Central extensions of groups of symplectomorphisms

Yuri A. Neretin

We construct canonically defined central extensions of groups of symplectomorphisms. We show that this central extension is nontrivial in the case of a torus of dimension $\geq 6$ and in the case of a two-dimensional surface of genus $\geq 3$.

1 Formulation of results

Central extensions of the groups of symplectomorphisms discussed in this paper appeared as a byproduct in [25]. Here we prove several nontriviality and triviality theorems concerning this cocycle.

1.1. Preliminaries. Cocycle on the symplectic group $\text{Sp}(2n, \mathbb{R})$. We define the real symplectic group $\text{Sp}(2n, \mathbb{R})$ as the group of real matrices
\[
g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}
\] (1.1)
preserving the standard skew-symmetric bilinear form $K := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, i.e.
\[g^t K g = K \] (1.2)
The complex symplectic group $\text{Sp}(2n, \mathbb{C})$ is the group of complex matrices satisfying the same condition (1.2).

Consider the block $(n + n) \times (n + n)$ matrix $J \in \text{Sp}(2n, \mathbb{C})$ given by
\[
J = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}
\] (1.3)
For $g \in \text{Sp}(2n, \mathbb{R})$, we consider the matrix $J^{-1}gJ \in \text{Sp}(2n, \mathbb{C})$, this matrix has the structure
\[J^{-1}gJ = \begin{pmatrix} \Phi & \Psi \\ \bar{\Psi} & \bar{\Phi} \end{pmatrix}\]
where bar means the element-wise complex conjugation. We denote
\[
\Phi = \Phi(g); \quad \Psi = \Psi(g)
\]
We define the Berezin cocycle
\[c : \text{Sp}(2n, \mathbb{R}) \times \text{Sp}(2n, \mathbb{R}) \to \mathbb{R}\]

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1Partially supported by the grant NWO.047.017.015 and the grant FWF, project P19064
by
c\left( g_1, g_2 \right) = \text{Im} \; \text{tr} \; \ln \left( \Phi(g_1)^{-1} \Phi(g_1 g_2) \Phi(g_2)^{-1} \right)

(1.4)

Below (Theorem 2.1) we show that the matrix in the brackets has the form
1 + Z, where \|Z\| < 1. Then \ln(1 + Z) := Z - Z^2/2 + Z^3/3 - \ldots \text{ and hence our expression is well defined.}

The cocycle \( c \) defines a central extension of \( \text{Sp}(2n, \mathbb{R}) \). In other words, the set \( \text{Sp}(2n, \mathbb{R}) \times \mathbb{R} \) with the multiplication

\((g, x) \cdot (h, y) = (gh, x + y + c(g, h))\)

is a group.

1.2. Groups of symplectomorphisms. Notation. Consider a \( 2n \)-dimensional symplectic manifold \( M \). Define the following groups

- \( \text{Symp}(M) \) is the group of all \( C^\infty \)-smooth \textit{compactly supported} symplectomorphisms of \( M \). If the manifold \( M \) itself is compact, then \( \text{Symp}(M) \) is the group of all symplectomorphisms of \( M \).
- \( \text{SSymp}(M) \) is the connected component of \( \text{Symp}(M) \) containing unit \( e \).
- \( \text{SSymp}^\sim(M) \) is the \textit{universal covering group} of \( \text{SSymp}(M) \).
- More generally, we denote the universal covering of any connected group \( G \) by \( G^\sim \).
- \( \text{Map}(M) \) is the \textit{mapping class group} \( \text{Symp}(M) / \text{SSymp}(M) \).

We have a natural topology on \( \text{Symp}(M) \) and hence we have a natural Borel structure on \( \text{Symp}(M) \). In particular, we have a notion of a \textit{measurable function} on \( \text{Symp}(M) \). Let \( F \) be a measurable function on \( \text{Symp}(M) \), let \( U \subset \mathbb{R}^N \) be an open domain and \( \psi : U \rightarrow \text{Symp}(M) \) be a smooth map. Then \( F \circ \psi \) is a measurable function on \( U \) in the usual sense.

1.3. Central extension of the group of symplectomorphisms. Equip the space \( \mathbb{R}^{2n} \) with the standard symplectic structure. Consider an open set \( \Omega \subset \mathbb{R}^{2n} \) and a symplectic embedding \( \iota : \Omega \rightarrow M \) such that the measure of \( M \setminus \iota(\Omega) \) is zero.

REMARKS. a) We admit disconnected sets \( \Omega \).

b) It is pleasant (but not necessary) to think that \( M \setminus \iota(\Omega) \) is a union of submanifolds.

Any element \( g \in \text{Symp}(M) \) induces a transformation \( \iota^{-1} g \iota \) of \( \Omega \) defined almost everywhere. Denote the group of all such transformations by \( \text{Symp}(M, \Omega, \iota) \).

By the definition, \( \text{Symp}(M, \Omega, \iota) \simeq \text{Symp}(M) \). This group contains \( \text{Symp}(\Omega) \) as a proper subgroup.

For \( q \in \text{Symp}(M, \Omega, \iota) \) and \( x \in \Omega \), we denote by \( q'(x) \) its Jacobi matrix at the point \( x \). We define the 2-cocycle \( C(q_1, q_2) \), where \( q_1, q_2 \in \text{Symp}(M, \Omega, \iota) \),

\[ C(q_1, q_2) = \text{Im} \; \text{tr} \; \ln \left( \Phi(q_1)^{-1} \Phi(q_1 q_2) \Phi(q_2)^{-1} \right) \]

(1.5)

In considerations of Subsection 1.2, this formula arises in a natural way.
by the formula\(^3\)

\[
C(q_1, q_2) = \\
= \int_{\Omega} \text{Im tr} \ln \left\{ \Phi^{-1}[q_1'(q_2(m))] \Phi[q_1'(q_2(m))q_2'(m)] \Phi^{-1}[q_2'(m)] \right\} dm = \\
\int_{\Omega} c(q_1'(q_2(m)), q_2'(m)) \, dm \quad (1.5)
\]

**Theorem 1.1** a) The expression \(C(q_1, q_2)\) defines an element of the second cohomology group \(H^2(\text{Symp}(M), \mathbb{R})\). In another words, the space \(\text{Symp}(M) \times \mathbb{R}\) with the product

\[
(q_1, x_1) \times (q_2, x_2) = (q_1 \circ q_2, x_1 + x_2 + C(q_1, q_2)) \quad (1.6)
\]

is a group.

b) This central extension of \(\text{Symp}(M)\) does not depend on a choice of the domain \(\Omega\) and the map \(i\).

1.4. Triviality results for the cocycle \(C\). We have a map from the universal covering group \(\text{SSymp}^\sim(M)\) to \(\text{Ssymp}(M)\) and hence we can consider the cocycle \(C\) as a cocycle on \(\text{SSymp}^\sim(M)\).

**Proposition 1.2** Let \(\Xi \subset \mathbb{R}^{2n}\) be an open domain. Then the central extension of \(\text{SSymp}^\sim(\Xi)\) defined by the cocycle \(C\) is trivial.

Let \(M\) be a symplectic manifold. Consider an almost complex structure on the tangent bundle of \(M\) compatible with the symplectic structure (in particular, we obtain \(n\)-dimensional complex vector bundle). Assume that the corresponding Hermitian metric is positive definite.

For the complex bundle obtained in this way, consider its \(n\)-th exterior power \(L\).

**Proposition 1.3** If the complex line bundle \(L\) on \(M\) is trivial, then the central extension of \(\text{SSymp}^\sim(M)\) defined by the cocycle \(C\) is trivial.

**Corollary 1.4** For a noncompact 2-dimensional surfaces \(M\) of a finite genus, our cocycle is trivial on \(\text{SSymp}(M)\).

Indeed, in this case, the group \(\text{SSymp}(M)\) is contractible, see \([10]\), hence \(\text{SSymp}^\sim(M) = \text{Ssymp}(M)\). Also, in this case, the line bundle \(L\) is trivial. Thus, by Proposition 1.3 our central extension is trivial.

1.5. Nontriviality results.

\[^3\]This cocycle is induced from a cocycle on the group of measurable currents, see below Subsection 2.4. Also, below we propose a coordinate-less description of the cocycle \(C\), see Subsection 2.9.
Theorem 1.5 For a two-dimensional oriented (compact or noncompact) surface $M_g$ of genus $g \geq 3$, our central extension of $\text{Symp}(M_g)$ is nontrivial in measurable cohomologies.

Further, consider the standard lattice $\mathbb{Z}^{2n}$ in the standard symplectic space $\mathbb{R}^{2n}$. Consider the torus $T^{2n} := \mathbb{R}^{2n}/\mathbb{Z}^{2n}$. Denote by $\text{Sp}(2n, \mathbb{Z})$ the subgroup in $\text{Sp}(2n, \mathbb{R})$ consisting of all matrices (1.1) with integer elements.

The action of $\text{Sp}(2n, \mathbb{Z})$ on $\mathbb{R}^{2n}$ induces the symplectic action of $\text{Sp}(2n, \mathbb{Z})$ on $T^{2n}$.

Theorem 1.6 The central extension of $\text{Symp}(T^{2n})$ defined by the cocycle $C$ is nontrivial for $n \geq 3$.

In fact, we prove that this extension is nontrivial on the countable subgroup $\text{Sp}(2n, \mathbb{Z}) \subset \text{Symp}(T^{2n})$. The latter statement is a kind of a rigidity theorem for lattices.

1.6. Some discussion.

a) Central extensions of of $\text{SSymp}(M)$ were discussed in several works, see Kostant [9], Brylinski [6], Ismagilov [16], [17], Haller, Vizman [14]. Our construction differs from these constructions.

b) Consider a surface $M_g$ of genus $\geq 3$. It seems that our central extension in this case must be related to the Harer central extension [15] of the mapping class group $\text{Map}(M_g)$, see also [11].

But this relation now is not clear. Indeed, $\text{Symp}(M_g)$ is not a semidirect product $\text{Map}(M_g) \ltimes \text{SSymp}(M_g)$ and hence our construction does not induce automatically an extension of $\text{Map}(M_g)$.

Second, the central extension of $\text{Map}(M_g)$ induces a central extension of $\text{Symp}(M_g)$. Unfortunately, I do not know, gives our construction the same result or not.

c) Symplectic mapping class groups in higher dimensions were discussed by Seidel, see [27]; for results and references on topology of groups of symplectomorphisms, see surveys [20], [21].

d) For a two-dimensional surface $M_g$, our central extension can be realized in a unitary representation of $\text{Symp}(M_g)$, this construction was obtained in [24]. I do not believe that a realization in a unitary representation is possible for dimensions $\geq 4$ (realizations in nonunitary representations exist).

