REMARKS ON $L^p$-VANISHING RESULTS IN GEOMETRIC ANALYSIS

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Abstract. We survey some $L^p$-vanishing results for solutions of Bochner or Simons type equations with refined Kato inequalities, under spectral assumptions on the relevant Schrödinger operators. New aspects are included in the picture. In particular, an abstract version of a structure theorem for stable minimal hypersurfaces of finite total curvature is observed. Further geometric applications are discussed.

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1. Introduction and some vanishing results

This paper originates from an attempt to understand the abstract content of a structure theorem for stable minimal hypersurfaces of finite total scalar curvature.

Theorem 1. Let $f : M^m \to \mathbb{R}^N$ be a complete, $m$-dimensional, minimal submanifold, $m \geq 2$. Assume that its second fundamental form $\Pi$ satisfies

(i) $|\Pi|^2 \in L^{m/2}(M)$
(ii) The “stability operator” $\mathcal{L} = -\Delta - |\Pi|^2$ has non-negative spectrum.

Then, $|\Pi| \equiv 0$, that is, $f(M)$ is an affine $m$-plane.

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Partial forms of this beautiful result, under various restrictions, have been obtained by R. Schoen, L. Simon and S.T. Yau, [27], M.P. do Carmo and C.K. Peng, [13], [14] and P. Bérard, [4]. In the above general form, it is due to M. Anderson, [1], and, Y.-B. Shen and X.-H. Zhu, [30].

In case $m = 2$, it is known from previous work by Bernstein (minimal graphs), do Carmo-Peng, [13], and D. Fisher-Colbrie and Schoen (general surfaces), [16], that condition (i) is unnecessary. Actually, it has been conjectured that this conclusion can be extended to every dimension $m \leq 7$. The upper bound $m = 7$ is due to the famous non-planar (and stable) minimal graphs by E. Bombieri, E. de Giorgi and E. Giusti. We also record that, in higher dimensions, condition (i) can be replaced by the next more general requirement, [35], [25],

$$(i)' \, |\mathbf{II}|^2 \in L^\gamma(M), \text{ for some } \gamma \geq m/2.$$

As we shall see in the next section, the approaches proposed by the above mentioned authors are very different from each others. However they have a key analytic ingredient in common, i.e., the fact that the second fundamental form satisfies the Simons equation

$$\frac{1}{2} \Delta |\mathbf{II}|^2 + |\mathbf{II}|^2 |\mathbf{II}|^2 = |D\mathbf{II}|^2 \geq \left(1 + \frac{2}{m}\right)|\nabla |\mathbf{II}||^2. \tag{1}$$

The last inequality is a special case of what is known in the literature as a “refined Kato inequality”, [5], [10]. In particular, from (1) it follows that

$$|\mathbf{II}| \left(\Delta |\mathbf{II}| + |\mathbf{II}|^2 |\mathbf{II}|\right) - \frac{2}{m} |\nabla |\mathbf{II}||^2 \geq 0, \tag{2}$$

weakly on $M$.

From the analytic viewpoint, vanishing results for $L^p$-solutions of PDEs, under spectral assumptions on the relevant Schrödinger operator, are quite well understood, [15], [3], [20], [22]. For instance, consider the following statement from [20].

**Theorem 2.** Let $(M, \langle , \rangle)$ be a complete Riemannian manifold and let $0 \leq \psi \in \operatorname{Lip}_{\operatorname{loc}}(M)$ be a distributional solution of the differential inequality

$$\psi \Delta \psi + a(x) \psi^2 + A |\nabla \psi|^2 \geq 0, \tag{3}$$

for some $a(x) \in C^0(M)$ and $A \in \mathbb{R}$. Assume that the bottom of the spectrum of the (modified) Schrödinger operator $\mathcal{L}_H = -\Delta - Ha(x)$ satisfies

$$\lambda_1(\mathcal{L}_H) \geq 0, \tag{4}$$

for some $H \geq A + 1 > 0$. If

$$\int_{+\infty}^{+\infty} \frac{dR}{\int_{\partial B_R(o)} \psi^{2p}} = +\infty, \tag{5}$$

for some origin $o \in M$ and for some $A + 1 \leq p \leq H$, then the following hold:

(i) If $H \geq p > A + 1$ then $\psi \equiv \text{const.}$ and either $a(x) \equiv 0$ or $\psi \equiv 0$. 

(ii) If $H = A + 1$ then $\psi$ satisfies (4) with the equality sign.

**Remark 3.** For future purposes, and for comparison with other vanishing results, we point out the following chain of (strict) implications:

$$\psi \in L^2 \Rightarrow \int_{B_R} \psi^{2p} \leq CR^2 \Rightarrow \int_{B_R} \frac{RdR}{\psi^{2p}} = +\infty \Rightarrow (5).$$

Thus, in the standard “stable” case $H = 1$, one can get triviality provided a small power (say $\leq 2$) of $\psi$ is integrable. It is important to observe that large powers of $\psi$ (like $p = m/2$ in the geometric case) are not allowed.

**Proof of Theorem 2 (sketch).** The method to prove Theorem 2 resembles the one used to prove the generalized maximum principle for the operator $\mathcal{L}_H$, [26]. Namely, following [16], one interprets (4) as the existence of a smooth solution $\varphi > 0$ of the linear equation

$$\mathcal{L}_H \varphi = 0.$$ 

Next, we combine $\psi$ and $\varphi$ to obtain a new function

$$u = \frac{\psi^p}{\varphi^{p/H}}$$

which is subharmonic with respect to a suitable diffusion-type operator, i.e.,

$$\Delta_\omega u := \omega^{-1} \operatorname{div} (\omega \nabla u) \geq 0,$$

with $\omega = \varphi^{2p/H}$. Assume $\psi \in L^2(M)$, the more general case where (5) holds true can be dealt similarly. Since $u \in L^2(M, \omega d\operatorname{vol})$, a version (for diffusion operators) of the classical theorem by Yau on positive, $L^{p>1}$-subharmonic functions implies that $u$ is constant. The remaining conclusions on $\psi$ now follows easily from considerations on the constants $A, H, p$. □

It is worth to point out that, actually, one can obtain triviality making a direct use of the spectral assumption with suitable test functions. This method, which was used in [27], [3], works particularly well in non-linear contexts, and vanishing results for the energy density of a $p$-harmonic map are nontrivial examples, [24]. The “direct” method has a further nice feature. It enables one to obtain vanishing for $\psi \in L^2$ even if $p > H$. This is shown in the following result that, in its full generality, has been never observed before.

