-t(b_1 + b_2)} ||Y^2||_{r} ||\phi_{-r}^+(z')||.
\]

Let \( \rho'_2 \) be an arbitrary positive constant. Then there is \( m_2 = \left[ r, \frac{b_2}{b} \right] + \rho'_2 > 0 \), we will have
\[
||(\phi_{-r})_+ \cdot Y^2||_{r} \leq e^{-t(b_2 + b)} ||Y^2||_{r} ||\phi_{-r}^+(z')|| \sup_{y \in \mathbb{R}^l} \left( \frac{1}{1 + ||\phi_{-r}^+(y)||^2 + m_2} \right).
\]

From Lemma 3.1, \( ||y|| e^{b_2 t} \leq || \phi_{-r}^+(y) || \), there is a constant \( C_2 > 0 \) such that
\[
||(\phi_{-r})_+ \cdot Y^2||_{r} \leq C_2 \cdot e^{-t \rho'_2 b_2} \cdot ||Y^2||_{r+m_2}.
\]

And as the integral \( \int_0^\infty e^{-t(\rho'_2 b_2)} dt \) is convergent, \( (\phi_{-r})_+ \) operates smoothly on \( U_2 \).
(b) Wave operators

Let

\[ \phi^{-}_t = \exp tX^{-}_0 \quad \text{and} \quad \psi^{-}_t = \exp tY^{-}_0 \]

( resp \( \phi^{+}_t = \exp tX^{+}_0 \) and \( \psi^{+}_t = \exp tY^{+}_0 \) )

and the diffeomorphism \( f^{-}_t (x) = (\phi^{-}_t \circ \psi^{-}_t)(x) \in G. \) According to expression (1) (in Sect. 3.1) of \( \psi^{-}_t (x) \), the diffeomorphism \( f^{-}_t \) becomes

\[ f^{-}_t (x) = x + \int_0^t \phi^{-}_s(Z^{-}_0 \circ \psi^{-}_s)(x)ds \]

(respectively,

\[ f^{+}_t (y) = y - \int_0^t \phi^{+}_s(Z^{+}_0 \circ \psi^{+}_s)(y)ds. \]

The wave operator is defined by

\[ f^{-} = \lim_{t \to -\infty} \phi^{-}_t \circ \psi^{-}_t \quad \text{(resp \( f^{+} = \lim_{t \to +\infty} \phi^{+}_t \circ \psi^{+}_t \)).} \]

**Lemma 3.6** (1) \( f^{-}_t, \ D^r f^{-}_t \) (respectively, \( f^{+}_t, \ D^r f^{+}_t \)), \( f^{-}_t \) and \( D^r f^{-}_t \) (respectively, \( f^{+}_t \) and \( D^r f^{+}_t \)) are infinite globally uniformly bounded \( \forall t > 0, \forall r \in \mathbb{N}. \)

(2) For all compact \( K_1 \subseteq \Omega_1 \) (respectively, \( K_2 \subseteq \Omega_2 \)), there is \( \varepsilon > 0 \) such that \( \forall t \in \mathbb{R}^+: \)

\[ \| f^{-}_t - id \|_{K_1} \leq \varepsilon \quad \text{and} \quad \| f^{+}_t - id \|_{K_2} \leq \varepsilon. \]

**Proof** (1) Let us show that \( f^{-}_t \) and \( D^r f^{-}_t \) (respectively, \( f^{+}_t \) and \( D^r f^{+}_t \)) are infinite globally uniformly bounded \( \forall t > 0, \forall r \in \mathbb{N}. \)

Let

\[ f^{-}_t (x) = x + \int_0^t \phi^{-}_s(Z^{-}_0 \circ \psi^{-}_s)(x)ds, \]

then for all \( r \in \mathbb{N}; \forall v \in \mathbb{R}^n \) such as \( \| v \| = 1 \), we have

\[ D^r_x (f^{-}_t - id)(x)v^r = \int_0^t e^{-A^{-}s}D_x^r(Z^{-}_0 \circ \psi^{-}_s(x)).v^r ds. \]

As

\[ D_x^r (Z^{-}_0 \circ \psi^{-}_s(x)).v^r = \sum_{\gamma=1}^r D_x^r Z^{-}_0(\xi) \sum_{l_1+\ldots+l_r=r} D^{l_1}\psi^{-}_s(x).v^{l_1} \cdots D^{l_r}\psi^{-}_s(x).v^{l_r}, \]

where \( \xi = \psi^{-}_s(x). \)

According to Lemma 3.2, there exist constants \( M_i > 0 \) such that \( \| \psi^{-}_s(x) \|^{K_i} \leq M_1 e^{-s\alpha_k} \) with \( K_i \) a compact included in \( \Omega_1 \) and \( \forall k_1 > 1 \); we will then have

\[ \| D_x^r Z^{-}_0(\xi) \| \leq M_\gamma \| \xi \|^{K_i} = M_\gamma \| \psi^{-}_s(x) \|^{K_i}, \]

from which,

\[ \| D_x^r (f^{-}_t - id)(x) \|^{K_i} \leq M_2 \int_0^t e^{aL}\| \psi^{-}_s(x) \|^{K_1} ds \]

\[ \leq M_3 \int_0^t e^{aL} \cdot e^{-s\alpha_k} ds \]

\[ \leq M_3 \int_0^t e^{-s(k_1\alpha_k-aL)} ds. \]
By hypothesis (I) in Sect. 2.3, \( \exists \rho_1 > 1/k_1 a_R - a_L \geq \rho_1 > 1 \), then

\[
\| D_x' \left( f_t^{-} - id \right)(x) \|^K_1 \leq M_3 \int_0^t e^{-s \rho_1} ds \leq \frac{M_3}{\rho_1},
\]

from which,

\[
\|(f_t^{-} - id)(x)\|_r^K \leq \frac{M_3}{\rho_1}; \quad \forall t > 0.
\]

It follows that \( f_t^{-} \) and \( D_f f_t^{-} \) are infinite-uniformly bounded \( \forall t > 0; \forall r \in \mathbb{N} \).

