Sobolev gradients of viscosity supersolutions

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Abstract. We investigate which elliptic PDEs have the property that every viscosity supersolution belongs to a Sobolev space \( W^{1,q}_{\text{loc}}(\Omega), \Omega \subseteq \mathbb{R}^n \). The asymptotic cone of the operator’s sublevel set seems to be essential. It turns out that much can be said if we know how the cone is related to the sublevel set of the dominative \( p \)-Laplacian, with \( p = \frac{n-1}{n} q + 1 \). We show that, in a certain sense, this is the minimal operator associated to the exponent \( q \).

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1. Introduction

A viscosity supersolution of an elliptic equation is, a priori, no more regular than lower semicontinuous. Its definition does not require any differentiability. However, some equations are known to impose a regularity on their supersolutions. For example, superharmonic functions have weak gradients that are locally integrable. If \( u \) is a viscosity supersolution to the Laplace equation \( \Delta u = 0 \) in some open set \( \Omega \subseteq \mathbb{R}^n, n \geq 2 \), then

\[
\int_D |\nabla u|^q \, dx < \infty
\]

whenever

\[
0 < q < \frac{n}{n-1}
\]

for every compact subset \( D \) of \( \Omega \). That is, \( u \) belongs to the Sobolev space \( W^{1,q}_{\text{loc}}(\Omega) \). The fundamental solution \( x \mapsto |x|^{2-n} \), or \( -\ln |x| \) in the case \( n = 2 \), shows that the bound on \( q \) is sharp. This result is generalized to \( p \)-superharmonic functions in [6]. The exponent \( q \) can be increased when \( p > 2 \). More precisely, if \( u \) is a viscosity supersolution to the \( p \)-Laplace equation

\[
\Delta_p u := \text{div}(|\nabla u|^{p-2} \nabla u) = 0
\]

(1.1)
for some $2 \leq p \leq n$, then $u \in W^{1,q}_{loc}(\Omega)$ for every $q$ such that

$$0 < q < \frac{n}{n-1}(p-1).$$

Again the bound is sharp, as shown by the fundamental solution $x \mapsto |x|^\frac{p-n}{n-p-1}$ or $-\ln |x|$ in the case $p = n$.\(^1\)

One may wonder what it is that characterizes a PDE whose supersolutions are in some first order Sobolev space. In this paper we consider equations that depend only on the second order partial derivatives

$$\mathcal{H}u := \left[ \frac{\partial^2 u}{\partial x_i \partial x_j} \right]_{i,j}.$$  \(^{2}\)

Given $q$, we ask the following question. For which operators $F: \mathcal{S}^n \to \mathbb{R}$ is every viscosity supersolution of the equation

$$F(\mathcal{H}u) = 0 \quad \text{in } \Omega$$

in $W^{1,q}_{loc}(\Omega)$? To our knowledge, this particular problem has not been addressed before.

Our theorems are presented in the next Section. Sufficient and necessary conditions are established in Theorems 1 and 2, respectively. The *dominative $p$-Laplacian*

$$D_p u := \Delta u + (p-2)\lambda_n(\mathcal{H}u)$$

will play a prominent role in our characterization. It was introduced in [3] in order to explain a superposition principle in the $p$-Laplace Eq. (1.1). The key property in that setting was its domination

$$|
abla u|^{2-p} \Delta_p u \leq D_p u$$

over the normalized $p$-Laplacian. In the present situation, a sort of opposite property will also be of importance. In [2] it is shown that $D_p$ is *minimal* in the class of sublinear elliptic operators that share its fundamental solution (3.5). The exact statement is found in Sect. 3, together with a simplified version of the proof. We shall also make use of the *canonical operator*, which is described in [4]. Some additional properties are proved in Sect. 4.

We recall that a function $u: \Omega \to (-\infty, \infty]$ is a viscosity supersolution of (1.2) if it is lower semicontinuous, and whenever a $C^2$ function $\phi$ touches $u$ from below at a point $x_0 \in \Omega$, then $F(\mathcal{H}\phi(x_0)) \leq 0$. In particular, $u \in C^2(\Omega)$ is a viscosity supersolution if and only if $F(\mathcal{H}u(x)) \leq 0$ for all $x \in \Omega$.

\(^1\)The viscosity supersolutions of (1.1) coincide with the traditional $p$-superharmonic functions defined by differentiating test functions under the integral sign. See [5].
2. The main theorems

In order to start the search for operators having only $W^{1,q}_{loc}$ supersolutions, we make some observations. Firstly, the properties we are looking for must depend only on the sublevel set

$$
\Theta = \Theta(F) := \{ X \in S^n \mid F(X) \leq 0 \}
$$

in the space $S^n$ of symmetric $n \times n$ matrices. Indeed, if two operators have the same sublevel set, then they also share the same set of supersolutions. Secondly, the properties should be invariant under translations of $\Theta$. This is because a function $u$ is a supersolution to $F(\mathcal{H}u) = 0$ if and only if $v(x) := u(x) + \frac{1}{2} x^\top X_0 x$ is a supersolution to the equation $F(\mathcal{H}v - X_0) = 0$. The sublevel set of $X \mapsto F(X - X_0)$ is the translation $\Theta + \{X_0\}$, and $u$ and $v$ are clearly in the same Sobolev space. Also, a linear transformation of the form

$$
B^\top \Theta B := \{ B^\top XB \mid X \in \Theta \}
$$

where $B$ is an invertible $n \times n$ matrix, should not matter: If $u$ is a supersolution to $F(\mathcal{H}u) = 0$, define the function $v(x) := u(Bx)$. Then $\mathcal{H}v(x) = B^\top \mathcal{H}u(Bx)B$ and $v$ is a supersolution to the equation $F(B^{-\top} \mathcal{H}vB^{-1}) = 0$ with sublevel set $B^\top \Theta B$. Again, $u \in W^{1,q}_{loc}$ if and only if $v \in W^{1,q}_{loc}$. Admittedly, we conducted this argument as if $u$ and $v$ were twice differentiable, but, as we shall see, the reasoning is sound because we can do the computations on the test functions. Finally, if a supersolution is not smooth, one can suspect that its Hessian matrix – at some point – has to run off to infinity in some direction in $\Theta$. It is perhaps only the shape of $\Theta$ for large $\|X\|$ that is significant for whether a supersolution is in $W^{1,q}_{loc}$ or not. We use the asymptotic cone

$$
ac(\Theta) := \left\{ Z \in S^n \mid \exists t_k \to \infty, \exists X_k \in \Theta \text{ with } \lim_{k \to \infty} \frac{X_k}{t_k} = Z \right\}
$$

to capture the behaviour of $\Theta$ at infinity.