**Theorem 4.** Let $(M, \langle \cdot, \cdot \rangle)$ be a complete Riemannian manifold and let $0 \leq \psi \in \text{Lip}_{\text{loc}}(M)$ be a distributional solution of the differential inequality (3) for some $0 \leq a(x) \in C^0(M)$ and $A \in \mathbb{R}$. Assume that the bottom of the spectrum of the (modified) Schrödinger operator $\mathcal{L}_H = -\Delta - Ha(x)$ satisfies (4) for some $H \geq A + 1 > 0$. Let $p_0 \leq H \leq p_1$ be the solutions of

$$q^2 - 2Hq + H(A + 1) = 0.$$
If
\[ \int_{B_R(o)} \psi^{2p} = o\left(R^2\right), \quad \text{as } R \to +\infty, \]
for some origin $o \in M$ and for some $p_0 \leq p \leq p_1$,

then

(i) If $H > A + 1$ and $p_0 < p < p_1$, then $\psi \equiv \text{const}$.

(ii) If $H = A + 1$ then $\psi$ satisfies (3) with equality sign on the open set $\Omega$ where $\psi^A > 0$. In particular, it holds on $M$ if $A \leq 0$.

Thus, as expected (and confirmed by computations), there is again an upper bound for the integrability exponent and this is related to the (Kato) constant $A$. For instance, in the minimal surface setting, this upper bound prevented Bérard to reach dimensions $m \geq 6$. Very recently, L.F. Tam and D. Zhou employed this method to give characterizations of higher-dimensional catenoids, [33]. It is also important to remark that with the direct method we are able to reach low exponents $p_0 < p \leq A + 1$, i.e., even smaller than those considered in Theorem 2. Apparently, one of the main drawbacks of the direct approach, when compared with Theorem 2, concerns with the case $H = A + 1$ where one obtains equality only on the set $\Omega$ where $\psi^A$ is positive. However in geometric applications this is not a serious problem, because, typically, $A$ arises from a refined Kato inequality and therefore it is negative.

**Proof of Theorem 4**. The proof aims at obtaining a Caccioppoli-type inequality for $\psi$. Define the distribution

\[ 0 \leq \mathcal{E} := \psi \Delta \psi + a\psi^2 + A|\nabla \psi|^2. \]

Note that since by assumption $\mathcal{E}$ is nonnegative, $\mathcal{E}$ is in fact a positive measure. By the weak formulation of inequality (6),

\[ 0 \leq \int \eta d\mathcal{E} = \int_M a\eta \psi^2 + \int_M A\eta|\nabla \psi|^2 - \int_M \langle \nabla (\eta \psi), \nabla \psi \rangle, \]

for each test function $0 \leq \eta \in Lip_c(M)$. From now on, until otherwise specified, we assume that $0 < p < 1$. The case $p \geq 1$ is easier and will be dealt with at the end of proof. We apply (7) to the function $\eta = \rho^2 \psi^{2p-2}$ where $0 \leq \rho \in C^\infty_c(M)$ is a cut-off function to be chosen later, and, for
every $\delta > 0$, $\psi_\delta = \psi + \delta > 0$, to get

\begin{equation}
0 \leq \int_M \rho^2 \psi_\delta^{2p-2} d\mathcal{E}
\end{equation}

\begin{align*}
&\leq \int_M A \rho^2 \psi_\delta^{2p-2} |\nabla \psi|^2 - (2p - 2) \int_M \rho^2 \psi_\delta^{2p-3} \psi |\nabla \psi|^2 \\
&\quad - \int_M \rho^2 \psi_\delta^{2p-2} |\nabla \psi|^2 - 2 \int_M \rho \psi_\delta^{2p-2} \psi \langle \nabla \rho, \nabla \psi \rangle.
\end{align*}

On the other hand, the spectral assumption on $L_H$ implies that, for each test function $\phi \in \text{Lip}_c(M)$,

\[ \int_M |\nabla \phi|^2 \geq \int_M H a \phi^2. \]

Choosing $\phi = \rho^{p} \psi_\delta$, we get

\[ \int_M a \rho^2 \psi_\delta^{2p-2} \psi^2 \leq \int_M a \rho^2 \psi_\delta^{2p} \]

\[ \leq \frac{p^2}{H} \int_M \rho^2 \psi_\delta^{2p-2} |\nabla \psi|^2 + \frac{1}{H} \int_M \psi_\delta^{2p} |\nabla \rho|^2 \\
+ \frac{2p}{H} \int_M \rho \psi_\delta^{2p-1} \langle \nabla \psi, \nabla \rho \rangle. \]

Inserting this latter in (8) gives

\begin{equation}
0 \leq \int_M \rho^2 \psi_\delta^{2p-2} d\mathcal{E}
\end{equation}

\begin{align*}
&\leq \left( \frac{p^2}{H} + A - 1 \right) \int_M \rho^2 \psi_\delta^{2p-2} |\nabla \psi|^2 - (2p - 2) \int_M \rho^2 \psi_\delta^{2p-3} \psi |\nabla \psi|^2 \\
&\quad + \frac{1}{H} \int_M \psi_\delta^{2p} |\nabla \rho|^2 + 2 \int_M \rho \left( \frac{p}{H} \psi_\delta - \psi \right) \psi_\delta^{2p-2} \langle \nabla \rho, \nabla \psi \rangle.
\end{align*}

Recalling that $0 < p < 1$, since $\psi < \psi + \delta = \psi_\delta$, from the above we get

\begin{equation}
\int_M \rho^2 \psi_\delta^{2p-2} d\mathcal{E} + \left( 2p - \frac{p^2}{H} - (A + 1) \right) \int_M \rho^2 \psi_\delta^{2p-2} |\nabla \psi|^2
\end{equation}

\begin{align*}
&\leq \frac{1}{H} \int_M \psi_\delta^{2p} |\nabla \rho|^2 + 2 \left( \frac{p}{H} - 1 \right) \int_M \rho \psi_\delta^{2p-1} \langle \nabla \rho, \nabla \psi \rangle \\
&\quad + 2 \delta \int_M \rho \psi_\delta^{2p-2} \langle \nabla \rho, \nabla \psi \rangle.
\end{align*}

