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Cosmological perturbations for two cold fluids in $\Lambda$CDM

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ABSTRACT
The cosmic large-scale structure of our Universe is comprised of baryons and cold dark matter (CDM). Yet it is customary to treat these two components as a combined single-matter fluid with vanishing pressure, which is justified only for sufficiently large scales and late times. Here, we go beyond the single-fluid approximation and develop the perturbation theory for two gravitationally coupled fluids while still assuming vanishing pressure. We mostly focus on perturbative expansions in powers of $D$ (or $D_1$), the linear structure growth of matter in a $\Lambda$CDM Universe with cosmological constant $\Lambda$. We derive in particular (1) explicit recursion relations for the two fluid densities, (2) complementary all-order results in the Lagrangian-coordinates approach, as well as (3) the associated component wavefunctions in a semiclassical approach to cosmic large-scale structure. In our companion paper, we apply these new theoretical results to generate novel higher order initial conditions for cosmological hydrodynamical simulations.

Key words: dark matter – large-scale structure of Universe – cosmology: theory.

1 INTRODUCTION
Analytical models for predicting the cosmic large-scale structure (LSS) are indispensable for interpreting cosmological observations, especially at high redshifts where cosmological perturbation theory (PT) is meaningful. In particular, accurate theoretical modelling is needed to extract and interpret cosmological data from the baryonic acoustic oscillation features imprinted in the statistics of the LSS (Beutler et al. 2011; Blake et al. 2011; Dawson et al. 2013; Slepian & Eisenstein 2015; Slepian et al. 2018), or from tracers of the intergalactic medium that can be probed through absorption lines in the Ly$\alpha$ forest (McDonald et al. 2006; Chang et al. 2010). Furthermore, having accurate PT predictions at hand is essential to reduce theoretical uncertainties in the initial conditions for cosmological simulations; see e.g. Crocce, Pueblas & Scoccimarro (2006), Garrison et al. (2016), and Michaux et al. (2020).

According to the standard model of cosmology, dubbed $\Lambda$CDM, our Universe is comprised of cold dark matter (CDM), baryons, dark energy, and relativistic species (radiation and neutrinos). After an initial inflationary phase of accelerated expansion, these species are effectively coupled to each other through gravito-electroweak interactions. In full generality, the evolution of the relativistic and non-relativistic species is governed by the Einstein–Boltzmann equations. Predicting the LSS thus amounts to solving for these highly non-linear equations, which at this stage is not feasible. Instead, it is customary to dissect the full problem into individual sub-problems and solve them for given temporal and spatial scales. For this analytical insight is of utmost importance, especially considering that some of the sub-problems are not decoupled, as we elucidate briefly in the following.

Dark energy is believed to be described by a cosmological constant $\Lambda$ – which has no spatial dependence, and thus affects the evolution of the matter species through the global expansion of the Universe (parametrized by the usual Friedmann equations). The peculiar motion of matter, superimposed on the global expansion, is, however, not decoupled from the Friedmann equations and manifest, for example, as the Hubble drag in the law of momentum conservation.

Also radiation – photons and massless neutrinos, affect the matter evolution. Before decoupling, radiation has a predominant effect, especially through Compton scattering with baryons. As the Universe expands, Compton scattering becomes ineffective and radiation largely decouples from the peculiar matter evolution. This is because the mean density of radiation decays faster than the one of matter as the Universe expands, such that the impact of radiation on matter becomes less significant at late times. This argument can be demonstrated rigorously: Indeed, within the weak-field limit of general relativity and by employing tailor-made coordinate transformations, Fidler et al. (2017) has shown that the non-linear general relativistic equations of motion for matter can be brought precisely into the form of the Newtonian equations, a.k.a. the Euler–Poisson equations, which do not possess any couplings to radiation fluctuations. Their analysis reveals that any residual coupling between radiation and matter can be incorporated into a coordinate transformation, implying that a Newtonian theory (or simulation) for the matter evolution is meaningful on the considered scales. Additionally, the approach of Fidler et al. (2017) provides explicit instructions how the residual couplings can be efficiently incorporated a posteriori, i.e. after the Newtonian evolution was solved for (e.g. through an $N$-body simulation). Surprisingly, similar simplifications apply also when massive neutrinos are included, which has been very recently...
demonstrated by Partmann et al. (2020). None the less, we remark that there are recent attempts for incorporating massive neutrinos in an active manner, see e.g. the numerical approaches of Brandbyge et al. (2017) and Tram et al. (2019) or the analytical approaches of Blas et al. (2014b) and Aviles & Banerjee (2020).

Thus, determining the LSS can be effectively reduced to solving the non-linear equations of baryons and CDM. The problem focused originally on solving not for the individual baryons and CDM but for a combined, single-matter fluid, governed by the Euler–Poisson equations for sufficiently early times; see e.g. Bernardeau et al. (2002) for a review. These single-fluid equations can be solved using PT, either in Eulerian or Lagrangian coordinates. For the former, limitations of Eulerian PT have been known for quite a while; see e.g. Blas, Garny & Konstandin (2014a), Barnabe (2019), and Chen & Pietroni (2020).

Our approach is possibly closer to the one of Somogyi & Smith (2010) and Bernardeau et al. (2012) who also work in the limit of vanishing pressure, thereby assuming effectively two separate sets of identical fluid equations for CDM and baryons that are connected via a shared gravitational potential. Furthermore, while Somogyi & Smith (2010) developed a renormalized PT for the two-fluid set-up – and Bernardeau et al. (2012) the eikonal approximation, which could potentially also model some shell-crossing effects, we are here focusing on times when shell-crossing dynamics are not yet dominant. One of our motivations is to develop the necessary tools to provide accurate initial conditions for two-fluid cosmologies, both in Eulerian and Lagrangian coordinates that we directly exploit and compare against similar numerical avenues (e.g. Angulo, Hahn & Abel 2013; Valkenburg & Villaescusa-Navarro 2017; Bird et al. 2020) in our companion paper (Hahn, Rampf & Uhlemann 2020).

In this paper, we develop various PT approaches for the cold two-fluid model. The general methodology aims to take all decaying modes into account. None the less, one of our main focuses here is to exploit certain boundary conditions that select, in the two-fluid case, the fastest growing modes as well as a constant mode in the difference of the linear fluid densities, which amounts to including the leading-order effects in the considered two-fluid model. For these growing-mode solutions, we are actually able to provide explicit recursion relations to all orders. In addition to the classical approaches in Eulerian and Lagrangian coordinates, we also extend here the semiclassical description of Uhlemann et al. (2019), called propagator perturbation theory (PPT), by generalizing their findings to a $\Lambda$CDM cosmology and to allow for two coupled fluids. Apart from circumventing some of the shortcomings of Eulerian PT, which is in particular resolving the inaccuracies of modelling convective motion (see Section 7 for further arguments), PPT is particularly suited for Ly$\alpha$ studies (Porqueres et al. 2020) as well as generating initial conditions for simulations that require Eulerian fields as input.

It is worth noting here an appropriate physical picture for our model. As we elucidate in detail later on, the common gravitational potential of the baryonic and CDM components is sourced by the sum of their weighted densities that, by definition, is the density of a total matter fluid. Now, if that single-matter source in the gravitational potential is described in terms of purely growing-mode solutions (as it is very common in the literature), then it becomes evident that the individual fluid motion of the baryonic and CDM components must be identical in the growing mode, simply as a consequence of Newton’s second law of motion. The component densities, however, generally differ, due to employing the boundary conditions that come with growing-mode solutions. Thus, in some sense, the employed approach for the growing-mode boils down to propagating initial density fluctuations along the paths of the respective fluids.

This paper is structured as follows. In Section 2, we begin with the Eulerian-coordinates approach for a single matter fluids, and explain the appropriate boundary conditions for selecting growing-mode solutions, which also are crucial for avenues beyond single fluid models. Section 3 generalizes the approach to two shared fluid components, where we report explicit all-order recursion relations for the difference of the fluid densities arising from non-decaying modes. In Section 4, we review the Lagrangian-coordinates approach for a single fluid, while we generalize to two fluids in Section 5. A variational approach to single and two fluids, which employs contact geometry (an extension of symplectic geometry) is introduced in Section 6, which largely serves as a classical analogue of the semiclassical description that we discuss in Section 7. Finally, we summarize our results and provide an outlook in Section 8.
1.1 Notation and nomenclature

We denote Eulerian coordinates with $x$ and Lagrangian coordinates with $q$. We use italic Latin letters for referring to spatial indices, summation over repeated indices is implicitly assumed, and we denote partial derivatives with a comma, i.e. $\nabla_x F = F_{,i}$. When there is risk of confusion, we reserve the comma notation for Lagrangian derivatives and the slash notation for Eulerian space derivatives, i.e. $\nabla_x G = G_{j}$. As regards to temporal derivatives, we use the overdot for denoting the Lagrangian (convective) time derivative with respect to the linear growth time $D$. When a single fluid is considered, we attach an ‘m’ to the fields, while in the two-fluid case the individual fluids have the roman labels ‘b’ and ‘c’, which are occasionally summarized with a (non-running!) Greek label $\alpha = b, c$.

2 SINGLE FLUID IN EULERIAN SPACE

We begin by introducing the basic equations for a single cosmological fluid with vanishing pressure in the Newtonian limit, which are usually called Euler–Poisson equations. Throughout this paper, we assume a $\Lambda$CDM Universe. After formulating the Euler–Poisson equations in suitable coordinates in Section 2.1, for which we employ the linear growth factor $D$ as the time variable, we discuss the used boundary conditions in Section 2.2 that justify power expansions around $D = 0$. Alternative perturbative expansions schemes are common in the literature and discussed in Section 2.3.

2.1 Basic equations

We define the peculiar velocity with $\mathbf{u}_m = \partial_t \mathbf{x}$ where $\mathbf{x} = r/a$ are the usual comoving coordinates and $a$ the cosmic scale factor that grows as $a(t) = \Omega_0^{1/3}$ for small $t$. We surmount dependent variables with a tilde when the cosmic time $t$ is used as an independent time variable. Further, we define the matter density contrast $\tilde{\delta}_m = (\tilde{\rho} - \bar{\rho}(t))/\bar{\rho}(t)$, where $\bar{\rho}(t) \sim a^{-3}$. In these variables, the Eulerian fluid equations for a $\Lambda$CDM Universe are (cf. Rampf, Villone & Frisch 2015)

$$\partial_t \mathbf{u}_m + \mathbf{u}_m \cdot \nabla \mathbf{u}_m = -2H\mathbf{u}_m - \frac{1}{a^2} \nabla \tilde{\phi}, \quad (1a)$$

$$\partial_t \tilde{\delta}_m + \nabla \cdot [(1 + \tilde{\delta}_m) \mathbf{u}_m] = 0, \quad (1b)$$

$$\nabla^2 \tilde{\phi} = 3 \frac{a^2}{H^2} \tilde{\delta}_m, \quad (1c)$$

where $H = (\partial_t a)/a$ is the Hubble parameter governed by the Friedmann equation, defined here with $H^2 = 1/a^2 + \Lambda$ where $\Lambda = \Omega_\Lambda/\Omega_m$. Linearizing the fluid equations, one finds the standard ODE

$$\partial_t^2 \tilde{\delta}_m + 2H\partial_t \tilde{\delta}_m = \frac{3}{2a^2} \tilde{\delta}_m. \quad (2)$$

Its solution is most easily obtained by changing from cosmic time to $a$-time: the growing-mode solution is

$$D(a) = a \sqrt{1 + \Lambda a^2/2F(3/2, 5/6, 11/6, -\Lambda a^2)} , \quad (3)$$

where $F$ is the Gauss hypergeometric function, while the other solution is decaying as $\sqrt{1 + \Lambda a^2 a^{-3/2}}$ (see e.g. Demianski, Golda & Woszczyna 2005).

Analytic solutions for arbitrary short times are only feasible when growing-mode solutions are selected. Indeed, the decaying solution blows up for $a \to 0$ invalidating linearization, while the growing-mode solution is analytic and has the small-$a$ expansion $D(a) = a - (2/11)\Lambda a^2 + O(a^2)$. When we later seek perturbative expansions in powers of the growing-mode $D$, it will turn out to be essential to change the temporal dependence in the fluid equations to the growing mode $D$. Defining a new velocity variable $v_m = \partial_D \mathbf{x} = \mathbf{u}_m/\partial_D D$ and setting $\delta_m(t) = \delta_m(D)$, we can recast the fluid equations to

$$\partial_D v_m + v_m \cdot \nabla v_m = \frac{3}{2D} (v_m + \nabla \tilde{\phi}), \quad (4a)$$

$$\partial_D \delta_m + \nabla \cdot [(1 + \delta_m) v_m] = 0, \quad (4b)$$

$$\nabla^2 \tilde{\phi} = \frac{\delta_m}{D}, \quad (4c)$$

where $\tilde{\phi} = 3D \tilde{\phi}/(2a)$, and we have defined

$$g = g(D) = (D/\partial_D D)^2 a^2 = 1 + \Lambda D^4/11 + O(D^6), \quad (5)$$

which is analytic for small $D$ (and for small $a$, too). Thus, $g \approx 1$ up to third order in PT (see also Fig. 1), justifying the validity of the following approximation of the ODE

$$\partial_D^2 \delta_m + \frac{3}{2D} \partial_D \delta_m - \frac{3}{2D^2} \delta_m = 0, \quad (6)$$

which has the general solution

$$\delta_m = D C^m + D^{-3/2} C^m . \quad (7)$$

Here, $C^m_0$ and $C^m$ are the spatial integration constants for the standard growing and decaying solutions of linear density fluctuations, which can be fixed by providing suitable boundary conditions to (6) at sufficiently early times $D_{ai}$. Buchert (1992) has shown that Zel’dovich-like solutions can be achieved with two types of boundary conditions, which either achieve $\delta_m(D_{ai}) = 0$ exactly, or to a very
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good approximation, assuming initial quasi-homogeneity; see also section 3 of Rampf & Buchert (2012) for further details. Henceforth, when discussing solutions including decaying modes we shall use the common assumption \( \delta_m(D_{ini}) = 0 \). This setting is actually essential for growing-mode solutions, as we elucidate in the following.

2.2 Growing-mode solutions and slaving

Observe that equations (4) are analytic for \( D \to 0 \) provided we impose the slaved boundary conditions (Brenier et al. 2003)

\[
\delta^{ini}_m = 0, \quad v^{ini}_m = -\nabla \varphi^{ini}_m, \quad (8)
\]

where ‘\( ini \)’ denotes evaluation at \( D = 0 \). As argued by Rampf et al. (2015), these boundary conditions imply initial quasi-homogeneity with zero vorticity (\( \nabla \times v = 0 \)). Furthermore, they provide the mathematical foundation for growing-mode solutions of the form

\[
\delta_m = \sum_{n=1}^{\infty} \delta_m^{(n)}(x) D^n, \quad \theta_m = \nabla \cdot v_m = -\sum_{n=1}^{\infty} \theta_m^{(n)}(x) D^{n-1}, \quad (9)
\]

where \( \delta^{(n)}_m \) and \( \theta^{(n)}_m \) are the temporal coefficients that can be easily determined. For reference, the first- and second-order solutions are

\[
\begin{align*}
\delta^{(1)}_m &= \nabla \varphi^{ini}_m, \\
\delta^{(2)}_m &= \frac{5}{7} \varphi^{ini}_m \varphi^{ini}_m + \varphi^{ini}_m \varphi^{ini}_m + \frac{4}{7} \varphi^{ini}_m \varphi^{ini}_m,
\end{align*}
\]

and likewise for the velocity (and displacement) field, where \( T \) denotes partial differentiation with respect to component \( x_i \), and summation over repeated indices is assumed. We again like to stress that in the present expansion scheme, assuming \( g = 1 \) is exact up to fourth order in \( PT \). Of course, other expansions may be employed, which we discuss next.

2.3 Alternative expansions in \( \Lambda CDM \)

While the perturbative solutions (10) are well known (e.g. Bernardeau et al. 2002), their derivation is usually not based on a strict \( D \) expansion. Instead, it is customary to either employ fitting functions, or to derive the results in the Einstein–de Sitter (EdS) approximation and replace the respective growth functions according to \( a \to D \), which yields fairly accurate approximations to the \( \Lambda CDM \) equations (see e.g. Pietroni 2008; Hiramatsu & Taruya 2009).

