AN EXISTENCE RESULT FOR DISCRETE DISLOCATION DYNAMICS IN THREE DIMENSIONS

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ABSTRACT. We present a mathematical framework within which Discrete Dislocation Dynamics in three dimensions is well–posed. By considering smooth distributions of slip, we derive a regularised energy for curved dislocations, and rigorously derive the Peach–Koehler force on the dislocation network via an inner variation. We propose a dissipative evolution law which is cast as a generalised gradient flow, and using a discrete–in–time approximation scheme, existence and regularity results are obtained for the evolution, up until the first time at which an infinite density of dislocation lines forms.

1. INTRODUCTION

In crystalline materials, plastic behaviour is characterised by the generation of slip, which is the process by which the planes of the material’s lattice structure are reordered. As the material deforms, slip is propagated via the motion of dislocations, which are topological line defects found in regions where the lattice mismatch required for slip to occur is most concentrated [45, 50]. From the very beginnings of the study of dislocations [63,69,76], linear continuum theories have been used to model these defects with great success [9,14,18,19,60,67,79], despite the fact that unphysical singularities are induced in the stress, strain and energy density fields at the dislocation lines. These singularities are a signature of the breakdown of the assumption that the material behaves as a continuum close to dislocation lines, and although no continuum theory is able to accurately capture the properties of dislocation cores, continuum approaches have nevertheless been highly successful at capturing bulk behaviour. Indeed, in a series of recent mathematical works it has been rigorously demonstrated that linear elastoplasticity theory provides an excellent prediction of the strain caused by defects on the atomistic scale [16,34,48,49].

Since dislocation motion determines the plastic behaviour of crystalline materials, one of the principal aims of studying these objects is to understand the physical laws which govern their microscopic motion, and consequently, to obtain an accurate description of the evolution of crystal plasticity on a macroscopic scale. To that end, two broad approaches to modelling dislocation motion have developed. Phase field models consider a continuous distribution of dislocations, an approach which has its roots in the classic Peierls–Nabarro model [62,68]; modern examples of such modelling approaches include [43,51,60]. A variety of mathematical results concerning models in this class have been obtained, including well–posedness [10,22], long–time asymptotics [64,66], and homogenisation results [28,39,40,55,56]. While these models have good mathematical structure, they are typically limited to considering only one family of slip planes at once; moreover, simulating phase field models on a microscopic scale is computationally intensive, since a high resolution mesh is required to accurately resolve the level sets corresponding to individual dislocations.

The second approach is Discrete Dislocation Dynamics (DDD), in which dislocations lines are described as curves within the crystal, and are driven by the action of the Peach–Koehler force [67]. The fact that the dislocations alone are tracked in this approach has the advantage of drastically reducing the computational complexity in comparison with phase field approaches, and as such, DDD has been used as a simulation technique for studying plasticity since the early 1990s [5,6,8,17,44,77]. While a significant mathematical literature has developed which considers one– and two–dimensional DDD models for the motion of straight dislocations [12,13,14,23,24,46,47,78], few mathematical results concerning DDD in a three–dimensional setting exist to date, and in part, this appears to be due to the lack of a clear mathematical statement of what the evolution problem for DDD should be in this setting.

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This paper therefore seeks to bring together many of the ideas already present in the literature in laying out a well-posed mathematical formulation of DDD which is both general enough to encompass evolution problems similar to those considered by Materials Scientists and Engineers simulating DDD in practice, and mathematically concrete enough to allow the development of further results concerning dislocations in three dimensions. In particular, we hope to open paths towards a deeper mathematical understanding of the numerical schemes used to simulate DDD in practice.

1.1. **A regularised theory of DDD in three dimensions.** As mentioned above, we seek to develop a well-posed mathematical theory of DDD in three dimensions. In order to be physically-relevant, practical for computation and amenable to mathematical analysis, we make several requirements of this theory:

1. Dislocations should be curved and satisfy the physically-necessary condition that they are the boundaries of regions of slip.
2. The stress, strain and internal energy density induced in the material by the presence of dislocations are required to be non-singular.
3. The energy of and Peach–Koehler force on a configuration of dislocations should take on explicit expressions in terms of an integral kernel which are computable with a quantifiable error.
4. The underlying material is assumed to be linearly elastic, but need not be isotropic.
5. Dislocation motion should dissipate internal energy.

The first of these conditions is a kinematic requirement for a theory of dislocations to make sense. In common with several recent mathematical works [27, 29, 73, 74], this condition is encoded by describing dislocations as **closed 1–currents**, which may be viewed closed oriented Lipschitz curves satisfying certain topological constraints.

In order to satisfy the second condition, we construct a regularised version of the classical linear theory via a similar approach to that used in [29]. Starting from a regularised distribution of slip and using the ideas of Mura [61], we derive an expression for the internal energy as a double integral which depends on the boundary of the slip surfaces alone. As this expression depends only on the boundaries of the surfaces over which slip has occurred, which correspond exactly to the dislocations in the material, this energy is furthermore consistent with the first requirement made above. Using the Fourier analytic ideas of [9], we study the energy, providing a computable expression for the integral kernel without requiring an explicit expression of the elastic Green’s function, which renders the theory broad enough to satisfy both the third and fourth conditions. By performing an inner variation of the energy with respect to the positions of dislocations, we are able to rigorously derive an expression for the Peach–Koehler force.

