LEVEL SET MEAN CURVATURE FLOW
WITH NEUMANN BOUNDARY CONDITIONS

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Abstract. We investigate the relation between the level set approach and the varifold approach for the mean curvature flow with Neumann boundary conditions. With an appropriate initial data, we prove that the almost all level sets of the unique viscosity level set solution satisfy Brakke’s inequality and a generalized Neumann boundary condition.

Keywords: mean curvature flow, Neumann boundary conditions, level set method, varifolds.

1. Introduction

Let \( \Omega \) be a bounded domain with a smooth boundary \( \partial \Omega \) in \( \mathbb{R}^N \). We consider a family of hypersurfaces \( \{ \Gamma_t \}_{t \geq 0} \) in \( \Omega \) which evolves according to its mean curvature with Neumann boundary conditions, i.e.

\[
\begin{align*}
V &= -H \quad \text{on } \Gamma_t, \\
\Gamma_t \perp \partial \Omega,
\end{align*}
\]

where \( V \) is the normal velocity of \( \Gamma_t \) and \( H \) is the mean curvature of \( \Gamma_t \). In the case of \( \Omega = \mathbb{R}^N \), i.e. without boundary, the problem has long been studied and several different approaches have been proposed. One of these approaches is called the level set method (see [2, 4]), another is based on geometric measure theory called Brakke flow (see [1]). The consistency between these approaches is studied by numerous researchers. In particular, we mention [5], which showed a relation between the level set method and the Brakke flow when \( \Omega = \mathbb{R}^N \), namely, with an appropriate initial data, they showed that almost all level sets of the unique viscosity solution are Brakke flows. In the present paper, we investigate the similar relation when the problem is supplemented by the Neumann boundary condition.

Here, we mention some works related to the mean curvature flow with boundary conditions. Sato [14] proved the existence of a unique viscosity level set solution of (1.1) and Giga and Sato [8] established the comparison principle. In the classical setting, Stahl [15] proved a short-time existence of a unique smooth mean curvature flow with Neumann boundary conditions. For a long-time existence result, Mizuno and Tonegawa [13] introduced a notion of the Neumann boundary condition for Brakke flow and proved its existence via the Allen-Cahn equation (see also [10]). In [3], Edelen also studied the Brakke flow with Neumann boundary conditions and established its existence and a short-time regularity via the elliptic regularization approach.

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In this paper, we investigate a connection between the level set solution of (1.1) and Brakke flow. First, we consider the following PDE:

\[
\begin{align*}
\partial_t u &= |\nabla u| \div \left( \frac{\nabla u}{|\nabla u|} \right) \quad \text{in } \Omega \times (0, \infty), \\
\nabla u \cdot \nu_{\partial \Omega} &= 0 \quad \text{on } \partial \Omega \times (0, \infty), \\
u_{\partial \Omega} &= g \quad \text{on } \Omega \times \{t = 0\},
\end{align*}
\]

where \( \nu_{\partial \Omega} \) denotes the outer unit normal of \( \partial \Omega \), and \( g : \mathbb{R}^N \to \mathbb{R} \) satisfies

\[
\begin{align*}
\Gamma_0 &= \{x \in \Omega \mid g(x) = 0\}, \\
\nabla g \cdot \nu_{\partial \Omega} &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

Heuristically, this PDE asserts that each level set of \( u \) evolves by (1.1), if \( u \) is smooth and \( |\nabla u| \neq 0 \). Mostly following the similar argument in [5], we show that almost all level sets of \( u \) are Brakke flows if \( g \) is chosen appropriately. The main interest of the present paper lies in the characterization of the boundary condition that the Brakke flow satisfies. It turned out that the first variation of the Brakke flow on \( \partial \Omega \) is bounded and perpendicular to \( \partial \Omega \) almost everywhere, which was not guaranteed in [13]. Moreover, the class of admissible test functions for Brakke’s inequality is strictly larger than those in [3, 13] in that our test functions \( \phi \) do not need to satisfy the Neumann boundary conditions \( \nabla \phi \cdot \nu_{\partial \Omega} = 0 \) which was required in [3, 13]. One intuitive reason for the difference is that the conclusion here is for almost all level sets and they have null \((N - 1)\)-dimensional measure on \( \partial \Omega \), while the Brakke flows in [3, 13] may have non-trivial measure on \( \partial \Omega \).

Since almost all level sets are in addition unit-density, they are smooth almost everywhere by the general regularity theory in [1, 11, 16]. The regularity up to the boundary is not known so far on the other hand.

The organization of the paper is as follows. In Section 2, we state the basic notation and main results. In Section 3, we obtain a crucial \( L^1 \) estimate of the curvature which is a localized version of [5]. This estimate leads to various convergence results, allowing the derivations of the first variation formula and Brakke’s inequality for the level set solution. In Section 4, we interpret formulae obtained in Section 3 into varifolds settings. We show that the first variation is bounded and conclude that almost all level sets of \( u \) are Brakke flows with Neumann boundary conditions. In Appendix, a technical construction of the initial data \( g \) is carried out.

2. Preliminaries and main results

2.1. Notation and assumption. Let \( N \geq 2 \) be an integer. For a \( C^1 \) vector field \( X \) in \( \mathbb{R}^N \), we write \( \nabla X := (\partial_i X_j)_{i,j} \). The symbol \( \mu|_A \) denotes the restriction of a measure \( \mu \) to a set \( A \). Throughout the paper, we make the following assumption on \( \Omega \) and initial surface \( \Gamma_0 \).

**Assumption 2.1.** Assume that \( U, \Omega \subseteq \mathbb{R}^N \) with \( U \cap \Omega \neq \emptyset \) are bounded open sets with smooth boundaries and we set the initial surface \( \Gamma_0 := \partial U \cap \Omega \). Assume that \( \partial U \cap \partial \Omega \neq \emptyset \) and that \( \Gamma_0 \) meets \( \partial \Omega \) orthogonally.

We write \( \nu_{\partial \Omega} \) for the outer unit normal of \( \partial \Omega \).

2.2. Varifolds. A set \( G_k(\mathbb{R}^N) \) denotes the Grassmannian, i.e. \( G_k(\mathbb{R}^N) \) is a collection of \( k \)-dimensional linear subspaces of \( \mathbb{R}^N \). We often identify \( S \in G_k(\mathbb{R}^N) \) with the orthogonal projection from \( \mathbb{R}^N \) onto \( S \). A \( k \)-varifold in \( \overline{\Omega} \) is a Radon measure on \( \overline{\Omega} \times G_k(\mathbb{R}^N) \). By
the Riesz representation theorem, $k$-varifolds in $\Omega$ correspond bijectively to bounded linear functionals on the space of continuous functions $C(\Omega \times G_k(\mathbb{R}^N))$. For a $k$-varifold $V$ in $\Omega$, $\|V\|$ denotes its weight measure, i.e.

\begin{equation}
\|V\|(\phi) := \int_{\Omega \times G_k(\mathbb{R}^N)} \phi(x) \, dV(x, S) \quad \text{for } \phi \in C(\Omega).
\end{equation}

We write $\delta V$ for the first variation of $V$, i.e.

\begin{equation}
\delta V(X) := \int_{\Omega \times G_k(\mathbb{R}^N)} \text{div}_S(X) \, dV(x, S) \quad \text{for } X \in C^1(\Omega; \mathbb{R}^N),
\end{equation}

where $\text{div}_S(X) := \text{tr}(S \nabla X)$. We write $\|\delta V\|(\Omega)$ for the total variation of $\delta V$ in $\Omega$, i.e.

\begin{equation}
\|\delta V\|(\Omega) := \sup \{ \delta V(X) \mid X \in C^1(\Omega; \mathbb{R}^N), \|X\|_{\infty} \leq 1 \}.
\end{equation}

A $k$-rectifiable set $\Gamma \subseteq \Omega$ induces a $k$-varifold in $\Omega$ as in

\begin{equation}
V_{\Gamma}(\phi) := \int_{\Gamma} \phi(x, T_x \Gamma) \, d\mathcal{H}^k(x) \quad \text{for } \phi \in C(\Omega \times G_k(\mathbb{R}^N)).
\end{equation}

Here, note that a $k$-rectifiable set $\Gamma$ has a unique approximate tangent space $T_x \Gamma$ for $\mathcal{H}^k$-a.e. $x \in \Gamma$. A varifold induced from a rectifiable set as above is said to be unit-density. We work mainly with unit-density varifolds.

