Newton polytope of good symmetric polynomials

Duc-Khanh Nguyen, Nguyen Thi Ngoc Giao, Dang Tuan Hiep, Do Le Hai Thuy

Abstract

We introduce a general class of symmetric polynomials that have saturated Newton polytope and their Newton polytope has integer decomposition property. The class covers numerous previously studied symmetric polynomials.

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1 Introduction

In combinatorics, if a convex polytope equals the convex hull of its integer points, we say that it is a lattice polytope. Studying lattice polytopes is important because of their connections in many other domains. For instance, in mathematical optimization, if a system of linear inequalities defines a polytope, then we can use linear programming to solve integer programming problems for this system (see [Bar17]). In algebraic geometry, lattice polytopes are used to study projective toric varieties (see [CLS11, Ful16]). The Newton polytope is a lattice polytope associated with a polynomial: it is the convex hull of exponent vectors. The Newton polytope is a central object in tropical geometry (see [KKE21]), and they are used to characterizing Grobner bases (see [Stu96]).

Lattice polytopes are studied by Ehrhart polynomials (see [Eug62]). Important properties of Ehrhart polynomials such as unimodality and log-concavity are related to the integer decomposition property (IDP) of the lattice polytope (see [OH06, BR07, SVL13]). In [BGH+21], the authors studied the Newton polytope of inflated symmetric Grothendieck polynomials. The saturated property (SNP) of inflated symmetric Grothendieck polynomials in [BGH+21] generalizes the SNP of symmetric Grothendieck polynomials in [EY17]. The SNP of the inflated symmetric Grothendieck polynomials is an important point to derive the IDP of their Newton polytope.

In this paper, we introduce a general class of symmetric polynomials that has SNP with Newton polytope has IDP (see Theorem 4.2 and Corollary 4.3). Our class covers symmetric polynomials in [EY17, MTY19, BGH+21, MMS22]: symmetric Grothendieck polynomials, inflated symmetric Grothendieck polynomials, Stembridge’s symmetric polynomials associated with totally nonnegative matrices, cycle index polynomials, Reutenauer’s symmetric polynomials, Schur $P$-polynomials and Schur $Q$-polynomials, Stanley’s symmetric polynomials, chromatic symmetric polynomials of co-bipartite graphs, indifference graphs of Dyck paths, incomparability graphs of $(3+1)$-free posets. It also covers other symmetric polynomials, for instance, dual Grothendieck polynomials in [LP07].

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2 Newton polytope

A polytope $P$ in $\mathbb{R}^m$ is the convex hull $\text{Conv}(v_1, \ldots, v_k)$ of finite many points $v_1, \ldots, v_k \in \mathbb{R}^m$. The vertex set of $P$ is the minimal set $V$ in $\mathbb{R}^m$ such that $P = \text{Conv}(V)$. Algebraically, a point $v \in P$ is a vertex if, $v = tw + (1-t)u$ for some $w, u \in P$, $t \in (0,1)$ implies $w = u = v$. We say that $P$ is a lattice polytope if $V$ is a subset of $\mathbb{Z}^m$.

**Example 2.1.** The convex hull $P$ of twelve points in $\mathbb{R}^3$ below is a lattice polytope.

\[(3,1,0), (3,0,1), (1,0,3), (0,1,3), (0,3,1), (1,3,0),
(2,2,0), (2,0,2), (0,2,2),
(2,1,1), (1,1,2), (1,2,1).\]

The permutations of $(3,1,0)$ are vertices of the polytope $P$. In the picture below, $P$ is the blue hexagon.

Let $P$ be a lattice polytope. For a positive integer $t$, let $tP = \{tv \mid v \in P\}$. We say that $P$ has integer decomposition property (IDP) if, for any positive integer $t$ and $p \in tP \cap \mathbb{Z}^m$, there are $t$ points $v_1, \ldots, v_t \in P \cap \mathbb{Z}^m$ such that $p = v_1 + \cdots + v_t$.