To fulfill the final condition, we choose to formulate an evolution law for DDD as a generalised gradient flow [4] by prescribing a dissipation potential expressed in terms of the velocity field perpendicular to the dislocation line. This framework enables us to prove our main result, that the evolution problem is well-posed.

1.2. **Outline.** §1.3 provides a record of the notation used throughout the paper for the reader’s convenience, and §2 is devoted to an exposition of the main results of the paper, which are Theorem [1] providing the properties of the regularised energy; Theorem [2] which provides properties of the configurational force on dislocations; and Theorem [3] which asserts the well-posedness of the model for DDD considered here.

The subsequent sections of the paper are then devoted to proving these results. §3 describes the regularisation procedure applied to dislocations and derivation of the explicit representation of the energy due to dislocations; §4 derives the Peach–Koehler (or configurational) force of a dislocation by inner variation; and §5 prove existence, uniqueness, and regularity results for the evolution.

As mentioned above, we consider configurations of slip and dislocations as integral currents [32, 35, 36, 57], since these are the correct mathematical objects to describe the topological restrictions on dislocations [7, 27, 29, 73, 74]. Currents generalise the notion of distributions [57] to a geometric setting, and while the theory of these objects can be forbidding, in our setting, the reader should always have in mind surfaces and curves. For convenience, Appendix A recalls the definitions and basic theory related to these objects that is used here.

1.3. **Notation.** The following notational conventions will be used throughout the paper.
1.3.1. Tensors.

- Important tensors which are fixed throughout (rather than variables) are generally denoted using sans serif fonts, e.g. $K^\epsilon$.
- Subscript indices always refer to components in Cartesian coordinates, e.g. $f_{abc}$.
- Subscript indices appearing after a comma denote partial derivatives: e.g. $f_{i,j} = \frac{\partial f_i}{\partial x_j}$.
- The Einstein summation convention is used throughout, so repeated indices within an expression are always summed, e.g. $a_{ijlk} b_{k} = \sum_{l,k=1}^3 a_{ijlk} b_{k}$.
- Tensor products of vectors are denoted $\otimes$.
- $\wedge$ denotes the usual alternating product acting on vectors and covectors (for further details, see §A).
- The dot products between vectors in $\mathbb{R}^3$ is denoted $\cdot$.
- $I \in \mathbb{R}^{3 \times 3}$ always denotes the identity matrix.
- $A \in \mathbb{R}^{3 \times 3, 3 \times 3}$ denotes the alternating tensor, which satisfies $A_{ijk} = \begin{cases} +1 & (ijk) \text{ is an even permutation of } (123) \\ -1 & (ijk) \text{ is an odd permutation of } (123) \\ 0 & \text{ otherwise.} \end{cases}$
- $C \in \mathbb{R}^{3 \times 3 \times 3 \times 3}$ denotes an elasticity tensor, which satisfies the major symmetry $C_{abcd} = C_{cdab}$, the minor symmetries $C_{abcd} = C_{bacd} = C_{abdc}$ and a Legendre–Hadamard condition, i.e. there exists $c_0 > 0$ such that $C_{abcd} v_a k_b v_c k_d \geq c_0 |v|^2 |k|^2$ for all $v, k \in \mathbb{R}^3$.
- $G : \mathbb{R}^3 \to \mathbb{R}^{3 \times 3}$ denotes the elastic Green’s function, i.e. the fundamental solution of the differential operator $-C_{abcd} v_c, v_d$, which solves

\begin{equation}
- C_{ijkl} G_{jk, kl} = I_{\alpha \beta} \delta_{0} \tag{1}
\end{equation}

in the sense of distributions.
- $G^\varepsilon := G * \varphi^\varepsilon$ denotes a regularised version of the elastic Green’s function, solving

\begin{equation}
- C_{ijkl} G^\varepsilon_{jk, kl} = I_{\alpha \beta} \varphi^\varepsilon \tag{2}
\end{equation}

in the sense of distributions, where $\varphi^\varepsilon$ is a smooth, positive, radially symmetric function satisfying \( \int_{\mathbb{R}^3} \varphi^\varepsilon \, dx = 1 \).

1.3.2. Sets, currents, measures and integration.

- $B(s)$ denotes the closed ball of radius $r$, centred at $s \in \mathbb{R}^3$.
- $\Sigma$ will denote a 2–rectifiable subset of $\mathbb{R}^3$ (i.e. a generalised surface) and $\Gamma$ will denote a 1–rectifiable subset of $\mathbb{R}^3$ (i.e. a generalised curve).
- $T$ and $S$ will denote currents, and $\partial$ is the usual boundary operator.
- $\mathcal{L}(\mathbb{R}^3; \mathcal{L})$ and $\mathcal{D}(\mathbb{R}^3; \mathcal{L})$ denote space of integral 1– and 2–currents with multiplicities in a given lattice $\mathcal{L} \subset \mathbb{R}^3$.
- $S \Subset A$ denotes the restriction of a current $S$ to $A$.
- $F \# S$ denotes the pushforward of the current $S$ by $F$, i.e. the current corresponding to the image $F(S)$.
- $\text{M}(S)$ denotes the mass of a current $S$, defined in §4.
- $\Theta(S)$ denotes the maximal mass ratio of a current $S$, defined in §6.
- $H^m$ denotes the $m$–dimensional Hausdorff measure on $\mathbb{R}^3$.
- $\int_A f(t) \, d\mu(t)$ denotes the Lebesgue integral of a Borel measurable function $f$ with respect to the measure $\mu$ restricted to a Borel–measurable set $A$.