\section{2.3. Level set solution.} Consider the following singular second order parabolic PDE:

\begin{equation}
\begin{aligned}
\partial_t u &= |\nabla u| \text{div} \left( \frac{\nabla u}{|\nabla u|} \right) \quad \text{in } \Omega \times (0, \infty), \\
\nabla u \cdot \nu_{\partial \Omega} &= 0 \quad \text{on } \partial \Omega \times (0, \infty), \\
u &= g \quad \text{on } \Omega \times \{ t = 0 \}.
\end{aligned}
\end{equation}

We approximate (2.5) by the following nonsingular PDE for $\varepsilon \in (0, 1)$:

\begin{equation}
\begin{aligned}
\partial_t u^\varepsilon &= \sqrt{|\nabla u^\varepsilon|^2 + \varepsilon^2} \text{div} \left( \frac{\nabla u^\varepsilon}{\sqrt{|\nabla u^\varepsilon|^2 + \varepsilon^2}} \right) \quad \text{in } \Omega \times (0, \infty), \\
\nabla u^\varepsilon \cdot \nu_{\partial \Omega} &= 0 \quad \text{on } \partial \Omega \times (0, \infty), \\
u^\varepsilon &= g \quad \text{on } \Omega \times \{ t = 0 \}.
\end{aligned}
\end{equation}

We construct a suitable initial data $g$ with $\{ g = 0 \} = \Gamma_0$ which is smooth and which has a compatible Neumann condition but we defer the description of $g$ at this point. Here, we note that the equation (2.5) is uniformly parabolic, provided $\|\nabla u^\varepsilon\|_{\infty}$ is finite. Thus the existence of a unique bounded solution $u^\varepsilon \in C^\infty(\Omega \times [0, \infty))$ follow by an apriori estimate for $\|\nabla u^\varepsilon\|_{\infty}$ and the theory of parabolic equations (see for instance [9, 12]).

Furthermore, one may have

\begin{equation}
\sup_{x \in \Omega} \|u^\varepsilon\|, \|\nabla u^\varepsilon\|, |\partial_t u^\varepsilon| \leq C_T
\end{equation}

with a constant $C_T > 0$ independent of $\varepsilon$. Indeed, like in [4], the bounds for $u^\varepsilon$ and $\partial_t u^\varepsilon$ follow by the maximum principle with Neumann conditions. Set $W := |\nabla u^\varepsilon|^2$ and $\sigma_{ij}(p) := \delta_{ij} - \frac{p_ip_j}{|p|^2 + \varepsilon^2}$, and let $d : \mathbb{R}^N \rightarrow [-1, 1]$ be a smooth function such that $\nabla d \cdot \nu_{\partial \Omega} \geq 1$ on $\partial \Omega$. Then we can see

\begin{equation}
\partial_t W = \sigma_{ij}(\nabla u^\varepsilon)\partial_{ij} W + \partial_k \sigma_{ij}(\nabla u^\varepsilon)\partial_{ij} u^\varepsilon \partial_k W - 2\sigma_{ij}(\nabla u^\varepsilon)\partial_{ik} u^\varepsilon \partial_{jk} u^\varepsilon \leq \sigma_{ij}(\nabla u^\varepsilon)\partial_{ij} W,
\end{equation}
where we use the following computation and the positive-definiteness of $\sigma_{ij}$:

$$
\partial_k \sigma_{ij}(\nabla u^\varepsilon) \partial_j u^\varepsilon \partial_k W = \left( \frac{2\partial_i u \partial_j u \partial_k u}{|\nabla u^\varepsilon|^2 + \varepsilon^2} - \frac{\delta_{ik} \partial_j u + \delta_{jk} \partial_i u}{|\nabla u^\varepsilon|^2 + \varepsilon^2} \right) \partial_j u^\varepsilon \partial_k W
$$

$$
= \frac{|\nabla u \cdot \nabla W|^2}{(W + \varepsilon^2)^2} - \frac{|\nabla W|^2}{W + \varepsilon^2} \leq 0.
$$

On the other hand, we calculate $\nabla W \cdot \nu_{\partial \Omega} \leq C|\nabla \nu_{\partial \Omega}|_{\infty} W$ on $\partial \Omega$ for some constant $C$ (see [7, Proposition 2.1]). Therefore, for sufficiently large $\alpha, \beta > 0$, $v := e^{-\alpha t - \beta d} W$ is a subsolution to a linear 2nd order parabolic PDE with a non-positive 0th order coefficient. Hence we conclude by the maximum principle,

$$
|\nabla u^\varepsilon(x, t)|^2 \leq e^{\alpha t + 2\beta} \sup_{x \in \Omega} |\nabla g|^2
$$

for $x \in \Omega, 0 \leq t \leq T$.

By the general theory of viscosity solutions, there exists a unique viscosity solution $u$ to (2.5), and as $\varepsilon \to 0$, $u^\varepsilon$ converges to $u$ locally uniformly on $\overline{\Omega} \times [0, \infty)$. See [3, 8, 14] for the existence and uniqueness of the viscosity solution in this case. Due to the degeneracy of (2.5), $u$ is expected to be merely Lipschitz continuous in time and space in general.

**Definition 2.2.** For the unique viscosity solution $u$ of (2.5), we set

$$
\Gamma^\gamma_t := \{ x \in \Omega \mid u(x, t) = \gamma \}
$$

for $(\gamma, t) \in \mathbb{R} \times [0, \infty)$, and define (note that $u$ is differentiable a.e. due to Rademacher’s theorem)

$$
\nu := \begin{cases} 
\nabla u & \text{if } |\nabla u| > 0, \\
0 & \text{if } |\nabla u| = 0,
\end{cases}
$$

$$
H := \begin{cases} 
\frac{\partial u}{|\nabla u|} & \text{if } |\nabla u| > 0, \\
0 & \text{if } |\nabla u| = 0.
\end{cases}
$$

By [5, Lemma 6.1], $\Gamma^\gamma_t$ is $(N - 1)$-rectifiable for a.e. $(\gamma, t) \in \mathbb{R} \times (0, \infty)$ and $\Gamma^\gamma_t$ induces a unit-density $(N - 1)$-varifold $V^\gamma_t$ for such $(\gamma, t)$.

**2.4. Main results.** Now we describe the main results in the present paper. We need to have a suitable initial data with the following property.

**Lemma 2.3.** There exists a smooth and bounded function $g : \overline{\Omega} \to \mathbb{R}$ such that

(i) $\Gamma_0 = \{ x \in \overline{\Omega} \mid g(x) = 0 \}$,

(ii) $\nabla g \cdot \nu_{\partial \Omega} = 0$ on $\partial \Omega$,

(iii) $\sup_{0 < \varepsilon < 1} \int_{\Omega} \left| \operatorname{div} \left( \frac{\nabla g}{\sqrt{|\nabla g|^2 + \varepsilon^2}} \right) \right| \, dx < \infty$.

With this $g$, we define all the relevant quantities (the proof of Lemma 2.3 is somewhat technical and is deferred to Appendix). Then we have the following boundedness of the first variation of $V^\gamma_t$.

