Calculating the image of the second Johnson-Morita representation

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This paper is dedicated, with respect, to Shigeyuki Morita.

Abstract

Johnson has defined a surjective homomorphism from the Torelli subgroup of the mapping class group of the surface of genus $g$ with one boundary component to $\wedge^3 H$, the third exterior product of the homology of the surface. Morita then extended Johnson’s homomorphism to a homomorphism from the entire mapping class group to $1/2 \wedge^3 H \rtimes \text{Sp}(H)$. This Johnson-Morita homomorphism is not surjective, but its image is finite index in $1/2 \wedge^3 H \rtimes \text{Sp}(H)$ [11]. Here we give a description of the exact image of Morita’s homomorphism. Further, we compute the image of the handlebody subgroup of the mapping class group under the same map.

1 Introduction

Let $S_g$ be a closed surface of genus $g$. We fix a closed disk $D$ in $S_g$, and by deleting its interior, obtain $S_{g,1}$, a genus $g$ surface with one boundary component, as illustrated in Figure 1. Let $\mathcal{M}_g$ (resp. $\mathcal{M}_{g,1}$) denote the mapping class group of the surface $S_g$ (resp. $S_{g,1}$). In the case of $\mathcal{M}_{g,1}$ we assume the boundary component is fixed pointwise.

![Figure 1: (a) A basis for $H_1(S_{g,1})$ (b) Generators for $\pi_1(S_{g,1})$](image)

We choose a base point on $\partial S_{g,1}$, and let $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$ denote the based loops illustrated in Figure 1(b). Let $a_1, \ldots, a_g, b_1, \ldots, b_g$ denote the corresponding homology classes, as in Figure 1(a). It will sometimes be convenient to denote these same homology classes by $x_1, \ldots, x_{2g}$ with the understanding that $x_i = a_i$ and $x_{i+g} = b_i$ for $1 \leq i \leq g$. Likewise, we will sometimes refer to the based loops $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$ by $\xi_1, \ldots, \xi_{2g}$ with the understanding that $\xi_i = \alpha_i$ and $\xi_{i+g} = \beta_i$ for $1 \leq i \leq g$.

Now, let $H = H_1(S_{g,1})$ be the free abelian group with generating set $\{a_1, \ldots, a_g, b_1, \ldots, b_g\}$ and $\pi = \pi_1(S_{g,1})$ which is a free group on the generating set $\{\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g\}$. The action of $\mathcal{M}_{g,1}$ on $\pi$ gives an injection $\mathcal{M}_{g,1} \hookrightarrow \text{Aut}(\pi)$. More generally, we can compose with the homomorphism $\text{Morita}(\mathcal{M}_{g,1}) : \mathcal{M}_{g,1} \to \text{Sp}(H)$.
Aut(π) → Aut(π/χ) for any characteristic subgroup χ ⊂ π. The lower central series of the free group π is a sequence of characteristic subgroups defined inductively by setting π(0) = π and π(k+1) = [π, π(k)]. We define the k-th Johnson-Morita representation to be the map

ρ_k : M_{g,1} → Aut(π/π(k))

We note that these maps were first studied by Johnson in [7, 6] and subsequently developed by Morita in a series of papers [11, 12, 13, 14].

Observe that the first Johnson-Morita map is just the classical symplectic representation ρ_1 : M_{g,1} → Sp(H) which is surjective ([4], in particular pp. 209-212). In [11] Theorem 4.8 Morita shows that the image of ρ_2 is isomorphic to a subgroup of finite index in \( \frac{1}{2} \wedge^3 H \rtimes Sp(H) \). Our first main result in this paper, given in Theorem 2.4, is to identify the precise image ρ_2(M_{g,1}) using a formulation due to Perron [16].

Let us now consider S_g as ∂X_g, where X_g is a genus g handlebody. Let H_g denote the handlebody subgroup of M_{g,1}, that is, the subgroup consisting of maps of S_g which extend to the handlebody X_g. There is a natural surjection M_{g,1} → M_g obtained by extending via the identity map along D. The kernel of this surjection is generated by two kinds of elements: the Dehn twist along the boundary curve, and “push” maps along elements of π_1(S_g,1) [1]. Note that any map in this kernel extends to X_g. Hence, we are justified in defining the handlebody subgroup H_{g,1} of M_{g,1} as the pullback of H_g.

The handlebody group arises naturally in a number of applications in 3-manifold topology, particularly through Heegaard splittings of 3-manifolds. Our second result in this paper is to compute ρ_2(H_{g,1}), given in Theorem 3.5.

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2 The second Johnson-Morita map

In this section we will describe Perron’s formulation [16] of the second Johnson-Morita representation. We will give a precise characterization of the image of the mapping class group under this map. First, it will be useful to review the image of the first Johnson-Morita representation, i.e., the symplectic group.

