ON SELF-MAPPING DEGREES OF $S^3$-GEOMETRY MANIFOLDS

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ABSTRACT. In this paper we determined all of the possible self mapping degrees of the manifolds with $S^3$-geometry, which are supposed to be all 3-manifolds with finite fundamental groups. This is a part of a project to determine all possible self mapping degrees of all closed orientable 3-manifold in Thurston’s picture.

1. Introduction and Results

For a closed oriented manifold $M$, deciding the mapping-degree set $D(M) = \{ \deg f : f : M \to M \}$ is a natural problem. When the dimension is 1 and 2, the answer is well-known. When the dimension is higher than 3, there are many interesting results in this topic. For classical discussions see [1], [2] and for quite recent papers see [3], [4] and [5]. But it is difficult to get general results, since there is no classification result for manifolds of dimension $n > 3$.

The case of dimension 3 becomes attractive in the topic and it becomes possible to calculate $D(M)$ for any closed oriented 3-manifold $M$. Since Thurston’s geometrization conjecture, which seems to be confirmed, implies that closed oriented 3-manifolds can be classified in a reasonable sense.

Thurston’s geometrization conjecture claims that the each Jaco-Shalen-Johanson decomposition piece of a prime 3-manifold supports one of eight geometries, which are $H^3$, $\widetilde{PSL}(2, \mathbb{R})$, $H^2 \times E^1$, $Sol$, $Nil$, $E^3$, $S^3$ and $S^2 \times S^1$ (for details see [6] and [7]). Call a closed orientable 3-manifold $M$ is geometric if it supports one of the eight geometries above.

Hence we should ask first

Question(*) How to determine $D(M)$ for a geometric 3-manifold $M$?

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Indeed, for most cases the answer to this question is known. When $M$ supports the geometry of $H^3$ or $\widetilde{PSL}(2,\mathbb{R})$, by using the Gromov volume or $\widetilde{PSL}(2,\mathbb{R})$ volume, $D(M)$ should be either $\{0,1,-1\}$ or $\{0,1\}$, depending on whether $M$ admits a self map of degree $-1$ or not (see [3] for details). For 3-manifolds admitting $E^3$ and $Sol$ structures and for some $Nil$ 3-manifolds, $D(M)$’s have been determined just recently in [9].

The remaining important case for question(*) is for the 3-manifolds supporting $S^3$-geometry, which are supposed to be all 3-manifolds with finite fundamental groups.

The question(*) for this case has several partial answers before. The answer was known for Poincare homology 3-spheres in [10], and for Quaternion spaces in [11]. More generally, for the degrees of the self-mappings inducing automorphisms of $\pi_1(M)$, the answer had been known in [12], which stated that $\{\deg f \mid f_\ast : \pi_1(M) \to \pi_1(M) \text{ is an automorphism}\} = \{k^2 \mid (k,|\pi_1(M)|) = 1\} + |\pi_1(M)| \cdot \mathbb{Z}$ ([12], Theorem 2.2).

In this paper, we will determine the degrees of the mappings inducing all possible endomorphisms of the fundamental groups to give a complete answer of question(*) for every 3-manifold $M$ admitting $S^3$-geometry.

According to [7], the fundamental group of a 3-manifold with $S^3$-geometry structure is among the following eight types: $\mathbb{Z}_p$, $D^*_{4n}$, $T^*_{24}$, $O^*_{48}$, $I^*_{120}$, $T'_{8,3^3}$, $D'_{n',2^9}$ ($2 \nmid n'$) and $\mathbb{Z}_m \times G$ where $G$ belongs to the previous seven ones and $|G|$ is coprime to $m$. $T^*_{24}$, $O^*_{48}$ and $I^*_{120}$ have intuitive explanations: they are the pre-images of the rigid rotation groups of the regular tetrahedron, octahedron and icosahedron under the double covering $S^3 \to SO(3)$. $D^*_{4n}$ is the pre-image of the dihedral group $D_{2n}$ under the same double covering. $T'_{8,3^3}$, $D'_{n',2^9}$ and $\mathbb{Z}_m \times G$ are discrete subgroups of $SO(4)$ which act on $S^3$. These groups have presentations as following:

| $\pi_1(M)$ | presentation |
|-------------|-------------|
| $\mathbb{Z}_p$ | $\langle a \mid a^p = 1 \rangle$ |
| $D^*_{4n}$ | $\langle a, b \mid a^2 = b^p = (ab)^2, a^4 = 1 \rangle$ |
| $T^*_{24}$ | $\langle a, b \mid a^2 = b^3 = (ab)^2, a^4 = 1 \rangle$ |
| $O^*_{48}$ | $\langle a, b \mid a^2 = b^3 = (ab)^4, a^4 = 1 \rangle$ |
| $I^*_{120}$ | $\langle a, b \mid a^2 = b^3 = (ab)^5, a^4 = 1 \rangle$ |
| $T'_{8,3^3}$ | $\langle i, j, k, w \mid i^2 = j^2 = k^2, i^4 = 1, ij = k, jk = i, ki = j, w^{3^3} = 1, w \cdot i \cdot w^{-1} = j; \ w \cdot j \cdot w^{-1} = k; \ w \cdot k \cdot w^{-1} = i \rangle$ |
| $D'_{n',2^9}$ | $\langle w, u \mid u^{w} = 1, w^{2^9} = 1, w \cdot u \cdot w^{-1} = u^{-1} \rangle$ |
| $\mathbb{Z}_m \times G$ | $G$ is in one of the seven types above and $(m,|G|) = 1$ |
Respectively, we have the following theorem on the self-mapping degrees.

