Rigidity and stability estimates for minimal submanifolds in the hyperbolic space

A. C. Bezerra and F. Manfio

2010 Mathematics Subject Classification: 35P15, 53C24, 53C42.

Key words: Super stability operator, Eigenvalues, Minimal submanifolds.

Abstract

In this paper we establish conditions on the length of the second fundamental form of a complete minimal submanifold $M^n$ in the hyperbolic space $\mathbb{H}^{n+m}$ in order to show that $M^n$ is totally geodesic. We also obtain sharp upper bounds estimates for the first eigenvalue of the super stability operator in the case of $M$ is a surface in $\mathbb{H}^4$.

1 Introduction

In the seminal work [16], J. Simons established a formula for the Laplacian of the second fundamental form of a minimal submanifold in a space form and important applications have been obtained, among which we highlight one: if $M^n$ is a closed minimal submanifold in the unit sphere $S^{n+m}$, with squared norm of the second fundamental form less than $n/(2 - 1/m)$, then $M$ is totally geodesic. Simon’s work has been of great interest to differential geometers, and in the last decade several interesting gap theorems for submanifolds have been successfully obtained. We refer the reader to [1], [2], [3], [4], [5], [7], [9], [10], [11], [13], [14], and the references therein.

A natural problem is to ask whether a Simon’s type pinching theorem holds for minimal submanifolds in other ambient spaces. For example, in the case of hyperbolic space, Xia-Wang [17] showed that the result is true if the $L^2$-norm on geodesic balls of the length of the second fundamental form of the minimal submanifold has less than quadratic growth and if the dimension of the submanifolds is not less than 5. More precisely, the condition established in [17] is given by

$$\sup_{x \in M} |A^2(x)| < \begin{cases} \frac{(n+2)(n-1)^2}{4n} - n, & \text{if } m = 1, \\ \frac{2}{3} \left( \frac{(mn+2)(n-1)^2}{4mn} - n \right), & \text{if } m \geq 2. \end{cases}$$

$$\tag{1.1}$$

The case of low dimension, not considered in [17], was studied by Oliveira-Xia [14]. However, the condition obtained by Oliveira-Xia on the norm of the
second fundamental form depends on more constants compared to condition (1.1), and it is required that $n^2 - 6n + 1 + 8/m > 0$.

Our first main result gives an improvement of that obtained by Xia-Wang [17] and Oliveira-Xia [14]. The key point is to make a suitable change in the condition (1.1) for submanifolds $M^n$ in the hyperbolic space, when $n \geq 6$. In fact, condition (1.3) below does not depend on the codimension of the submanifold, but only on the dimension. The $L^2$-norm on geodesic balls of the length of the second fundamental form of $M$ was replaced by $L^d$-norm, where $d$ was chosen in an appropriate interval. Here, $A$ and $B_p(R)$ denote the second fundamental form of $M$ and the geodesic ball of radius $R$ centered at $p \in M$, respectively.

**Theorem 1.1.** Let $M^n$, $n \geq 6$, be a complete immersed minimal submanifold in the hyperbolic space $\mathbb{H}^{n+m}$. Suppose that there exists a constant $d \in 2(1 - \sqrt{2}/mn, 1 + \sqrt{2}/mn)$ such that
\[ \lim_{R \to +\infty} \frac{1}{R^2} \int_{B_p(R)} |A|^d = 0. \] (1.2)
If the length of the second fundamental form $A$ of $M$ satisfies
\[ \sup_{x \in M} |A|^2(x) < C(n) := \begin{cases} \frac{(n-1)^2}{4} - n, & \text{if } m = 1, \\ \frac{(n-1)^2}{6} - \frac{2}{3}n, & \text{if } m \geq 2, \end{cases} \] (1.3)
then $M$ is total geodesic.

An interesting related problem is the study of the stability operator on minimal submanifolds in the hyperbolic space $\mathbb{H}^m$. We briefly describe some basic facts.

Given a complete noncompact Riemannian manifold $M^n$, fix a continuous function $\beta : M \to \mathbb{R}$ and consider the Laplacian operator $\Delta$ acting on the space $C^\infty_0(M)$. We denote by $\lambda_1(L_\beta, M)$ the first eigenvalue of the operator $L_\beta = \Delta + \beta$, which is defined by
\[ \lambda_1(L_\beta, M) = \inf_{f \in C^\infty_0(M), f \neq 0} \frac{\int_M (|\nabla f|^2 - \beta f^2)}{\int_M f^2}. \] (1.4)
Note that, when $\beta = 0$, $\lambda_1(L_0, M)$ recover the usual first eigenvalue of $M$.

