ASYMPTOTIC BEHAVIOR OF COUPLED INCLUSIONS WITH VARIABLE EXPO NENTS

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Abstract. This work concerns the study of asymptotic behavior of the solutions of a nonautonomous coupled inclusion system with variable exponents. We prove the existence of a pullback attractor and that the system of inclusions is asymptotically autonomous.

1. Introduction. Nonlinear reaction-diffusion equations have been studied extensively in recent years and a special attention has been given to coupled reaction-diffusion equations from various fields of applied sciences arising from epidemics, biochemistry and engineering [18]. Reaction-diffusion systems are naturally applied in chemistry where the most common is the change in space and time of the concentration of one or more chemical substances. One interest in chemical kinetics is the construction of mathematical models that can describe the characteristics of a chemical reaction. Mathematical models for electrorheological fluids were considered in [19, 20, 21] and variable exponents do appear in the diffusion term (see also [7, 9]). Reaction-diffusion systems can be perturbed by discontinuous nonlinear terms, which leads to study differential inclusions rather than differential equations, for example, evolution differential inclusion systems with positively sublinear upper semicontinuous multivalued reaction terms \( F \) and \( G \) (see [6]).

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This work concerns the coupled system of inclusions:

\[
\begin{cases}
\frac{\partial u_1}{\partial t} - \text{div}(D_1(t, \cdot) |\nabla u_1|^{p(\cdot)-2} \nabla u_1) + |u_1|^{q(\cdot)-2} u_1 \in F(u_1, u_2) & t > \tau \\
\frac{\partial u_2}{\partial t} - \text{div}(D_2(t, \cdot) |\nabla u_2|^{p(\cdot)-2} \nabla u_2) + |u_2|^{q(\cdot)-2} u_2 \in G(u_1, u_2) & t > \tau \\
\frac{\partial u_1}{\partial n}(t, x) = \frac{\partial u_2}{\partial n}(t, x) = 0 & \text{in } \partial \Omega, \quad t \geq \tau \\
(u_1(\tau), u_2(\tau)) = (u_{0,1}, u_{0,2}) & \text{in } L^2(\Omega) \times L^2(\Omega),
\end{cases}
\]

on a bounded domain \( \Omega \subset \mathbb{R}^n, n \geq 1 \), with smooth boundary, where \( F \) and \( G \) are bounded, upper semicontinuous and positively sublinear multivalued maps and the exponents \( p(\cdot), q(\cdot) \in C(\Omega) \) satisfy

\[
p^+ := \max_{x \in \Omega} p(x) > p^- := \min_{x \in \Omega} p(x) > 2, \quad q^+ := \max_{x \in \Omega} q(x) > q^- := \min_{x \in \Omega} q(x) > 2.
\]

In addition, the diffusion coefficients \( D_1, D_2 \) are assumed to satisfy:

**Assumption D.** \( D_1, D_2 : [\tau, T] \times \Omega \to \mathbb{R} \) are functions in \( L^\infty([\tau, T] \times \Omega) \) satisfying:

1. There is a positive constant \( \beta \) such that \( 0 < \beta \leq D_i(t, x) \) for almost all \((t, x) \in [\tau, T] \times \Omega, i = 1, 2\).
2. \( D_i(t, x) \geq D_i(s, x) \) a.a. \( x \in \Omega \) and \( t \leq s \) in \([\tau, T]\), \( i = 1, 2\).

In this work we extend the results in [15] for a single inclusion to the case of a coupled inclusion system. We will prove that the strict generalized process (see Definition 2.7 in Section 2) defined by (S) possesses a pullback attractor. Moreover, we prove that the system (S) is in fact asymptotically autonomous. It makes use of a collection of ideas and results of some recent, distinct previous works [15, 22, 23, 27] of the authors, which are applied here to a new problem to yield interesting new results. Regarding [13, 14, 15] where an equation and a single inclusion of this type of problems were considered, the coupled system can not be treated in the same way as the single case, the principal additional technical difficulty is to adjust the results considering two inclusions, in this sense, the main technical difficulty appears to prove dissipativity.

The paper is organized as follows. First, in Section 2 we provide some definitions and results on existence of global solutions and generalized processes. In Section 3 we prove the existence of the pullback attractor for the system (S). In Section 4 we say some words about forward attraction and in the last section we prove that the system (S) is asymptotically autonomous.

### 2. Preliminaries, existence of global solutions and generalized processes.

Consider now the system (S) in the following abstract form

\[
(S2) \quad \begin{cases}
\frac{du}{dt}(t) + A(t)u(t) \in F(u(t), v(t)) & t > \tau \\
\frac{dv}{dt}(t) + B(t)v(t) \in G(u(t), v(t)) & t > \tau \\
(u(\tau), v(\tau)) = (u_0, v_0) \in H \times H,
\end{cases}
\]

where \( F \) and \( G \) are bounded, upper semicontinuous and positively sublinear multivalued maps (see Definitions 2.4, 2.3 and 2.5, respectively) and, for each \( t > \tau \), \( A(t) \) and \( B(t) \) are univalued maximal monotone operators of subdifferential type in a real separable Hilbert space \( H \). Specifically, \( A(t) = \partial \varphi^t \) and \( B(t) = \partial \psi^t \) for
nonnegative mappings \( \varphi^t, \psi^t \) with \( \partial \varphi^t(0) = \partial \psi^t(0) = 0, \forall t \in \mathbb{R} \) and the mappings \( \varphi^t, \psi^t \) satisfy:

**Assumption A.** Let \( T > \tau \) be fixed.

(A.1) There is a set \( Z \subset (\tau, T) \) of zero measure such that \( \varphi^t \) is a lower semicontinuous proper convex function from \( H \) into \((-\infty, \infty] \) with a nonempty effective domain for each \( t \in [\tau, T] \setminus Z \).

(A.2) For any positive integer \( r \) there exist a constant \( K_r > 0 \), an absolutely continuous function \( g_r : [\tau, T] \rightarrow \mathbb{R} \) with \( g'_r \in L^\beta(\tau, T) \) and a function of bounded variation \( h_r : [\tau, T] \rightarrow \mathbb{R} \) such that if \( t \in [\tau, T] \setminus Z, w \in D(\varphi^t) \) with \( |w| \leq r \) and \( s \in [t, T] \setminus Z \), then there exists an element \( \tilde{w} \in D(\varphi^s) \) satisfying

\[
|\tilde{w} - w| \leq |g_r(s) - g_r(t)|(\varphi^t(w)) + K_r, \\
\phi^s(\tilde{w}) \leq \varphi^t(w) + |h_r(s) - h_r(t)|(\varphi^t(w) + K_r),
\]

where \( \alpha \) is some fixed constant with \( 0 \leq \alpha \leq 1 \) and

\[
\beta := \begin{cases} 
2 & \text{if } 0 \leq \alpha \leq \frac{1}{2}, \\
\frac{1}{1 - \alpha} & \text{if } \frac{1}{2} \leq \alpha \leq 1 
\end{cases}
\]

Let us first review some concepts and results from the literature, which will be useful in the sequel. We refer the reader to [2, 3, 29] for more details about multivalued analysis theory.

