Partial Shattering for Permutations

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Abstract

Let $S_n$ be the set of all permutations of $[n] = \{1, \ldots, n\}$ which we think of as fixed orderings of the elements $1, \ldots, n$. We say $S \subseteq S_n$ shatters a $k$-set $X \subseteq [n]$ if all of the $k!$ linear orderings of $X$ appear in permutations from $S$. That is, there are $k!$ permutations in $S$ such that, when we restrict them to the elements of $X$, we get all possible linear orders of $X$. It is known, via a probabilistic argument, that the smallest $S$ which shatters every $k$-set has size $\Theta(\log n)$.

We introduce a relaxation to determine the size of a family that covers at least $t$ orders of every $k$-tuple for different values of $t \leq k!$. We show that when $k = 3$ there are $3$ possible size classes depending on $t$: constant, $\Theta(\log \log n)$, and $\Theta(\log n)$. We give a partial result on the analogue of this for $k > 3$, showing similar behaviour when $t \notin [k + 1, 2(k - 1)!]$.

We introduce a further variant to determine the largest number of $k$-sets that can be totally shattered by a family with given size. For instance we show that with $6$ permutations from $S_n$, the proportion of triples shattered lies between $\frac{2}{5}$ and $\frac{4}{5}$ for $n \geq 5$.

Finally we give an explicit construction for a family which shatters every $k$-tuple and has size $(\log n)^{\frac{1}{k}}$.

1 Introduction

1.1 Background

Let $S_n$ be the set of all permutations of $\{1, \ldots, n\}$ thought of as ordered $n$-tuples. Our aim is to study properties of families of permutations from $S_n$ inspired by concepts of shattering from extremal set theory. We begin with the notion of shattering for sets. Let $\mathcal{F}$ be a family of subsets of $[n] = \{1, 2, \ldots, n\}$ and let $A \subseteq [n]$, we say that $A$ is shatted by $\mathcal{F}$ if for each $B \subseteq A$ there exists a set $S \in \mathcal{F}$ such that $A \cap S = B$. The notion of shattered sets has uses throughout combinatorics and computer science, with the focus being on the size of families that shatter certain sets. For examples of work in this area see [12], [3], [9], [5], and [1].

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A family $\mathcal{R}$ of subsets of $[n]$ is $k$-independent if any $R_1, R_2, \ldots, R_k \in \mathcal{R}$ have the property that all $2^k$ intersections $\cap_{i=1}^k J_i$ are non-empty, where $J_i$ takes on either $R_i$ or its compliment $R_i^c$. A large $k$-independent family $\mathcal{R}$ gives rise to a small family $\mathcal{F}$, where $\mathcal{F} \subseteq 2^{[r]}$ is a family that shatters all the $k$-subsets of $[r]$ and $|\mathcal{R}| = r$. For each $x \in [n]$ we can define a set $F(x) \subseteq [r]$ by setting $i \in F(x)$ if and only if $x \in R_i$. Since $\mathcal{R}$ is $k$-independent, a family consisting of all such $F(x)$ sets must shatter all $k$-subsets of $[r]$. Set $\mathcal{F} = \{F(x) : x \in [n]\}$ then clearly $|\mathcal{F}| = n$, so the bigger $r$ is, the greater the number of $k$-subsets shattered by $n$ sets. Kleitman and Spencer [5] posed the question ‘how large can a family of $k$-independent sets be?’ which is equivalent to the question ‘how small can a family that shatters every $k$-subset of $[n]$ be?’

**Theorem 1.1.** (Kleitman and Spencer [5]) For fixed $k$ and $n$ sufficiently large there are absolute constants $d_1$ and $d_2$ such that

$$\frac{2^k}{d_2} \log n \leq g_k(n) \leq \frac{k2^k}{d_1} \log n$$

where $g_k(n)$ is the size of the smallest family from $2^{[n]}$ that shatters every $k$-subset of $[n]$.

Our starting point is an analogue of this family size problem using permutations in place of sets. This has been studied under a variety of names ([3], [7], [11], [10]): completely mixing permutation families, mixing permutations, and sequence covering arrays. We first formalise the problem, establish notation, and summarise the previous work.

Consider the set $S_n$ of all permutations of $[n]$, any permutation $P \in S_n$ corresponds to a particular linear order of the elements of $[n]$, so $P$ can be written as $P = (p_1, p_2, \ldots, p_n)$ where $\{p_1, p_2, \ldots, p_n\} = [n]$. Note that this is not cycle notation, rather it can be thought of as the second line of two-row notation. Then we can define a function that gives the position of each element for each $P$. Let $pos(P, a) : S_n \times [n] \rightarrow [n]$ be such that $pos(P, a) = i$ when $a = p_i$, so $a$ has exactly $i - 1$ elements to its left in $P$.

Suppose $R$ is a permutation of $[k]$, we say that a $k$-tuple $\{a_1, \ldots, a_k\} \subseteq [n]$ with $a_1 < \cdots < a_k$ follows the pattern $R \in S_k$ in some $P \in S_n$ if both $pos(P, a_i) < pos(P, a_j)$ and $pos(R, i) < pos(R, j)$ for all $i, j \in [k]$. We can express our shattering condition through permutation patterns.

Let $X$ be a $k$-tuple from $[n]$ and let $P \in S_n$. Let $P(X)$ be the restriction of $P$ to the elements of $X$, then $P(X)$ is simply a permutation of $X$ where the elements appear in the same order that they appear in $P$. Then consider the permutation $P_X \in S_k$ which is the permutation pattern followed by $X$ in $P$, the two permutations $P_X$ and $P(X)$ are order-isomorphic but do not permute the same elements.

**Definition 1.2.** We say that a family $\mathcal{S} \subseteq S_n$ shatters $X$ if $\{P_X : P \in \mathcal{S}\} = S_k$.

In other words, $\mathcal{S}$ shatters $X$ if every possible ordering of the elements of $X$ appears in the permutations of $\mathcal{S}$.

We are interested in families that shatter many different sets at the same time, in particular when all sets of a particular size are shattered by one family. Let $f_k(n)$ be the smallest integer such that there exists a family $\mathcal{S}$ of permutations from $S_n$ that shatters
every $k$-tuple from $[n]$ and has $|S| = f_k(n)$. Throughout, if a family is said to shatter
every $k$-tuple in $[n]$, then it is implied that said family is a subset of $S_n$.

Clearly we have that $f_k(n) \geq |S_k|$, otherwise we certainly cannot shatter any $k$-tuple. It
is also plain that $f_k(k) = k!$ since the family $S_k$ is suitable. Interestingly when $k = 3$
we have that $f_3(4) = 6$ as can be seen in the following, which we call a perfect family
$Q_3(4)$,

$$
Q_1 = (1, 2, 3, 4) \quad Q_2 = (2, 4, 1, 3) \quad Q_3 = (3, 4, 1, 2) \\
Q_4 = (1, 4, 3, 2) \quad Q_5 = (4, 2, 3, 1) \quad Q_6 = (3, 2, 1, 4).
$$

The order of magnitude of the value $f_k(n)$ is known asymptotically. The upper bound
given by Spencer [10] can be seen with a simple probabilistic argument, the lower bound
is more involved and was shown by Radhakrishnan [7]

$$
1 + \frac{2n(k - 1)!}{(2n - k + 1)\log(e)} \log(n - k + 2) \leq f_k(n) \leq \frac{k}{\log(k!) - \log(k! - 1)} \log(n).
$$

In the case where $k = 3$ the best upper bound is actually given by a construction rather
than using the probabilistic method. The construction is given by Tarui in [11] and the
lower bound from Füredi [4]

$$
\frac{2}{\log(e)} \log(n) \leq f_3(n) \leq 2 \log(n) + (1 + o(1)) \log \log(n).
$$

For $k > 3$ no explicit construction that asymptotically matches the probabilistic upper
bound is known.