The dominative $p$-Laplacian $\mathcal{D}_p : C^2(\Omega) \to C(\Omega)$ can be written as $\mathcal{D}_p u(x) = F_p(\mathcal{H}u(x))$ where $F_p : S^n \to \mathbb{R}$ is given by $F_p(X) = \text{tr} \, X + (p - 2) \lambda_n(X)$. For computational convenience we shall in this paper multiply the operator with a practical, but otherwise insignificant, scaling constant. We define

$$
F_p(X) := \frac{1}{1 + p - 2} \left( \text{tr} \, X + (p - 2) \lambda_n(X) \right) \quad \text{for } 2 \leq p < \infty, \quad \text{and}
$$

$$
F_\infty(X) := \lambda_n(X),
$$

the largest eigenvalue of $X$.

The normalization makes

$$
F_p(X + \tau I) = F_p(X) + \tau
$$

for all $p \in [2, \infty]$, $X \in S^n$, and $\tau \in \mathbb{R}$. We let

$$
\Theta_p := \Theta(F_p) = \{ X \in S^n \mid F_p(X) \leq 0 \}
$$

denote the sublevel set of $F_p$. It can be verified that $\Theta_p$ is a closed convex cone in $S^n$. At $p = \infty$,

$$
\Theta_\infty = \{ X \in S^n \mid \lambda_n(X) \leq 0 \} = \{ X \in S^n \mid X \leq 0 \}$$
is the set $S^n$ of negative semidefinite matrices. When $p$ decreases, the cone $\Theta_p$ gradually opens up, and eventually flattens out to the half-space

$$\Theta_2 = \{ X \in S^n \mid \text{tr} \, X \leq 0 \} = \{ X \in S^n \mid \langle I, X \rangle \leq 0 \}.$$ 

See Fig. 1a. For $2 \leq p' < p \leq \infty$, one can check that

$$\Theta_{\infty} \subseteq \Theta_p \subseteq \Theta_{p'} \subseteq \Theta_2 \quad \text{and} \quad \partial \Theta_p \cap \partial \Theta_{p'} = \{0\}. \quad (2.1)$$

The following fact is an immediate consequence of (1.3).

**Proposition 2.1.** Let $2 \leq p \leq \infty$. A function $u \colon \Omega \to (-\infty, \infty]$ is $p$-superharmonic whenever it is dominative $p$-superharmonic.

See Proposition 5 in [3]. Next, Theorem 5.18 in [6] reads

**Proposition 2.2.** Let $p > 2 - 1/n$. If $u \colon \Omega \to (-\infty, \infty]$ is $p$-superharmonic, then the Sobolev gradient $\nabla u = [\partial u / \partial x_1, \ldots, \partial u / \partial x_n]$ exists and

$$\int_D |\nabla u|^q \, dx < \infty, \quad D \subset \subset \Omega,$$

for

$$\begin{cases} 
0 < q < \frac{n}{n-1} (p-1), & \text{in the case } p \leq n, \\
0 < q \leq p & \text{in the case } p > n.
\end{cases}$$

Thus, if we can find an invertible $B$ and an $X_0 \in S^n$ such that

$$B^\top \Theta B \cap \{X_0\} \subseteq \Theta_p \quad (2.2)$$

for some $2 \leq p \leq n$, then every supersolution $u$ of $F(\mathcal{H} u) = 0$ is, by a change of variables and by subtracting a quadratic, $p$-superharmonic. We conclude from Proposition 2.2 that $u \in W^{1,q}_{\text{loc}}(\Omega)$ for every $0 < q < \frac{n}{n-1} (p-1)$. We shall show that

$$\text{ac}(B^\top \Theta B) \subseteq \Theta_p \quad (2.3)$$

is a strictly weaker condition than (2.2), but still sufficient in order to ensure $u \in W^{1,q}_{\text{loc}}(\Omega)$ in the case $2 < p \leq n$. We find it rather interesting that (2.3) turns out to also be necessary under some common assumptions on $F$.

**Theorem 1.** (Sufficient condition) Let $p \in (2, n]$, $\Omega \subseteq \mathbb{R}^n$ be open, and let $\Theta \subseteq S^n$ be the sublevel set of an operator $F \colon S^n \to \mathbb{R}$.\footnote{\(\Theta\) is well-defined even if $F$ takes values in the extended real line $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm \infty\}$.}

If there is an invertible $n \times n$ matrix $B$ such that

$$\text{ac}(B^\top \Theta B) \subseteq \Theta_p$$

then every viscosity supersolution of

$$F(\mathcal{H} u) = 0 \quad \text{in } \Omega$$

is $W^{1,q}_{\text{loc}}(\Omega)$ for all $q$ such that

$$0 < q < \frac{n}{n-1} (p-1).$$
When \( p = 2 \) the condition (2.3) is necessary (under our assumptions), but our method of proof for sufficiency does no longer work. In this case we have to assume (2.2), which can be rephrased as \( \Theta \) being confined to an affine half-space with positive definite outer normal.

**Proposition 2.3.** (Sufficient condition. \( p=2 \)) If there is a \( \tau \in \mathbb{R} \) and a positive definite \( n \times n \) matrix \( A \) such that

\[
\Theta(F) \subseteq \{ X \in S^n \mid \langle X, A \rangle \leq \tau \}, \tag{2.4}
\]

then every viscosity supersolution of \( F(Hu) = 0 \) is \( W^{1,q}_{loc} \) for all \( 0 < q < \frac{n}{n-1} \).

The condition (2.4) is probably not necessary as indicated by the following example. Consider the equation

\[
\lambda_1(Hu) + \lambda_2(Hu) - 2\sqrt{1 + \lambda_2(Hu)} - 2 = 0 \tag{2.5}
\]

in the unit disk \( D \) in \( \mathbb{R}^2 \). One can check that radial solutions \( w(x) = U(|x|) \) with \( U(0) = +\infty \) are on the form \( w(x) = -\frac{1}{2}|x|^2 + 2c|x| - c^2 \ln |x|, \ c \geq 1 \), and are thus \( W^{1,q}_{loc}(D) \) for all \( q < n/(n-1) = 2 \). However, (2.4) does not hold since the trace of the Hessian matrix of \( w \) is not bounded above as \( x \to 0 \).