Apart from a strict expansion in the growth of the mode, one could also solve for the temporal coefficients at each order separately. In our language, this amounts to incorporating the late-time evolution of \( g \) as it could be already relevant at low perturbative orders. Such avenues may become relevant particularly when decaying modes are taken into account. In this scenario, one may impose a perturbative expansion in the ‘weak’ sense, i.e. to not fix a physical expansion parameter and instead assume a certain smallness in the fields

\[
\delta_m = \epsilon \delta^{(1)}_m + \epsilon^2 \delta^{(2)}_m + \ldots, \quad (11)
\]

and likewise for the velocity (and displacement) field, where \( \epsilon \) is a small perturbation parameter that may be set to unity, once the perturbative equations are determined. Calculational details are provided in Appendix A, while the results for the densities are

\[
\begin{align*}
\delta^{(1)}_m &= D \nabla^2 \varphi^{ini}_m, \\
\delta^{(2)}_m &= \frac{D^2 - E}{2} \varphi^{ini}_m \varphi^{ini}_m + D^2 \varphi^{ini}_m \varphi^{ini}_m + \frac{D^2 + E}{2} \varphi^{ini}_m \varphi^{ini}_m. 
\end{align*}
\]

where \( E \) is a second-order temporal coefficient, subject to the ODE

\[
\frac{\partial}{\partial D} E + \frac{3g}{2D} \frac{\partial}{\partial D} E - \frac{3g}{2D^2} E = -\frac{3g}{2}. \quad (13)
\]

When \( g = 1 \), as employed for growing-mode expansions, the solution to this ODE can be analytically determined, with the fastest growing mode \( E \to -(3/7)D \). Numerical results to equation (13) can be found in Bouchet et al. (1995), while analytical solutions involving hypergeometric functions are provided by Matsubara (1995). In Fig. 1, we show that, for the two temporal coefficients involved, the analytical solutions for \( g = 1 \), against their respective numerical solutions where \( g \) is fully taken into account. The agreement between the analytical ‘approximations’ and the numerical solutions is excellent. We thus conclude that \( g \) can be safely set to unity, at least for the present task. Similar conclusions have been drawn by Tram et al. (2016) who also investigated numerically the solution of the ODE (13); however, their numerical result departs at late times close to \( a \approx 1 \) from the analytical prediction by a few percent (see their fig. 6), which we speculate may be of numerical nature.

We note again that when (all) decaying modes are included in the analysis, which is not our main focus, the late-time evolution of \( g \) could eventually become important at successive higher orders.

3 TWO COLD FLUIDS IN EULERIAN SPACE

We now turn to the Eulerian formulation for two fluids. These two fluids can be thought of as modelling the individual evolution of baryons and CDM in our Universe, hence the labels ‘b’ and ‘c’ that we use throughout this paper. None the less, we remark that we do not consider the effects of baryonic pressure, which limits our theoretical predictions to scales larger than the Jeans length. After introducing the relevant equations and linearized solutions including decaying modes in Section 3.1, we generalize the slaving conditions to the two-fluid case and derive all-order recursion solution in Section 3.2. Note that the present considerations can be easily generalized to more than two fluids; we come back to this in Section 8.

3.1 General formalism

Consider two fluids \( \alpha = b, c \) that are gravitationally coupled via

\[
\begin{align*}
\partial_D v_a + v_a \cdot \nabla v_a &= -\frac{3g}{2D} (v_a + \nabla \varphi), \\
\partial_D \delta_a + \nabla \cdot [(1 + \delta_a) v_a] &= 0, \\
\nabla^2 \varphi &= \frac{\delta_m}{D},
\end{align*}
\]

where

\[
\delta_m = f_b \delta_b + f_c \delta_c, \quad f_b + f_c = 1, \quad (15)
\]

with the present (baryon) fraction \( f_b = \Omega_b/\Omega_m \), with \( \Omega_m = \Omega_b + \Omega_c \). To proceed, it is instructive to work with the sum and difference of the Euler equation (14a) for the fluid components. A ‘sum’ equation for (14a) is obtained by first multiplying the equation for \( b \) by \( f_b \) and \( c \) by \( f_c \), and sum up those equations. Similarly, one proceeds with (14b). Linearizing the sum and difference equations we have

\[
\begin{align*}
\partial_D v_m &= -
\frac{3g}{2D} (v_m + \nabla \varphi), \quad \partial_D \delta_m + \nabla \cdot v_m = 0, \\
\partial_D v_{bc} &= -
\frac{3g}{2D} v_{bc}, \quad \partial_D \delta_{bc} + \nabla \cdot v_{bc} = 0,
\end{align*}
\]

\[
\nabla^2 \varphi = \frac{\delta_m}{D}. \quad (16c)
\]

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(see App. B3 for a non-perturbative generalization of these equations), where \( \delta_{bc} = \delta_{b} - \delta_{c} , v_{bc} = v_{b} - v_{c} , \) and \( w_{bc} = f_{b}w_{b} + f_{c}w_{c}. \) The linearized equations for the sum and difference decouple (cf. Schmidt 2016), leading to the separate evolution equations
\[
\frac{\partial^2 \delta_{m}}{\partial D^2} + \frac{3g}{2D} \partial_{D} \delta_{m} = \frac{3g}{2D^2} \delta_{m},
\]
\[
\frac{\partial^2 \delta_{bc}}{\partial D^2} + \frac{3g}{2D} \partial_{D} \delta_{bc} = 0.
\]
For \( g = 1 \) (justified in Section 2.3 and through Fig. 1), the general analytical solutions are
\[
\delta_{m} = DC_{m}^{+} + D^{-1/2}C_{m}^{-}, \quad \delta_{bc} = \delta_{bc}^{ini} - 2D^{-1/2}C_{bc}^{-}.
\]
which likewise can be used to express the general solutions for the components at first order:
\[
\delta_{b} = DC_{b}^{+} + D^{-1/2}C_{b}^{-} - 2D^{-1/2}f_{b}C_{bc}^{+} + \delta_{bc}^{ini},
\]
\[
\delta_{c} = DC_{c}^{+} + D^{-1/2}C_{c}^{-} + 2D^{-1/2}f_{c}C_{bc}^{+} + \delta_{bc}^{ini},
\]
where the initial densities are given by \( \delta_{b}^{ini} = f_{b}\delta_{bc}^{ini} \) and \( \delta_{c}^{ini} = -f_{c}\delta_{bc}^{ini}. \) As we shall show in the following, these initial densities cannot be set to zero in general, especially not when growing-mode solutions are considered.

3.2 Slaving and all-order growing-mode solutions
Similar to the single-fluid case, we can deduce from the fluid equations (16) the necessary conditions that guarantee their regularity for \( D \to 0 \). Indeed, we find that in the two-fluid case the slaved boundary conditions at \( D = 0 \) are
\[
\nu_{bc}^{ini} \to 0, \quad \delta_{m}^{ini} \to 0, \quad v_{m}^{ini} = -\nabla \psi_{m}^{ini},
\]
which, from (19), translate into ‘growing-’ and ‘constant/persisting modes’ for the two fluids; at first order they read
\[
\delta_{b} = D\nabla \psi_{b}^{ini} + \delta_{bc}^{ini}, \quad \delta_{c} = D\nabla \psi_{c}^{ini} + \delta_{bc}^{ini},
\]
where we have used that \( C_{m} = \nabla^{2} \psi_{m}^{ini} \) (cf. equation 10). It is crucial to note that in the presence of two (shared) fluids, the weighted sum \( \delta_{m} = f_{b}\delta_{b} + f_{c}\delta_{c} \) must sum up to zero initially, which requires a compensating relationship between \( \delta_{bc}^{ini} \) and \( \delta_{bc}^{ini}. \) Thus, in general the initial densities \( \delta_{bc}^{ini} \) must be non-zero, of course except in special points where both densities are zero on their own.

We remark that the evaluation of the fields at \( D = 0 \) does not imply that we ignore the inflationary and recombination dynamics; instead we actually reduce all physics prior to recombination to an infinitely thin boundary layer. As is discussed in detail by, e.g. Michaux et al. (2020), this procedure is actually implicitly assumed when initializing Newtonian simulations in the growing mode, which is the standard in numerical cosmology. In the literature, the respective approach for generating the initial fields is usually called ‘backscaling’; for details and further remarks we kindly refer to section 2.5 in our companion paper.

3.2.1 Growing-mode contributions
We call the contributions from \( \delta_{bc} \) as instructed through (26); it appears that the \( \delta_{bc}^{ini} \)'s can be easily determined to any desired level of accuracy, namely by iteratively solving for \( \delta_{bc} \) through the two identities \( \delta_{m} = f_{b}\delta_{b} + f_{c}\delta_{c} \) and \( \delta_{bc} = \delta_{bc}^{ini} - \delta_{bc} \), leading to
\[
\delta_{bc} = \delta_{m} + f_{b}\delta_{b} = \sum_{n=1}^{\infty} \left[ \delta_{m}^{(n)}(x) D^{n} + f_{b}\delta_{b}^{(n)}(x) D^{n-1} \right],
\]
\[
\delta_{c} = \delta_{m} - f_{c}\delta_{c} = \sum_{n=1}^{\infty} \left[ \delta_{m}^{(n)}(x) D^{n} - f_{c}\delta_{c}^{(n)}(x) D^{n-1} \right].
\]
for \( g = 1 \), which has the solution \( \delta_{m} = D\nabla^{2} \psi_{m}^{ini} - 2C_{m}^{+}D^{-1/2} + C_{m}^{-} \). Clearly, in the case of slaved boundary conditions that only select non-decaying terms, \( C_{m}^{+} \) must vanish. Similarly, by identification with the general solution (19), we have \( C_{m}^{-} = \delta_{m}^{ini}. \) Thus, we arrive at the identical result as above, without first solving separately for the sum and difference variables.

Physically, this procedure works as in this case with growing-mode solutions, we are essentially just splitting up a single matter fluid into two shared components. While the shared fluid components must begin their evolution with non-zero initial densities (due to slaving), the evolved matter density in the growing mode must be equal to the growing mode of the sum of the two shared component densities.

This simplification carries over to all orders. For example, truncated to second order we find the following solutions (\( \alpha = b, c \))
\[
\delta_{b}(x, D) = D\delta_{m}^{(1)} + \delta_{m}^{ini} + D^{2}\delta_{bc}^{(2)} + D D^{2}\delta_{b}^{(2)} + D\left[ \delta_{m}^{ini} + \delta_{c}^{ini} \right],
\]
where the growing-mode solutions for the total matter density and velocity are given in equations (10). Furthermore, imposing
\[
\delta_{bc} = \sum_{n=1}^{\infty} \delta_{bc}^{(n)}(x) D^{n-1},
\]
we find simple recursion relations for the density difference,
\[
\delta_{bc}^{(n)} = \frac{1}{n-1} \sum_{0<k<n} \nabla \cdot \left[ \delta_{m}^{(k)} \nabla \cdot \nabla \delta_{bc}^{(n-k)} \right]
\]
for \( n > 1 \), and \( \delta_{bc}^{(1)} = \delta_{bc}^{ini} \) for \( n = 1 \), where the coefficients \( \delta_{b}^{(n)} \) are defined in equation (9). See Appendix B for calculational details, where we also provide explicit formulas to determine the power spectrum of \( \delta_{bc} \) to one-loop accuracy. Using the well-known recursion relations for the matter density and velocity from standard PT, as well as the recursion relation for the difference density, it is clear that the \( \delta_{b}^{(n)} \)'s can be easily determined to any desired level of accuracy, namely by iteratively solving for \( \delta_{bc} \) through the two identities \( \delta_{m} = f_{b}\delta_{b} + f_{c}\delta_{c} \) and \( \delta_{bc} = \delta_{bc}^{ini} - \delta_{bc} \), leading to
\[
\delta_{bc} = \delta_{m} + f_{b}\delta_{b} = \sum_{n=1}^{\infty} \left[ \delta_{m}^{(n)}(x) D^{n} + f_{b}\delta_{b}^{(n)}(x) D^{n-1} \right],
\]
\[
\delta_{c} = \delta_{m} - f_{c}\delta_{c} = \sum_{n=1}^{\infty} \left[ \delta_{m}^{(n)}(x) D^{n} - f_{c}\delta_{c}^{(n)}(x) D^{n-1} \right].
\]
et al. (2013), Porto et al. (2014), Vlah et al. (2015), Aviles & Banerjee (2020), and many others.

Let \( q \mapsto x(q, D) = q + \xi^m(q, D) \) be the Lagrangian map, from initial \((D = 0)\) position \( q \) to the current position \( x \) at time \( D \). In the Lagrangian representation, the velocity is defined by \( \partial_D^m x = \dot{x} = v_m \), where \( \partial_D^m \) is the Lagrangian (convective) \( D \)-time derivative, which is here and in the following also denoted by an overdot. The Lagrangian time derivative commutes with the Lagrangian space derivative, however, not with the Eulerian one. In the single fluid case, and before shell-crossing, mass conservation reads exactly

\[
\delta_m(x(q, D)) = \frac{1}{\Omega} - 1,
\]

(27)

where \( \Omega \) is the Jacobian

\[
\Omega = \det[x_{ij}] = 1 + \xi^m + \frac{1}{2} [\xi_j^m \xi^m_j - \xi_j^{mm} \xi_j^{m}] + \text{det} [\tilde{\xi}_j^m] ,
\]

(28)

which, due to the choice of Lagrangian coordinates, is unity at initial time. To get Lagrangian evolution equations, one usually takes the Eulerian divergence of the Euler equation (4a) and considers the vanishing of the Eulerian vorticity, which, respectively, lead to

\[
\nabla \cdot R_\Omega x = \frac{3g}{2D^2} \delta_m(x(q, D)) ,
\]

(29a)

\[
\nabla \times \dot{x} (q, D) = 0 ,
\]

(29b)

where we have used equation (4c) and defined

\[
R_\Omega = (\partial q)^2 + [3g/2D^2] \partial^2 q_j^m .
\]

(30)

Equations (29) are not yet fully written in Lagrangian space since the density should be expressed in terms of the Jacobian; furthermore, there are still remaining Eulerian derivatives. Regarding the latter, Buchert & Goetz (1987) suggested to proceed as follows: convert equation (31b) can be generalized to allow for initial vorticity (cf. the conservation of the Eulerian zero-vorticity condition.

All indices in equation (31a) are contracted, hence it is a scalar equation. By contrast, equation (31b) is a vector equation that states the vanishing of the Eulerian vorticity, for example, that we have now two Poisson equations, one for each Lagrangian \( R \)-fluids. The solutions for the two-fluid displacements are particularly simple, however, only when the initial densities are transported along the fluid paths. For numerical applications, it may be instructive to absorb these initial densities in the displacements, which is discussed in Section 5.4.

5 TWO FLUIDS IN LAGRANGIAN SPACE

Now we switch to the Lagrangian-coordinates approach for two fluids. The governing equations are provided in Section 5.1 and general perturbative solutions developed in Section 5.2. In Section 5.3, we provide a simplified derivation that is particularly suited for deriving all-order solutions in the growing modes. The solutions for the two-fluid displacements are particularly simple, however, only when the initial densities are transported along the fluid paths. For numerical applications, it may be instructive to absorb these initial densities in the displacements, which is discussed in Section 5.4.

5.1 General formalism

Let \( q \mapsto x^\alpha(q, D) = q + \xi^\alpha(q, D) \) be the respective Lagrangian maps for the two fluids with corresponding displacements \( \xi^\alpha \), for \( \alpha = b, c \). Without loss of generality, we require that both component displacements vanish initially, except in Section 5.4, where we provide a complementary derivation where the component fluids are initialized in a ‘perturbed’ Lagrangian coordinate system.

