ON NODAL QUINTIC FOURFOLD

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Abstract. We use the Shokurov connectedness principle and the Corti inequality to prove the birational superrigidity of a nodal hypersurface in $\mathbb{P}^5$ of degree 5.

We assume that all varieties are projective, normal and defined over $\mathbb{C}$.

1. Introduction.

The following result is proved in [6].

Theorem 1. Every smooth hypersurface in $\mathbb{P}^4$ of degree 4 is birationally superrigid\(^1\).

The following generalization of Theorem 1 is proved in [9].

Theorem 2. Every smooth hypersurface in $\mathbb{P}^5$ of degree 5 is birationally superrigid.

The following generalization of Theorem 1 is proved in [10] and [8].

Theorem 3. Every $\mathbb{Q}$-factorial nodal\(^2\) hypersurface in $\mathbb{P}^4$ of degree 4 is birationally rigid.

In this paper we prove the following result.

Theorem 4. Every nodal hypersurface in $\mathbb{P}^5$ of degree 5 is birationally superrigid.

It must be pointed out, that the proof of Theorem 4 is based on the Shokurov connectedness principle (see [12] and Theorems 7.4 and 7.5 in [7]) and the Corti inequality (see Theorem 3.1 in [4]). The proof of Theorem 4 and technique used in [2] imply that every nodal quintic fourfold can not be birationally transformed into an elliptic fibration.

2. Smooth points.

Let $X$ be a fourfold, $\mathcal{M}$ be a linear system on $X$ that does not have fixed components, and $O$ be a smooth point of $X$ such that $O$ is a center of canonical singularities of the movable log pair $(X, \lambda \mathcal{M})$, but the singularities of the log pair $(X, \lambda \mathcal{M})$ are log terminal in a punctured neighborhood of the point $O$, where $\lambda$ is a positive rational number.

Let $\pi : V \to X$ be a blow up of the point $O$, and $E$ be the $\pi$-exceptional divisor. Then

$$K_V + \lambda \mathcal{B} \sim_\mathbb{Q} \pi^* (K_X + \lambda \mathcal{M}) + (3 - m) E,$$

where $\mathcal{B}$ is a proper transform of $\mathcal{M}$ on the fourfold $V$, and $m$ is a positive rational number such that $m/\lambda$ is the multiplicity of a general divisor of $\mathcal{M}$ in the point $O$. Then $m > 1$.

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\(^2\)Let $V$ be a Fano variety such that the singularities of the variety $V$ are at most terminal and $\mathbb{Q}$-factorial singularities, and the equality $\text{rk Pic}(V) = 1$ holds. Then $V$ is called birationally rigid if it cannot be fibred into uniruled varieties by a non-trivial rational map, and $V$ is not birational to a Fano variety with terminal $\mathbb{Q}$-factorial singularities of Picard rank 1 not biregular to $V$. The variety $V$ is called birationally superrigid if it is birationally rigid and every birational automorphism of $V$ is biregular.

\(^3\)A variety is called nodal if it has at most isolated ordinary double points.
Remark 5. In fact, the inequality \( \text{mult}_O(M_1 \cdot M_2) \geq 4/\lambda^2 \) holds (see Corollary 3.4 in [4]), where \( M_1 \) and \( M_2 \) are sufficiently general divisors of the linear system \( \mathcal{M} \).

In this section we prove the following result.

**Theorem 6.** There is a line \( L \subset E \cong \mathbb{P}^3 \) such that

\[
\text{mult}_O(M_1 \cdot M_2 \cdot Y) \geq \frac{8}{\lambda^2},
\]

where \( M_1 \) and \( M_2 \) are sufficiently general divisors in the linear system \( \mathcal{M} \), and \( Y \) is an effective divisor on \( X \) such that \( \dim(\text{Supp}(Y) \cap \text{Supp}(M_1 \cdot M_2)) = 1 \) and \( L \subset \text{Supp}(\hat{Y}) \), where \( \hat{Y} \) is the proper transform of the threefold \( Y \) on the fourfold \( V \).

The claim of Theorem 6 is obvious when \( m \geq 3 \). Moreover, it follows from Theorem 7.4 and the proof of Corollary 3.5 in [4] that one of the following possibilities holds:

- the inequality \( m \geq 3 \) holds;
- there is a surface \( S \subset E \) such that \( S \) is a center of log canonical singularities of the log pair \( (V, \lambda \mathcal{B} + (m - 2)E) \);
- there is a line \( L \subset E \cong \mathbb{P}^3 \) such that \( L \) is a center of log canonical singularities of the log pair \( (V, \lambda \mathcal{B} + (m - 2)E) \).

**Lemma 7.** Suppose that there is a surface \( S \subset E \) such that \( S \) is a center of log canonical singularities of the log pair \( (V, \lambda \mathcal{B} + (m - 2)E) \). Then

\[
\text{mult}_O(M_1 \cdot M_2) \geq \frac{8}{\lambda^2},
\]

where \( M_1 \) and \( M_2 \) are general divisors in the linear system \( \mathcal{M} \).

**Proof.** Let \( B_i \) be a proper transform of the divisor \( M_i \) on the fourfold \( V \). Then

\[
\text{mult}_S(B_1 \cdot B_2) \geq \frac{4(3 - m)}{\lambda^2}
\]

by Theorem 3.1 in [4]. Therefore, we have

\[
\text{mult}_O(M_1 \cdot M_2) \geq \text{mult}_O^2(M_i) + \text{mult}_S(B_1 \cdot B_2) \geq \frac{m^2 + 4(3 - m)}{\lambda^2} \geq \frac{8}{\lambda^2},
\]

which concludes the proof. \( \square \)

Now we suppose that \( m < 3 \) and there are no two-dimensional centers of log canonical singularities of the log pair \( (V, \lambda \mathcal{B} + (m - 2)E) \) that are contained in the \( \pi \)-exceptional divisor \( E \). Therefore, there is a line \( L \subset E \cong \mathbb{P}^3 \) such that \( L \) is a center of log canonical singularities of the log pair \( (V, \lambda \mathcal{B} + (m - 2)E) \).

Let \( \eta : W \rightarrow V \) be a blow up of the curve \( L \), and \( F \) be the \( \eta \)-exceptional divisor. Then

\[
K_W + \lambda \mathcal{D} + (m - 3)\hat{E} + (m + n - 5)F \sim_{\mathbb{Q}} (\pi \circ \eta)^* (K_X + \lambda \mathcal{M})
\]

where \( \mathcal{D} \) and \( \hat{E} \) are proper transforms of the linear system \( \mathcal{M} \) and the \( \pi \)-exceptional divisor \( E \) on the fourfold \( W \) respectively, and \( n \) is a positive rational number such that the number \( n/\lambda \) is the multiplicity of a general divisor of the linear system \( \mathcal{B} \) in a general point of the curve \( L \). Therefore, we have

\[
K_W + \lambda \mathcal{D} + \hat{H} + (m - 2)\hat{E} + (m + n - 4)F \sim_{\mathbb{Q}} (\pi \circ \eta)^* (K_X + \lambda \mathcal{M} + H),
\]
where $H$ is a sufficiently general hyperplane section of the fourfold $X$ passing through the point $O$, and $\bar{H}$ is a proper transform of $H$ on the fourfold $W$. Moreover, we have

$$K_W + \lambda D + \bar{Y} + (m - 2) \bar{E} + (m + n - 3) F \sim_{\mathbb{Q}} \left( \pi \circ \eta \right)^* \left( K_X + \lambda M + Y \right),$$

where $Y$ is a general hyperplane section of $X$ such that $L$ is contained in the proper transform of $Y$ on the fourfold $V$, and $\bar{Y}$ is a proper transform of $Y$ on the fourfold $W$.

