On the Hamiltonian Description of Fluid Mechanics

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Abstract

We suggest the Hamiltonian approach for fluid mechanics based on the dynamics, formulated in terms of Lagrangian variables. The construction of the canonical variables of the fluid sheds a light on the origin of Clebsh variables, introduced in the previous century. The developed formalism permits to relate the circulation conservation (Tompson theorem) with the invariance of the theory with respect to special diffeomorphisms and establish also the new conservation laws. We discuss also the difference of the Eulerian and Lagrangian description, pointing out the incompeteness of the first. The constructed formalism is also applicable for ideal plasma. We conclude with several remarks on the quantization of the fluid.

1 Introduction

At the present time almost all fundamental physical phenomena could be formulated in the frameworks of either classical or quantum mechanics. That means that these phenomena admits the Hamiltonian description, which due to its long history developed many powerful methods of analysis of the general properties of evolution of the systems and the tools for the solutions of partial problems. The Hamiltonian formalism also provides the unique way of transition from classical to quantum description of the systems.

In this respect the fluid mechanics stands aside (in spite of its name) from orthodox mechanics. The reasons for that is not only the infinite number of degrees of freedom of fluid. We have already learned how to formulate the theory of classical and quantum fields. The main difference between conventional field theory and the fluid is that in the first case we can speak about the dynamics of the field at one point in space (which of course interacts with the field at the neighbouring points), while in the case of the
fluid, describing the interaction of the neighbouring particles which constitute the fluid we are loosing its position in the space due to the motion of fluid. In the same time the objective of usual problems of hydrodynamics is to define the velocity, density and a thermodynamical variable (pressure or entropy) as the functions of the coordinates $\vec{x}$ and time $t$ for the appropriate boundary conditions and/or initial data [1]. The similar problems also appear for magnetohydrodynamics dealing with sufficiently dense plasma [2]. For the developing of the Hamiltonian formalism we need to start with more detailed description based initially on the trajectories of the particles which constitutes the fluid or plasma. This description is especially important for plasma, because the fundamental electromagnetic interaction could be formulated only in terms of the trajectories of the charges. Needless to say that some aspects of this approach was extensively studied in the series of papers by J.Marsden, A.Weinstein, P.Kupershmidt, D.Nolm , T.Ratiu and C.Levermore [4] especially in the context of stability problem for ideal fluid. In contrast to these papers here we deliberately avoid, when it is possible, the use of language of modern differential geometry to make our paper accessible to physicists.

Of course not all properties of fluid could be formulated in the frameworks of the Hamiltonian approach. For example, we leave open the question of the energy dissipation, viscosity et cetera.

2 The Lagrangian equations of motion

In fluid mechanics there are two different pictures of description. The first, usually refereed as Eulerian, uses as the coordinates the space dependent fields of velocity, density and some thermodynamic variable. The second, Lagrangian description, uses the coordinates of the particles $\vec{x}(\xi, t)$ labeled by the set of the parameters $\xi$, which could be considered as the initial positions $\vec{\xi} = \vec{x}(t = 0)$ and time $t$. These initial positions $\vec{\xi}$ as well, as the coordinates $\vec{x}(\xi, t)$ belong to some domain $D \subseteq \mathbb{R}^3$. In sequel we shall consider only conservative systems, where the paths of different particles do not cross, therefore it is clear that the functions $\vec{x}(\xi, t)$ define a diffeomorphism of $D \subseteq \mathbb{R}^3$ and the inverse functions $\vec{\xi}(x, t)$ should also exist.

\[ x_j(\xi_i, t) \big|_{\xi = \vec{\xi}(x_i, t)} = x_j, \]
\[ \xi_j(x_i, t) \big|_{\vec{x} = \vec{x}(\xi_i, t)} = \xi_j. \]  

(1)

The density of the particles in space at time \( t \) is

\[ \rho(\vec{x}, t) = \int d^3\xi \rho_0(\xi_i) \delta(\vec{x} - \vec{x}(\xi_i, t)), \]  

(2)

where \( \rho_0(\xi) \) is the initial density at time \( t = 0 \). The velocity field \( \vec{v} \) as a function of coordinates \( \vec{x} \) and \( t \) is:

\[ \vec{v}(x_i, t) = \dot{\vec{x}}(\xi_i(x_i, t), t), \]  

(3)

where \( \xi(x, t) \) is the inverse function (1). The velocity also could be written in the following form:

\[ \vec{v}(x_i, t) = \int d^3\xi \rho_0(\xi_i) \dot{\vec{x}}(\xi_i, t) \delta(\vec{x} - \vec{x}(\xi_i, t)), \]  

(4)

or

\[ \rho(x_i, t)\vec{v}(x_i, t) = \int d^3\xi \rho_0(\xi_i) \dot{\vec{x}}(\xi_i, t) \delta(\vec{x} - \vec{x}(\xi_i, t)). \]  

(5)

Let us calculate the time derivative of the density using its definition (2):

\[ \dot{\rho}(x_i, t) = \int d^3\xi \rho_0(\xi_i) \frac{\partial}{\partial t} \delta(\vec{x} - \vec{x}(\xi_i, t)) \]

\[ = \int d^3\xi \rho_0(\xi_i) \left( -\dot{\vec{x}}(\xi_i, t) \right) \frac{\partial}{\partial \vec{x}} \delta(\vec{x} - \vec{x}(\xi_i, t)) \]

\[ = -\frac{\partial}{\partial \vec{x}} \rho(x_i, t)\vec{v}(x_i, t) \]

(6)

In such a way we verify the continuity equation of fluid dynamics:

\[ \dot{\rho}(x_i, t) + \vec{\partial}\left(\rho(x_i, t)\vec{v}(x_i, t)\right) = 0. \]  

(7)

Using the coordinates \( \vec{x}(\xi_i, t) \) as a configurational variables we can consider the simplest motion of the fluid described by the Lagrangian

\[ L = \int d^3\xi \frac{m\dot{x}^2}{2}(\xi_i, t). \]  

(8)
The equations of motion which follow from (8) apparently are

\[ m \ddot{x}(\xi_i, t) = 0 \] (9)

Now let us find what does this equation mean for the density and velocity of the fluid. For that we shall differentiate both sides of (5) with respect to time

\[
\frac{\partial}{\partial t} \rho(x_i, t) \vec{v}(x_i, t) = \int d^3 \xi \rho_0(\xi_i) \ddot{x}(\xi_i, t) \delta(\vec{x} - \vec{x}(\xi_i, t)) \\
+ \int d^3 \xi \rho_0(\xi_i) \dot{x}(\xi_i, t) \frac{\partial}{\partial t} \delta(\vec{x} - \vec{x}(\xi_i, t))
\] (10)