2.1 The symplectic group

The group H = H_1(S_{g,1}) is free abelian with free basis a_1,...,a_g, b_1,...,b_g, as in Figure 1(a), and has a symplectic intersection form given by signed intersection of curves which is preserved by every mapping class f ∈ M_{g,1}. In the basis above, the intersection form is given by the the matrix J with g × g block form

\[ J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \]

(1)

The intersection form got by acting by the linear transformation M on an intersection form with matrix L is given by \( ML\overline{M} \) where \( \overline{M} \) denotes the transpose of M. Hence for every M in the image of the mapping class group

\[ MJ\overline{M} = J, \quad \text{or equivalently} \quad \overline{M}JM = J \] (2)

In fact (2) is a sufficient condition for M to be in the image of the mapping class group under ρ_1. It is sometimes useful to write a symplectic matrix M in g × g block form as

\[ M = \begin{pmatrix} S & T \\ P & Q \end{pmatrix} \]

A convenient consequence of (2) is that \( M^{-1} = J\overline{M}J^{-1} \). In block form this becomes

\[ \begin{pmatrix} S & T \\ P & Q \end{pmatrix}^{-1} = \begin{pmatrix} \overline{Q} & -\overline{T} \\ -\overline{P} & \overline{S} \end{pmatrix} \]
The group of such matrices form the symplectic group. Writing $M$ and $\overline{M}$ in $g \times g$ block form

$$M = \begin{pmatrix} S & T \\ P & Q \end{pmatrix}, \quad \overline{M} = \begin{pmatrix} S & \overline{T} \\ \overline{P} & \overline{Q} \end{pmatrix}$$

we derive the symplectic constraints, which follow directly from the condition in (2):

\[(i) \quad Q\overline{S} - P\overline{T} = I, \quad (ii) \quad S\overline{T} \text{ symmetric}, \quad (iii) \quad P\overline{Q} \text{ symmetric}.\]

(3)

### 2.2 Perron’s formulation of $\rho_2$

The Torelli group $\mathcal{I}_{g,1}$ is the kernel of the symplectic representation $\rho_1 : \mathcal{M}_{g,1} \rightarrow \text{Sp}(H)$. Johnson proved, in [5], that the image of the Torelli group under $\rho_2$ is $\wedge^3 H$. In the next section we will go a step further, and describe, in Theorem 2.4, the image of the full mapping class group $\rho_2$ noting that Morita [11, Theorem 4.8] has already identified this image as being finite index in $\frac{1}{2} \wedge^3 H \simeq \text{Sp}(H)$. We begin by summarizing Morita’s explicit description of $\rho_2$ as given in [11, Section 4]. Consider the 2-step nilpotent group

$$\Phi_2 = \left\{ (\eta, y) \big| \eta \in \frac{1}{2} \wedge^2 H, \; y \in H \right\}$$

with multiplication in $\Phi_2$ given by $(\eta, y)(\nu, z) = (\eta + \nu + \frac{1}{2} y \wedge z, y + z)$. It contains a subgroup of finite index which can be identified (see [8, Sec. 5.5]) with the second nilpotent quotient $\pi/\pi^{(2)} = \pi/[[\pi, [\pi, \pi]]]$ of our surface group via the homomorphism $\phi_2 : \pi \rightarrow \Phi_2$

$$\phi_2(\xi_i) = (0, x_i)$$

where $\{\xi_1, \cdots, \xi_{2g}\}$ generate $\pi = \pi_1(S_{g,1})$ and $\{x_1, \cdots, x_{2g}\}$ is our basis for $H = H_1(S_{g,1})$ (see Figure [11, a-b]). The group $\Phi_2$ can be viewed as a subgroup of the Mal’cev completion of the nilpotent group $\pi/\pi^{(2)}$. Any automorphism of $\pi/\pi^{(2)}$ extends to the Mal’cev completion and preserves $\Phi_2$ so we may think of $\mathcal{M}_{g,1}$ as acting on $\Phi_2$ [11, Proposition 2.5].

In [11, Section 3] Morita describes a function $\mathcal{M}_{g,1} \rightarrow \text{Hom}(H, \frac{1}{2} \wedge^2 H)$. An automorphism $f$ of $\Phi_2$ coming from an automorphism of the Mal’cev completion of $\pi/\pi^{(2)}$ can be specified by the images

$$f(0, x_i) = (w_i, h_i) \quad w_i \in \frac{1}{2} \wedge^2 H, \; h_i \in H$$

for each $x_i$. The homomorphism $\rho_1(f) : H \rightarrow H$ given by $\rho_1(f)(x_i) = h_i$ is just the image of $f$ under the symplectic representation. Johnson looks at the homomorphism $\tilde{\tau}_2(f) : H \rightarrow \frac{1}{2} \wedge^2 H$ given by

$$\tilde{\tau}_2(f)(x_i) = w_i$$

The function $\tilde{\tau}_2 : \mathcal{M}_{g,1} \rightarrow \text{Hom}(H, \frac{1}{2} \wedge^2 H)$ is a homomorphism when restricted to the kernel $\mathcal{I}_{g,1}$ of the symplectic representation. Johnson [5, Theorem 1] identifies its image as $\wedge^3 H \subset \text{Hom}(H, \frac{1}{2} \wedge^2 H)$, where $x_i \wedge x_j \wedge x_k \in \wedge^3 H$ is understood to be the homomorphism

$$(x_i \wedge x_j \wedge x_k)(y) = \langle y, x_k \rangle x_i \wedge x_j + \langle y, x_i \rangle x_j \wedge x_k + \langle y, x_j \rangle x_k \wedge x_i$$

(4)

where $\langle , \rangle$ gives the symplectic pairing for vectors in $H$. The map $\mathcal{I}_{g,1} \rightarrow \wedge^3 H \subset \text{Hom}(H, \frac{1}{2} \wedge^2 H)$ is usually referred to as the Johnson homomorphism.