In this direction, one significant contribution is due McKean [12], who proved that if $M$ is simply connected and its sectional curvature satisfies $K_M \leq -1$, then
\[ \lambda_1(M) \geq \frac{(n-1)^2}{4} = \lambda_1(\mathbb{H}^n). \] (1.5)
In the context of submanifolds, Cheng and Leung [8] proved that if $M^n$ is a complete minimal submanifold of $\mathbb{H}^m$, then
\[ \lambda_1(M) \geq \frac{(n-1)^2}{4}. \] (1.6)
Motivated by the second variation formula for the volume of minimal submanifolds in the hyperbolic space, Seo [15] introduced the concept of super stability of such submanifolds. More precisely, a complete minimal submanifold $M^n$ of the hyperbolic space $\mathbb{H}^{n+m}$ is said to be super stable if

$$\int_M (|\nabla f|^2 - (|A|^2 - n)f^2) \geq 0,$$

for all $f \in C_0^\infty(M)$. We point out that for the case of hypersurfaces, the concept of super stability is the same as the usual definition of stability. Recall that the stability operator of a complete minimal hypersurface $M^n$ of $\mathbb{H}^{n+1}$ is $L|A|^2 - n$, where $A$ is the second fundamental form of $M^n$. Moreover, it follows from (1.4) and (1.6) that the first eigenvalue of the stability operator of a complete totally geodesic hypersurface of $\mathbb{H}^{n+1}$ is

$$\lambda_1(L|A|^2 - n, M) = \frac{(n-1)^2}{4} + n. \quad (1.7)$$

In this direction, the first author and Wang studied in [3] the stability operator for complete minimal submanifolds $M^n$ in $\mathbb{H}^{n+m}$. They showed that if the condition (1.2) is satisfied and the first eigenvalue of the super stability operator is greater than a certain constant, then a Simon’s type theorem holds if $2 \leq n \leq 5$, and for all $n \neq 3$ if $m = 1$. They also obtain upper estimates for the first eigenvalue this operator.

In our next result we present a version of [3, Theorem 1.2] for minimal submanifolds $M^n$ of the hyperbolic space $\mathbb{H}^{n+m}$, with a condition involving the norm of the second fundamental form of $M^n$ and the first eigenvalue of the super stability operator for dimensions $n \geq 6$. We will denote the first eigenvalue $\lambda_1(L|A|^2 - n, M)$ of the super stability operator of $M$ by $\overline{\lambda}_1$.

**Theorem 1.2.** Let $M^n, n \geq 6$, be a complete minimal submanifold of the hyperbolic space $\mathbb{H}^{n+m}$. If the condition (1.2) is satisfied and

$$\sup_{x \in M} |A|^2(x) < 2(\overline{\lambda}_1 - 2n), \quad (1.8)$$

then $M$ is total geodesic.

Finally, the next result is an improvement of [3, Theorem 1.1] by changing a pinching constant for an optimal one. More precisely, for the case of complete minimal surfaces in $\mathbb{H}^4$, if the condition (1.2) is satisfied for a certain constant $d$, we found an upper bound for $\overline{\lambda}_1$ by an optimal constant that depends only on $d$.

**Theorem 1.3.** Let $M^n, n \geq 2$ and $n \neq 3$, be a complete minimal submanifold of the hyperbolic space $\mathbb{H}^{n+m}$. Suppose that there exists a constant $d$ such that (1.2) is satisfied. Then we have the following situations:
(i) If \( m = 1 \),
\[
\lambda_1 > \frac{n^2 d^2}{4(n(d-1) + 2)} + n, \tag{1.9}
\]
and
\[
d \in \begin{cases} 
(0, 1/2), & \text{if } n = 2, \\
((n-1)/n, (n-1)(n-2)/n), & \text{if } n = 4 \text{ or } 5, \\
(2 - 2\sqrt{2}/n, 2 + 2\sqrt{2}/n), & \text{if } n \geq 6,
\end{cases} \tag{1.10}
\]
then \( M \) is total geodesic.