**Lemma 2.4.** For a.e. $(\gamma, t) \in \mathbb{R} \times (0, \infty)$, we have $\|\delta V^\gamma_t\|((\Omega)) < \infty$ and for a.e. $\gamma \in \mathbb{R}$ and any $T > 0$, we have

$$
\int_0^T \|\delta V^\gamma_t\|((\Omega)) \, dt < \infty.
$$

By Lemma 2.4, we may define the tangential component of the first variation $\delta V^\gamma_t$ on $\partial \Omega$ as in [13, Definition 2.4]:
**Definition 2.5.** For a.e. \((\gamma, t) \in \mathbb{R} \times (0, \infty)\) such that \(\|\delta V_t^\gamma\|_*(\overline{\Omega}) < \infty\), set
\[
(2.10) \quad \delta V_t^\gamma|_{\partial \Omega}(X) := \delta V_t^\gamma|_{\partial \Omega}(X - (X \cdot \nu_{\partial \Omega})\nu_{\partial \Omega}) \quad \text{for } X \in C(\partial \Omega; \mathbb{R}^N).
\]

With this definition, we have

**Theorem 2.6.** For a.e. \((\gamma, t) \in \mathbb{R} \times (0, \infty)\), we have
(i) \(\delta V_t^\gamma|_{\Omega} = H\nu\|V_t^\gamma\|_1\),
(ii) \(H \in L^2(\|V_t^\gamma\|_1)\),
(iii) \(\delta V_t^\gamma|_{\partial \Omega} = 0\).

The properties (i) and (ii) show that the first variation is absolutely continuous with respect to \(\|V_t^\gamma\|_1\) inside \(\Omega\) with \(L^2\) generalized mean curvature, and (iii) shows that the first variation on \(\partial \Omega\) is parallel to \(\nu_{\partial \Omega}\) as a measure. The latter may be described that the \(V_t^\gamma\) satisfies the Neumann boundary condition in a generalized sense. Finally, \(V_t^\gamma\) satisfies the following integral inequality called Brakke’s inequality.

**Theorem 2.7.** For a.e. \(\gamma \in \mathbb{R}\), we have for a.e. \(0 \leq t_1 < t_2 < \infty\) and for any non-negative \(\phi \in C^1(\overline{\Omega} \times [0, \infty))\) the inequality
\[
(2.11) \quad \int \phi(\cdot, t_2) d\|V_{t_2}^\gamma\| - \int \phi(\cdot, t_1) d\|V_{t_1}^\gamma\| \leq \int_{t_1}^{t_2} \int_{\Omega} \partial_t \phi - H(\nu \cdot \nabla \phi) - \phi H^2 d\|V_t^\gamma\| dt.
\]

As stated in Section 1, it is interesting to observe that we do not need to assume \(\nabla \phi \cdot \nu_{\partial \Omega} = 0\) on \(\partial \Omega\), which was additionally required in \([3, 13]\). We also note that for a.e. \(\gamma\), \(V_t^\gamma\) has the semi-decreasing property, namely, the mapping \(t \mapsto \|V_t^\gamma\|(\phi) - C_\phi t\) is non-increasing for any non-negative \(\phi \in C^2(\overline{\Omega})\). From this property, we may define \(\|V_t^\gamma\|\) for all \(t \in [0, \infty)\) so that \(t \mapsto \|V_t^\gamma\|\) is left-continuous, and one may prove Theorem 2.7 is still valid for all \(0 \leq t_1 < t_2 < \infty\).

3. \(L^1\) estimate on the mean curvature and geometric formulae

We first show a crucial \(L^1\) estimate for the approximate mean curvature \(H^\varepsilon\). This estimate plays an important role in proceeding convergence arguments based on compensated compactness. In the latter part of Section 3, we establish the level set version of the first variation formula and Brakke’s inequality.

**Definition 3.1.** We define
\[
(3.1) \quad \nu^\varepsilon := \frac{\nabla \nu^\varepsilon}{\sqrt{|\nabla \nu^\varepsilon|^2 + \varepsilon^2}}, \quad H^\varepsilon := \text{div}(\nu^\varepsilon), \quad A^\varepsilon := I - \nu^\varepsilon \otimes \nu^\varepsilon.
\]

**Theorem 3.2.** We have
\[
(3.2) \quad \sup_{0 < \varepsilon < 1} \sup_{t \geq 0} \int_{\Omega} |H^\varepsilon(x, t)| dx < \infty.
\]

Moreover, for each \(\varepsilon \in (0, 1)\), the mapping \(t \mapsto \int_{\Omega} |H^\varepsilon(x, t)| dx\) is non-increasing.

**Proof.** Let \(\eta \in C^\infty(\mathbb{R})\) be convex and \(|\eta'| \leq 1\). For any \(T > 0\) and \(t \in [0, T]\), we compute
\[
(3.3) \quad \frac{d}{dt} \int_{\Omega} \eta(H^\varepsilon) dx = \int_{\Omega} \eta'(H^\varepsilon) \partial_t H^\varepsilon dx = \int_{\Omega} \eta'(H^\varepsilon) \text{div}(\partial_t \nu^\varepsilon) dx
\]
From the boundary conditions of \( u^\varepsilon \), we see \( \partial_t \nu^\varepsilon \cdot \nu_{\partial \Omega} = \partial_t (\nu^\varepsilon \cdot \nu_{\partial \Omega}) = 0 \) on \( \partial \Omega \). On the other hand, since \( \eta \) is convex and \( A^\varepsilon \) is positive semi-definite,

\[
\eta''(H^\varepsilon) \nabla H^\varepsilon \cdot \partial_t \nu^\varepsilon = \eta''(H^\varepsilon) \nabla H^\varepsilon \cdot \left( A^\varepsilon \nabla H^\varepsilon + H^\varepsilon A^\varepsilon \frac{\nabla (\sqrt{\| \nabla u^\varepsilon \|^2 + \varepsilon^2})}{\sqrt{\| \nabla u^\varepsilon \|^2 + \varepsilon^2}} \right)
\]

(3.4)

\[
\geq \eta''(H^\varepsilon) H^\varepsilon \nabla H^\varepsilon \cdot \left( A^\varepsilon \frac{\nabla (\sqrt{\| \nabla u^\varepsilon \|^2 + \varepsilon^2})}{\sqrt{\| \nabla u^\varepsilon \|^2 + \varepsilon^2}} \right)
\]

\[
\geq -C_{\varepsilon,T} \eta''(H^\varepsilon) |H^\varepsilon| |\nabla H^\varepsilon|,
\]

where \( C_{\varepsilon,T} > 0 \) is a constant. Therefore,

\[
\frac{d}{dt} \int_\Omega \eta(H^\varepsilon) \, dx \leq C_{\varepsilon,T} \int_\Omega \eta''(H^\varepsilon) |H^\varepsilon| |\nabla H^\varepsilon| \, dx.
\]

Integrating in \( t \), we have

\[
\left[ \int_{t_1}^{t_2} \eta(H^\varepsilon) \, dx \right]_{t=t_1}^{t_2} \leq C_{\varepsilon,T} \int_{t_1}^{t_2} \int_\Omega \eta''(H^\varepsilon) |H^\varepsilon| |\nabla H^\varepsilon| \, dx \, dt
\]

(3.6)

for any \( 0 \leq t_1 < t_2 < T \). For each \( \delta > 0 \), choose \( \eta = \eta_\delta \) such that

\[
\begin{align*}
\eta_\delta(s) &\rightarrow |s| \text{ locally uniformly in } \mathbb{R} \text{ as } \delta \rightarrow 0, \\
\text{spt } \eta_\delta' &\subseteq [-\delta, \delta], \quad 0 \leq \eta_\delta'' \leq \frac{C}{\delta},
\end{align*}
\]

where \( C > 0 \) is a constant independent of \( \delta > 0 \). Then

\[
\left[ \int_{t_1}^{t_2} \eta_\delta(H^\varepsilon) \, dx \right]_{t=t_1}^{t_2} \leq C_{\varepsilon,T} \int_{t_1}^{t_2} \int_\Omega |H^\varepsilon| \leq \delta |\nabla H^\varepsilon| \, dx \, dt
\]

(3.7)

\[
\leq C_{\varepsilon,T} \int_{t_1}^{t_2} \int_\Omega |\nabla H^\varepsilon| \, dx \, dt.
\]

Letting \( \delta \rightarrow 0 \), we obtain

\[
\left[ \int_{t_1}^{t_2} |H^\varepsilon| \, dx \right]_{t=t_1}^{t_2} \leq 0.
\]

Since the above integral is bounded at \( t = 0 \) due to Lemma 2.3(iii), this concludes the proof. \( \square \)

The estimate (3.2) allows us to take limits of various geometric quantities as \( \varepsilon \rightarrow 0 \). As a consequence, one may obtain the following.

