FUTURE STABILITY OF THE FLRW SPACETIME FOR A LARGE CLASS OF PERFECT FLUIDS

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Abstract. We establish the future non-linear stability of Friedmann-Lemaître-Robertson-Walker (FLRW) solutions to the Einstein–Euler equations of the universe filled with a large class of Makino-type fluids (the equations of state are allowed to be certain nonlinear or linear types both). Several previous results as specific examples can be covered in the results of this article. We emphasize that the future stability of FLRW metric for polytropic fluids with positive cosmological constant has been a difficult problem and can not be directly generalized from the previous known results. Our result in this article has not only covered this difficult case for the polytropic fluids, but also unified more types of fluids in a same scheme of proofs.

Keywords: Einstein–Euler system; FLRW metric; future non-linear stability; polytropic gas; Chaplygin gas

Mathematics Subject Classification: Primary 35A01; Secondary 35Q31, 35Q76, 83C05, 83F05

1. Introduction

Cosmological observations predict that our universe is currently undergoing an accelerated expansion which is potentially achieved by various models. Candidates such as positive cosmological constant, quintessence of dark energy have been widely studied, for example, in [2–4, 8, 13–15, 31, 33, 35, 36, 44, 46]. During these candidates, a well-known family of Friedmann-Lemaître-Robertson-Walker (FLRW) solutions are often used by cosmologists to model a fluid-filled, spatially homogeneous and isotropic universe. In mathematics, the future non-linear stability of perturbations of FLRW solutions to the Einstein–Euler equations with a positive cosmological constant and a linear equation of state \( p = K \rho \) has been well studied. However, in reality, the equations of state of the fluids can not be precisely linear. A natural question arises: if the equation of state deviates from the linear one, what happens to the longtime behavior of perturbations of FLRW metrics? Or, more generally, how the equations of state of the perfect fluids influence the future non-linear stability of FLRW solutions. This question is attractive to us because if small deviations from the linear equation of state of fluids destroy the non-linear future stability of FLRW metrics, then the FLRW metric with positive cosmological constant and the linear model of equation of state of fluids is not decent to predict the future of the universe due to the instability of this model with respect to the equation of state of the filled fluids. This article aim to solve the proposed question partially. However, our results can not answer above questions completely and it is very difficult to investigate this question directly, since the equation of state affects the system in very complex ways. In fact, we attempt to investigate it by asking firstly what types of fluids can guarantee the validity of the future non-linear stability of FLRW solutions to the Einstein–Euler system with a positive cosmological constant. In this article, we construct a large class of fluids from the mathematical point of view, called Makino-type fluids, and demonstrate several common and frequently used fluids (the equations of state are allowed to be certain nonlinear or linear types both) are in this class and prove that Makino-type fluids do make sure the target stability holds. On the other
hand, another direct motivation for us is to investigate the evolution of the general Chaplygin fluids and polytropic fluids in accelerated expanding spacetime, which is not studied so far to the authors’ knowledge. In fact, one would see the Makino-type fluids proposed in this article include certain cases of general Chaplygin fluids and polytropic fluids. We emphasize that the future stability of FLRW metric for polytropic fluids with positive cosmological constant has been a difficult problem and can not be directly generalized from the previous known results. Our result in this article has not only covered this difficult case for the polytropic fluids, but also unified more types of fluids in a same scheme of proofs.

The dimensionless Einstein-Euler system is given by

\[ \tilde{G}^{\mu\nu} + \Lambda \tilde{g}^{\mu\nu} = \tilde{T}^{\mu\nu}, \]  
\[ \nabla_\mu \tilde{T}^{\mu\nu} = 0, \]  

where \( \tilde{G}^{\mu\nu} = \tilde{R}^{\mu\nu} - \frac{1}{2} \tilde{R} \tilde{g}^{\mu\nu} \) is the Einstein tensor of the metric \( \tilde{g} = \tilde{g}_{\mu\nu} dx^\mu dx^\nu \), and \( \tilde{T}^{\mu\nu} = (\rho + p) \tilde{u}^\mu \tilde{u}^\nu + p \tilde{g}^{\mu\nu} \) is the stress energy tensor of the perfect fluid. Here, \( \tilde{R}_{\mu\nu}, \tilde{R} \) are the Ricci and scalar curvature of the metric \( \tilde{g} \) respectively, \( \nabla_\mu \) is the covariant derivative of \( \tilde{g} \), and \( \rho, p = p(\rho) \) denote the energy density and pressure of the perfect fluid, respectively. We require that \( p(0) = 0 \) and \( p(\rho) \) is analytic on an compact set \( I_\rho \subset [0, +\infty] \). \( \tilde{u}^\mu \) is the fluid four-velocity, which we assume is normalized by

\[ \tilde{g}_{\mu\nu} \tilde{u}^\mu \tilde{u}^\nu = -1. \]  

A well-known cosmological model is the family of Friedmann-Lemaître-Robertson-Walker (FLRW) solution to (1.1)–(1.2) representing a homogeneous, fluid filled universe that is undergoing accelerated expansion. We use \( x^i \) \( (i = 1, 2, 3) \) to denote the standard periodic coordinates on the 3-torus \( \mathbb{T}^3 \) and \( \tau = x^0 \) a time coordinate on the interval \((0, 1] \), then the FLRW solutions on the manifold \( \mathcal{M} = (0, 1] \times \mathbb{T}^3 \) are defined by

\[ \tilde{\eta} = \frac{1}{\tau^2} \left( -\frac{1}{\omega^2(\tau)} d\tau^2 + \delta_{ij} dx^i dx^j \right), \]  
\[ \tilde{u} = -\tau \omega \partial_\tau, \]  

and the corresponding density of the fluid \( \tilde{\rho} \) verifies the estimate (see (2.6) later)

\[ \tau^4 \tilde{\rho}(1) \leq \tilde{\rho}(\tau) \leq \tau^3 \tilde{\rho}(1), \]  

where \( \omega(\tau) \in C^2([0, 1]) \) and the initial proper density \( \tilde{\rho}(1) \) can be freely specified.

**Remark 1.1.** We emphasize that, as is pointed out at Remark 1.2 in [20], the expression (1.4) of FLRW solutions is not the standard one because of the choice of the time coordinate which compactifies the time interval from infinity interval \([0, \infty) \) in the standard presentation to \((0, 1] \).
in the coordinates used here. In order to recover the standard presentation from this expression, let us define a new time coordinate \( t \) in term of \( \tau \in (0, 1] \),
\[
t := t(1/\tau) := - \int_{1}^{\tau} \frac{1}{y \omega(y)} dy > 0.
\] (1.7)

It is evident that \( t \) is a strictly increasing function of \( 1/\tau \) due to the strictly positive integrand. Therefore, there exists an inverse function \( t^{-1} \), which we denote by \( a(t) \), such that
\[
a(t) := t^{-1}(t) = \frac{1}{\tau}.
\] (1.8)

According to our choice of time coordinate \( \tau \), the future lies in the direction of decreasing \( \tau \) and timelike infinity is located at \( \tau = 0 \). Transforming the time coordinate \( \tau \) to \( t \) via (1.7)–(1.8), the FLRW metric (1.4) recovers to the standard one
\[
ds^2 = -dt^2 + a^2(t) \delta_{ij} dx^i dx^j,
\]
which can be found in a variety of references, for instance [45].

**Remark 1.2.** The fluid-four velocity \( \tilde{u}^\mu \) is assumed to be future oriented, which is equivalent to the condition
\[
\tilde{u}^0 < 0.
\]

Before stating the main theorem of this article, we fix notations and conventions first. A number of new variables and preliminary concepts are introduced as well.

1.1. **Notations.**

1.1.1. **Indices and coordinates.** Unless stated otherwise, our indexing convention will be as follows: we use lower case Latin letters, e.g. \( i; j; k \), for spatial indices that run from 1 to 3, and lower case Greek letters, e.g. \( \alpha, \beta, \gamma \), for spacetime indices that run from 0 to 3. We will follow the Einstein summation convention, that is, repeated lower and upper indices are implicitly summed over. We use \( x^i (i = 1, 2, 3) \) to denote the standard periodic coordinates on the 3-torus \( T^3 \) and \( \tau = x^0 \) a time coordinate on the interval \( (0, 1] \).

1.1.2. **Descriptions of background and perturbed manifolds.** Throughout this article, we use \( \tilde{g} \) and \( \tilde{\eta} \) to denote the original metric and original background FLRW metric respectively; we also use \( g \) and \( \eta \) to denote the conformal metric and the conformal background metric. \( \tilde{\Gamma}, \tilde{\gamma}, \Gamma \) and \( \gamma \) denote the Christoffel symbols with respect to \( \tilde{g}, \tilde{\eta}, g \) and \( \eta \), respectively, similar conventions are used for all kinds of the curvature tensors \( \tilde{R}, \tilde{\mathcal{R}}, \mathcal{R} \).

1.1.3. **Derivatives.** Partial derivatives with respect to coordinates \( (x^\mu) = (\tau, x^i) \) will be denoted by \( \partial_\mu = \partial / \partial x^\mu \) which are the partial derivatives of the conformal spacetime. \( \nabla \) and \( \bar{\nabla} \) are the covariant derivatives of the original physical spacetime and the conformal spacetime, respectively. We use \( Du = (\partial_j u) \) and \( \bar{D}u = (\partial_\mu u) \) to denote the spatial and spacetime gradients, respectively. \( f'(g) \), or simply \( f' \) if there is no confusion, will be used to denote \( f'(g) := df(g)/dg \).

Greek letters will also be used to denote multi-indices, e.g. \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}, \) and we will employ the standard notation \( D^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_n^{\alpha_n} \) for spatial partial derivatives. It will be clear from context whether a Greek letter stands for a spacetime coordinate index or a

\[^1\text{note the following identity implies}
\]
\[
\frac{dt}{d\tau} = - \frac{1}{\tau \omega(\tau)}.
\]
multi-index. Furthermore, we will use $D^k u = \{D^\alpha u \mid |\alpha| = k\}$ to denote the collection of partial derivatives of order $k$.

Given a vector-valued map $f(u)$, where $u$ is a vector, we use $Df$ and $D_u f$ interchangeably to denote the derivative with respect to the vector $u$, and use the standard notation

$$D f(u) \cdot \delta u := \left| \frac{d}{dt} \right|_{t=0} f(u + t\delta u)$$

for the action of the linear operator $D f$ on the vector $\delta u$. For vector-valued maps $f(u, v)$ of two (or more) variables, we use the notation $D_1 f$ and $D_u f$ interchangeably for the partial derivative with respect to the first variable, i.e.

$$D_1 f(u, v) \cdot \delta u := \left| \frac{d}{dt} \right|_{t=0} f(u + t\delta u, v),$$

and a similar notation for the partial derivative with respect to the other variable.

1.1.4. Function spaces. For a function $u(\tau, x)$, we define the following standard Sobolev norms

$$\|u(\tau, x)\|_{L^2(\mathbb{T}^n)} := \left( \int_{\mathbb{T}^n} |u(\tau, x)|^2 d^n x \right)^{\frac{1}{2}},$$

$$\|u(\tau, x)\|_{H^k(\mathbb{T}^n)} := \sum_{|\alpha| = 0}^k \|D^\alpha u(\tau, x)\|_{L^2(\mathbb{T}^n)},$$

and

$$\|u(\tau, x)\|_{L^\infty(\mathbb{T}^n)} := \text{ess sup}_{x \in \mathbb{T}^n} |u(\tau, x)|.$$

1.1.5. Remainder terms. In order to simplify the handling of remainder terms whose exact form is not important, we will use, unless otherwise stated, upper case calligraphic letters, e.g., $S(\tau, \xi)$, $T(\tau, \xi)$ and $U(\tau, \xi)$, to denote vector-valued maps that are elements of the space $C^1([0,1], C^\infty(\mathbb{R}^M))$ for $\xi \in \mathbb{R}^M$ and upper case letters in typewriter font, e.g., $s(\tau, \xi)$, $t(\tau, \xi)$ and $u(\tau, \xi)$, to denote vector-valued maps that are elements of the space $C^0([0,1], C^\infty(\mathbb{R}^M))$. We also remark that because the exact forms of these calligraphic or typewriter font remainders are not important, the remainders of the same letter may change form to line to line.

We will say that a function $f(x, y)$ vanishes to the $n^{th}$ order in $y$ if it satisfies $f(x, y) \sim O(y^n)$ as $y \to 0$, that is, there exists a positive constant $C$ such that $|f(x, y)| \leq C|y|^n$ as $y \to 0$.

1.1.6. Intermediate point. We use $\alpha_{K_\ell}$ to denote the intermediate point between $\bar{\alpha}$ and $\alpha$ measured by $K_\ell$ in a linear fashion, that is,

$$\alpha_{K_\ell} := \bar{\alpha} + K_\ell (\alpha - \bar{\alpha})$$

for some constant $K_\ell \in (0,1)$ ($\ell = 1, 2, \cdots$).

1.2. Makino-type fluids. Throughout this article, we concentrate on a type of perfect fluids with the equation of state $p = p(\rho)$ where $p(0) = 0$ and $p(\rho)$ is analytic on a compact set $I_\rho \subset [0, +\infty]$, and we also require that for this fluid, there exists an invertible transformation $\alpha = \mu^{-1}(\rho)$ determined by a set of quantities $\{\mu(\alpha), \varrho, \varsigma, \beta(\tau)\}$ satisfying the following Assumptions 1.3–1.5, which rephrase this fluid in terms of a new density variable $\alpha$, and $\{\varrho, \varsigma, \beta(\tau)\}$ restrict
the property of the transformation $\mu$. Only for convenience of expression, in this article, we will always mention such fluids by Makino-type fluids\textsuperscript{2}.

**Assumption 1.3.** (The symmetrization condition of Makino-type variable $\alpha$) There exists an invertible transformation

$$ C^2 \ni \mu : \quad I_\alpha \to I_\rho, \quad \alpha(x^\mu) \mapsto \rho(x^\mu), $$

that is, $\mu(\alpha) = \rho$, where $I_\alpha \subset [-\infty, +\infty]$ is a compact set, such that a quantity $\lambda(\alpha)$ constructed by $\mu$ and the pressure of the fluid $p$ is uniformly bounded and away from 0. That is, there is a constant

$$ \delta \in \left(0, \min\left\{ \frac{3}{4}, \frac{3}{3 + \Lambda} \left(1 + \sqrt{\frac{3}{\Lambda}} \right), \frac{1}{3 + \Lambda} \left(1 + \sqrt{\frac{3}{\Lambda}} \right), 1 \right\} \right), \quad (1.9) $$

such that

$$ \lambda(\alpha) := \frac{s(\alpha)}{\mu(\alpha) + \mu^* p(\alpha)} \frac{d\mu(\alpha)}{d\alpha} \in [\delta, 1/\delta] \quad (1.10) $$

and $\lambda \in C^2(I_\alpha)$, where

$$ s := \mu^* c_s \quad \text{and} \quad c_s := \sqrt{\frac{dp}{d\rho}} $$
describe the sound speed and $\mu^*$ is the pullback of $\mu$.

**Assumption 1.4.** Suppose $\bar{\rho} = \bar{\rho}(\tau)$ is the density of the homogeneous, isotropic fluid (for example, in this article, we take it to be the background FLRW solution (1.4)–(1.6)), then we denote

$$ \bar{\alpha} := \mu^{-1}(\bar{\rho}). $$

Assume there exists an function $\varrho \in C([0, 1], C^\infty(\mathbb{R}))$ satisfying $\varrho(\tau, 0) = 0$ and a rescaling function $\beta(\tau) \in C([0, 1]) \cap C^1(0, 1)$ of $\alpha$ such that

$$ \mu(\alpha) - \mu(\bar{\alpha}) = \tau^\varsigma \varrho(\tau, \beta^{-1}(\tau)(\alpha - \bar{\alpha})), \quad \varsigma \geq 2. \quad (1.11) $$

**Assumption 1.5.** Let $q(\alpha) := s(\alpha)/\lambda(\alpha)$, then $\bar{q} := q(\bar{\alpha}) = \bar{s}/\bar{\lambda}$ where $\bar{s} := s(\bar{\alpha})$, $\bar{\lambda} := \lambda(\bar{\alpha})$.

Suppose

$$ \bar{s} \lesssim \beta(\tau), \quad \lambda(\bar{\alpha}) \partial_\tau \beta(\tau) \lesssim 1 \quad \text{and} \quad \frac{\bar{s}}{\tau} \lambda'(\bar{\alpha}) \lesssim 1, \quad (1.12) $$

and one of the following two conditions holds,

1. If there is a positive constant $\delta$ given by (1.9), such that $\beta(\tau)$ is bounded by

$$ \frac{1}{C^* \delta^\tau} \leq \beta(\tau) \leq \frac{1}{C^* \delta^\tau} \sqrt{\tau} \quad \text{and} \quad \chi(\tau) := \tau \partial_\tau \ln \beta(\tau) \geq 0, \quad (1.13) $$

where $C^* := \left(\sqrt{\frac{3}{\Lambda}} + 1 \right)^{-1} \left(\frac{3}{4 \Lambda x^2} + 2 \right)^{-\frac{x}{2}}$ and $\chi(\tau)$ satisfies

$$ 1 - 3s^2 \geq \chi(\tau) + \delta \quad (1.14) $$

and

$$ \frac{1}{3} \chi(\tau) + \frac{1}{\delta} \geq q'(\bar{\alpha}) \geq \frac{1}{3} \chi(\tau) + \delta, \quad (1.15) $$

\textsuperscript{2}We name the fluid as this mainly because for polytropic fluids, the Euler equations become degenerated, the key idea to overcome such degeneracy comes from the Makino’s well-known transformation [25]. Inspired by this idea, we give this more abstract fluid to ensure it compatible with the longtime problem and include more situations as well.
for all $\tau \in [0, 1]$.

(2) If $\beta \equiv \text{constant} > 0$, $s$ and one of the following cases happens
(a) $q = \bar{q}$ and $\delta \leq 1 - 3s^2 \leq 1 - 3\tilde{\delta}^2$ where $\tilde{\delta}$ is given by (1.9);
(b) $q = \bar{q}$ and $1 - 3s^2 = 0$;
for all $\tau \in [0, 1]$.

Furthermore, Assumption 1.5.(1) and (2) separate Makino-type fluids into two classes according to their different proofs. We name $\textit{Makino-type (1) fluids}$ if Assumption 1.5.(1) holds and the other one is $\textit{Makino-type (2) fluids}$ which satisfy Assumption 1.5.(2).

**Remark 1.6.** With the help of that $s^2 \geq 0$, (1.13) and (1.14) imply
\[ 0 \leq \chi(\tau) \leq 1 - \tilde{\delta}. \tag{1.16} \]
In addition, (1.13) and (1.14) and Assumption 1.5.(2), with the fact that $c_s(\rho) = s(\alpha)$, yield that
\[ 0 \leq s^2 \leq \frac{1}{3} \quad \text{and} \quad 0 \leq c_s^2 := c_s^2(\rho) \leq \frac{1}{3}. \tag{1.17} \]
By (1.13) and (1.16), we arrive at
\[ \bar{s}\partial_\tau \beta \lesssim \beta \partial_\tau \beta \lesssim \frac{\beta^2}{\tau} \lesssim 1 \quad \text{and} \quad \bar{s}\beta \lesssim \tau \quad (\text{i.e. } \bar{q}\beta \lesssim \tau) \tag{1.18} \]
for $\tau \in [0, 1]$.

**Remark 1.7.** We give three examples of Makino fluids in §2.2 which guarantee that the set of Makino-type fluids is not empty. These three examples are
(1) Fluids with the linear equation of state $p = K\rho$ ($K \in (0, \frac{1}{3}]$). This is a standard fluid model in cosmology and the future stability has been well investigated in [20, 21, 30, 40, 43]. Especially, [20, 21] also conclude the statements of cosmological Newtonian limits on large scale based on such a linear model of perfect fluids;
(2) Chaplygin gas\(^3\) with the equation of state that
\[ p = -\frac{A^{1+\vartheta}}{(\rho + A)^\vartheta} + \Lambda \quad (\vartheta \in \left(0, \sqrt{\frac{1}{3}}\right)); \tag{1.20} \]
is a revised version of Chaplygin gases which is equivalent to the original one if the positive cosmological constant is explicitly expressed in the Einstein equation. The original definition of Chaplygin gas is a cosmological gas model for dark energy. In other words, it essentially plays the role of positive cosmological constant. [16] adopts the method of [30] deriving the long time stability of irrotational Chaplygin gas filled universe.
(3) Polytropic gas with $p = K\rho^{\frac{n+1}{n}}$ ($n \in (1, 3)$). The polytropic gas is also a well-known fluid model in stellar system. However, there is no results known about polytropic fluids filled general relativistic universe to the best of our knowledge. In fact, this article, after proving the future stability of Makino-type fluids, implies the corresponding results of polytropic ones.

\(^3\)This equation of state does not consist with the standard Chaplygin gas
\[ p = -\frac{A}{\rho^\vartheta}. \tag{1.19} \]
Due to presence of positive cosmological constant in the Einstein equations (1.1)–(1.2), with the help of simple calculations, one can derive (1.20) with positive cosmological constant in Einstein equation is equivalent to (1.19) without positive cosmological constant in Einstein equation by letting $A = \Lambda^{1+\vartheta}$.
Hence, Makino-type fluid is a decent model at least containing all these common models in astrophysics.

**Remark 1.8.** Assumption 1.3 consists with the standard symmetrization condition of the transformation of the Makino-type variable (see, for example, [5, Page 114]).

**Remark 1.9.** Assumption 1.4 implies that
\[
\frac{\rho - \bar{\rho}}{\tau^2} = \tau^{c-2} g(\tau, \beta^{-1}(\alpha - \bar{\alpha})) \quad \text{and} \quad \frac{p - \bar{p}}{\tau^2} = \tau^2 (\rho K_8) \tau^{c-2} g(\tau, \beta^{-1}(\alpha - \bar{\alpha})),
\]
which will be used in (3.42) to make sure $\frac{\rho - \bar{\rho}}{\tau^2}$ and $\frac{p - \bar{p}}{\tau^2}$ are not $\tau$-singular terms in the remainder terms of the Einstein equation (see (3.42) for details).

**Remark 1.10.** Let us state an alternative but slightly stronger expression of Assumption 1.4.(1.11) which is given by
\[
|\mu'(\alpha)| \lesssim \frac{\tau^c}{\beta}, \quad \varsigma \geq 2,
\]
for $\alpha \in I_\alpha$. Noting the expansion of $\mu$,
\[
\mu(\alpha) = \mu(\bar{\alpha}) + \beta \mu'(\alpha K_8) \beta^{-1}(\alpha - \bar{\alpha})
\]
for some constant $K_8 \in (0, 1)$, we can, with the help of (1.22), arrive at Assumption 1.4.(1.11) readily.

**Remark 1.11.** Once we rewrite Einstein–Euler equations in terms of the singular symmetric hyperbolic formulations (A.1) given in Appendix A, then Assumption 1.5.(1) and (2) ensure the positivity of the matrix $B$ of the singular term in the singular hyperbolic system (A.1). While Assumption 1.5.(1.12) ensures that the coefficient matrix $B^0 \in C^1([0, 1], C^\infty(\mathbb{R}^M))$ in the model equation (A.1).