Now, assume $H > A + 1$. Since $\delta \leq \psi_\delta$, $\mathcal{E} \geq 0$, recalling the Young inequality $2ab \leq \varepsilon a^2 + \varepsilon^{-1} b^2$, valid for each $a, b \geq 0$ and $\varepsilon > 0$, and applying Schwarz
inequality, we deduce that
\[
2 \left( \frac{p}{H} - 1 \right) \int_M \rho \psi^2 \partial - 1 (\nabla \psi, \nabla \psi) + 2 \delta \int_M \rho \psi^2 \partial - 2 (\nabla \psi, \nabla \psi) \\
\leq \varepsilon \int_M |\nabla \psi|^2 \rho^2 \psi^2 \partial - 2 + \frac{1}{\varepsilon} \left( \left| \frac{p}{H} - 1 \right| + 1 \right)^2 \int_M \psi^2 |\nabla \psi|^2.
\]
Using this inequality into (9) we finally obtain
\[
(10) \quad F \int_M |\nabla \psi|^2 \rho^2 \psi^2 \partial - 2 \leq G \int_M \psi^2 |\nabla \psi|^2,
\]
where
\[
F := 2p - \frac{p^2}{H} - (A + 1) - \varepsilon, \\
G := \frac{1}{H} + \frac{1}{\varepsilon} \left( \left| \frac{p}{H} - 1 \right| + 1 \right)^2.
\]
Note that, by assumption,
\[
p^2 - 2Hp + H (A + 1) < 0,
\]
therefore we can take \( \varepsilon \) small enough so that \( F > 0 \). Moreover, using monotone and dominated convergence we see that (10) holds even in the limit \( \delta \rightarrow 0 \). Thus, if we choose \( \rho = \rho_R \in C_c^\infty (B_R (o)) \) in such a way that \( |\nabla \rho_R| \leq R^{-1} \) and \( \rho_R \equiv 1 \) on \( B_{R/2} (o) \), we get
\[
F \int_{B_{R/2}} |\nabla \psi|^2 \psi^2 \partial - 2 \leq \frac{G}{R^2} \int_{B_R} \psi^2,
\]
and letting \( R \rightarrow +\infty \) we conclude that \( \psi \) is constant on the set
\[
\Omega_{p-1} = \left\{ x \in M : \psi^{2(p-1)} (x) > 0 \right\}.
\]
Since \( 0 < p < 1 \), then \( \Omega_{p-1} = M \) and \( \psi \) is constant on \( M \). This completes the proof in the case \( H > A + 1 \).

Now suppose that \( H = A + 1 \). Since, in this case, \( p_0 = p_1 = A + 1 = H \), (9) reads
\[
(11) \quad 0 \leq \int_M \rho^2 \psi^2 \partial A d\mathcal{E} \leq \frac{1}{H} \int_M \psi^2 \partial A + 2 |\nabla \rho|^2 + 2 \delta \int_M \rho \psi^2 \partial A (\nabla \rho, \nabla \psi) \\
\leq \frac{1}{H} \int_M \psi^2 \partial A + 2 |\nabla \rho|^2 + 2 \delta \int_M \rho \psi^2 \partial A |\nabla \rho| |\nabla \psi|.
\]
We claim that
\[
(12) \quad 2 \delta \int_M \rho \psi^2 \partial A |\nabla \rho| |\nabla \psi| \rightarrow 0, \text{ as } \delta \rightarrow 0.
\]
Indeed, since $0 < \delta \leq \psi_\delta$, recalling also that we are in the case $-1 < A = p - 1 < 0$, we can compute

$$2 \int_M \delta \rho \psi_\delta^{2A} |\nabla \rho| |\nabla \psi| = 2 \int_M |\nabla \rho| \{\delta \psi_\delta^{2A}\} \{\rho \psi_\delta^{2A} |\nabla \psi|\} \leq 2 \delta^{A+1} \int_M |\nabla \rho| \{\rho \psi_\delta^{2A} |\nabla \psi|\} \leq \delta^{A+1} \left\{ \int_M |\nabla \rho|^2 + \int_M \rho^2 \psi_\delta^{2A} |\nabla \psi|^2 \right\}.$$  

Now, we recall from the regularity theory for Bochner type inequalities in the presence of a refined Kato inequality ([20] or Lemma 4.12 and Lemma 4.13 in [22]) that

$$\psi^{A+1} \in W^{1,2}_{loc}(M)$$

with

$$\nabla \psi^{A+1} = (A + 1) \psi^A \nabla \psi.$$  

Since

$$\int_M \rho^2 \psi_\delta^{2A} |\nabla \psi|^2 \leq \int_M \rho^2 \psi^{2A} |\nabla \psi|^2 = \int_M \frac{\rho^2}{(A + 1)^2} |\nabla \psi^{A+1}|^2 < +\infty$$

we deduce that the quantity

$$\left\{ \int_M |\nabla \rho|^2 + \int_M \rho^2 \psi_\delta^{2A} |\nabla \psi|^2 \right\}$$

on the right hand side of (13) is bounded uniformly in $\delta$. The validity of (12) now follows by letting $\delta \to 0$ in (13), thus proving the claim. With this preparation, we can now take the limits as $\delta \to 0$ in (11) and conclude

$$0 \leq \int_M \rho^2 \psi_\delta^{2A} dE \leq \frac{1}{H} \int_M \psi^{2A+2} |\nabla \rho|^2.$$  

Then, choosing as before $\rho = \rho_R$ and letting $R \to \infty$ we obtain that $E \equiv 0$ on the set $\Omega_A$ where $\psi^A > 0$, thus completing the proof of the Theorem for $0 < p < 1$.

If $p \geq 1$, all the above arguments can be greatly simplified because one can obtain the validity of inequality (8) with $\delta = 0$. This is obviously true if $2p - 2 \geq 1$, i.e., $p \geq 3/2$. On the other hand, if $1 \leq p < 3/2$, then

$$\rho^2 \psi_\delta^{2p-3} |\nabla \psi|^2 \leq \rho^2 \psi^{2p-2} |\nabla \psi|^2 \in L^1.$$  

Accordingly, we can take the limits as $\delta \to 0$ and use dominated convergence to deduce that inequality (8) persists with $\delta = 0$ and $\psi_\delta = \psi$, as claimed. Now, in case $H > A + 1$ we get that $\psi$ is constant on $\Omega_{p-1}$. Using a connectedness argument, it follows easily that either $\Omega_{p-1} = \emptyset$ or $\Omega_{p-1} = M$. In any case, $\psi$ is constant. Finally, in case $H = A + 1(\geq 1)$, we immediately get the validity of (14) and the desired conclusion follows.  