**Lemma 3.3.** We have the following convergence as \( \varepsilon \rightarrow 0 \) for each time \( t \geq 0 \):

(i) \( \sqrt{\| \nabla u^\varepsilon \|^2 + \varepsilon^2 \Delta} \| \nabla u \| \) in \( L^\infty(\Omega) \),

(ii) \( \nu^\varepsilon \rightarrow \nu = \frac{\nabla u}{|\nabla u|} \) strongly in \( L^2(\{ |\nabla u| > 0 \}) \).

**Proof.** The proof is identical to [5] Theorem 3.1-3.3]. The argument is based on the compensated compactness and the estimate (3.2). \( \square \)

We also prove an \( L^2 \) estimate for the approximate mean curvature.

**Lemma 3.4.** For any \( T > 0 \),

\[
\sup_{0 < \varepsilon < 1} \int_0^T \int_\Omega |H^\varepsilon|^2 \sqrt{\| \nabla u^\varepsilon \|^2 + \varepsilon^2} \, dx \, dt < \infty.
\]

(3.8)
Theorem 3.6. The proofs are similar to [5], hence we simply sketch their proofs. To prove the first variation formula, we next establish the first variation formula and Brakke’s inequality of the level set solution.

Proof. Recalling that \( \nu^\varepsilon \cdot \nu_{\partial \Omega} = 0 \) on \( \partial \Omega \), we compute

\[
\frac{d}{dt} \int_\Omega \sqrt{\nabla u^\varepsilon |^2 + \varepsilon^2} \, dx = \int_\Omega \nu^\varepsilon \cdot \nabla (\partial_t u^\varepsilon) \, dx = \int_{\partial \Omega} \partial_t u^\varepsilon (\nu^\varepsilon \cdot \nu_{\partial \Omega}) \, d\mathcal{H}^{N-1} - \int_\Omega H^\varepsilon \partial_t u^\varepsilon \, dx
\]

Integrating in \( t \), we have

\[
(3.9) \quad \int_\Omega \sqrt{\nabla u^\varepsilon (\cdot, T)} |^2 + \varepsilon^2 \, dx + \int_0^T \int_\Omega |H^\varepsilon|^2 \sqrt{\nabla u^\varepsilon |^2 + \varepsilon^2} \, dx \, dt = \int_\Omega \sqrt{\nabla g |^2 + \varepsilon^2} \, dx.
\]

Since the right-hand side may be bounded independent of \( \varepsilon \in (0, 1) \), we conclude the proof. \( \square \)

The proof of the following claim is identical to [5, Lemma 4.2].

Lemma 3.5. \( \partial_t u = 0 \) a.e. on \( \{ |\nabla u | = 0 \} \subset \Omega \times (0, \infty) \) and

\[
(3.10) \quad H^\varepsilon \sqrt{\nabla u^\varepsilon |^2 + \varepsilon^2} \to H |\nabla u | \quad \text{in} \quad L^\infty (\Omega \times (0, \infty)).
\]

Then we claim the \( L^2 \) estimate on the mean curvature \( H \). Theorem 3.6 may be derived from Lemma 3.4. See [5, Lemma 4.3] for the details.

Theorem 3.6. We have

\[
(3.11) \quad \int_0^\infty \int_\Omega H^2 |\nabla u | \, dx \leq \int_\Omega |\nabla g | \, dx < \infty.
\]

We next establish the first variation formula and Brakke’s inequality of the level set solution. The proofs are similar to [5], hence we simply sketch their proofs. To prove the first variation formula, we deduce the \( \varepsilon \)-version of the first variation formula for \( u^\varepsilon \).

Lemma 3.7. For any \( X \in \text{Lip}(\Omega; \mathbb{R}^N) \),

\[
(3.12) \quad \int_\Omega \text{tr} \left( (I - \nu^\varepsilon \otimes \nu^\varepsilon) \nabla X \right) \sqrt{\nabla u^\varepsilon |^2 + \varepsilon^2} \, dx = \int_\Omega H^\varepsilon (\nu^\varepsilon \cdot X) \sqrt{\nabla u^\varepsilon |^2 + \varepsilon^2} \, dx + \int_{\partial \Omega} (X \cdot \nu_{\partial \Omega}) \sqrt{\nabla u^\varepsilon |^2 + \varepsilon^2} \, d\mathcal{H}^{N-1}.
\]

Proof. Recalling that \( \nu^\varepsilon \cdot \nu_{\partial \Omega} = 0 \), we compute

\[
\int_\Omega H^\varepsilon (\nu^\varepsilon \cdot X) \sqrt{\nabla u^\varepsilon |^2 + \varepsilon^2} \, dx = \int_\Omega H^\varepsilon (\nabla u^\varepsilon \cdot X) \, dx
\]

\[
= \int_{\partial \Omega} (\nu^\varepsilon \cdot \nu_{\partial \Omega}) (\nabla u^\varepsilon \cdot X) \, d\mathcal{H}^{N-1} - \int_\Omega t_{\nu^\varepsilon} \nabla^2 u^\varepsilon X + t^\varepsilon \nabla X \nabla u^\varepsilon \, dx
\]

\[
= - \int_\Omega (X \cdot \nu_{\partial \Omega}) \sqrt{\nabla u^\varepsilon |^2 + \varepsilon^2} \, d\mathcal{H}^{N-1} + \int_\Omega (\nabla (X - t_{\nu^\varepsilon} \nabla X \nu^\varepsilon)) \sqrt{\nabla u^\varepsilon |^2 + \varepsilon^2} \, dx
\]

\[
= - \int_{\partial \Omega} (X \cdot \nu_{\partial \Omega}) \sqrt{\nabla u^\varepsilon |^2 + \varepsilon^2} \, d\mathcal{H}^{N-1} + \int_\Omega \text{tr} \left( (I - \nu^\varepsilon \otimes \nu^\varepsilon) \nabla X \right) \sqrt{\nabla u^\varepsilon |^2 + \varepsilon^2} \, dx.
\]

\( \square \)

We take the limit of the formula (3.12) to obtain the following, which may be regarded as the first variation formula of the level set solution \( u \).
**Theorem 3.8.** For a.e. $t \geq 0$, we have the following equality for each $X \in \text{Lip}(\Omega; \mathbb{R}^N)$ with $X \cdot \nu_{\partial \Omega} = 0$ on $\partial \Omega$:

\[
(3.13) \quad \int_{\Omega} \text{tr} \left( (I - \nu \otimes \nu) \nabla X \right) |\nabla u| \, dx = \int_{\Omega} H(\nu \cdot X) |\nabla u| \, dx.
\]

**Proof.** By Lemma 3.7, we have

\[
\int_{\Omega} \text{tr} \left( (I - \nu^\varepsilon \otimes \nu^\varepsilon) \nabla X \right) \sqrt{|\nabla u|^2 + \varepsilon^2} \, dx = \int_{\Omega} H^\varepsilon(\nu^\varepsilon \cdot X) \sqrt{|\nabla u|^2 + \varepsilon^2} \, dx.
\]

Hence for any $T > 0$ and any $\phi \in L^\infty(0, T)$,

\[
(3.14) \quad \int_0^T \phi \int_{\Omega} \text{tr} \left( (I - \nu^\varepsilon \otimes \nu^\varepsilon) \nabla X \right) \sqrt{|\nabla u|^2 + \varepsilon^2} \, dx \, dt = \int_0^T \phi \int_{\Omega} H^\varepsilon(\nu^\varepsilon \cdot X) \sqrt{|\nabla u|^2 + \varepsilon^2} \, dx \, dt.
\]

Now separate the integration into $\{|\nabla u| > 0\}$ and $\{|\nabla u| = 0\}$, and take the limit of each part as $\varepsilon \to 0$ using Lemma 3.3 and 3.5 to obtain

\[
(3.15) \quad \int_0^T \phi \int_{\Omega} \text{tr} \left( (I - \nu \otimes \nu) \nabla X \right) |\nabla u| \, dx \, dt = \int_0^T \phi \int_{\Omega} H(\nu \cdot X) |\nabla u| \, dx \, dt.
\]

See [5 Theorem 5.1] for the details. \hfill \Box

In a similar manner, one may prove the following Brakke’s inequality.