1.7. Structure of the paper. Details of construction of the cocycle $C$ and proofs of Theorem 1.1 and Propositions 1.2–1.3 are contained in Section 2. Theorems 1.5 and 1.6 are proved in Sections 3 and 4 respectively.

Acknowledgments. I am very grateful to V.Fock who explained me how to translate the measure-theoretic construction of [25] to the language of vector bundles (see Subsection 2.9). I thank R.S.Ismagilov who simplified the proof of Theorem 1.4 respectively the preprint variant of the work.

The main part of this work was done during my visits to Lyon and Grenoble in December 1999 and to Vienna in December 2003. I thank C.Roger,
2 Constructions of cocycles

2.1. Central extensions. Preliminaries. A. Let \( G \) be a group, let \( A \) be an Abelian group (in our work, \( A \) is the additive group of \( \mathbb{R}, \mathbb{Z} \), or the circle \( T := \mathbb{R}/\mathbb{Z} \); we write the operation in \( A \) in the additive form). A central extension of \( G \) by the group \( A \) (or \( A \)-extension of \( G \)) is a group \( \tilde{G} \) such that \( A \) is a central subgroup in \( \tilde{G} \) and \( \tilde{G}/A \cong G \).

The set \( \tilde{G} \) can be identified noncanonically with the product \( G \times A \) and the homomorphism \( \tilde{G} \rightarrow G \) can be identified with the projection \( G \times A \rightarrow G \). Then the multiplication in \( G \times A \) must have the form

\[
(g_1, a_1) \cdot (g_2, a_2) = (g_1 g_2, a_1 + a_2 + c(g_1, g_2))
\]

(2.1)

where the function \( c : G \times G \rightarrow A \) (a 2-cocycle) satisfies the identities

\[
c(e, g) = c(g, e) = 0 \quad \text{for all } g ;
\]

(2.2)

\[
c(g_1, g_2) + c(g_1 g_2, g_3) = c(g_1, g_2 g_3) + c(g_2, g_3)
\]

(2.3)

where \( e \) is the unit of \( G \). The first condition means that \((e, 0)\) is the unit of \( \tilde{G} \). The second condition is equivalent to the associativity of the multiplication (2.1).

B. If we change the identification \( G \times A \) and \( \tilde{G} \), then \( c(g_1, g_2) \) is changed according the rule

\[
c(g_1, g_2) \mapsto c(g_1, g_2) - \gamma(g_1) - \gamma(g_2) + \gamma(g_1 g_2)
\]

(2.4)

where \( \gamma : G \rightarrow A \) is some function such that \( \gamma(e) = 0 \).

The central extension is trivial if \( c(g_1, g_2) \) can be transformed to 0 by the operation (2.4), i.e.,

\[
c(g_1, g_2) = \gamma(g_1) + \gamma(g_2) - \gamma(g_1 g_2)
\]

(2.5)

In this case \( \tilde{G} = G \times A \) and we say that \( \gamma(g) \) is a trivializer of \( c \).

C. If \( \gamma, \gamma' \) are two trivializers of the same cocycle \( c \), then \( \gamma - \gamma' \) is a homomorphism \( G \rightarrow A \).

D. The additive group of functions \( c(g_1, g_2) \) satisfying (2.2)–(2.3) is denoted by \( C^2(G, A) \) (group of cocycles); the group of functions having the form (2.5) is denoted by \( B^2(G, A) \) (the group of coboundaries). The second cohomology group is the factor-group

\[
H^2(G, A) := C^2(G, A)/B^2(G, A)
\]

E. Let \( B \) be an Abelian group, \( A \) be its subgroup, \( C = B/A \) be the factor-group. Then we have the obvious maps

\[
C^2(G, A) \rightarrow C^2(G, B) \rightarrow C^2(G, C);
\]

(2.6)
The first map means that an $A$-valued function $c$ is also a $B$-valued function; considering a composition of a function $G \times G \to B$ and the homomorphism $B \to C$, we obtain the second map. We also have the corresponding map in cohomologies

$$H^2(G, A) \to H^2(G, B) \to H^2(G, C)$$

F. Let $G$, $G'$ be groups, let $\theta : G' \to G$ be a homomorphism. Then $\theta$ induces a natural map $C^2(G, A) \to C^2(G', A)$, i.e., for a cocycle $c \in C^2(G, A)$ we consider the cocycle $c(\theta(g'_1), \theta(g'_2)) \in C^2(G', A)$. Hence we also have a map of cohomologies

$$H^2(G, A) \to H^2(G', A)$$

G. Let $G$ be a group, fix $h \in G$.

The cocycle $c(h^{-1}g_1 h, h^{-1}g_2 h)$ is equivalent to $c(g_1, g_2)$, see [5], III.8.1.

H. Now let $G$, $A$ be topological groups. We say that a cocycle $c \in H^2(G, A)$ is nontrivial in measurable cohomologies if it can not be trivialized (see (2.6)) by a measurable trivializer $\gamma$.

2.2. A model of $\text{Sp}(2n, \mathbb{R})$. In [11] we realized the group $\text{Sp}(2n, \mathbb{R})$ as the group of complex matrices having the block structure

$$g = \begin{pmatrix} \Phi & \Psi \\ \overline{\Psi} & \overline{\Phi} \end{pmatrix}$$

(2.8)

preserving the skew-symmetric bilinear form $K$ with the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Then the matrix (2.8) also

--- preserves the indefinite Hermitian form $M$ with the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, i.e.,

$$g \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g^* = g^* \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

(2.9)

--- preserves the real subspace $V \subset \mathbb{C}^n$, consisting of the vectors $(h, \overline{h})$, moreover, matrices (2.8) commute with the antilinear operator $(p, q) \mapsto (\overline{q}, p)$.

Now let us give a coordinateless description of this realization of $\text{Sp}(2n, \mathbb{R})$. Consider an $n$-dimensional complex space $V$ equipped with a positive definite Hermitian form $H(\cdot, \cdot)$. Denote by $V_\mathbb{R}$ the same space considered as $2n$-dimensional linear space over $\mathbb{R}$. The operator $v \mapsto iv$ in $V$ is also a linear operator in $V_\mathbb{R}$, we denote it by $I$.

Denote by $\{\cdot, \cdot\}$ the imaginary part of the Hermitian form $H$, it is a skew-symmetric bilinear form on $V_\mathbb{R}$. The group preserving this form is $\text{Sp}(2n, \mathbb{R})$.

Consider the complexification $(V_\mathbb{R})_\mathbb{C}$ of the space $V_\mathbb{R}$. It is a $2n$-dimensional complex linear space equipped with several additional structures

1) We have an operator $I$ such that $I^2 = -1$.

2) Since $I^2 = -1$, the eigenvalues of $I$ are $\pm i$. Denote by $V_\pm$ the corresponding eigenspaces. Then $V = V_+ \oplus V_-$. 

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3) We have the operation $Q$ of complex conjugation $Q : v + iw \mapsto v - iw$, where $v, w \in V$. It satisfies $QV_\pm = V_\mp$.

Now we consider the action of $\text{Sp}(2n, \mathbb{R})$ in $(V_{\mathbb{R}})_C$. An operator $g \in \text{Sp}(2n, \mathbb{R})$ preserves the bilinear form in $(V_{\mathbb{R}})_C$ and commutes with the complex conjugation.

Representing it as a block operator $V_+ \oplus V_- \rightarrow V_+ \oplus V_-$, we obtain the block matrix representation (2.8)

The Hermitian form $M$ on $(V_{\mathbb{R}})_C$ is

$$M(v + iw, v' + iw') = \{v, w'\} - \{w, v'\} + i\{v, v'\} + i\{w, w'\}$$

it is more natural to say that we extend $i\{\cdot, \cdot\}$ as an Hermitian form from $V_{\mathbb{R}}$ to $(V_{\mathbb{R}})_C$.

2.3. The Berezin cocycle on $\text{Sp}(2n, \mathbb{R})$.

Theorem 2.1  

a) The function $\text{Sp}(2n, \mathbb{R}) \times \text{Sp}(2n, \mathbb{R}) \rightarrow \mathbb{R}$ given by

$$c(g_1, g_2) = \text{Im} \; \text{tr} \; \ln \left[ \Phi(g_1)^{-1}\Phi(g_1g_2)(\Phi(g_2)^{-1} \right]$$

is well-defined.

b) The function $c(g_1, g_2)$ is a 2-cocycle.

c) The $\mathbb{R}$-valued cocycle $\frac{1}{2}c(g_1, g_2)$ can be reduced to a $\mathbb{Z}$-valued cocycle. The corresponding $\mathbb{Z}$-extension of $\text{Sp}(2n, \mathbb{R})$ coincides with the universal covering group of $\text{Sp}(2n, \mathbb{R})$.

d) The cocycle $c$ is uniformly bounded on $\text{Sp}(2n, \mathbb{R}) \times \text{Sp}(2n, \mathbb{R})$

$$|c(g_1, g_2)| < n\pi/2$$

Remark. In [3], Berezin wrote the following $\mathbb{T}$-cocycle on $\text{Sp}(2n, \mathbb{R})$ and on its infinite-dimensional analogue

$$\sigma(g_1, g_2) = \det(1 + \Phi(g_1)^{-1}\Psi(g_1) \cdot \overline{\Psi}(g_2) \Phi(g_2)^{-1})^{-1/2}$$

Trivial calculation shows that

$$\sigma(g_1, g_2) = \exp\{-c(g_1, g_2)/2\}$$

This cocycle can be trivialized on the two-sheet covering of $\text{Sp}(2n, \mathbb{R})$. For $n = 1$, an explicit formula for $\mathbb{R}$-valued cocycle $c$ was written by Guichardet [13].