**Lemma 8.** Suppose that either $m + n \geq 4$ or there is a surface $S \subset F$ that is a center of log canonical singularities of $(W, \lambda D + (m - 2) \bar{E} + (m + n - 3) F)$ and $\eta(S) = L$. Then

$$\text{mult}_O \left( M_1 \cdot M_2 \cdot Y \right) \geq \frac{8}{\lambda^2},$$

where $M_1$ and $M_2$ are general divisors in the linear system $\mathcal{M}$.

**Proof.** Let $\bar{Y}$ be a proper transform of the divisor $Y$ on the fourfold $V$. Then

$$K_V + \lambda B + \bar{Y} + (m - 2) E \sim_{\mathbb{Q}} \pi^* \left( K_X + \lambda M + Y \right)$$

and $L$ is a center of log canonical singularities of the log pair $(V, \lambda B + \bar{Y} + (m - 2) E)$. The morphism $\pi|_Y : \bar{Y} \to Y$ is a blow up of the point $O$, the linear system $B|_Y$ does not have fixed components due to the generality in the choice of the divisor $Y$, and we can identify the divisor $G$ with the $\pi|_Y$-exceptional divisor. Let us show that $L$ is a center of log canonical singularities of the log pair $(\bar{Y}, \lambda B|_Y + (m - 2) E|_Y)$.

In the case when $m + n \geq 4$, the equivalence

$$K_{\bar{Y}} + \lambda D|_{\bar{Y}} + (m - 2) E|_{\bar{Y}} \sim_{\mathbb{Q}} \left( \eta|_{\bar{Y}} \right)^* \left( K_Y + \lambda B|_Y + (m - 2) E|_Y \right) + (3 - m - n) F|_{\bar{Y}}$$

shows that $L$ is a center of log canonical singularities of $(\bar{Y}, \lambda B|_Y + (m - 2) E|_Y)$.

Suppose that there is a surface $S \subset F$ that is a center of log canonical singularities of the log pair $(W, (m - 2) \bar{E} + (m + n - 3) F)$ and $\eta(S) = L$. Then every irreducible component of $S \cap \bar{Y}$ is a center of log canonical singularities of the log pair

$$\left( \bar{Y}, \lambda D|_Y + (m - 2) E|_Y + (3 - m - n) F|_Y \right),$$

which implies that $L$ is a center of log canonical singularities of $(\bar{Y}, \lambda B|_Y + (m - 2) E|_Y)$, because every irreducible component of the intersection $S \cap \bar{Y}$ dominates the curve $L$.

Let $B_i$ be a proper transform of the divisor $M_i$ on the fourfold $V$. Then

$$\text{mult}_L \left( B_1|_Y \cdot B_2|_Y \right) \geq \frac{4(3 - m)}{\lambda^2}$$

by Theorem 3.1 in [4]. Therefore, we have

$$\text{mult}_O \left( M_1 \cdot M_2 \cdot Y \right) = \text{mult}_O \left( M_1|_Y \cdot M_2|_Y \right) \geq \frac{m^2}{\lambda^2} + \text{mult}_L \left( B_1|_Y \cdot B_2|_Y \right) \geq \frac{8}{\lambda^2},$$

which concludes the proof. \qed

Suppose that $m + n < 4$. Then in order to prove Theorem 3.1 we must show that there is a surface $S \subset F$ such that $Z$ is a center of log canonical singularities of the log pair

$$\left( W, \lambda D + (m - 2) \bar{E} + (m + n - 3) F \right)$$

and $\eta(S) = L$. However, the last assertion is local and we may assume that $X \cong \mathbb{C}^4$. 

3
The singularities of the log pair \((H, \lambda M|_H)\) are log terminal in a punctured neighborhood of the point \(O\). Moreover, the point \(O\) is a center of log canonical singularities of the log pair \((H, \lambda M|_H)\) by Theorem 7.5 in [7]. Therefore, the equivalence

\[
K_H + \lambda D|_H + (m - 2)\bar{E}|_H + (m + n - 4)F|_H \sim_Q (\pi \circ \eta|_H)^*\left(K_H + \lambda M|_H\right);
\]

and Theorem 7.4 in [7] imply that we have the following possibilities:

- there is a curve \(C \subset F|_H\) such that \(C\) is a center of log canonical singularities of the log pair \((H, \lambda D|_H + (m - 2)\bar{E}|_H + (m + n - 4)F|_H)\);
- there is a point \(P \in F \cap H\) such that \(P\) is a center of log canonical singularities of the log pair \((\bar{H}, \lambda D|_H + (m - 2)\bar{E}|_H + (m + n - 4)F|_H)\), and there are no other centers of log canonical singularities of the log pair \((\bar{H}, \lambda D|_H + (m - 2)\bar{E}|_H + (m + n - 4)F|_H)\) except the point \(P\) that are contained in the intersection \(F \cap \bar{H}\).

**Remark 9.** In the case when there is a curve \(C \subset F \cap H\) such that \(C\) is a center of log canonical singularities of \((\bar{H}, \lambda D|_H + (m - 2)\bar{E}|_H + (m + n - 4)F|_H)\), the curve \(C\) is an intersection of the divisor \(H\) with a surface \(S \subset F\) such that \(S\) is a center of log canonical singularities of \((W, \lambda D + (m - 2)\bar{E} + (m + n - 4)F)\) and \(\eta(S) = L\).

To prove Theorem 9 we may assume that there is a point \(P \in F \cap H\) that is a center of log canonical singularities of the log pair

\[
(\bar{H}, \lambda D|_H + (m - 2)\bar{E}|_H + (m + n - 4)F|_H);
\]

but singularities of the log pair \((\bar{H}, \lambda D|_H + (m - 2)\bar{E}|_H + (m + n - 4)F|_H)\) are log terminal in a punctured neighborhood of the point \(P \in \bar{H}\).

**Remark 10.** The morphism \(\eta|_F : F \to L\) is a \(\mathbb{P}^2\)-bundle, and the intersection \(\bar{H} \cap F\) is just a fiber of \(\eta|_F\). Hence, the generality of in the choice of the divisor \(H\) implies the existence of a curve \(Z \subset F\) such that \(P = Z \cap H\) and \(Z\) is a center of log canonical singularities of the log pair \((W, \lambda D + (m - 2)\bar{E} + (m + n - 4)F)\).