**Example 2.2.** Let $P$ be the lattice polytope in Example 2.1. It is known that $P$ has IDP ([BGH+21, Proposition 11]). For instance, $3P$ is the convex hull of six points

\[(9,3,0), (9,0,3), (3,0,9), (0,3,9), (0,9,3), (3,9,0).\]

We see that $(9,2,1) \in 3P \cap \mathbb{Z}^3$ and is the sum of three points in $P \cap \mathbb{Z}^3$.

\[(9,2,1) = (3,1,0) + (3,1,0) + (3,0,1).\]

**Example 2.3.** Let $G$ be convex hull of four points

\[(0,0,0), (1,0,0), (0,0,1), (1,2,1).\]
The elements in \( \mathcal{G} \cap \mathbb{Z}^3 \) are
\[
(0,0,0), (1,0,0), (0,0,1), (1,2,1).
\]
We have \((1,1,1) \in 2\mathcal{G} \cap \mathbb{Z}^3 \), but it cannot be written as a sum of two points in \( \mathcal{G} \cap \mathbb{Z}^3 \). So \( \mathcal{G} \) does not have IDP.

Let \( f(x) = \sum_{\alpha \in \mathbb{Z}^m_{\geq 0}} c_\alpha x^\alpha \in \mathbb{C}[x_1, \ldots, x_m] \). The support of \( f \) is defined by
\[
\text{Supp}(f) = \{ \alpha \in \mathbb{Z}^m_{\geq 0} \mid c_\alpha \neq 0 \}.
\]
The Newton polytope of \( f \) is defined by
\[
\text{Newton}(f) = \text{Conv}(\text{Supp}(f)).
\]
We say that \( f \) has saturated Newton polytope (SNP) if \( \text{Newton}(f) \cap \mathbb{Z}^m = \text{Supp}(f) \).

**Example 2.4.** Let \( f(x_1, x_2, x_3) \) be the polynomial
\[
\begin{align*}
& x^{(3,1,0)} + x^{(3,0,1)} + x^{(1,0,3)} + x^{(0,1,3)} + x^{(0,3,1)} + x^{(1,3,0)} \\
& + x^{(2,2,0)} + x^{(2,0,2)} + x^{(0,2,2)} \\
& + 2x^{(2,1,1)} + 2x^{(1,1,2)} + 2x^{(1,2,1)}.
\end{align*}
\]
The set \( \text{Supp}(f) \) contains twelve points in Example 2.1. Then \( \text{Newton}(f) \) is the polytope \( \mathcal{P} \) in Example 2.1. Since \( \text{Newton}(f) \cap \mathbb{Z}^3 = \text{Supp}(f) \), \( f \) has SNP.

**3 Schur polynomials**

A partition with at most \( m \) parts is a sequence of weakly decreasing nonnegative integers \( \lambda = (\lambda_1, \ldots, \lambda_m) \). The size of partition \( \lambda \) is defined by \( |\lambda| = \sum_{i=1}^{m} \lambda_i \). Each partition \( \lambda \) is presented by a **Young diagram** \( Y(\lambda) \) that is a collection of boxes such that the leftmost boxes of each row are in a column, and the numbers of boxes from the top row to bottom row are \( \lambda_1, \lambda_2, \ldots, \) respectively. A **semistandard Young tableau** of shape \( \lambda \) with entries from \( \{1, \ldots, m\} \) is a filling of the Young diagram \( Y(\lambda) \) by the ordered alphabet \( \{1 < \cdots < m\} \) such that the entries in each column are strictly increasing from top to bottom, and the entries in each row are weakly increasing from left to right. A Young tableau \( T \) is said to have **content** \( \alpha = (\alpha_1, \alpha_2, \ldots) \) if \( \alpha_i \) is the number of entries \( i \) in the tableau \( T \). We write
\[
x^T = x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots.
\]
For each partition \( \lambda \) with at most \( m \) parts, the **Schur polynomial** \( s_\lambda(x_1, \ldots, x_m) \) is defined as the sum of \( x^T \), where \( T \) runs over the semistandard Young tableaux of shape \( \lambda \) with filling from \( \{1, \ldots, m\} \).