(ii) If \( n = m = 2 \) and \( d \in (2/3, 2) \), then
\[
\lambda_1 \leq n d^2 + 2.
\]

2 Preliminaries

In this section we establish an inequality that will be used throughout the paper. Given a minimal submanifold \( M^n \) in the hyperbolic space \( \mathbb{H}^{n+m} \), we have the following Kato-type inequality
\[
|A| \Delta |A| + \beta(m)|A|^4 + n|A|^2 \geq \frac{2}{nm} |\nabla |A||^2, \tag{2.1}
\]
with \( \beta(1) = 1 \) and \( \beta(m) = \frac{2}{m} \), if \( m \geq 2 \) (cf. [17]). For every constant \( \alpha > 0 \), and taking into account (2.1), we obtain the following inequality:
\[
|A|^\alpha \Delta |A|^\alpha \geq \left( 1 - \frac{mn - 2}{nm\alpha} \right) |\nabla |A||^\alpha - \beta(m)\alpha|A|^{2\alpha + 2} - n\alpha|A|^{2\alpha}. \tag{2.2}
\]
Let \( q \) be a nonnegative constant and let \( f \in C_0^\infty(M) \). Multiplying the inequality (2.2) for \( |A|^{2\alpha} f^2 \) and integrating over \( M \), we have
\[
\left( 1 - \frac{mn - 2}{nm\alpha} \right) \int_M |\nabla |A||^2 |A|^{2\alpha} f^2 \leq \beta(m)\alpha \int_M |A|^{2(q+1)\alpha + 2} f^2 + n\alpha \int_M |A|^{2(q+1)\alpha} \nabla |f||^2 + \int_M |A|^{2(q+1)\alpha} |A|^\alpha f^2. \tag{2.3}
\]
By integration by parts in the last term of (2.3), and using the Schwarz inequality and the Young inequality, we obtain
\[
\left( 2(q + 1) - \frac{mn - 2}{nm\alpha} - \epsilon \right) \int_M |\nabla |A||^\alpha |A|^{2\alpha} f^2 \leq \frac{1}{\epsilon} \int_M |A|^{2(q+1)\alpha} |\nabla f||^2 + \beta(m)\alpha \int_M |A|^{2(q+1)\alpha + 2} f^2 + n\alpha \int_M |A|^{2(q+1)\alpha} f^2. \tag{2.4}
\]
3 Proof of Theorems

In this section we will proof the main results of our paper.

Proof of Theorem 1.1. Setting \( d = 2(q+1) \) and taking \( \alpha = 1 \), the inequality \((2.4)\) becomes

\[
\left( d - \frac{mn - 2}{mn} - \epsilon \right) \int_M |\nabla|A||^2 |A|^{2q} f^2 \leq \beta(m) \int_M |A|^{d+2} f^2 \\
+ n \int_M |A|^{d} f^2 + \frac{1}{\epsilon} \int_M |A||\nabla f|^2. \tag{3.1}
\]

On the other hand, it follows directly from the definition of \( \lambda_1(M) \) that

\[
\int_M |\nabla f|^2 \geq \lambda_1 \int_M f^2, \quad \forall f \in C^\infty_0(M). \tag{3.2}
\]

Plugging \( f|A|^{(q+1)} \) in \((3.2)\) and using Young’s inequality, we obtain

\[
\lambda_1 \int_M |A|^d f^2 \leq \left( 1 + \frac{q + 1}{\epsilon} \right) \int_M |A|^d |\nabla f|^2 \\
+ (q + 1)(q + 1 + \epsilon) \int_M |\nabla|A||^2 |A|^{2q} f^2. \tag{3.3}
\]

Recall that, as \( M^n \) is a minimal submanifold of the hyperbolic space \( \mathbb{H}^{n+m} \), the first eigenvalue \( \lambda_1 \) satisfies \( \lambda_1 \geq \frac{(n-1)^2}{4} \) (cf. [8, Corollary 3]), and thus the inequality \((3.3)\) becomes

\[
\frac{(n-1)^2}{4} \int_M |A|^d f^2 \leq \left( 1 + \frac{q + 1}{\epsilon} \right) \int_M |A|^d |\nabla f|^2 \\
+ \left( d^2 + (q + 1)\epsilon \right) \int_M |\nabla|A||^2 |A|^{2q} f^2. \tag{3.4}
\]