Nevertheless, \( \text{ac}(\Theta) \subseteq \Theta_2 \), because if \( X_k \in \Theta \) and \( t_k \to \infty \) are sequences such that \( X_k/t_k \to Z \in \text{ac}(\Theta) \), then \( t_k \) must be comparable to \( \lambda_n(X_k) \) for large \( k \) and

\[
\text{tr } Z = \lim_{k \to \infty} \frac{\text{tr } X_k}{t_k} \leq \lim_{k \to \infty} \frac{2\sqrt{1 + \lambda_n(X_k)} - 2}{t_k} = 0.
\]

We conjecture that Theorem 1 is valid also for \( p = 2 \).

In order to state the necessary conditions, we establish some terminology. An operator \( F: S^n \to \mathbb{R} \) is said to be *rotationally invariant* (also called *spectrally defined*) if it only depends on the eigenvalues of the argument. This is equivalent to

\[
F(X) = F(Q^T X Q) \text{ for all } X \in S^n \text{ and all } Q \in O^n,
\]

where \( O^n \) is the set of \( n \times n \) orthogonal matrices. The Laplacian, the dominantative \( p \)-Laplacian, the Pucci operators, and Monge-Amprère operators are typical examples from this class. Linear operators \( F(X) = \text{tr}(AX) \) are counterexamples provided \( A \) is not a scaling of the identity matrix. However, if \( A \) is positive definite, then \( F \) can be made rotationally invariant by a linear change of variables. Indeed, \( X \mapsto F(\sqrt{A}^{-1}X\sqrt{A}^{-1}) \) is the Laplacian.

**Definition 2.1.** An operator \( F: S^n \to \mathbb{R} \) is *essentially rotationally invariant* if there is an invertible matrix \( B \) such that

\[
X \mapsto F(BX B^\top)
\]

is rotationally invariant.

An operator is *elliptic* if

\[
X \leq Y \text{ implies } F(X) \leq F(Y). \tag{2.6}
\]
Figure 1. a Sublevel sets of the dominative $p$-Laplacian in a simplified model of $S^n$. b The sublevel set $\Theta$ of Eq. (2.5) together with its asymptotic cone $ac(\Theta) = \Theta_2$

As always, $X \leq Y$ is the standard partial ordering in $S^n$ and means $X - Y \in S^n$. Ellipticity ensures that the sublevel set $\Theta$ of $F$ is a (negative) elliptic set. That is,

\[ X \leq Y \text{ and } Y \in \Theta \quad \text{implies} \quad X \in \Theta. \]

An equivalent statement can be made in terms of Minkovski addition,

\[ \Theta + S^n = \Theta. \]

In terms of the model of $S^n$ used in Fig.1, the boundary of an elliptic set will appear as the graph of a nonincreasing function. Next, if $F$ is rotationally invariant, then $\Theta$ is a rotationally invariant set. That is, $\Theta = \text{rot } \Theta$ where

\[ \text{rot } \Theta := \{ Q^T X Q \mid X \in \Theta, Q \in O^n \}. \]

Similarly, if $F$ is essentially rotationally invariant, then $\Theta$ is essentially rotationally invariant, meaning that there is an invertible $B$ so that $B^T \Theta B$ is a rotationally invariant set. Finally, $\Theta$ is convex whenever $F$ is convex, but also, for example, if $F$ is merely quasiconvex.

Theorem 2. (Necessary condition) Let $p \in [2, \infty]$, $\Omega \subseteq \mathbb{R}^n$ be open, and assume that $\Theta = \Theta(F) \subseteq S^n$ is an elliptic, essentially rotationally invariant, and convex sublevel set of an operator $F : S^n \to \mathbb{R}$.

If every viscosity supersolution of $F(\mathcal{H}u) = 0$ in $\Omega$

is $W^{1,q}_{loc}(\Omega)$ for all $q$ with

\[ 0 < q < \frac{n}{n-1}(p - 1), \]

then there is an invertible $n \times n$ matrix $B$ such that

\[ ac(B^T \Theta B) \subseteq \Theta_p. \]
The proofs of Theorem 1 and Proposition 2.3 follow in the next Section. The proof of Theorem 2 is postponed until Sect. 4.

3. Rotationally invariant sublinear elliptic operators

An operator $G: S^n \to \mathbb{R}$ is sublinear if it is positive homogeneous (of order 1) and subadditive. That is, for $X, Y \in S^n$ and positive numbers $c$ we have $G(cX) = cG(X)$ and $G(X + Y) \leq G(X) + G(Y)$. This class of operators is nothing but the family of support functions in $S^n$. See Theorem 1.7.1 and the foregoing discussion in [7]. There is thus a unique non-empty, compact and convex subset (i.e. a convex body) $K = K(G) \subseteq S^n$ such that

$$G(X) = \max_{A \in K} \langle A, X \rangle.$$  

It is not difficult to show that

$$K(G) = \{ A \in S^n \mid \langle A, X \rangle \leq G(X) \ \forall X \in S^n \}.$$  

It can be checked that the sublinear operator is elliptic if and only if $K$ is a subset of $S^n_+ := \{ A \in S^n \mid A \geq 0 \}$, and it is rotationally invariant if and only if

$$K = \text{rot} K := \{ QAQ^\top \mid A \in K, Q \in O^n \}.$$  

Furthermore, we label $G$ as non-totally degenerate if $0 \notin K$. Though not strictly necessary, this pragmatic assumption simplifies the exposition. It is a rather natural condition because $0 \in K \Rightarrow G \geq 0$ in $S^n$.

To each such operator we assign a number $p \in [2, \infty]$.

**Definition 3.1.** The body cone aperture to a non-totally degenerate rotationally invariant sublinear elliptic operator $G: S^n \to \mathbb{R}$ is

$$p = p(G) := \begin{cases} \frac{n+\alpha-2}{\alpha-1}, & \text{if } 1 < \alpha \leq n, \\ \infty, & \text{if } \alpha = 1, \end{cases}$$

where

$$\alpha = \alpha(G) := \min_{A \in K(G)} \frac{\text{tr} A}{\lambda_n(A)}.$$  

(3.1)

Observe that $p$ and $\alpha$ are well-defined. In fact, the trace and the largest eigenvalue $\lambda_n(A) > 0$ are continuous functions of $A$, and $K$ is compact. Additionally,

$$1 = \frac{\lambda_n(A)}{\lambda_n(A)} \leq \frac{\text{tr} A}{\lambda_n(A)} \leq \frac{n\lambda_n(A)}{\lambda_n(A)} = n$$

for all $A \in S^n_+ \setminus \{0\}$. The numbers are duals in the sense $(\alpha - 1)(p - 1) = n - 1$. Since $K \subseteq S^n_+$, one can also note that (3.1) is a minimum of a ratio of the norms

$$\|X\|_1 := \sum_{i=1}^n |\lambda_i(X)| \quad \text{and} \quad \|X\|_{\infty} := \max\{-\lambda_1(X), \lambda_n(X)\}.$$
Furthermore, $\alpha$ is invariant under positive scalings of the convex body. The body cone aperture is therefore – as suggested by its name – determined by the convex cone \( \{ cA \mid A \in \mathcal{K}, c \geq 0 \} \) in \( S^n \).