And by a similar reasoning, with \( K_2 \) a compact included in \( \Omega_2 \) and with the hypothesis (II) in Sect. 2.3 \( \exists \rho_2 > 1/k_2 b_L - b_R \geq \rho_2 > 1 \), we can demonstrate that

\[
\| (f_t^{+} - id)(y) \|_{r'}^K \leq \frac{M_4}{\rho_2},
\]

from which \( f_t^{+} \) is infinite-globally bounded \( \forall t > 0 \), i.e.

\[
\| D_x'(f_t^{-} - id)(y) \|_{r'}^K \leq M_4 \int_0^t e^{-s(k_2 b_L - b_R)} ds \leq M_4 \int_0^t e^{-s \rho_2} ds = M_4 \frac{1 - e^{-\rho_2}}{\rho_2} \leq \frac{M_4}{\rho_2},
\]

where \( k_2, b_L - b_R \geq \rho_2 > 1; k_2 > 1 \).

Let us show that \( f_t^{-} \) and \( D^r(f_t^{-}) \) (respectively, \( f_t^{+} \) and \( D^r(f_t^{+}) \)) are infinite globally uniformly bounded \( \forall t > 0, \forall r' \in \mathbb{N} \).

Let

\[
f_t^{-}(x) = x - \int_0^t \phi_s(Z_0^{-} \circ \psi_{-s})(x) ds,
\]

then, for all \( r \in \mathbb{N}; \forall v \in \mathbb{R}^n \) such that \( \|v\| = 1 \), we have

\[
D_x'(f_t^{-} - id)(x) v^r = -\int_0^t e^{A_s} D_x'(Z_0^{-} \circ \psi_{-s}(x)) . v^r ds.
\]

As

\[
D_x'(Z_0^{-} \circ \psi_{-s}(x)) . v^r = \sum_{y=1}^{r} \sum_{\prod I_y = r} D_x' Z_0^{-}(\xi') \sum_{l_y} D_l \psi_{-s}(x) . v^{l_1} \ldots D^{l_y} \psi_{-s}(x) . v^{l_y}, \quad \text{where} \quad \xi' = \psi_{-s}(x).
\]

According to Lemma 3.2, \( \| \psi_{-s}(x) \|_r^K \leq M e^{r a_L} \), we deduce that there is a constant \( M' > 0 \) such that

\[
\| D_x'(f_t^{-} - id)(x) \|_r^K \leq M' \int_0^t e^{-s a_R} \| Z_0^{-} \|_r^{\psi_{-s}(K)} \sup_{x \in K} \left( \frac{1}{1+\| \psi_{-s}(x) \|^2} \right)^{m'/2} e^{r a_L} ds
\]

\[
\leq M' \int_0^t e^{-s((1+m')a_R-(k_1+r)a_L)} ds.
\]

We choose

\[
m' = (r + k_1) \left( \frac{a_L}{a_R} + 1 \right) + \rho > 0
\]

with \( \rho \) any positive constant.

\[
\| D_x'(f_t^{-} - id)(x) \|_r^K \leq M' \int_0^t e^{-s(r+1)a_R} ds = M' \frac{1 - e^{-t(r+1)}}{(r+1)a_R}
\]

As \( K_1 \) is arbitrary, we can find \( \epsilon_1 > 0 \) and \( \rho > 0 \) such that

\[
\| D_x'(f_t^{-} - id)(x) \|_r^K \leq \epsilon_1; \quad \forall t > 0. \quad (3.1)
\]
It follows that $f_{-t}$ and $D^r f_{-t}$ are infinite-uniformly bounded $\forall t > 0; \forall r \in \mathbb{N}$. And by a similar reasoning, there is a constant $M'' > 0$ such that

$$\|D'_y (f_t - id) (y)\|^{K_2} \leq M'' \int_0^t e^{-s((m'_2 + 1)b_L - (r + k_2)b_R)} ds.$$ 

By choosing

$$m'_2 = (r + k_2) \left[ \frac{b_R}{b_L} + 1 \right] + \rho' > 0$$

with $\rho'$ an arbitrary positive constant, we deduce that

$$\|D'_y (f_t^+ - id) (y)\|^{K_2} \leq M'' \int_0^t e^{-s(\rho' + 1)b_R} ds.$$ 

As $K_2$ is arbitrary, we can find $\varepsilon_2 > 0$ and $\rho' > 0$ such that

$$\|D'_y (f_t^+ - id) (y)\|^{K_2} \leq \varepsilon_2; \quad (3.2)$$

from which $f_t^+$ and $D^r f_t^+$ are infinite-globally bounded $\forall t \geq 0$. \hfill $\Box$

2) From Eqs. (3.1) and (3.2), $\forall t > 0; \exists \varepsilon = max\{\varepsilon_1, \varepsilon_2\}$:

$$\|f_{-t} - id\|^{K_1}_r \leq \varepsilon; \quad \text{and} \quad \|f_t^+ - id\|^{K_2}_r \leq \varepsilon.$$

**c) Estimation of** $(\exp t Y_0^+)_*$ and $(\exp t Y_0^-)_*$

Let $B_\varepsilon = B(0, \varepsilon)$ be the ball centered at the origin with radius a certain $\varepsilon > 0$.