The first term in the r.h.s. of (10) vanishes due to the equations of motion (9) and transforming the second in the same way, as we did in (3) we arrive at

\[
\frac{\partial}{\partial t} \rho(x_i, t) \vec{v}(x_i, t) + \frac{\partial}{\partial x_k} \left( \rho(x_i, t) \vec{v}(x_i, t) v_k(x_i, t) \right) = 0
\] (11)

Let us rewrite (11) in the following form:

\[
\vec{v}(x_i, t) \left[ \dot{\rho}(x_i, t) + \frac{\partial}{\partial x_k} \left( \rho(x_i, t) v_k(x_i, t) \right) \right] \\
+ \rho(x_i, t) \left[ \ddot{\vec{v}}(x_i, t) + v_k(x_i, t) \frac{\partial}{\partial x_k} \vec{v}(x_i, t) \right] = 0.
\] (12)

The first term in (12) vanishes due to the continuity equation, while the second gives Euler’s equation in the case of the free flow:

\[ \ddot{\vec{v}}(x_i, t) + v_k(x_i, t) \frac{\partial}{\partial x_k} \vec{v}(x_i, t) = 0 \] (13)

To move further we need to introduce into the Lagrangian (8) the "potential energy" term which will give rise to the internal pressure field in Euler’s equation. As we have mentioned above, the functions \( \vec{x}(\xi_i, t) \) define a diffeomorphism in \( \mathbb{R}^3 \) therefore the Jacobean matrix

\[ A_j^i(\xi_i, t) = \frac{\partial x_j(\xi_i, t)}{\partial \xi_k} \] (14)
is nondegenerate for any $\xi_i$ and $t$. The integral (1) may be expressed via the Jacobean determinant of $A_j^k(\xi_i, t)$:

$$\rho(\vec{x}, t) = \left. \rho_0(\xi_i) \right|_{\vec{\xi} = \vec{u}(x_i, t)} \frac{\det A(\xi_i, t)}{\det A(\xi_i, t)} |_{\vec{\xi} = \vec{u}(x_i, t)}. \tag{15}$$

Assume for the simplicity that the initial density $\rho_0(\xi_i)$ is uniform in $D \subseteq \mathbb{R}^3$, and effectively normalize the density field $\rho(\xi, t)$ by putting $\rho_0(\xi_i) = 1$ (one particle in the elementary volume). Then (15) becomes

$$\rho(\vec{x}, t) = \frac{1}{\det A(\xi_i, t)} \left. \rho_0(\xi_i) \right|_{\vec{\xi} = \vec{u}(x_i, t)}. \tag{16}$$

Consider as the "potential energy" the functional of $\det A$:

$$L = \int d^3\xi \left[ \frac{m^2 \vec{\dot{x}}_j(\xi_i, t)^2}{2} - f'(\det A(\xi_i, t)\det A) \right]. \tag{17}$$

Now the equations of motion become

$$m\ddot{x}_j(\xi_i, t) - \frac{\partial}{\partial \xi_k} \left( (A^{-1})_j^k(\xi_i, t) f'(\det A) \det A \right) = 0 \tag{18}$$

Substituting $\ddot{x}_j(\xi_i, t)$ from (18) to the equation (10) and acting as we did in the derivation of the equation (13), we obtain

$$m\rho(x_i, t) \left( \frac{\partial}{\partial t} + v_k(x_i, t) \frac{\partial}{\partial x_k} \right) v_j(x_i, t) - \frac{\partial}{\partial x_j} \left. \left( f'(\det A(\xi_i, t)) \right|_{\vec{\xi} = \vec{u}(x_i, t)} \right) = 0 \tag{19}$$

It is now obvious that if we identify the $-1/mf'(\det A(\xi_i, t))|_{\vec{\xi} = \vec{u}(x_i, t)}$ with pressure $p(x_i, t)$, the equation (14) takes the form of usual Euler equation without viscosity:

$$\rho(x_i, t) \left( \frac{\partial}{\partial t} + v_k(x_i, t) \frac{\partial}{\partial x_k} \right) v_j(x_i, t) = -\frac{\partial}{\partial x_j} p(x_i, t). \tag{20}$$

Note, that the pressure $p(x_i, t)$ which appeared here is not due to an external force, but the result of interaction between particles. The further complication of the Lagrangian is not necessary for the moment. One comment nevertheless should be done. No modification of the Lagrangian will give us the Navier-Stokes equation because the later includes the dissipation effects, which is not time symmetric....
3 Hamiltonian formalism

The Hamiltonian formalism of classical mechanics, which among other adventures gives us the unique way for construction of the quantum theory. In this respect the hydrodynamics stands aside because its fundamental variables - local velocity, density and thermodynamical function does not admit an immediate Hamiltonian interpretation, in spite of quite often in the text-book one can meet the terms like "momentum density" (see e.g. [1]).

We shall introduce the canonical variables according to the Lagrangian (17), so our canonical coordinates will be the functions \( \vec{x}(\xi_i, t) \) and its conjugated momenta are defined as the derivatives of the Lagrangian with respect to the velocities \( \dot{\vec{x}}(\xi_i, t) \):

\[
\vec{p}(\xi_i, t) = \frac{\delta L}{\delta \dot{\vec{x}}(\xi_i, t)} = m\dot{\vec{x}}(\xi_i, t),
\]

The Hamiltonian is given by the Legendre transform of the Lagrangian:

\[
H = \int d^3\xi \left( \frac{1}{2m} \vec{p}^2(\xi_i, t) + f(detA(\xi_i, t)) \right).
\]

The canonical Poisson brackets is defined by

\[
\{p_j(\xi_i), x_k(\xi'_i)\} = \delta_{jk}\delta^3(\xi_i - \xi'_i). \tag{23}
\]

Apparently the Poisson brackets (23) and the Hamiltonian (22) define the equations of motion for the canonical variables, which are equivalent to the Lagrange equations. The phase space \( \Gamma \) of the fluid is formed by \( \vec{x}(\xi_i, t) \) and \( \vec{p}(\xi_i, t) \). What is missing at the moment is the space interpretation of the variables \( \vec{x}(\xi_i, \cdot) \) and \( \vec{p}(\xi_i) \). To get one let us introduce the new objects using the same averaging, as we used in the previous section

\[
\bar{l}(x) = \int d^3\xi \vec{p}(\xi_i)\delta(\vec{x} - \vec{x}(\xi_i)) = \rho(x)\vec{p}(\xi_i(x)).
\]