As examples, we mention the maximal Pucci operator \( F_{\lambda,\Lambda} \), defined by the convex body
\[
\mathcal{K}_{\lambda,\Lambda} := \{ A \in S^n \mid \lambda I \leq A \leq \Lambda I \}, \quad 0 < \lambda \leq \Lambda,
\]
and the dominative \( p \)-Laplace operator
\[
F_p(X) = \frac{1}{n+p-2} \left( \text{tr} \, X + (p-2)\lambda_n(X) \right), \quad p \in [2, \infty),
\]
\[
F_\infty(X) = \lambda_n(X).
\]
Here, \( F_p(X) = \max_{A \in \mathcal{K}_p} \langle A, X \rangle \) where \( \mathcal{K}_p := \mathcal{K}(F_p) \) must be the convex hull of the compact subset
\[
E_p := \left\{ \frac{I + (p-2)\xi \xi^T}{n+p-2} \mid \xi \in S^{n-1} \right\}, \quad p \in [2, \infty),
\]
\[
E_\infty := \{ \xi \xi^T \mid \xi \in S^{n-1} \}.
\]
A computation will reveal that
\[
\mathcal{K}_p = \{ A \in S^n \mid \frac{1}{n+p-2} I \leq A \leq \frac{p-1}{n+p-2} I, \text{tr} \, A = 1 \}, \quad p \in [2, \infty),
\]
\[
\mathcal{K}_\infty = \{ A \in S^n \mid 0 \leq A \leq I, \text{tr} \, A = 1 \},
\]
and thus,
\[
\alpha(F_p) = \min_{A \in \mathcal{K}_p} \frac{\text{tr} \, A}{\lambda_n(A)} = \begin{cases} \frac{n+p-2}{p-1}, & p \in [2, \infty), \\ 1, & p = \infty, \end{cases}
\]
which implies \( p(F_p) = p \). By way of illustration,
\[
\alpha(F_{\lambda,\Lambda}) = \min_{A \in \mathcal{K}_{\lambda,\Lambda}} \frac{\text{tr} \, A}{\lambda_n(A)} = \min_{A \in \mathcal{K}_{\lambda,\Lambda}} \frac{\lambda_1(A) + \cdots + \lambda_{n-1}(A)}{\lambda_n(A)} + 1 = \left( \frac{n-1}{\Lambda} \right) + 1
\]
and \( p(F_{\lambda,\Lambda}) = \Lambda/\lambda + 1 \).

We now prove that \( ac(B^T \Theta(F)B) \subseteq \Theta_p \) implies that every supersolution of \( F(\mathcal{H}u) = 0 \) have a Sobolev gradient.

**Proof of Theorem 1.** Let \( 2 < p \leq n \). Recall that \( \mathcal{K}_p \) – the associated convex body to the dominative \( p \)-Laplacian – is the unique convex and compact subset of \( S^n \) such that \( F_p \) is the support function of \( \mathcal{K}_p \). That is,
\[
F_p(X) = \max_{A \in \mathcal{K}_p} \langle A, X \rangle.
\]
Introduce the short-hand
\[
\Theta^B := B^T \Theta B
\]
and suppose \( ac(\Theta^B) \subseteq \Theta_p \) for an invertible matrix \( B \). Let \( 0 < q < \frac{n}{n-1}(p-1) \), and choose \( p' \in [2, p) \) so that we still have \( q < \frac{n}{n-1}(p' - 1) \). Now,
\[
\mathcal{K}_{p'} \subseteq \mathcal{K}_p.
\]
and we claim that
\[
\sup_{A \in \mathcal{K}_{p'}} \langle A, X \rangle \quad (3.3)
\]
is finite. Suppose it is not. It is $-\infty$ only if $\Theta$ is empty, but then there are no supersolutions and nothing to prove in the Theorem. There are therefore sequences $A_k \in \mathcal{K}_{p'}$ and $X_k \in \Theta^B$ with $\langle A_k, X_k \rangle \to \infty$ as $k \to \infty$. By compactness, we may assume $A_k$ to converge to some $A_0 \in \mathcal{K}_{p'}$ and $\hat{X}_k := X_k/\|X_k\|$ to converge to some $Z$ in the unit sphere in $S^n$. Obviously, $\limsup_{k \to \infty} \|X_k\| = \infty$, and $\langle A_k, \hat{X}_k \rangle$ is eventually non-negative. Thus,
\[
0 \leq \langle A_0, Z \rangle \leq \max_{A \in \mathcal{K}_{p'}} \langle A, Z \rangle \leq \max_{A \in \mathcal{K}_{p'}} F_p(Z) \leq 0
\]
since $Z \in \text{ac}(\Theta^B)$ and by the hypothesis $\text{ac}(\Theta^B) \subseteq \Theta_p$. Therefore, $F_{p'}(Z) = 0 = F_p(Z)$, which leads to the contradiction $Z = 0$ by (2.1).

Let $u$ be a supersolution to $F(Hu) = 0$ in some $\Omega \subseteq \mathbb{R}^n$. Define the function
\[
v(x) := u(Bx) - \frac{\tau}{2} |x|^2
\]
where $\tau$ is the number (3.3). Suppose $\phi$ is a test function touching $v$ from below at $x_0 \in \Omega^B := \{x \mid Bx \in \Omega\}$, and consider the test function
\[
\psi(y) := \phi(B^{-1}y) + \frac{\tau}{2} y^\top B^{-\top} B^{-1} y.
\]
Then, $\psi(y) \leq u(y)$, and at $y_0 := Bx_0$ we have $\psi(y_0) = u(y_0)$. Thus $F(H\psi(y_0)) \leq 0$ and $\Theta \ni \mathcal{H}\psi(y_0) = B^{-\top} \mathcal{H}\phi(x_0) B^{-1} + \tau B^{-\top} B^{-1}$.