The velocities are defined via \( v^\alpha = x^\alpha(q, D) \). Using similar algebraic manipulations as above, we transfer the Eulerian equations (14) to Lagrangian space. We have (cf. Chen et al. 2019)

\[
R_\Omega x^\alpha = -3g(\nabla \cdot \varphi^\alpha)/2D^2 ,
\]

(33a)

\[
\delta^\alpha_\nu (x^\alpha(q, D)) = 1 + \delta^\alpha_\nu(\xi^\alpha(q, D))/\text{det} [\nabla x^\alpha(q, D)] - 1 ,
\]

(33b)

\[
\nabla^2 \varphi^\alpha = \begin{bmatrix} f_b (x^b(q, D)) + f_c (x^c(q, D)) \end{bmatrix} ,
\]

(33c)

where \( g = 1 + \Delta D/11 + O(D^2) \) (see equation 5.5), while the initial density \( \delta^\alpha_\nu \) and the temporal operator \( R_\Omega \) are, respectively, given in equations (19) and (30). Furthermore, we have defined the abbreviation \( (\nabla x)^\alpha = \nabla x(x^\alpha(q, D)) \), and, for notational simplicity, we have suppressed some obvious dependencies in (33).

Note that we have now two Poisson equations, one for each Lagrangian map, they read (Frisch & Sobolevskii 2015)

\[
(\nabla^2 \varphi^\alpha) = \frac{1}{D} \begin{bmatrix} f_b (1 + \delta^b_\nu(q)) + f_c (1 + \delta^c_\nu(q)) \end{bmatrix} - 1 ,
\]

(34a)

\[
(\nabla^2 \varphi^\alpha) = \frac{1}{D} \begin{bmatrix} f_b (1 + \delta^b_\nu(q)) + f_c (1 + \delta^c_\nu(q)) \end{bmatrix} - 1 .
\]

(34b)
where $J^\mu(q) = \text{det}[\delta^\mu_\nu(q)]$. Furthermore, we employ the composition $x_c^{-1} \circ x_b = x_c^{-1}(x_b)$, where $x_c^{-1} = q^c(x)$ is the inverse Lagrangian map defined such that $x^c(q^c(x)) = x$.

Compositions such as $x_c^{-1} \circ x_b$ have the purpose of pulling the field of particles of species $c$ back to its initial value, and then forward it to the current time such that the Poisson equation of species $b$ takes the gravitational interactions of $c$ at the current position $x_b$ into account.

Finally, these equations can be combined leading to the Lagrangian fluid equations for the two-fluid system:

\begin{equation}
\begin{aligned}
\xi_{\alpha} & = \xi^{(1)} + \xi^{(2)} + \ldots \, , \\
\end{aligned}
\end{equation}

In the following, after providing explicit perturbation equations for the component displacements, we report the results for the fastest growing mode to first and second order.

5.2 Perturbative solutions

To solve the above (35) in the most general way, let us expand the component displacements perturbatively according to

\begin{equation}
\xi_{\alpha}^{(1)} + \xi_{\alpha}^{(2)} + \ldots
\end{equation}

where we remind the reader that $\alpha = b$, $c$ are non-running fluid labels, while summation over Latin indices is implicitly assumed. In the following we solve equations (35) perturbatively.

5.2.1 First order

Formally linearizing the Lagrangian evolution equations, we obtain, to first order, a trivial identity from (35b) implying that the fluid motion is potential in Lagrangian space. From equation (35a), we get at first order

\begin{equation}
\begin{aligned}
\nabla D \xi_{\alpha}^{(1)} & = 3g \frac{\nabla^2}{D^2} \left[ f_{\alpha} \xi_{\alpha}^{(1)} + f_{\alpha} \xi_{\alpha}^{(1)} \right] , \\
\end{aligned}
\end{equation}

where we have used $\xi_{\alpha}^{(1)} = \xi_{\alpha}^{(1)} + \xi_{\alpha}^{(1)}$ leading to $J^\mu(q_{\xi_{\alpha}^{(1)}}) = J^\mu(q_{\xi_{\alpha}^{(1)}})$. Similarly for the term appearing in the evolution equation for component $c$. To arrive at equation (37a) we have used the boundary condition $\delta_n(D_m) = \delta_{mi}$ (cf. equation 19), which implies that $\delta_n(D_m) \approx 0$, a boundary condition that leads to Zel’Dovich-type solutions; see e.g. Buchert (1992) and Rampf & Buchert (2012). Of course, when slaving is applied which is the main focus in this paper, then $\delta_n(D = 0) = 0$ holds exactly. Furthermore, we have assumed that $\delta_{mi}$ are perturbations in the same sense as generic first-order displacement perturbations. For further calculational details, see Appendix C1.

The differential equation (37a) can be solved by considering difference and weighted sum displacements that we define, respectively, as follows:

\begin{equation}
\begin{aligned}
\xi_{\alpha}^{(1)} & = f_b \xi_{\alpha}^{(b)} + f_c \xi_{\alpha}^{(c)} , \\
f_b \xi_{\alpha}^{(b)} & = \xi_{\alpha}^{(b)} - \xi_{\alpha}^{(c)} .
\end{aligned}
\end{equation}

At first order, we then obtain from (37a) that

\begin{equation}
\begin{aligned}
\nabla D \xi_{\alpha}^{(1)} & = 3g \frac{\nabla^2}{D^2} \left[ f_{\alpha} \xi_{\alpha}^{(1)} + f_{\alpha} \xi_{\alpha}^{(1)} \right] , \\
\nabla D \xi_{\alpha}^{(1)} & = 0 .
\end{aligned}
\end{equation}

The general solutions of these first-order equations can be analytically obtained for $g = 1$, leading to

\begin{equation}
\begin{aligned}
\xi_{\alpha}^{(1)} & = D C_+^\alpha + D^{-3/2} C_-^\alpha , \\
\xi_{\alpha}^{(2)} & = C_c^\alpha - D^{-1/2} C_c^\alpha ,
\end{aligned}
\end{equation}

where the $C$s are integration constants. Upon identification with the linearized Eulerian solutions, we have $C_+^b = -C_+^c$, $C_-^b = -C_-^c$, $C_c^{bc} = -C_c^{bc}$, and $C_c^{bc} = 0$. From this, one can easily determine the general first-order solutions for the components $\alpha = b, c$.

5.2.2 Second order

Next, we consider the second-order perturbations. For this, it is useful to define the second-order invariants for arbitrary displacements $\xi$ and scalar $\phi$:

\begin{equation}
\begin{aligned}
\mu^2_{\phi,\phi} & = \frac{1}{2} \left[ \xi_{\phi,\phi} \xi_{\phi,\phi} - \xi_{\phi,\phi} \xi_{\phi,\phi} \right] , \\
\mu^2_{\phi,\phi} & = \frac{1}{2} \left[ (\phi_{,\phi})^2 - (\phi_{,\phi})^2 \right] ,
\end{aligned}
\end{equation}

where, in the following, $\beta$ and $\gamma$ are either $m, b$, or $c$. Keeping only the second-order terms in equation (35a), we find the following Lagrangian perturbation equations for the two-fluid system:

\begin{equation}
\begin{aligned}
\nabla D \xi_{\alpha}^{(b)} & = \frac{3g}{2D} \left[ f_b \xi_{\alpha}^{(b)} - \mu_{b,\phi} \right] + f_c \left( \xi_{\alpha}^{(c)} + \xi_{\alpha}^{(c)} - 2\mu_{b,\phi} \right) \\
& - f_c \left( \xi_{\alpha}^{(c)} - \xi_{\alpha}^{(c)} \right) \delta_j - \xi_{\alpha}^{(c)} \left( \xi_{\alpha}^{(b)} - \xi_{\alpha}^{(b)} \right) \right] ,
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
\nabla D \xi_{\alpha}^{(c)} & = \frac{3g}{2D} \left[ f_b \xi_{\alpha}^{(b)} - \mu_{b,\phi} \right] + f_b \left( \xi_{\alpha}^{(c)} + \mu_{b,\phi} - 2\mu_{b,\phi} \right) \\
& + f_b \left( \xi_{\alpha}^{(b)} - \xi_{\alpha}^{(b)} \right) \delta_j - \xi_{\alpha}^{(b)} \left( \xi_{\alpha}^{(c)} - \xi_{\alpha}^{(c)} \right) \right] .
\end{aligned}
\end{equation}
while, using the first-order results (37e) including decaying modes, the second-order part of (35b) yields
\[
\varepsilon_{ijk} \xi_{i,j,k}^{(2)} = \varepsilon_{ijk} \left\{ \frac{5}{2} \hat{C}_{m}^{i} \hat{C}_{ji} D^{-3/2} + 2 \hat{C}_{m}^{i} \hat{C}_{ji}^{bc} \right\},
\]
where for the constants we employ a hat notation according to \( \hat{C} = \nabla^{-2} C \), and we have defined \( c^a \) with \( c^b = 1 - f_b \) and \( c^c = -f_c \). Calculational details to equations (40) are provided in Appendix C1.

5.2.3 Fastest growing modes

The second-order perturbation equations (40) can be easily integrated if needed. In the following, we report the results for the fastest growing modes, thus taking slaved boundary conditions into account. For this observe that in the absence of decaying modes, we have \( \xi^{(1)} = \xi^{(2)} \) implying also \( \mu_{ij}^{(2)} = \mu_{mn}^{(2)} \) and, of course \( C_{m,n} \rightarrow 0 \), \( C_{bc} \rightarrow 0 \); thus, all terms in curly brackets in the equations (40) vanish, which implies for equation (40c) a vanishing source. The evolution equations for the fastest growing modes thus simplify to
\[
\varepsilon_{ij} \xi_{i,j}^{(2)} = \frac{3g}{2D^2} \left[ f_{b} \xi_{i,b}^{(2)} + f_{c} \xi_{i,c}^{(2)} - D^2 \mu_{2}(\psi^{(m)}) \right],
\]
which after suitable algebraic manipulations leads to
\[
\varepsilon_{ij} \xi_{i,j}^{(2)} = 0,
\]
where \( \mu_{2} \) is defined in (39b). Equation (41b) states the potential character for the first time derivative of the component displacements, which, by virtue of the used boundary conditions, leads trivially to \( \varepsilon_{ij} \xi_{i,j}^{(2)} = 0 \). Equations (41) are easily solved, e.g. by employing the weighted sum and difference displacements
\[
\xi^{(m)} = f_{b} \xi_{b}^{(2)} + f_{c} \xi_{c}^{(2)} = \xi^{(2)} - \xi^{(2)} \]
which, of course, agrees with the result given in the single-fluid (Section 4). Since the second-order difference \( \xi_{i,j}^{(2)} \) is vanishing, the second-order growing-mode solution for the components are simply
\[
\xi^{(2)}(\mathbf{q}, D) = \xi^{(2)}(\mathbf{q}, D).
\]

We remind the reader that the results of this section are to be used with the mass conservation law (33b), where the appearing \( \delta^{(m)} \mathbf{q} \) is with our choice of coordinates and boundary conditions non-negligible.

Thus, the component displacements formally agree with the single fluid displacement (at least to third order, but see below), however, only if the initial density perturbation \( \delta^{(m)} \mathbf{q} \) is kept in the mass conservation law according to (33b). Of course, that initial density perturbation affects the density to all orders. Indeed, expanding (33b) to second order using the results (38) and (45), one finds firstly
\[
\delta_{\alpha}(\mathbf{q}) = \delta_{\alpha}^{(m)}(\mathbf{q}) + D \psi_{\alpha}^{(m)}(\mathbf{q}) + D \delta \omega_{\alpha}^{(m)} + D^2 \left[ \frac{5}{7} \psi_{\alpha}^{(m)} + \frac{2}{7} \psi_{\alpha}^{(m)} \right] + O(3).
\]

For direct comparison with the Eulerian result, we need to evaluate all terms in the last expression at the identical (Eulerian) position, for which we use the ‘pullback’ \( q(x) = x - \xi \) first order in functions \( F \) that depend on \( q \), i.e. \( F(q(x)) = F(x - \xi) = F(x) - F_{x} \xi \) to first order. As a consequence, the first two terms on the right-hand side of (46) generate higher order perturbations due to the pullback, leading to the ‘Eulerian’ density
\[
\delta_{\alpha}(x) = \delta_{\alpha}^{(m)}(x) + D \psi_{\alpha}^{(m)}(x) + D \left[ \frac{5}{7} \psi_{\alpha}^{(m)} + \frac{2}{7} \psi_{\alpha}^{(m)} \right] + O(3).
\]

which agrees with the one obtained from the Eulerian calculation in Section 3.2. Furthermore, as shown in Appendix C2, the component velocity corresponding to the above reported component displacement evaluated at the Eulerian position agrees with the one from the Eulerian two-fluid result (equation 23).

Alternatively, for numerical applications such as generating initial conditions for simulations, the initial densities can also be incorporated in the component displacements. As we will elucidate in Section 5.4, the resulting component displacements differ substantially from the above.

5.3 Simplified derivation of growing-mode solutions

To obtain growing-mode solutions we can alternatively apply a similar simplification to the calculations as outlined in the Eulerian Section (see around equation 22): for this we express the Poisson source of the shared two-fluid system by its counterpart in the single-fluid case. We thus replace in (35a) the term \( (\nabla_{i} \psi) \), by the much simpler \( \nabla_{i} \psi(x(\mathbf{q})) \), and express the latter by means of the single-fluid displacement (32) up to second order, i.e.
\[
\nabla_{i} \psi(x(\mathbf{q})) = \frac{1}{D} (\nabla_{i} \psi^{(m)}) - \frac{1}{D} \psi_{\alpha}^{(m)}(\mathbf{q}) + D \left[ \frac{5}{7} (\psi_{\alpha}^{(m)})^2 + \frac{2}{7} (\psi_{\alpha}^{(m)})^2 \right].
\]

Doing so, equation (35a) becomes an ODE with an inhomogeneous term \( \frac{1}{D} \psi^{(m)} \) and reads at first order
\[
\varepsilon_{ij} \psi_{i,j}^{(m)} = -\frac{3g}{2D^2} \psi^{(m)}.
\]

The analytic solution for non-decaying modes is \( \psi_{i,j}^{(1)} = -D \psi_{i,j}^{(m)} + C_{i} \psi_{i}^{(m)} \) for \( g = 1 \). Here, we have a choice as regards to the setting of \( C_{i} \). Assuming that mass conservation for the components is
\[
\delta_{\alpha} = \frac{1}{\det[\delta_{ij} + \xi_{ij}^{(m)}]} - 1,
\]
and expanding this to first order and set it equal to the Eulerian result for the component density (equation 21), one easily establishes that \( C_{i} \rightarrow 0 \) in this setting. Similarly, if mass conservation is assumed to be \( \delta_{\alpha} = 1/\det[\delta_{ij} + \xi_{ij}^{(m)}] - 1 \) then one finds that \( C_{i} \rightarrow -\delta_{\alpha} \), of course in that setting the Jacobian departs from unity already at initial time, contrary to what is assumed for (50). While both realizations are consistent, for the rest of the section we (continue to) assume mass conservation according to (50) instead and thus set \( C_{i} = 0 \), while the other setting is executed in the following subsection.

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Iterating to the next order, we use again (48) and obtain from equation (35a) at second order
\[ \mathbf{R}_{D\mathbf{x}_{n}}^{(2)} = -\frac{15}{7} \mu^{2} \mathbf{q}^{(m)}(0), \] (51)
where \( \mu^{2} \mathbf{q}^{(m)} \) is defined in equation (39b). The non-decaying solution of equation (51) is \( \mathbf{q}_{n}^{(2)} = (-3/7) \mathbf{D}^{2} \mathbf{q}_{n}^{(1)} \), where an occurring integration constant can be safely set to zero because of the boundary conditions and definition of the Lagrangian map.

Summing up, we find for the fluid components, truncated to second order in the growing modes, the simple result
\[ \mathbf{x}^{u}(q, D) = D \mathbf{q}^{(m)(1)}(q) + D^{2} \mathbf{q}^{(m)(2)}(q), \] (52)
where the spatial functions \( \mathbf{q}^{(m)(1)} \) and \( \mathbf{q}^{(m)(2)} \) are given in equations (32b).