This new object will serve as a part of the coordinates in the phase space \( \Gamma \), therefore we need to calculate its Poisson brackets, using (23). The result has the following form:

\[
\{l_j(x_i), l_k(y_i)\} = \left[l_k(x_i)\frac{\partial}{\partial x_j} + l_j(y_i)\frac{\partial}{\partial x_k}\right]\delta(\vec{x} - \vec{y}). \tag{25}
\]
The Poisson brackets was introduced geometrically as "the hydrodynamic-type brackets" long ago in the papers [5] without physical derivation. The present discussion reveals the origin of these brackets. The commutation relation (25) coincides with algebra of 3-dimensional diffeomorphisms, where the $l_j(x_i)$ serves as the generators. In other words, with the help of these generators we can realize the finite diffeomorphism $x_j \rightarrow \phi_j(x_i)$ of any $x$-dependent dynamical variable. It should be mentioned that the group of diffeomorphisms, generated by $l_j(x_i)$ is not a gauge symmetry in case of fluid mechanics, as it is in the case of e.g. relativistic string or membrane. In the same time in fluid mechanics there is an infinite dimensional symmetry with respect to special (i.e. volume preserving) diffeomorphisms $SDiff$, which will be considered in the next section.

As we mentioned above, the $l_j(x_i)$ is only a part of $x$-dependent coordinates in the phase space $\Gamma$, one may consider it as the "$x$-dependent momenta". Now we should define the corresponding "$x$-dependent coordinates". For example we may choose for this role the functions $\xi_j(x_i)$, which are the inverse to $x_j(\xi_i)$ functions (1). Differentiating the first equation (1) with respect to $x$ we obtain:

$$A^j_k(\xi_i(x_k)) \frac{\partial \xi_k(x_i)}{\partial x_l} = \delta^j_l,$$

in other words, the matrix

$$a^k_i(x_i) = \frac{\partial \xi_k(x_i)}{\partial x_i}$$

is inverse to $A^j_k(\xi_i(x_k))$. Therefore, from (14) follows that

$$\rho(x_i) = \text{det}(x_i)$$

To simplify the calculation of the Poisson brackets we can express $\xi_j(x_i)$ in the following form:

$$\xi_j(x_i) = \frac{\int d^3 \xi_j \delta(\vec{x} - \vec{x}(\xi_i, t))}{\int d^3 \delta(\vec{x} - \vec{x}(\xi_i, t))}.$$  

From (29) we easily obtain

$$\{\xi_j(x_i), \xi_k(y_i)\} = 0$$
The calculation of the Poisson brackets between \( \xi_j(x_i) \) and \( l_j(x_i) \) is more involved, but the result is simple:

\[
\{ l_j(x_i), \xi_k(y_i) \} = -\frac{\partial \xi_k(x_i)}{\partial x_j} \delta(\vec{x} - \vec{y})
\]  
(31)

In such a way we have constructed the set of \( x \)-dependent coordinates \((l_j(x_i), \xi_j(x_i))\) in the phase space \( \Gamma \), but the transformation

\[
(p_j(\xi_i), x_j(\xi_i)) \rightarrow (l_j(x_i), \xi_j(x_i))
\]
(32)

is not canonical. The set of canonical \( x \)-dependent coordinates in \( \Gamma \) could be obtained in the following way. Let us multiply both sides of (31) by matrix \( A^j_m(\xi_i(x_k)) \):

\[
A^j_m(\xi_i(x_k))\{l_j(x_i), \xi_k(y_i)\} = -\delta^k_m \delta(\vec{x} - \vec{y}),
\]  
(33)

where we have used (26). Further, due to the relation (30), we can put

\[
\{\pi_m(x_i), \xi_k(y_i)\} = \delta^k_m \delta(\vec{x} - \vec{y})
\]
(34)

where

\[
\pi_m(x_i) = -A^j_m(\xi_i(x_k))l_j(x_i)
\]  
(35)

By direct calculation we also obtain

\[
\{\pi_m(x_i), \pi_k(y_i)\} = 0
\]
(36)

so the set \((\pi_m(x_i), \xi_k(y_i))\) is formed by the canonical variables. In terms of these canonical variables the generators of the group of diffomorphisms \( l_j(x_i) \) has the following form:

\[
l_j(x_i) = -\frac{\partial \xi_k(x_i)}{\partial x_j} \pi_k(x_i).
\]
(37)

For the Lagrangian considered, from (21) and (24) follows that \( l_j(x_i) \) is given by

\[
l_j(x_i) = m\rho(x_i)v_j(x_i)
\]
(38)

and the representation (37) is very similar to Clebsh parametrization of the velocity. The distinction of (37) from the original Clebsh paramerization
is the appearance of three ”potentials”, instead of two, for the 3-dimensional fluid. The reason of this difference will be discussed in the end of this section.

The above given construction of the canonical $x$-dependent variables is the most simple one. For some problems another set might be useful. Let us construct one denoting by $\zeta_j(x_i)$ the canonical coordinates:

$$\zeta_j(x_i) = \int d^3 \xi_j \delta(\vec{x} - \vec{x}(\xi_i, t)).$$

(39)

Calculating the Poisson brackets of $\zeta_j(x_i)$ and $l_j(x_i)$ we obtain:

$$\{l_j(x_i), \zeta_k(y_i)\} = \zeta_k(x_i) \frac{\partial}{\partial x_j} \delta(\vec{x} - \vec{y}).$$

(40)

From (40) we conclude that $l_j(x_i)$ should have the following representation

$$l_j(x_i) = \zeta_k(x_i) \frac{\partial}{\partial x_j} \eta_k(x_i),$$

(41)

where $\eta_k(x_i)$ is the variable, canonically conjugated to $\zeta_j(x_i)$:

$$\{\eta_j(x_i), \zeta_k(y_i)\} = \delta_{jk} \delta(\vec{x} - \vec{y}).$$

(42)

The canonical Hamiltonian (22) should now be expressed in the terms of new variables $(\pi_m(x_i), \xi_k(y_i))$. Indeed, let us insert the unity

$$1 = \int d^3 x \delta(\vec{x} - \vec{x}(\xi_i))$$

(43)

into the integrand (22) and change the order of integration:

$$H = \int d^3 x \int d^3 \xi \left( \frac{1}{2m} \rho^2(\xi_i, t) + f(detA(\xi_i)) \right) \delta(\vec{x} - \vec{x}(\xi_i)),$$