**Theorem 3.9.** For any $0 \leq t_1 < t_2 < \infty$ and any $\phi \in C^1(\Omega \times [0, \infty))$ with $\phi \geq 0$,

\[
(3.16) \quad \left[ \int_{\Omega} \phi |\nabla u| \, dx \right]_{t=t_1}^{t=t_2} \leq \int_{t_1}^{t_2} \int_{\Omega} \left( \partial_t \phi - H(\nu \cdot \nabla \phi) - \phi H^2 \right) |\nabla u| \, dx \, dt.
\]

**Proof.** Recalling that $\nu^\varepsilon \cdot \nu_{\partial \Omega} = 0$ on $\partial \Omega$, we compute

\[
\frac{d}{dt} \int_{\Omega} \phi \sqrt{|\nabla u^\varepsilon|^2 + \varepsilon^2} \, dx = \int_{\Omega} \partial_t \phi \sqrt{|\nabla u^\varepsilon|^2 + \varepsilon^2} + \phi (\nu^\varepsilon \cdot \nabla (\partial_t u^\varepsilon)) \, dx
\]

\[
= \int_{\Omega} \partial_t \phi \sqrt{|\nabla u^\varepsilon|^2 + \varepsilon^2} - \partial_t u^\varepsilon (\nu^\varepsilon \cdot \nabla \phi) - \phi \partial_t u^\varepsilon \cdot H^\varepsilon \, dx + \int_{\partial \Omega} \phi (\partial_t u^\varepsilon (\nu^\varepsilon \cdot \nu_{\partial \Omega})) \, d\mathcal{H}^{N-1}
\]

\[
= \int_{\Omega} \partial_t \phi \sqrt{|\nabla u^\varepsilon|^2 + \varepsilon^2} - H^\varepsilon (\nu^\varepsilon \cdot \nabla \phi) \sqrt{|\nabla u^\varepsilon|^2 + \varepsilon^2} - \phi |H^\varepsilon|^2 \sqrt{|\nabla u^\varepsilon|^2 + \varepsilon^2} \, dx.
\]

Integrating in $t$, we have

\[
(3.17) \quad \left[ \int_{\Omega} \phi \sqrt{|\nabla u^\varepsilon|^2 + \varepsilon^2} \, dx \right]_{t=t_1}^{t=t_2} = \int_{t_1}^{t_2} \int_{\Omega} \partial_t \phi \sqrt{|\nabla u^\varepsilon|^2 + \varepsilon^2} - H^\varepsilon (\nu^\varepsilon \cdot \nabla \phi) \sqrt{|\nabla u^\varepsilon|^2 + \varepsilon^2} - \phi |H^\varepsilon|^2 \sqrt{|\nabla u^\varepsilon|^2 + \varepsilon^2} \, dx \, dt.
\]

By the same argument as in [5 Theorem 5.2], one may obtain (3.16). \hfill \Box

4. **Interpretation to varifolds**

In Section 4, we interpret formulae in Section 3 in the language of varifolds. We shall prove that almost all level sets of $u$ are unit-density Brakke flows with appropriate boundary conditions. First of all, we collect basic properties of $\Gamma^\gamma = \{ x \in \Omega \mid u(x, t) = \gamma \}$.

**Lemma 4.1.** For a.e. $(\gamma, t) \in \mathbb{R} \times (0, \infty)$, we have the following properties.
The coarea formula and Theorem 3.6 show the following.

Take any $\phi \in C^\infty(\mathbb{R})$ with $\phi' > 0$. Then $\tilde{u} := \phi(u)$ is a unique viscosity solution to (2.5) with an initial data $\tilde{g} := \eta(g)$. From (4.2) for $\tilde{u}$ and $\tilde{g}$, we see

$$\int_\Omega |\nabla u| \, dx \leq \int_\Omega |\nabla g| \, dx.$$

Proof. The properties (i) and (iii) follow from the coarea formula (see [5, Lemma 6.1] for the detail) and (iv) follows from the differentiability of $\eta$ and the well-known characterization of rectifiability. We only verify (ii). From (3.9), we see

$$\delta V^\gamma_t(X) = \int_{\Gamma^\gamma_t} \text{tr}((I - \nu \otimes \nu) \nabla X) \, dH^{N-1}$$

for $X \in C^1(\overline{\Omega}; \mathbb{R}^N)$.

The coarea formula and Theorem 3.6 show the following.

Lemma 4.2. For a.e. $\gamma \in \mathbb{R}$, we have

$$\int_0^\infty dt \int_\Omega H^2 \, d\|V^\gamma_t\| < \infty.$$

Next we prove Lemma 2.4.
Proof of Lemma 2.4. Take any \( \phi \in L^\infty(0, T), \; \eta \in \text{Lip}(\mathbb{R}) \) with \( \phi \geq 0, \; \eta \geq 0. \) From Lemma 3.7, we have for any \( X \in \text{Lip}(\mho; \mathbb{R}^N), \)

\[
\lim_{\varepsilon \to 0} \int_0^T \phi(t) \int_{\partial \Omega} (X \cdot \nu_{\varepsilon \Omega}) \sqrt{|\nabla u|^2 + \varepsilon^2} d\mathcal{H}^{N-1} dt
\]

(4.7)

\[
= \int_0^T \phi(t) \int_{\Omega} \{ \text{tr} ((I - \nu \otimes \nu) \nabla X) - H(\nu \cdot X) \} |\nabla u| dx dt
\]

by the same argument as in Theorem 3.8. We replace \( X \) by \( \eta(t)X \) and apply the coarea formula to obtain

\[
\lim_{\varepsilon \to 0} \int_0^T \phi(t) \int_{\partial \Omega} \eta(t) (X \cdot \nu_{\varepsilon \Omega}) \sqrt{|\nabla u|^2 + \varepsilon^2} d\mathcal{H}^{N-1} dt
\]

(4.8)

\[
\leq C \int_0^T \int_{\mathbb{R}} \phi(t) \eta(t) \left\{ \mathcal{H}^{N-1}(\Gamma^t_\gamma) + \|H\|_{L^1(\Gamma^t_\gamma)} \right\} d\gamma dt,
\]

where \( C = C(\|X_0\|_{C^1}, N) > 0 \) is a constant. Combining (4.8) and (4.9), we have

\[
\int_0^T \int_{\mathbb{R}} \phi(t) \eta(t) \left\{ \mathcal{H}^{N-1}(\Gamma^t_\gamma) + \|H\|_{L^1(\Gamma^t_\gamma)} \right\} d\gamma dt
\]

(4.10)

\[
\leq (C + 1)\|X\|_{\infty} \int_0^T \int_{\mathbb{R}} \phi(t) \eta(t) \left\{ \mathcal{H}^{N-1}(\Gamma^t_\gamma) + \|H\|_{L^1(\Gamma^t_\gamma)} \right\} d\gamma dt.
\]

By the arbitrariness of \( \phi \) and \( \eta, \) we have for a.e \( (\gamma, t), \)

\[
\int_{\Gamma^t_\gamma} \text{tr} ((I - \nu \otimes \nu) \nabla X) d\mathcal{H}^{N-1} \leq (C + 1)\|X\|_{\infty} \left( \mathcal{H}^{N-1}(\Gamma^t_\gamma) + \|H\|_{L^1(\Gamma^t_\gamma)} \right).
\]

Therefore,

\[
\|\delta V_t^\gamma\|_{(\mho)} \leq (C + 1) \left( \mathcal{H}^{N-1}(\Gamma^t_\gamma) + \|H\|_{L^1(\Gamma^t_\gamma)} \right).
\]

By Lemma 4.1(ii) and Lemma 4.2, we conclude the proof. \( \square \)

By Lemma 2.4, we may define the first variation on \( \partial \Omega \) and its tangential component as in Definition 2.5. Now we prove Theorem 2.6.