Actually, the above considerations carry over trivially to arbitrary high orders. Indeed, it is easily checked that when expressing the evolution equations for the component displacement in terms of the single-fluid displacement \( \mathbf{x} \), i.e.
\[ \begin{align*}
\mathbf{f}_{ijkl} \mathbf{y}^{u}_{n} \mathbf{x}^{u}_{n} \mathbf{R}_{D\mathbf{x}_{n}}^{(2)} &= -\frac{3}{D^{2}} f^{(m)} \left( \frac{1}{\mathbf{j}(\mathbf{x}^{(m)})} - 1 \right), \\
\mathbf{f}_{ijkl} \mathbf{y}^{u}_{n} \mathbf{x}^{u}_{n} \mathbf{R}_{D\mathbf{x}_{n}}^{(2)} &= 0,
\end{align*} \] (53)
and uses the known recursion relations for \( \mathbf{x}^{u} = \sum_{n=1}^{\infty} \mathbf{q}^{(m)(n)} D^{n} \) (see e.g. Zheligovsky & Frisch 2014), then one finds the following simple result for all coefficients \( 1 \leq n < \infty \) of the component displacement,
\[ \mathbf{q}^{(m)(n)} = \mathbf{q}^{(m)(n)} \cdot \] (54)
We have explicitly verified that the simplified equations (53) provide identical results with the approaches of Sections 3.2 and 5.2 up to third order; see Appendix B3 for details. Furthermore, in Appendix C5 we recursively prove that
\[ \mathbf{q}^{(m)(n+1)} = \mathbf{q}^{(m)(n+1)} - \mathbf{q}^{(m)(n)} = 0, \] (55)
from which it follows that equation (54) actually holds at all orders.

5.4 Incorporating initial density perturbations in component displacements

We have seen that for the two-fluid case with growing-mode solutions, the initial density perturbations \( \mathbf{y}^{m} \) are non-negligible; ignoring them would induce quasi-singular behaviour for \( D \rightarrow 0 \). This should be contrasted to similar derivations in the single-fluid case, where initial density perturbations are usually ignored; for a discussion see e.g. Rampf & Buchert (2012). We note, however, that initial density perturbations can also not be ignored in Lagrangian bias expansions since biased/observed objects are generally not uniformly distributed, whereas the spatial functions \( \mathbf{q}^{(m)(1)}(q) \) and \( \mathbf{q}^{(m)(2)}(q) \) are given in equations (32b). Using the methods described above, it is easily shown that this composite map exactly produces the same second-order component density (equation 47) as in the approaches of Sections 3.2 and 5.3. Furthermore, in Appendix C2 we show that the associated velocity at the current time instant is defined in equation (39b). The non-decaying second-order component density
\[ \mathbf{q}^{(m)(2)}(q) = -\mathbf{q}^{(1)}(q) - \mathbf{q}^{(2)}(q) + \mathbf{q}^{(3)}(q) + (\nabla^{2} \mathbf{q})^{2}, \] (60)
to second-order accuracy. Clearly, the first-order solution is simply
\[ \mathbf{q}^{(m)(1)}(q) = -\mathbf{q}^{(1)}(q) - \mathbf{q}^{(2)}(q) + (\nabla^{2} \mathbf{q})^{2}, \] (61)
where we have used the fact that the initial density perturbation is, to the leading order, evaluated at the position \( q \). At second order, however, the actual functional dependence of \( \mathbf{y}^{(m)(2)}(q) = \mathbf{q}^{(m)(2)}(q) \) must be taken into account, which generates a second-order term – similarly as discussed in the previous section. It is then straightforward to determine \( \mathbf{q}^{(m)} \); truncated to second order it reads
\[ \mathbf{q}^{(m)}(q) = \mathbf{q}^{(1)}(q) + \mathbf{q}^{(2)}(q) + (\nabla^{2} \mathbf{q})^{2} + O(3). \] (62)
Using this in (57), we finally obtain the composite map
\[ \mathbf{x}^{(m)}(q) = q + \mathbf{D} \mathbf{q}^{(m)(1)}(q) + D^{2} \mathbf{q}^{(m)(2)}(q) - D \mathbf{q}^{(m)(1)} \nabla \mathbf{q}^{(m)(1)} + \mathbf{v}^{(1)}(q) \] (63)
with \( \mathbf{v}^{(1)} = -\nabla^{-1} \mathbf{q}^{(1)}(q) \) and \( \mathbf{v}^{(2)}(q) \) are given in equations (32b). Using the methods described above, it is easily shown that this composite map exactly produces the same second-order component density (equation 47) as in the approaches of Sections 3.2 and 5.3. Furthermore, in Appendix C2 we show that the associated velocity at the current (Eulerian) position agrees as well with the Eulerian result.

In summary, on a theoretical footing, the method presented here agrees exactly with the one of Section 5.3. For the numerical application, the methods are, however, fairly distinct: While for the method of Section 5.3 we should incorporate the initial density perturbations in the growing modes, the simple result
\[ \mathbf{x}^{(m)}(q) = q + \mathbf{q}^{(1)}(q) + \mathbf{q}^{(2)}(q), \] (64)
and uses the known recursion relations for \( \mathbf{q}^{(m)}(0) \) defined in equation (39b). The non-decaying solution of equation (51) is \( \mathbf{q}_{n}^{(2)} = (-3/7) \mathbf{D}^{2} \mathbf{q}_{n}^{(1)} \), where an occurring integration constant can be safely set to zero because of the boundary conditions and definition of the Lagrangian map.

\[ \begin{align*}
\mathbf{x}^{(m)}(q) &= q + \mathbf{q}^{(1)}(q) + \mathbf{q}^{(2)}(q), \\
\mathbf{x}^{(m)}(q, D) &= q + \mathbf{q}^{(1)}(q, D),
\end{align*} \] (56)
where \( \nabla \mathbf{q}^{(1)}(q) \) is a time independent yet non-perturbative displacement generated by the initial density perturbation \( \mathbf{y}^{(1)} \), and the component displacements \( \mathbf{y}^{(2)} \) are given in equation (52). To evolve the fluid system for these two consecutive maps, we use the composition
\[ \begin{align*}
\mathbf{x}^{(1)}(q) &= q + \mathbf{q}^{(1)}(q), \\
\mathbf{x}^{(2)}(q, D) &= q + \mathbf{q}^{(1)}(q, D) + \mathbf{q}^{(2)}(q, D),
\end{align*} \] (57)
where \( \mathbf{q}^{(2)}(q) = \mathbf{q}^{(1)}(q) + \mathbf{q}^{(2)}(q) + (\nabla^{2} \mathbf{q})^{2} + O(3). \] (58)
Clearly, the first-order solution is simply
\[ \mathbf{x}^{(1)}(q) = q + \mathbf{q}^{(1)}(q) + \mathbf{q}^{(2)}(q) + O(3). \] (59)
Using this in (57), we finally obtain the composite map
\[ \mathbf{x}^{(2)}(q) = q + \mathbf{q}^{(1)}(q) + \mathbf{q}^{(2)}(q) + (\nabla^{2} \mathbf{q})^{2} + O(3). \] (60)

6 VARIATIONAL APPROACH FOR THE LSS

Complementary to the Eulerian and Lagrangian fluid approaches, recently a semiclassical formalism has been put forward by Uhlemann.

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et al. (2019). This formalism is related to a classical Hamiltonian theory, however, only once the corresponding momentum variable is non-canonically transformed – this is the essence that leads to a direct relationship between Hamiltonian theory and the cosmological fluid description (see e.g. Bartelmann 2015). Here, we reconsider this relationship and show that the classical Hamiltonian can be transformed to a ‘new’ (contact) Hamiltonian that lives on a special manifold on an extended phase-space. We will show that, on the one hand, perturbative solution techniques are amenable to this transformed Hamiltonian – which is not at all straightforward within the classical Hamiltonian theory. On the other hand, this Hamiltonian builds the basis for the semiclassical approach discussed in Section 7, whose perturbative solutions in terms of the propagator are related to the classical action along the phase-space trajectory. We begin with the Hamiltonian in $\Lambda CDM$ in cosmic time that is

$$H(x, p, t) = \frac{p^2}{2ma^2(t)} + \psi(x),$$  

(64)

where $p = ma^2 \dot{a}$ is the canonically conjugated momentum, and we have employed the same (tilde) notation as in Section 2. To change the time variable from cosmic time to $D$-time, recall that $dD/dt = (\dot{a} D)dD/dt$, and consider the action

$$S_D = \int \mathcal{L} dt = \int \left( p \cdot m \frac{dx}{dD} - \frac{p^2}{2ma^2(\dot{a}, D)} - \frac{3D}{2a(\dot{a}, D)} \psi \right) dD,$$

(65)

where $\dot{\psi} = 3D \psi/(2a)$, with $\psi$ governed by the Poisson equation (4c). From the action, we can read off the Hamiltonian in $D$-time:

$$H(x, p, D) = \frac{p^2}{2ma^2(\dot{a}, D)} + \frac{3D}{2a(\dot{a}, D)} \psi(x).$$

(66)

It is easily checked that the corresponding Hamiltonian evolution equations are not compatible with slaving (cf. Section 2.2), which, on a technical level stems from the fact that in a Hamiltonian theory one is forced to employ canonically conjugate variables. Liouville’s theorem forces momentum space to expand indefinitely, while comoving coordinate space contracts to a point as $a \to 0$. To proceed, and to make the connection to the fluid approach transparent, one may express the Hamiltonian in terms of the velocity $v$ instead of $p$, which is a non-canonical transformation.

An ‘alternative’ of this is using the so-called contact geometry, where the above Hamiltonian can be ‘contact transformed’ – which is a generalization of canonical transformations; see e.g. Arnold & Novikov (2001) and Bravetti, Cruz & Tapias (2017). Historically, the early development of contact geometry traces back to Sophus Lie (cf. Etnyre 2001), and was later on applied to geometrize thermodynamics. More recently, contact geometry has been employed in Hamiltonian dynamics, with important fundamental work performed by Vladimir Arnold in the late 80s. In the following, we shall employ the contact approach, while calculational details and further results are provided in Appendix D.

Expressing (66) in terms of the velocity $v$ (we set $m = 1$) which is here an independent variable and the contact conjugate to $x$, we obtain the contact Hamiltonian in $\Lambda CDM$, derived in Appendix D,

$$\mathcal{H}(x, v, S, D) = \frac{v^2}{2} + \psi, \quad \mathcal{V}(x, S, D) = \frac{3g}{2D} (\psi(x) + S),$$

(67a)

where $S = \int v \cdot dx - \mathcal{H}dD$ is the corresponding action (up to an integration constant). We remark that the corresponding Hamiltonian equations of motion for (67a) are actually independent of $S$, they read (see Appendix D)

$$\frac{dx}{dD} = v, \quad \frac{dr}{dD} = -\frac{3g}{2D} (v + \nabla \psi)$$

(67b)

for the position and velocity variables, while $S$ is determined through $dS/dD = v^2/2 - 3g(\psi + S)/2D$. In Section 7, we show how the contact Hamiltonian relates to a Schrödinger equation.

Now, defining the generating function $W(x', D'; x, D)$, which physically corresponds to the action along the phase-space trajectory from $(x', D')$ to $(x, D)$, with

$$S_{x', D'} : (x, D) \mapsto W(x', D'; x, D) = \int_{x', D'}^{x, D} v \cdot dx - \mathcal{H}dD,$$

(68)

we find that the generating function is governed by the Hamilton–Jacobi equation

$$\partial_D S_{x', D'} + \mathcal{H}(x, \nabla, S_{x', D'}, \mathcal{V}_{x', D'}) = 0,$$

(69)

for fixed (initial) coordinates $(x', D')$. Using $\nabla, S_{x', D'}$ as the velocity variable confines the phase-space trajectories on the so-called Legendrian submanifold (cf. Ehlers & Newman 2000), which is the contact analogue to the Lagrangian submanifold in symplectic geometry, and thus of importance when determining the phase-space of infinitely cold matter (see e.g. Abel, Hahn & Kaehler 2012).

The Hamiltonian (67a), as well as the Hamilton–Jacobi equation (69), comprise individual starting points for investigating perturbative solutions in phase-space. For this, it is important to note that both (67a) and (69) remain regular at $D = 0$ provided we use the slaving conditions:

$$\lim_{D' \to 0} S_{x', D'} = S^{\text{ini}}(x') = -\psi^{\text{ini}},$$

(70)

which, as in the previous sections, should be supplemented with the statement of initial quasi-homogeneity. It is also worthwhile to point out that the Hamilton–Jacobi equation (69) admits solutions to the corresponding Cauchy problem (cf. de Gossos 2017), it reads

$$S_{x', D' = 0}(x, D) = S^{\text{ini}}(x') + \mathcal{V}(x', x; 0, D),$$

(71)

which, in the cosmological case, should remain meaningful at least until the instance of the first shell-crossing. Perturbative solutions in terms of the growing-mode $D$ are discussed in the following, while the inclusion of decaying modes will be kept for future work.

### 6.1 Single-fluid case

As motivated above, the generating function $W$ is governed by the Hamilton–Jacobi equation

$$\partial_D W(q, 0; x, D) + \frac{(\nabla, W(q, 0; x, D))^2}{2} + \mathcal{V} = 0,$$

(72)

where, instead of $x'$, from here on we use $q$ to denote the initial (fixed) coordinate. From conventional PT, which is valid before shell-crossing, we know that, to the leading order and with the appropriate choice of coordinates, the fluid motion in an expanding Universe is ballistic with prescribed initial velocities. At the level of the Hamiltonian (67a), this statement implies that $\mathcal{V} \approx 0$ to first order, which translates into the following potential-free Hamilton–Jacobi equation:

$$\partial_D W_{\text{free}} + (\nabla, W_{\text{free}})^2/2 = 0.$$  

(73)
It is elementary to solve such an equation in the context of classical and quantum mechanics (see e.g. de Gosson 2017), leading to
\[ \mathcal{W}_{\text{free}}(q, 0, x, D) = \frac{(x - q)^2}{2D} , \]  
(74)
the so-called free-particle generating function.

To incorporate \( V \neq 0 \) and thus to solve the Hamilton–Jacobi equation (72) to second-order accuracy, we adopt the methodology of de Gosson (2017) and impose a power-law Ansatz for the generating functional
\[ \mathcal{W}(q, 0, x, D) = \mathcal{W}_{\text{free}}(q, 0, x, D) + \sum_{n=2}^{\infty} \mathcal{W}^{(n-1)}(q, x) D^{-n} , \]  
(75)
which is an eligible Ansatz provided that \( V \) can be represented in terms of a power series in \( D \) – as it is the case in PT where \( V = \sum_{n=n}^{\infty} V^{(n)} D^{-n} \) (Uehlemann et al. 2019). Plugging this into (72) together with the leading-order result (74), we obtain the following condition, at \( n = 2 \),
\[ \mathcal{W}^{(2)}(q, x) = - \int_0^1 \mathcal{W}^{(2)}(q + s(x - q)) \, ds , \]  
(76)
which is solved exactly by the following second-order result
\[ \mathcal{W}^{(2)}(q, x) = -(x - q) \cdot \nabla q \mathcal{W}^{(2)} + \mathcal{W}^{(2)} = 0 , \]  
(77)
for practical applications, such as for the Schrödinger approach employed in Section 7, one may also use the two-endpoint approximation for the above, which is \( \mathcal{W}^{(2)} \approx - \left[ \mathcal{W}^{(2)}(q) + \mathcal{W}^{(2)}(x) \right] / 2 \).

Observe that the above result for the generating functional, which is essentially a result of the Cauchy problem (71), does not require explicit solutions of the Poisson equation. In fact this independence is one of the key known advantages of Hamilton–Jacobi. By contrast, in the Hamiltonian approach any progress – be it analytical or numerical, requires either solving for the equations of motion, or by considering invariants that are constants of motion along the path; thus, either way, the Hamiltonian approach requires the explicit knowledge of the Poisson equation.

