Abstract. In this paper we prove that a capillary minimal surface outside the unit ball in \( \mathbb{R}^3 \) with one embedded end and finite total curvature must be either part of the plane or part of the catenoid. We also prove that a capillary minimal surface outside the unit ball with one end asymptotic to the end of the Enneper’s surface and finite total curvature cannot exist if the flux vector vanishes on the first homology class of the surface. Furthermore, we prove that a capillary minimal surface outside the convex domain bounded by several spheres with one embedded end and finite total curvature must be part of the plane.

1. Introduction

The plane and the catenoid are two important examples in minimal surface theory. The plane is the only totally umbilic minimal surface in \( \mathbb{R}^3 \) and the catenoid is the only rotational minimal surface in \( \mathbb{R}^3 \). Beside these properties, various characterizations of the plane and the catenoid have been studied in many years. For example, the plane is the only entire minimal graph in \( \mathbb{R}^3 \) [2]. The complete minimal surface of finite total curvature with two annular ends must be the catenoid [21]. In fact, the catenoid is the only embedded complete minimal annulus with finite total curvature [11]. It is also known that complete minimal annulus in \( \mathbb{R}^3 \) whose intersection with every \( z \)-planes are Jordan curves must be the catenoid [5]. There is also a variational characterization of the catenoid in view of stability of the surface [1]. In [1], it was proved that among all embedded minimal annuli in a slab maximally stable part of the catenoid in the same slab has the minimum area. Such the catenoidal waist is said to be maximally stable since any proper subset of the waist is stable and any subset of the catenoid containing such a waist is unstable [1]. Among different maximally stable parts of the catenoid in a slab, the surface is called a critical catenoid if a center of the slab becomes the point of symmetry of the surface.

Various results on characterizations of capillary minimal surfaces are also known. Capillary minimal surface \( \Sigma \) in a domain \( U \) is a minimal surface \( \Sigma \) which meets \( \partial U \) in a constant contact angle along \( \partial U \). When the contact angle is 90°, the surface is called the free boundary minimal surface. Nitsche [17] proved that the disk type capillary minimal surface inside a unit ball must be an equatorial disk. Ros and Souam [20] generalized the result to space forms \( \mathbb{H}^3, \mathbb{S}^3 \). Fraser and Schoen [8] showed that the free boundary minimal disk in a higher dimensional ball in space forms must be a planar one. The critical catenoid meets the boundary of the unit ball perpendicularly. Park and Pyo [19] showed that the immersed minimal annulus with two planar boundary curves along which the surface has constant contact angle must be a part of the catenoid. A famous conjecture that asks if the embedded free boundary minimal annulus inside the unit ball would necessarily be a critical catenoid still remains unanswered.
In this paper, we characterize the plane and the catenoid as capillary minimal surfaces with embedded ends. Instead of looking at the capillary minimal surface in the unit ball $B^3$, we consider the exterior problem that generalizes the Nitsche’s theorem. We study a capillary minimal surface in $\mathbb{R}^3 \setminus B^3$ with one embedded end and with boundary lying on the unit sphere $S^2$.

Let $D$ denote the open disk $\{ z \in \mathbb{C} | |z| < 1 \}$, $D'$ the punctured unit disk $\{ z \in \mathbb{C} | 0 < |z| < 1 \}$ and $D'$ the punctured closed unit disk $\{ z \in \mathbb{C} | 0 < |z| \leq 1 \}$. Then the minimal surface with finite total curvature and one embedded end can be conformally parameterized by $z \in D'$ with the puncture corresponding to the end of $\Sigma$ and $\{ z \in \mathbb{C} | |z| = 1 \}$ to the boundary $\partial \Sigma$. By the help of Osserman’s theorem [18], a complete minimal surface is conformally equivalent to a compact Riemann surface with finite number of punctures. If the metric of the surface diverges at a puncture, it becomes a complete end of the minimal surface. Schoen [21] showed that if a complete minimal surface with finite total curvature has an embedded end, the surface is regular at infinitely and that the end must be planar or catenoidal one. Many other geometric results on topological properties of the ends of minimal surfaces were studied by Jorge and Meeks [11].

We have the following theorem.

**Theorem 1.** Let $X \in (C^2(D', \mathbb{R}^3) \cap C^1(D', \mathbb{R}^3))$ be a capillary minimal surface in $\mathbb{R}^3 \setminus B^3$ with finite total curvature and one embedded end. Then it must be either part of the plane or part of the catenoid.

The Enneper's surface is a minimal immersion of $\mathbb{R}^2$ in $\mathbb{R}^3$ that has one immersed end. It is well known that the only complete orientable minimal surfaces with the total curvature $4\pi$ are the catenoid and Enneper's surface. So it can be said that that the Enneper's surface is the simplest complete minimal surfaces in $\mathbb{R}^3$ whose end is not embedded. The end of the Enneper's surface has multiplicity 3 and its height function has quadratic growth near the end. We will say the surface has an end asymptotic to the end of the Enneper's surface if its end has multiplicity 3 and the height function of the surface has quadratic growth at the end. There are a lot of examples of surfaces whose end is asymptotic to the end of the Enneper's surface because a slight change in the Weierstrass data would give us uncountable number of those surfaces. Chern-Gackstatter surface is an example with Enneper type end which is obtained by adding arbitrary number of handles to the Enneper's surface [3]. There are also higher order Enneper surfaces in $\mathbb{R}^3$ whose end have odd numbered multiplicity and quadratic growth. Now recall that the flux vector is defined as follows.