Equivalently,
\[
\mathcal{H}\phi(x_0) + \tau I \in \Theta_B.
\]
Now,
\[
F_{p'}(H\phi(x_0)) = F_{p'}(\mathcal{H}\phi(x_0) + \tau I) - \tau
= \max_{A \in \mathcal{K}_{p'}} \langle A, \mathcal{H}\phi(x_0) + \tau I \rangle - \tau
\leq \sup_{A \in \mathcal{K}_{p'}} \langle A, X \rangle - \tau
\leq 0,
\]
which proves that $v$ is dominative $p'$-superharmonic in $\Omega^B$. This implies $p'$-superharmonicity by Proposition 2.1 and $v$ is then $W^{1,q}_{loc}(\Omega^B)$ by Proposition 2.2. It follows that $u(x) = v(B^{-1}x) + \frac{\tau}{2} x^\top B^{-\top} B^{-1} x$ is $W^{1,q}_{loc}(\Omega)$ as well.

\begin{proof}[Proof of Proposition 2.3]
As in the above proof, a change of variables in the test functions will show that
\[
v(x) := u(\sqrt{A}x) - \frac{\tau}{2n} |x|^2
\]
is superharmonic whenever $u$ is a supersolution of $F(Hu) = 0$.
\end{proof}
We now prepare the ground for the proof of Theorem 2. The dominative $p$-Laplacian holds the special position of being the minimal operator of its class.

**Proposition 3.1.** (Minimal operator) Let $G: S^n \to \mathbb{R}$ be a non-totally degenerate rotationally invariant sublinear elliptic operator. Then there exists a constant $c > 0$ such that

$$c F_p(G)(X) \leq G(X) \quad \forall X \in S^n. \quad (3.4)$$

In particular, $\Theta(G) \subseteq \Theta_p(G)$.

For $p \in [2, \infty]$ we define the lower semicontinuous function $w_{n,p}: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ as

$$w_{n,p}(x) := \begin{cases} \frac{p-1}{p-n} |x|^{\frac{p-n}{p-1}}, & 2 \leq p \neq n, \\ -\ln |x|, & p = n, \\ -|x|, & p = \infty, \end{cases} \quad (3.5)$$

with the interpretation $w_{n,p}(0) = \infty$ for $p \leq n$. In $\mathbb{R}^n \setminus \{0\}$, it is a solution to both the dominative $p$-Laplace equation $\mathcal{D}_p w = 0$ and the 'ordinary' $p$-Laplace equation $\Delta_p w = 0$. It is a viscosity supersolution in $\mathbb{R}^n$. We show next that the same is true for the equation $G(\mathcal{H} w) = 0$ when $p = p(G)$.

**Proposition 3.2.** (Existence of fundamental solution) Let $G: S^n \to \mathbb{R}$ be a non-totally degenerate rotationally invariant sublinear elliptic operator and let $p \in [2, \infty]$ be its body cone aperture. Then $w_{n,p}$ is a solution to the equation $G(\mathcal{H} w) = 0$ in $\mathbb{R}^n \setminus \{0\}$ and a viscosity supersolution in $\mathbb{R}^n$.

Note that this means the bound (3.4) is sharp.

The key ingredient in the proof of Proposition 3.1 is established in the following Lemma. Due to rotational invariance, it can be conducted in $\mathbb{R}^n$ rather than in $S^n$. The standard basis vectors are denoted by $e_1, \ldots, e_n$, and we write $1 := [1, \ldots, 1]^\top = e_1 + \cdots + e_n \in \mathbb{R}^n$.

**Lemma 3.1.** Let $p \in [2, \infty]$ and set $p \in \mathbb{R}^n$ to be

$$p := \begin{cases} \frac{1}{n+p-2} [1, \ldots, 1, p - 1]^\top, & \text{if } p \in [2, \infty), \\ [0, \ldots, 0, 1]^\top = e_n, & \text{if } p = \infty. \end{cases} \quad (3.6)$$

Suppose $a = [a_1, \ldots, a_n]^\top \in \mathbb{R}^n$ is a vector such that the sum of its elements equals the sum of the elements in $p$, and such that $a_n$ is equal to the last entry of $p$. i.e.,

$$1^\top a = 1 = 1^\top p \quad \text{and} \quad e_n^\top a = e_n^\top p. \quad (3.7)$$

Then $p$ is in the convex hull of the set of vectors in $\mathbb{R}^n$ obtained by permuting the elements in $a$. In symbols,

$$p \in \text{conv}\{Pa \mid P \in \mathcal{P}^n\}$$

where $\mathcal{P}^n$ is the set of $n \times n$ permutation matrices.
Proof. Let \( \tilde{P} \in \mathcal{P}^{n-1} \) be a permutation with no cycles of order less than \( n - 1 \). For example,
\[
\tilde{P} = \begin{bmatrix}
0 & 0 & \cdots & 0 & 1 \\
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0
\end{bmatrix} \in \mathbb{R}^{(n-1) \times (n-1)}.
\]
Then \( \tilde{P}^{n-1} = I_{n-1} \) and
\[
\tilde{P} + \tilde{P}^2 + \cdots + \tilde{P}^{n-1} = \mathbb{1} \mathbb{1}^\top
\]
is the \((n - 1) \times (n - 1)\) matrix with all ones. Here, \( \mathbb{1} := [1, \ldots, 1]^\top \in \mathbb{R}^{n-1} \).
Define
\[
P := \begin{bmatrix}
\tilde{P} & 0 \\
0 & 1
\end{bmatrix} \in \mathcal{P}^n,
\]
and write \( \tilde{a} := [a_1, \ldots, a_{n-1}]^\top \). Now \( P^k = \begin{bmatrix} \tilde{P}^k & 0 \\
0 & 1
\end{bmatrix} \in \mathcal{P}^n \) so,
\[
\text{conv}\{Pa \mid P \in \mathcal{P}^n\} \ni \sum_{k=1}^{n-1} \frac{1}{n-1} P^k a
\]
By (3.7), this equals \( p \): If \( p = \infty \), then \( \mathbb{1}^\top \tilde{a} = 1 - a_n = 0 \), and when \( p < \infty \),
\[
\frac{\mathbb{1}^\top \tilde{a}}{n-1} = \frac{1 - a_n}{n-1} = \frac{1 - \frac{p-1}{n+p-2}}{n-1} = \frac{1}{n+p-2}.
\]
\( \square \)