Alternatively to a full-fledged PT in phase-space, one may also employ conventional PT to express some of the required functions approximatively. For example, the Hamiltonian (67a) valid to second-order within a \( D \) expansion is
\[ \mathcal{H} = \mathcal{V}_{\text{eff}} = \frac{\mathcal{V}^{(2)}}{2} + \mathcal{V}_{\text{eff}}^{(2)} = \frac{3}{7} \nabla^{-2} \left[ q_{\text{eff}}^{(2)} + V_{\text{eff}}^{(2)}(x) \right] . \]  
(78)
Calculational details about how \( V_{\text{eff}} \) is determined are given in Appendix B.

### 6.2 Two-fluid case

Consider now the two-fluid (contact) Hamiltonian
\[ \mathcal{H}(x_a, v_a, S_a, D) = \frac{v_a^2}{2} + \frac{3g}{2D} \left[ S_a + \varphi(x_a) \right] , \]  
(79)
where the common potential is governed by the Poisson equation
\[ \left( \nabla^2 \varphi \right)_a = \left[ \int f_b \left[ 1 + \delta_0^{(i)}(q) \right] \delta_0^{(3)}(x^a - x_b(q, D)) \right] \left[ 1 + \delta_0^{(i)}(q) \right] \delta_0^{(3)}(x^a - x_b(q, D)) ] \, d^3 q - 1 \right]/D , \]  
(80)
for two-fluid case, where \( \delta_0^{(3)} \) is the Dirac delta and we have introduced the sought solutions in phase-space \( x_c(q, D) \) in parametric form. To guarantee regularity at arbitrary short times, we should impose that equation (80) remains finite for \( D \to 0 \), as well as demand the following slaving condition on the individual generating functionals
\[ \lim_{D' \to 0} S_{\alpha}^{(i)}(D') = S_{\alpha}^{(i)}(x') = -\varphi^{(i)} . \]  
(81)
Similarly, as in the single-fluid case these equations could be solved perturbatively using standard methods known from symplectic geometry. However, for the purpose for applying the above to a phenomenological Schrödinger equation (Section 7.2), it is useful to determine the two-fluid Hamiltonian (79) valid to second-order accuracy in the strict \( D \)-expansion. Calculations details for determining the effective potential \( V_{\text{eff}}^{(2)} \) are provided in Appendix B, leading to the Hamiltonian for fluid component \( \alpha = b, c \),
\[ \mathcal{H}(x_{\alpha}, v_{\alpha}, D) = \frac{v_{\alpha}^2}{2} + V_{\text{eff}}^{(2)}(x_{\alpha}) , \]  
(82a)

where
\[ V_{\text{eff}}^{(2)}(x) = \frac{3}{7} \nabla^{-2} \left[ q_{\text{eff}}^{(2)} + V_{\text{eff}}^{(2)}(x) \right] , \]  
(82b)
to second-order accuracy.

### 7 SEMICLASSICAL APPROACH FOR THE LSS

The perturbative contact Hamiltonian (66) discussed in the previous section can be used to formulate a Schrödinger wave equation. Based on this, we generalize the semiclassical approach from Uehlemann et al. (2019; an extension of the free-particle approximation from Short & Coles 2006a,b) for a single fluid in EdS to two fluids in \( \Lambda \)CDM. In so-called PPT, we obtain perturbative solutions for the wavefunction, which straightforwardly predict Eulerian fluid observables while simultaneously implementing a semiclassical analogue of Lagrangian PT.

Before proceeding, we remark that formulating a wave equation using the canonical Hamiltonian (64) would lead to the Schrödinger–Poisson equation describing structure formation for fuzzy dark matter (see Hui et al. 2017 for a review), or an approximate treatment of standard CDM (see e.g. Widrow & Kaiser 1993; Uehlemann, Kopp & Haugg 2014; Kopp, Vattis & Skordis 2017). Here, we do not follow this idea because formulating a PT for the Hamiltonian (64) is hampered by the distinct time dependencies of the kinetic and potential terms in the Hamiltonian. Furthermore, expanding the wavefunction in amplitude and phase, as done in wave PT presented by Li, Hui & Bryan (2019), appears to be even more limited than Eulerian PT.

#### 7.1 Single-fluid case

The wavefunction analogue of the fluid equations for a single fluid is obtained as a solution to the Schrödinger equation
\[ i\hbar \partial_q \psi(x, D) = \mathcal{H} \psi(x, D) , \quad \mathcal{H} = -\frac{\hbar^2}{2} \nabla^2 + V(x) , \]  
(83a)
supplemented with the Poisson equation
\[ \nabla^2 \varphi = \frac{|\psi|^2 - 1}{D} . \]  
(83b)
Here, $\hat{H}$ may be viewed as the contact-Hamiltonian given in equation (78) in operational form, where $S$ is the solution of a Bernoulli-type equation (see equation D7c), and is thus intrinsically associated with the velocity potential (before shell-crossing).

Before investigating the non-linear Schrödinger theory, it is useful to establish the necessary boundary conditions for growing-mode solutions. It is easily verified that equations (83) remain regular for $D \to 0$ provided we use the slaving conditions, which in the present context translate to

$$|\psi^{\text{ini}}|^2 = 1, \quad S^{\text{ini}} = -q^{\text{ini}},$$  \hspace{1cm} (84)

where ‘ini’ denotes as before evaluation at $D = 0$. These slaving conditions imply for the wavefunction initially

$$\psi^{\text{ini}}(q) = \exp \left[ \frac{1}{\hbar} q^{\text{ini}}(q) \right],$$  \hspace{1cm} (85)

where, in accordance with our used notation in this paper, $q$ can be viewed as the initial particle coordinate. Supplemented with this initial condition, Uhlemann et al. (2019) suggested to solve the Schrödinger equation by employing the PPT that has at its central object the propagator $K(q, 0 | x, D) = K(q, x; D)$, which propagates a wavefunction $\psi$ from its initial state to the final (or current) state at time $D$, i.e.

$$\psi(x; D) = \int dq K(q, x; D) \psi^{\text{ini}}(q).$$  \hspace{1cm} (86)

Knowing the propagator and the initial conditions implies knowing the solution to the wavefunction.

In PPT, the quantity $V_{\text{eff}} \equiv 3g(S + \varphi)/(2D)$ is taken to be an external potential, which can be easily determined using conventional PT. The Schrödinger equation (83) then becomes

$$i\hbar \partial_D \psi(x, D) = \left[ -\frac{\hbar^2}{2} \nabla^2_x + V_{\text{eff}}(x, D) \right] \psi(x, D),$$  \hspace{1cm} (87)

with $V^{(1)}_{\text{eff}} \equiv 0$ and $V^{(2)}_{\text{eff}}(x) = (3/7) \nabla^2 \left[ \psi^{\text{ini}} \psi^{\text{ren}} - \psi^{\text{ini}} \psi^{\text{ren}} \right]$, which are evidently time independent at the considered orders. In the following, we solve equation (87) using PPT.

### 7.1.1 Leading order

Since $V^{(1)}_{\text{eff}} = 0$, the Schrödinger equation (83a) is potential free at the leading order, i.e.

$$i\hbar \partial_D \psi^{\text{free}}(x, D) = -\frac{\hbar^2}{2} \nabla^2_x \psi^{\text{free}}(x, D).$$  \hspace{1cm} (88)

Solving for the associated propagator $K^{\text{free}}$ for the free wavefunction $\psi^{\text{free}}$, one easily finds

$$K^{\text{free}}(q, x; D) = (2\pi i \hbar D)^{-3/2} \exp \left[ \frac{i}{\hbar} \frac{(x - q)^2}{2D} \right].$$  \hspace{1cm} (89)

where the normalization factor is introduced such that the propagator amounts to the Dirac delta $\delta_0^D(x - q)$ for $D \to 0$, and thus, equation (86) returns the initial wavefunction ($85$) at $D = 0$.

Observe that, apart from the pre-factor $i/\hbar$, the exponential in (89) is nothing but the free generating functional $W^{\text{free}}$, equation (74), in the Hamilton–Jacobi approach.

### 7.1.2 Beyond leading order

Beyond the leading order, the effective potential $V^{(2)}_{\text{eff}}$ needs to be included in the analysis. To iterate to arbitrary high orders, we impose the Ansatz for the propagator

$$K(q, x; D) = K^{\text{free}}(q, x; D) \exp \left[ \frac{i}{\hbar} \sum_{n=2}^{\infty} \frac{1}{n!} M^{(n-1)}(q, x) D^{n-1} \right].$$  \hspace{1cm} (90)

The first new unknown $M^{(2)}$ is easily determined by first plugging the Ansatz (90) into the Schrödinger equation (83a) and keeping only terms at second order in PPT, which leads to the equation

$$M^{(2)} + (x - q) \cdot \nabla_x M^{(2)} + V^{(2)}_{\text{eff}} = 0,$$  \hspace{1cm} (91a)

with solution

$$M^{(2)}(q, x) = -\int_0^1 V^{(2)}_{\text{eff}}(q + s(x - q)) \, ds,$$  \hspace{1cm} (91b)

which coincides precisely with the corresponding second-order solution for the generating functional $W^{(2)}$ in equation (77) in the Hamilton–Jacobi approach (Section 6.1). Uhlemann et al. (2019) advocated to use the two-endpoint approximation for the 2PPT kernel (91b), for which the propagator $K = K^{\text{2PPT}}$ becomes

$$K^{\text{2PPT}}(q, x; D) = K^{\text{free}}(q, x; D) \exp \left[ \frac{iD}{2\hbar} \left( V^{(2)}_{\text{eff}}(q) + V^{(2)}_{\text{eff}}(x) \right) \right],$$  \hspace{1cm} (92)

implying that the effective potential (82b) is evaluated at the initial and final positions that resembles a numerical kick-drift-kick scheme; see Uhlemann et al. (2019) for details and Hahn et al. (2020) for an application of this method to initial conditions for $N$-body simulations.

### 7.2 Two-fluid case

For two gravitationally coupled fluids, the component Schrödinger equation reads ($\alpha = b, c$)

$$i\hbar \partial_D \psi^{\alpha}(x, D) = -\frac{\hbar^2}{2} \nabla_x^2 \psi^{\alpha} + 3g \left( S^{\alpha} + \varphi \right) \psi^{\alpha},$$  \hspace{1cm} (93a)

which is to be supplemented with the Poisson equation

$$\nabla^2 \varphi = f_b |\psi_b|^2 + f_c |\psi_c|^2 - 1 \quad \frac{D}{D},$$  \hspace{1cm} (93b)

For simplicity, we focus on purely growing-mode solutions for which the necessary slaving conditions read at $D = 0$:

$$f_b |\psi_b^{\text{ini}}|^2 + f_c |\psi_c^{\text{ini}}|^2 = 1 \quad \text{and} \quad S^{\text{ini}} = -q^{\text{ini}}.$$  \hspace{1cm} (94)

Similarly, as in the two-fluid case, the initial component densities must be non-vanishing to avoid quasi-singular behaviour at arbitrary short times; furthermore, from the second boundary condition it is clear that the initial phase for the two fluids coincide. The initial wavefunction is thus

$$\psi^{\text{ini}}_\alpha(q) = \sqrt{1 + S^{\text{ini}}_\alpha} \exp \left[ \frac{i}{\hbar} \psi^{\text{ini}}(q) \right].$$  \hspace{1cm} (95)

As it was the case for a single fluid, we can use standard PT to obtain solutions to the effective potentials $V^{(\text{eff})}_{\alpha, \alpha} = 3g(S^{\alpha} + \varphi)/(2D)$ entering in the Schrödinger equations (93). For the non-decaying mode initial conditions specified here, the velocities of the two fluid species agree at all orders, as we demonstrate in Appendix B. This means that the perturbative effective potentials of the two fluid species are identical with the single fluid effective potential, i.e. $V^{(\text{eff})}_{\alpha, \alpha} \equiv V^{(\text{eff})}_{\alpha}$ (cf. after equation 87) and hence

$$i\hbar \partial_D \psi^{\alpha}(x, D) = \left[ -\frac{\hbar^2}{2} \nabla_x^2 + V^{\alpha}(x, D) \right] \psi^{\alpha}(x, D).$$  \hspace{1cm} (96)
At the leading order, we have $V^{(1)}_{\alpha\alpha} = 0$ and thus the solution to the free Schrödinger equation for component $\alpha = b, c$ is given by

$$\psi^{\text{free}}_\alpha(x; D) = \int d^3q \ K_{\text{free}}(q, x; D) \psi^{\text{ini}}_\alpha(q),$$

(97)

where the free propagator is given by equation (89). Thus, apart from the initial density fluctuations inherent to the component fluids, the solution coincides with the one obtained in the single-fluid case. Similarly, the derivations at next-to-leading order are essentially identical as outlined above, with the solution for the wavefunction

$$\psi^{\text{2PPT}}_\alpha(x; D) = \int d^3q \ K_{\text{2PPT}}(q, x; D) \psi^{\text{ini}}_\alpha(q),$$

(98)

where $K_{\text{2PPT}}$ is given by equation (92) in terms of the second-order effective potential (82b).

Having obtained the component wavefunction to the desired order in PPT, the corresponding Eulerian density $\rho_\alpha = 1 + \delta_\alpha$ and momentum density $\pi_\alpha = \rho_\alpha v_\alpha$ are, respectively, given by

$$\rho_\alpha = \psi_\alpha \bar{\psi}_\alpha,$$

(99a)

$$\pi_\alpha = \frac{i\hbar}{2} \left( \psi_\alpha \nabla \bar{\psi}_\alpha - \bar{\psi}_\alpha \nabla \psi_\alpha \right),$$

(99b)

where an overline denotes complex conjugation.

Finally, we determine the classical limits of the derived PPT solutions for the component fluids. Actually, since the propagator for the component fluids agrees with the one for the single fluid, the classical limit can be performed precisely with the same methodology as outlined in section VI of Uhlemann et al. (2019), however, now generalized to $\Lambda$CDM. In the classical limit $\hbar \to 0$, we find the following displacement and corresponding velocity valid up to 2PPT

$$\zeta^{\text{2PPT}}_\alpha = D \zeta^{(1)} + D^2 \zeta^{(2)},$$

(100a)

$$v^{\text{2PPT}}_\alpha = \frac{\zeta^{(1)}}{2} + 2D \zeta^{(2)} + D^2 \zeta^{(1)} \zeta^{(2)} + \zeta^{(2)} \zeta^{(0)},$$

(100b)

where the purely spatial functions $\zeta^{(1)}$ and $\zeta^{(2)}$ are given in equations (32b), and, similarly as in the classical case, the corresponding mass conservation law for the above displacement is $\delta_\alpha = (1 + \delta_\alpha)/\det[\delta_{ij} + \zeta^{\text{2PPT}}_{ij}] - 1$. We remark that the ‘additional’ term $\sim D^2$ in (100b) would be of third order in the classical Lagrangian-coordinates approach, but here arises naturally in order to preserve the underlying Hamiltonian structure in the present approach. The appearance of this term was first noted by Uhlemann et al. (2019), where it was also demonstrated that this term is actually needed to preserve the assumed zero-vorticity condition (cf. their fig. 6).

For numerical applications, keeping $\hbar$ non-zero is crucial as the numerical complexity becomes very demanding in the limit $\hbar \to 0$, due to strong oscillations of the complex wavefunction. For non-zero $\hbar$ that effectively controls the resolution in the phase-space, PPT has significant advantages as compared to classical Eulerian perturbative schemes: On the one hand, the density and velocity fields in PPT are essentially derived by propagating initial fields along the fluid flows – in a fairly similar way as one follows fluid particles in classical Lagrangian-coordinate approaches. Since Lagrangian perturbative approaches are naturally very efficient in resolving convective motion, roughly the same is true for PPT. On the other hand, PPT outputs directly Eulerian fields and thus does not require any $N$-body particle realization which can lead to discretization errors (see e.g. Michaux et al. 2020). Not relying on particle sampling is a significant advantage particularly for hydrodynamical simulations that usually require Eulerian fields for their initialization. For further details and numerical implementation of PPT, see our companion paper (Hahn et al. 2020).

8 SUMMARY AND OUTLOOK

Given two gravitationally coupled fluids that are governed by the equations (14), it becomes evident that relative effects between the fluids are mathematically described by all but the fastest growing modes. Therefore, as a first approximation, one may include just the strongest of the sub-leading (decaying) modes in the analysis. As it turns out, the most persisting sub-leading mode stems from initially prescribed density perturbations of the two fluids, which are constant in linear theory but none the less grow non-linearly in time.