\[
\text{Flux}(\gamma) = \int_\gamma \nu(s)ds.
\]

Here, $\gamma$ is the closed curve on the surface and $\nu$ is the outward pointing unit normal vector field along $\gamma$. Since two homologous closed oriented curves on the surface have the same flux vector, we can see the flux vector as a flux homomorphism defined on the first homology class of the surface. Since the Enneper's surface is simply connected, the flux vector vanishes on the first homology class of the surface. But as there exists a complete minimal annulus in $\mathbb{R}^3$ with two ends asymptotic to the end of the Enneper's surface joined by the catenoidal neck ([12, p.40]), it is not necessarily true that the surface with the end asymptotic to the end of the Enneper's surface has a vanishing flux vector along the closed curve around the end. Consider a capillary minimal surface in $\mathbb{R}^3 \setminus B^3$ with finite total curvature and one end asymptotic to the end of the Enneper's surface. With the additional assumption that
the flux vector vanishes on the first homology class of the surface, we have the following nonexistence theorem.

**Theorem 2.** Let \( X \in (C^2(D', \mathbb{R}^3) \cap C^1(\overline{D'}, \mathbb{R}^3)) \) be a capillary minimal surface in \( \mathbb{R}^3 \setminus B^3 \) with finite total curvature and one end asymptotic to the end of the Enneper’s surface. We further assume that the flux vector vanishes on the first homology class of the surface. Then such a surface cannot exist.

We exploit Hopf’s methods ([4], [9]) to prove the above theorem. In [4], characterization of constant mean curvature capillary surfaces were studied by generalizing Hopf’s methods. It generalized Nitsches theorem to domains bounded by several number of spheres or planes. Motivated by this work, we generalize Theorem 1 to the domain exterior to convex domains bounded by several spheres. In this domain, the contact angle may be distinct along each component of the spheres. We have the following theorem.

**Theorem 3.** Let \( U \) be a convex domain in \( \mathbb{R}^3 \) which is bounded by \( k(k \geq 1) \) spheres. Let \( X \in C^{2,\alpha}(\overline{D'}, \mathbb{R}^3) \) be a capillary minimal surface in \( \mathbb{R}^3 \setminus U \) with finite total curvature and one embedded end. If \( k > 1 \), the surface with a catenoidal end cannot exist. Furthermore, the surface with a planar end must be part of the plane.

2. Preliminaries

In this section we present the Weierstrass representation formula for minimal surfaces in \( \mathbb{R}^3 \). We know that the Gauss map of a complete minimal surface of finite total curvature \( \Sigma \) can be considered as a meromorphic function \( g(z) : \Sigma \to \mathbb{C} \cup \infty \) where \( z \) is the conformal coordinate on the surface. Taking third coordinate function \( x_3 \) defined on the surface, we have the holomorphic differential \( dh = dx_3 + idx_3^* \). Here, we have to be aware that the harmonic conjugate \( x_3^* \) can be a multi-valued function since the conformal coordinate could be defined on a multiply connected domain. The pair \((g, dh)\) is called the Weierstrass data and the minimal surface \( X : \Sigma \to \mathbb{R}^3 \) can be represented by the real part of the holomorphic curve as follows.

\[
X(p) = \text{Re} \int_{p_0}^{p} \left( \frac{1}{2} \left( \frac{1}{g} - g \right), i \frac{1}{2} \left( \frac{1}{g} + g \right), 1 \right) dh.
\]

Conversely if the Weierstrass data \((g, dh)\) is given, the above representation give rise to a conformal minimal immersion if zeroes of \( dh \) coincide with the zeroes and poles of \( g \) with the same order, and for any closed curve \( \gamma \in \Sigma \),

\[
\int_{\gamma} \overline{g} dh = \int_{\gamma} \frac{dh}{g}, \quad \text{Re} \int_{\gamma} dh = 0
\]

holds.

With the Weierstrass data, various geometric invariants can also be represented by the data. In particular, the first and second fundamental forms of the surface are given by

\[
ds^2 = \left( \frac{1}{2} (|g| + |g|^{-1}) |dh| \right)^2 = \Lambda^2 |dz|^2, \tag{2.1}
\]

\[
b(v, v) = \text{Re} \left( \frac{dg}{g} (v) \cdot dh(v) \right) \tag{2.2}
\]
where \( v \) is a tangent vector to the surface \( \Sigma \). Also, the Gaussian curvature of the surface is given by

\[ K = -\left( \frac{4|dg|}{(|g| + |g^{-1}|^2|dh|)} \right)^2. \]

(2.3)

Exploiting the Weierstrass data, minimal surface theory flourishes in \( \mathbb{R}^3 \). For example, the Enneper’s surface and the higher order Enneper’s surface have the following Weierstrass data defined on \( \mathbb{C} \).

\[ g(z) = z^k \]
\[ dh = z^kdz. \]

We can deform the Gauss map to get the surface as follows.