Proof. a) We have

$$\Phi(g_1g_2) = \Phi(g_1)\Phi(g_2) + \Psi(g_1)\overline{\Psi}(g_2)$$

Hence

$$\Phi(g_1)^{-1}\Phi(g_1g_2)\Phi(g_2)^{-1} = 1 + \Phi(g_1)^{-1}\Psi(g_1) \cdot \overline{\Psi}(g_2)\Phi(g_2)^{-1}$$
Equations (2.9) imply
\[ \Phi \Phi^* - \Psi \Psi^* = 1; \quad \Phi^* \Phi - \Psi^T \Psi = 1 \] (2.13)
Hence \( \Phi \) is invertible and we have the following inequalities for norms
\[ \| \Phi^{-1} \Psi \| < 1, \quad \| \Psi \Phi^{-1} \| < 1 \] (2.14)
Thus,
\[ \| \Phi(g_1)^{-1} \Psi(g_1) \cdot \overline{\Psi(g_2)} \Phi(g_2)^{-1} \| < 1 \]
We define the logarithm of (2.12) by
\[ \ln(1 + Z) := \sum_{n=1}^{\infty} (-1)^{n+1} Z^n / n \]
Since the norm of our \( Z \) is < 1, our series converges.

b) Let \( A, B \) be \( n \times n \) matrices and \( \| A - 1 \|, \| B - 1 \| \) be sufficiently small. Then
\[ \text{tr} \ln(AB) = \text{tr} \ln A + \text{tr} \ln B \]
Hence, for \( g_1, g_2 \) lying in a small neighborhood of the unit we can write
\[ \text{tr} \ln \left[ \Phi(g_1)^{-1} \Phi(g_1 g_2) \Phi(g_2)^{-1} \right] = -\text{tr} \ln \Phi(g_1) + \text{tr} \ln \Phi(g_1 g_2) - \text{tr} \ln \Phi(g_2) \]
Now the cocycle identity (2.3) became trivial for \( g_1, g_2, g_3 \) lying in a sufficiently small neighborhood of unit.

But all our expressions are real analytic and the group \( \text{Sp}(2n, \mathbb{R}) \) is connected. Hence the cocycle identity (2.3) is valid for all \( g_1, g_2, g_3 \in \text{Sp}(2n, \mathbb{R}) \).

A verifying of (2.2) is trivial.

c) As we have seen \( \det \Phi \neq 0 \). Assume
\[ \gamma(g) := \text{Im} \text{tr} \ln \Phi(g) = \arg \ln \det \Phi(g) \] (2.15)
where \( 0 \leq \arg z < 2\pi \). Obviously,
\[ \alpha := \frac{1}{2\pi} \left( c(g_1, g_2) + \gamma(g_1) + \gamma(g_2) - \gamma(g_1 g_2) \right) \in \mathbb{Z} \]
since \( \exp(2\pi \alpha) = 1 \), and the first statement is proved.

The \( \mathbb{Z} \)-central extension of \( \text{Sp}(2n, \mathbb{R}) \) obtained in this way can be considered as the subset \( \text{Sp}(2n, \mathbb{R})^\sim \subset \text{Sp}(2n, \mathbb{R}) \times \mathbb{R} \) consisting of pairs
\[ (g, x), \quad \text{where } g \in \text{Sp}(2n, \mathbb{R}), \ x \in \mathbb{R} \text{ and } \det \Phi(g) / |\det \Phi(g)| = e^{ix} \]
the multiplication is given by (2.1). Obviously, the projection \( \text{Sp}(2n, \mathbb{R})^\sim \to \text{Sp}(2n, \mathbb{R}) \) is a covering map. For \( y \in \mathbb{R} \) consider the matrix \( g(y) \) with \( \Psi(y) = 0 \) and \( \Phi \) being the diagonal matrix with entries \( e^{iy}, 1, \ldots, 1 \). The map \( y \mapsto \)
(g(y), y) is a continuous map $\mathbb{R} \to \text{Sp}(2n, \mathbb{R})$ and hence $\text{Sp}(2n, \mathbb{R})$ is connected. Thus $\text{Sp}(2n, \mathbb{R})$ is a covering group for $\text{Sp}(2n, \mathbb{R})$.

It remains to notice that the unitary group $U(n)$ is a deformation retract of $\text{Sp}(2n, \mathbb{R})$ and the loop $g(y)$, $y \in [0, 2\pi]$, is a generator of the fundamental group $\pi_1(U(n))$. Hence $\text{Sp}(2n, \mathbb{R})$ is connected. Thus $\text{Sp}(2n, \mathbb{R})$ is a covering group for $\text{Sp}(2n, \mathbb{R})$.

d) Let $Z$ be a matrix satisfying $\|Z\| < 1$. Let $\lambda_j(Z)$ be its eigenvalues. Obviously, $|\lambda_j| < 1$. Let us prove that

$$\text{tr} \ln(1 + Z) = \sum \ln(1 + \lambda_j(Z)) \quad (2.16)$$

For a self-adjoint matrix $Z$ this identity is obvious. The both sides of (2.16) are complex analytic in $Z$ and hence this identity is valid if $\|Z\| < 1$.

But $|\lambda_j(Z)| < 1$ implies $|\text{Im}(1 + \lambda_j)| < \pi/2$ and we obtain (2.11). □

Remark. Obviously, the expression

$$\text{Re} \text{tr} \ln \left[ \Phi(g_1)^{-1} \Phi(g_1g_2)(\Phi(g_2)^{-1} \right]$$

also satisfies the cocycle equation (2.3), but this cocycle is trivial, since its trivializer

$$\gamma(g) = \text{Re} \text{tr} \ln \Phi(g) = \ln |\det(\Phi(g))|$$

is well defined (but similar expression (2.15) is multivalued).

2.4. Groups $\mathfrak{B}(X, G)$. Denote by $X$ some Lebesgue space with a continuous measure $\mu$. For example, we can consider $X$ being an arbitrary symplectic manifold. Denote by $\text{Ams}(X)$ the group of all measure-preserving maps from the space $X$ to itself (by definition, these maps are defined almost everywhere).

Let $G$ be an arbitrary group (below $G = \text{Sp}(2n, \mathbb{R})$). Denote by $\mathcal{F}(X, G)$ the group of all measurable functions $f : X \to G$ such that

$$f(x) = 1 \quad \text{outside a set of finite measure}$$

If $X$ itself has a finite measure, then we can forget the last condition.

Consider the semidirect product

$$\mathfrak{B}(X, G) = \text{Ams}(X) \ltimes \mathcal{F}(X, G)$$

Elements of the group $\mathfrak{B}(X, G)$ are pairs $\{p, h\} \in \text{Ams} \times \mathcal{F}(X, G)$ and the product is given by

$$\{p_1(x), h_1(x)\} \ast \{p_2(x), h_2(x)\} := \{p_1(p_2(x)), h_1(p_2(x))h_2(x)\}$$

Remark. Let the group $G$ acts on a manifold $Y$ by transformations $y \mapsto yg$. Consider the space $\mathcal{L}(X, Y)$ of all $Y$-valued measurable functions on $X$. The group $\mathfrak{B}(X, G)$ acts in this space by the transformations

$$T(p, h)f(x) = f(p(x))h(x) \quad (2.17)$$

In fact, any such space is equivalent to a segment $[a, b] \subset \mathbb{R}$ equipped with the Lebesgue measure or to the whole line $\mathbb{R}$. 9
2.5. A central extension of $\mathcal{B}(X,G)$. Let $c \in H^2(G,\mathbb{R})$ be a bounded cocycle. We define the function $C: \mathcal{B}(X,G) \times \mathcal{B}(X,G) \to \mathbb{R}$ by

$$C(\{p_1, h_1\}, \{p_2, h_2\}) = \int_X c(h_1 \circ p_2(x), h_2(x)) d\mu(x) \quad (2.18)$$

**Theorem 2.2** The function $C$ is a 2-cocycle on $\mathcal{B}(X,G)$.

We denote the corresponding central extension of $\mathcal{B}(X,G)$ by $\tilde{\mathcal{B}}(X,G)$.

**Proof.** We directly verify the cocycle identity (2.3). First,

$$C\left(\{p_1, h_1\}, \{p_2, h_2\}\right) + C\left(\{p_1, h_1\} \ast \{p_2, h_2\}, \{p_3, h_3\}\right) = \int_X c(h_2 \circ p_3)(x), h_3(x)) d\mu(x) + \int_X c(h_1 \circ p_2)(x), h_2(x)) d\mu(x) \quad (2.19)$$

Secondly,

$$C\left(\{p_2, h_2\}, \{p_3, h_3\}\right) + C\left(\{p_1, h_1\} \ast \{p_2, h_2\}, \{p_3, h_3\}\right) = \int_X c(h_2 \circ p_3)(x), h_3(x)) d\mu(x) + \int_X c(h_1 \circ p_2)(x), h_2(x)) d\mu(x) \quad (2.20)$$

We substitute $x = p_3(y)$ to (2.20) and replace the summand (2.20) by

$$\int_X c(h_1 \circ p_2 \circ p_3)(x), h_3(x)) d\mu(x)$$

We obtain that (2.19) equals (2.21) due the cocycle identity (2.3) for $c(\cdot, \cdot)$ with

$$g_1 = h_1 \circ p_2 \circ p_3(x), \quad g_2 = h_2 \circ p_3(x), \quad g_3 = h_3(x)$$

**Remark.** To apply this construction, we need in a group $G$, having a non-trivial $\mathbb{R}$-central extension. Some reasonable examples are $G = \mathbb{R}/\mathbb{Z}$, $U(p, q)$, $\text{SO}^*(2n)$, $\text{SO}(n, 2)$, and the group $\text{SDiff}(S^1)$ of orientation preserving diffeomorphisms of the circle.

2.6. Another explanation of the central extension of $\mathcal{B}(X,G)$. Denote by $\tilde{\mathcal{B}}$ the central extension of $G$ defined by the cocycle $c$, by definition $\tilde{\mathcal{B}} \supset \mathbb{R}$. Consider the group $\mathcal{B}(X,\tilde{\mathcal{B}})$ and its central Abelian subgroup $\mathcal{F}(X,\mathbb{R})$. Denote by $Q$ the subgroup of $\mathcal{F}(X,\mathbb{R})$ consisting of functions $f$ such that

$$\int_X f(x) d\mu(x) = 0$$
Obviously, $Q$ is a normal subgroup in $\mathfrak{B}(X, \tilde{G})$. It can readily checked that

\[
\mathfrak{B}(X, G) = \mathfrak{B}(X, \tilde{G})/Q
\]

2.7. **Embedding** $\text{Symp}(M) \to \mathfrak{B}(M, \text{Sp}(2n, \mathbb{R}))$. Now let $X \simeq M$ be a $2n$-dimensional symplectic manifold, let $\Omega \subset \mathbb{R}^{2n}$ be an open domain, and let $\iota: \Omega \to M$ be a symplectic embedding such that the set $M \setminus \iota(\Omega)$ has a zero measure. For $g \in \text{Symp}(M)$, consider the map $q := \iota^{-1}g\iota : \Omega \to \Omega$ defined almost sure. Denote by $q'(x)$ the Jacobi matrix of $g$ at point $x$.