### 1.3. Conformal Einstein-Euler system.

The main tool of this article is Oliynyk’s conformal singular system which was first established by T. Oliynyk [30], then developed by [20, 21] to prove the cosmological Newtonian limits on large scales and applied by [16] to future stability of Chaplygin gas filled universe. Instead of describing the spacetime in terms of physical metric $\tilde{g}$ directly, we turn to the conformal one
\[
g_{\mu\nu} = e^{-2\Phi} \tilde{g}_{\mu\nu}, \quad \text{(i.e.} g^{\mu\nu} = e^{2\Phi} \tilde{g}^{\mu\nu})
\]
and the fluid four velocity governed by
\[
u^\mu = e^\Phi \tilde{u}^\mu,
\]
where, throughout this article, we take explicitly
\[
\Phi = -\ln(\tau).
\]
In addition, by (1.3), there is a normalization relation of the conformal four velocity,
\[
u^\mu u_\mu = -1.
\]

Under the conformal transformation (1.23)–(1.25), the conformal background metric becomes
\[
\eta = -\frac{1}{\omega^2(\tau)} d\tau^2 + \delta_{ij} dx^i dx^j,
\]
and recalling identity
\[
\bar{R}_{\mu\nu} - R_{\mu\nu} = -g_{\mu\nu} \Box \Phi - 2 \nabla_\mu \nabla_\nu \Phi + 2 (\nabla_\mu \nabla_\nu \Phi - |\nabla \Phi|^2 g_{\mu\nu}),
\]
where $\nabla$ and $R_{\mu
u}$ are the covariant derivative and the Ricci tensor of $g_{\mu\nu}$, respectively, $\Box = \nabla_\nu \nabla^\nu$ and $|\nabla \Phi|^2 = g^{\mu\nu} \nabla_\mu \Phi \nabla_\nu \Phi$, and then the equations (1.1)–(1.2) becomes conformal Einstein-Euler equations,

$$ G^{\mu\nu} = T^{\mu\nu} := e^{4\Phi} \tilde{T}^{\mu\nu} - e^{2\Phi} \Lambda g^{\mu\nu} + 2(\nabla^\mu \nabla^\nu \Phi - \nabla^\nu \Phi \nabla^\mu \Phi) - (2\Box_g \Phi + |\nabla \Phi|^2)g^{\mu\nu}, \quad (1.27) $$

$$ \nabla_\mu \tilde{T}^{\mu\nu} = -6\tilde{T}^{\mu\nu} \nabla_\mu \Phi + g_{\kappa\lambda} \tilde{T}^{\kappa\lambda} g^{\mu\nu} \nabla_\mu \Phi, \quad (1.28) $$

where here and in the following, unless otherwise specified, we raise and lower all coordinate tensor indices using the conformal metric $g_{\mu\nu}$. Note that (1.28) can be derived due to the identity of the difference $\tilde{\Gamma}^\gamma_{\mu\nu} - \Gamma^\gamma_{\mu\nu}$,

$$ \tilde{\Gamma}^\gamma_{\mu\nu} - \Gamma^\gamma_{\mu\nu} = g^{\gamma\alpha} (g_{\alpha\mu} \nabla_\nu \Phi + g_{\nu\alpha} \nabla_\mu \Phi - g_{\mu\nu} \nabla_\alpha \Phi). $$

Contracting the free indices of (1.27) gives $R = 4\Lambda - T$, where $T = g_{\mu\nu} T^{\mu\nu}$ and $R$ is the Ricci scalar of the conformal metric. Using this and the definition $G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu}$ of the Einstein tensor, we can write (1.27) as

$$ -2R^{\mu\nu} = -4\nabla^\mu \nabla^\nu \Phi + 4\nabla^\mu \Phi \nabla^\nu \Phi - 2[\Box_g \Phi + 2|\nabla \Phi|^2] $$

$$ + \left(\frac{\rho - p}{2} + \Lambda\right)g^{\mu\nu} - 2e^{2\Phi} (\rho + p) u^\mu u^\nu, \quad (1.29) $$

which we call as the conformal Einstein equations.

1.4. The wave gauge. In order to rewrite the conformal Einstein equation into a hyperbolic system, wave gauge is a useful technique. In this section, we introduce the wave gauge in the spirit of [16, 20, 21, 30]. Straightforward calculations show that the Christoffel symbols, contracted Christoffels and Ricci tensors are

$$ \gamma^\lambda_{\mu\nu} = \frac{1}{2} \eta^{00}(\partial_\tau \eta_{00}) \delta^\lambda_\alpha \delta^\sigma_\mu \delta^\nu_\nu = -\frac{\partial_\tau \omega}{\omega} \delta^\lambda_\alpha \delta^\sigma_\mu \delta^\nu_\nu, \quad \gamma^\mu = -\frac{\Omega}{\tau} \delta^\mu_\nu $$

and

$$ R_{\mu\nu} = \partial_\alpha \gamma^\alpha_{\mu\nu} - \partial_\mu \gamma^\alpha_{\alpha\nu} + \gamma^\alpha_{\alpha\lambda} \gamma^\lambda_{\mu\nu} - \gamma^\alpha_{\mu\lambda} \gamma^\lambda_{\alpha\nu} = 0. $$

where for the simplicity of notations, we denote

$$ \Omega(\tau) = -\tau \omega \partial_\tau \omega. \quad (1.30) $$

Define the wave coordinates as

$$ Z^\mu = 0, \quad (1.31) $$

where

$$ Z^\mu = X^\mu + Y^\mu \quad (1.32) $$

with

$$ X^\mu := \Gamma^\mu - \gamma^\mu = -\partial_v g^{\mu\nu} + \frac{1}{2} g^{\mu\nu} g_{\alpha\beta} \partial_\sigma g^{\alpha\beta} + \frac{\Omega}{\tau} \delta^\mu_0 \quad (1.33) $$

and

$$ Y^\mu := -2(g^{\mu\nu} - \eta^{\mu\nu}) \nabla_\nu \Phi = -2g^{\mu\nu} \nabla_\nu \Phi + 2\frac{\omega^2}{\tau} \delta^\mu_0. \quad (1.34) $$

Then,

$$ Z^\mu = X^\mu + Y^\mu = \Gamma^\mu + \frac{2}{\tau} \left( g^{\mu\nu} + \left(\omega^2 + \frac{\Omega}{2}\right) \delta^\mu_0 \right) = \Gamma^\mu + \frac{2}{\tau} \left( g^{\mu\nu} + \frac{\Omega(\tau)}{3} \delta^\mu_0 \right) = 0, \quad (1.35) $$
where we, for simplicity, denote
\[ \frac{\psi(\tau)}{3} := \omega^2 + \frac{\Omega}{2}. \] (1.36)

**Remark 1.12.** A well known result from Zenginoğlu [47] implies if initially \( Z^\mu = 0 \), then \( Z^\mu \equiv 0 \) in the whole evolution of the Einstein-Euler system.

1.5. **Transformed field variables.** The gravitational and matter field variables, such as \( \{g^{\mu\nu}(x^\alpha), \rho(x^\alpha), u^\mu(x^\alpha)\} \), as they stand for, are not suitable for establishing the global existence of solutions. In order to obtain suitable variables, we employ the following field variables, which are adopted firstly by [30] and then used in [16, 20, 21] via their variations.

Define the densitized three-metric \( g_{ij} = \text{det}(\tilde{g}_{lm})^{\frac{1}{3}}g^{ij} \), where \( \tilde{g}_{lm} = (g^{lm})^{-1} \), and introduce
\[ q = g^{00} - \eta^{00} + \frac{\eta^{00}}{3} \ln(\text{det}(g^{ij})). \] (1.37)
Then let
\[ u^{\nu} = g^{0\nu} - \eta^{0\nu}, \] (1.38)
\[ u_0^{0\nu} = \partial_\tau (g^{0\nu} - \eta^{0\nu}) - \frac{3(g^{0\nu} - \eta^{0\nu})}{2\tau}, \] (1.39)
\[ u_i^{0\nu} = \partial_i (g^{0\nu} - \eta^{0\nu}), \] (1.40)
\[ u^{ij} = g^{ij} - \delta^{ij}, \] (1.41)
\[ u_\mu^{ij} = \partial_\mu g^{ij}, \] (1.42)
\[ u = q, \] (1.43)
\[ u_\mu = \partial_\mu q, \] (1.44)
\[ \alpha = \beta(\tau)\zeta, \] (1.45)
\[ \bar{\alpha} = \beta(\tau)\bar{\zeta}, \] (1.46)
\[ u^i = \beta(\tau)u^i, \] (1.47)
\[ \delta\zeta = \zeta - \bar{\zeta} \] (1.48)

where \( \beta(\tau) \) comes from the definition of Makino-type fluid (see Assumption 1.4).

1.6. **Initial data.** It is well known that the initial data for the conformal Einstein–Euler equations cannot be chosen freely on the initial hypersurface
\[ \Sigma_1 = \{1\} \times T^3 \subset M = (0, 1] \times T^3. \]
Indeed, a number of constraints, which we can separate into gravitational, gauge and velocity normalization, must be satisfied on \( \Sigma_1 \). In specific, the initial data are governed by the constraint equations which are essentially an elliptic system. We have to specify some part of the data, then the other data will be derived from these free ones via constraints. Let
\[ g^{\mu\nu}|_{\tau=1} = g_0^{\mu\nu}(x), \quad \partial_\tau g^{\mu\nu}|_{\tau=1} = g_1^{\mu\nu}(x), \quad \rho|_{\tau=1} = \rho_0(x), \quad \nu^\alpha|_{\tau=1} = \nu^\alpha(x). \]
Above initial data set \( (g_0^{\mu\nu}(x), g_1^{\mu\nu}(x), \rho_0(x), \nu^\alpha(x)) \) can not be chosen arbitrarily. They must satisfy the Gauss-Codazzi equations, which are equivalent to \( (G^{\mu\nu} - T^{\mu\nu})|_{\tau=1} = 0 \). Moreover, they also satisfy the wave coordinates condition \( Z^\mu|_{\tau=1} = 0 \), the precise definition of \( Z^\mu \) can be found in Section 1.4.

There are a number of distinct methods available to solve these constraint equations. However, we do not intent to state or prove the exact initialization theorem in the current article.
One can always use the similar method in [20, 21, 27, 28] which is an adaptation of the method introduced by Lottermoser in [22] to recover the statements and proofs. The data exists but may not be uniquely selected. In this article, we assume the data has already been selected properly and only focus on the further evolutions as $\tau \searrow 0$.

### 1.7. Main Theorem.

With above notations, our main result is stated here and the proof will be given in §5.

**Theorem 1.13.** Suppose $k \in \mathbb{Z}_{\geq 3}$, $\Lambda > 0$, $g^\mu_0, g^\mu_1, \rho_0, \nu^\alpha \in H^k(\mathbb{T}^3)$, $\rho_0 > 0$ for all $x \in \mathbb{T}^3$ and the unknowns are determined by the data on the initial hypersurface that

$$
(g^\mu, \partial_\tau g^\mu, \rho, u^i)\big|_{\tau=1} = (g^\mu_0, g^\mu_1, \rho_0, \nu^\alpha)
$$

which solves the constraint equations

$$
(G^\mu_{\nu} - T^\mu_{\nu})\big|_{\tau=1} = 0 \quad \text{and} \quad Z^\mu\big|_{\tau=1} = 0.
$$

The fluids for the Einstein-Euler system are the Makino-type fluids satisfying Assumption 1.3–Assumption 1.5. Then there exists a constant $\sigma > 0$, such that if

$$
\|g^\mu_0 - \eta^\mu(1)\|_{H^{k+1}} + \|g^\mu_1 - \partial_\tau \eta^\mu(1)\|_{H^k} + \|\rho_0 - \bar{\rho}(1)\|_{H^k} + \|\nu^i\|_{H^k} < \sigma,
$$

there exists a unique classical solution $g^\mu \in C^2((0,1] \times \mathbb{T}^3)$, $\rho, \upsilon^i \in C^1((0,1] \times \mathbb{T}^3)$ to the conformal Einstein-Euler system (1.27)–(1.28) that satisfies the initial data (1.49), the wave gauge $Z^\mu = 0$ in $(0, 1] \times \mathbb{T}^3$ and the following regularity conditions

$$
(g^\mu, u^\mu, \rho) \in \bigotimes_{\ell=0}^2 C^\ell((T_1,1], H^{k+1-\ell}(\mathbb{T}^3)) \times \bigotimes_{\ell=0}^1 C^\ell((T_1,1], H^{k-\ell}(\mathbb{T}^3))
$$

and the estimates that

$$
\|g^\mu(\tau) - \eta^\mu(\tau)\|_{H^{k+1}} + \|\partial_\tau g^\mu(\tau) - \partial_\tau \eta^\mu(\tau)\|_{H^k} + \|\rho(\tau) - \bar{\rho}(\tau)\|_{H^k} + \|u^i(\tau)\|_{H^k} \lesssim \sigma.
$$

**Remark 1.14.** We do not include the asymptotics of the solutions in this article, but they can be derived in a similar way to [30] involving some calculations of decay exponents.

### 1.8. Prior and related work.

The stability problems of certain exact solutions to Einstein–matter systems is important in mathematical general relativity and there are a lot of works related to them. We only mention some of them which directly related to our current article, readers can find more in these references. First, a well known and groundbreaking work by D. Christodoulou and S. Klainerman is the stability of Minkowski spacetime [7] as a solution to the Einstein-vacuum equations. Another remarkable approach based on the wave coordinates is given by H. Lindblad and I. Rodnianski [17, 18].

To serve our purpose, we turn to the fully nonlinear future stability of Friedmann-Lemaître-Robertson-Walker (FLRW) solutions with the positive cosmological constant which has been well studied during this decade. H. Ringström [38] firstly investigate the future global non-linear stability in the case of Einstein’s equations coupled to a non-linear scalar field $\tilde{T}^{\mu\nu} = \tilde{\nu}^{\mu}\tilde{\nu}^{\nu} - \frac{1}{2} \bar{g}^{\mu\nu} \tilde{\nu}^{\rho} \tilde{\nu}^{\sigma} + V(\bar{\Psi}) \bar{g}^{\mu\nu}$ where, under the assumption that $V(0) > 0$, $V'(0) = 0$, $V''(0) > 0$, $V(\bar{\Psi})$ plays the role of the positive cosmological constant which appears in the

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4This constant $\sigma$ can be considered small eventually from the proof of this Theorem.
most of the following works. The main observation of this paper is two-fold: First, the Einstein-
nonlinear scalar field system can be formulated as a system of nonlinear wave equations if
introducing generalized wave coordinates; Second, the problem under consideration describes
the accelerated expansion of the universe, and the expansion provides dispersive terms, which
leads to exponential decay for solutions. Inspired by H. Ringström’s work, I. Rodnianski and
J. Speck [40] established the future non-linear stability of these FLRW solutions with positive
cosmological constant and linear equation of state under the condition \(0 < K < 1/3\) and the
assumption of zero fluid vorticity. After that, M. Hadzić and J. Speck [12] and J. Speck
[43] answered that this future non-linear stability result remains true for fluids with non-zero
vorticity and also for the equation of state parameter values \(K = 0\). By employing the conformal
method developed by H. Friedrich [9, 10], C. Lübbe and J. A. V. Kroon [23] proved the
above question for the equation of state parameter values \(K = 1/3\) that is the pure radiation
universe. After these, T. Oliynyk [30] gave an alternative proof for non-linear future stability
problems of FLRW solutions based on conformal singular hyperbolic formulations of Einstein–
Euler equations, a completely different method comparing with above works, which provide
the basic tool of the current article for us. One advantage of the this method is that, under a
conformal transformation, by choosing the conformal factor and the source term of the wave
gauge and variables judiciously, the whole Einstein-Euler system can be turned into a symmetric
hyperbolic system with singular in time terms (with “good” sign) and solutions defined on
finite interval of time. By a variation of standard energy estimates, one can then get the global
nonlinear stability of a family of FLRW solutions and established the asymptotic behavior of
perturbed solutions in the far future. Finally, we point out that, in the regime \(K > 1/3\), [34] has
found some evidence for instability.

Inspired by the structure of the conformal singular hyperbolic equations in [30], C. Liu and
T. Oliynyk [19–21] answer a fundamental question: On what scales can Newtonian cosmological
simulations be trusted to approximate realistic relativistic cosmologies? They investigated the
fully nonlinear long time behavior of our universe basing on a small perturbation of the FLRW
metric with perfect fluid of linear equation of state and positive cosmological constant, and
rigorously proving the errors of solutions between Newtonian gravity (governed by Poisson–Euler
equations) and general relativity (governed by Einstein–Euler equations) are well controlled
small for the long time future direction in suitable norms by judicious initial data selections.
That is, they establish the existence of 1-parameter families of \(\epsilon\)-dependent solutions to the
Einstein–Euler equations with a positive cosmological constant \(\Lambda > 0\) and a linear equation of
state \(p = \epsilon^2 K \rho, 0 < K \leq 1/3\), for the parameter values \(0 < \epsilon < \epsilon_0\). These solutions exist glo-
ally to the future, converge as \(\epsilon \searrow 0\) to solutions of the cosmological Poisson–Euler equations
of Newtonian gravity, and are inhomogeneous nonlinear perturbations of FLRW fluid solutions.
This cosmological Newtonian limit problem have been investigated for both the isolated version
(periodic universe) of cosmology [21] and the cosmological large space-time scales (multi-body
version of universe) [20]. These two results answered the feasibility of Newtonian approxima-
tions used in astrophysics for 100 years old with rigid mathematical proofs (astrophysicists use
this approximation as a hypothesis for a century) and intuitively, this question can be viewed as
a stability problem of general relativistic cosmological solution around the Newtonian gravity
in large scales.

We emphasize that the above results are obtained by assuming linear equation of state of
the perfect fluids. P. LeFloch and C. Wei [16] investigated the Einstein-Chaplygin fluids with
equation of state \(p = -\frac{\Lambda}{\rho}^{\nu}\) with \(\Lambda = 0\). This model is widely investigated by physicians as a
candidate for the dark energy. In [16], they only considered the irrotational fluids, such that the
matter is indeed a scalar field, under which the mechanism for the accelerated expansion of the spacetime is the negativity of the pressure (an analogue of positive cosmological constant). The main idea comes from T. Oliynyk’s conformal singular hyperbolic equations [30] and the main difference is that they choose different conformal factor to make sure that the equations satisfied by the fluids are regular. However, this technique can not be applied to the general system due to the normalization condition $\tilde{u}^\mu \tilde{u}_\mu = -1$. Thus, one of the main motivations of this article is to investigate the evolution of the general Chaplygin fluids in accelerated expanding spacetime.

It is worth noting that in 1994, U. Brauer, A. Rendall and O. Reula [6] proved a fully non-linear future stability problem on the Newtonian cosmological model, which laid the foundation on the corresponding stability problem in the Newtonian setting. By the classical energy estimate, they showed the fluids with equation of state $p = K \rho^{\frac{n+1}{n}}$ is globally stable in accelerated expanding universe when the initial data are small.

On the other direction, I. Rodnianski and J. Speck [41, 42] has proven in the collapsing direction $t \searrow 0$ the FLRW solution is globally non-linearly stable under small perturbations of its initial data at $t = 1$. They formulate their results in the constant mean curvature (CMC)-transported spatial coordinates gauge. Along this direction, we point out another two important works by L. Andersson and A. Rendall [1], and H. Rinström [37], respectively.

1.9. Overview of the method.

1.9.1. Motivations of Makino-type fluids. We firstly introduce a new class of fluids, Makino-type fluids (§1.2), which includes several common and frequently used fluid models such as, by direct verifying the definition in the next section, fluids with linear equation of state, Chaplygin gases and polytropic gases, etc. As we have pointed out at the beginning of this paper, our direct purpose of current article is to find out what types of fluids can guarantee the validity that the future non-linear stability of FLRW solutions to the Einstein-Euler system with a positive cosmological constant and try to identify as many fluids as possible. Once this type of fluids cover a large class of reasonable set of fluids, then the stability problem of FLRW metric is “stable” with respect to equations of state of this class of fluids in some sense. The assumptions of Makino-type fluids come from the proof of the future non-linear stability of FLRW solutions to the Einstein-Euler system with a positive cosmological constant. We carefully give these assumptions to make sure the stability proof holds and contain as many types of fluids as possible.

1.9.2. Conformal singular formulations of Einstein equations. The main tool of the proof of the main Theorem 1.13 is the conformal singular hyperbolic formulation of [30]. The idea of writing conformal Einstein equations as this formulations belongs to [30] and is based on three main ingredients:

(1) Conformal transformation with certain conformal factor $\Phi$ defined by (1.25). This is reasonable because the conformal FLRW metric is close to the Minkowski metric which simplify the geometry and calculations, and the conformal time coordinate $\tau \in (0, 1]$ compatible with this conformal transformation compacifies the original time $t \in [0, \infty)$ to a finite interval, which converts a long time problem to a short finite time one. The cost of this compactification of time is the equation becomes singular in time but singular in a “good” way.

(2) Wave gauge defined by (1.31)–(1.35). This is not surprising, because this wave gauge comes from the standard one without source term via conformal transformations, and this new source term eliminates and regulates the bad terms of the remainders to the
suitable singular terms and regular terms consistent with the singular hyperbolic system (A.1).

(3) Good analyzable variables given by (1.37)–(1.48) are the key part of writing Einstein equations into the appropriate singular formulations (A.1).

By using these three ingredients, it is not hard to rewrite conformal Einstein system (1.29) into a symmetric hyperbolic system with \( \tau \)-singular term like (A.1).

1.9.3. Conformal singular formulations of Euler equations. In order to rewrite the Euler equations as the target singular equation, more attentions are required than the one with the linear equation of state or Chaplygin gases due to the generality of this large class of fluids, and these attentions constitute the main innovations of this article. Let us briefly point out the difficulties here and address the key ideas about how to overcome these difficulties. Because of differences between the structures of Makino-type (1) and (2) fluids, we have to write them to the aimed singular system (A.1)–(A.2) in different ways.

Let us first concentrate on the Makino-type (1), and there are four steps to arrive at the expect formulations. The first step is standard, by using the normalization relation of velocity \( u^\mu u_\mu = -1 \), Rendall’s projection (4.1) and reflection operator \( M_{ki} \) defined in (4.4) to write the conformal Euler equations (1.28) to a symmetric hyperbolic system (4.2)–(4.3) (this formulation is widely used and can be found for example, in [20, 21, 29, 32]). For the problem away from \( \tau = 0 \), this formulation is symmetric hyperbolic equation. However, when \( \tau \) is close to 0, the coefficient matrix will degenerate, which destroys the structure of hyperbolicity of this system. In specific, multiplying suitable factors, for example, \( s^2/(\rho + p) \) on both sides of (4.2) and \( \rho + p \) on both sides of (4.3), then (4.2)–(4.3) become symmetric. However, from the behavior of the background density \( \bar{\rho} \searrow 0 \) and pressure \( \bar{p} \searrow 0 \), as \( \tau \to 0 \) (see (2.5)–(2.6)), the expecting behaviors of \( \rho \) and \( p \) due to small perturbation of initial data \( \bar{\rho}(1) \) and \( \bar{p}(1) \) are also \( \rho \searrow 0 \) and \( p \searrow 0 \) as \( \tau \to 0 \), which leads some of elements of the coefficient matrix of above symmetric hyperbolic system tend to 0 or \( \infty \) as \( \tau \to 0 \). The degenerated coefficients when time is close to 0 violate the Condition (V) in Appendix A.