### 2.1. Setvalued mappings.

Let \( X \) be a real Banach space and \( M \) a Lebesgue measurable subset in \( \mathbb{R}^q \), \( q \geq 1 \).

**Definition 2.1.** The map \( G : M \rightarrow P(X) \) is called measurable if for each closed subset \( C \) in \( X \) the set

\[
G^{-1}(C) = \{ y \in M; G(y) \cap C \neq \emptyset \}
\]

is Lebesgue measurable.

If \( G \) is a univalued map, the above definition is equivalent to the usual definition of a measurable function.

**Definition 2.2.** By a selection of \( E : M \rightarrow P(X) \) we mean a function \( f : M \rightarrow X \) such that \( f(y) \in E(y) \) a.e. \( y \in M \), and we denote by \( \text{Sel}E \) the set

\[
\text{Sel}E \doteq \{ f, f : M \rightarrow X \text{ is a measurable selection of } E \}.
\]

**Definition 2.3.** Let \( U \) be a topological space. A mapping \( G : U \rightarrow P(X) \) is called upper semicontinuous [weakly upper semicontinuous] at \( u \in U \), if

(i) \( G(u) \) is nonempty, bounded, closed and convex.

(ii) For each open subset [open set in the weak topology] \( D \) in \( X \) satisfying \( G(u) \subset D \), there exists a neighborhood \( V \) of \( u \), such that \( G(v) \subset D \), for each \( v \in V \).

If \( G \) is upper semicontinuous [weakly upper semicontinuous] at each \( u \in U \), then it is called upper semicontinuous [weakly upper semicontinuous] on \( U \).
Definition 2.4. $F, G : H \times H \to P(H)$ are said to be bounded if, whenever $B_1, B_2 \subset H$ are bounded, then $F(B_1, B_2) = \bigcup_{(u,v) \in B_1 \times B_2} F(u,v)$ and $G(B_1, B_2) = \bigcup_{(u,v) \in B_1 \times B_2} G(u,v)$ are bounded in $H$.

In order to obtain global solutions we impose the following suitable conditions on terms $F$ and $G$.

Definition 2.5 ([24]). The pair $(F, G)$ of maps $F, G : H \times H \to P(H)$, which takes bounded subsets of $H \times H$ into bounded subsets of $H$, is called positively sublinear if there exist $a > 0, b > 0, c > 0$ and $m_0 > 0$ such that for each $(u,v) \in H \times H$ with $\|u\| > m_0$ or $\|v\| > m_0$ for which either there exists $f_0 \in F(u,v)$ satisfying $(u, f_0) > 0$ or there exists $g_0 \in G(u,v)$ with $(v, g_0) > 0$, then both

$$\|f\| \leq a\|u\| + b\|v\| + c \quad \text{and} \quad \|g\| \leq a\|u\| + b\|v\| + c$$

hold for each $f \in F(u,v)$ and each $g \in G(u,v)$.

2.2. Generalized processes. In order to study the asymptotic behavior of the solutions of the system (S) we will work with a multivalued process defined by a generalized process. We will review these concepts which had been considered in [22, 23] and can be used in the study of infinite dimensional dynamical systems.

Definition 2.6. Let $(X, \rho)$ be a complete metric space. A generalized process $\mathcal{G} = \{\mathcal{G}(\tau)\}_{\tau \in \mathbb{R}}$ on $X$ is a family of function sets $\mathcal{G}(\tau)$ consisting of maps $\varphi : [\tau, \infty) \to X$, satisfying the conditions:

C1) For each $\tau \in \mathbb{R}$ and $z \in X$ there exists at least one $\varphi \in \mathcal{G}(\tau)$ with $\varphi(\tau) = z$;
C2) If $\varphi \in \mathcal{G}(\tau)$ and $s \geq 0$, then $\varphi^{+s} \in \mathcal{G}(\tau + s)$, where $\varphi^{+s} := \varphi|_{[\tau + s, \infty)}$;
C3) If $\{\varphi_j\}_{j \in \mathbb{N}} \subset \mathcal{G}(\tau)$ and $\varphi_j(\tau) \to z$, then there exists a subsequence $\{\varphi_{j \mu}\}_{\mu \in \mathbb{N}}$ of $\{\varphi_j\}_{j \in \mathbb{N}}$ and $\varphi \in \mathcal{G}(\tau)$ with $\varphi(\tau) = z$ such that $\varphi_{j \mu}(t) \to \varphi(t)$ for each $t \geq \tau$.

Definition 2.7. A generalized process $\mathcal{G} = \{\mathcal{G}(\tau)\}_{\tau \in \mathbb{R}}$ which satisfies the condition

C4) (Concatenation) If $\varphi, \psi \in \mathcal{G}$ with $\varphi \in \mathcal{G}(\tau)$, $\psi \in \mathcal{G}(\tau)$ and $\varphi(s) = \psi(s)$ for some $s \geq r \geq \tau$, then $\theta \in \mathcal{G}(\tau)$, where $\theta(t) := \begin{cases} \varphi(t), t \in [\tau, s] \\ \psi(t), t \in (s, \infty) \end{cases}$

is called an exact (or strict) generalized process.

A multivalued process $\{U_\mathcal{G}(t, \tau)\}_{t \geq \tau}$ defined by a generalized process $\mathcal{G}$ is a family of multivalued operators $U_\mathcal{G}(t, \tau) : P(X) \to P(X)$ with $-\infty < \tau \leq t < +\infty$, such that for each $\tau \in \mathbb{R}$

$$U_\mathcal{G}(t, \tau)E = \{\varphi(t) : \varphi \in \mathcal{G}(\tau), \ \text{with} \ \varphi(\tau) \in E\}, \ t \geq \tau.$$

Theorem 2.8 ([22, 23]). Let $\mathcal{G}$ be an exact generalized process. If $\{U_\mathcal{G}(t, \tau)\}_{t \geq \tau}$ is a multivalued process defined by $\mathcal{G}$, then $\{U_\mathcal{G}(t, \tau)\}_{t \geq \tau}$ is an exact multivalued process on $P(X)$, i.e.,

1. $U_\mathcal{G}(t, t) = Id_{P(X)}$,
2. $U_\mathcal{G}(t, \tau) = U_\mathcal{G}(t, s)U_\mathcal{G}(s, \tau)$ for all $-\infty < \tau \leq s \leq t < +\infty$.

A family of sets $K = \{K(t) \subset X : t \in \mathbb{R}\}$ will be called a nonautonomous set. The family $K$ is closed (compact, bounded) if $K(t)$ is closed (compact, bounded) for all $t \in \mathbb{R}$. The $\omega$–limit set $\omega(t, E)$ consists of the pullback limits of all converging sequences $\{\xi_n\}_{n \in \mathbb{N}}$ where $\xi_n \in U_\mathcal{G}(t, s_n)E, s_n \to -\infty$. Let $A = \{A(t)\}_{t \in \mathbb{R}}$ be a family of subsets of $X$. We have the following concepts of invariance:
Definition 2.9. Let $t \in \mathbb{R}$.