\[ g(z) = z^k + P(z) \]
\[ dh = g(z)dz \]

where \( P(z) \) is a polynomial of degree \( n \leq k - 1 \). Then the surface wouldn’t have rotationally symmetric metric but still has the end asymptotic to the end of the Enneper’s surface. As in [12], we can also consider the following Weierstrass data defined on \( \mathbb{C} \setminus \{0\} \).

\[ g(z) = \frac{z^2 - R^2}{R^2 z^2 - 1} \]
\[ dh = \left(1 - \frac{1}{R^2 + R^{-2}(z^2 + z^{-2})}\right) \, \frac{dz}{z}. \]

The above Weierstrass data represents two Enneper’s surfaces joined by the catenoidal neck. As it is geometrically observed that the flux vector along the closed curve on the catenoidal neck is a nonvanishing vector Weierstrass data gives the same observation. Indeed, the above height differential in the Weierstrass data shows that the flux vector cannot be trivial since it has nonzero residue around the origin [10, p31].

Now we review some facts in [9] for completeness. A field of line elements is a vector field defined on a region that is a family of curves such that at each point the vector field is tangent to a curve through that point. We assume all fields of line elements in this paper is smooth. In case a field of line elements cannot be extended to a single point \( p \) then the point \( p \) is called a singularity of the field of line elements. An index of an isolated singularity \( p \) of the field of line elements is defined as \( j \) where \( 2\pi j \) is a rotation number around a singular point \( p \). We can see that \( j \) is of the form \( \frac{n}{2} \) where \( n \) is an integer. The index number does not depend on what curve we choose or what metric we use on the surface. Figure 1 shows that for arbitrary integer \( n \), we can consider a field of line elements with isolated singularity of index \( \frac{n}{2} \). We review two important theorems.

**Theorem** ([9]). Let \( S \) be a closed, orientable surface of genus \( g \) with a Riemannian metric defined on \( S \) and denote \( K \) as the Gaussian curvature defined on \( S \). Given a field of line elements \( F \) on \( S \), assume that it has finite number of singularities \( p_i \) for \( i \in I \) where \( I \) is a finite index set. And denote \( j_{p_i} \) as the index of \( p_i \). then

\[ \int \int_S K \, dA = 2\pi \sum_{i \in I} j_{p_i}. \]
Using the above Theorem, we have

**Poincaré-Hopf Theorem** ([9]). If $F$ is a field of line elements on a closed surface $S$ of genus $g$ with finite number of singularities. If we use same notations as Theorem 2.1, then

\[
\sum_{i \in I} j_{p_i} = 2 - 2g.
\]

(2.4)

### 3. Proof of Theorem 1

In this section, we prove the Theorem 1. As mentioned in the section 1, we parameterize the surface with a conformal parameter $z = u + iv$ in a punctured disk $D' = \{ z \in \mathbb{C} | 0 < |z| < 1 \}$ so that we get a conformal immersion $X : D' \to \Sigma$.

Since the surface $D' \to \Sigma$ has finite total curvature, we know the fact that the Gauss map of the surface can be meromorphically extended to the puncture [18]. Indeed, we see from [21] that the surface must have a catenoidal end or a planar end. After rotation, we can assume that the Gauss map points the south pole so the meromorphic function $g(z) : D' \to \mathbb{C}$ has the value $0$ on $z = 0$.

Following Nitsche’s method as in [17], we can consider the Hopf differential $\Phi(z)dz^2$ on the surface. It is well known that the complex function $\Phi(z)$ is holomorphic on the minimal surface [9]. Estimating the Gaussian curvature near the end of the surface, we have the following lemma.

**Lemma 1.** Let a minimal surface $\Sigma$ be as in the main theorem. Then the harmonic functions $\alpha, \beta$ satisfies

\[ \alpha = \beta = 0 \]

on the whole surface when the surface has the planar end. In case the surface has the catenoidal end,

\[ \alpha = A, \ \beta = 0 \]

for some nonzero constant $A$.

**Proof.** Since conformal parameter $z = u + iv$ is given on the surface, we can consider the function $f(z) = z^2\Phi(z) = \alpha + i\beta$ as in [17]. The Terquem-Joachimsthal theorem [22] shows...
that the boundary meeting $\partial B^3$ with constant contact angle implies that the $\partial \Sigma$ is the line of curvature. When we put $z = \rho e^{i \theta}$, we get by direct calculation that

$$
\frac{1}{\rho} N_\theta = \frac{\beta}{\rho^2 \Lambda} X_\rho + \frac{1}{\Lambda} \left( \frac{\alpha}{\rho^2} - \Lambda H \right) \frac{1}{\rho} X_\theta
$$

where $N$ is the Gauss map of the surface and $\Lambda$ is defined as in (2.1). Since the boundary of the surface becomes the line of curvature, we have $\beta = 0$ along the boundary of the surface.

We now describe the surface in terms of the Weierstrass data. To be more specific, the Weierstrass data of the surface is given by

$$
g(z) = z^n + c_{n-1} z^{n-1} + \cdots + c_1 z \quad (n \geq 1)
$$

$$
dh/g = \frac{d-2}{z^2} + \frac{d-1}{z} + d_0 + d_1 z + \cdots.
$$

The reason that the Gauss map can be represented as a holomorphic function of $z$ is that we assumed the Gauss map points the south pole at the puncture. Note that the surface has the planar end for $n > 1$, the catenoidal end when $n = 1$. Representation of $dh$ follows from the growth of the height function, $x_3$, which is given by

$$
x_3(z) = \text{Re} \int dh
$$

in the Weierstrass representation formula.