The map

\[
g \mapsto (q, q')
\]

is an embedding $\text{Symp}(M) \to \mathfrak{B}(M, \text{Sp}(2n, \mathbb{R}))$. Our cocycle (2.18) on $\mathfrak{B}(M, \text{Sp}(2n, \mathbb{R}))$.

2.8. **Groups of automorphisms** $\text{Aut}(R, M)$ of bundles. Now let $X = M$ be an $m$-dimensional manifold equipped with a volume form $\Omega$. Consider an $2n$-dimensional real vector bundle $R \to M$ on $M$ with fibers $R_x$, $x \in M$. Assume that fibers $R_x$ are equipped with skew-symmetric bilinear forms $\{\cdot, \cdot\}_x$.

Denote by $\text{Aut}(R, M)$ the group of smooth maps $\Theta : R \to R$ satisfying the conditions

1. An image of any fiber $R_x$ is some fiber $R_{\Theta(x)}$, and the map $\Theta$ induces a linear map $R_x$ to $R_{\Theta(x)}$ preserving the skew-symmetric form, i.e.,

\[
\{\Theta v, \Theta w\}_{\Theta(x)} = \{v, w\}_x
\]

where $v, w \in R_x$.

2. The map $x \mapsto \Theta(x)$ is a diffeomorphism of the base $M$ preserving the volume form $\Omega$.

3. For a noncompact manifold $M$, we have an additional conditions: there is a compact subset $L \subset M$ such that for any $x \notin L$ we have $\Theta(x) = x$ and $\Theta v = v$ for any $v \in R_x$.

The group $\text{Aut}(R, M)$ admits an obvious embedding to the group $\mathfrak{B}(M, \text{Sp}(2n, \mathbb{R}))$. Indeed, consider an arbitrary trivialization of the bundle $R$ over an arbitrary open set $X \subset M$ of a complete measure. Then elements of $\text{Aut}(R, M)$ induce transformations of $\mathcal{L}(X, \mathbb{R}^{2n})$ of the type (2.17).

The embedding $\text{Aut}(R, M) \to \mathfrak{B}(M, \text{Sp}(2n, \mathbb{R}))$ described above is not canonical, but any two embeddings $\eta_1, \eta_2$ of this type are conjugated by some element $r \in \mathfrak{B}(M, \text{Sp}(2n, \mathbb{R}))$:

\[
\eta_2(\Theta) = r^{-1} \eta_1(\Theta) r \tag{2.23}
\]

Thus, the central extension of $\mathfrak{B}(M, \text{Sp}(2n, \mathbb{R}))$ induces a central extension of the group $\text{Aut}(R, M)$. The formula for the cocycle depends on the set $X \subset M$ and on a trivialization of the bundle. But due (2.23) all these cocycles are equivalent. Thus, our central extension is canonical.

2.9. **A geometric construction of the central extension of** $\text{Aut}(R, M)$. Let $R, M$ be the same as above.
Consider an almost complex structure on the bundle $R$. Recall that this is an operator $J_x : R_x \to R_x$ depending on $x$ smoothly and satisfying

$$J_x^2 = -1; \quad \{J_x v, J_x w\}_x = \{v, w\}_x$$

We also assume that the symmetric bilinear form

$$S_x(v, w) = \{J_x v, w\}_x$$

is positive definite on each fiber. Such almost complex structures exist, see, for instance, [22] (evidently, such structure is not unique). In particular, the tangent bundle $TM$ became a complex $n$-dimensional bundle, but in this moment we prefer to consider $TM$ as a real bundle with an additional structure.

Consider the fiber-wise complexification $R^C$ of the vector bundle $R \to M$. In each fiber $(R_x)^C$ of $R^C$, we have two canonically defined subspaces

$$R^\pm_x := \ker(J_x \mp i), \quad (R_x)^C = R^+_x \oplus R^-_x$$

Thus we can represent any real symplectic linear operator $h : R_x \to R_y$ as a complex block operator

$$h : R^+_x \oplus R^-_x \to R^+_y \oplus R^-_y$$

We denote by $\Phi(h)$ the block corresponding $R^+_x \to R^+_y$.

The formula for the 2-cocycle is

$$C(\Theta^{(1)}, \Theta^{(2)}) = \int_M c\left(\Theta^{(1)}\big|_{R_{\Theta^{(2)}}(x)}, \Theta^{(2)}\big|_{R^+_x}\right)\Omega(x)$$

2.10. Groups of symplectomorphisms. Now let $M$ be a $2n$-dimensional symplectic manifold. Denote by $R$ the tangent bundle to $M$. We have the natural embedding

$$\text{Symp}(M) \subset \text{Aut}(R, M)$$

and hence we can induce a 2-cocycle from the group $\text{Aut}(R, M)$.

We also can put $\text{Symp}(M)$ to $\mathfrak{B}(M, \text{Sp}(2n, \mathbb{R}))$ directly, as it was done above in [13].

The cohomological class of the induced cocycle do not depend on the embeddings since all these embeddings are conjugated by interior automorphisms of $\mathfrak{B}(M, \text{Sp}(2n, \mathbb{R}))$.

2.11. Proof of Proposition 1.2. Let $\Xi$ be a domain in $\mathbb{R}^{2n}$. Then

$$\nu : (g, x) = \frac{\det \Phi(g'(x))}{\det \Phi(g'(x))}$$

is a well defined function

$$\text{SSymp}(\Xi) \times \Xi \to \mathbb{T}$$

where $\mathbb{T}$ is the group of complex numbers $z$ such that $|z| = 1$. 

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By the covering homotopy theorem, this map can be lifted to the map
\[ \tilde{\nu} : \text{SSymp}^{\sim}(\Xi) \times \Xi \to \mathbb{R} \]
such that
\[ \exp(i\tilde{\nu}(g, x)) = \nu(g, x), \quad \tilde{\nu}(e, x) = 0. \]
The function \( \tilde{\nu} \) is precisely a single-valued branch of
\[ \text{Im } \ln \det \Phi(g'(x)) = \ln \left[ \det \Phi(g'(x))/|\det \Phi(g'(x))| \right] \]
Then
\[ \Gamma^\circ(g) := \int_{\Xi} \tilde{\nu}(g, x) \, dx \quad (2.25) \]
is a trivializer of the 2-cocycle \( C(\cdot, \cdot) \) on \( \text{SSymp}(\Xi) \), i.e.
\[ C(g_1, g_2) = \Gamma^\circ(g_1 g_2) - \Gamma^\circ(g_1) - \Gamma^\circ(g_2) \]
Indeed, the right hand side is
\[ \int \text{Im } \ln \Phi([g'_1 \circ g_2][x] \cdot g'_2(x)) \, dx - \int \text{Im } \ln \Phi(g'_1(x)) \, dx - \int \text{Im } \ln \Phi(g'_2(x)) \, dx \]
We change the variable \( x \mapsto g_2(x) \) in the second summand and transform the expression to the form
\[ \int \text{Im } \text{tr } \ln \left[ \Phi([g'_1 \circ g_2][x])^{-1} \Phi([g'_1 \circ g_2][x] \cdot g'_2(x)) \Phi(g'_2(x)^{-1}) \right] \, dx \]
This proves Proposition 1.2.

Remark. Generally, this trivializer is not unique, since (see 2.1) there are nontrivial homomorphisms \( \text{SSymp}(\Xi)^{\sim} \to \mathbb{R} \), namely the flux homomorphisms and the Calabi invariant (see, for instance \cite{1}, \cite{22}, we discuss them below in a special case). By the Banyaga Theorem \cite{1}, this list exhaust all the homomorphisms from \( \text{SSymp}^{\sim} \) to Abelian groups.

Corollary 2.3 Let \( \Xi \subset \mathbb{R}^4 \) be an open domain. Then our central extension of the (disconnected) group \( \text{Symp}(\Xi) \) is trivial.

Indeed, the group \( \text{Symp}(\mathbb{R}^4) \) is connected and contractible (Gromov \cite{12}). By Proposition 1.2, its central extension is trivial. But \( \text{Symp}(\Xi) \subset \text{Symp}(\mathbb{R}^4) \).

2.12. Proof of Proposition 1.3. In the case of general symplectic manifolds \( M \), the sense of the expression \( \text{Im } \ln \Phi(\cdot) \) is not clear, since an operator \( \Phi(\cdot) \) maps one fiber of a bundle to another one and hence its determinant is not well defined.

Consider an Hermitian structure on the tangent bundle \( TM \) to \( M \) compatible with the symplectic form (as in \cite{239}). Then \( TM \) becomes a complex bundle,
denote it by $TM$ (it is also isomorphic to the subbundle $R^+ \subset R^C$ defined above). Consider its maximal exterior power $\wedge^n TM$.

Assume that the bundle $\wedge^n TM$ is trivial. Fix its trivialization. This precisely means that we have defined determinants of the operators $\Phi(\cdot)$ connecting different fibers.

Then we repeat our argument with a covering homotopy and obtain Proposition 1.3.

3 Two-dimensional surfaces

Here we prove nontriviality of the cocycle $C$ in the case of a two-dimensional oriented surface $M$ of genus $g \geq 3$. Evidently, in 2-dimensional case the group $\text{SSymp}(M)$ coincides with the group of volume preserving and orientation preserving diffeomorphisms. Also, $\text{Sp}(2, \mathbb{R}) = \text{SL}(2, \mathbb{R})$.

We fix the following notation

— $\Lambda$ is a simply connected domain in $\mathbb{R}^2$ equipped with the form $\omega = dx \wedge dy$

It convenient to think that $\Lambda$ is a standard disk.

— $\Delta$ is a multiconnected domain in $\mathbb{R}^2$. It is convenient to think, that $\Delta$ is a disk with $k > 0$ holes and the exterior boundary of $\Delta$ is the boundary of $\Lambda$.

— $\sigma(S)$ is the area of a domain $S \subset \mathbb{R}^2$.

The group $\text{Symp}(\Lambda) = \text{SSymp}(\Lambda)$ is connected and contractible (Smale). The groups $\text{Symp}(\Delta)$ are disconnected (this is obvious). The identity component $\text{SSymp}(\Delta)$ is contractible (Earle, Eells [10]).

