Bouncing Universe in loop quantum gravity: full theory calculation

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Received 11 December 2022; revised 10 August 2023
Accepted for publication 21 August 2023
Published 8 September 2023

Abstract
In loop quantum gravity mathematically rigorous models of full quantum gravity were proposed. In this paper we will study a cosmological sector of one of the models describing quantum gravity with positive cosmological constant coupled to massless scalar field. In our previous research we introduced a method to reduce the model to homogeneous-isotropic sector at the quantum level. In this paper we propose a method to restrict our homogeneous-isotropic model to the spatially flat sector. After this restriction the number of degrees of freedom gets substantially reduced. This allows us to make numerical and analytical calculations. Remarkably, the resulting model shares some structural similarities with the loop quantum cosmological models and therefore sheds some new light on the relation between loop quantum gravity and loop quantum cosmology. According to our model the evolution of the Universe is periodic. The quantum gravity effects resolve the Big Bang singularity leading to a Big Bounce and cause the Universe to contract after a classical expansion phase.

Keywords: loop quantum gravity, loop quantum cosmology, Big Bounce, cosmological constant

1. Introduction

Loop quantum gravity (LQG) [1–7] is rapidly developing research field. In particular, mathematically sound quantum gravity models were proposed [8, 9] and the challenge is now to model physical phenomena. Remarkable results were obtained for symmetry reduced models, such as quantum resolution of the Big Bang singularity. In the early times most efforts were focused on quantizing models symmetry reduced at the classical level. This lead to a research
field called loop quantum cosmology (LQC) [10–13]. Recently much progress has been made to study the symmetry reduction at the quantum level [14–31]. This paper continues our efforts in the second direction. An advantage of our approach is that it not only derives a quantum cosmological model from the full theory but also contributes to our understanding of the relation between LQC and LQG.

As argued in [32–34] a proper correspondence between LQC and LQG requires an embedding of the LQC states into the LQG states such that the matrix elements of the corresponding geometric operators (in particular the quantum physical Hamiltonian) in LQC and LQG are approximately equal. Our approach is to start with a model of loop quantum gravity coupled to the massless scalar field that was proposed in [9] and derive a simple quantum cosmological model. This will be achieved by recognizing cosmological states in the full theory and defining simple Hamiltonian operator which matrix elements between our states are approximately equal to the matrix elements of the full theory Hamiltonian.

Our approach is based on the symmetry reduction of states that we developed in [31]. In contrast to [14–18] we did not restrict the space of connections but rather the graphs, spin labels, and intertwiners. We restricted the Hilbert space to a cubical lattice with loops—in addition to the sides of the cubical lattice, we included also links starting and ending at the same node, called loops. Such loops are created and annihilated by the gravitational part of the scalar constraint. We constructed homogeneous-isotropic states by averaging over discrete translations and discrete rotations that are symmetries of the cubical lattice. The quantum Hamiltonian proposed in [35–38] commutes with the symmetries and therefore descends to the symmetry reduced space. We introduced a cut-off in the number of loops in order to make numerical calculations. We observed that after the symmetry reduction the rank of the truncated matrix gets approximately 24 times smaller. On the one hand, this opened a possibility of studying the model further using the same computing resources. On the other hand, the reduction of the degrees of freedom was still very small compared to the drastic reduction at the classical level. In this paper we will extract the zero intrinsic curvature sector of our homogeneous-isotropic model. This restriction will be done approximately using the Livine–Speziale coherent states. It leads to substantial reduction of the degrees of freedom at the quantum level and allows numerical and analytical study of the model.

In this paper we will consider spin-networks in which all links are labeled with spin \( \frac{1}{2} \). The quantum Hamiltonian from [35–38] restricted to the spaces of monochromatic spin \( \frac{1}{2} \) intertwiners has a property which makes it very similar to the quantum Hamiltonian in Loop Quantum Cosmology. In [39] we noticed that the Rovelli–Smolin volume operator [40] restricted to the spaces of monochromatic spin \( \frac{1}{2} \) intertwiners is proportional to identity:

\[
\hat{V} = \frac{\kappa_0}{8} \left( \frac{8\pi G \hbar}{c^3} \right)^{\frac{1}{2}} \sqrt{\frac{\sqrt{3}}{3!}} (N - 2)N(N + 2) \cdot \mathbf{1}.
\]

The proportionality factor depends only on the valence of the intertwiner space \( N \). There are three terms in the gravitational part of the scalar constraint [35–38]: one adding a loop, one removing a loop and one which does not change the graph. Adding a loop increases the valence of the intertwiner by 2 and removing a loop decreases the valence by the same number. As a result, the gravitational part of the scalar constraint, in both loop quantum cosmology and in loop quantum gravity models proposed in [35–38], acting on a volume eigenvector produces a state which is a linear combination of three volume eigenstates: one with a higher value,
one with a lower volume and one with the same volume as the original state. This analogy will allows us to build a quantum Hamiltonian in our model which resembles the LQC Hamiltonian.

1.1. Outline of the paper

Section 2 is a short review of the loop quantum gravity model used in this paper. We restrict to cubical lattices with loops but allow them to be very fine (we control the lattice refinement with a parameter \( L \)). We assume that all links of the lattice are labeled with spin \( \frac{1}{2} \). In section 3 we construct our coherent states. They are suitably chosen Livine–Speziale coherent states projected onto the homogeneous-isotropic states introduced in [31]. In section 4 we show that the states describe flat geometries. Our states depend only on the number of loops \( k \) at a node and the refinement parameter \( L \) (since the state is homogeneous-isotropic, the number of loops does not depend on the node)—we will denote them by \( |k, L\rangle \). The resulting states are eigenstates of the volume operator. A volume of a node of the lattice is proportional to \( k^2 \). This allows us to expect that the interpretation of \( k \) is in terms of the cosmological variable \( p \) (proportional to the square of the cosmological scale factor). In section 6.2 we settle this correspondence. In section 6.1 we define the Hamiltonian of our model by

\[
\hat{H} = \sqrt{|-2\hat{\Theta}_\Lambda|} +, \quad (2)
\]

where \( |\cdot|_+ \) denotes a positive part of an operator and the matrix elements of \( \hat{\Theta}_\Lambda \) are:

\[
\langle k', L | \hat{\Theta}_\Lambda | k, L \rangle \approx \langle k', L | \sqrt{qC^E} | k, L \rangle.
\]

Thanks to this property our Hamiltonian satisfies a requirement of the proper correspondence between LQC and LQG from [32–34]. In section 6.3 we calculate the effective dynamics and show that the model predicts a Big Bounce and a recontraction phase. Since the paper is technical we summarize the main steps in section 7. We end with discussion and outlook (section 8).

In the derivation we made the following assumptions:

- We restricted to lattices with loops. This can be considered as a cut-off of the full theory similar to lattice approaches to quantum field theory (let us note similar restriction has been proposed for example in [41]).
- We fixed each spin to be equal \( \frac{1}{2} \). Similar assumption was made in loop quantum cosmology [11]. This choice is motivated by the fact that we expect that a physically realistic scenario will involve very refined lattice such that each node has a very small volume. An additional advantage of this assumption is that it simplifies our calculations. In particular, we use our result from [39] stating that the Rovelli–Smolin volume operator [40] restricted to the spaces of monochromatic spin \( \frac{1}{2} \) intertwiners is proportional to identity (see (1)).
- We chose specific states formed from certain Livine–Speziale coherent intertwiners and projected them further onto the space of homogeneous-isotropic states proposed in [31]. The choice of the Livine–Speziale coherent states is motivated by successful applications of such states in the spin-foam theory [42–47] in particular in spin-foam cosmology [48–51] and canonical loop quantum gravity (via the Thiemann’s coherent states) [28, 29, 52]. They have a clear geometrical interpretation in terms of polyhedra [53]. The projection onto the space of homogeneous-isotropic states proposed in [31] guarantees that the resulting states are homogeneous-isotropic. Since we fixed each spin to be equal \( \frac{1}{2} \) the Livine Speziale coherent intertwiners used in this paper are monochromatic spin \( \frac{1}{2} \) intertwiners. Thanks to the result [39], we know that they are eigenstates of the Rovelli–Smolin volume operator [40].
We neglected the Lorentzian part of the scalar constraint, because in section 4 we showed that our states describe (approximately) flat geometries. The higher the number of loops \( k \), the better the approximation.

We defined the dynamics of the cosmological model using the Hamiltonian for which the relation (2) holds only approximately. The higher the number of loops \( k \), the better the approximation.

In our calculations we restrict to the leading order in the number of loops \( k \). The low number of loops corrections can be treated as the next to the leader order quantum gravity corrections. In similar manner the low spin corrections are treated as the next to the leading order quantum gravity corrections in the graviton propagator calculations from [54] and in the spin-foam framework [55–60].

Let us notice that there is a limitation for the value of \( k \) corresponding to the current state of the Universe, because a volume of a node is proportional to \( k^2 l_P^3 \) (where \( l_P \) is the Planck length) and we assume that the space is formed from a large number of small cubes. The number of loops \( k \) should be small enough for the volume of a cube to be well below the size of a quark. Since our approximations work well already for moderately high values of \( k \), this can be easily achieved in our approach.

2. Quantum gravity model

Our starting point is the quantum gravity model described in [9, 36]. In this section we will recall its mathematical formulation which will closely follow the presentation in [31].

2.1. Kinematical Hilbert space of loop quantum gravity

In the construction of the kinematical Hilbert space of loop quantum gravity \( \mathcal{H}_{\text{kin}} \) fundamental role is played by graphs embedded in the spatial manifold \( \Sigma \) which are build from 1-dimensional submanifolds (intervals or circles), called links, that intersect at points of the manifold. Such intersection points are called nodes. In our paper we will denote by \( \text{Links}(\Gamma) \) the set of the links of a graph \( \Gamma \) and by \( \text{Nodes}(\Gamma) \) the set of the nodes of the graph \( \Gamma \):

\[
\text{Links}(\Gamma) = \{ \ell_1, \ldots, \ell_N \}, \quad \text{Nodes}(\Gamma) = \{ n_1, n_2, \ldots, n_M \}.
\]  

The states in \( \mathcal{H}_{\text{kin}} \) are cylindrical functions of SU(2) connections on spatial manifold \( \Sigma \). A function \( \Psi \) is called cylindrical with respect to a graph \( \Gamma \) (embedded in \( \Sigma \)) if there is suitably regular function \( \psi : \text{SU}(2)^N \rightarrow \mathbb{C} \) such that

\[
\Psi(A) = \psi(A_{\ell_1}, \ldots, A_{\ell_N}),
\]

where \( A_\ell \) is a holonomy of the connection \( A \) along the link \( \ell \). If a function \( \Psi \) is cylindrical with respect to a graph \( \Gamma \) and a function \( \Psi' \) is cylindrical with respect to a graph \( \Gamma' \) then it is always possible to find a graph \( \Gamma'' \) such that both \( \Psi \) and \( \Psi' \) are cylindrical with respect to it. Using this property, it is possible to define a scalar product in the space of cylindrical functions:

\[
\langle \Psi | \Psi' \rangle = \int du_1 \ldots du_{N''} \overline{\Psi}(u_1, \ldots, u_{N''}) \psi'(u_1, \ldots, u_{N''}),
\]

where \( N'' \) is the number of links in \( \Gamma'' \). The kinematical Hilbert space is the Cauchy completion of the space of cylindrical functions with respect to the scalar product (5):

\[
\mathcal{H}_{\text{kin}} := \overline{\text{Cyl}}.
\]
2.2. Vertex Hilbert space

In the model that we consider, the quantum constraint operators are defined on the so-called vertex Hilbert space, which is a space of solutions to the Gauss constraint and partial solutions to the vector constraint.

The Gauss constraint generates SU(2) gauge transformations. Every such gauge transformation \( g : \Sigma \to SU(2) \) induces a transformation of the cylindrical functions

\[
\Psi(A) \mapsto \Psi(gAg^{-1} + g^{-1} dg).
\]  

(7)

The space of solutions to the Gauss constraint \( \mathcal{H}^G \) is the Cauchy completion of the space \( \text{Cyl}^G \) of cylindrical functions invariant under the action of all SU(2) gauge transformations:

\[
\mathcal{H}^G = \overline{\text{Cyl}^G}.
\]  

(8)

Let us denote by \( \mathcal{H}_{\Gamma}^G \) the subspace of \( \mathcal{H}^G \) formed by all gauge invariant functions cylindrical with respect to \( \Gamma \).

The vector constraint generates diffeomorphisms of \( \Sigma \). Every analytic diffeomorphism \( f \in \text{Diff}(\Sigma) \) defines a unitary operator \( U_f : \mathcal{H}_{\text{kin}} \to \mathcal{H}_{\text{kin}} \):

\[
U_f \Psi(A) = \Psi(f^* A).
\]  

(9)

Let us consider a graph \( \Gamma \), the diffeomorphisms \( \text{Diff}^{\text{Nodes}}(\Gamma) \) that act trivially on the nodes of the graph \( \Gamma \) and a set \( \text{TDiff}_\Gamma \) formed by diffeomorphisms acting trivially on \( \mathcal{H}_{\Gamma}^G \). We will consider a rigging map \( \eta \)

\[
\mathcal{H}_{\Gamma} \ni \Psi \mapsto \eta(\Psi) \in \text{Cyl}^* \]  

(10)

defined by

\[
\eta(\Psi) := \frac{1}{N_\Gamma} \sum_{[f] \in \text{Diff}^{\text{Nodes}}(\Gamma) / \text{TDiff}_\Gamma} \langle U_f \Psi \rangle.
\]  

(11)

The factor \( N_\Gamma \) can be arbitrary but it is convenient to set it equal to order of the group

\[
\text{Sym}_\Gamma = \{ f \in \text{Diff}^{\text{Nodes}}(\Gamma) : f(\Gamma) = \Gamma \} / \text{TDiff}_\Gamma
\]  

(12)

of symmetries of the graph \( \Gamma \) fixing each node of the graph:

\[
N_\Gamma = \#\text{Sym}_\Gamma.
\]  

(13)

Let us notice that \( \eta(\Psi) \) is invariant under the action of the diffeomorphisms \( \text{Diff}^{\text{Nodes}}(\Gamma) \) that have trivial action on the nodes of the graph \( \Gamma \). The map \( \eta \) is defined for any graph \( \Gamma \) and extended by linearity to the whole space \( \text{Cyl}^G \). The vertex Hilbert space is the Cauchy completion of \( \eta(\text{Cyl}^G) \):

\[
\mathcal{H}_{\text{vtx}} = \overline{\eta(\text{Cyl}^G)}
\]  

(14)

under the scalar product:

\[
\langle \eta(\Psi) | \eta(\Psi') \rangle := \eta(\Psi)(\Psi').
\]  

(15)

Let us denote by \( \text{FS}(\Sigma) \) the set of finite subsets of \( \Sigma \). The space \( \mathcal{H}_{\text{vtx}} \) has an orthogonal decomposition:

\[
\mathcal{H}_{\text{vtx}} = \bigoplus_{V \in \text{FS}(\Sigma)} \mathcal{H}_V^G
\]  

(16)

where \( \mathcal{H}_V^G \) is the subspace formed by states invariant under the action of the group \( \text{Diff}_V \) of diffeomorphism that act trivially on the set \( V \in \text{FS}(\Sigma) \).
2.3. Scalar field Hilbert space

We consider a model coupled to the massless scalar field. The scalar field Hilbert space will be a polymer Hilbert space studied in [61], which is a direct sum

$$\mathcal{H}^{\text{mat}}_V = \bigoplus \mathcal{H}^{\text{mat}}_V.$$  \hspace{1cm} (17)

A space $\mathcal{H}^{\text{mat}}_V$ is spanned by functionals $|\pi\rangle$ of a real-valued scalar field $\phi : \Sigma \rightarrow \mathbb{R}$ each defined by a function $\pi : \Sigma \rightarrow \mathbb{R}$ with finite support $\text{supp}(\pi) = \{x_1, \ldots, x_n\}$:

$$|\pi\rangle(\phi) := U_\pi(\phi) = e^{i(\pi(x_1)\phi(x_1) + \ldots + \pi(x_n)\phi(x_n))}.$$  \hspace{1cm} (18)

The polymer Hilbert $\mathcal{H}^{\text{mat}}_V$ is equipped with a scalar product:

$$\langle \pi | \pi' \rangle = \delta_{\pi, \pi'},$$

where $\delta$ is the Kronecker delta. The states are eigenvectors of the momentum operator

$$\hat{\pi}(V)|\pi\rangle = \sum_{x \in V} \pi(x) |\pi\rangle.$$  \hspace{1cm} (19)

2.4. The scalar constraint operator of gravity and massless scalar field

We consider the Hilbert space

$$\mathcal{H}^{\text{tot}} = \bigoplus V \mathcal{H}_V,$$  \hspace{1cm} (20)

where

$$\mathcal{H}_V = \mathcal{H}^{\text{mat}}_V \otimes \mathcal{H}^{\text{gr}}_V.$$  \hspace{1cm} (21)

The dynamics of the model is given by the scalar constraint:

$$\hat{C}_x = \frac{1}{2} \hat{\pi}_x^2 + \sqrt{4\pi \hat{C}^G}.$$  \hspace{1cm} (22)

The operator $\hat{C}^G$ is the gravitational part of the scalar constraint operator. It can be written in terms of the so-called Lorentzian part $\hat{C}_L$, Euclidean part $\hat{C}_E$ and the cosmological constant term:

$$\sqrt{4\pi \hat{C}^G} = - \frac{E_P l_P^3}{16\pi \gamma^2} \left( (1 + \gamma^2) \hat{C}_L + \frac{1}{2} \left( \hat{C}_E^\dagger + \hat{C}_E \right) \right) \pm \frac{\Lambda E_P}{8\pi l_P} V^2.$$  \hspace{1cm} (23)

In the formula above: $E_P$ is the Planck energy, $l_P$ is the Planck length, $\gamma$ is the Barbero–Immirzi parameter, $\Lambda$ is the cosmological constant. Let us recall that $V$ is the volume operator. We use the operators $\hat{C}_L$ and $\hat{C}_E$ from [36] and the operator $V$ from [40]. The necessity to use the vertex Hilbert space comes from the fact that $\hat{C}_E$ is defined on such space. The operator $\hat{C}_E^\dagger$ adds loops tangential to two different links of the graph and the operator $\hat{C}_E$ is its hermitian conjugate and removes such loops (the operators $\hat{C}_L, \hat{C}_E$ are graph changing). The new loop is labelled with fixed representation label (in our case it will be $\frac{1}{2}$). The operator $\hat{C}_L$ neither changes the graph nor the representation labels (but acts non-trivially in the intertwiner spaces). In this aspect it is similar to the volume operator $V$.

