Chapter 1

Random matrix representations of critical statistics.

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Abstract

We consider two random matrix ensembles which are relevant for describing critical spectral statistics in systems with multifractal eigenfunction statistics. One of them is the Gaussian non-invariant ensemble which eigenfunction statistics is multifractal, while the other is the invariant random matrix ensemble with a shallow, log-square confinement potential. We demonstrate a close correspondence between the spectral as well as eigenfunction statistics of these random matrix ensembles and those of the random tight-binding Hamiltonian in the point of the Anderson localization transition in three dimensions. Finally we present a simple field theory in 1+1 dimensions which reproduces level statistics of both of these random matrix models and the classical Wigner-Dyson spectral statistics in the framework of the unified formalism of Luttinger liquid. We show that the (equal-time) density correlations in both random matrix models correspond to the finite-temperature density correlations of the Luttinger liquid. We also present a mechanism of the finite-temperature generation with breaking the translational invariance in space by a metric with the event horizon, similar to the problem of Hawking radiation in the black holes.
1.1 Introduction

It was known since the pioneer’s work by F. Wegner \[1\] that the eigenfunctions $\psi_i(r)$ of the random Schrödinger operator $\hat{H}\psi_i(r) = E_i\psi_i(r)$ at the mobility edge $E_i = E_m$ corresponding to the critical point of the Anderson localization transition, possess the property of multifractality. In particular, at $E = E_m$ the moments of the inverse participation ratio:

\[ P_n(E) = \sum_r \sum_i \langle |\psi_i(r)|^{2n} \delta(E_i - E) \rangle \propto L^{-d_n(n-1)}, \tag{1.1.1} \]

scale as a certain power-law with the total size $L$ of the system. This power is a critical exponent which depends only on the basic symmetry (see Ref.\[2\] and Chapter 3 of this book) of the Hamiltonian $\hat{H}$ and on the dimensionality of space $d$. The true extended states of the Schrödinger operator are characterized by all $d_n = d$. This allows to interpret $d_n < d$ as a certain \textit{fractal dimension} which depends on the order $n$ of the moment. As a matter of fact the statistics of critical eigenfunctions are described by a \textit{set} of fractal dimensions $d_n$ which justifies the notion of \textit{multi-fractality}.

Another aspect of criticality is the scaling with the energy difference $\omega = |E_i - E_j|$ between \textit{two} eigenvalues. It is similar to the \textit{dynamical scaling} and is relevant for the correlation functions of \textit{different} eigenfunctions $\psi_i(r)$ and $\psi_j(r)$. The most important of them is the local density of states correlation function:

\[ C(\omega, R) = \sum_r \sum_{i \neq j} \langle |\psi_i(r)|^2 |\psi_j(r + R)|^2 \delta(E - E_i)\delta(E + \omega - E_j) \rangle. \tag{1.1.2} \]

This correlation function is relevant for the matrix elements of the two-body interaction. The dynamical scaling connects the power law behavior of $C(\omega, 0) \propto \omega^{-\mu}$ with that of $C(0, R) \propto R^{-(d-d_2)}$ by a conjecture \[3\] on the dynamical exponent:

\[ R^{d} \rightarrow \omega, \quad \mu = 1 - \frac{d_2}{d}. \tag{1.1.3} \]

Although there is an extensive numerical evidence in favor of conjecture Eq.\[1.1.3\] its rigorous proof has been lacking so far.

Last but not least, there is a growing interest to the spectral (level) statistics in quantum systems whose classical counterparts are in between of chaos and integrability \[4\], the simplest of them being the \textit{two-level spectral correlation function} (TLS CF) $R(\omega) \propto \sum_R C(\omega, R)$. Some of them apparently share the characteristic features of spectral statistics at the Anderson localization transition, e.g. the finite level compressibility $0 < \chi < 1$ \[5, 31\], and the Poisson tail of the level spacing distribution function $P(s) \propto e^{-s/\chi}$ combined with the Wigner-Dyson level repulsion $P(s) \propto s^\beta$ at small level separation $s \ll 1$ \[7\]. There is a conjecture that these features are also related with the multifractality of critical eigenfunctions, however its exact formulation has not been
developed beyond the limit of weak multifractality $\chi \ll 1$, where one can show that $\chi = \frac{1}{2}(1 - d_2/d)$ [31].

The main reason for scarcity of rigorous knowledge about multifractality of critical eigenfunctions is a lack of exactly solvable models of sufficient generality. The most popular three-dimensional (3D) Anderson model of localization [8] is quite efficient for numerical simulations but so far evaded any rigorous analytical treatment in the critical region. More perspective seemed to be the Chalker’s network model for the Quantum Hall transition [9] and its generalizations. However, numerous proposals for the critical field theory were not successful so far [10]. In this situation the random matrix theory may prove to be the simplest, universal and representative tool to obtain a rigorous knowledge about the critical eigenfunction and spectral statistics [11].

1.2 Non-invariant Gaussian random matrix theory with multifractal eigenvectors.

Guided by the idea that multifractality of eigenstates is the hallmark of criticality, we introduce the Gaussian random matrix ensemble [12, 11] which eigenvectors obey Eq. (1.1.1) with $L$ being replaced by the matrix size $N$. This random matrix theory and its modifications describes very well not only the critical eigenfunction statistics at the Anderson localization transition in three-dimensional (3D) Anderson model but also the off-critical states close to the transition [13]. The critical random matrix ensemble (CRMT) suggested in [12, 11] is manifest non-invariant, and is defined as follows:

$$
\langle H_{nm} \rangle = 0, \quad \langle |H_{nm}|^2 \rangle = \begin{cases} 
\frac{\beta}{2} \left[ 1 + \frac{(n-m)^2}{b^2} \right]^{-1}, & n = m \\
\frac{\beta}{2} \left[ 1 + \frac{(n-m)^2}{b^2} \right]^{-1}, & n \neq m 
\end{cases}
$$

(1.2.1)

where $H_{nm}$ is the Hermitian $N \times N$ random matrix with entries $H_{nm}$ ($n > m$) being independent Gaussian random variables; $\beta = 1, 2, 4$ for the Dyson orthogonal, unitary and symplectic symmetry classes, and $b > 0$ is the control parameter. This CRMT can be considered as a particular deformation of the Wigner-Dyson RMT which corresponds to $b = \infty$.