Morita [11, Section 3] begins by considering this map $\tilde{\tau}_2 : \mathcal{M}_{g,1} \rightarrow \text{Hom}(H, \frac{1}{2} \wedge^2 H)$ (in Morita’s notation this is the map $k$). While not a homomorphism it is a crossed homomorphism with respect to the symplectic action of the mapping class group on $\text{Hom}(H, \frac{1}{2} \wedge^2 H)$. In other words, the map $\tilde{\tau}_2$ satisfies:

$$\tilde{\tau}_2(fg) = \tilde{\tau}_2(f) + \rho_1(f)\tilde{\tau}_2(g) \quad f, g \in \mathcal{M}_{g,1}$$

3
Choose $R \in \text{Sp}(H)$, $y \in H$, and $m \in \text{Hom}(H, \frac{1}{2} \wedge^2 H)$. We note that the action of $\text{Sp}(H)$ on $\text{Hom}(H, \frac{1}{2} \wedge^2 H)$ in the equation above (and in the remainder of this paper) is the natural “change-of-basis” action:

$$(Rm)(y) = Rm(R^{-1}y)$$

The crossed homomorphism property is exactly what is needed for the map $\tilde{\rho}_2 : \mathcal{M}_{g,1} \to \text{Hom}(H, \frac{1}{2} \wedge^2 H) \times \text{Sp}(H)$ given by

$$\tilde{\rho}_2(f) = (\tilde{\tau}_2(f), \rho_1(f))$$

to be a homomorphism. The homomorphism $\tilde{\rho}_2$ gives the action of $\mathcal{M}_{g,1}$ on $\phi_2(\pi) \subset \Phi_2$, via the action of $(r, R) \in \text{Hom}(H, \frac{1}{2} \wedge^2 H) \times \text{Sp}(H)$ on $\Phi_2$:

$$(r, R) \ast (\eta, y) = (r(Ry) + R\eta, Ry)$$

Morita shows that by modifying the crossed homomorphism $\tilde{\tau}_2 : \mathcal{M}_{g,1} \to \text{Hom}(H, \frac{1}{2} \wedge^2 H)$, one obtains a crossed homomorphism $\tilde{\tau}_2'$ (Morita denotes this map by $\tilde{k}'$ in [11] Section 4) from $\mathcal{M}_{g,1}$ to the submodule $\frac{1}{2} \wedge^3 H$ of $\text{Hom}(H, \frac{1}{2} \wedge^2 H)$ which extends the Johnson homomorphism. We will modify $\tilde{\tau}_2$ to get a different crossed homomorphism $\tau_2 : \mathcal{M}_{g,1} \to \frac{1}{2} \wedge^3 H$ extending the Johnson homomorphism. Our map $\tau_2$ is a trivial modification of Morita’s map $\tilde{\tau}_2'$ which will lend itself to later calculations.

For any $m \in \text{Hom}(H, \frac{1}{2} \wedge^2 H)$, the map $\sigma_m : \mathcal{M}_{g,1} \to \text{Hom}(H, \frac{1}{2} \wedge^2 H)$ given by

$$\sigma_m(f) = m - \rho_1(f)m$$

is a crossed homomorphism. Such a crossed homomorphism is called principal; two crossed homomorphisms are cohomologous if they differ by a principal crossed homomorphism [3 Chapter IV.2].

Let $\kappa \in \text{Hom}(H, \frac{1}{2} \wedge^2 H)$ be the homomorphism

$$\kappa(a_i) = \frac{1}{2} a_i \wedge b_i \quad \kappa(b_i) = -\frac{1}{2} a_i \wedge b_i$$

or equivalently

$$\kappa(x_i) = \frac{1}{2} x_i \wedge Cx_i$$

where $C$ is the $2g \times 2g$ matrix with $g \times g$ block form $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$. Define

$$\tau_2(f) = \tilde{\tau}_2(f) + \kappa - \rho_1(f)\kappa$$

This is the crossed homomorphism that Perron [16 Remark 5.5] denotes $-\frac{1}{8} \tilde{A}_1$. We note that by comparing the above with [11 Proposition 4.7], it is straightforward to see that Morita’s crossed homomorphism $\tilde{\tau}_2'$ can be expressed as

$$\tilde{\tau}_2'(f) = \tau_2(f) + m - \rho_1(f)m$$

where $m = -\frac{1}{2}(\sum_{i=1}^{g} a_i + b_i) \wedge (\sum_{i=1}^{g} a_i \wedge b_i)$. In other words, the map $\tau_2$ and Morita’s original map $\tilde{\tau}_2'$ are cohomologous, that is, they represent the same element of $H^1(\mathcal{M}_{g,1}, \frac{1}{2} \wedge^3 H)$.