Proof of Theorem 2.6. The properties (i) and (ii) follow from Lemma 4.2 (4.5) and another application of the coarea formula to Theorem 3.8. It remains to demonstrate (iii). Take any \( X \in C^1(\partial \Omega; \mathbb{R}^N), \) so that \( Y = X - (X \cdot \nu_{\partial \Omega})\nu_{\partial \Omega} \) on \( \partial \Omega. \) We apply the coarea formula to Theorem 3.8 to obtain \( \delta V_t^\gamma(Y) = \int_\Omega H\nu \cdot X d||V_t^\gamma|| \) for a.e. \( (\gamma, t). \) By (i) and Lemma 4.1(iii), we deduce \( \delta V_t^\gamma(Y) = \delta V_t^\gamma\big|_{\Omega}(Y). \) Therefore \( \delta V_t^\gamma\big|_{\partial \Omega}(X) = \delta V_t^\gamma\big|_{\partial \Omega}(Y) = 0. \) \( \square \)

Lastly, we show Theorem 2.7.
Proof of Theorem 2.7. Take any \( \eta \in C^\infty(\mathbb{R}) \) with \( \eta' > 0 \). Then \( \tilde{u} := \eta(u) \) is a unique viscosity solution to (2.5) with an initial data \( \tilde{g} := \eta(g) \). By Theorem 3.9 for \( \tilde{u} \), we obtain

\[
\int_{\Omega} \phi' (u) |\nabla u| \, dx \leq \int_{t_1}^{t_2} \int_{\Omega} \left( \partial_t \phi - H(\nu \cdot \nabla \phi) - \phi H^2 \right) \eta'(u) |\nabla u| \, dx \, dt.
\]

Using the coarea formula, we have

\[
\int_{\mathbb{R}} \eta' \, d\gamma \left[ \int_{\Gamma_t^s} \phi \, d\mathcal{H}^{N-1} \right]_{t_1}^{t_2} \leq \int_{\mathbb{R}} \eta' \, d\gamma \int_{t_1}^{t_2} \int_{\Gamma_t^s} \left( \partial_t \phi - H(\nu \cdot \nabla \phi) - \phi H^2 \right) \, d\mathcal{H}^{N-1} \, dt.
\]

Since \( \eta \) is arbitrary, we conclude the proof.

As stated in Section 2, for a.e. \( \gamma \), one may verify that \( V_t^\gamma \) has the semi-decreasing property, i.e. the mapping \( t \mapsto \|V_t^\gamma\|_\phi - C_\phi t \) is non-increasing for non-negative \( \phi \in C^2(\Omega) \) with \( C_\phi := \mathcal{H}^{N-1}(\Gamma_t^\gamma) \sup_{\phi > 0} \frac{\|\nabla \phi\|}{\phi} \). Thus we may define \( \|V_t^\gamma\|_\phi \) left-continuously for all \( t \in [0, \infty) \). Moreover, by Lemma 4.1(ii), \( \|V_t^\gamma\|_\phi \leq \|\phi\| \mathcal{H}^{N-1}(\Gamma_0^\gamma) \). Therefore, we may define a Radon measure \( \|V_t^\gamma\| \) for all \( t \in [0, \infty) \). Since \( t \mapsto \|V_t^\gamma\| \) is left-continuous, Theorem 2.7 is still valid for any \( 0 \leq t_1 < t_2 < \infty \).

APPENDIX: PROOF OF LEMMA 2.3

In this appendix, we carry out a technical construction of the appropriate initial data \( g \) as in Lemma 2.3. For the simplicity of notation, we will define the initial data \( g \) on the whole space \( \mathbb{R}^N \).

Proof of Lemma 2.3. Step1. For each \( A \subseteq \mathbb{R}^N \), we define the signed distance by \( d_A(x) := \text{dist}(x, A) - \text{dist}(x, A^c) \) for \( x \in \mathbb{R}^N \). Since \( U, \Omega \) are smooth, there exists \( R > 0 \) such that \( d_U \) is smooth in \( \{-2R < d_U < 2R\} \) and that \( d_\Omega \) is smooth in \( \{-2R < d_\Omega < 2R\} \). Choose \( \eta \in C^\infty_c(\mathbb{R}) \) to satisfy

\[
\eta(s) = 1 \text{ if } |s| < R, \quad \eta(s) = 0 \text{ if } |s| \geq \frac{3}{2}R, \quad 0 \leq \eta \leq 1.
\]

Define \( X \in C^\infty(\mathbb{R}^N; \mathbb{R}^N) \) by

\[
X := \eta(d_U) \eta(d_\Omega) \{(I - \nabla d_\Omega \otimes \nabla d_\Omega) \nabla d_U + d_\Omega X_0\},
\]

where \( X_0 \) is a vector field to be defined later. Here, remark that \( X|_{\partial \Omega} \) is a tangent vector field of \( \partial \Omega \). Let \( \Phi : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}^N \) be a 1-parameter group generated by \( X \), i.e.

\[
\begin{cases}
\frac{d}{ds} \Phi(x, s) = X(\Phi(x, s)), \\
\Phi(x, 0) = x.
\end{cases}
\]

By Assumption 2.1, we compute

\[
\frac{d}{ds} d_U(\Phi(x, s)) \bigg|_{s=0} = 1 - (\nabla d_U \cdot \nabla d_\Omega)^2 = 1 \neq 0 \text{ on } \partial U \cap \partial \Omega.
\]

By the implicit function theorem, there exist a neighborhood \( W \) of \( \partial U \cap \partial \Omega \), a positive number \( s_0 > 0 \) and a smooth function \( f : W \to \mathbb{R} \) such that

\[
\{(x, s) \in W \times (-s_0, s_0) \mid f(x) = s\} = \{(x, s) \in W \times (-s_0, s_0) \mid d_U(\Phi(x, -s)) = 0\}.
\]

By differentiating \( d_U(\Phi(x, -f(x))) = 0 \), we compute

\[
\nabla f(x) = \frac{\nabla \Phi(x, -f(x)) \nabla d_U(\Phi(x, -f(x)))}{X(\Phi(x, -f(x))) \cdot \nabla d_U(\Phi(x, -f(x)))}.
\]
Moreover, we may assume the following conditions:

(F1) \( W \subseteq \{ -R < d_U < R \} \cap \{ -R < d_\Omega < R \} \).

(F2) \( \nabla f \cdot \nabla d_U > 0 \) in \( W \).

(F3) \( d_U \) and \( f \) have the same sign in \( W \).

(F4) \( \Phi(\partial U \cap \partial \Omega, (-s_0, s_0)) = \partial \Omega \cap W \).

(F5) \( \nabla f \cdot \nabla d_\Omega = 0 \) on \( \partial \Omega \cap W \).