1. A set $\mathcal{A}(t) \subset X$ pullback attracts a set $B \in X$ at time $t$ if
   \[ \text{dist}(U_{\mathcal{A}}(t,s), \mathcal{A}(t)) \to 0 \text{ as } s \to -\infty. \]

2. A family $\mathcal{A} = \{ \mathcal{A}(t) \}_{t \in \mathbb{R}}$ pullback attracts bounded sets of $X$ if $\mathcal{A}(\tau) \subset X$ pullback attracts all bounded subsets at $\tau$, for each $\tau \in \mathbb{R}$. In this case, we say that the nonautonomous set $\mathcal{A}$ is pullback attracting.

3. A set $\mathcal{A}(t) \subset X$ pullback absorbs bounded subsets of $X$ at time $t$ if, for each bounded set $B$ in $X$, there exists $T = T(t, B) \leq t$ such that $U_{\mathcal{A}}(t, \tau) B \subset \mathcal{A}(t)$ for all $\tau \leq T$.

4. A family $\{ \mathcal{A}(t) \}_{t \in \mathbb{R}}$ pullback absorbs bounded subsets of $X$ if for each $t \in \mathbb{R}$, $\mathcal{A}(t)$ pullback absorbs bounded sets at time $t$.

2.3. Strong solutions. Consider the following initial value problem:

\[
(P_1) \quad \begin{cases} 
\frac{du}{dt}(t) + A(t)u(t) \ni f(t), \quad t > \tau \\
u(\tau) = u_0
\end{cases}
\]

where for each $t > \tau$, $A(t)$ is maximal monotone in a Hilbert space $H$, $f \in L^1(\tau, T; H)$ and $u_0 \in H$. Moreover, suppose $D(A(t)) = D(A(\tau))$, $\forall t, \tau \in \mathbb{R}$ and $\overline{D(A(t))} = H$, for all $t \in \mathbb{R}$.

Definition 2.10. A function $u : [\tau, T] \to H$ is called a strong solution of $(P_1)$ on $[\tau, T]$ if

(i) $u \in C([\tau, T]; H)$;
(ii) $u$ is absolutely continuous on any compact subset of $(\tau, T)$;
(iii) $u(t)$ is in $D(A(t))$ for a.e. $t \in [\tau, T]$, $u(\tau) = u_0$ and satisfies the inclusion in $(P_1)$ for a.e. $t \in [\tau, T]$.

Definition 2.11. A strong solution of $(S2)$ is a pair $(u, v)$ satisfying: $u, v \in C([\tau, T]; H)$ for which there exist $f, g \in L^1(\tau, T; H)$, $f(t) \in F(u(t), v(t))$, $g(t) \in G(u(t), v(t))$ a.e. in $(\tau, T)$, and such that $(u, v)$ is a strong solution (see Definition 2.10) over $(\tau, T)$ to the system $(P_1)$ below:

\[
(P_1) \quad \begin{cases} 
\frac{du}{dt} + A(t)u = f \\
\frac{dv}{dt} + B(t)v = g \\
u(\tau) = u_0, v(\tau) = v_0
\end{cases}
\]

Theorem 2.12 ([27]). Let $A = \{ A(t) \}_{t \geq \tau}$ and $B = \{ B(t) \}_{t \geq \tau}$ be families of univalued operators $A(t) = \partial \varphi^t$, $B(t) = \partial \psi^t$ with $\varphi^t$, $\psi^t$ non-negative maps satisfying Assumption A with $\partial \varphi^t(0) = \partial \psi^t(0) = 0$. Also suppose each one of $A$ and $B$ generates a compact evolution process, and let $F, G : H \times H \to P(H)$ be upper semicontinuous and bounded multivalued maps. Then given a bounded subset $B_0 \subset H \times H$, there exists $T_0 > 0$ such that for each $(u_0, v_0) \in B_0$ there exists at least one strong solution $(u, v)$ of $(S2)$ defined on $[\tau, T_0]$. If, in addition, the pair $(F, G)$ is positively sublinear, given $T > \tau$, the same conclusion is true with $T_0 = T$. 

Let $D(u(\tau), v(\tau))$ be the set of solutions of (S2) with initial data $(u_\tau, v_\tau)$ and define $G(\tau) := \bigcup_{(\alpha, \nu) \in H \times H} D(u(\tau), v(\tau))$. Consider $G := \{G(\tau)\}_{\tau \in \mathbb{R}}$.

**Theorem 2.13** ([27]). Under the conditions of Theorem 2.12, $G$ is an exact generalized process.

Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be a bounded smooth domain and write $H := L^2(\Omega)$ and $Y := W^{1,p(\cdot)}(\Omega)$ with $p^* > 2$. Then $Y \subset H \subset Y^*$ with continuous and dense embeddings. We refer the reader to [7, 8] and references therein to see properties of the Lebesgue and Sobolev spaces with variable exponents. In particular, with

$$L^{p(\cdot)}(\Omega) := \left\{ u : \Omega \to \mathbb{R} : u \text{ is measurable, } \int_\Omega |u(x)|^{p(x)}dx < \infty \right\}$$

and $L^\infty_+(\Omega) := \{ q \in L^\infty(\Omega) : \text{ess inf } q \geq 1 \}$, define

$$\rho(u) := \int_\Omega |u(x)|^{p(x)}dx, \quad \|u\|_{L^{p(\cdot)}(\Omega)} := \text{inf} \left\{ \lambda > 0 : \rho\left(\frac{u}{\lambda}\right) \leq 1 \right\}.$$ 

for $u \in L^{p(\cdot)}(\Omega)$ and $p \in L^\infty_+(\Omega)$.

Consider the operator $A(t)$ defined in $Y$ such that for each $u \in Y$ is associated the following element of $Y^*$, $A(t)u : Y \to \mathbb{R}$ given by

$$A(t)u(v) := \int_\Omega D_1(t, x)|\nabla u(x)|^{p(x)-2}\nabla u(x) \cdot \nabla v(x)dx + \int_\Omega |u(x)|^{p(x)-2}u(x)v(x)dx.$$ 

The authors proved in [13] that:

- For each $t \in [\tau, T]$ the operator $A(t) : Y \to Y^*$, with domain $Y = W^{1,p(\cdot)}(\Omega)$, is maximal monotone and $A(t)(Y) = Y^*$.
- The realization of the operator $A(t)$ in $H = L^2(\Omega)$, i.e.,

$$A_H(t)u = -\text{div}(D_1(t)|\nabla u(t)|^{p(x)-2}\nabla u(t)) + |u(t)|^{p(x)-2}u(t),$$

is maximal monotone in $H$ for each $t \in [\tau, T]$.
- The operator $A_H(t)$ is the subdifferential $\partial \varphi_{p(\cdot)}^t$ of the convex, proper and lower semicontinuous map $\varphi_{p(\cdot)}^t : L^2(\Omega) \to \mathbb{R} \cup \{+\infty\}$ given by

$$\varphi_{p(\cdot)}^t(u) = \left\{ \begin{array}{ll} \int_\Omega \frac{D_1(t, x)}{p(x)}|\nabla u|^{p(x)}dx + \int_\Omega \frac{1}{p(x)}|u|^{p(x)}dx & \text{if } u \in Y \\ +\infty, & \text{otherwise.} \end{array} \right.$$ 

Using the following elementary assertion we can obtain estimates on the operator considering only two cases.