As is clear from the definition Eq. (1.2.1) the variance $\langle |H_{nm}|^2 \rangle$ is non-invariant under unitary transformation $\hat{H} \rightarrow U \hat{H} U^\dagger$. The existence of the preferential basis is natural, as this RMT mimics the properties of the Anderson model of localization which happens in the co-ordinate space, and not necessarily e.g. in the momentum space.

The critical nature of the CRMT is encoded in the power-law decay of the variance matrix, the typical off-diagonal entry being proportional to $|n - m|^{-1}$ in the absolute value. This is exactly the decay with the power equal to the dimensionality of space ($d = 1$ in the case of matrices). In contrast to
the Wigner-Dyson RMT which probability distribution is parameter-free, the CRMT is a \textit{one-parameter family}. The parameter $b$ controls the spectrum of fractal dimensions $d_n = d_n(b)$. One can show \cite{12, 14} that both at $b \gg 1$ and $b \ll 1$ the scaling relation Eq.(1.1.1) holds true, and the basic fractal dimension is equal to:

$$d_2 = \begin{cases} 1 - c_\beta B^{-1}, & B \gg 1 \\ c_\beta B, & B \ll 1 \end{cases}$$  \hspace{1cm} (1.2.2)

where $c_\beta = \frac{\pi^{\frac{3}{2}}}{\beta} - 2^{\frac{1}{2}}$, and $B = b \pi^{\frac{1}{2}} 2^{\frac{1}{4}}$.

Eq.(1.2.2) can be cast in a form of the \textit{duality relationship}:

$$d_2(B) + d_2(B^{-1}) = 1.$$  \hspace{1cm} (1.2.3)

The duality relation has been checked numerically \cite{15} for the unitary CRMT

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig1}
\caption{Numerical verification of the duality relation Eq.(1.2.3), done by the box counting method \cite{15} for the unitary CRMT ensemble Eq.(1.2.1). The deviations from 1 in Eq.(1.2.3) do not exceed 1\% in the entire region of $B \in [0, 1]$.}
\end{figure}

Eq.(1.2.1). The results are presented in Fig.1.1. In particular, it was found that $2d_2(B = 1) = 1.003 \pm 0.004$. The fact that the deviation of the sum $d_2(B) + d_2(1/B)$ from 1 does not exceed 1\% in the entire region of $B \in [0, 1]$ looks extremely interesting. However, it is not known yet whether an exact
function $d_2(B)$ obeys this remarkable duality relationship which suggests that
\[ d_2 = \frac{1}{2} \text{ at } b = \frac{1}{\sqrt{2\pi \beta}}. \]

There was vast numerical evidence that the CRMT Eq.(1.2.1) reproduces
the main qualitative features of the critical eigenfunction and spectral statistics,
including the power-law behavior of the DoS correlation function $C(\omega, 0)$ [see
Ref.[13] and references therein], distributions of moments of inverse participation
ratio $P_q$ [14], role of rare realizations and multifractality spectrum close to
termination point [16], and the hybrid Wigner-Dyson & Poisson level spacing
distribution [7].

The very fact that by a choice of control parameter $B$ one can fit quite
accurately the critical statistics of both eigenvalues and eigenfunctions of the
Anderson localization model, is extremely encouraging.

### 1.3 Invariant RMT with log-square confinement

Quite remarkably, there is an invariant (but non-Gaussian) RMT whose
two-level spectral correlation function $R_N(s, s')$ is closely related with that of the
CRMT discussed in the previous section. Namely, in the unfolded energy vari-
ables $s$ in which the mean spectral density ("global Density of States") is equal
to unity, and in the large $B$ limit one finds:

\[ R_\infty(s - s') \big|_{\text{inv}} = R_\infty(s - s') \big|_{\text{non-inv}}, \quad (1.3.1) \]

where $R_\infty(s - s') = \lim_{s \to \infty}(\lim_{N \to \infty} R_N(s, s'))$.

For the unitary symmetry class one can show [11] that:

\[ R_\infty(s - s') \big|_{\text{non-inv}} = 1 - \pi^2 \kappa^2 \frac{\sin^2(\pi(s - s'))}{\sinh^2[\pi^2 \kappa(s - s')]} , \quad \kappa = \frac{\beta}{2} \chi(b), \quad (1.3.2) \]

where $\kappa$ is related with the level compressibility $\chi(b)$ (see Eq.(1.8.3),(1.8.6)
below). This suggests [2] the form of the kernel $K(s - s')$ of the invariant
RMT which is the only input one needs to compute all many-point spectral
correlation functions:

\[ K(s - s') = \pi \kappa \frac{\sin(\pi(s - s'))}{\sinh[\pi^2 \kappa(s - s')]} . \quad (1.3.3) \]

Thus a remarkable correspondence between an invariant and a non-invariant
RMT can be conjectured [11] which allows to use the full power of the unitary
invariance for calculation of spectral statistics.