□
What makes Theorem 2 and Theorem 4 interesting is that, due to their abstract formulations, they can be adapted to a number of different geometric situations where a relevant function, of geometric content, obeys an inequality of type (3). For instance, this is the case if $\psi = |df|$ where $f : M \to N$ is a harmonic map or $\psi = |Ric|$ where $Ric$ is the Ricci tensor of a locally conformally flat manifold $M$ with zero scalar curvature. However, in many geometric applications, this function is required to satisfy (scale-invariant) integral conditions that cannot be captured in the above theorem, unless more restrictive spectral assumptions are imposed. Thus, we are led to ask the following

**Problem 1.** Is there an abstract formulation of Theorem 4, i.e., a vanishing result for solutions of (3) with higher integrability exponents?

2. **Statement of the new abstract result**

Note that, in the special situation of minimal submanifolds, it happens that the solution to the PDE, the potential term of the Schrödinger (stability) operator, the bound of the curvature tensor (Gauss equations) etc... coincide. This is the reason why the result looks so elegant: one $L^p$ assumption controls so many different objects! Furthermore, it is well known that Euclidean minimal submanifolds enjoy an isoperimetric (hence Sobolev) inequality and much more. In particular, volumes are at least Euclidean. Obviously all this is, in general, no longer true in the abstract setting and some choices have to be made. In this note we propose the following answer to Problem 1.

**Theorem 5.** Let $(M, \langle \cdot, \cdot \rangle)$ be a complete, $m(\geq 3)$-dimensional Riemannian manifold supporting the Euclidean Sobolev inequality

$$\|v\|_{L^\frac{2m}{m-2}} \leq S \|
abla v\|_{L^2},$$

for every $v \in C^\infty_c(M)$ and for some constant $S > 0$. Assume also that

$$\text{vol}(B_R) = O(R^m).$$

Let $0 \leq \psi \in Lip_{\text{loc}}$ be a distributional solution of the differential inequality (3) where $0 \leq a(x) \in L^{m/2}$ satisfies the uniform decay estimate

$$\sup_{M \setminus B_R} a(x) = O\left(\frac{1}{R^2}\right).$$

Suppose that

$$\psi \in L^{2p}(M),$$

for some $p > (1 + A)/2$. Then:

(i) If the spectral condition (4) holds for some $H > 1 + A > 0$

then either $a \equiv 0$ or $\psi \equiv 0$. 
(ii) If the spectral condition (4) holds with
\[ H = 1 + A > 0, \]
then \( \psi \) satisfies (3) with the equality sign.

In geometric situations, the potential term \( a(x) \) has a geometric content (the norm of the second fundamental form, of a curvature tensor etc.) and satisfies some semi-linear elliptic inequality. It follows that the uniform decay estimate required on \( a(x) \) is a consequence of the assumption \( a \in L^{m/2} \); see e.g. [8], [25], [34] and references therein.

It is well known that, in the presence of a Sobolev inequality, the spectral assumption is implied by the requirement that the potential term \( a(x) \) is small in a suitable integral sense when compared with the Sobolev constant. Using this point of view, the vanishing part of Theorem 5 can be formulated in the following elegant form.

**Corollary 6.** Let \((M, \langle \cdot, \cdot \rangle)\) be a complete, \(m(\geq 3)\)-dimensional Riemannian manifold supporting the Euclidean Sobolev inequality
\[ \|v\|_{L^{2m/(m-2)}} \leq S \|\nabla v\|_{L^2}, \]
for every \( v \in C_c^{\infty}(M) \) and for some constant \( S > 0 \). Assume also that
\[ \text{vol}(B_R) \leq O(R^m). \]
Let \( 0 \leq \psi \in \text{Lip}_\text{loc} \) be a distributional solution of the differential inequality (3) where \( 0 \leq a(x) \in L^{m/2} \) satisfies
\[ \sup_{M \setminus B_R} a(x) = O \left( \frac{1}{R^2} \right) \]
and
\[ \|a\|_{L^{2p}} \leq (H^2S^{-1}), \]
for some
\[ H > 1 + A > 0. \]
If \( \psi \in L^{2p} \) for some \( p > (1 + A)/2 \), then either \( a \equiv 0 \) or \( \psi \equiv 0 \).

We note that, actually, the upper bound on \( \|a\|_{L^{2p}} \) can be improved bypassing Theorem 2 and using directly the Sobolev inequality as explained in [23], [22]. We leave the corresponding statement to the interested reader.

### 3. Back to minimal hypersurfaces

Before proceeding in our analysis, let us observe how Theorem 1 follows from the vanishing part of Theorem 5. Actually, it improves the original version. This generalization of Shen-Zhou result has been recently observed also by H. Fu and Z. Li, [19]. However, their approach is less direct than ours and relies on a structure theorem by Anderson, [1]. Accordingly, they show that, in the above assumptions, the minimal submanifold has only one end; see also Section 4. Recall that the immersed, minimal submanifold is
said to be \( H \)-stable if the “modified stability operator” \( \mathcal{L}_H = -\Delta - H |\Pi|^2 \) satisfies \( \lambda_1 (\mathcal{L}_H) \geq 0 \).

**Corollary 7.** Let \( f : M^m \to \mathbb{R}^N \) be a complete, \( m \)-dimensional, minimal submanifold of finite total curvature \( |\Pi|^2 \in L^{m/2} (M), m \geq 3 \). If \( M \) is \( H \)-stable for some \( H > (m - 2) / m \) then \( f (M) \) is an affine \( m \)-plane.

**Proof.** By the Simons equation, \( \psi = |\Pi|^2 \) satisfies (3) with \( a (x) = |\Pi|^2 \in L^{m/2} \) and \( A = -(m + 2) / m \).

Anderson curvature estimates (see also [25]) state that \( |\Pi|^2 = o \left( \frac{1}{R^2} \right) \) as \( R \to +\infty \).

Since \( 0 < 1 + A = (m - 2) / m \), \( \psi \in L^{m+1} \), and \( \lambda_1 (\mathcal{L}_H) \geq 0 \), for some \( H > (m - 2) / m \) then, by Theorem 5 (i), either \( a = |\Pi|^2 \equiv 0 \) or \( \psi = |\Pi| \equiv 0 \). In any case, the submanifold is planar. \( \square \)

**Remark 8.** The geometric setting of minimal submanifolds allows us to test the sharpness of our abstract theorem. In the recent paper [33] it is shown that the \( m \)-dimensional catenoid is a (non-planar) minimal hypersurface of \( \mathbb{R}^{m+1} \) satisfying \( \lambda_1 (\mathcal{L}_H) \geq 0 \) with \( H = (m - 2) / m \). A direct computation also shows that its total curvature is finite. Noting that if \( a (x) \geq 0 \) and \( H_1 > H_2 \) then \( \lambda_1 (\mathcal{L}_{H_2}) \geq \lambda_1 (\mathcal{L}_{H_1}) \), it follows from the discussion in the previous proof, that the condition \( H > (1 + A) \) in Theorem 5 cannot be relaxed.