We can now define a homomorphism $\rho_2 : \mathcal{M}_{g,1} \to \frac{1}{2} \wedge^3 H \times \text{Sp}(H)$ as follows:

$$\rho_2(f) = (\tau_2(f), \rho_1(f))$$

Using (9), (10), (11), and (12), we obtain the correct action of $\rho_2(\mathcal{M}_{g,1})$ on $\Phi_2$:

$$\left( \sum r_{ijk}x_i \wedge x_j \wedge x_k, R \right) \ast (\eta, y) = \left( R\eta - \kappa(Ry) + R(\kappa(y)) + r(y), Ry \right)$$

where $\langle , \rangle$ is the symplectic pairing on $H$ and the sums are taken over $1 \leq i < j < k \leq 2g$. 

2.3 Calculating the image of the mapping class group

In this section we compute \(\rho_2(M_{g,1})\). See Theorem 2.4 below.

Recall the map \(\phi_2 : \pi \to \Phi_2\) given in the previous section. It will be helpful for us to identify \(\phi_2(\pi) \subset \Phi_2\) precisely. The gist of the following lemma is that for pairs in the image of \(\phi_2\), the second coordinate determines the first coordinate modulo 1.

**Lemma 2.1.** The image of \(\pi\) under the map \(\phi_2\) is given as follows.

\[
\phi_2(\pi) = \left\{ \left( \sum_{1 < i < j < 2g} \left( n_{ij} + l_i l_j \right) x_i \wedge x_j, \sum_{i=1}^{2g} l_i x_i \right) \mid n_{ij}, l_i \in \mathbb{Z} \right\}
\]

**Proof.** Let \(G \subset \Phi_2\) denote the set on the right-hand side of the equation in the lemma. We claim that the set \(G\) is a subgroup of \(\Phi_2\). First, \(G\) is closed under inversion since \((\eta, y)^{-1} = (-\eta, -y)\). For closure under products consider

\[
\left( \sum_{1 < i < j < 2g} \left( n_{ij} + l_i l_j \right) x_i \wedge x_j, \sum_{i=1}^{2g} l_i x_i \right) \cdot \left( \sum_{1 < i < j < 2g} \left( n'_{ij} + l'_i l'_j \right) x_i \wedge x_j, \sum_{i=1}^{2g} l'_i x_i \right) = \left( \sum_{1 < i < j < 2g} \left( n_{ij} + n'_{ij}, l_i l_j + l'_i l'_j \right) x_i \wedge x_j, \sum_{i=1}^{2g} (l_i + l'_i) x_i \right)
\]

This product is in \(G\) because \(l_i l_j + l'_i l'_j + l_i l'_j - l_j l'_i \equiv (l_i + l'_i)(l_j + l'_j) \mod 2\).

Clearly, \(G\) contains each generator \(\phi_2(\xi_i) = (0, x_i)\) of \(\phi_2(\pi)\). For the reverse inclusion, note that any element of the form

\[
(0, x_i)(0, x_j)(0, -x_i)(0, -x_j) = (x_i \wedge x_j, 0)
\]

lies in \(\phi_2(\pi)\). In fact such an element is in the center of \(G\). Now, any element of \(G\) can be written as a product of \((0, x_i)\)’s to get the correct second coordinate, followed by a product of \((x_i \wedge x_j, 0)\)’s to get the correct first coordinate. Hence \(G \subset \phi_2(\pi)\).

We are almost ready to characterize the subgroup \(\rho_2(M_{g,1}) \subset \frac{1}{2} \wedge^3 H \rtimes \text{Sp}(H)\). We begin with a simple yet fundamental observation.

**Remark 2.2.** Suppose \(R\) is a symplectic matrix and \((r_1, R), (r_2, R) \in \rho_2(M_{g,1})\). Then \((r_1, R)^{-1} = (-R^{-1} r_1, R^{-1}) \in \rho_2(M_{g,1})\) so

\[
(r_2, R)(-R^{-1} r_1, R^{-1}) = (r_2 - r_1, I) \in \rho_2(M_{g,1}).
\]

In other words, we have that \((r_2 - r_1, I) \in \rho_2(I_{g,1})\). Using Johnson’s characterization of \(\tau_2(I_{g,1})\) [3] Theorem 1] we conclude that if two elements of \(\rho_2(M_{g,1})\) have identical symplectic matrices, then their \(\frac{1}{2} \wedge^3 H\) coordinate must differ by an integral element of \(\wedge^3 H\).

As a consequence of this observation, we expect that the symplectic matrix \(R\) will determine the coefficients of \(r_1\) and \(r_2\) modulo 1. Theorem 2.4 makes this precise and gives the characterization of \(\rho_2(M_{g,1})\). First we give a short definition.