Since \( d \in 2(1 - \sqrt{2/mn}, 1 + \sqrt{2/mn}) \), we have

\[
\frac{d^2}{4} - d + \frac{mn - 2}{mn} < 0, \tag{3.5}
\]

and therefore we can choose \( \epsilon > 0 \) in such a way that, by using \((3.1)\) and \((3.4)\), we obtain

\[
\frac{(n-1)^2}{4} \int_M |A|^d f^2 \leq \left( 1 + \frac{q + 1}{\epsilon} \right) \int_M |A|^d |\nabla f|^2 + \beta(m) \int_M |A|^{d+2} f^2 \\
+ n \int_M |A|^d f^2 + \frac{1}{\epsilon} \int_M |A||\nabla f|^2, \tag{3.6}
\]

(5)
that is,
\[
\int_M \left( \frac{(n-1)^2}{4} - n - \beta(m)|A|^2 \right) |A|^d f^2 \leq C \int_M |A|^d |\nabla f|^2, \tag{3.7}
\]

where \(C\) is a positive constant. Now let \(f\) be a smooth function defined on \([0, \infty)\) such that \(f \geq 0\), \(f \equiv 1\) in \([0, R]\), \(f \equiv 0\) in \([2R, +\infty)\), and with \(|f'| \leq \frac{2}{R}\). Consider the composition \(f \circ r\), where \(r\) is the distance function from the point \(p\). It follows from (3.7) that
\[
\int_{B_p(R)} \left( \frac{(n-1)^2}{4} - n - \beta(m)|A|^2 \right) |A|^d \leq 4C R^2 \int_{B_p(2R)} |A|^d.
\]

By letting \(R \to +\infty\) and using (1.2), we have
\[
\int_M \left( \frac{(n-1)^2}{4} - n - \beta(m)|A|^2 \right) |A|^d \leq 0.
\]

In view of the hypothesis on \(M\), given by (1.3), we have
\[
\frac{(n-1)^2}{4} - n - \beta(m)|A|^2 > 0,
\]
that implies that \(|A| = 0\), that is, \(M\) is totally geodesic. \(\Box\)

**Proof of Theorem 1.2.** It follows from the definition of \(\lambda_1\) that
\[
\int_M |\nabla f|^2 \geq \int_M |A|^2 f^2 - n \int_M f^2 + \lambda_1 \int_M f^2, \tag{3.8}
\]
for all \(f \in C^\infty_0(M)\). Plugging \(f|A|^{(q+1)}\) in the inequality (3.8) and using Young’s inequality, we obtain
\[
\int_M |A|^{2(q+1)+2} f^2 + (\lambda_1 - n) \int_M |A|^{2(q+1)} f^2 \leq \frac{\epsilon + q + 1}{\epsilon} \int_M |A|^{2(q+1)} |\nabla f|^2
\]
\[
+ (q + 1)(q + 1 + \epsilon) \int_M |\nabla|A||^2 |A|^{2q} f^2. \tag{3.9}
\]

Under the hypothesis (1.8), we have
\[
2n < \lambda_1 - \frac{1}{2} |A|^2,
\]
what replacing in inequality (3.1) gives us
\[
\left( d - \frac{mn-2}{mn} - \epsilon \right) \int_M |\nabla|A||^2 |A|^{2q} f^2 \leq \beta(m) \int_M |A|^{d+2} f^2
\]
\[
+ (\lambda_1 - n) \int_M |A|^{d} f^2 \leq \frac{1}{2} \int_M |A|^{d+2} f^2 + \frac{1}{\epsilon} \int_M |A|^{d} |\nabla f|^2. \tag{3.10}
\]
Regrouping the terms in (3.10), we can see that
\[
\left( d - \frac{mn - 2}{mn} - \epsilon \right) \int_M |\nabla|A|^{2}|A|^{2q}f^2 \leq \left( \beta(m) - \frac{1}{2} \right) \int_M |A|^{d+2}f^2 \\
+ (\lambda_1 - n) \int_M |A|^d f^2 + \frac{1}{\epsilon} \int_M |A|^d |\nabla f|^2.
\] (3.11)

Recall that \( \beta(1) = 1 \) and \( \beta(m) = \frac{3}{2} \) if \( m \geq 2 \). Thus, we can rewrite the inequality above as
\[
\left( d - \frac{mn - 2}{mn} - \epsilon \right) \int_M |\nabla|A|^{2}|A|^{2q}f^2 \leq \int_M |A|^d f^2 \\
+ (\lambda_1 - n) \int_M |A|^d f^2 + \frac{1}{\epsilon} \int_M |A|^d |\nabla f|^2.
\] (3.12)