**Example 3.1.** Vector \((3,1,0)\) is a partition. The Young diagram of \((3,1,0)\) is

\[
\begin{array}{ccc}
& & \\
& & \\
& & \\
\end{array}
\]

The following filling is a semistandard tableau of shape \((3,1,0)\) and content \((1,2,1)\).

\[
\begin{array}{ccc}
1 & 2 & 3 \\
2 & & \\
& & \\
\end{array}
\]

Schur polynomial \( s_{(3,1,0)}(x_1, x_2, x_3) \) is the polynomial \( f \) in Example 2.4.
4 Good symmetric polynomials

Let $\alpha$ and $\beta$ be partitions with at most $m$ parts. We say $\beta$ is bigger than $\alpha$ and write $\beta \geq \alpha$ if and only if $\beta_i \geq \alpha_i$ for all $i$. If $\alpha$, $\beta$ are partitions of the same size, we say $\beta$ dominates $\alpha$ and write $\beta \trianglerighteq \alpha$ if $\sum_{i=1}^{j} \beta_i \geq \sum_{i=1}^{j} \alpha_i$ for all $j \geq 1$.

Example 4.1. $(3, 1, 0) < (3, 3, 3)$ and $(3, 2, 0) \trianglerighteq (3, 1, 1)$.

Let $F(x_1, \ldots, x_m)$ be a linear combination of Schur polynomials associated to partitions with at most $m$ parts. We can collect Schur polynomials appearing in $F$ associated with partitions of the same size to a bracket. We say that $F$ is good if it satisfies the following conditions:

(a) The support of each bracket equals the union of supports of its Schur elements.

(b) Suppose that there are $l + 1$ brackets in condition (a). In each bracket, there is a unique $\trianglerighteq$-maximum partition. These $\trianglerighteq$-maximum partitions have a form

$$\alpha = \lambda^0 < \cdots < \lambda^l = \beta,$$

where $\alpha \leq \beta$ are fixed partitions and for each $i > 0$, $\lambda^i$ is obtained from $\lambda^{i-1}$ by adding a box in the northmost row of $\lambda^{i-1}$ such that the addition gives a Young diagram, $\alpha < \lambda^i \leq \beta$.

Theorem 4.2. Let $F$ be a good linear combination of Schur polynomials. Then $F$ has SNP and Newton($F$) has IDP.

Corollary 4.3. Let $F$ be a linear combination of Schur polynomials such that the condition (a) is replaced by (a') or the condition (b) is replaced by (b') below:

(a') any two Schur polynomials in the same bracket of $F$ have the same sign,

(b') there exists partitions $\tilde{\lambda}, \hat{\lambda}$ so that $s_\mu$ appears in $F$ if and only if $\tilde{\lambda} \leq \mu \leq \hat{\lambda}$.

Then $F$ is a good polynomial. In particular, $F$ has SNP and Newton($F$) has IDP.

Proof. The condition (a'), (b') are particular cases of condition (a), (b), respectively. Moreover, the partitions $\alpha, \beta$ in (b') are $\tilde{\lambda}, \hat{\lambda}$, respectively. \qed

Example 4.4. Let $F(x_1, x_2, x_3)$ be

$$s_{(3,1,0)} - (3s_{(3,2,0)} + 6s_{(3,1,1)}) + (3s_{3,3,0} + 18s_{(3,2,1)}) - (18s_{(3,3,1)} + 4s_{(3,2,2)}) + 44s_{(3,3,2)} - 55s_{(3,3,3)}.$$

Schur polynomials in the same bracket have the same sign. The $\trianglerighteq$-maximum partitions $\lambda^i$ for $i = 0, \ldots, 5$ chosen from brackets have form

$$\alpha = (3, 1, 0) < (3, 2, 0) < (3, 3, 0) < (3, 3, 1) < (3, 3, 2) < (3, 3, 3) = \beta.$$