**Definition 2.3.** Given three \(n\)-dimensional vectors \(\vec{w} = (w_1, \ldots, w_n), \vec{y} = (y_1, \ldots, y_n), \vec{z} = (z_1, \ldots, z_n)\) in basis \(B\), their \(B\)-triple dot product is the scalar

\[
\bullet_B(\vec{w}, \vec{y}, \vec{z}) = \sum_{i=1}^{n} w_i y_i z_i.
\]

When the basis \(B\) is clear, we will write \(\bullet(\vec{w}, \vec{y}, \vec{z})\).
Recall that $J$ is the matrix given in (11).

Theorem 2.4. Let $R \in \text{Sp}(2g, \mathbb{Z})$ be an arbitrary symplectic matrix. Let $r$ be any element of $\frac{1}{2} \wedge^3 H$ with $r = \sum_{1 \leq i < j < k \leq 2g} r_{ijk} x_i \wedge x_j \wedge x_k$. Then $(r, R) \in \rho_2(M_{g,1})$ if and only if

$$r_{ijk} \equiv \frac{E_{ijk}}{2} \mod 1$$

where

$$E_{ijk} = \cdot (\text{row}_i(R), \text{row}_j(R), \text{row}_k(R)) - \cdot (\text{row}_i(R), \text{row}_j(R), \text{row}_k(R)) + \cdot (\text{row}_i(R), \text{row}_j(R), \text{row}_k(R))$$

for all $1 \leq i < j < k \leq 2g$.

Proof. Let $(r, R) \in \rho_2(M_{g,1})$, and let

$$r = \sum_{1 \leq i < j < k \leq 2g} r_{ijk} x_i \wedge x_j \wedge x_k.$$ 

For $1 \leq i, j, k \leq 2g$ we set $r_{ijk} = 0$ unless $i < j < k$. The group $\rho_2(M_{g,1})$ preserves $\phi_2(\pi)$, described in Lemma 2.1. Let $x_n$ be an arbitrary basis element of $H$, and consider the action of $(r, R)$ on $(0, x_n)$. We will use the standard notation $M_{ij}$ to denote the entry in the $i$th row and $j$th column of a matrix $M$ throughout. By (10), we get that the second coordinate of $(r, R) \ast (0, x_n)$ is simply $Rx_n$, which we can write as $\sum_{i=1}^{2g} R_{in} x_i$, with an eye on eventually applying Lemma 2.1. Using (10) and (7), we obtain the following for the first coordinate of $(r, R) \ast (0, x_n)$:

$$-\kappa(Rx_n) + R(\kappa(x_n)) + \sum_{1 \leq i < j < k \leq 2g} r_{ijk} \begin{pmatrix} \langle Rx_n, x_k \rangle x_i \wedge x_j + \langle Rx_n, x_i \rangle x_j \wedge x_k & -\langle Rx_n, x_j \rangle x_i \wedge x_k \\ \langle Rx_n, x_i \rangle x_j \wedge x_k & +\langle Rx_n, x_j \rangle x_i \wedge x_k \end{pmatrix}$$

Notice that under the symplectic pairing $\langle Rx_n, x_k \rangle = (JR)_{kn}$ so the above can be rewritten:

$$-\kappa \left( \sum_{i=1}^{2g} R_{in} x_i \right) + R \left( \frac{1}{2} x_n \wedge Cx_n \right) + \sum_{1 \leq i < j < k \leq 2g} r_{ijk} \begin{pmatrix} ((JR)_{kn}) x_i \wedge x_j + ((JR)_{in}) x_j \wedge x_k & -((JR)_{jn}) x_i \wedge x_k \\ ((JR)_{in}) x_j \wedge x_k & +((JR)_{jn}) x_i \wedge x_k \end{pmatrix}$$

$$= -\sum_{i=1}^{2g} \frac{R_{in} x_i \wedge Cx_i}{2} + \left( \sum_{1 \leq i, j \leq 2g} \frac{R_{in}(RC)_{jn} x_i \wedge x_j}{2} \right) + \sum_{1 \leq i < j < k \leq 2g} r_{ijk} \begin{pmatrix} ((JR)_{kn}) x_i \wedge x_j + ((JR)_{in}) x_j \wedge x_k & -((JR)_{jn}) x_i \wedge x_k \\ ((JR)_{in}) x_j \wedge x_k & +((JR)_{jn}) x_i \wedge x_k \end{pmatrix}$$

$$= \sum_{i=1}^{g} \frac{(CR)_{in} - R_{in}}{2} x_i \wedge x_{i+g}$$

$$+ \left( \sum_{1 \leq i < j \leq 2g} \frac{R_{in}(RC)_{jn} - R_{jn}(RC)_{in} x_i \wedge x_j}{2} \right) + \sum_{1 \leq i < j < k \leq 2g} r_{ijk} \begin{pmatrix} ((JR)_{kn}) x_i \wedge x_j + ((JR)_{in}) x_j \wedge x_k & -((JR)_{jn}) x_i \wedge x_k \\ ((JR)_{in}) x_j \wedge x_k & +((JR)_{jn}) x_i \wedge x_k \end{pmatrix}$$
Now, applying Lemma 2.1 to the coefficient of \( x_p \wedge x_q \), where \( p < q \), gives

\[
\frac{\delta_{q,p+g}((CR)_{pn} - R_{pn}) + R_{pn}(RC)_{qn} - R_{qn}(RC)_{pn}}{2} + \sum_{i=1}^{2g} (r_{ipq}(JR)_{in} - r_{piq}(JR)_{in} + r_{pqi}(JR)_{in}) \equiv \frac{R_{pn}R_{qn}}{2} \bmod 1
\]