The curve \(Z\) is a section of the \(\mathbb{P}^2\)-bundle \(\eta|_F\) and a center of log canonical singularities of the log pair \((W, \lambda D + (m - 2)\bar{E} + (m + n - 3)F)\). It follows from Lemma 8 that we may assume that \(F\) does not contains surfaces dominating \(L\) that are centers of log canonical singularities of \((W, \lambda D + (m - 2)\bar{E} + (m + n - 3)F)\).

It follows from Theorem 7.5 in [7] that the point \(O\) is an isolated center of log canonical singularities of the log pair \((Y, \lambda M|_Y)\). Therefore, the equivalence

\[
K_Y + \lambda D|_Y + (m - 2)\bar{E}|_Y + (m + n - 3)F|_Y \sim_Q (\pi \circ \eta|_Y)^*\left(K_Y + \lambda M|_Y\right);
\]

and Theorem 7.4 in [7] imply that we have the following possibilities:

- the curve \(Z\) is contained in the threefold \(\bar{Y}\) and \(Z\) is a center of log canonical singularities of the log pair \((\bar{Y}, \lambda D|_Y + (m - 2)\bar{E}|_Y + (m + n - 3)F|_Y)\);
- the intersection \(Z \cap \bar{Y}\) consists of a single point that is a center of log canonical singularities of the log pair \((\bar{Y}, \lambda D|_Y + (m - 2)\bar{E}|_Y + (m + n - 3)F|_Y)\).

**Corollary 11.** Either \(Z \subset \bar{Y}\), or the intersection \(Z \cap \bar{Y}\) consists of a single point.

By construction we have \(L \cong Z \cong \mathbb{P}^1\) and

\[
F \cong \text{Proj}\left(O_{\mathbb{P}^1}(-1) \oplus O_{\mathbb{P}^1}(1) \oplus O_{\mathbb{P}^1}(1)\right),
\]
but the equivalence \( \tilde{Y}|_F \sim B + D \) holds, where \( B \) is the tautological line bundle on the threefold \( F \), and \( D \) is a fiber of the natural projection \( \eta|_F : F \to L \cong \mathbb{P}^1 \).

**Lemma 12.** The equality \( h^1(\mathcal{O}_W(\tilde{Y} - F)) \) holds.

**Proof.** The divisor \(-\eta^*(E) - F\) intersects every curve contained in \( \tilde{E} \) non-negatively and

\[
( - \eta^*(E) - F)|_F \sim B + D,
\]

which implies that \(-4\eta^*(E) - 4F\) is \((\pi \circ \eta)\)-big and \((\pi \circ \eta)\)-nef. However, we have

\[
K_W - 4(\pi \circ \eta)^* (E) - 4F \sim \tilde{Y} - F
\]

and \( X \cong \mathbb{C}^4 \), which implies that \( h^1(\mathcal{O}_W(\tilde{Y} - F)) = 0 \) by the Kawamata-Viehweg vanishing theorem (see Theorem 2.3 in [4]). \( \square \)

Thus, the restriction map \( H^0(\mathcal{O}_W(\tilde{Y})) \to H^0(\mathcal{O}_F(\tilde{Y}|_F)) \) is surjective, but the complete linear system \(|\tilde{Y}|_F|\) does not have base points.

**Corollary 13.** The intersection \( \tilde{Y} \cap Z \) consists of a single point.

Let \( \mathcal{I}_Z \) be an ideal sheaf of \( Z \) on \( F \). Then \( R^1(\eta|_F)_*(B \otimes \mathcal{I}_Z) = 0 \) and there is a surjective map \( \psi : \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \to \mathcal{O}_{\mathbb{P}^1}(k) \), where \( k = B \cdot Z \). The map \( \psi \) is given by a an element of the group

\[
H^0\left( \mathcal{O}_{\mathbb{P}^1}(k + 1) \right) \oplus H^0\left( \mathcal{O}_{\mathbb{P}^1}(k - 1) \right) \oplus H^0\left( \mathcal{O}_{\mathbb{P}^1}(k - 1) \right),
\]

which implies that \( k \geq -1 \).

**Lemma 14.** The equality \( k = 0 \) is impossible.

**Proof.** Suppose \( k = 0 \). Then \( \psi \) is given by matrix \((ax + by, 0, 0)\), where \( a \) and \( b \) are constants and \((x : y)\) are homogeneous coordinates on \( L \cong \mathbb{P}^1 \). Thus \( \psi \) is not surjective over the point of \( L \) at which \( ax + by \) vanishes. \( \square \)

Therefore, the divisor \( B \) can not have trivial intersection with \( Z \). Hence the intersection of the divisor \( \tilde{Y} \) with the curve \( Z \) is either trivial or consists of more than one point, but the intersection \( \tilde{Y} \cap Z \) consists of one point. The obtained contradiction proves Theorem 6.

3. Singular points.

Let \( X \) be a fourfold, \( \mathcal{M} \) be a linear system on the fourfold \( X \) that does not have fixed components, and \( O \) be an isolated ordinary double point of \( X \) such that \( O \) is a center of canonical singularities of the log pair \((X, \lambda \mathcal{M})\), but \((X, \lambda \mathcal{M})\) has log terminal singularities in a punctured neighborhood of the point \( O \), where \( \lambda \) is a positive rational number.

Let \( \pi : V \to X \) be a blow up of the point \( O \), and \( E \) be the exceptional divisor of the birational morphism \( \pi \). Then \( E \) is a smooth quadric hypersurface in \( \mathbb{P}^4 \) and

\[
K_V + \lambda B \sim_\mathbb{Q} \pi^* \left( K_X + \lambda \mathcal{M} \right) + (2 - m)E,
\]

where \( B \) is a proper transform of \( \mathcal{M} \) on the fourfold \( V \), and \( m \) is a positive rational number. It follows from Theorem 3.10 in [4] and Theorem 7.5 in [4] that \( m > 1 \).

Let \( M_1 \) and \( M_2 \) be general divisors in \( \mathcal{M} \), and \( Y \) be a hyperplane section of \( X \). Then

\[
B_i \sim \pi^* (M_i) - \frac{m}{\lambda} E
\]

and \( \tilde{Y} \sim \pi^*(Y) - E \), where \( B_i \) and \( \tilde{Y} \) are proper transforms of the divisors \( M_i \) and \( Y \) on the fourfold \( V \) respectively. Suppose in addition that the following conditions hold:
Theorem 15. There is a line \( L \subset C \subset \mathbb{P}^4 \) such that \( \nabla > 6n^2 \) whenever \( L \subset \text{Supp}(\tilde{Y}) \).

Let \( H' \) be a sufficiently general hyperplane section of the fourfold \( X \) that passes through the point \( O \), and \( \Sigma' = \text{Supp}(B_1 \cdot B_2 \cdot \tilde{Y}' \cdot \tilde{H}) \cap \text{Supp}(E) \), where \( \tilde{H}' \) is a proper transform of the divisor \( H' \) on the fourfold \( V \). Then \( |\Sigma'| < +\infty \). Put

\[
\nabla' = 2m^2 \frac{1}{\lambda^2} + \sum_{P \in \Sigma'} \text{mult}_P \left( B_1 \cdot B_2 \cdot \tilde{Y}' \cdot \tilde{H} \right),
\]

which implies that the inequality \( \nabla > \nabla' \) holds.