**Theorem 1.1.** The sets of self-mapping degrees of \( S^3 \)-geometry manifolds are listed as follow:

| \( \pi_1(M) \) | \( D(M) \) |
|----------------|----------------|
| \( \mathbb{Z}_p \) | \( \{ k^2 \mid k \in \mathbb{Z} \} + p\mathbb{Z} \) |
| \( D_{4n}^* \) | \( \{ h^2 \mid h \in \mathbb{Z}; 2 \nmid h \text{ or } h = n \text{ or } h = 0 \} + 4n\mathbb{Z} \) |
| \( T_{24}^* \) | \( \{ 0, 1, 16 \} + 24\mathbb{Z} \) |
| \( O_{48}^* \) | \( \{ 0, 1, 25 \} + 48\mathbb{Z} \) |
| \( I_{120}^* \) | \( \{ 0, 1, 49 \} + 120\mathbb{Z} \) |
| \( T'_{8,3q} \) | \( \{ k^2 \cdot (3^{2q-2p} - 3^q) \mid 3 \nmid k, q \geq p > 0 \} + 8 \cdot 3^q \mathbb{Z} \) \( (2 \mid q) \) |
| | \( \{ k^2 \cdot (3^{2q-2p} - 3^{q+1}) \mid 3 \nmid k, q \geq p > 0 \} + 8 \cdot 3^q \mathbb{Z} \) \( (2 \nmid q) \) |
| \( D'_{n',2q} \) | \( \{ k^2 \cdot [1 - (n')^{2-1}]^i \cdot [1 - 2^{(2p-q)(n'-1)}]^j] \mid i,j,k,p \in \mathbb{Z}, q \geq p > 0 \} + n'2^q \mathbb{Z} \) |
| \( \mathbb{Z}_m \times G \) | \( d \equiv h \pmod{|G|} (h \in D(N), N \text{ is the } \mathbb{S}^3\text{-geometry manifold with } \pi_1(N) = G) \) |
| | \( d \equiv k^2(k \in \mathbb{Z})(\text{mod } m) \) |

The proof of this theorem will be divided into two parts. Firstly, since Proposition 2.1 (see below) states that for each spherical 3-manifold, there is a well defined mapping from the set of endomorphisms of the fundamental group to the \( |\pi_1| \) classes of the self-mapping degrees, we need to find out all possible endomorphisms of the fundamental groups of the above eight types. Secondly, for each endomorphism, we need to calculate the degree of some self-mapping realizing it.

### 2. Preliminaries

Since the second homotopy group of a \( S^3 \)-geometry manifold is trivial, the existences of self-mappings can be detected by the obstruction theory. P. Olum showed in [1] the first and in [2] the second part of the following proposition.

**Proposition 2.1** (Olum). Let \( M \) be an orientable 3-manifold with finite fundamental group and trivial \( \pi_2(M) \). Every endomorphism \( \phi : \pi_1(M) \to \pi_1(M) \) is induced by a (basepoint preserving) continuous map \( f : M \to M \). Furthermore, if \( g \) is also a continuous self-mapping of \( M \) such that \( f_* = g_* = \phi \) then \( \deg f \equiv \deg g \pmod{|\pi_1(M)|} \).

According to this proposition, the self-mapping degrees of a spherical 3-manifold \( M \) are closely related to the endomorphisms of \( \pi_1(M) \).
The statement and proof of the next lemma is elementary (see [12] lemma 3.4). This lemma is very useful in the calculation of the degree of a self-mapping corresponding to a given endomorphism on the fundamental group.

**Lemma 2.2** (Hayat-Kudryavtseva-Wang-Zieschang). Let \( f \) be a self-mapping of \( M \), \( G = \pi_1(M) \) contain a subgroup \( H \) such that \( f_*(H) \subseteq gHg^{-1} \) for some \( g \in G \). Consider the covering \( p : \tilde{M} \to M \) corresponding to \( H \), that is, \( H = p_*(\pi_1(\tilde{M})) \). Then there is a map \( \tilde{f} : \tilde{M} \to \tilde{M} \) and a homeomorphism \( J : M \to M \) isotopic to \( \text{id}_M \) such that the following diagram is commutative:

\[
\begin{array}{ccc}
\tilde{M} & \xrightarrow{\tilde{f}} & \tilde{M} \\
p \downarrow & & \downarrow J \circ p \\
M & \xrightarrow{f} & M.
\end{array}
\]

A consequence is that \( \deg(\tilde{f}) = \deg f \).