For some additional details on the basic theory of currents, see Appendix A.

1.3.3. Functions and function spaces.

- Lebesgue, Sobolev and Hölder spaces are all given standard notation, i.e. $L^p$, $H^k$, $C^k$, as are their norms.
- The $C^{0,\gamma}$ Hölder seminorm is denoted $[F]_{\gamma}$.
- The space of smooth $m$–forms is denoted $\mathcal{D}^m(\mathbb{R}^3)$ (see §A.2 for a full definition).
- We set $H^1(\mathbb{R}^3) := \{ \nabla u \in L^2(\mathbb{R}^3) \}$, which is equipped with the seminorm $u \mapsto \| \nabla u \|_{L^2(\mathbb{R}^3)}$. 

The space of bounded linear operators mapping a Banach space $X$ to a Banach space $Y$ is denoted $L(X, Y)$.

The $n$th Frechet derivative of a function $f$ at a point $x$ is denoted $D^m f$, and its (multilinear) action on vector $v_1, \ldots, v_m \in \mathbb{R}^3$ is denoted $D^m f(x)[v_1, \ldots, v_m]$.

The pullback of a function $G$ by $F$ is denoted $F^* G$ (see A.4 for a full definition).

The identity mapping on $\mathbb{R}^3$ is denoted id.

The Fourier transform is of a function $f$ is denoted $\hat{f}$, and (for $f \in L^1(\mathbb{R}^3)$) we use the definition

$$\hat{f}(k) := \int_{\mathbb{R}^3} f(x)e^{-i(k,x)} \, dx, \quad \text{so that} \quad f(x) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \hat{f}(k)e^{i(k,x)} \, dk.$$

For numbers, a bar denotes complex conjugation, i.e. if $z = x + iy$ with $x, y \in \mathbb{R}$, then $\bar{z} = x - iy$. The duality relation between a vector space and its dual is denoted with angular brackets, e.g. $(T, \phi), \langle e^i, e_j \rangle$.

Inner products are denoted with parentheses, e.g. $(u, v)$.

2. Main results

2.1. Modelling slip and dislocations as currents. Dislocations are usually modelled as being described by [45,50]

- their position, a curve $\Gamma \subset \mathbb{R}^3$,
- their orientation, fixed by defining a tangent field $\tau : \Gamma \to \Lambda_1 \mathbb{R}^3$, and
- their topological ‘charge’, known as the Burgers vector $b \in \mathbb{R}^3$.

Very often, dislocations are presented as simply being described by the quantities described above, but there is a further topological restriction on possible dislocation configurations, which arises since dislocations must always be the boundary of a region of slip [27]. In crystal plasticity, the term slip refers to a displacement across a surface inside a crystal such that the lattice matches perfectly on either side of it: as a consequence, slip is characterised by a lattice vector.

To illustrate the action of slip, suppose $B \in \mathbb{R}^{3 \times 3}$ is an invertible matrix, define a fixed lattice $\mathcal{L} := B \mathbb{Z}^3 \subset \mathbb{R}^3$ which describes the structure of the material considered which satisfies the property that the shortest non–zero lattice vector is of length 1, i.e. $\min\{|b| : b \in \mathcal{L} \setminus \{0\}\} = 1$. Suppose also that $\Sigma \subset \mathbb{R}^3$ is a compactly–supported oriented surface with a normal field $\nu$ across which a slip has occurred. The plastic distortion corresponding to a slip vector $b \in \mathcal{L}$ across $\Sigma$ is then the strain field

$$z = b \otimes \nu \mathcal{H}^2 \mathcal{L} \Sigma,$$

where $\mathcal{H}^2 \mathcal{L} \Sigma$ is the 2-dimensional Hausdorff measure restricted to $\Sigma$.

This description of $z$ as a plastic distortion is motivated by the following remark, which is illustrated in Figure [1]. Suppose that $s \in \mathbb{R}^3 \setminus \partial \Sigma$, and $\delta > 0$ is taken such that $B_\delta(s)$ is disjoint from $\partial \Sigma$. Assuming that $\Sigma \cap B_\delta(s)$ is sufficiently regular, we may partition $B_\delta(s) = (\Sigma \cap B_\delta(s)) \cup \Sigma^+ \cup \Sigma^-$, where $\Sigma^+$ and $\Sigma^-$ are disjoint open sets. Define the BV vector field $u^0 : B_\delta(s) \to \mathbb{R}^3$ such that

$$u^0(x) = \begin{cases} b & x \in \Sigma^+, \\ 0 & x \in \Sigma^- \end{cases}, \quad \text{for which} \quad Du^0 = b \otimes \nu \mathcal{H}^2 \mathcal{L} (\Sigma \cap B_\delta(s)).$$

The function $u^0$ represents a jump in the displacement of $b$ across the surface $\Sigma \cap B_\delta(s)$, and we can perform this construction for any $s \notin \partial \Sigma$; however, the same construction fails for $s \in \partial \Sigma$, which indicates that $z$ has non–trivial distributional curl concentrated on $\partial \Sigma$; indeed, the fact that $z$ therefore cannot be globally represented as a gradient is precisely the reason that $z$ is a plastic distortion.