Hence, $F$ is a good symmetric polynomial. Newton($F$) is the convex hull of six different color polygons in the picture below. Each polygon is the Newton polytope of each bracket. In fact, $F$ is the inflated symmetric Grothendieck polynomial $G_{2,(3,1,0)}$ in $[BGH^+ 21]$. Hence, $F$ has SNP and Newton($F$) has IDP by $[BGH^+ 21$, Proposition 21, Theorem 27].
The following examples tell us that when Theorem 4.2 does not apply, we may not have a definite affirmation of SNP and IDP.

**Example 4.5.** When the condition (a) fails, for instance:

- Let $F(x_1, x_2, x_3)$ be $s_{(3,1,0)} - s_{(2,2,0)}$. Then $F$ does not have SNP because $(2, 2, 0) \notin \text{Supp}(F)$, but $\text{Newton}(F) = \text{Newton}(s_{(3,1,0)})$ still has IDP.

  When adding blocks to $\alpha$ in a wrong order in (b), for instance:

- Let choose $\alpha = (3, 1, 0) < (3, 1, 1) < (3, 2, 1) = \beta$ and let $F(x_1, x_2, x_3)$ be $s_{(3,1,0)} + s_{(3,1,1)} + s_{(3,2,1)}$. Then $F$ has SNP.

- Let choose $\alpha = (6, 4, 0) < (6, 4, 1) < (6, 4, 2) < (6, 4, 3) < (6, 5, 3) < (6, 6, 3) = \beta$ and let $F(x_1, x_2, x_3)$ be $s_{(6,4,0)} + s_{(6,4,1)} + s_{(6,4,2)} + s_{(6,4,3)} + s_{(6,5,3)} + s_{(6,6,3)}$. Since $(6, 5, 2) \in \text{Newton}(F) \cap \mathbb{Z}^3 \setminus \text{Supp}(f)$, then $F$ does not has SNP.

We are not sure if there exists a symmetric polynomial that has SNP, but its Newton polytope does not have IDP.

We need the following facts to prove Theorem 4.2.

**Proposition 4.6.** ([Rad52, Proposition 2.5]) Let $\alpha, \beta$ be partitions of the same size. Then, $\text{Newton}(s_\alpha) \subseteq \text{Newton}(s_\beta)$ if and only if $\alpha \subseteq \beta$.

**Lemma 4.7.** ([EY17, Theorem 0.1]) Let $\alpha$ be a partition with at most $m$ parts. Then $s_\alpha$ has SNP with Newton polytope being the convex hull of the $S_m$-orbit of $\alpha$.

**Proof of Theorem 4.2.** We first prove that $F$ has SNP. We use the trick from [EY17].

1. Let $F = \sum_{\mu} C_\mu s_\mu$ with $C_\mu \neq 0$. By condition (a) of $F$, we have

   \[ \text{Supp}(F) = \bigcup_{\mu} \text{Supp}(s_\mu). \tag{2} \]

Then

\[ \text{Newton}(F) = \text{Conv}(\bigcup_{\mu} \text{Supp}(s_\mu)). \tag{3} \]

Let $\alpha = \lambda^0 < \lambda^1 < \cdots < \lambda^t = \beta$ be the $\gtrsim$-maximum partitions in condition (b) of $F$. By Proposition 4.6, the right-hand side of (2) is

\[ \bigcup_{\mu} \text{Supp}(s_\mu) = \bigcup_{i=0}^{t} \text{Supp}(s_{\lambda^i}). \tag{4} \]
Therefore, by (2), (4),
\[ \text{Supp}(F) = \bigcup_{i=0}^{l} \text{Supp}(s_{\lambda_i}). \] (5)