As a straightforward application of the non-vanishing part of Theorem 5 we get the following characterization of the higher dimensional catenoids.

**Corollary 9.** Let \( f : M^m \to \mathbb{R}^{m+1} \) be a complete, non-planar, minimal hypersurface of finite total curvature \( |\Pi|^2 \in L^{m/2} (M), m \geq 3 \). If \( M \) is \( \frac{m-2}{m} \)-stable then it is a catenoid.

**Proof.** According to a nice result in [33] it suffices to show that
\[
|\Pi| \left( \Delta |\Pi| + |\Pi|^2 |\Pi| \right) = \frac{2}{m} |\nabla |\Pi||^2.
\]
But this follows immediately from Theorem 5 (ii) applied with the choices \( a (x) = |\Pi|^2 \in L^{m/2}, A = -(m + 2) / m, H = A + 1 = (m - 2) / m \) and \( \psi = |\Pi| \).

\( \square \)

We note that the same conclusion can be deduced e.g. by [9] where the authors prove a structure theorem for \( \frac{m-2}{m} \)-stable minimal hypersurfaces with bounded second fundamental form. More precisely, they show that, in these assumptions, the hypersurface is a catenoid provided it has at least two ends. Since, by Anderson, non-planar minimal hypersurfaces with finite total curvature have bounded second fundamental form and more than one end, the asserted characterization follows. As the above proof shows, our argument is much more straightforward.
4. Analysis of the proofs of the geometric theorem

In order to prove Theorem 1, the arguments supplied by Schoen-Simon-Yau, do Carmo-Peng and Bérard on the one hand, and Anderson and Shen-Zhu on the other hand are very different. More precisely, both start from uniform decay estimates for $|\mathcal{II}|$ and use upper volume estimates in a crucial way. The decay estimates follow from Moser-iteration arguments once it is observed that $v = |\mathcal{II}|^2$ satisfies the Simons (semi-linear) inequality $\Delta v + 2v^2 \geq 0$. Thus $\sup_{M \setminus B_R} |\mathcal{II}|^2 = o (R^{-2})$. Granted this, the former approach is very analytic and abstract in nature. Using directly, and from the very beginning, the spectral assumption in combination with equation (1) and suitable test functions, it produces an estimate of the $L_m^2$-norm of $|\mathcal{II}|^2$ over large balls. In contrast, the second approach is very geometric and relies on smooth convergence of Riemannian manifolds. The spectral assumption appears only at the final step and in relation with the $L^1$-norm of $|\mathcal{II}|^2$. More precisely, Anderson deduces from (i) of Theorem 1 that $M$ is planar provided it has only one end and this is the case because of condition (ii). Indeed, the potential theoretic characterization of the ends by H.-D. Cao, Y. Shen and S. Zhu, [11], shows that all the ends of $M$ are non-parabolic so that, by harmonic function theory, [17], [18], condition (ii) of Theorem 1 forces $M$ to have only one end; see also [22]. Unlike Anderson, Shen-Zhu show that (i) of Theorem 1 forces a uniform decay of $|\mathcal{II}|^2$ that is faster than expected, i.e., $\sup_{M \setminus B_R} |\mathcal{II}|^2 = O (R^{-m})$. This is obtained via a theorem by Schoen on the curvature decay of minimal graphs of bounded slope. Now, as established by Anderson, $M$ has Euclidean volume growth. It follows that $|\mathcal{II}|^2 \in L^1(M)$ and the conclusion is reduced on the $L^2$-vanishing result by do Carmo-Peng, [14].

5. Proof of the abstract result

In a certain sense, the strategy by Shen-Zhu looks promising. According to Theorem 2, on noting also that Moser-type arguments work for general PDEs, what we need is a way to improve the uniform decay of the solution of (3). It is reasonable that this can be obtained thanks to the presence of the gradient term, that corresponds to a refined Kato inequality. But this kind of arguments are already known in an ambient that only apparently is far from the minimal surface theory. Namely, the theory of ALE ends of conformally flat, half-conformally or, more generally, Bach-flat 4-manifolds, see [7], and e.g. [8], [31], [32]. In fact, we have the following result by S. Bando, A. Kasue and H. Nakajima; see Section 4 in [7].

**Theorem 10.** Let $(M, \langle \cdot, \cdot \rangle)$ be a complete, $m(\geq 3)$-dimensional Riemannian manifold supporting the Euclidean Sobolev inequality

$$\|v\|_{L^2_{\frac{2m}{m-2}}} \leq S \|\nabla v\|_{L^2},$$

where $S = S(m)$. Then $M$ is of finite type.
for every \( v \in C^\infty_c (M) \) and for some constant \( S > 0 \). Assume also that
\[
\text{vol} (B_R) = O (R^m).
\]
Let \( 0 \leq u \in L^{2p}, \ p > 1/2 \), be a weak solution of the differential inequality
\[
\Delta u + a (x) u \geq 0.
\]
If \( a \in L^{m/2} \cap L^q \) for some \( q > m/2 \) and
\[
\int_{M \setminus B_R} a^q = O \left( \frac{1}{R^{2q-m}} \right),
\]
then
\[
\sup_{M \setminus B_R} u = O \left( \frac{1}{R^\alpha} \right),
\]
for every \( \alpha < m - 2 \).

