Unifying distribution functions: some lesser known distributions

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We show that there is a way to unify distribution functions that describe simultaneously a signal in space and (spatial) frequency. Probably the most known of them is the Wigner distribution function. Here we show how to unify functions of the Cohen class, Rihaczek’s complex energy function, Husimi and Glauber-Sudarshan distribution functions.

I. INTRODUCTION

Distribution functions are widely used in optical physics (see [1] for a review) and in quantum mechanics where they are usually called quasiprobability distribution functions [2, 3, 4]). Probably the best known quasiprobability distribution is the Wigner function [2, 5] with applications in reconstruction of signals [6] and resolution [7] in the classical world and reconstruction of quantum states of different systems such as ions [8] or quantized fields [4, 9, 10, 11] in the quantum world. In this contribution we would like to re-introduce a lesser known quasiprobability distribution function, namely the Kirkwood-Rihaczek function [1, 12, 13, 14, 15], show how it can be related to the Wigner function, and express it as an expectation value in some eigenbasis, just as the other quasiprobability functions may be also expressed.

II. BEST KNOWN QUASIPROBABILITY DISTRIBUTION FUNCTIONS

A. Wigner function

We start by introducing the Wigner function, probably the best known. It may be written in two forms: series representation (see for instance [16]), and integral representation

\[ W(q, p) = \frac{1}{2\pi} \int du e^{iup} \langle q + \frac{u}{2} | \rho | q - \frac{u}{2} \rangle, \]

for simplicity we use the Dirac notation here (see the appendix). In the above equation, \( \rho \) is the so-called density matrix. In 1932, Wigner introduced this function \( W(q, p) \), known now as his distribution function [2, 5] and contains complete information about the state of the system (\( \rho = |\psi\rangle\langle\psi| \)).

It may be written also as in terms of the (double) Fourier transform of the characteristic function

\[ W(\alpha) = \frac{1}{4\pi^2} \int \exp(\alpha \beta^* - \alpha^* \beta) C(\beta) d^2 \beta, \]

with \( \alpha = (q + ip)/\sqrt{2} \) and where \( C(\beta) \) in terms of annihilation and creation operators is given by

\[ C(\beta) = Tr\{ \hat{\rho} \exp(\beta \hat{a}^\dagger - \beta^* \hat{a}) \}, \]

also known as ambiguity function in classical optics [17].

B. Q-function

The Q or Husimi function [18], which is expressed as the coherent state expectation value of the density operator

\[ Q(\alpha) = \frac{1}{\pi} \int \exp(\alpha \beta^* - \alpha^* \beta) Tr\{ \hat{\rho} \exp(-\beta^* \hat{a}) \exp(\beta \hat{a}^\dagger) \} d^2 \beta, \]

the alternative form is

\[ Q(\alpha) = \frac{1}{\pi} \langle \alpha | \hat{\rho} | \alpha \rangle . \]
C. Relating Q and Wigner functions in a differential form

It is possible two group the Wigner and the Husimi functions

\[ F(\alpha, s) = \frac{1}{\pi} \int C(\beta, s) \exp(\alpha \beta^* - \alpha^* \beta) d^2 \beta \]  

where \( C(\beta, s) \) is the characteristic function of order \( s \)

\[ C(\beta, s) = Tr \{ \hat{D}(\beta) \hat{\rho} \} \exp(s |\beta|^2 / 2) \]  

with \( s \) a parameter that defines which is the function we are looking at. For \( s = 1 \) it is obtained the \( P \)-function, for \( s = 0 \) the Wigner function, and for \( s = -1 \) the \( Q \)-function.