**Lemma 3.7** For all $Y^1 \in U_1$ and $\forall t \geq 0$, there exists $C', C''$ positive constants such that

$$\|(\exp t Y_0^-)_* \cdot Y^1\|_{r+1}^{\Omega_1} \leq C' \cdot e^{-t\rho'_2 \cdot b_R} \cdot \|Y^1\|_{r+m_1}^{\Omega_1}$$

and, respectively,

$$\|(\exp -t Y_0^+)_* \cdot Y^2\|_{r+1}^{\Omega_2} \leq C'' \cdot e^{-t\rho'_2 \cdot b_L} \cdot \|Y^2\|_{r+m_2}^{\Omega_2}.$$ 

**Proof** Let $Y^1 \in U_1$ and $\forall t \geq 0$, we have

$$(\exp t Y_0^-)_* \cdot Y^1 = (\psi^-)_* Y^1 = (\phi^-_1 \circ f^-_1)_* \cdot Y^1$$

$$= (D(\phi^-_1 \circ f^-_1) \cdot Y^1) \circ (\phi^-_1 \circ f^-_1)^{-1}$$

$$= D\phi^-_1 (f^-_1) \cdot (f^-_1)^{-1} \circ \phi^-_1 \cdot Y^1((f^-_1)^{-1} \circ \phi^-_1). Df^-_1 ((f^-_1)^{-1} \circ \phi^-_1)$$

$$= D\phi^-_1 (\phi^-_1) \cdot Y^1(\psi^-). Df^-_1 (\psi^-)$$

Let $K_1$ be a compact of $\Omega_1$, we then deduce the following estimate:

$$\|(\exp t Y_0^-)_* \cdot Y^1\|^{K_1}_r \leq \|D\phi^-_1 (\phi^-_1) \cdot Y^1(\phi^-_1 \circ f^-_1)\|^{K_1}_r \cdot \|Df^-_1 (\phi^-_1 \circ f^-_1)\|^{K_1}_r.$$ 

According to Lemma 3.6, $Df^-_1$ and $f^-_1$ are uniformly bounded, then there is a constant $C > 0$ such that

$$\|(\exp t Y_0^-)_* \cdot Y^1\|^{K_1}_r \leq C \cdot (\phi^-_1)_* \cdot Y^1 \|f^-_1(K_1) \|^{K_1}_r \quad \forall r \geq 0 \quad (*).$$

As

$$\|f^-_1\|^{K_1}_r - \|id\|^{K_1}_r \leq \|f^-_1 - id\|^{K_1}_r \leq \varepsilon.$$
then
\[ \| id \|_{K_1}^1 - \varepsilon \leq \| f_{\omega}^1 \|_{K_1} \leq \| id \|_{K_1} + \varepsilon; \]
hence,
\[ K_1 - B_\varepsilon \subseteq f_{\omega}^{-1}(K_1) \subseteq K_1 + B_\varepsilon \quad (**) \]
We will then have, from equations (*) and (**).
\[ \| (\exp t Y_0^-) \cdot Y \|_{f_{\omega}^{-1}(K_1)} \leq C(e^{-a t Y} \cdot Y) \leq C(e^{-a t Y} \cdot Y) \,
\]
from which,
\[ \Phi_{\omega}^{-1}(f_{\omega}^{-1}(K_1)) \subseteq \Omega_1^\varepsilon. \]
We will then have
\[ \| (\exp t Y_0^-) \cdot Y \|_{\Omega_1^n} \leq C(e^{-a t Y} \cdot Y) \leq C(e^{-a t Y} \cdot Y) \,
\]
As \( K_1 \) is arbitrary on \( \Omega_1 \), so
\[ \| (\exp t Y_0^-) \cdot Y \|_{\Omega_1^n} \leq C(e^{-a t Y} \cdot Y) \leq C(e^{-a t Y} \cdot Y) \,
\]
We proceed in the same way to demonstrate
\[ ||(\exp -t Y_0^+ \cdot Y)\|_{\Omega_2^n} \leq C' \cdot e^{-t Y_0^+ \cdot Y \cdot Y} \leq C' \cdot e^{-t Y_0^+ \cdot Y \cdot Y} \,
\]
\[ \square \]

4 Admissible algebra with hyperbolic structure

4.1 (\( \exp t X_0 \)) \( \cdot Y \) decays tamely

**Lemma 4.1**

(i) \( (\phi_t)_s \) is invariant on \( U \); \( \forall t \in \mathbb{R} \).
(ii) \( (\phi_t)_s \) decays tamely on \( U_1 \) and \( (\phi_{-t})_s \) decays tamely on \( U_2 \) for all \( t \geq 0 \). That is to say, \( \forall t \geq 0 \), we have
\[ \| (\exp t X_0) \cdot Y \|_{\Omega_1^n} \leq e^{-ot} \cdot || Y \|_{\Omega_1^n} \quad \forall \omega \in \mathbb{R}^{+}, \forall Y \in U \]
and
\[ ||(\exp -t X_0) \cdot Y \|_{\Omega_2^n} \leq e^{-ot} \cdot || Y \|_{\Omega_2^n} \quad \forall \omega' \in \mathbb{R}^{+}, \forall Y \in U. \]