Our main arguments for a proof are: continuity of the cocycle in the topology of convergence in measure, formula (2.25) for a trivializer in a flat case, a certain Dehn relation in the Teichmüller group, and the Banyaga Theorem.

3.1. Global angle of rotation. For $\psi \in \mathbb{R}$ consider the unit vector $v_\psi = \cos \psi e_1 + \sin \psi e_2$

Denote by $S^1$ the set of all unit vectors in $\mathbb{R}^2$.

Let $q \in \text{Symp}(\Lambda)$. Consider a point $x \in \Lambda$ and a unit vector $v_\psi$ applied at this point. Consider the image $w = q'(x)v$ of $v$ under the Jacobi matrix $q'(x)$. The normalized vector $w/||w||$ has a form $v_x$. We assume $\text{ang}(q, x, v) := \varphi - \psi$ i.e., $\text{ang}(\cdot)$ is the angle of turning of a vector under a diffeomorphism.
Lemma 3.1  a) There exists a unique continuous function (global turning angle)
\[ \text{Ang} : \text{SSymp}(\Lambda) \times \Lambda \times S^1 \to \mathbb{R} \]
such that

1) The composition of $\text{Ang}()$ and the map $\mathbb{R} \to \mathbb{R}/2\pi\mathbb{Z}$ is $\text{ang}()$.
2) $\text{Ang}(e, x, v) = 0$.
3) Let $U$ be a neighborhood of boundary of $\Lambda$, where $q(x) = x$. Then
\[ \text{Ang}(q, x, v) = 0 \quad \text{for all } x \in U \text{ and all } v \in S^1 \]

b) The function $\text{Ang}$ satisfies the identity
\[ \text{Ang}(q_1 \circ q_2, x, v) = \text{Ang}(q_2, x, v) + \text{Ang}(q_1, q_2(x), q_2'(x)v) \quad (3.1) \]

Proof. a) Consider a map
\[ f : \text{SSymp}(\Lambda) \times \Lambda \times S^1 \to S^1 \]
defined in the following way:
\[ f(q, x, v) = \frac{q'(x)v}{\|q'(x)v\|} \]
Next, we consider the covering map
\[ \tilde{f} : \text{SSymp}(\Lambda) \times \Lambda \times \mathbb{R} \to \mathbb{R}; \quad \tilde{f}(e, x, \varphi) = \varphi \]
defined by the covering homotopy Theorem (recall that SSymp($\Lambda$) is simply connected). Then
\[ \text{Ang}(q, x, v_\psi) = \tilde{f}(q, x, \psi) - \psi \]

b) is obvious. \(\square\)

Corollary 3.2 The global turning angle is well defined on the (disconnected) group $\text{Symp}(\Delta)$.

Proof. Indeed, each compactly supported symplectomorphism of $\Delta \subset \Lambda$ is also a symplectomorphism of $\Lambda$. \(\square\)

In particular, Lemma 3.1 gives the following geometrically visual way of evaluation of $\text{Ang}()$.

Lemma 3.3 Let $q \in \text{SSymp}(\Lambda)$. Consider a point $z$ near the boundary of $\Lambda$, where $q(z) = z$. Let $\ell(t)$ be a smooth curve, $\ell(0) = z$, $\ell(1) = x$, $\frac{d\ell}{dt}(1) = v$. Then
\[ \text{Ang}(q, x, v) = \{\text{total turning of vector } \frac{d}{dt}(q(\ell(t)))\} - \{\text{total turning of vector } \frac{d}{dt}\ell(t)\} \]
Proof. Indeed, \( \text{Ang}(\cdot) \) must be continuous along the curve
\[
\xi(t) := (q, \ell(t), \dot{\ell}(t)) \in \text{SSymp} \times \Lambda \times S^1
\]

Let \( \Phi(h) \), where \( h \in \text{SL}(2, \mathbb{R}) \), be the same as above. In our case \( \Phi(h) \) is an element of \( \mathbb{C} \) and \( |\Phi(h)| \geq 1. \) Let
\[
h = SA,
\]
be the polar decomposition of \( h \), where \( A \) is a rotation by some angle \( \theta \) and \( S \) is a contraction-dilatation with respect to two orthogonal axes. Then
\[
\Phi(h)/|\Phi(h)| = e^{i\theta}
\]

Lemma 3.4 Consider a continuous branch of the function
\[
\gamma^\circ(g, x) := \text{Im} \ln \Phi(g'(x))
on\text{SSymp}(\Lambda) \times \Lambda)
\]
such that \( \gamma(e, x) = 0. \) We have
\[
|\text{Ang}(q, x, v) - \text{Im} \ln \Phi(g'(x))| < \pi/2
\]
for all \( q \in \text{SSymp}(\Lambda), x \in \Lambda, v \in S^1. \)

Proof. We can define the both functions in the following way. Consider the map
\[
F : \text{SSymp}(\Lambda) \times \Lambda \rightarrow \text{SL}(2, \mathbb{R})
given by \((q, x) \mapsto q'(x).\) Consider the covering map
\[
\tilde{F} : \text{SSymp}(\Lambda) \times \Lambda \rightarrow \text{SL}(2, \mathbb{R})^\sim
\]
Let \( h \) ranges in \( \text{SL}(2, \mathbb{R}). \) The function \( \alpha(h) := \text{Im} \ln \Phi(h) \) is a function on \( \text{SL}(2, \mathbb{R})^\sim. \) Also the \( \mathbb{R} \)-valued angle of turning of a unit vector \( v \in \mathbb{R}^2 \) under \( h \in \text{SL}(2, \mathbb{R})^\sim \) is a function on \( \text{SL}(2, \mathbb{R})^\sim \) (denote it by \( \beta_v(h). \)) We have
\[
\text{Ang}(g, x, v) = \beta_v(\tilde{F}(g'(x))), \quad \gamma^\circ(g, x) = \alpha(\tilde{F}(g'(x)))
\]
Hence it is sufficient to show that
\[
|\alpha(h) - \beta_v(h)| < \pi/2
\]
This is a corollary of the following statement:
- Let \( h \in \text{SL}(2, \mathbb{R}), \) let \( h = SA \) be its polar decomposition \((3.2).\) Then the angle between \( hv \) and \( Av \) is less than \( \pi/2. \)
The latter statement is obvious. Passing to the covering group \( \text{SL}(2, \mathbb{R})^\sim, \) we obtain \((3.6);\) applying \((3.5), \) we obtain \((3.4). \)

Lemma 3.5
\[
\max_{x,v} |\text{Ang}(q_1 \circ q_2, x, v)| \leq \max_{x,v} |\text{Ang}(q_1, x, v)| + \max_{x,v} |\text{Ang}(q_2, x, v)|
\]
Figure 1: A standard twist.

Proof. This follows from (3.1).

3.2. Twists. Let $r > 0$, $\varphi$ be the polar coordinates on the plane. Consider a ring $S : a \leq r \leq b$. A standard right twist is a diffeomorphism $q : S \rightarrow S$ that fix the both boundary circles, i.e., $q(a, \varphi) = (a, \varphi)$, $q(b, \varphi) = (b, \varphi)$, and

$$q(r, \varphi) = q(r, \varphi + \mu(r))$$

where $\mu$ is an arbitrary smooth increasing function on $[0, \infty)$ such that

$$\mu(x) = 0 \quad \text{for} \ x < a + \delta; \quad \mu(r) = 2\pi \quad \text{for} \ x > b - \delta$$

for some $\delta > 0$, see Fig. 1.

Remark. The image of a right twist under the orientation preserving map $(r, \varphi) \mapsto (ab/r, -\varphi)$ is a right twist again. A diffeomorphism inverse to a right twist is not a right twist (it is called a left twist).

Consider a closed smooth non-self-intersecting curve $C$ on the surface $M$. Consider a ‘small’ neighborhood $U$ of $C$. For some standard ring $S \subset \mathbb{R}^2$ consider some area-preserving and orientation preserving diffeomorphism $p : S \rightarrow U$. Consider a diffeomorphism $h \in \text{Symp}(M)$ having the form

- $h(m) = m$ if $m \notin U$
- $p^{-1}hp$ is a standard right twist of $S$.

We call such diffeomorphism by a twist about the curve $C$ supported by the neighborhood $U$.

A Dehn twist is a twist about a nonseparating curve $B$ (i.e., the domain $M \setminus B$ is connected). For any two nonseparating curves $B_1$, $B_2$ there is a diffeomorphism $q \in \text{Symp}(M)$ such that $q(B_1) = B_2$ (see, for instance, [18]). If $h$ is a right twist about $B_1$, then $q^{-1}hq$ is a right twist about $B_2$.

3.3. $\varepsilon$-twists. Below we use families of twists depending on parameters and we are need in some uniform estimates in parameters. By this reason, we give more rigid definitions.

Fix a function $\nu$ on $(-\infty, \infty)$ satisfying the conditions

- $\nu = 0$ on the ray $(-\infty, 0)$ and $\nu = 2\pi$ on the ray $x \geq 1/2$
- $\nu$ is $C^\infty$-smooth and increasing.

This function remains fixed until the end of this section.

A standard $\varepsilon$-twist is a family $q(\varepsilon)$ of diffeomorphisms of the ring $1/2 < r < 3/2$ depending in a parameter $\varepsilon$, they have the form

$$(r, \varphi) \mapsto (r, \varphi + \nu(\frac{r-1}{\varepsilon})), \quad 0 < \varepsilon < 1/2$$

(in particular, this map is identical outside the ring $1 < r < 1 + \varepsilon/2$).

Now let $C$ be a closed non-self-intersecting curve on $M$. Let $h$ be an embedding of some ring $1 - \delta < r < 1 + \delta$ to $M$ such that the $C$ is the image of the circle $r = 1$. We define an $\varepsilon$-twist about $C$ as a family of diffeomorphisms of $M$ depending on $\varepsilon$ and given by

$$m \mapsto h \circ q(\varepsilon) \circ h^{-1}(m), \quad \text{for } m \in U$$

$$m \mapsto m, \quad \text{for } m \notin U$$

The parameter $\varepsilon$ ranges in the interval $(0, \delta)$.

We fix some notation

- $T_C(\varepsilon)$ for an $\varepsilon$-twist about $C$; we omit $h$ from our notation, but we remember that $h$ is fixed.

- $U(\varepsilon) = U_C(\varepsilon)$ for the support of the twist $T_C(\varepsilon)$. The area of $U_C(\varepsilon)$ tends to 0 as $\varepsilon \to 0$, more precisely, $\sigma(U_C(\varepsilon)) = O(\varepsilon)$.