Let us denote $\begin{pmatrix} L & M \\ M & N \end{pmatrix}$ the second fundamental form of the surface as usual. Here,

$$
L = -\langle N_u, X_u \rangle, \quad M = -\langle N_v, X_u \rangle, \quad N = -\langle N_v, X_v \rangle
$$

where $N : \Sigma \to S^2$ is the Gauss map of the surface. In terms of the above notation, the holomorphic function $\Phi$ can be written as

$$
\Phi(z) = \frac{L - N}{2} + iM.
$$

Since $\begin{pmatrix} L & M \\ M & N \end{pmatrix}$ is similar to the matrix $\begin{pmatrix} \Lambda^2 \kappa_1 & 0 \\ 0 & \Lambda^2 \kappa_2 \end{pmatrix}$, it follows that

$$
|f(z)|^2 \leq |z^2 \Phi(z)|^2 \leq C |z|^4 \Lambda^4 (\kappa_1^2 + \kappa_2^2) \leq 2C |z|^4 \Lambda^4 |K|
$$

for some fixed constant $C$.

Now we restrict the surface to near the boundary or near the puncture in $D'$. By (2.1),(2.3),(3.5),(3.6) and (3.7) we have

$$
|f(z)|^2 \leq 2C |z|^4 \Lambda^4 |K| \leq C_1 |z|^4 |z|^{-8} |z|^{2n+2} = C_1 |z|^{2n-2}
$$

near $z = 0$.

When the surface has the planar end or, $n > 1$, (3.8) implies that $|f(z)|^2 \to 0$ as $z \to 0$. For the case $n = 1$, we see that $|f(z)|^2$ is bounded as $z \to 0$. It means that the harmonic function $\alpha$ and $\beta$ are bounded in a punctured disk so that $\alpha, \beta$ have removable singularity on the puncture.

As a result, for arbitrary $n \geq 1$, we get $\beta \equiv 0$. Furthermore, we see that $\alpha \equiv 0$ when the end of the surface is planar, and $\alpha \equiv A$ in case the surface has the catenoidal end for some nonzero constant $A$. This completes the proof of the lemma.

□
Remark 3.1. We note here that in case the surface has an end with multiplicity, \( f(z) \) could have a pole in the puncture. In fact, if the surface has the end asymptotic to the end of Enneper’s surface, \( f(z) \) has a pole of order 2.

Lemma 2. Let a minimal surface \( D' \to \Sigma \) be as in the Theorem 1 and assume that it has the catenoidal end. Then all concentric circles centered at the origin of \( D' \) become lines of curvature of the surface and there is no umbilic point on the surface.

Proof. If there were an umbilic point \( X(z_1) \), we would have \( \Phi(z_1) = 0 \) since the surface has mean curvature 0 and \( z \) is the conformal coordinate on the surface. Since we have \( z^2 \Phi(z_1) \equiv A \neq 0 \) as seen in the Lemma 1, it follows that the surface has no umbilic point.

Concentric circles being lines of curvature can be seen easily. Lemma 1 implies that (3.5) becomes

\[
\frac{1}{\rho} N_\theta = \left( \frac{\alpha}{\rho^3 A} \right) X_\theta.
\]

It means that the vector for all \( p \in D' \), \( N_\theta|_p \) is parallel to the vector \( X_\theta|_p \) which implies that every circle centered at the origin becomes the line of curvature. \( \square \)

Remark 3.2. Note that the Enneper’s surface \( X: \mathbb{C} \to \Sigma \) has the Weierstrass data \( (g,dh) = (z,zdz) \) defined on \( \mathbb{C} \). As (2.2) shows, the second fundamental form of the surface is given by \( \text{Re} (dz(v) \cdot dz(v)) \). Thus two perpendicular rays \( X(re^{i\theta}), X(re^{i\frac{\pi}{2}}) \) become lines of curvature and the harmonic function \( \beta \) is zero on those rays.

Remark 3.3. Regarding the Gauss map \( g(z) = \pi \circ N(X(z)) : D' \to \mathbb{C} \) of the surface, there is an important conclusion that follows from the above lemma. Since the surface in the Lemma 1 is foliated by lines of curvature and on each point \( X(z) \), \( \frac{\partial}{\partial \theta}|_z, \frac{\partial}{\partial r}|_z \) become eigenvectors. It means that since two vector fields form an orthonormal basis in each tangent space, \( dN: T_X(Z) \to T_N(X(z)) S^2 \neq 0 \) for all \( z \in D' \). It means that there is no such a point that makes the derivative of the Gauss map zero i.e. \( g'(z) \neq 0 \) for all \( z \in D' \).

Proof of Theorem 1. We start with the planar end case. By Lemma 1, we know that the harmonic functions \( \alpha \) and \( \beta \) both vanish throughout the whole surface. It implies that the surface is totally umbilic. By the condition that the surface being minimal and the boundary angle being constant, the surface must be a part of the plane through a great circle of the unit sphere.

