π − π Scattering Lengths and Chiral Condensate in NJL Model

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Within the general framework of SU(2) Nambu-Jona-Lasinio model beyond mean-field approximation, the S-wave π − π scattering lengths \( a_0^0 \) and \( a_0^2 \) are calculated. The numerical results are in agreement with that analyzed from recent experiment \( \pi N \rightarrow \pi \pi N \) data by using Chew-Low-Goebel technique, and the corresponding chiral condensate \( \langle \bar{q}q \rangle \) is around \(-(250\text{MeV})^3\), which is close to the ones obtained from lattice and sum rules.
I. INTRODUCTION

Spontaneous chiral symmetry breaking ($S\chi$SB) is one of the important characteristics of QCD. The standard scenario assumes that the mechanism of $S\chi$SB is triggered by the formation of a large $<\bar{q}q>$ in the QCD vacuum. And there is an alternative scenario regarding that $S\chi$SB can be realized even with a small or vanishing $<\bar{q}q>$ in confining vector-like gauge theories [1] [2].

$\pi - \pi$ scattering at threshold will provide a test to the two pictures of $S\chi$SB. For a summary of current experimental information can refer Ref. [3]. In the near future, there will be several experiments dedicated to the precise determination of the $\pi - \pi$ scattering lengths, including semileptonic $K_{e4}$ decays from E865 [4] and KLOE [5], and the measurement of the pionium lifetime (DIRAC) [6], which directly determines the S-wave $\pi - \pi$ scattering length difference $a_0^0 - a_0^2$.

On the theoretical side, the S-wave $\pi - \pi$ scattering lengths $a_0^0$ and $a_0^2$ have been investigated in the standard scenario by using the standard chiral perturbation theory up to two-loops [7] [8] and the Nambu-Jona-Lasinio (NJL) model to the leading order of $1/N_c$ expansion where $N_c$ is the number of color freedoms [9] - [11]. Also $a_0^0$ and $a_0^2$ have been calculated in the alternative scenario [12] by using the generalized chiral perturbation theory formulated by Orsay group [13].

The standard scenario of $S\chi$SB is usually motivated by the NJL model [15], in which $S\chi$SB is associated with large quark condensate. The effect of explicit chiral symmetry breaking induced by a small current quark mass $m_0$ which is important to discuss the process of $\pi - \pi$ scattering at threshold. As we have analyzed in [16], only beyond mean-field approximation, can the effect of massive pions be seen clearly. Our interest in this paper is to study the S-wave $\pi - \pi$ scattering lengths $a_0^0$ and $a_0^2$ in a self-consistent SU(2) NJL model [17] with explicit chiral symmetry breaking [16].
II. SU(2) NJL MODEL BEYOND MEAN-FIELD APPROXIMATION

The two-flavor NJL model is defined through the Lagrangian density,

\[ \mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m_0)\psi + G[(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\vec{\tau}\psi)^2], \]  

(1)

where \( G \) is the effective coupling constant with dimension GeV\(^{-2} \), \( m_0 \) the current quark mass, and \( \psi, \bar{\psi} \) quark fields with flavor, colour and spinor indices suppressed, assuming isospin degeneracy of the \( u \) and \( d \) quarks.

We first briefly review the general scheme of the NJL model beyond mean-field approximation \[10\]. Including current quark mass explicitly, the quark self-energy \( m \) expanded to \( O(1/N_c) \) order can be expressed as

\[ m = m_0 + m_H + \delta m, \]  

(2)

where \( m_H \) and \( \delta m \) are the leading \( O(1) \) and subleading \( O(1/N_c) \) contributions shown in Fig. 1. The solid lines in Fig. 1 indicate quark propagator \( S(p) = 1/(p - m) \) with full \( m \). The quark condensate \( <\bar{q}q> \) is a one-loop quark integral

\[ <\bar{q}q> = \frac{1}{N_f} <\bar{\psi}\psi> = -4iN_cm \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 - m^2}. \]  

(3)

The corresponding meson propagator \( D_M(k) \) (\( M \) means \( \pi \) or \( \sigma \)) has the form

\[ -iD_M(k) = \frac{2iG}{1 - 2G\Pi_M(k)}, \]

\[ \Pi_M(k) = \Pi^{(RPA)}_M(k) + \delta\Pi^b_M(k) + \delta\Pi^c_M(k) + \delta\Pi^d_M(k), \]  