3.4. Trivializer of the cocycle $C$ on a twist. The cocycle $C(\cdot, \cdot)$ on the group $\text{SSymp}(\Lambda)$ is trivial, its canonical trivializer $\Gamma(\cdot)\circ q(\cdot) = \int_\Lambda \text{Im} \ln \Phi(q'(x)) dx$ (3.7)

was defined above in 2.11.

**Lemma 3.6** Let $B$ be a smooth non self-intersecting curve in $\Lambda$ surrounding the domain $S$. Then

$$\Gamma^\circ(U_B(\varepsilon)) = 2\pi \sigma(S) + O(\varepsilon), \quad \text{as } \varepsilon \to 0$$

**Proof.** The strip $U(\varepsilon)$ separates $\Lambda$ into two domains, i.e., the exterior domain $W^{ext}$ and the interior domain $W^{int}$. Let $\ell(t)$ be a (short) curve intersecting the strip $U(\varepsilon)$, let $\ell(0) \in W^{ext}$, $\ell(\delta) \in W^{int}$. The Jacobi matrix $[\Sigma_B'(\ell(t))]$ is 1 at $t = 0$ and at $t = \delta$. But the curve $[\Sigma_B' \circ \ell(t)]$ is noncontractible in $\text{SL}(2, \mathbb{R})$ and moreover, it is a generator of the fundamental group $\pi_1(\text{SL}(2, \mathbb{R}))$ (it is sufficient to verify that the vector $[\Sigma_B' \circ \ell(t)]\dot{\ell}(t)$ is turning by the angle $2\pi$ as we pass $\ell(t)$).

Hence $\ln \Phi([\Sigma_B'(\ell(\delta))] = 2\pi$, and thus $\ln \Phi([\Sigma_B'(x)]) = 2\pi$ for $x \in W^{int}$.

Further,

$$\Gamma^\circ(U_B(\varepsilon)) = \int_\Lambda \text{Im} \ln \Phi([\Sigma_B'(x)]) dx = \int_{W^{ext}} + \int_{U(\varepsilon)} + \int_{W^{int}}$$
The integrand in the first summand is 0, the integrand in the last summand is $2\pi$ (and hence the integral over $W^{\text{int}}$ is $2\pi \sigma(S) + O(\varepsilon)$). Thus it is sufficient to show that $\gamma^0 = \text{Im ln} \Phi(\Sigma_B(x))$ is bounded in the thin strip $\Omega(\varepsilon)$. By Lemma 3.3, it is sufficient to show the boundedness of $\text{Ang}(\Sigma_B, x, v)$, and by Lemma 3.3, the latter statement is obvious. \qed

3.5. Flux homomorphisms. Now, let $\Lambda \subset \mathbb{R}^2$ be a disk (or simply connected domain) bounded by a curve $Z_0$. Let $\Delta \subset \Lambda$ be a multi-connected domain bounded by smooth curves $Z_0$ (the exterior boundary) and $Z_1, \ldots, Z_k$ (boundaries of the holes). Denote by $\omega$ the standard symplectic form $dx \wedge dy$ on $\mathbb{R}^2$. Let $\lambda$ be a 1-form on $\mathbb{R}^2$ such that $d\lambda = \omega$ (for instance, $\lambda = x \, dy$).

For each $j$, fix a non-self-intersecting curve $u_j(t)$ connecting $Z_0$ and $Z_j$, let $u_j(0) \in Z_0$, $u_j(1) \in Z_j$. We define the function $\tau_j(g)$ in the variable $g \in \text{Symp}(\Delta)$ by

$$\tau_j(g) = \int_{u_j} \lambda - \int_{gu_j} \lambda = \int_{D_j} \omega$$

(3.8)

where $D_j$ is a 2-cycle whose boundary is $u_j - gu_j$. Non-formally, $\tau_j(g)$ is the oriented area of the domain bounded by the curves $u_j(t)$ and $g(u_j(t))$.

Lemma 3.7

a) $\tau_j$ does not depend on a choice of $u_j$.

b) $\tau_j$ is a homomorphism $\text{Symp}(\Delta) \to \mathbb{R}$.

The maps $\tau_j$ are called by flux homomorphisms, for definitions and properties in a general symplectic case, see Banyaga [1], McDuff, Salamon [22].

Proof. a) Let $u_j'$ be another curve, let $\tau_j'$ be another map. Let $R$ be a 2-cycle on $\mathbb{R}^2$ with boundary $u_j - u_j'$,

$$\tau_j(g) - \tau_j'(g) = \left( \int_{u_j} \lambda - \int_{u_j'} \lambda \right) - \left( \int_{gu_j} \lambda - \int_{gu_j'} \lambda \right) = \int_R \omega - \int_{gR} \omega = 0$$

b)

$$\tau_j(g_1 g_2) = \int_{u_j} \lambda - \int_{g_1 g_2 u_j} \lambda = \int_{u_j} \lambda - \int_{g_2 u} \lambda + \int_{g_2 u} \lambda - \int_{g_1 (g_2 u)} \lambda = \tau_j(g_2) + \tau_j(g_1)$$

Remark. In a general case, flux homomorphisms are defined on the connected group $\text{SSymp}$. In our case, we obtain homomorphisms $\tau_j$ of group $\text{Symp}(\Delta)$, these homomorphisms depend on an embedding of the symplectic manifold $\Delta$ to $\mathbb{R}^2$, since the areas of the holes $Z_j$ participate in formula (3.8).

3.6. Values of fluxes on twists. We preserve the notation of the previous subsection. The following statement is obvious.

Lemma 3.8 Let $C$ be a Jordan contour in $\Delta$ surrounding a domain $S$. Then

$$\tau_j(\Sigma_C(\varepsilon)) = \begin{cases} 
\sigma(S) + O(\varepsilon), & \text{if } Z_j \subset S \\
0, & \text{otherwise}
\end{cases}$$
3.7. The Calabi homomorphism. We preserve the notation of the previous subsection. Consider a 1-form $\lambda$ on $\mathbb{R}^2$ such that $d\lambda = \omega$.

For $g \in \text{SSymp}(\Lambda)$, the form $g^*\lambda - \lambda$ is closed and hence it is exact. Hence

$$g^*\lambda - \lambda = dF$$

(3.9)

The function $F$ is defined up to an additive constant. We assume $F = 0$ on $Z_0$. Then

$$\varkappa(g) := \int_{\Lambda} F \, dx \, dy$$

(3.10)

is a homomorphism $\text{SSymp}(\Lambda) \to \mathbb{R}$. It can easily be checked that $\varkappa(g)$ does not depend on a choice of a potential $\lambda$. The homomorphism $\varkappa(g)$ is called by the Calabi invariant.

Next, we restrict the Calabi invariant to the group $\text{Symp}(\Delta) \subset \text{SSymp}(\Lambda)$ and hence we obtain the homomorphisms $\text{Symp}(\Delta) \to \mathbb{R}$.

Remark. In general situation, the Calabi invariant is defined on the kernel of all the flux homomorphisms $\subset \text{Symp}$. In our case, it is defined globally, but its definition is not canonical, it depends on an embedding of the symplectic domain $\Delta$ to $\mathbb{R}^2$. Indeed, formula (3.10) includes integration over the holes, this operation is not invariently defined on the manifold $\Delta$. But these integrals over the holes give a linear combination of flux homomorphisms.

By the Banyaga Theorem (see [1], see also some useful additions in Rousseau [20], see also [2]), the intersection of the kernels of all the flux homomorphisms $\subset \text{Symp}$ and of the kernel of the Calabi invariant is a perfect group (i.e., it has no homomorphisms to Abelian groups; moreover, this intersection is a simple group).

Thus, in our case, each measurable homomorphism $\text{SSymp}(\Delta) \to \mathbb{R}$ is a linear combination of $k$ flux homomorphisms $\tau_j$ and of the Calabi invariant $\varkappa$.

3.8. Values of the Calabi invariants on twists.

Lemma 3.9 Let $B$ be a smooth non self-intersecting curve in $\Lambda$ surrounding the domain $S$. Then

$$\varkappa(\Sigma_B(\varepsilon)) = \sigma(S)^2 + O(\varepsilon), \quad \varepsilon \to 0$$

(3.11)

Proof. We preserve the notation from the proof of previous Lemma 3.6. Let us choose $\lambda = \frac{1}{2}(x \, dy - y \, dx)$. Let $F$ be the function (3.9).

$$F(\ell(1)) = \int_{\ell} (\Sigma_B^* \lambda - \lambda) = \int_{\Sigma_B \ell} \lambda - \int_{\ell} \lambda$$

We obtain an integral over a closed curve $Q$ composed from $\ell$ and $\Sigma_B \ell$.

This curve lies in the strip $\Delta_B(\varepsilon)$ and surrounds $W^{int}$. By the Green formula, $F \simeq \sigma(S)$ in $W^{int}$.

---

6The Choice Axiom implies an “existence” of non-measurable homomorphisms $\psi : \mathbb{R} \to \mathbb{R}$ (number of such homomorphisms is $2^{\text{continuum}}$). A composition of $\psi$ and a flux homomorphism is non-measurable homomorphism $\text{SSymp} \to \mathbb{R}$. 20
We have
\[ \kappa(T_B(\varepsilon)) = \int A F dx dy = \int_{W_{\text{int}}} + \int_{W_{\text{ext}}} + \int_{M_B(\varepsilon)} \]
The first summand gives \( \sigma(S)^2 + O(\varepsilon) \), the second summand is 0. It remains to show that \( F \) is uniformly bounded in \( x \) and \( \varepsilon \) in the strip \( \Omega(\varepsilon) \).

The value of \( F(\ell(s)) \) for \( 0 < s < 1 \) is the integral of \( \lambda \) over a nonclosed curve \( L \) composed from \( \ell(t) \) and \( T(\ell(t)) \) with \( 0 < t < s \) \( (\ell(t) \) is passed in the inverse direction). We include this curve into a closed contour \( C \) adding the direct segments \([0, \ell(0)], [0, \ell(s)]\). The integral of \( \lambda \) over these segments vanishes. Hence, by the Green formula, \( F(\ell(s)) \) is the oriented area of the curvilinear sector bounded by the contour \( C \). Obviously, this area is uniformly bounded.

\[ \square \]

3.9. Preliminary remarks on trivializers. First, let us consider a domain \( \Omega \subset \mathbb{R}^2 \) and a map \( \iota : \Omega \to M \) as in 1.3. This allows to fix an explicit expression (1.5) for the cocycle \( C \). Below we will choose \( \Omega \) and \( \iota \) in a certain appropriate way.