The \( Q \)-function is then

\[ Q(\alpha) = \int G(\beta) \exp(\alpha \beta^* - \alpha^* \beta) d^2 \beta \]  

and for \( s = 0 \) the Wigner function

\[ W(\alpha) = \int G(\beta) \exp(\alpha \beta^* - \alpha^* \beta) \exp(|\beta|^2 / 2) d^2 \beta \]  

where

\[ G(\beta) = \frac{1}{\pi^2} Tr \{ D(\beta) \hat{\rho} \} \exp(-|\beta|^2 / 2), \]  

The equation above may be written as an infinite (Taylor) series and inserted into \( 9 \) to obtain

\[ W(\alpha) = \sum_{n=0}^{\infty} \frac{2^{-n}}{n!} \int G(\beta) \exp(\alpha \beta^* - \alpha^* \beta) |\beta|^{2n} d^2 \beta. \]  

Considering the equality

\[ \frac{\partial}{\partial \alpha} \frac{\partial}{\partial \alpha^*} \exp(\alpha \beta^* - \alpha^* \beta) = -|\beta|^2 \exp(\alpha \beta^* - \alpha^* \beta) \]  

we can cast equation \( 11 \) into

\[ W(\alpha) = \sum_{n=0}^{\infty} \frac{2^{-n}}{n!} \left( - \frac{\partial}{\partial \alpha} \frac{\partial}{\partial \alpha^*} \right)^n Q(\alpha), \]  

or, finally

\[ W(\alpha) = \exp \left( -\frac{1}{2} \frac{\partial}{\partial \alpha} \frac{\partial}{\partial \alpha^*} \right) Q(\alpha). \]  

The analysis just done will help us to relate the Kirkwood-Rihaczek function with the Wigner function.
For the sake of completeness, we introduce another well-known distribution function: the Glauber-Sudarshan $P$-function [19, 20]. First let us note that by using the coherent state eigenbasis, we can express the density matrix as the following double integral

$$\hat{\rho} = \frac{1}{\pi^2} \int \int \langle \alpha | \hat{\rho} | \beta \rangle | \alpha \rangle d^2 \alpha d^2 \beta. \quad (15)$$

This representation involves off-diagonal elements $\langle \alpha | \hat{\rho} | \beta \rangle$, and two integrations in phase space. The next diagonal representation was introduced independently by Glauber and Sudarshan [19, 20]

$$\hat{\rho} = \int P(\alpha) | \alpha \rangle d^2 \alpha \quad (16)$$

and involves only one integration. Using the equation above, we can write the Glauber-Sudarshan $P$-function in the form

$$P(\alpha) = \frac{1}{\pi} \int \exp(\alpha \beta^* - \beta^* \alpha) \text{Tr} \{ \hat{\rho} \exp(\beta \hat{a}^\dagger) \exp(-\beta^* \hat{a}) \} d^2 \beta, \quad (17)$$

or

$$P(\alpha) = F(\alpha, 1) = \frac{1}{\pi} \int C(\beta, 1) \exp(\alpha \beta^* - \beta^* \alpha) d^2 \beta \quad (18)$$

### E. Cohen-class distribution functions

A function of the Cohen class is described by the general formula [1, 22]

$$W_C = \frac{1}{2\pi} \int \int \int \phi(y + \frac{1}{2} x') \phi(y - \frac{1}{2} x') k(x, u, x', u') e^{-i(ux' - u'x + u'y)} dx dx' du \quad (19)$$

and the choice of the kernel $k(x, u, x', u')$ selects one particular function of the Cohen class. The Wigner function, for instance arises for $k(x, u, x', u') = 1$, whereas the ambiguity function is obtained for $k(x, u, x', u') = 2\pi \delta(x - x') \delta(u - u')$.

### III. LESSER KNOWN DISTRIBUTION FUNCTION: THE KIRKWOOD-RIHACEK QUASIDISTRIBUTION FUNCTION

Now we turn our attention to a lesser known distribution, the Kirkwood-Rihaczek function, that may be written using the notation above as [24]

$$K(\beta) = \int d^2 \alpha e^{3\alpha^* - \beta^* \alpha} e^{-\frac{1}{4}(\alpha^* \alpha + \beta^* \beta)} C(\alpha). \quad (20)$$

This equation has been obtained from an equation similar to equations (2), (4) and (17), i.e. taking the double Fourier transform

$$K(q, p) = \int du dv e^{-iup} e^{ivq} \text{Tr} \{ \rho e^{iv \hat{q}} e^{-iup} \} \quad (21)$$

and taking the trace as in equation (56) in the appendix.