Setting \( d = 2(q + 1) \) in the inequality (3.9) and relating (3.12), we get
\[
\left( d - \frac{mn - 2}{mn} - \epsilon \right) \int_M |\nabla|A|^{2}|A|^{2q}f^2 \leq \left( 1 + \frac{q + 2}{\epsilon} \right) \int_M |A|^d |\nabla f|^2 \\
+ \left( \frac{d^2}{4} + (q + 1)\epsilon \right) \int_M |\nabla|A|^{2}|A|^{2q+2}f^2,
\]
that is
\[
\left( -\frac{d^2}{4} + d - \frac{mn - 2}{mn} - (q + 2)\epsilon \right) \int_M |\nabla|A|^{2}|A|^{2q}f^2 \leq C \int_M |A|^d |\nabla f|^2.
\]

Since (3.5) also holds here, we can choose \( \epsilon > 0 \) such that
\[
-\frac{d^2}{4} + d - \frac{mn - 2}{mn} - (q + 2)\epsilon > 0.
\]

Therefore, for such \( \epsilon > 0 \), we have
\[
\int_M |\nabla|A|^{2}|A|^{2q}f^2 \leq C \int_M |A|^d |\nabla f|^2,
\] (3.13)

where \( C \) is a positive constant that depends on \( d, m, n, q \) and \( \epsilon \). Proceeding as in the proof of Theorem 1.1, we can choose a nonnegative smooth function \( f \) such that
\[
\int_{B_p(R)} |\nabla|A|^{2}|A|^{2q} \leq \frac{4C}{R^2} \int_{B_p(2R)} |A|^d.
\] (3.14)

Taking \( R \to +\infty \) and taking into account (1.2), we conclude that the second fundamental form \( A \) satisfies \( |A| = c = const. \) If \( c \neq 0 \), we know from (1.2) that
\[
\lim_{R\to+\infty} \frac{Vol[B_p(R)]}{R^2} = 0.
\] (3.15)

It then follows from (3) that \( \lambda_1(M) = 0 \) which contradicts with (1.4). Therefore \( |A| = 0 \), and this concludes the proof.
Proof of Theorem 1.3. By setting \( d := 2(q + 1)\alpha \), the inequality (2.4) becomes

\[
\frac{mn(d - 1 - \alpha \epsilon)}{mn\alpha} + 2 \int_M \nabla |A|^\alpha |A|^{2q} f^2 \leq \beta \alpha \int_M |A|^{d+2} f^2 \\
+ n\alpha \int_M |A|^d f^2 + \frac{1}{\epsilon} \int_M |\nabla f|^2 |A|^d,
\]

where \( \beta = \beta(m) \) is such that \( \beta(1) = 1 \) and \( \beta(m) = \frac{3}{2} \), if \( m \geq 2 \). Plugging \( f|A|^{(q+1)\alpha} \) in (3.8) with constants \( q \geq 0 \) and \( \alpha > 0 \), and using Young’s inequality, we obtain

\[
\int_M |A|^{d+2} f^2 \leq \left( 1 + \frac{q + 1}{\epsilon} \right) \int_M |A|^d |\nabla f|^2 \\
+ \left( \frac{d^2}{4\alpha^2} + (q + 1)\epsilon \right) \int_M |\nabla |A|^\alpha|A|^{2q}\alpha^2 f^2.
\]

Multiplying (3.16) by \( \frac{d^2 + 4(q + 1)\alpha^2}{4\alpha^2} \) and (3.17) by \( \frac{mn(d - 1 - \alpha \epsilon) + 2}{mn\alpha} \), and joining these new inequalities, we get

\[
\frac{mn(d - 1 - \alpha \epsilon)}{mn\alpha} + 2 \left[ \int_M |A|^{d+2} f^2 + (\lambda_1 - n) \int_M |A|^d f^2 \right] \\
\leq \frac{d^2 + 4(q + 1)\alpha^2}{4\alpha^2} \left[ \beta \int_M |A|^{d+2} f^2 + n \int_M |A|^d f^2 \right]
\]

where \( C \) is a positive constant that depends only on \( m, n, q, d, \epsilon \) and \( \alpha \).