### 1.2 Definitions and alternate shattering problems

Our main aim is to introduce and investigate variations of the original family size problem.
In particular we ask about the size of families satisfying relaxed versions of the shattering
property.

There are a few reasonable definitions for this relaxation. We can either weaken the
definition of shattering a $k$-tuple to some notion of partial shattering or we can ask about
totally shattering some partial collection of $k$-tuples. There are two different problems we
will consider.

- Find the smallest family of permutations from $S_n$ which ensures each $k$-tuple appears
  in at least $t \leq k!$ orders.

- Find the smallest family that shatters at least $\alpha \binom{n}{k}$ of all $k$-tuples for $\alpha \in [0, 1]$.

Note that there is another possible definition of partial shattering. The problem is to
find the smallest family of permutations from $S_n$ such that every $k$-tuple follows all of the
patterns in $T$, where $T \subseteq S_k$ is some fixed set of patterns. It turns out that for any $S \subseteq S_n$
in which all $k$-tuples follow a specific non-monotone pattern, we have $|S| = O(\log(n))$
(see Lemma 2.3). Since this matches the lower bound for total shattering this problem is
uninteresting.

For our first problem we formally define partial shattering as follows, using the same
notation used in Definition 1.2
Definition 1.3. We say that a family $S \subseteq S_n$ partially shatters $X$ with $t$ orders if $|\{P_X : P \in S\}| \geq t$. We define $f_k(n, t)$ to be the smallest integer such that there exists a family $S$ that partially shatters every $k$-tuple in $[n]$ with $t$ orders, and $|S| = f_k(n, t)$.

Clearly $f_k(n, k!)$ is the size of the smallest shattering family on $k$-tuples, $f_k(n, k!) = f_k(n)$. We also always have the trivial cases $f_k(n, 1) = 1$ and $f_k(n, 2) = 2$ which can be seen by taking only monotone permutations.

For our second problem it is more natural to ask the question in reverse than it is to fix the fraction. We ask for the maximum number of shattered $k$-tuples from a family of fixed size.

Definition 1.4. Let $F_k(n, m)$ be the largest $\alpha \in [0, 1]$ such that there exists a collection $R$ of exactly $m$ permutations from $S_n$ with the property that $\alpha \binom{n}{k}$ $k$-tuples are totally shattered by $R$. We call this fractional shattering of $n$ with $m$ permutations.

It is plain that $F_k(n, m) = 0$ whenever $m < k!$ and that $F_k(k, k!) = 1$. In fact, by a result of Levenshtein [6] we have that $F_k(k + 1, k!) = 1$.

The remainder of this paper will be structured as follows. Starting with partial shattering we begin by showing that fixing a non-monotone pattern which all $k$-tuples follow, results in a large family of permutations. We then consider the case where $k = 3$ and classify the size of $f_3(n, t)$ asymptotically for all values of $t$.

Theorem 1.5. For large $n$ we have the following bounds on $f_3(n, t)$

$$f_3(n, t) = \begin{cases} t & \text{for } t = 1, 2 \\ \Theta(\log \log n) & \text{for } t = 3, 4 \\ \Theta(\log n) & \text{for } t = 5, 6. \end{cases}$$

We follow up with the extension to larger values of $k$. We see that the same structure appears for small $t$ and large $t$ but there is the possibility that another size class exists for $t \in [k + 1, 2(k - 1)!]$.

Theorem 1.6. For large $n$ we have the following bounds on $f_k(n, t)$

$$f_k(n, t) = \begin{cases} t & \text{when } t = 1, 2 \\ \Theta(\log \log n) & \text{when } t \in [3, k] \\ \Theta(\log n) & \text{when } t \in [2(k - 1)! + 1, k]. \end{cases}$$

We then move on to the fractional version of the problem. Again we focus on the case where $k = 3$, the first interesting case is $F_3(n, 6)$ where we get the following bounds.

Theorem 1.7. For any $n \geq 5$ we have

$$\frac{2}{5} \leq F_3(n, 6) \leq \frac{4}{5}.$$

We also show that in general the value of $F_k(n, m)$ is decreasing as $n$ increases, meaning that the limit for $F_k(n, m)$ exists for fixed $k$, $m$ and as $n$ tends to infinity.
Finally we have a construction for the original total shattering problem, that is an upper bound for $f_k(n)$. The best bound, of order $\log n$, is given by a probabilistic argument and no explicit construction of this size is known. Our construction gives a family with a power of $\log n$ permutations, this is above the known upper bound for $f_k(n)$ but is constructive.

We finish with some open problems about all the topics covered. To simplify the notation throughout, $\log_2(n) = \log n$ unless otherwise stated.

## 2 Partially shattering every $k$-tuple

The following are two well known but useful results that will be used throughout.

**Lemma 2.1.** Let $(A, B)$ be a partition of $[n]$, so $A \cup B = [n]$ and $A \cap B = \emptyset$. Any family $U$ of such partitions with the property that, for every $x, y \in [n]$ there exists $(A, B) \in U$ where exactly one of $x$ and $y$ is in $A$ and the other is in $B$, also satisfies $|U| \geq \lceil \log n \rceil$. Furthermore there exists such a family $U$ where the bound holds with equality.

Throughout $C := \frac{1}{2} + o(1)$.

**Lemma 2.2.** (Chung, Graham, and Winkler [2]) Let $(A, B)$ be a partition of $[n]$, so $A \cup B = [n]$ and $A \cap B = \emptyset$. Any family $U$ of these partitions with the property that, for every $x, y \in [n]$ there exists $(A, B) \in U$ where $x \in A$ and $y \in B$, must satisfy $|U| \geq \lfloor \log n + C \log \log n \rfloor$. Moreover, there exists such a family with $|U| = \lfloor \log n + C \log \log n \rfloor$.

Below is a useful lemma showing that whenever we require a family of permutations to follow a fixed non-monotone pattern, that family has the same order of magnitude as a totally shattering family. This is the reason we have chosen to define partially shattering as in Definition 1.3.

**Lemma 2.3.** Let $n \geq 3$ and $R \in S_k$ be any non-monotone permutation pattern. If $S \subseteq S_n$ is a family of permutations for which every $k$-tuple follows $R$ in at least one $P \in S$, then we must have $|S| \geq \log(n - k + 2)$.

**Proof.** Let $S$ and $R$ be as described in the statement of the lemma. Note that whenever $R$ is non-monotone there is some element $x \in [k]$ such that when $R$ is restricted to $\{x, x+1, x+2\}$ the triple is non-monotone.

If $R$ induces $(x+1, x, x+2)$ or $(x+2, x, x+1)$ then set $y = x$. If $R$ induces $(x, x+2, x+1)$ or $(x+1, x+2, x)$ then set $y = n - k + x + 2$.