If \( H \) is a Sylow subgroup of \( \pi_1(M) \), \( H \) satisfies the condition of the above lemma by the second Sylow theorem ([13], 1.13).

3. Proof of Theorem 1.1

At first we will find out all the possible endomorphisms of the fundamental groups of the spherical 3-manifolds.

**Lemma 3.1.** Two endomorphisms of the fundamental group of a spherical 3-manifold having isomorphic kernels can be transformed to each other by a composition with an isomorphism of the fundamental group. The kernels of the non-trivial endomorphisms of the fundamental groups of the spherical 3-manifolds are listed as follow:

| \( \pi_1(M) \) | non-trivial kernels of endomorphisms of \( \pi_1(M) \) |
|-----------------|------------------------------------------------|
| \( \mathbb{Z}_p \) | \( \mathbb{Z}_{p/k} \) |
| \( D_{4n}^* \) | \( D_{42}^* \) (if \( 2 \mid n \)), \( \mathbb{Z}_{2n} \) and \( \mathbb{Z}_h \) (2 \( \nmid h \)) |
| \( T_{24}^* \) | \( Q_8 \) |
| \( O_{48}^* \) | \( T_{24}^* \) |
| \( I_{120}^* \) | none |
| \( T_{8:3}^* = Q_8 \times \mathbb{Z}_{3n} \) | \( Q_8 \times \mathbb{Z}_{q_3 \cdot p} \) (\( q \geq p > 0 \)) |
| \( D_{2q:2}^* = \mathbb{Z}_{n'} \times \mathbb{Z}_{2q} \) | \( \mathbb{Z}_{n'} \times \mathbb{Z}_{q_3 \cdot p} \) (\( q \geq p > 0 \)) and \( \mathbb{Z}_{n'/n''} \times \mathbb{Z}_{2q} \) (\( n'' \mid n', n'' > 1 \)) |
| \( \mathbb{Z}_m \times G \) | \( \mathbb{Z}_{m'} \times H \) (\( m' \mid m, H \subseteq G \)) |
Proof. Case I:

Since \( \mathbb{Z}_p = \langle a \mid a^p = 1 \rangle \) is a cyclic group, the result is obvious. The endomorphism is like \( \phi : a \mapsto a^t \) and its kernel is \( \langle a^{\lceil t/p \rceil} \rangle \cong \mathbb{Z}_{t/p} \).

Case II:

\( D_{2n}^* \) has a representation as \( \langle a, b \mid a^2 = b^n = (ab)^2 = 1 \rangle, n = 2^qn', n' \) is an odd number (see [12]).

First note that \( ab^k a^{-1} = b^{2n-k} \), so \( \langle b^k \rangle \cong \mathbb{Z}_{2n}^{(2n,k)} \). Secondly, The elements in \( D_{2n}^* \langle b \rangle \) are like \( ab^l \) and have order 4. We will discuss the normal subgroup according to whether it includes such elements or not.

(I) If the normal subgroup \( H \) includes some element in \( D_{2n}^* \langle b \rangle \), because \( b(ab^l)b^{-1} = ab^{l-2} \), we have \( b^2 \in H \), \( H \supseteq \langle ab^l, b^2 \rangle \). In fact, \( \langle ab^l, b^2 \rangle \) is a proper subgroup of \( D_{2n}^* \) if and only if \( 2 \nmid n \). When \( 2 \nmid n \), we have \( \langle ab^l, b^2 \rangle \cong D_{2n}^* \) and \( D_{2n}^*/\langle ab^l, b^2 \rangle = D_{4n}^*/D_{4n}^* \cong \mathbb{Z}_2 \leq D_{4n}^* \). The corresponding endomorphism is like \( \phi : a \mapsto (-1)^l, b \mapsto -1 \).

(II) If the normal subgroup \( H \) is only generated by some \( b^k \), then

\( \langle b^k \rangle \neq \langle b \rangle \)

(A) When \( 2 \mid \frac{2n}{(2n,k)} \), we have \( -1 \in \langle b^k \rangle \). The quotient group \( D_{2n}^*/\langle b^k \rangle \) is isomorphic to the dihedral group \( D_{2(2n,k)} \) and also a subgroup of \( D_{2n}^* \).

However, there are \( (2n, k) \) elements of order 2 in \( D_{2(2n,k)} \) while there is only one in \( D_{2n}^* \). This is absurd when \( 2n > 1 \). So the kernels of endomorphisms of \( D_{2n}^* \) cannot be \( \langle b^k \rangle \) with \( (2n, k) > 1 \). When \( (2n, k) = 1 \),

\( \langle b^k \rangle = \langle b \rangle \), we have \( D_{4n}^*/\langle b \rangle = D_{2n}^*/\langle b \rangle \cong \mathbb{Z}_2 \leq D_{4n}^* \). So there exists only one endomorphism of \( D_{2n}^* \) the kernel of which is \( \langle b \rangle \cong \mathbb{Z}_2 \).