Proof of Proposition 3.1. We have
\[
\alpha = \alpha(G) = \min_{A \in K} \frac{\text{tr} A}{\lambda_n(A)} = \frac{\text{tr} A'}{\lambda_n(A')}
\]
for some \( A' \) in the associated convex body \( K \) of \( G \). Set
\[
c := \text{tr} A' > 0
\]
and let \( A_0 := A'/c \). Then \( \text{tr} A_0 = 1 \) and the vector
\[
a := [\lambda_1(A_0), \ldots, \lambda_n(A_0)]^\top
\]
satisfies \( \mathbb{1}^\top a = 1 \). Moreover, \( \lambda_n(A_0) = 1/\alpha \) and when \( p = p(G) \in [2, \infty] \) is the body cone aperture of \( G \), then
\[
e_n^\top a = \frac{1}{\alpha} = \begin{cases}
\frac{p-1}{n+p-2}, & p \in [2, \infty), \\
1, & p = \infty,
\end{cases}
\]
which equals the last element of the vector \( p \in \mathbb{R}^n \) given by (3.6) in Lemma 3.1. Thus,

\[
p \in \text{conv}\{Pa \mid P \in \mathcal{P}^n\}.
\]

(3.8)

By using the standard property

\[
\text{diag}(Pz) = P(\text{diag } z)P^T \quad \forall z \in \mathbb{R}^n,
\]

of permutation matrices \( P \), we want to show that the dominative body \( \mathcal{K}_p \) is a subset of \( \frac{1}{c}\mathcal{K} \). Of course, \( \text{diag}: \mathbb{R}^n \to S^n \) is the linear mapping \( \text{diag } z := \sum_{k=1}^n z_k e_k e_k^T \).

Since \( p = [\lambda_1(E), \ldots, \lambda_n(E)]^T \) for every \( E \in \mathcal{E}_p \) (see formula (3.2)), we can choose \( Q \in \mathcal{O}^n \) such that \( Q^T EQ = \text{diag } p \). By (3.8) we can write \( p \) as a convex combination \( \sum_i \alpha_i P_i a \). Thus,

\[
Q^T EQ = \text{diag} \left( \sum_i \alpha_i P_i a \right)
= \sum_i \alpha_i \text{diag } (P_i a)
= \sum_i \alpha_i P_i (\text{diag } a) P_i^T
= \frac{1}{c} \sum_i \alpha_i P_i U^T A' U P_i^T
\]

for some \( U \in \mathcal{O}^n \) diagonalizing \( A' \in \mathcal{K} \). The orthogonal matrices \( Q_i := QP_i U^T \) then makes

\[
E = \frac{1}{c} \sum_i \alpha_i Q_i A' Q_i^T \in \frac{1}{c} \text{ conv rot}\{A'\} \subseteq \frac{1}{c} \text{ conv rot } \mathcal{K} = \frac{1}{c} \mathcal{K}.
\]

That is, \( \mathcal{E}_p \subseteq \frac{1}{c} \mathcal{K} \) and

\[
\mathcal{K}_p = \text{conv } \mathcal{E}_p \subseteq \frac{1}{c} \mathcal{K}
\]
as well. Thus, for any \( X \in S^n \),

\[
cF_p(X) = c \max_{A \in \mathcal{K}_p} \text{tr}(AX) = \max_{A \in \mathcal{K}_p} \text{tr}(AX) \leq \max_{A \in \mathcal{K}} \text{tr}(AX) = G(X).
\]

□

Proof of Proposition 3.2. For \( p \in [2, \infty] \) one can check that the Hessian matrix \( \mathcal{H}_{w_{n,p}}: \mathbb{R}^n \setminus \{0\} \to S^n \) of the fundamental solution is

\[
\mathcal{H}_{w_{n,p}}(x) = |x|^{-\alpha} \left( (\alpha - 1)\hat{x}\hat{x}^T - (I - \hat{x}\hat{x}^T) \right), \quad \hat{x} := \frac{x}{|x|},
\]

where \( \alpha \in [1, n] \) is related to \( p \) as in Definition 3.1. Setting

\[
\Lambda_\alpha := \text{diag}(\alpha - 1, -1, \ldots, -1) = \alpha e_1 e_1^T - I,
\]
produces a diagonalization \( |x|^{-\alpha} \Lambda_\alpha \) of \( \mathcal{H}_{w_{n,p}}(x) \). Since \( G \) is rotational invariant and positive homogeneous,

\[
G(\mathcal{H}_{w_{n,p}}(x)) = |x|^{-\alpha} G(\Lambda_\alpha)
\]

and will vanish independently of $x \neq 0$ if we can show that $G(\Lambda_\alpha) = 0$. Indeed, when $p$ is the body cone aperture of $G$, Proposition 3.1 and the fact $\text{tr } A/\lambda_n(A) \geq \alpha$ for all $A \in \mathcal{K}$ yields

$$0 = cF_p(\Lambda_\alpha) \leq G(\Lambda_\alpha) = \max_{A \in \mathcal{K}} \text{tr}(AA_\alpha)$$

$$= \max_{A \in \mathcal{K}} \alpha e_1^TAe_1 - \text{tr } A$$

$$\leq \max_{A \in \mathcal{K}} \alpha \lambda_n(A) - \text{tr } A \leq 0,$$

and hence $w_{n,p}$ is a smooth solution of $G(\mathcal{H}w) = 0$ in $\mathbb{R}^n \setminus \{0\}$. There are no test functions touching the fundamental solution from below at $x = 0$, and $w_{n,p}$ is therefore also a viscosity supersolution in $\mathbb{R}^n$.

We conclude this Section with an observation regarding uniformly elliptic operators,

$$\lambda \text{ tr } A \leq F(X + A) - F(X) \leq \Lambda \text{ tr } A, \quad \forall A \geq 0.$$  

Here, $0 < \lambda \leq \Lambda \in \mathbb{R}$ are the ellipticity constants of $F$. The above is equivalent to

$$F(X + Y) - F(X) \leq F_{\lambda,\Lambda}(Y), \quad \forall X, Y \in S^n,$$  

(3.9)

where $F_{\lambda,\Lambda}$ is the maximal Pucci operator. Although uniform ellipticity is in many settings a desirable property of $F$, it has a negative impact on the question raised in this paper. The “more” uniformly elliptic the equation is, the “less” integrable are the gradients of the supersolutions. In fact, since the body cone aperture of $F_{\lambda,\Lambda}$ is $p := \frac{\Lambda}{\lambda} + 1$, Proposition 3.2 and (3.9) implies that $w(x) := w_{n,p}(x) + \frac{1}{2}x^TX_0x$, $X_0 \in \Theta(F)$, is a viscosity supersolution of $F(\mathcal{H}u) = 0$, which is not in $W^{1,q}_{loc}(\mathbb{R}^n)$ for

$$q \geq \frac{n}{n-1}(p-1) = \frac{n}{n-1} \frac{\Lambda}{\lambda}.$$  

4. Canonical operators and the proof of Theorem 2

Let $\Theta$ be a negative elliptic and proper subset of $S^n$, i.e.,

$$\emptyset \neq \Theta \neq S^n \quad \text{and} \quad \Theta + S^n = \Theta.$$

The canonical operator to $\Theta$ is the function $F$ defined on $S^n$ as

$$F(X) := -\sup\{t \in \mathbb{R} \mid X + tI \in \Theta\}. \quad (4.1)$$

In [4] we prove a more general version of the following.