**Proposition 1** ([1]). Let $\lambda, \mu$ be arbitrary nonnegative numbers. For every positive $\alpha, \theta$, $\alpha \geq \theta$,

$$\lambda^\alpha + \mu^\theta \geq \frac{1}{2^\alpha} \begin{cases} (\lambda + \mu)^\alpha & \text{if } \lambda + \mu < 1, \\ (\lambda + \mu)^\theta & \text{if } \lambda + \mu \geq 1. \end{cases}$$

Then it is easy to show that for every $u \in Y$

$$\langle A(t)u, u \rangle_{Y^*, Y} \geq \frac{\min\{\beta, 1\}}{2^{p^*}} \begin{cases} \|u\|_{Y^*}^{p^*} & \text{if } \|u\|_Y < 1, \\ \|u\|_{Y^*}^{p^*} & \text{if } \|u\|_Y \geq 1. \end{cases}$$ 

From Example 4.4 in the last section of [27] we can apply Theorem 2.12 and Theorem 2.13 for $A(t)u = -\text{div}(D_1(t, \cdot)|\nabla u|^{p(\cdot)-2}\nabla u) + |u|^{p(\cdot)-2}u$ and $B(t)v =$
3. Existence of the pullback attractor. First, we provide estimates on the solutions in the spaces $H \times H$ and $Y \times Y$.

Lemma 3.1. Let $(u_1, u_2)$ be a solution of problem (S). Then there exist a positive number $r_0$ and a constant $T_0$ which do not depend on the initial data, such that

\[
\|(u_1(t), u_2(t))\|_{H \times H} \leq r_0, \quad \forall \, t \geq T_0 + \tau.
\]

Proof. Let $\varphi = (u_1, u_2) \in \mathcal{G}$ be a solution of (S). Then there exists a pair $(f, g) \in \text{Sel} \, F(u_1, u_2) \times \text{Sel} \, G(u_1, u_2)$ with $f, g \in L^1(\tau, T; H)$ for each $T > \tau$ such that $u_1$, $u_2$ satisfy the problem

\[
\begin{aligned}
&\frac{du_1}{dt} + A(t)(u_1) = f \quad \text{in} \quad (\tau, T) \times \Omega, \\
&\frac{du_2}{dt} + B(t)(u_2) = g \quad \text{in} \quad (\tau, T) \times \Omega, \\
&u_1(\tau, x) = u_{1,0}(x), \quad u_2(\tau, x) = u_{2,0}(x) \quad \text{in} \quad \Omega.
\end{aligned}
\]

Using (5) in (4) we obtain

\[
\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|u_1(t)\|_{H}^2 &\leq \left\{ \begin{array}{ll}
- \frac{\sigma}{\alpha p^+} \|u_1(t)\|_{H}^{p^+} + \langle f(t), u_1(t) \rangle_{H} & \text{if} \quad t \in I_1, \\
- \frac{\sigma}{\alpha p^-} \|u_1(t)\|_{H}^{p^-} + \langle f(t), u_1(t) \rangle_{H} & \text{if} \quad t \in I_2,
\end{array} \right.
\end{aligned}
\]

where

\[
I_1 := \{ t \in (\tau, T) : \|u_1(t)\|_{Y} < 1 \}, \quad I_2 := \{ t \in (\tau, T) : \|u_1(t)\|_{Y} \geq 1 \},
\]

and

\[
\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|u_2(t)\|_{H}^2 &\leq \left\{ \begin{array}{ll}
- \frac{\sigma}{\alpha q^+} \|u_2(t)\|_{H}^{q^+} + \langle g(t), u_2(t) \rangle_{H} & \text{if} \quad t \in \tilde{I}_1, \\
- \frac{\sigma}{\alpha q^-} \|u_2(t)\|_{H}^{q^-} + \langle g(t), u_2(t) \rangle_{H} & \text{if} \quad t \in \tilde{I}_2,
\end{array} \right.
\end{aligned}
\]

where

\[
\tilde{I}_1 := \{ t \in (\tau, T) : \|u_2(t)\|_{Y} < 1 \}, \quad \tilde{I}_2 := \{ t \in (\tau, T) : \|u_2(t)\|_{Y} \geq 1 \}.
\]

Now, define $r := \frac{p^+}{p} > 1$ and let $r'$ be such that $\frac{1}{r'} + \frac{1}{r} = 1$. Then, by Young's inequality,

\[
- \frac{\sigma}{\alpha p^+} \|u_1(t)\|_{H}^{p^+} \leq r \left( - \frac{\sigma}{\alpha q^+} \|u_1(t)\|_{H}^{q^+} + \frac{\sigma}{\alpha p^+} r' \right). \tag{5}
\]

Using (5) in (4) we obtain

\[
\frac{1}{2} \frac{d}{dt} \|u_1(t)\|_{H}^2 \leq -C_2 \|u_1(t)\|_{H}^{p^-} + \langle f(t), u_1(t) \rangle_{H} + C_1 \forall \, t \in I := (\tau, T), \tag{6}
\]

where $C_1 := \frac{L}{\alpha p^-}$ and $C_2 := \frac{\min\{1, \beta\}}{(2\alpha)^2}$ with $L := \max\{p^+, q^+\}$.

In an analogous way, taking $\tilde{r} := \frac{q^+}{q} > 1$ and $\tilde{r}'$ such that $\frac{1}{\tilde{r}} + \frac{1}{\tilde{r}'} = 1$ we have

\[
\frac{1}{2} \frac{d}{dt} \|u_2(t)\|_{H}^2 \leq -\tilde{C}_2 \|u_2(t)\|_{H}^{q^-} + \langle g(t), u_2(t) \rangle_{H} + \tilde{C}_1 \forall \, t \in \tilde{I},
\]

where $\tilde{C}_1 := \frac{L}{\alpha q^-}$ and $\tilde{C}_2 \equiv C_2 = \frac{\min\{1, \beta\}}{(2\alpha)^2}$. 