In order to prove Theorem 15 we may assume that \( m < 2 \). Then the singularities of the log pair \((V, \lambda B + (m-1)E)\) are not log terminal in the neighborhood of \( E \).

Lemma 16. Suppose that there is a surface \( S \subset E \) such that \( S \) is a center of log canonical singularities of the log pair \((V, \lambda B + (m-1)E)\). Then \( \nabla > 6/\lambda^2 \).

Proof. It follows from Theorem 3.1 in [4] that the inequality

\[
\text{mult}_S \left( B_1 \cdot B_2 \right) \geq \frac{4(2-m)}{\lambda^2}
\]

holds. Therefore, we have

\[
\nabla' = 2m^2 \frac{1}{\lambda^2} + \sum_{P \in \Sigma'} \text{mult}_P \left( B_1 \cdot B_2 \cdot \tilde{H}' \cdot \tilde{H} \right) \geq 2m^2 \frac{1}{\lambda^2} + \text{mult}_S \left( B_1 \cdot B_2 \right) \geq \frac{2m^2 + 4(2-m)}{\lambda^2} \geq 6/\lambda^2,
\]

which concludes the proof. \( \square \)

Therefore, in order to prove Theorem 15 we may assume that the set of centers of log canonical singularities of the log pair \((V, \lambda B + (m-1)E)\) does not contain surfaces that are contained in \( E \). Then the claim of Theorem 7.4 in [7] together with the equivalences

\[
K_V + \tilde{H}' + \lambda B + (m-1)E |_{\tilde{H}} \sim_{\tilde{H}} \pi^* \left( K_X + H' + \lambda M \right)
\]

and \( K_{\tilde{H}'} + \lambda B |_{\tilde{H}'} + (m-1)E |_{\tilde{H}'} \sim_{\tilde{H}'} \pi^* \left( K_{\tilde{H}'} + \lambda M |_{\tilde{H}'} \right) \) imply that there is a line \( L \subset C \subset \mathbb{P}^4 \) such that \( L \) is the unique center of log canonical singularities of \((V, \lambda B + (m-1)E)\) that is contained in \( E \), because \( H' \) is sufficiently general, but the point \( O \) is a center of log canonical singularities of the log pair \((H', \lambda M |_{\tilde{H}})\) by Theorem 7.5 in [7].

Now we suppose that \( L \subset \text{Supp}(\tilde{Y}) \). Then \( L \) is a center of log canonical singularities of the log pair \((\tilde{Y}, \lambda B |_{\tilde{Y}} + (m-1)E |_{\tilde{Y}})\) by Theorem 7.5 in [7]. Hence, we have

\[
\text{mult}_L \left( B_1 \cdot B_2 \cdot \tilde{Y} \right) = \text{mult}_L \left( B_1 |_{\tilde{Y}} \cdot B_2 |_{\tilde{Y}} \right) \geq \frac{4(2-m)}{\lambda^2}
\]
by Theorem 3.1 in \cite{4}. Therefore, we have
\[
\nabla = 2\frac{m^2}{\lambda^2} + \sum_{p \in \Sigma} \text{mult}_p \left( B_1 \cdot B_2 \cdot \tilde{Y} \cdot \tilde{H} \right) \geq 2\frac{m^2}{\lambda^2} + \text{mult}_L \left( B_1 \cdot B_2 \cdot \tilde{Y} \right) \geq \frac{2m^2 + 4(2 - m)}{\lambda^2} \geq 6/\lambda^2,
\]
which concludes the proof of Theorem \cite{4}.

4. Birational rigidity.

In this section we prove Theorem \cite{4}. Let $X$ be a hypersurface in $\mathbb{P}^5$ of degree 5 with at most isolated ordinary double points. Then the group $\text{Cl}(X)$ is generated by the class of a hyperplane section (see \cite{11}). Suppose that the quintic fourfold $X$ is not birationally superrigid. Then there is a linear system $\mathcal{M}$ on the fourfold $X$ that does not have fixed components, but the singularities of the log pair $(X, \frac{1}{n} \mathcal{M})$ are not canonical (see \cite{3}), where $n$ is a natural number such that the equivalence $\mathcal{M} \sim -nK_X$ holds.

Let $Z$ be an irreducible subvariety of the fourfold $X$ having maximal dimension such that the singularities of the log pair $(X, \frac{1}{n} \mathcal{M})$ are not canonical in a general point of the subvariety $Z$. Then $\text{mult}_Z(\mathcal{M}) > n$, which implies that $\dim(Z) \leq 1$ due to \cite{11}.

**Lemma 17.** The subvariety $Z$ is not a smooth point of the hypersurface $X$.

**Proof.** Suppose that $Z$ is a smooth point of the hypersurface $X$. Let $\pi : V \to X$ be a blow up of the point $Z$, and $E$ be the exceptional divisor of the morphism $\pi$. Then

\[
K_V + \frac{1}{n}B \sim_{\varphi} \pi^* \left( K_X + \frac{1}{n} \mathcal{M} \right) + \left( 3 - \text{mult}_Z(\mathcal{M})/n \right) E \sim_{\varphi} \left( 3 - \text{mult}_Z(\mathcal{M})/n \right) E,
\]

where $B$ is a proper transform of the linear system $\mathcal{M}$ on the variety $V$.

Let $M_1$ and $M_2$ be general divisors in $\mathcal{M}$, and $H_1$ and $H_2$ be general hyperplane sections of the hypersurface $X$ passing through the point $Z$. Then

\[
5n^2 = M_1 \cdot M_2 \cdot H_1 \cdot H_2 \geq \text{mult}_Z \left( M_1 \cdot M_2 \right) \geq \text{mult}_Z^2(\mathcal{M}) > n^2,
\]

which implies that $\text{mult}_Z(\mathcal{M}) \leq \sqrt{5}n < 3n$.

Now it follows from Theorem \cite{4} that there is a line $L \subset E \cong \mathbb{P}^3$ such that

\[
\text{mult}_Z \left( M_1 \cdot M_2 \cdot Y \right) > 8n^2,
\]

where $Y$ is a hyperplane section of the hypersurface $X$ such that

\[
\dim \left( \text{Supp}(Y) \cap \text{Supp}(M_1 \cdot M_2) \right) = 1
\]

and $L \subset \text{Supp}(\tilde{Y})$, where $\tilde{Y}$ is the proper transform of $Y$ on the fourfold $V$.

Let $\mathcal{D}$ be a linear subsystem in $|\mathcal{O}_{\mathbb{P}^5}(1)|_{|X|}$ such that

\[
D \in \mathcal{D} \iff L \subset \text{Supp}(\tilde{D}) \text{ or } \text{mult}_Z(D) \geq 2,
\]

where $\tilde{D}$ is a proper transform of $D$ on the fourfold $V$. Then there is a two-dimensional linear subspace $\Pi \subset \mathbb{P}^5$ such that the base locus of $\mathcal{D}$ consists of $X \cap \Pi$.