This endomorphism is like \( \phi : a \mapsto -1, b \mapsto 1 \).

(B) When \( 2 \nmid \frac{2n}{(2n,k)} \) (i.e. \( 2n+1 \mid k \)), we have \( -1 \not\in \langle b^k \rangle \) and \( D_{2n}^*/\langle b^k \rangle \cong D_{4n}^*/\mathbb{Z}_{2n}^{(2n,k)} \leq D_{4n}^* \). So there exists an endomorphism of \( D_{4n}^* \) the kernel of which is \( \langle b^k \rangle \). The endomorphisms having this kernel are like

\( \phi : a \mapsto b^sab^{-s}, b \mapsto b^{2(2n,s)} \) (\( s, t \in \mathbb{Z}, (t,k) = 1 \)), which are conjugate to \( \psi : a \mapsto a, b \mapsto b^{(2n,t)} \).

By the composition with the isomorphism \( a \mapsto a, b \mapsto b^l \) of \( D_{4n}^* \), they can be transformed to \( \phi_0 : a \mapsto a, b \mapsto b^{(2n,s)} \).

Case III:

\( T_{24}^* \) has a representation as \( \langle a, b \mid a^2 = b^3 = (ab)^3 = 1 \rangle \).

The normal subgroups of \( T_{24}^* \) are:

(I) \( \{ 1, -1 \} \). But \( T_{24}^*/\{ 1, -1 \} \cong T_{12} = A_4 \) is not a subgroup of \( T_{24}^* \);

(II) \( \langle a, bab^{-1} \rangle \cong Q_8 \). In this case \( T_{24}^*/\langle a, bab^{-1} \rangle \cong \mathbb{Z}_4 \) and subgroups of \( T_{24}^* \) isomorphic to \( \mathbb{Z}_4 \) have the forms like \( \langle 1, x^2, x^3, x^4 \rangle \).

\( T_{24}^* \) has no other normal subgroup. In fact, if \( H \) is a normal subgroup of \( T_{24}^* \) including \( a^2 = -1 \), then \( H/\{ 1, -1 \} \) is isomorphic to a normal subgroup

\( T_2^* \), which is isomorphic to a normal subgroup of \( T_{24}^* \).
of the tetrahedron group $T_{12} \cong A_4$. But the only normal subgroup of $T_{12}$ is $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. Its pre-image in $T_{24}^*$ is $Q_8$. If $H$ does not include $-1$, then $H$ must include an element of $T_{24}^*$ of odd order. The odd-order elements of $T_{24}^*$ conjugate to $b^2$. So $b^2 \in H$, $ab^2a^{-1}b^2 \in H$. But $ab^2a^{-1}b^2$ is an element of order six. It is conjugated to $b$. This implies $b \in H$, $-1 = b^3 \in H$, contradiction.

The endomorphisms corresponding to the normal subgroup $Q_8$ are like $\phi : a \mapsto 1$, $b \mapsto xb^2x^{-1}$. They are conjugate to $\phi_0 : a \mapsto 1$, $b \mapsto b^2$.

**Case IV:**

$O_{48}^*$ has a representation as $\langle a, b \mid a^4 = b^3 = (ab)^2 = -1 \rangle$.

The normal subgroups of $O_{48}^*$ are:

(I) $\{1, -1\}$. But $O_{48}^*/\{1, -1\} \cong O_{24} = S_4$ is not a subgroup of $O_{48}^*$;

(II) $\langle a^2, b \rangle \cong T_{24}^*$. In this case $O_{48}^*/T_{24}^* \cong \mathbb{Z}_2 \leq O_{48}^*$.

Similar to the last case, $O_{48}^*$ has no other normal subgroup.

The endomorphism corresponding to the normal subgroup $T_{24}^*$ is like $\phi : a \mapsto -1$, $b \mapsto 1$.

**Case V:**

$I_{120}^* = \langle a, b \mid a^2 = b^3 = (ab)^5 = -1 \rangle$.

The only non-trivial normal subgroup of $I_{120}$ is $\{1, -1\}$. But $I_{120}/\{1, -1\} \cong I_{60} = A_5$ which is not a subgroup of $I_{120}^*$. So the $I_{120}^*$ only admits trivial endomorphism.

**Case VI:**

$T_{8,3^q}^\prime = Q_8 \times \mathbb{Z}_{3^q}, Q_8 = \{ \pm 1, \pm i, \pm j, \pm k \}, \mathbb{Z}_{3^q} = \langle w \rangle$, $w \cdot i \cdot w^{-1} = j$; $w \cdot j \cdot w^{-1} = k$; $w \cdot k \cdot w^{-1} = i$.

