A direct comparison between the mixing time of the interchange process with “few” particles and independent random walks

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Abstract

We consider the interchange process with \( k \) particles (denoted IP(\( k \))) on \( n \)-vertex hypergraphs in which each hyperedge \( e \) rings at rate \( r_e \). When \( e \) rings, the particles occupying it are permuted according to a random permutation from some arbitrary law, where our only assumption is that IP(2) has uniform stationary distribution. We show that \( t_{\text{mix}}^{\text{IP}(k)}(\varepsilon) = O(\varepsilon^{b/(k^2)}) \), provided that \( kn^{-2}R_{\text{mix}}^{\text{IP}(2)}(\varepsilon/k) = O((\varepsilon/k)^b) \) for some \( b > 0 \), where \( R = \sum_e r_e |e| (|e| - 1) \) is \( n(n - 1) \) times the particle-particle interaction rate at equilibrium.

This has some consequences concerning the validity (in this regime) of conjectures of Oliveira about comparison of the \( \varepsilon \) mixing time of IP(\( k \)) to that of \( k \) independent particles, each evolving according to IP(1), denoted RW(\( k \)), and of Caputo about comparison of the spectral-gap of IP(\( k \)) to that of a single particle IP(1) = RW(1).

We also show that \( t_{\text{mix}}^{\text{IP}(k)}(\varepsilon) \approx t_{\text{mix}}^{\text{RW}(1)}(\varepsilon) \approx t_{\text{mix}}^{\text{RW}(k)}(\varepsilon k/4) \) for all \( k \lesssim n^{1-\Omega(1)} \) and all \( \varepsilon \leq \frac{1}{k} \wedge \frac{1}{4} \) for vertex-transitive graphs of constant degree, as well as for general graphs satisfying a mild (“transience-like”) heat-kernel condition.

In the special case where the particles occupying a hyperedge \( e \) are permuted uniformly at random (in \( e \)) when \( e \) rings, we obtain results bounding the spectral gap of IP(\( k \)) in terms of that RW(1).

In contrast to recent works on mixing times of IP(\( k \)), the proof does not use Morris’ chameleon process. It can be seen as a rigorous and direct way of arguing that when the number of particles is fairly small, the system behaves similarly to \( k \) independent particles, due to the small amount of interaction between particles.

1 Introduction

The interchange process IP(\( k \)) on a finite, connected graph \( G = (V, E) \) is the following continuous-time Markov process. In a configuration, each vertex is either occupied by a labelled black particle, or by an unlabelled white particle such that the number of black particles equals \( k \leq |V| = n \). We label the black particles by the set \( [k] := \{1, \ldots, k\} \).

For each edge \( e \) independently, at the times of a Poisson process of rate \( r_e > 0 \), switch the particles on the endpoints of \( e \). The exclusion process EX(\( k \)) is similarly defined, except the black particles are also unlabelled. We will further denote by RW(\( k \)) the process of \( k \)

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independent continuous-time random walks on $G$, each with the same transition rates $\{r_e\}_{e \in E}$ (i.e. $\text{RW}(1)$ equals $\text{IP}(1)$ and $\text{RW}(k)$ are $k$ independent $\text{IP}(1)$). Motivated by conjectures of Oliveira and Caputo (stated below) we are interested in comparing the mixing times and spectral gaps of $\text{IP}(k)$ with those of $\text{RW}(k)$ and $\text{RW}(1)$.

Our interest extends also to $\text{IP}(k)$ on hypergraphs. In this process each (hyper)edge $e$ rings independently at rate $r_e$. When an edge rings, some random permutation (not necessarily uniformly distributed) of the vertices in $e$ is applied to the current configuration (that is, the particles currently occupying $e$, including the white ones, are permuted). We stress that we make no assumption on the law of this random permutation of $e$, other than it being the same law at every ring of $e$. We also make the global assumption that the process $\text{IP}(2)$ is irreducible (although when this fails our result holds trivially) and has uniform stationary distribution. We suppose throughout that $n \geq 3$. By abuse of terminology, we shall often use the term ‘hypergraph’ to refer also to the associated rates and the rules of the dynamics associated with the edge rings.

Of particular interest is the case in which, when a hyperedge $e$ rings, the particles currently occupying $e$ are permuted uniformly at random (in $e$). We refer to this setup as a uniform interchange process. Caputo conjectures (see [6]) that the spectral gaps of $\text{IP}(k)$ and $\text{RW}(1)$ are the same for the uniform interchange processes (for all $k$) – this is the hypergraph version of the Caputo, Liggett and Richthammer Theorem [5], a.k.a. Aldous’ spectral gap conjecture. This provides motivation for comparing the spectral gaps of $\text{IP}(k)$ and $\text{RW}(1)$ for uniform interchange processes on hypergraphs (and more generally (in the non-uniform case) with that of $\text{IP}(2)$). We note that a uniform interchange process on a hypergraph is reversible w.r.t. the uniform distribution for all $k$. More generally, the same holds whenever the law of the permutation associated with each hyperedge $e$ gives each permutation and its inverse the same probability, which is in particular the case when the law is constant on conjugacy classes.

On graphs, the generator of $\text{IP}(k)$ is symmetric and so when $\text{IP}(k)$ is irreducible its time-$t$ law converges as $t \to \infty$ to uniform on the set of possible configurations. We seek to upper-bound the rate of this convergence, measured using the total-variation distance. While one should expect the case when $k$ is small to be easier, as there are fewer interactions between different particles in this case, this historically has not been the case.\[\text{[1]}\] The order of the mixing time of $\text{IP}(n^d)$ on a $d$-dimensional torus $\mathbb{Z}_n^d$ of side length $n$ was first determined by Yau [20] by estimating the log-Sobolev constant (this gives an upper bound; a lower bound, matching up to a constant factor, was first proven by Wilson [19] for $d = 1, 2$, and later by Morris in [14] for all $d$; see [10, Theorem 1.4] for a general lower bound which combines ideas from the aforementioned two proofs, together with negative correlation). In [14], Morris introduces the ingenious chameleon process, a process similar to the evolving sets process [15], tailored to handle the complicated dependencies between the particles in the interchange process. This allows determination of the order of the mixing time of $\text{IP}(k)$ on $\mathbb{Z}_n^d$ for all $k$, which offers an improved bound for $k = n^{o(1)}$ compared with [20]. For larger $k$ it also offers some improvement, but only by a constant factor, as well as a better constant dependence on the dimension (from linear to logarithmic).

While obtaining refined bounds on $t_{\text{mix}}^{\text{IP}(k)} := t_{\text{mix}}^{\text{IP}(k)}(1/4)$, the $1/4$ mixing time of $\text{IP}(k)$, when

\[\text{[1]}\] Of course an upper bound on the mixing time of $t_{\text{mix}}^{\text{IP}(n)}$ provides an upper bound also on $t_{\text{mix}}^{\text{IP}(k)}$ for $k < n$ by the contraction principle. However, obtaining more refined bounds on $t_{\text{mix}}^{\text{IP}(k)}$ for small $k$ has proven challenging.
k is small is one of the main motivations in [14], using the chameleon process on other graphs to obtain refined bounds for small k has proven challenging. Oliveira [16] generalises Morris’ argument to arbitrary graphs and rates by an elegant use of the negative correlation property enjoyed by the exclusion process on graphs. Alas, his method gives the same upper bound on $t_{\text{mix}}^{\text{IP}(k)}$ for all k. The analysis in [10], which refines that of [16] (other than the fact that the setup in [16] is more general), also relies on the chameleon process. More specifically, the chameleon process is analysed using $L_2$ techniques and by exploiting a certain negative association property of the exclusion process on graphs.\(^{[2]}\) The most challenging proof in [10] is of the refined bound on $t_{\text{mix}}^{\text{IP}(k)}$ for small k; namely, the proof that $t_{\text{mix}}^{\text{IP}(k)}$ is upper bounded by the upper bound on the $1/k$ $L_2$ mixing time of RW(1) given by the spectral-profile. We do not believe it is possible to prove a stronger result using the chameleon process.\(^{[3]}\)

1.1 Results

For a continuous-time Markov process $Q$ we will denote by $t_{\text{rel}}^Q$ and $t_{\text{mix}}^Q(\varepsilon)$ the inverse of the spectral-gap of the process and its $\varepsilon$ total-variation mixing time, respectively. When $\varepsilon = 1/4$ we omit it from this notation.

In this paper we present a simple way of analysing $t_{\text{mix}}^{\text{IP}(k)}(\varepsilon)$ for small k which does not rely on the chameleon process. Besides its simplicity, it has the advantage of applying also for hypergraphs for arbitrary rates $\{r_e\}_e$ and for arbitrary rules for the law of the random permutation of a hyperedge $\varepsilon$ when it rings. The argument can be seen as a direct way of making rigorous the intuition that when particles rarely interact with one another the system should evolve similarly to $k$ independent particles. We emphasize that our approach goes beyond a more naïve version of such an argument which requires $k$ to be small enough that with probability bounded away from zero, no particles interact with other particles until they are mixed (after a certain initial burn in period). Instead, by considering the behaviour of the $k$th particle conditioned on the rest and exploiting a certain submultiplicativity property (presented in Lemma 3.3) we are able to extend the result to much larger values of $k$.

Before stating our main (and more general) theorem (see Theorem 1.4), we first present some more lucid results. The first concentrates on the case of vertex-transitive graphs, for which we present bounds on the $\varepsilon$ mixing time of IP(k) for $\varepsilon \leq 1/k$. In fact, our argument gives the same bounds (up to a constant factor) on the $1/4$ and the $1/k$ mixing times of IP(k) (this applies to all of our results). The fact that we bound also $t_{\text{mix}}^{\text{IP}(k)}(1/k)$ rather than just $t_{\text{mix}}^{\text{IP}(k)} = t_{\text{mix}}^{\text{IP}(k)}(1/4)$ will be important later on in order to derive an upper bound on the relaxation-time and is also used to derive a comparison with $t_{\text{mix}}^{\text{RW}(k)}$.

We write $o(1)$ for terms which vanish as $n \to \infty$ and $O(1)$ for terms which are bounded from above by a constant. We write $f_n \lesssim g_n$ if $|f_n|/|g_n| = O(1)$ and $f_n \asymp g_n$ if $g_n \lesssim f_n \lesssim g_n$. We write $f_n \asymp g_n$ if the implicit constant depends on $\eta$. Similarly, we write $C(\eta)$ or $c_\eta$ for positive constants depending only on $\eta$. We also write $a \land b := \min\{a, b\}$.

\(^{[2]}\)Unfortunately, this property fails to hold for exclusion on hypergraphs. This prevents one from extending the analysis from [10] from graphs to hypergraphs and is one of the main obstacles encountered in [7].

\(^{[3]}\)Recalling that the chameleon process is a variant of the evolving sets process, and that the spectral-profile bound on the mixing time refines the evolving sets isoperimetric-profile bound [9].
Hence it suffices to consider the case that HK- 
below. (We note that similar reasoning has previously been used by Tessera and Tointon [4] to prove some related results).

The proof involves a certain case analysis. Let $D$ be the diameter of the vertex-transitive graph. If $D$ is at least polynomial in $n$ then using an inspired recent approximate group theoretical result of Tessera and Tointon [18], providing finitary quantitative forms of Gromov’s and Trofimov’s Theorems, it follows that the graph satisfies a certain technical condition due to Diaconis and Saloff-Coste [8], called “moderate growth”, and this case is already covered by [10, Prop. 11.1]. Hence it suffices to consider the case that $D \leq n^{1/3}$. To treat this case we appeal to a result in an upcoming work of the first author with Berestycki and Teyssier, which relies on an isoperimetric inequality due Tessera and Tointon [17] (which in turn, follows from their estimates on growth of balls from [18]), in order to verify the conditions in Theorem 1.2 below. (We note that similar reasoning has previously been used by Tessera and Tointon in [17] to prove some related results).