Since the slip $z$ is concentrated on a two–dimensional set $\Sigma$, following [27],[29], it is natural to define an associated vector–valued integral 2–current $T$, for which

$$\langle T, \phi \rangle := \int_{\Sigma} (u, \phi) b \, d\mathcal{H}^2 \quad \text{for any} \ \phi \in \mathcal{D}^2(\mathbb{R}^3).$$

The space of such 2–currents will be denoted $\mathcal{J}_2(\mathbb{R}^3; \mathcal{L})$, and can be endowed with an additive structure, which arises by taking the union of the support, and the sum of the corresponding fields $b$.
If $T$ is an integral current, then its boundary $\partial T$ is also an integral current, and has a consistently oriented tangent field $\tau: \partial \Sigma \to \mathbb{R}^3$; the equivalent of Stokes Theorem then implies that
$$\int_{\partial \Sigma} \langle \tau, \phi \rangle b \, d\mathcal{H}^1 = \langle \partial T, \phi \rangle = \langle T, d\phi \rangle = \int_{\Sigma} \langle \nu, d\phi \rangle b \, d\mathcal{H}^2$$
for all $\phi \in \mathcal{D}^1(\mathbb{R}^3)$.

The integral 1–current $\partial T$ encodes a configuration of dislocations supported on $\Gamma = \partial \Sigma$, with Burgers vector $b: \Gamma \to \mathcal{L}$, and line direction fixed by the tangent field $\tau$. In analogue with the previous case, the space of all 1–currents taking values in $\mathcal{L}$ is denoted $\mathcal{I}_1(\mathbb{R}^3; \mathcal{L})$; we note that Theorem 2.5 of [27] describes the structure of $\mathcal{I}_1(\mathbb{R}^3; \mathcal{L})$.

We call the vector–valued current $T$ a slip configuration, and $S = \partial T$ the corresponding dislocation configuration. It is clear that there are many possible slip configurations corresponding to the same dislocation configuration, since there are many surfaces with the same boundary. The space of admissible dislocation configurations is then defined to be
$$\mathcal{A} := \left\{ S \in \mathcal{I}_1(\mathbb{R}^3; \mathcal{L}) \mid S = \partial T \text{ for some } T \in \mathcal{I}_2(\mathbb{R}^3; \mathcal{L}) \text{ s.t. } \langle T, \phi \rangle = \int_{\Sigma} \langle \nu, \phi \rangle b \, d\mathcal{H}^2 \text{ for all } \phi \in \mathcal{D}^2(\mathbb{R}^3) \right\}.$$

It is straightforward to check that $\mathcal{A}$ is an additive subspace of $\mathcal{I}_1(\mathbb{R}^3; \mathcal{L})$, since $\partial^2 T = 0$ for any $T \in \mathcal{I}_2(\mathbb{R}^3; \mathcal{L})$.

The choice to require that each $S \in \mathcal{A}$ is the boundary of a 2–current naturally encodes the fact that dislocation must be the boundary of a region of slip, and this requirement is equivalent to that dislocation configurations satisfy a 'divergence–free' condition, as discussed in [27], which is in turn equivalent to the principle of the conservation of Burgers vector (see [45, 50, 62]). The additive structure of integral currents allows for the description of complicated configurations of dislocations via a superposition of corresponding elementary currents.

We now define two useful functions which quantify aspects of the geometry of dislocation configurations. The mass of $S \in \mathcal{I}_1(\mathbb{R}^3; \mathcal{L})$ is defined to be
$$M(S) := \sup \{ |\langle S, \phi \rangle| \mid \phi \in \mathcal{D}^3(\mathbb{R}^3) \text{ with } |\phi(x)| \leq 1 \text{ for all } x \in \mathbb{R}^3 \}.$$  

In the case where $S \in \mathcal{A}$ is characterised as above, this is equivalent to the formula
$$M(S) = \int_{\Gamma} |b| \, d\mathcal{H}^1,$$
which corresponds physically to the total length of dislocation, weighted by the Burgers vector. In analogy with §9.2 in [57], we also define the mass ratio for $S \in \mathcal{A}$ as

$$
\Theta(S) := \sup \left\{ \frac{M(S \setminus B_r(s))}{r} \mid s \in \text{supp}(S), r > 0 \right\};
$$

here $B_r(s)$ is the closed ball of radius $r > 0$ centred at $s \in \mathbb{R}^3$, and $S \setminus A$ means the restriction of a current $S$ to a set $A$. The mass ratio should be viewed as a way of measuring the maximal spatial density of a current; for currents of fixed mass, $\Theta(S)$ can be arbitrarily large (see Figure 2 for an explanation). Note however that as long as $S \neq 0$, then it follows that $\Theta(S) \geq 1$: If $S$ is supported on $\Gamma$, then the definition of the one–dimensional density of $S$ at $s \in \Gamma$, defined in analogy with the definition in §9.2 of [57], must satisfy

$$
\Theta(S) \geq \limsup_{r \to 0} \frac{M(S \setminus B_r(s))}{r} \geq \min_{b \in \mathcal{L} \setminus \{0\}} |b| = 1.
$$

2.2. Functions defined on dislocation configurations. In order to formulate an appropriate setting in which to consider DDD, we will need to consider fields which are defined on dislocations themselves. As such, we will consider functions which are differentiable ‘along’ a dislocation configuration $S$. This section lays out the basic definitions we use, which allow us to state our main results.