We can suppose, without losing generality that \( p^- \geq q^- \). If \( p^- = q^- \) we obtain a similar expression as (6) with \( q^- \) in the place of \( p^- \). If \( p^- > q^- \), taking \( \theta := \frac{q^-}{q} > 1, \theta' \) such that \( \frac{1}{\theta} + \frac{1}{\theta'} = 1 \) and \( \epsilon > 0 \) we have

\[
\|u_1(t)\|_H^{q^-} = \frac{e}{\epsilon} \|u_1(t)\|_H^{q^-} \leq \frac{1}{\theta'} \epsilon^{\theta'} + \frac{1}{\theta} \epsilon^{\theta} \|u_1(t)\|_H^{p^-}
\]

and then

\[
-C_2 \|u_1(t)\|_H^{p^-} \leq \frac{\epsilon}{\theta'} \left[ \frac{C_2}{\theta'} \epsilon - C_2 \|u_1(t)\|_H^{q^-} \right].
\]

Thus we obtain

\[
\begin{cases}
\frac{1}{2} \frac{d}{dt} \|u_1(t)\|_H^2 \leq -\frac{C_2 \theta}{\epsilon} \|u_1(t)\|_H^{q^-} + \langle f(t), u_1(t) \rangle_H + C_1 + \frac{\theta C_2}{\theta'} \epsilon \|u_1(t)\|_H^{p^-} \\
\frac{1}{2} \frac{d}{dt} \|u_2(t)\|_H^2 \leq -\tilde{C}_2 \|u_2(t)\|_H^{q^-} + \langle g(t), u_2(t) \rangle_H + \tilde{C}_1
\end{cases}
\]

(7)

We estimate \( \langle f(t), u_1(t) \rangle_H \) and \( \langle g(t), u_2(t) \rangle_H \) using the assumption that \( (F,G) \) is positively sublinear (see Definition 2.5) and Young’s inequality. Choosing a convenient, sufficiently small \( \epsilon \) we obtain

\[
\frac{1}{2} \frac{d}{dt} \left( \|u_1(t)\|_H^2 + \|u_2(t)\|_H^2 \right) \leq -C_5 \left( \|u_1(t)\|_H^{q^-} + \|u_2(t)\|_H^{q^-} \right) + C_6
\]

\[
\leq -\frac{C_5}{2} \left( \|u_1(t)\|_H^2 + \|u_2(t)\|_H^2 \right)^{\frac{q^-}{2}} + C_6,
\]

where \( C_5, C_6 > 0 \) are constants that depend on the numbers \(|\Omega|, \beta, p^-, p^+, q^-, q^+, a, b, c \) and \( m_0 \).

Hence, the function \( y(t) := \|u_1(t)\|_H^2 + \|u_2(t)\|_H^2 \) satisfies the inequality

\[
y(t) \leq -\frac{2C_5}{2} y(t)^{\frac{q^-}{2}} + 2C_6, \quad t > 0.
\]

From Lemma 5.1 in [28] we obtain

\[
y(t) \leq \left( \frac{C_6}{C_5} \right)^{2/q^-} \left( \frac{2C_5}{2} \left( q^-/2 - 1 \right)(t - \tau) \right)^{-1/(q^-/2 - 1)}.
\]

Let \( T_0 > 0 \) be such that \( \left[ \frac{2C_5}{2} \left( \frac{q^-}{2} - 1 \right) T_0 \right]^{-1/(q^-/2 - 1)} \leq 1 \). Then,

\[
\|u_1(t)\|_H^2 + \|u_2(t)\|_H^2 \leq \kappa_0 := \left( C_0 2^{1/2} C_6 \right)^{2/q^-} + 1 \text{ for all } t \geq T_0 + \tau.
\]

\[ \Box \]

**Lemma 3.2.** Let \( (u_1, u_2) \) be a solution of problem (S). Then there exist positive constants \( r_1 \) and \( T_1 > T_0 \), which do not depend on the initial data, such that

\[
\|u_1(t), u_2(t)\|_{Y \times Y} \leq r_1, \quad \forall t \geq T_1 + \tau.
\]

**Proof.** Take \( T_1 > T_0 \). Since \( (u_1, u_2) \) is a solution of (S) there exists a pair \( (f, g) \in \text{Sel } F(u, v) \times \text{Sel } G(u, v) \) with \( f, g \in L^1(\tau, T; H) \) such that \( u \) and \( v \) satisfy the problem

\[
\begin{cases}
\frac{du_1}{dt} + A(t)(u_1) = f \quad \text{in } (\tau, T) \times \Omega, \\
\frac{du_2}{dt} + B(t)(u_2) = g \quad \text{in } (\tau, T) \times \Omega.
\end{cases}
\]
Consider $\varphi^t_{\mathcal{P}(\cdot)}$ as in (1). Using Assumption D (ii),
\[
\frac{d}{dt} \varphi^t_{\mathcal{P}(\cdot)}(u_1(t)) \leq \left( \partial \varphi^t_{\mathcal{P}(\cdot)}(u_1(t)), \frac{du_1}{dt}(t) \right)
\]
and then we obtain
\[
\frac{d}{dt} \varphi^t_{\mathcal{P}(\cdot)}(u_1(t)) + \frac{1}{2} \left\| f(t) - \frac{du_1}{dt}(t) \right\|^2_H \leq \frac{1}{2} \left\| f(t) \right\|^2_H.
\]
Now by Lemma 3.1 and the fact that $F$ and $G$ are bounded, there exists a positive constant $C_0$ such that $\|f(t)\|_H \leq C_0$ for all $t \geq T_0 + \tau$. Then, by the definition of a subdifferential and the Uniform Gronwall Lemma (see [28]), there exists a positive constant $C_1$ such that $\varphi^t_{\mathcal{P}(\cdot)}(u_1(t)) \leq C_1$ for all $t \geq T_1 + \tau$. Consequently, there exists a positive constant $K_1$ such that $\|u_1(t)\|_Y \leq K_1$ for all $t \geq T_1 + \tau$.

In a similar way, we conclude $\|u_2(t)\|_Y \leq K_2$ for all $t \geq T_1 + \tau$ for a positive constant $K_2$. The assertion of the lemma then follows.

Let $U_G$ be the multivalued process defined by the generalized process $\mathbb{G}$. We know from [23] that for all $t \geq s$ in $\mathbb{R}$ the map $x \mapsto U_G(t,s)x \in P(H \times H)$ is closed, so we obtain from Theorem 18 in [4] the following result.

**Theorem 3.3.** If for any $t \in \mathbb{R}$ there exists a nonempty compact set $D(t)$ which pullback attracts all bounded sets of $H \times H$ at time $t$, then the set $A = \{A(t)\}_{t \in \mathbb{R}}$ with $A(t) = \bigcup_{B \in \mathcal{B}(H \times H)} \omega(t,B)$, is the unique compact, negatively invariant pullback attracting set which is minimal in the class of closed pullback attracting nonautonomous sets. Moreover, the sets $A(t)$ are compact.

**Theorem 3.4.** The multivalued evolution process $U_G$ associated with system (S) has a compact, negatively invariant pullback attracting set $\mathcal{A} = \{A(t)\}_{t \in \mathbb{R}}$ which is minimal in the class of closed pullback attracting nonautonomous sets. Moreover, the sets $A(t)$ are compact.

**Proof.** By Lemma 3.2 we have that the family $D(t) := \overline{B_{Y \times Y}(0, r_1)}^{H \times H}$ of compact sets of $H \times H$ is attracting. The result thus follows from Theorem 3.3.