**Proof**
Let \( Y \in U = U_1 \oplus U_2 \); then \( \exists! Y^i \in U_i (i = 1, 2) \) such that \( Y = Y^1 + Y^2 \); and let \( t \in \mathbb{R}, \forall (x, y) \in \mathbb{R}^n = \Omega_1 \cup \Omega_2 \), we have
\[ (\phi_t)_s Y(x, y) = ((\phi_t^1)_s Y^1(x), (\phi_t^2)_s Y^2(y)). \]
(i) We will show that \( (\phi_t)_s Y \in U, \forall t \in \mathbb{R} \).
- We project \( (\phi_t)_s Y \) on \( U_1 \), So
\[ (\phi_t)_s Y(x, y) = ((\phi_t^1)_s Y^1(x), 0), \]
from which \( (\phi_t)_s Y \in U \).
- Similarly, we project \( (\phi_t)_s Y \) on \( U_2 \), So
\[ (\phi_t)_s Y(x, y) = (0, (\phi_t^2)_s Y^2(x)), \]
from which \( (\phi_t)_s Y \in U \).
We deduce that \( (\phi_t)_s \) is invariant on \( U, \forall t \in \mathbb{R} \).
Lemma 4.2

(i) Let us first show that \((\phi_t)_s\) decays tamely on \(U_1; \forall t > 0\):
\[
||(\phi_t)_s \cdot Y||_{r_1}^{\Omega_1} = ||(\phi_t^{-})_s \cdot Y^1||_{r_1}^{\Omega_1}
\]
and for all \(\rho_1' > 0\), we have
\[
||(\phi_t)_s \cdot Y||_{r_1}^{\Omega_1} \leq C_1 \cdot e^{-\rho_1' a_R} ||Y^1||_{r_1+m_1}^{\Omega_1} \quad \text{(see Lemma 3.5)}
\]
\[
\leq C_1 \cdot e^{-\omega t} ||Y||_{r_1+m_1}^{\Omega_1}
\]
with \(\omega = \rho_1' a_R\).

Then we show that \((\phi_t)_s\) decays tamely on \(U_2; \forall t > 0\):
\[
||(\phi_t)_s \cdot Y||_{r_2}^{\Omega_2} = ||(\phi_t^{-})_s \cdot Y^2||_{r_2}^{\Omega_2}
\]
and for all \(\rho_2' > 0\), we have
\[
||(\phi_t)_s \cdot Y||_{r_2}^{\Omega_2} \leq C_2 \cdot e^{-\rho_2' b_L} ||Y^2||_{r_2+m_2}^{\Omega_2} \quad \text{(see Lemma 3.5)}
\]
\[
\leq C_2 \cdot e^{-\omega' t} ||Y||_{r_2+m_2}^{\Omega_2}
\]
with \(\omega' = \rho_2' b_L\).

\(\square\)

4.2 \((\text{expt} Y_0)_s\) decays tamely

Lemma 4.2

(i) \((\psi_t)_s\) is invariant on \(U, \forall t \in \mathbb{R}\).

(ii) The diffeomorphism \((\psi_t)_s\) decays tamely on \(U_1^\varepsilon\) and \((\psi_t)_s\) decays tamely on \(U_2^\varepsilon\), that is to say, \(\forall t \geq 0\), there exist \(\omega \in \mathbb{R}^{+}\) and \(\omega' \in \mathbb{R}^{+}\) such that
\[
||(\psi_t)_s \cdot Y||_{r}^{\Omega} \leq e^{-\omega t} ||Y||_{r+m}^{\Omega},
\]
respectively,
\[
||(\psi_t)_s \cdot Y||_{r}^{\Omega} \leq e^{-\omega' t} ||Y||_{r+m}^{\Omega}.
\]

Proof Let \(Y \in U = U_1 \cup U_2\) then \(\exists! Y^i \in U_i (i = 1, 2)\) such that \(Y = Y^1 + Y^2\).
Let \(t \in \mathbb{R}, \forall (x, y) \in \mathbb{R}^n = \Omega_1 \cup \Omega_2\)

\[
(\psi_t)_s Y(x, y) = \left( ((\psi_t^{-})_s Y^1(x), (\psi_t^+_s) Y^2(y) \right).
\]

(i) We will prove that \((\psi_t)_s Y \in U, \forall t \geq 0\).
- We project on \(U_1 \Rightarrow Y(x, y) = (Y^1(x), 0)\).

Therefore,
\[
(\psi_t)_s Y(x, y) = \left( (\psi_t^{-})_s Y^1(x), 0 \right),
\]
from which \((\psi_t)_s Y \in U\)
- Similarly, if we project on \(U_2 \Rightarrow Y(x, y) = (0, Y^2(x))\),
then
\[
(\psi_t)_s Y(x, y) = \left( 0, (\psi_t^+_s) Y^2(x) \right),
\]
from which \((\psi_t)_s Y \in U\). We deduce that \((\psi_t)_s\) is invariant on \(U, \forall t \in \mathbb{R}\).