For each $P \in S$ we generate a partition of the set $W := [n]\setminus([x-1]\cup[n-k+x+3, n]\cup y)$ into two parts $A_P$ and $B_P$, where $A_P$ contains every $w \in W$ such that $pos(P, w) > pos(P, y)$ and $B_P = W \setminus A_P$.

Note that when $y = x$ the set $W$ contains only elements larger than $y$, and when $y = n - k + x + 2$ the set $W$ only contains smaller elements. Hence for any pair $a, b \in W$ we must have one of the orders $(a, y, b)$ or $(b, y, a)$ appearing in $S$ since exactly one of them follows $R$ as part of the $k$-tuple $\{1, 2, \ldots, x-1, y, x, a, b, x+2-k+x+3, \ldots, n\}$. To see this note that the elements that correspond to $\{x, x+1, x+2\}$ are exactly $\{y, a, b\}$.
Then we have satisfied the conditions for Lemma 2.2 and we must have that $|S| \geq \log |W|$, which gives the result.

Our aim will be to prove the following classification of $f_3(n,t)$.

**Theorem 1.5.** For large $n$ we have the following bounds on $f_3(n,t)$

$$f_3(n,t) = \begin{cases} 
    t & \text{for } t = 1, 2 \\
    \Theta(\log \log n) & \text{for } t = 3, 4 \\
    \Theta(\log n) & \text{for } t = 5, 6.
\end{cases}$$

The upper bound when $t = 6$ comes from the total shattering bounds for $f_k(n)$. The value of $f_k(t)$ whenever $t = 1, 2$ is trivial. Indeed note that any one single permutation requires each $k$-tuple to follow some pattern (not necessarily the same pattern). So we get that $f_k(n,1) = 1$ simply by choosing any $P \in S_n$. We call $\overline{P}$ the reverse permutation of $P$ if $\text{pos}(P,a) = n + 1 - \text{pos}(\overline{P},a)$, then we must have that all $k$-tuples follow a different pattern in $\overline{P}$ as they did in $P$. Therefore we must have that $f_k(n,2) = 2$.

To get a lower bound we need to work a little harder to get anything for $t \geq 3$ however, observe that taking $P$ to be the increasing permutation $(1,2,\ldots,n-1,n)$ we get that $\overline{P}$ must be the decreasing permutation. Now consider a third permutation $P'$, any $k$-tuple contained in a monotone subpermutation of $P'$ will not follow a new pattern. We know by the Erdős-Szekeres Theorem that when $n$ is large we must have some reasonably large monotone subpermutation. We use this idea to get the lower bound in this case.

**Theorem 2.4.** (Erdős-Szekeres Theorem) Let $r,s \in \mathbb{N}$, then any sequence of real numbers with length at least $n = rs + 1$ contains an increasing subsequence of length at least $r + 1$ or a decreasing subsequence with length at least $s + 1$.

**Theorem 2.5.** For any $n \geq 3$ and every $t \geq 3$, we have

$$f_k(n,t) \geq \log \log (n-1) - \log \log (k-1) + t - 3.$$  

**Proof.** Let $S$ be a family of permutations of $[n]$ such that every $k$-tuple appears in at least $t$ orders across $S$. Suppose for a contradiction that $|S| \leq \log \log (n-1) - \log \log (k-1) + t - 4$.

Choose any $t-3$ permutations $R_1,\ldots,R_{t-3} \in S$, and set $S' := S \setminus \{R_1,\ldots,R_{t-3}\}$. This is a family of permutations from $S_n$ such that every $k$-tuple appears in at least 3 orders. Note that $|S'| \leq \log \log (n-1) - \log \log (k-1) - 1 = m$.

Take any $P_1 \in S'$, then by the Erdős-Szekeres Theorem $P_1$ must contain an increasing subsequence of length $r = \lfloor (n-1)^{\frac{t}{r}} \rfloor + 1$ (or a decreasing subsequence of length $r$). Let the elements in this increasing (or decreasing) subsequence be written as $X_1 = \{x_1,\ldots,x_r\}$, then $P_1(X_1)$ is the restriction of $P_1$ to the elements of $X_1$.

Look at another permutation $P_2 \in S'$ restricted to the elements of $X_1$, $P_2(X_1)$. Applying Erdős-Szekeres again, this time to $P_2(X_1)$, we see that there must be an monotonic subsequence of length $\lfloor (r-1)^{\frac{t}{r}} \rfloor + 1$. Let $X_2 \subseteq X_1$ be the set of elements in this subsequence. Then consider $P_3(X_2)$ and generate $X_3 \subseteq X_2$ in an analogous manner.
Take each permutation from $\mathcal{S}'$ into consideration one by one, at step $i$ generate a set of ‘bad’ elements $X_i \subseteq X_{i-1}$ by applying Erdős-Szekeres to $P_i(X_{i-1})$ and finding a monotone subsequence of length at least $\left\lceil (|X_{i-1}| - 1) \frac{3}{2} \right\rceil + 1$.

Consider the set $X_m$, it must contain elements that appear in a monotone subsequence of every permutation in $\mathcal{S}'$. In other words, the elements of $X_m$ appear in a maximum of 2 possible orders. Therefore if $|X_m| \geq k$ then $X_m$ contains a $k$-tuple that does not appear in 3 orders across $\mathcal{S}'$. Hence, from our initial conditions we must have $|X_m| < k$.

Note that the number of elements we are restricting to in the final stage is at most

$$(n - 1)^{\frac{1}{2^m}} + 1.$$ 

Then observe

$$(n - 1)^{\frac{1}{2^m}} + 1 < k$$

$$\log \log(n - 1) - \log \log(k - 1) < m.$$ 

On the other hand, based on the assumed size of $\mathcal{S}$ we have that $m = \log \log(n - 1) - \log \log(k - 1) - 1$, a contradiction. Therefore we must have

$$|\mathcal{S}| \geq \log \log(n - 1) - \log \log(k - 1) + t - 3.$$

From this we can see that if $t \geq 3$ then $f_k(n, t)$ is always between $\Omega(\log \log n)$ and $O(\log n)$. In the case where $k = 3$, Theorem 1.5 shows that the size of $f_3(n, t)$ always falls into one of these size categories.

The next result gives an upper bound on $f_3(n, 4)$, first we show the bound using a recursion (2.7), then we given the construction that provides the recursion (2.6).

**Lemma 2.6.** For $n \geq 3$ we have that $f_3(n^n, 4) \leq f_3(n, 4) + \lceil \log n + C \log \log n \rceil + 1$.

Therefore we get the following bound.

**Theorem 2.7.** For large $n$ we have $\log \log(n - 1) + 1 \leq f_3(n, 4) \leq 2 \log \log(n)$

**Proof.** The lower bound is directly from Lemma 2.5 with $k = 3$ and $t = 4$.

For the upper bound, write $n = m^m$ for some real number $m$, and similarly $m = p^p$. Then from Lemma 2.6 we have

$$f_3(n, 4) \leq f_3(\lceil m \rceil, 4) + \lceil \log m \rceil + \log \log \lceil m \rceil + 1$$

$$\leq f_3(m + 1, 4) + \log m + \log \log m + 4$$

$$\leq f_3(p + 1, 4) + \log p + \log \log p + 4 + \log \log n + 4$$

$$\leq \log \log n + 8 + \log \log m + 1 + f_3(p + 1)$$

$$\leq \log \log n + 4 \log \log m + 15 \leq 2 \log \log n.$$ 

□
Now we see the construction that gives the recursion.