(44)

Fulfilling the integration over $\xi$ with the help of (2) and (24) we obtain:

$$H = \int d^3 x \left( \frac{1}{2m \rho(x_i)} \vec{p}^2(x_i) + \rho(x_i) f\left( \frac{1}{\rho(x_i)} \right) \right).$$

(45)

Making use of (37) we can express $H$ via canonical variables $(\pi_m(x_i), \xi_k(y_i))$:

$$H = \int d^3 x \left( \frac{1}{2m \rho(x_i)} \frac{\partial \xi_k(x_i)}{\partial x_j} \frac{\partial \xi_m(x_i)}{\partial x_j} \pi_k(x_i) \pi_m(x_i) + V(\rho(x_i)) \right).$$

(46)
where we have introduced a notation $V(\rho(x_i))$ for the ”potential” part of the energy. This term represents the internal energy of the fluid and it should vanish (up to inessential constant) for homogeneous density distribution $\rho(x_i) = \rho_0$. A phenomenological expression for $V(\rho(x_i))$ could be written as follows [7]:

$$V(\rho(x_i)) = \frac{\kappa}{2\rho_0}(\delta\rho(x_i))^2 + \lambda(\nabla\rho(x_i))^2 + ... , \quad (47)$$

where $\delta\rho(x_i)$ is the deviation of the density from its homogeneous distribution:

$$\delta\rho(x_i) = \rho(x_i) - \rho_0. \quad (48)$$

The first term in (47) is responsible for the sound wave in the fluid ($\kappa$ is the velocity of sound), the second term in (47) describes the dispersion of the sound waves.

In order to reveal the relation of the Hamiltonian flow, generated by (46) with geodesic flow [3] let us introduce the metric tensor $g_{jk}(x_i)$:

$$g_{jk}(x_i) = A^m_j(\xi_i(x_k))A^m_k(\xi_i(x_k)). \quad (49)$$

With this notation (46) takes the form:

$$H = \int d^3x \left( \frac{1}{2m}\sqrt{g(x_i)}g^{km}(x_i)\pi_k(x_i)\pi_m(x_i) + V(\frac{1}{\sqrt{g(x_i)}}) \right). \quad (50)$$

The metric tensor with upper indices $g^{jk}(x_i)$ denotes, as usually the inverse matrix and

$$g(x_i) = det g_{jk}(x_i) = \frac{1}{\rho^2(x_i)} \quad (51)$$

The representation (50) permit us to consider the hydrodynamics as the geodesic flow on the dynamical manifold with metric $g_{jk}(x_i)$ and many general properties of the hydrodynamics could be derived from this fact (see e.g. [3]).

We shall complete this section with one important comment concerning our description of hydrodynamics. In the present discussion the phase space of the 3-dimensional fluid is 6-dimensional, as it naturally follows from our approach. This could be compared with recent paper [3], where (in our
notations) only $\rho(x_i)$ and $v_j(x_i)$ are regarded as the coordinates in the phase space (this point of view of could be found also in different text books and articles) see for example [3], [7], [10].

According to the conventional point of view the state of the fluid is determined by its velocity and density and therefore all other variables like ours $\vec{x}(\xi)$ are not needed. Indeed, solving the Euler equations of motion we can express the velocity $\vec{v}(x_i, t)$ and the density $\rho(x_i, t)$ at time $t$ via the initial data $\vec{v}(x_i, 0)$ and $\rho(x_i, 0)$. Then, from the definition of $\vec{v}(x_i, t)$ (3) follows

$$\vec{v}(x(\xi, t), t) = \dot{x}(\xi, t).$$

(here we have suppressed the indices of the arguments for brevity). Apparently we can solve these equations with respect to $\vec{x}(\xi, t)$, provided we know the initial data $\vec{x}(\xi, 0)$. Therefore it may seem that the variables $\vec{x}(\xi, t)$ are unnecessary, as it could be obtained through the others. But the initial data $\vec{x}(\xi, 0)$ are half of the canonical variables in Hamiltonian formalism. So, in our approach we indeed need more variables. These additional variables provide the complete description of the fluid in a sense that solving the equations of motion we define not only the $\vec{v}(x_i, t)$ and $\rho(x_i, t)$ but also the trajectories of the particles which could not be obtained in the conventional formalism. The situation is analogous to the rigid body rotation: here the phase space $\Gamma$ is formed by 3 angles and 3 components of the angular momentum $J_i$. As the Hamiltonian depends on the components of $J_i$ only, we can consider separately the evolution of the angular momentum. This is incomplete description for this mechanical system. The complete one certainly should includes the evolution of the angles, which define the location of the rigid body in space.

In the case of fluid dynamics the complete description is to be done in the 6-dimensional phase space $\Gamma$ formed by $\vec{x}(\xi)$ and $\vec{p}(\xi)$ (or $\vec{L}(x)$ and $\vec{l}(x)$). If, as it is usually the case, the Hamiltonian depends only on $\vec{L}(x)$ and $\rho(x)$, partial description in terms of the velocities and densities considered as "relevant" variables is possible. This means that we do not care about the evolution of the coordinates of the phase space which are considered as inessential. The "relevant" part of the coordinates does not necessarily form a simplectic subspace in $\Gamma$. The nondegenerate Poisson brackets in $\Gamma$ could become degenerate on the subset of $\Gamma$, corresponding to the "relevant" variables. This is indeed the case in the rigid body and in the conventional fluid dynamics.
In the first case the degeneracy of the algebra of Poisson brackets for the "relevant" variables — angular momentum is well known. Its center element is Casimir operator of the rotation group. In the case of fluid dynamics the algebra of the "relevant" variables — velocities and densities has the following form:

\[
\{v_j(x_i), v_k(y_i)\} = -\frac{1}{\rho(x)} \left( \nabla_j v_k(x_i) - \nabla_k v_j(x_i) \right) \delta(\vec{x} - \vec{y})
\]

\[
\{v_j(x_i), \rho(y_i)\} = \nabla_j \delta(\vec{x} - \vec{y})
\]

\[
\{\rho(x_i), \rho(y_i)\} = 0
\]

This algebra could be obtained from equations (25), (30) and (31). The center of this algebra is infinite-dimensional and its structure depends on the dimension of \(x\)-space. In 2-dimensional case the center is formed by:

\[
I_k = \int dxdy \left( \frac{\partial v_1(x, y)}{\partial y} - \frac{\partial v_2(x, y)}{\partial x} \right)^k \rho^{1-k}(x, y),
\]

see [3], [9] for the discussion. In 3-dimentional case the general answer is not known to our knowledge and its construction is out of the scope of our paper. As an example we mention that Hopf invariant

\[
Q = \int d^3x \epsilon_{jkl} v_j(\vec{x}) \nabla_k v_l(\vec{x})
\]

belongs to this center.