Note that for fixed \( i,p,q \), at most one of the \( r \)-coefficients in the above summation is nonzero. For bookkeeping purposes, when \( 1 \leq j < r \leq 2g \) we define \( \vec{r}_{jk} \) be the 2\( g \)-dimensional column vector whose \( i \)th entry is \( r_{ijk} \) if \( i < j \), \(-r_{jik} \) if \( j < i < k \), \( r_{jki} \) if \( k < i \), and 0 otherwise. If \( \col_n(M) \) denotes the \( n \)th column vector of \( M \), we may rewrite this to obtain that \( \col_n(JR) \cdot \vec{r}_{pq} \) is congruent (mod 1) to

\[
\frac{\delta_{q,p+g}(R_{pn} - (CR)_{pn}) + R_{pn}R_{qn} - R_{pn}(RC)_{qn} + R_{qn}(RC)_{pn}}{2}
\]

In order to write this a bit more compactly, for \( 1 \leq j < k \leq 2g \), we define \( \vec{r}_{jk} \) to be the 2\( g \)-dimensional column vector whose \( i \)th entry is \( \delta_{i,j+k}(R_{ji} - (CR)_{ji}) + R_{ji} - R_{ji}(RC)_{ji} + R_{ki}(RC)_{ji} \). Combining the equations above for all \( 1 \leq n \leq 2g \) we get:

\[
\vec{J}R\vec{r}_{pq} \equiv \frac{\vec{r}_{pq}}{2} \bmod 1 \quad \forall 1 \leq p < q \leq 2g
\]

Solving for \( \vec{r}_{pq} \), we obtain:

\[
\vec{r}_{pq} \equiv \frac{(\vec{J}R)^{-1}\vec{r}_{pq}}{2} \bmod 1
\]

Since \( R \) is assumed to be symplectic, we can rewrite this as:

\[
\vec{r}_{pq} \equiv \frac{RJ\vec{r}_{pq}}{2} \bmod 1
\]

Observe that the \( i \)th entry of the vector on the right-hand side is

\[
\frac{1}{2}\delta_{q,p+g}\row_i(RJ) \cdot (\row_p(R) - \row_p(CR)) + \frac{1}{2} \cdot (\row_i(RJ), \row_p(R), \row_q(R)) - \frac{1}{2} \cdot (\row_i(RJ), \row_p(R), \row_q(RC)) + \frac{1}{2} \cdot (\row_i(RJ), \row_p(RC), \row_q(R)) \tag{11}
\]

We are interested in calculating the coefficients \( r_{ipq} \) for \( 1 \leq i < p < q \leq 2g \). Thus we are interested in the \( i \)th entry of \( \vec{r}_{pq} \) when \( 1 \leq i < p < q \leq 2g \). If \( q \neq p + g \) then \( \delta_{q,p+g} = 0 \). Assume that \( q = p + g \).

Then \( 1 \leq i < p \leq g \), and if we write \( R = \begin{pmatrix} S & T \\ P & Q \end{pmatrix} \), we have

\[
\row_i(RJ) \cdot (\row_p(R) - \row_p(CR)) = \row_i(T) \cdot \row_p(S) - \row_i(S) \cdot \row_p(T) - \row_i(T) \cdot \row_p(P) + \row_i(S) \cdot \row_p(Q) = (T\overline{S})_{ip} - (\overline{T}S)_{ip} - (T\overline{P})_{ip} + (S\overline{Q})_{ip}
\]

\[
= 0 - 0
\]
The last equality results from using the symplectic conditions (3.i,ii) and by our assumption that \(i \neq p\). Thus we may drop the first term of (11). In other words, for \(1 \leq i < p < q \leq 2g\) the \(i^{th}\) entry of \(\vec{r}_{pq} \pmod{1}\) is given by

\[
\begin{align*}
\frac{1}{2} & \cdot (\text{row}_i(RJ), \text{row}_p(R), \text{row}_q(R)) \\
- \frac{1}{2} & \cdot (\text{row}_i(RJ), \text{row}_p(R), \text{row}_q(RC)) \\
+ \frac{1}{2} & \cdot (\text{row}_i(RJ), \text{row}_p(RC), \text{row}_q(R)) \pmod{1}
\end{align*}
\]

For aesthetic reasons we rewrite the expression above more symmetrically to show that \(i^{th}\) entry of \(\vec{r}_{pq} \pmod{1}\) is:

\[
\begin{align*}
\frac{1}{2} & \cdot (\text{row}_i(RJ), \text{row}_p(R), \text{row}_q(R)) \\
- \frac{1}{2} & \cdot (\text{row}_i(R), \text{row}_p(R), \text{row}_q(R)) \\
+ \frac{1}{2} & \cdot (\text{row}_i(R), \text{row}_p(RC), \text{row}_q(RJ)) \pmod{1}
\end{align*}
\]