If $K \leq T_{8,3^q}^\prime$, for the symmetry of $i$, $j$ and $k$, suppose $w^\alpha i \in K$, then $w^{-1}(w^\alpha i)w = w^\alpha w^{-1}(wk) = w^\alpha k \in K$. So $i^{-1}k = j \in K$, similarly $k, i \in K$ and $Q_8 \subset K$.

Since $w^3$ commutes with the elements in $Q_8$, the normal subgroups of $T_{8,3^q}^\prime$ should be like $Q_8 \times \langle w^3 \rangle \cong Q_8 \times \mathbb{Z}_{3^q-p}$ ($q \geq p > 0$) and $(Q_8 \times \mathbb{Z}_{3^q})/(Q_8 \times \mathbb{Z}_{3^q-p}) \cong \mathbb{Z}_{3^p} \cong \langle w^3 \rangle$ which is isomorphic to a subgroup of $T_{8,3^q}^\prime$.

The endomorphisms corresponding to the normal subgroup $Q_8 \times \langle w^3 \rangle$ are like $\phi : i \mapsto 1$, $j \mapsto 1$, $k \mapsto 1$, $w \mapsto w^{h \cdot 3^q-p}$.

**Case VII:**

$D_{n' \cdot 2^q} = \mathbb{Z}_{n'} \times \mathbb{Z}_{2^q}$, $2 \not\mid n'$, $\mathbb{Z}_{n'} = \langle u \rangle$, $\mathbb{Z}_{2^q} = \langle w \rangle$, $w \cdot u \cdot w^{-1} = u^{-1}$.

Any element in $\mathbb{Z}_{n'} \times \mathbb{Z}_{2^q}$ can be written as $w^\alpha u^\beta$. If $K$ is a normal subgroup, for $w^\alpha u^\beta \in K$, we have $w^{-1}(w^\alpha u^\beta)w = w^\alpha(w^{-1}u^\beta w) = w^\alpha u^{-\beta} \in KH_4$. But $KH_4 = KH_4$. Thus $K$ is a subgroup of $KH_4$.
If some \( w^\alpha u^\beta \) satisfies \( 2 \mid \alpha \), then \( w = w(\alpha, 2^\beta) \in K \), \( u(w^\alpha u^\beta)u^{-1} = w^\alpha u(-1)^\alpha u^{-1} = w^\alpha u^{-\beta-2} \in K \). Hence \( u^2 \in K \) and \( u = u(2, n') \in K \). Then \( K = \langle u, w \rangle \) is a trivial subgroup.

If every element \( w^\alpha u^\beta \) in \( K \) satisfies \( 2 \mid \alpha \), let \( \alpha_o \) be the greatest common divisor of such \( \alpha \)’s. We have \( w^\alpha_0 \in K \), \( \langle w(\alpha_0, 2^\beta) \rangle = \langle w^2 \rangle \subset K \) (\( q \geq p > 0 \)). Note that \( w^2 \) commutes with the elements in \( \mathbb{Z}_{n'} \times \mathbb{Z}_{2^q} \). Hence \( K \cong \langle w^2 \rangle \times \langle w^2 \rangle \cong \mathbb{Z}_{n'} \times \mathbb{Z}_{2^q} \times \mathbb{Z}_{2^{q-p}}(n''|n') \) and \( (\mathbb{Z}_{n'} \times \mathbb{Z}_{2^q})/K \cong \mathbb{Z}_{n''} \times \mathbb{Z}_{2^p} / \mathbb{Z}_{2^q} \).

This quotient group is isomorphic to a subgroup of \( \mathbb{Z}_{n'} \times \mathbb{Z}_{2^q} \) if and only if (A) \( n'' = 1 \) and \( q \geq p > 0 \) or (B) \( n'' > 1 \) and \( p = 0 \). So endomorphisms \( \phi \) of \( \mathbb{Z}_{n'} \times \mathbb{Z}_{2^q} \) have the forms (A) \( \phi(w) = w^{h \cdot 2^{q-p}}(2 \mid h, q \geq p > 0) \) or (B) \( \phi(w) = w^{l \cdot n''/n'}, \phi(w) = 1 \) (\( n'' > 1, (n'', l) = 1 \)).

**Case VIII:**

Since \( \mathbb{Z}_m \times G \) is a direct product, the result is obvious. \( \square \)

In the following part we will calculate the degree of the self-mappings corresponding to the above endomorphisms. For the cases \( \pi_1(M) = \mathbb{Z}_p, D^*_4n \) and \( T^*_{24} \), by using the coordinate system of \( SU(2) \), we construct the concrete mappings. For the remaining cases, by using lemma 2.2, we reduce the problem to the known cases.

We denote \( S^3 = SU(2) \equiv \{(z_1, z_2) \in \mathbb{C}^2 | |z_1|^2 + |z_2|^2 = 1 \} \).