The next result concerns general graphs satisfying a certain heat-kernel condition. Loosely speaking, this is the condition that the spectral-dimension is at least $2+\varepsilon$. Such a “transience-like” condition is consistent with the general theme of this paper of bounding the mixing time in regimes in which there are few interactions between particles. We write $p_t(x, y)$ for the time $t$ transition probability from $x$ to $y$ of $RW(1)$.

**Theorem 1.2** (Bound under a ‘transience-like’ heat-kernel condition). For every $d \in \mathbb{N}$, $\theta, a \in (0, 1/2)$ and $c > 0$ there exists $n_0 = n_0(a, d)$ and $C = C(c, d, \theta)$ (both independent of $k, G$ and $n$) such that for every connected graph $G = (V, E)$ equipped with rates $r_e \equiv 1$, of size $|V| = n \geq n_0$, of maximal degree $d$, satisfying $t_{rel}^{RW(1)} \leq n^{1-2a}$ and

$$\max_x p_t(x, x) - \frac{1}{n} \leq \frac{c}{t^{1+\theta}} \quad \forall t \geq t_{rel}^{RW(1)},$$

we have

$$t_{mix}^{IP(k)}(\varepsilon) \leq C t^{RW(1)}_{mix}(\varepsilon) \asymp t_{mix}^{RW(k)}(\varepsilon k/4)$$

for all $3 \leq k \leq n^a$ and $\varepsilon \leq \frac{1}{k} \land \frac{1}{4}$.

It easily follows that under the assumptions (HK-(\theta)) and $t_{rel}^{RW(1)} \leq n^{1-2a}$ we have that

$$t_{mix}^{IP(k)}(\varepsilon) \lesssim_{c, d, \theta} t_{mix}^{RW(k)}(\varepsilon) \quad \text{(uniformly for all } \varepsilon \leq 1/4 \text{ and all } 3 \leq k \leq n^a),$$

provided that $t_{mix}^{IP(n)} \asymp t_{mix}^{IP(1)}(1/n) \asymp t_{mix}^{IP(n)}(1)$ (the implicit constant in the first $\asymp$ depends on $a, b, c, d$, where $D = cn^b$).

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[4]: The Cayley graph case is due to Breuillard and Tointon [4]. A finitary version of Gromov’s Theorem was first proved by Breuillard, Green, and Tao [3]. A recent result of Alon and Kozma [2] is that for graphs with general rates $\{r_e\}_{e}$, under mild conditions $t_{mix}^{IP(n)} \lesssim t_{mix}^{IP(1)} \log n$. In the case of vertex transitive graphs of moderate growth, this bound combined with the analysis from [10, Thm 1.4 & §11] implies that $t_{mix}^{IP(n)} \asymp t_{mix}^{IP(1)}(1/n) \asymp t_{mix}^{IP(n)}(1)$ (the implicit constant in the first $\asymp$ depends on $a, b, c, d$, where $D = cn^b$). We also note that while the results in [10] are stated for $EX(k)$, they are all proven for $IP(k)$ for $k \leq n/2$, and are in fact valid for $IP(k)$ when $k \leq (1 - \delta)n$, but with additional dependence on $\delta$ of some constants.
n ≥ n_0(a, b, d). This is yet another partial progress on Oliveira’s conjecture. We remark that we could have instead assumed that \( t_{\text{mix}}^{\text{RW}(1)} \leq n^{1-a-b} \) for some \( a, b > 0 \). This allows one to consider larger values of \( k \), as \( k \leq n^a \) and the above allows to take a larger value for \( a \). However the case that \( n^c \leq k \leq n^{1-c} \) for some \( c \in (0, \frac{1}{2}) \) is already covered in [10]. We also note that the proof of Theorem 1.2 uses the negative correlation property of the exclusion process – a property which does not hold for the process on hypergraphs. As a result, one cannot easily extend this result to hypergraphs.

Remark 1.3. Our proof shows that for the last inequality to hold for a certain \( 3 \leq k \leq n^a \) and \( 0 < \varepsilon \leq \frac{1}{k} \wedge \frac{1}{k} \) we only require \((\text{HK}-(\theta))\) to hold at a time \( t = 2\alpha t_{\text{mix}}^{\text{RW}(1)}(\varepsilon) \) for some constant \( \alpha = \alpha(c, d, \theta) \) which is chosen in the proof.

We introduce some notation before presenting our main result. For a size \( n \) hypergraph with rates \( \{r_e\}_e \) we set

\[
R := \sum_e r_e |\{e\}|(|\{e\}| - 1).
\]

The quantity \( R/[n(n - 1)] \) is the rate of particle-particle interaction for two particles at equilibrium.[5] The appearance of \( \delta \) in the below theorem may at first appear cumbersome, however it arises naturally in the proof as a bound on the expected number of interactions of a certain particle with the rest of the particles during a time interval of length \( t_{\text{mix}}^{\text{IP}(2)}(\frac{\varepsilon}{8k}) \), after an initial burn-in period of length \( t_{\text{mix}}^{\text{IP}(2)}(\frac{\varepsilon}{8k}) \). The quantity \( t_{\text{mix}}^{\text{IP}(2)}(\varepsilon) \) for \( \varepsilon \in (0, \frac{1}{k} \wedge \frac{1}{k}) \) appearing below has a natural interpretation. It is up to some universal constant comparable to the \( \varepsilon k/4 \) mixing time of \( \lfloor k/2 \rfloor \) independent realizations of \( \text{IP}(2) \), cf. [10]. Recall that we make the global assumption that the process \( \text{IP}(2) \) is irreducible and has uniform stationary distribution.

**Theorem 1.4 (Bound for hypergraphs).** There exists a universal constant \( C > 0 \) such that for every size \( n \) hypergraph and for each \( k \geq 3 \) and \( \varepsilon \in (0, \frac{1}{k} \wedge \frac{1}{k}) \) satisfying \( \delta = \delta(\varepsilon, k) := 8Rkn^{-2}t_{\text{mix}}^{\text{IP}(2)}(\frac{\varepsilon}{8k}) < 1 \) we have that

1. if \( \varepsilon k^{-1} \geq 2\delta \) then
   \[
t_{\text{mix}}^{\text{IP}(k)}(\varepsilon) \leq C t_{\text{mix}}^{\text{IP}(2)}(\varepsilon),
   \]

2. if \( \varepsilon k^{-1} < 2\delta \) then
   \[
t_{\text{mix}}^{\text{IP}(k)}(\varepsilon) \leq C t_{\text{mix}}^{\text{IP}(2)}(\varepsilon) \log_{1/\delta}(k/\varepsilon).
   \]

Equivalently, for all \( a > 0 \), if \( \delta(\varepsilon, k) \leq (\varepsilon/k)^a \) then \( t_{\text{mix}}^{\text{IP}(k)}(\varepsilon) \leq C \frac{1+a}{a} t_{\text{mix}}^{\text{IP}(2)}(\varepsilon) \).

Moreover, there exists \( C' > 0 \) such that for all \( b \in (0, 1] \) there exists \( n_0(b) \) such that for every size \( n \geq n_0(b) \) hypergraph, for each \( k \geq 3 \), if \( Rkn^{-2}t_{\text{mix}}^{\text{IP}(2)}(n^{-b}) \leq n^{-b} \) then

\[
t_{\text{mix}}^{\text{IP}(k)}(\varepsilon) \leq C' b^{-1} t_{\text{mix}}^{\text{IP}(2)}(\varepsilon) \quad \text{for all } 0 < \varepsilon \leq \frac{1}{4} \wedge \frac{1}{k},
\]

and if in addition \( \text{IP}(2) \) and \( \text{IP}(k) \) are also reversible then

\[
t_{\text{mix}}^{\text{IP}(k)}(\varepsilon) \leq C' b^{-1} t_{\text{mix}}^{\text{IP}(2)}(\varepsilon).
\]

[5]Note that for \( d \)-regular hypergraphs with all hyperedges of size \( L \) if \( r_e \equiv 1/d \), we have that \( Rn^{-1} = L - 1 \). For hypergraphs with maximal hyperedge size \( L \) and maximal degree \( \Delta \), if \( r_e \equiv 1 \) then \( Rn^{-1} \leq \Delta L \).
Remark 1.5. 1. As we make no assumption on the law of the permutations associated with the hyperedges, other than IP(2) being irreducible with uniform stationary distribution, it need not be the case that IP(n) is irreducible. For instance, consider the case \( n = 4 \) where there is a single hyperedge containing all 4 vertices, and when it rings a random 3-cycle is applied. Since 3-cycles are even permutations IP(4) is reducible. The irreducibility of IP(k) under the assumption \( \delta < 1 \) is thus a non-trivial consequence of Theorem 1.4.

2. The condition \( Rkn^{-2}t^{IP(2)}_\text{mix}(n^{-b}) \leq n^{-b} \) may seem strong, for example if \( k = n \) and \( r_e \equiv 1 \) for all \( e \), there are no regular hypergraphs for which the condition holds. However, our main focus is not in this regime; instead we are interested in much smaller values of \( k \) for which there do exist hypergraphs satisfying this condition.

3. It follows by the contraction principle (see [1]) that the same bounds as in Theorem 1.4 hold for the exclusion process for the same values of \( k \).\[^6\]

For hypergraphs, \( t^{IP(2)}_\text{mix}(1/4) \) and \( t^{RW(1)}_\text{mix}(1/4) \) can be of different orders, see the example in [7, Remark 1.5]. That example also demonstrates that one cannot replace \( t^{IP(2)}_\text{mix}(\varepsilon) \) in Theorem 1.4 by \( t^{RW(1)}_\text{mix}(\varepsilon) \). Even for graphs, the proof that \( t^{IP(2)}_\text{mix}(1/4) \) and \( t^{RW(1)}_\text{mix}(1/4) \) are comparable is surprisingly difficult \([16]\), and it is not known if \( t^{IP(2)}_\text{mix}(\varepsilon) \) and \( t^{RW(1)}_\text{mix}(\varepsilon) \) are comparable, uniformly for all \( \varepsilon \leq 1/4 \). One exception in the graph setup is the case that \( \varepsilon \leq n^{-\Omega(1)} \) where both quantities are comparable up to a constant factor to \( t^{RW(1)}_{\text{rel}} \log(1/\varepsilon) \) (using (12) combined with \( t^{IP(2)}_{\text{rel}} = t^{RW(1)}_{\text{rel}} \) (the Caputo, Liggett and Richthammer Theorem \([5]\))). Using (5) below one can show that \( t^{IP(2)}_\text{mix}(\varepsilon) \) and \( t^{RW(1)}_\text{mix}(\varepsilon) \) are comparable also for uniform hypergraphs, when \( \varepsilon \leq n^{-\Omega(1)} \).

To complement Theorem 1.4, we are interested in finding general conditions under which the conditions of Theorem 1.4 hold, and under which \( t^{IP(2)}(\varepsilon) \asymp t^{RW(1)}(\varepsilon) \) or \( t^{IP(2)}(\varepsilon) \asymp t^{RW(1)}(\varepsilon) \). Theorem 1.2 gives one such case, and the following result gives another.