We have just shown that the \(\binom{2g}{3}\) equations in the statement of the lemma are necessary for \((r, R)\) to be an element of \(\rho_2(\mathcal{M}_{g,1})\). Since the symplectic representation \(\rho_1\) is surjective, \(\rho_2(\mathcal{M}_{g,1})\) contains an element of the form \((r, R)\) for any given \(R\). Johnson \([5, \text{Theorem 1}]\) showed that any element of the form \((w, I)\) with \(w \in \wedge^3 H\) is in \(\rho_2(\mathcal{M}_{g,1})\). Then if \((r, R) \in \rho_2(\mathcal{M}_{g,1})\), so is \((w, I)(r, R) = (w + r, R)\) for any \(w \in \wedge^3 H\). Hence we can hit any other possible choice of the coefficients \(r_{ijk}\) satisfying the “mod 1” conditions imposed by \(R\) by composing our map with different choices of Torelli elements. This shows sufficiency. \(\square\)

3 The handlebody group

Our primary goal in this section is to compute \(\rho_2(\mathcal{H}_{g,1})\) explicitly. We will begin with some known algebraic characterizations of \(\mathcal{H}_{g,1}\) and of \(\rho_1(\mathcal{H}_{g,1})\) which will be helpful to us, and use them to derive an analogous characterization at the second level. Thus equipped, we derive an explicit formulation of \(\rho_2(\mathcal{H}_{g,1})\) in Section 3.2.

3.1 Algebraic characterizations of the handlebody subgroup

Let \(b\) denote the normal closure in \(\pi\) of \(\{\beta_1, \ldots, \beta_g\}\). Note that \(b\) is also the kernel of the homomorphism \(\pi \to \pi_1(X_g)\) induced by inclusion.

The following proposition was first proved by McMillan \([9]\). The proof given here was suggested to the authors by Saul Schleimer.

**Proposition 3.1.** The handlebody subgroup \(\mathcal{H}_{g,1}\) of the mapping class group \(\mathcal{M}_{g,1} \subset \text{Aut}(\pi_1(S_{g,1}))\) is precisely the subgroup which preserves \(b\).

**Proof.** One direction is immediate; in order for a mapping class in \(\mathcal{M}_{g,1}\) to extend to the \(X_g\) it must preserve \(b\). Now suppose \(f\) is a mapping class which preserves \(b\). Then \(f\) sends each \(\beta_i\) to a loop that can be represented by a simple closed curve which is trivial in \(\pi_1(X_g)\). Dehn’s Lemma \([15]\) shows that these curves bound disks in \(X_g\) that can be made disjoint. By matching these disks to the ones bounded by each \(\beta_i\) we may construct a homeomorphism from \(X_g\) to itself restricting to \(f\) on its boundary. \(\square\)
Moving on to level one of the Johnson-Morita representations, Birman has shown that the image of the handlebody group in $\text{Sp}(2g, \mathbb{Z})$ is particularly nice [2, Lemma 2.2]. All subblocks are $g \times g$ matrices.

**Proposition 3.2 (Birman).** The image of the handlebody group under the symplectic representation is characterized by a $g \times g$ block of zeroes in the upper-right corner. That is,

$$\rho_1(\mathcal{H}_g, 1) = \left\{ M \in \text{Sp}(2g; \mathbb{Z}) \bigg| M \text{ has block form } \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \right\}$$

Sufficiency is shown in [2] by exhibiting generators for $\rho_1(\mathcal{H}_g, 1)$ which are in the image of the handlebody group. The necessity of this condition for membership in $\rho_1(\mathcal{H}_g, 1)$ follows from the observation that in the handlebody $X_g$, the homology classes of the generators of type $b_i$ are all 0. Any homeomorphism of $S_g$ which extends to $X_g$ must take trivial elements in the homology of the handlebody to trivial elements in the homology of the handlebody. In other words, $\rho_1(\mathcal{H}_g, 1)$ is characterized by the property that its elements must preserve the subgroup of $H$ generated by the $b_i$'s.

We will now give a second-level analogue of these characterizations by describing a subgroup of $\pi/\pi^{(2)}$ which must be preserved by $\rho_2(\mathcal{H}_g, 1)$, thus giving a restriction on the image of the handlebody group.

The second Johnson-Morita homomorphism is given by the action of $\mathcal{M}_{g,1}$ on the nilpotent quotient $\pi/\pi^{(2)}$. Let $b \subset \pi$ be as above, and recall from Section 2.2 the map $\phi_2 : \pi \to \Phi_2$ be as above. The following lemma computes $\phi_2(b)$.