Suppose that $\Pi \nsubseteq \text{Supp}(M_1 \cdot M_2)$. Let $D$ be a general divisor in $\mathcal{D}$. Then

\[
5n^2 = M_1 \cdot M_2 \cdot D \cdot H_1 \geq \text{mult}_Z \left( M_1 \cdot M_2 \cdot D \right) > 8n^2,
\]

which is a contradiction. In particular, the quintic $X$ contains the plane $\Pi$. 

Let $\bar{X}$ be a general hyperplane section of $X$ containing $\Pi$. Then $\bar{X}$ is a quintic hypersurface with isolated singularities in $\mathbb{P}^4$ that is smooth at $Z$. Let $\bar{\pi} : \bar{V} \to \bar{X}$ be a blow up of the point $Z$, and $\bar{E}$ be the $\bar{\pi}$-exceptional divisor. There is a commutative diagram

\[ \begin{array}{ccc} \bar{V} & \xrightarrow{\bar{\pi}} & V \\
\downarrow & & \downarrow \pi \\
\bar{X} & \xrightarrow{\pi} & X, \end{array} \]

where we identify $\bar{V}$ with the proper transform of $\bar{X}$ on the fourfold $V$, and $\bar{E} = E \cap \bar{V}$.

The plane $\Pi$ is a fixed component of the linear system $\mathcal{M}|_{\bar{X}}$. Moreover, we have

\[ \mathcal{M}|_{\bar{X}} = \mathcal{R} + \alpha \Pi, \]

where $\mathcal{R}$ is a linear system on $\bar{X}$ that does not have fixed components, and $\alpha$ is a multiplicity of a general divisor of the linear system $\mathcal{M}$ in a general point of the plane $\Pi$.

Let $\mathcal{L}$ and $\bar{\Pi}$ be the proper transforms of the linear system $\mathcal{R}$ and the plane $\Pi$ on the threefold $\bar{V}$ respectively. Then $L \subset \bar{\Pi}$ and

\[ K_{\bar{V}} + \frac{1}{n}(\mathcal{L} + \alpha \bar{\Pi}) \sim_{\mathbb{Q}} \bar{\pi}^*(K_{\bar{X}} + \frac{1}{n}\mathcal{R} + \frac{\alpha}{n}\Pi) + (2 - \text{mult}_Z(\mathcal{R})/n - \alpha/n)\bar{E}, \]

but the proof of Theorem \[\text{Lemma 18.} \]

The subvariety $Z$ is not a singular point of the hypersurface $X$.

**Proof.** Suppose that $Z$ is a singular point of the quintic $X$. Let $\pi : V \to X$ be a blow up of the point $Z$, and $E$ be the exceptional divisor of $\pi$. Then $E$ is a quadric in $\mathbb{P}^4$ and

\[ K_V + \frac{1}{n}B \sim_{\mathbb{Q}} \pi^*(K_X + \frac{1}{n}\mathcal{M}) + (2 - m)E, \]

where $B$ is a proper transform of $\mathcal{M}$ on the fourfold $V$, and $m$ is a positive rational number. The inequality $m > 1$ holds by Theorem 3.10 in [4] and Theorem 7.5 in [7].

Let $M_1$ and $M_2$ be general divisors of the linear system $\mathcal{M}$, and $B_i$ be a proper transform of the divisor $M_i$ on the fourfold $V$. Then

\[ B_1 \cdot B_2 \cdot \bar{H}_1 \cdot \bar{H}_2 \leq 5n^2 - 2m^2n^2, \]
where \( \bar{H}_1 \) and \( \bar{H}_2 \) are proper transforms on the fourfold \( V \) of hyperplane sections of the hypersurface \( X \) that pass through the point \( Z \) such that the equality
\[
\dim \left( \text{Supp}(B_1 \cdot B_2) \cap \text{Supp}(\bar{H}_1) \cap \text{Supp}(\bar{H}_2) \right) = 0
\]
holds. In particular, the inequality \( m \leq \sqrt{5/2} \) holds.

It follows from Theorem 15 that there is a line \( L \subset E \subset \mathbb{P}^4 \) such that
\[
B_1 \cdot B_2 \cdot \bar{Y} \cdot \bar{H} > 6n^2 - 2m^2n^2,
\]
where \( \bar{H} \) is a proper transform on \( V \) of a general hyperplane sections of \( X \) that passes through the point \( Z \), and \( \bar{Y} \) is a proper transform on \( V \) of a hyperplane sections of the fourfold \( X \) passing through \( Z \) such that \( Z \) is an ordinary double point of \( \pi(\bar{Y}) \), the equality
\[
\dim \left( \text{Supp}(B_1 \cdot B_2) \cap \text{Supp}(\bar{Y}) \right) = 1
\]
holds and \( L \subset \text{Supp}(\bar{Y}) \).

Let \( D \) be a linear subsystem in \( |O_{\mathbb{P}^5}(1)|_X \) spanned by the divisors whose proper transforms on \( V \) contain \( L \). Then there is a two-dimensional linear subspace \( \Pi \subset \mathbb{P}^5 \) such that the base locus of \( D \) consists of \( X \cap \Pi \). Therefore, we have \( \Pi \subset \text{Supp}(M_1 \cdot M_2) \).

Let \( \bar{X} \) be a general hyperplane section of \( X \) that contains \( \Pi \). Then \( \bar{X} \) is a quintic hypersurface in \( \mathbb{P}^4 \) having isolated singularities, and \( Z \) is an isolated ordinary double point of the quintic \( \bar{X} \), which has at most canonical singularities (see Corollary 4.9 in [4]).

The plane \( \Pi \) is a fixed component of the linear system \( \mathcal{M}|_{\bar{X}} \). Moreover, we have
\[
\mathcal{M}|_{\bar{X}} = \mathcal{R} + \alpha \Pi,
\]
where \( \mathcal{R} \) is a linear system on \( \bar{X} \) that does not have fixed components, and \( \alpha \) is a multiplicity of a sufficiently general divisor of the linear system \( \mathcal{M} \) in a general point of the plane \( \Pi \). The plane \( \Pi \) and a general surface of \( \mathcal{R} \) are not \( \mathbb{Q} \)-Cartier divisors on \( \bar{X} \).

Let \( \bar{\pi} : \bar{V} \to \bar{X} \) be a composition of the blow up of \( Z \) with a subsequent \( \mathbb{Q} \)-factorialization, and \( \bar{E} \) be the \( \bar{\pi} \)-exceptional divisor. Then \( \bar{E} \) is a smooth quadric surface and
\[
K_{\bar{V}} + \frac{1}{n} \mathcal{L} + \frac{\alpha}{n} \bar{\Pi} \sim_{\mathbb{Q}} \bar{\pi}^* \left( K_{\bar{X}} + \frac{1}{n} \mathcal{R} + \frac{\alpha}{n} \Pi \right) + (1 - m) \bar{E},
\]
where \( \mathcal{L} \) and \( \bar{\Pi} \) are proper transforms of \( \mathcal{R} \) and \( \Pi \) on the threefold \( \bar{V} \) respectively.