Suppose that $S \in \mathcal{A}$, with corresponding 1-rectifiable set $\Gamma$, Burgers vector $b : \Gamma \to \mathcal{L}$ and tangent field $\tau$. Consider $g : \Gamma \to \mathbb{R}^3$ which is measurable with respect to $\mathcal{H}^1 \setminus \Gamma$. We will say that $\nabla_\tau g : \Gamma \to \mathbb{R}^3$, a $\mathcal{H}^1 \setminus \Gamma$–measurable function is a weak derivative of $g$ along $S$ if, for any $C^1$ function $f : \mathbb{R}^3 \to \mathbb{R}^3$, we have

$$
\int_{\Gamma} f \cdot \nabla_\tau g \, d\mathcal{H}^1 = -\int_{\Gamma} Df[\tau] \cdot g \, d\mathcal{H}^1,
$$

where $Df \in \mathcal{L}(\mathbb{R}^3; \mathbb{R}^3)$ denotes the Frechet derivative of $f$. We note that if $G \in C^1(\mathbb{R}^3, \mathbb{R}^3)$ and $g := G[\tau], \nabla_\tau g = DG[\tau]$, the fact that $\partial S = 0$, i.e. $S$ has no boundary, ensures that no ‘boundary terms’ are required in the definition above. It is straightforward to show that weak derivatives defined in this way are unique when they exist by using the fact that $\Gamma$ is expressible as a union of images of $\mathbb{R}$ under Lipschitz maps, pulling back, and applying the Fundamental Lemma of the Calculus of Variations on $\mathbb{R}$.

For $1 \leq p < +\infty$, in analogy with the usual definitions, we set

$$
L^p(S; \mathbb{R}^N) := \left\{ g : C \to \mathbb{R}^N \mid \|g\|_p < +\infty \right\} \quad \text{with norm} \quad \|g\|_{L^p} := \left( \int_{\Gamma} |g|^p \, d\mathcal{H}^1 \right)^{1/p},
$$

$$
L^\infty(S, \mathbb{R}^N) := \left\{ g : C \to \mathbb{R}^N \mid \|g\|_{L^\infty} < +\infty \right\} \quad \text{with norm} \quad \|g\|_{L^\infty} := \text{ess sup} \{ |g(s)| \mid s \in \Gamma \},
$$

where the essential supremum in the latter definition is taken up to $\mathcal{H}^1$ null sets.

We also define the space

$$
H^1(S; \mathbb{R}^3) := \left\{ g \in L^2(S; \mathbb{R}^3) \mid \nabla_\tau g \in L^2(S; \mathbb{R}^3) \right\},
$$
which has a real Hilbert space structure when endowed with the inner product
\[(f,g) = \int_T [f \cdot g + \nabla_T f \cdot \nabla_T g] \, dH^1.\]

As usual, we will denote the dual space of $H^1(S;\mathbb{R}^3)$ as $H^1(S;\mathbb{R}^N)^*$. 

2.3. Energy of regularised slip distributions. For any slip configuration $T \in \mathcal{A}(\mathbb{R}^3; \mathcal{L})$, we now define a regularised energy by a procedure similar to that considered in §2.4.2 of [29], distributing slip about the set $\Sigma$ on which $T$ is supported by mollifying. Physically—speaking, our choice to smooth distributions of slip can be justified by noting that a definition of the lattice plane over which slip has occurred cannot be given with a precision greater than that of a single lattice spacing, and similarly, the position of a dislocation core cannot be ascertained to a precision of less than a few lattice spacings. As such, regularising defines the lengthscale at which linear elasticity is invalid, and as stated in §1.1 this choice has the convenient mathematical benefit that all fields considered are non–singular, in common with the reality on the atomistic scale.

To this end, we suppose that $\varphi^1 \in C^\infty(\mathbb{R}^3)$ is a function which is rapidly decreasing (in the Schwartz sense), is radially symmetric, and satisfies $\int_{\mathbb{R}^3} \varphi^1(x) \, dx = 1$, such as the Gaussian
\[(8) \quad \varphi^1(x) = \frac{1}{(2\pi)^{3/2}} \exp \left(-\frac{1}{2} |x|^2 \right) .\]

For any $\varepsilon > 0$, we then define $\varphi^\varepsilon(x) := \varepsilon^{-3} \varphi^1(x/\varepsilon)$.

If $T \in \mathcal{A}(\mathbb{R}^3; \mathcal{L})$ is a slip configuration supported on $\Sigma$, with corresponding normal field $\nu$, and slip $b : \Sigma \to \mathcal{L}$, in analogy with the plastic deformation considered in (3), we define a smoothed plastic distortion $z_T^\varepsilon : \mathbb{R}^3 \to \mathbb{R}^{3 \times 3}$ to be
\[(9) \quad z_T^\varepsilon(x) := \int_\Sigma b(s) \otimes \nu(s) \varphi^\varepsilon(x - s) \, dH^2(s).\]

Here $H^2$ is the 2–dimensional Hausdorff measure (i.e. the surface area measure on $\Sigma$), and the field $z_T^\varepsilon$ is well–defined and smooth since $\varphi^\varepsilon$ is assumed to be $C^\infty$.