(ii) We note that
\[
||(\psi_t)_s \cdot Y||_{r}^{\Omega_1} = ||(\psi_t^{-})_s \cdot Y^1||_{r}^{\Omega_1}
\]
\[
\leq C' \cdot e^{-\rho_1' a_R} ||Y^1||_{r+m}^{\Omega_1} \quad \text{(cf Lemma 3.7)}
\]
\[
\leq C'_1 \cdot e^{-\omega t} ||Y||_{r+m}^{\Omega_1}
\]
with \(\omega = \rho_1' a_R\).

This means that \((\psi_t)_s\) decays tamely on \(U_1^\varepsilon\).
- Similarly,
\[
||(\psi_t)_s \cdot Y||_{r}^{\Omega_2} = ||(\psi_t^{-})_s \cdot Y^2||_{r}^{\Omega_2}
\]
\[
\leq C'' \cdot e^{-\rho_2' b_L} ||Y^2||_{r+m}^{\Omega_2} \quad \text{(cf Lemma 3.7)}
\]
\[
\leq C''_1 \cdot e^{-\omega' t} ||Y||_{r+m}^{\Omega_2}
\]
with \(\omega' = \rho_2' b_L\).

We deduce that \((\psi_t)_s\) decays tamely on \(U_2^\varepsilon\).
Theorem 4.3 The admissible algebra U admits a hyperbolic structure for the flow \( \psi_t \).

Proof According to Lemma 4.2, we deduce that U has a hyperbolic structure for \( \psi_t \). \( \square \)

Part II: applications

We will show, as an application, that the ideal of finite codimension extends over all the hyperbolic-type admissible algebra, and this using the following lemma:

Fundamental lemma: ([8]) Let V be a finite codimension subspace of a \( \mathbb{R} \)-vector space E and an endomorphism \( \psi \) on E such that
1. \( \psi (V) \subset V \);
2. \( \psi + b \cdot id \) is surjective on V for all \( b \in \mathbb{R} \);
3. for all numbers \( b, c \in \mathbb{R} \) such as \( b^2 - 4c < 0 \), the operator \( \psi^2 + b\psi + c \cdot id \) is surjective on V.

Then \( V = E \).

5 Surjectivity of the operator \( adY_0 + bI \)

In this section, we study the surjectivity of some linear operators. Note by \( \varphi = (ad_{-Y_0}, ad_{Y_0}^+) \), where \( ad_{-Y_0} = \varphi_1 \) and \( ad_{Y_0}^+ = \varphi_2 \) two adjoint endomorphisms and \( id \) the identity application.

Lemma 5.1 For all \( b \in \mathbb{R} \), the operator \( \varphi_1 + b \cdot id_{\mathbb{R}} \) (respectively, \( \varphi_2 + b \cdot id_{\mathbb{R}} \)) is surjective on \( U_1^c \) (respectively, on \( U_2^c \)).

Proof Let \( Y^i \in U_i (i = 1, 2) \) such that
\[
\begin{cases}
Y^1 = \sum_{i=1}^{k} f_i(x) \frac{\partial}{\partial x_i} \in U_1 \\
Y^2 = \sum_{j=1}^{l} g_j(y) \frac{\partial}{\partial y_j} \in U_2.
\end{cases}
\]

By putting
\[
\begin{cases}
W_1 = \int_0^{\infty} R(t)(\psi_d^-)_s Y^1 dt \in U_1^c \\
W_2 = \int_0^{\infty} R(t)(\psi_d^+)_s Y^2 dt \in U_2^c
\end{cases}
\]

- First stage: Let us prove that \( W_i \) is a solution of the equation:

\[
(ad_{-Y_0})(W_1) = (\varphi_1 + b \cdot id)(W_1) = Y^i, \quad (i = 1, 2), \quad \forall b \in \mathbb{R}
\]

\[
ad_{-Y_0}(W_1) = \lim_{s \to 0} \frac{d}{ds}(\psi_s^-) \left( \int_0^{\infty} R(t)(\psi_d^-)_s Y^1 dt \right) = \lim_{s \to 0} \int_0^{\infty} R(t) \frac{d}{ds} (\psi_d^-) Y^1 dt
\]

\[
= \int_0^{\infty} R(\tau) \frac{d}{d\tau} (\psi_{\tau}^-) Y^1 d\tau, \quad \text{where} \quad \tau = t + s
\]

\[
= R(\tau)(\psi_{\tau}^-)_s Y^1 |_{\tau = 0}^{\infty} - \int_0^{\infty} (\psi_{\tau}^-)_s Y^1 d\tau = R(\tau) - R'(\tau) \quad \text{where} \quad \tau = t + s
\]

\[
ad_{-Y_0}(W_1) + bW_1 = R(\tau)(\psi_{\tau}^-)_s Y^1 |_{\tau = 0}^{\infty} + \int_0^{\infty} (bR(\tau) - R'(\tau))(\psi_{\tau}^-)_s Y^1 d\tau.
\]

We put
\[
\begin{cases}
R'(\tau) - bR(\tau) = 0 \\
R(0) = -1
\end{cases}
\]
\[
then \quad R(\tau) = -e^{bt}
\]
and
\[
\lim_{\tau \to +\infty} \| R(\tau)(\psi^-_\tau)_* \cdot Y^1 \|_{r} \leq C' \lim_{\tau \to +\infty} e^{b\tau} e^{-\tau \rho'_1 a_R} \| Y^1 \|_{r + m_1}^{\Omega_1}
\]
\[
\leq C' \lim_{\tau \to +\infty} e^{-\tau (-b + \rho'_1 a_R)} \| Y^1 \|_{r + m_1}^{\Omega_1}.
\]