In step two, the idea to overcome above difficulty is to generalize the idea of the non-degenerated symmetrization of Euler equations originated by Makino [25] who firstly came up this formulation for polytropic fluids in a compact set in Newtonian setting. We extract the key properties of his idea which is enclosed in the definition of Makino-type fluids §1.2 to write above degenerated equations to a non-degenerated hyperbolic system by introducing a new density variable \( \alpha \) defined by Assumption 1.3. However, this is still not enough to apply the theorem in Appendix A, because the current variable \( \alpha - \bar{\alpha} \) can not make sure some of the remainders in Einstein equation to be regular in \( \tau \) (eventually, we need the complete Einstein-Euler system to be the target singular system in Appendix A). In other words, The variable \( \alpha - \bar{\alpha} \) in above system is not a suitable one of the singular hyperbolic system.

In order to eliminate the singular terms in Einstein equation, we choose, in step three, \( \delta \zeta \) as the variable describing the density instead of \( \alpha - \bar{\alpha} \). With the help of Assumption 1.4.(1.11), the remainder terms including \( (\rho - \bar{\rho})/\tau^2 \) and \( (p - \bar{p})/\tau^2 \) in the Einstein equations are regular in \( \tau \) and analytic in \( \delta \zeta \), see (3.42). After changing \( \alpha - \bar{\alpha} \) to \( \delta \zeta \), \( u^i \) have to been changed to \( v^i \) defined by \( u^i = \beta(\tau)v^i \) to make sure this system to be symmetric. Hence, we have to change the variables of fluids from \( (\alpha - \bar{\alpha}, u^i) \) to \( (\delta \zeta, v^i) \) to rewrite the equations.

However, rescaling the velocity to \( v^i \) by \( \beta \) brings a new singular term \( \frac{\partial u^0}{\partial \beta(\tau)} \) in Euler equations. Although equations (4.19)–(4.20) seem to be consistent with the non-degenerated singular hyperbolic system given in Appendix A, the remainders involves \( \tau \)-singular terms of \( u^0 \),
in a “bad” way in (4.29), which destroys the structure of the singular term in the system of Appendix A. In order to conquer this difficulty, we introduce a new variable \( v^k = v^k - Ag^{0k} = v^k - 2\tau Au^{0k} \), where \( A = -\frac{3\omega^2}{\omega(\tau)} \). This new variable adjusts the relations of \( u^0 \) and \( u_0^0 \) to a “good” form, which leads to the “right” formulation of the full set of Einstein–Euler system agreeing with the model (A.1) in Appendix A.

For Makino-type (2) fluids that implies \( \beta \equiv \text{constant} \), by (1.11) in Assumption 1.4, we only need to proceed the first and second steps as above Makino-type (1) fluids, but give up Step 3–4. However, we have to change \( w^j \) to \( u_q \), otherwise, Condition (VI) in Appendix A fails due to the degeneracy of \( BP \) and \( Pu \) is the velocity \( w^j \), by letting \( Pu = 0 \) in \( P^\perp B(t,u)P \) and noting \( u_q = g_{q0} u^0 + g_{qi} u^i \), then \( P^\perp B(t,P^\perp u)P = PB(t,P^\perp u)P^\perp \neq 0 \). A good expression of the Euler equations verifying Condition (VI) in Appendix A relies on the variable \( u_q \) instead of \( w^j \). We only need to rewrite the Euler equations in terms of variables \( (\delta\zeta, u_q) \) further to obtain the system consistent with the model equation (A.1) in Appendix A.

1.10. **Paper outline.** In §2, we first analyze some key asymptotic properties of the background solutions, and then verify that several common and frequently used examples belong to Makino-type fluids.

In §3, we employ the variables (1.37)–(1.48) and the wave gauge (1.31)–(1.35) to write the conformal Einstein system, given by (1.29), as a symmetric hyperbolic system, see (3.23)–(3.25), that is singular in \( \tau \). The method of this transformation comes from [30] originally.

In §4, we further write conformal Euler equation (1.28) as the non-degenerated singular symmetric hyperbolic system satisfying the conditions in the Appendix A for Makino-type (1) and (2) fluids respectively. This section is the main innovation of this article and the goal of this section is to arrive at equations (4.38) and (4.47). The key to achieve this goal is to choose the right variables. For Makino-type (1), we propose a four-step process to reach it and for Makino-type (2), the appropriate variable of velocity is \( u_i \) instead of \( w^j \).

In §5, we complete the proof of Theorem 1.13 by using the results from §3 to §4 to verify that all the conditions from the model equation (A.1) in Appendix A hold for the formulation (5.2) and (5.24) of the conformal Einstein-Euler equations. This allows us to apply Theorem A.1 to obtain the desired conclusion.

In the Appendix A, The slightly revised version of Theorem of singular hyperbolic equations has been introduced and pointed how to revise the proof due to the revision of the statement.

## 2. Examples of Makino-type fluids

Before proceeding to the examples of Makino-type fluids, let us first derive some general properties of the background solutions which will play the role as the center of perturbations.

2.1. **Analysis of FLRW spacetimes.** Einstein equations for the FLRW solutions reduce to the Friedman equations

\[
-3\omega^2 - \Omega + \left( \frac{\dot{\rho} - \ddot{\rho}}{2} + \Lambda \right) = 0
\]

and

\[
-6\Omega - 6\omega^2 + 2\Lambda - (\dot{\rho} + 3\ddot{\rho}) = 0
\]

where we recall \( \Omega(\tau) = -\tau \omega \partial_\tau \) is defined by (1.30). The background solution verifies the Euler equations that

\[
\partial_0 \dot{\rho} = \frac{3}{\tau} (\dot{\rho} + \ddot{\rho}).
\]
In fact, equations (2.1), (2.2) and (2.3) form the Einstein-Euler system for the background FLRW solutions.

Let us estimate several crucial quantities which characterize the asymptotic behaviors of the background. First note that (2.1) and (2.2) yield
\[ \omega^2 = \frac{1}{3}(\Lambda + \bar{\rho}) \quad \text{and} \quad \Omega = -\frac{\bar{\rho} + \bar{\rho}}{2}. \] (2.4)

Due to the fact that \( p = p(\rho) = c_s^2(K_0\rho)\rho \) where \( K_0 \in (0, 1) \), with the help of (1.17), we derive
\[ 0 \leq \bar{\rho} \leq \frac{1}{3} \bar{\rho}. \] (2.5)

Then integrating (2.3), with the help of (2.5) and (2.4), yields
\[ \tau^4 \bar{\rho}(1) \leq \bar{\rho}(\tau) \leq \tau^3 \bar{\rho}(1), \] (2.6)
\[ \frac{1}{3} \bar{\rho}(1) \tau^4 \leq \omega^2 - \frac{\Lambda}{3} \leq \frac{1}{3} \bar{\rho}(1) \tau^3, \] (2.7)
\[ -\frac{2}{3} \tau^3 \bar{\rho}(1) \leq \Omega \leq -\frac{1}{2} \tau^4 \bar{\rho}(1) \] (2.8)
and
\[ 3 \tau^3 \bar{\rho}(1) \leq \partial_\tau \bar{\rho} \leq 4 \tau^2 \bar{\rho}(1). \] (2.9)

Moreover, recalling \( \psi(\tau) \) in (1.36), let us estimate its time derivative which will be used in the derivations of reduced conformal Einstein equations. By (2.4), we arrive at
\[ \partial_\tau \psi(\tau) = \partial_\tau \left(3 \omega^2 + \frac{3}{2} \Omega\right) = \partial_\tau \bar{\rho} - \frac{3}{4} \partial_\tau(\bar{\rho} + \bar{p}) = \frac{1}{4} \partial_\tau \bar{\rho} - \frac{3}{4} \partial_\tau \bar{p}. \] (2.10)

Note that
\[ \partial_\tau \bar{p} = c_s^2 \partial_\tau \bar{\rho} \in [0, \frac{1}{3} \partial_\tau \bar{\rho}]. \] (2.11)

Gather (2.9)–(2.11) together, we arrive at
\[ 0 \leq \partial_\tau \psi(\tau) \leq \tau^2 \bar{\rho}(1). \] (2.12)

In addition, direct calculations show that
\[ \partial^2_\tau \omega = \frac{1}{\tau^2 \omega} \frac{\bar{\rho} + \bar{p}}{2} \left(\frac{\Omega}{\omega^2} - 1\right) + \frac{1}{\tau^2 \omega} \frac{\partial_\tau \bar{\rho} + \partial_\tau \bar{p}}{2} \] (2.13)
with the help of (2.6)–(2.12), (2.13) implies that \( \omega \in C^2([0, 1]) \).

2.2. Examples of Makino-type fluids. In this section, we give three examples of Makino-type fluids, which demonstrates that the set of Makino-type fluids is non-empty.

2.2.1. The polytropic gas. Suppose the equation of state of the polytropic gas is given by
\[ p = K\rho^{\frac{n+1}{n}} \] (2.14)
for \( n \in (1, 3) \). We also assume the initial homogeneous, isotropic density is bounded by
\[ 0 < \bar{\rho}(1) \leq \frac{1}{(4Kn(n+1))^n} \left(2n \sqrt{\frac{1}{3} \left(1 - \frac{3}{2n}\right)}\right)^{2n}. \] (2.15)
We introduce the standard relationship between $\rho$ and $\alpha$ to define Makino density $\alpha$ which are (for details, see, for example, [26, 32])

$$\rho = \mu(\alpha) = \frac{1}{(4Kn(n+1))^n} \alpha^{2n}$$

(2.16)

and

$$\lambda = \lambda(\alpha) = \left(1 + \frac{1}{4n(n+1)} \alpha^2\right)^{-1}.$$  

(2.17)

Direct calculations imply that

$$p = \frac{K}{(4Kn(n+1))^{n+1}} \alpha^{2(n+1)}$$

and

$$s = \frac{\alpha}{2n}.$$  

(2.18)

Furthermore,

$$\frac{d\mu(\alpha)}{d\alpha} = \frac{2n}{(4Kn(n+1))^{n-1}} \alpha^{2n-1} = \frac{\lambda(\alpha)(\rho + p)}{s(\alpha)},$$

which verifies Assumption 1.3.

In order to verify Assumption 1.4 and 1.5, we first calculate the background density $\bar{\alpha}$. Integrating (2.3) yield

$$\int_{\bar{\rho}(1)}^{\hat{\rho}(\tau)} \frac{d\xi}{\xi + \rho(\xi)} = \ln \tau^3.$$  

(2.19)

Combining equation of state (2.14), transformation (2.16) and (2.19) yields

$$2n \int_{\bar{\rho}(1)}^{\hat{\rho}(\tau)} \frac{dy}{y(1 + \frac{1}{4n(n+1)} y^2)} = \ln \tau^3$$

which, in turn, implies

$$\hat{\alpha}(\tau) = \tau^{\frac{3}{2n}} Q(\tau) \in [0, \bar{\alpha}(1)]$$  

(2.20)

where

$$Q(\tau) = \left(\frac{1}{\alpha^2(1)} + \frac{1}{4n(n+1)} - \frac{1}{4n(n+1) \tau^{\frac{3}{2n}}}\right)^{\frac{1}{2}}$$

for $\tau \in [0, 1]$. Now let us continue to verify Assumptions 1.4–1.5.

For every $n \in (1, 3)$ there exists a $\varepsilon \in (0, 1)$ such that $n \in (\frac{3}{2n} - \varepsilon, 3 - \varepsilon]$, we take $\zeta = 3 - \varepsilon$, $\beta(\tau) = (C^* \hat{\delta})^{-1} \tau^{(3-\varepsilon)/(2n)} \in C[0,1] \cap C^1(0,1)$ (where, we recall, $C^*$ and $\hat{\delta}$ are defined in Assumption 1.5.(1) and 1.3, respectively) and

$$\varrho(\tau, y) = \frac{(C^* \hat{\delta})^{-2n}}{(4Kn(n+1))^{n}} \left((y + (C^* \hat{\delta}) \tau^{-\frac{3-\varepsilon}{2n}} \bar{\alpha})^2 - ((C^* \hat{\delta}) \tau^{-\frac{3-\varepsilon}{2n}} \bar{\alpha})^2\right),$$

(2.21)

then $\varrho \in C([0,1], C^\infty(\mathbb{R}))$, $\varrho(\tau, 0) = 0$, $\tau^{(3-\varepsilon)/(2n)} \gtrsim \tau$ and $\chi(\tau) = (3 - \varepsilon)/(2n)$. Moreover, transformation (2.21) yields

$$\mu(\alpha) - \mu(\bar{\alpha}) = \tau^{3-\varepsilon} [\mu(\tau^{-(3-\varepsilon)/(2n)} \alpha) - \mu(\tau^{-(3-\varepsilon)/(2n)} \bar{\alpha})]$$

$$= \tau^{3-\varepsilon} \varrho(\tau, (C^* \hat{\delta}) \tau^{-(3-\varepsilon)/(2n)}(\alpha - \bar{\alpha})),$$

which verifies Assumption 1.4.
Equations (2.17), (2.18) and (2.20) lead to
\[
\tilde{\lambda} = \lambda(\tilde{\alpha}) = \left(1 + \frac{1}{4n(n+1)}\tilde{\alpha}^2\right)^{-1} \quad \text{and} \quad \tilde{s} = s(\tilde{\alpha}) = \frac{\tilde{\alpha}}{2n}.
\]
It is clear that \(\tilde{s} \lesssim \beta(\tilde{\tau})\) by (2.20). Noting that \(n \in \left(\frac{3-\varepsilon}{2}, 3-\varepsilon\right]\) leads to \(\tau^{\frac{3}{4n}} < \tau^{\frac{\varepsilon}{2n}} \lesssim \tau^{\frac{\varepsilon}{2n}} \lesssim 1\) and \(\tau^{1+\frac{\varepsilon}{n}} \lesssim \tau^{\frac{3}{4n}} \lesssim 1\), we calculate the quantity
\[
\lambda'(\tilde{\alpha}) \partial_{\tau} \beta(\tau) = -\frac{1}{2n(n+1)}\left(\frac{3-\varepsilon}{2n}\right)Q(\tau)\left(1 + \frac{1}{4n(n+1)}\tilde{\alpha}^2\right)^{-2}(C^*\delta)^{-1} \lesssim 1
\]
and
\[
\frac{\tilde{s}}{\tau} \lambda'(\tilde{\alpha}) = -\frac{1}{2n(n+1)}\left(\frac{3-\varepsilon}{2n}\right)Q(\tau)^2\left(1 + \frac{1}{4n(n+1)}\tilde{\alpha}^2\right)^{-2} \lesssim 1.
\]
Then, calculate quantity
\[
\frac{1}{3} \chi(\tau) + \frac{\varepsilon}{6n} = \frac{1}{2n} \leq q'(\tilde{\alpha}) = \frac{1}{2n} + \frac{3\tilde{\alpha}^2}{8n^2(n+1)} \leq \frac{1}{3} \chi(\tau) + \frac{\varepsilon}{6n} + \frac{3\tilde{\alpha}^2(1)}{8n^2(n+1)},
\]
and noting \(n \in \left(\frac{3-\varepsilon}{2}, 3-\varepsilon\right]\) implies \(1 - \frac{3-\varepsilon}{2n} > 0\), then (2.15) implies
\[
0 < \tilde{s}(1) \leq 2n \sqrt{\frac{1}{3} \left(1 - \frac{3}{2n}\right)},
\]
which, in turn, yields
\[
1 - 3\tilde{s}^2 - \chi(\tau) = 1 - \frac{3\tilde{\alpha}^2}{4n^2} - \frac{3 - \varepsilon}{2n} \geq 1 - \frac{3\tilde{\alpha}^2(1)}{4n^2} - \frac{3 - \varepsilon}{2n} \geq \frac{3}{2n} - \frac{3 - \varepsilon}{2n} = \frac{\varepsilon}{2n}.
\]
It is evident that (1.13) holds and then, Assumption 1.5 can be concluded. We have verified all the Assumptions 1.3–1.5 now and this means this type of polytropic fluids is Makino-type.

2.2.2. Fluids with the linear equation of state. Suppose the equation of state of this fluid is
\[
p = K \rho \quad (2.22)
\]
for \(K \in \left(0, \frac{1}{3}\right]\). Using (2.22) and (2.3) arrive at
\[
\rho = \tilde{\rho}(1)\tau^{3(1+K)}.
\]
Let the set \(\{\lambda(\alpha), \varrho, \varsigma, \beta(\tau)\}\) be
\[
\alpha = \mu^{-1}(\rho) = \int_{\rho(1)}^{\rho} \frac{d\xi}{\xi + p(\xi)} = \frac{1}{1 + K} \ln \frac{\rho}{\rho(1)}, \quad \lambda(\alpha) \equiv \sqrt{K}, \quad \beta(\tau) \equiv 1,
\]
\[
\varsigma = 3(1 + K) \quad \text{and} \quad \varrho(\tau, y) = \tilde{\rho}(1)[e^{(1+K)y} - 1].
\]
Then \(\varrho \in C([0, 1], C^\infty(\mathbb{R}))\) and \(\varrho(\tau, 0) = 0\). It is easy to calculate that
\[
\tilde{\alpha} = \mu^{-1}(\tilde{\rho}) = \int_{\tilde{\rho}(1)}^{\tilde{\rho}} \frac{d\xi}{\xi + p(\xi)} = \frac{1}{1 + K} \ln \frac{\tilde{\rho}}{\tilde{\rho}(1)} = \ln \tau^3,
\]
\[
s(\alpha) \equiv \lambda(\alpha) \equiv \sqrt{K}, \quad \text{and} \quad \mu(\alpha) = \tilde{\rho}(1)e^{(1+K)\alpha},
\]
Direct calculations, with the help of \(\tilde{s} = s(\tilde{\alpha}) = \tilde{\lambda} = \lambda(\tilde{\alpha}) = \sqrt{K}\), show that \(\tilde{s} \lesssim \beta(\tau)\), and
\[
\frac{d\mu(\alpha)}{d\alpha} = (1 + K)\mu(\alpha) = (1 + K)\tilde{\rho}(1)e^{(1+K)\alpha} = \frac{\lambda(\alpha)(\rho + p)}{s(\alpha)},
\]
\[
\mu(\alpha) - \mu(\tilde{\alpha}) = \tau^{3(1+K)}(\tilde{\rho}(1)e^{(1+K)(\alpha - \tilde{\alpha})} - \tilde{\rho}(1)) = \tau^{3(1+K)}\varrho(\tau, \alpha - \tilde{\alpha}),
\]
and
\[ \lambda'(\alpha) \partial_\tau \beta(\tau) \equiv 0, \quad \bar{\lambda}'(\bar{\alpha}) \equiv 0, \]
which is enough to conclude Assumptions 1.3, 1.4, (1.12) in Assumption 1.5.

Note that \( q \equiv 1 \), and
\[ 1 - 3s^2 = 1 - 3K = \begin{cases} 0 & \text{if } K = \frac{1}{3}, \\ 1 - 3K > 0 & \text{if } 0 < K < \frac{1}{3}, \end{cases} \]
which verifies Assumption 1.5. Therefore, we conclude that this fluid is Makino-type.

2.2.3. Chaplygin gases. Suppose the equation of state of Chaplygin gases is expressed by
\[ p = -\frac{\Lambda^{1+\vartheta}}{(\rho + \Lambda)\vartheta} + \Lambda \]
for \( \vartheta \in (0, \sqrt{\frac{3}{2}}) \). It is easy to check that \( p(0) = 0 \). Equation (2.19), with the help of (2.23), leads to
\[ (\bar{\rho} + \Lambda)^{\vartheta+1} - \Lambda^{\vartheta+1} = \tau^{3(1+\vartheta)}[ (\bar{\rho}(1) + \Lambda)^{\vartheta+1} - \Lambda^{\vartheta+1}]. \]

We take \( \{\lambda(\alpha), \varrho, \varsigma, \beta(\tau)\} \) as
\[ \alpha = \mu^{-1}(\rho) = \int_{\rho(1)}^{\rho} \frac{d\xi}{\xi + \rho(\xi)} = \frac{1}{\vartheta + 1} \ln \frac{(\rho + \Lambda)^{\vartheta+1} - \Lambda^{\vartheta+1}}{(\bar{\rho}(1) + \Lambda)^{\vartheta+1} - \Lambda^{\vartheta+1}}, \]
\[ \lambda(\alpha) = \frac{\Lambda^{\vartheta+1} + [(\bar{\rho}(1) + \Lambda)^{\vartheta+1} - \Lambda^{\vartheta+1}]e^{(\vartheta+1)\alpha}}{\vartheta+1}, \]
\[ \beta(\tau) = 1, \quad \varsigma = 3(1+\vartheta) \quad \text{and} \quad \varrho(\tau, y) = \tau^{-3(1+\vartheta)}(\mu(y + \alpha) - \mu(\bar{\alpha})). \]

It is easy to calculate that
\[ \bar{\alpha} = \mu^{-1}(\bar{\rho}) = \frac{1}{\vartheta + 1} \ln \frac{(\bar{\rho} + \Lambda)^{\vartheta+1} - \Lambda^{\vartheta+1}}{(\bar{\rho}(1) + \Lambda)^{\vartheta+1} - \Lambda^{\vartheta+1}} = \ln \tau^3, \]
\[ \mu(\alpha) = \Lambda \left\{ 1 + \left[ (1 + \frac{\bar{\rho}(1)}{\Lambda})^{\vartheta+1} - 1 \right] e^{(\vartheta+1)\alpha} \right\}^{\frac{1}{\vartheta+1}}, \]
\[ \mu(\bar{\alpha}) = \Lambda \left\{ 1 + \left[ (1 + \frac{\bar{\rho}(1)}{\Lambda})^{\vartheta+1} - 1 \right] e^{(\vartheta+1)} \right\}^{\frac{1}{\vartheta+1}} = \Lambda, \]
and
\[ s(\alpha) = \lambda(\alpha) = \sqrt{\rho'(\rho)} = \frac{\vartheta \Lambda^{1+\vartheta}}{\Lambda^{\vartheta+1} + [(\bar{\rho}(1) + \Lambda)^{\vartheta+1} - \Lambda^{\vartheta+1}]e^{(\vartheta+1)\alpha}} \]
for \( |\alpha| \leq K_1 \) (recall the fact that \( \alpha \in I_\alpha \) and \( I_\alpha \) is compact). Then, (2.28), (2.29) and (2.26), with the help of mean value theorem, implies
\[ \varrho(\tau, \alpha - \bar{\alpha}) = \tau^{-3(1+\vartheta)}(\mu(\alpha) - \mu(\bar{\alpha})) \]
\[ = e^{-3\vartheta+1}\Lambda \left[ (1 + K_3 e^{(\vartheta+1)\alpha})^{\frac{1}{\vartheta+1}} - (1 + K_3 e^{(\vartheta+1)\bar{\alpha}})^{\frac{1}{\vartheta+1}} \right] \]
\[ = \Lambda(1 + K_3 \tau^{3(1+\vartheta)} e^{(\vartheta+1)K_2(\alpha - \bar{\alpha})})^{\frac{1}{\vartheta+1}} - \Lambda(1 + K_3 \tau^{3(1+\vartheta)} e^{(\vartheta+1)K_2(\alpha - \bar{\alpha})})^{\frac{1}{\vartheta+1}} \]
for some $K_2 \in (0, 1)$, where $K_3 = [(1 + \frac{\dot{\rho}(1)}{\Lambda})^{\vartheta + 1} - 1]$. Therefore, $\varrho \in C([0, 1], C^\omega(\mathbb{R}))$ and $\varrho(\tau, 0) = 0$.