The solutions to the scalar constraint are of the following form [9]:

$$\Psi(A, \phi) = \exp \left( \frac{i}{E_P l_P} \sum_{x \in \Sigma} \sqrt{\left| 2\sqrt{4\pi \hat{C}^G}_x + \phi(x) \right|} \Psi^{ER}(A) \right),$$  \hspace{1cm} (24)
where $\Psi^\Gamma(A)$ is an element of the vertex Hilbert space and $|\cdot|_+$ denotes the positive part of an operator.

2.5. Restriction to the lattice with loops

In order to make practical calculations we continue our approach started in [31] and restrict the theory to cubical lattices with loops. This subspace was build by considering a cubical lattice $\Gamma$ and adding loops (links starting and ending at a node) tangential to two different sides of the lattice. We will assume that the nodes of our lattice $\Gamma$ are given by $(\frac{q}{L}, \frac{r}{L}, \frac{s}{L})$ for $q, r, s \in \mathbb{Z}, L \in \mathbb{N}, L \geq 2$. The parameter $L$ will control the refinement of our lattice. Let us denote by $\ell_1, \ldots, \ell_6$ the links of the graph $\Gamma$ meeting at the node $n$—the six sides of the cubical lattice at the node $n$ (see figure 1). Since $\Gamma$ is a graph and a lattice at the same time we will call the links of $\Gamma$ the sides. This will make it easier to distinguish them from the loops which will be added to the lattice. Let us consider a map:

$$I_0 : \{ (I, J) : I < J, \epsilon(\hat{\ell}_I, \hat{\ell}_J) \neq 0 \} \rightarrow \mathbb{N},$$

where

$$\epsilon(\hat{\ell}_I, \hat{\ell}_J) = \begin{cases} 0, & \text{if the vectors } \hat{\ell}_I, \hat{\ell}_J \text{ are colinear}, \\ 1, & \text{otherwise.} \end{cases}$$

In the formula above we denoted by $\hat{\ell}$ a vector tangent to link $\ell$ at the node $n$. We will call $I_n$ a loop configuration at the node $n$. We will denote by $\mathcal{L}(I_n)$ the total number of loops at the node $n$:

$$\mathcal{L}(I_n) = \sum_{I,J} I_n(I,J).$$

Let us denote by $\Gamma_1$ a graph obtained from $\Gamma$ by adding at each node $\mathcal{L}(I_n)$ loops such that there are $I_n(\ell_I, \ell_J)$ loops tangential to links $\ell_I$ and $\ell_J$—see figure 1. In [31] we made an additional assumption about tangentiality orders of the loops to the sides of the lattice to make the loops tangent to links $\ell_I$ and $\ell_J$ distinguishable (for each $I, J$) after an averaging with respect to all diffeomorphisms fixing each node of the graph $\Gamma$. Next, we considered spaces $\mathcal{H}^{\text{loop}}_{\Gamma, j}$ invariant under the action of the gravitational part of the scalar constraint. Such spaces are spanned by spin networks $(\Gamma_1, \rho, \iota)$ such that:

- $\rho_\ell$ are SU(2) representations $\forall \ell \in \text{SU}(2) \rho_\ell(u) : \mathcal{H}_{\rho_\ell} \rightarrow \mathcal{H}_{\rho_\ell}$ such that $\rho_\ell \equiv \rho_{\ell'}$ for the sides of the lattice $\Gamma$ and $\rho_\ell$ is some fixed representation $\rho(u)$ for each loop of $\Gamma_1$,
- $\iota$ are tensors invariant under the action of the SU(2) group (intertwiners):

$$\iota_n \in \mathcal{H}_{\iota} : \text{Inv} \left( \mathcal{H}_{\rho_{\ell_1}} \otimes \cdots \otimes \mathcal{H}_{\rho_{\ell_M}} \otimes \mathcal{H}_{\rho_{\ell_{M+1}}} \otimes \cdots \otimes \mathcal{H}_{\rho_{\ell_N}} \right),$$

where links $\ell_1, \ldots, \ell_M$ are incoming to the node $n$ and links $\ell_{M+1}, \ldots, \ell_N$ are outgoing.

Each space $\mathcal{H}^{\text{loop}}_{\Gamma, j}$ has the (standard) spin-network decomposition (following from the Peter–Weyl theorem) [1]:

$$\mathcal{H}^{\text{loop}}_{\Gamma, j} \cong \bigoplus_{n \in \text{Nodes}(\Gamma)} \mathcal{H}_n.$$

We exchange the tensor product and direct sum:

$$\mathcal{H}^{\text{loop}}_{\Gamma, j} \cong \bigotimes_{n \in \text{Nodes}(\Gamma)} \bigoplus_{\ell} \mathcal{H}_n.$$
This trick is an important technical and conceptual step. In the standard spin-network picture, to each node of the graph there is assigned a space of invariant tensors $\mathcal{H}_{l_n}$, in our picture to each node of the graph there is assigned a tower

$$\mathcal{H}_{\Gamma,j,n}^{\text{loop}} := \bigoplus_{l_n} \mathcal{H}_{l_n}$$

of spaces of intertwiners. The standard picture requires further to consider a direct sum over possible graphs, in our picture this is incorporated in the tower $\bigoplus_{l_n} \mathcal{H}_{l_n}$. This reinterpretation has import applications. Firstly, this allows us to consider the quantum gravitational part of the scalar constraint at each node separately. For example, one can diagonalize the quantum operator at each node separately and obtain a complete eigenstate by tensor multiplying eigenstates calculated for the nodes. Secondly, it allows to incorporate homogeneity by restricting to elements of $\mathcal{H}_{\Gamma,j,n}^{\text{loop}}$. In order to implement homogeneity and isotropy, we restrict also the representation labels $\rho_{l_n}$ to be all the same for each side of the lattice. In fact in this paper we will fix this representation label to be the spin $\frac{1}{2}$ representation and we will choose a regularization of the gravitational part of the scalar constraint such that $\rho_{l_n}$ is also the spin $\frac{1}{2}$ representation.

Let us underline a subtle difference between the representations of the spin-network decomposition of $\mathcal{H}_{\Gamma,j,n}^{\text{loop}}$ from [31] (illustrated on figure 1(a)) and the spin-network decomposition of $\mathcal{H}_{\Gamma,j,n}^{\text{loop}}$ in this paper (illustrated on figure 1(b)). In this paper we use an equivalent spin network (in the sense of cylindrical equivalence—see for example [1]) such that each loop is divided into two links by adding a bi-valent node and all links at the original nodes (at least 6-valent) are outgoing. In [31] we chose a spin $j$ labeling the sides of the cubical lattice and a spin $l$ labeling the loops. Let us recall that in this paper we limit to the case $j = l = \frac{1}{2}$ and we will denote by $\mathcal{H}_{l_n}$ the spin $\frac{1}{2}$ representation space of the SU(2) group. Let us notice that in the neighbourhood of the node $n$ there are $2\Omega(l_n)$ bivalent nodes and one node of valence $2\Omega(l_n) + 6$ (see figure 1). With the node $n$ and the loop configuration $l_n$ we associate a space of intertwiners:

$$\mathcal{H}_{l_n} = \text{Inv} \left( \mathcal{H}_{l_n} \otimes \ldots \otimes \mathcal{H}_{l_n} \right) \otimes \text{Inv} \left( \mathcal{H}_{l_n}^* \otimes \mathcal{H}_{l_n}^* \right) \otimes \ldots \otimes \text{Inv} \left( \mathcal{H}_{l_n}^* \otimes \mathcal{H}_{l_n}^* \right).$$

The Hilbert spaces $\text{Inv} \left( \mathcal{H}_{l_n}^* \otimes \mathcal{H}_{l_n}^* \right)$ correspond to bivalent nodes and the Hilbert space

$$\text{Inv} \left( \mathcal{H}_{l_n} \otimes \ldots \otimes \mathcal{H}_{l_n} \right)$$

(31)

corresponds to the node of the original cubical lattice (with loops).

In this paper we will fix a node of the cubical lattice $n$ and omit the subscript $n$ in $l_n$. A convenient basis of our space $\mathcal{H}_{\Gamma,j}^{\text{loop}}$ is of the form

$$|l,\iota\rangle,$$

(32)

where $l$ runs through set of loop configurations and $\iota$ runs through a basis of the intertwiners $\mathcal{H}_{l_n}$. In particular:

$$\langle l',\iota' | l,\iota \rangle = \delta_{l',l} \langle \iota' | \iota \rangle.$$
Figure 1. A loop configuration $l_n$ encodes the information about the loops tangent to the sides of the cubical lattice $\ell_1, \ldots, \ell_6$. On this figure we drew two equivalent neighborhoods of a node $n$ corresponding to loop configuration $l_n$ such that $l_n(1, 3) = 1, l_n(2, 5) = 2, l_n(I, J) = 0$ if $(I, J) \notin \{(1, 3), (2, 5)\}$. In this paper we prefer to use the representation from figure (b). We denote by $\mathcal{L}(l_n)$ the total number of loops at the node $n$. In this example $\mathcal{L}(l_n) = 3$. (a) A neighbourhood of the node $n$ of a lattice with loops corresponding to $l_n$. This representation was used in [31]. (b) A neighborhood of the node $n$ of an equivalent lattice with loops corresponding to $l_n$. This representation is used in this paper.

3. Homogeneous-isotropic coherent states

In this section we will define states that will form our basis of the cosmological Hilbert space. We will construct them using the Livine–Speziale coherent states. The resulting states, called homogeneous-isotropic coherent states, will lie in our space of homogeneous-istropic space constructed in [31].

3.1. Coherent intertwiners

Our coherent states are built from Livine–Speziale coherent states. The Livine–Speziale coherent intertwiner was introduced in [42]. It is an element of the space $\text{Inv}(\mathcal{H}_{j_1} \otimes \cdots \otimes \mathcal{H}_{j_N})$ defined by a set of normalized vectors $\{\vec{n}_1, \ldots, \vec{n}_N\}$. It will be denoted by $|j_1, \ldots, j_N; \vec{n}_1, \ldots, \vec{n}_N\rangle$. It is obtained by projecting a tensor product of Perelomov coherent states $|j_i, \vec{n}_i\rangle$ [62] onto the subspace of SU(2) invariant tensors. Let us denote by $\rho_j$ a unitary irreducible representation of the SU(2) group acting in the space $\mathcal{H}_j$. The Livine–Speziale coherent state is:

$$|j_1, \ldots, j_N; \vec{n}_1, \ldots, \vec{n}_N\rangle = \int_{SU(2)} du \prod_{i=1}^{N} \rho_{j_i}(u) \otimes |j_i, \vec{n}_i\rangle. \quad (34)$$

There are two types of nodes in the graph: the nodes that are also nodes of the lattice with loops (they are at least 6-valent) and the nodes that are obtained by splitting loops into two links (they are at 2-valent). For each $l$ we will construct a tensor $\iota_l$ which is a tensor product
of some specific Livine–Speziale coherent intertwiners at the 2-valent nodes and a specific Livine–Speziale coherent intertwiner at the at least 6-valent node \( n_0 \):

\[
\iota_1 = \left( \bigotimes_{n, \text{Valence}(n)=2} \iota_n^1 \right) \otimes \iota_{n_0}.
\]

(35)

We will call \( \iota_1 \) a coherent intertwiner. In the next subsections we will describe the intertwiners \( \iota_n \) and \( \iota_{n_0} \).

3.1.1. At least 6-valent node. Let us consider a small neighbourhood of at least 6-valent node \( n_0 \) of the graph. This neighbourhood is isomorphic with a small neighborhood of the graph \( \tilde{\Gamma} \) in \( \mathbb{R}^3 \). Therefore the possible directions of vectors tangent to the links of the graph at the node \( n_0 \) are \( \partial_x, -\partial_x, \partial_y, -\partial_y, \partial_z, -\partial_z \). To each link \( \ell \) at the node \( n_0 \) there corresponds a Perelomov coherent state \( |1/2, \hat{\ell} \rangle \), where \( \hat{\ell} \) is the vector tangent to \( \ell \) at the node \( n \). Therefore the tensor assigned to the node \( n_0 \) is defined by Livine–Speziale coherent intertwiner:

\[
|l, \vec{n}_1 \ldots \vec{n}_6 \rangle := \int_{\text{SU}(2)} \text{d}u \rho_{1/2}(u)^{\otimes(6)} \otimes |1/2, \vec{n}_j \rangle^{\otimes N_l},
\]

(36)

where \( N_l = \sum_{j=1}^{l-1} l(J, I) + \sum_{j=l+1}^{6} l(I, J) + 1 \) is the number of links starting at the node \( n_0 \) whose tangent direction is \( \vec{n}_l \),

\[
\vec{n}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{n}_2 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{n}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{n}_4 = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \quad \vec{n}_5 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \vec{n}_6 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}.
\]

(37)

In our notation,

\[
|1/2, \vec{n}_j \rangle^{\otimes N_l} := \underbrace{|1/2, \vec{n}_1 \rangle \otimes \cdots \otimes |1/2, \vec{n}_j \rangle}_{N_l},
\]

(38)

It is straightforward to notice that \( |1/2, \vec{n} \rangle^{\otimes N} \) is the same state as the Perelomov coherent state with spin \( N/2 \) and direction vector \( \vec{n} \) (under the standard isomorphism mapping spinors into the corresponding states in the magnetic number basis):

\[
|1/2, \vec{n} \rangle^{\otimes N} = |N/2, \vec{n} \rangle.
\]

(39)

Under this isomorphism we have:

\[
|l, \vec{n}_1 \ldots \vec{n}_6 \rangle = |j_1, \ldots, j_6; \vec{n}_1 \ldots \vec{n}_6 \rangle,
\]

(40)

where \( j_i = N_i/2 \) and \( |j_1, \ldots, j_6; \vec{n}_1 \ldots \vec{n}_6 \rangle \) is the Livine–Speziale coherent intertwiner:

\[
|j_1, \ldots, j_6; \vec{n}_1 \ldots \vec{n}_6 \rangle = \int_{\text{SU}(2)} \text{d}u \bigotimes_{l=1}^{6} \rho_{j_l}(u) |j_l, \vec{n}_l \rangle.
\]

(41)

We will assign to the node \( n_0 \) the normalized Livine–Speziale intertwiner:

\[
\iota_{n_0} = \frac{1}{\sqrt{|l, \vec{n}_1 \ldots \vec{n}_6 |l, \vec{n}_1 \ldots \vec{n}_6 \rangle}} |l, \vec{n}_1 \ldots \vec{n}_6 \rangle.
\]

(42)

3.1.2. Two-valent nodes. The 2-valent nodes are added to the graph to split the loops of our lattice with loops. As a result, the node is an endpoint of two links which are tangent to two different sides of the lattice \( \ell_l \) and \( \ell_f \). We can associate with such node an intertwiner \( \iota_n^1 \).
where \( t_n \) is a normalized Livine–Speziale coherent intertwiner. Strictly speaking, we consider a Livine–Speziale coherent intertwiner:

\[
|1/2, 1/2; \vec{n}_i, \vec{n}_j\rangle := \int_{\text{SU}(2)} du \rho_{1/2}(u) \otimes \rho_{1/2}(u) |1/2, \vec{n}_i\rangle \otimes |1/2, \vec{n}_j\rangle. \tag{43}
\]

In the formula above the vectors \( \vec{n}_i \) and \( \vec{n}_j \) are two vectors parallel to the sides of the lattice \( \ell_i \) and \( \ell_j \). We fix the ambiguity in the definition of the state by requiring that \( I < J \). As previously, we will associate with the node an intertwiner

\[
\hat{\text{SU}}(2) d u \rho_{1/2}(u) \rho_{1/2}(u) |1/2, \vec{n}_i\rangle \otimes |1/2, \vec{n}_j\rangle. \tag{44}
\]

In appendix A.1 we calculate the explicit form of the 2-valent intertwiner:

\[
\iota_{AB} = \epsilon(\vec{n}_i, \vec{n}_j) \epsilon^{AB}. \tag{45}
\]

### 3.2. Homogeneous-isotropic states

It will be convenient to introduce a notation

\[
|l\rangle := |I, t_l\rangle. \tag{46}
\]

Our homogeneous-isotropic states are obtained by averaging states \(|l\rangle\) over the group of orientation preserving symmetries of a cube \(O_{\text{cube}}\). Let us denote by \(O_l\) the orbit of the action of the group \(O_{\text{cube}}\) on a loop configuration \(l\). The homogeneous-isotropic state corresponding to \(|l\rangle\) is:

\[
||l|| = \frac{1}{\sqrt{\#O_l}} \sum_{l' \in O_l} |l'\rangle. \tag{47}
\]

Thanks to the numerical factor in front, the states are normalized. Let us denote by \(S_l\) the stabilizer group of \(O_{\text{cube}}\) with respect to \(l\). Using the transformation properties of the states \(|l\rangle\) discussed in appendix A.2 the formula above can be written using the group action explicitly:

\[
||l|| = \frac{1}{\sqrt{\#O_l \#S_l}} \sum_{g \in O_{\text{cube}}} U_g |l\rangle = \frac{1}{\sqrt{24 \#S_l}} \sum_{g \in O_{\text{cube}}} U_g |l\rangle. \tag{48}
\]

### 3.3. The basis states

We will further restrict the space to states which are defined by the number of loops only

\[
|k\rangle := ||l_k||, \tag{49}
\]

where \(l_k\) is loop configuration defined in the following way. Let us split \(k\) into a quotient \(n\) and a remainder \(r\) with respect to division by 3: \(k = 3n + r\), where \(n, r \in \mathbb{N}\) and \(r < 3\). We consider three cases:

- \(r = 0\). In this case:
  \[
l_k(1, 3) = l_k(2, 6) = l_k(4, 5) = n. \tag{50}
  \]
- \(r = 1\). In this case:
  \[
l_k(1, 3) = n + 1, \quad l_k(2, 6) = l_k(4, 5) = n. \tag{51}
  \]
• $r = 2$. In this case:

$$l_k(1, 3) = l_k(2, 6) = n + 1, \quad l_k(4, 5) = n.$$  \hspace{1cm} (52)

All the remaining $l(I, J)$ are 0.