By Proposition 4.6,
\[ \text{Conv}(\text{Supp}(s_{\mu})) = \text{Newton}(s_{\mu}) \subseteq \text{Newton}(s_{\lambda_i}) = \text{Conv}(\text{Supp}(s_{\lambda_i})) \]

for some \( i \). It implies that the right-hand side of (3) is
\[ \text{Conv}(\bigcup_{\mu} \text{Supp}(s_{\mu})) = \text{Conv}(\bigcup_{i=0}^{l} \text{Newton}(s_{\lambda_i})). \] (6)

Hence by (3), (6), we have
\[ \text{Newton}(F) = \text{Conv}(\bigcup_{i=0}^{l} \text{Newton}(s_{\lambda_i})). \] (7)

2. Let \( p \) be a point in \( \text{Newton}(F) \cap \mathbb{Z}^m \). By (7), \( p \) has form \( p = \sum_{i=0}^{l} c_i v^i \) for some \( v^i \in \text{Newton}(s_{\lambda_i}) \), and some \( c_i \in \mathbb{R}_{\geq 0} \), \( \sum_{i=1}^{l} c_i = 1 \). We see that \( v^i \) is not a partition in general. However, if we denote the sum of its coordinates by \( |v^i| \), then \( |v^i| = |\lambda^i| \).

Then \( |p| = \sum_{i=0}^{l} c_i |\lambda^i| \) is between \( |\lambda^0| \) and \( |\lambda^l| \), because of (1). Thus \( |p| = |\lambda^j| \) for some \( j \in [0, l] \), because \( \lambda^i \) is obtained from \( \lambda^{i-1} \) by adding a box. Let \( \overline{p} \) be \( \sum_{i=0}^{l} c_i \lambda^i \) and \( p^i \) be the rearrangement of the components of \( p \) into decreasing order. It was proven in [EY17] that \( p^i \leq (\overline{p})^i \) (Claim B) and \( (\overline{p})^i \leq \lambda^i \) (Claim C). So \( p^i \leq \lambda^i \).

By Lemma 4.7, Proposition 4.6, \( p \) is a point in
\[ \text{Newton}(s_{\mu}) \cap \mathbb{Z}^m \subseteq \text{Newton}(s_{\lambda_i}) \cap \mathbb{Z}^m = \text{Supp}(s_{\lambda_i}) \subseteq \text{Supp}(F). \] (8)

Therefore we conclude that \( F \) has SNP.

Now we show that \( \text{Newton}(F) \) has IDP. We use the trick from [BGH+21].

1. We have proven that \( F \) has SNP. Then by (5), Lemma 4.7, we have
\[ \text{Newton}(F) \cap \mathbb{Z}^m = \text{Supp}(F) = \bigcup_{i=0}^{l} \text{Supp}(s_{\lambda_i}) = \bigcup_{i=0}^{l} \text{Newton}(s_{\lambda_i}) \cap \mathbb{Z}^m. \] (9)

2. Suppose that \( \alpha = (\alpha_1, \ldots, \alpha_m) \) and \( \beta = (\beta_1, \ldots, \beta_m) \). For \( i = 1, \ldots, m - 1 \), set \( \lambda^{(i)} = (\beta_1, \ldots, \beta_i, \alpha_{i+1}, \ldots, \alpha_m) \). Set \( \lambda^{(0)} = \alpha, \lambda^{(m)} = \beta \). Then \( \alpha = \lambda^{(0)} < \cdots < \lambda^{(m)} = \beta \) is a subchain of (1). We have
\[ \text{Newton}(F) = \text{Conv}(\bigcup_{i=0}^{m} \text{Newton}(s_{\lambda^{(i)}})). \] (10)

Indeed, \( \text{Newton}(F) \) is the convex hull of its vertex set. We can get (10) from (7) by showing that a partition \( \lambda^j \) not of form \( \lambda^{(i)} \) is not a vertex of \( \text{Newton}(F) \). It is trivial because \( \lambda^j = \frac{1}{2}(\lambda^{j-1} + \lambda^{j+1}) \).
3. For a positive integer \( t \), we construct a chain of form (11)

\[
    t\alpha = \Lambda^0 < \cdots < \Lambda^L = t\beta. 
\]