We will now do an analysis similar to the one done in subsection 2.D. We relate the Kirkwood-Rihaczek function to the Wigner function by using [20], via the following exponential of derivatives

$$K(\beta) = e^{-\frac{i\beta^2}{4} \sigma^2} e^{\frac{i}{4} \beta^2 \sigma^2} W(\beta). \quad (22)$$

We now use the non-integral expression for the Wigner function [16]

$$W(\beta) = \text{Tr} \left[ (-1)^\hat{\mathbf{D}} \hat{D} \right] \hat{\rho} \hat{\mathbf{D}} (\beta), \quad (23)$$
Therefore, we have that the Kirkwood-Rihaczek function may be written as

\[ W(\beta) = \text{Tr} \left[ (-1)^{\hat{n}} \hat{D}(2\beta) \right], \]

where we have used the trace property \( \text{Tr}(AB) = \text{Tr}(BA) \) and the following identities \((-1)^{\hat{n}} \hat{D}(\alpha) = \hat{D}(\beta) (-1)^{\hat{n}} \).

Now we use the factorized form of the Glauber displacement operator \[21\] \( \hat{D}(2\beta) = e^{-|\beta|^2} e^{2\beta \hat{a}^*} e^{-2\beta^* \hat{a}} \) to obtain

\[ W(\beta) = \text{Tr} \left[ (-1)^{\hat{n}} \hat{D}(2\beta) \right] e^{-2|\beta|^2} e^{2\beta \hat{a}^*} e^{-2\beta^* \hat{a}}. \]  

Therefore, we have that the Kirkwood-Rihaczek function may be written as

\[ K(\beta, \beta^*) = e^{-\frac{1}{2} \frac{\partial^2}{\partial \beta^2}} e^{\frac{1}{2} \frac{\partial^2}{\partial \beta^* \partial \beta^*}} W(\beta, \beta^*) \]

\[ = \text{Tr} \left[ (-1)^{\hat{n}} \hat{D}(2\beta) \right]. \]

The calculation of the exponential of derivatives of the Glauber operator will be tedious but straightforward. We will just write the main steps to obtain the final form, for instance, it is not difficult to show that

\[ e^{\frac{1}{2} \frac{\partial^2}{\partial \beta^2}} \hat{D}(2\beta) = e^{-\beta^2} e^{2\beta(\hat{a}^* + \beta^*)} e^{\beta \hat{a}^2} e^{-2\beta^* \hat{a}}. \]

By using that \[25\]

\[ e^{-t^2 + 2tx} = \sum_{k=0}^{\infty} H_k(x) \frac{t^k}{k!} \]

we can express the above equation as

\[ e^{\frac{1}{2} \frac{\partial^2}{\partial \beta^2}} \hat{D}(2\beta) = \sum_{k=0}^{\infty} H_k(\beta) (\hat{a}^* + \beta^*)^k e^{\beta^2} e^{-2\beta^* \hat{a}}, \]

with \( H_k(x) \) the Hermite polynomials. From the above equation, is easy to obtain

\[ \frac{\partial^{2n}}{\partial \beta^{2n}} \sum_{k=0}^{\infty} H_k(x) \frac{\beta^k}{k!} = \sum_{k=0}^{\infty} H_{k+2n}(x) \frac{\beta^k}{k!} \]

and therefore

\[ e^{-\frac{1}{2} \frac{\partial^2}{\partial \beta^2}} e^{\frac{1}{2} \frac{\partial^2}{\partial \beta^* \partial \beta^*}} \hat{D}(2\beta) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \left( \frac{-1}{n!} \right)^n H_{k+2n}(\beta) \frac{\beta^k}{k!} e^{\beta^2} e^{-2\beta^* \hat{a}}. \]