Proof of Lemma 2.6. Let \( S \) be a partial-shattering family for \([n]\) where every triple has at least 4 orders covered and \(|S| = f_3(n, 4)\). We will use this family to construct a new family from \( S_n \) that partially shatters every triple with 4 orders. In this proof we will say ‘partially shatters’ or ‘partial shattering’ to mean partially shatters with 4 orders.

Let \( x \in [n^n] \) and note that we may write \( x \) uniquely in the form
\[
x = (x_1 - 1)n^{n-1} + (x_2 - 1)n^{n-2} + (x_3 - 1)n^{n-3} + \cdots + (x_{n-1} - 1)n + (x_n - 1) + 1
\]
where \( x_i \in [n] \) for all \( i \in [n] \). Then we are able to identify each element \( x \in [n^n] \) with a string from \([n]^n\) by assigning \( x \) to the unique string \((x_1, \ldots, x_n)\). We call \((x_1, \ldots, x_n)\) the code (or unique code) for \( x \).

First we section the elements of \([n^n]\) into blocks as follows. Partition \([n^n]\) into \( n \) blocks \( B_1(1), B_1(2), \ldots, B_1(n) \) each of size \( n^{n-1} \), so that every element \( x \in [n^n] \) is put into block \( B_1(x_1) \).

We then partition each of these blocks according to the second coordinate of the codes. We have \( n \) blocks
\[
B_2(i, 1), B_2(i, 2), \ldots, B_2(i, n)
\]
that partition each \( B_1(i) \) and have size \( n^{n-2} \), where
\[
B_2(i, j) = \{ x \in [n^n] : x_1 = i \text{ and } x_2 = j \}.
\]

Continue in this way, defining blocks \( B_k(i_1, i_2, \ldots, i_k) \) containing all elements that have appropriate coordinates in their unique code, until we have defined blocks \( B_n(i_1, i_2, \ldots, i_n) \) that each contain only one element. All blocks of the form
\[
B_k(i_1, i_2, \ldots, i_k) = \{ x \in [n^n] : x_1 = i_1, x_2 = i_2, \ldots, x_k = i_k \}
\]
we call level \( k \) blocks, notice that each level \( k \) block contains exactly \( n \) level \( k-1 \) blocks.

Now we generate two types of permutations on \([n^n]\).

**Type 1.** Here we apply permutations from \( S \). We will generate one permutation \( P' \) of \([n^n]\) from each permutation \( P \in S \).

Consider any \( P \in S \), we can apply \( P \) to any set of \( n \) objects. In particular we can apply \( P \) to the level 1 blocks \( B_1(1), B_1(2), \ldots, B_1(n) \), where \( B_1(i) \) takes the place of \( i \in [n] \) in \( P \). That is, in \( P' \) an element \( x \) will appear before \( y \) whenever \( x_1 \) appears before \( y_1 \) in \( P \).

We keep applying \( P \) in this way to all the blocks, level by level. So in the next stage apply \( P \) to each set
\[
\{ B_2(1, 1), B_2(1, 2), \ldots, B_2(1, n) \}, \ldots, \{ B_2(n, 1), B_2(n, 2), \ldots, B_2(n, n) \}.
\]
The result is that for elements \( x, y \in [n^n] \), with \( i \in [n] \) the smallest integer such that \( x_i \neq y_i \), we have that \( x \) precedes \( y \) in \( P' \) if and only if \( x_i \) precedes \( y_i \) in \( P \).
We do this for all \( P \in S \) which gives us \(|S| = f_3(n, 4)\) permutations of \([n^n]\), call this collection of permutations \( S' \).

To see which triples are now partially shattered, consider the triple \( \{x, y, z\} \) with codes \((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)\) and \((z_1, z_2, \ldots, z_n)\) respectively. Suppose all three elements agree in the first \( k \) coordinates, so \( x_i = y_i = z_i \) for \( i \leq k \), and further suppose that none agree in coordinate \( k + 1 \), that is \( d(x, y) = d(x, z) = d(y, z) = k + 1 \). Then note that \( B_k(x_1, x_2, \ldots, x_k) \) contains \( x, y \) and \( z \), but each element is contained in a different level \( k + 1 \) block \( x \in B_x := B_{k+1}(x_1, x_2, \ldots, x_k, x_{k+1}), y \in B_y := B_{k+1}(x_1, x_2, \ldots, x_k, y_{k+1}) \) and \( z \in B_z := B_{k+1}(x_1, x_2, \ldots, x_k, z_{k+1}) \). Since \( S \) partially shatters triples in \([n^n]\) there must be permutations \( P_1, P_2, P_3, P_4, P_5, P_6 \in S \) that partially shatter \( \{x_{k+1}, y_{k+1}, z_{k+1}\} \), then \( P'_1, P'_2, P'_3, P'_4, P'_5, P'_6 \) must partially shatter \( \{x, y, z\} \) as the blocks \( B_x, B_y, B_z \) appear in \( 4 \) orders since they are all within the same level \( k \) block.

The only triples that do not have \( 4 \) orders covered by permutations in \( S' \) are those that have two elements that agree in the first \( k \) coordinates of their unique code and the final element only agrees in the first \( r \) coordinates where \( r < k \). This is equivalent to triples \( \{x, y, z\} \) with \( d(x, y) = k \) and \( d(x, z) = d(y, z) = r \), we will call such triples ‘bad’.

Note that for a ‘bad’ triple \( \{x, y, z\} \) where \( x < y < z \) we must have either \( d(x, y) = k \) and \( d(x, z) = d(y, z) < k \) or \( d(y, z) = k \) and \( d(x, y) = d(x, z) < k \). It cannot be that \( d(x, z) = k \) and \( d(x, y) = d(y, z) = r < k \) because \( x < y < z \) means that \( x_r < y_r < z_r \) but \( d(x, z) = k \) implies \( x_r = z_r \).

We can assume without loss of generality that \( S \) contains the monotone increasing order, therefore we can assume \( S' \) contains it. We now construct the other collection of permutations to cover orders on these ‘bad’ triples.

**Type 2.** By Lemma 2.2 we are able to find \([\log n + C \log \log n]\) partitions of \([n]\) into two sets \( I \) and \( D \) that satisfy the conditions in 2.2. In this section we will view \([n]\) as the set of all block levels, and therefore \( I \) and \( D \) as partitions of block levels. We now construct one permutation for each of these partitions in the following way.

Let \((I, D)\) be any of our partitions and start at level \( 1 \). If \( 1 \in I \) then we order the level \( 1 \) blocks in increasing order, namely all elements in \( B_1(i) \) appear before elements in \( B_1(j) \) whenever \( i < j \), otherwise order in decreasing order. The order of elements within \( B_1(i) \) will be determined in the later stages. Next look at level \( 2 \), if \( 2 \in I \) order the level \( 2 \) blocks increasing, otherwise order them decreasing. Continue in this way until all levels have been ordered. Formally, consider \( x, y \in [n^n] \) and let \( d(x, y) = i \), we have that \( x \) precedes \( y \) if

\[
\begin{align*}
\{ xi < yi & \quad \text{ and } i \in I \\
xi > yi & \quad \text{ and } i \in D.
\end{align*}
\]

We use the above process to construct \([\log n + C \log \log n]\) permutations with the property that for any two block levels \( i \) and \( j \) we can always find two permutations such that one has \( i \) increasing and \( j \) decreasing and the other has \( i \) decreasing and \( j \) increasing.