**Theorem 1.6.** There exists an absolute constant \( C > 0 \) such that for all \( b \in (0,1] \) there exists \( n_0(b) \) such that for all uniform interchange processes on a size \( n \geq n_0(b) \) hypergraph satisfying \( Rkn^{-2}t^{RW(1)}_\text{mix}(n^{-b}) \leq n^{-b} \) we have that

\[
t^{IP(k)}_{\text{rel}} \leq Cb^{-1}t^{RW(1)}_{\text{rel}}. \tag{4}
\]

Moreover, regardless of the value of \( R \), for a uniform interchange process on a finite hypergraph we always have that

\[
t^{IP(2)}_{\text{rel}} \leq Ct^{RW(1)}_{\text{rel}}. \tag{5}
\]

Equation (4) can be seen as a partial progress on a conjecture of Caputo \([6]\) that for uniform interchange process on hypergraphs, we have that \( t^{IP(n)}_{\text{rel}} = t^{RW(1)}_{\text{rel}} \). We note that in (3) we consider a more general class of interchange processes, by not requiring the permutations chosen to be uniformly distributed. However, (3) and (4) are of course weaker than Caputo’s conjecture, as they do not apply for all \( k \) and include some absolute constant.

\[^6\]In fact, an inspection of the proof reveals that for \( \text{EX}(k) \) we can replace \( t^{IP(2)}_\text{mix}(\cdot) \) by \( t^{EX(2)}_\text{mix}(\cdot) \lor t^{RW(1)}_\text{mix}(\cdot) \) (both in our upper bounds and in the definition of \( \delta \) from Theorem 1.4).
The proof of (5) uses a comparison of Dirichlet forms. One obstacle is that IP(2) and RW(2) (as well as RW(1)) do not have the same state space. Moreover, some care is required to avoid dependence on the maximal degree, the maximal size of a hyperedge and on \( \max_{e,e' \in E} \frac{r_e}{r_{e'}} \) in the constant in the right-hand side of (5).

1.2 Oliveira’s conjecture - comparing with independent particles

In [16] Oliveira showed the existence of a universal constant \( C \) such that for general graphs (but not hypergraphs) and rates \( \max_{k \leq n/2} t_{\text{mix}}^{\text{EX}(k)}(\varepsilon) \leq C t_{\text{mix}}^{\text{RW}(1)}(\varepsilon) \log(n/\varepsilon) \) for all \( \varepsilon \in (0, 1) \), and conjectured that for all \( \varepsilon \in (0, 1) \) and \( k \leq n \),

\[
t_{\text{mix}}^{\text{EX}(k)}(\varepsilon) \leq C t_{\text{mix}}^{\text{RW}(k)}(\varepsilon)
\]  

(see [11, Conjecture 2] and [10, Question 1.5] for related problems; see Footnote [4] for a recent related result). As mentioned above, our results verify this in various setups for ‘small’ \( k \).

In [10] we proved an upper-bound on \( t_{\text{mix}}^{\text{EX}(k)} \) that was within a multiplicative factor of \( \log \log n \) of this conjecture for all \( k = n^{\Omega(1)} \) such that \( k \leq n/2 \) (for regular graphs with \( r_e \equiv 1 \)). Further, we demonstrated in certain situations that the conjecture holds; for example if \( n^c \lesssim k \lesssim n^{1-c} \) for \( c \in (0, 1/2) \) (in this case the constant \( C \) in (6) depends on \( c \)). Other examples when \( k \) is not assumed to be small are when the spectral-gap is at most \( (\log n)^{-4} \) or when the degree is at least logarithmic. These bounds (and in fact all results on \( \text{EX}(k) \) in [10]) are valid also for \( t_{\text{mix}}^{\text{IP}(k)} \) when \( k \leq n(1-c) \) (for any constant \( c \in (0, 1/2] \), possibly with additional dependence of the constant \( C \) on \( c \)). It is interesting to note that the graphs for which [10] does not offer sharp bounds for large \( k \) are, as said above, ones with fairly large spectral-gaps (at least \( (\log n)^{-4} \)). For such graphs the condition on \( k \) in Theorems 1.2 and 1.4 is milder.

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2 Preliminaries

2.1 Notation and basic definitions

For a finite set \( \Omega \) we denote by \( S_\Omega \) the group of permutations of elements in \( \Omega \). For \( k \leq |\Omega| \), we write \( (\Omega)_k \) for the set of \( k \)-tuples of distinct elements from \( \Omega \). For \( x \in (\Omega)_k \) we denote \( \mathcal{O}(x) = \{ x_i : i \in [k] \} \). For a random variable \( X \) we write \( \mathcal{L}[X] \) for the law or distribution of \( X \). The total-variation distance between two distributions \( \mu \) and \( \nu \) is defined as

\[
\|\mu - \nu\|_{\text{TV}} := \sum_a (\mu(a) - \nu(a))_+ = \frac{1}{2} \sum_a |\mu(a) - \nu(a)| = \inf_{(X,Y):\mathcal{L}[X]=\mu,\mathcal{L}[Y]=\nu} \mathbb{P}[X \neq Y],
\]

where, in the last equality, the infimum is over all couplings \( (X,Y) \) of \( (\mu, \nu) \).
For ω ∈ Ω and a continuous-time irreducible Markov process (X_τ^ω)_{t≥0} with state space Ω and satisfying X_0^ω = ω, we define its ε total-variation mixing time as
\[ t_{\text{mix}}^X(ε) := \inf \{ t ≥ 0 : \max_{ω ∈ Ω} ||\mathcal{L}[X_t^ω] - \pi||_{TV} ≤ ε \}, \]
where π denotes the stationary distribution of the process.

### 2.2 Graphical construction

We present a construction of a random walk and the interchange process. One important feature of this construction is it places these processes on the same probability space which allows us to directly relate them. Graphical construction for the interchange process on graphs is classical, see [13]. Our construction for this process on hypergraphs is the same as that appearing in Connor-Pymar.

We take the state space to be \( S_V \). In this notation, the particles are labeled by the set \( V \). We think of \( \sigma(v) \) for \( \sigma ∈ S_V \) and \( v ∈ V \) as the location of the particle labelled \( v \) in configuration \( \sigma \).

The first step is to construct a sequence of independent edge choices, that is, a sequence \((ε_n)_{n ∈ \mathbb{N}}\) with the property that for each \( ε ∈ E \), \( \mathbb{P}[ε_n = ε] \propto r_ε \). Next, we require a sequence of permutations choices. Given \((ε_n)_{n ∈ \mathbb{N}}\), we construct a sequence \((σ_n)_{n ∈ \mathbb{N}}\) such that for each \( n ∈ \mathbb{N} \), \( σ_n ∈ S_{e_n} \). Finally we determine the jump times. Let \( Λ \) be a Poisson process of rate \( \sum_ε r_ε \). For \( 0 < s < t \) denote by \( Λ[s, t] \) the number of points of \( Λ \) in interval \( [s, t] \) and define a permutation \( I_{[s, t]} : V → V \) associated with time interval \([s, t]\) to be the composition of permutations occurring during this time:

\[ I_{[s, t]} = σ_{Λ[0, t]} ∘ σ_{Λ[0, t−1]} ∘ \cdots ∘ σ_{Λ[0, s]} + 1. \]

We set \( I_t := I_{[0, t]} \) for each \( t > 0 \). These functions can be lifted to functions on \((V)_k\) by setting \( I_{[s, t]}(x) = (I_{[s, t]}(x_1), I_{[s, t]}(x_2), \ldots, I_{[s, t]}(x_n)) \), for \( x ∈ (V)_k \). In this construction, \( I_{[s, t]}(a) \) is the location at time \( t \) of the particle that occupied \( a \) at time \( s \). We have the following consequence:

**Proposition 2.1** (Proof omitted). Fix \( s ≥ 0 \). Then

1. for each \( x ∈ V \), the process \((I_{[s, t]}(x))_{t≥0}\) is a random walk started from \( x \),
2. for each \( x ∈ (V)_k \), the process \((I_{[s, t]}(x))_{t≥0}\) is a \( k \)-particle interchange process started from \( x \).

### 2.3 Some auxiliary results

**Lemma 2.2.**

\[ ∀k ≥ 3, \ ε ∈ (0, 1/4), \ \frac{1}{2} t_{\text{mix}}^{\text{RW}(1)}(4ε/k) ≤ t_{\text{mix}}^{\text{RW}(k)}(ε) ≤ t_{\text{mix}}^{\text{RW}(1)}(ε/k). \]  

Moreover, for an interchange process on a size \( n \) hypergraph such that \( \text{RW}(1) \) is reversible and has uniform stationary distribution, if for some \( b ∈ (0, 1] \) such that \( n−b ≤ 1/4 \) and \( C ≥ 1 \) we have that \( t_{\text{mix}}^{\text{IP}(k)}(n−b) ≤ C t_{\text{mix}}^{\text{RW}(1)}(n−b) \) then for all \( ε ∈ (0, n−b) \)

\[ t_{\text{mix}}^{\text{IP}(k)}(ε) ≤ 32Cb−1 t_{\text{mix}}^{\text{RW}(1)}(ε), \]
and provided that $IP(k)$ is also reversible then we have that
\[ t_{rel}^{IP(k)} \leq 32Cb^{-1}t_{rel}^{RW(1)}. \]  
(9)

Similarly, for an interchange process on a size $n$ hypergraph such that $IP(2)$ is reversible, irreducible and has uniform stationary distribution, if for some $b \in (0, 1]$ such that $n^{-b} \leq 1$ and $C \geq 1$ we have that $t_{mix}^{IP(k)}(n^{-b}) \leq Ct_{mix}^{IP(2)}(n^{-b})$ then for all $\varepsilon \in (0, n^{-b})$
\[ t_{mix}^{IP(k)}(\varepsilon) \leq 64Cb^{-1}t_{mix}^{IP(2)}(\varepsilon), \]  
and provided that $IP(k)$ is also reversible then we have that
\[ t_{rel}^{IP(k)} \leq 64Cb^{-1}t_{rel}^{IP(2)}. \]  
(11)

Proof. The first display is Equation (18) from [10]. For (8) we use the general relations between $t_{mix}$ and $t_{rel}$ (here we rely on reversibility and on the uniform distribution being stationary)
\[ t_{rel}^{RW(1)} \log \left( \frac{1}{2\varepsilon} \right) \leq t_{mix}^{RW(1)}(\varepsilon) \leq t_{rel}^{RW(1)} \log(n/\varepsilon) \]  
(12)

[12, Lemma 20.11, Theorem 20.6 and (4.43)] as well as submultiplicativity of mixing times [12, (4.29)] (i.e. $t_{mix}(\delta^l) \leq \ell t_{mix}(\delta/2)$ for all $\delta > 0$ and $\ell \in \mathbb{N}$) to deduce that for all $m \in \mathbb{N},$
\[ t_{mix}^{RW(1)} \left( \frac{1}{2n^2mc(1+b)} \right) \leq t_{mix}^{RW(1)} \left( \frac{1}{3nmc(1+b)} \right) \leq 8Cb^{-1}t_{mix}^{RW(1)} \left( \frac{1}{2n^m} \right) \leq 16Cb^{-1}t_{mix}^{RW(1)}(n^{-mb}), \]  
and that
\[ t_{mix}^{IP(k)}(n^{-mb}) \leq mt_{mix}^{IP(k)}(n^{-b}/2) \leq 2mt_{mix}^{IP(k)}(n^{-b}) \leq 2mc^1t_{mix}^{RW(1)}(n^{-b}) \leq 2mt_{mix}^{RW(1)}(n^{-b}/2) \leq 16Cb^{-1}t_{mix}^{RW(1)}(n^{-mb}). \]  
(13)

Finally, if $\varepsilon \in (n^{-mb}, n^{-(m+1)b})$ then by monotonicity and submultiplicativity
\[ t_{mix}^{RW(k)}(\varepsilon) \leq t_{mix}^{RW(k)}(n^{-(m+1)b}) \leq 2t_{mix}^{RW(k)}(n^{-mb}) \leq 32Cb^{-1}t_{mix}^{RW(1)}(n^{-mb}) \leq 32Cb^{-1}t_{mix}^{RW(1)}(\varepsilon). \]

This concludes the proof of (8). We now prove (9). By the general relations between $t_{mix}$ and $t_{rel},$ and equation (8),
\[ t_{rel}^{IP(k)} \leq \frac{t_{mix}^{IP(k)}(\varepsilon)}{|\log(2\varepsilon)|} \leq \frac{32Cb^{-1}t_{mix}^{RW(1)}(\varepsilon)}{|\log(2\varepsilon)|} \leq \frac{32Cb^{-1}t_{rel}^{RW(1)}(\varepsilon)}{|\log(2\varepsilon)|}. \]

Taking the limit as $\varepsilon \to 0$ concludes the proof of (9).