Now it is time to specify the invariant RMT whose counterpart in Eq.(1.3.1)
is the critical RMT with multifractal eigenstates. The probability distribution
for the random matrix Hamiltonian $\hat{H}$ is [17, 11]:

$$
P(\hat{H}) \propto \exp \left[ -\beta \text{tr} \, V(\hat{H}) \right], \quad V(x) = \sum_{n=0}^{\infty} \ln \left[ 1 + 2q^{n+1}(1 + 2x^2) + q^{2(n+1)} \right],
$$

(1.3.4)

where $1 < q < 0$ is a control parameter.

It is extremely important that the ”confinement potential” $V(x)$ at large $|x|$ behaves like:

$$
V(x) \to A \ln^2 |x|, \quad A = \frac{2}{\ln(q^{-1})}.
$$

(1.3.5)

thus being an example of a ”shallow” confinement. The correspondence Eq.(1.3.1) holds for $\ln q^{-1} << 1$, while for $\ln q^{-1} > \pi^2$ an interesting phenomenon of ”crystallization” of eigenvalues happens with the TLCF taking a triangular form [17, 11, 18].

The form of the confinement potential in Eq.(1.3.4) is quite specific, even within the class of shallow potentials with a double-logarithmic asymptotic behavior Eq.(1.3.5). It has been chosen in Ref.[17] in order to enable an exact solution in terms of the $q$-deformed Hermite polynomials. However, there exists another, more simple argument why this particular form leads to an exact solution [19]. The point is that the measure Eq.(1.3.4) can be considered as a generalized Cauchy distribution:

$$
V(E_i) = \prod_{n=1}^{\infty} \frac{1}{4q^{n+1}} \prod_{i=1}^{N} \frac{1}{E_i^2 + \Gamma_n}, \quad \Gamma_n = \frac{1 + q^{n+1}}{2q^{n+1/2}}.
$$

(1.3.6)

1.4 Self-unfolding and not self-unfolding invariant RMT.

As we will see below, the cause of all the peculiarities of the RMT with double-logarithmic confinement is the fact that the mean density of states $\rho_{\infty}(E) = \lim_{N \to \infty} \sum_{i=1}^{N} \langle \delta(E - E_i) \rangle$ approaches some stable non-trivial form in the bulk of the spectrum as the size of matrix $N$ tends to infinity. This is in contrast to the Wigner-Dyson classical RMT where $\lim_{N \to \infty} \rho_N(E)/\sqrt{2N} = 1$ is independent of $E$. This is the reason why the Wigner-Dyson RMT can be referred to as self-unfolding, while the RMT with the double-logarithmic confinement is not self-unfolding.

In order to find a criterion for an invariant RMT to be self-unfolding, let us apply the Wigner-Dyson plasma analogy [2]. According to this approximation the mean density of states $\rho_N(E)$ obeys the integral equation of the equilibrium classical plasma with logarithmic interaction subject to the confining force
1.4. SELF-UNFOLDING AND NOT SELF-UNFOLDING INVARIANT RMT.

\[ -\frac{dV}{dE} = v.p. \int_{-\infty}^{+\infty} \rho_N(E') \frac{dE'}{E - E'} = \frac{dV}{dE} \equiv f(E), \]  

(1.4.1)

where \( v.p. \) denotes the principle value of the integral. The solution to this equation in the case of symmetric confining potential is:

\[ \rho_N(E) = \frac{1}{\pi^2} \sqrt{D^2_N - E^2} v.p. \int_{-D_N}^{D_N} \frac{f(E')}{\sqrt{D^2_N - E'^2}} \frac{dE'}{E' - E}, \]  

(1.4.2)

where the band-edge \( D_N \) should be determined from the normalization condition

\[ \int_{-D_N}^{D_N} \rho_N(E') dE' = N. \]

One can see that if \(|f(E')|\) increases slower than \(|E'|\) as \(|E'| \to \infty\), the integral converges to a non-trivial function as \(N \) and \(D_N \to \infty\):

\[ \rho_\infty(E) = \frac{1}{\pi^2} v.p. \int_{-\infty}^{+\infty} dE' \frac{f(E')}{E' - E}. \]  

(1.4.3)

Otherwise, the large values of \(|E'| \sim D_N\) dominate the integral in Eq.(1.4.2), so that at a fixed \(E\) and \(N \to \infty\) (when \(D_N \gg |E|\)) the dependence of \(\rho_N(E)\) on \(E\) disappears. We conclude therefore [20, 21] that the criterion of a non-self-unfolding RMT is:

\[ \lim_{|x| \to \infty} \frac{V(x)}{|x|} = 0. \]  

(1.4.4)

Let us consider the shallow confining potential with the power-law large-\(x\) asymptotic behavior

\[ V(x) = A|x|^\alpha, \quad (\alpha < 1). \]  

(1.4.5)

Then at large \(|E| \gg 1\) we have [20 21]:

\[ \rho_\infty(E) = \frac{A\alpha}{\pi} \tan\left(\frac{\pi\alpha}{2}\right) \frac{1}{|E|^{1-\alpha}}. \]  

(1.4.6)

In particular for the log-square confining potential

\[ V(x) = A \ln^2 |x| = A \lim_{\alpha \to 0} \left( \frac{|x|^\alpha - 1}{\alpha} \right)^2 = A \lim_{\alpha \to 0} \frac{|x|^{2\alpha} - 2|x|^\alpha + 1}{\alpha^2}. \]  

(1.4.7)

one finds using the linear dependence of Eq.(1.4.3) \(\rho_\infty(E)\) on \(f(E)\) and Eq.(1.4.5):

\[ \rho_\infty(E) = A \frac{|E|}{|E|}. \]  

(1.4.8)

There is a qualitative and far-reaching difference between the shallow power-law confinement potential with \(0 < \alpha < 1\), and the log-square confinement.