Let \( T \) be the set of these **Type 2** permutations along with the permutation of \([n^n]\) that is
totally decreasing (if this is not already included by the two types), then $|T| \leq \lceil \log n + C \log \log n \rceil + 1$.

To identify the triples that are partially shattered by $T$, consider a triple $\{x, y, z\}$ where $x < y < z$ that was not partially shattered by $S'$. As discussed after Type 1 there are two cases, either $d(x, y) = i$ and $d(x, z) = d(y, z) = j$ where $j < i$, or $d(y, z) = i$ and $d(x, y) = d(x, z) = j$ for $j < i$.

Suppose $d(x, y) = i$ and $d(x, z) = d(y, z) = j$ where $j < i$. We know from our assumption that the order $(x, y, z)$ appears in $S'$, we also know that $(z, y, x)$ appears in $T$ since we included the decreasing permutation here. Furthermore we know that there is some permutation in $T$ where $i$ is increasing and $j$ is decreasing. Since $x < z$ and $d(x, z) = j$ we must have $x_j < z_j$ from the construction of the unique codes, similarly we have $x_i < y_i$. Then for any permutation with $i$ increasing, $x$ must appear before $y$, and $j$ decreasing means $x$ (and $y$ since $x_j = y_j$) comes after $z$. This means that the order $(z, x, y)$ is covered. Similarly there is a permutation where $i$ is decreasing and $j$ is increasing, giving the order $(y, x, z)$. By the same reasoning, if we are in the second case where $d(y, z) = i$ and $d(x, y) = d(x, z) = j$, then we find the orders $(x, y, z)$, $(z, y, x)$, $(y, z, x)$ and $(x, z, y)$.

We now have our desired partial shattering condition, the family $S' \cup T$ covers 4 orders for each triple in $[n^n]$.

Thus

$$f_3(n^n, 4) \leq |S'| + |T| = f_3(n, 4) + \lceil \log n + C \log \log n \rceil + 1.$$  

We now have all the ingredients needed to prove Theorem 1.5.

**Proof of Theorem 1.5.** Clearly $f_3(n, 1) = 1$ as any $P \in S_n$ forces all triples from $[n]$ to appear in one order. Recall $P$ is the reverse permutation of $P$, then we must have that all triples follow a different pattern in $P$ as they did in $P$, hence $f_3(n, 2) = 2$.

The lower bound $f_3(n, 3) \geq \log \log (n - 1)$ comes directly from Theorem 2.5. The upper bound $f_3(n, 4) \leq 2 \log \log (n)$ comes directly from Theorem 2.7. This gives us $f_3(n, t) = \Theta(\log \log n)$ when $t = 3, 4$.

To see $f_3(n, 5) \geq \log (n - 1)$, let $S$ be a family of permutations from $S_n$ that partially shatter every triple with 5 orders. Consider any triples of the form $\{n, x, y\}$, we must have at least one of the orders $(x, n, y)$ and $(y, n, x)$ appearing in some permutation from $S$ otherwise we have at most 4 orders for $\{n, x, y\}$. For each $P \in S$ generate a partition of $[n - 1]$ by having

$$A_P := \{x \in [n - 1] : x \text{ appears after } n \text{ in } P\}$$
$$B_P := \{x \in [n - 1] : x \text{ appears before } n \text{ in } P\}.$$

Then using Lemma 2.4 in order to ensure at least one of the orders $(x, n, y)$ and $(y, n, x)$ is seen we must have at least $(n - 1)$ permutations in $S$.

Finally we have that $f_3(n, 6) \leq 2 \log n + o(1) \log n$ from [11] since $f_3(n, 6) = f_3(n)$. 

\[ \square \]
For triples, the different values of $t$ feed equally into the three size classifications. In the general $k$-tuple case, we actually have that for most values of $t$ we require $O(\log n)$ permutations.

**Theorem 1.6.** For large $n$ we have the following bounds on $f_k(n, t)$

$$f_k(n, t) = \begin{cases} 
  t & \text{when } t = 1, 2 \\
  \Theta(\log \log n) & \text{when } t \in [3, k] \\
  \Theta(\log n) & \text{when } t \in [2(k - 1)! + 1, k!]. 
\end{cases}$$

The value of $f_k(n, t)$ for $t \in [k + 1, 2(k - 1)!]$ is unknown but does lie between $\log \log n$ and $O(\log n)$. An interesting further question here is if the cases always split into these three sizes, or is there a different behaviour for some $t \in [k + 1, 2(k - 1)!]$?

We again have the trivial cases $t = 1, 2$. Since Theorem 2.5 was for general $k$ that result is still giving us the lower bound when $t \neq 3$.

It is a direct consequence of a result of Spencer [10] that $f_k(n, k) = O(\log \log n)$. In fact Spencer proved the stronger claim that there exists a family $F$ of permutations from $S_n$ with size $O(\log \log n)$ such that for every $k$-tuple $X$, and every $x \in X$, there is some $P \in F$ with $\text{pos}(P, x) < \text{pos}(P, y)$ for all $y \in X \setminus x$. That is, not only does any $k$-tuple appear in at least $k$ orders, but each element in the $k$-tuple appears first in at least one order.

That leaves us with only the following result left to prove Theorem 1.6.

**Theorem 2.8.** For fixed $k$ and when $n$ is large, we have that $f_k(n, t) = \Theta(\log n)$ whenever $t > \frac{2(k!)}{k}$.

**Proof.** Let $S$ be a family that partially shatters every $k$-tuple with $2(k - 1)! + 1$ different orders. Consider $k$-tuples of the form $X := \{x, y, n - k + 3, \ldots, n - 1, n\}$ for any $x, y \in [n - k + 2]$.

Note that $x$ and $y$ must be split by at least one of $\{n - k + 3, \ldots, n\}$ in some $P \in S$. Indeed, there are only $2(k - 1)!$ ways to order $X$ such that $x$ and $y$ are consecutive, yet we know that $X$ appears in at least $2(k - 1)! + 1$ orders in $S$.

Consider the following sets

$$A_x^i := \{P \in S : x \text{ appears after } i\}$$

where $i \in [n - k + 3, n]$. Then for any $x$ and $y$ there exists an $i \in [n - k + 3, n]$ such that $A_x^i \neq A_y^i$.

For each $i \in [n - k + 3, n]$ we define a partition of $[n - k + 2]$ into at most $m$ parts, where elements $x, y \in [n - k + 2]$ are in the same part if and only if $A_x^i = A_y^i$. Label the parts arbitrarily with labels $B_1^i, \ldots, B_m^i$ noting that some labels may not be used at all. Then we must have that

$$m^{k-2} \geq n - k + 2.$$
Indeed, we are able to write each element $x$ uniquely as a $k - 2$ length string from $[m]$, $x = x_1x_2 \ldots x_{k-2}$ where $x_i = r$ if $x \in B^i_r$. To see that this does create a unique identification, consider a pair $x, y$ with the same string. We must have that $x$ and $y$ are in the same $B^i$ part for all $i$, then from the definition of $B^i$'s that means $A^i_y = A^i_x$ for all $i$. We have already established that distinct $x, y$ must have $A^i_y \neq A^i_x$ for some $i$ so conclude that $y = x$.