4. **Forward attraction.** Pullback attractors contain all of the bounded entire solutions of the nonautonomous dynamical system [11, 12]. Simple counterexamples show that a pullback attractor need not be attracting in the forward sense [11]. However, since the pullback absorbing set $D$ above is also forward absorbing (the absorption time is independent of the initial time $\tau$), the forward omega limit sets $\omega_f(\tau,D)$ of the multivalued process starting at time $\tau$ are nonempty and compact subsets of the compact set $D$. Moreover, it follows by the positive invariance of the $D$ and the two-parameter semi-group property that they are increasing in time. The forward limiting dynamics thus tends to the nonempty compact subset $\omega^f_\infty(D) = \cup_{\tau \geq 0} \omega_f(\tau,D) \subset D$, which was called the forward attracting set in [16]. (It is related to the Vishik uniform attractor, when that exists, but can be smaller since the attraction here need not be uniform in the initial time).

As shown in Proposition 8 of [16] (in the context of single valued difference equations, but a similar proof holds here) the forward attracting set $\omega^f_\infty(D)$ is asymptotically positively invariant with respect to the set valued process $U_G(t,\tau)$.
i.e., if for any monotone decreasing sequence \( \varepsilon_p \to 0 \) as \( p \to \infty \) there exists a monotone increasing sequence \( T_p \to \infty \) as \( p \to \infty \) such that for each \( \tau \geq T_p \)

\[
U_G(t, \tau) \omega_f^\infty(D) \subset B_{\varepsilon_p}(\omega_f^\infty(D)), \quad t \geq \tau,
\]

where \( B_{\varepsilon_p}(\omega_f^\infty(D)) := \{ x \in H \times H : \text{dist}_{H \times H} (x, \omega_f^\infty(D)) < \varepsilon_p \} \).

Simple counterexamples show that the set \( \omega_f^\infty(D) \) need not be invariant or even positive invariant, although it may be in special cases depending on the nature of the time varying terms in the system. For asymptotically autonomous systems \( \omega_f^\infty(D) \) is contained in the global attractor \( A_\infty \) for the multivalued semigroup \( G \) associated with the limiting autonomous system.

Moreover, it is possible to compare the global attractor \( A_\infty \) with the limit-set \( A(\infty) \) defined by \( A(\infty) := \bigcap_{t \in \mathbb{R}} \bigcup_{t \geq t} A(\tau) \) and which can be characterized by

\[
\bigcup_{r_n \nearrow \infty} \{ x \in X : \exists x_n \in A(r_n) \text{ s. t. } x_n \to x \}.
\]

This kind of comparison was done in [26] for the multivalued context.

**Theorem 4.1** ([26]). Suppose the pullback attractor \( A \) is forward compact, i.e., \( \bigcup_{t > \tau} A(\tau) \) is precompact for each \( t \in \mathbb{R} \). Moreover, suppose that for each solution \( u \) of problem (8) there exists a solution \( v \) of problem (9) such that \( u(t + \tau) \to v(t) \) in \( X \) as \( \tau \to +\infty \) for each \( t \geq 0 \) whenever \( \psi_\tau \in A(\tau) \) and \( \psi_\tau \to \psi_0 \) in \( X \) as \( \tau \to +\infty \). Then \( A_\infty \supset A(\infty) \).

To obtain the equality \( A_\infty = A(\infty) \) we need to assume stronger conditions as in the next result.

**Theorem 4.2** ([26]). Under the same assumptions of Theorem 4.1, we have \( A_\infty = A(\infty) \) if we further assume the following conditions:

(a) \( A(\infty) \) forward attracts \( A_\infty \) by \( U_G(\cdot, 0) \), i.e.,

\[
\lim_{t \to +\infty} \text{dist}(U_G(t, 0) A_\infty, A(\infty)) = 0;
\]

(b) \( \lim_{t \to +\infty} \sup_{x \in A_\infty} \text{dist}(G(t)x, U_G(t, 0)x) = 0 \).

5. Asymptotic upper semicontinuity. In this section we establish the asymptotic upper semicontinuity of the elements of the pullback attractor. Specifically, we prove that the system (S) is asymptotically autonomous.

5.1. **Theoretical results.** In this subsection motivated by problem (S), we study the asymptotic behavior of an abstract nonautonomous multivalued problem in a Hilbert space \( H \) of the form

\[
\begin{cases}
\frac{d u_1}{d t}(t) + A(t) u_1(t) \in F(u_1(t), u_2(t)) & t > \tau \\
\frac{d u_2}{d t}(t) + B(t) u_2(t) \in G(u_1(t), u_2(t)) & t > \tau \\
(u_1(\tau), u_2(\tau)) = (\psi_{1, \tau}, \psi_{2, \tau}) =: \psi_\tau,
\end{cases}
\]

compared with that of an autonomous multivalued problem of the form

\[
\begin{cases}
\frac{d v_1}{d t}(t) + A_\infty v_1(t) \in F(v_1(t), v_2(t)) & t > 0 \\
\frac{d v_2}{d t}(t) + B_\infty v_2(t) \in G(v_1(t), v_2(t)) & t > 0 \\
(v_1(0), v_2(0)) = (\psi_{1,0}, \psi_{2,0}) =: \psi_0,
\end{cases}
\]
where $A(t)$, $B(t)$, $A_\infty$ and $B_\infty$ are univalued operators in $H \times H$ and $F$, $G : H \times H \to P(H \times H)$ are multivalued maps.

Under appropriate relationships between the operators $A(t)$, $A_\infty$ and $B(t)$, $B_\infty$, the autonomous problem (9) is the asymptotic autonomous version of the nonautonomous problem (8) . In particular, we establish the convergence in the Hausdorff semi-distance of the component subsets of the pullback attractor of the nonautonomous problem (8) to the global autonomous attractor of the autonomous problem (9).

Some definitions on multivalued semigroups are recalled here, see for example [5, 17, 24] for more details.

**Definition 5.1.** Let $X$ be a complete metric space. The map $G : \mathbb{R}^+ \times X \to P(X)$ is called a multivalued semigroup (or $m$-semiflow) if 
(1) $G(0, \cdot) = 1$ is the identity map; 
(2) $G(t_1 + t_2, x) \subset G(t_1, G(t_2, x))$, for all $x \in X$ and $t_1, t_2 \in \mathbb{R}^+$.

It is called strict (or exact) if $G(t_1 + t_2, x) = G(t_1, G(t_2, x))$, for all $x \in X$ and $t_1, t_2 \in \mathbb{R}^+$.

**Definition 5.2.** Let $G$ be a multivalued semigroup on $X$. The set $A \subset X$ attracts the subset $B$ of $X$ if $\lim_{n \to \infty} \text{dist}_H(G(t, B), A) = 0$. The set $M$ is said to be a global $B$-attractor for $G$ if $M$ attracts any nonempty bounded subset $B \subset X$.