\[ ^{\text{1For } V(x) \propto \ln^d |x| \text{ with } d \geq 2 \text{ the leading term at large } E \text{ is } \rho_\infty(E) \propto \frac{\ln^{d-2} |E|}{|E|}.} \]
Although both lead to a non-self-unfolding RMT, the case of finite $\alpha$ can be called \emph{weakly non-self unfolding}, because for large enough $E$ the variation of the mean density at a scale of the mean level spacing $\Delta = \rho^{-1}_\infty$ is much smaller than the density itself:

\[
\frac{\rho_\infty(E + \Delta) - \rho_\infty(E)}{\rho_\infty(E)} = \frac{\rho'_\infty}{\rho^2_\infty} \propto |E|^{-\alpha} \to 0. \tag{1.4.9}
\]

In contrast to that, the log-square confinement is \emph{strongly non-self unfolding}, since the ratio in Eq.\,(1.4.9) is always finite\footnote{We put the confinement potentials $\ln^d |E|$ with $d > 2$ to the class of strongly non-self unfolding potentials even though the ratio Eq.\,(1.4.9) logarithmically vanishes at large $E$.}.

We conclude that depending on the steepness of the confinement potential at large $E$ there are three qualitatively different cases\,[20, 21]::

- \emph{self-unfolding RMT} for $\lim_{x \to \infty} V(x)/|x| > 0$
- \emph{weakly non-self-unfolding RMT} for $\lim_{x \to \infty} V(x)/|x| = 0$ but $\exists \alpha > 0$ such that $\lim_{x \to \infty} V(x)/|x|^\alpha > 0$
- \emph{strongly non-self unfolding RMT} if $\forall \alpha > 0$ holds $\lim_{x \to \infty} V(x)/|x|^\alpha = 0$

The first case is characterized by the Wigner-Dyson universality of the spectral correlation functions. In the second case this universality holds only approximately for sufficiently large distance from the origin, while if one or two energies are close to the origin, the Wigner-Dyson universality is no longer valid, even after unfolding. In the third case, the Wigner-Dyson universality is not valid also in the bulk of the spectrum.

\section*{1.5 Unfolding and the spectral correlations}

Let us consider the power-law confinement Eq.\,(1.4.5) and find the corresponding unfolding co-ordinates $s(E)$ in which the spectral density is 1:

\[
s(E) = \sgn(E) \int_0^E \rho_\infty(E') dE' = \frac{A}{\pi} \tan \left( \frac{\pi \alpha}{2} \right) \sgn(E) |E|^{\alpha}. \tag{1.5.1}
\]

Note that for large enough $E$ the unfolding co-ordinates are given by Eq.\,(1.5.1) even if $V(E)$ is not a pure power-law at small $E$ but is rather deformed to have a regular behavior at the origin.

The corresponding unfolding co-ordinates for the log-square potential Eq.\,(1.4.7) are:

\[
s(E) = A \sgn(E) \ln(c|E|), \quad cE(s) = \sgn(s) e^{\frac{|s|}{\pi}}. \tag{1.5.2}
\]

where $c$ is a constant which depends on the regularization of the log-square potential close to the origin.
1.5. UNFOLDING AND THE SPECTRAL CORRELATIONS

As we will see, the exponential change of co-ordinates Eq. (1.5.2) leads to dramatic consequences for two-level spectral correlations, and has even a far-reaching analogy in the physics of black holes [20, 22].

In order to show how a non-trivial unfolding changes the form of the spectral correlations consider the spectral kernel $K_N(E, E')$ given in terms of the orthogonal polynomials by the Christoffel-Darboux formula [2]:

$$K_N(E, E') = \left| \frac{\varphi_{N-1}(E)\varphi_N(E') - \varphi_{N-1}(E')\varphi_N(E)}{E - E'} \right|, \quad (1.5.3)$$

where $\varphi_N(E)$ is the (properly normalized) "wave-function" which is related with the orthogonal polynomials $p_n$:

$$\varphi_N(E) = p_n(E) e^{-\beta V(E)/2}, \quad \int_{-\infty}^{+\infty} p_n(E') p_m(E') e^{-\beta V(E')} dE' = \delta_{nm}. \quad (1.5.4)$$

Any spectral correlation function can be expressed [2] in terms of the kernel Eq. (1.5.3). In particular, the $n$-point Density of States correlation function at $\beta = 2$ takes the form [2]:

$$R_{\infty}(E_1, ... E_n) = \det [K_{\infty}(E_i, E_j)] \prod_{i=1}^{n} K^{-1}(E_i, E_i). \quad (1.5.5)$$

Below we derive a semi-classical form of the kernel which is valid in the $N \to \infty$ limit provided that the coefficient $A$ in Eqs. (1.4.5), (1.4.7) is large. In particular for the RMT with log-square confinement given by Eq. (1.3.4) the condition of applicability of the analysis done below is $\ln q^{-1} < 2\pi$ [17].

In this semiclassical limit the "wave functions" take the form (we assume that the confinement potential is an even function of $E$):

$$\varphi_{N-1}(E) = \sin(\pi s(E)), \quad \varphi_N = \cos(\pi s(E)), \quad (1.5.6)$$

if $N$ is even and $\cos \to \sin$ if $N$ is odd. Then one finds in the unfolding co-ordinates:

$$K_{\infty}(E, E') = \frac{\sin(\pi(s - s'))}{E(s) - E(s')}. \quad (1.5.7)$$

Equation Eq. (1.5.7) demonstrates that as long as the semiclassical approach applies, the non-trivial unfolding $E = E(s)$ is the only source of deformation of the kernel and its deviation from the universal Wigner-Dyson form.

The main conclusion one can draw from Eq. (1.5.7) is that the translational invariance is lost in the $N \to \infty$ limit for all non-self-unfolding invariant RMT. It is not sufficient to have the mean density $\rho_{\infty}(s)$ equal to unity for all values of $s$ in order to have all the correlation functions of the universal Wigner-Dyson form.
1.6 Ghost correlation dip in RMT and Hawking radiation.