By choosing the smallest possible $m$ we can assume that there is some $i$ such that the partition has exactly $m$ parts, that is

$$|\{B^i_1, \ldots, B^i_m\}| = m.$$ 

Notice that the set

$$\{A^i_x : x \in [n - k + 2]\}$$

must also have size $m$ since each $x \in B^i_r$ gives rise to the same set $A^i_x$. Therefore this set has size at least $(n - k + 2)^{\frac{1}{k-2}}$ by our above bound on $m$ and hence

$$(n - k + 2)^{\frac{1}{k-2}} \leq 2^{|S|}.$$ 

Giving us the result

$$|S| \geq \frac{1}{k-2} \log(n - k + 2).$$

\[ \square \]

3 Totally shattering a fraction of all $k$-tuples

For this problem we have a fixed number of permutations and wish to know the largest proportion of $k$-tuples that can be shattered. Recall that $F_k(n, m)$ is the maximum proportion $\alpha$ such that there is a family of size $m$ which completely shatters $\alpha \binom{n}{k}$ $k$-tuples from $[n]$.

The function $F_k(n, m)$ is weakly decreasing in $n$ when $k$ and $m$ are fixed such that $m \geq k!$.

**Theorem 3.1.** For fixed $k$ and fixed $m \geq k!$ we have $F_k(n, m) \geq F_k(n + 1, m)$.

**Proof.** Suppose $F_k(n, m) = \alpha$ and consider a family $S$ of $m$ permutations from $S_{n+1}$. Let $X \subseteq [n+1]$ with $|X| = n$, then by taking the permutations of $S$ restricted to the elements of $X$, we can only shatter at most $\alpha \binom{n}{k}$ $k$-tuples. In other words, at least $(1 - \alpha) \binom{n}{k}$ $k$-tuples from $X$ remain un-shattered by $S$. Since this is true for any such $X$ we get that the number of un-shattered $k$-tuples in $[n+1]$ is at least

$$\frac{(1 - \alpha) \binom{n}{k} \binom{n+1}{n}}{\binom{n+1-k}{n-k}} = (1 - \alpha) \binom{n+1}{k}.$$ 

Therefore the number of $k$-tuples that are shattered by $S$ is at most $\alpha \binom{n+1}{k}$. Therefore, $F_k(n + 1, m) \leq \alpha$. 

\[ \square \]
This tells us that $F_k(n, m)$ tends to a limit between 0 and 1, an open question then is ‘what is this limit?’. A direct consequence of the weakly decreasing behaviour of $F_k(n, m)$ is that if $F_k(n, m)$ is known for certain values, then we have a lower bound on $F_k(N, m)$ where $N \geq n$.

We now consider the case when $k = 3$. In particular we fix our family size at 6 since this is the first non-trivial case.

**Corollary 3.2.** For any $n \geq 5$ we have that $F_3(n, 6) \leq \frac{4}{5}$.

The above comes from the fact that a family of 6 permutations of $[5]$ can shatter at most 8 triples, so $F_3(5, 6) = \frac{1}{5}$, which we have checked by hand (see Appendix A). Any family with a smaller proportion will of course give a better upper bound for large $n$. The idea for the upper bound is based on finding an inefficient family and showing that it effects larger families in a similar way. For the lower bound, we do the opposite, using an efficient family to define permutations in order to construct a new family that shatters as many triples as possible.

**Theorem 3.3.** For any $n$ we have $F_3(n, 6) \geq \frac{2}{5}$.

*Proof.* Since $F_3(n, 6)$ is decreasing in $n$ by [3.1] we may assume that $n = 4^r$ for some integer $r$. We will base this construction on an iterated version of the family $Q_3(4)$ [Section 1.1] which shatters every triple in $[4]$ using 6 permutations.

Let $x \in [n]$ and note that we may write $x$ uniquely in the form

$$x = (x_1 - 1)4^{r-1} + (x_2 - 1)4^{r-2} + (x_3 - 1)4^{r-3} + \cdots + (x_r - 1)4 + (x_r - 1) + 1$$

where $x_i \in [4]$ for all $i \in [r]$. As in the proof of [2.6] we will use this to identify each element with a unique code and define $r$ levels of blocks. We will use the same block notation here as in [2.6], but note that there are $r$ block levels here and each level $k$ block contains exactly 4 level $k-1$ blocks. As before, the final level blocks each contain a single element.

Now we will define our permutations. Let $P_1 = (1, 2, \ldots, n)$ be the increasing order of $[n]$.

Let $P_2$ be such that each block (on every level) follows the permutation pattern $Q_2 = (2, 4, 1, 3)$. That is, start by ordering the level 1 blocks as given by the pattern $(B_1(2), B_1(4), B_1(1), B_1(3))$, these block positions are fixed but the order of the elements within the blocks will be fixed later. Now we order level 2, again following the given pattern, within each $B_1(i)$ we fix the order $B_2(i, 2), B_2(i, 4), B_2(i, 1), B_2(i, 3)$. Continuing in this manner we get one ordering of $[n]$ which we call $P_2$.

Analogously, the permutation $P_3$ follows pattern $Q_3 = (3, 4, 1, 2), P_4$ follows $Q_4 = (1, 4, 3, 2), P_5$ follows $Q_5 = (4, 2, 3, 1)$, and $P_6$ follows $Q_6 = (3, 2, 1, 4)$.

Note that any triple whose elements have unique codes which differ for the first time in the same coordinate, is shattered by the collection $P_1, \ldots, P_6$. Indeed, consider $a, b, c \in [n]$ and write $a = (a_1, \ldots, a_r), b = (b_1, \ldots, b_r), c = (c_1, \ldots, c_r)$ for the unique codes. Then if there exists a coordinate $s \in [r]$ such that $a_j = b_j = c_j$ whenever $j < s$ and where $a_s, b_s, c_s$ are all distinct, then the triple $(a, b, c)$ will have its order defined by the order
of \((a_s, b_s, c_s)\) in the pattern followed by \(P_i\). Since \((a_s, b_s, c_s)\) is a triple from \([4]\) and the patterns were chosen from \(Q_3(4)\), we have that \((a, b, c)\) is shattered.

Therefore, we may count the minimum number of shattered triples by counting exactly those triples whose codes have the above property. Firstly, the number of such triples containing some fixed element \(x\) is given by

\[
3 \left( \frac{n}{4} \right)^2 + 3 \left( \frac{n}{4^2} \right)^2 + 3 \left( \frac{n}{4^3} \right)^2 + \cdots + 3 \left( \frac{n}{4^r} \right)^2 = 3n^2 \sum_{i=1}^{r} \left( \frac{1}{4^i} \right)^i = \frac{n^2 - 1}{5}.
\]

So the total number of these triples, after adjusting for over-counting, is \(\frac{n(n^2-1)}{15}\). This gives

\[
F_3(n, 6) \geq \frac{n(n^2-1)}{15} \left( \frac{n}{3} \right)^{-1} = \frac{2(n^2-1)}{5(n-1)(n-2)} \geq \frac{2}{5}.
\]

For the general \(k\)-tuple setting, it is possible to adapt both of the results of this section to give bounds for \(F_k(n, m)\), however bounds obtained in this way become less significant for larger values of \(k\).