Now we use the integral form of the Hermite polynomials \[25\]

\[ H_p(x) = \frac{2^p}{\sqrt{\pi}} \int_{-\infty}^{\infty} (x + it)^p e^{-t^2} dt \]

to obtain

\[ K(\beta, \beta^*) = e^{-\beta^2} e^{-2\beta^* \hat{a}} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dt e^{-(2\sqrt{2}x + 2\beta^* + 2\beta)t} e^{-2x^2} e^{2\sqrt{2}x(\beta + \beta^*)} \langle x \vert e^{\hat{a}^2} e^{-2\beta^* \hat{a}} (-1)^{\hat{n}} \hat{p} \vert x \rangle \]

by using

\[ \int_{-\infty}^{\infty} e^{-ikt} dt = 2\pi \delta(k) \]
and taking \( k = 2\sqrt{2} - 2\beta^* - 2\beta \) we have
\[
K(\beta, \beta^*) = 2\sqrt{\pi} e^{-\beta^*} e^{-2\beta\beta^*} \int_{-\infty}^{\infty} dx \delta \left( 2\sqrt{2} - 2\beta^* - 2\beta \right) e^{-2x^2} e^{2\sqrt{\pi}(\beta^* + \beta)}
\]
\[
\times \langle x | e^{\hat{a}^2} e^{-2\beta^* \hat{a}} (-1)^{\hat{a}^2} \rho | x \rangle. 
\]
Making use of the identity \( \delta(\alpha x) = \delta(x) |\alpha| \) we finally obtain
\[
K(\beta, \beta^*) = e^{-2\beta^*} e^{-2\beta\beta^*} \sqrt{\pi} \frac{\beta^* + \beta}{\sqrt{2}} \langle \beta^* + \beta | \hat{a}^2 | -\beta \rangle
\]
\[
or
\]
\[
K(\beta, \beta^*) = \sqrt{\frac{\pi}{2}} e^{2\beta^* - 2\beta\beta^*} \langle X | D^\dagger(-\beta^*) e^{\hat{a}^2} D(-\beta^*) (-1)^{\hat{a}^2} \rho | X \rangle
\]
with \( X = \frac{\beta^* + \beta}{\sqrt{2}} \).

In this form, we have succeeded in obtaining the Kirkwood-Rihaczek function as an expectation value, in terms of position eigenstates, just as the \( Q \)-function, in terms of coherent states [see equation (5)], the Wigner and Glauber-Sudarshan functions in terms of number states [16].

IV. CONCLUSIONS

We have shown that some distribution functions may be related through a method that allows the construction of some quasi-probability functions such as the Wigner, Glauber-Sudarshan and Husimi functions [23]. This method consists of obtaining the distribution functions from a double Fourier transform of an averaged exponential operator. If we use the exponential operator in terms of creation and annihilation operators we construct the already mentioned distribution functions. Via this method, but leaving the exponential operator in terms of position and momentum, and ordering (factorizing) the exponential in a convenient way, another function lesser used in classical optics, namely, Kirkwood-Rihacek’s distribution function may be obtained. This function was recently introduced in quantum mechanics by Praxmeyer and Wódkiewicz [14, 15] to have a phase representation of the Hydrogen atom. The connection between Glauber-Sudarshan and Husimi functions and functions of the Cohen class has been given, i.e. the adequate kernels. Finally, the Kirkwood-Rihacek has been given in term of an expectation value, in terms of position eigenstates, just as other distribution functions may also be given in terms of (sums of) expectation values. This may be of interest because it has been already exploited the fact that these forms allow reconstruction of quasiprobability distribution functions [10, 11].

V. APPENDIX

In Dirac notation, we denote functions ”\( f \)” by means of ”kets” \( |f \rangle \). For instance an eigenfunction of the harmonic oscillator [25]
\[
\psi_n(x) = \frac{\pi^{-1/4}}{\sqrt{2^n n!}} e^{-x^2/2} H_n(x).
\]
is represented by the ket \( |n \rangle \), with \( n = 0, 1, 2, ..., \) In quantum mechanics, these states are called number or Fock states (see for instance [3]). Any function can be expanded in terms of eigenfunctions of the harmonic oscillator
\[
f(x) = \sum_{n=0}^{\infty} c_n \psi_n(x)
\]
where
\[
c_n = \int_{-\infty}^{\infty} f(x) \psi_n(x) dx
\]
and in the same way any ket may be expanded in terms of $|n\rangle$'s

$$|f\rangle = \sum_{n=0}^{\infty} c_n |n\rangle$$  \hfill (42)

where the orthonormalization relation

$$\langle m|n \rangle = \int_{-\infty}^{\infty} \psi_m^*(x) \psi_n(x) dx = \delta_{mn}$$  \hfill (43)

has been used. The quantity $\langle m|$ is a so-called "bra".