**Proposition 4.1.** The canonical operator $F$ to a negative elliptic and proper subset $\Theta$ is finite and elliptic (in the standard sense (2.6)). It is 1-Lipschitz and has the nondegeneracy

$$F(X + \tau I) - F(X) = \tau$$

and

$$0 = cF_p(\Lambda_\alpha) \leq G(\Lambda_\alpha) = \max_{A \in \mathcal{K}} \text{tr}(AA_\alpha)$$

$$= \max_{A \in \mathcal{K}} \alpha e_1^TAe_1 - \text{tr } A$$

$$\leq \max_{A \in \mathcal{K}} \alpha \lambda_n(A) - \text{tr } A \leq 0,$$
for all $X \in S^n$, $\tau \in \mathbb{R}$. Moreover, if $\Theta$ is a sublevel set of an operator $F$, then every viscosity supersolution of $F(\mathcal{H}u) = 0$ is also a viscosity supersolution of $\overline{F}(\mathcal{H}u) = 0$. The opposite inclusion holds if $\Theta$ is closed.

The canonical operator is in fact the signed distance function

$$
\overline{F}(X) = \begin{cases} 
\text{dist}(X, \partial \Theta), & X \notin \Theta, \\
-\text{dist}(X, \partial \Theta), & X \in \Theta,
\end{cases}
$$

from the boundary of $\Theta$ when the distance $\text{dist}(X, \partial \Theta) := \inf_{W \in \partial \Theta} \|X - W\|_{\infty}$ is measured in the infinity norm $\|X\|_{\infty} := \max\{-\lambda_1(X), \lambda_n(X)\}$. For $X \in S^n$, $\overline{F}(X)$ is the unique number such that

$$
X - \overline{F}(X)I \in \partial \Theta.
$$

In addition to the ellipticity and uniform continuity, the canonical operator (4.1) can have desirable global properties that may not be present in the original operator $F$.

**Proposition 4.2.** Let $\emptyset \neq \Theta \neq S^n$ be an elliptic set. Then the following hold.

(a) If $\Theta$ is convex, then $\overline{F}$ is convex.

(b) If $S^n \setminus \Theta$ is convex, then $\overline{F}$ is concave.

(c) If $\partial \Theta$ is a (hyperplane/subspace) in $S^n$, then $\overline{F}$ is (affine/linear).

(d) If $\Theta$ is a cone, then $\overline{F}$ is positively homogeneous, i.e.,

$$
\overline{F}(cX) = c\overline{F}(X) \quad \forall c > 0, X \in S^n.
$$

(e) If $\Theta$ is a convex cone, then $\overline{F}$ is sublinear.

(f) If $\Theta$ is a rotationally invariant set, then $\overline{F}$ is rotationally invariant.

**Proof.** (a): Since the closure of a convex set is convex, we may assume that $\Theta$ is closed. Let $X, Y \in S^n$ and let $\gamma \in [0, 1]$. Then $Z := \gamma X + (1 - \gamma)Y \in \Theta$ and $X - \overline{F}(X)I \in \partial \Theta, Y - \overline{F}(Y)I \in \partial \Theta$. Thus,

$$
Z - (\gamma \overline{F}(X) + (1 - \gamma)\overline{F}(Y))I = \gamma (X - \overline{F}(X)I) + (1 - \gamma)(Y - \overline{F}(Y)I) \in \Theta,
$$

and

$$
-\overline{F}(Z) = \sup\{t \mid Z + tI \in \Theta\} \geq -(\gamma \overline{F}(X) + (1 - \gamma)\overline{F}(Y)).
$$

(b): One can show that $\overline{\Theta} := -(S^n \setminus \Theta)$ is elliptic. Then since

$$
X \mapsto -\overline{F}(-X) = \sup\{t \mid -X + tI \in \Theta\}
$$

$$
= \inf\{t \mid -X + tI \notin \Theta\}
$$

$$
= -\sup\{t \mid -X - tI \notin \Theta\}
$$

$$
= -\sup\{t \mid X + tI \notin \Theta\}
$$

is convex by (a), it follows that $\overline{F}$ is concave.

(c): By ellipticity, $\partial \Theta$ is necessarily a hyperplane $\partial \Theta = \{X \mid \langle A, X \rangle = \tau\}$ for some nonzero positive semidefinite matrix $A$ and $\tau \in \mathbb{R}$. Thus by (4.2),

$$
\langle A, X - \overline{F}(X)I \rangle = \tau
$$

for all $X$ and

$$
\overline{F}(X) = \frac{1}{\text{tr } A} \text{tr}(AX) - \frac{\tau}{\text{tr } A}.
$$
(d): Since $X - \bar{F}(X)I \in \partial \Theta$, then also $cX - c\bar{F}(X)I \in \partial \Theta$ when $\Theta$ is a cone. That is, 

$$0 = \bar{F}(cX - c\bar{F}(X)I) = \bar{F}(cX) - c\bar{F}(X).$$

(e): This is immediate from (a) and (d).

(f): $\bar{F}$ is rotationally invariant since 

$$\{t \mid Q^\top XQ + tI \in \Theta\} = \{t \mid Q^\top (X + tI)Q \in \Theta\} = \{t \mid X + tI \in Q\partial \Theta Q^\top\} = \{t \mid X + tI \in \Theta\}.$$

□

A final lemma is needed before we can prove the necessity of the condition $\text{ac}(\Theta^B) \subset \Theta_p$.

**Lemma 4.1.** Let $\Theta \subseteq S^n$ be a proper negative elliptic set, and let $B \in \mathbb{R}^{n \times n}$ be invertible. Then $\Theta^B := B^\top \Theta B$ and $\text{ac}(\Theta)$ are again proper negative elliptic sets. If $\Theta$ is rotationally invariant, then so is $\text{ac}(\Theta)$.

**Proof.** Let $X, Y \in S^n$ with $X \leq Y$ and $Y \in \Theta^B$. Then $Y = B^\top \bar{Y}B$ for some $\bar{Y} \in \Theta$. Thus, 

$$\bar{X} := B^{-\top}XB^{-1} \leq B^{-\top}YB^{-1} = \bar{Y}$$

and $\bar{X} \in \Theta$ since $\Theta$ is an elliptic set. It follows that $X = B^\top \bar{X}B \in \Theta^B$. The properness is clear.