In the rest part of Step 1, we verify these properties (F1)-(F5). At first, it is easy to check the property (F1). From (4.19), for \( x_0 \in \partial U \cap \partial \Omega \),

\[
\nabla f(x_0) \cdot \nabla d_U(x_0) = \frac{\nabla \Phi(x_0, 0)}{X(x_0)} \cdot \nabla d_U(x_0) = \frac{\nabla d_U(x_0)}{\nabla d_U(x_0) \cdot \nabla d_U(x_0)} \cdot \nabla d_U(x_0) = 1.
\]

Therefore, taking \( W \) smaller, we may assume (F2). Next we verify (F3). On a neighborhood of \( (\partial U \cap \partial \Omega) \times \{0\} \), we compute

\[
\begin{align*}
d_U(x) &= d_U(\Phi(\Phi(x, -f(x)), f(x)) = \int_{0}^{f(x)} \frac{d}{ds} [d_U(\Phi(\Phi(x, -f(x)), s)] \, ds \\
&= \int_{0}^{f(x)} \nabla d_U(\Phi(\Phi(x, -f(x)), s)) \cdot X(\Phi(\Phi(x, -f(x)), s)) \, ds.
\end{align*}
\]

(4.20)

Here we may assume the integrand of (4.20) is positive since

\[ \nabla d_U(\Phi(\Phi(x, -f(x)), s)) \cdot X(\Phi(\Phi(x, -f(x)), s)) = \nabla d_U(x) \cdot \nabla d_U(x) = 1 \]

on \( (\partial U \cap \partial \Omega) \times \{0\} \). Thus, the property (F3) holds true. Moreover, choosing \( s_0 \) smaller, we may assume \( \Phi(\partial U \cap \partial \Omega, (-s_0, s_0)) \subseteq W \). Recalling that \( X \) is a tangent vector field of \( \partial \Omega \), it is easy to see

\[ \Phi(\partial U \cap \partial \Omega, (-s_0, s_0)) = \partial \Omega \cap W \cap f^{-1}((-s_0, s_0)). \]

Thus we obtain (F4) by replacing \( W \) with \( W \cap f^{-1}((-s_0, s_0)) \). Finally, we verify (F5). From (F4), it suffices to prove that for every \( x_0 \in \partial U \cap \partial \Omega \),

\[
\alpha(s) := \nabla f(\Phi(x_0, s)) \cdot \nabla d_\Omega(\Phi(x_0, s))
\]

(4.21)
is constantly 0 on \((-s_0,s_0)\). By the group property of \(\Phi\), we have \(\nabla \Phi(\Phi(x,s),-s) = \{\nabla \Phi(x,s)\}^{-1}\). Recalling (4.19) and \(|d_U| = 1\), we compute
\[
\alpha(s) = \frac{\{\nabla \Phi(x_0,s)\}^{-1}\nabla d_U(x_0) \cdot \nabla d_\Omega(\Phi(x_0,s))}{X(x_0) \cdot \nabla d_U(x_0)}
\]
\[
= \frac{\{\nabla \Phi(x_0,s)\}^{-1}\nabla d_U(x_0) \cdot \nabla d_\Omega(\Phi(x_0,s))}{\nabla d_U(x_0) \cdot \nabla d_U(x_0)}
= \{\nabla \Phi(x_0,s)\}^{-1}\nabla d_U(x_0) \cdot \nabla d_\Omega(\Phi(x_0,s)).
\]
In particular, \(\alpha(0) = (\{\nabla \Phi(x_0,0)\}^{-1}\nabla d_U(x_0)) \cdot \nabla d_\Omega = \nabla d_U(x_0) \cdot \nabla d_\Omega(x_0) = 0\) since \(\nabla \Phi(x_0,0) = \text{id}_{\mathbb{R}^N}\). Additionally, we calculate
\[
\frac{d}{ds}\{\nabla \Phi\}^{-1} = -\{\nabla \Phi\}^{-1} \left\{ \frac{d}{ds}\{\nabla \Phi\} \right\} \{\nabla \Phi\}^{-1}
= -\{\nabla \Phi\}^{-1} \{\nabla (X \circ \Phi)\} \{\nabla \Phi\}^{-1} = -(\nabla X \circ \Phi)\{\nabla \Phi\}^{-1}.
\]
Therefore,
\[
\alpha'(s) = \frac{d}{ds}\left[ \{\nabla \Phi(x_0,s)\}^{-1}\nabla d_U(x_0) \cdot \nabla d_\Omega(\Phi(x_0,s)) \right]
= \left\{\nabla \Phi(x_0,s)\}^{-1}\nabla d_U(x_0) \cdot \left(\nabla^2 d_\Omega(\Phi(x_0,s))X(\Phi(x_0,s))\right)
\]
\[
- \left(\nabla X(\Phi(x_0,s))\{\nabla \Phi(x_0,s)\}^{-1}\nabla d_U(x_0) \cdot \nabla d_\Omega(\Phi(x_0,s)) \right)
= \left\{\nabla \Phi(x_0,s)\}^{-1}\nabla d_U(x_0) \cdot \left(\nabla^2 d_\Omega(y)X(y) - (\nabla X(y))^T \nabla d_\Omega(y) \right),
\]
here we write \(y := \Phi(x_0,s) \in \partial \Omega \cap W\). On the other hand, we compute
\[
(\nabla X)^T = [\nabla \{(I - \nabla d_\Omega \otimes \nabla d_\Omega)\nabla d_U + d_\Omega X_0\}]^T
= -(\nabla d_\Omega \cdot \nabla d_U)^2 d_\Omega - (\nabla d_\Omega \otimes \nabla d_U)\nabla^2 d_\Omega + (I - \nabla d_\Omega \otimes \nabla d_\Omega)\nabla^2 d_U
+ X_0 \otimes \nabla d_\Omega + d_\Omega(\nabla X_0)^T
\]
in \(W\). Recalling that \(d_\Omega = 0\) on \(\partial \Omega\), \(|\nabla d_\Omega| = 1\) and \((\nabla^2 d_\Omega)\nabla d_\Omega = 0\), we calculate
\[
(\nabla^2 d_\Omega)X - (\nabla X)^T \nabla d_\Omega
= (\nabla^2 d_\Omega)(I - \nabla d_\Omega \otimes \nabla d_\Omega)\nabla d_U + \nabla^2 d_\Omega(d_\Omega X_0) + (\nabla d_\Omega \cdot \nabla d_U)(\nabla^2 d_\Omega)\nabla d_\Omega
+ (\nabla d_\Omega \otimes \nabla d_U)(\nabla^2 d_\Omega)\nabla d_\Omega - (I - \nabla d_\Omega \otimes \nabla d_\Omega)(\nabla^2 d_U)\nabla d_\Omega
- (X_0 \otimes \nabla d_\Omega)\nabla d_\Omega - d_\Omega(\nabla X_0)^T \nabla d_\Omega
= (\nabla^2 d_\Omega)\nabla d_U - (I - \nabla d_\Omega \otimes \nabla d_\Omega)(\nabla^2 d_U)\nabla d_\Omega - X_0
\]
on \(\partial \Omega \cap W\). Hence we put \(X_0 := (\nabla^2 d_\Omega)\nabla d_U - (I - \nabla d_\Omega \otimes \nabla d_\Omega)(\nabla^2 d_U)\nabla d_\Omega\) to obtain
\((\nabla^2 d_\Omega)X - (\nabla X)^T \nabla d_\Omega = 0\) on \(\partial \Omega \cap W\). Then \(\alpha'(s) = 0\) on \((-s_0,s_0)\).

Step 2. We shall extend \(f\) to \(\mathbb{R}^N\). Take \(r > 0\) so that
\[
\{ -2r < d_U < 2r \} \cap \{ -2r < d_\Omega < 2r \} \subseteq W.
\]
Choose \(\zeta \in C^\infty_c(\mathbb{R})\) satisfying
\[
\zeta(s) = 1 \text{ if } |s| < r, \quad \zeta(s) = 0 \text{ if } |s| \geq \frac{3}{2}r, \quad 0 \leq \zeta \leq 1.
\]
We estimate the first term as follows.

(4.26) \[ g_0 := (1 - \zeta \circ d\Omega) du + (\zeta \circ d\Omega) f. \]

From (F1)-(F5), one may verify the following:

(G1) There exists \( \delta > 0 \) such that \( V := \{ -\delta \leq g_0 \leq \delta \} \subseteq \{ -r < du < r \} \).

(G2) For \( x \in \{-2r < du < 2r\} \), \( g_0(x) = 0 \) if and only if \( du(x) = 0 \).