Curiously, there is no zeroth-order approximation in the present case, and we must keep those initial density perturbations in the fluids. Indeed, if we had ignored those initial densities and just kept the very fastest growing modes, a quick analysis would have revealed mathematical inconsistencies that are accompanied with quasi-singular irregularities in the governing equations.

The rigorous argument of the above implies certain boundary conditions on the initial conditions, which are compatible with the requirement of initial quasi-homogeneity. Actually, these boundary conditions are known but, so far, have been exploited only for the single-matter fluid where they build the mathematical foundation of perturbative solutions in powers of the linear structure growth. In this paper, we have generalized the boundary conditions to allow for multiple fluids (equations 20), thereby providing the stepping stone for initializing two-fluid numerical simulations in the growing and persisting modes which we discuss in detail in our companion paper (Hahn et al. 2020).

Even more, these boundary conditions translate straightforwardly into explicit all-order solutions for the difference density in Eulerian coordinates (equation 25), and to the displacement fields for the two fluids in Lagrangian coordinates. We show that, with a suitable choice of Lagrangian coordinates, the two-fluid displacements (equation 54) actually coincide with the standard ones for the single-matter fluid. Essentially, in those coordinates, the initial fluid densities are just transported along their fluid paths (cf. equation 50).

Alternatively, one may absorb the initial densities by means of a redefinition of the Lagrangian coordinate system, which, however, clutters the solutions for the fluid displacements (equation 63). Furthermore, as we elucidate in Hahn et al. (2020), absorbing the initial density in the displacements leads to the excitation of large discreteness errors in the numerical solution. We remark that previous approaches for initializing two-fluid simulations, such as the ones of Hahn & Abel (2011) and Valkenburg & Villaescusa-Navarro (2017), implicitly perform such an operation.

We have also considered a semiclassical approach, which largely builds on the work of Uhlemann et al. (2019) that we have generalized here to a $\Lambda$CDM cosmology for two fluids. We have motivated the semiclassical approach by a variational principle employing the so-called contact geometry (Section 6), which may be viewed as an extension to the symplectic geometry known from standard Hamiltonian theory. In the semiclassical approach, we establish complementary results using the PPT up to second order, which delivers wavefunctions for the two coupled fluids (equation 98) that reproduce in the classical limit the component displacements.

There are several avenues that could be considered in future works. In this study, we ignore baryonic pressure, which hampers our theoretical prediction close to the Jeans scale. Incorporating pressure, possibly along the ways of the single-fluid Lagrangian approach of Tatekawa et al. (2002) or of the two-fluid Eulerian approach of Shoji & Komatsu (2009), could comprise promising starting points.
for initializing $N$-body or hydrodynamical simulations accurately on much smaller scales than anticipated in this work.

Another interesting application of our approach relates to incorporating all decaying modes, which would allow the initialization of several fluids with distinct velocities. This involves in particular the accurate modelling of the advection of small-scale perturbations by large-scale cosmological flows, as has been investigated by Tseliakovitch & Hirata (2010) – a problem for which our Lagrangian and semiclassical approaches might be ideally suited. None the less, while our general framework is capable of incorporating all decaying modes (Section 5.1), at this stage it is unclear how respective higher order initial conditions for simulations could be consistently implemented and thus, such issues require future investigations.

Our two-fluid approach could be also straightforwardly extended to redshift space. Indeed, it is well known that redshift-space distortions are easily incorporated in Lagrangian coordinates (see e.g. Matsubara 2008), and, as shown recently by Porqueres et al. (2020), the same is also true for PPT. Once incorporated for multiple fluids in Lagrangian coordinates, our formalism could be used to determine the power spectra in redshift space to arbitrary high order. For the semiclassical approach, a PPT extension to redshift space could provide accurate theoretical predictions that relates the matter distribution to quasars in the Ly$\alpha$ forest, particularly including the correct description of the scale-dependent bias of baryons relative to the total matter distribution. Our work thus provides encouraging starting points to account more accurately for non-linear effects in the fluids, which is essential for both the forward modelling as well as the reconstruction problem based on observations of the high-redshift intergalactic medium.

Finally, while we have focused on two fluids, generalizations to more fluids are straightforward. We remark, however, that the ‘obvious’ case of three fluids, i.e. including baryons, CDM and massive neutrinos might be best tackled by marrying a two-fluid approach for baryons and CDM with a relativistic description of, e.g. Brandbyge et al. (2017), Tram et al. (2019), Zennaro et al. (2019), and Partmann et al. (2020) that effectively incorporates massive neutrinos (in simulations) by means of suitable coordinate transformations. We will come back to this problem in a forthcoming work.

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APPENDIX A: SINGLE-FLUID SOLUTIONS IN WEAK EXPANSION

Here we review the standard results for a single fluid in $\Lambda$CDM, assuming a weak perturbation expansion of the form

$$\delta_m = \epsilon \delta_1^{(m)} + \epsilon^2 \delta_2^{(m)} + \ldots \quad \text{(A1)}$$

Naturally one could assume that each order $\delta^{(m)}_1$ factorizes into a purely space and time dependent part, at least for the fastest growing-mode solutions. However, Bouchet et al. (1995) and Matsubara (1995) showed that this is not the case for the weak expansion of the Eulerian density in $\Lambda$CDM; see Villa & Rampf (2016) for a more recent discussion in a relativistic context.

To make progress, even for determining the Eulerian density, it is – not necessary but advantageous – to solve the problem firstly in Lagrangian space, and then subsequently transform those results to Eulerian space. To do so, we solve the Lagrangian evolution equations (32b) with the following weak expansion Ansatz for the displacement (the dynamical quantity in Lagrangian space):

$$\xi_m = \epsilon \partial_t \xi_1^{(m)}(q) + \epsilon^2 E(t) \xi_2^{(m)}(q) + \ldots \quad \text{ (A2)}$$

where we have assumed (correctly) that the Lagrangian solutions factorize, and we have added the bar on top of some quantities to distinguish between the different expansion schemes used in this paper. It is also useful to provide the corresponding Jacobian up to second order:

$$J = \det \left[ \delta_{ij} + \xi^{(m)}_{i,j} \right] = 1 + \epsilon \partial \xi_1^{(m)} + \epsilon^2 E \xi_2^{(m)} + \epsilon^3 D^2 \mu_2^{(m)} + \ldots$$

At first order in $\epsilon$, the space part is determined using the boundary conditions (see Section 4), leading to $\xi_1^{(m)} = -\nabla q_m$, while one gets an ODE for the temporal coefficient

$$\epsilon^2 \partial_t \bar{D} = -\frac{3g}{2D^2} \bar{D} = 0 \quad \text{ (A4)}$$

where as before $\epsilon^2 \partial_t \bar{D} = (\partial_t \bar{D})^2 + [3g/(2D)] \partial_l \bar{D}$. Due to the appearance of the time-dependent factor $g(D)$, this ODE is most easily solved numerically. We show the numerical solution in Fig. 1, which agrees with the standard analytical solution (3) to machine precision, thus suggesting that effectively $g = 1$ at this order, and thus we set from here on $\bar{D} = D$.

To get the second-order solution, we truncate all terms $O(\epsilon^3)$ in (32b) and get for the spatial part $\xi_2^{(m)} = \mu_2^{(m)}$ that again coincides with the standard result. By contrast, for the temporal coefficient we obtain the ODE

$$\epsilon^2 \partial_t \bar{E} = -\frac{3g}{2D^2} \bar{E} = -\frac{3g}{2} \quad \text{ (A5)}$$

While in EdS, this equation can still be solved analytically ($\bar{E} \sim -(3/7)a^3$), there is no known analytical solution in $\Lambda$CDM – as opposed to the derived analytical solution derived in terms of the $\bar{D}$ expansion where $\bar{E} \sim -(3/7)D^2$. However, the numerical solution for (A5) is displayed in Fig. 1, and the comparison against $\bar{E} \sim -(3/7)D^2$ reveals excellent agreement. Therefore, we set in the following $\bar{D} = D$ and $\bar{E} = E \sim -(3/7)D^2$.

To get an expression for the density in terms of the Lagrangian mass, we Taylor expand the Lagrangian mass density $\delta = 1/J - 1$ to second order, yielding firstly

$$\delta_m(q) = -D q_{ll}^{\text{ini}} + \frac{D^2}{2} q_{ll}^{\text{ini}} q_{mm}^{\text{ini}} + \frac{D^2}{2} q_{lm}^{\text{ini}} q_{lm}^{\text{ini}} \quad \text{ (A6)}$$

Finally, to obtain the density evaluated at the current (Eulerian) position, we use the inverse map $q(x) = x - \xi$ to first order in $\delta_m(q(x)) = \delta_m(x - \xi)$. As a consequence, the transported first-order solution generates a second-order term (a.k.a. the convective term), leading to the second-order result for the Eulerian density

$$\delta_2^{(m)}(x) = \frac{D^2}{2} q_{ll}^{\text{ini}} q_{mm}^{\text{ini}} + \frac{D^2}{2} q_{lm}^{\text{ini}} q_{lm}^{\text{ini}} + \frac{D^2}{2} q_{mm}^{\text{ini}} q_{lm}^{\text{ini}} \quad \text{ (A7)}$$

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which agrees with the reported result given in equation (12b). From the solution (A7) it is clear that, in Eulerian coordinates, spatial and temporal dependencies do not factorize at second order in $\Lambda CDM$, a fact that was first noted by Matsubara (1995).

**APPENDIX B: TWO-FLUID SOLUTIONS IN SECOND-ORDER PERTURBATION THEORY**

**B1 Two-fluids assuming single-fluid Poisson source**

Set $\delta_a = \delta_a^{(1)} + \delta_a^{(2)}$ and $\phi_a = \phi_a^{(1)} + \phi_a^{(2)}$, where $\phi_a \equiv -\nabla \phi_a$, for $\alpha = b, c$. The first-order growing-mode solutions are

$$\delta_a^{(1)} = D \nabla^2 \phi_a^{(i)} + \delta_a^{(i)},$$

$$\phi_a^{(i)} = \phi_a^{(i)},$$  \hspace{1cm}  (B1)

where

$$\delta_a^{(i)} = \begin{cases} (1 - f_0) \delta_a^{(i)}, & \alpha = b, \\ -f_0 \delta_a^{(i)}, & \alpha = c. \end{cases}$$  \hspace{1cm}  (B2)

The fluid equations for the components $\alpha = b, c$ can be written as

$$\partial_D \phi_a - \frac{1}{2} \left( \nabla \phi_a \right)^2 = \frac{3g}{2D} (\phi - \phi_a),$$

$$\partial_D \delta_a - \nabla \cdot (1 + \delta_a) \nabla \phi_a = 0,$$

$$\nabla^2 \phi_a = \delta_a.$$  \hspace{1cm}  (B3)

At second order we have $g \simeq 1$, and thus these equations become

$$\partial_D \phi_a^{(2)} - \frac{1}{2} \left( \nabla \phi_a^{(1)} \right)^2 = \frac{3g}{2D} (\phi^{(2)} - \phi_a^{(2)}),$$

$$\partial_D \delta_a^{(2)} - \nabla \cdot (\delta_a^{(1)} \nabla \phi_a) = 0,$$  \hspace{1cm}  (B7)

$$\nabla^2 \phi_a^{(2)} = \frac{\delta_a^{(2)} (x, D)}{D}.$$  \hspace{1cm}  (B8)

In these equations, all quantities with a perturbation index of 1 are already determined; furthermore we have already derived $\delta_a^{(2)} (x, D) = D^2 \delta_a^{(2)} (x)$ in the main text, see equation (10), implying that we also know already $\psi^{(2)}$. Rewriting (B7) in terms of $\phi_a^{(1)}$ and plugging it into (B6), we get

$$\partial_D \delta_a^{(2)} + \frac{3g}{2D} \partial_D \delta_a^{(2)} = 5 \delta_a^{(2)} (x) + \frac{3g}{2D} \left[ \phi^{(1)} \psi^{(i)}_{mm} + \phi^{(2)} \psi^{(i)}_{mm} \right].$$  \hspace{1cm}  (B9)

The general solution to this equation is

$$\delta_a^{(2)} = D^2 \delta_a^{(2)} - 2C_1 D^{-1/2} + C_2 + D \left[ \phi^{(1)} \psi^{(i)}_{mm} + \phi^{(2)} \psi^{(i)}_{mm} \right],$$  \hspace{1cm}  (B10)

where $C_1$ and $C_2$ are integration constants; actually the term involving $C_1$ is a decaying mode and not compatible with slaving, thus $C_1 = 0$. The other term is physically redundant as it can be absorbed into $\delta_a^{(2)}$, thus we can set $C_2 = 0$. Plugging the solution for $\delta_a^{(2)}$ into (B7) we find the solution for the second-order velocity potential, i.e.

$$\nabla^2 \phi_a^{(2)} = \frac{\delta_a^{(2)} (x, D)}{D}.$$  \hspace{1cm}  (B11)

where, evidently, all terms involving $\delta_a^{(i)}$ have cancelled out. Finally, using these results, we find that the so-called effective potential for the components $\alpha = b, c$ is

$$V_{\text{eff}} \equiv \frac{3g}{2D} (\phi - \phi_a) = \frac{3g}{7} \nabla^{-2} \left[ \phi^{(1)} \psi^{(i)}_{mm} - \phi^{(2)} \psi^{(i)}_{mm} \right] + O(3).$$  \hspace{1cm}  (B12)

This effective potential is the necessary input to determine the NLO propagator in PPT.

**B2 Growing-mode solutions for sum and difference variables**

Here, we are concerned with deriving the second-order solutions by employing the weighted sum and difference variables

$$\delta_m = f_b \delta_b + f_c \delta_c,$$

$$v_m = f_b v_b + f_c v_c,$$

at first and second order in PT, i.e.

$$\delta_m = \delta_b^{(1)} + \delta_c^{(2)} + \ldots,$$

$$v_m = v_b^{(1)} + v_c^{(2)} + \ldots,$$

$$\delta_b = \delta_b^{(1)} + \delta_b^{(2)} + \ldots,$$

$$v_b = v_b^{(1)} + v_b^{(2)} + \ldots,$$

$$\delta_c = \delta_c^{(1)} + \delta_c^{(2)} + \ldots,$$

$$v_c = v_c^{(1)} + v_c^{(2)} + \ldots.$$  \hspace{1cm}  (B14)

From the main text, we have already derived the growing-mode solutions at first order, which we summarize here for convenience

$$\delta_m^{(1)} = D \nabla^2 \phi_m^{(i)},$$

$$\delta_b^{(1)} = \phi_b^{(i)},$$

$$v_m^{(1)} = -\nabla \phi_m^{(i)},$$

$$v_b^{(1)} = 0,$$  \hspace{1cm}  (B15)

$$\delta_b^{(1)} = \phi_b^{(i)},$$

$$\delta_c^{(1)} = \phi_c^{(i)},$$

$$v_b^{(1)} = \psi_m^{(1)},$$

$$v_c^{(1)} = \psi_m^{(1)}.$$  \hspace{1cm}  (B16b)

where $\delta_m^{(1)} = (1 - f_0) \phi_b^{(i)}$ and $\delta_c^{(1)} = -f_0 \phi_c^{(i)}$. As before these growing-mode results employ implicitly the slaving conditions (20). Using these results in the component fluid equations (14), we have at second order for the components

$$\partial_D v_m^{(2)} + v_m \cdot \nabla v_m + O(3) = -\frac{3g}{2D} (v_b^{(2)} + v_c^{(2)}),$$  \hspace{1cm}  (B16a)

$$\partial_D \delta_m^{(2)} + \nabla \cdot (\delta_b^{(1)} v_b^{(1)} + \delta_c^{(1)} v_c^{(1)}) + \nabla \cdot \left[ \psi_m^{(1)} \psi_m^{(i)} \right] = 0,$$  \hspace{1cm}  (B16b)

$$\nabla^2 \psi_m^{(2)} = \frac{\delta_m^{(2)}}{D}.$$  \hspace{1cm}  (B16c)

where we have expressed some first-order component variables in terms of the single fluid variables. These equations can be written in terms of the weighted sum and difference variables, we find truncated up to second order for the growing modes:

$$\partial_D v_m + v_m \cdot \nabla v_m + O(3) = \frac{3g}{2D} (v_m + \nabla \phi),$$

$$\partial_D \delta_m + \nabla \cdot (1 + \delta_m) v_m + O(3) = 0,$$  \hspace{1cm}  (B17b)

$$\partial_D v_b + \frac{3g}{2D} v_b + O(3) = 0,$$  \hspace{1cm}  (B17c)

$$\partial_D \delta_c + \nabla \cdot v_c + O(3) = 0.$$  \hspace{1cm}  (B17d)

To arrive at (B17b) we have explicitly used the slaving condition $\delta_m^{(2)} = 0$. Crucially, the sum and difference equations still decouple effectively at second order. While the truncated sum equations are identical with the standard equations for a single fluid, the continuity equation for the difference variables receives a second-order correction. Specifically, combining (B17c)-(B17d) we obtain at second order the ODE

$$\partial_D \psi_m^{(2)} + \frac{3g}{2D} \delta_m^{(2)} = \frac{3g}{2D} \nabla \cdot \left( \psi_m^{(2)} \nabla \phi \right),$$  \hspace{1cm}  (B18)

which for $g \simeq 1$ has the analytic solution

$$\delta_m^{(2)} = D \nabla \cdot \left( \delta_m^{(2)} \nabla \phi \right) - \frac{2C_1}{\sqrt{D}} + C_2,$$  \hspace{1cm}  (B19)

where $C_1 \to 0$ due to slaving, and $C_2$ can be set to zero (or, equivalently, absorbed in the first-order constant). It is easily checked that having the sum and difference solutions, one re-derives the
identical solution (B10) from \( \delta_c^{(2)} = \delta_m^{(2)} + (1 - f_b) \delta_{bc}^{(2)} \) and \( \delta_c^{(2)} = \delta_m^{(2)} - f_b \delta_{bc}^{(2)} \).