Recall that the Gauss map of the surface can be meromorphically extended to the puncture [18]. And we can assume that the limit normal of the end points the south pole of the unit sphere. It means that we can assume the stereographically projected Gauss map \( g(w) \) is holomorphic. Since we have that the derivative of the Gauss map does not vanish at \( z = 0 \), i.e. \( g'(z) \neq 0 \) at \( z = 0 \), we can set a new conformal coordinate \( w \) so that we have \( g(z) = w, \forall z \in \partial D_{\epsilon} \) where \( D_{\epsilon} = \{ z : |z| \leq \epsilon \} \). We can now write down the Weierstrass data of the surface as follows.

\[
g(z) = z + b_2 z^2 + b_3 z^3 \ldots,
\]

\[
dx = \left( \frac{e_{-2}}{z^2} + \frac{e_{-1}}{z} + e_0 + e_1 z + \cdots \right) dz.
\]

We can also put

\[
g^{-1}(w) = w + a_2 w^2 + a_3 w^3 \ldots
\]
so that with new coordinate \( w \), we can express the surface with the new Weierstrass data as follows.

\[
g(w) = w,
\]

\[
\frac{dh}{g} = \left( \frac{d_{-2}}{w^2} + \frac{d_{-1}}{w} + d_0 + d_1w + \cdots \right) dz.
\]

Here we can see that the above coefficient \( d_{-2} \in \mathbb{R} \) in order for the surface not to have real period therefore we can assume that \( d_{-2} > 0 \). Also note that by Lemma 2 we have that for every \( \epsilon > 0 \), \( X(\partial D_\epsilon) \) on \( z \)-plane becomes a line of curvature of the surface \( \Sigma \). It follows that for every \( \epsilon > 0 \), \( g(\partial D_\epsilon) \) on \( w \)-plane also represents the line of curvature via the Weierstrass data given above. We now calculate tangent vectors on the lines of curvature with the new coordinate \( w \). From elementary calculation we have

\[
g_\theta = izg'(z).
\]

It means that the gauss map pushes forward a tangent vector \( \frac{\partial}{\partial \theta} \) on \( z \)-plane to a tangent vector \( \vec{v}_z \) of the line of curvature \( g(\partial D_\epsilon) \) which has following expression.

\[
\vec{v}_z = \text{Re} \left( \frac{g}{|g|} izg'(z) \right) \frac{\partial}{\partial \rho} + \text{Re} \left( \frac{-i\overline{g}}{|g|} izg'(z) \right) \frac{\partial}{\partial \phi},
\]

\[
(3.9)
\]

Here \( w = \rho e^{i\phi} \) is a polar coordinate in the \( w \)-plane. Since we have

\[
g'(g^{-1}(w)) = 1 + 2b_2w + \cdots,
\]

we get

\[
\frac{\overline{g}}{|g|} izg'(z) = \left| \frac{w}{|w|} \right| i(w + a_2w^2 + \cdots)(1 + 2b_2w + \cdots)
\]

\[
= |w|i(1 + a_2w + \cdots)(1 + 2b_2w + \cdots)
\]

\[
= |w|i(1 + c_1w + c_2w^2 + \cdots).
\]

For the later convenience, let us denote the above holomorphic functions as follows.

\[
K(w) = i(1 + c_1w + c_2w^2 + \cdots),
\]

\[
H(w) = d_{-2} + d_{-1}w + d_0w^2 + \cdots.
\]

It follows from (3.9) that

\[
(3.10)
\]

\[
\vec{v}_z = |w| \text{Re}(K(w)) \frac{\partial}{\partial \rho} + |w| \text{Im}(K(w)) \frac{\partial}{\partial \phi}.
\]

Recall that the second fundamental form is given by (2.2). So we have that for every point \( p = g(z) \) on \( g(\partial D_\epsilon) \),

\[
(3.11)
\]

\[
\text{Im} \left( \frac{d g}{g} \left( \vec{v}_z \right) \bigg|_p \cdot dh \left( \vec{v}_z \right) \bigg|_p \right) = 0.
\]
By direct calculation we have
\[
\frac{1}{g} \, dgdh = \frac{1}{w} (\frac{d-2}{w} + d_{-1}w + d_0w^2 + \cdots) dw^2 \\
= \frac{1}{w} (\frac{d-2}{w} + d_{-1}w + d_0w^2 + \cdots)(e^{i\theta} d\rho + i \epsilon e^{i\theta} d\phi)^2 \\
= \frac{1}{w} (\frac{d-2}{w} + d_{-1}w + d_0w^2 + \cdots)(\frac{w}{|w|} d\rho + i \epsilon d\phi)^2 \\
= H(w)((\frac{1}{|w|} d\rho + i \epsilon d\phi)^2 \\
= H(w)((\frac{1}{|w|} d\rho + 2i \epsilon d\rho - d\phi^2).
\]

The above calculation together with (3.10) give us
\[
\frac{dg}{g} \, (\tilde{v}_z) \bigg|_p \cdot dh(\tilde{v}_z) \bigg|_p = H(w)((\text{Re}(K(w)))^2 - |w|^2(\text{Im}(K(w)))^2 + 2|w|\text{Re}(K(w))\text{Im}(K(w))).
\]

Now we have by (3.11),
\[
(3.12) \quad \text{Im}(H(w))((\text{Re}(K(w)))^2 - |w|^2(\text{Im}(K(w)))^2) + 2\text{Re}(H(w))(|w|\text{Re}(K(w))\text{Im}(K(w))) = 0.
\]