Rearranging the terms in (3.18), we obtain

\[
\frac{mn(d - 1 - \alpha \epsilon)}{mn\alpha} + 2 \left[ (\lambda_1 - n) - \frac{nd^2}{4\alpha} - (\lambda_1 - n + n\alpha(q + 1))\epsilon \right] \int_M |A|^d f^2 \\
+ \left[ \frac{mn(d - 1) + 2}{mn\alpha} - \frac{\beta d^2}{4\alpha} - (\beta\alpha(q + 1) + 1)\epsilon \right] \int_M |A|^{d+2} f^2 \\
\leq C \int_M |A|^d |\nabla f|^2.
\]

We consider separately two possible cases:

Case (i) : \( m = 1 \). In this case, we have \( \beta(1) = 1 \), and the constant that appears in the first term on the left side of (3.19) becomes

\[
\frac{n(d - 1) + 2}{n\alpha} - \frac{d^2}{4\alpha} - (\alpha(q + 1) + 1)\epsilon.
\]

Moreover, by (1.10) we can see that \( d \in 2(1 - \sqrt{2/n}, 1 + \sqrt{2/n}) \), for all \( n \geq 2 \). Similarly, the constant that appears in the second term on the left
side of (3.19) becomes
\[
\left( \frac{n(d-1)+2}{n\alpha} \right) (\lambda_1 - n) - \frac{nd^2}{4\alpha} - (\lambda_1 - n + n\alpha(q+1))\epsilon. \tag{3.21}
\]
Because of (1.9), we have
\[
\bar{\lambda}_1 > \frac{n^2d^2}{4(n(d-1)+2)} + n.
\]
Thus, we can find \(\epsilon > 0\) such that (3.20) and (3.21) are both positive. On the other hand, from (3.19), we obtain
\[
\int_M |A|^{d+2}f^2 \leq C \int_M |A|^2|\nabla f|^2, \tag{3.22}
\]
where \(C\) is a positive constant that depends only on \(n, d, q, \alpha\) and \(\epsilon\). Using (1.2) and arguing as in the end of the proof of Theorem 1.1, we conclude that \(|A| = 0\) along \(M\), that is, \(M\) is total geodesic. Note, furthermore, that the hypothesis (1.9) implies that (1.7) is satisfied.

Case (ii) : \(m = n = 2\). In this case, we have \(\beta(2) = \frac{3}{2}\), and the constants that appears on the left side of (3.19) become
\[
\frac{4(d-1)+2}{4\alpha} - \frac{3d^2}{8\alpha} - \left( \frac{3}{2} \alpha(q+1) + 1 \right)\epsilon \tag{3.23}
\]
and
\[
\frac{4(d-1)+2}{4\alpha} (\lambda_1 - 2) - \frac{d^2}{2\alpha} - (\lambda_1 - 2 + 2\alpha(q+1))\epsilon. \tag{3.24}
\]
On the other hand, if
\[
\bar{\lambda}_1 > \frac{d^2}{2d-1} + 2, \tag{3.25}
\]
and since \(d \in (2/3, 2)\), we can find \(\epsilon > 0\) such that the constants (3.23) and (3.24) are both positive. Arguing as in the end of case (i), we conclude that \(|A| = 0\) along \(M\), that is, \(M\) is total geodesic. Now, since the first eigenvalue of the super stability operator of a totally geodesic submanifold in \(\mathbb{H}^{n+m}\) is given by (1.7), we conclude in this case that \(n = 2\), and thus \(\lambda_1 = \frac{9}{4}\).
Therefore, (3.25) makes sense if only if
\[
\frac{d^2}{2d-1} + 2 < \frac{9}{4} \iff 4d^2 - 2d + 1 < 0.
\]
Since the last inequality above has negative discriminant, we have
\[
\bar{\lambda}_1 = \lambda_1(L|A|^{2-2}, M) \leq \frac{d^2}{2d-1} + 2,
\]
and this concludes the proof. \(\square\)
Remark 3.1. The condition (1.2) can be replaced by
\[ \lim_{R \to +\infty} \frac{1}{R^k} \int_{B_p(R)} |A| = 0. \]  
(3.26)