Following the ideas of Kröner and Mura [52, 53, 61], we suppose that the system equilibrates elastically in response to this plastic strain. Using the additive decomposition of the strain, the total distortion $\beta^\varepsilon$ is assumed to take the form
\[\beta^\varepsilon = z_T^\varepsilon + D\varepsilon,\]

where $\varepsilon : \mathbb{R}^3 \to \mathbb{R}^3$, and recalling the definition of $\hat{H}^1(\mathbb{R}^3)$ given in §1.3, the energy at equilibrium due to a configuration of slip described by $T \in \mathcal{A}(\mathbb{R}^3; \mathcal{L})$ is therefore
\[(10) \quad \mathcal{E}^\varepsilon(T) := \min_{\varepsilon \in \hat{H}^1(\mathbb{R}^3)} \mathcal{I}(z_T^\varepsilon + D\varepsilon),\]

where the total internal energy $\mathcal{I} : L^2(\mathbb{R}^{3 \times 3}) \to \mathbb{R}$, is defined to be
\[(11) \quad \mathcal{I}(\beta) := \int_{\mathbb{R}^3} \frac{1}{2} \beta : \mathcal{C} \beta \, dx = \int_{\mathbb{R}^3} \frac{1}{2} \epsilon \epsilon_{ijkl} \beta_{ij} \beta_{kl} \, dx.\]

The fundamental insight in the work of Kröner and Mura is that while dislocations must be the boundary of a region of slip, the precise surface over which slip has occurred is irrelevant to the energy and mechanical response of the system, i.e. although (10) appears to depend on $T$, in fact it only depends upon $\partial T$. This is compatible with the experimental observation that dislocations moving through a crystal leave no trace of having passed, since the crystal 'heals' perfectly after slip occurs. Using these insights, we prove the following theorem which encodes the dependence of the energy on $\partial T$ alone, and moreover provides an expression for the internal energy given directly in terms of an integral over the dislocation configuration itself.

**Theorem 1.** If $T \in \mathcal{A}(\mathbb{R}^3; \mathcal{L})$ is compactly supported, the energy $\mathcal{E}^\varepsilon$ defined in (10) depends only on $S = \partial T$, so that we may define $\Phi^\varepsilon : \mathcal{A} \to \mathbb{R}$ with
\[\Phi^\varepsilon(S) := \mathcal{E}^\varepsilon(T) \quad \text{for any } S \in \mathcal{A} \text{ where } S = \partial T.\]
Further, if \( T \in \mathcal{S}^2 \) and \( S = \partial T \) respectively take the form \( \langle T, \phi \rangle = \int_\Sigma (v, \phi) b d\mathcal{H}^2 \) for \( \phi \in \mathcal{D}^2(\mathbb{R}^3) \) and \( \langle S, \eta \rangle = \int_\Gamma (\tau, \eta) b d\mathcal{H}^1 \) for \( \eta \in \mathcal{D}^1(\mathbb{R}^3) \), then the energy functionals \( \mathcal{E}(T) \) and \( \Phi(S) \) may be written as

\[
\begin{align*}
\mathcal{E}(T) &= \int_{\Sigma \times \Sigma} \frac{1}{2} J_{s,t}^e(s-t) b_a(s) v_b(s) b_c(t) \nu_d(t) d(\mathcal{H}^2 \otimes \mathcal{H}^2)(s, t) \\
\Phi(S) &= \int_{\Gamma \times \Gamma} \frac{1}{2} K_{s,t}^e(s-t) b_a(s) \tau_b(s) b_c(t) \tau_d(t) d(\mathcal{H}^1 \otimes \mathcal{H}^1)(s, t),
\end{align*}
\]

where the kernels \( J^e, K^e : \mathbb{R}^3 \to \mathbb{R}^{3 \times 3 \times 3 \times 3} \) are defined to be

\[
\begin{align*}
J_{k,m,p,q}^e(s) &= \int_{\mathbb{R}^3} C_{s,t}^e A_{k,m} C_{i,j}^e G_{a,i,a,j}(x-s) A_{p,m} A_{d,q} G_{s,f,s,f}(x) A_{q,r} s \, dx, \\
K_{k,p,q,q}^e(s) &= \int_{\mathbb{R}^3} C_{s,t}^e A_{k,m} C_{i,j}^e G_{a,i,a,j}(x-s) A_{d,q} G_{s,f,s,f}(x) d_x,
\end{align*}
\]

where \( A \) is the alternating tensor as defined in (13) and \( G^e := G * \varphi^e \), i.e. the convolution of the elastic Green’s function with \( \varphi^e \). Moreover, \( J^e \) and \( K^e \) satisfy the following properties.

1. \( J_{s,t}^{e} = J_{-s,t}^{e} \) and \( K_{s,t}^{e} = K_{-s,t}^{e} \) for any \( s \in \mathbb{R}^3 \);
2. \( J \) and \( K \) are smooth.
3. For any \( m \in \mathbb{N}, 0 \leq j \leq m, \) and vectors \( v_1, \ldots, v_j \in \mathbb{S}^2 \), there exists a constant \( C_{m,j} \) such that for all \( s \in \mathbb{R}^3 \),

\[
\left| D^m K^e(s) : \left[ v_1, \ldots, v_j, \frac{s}{|s|}, \ldots, \frac{s}{|s|} \right] \right| \leq \frac{C_{m,j}}{\sqrt{\varepsilon^{2m+2} + \varepsilon^{2j}} |s|^{2m+2-2j}}.
\]

The proof of this theorem is given in [13] and proceeds by verifying that \( \mathcal{E}(T) \) as given in (10) is well-defined, before characterising the solution of the minimisation problem by applying the elastic Green’s function and the ideas used to derive Mura’s formula [61]. Ideas from [9] (closely related to ideas used in proving Theorem 4.1 in [29]) are then used to derive a Fourier characterisation, which allow us to deduce the asserted properties of the kernels \( J^e \) and \( K^e \).