As \( \rho'_1 \) is arbitrary, then
\[
\forall b \in \mathbb{R}, \exists \rho'_{1,b} > 0 / -b + \rho'_{1,b} a_R > 0,
\]
then
\[
\lim_{\tau \to +\infty} \| R(\tau)(\psi^-_\tau)_* Y^1 \|_{r} = 0.
\]

It results
\[
ad_{\psi^+_\tau} W_1 + b W_1 = R(\tau)(\psi^-_\tau)_* Y^1|_0^{+\infty} = Y^1.
\]

Similarly,
\[
ad_{\psi^+_\tau} (W_2) = \lim_{s \to 0} \frac{d}{ds} (\psi^+_s)_* \left( \int_0^{+\infty} R(\tau)(\psi^+_\tau)_* Y^2 d\tau \right)
\]
\[
= \lim_{s \to 0} \int_0^{+\infty} R(\tau) \frac{d}{d\tau} ((\psi^+_\tau)_* Y^2 d\tau
\]
\[
= \int_0^{+\infty} R(\tau) (\psi^+_\tau)_* Y^2 d\tau \quad \text{where} \quad \tau = s + t
\]
\[
= R(\tau)(\psi^-_\tau)_* Y^2|_0^{+\infty} - \int_0^{+\infty} ((\psi^-_\tau)_* Y^2) \frac{d}{d\tau} R(\tau) d\tau.
\]

By adding \( b W_2 \) in both members, we will have
\[
ad_{\psi^+_\tau} (W_2) + b W_2 = R(\tau)(\psi^-_\tau)_* Y^2|_0^{+\infty} + \int_0^{+\infty} (-R'(\tau) + b R(\tau))(\psi^-_\tau)_* Y^2 d\tau.
\]

We put
\[
\begin{cases}
-R'(\tau) + b R(\tau) = 0 \\
R(0) = -1
\end{cases}
\]

then \( R(\tau) = -e^{bt} \)

We have
\[
\lim_{\tau \to +\infty} \| R(\tau)(\psi^-_\tau)_* Y^2 \|_{r} \leq C'' \lim_{\tau \to +\infty} e^{b\tau} e^{-\tau \rho'_2 b_L} \| Y^2 \|_{r + m_2}^{\Omega_2}
\]
\[
\leq C'' \lim_{\tau \to +\infty} e^{-\tau (-b + \rho'_2 b_L)} \| Y^2 \|_{r + m_2}^{\Omega_2}.
\]

As \( \rho'_2 \) is arbitrary, then
\[
\forall b \in \mathbb{R}, \exists \rho'_{2,b} > 0 / -b + \rho'_{2,b} b_L > 0,
\]
from which
\[
\lim_{\tau \to +\infty} \| R(\tau)(\psi^-_\tau)_* Y^2 \|_{r} = 0.
\]

Therefore,
\[
ad_{\psi^+_\tau} (W_2) + b W_2 = R(\tau)(\psi^-_\tau)_* Y^2|_0^{+\infty} = Y^2.
\]
- **Second stage:** Let us prove that \( W_1 \) is of class \( C^\infty \) on any compact.

Let \( K_1 \) be a compact of \( \Omega_1 \), and as

\[
W_1 = \int_0^{+\infty} R(t)(\psi_{i}^-)_+ Y_1 \, dt,
\]

according to Lemma 3.7, we will have

\[
\| R(t)(\text{expt}_0^-) Y_1 \|_{K_1}^c \leq C \varepsilon t^{-\left( -b + \rho'_1 a_R \right)} \| Y_1 \|_{\Omega_1^c}^r.
\]

As \( \rho'_1 \) is arbitrary, then

\[
\forall b \in \mathbb{R}, \exists \rho'_{1,b} > 0, -b + \rho'_{1,b} a_R > 0, \int_0^{+\infty} e^{-t(b + \rho'_{1,b} a_R)} \, dt \text{ converges } \forall b \in \mathbb{R}.
\]

It follows that \( W_1 \) converges uniformly, \( \forall x \in \Omega_1 \), where ultimately \( W_1 \) is of class \( C^\infty \) on any compact of \( \Omega_1 \). The same reasoning is valid to prove that \( W_2 \) is of class \( C^\infty \) on any compact of \( \Omega_2 \).

\[\square\]

**Lemma 5.2** For all \( b \in \mathbb{R}, \varphi + b.\text{id} \) is surjective on \( U \).

**Proof** Let \( Y \in U \), seeking a \( W \in U \) such that \( \forall (x, y) \in \mathbb{R}^n; Y(x, y) = (\varphi + b.\text{id}) W(x, y), \forall b \in \mathbb{R} \).

As \( Y \in U = U_1 \oplus U_2, \exists ! Y^i \in U_i \) such that

\[
Y(x, y) = (Y^1(x), Y^2(y)) = (\varphi_1 W_1(x) + b W_1(x), \varphi_2 W_2(y) + b W_2(y)).
\]

We project on \( U_i \):

- On \( U_1 \), we have \( Y(x, y) = (Y^1(x), 0) = (\varphi_1 W_1(x) + b W_1(x), 0) \), and according to Lemma 5.1 \( \exists \varepsilon > 0 \) and \( W_1 \), such that \( W_1(x) = -\int_0^{+\infty} e^{bt}(\psi_{i}^-)_+ Y^1(x) \, dt \in \Omega_1^c \).