(4)

where \( \Pi_M \) is the meson polarization function, which includes the leading order \( \Pi^{(RPA)}_M \) and subleading order \( \delta\Pi^{(b,c,d)}_M \), shown in Fig. 1. The above constituent quark mass and the meson propagator, namely the Eqs. (2) and (4), or the Feynman diagrams in Fig. 1, form a self-consistent description of the SU(2) NJL model to the subleading order of \( 1/N_c \) expansion, which is different from the earlier calculations \[3 - 11\] where only the mean-field quark mass \( m_H \) and the Random-Phase-Approximation (RPA) meson polarization function \( \Pi^{(RPA)}_M \) are considered.
The meson mass $m_M$ satisfies the total meson propagator’s pole condition

$$1 - 2G\Pi_M(k^2 = m_M^2) = 0,$$

and the meson-quark coupling constant $g_{Mqq}$ is determined by the residue at the pole

$$g_{Mqq}^{-2} = (\partial \Pi_M(k)/\partial k^2)^{-1}|_{k^2 = m_M^2}.$$

Another important quantity in the meson sector is the pion decay constant $f_\pi$ which generally satisfies

$$\frac{m_\pi^2 f_\pi}{g_{\pi qq}} = \frac{m_0}{2G}.$$  \hfill (7)

In the chiral limit, $f_\pi$ satisfies the Goldberger-Treiman relation $f_\pi(k)g_{\pi qq}(k) = m$.

![Feynman diagrams](image)

**FIG. 1.** Feynman diagrams for leading and subleading order quark self-energy $m_H$ and $\delta m$, and meson polarization functions $\Pi_M^{(RPA)}(k)$, $\delta\Pi_M^{(b)}(k)$, $\delta\Pi_M^{(c)}(k)$, and $\delta\Pi_M^{(d)}(k)$. The solid and dashed lines indicate the quark internal meson propagators respectively.
III. \(\pi - \pi\) SCATTERING AT THRESHOLD

Now we turn to the calculation of the S-wave \(\pi - \pi\) scattering length \(a_0^0\) and \(a_0^2\). The invariant scattering amplitude can be generally written as

\[
T_{ab,cd} = A(s, t, u)\delta_{ab}\delta_{cd} + B(s, t, u)\delta_{ac}\delta_{bd} + C(s, t, u)\delta_{ad}\delta_{bc},
\]

where \(a, b\) and \(c, d\) are the isospin labels of the initial and final states respectively, and \(s, t\) and \(u\) are the Mandelstam variables, \(s = (p_a + p_b)^2\), \(t = (p_a - p_c)^2\), \(u = (p_a - p_d)^2\), where \(p_a \sim p_d\) are initial and final isospin momentum. Using perfect crossing symmetry, one can project out isospin amplitudes

\[
A_0 = 3A + B + C, \quad A_1 = B - C, \quad A_2 = B + C. \tag{9}
\]

In the limit of scattering at threshold, \(\sqrt{s} = 2m_\pi\), \(t = u = 0\), the S-wave scattering lengths \(a_I^I\) (given in units of \(m_\pi^{-1}\)) are:

\[
a_I^I = \frac{1}{32\pi} A_I(s = 4m_\pi^2, t = 0, u = 0), \quad I = 0, 2. \tag{10}
\]

We will calculate the scattering length to the lowest order in \(1/N_c\) expansion of the process \(\pi\pi \rightarrow \pi\pi\) in the NJL model with quark mass \(m\) and meson polarization function \(\Pi_M\) including the subleading order of \(1/N_c\) expansion. The Feynman diagrams include box and \(\sigma\)-exchange diagrams, and are the same as those in [11], but in [11] quark mass \(m_H\) in mean-field approximation and meson polarization function in RPA \(\Pi_{\text{RPA}}^M\).

To keep the lowest order of the \(\sigma\) exchange diagrams, the internal \(\sigma\) propagator should be in leading \(O(1/N_c)\) order, which has the same form as that in RPA \(\Pi_{\text{RPA}}^M\). So what we need to do is using the expression formulae given in [11] directly, and replacing the quantities in the mean-field approximation with those beyond mean-field approximation, i.e., the external pion propagator becomes the total propagator Eq. (4), the pole for quark propagator becomes the total quark self-energy \(m\) defined in Eq. (2), and the coupling constant \(g_{\pi qq}\) now is the one expressed in Eq. (6).
IV. NUMERICAL RESULTS

For the numerical calculations, we adopt the external momentum expansion method as discussed in detail in [16]. We introduce a quark momentum cut-off $\Lambda_f$ in Pauli-Villars regularization and a meson momentum cut-off $\Lambda_b$ in covariant regularization for the divergent momentum integrals. Using only the two experimental observables $m_\pi = 139\text{MeV}$ and $f_\pi = 92.4\text{MeV}$ and the reasonable quark condensate $<\bar{q}q>^{1/3}$ in the empirical range of $-300\text{MeV} \sim -200\text{MeV}$ as input, we can not give fixed values of the four parameters in the model, namely the current quark mass $m_0$, coupling constant $G$, and the two momentum cuts $\Lambda_f$ and $\Lambda_b$. So we should introduce one more free parameter $z = \Lambda_b/\Lambda_f$, which characterizes the meson cloud contributions. Especially, in the limit of $z = 0$, the model goes back to the mean-field approximation automatically.