The break-down of translational invariance takes especially dramatic form in case of the log-square confinement. We will show below that in this case a ghost correlation dip appears in the two-level spectral correlation function which position at \( s \approx -s' \) is mirror reflected relative to the position of the translational-invariant correlation dip at \( s \approx s' \).

Indeed let us consider the semiclassical kernel Eq.(1.5.7) with the exponential unfolding Eq.(1.5.2) for \( s \gg 1 \) and \( |s-s'| \ll |s| \). Then after a simple algebra we obtain for \( ss' > 0 \) the Two-level Spectral Correlation Function given by Eq.(1.3.2) (\( \beta = 2 \)):

\[
R_\infty(ss' > 0) = 1 - \pi^2 \kappa^2 \frac{\sin^2(\pi(s-s'))}{\sinh^2(\pi^2 \kappa(s-s'))}, \quad \kappa = \frac{1}{2\pi^2 A} = \frac{\ln q^{-1}}{4\pi^2} \ll 1.
\] (1.6.1)

This correlation functions exhibits a dip of anti-correlation (level repulsion) at small \( s-s' \) and approaches exponentially fast the uncorrelated asymptotic \( R_\infty(s,s') = 1 \) for \( |s-s'| > 1/\pi^2 \kappa \). An amazing fact is that the anti-correlation revives again when \( s+s' \) becomes small. Considering \( ss' < 0 \) and plugging the exponential unfolding Eq.(1.5.2) into Eq.(1.5.7) we obtain for the unitary case \( \beta = 2 \):

\[
R_\infty(ss' < 0) = 1 - \pi^2 \kappa^2 \frac{\sin^2(\pi(s-s'))}{\cosh^2(\pi^2 \kappa(s+s'))}.
\] (1.6.2)

This is the ghost anti-correlation peak discovered in Ref.[20] but also present in the exact solution [17].

The existence of such a ghost correlation dip is requested by the normalization sum rule [20] which for the invariant RMT survives taking the limit \( N \to \infty \):

\[
\int_{-\infty}^{+\infty} (1 - R_\infty(s,s')) ds' = 1.
\] (1.6.3)

Substituting in Eq.(1.6.3) the sum of the translational invariant and the ghost peak found from Eqs.(1.6.1),(1.6.2) and doing the integral one obtains:

\[
\coth(\kappa^{-1}) - \frac{1}{\sinh(\kappa^{-1})} \cos(4\pi s) \approx 1.
\] (1.6.4)

This expression is equal to 1 with the exponential accuracy in \( e^{-\frac{1}{\kappa}} \ll 1 \), which is exactly the accuracy of the semiclassical approximation Eqs.(1.6.1),(1.6.2).

It appears that the situation with the ghost correlation dip has an important physical realization [22]. Let us consider the sonic analogue of the black hole [23,24] described in Fig.1.3. The stream of cold atoms moves with the constant velocity \( v_0 \), while the interaction between atoms is tuned so that the sound...
1.6. GHOST CORRELATION DIP IN RMT AND HAWKING RADIATION.

Figure 1.2: The irreducible part \( R_\infty(s, s') - 1 \) of the TLSCF obtained \(^{[21]}\) by the classical Monte-Carlo simulations at a temperature \( T = \frac{1}{2} \) on the one-dimensional plasma with logarithmic interaction in the log-squared confinement potential Eq.(1.4.7) with \( A = 0.5 \). Two dips of the anti-correlations are well seen with the smaller ghost dip shown in detail in the insert. The edges of the correlation function correspond to the finite number of particles (energy levels) \( N = 101 \).

velocity is larger than the stream velocity for \( x < -\delta \) and smaller than the stream velocity for \( x > \delta \). The latter region where no phonon may escape from, is analogous to the interior of a black hole. The point in space where \( c(x) = v_0 \) is analogous to the horizon in the black hole physics and the phonon radiation emerging when the horizon is formed, is analogous to the Hawking radiation \(^{[25]}\).

As it was shown in Ref.\(^{[24]}\), the Hawking radiation possesses a peculiar correlation of photons (phonons): despite being bosons, they statistically repel each other forming a dip in the density correlation function not only at \( x \approx x' \) but also near the mirror point \( x \approx -x' \), with the envelopes of the normal and ghost dips proportional to sinh\(^{-2}\) and cosh\(^{-2}\), respectively. As in the case of
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Figure 1.3: The sonic analogue of a black hole (courtesy of I. Carusotto): the stream of cold atoms with the velocity $v_0$ and the phonon velocity $c(x)$ changing near the origin from the value $c_1 > v_0$ to the value $c_2 < v_0$. This region of the super-sonic flow is analogous to the interior of the black hole where light cannot escape from.

RMT with log-square confinement, the ghost correlation dip appears in the absence of any symmetry that would require the eigenvalues $E_i$ or the Hawking bosons to appear in pairs simultaneously at points $\pm E_i (\pm x_i)$. More close inspection shows that the mechanism of its appearance in quantum gravity and in the random matrix theory is very similar: it is the exponential change of variables similar to Eq. (1.5.2) which is aimed to make flat the mean DoS (unfolding) in RMT or to make flat the metric of space-time in the quantum gravity (exponential red-shift) \[22\]. Thus formation of the strong non-self-unfolding RMT with the log-square confinement appears to be analogous to formation of a horizon in the general relativity.