Direct calculations using (2.19), (2.26), with the help of (by (2.27) and (2.30))

$$s = s(\bar{\alpha}) = \bar{\lambda} = \lambda(\bar{\alpha}) = \frac{\partial \Lambda^{1+\vartheta}}{\Lambda^{\vartheta+1} + \tau^{3(1+\vartheta)}[(\dot{\rho}(1) + \Lambda)^{\vartheta+1} - \Lambda_{\vartheta+1}]} < \vartheta \lesssim \beta$$

and

$$\lambda'(\bar{\alpha}) = \left(1 + \vartheta\right) \frac{\partial \Lambda^{1+\vartheta}}{\Lambda^{\vartheta+1} + \tau^{3(1+\vartheta)}[(\dot{\rho}(1) + \Lambda)^{\vartheta+1} - \Lambda_{\vartheta+1}]}$$

show that

$$d\mu(\alpha) = \frac{\lambda(\alpha)(\rho + p)}{s(\alpha)} = \rho + p,$$

$$\mu(\alpha) - \mu(\bar{\alpha}) = \tau^{3(1+\vartheta)} \varrho(\tau, \alpha - \bar{\alpha}),$$

and $\lambda'(\bar{\alpha}) \partial_{\tau, \beta}(\tau) \equiv 0$,

$$\frac{\bar{s}}{\tau} \lambda'(\bar{\alpha}) = \left(1 + \vartheta\right) \frac{\partial \Lambda^{1+\vartheta}}{\Lambda^{\vartheta+1} + \tau^{3(1+\vartheta)}[(\dot{\rho}(1) + \Lambda)^{\vartheta+1} - \Lambda_{\vartheta+1}]} \lesssim 1,$$

which is enough to conclude Assumptions 1.3, 1.4, (1.12) in Assumption 1.5, and note that $q \equiv 1$, and because $0 < \vartheta < \sqrt{\frac{4}{7}}$,

$$1 - 3s^2 = 1 - 3 \left(\frac{\partial \Lambda^{1+\vartheta}}{\Lambda^{\vartheta+1} + \tau^{3(1+\vartheta)}[(\dot{\rho}(1) + \Lambda)^{\vartheta+1} - \Lambda_{\vartheta+1}]}\right)^2 \geq 1 - 3\vartheta^2 > 0$$

for $\tau \in [0, 1]$, which verifies Assumption 1.5. Therefore, we are able to conclude that the generalized Chaplygin gases are Makino-type.

3. SINGULAR HYPERBOLIC FORMULATIONS OF THE CONFORMAL EINSTEIN EQUATIONS

This section contributes to derive the singular hyperbolic formulations of the conformal Einstein equations using the method initiated in [30], and applied in [16, 20, 21]. Readers who are familiar with this approach can quickly browse but pay more attention to the next section on conformal Euler equations which involve main difficulties and contribute to the main innovations of this article.

3.1. The reduced conformal Einstein equations. With the help of the wave coordinates $Z^\mu$ defined by (1.35), we transform conformal Einstein equation to the gauge reduced one,

$$-2R_{\mu\nu} + 2\nabla(\mu)Z^\nu + A_{\kappa}^{\mu\nu}Z^\kappa = -4\nabla\mu\nabla\nu\Phi + 4\nabla\mu\Phi\nabla\nu\Phi - 2\left[\Box_g \Phi + 2|\nabla\Phi|^2_g + \frac{(\rho - p)}{2} + \Lambda e^{2\Phi}\right] g^{\mu\nu} - 2e^{2\Phi}(\rho + p)u^\mu u^\nu, \quad (3.1)$$

where

$$A_{\kappa}^{\mu\nu} = -X^{(\mu}\delta^{\nu)} + Y^{(\mu; \delta^{\nu)}.\quad (3.2)$$

Direct calculations yield

$$2\nabla(\mu)Z^\nu = 2\nabla(\mu\Gamma^\nu) + \partial_\tau \left(\frac{2\psi(\tau)}{3\tau}\right) (g^{\mu\nu} \delta^\nu_0 + g^{\mu\nu} \delta^\nu_0 - \frac{2\psi(\tau)}{3\tau} \partial_\tau g^{\mu\nu} - 4\nabla\mu\nabla\nu\Phi) \quad (3.2)$$
and

\[ A^\mu_\nu Z^\kappa = -\Gamma^{(\mu \Gamma \nu)} + 4 \nabla^\mu \Phi \nabla^\nu \Phi - \frac{4\psi(\tau)}{3\tau} (\nabla^\mu \Phi \delta^\nu_0 + \nabla^\nu \Phi \delta^\mu_0) + \frac{4\Lambda^2(\tau)}{9\tau^2} \delta^\mu_0 \delta^\nu_0. \]  

(3.3)

Reduced Einstein equations (3.1) and (3.2)–(3.3) lead to

\[ -2R^\mu_\nu + 2\nabla(\nu \Gamma \nu) - \Gamma^{\mu \nu} = \frac{2\psi(\tau)}{3\tau} \partial_\rho g^{\mu \nu} - \frac{2\partial_\rho \psi(\tau)}{3\tau} (g^{\rho \nu} \delta^\mu_0 + g^{\rho \mu} \delta^\nu_0) \]

\[ - \frac{4\psi(\tau)}{3\tau^2} \left( g^{00} + \frac{\psi(\tau)}{3} \right) \delta^\mu_0 \delta^\nu_0 - \frac{4\psi(\tau)}{3\tau^2} g^{0i} \delta^\mu_i \delta^\nu_0 - \frac{2}{\tau^2} g^\mu_\nu \left( g^{00} + \frac{\psi(\tau)}{3} \right) \]

\[ + \frac{2}{\tau^2} \left( \psi(\tau) - \frac{\rho - \bar{\rho}}{2} \right) g^{\mu \nu} - 2e^{2\Phi}(\rho + p)u^\mu u^\nu. \]  

(3.4)

Recalling the formula (e.g., see [11, 39])

\[ R^{\mu \nu} = \frac{1}{2} g^{\lambda \sigma} \partial_\lambda \partial_\sigma g^{\mu \nu} + \nabla(\mu \Gamma \nu) + \frac{1}{2}(Q^{\mu \nu} - X^\mu X^\nu), \]  

(3.5)

where

\[ Q^{\mu \nu}(g, \partial g) = g^{\lambda \sigma} \partial_\lambda (g^{\alpha \mu} g^{\nu \sigma}) \delta_\sigma \delta_\rho + 2 g^{\alpha \mu} \Gamma^{\eta}_{\lambda \alpha} g^{\eta \beta} g^{\lambda \nu} \Gamma^{\delta}_{\rho \gamma} \]

\[ + 4 \Gamma^{\lambda}_{\delta \eta} g^{\delta \chi} \Gamma^{\chi}_{\rho \gamma} g^{\eta \mu} g^{\nu \sigma} + (\Gamma^{\mu} - \bar{\gamma}^{\mu}) (\bar{\gamma}^{\nu} - \bar{\gamma}^{\nu}), \]  

(3.6)

and inserting (3.5) and (3.6) into (3.4), with the help of (1.36), that is, \( \frac{\psi(\tau)}{3} = \omega^2 + \frac{\Omega}{2} \), yields

\[ -g^{\kappa \lambda} \partial_\kappa \partial_\lambda (g^{\mu \nu} - \eta^{\mu \nu}) = \frac{2\omega^2}{\tau} \partial_\tau g^{\mu \nu} - \frac{4\omega^2}{\tau^2} (g^{00} + \omega^2) \delta^\mu_0 \delta^\nu_0 - \frac{4\omega^2}{\tau^2} g^{0i} \delta^\mu_i \delta^\nu_0 \]

\[ - \frac{2}{\tau^2} g^{\mu \nu} (g^{00} + \omega^2) - \frac{2\Omega}{\tau^2} (g^{00} + \omega^2 + \frac{\Omega}{2}) \delta^\mu_0 \delta^\nu_0 - \frac{\Omega}{\tau} \partial_\tau g^{\mu \nu} \]

\[ - \frac{2}{\tau^2} \omega^2 \Omega \delta^\mu_0 \delta^\nu_0 - \frac{2\Omega}{\tau^2} g^{0i} \delta^\mu_i \delta^\nu_0 - \frac{\Omega}{\tau^2} g^{\mu \nu} - \frac{2}{\tau^2} \psi(\tau) (g^{00} \delta^\nu_0 + g^{00} \delta^\mu_0) \]

\[ + \frac{2}{\tau^2} \left( 3\omega^2 + \Omega - \frac{\rho - \bar{\rho}}{2} \right) g^{\mu \nu} \]

\[ - 2e^{2\Phi}(\rho + p)u^\mu u^\nu + Q^{\mu \nu}(g, \partial g), \]

which is equivalent to

\[ -g^{\kappa \lambda} \partial_\kappa \partial_\lambda (g^{\mu \nu} - \eta^{\mu \nu}) = \frac{2\omega^2}{\tau} \partial_\tau (g^{\mu \nu} - \eta^{\mu \nu}) - \frac{4\omega^2}{\tau^2} (g^{00} + \omega^2) \delta^\mu_0 \delta^\nu_0 \]

\[ - \frac{4\omega^2}{\tau^2} g^{0i} \delta^\mu_i \delta^\nu_0 - \frac{2}{\tau^2} g^{\mu \nu} (g^{00} + \omega^2) + \Phi^{\mu \nu}, \]  

(3.7)

where

\[ \Phi^{\mu \nu} = (g^{\kappa \lambda} - \eta^{\kappa \lambda}) \partial_\kappa \partial_\lambda \eta^{\mu \nu} - \frac{2\Omega}{\tau^2} (g^{00} + \omega^2) \delta^\mu_0 \delta^\nu_0 - \frac{\Omega}{\tau} \partial_\tau (g^{\mu \nu} - \eta^{\mu \nu}) - \frac{2\Omega}{\tau^2} g^{0i} \delta^\mu_i \]

\[ - \frac{\Omega}{\tau^2} (g^{\mu \nu} - \eta^{\mu \nu}) - \frac{2\partial_\tau \psi(\tau)}{3\tau} \left( (g^{00} - \eta^{00}) \delta^\nu_0 + (g^{00} - \eta^{00}) \delta^\mu_0 \right) \]

\[ - \frac{1}{\tau^2} (\rho - \bar{\rho} - p + \bar{p}) (g^{\mu \nu} - \eta^{\mu \nu}) - \frac{2}{\tau^2} \left( \rho - \bar{\rho} - \frac{p - \bar{p}}{2} \right) \eta^{\mu \nu} \]

\[ - \frac{2}{\tau^2} \left[ (\rho - \bar{\rho} - p + \bar{p}) u^\mu u^\nu + (\bar{p} + p) (u^\mu u^\nu - \bar{u}^\mu \bar{u}^\nu) \right] \]

\[ + Q^{\mu \nu}(g, \partial g) - Q^{\mu \nu}(\eta, \partial \eta), \]
and we have applied the identity $3\omega^2 - \Lambda + \Omega - \frac{\rho - p}{2} = -\frac{1}{2}(\rho - \bar{p} - p + \bar{p})$ due to (2.4).

**Remark 3.1.** Note that, by (3.6), $Q^{\mu\nu}(g, \partial g)$ are quadratic in $\partial g = (\partial g^{\mu\nu})$ and analytical in $g = (g^{\mu\nu})$.

**Remark 3.2.** Note that terms with $\frac{2\partial \psi(\tau)}{\partial \tau}$ is regular in $\tau$ due to the estimate (2.12).

Definitions (1.38)–(1.40) imply

$$g^{0\mu} - \eta^{0\mu} = 2\tau u^{0\mu}, \quad \partial_\tau(g^{0\mu} - \eta^{0\mu}) = u^{0\mu}_\tau \quad \text{and} \quad \partial_\tau(g^{0\mu} - \eta^{0\mu}) = u^{0\mu}_\tau + 3u^{0\mu}. \tag{3.8}$$

Then, by letting $\nu = 0$, with the help of (3.8), the equation (3.7) becomes

$$-g^{00}\partial_\tau u^{0\mu}_0 - 2g^{0i}\partial_i u^{0\mu}_0 - g^{ij}\partial_j u^{0\mu}_i = \frac{1}{\tau} \left[ -\frac{g^{00}}{2}(u^{0\mu}_0 + u^{0\mu}) \right] + 6u^{0i}u^{0\mu}_i + 4u^{00}u^{0\mu}_0 - 4u^{00}u^{0\mu} + \mathcal{F}^{0\mu}. \tag{3.9}$$

Differentiating (1.40), which is the definition of $u^{0\mu}_j$, implies

$$\partial_\tau u^{0\mu}_j = \partial_\tau(g^{0\mu} - \eta^{0\mu}) = \partial_j(u^{0\mu}_0 + 3u^{0\mu}) = \partial_j u^{0\mu}_0 + \frac{3}{2\tau}u^{0\mu}_j,$$

namely

$$\partial_\tau u^{0\mu}_j - \partial_j u^{0\mu}_0 = \frac{3}{2\tau}u^{0\mu}_j. \tag{3.10}$$

Then, differentiating (1.38) yields

$$\partial_\tau u^{0\mu} = \partial_\tau \left( \frac{g^{0\mu} - \eta^{0\mu}}{2\tau} \right) = \frac{u^{0\mu}_0 + 3u^{0\mu}_\tau}{2\tau} - \frac{g^{0\mu} - \eta^{0\mu}}{2\tau^2} = \frac{1}{2\tau}(u^{0\mu}_0 + u^{0\mu}). \tag{3.10}$$

For the spatial components, a more delicate transformation is applied to the $\mu = i, \nu = j$ components of (3.7) in order to rewrite those equations into the desired singular hyperbolic form. The first step is to contract $(\mu, \nu) = (i, j)$ components of (3.7) with $\tilde{g}_{ij}$, where we recall that $(\tilde{g}_{pq}) = (g^{pq})^{-1}$. A straightforward calculation, using the identity $\tilde{g}_{pq}\partial_p g^{pq} = \det(g^{pq})\partial_\mu \det(g^{pq})^{-1}$ and (3.7) with $(\mu, \nu) = (0, 0)$, recalling the definition of $q$ in (1.37), which is

$$q = g^{00} - \eta^{00} + \frac{\eta^{00}}{3} \ln(\det(g^{ij})),$$

leads to

$$\partial_\lambda q = \partial_\lambda(g^{00} + \omega^2) - \frac{\omega^2}{3}\tilde{g}_{pq}\partial_\lambda g^{pq} - \frac{2\omega \delta^0_3 \partial_\tau \omega}{3} \ln(\det(g^{pq})). \tag{3.11}$$

Then differentiating (3.11), we arrive at

$$\partial_\mu \partial_\lambda q = \partial_\mu \partial_\lambda(g^{00} + \omega^2) - \frac{\omega^2}{3}\tilde{g}_{pq}\partial_\mu \partial_\lambda g^{pq} + \bar{\mathcal{F}}_{\kappa\lambda}, \tag{3.12}$$

where

$$\bar{\mathcal{F}}_{\kappa\lambda} = -\frac{\omega^2}{3}\partial_\kappa \tilde{g}_{pq}\partial_\lambda g^{pq} - 2\omega \partial_\kappa \omega \delta^0_3 \tilde{g}_{pq}\partial_\lambda g^{pq} - \frac{2(\partial_\tau \omega)^2}{3}\delta^0_3 \delta^0_3 \ln(\det(g^{pq}))$$

$$- \frac{2\omega \partial_\tau \omega}{3} \delta^0_3 \delta^0_3 \ln(\det(g^{pq})) - \frac{2\omega \partial_\tau \omega}{3} \delta^0_3 \tilde{g}_{pq} \partial_\kappa g^{pq}.$$
\[
\frac{2\omega^2}{\tau} \partial_\tau (g^{00} + \omega^2) - \frac{4\omega^2}{\tau^2} (g^{00} + \omega^2) - \frac{2}{\tau^2} g^{00}(g^{00} + \omega^2) + \tilde{\mathcal{H}}^{00} \\
- \frac{\omega^2}{3} \tilde{g}_{pq} \left( \frac{2\omega^2}{\tau} \partial_\tau g^{pq} - \frac{2}{\tau^2} g^{pq}(g^{00} + \omega^2) + \tilde{\mathcal{H}}^{pq} \right) - g^{\kappa\lambda} \tilde{\mathcal{H}}_{\kappa\lambda} \\
= - \frac{2}{\tau} g^{00} \partial_\tau q + 4u^{00} \partial_\tau q - 8(u^{00})^2 + \tilde{\mathcal{H}}, \quad (3.13)
\]

where
\[
\tilde{\mathcal{H}} = \tilde{\mathcal{H}}^{00} - \frac{\omega^2}{3} \tilde{g}_{pq} \tilde{\mathcal{H}}^{pq} + \frac{4\omega^2 \partial_\tau \omega}{3\tau} \ln(\det(g^{pq})) - g^{\kappa\lambda} \tilde{\mathcal{H}}_{\kappa\lambda}.
\]

Furthermore, by (1.44), the equation (3.13) is equivalent to
\[
-g^{00} \partial_\tau u_0 - 2g^{0i} \partial_i u_0 - g^{ij} \partial_j u_j = -\frac{2}{\tau} g^{00} u_0 + 4u^{00} u_0 - 8(u^{00})^2 + \tilde{\mathcal{H}}, \quad (3.14)
\]

Using (1.43)–(1.44) and differentiating them imply that
\[
\partial_\tau u_j = \partial_\tau \partial_j q = \partial_j \partial_\tau q = \partial_j u_0, \quad (3.15)
\]

and
\[
\partial_\tau u = \partial_\tau q = u_0. \quad (3.16)
\]

Introduce an operator
\[
L_{ij}^{lm} = \delta_i^j \delta_m^l - \frac{1}{3} \tilde{g}_{lm} g^{ij}. \quad (3.17)
\]

Direct calculations yield
\[
\partial_\mu g^{ij} = (\det(\tilde{g}_{pq}))^{\frac{1}{2}} L_{im}^{ij} \partial_\mu g^{lm} \quad \text{and} \quad L_{im}^{ij} g^{lm} = 0. \quad (3.18)
\]

Let us turn to \( g^{ij} - \delta^{ij} \). Applying \((\det(\tilde{g}_{pq}))^{\frac{1}{2}} L_{im}^{ij}\) to (3.7) with \((\mu, \nu) = (l, m), \) with the help of (3.17)–(3.18), direct calculations give
\[
-g^{\kappa\lambda} \partial_\kappa \partial_\lambda g^{ij} = -g^{\kappa\lambda} \partial_\kappa [(\det(\tilde{g}_{pq}))^{\frac{1}{2}} L_{im}^{ij} \partial_\lambda g^{lm}] \\
= (\det(\tilde{g}_{pq}))^{\frac{1}{2}} L_{im}^{ij} (-g^{\kappa\lambda} \partial_\kappa \partial_\lambda g^{lm}) - g^{\kappa\lambda} \partial_\kappa [(\det(\tilde{g}_{pq}))^{\frac{1}{2}} L_{im}^{ij} \partial_\lambda g^{lm}] \\
= (\det(\tilde{g}_{pq}))^{\frac{1}{2}} L_{im}^{ij} \left( \frac{2\omega^2}{\tau} \partial_\tau (g^{lm} - \eta^{lm}) - \frac{2}{\tau^2} g^{lm}(g^{00} + \omega^2) + \tilde{\mathcal{H}}^{lm} \right) \\
- g^{\kappa\lambda} \partial_\kappa [(\det(\tilde{g}_{pq}))^{\frac{1}{2}} L_{im}^{ij} \partial_\lambda g^{lm}] \\
= \frac{2\omega^2}{\tau} \partial_\tau g^{ij} + \mathcal{R}^{ij}, \quad (3.19)
\]

where
\[
\mathcal{R}^{ij} = (\det(\tilde{g}_{pq}))^{\frac{1}{2}} L_{im}^{ij} \tilde{\mathcal{H}}^{lm} - g^{\kappa\lambda} \partial_\kappa [(\det(\tilde{g}_{pq}))^{\frac{1}{2}} L_{im}^{ij} \partial_\lambda g^{lm}].
\]

Then, inserting (1.41)–(1.42) into (3.19), we arrive at
\[
-g^{00} \partial_\tau u_0^{ij} - 2g^{0i} \partial_0 u_0^{ij} - g^{pq} \partial_p u_q^{ij} = -\frac{2}{\tau} g^{00} u_0^{ij} + 4u^{00} u_0^{ij} + \mathcal{R}^{ij}. \quad (3.20)
\]

Direct differentiating (1.42) and (1.41) also gives
\[
\partial_\tau u_j^{lm} = \partial_\tau \partial_j g^{lm} = \partial_j \partial_\tau g^{lm} = \partial_j u_0^{lm} \quad (3.21)
\]

and
\[
\partial_\tau u_j^{lm} = \partial_\tau (g^{lm} - \delta^{lm}) = \partial_j g^{lm} = u_0^{lm}. \quad (3.22)
\]
Collecting (3.9)–(3.10), (3.20)–(3.22) and (3.14)–(3.16) together, we transform the reduced conformal Einstein equations into the following singular symmetric hyperbolic formulation,

\[ A^\kappa \partial_\kappa \begin{pmatrix} u_0^0 \\ u_i^0 \\ u_0^\mu \\ u_i^\mu \\ u_0^m \\ u_i^m \end{pmatrix} = \frac{1}{\tau} AP^* \begin{pmatrix} u_0^0 \\ u_i^0 \\ u_0^\mu \\ u_i^\mu \\ u_0^m \\ u_i^m \end{pmatrix} + F_1, \]  

(3.23)

\[ A^\kappa \partial_\kappa \begin{pmatrix} u_0^m \\ u_i^m \end{pmatrix} = \frac{1}{\tau} (-2g^{00}) \Pi \begin{pmatrix} u_0^m \\ u_i^m \end{pmatrix} + F_2, \]  

(3.24)

and

\[ A^\kappa \partial_\kappa \begin{pmatrix} u_0 \\ u_j \end{pmatrix} = \frac{1}{\tau} (-2g^{00}) \Pi \begin{pmatrix} u_0 \\ u_j \end{pmatrix} + F_3, \]  

(3.25)

where

\[ A^0 = \begin{pmatrix} -g^{00} & 0 & 0 \\ 0 & g^{ij} & 0 \\ 0 & 0 & -g^{00} \end{pmatrix}, \quad A^k = \begin{pmatrix} -2g^{0k} & -g^{ik} & 0 \\ -g^{ik} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \]

\[ P^* = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \delta^j_i & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}, \quad A = \begin{pmatrix} -g^{00} & 0 & 0 \\ 0 & \frac{3}{2}g^{ij} & 0 \\ 0 & 0 & -g^{00} \end{pmatrix}, \]

\[ \Pi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad F_1 = \begin{pmatrix} 6u^0_i u_i^0 + 4u^{00} u_0^0 + 4u^{00} u_0^\mu + 4u^{00} u_i^\mu \\ 0 \\ 0 \end{pmatrix}, \]

and

\[ F_2 = \begin{pmatrix} 4u^{00} u_0^i + 2\mathcal{R}^{ij} \\ 0 \\ -g^{00} u_0^m \end{pmatrix}, \quad F_3 = \begin{pmatrix} 4u_0^{00} - 8(u^{00})^2 + \mathfrak{F} \\ 0 \\ -g^{00} u_0 \end{pmatrix}. \]

3.2. Regular remainder terms. In order to apply the singular symmetric hyperbolic system of Theorem A.1, we have to verify that \( \mathcal{R}^{ij}, \mathfrak{R}^{ij}, \mathfrak{F}^{ij}, \mathcal{F} \) belong to \( C^0([0, 1], C^\infty(V)) \). The similar examinations and the key calculations have appeared in [16, 20, 21]. We only state the main ideas and list the key results which are minor variations of the corresponding quantities in [20, 21] and the similar calculations apply.