Suppose that the multivalued evolution process $\{U(t, \tau) : t \geq \tau\}$ in $H \times H$ associated with problem (8) has a pullback attractor $\mathfrak{A} = \{\mathfrak{A}(t) : t \in \mathbb{R}\}$ and that the multivalued semigroup $G : \mathbb{R}^+ \times H \times H \to P(H \times H)$ associated with problem (9) has a global autonomous $B$-attractor $\mathcal{A}_\infty$ in the Hilbert space $H \times H$. The following result will be used later to establish the convergence in the Hausdorff semi-distance of the component subsets $\mathcal{A}(t)$ of the pullback attractor $\mathfrak{A}$ to $\mathcal{A}_\infty$ as $t \to \infty$.

**Theorem 5.3.** Suppose that $\mathcal{C} := \overline{\bigcup_{\tau \geq 0} \mathcal{A}(\tau)}$ is a compact subset of $H \times H$. In addition, suppose that for each solution $u$ of problem (8) there exists a solution $v$ of problem (9) with initial values $\psi_\tau$ and $\psi_0$, respectively, such that $u(t + \tau) \to v(t)$ in $H \times H$ as $\tau \to +\infty$ for each $t \geq 0$ whenever $\psi_\tau \in \mathcal{A}(\tau)$ and $\psi_\tau \to \psi_0$ in $H$ as $\tau \to +\infty$. Then

$$\lim_{t \to +\infty} \text{dist}_{H \times H}(\mathcal{A}(t), \mathcal{A}_\infty) = 0.$$  

**Proof.** Suppose that this is not true. Then there would exist an $\epsilon_0 > 0$ and a real sequence $\{\tau_n\}_{n \in \mathbb{N}}$ with $\tau_n \nearrow +\infty$ such that $\text{dist}_{H \times H}(\mathcal{A}(\tau_n), \mathcal{A}_\infty) \geq 3\epsilon_0$ for all $n \in \mathbb{N}$. Since the sets $\mathcal{A}(\tau_n)$ are compact, there exist $a_n \in \mathcal{A}(\tau_n)$ such that

$$\text{dist}_{H \times H}(a_n, \mathcal{A}_\infty) = \text{dist}_{H \times H}(\mathcal{A}(\tau_n), \mathcal{A}_\infty) \geq 3\epsilon_0,$$

for each $n \in \mathbb{N}$. By the attraction property of the multivalued semigroup, we have

$$\text{dist}_{H \times H}(G(\tau_0, \mathcal{C}), \mathcal{A}_\infty) \leq \epsilon_0$$

for $\tau_0 > 0$ large enough. Moreover, by the negative invariance of the pullback attractor there exist $b_n \in \mathcal{A}(\tau_n - \tau_0) \subset \mathcal{C}$ for $n > n_0$ such that $a_n \in U(\tau_n, \tau_n - \tau_0) b_n$ for each $n > n_0$. Since $\mathcal{C}$ is compact, there is a convergent subsequence $b_{n'} \to b \in \mathcal{C}$. Since $a_{n'} \in U(\tau_{n'}, \tau_{n'} - \tau_0) b_{n'}$ there exists
For each Assumption G, such that $a_{n'} = u_{n'}(\tau_{n'})$.

Writing $\tau_{n'} = \tau_{n_0} + (\tau_{n'} - \tau_{n_0})$ and using the hypotheses with $t = \tau_{n_0}$ and $\tau = \tau_{n'} - \tau_{n_0} \to +\infty$ (as $n' \to +\infty$), there exists a solution $v_{n'}$ of

$$\begin{cases}
\frac{dv_{1n'}}{dt}(t) + A\infty v_{1n'}(t) \in F(v_{1n'}(t),v_{2n'}(t)) \\
\frac{dv_{2n'}}{dt}(t) + B\infty v_{2n'}(t) \in G(v_{1n'}(t),v_{2n'}(t)) \\
v_{n'}(0) = b,
\end{cases}$$

such that $\|u_{n'}(\tau_{n'}) - v_{n'}(\tau_{n_0})\|_{H \times H} < \epsilon_0$

for $n'$ large enough. Hence,

$$\text{dist}_{H \times H}(a_{n'}, A\infty) = \text{dist}_{H \times H}(u_{n'}(\tau_{n'}), A\infty) \leq \|u_{n'}(\tau_{n'}) - v_{n'}(\tau_{n_0})\|_{H \times H} + \text{dist}_{H \times H}(v_{n'}(\tau_{n_0}), A\infty) \leq 2\epsilon_0,$$

which contradicts (10).

The next result is very useful for checking that the hypothesis of asymptotic continuity of the nonautonomous flow in the preceding theorem for problems like (8) holds. In order to obtain the result we suppose that the operators $A(t)$ and $A\infty$ satisfy the following assumption.

**Assumption G.** For each $\tau \in \mathbb{R}$ there exist non increasing functions $g_{1,\tau}$, $g_{2,\tau} : [0, +\infty) \to [0, +\infty)$ such that $g_{i,\tau}(t) \to 0$ as $\tau \to +\infty$, for each $t \geq 0$, $i = 1, 2$, and

$$\langle A(t + \tau)u_1(t + \tau) - A\infty v_1(t), u_1(t + \tau) - v_1(t) \rangle \geq -g_{1,\tau}(t), \text{ for all } t \in \mathbb{R}^+, \tau \in \mathbb{R},$$

and

$$\langle B(t + \tau)u_2(t + \tau) - B\infty v_2(t), u_2(t + \tau) - v_2(t) \rangle \geq -g_{2,\tau}(t), \text{ for all } t \in \mathbb{R}^+, \tau \in \mathbb{R},$$

for any solution $u = (u_1, u_2)$ of (8) and $v = (v_1, v_2)$ of (9).

**Lemma 5.4.** Suppose that Assumption G is satisfied. If $\psi_{\tau} = (\psi_{1,\tau}, \psi_{2,\tau}) \to \psi_0 = (\psi_{1,0}, \psi_{2,0})$ in $H \times H$ as $\tau \to +\infty$, then for each solution $u$ of (8) there exists a solution $v$ of (9) such that $u(t + \tau) \to v(t)$ in $H \times H$ as $\tau \to +\infty$ for each $t \geq 0$. 

\[\square\]
Proof. Let \( u \) be a solution of (8) then there exists \( f = (f_1, f_2) \) with \( f_1, f_2 \in L^2([\tau, T]; H) \) such that 
\[
\begin{align*}
\frac{du_1}{dt}(t) + A(t)u_1(t) &= f_1(t), \quad \text{a.e in } (\tau, T], \\
\frac{du_2}{dt}(t) + B(t)u_2(t) &= f_2(t), \quad \text{a.e in } (\tau, T], \\
u(\tau) &= \psi_r. 
\end{align*}
\]
(11)

Consider \( g \in L^2([0, T]; H \times H) \) such that \( g(t) = f(t + \tau) \) and let \( v \) be the unique solution of the problem
\[
\begin{align*}
\frac{dv_1}{dt}(t) + A_{\infty}v_1(t) &= g_1(t), \quad \text{a.e in } (0, T], \\
\frac{dv_2}{dt}(t) + B_{\infty}v_2(t) &= g_2(t), \quad \text{a.e in } (0, T], \\
v(0) &= \psi_0.
\end{align*}
\]
(12)