2.4. The Peach–Koehler force. The power of the expression for the energy given in (12b) is that it allows us to take explicit variations of the energy, and thereby to derive the equivalent of the configurational Peach–Koehler force in this model. Since we are varying the set on which the Burgers vector \( b \) and line direction \( \tau \) are defined, an appropriate notion of variation is that of inner variation (see Chapter 3 of [41]) or variation of the reference state (as described in §2.1.5 of [12]). To construct an inner variation, we use the notion of pushforward (defined in [A.4]) if \( g \in C^m(\mathbb{R}^3, \mathbb{R}^3) \), we define the inner variation of \( \Phi^e \) at \( S \in \mathcal{S} \) in the direction \( g \) to be the linear functional

\[
\langle D\Phi^e(S), g \rangle := \frac{d}{d\delta} \Phi^e((\text{id} + \delta g) \# S) \bigg|_{\delta = 0}
\]

where \( \text{id} : \mathbb{R}^3 \to \mathbb{R}^3 \) is the identity mapping \( \text{id}(x) := x \). The result of taking this variation and properties of the resulting functional are encoded in the following theorem, which provides an expression...
of the variation as a field defined on the dislocation configuration, as well as a uniform bound and a form of continuity result under deformation.

**Theorem 2.** If \( S \in \mathcal{S} \), the inner variation of the energy at \( S \) is given by

\[
(D\Phi^\varepsilon(S), g) = -\int_I f^{PK}_{\varepsilon}(s, S)g(s)\,d\mathcal{H}^1(s), \quad \text{where} \quad f^{PK}_{\varepsilon}(s, S) := G(s, S) \wedge \tau(s)
\]

and

\[
G_k(s, S) := \int_I A_{klm}K_{\varepsilon,kcd,m}(s-t)b_a(s)b_c(t)\tau_d(t)\,d\mathcal{H}^1(t).
\]

If \( S = \partial T \), where \( T \in \mathcal{J}_2(\mathbb{R}^3; \mathcal{L}) \) is supported on \( \Sigma \) with Burgers vector \( b : \Sigma \rightarrow \mathcal{L} \) and normal field \( \nu \), \( G \) can alternatively be written

\[
G_k(s, S) = -\int_{\Sigma} A_{cd}A_{klm}K_{\varepsilon,kcd,mb}(s-t)b_a(s)b_c(t)\nu_f(t)\,d\mathcal{H}^2(t).
\]

Moreover, recalling the definitions of \( \mathcal{M} \) and \( \Theta \) given respectively in (4) and (8), we have the bound

\[
\|f^{PK}(S)\|_{L^\infty} \leq \frac{C}{\varepsilon}\|b\|_{L^\infty}\Theta(S) \log \left( 1 + \frac{2\mathcal{M}(S)}{\varepsilon\Theta(S)} \right),
\]

and if \( g : S \rightarrow \mathbb{R}^3 \) is a Lipschitz map and \( F := \text{id} + g \), then

\[
\|F^*f^{PK}(F#S) - f^{PK}(S)\|_{L^\infty} \leq (1 + C\mathcal{M}(S))\|
abla_{PS}g\|_{L^\infty} + C\mathcal{M}(S)\|g\|_{L^\infty},
\]

where \( C \) is a constant independent of \( v \) and \( S \).

A proof of this Theorem 2 is given in [4]. The main achievement of this result is in obtaining the bound (17), which is crucial to proving the long–time existence result for the formulation of DDD considered in the following sections. This bound is a significant improvement over the more na"ive estimate \( \|f^{PK}(S)\|_{L^\infty} \leq C\mathcal{M}(S) \), which can be deduced directly from the fact that \( \mathcal{K}^\varepsilon \) and \( D\mathcal{K}^\varepsilon \) are uniformly bounded, proved in Theorem 1. Instead, the estimate is obtained by a rearrangement argument, using the limited geometric information \( \Theta(S) \) provides to guarantee this weaker dependence on the mass.

The bound (18) provides a form of continuity for the Peach–Koehler force, demonstrating that under a Lipschitz variation of \( S \) which is close to the identity, the Peach–Koehler force of the new configuration \( F#S \) is close to that obtained on the initial configuration. Since these fields are defined on different sets, in order to measure the difference we must pull back this new field onto the initial configuration.

### 2.5. Dislocation mobility

It is widely believed that moving dislocations dissipate energy via phonon radiation (see for example §3.5 in [50], and §7-7 in [45]): the movement of a dislocation through a ‘rough’ landscape of local minima generates high–frequency lattice waves which radiate away as heat, leading to drag. When modelling dislocation motion at low to moderate strain rates, it is typically assumed that drag dominates inertial effects, which are therefore neglected. This assumption has been supported by microscopic simulations of dislocation motion (see for example the discussion in §4.5 and §10.2 of [17], and [11, 25, 26]). Neglecting inertia inevitably entails that dislocation motion (and hence plastic distortion) is modelled as a dissipation–dominated process; as such, a natural mathematical framework for modelling dislocation motion is that of a generalised gradient flow [4].

At low temperatures, the process of slip is dominated by glide, a process which allows dislocations to move while conserving lattice volume [7, 45, 50]. Requiring that dislocations undergo glide motion only is equivalent to requiring that slip can only evolve on planes which contain \( b \). The evolution of slip in directions parallel to the Burgers vector is called climb, and requires mass transport via point defect diffusion; at low temperatures this is a much slower process than that of glide.