Assume, that our cocycle \( C(q_1, q_2) \) is trivial on \( \text{Symp}(M) = \text{Symp}(M, \Omega, \iota) \), let \( \Gamma(q) \) be its trivializer. In other words, consider the space \( \text{Symp}(M, \Omega, \iota) \times \mathbb{R} \) with the multiplication (1.6). For each \( q \in \text{Symp}(M, \Omega, \iota) \), consider the element \( \tilde{q} := (q, \Gamma(q)) \in \text{Symp}(M, \Omega, \iota) \times \mathbb{R} \)

Then
\[ \tilde{q}_1 \tilde{q}_2 = \tilde{q}_1 \tilde{q}_2 \]

**Lemma 3.10** For a diffeomorphism \( q \in \text{Symp}(M, \Omega, \iota) \) consider the set \( \text{Move}(q) \) of all \( x \in \Omega \) such that \( q(x) \neq x \). Then, for each \( r \in \text{Symp}(M, \Omega, \iota) \),
\[ |C(q, r)| < \frac{\pi}{2} \sigma(\text{Move}(q)), \quad |C(r, q)| < \frac{\pi}{2} \sigma(\text{Move}(q)) \]

In particular, the value of the cocycle \( C(q_1, q_2) \) is \( O(\varepsilon) \) if one of the arguments \( q_1, q_2 \) is an \( \varepsilon \)-twist.

**Proof.** Let \( g_1, g_2 \in \text{Sp}(2n, \mathbb{R}) \). If \( g_1 = 1 \) or \( g_2 = 1 \), then \( c(g_1, g_2) = 0 \), see formula (2.10). It remains to apply Theorem 2.1.d.

**Corollary 3.11**
\[ |\Gamma(g_1 \cdots g_k) - (\Gamma(g_1) + \cdots + \Gamma(g_k))| \leq \frac{\pi}{2} \sum \sigma_i(\text{Move}(g_i)) \]

**Lemma 3.12** There is a constant \( H \) such that
\[ \Gamma(\Xi_C(\varepsilon)) = H + O(\varepsilon), \quad \varepsilon \to 0 \]

for each Dehn \( \varepsilon \)-twist.
Proof. Asymptotics for a single twist. Consider a nonseparating non-self-intersecting curve \( C \) on the surface. Consider a ring \( \Theta \supset C \), identify this ring with a subdomain in the flat circle \( \Lambda \). Our cocycle must be trivial on the group \( \text{Symp}(\Theta) \) of symplectomorphisms of \( \Theta \). On another side, our central extension admits a canonical trivialization \( \Gamma \circ x \) on the group \( \text{SSymp}(\Lambda) \supset \text{Symp}(\Theta) \). Hence

\[ \Gamma(q) = \Gamma^o(q) + u(q), \quad q \in \text{Symp}(\Theta) \]

where \( u \) is a homomorphism \( \text{Symp}(\Theta) \to \mathbb{R} \).

The group \( \text{Symp}(\Theta) \) is a semidirect product \( \mathbb{Z} \rtimes \text{SSymp}(\Theta) \). Let \( \xi \in \text{Symp}(\Theta) \) be a generator of the mapping class group \( \text{Symp}(\Theta) / \text{SSymp}(\Theta) \). Then \( q = \xi^n \circ q^* \), where \( q^* \in \text{SSymp}(\Theta) \). Hence the homomorphism \( u(q) \) must have a form

\[ u(q) = n \cdot u(\xi) + a\tau(q^*) + b\kappa(q) \]

Now let \( q = T_C(\varepsilon) \). Then all the terms of \( \Gamma(q) \) have asymptotics of the form \( \text{const} + O(\varepsilon) \) (for \( \Gamma \circ x \), see Lemma 3.6; for \( \kappa \), see Lemma 3.9; for \( \tau \) this is more-or-less obvious).

Coincidence of asymptotics for different twists. Let \( B, C \) be two nonseparating non self-intersecting curves on \( M \). Let us show, that there exists \( r \in \text{Symp}(M) \) such that

\[ T_C(\varepsilon) = r T_B(\varepsilon) r^{-1} \]

for sufficiently small \( \varepsilon \). Indeed, by definition of \( \varepsilon \)-twists, we have fixed diffeomorphisms \( h_B, h_C \) from some ring \( 1 - \delta < r < 1 + \delta \) to some strips \( \mathcal{U}_B(\delta), \mathcal{U}_C(\delta) \) about \( B, C \). For \( m \in \mathcal{U}_B(\delta) \), we define \( r \) as \( r(m) = h_C \circ h_B^{-1}(m) \). After this we extend \( r \) to \( M \setminus \mathcal{U}_B \) in an arbitrary way. This is possible since \( M \setminus \mathcal{U}_B \) and \( M \setminus \mathcal{U}_C \) are symplectomorphic.

Now consider the corresponding elements of \( \text{Symp}(M, \Omega, \iota) \). We preserve for them the same notation. We have

\[ \tilde{T}_C(\varepsilon) \tilde{r} = \tilde{T}_C(\varepsilon) r = r \tilde{T}_B(\varepsilon) = \tilde{r} \tilde{T}_B(\varepsilon) \]

Evaluating the trivializer for the first and the last terms of this chain we obtain

\[ \Gamma(\tilde{T}_C(\varepsilon)) + \Gamma(r) + C(\tilde{T}_C(\varepsilon), r) = \Gamma(r) + \Gamma(\tilde{T}_B(\varepsilon)) + C(r, \tilde{T}_B(\varepsilon)) \]

By Lemma 3.10, we have \( \Gamma(\tilde{T}_C(\varepsilon)) - \Gamma(\tilde{T}_B(\varepsilon)) = O(\varepsilon) \).

3.10. Proof of Theorem 1.5. Now consider an open domain \( \Delta \) on \( M \) homeomorphic to a disk with 3 holes. Assume also that the set \( M \setminus \Delta \) is connected. Since the genus is \( \geq 3 \), this is possible. Now we define more precisely the set \( \Omega \subset \mathbb{R}^2 \) and the map \( \iota \). Let \( \iota \) identify symplectomorphically \( \Delta \) and some disk \( \Delta \) with 3 holes on \( \mathbb{R}^2 \); denote by \( \Lambda \) the simply connected domain inside the exterior boundary of \( \Delta \).

On \( M \setminus \Delta \) we can choose the map \( \iota \) in an arbitrary way.

Consider 7 curves \( V_0, V_1, V_2, V_3, W_1, W_2, W_3 \) as on Fig.22.
Figure 2: Notation for curves. The domain $\Delta$ is bounded by 4 dotted circles $Z_j$. The domain $\Lambda$ is the disk bounded by $Z_0$. 
Consider the corresponding $\varepsilon$-twists. The diffeomorphism
\[
p(\varepsilon) := \sum_{i=0}^{3} T_{V_i}^{-1} T_{W_i}^{-1} T_{V_i} T_{W_i} T_{V_i}
\]
is isotopic to the identity map, and moreover, the isotopy can be done inside
the domain $\Delta$; this statement is the Dehn’s lantern relation in Teichmüller group
rediscovered by Johnson, see [8], [19], [18].

All our curves are nonseparating in $\mathcal{M}$, and by Lemma 3.12 the trivializer $\Gamma$
on each of our 7 twists is $H + O(\varepsilon)$. By Lemma 3.10,
\[
\Gamma(p) = H + O(\varepsilon), \quad \varepsilon \to 0
\]
In particular, this means that the main term of the asymptotics of $\Gamma(p(\varepsilon))$ is
invariant with respect to deformations of our seven curves $V_0, \ldots$

On another hand, we have the trivializer $\Gamma^o$ of $C(\cdot, \cdot)$ on $\text{SSym}(\Delta)$ described
in 2.11 and given by (3.7). The difference of two trivializers is a homomorphism $\text{SSym}(\Delta) \to \mathbb{R}$. Hence it must be a linear combination of 3 flux homomorphisms $\tau_j$ and the Calabi invariant. Thus, we have
\[
-H + \Gamma^o(p) + \sum_{j=1}^{3} a_j \tau_j(p) + b \cdot \chi(p) = O(\varepsilon)
\]
where $a_j, b$ are some real constants. We evaluate $\tau_j(p)$ and $\chi(p)$ using Lemmas 3.8 and 3.9. Denote by $\sigma[V_i]$ (resp. $\sigma[W_j]$) the area surrounded by a contour $V_i$ (resp. $W_j$). Then
\[
-H + 2\pi \sum_i \sigma[V_i] - 2\pi \sum_j \sigma[W_j] +
\]
\[
+ a_1(\sigma[V_0] + \sigma[V_1] - \sigma[W_1] - \sigma[W_2]) + a_2(\sigma[V_0] + \sigma[V_2] - \sigma[W_2] - \sigma[W_3]) +
\]
\[
+ a_3(\sigma[V_0] + \sigma[W_3] - \sigma[W_3] - \sigma[V_1]) +
\]
\[
+ b(\sum_i \sigma[V_i]^2 - \sum_j \sigma[W_j]^2) = O(\varepsilon)
\]
We can vary the areas $\sigma[\cdot]$ in arbitrary way in certain small intervals. This is a contradiction.

4 Nontriviality of the extension in the case of tori

4.1. Formulation of result. Consider the torus $\mathbb{T}^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$. In notation
of 1.3 it is natural to consider $\Omega = (0, 1)^{2n}$ and the identical embedding $\Omega \to \mathbb{R}^n \to \mathbb{T}^{2n}$. Then Theorem 1.6 is a corollary of the following theorem.

Theorem 4.1 For $n > 2$, the $\mathbb{R}$-valued cocycle $c \in H^2(\text{Sp}(2n, \mathbb{R}))$ is nontrivial
on the group $\text{Sp}(2n, \mathbb{Z})$.  

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Fix $\alpha > 0$. Consider the homomorphism $\mathbb{R} \to \mathbb{R}/2\pi\alpha\mathbb{Z}$ and consider the image $c_\alpha$ of $c$ under this map.

**Theorem 4.2** For $n > 2$ and a noninteger $\alpha > 0$, the $\mathbb{R}/2\pi\alpha\mathbb{Z}$-valued cocycle $c_\alpha$ is nontrivial on the group $\text{Sp}(2n, \mathbb{Z})$.

Theorem 4.2 implies Theorem 4.1 thus it is sufficient to prove Theorem 4.2.