For (10) we use the general relations between $t_{mix}$ and $t_{rel}$ (here again we rely on reversibility and on the uniform distribution being stationary),
\[ t_{rel}^{IP(2)} \log \left( \frac{1}{2\varepsilon} \right) \leq t_{mix}^{IP(2)}(\varepsilon) \leq t_{rel}^{IP(2)} \log(n^2/\varepsilon) \]  
(14)

as well as submultiplicativity of mixing times to deduce that for all $m \in \mathbb{N},$
\[ t_{mix}^{RW(k)}(n^{-mb}) \leq 2mt_{mix}^{RW(1)}(n^{-b}) \leq 2mt_{rel}^{IP(2)}(n^2+b) \leq 32Cb^{-1}t_{mix}^{IP(1)}(n^{-mb}). \]  
(15)

The proof of (10) is concluded as that of (8). Finally, the proof of (11) is analogous to that of (9) and is hence omitted.
3 Proof of bound for hypergraphs: Theorem 1.4

We shall say that two particles interact at time $t$ if they occupy some vertices $u$ and $v$ at time $t$ (i.e. at some time interval $[t \pm \varepsilon, t]$) and at time $t$ an edge containing $u$ and $v$ rings. For $s > 0$ and $z \in (V)_{k-1}$, we denote by $\Gamma_s$ the set of càdlàg sample paths of a $(k-1)$-particle interchange process up to time $s$ and by $\Gamma^a_s \subseteq \Gamma_s$ those paths which start at configuration $a \in (V)_{k-1}$. We also let $J_s = J_s(x)$ be the event that the $k$th particle avoids interacting with the other $k-1$ particles during time interval $[s, 2s]$ when we initialise from configuration $x \in (V)_k$.

For $s \geq 0$, $z \in (V)_{k-1}, \gamma \in \Gamma^a_s$, $x \in V \setminus O(z)$ and $c \in V$, we define a law

$$\mu_s(\bullet) = \mu_s^{\gamma, x, c}(\bullet) = \mathbb{P}[I_{2s}(x) \in \bullet \mid I_s(x) = c, (I_t(z))_{0 \leq t \leq 2s} = \gamma],$$

that is, $\mu_s$ is the law of the $k$th particle at time $2s$ of a $k$-particle interchange process conditioned on the trajectory of the first $k-1$ particles, and on the location of the $k$th particle at time $s$. (Note that, by the Markov property, $\mu_s^{\gamma, x, c}$ does in fact not depend on $x$.)

**Lemma 3.1.** For all $s \geq 0$, $z \in (V)_{k-1}, \gamma \in \Gamma^a_s$, $x \in V \setminus O(z)$ and $c \in V$,

$$||\mu_s^{\gamma, x, c} - \mathcal{L}[I_s(c)]||_{TV} \leq 1 - \mathbb{P}[J_s((z, x)) \mid I_s(x) = c, (I_t(z))_{0 \leq t \leq 2s} = \gamma].$$

**Proof.** From the definition of total-variation and using that for $a, b, c \in \mathbb{R}_+$,

$$(a + b - c)_+ \leq a + (b - c)_+,$$

$$||\mu_s^{\gamma, x, c} - \mathcal{L}[I_s(c)]||_{TV} = \sum_{a \in V} (\mu_s^{\gamma, x, c}(a) - \mathbb{P}[I_s(c) = a])_+ \leq \sum_{a \in V} \left[\mathbb{P}[I_{2s}(x) = a, J_s((z, x)) \mid I_s(x) = c, (I_t(z))_{0 \leq t \leq 2s} = \gamma] - \mathbb{P}[I_s(c) = a]\right]_+$$

$$+ 1 - \mathbb{P}[J_s((z, x)) \mid I_s(x) = c, (I_t(z))_{0 \leq t \leq 2s} = \gamma],$$

where the last term comes from $\sum_{a \in V} \mathbb{P}[I_{2s}(x) = a, J_s((z, x)) \mid I_s(x) = c, (I_t(z))_{0 \leq t \leq 2s} = \gamma]$.

Next we argue that $\mathbb{P}[I_{2s}(x) = a, J_s((z, x)) \mid I_s(x) = c, (I_t(z))_{0 \leq t \leq 2s} = \gamma] \leq \mathbb{P}[I_s(c) = a]$ for all $a$. The intuition is that having to avoid interacting with the trajectory $\gamma$ during $[s, 2s]$ imposed by $J_s((z, x))$ and the conditioning $(I_t(z))_{0 \leq t \leq 2s} = \gamma$ can only decrease the chance of reaching any given target vertex $a$ at time $2s$. To prove this we need some additional notation. For $a, b \in V$ and $s > 0$, let $\Gamma^RW_s(a, b)$ be the set of càdlàg sample paths of a random walk up to time $s$ which starts at $a$ and terminates at $b$. Further, for any $\gamma \in \Gamma_s$, let $\Gamma^{RW}_s(\gamma)(a, b) \subseteq \Gamma^RW_s(a, b)$ be those sample paths which avoid interacting with $\gamma$ (that is, trajectories of the random walk which do not interact with any of the $k-1$ particles moving according to $\gamma$). Then for $\gamma \in \Gamma_s$ and $a, c \in V$, we have

$$\mathbb{P}[I_{2s}(x) = a, J_s((z, x)) \mid I_s(x) = c, (I_t(z))_{0 \leq t \leq 2s} = \gamma] = \mathbb{P}[(I_{s,t}(c))_{s \leq t \leq 2s} \in \Gamma^RW_s(\gamma)(c, a)] \leq \mathbb{P}[(I_{s,t}(c))_{s \leq t \leq 2s} \in \Gamma^RW_s(c, a)]$$

$$= \mathbb{P}[I_s(c) = a].$$

Plugging this into (16) gives the claimed inequality. \qed
For Lemma 3.1 to be useful we need to lower-bound the probability of $J_s$:

**Lemma 3.2.** Fix $\varepsilon \in (0, 1)$ and let $s = t_{\text{mix}}^{\text{IP}(2)}(\varepsilon^{16k}/16k^2)$. Then for all $k \geq 2$,

$$\min_{x \in (V)_k} \mathbb{P}[J_s(x)] \geq 1 - \frac{\varepsilon}{16k} - \frac{sk}{n^2} \sum_{e} r_e |(|e| - 1)|.$$

**Proof.** By a union bound

$$\mathbb{P}[J_s^c] \leq \sum_{i=1}^{k-1} \mathbb{P}[J_{s,i}^c],$$

(17)

where $J_{s,i}$ is the event that the $k$th particle avoids interacting with the $i$th particle during time interval $[s, 2s]$. We will use a coupling argument to upper-bound $\mathbb{P}[J_{s,i}^c]$ for each $i \in \{1, \ldots, k - 1\}$. Specifically, we couple the pair ($i$th and $k$th particles) with a pair started from time 0 according to the stationary distribution of process IP(2). The chosen coupling is one which satisfies the coupling equality in the definition of total-variation. Observe that, crucially, by using a union bound, we can use $k - 1$ different couplings (which need not be related to one another in any way), each of which involves just 2 particles. Let $A_{s,i}$ denote the event that the coupling of the $i$th and $k$th particles is successful at time $s$. Then we can write

$$\mathbb{P}[J_{s,i}^c] \leq \mathbb{P}[J_{s,i}^c, A_{s,i}] + \mathbb{P}[A_{s,i}^c].$$

(18)

Let $(y, x)$ be the initial location of the $(i, k)$th particles. Since $s = t_{\text{mix}}^{\text{IP}(2)}(\varepsilon^{16k}/16k^2)$, for $k \geq 3$ we have

$$\mathbb{P}[A_{s,i}^c] = \|L[I_s((y, x))] - \pi^{\text{IP}(2)}\|_{TV} \leq \frac{\varepsilon}{16k^2}.$$

We also need to upper-bound $\mathbb{P}[J_{s,i}^c, A_{s,i}]$. For $x, y \in V$, let $T_s(x, y)$ denote the number of times that two particles evolving as IP(2) started from vertices $x$ and $y$ interact during time interval $[0, s]$. By Markov’s inequality we have the bound

$$\mathbb{P}[J_{s,i}, A_{s,i}] \leq \sum_{(x,y) \in (V)_2} \frac{1}{n(n-1)} \mathbb{E}[T_s(x, y)].$$

We can bound this expectation via:

$$\sum_{(x,y) \in (V)_2} \frac{1}{n(n-1)} \mathbb{E}[T_s(x, y)] = \sum_{(x,y) \in (V)_2} \frac{1}{n(n-1)} \mathbb{E} \left[ \int_0^s \sum_{I_t(x), I_t(y) \in e} r_e \, dt \right]$$

$$= \sum_{e} r_e \frac{1}{n(n-1)} \int_0^s \mathbb{E} \left[ \sum_{x \in V} \sum_{y \in V} 1_{I_t(x) \in e} \sum_{y \neq x} 1_{I_t(y) \in e} \right] \, dt$$

$$\leq \sum_{e} r_e \frac{1}{n(n-1)} \int_0^s |e|(|e| - 1) \, dt$$

$$= \frac{s}{n(n-1)} \sum_{e} r_e |(|e| - 1)|.$$
Putting the two bounds into (18) and using (17), we obtain
\[ P[J^*_k] \leq (k-1) \left( \frac{\varepsilon}{16k^2} + \frac{s}{n(n-1)} \sum e r_e |e| (|e| - 1) \right) \leq \frac{\varepsilon}{16k} + \frac{sk}{n^2} \sum e r_e |e| (|e| - 1). \]

Before stating the next lemma, we define
\[ \tilde{d}_k(t) := \max_{w \in (V)_{k-1}, u, v \in V \setminus O(w)} [\|\mathcal{L}[I_t((w, u))] - \mathcal{L}[I_t((v, u))]\|_{TV}]. \]

In the case \( k = 1 \) this reduces to \( \max_{u, v \in V} [\|\mathcal{L}[I_t(u)] - \mathcal{L}[I_t(v)]\|_{TV}] \).

The next lemma formalises the following idea: if the \( k \)th particle in an interchange process is unlikely to interact with any of the other \( k - 1 \) particles for time \( s \) sufficiently large then, conditionally on the trajectory of the first \( k - 1 \) particles, the \( k \)th particle will be close to mixed. The idea of the lemma is that the usual submultiplicativity property of the worst case distance from equilibrium can be extended to the notion \( \tilde{d}_k(t) \), provided one only considers couplings of \( \mathcal{L}[I_t((w, u))] \) and \( \mathcal{L}[I_t((w, v))] \) which take the same value in the first \( k - 1 \) coordinates.