The space of such states will be denoted by $\mathcal{H}_{\text{flat}}$.

4. Internal flatness of the quantum geometry

In order to investigate the internal geometry of the states proposed in the previous section, we will study the expectation values of the area and angle operators. We will consider a small cube whose center is at the node $n_0$ and such that:

• the outward pointing normals to the faces of the cube are $\vec{n}_1, \ldots, \vec{n}_6$,
• each link intersects (transversally) only one face of the cube—the face with the outward normal pointing in the same direction as the vector tangent to the link at the node $n_0$.

We will denote the faces of the cube by $S_I, I = 1, \ldots, 6$.

We will argue that the homogeneous-isotropic states proposed in [31] have the property that the expectation values of the area operator is the same for any face of the cube and the dihedral angle operator is the same for any pair of non-parallel faces of the cube. For the states proposed in this paper, this dihedral angle will be approximately $\frac{\pi}{2}$ and will reach this value in the semi-classical limit (large volume limit). As a result the corresponding quantum polyhedron (see [53] for a discussion of quantum polyhedra) is a quantum cube. Therefore the (semi-classical) deficit angle (in the sense of Regge calculus) corresponding to the quantum geometry is zero. This argument allows us to interpret the space of states considered in this paper as a subspace of homogeneous-isotropic states corresponding to internally flat geometries.

4.1. Areas

We refer our reader to [1] for a detailed presentation of the area operator in loop quantum gravity. In this paper we consider the particular case described in the introductory part of this section. For the calculations in this paper it is enough to know some general properties of this operator. First property is that the state $|l\rangle$ is an eigenstate of the area operators $\hat{A}_I$ corresponding to the surfaces $S_I$. Second property is that the eigenvalue of the operator $\hat{A}_I$ corresponding to state $|l\rangle$ is equal to the number of times $S_I$ is punctured by the links of the graph times the so-called area gap $\Delta l_I^2$ [11], where $\Delta = 4\sqrt{3}\pi \gamma$, $\gamma$ is the Barbero–Immirzi parameter:

$$\hat{A}_I|l\rangle = \left(\text{number of times } S_I \text{ is punctured by the links of the graph}\right) \cdot \Delta l_I^2 |l\rangle.$$  \hspace{1cm} (53)

The second property is a consequence of our choice to label all links of the graph with spin $\frac{1}{2}$. Let us notice that the right-hand side of (53) depends on $l$ through the graph (in fact, only local information about the loop configuration at the node $n_0$ is needed). Using these properties, we can calculate the expectation value of the area operator $\hat{A}_I$ corresponding to the surface $S_I$ in the state $|l\rangle$:
\[
\langle |l| \hat{A}_l |l \rangle = \frac{1}{24 \# S_6} \sum_{g,g' \in O_{cube}} \langle l | U_g^t \hat{A}_l U_{g'} | l \rangle = \frac{1}{24} \sum_{g \in O_{cube}} \langle l | U_g^t \hat{A}_l U_g | l \rangle \\
= \frac{1}{6} \sum_{l=1}^6 \langle l | \hat{A}_l | l \rangle = \frac{1}{6} (2\Sigma(l) + 6) \cdot 4\sqrt{3} \pi \gamma \vec{P}^2.
\] (54)

The second equality comes from the fact that \(|l|\) is an eigenstate of \(\hat{A}_l\) and from the orthogonality condition (33). The third equality comes from the fact that \(A_l\) commutes with any transformation that does not change the links intersecting \(S_l\) and their spin labels, in particular with any element of \(O_{cube}\) that leaves \(\vec{n}_l\) invariant (\(\# S_6 = 4\)). The last equality comes from the fact that \(\sum_{l=1}^6 \hat{A}_l\) is the operator corresponding to a quantum measurement of the total surface area of the cube (i.e. an area of the surface formed by all faces \(S_l\)). Since each link is labelled with spin \(\frac{1}{2}\), each puncture contributes with area \(4\sqrt{3} \pi \gamma \vec{P}^2\). As a result the total surface area of the cube is equal to \(4\sqrt{3} \pi \gamma \vec{P}^2\) times \(2\Sigma(l) + 6\). We interpret the formula in the following way: each side of the lattice contributes in average with an area \(\Delta l^2 := 4\sqrt{3} \pi \gamma \vec{P}^2\) and each loop contributes in average with an area \(\frac{1}{2} \Delta l^2 = \frac{3}{2} \Delta l^2\) (each loop punctures the surface \(S_l\) in average \(\frac{1}{2}\) times—a loop is tangent to two different sides of the lattice and therefore appears in this formula 2 times and the average is over six sides of the cube, which leads to the factor \(\frac{3}{2}\) in front).

Let us notice that from (54) it follows that \(\langle |l| \hat{A}_l |l \rangle\) does not depend on \(l\). This is a sign of isotropy of the quantum geometry. Therefore, we can introduce a notation:

\[
A(l) := \langle |l| \hat{A}_l |l \rangle.
\] (55)

4.2. Angles

We will use the (dihedral) angle operator defined in [63, 64]. Consider a sequence of representations \(\rho_1, \ldots, \rho_n\) and a sequence of their representation spaces \(H_1, \ldots, H_n\). In the space \(H_1 \otimes \cdots \otimes H_n\) we define operators \(J_r, r \in \{1, \ldots, N\}, i \in \{1,2,3\}\) by the following formula:

\[
J_r := I \otimes I \otimes \rho_i^r (\gamma_i) \otimes I \otimes \cdots \otimes I,
\] (56)

where \(\gamma_i = -\frac{i}{2}\sigma_i\) form the su(2) Lie algebra basis defined by the Pauli matrices \(\sigma_i\) and \(\rho_i^r\) is the su(2) representation corresponding to \(\rho_i\). Following [64] we introduce an angular momentum operator \(J_r\) assigned to the surface \(S_l\), or equivalently the side \(\ell_l\) of the lattice, which is a sum of angular momentum operators for each link of the graph (lattice with loops) intersecting the surface \(S_l\) (or in other words a sum over angular momentum operators corresponding to links parallel of \(\ell_l\)):

\[
\hat{J}_r = \sum_{r \in \ell_l, ||l||} J_r.
\] (57)

For each pair \(\{I,J\}\) such that \(\vec{n}_I \cdot \vec{n}_J = 0\) we consider an operator corresponding to a dihedral angle between the surfaces \(S_I\) and \(S_J\):

\[
\cos \hat{\theta}_{IJ} = \frac{-\hat{J}_I^J \hat{J}_J^I \delta_{ij}}{\sqrt{-\hat{J}_I^I \hat{J}_I^I \delta_{ij} \hat{J}_J^J \hat{J}_J^J \delta_{ij}}}
\] (58)
Due to the invariance of the states under $O_{\text{cube}}$ transformations the expectation value
\[
\langle |l| \cos \hat{\theta}_{IJ} |l| \rangle
\] does not depend on $I, J$. Indeed,
\[
\forall g \in O_{\text{cube}} \quad \langle |l| \cos \hat{\theta}_{\sigma_e^{-1}(I)\sigma_e^{-1}(J)} |l| \rangle = \langle |l| U_g^* \cos \hat{\theta}_{IJ} U_g |l| \rangle = \langle |l| \cos \hat{\theta}_{IJ} |l| \rangle.
\] (60)

This means that the dihedral angle is the same between any pair of non-parallel faces $S_I, S_J$ and is a clear sign of isotropy of the quantum geometry. The expectation value of the dihedral angle operator in the state $|l\rangle$ can be expressed in terms of an expectation value of the dihedral operator in the state $|l\rangle$:
\[
\langle |l| \cos \hat{\theta}_{IJ} |l| \rangle = \frac{1}{24 \cdot \#S_I} \sum_{g, g' \in O_{\text{cube}}} \langle |l| U_g^* \cos \hat{\theta}_{IJ} U_{g'} |l| \rangle = \frac{1}{24} \sum_{g \in O_{\text{cube}}} \langle |l| U_g^* \cos \hat{\theta}_{IJ} U_g |l| \rangle
\] (61)

Let us introduce a notation:
\[
\cos \theta(k) := \langle k | \cos \hat{\theta}_{IJ} |k \rangle.
\] (62)

We calculated the expectation values of the cosine of the dihedral angle operator $\langle k | \cos \hat{\theta}_{IJ} |k \rangle$ for different values of number of loops $k$. As $k$ increases the values grow and seem to reach zero asymptotically (see figure 2). The plot (b) indicates that they tend to zero as an inverse polynomial in $k$. The fact that the asymptotic value is zero can be calculated using the (extended) stationary phase method. This calculation will be done in the next subsection.

4.3. Large volume limit

A large volume limit has been used for example in [48] to study a semiclassical limit of a quantum cosmological model based on a spin-foam theory. In [48] the Universe is triangulated with 2 tetrahedra and large volume transfers directly to large spin limit, which allows the
authors to make their calculations. In this paper the Universe will be defined on a lattice with a large number of cubes and each cube will be large. The large volume limit will be used as an approximate tool to simplify our calculations. In fact, the volume at the node does not need to be astronomically large but just large enough to make our approximations hold. It will be enough if the volume at the node is just a couple of orders of magnitude higher than the Planck volume.

In our approach, the volume operator is the Rovelli–Smolin volume operator, which for spin $1/2$ $N$-valent intertwiners is proportional to the identity. In [39] we have shown that:

$$V_j = \frac{\kappa_0}{8} \left( \frac{8\pi G \hbar \gamma}{c^3} \right)^{\frac{3}{2}} \sqrt{\frac{\sqrt{3}}{3t}} (N-2)N(N+2) \cdot \mathbb{1}, \quad (63)$$

where $\mathbb{1}$ encodes the spin labels of the $N$-valent intertwiner.

In this paper, the valence $N$ of an intertwiner at the node $n_0$ is

$$N = 2k + 6, \quad (64)$$

where $k$ is the number of loops at $n_0$. Therefore, the large volume limit coincides with the limit of large number of loops. We will therefore consider the limit of large $n$, where the number of loops is $k = 3n + r$. As we argued earlier, the intertwiners at the node $n_0$ can be described by Livine–Speziale coherent intertwiners

$$|j_1, \ldots, j_6; \tilde{n}_1, \ldots, \tilde{n}_6\rangle, \quad (65)$$

where $j_l = N_l/2$ and $N_l$ is the number of links starting at $n_0$ whose tangent direction is $\tilde{n}_l$ (see (40)). Due to the particular choice of loop configurations in the definition of the states we have the following values of spins:

- for $k = 3n$ the spins are $j_1 = \ldots = j_6 = \frac{n+1}{2}$,
- for $k = 3n + 1$ the spins are $j_1 = j_3 = \frac{n+2}{2}, j_2 = j_4 = j_6 = \frac{n+1}{2}$,
- for $k = 3n + 2$ the spins are $j_1 = j_2 = j_3 = j_6 = \frac{n+2}{2}, j_4 = j_5 = \frac{n+1}{2}$.

Therefore, the large volume limit translates into the limit of large spins $j_l$.

We will calculate the expectation value

$$\langle l_k | \cos \theta_{ij} | l_k \rangle = \langle l_k | \frac{-j_l^*j_j \delta_{ij}}{\sqrt{-j_l^*j_l \delta_{ij}} \sqrt{-j_j^*j_j \delta_{ij}}} \rangle | l_k \rangle = \frac{-\delta_{ij} \beta_{ij}^\mu}{\sqrt{j_l(j_l + 1) \sqrt{j_j(j_j + 1)} N_k^2}},$$

where $N_k$ is the factor coming from normalization of the Livine–Speziale coherent intertwiners

$$N_k = \sqrt{\langle l_k, \tilde{n}_1, \ldots, \tilde{n}_6 | l_k, \tilde{n}_1, \ldots, \tilde{n}_6 \rangle} \quad (67)$$

and the remaining factor is

$$\beta_{ij}^\mu = \int_{SU(2)} du \langle j_l, \tilde{n}_l | j_j \rho_j(u) | j_l, \tilde{n}_l \rangle \langle j_j, \tilde{n}_j | j_j \rho_j(u) | j_j, \tilde{n}_j \rangle \cdot \prod_{K \in \{l,j\}} \langle j_K, \tilde{n}_K | \rho_K(u) | j_K, \tilde{n}_K \rangle \cdot (68)$$
Let us notice that
\[ \beta_{jl}^{\mu} = \int_{\text{SU}(2)} du B^\mu_{li}(u) \prod_{k=1}^{6} \langle j_k, \vec{n}_k | \rho_{j_k}(u) | j_k, \vec{n}_k \rangle, \]  
(69)

where
\[ B^\mu_{li}(u) = \langle 1/2, \vec{\mu}_i | \tau^\mu \rho_{1/2}(u) | 1/2, \vec{\nu}_j \rangle \langle 1/2, \vec{\nu}_j | \tau^\nu \rho_{1/2}(u) | 1/2, \vec{\mu}_i \rangle. \]  
(70)

The expression (69) can be written in the following form:
\[ \beta_{jl}^{\mu} = \int_{\text{SU}(2)} du B^\mu_{li}(u) e^{S_j(u)}, \]  
(71)

where
\[ S_j(u) = \sum_{k=1}^{6} 2j_k \ln \left( \langle 1/2, \vec{n}_k | \rho_{1/2}(u) | 1/2, \vec{n}_k \rangle \right). \]  
(72)

Let us consider the case \( k = 3n \). We scale \( j_I = n + \frac{1}{2} \) by an overall constant \( \lambda : j_I \mapsto \lambda j_I \). Under this scaling, the expression (69) transforms as
\[ (\beta_{\lambda j})_{jl}^{\mu} = \lambda^2 \int_{\text{SU}(2)} du B^\mu_{li}(u) e^{\lambda S_j(u)}. \]  
(73)

In order to evaluate the integral, we will use the extended saddle point analysis (see for example [44]). We notice, that the integral
\[ \int_{\text{SU}(2)} du e^{S_j(u)} \]  
(74)

is the norm of the Livine–Speziale intertwiner \( |j_1, \ldots, j_6; \vec{n}_1, \ldots, \vec{n}_6 \rangle \), where \( j_1 = \ldots = j_6 = n + \frac{1}{2} \).