**Case I:**

When \( M \) is a lens space and \( \pi_1(M) = \mathbb{Z}_p = \langle a | a^p = 1 \rangle \), the action of \( \mathbb{Z}_p \) on \( S^3 \) can be realized as \( a \circ (z_1, z_2) = (\zeta z_1, \zeta^p z_2) \), here \( \zeta \) is a \( p^{th} \) root of 1.

Let \( \phi : \mathbb{Z}_p \to \mathbb{Z}_p, a \mapsto a^k \) be an endomorphism with the arbitrary integer \( k \). Then take \( \bar{f}^i : S^3 \to \mathbb{C}^2, (z_1, z_2) \mapsto (z_1^k, z_2^k) \) and \( \bar{f} = \pi \circ \bar{f}^i : S^3 \to \mathbb{C}^2 \), here \( \pi : \mathbb{C}^2 \to \mathbb{C}^2 \) is the standard radial projection.

It is easy to check \( \bar{f} \circ a = a^k \circ \bar{f} \). This means we can well-define \( f : M \to M \) with \( f_* = \phi : \pi_1(M) \to \pi_1(M) \) and \( \text{deg} f = \text{deg} \bar{f} = k^2 \).

So \( D(L_{p, q}) = \{ k^2 | k \in \mathbb{Z} \} + p \mathbb{Z} \).

**Case II:**

\( \pi_1(M) = D^*_4n = \langle a, b | a^2 = b^2 = (ab)^2 = -1 \rangle, n = 2^n n', n' \) is an odd number.

\( D^*_4n \) is the double cover of the dihedral group \( D_{2n} \subset SO(3) \) in \( S^3 \). To calculate the endomorphisms and construct the corresponding mappings, take a representation \( \rho : D^*_4n \to SU(2) \) as
\[a \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad b \mapsto \begin{pmatrix} \zeta & 0 \\ 0 & \zeta \end{pmatrix}, \quad \text{here } \zeta^n = -1.\]

Then the left actions of elements of \(D_{4n}^*\) on \(SU(2)\) are \(a \circ (z_1, z_2) = (-\bar{z}_2, \bar{z}_1), \quad b \circ (z_1, z_2) = (\zeta z_1, \zeta z_2)\).  

When the self-mapping induces an endomorphism having the kernel as \(\mathbb{Z}_{2n}\), take \(f: S^3/\mathbb{Z}_{2n} \to S^3/\mathbb{Z}_{2n}\) with ker\(f_s \cong \mathbb{Z}_{2n}\) and deg\(f = n^2\).

When the self-mapping induces an endomorphism having the kernel as \(\mathbb{Z}_{2n}^{(2n, k)}\), according to the proof of lemma 3.1, \(2n^{(2n, k)}\) should be an odd number. Take \(f: S^3/\mathbb{Z}_{2n}^{(2n, k)} \to S^3/\mathbb{Z}_{2n}^{(2n, k)}\) and let \(\bar{f} = \pi \circ f: S^3 \to S^3\). Then \(\bar{f} \circ a = b^n \circ \bar{f} = -\bar{f}, \bar{f} \circ b = \bar{f}\), which induces \(f: S^3/D_{4n}^* \to S^3/D_{4n}^*\) with ker\(f_s \cong \mathbb{Z}_{2n}^{(2n, k)}\) and deg\(f = (2n)^2 = 4n^2 \equiv 0 \pmod{4n}\).

Case III:

\[\pi_1(M) = T_{24}^* = \langle a, b \mid a^2 = b^3 = (ab)^3 = -1 \rangle .\]

Take \(\rho: T_{24}^* \to SU(2)\) as \(a \mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad b \mapsto \begin{pmatrix} \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i & \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i \\ -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i & \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i \end{pmatrix}, \quad \text{here } \zeta^4 = -1.\]

Then the left actions of elements of \(T_{24}^*\) on \(SU(2)\) are \(a \circ (z_1, z_2) = (iz_1, iz_2), \quad b \circ (z_1, z_2) = (\frac{\sqrt{2}}{2} \zeta z_1 - \frac{\sqrt{2}}{2} \zeta \bar{z}_2, \frac{\sqrt{2}}{2} \zeta \bar{z}_1 + \frac{\sqrt{2}}{2} \zeta z_2).\)
In lemma 3.1 we had proved that the non-trivial endomorphism of $T_{24}^*$ has the kernel isomorphic to $Q_8$. Take
\[ \tilde{f} : S^3 \to \mathbb{C}^2, (z_1, z_2) \mapsto (z_1^4 + \bar{z}_2^4 + 2iz_1^2\bar{z}_2^2, 2\sqrt{2}z_1z_2^2) \]
and let $\tilde{f} = \pi \circ \tilde{f} : S^3 \to S^3$. Then $\tilde{f} \circ a = \tilde{f}$, $\tilde{f} \circ b = b^2 \circ \tilde{f}$, which induces $f : S^3/T_{24}^* \to S^3/T_{24}^*$. $\text{Ker}f^* \cong Q_8$, $\text{deg}f = 16$.