- On \( U_2 \), we have \( Y(x, y) = (0, Y^2(y)) = (0, \varphi_2 W_2(y) + b W_2(y)) \), and according to Lemma 5.1 \( \exists \varepsilon > 0 \) and \( W_2 \), such that \( W_2(y) = -\int_0^{+\infty} e^{bt}(\psi_{i}^-)_+ Y^2(y) \, dt \in \Omega_2^c \).

It follows that

\[
W_i \in U_i^\varepsilon
\]

We put \( W(x, y) = (W_1(x), W_2(y)) \in \mathbb{R}^k \times \mathbb{R}^l = \mathbb{R}^n \).

As \( W = W_1 + W_2 \in U_1^\varepsilon + U_2^\varepsilon = U \), it follows that \( \varphi + b.\text{id} \) is surjective on \( U \), for all \( b \in \mathbb{R} \).

\[\square\]

6 Surjectivity of the operator \((a \varphi_0)^2 + b.a \varphi_0 + cI\)

**Lemma 6.1** For all numbers \( b, c \in \mathbb{R} \) such that \( b^2 - 4c < 0 \), the operator \( \varphi_1^2 + b \varphi_1 + c.\text{id}_{\mathbb{R}^n} \), (respectively, \( \varphi_2^2 + b \varphi_2 + c.\text{id}_{\mathbb{R}^l} \)) is surjective on \( U_1^\varepsilon \) (respectively, on \( U_2^\varepsilon \)).

**Proof** Let \( Y^i \in U_i \) such that

\[
Y^1 = \sum_{i=1}^{k} f_i(x) \frac{\partial}{\partial x_i} \in U_1
\]

\[
Y^2 = \sum_{i=1}^{l} g_i(y) \frac{\partial}{\partial y_i} \in U_2.
\]
such that \((\varphi_1^2 + b \varphi_1 + c d_{x^4}) W_1 = [-Y_0^-, [-Y_0^-, W_1]] + b[-Y_0^-, W_1] + c W_1\) and \((\varphi_2^2 + b \varphi_2 + c d_{x^2}) W_2 = [-Y_0^+, [-Y_0^+, W_2]] + b[-Y_0^+, W_2] + c W_2\).

\[
\begin{align*}
W_1 &= \int_0^\infty R(t) (\psi_0^-)_t Y^1 \, dt \in U^c_1 \\
W_2 &= \int_0^\infty R(t) (\psi_0^+)_t Y^2 \, dt \in U^c_2.
\end{align*}
\]

With

\[
R(t) = \frac{2\exp(b t)}{\sqrt{4c - b^2}} \sin(\sqrt{4c - b^2}/2)t.
\]

- **First stage:** Let us show that \(W_i\) is a solution of the equation \((\varphi_i^2 + b \varphi_i + c d) W_i = Y^i\).

If we put \(Z_1 = [-Y_0^-, W_1] = a d_{-Y_0^-}(W_1)\), then

\[
\begin{align*}
Z_1 &= \lim_{s \to 0} \frac{d}{ds} (\psi_0^-)_s \left( \int_0^{+\infty} R(t) (\psi_0^-)_t Y^1 \, dt \right) \\
&= R(\tau) (\psi_0^-)_s Y^1 \mid_{0}^{+\infty} - \int_0^{+\infty} (\psi_0^-)_s Y^1 \frac{d}{d\tau} R(\tau) \, d\tau \\
&= R(\tau) (\psi_0^-)_s Y^1 \mid_{0}^{+\infty} - \int_0^{+\infty} R'(\tau) (\psi_0^-)_s Y^1 \, d\tau.
\end{align*}
\]

As \(R(t) = \frac{2\exp(b t)}{\sqrt{4c - b^2}} \sin(\sqrt{4c - b^2}/2)t\) and

\[
\| R(\tau) (\psi_0^-)_s Y^1 \|_{r} \leq C' e^{|b| \tau} e^{-\tau \rho'_1 a_R} \| Y^1 \|_{r + m_1} \mid_{r + m_1}
\]

\(\rho'_1\) being arbitrary; therefore, \(\forall b \in \mathbb{R}, \exists \rho'_1, b > 0 \): \(-b/2 + \rho'_1 a_R > 0\), where ultimately

\[
\lim_{\tau \to +\infty} \| R(\tau) (\psi_0^-)_s Y^1 \|_{r} = 0,
\]

then the first member vanishes and \(Z_1\) becomes

\[Z_1 = -\int_0^{+\infty} R'(\tau) (\psi_0^-)_s Y^1 \, d\tau.\]