In this case the saddle point analysis has been performed in [65] and we can use the result from this paper to conclude that the only critical point of \( S_j \) is \( u = \mathbb{1} \). In the saddle point approximation [65]
\[ \beta_{\lambda j}^{\mu} \approx \lambda^2 B^\mu_{li}(\mathbb{1}) \frac{1}{\sqrt{\pi} \lambda \det H}, \]  
(75)

where \( H \) is the Hessian of \( S_j \). Its explicit form will not be important here, because it gets cancelled with the Hessian from \( \frac{1}{N_{\lambda j}^2} \):
\[ \frac{\beta_{\lambda j}^{\mu}}{N_{\lambda j}^2} \approx \lambda^2 B^\mu_{li}(\mathbb{1}). \]  
(76)

It is straightforward to calculate \( B^\mu_{li}(\mathbb{1}) \):
\[ B^\mu_{li}(\mathbb{1}) = -j_l j_j \rho_{il} \rho_{jl}. \]  
(77)

Since \( \vec{n}_i \cdot \vec{n}_j = 0 \):
\[ \delta_{ij} B^\mu_{li}(\mathbb{1}) = 0. \]  
(78)

We conclude that
\[ \langle k | \cos \hat{\theta}_{ij} | k \rangle \rightarrow 0 \]  
(79)
in the large volume limit (the limit of large number of loops). The cases \( k = 3n + 1 \) and \( k = 3n + 2 \) are analogous. As a result the faces \( S_i \) and \( S_j \) are asymptotically orthogonal and the deficit angle (in the sense of Regge) is 0. Therefore the geometries are flat.

5. The Euclidean part of the Hamiltonian operator

We will start this section with recalling some properties of the quantum Hamiltonian operator. We will use the version of the operator from [38] that corresponds to the model of quantum gravity coupled to massless scalar field. In particular, we will see that thanks to the results from [39] about the Rovelli–Smolin volume operator in the space of spin \( \frac{1}{2} \) monochromatic intertwiners and due to internal flatness of the quantum geometries (which we discuss in section 4) the only non-trivial part is the Euclidean part of gravitational scalar constraint operator. We will study it in detail in this section.

5.1. Quantum Hamiltonian operator

For the class of operators considered in [35–38] it is enough to diagonalize the operator at each node. The eigenvectors of the full operator are tensor product of eigenvector of operators restricted to each node. In the homogeneous-isotropic sector the situation is even simpler, because the isotropic eigenvectors of the full operator are tensor products of one and the same eigenvector corresponding to a fixed node. Furthermore, we argued in [31] that it is enough to restrict to a subspace (see also (29))

\[
\mathcal{H}_{\Gamma,j,n}^{\text{loop}} = \bigoplus_{l} \mathcal{H}_{l}
\]

(80)

and treat the quantum Hamiltonian as acting in the direct product of the intertwiner spaces. We consider a model of gravity coupled to a massless scalar field and use the Hamiltonian in the form proposed in [38]:

\[
\hat{H} = \sqrt{-2\sqrt{q}C^0} = \sqrt{2\frac{E_P l_P^3}{16\pi \gamma^2} \left( 1 + \gamma^2 \right) \hat{C}_L + \frac{1}{2} \left( \hat{C}_E + \hat{C}_E^\dagger \right) } - \frac{\Lambda E_P}{8\pi l_P} \hat{V} |+|
\]

(81)

Let us recall that in the formula above: \( E_P \) is the Planck energy, \( l_P \) is the Planck length, \( \gamma \) is the Barbero–Immirzi parameter, \( \Lambda \) is the cosmological constant and \( |+| \) denotes the positive part of an operator. The operator \( \hat{V} \) is the volume operator which preserves each space \( \mathcal{H}_l \) and in fact is proportional to the identity operator when restricted to each \( \mathcal{H}_l \) (see [39] for proof). \( \hat{C}_L \) is a quantum operator corresponding to classical term \( \int d^3x (\sqrt{q})^2 R^{(3)} \), where \( q \) is the spatial (3d) metric and \( R^{(3)} \) is the corresponding Ricci scalar. In the proposal [36–38] this operator does not change the graph—in our notation this means that \( \hat{C}_L \mathcal{H}_l \subset \mathcal{H}_l \). We will not use this operator in this paper, because as we argue in section 4 the internal geometries are flat and we expect that with proper definition this part (called in [36–38] Lorentzian part) vanishes on our states or at least is negligibly small compared to the Euclidean part and cosmological constant term.

The nontrivial operator is the Euclidean part: \( \hat{C}_E \) and \( \hat{C}_E^\dagger \). The operator \( \hat{C}_E^\dagger \) adds loops and the operator \( \hat{C}_E \) is its hermitian adjoint, in particular it subtracts loops. The operator \( \hat{C}_E^\dagger \) is further defined by a sum of operators

\[
\hat{C}_E^\dagger = \sum_{r,s} \epsilon(r_s \ell_r \ell_s) \hat{C}_{E_{rs}}^\dagger,
\]

(82)
where \( \epsilon(\ell_r, \ell_s) \) was defined in (24) and the sum runs over all possible pairs of links \((\ell_r, \ell_s)\) at the at least 6-valent node. The operator \( \hat{C}_{E_{\kappa}} \) maps elements of \( \mathcal{H}_I \) into elements of \( \mathcal{H}_{\kappa r s} \), where \( \Sigma(I_{\kappa r s}) = \Sigma(I) + 1 \). Let us underline that with \( I, J, R, S = 1, \ldots, 6 \) we label the sides of the cubical lattice and with \( r, s = 1, \ldots, 2\Sigma(I) + 6 \) we label the links outgoing from the fixed node \( n \). The loop configuration \( l_r \) is constructed in the following way. Let \( \ell_k \) be a side of the cubical lattice to which \( \ell_r \) is tangent and \( \ell_s \) be a side of the cubical lattice to which \( \ell_s \) is tangent (a link is tangent to itself). The loop configuration \( l_r \) has the following form:

\[
 l_r(I) = \begin{cases} 
 l(I, J) + 1, & \text{if } (I, J) = (R, S) \text{ or } (I, J) = (S, R), \\
 l(I, J), & \text{otherwise}. 
\end{cases}
\]  

The action of the operator \( \hat{C}_{E_{\kappa}} \) on the intertwiner spaces is given in the next subsection.

5.2. The action of the Euclidean part in the intertwiner spaces

As we discussed in detail in [66], the Euclidean part of the Hamiltonian operator is defined by a family of operators

\[
 \hat{C}_{E_{\kappa}} : \mathcal{H}_{J_1} \otimes \cdots \otimes \mathcal{H}_{J_n} \to \mathcal{H}_{J_1} \otimes \cdots \otimes \mathcal{H}_{\kappa} \otimes \mathcal{H}_{I_{1/2}} \cdots \otimes \mathcal{H}_{I_{1/2}} \otimes \mathcal{H}_{J_1} \otimes \cdots \otimes \mathcal{H}_{J_n} 
\]

such that (repeated indices are summed over)

\[
 \hat{C}_{E_{\kappa}} = 8\kappa_I \epsilon_I \tau_I \hat{J}_I \hat{J}_I, 
\]

where the angular momentum operators \( \hat{J}_I \) are defined in (56). In this formula we treat \( \tau_I \) as a tensor in \( \mathcal{H}_{I_{1/2}} \otimes \mathcal{H}_{I_{1/2}} \). The coefficient \( \kappa_I \) is a real number which value can depend only on the valence of the intertwiners [36]. We will denote by \( C_{E_{\kappa}} \) the hermitian adjoint of \( \hat{C}_{E_{\kappa}} \).

In this paper we split each loop created by the Euclidean part into two links by adding a 2-valent node \( n \) and re-orienting the links such that all links are outgoing from the node which is at least 6-valent. Therefore we will consider an equivalent family of operators (which will be denoted by the same symbol):

\[
 \hat{C}_{E_{\kappa}} : \mathcal{H}_{J_1, \ldots, J_n} \to \mathcal{H}_{J_1, \ldots, J_n, 1/2, \ldots, 1/2, \ldots, J_n} \otimes \left( \mathcal{H}_{I_{1/2}} \otimes \mathcal{H}_{I_{1/2}} \right). 
\]

In the formula above, the Hilbert spaces \( \mathcal{H}_{J_1, \ldots, J_n} \) and \( \mathcal{H}_{J_1, \ldots, J_n, 1/2, \ldots, 1/2, \ldots, J_n} \) correspond to the at least 6-valent node \( n_0 \) and the Hilbert space \( \mathcal{H}_{I_{1/2}} \otimes \mathcal{H}_{I_{1/2}} \) corresponds to the bivalent node \( n \). The operator \( \hat{C}_{E_{\kappa}} \) becomes:

\[
 \hat{C}_{E_{\kappa}} = 8\kappa_I \epsilon_I \tau_I \hat{J}_I \hat{J}_I. 
\]

In the formula above \( \epsilon : \mathcal{H}_{I_{1/2}} \to \mathcal{H}_{I_{1/2}} \) is the canonical intertwiner between the spin \( 1/2 \) representation and its dual, and \( \tilde{\tau}_I = \left( \mathbb{1} \otimes \epsilon^{-1} \right) \tau_I \). In the index notation

\[
 \epsilon_{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (\epsilon^{-1})_{AB} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \tilde{\tau}_{AB} = (\epsilon^{-1})_{BC} \tau_{BC}. 
\]

5.3. Contributions for parallel links

Let us denote by \( \ell_I, I \in \{1, \ldots, 6\} \) the segments of the lattice (without loops) intersecting at the node \( n_0 \). Let us introduce operators

\[
 \hat{C}_{E_{I, I}} = 8\kappa_I \epsilon_I \hat{\tau}_I \left( \sum_{r \ell_I \parallel \ell_I} \hat{J}_r \right) \left( \sum_{s \ell_I \parallel \ell_I} \hat{J}_s \right) \otimes \epsilon, 
\]

\[
 (59) 
\]
where the first sum is over links $\ell_r$ tangent to $\ell_I$ and the second sum is over $\ell_J$ tangent to link $\ell_J$. The operator $\hat{C}_{E}^{\dagger}$ can now be written in the following form:

$$\hat{C}_{E}^{\dagger} = \sum_{i=1}^{6} \sum_{J=1}^{6} \epsilon_{IJ} \hat{C}_{I J}^{\dagger}.$$  \hspace{1cm} (90)

The $\epsilon_{IJ}$ is 0 if the sides of the lattice $\ell_I$ and $\ell_J$ are anti-parallel and 1 otherwise. This expression is especially useful in our coherent state basis build from Perelomov coherent states. Let us recall that we assign to each link $\ell_r$ tangent to $\ell_I$ a coherent state $j_{1/2}, \vec{n}_I$. We can use the standard isomorphism mapping spinors into the corresponding states in the magnetic number representation to obtain (compare (39)):

$$|1/2, \vec{n}_I\rangle^{\otimes N_I} = |j_I, \vec{n}_I\rangle,$$  \hspace{1cm} (91)

where $j_I = N_I/2$. Under this isomorphism we can write:

$$\sum_{r: \ell_r \parallel \ell_I} \hat{J}_{r}^{I} \otimes |1/2, \vec{n}_r\rangle = \hat{J}^{I}_{J} |N_I/2, \vec{n}_I\rangle,$$  \hspace{1cm} (92)

where $\hat{J}^{I}_{J} = \rho_{I}^{J}(\tau^{I})$ (and coincides with the operator $\hat{J}^{I}_{J}$ defined in (57)). With this notation, we can write $\hat{C}_{E}^{IJ}$ in the following form:

$$\hat{C}_{E}^{IJ} = 8 \kappa_1 \epsilon_{ijk} \tilde{\tau}_i \hat{J}_{J}^{I} \otimes \epsilon.$$  \hspace{1cm} (93)

This form will be useful for calculating matrix elements between our coherent basis states.

In this paper we will use non-trivially the fact the $\kappa_1$ may depend on the valence of the intertwiners (and therefore on the number of loops $k$) [36]. This property is studied in detail at the end of section III D 1 in [36]. Let us briefly summarize the argument. In [36], the authors firstly define an $\epsilon$-dependent Euclidean part of the scalar constraint operator on the space of cylindrical functions, where $\epsilon$ controls the coordinate size of the loop. The authors argue that every value of $\kappa_1$ can be achieved by suitable choice of the shape and size of the loops. The parameter $\epsilon$ is trivially dropped when the operator is transferred to the space of partial solutions to the vector constraint (called the vertex Hilbert space). With an appropriate definition of the operator, which avoids the problem of cylindrical consistency, $\kappa_1$ can depend on the valence of the node.

We will assume that $\kappa_1$ is the following function of $k$:

$$\kappa_1(k) = \begin{cases} \frac{\kappa_E}{4\sqrt{3}}, & \text{if } k = 0, \\ \frac{\kappa_E}{2\sqrt{3}}, & \text{if } k = 3n + 2 \text{ or } k = 3n + 3, \\ \frac{\kappa_E}{4}, & \text{if } k = 3n + 1, \end{cases}$$  \hspace{1cm} (94)

where $\kappa_E$ does not depend on $k$ and $n \in \mathbb{N}$.

5.4. Matrix elements of the Euclidean part between our coherent states

We will calculate now explicitly the matrix elements on the right hand side of equations (B8), (B13) and (B15). Let us notice that they are given by matrix elements of an SU(2) invariant operator between Livine–Speziale coherent intertwiners. Each of the Livine–Speziale coherent intertwiner (in and out) is obtained by averaging a tensor product of Perelomov coherent states over the SU(2) group. Due to the invariance of the operator, we can omit one such averaging.
Using this property and the symmetry properties of the matrix elements of the Euclidean part from appendix B, we can write the expression for matrix elements in the following form:

\[
\langle k+1|\hat{\mathcal{C}}_{E}^{\dagger}|k\rangle = 8\kappa_{E} \sum_{i,j,l} \epsilon_{i\ell} \frac{\alpha_{ij}^{\ell}}{N_{k}N_{k+1}} \cdot \beta_{ij}^{\ell}
\]

(95)

In the formula above, \(\alpha_{ij}^{\ell}\) is combining two factors: a factor coming from a contraction of indices of \(\bar{\tau}_{i}\) with Perelomov coherent states \(|1/2, \bar{n}_{i}\rangle, |1/2, \bar{n}_{j}\rangle\) in the out state and a factor coming from the contraction of \(\epsilon\) assigned to bivalent node in (93) with the Livine–Speziale coherent intertwiner assigned to the bivalent node (see section 3.1):

\[
\alpha_{ij}^{\ell} = 2 \epsilon(\bar{n}_{i}, \bar{n}_{j}) \cdot \langle[(1/2, \bar{n}_{i}) \otimes (1/2, \bar{n}_{j})], \bar{\tau}_{i}\rangle.
\]

(96)

The factor \(N_{k}\) is the normalization factor of the Livine–Speziale coherent intertwiners (see also (67)):

\[
N_{k} = \sqrt{\langle k, \bar{n}_{1}, ..., \bar{n}_{6}|k, \bar{n}_{1}, ..., \bar{n}_{6}\rangle}.
\]

(97)

The remaining factor is \(\beta_{ij}^{\ell}\) (see also (69)):

\[
\beta_{ij}^{\ell} = \int_{SU(2)} \, du \langle j, \bar{n}_{i}|\hat{\mathcal{Y}}_{j}(u)|j, \bar{n}_{i}\rangle \cdot \langle j, \bar{n}_{j}|\hat{\mathcal{Y}}_{j}(u)|j, \bar{n}_{j}\rangle \cdot \prod_{k \notin \{I,J\}} \langle j, \bar{n}_{k}|\rho_{k}(u)|j, \bar{n}_{k}\rangle.
\]

(98)

Let us calculate first \(\alpha_{ij}^{\ell}\). In the index notation:

\[
\alpha_{ij}^{\ell} = 2 \bar{n}_{i}^{2} \epsilon_{iB}^{\mathcal{M}} \cdot \bar{n}_{A} \cdot \mathcal{C}^{E} \cdot \mathcal{M}_{JB} = -2 \bar{n}_{i}^{2} \epsilon_{iB}^{\mathcal{M}} \cdot \bar{n}_{A} \cdot \mathcal{C}^{E} \cdot \mathcal{M}_{JB}
\]

\[
= \frac{i}{4} \text{Tr} \left( \epsilon(\mathbb{1} + \bar{n}_{i} \cdot \hat{\sigma}) \epsilon(\mathbb{1} + \bar{n}_{j} \cdot \hat{\sigma}) \sigma_{i} \right) = \frac{i}{4} \text{Tr} \left( (\mathbb{1} + \bar{n}_{i} \cdot \hat{\sigma})(\mathbb{1} + \bar{n}_{j} \cdot \hat{\sigma}) \sigma_{i} \right)
\]

\[
= \frac{1}{2} i(n_{i}^{j} - n_{i}^{j}) - \frac{1}{2} \epsilon_{iB} \epsilon_{jA} \bar{n}_{i}^{2}.
\]

(99)

Let us recall that in section 4.3 we argued that the large volume limit translates in our model into a limit of large number of loops \(k\). We can directly apply the results of the previous section to calculate the factor \(\frac{\beta_{ij}^{\ell}}{N_{k}N_{k+1}}\) in this limit:

\[
\frac{\beta_{ij}^{\ell}}{N_{k}N_{k+1}} \approx -\frac{(n+1)^{2}}{16} \bar{n}_{i}^{2} \bar{n}_{j}^{2}.
\]

(100)

It is now straightforward to calculate \(\langle k+1|\hat{\mathcal{C}}_{E}^{\dagger}|k\rangle\) in the limit of large volumes:

\[
\langle k+1|\hat{\mathcal{C}}_{E}^{\dagger}|k\rangle \approx -8\kappa_{E} \sum_{i,j,l} \epsilon_{i\ell} \left( \frac{i}{2} (n_{i}^{j} - n_{i}^{j}) - \frac{1}{2} \epsilon_{iB} \epsilon_{jA} \bar{n}_{i}^{2} \bar{n}_{j}^{2} \right) \frac{(n+1)^{2}}{16} \bar{n}_{i}^{2} \bar{n}_{j}^{2} = -\kappa_{E} \frac{(n+1)^{2}}{4}.
\]