4 Infinite-dimensional symmetry and integrals of motion.

As we have mentioned above, the Lagrangian (17) possesses the invariance with respect to the "volume preserving" group of diffeomorphisms \(SDiff[D]\), where \(D \subseteq \mathbb{R}^3\). Indeed, let us write the Lagrangian (17) in terms of the Lagrangian density

\[
L = \int d^3\xi L(\xi_i)
\]

and consider the transformations from \(SDiff[D]\) of the coordinates \(\xi_i \in D\)

\[
\xi_j \rightarrow \xi'_j = \phi_j(\xi_i),
\]
\[ \det \frac{\partial \phi_j(\xi_i)}{\partial \xi_k} = 1. \] 

(58)

Apparently, due to (58) we obtain:

\[ L = \int d^3 \phi(\xi_i) L(\phi(\xi_i)) = \int d^3 \xi L(\phi(\xi_i)) \] 

(59)

and according to Noether’s theorem this invariance results in the existence of an infinite set of integrals of motion. To obtain these integrals we first need to find the parametrization of the transformations (57), (58) in the vicinity of identity transformation:

\[ \phi_j(\xi_i) = \xi_j + \alpha_j(\xi_i). \] 

(60)

From (58) follows the equation for \( \alpha_j(\xi_i) \):

\[ \frac{\partial \alpha_j(\xi_i)}{\partial \xi_j} = 0. \] 

(61)

Further we must explicitly take into account that the volume preserving diffeomorphism (57) leaves the boundary of \( D \) invariant. We shall limit ourself with the case when \( D \) is formed by extraction of the domain with the differentiable boundary given by

\[ g(\xi_i) = 0 \] 

(62)

from \( R^3 \). Physically that means that we put in the fluid the fixed body, the shape of which is given by (62). The condition that the infinitesimal diffeomorphism (63) preserves \( D \) in this case is

\[ g(\xi_j + \alpha_j(\xi_i)) \bigg|_{g(\xi_i)=0} = 0, \] 

(63)

or

\[ \alpha_j(\xi_i) \nabla_j g(\xi_i) \bigg|_{g(\xi_i)=0} = 0. \] 

(64)

Geometrically equation (64) means that the vector \( \vec{\alpha}(\xi_i) \) is tangent to the surface, defined by (62), because the vector \( \nabla_j g(\xi_i) \bigg|_{g(\xi_i)=0} \) is proportional to the normal \( n_j(\xi_i) \) of the surface (62) at the point \( \xi_i \).
From Noether’s theorem we obtain that the invariance of the Lagrangian with respect to the transformation (60) gives the following conservation law:

$$\frac{\partial}{\partial t} \int_D d^3 \xi p_m(\xi_i) \frac{\partial x_m(\xi_i)}{\partial \xi_l} \alpha_l(\xi_i) = 0,$$

(65)

where $\alpha_l(\xi_i)$ satisfies to the conditions (61) and (64). The existence of these conditions forbids to take the variation of the l.h.s. of (65) over $\alpha_l(\xi_i)$ and obtain the local form of integrals of motion. For that we need to extract from (61) and (64) the integral properties of $\alpha_l(\xi_i)$.

Consider an arbitrary, single-valued, differentiable in $D$ function $\beta(\xi_i)$. Then the following equations are valid:

$$0 = \int_D d^3 \xi \beta(\xi_i) \frac{\partial \alpha_j(\xi_i)}{\partial \xi_j} = \int_D d^3 \xi \frac{\partial}{\partial \xi_j} \left( \beta(\xi_i) \alpha_j(\xi_i) \right) - \int_D d^3 \xi \alpha_j(\xi_i) \frac{\partial \beta(\xi_i)}{\partial \xi_j},$$

(66)

The first equality is valid due to condition (61). Using Stokes theorem we can transform the integral of total derivative in the last equality (66):

$$\int_D d^3 \xi \frac{\partial}{\partial \xi_j} \left( \beta(\xi_i) \alpha_j(\xi_i) \right) = \int_{\partial D} dS_j \left( \beta(\xi_i) \alpha_j(\xi_i) \right) = 0,$$

(67)

where $\partial D$ denotes the boundary of $D$. The last integral in (67) vanishes due to condition (64) because the differential $dS_j$ is proportional to the normal vector of the surface $\partial D$, defined by (62). From (66) and (67) we conclude that

$$\int_D d^3 \xi \alpha_j(\xi_i) \frac{\partial \beta(\xi_i)}{\partial \xi_j} = 0$$

(68)

for any smooth, differentiable $\beta(\xi_i)$. Taking this property of $\alpha_j(\xi_i)$ into account we obtain from the conservation laws (65) that the quantities

$$J_k(\xi_i) = p_m(\xi_i) \frac{\partial x_m(\xi_i)}{\partial \xi_k}$$

(69)

are conserved modulo some term which is the gradient of a scalar. In particular, that means that

$$R_j(\xi_i) = \epsilon_{jkl} \frac{\partial}{\partial \xi_k} J_l(\xi_i) = \epsilon_{jkl} \frac{\partial}{\partial \xi_k} \left( p_m(\xi_i) \frac{\partial x_m(\xi_i)}{\partial \xi_l} \right).$$

(70)
is the integrals of motion. Note, that as the group of invariance is infinite-dimensional, we obtain an infinite number of integrals of motion. With respect to Poisson brackets (23) the \( R_j(\xi) \)'s form an algebra. This algebra could be written in a compact form for integrated objects

\[
R[\phi] = \int d^3\xi \phi_j(\xi) R_j(\xi),
\]

(71)

where \( \phi_j(\xi) \) are smooth, rapidly decreasing functions. The algebra of \( R[\phi] \) induced by Poisson brackets (23) has the form:

\[
\{ R[\phi], R[\psi] \} = R[\text{curl}\phi \times \text{curl}\psi]
\]

(72)