We will have

\[\varphi_1^2(W_1) = [-Y_0^-, [-Y_0^-, W_1]] = [-Y_0^-, Z_1]\]

\[
\begin{align*}
&= -\lim_{s \to 0} \frac{d}{ds} (\psi_0^-)_s \int_0^{+\infty} R'(\tau) (\psi_0^-)_t Y^1 \, d\tau \\
&= -\lim_{s \to 0} \int_0^{+\infty} R'(\tau) \frac{d}{ds} (\psi_0^-)_s Y^1 \, d\tau \\
&= -\left[ R'(\tau) (\psi_0^-)_s Y^1 \right]_{0}^{+\infty} + \int_0^{+\infty} R'(\tau) (\psi_0^-)_s Y^1 \, d\tau \quad \text{where} \ t = s + \tau \\
&= Y^1 + \int_0^{+\infty} R''(\tau) (\psi_0^-)_s Y^1 \, d\tau.
\end{align*}
\]
Consequently,

\[(\varphi_1^2 + b\varphi_1 + c.id_{\mathbb{R}^2})W_1 = [-Y_0^-, [-Y_0^-, W_1]] + b[-Y_0^-, W_1] + cW_1\]

\[= Y^1 + \int_0^{+\infty} (R''(t) - bR'(t) + cR(t))(\psi_1^-)_x Y dt\]

\[= Y^1,\]

where \(R(t) = \frac{2\exp\left(\frac{b}{2}t\right)}{\sqrt{4c - b^2}} \sin(\sqrt{4c - b^2}/2)t\) is solution of the Cauchy problem:

\[
\begin{align*}
R''(t) - bR'(t) + cR(t) &= 0 \\
R(0) &= 0 \\
R'(0) &= 1.
\end{align*}
\]

The operator \(\varphi_1^2 + b\varphi_1 + c.id_{\mathbb{R}^2}\) is, therefore, surjective on \(U_1^\varepsilon\). The proof of the surjectivity of the operator \(\varphi_1^2 + b\varphi_2 + c.id_{\mathbb{R}^2}\) on \(U_2^\varepsilon\) is made in the same way.

**Second stage:** According to the previous lemma, \(W_i, (i = 1, 2)\) are of class \(C^\infty\) on all compact.

**Lemma 6.2** For all \(b, c \in \mathbb{R}, b^2 - 4.c < 0\) the operator \(\varphi_2^2 + b.\varphi + c.id\) is surjective in \(U\).

**Proof** Let \(Y \in U\), seeking a \(W \in U\) such that

\[\forall (x, y) \in \mathbb{R}^n, Y(x, y) = (\varphi^2 + b.\varphi + c.id)W(x, y), \text{ for all } b, c \in \mathbb{R}, b^2 - 4.c < 0.\]

As \(Y \in U = U_1 \oplus U_2, \exists! Y_i \in U_i\) such that

\[Y(x, y) = (Y^1(x), Y^2(y)) = \left(\varphi_1^2 W_1(x) + b.\varphi_1 W_1(x) + c.W_1(x), \varphi_2^2 W_2(y) + b.\varphi_2 W_2(y) + c.W_2(y)\right).\]

We project on \(U_i\):

- On \(U_1\), we have \(Y(x, y) = (Y^1(x), 0) = \left(\varphi_1^2 W_1(x) + b.\varphi_1 W_1(x) + c.W_1(x), 0\right),\) and according to Lemma 6.1, \(\exists \varepsilon > 0\) and \(W_1\), such that \(W_1(x) = \int_0^{+\infty} R(t)(\psi_1^-)_x Y^1(x) dt \in \Omega_1^\varepsilon\) with

\[R(t) = \frac{2\exp\left(\frac{b}{2}t\right)}{\sqrt{4c - b^2}} \sin(\sqrt{4c - b^2}/2)t\]

- On \(U_2\), we have \(Y(x, y) = (0, Y^2(y)) = \left(0, \varphi_2^2 W_2(y) + b.\varphi_2 W_2(y) + c.W_2(y)\right),\) and according to Lemma 6.1, \(\exists \varepsilon > 0\) and \(W_2\) such that \(W_2(y) = \int_0^{+\infty} R(t)(\psi_2^-)_x Y^2(y) dt \in \Omega_2^\varepsilon \subset \Omega_2\) with

\[R(t) = \frac{2\exp\left(\frac{b}{2}t\right)}{\sqrt{4c - b^2}} \sin(\sqrt{4c - b^2}/2)t.\]

It follows that

\[W_i \in U_i^\varepsilon.\]

We put \(W(x, y) = (W_1(x), W_2(y)) \in \mathbb{R}^k \times \mathbb{R}^l = \mathbb{R}^n\). Since \(W = W_1 + W_2 \in U_1^\varepsilon + U_2^\varepsilon = U\). Consequently, \(\varphi_2^2 + b.\varphi + c.id\) is surjective on \(U\), for all \(b, c \in \mathbb{R}, b^2 - 4.c < 0.\)

\[\square\]
7 Ideals of finite codimension in hyperbolic-type algebra

Theorem Let $U$ be the admissible algebra having a hyperbolic structure for the flow:
$$(\psi_t)_* = (\text{expt}Y_0)_* = (\text{expt}(X_0 + Z_0))_*,$$
if $\text{dim}(E - U)$ is finite, and $\varphi = (ad_{Y_0}, ad_{X_0})$ an endomorphism
on $U$, such that

(i) $\varphi(U) \subset U$,
(ii) $\varphi + b.\text{id}_{R^n}$ is surjective on $U$; $\forall b \in \mathbb{R}$,
(iii) $\varphi^2 + b\varphi + c.\text{id}_{R^n}$ is surjective on $U$; $\forall b, c \in \mathbb{R} / b^2 - 4c < 0$ Then $U = E$.

Proof The hypotheses of the fundamental lemma is satisfied thanks to Lemmas 4.2, 5.2 and 6.2, and, if in
addition we have $\text{dim}(E - U)$ finite, then we deduce

$$U = E.$$

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