(101)

5.5. Numerical analysis

We have calculated the matrix elements \(\frac{1}{\kappa_{E}}\langle k+1|\hat{\mathcal{C}}_{E}^{\dagger}|k\rangle\) numerically for \(k \in \{0, 1, ..., 50\}\). The results are summarized in figure 3. We investigated the dependence of \(\frac{1}{\kappa_{E}}\langle k+1|\hat{\mathcal{C}}_{E}^{\dagger}|k\rangle\) as a function of \(n\). Using gnuplot software we fitted a quadratic function \(\frac{1}{\kappa_{E}}\langle k+1|\hat{\mathcal{C}}_{E}^{\dagger}|k\rangle = an^{2} + bn + c\) in 3 cases \(k = 3n, k = 3n + 1, k = 3n + 2\) separately (see figure 3). We obtained the following results
Figure 3. Matrix elements $\frac{1}{\kappa_E} \langle k + 1 | \hat{C}_E^\dagger | k \rangle$ as a function of $n$. We plotted the 3 cases $k = 3n, k = 3n + 1, k = 3n + 2$ and fitted a quadratic function $\frac{1}{\kappa_E} \langle k + 1 | \hat{C}_E^\dagger | k \rangle = an^2 + bn + c$ in each case separately. The fitted function is plotted with a dotted line.

\begin{align*}
    a_{3n} &= -0.250164 \pm 0.000034, b_{3n} = -0.527623 \pm 0.0005636, c_{3n} = -0.275961 \pm 0.001944, \\
    a_{3n+1} &= -0.249966 \pm 0.000017, b_{3n+1} = -0.563023 \pm 0.0002825, c_{3n+1} = -0.321144 \pm 0.0009746, \\
    a_{3n+2} &= -0.249991 \pm 0.000002, b_{3n+2} = -0.593374 \pm 0.0000381, c_{3n+2} = -0.308724 \pm 0.0001085.
\end{align*}

This confirms our asymptotic result (101) predicting that in the limit of large $n$:

$$\langle k + 1 | \hat{C}_E^\dagger | k \rangle \approx -\frac{(n + 1)^2}{4} \kappa_E.$$  \hspace{1cm} (102)

The coefficient of the quadratic term in the fitted functions are in good accordance with the asymptotic value $-0.25$. The appearance of an additional linear term is expected, because our formula (101) holds in the limit of large $n$ only. We expect that the values of the linear coefficients can be calculated by considering subleading order contributions in the saddle point analysis. In this paper, we determine them from our numerical experiment. Strictly speaking, we approximate the matrix elements (for any $n$) with

$$\frac{1}{\kappa_E} \langle k + 1 | \hat{C}_E^\dagger | k \rangle \approx -\frac{(n + 1)^2}{4} - Bn - C.$$  \hspace{1cm} (103)
The coefficients $B$ and $C$ are obtained by fitting the function on the right hand side to our numerical data in the 3 cases $k = 3n$, $k = 3n + 1$, $k = 3n + 2$ separately. We obtained:

$$B_{3n} = 0.0302524 \pm 0.0002339, \quad C_{3n} = 0.0193867 \pm 0.002194,$$

$$B_{3n+1} = 0.0624754 \pm 0.0000814, \quad C_{3n+1} = 0.0725132 \pm 0.0007639,$$

$$B_{3n+2} = 0.0938279 \pm 0.000013, \quad C_{3n+2} = 0.0590887 \pm 0.0001219.$$ 

We calculated the matrix elements by a straightforward implementation of formula (95). In order to calculate $\beta^i_{ij}$ and $N_k$ we constructed a tree basis of intertwiners by contracting 3j-symbols. Using this tree basis we constructed a projection onto invariant tensors. Next, we projected a tensor product of Perelomov coherent states and obtained the Livine–Speziale coherent intertwiners. The implementation of angular momentum operators was straightforward as it is a standard textbook formula. The coefficients $\beta^i_{ij}$ and $N_k$ were obtained by further contractions. In our algorithm the number of contractions grows very fast with increasing the number of loops. As a result we were able to calculate the matrix elements up to 50 loops (it took around 1 week on 1 node equipped with 40 CPUs and 128 GB RAM). While we expect that it is possible to derive a computationally faster method, it will be enough for us to use the approximate formula (103).

6. Cosmological model derived from the full loop quantum gravity

6.1. Cosmological Hamiltonian

In our model we consider a Hilbert space $\mathcal{H}^{flat}$ spanned by states $|k\rangle$. It is a subspace of homogeneous-isotropic states proposed in [31] which we interpret as a space of quantum geometries with zero intrinsic curvature. Indeed, in section 4 we discuss the internal geometry of the states and conclude that the geometries are approximately flat. Therefore the quantum operator corresponding to the term $\int d^3x \sqrt{q} R^{(3)}$ should annihilate the states or at least it is expectation values in such states should be negligibly small compared to the expectation values of the Euclidean part and the cosmological constant term (in the limit of large volumes). The gravitational part of the scalar constraint operator $\sqrt{q} C^\varphi$ restricted to this space has only diagonal and subdiagonal entries. The diagonal entries are given by the cosmological constant term (in the case of zero intrinsic curvature):

$$\langle k | \sqrt{q} C^\varphi | k \rangle \approx \frac{\Lambda}{8\pi I_p} E_p \langle k | \hat{V}^2 | k \rangle = \left( \Lambda I_p^2 \right) \frac{\sqrt{3} \pi^2 \gamma^3 \kappa_0^2}{6} (N - 2)N(N + 2)E_p I_p^2,$$

(104)

where $N = 2k + 6$. In the formula above $\gamma$ is the Barbero–Immirzi parameter, $\kappa_0$ is a free constant in the definition of the volume operator. The subdiagonal elements are $\langle k + 1 | C^\varphi | k \rangle$ which we approximate according to formula (103):

$$\langle k + 1 | \sqrt{q} C^\varphi | k \rangle = \frac{E_p I_p K_E}{32\pi \gamma^2} \left( \frac{n + 1}{2} + B_kn + C_k \right).$$

(105)

The operator $\sqrt{ - 2 \hat{\Theta}_\lambda |_+}$ plays a role of true Hamiltonian in a deparametrized model of loop quantum gravity coupled with a massless scalar field [9] (see also section 2.4). In this paper we will consider a simplified model which evolution is governed by Hamiltonian

$$\hat{H} = \sqrt{ - 2 \hat{\Theta}_\lambda |_+},$$

(106)
where $\hat{\Theta}_\Lambda$ is obtained from $\sqrt{qC^{\text{flat}}}$ by projecting it onto $\mathcal{H}_{\text{flat}}$. Strictly speaking, our operator $\hat{\Theta}_\Lambda$ is defined by the following relations:

$$
\langle k | \hat{\Theta}_\Lambda | k \rangle = E_p l_P^3 \left( \Lambda l_P^2 \right) \frac{4 \sqrt{3} \pi^2 \gamma^3 \kappa_0^2}{3} k^3,
$$  
(107)

$$
\langle k | \hat{\Theta}_\Lambda | k + 1 \rangle = \langle k + 1 | \hat{\Theta}_\Lambda | k \rangle = E_p l_P^3 \kappa_0 \kappa_s \frac{k^2}{32 \pi^2 \gamma^2 36},
$$  
(108)

$$
\langle k | \hat{\Theta}_\Lambda | k' \rangle = 0 \text{ if } |k' - k| \neq 1.
$$  
(109)

Let us notice that the matrix elements of $\hat{\Theta}_\Lambda$ agree with the matrix elements of $\sqrt{qC^{\text{flat}}}$ only approximately. The higher $k$ the better the approximation.

### 6.2. Cosmological Hamiltonian in terms of $\hat{p}$ and $\hat{c}$ operators

In this section we will express our cosmological Hamiltonian in terms of the standard operators $\hat{p}$ and $\hat{c}$ used in loop quantum cosmology (see for example [11] [67]). They have a meaning of collective (coarse-grained) variables. Following the standard LQC approaches [11] we consider a fundamental cell $V_0$. Let us recall that we assume that the nodes of our lattice $\Gamma$ are given by $(\frac{q}{L^1}, \frac{r}{L^2}, \frac{s}{L^3})$ for $q, r, s \in \mathbb{Z}, L \in \mathbb{N}, L \geq 2$. The fundamental cell will be a cube

$$
V_0 := \{ (x^1, x^2, x^3) : \forall i \in \{1, 2, 3 \} - \epsilon \leq x^i \leq 1 + \epsilon \}
$$

(110)

for sufficiently small $\epsilon$. The fundamental cell contains $L^3$ nodes. In order to underline that the lattice depends now on the parameter $L$ we will denote the states $|k\rangle$ by $|k, L\rangle$. Let us recall that a state $|k, L\rangle$ at a single node defines a homogeneous state on the lattice which is obtained by assigning to each node of the lattice a copy of $|k, L\rangle$ and tensor multiplying the copies (see section 3 and [31] for more detailed discussion how states $|k, L\rangle$ are related to states on the lattice). Since we study only homogeneous geometries in this paper, we will use the same notation for the state at the node and the corresponding homogeneous state on the lattice. The volume of the fundamental cell is:

$$
\hat{V}(V_0)(k, L) = L^3 \sqrt{\frac{\sqrt{6} \kappa_0}{6}} \Delta^2 \sqrt{(k + 2)(k + 3)(k + 4)} l_P^3 |k, L\rangle,
$$

(111)

where $\Delta = 4 \sqrt{3} \pi \gamma (\Delta l_P^2$ is the area gap (see [11])).

In the following, we will use approximate expressions which gets better when $k$ becomes larger. Let us recall that we have already made similar simplifications in the previous sections, for example we excluded the Lorentzian part. We will approximate the volume of the fundamental cell by

$$
\hat{V}(V_0)(k, L) \approx L^3 \sqrt{ \frac{\sqrt{2} \kappa_0}{6}} \Delta^2 k^2 l_P^3 |k, L\rangle.
$$

(112)

Let us calculate the expectation value of the area operator of $\partial V_0$, i.e. the boundary of $V_0$. Let us notice that depending on $\epsilon$ in (110) the boundary of the cell $\partial V_0$ intersects the loops at the nodes $\{ (x^1, x^2, x^3) : x^1 = \pm 1 \lor x^2 = \pm 1 \lor x^3 = \pm 1 \}$ zero, one or two times. We choose $\epsilon$ small enough that the number of intersection is maximal. In this case:

$$
\frac{1}{6} \langle k, L | \hat{A}(\partial V_0) | k, L \rangle \approx \frac{2}{3} L^2 \Delta k l_P.
$$

(113)

In the formula above we used the result (54) $A(l) = \frac{1}{8} (2k + 6) \Delta l_P \approx \frac{1}{8} k \Delta l_P$ (see section 4.1). Let us recall that the factor $\frac{1}{8}$ in (54) comes from the fact that $A(l)$ is an area of an infinitesimal square which is intersected in average by $\frac{1}{8} k$ loops. For each node, the side
of the fundamental cell is intersected twice more often than an infinitesimal square. As a result, the side of the fundamental cell is intersected in average $\frac{1}{2} k$ times per each node.

We will use the classical relation between the area of the side of the cube and its volume to fix the parameter $\kappa_0$:

$$\langle k, L | \hat{V}(\nu_0) | k, L \rangle = \frac{1}{6^2} (k, L) \hat{A}(\partial \nu_0)(k, L)^2. \quad (114)$$

This condition gives $\kappa_0 = \frac{4}{\sqrt{3}}$.

Classically, $p$ can be identified with an area of a side of the fundamental cell [11]. Motivated by this identification, we will define an operator $\hat{p}$ by the following relation:

$$\hat{p}|k, L\rangle = \frac{2}{3} L^2 \Delta k \ell_P |k, L\rangle. \quad (115)$$

It is clear that this operator has the property that

$$\langle k, L | \hat{p} | k, L \rangle = \frac{1}{6} \langle k, L | \hat{A}(\partial \nu_0) | k, L \rangle,$$

i.e. the expectation value of $\hat{p}$ in the state $|k, L\rangle$ is equal to the expectation value of the quantum area of a side of the fundamental cell in the same state $|k, L\rangle$. In this sense it corresponds to the operator $\hat{p}$ from the LQC literature [11]. By construction, the states $|k, L\rangle$ are eigenvectors of the operator $\hat{p}$ with eigenvalues

$$p = \frac{8 \pi \gamma \ell_P^2}{3} \mu k, \quad \text{where } \mu := \sqrt{3} L^2. \quad (117)$$

It is therefore justified to denote the states $|k, L\rangle$ by $|p\rangle$.

Let us notice, that for different parameters $L$ the lattice is different $\Gamma^L$ and therefore our Hilbert space $\mathcal{H}^L_{\text{flat}}$ is different. We will denote by $\hat{\Theta}^L_{\Lambda}$ an operator $\hat{\Theta}_{\Lambda}$ (densely) defined in the space $\mathcal{H}^L_{\text{flat}}$ such that the matrix elements of $\hat{\Theta}^L_{\Lambda}$ in the basis $|k, L\rangle$ agree with the matrix elements of $\hat{\Theta}_{\Lambda}$ in the basis $|k\rangle$ (see section 6.1 for the definition of $\hat{\Theta}_{\Lambda}$):

$$\langle p | \hat{\Theta}^L_{\Lambda} | p \rangle = \frac{E_p \ell_P^3}{8 \pi} \Lambda_p \left( \frac{p}{\ell_P} \right)^3 \quad (118),$$

$$\langle p | \hat{\Theta}^L_{\Lambda} | p + \frac{8 \pi \gamma \ell_P^2}{3} \mu \rangle = \left( p + \frac{8 \pi \gamma \ell_P^2}{3} \mu \right) \langle p | \hat{\Theta}^L_{\Lambda} | p \rangle = \frac{E_p \ell_P^3 \kappa_E}{6 \cdot 16^3 \pi^3 \gamma^4 \mu^2} \left( \frac{p}{\ell_P} \right)^3. \quad (119)$$

Let us underline that in our basis the $\hat{p}$ operator acts as multiplication operator (see (115)).

In analogy to loop quantum cosmology [11] we define the operator $\hat{c}$ through a shift operator $e^{-i\xi \hat{c}}$:

$$e^{-i\xi \hat{c}} |p\rangle = | p + \frac{8 \pi \gamma \ell_P^2}{3} \mu \rangle, \quad (120)$$

where $\xi = \mu n, n \in \mathbb{N}$. Let us notice, that the factor $\frac{8 \pi \gamma \ell_P^2}{3}$ in (120) comes from the fact that in the classical theory the Poisson bracket between $c$ and $p$ is $\{c, p\} = \frac{8 \pi \gamma \ell_P^2}{3}$. The operator $\hat{\Theta}^L_{\Lambda}$ can be expressed using the operators $\hat{p}$ and $\hat{c}$:

$$\hat{\Theta}^L_{\Lambda} = \frac{E_p \ell_P^3 \kappa_E}{6 \cdot 16^3 \pi^3 \gamma^4 \mu^2} \left( e^{-i \hat{c} \hat{p}^2} + \hat{p}^3 e^{i \hat{c} \hat{p}^2} \right) + \frac{E_p \ell_P^3 \Lambda}{8 \pi} \frac{3 \sqrt{3}}{\mu^2} \hat{p}^3. \quad (121)$$

6.3. Effective dynamics

In the following, we will use Planck units $\hbar = c = G = 1$.
Following [68] we consider Gaussian coherent states. Firstly, we define a Gaussian coherent state at a single node:

$$|\Psi_{\phi',p',c',p'}\rangle = \frac{1}{A} \sum_{p} e^{-\frac{1}{2} \mu_{\phi}^2 (\pi-p')^2} e^{i \phi' (\pi-p')^2} e^{i \frac{1}{2} \mu_{c}^2 (p-p')^2} e^{i \frac{1}{2} \mu_{\pi}^2 (p-p')^2} |\pi\rangle \otimes |p\rangle.$$  \hspace{1cm} (122)

In the formula above the sum over $\pi$ is over all values of the form $\delta \pi n$, where $n \in \mathbb{Z}$ and $\delta \pi$ is very small (this is a form of a regularization of a sum over real line), $A$ is the normalization factor. The sum over $p$ is the sum over all allowed values. The coherent state for the whole lattice is obtained by tensor multiplying the same state $|\Psi_{\phi',p',c',p'}\rangle$ for each node. Let us notice that the gravitational part of these coherent states are related to the coherent states from [69]. We expect that with the vacuum state defined by the Livine–Speziale coherent states for lattice with loops, the coherent states (122) approximate the coherent states from [69].