**Lemma 3.3.** For any \( s, t \geq 0 \),
\[ \tilde{d}_k(s + t) \leq \tilde{d}_k(t) \left( 2 \max_{x \in (V)_{k}} (1 - P[J_{s/2}(x)]) + \tilde{d}_1(s/2) \right). \]

**Proof.** Let \( z \in (V)_{k-1}, x, y \in V \setminus O(z) \) and write \( x = (z, x), y = (z, y) \). Then for any \( s, t \geq 0 \),
\[ [\|\mathcal{L}[I_{s+t}(x)] - \mathcal{L}[I_{s+t}(y)]\|_{TV} \leq \mathbb{E} [\|\mathcal{L}[I_t(x(s))] - \mathcal{L}[I_t(y(s))]\|_{TV} \]
for any coupling \((x(s), y(s)) = ((z(s), x(s)), (z(s), y(s))) \) where \( \mathcal{L}[x(s)] = \mathcal{L}[x(s)] \) and \( \mathcal{L}[y(s)] = \mathcal{L}[y(s)] \). The coupling we choose will be one which keeps the first \( k - 1 \) coordinates matched in the two processes (i.e evolves the first \( k - 1 \) particles identically) and moreover, this coupling will depend on the trajectory of the first \( k - 1 \) particles.\(^7\) Note that the quantity inside the expectation is zero if \( x(s) = y(s) \) and is always bounded by \( \tilde{d}_k(t) \) (as we keep the first \( k - 1 \) coordinates equal). Hence
\[ \tilde{d}_k(s + t) \leq \tilde{d}_k(t) P[x(s) \neq y(s)]. \]

By our choice of coupling we can write
\[ P[x(s) \neq y(s)] = \mathbb{E} [P[x(s) \neq y(s)] | x(s/2), y(s/2), (z(t))_{0 \leq t \leq s}] \]
\[ = \mathbb{E} [P[x(s) \neq y(s)] | x(s/2), y(s/2), (z(t))_{0 \leq t \leq s}]. \]

Given the trajectory \((z(t))_{0 \leq t \leq s}\), the coupling we choose is that which attains equality in the definition of total-variation, that is, the one which allows us to write the above as
\[ P[x(s) \neq y(s)] | x(s/2), y(s/2), (z(t))_{0 \leq t \leq s}] = \mathbb{E} [\|\mu - \nu\|_{TV}], \]

\(^7\)We clarify that we do not couple the dynamics performed by the \( k \)th particle in the two systems by time \( s \). We only couple them at time \( s \), in a manner that depends on the trajectories of the rest of the \( k - 1 \) particles by time \( s \).
where \( \mu = \mu_s^{(z(t))_{0 \leq t \leq s}, x, x(s/2)} \) and \( \nu = \mu_s^{(z(t))_{0 \leq t \leq s}, y, y(s/2)} \). Next, by the triangle inequality we have
\[
\mathbb{E} [\|\mu - \nu\|_{TV}] \leq \mathbb{E} [\|\mu - \mathcal{L}[I_{s/2}(x(s/2))]\|_{TV}] + \mathbb{E} [\|\nu - \mathcal{L}[I_{s/2}(y(s/2))]\|_{TV}]
+ \max_{x, y} \|\mathcal{L}[I_{s/2}(x)] - \mathcal{L}[I_{s/2}(y)]\|_{TV}.
\]
(19)
The first two expectations on the right-hand side can be bounded using Lemma 3.1:
\[
\mathbb{E} [\|\mu - \mathcal{L}[I_{s/2}(x(s/2))]\|_{TV}] \leq 1 - \mathbb{E} [\mathbb{P}[J_{s/2}((z, x)) | x(s/2), (z(t))_{0 \leq t \leq s}]]
= 1 - \mathbb{P}[J_{s/2}((z, x))],
\]
and similarly
\[
\mathbb{E} [\|\nu - \mathcal{L}[I_{s/2}(y(s/2))]\|_{TV}] \leq 1 - \mathbb{P}[J_{s/2}((z, y))].
\]
The third term on the right-hand side of (19) is simply \( \bar{d}_1(s/2) \).

Proof of Theorem 1.4. For the first part of the statement, we need to show that if \( s \geq \delta_{\text{mix}}^{IP(2)}(\varepsilon) \) in the case \( \varepsilon k^{-1} \geq 2\delta \) and if \( s \geq \delta_{\text{mix}}^{IP(2)}(1/k) \log_{1/\delta}(k/\varepsilon) \) in the case \( \varepsilon k^{-1} < 2\delta \), for some universal \( c > 0 \), then
\[
\max_{x, y \in (V)_k} \|\mathcal{L}[I_s(x)] - \mathcal{L}[I_s(y)]\|_{TV} \leq \varepsilon.
\]
(20)
In order to apply Lemma 3.3 we must reduce the above total-variation distance to one between initial configurations which differ in a single coordinate. This is achieved via the triangle inequality.

Suppose \( x, y \in (V)_k \) are arbitrary. Note that there exists a sequence \( x =: x_0, x_1, \ldots, x_r := y \) for some \( r \leq k \) with \( x_i \in (V)_k \) and such that \( x_i \) differs in one coordinate from \( x_{i+1} \) for all \( 0 \leq i < r \). Hence for any \( s > 0 \),
\[
\|\mathcal{L}[I_s(x)] - \mathcal{L}[I_s(y)]\|_{TV} \leq \sum_{i=1}^{r} \|\mathcal{L}[I_s(x_{i-1})] - \mathcal{L}[I_s(y_{i-1})]\|_{TV}.
\]
(21)
So now suppose that \( x = (z, x) \), \( y = (z, y) \in (V)_k \) differ in just one coordinate, which, without loss of generality, we assume is the \( k \)th coordinate. Then by repeated application of Lemma 3.3, for any \( m \in \mathbb{N} \) and \( t > 0 \),
\[
\|\mathcal{L}[I_{mt}(x)] - \mathcal{L}[I_{mt}(y)]\|_{TV} \leq \left( 2 \max_{x \in (V)_k} \mathbb{P}[J_{t/2}^r(x)] + \bar{d}_1(t/2) \right)^m.
\]
We now set \( t = 2\delta_{\text{mix}}^{IP(2)}(\varepsilon) \) (which by submultiplicativity of mixing times is at most \( 20\delta_{\text{mix}}^{IP(2)}(\varepsilon) \), since \( \varepsilon \leq \frac{1}{k} \wedge \frac{1}{4k} \) so that \( \bar{d}_1(t/2) \leq \varepsilon/(8k^2) \). By Lemma 3.2 we thus have
\[
\|\mathcal{L}[I_{mt}(x)] - \mathcal{L}[I_{mt}(y)]\|_{TV} \leq \left( \frac{\varepsilon}{4k} + \frac{t(k+\varepsilon)}{n^2} \sum \mathbb{E}[|e|(|e|-1)] \right)^m \leq \left( \frac{\varepsilon}{4k} + \frac{\delta}{2} \right)^m,
\]
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where the last inequality follows by noting
\[
\frac{\delta}{2} = 4kn^{-2}t_{\text{mix}}^{\text{IP}(2)}(\varepsilon/(8k)) \sum_e r_e e(\|e\| - 1) \geq 2kn^{-2}t_{\text{mix}}^{\text{IP}(2)}(\varepsilon/(16k^2)) \sum_e r_e e(\|e\| - 1)
\]
\[
= tkn^{-2} \sum_e r_e e(\|e\| - 1).
\]
If \(\varepsilon k^{-1} \geq 2\delta\) we take \(m = 1\). Otherwise we take \(m = \lceil \log_{1/\delta}(k/\varepsilon) \rceil\) (recall our assumption that \(\delta \leq 1\)). In each case taking \(s = mt\) in (21) we deduce that for arbitrary \(x, y \in (V)_k\),
\[
\|\mathcal{L}[I_s(x)] - \mathcal{L}[I_s(y)]\|_{\text{TV}} \leq \varepsilon,
\]
which completes the proof of (20).

It remains to prove (2) and (3). Using submultiplicativity of mixing times, provided that \(n \geq n_0(b)\) the condition \(Rkn^{-2}t_{\text{mix}}^{\text{IP}(2)}(n^{-b}) \leq n^{-b}\) implies that
\[
8Rkn^{-2}t_{\text{mix}}^{\text{IP}(2)}(8k) \leq 8Rkn^{-2}t_{\text{mix}}^{\text{IP}(2)}\left(\frac{1}{8n^{1+b/2}}\right) \leq n^{-b/2}. \tag{22}
\]
Hence by (20) we have that
\[
t_{\text{mix}}^{\text{IP}(k)}(n^{-b/2}) \leq C t_{\text{mix}}^{\text{IP}(2)}(n^{-b/2}).
\]
Hence (2) for \(\varepsilon \leq n^{-b/2}\) follows from (10) and (3) follows from (11). To obtain (2) for \(\varepsilon \in (n^{-b/2}, 1/4 \land 1/2)\), we note that similarly to (22) we have that if \(n \geq n_0(b)\) then
\[
8Rkn^{-2}t_{\text{mix}}^{\text{IP}(2)}\left(\frac{\varepsilon}{8k}\right) \leq n^{-b/2}.
\]
Hence by (20) we have that \(t_{\text{mix}}^{\text{IP}(k)}(\varepsilon) \leq C t_{\text{mix}}^{\text{IP}(2)}(\varepsilon)\).

Remark 3.4. Under stronger conditions on \(k\) the argument just presented could be simplified. Consider, for example, a burn-in period of duration \(s = Ct_{\text{mix}}^{\text{IP}(2)}(1/k)\). Then it can be shown that if \(k\delta \leq \frac{1}{32}\), with probability bounded away from zero, no pair of particles interact during time interval \([s, 2s]\). This would lead to the bound \(t_{\text{mix}}^{\text{IP}(k)} \lesssim t_{\text{mix}}^{\text{IP}(2)}(1/k)\).

4 Proof of bound for vertex-transitive graphs: Theorem 1.1

Proof of Theorem 1.1. If \(n^{1/12} < k \leq n^a\) we appeal to Theorem 1.2 of [10] (which holds also for interchange provided \(k \leq n/2\), despite being stated for the exclusion process) to obtain \(t_{\text{mix}}^{\text{IP}(k)}(\varepsilon) \lesssim t_{\text{rel}} \log(n/\varepsilon)\) for all \(n\) sufficiently large (depending on \(a\) – to guarantee \(n^a \leq n/2\)). On the other hand we have (using (12)) \(t_{\text{mix}}^{\text{RW}(1)}(\varepsilon) \gtrsim t_{\text{rel}} |\log(2\varepsilon)| \gtrsim t_{\text{rel}} \log(n/\varepsilon)\), for \(\varepsilon \leq k^{-1}\) and \(k > n^{1/12}\) which completes the proof in this regime. So for the rest of the proof we suppose that \(k \leq n^{1/12}\).