Let us denote

\[ \mathcal{U} := (u_0^0, u_i^0, u_0^\mu, u_i^\mu, u_0^m, u_i^m, u_0, u_i, u)^T \quad \text{and} \quad \mathcal{V} := (\delta_i^j, v^i) \]  

(3.26)

and

\[ \mathcal{V} = \mathbb{R}^4 \times \mathbb{R}^{12} \times \mathbb{R}^4 \times S_3 \times (S_3)^3 \times S_3 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^3, \]  

(3.27)

First note

\[ \det(g^{ij}) = e^{\frac{3}{2}a(\tau, u^{00} - u)}, \]

(3.28)

then we have the following expansions

\[ g^{ij} = \delta^{ij} + u^{ij} + \frac{\eta^{ij}}{\omega^2 (2\tau u^{00} - u)} + S^{ij}(\tau, u, u^\mu), \]  

(3.29)

---

5Let \( \epsilon = v_T/c \equiv 1 \). For more details please refer to [20, 21].
\[ \partial_\lambda g^{ij} = u_\lambda^{ij} + \frac{\eta^{ij}}{\omega^2} (2u_0^{00} \delta^0_\lambda + 2\tau u_0^{00} \delta^j_\lambda + (u_0^{00} + u^{00}) \delta^0_\lambda - u_\lambda^{ij}) \\
+ \frac{2\Omega^j_\lambda \delta^0_\lambda (2\tau u_0^{00} - u) + S^j_\lambda (\tau, u, u^{\mu\nu}, u_\sigma^{\alpha\beta}, u_\sigma)}{\tau \omega^4}, \tag{3.30} \]

By Assumption 1.4, we have identity

\[ g_{\mu\nu} = \eta_{\mu\nu} + S_{\mu\nu}(\tau, u, u^{\mu\nu}), \tag{3.31} \]

\[ \dot{g}_{ij} = \delta_{ij} + S_{ij}(\tau, u, u^{\mu\nu}), \tag{3.32} \]

\[ g^{0\mu} = \eta^{0\mu} + 2\tau u_0^{0\mu}, \tag{3.33} \]

\[ \partial_\tau g^{0\mu} = u_0^{0\mu} + 3u_0^{\mu0} + 2\omega \partial_\omega \delta^\mu_0, \tag{3.34} \]

\[ \partial_\sigma g_{\mu\nu} = \partial_\sigma \eta_{\mu\nu} + S_{\mu\nu\sigma}(\tau, u^{\alpha\beta}, u, u_\gamma^{\alpha\beta}, u_\gamma), \tag{3.35} \]

\[ g_{00} = \eta_{00} + \tau S_{00}(\tau, U), \tag{3.36} \]

\[ g_{0i} = \tau S_{0i}(\tau, U), \tag{3.37} \]

\[ u_0 = \frac{1}{\sqrt{-\eta^{00} + \tau S(\tau, U) + \beta^2(\tau) W(\tau, U, V) + \tau \beta(\tau) V(\tau, U, V)}, \tag{3.38} \]

\[ u_i = \beta(\tau) g_{ij} v^j + 2\tau u_0^{0j} \frac{g_{kj} u^0}{g_{0k} g^{0i} - g^{00}} \]

\[ = \beta(\tau) g_{ij} v^j + \tau S_i(\tau, U) + \beta^2(\tau) W_i(\tau, U, V) + \tau \beta(\tau) V_i(\tau, U, V), \tag{3.39} \]

where we recall that the upper case calligraphic letters above and below obey the convention in §1.1.5 and they vanish in \( \xi = 0 \). We also remark that because the exact forms of these calligraphic remainders are not important, the remainders of the same letter may change from line to line. In specific, in above (3.38)–(3.40), terms like \( V \) and \( W \) vanish in \( V = 0 \) as well.

The similar arguments to [21, Proposition 2.3] can be applied to

\[ Q^{\mu\nu}(g, \partial g) - Q^{\mu\nu}(\eta, \partial \eta) = S^{\mu\nu}(\tau, u^{\alpha\beta}, u, u_\sigma^{\alpha\beta}, u_\sigma). \tag{3.41} \]

By Assumption 1.4, we have identity

\[ \rho - \bar{\rho} = \frac{\rho - \bar{\rho}}{\tau^2} = \tau^{-2} g(\tau, \delta \zeta) \quad \text{and} \quad \frac{p - \bar{p}}{\tau^2} = c_s^2(\rho_{K_1}) \tau^{-2} g(\tau, \delta \zeta), \tag{3.42} \]

where \( \rho_{K_1} := \bar{\rho} + K_1(\rho - \bar{\rho}) \) for some constant \( K_1 \), where \( g \) is defined in Assumption 1.4 and we have used the mean value theorem above.

By the definitions of field variables (1.38)–(1.47) and expansions (3.28) and (3.29)–(3.40), with the help of (2.5)–(2.12), (3.41) and (3.42), we conclude that \( \mathcal{H}^{\mu\nu}, \mathcal{H}^{ij}, \mathbb{F} \) belong to \( C^0([0, 1], C^\infty(\mathbb{V})) \) and can be expressed as

\[ \mathcal{H}^{\mu\nu} + \mathcal{H}^{ij} + \mathbb{F} = S(\tau, u^{\alpha\beta}, u, u_\sigma^{\alpha\beta}, u_\sigma, \delta \zeta, v^i) \tag{3.43} \]

where \( S(\tau, 0, 0, 0, 0, 0, 0, 0) = 0 \).

4. Singular hyperbolic formulations of the conformal Euler equations

The main goal of this section is to derive the conformal Euler equations (4.38) and (4.47). The idea of this section is new and contribute to the main innovation of this paper.
4.1. Formulation of conformal Euler equations for Makino-type (1) fluids. In this section, we rewrite the conformal Euler equation (1.28) as a non-degenerated singular symmetric hyperbolic system for Makino-type (1) fluids. To achieve that, a series of transformations have been applied and we demonstrate them in the following four steps to make sure the difficulties and the ideas how to overcome them clear for every step.

4.1.1. Step 1: symmetric hyperbolic formulations. Recall the conformal Euler equations (1.28)
\[ \nabla_\mu \tilde{T}^{\mu \nu} = -6 \tilde{T}^{\mu \nu} \nabla_\mu \Phi + g_{\kappa \lambda} \tilde{T}^{\kappa \lambda} g^{\mu \nu} \nabla_\mu \Phi. \]

First note that differentiating (1.26) yields
\[ u_\mu \nabla_\nu u^\mu = 0 \quad \text{(that is} \quad \nabla_\nu u^0 = -\frac{u_i}{u_0} \nabla_\nu u^i), \]
and define
\[ L_i^\mu = \delta_i^\mu - \frac{u_i}{u_0} \delta_0^\mu \quad \text{and} \quad L_{k \nu} = g_{\nu \lambda} L_k^\lambda. \] (4.1)

Then using the conformal fluid four velocity \( u^\mu \) and \( L_{k \nu} \) acting on above conformal Euler equations (1.28), respectively yields the following formulation of conformal Euler equations (for more details, we refer to [29, §2.2])
\[ u^\mu \partial_\mu \rho + (\rho + p) L_i^\mu \nabla_\mu u^i = -3(\rho + p) u^\mu \nabla_\mu \Phi, \] (4.2)
\[ \frac{c_s^2}{\rho + p} L_i^\mu \partial_\mu \rho + M_{k i} u^\mu \partial_\mu u^i = -L_k^\mu \partial_\mu \Phi, \] (4.3)

where \( c_s^2 = \frac{d p}{d \rho} \) and
\[ M_{k i} = g_{k i} - \frac{u_i}{u_0} g_{0 k} - \frac{u_k}{u_0} g_{0 i} + \frac{u_i u_k}{u_0^2} g_{00}. \] (4.4)

Then, by the normalization
\[ g_{00}(u^0)^2 + 2 g_{0 i} u^0 u^i + g_{ij} u^i u^j = -1, \]
we are able to recover \( u^0 \) given by
\[ u^0 = -\frac{g_{0 i} u^i - \sqrt{(g_{0 i} u^i)^2 - g_{00}(g_{ij} u^i u^j + 1)}}}{g_{00}}. \]

4.1.2. Step 2: non-degenerated symmetric hyperbolic formulations. We expect to rewrite (4.2)–(4.3) into a singular symmetric hyperbolic system. Once multiplying suitable factors, for example, \( s^2/(\rho + p) \) on both sides of (4.2) and \( \rho + p \) on both sides of (4.3), then (4.2)–(4.3) become symmetric. However, since the behavior of the background density \( \bar{\rho} \searrow 0 \) and pressure \( \bar{p} \searrow 0 \), as \( \tau \to 0 \) (see (2.5)–(2.6)), the expecting behaviors of \( \rho \) and \( p \) due to small perturbation of initial data \( \bar{\rho}(1) \) and \( \bar{p}(1) \) also satisfy that \( \rho \searrow 0 \) and \( p \searrow 0 \) as \( \tau \to 0 \), which leads some of elements of the coefficient matrix of above symmetric hyperbolic system tend to 0 or \( \infty \) as \( \tau \to 0 \). The aim of our method is to use the Theorem A.1 of the singular symmetric hyperbolic equation in Appendix A, but the probably degenerated coefficients around \( \tau = 0 \) violate the condition (V) in Appendix A. Hence it is not possible to represent this system in a non-degenerate form by multiplying these factors.

In order to overcome this difficulty, we adopt and generalize the idea of the non-degenerated symmetrization of Euler equations originated by Makino\(^6\) [25] (the idea is also clearly stated

\(^6\)The idea has been generalized to the Makino-type fluids in this article rather than the polytropic ones.
in, for example, [5, 26]). The key point is, for Makino-type fluids defined in §1.2, to introduce a new density variable $\alpha$ defined by Assumption 1.3, such that (4.2)–(4.3) become

$$u^\mu \frac{d\mu}{d\alpha} \partial_\mu \alpha + (\mu + \mu^* p) L_i^\mu \nabla_\mu u^i = -3(\mu + \mu^* p) u^\mu \nabla_\mu \Phi,$$

(4.5)

$$M_{ki} u^\mu \partial_\mu u^k + s^2 L_i^\mu \frac{d\mu}{\mu + \mu^* p} \partial_\mu \alpha = -L_k^\mu \partial_\mu \Phi.$$  

(4.6)

Recalling the non-degenerate function $\lambda(\alpha)$ of Makino fluids defined in Assumption 1.3 and multiplying both sides of (4.5) by $\lambda^2(\alpha) \frac{d(\mu^{-1})}{d\rho}$, with the help of that $\frac{d\mu(\alpha)}{d\alpha} \frac{d(\mu^{-1})}{d\rho} = 1$, we obtain

$$\lambda^2 u^\mu \partial_\mu \alpha + \lambda^2 \frac{d(\mu^{-1})}{d\rho} (\mu + \mu^* p) L_i^\mu \nabla_\mu u^i = -3(\mu + \mu^* p) \lambda^2 \frac{d(\mu^{-1})}{d\rho} u^\mu \nabla_\mu \Phi,$$

(4.7)

$$M_{ki} u^\mu \partial_\mu u^k + \frac{s^2 L_i^\mu}{\mu + \mu^* p} \frac{d\mu}{d\alpha} \partial_\mu \alpha = -L_k^\mu \partial_\mu \Phi.$$  

(4.8)

The relation (1.10) in Assumption 1.3 which is equivalent to

$$\frac{d\mu(\alpha)}{d\alpha} = \frac{\lambda(\alpha)(\mu + \mu^* p)(\alpha)}{s(\alpha)}$$

(4.9)

ensures that the above system (4.7)–(4.8) is symmetric. This is because (4.9) implies

$$\lambda^2 (\mu + \mu^* p) \frac{d(\mu^{-1})}{d\rho} = \frac{s^2 d\mu/d\alpha}{\mu + \mu^* p} = \lambda s.$$  

(4.10)

In view of (4.10), equations (4.7)–(4.8) turn to

$$\lambda^2 u^\mu \partial_\mu \alpha + \lambda s L_i^\mu \nabla_\mu u^i = -3\lambda s u^\mu \nabla_\mu \Phi,$$

(4.11)

$$\lambda s L_i^\mu \partial_\mu \alpha + M_{ki} u^\mu \nabla_\mu u^k = -L_k^\mu \partial_\mu \Phi,$$

(4.12)

which is a non-degenerated symmetric hyperbolic equation and the coefficient matrix does not include the trouble variables $\rho$ and $p$.

**Remark 4.1.** In [29] (also see in [16, 20, 21, 30]), the author took another non-degenerated symmetric hyperbolic formulation for (4.2)–(4.3), in which the new density variable is defined by $\xi = \xi(\rho) = \int_0^\rho \frac{dy}{y + p(y)}$. Under this variable transformation, (4.2)–(4.3) become

$$s^2 w^\mu \partial_\mu \xi + s^2 L_i^\mu \nabla_\mu u^i = -3s^2 u^\mu \partial_\mu \Phi,$$

(4.13)

$$s^2 L_i^\mu \partial_\mu \xi + M_{ij} u^\mu \nabla_\mu u^j = -L_i^\mu \partial_\mu \Phi.$$  

(4.14)

It is evident that (4.11)–(4.12) coincide with (4.13)–(4.14) by choosing $\lambda = s$ and $\alpha = \xi$ provided $s$ is non-degenerate (in fact, $s = \sqrt{K}$ in [29]).

It is evident that (4.11)–(4.12) admit a homogeneous solution $(\alpha, w^\mu) = (\bar{\alpha}(\tau), -\omega \bar{\delta}_0^\mu)$, which leads to

$$\partial_\tau \bar{\alpha} = \frac{3}{\tau} \bar{q}.$$  

(4.15)

Subtracting (4.11)–(4.12) by (4.15), we obtain the formulation of conformal Euler equations as follows,

$$\lambda^2 w^\mu \partial_\mu (\alpha - \bar{\alpha}) + \lambda s L_i^\mu \partial_\mu u^i = \frac{3}{\tau} \lambda s u^0 - \frac{3}{\tau} \lambda^2 u^0 \bar{q} - \lambda s L_i^\mu \Gamma_i^\nu_\mu u^\nu,$$

(4.16)

$$M_{ki} w^\mu \partial_\mu u^k + \lambda s L_i^\mu \partial_\mu (\alpha - \bar{\alpha}) = L_0^\mu \frac{1}{\tau} - \frac{3}{\tau} \lambda s L_i^\mu \bar{q} - M_{ki} u^\mu \Gamma_{\mu\lambda}^k u^\nu.$$  

(4.17)
4.1.3. Step 3: non-degenerated symmetric hyperbolic formulations of new “good” variables. Recall (1.45)–(1.48) which are
\[ \alpha = \beta(\tau)\zeta, \quad \bar{\alpha} = \beta(\tau)\bar{\zeta}, \quad u^i = \beta(\tau)v^i \quad \text{and} \quad \delta \zeta = \zeta - \bar{\zeta}, \]
then re-express (4.16)–(4.17) in terms of these new variables \( (\delta \zeta, v^i) \). They become
\[ \lambda^2 u^\mu \partial_\mu \delta \zeta + \lambda s L^i_\zeta \partial_\mu v^i = S, \]
\[ \lambda s L^i_\zeta \partial_\mu \delta \zeta + M_{ki} u^\mu \partial_\mu v^k = S_i, \]
where
\[ S = -\frac{1}{\tau} 3\lambda^2 u^0(q - \bar{q}) - \frac{1}{\tau} \chi(\tau)\lambda^2 u^0 \delta \zeta \]
\[ \quad \quad \quad + \frac{1}{\tau} \chi(\tau) \left( \frac{\lambda sg_{ij} \beta(\tau)v^i u^j}{u_0} \right) v^i + \frac{\lambda s \beta'(\tau)g_{0i}u^0}{\beta(\tau)u_0} v^i + \frac{1}{\beta(\tau)}(\lambda s L^i_\zeta \Gamma^i_{\mu\nu} u^\nu), \]
and
\[ S_i = \frac{1}{\tau} \left( \frac{g_{ik}}{u_0} \left( 1 - 3s^2 \bar{q} \right) - \chi(\tau)\beta(\tau)\lambda s \delta \zeta \right) - \chi(\tau) M_{ki} u^0 \]
\[ \quad \quad \quad + \frac{1}{\tau} \left( \frac{1}{\beta(\tau)}(1 - 3s^2) \right) g_{0i} u^0 + \frac{1}{\tau} \left( \frac{3(\lambda s - \bar{\lambda} s)\bar{s}}{\lambda \beta(\tau)} + \chi(\tau)\lambda s \delta \zeta \right) \frac{g_{0i}u^0}{u_0} \]
\[ \quad \quad \quad - \frac{1}{\beta(\tau)}(M_{ki} u^\mu \Gamma^k_{\mu\nu} u^\nu). \]

**Remark 4.2.** The variable \( \alpha - \bar{\alpha} \) in above system is not suitable for the purpose of formulating the target singular hyperbolic system (A.1) in Appendix A, since, with the help of (1.11), the remainder terms including \( (\rho - \bar{\rho})/\tau^2 \) and \( (p - \bar{p})/\tau^2 \) in the Einstein equations are regular in \( \tau \) and analytic in \( \delta \zeta \) instead of \( \alpha - \bar{\alpha} \), see (3.42), and the remainders have been represented by (3.43) in terms of variables \( (\tau, u^\alpha, u, u^\sigma, u^\tau, \delta \zeta, v^i) \). In other words, if one use \( \alpha - \bar{\alpha} \) as the variable, the remainders of the Einstein equation contain the \( \frac{1}{\tau} \)-singular terms. After changing \( \alpha - \bar{\alpha} \) to \( \delta \zeta \), \( u^i \) have to been changed to \( v^i \) to make sure this system to be symmetric. Hence, we have to change the variables of fluids from \( (\alpha - \bar{\alpha}, u^i) \) to \( (\delta \zeta, v^i) \). In summary, we emphasize that use \( \delta \zeta \) instead of \( \alpha - \bar{\alpha} \) as the variable of the target equation (A.1) due to the requirement of the regular remainders in the Einstein equations and adopt \( v^i \) rather than \( u^i \) because of the requirement of the symmetric coefficient matrix of the Euler equations.

Next, in order to apply the singular hyperbolic system (A.1) in Appendix A, we have to distinguish the singular (in \( \tau \)) and the regular terms in \( S \) and \( S_i \). To achieve this, we expand the following key quantities by direct calculations. With the help of (3.29)–(3.40), (4.4) becomes
\[ M_{ik} = g_{ki} + \beta^2(\tau)W_{ki}(\tau, U, V) + \beta^3(\tau)U_{ki}(\tau, U, V) \]
\[ + \tau \beta(\tau)\nu_{ki}(\tau, U, V) + \tau S_{ki}(\tau, U, V). \]
Note that \( \tau \beta, \beta^3 \) and \( \beta^2 \in C^1([0, 1]) \) by Assumption 1.4, 1.5 and (1.18)) which surely can be absorbed into the Calligraphic remainders. However, later on we expect \( \frac{1}{\tau}(M_{ik} - g_{ik}) \) is regular in \( \tau \) and since \( \beta^2, \beta^3 \) and \( \beta \beta \lesssim \tau \) (due to the fact that \( \beta(\tau) \lesssim \sqrt{\tau} \)), it is better to expand \( M_{ik} \) in this form. Similarly, using (3.29)–(3.40), we expand
\[ L^0_i = -\omega \beta \delta_{ij} v^j + \beta T_{i}(\tau, U, V) + \tau S_i(\tau, U, V) \]
\[ + \beta^2 W_i(\tau, U, V) + \tau \beta \nu_i(\tau, U, V). \]
Calculating $\Gamma_{00}^i$ yields
\[
\Gamma_{00}^i = \frac{1}{2} g^{i0} \partial_0 g_{00} + g^{ij} \partial_0 g_{j0} - \frac{1}{2} g^{ij} \partial_j g_{00} = \tau S_{00}^i (\tau, u_\sigma^{\mu \nu}, u^{\mu \nu}) - g_{jk} g^{ij} \eta_{00} (u_0^k + 3 u^{0k}) + \frac{1}{2} g^{ij} (\eta_{00})^2 u_j^{00}.
\]

Then by (3.29)–(3.40), further expand gives
\[
L^\mu_i \Gamma_{\mu \nu} u^\nu = - \frac{u_i}{u_0} \Gamma_{00}^i u^0 - \frac{u_i}{u_0} \Gamma_{ij}^i u^j + \delta_1^i \Gamma_{j0}^i u^0 + \delta_1^i \Gamma_{jk}^i u^k
= \frac{u_0}{2} g^{ik} (\partial_k g_{00} + \partial_\tau g_{ki} - \partial_k g_{i0}) + \beta(\tau) S(\tau, U, V). \tag{4.23}
\]

Noting
\[
\partial_\lambda g_{\mu \nu} = - g_{\mu \alpha} g_{\beta \nu} \partial_\lambda g^{\alpha \beta},
\]
we derive that
\[
u^\nu \Gamma_{\mu \nu} u^\nu = u_0^0 \Gamma_{00}^k u^0 + 2 u_i \Gamma_{ij}^k u^0 + u^i \Gamma_{ij}^i u^i
= g^{kj} g_{ji} (u_0^k + 3 u^{0k}) - \frac{u_0}{2} \eta_{00} u_j^{00} + \beta(\tau) S^{k} (\tau, U, V). \tag{4.24}
\]

By the mean value theorem, express $s(\alpha) - s(\tilde{\alpha})$ and $\lambda(\alpha) - \lambda(\tilde{\alpha})$ in terms of $\delta \zeta$,
\[
s(\alpha) - s(\tilde{\alpha}) = s'(\alpha_{K\ell}) (\alpha - \tilde{\alpha}) = \beta(\tau) s'(\alpha_{K\ell}) \delta \zeta \tag{4.25}
\]
and
\[
\lambda(\alpha) - \lambda(\tilde{\alpha}) = \lambda'(\alpha_{K\ell}) (\alpha - \tilde{\alpha}) = \beta(\tau) \lambda'(\alpha_{K\ell}) \delta \zeta, \tag{4.26}
\]
where $\alpha_{K\ell}$ ($\ell = 5, 6$) are intermediate points defined in §1.1.6 for some constants $K_5, K_6 \in (0, 1)$. Since $\alpha - \tilde{\alpha} := \beta(\tau) \delta \zeta$, then
\[
\alpha_{K\ell} := \tilde{\alpha} + K_\ell (\alpha - \tilde{\alpha}) = \tilde{\alpha} + K_\ell \beta(\tau) \delta \zeta = S(\tau, \delta \zeta). \tag{4.27}
\]