The expectation values of the quantum scalar constraint at the node of the lattice $x$

$$\hat{C}_x = \frac{1}{2} \hat{\mu}_x^2 + \sqrt{q} \hat{C}_x ^{gr}$$  \hspace{1cm} (123)

in a coherent state $|\Psi_{\phi,p,c,p}\rangle$ is:

$$\langle \Psi_{\phi,p,c,p} | \hat{C}_x | \Psi_{\phi,p,c,p} \rangle = \frac{1}{2} p_\phi^2 + \frac{\kappa_E}{16 \pi^3 \gamma^4 \mu^2} p^2 \cos(\mu c) + \frac{3 \sqrt{3}}{8 \pi} \frac{\Lambda}{\mu} p^3 + O(\epsilon).$$  \hspace{1cm} (124)

After applying this to the constraint equation $\hat{C}_x = 0$ and omitting the $O(\epsilon)$ terms we arrive at an effective constraint equation:

$$\frac{1}{2} p_\phi^2 + \frac{\kappa_E}{16 \pi^3 \gamma^4 \mu^2} p^2 \cos(\mu c) + \frac{3 \sqrt{3}}{8 \pi} \frac{\Lambda}{\mu} p^3 = 0.$$  \hspace{1cm} (125)

It can be written in the following form:

$$\frac{1}{2} p_\phi^2 - \frac{2 \kappa_E}{16 \pi^3 \gamma^4 \mu^2} p^2 \sin^2 \left(\frac{\mu}{2} c\right) + \frac{3 \sqrt{3}}{8 \pi} \frac{\Lambda}{\mu} p^3 + \frac{\kappa_E}{16 \pi^3 \gamma^4 \mu^2} p^2 = 0.$$  \hspace{1cm} (126)

Let us divide the equation by $p^2$:

$$- \frac{2 \kappa_E}{16 \pi^3 \gamma^4 \mu^2} p^2 \sin^2 \left(\frac{\mu}{2} c\right) + \frac{p_\phi^2}{2p^2} + \frac{3 \sqrt{3}}{8 \pi} \frac{\Lambda}{\mu} p^2 + \frac{\kappa_E}{16 \pi^3 \gamma^4 \mu^2} p^2 = 0.$$  \hspace{1cm} (127)

Following [70] we will call:

$$H_{eff} := - \frac{2 \kappa_E}{16 \pi^3 \gamma^4 \mu^2} p^2 \sin^2 \left(\frac{\mu}{2} c\right) + \frac{p_\phi^2}{2p^2} + \frac{3 \sqrt{3}}{8 \pi} \frac{\Lambda}{\mu} p^2 + \frac{\kappa_E}{16 \pi^3 \gamma^4 \mu^2} p^2$$  \hspace{1cm} (128)

an effective Hamiltonian constraint (see appendix B in [70]). The first three terms look very similar to the three terms of the effective Hamiltonian constraint in loop quantum cosmology [70] (equation (B1)):

- we can make the first term the same as in [70] if we replace $\mu$ with $\pi$ and choose $\kappa_E$ appropriately (we will fix $\kappa_E$ shortly by another argument);
- we can make the third term the same as in [70] if we make $\Lambda$ a running coupling constant (in the sense of parameter flow in the renormalization group, where $L$ is the cut-off):

$$\Lambda_L(L) = \frac{3 \sqrt{3}}{\mu^2(L)} = \frac{\Lambda}{L^2}.$$  \hspace{1cm} (129)
The fourth term is only present in our approach. Let us notice that \( \tilde{\mu} \) is very different from \( \mu \) from loop quantum cosmology. In loop quantum cosmology \( \mu = \left( \frac{\Delta l}{p} \right)^2 \) clearly depends on \( p \). In our model \( \tilde{\mu} \) does not depend on \( p \) (it depends only on the cut-off parameter \( L \)). In fact, we did not obtain the \( \mu \) scheme but the \( \mu_0 \) scheme, where \( \mu_0 \) is constant—see for example [71]. This will have a consequence for the effective equations. However, as we will show, the model still has Friedmann dynamics as a classical limit, experiences Big Bounce and recollapse. Let us underline that in our model the cosmological constant is running towards 0. It seems to be reasonable to assume that the number \( L \) is very large but not infinite. This behavior is compatible with the fact that the value of cosmological constant measured in the experiments is very small (in Planck units).

As in [70] we calculate time evolution of the scalar field energy density:

\[
\rho = \frac{p_\phi^2}{2p^3} \quad (130)
\]

We will be interested in the evolution equation for \( \rho \) as a function of the scalar field \( \phi \). We will denote the derivative with respect to the coordinate time by dots and the derivatives with respect to the scalar field by primes. In particular

\[
\rho' = \frac{\dot{\rho}}{\dot{\phi}}. \quad (131)
\]

In complete analogy to [70] we calculate the coordinate time derivatives of \( \rho \) and \( \phi \) by taking their Poisson bracket with \( H_{\text{eff}} \) and use \( H_{\text{eff}} \approx 0 \) to get:

\[
\rho' = \pm \frac{1}{16\gamma} \sqrt{\frac{\kappa_E}{\pi}} \sqrt{\rho \rho_{\text{tot}} \left( 1 - \frac{\rho_{\text{tot}}}{\rho_{\text{max}}} \right)}, \quad (132)
\]

where

\[
\tilde{\rho}_{\text{tot}} = \rho + \frac{\Lambda_E}{8\pi} \frac{\rho_{\text{max}}}{2}, \quad \rho_{\text{max}}(p) = \frac{K}{p}, \quad K = \frac{\kappa_E}{2^{1/3} \gamma^4 \tilde{\mu}^2}. \quad (133)
\]

The resulting evolution equation looks the same as the evolution equation (B2) in [70] except that \( \kappa_E \) in [70] is fixed appropriately, \( \rho_{\text{max}} \) depends on \( p \) (this is a consequence of the fact that \( \tilde{\mu} \), in contrast to \( \mu \), does not depend on \( p \)), \( \rho_{\text{tot}} = \rho + \frac{\Delta E}{8\pi} \) is replaced by \( \tilde{\rho}_{\text{tot}} \), where

\[
\tilde{\rho}_{\text{tot}} = \rho_{\text{tot}} + \frac{\rho_{\text{max}}}{2}. \quad (134)
\]

However, it will be more practical for us to write equation (132) in an equivalent form:

\[
\rho' = \pm \frac{1}{16\gamma} \sqrt{\frac{\kappa_E}{\pi}} \sqrt{\rho \rho_{\text{tot}} \left( \frac{1}{2} - \frac{\rho_{\text{tot}}}{\rho_{\text{max}}} \right)}. \quad (135)
\]

We will now study the classical limit of the model. It will make it possible for us to interpret the additional term and fix the constant \( \kappa_E \). The classical region is the region where the total energy density is small:

\[
\rho_{\text{tot}} \ll \rho_{\text{max}}. \quad (136)
\]

In this region:

\[
\rho' \approx \pm \frac{1}{16\gamma} \sqrt{\frac{\kappa_E}{2\pi}} \sqrt{\rho \rho_{\text{tot}}}. \quad (137)
\]
This equation is equivalent to the Friedmann equation. Let us notice, that \( \rho \sim a^{-6} \), where \( a \) denotes the cosmological scale factor. Using this relation and the fact that \( \rho' = \frac{a^4}{\sqrt{\kappa_E}} \) it is easy to show that:

\[
-6 \frac{\dot{a}}{a} = \frac{\dot{\rho}}{\rho} = \sqrt{2} \rho'.
\]  
(138)

As a result, in the classical limit we get:

\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{1}{18} \frac{(\rho')^2}{\rho} = \frac{\kappa_E}{36 \cdot 16^2 \gamma^2 \pi} \left( \rho + \frac{\Lambda_R}{8 \pi} + \frac{\rho_{\text{max}}}{2} \right).
\]  
(139)

Therefore, if we set \( \kappa_E = 96 \cdot 16^2 \gamma^2 \pi^2 \), we get:

\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{8 \pi}{3} \left( \rho + \frac{\Lambda_R}{8 \pi} + \frac{\rho_{\text{max}}}{2} \right).
\]  
(140)

This expression looks like the classical Friedmann equation except for the term with \( \rho_{\text{max}} \). Let us recall, that

\[
\rho_{\text{max}} = \frac{K}{p}.
\]  
(141)

Since \( p \sim a^2 \), we get:

\[
\rho_{\text{max}} = \frac{\tilde{K}}{a^2},
\]  
(142)

for some positive constant \( \tilde{K} \). If we move this term to the left-hand side, we get:

\[
\left( \frac{\dot{a}}{a} \right)^2 + \frac{\mathcal{R}}{a^2} = \frac{8 \pi}{3} \left( \rho + \frac{\Lambda_R}{8 \pi} \right),
\]  
(143)

where \( \mathcal{R} = -\frac{8 \pi \tilde{K}}{a^2} \). Clearly, \( \mathcal{R} \) depends on the coordinate size of our fundamental cell. We chose it to be equal 1, but of course, the model does not depend on this parameter. We can change our coordinates and choose:

\[
\mathcal{R} = -1.
\]  
(144)

As a result, in the classical limit, our model describes an evolution of the Universe with positive cosmological constant and negative intrinsic curvature. This is very surprising, because according to our study in section 4, the quantum geometries are approximately flat. Let us also notice that the only scale dependent term (term dependent on \( L \)) in the resulting classical equations is the cosmological constant term. The measurable quantity is \( \Lambda_R \), not \( \Lambda \).

Having fixed the parameter \( \kappa_E \), we can now derive the effective Friedmann equations with quantum corrections and study the dynamics of our model. Previously, we calculated the Friedmann equations in the classical limit \( \rho_{\text{max}} \ll \rho_{\text{max}} \). It is straightforward to repeat the steps for the generic case and arrive at effective Friedmann equations with quantum corrections:

\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{8 \pi}{3} \left( \rho + \frac{\Lambda_R}{8 \pi} + \frac{\rho_{\text{max}}}{2} \right) \left( 1 - 2 \frac{\rho + \frac{\Lambda_R}{8 \pi}}{\rho_{\text{max}}} \right),
\]  
where \( \rho_{\text{max}} = \frac{3}{4 \pi} \frac{1}{a^2} \).

\]  
(145)

Our study the effective quantum dynamics of the model, we will not use be based on the Friedmann equation in the form (145) but rather in the form (135). The equation (135) is not
Figure 4. An evolution of the energy density $\rho$ predicted by the effective Hamiltonian (128)—a solution to the differential equation (146) with $\Lambda_R = 10^{-3}$, $\rho_{\text{max}} = 10^5 \rho_1^3$. The evolution is periodic (a). On figure (b) the quantum evolution (solid line) is compared with (expanding and contracting branch of) the classical evolution (dotted lines).

convenient for numerical calculations. Therefore, following [70], we will use a second order equation derived from (135):

$$\rho'' = 48\pi \left( \frac{1}{3} \rho_{\text{max}} - \frac{2}{3} \rho_{\text{tot}}^2 - \frac{2 \rho_{\text{tot}}}{\rho_{\text{max}}} \right). \quad (146)$$

Figure 4 illustrates a solution to (146) calculated using the Runge–Kutta method of order 5(4) from the scipy package (scipy.integrate.RK45 routine) for $\Lambda_R = 10^{-3}, \rho_{\text{max}} = 10^5 \rho_1^3$. Let us underline that the energy density is bounded and the evolution is periodic. As expected, the quantum gravity effects resolve the Big Bang singularity—the model predicts Big Bounce. It also predicts a recontraction phase after the expansion. Let us estimate the values of $\rho$ at the Big Bounce and the recontraction. At these points $\rho'$ is equal 0. From (135) we infer that $\rho' = 0$ if $\rho = 0$ or $\rho_{\text{max}} = 2 \rho_{\text{tot}}$. The first condition is achieved only for the trivial solution $\rho \equiv 0$ (this follows from the uniqueness of the solution of the ordinary differential equation (135)). The second condition leads to:

$$K \rho^\frac{4}{3} = \frac{2 \Lambda_R}{4\pi}, \quad (147)$$

where we have used the fact that $\rho_{\text{max}} \sim \rho^\frac{4}{3}$ and denoted by $K$ the proportionality factor. For positive $\rho$, this equation has 2 solutions:

1. One solution is for $\rho$ very large. It corresponds to the Big Bounce. Let us denote this solution by $\rho_{BB}$. In this case we can omit the cosmological term and we get:

$$K \rho_{BB}^\frac{4}{3} \approx 2 \rho_{BB} \implies \rho_{BB} \approx \left( \frac{K}{2} \right)^\frac{3}{4}. \quad (148)$$

For the case depicted on figure 4 the value is $\rho_{BB} \approx 1.12 \cdot 10^7$. 
Another solution is for $\rho$ very small, which corresponds to the recontraction. Let us denote this solution by $\rho_{RC}$. In this case we can omit the term proportional to $\rho$ and get:

$$K_{\rho} \frac{1}{4\pi} \approx \frac{\Lambda_{R}}{4\pi} \Rightarrow \rho_{RC} \approx \left( \frac{\Lambda_{R}}{4\pi K} \right)^{3}.$$  \hspace{1cm} (149)

For the case depicted on figure 4 the value is $\rho_{RC} \approx 5.04 \cdot 10^{-28}$.

Far away from the turning points, the evolution is well approximated by the classical solution (for the classical solution $\rho + \frac{\Lambda_{R}}{8\pi} \ll \rho_{\text{max}}$).

Let us study the sign of the second derivate at the turning points. At the turning points we have:

$$\rho'' = 48\pi \left( \frac{\Lambda_{R}}{24\pi} - \frac{2}{3} \rho \right).$$  \hspace{1cm} (150)

1. At the Big Bounce the function should have a maximum $\rho''(\phi) < 0$:

$$\frac{\Lambda_{R}}{24\pi} < \frac{2}{3} \rho_{BB} \approx \frac{2}{3} \left( \frac{K}{2} \right)^{2}.$$  \hspace{1cm} (151)

This leads to a condition:

$$\Lambda_{R} < 4\sqrt{2\pi} \left( K \right)^{2}.$$  \hspace{1cm} (152)

2. At the recolapse the function should have a minimum $\rho''(\phi) > 0$:

$$\frac{\Lambda_{R}}{24\pi} \frac{2}{3} \rho_{RC} \approx \frac{2}{3} \left( \frac{\Lambda_{R}}{4\pi K} \right)^{3}.$$  \hspace{1cm} (153)

This leads to a condition:

$$\Lambda_{R} < 2\pi \left( K \right)^{3}.$$  \hspace{1cm} (154)

Let us notice that the conditions (152) and (154) are in fact milder than the conditions implicitly assumed in the approximations: $\rho_{BB} \gg 1, \rho_{RC} \ll 1$. If the conditions are satisfied the derivative $\rho'$ changes the sign at the Big Bounce and the recolapse. This means that at the turning points a solution of the equation (135) with the positive derivative (overall plus sign on the right-hand-side) is glued with the solution of the equation (135) with the negative derivative (overall minus sign on the right-hand-side). Let us start for example with $\rho(0) = \rho_{RC}$. We follow the solution of the equation (135) with the plus sign:

$$\rho' = 4\sqrt{6\pi} \sqrt{\left( \frac{\rho_{\text{max}}}{4} - \frac{\rho_{\text{tot}}^{2}}{\rho_{\text{max}}} \right)}$$  \hspace{1cm} (155)

until we reach $\rho_{BB}$. Next, we follow the solution of the equation (reverse time solution):

$$\rho' = -4\sqrt{6\pi} \sqrt{\left( \frac{\rho_{\text{max}}}{4} - \frac{\rho_{\text{tot}}^{2}}{\rho_{\text{max}}} \right)}$$  \hspace{1cm} (156)

until we reach $\rho_{RC}$ again. We repeat the steps to get the evolution. The resulting function $\rho(\phi)$ is periodic. We can integrate (155) to get that the period of the evolution:

$$T = 2 \int_{\rho_{RC}}^{\rho_{\text{inf}}} \frac{d\rho}{4\sqrt{6\pi} \sqrt{\left( \frac{\rho_{\text{max}}(\rho)}{4} - \frac{\rho_{\text{tot}}^{2}(\rho)}{\rho_{\text{max}}(\rho)} \right)}}.$$  \hspace{1cm} (157)
7. Summary

We selected the zero internal curvature sector of a homogeneous-isotropic space proposed in [31]. The space can be constructed in the following way. In [31] we introduced a space $\mathcal{H}_{\text{loops}}^{\text{loops}}$, which is spanned by vectors of the form:

$$|l, \iota\rangle,$$

where $l$ runs through all possible loop configurations (see section 3, in particular figure 1) and $\iota$ runs through a basis of certain intertwiners. In our approach each link of the corresponding spin-network is labeled with spin $\frac{1}{2}$ (i.e. each side of the lattice and each loop). Let us underline, that the states $|l, \iota\rangle$ corresponding to different loop configurations are orthogonal

$$\langle l', \iota'|l, \iota\rangle = \delta_{l',l} \delta_{\iota',\iota}.$$  

For each $l$ we construct a unique intertwiner $\iota_l$ which is built from certain Livine–Speziale intertwiners. This leads us to a subspace spanned by vectors

$$|l\rangle = |l, \iota_l\rangle.$$  

The states do not have the necessary symmetry properties. Following [31] we project them onto a subspace of homogeneous-isotropic states by averaging over a group of discrete rotations $O_{\text{cube}}$ (orientation preserving symmetries of a cube):

$$|\{l\}\rangle = \frac{1}{\sqrt{|\mathcal{O}_l|}} \sum_{l' \in \mathcal{O}_l} |l'\rangle,$$  

where $\mathcal{O}_l$ denotes the orbit of the loop configuration $l$. Finally, for each $k \in \mathbb{N}$ we specify in a unique way a loop configuration $l_k$ with $k$ loops (see section 3.3), $\mathcal{O}(l_k) = k$. The choice is made in a way which makes the calculations easier. Finally, our cosmological Hilbert space, which we denoted by $\mathcal{H}_{\text{flat}}$, is spanned by the following states:

$$|k\rangle := |\{l_k\}\rangle.$$  

In section 4 we argued that the states encode quantum geometries with zero intrinsic curvature (which is shown by studying the expectation values of the area and dihedral angle operators). The resulting Hilbert space resembles the loop quantum cosmology Hilbert space. In fact, thanks to the result from [39] the states are eigenstates of the volume operator:

$$\hat{V}|k\rangle = \frac{\kappa_0}{8} \left( \frac{8\pi G\hbar}{c^3} \right)^2 \sqrt{\frac{4\sqrt{3}}{3}} (k+2)(k+3)(k+4) |k\rangle.$$  

After constructing the states, we project the quantum Hamiltonian of Loop Quantum Gravity to our space. Strictly speaking, we consider an operator $\hat{\Theta}_\Lambda$ defined on (a dense domain of) $\mathcal{H}_{\text{flat}}$. It has the following defining property. For any pair of states $|\Psi\rangle$ and $|\Psi'\rangle$ which are finite linear combinations of the states $|k\rangle$ the matrix elements of the operator $\hat{\Theta}_\Lambda$ coincide with the matrix elements of the gravitational part of scalar constraint operator $\sqrt{q}C^\Lambda$:

$$\langle \Psi' | \hat{\Theta}_\Lambda | \Psi \rangle \approx \langle \Psi' | \sqrt{q}C^\Lambda | \Psi \rangle.$$  

Our Hamiltonian is

$$\hat{H} = \sqrt{|-2\hat{\Theta}_\Lambda|}_+, \tag{165}$$

where $|\cdot|_+$ denotes the positive part of an operator.
The correspondence (164) looks very similar to the requirement for a relation between loop quantum cosmology and loop quantum gravity proposed in [32–34]. We intentionally used the symbol \( \approx \) in (164) to underline that the equality holds only approximately. We made some simplifying assumptions, which are discussed in section 5. Let us recall what simplifications we made. We omitted the so-called Lorentzian part of the gravitational scalar constraint operator corresponding to operator \( \int d^3x\sqrt{-g} R^{(3)} \) (because the quantum geometries are intrinsically flat—see section 4 in particular figure 2), we approximated the matrix elements of the Euclidean part using a formula based on the stationary phase analysis (see sections 5.4 and 5.5, in particular figure 3) and we approximated the cosmological term with the highest order term (proportional to \( k^3 \)).