Set \( F_t = \sum_{i=0}^{L} s_{\Lambda^i} \). Then \( F_t \) is a good linear combination of Schur polynomials and \( \Lambda^{(i)} = t\lambda^{(i)} \) for each \( i = 0, \ldots, m \). By (10), we have

\[
    \text{Newton}(F_t) = \text{Conv}(\bigcup_{i=0}^{m} \text{Newton}(s_{\Lambda^{(i)}})) = t\text{Conv}(\bigcup_{i=0}^{m} \text{Newton}(s_{\Lambda^{(i)}})) = t\text{Newton}(F). 
\]

4. Let \( p \) a point in \( t\text{Newton}(F) \cap \mathbb{Z}^m \). By (12), \( p \) is a point in \( \text{Newton}(F_t) \cap \mathbb{Z} \). Since \( F_t \) has SNP, by (9), it is a point in \( \text{Newton}(s_{\Lambda^t}) \cap \mathbb{Z} \) for some \( \Lambda^t \) in (11). Hence, \( p \) is the content of some semistandard tableau \( T \) of shape \( \Lambda^t \) with filling from \( \{1, \ldots, m\} \). For \( j = 1, \ldots, t \), let \( T_j \) be the semistandard tableau obtained by taking \( j' \)-th column of \( T \) for \( j' \equiv j \mod t \). Let \( \theta(j) \) be the shape of tableau \( T_j \). Let \( v_j \) be the content of tableau \( T_j \). Then \( p = v_1 + \cdots + v_t \). We also have \( \alpha \leq \theta(j) \leq \beta \). So there is a unique partition \( \lambda^k \) in chain (1) such that \( \theta(j) \subseteq \lambda^k \). Then by Proposition 4.6, \( v_j \) is a point in

\[
    \text{Newton}(s_{\theta(j)}) \cap \mathbb{Z}^m \subseteq \text{Newton}(s_{\lambda^k}) \cap \mathbb{Z}^m. 
\]

So by (9), \( v_j \) is a point of \( \text{Newton}(F) \cap \mathbb{Z}^m \). Therefore we conclude that \( \text{Newton}(F) \) has IDP.

\[\Box\]

**Example 4.8.** In Example 4.4, the subchain \( \lambda^{(i)} \) for \( i = 0, \ldots, 3 \) in the proof of Theorem 4.2 is

\[
    \alpha = (3,1,0) = (3,1,0) < (3,3,0) < (3,3,3) = \beta.
\]

In this case, \( \lambda^{(0)} = \lambda^{(1)} \). The vertex set of \( \text{Newton}(F) \) is the union of \( S_3 \)-orbits of partitions \( (3,1,0), (3,3,0), (3,3,3) \).
5 Applications

Theorem 4.2, Corollary 4.3 cover the following cases. Known results are:

- SNP and IDP of inflated symmetric Grothendieck polynomials \(G_{h,\lambda}\) (see [EY17, Theorem 0.1], [BGH+21, Proposition 21, Theorem 27]). Indeed, by definition

\[G_{h,\lambda} = \sum_{\mu} (-1)^{|\mu/\lambda|} b_{h,\lambda \mu} s_\mu,\]

where \(b_{h,\lambda \mu}\) is the number of fillings satisfying certain conditions. So, all Schur elements in the same bracket with \(s_\mu\) have the same sign \((-1)^{|\mu/\lambda|}\), and then the condition (a) is valid. By [BGH+21, Lemma 18 (c)], \(b_{h,\lambda \mu}\) is nonzero if and only if \(\lambda \leq \mu \leq \lambda(N)\). Hence, by Corollary 4.3, the condition (b) is valid with \(\alpha = \lambda\) and \(\beta = \lambda(N)\).