With the help of $q = s/\lambda$, we have
\[
1 - 3 s^2 \frac{q}{q} = 1 - 3 \lambda - \lambda s q - 3 (s - \bar{s}) \bar{s} - 3 s^2
= 1 - 3 s^2 - 3 \beta(\tau) \lambda'(\alpha_{K\ell}) sq \delta \zeta - 3 \beta(\tau) s'(\alpha_{K\ell}) \bar{s} \delta \zeta,
\]
and
\[
q - \bar{q} = \beta'(\alpha) \delta \zeta + \frac{1}{2} \beta'' q^0 (\alpha_{K\ell}) (\delta \zeta)^2,
\]
where $\alpha_{K\ell} := \tilde{\alpha} + K_\ell (\alpha - \tilde{\alpha})$ for constant $K_\ell \in (0, 1)$. With the help of identities
\[
g_{ik} g_{ij} = \delta_{ik} - g_{0k} g^0_{ij} \quad \text{and} \quad g_{ik} g_{ij} g^0_{ij} = g_{ik} - g_{ik} g_{0i} g^0_{0i},
\]
we can rewrite $S$ and $S_i$ as follows
\[
S = \frac{1}{\tau} \lambda^2 u^0 \left[ 3 q'(\alpha) + \frac{3}{2} q'' \left( \alpha_{K\ell} \right) \beta^2 \delta \zeta - \chi(\tau) \right] \delta \zeta + \frac{1}{\tau} \chi(\tau) \left( \frac{\lambda s g_{ij} \beta(\tau) v^j}{u_0} \right) v^i
+ \frac{\lambda s \beta(\tau) g_{0k} u^0}{\beta(\tau) u_0} v^i - \frac{s}{\beta(\tau)} \frac{u^0}{2} g^{ik} (\partial_i g_{00} + \partial_\tau g_{ki} - \partial_k g_{i0}) - S(\tau, U) \tag{4.28}
\]
and
\[
S_i = \frac{1}{\tau} \left( - \frac{g_{ik}}{u_0} \left( 1 - 3 s^2 \frac{q}{q} - \chi(\tau) \beta(\tau) \lambda s \delta \zeta \right) - \chi(\tau) M_{ki} u^0 \right) v^k
\]


\[-2 \left( \frac{3(\lambda s - \bar{s})\bar{s}}{\lambda \beta(\tau)} + \chi(\tau)\lambda \delta \zeta \right) g_{ij}u^{0j} - \frac{1}{\tau \beta(\tau)} \left( (1 + 6\bar{s}^2)u^{0j} + u_0^{0j} \right) \]

\[+ \frac{\eta_{0i}u_{i0}^{00}}{2\beta(\tau)} + S_i(\tau, U, V). \quad (4.29)\]

4.1.4. **Step 4:** non-degenerated symmetric hyperbolic formulations of new “better” variables. As mentioned in Remark 4.2, we need to rescale the velocity to \( \nu^i \) by \( \beta \) to ensure the system to be symmetric. However, this brings a new singular term \( \frac{\partial \nu^i}{\partial (\nu^i)} \) in \( S_i \). Although equations (4.19)–(4.20) seem to be consistent with the non-degenerated singular hyperbolic system given in Appendix A, \( S_i \) involves \( \tau \)-singular terms of \( u_0^{00}, u_0^{0j} \) and \( u_0^{0j} \) in a “bad” way in (4.29), which destroys the structure of the singular term in the system of Appendix A. In order to overcome this difficulty, we introduce a new variable

\[v^k = v^k - Ag^{0k} = v^k - 2\tau A u^{0k}, \quad (4.30)\]

where

\[A = A(\tau) = -\frac{3s^2(\bar{a}(\tau))}{\sqrt{-\eta^{00}\beta(\tau)}} = -\frac{3s^2}{\omega \beta(\tau)}. \quad (4.31)\]

This new variable adjusts the relations between \( u_0^{0j} \) and \( u_0^{0j} \) to a “good” form via subtracting \( 2\tau A u^{0k} \) (hence the equations at the end of this section will precisely agree with the one in Appendix A). Note that by Assumption 1.5, \( \bar{s} \lesssim \beta \), then \( A \lesssim 1 \) is a regular term.

Expressing (4.20) in terms of \( v^k \) yields

\[M_{ki} u^\mu \partial_\mu (v^k) + \lambda s L^\mu_i \partial_\mu \delta \zeta = M_{ki} u^\mu \partial_\mu (Ag^{0k}) + S_i. \quad (4.32)\]

Then direct calculations give

\[M_{ki} u^\mu \partial_\mu (Ag^{0k}) + S_i = \frac{1}{\tau} \left( -\frac{g_{ik}}{u_0} \left( 1 - 3s^2 \bar{g} \right) - (1 + \frac{1}{\bar{g}} \beta(\tau) \lambda \delta \zeta) - \chi(\tau) g_{ki} u^0 \right) v^k \]

\[- g_{ij} \left( \frac{1}{\beta(\tau)} \left( (1 + 6\bar{s}^2)u^{0j} + u_0^{0j} \right) + \sqrt{-\eta^{00}} A(u_0^{0j} + 3u_0^{0j}) \right) \]

\[+ M_{ki} u^0 (\partial_\tau A) g^{0k} + M_{ki} u^{j0} \partial_j (Ag^{0k}) + \frac{\eta_{0i}u_{i0}^{00}}{2\beta(\tau)} + \hat{S}_i(\tau, U, \dot{V}). \quad (4.33)\]

Let us focus on some dangerous terms on the right hand side of (4.33). By (4.31) and the equation (4.15), we have

\[(\partial_\tau A) g^{0k} = -2\tau u^{0k} \left( \frac{3\bar{s}}{\omega \beta(\tau)} \right) + 2\chi u^{0k} \left( \frac{3s^2}{\omega \beta} \right) \]

\[= -u^{0k} \frac{36s^2}{\omega \beta(\tau)} \bar{s}(\bar{a}) - u^{0k} \frac{6\Omega s^2}{\omega \beta(\tau)} + 2\chi u^{0k} \left( \frac{3s^2}{\omega \beta} \right). \quad (4.34)\]

Assumption 1.5 and (2.8) imply \( (\partial_\tau A) g^{0k} \) is not singular in \( \tau \). Then, with the help of identities

\[1 + 6s^2 + 3\sqrt{-\eta^{00}} \beta(\tau) A = 1 - 3s^2, \quad A \partial_\tau g^{0k} = A(u_0^{0k} + 3u_0^{0k}) \quad \text{and} \quad \partial_j (Ag^{0k}) = A \partial_j g^{0k} = A u_j^{0k}, \]

we derive a further expression of (4.32)–(4.33),

\[\lambda s L^\mu_i \partial_\mu \delta \zeta + M_{ki} u^\mu \partial_\mu (v^k) = \frac{1}{\tau} \left( -\frac{g_{ik}}{u_0} \left( 1 - 3s^2 \bar{g} \right) - \chi(\tau) \beta(\tau) \lambda \delta \zeta - \chi(\tau) g_{ki} u^0 \right) v^k \]
where $\hat{S}_i$ satisfies $\hat{S}_i(\tau, 0, 0) = 0$.

Next, let us express (4.19) in terms of $v^k$. Direct calculation gives

$$
\lambda^2 u^\mu \partial_\mu \delta \zeta + \lambda s L_\nu^\mu \partial_\mu v^i = S + \lambda s L_\nu^\mu \partial_\mu (Ag^{0i}).
$$

Note that

$$
\partial_\mu (Ag^{0i}) = \delta_\mu^0 (\partial_\tau A) g^{0i} + A \delta_\mu^0 (u_0^{0i} + 3u^{0i}) + A \delta_\mu^i u_0^{0i}
$$

is regular in $\tau$ due to (4.34), and using (4.25) and Assumption 1.5.(1) that $\bar{s} \lesssim \beta(\tau)$,

$$
\frac{s}{\beta(\tau)} = \frac{s - \bar{s}}{\beta(\tau)} + \frac{\bar{s}}{\beta(\tau)} = s'(\alpha_{K_6}) \delta \zeta + \frac{\bar{s}}{\beta(\tau)}
$$

is also regular in $\tau$, which can be absorbed into $\hat{F}(\tau, U, \tilde{V})$ in the following equation. Thus, eventually, the equation (4.36) becomes

$$
\lambda^2 u^\mu \partial_\mu \delta \zeta + \lambda s L_\nu^\mu \partial_\mu v^i = \frac{1}{\tau} \lambda^2 u_0^0 \left[ 3q'(\bar{\alpha}) + \frac{3}{2} q''(\alpha_{K_7}) \beta \delta \zeta - \chi(\tau) \right] \delta \zeta

+ \frac{\chi(\tau)}{\tau} \left( \frac{\beta(\tau) \lambda s g_{ij} v^j}{u_0} \right) v^i + \tilde{F}(\tau, U, \tilde{V}),
$$

where $\tilde{F}(\tau, 0, 0) = 0$.

Gathering (4.35) and (4.37) together, we bring them to a matrix form

$$
N^\mu \partial_\mu \tilde{V} = \frac{1}{\tau} N P^\dagger \tilde{V} + \frac{1}{\tau} (E_0 \delta_\mu^0 + E_q \delta_\mu^q) U^\mu + F(\tau, \tilde{V}, U),
$$

where

$$
\tilde{V} = (\delta \zeta, v^p)^T, \quad U^\mu = (u_0^\mu, u_j^0, u_0^0)^T
$$

and

$$
N^\mu = \begin{pmatrix}
\lambda^2 u^\mu & \lambda s L_\nu^\mu \\
\lambda s L_\nu^\mu & M_\nu u^\mu
\end{pmatrix}, \quad E_0 = \frac{\tau u_0}{2 \beta(\tau)} \begin{pmatrix}
0 & 0 & 0 \\
0 & \delta_\nu^0 & 0
\end{pmatrix},
$$

$$
N = \begin{pmatrix}
\lambda^2 u_0^0 \left[ 3q'(\bar{\alpha}) + \frac{3}{2} q''(\alpha_{K_7}) \beta \delta \zeta - \chi(\tau) \right] & \frac{1}{u_0} \chi(\tau) \beta(\tau) \lambda s g_{ij} v^j \\
\frac{1}{u_0} \chi(\tau) \beta(\tau) \lambda s g_{ij} v^j & -\frac{2u_0}{q} (1 - 3s^2) \left( 1 - 3s^2 \bar{q} \right) - \chi(\tau) g_{ri} u_0^0
\end{pmatrix},
$$

$$
E_q = -\frac{\tau}{\beta(\tau)} (1 - 3s^2) \begin{pmatrix}
0 & 0 & 0 \\
g_{rq} & 0 & g_{rq}
\end{pmatrix}, \quad P^\dagger = \begin{pmatrix}
1 & 0 & 0 \\
0 & \delta_\nu^0 & 0
\end{pmatrix},
$$

and $F = (\hat{F}, \hat{S}_i)^T$. 
4.2. Formulation of conformal Euler equations for Makino-type (2) fluids. When $\beta \equiv$ constant, by (1.11) in Assumption 1.4, the suitable variable of density is $\delta \zeta = \alpha - \bar{\alpha}$ (see Remark 4.2). Therefore, once obtaining (4.16)–(4.17), we do not need to proceed Step 3–4, that is, §4.1.3–§4.1.4, which are performed due to the bad extra decay rate ($\beta(\tau) \lesssim \sqrt{\tau}$) of density and velocity in that case. However, another procedure has been carried out if $q$ is conserved with respect to the perturbations, that is

$$q = \bar{q}.$$  

We have to change $u^j$ to $u_q$, otherwise, Condition (VI) in Appendix A can not be satisfied due to the degeneracy of BP and $P_u$ is the velocity $u^j$, by letting $P_u = 0$ in $P^\perp B(t, u) P$ and noting

$$u_q = g_{00} u^0 + g_{0i} u^i,$$  

then $P^\perp B(t, P^\perp u) P = P B(t, P^\perp u) P^\perp \neq 0$. For more details of this case, see [16, 20, 21, 30]. In other words, a good expression of the Euler equations verifying Condition (VI) in Appendix A relies on the variable $u_q$ instead of $u^i$. In this section, we rewrite the Euler equations in terms of variables $(\delta \zeta, u_q)$ first.

We express $u^k$ in terms of $(g^{\mu \nu}, u_q)$ by performing a change of variables from $u^i$ to $u_q$, which are related via the map $u^i = u^i(u_q, g^{\mu \nu})$ given by

$$u^k = g^{ki} u_i + g^{k0} u_0 = g^{ki} u_i + g^{k0} - g^{0i} u_i - \sqrt{(g^{0i} u_i)^2 - g^{00}(1 + g^{ij} u_j u_j)}.$$  

In above derivation, we have used the normalization (1.26), which is $g^{\alpha \beta} u_\alpha u_\beta = -1$, to obtain

$$u_0 = -g^{0i} u_i - \sqrt{(g^{0i} u_i)^2 - g^{00}(1 + g^{ij} u_j u_j)}.$$  

Denote $J^{ij}$ the Jacobian of $u^i = u^i(u_j, g^{\mu \nu})$. Differentiating (4.42) with respect to $u_j$, we calculate

$$J^{ij} = \frac{\partial u^i}{\partial u_j} = g^{ij} + \frac{g^{0i}}{g^{00}} \left( -g^{0j} - \frac{2g^{0j} g^{0k} u_k - g^{00} g^{kj} u_k}{\sqrt{(g^{0k} u_k)^2 - g^{00}(1 + g^{kl} u_k u_l)}} \right)$$

$$= \delta^{ij} + u^{ij} + \frac{\eta^{ij}}{\omega^2} (2\tau u^{00} - u) + S^{ij}(\tau, u, u^{\mu \nu}) + \frac{2\tau u^{0i}}{\eta^{00} + 2\tau u^{00}} (2\tau u^{0j})$$

$$- \frac{4\tau^2 u^{0j} u^{0k} u_k - (\eta^{00} + 2\tau u^{00}) g^{kj} u_k}{2\sqrt{(2\tau u^{0k} u_k)^2 - (\eta^{00} + 2\tau u^{00})(1 + g^{kl} u_k u_l)}}$$

$$= \delta^{ij} + J^{ij}(\tau, u^{0i}, u_k).$$  

Where $J^{ij}(\tau, 0, 0) = 0$. We differentiate $u^i$ with respect to $x^\mu$, then a simple chain rule gives

$$\partial_\mu u^i = J^{ij} \partial_\mu u_j + \frac{\partial u^i}{\partial g^{\alpha \beta}} \partial_\mu g^{\alpha \beta}.$$  

Inserting (4.44) into (4.16)–(4.17), and multiplying both sides of (4.17) by $J^{ij}$, we can rewrite the Euler equations as

$$\lambda^2 u^\mu \partial_\mu (\alpha - \bar{\alpha}) + \lambda s L_\mu^i J^{iq} \partial_\mu u_q = \frac{3}{\tau} \lambda^2 u^0 (q - \bar{q})$$

$$- \lambda s L_\mu^i \left( \frac{\partial u^i}{\partial (g^{\alpha \beta})} \partial_\mu g^{\alpha \beta} + \Gamma^i_{\mu \nu} u^\nu \right),$$  

where $\Gamma^i_{\mu \nu} = \frac{1}{2} g^{ij} \partial_\mu g_{ij}$.
\[
M_{ki}u^\mu J^{kj}J^{iq}\partial_\mu u_q + \lambda s J^{ij}L^{\mu}_i \partial_\mu (\alpha - \bar{\alpha}) = J^{jq}\left[ -\frac{1}{\tau} u_0 (1 - 3s^2\frac{\bar{q}}{q}) u_q - M_{ki}u^\mu \left( \frac{\partial u^k}{\partial (g^{\alpha\beta})} \partial_\mu g^{\alpha\beta} + \Gamma^k_{\mu\nu}u^\nu \right) \right].
\]

(4.46)

Multiplying \(\frac{1}{\lambda^2 u^0}\) on the both sides of above Euler equations (4.45)–(4.46) (this step is necessary in order to satisfy Condition (VII) of the singular hyperbolic system (A.1), see (5.38) for the examination of this condition) and rephrasing them to a more compact matrix form yields that

\[
\hat{N}^\mu \partial_\mu \hat{V} = \frac{1}{\tau} \hat{N} \hat{P}^\dagger \hat{V} + \hat{H}(\tau, U, \hat{V}),
\]

(4.47)

where \(\hat{V} = (\delta \zeta, u_q)^T\), \(\hat{N}^\mu\) and \(\hat{H}\) are given by

\[
\hat{N}^\mu = \begin{pmatrix}
\frac{u^\mu}{u^0} & q J^{ij} L^{\mu}_i & q J^{jq} L^{\mu}_i \\
J^{ij} L^{\mu}_i & 0 & \frac{q^2}{s^2} M_{ki} J^{kj} J^{iq} L^{\mu}_i u^\mu \\
0 & \frac{q^2}{s^2} M_{ki} J^{kj} J^{iq} L^{\mu}_i & \frac{q^2}{s^2} M_{ki} J^{kj} J^{iq} L^{\mu}_i
\end{pmatrix},
\]

and

\[
\hat{H} = \frac{1}{\lambda^2 u^0} \begin{pmatrix}
-\lambda s L^{\mu}_i \left( \frac{\partial u^i}{\partial (g^{\alpha\beta})} \partial_\mu g^{\alpha\beta} + \Gamma^i_{\mu\nu} u^\nu \right) & -J^{ij} M_{ki} u^\mu \left( \frac{\partial u^k}{\partial (g^{\alpha\beta})} \partial_\mu g^{\alpha\beta} + \Gamma^k_{\mu\nu} u^\nu \right)
\end{pmatrix}.
\]

In order to use Theorem A.1, we list two cases to choose \(\hat{N}\) and \(\hat{P}^\dagger\):

1. If \(q = \bar{q}\) and \(1 - 3s^2 \geq \hat{\delta}\) hold, then set

\[
\hat{N} = \begin{pmatrix}
1 & 0 \\
0 & \frac{1}{\lambda^2 u^0} (1 - 3s^2) J^{ij}
\end{pmatrix} \quad \text{and} \quad \hat{P}^\dagger = \begin{pmatrix}
0 & 0 \\
0 & \delta^q_i
\end{pmatrix}.
\]

(4.48)

2. If \(q = \bar{q}\) and \(1 - 3s^2 \equiv 0\) hold, then set

\[
\hat{N} = \begin{pmatrix}
1 & 0 \\
0 & \frac{1}{\lambda^2 \delta^q_i}
\end{pmatrix} \quad \text{and} \quad \hat{P}^\dagger = \begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix}.
\]

(4.49)

Now we have transformed the Einstein–Euler system to the target singular symmetric hyperbolic form. In the next section, we focus on examining the Conditions in Appendix A are satisfied by above formulations.

5. Proof of the main Theorems

In this section, we prove the main theorem based on Theorem A.1.

5.1. Proof of Theorem 1.13.

5.1.1. Local existence and continuation of reduced conformal Einstein-Euler equations. The local existence and continuation of reduced conformal Einstein-Euler equations can be derived using standard local existence and continuation results for symmetric hyperbolic systems (see, e.g. Theorems 2.1 and 2.2 of [24]), as long as the reduced conformal Einstein–Euler equations are well-defined when they are written as a symmetric hyperbolic system, that is, the conformal metric \(g^{\mu\nu}\) remains non-degenerate and the conformal fluid four-velocity remains future directed,

\[
det(g^{\mu\nu}) < 0 \quad \text{and} \quad u_0 < 0.
\]

and \(\rho\) remains strictly positive, and the new variables \((U, V)\) or \((\hat{U}, \hat{V})\) are equivalent to the original ones \((g^{\mu\nu}, \partial_\sigma g^{\mu\nu}, \rho, u^\sigma)\). We omit the details for which we refer readers to [21, §3] and [20, §4], but give the proposition to state this result
Proposition 5.1. Suppose $k \in \mathbb{Z}_{\geq 3}$, $\Lambda > 0$, $(g_0^{\mu \nu}) \in H^{k+1}(T_3^3, S_4)$, and $(g_1^{\mu \nu}) \in H^k(T_3^3, S_4)$, $\nu^\alpha \in H^k(T_3^3, \mathbb{R}^4)$ and $\rho_0 \in H^k(T_3^3)$, where $\nu^\alpha$ is normalized by $g_{\alpha \beta} \nu^\alpha \nu^\beta = -1$, and $\det(g_0^{\mu \nu}) < 0$ and $\rho_0 > 0$ on $T_3$. Then there exists a $T_1 \in (0, 1]$ and a unique classical solution

$$(g^{\mu \nu}, u^\mu, \rho) \in \bigcap_{\ell=0}^2 C^\ell((T_1, 1], H^{k+1-\ell}(T_3^3)) \times \bigcap_{\ell=0}^1 C^\ell((T_1, 1], H^{k-\ell}(T_3^3))$$

of the conformal Einstein–Euler equations, given by (1.27) and (1.28), on the spacetime region $(T_1, 1] \times T_3$ that satisfies

$$(g^{\mu \nu}, \partial_\tau g^{\mu \nu}, \rho, u^\alpha)|_{\tau=1} = (g_0^{\mu \nu}, g_1^{\mu \nu}, \rho_0, \nu^\alpha).$$

Moreover,

(i) the vector $(U, V)$, see (3.26), is well-defined, lies in the space

$$(U, V) \in \bigcap_{\ell=0}^1 C^\ell((T_1, 1], H^{k-\ell}(T_3^3, V)),$$

where

$$V = \mathbb{R}^4 \times \mathbb{R}^{12} \times \mathbb{R}^4 \times S_3 \times (S_3)^3 \times S_3 \times \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^3,$$

and solves (3.23)–(3.25) and the Euler equations (4.38) basing on a transformation (4.30), or (4.47) via transformation (4.41) on the spacetime region $(T_1, 1] \times T_3$, and

(ii) there exists a constant $\sigma > 0$, independent of $T_1 \in (0, 1)$, such that if $(U, V)$ satisfies

$$\|(U, V)\|_{L^\infty((T_1, 1], H^s(T_3^3))} < \sigma,$$

then the solution $(g^{\mu \nu}, u^\mu, \rho)$ can be uniquely continued as a classical solution with the same regularity to the larger spacetime region $(T_1^*, 1] \times T_3^*$ for some $T_1^* \in (0, T_1)$.