Note that when $|z|$ is sufficiently small, $|w|$ is also sufficiently small since $g(z)$ is a conformal map near the origin. Therefore if $H(w)$ and $K(w)$ were not constant functions, we have $\text{Re}(H(w)) > 0$ and $\text{Im}(K(w)) > 0$ for all $w \in g(D_\epsilon)$ by taking $\epsilon$ small enough. Observing (3.13), we see that whenever $\text{Re}(K(w)) = 0$, (3.13) reduces to
\[
-|w|^2\text{Im}(K(w))^2\text{Im}(H(w)) = 0.
\]

Since $\text{Im}(K(w)) > 0$, we have $\text{Im}(H(w)) = 0$ if $\text{Re}(K(w)) = 0$. We easily see that the converse is also true. It means that when we go around the closed curve around the origin $g(0) = 0$ in $g(D_\epsilon)$, the holomorphic function $K(w)$ and $H(w)$ change sign simultaneously. In that sense, we can also see that
\[
(3.13) \quad \text{Im}(H(w)) > 0 \iff \text{Re}(K(w)) > 0
\]
holds since $((\text{Re}(K(w)))^2 - |w|^2(\text{Im}(K(w)))^2)$ can obviously have negative values. in fact, the term $((\text{Re}(K(w)))^2 - |w|^2(\text{Im}(K(w)))^2)$ must only have negative values by (3.12) and (3.13).

Note that two functions $\text{Im}(H(w))$ and $\text{Re}(K(w))$ are harmonic functions and their zero set coincide. It means that when $\text{Re}(K(w)) > 0$, $\text{Re}(K(w)) \neq \text{Im}(H(w))$ so that either $\text{Re}(K(w)) > \text{Im}(H(w))$ or $\text{Re}(K(w)) > \text{Im}(H(w))$ holds for all $w \in \{w|\text{Re}(K(w)) > 0\}$. Indeed, one can observe that the function $\text{Re}(K(w)) - \text{Im}(H(w))$ is again a real part of a holomorphic function defined on a disk $D$ and that the zero set of such a function consists of even numbered lines through the origin by which the region near the origin is divided into pie-like regions such that in any two adjacent regions the harmonic function $\text{Re}(K(w)) - \text{Im}(H(w))$ has opposite signs. Among the pie-like regions, we want to find a region $U$ that for $w \in U$, $\text{Re}(K(w)) > 0$ and $\text{Re}(K(w)) > \text{Im}(H(w))$ hold. If such a region does not exist, we can fix $w_0$ satisfying $\text{Re}(K(w_0)) > 0$ and choose some positive constant $\epsilon$ such that the function $\text{Re}(K(w)) - \epsilon(\text{Im}(H(w)))$ becomes $0$ at $w_0$. When we observe that the zero set of the function $\text{Re}(K(w)) - \epsilon(\text{Im}(H(w)))$ has additional even numbered lines where
the function vanishes so that one of two adjacent regions whose boundary line includes \(w_0\) satisfies \(\text{Re}(K(w)) > 0\) and \(\text{Re}(K(w)) > \epsilon(\text{Im}(H(w)))\) for all points \(w\) in the region. Then for a fixed positive real constant \(\epsilon\), we rearrange (3.12) to get an inequality for \(w\in U\) as follows.

\[
\text{Im}(H(w))((\text{Re}(K(w)))^2 - |w|^2(\text{Im}(K(w)))^2) + 2\text{Re}(H(w))(|w|\text{Re}(K(w))\text{Im}(K(w))) \\
\geq \frac{1}{\epsilon}\text{Re}(K(w))((\text{Re}(K(w)))^2 - |w|^2(\text{Im}(K(w)))^2) + 2\text{Re}(H(w))(|w|\text{Re}(K(w))\text{Im}(K(w))) \\
= \text{Re}(K(w))((\frac{1}{\epsilon}\text{Re}(K(w)))^2 - \frac{1}{\epsilon}|w|^2(\text{Im}(K(w)))^2 + 2|w|\text{Re}(H(w))\text{Im}(K(w))).
\]

But as \(\text{Im}(K(w))\) and \(\text{Re}(H(w))\) are bounded for all \(w \in \{w|\text{Re}(K(w)) > 0\}\), we can observe that the term

\[
(\frac{1}{\epsilon}\text{Re}(K(w)))^2 - \frac{1}{\epsilon}|w|^2(\text{Im}(K(w)))^2 + 2|w|\text{Re}(H(w))\text{Im}(K(w)))
\]

must be positive when \(|w|\) is sufficiently small. Indeed, we can compare the order of \(|w|\) in negative and positive valued terms in (3.16), the negative valued term, \(\frac{1}{\epsilon}|w|^2(\text{Im}(K(w)))^2\), has \(O(|w|^2)\) decay and the other two positive valued terms have \(O(|w|)\) decay as \(|w|\) becomes 0. We just deduced the fact that (3.14) can have positive values as \(|w|\) becomes 0 which yields a contradiction to the equation (3.12).