**Lemma 3.3.**

$$\phi_2(b) = \left\{ \left( \sum_{i \leq j < k \leq g} m_{ijk} a_i \wedge b_j + \sum_{i < j < k} l_i b_i \bigg| \sum_{i=1}^g l_i b_i \right) \bigg| m_{ij}, n_{ij}, l_i \in \mathbb{Z} \right\}$$

**Proof.** In light of Lemma 2.1, the right-hand side above is clearly the kernel of the quotient homomorphism $\pi/\pi^{(2)} \to \pi_1(X_g)/\pi_1(X_g)^{(2)}$.

Now that we have identified $\phi_2(b)$ we will describe $\rho_2(\mathcal{H}_g, 1)$.

### 3.2 Image of the handlebody subgroup under $\rho_2$

Theorem 2.4 above gives $\rho_2(\mathcal{M}_{g,1})$. The missing ingredient for a characterization of $\rho_2(\mathcal{H}_g, 1)$ is $\rho_2(\mathcal{I}_g, 1 \cap \mathcal{H}_g, 1)$ which was computed by Morita.

**Proposition 3.4 ([10, Lemma 2.5]).** $\rho_2(\mathcal{I}_g, 1 \cap \mathcal{H}_g, 1)$ is the free abelian group with free basis:

$$(b_i \wedge b_j \wedge b_k, I), \quad (a_i \wedge b_j \wedge b_k, I), \quad \text{and} \quad (a_i \wedge a_j \wedge b_k, I) \quad 1 \leq i, j, k \leq g.$$  

Now we have the tools to assemble a description of $\rho_2(\mathcal{H}_g, 1)$. The following theorem gives a complete characterization of $\rho_2(\mathcal{H}_g, 1)$; it says that an element is in this image if and only if its first factor has no “triple-a” terms and its second factor has the form of Proposition 3.2.

**Theorem 3.5.** Let $R \in \text{Sp}(2g, \mathbb{Z})$ be an arbitrary symplectic matrix. Let $r$ be any element of $\frac{1}{2} \lambda^3 H$ with $r = \sum_{1 \leq i < j < k \leq 2g} r_{ijk} x_i \wedge x_j \wedge x_k$. Then $(r, R) \in \rho_2(\mathcal{H}_g, 1)$ if and only if all of the following three conditions hold:

1. $R$ has $g \times g$ block form $\begin{pmatrix} * & 0 \\ * & * \end{pmatrix}$

2. $r_{ijk} \equiv \frac{1}{2} E_{ijk} \mod 1$ for all $1 \leq i < j < k \leq 2g$.  


3. $r_{ijk} = 0$ for all $i, j, k$ with $0 \leq i < j < k \leq g$. (i.e. $r$ contains no terms of the form $a_i \wedge a_j \wedge a_k$.)

We refer the reader to Theorem 2.4 for the definition of $E_{ijk}$, which depends on the matrix $R$.

Proof. The necessity of condition 1 has already been established in [2, Lemma 2.2]. We claim that only elements of $\frac{1}{2} \wedge^3 H \times \text{Sp}(H)$ satisfying condition 3 above preserve $\phi_2(b)$ under the action of $\text{Sp}(H)$. Suppose $R^{-1}$ must satisfy condition 1 above and using Lemma 3.3, there is an element $(\nu, R^{-1}b_i) \in \phi_2(b)$ where $\nu$ has only terms of the form $\frac{1}{2}b_n \wedge b_m$. Applying (9) we get

$$(r, R) * (\nu, R^{-1}b_i) =
\begin{align*}
&= (R(\nu) + \kappa(R^{-1}b_i) + R\kappa(R^{-1}b_i) + r(R^{-1}b_i), RR^{-1}b_i) \\
&= (R(\nu) + \kappa(b_i) + R\kappa(R^{-1}b_i) + r(b_i), b_i)
\end{align*}$$

Consider each of the terms in the first coordinate of the ordered pair above. Since $\nu$ only has terms of the form $\frac{1}{2}b_n \wedge b_m$ and the matrix $R$ has the block form given in condition 1, we must have that $R(\nu)$ contains no terms of the form $a_j \wedge a_k$. The image of the homomorphism $\kappa$ has no $a_j \wedge a_k$ terms so neither $\kappa(b_i)$ nor $\kappa(R^{-1}b_i)$ contains any $a_j \wedge a_k$ terms. Application of the matrix $R$ preserves this quality; hence $R\kappa(R^{-1}b_i)$ contains no $a_j \wedge a_k$ terms. We can see using (1) that $r(b_i)$ will contain a term of the form $-ca_j \wedge a_k$ by construction. Then Lemma 3.3 implies that $c = 0$. It follows that the two conditions of the corollary are necessary.

For each $R$ satisfying 1 there is some mapping class $f \in \mathcal{H}_{g,1}$ with $\rho_1(f) = R$ as shown in [2, Lemma 2.2]. We have shown that $\rho_2(f)$ satisfies conditions 1 and 2. Applying Proposition 2.4 we can get every other element of the form $(w, R)$ satisfying 1 and 2 as a product $(z, I)\rho_2(f)$ where $(z, I) \in \rho_2(I_{g,1} \cap \mathcal{H}_{g,1}).$ This establishes sufficiency. □

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