In order to make the relation with loop quantum cosmology closer, in section 6.2 we identified the standard operators \( \hat{p} \) and \( \hat{c} \). We introduced a fundamental cell \( \mathcal{V}_0 \) and considered a lattice with \( L^3 \) cubes in \( \mathcal{V}_0 \). In this case we denoted the states \( |k\rangle \) by \( |k,L\rangle \) to indicate lattice dependence. We introduced an operator \( \hat{p} \) which acts as multiplication operator on the states \( |k,L\rangle \) and has the property that its expectation value in a state \( |k,L\rangle \) is equal to the expectation value of the area of a side of the fundamental cell in the same state \( |k,L\rangle \):

\[
\langle k,L|\hat{p}|k,L\rangle = \frac{1}{6} \langle k,L|\hat{A}(\mathcal{V}_0)|k,L\rangle.
\] (166)

We also introduced a shift operator \( e^{-i\hat{c}\hat{\tilde{c}}} \) (see (120)) and expressed the operator \( \hat{\Theta}_A \) (the operator \( \Theta_A \) defined on a refined lattice) in the form which makes the comparison with loop quantum cosmology easier:

\[
\hat{\Theta}_A = \frac{3}{4\pi^2} \left( e^{-i\hat{c}\hat{\tilde{c}}\frac{p^2}{\hat{\tilde{c}}}} + \hat{p}^2 e^{i\hat{c}\hat{\tilde{c}}} \right) + \frac{\Lambda_R}{8\pi p^3},
\] (167)

where \( \hat{\mu} = \sqrt{3}L^2 \), \( \Lambda_R = \frac{\Lambda}{2\pi} \).

From (167) we calculated the effective Friedman equation (see section 6.3):

\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi}{3} \left( \rho + \frac{\Lambda_R}{8\pi} + \frac{\rho_{\max}}{2} \right) \left( 1 - 2 \frac{\rho + \frac{\Lambda_R}{8\pi}}{\rho_{\max}} \right), \quad \text{where } \rho_{\max} = \frac{3}{4\pi^2} \frac{1}{a^2}.
\] (168)

We have analyzed the dynamics of the effective model and showed that it predicts a Big Bang singularity resolution (in the form of Big Bounce) and a recontraction after the expansion phase. The resulting evolution is periodic. An example is shown on figure 4. The Big Bounce and recontraction are at the points where \( \rho + \frac{\Lambda_R}{8\pi} = \frac{1}{2}\rho_{\max} \). In the region, where \( \rho + \frac{\Lambda_R}{8\pi} \ll \rho_{\max} \) we recover a classical Friedman equation:

\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{1}{a^2} = \frac{8\pi}{3} \left( \rho + \frac{\Lambda_R}{8\pi} \right),
\] (169)

where \( a \) denotes the cosmological scale factor. This is a Friedmann equation with negative spatial curvature. This curvature term is unexpected (as we show in section 4 the quantum geometries are flat) and may be traced back to the appearance of the fourth (unwanted) term in (128).

8. Discussion and outlook

The results from this paper have important conceptual consequences. In our model the symmetry reduction is done at the quantum level. As a result, any test of the reduced theory would be a test of the full loop quantum gravity theory. Let us notice that any physical prediction of
loop quantum cosmology (such as deviations in the classical CMB power spectrum [72–79]) supports or denies LQC but has only indirect consequences for loop quantum gravity. In addition, loop quantum cosmology applies to cosmological models and has limited application to other symmetry reduction schemes (used for example in the study of black-hole physics). We expect that the techniques developed in our research can be applied to other symmetry reduced models, possibly leading to further tests of the same full theory. It would be very interesting to extend the proposal to inhomogeneous geometries. In such setting it could be tested if the quantum Hamiltonian creating ultra-local loops can develop long-range correlations. In the current approach this was not an issue because of the very strong symmetry conditions.

The results have also important technical consequences. Although we restrict to homogeneous-isotropic sector, inhomogeneities and anisotropies can be taken into account. For example considering small perturbations of the homogeneous-isotropic geometries could be done now in a single scheme. Let us recall in the recent proposal in loop quantum cosmology [72–75] the background is quantized using loop quantum cosmology quantization but perturbations are quantized using the Fock quantization. Moreover, our approach introduces a natural splitting of the quantum Hamiltonian into homogeneous-isotropic part and the remaining part, which suggests that in our approach the technical tool appropriate for this problem is the perturbation theory of operators. Recently there has been some progress in applying the perturbation theory of operators in loop quantum cosmology [80–82] and loop quantum gravity [38, 66].

In our model the cosmological constant, becomes a running coupling constant. The renormalized cosmological constant \( \Lambda_R = \frac{\Lambda}{L^6} \) runs to zero as \( L \) goes to infinity or in other words the lattice becomes finer and finer (\( L^3 \) is the number of cubes in the fundamental cell \( V_0 \), see section 6.2). This means that our model is in accord with observations, where the cosmological constant is very small in Planck units. This can be explained by taking very fine lattice, i.e. very large but not infinite number \( L \) of cubes in a fundamental cell. The scaling of the cosmological constant with the number of geometry quanta is qualitatively similar to the scaling obtained [83, 84]. In [83, 84] the authors relate the very small value of the cosmological constant to the very large cardinality of the causal set (quanta of space-time). In similar manner we relate the very small value of the cosmological constant to the very large number of (quantum) cubes (quanta of space).

We shed some new light on the relation between LQC and LQG. In this paper we projected the full loop quantum gravity Hamiltonian onto our space of homogeneous-isotropic states corresponding to zero intrinsic curvature geometries. The resulting operator resembles the loop quantum cosmology Hamiltonian. The advantage of our approach is that the states are naturally embedded in the full LQG Hilbert space and the matrix elements of the full loop quantum gravity Hamiltonian coincide with matrix elements of the quantum Hamiltonian in the reduced theory. These are the conditions proposed in [32–34] as a proper correspondence between LQC and LQG.

In the literature, when the cosmological sector is considered, one either considers a Universe composed of many small fundamental building blocks (cubes, tetrahedra), for example in LQC [10–13] and in group field theory approach to cosmology [26, 27], or a Universe composed of a few fundamental building blocks, for example in spin-foam cosmology [48]. In the second case classical dynamics arises in a large spin limit. Our approach is in the middle. We consider large number of cubes in the fundamental cell but at the same time we consider a large number of loops approximation (which actually translates into large spin approximation). We do not require the volume of each fundamental building block to be astronomically large but large enough for our approximation to hold (a couple of orders of magnitude higher than the Planck volume).
We restricted our space of homogeneous-isotropic states to a subspace of states intrinsically flat. Let us notice, that similar restriction on the intertwiner spaces has been considered in [26, 27]. In [26, 27] each intertwiner space is reduced to 1-dimensional subspace spanned by the highest volume eigenvalue, in our approach we restrict to 1-dimensional subspace spanned by the Livine–Speziale coherent intertwiner (for a chosen set of normalized vectors $\hat{n}_I$).

We used the Rovelli–Smolin volume operator [40] in order to simplify our calculations. Let us recall that in [39] we have shown that on the spaces of spin $\frac{1}{2}$ intertwiners it is proportional to the identity operator and we calculated the proportionality factor. We expect that if the Ashtekar–Lewandowski volume operator [85] was use instead, the results would be similar, because we are effectively concerned with the matrix elements of the operator between Livine–Speziale coherent states in the large spin limit (at the technical level, the large number of loops translates into the large spin limit—see section 4.3).

There are a number of technical problems which need a further study. We will name some of them:

- In the classical limit we recovered Friedmann equations with positive cosmological constant and negative spatial curvature. The appearance of the spatial curvature term is unexpected and may be traced back to an unwanted fourth term in (128), which is quadratic in the momentum. This term could be mitigated by modifying the Euclidean part of the scalar constraint $\frac{1}{2}(C_E + C_E^\dagger)$ into:

$$\frac{1}{2}(C_E + C_E^\dagger - \sqrt{C_E^\dagger C_E}),$$

but we do not see how such additional term could be derived from fundamental principles. Let us only notice that in the literature there is no consensus how the symmetric part should be extracted from the operator $C_E$. For example in [86] the authors consider $\frac{1}{2}(C_E - C_E^\dagger)$ as the Euclidean part.

- We made some simplifying assumptions, for example we approximated the gravitational part of the scalar constraint operator in the large $k$ approximation. It would be important to study how the lower order terms affect the dynamics.

- We studied the dynamics through an effective Hamiltonian constraint. A next step is to study an evolution of Gaussian states, which can be constructed in complete analogy to the Gaussian states in LQC [13, 70], and compare their evolution with our effective dynamics.

- Our zero-intrinsic curvature space $\mathcal{H}_{\text{flat}}$ is not preserved by the quantum Hamiltonian of the full theory. It would be interesting to verify if the space is preserved in some approximate sense. For example, it would be interesting to check if the eigenvectors from the reduced theory solve the full eigenvalue equation approximately.

- For each $k \in \mathbb{N}$ we chose a loop configuration $l_k$ with $k$ loops (see section 3.3). This choice was made to simplify the calculations. An alternative would be to consider states $|l\rangle$ as some other combination of states $|l\rangle$ such that $\Sigma(l) = k$. We believe that one proposal is particularly worth investigating. Let us describe it shortly. Let us notice that there are 15 pairs of sides of a cubical lattice meeting at a given node $n$ but 3 pairs are antiparallel. This gives 12 pairs $(I, J), I < J$ that appear in the domain of a loop configuration. We can order them (lexicographically) and number with indices in the set $\{1, \ldots, 12\}$. Let us denote by $\alpha$ a function that assigns to a links its corresponding index in $\{1, \ldots, 12\}$. Let us consider the following function:

$$f_k(x_1, x_2, \ldots, x_{12}) = (x_1 + \ldots + x_{12})^k = \sum_{n_1, \ldots, n_{12} \geq 0, n_1 + \ldots + n_{12} = k} \left(\begin{array}{c} k \\ n_1, \ldots, n_{12} \end{array}\right) \cdot x_1^{n_1} \cdot x_2^{n_2} \cdots \cdot x_{12}^{n_{12}}.$$
Clearly, in the formula above:

\[ \binom{k}{n_1, \ldots, n_{12}} = \frac{k!}{n_1! n_2! \ldots n_{12}!} \]  

(172)

is the multinomial coefficient. We construct an alternative state to our state \(|j\rangle|k\rangle_i\) in the following way. Firstly, we construct a state

\[ |k\rangle = \sum_{n_1, \ldots, n_{12} : n_1 + \cdots + n_{12} = k} \binom{k}{n_1, \ldots, n_{12}} |l_{n_1, \ldots, n_{12}}\rangle, \]  

(173)

where \(l_{n_1, \ldots, n_{12}}\) is a loop configuration such that \(l_{n_1, \ldots, n_{12}}(I,J) = n_{\alpha(I,J)}\). The alternative state to \(|k\rangle\) is obtained by normalizing the state \(|k\rangle\). The resulting state has interesting properties. First of all, it is homogeneous-isotropic. The action of element \(g \in O_{\text{cube}}\) induces an action of \(g\) on the pairs of links which results in a permutation of the numbers \(1, \ldots, 12\). Since the function \(f_k\) is permutation invariant, the resulting state is homogeneous-isotropic. Moreover,

\[ f_{k+1}(x_1, x_2, \ldots, x_{12}) = f_k(x_1, x_2, \ldots, x_{12}) \cdot (x_1 + \cdots + x_{12}). \]  

(174)

This has an effect that each \(\hat{C}^\dagger_{IJ}\) contributes with the same factor to the matrix elements of \(\hat{C}^\dagger\) in such states. Let us recall that in the proposal studied in this paper only certain \(\hat{C}^\dagger_{IJ}\) contributed and we needed to consider the three cases \(k = 3n, k = 3n + 1, k = 3n + 2\) separately.

We expect that the resulting cosmological Hamiltonian will not differ much but the Hilbert space \(\mathcal{H}_{\text{flat}}\) will be different and its properties may change. This is important for example in the study of the previous technical problem, i.e. the study of approximate invariance of \(\mathcal{H}_{\text{flat}}\). Such choices probably lead to more complicated calculations and we leave them for future research.

Data availability statement

The data cannot be made publicly available upon publication because no suitable repository exists for hosting data in this field of study. The data that support the findings of this study are available upon reasonable request from the authors.

Acknowledgments

I would like to thank Simone Speziale and Alejandro Perez from Centre de Physique Theorique, Aix-Marseille University for warm hospitality during my research visit and for stimulating discussions. This work was supported by the National Science Centre, Poland Grant No. 2018/28/C/ST9/00157.