Assume now that $X \leq Y \in \text{ac}(\Theta)$. Let $t_k \to \infty$ and $Y_k \in \Theta$ be such that $Y_k/t_k \to Y$. Since $\Theta$ is elliptic and $t_k(X - Y) \leq 0$, we have $Y_k + t_k(X - Y) \in \Theta$ for each $k$. Thus, 

$$X = \lim_{k \to \infty} \frac{Y_k + t_k(X - Y)}{t_k} \in \text{ac}(\Theta).$$

The asymptotic cone is nonempty. In fact, $S^n_- \subseteq \text{ac}(\Theta)$ because if $X \leq 0$ and $Y \in \Theta$, then $Y + kX \in \Theta$ for all $k = 1, 2, \ldots$ and 

$$X = \lim_{k \to \infty} \frac{Y + kX}{k} \in \text{ac}(\Theta).$$

On the other hand, $\text{int} S^n_+ \subseteq (S^n \setminus \text{ac}(\Theta))$. Because if $Z \in \text{ac}(\Theta)$ and say, $Z \geq \epsilon I$, then there are $X_k \in \Theta$ and $t_k \to \infty$ so that $X_k/t_k \geq (\epsilon/2)I$ for all sufficiently large $k$. Thus, eventually 

$$X_k \geq \frac{\epsilon t_k}{2} I \geq Y$$

for any $Y \not\in \Theta$. A contradiction.

Finally, for $Z \in \text{ac}(\Theta)$ and $Q \in O^n$ let $t_k \to \infty$ and $X_k \in \Theta$ be sequences so that $X_k/t_k \to Z$. Since $\Theta$ is rotationally invariant, $Q^\top X_kQ \in \Theta$ for each $k$, and since $Q^\top X_kQ/t_k \to Q^\top ZQ$ it follows that $Q^\top ZQ \in \text{ac}(\Theta)$. Therefore, 

$$\text{rot \ ac}(\Theta) := \{Q^\top ZQ \mid Z \in \text{ac}(\Theta), Q \in O^n\} = \text{ac}(\Theta).$$

□
Proof of Theorem 2. The asymptotic cone $ac(\Theta^B) = ac(B^T \Theta B)$ is easily seen to be a closed cone in $S^n$. As $\Theta$ is assumed to be convex, $\Theta^B$ is convex and so is $ac(\Theta^B)$. See Section 2.1 and 2.2, and in particular, Proposition 2.1.5 in [1]. Moreover, we are there given the equivalent formulations
\[\begin{align*}
ac(\Theta^B) &= \{ Z \in S^n \mid X + tZ \in \text{cl} \Theta^B \text{ for all } X \in \Theta^B \text{ and all } t \geq 0 \} \\
&= \{ Z \in S^n \mid X + tZ \in \text{cl} \Theta^B \text{ for some } X \in \Theta^B \text{ and all } t \geq 0 \}.
\end{align*}\]
Here, $\text{cl} \Theta^B$ denotes the closure of $\Theta^B$. For convex sets the above is also called the recession cone.

The available alternative formula for $ac(\Theta^B)$ implies $ac(\Theta^B) + \{ X \} \subseteq \text{cl} \Theta^B$ for all $X \in \Theta^B$. In particular, since $\Theta^B$ is elliptic by Lemma 4.1 above and since $Y \in \text{cl} \Theta^B \Rightarrow Y - \epsilon I \in \text{int} \Theta^B$ for all $\epsilon > 0$ by Lemma 3.1 (2) in [4], we can pick a large enough $\tau \in \mathbb{R}$ so that
\[ac(\Theta^B) - \tau \{ I \} \subseteq \Theta^B.\] (4.3)

We now choose $B$ so that $\Theta^B$ is rotationally invariant and let $G$ denote the canonical operator to $ac(\Theta^B)$. That is,
\[G(X) := -\sup \{ t \mid X + tI \in ac(\Theta^B) \} .\]
By Lemma 4.1 and the discussion above, $ac(\Theta^B)$ is a proper and rotationally invariant elliptic convex cone. Therefore, by Proposition 4.2 (e) and (f), $G : S^n \to \mathbb{R}$ is a rotationally invariant sublinear elliptic operator. By Proposition 3.1, and as $ac(\Theta^B)$ is closed, we get
\[ac(\Theta^B) = \Theta(G) \subseteq \Theta_g\] (4.4)
where we have set $g := p(G) \in [2, \infty]$ to be the body cone aperture of $G$. Recall from Proposition 3.2 that $G(\mathcal{H}w_{n,g}(x)) = 0$ and thus $\mathcal{H}w_{n,g}(x) \in ac(\Theta^B)$ for $x \neq 0$.

Let $p \in [2, \infty]$ and suppose every supersolution of $F(\mathcal{H}w) = 0$ is $W^{1,q}_{\text{loc}}(\Omega)$ for $q < \frac{n}{n-1}(p-1)$ (or $q < \infty$ in the case $p = \infty$). Assume for the sake of contradiction that $ac(\Theta^B)$ is not a subset of $\Theta_p$. Then we must have $2 \leq q < p$ by (4.4) and (2.1). Let $\tau \in \mathbb{R}$ be the constant from (4.3) and define the lower semicontinuous function $v : \Omega^B \to (-\infty, \infty]$ as
\[v(x) := w_{n,g}(x - x_0) - \frac{\tau}{2} |x|^2,\]
for some $x_0 \in \Omega^B$. This function is not $W^{1,q}_{\text{loc}}(\Omega^B)$ for
\[q := \frac{n}{n-1}(g-1) < \frac{n}{n-1}(p-1).\]
In $\Omega^B \setminus \{x_0\}$, $v$ is smooth with Hessian matrix
\[\mathcal{H}v(x) = \mathcal{H}w_{n,g}(x - x_0) - \tau I \subseteq ac(\Theta^B) - \tau \{ I \} \subseteq \Theta^B\]
by (4.3). The change of variables $u(x) := v(B^{-1}x)$ produces a contradiction since the are no test functions from below at $Bx_0$ and
\[\mathcal{H}u(x) = B^{-T} \mathcal{H}v(B^{-1}x)B^{-1} \subseteq \Theta\]
for $x \neq Bx_0$. That is, $u \notin W^{1,q}_{\text{loc}}(\Omega)$ is a supersolution to $F(\mathcal{H}w) = 0$. \qed
There is probably some room for improvement in Theorem 2. In particular, it should be possible to relax the convexity assumption on $\Theta$.

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