We consider two cases depending on the growth rate of the diameter \(D\) of the vertex-transitive graph \(G\). Suppose first that \(D > n^{1/3}\) so that for all \(n\) sufficiently large (depending on \(d\),
we have \( D \geq (n/d)^{1/4}. \)[8] Then by Corollary 2.8 of [18] we know that there exist constants \( A, B > 0 \) such that (provided \( n \geq n_0(d) \)) \( G \) has \((A,B)\)-moderate growth (in the sense described in [8]). It then follows from Proposition 11.1 of [10] (the proposition as stated there is for the 1/4-mixing time of the exclusion process but the upper-bound holds also for interchange on \( k \leq n/2 \) particles and the proof carries over to the \( \varepsilon \)-mixing time for \( \varepsilon \leq k^{-1} \)) that, uniformly in \( k \leq n^{1/12} \), \( t_{\text{mix}}^{\text{IP}}(\varepsilon) \lesssim_d D^2 \log(1/\varepsilon) \approx t_{\text{rel}} \log(1/\varepsilon) \lesssim_d t_{\text{mix}}^{\text{RW}}(1)(\varepsilon) \) provided \( n \geq n_0(d) \).

Now suppose that \( D \leq n^{1/3} \). We use the following result on vertex transitive graphs (to appear in a future work of Nathanaël Berestycki, the first author, and Lucas Teyssier; as mentioned in the introduction, the credit for this result is due to Tessera and Tointon [17], as this bound is a consequence of their bound on the isoperimetric profile and the generic evolving sets bound on the return probability [15]):

**Proposition 4.1** (Berestycki, Hermon, Teyssier). There exist \( C(d,m) \) such that for over all vertex-transitive graphs of size \( n \) and degree \( d \) satisfying that \( n \geq D^q \) with \( |q| = m \), writing \( R := D^{q-m} \), for all \( t \leq D^2 \) and every vertex \( x \) we have that

\[
p_t(x,x) \leq C(d,m) \left( \frac{1}{t^{(m+1)/2}} \vee \frac{1}{R^{m/2}} \right).
\]

In particular, if \( n \geq D^3 \), then uniformly over \( t \leq D^2 \), \( p_t(x,x) \lesssim d t^{-3/2} \).

In order to apply Theorem 1.2 we also need to verify that \( t_{\text{rel}}^{\text{RW}(1)} \leq n^{1-2a} \) for some \( a > 0 \). The diameter bound on the relaxation time gives \( t_{\text{rel}}^{\text{RW}(1)} \leq 2D^2 \leq 2n^{2/3} \leq n^{3/4} \) for \( n \) sufficiently large [12, Theorem 13.26] (because \( r_e \equiv 1 \) rather than \( r_e \equiv 1/d \) the factor \( d \) in the reference can be removed). Thus by Theorem 1.2 (and using the remark that follows it) we deduce that if \( D^2 \geq 2a_{\text{mix}}^{\text{RW}(1)}(\varepsilon) \) (where \( a = \alpha(d) \) is determined in the proof of Theorem 1.2) then for all \( n \) sufficiently large (depending on \( d \)), and all \( 3 \leq k \leq n^{1/12} \) we have \( t_{\text{mix}}^{\text{IP}(k)}(\varepsilon) \lesssim_d t_{\text{mix}}^{\text{RW}(1)}(\varepsilon) \). On the other hand, if \( D^2 < 2a_{\text{mix}}^{\text{RW}(1)}(\varepsilon) =: 2s \), then we apply Proposition 4.1 at time \( D^2 \) (namely, \( \max_x (p_{D^2}(x,x) \lesssim D^{-3}) \) together with the Poincaré inequality (between times \( D^2 \) and \( s \wedge D^2 \)) and the diameter upper bound on the relaxation time to deduce that

\[
\max_x (p_{2s}(x,x) - 1/n) \lesssim_d \left( 1 \wedge e^{-2(s-D^2)/t_{\text{rel}}} \right) \lesssim_d \left( 1 \wedge e^{-3(s-D^2)/D^2} \right) D^{-3}.
\]

To complete the proof it suffices to show that the right-hand side above is \( \lesssim_d (2s)^{-1(1+\theta)} \) for some \( \theta > 0 \). If \( s < D^2 \) then \( D^{-3} \lesssim (2s)^{-3/2} \) as needed. So suppose \( s > D^2 \). We can work on the additional assumption that \( D^2 > D_0 \) for any constant \( D_0 \) since, for fixed \( d \), there are infinitely many graphs with \( D^2 \leq D_0 \) and (as \( r_e \equiv 1 \)) finitely many Markov processes, and so the result is trivial in this case. It remains to show

\[
(1 + \theta) \log s \leq \frac{s}{D^2} + 3\log D,
\]

for some \( \theta > 0 \) and \( D \) sufficiently large. However this is clear: if \( 3\log D \geq sD^{-2} \) then \( \log s \leq 2\log D + 3\log D \leq \frac{5}{4} \log D \) (for \( D \) sufficiently large), so \( \frac{5}{4} \log s \leq 3\log D \leq sD^{-2} + 3\log D \); if \( 3\log D < sD^{-2} \) then (for \( D \) sufficiently large)

\[
\frac{5}{4} \log s < \frac{s \log (3D^2 \log D)}{4 - 3D^2 \log D} \leq \frac{s}{D^2}.
\]

\[\text{[8]}\]If \( d \geq n^{1/3} \) one can bound the mixing time e.g. using the bound from [2], and for such large \( d \) this bound can be completely absorbed into the constant which depend on \( d \).
5 Proof of Theorem 1.2

Proof of Theorem 1.2. We first consider the case \( \varepsilon \in [n^{-a}, \frac{1}{k} \wedge \frac{1}{A}] \). In order to apply Theorem 1.4, we show that \( \delta \leq \varepsilon/(2k) \), where (as defined in the statement of Theorem 1.4) \( \delta := 8Rkn^{-2}t_{\text{mix}}^{(2)}(\varepsilon/8k) \). Under the maximal degree assumption, we have the bound \( R := n^{-1} \sum e \leq 2d \) and so \( \delta \leq 16dkn^{-2}t_{\text{mix}}^{(2)}(\varepsilon/8k) \). Using the general relations between \( t_{\text{mix}} \) and \( t_{\text{rel}} \) (i.e. for reversible irreducible chains \( t_{\text{mix}}(\varepsilon) \leq t_{\text{rel}}(\varepsilon) \), and the Caputo, Liggett and Richthammer Theorem \([5]\) we have \( t_{\text{mix}}^{(2)}(\varepsilon/8k) \leq t_{\text{rel}}^{(2)}(\varepsilon) = t_{\text{rel}}^{(1)}(\varepsilon) = Cn^{-2a} \). Now \( k \leq n^a \), and \( \varepsilon \geq n^{-a} \), hence \( \delta \leq 16dk^{-1}n^{-1} \leq \varepsilon/(2k) \), provided \( n \) is sufficiently large (depending on \( d \) and \( a \)). Thus it follows by Theorem 1.4 that there exists a universal \( C > 0 \) such that for each \( \varepsilon \in [n^{-a}, \frac{1}{k} \wedge \frac{1}{A}] \),

\[
t_{\text{mix}}^{(k)}(\varepsilon) \leq Ct_{\text{mix}}^{(2)}(\varepsilon).
\] (23)

Next, set \( \alpha := \alpha_{\text{mix}}^{(k)}(\varepsilon) \) for some \( \alpha = \alpha(c, d, \theta) \) to be determined, and let \( N_{(t_1, t_2)}(a, b) \) denote the number of interactions during time interval \((t_1, t_2)\) between particles started (at time \(0\)) from vertices \(a\) and \(b\). Similarly let \( \bar{N}_{(t_1, t_2)}(a, b) \) denote the time that these particles are adjacent during interval \((t_1, t_2)\). Then as edges ring at rate \(1\), \( \mathbb{E}[N_{(t_1, t_2)}(a, b)] = \mathbb{E}[\bar{N}_{(t_1, t_2)}(a, b)] \) (this follows by the same reasoning as a similar statement in the proof of Lemma 5.9 in \([10]\), i.e. by noticing that interactions between \(a\) and \(b\) do not affect the unordered pair of trajectories \(\{I_t(a), I_t(b)\}\)). Thus for any \(a, b \in V\),

\[
\mathbb{E}[N_{(s, 2s)}(a, b)] = \mathbb{E} \left[ \int_s^{2s} \sum_x \sum_{y \sim x} 1_{\{I_t(a) = (x, y)\}} dt \right] \\
\leq \int_s^{2s} \sum_x \sum_{y \sim x} \mathbb{P}[I_t(a) \in \{x, y\}, I_t(b) \in \{x, y\}] dt \\
\leq \int_s^{2s} \sum_x \sum_{y \sim x} \mathbb{P}[I_t(a) \in \{x, y\}] \mathbb{P}[I_t(b) \in \{x, y\}] dt \\
\leq \int_s^{2s} \sqrt{\left( \sum_x \sum_{y \sim x} \mathbb{P}[I_t(a) \in \{x, y\}]^2 \right) \left( \sum_x \sum_{y \sim x} \mathbb{P}[I_t(b) \in \{x, y\}]^2 \right)} dt \\
\leq \int_s^{2s} \sqrt{\sum_x \sum_{y \sim x} (2p_t(a, x)^2 + 2p_t(a, y)^2) \left( \sum_x \sum_{y \sim x} (2p_t(b, x)^2 + 2p_t(b, y)^2) \right)} dt
\]
where the second inequality follows from the negative correlation property of the exclusion process and the third is by Cauchy-Schwarz. Now we observe that by reversibility for each \(a \in V\),

\[
\sum_x \sum_{y \sim x} (p_t(a, x)^2 + p_t(a, y)^2) = d \sum_x p_t(a, x)^2 + d \sum_y p_t(a, y)^2 = 2dp_{2t}(a, a).
\]
Using this in the previous display we obtain
\[
\mathbb{E}[N_{(s, 2s)}(a, b)] \leq 4d \int_s^{2s} \max_z p_{2t}(z, z) dt \leq 4d \left( \frac{s}{n} + s \left( \max_z p_{2s}(z, z) - \frac{1}{n} \right) \right).
\] (24)

By assumption HK-(\theta),
\[
2s \leq c^{(1+\theta)^{-1}} \left( \max_z p_{2s}(z, z) - \frac{1}{n} \right)^{-(1+\theta)^{-1}}
\]
from which we obtain
\[
2s \left( \max_z p_{2s}(z, z) - \frac{1}{n} \right) \leq c^{(1+\theta)^{-1}} \left( \max_z p_{2s}(z, z) - \frac{1}{n} \right)^{\theta(1-\theta)}.
\]
The Poincaré inequality gives that for reversible Markov chains, if \( t \geq c_1 t_{\text{rel}} \log(1/\varepsilon) \) then
\[
\max_x p_t(x, x) - \frac{1}{n} \leq \varepsilon^{c_1}.
\]
It follows that, as \( 2s = 2\alpha t_{\text{mix}}^{\text{RW}(1)}(\varepsilon) \geq 2\alpha t_{\text{rel}}^{\text{RW}(1)} \log(1/\varepsilon) \), we have
\[
\max_z p_{2s}(z, z) - \frac{1}{n} \leq \varepsilon^{4\alpha}
\]
and so
\[
2s \left( \max_z p_{2s}(z, z) - \frac{1}{n} \right) \leq c^{(1+\theta)^{-1}} \varepsilon^{4\alpha \theta(1-\theta)}.
\]
Using this in (24) we obtain
\[
\mathbb{E}[N_{(s, 2s)}(a, b)] \leq \frac{4 ds}{n} + 2d c^{(1+\theta)^{-1}} \varepsilon^{4\alpha \theta(1-\theta)} = \frac{4d}{n} \alpha t_{\text{mix}}^{\text{RW}(1)}(\varepsilon) + 2d c^{(1+\theta)^{-1}} \varepsilon^{4\alpha \theta(1-\theta)}
\]
\[
\leq \frac{4d}{n} \alpha t_{\text{mix}}^{\text{RW}(1)} \log(n/\varepsilon) + \frac{\varepsilon}{16} \leq \frac{4d}{n} \alpha n^{1-2\alpha} \log(n/\varepsilon) + \frac{\varepsilon}{16}
\]
\[
\leq \frac{\varepsilon}{8},
\]
using \( \varepsilon \geq n^{-a} \), choosing an appropriately large \( \alpha = \alpha(c, d, \theta) \), and provided \( n \) is sufficiently large (depending on \( a \) and \( d \)).