The construction of the \( x \)-dependent object, corresponding to \( R_j(\xi) \) is not an easy task, because our ”averaging” with \( \delta(\vec{x} - \vec{x}(\xi)) \) will introduce time dependence and instead of conserved object we shall obtain a density, whose time derivative gives a divergence of a ”current”. Therefore we need to introduce another kind of averaging without explicit refering to the \( \vec{x}(\xi) \) coordinates. For that recall that under diffeomorphism a closed loop transforms into closed loop. Then let us consider such a loop \( \lambda \) and a surface \( \sigma \) whose boundary is \( \lambda \). The integral

\[
V = \int_\sigma dS_j R_j(\xi)
\]

(73)

where the vector \( dS_j \) is as usually the area element times the vector, perpendicular to the surface, is conserved, because of the conservation of \( R_j(\xi) \). Further, from Stokes theorem we have:

\[
V = \oint_\Lambda dx_j p_m(\xi) \frac{\partial x_m(\xi)}{\partial \xi_j}.
\]

(74)

Changing the variables in (74) we obtain:

\[
V = \oint_\Lambda dx_j \frac{l_j(x_i)}{\rho(x_i)} = \oint_\Lambda dx_j v_j(x_i),
\]

(75)

where \( \Lambda \) is the image of the loop \( \lambda \) under diffeomorphism \( \xi_j \rightarrow x_j(\xi_i) \). The object (75) is very well known in hydrodynamics as the ”circulation” and its conservation is known as W.Thompson theorem \[8\]. The relation of the
circulation conservation with the invariance under special diffeomorphisms was first explicitly established in [11], though it also could be extracted from general discussion in Appendix 2 of [3].

Conservation of circulation is not the only consequence of (65). Consider for example the case of 2-dimensional space. Here, instead of the conserved vector \( R_j(\xi_i) \) we shall have the conserved scalar

\[
R(\xi_i) = \epsilon_{kl} \frac{\partial}{\partial \xi_k} J_l(\xi_i) = \epsilon_{kl} \frac{\partial}{\partial \xi_k} \left( p_m(\xi_i) \frac{\partial x_m(\xi_i)}{\partial \xi_l} \right),
\]

(76)

This scalar defines the following integrals of motion:

\[
I_n = \int_D d^2 \xi R^n(\xi_i) = \int_D d^2 \xi \left( \epsilon_{kl} \frac{\partial}{\partial \xi_k} \left( p_m(\xi_i) \frac{\partial x_m(\xi_i)}{\partial \xi_l} \right) \right)^n.
\]

(77)

Changing variables in (77) \( \xi_j \rightarrow \xi_j(x_i) \) which is possible, because \( \xi_j(x_i) \) is a diffeomorphism of \( D \), we obtain:

\[
I_n = \int_D d^2 x \rho(x_i) \left( \epsilon_{kl} A^j_k(\xi(x)) A^m_l(\xi(x)) \frac{\partial p_m(\xi(x))}{\partial x_j} \right)^n,
\]

(78)

where the matrix \( A^j_k(\xi) \) was defined in (14). We can present \( I_n \) as

\[
I_n = \int_D d^2 x \rho(x_i) \left( \epsilon_{jm} \frac{\partial}{\partial x_j} l^m_m(x_i) \right)^n.
\]

(79)

using the following property of 2-dimensional matrix \( A^j_k(\xi) \):

\[
\epsilon_{kl} A^j_k(\xi(x)) A^m_l(\xi(x)) = \epsilon_{jm} det A(\xi(x)) = \epsilon_{jm} \frac{1}{\rho(x_i)},
\]

(80)

Recall that in the case of the Lagrangian (17), we are considering

\[
\frac{l_j(x_i)}{\rho(x_i)} = m v_j(x_i)
\]

(81)

Therefore the integrals \( I_n \) coincide with the center (54) of the Poisson algebra (53).
In the case of 3-dimensional space we can construct the analogous integrals of motion, integrating the products of the vector (70):

\[ K_{j_1,j_2...j_n} = \int_D d^3\xi R_{j_1}(\xi_i) R_{j_2}(\xi_i) ... R_{j_n}(\xi_i) \]  \hspace{1cm} (82)

Changing variables \( \xi_j \to \xi_j(x_i) \) as above we shall obtain:

\[ R_j(\xi) \to R_j(\xi(x)) = \epsilon_{jkl} A^n_k(\xi(x)) A^n_l(\xi(x)) \frac{\partial p_n(\xi(x))}{\partial x_m}. \]  \hspace{1cm} (83)

In the 3-dimensional case the matrices \( A^m_j(\xi(x)) \) satisfy the equation:

\[ \epsilon_{jkl} A^m_k(\xi(x)) A^m_l(\xi(x)) = \epsilon_{mnr} \frac{\partial \xi_j(x)}{\partial x_r} \text{det} A(\xi(x)), \]  \hspace{1cm} (84)

Therefore (83) takes the form:

\[ R_j(\xi(x)) = \frac{1}{\rho(x_i)} \epsilon_{mnr} \frac{\partial \xi_j(x)}{\partial x_r} \frac{\partial p_n(\xi(x))}{\partial x_m}. \]  \hspace{1cm} (85)

The integrals \( K_n \) in (82) become

\[ K_{j_1,j_2...j_n} = \int_D d^3x \rho(x_i) R_{j_1}(\xi(x)) R_{j_2}(\xi(x)) ... R_{j_n}(\xi(x)) \]  \hspace{1cm} (86)

Apparently, due to the presence of \( \frac{\partial \xi_j(x)}{\partial x_r} \) in (85), we can not in this case express (86) only in terms of velocity and density, i.e. the Eulerian description does not admit this kind of integrals of motion.

In the 3-dimensional case there is one more integral, which does not exist in any other dimension. Recall that the vector \( J_k(\xi_i) \), given by (69) is conserved modulo gradient, therefore the integral

\[ Q = \int_D d^3\xi J_k(\xi_i) R_k(\xi_i) \]  \hspace{1cm} (87)

is conserved because \( \vec{R}(\xi_i) = \text{curl} \vec{J}(\xi_i) \). Transforming the \( \xi_j \)-dependent variables to the \( x_j \)-dependent ones in (87) we obtain Hopf invariant:

\[ Q = \int_D d^3x \epsilon_{jkl} \rho J_j(\vec{x}(\xi(x))) \frac{\partial p_k(\xi(x))}{\partial x_l}. \]  \hspace{1cm} (88)
5 Inclusion of the electromagnetic interaction. Plasma.