Let \( \bar{L} = \bar{E} \cap \bar{\Pi} \). Then the curve \( \bar{L} \) is a line on the quadric \( \bar{E} \). Moreover, it follows from the proof of Theorem 15 that the singularities of the log pair
\[
\left( \bar{V}, \frac{1}{n} \mathcal{L} + \frac{\alpha}{n} \bar{\Pi} + (m - 1) \bar{E} \right)
\]
are not log canonical in a general point of \( \bar{L} \). We have \( \text{mult}_L(\mathcal{L}) > 2n - \alpha - mn \), but
\[
\text{mult}_L \left( L_1 \cdot L_2 \right) > 4(2n - mn)(n - \alpha)
\]
by Theorem 3.1 in [4], where \( L_1 \) and \( L_2 \) are general surfaces in \( \mathcal{L} \). The inequality
\[
3\alpha - mn + n > \text{mult}_L(\mathcal{L}) \geq 0
\]
holds, because \( \bar{C} \cdot L_i \geq \text{mult}_L(\mathcal{L}) \) and \( \bar{C} \cdot \bar{\Pi} = -3 \), where \( \bar{C} \) is a proper transform on the threefold \( \bar{V} \) of a sufficiently general line contained in \( \Pi \). Thus, we have \( \alpha > n/4 \).

Let \( \psi : \bar{X} \to \mathbb{P}^1 \) be a projection from the plane \( \Pi \). Then \( \psi \) is not defined in the points where the plane \( \Pi \) is not a Cartier divisor on \( \bar{X} \). In particular, the rational map \( \psi \) is not
defined in the point \( Z \). However, we may assume that the birational morphism \( \bar{\pi} \) resolves the indeterminacy of the rational map \( \psi \). Therefore, we have a commutative diagram

\[
\begin{array}{ccc}
\bar{V} & \xrightarrow{\bar{\pi}} & \bar{X} \\
\downarrow{\eta} & & \downarrow{\psi} \\
\psi & \rightarrow & \mathbb{P}^2,
\end{array}
\]

where \( \eta \) is a morphism. Let \( S \) be a sufficiently general fiber of \( \eta \) and \( H \) be a hyperplane section of \( \bar{X} \). Then \( \bar{\pi}(S) \) is a quartic surface in \( \mathbb{P}^4 \) and \( S \sim \bar{\pi}^*(H) - \bar{E} - \bar{\Pi} \), which implies

\[
(19) \quad \left( \bar{\pi}^*(H) - \bar{E} - \bar{\Pi} \right) \cdot \left( \bar{\pi}^*(nH) - nm\bar{E} - \alpha\bar{\Pi} \right)^2 > 4(2n - mn)(n - \alpha),
\]

because \( S \cdot \bar{L} = 1 \) and \( L_i \sim \bar{\pi}^*(nH) - nm\bar{E} - \alpha\bar{\Pi} \).

We have \( H^3 = 5, \bar{E}^3 = 2, \bar{\pi}^*(H) \cdot \bar{\Pi}^2 = -3, \bar{E} \cdot \bar{\Pi}^2 = 0, \bar{\Pi} \cdot \bar{E}^2 = -1 \), and

\[
\bar{\Pi}^3 = \left( \bar{\pi}^*(H) - \bar{E} - S \right) \cdot \bar{\Pi}^2 = -3 - S \cdot \bar{\Pi}^2,
\]

but \( \bar{\pi}(S) \cap \Pi \) is a hyperplane section of the surface \( \bar{\pi}(S) \subset \mathbb{P}^4 \). Moreover, the generality in the choice of the threefold \( \bar{X} \) implies that the threefold \( \bar{X} \) has isolated ordinary double points, the quartic surface \( \bar{\pi}(S) \) is smooth, and the birational morphism \( \bar{\pi}|_S \) is a blow up of a point \( Z \) on the surface \( \bar{\pi}(S) \). Therefore, we have \( S \cdot \bar{\Pi}^2 = 3 \), which gives \( \bar{\Pi}^3 = -6 \).

The inequality (19) implies that the inequality

\[
4n^2 - 9\alpha^2 + 4n\alpha - 2mn\alpha - m^2n^2 > 4(2n - mn)(n - \alpha)
\]

holds. Therefore, we have

\[
0 > 4n^2 + 9\alpha^2 - 12n\alpha + 6mn\alpha - 4mn^2 + m^2n^2 = \left( 3\alpha - 2n + mn \right)^2,
\]

which is a contradiction. \( \square \)

**Lemma 20.** The curve \( Z \) is a line.

**Proof.** Suppose that \( Z \) is not a line. Let \( P_1 \) and \( P_2 \) be sufficiently general points of the curve \( Z \), and \( L \) be a line in \( \mathbb{P}^5 \) that passes through the points \( P_1 \) and \( P_2 \). Then \( L \not\equiv Z \).

Let \( H_1 \) and \( H_2 \) be sufficiently general hyperplane section of the fourfold \( X \) that pass through \( P_1 \) and \( P_2 \). Put \( S = H_1 \cap H_2 \). Then the singularities of the log pair \( (S, \frac{1}{n}\mathcal{M}|_S) \) are not log canonical in the points \( P_1 \) and \( P_2 \) by Theorem 7.5 in [8], but the singularities of the log pair \( (S, \frac{1}{n}\mathcal{M}|_S) \) are log canonical in punctured neighborhoods of these points, because the secant variety of the curve \( Z \) is at least two-dimensional. Put

\[
\mathcal{M}|_S = \mathcal{B} + \gamma L,
\]

where \( \mathcal{B} \) is a linear system on \( S \) that does not have fixed components, and \( \gamma \) is the multiplicity of a general divisor of \( \mathcal{M} \) in a general point of the line \( L \). Then

\[
\text{mult}_{P_1}(B_1 \cdot B_2) > 4(n^2 - \gamma n)
\]

by Theorem 3.1 in [8], where \( B_1 \) and \( B_2 \) are general divisors in \( \mathcal{B} \). Thus, we have

\[
5n^2 - 2\gamma n - 3\gamma^2 = B_1 \cdot B_2 \geq \text{mult}_{P_1}(B_1 \cdot B_2) + \text{mult}_{P_2}(B_1 \cdot B_2) > 8(n^2 - \gamma n),
\]

which is a contradiction. \( \square \)
Let $Y$ be a sufficiently general hyperplane section of the fourfold $X$ that passes through the line $Z$, and $\mathcal{B} = \mathcal{M}|_Y$. Then $Y$ is a quintic threefold in $\mathbb{P}^4$, the linear system $\mathcal{B}$ does not have fixed components, but the singularities of the log pair $(Y, \frac{1}{n}\mathcal{B})$ are not log canonical in a general point of the curve $Z$ by Theorem 7.5 in [7].

The line $Z$ contains a singular point of the threefold $Y$ due to [11], but $Z$ contains at most 4 singular points of the threefold $Y$. Put

$$ Z \cap \text{Sing}(Y) = \{P_1, \ldots, P_k\}, $$

where $P_i$ is a singular point of the threefold $Y$ and $k \leq 4$. Then the point $P_i$ is an isolated ordinary double point of the threefold $Y$, and the group $\text{Cl}(Y)$ is generated by the class of a hyperplane section (see [5]).