### B3 All-order recursions for the difference density

From the above considerations, it is clear that for growing-mode initial conditions we have at least to second order \( \mathbf{v}_bc = 0 \) and thus the component velocities coincide. Actually, using this and iterating the expressions (B17) to third order, one finds that \( \mathbf{v}_bc = 0 \) also at third order. Further iterations then lead to the conclusions that \( \mathbf{v}_bc = 0 \) must be zero to all orders in PT, essentially since all appearing terms at higher orders in (B17a) are quadratic combinations of the lower-order velocity components stemming from convective terms such as \( \mathbf{v}_n \cdot \nabla \mathbf{v}_m \); but since those lower order velocity components achieve \( \mathbf{v}_bc = 0 \), this also implies that \( \mathbf{v}_n = \mathbf{v}_m = \mathbf{v}_0 \) to all orders. Thus, the convective term in the evolution equation for \( \mathbf{v}_bc \) drops out to all orders.

In summary, the sum and difference fluid equations are non-perturbatively for the growing mode:

\[
\begin{align*}
\partial_D \mathbf{v}_m + \mathbf{v}_m \cdot \nabla \mathbf{v}_m &= - \frac{3g}{2D} (\mathbf{v}_m + \nabla \psi), \quad (B20a) \\
\partial_D \delta_m + \nabla \cdot (\{1 + \delta_m\} \mathbf{v}_m) &= 0, \quad (B20b) \\
\partial_D \mathbf{v}_bc + \frac{3g}{2D} \mathbf{v}_bc &= 0, \quad (B20c) \\
\partial_D \delta_{bc} + \nabla \cdot [\delta_{bc} \mathbf{v}_m] &= 0. \quad (B20d)
\end{align*}
\]

Two remarks are in order. First, as mentioned above, from (B20c) follows that the growing-modes of \( \mathbf{v}_bc \) must be zero at all orders. Secondly, since \( \mathbf{v}_n = \mathbf{v}_m \) non-perturbatively, from the mass conservation (B20d) it is clear that \( \delta_{bc} \) couples only to the sum velocity \( \mathbf{v}_m \). For the latter there exist explicit all-order recursion relations for the growing mode, they can be written as

\[
\begin{align*}
\delta_m &= - \sum_{n=1}^{\infty} \frac{d^{(n)}m}{D^{n-1}} = - \sum_{n=1}^{\infty} \nabla \cdot \mathbf{v}^{(n)}m D^{n-1} = \nabla \cdot \mathbf{v}_m, \quad (B21)
\end{align*}
\]

where \( d^{(n)}m \) is the \( n \)-th order perturbation kernel which in the literature are usually formulated in Fourier space and then denoted with \( G_n \) (see, however, equation 10 of Taruya, Nishimichi & Jeong 2018 for a real-space version). From this, it is easily checked that the appropriate Ansätze for \( \delta_{bc} \) is

\[
\begin{align*}
\delta_{bc} &= \sum_{n=1}^{\infty} \frac{d^{(n)}bc}{D^{n-1}}. \quad (B22)
\end{align*}
\]

Plugging the Ansätze for \( \delta_{bc} \) and \( \mathbf{v}_m \) into (B20) and using that \( \theta_{bc} = 0 \), we obtain the following all-order recursion relation

\[
\begin{align*}
\mathcal{D}^{(n)}bc = \frac{1}{n-1} \sum_{i+j=n} \nabla \cdot \mathbf{D}^{(i)}bc \mathbf{v}_m \quad (n \geq 1), \quad (B23)
\end{align*}
\]

for \( n > 1 \), and \( \mathcal{D}^{(1)}bc = \delta^{(1)}bc \) for \( n = 1 \).

We remark that a similar relation for \( \delta_{bc} \) can also be formulated in Lagrangian space: Writing \( \delta^{(n)}(x(q)) = (1 + \delta^{(m)}m(q)/J^m(q) - 1 \) and noting that for the growing modes we have \( F^m = P^m \), we can obtain the difference density by simply subtracting the two definitions of mass conservation, i.e.

\[
\begin{align*}
\delta^{(n)}(x(q)) &= \delta^{(n)}(x(q)) - \delta^{(n)}(x(q)) \quad (B24)
\end{align*}
\]

which, when evaluated at the Eulerian position to fixed order, delivers identical results as from (B23). We remark that to verify the agreement to order \( n = 3 \), one needs to evaluate the fields at the position \( q_i(x) = x_i - \delta^{(m)(1)}i(x) = \delta^{(m)(2)}i + \delta^{(m)(1)}i \delta^{(m)(1)}j \), where the slash denotes differentiation with respect to Eulerian coordinates.

Finally, let us provide these recursion relations in Fourier space, we find

\[
\begin{align*}
\delta^{(n)}bc(k) &= \int \frac{d^3k_1 \cdots d^3k_n}{(2\pi)^n} \delta^{(n)}bc(k_{1:m} - k) F^{(n)}bc(k_1, \ldots, k_n)
\times \delta^{(n)}bc(k_1, \ldots, k_n) \delta^{(n)}bc(k_{1:m}), \quad (B25)
\end{align*}
\]

where \( k_{1:m} = k_1 + k_2 + \cdots + k_m \), and the first kernels are

\[
\begin{align*}
F^{(1)}bc &= 1, \quad (B26)
F^{(2)}bc &= \frac{k_1 \cdot k_2}{k_2^2}, \quad (B27)
F^{(3)}bc &= \frac{1}{2} k_{1:3} \cdot k_{23} \left[ \frac{3}{7} + k_3 \cdot k_3 \frac{k_2 \cdot k_3}{k_3^2} \frac{k_1 + k_2}{k_2} + \frac{4 (k_2 \cdot k_3)^2}{7 k_2 k_3^2} \right]
\times \frac{k_1 k_2 k_3}{k_2^2} k_{1:3}, \quad (B28)
\end{align*}
\]

where \( k = |k| \). These kernels are to be symmetrized in their arguments \( k_2 - k_1 \), but not in its first argument \( k_1 \); this is a consequence of the Fourier kernels (B25) within the integrals. Fairly similar to the well-known density and velocity kernels in SPT (see e.g. Bernardaeu et al. 2002), the kernels \( F^{(n)}bc \) are well behaved when the sum of some of its arguments cancel, but there are (the known) infrared divergences when one or more of its arguments go to zero. Furthermore and in contrast to the standard SPT kernels, for \( k = k_{1:3} = 0 \), the kernels \( F^{(n)}bc \) do not asymptote \( k^2 \) but vanish instead; this so due to the appearance of the overall divergence in the recursion relation (B23).

### B4 One-loop power spectrum for the density difference

Define the linear power and cross spectra with

\[
\begin{align*}
\langle \tilde{\delta}^{(1)}m k \rangle \langle \tilde{\delta}^{(1)}m k \rangle &= (2\pi)^3 \delta^3(k_1) P^{lin}_m(k_1), \quad (B29a)
\langle \tilde{\delta}^{(1)}bc k \rangle \langle \tilde{\delta}^{(1)}bc k \rangle &= (2\pi)^3 \delta^3(k_1) P_{bc,lin}(k_1), \quad (B29b)
\langle \tilde{\delta}^{(1)}bc k \rangle \langle \tilde{\delta}^{(1)}bc k \rangle &= (2\pi)^3 \delta^3(k_1) P_{bc,lin}(k_1), \quad (B29c)
\end{align*}
\]

Here, we like to determine the power spectrum for \( P_{bc, lin} \) to one-loop accuracy, i.e. approximate \( \delta_{bc} = \delta^{(1)}bc + \delta^{(2)}bc + \delta^{(3)}bc \) and derive

\[
\begin{align*}
P_{bc, lin}(k) &= P^{lin}_{bc, lin}(k) + P^{one-loop}_{bc, lin}(k) \quad (B30)
\end{align*}
\]

with

\[
\begin{align*}
P^{one-loop}_{bc, lin}(k) &= P^{(2,1)}bc, lin(k) + P^{(1,3)}bc, lin(k) \quad (B31)
\end{align*}
\]

and

\[
\begin{align*}
\langle \tilde{\delta}^{(1)}bc k \rangle \langle \tilde{\delta}^{(3)}bc k \rangle &= (2\pi)^3 \delta^3(k_1) P^{(1,3)}bc, lin(k_1), \quad (B32)
\langle \tilde{\delta}^{(2)}bc k \rangle \langle \tilde{\delta}^{(2)}bc k \rangle &= (2\pi)^3 \delta^3(k_1) P^{(2,2)}bc, lin(k_1), \quad (B33)
\end{align*}
\]

Having the explicit expressions for \( \delta^{(n)}bc \), it is straightforward to determine these one-loop corrections by applying Wick’s theorem (for similar derivation for the matter power spectrum, see e.g. Crocce
& Scoccimarro 2006). We find the connected parts

\[ P_{bc,bc}^{1,3}(k) = P_{bc,bc}^{\text{lin}}(k) \int \frac{d^3 p}{(2\pi)^3} \bar{F}^{(3)}_{bc}(k, p, -p) P_{m,m}^{\text{lin}}(p) \]

\[ + 2P_{bc,bc}^{\text{lin}}(k) \int \frac{d^3 p}{(2\pi)^3} \bar{F}^{(3)}_{bc}(p, -p, k) P_{bc,m}^{\text{lin}}(p). \]  

(B34)

\[ P_{bc,bc}^{2,2}(k) = \int \frac{d^3 p}{(2\pi)^3} \left( \bar{F}^{(2)}_{bc}(k - p, p) \right)^2 P_{bc,bc}^{\text{lin}}(k - p) P_{bc,m}^{\text{lin}}(p) \]

\[ + \int \frac{d^3 p}{(2\pi)^3} \bar{F}^{(2)}_{bc}(p, k - p) \bar{F}^{(2)}_{bc}(k - p, p) \times P_{bc,bc}^{\text{lin}}(k - p) P_{bc,m}^{\text{lin}}(p). \]

(B35)

**APPENDIX C: TWO-FLUID SOLUTIONS IN LAGRANGIAN PERTURBATION THEORY**

### C1 Explicit derivations in the general formalism

Here we provide the general second-order perturbation equations for the two-fluid system in Lagrangian coordinates. For this we assume a weak expansion in the displacement according to

\[ x^\mu - q = \xi^\mu + \xi^\mu_{(2)} + \ldots, \]  

(C1)

and assume, as before, that \( \xi^\mu_{(n)} \) is not larger than typical first-order perturbations. Let us define

\[ \mu_2^{\alpha,\beta} = \frac{1}{2} \left[ \xi^{\alpha}_{i,j} \xi^{\beta}_{l,j} - \xi^{\beta}_{i,j} \xi^{\alpha}_{l,j} \right] \]

\[ = \frac{1}{2} \left[ \xi^{(1)\mu}_{i,j} \xi^{(1)\nu}_{l,j} - \xi^{(1)\nu}_{i,j} \xi^{(1)\mu}_{l,j} \right] + O(3), \]

where \( \alpha \) and \( \beta \) are either \( b \), \( c \), and provide some of the related expressions that are needed later on, valid until second order,

\[ J^\mu(q) = 1 + \xi^{(1)\mu}_{l,j}(q) + \xi^{(2)\mu}_{l,j} + \mu_2^{\mu,\nu}, \]  

(C3)

\[ x^\mu_\nu(q) = q_\mu + \xi^{(1)\nu}_{l,j}(q_\nu) + \xi^{(2)\nu}_{l,j}(q_\nu), \]  

(C4)

\[ q^\mu_{(2)}(x) = x_\mu + \xi^{(1)\mu}_{l,j}(x) - \xi^{(2)\mu}_{l,j}(x), \]  

(C5)

\[ q^\mu_{(b)}(x) = q_\mu + \xi^{(1)\mu}_{l,j}(x) - \xi^{(2)\mu}_{l,j}(x) + \xi^{(1)c}_{l,j} + \xi^{(2)c}_{l,j}, \]  

(C6)

\[ J^\mu(q_{(2)}(x)) = 1 + \xi^{(1)\mu}_{l,j} + \xi^{(1)c}_{l,j} \left( \xi^{(1)b}_{l,j} - \xi^{(1)c}_{l,j} \right) + \xi^{(2)c}_{l,j} + \mu_2^{\mu,c}, \]  

(C7)

and similarly for the nontrivial terms appearing in the evolution equation (35a) of the \( c \) component. Here, \( F_{ij} \) denotes partial differentiation of an arbitrary function \( F \) with respect to Eulerian component \( x_i \).

Using these identities, it is straightforward to determine the second-order part of equation (35a), we find

\[ \mathcal{N}_D \xi^{(2)}_{b,j} = \frac{3g}{2D^2} \left[ f_a \left( \xi^{(2)}_{b,j} - \mu_2^{b,c} \right) + f_c \left( \xi^{(2)}_{b,j} + \mu_2^{b,c} - 2\mu_2^{b,c} \right) \right] \]

\[ - f_b \left( \left( \xi^{(1)\mu}_{l,j} - \xi^{(1)c}_{l,j} \right) \partial_j - \xi^{(1)b}_{l,j} \right) \left( \xi^{(1)b}_{l,j} - \xi^{(1)c}_{l,j} \right), \]  

(C8a)

\[ \mathcal{N}_D \xi^{(2)}_{c,j} = \frac{3g}{2D^2} \left[ f_a \left( \xi^{(2)}_{c,j} - \mu_2^{b,c} \right) + f_b \left( \xi^{(2)}_{c,j} + \mu_2^{b,c} - 2\mu_2^{b,c} \right) \right] \]

\[ + f_b \left( \delta^{(1)i}_{l,j} - \xi^{(1)b}_{l,j} \right) \partial_j - \xi^{(1)b}_{l,j} \right) \left( \xi^{(1)b}_{l,j} - \xi^{(1)c}_{l,j} \right), \]  

(C8b)

In deriving (C8a), we have simplified first-order expressions of the kind \( \mathcal{N}_D \xi^{(1)}_{b,j} \) and \( \mathcal{N}_D \xi^{(1)}_{c,j} \) with their respective right-hand side’s, as instructed through equation (37a); here occurring integration constants can be safely ignored.