To complete the proof for this case, we apply Lemma 3.3 (taking the \( k \) there to be 2), which gives that
\[
d_2(2s) \leq 2 \max_{x \in (V)} (1 - \mathbb{P}[J_s(x)]) + d_1(s) \leq 2 \max_{x \in (V)} \mathbb{E}[N_{(s, 2s)}(x)] + \frac{\varepsilon}{4} \leq \frac{\varepsilon}{2},
\]
where we have used Markov’s inequality in the second inequality above and possibly increased \( \alpha \) so that by submultiplicativity \( d_1(s) \leq \varepsilon/2 \). It follows that \( t_{\text{mix}}^{\text{IP}(2)}(\varepsilon/2) \leq 2s \) and combining with (23) (and using submultiplicativity again) we deduce that there exists a constant \( C' = C'(\alpha) \) such that provided \( n \) is sufficiently large (depending on \( a \) and \( d \)),
\[
t_{\text{mix}}^{\text{IP}(k)}(\varepsilon) \leq C' t_{\text{mix}}^{\text{RW}(1)}(\varepsilon),
\]
for each \( \varepsilon \in [n^{-a}, \frac{1}{k} \wedge \frac{1}{4}] \). It remains to consider the case \( \varepsilon < n^{-a} \). By submultiplicativity, \( t_{\text{mix}}^{\text{IP}(k)}(\varepsilon) \leq t_{\text{mix}}^{\text{IP}(k)}(n^{-a}) \log_n(1/\varepsilon) \), and so by the result just demonstrated, we have \( t_{\text{mix}}^{\text{IP}(k)}(\varepsilon) \leq \alpha t_{\text{mix}}^{\text{RW}(1)}(n^{-a}) \log_n(1/\varepsilon) \leq t_{\text{rel}} \log n \log_n(1/\varepsilon) \leq \Omega(1) \leq t_{\text{mix}}^{\text{IP}(1)}(\varepsilon) \) for \( n \) sufficiently large. \( \square \)
6 Comparison of Dirichlet forms: Proof of Theorem 1.6

The majority of this section is devoted to the proof of (5). Once we have this, (4) follows easily by combining it with (3). So we can now focus on the proof of (5).

Since RW(1) and RW(2) have the same spectral-gap, our goal is to compare the Dirichlet forms associated with IP(2) and RW(2). However the two do not have the same state space. To rectify this, we consider the auxiliary process Q(2) – the process obtained from RW(2) by observing it only when particles are at different locations (that is, we remove the times at which they are at the same location). We shall compare Dirichlet forms associated with IP(2) and Q(2). This suffices because of the following lemma.

Lemma 6.1. For all finite hypergraphs $G = (V, E)$ and all rates $(r_e : r \in E)$ such that the associated uniform interchange process with two particles is irreducible we have that $Q(2)$ is reversible and its stationary distribution is the uniform distribution on $(V)_2$. Moreover

$$t^{Q(2)}_{\text{rel}} \leq t^{\text{RW}(2)}_{\text{rel}}.$$  \hfill (25)

Proof. It is easy to check that the generator of $Q(2)$ is symmetric and hence indeed it is reversible w.r.t. the uniform distribution on $V_2$, which below we denote by $\mu$.

For (25) cf. [12, Theorem 13.16] (the proof is written in discrete time, but only minor adaptations are needed for the continuous time case).

Write $q^{\text{IP}(2)}(\cdot, \cdot)$ for the transition rates of IP(2). We first note that the Dirichlet form associated with IP(2) can be written as

$$E^{\text{IP}(2)}(f, f) = \frac{1}{2} \sum_{a \in (V)_2} \sum_{b \in (V)_2} \frac{1}{(n/2)!} q^{\text{IP}(2)}(a, b) (f(a) - f(b))^2$$

$$= \frac{1}{2} \sum_{a \in (V)_2} \sum_{b \in (V)_2} \frac{1}{(n/2)!} (f(a) - f(b))^2 \sum_{e : a, b \in (e)_2} \frac{r_e}{|e|(|e| - 1)}$$ \hfill (26)

$$+ \frac{1}{2} \sum_{a \in (V)_2} \sum_{b \in (V)_2} \frac{1}{(n/2)!} (f(a) - f(b))^2 \sum_{e : a(1) \in e, a(2), b(2) \in e} \frac{r_e}{|e|},$$ \hfill (27)

$$+ \frac{1}{2} \sum_{a \in (V)_2} \sum_{b \in (V)_2} \frac{1}{(n/2)!} (f(a) - f(b))^2 \sum_{e : a(2) \in e, a(1), b(1) \in e} \frac{r_e}{|e|},$$ \hfill (28)

Write $q^{Q(2)}(\cdot, \cdot)$ for the transition rates of Q(2). The process Q(2) is reversible with uniform
stationary distribution and Dirichlet form

\[ \mathcal{E}^{Q(2)}(f,f) = \frac{1}{2} \sum_{x \in (V)_{2}} \sum_{y \in (V)_{2}} \sum_{y \neq x} \frac{1}{n(2)} q^{Q(2)}(x,y) (f(x) - f(y))^2 \]

\[ \leq \frac{1}{2} \binom{n}{2} \sum_{x \in (V)_{2}} \sum_{y \in (V)_{2}} q^{Q(2)}(x,y) (f(x) - f(y))^2 \]

\[ + \frac{1}{2} \binom{n}{2} \sum_{x \in (V)_{2}} \sum_{y \in (V)_{2}} q^{Q(2)}(x,y) (f(x) - f(y))^2 \]

\[ + \frac{1}{2} \binom{n}{2} \sum_{x \in (V)_{2}} \sum_{y \in (V)_{2}} q^{Q(2)}(x,y) (f(x) - f(y))^2 \]

\[ + \frac{1}{2} \binom{n}{2} \sum_{x \in (V)_{2}} \sum_{y \in (V)_{2}} q^{Q(2)}(x,y) (f(x) - f(y))^2. \]

\[ \text{(29)} \]

(this is an inequality rather than equality because we have, in the last two lines, double-counted the terms where \( y = (x(2),x(1)) \). We show that each of these terms can be upper-bounded by a linear combination of the terms (26)-(28).

6.1 Bounding terms (29) and (30)

The two terms (29) and (30) can be dealt with similarly (likewise for (31) and (32) in the next subsection). Term (29) is equal to

\[ \frac{1}{2} \binom{n}{2} \sum_{x \in (V)_{2}} \sum_{y \in (V)_{2}} q^{Q(2)}(x,y) (f(x) - f(y))^2 \]

\[ = \frac{1}{2} \binom{n}{2} \sum_{x \in (V)_{2}} \sum_{y \in (V)_{2}} (f(x) - f(y))^2 \sum_{e: x(1) \neq e} \frac{r_e}{|e|} \]

\[ + \frac{1}{2} \binom{n}{2} \sum_{x \in (V)_{2}} \sum_{y \in (V)_{2}} (f(x) - f(y))^2 \sum_{e: x(1) \in e} \frac{r_e}{|e|} \]

\[ + \frac{1}{2} \binom{n}{2} \sum_{x \in (V)_{2}} \sum_{y \in (V)_{2}} (f(x) - f(y))^2 \sum_{e: x(2) \in e} \frac{r_{e'}}{|e'|} \]

\[ \text{(30)} \]

\[ \text{(31)} \]

\[ \text{(32)} \]

\[ \text{(33)} \]

\[ \text{(34)} \]

\[ \text{(35)} \]
Terms (33) and (34) correspond to the situation in which particle 2 jumps directly from \(x(2)\) to \(y(2)\). Note that (33) is the same as (27). Term (35) corresponds to particle 2 jumping first to the vertex occupied by particle 1 (which is not observed by \(Q(2)\) as at this point the two particles are at the same location), and then the next edge that rings containing \(x(1)\) (in \(RW(2)\)) is an edge which also has \(y(2)\) on it, and when it rings particle 2 jumps to \(y(2)\).

Term (34) requires a little manipulation. We can write it as

\[
\frac{1}{2} \left( \binom{n}{2} \right) \sum_{x \in (V)} \sum_{y \in (V)} \sum_{e : x(1) \in e \neq x(2), y(2) \in e} \frac{r_e}{|e|} \left( \binom{|e|}{2} \right) \sum_{z \in (e)} (f(x) - f(z) + f(z) - f(y))^2
\]

\[
\leq \frac{1}{2} \left( \binom{n}{2} \right) \sum_{x \in (V)} \sum_{y \in (V)} \sum_{e : x(1) \in e \neq x(2), y(2) \in e} \frac{r_e}{|e|} \left( \binom{|e|}{2} \right) \sum_{z \in (e)} 2[(f(x) - f(z))^2 + (f(z) - f(y))^2]
\]

\[
= \frac{1}{2} \left( \binom{n}{2} \right) \sum_{x, x \in (V)} \sum_{e : x, z \in (e)} \frac{r_e}{|e|} \left( \binom{|e|}{2} \right) 2(f(x) - f(z))^2(|e| - 1)
\]

\[
= \frac{4}{2} \left( \binom{n}{2} \right) \sum_{x, z \in (V)} (f(x) - f(z))^2 \sum_{e : x, z \in (e)} \frac{r_e}{|e|^2}
\]

Observe that this is at most 8 times (26).

Term (35) requires even more manipulation. For fixed \(x, y\), set \(w = (y(2), x(2))\), \(z = (y(2), y(1))\), and \(z = (z(2), z(1))\). Using that

\[
(f(x) - f(y))^2 \leq 3 \left[ (f(x) - f(w))^2 + (f(w) - f(z))^2 + (f(z) - f(y))^2 \right]
\]

we decompose (35) into three terms:

\[
\frac{3}{2} \left( \binom{n}{2} \right) \sum_{x \in (V)} \sum_{w \in (V)} (f(x) - f(w))^2 \sum_{e : x \in (e)} \frac{r_e}{2|e|} \sum_{e' : x(1) \in e'} \frac{r_{e'}}{|e'| - 1} \left( \sum_{e : x(1) \in e} r_e \right)^{-1}
\]

\[
+ \frac{3}{2} \left( \binom{n}{2} \right) \sum_{w \in (V)} \sum_{z \in (V)} (f(w) - f(z))^2 \sum_{e : w(2) \in e} \frac{r_e}{2|e|} \sum_{e' : z \in (e')} \frac{r_{e'}}{|e'| - 1} \left( \sum_{e : z(2) \in e} r_e \right)^{-1}
\]

\[
+ \frac{3}{2} \left( \binom{n}{2} \right) \sum_{z \in (V)} \sum_{x(2) \in V} (f(z) - f(z))^2 \sum_{e : x(2) \in e} \frac{r_e}{2|e|} \sum_{e' : z(2) \in e'} \frac{r_{e'}}{|e'| - 1} \left( \sum_{e : z(2) \in e} r_e \right)^{-1}
\]
Terms (36) and (37) can be treated similarly. Firstly, (36) can be split into two terms depending on whether \( x(2) \) is in \( e' \):

\[
\frac{3}{2} \binom{n}{2} \sum_{x \in (V)_2} \sum_{w \in (V)_2} (f(x) - f(w))^2 \sum_{e : x \in (e)_2} \frac{r_e}{2 |e|} \sum_{e' : x(1), w(1) \in e'} \frac{r_{e'}}{|e'| - 1} \left( \sum_{\bar{e} : x(1) \in \bar{e}} r_{\bar{e}} \right)^{-1}
\]

\[
+ \frac{3}{2} \binom{n}{2} \sum_{x \in (V)_2} \sum_{w \in (V)_2} (f(x) - f(w))^2 \sum_{e : x \in (e)_2 \neq x(2)} \frac{r_e}{2 |e|} \sum_{e' : x(1), w(1) \in e'} \frac{r_{e'}}{|e'| - 1} \left( \sum_{\bar{e} : x(1) \in \bar{e}} r_{\bar{e}} \right)^{-1}
\]

\[
\leq \frac{3}{4} \binom{n}{2} \sum_{x \in (V)_2} \sum_{w \in (V)_2 \neq x(2)} (f(x) - f(w))^2 \sum_{e' : x(1), w(1) \in e'} \frac{r_{e'}}{|e'| - 1}
\]