As we discussed in [16], for each $z$, quark condensate $-<\bar{q}q>^{1/3}$ has a minimum $(-<\bar{q}q>^{1/3})_{\text{min}}$, around which there is a region quark condensate changes slowly with quark mass $m$, and in this region of $m$, $m_0$ and $\Lambda_f$ are in plateaus too. In Table. 1, we list the values of $m$, $m_0$ and $\Lambda_f$ corresponding to $(-<\bar{q}q>^{1/3})_{\text{min}}$ (in units of GeV).

|            | $z = 0$ | $z = 0.5$ | $z = 1$ | $z = 1.5$ |
|------------|---------|-----------|---------|-----------|
| $-<\bar{q}q>^{1/3}$ | 0.209   | 0.216     | 0.237   | 0.256     |
| $m$        | 0.49    | 0.47      | 0.36    | 0.34      |
| $m_0$      | 0.0089  | 0.0086    | 0.0075  | 0.0067    |
| $\Lambda_f$| 0.615   | 0.63      | 0.73    | 0.82      |

Table. 1. The values of $m$, $m_0$ and $\Lambda_f$ corresponding to $(-<\bar{q}q>^{1/3})_{\text{min}}$ (in units of GeV) for different $z = \Lambda_b/\Lambda_b$.

For each $z$, with changing $<\bar{q}q>$, we get a series of $m$ and $g_{\pi qq}$, from which we can calculate the S-wave $\pi - \pi$ scattering length $a_0^0$ and $a_0^2$. In the quark condensate plateau region around $-<\bar{q}q>^{1/3}$, there is also a plateau for $a_0^0$ and $a_0^2$ respectively. We choose the values of $a_0^0$ and $a_0^2$ corresponding to $(-<\bar{q}q>^{1/3})_{\text{min}}$ to compare with those analyzed from experimental data and predicted by other theories, which are listed in Table. 2, where
all the values are in units of $m^{-1}_\pi$. In this table, the experimental values of $a_0^0$ and $a_2^0$ without star were obtained in 1979 in a comprehensive phase shift analysis of peripheral $\pi N \rightarrow \pi \pi N$ reactions and $K_{e4}$ decays \[19\]; those marked with star were from the Chew-Low-Goebel analyses of the recent CHAOS data $\pi^- p \rightarrow \pi^+ \pi^0 n$, and LAMPF E1179 data $\pi^+ p \rightarrow \pi^+ \pi^0 p$ \[20\]. And the experimental value of $a_0^0 - a_2^0$ was obtained from the universal curve in \[8\].

| NJL($z = 0, z = 0.5, z = 1, z = 1.5$) | $exp$          | $a$  | $b$  |
|--------------------------------------|----------------|------|------|
| $a_0^0$                              | 0.16           | 0.26±0.05 | 0.217 | 0.263 |
|                                      | 0.17           | * 0.206±0.013 |      |      |
| $a_2^0$                              | -0.044         | -0.028±0.012 | -0.0413 | -0.027 |
|                                      | -0.046         | * -0.055±0.021 |      |      |
| $a_0^0 - a_2^0$                       | 0.20           | 0.29±0.04 | 0.258 | 0.29  |

Table. 2. Scattering lengths predicted by NJL model in different $z$ compare with the experimental values ($exp$) and those calculated from chiral perturbation theory ($a$) \[8\] and generalized chiral perturbation theory ($b$) \[13\] up two-loop.

From Table. 2 we can see that, $a_0^0$ and $a_2^0$ calculated from NJL model at a certain $z$ between 1 ∼ 1.5 are in agreement with the recent experimental values marked with star, and the scattering lengths at $z = 0$ are the same as the Weinberg values $a_0 = 0.16$ and $a_2 = -0.044$ \[21\]. Comparing with those at $z = 0$, the scattering lengths at $1 < z < 1.5$ are all modified by 20 ∼ 40%.

In the region of $z$ between 1 ∼ 1.5 in Table. 1, the quark mass $m$ is about 333 MeV, and current quark mass $m_0$ is in the empirical region $5MeV ∼ 7MeV$, and the quark condensate $- < \bar{q}q >$ is between $(237MeV)^3 ∼ (256MeV)^3$, which is in agreement with that $(245 ± 4 ± 9 ± 7MeV)^3$ from the lattice simulation \[22\] and that $(242 ± 9MeV)^3$ from the sum rule calculation (see \[23\] and references therein).

It is can be seen from Table. 2, that the recent experimental results support the standard chiral mechanism, i.e., $S\chi SB$ is triggered by the formation of large quark condensate in QCD vacuum. While the scattering lengths predicted by two-loop generalized chiral perturbation theory, in which the quark condensate $< \bar{q}q >$ is around $(-100MeV)^3$, are in agreement with
the old experimental values without star. Anyway, which SχSB mechanism is right, should be answered after the analyses of the new experimental data in the near future.

If the elastic pion-pion scattering experimental data in the near future support the mechanism of large quark condensate, the free parameter \( z \) in the NJL model introduced to characterize the mesonic contributions can be fixed by the scattering lengths. Then one can determine the four parameters \( m_0, G, \Lambda_f \) and \( \Lambda_b \). From the above analyses, it seems that \( z \) is between 1 ∼ 1.5 for the NJL model.

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