We shall consider plasma as a fluid of two components, namely electrons with mass \( m \) and electric charge \((-e)\) and ions with mass \( M \) and charge \((+e)\). The coordinates of electrons we shall denote as \( \vec{x}(\xi_i) \), while the coordinates of ions will be \( \vec{X}(\Xi_i) \). The interaction of the components of the plasma with the electromagnetic field \( A_\mu(x) \) is governed by the following Lagrangian:

\[
L = L^e_0 + L^i_0 + \int d^3\xi \left[ \frac{m}{2} \ddot{\vec{x}}(\xi_i, t) - f^e(\text{det}\partial_x(\xi_i, t)) \right] - e \left( A_0(\vec{x}(\xi_i, t)) - \dot{\vec{x}}(\xi_i, t) \cdot A(\vec{x}(\xi_i, t)) \right) - \frac{1}{4} \int d^3x F^{\mu\nu}(x,t) F_{\mu\nu}(x,t),
\]

(89)

where \( L^e_0 \) and \( L^i_0 \) are "free" Lagrangians

\[
L^e_0 = \int d^3\xi \left[ \frac{m\ddot{x}_i(\xi_i, t)}{2} - f^e(\text{det}\partial_x(\xi_i, t)) \right],
\]

(90)

\[
L^i_0 = \int d^3\Xi \left[ \frac{M\ddot{X}_i(\Xi_i, t)}{2} - f^i(\text{det}\partial_X(\Xi_i, t)) \right],
\]

and \( F_{\mu\nu}(x,t) \) denotes the electromagnetic field tensor:

\[
F_{\mu\nu}(x,t) = \partial_\mu A_\nu(x,t) - \partial_\nu A_\mu(x,t)
\]

(91)

The advantage of Lagrangian description of fluid (plasma) is clear. Using of the coordinates of charged particles as the fundamental variables makes explicit the interaction with electromagnetic field.

The Lagrangian (89) possesses \( U(1) \) gauge invariance. We shall use the usual Hamiltonian formalism for the constraint system [12]. The canonical variables are:

\[
\vec{x}(\xi_i), \quad \vec{p}(\xi_i) = \frac{\delta L}{\delta \dot{\vec{x}}(\xi_i)};
\]

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\[ \vec{X}(\Xi_i), \quad \vec{P}(\Xi_i) = \frac{\delta L}{\delta \vec{X}(\Xi_i)}; \]
\[ \vec{A}(x_i), \quad \vec{P}_{em}(x_i) = \frac{\delta L}{\delta \vec{A}(x_i)} = -\vec{E}(x_i); \]
\[ A_0(x_i), \quad P^0_{em}(x_i) = 0, \quad (92) \]

where \( \vec{E}(x_i) \) is the electric field strength:
\[ \vec{E}(x_i) = -\nabla A_0(x_i) - \vec{A}(x_i). \quad (93) \]

The last of the equations \((92)\) is actually the primary constraint. The Legendre transformation of the Lagrangian \((89)\) gives us the canonical Hamiltonian:
\[ H = \int d^3x \left[ \frac{1}{2} \left( \vec{P}^2_{em}(x_i) + \vec{H}^2(x_i) \right) + A_0(x_i) \nabla \Pi(x_i) \right] + \int d^3\xi \left[ \frac{1}{2m} \left( \vec{p}(\xi_i) + e\vec{A}(x(\xi_i)) \right)^2 - eA_0(x(\xi_i)) + f_{el}(det \frac{\partial x(\xi_i, t)}{\partial \xi_k}) \right] + \int d^3\Xi \left[ \frac{1}{2m} \left( \vec{P}(\Xi_i) + e\vec{A}(X(\Xi_i)) \right)^2 + eA_0(X(\Xi_i)) + f_{ion}(det \frac{\partial X(\Xi_i, t)}{\partial \Xi_k}) \right], \quad (94) \]

where \( \vec{H}(x_i) = \text{curl} \vec{A}(x_i) \) is the magnetic field strength. The requirement of the conservation of the primary constraint \( \Pi_0(x_i) = 0 \) gives the secondary constraint (Gauss law):
\[ \nabla_j (P_{em}(x_i))_j - e(\rho_{el}(x_i) - \rho_{ion}(x_i)) = 0. \quad (95) \]

Now we can add to the primary constraint \( \Pi_0(x_i) = 0 \) the gauge fixing condition \( A_0(x_i) = 0 \) and eliminate these variables from consideration. Introducing the \( x \)-dependent functions instead of \( u \) and \( U \)-dependent, as we did in the 2-nd section we obtain the Hamiltonian of the plasma in the following form:
\[ H = \int d^3x \left[ \frac{1}{2} \left( \vec{P}^2_{em}(x_i) + \vec{H}^2(x_i) \right) + \frac{\left( \vec{l}(x_i) + e\rho_{el}(x_i)\vec{A}(x_i) \right)^2}{2m\rho_{el}(x_i)} \right] + \frac{\left( \vec{L}(x_i) + e\rho_{ion}(x_i)\vec{A}(x_i) \right)^2}{2m\rho_{ion}(x_i)} + v_{el}(\rho_{el}(x_i)) + v_{ion}(\rho_{ion}(x_i)) \],
\[ (96) \]

where we have introduced
\[ \vec{l}(x_i) = \int d^3\xi \vec{p}(\xi_i) \delta(\vec{x} - \vec{x}(\xi_i)) \]
\[ \vec{L}(x_i) = \int d^3\Xi \vec{P}(\Xi_i) \delta(\vec{x} - \vec{X}(\Xi_i)) \quad (97) \]
This Hamiltonian is gauge invariant with respect to the transformations, generated by the constraint \( (15) \). Further, we impose the Coulomb gauge condition on the electromagnetic field

\[
\nabla_j A_j(x_i) = 0
\]

and following the usual procedure will eliminate the longitudinal components of \( \vec{A}(x_i) \) and \( \vec{P}_{em}(x_i) \). As a result of the gauge fixing, the longitudinal part of \( \vec{P}_{em}(x_i) \) give rise to the Coulomb term in the Hamiltonian

\[
H_{Col} = -e^2 \int d^3x \left( \rho_{el}(x_i) - \rho_{ion}(x_i) \right) \left( -\frac{1}{4\pi|\vec{x} - \vec{y}|} \right) \left( \rho_{el}(y_i) - \rho_{ion}(y_i) \right).
\]

and the whole Hamiltonian takes the following form:

\[
H = H_{Col} + \int d^3x \left[ \frac{1}{2} \left( \vec{P}_{em}^2(x_i) + \vec{H}^2(x_i) \right) + \frac{\left( \tilde{L}(x_i) + e\rho_{el}(x_i)\vec{A}_\perp(x_i) \right)^2}{2m\rho_{el}(x_i)} + v_{el}(\rho_{el}(x_i)) \right.
\]
\[
+ \left. \frac{\left( \tilde{E}(x_i) + e\rho_{ion}(x_i)\vec{A}_\perp(x_i) \right)^2}{2m\rho_{ion}(x_i)} + v_{ion}(\rho_{ion}(x_i)) \right],
\]

where the subscript \( \perp \) denotes the transverse components of electromagnetic variables, for which the Poisson (Dirac) brackets are given by \( (12) \)

\[
\{\pi_j(x_i), \xi_k(y_i)\} = \delta_{jk} - \frac{1}{\Delta} \partial_j \partial_k \delta(\vec{x} - \vec{y}),
\]
\[
\{\Pi_j(x_i), \Xi_k(y_i)\} = \delta_{jk} \delta(\vec{x} - \vec{y}),
\]

As usually, the Coulomb term \( (3.11) \) contains an infinite additive part of "self interaction" which should be subtracted. In equilibrium plasma the Coulomb interaction is known to be screened by the cloud and the residual Debye interaction is a short range one (see e.g. \( [13] \)). In \( [14] \) it was stated that Coulomb term is reduced to the local functional of the difference of charge densities \( (\rho_{el}(x_i) - \rho_{ion}(x_i)) \).

Following our consideration of a fluid in the Section 3, we can introduce the canonical coordinates \( \vec{\xi}(x_i), \vec{\pi}(x_i) \) for electron and \( \vec{\Xi}(x_i), \vec{\Pi}(x_i) \) for ion components of the plasma:

\[
\{\pi_j(x_i), \xi_k(y_i)\} = \delta_{jk}\delta(\vec{x} - \vec{y}),
\]
\[
\{\Pi_j(x_i), \Xi_k(y_i)\} = \delta_{jk}\delta(\vec{x} - \vec{y}),
\]

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for which

\[ l_j(x_i) = \xi(x_i) \frac{\partial \pi(x_i)}{\partial x_j}, \]

\[ L_j(x_i) = \Xi(x_i) \frac{\partial \Pi(x_i)}{\partial x_j}. \] (103)

Substituting (103) into (100) we obtain the Hamiltonian in terms of the canonical variables.

The Lagrangian (89) is invariant with respect to the volume preserving diffeomorphisms of both components of plasma separately, so we shall have in this case two sets of conservation laws – one for electrons, the other for ions. Applying the procedure, which we have described in the previous section we shall construct the conserved circulations:

\[ V_{el}^{\Lambda} = \oint_{\Lambda} dx_j \frac{l_j(x_i)}{\rho_{el}(x_i)}, \]

\[ V_{ion}^{\Lambda} = \oint_{\Lambda} dx_j \frac{L_j(x_i)}{\rho_{ion}(x_i)}, \] (104)

where \( l_j(x) \) and \( L_j(x) \) are given by (103). In the same way we can construct the analogues of the integrals (86) and (88) for this case. Note that in the case of plasma the equation (38) is not valid due to the presence of the electromagnetic field, instead we have:

\[ \vec{l}(x) = \rho_{el}(x) \left( m \vec{v}_{el}(x) - e \vec{A}(x) \right), \]

\[ \vec{L}(x) = \rho_{ion}(x) \left( M \vec{v}_{ion}(x) + e \vec{A}(x) \right), \] (105)

therefore the equations (104) are not the circulations of the velocities. In the same time, adding circulations \( V_{el}^{\Lambda} \) and \( V_{ion}^{\Lambda} \) on the common contour \( \Lambda \) we obtain:

\[ V_{\Lambda} = V_{el}^{\Lambda} + V_{ion}^{\Lambda} = \oint_{\Lambda} dx_j \left( m v_{el}(x) + M v_{ion}(x) \right), \] (106)

so this linear combination of circulations of electons and ions velocities is conserved even in the presence of the electromagnatic interaction.

Summarizing we can say that the phase space of plasma \( \Gamma \) in Coulomb gauge is the space with coordinates \( \vec{A}_\perp(x), \vec{P}_{em}_\perp(x); \vec{\xi}(x_i), \vec{\pi}(x_i); \vec{\Xi}(x_i), \vec{\Pi}(x_i) \) with Poisson brackets given by (101) and (102). The evolution of a state in \( \Gamma \) is defined by the Hamiltonian (100).
6 Concluding remarks

The Hamiltonian approach for ideal fluid and plasma, considered in the present paper in some aspects resembles the Hamiltonian formulation of a classical field. It deals with the system with infinite number of degrees of freedom – ”particles” which constitutes a continuous media. These particles interact with nearest neighboors due to the potential part of the Lagrangian or Hamiltonian, as the field amplitudes in conventional field theory. The difference with the later, which we emphasized in the course of the paper is that the constituents of the fluid change their position in space while evolution and to have a local description of fluid in terms of $x$-dependent variables we need the projection realized by e.g. equation (4). The fluid dynamics could be also compared with the mechanics of the extended objects like $n$-branes. These objects are also $n$-dimensional media embeded in the space-time, but its constituents are indistinguishable and due to this property the general diffiomorphisms of its coordinates appear as the gauge invariance of the theory. In the case of the fluid it is supposed that the constituents are distinguishable and, as a result only special diffiomorphisms preserve the Lagrangian giving rise to the Noether’s integrals of motion.

So far we have considered only classical mechanics of the fluid and plasma, but as we know any classical Hamiltonian system could be quantized. Here as a quantization we mean a formal procedure of the transition from classical canonical coordinates to the operators, satisfying commutation relations inherited from Poisson brackets and the construction of a represen- tation of these operators in an appropriate space. Of course this procedure has no physical meaning in the case of plasma, but for a fluid it could be quite reasonable providing us with the theory of an ideal quantum fluid. The quantum fluid could be considered as the new representation of a quantum field in which the eigenvalue of the operator of density of particles may continuos function of $x$. The matter is that in the case of usual quantum field we know only Fock representation [13], for which the operator of density of particles in space (or in momentum space) has the eigenvalues which are the superposition of $\delta$-functions, that means that the average density is zero and to consider the states with finite density we need to go outside the Fock space.

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