Let $\pi : V \to Y$ be a blow up of the points $\{P_1, \ldots, P_k\}$, and $E_i$ be an exceptional divisor of the morphism $\pi$ such that $\pi(E_i) = P_i$. Then

$$ D \sim \pi^*(\mathcal{O}_{\mathbb{P}^4}(n)|_Y) - \sum_{i=1}^{4} m_i E_i, $$

where $D$ is a proper transform on $V$ of a general divisor in $\mathcal{B}$, and $m_i$ is a natural number.

Let $\bar{Z}$ be a proper transform of $Z$ on the fourfold $V$. Then $V$ is smooth and

$$ 2m_i \geq \text{mult}_{\bar{Z}}(D) = \text{mult}_{\bar{Z}}(\mathcal{B}) = n, $$

where $Q_i = Z \cap E_i$. Hence, we have $m_i \geq \text{mult}_{\bar{Z}}(\mathcal{B})/2 \geq n/2$.

Let $\Pi$ be a general plane in $\mathbb{P}^4$ that contains the line $Z$, and $C$ be a quartic curve in the plane $\Pi$ such that $\Pi \cap Y = L \cup C$. Then $|C \cap Z| = 4$ and the curve $C$ contains all singular points of the threefold $Y$ that is contained in $Z$. Let $\bar{C}$ be a proper transform of the curve $C$ on the threefold $V$. Then $\bar{C} \not\subset \text{Supp}(D)$ and

$$ 0 \leq D \cdot \bar{C} = 4n - \sum_{i=1}^{k} m_i \leq 4n - 2k\text{mult}_{\bar{Z}}(\mathcal{B}) + (4 - k)\text{mult}_{\bar{Z}}(\mathcal{B}), $$

which implies that $\text{mult}_{\bar{Z}}(\mathcal{B}) \leq 2n$.

Let $\omega : W \to V$ be a blow up of $\bar{Z}$, and $G$ be the $\omega$-exceptional divisor. Then

$$ K_W + \frac{1}{n}\mathcal{D} \sim (\pi \circ \omega)^* (K_Y + \frac{1}{n}\mathcal{B}) + \sum_{i=1}^{k} \left(1 - \frac{m_i}{n}\right) \bar{E}_i + \left(1 - \text{mult}_{\bar{Z}}(\mathcal{B})/n\right)G, $$

where $\bar{E}_i$ and $\mathcal{D}$ are proper transforms of the divisor $E_i$ and the linear system $\mathcal{B}$ on the threefold $W$ respectively. Therefore, the inequality $\text{mult}_{\bar{Z}}(\mathcal{B}) < 2n$ implies the existence of an irreducible curve $L \subset G$ such that $\omega(L) = \bar{Z}$, but the singularities of the log pair

$$ \left(W, \frac{1}{n}\mathcal{D} + \frac{\text{mult}_{\bar{Z}}(\mathcal{B}) - n}{n}G\right) $$

are not log canonical in a general point of $L$. Thus, we have $\text{mult}_L(\mathcal{D}) + \text{mult}_Z(\mathcal{B}) > 2n$.

The surface $E_i$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Let $A_i$ and $B_i$ be the fibers of the projections of the surface $E_i$ to $\mathbb{P}^1$ that pass through the point $Q_i$, and $\bar{A}_i$ and $\bar{B}_i$ be proper transforms of the curves $A_i$ and $B_i$ on the threefold $W$ respectively. Then

$$ \mathcal{N}_{W/A_i} \cong \mathcal{N}_{W/B_i} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1), $$

which implies that we can flop the curves $\bar{A}_i$ and $\bar{B}_i$. Namely, let $\xi : U \to W$ be a blow up of the curves $\bar{A}_1, \bar{B}_1, \ldots, \bar{A}_k, \bar{B}_k$, and $F_i$ and $H_i$ be the exceptional divisors of the morphism $\xi$ such that $\xi(F_i) = \bar{A}_i$ and $\xi(H_i) = \bar{B}_i$. Then $F_i \cong H_i \cong \mathbb{P}^1 \times \mathbb{P}^1$ and there is
a birational morphism $\xi' : U \to W'$ such that $\xi'(F_i)$ and $\xi'(H_i)$ are rational curves, but the map $\xi' \circ \xi^{-1}$ is not an isomorphism in the neighborhood of the curves $\bar{A}_i$ and $\bar{B}_i$.

Let $E'_i$ be a proper transform of $E_i$ on the threefold $W'$. Then $E'_i \cong \mathbb{P}^2$ and we can contract the surface $E'_i$ to a singular point of type $\frac{1}{2}(1, 1, 1)$, because

$$\mathcal{N}_{W'/E'_i} \cong \mathcal{O}_{E'_i}(E'_i|E'_i) \cong \mathcal{O}_{\mathbb{P}^2}(-2).$$

Let $\omega' : W' \to V'$ be a contraction of $E'_1, \ldots, E'_k$, and $G'$ be a proper transform of the surface $G$ on the threefold $V'$. Then there is a birational morphism $\pi' : V' \to Y$ that contracts the divisor $G'$ to the line $Z$. Hence, we constructed the commutative diagram

![Diagram](https://via.placeholder.com/150)

such that $V'$ is projective and $\mathbb{Q}$-factorial, and $\text{rk} \text{Pic}(V') = 2$. Therefore, the birational morphism $\pi' : V' \to Y$ is an extremal terminal divisorial contraction (see [3]). We have

$$K_{V'} + \frac{1}{n} \mathcal{R} \sim_{\mathbb{Q}} \pi'^* \left(K_Y + \frac{1}{n} \mathcal{B} \right) + \left(1 - \text{mult}_Z \mathcal{B} / n \right) G',$$

where $\mathcal{R}$ is a proper transform of $\mathcal{B}$ on the threefold $V'$. Let $L'$ be a proper transform of the curve $L$ on the threefold $V'$. Then $(V', \frac{1}{n} \mathcal{R} + (\text{mult}_Z \mathcal{B} / n - 1)G')$ is not log canonical in a general point of the curve $L'$. Hence, the inequality

$$\text{mult}_{L'}(R_1 \cdot R_2) > 4n \left(2n - \text{mult}_Z \mathcal{B} \right)$$

holds by Theorem 3.1 in [4], where $R_1$ and $R_2$ are general surfaces in $\mathcal{R}$.

Let $\bar{H}$ be a hyperplane section of the threefold $Y$, and $P'_i = \pi'(E'_i)$. Then the base locus of the linear system $|\pi'^*(H) - G'|$ consists of the points $P'_1, \ldots, P'_k$. The construction of the morphism $\pi'$ implies that $G^3 = 2 - k/2$. We have $(\pi'^*(H) - G') \cdot L' = 0$, because

$$2 \left(\pi'^*(H) - G' \right) \cdot R_1 \cdot R_2 = 2 \left(\pi'^*(H) - G' \right) \cdot \left(\pi'^*(nH) - \text{mult}_Z \mathcal{B}G' \right)^2 < 4n \left(2n - \text{mult}_Z \mathcal{B} \right)$$

and $2(\pi'^*(H) - G')$ is a Cartier divisor. In particular, we have $L' \cap \{P'_1, \ldots, P'_k\} = \emptyset$.