The basis set of kets $|n\rangle$ is a discrete one. However, there are also continuous basis. We can form one continuous basis for example with the function $e^{ipq}/\sqrt{2\pi}$ and the corresponding ket $|p\rangle$. First note that

$$\langle p'|p \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i(p-p')q} dx = \delta(p - p'),$$  \hfill (44)

so that

$$e^{ip'q}/\sqrt{2\pi} = \int_{-\infty}^{\infty} \delta(p - p') e^{ipq}/\sqrt{2\pi} dp,$$  \hfill (45)

or, in bra-ket notation we have

$$|p'\rangle = \int_{-\infty}^{\infty} \delta(p - p') |p\rangle dp = \int_{-\infty}^{\infty} \langle p'|p \rangle |p\rangle dp,$$  \hfill (46)

rearranging terms we have

$$|p'\rangle = \left( \int_{-\infty}^{\infty} |p\rangle \langle p| dp \right) |p'\rangle = 1 |p'\rangle,$$  \hfill (47)

i.e. we have the completeness relation

$$\int_{-\infty}^{\infty} |p\rangle \langle p| dp = 1.$$  \hfill (48)

Finally note that the function $e^{ipq}/\sqrt{2\pi}$ is an eigenfunction of the operator $-\frac{\hat{a}^2}{2}$ with eigenvalue $p$. For position, an "eigenket" of $\hat{q}$ is

$$\hat{q}|q\rangle = q|q\rangle$$  \hfill (49)

and an "eigenbra"

$$\langle q'|q \rangle = \langle q' |\hat{q}\rangle.$$  \hfill (50)

We therefore find

$$\langle q'|q \rangle (q' - q) = 0,$$  \hfill (51)

that has as solution $\delta(q' - q)$.  \hfill (52)

We then can express the completeness relation

$$1 = \int_{-\infty}^{\infty} |q\rangle \langle q| dq,$$  \hfill (53)

such that

$$|\Psi\rangle = \int_{-\infty}^{\infty} |q\rangle \langle q| \Psi\rangle dq = \int_{-\infty}^{\infty} \Psi(q) |q\rangle dq$$  \hfill (54)
where $\Psi(q) = \langle q | \Psi \rangle = \langle q | \Psi \rangle$. The density matrix $\rho$ is defined simply as the ket-bra operator $\rho = |\Psi\rangle \langle \Psi|$.

The completeness relation serve us among other things to calculate averages, for instance

$$
\langle \Psi | \hat{A} | \Psi \rangle = \langle \Psi | \hat{A} | \Psi \rangle = \langle \Psi | \hat{A} | q \rangle \langle q | \Psi \rangle dq = \int_{-\infty}^{\infty} \langle \Psi | \hat{A} | q \rangle \langle q | \Psi \rangle dq
$$

or finally,

$$
\langle \Psi | \hat{A} | \Psi \rangle = \int_{-\infty}^{\infty} \langle q | \Psi \rangle \langle \Psi | \hat{A} | q \rangle dq.
$$

Note that in the above equation we are simply adding ''diagonal'' elements, i.e. we have the trace of the the operator $|\Psi\rangle \langle \Psi| \hat{A}$. As the trace is independent of the basis, we can have it in terms of the discrete basis $|n\rangle$

$$
\langle \Psi | \hat{A} | \Psi \rangle = \sum_{n=0}^{\infty} \langle n | \Psi \rangle \langle \Psi | \hat{A} | n \rangle \equiv Tr\{\rho \hat{A}\}.
$$

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