Now, suppose we are in the assumptions of Theorem 5 so that \( \psi \) satisfies inequality \( (3) \) and \( \psi \in L^{2p} \) for some \( p > (1 + A)/2 \). As we have already recalled during the proof of Theorem 4 we have that \( \psi^{1+A} \in W^{1,2}_{loc}, \ [21], [22] \). Therefore, using the test function \( (\psi + \varepsilon)^{4-1} \rho, \ 0 \leq \rho \in \text{Lip}_c (M) \), in the weak formulation of \( (3) \) and letting \( \varepsilon \to 0 \) yields that \( u = \psi^{1+A} \) satisfies
\[
\Delta u + bu \geq 0,
\]
weakly on \( M \), where we have set \( b (x) = (1 + A) a (x) \geq 0 \); see page 209 in [22]. Since \( b \in L^{m/2} \) and
\[
\sup_{M \setminus B_R} b = O \left( \frac{1}{R^2} \right),
\]
we deduce that, for every \( q > m/2 \),
\[
\int_{M \setminus B_R} b^q \leq \sup_{M \setminus B_R} b^{q-m/2} \|b\|^{m/2}_{L^{m/2}}
\]
\[
= O \left( \frac{1}{R^{2q-m}} \right).
\]
Therefore, we can use Theorem 10 to deduce
\[
\sup_{M \setminus B_R} u = O \left( \frac{1}{R^\alpha} \right),
\]
for every \( \alpha < m - 2 \). This means that the original solution \( \psi \) of \( (3) \) satisfies
\[
\sup_{M \setminus B_R} \psi = O \left( \frac{1}{R^{\frac{m}{4}+A-\varepsilon}} \right),
\]
for every $0 < \varepsilon << 1$. Thus, using the co-area formula, the volume assumption, the fact that $H \geq A + 1$, and integrating by parts,

$$\int_{B_R \setminus B_1} \psi^{2H} = \int_1^R \int_{\partial B_t} \psi^{2H}$$

$$\leq C \int_1^R \frac{\text{Area}(\partial B_t)}{t^{2H(\frac{m-2}{2-H}) - \varepsilon}}$$

$$\leq C \int_1^R \frac{\text{Area}(\partial B_t)}{t^{2m-4-\varepsilon}}$$

$$\leq C_1 \left\{ \frac{\text{Vol}(B_R)}{R^{2m-4-\varepsilon}} + \int_1^R \frac{\text{Vol}(B_t)}{t^{2m-4-\varepsilon}} + 1 \right\}$$

$$\leq C_2 \left\{ R^{-m+4+\varepsilon} + \int_1^R t^{-m+3+\varepsilon} + 1 \right\}$$

$$\leq C_3 \left\{ R^{-m+4+\varepsilon} + 1 \right\}.$$ 

It follows that

$$\int_{B_R(\phi)} \psi^{2H} = o\left( R^2 \right), \text{ as } R \to +\infty,$$

for every $m \geq 3$. Now, if $H > (1 + A)$, since $\lambda_1(\mathcal{L}_H) \geq 0$, application of Theorem 2 (i) or Theorem 4 (i) yields that either $a \equiv 0$ or $\psi \equiv 0$. On the other hand, if $H = (1 + A)$ we can apply Theorem 2 (ii) or Theorem 4 (ii) to deduce that $\psi$ satisfies (3) with the equality sign.

6. Further applications

6.1. Topology of $H$-stable minimal hypersurfaces. As observed in the introduction, the “direct” approach has been proposed to face the problem of the triviality of minimal surfaces with finite total curvature, [4]. We also pointed out that this method is not suitable to obtain Theorem 1 for higher dimensions $m$ because an upper bound for the integrability exponent arises. Nevertheless, since it permits to consider the case $H < p$, it reveals useful to obtain topological informations on minimal surface immersed in non-negatively curved manifolds without integrability assumptions on $|\nabla|$, provided $M$ is $H$-stable. Namely, in the spirit of [28], we obtain the following

**Theorem 11.** Let $M^m$ be a complete non-compact minimally immersed hypersurface in a manifold of non-negative sectional curvature. Suppose that $M$ is $H$-stable, for some $H > (m-1)/m$. If $D \subset M$ is a compact domain in $M$ with smooth, simply connected boundary, then there is no non-trivial homomorphism of $\pi_1(D)$ into the fundamental group of a compact manifold with non-positive sectional curvature.

The proof relies on a suitable use of harmonic maps; see also Remark 6.22 in [22]. Consider a complete Riemannian manifold $M$ whose Ricci curvature
satisfies
\[ M \text{Ric} \geq -k(x), \]
for some continuous function \( k \geq 0 \) and let \( h : M \to N \) be a harmonic map with finite energy \( |dh| \in L^2 \). In \( N \) has non-positive curvature, then the energy density satisfies the Bochner inequality
\[ \frac{1}{2} \Delta |dh|^2 + k(x) |dh|^2 \geq |Ddh|^2. \]
Furthermore, the following refined Kato inequality holds true, \cite{5, 10},
\[ |Ddh|^2 \geq \frac{m}{m-1} |\nabla |dh||^2. \]
Therefore \( \psi = |dh| \) satisfies \cite{3} with \( A = -1/(m-1) \). Since every map \( f : M \to N \) with finite energy \( |df| \in L^2 \) is homotopic to a harmonic map \( h : M \to N \) of finite energy, applying to \( h \) the vanishing result of Theorem 4, we get

**Proposition 12.** Let \( f : M \to N \) be a continuous map from a complete, \( m \)-dimensional manifold \((M, \langle \cdot, \cdot \rangle)\) with Ricci curvature satisfying \( (16) \) into a compact manifold of non-positive sectional curvature. Assume that \( f \) has finite energy \( |df|^2 \in L^1(M) \). If
\[ \lambda_1 (-\Delta - Hk(x)) \geq 0, \]
for some
\[ H > (m-1)/m, \]
then \( f \) is homotopic to a constant.

Now, suppose that \( M \) is isometrically immersed as complete, \( H \)-stable, minimal hypersurface into a space \( Q \) with \( \text{Sect} Q \geq 0 \). According to Gauss equations, \( M \text{Ric} \geq -|\Pi|^2 \). Moreover, by assumption, the operator \( L_H = -\Delta - H|\Pi|^2 \) satisfies \( \lambda_1 (L_H) \geq 0 \). Hence, we are precisely in the assumptions of Proposition 12, thus obtaining that each harmonic map with finite energy from \( M \) into a non-positively curved \( N \) is homotopic to a constant. Starting from this point, we can conclude the proof of Theorem 11 proceeding as in \cite{28}. See also \cite{22}, Theorem 6.21.

6.2. **Locally conformally flat manifolds.** As expected, Theorem 5 works perfectly in the setting of conformally flat manifolds. Recall that the \( m \)-dimensional Riemannian manifold \((M^m, \langle \cdot, \cdot \rangle)\) is said to be locally conformally flat if every point of \( M \) has a neighborhood which is conformally immersed into the standard sphere \( S^m \). We have the following result which, in our opinion, fits very well in the nice works by G. Carron and M. Herzlich, \cite{8}, and, G. Tian and J. Viaclovsky, \cite{31, 32}.
Theorem 13. Let \((M, \langle , \rangle)\) be a complete, simply connected, scalar flat, locally conformally flat Riemannian manifold of dimension \(m \geq 3\). Assume that \(|\text{Ric}| \in L^{m/2}\) and that the Schrödinger operator

\[ \mathcal{L}_H = -\Delta - H \sqrt{\frac{m}{m-1}} |\text{Ric}| \]

satisfies \(\lambda_1(\mathcal{L}_H) \geq 0\) for some \(H > (m-2)/m\). Then:

(i) If \(H > (m-2)/m\) then \(M = \mathbb{R}^m\).