1.7 Invariant-noninvariant correspondence.

Note that the translational-invariant part of spectral correlations described by Eq. (1.6.1) is well separated from the translational non-invariant ghost dip if $|s| \gg |s - s'|$. As correlations exponentially decrease at $|s - s'| > (\pi^2 \kappa)^{-1}$, the scale separation takes place when $|s| \gg (\pi^2 \kappa)^{-1}$. Essentially what happens
1.7. INVARIANT-NONINVARIANT CORRESPONDENCE.

because of the scale separation is that (like in a black hole) the world is divided in two parts, and in each of them local spectral correlations (which are approximately translational-invariant) are not affected by the presence of a "parallel world". Moreover, the local correlations in the invariant RMT with log-square confinement and \( \kappa \ll 1 \) were conjectured (for the two-level correlations this conjecture has been proven in \[11\]) to be the same as in the non-invariant critical RMT \[11, 26\]. This conjecture is related with the idea of the spontaneous breaking of unitary invariance in RMT with shallow confinement \[20\]. Although this idea has never been proven or even convincingly demonstrated numerically, it seems to be the only physically reasonable cause of the invariant-noninvariant correspondence like the one given by Eq. (1.3.1).

The invariant-noninvariant correspondence allows to use the invariant RMT with log-square confinement for computing spectral correlations of non-invariant RMT considered in sec.1.2. The idea of such calculations is to use the kernel in the form Eq. (1.3.3) to be plugged into the conventional machinery of invariant RMT \[2\]. Here we present some results of such calculations taken from Ref. \[27\]. One can see from Fig.1.4 that the choice of one single parameter \( A \) in

![Figure 1.4: Level spacing distribution function (after Nishigaki \[27\]) for the orthogonal and the unitary ensembles with log-square confinement (black solid lines) and the corresponding distributions for the 3D Anderson localization model with critical disorder (data points). Grey solid lines are the Wigner-Dyson and the Poisson distributions. The same values of \( A = 0.34 \) (orthogonal ensemble) and \( A = 0.14 \) (unitary ensemble) allow to fit well both the body and the far tail of the distribution shown in inserts.](image)

the invariant RMT with log-square confinement allows to fit well both the body and the exponential tail of the level spacing distribution function obtained by numerical diagonalization of the 3D Anderson localization model at criticality. The corresponding parameters \( b \) of the non-invariant CRMT Eq. (1.2.1) should
be chosen so that the parameter \( \kappa \) in the kernel Eq. (1.3.3) expressed through \( A \) and \( b \) is the same. Comparing expressions for \( \kappa \) in Eqs. (1.3.2), (1.6.1) we conclude that \( b \) is a solution to the equation:

\[
\frac{1}{\pi^2 A} = \beta \chi(b).
\] (1.7.1)

For the unitary ensemble the function \( \chi(b) \) in the range \( \chi = (2\pi^2 A)^{-1} \approx 0.36 \) is well described \([34]\) by the large-\( b \) asymptotic formula \([14]\) \( \chi(b) = (4\pi b)^{-1} \). This gives an estimate \( b = 0.22 \). Numerical diagonalization of the CRMT shows that \( b = 0.22 \) corresponds to \( d_2 \approx 0.46 \pm 0.01 \). To establish the correspondence with the Anderson localization model of unitary symmetry at critical disorder we note that the fractal dimension in the 1d CRMT corresponds to the reduced dimension \( d_2/3 \) of the three-dimensional Anderson model. Thus the fractal dimension of the Anderson model should be compared with \( 0.46 \times 3 = 1.38 \).

It appears to be in an excellent agreement with the direct diagonalization of the Anderson model of unitary symmetry which gives \([35]\) \( d_2 = 1.37 \). This example demonstrates that the values of parameters \( A \) and \( b \) found from fitting the spectral statistics of RMT to that of the 3D Anderson localization model, automatically give an excellent fitting of the eigenfunction statistics.

### 1.8 Normalization anomaly, Luttinger liquid analogy and the Hawking temperature.

As has been already mentioned, the local spectral correlations with \( |s-s'| \ll |s| \) are well described by the translational-invariant kernel Eq. (1.3.3). An important difference between this kernel and that of the conventional Wigner-Dyson theory is that it contains a second energy scale \( \kappa^{-1} \) in addition to the mean level spacing \( \Delta = 1 \). Moreover, the way this scale appears through \( s - s' \rightarrow \sinh(\pi^2 \kappa (s - s')) \) suggests an analogy with the system of one-dimensional fermions at a finite temperature. Indeed, the density correlations of non-interacting one-dimensional fermions (whose ground state correlations are equivalent to the Wigner-Dyson spectral statistics at \( \beta = 2 \)) is described by Eq. (1.3.2), where

\[
T = \pi \kappa.
\] (1.8.1)

Given the kernel Eq. (1.3.3) it is not difficult to obtain the level density correlation functions for \( \beta = 1, 4 \) using the standard formulae of the Wigner-Dyson RMT \([2]\). It appears that both in the region of \( |s-s'| \gg 1 \) and in the region of \( |s-s'| \ll \kappa^{-1} \) they coincide with the generalization of Eq. (1.3.2) to \( \beta = 1, 4 \) obtained \([26]\) directly from the non-invariant CRMT Eq. (1.2.1). For \( \kappa \ll 1 \) these regions overlap on a parametrical large interval, and one can again demonstrate the invariant-noninvariant correspondence.
1.8. NORMALIZATION ANOMALY, LUTTINGER LIQUID ANALOGY AND THE HAWKING TEMPERATURE

It is important to note that breaking the unitary invariance explicitly by a finite $b$ in Eq. (1.2.1) leads to a normalization anomaly. Namely, the normalization sum rule [20] which exactly holds for the finite size of matrix $N$:

$$2 \int_{s>0}^{+\infty} (1 - R_N(s' - s)) ds' = 1,$$  \hfill (1.8.2)

gets violated if the limit $N \to \infty$ is taken prior to doing the integral. For instance at $\beta = 2$ we have with exponential accuracy $e^{-1/\kappa} \ll 1$:

$$\eta = 2 \int_{s>0}^{\infty} (1 - R_N(s' - s)) ds' = \coth(\kappa^{-1}) - \kappa \approx 1 - \kappa.$$  \hfill (1.8.3)