**Theorem 3.4.** If there exists a perfect family \(Q_k(m)\) that shatters every \(k\)-tuple from \([m]\) with \(|Q_k(m)| = k!\), then

\[
F_k(n, k!) \geq \frac{(m - 1)!}{(m - k)!(m^{k-1} - 1)}.
\]

This result follows the same method from Theorem 3.3 assuming \(n = m^r\) and copying permutations from \(Q_k(m)\). The number of shattered \(k\)-tuples are then counted in the same manner. We know from Levenshtein \([4]\) that \(Q_k(k+1)\) exists, which gives the following result for all \(k\).

**Corollary 3.5.** We have that

\[
F_k(n, k!) \geq \frac{k!}{(k + 1)^{k-1} - 1}.
\]

### 4 Constructions for total Shattering

Although there is a probabilistic argument showing that \(f_k(n) = O(\log n)\), there is only a construction matching this size for \(k = 3\). We have not been able to extend any of the separation ideas that work in the special \(k = 3\) case to give an \(O(\log n)\) construction for arbitrary values of \(k\). Therefore finding such a construction is still an open problem. This section gives an iterative construction for small shattering families and applies to any value of \(k\). The main idea is to identify each element of \([n]\) with a point in the \(k\)-dimensional lattice, and take permutations by grouping the points in each of the \(k\) directions. Unfortunately the bound this gives is a power of \(\log n\) rather than the known \(O(\log n)\) bound.
Lemma 4.1. Given a family that shatters every $k$-tuple in $[n^{k-1}]$ and has size $S$, we can give an explicit construction of a family that shatters every $k$-tuple from $[n^k]$ and has size $kS$.

Proof. Let $S$ be a shattering family for $[n^{k-1}]$, we use the permutations in this family to construct permutations of $[n^k]$. For simplicity, let $m = n^{k-1}$.

We consider the elements of $[n^k]$ viewed geometrically as points on an integer lattice of dimension $k$. Label each element $x \in [n^k]$ by the string $(i_1, \ldots, i_k)$ with all $i_j \in [n]$. Without loss of generality, we may assume that all elements of $[m]$ are labelled with $(1, i_2, \ldots, i_k)$ and hence can be thought of instead as the $k-1$ string $(i_2, \ldots, i_k)$. This means every $k-1$ string is associated to an element of $[m]$.

Let $r_j(x) = (i_1, \ldots, i_{j-1}, i_{j+1}, \ldots, i_k)$ be $x$ with the $j$th coordinate omitted. Note that $r_j(x)$ is therefore associated to an element of $[m]$.

For each $P \in S$ we will create $k$ permutations of $[n^k]$, $P'_1, \ldots, P'_k$. To generate the permutation $P'_j$ order $x$ before $y$ if and only if $r_j(x)$ appears before $r_j(y)$ in $P$, if $r_j(x) = r_j(y)$ then order arbitrarily.

This generates $k|S|$ permutations of $[n^k]$, it is left to show that these are sufficient to shatter all $k$-tuples.

Given $k$ points in $[n]^k$, $A = \{a_1, \ldots, a_k\}$, there exists a direction $j \in [k]$ such that the projection of $A$ in direction $j$ has $k$ points.

Indeed, suppose not for a contradiction. For all $j \in [k]$ there is some pair $a_\ell, a_t \in A$ such that $r_j(a_\ell) = r_j(a_t)$. Define a graph $G$ with vertex set $A$, and with one edge for each direction $j \in [k]$ between some pair of vertices $u, v \in A$ with $r_j(u) = r_j(v)$. By our assumption there is at least one such pair for each $j \in [k]$, if there is a choice then pick arbitrarily. Note that if $r_j(u) = r_j(v)$ for some $j$ then we cannot have $r_\ell(u) = r_\ell(v)$ for any $\ell \in [k] \setminus j$ by definition, so we can never pick the same edge more than once. Hence our graph on $k$ vertices has exactly $k$ edges, therefore $G$ must contain a cycle. Let $v_1, \ldots, v_t$ be a cycle in $G$, then the edge $v_1v_2$ demonstrates a change in one coordinate, say $j_1$. Similarly edge $v_2v_3$ demonstrates a change in coordinate $j_2$. Observe that $j_1$ and $j_2$ are distinct since there is only one edge for each direction. Continuing, we find $t$ distinct directions $j_1, \ldots, j_t$ where $j_t$ is the direction of the edge $v_tv_1$. This is equivalent to starting with $v_1$, changing $t$ different coordinates and ending up back at $v_1$. Clearly this cannot happen and therefore we have a contradiction.

We have shown that for any $k$-tuple $A = \{a_1, \ldots, a_k\}$, there is a coordinate $j$ such that $r_j(a_\ell) \neq r_j(a_q)$ for all $\ell, q \in [k]$. Hence $A$ is shattered by the collection of permutations of the form $P'_j$ where $P \in S$.

Therefore our collection of $P$’s does indeed shatter all the $k$-tuples in $[n^k]$, and we used $k|S|$ permutations in total.}

Repeatedly applying the construction in Lemma 4.1 gives an upper bound $f_k(n) \leq (\log n)^{c_k}$ where $c_k \approx \frac{\log(k)}{\log(k) - \log(k-1)}$.  

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5 Open Problems

Our main question pertains to the partial shattering variant, does the size of \( f_k(n, t) \) always fall into one of the three sizes as classified in Theorem 1.5 and 1.6? We know that there are values of \( t \) that put \( f_k(n, t) \) in each of these regimes, but is there another size bracket in between \( \log \log n \) and \( \log n \)? In particular we ask:

**Question 5.1.** When \( k > 3 \), what is the value of \( f_k(n, t) \) for \( k + 1 \leq t \leq 2(k-1)! \)?

It is not difficult to see that when \( t \) is odd we can bound \( f_k(n, t+1) \leq 2f_k(n, t) \) by taking the family that realises \( f_k(n, t) \) along with all its reverse permutations. This means that for odd \( k \) we know \( f_k(n, k+1) \) is \( O(\log \log n) \). We also ask the slightly weaker question:

**Question 5.2.** Is it true that \( f_k(n, t) \) is one of three sizes \( \Theta(\log n) \), \( \Theta(\log \log n) \), or constant for all values of \( t \)?

Thinking about fractional shattering, where \( \alpha(n) \) \( k \)-tuples are completely shattered, we ask:

**Question 5.3.** What is the value of \( F_k(n, k!) \)? In particular what is the value of \( F_3(n, 6) \)?

We saw that \( F_k(n, m) \) is decreasing, and by choosing \( m \geq k! \) we know that the limit as \( n \) increases exists and is strictly between 0 and 1.

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Appendix A

To show that $F_3(5, 6) = \frac{4}{5}$ we must show that any 6 permutations from $S_5$ shatter at most 8 triples out of a possible 10. Therefore it is sufficient to show that in any 6 permutations there are at least 2 distinct triples that are not shattered, and hence have a repeated pattern.