(ii) If \(H = (m-2)/m\) then, either \(M = \mathbb{R}^m\) or \(|\text{Ric}| > 0\) and, in a neighborhood \(U\) of each point \(x \in \{\text{DRic} \neq 0\} \neq \emptyset\), there is an isometric splitting \(U = (-\varepsilon, \varepsilon) \times f\mathbb{N}\) where \(\mathbb{N}\) has constant curvature \(K\) and \(f\) satisfies

\[ (m-2) f'^2 + 2ff'' - K(m-2) = 0. \]

Proof. Our basic reference for conformally flat manifolds is Chapter 6 in Schoen and Yau book [29]. See also Section 9 in [22] for the relevant PDEs involving the traceless Ricci tensor of a conformally flat manifold.

It is a classical result by N. Kuiper that the simply connected, locally conformally flat \(M^m\) has a conformal immersion into \(S^m\). Therefore, according to Schoen and Yau, the Yamabe constant of \(M\) satisfies \(Q(M) = Q(S^m) > 0\). Since \(M\) is scalar-flat, this means that \(M\) enjoys the Sobolev inequality

\[ \|v\|_{L^\frac{2m}{m-2}} \leq S \|\nabla v\|_{L^2}, \]

with

\[ S = \sqrt{\frac{4(m-1)}{Q(S^m)(m-2)}}. \]

Now, the norm of the Ricci tensor of \(M\) satisfies the Simons-type identity

\[ \frac{1}{2} \Delta |\text{Ric}|^2 = |\text{DRic}|^2 + \frac{m}{m-2} \text{tr} \left( \text{Ric}^{(3)} \right), \]

where \(\text{Ric}^{(3)}\) denotes the third composition power of the Ricci tensor. Moreover, since \(\text{Ric}\) is scalar flat we have the classical Okumura inequality

\[ \text{tr} \left( \text{Ric}^{(3)} \right) \geq -\frac{m-2}{\sqrt{m(m-1)}} |\text{Ric}|^3, \]

the equality holding at some point \(x\) if and only if \((m-1)\) eigenvalues of \(\text{Ric}(x)\) coincide, [2]. Finally, since \(\text{Ric}\) is a Codazzi tensor the refined Kato inequality

\[ |\text{DRic}|^2 \geq \frac{m+2}{m} |\nabla |\text{Ric}||^2, \]

holds. Summarizing, we have

\[ |\text{Ric}| \left( \Delta |\text{Ric}| + \sqrt{\frac{m}{m-1}} |\text{Ric}|^2 \right) \geq \frac{2}{m} |\nabla |\text{Ric}||^2, \]

(18)
weakly on $M$. In particular, $u = |\text{Ric}|$ satisfies the semilinear elliptic inequality
\[ \Delta u + \sqrt{m/(m-1)}u^2 \geq 0. \]
When combined with the Sobolev inequality this yields that $|\text{Ric}| = o\left(R^{-2}\right)$ as $R \to +\infty$ (see, e.g. [8], Lemma 5.2). Since $M$ is locally conformally flat and scalar flat, by the decomposition of the Riemann tensor we also have $|\text{Riem}| = o\left(R^{-2}\right)$. It follows from the volume growth estimates by Tian-Viaclovsky that $\text{vol}(B_R) = O\left(R^m\right)$. Therefore, if $H > (m-2)/m$, we can apply Theorem 5 (i) with the choices $\psi = |\text{Ric}|$, $a(x) = \sqrt{m/(m-1)}|\text{Ric}|$, $A = -2/m$ to conclude that either $a \equiv 0$ or $\psi \equiv 0$. In any case $\text{Ric} = 0$ which, in turn, forces $\text{Riem} = 0$. The desired conclusion $M = \mathbb{R}^m$ then follows from the Hopf classification theorem. On the other hand, suppose $H = (m-2)/m$. Assume also that $M \neq \mathbb{R}^m$, for otherwise there is nothing to prove. Application of Theorem 4 (ii) yields the equality in (18), i.e.,
\[ |\text{Ric}| \left(\Delta |\text{Ric}| + \sqrt{m/m-1}|\text{Ric}|^2\right) = \frac{2}{m} |\nabla |\text{Ric}||^2. \]
From the equality case in Okumura inequality we obtain that $\text{Ric}$ has two (possibly equal) eigenvalues $\mu$ and $-\mu/(m-1)$ of multiplicity 1 and $(m-1)$, respectively. We claim that they are everywhere distinct. Indeed, using (19) and arguing as in Section 5 we see that $v = |\text{Ric}|^{(m-2)/m}$ is a weak solution of
\[ \Delta v + b(x)v = 0, \]
where $b(x) = (1-2/m)\sqrt{m/(m-1)}|\text{Ric}|$. It follows from the strong minimum principle that either $v > 0$ or $v \equiv 0$. The second possibility cannot occur because, in this case, $\text{Riem} = 0$ and $M = \mathbb{R}^m$, against our assumption. Thus $|\text{Ric}| > 0$ and this forces $\mu \neq -\mu/(m-1)$, as claimed.

Now, we observe that $\text{Ric}$ is not parallel. Indeed, suppose the contrary. Since $|\text{Ric}|$ is constant, $|\text{Ric}| \in L^{m/2}$ and, due to the Sobolev inequality (17), $\text{vol}(M) = +\infty$, then $\text{Ric} = 0$. As above, this implies $M = \mathbb{R}^m$, contradicting our initial assumption. Therefore, $\text{Ric}$ is not parallel. Since $\text{Ric}$ is a Codazzi tensor with constant (zero) trace and two distinct eigenvalues, we can apply a result by A. Derdzinski, [12], and conclude that every point $x \in \{D\text{Ric} \neq 0\}$ has a neighborhood of the form $U = (-\varepsilon, \varepsilon) \times f\Sigma$, with $f$ nonconstant. Since $U$ is locally conformally flat then the $(m-1)$-dimensional manifold $\Sigma$ must be of constant curvature, [9], say $\Sigma\text{Sec} = K$. Computing the scalar curvature of the warped product $U$, and recalling that $M$ is scalar flat, we get
\[ (m-2) (f')^2 + 2ff'' - K(m-2) = 0, \]
thus completing the proof. \qed

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References

[1] M. Anderson, The compactification of a minimal submanifold by its Gauss map. Available at http://www.math.sunysb.edu/~anderson/papers.html

[2] H. Alencar, M. do Carmo, Hypersurfaces with constant mean curvature in spheres. Proc. Amer. Math. Soc. 120 (1994), 1223–1229.