**Corollary 21.** The cone $\overline{NE}(V')$ is generated by the curves $\xi'(F_i) \equiv \xi'(H_i)$ and $L'$.

Let $\bar{H}$ be a proper transform on the threefold $V$ of a sufficiently general hyperplane section of the threefold $Y$ that contains the line $Z$. Then $\bar{H}$ is a smooth surface such that the equality $Z^2 = -3$ holds on the surface $\bar{H}$. One the other hand, we have

$$c_1 \left(\mathcal{N}_{V/Z} \right) = -2 - K_V \cdot \bar{Z} = -2 - k,$$

which implies that $\mathcal{N}_{V/Z} \cong \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)$, where $a$ and $b$ are integer numbers such that the inequality $a \geq b \geq -3$ holds and $a + b = -2 - k$.

Let $\bar{H}$ be the proper transform of the divisor $\bar{H}$ on the threefold $W$. Then

$$\bar{H} \sim (\pi \circ \omega)^* \left(\mathcal{O}_{\mathbb{P}^1}(1)|_Y \right) - \sum_{i=1}^{4} \bar{E}_i - G$$

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and $G^3 = 2 + k$. Elementary calculations implies that $\tilde{H}$ intersects the curve $L$ in its general point in the case when $L$ is not the exceptional section of the projection

$$\varpi|_G : G \cong \mathbb{P}_{a-b} \to \bar{Z}$$

or $b \neq -3$. Thus, we have $b = -3$ and the curve $L$ is the exceptional section of the ruled surface $G \cong \mathbb{P}_{4-k}$. Moreover, the construction of the map $\varpi' \circ \xi' \circ \xi^{-1}$ implies that

$$L \cap \left( A_i \cup B_i \right) \neq \emptyset$$

for every $i = 1, \ldots, k$, because otherwise $P'_i \in L'$ and $(\pi''(H) - G') \cdot L' \neq 0$.

**Lemma 22.** The inequality $k \leq 3$ holds.

*Proof.* Suppose that $k = 4$. Then $|\pi''(H) - 2G'|$ contains a divisor $T$ such that $\pi'(T)$ is a hyperplane section of $Y$ that tangents $Y$ along $Z$. The cycle $T \cdot R_i$ must be effective, but

$$T \cdot R_i \equiv \left( 6n - 6\text{mult}_Z(B) \right) \xi'(F_i) + \left( 5n - 2\text{mult}_Z(B) \right) L',$$

which implies that $\text{mult}_Z(B) \leq n$. Contradiction. \hfill \Box

Let $H'$ be a proper transform on $V'$ of a general hyperplane section of $Y$. Then

$$\left\{ \begin{array}{l} H' \cdot H' \equiv 10\xi'(F_i) + 5L', \\ H' \cdot G' \equiv 2\xi'(F_i) \equiv \xi'(F_i) + \xi'(H_i), \\ G' \cdot G' \equiv (k - 6)\xi'(F_i) - L', \end{array} \right.$$  

which implies that the equivalence

$$R_1 \cdot R_2 \equiv \left( 10n^2 - 4n\text{mult}_Z(B) + (k - 6)\text{mult}_Z(B)^2 \right) \xi'(F_i) + \left( 5n^2 - \text{mult}_Z(B)^2 \right) L'$$

holds. Thus, we have $10n - 4n\text{mult}_Z(B) + (k - 6)\text{mult}_Z(B)^2 \geq 0$, which implies that

$$n < \text{mult}_Z(B) \leq \frac{\sqrt{34} - 2}{3} n < \frac{32}{25} n < \frac{4}{3} n.$$

Let $\pi'' : V'' \to V'$ be a blow up of $L'$, and $G''$ be the $\pi''$-exceptional divisor. Then

$$K_{V''} + \frac{1}{n} L + \frac{\text{mult}_Z(B) - n}{n} \tilde{G}' + \frac{\text{mult}_{L'}(R) + \text{mult}_Z(B) - 2n}{n} G'' \sim \left( \pi' \circ \pi'' \right)^* \left( K_Y + \frac{1}{n} B \right),$$

where $L$ and $\tilde{G}'$ are proper transforms of $R$ and $G'$ on the threefold $V''$ respectively, which implies that either the inequality $\text{mult}_{L'}(R) + \text{mult}_Z(B) > 3n$ holds, or the log pair

$$(23) \quad \left( V'', \frac{1}{n} L + \left( \text{mult}_Z(B)/n - 1 \right) \tilde{G}' + \left( \text{mult}_{L'}(R)/n + \text{mult}_Z(B)/n - 2 \right) G'' \right)$$

are not log canonical in a general point of a curve dominating the curve $L'$. We have

$$\text{mult}_{L'}(R) \leq \text{mult}_Z(B) \leq \frac{\sqrt{34} - 2}{3} n < \frac{4}{3} n,$$

which implies the inequality $\text{mult}_{L'}(R) + \text{mult}_Z(B) < 3n$. Therefore, there is an irreducible curve $L'' \subset G''$ such that $\pi''(L'') = L'$, but the singularities of the log pair $23$ are not log canonical in a general point of the curve $L''$. In particular, the inequality

$$\text{mult}_{L''}(L) > 4n - \text{mult}_{L'}(R) - 2\text{mult}_Z(B)$$

holds, because the inequality $\text{mult}_{L'}(R) > 2n - \text{mult}_Z(B)$ holds, but the singularities of the log pair $23$ are not canonical in a general point of the curve $L''$. 

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Lemma 24. The curve \( L'' \) is not contained in the divisor \( \bar{G}' \).

Proof. Suppose that \( L'' = G'' \cap \bar{G}' \). Then taking the intersection of a general surface in the linear system \( L \) and a general fiber of the morphism \( (\pi' \circ \pi'')|_{\bar{G}'} \) we see that

\[
\text{mult}_Z(B) - \text{mult}_L(R) > 4n - \text{mult}_L(R) - 2\text{mult}_Z(B),
\]

which implies that \( \text{mult}_Z(B) > 4n/3 > n(\sqrt{34} - 2)/3 \). Contradiction. \( \square \)

Let \( L_1 \) and \( L_2 \) be general surfaces in the linear system \( L \), and \( H'' \) be a proper transform of a general hyperplane section of the threefold \( Y \) on the threefold \( V'' \). Then

\[
5n^2 - \text{mult}_L^2(R) - \text{mult}_Z^2(B) = H'' \cdot L_1 \cdot L_2 > 4n \left( 3n - \text{mult}_L(R) - \text{mult}_Z(B) \right)
\]

by Theorem 3.1 in [4]. Therefore, we have

\[
n^2 > \left( \text{mult}_L(R) - 2n \right)^2 + \left( \text{mult}_Z(B) - 2n \right)^2 \geq \frac{2n^2(\sqrt{34} - 8)^2}{3} > n^2,
\]

which is a contradiction. The obtained contradiction concludes the proof of Theorem 4.

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