For $\beta = 1$ the corresponding expression is:

$$\eta = 2 \cot(\kappa^{-1}) - \tanh(1/2\kappa) - 2\kappa \approx 1 - 2\kappa.$$  \hfill (1.8.4)

This leads to the finite level compressibility [31]:

$$\chi \equiv \frac{d}{dn} \langle (n - \bar{n})^2 \rangle = 1 - \eta, \quad N \gg \bar{n} = \langle n \rangle \gg 1,$$  \hfill (1.8.5)

and the exponential (instead of the Gaussian in the Wigner-Dyson RMT) tail of the level spacing distribution function:

$$\ln P(s) \approx s^{2\chi}.$$  \hfill (1.8.6)

Both properties Eq. (1.8.5) and Eq. (1.8.6) are the signatures of criticality [7]. Due to invariant-noninvariant correspondence the normalization anomaly Eq. (1.8.3) holds also for the invariant RMT with log-square confinement thus again raising a question on the spontaneous breaking of unitary invariance.

In the absence of the formal proof of this conjecture, we present here a simple theory which may unify both of the ensembles. The idea of such a theory stems from the fact [28] that the classical Wigner-Dyson RMT [2] is equivalent to the ground state of the one-dimensional Calogero-Sutherland model [29] of fermions with inverse square interaction of the strength $\frac{\beta}{2} \left( \frac{\beta}{2} - 1 \right)$ in a harmonic confinement potential. The large-scale properties of such a model are described by the Luttinger liquid phenomenology [30]. Its simplest finite-temperature formulation [32] is in terms of the free-bosonic field in a 1 + 1 space-time $\Phi(s, \tau) = \frac{1}{2} [\Phi_R(s, \tau) + \Phi_L(s, \tau)]$ with the quadratic action:

$$S_T[\Phi] = \frac{\beta}{4\pi K} \int_0^{1/T} d\tau \int_{-\infty}^{+\infty} ds \left[ (\partial_s \Phi)^2 + (\partial_\tau \Phi)^2 \right], \quad \Phi(s, \tau) = \Phi(s, \tau + 1/T).$$  \hfill (1.8.7)
The density correlation functions are given by the functional averages of the density operator:

\[ \rho(s, \tau) = \rho_0 + \frac{1}{\pi} \partial_s \Phi(s, \tau) + A_1 \cos[2\pi s + 2\Phi(s, \tau)] + A_2 \cos[4\pi s + 4\Phi(s, \tau)] + \ldots, \]

(1.8.8)

where \( A_k \) are structural constants which are determined by details of interaction at small distances and take some fixed values for the Calogero-Sutherland model corresponding to the symmetry class \( \beta \). Using Eq. (1.8.8) one may express the density correlation functions in terms of the fundamental Green’s function:

\[ \langle \Phi(s, \tau) \Phi(s', \tau) \rangle_S - \frac{1}{2} \langle \Phi(s, \tau) \Phi(s, \tau) \rangle_S - \frac{1}{2} \langle \Phi(s', \tau) \Phi(s', \tau) \rangle_S = \frac{\pi}{\beta} G(s, s'), \]

(1.8.9)

where \( \langle \ldots \rangle_S \) is the functional average with the action \( S[\Phi] \). For example, the two-level spectral correlation functions are equal to:

\[ R_\infty(s, s') = 1 + \frac{2}{\pi \beta} \partial_s \partial_{s'} G(s, s') + \frac{2}{\beta (2\pi^2)^3} \cos(2\pi s) e^{8\pi \beta s} G(s, s'). \]

(1.8.10)

For \( T = 0 \) the Green’s function is \( G(s - s') = -\frac{1}{4\pi} \ln(s - s')^2 \) at \( |s - s'| \gg 1 \) and one reproduces the asymptotic form of the Wigner-Dyson correlations.

In order to reproduce by the same token the corresponding correlation functions \([26]\) for the non-invariant CRMT Eq. (1.2.1) one merely substitutes in Eq. (1.8.7) the finite \( T \) given by Rq. (1.8.11) and replaces the zero-temperature Green’s function by the finite-temperature one:

\[ G(s - s') = \frac{1}{2\pi} \ln \left( \frac{\pi T}{\sinh(\pi T |s - s'|)} \right). \]

(1.8.11)

Thus we conclude that the deformation of the Wigner-Dyson RMT given by Eq. (1.2.1) with large but finite \( b \), retains the analogy with the Calogero-Sutherland model. However, instead of the ground state, the non-invariant CRMT with \( \kappa \ll 1 \) is equivalent to the Calogero-Sutherland model at a small but finite temperature Eq. (1.8.1).