(G3) \( g_0 \) and \( du \) have the same sign in \( \{-2r < du < 2r\} \).

(G4) \( \nabla g_0 \cdot \nabla d\Omega = 0 \) on \( \partial \Omega \cap \{ -2r < du < 2r \} \).

(G5) Choosing \( \delta > 0 \) even smaller, \( \nabla g_0 \neq 0 \) on \( V \).

Now take any \( \phi \in C^\infty(\mathbb{R}) \) such that

\[
\begin{cases}
\phi(-s) = -\phi(s) & \text{for any } s \in \mathbb{R}, \\
\phi'(s) > 0, \phi''(s) \geq 0 & \text{if } -\delta < s \leq 0, \\
\phi(s) = -1 & \text{if } s \leq -\delta.
\end{cases}
\]

Then we set \( g : \mathbb{R}^N \to \mathbb{R} \) by

(4.27) \[ g(x) := \begin{cases}
\phi(g_0(x)) & \text{if } -2r < du(x) < 2r, \\
1 & \text{if } du(x) > r, \\
-1 & \text{if } du(x) < -r.
\end{cases} \]

Step 3. From (G1)-(G4), we see that \( g \in C^\infty(\mathbb{R}^N) \) with \( \text{spt } \nabla g \subseteq V \) and the properties (i) and (ii). It remains to verify the property (iii). By (G5), we have \( c_0 := \inf_{V} |\nabla g_0| > 0 \). We compute

\[
\begin{align*}
\text{div} \left( \frac{\nabla g}{\sqrt{|\nabla g|^2 + \varepsilon^2}} \right) &= \frac{1}{\sqrt{|\nabla g|^2 + \varepsilon^2}} \text{tr} \left( I - \frac{\nabla g \otimes \nabla g}{|\nabla g|^2 + \varepsilon^2} \right) \nabla^2 g \\
&= \frac{1}{\sqrt{|(\overline{\phi})|^2 |\nabla g_0|^2 + \varepsilon^2}} \text{tr} \left( I - \frac{(\overline{\phi}')^2 \nabla g_0 \otimes \nabla g_0}{(\overline{\phi}')^2 |\nabla g_0|^2 + \varepsilon^2} \right) \phi' \nabla^2 g_0 \\
&\quad + \frac{1}{\sqrt{|(\overline{\phi})|^2 |\nabla g_0|^2 + \varepsilon^2}} \text{tr} \left( I - \frac{(\overline{\phi}')^2 \nabla g_0 \otimes \nabla g_0}{(\overline{\phi}')^2 |\nabla g_0|^2 + \varepsilon^2} \phi'' \nabla g_0 \otimes \nabla g_0 \right).
\end{align*}
\]

We estimate the first term as follows.

\[
\left| \frac{1}{\sqrt{|(\overline{\phi})|^2 |\nabla g_0|^2 + \varepsilon^2}} \text{tr} \left( I - \frac{(\overline{\phi}')^2 \nabla g_0 \otimes \nabla g_0}{(\overline{\phi}')^2 |\nabla g_0|^2 + \varepsilon^2} \right) \phi' \nabla^2 g_0 \right| \leq \frac{\phi'' |\nabla^2 g_0|}{\sqrt{(|(\overline{\phi}')|^2 |\nabla g_0|^2 + \varepsilon^2)^{5/2}}} \leq \frac{1}{c_0 |\nabla^2 g_0|}.
\]

For the second term,

\[
\left| \frac{1}{\sqrt{|(\overline{\phi})|^2 |\nabla g_0|^2 + \varepsilon^2}} \text{tr} \left( I - \frac{(\overline{\phi}')^2 \nabla g_0 \otimes \nabla g_0}{(\overline{\phi}')^2 |\nabla g_0|^2 + \varepsilon^2} \phi'' \nabla g_0 \otimes \nabla g_0 \right) \right| \leq \frac{\varepsilon^2 |\nabla g_0|^2}{(|(\overline{\phi}')|^2 |\nabla g_0|^2 + \varepsilon^2)^{3/2}} \leq \frac{\varepsilon^2 c_1 |\phi''|}{(c_0 |\nabla g_0| + \varepsilon^2)^{3/2}} |\nabla g_0|,
\]
where $c_1 := \sup_V |\nabla g_0| < \infty$. Therefore, by the coarea formula,

$$
\int_{\mathbb{R}^N} \left| \operatorname{div} \left( \frac{\nabla g}{\sqrt{|\nabla g|^2 + \varepsilon^2}} \right) \right| \, dx \leq C \int_V |\nabla^2 g_0| + \frac{\varepsilon^2 |\phi''|}{(c_0^2 (\phi')^2 + \varepsilon^2)^{3/2}} |\nabla g_0| \, dx
$$

(4.28)

$$
\leq C' \left( 1 + \int_{-\delta}^{0} \frac{\varepsilon^2 \phi''(s)}{(c_0^2 (\phi'(s))^2 + \varepsilon^2)^{3/2}} \, ds \right) = C' \left( 1 + \int_{-\delta}^{0} \frac{d}{ds} \left( \frac{\phi'(s)}{\sqrt{c_0^2 (\phi'(s))^2 + \varepsilon^2}} \right) \, ds \right)
$$

$$
\leq C' \left( 1 + \frac{\phi'(0)}{\sqrt{c_0^2 (\phi'(0))^2 + \varepsilon^2}} \right) < \infty,
$$

where $C, C' > 0$ are some constants independent of $\varepsilon$. \hfill \Box

**References**

[1] Brakke K. A., *The motion of a surface by its mean curvature*. Mathematical Notes, vol. 20, Princeton University Press, Princeton, 1978.

[2] Chen Y.-G., Giga Y. and Goto S., *Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations*. J. Differential Geom. 33 (1991), no. 3, 749-786.

[3] Edelen N., *The free-boundary Brakke flow*. J. Reine Angew. Math. 758 (2020), 95-137.

[4] Evans L. C. and Spruck J., *Motion of level sets by mean curvature*. I. J. Differential Geom. 33 (1991), no. 3, 635-681.

[5] Evans L. C. and Spruck J., *Motion of level sets by mean curvature*. IV. J. Geom. Anal. 5 (1995), no. 1, 77-114.

[6] Giga Y., *Surface evolution equations: a level set approach*. Monographs in Mathematics vol. 99, Birkhäuser Verlag, Basel, 2006.

[7] Giga Y., Ohmura M. and Sato M.-H., *On the Strong Maximum Principle and the Large Time Behavior of Generalized Mean Curvature Flow with the Neumann Boundary Condition*. J. Differential Equations 154 (1999), 107-131.

[8] Giga Y. and Sato M.-H., *Neumann problem for singular degenerate parabolic equations*. Differential Integral Equations 6 (1993), 1217-1230.

[9] Huisken G., *Non-parametric Mean Curvature Evolution with Boundary Conditions*. J. Differential Equations 77 (1989), 369-378.

[10] Kagaya T., *Convergence of the Allen-Cahn equation with a zero Neumann boundary condition on non-convex domains*. Math. Ann. 373 (2019), no. 3, 1485-1528.

[11] Kasai K. and Tonegawa Y., *A general regularity theory for weak mean curvature flow*. Calc. Var. PDE. 50 (2014), 1-68.

[12] Lieberman G. M., *Second order parabolic differential equations*. World Scientific Publishing, River Edge, 1996.

[13] Mizuno M. and Tonegawa Y., *Convergence of the allen-cahn equation with Neumann boundary conditions*. SIAM J. Math. Anal. 47 (2015), 1906-1932.

[14] Sato M.-H., *Interface evolution with Neumann boundary condition*. Adv. Math. Sci. Appl. 4 (1994), 249-264.

[15] Stahl A., *Regularity estimates for solutions to the mean curvature flow with a Neumann boundary condition*. Calc. Var. PDE. 4 (1996), 385-407.

[16] Tonegawa Y., *A second derivative Hölder estimate for weak mean curvature flow*. Adv. Calc. Var. 7 (2014), 91-138.

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