Appendix A. Some properties of the coherent intertwiners

In this section we show some basic properties of the coherent intertwiners that we use in the paper.
A.1. Explicit form of the 2-valent Livine–Speziale coherent intertwiner

Let us notice that any element of the space \( \text{Inv} \{ \mathcal{H}_{1/2} \otimes \mathcal{H}_{1/2} \} \) is proportional to \( \frac{1}{\sqrt{2}} \epsilon^{AB} \). Since \( \epsilon_n \) is normalized, it can only differ from \( \frac{1}{\sqrt{2}} \epsilon^{AB} \) by phase. Let \( \vec{n}_I^J \) be the components of the vector \( |1/2, \vec{n}\rangle \) in the spinor basis \(|1/2, A\rangle, A \in \{ -\frac{1}{2}, \frac{1}{2} \} \):

\[
|1/2, \vec{n}\rangle = \vec{n}_I^J |1/2, 1/2\rangle + \vec{n}_I^{-J} |1/2, -1/2\rangle.
\]

Let us denote by

\[
\epsilon(\vec{n}_I, \vec{n}_J) = \epsilon_{AB} \vec{n}_I^A \vec{n}_J^B,
\]

where \( A, B \subseteq \{ -\frac{1}{2}, \frac{1}{2} \} \) are spinor indices. Using this notation

\[
\epsilon^{AB}_n = \frac{1}{\sqrt{2}} \frac{\epsilon(\vec{n}_I, \vec{n}_J)}{|\epsilon(\vec{n}_I, \vec{n}_J)|} \epsilon^{AB}.
\]

For completeness, let us calculate \( |\epsilon(\vec{n}_I, \vec{n}_J)| \):

\[
|\epsilon(\vec{n}_I, \vec{n}_J)|^2 = \vec{n}_I^A \epsilon_{AB} \vec{n}_J^B \vec{n}_I'^A \epsilon_{AB} \vec{n}_J'^B = -\vec{n}_I^A \epsilon_{AB} \vec{n}_J^B \vec{n}_I'^A \epsilon_{AB} \vec{n}_J'^B
\]

\[
= -\frac{1}{4} \text{Tr} \left( |1/2, \vec{n}\rangle \langle 1/2, \vec{n}| \epsilon |(1/2, \vec{n}\rangle \langle 1/2, \vec{n}| \epsilon \right)
\]

\[
= -\frac{1}{4} \text{Tr} \left( (1 + \hat{n}_I \cdot \hat{\sigma})\epsilon (1 + \hat{n}_J \cdot \hat{\sigma})\epsilon \right) = -\frac{1}{4} \text{Tr} \left( (1 + \hat{n}_I \cdot \hat{\sigma})(-1 + \hat{n}_I \cdot \hat{\sigma}) \right)
\]

\[
= \frac{1}{2} (-2 + 2 \vec{n}_I \cdot \vec{n}_I) = \frac{1}{2}.
\]

Inserting this result into (A3) we obtain:

\[
\epsilon^{AB}_n = \epsilon(\vec{n}_I, \vec{n}_J) \epsilon^{AB}.
\]

A.2. Invariance of the coherent intertwiners

The action of the group of orientation preserving isometries of a cube \( O_{\text{cube}} \) induces an action of the permutation group on the links of the graph. This permutation induces an action of the group on loop configurations \( \mathcal{L} \) and on intertwiners. The states \( |I, \tau_I \rangle \) transform in the following way:

\[
U_g |I, \tau_I \rangle = \text{sgn}(g) |g \cdot I, R(g) \tau_I \rangle,
\]

where the action of \( g \) on \( I \) is the following [31]:

\[
(g \cdot I)(J) = \begin{cases} \text{sgn}(g) |g^{-1}(I), g^{-1}(J)\rangle, & \text{if } g^{-1}(I) < g^{-1}(J) \\ |g^{-1}(J), g^{-1}(I)\rangle, & \text{otherwise.} \end{cases}
\]

\[
R(g) \text{ acts by permuting the indices of the intertwiners:}
\]

\[
R(g) |\vec{j}_1, \ldots, \vec{j}_6; \vec{n}_1 \ldots \vec{n}_6 \rangle = |\vec{j}_{\sigma^{-1}_1(1)}, \ldots, \vec{j}_{\sigma^{-1}_6(6)}; \vec{n}_{\sigma^{-1}_1(1)}, \ldots, \vec{n}_{\sigma^{-1}_6(6)} \rangle.
\]

In appendix A.2 we show that:

\[
U_g |I, \tau_I \rangle = |g \cdot I, \tau_{g \cdot I} \rangle.
\]

We will show now that

\[
U_g |I, \tau_I \rangle = |g \cdot I, \tau_{g \cdot I} \rangle.
\]
In the proof the fundamental role is played by the transformation property of the Perelomov coherent states under the action of an SU(2) group:
\[ \rho_j(u)|j, n\rangle = e^{\Phi(j, n, u)}|j, u, n\rangle, \quad (A11) \]
where \( u \cdot n \) is the unit vector obtained by rotating the vector \( n \) with an SO(3) group element corresponding to \( u \), the phase \( \Phi(j, n, u) \) is given in \([87]\) and its particular form will not be used explicitly here.

Let us consider the transformation property of the intertwiner at the node \( n_0 \). In this case we can write the transformation (A8) in the following form:
\[ R(g)|1/2, 1/2; n_1, n_2\rangle = |g \cdot 1/2, 1/2; g \cdot n_1, g \cdot n_2\rangle. \quad (A12) \]
From the property (A11) and from the invariance of the Haar measure it follows that
\[ |g \cdot 1; g \cdot n_1 \ldots g \cdot n_6\rangle \sim |g \cdot 1; n_1 \ldots n_6\rangle, \quad (A13) \]
where \( \sim \) denotes proportionality up to a phase.

Let us consider the transformation property of an intertwiner at a bivalent node \( n \) splitting a loop into two links. In this case
\[ R(g)|1/2, 1/2; n_1, n_2\rangle = \begin{cases} |1/2, 1/2; n_1, n_1\rangle, \quad \text{if } g \text{ flips the loop,} \\ |1/2, 1/2; n_1, n_2\rangle, \quad \text{otherwise.} \end{cases} \quad (A14) \]
After the action of the group element \( g \), the loop is between links \( I' \) and \( J' \), where:
\[ \sigma_g(I) = I', \quad \sigma_g(J) = J'. \quad (A15) \]
There are two cases:

- **\( I' < J' \)**. In this case
\[ R(g)|1/2, 1/2; n_1, n_2\rangle = |1/2, 1/2; n_1, n_1\rangle \\
= |1/2, 1/2; n_{\sigma_{g^{-1}(I')}}(I'), n_{\sigma_{g^{-1}(J')}}(J')\rangle \sim |1/2, 1/2; n_{I'}, n_{I'}\rangle. \quad (A16) \]

- **\( I' > J' \)**. In this case
\[ R(g)|1/2, 1/2; n_1, n_2\rangle = |1/2, 1/2; n_1, n_1\rangle \\
= |1/2, 1/2; n_{\sigma_{g^{-1}(I')}}(I'), n_{\sigma_{g^{-1}(J')}}(J')\rangle \sim |1/2, 1/2; n_{I'}, n_{I'}\rangle. \quad (A17) \]
The phases in (A13), (A16) and (A17) can be calculated using formula (A11). Since 2-valent intertwiners come with dagger into the definition of the coherent state (35), there is substantial cancellation of phases when \( R(g) \) is applied. As a result, to the overall phase contribute only the Perelomov coherent states that correspond to the sides of the lattice and
\[ R(g)|l, \iota_1\rangle = \text{sgn}(g)|l, \iota_1\rangle. \quad (A18) \]
Combining this result with the transformation property of our coherent states (35) gives the covariance property of our states:
\[ U_\iota|l, \iota_1\rangle = |g \cdot 1, l_{g^{-1}}\rangle. \quad (A19) \]
It will be convenient to introduce a notation
\[ |l\rangle := |l, \iota_1\rangle. \quad (A20) \]
Let us notice that from the transformation property (A19) it follows that:
\[ U_g |l\rangle = |g \cdot l\rangle. \]
In particular a state $|l\rangle$ is invariant under the stabilizer group of $l$.

### Appendix B. Symmetries of the matrix elements of the Euclidean part of the scalar constraint

In [31] we studied in detail the consequences of an the invariance of the Hamiltonian operator. In particular, for the operator $\tilde{C}_E$ we have:
\[ \forall g \in O_{\text{cube}}, U_g^\dagger \tilde{C}_E U_g = \tilde{C}_E. \]  
By using this property we will reduce the problem of calculating matrix elements of $\tilde{C}_E$ between states $|k\rangle$ into a simpler problem of calculating matrix elements of $\tilde{C}_{EIJ}$ for some $I,J$ between states $|l_k\rangle$. In particular, from (B1) and (48) it follows that
\[ \langle k+1| \tilde{C}_E^\dagger |k\rangle = \frac{1}{24 \cdot \sqrt{\#S_{k+1} \#S_k}} \sum_{g \in O_{\text{cube}}} \langle k+1| U_g^\dagger \tilde{C}_E U_g |k\rangle \]
\[ = \frac{1}{\sqrt{\#S_{k+1} \#S_k}} \sum_{g \in O_{\text{cube}}} \langle k+1| U_g^\dagger \tilde{C}_E^\dagger |k\rangle. \]  
Let us recall that in the formula above $\#S_k$ is the number of elements in the stabilizer group of $l_k$. We have:
\[ \#S_{l_n} = \begin{cases} 24, & \text{if } n = 0, \\ 6, & \text{if } n > 0. \end{cases} \]  
\[ \#S_{l_{n+2}} = \#S_{l_{n+2}} = 2. \]  
Let us notice that the only non-zero contribution in the sum in (B2) comes from elements $g'$ that map links $\{\ell_1, \ell_3\}$ into: $\{\ell_1, \ell_5\}$ or $\{\ell_2, \ell_6\}$ or $\{\ell_4, \ell_3\}$. Since $g'$ is orientation preserving isometry of a cube, it is completely determined by its action on $e_1$ and $e_2$. Therefore there are 6 possibilities:
1. $g'(1) = 1, g'(3) = 3$,
2. $g'(1) = 3, g'(3) = 1$,
3. $g'(1) = 2, g'(3) = 6$,
4. $g'(1) = 6, g'(3) = 2$,
5. $g'(1) = 4, g'(3) = 5$,
6. $g'(1) = 5, g'(3) = 4$.

Each $g'$ from the list above is a different element of $O_{\text{cube}}$. Direct calculation shows that the elements form a group—the stabilizer of loop configuration $l_n, n \in \mathbb{N}$. Taking into account that $\tilde{C}_E$ is a sum of operators $\tilde{C}_{EIJ}$ and that $g'$ is in the stabilizer group of $l_n$:
\[ \langle k+1| U_{g'} \tilde{C}_E \dagger |k\rangle = \sum_{I,J} \epsilon_{IJ} \langle k+1| U_{g'} \tilde{C}_{EIJ} \dagger |k\rangle = 2 \langle k+1| U_{g'} (\tilde{C}_{E13}^\dagger + \tilde{C}_{E25}^\dagger + \tilde{C}_{E34}^\dagger) |k\rangle. \]  
The factor of 2 in the last equality comes from the fact that $\tilde{C}_{EIJ} = \tilde{C}_{EJI}$.

There are three cases:
1. \( k = 3n \). Since \( g' \) is in the stabilizer group of \( l_k \) and \( \hat{C}_E \) transforms according to (B1):
\[
\langle k+1 | U_{g'} \hat{C}_E^\dagger | l_k \rangle = \langle k+1 | \hat{C}_E^\dagger | l_k \rangle = 2 \langle k+1 | \hat{C}_{E_{13}}^\dagger | l_k \rangle. \tag{B6}
\]
As a result:
\[
\langle k+1 | \hat{C}_E^\dagger | k \rangle = \frac{2 \cdot \#S_{l_n}}{\sqrt{\#S_{l_n} \cdot \#S_{l_{n+1}}}} \langle k+1 | \hat{C}_{E_{13}}^\dagger | l_k \rangle. \tag{B7}
\]
After inserting the values from (B3) and (B4) we obtain:
\[
\langle 3n+1 | \hat{C}_E^\dagger | 3n \rangle = \begin{cases} 4 \sqrt{3} \langle k+1 | \hat{C}_{E_{13}}^\dagger | l_k \rangle, & \text{if } n = 0, \\ 2 \sqrt{3} \langle k+1 | \hat{C}_{E_{13}}^\dagger | l_k \rangle, & \text{if } n > 0. \end{cases} \tag{B8}
\]
2. \( k = 3n + 1 \). In this case \( \langle k+1 | U_{g'} \hat{C}_E^\dagger | l_k \rangle = 0 \), because after the action of \( \hat{C}_{E_{13}} \) there would be \( n + 2 \) loops between links \( l_1, l_3 \) and it is not possible to bring such loop configuration to \( l_{k+1} \) by acting with an orientation preserving symmetry of a cube. Therefore,
\[
\langle k+1 | U_{g'} \hat{C}_E^\dagger | l_k \rangle = 2 \langle k+1 | U_{g'} (\hat{C}_E^\dagger_{26} + \hat{C}_E^\dagger_{45}) | l_k \rangle. \tag{B9}
\]
Let us notice that the transformation \( h \) such that \( h(1) = 3, h(3) = 1 \) is a symmetry of the loop configuration \( l_{3n+1} \). Moreover: \( h(2) = 4, h(6) = 5 \) and \( h(4) = 2, h(5) = 6 \). As a result,
\[
\langle k+1 | U_{g'} \hat{C}_{E_{26}}^\dagger | l_k \rangle = \langle k+1 | U_{g'} U_2 \hat{C}_{E_{26}} U_2^\dagger | l_k \rangle = \langle k+1 | U_{g'} \hat{C}_{E_{26}}^\dagger | l_k \rangle. \tag{B10}
\]
Let us notice further that there are two possible \( g \) such that
\[
\langle k+1 | U_{g'} \hat{C}_{E_{26}}^\dagger | l_k \rangle \neq 0. \tag{B11}
\]
First is \( g = \text{id} \), second is \( g \) such that \( g(4) = 5, g(5) = 4 \). They both are symmetries of \( l_{k+1} \). Therefore,
\[
\langle k+1 | U_{g'} \hat{C}_{E_{26}}^\dagger | l_k \rangle = 0 \implies \langle k+1 | U_{g'} \hat{C}_{E_{26}}^\dagger | l_k \rangle = \langle k+1 | \hat{C}_{E_{26}}^\dagger | l_k \rangle. \tag{B12}
\]
As a result,
\[
\langle 3n+2 | \hat{C}_E^\dagger | 3n+1 \rangle = \frac{2 \cdot 2 \cdot \#S_{l_{n+2}}}{\sqrt{\#S_{l_{n+2}} \cdot \#S_{l_{n+1}}}} \langle l_{n+2} | \hat{C}_{E_{26}}^\dagger | l_{n+1} \rangle
\]
\[
= 4 \cdot \langle l_{n+2} | \hat{C}_{E_{26}}^\dagger | l_{n+1} \rangle. \tag{B13}
\]
3. \( k = 3n + 2 \). In this case \( g' \) is in the stabilizer group of \( l_{k+1} \) and
\[
\langle k+1 | U_{g'} \hat{C}_E^\dagger | l_k \rangle = \langle k+1 | \hat{C}_E^\dagger | l_k \rangle = 2 \langle k+1 | \hat{C}_{E_{45}}^\dagger | l_k \rangle. \tag{B14}
\]
As a result,
\[
\langle k+1 | \hat{C}_E^\dagger | k \rangle = \frac{2 \cdot \#S_{l_{n+3}}}{\sqrt{\#S_{l_{n+3}} \cdot \#S_{l_{n+2}}}} \langle l_{n+3} | \hat{C}_{E_{45}}^\dagger | l_{n+2} \rangle
\]
\[
= 2 \sqrt{3} \langle l_{n+3} | \hat{C}_{E_{45}}^\dagger | l_{n+2} \rangle. \tag{B15}
\]
Appendix C. Derivation of the second order equation for the energy density $\rho$ from the first order equation

In this section we derive the equation (146) from the equation (135). Let us apply the value of $\kappa_E = 96 \cdot 16^2 \gamma^2 \pi^2$ and write the equation (135) in an equivalent form:

$$\rho' = \pm 4\sqrt{6\pi} \left( \frac{\frac{\rho_{\text{max}}}{\rho_{\text{max}}} \left( \frac{\rho_{\text{tot}}}{\rho_{\text{tot}}} + \rho_{\text{max}} \right)}{2} - \rho_{\text{tot}} \right)$$

$$= \pm 4\sqrt{6\pi} \left( \frac{\rho_{\text{max}}}{\rho_{\text{max}}} \left( \frac{\rho_{\text{tot}}^2}{4} - \rho_{\text{tot}}^2 \right) = \pm 4\sqrt{6\pi} \sqrt{\frac{2\rho_{\text{tot}}}{\rho_{\text{max}}} - \frac{3\rho_{\text{tot}}}{\rho_{\text{max}}} \rho_{\text{tot}}^2} \right). \quad (C1)$$

Let us apply a derivative to both sides of the equation:

$$\rho'' = \pm 2\sqrt{6\pi} \left( \frac{\rho_{\text{max}}}{4} - \frac{\rho_{\text{tot}}^2}{\rho_{\text{max}}} \right)^{-\frac{1}{2}} \left( \frac{\rho'_{\text{max}}}{4} + \frac{\rho_{\text{tot}}}{\rho_{\text{max}}} \right) - \frac{\rho''_{\text{tot}}}{\rho_{\text{max}}} - \frac{\rho_{\text{tot}}}{\rho_{\text{max}}} \right) - \frac{2\rho_{\text{tot}}}{\rho_{\text{max}}} \rho_{\text{tot}}'' + \frac{\rho_{\text{tot}}''}{\rho_{\text{max}}^2} \right). \quad (C2)$$

In the equation above we used the fact that $\rho_{\text{tot}}' = \rho'$. Since $\rho_{\text{max}} \sim \rho$, we get:

$$\frac{\rho'_{\text{max}}}{\rho_{\text{max}}} = \frac{\rho'}{3}. \quad (C3)$$

Using this relation we get:

$$\rho'' = \pm 2\sqrt{6\pi} \left( \frac{\rho_{\text{max}}}{4} - \frac{\rho_{\text{tot}}^2}{\rho_{\text{max}}} \right)^{-\frac{1}{2}} \left( \frac{\rho'_{\text{max}}}{4} + \frac{\rho_{\text{tot}}}{\rho_{\text{max}}} \right) - \frac{\rho''_{\text{tot}}}{\rho_{\text{max}}} - \frac{\rho_{\text{tot}}}{\rho_{\text{max}}} \right) - \frac{2\rho_{\text{tot}}}{\rho_{\text{max}}} \rho_{\text{tot}}'' + \frac{\rho_{\text{tot}}''}{\rho_{\text{max}}^2} \right). \quad (C4)$$

Taking $\rho'$ in front and using the relation (C1) we arrive at equation (146):

$$\rho'' = 48\pi \left( \frac{\rho_{\text{max}}}{3} - \frac{2\rho_{\text{tot}}}{3\rho_{\text{max}}} - \frac{2\rho_{\text{tot}}}{\rho_{\text{max}}} \right). \quad (C5)$$

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