It is remarkable that there exists a deformation of the free-boson functional Eq. (1.8.7) capable of reproducing the two-level correlation function for the invariant RMT with log-square confinement, including the ghost correlation dip. All one has to do for that is to replace the action Eq. (1.8.7) defined on a cylinder by that defined on a curved space-time with a horizon:

\[ \int_0^\infty d\tau \int_{-\infty}^{+\infty} ds [ (\partial_s \Phi)^2 + (\partial_\tau \Phi)^2 ] \rightarrow \int d^2 \xi \sqrt{-g(\xi)} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi, \]

(1.8.12)

where \( g \equiv \det g_{\mu\nu} \) with \( g_{\mu\nu} \) being the metric, i.e. \( ds^2 = g_{\mu\nu} dx^\mu \, dx^\nu \) with \( x^1 = x \) and \( x^0 = t = -i\tau \). The main requirement for the metric is that
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the transformation of co-ordinates to the frame $(\bar{x}, \bar{t})$ where the metric is flat ("Minkovski space" with $ds^2 = d\bar{x}^2 - dt^2$), is exponential for large enough $x$ with a factorized $x$ and $t$ dependence, e.g.

$$\bar{x} = \varphi(x) \cosh(t/A), \quad \bar{t} = \varphi(x) \sinh(t/A), \quad (1.8.13)$$

$$\varphi(x) \sim \text{sgn}(x) e^{\frac{|x|}{A}} \quad \text{at} \quad |x| \gg A = \frac{2\pi}{T}. \quad (1.8.14)$$

In order to represent the invariant RMT with an even confinement potential, the function $\varphi(x)$ should be odd. This last requirement plus the continuity of the function at $x = 0$ necessarily implies $g_{00}(x = 0) = 0$, that is $x = 0$ is the horizon. One example of such a metric is:

$$\bar{x} = A \sinh(x/A) \cosh(t/A), \quad \bar{t} = A \sinh(x/A) \sinh(t/A) \quad (1.8.15)$$

$$ds^2 = \cosh^2(x/A) dx^2 - \sinh^2(x/A) dt^2 \quad (1.8.16)$$

$$\sqrt{-g(\xi) g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi} = |\tanh(x/A)| (\partial_x \Phi)^2 + |\coth(x/A)| (\partial_\tau \Phi)^2. \quad (1.8.17)$$

It maps the strip $x \geq 0$, $0 < \tau < 2\pi A$ and the strip $x \leq 0$, $0 < \tau < 2\pi A$ separately onto the entire plane $(\bar{x}, \bar{t})$.

In order to compute the Green’s function on the curved space-time corresponding to Eq.(1.8.13) we use the well known formula [33]:

$$G(z, z') = -\frac{1}{2\pi} \ln |\tilde{z}(z) - \tilde{z}(z')| - \frac{1}{4\pi} \ln |\partial_z \tilde{z}(z) \partial_{z'} \tilde{z}(z')|, \quad (1.8.18)$$

where $\tilde{z} = \bar{x} \pm i\bar{t}$, $z = x \pm i\tau$ and $\partial_z = \partial_x \mp i\partial_\tau$ with $\pm = \text{sgn}(x)$. Then from Eq.(1.8.13) for $x, x'$ sufficiently far from the origin one easily obtains:

$$G(x, x') = -\frac{1}{4\pi} \ln \left[ \frac{(\varphi(x) - \varphi(x'))^2}{4|\varphi(x) \varphi(x')|} \right]$$

$$= -\frac{1}{2\pi} \left\{ \ln \left[ 2A \sinh\left(\frac{(x - x')/A}{2}\right) \right], \quad x x' > 0 \right. \right.$$}

The origin of sinh and cosh in Eq.(1.8.19) is very much the same as that in Eqs. (1.6.1), (1.6.2). The exponential transformation of coordinates Eq.(1.8.13) plays the same role as the exponential unfolding Eq.(1.5.2). Finally substituting Eq.(1.8.19) into the expression for the two-level correlation function Eq.(1.8.10) one reproduces Eqs. (1.6.1), (1.6.2).

Note that the new scale $A = (2\pi T)^{-1}$ sets the temperature scale $T$ given by Eq.(1.8.1) which has a meaning of the Hawking temperature in the black hole analogy. While the finite temperature arises because of the periodicity of transformations Eq.(1.8.15) in the imaginary time $\tau = i t$, this compactification is different from the standard one: each of the semi-strips $x \geq 0$ and $x \leq 0$ are mapped on the entire plane $(\bar{x}, \bar{t})$ independently of each other. This is essentially the effect of the horizon.
1.9 Conclusion

I would like to close this Chapter by some concluding remarks. First of all, it is by no means a comprehensive review but rather some introduction to the subject written by a physicist motivated by the physics applications and not by the formal rigor. My goal was to show that the subject is rich and poorly explored and that the efforts are likely to be rewarded by non-trivial discoveries. Let me formulate in the end the (highly subjective) list of open problems as I see them.

- **Further study of the non-invariant Gaussian critical RMT**
  Some progress has been made in this direction by development of the regular expansion in $b \ll 1$, the so called *virial expansion method* [36, 37, 38], which extends ideas of Refs. [39, 14]. This approach made it possible to compute analytically the level compressibility up to $b^2$ terms and find an extremely good and simple approximation [38, 13] to the LDoS correlation function Eq.(1.1.2). These works are the basis for a perturbative proof of the dynamical scaling Eq.(1.1.3) [3] and the duality relation Eq.(1.2.3).

- **Non-perturbative solution to the non-invariant critical RMT**
  Some nice relationships such as the duality relation Eq.(1.2.3), raise a question about a possibility of exact solution to the CRMT. In our opinion this possibility does exist.

- **Invariant-noninvariant correspondence**
  This is a very interesting issue with lots of applications in case the origin of this correspondence is understood. It is also important to invest some efforts to study the issue of the correspondence between the spectral and eigenvector statistics, in particular the possible spontaneous breaking of unitary invariance and emergence of a preferential basis [20].

- **Level statistics in weakly non-self-unfolding RMT**
  This is a broad class of invariant RMT where the Wigner-Dyson universality is broken. The spectral statistics in such RMT exhibits unusual features like super-strong level repulsion near the origin.

- **Crystallization of eigenvalues**
  This phenomenon takes place in the non-perturbative regime $\kappa > 1$ of RMT with log-square confinement and has its counterpart in the "crystallization" of roots of orthogonal polynomials [18]. So far it does not have physical realization, with Calogero-Sutherland model being a clear candidate [40].

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