In case (ii) of Theorem 1.3, we obtain
\[ \lambda_1 \leq \frac{(k-1)^2}{2k-3} + 2, \]  
(3.27)

with \( k \in (5/3, 3) \) and bearing in mind that \( d = 1 \). From this, we can replace \( k = d + 1 \) in (3.22) in order to get
\[ \int_M |A|^{k+1} f^2 \leq C \int_M |A|^{k-1} |\nabla f|^2, \]  
(3.28)

for all \( f \in C^\infty_0(M) \), where \( C \) is a positive constant. Changing \( f \) by \( f^{\frac{k}{2}} \) in (3.28), it follows from Hölder inequality that
\[ \int_M |A|^{k+1} f^k \leq C \left( \int_M |A|^{k-1} f^{k-2} |\nabla f|^2 \right)^{\frac{k}{k-2}} \left( \int_M |A||\nabla f|^k \right)^{\frac{2}{k}}. \]

Arguing as in the end of the proof of Theorem 1.1, we can choose a nonnegative smooth function \( f \) such that
\[ \left( \int_{B_p(R)} |A|^{k+1} \right)^{\frac{2}{k}} \leq C \left( \frac{1}{R^k} \int_{B_p(R)} |A| \right)^{\frac{2}{k}}, \]
and the proof follows as in the end of the proof of Theorem 1.3.

References

[1] H. Alencar, M. P. do Carmo, Hypersurfaces with constant mean curvature in spheres. Proc. Amer. Math. Soc. 120 (1994), no. 4, 1223–1229.

[2] P. Bérard, Remarques sur l’équation de J. Simons. Differential Geometry, 47–57, Pitman Monogr. Surveys Pure Appl. Math., 52, Longman Sci. Tech., Harlow, 1991.

[3] A. C. Bezerra, Q. Wang, Rigidity theorems for minimal submanifolds in a hyperbolic space. Ann. Acad. Sci. Fenn. Math. 42 (2017), no. 2, 905–920.

[4] M. P. Cavalcante, F. Manfio, On the fundamental tone of immersions and submersions. Proc. Amer. Math. Soc. 146 (2018), no. 7, 2963–2971.
[5] Q.-M. Cheng, S. Ishikawa, *A characterization of the Clifford torus*. Proc. Amer. Math. Soc. **127** (1999), no. 3, 819–828.

[6] S. Y. Cheng, S. T. Yau, *Differential equations on Riemannian manifolds and their geometric applications*. Comm. Pure Appl. Math. **28** (1975), no. 3, 333–354.

[7] S. S. Chern, M. P. do Carmo, S. Kobayashi, *Minimal submanifolds of a sphere with second fundamental form of constant length*. Functional Analysis and Related Fields, pp. 59–75, Springer, New York.

[8] L.-F. Cheung, P. F. Leung, *Eigenvalue estimates for submanifolds with bounded mean curvature in the hyperbolic space*. Math. Z. **236** (2001), no. 3, 525–530.

[9] S. C. de Almeida, F. Brito, *Minimal hypersurfaces of $S^4$ with constant Gauss-Kronecker curvature*. Math. Z. **195** (1987), no. 1, 99–107.

[10] G. de Oliveira Filho, *Compactifications of minimal submanifolds of hyperbolic space*. Comm. Anal. Geom. **1** (1993), no. 1, 1–29.

[11] M. F. Elbert, B. Nelli, H. Rosenberg, *Stable constant mean curvature hypersurfaces*. Proc. Amer. Math. Soc. **135** (2007), no. 10, 3359–3366.

[12] H. P. McKean, *An upper bound to the spectrum of $\Delta$ on a manifold of negative curvature*. J. Differential Geometry **4** (1970), 359–366.

[13] N. M. Neto, Q. Wang, *Some Bernstein-type rigidity theorems*. J. Math. Anal. Appl. **389** (2012), no. 1, 694–700.

[14] H. P. de Oliveira, C. Xia, *Rigidity of complete minimal submanifolds in a hyperbolic space*. Manuscripta Math. **158** (2019), no. 1-2, 21–30.

[15] K. Seo, *$L^2$ harmonic 1-forms on minimal submanifolds in hyperbolic space*. J. Math. Anal. Appl. **371** (2010), no. 2, 546–551.

[16] Simons, J., *Minimal varieties in Riemannian manifolds*, Ann. of Math. (2) 88 (1968), 62–105.

[17] C. Xia, Q. Wang, *Gap theorems for minimal submanifolds of a hyperbolic space*. J. Math. Anal. Appl. **436** (2016), no. 2, 983–989.

Instituto Federal Goiano, Campus Trindade, Brazil
E-mail address: adriano.bezerra@ifgoiano.edu.br

Universidade de São Paulo, São Carlos, Brazil
E-mail address: manfio@icmc.usp.br