In [12] we know that the set of degrees of self-mappings inducing automorphisms of $T_{24}^*$ is $\{1\} + 24\mathbb{Z}$. Hence,
\[ D(S^3/T_{24}^*) = \{0, 1, 16\} + 24\mathbb{Z}. \]

**Case IV:**
\[ \pi_1(M) = O_{48}^* = \langle a, b | a^4 = b^3 = (ab)^2 = -1 \rangle. \]

According to Lemma 2.2, there exists some mapping $f : S^3/O_{48}^* \to S^3/O_{48}^*$ with $\text{Ker}f^* = T_{24}^*$, $f^*: a \mapsto -1$, $b \mapsto 1$.

For the Sylow 2-subgroup $H = \langle a, ba^2b^{-1} \rangle \cong D_{4,2}^*$ of $O_{48}^*$, construct covering map $p: \hat{M} \cong S^3/D_{4,2}^* \to S^3/O_{48}^*$ such that $p_*\pi_1(\hat{M}) = H$.

Then $f$ can be lifted to $Hf : \hat{M} \to \hat{M}$:
\[
\begin{array}{ccc}
\hat{M} & \xrightarrow{\hat{f}} & \hat{M} \\
\downarrow p & & \downarrow p \\
S^3/O_{48}^* & \xrightarrow{f} & S^3/O_{48}^*.
\end{array}
\]

Now $\pi_1(\hat{M})$ is $D_{4n}^*$ type and $\text{Ker}(\hat{f})^* \cong Q_8 \cong D_{4,2}^* = \langle a^2, ba^2b^{-1} \rangle$. We had discussed this case in Case II. So $\text{deg}Hf = 4^2 \equiv 0 \pmod{16}$ and $\text{deg}f \equiv 0 \pmod{16}$.

For the Sylow 3-subgroup $K = \langle b^2 \rangle \cong \mathbb{Z}_3$, construct the covering map $q : \hat{M} \to S^3/O_{48}^*$ and lift $f$ to $Kf : \hat{M} \to \hat{M}$. It is easy to check $(Kf)^*$ is a constant endomorphism. So $\text{deg}f \equiv 0 \pmod{3}$. Together with $\text{deg}f \equiv 0 \pmod{16}$ we can get $\text{deg}f \equiv 0 \pmod{48}$.

So non-trivial endomorphisms of $O_{48}^*$ give no new self-mapping degree of $S^3/O_{48}^*$ other than the trivial ones. Hence, together with the results of the degrees yielded by mappings inducing automorphisms of $O_{48}^*$ (see [12]), we have
\[ D(S^3/O_{48}^*) = \{0, 1, 25\} + 48\mathbb{Z}. \]

**Case V:**
\[ \pi_1(M) = I_{120}^* = \langle a, b | a^2 = b^3 = (ab)^5 = -1 \rangle. \]
$I^*_{120}$ has no non-trivial endomorphisms and the set of degrees of self-mapping inducing isomorphisms is $\{1, 49\} + 120\mathbb{Z}$ (see [12]). Hence,

$$D(S^3/I^*_{120}) = \{0, 1, 49\} + 120\mathbb{Z}.$$  

**Case VI:**

$$\pi_1(M) = T'_{8,3q} = Q_8 \rtimes \mathbb{Z}_{3q}, \quad Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}, \quad \mathbb{Z}_{3q} = \langle w \rangle,$$

$$w \cdot i \cdot w^{-1} = j; \quad w \cdot j \cdot w^{-1} = k; \quad w \cdot k \cdot w^{-1} = i.$$  

If a self-mapping $f$ of $S^3/T'_{8,3q}$ induces an endomorphism of $T'_{8,3q}$ with the kernel $K = Q_8 \times \langle w^{3p} \rangle$, then $f_*(Q_8) = 1$, $f_*(w) = w^{k \cdot 3^q - p}$ ($3 \nmid k, q \geq p > 0$). Its lifting $(\overline{Q_8f}) : \overline{Q_8M} \to \overline{Q_8M}$ is the covering space of $M$ with the fundamental group equals $Q_8$ will satisfy $(\overline{Q_8f})_* = 1$ and its lifting $\overline{z_{3q}f} : \overline{z_{3q}M} \to \overline{z_{3q}M} \cong S^3/\mathbb{Z}_{3q}$ will satisfy $(\overline{z_{3q}f})_* : w \mapsto w^{k \cdot 3^q - p}$. So

$$\begin{align*}
\deg f &\equiv \deg \overline{Q_8f} \equiv 0 \pmod{8} \\
\deg f &\equiv \deg \overline{z_{3q}f} \equiv (k \cdot 3^q - p)^2 \pmod{3^q}. 
\end{align*}$$

The solution is

$$\deg f \equiv \begin{cases} 
k^2 \cdot (3^{2q} - 2p - 3^q) & (2 \mid q) \\
k^2 \cdot (3^{2q} - 2p - 3^q + 1) & (2 \nmid q) \pmod{8 \cdot 3^q} \quad (3 \nmid k, q \geq p > 0). 
\end{cases}$$