We conclude that

\[
\text{Re}(K(w)) = 0, \text{Im}(H(w)) = 0.
\]

By (3.16), we get the Weierstrass data for the surface as follows.

\[
g(w) = w, \\
\frac{dh}{g} = \frac{d-2}{w^2} dw.
\]

We now can finish the proof since (3.17) shows that the surface is indeed part of the catenoid near the end of the surface implying that the whole surface must be part of the catenoid.

\[\Box\]

**Remark 3.4.** In fact, the coefficient \(d_{-1}\) is actually zero since coordinate functions of the surface in \(\mathbb{R}^3\) are well defined functions.

**Remark 3.5.** Recently Park and Pyo obtained a similar result assuming the boundary and the end of the minimal surface to be in the same half space and imposing embeddedness condition to the surface [19].

**Remark 3.6.** The proof strongly depends on the facts that the surface is foliated by lines of curvature and that the parameterizing domain has isolated singularity at the puncture because we could reparameterize the surface near the puncture with the inverse of the Gauss map. We can see that the capillary condition on the boundary is very strong that it controls the first and second fundamental forms of the surface.

**Remark 3.7.** Instead of taking enbedded end, we can also consider a capillary minimal surface in \(\mathbb{R}^3 \setminus B^3\) with finite total curvature and one end asymptotic to the end of the Enneper’s surface as described in the introduction. In the following section we prove the nonexistence of the surface with the assumption that the surface has zero flux vector along
the closed curve around the end of the surface. It is yet unknown that such surface could exist in general cases. When the surface has a nonzero flux vector, it shows much more complicated behavior along the boundary of the surface.

**Remark 3.8.** The famous conjecture regarding this problem is whether the annular free boundary minimal surface inside the unit ball is necessarily a part of the catenoid. Difference between the above theorem and the conjecture is that the assumption with the embedded end. When we parameterize the surface with a complex coordinate around the end of the surface, the embedded end corresponds to the removable singularity of complex functions defined on the surface.

4. **Proof of Theorem 2**

In this section, we prove the Theorem 2.

**Proof of Theorem 2.** We begin with Hopf’s arguments ([4], [9]). Let $\kappa$ denote the principal curvature of the surface and $\vec{v}$ the corresponding principal vector. Then it satisfies

$$\left( \begin{array}{cc} \mathcal{L} & \mathcal{M} \\ \mathcal{M} & \mathcal{N} \end{array} \right) \vec{v} = \Lambda^2 \kappa \vec{v}.$$  

Given the complex coordinate $z = u + iv$ on the surface, the complex function

$$\Phi(z) = \frac{\mathcal{L} - \mathcal{N}}{2} + i\mathcal{M}$$

(4.18)

is holomorphic on the surface since the surface is minimal [9]. Then we get the differential equation of the lines of curvature as follows.

$$\mathcal{M}du^2 + (\mathcal{N} - \mathcal{L})dudv - \mathcal{M}dv^2 = 0.$$  

Here, $\left( \begin{array}{c} du \\ dv \end{array} \right)$ is set to be the infinitesimal tangent vector. Indeed, the above equation can be represented in the form

$$\text{Im}(\Phi dz^2) = 0.$$  

On an isolated umbilic point $p$, the rotation index of the lines of curvature is given by

$$I_{\text{rot}}(p) = -\frac{1}{4\pi} \delta(\arg \Phi).$$

Note that if $\Phi(z)$ has a zero (a pole, respectively) of order $m$ ($-m > 0$, respectively) at $p$,

$$I_{\text{rot}}(p) = -\frac{m}{2}.$$  

(4.19)

The surface having finite total curvature and the end asymptotic to the end of the Enneper’s surface allows us to write down the Weierstrass data of the surface as follows.

$$g(z) = z + b_2 z^2 + b_3 z^3 \cdots,$$

$$\frac{dh}{g} = \left( \frac{f_{-4}}{z^4} + \frac{f_{-3}}{z^3} + \cdots \right)dz =: F(z)dz.$$  

Note that the function $g(z)$ can be expressed as the holomorphic function since we can assume that the Gauss map points southpole at the puncture. And we also can see that the holomorphic function $g(z)$ has the zero of order 1 at the puncture as the surface has the end asymptotic to the end of the Enneper’s surface. Furthermore, the differential $dh$ is set for the surface to have the quadratic growth at the end. We know from (4.19) that if there is
an interior umbilic point, the rotation index must be negative. As for the boundary umbilic points, the rotation index must be negative as well. (see [4] for more details). In fact, the rotation indices of boundary umbilic points must either be $-\frac{1}{4}$ or $-\frac{1}{2}$ ([4, Lemma 2]). From (2.4), we must have sum of the rotation indices of the surface equal to 1. Note that (4.17) shows that the rotation index at the puncture becomes 2 since the function $\Phi(z)$ has the pole of order 4 at the puncture and $\Phi(z)$ does not have any other singularity on the closed disk.

\begin{center}
\includegraphics[width=0.3\textwidth]{figure2.png}
\end{center}

**Figure 2.** Behavior of the line of curvature near the boundary umbilic point $P$ with rotation index $-\frac{1}{4}$.

It can be first shown that there cannot be a boundary umbilic point of rotation index $-\frac{1}{4}$. Suppose there is a boundary umbilic point $P \in \partial D$ with rotation index $-\frac{1}{4}$. Since the line of curvature either becomes parallel or orthogonal to the boundary, the line of curvature at the umbilic point $P$ must be like as presented in the figure 2. We instantly find out that there are umbilic points in Part A, Part B and Part C respectively. Umbilic points in Part A and Part C have positive rotation indices. It is a contradiction since there cannot be any other umbilic point with positive rotation index other than the puncture otherwise implying that $\Phi(z)$ has a pole on the closed disk other than the puncture.