5.1.2. Proof for Makino-type (1) fluids. Let us first gather the Einstein equations (3.23)–(3.25) and the Euler equations (4.38) together, recalling (3.26) and (4.39) that

$$U := (u_0^\mu, u_j^\mu, u_0^\nu, u_j^\nu, u_0^{lm}, u_j^{lm}, u_0, u_j, u)^T$$
and $$\tilde{V} = (\delta_\zeta, \nu^\rho)^T,$$

(5.1) to get the complete non-degenerated singular symmetric hyperbolic system

$$B^\mu \partial_\mu \begin{pmatrix} U \\ V \end{pmatrix} = \frac{1}{\tau} BP \begin{pmatrix} U \\ V \end{pmatrix} + H,$$

(5.2)

where

$$B = \begin{pmatrix}
A & 0 & 0 & 0 \\
0 & -2g^{00} \mathbb{I} & 0 & 0 \\
0 & 0 & -2g^{00} \mathbb{I} & 0 \\
-(E_0 \delta_\mu^0 + E_q \delta_\mu^q) & 0 & 0 & -N \\
\end{pmatrix},$$

$$B^\mu = \begin{pmatrix}
A^\mu & 0 & 0 & 0 \\
0 & A^\mu & 0 & 0 \\
0 & 0 & A^\mu & 0 \\
0 & 0 & 0 & -N^\mu \\
\end{pmatrix},$$

$$P = \begin{pmatrix}
P^* & 0 & 0 & 0 \\
0 & \Pi & 0 & 0 \\
0 & 0 & \Pi & 0 \\
0 & 0 & 0 & P^\dagger \\
\end{pmatrix},$$

and $H = (F_1, F_2, F_3, -F)^T$. 
Then by simply reversing the time $\tau \to -\tau$ on above equations or the model equation in Appendix A, we can apply Theorem A.1 to our above equations (5.2) directly.

We need to verify all the conditions in Appendix A. **Conditions** (I)-(III) are evident and with the help of (4.27), it is direct to verify $H, B^p, B \in C^0([T_0, 0], C^\infty(\mathbb{V}))$.

The key to **Condition** (IV) are $B^0 \in C^1([T_0, 0], C^\infty(\mathbb{V}))$ and $[P, B] = PB - BP = 0$. We now verify them respectively.

(1) **Verification of** $B^0 = \text{diag}\{A^0, A^0, A^0, -N^0\} \in C^1([0, 1], C^\infty(\mathbb{V}))$; $\partial_\tau A^0$ has been investigated in [20, 21, 30], we need to verify $N^0 \in C^1([0, 1], C^\infty(\mathbb{V}))$. First we prove that $\tau A \in C^1([0, 1])$ which can be verified by noting that $A \in C^0([0, 1])$ (recall that $A$ is defined by (4.31)) and by (4.31) and the equation (4.15),

$$\partial_\tau (\tau A) = - \frac{3s^2}{\omega\beta} - \frac{18s^2}{\omega\beta\lambda} s'(\bar{\alpha}) - \frac{3\Omega s^2}{\omega^3\beta} + \chi \left( \frac{3s^2}{\omega\beta} \right) \lesssim 1 \quad \text{(by Assumption 1.5, } s \lesssim \beta),$$

for $\tau \in [0, 1]$. Therefore, (4.30) can be expressed as $v^j = v^j + \tau A(\tau) \hat{S}^j(\tau, U)$ where, as our convention of notations, $\hat{S}^j(\tau, U) \in C^1([0, 1], C^\infty(\mathbb{V}))$ and $\hat{S}^j(\tau, 0) = 0$. Applying (3.38)–(3.40), (4.15), (4.18), (4.21) and (4.22), we calculate $D_\tau N^0(\tau, U, \bar{V})$ (recall the convention of notation in §1.1.3, the derivative operator $D_\tau$ is the partial derivative with respect to the first variable $\tau$) and try to see if it is continuous in $[0, 1]$. Directly expanding all the derivatives in $D_\tau N^0$, with the help of that $\tau A \in C^1([0, 1])$, we find the key quantities in $D_\tau N^0$ (the other terms are easy to bound in $[0, 1]$) are

$$D_\tau u^0(\tau, U, \bar{V}) = \frac{1}{\Omega/\tau} + S(\tau, U, \bar{V}), \quad (5.3)$$
$$D_\tau u^0(\tau, U, \bar{V}) = \frac{1}{\Omega/\tau} + T(\tau, U, \bar{V}), \quad (5.4)$$
$$D_\tau M_{ki}(\tau, U, \bar{V}) = F_{ki}(\tau, U, \bar{V}) \quad (5.5)$$

where, by (2.7)–(2.8), $|\Omega/\tau| \lesssim 1$ for $\tau \in [0, 1]$ and the typewriter fonts remainders $S, T, F_{ki} \in C^0([0, 1], C^\infty(\mathbb{V}))$ agree with the conventions in §1.1.5 and $S(\tau, 0, 0) = T(\tau, 0, 0) = F_{ki}(\tau, 0, 0) = 0$. Also note that

$$sD_\tau u_1(\tau, U, \bar{V}) = (s + s'(\alpha_K_5)\beta\delta\zeta)\partial_\tau \beta(\tau)g_{ij}v^j + L_i(\tau, U, \bar{V}),$$

and another crucial quantities are (view $\lambda(\alpha)$ and $s(\alpha)$ are functional of $(\tau, U, \bar{V})$)

$$D_\tau \lambda(\alpha) = D_\tau \lambda(\bar{\alpha}(\tau) + \beta(\tau)\delta\zeta) = \lambda'(\alpha) \left[ \delta\zeta\partial_\tau \beta(\tau) + \frac{3}{\tau} \bar{q} \right],$$

$$= \lambda'(\bar{\alpha}) \left[ \delta\zeta\partial_\tau \beta(\tau) + \frac{3s}{\tau}\delta\zeta \right] + \lambda''(\alpha_K_5)\delta\zeta \left[ \delta\zeta\partial_\tau \beta(\tau) + \frac{3}{\tau} \beta(\tau)\bar{q} \right], \quad (5.6)$$

and

$$\beta(\tau)D_\tau s(\alpha) = \beta(\tau)D_\tau \lambda(\alpha) = \beta(\tau) \delta\zeta\partial_\tau \beta(\tau) + \frac{3}{\tau} \bar{q}$$

$$= \beta(\tau)\left( s'(\bar{\alpha}) + s''(\alpha_K_5)\beta\delta\zeta \right) \left[ \delta\zeta\partial_\tau \beta(\tau) + \frac{3}{\tau} \bar{q} \right], \quad (5.7)$$

where $K_5, K_9 \in (0, 1)$. Then Assumption 1.5.(1.12) and inequalities (1.18) guarantee $N^0 \in C^1([0, 1], C^\infty(\mathbb{V}))$ and we conclude this condition.
(2) Verification of \([P, B] = PB - BP = 0\): To verify this condition, we only need to examine the following three relations,

\[
P^*A = AP^*, \tag{5.8}
\]

\[
P^\dagger N = NP^\dagger \tag{5.9}
\]

and

\[
P^\dagger (E_0 \delta^0_\mu + E^q_\delta^q_{\mu}) = (E_0 \delta^0_\mu + E^q_\delta^q_{\mu}) P^*. \tag{5.10}
\]

It is evident that (5.8) and (5.9) hold by direct calculations. Let us focus on (5.10) and calculate

\[
(E_0 \delta^0_\mu + E^q_\delta^q_{\mu}) P^*
\]

\[
= \begin{pmatrix}
\frac{\tau \eta_{00}}{2\beta(\tau)} & 0 & 0 & 0 \\
0 & \frac{\tau}{\beta(\tau)} \delta^0_\mu - \frac{\tau}{\beta(\tau)} (1 - 3s^2) & 0 & \frac{\tau}{\beta(\tau)} (1 - 3s^2) g_{rq} & 0 \\
0 & 0 & \frac{\tau}{\beta(\tau)} & 0 & g_{rq} \\
0 & \frac{\tau}{\beta(\tau)} & 0 & \frac{\tau}{\beta(\tau)} & 0
\end{pmatrix}
\]

\[
= \begin{pmatrix}
1 & 0 & 0 & \delta^0_\mu - \frac{\tau}{\beta(\tau)} (1 - 3s^2) g_{iq} & 0 \\
0 & \frac{\tau}{\beta(\tau)} & 0 & \frac{\tau}{\beta(\tau)} & 0 \\
0 & 0 & \frac{\tau}{\beta(\tau)} & 0 & g_{iq}
\end{pmatrix}
\]

\[
= P^\dagger (E_0 \delta^0_\mu + E^q_\delta^q_{\mu}).
\]

Now we complete the verification of **Condition** (IV).

Next, we verify **Condition** (V). In order to state this concisely, we address it by using the following simple Lemma,

**Lemma 5.2.** Suppose a real block matrix

\[
Q = \begin{pmatrix}
\tilde{A} & 0 \\
E & N
\end{pmatrix}
\]

where \(\tilde{A}\) is a \(m \times m\) symmetric matrix, \(N\) a \(n \times n\) symmetric matrix and \(E\) a \(n \times m\) matrix. If there is a constant \(\epsilon > 0\), such that

\[
\tilde{A} - \frac{1}{2\epsilon} I \quad \text{and} \quad N - \frac{\epsilon}{2} EE^T
\]

are both positive definite, then \(Q\) is positive definite, that is

\[
(X^T, Y^T) Q \begin{pmatrix} X \\ Y \end{pmatrix} \geq 0
\]

for any \(X \in \mathbb{R}^m\) is a \(m \times 1\) matrix and \(Y \in \mathbb{R}^n\) a \(n \times 1\) matrix.

**Proof.** For any \(X \in \mathbb{R}^m\) is a \(m \times 1\) matrix and \(Y \in \mathbb{R}^n\) a \(n \times 1\) matrix, note there is an inequality that

\[
-\frac{\epsilon}{2} Y^T EE^T Y - \frac{1}{2\epsilon} X^T X \leq Y^T EX \leq \frac{\epsilon}{2} Y^T EE^T Y + \frac{1}{2\epsilon} X^T X
\]

for some constant \(\epsilon > 0\). Then since

\[
(X^T, Y^T) \begin{pmatrix}
\tilde{A} & 0 \\
E & N
\end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = X^T \tilde{A} X + Y^T N Y + Y^T E X,
\]
we arrive at
\[
X^T \left( \tilde{A} - \frac{1}{2\epsilon} I \right) X + Y^T \left( N - \frac{\epsilon}{2} EE^T \right) Y \leq \left( X^T, Y^T \right) \left( \begin{array}{cc} \tilde{A} & 0 \\ E & N \end{array} \right) \left( \begin{array}{c} X \\ Y \end{array} \right) \\
\leq X^T \left( \tilde{A} + \frac{1}{2\epsilon} I \right) X + Y^T \left( N + \frac{\epsilon}{2} EE^T \right) Y.
\]

We then complete the proof of this Lemma due to the positive definite properties of \( \tilde{A} - \frac{1}{2\epsilon} I \) and \( N - \frac{\epsilon}{2} EE^T \) for some \( \epsilon > 0 \).

**Remark 5.3.** Note that if \( E = (a_{ij}) \), \( E^T \) denote \( (a_{kl}^T) \) where \( a_{kl}^T = a_{lk} \), then we can calculate the element \( b_{il} \) of \( EE^T \) is equal to \( \delta_{kj} a_{ik} a_{lj} \).

Now let us use this Lemma to examine the Condition (V). Firstly, we introduce a notation
\[
\text{Circ} [M] := \hat{M}(\tau) \quad \text{and} \quad \text{Tild} [M] := \tilde{M}(\tau, U, \tilde{V})
\]
respectively, provided there is a decomposition of a functional \( M(\tau, U, \tilde{V}) = \hat{M}(\tau) + \tilde{M}(\tau, U, \tilde{V}) \) where \( \hat{M}(\tau, 0, 0) = 0 \). Then, it is easy to conclude simple but crucial identities that
\[
\text{Circ} [M] = \hat{M}(\tau) = M(\tau, 0, 0), \quad \text{Circ}[M_1 M_2] = \text{Circ}[M_1] \text{Circ}[M_2] = \hat{M}_1 \hat{M}_2
\]
and
\[
\text{Circ}[M_1 + M_2] = \text{Circ}[M_1] + \text{Circ}[M_2] = \hat{M}_1 + \hat{M}_2
\]
for \( M_\ell = \hat{M}_\ell(\tau) + \tilde{M}_\ell(\tau, U, \tilde{V}) \) \((\ell = 1, 2)\).

In order to verify (A.6), we only concern
\[
Q = \frac{1}{\kappa} \mathbf{B} - \hat{B}^0 = \text{Circ} \left[ \frac{1}{\kappa} \mathbf{B} - \hat{B}^0 \right] = \frac{1}{\kappa} \mathbf{B}(\tau, 0, 0) - B^0(\tau, 0, 0),
\]
due to the fact that evidently \( \hat{B}^0 \geq \frac{1}{\gamma_1} \mathbb{I} \) for some constant \( \gamma_1 > 0 \) and \( \mathbf{B} \) is bounded. We only need to verify \( Q \) is positive definite, which implies \( \hat{B}^0 \leq \frac{1}{\kappa} \mathbf{B} \). Then, the corresponding matrices in the Lemma are selected by
\[
\tilde{A} = \text{Circ} \left( \frac{1}{\kappa} \mathbf{A} - A^0 \right)
\]
\[
N = \text{Circ} \left( -\frac{1}{\kappa} (\mathbf{N} - N^0) \right)
\]
and
\[
E = \text{Circ} \left( -\frac{1}{\kappa} (E_0 \delta^0_\mu + E_q \delta^q_\mu) \quad 0 \quad 0 \right).
\]
Then by Lemma 5.2, we only need to concentrate on examining
\[
\tilde{A} - \frac{1}{2\epsilon} \mathbb{I} = \text{Circ} \left( \frac{1}{\kappa} \mathbf{A} - A^0 - \frac{1}{2\epsilon} \mathbb{I} \right)
\]
\[
\frac{1}{\kappa} g^{00} \quad 0 \quad \frac{1}{\kappa} g^{00} \quad 0
\]
\[
0 \quad 0 \quad 0 \quad -\frac{1}{\kappa} g^{00} \quad 0
\]
and
\[
N - \frac{\epsilon}{2} EE^T = \text{Circ} \left( -(\frac{1}{\kappa} \mathbf{N} - N^0) - \frac{\epsilon}{2\kappa} (E_0 \delta^0_\mu + E_q \delta^q_\mu) (E_0 \delta^0_\nu + E_q \delta^q_\nu)^T \delta^{\mu\nu} \right)
\]
are both positive definite. Let us, with the help of (5.11), calculate

\[
\text{Circ}\left[\frac{1}{\kappa}A - A^0 - \frac{1}{2\epsilon}\mathbb{I}\right] = \begin{pmatrix}
\left(\frac{1}{\kappa} - 1\right)\omega^2 - \frac{1}{2\epsilon} & 0 & 0 \\
0 & \left(\frac{3}{2\kappa} - 1 - \frac{1}{2\epsilon}\right)\delta^{ij} & 0 \\
0 & 0 & \left(\frac{1}{\kappa} - 1\right)\omega^2 - \frac{1}{2\epsilon}
\end{pmatrix},
\]

(5.19)

and by noting that Circ [u^0] = -\omega and Circ [u_0] = \frac{1}{\omega}, and since \(q''(\alpha) \lesssim 1\), then

\[
\text{Circ}\left[-\frac{1}{\kappa}N - N^0\right] - \frac{\epsilon}{2\kappa^2}(E_0\delta_\mu^0 + E_q\delta^q_\mu)(E_0\delta_\nu^0 + E_q\delta^q_\nu)^T \delta^{uv} = \begin{pmatrix}
\frac{1}{\kappa}\lambda^2 \omega(3q'(\bar{\alpha}) - \chi(\tau)) - \lambda^2 \omega & 0 \\
0 & \frac{1}{\kappa}(1 - 3s^2) - \frac{1}{\kappa}\chi(\tau) - 1 & \delta_{ri}\omega - \frac{\epsilon}{2\kappa^2}S(\tau)\delta_{ri}
\end{pmatrix}
\]

(5.21)

where

\[
S(\tau) = \left(-\frac{1}{2\omega^2}\right)^2 + 2(1 - 3s^2)^2.
\]

Now we examine all the elements in above (5.19)–(5.21) are positive by recalling \(\hat{\delta}\) defined by (1.9) which is \(\hat{\delta} \in (0, (1 + \sqrt{\frac{3}{\Lambda}}) \min\{\frac{3}{4}, \frac{\Lambda}{3+\Lambda}\})\) and taking

\[
\kappa = \frac{1}{2}\left(1 + \sqrt{\frac{3}{\Lambda}}\right)^{-1} \hat{\delta} \quad \text{and} \quad \epsilon = \frac{1}{2},
\]

which implies (5.17) and (5.18) are both positive definite. With the help of (2.7) which implies \(\omega^2 \geq \frac{1}{3}\), we arrive at

\[
\left(\frac{1}{\kappa} - 1\right)\omega^2 - 1 \geq \left[\frac{2}{\delta}(1 + \sqrt{\frac{3}{\Lambda}}) - 1\right]A^3 - 1 \geq 1 + \frac{A}{3} > 0 \quad \text{(by } \hat{\delta} < (1 + \sqrt{\frac{3}{\Lambda}}) \frac{A}{3+\Lambda}),
\]

\[
\frac{3}{2\kappa} - 1 - 1 = 2\left(1 + \sqrt{\frac{3}{\Lambda}}\right)\frac{3}{2\delta} - 2 > 2 > 0 \quad \text{(by } \hat{\delta} < (1 + \sqrt{\frac{3}{\Lambda}}) \frac{3}{4}),
\]

\[
\frac{2}{\kappa}\omega^2 - 1 \geq 2\left(1 + \sqrt{\frac{3}{\Lambda}}\right)\frac{2\Lambda}{3\delta} - 2 > 2 + \frac{4}{3}\Lambda > 0 \quad \text{(by } \hat{\delta} < (1 + \sqrt{\frac{3}{\Lambda}}) \frac{A}{3+\Lambda}),
\]

with the help of Assumption 1.5.(1), we obtain

\[
\left[\frac{1}{\kappa}(1 - 3s^2 - \chi(\tau)) - 1\right]\omega - \frac{\tau^2}{4\kappa^3} = \frac{\tau^2}{\beta^2}S(\tau) \geq \left(\frac{\hat{\delta}}{\kappa} - 1\right)\sqrt{\frac{A}{3}} - \frac{1}{4\kappa^2}(C^*\hat{\delta})^2\left(\frac{9}{4\Lambda^2} + 2\right)
\]

\[
\geq \left(2\left(1 + \sqrt{\frac{3}{\Lambda}}\right)\frac{\hat{\delta}}{\delta} - 1\right)\sqrt{\frac{A}{3}} - 1 = 1 + \sqrt{\frac{A}{3}} > 0
\]

for any \(\tau \in [0, 1]\). Then we verified Condition (V).
Let us turn to **Condition (VI).** First calculate
\[
P^\perp = \begin{pmatrix}
(P^\perp) & 0 & 0 & 0 \\
0 & \Pi^\perp & 0 & 0 \\
0 & 0 & \Pi^\perp & 0 \\
0 & 0 & 0 & (P^\perp)^\perp
\end{pmatrix}.
\]
Then this condition means we need to examine
\[
(P^\perp)^\perp N^0(\tau, P^\perp(U, \tilde{V})^T)(P^\perp) = (P^\perp)^N(\tau, P^\perp(U, \tilde{V})^T)(P^\perp)^\perp = 0.
\]
This always holds since \((P^\perp)^\perp \equiv 0\). The other parts are easy to check and the same as the corresponding part in [30].

The last **Condition (VII) can be examined by noting** \((P^\perp)^\perp \equiv 0\) and using the derivations in [21, §7.1], which we omit here and readers can consult [21, §7.1] for the details.

Having verified that all of the hypotheses of Theorem A.1 are satisfied, we conclude that there exists a constant \(\sigma > 0\), such that if
\[
\|g^{\mu\nu}_0 - \eta^{\mu\nu}(1)\|_{H^{k+1}} + \|g^{\mu\nu}_0 - \partial_\mu \eta^{\nu}(1)\|_{H^k} + \|\rho_0 - \tilde{\rho}(1)\|_{H^k} + \|\nu^\parallel\|_{H^k} < \sigma,
\]
which, by Assumption 1.3, (1.37)–(1.48), (4.30)–(4.31) and \(\delta \zeta = \beta^{-1}(\mu^{-1})'(\rho K_{10})(\rho - \tilde{\rho})\), implies
\[
\|(U, \tilde{V})\|_{\tau = 1} ||_{H^k} \leq C\sigma,
\]
then by Theorem A.1, there exists a \(T_* \in (0, 1)\), and a unique classical solution \((U, \tilde{V}) \in C^1([T_*, 1] \times \mathbb{T}^3)\) that satisfies
\[
(U, \tilde{V}) \in C^0([T_*, 1], H^k) \cap C^1([T_*, 1], H^{k-1}),
\]
and the energy estimate
\[
\|(U, \tilde{V})\|_{H^k} \leq C\|(U, \tilde{V})\|_{\tau = 1} ||_{H^k} \leq C\sigma
\]
for all \(1 \geq \tau > T_*\), and can be uniquely continued to a larger time interval \((T^*, 1)\) for all \(T^* \in [0, T_*)\). Furthermore, above bound leads to the solutions \((U, \tilde{V})\) exist globally on \(\mathfrak{M} = (0, 1] \times \mathbb{T}^3\) and satisfy the estimates (5.23) with \(T_* = 0\). In particular, this implies, via the definition (5.1) of \((U, \tilde{V})\), that
\[
\|(u_0^0, u_j^0, u_0^0 u_j^0, u_0^0 u_j^0 u_j^0, u_0^0 u_j^0, u_0^0, u_j^0, \delta \zeta, v^p)\|_{H^k} \leq C\sigma.
\]
Furthermore, by (3.29)–(3.34), we obtain
\[
\|g^{\mu\nu} - \eta^{\mu\nu}\|_{H^{k+1}} + \|\partial_\mu g^{\nu\nu} - \partial_\nu \eta^{\mu\nu}\|_{H^k} \leq C\|U\|_{H^k} \leq C\sigma,
\]
and the transformations in Assumption 1.3 and 1.4 imply there exists a function \(\varrho \in C([0, 1], C^\infty(\mathbb{R}))\) satisfying \(\varrho(\tau, 0) = 0\) such that
\[
\rho - \tilde{\rho} = \tau^\varsigma \varrho(\tau, \beta^{-1}(\alpha - \tilde{\alpha})), \quad \varsigma \geq 2.
\]
Then Moser estimates for composition of functions (see, e.g. [21, Lemma A.3]) yield
\[
\|\rho - \tilde{\rho}\|_{H^k} \leq C\beta^{-1}(\alpha - \tilde{\alpha})\|_{H^k} \leq C\sigma,
\]
and using (4.30)–(4.31), that is, \(u^p = \beta(\tau)(v^p + 2\tau A u^0 p)\), we derive that
\[
\|u^p\|_{H^k} \leq \beta(\tau)\|v^p\|_{H^k} + 2\tau \beta(\tau) A(\tau)\|u^0 p\|_{H^k} \leq C\sigma
\]
for \(\tau \in (0, 1)\). In addition, (5.22) with above transformations and estimates yield (1.50). Then, we complete the proof of the main Theorem 1.13 for Makino-type (1) fluids.
5.1.3. **Proof for Makino-type (2) fluids.** As the previous section §5.1.2, Let us first gather the Einstein equations (3.23)–(3.25) and the Euler equations (4.47) together, recalling the definition of

\[ \mathbf{U} := (u^0, u^1, u^2, u^3, u_0, u_1, u_2, u_3)^T \quad \text{and} \quad \mathbf{\hat{V}} := (\delta \zeta, u_q)^T, \]

to get the complete non-degenerated singular symmetric hyperbolic system

\[ B^\mu \partial_\mu \left( \frac{\mathbf{U}}{\mathbf{\hat{V}}} \right) = \frac{1}{\tau} \mathbf{B} \mathbf{P} \left( \frac{\mathbf{U}}{\mathbf{V}} \right) + H \quad (5.24) \]

where

\[ \mathbf{B} = \begin{pmatrix} \mathbf{A} & 0 & 0 & 0 & 0 \\ 0 & -2g^{00} & 0 & 0 \\ 0 & 0 & -2g^{00} & 0 \\ 0 & 0 & 0 & \hat{\mathbf{N}} \end{pmatrix}, \quad \mathbf{P} = \begin{pmatrix} \mathbf{P}^* & 0 & 0 & 0 \\ 0 & \mathbf{I} & 0 & 0 \\ 0 & 0 & \mathbf{I} & 0 \\ 0 & 0 & 0 & \hat{\mathbf{P}}^\dagger \end{pmatrix} \]

and \( H = (\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3, \mathbf{H})^T \). To prove the main theorem in this case, our purpose is still applying Theorem A.1 in Appendix A. Thus we need to examine Conditions (I)–(VII) for equation (5.24).