After multiplied by a square of integer coprime to $8 \cdot 3^q$ which is a feasible degree of some mapping inducing an automorphism of $\pi_1(M)$, the possible self-mapping degrees remain the same. Hence,

$$D(S^3/T'_{8,3q}) = \{k^2 \cdot (3^{2q} - 2p - 3^q) \mid 3 \nmid k, q \geq p > 0\} \cup 8 \cdot 3^q \mathbb{Z} \cup \{k^2 \cdot (3^{2q} - 2p - 3^q + 1) \mid 3 \nmid k, q \geq p > 0\} \cup 8 \cdot 3^q \mathbb{Z} \cup \{2 \mid q\}.$$  

**Case VII:**

$$\pi_1(M) = D'_{n',2q} \rtimes \mathbb{Z}_{2q}, \quad \mathbb{Z}_{2q} = \langle u \rangle, \quad \mathbb{Z}_{2q} = \langle w \rangle, \quad w \cdot u \cdot w^{-1} = u^{-1}. $$

Given $f : S^3/D'_{n',2q} \to S^3/D'_{n',2q}$ with $f_*= \phi : u \mapsto 1, \quad w \mapsto w^{h \cdot 2^{q-p}}$ ($2 \nmid h, q \geq p > 0$), the liftings $\overline{z_{n'}f} : \overline{z_{n'}M} \to \overline{z_{n'}M}$ and $\overline{z_{2q}f} : \overline{z_{2q}M} \to \overline{z_{2q}M}$ will satisfy $(\overline{z_{n'}f})_* : u \mapsto 1$ and $(\overline{z_{2q}f})_* : w \mapsto w^{h \cdot 2^{q-p}}$; Given $f : S^3/D'_{n',2q} \to S^3/D'_{n',2q}$ with $f_*= \phi : u \mapsto u^{l \cdot n'/n''}, \quad w \mapsto 1$ $(n'' > 1, (l, n'') = 1)$, these liftings will satisfy $(\overline{z_{n'}f})_* : u \mapsto u^{l \cdot n'/n''}$ and $(\overline{z_{2q}f})_* : w \mapsto 1$.

The equations

$$\begin{align*}
\deg f &\equiv \deg \overline{z_{n'}f} \equiv 0 \pmod{n'} \\
\deg f &\equiv \deg \overline{z_{2q}f} \equiv (h \cdot 2^{q-p})^2 \pmod{2^q} \quad (2 \mid h, q \geq p > 0)
\end{align*}$$

imply

$$\deg f \equiv h^2 \cdot 2^{2(q-p) \cdot [1 - 2^{(2p-q)(n'-1)}]} \pmod{n' \cdot 2^q},$$
and the equations
\[
\begin{align*}
\deg f &\equiv \deg \tilde{Z}_m f \equiv (l \cdot n'/n'')^2 \quad (l,n'') = 1 \pmod{n'}, \\
\deg f &\equiv \deg \tilde{Z}_p f \equiv 0 \pmod{2^q}
\end{align*}
\]

imply
\[
\deg f \equiv (l \cdot n'/n'')^2 [1 - (n')^{2^q-1}] \pmod{n' \cdot 2^q}.
\]

After considering the composition of the above mappings, we know that the self-mapping degree should be the multiplication of such numbers. Hence
\[
D(S^3/D_{n',2q}) = \{ k^2 \cdot [1 - (n')^{2^q-1}] \cdot ([1 - 2^{(2p-q)(n' - 1)}])^j | i,j,k,p \in \mathbb{Z}, q \geq p > 0 \} + n'2^q \mathbb{Z}.
\]

Case VIII:
\[
\pi_1(M) = \mathbb{Z}_m \times G, (m,|G|) = 1, G \text{ is some group of the above seven cases.}
\]

For a self mapping \( f \) of \( M \), construct the covering space of \( \tilde{G}M \) and \( \tilde{Z}_m \tilde{M} \) correspondence to subgroups \( G \) and \( Z_m \) of \( \pi_1(M) \). Similarly, consider the liftings \( \tilde{g}f : \tilde{G}M \to \tilde{G}M \) and \( \tilde{z}_m f : \tilde{Z}_m \tilde{M} \to \tilde{Z}_m \tilde{M} \). We have
\[
\begin{align*}
\deg f &\equiv \deg \tilde{G}f \equiv a \quad (a \in D(S^3/G)) \pmod{|G|} \\
\deg f &\equiv \deg \tilde{Z}_m f \equiv k^2 \quad (k \in \mathbb{Z}) \pmod{m}
\end{align*}
\]

When \( G \) is \( D_{n',2q}' \) and \( T_{b,3q}' \), the solution do not have neat expressions. Nevertheless, the set of self mapping degree of \( M \) can be always completely determined by the above equations.

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