Now, assume that the boundary umbilic point has a rotation index $-\frac{1}{2}$. We denote $z^2 \Phi(z) = \alpha + i\beta$ the holomorphic function defined on the surface as in (4.18). We see that the $\Phi(z)$ acts like it has a double zero at the boundary umbilic point. In other words, if we extend $\Phi(z)$ holomorphically to $\Phi(z)$ across the boundary near the umbilic point, $\Phi(z)$ exactly has a double zero at the umbilic point ([4]). It follows that the points on the boundary near the umbilic point is mapped under $\Phi(z)$ to the real axis in a manner that the function $\alpha$ does not change near the umbilic point. We just showed that the harmonic function $\alpha$ does not change sign along the boundary.

Elementary calculation shows that the boundary curve $X(\partial D) \subset S^3$ has constant normal curvature 1 with respect to the unit normal vector field on the sphere. In fact, any smooth curve on the unit sphere shares the above property since we can develop the curve $C \subset S^3$ in a plane to get a circular arc $\tilde{C}$, and the normal curvature of the curve $C$ corresponds to the curvature of the planar curve $\tilde{C}$. Also, (3.5) implies that the geodesic curvature of the curve $X(\partial D) \subset S^3$ is given by $\frac{\alpha}{\rho}$. It means that the geodesic curvature of the boundary curve of the surface does not change sign. However it is well known that the smooth curve on the sphere with positive geodesic curvature only turns left with respect to the positively oriented tangent great circle at every point. We now conclude that the boundary curve of
the surface stays in a certain hemisphere, namely the hemisphere whose boundary curve is the great circle tangent to the \( X(\partial D) \) at a point on which \( \frac{\alpha}{\Lambda} \) attains its minimum. Finally, recall that we are assuming the surface has zero flux vector. However, \( X(\partial D) \) being in the hemisphere and the capillary surface implies that the flux vector has a nonzero component parallel to the vector pointing the north pole of the hemisphere. It’s a contradiction and we are done.

**Remark 4.1.** We can also generalize the above theorem to surfaces whose ends are asymptotic to higher order Enneper’s ends. Note that higher order Enneper’s surfaces can have odd numbered multiplicities at the end. Since the function \( \Phi(z) \) defined on the surface has a pole of degree \( m+3 \) when the surface has multiplicity \( 2m+1 \) at the end, similar theorem holds. Namely, a capillary minimal surface outside the unit ball with finite total curvature whose end is asymptotic to the end of the higher order Enneper’s surfaces cannot exist in case the flux vector vanishes on the first homology class of the surface.

\[ \square \]

5. Proof of Theorem 3

In this section, we prove the Theorem 3.

**Proof of Theorem 3.** We continue with the notations of the holomorphic function \( \Phi(z) \) and the rotation index \( I \) given in the previous section. Let us look at the order of the holomorphic function \( \Phi(z) \) at the puncture. As in the proof of the Lemma 1 in the previous section, we have

\[
(5.20) \quad |\Phi(z)| \sim |z|^{n-3}
\]

near the puncture.

Assume that the surface has a catenoidal end. Since \( n = 1 \) in this case, we have \( |\Phi(z)| \sim |z|^{-2} \) near the puncture by (5.20). Thus we have that

\[
(5.21) \quad I_{rot}(z = 0) = 1.
\]

As in the proof of Hopf’s theorem [9], at an isolated interior umbilic point \( q \),

\[
(5.22) \quad I_{rot}(q) \leq -\frac{1}{2}
\]

since \( \Phi(z) \) has a zero on \( z = q \).

Let \( t \in \partial \Sigma \) denote the boundary umbilic point or the vertex point of the boundary. By the estimation of the rotation index of the boundary umbilic points([4], Lemma 2), we have

\[
(5.23) \quad I_{rot}(t) \leq -\frac{1}{4}
\]

since domain \( S \) is convex.

From (2.4), we have that the sum of rotation indices of umbilic points in \( \overline{D} = \{ z \in \mathbb{C} | 0 \leq z \leq 1 \} \) is equal to the Euler characteristic \( \chi(\Sigma) = 1 \) if the set of umbilic points is finite. Since the surface cannot be totally umbilic, set of umbilic points which is a descrete set in a compact set is a finite set. (5.21),(5.22) together with (5.23) imply that if \( k > 1 \),

\[
\sum_p I_p \leq 1 - \frac{1}{4} < 1.
\]
This is a contradiction thus it follows that if $k > 1$, the surface with catenoidal end cannot exist.

Now assume that the surface has a planar end. Since $n > 1$, (5.20) implies that $|z\Phi(z)| \leq C$ for some fixed constant $C$. We see that

$$I_{\text{rot}}(z = 0) \leq \frac{1}{2}.$$ 

In the same way, the surface cannot exist unless it is totally umbilic. This completes the proof. \hfill \square

**Remark 5.1.** From the fact that the zeroes and poles of the nonzero holomorphic function $\Phi(z)$ give isolated umbilic points, we can geometrically see that the upper bound of the order of the pole of $f(z)$ as calculated in (3.8) actually becomes the exact order.

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