So we look for a family of 6 permutations from $S_5$ with at most 1 triple un-shattered (i.e. with a pattern repeated), if we cannot find such a family then we have proved the statement.

If such a family exists, we may assume that the triple that is not shattered contains the element 5. From this we see that the family of 6 permutations of $S_5$ generated by omitting 5 must shatter every triple from [4]. Since the family $Q_k(4)$ is unique up to isomorphism, to prove the statement we show that adding 5 in any position to permutations from $Q_k(4)$ results in a family that leaves 2 triples un-shattered.

Here is the family $Q_k(4)$:

\[
\begin{align*}
Q_1 &= (1,2,3,4) & Q_2 &= (2,4,1,3) & Q_3 &= (3,4,1,2) \\
Q_4 &= (1,4,3,2) & Q_5 &= (4,2,3,1) & Q_6 &= (3,2,1,4).
\end{align*}
\]

The permutation generated by adding 5 into $Q_i$ in some position will be denoted $Q_i'$.

We split into 5 cases, one for each location of element 5 in $Q_1$.

Case 1: $Q_1' = (1,2,3,4,5)$.

So we have fixed $Q_1'$, consider the options for $Q_2'$.

If $Q_2' = (2,4,1,3,5)$ the triples $\{1,3,5\}$ and $\{2,4,5\}$ appear in the same order in both $Q_1'$ and $Q_2'$. Any family that contains $Q_1'$ and $Q_2' = (2,4,1,3,5)$ is not the family we search for, so we look at a different option for $Q_2'$.

If $Q_2' = (2,4,1,5,3)$ then the triple $\{2,4,5\}$ appears in the same order in $Q_1'$ and $Q_2'$. Consider now $Q_3' = (3,4,\ast,1,\ast,2,\ast)$ where 5 appears in any location denoted by $\ast$, the triple $\{3,4,5\}$ appears in the same order in $Q_1'$ and $Q_3'$. This forces both triples $\{2,4,5\}$ and $\{3,4,5\}$ to repeat. So we assume instead that $Q_3' = (\ast,3,\ast,4,1,2)$.

So we have fixed $Q_1' = (1,2,3,4,5)$, $Q_2' = (2,4,1,5,3)$, and $Q_3' = (\ast,3,\ast,4,1,2)$ and we know that $\{2,4,5\}$ is repeated already. Consider $Q_5' = (4,2,3,\ast,1,\ast)$, then $\{2,3,5\}$
appears in the same order in $Q'_1$ and $Q'_5$, meaning both $\{2, 4, 5\}$ and $\{2, 3, 5\}$ are not shattered. Similarly if $Q'_5 = (4, *, 2, *, 3, 1)$ the triple $\{3, 4, 5\}$ is copied in $Q'_2$ and $Q'_5$, meaning $\{2, 4, 5\}$ and $\{3, 4, 5\}$ are not shattered. Finally, if $Q'_5 = (5, 4, 2, 3, 1)$ we have $\{1, 4, 5\}$ appearing in the same pattern in $Q'_3$ and $Q'_5$ giving the pair $\{2, 4, 5\}$ and $\{1, 4, 5\}$.

We record the above information as follows.

$$Q'_2 = (2, 4, 1, 5, 3) \quad \{2, 4, 5\}$$

$$Q'_3 = (3, 4, *, 1, *, 2, *) \quad \{3, 4, 5\}$$

$$Q'_5 = (4, 2, 3, *, 1, *, 2, *) \quad \{2, 3, 5\}$$

This means that the family we search for cannot contain $Q'_1$ and $Q'_5 = (2, 4, 1, 5, 3)$, so we assume next that $Q'_2 = (2, 4, 5, 1, 3)$. It happens that for $Q'_2 = (2, 4, 5, 1, 3)$ the case is the same as that of $Q'_2 = (2, 4, 1, 5, 3)$.

For the remaining two options for $Q'_2$ we have the following.

$$Q'_2 = (2, 5, 4, 1, 3)$$

$$Q'_4 = (1, 4, 3, *, 2, *) \quad \{1, 4, 5\}, \{1, 3, 5\}$$

$$Q'_5 = (4, 2, 3, *, 1, *, 2, *) \quad \{2, 3, 5\}$$

$$Q'_6 = (\ast, 4, *, 2, *, 3, 1) \quad \{3, 4, 5\}$$

$$Q'_3 = (3, 4, 1, *, 2, *) \quad \{1, 2, 5\}$$

$$Q'_4 = (1, 5, 4, 3, 2) \quad \{3, 4, 5\}$$

$$Q'_5 = (4, 2, 3, *, 1, *, 2, *) \quad \{2, 3, 5\}$$

$$Q'_6 = (\ast, 4, *, 2, *, 3, 1) \quad \{3, 4, 5\}$$

$$Q'_3 = (3, 4, 5, 1, 2)$$

$$Q'_4 = (1, 5, 4, 3, 2) \quad \{3, 4, 5\}$$

$$Q'_5 = (\ast, 4, *, 2, *, 3, 1) \quad \{1, 4, 5\}$$

$$Q'_6 = (\ast, 4, *, 2, *, 3, 1) \quad \{1, 4, 5\}$$

$$Q'_3 = (\ast, 4, *, 2, *, 3, 1) \quad \{1, 4, 5\}$$

$$Q'_4 = (5, 1, 4, 3, 2) \quad \{3, 4, 5\}, \{1, 3, 5\}$$

$$Q'_2 = (5, 2, 4, 1, 3)$$

$$Q'_4 = (1, 4, 3, *, 2, *) \quad \{1, 4, 5\}, \{1, 3, 5\}$$

$$Q'_5 = (4, 2, 3, *, 1, *) \quad \{2, 3, 5\}$$

$$Q'_6 = (\ast, 4, *, 2, *, 3, 1) \quad \{3, 4, 5\}$$

$$Q'_3 = (3, 4, 5, 1, 2)$$

$$Q'_4 = (1, 5, 4, 3, 2) \quad \{3, 4, 5\}$$

$$Q'_5 = (\ast, 4, *, 2, *, 3, 1) \quad \{1, 2, 5\}$$

$$Q'_6 = (\ast, 4, *, 2, *, 3, 1) \quad \{1, 4, 5\}$$

$$Q'_3 = (3, 4, 1, *, 2, *) \quad \{2, 3, 5\}$$

$$Q'_4 = (1, 5, 4, 3, 2) \quad \{3, 4, 5\}$$

$$Q'_5 = (\ast, 4, *, 2, *, 3, 1) \quad \{1, 2, 5\}$$

$$Q'_6 = (\ast, 4, *, 2, *, 3, 1) \quad \{1, 4, 5\}$$
\[ Q'_4 = (5, 1, 4, 3, 2) \quad \{3, 4, 5\}, \{1, 3, 5\} \]

These show that if \( Q'_2 = (2, 5, 4, 1, 3) \) or \( Q'_2 = (5, 2, 4, 1, 3) \), there is no position 5 can take in \( Q'_4 \) without causing a pair of un-shattered triples. With all of these pieces we see that no matter where 5 is found in \( Q'_2 \) it leads to a family with at most 8 triples shattered.

We continue like this for the remaining cases, we omit the details since they are straightforward and unenlightening. None of the cases provide a family that shatters more than 8 triples, and hence we have \( F_3(5, 6) = \frac{4}{5} \).