Subtracting the equations in (11) from the equations in (12) gives
\[
\frac{d}{dt}(u_1(t + \tau) - v_1(t)) + A(t + \tau)u_1(t + \tau) - A_{\infty}v_1(t) = f_1(t + \tau) - g_1(t)
\]
and
\[
\frac{d}{dt}(u_2(t + \tau) - v_2(t)) + B(t + \tau)u_2(t + \tau) - B_{\infty}v_2(t) = f_2(t + \tau) - g_2(t)
\]
for a.e. \( t \in [0, T] \). Multiplying by \( u_i(t + \tau) - v_i(t) \) and taking the inner product, then using Assumption G, we obtain
\[
\frac{1}{2} \frac{d}{dt} \|u_i(t + \tau) - v_i(t)\|_H^2 \leq g_{i, \tau}(t), \quad i = 1, 2.
\]
Integrating this last inequality from 0 to \( t \), gives
\[
\|u_i(t + \tau) - v_i(t)\|_H^2 \leq \|\psi_{i, \tau} - \psi_{i, 0}\|_H^2 + 2\int_0^t g_{i, \tau}(s) ds.
\]
Since \( \psi_{i, \tau} \rightarrow \psi_{i, 0} \) in \( H \) and \( g_{i, \tau}(0) \rightarrow 0 \) as \( \tau \rightarrow +\infty \), the result follows.

5.2. Application to system \((S)\). The results in Subsection 5.1 are applied here to the nonlinear system of inclusions with spatially variable exponents \((S)\) in the Hilbert space \( H = H \times H \), with \( H := L^2(\Omega) \).

We assume that the diffusion coefficients satisfy Assumption D and the additional Assumption D (iii) that follows:

**Assumption D (iii).** For each \( t \geq 0 \), \( D_i((t + \tau, \cdot) \rightarrow D_i^*(\cdot) \) in \( L^\infty(\Omega) \) as \( \tau \rightarrow +\infty \), for \( i = 1, 2 \).

Assumptions D (i)–D (ii) imply that the pointwise limit \( D_i^*(x) \) as \( t \rightarrow \infty \) exists and satisfies \( 0 < \beta \leq D_i^*(x) \) for almost all \( x \in \Omega \), \( i = 1, 2 \). Then the problem \((S)\) with \( D^*(x) = (D_1^*(x), D_2^*(x)) \) is autonomous and has a global autonomous \( B \)-attractor as a particular case of the results in Section 3 (see also a direct proof in [25] for the autonomous system of inclusions without the nonlinear perturbation \( |u|^{p(c)-2}u \)).

We will show that the dynamics of the original nonautonomous problem is asymptotically autonomous and its pullback attractor converges upper semicontinuously.
to the autonomous global $B$-attractor $A_\infty$ of the problem
\begin{align}
\frac{\partial v_1}{\partial t}(t) & - \text{div} \left(D_1^*|\nabla v_1(t)|^{p(x)-2}\nabla v_1(t)\right) + |v_1(t)|^{p(x)-2}v_1(t) \in F(v_1(t), v_2(t)), \\
\frac{\partial v_2}{\partial t}(t) & - \text{div} \left(D_2^*|\nabla v_2(t)|^{q(x)-2}\nabla v_2(t)\right) + |v_2(t)|^{q(x)-2}v_2(t) \in G(v_1(t), v_2(t)), \\
v(0) & = \psi_0.
\end{align}
(13)

In particular, we consider the operators
\begin{align*}
A(t)u_1 & := -\text{div} \left(D_1(t)|\nabla u_1|^{p(x)-2}\nabla u_1\right) + |u_1|^{p(x)-2}u_1, \\
B(t)u_2 & := -\text{div} \left(D_2(t)|\nabla u_2|^{q(x)-2}\nabla u_2\right) + |u_2|^{q(x)-2}u_2, \\
A_\infty v_1 & := -\text{div} \left(D_1^*|\nabla v_1|^{p(x)-2}\nabla v_1\right) + |v_1|^{p(x)-2}v_1, \\
B_\infty v_2 & := -\text{div} \left(D_2^*|\nabla v_2|^{q(x)-2}\nabla v_2\right) + |v_2|^{q(x)-2}v_2.
\end{align*}

Applying Lemma 3.1, there exist positive constants $T_0$, $B_0$ such that
\[\|u(t)\|_{H \times H} \leq B_0, \quad \forall t \geq T_0 + \tau.\]
Moreover, applying Lemma 3.2 for $Y = W^{1,p(x)}(\Omega)$, there exist positive constants $T_1$, $B_1$ such that
\[\|u(t)\|_{Y \times Y} \leq B_1, \quad \forall t \geq T_1 + \tau.\]
(14)
Since also $\|v(t)\|_{Y \times Y} \leq B_1$ for all $t \geq T_1 + \tau$ and $Y \subset H$ with compact embedding, it follows

**Corollary 1.** $\bigcup_{\tau \in \mathbb{R}} A(\tau)$ is a compact subset of $H \times H$.

Using estimate (14), the proof of the next result follows the same lines as the proof of Theorem 4.2 of [14], and therefore is omitted here.

**Theorem 5.5.** If $\{\psi_\tau : \tau \in \mathbb{R}\}$ is a bounded set in $Y \times Y$ and $\psi_\tau \to \psi_0$ in $H \times H$ as $\tau \to +\infty$, then Assumption G is satisfied with $g_{0,\tau}(t) = K \|D_i(t+\tau, \cdot) - D_i^*(\cdot)\|_{L^\infty(\Omega)}$, $(i = 1, 2)$ where $K$ is a positive constant.

Observe that by Assumption D (iii) the function $g_{0,\tau} : [0, +\infty) \to [0, +\infty)$ given in Theorem 5.5 satisfies $g_{0,\tau}(t) \to 0$ as $\tau \to +\infty$ for each $t \geq 0$. The next result gives the desired asymptotic upper semi-continuous convergence.

**Theorem 5.6.** $\lim_{t \to +\infty} \text{dist}_{H \times H}(A(t), A_\infty) = 0$.

**Proof.** Suppose that $\psi_\tau \in A(\tau)$ and $\psi_\tau \to \psi_0$ in $H \times H$. Using the negatively invariance of the pullback attractor and the estimate (14) it follows that $\{\psi_\tau : \tau \in \mathbb{R}\}$ is a bounded set in $Y \times Y$. Theorem 5.5 then guarantees that Assumption G is satisfied. Thus, from Lemma 5.4, for each solution $u = (u_1, u_2)$ of (S) there exists a solution $v = (v_1, v_2)$ of (13) such that $u(t+\tau) \to v(t)$ in $H \times H$ as $\tau \to +\infty$ for each $t \geq 0$. Theorem 5.3 then yields $\lim_{t \to +\infty} \text{dist}(A(t), A_\infty) = 0$. \qed

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