### C2 Component velocity in the Lagrangian approaches

In the main text, we have derived the component displacement in two complementary approaches. Although these displacements appear substantially different, we have already shown in the main text that both displacement achieve the same Eulerian density, which is an important consistency check. Here, we show that the same is also true for the Eulerian velocity.

For both Lagrangian-coordinate approaches, it is useful to determine the Eulerian velocity by considering the Eulerian continuity equation for the components given in equation (14b), which can equivalently be written as

\[ \delta_a \mu_2^{\nu,\rho} + (1 + \delta_\nu) \theta_\rho = 0, \]  

(C9)

where \( \theta_\rho = \nabla_\rho - \nabla_\rho \), and \( \delta_\nu \) denotes, as before, the convective time derivative. Since we have already derived the density in both Lagrangian approaches parametrized through the Lagrangian coordinate \( q \), we use the above equation to determine the corresponding velocity divergence. For this we consider a weak expansion of all involved fields of the form

\[ \delta_a = \delta_a^{(1)} + \delta_a^{(2)}, \quad \theta_\rho = -\delta_a^{(1)} - \delta_a^{(2)} \]  

(C10)

(note the minus sign in the second Ansatz due to convention), which leads to the following constraint equations for the velocity at first and second order in parametrized form (i.e. depending on \( q \)),

\[ \delta_a^{(1)}(x(q)) = \partial_\rho \delta_a^{(2)}(x(q)), \]  

(C11a)

\[ \delta_a^{(2)}(x(q)) = \partial_\rho \delta_a^{(2)}(x(q)) - \delta_a^{(1)}(x(q)) \partial_\rho \delta_a^{(1)}(x(q)). \]  

(C11b)

Given the map and associated mass conservation law, we will solve these equations in the two Lagrangian approaches. We also note that, alternatively, the Eulerian velocity divergence could also be determined by considering the convective time derivative of the Lagrangian displacement field, from which one subsequently needs to take the Eulerian divergence; for explicit instructions in the single-fluid case, see e.g. section 6.3 of Rampf & Buchert (2012).

### C3 Eulerian velocity in the Lagrangian approach of Section 5.3

In the most straightforward implementation of LPT, the component displacement reads

\[ x^\mu(q, D) - q = \xi^\mu(q, D) = D \xi^{(1)}_{\mu,\nu}(q) + D^2 \xi^{(2)}_{\mu,\nu}(q), \]  

(C12)

where mass conservation reads in this case

\[ \delta_a(x^\mu(q, D)) = \frac{1 + \delta_a^{(1)} + \delta_a^{(2)}}{\det(\delta_{ij} + \delta_a^{(2)})} - 1. \]  

(C13)

For reference \( \xi^{(1)}_{\mu,\nu}(q) \) and \( \xi^{(2)}_{\mu,\nu}(q) \) coincide in the present approach with the single-fluid displacement, and are given in equations (32b). Expanding equation (C13) gives

\[ \delta_a(x^\mu(q, D)) = \delta_a^{(1)}(q) + D \delta_a^{(2)}(q) + 2D^2 \delta_a^{(1)}(q) + \frac{3}{2} \delta_a^{(1)}(q) + \delta_a^{(2)}(q) + 3 \delta_a^{(1)}(q) + \delta_a^{(2)}(q) \]

\[ + D^2 \left[ \frac{5}{2} \delta_a^{(1)}(q) + \frac{1}{2} \delta_a^{(2)}(q) + \frac{3}{2} \delta_a^{(1)}(q) + \delta_a^{(2)}(q) \right] + O(3). \]  

(C14)

Here, it is important to note that the convective time derivative does not commute with the Eulerian derivative/position; since we have
chosen to formulate the continuity equation using the convective time derivative, pullback operations must be performed after the temporal derivatives are evaluated. Keeping this in mind, one obtains from the above expressions and from equation (C11a) at first order
\[ \theta^{(1)}_a(x(q), D) = \frac{\partial}{\partial x^i} \phi^{(1)}_j(x(q)) = \phi^{(1)}_{ij}(q), \]  
(C15)
and subsequently at second order, using equation (C11b),
\[ \theta^{(2)}_a(x(q), D) = D \left[ \frac{3}{7} \phi^{(2)}_{im} \phi^{(2)}_{mn} + 4 \phi^{(2)}_{im} \phi^{(2)}_{lm} + \frac{4}{7} \phi^{(2)}_{im} \phi^{(2)}_{ln} \right]. \]  
(C16)

What is left is evaluating all terms at the current position, which induces a second-order term stemming from \( \theta^{(1)}_a(x(q)) = \phi^{(1)}_{ij}(q) \). The velocity divergence truncated to second order is then
\[ \theta_a = -\phi^{(2)}_{ii} = D \left[ \frac{3}{7} \phi^{(2)}_{im} \phi^{(2)}_{mn} + 4 \phi^{(2)}_{im} \phi^{(2)}_{lm} + \frac{4}{7} \phi^{(2)}_{im} \phi^{(2)}_{ln} \right], \]  
(C17)
which, as anticipated, agrees with the velocity divergence in the single-fluid case; cf. equation (10).

**C4 Eulerian velocity in the Lagrangian approach of Section 5.4**

Also in this approach, we need to verify whether the fastest growing mode of velocity divergence for the components agrees with the one from the single fluid. In the approach of Section 5.4, the map was chosen to formulate the continuity equation using the convective time derivative, pullback operations must be performed after the temporal derivatives are evaluated. Keeping this in mind, one obtains from the above expressions and from equation (C11a) at first order
\[ \theta^{(1)}_a(x(q), D) = \frac{\partial}{\partial x^i} \phi^{(1)}_j(x(q)) = \phi^{(1)}_{ij}(q), \]  
(C15)
and subsequently at second order, using equation (C11b),
\[ \theta^{(2)}_a(x(q), D) = D \left[ \frac{3}{7} \phi^{(2)}_{im} \phi^{(2)}_{mn} + 4 \phi^{(2)}_{im} \phi^{(2)}_{lm} + \frac{4}{7} \phi^{(2)}_{im} \phi^{(2)}_{ln} \right]. \]  
(C16)

What is left is evaluating all terms at the current position, which induces a second-order term stemming from \( \theta^{(1)}_a(x(q)) = \phi^{(1)}_{ij}(q) \). The velocity divergence truncated to second order is then
\[ \theta_a = -\phi^{(2)}_{ii} = D \left[ \frac{3}{7} \phi^{(2)}_{im} \phi^{(2)}_{mn} + 4 \phi^{(2)}_{im} \phi^{(2)}_{lm} + \frac{4}{7} \phi^{(2)}_{im} \phi^{(2)}_{ln} \right], \]  
(C17)
which, as anticipated, agrees with the velocity divergence in the single-fluid case; cf. equation (10).

**C5 Proof of zero relative displacement for growing modes**

Here we show that the Lagrangian equations of motion (35) for the case of growing modes, where \( \xi^{bc}(1) = 0 \) and \( \delta^{im}(0) = 0 \), predict vanishing relative displacements \( \xi^{bc}(n) = \xi^{bc}(0) = 0 \) at all orders in LPT. We will prove this by induction, showing that if \( \xi^{bc}(k) = 0 \) for all \( k < n \) then \( \xi^{bc}(n) = 0 \).

We begin by computing the difference of the evolution equations for the component displacements (35a) and replacing \( x^{(i)}_{ij} = \delta_j + \xi^{(n)}_{ji} \) on the LHS, which leads to
\[ \begin{align*}
2\delta_j \xi^{bc}_{ij} &+ 2\xi^{bc}_{ik} \xi^{bc}_{jk} \left( \xi^{bc}_{im} \partial D \xi^{bc}_{mj} - \xi^{bc}_{nm} \partial D \xi^{bc}_{ij} \right)^{(n)} \\
+ \xi^{bc}_{ik} \xi^{bc}_{jm} & \left( \xi^{bc}_{im} \partial D \xi^{bc}_{mj} - \xi^{bc}_{nm} \partial D \xi^{bc}_{ij} \right)^{(n)} \\
& = -\frac{3g}{D} \left[ J^b \left( \nabla_x^2 \psi \right)_b - J^c \left( \nabla_x^2 \psi \right)_c \right]^{(n)}.
\end{align*} \]  
(C23)

Evidently, the round bracketed term on the LHS contains terms quadratic and cubic in the displacements. Since their orders need to sum to \( n \), all displacements involved will be evaluated at an order \( k < n \), where it has already been established that \( \xi^{bc}(k) = \xi^{bc}(k) \) for \( k < n \) and thus, these terms cancel out. This leads to
\[ \begin{align*}
\mathcal{R}_D \xi^{bc}_{ij} & = -\frac{3g}{D} \left[ J^b \left( \nabla_x^2 \psi \right)_b - J^c \left( \nabla_x^2 \psi \right)_c \right]^{(n)} \\
& = -\frac{3g}{D} \left[ f_0 \left( 1 + \delta^{im}_{b} \right) + f_c \left( 1 + \delta^{im}_{b} \right) \right] J^b(q) - J^b \\
& - f_0 \left( 1 + \delta^{im}_{b} \right) J^b(q) - f_c \left( 1 + \delta^{im}_{b} \right) + J^c \right]^{(n)}.
\end{align*} \]  
(C24)

where \( \tilde{q}_i = x_i^{-1} \circ x^c(q) \) and \( \tilde{q}_j = x_i^{-1} \circ x^c(q) \). To arrive at the last equality, we have used the Poisson equations (34) for the two fluids. To proceed note that the nth-order approximations for the Jacobians can be exactly written as
\[ \begin{align*}
J^{a}(q) - 1 & = f_0^{(n)}(q) + \delta F^{a(0)}(q), \\
\text{and specifically}
J^{a}(q) - 1 & = f_0^{(n)}(q) + \delta F^{a(0)}(q).
\end{align*} \]  
(C25)

**APPENDIX D: CONTACT HAMILTONIAN**

Here, we derive the so-called contact Hamiltonian (67a), as well as provide some related tools. Many of the results presented here follow the methodology of Bravetti et al. (2017), applied to the cosmological problem; for more mathematical details we refer to their work and references therein.

Contact Hamiltonians employ contact geometry, a concept introduced through Arnold (1989). Contact Hamiltonians are well
suited for dissipative systems and thus, due to the continuous energy extraction because of the comoving expansion, are ideally suited for our purpose. One of the central ideas of contact Hamiltonians is to extend the symplectic phase-space (3 + 3 dimensions) by an extra dimension. Historically, such extended phase-spaces incorporated the time as the additional dimension, however, within the contact formalism one chooses instead a non-trivial dynamical variable. Up to an additive constant, this variable is the time-dependent contact transformation

\[ S(x, D) = \int L(x, \dot{x}, D) \, dD = \int \left( \frac{\partial L}{\partial \dot{\mathbf{x}}} \cdot \mathbf{\dot{x}} - H \right) \, dD \]  

(D1)

of the system, where one can rewrite the first term using \( \dot{x} \, d\mathbf{D} = dx \).

Bravetti et al. (2017) derived the contact transformations, which are essentially the counterpart of canonical transformations in standard Hamiltonian mechanics; for an arbitrary transformation of coordinates \((x', p', S) \rightarrow (\tilde{x}', \tilde{p}', \tilde{S})\) they are (cf. equations 60–62 of Bravetti et al. 2017):

\[
\begin{align*}
\frac{\partial \tilde{S}}{\partial S} &- \tilde{p}_i \frac{\partial \tilde{x}^i}{\partial S} = \tilde{f}, \\
\frac{\partial \tilde{S}}{\partial x^i} - \tilde{p}_i \frac{\partial \tilde{x}^i}{\partial x^i} = -\tilde{f} p_i, \\
\frac{\partial \tilde{S}}{\partial p_i} - \tilde{p}_i \frac{\partial \tilde{x}^i}{\partial p_i} &= 0,
\end{align*}
\]  

(D2a–D2c)

while the contact transformation of the Hamiltonian \( H \rightarrow \tilde{H} \) is (cf. equations 82 of Bravetti et al. 2017)

\[
\frac{\partial \tilde{H}}{\partial S} - \tilde{p}_i \frac{\partial \tilde{x}^i}{\partial S} + \tilde{H} = \tilde{f} H,
\]  

(D2d)

where \( \tilde{f} \) is a (time-dependent) parameter that needs to be determined.

Now we are equipped to determine the contact Hamiltonian in \( \Lambda \)CDM. For this let us begin with the Hamiltonian in D-time (equation 67a) formulated in the extended phase-space, i.e. including the functional dependence of \( S \),

\[
\mathcal{H}(x, p, S, D) = \frac{p^2}{2m(aD)} + \frac{3D}{2a(aD)} \psi(x),
\]  

(D3)

and consider the time-dependent contact transformation

\[(x, p, S, D) \rightarrow (\tilde{x}, \tilde{p}, \tilde{S}, D).\]

(D4)

It is easily checked that the following transformation \( x = \tilde{x}, p = a^2(\tilde{a}D) \tilde{p}, \) and \( S = a^2(\tilde{a}D) \tilde{S} \) is in accordance with equations (D2) and thus, is indeed a contact transformation. This particular transformation is simple in the sense that the first three equations from (D2) just define \( \tilde{f} = [a^2(\tilde{a}D)]^{-1} \) and the contact-transformed Hamiltonian is given by \( \tilde{H} = \tilde{f} H - \tilde{a} \tilde{S} / \tilde{p} D \) following equation (D2d). The latter term can be simplified using the equation (2) for the linear growth rate which is equivalently \( \partial_i (a^2(\tilde{a}D) \tilde{p}) = 3D/2a; \) c.f. Brenier et al. (2003), where \( g = (D(\tilde{a}D)) \tilde{a}^{-3} \). Hence, the Hamiltonian (D3) turns into the contact transformed Hamiltonian

\[
\tilde{H}(\tilde{x}, \tilde{p}, \tilde{S}, D) = \frac{\tilde{p}^2}{2m} + \frac{3}{2D} [\tilde{S} + \psi(x)],
\]  

(D5)

which, supplemented with the replacement \( \tilde{p} \rightarrow v \) and \( m = 1 \), agrees with the one in the main text, where we remove all tildes to avoid unnecessary cluttering.

Finally, for reasons of completeness let us report the equations of motions for the contact Hamiltonian, governed not by the usual Hamilton equations, but instead by

\[
\begin{align*}
\frac{dx^i}{dt} &= \frac{\partial \tilde{H}}{\partial \tilde{p}_i}, \\
\frac{dp_i}{dt} &= -\frac{\partial \tilde{H}}{\partial x^i} - \tilde{p}_i \frac{\partial \tilde{S}}{\partial \tilde{p}_i}, \\
\frac{d\tilde{S}}{dt} &= p_i \frac{\partial \tilde{H}}{\partial p_i} - \tilde{H},
\end{align*}
\]  

(D6a–D6c)

(cf. equations 37–39 of Bravetti et al. 2017), which in the case of (D5), for \( p \rightarrow v \) and \( m = 1 \), respectively, lead to equations (67b) in the main text, which we repeat here for convenience,

\[
\begin{align*}
\frac{dx}{dt} &= v, \\
\frac{dv}{dt} &= -\frac{3}{2D} (v + \nabla \psi), \\
\frac{d\psi}{dt} &= \frac{v^2}{2} - \frac{3}{2D} (\psi + S).
\end{align*}
\]  

(D7a–D7c)

These equations, which remain regular at \( D \to 0 \) for slaved boundary conditions (8), may be solved perturbatively in a similar fashion as outlined in the main text, which then lead to a contact Hamiltonian-style PT – an avenue that so far has not been reported in the cosmological literature. Furthermore, from (D7c) one may determine the action perturbatively.

Such a contact-Hamiltonian perturbative expansion would be similar, yet fairly distinct to the approaches of Bartelmann (2015), Floerchinger et al. (2017), McDonald & Vlah (2018), Lilow et al. (2019), and Geiss et al. (2019), who perform expansions around (parts of) the Hamiltonian (or action), which is, however, incompatible with slaving boundary conditions that we employ in this paper. Indeed, we have shown that not the Hamiltonian but the contact transformed Hamiltonian allows for growing-mode solutions, in accordance with slaving.

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