- SNP and IDP of the following symmetric polynomials in [MTY19]: Stembridge’s symmetric polynomials associated with totally nonnegative matrices (Theorem 2.28), cycle index polynomials (Theorem 2.30), Reutenauer’s symmetric polynomials (Theorem 2.32), Schur \(P\)-polynomials and Schur \(Q\)-polynomials (Proposition 3.5), Stanley’s symmetric polynomials (Theorem 5.8). They are particular cases of [MTY19, Propositions 2.5 (III)]. The proposition considers homogenous symmetric polynomials of degree \(d\)

\[f = \sum_{|\mu|=d} c_\mu s_\mu\]

with suppose that there exists \(\lambda\) so that \(c_\lambda \neq 0, c_\mu \neq 0\) only if \(\mu \preceq \lambda\), and \(c_\mu \geq 0\) for all \(\mu\). So, condition (a) is valid. The condition (b) is valid with \(\alpha = \beta = \lambda\). More precisely, the Schur expansion of those polynomials have nonnegative coefficients by [Ste91], [Sta99, page 396], [MTY19, page 12], [Ste89], [Sta84, Theorems 3.2, 4.1], respectively. Hence, condition (b) is valid with \(\alpha = \beta\) and they can be found in the proofs of corresponding theorems in [MTY19].

- SNP and IDP of the following symmetric polynomials in [MMS22]: chromatic symmetric polynomials of co-bipartite graphs (Proposition 3.1), indifference graphs of \((3+1)\)-free posets (Theorem 5.7). They are also particular cases of [MTY19, Proposition 2.5 (III)] above. More precisely, the Schur-expansion of those polynomials have nonnegative coefficients by [Sta95, Corollary 3.6], [SS93], [Gas96], respectively. Hence, condition (a) is valid. The condition (b) is valid with \(\alpha = \beta\) and they are \(\lambda(G), \lambda^{gr}(d), \lambda^{gr}(P)\), respectively.

Unknown results are:

- SNP and IDP of dual Grothendieck polynomials \(g_\lambda\) in [LP07]. Indeed, [LP07, Theorem 9.8] states that

\[g_\lambda = \sum f_\lambda^\mu s_\mu,\]

where \(f_\lambda^\mu\) is the number of semistandard tableaux of the skew shape \(\lambda/\mu\) with entries of the \(i\)-th row lie in \([1, i - 1]\). So, all nonzero coefficients \(f_\lambda^\mu\) have same sign, and then the condition (a) is valid. Moreover, \(f_\lambda^\mu\) is nonzero if and only if \((\lambda_1) \leq \mu \leq \lambda\). Hence, by Corollary 4.3, the condition (b) is valid with \(\alpha = (\lambda_1)\) and \(\beta = \lambda\).

**Remark 5.1.** Though Theorem 4.2 covers [BGH+21, Theorem 27], inside the proofs we do not need to choose \(F_t\) as a generalization of \(G_{th,1\lambda}\). The key point is to choose a set-up for

\[\]
$F_t$ so that it has SNP and $\text{Newton}(F_t) = t \text{Newton}(F)$ for any $t$. For this purpose, there are many choices for $F_t$, for instance $\sum_{i=0}^k s_{\Lambda^i}$, or $\sum_{i=0}^k (-1)^i s_{\Lambda^i}$, or $G_{th,t\lambda}$ when $F = G_{h,\lambda}$, etc. Our first choice $F_t = \sum_{i=0}^L s_{\Lambda^i}$ is the simplest.

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Department of Mathematics and Statistics, University at Albany, Albany, NY 12222, USA. E-mail: khanh.mathematic@gmail.com

Faculty of Advanced Science and Technology, University of Science and Technology - The University of Da Nang, 54 Nguyen Luong Bang, Da Nang, Vietnam. E-mail: ngocgiaol85@gmail.com

Department of Mathematics, Dalat University, 1 Phu Dong Thien Vuong, Ward 8, Dalat City, Lam Dong, Vietnam. E-mail: hiepdt@dlu.edu.vn

Institute of Mathematics, Vietnam Academy of Science and Technology, 18 Hoang Quoc Viet, Cau Giay, Hanoi, Vietnam. E-mail: cbl.dolehaithuy@gmail.com