**Remark.** For integer $\alpha$ the cocycle $c_\alpha$ is trivial on the whole group $\text{Sp}(2n, \mathbb{R})$.

**Remark.** Consider the subgroup $\Gamma_{1,2} \subset \text{Sp}(2n, \mathbb{Z})$ consisting of matrices $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ such that diagonals entries of the matrices $A^tC, B^tD$ are even.

The cocycle $c_{1/2}$ is trivial on $\Gamma_{1,2}$ (see [23], Section II.5).

### 4.2. Realization of the cocycle $c_\alpha$ in a representation.

Denote by $B_n$ the set of complex symmetric ($z = z^t$) matrices with norm $< 1$ (the Cartan matrix ball).

The symplectic group acts on $B_n$ by the linear-fractional transformations

$$z \mapsto z^g = (\Phi + z\Psi)^{-1}(\Psi + z\Phi)$$

Fix $\alpha > 0$. Consider the representation $\widetilde{T}_\alpha$ of $\text{Sp}(2n, \mathbb{R})$ in the space of holomorphic functions on $B_n$ given by

$$\widetilde{T}_\alpha(g) f(z) = f(z^g) \det(\Phi + z\Psi)^{-\alpha}$$

It is a standard formula for a highest weight representation of the group $\text{Sp}(2n, \mathbb{R})$.

For $z \in B_n$, the matrix $\Phi + z\Psi$ is nondegenerat, see (2.14). Hence the expression

$$\det(\Phi + z\Psi)^{-\alpha} = \det(1 + z\Phi^{-1})^{-\alpha}$$

has countable number of branches; they are enumerated by values of $(\det \Phi)^{-\alpha}$. Thus $\widetilde{T}_\alpha$ is a projective representation of $\text{Sp}(2n, \mathbb{R})$ (or representation of the universal covering group $\text{Sp}(2n, \mathbb{R})^\sim$).

Nevertheless, we interpret the standard formula (4.1) in the following slightly nonstandard way. Let us consider the normalized operators $T_\alpha(g)$ given by

$$T_\alpha(g) = f(z^g) \det(1 + z\Phi^{-1})^{-\alpha}$$

The expression

$$(1 + z\Phi^{-1})^{-\alpha} = \sum_{k=0}^{\infty} \frac{(\alpha)_k z^k}{k!} (-\Phi^{-1})^k$$

is well defined as a sum of series. Thus its determinant is well defined.

**Proposition 4.3**

$$T_\alpha(g_1)T_\alpha(g_2) = \sigma_\alpha(g_1, g_2)T_\alpha(g_1g_2)$$

where

$$\sigma_\alpha(g_1, g_2) = \det[\Phi(g_1)^{-1}\Phi(g_1g_2)^{-1}]^{-\alpha} = \exp\{-\alpha c(g_1, g_2)\}$$

\[25\]
Proof. For $g_1, g_2$ near 1, it is proved by a trivial calculation. After this we consider the analytic continuation in $g_1, g_2$. □

The cocycles $\sigma_\alpha$ are precisely the cocycles $c_\alpha$ in multiplicative notation. We intend to prove Theorem 4.2 in the following form:

the restriction of the representation $T_\alpha$ to $\text{Sp}(2n, \mathbb{Z})$ cannot be reduced to a linear representation of $\text{Sp}(2n, \mathbb{Z})$ by a correction of the form

$$T_\alpha(g) \mapsto \gamma(g)T_\alpha, \quad \gamma(g) \in \mathbb{C}^*$$

where $g$ ranges in $\text{Sp}(2n, \mathbb{Z})$.

4.3. Another model of the same representation. By $W_n$ we denote the Siegel wedge, i.e., the set of complex symmetric matrices satisfying $z$ with $\text{Im } z > 0$.

The group $\text{Sp}(2n, \mathbb{R})$ acts on $W_n$ by the transformations

$$z \mapsto z[g] := (a + zc)^{-1}(b + zd)$$

where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a real symplectic matrix in the usual notation. The action of $\text{Sp}(2n, \mathbb{R})$ on $W_n$ is given by

$$S_\alpha(g)f(z) = f(z[g])\det(a + zc)^{-\alpha} \quad (4.4)$$

It is well-known, that the (projective) representation $S_\alpha$ is equivalent to representation $T_\alpha$ defined above. The intertwining operator is given by the transformation

$$T_\alpha \left[ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \right] f(z) := f((1 + iz)^{-1}(i + z)) \cdot \det(1 + iz)^{-\alpha} \cdot 2^\alpha/2$$

where $z \in B_n$ (then its Cayley transform $(1 + iz)^{-1}(i + z)$ is an element of $W_n$).

4.4. Linearization of representation on a upper triangular subgroup. Consider the subgroup $\mathcal{B}(\mathbb{Z}) \subset \text{Sp}(2n, \mathbb{Z})$ consisting of the matrices

$$\begin{pmatrix} A & B \\ 0 & A^{t-1} \end{pmatrix}; \quad \det A = 1$$

The cocycle $\sigma_\alpha$ is equivalent to trivial cocycle on this subgroup. Indeed, the operators

$$S_\alpha \begin{pmatrix} A & B \\ 0 & A^{t-1} \end{pmatrix} f(z) = f(A^{-1}(B + zA^{t-1})) \quad (4.5)$$

define a linear representation of $\mathcal{B}(\mathbb{Z})$.

Lemma 4.4 Formula (4.5) gives a unique possible linearization of the representation $S_\alpha$ on the subgroup $\mathcal{B}(\mathbb{Z})$.

This follows from the next lemma.
Lemma 4.5 The group $B(Z)$ has no homomorphisms to $C^*$. In particular, it has no homomorphisms to $Z$ and $Z_k$.

Proof. Let $\chi : B(Z) \to C^*$ be a homomorphism.

First, the group $B(Z)$ contains the group $SL(n, Z)$ consisting of matrices
$$ \begin{pmatrix} A & 0 \\ 0 & A^{t-1} \end{pmatrix}.$$ This group has no Abelian quotients. Hence, $\chi = 1$ on $SL(n, Z)$.

Also, $B(Z)$ contains the group $N(Z)$ consisting of matrices $\nu(B) = \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix}$; the product in this group corresponds to the sum of the matrices $B$:
$$ \begin{pmatrix} 1 & B_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & B_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & B_1 + B_2 \\ 0 & 1 \end{pmatrix}$$

Also
$$ \begin{pmatrix} A & 0 \\ 0 & A^{t-1} \end{pmatrix} \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & A^{t-1} \end{pmatrix}^{-1} = \begin{pmatrix} 1 & ABA^t \\ 0 & 1 \end{pmatrix}$$

Thus, we must prove, that the there is no characters
$$ \chi(B_1 + B_2) = \chi(B_1)\chi(B_2)$$
on the additive group of symmetric integer matrices $B$ such that
$$ \chi(ABA^t) = \chi(B) \quad \text{for} \quad A \in SL(n, Z)$$

Any character of $N(Z)$ has the form
$$ \chi(B) = \prod_{i \geq j} y_{i,j}^{b_{ij}}, \quad y_{i,j} \in C^*$$

Since the group $SL_n(Z)$ contains all the even permutations of coordinates, our character has the form
$$ \chi(B) = u^{tr \cdot B} \cdot v^{\sum_{i>j} b_{ij}}$$

Next,
$$ \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$
$$ = \begin{pmatrix} a_{11} + 2a_{12} + a_{22} & a_{12} + a_{22} & a_{13} + a_{23} \\ a_{12} + a_{22} & a_{22} & a_{23} \\ a_{13} + a_{23} & a_{23} & a_{33} \end{pmatrix}$$

Hence, for all $a_{ij} \in Z$ we have
$$ u^{2a_{12} + a_{22} + a_{23}} = 1$$

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Thus \( u = v = 1 \).

### 4.5. Proof of Theorem 4.2

Assume that we have some linearization \( S_\alpha \) of the representation \( S_\alpha \) on \( \text{Sp}(2n, \mathbb{R}) \). By Lemma 4.4, this linearization is rigidly defined on the matrices \( \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} \) by the formula

\[
S_\alpha \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} f(z) = f(z + B) \tag{4.6}
\]

Now let us consider the subgroup \( \text{SL}(2, \mathbb{Z}) = \text{Sp}(2, \mathbb{Z}) \subset \text{Sp}(2n, \mathbb{Z}) \) consisting of the \( [1 + (n - 1) + 1 + (n - 1)] \times [1 + (n - 1) + 1 + (n - 1)] \) matrices

\[
\begin{bmatrix}
a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 
\end{bmatrix}
\]

To be short, below we will write \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \). Let us show that having condition (4.6), we cannot trivialize the cocycle on \( \text{SL}(2, \mathbb{Z}) \).

Denote

\[
I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \quad K = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}
\]

It can be easily checked that

\[
I^4 = 1, \quad J^3 = 1, \quad IK = J \tag{4.7}
\]

Our operators \( S_\alpha(K) \) have the form

\[
S_\alpha(K) f(z) = f(z - 1),
\]

\[
S_\alpha(I) f(z) = \theta \cdot f(-1/z)z^{-\alpha},
\]

\[
S_\alpha(J) f(z) = \theta' \cdot f(-1 - 1/z)z^{-\alpha}
\]

where

\[
z^{-\alpha} = |z|^{-\alpha}\exp(-i\alpha \arg z); \quad 0 < \arg z < \pi
\]

and \( \theta, \theta' \in \mathbb{C}^* \) are some unknown constants. The equation \( IK = J \) implies \( \theta' = \theta \). Also

\[
S_\alpha(I)^4 = \theta^4 \exp(-2\alpha \pi i), \quad S_\alpha(J)^3 = \theta^3 \exp(-2\alpha \pi i) \tag{4.8}
\]

Since the both these operators equal 1, \( \theta = 1, \exp(-2\alpha \pi i) = 1 \). We obtain a contradiction, since \( \alpha \notin \mathbb{Z} \).

**Remark.** An evaluation of powers in (4.8) can be simplified in the following way. First, the point \( i \) is a fix point of the transformation \( z \mapsto -1/z \). Hence we can follow only the values of \( S_\alpha(I)^k f(z) \) in this point. In the second case the fixed point is \( \lambda = \exp(2\pi i/3) \).
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Math.Phys. Group, Institute of Theoretical and Experimental Physics, B.Chemushkinskaya, 25, Moscow 117259
& University of Vienna, Math. Dept., Nordbergstrasse, 15, Vienna 1090, Austria
neretin@mccme.ru