[3] P. Bérard, A note on Bochner type theorems for complete manifolds. Manuscripta Math. 69 (1990), no. 3, 261–266.

[4] P. Bérard, Remarques sur l’équations de J. Simons, Differential Geometry (A symposium in honor of M. do Carmo on his 60th birthday), Pitman Monographs Surveys Pure Appl. Math, 52 Longman Sci. Tech. Harlow, 1991, 47–57.

[5] T. Branson, Kato constants in Riemannian geometry. Math. Res. Lett. 7 (2000), no. 2-3, 245–261.

[6] M. Brozos-Vázquez, E. García-Río, R. Vázquez-Lorenzo, Some remarks on locally conformally flat static space-times. J. Math. Phys. 46 (2005), 11 pp.

[7] S. Bando, A. Kasue, H. Nakajima, On a construction of coordinates at infinity on manifolds with fast curvature decay and maximal volume growth. Invent. Math. 97 (1989), 313–349.

[8] G. Carron, M. Herzlich, The Huber theorem for non-compact conformally flat manifolds. Comment. Math. Helv. 77 (2002), 192–220, and 82 (2007), 451–453.

[9] X. Cheng, D. Zhou, Manifolds with weighted Poincaré inequality and uniqueness of minimal hypersurfaces. Comm. Anal. Geom. 17 (2009), 3–15.

[10] D.M.J. Calderbank, P. Gauduchon, M. Herzlich, Refined Kato inequalities and conformal weights in Riemannian geometry. J. Funct. Anal. 173 (2000), no. 1, 214–255.

[11] H.-D. Cao, Y. Shen, S. Zhu, The structure of stable minimal hypersurfaces in $\mathbb{R}^{m+1}$. Math. Res. Lett. 4 (1997), no. 5, 637–644.

[12] A. Derdzinski, Some remarks on the local structure of Codazzi tensors. Lecture Notes in Mathematics, vol. 838 (1981), 251-255.

[13] M.P. do Carmo, C.K. Peng, Stable complete minimal surfaces in $\mathbb{R}^3$ are planes. Bull. Amer. Math. Soc. (N.S.) 1 (1979), 903–906.

[14] M.P. do Carmo, C.K. Peng, Stable complete minimal hypersurfaces. Proceedings of the 1980 Beijing Symposium on Differential Geometry and Differential Equations, Vol. 1, 2, 3 (Beijing, 1980), 1349–1358, Science Press, Beijing, 1982.

[15] K.D. Elworthy, S. Rosenberg, Generalized Bochner theorems and the spectrum of complete manifolds. Acta Appl. Math. 12 (1988), 1–33.

[16] D. Fischer-Colbrie, R. Schoen, The structure of complete stable minimal surfaces in 3-manifolds of non-negative scalar curvature, Comm. Pure Appl. Math. XXXIII (1980), 199–211.

[17] P. Li, L.-F. Tam, Harmonic functions and the structure of complete manifolds. J. Differential Geom. 35 (1992), 359–383.

[18] P. Li, J. Wang, Minimal hypersurfaces of finite index, Math. Res. Let. 9 (2002), 95–103.

[19] H. Fu, Z. Li, The structure of complete manifolds with weighted Poincare’ inequalities and minimal hypersurfaces. International J. Math. 21 (2010), 1–8.

[20] S. Pigola, M. Rigoli, A.G. Setti, Vanishing theorems on Riemannian manifolds and geometric applications. J. Funct. Anal. 229 (2005), 424–461.

[21] S. Pigola, M. Rigoli, A.G. Setti, A finiteness theorem for the space of $L^p$ harmonic sections. Rev. Mat. Iberoam. 24 (2008), no. 1, 91–116.

[22] S. Pigola, M. Rigoli, A.G. Setti, Vanishing and finiteness results in geometric analysis: a generalization of the Bochner technique. Progress in Mathematics 266 (2008), Birkhäuser.

[23] S. Pigola, M. Rigoli, A.G. Setti, Some characterizations of space-forms Trans. Amer. Math. Soc. 359 (2007), 1817-1828.
[24] S. Pigola, G. Veronelli, *On the homotopy class of maps with finite p-energy into non-positively curved manifolds*. Geom. Dedicata 143 (2009), 109–116.

[25] S. Pigola, G. Veronelli, *Uniform decay estimates for finite energy solutions of semi-linear elliptic inequalities and geometric applications*. Diff. Geom. Appl. 29 Issue 1 (2011), 35–54.

[26] M.H. Protter, H.F. Weinberger, *Maximum principles in differential equations*. Springer-Verlag, New York, 1984.

[27] R. Schoen, R.; L. Simon, L.; S.T. Yau, *Curvature estimates for minimal hypersurfaces*. Acta Math. 134 (1975), 275–288.

[28] R. Schoen and S.T. Yau, *Harmonic Maps and the Topology of Stable Hypersurfaces and Manifolds with Non-negative Ricci Curvature*. Comm. Math. Helv. 51 (1976), 333-341.

[29] R. Schoen, S.T. Yau, *Lectures on differential geometry*. Conference Proceedings and Lecture Notes in Geometry and Topology, I. International Press, Cambridge, MA, 1994.

[30] Y.-B. Shen, X.-H. Zhu, *On stable complete minimal hypersurfaces in $\mathbb{R}^{m+1}$*. Amer. J. Math. 120 (1998), 103–116.

[31] G. Tian, J. Viaclovsky, *Bach-flat asymptotically locally Euclidean metrics*. Invent. Math. 160, 357-415.

[32] G. Tian, J. Viaclovsky, *Volume growth, curvature decay, and critical metrics*. Comment. Math. Helv. 83 (2008), 889–911.

[33] L.-F. Tam, D. Zhou, *Stability properties for the higher dimensional catenoid in $\mathbb{R}^{m+1}$*. Proc. Amer. Math. Soc. 137 (2009), 3451–3461.

[34] G. Veronelli, *Uniform decay estimates for solutions of the Yamabe type equation*. Geom. Dedicata (to appear).

[35] S. Xu, Q. Deng, *On complete noncompact submanifolds with constant mean curvature and finite total curvature in Euclidean spaces*. Arch. Math. 87 (2006), 60–71.

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