(39)

\[
+ \frac{3}{4} \binom{n}{2} \sum_{x \in (V)_2} \sum_{w \in (V)_2 \neq x(2)} (f(x) - f(w))^2 \sum_{e' : x(1), w(1) \in e'} \frac{r_{e'}}{|e'| - 1}
\]

(40)

Now, (39) can be bounded in exactly the same way as (34), and so (since we have a factor of \( \frac{3}{4} \) instead of \( \frac{1}{2} \) as in (34)) it is at most 12 times (26). Term (40) is at most 3 times (28), and thus (36) is at most \( 12 \mathcal{E}_{\text{IP}}(2) (f, f) \). A similar argument gives the same bound for (37) (splitting it depending on whether \( w(1) \) is in \( e \)).

For (38), we can write it as

\[
\frac{3}{2} \binom{n}{2} \sum_{z \in (V)_2} (f(z) - f(\bar{z}))^2 \sum_{e : z(2) \in e} \frac{r_e}{2} \sum_{e' : z \in (e')_2} \frac{r_{e'}}{|e'| - 1} \left( \sum_{\bar{e} : z(2) \in \bar{e}} r_{\bar{e}} \right)^{-1}
\]

\[
= \frac{3}{4} \binom{n}{2} \sum_{z \in (V)_2} (f(z) - f(\bar{z}))^2 \sum_{e : z \in (e)_2} \frac{r_e}{|e| - 1} \sum_{v \in (e)_2} \left\{ (f(z) - f(v))^2 + (f(v) - f(\bar{z}))^2 \right\}
\]

\[
= \frac{3}{2} \binom{n}{2} \sum_{z \in (V)_2} \sum_{v \in (V)_2 \neq z} \frac{r_e}{|e| - 1} \left( \binom{|e|}{2} \right)
\]

which is clearly bounded by 12 times (26).

Adding, we see that we can bound (35) by \( 36 \mathcal{E}_{\text{IP}}(2) (f, f) \) and thus overall term (29) is at most \( 44 \mathcal{E}_{\text{IP}}(2) (f, f) \). The same is clearly true for (30).
6.2 Bounding terms (31) and (32)

It remains to consider terms (31) and (32). Term (31) is

\[
\frac{1}{2} \binom{n}{2} \sum_{x \in (V)_2} \sum_{y \neq x \in (V)_2, y \neq x, y(2) = x(1)} q^{Q(2)}(x, y)(f(x) - f(y))^2
\]

\[
= \frac{1}{2} \binom{n}{2} \sum_{x \in (V)_2} \sum_{y \neq x \in (V)_2, y(2) = x(1)} (f(x) - f(y))^2 \sum_{e: x \in (e)_2} \frac{r_e}{|e|} \sum_{e': y \in (e')_2} \frac{r_{e'}}{|e'| - 1} \left( \sum_{\bar{e}: x(1) \in \bar{e}} r_{\bar{e}} \right)^{-1}. \quad (41)
\]

We again must manipulate term (41) a little. For each fixed \( x, y \) appearing in the sums, we define \( z = (y(1), x(2)) \), and then we can upper-bound (41) by

\[
\frac{1}{2} \binom{n}{2} \sum_{x \in (V)_2} \sum_{y \neq x \in (V)_2, y(2) = x(1)} 2 \left[ (f(x) - f(z))^2 + (f(z) - f(y))^2 \right] \sum_{e: x \in (e)_2} \frac{r_e}{|e|} \sum_{e': y \in (e')_2} \frac{r_{e'}}{|e'| - 1} \left( \sum_{\bar{e}: x(1) \in \bar{e}} r_{\bar{e}} \right)^{-1}
\]

\[
= \frac{1}{2} \binom{n}{2} \sum_{x \in (V)_2} \sum_{y \neq x \in (V)_2, y(2) = x(2)} 2 (f(x) - f(z))^2 \sum_{e: x \in (e)_2} \frac{r_e}{|e|} \sum_{e': y \in (e')_2} \frac{r_{e'}}{|e'| - 1} \left( \sum_{\bar{e}: x(1) \in \bar{e}} r_{\bar{e}} \right)^{-1}
\]

\[
+ \frac{1}{2} \binom{n}{2} \sum_{x \in (V)_2} \sum_{y \neq x \in (V)_2, y(1) = z(1)} 2 (f(z) - f(y))^2 \sum_{e: y(2), x(2) \in e} \frac{r_e}{|e|} \sum_{e': y \in (e')_2} \frac{r_{e'}}{|e'| - 1} \left( \sum_{\bar{e}: y(2) \in \bar{e}} r_{\bar{e}} \right)^{-1}
\]
\[ \frac{1}{2} \frac{1}{n} \sum_{x \in (V)_{2}} \sum_{z \in (V)_{2}} 2(f(x) - f(z))^2 \sum_{e : x \in (e)_{2}} \frac{r_{e}}{|e|} \sum_{e' : x(1), x(1) \in e'} \frac{r_{e'}}{2(|e'| - 1)} \left( \sum_{e : x(1) \in e} r_{e} \right)^{-1} \]

\[ + \frac{1}{2} \frac{1}{n} \sum_{x \in (V)_{2}} \sum_{z \in (V)_{2}} 2(f(x) - f(z))^2 \sum_{e : x \in (e)_{2}} \frac{r_{e}}{|e|} \sum_{e' : z(2), x(2) \in e'} \frac{r_{e'}}{2(|e'| - 1)} \left( \sum_{e : x(1) \in e} r_{e} \right)^{-1} \]

\[ + \frac{1}{2} \frac{1}{n} \sum_{z \in (V)_{2}} \sum_{y \in (V)_{2}} 2(f(z) - f(y))^2 \sum_{e : y(2), z(2) \in e} \frac{r_{e}}{|e|} \sum_{e' : y(1), y(1) \in e'} \frac{r_{e'}}{2(|e'| - 1)} \left( \sum_{e : y(2) \in e} r_{e} \right)^{-1} \]

\[ + \frac{1}{2} \frac{1}{n} \sum_{z \in (V)_{2}} \sum_{y \in (V)_{2}} 2(f(z) - f(y))^2 \sum_{e : y(2), z(2) \in e} \frac{r_{e}}{|e|} \sum_{e' : y(1), y(1) \notin e'} \frac{r_{e'}}{2(|e'| - 1)} \left( \sum_{e : y(2) \in e} r_{e} \right)^{-1}. \]

Term (44) is at most (34) which we have already established is at most 8 times (26). By similar arguments we can obtain the same bound on (42). Term (43) is at most (28) and term (45) is at most (27). We deduce that (41), and hence (31) is at most 16EIP(2)(f, f). The same bound can be obtained for (32).

Putting the bounds on (29)-(32) together, we obtain that \( E^{Q(2)}(f, f) \leq 120EIP(2)(f, f) \).

References

[1] David Aldous and Jim Fill. Reversible Markov chains and random walks on graphs, 2002. Unfinished manuscript. Available at http://www.stat.berkeley.edu/~aldous/RWG/book.html.

[2] Gil Alon and Gady Kozma. Comparing with octopi. Ann. Inst. Henri Poincaré Probab. Stat., 56(4):2672–2685, 2020. MR4164852.

[3] Emmanuel Breuillard, Ben Green, and Terence Tao. The structure of approximate groups. Publ. Math. Inst. Hautes Études Sci., 116:115–221, 2012. MR3090256.

[4] Emmanuel Breuillard and Matthew C. H. Tointon. Nilprogressions and groups with moderate growth. Adv. Math., 289:1008–1055, 2016. MR3439705.

[5] Pietro Caputo, Thomas M. Liggett, and Thomas Richthammer. Proof of Aldous’ spectral gap conjecture. J. Amer. Math. Soc., 23(3):831–851, 2010. MR2629990.
[6] Filippo Cesi. A few remarks on the octopus inequality and Aldous’ spectral gap conjecture. *Comm. Algebra*, 44(1):279–302, 2016. MR3413687.

[7] Stephen B. Connor and Richard J. Pymar. Mixing times for exclusion processes on hypergraphs. *Electron. J. Probab.*, 24:48 pp., 2019. MR3978223.

[8] P. Diaconis and L. Saloff-Coste. Moderate growth and random walk on finite groups. *Geom. Funct. Anal.*, 4(1):1–36, 1994. MR1254308.

[9] Sharad Goel, Ravi Montenegro, and Prasad Tetali. Mixing time bounds via the spectral profile. *Electron. J. Probab.*, 11:no. 1, 1–26, 2006. MR2199053.

[10] Jonathan Hermon and Richard Pymar. The exclusion process mixes (almost) faster than independent particles. 2018. Ann. Probab. to appear. arxiv preprint.

[11] Jonathan Hermon and Justin Salez. The interchange process on high-dimensional products. *Ann. Appl. Probab.*, 31(1):–, 2021. MR4254474.

[12] David A. Levin and Yuval Peres. *Markov chains and mixing times*. American Mathematical Society, Providence, RI, 2005. Second edition of [MR2466937], with contributions by Elizabeth L. Wilmer. MR3726904.

[13] Thomas M. Liggett. *Interacting particle systems*. Classics in Mathematics. Springer-Verlag, Berlin, 2005. Reprint of the 1985 original. MR2108619.

[14] Ben Morris. The mixing time for simple exclusion. *Ann. Appl. Probab.*, 16(2):615–635, 2006. MR2244427.

[15] Ben Morris and Yuval Peres. Evolving sets, mixing and heat kernel bounds. *Probab. Theory Related Fields*, 133(2):245–266, 2005. MR2198701.

[16] Roberto Imbuzeiro Oliveira. Mixing of the symmetric exclusion processes in terms of the corresponding single-particle random walk. *Ann. Probab.*, 41(2):871–913, 2013. MR3077529.

[17] Romain Tessera and Matthew Tointon. Sharp relations between volume growth, isoperimetry and resistance in vertex-transitive graphs. 2020. arxiv preprint.

[18] Romain Tessera and Matthew Tointon. A finitary structure theorem for vertex-transitive graphs of polynomial growth. *Combinatorica*, 41(1):263–298, 2021. MR4253426.

[19] David Bruce Wilson. Mixing times of Lozenge tiling and card shuffling Markov chains. *Ann. Appl. Probab.*, 14(1):274–325, 2004. MR2023023.

[20] Horng-Tzer Yau. Logarithmic Sobolev inequality for generalized simple exclusion processes. *Probab. Theory Related Fields*, 109(4):507–538, 1997. MR1483598.