**Conditions** (I)–(II) is easy to verify as the previous section when we apply Theorem A.1.

Next, in order to verify Condition (III), we demonstrate \( H \) is regular in all the variables, that is \( H \in C^0([0, 1], C^\infty(\mathcal{V})) \), by expanding the following crucial quantities. Differentiating (4.42) with respect to \( g^{\mu \nu} \), we arrive at

\[
\frac{\partial u^k}{\partial (g^{00})} = g^{k0} \left( \frac{1 + g_{ij}u_iu_j}{2g^{00}\sqrt{(g^{00})^2 - g^{00}(1 + g_{ij}u_iu_j)}} + \frac{\sqrt{(g^{00})^2 - g^{00}(1 + g_{ij}u_iu_j)}}{(g^{00})^2} \right)
\]

\[
= \tau \mathcal{M}_{00}^k(\tau, \mathbf{U}, \mathbf{\hat{V}}),
\]

\[
\frac{\partial u^k}{\partial (g^{0i})} = \delta_i^k \left( -g^{0i}u_q - \sqrt{(g^{0i}u_q)^2 - g^{00}(1 + g_{ij}u_iu_j)} \right)
\]

\[
+ g^{k0} \left( -u_i - \frac{g^{0i}u_i^2}{g^{00}\sqrt{(g^{0i}u_q)^2 - g^{00}(1 + g_{ij}u_iu_j)}} \right)
\]

\[
= - \frac{\delta_i^k}{\sqrt{-\eta^{00}}} + \mathcal{M}_{0i}^k(\tau, \mathbf{U}, \mathbf{\hat{V}}),
\]

and

\[
\frac{\partial u^k}{\partial (g^{ij})} = \delta_j^k u_i + \frac{g^{k0}u_iu_j}{2\sqrt{(g^{0j}u_q)^2 - g^{00}(1 + g_{ij}u_iu_j)}} = \mathcal{M}_{ij}^k(\tau, \mathbf{U}, \mathbf{\hat{V}}),
\]

where \( \mathcal{M}_{ij}^k(\tau, \mathbf{U}, \mathbf{\hat{V}}) \) agree with the convention of notation of remainders in §1.1.5 and satisfy \( \mathcal{M}_{ij}^k(\tau, 0, 0) = 0 \). Then by (3.29)-(3.40) and letting \( \beta(\tau) = 1 \), we easily get

\[
L_k^\mu \frac{\partial u^k}{\partial (g^{0\beta})} \partial_\mu g^{0\beta} = \mathcal{N}(\tau, \mathbf{U}, \mathbf{\hat{V}}), \quad (5.25)
\]
where \( \mathcal{N}(\tau, \mathbf{U}, \mathbf{V}) \) agrees with the convention of notation of remainders and satisfies \( \mathcal{N}(\tau, 0, 0) = 0 \). Next let us calculate terms involving the Christoffel terms. Direct calculations, with the help of that

\[
u^0 = g^{00}u_0 + g^{k0}u_k = -\sqrt{(g^{0i}u_i)^2 - g^{00}(1 + g^{ij}u_iu_j)},
\]
yield

\[
\Gamma^k_{\mu\nu}u^\mu u^\nu = \Gamma^k_{00}u^0 u^0 + 2\Gamma^k_{0i}u^0 u^i + \Gamma^k_{lm}u^m
- 2\Gamma^k_{0i}(g^{0i}u_i)^2 - g^{00}(1 + g^{ij}u_iu_j)(g^{00}u_0 + g^{im}u_m)
\]

\[
+ \Gamma^k_{lm}(g^{l0}u_0 + g^{lq}u_q)(g^{m0}u_0 + g^{mq}u_q)
= \mathcal{N}^k(\tau, \mathbf{U}, \mathbf{V}),
\]

(5.26)

where \( \mathcal{N}^k(\tau, \mathbf{U}, \mathbf{V}) \) agree with the convention of notation of remainders and satisfy \( \mathcal{N}^k(\tau, 0, 0) = 0 \). Some direct expansions, with the help of (4.21)-(4.24), (4.42), (4.43), (5.25)-(5.26), lead to \( H \in C^0([0, 1], C^{\infty}(\mathcal{V})) \).

As the previous section, in order to verify Condition (IV), the crucial condition is \( B^0 \in C^1([0, 1], C^{\infty}(\mathcal{V})) \) and we only need to check \( \dot{N}^0 \in C^1([0, 1], C^{\infty}(\mathcal{V})) \). Using (1.12) in Assumption 1.5, with the help of (4.42) which implies \( u^i(\tau, \mathbf{U}, \mathbf{V}) \in C^1([0, 1], C^{\infty}(\mathcal{V})) \), the similar derivations to the previous section (5.3)-(5.7) yield,

\[
D_\tau u^0(\tau, \mathbf{U}, \mathbf{V}) = \frac{1}{\omega} \frac{\Omega}{\tau} + S(\tau, \mathbf{U}, \mathbf{V}),
\]

(5.27)

\[
D_\tau u_0(\tau, \mathbf{U}, \mathbf{V}) = \frac{1}{\omega^3} \frac{\Omega}{\tau} + T(\tau, \mathbf{U}, \mathbf{V}),
\]

(5.28)

\[
D_\tau M_{ki}(\tau, \mathbf{U}, \mathbf{V}) = \Omega_{ki}(\tau, \mathbf{U}, \mathbf{V}),
\]

(5.29)

\[
D_\tau q(\alpha) = q'(\alpha) \frac{3}{\tau} \bar{q} \equiv 0 \quad \text{(Because } q \equiv \bar{q}),
\]

(5.30)

and by \( q \equiv \bar{q} \), Assumption 1.3.(1.10) and 1.4.(1.11), (4.15), (4.21), the bounds (2.6) of \( \rho(\tau) \), and

\[
(s^2)'(\alpha) = \frac{d}{d\alpha} \mu^*(dp/d\rho) = \frac{d^2p}{d\rho^2{|_{\mu(\alpha)}}} \frac{d\mu}{d\alpha} \bigg{|}_{\alpha},
\]

(5.31)

we arrive at

\[
D_\tau s^2(\alpha) = (s^2)'(\alpha) \frac{3}{\tau} \bar{q} = p''_{|_{\mu(\alpha)}} \frac{d\mu}{d\alpha} \bigg{|}_{\alpha} \frac{3}{\tau} \bar{q} = p''_{|_{\mu(\alpha)}} \frac{\mu(\alpha) + \mu^*(p(\alpha))}{q(\alpha)} \frac{3}{\tau} \bar{q}
\]

\[
= p''_{|_{\mu(\alpha)}} \frac{\mu(\alpha) - \mu(\bar{\alpha}) + \mu^*(p(\alpha) - \mu^*(p(\alpha))}{q(\alpha)} \frac{3}{\tau} \bar{q} + p''_{|_{\mu(\alpha)}} \frac{\mu(\bar{\alpha}) + \mu^*(p(\alpha))}{q(\alpha)} \frac{3}{\tau} \bar{q}
\]

\[
= 3p''_{|_{\mu(\alpha)}} (1 + c^2_s(\rho K_i)) \tau^c \bar{q} \leq 1,
\]

(5.32)

for \( c \geq 2 \) and \( \tau \in [0, 1] \). Note that (4.43) implies

\[
D_\tau J^{iq}(\tau, u^0, u_k) = J^{iq}(\tau, u^0, u_k),
\]

(5.33)

in other words, \( J^{ij} \in C^1([0, 1], C^{\infty}(\mathcal{V})) \). Assisted with above preparations (5.27)-(5.33), we can directly expand \( D_\tau \bar{N}^0 \) and it is evident to conclude \( B^0 \in C^1([0, 1], C^{\infty}(\mathcal{V})) \). The other parts of this conditions are easy to examine and we omit the details.
Then, let us turn to **Condition** (V). By (5.11), to verify (A.6), as before, we only concern

\[ Q = \frac{1}{\kappa} B - B^0 = \text{Circ} \left[ \frac{1}{\kappa} B - B^0 \right] = \frac{1}{\kappa} B(\tau, 0, 0) - B^0(\tau, 0, 0), \]

and to verify \( Q \) is positive definite. We focus on one element of \( Q \) that is \( \frac{1}{\kappa} \hat{N}(\tau, 0, 0) - \hat{N}^0(\tau, 0, 0) \).

There are two cases to proceed:

1. If \( \hat{N} \) and \( \hat{N}^0 \) are defined by (4.48), then, with the help of (4.21), (4.22), (4.42) and (4.43) (Note that \( u_q = 0 \) and \( u^{(i} = 0 \) imply \( v^i = 0 \) and further \( v^i = 0 \) by (4.30) and (4.42)),

\[
\frac{1}{\kappa} \hat{N}(\tau, 0, 0) - \hat{N}^0(\tau, 0, 0) = \left( \begin{array}{cc} (\frac{1}{\kappa} - 1) & 0 \\ 0 & \frac{1}{\kappa^2} (1 - 3\bar{s}^2) - 1 \end{array} \right) \delta^{ij}.
\]

2. If \( \hat{N} \) and \( \hat{N}^0 \) are defined by (4.49), then,

\[
\frac{1}{\kappa} \hat{N}(\tau, 0, 0) - \hat{N}^0(\tau, 0, 0) = \left( \begin{array}{cc} (\frac{1}{\kappa} - 1) & 0 \\ 0 & \frac{1}{\kappa^2} (1 - 1) \delta^{ij} \end{array} \right).
\]

By taking

\[
\hat{\kappa} := \left( 1 + \sqrt{\frac{3}{\Lambda}} \right)^{-1},
\]

above \( \frac{1}{\kappa} \hat{N}(\tau, 0, 0) - \hat{N}^0(\tau, 0, 0) \), by noting \( \hat{\kappa} < 3/4 \) due to (1.9), with the help of (1.10) and Assumption 1.5.(2), is positive definite in both cases. This, with the help of (5.14), (5.17), (5.19) and (5.20) and the derivations in the previous section §5.1.2, implies \( Q \) is positive definite, which, in turn, implies **Condition** (V).

For **Conditions** (VI) and (VII), if \( \hat{N} \) and \( \hat{P}^\dagger \) are defined by (4.49), Then Conditions (VI) and (VII) hold evidently due to \( \hat{P}^\dagger \equiv 0 \). We only need to examine the case when \( \hat{N} \) and \( \hat{P}^\dagger \) are defined by (4.48). In this case,

\[
P(U, \hat{V})^T = \left( \frac{1}{2} u^0, \mu, \frac{1}{2} u^0, \mu, \frac{1}{2} u^0, \mu \right)^T.
\]

To verify Condition (VI) is equivalent to verifying

\[
(P^\ast)\Pi A^0(\tau, P^\dagger(U, \hat{V})^T) &= P^\ast A^0(\tau, P^\dagger(U, \hat{V})^T)(P^\ast)^\dagger = 0,
\]

\[
\Pi A^0(\tau, P^\dagger(U, \hat{V})^T) \Pi &= \Pi A^0(\tau, P^\dagger(U, \hat{V})^T) \Pi = 0,
\]

\[
(P^\dagger)\Pi N^0(\tau, P^\dagger(U, \hat{V})^T) \Pi^\dagger &= \Pi N^0(\tau, P^\dagger(U, \hat{V})^T) \Pi^\dagger = 0.
\]

To obtain \( A^0(\tau, P^\dagger(U, \hat{V})^T) \) and \( N^0(\tau, P^\dagger(U, \hat{V})^T) \), we set \( P(U, \hat{V})^T = 0 \) in the variables \( (U, \hat{V})^T \) of \( A^0 \) and \( N^0 \). Direct calculations yield

\[
(P^\ast)\Pi A^0(\tau, P^\dagger(U, \hat{V})^T)P^\ast = P^\ast A^0(\tau, P^\dagger(U, \hat{V})^T)(P^\ast)^\dagger
\]

\[
= \left( \begin{array}{ccc} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & g^{00} & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{array} \right) \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \delta^j_i & 0 \end{array} \right) = 0,
\]

\[
\Pi A^0(\tau, P^\dagger(U, \hat{V})^T) \Pi &= \Pi A^0(\tau, P^\dagger(U, \hat{V})^T) \Pi = 0.
\]

\[
= \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & g^{ij} & 0 \\ 0 & 0 & -g^{00} \end{array} \right) \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) = 0
\]

(5.35) (5.36)
and
\[
(\hat{\mathbf{P}}^\dagger)^\perp \hat{\mathcal{N}}^0(\tau, \mathbf{P}^\perp(\mathbf{U}, \mathbf{V})^T) \hat{\mathbf{P}}^\dagger = \hat{\mathbf{P}}^\dagger \hat{\mathcal{N}}^0(\tau, \mathbf{P}^\perp(\mathbf{U}, \mathbf{V})^T)(\hat{\mathbf{P}}^\dagger)^\perp
\]
\[
= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{N^2} M_{ki} J^{kj} J^{iq} \end{pmatrix} \bigg|_{\mathbf{P}(\mathbf{U}, \mathbf{V})^T=0} \begin{pmatrix} 0 & 0 \\ 0 & \delta^q_i \end{pmatrix} = 0.
\]
(5.37)

Note that we have set \(u_i = 0\) (from (5.34) and \(\mathbf{P}(\mathbf{U}, \mathbf{V})^T = 0\)) in (5.37), which leads to \(L^0_0 = 0\) in \(\hat{\mathcal{N}}^0\). This completes the verification of Condition (VI).

At the end, the examination of Condition (VII) is the same as the previous work in [30, §3] and [21, §7]. To avoid repeating the examination, we only note a crucial identity in the proof that is
\[
(\hat{\mathbf{P}}^\dagger)^\perp[\mathcal{D}(\mathbf{U}, \mathbf{V})^T \hat{\mathcal{N}}^0(\tau, \mathbf{U}, \mathbf{V})^T] (\hat{\mathbf{P}}^\dagger)^\perp \equiv 0,
\]
(5.38)
for any \(\mathbf{W} \in \mathbb{V}\). The same derivations, with the help of the identity (5.38), conclude Condition (VII).

Having verified that all of the hypotheses of Theorem A.1 are satisfied, we can conclude the main Theorem 1.13 via the similar arguments to the previous section §5.1.2, but only noting that, in this case, the invertible transformation between \(u_q\) and \(u^i\) is given by (4.41) and (4.42), and then they can be controlled by each other with the gravitational variables,
\[
\|u_q\|_{H^k} \lesssim \|(\mathbf{U}, u^i)\|_{H^k} \quad \text{and} \quad \|u^i\|_{H^k} \lesssim \|(\mathbf{U}, u_q)\|_{H^k}.
\]

We omit the reduplicate details and complete the proof of the main Theorem 1.13 for Makino-type (2) fluids.

**Appendix A. A class of symmetric hyperbolic systems**

In this Appendix, we introduce the main tool for this article which is a variation of the theorem originally established in [30, Appendix B]. The proof of it has been omit, but readers can see the details\(^7\) in [30], and its generalizations in [21] and [20].

Consider the following symmetric hyperbolic system.
\[
B^\mu \partial_\mu u = \frac{1}{t} \mathbf{B} \mathbf{P} u + H \quad \text{in } [T_0, T_1] \times \mathbb{T}^n, \\
u = u_0 \quad \text{in } T_0 \times \mathbb{T}^n,
\]
(A.1)
(A.2)

where we require the following Conditions:

(I) \(T_0 < T_1 \leq 0\).

(II) \(\mathbf{P}\) is a constant, symmetric projection operator, i.e., \(\mathbf{P}^2 = \mathbf{P}\), \(\mathbf{P}^T = \mathbf{P}\) and \(\partial_\mu \mathbf{P} = 0\).

(III) \(u = u(t, x)\) and \(H(t, u)\) are \(\mathbb{R}^N\)-valued maps, \(H \in C^0([T_0, 0], C^\infty(\mathbb{R}^N))\) and satisfies \(H(t, 0) = 0\).

(IV) \(B^\mu = B^\mu(t, u)\) and \(\mathbf{B} = \mathbf{B}(t, u)\) are \(\mathbb{M}_{N \times N}\)-valued maps, and \(B^\mu, \mathbf{B} \in C^0([T_0, 0], C^\infty(\mathbb{R}^N))\), \(B^0 \in C^1([T_0, 0], C^\infty(\mathbb{R}^N))\) and they satisfy
\[
(B^\mu)^T = B^\mu, \quad [\mathbf{P}, \mathbf{B}] = \mathbf{P} \mathbf{B} - \mathbf{B} \mathbf{P} = 0.
\]
(A.3)

\(^7\)A minor revision and improvement about the condition (VII) has been included in arXiv:1505.00857v4 and only (A.7) is necessary in our case. An alternative expression of this condition is given in [21] and [20].
(V) Suppose
\[ B^0 = \bar{B}^0(t) + \bar{B}^0(t, u) \]  
(A.4)

and
\[ B = \bar{B}(t) + \bar{B}(t, u) \]  
(A.5)

where \( \bar{B}^0(t, 0) = 0 \) and \( \bar{B}(t, 0) = 0 \). There exists constants \( \kappa, \gamma_1, \gamma_2 \) such that
\[ \frac{1}{\gamma_1} \leq \bar{B}^0 \leq \frac{1}{\kappa} B \leq \gamma_2 \]  
(A.6)

for all \( t \in [T_0, 0] \).

(VI) For all \((t, u) \in [T_0, 0] \times \mathbb{R}^N\), we have
\[ P^\perp B^0(t, P^\perp u) P = PB^0(t, P^\perp u) P^\perp = 0, \]
where \( P^\perp = I - P \) is the complementary projection operator.

(VII) There exists constants \( \zeta, \beta_1 \) and \( \varpi > 0 \) such that
\[ |P^\perp[D_u B^0(t, u)(B^0)^{-1} BP u]P^\perp|_{op} \leq |t| \zeta + \frac{2\beta_1}{\varpi + |P^\perp u|^2} |Pu|^2, \]  
(A.7)

**Theorem A.1.** Suppose that \( k \geq \frac{n}{2} + 1 \), \( u_0 \in H^k(\mathbb{T}^n) \) and conditions (I)–(VII) are fulfilled. Then there exists a \( T_* \in (T_0, 0) \), and a unique classical solution \( u \in C^1([T_0, T_*], H^k) \cap C^1([T_0, T_*], H^{k-1}) \) and the energy estimate
\[ \|u(t)\|_{H^k}^2 - \int_{T_0}^t \frac{1}{2} \|Pu\|_{H^k}^2 d\tau \leq Ce^{C(t-T_0)}(\|u(T_0)\|_{H^k}^2) \]
for all \( T_0 \leq t < T_* \), where \( C = C(\|u\|_{L^\infty([T_0, T_*], H^k)}), \gamma_1, \gamma_2, \kappa) \), and can be uniquely continued to a larger time interval \([T_0, T^*] \) for all \( T^* \in (T_0, 0) \) provided \( \|u\|_{L^\infty([T_0, T_*], W^{1, \infty})} < \infty \).

Let us end this Appendix with a remark of the proof although we omit the detailed proof.

**Remark A.2.** We give the key revision in the proof due to the changing of Condition (V), (i.e. (A.4)–(A.6)). As in [21, Page 2203], we rewriting \( \bar{B}^0 \), as \( \bar{B}^0 = (\bar{B}^0)^{1/2}(\bar{B}^0)^{1/2} \), which we can do since \( \bar{B}^0 \) is a real symmetric and positive-definite, we see from (A.6) that
\[ (\bar{B}^0)^{-\frac{1}{2}} B (\bar{B}^0)^{-\frac{1}{2}} \geq \kappa I. \]  
(A.8)

Since, by (A.3)–(A.5)
\[ \frac{2}{t} \langle D^\alpha u, BD^\alpha Pu \rangle = \frac{2}{t} \langle D^\alpha Pu, (\bar{B})^{1/2}(\bar{B}^{1/2} B (\bar{B}^{1/2}) D^\alpha Pu) + \frac{2}{t} \langle D^\alpha Pu, \bar{B} D^\alpha Pu \rangle \]
\[ \leq \frac{2\kappa}{t} \langle D^\alpha Pu, \bar{B} D^\alpha Pu \rangle + \frac{2}{t} \langle D^\alpha Pu, \bar{B} D^\alpha Pu \rangle \]
\[ = \frac{2\kappa}{t} \langle D^\alpha Pu, B D^\alpha Pu \rangle - \frac{2\kappa}{t} \langle D^\alpha Pu, \bar{B} D^\alpha Pu \rangle + \frac{2}{t} \langle D^\alpha Pu, \bar{B} D^\alpha Pu \rangle. \]

---

\footnote{This variation of the original condition (v) and (B.3) in [30] facilitate the examinations of the conditions of this theorem, and the proof is easy to recover by minor corrections. Note that the \( \tau \)-singular terms caused by \( \bar{B}(t, u) \) and \( B(t, u) \) can be absorbed into the principle singular term with the good sign. We give a Remark A.2 on the key revision in the proof. See the proof in [30] or more details in [21, §5] involving such variations.}
It follows immediately from (A.8) that
\[
\frac{2}{t} \sum_{0 \leq |\alpha| \leq s-1} \langle D^\alpha u, B D^\alpha P u \rangle \leq \frac{2\kappa}{t} \langle \|P u\|_{H^{s-1}}^2 \rangle_I - \frac{1}{t} C \langle \|u\|_{H^{s-1}} \|P u\|_{H^{s-1}}^2 \rangle_{II},
\] (A.9)
where
\[
\|u\|_{H^k}^2 := \sum_{0 \leq |\alpha| \leq k} \langle D^\alpha u, B^0(t, u(t)) D^\alpha u \rangle.
\]
Then, the following proof are the same as [30, Appendix B] or [21, §5] just noting that that the term II in (A.9) can be absorbed by I in the rest of estimates provided the data is small enough.

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