To model these phenomena in DDD simulations, various constitutive assumptions on dislocation mobility are available; for various examples, see [8, 21, 50]. Usually, the velocity \( v \) of a segment of dislocation is related to the configurational force on the dislocation line via a mobility function \( \mathcal{M} \), which depends locally on the Burgers vector \( b \), the dislocation orientation \( \tau \), and the Peach–Koehler force \( f^{PK} \):

\[
v = \mathcal{M}(b, \tau, f^{PK}).
\]

Such mobilities are informed by molecular dynamics simulations or experiment, and are generally linear, power laws, or possibly include some frictional threshold before the onset of dislocation motion, mimicking the Peierls barrier. In common with many geometric evolution problems, it is usually
assumed that dislocations have velocity only in normal directions, i.e. \((M(b, \tau, f), \tau) = 0\) for any \(b, \tau\) and \(f\).

In order to define a generalised gradient flow framework which encompasses mobilities of the type referred to above, we restrict ourselves to considering mobilities which take the form

\[
M(b, \tau, f) := -\nabla \psi^*(b, \tau, -f),
\]

where \(\psi^*\) is an entropy production which is convex in \(f\), describing the rate at which entropy is produced by the force \(f\). Requiring the existence of \(\psi^*\) is not particularly restrictive, as it ensures that energy is conserved in a closed system, and to the author’s knowledge, all mobility laws for DDD used in practice take this form.

When an entropy production is defined, it is natural to define a conjugate dissipation potential \(\psi\), which describes the rate at which energy is lost through the variation of a dislocation configuration, and is given as the Legendre–Fenchel transform of \(\psi^*\), i.e.

\[
(19) \quad \psi(b, \tau, v) := \sup_{f \in \mathbb{R}^3} \left\{ \langle v, f \rangle - \psi^*(b, \tau, f) \right\}.
\]

The frictional force resulting from a velocity \(v\) is then \(-\nabla \psi(v)\); this follows from the fact that for convex conjugate functions,

\[
(20) \quad v = -\nabla \psi^*(-f) \quad \text{if and only if} \quad f = -\nabla \psi(-v) \quad \text{if and only if} \quad \langle f, v \rangle = \psi(-v) + \psi^*(-f).
\]

As an example, in the case of a linear relationship between configurational force and velocity, \(v = B(b, \tau)f\), it is straightforward to check that we may define

\[
\psi^*(f) = \frac{1}{2} \langle f, B(b, \tau)f \rangle \quad \text{and} \quad \psi(v) = \frac{1}{2} \langle v, B(b, \tau)^{-1}v \rangle,
\]

where \(B^{-1}\) denotes the matrix inverse of \(B\).

In practice, we find that requiring that the dissipation potential is uniquely a function of the dislocation velocity as in (19) is insufficient to guarantee a well-defined evolution. As an illustration, consider the possible scenario in Figure 3. While the dislocation is initially smooth, and \(f^{PK}\) is smooth up until the final time, \(v = B(b, \tau)f^{PK}\) loses regularity exactly as a jump in \(\tau\) develops. At the final time, the dynamics could require \(v\) to be discontinuous, resulting in segments of dislocation 'ripping apart', breaking the physical requirement that dislocations are the boundary of regions of slip, and leading to a blow–up of the evolution.

To avoid this possibility, in the following section, we introduce assumptions requiring that the energy dissipated by dislocation motion also depends on the rate of change of the tangent to the dislocation.

### 2.6. Dissipation potential

Motivated by the discussion in \([2, 5]\), we suppose that the velocity of a dislocation is described by a field along its length, \(v\), and assume that the dissipation potential for a dislocation configuration \(S\) supported on \(\Gamma \subset \mathbb{R}^3\) with a velocity field \(v : \Gamma \to \mathbb{R}^3\) is expressed as

\[
\Psi(S, v) = \int_{\Gamma} \left( \frac{1}{2} \nabla_\tau v \cdot A(b, \tau) \nabla_\tau v + \psi(b, \tau, v) \right) d\mathcal{H}^1
\]
where, $\nabla_{\tau} v$ denotes the weak derivative of the velocity along the dislocation line as defined in (2.2), and as usual, $\tau : \Gamma \to S^2$ is the tangent field to $S$. We make the following constitutive assumptions on $A$ and $\psi$.

(C1) $A : \mathcal{L} \times S^2 \to \mathbb{R}^{3 \times 3}$ is a matrix-valued function which is symmetric and strictly positive definite everywhere, and there exists $\alpha > 0$ such that

$$w \cdot A(b, \tau) w \geq \alpha |w|^2 \quad \text{for any } (b, \tau, w) \in (\mathcal{L} \setminus \{0\}) \times S^2 \times \mathbb{R}^3.$$

(C2) $\psi : \mathcal{L} \times S^2 \times \mathbb{R}^3 \to [0, +\infty]$ is a positive function which is strictly convex in its third argument, satisfying $\psi(b, \tau, 0) = 0$ for any $(b, \tau) \in \mathcal{L} \times S^2$. Moreover,

$$\psi(b, \tau, v) = +\infty \quad \text{if} \quad v \cdot \tau \neq 0.$$

(R) For any $b \in \mathcal{L}$, $\tau \mapsto A(b, \tau)$ is smooth and bounded on $S^2$, and $(\tau, v) \mapsto \psi(b, \tau, v)$ is smooth on the set $\{v \cdot \tau \in S^2 \times \mathbb{R}^3 : v \cdot \tau \neq 0\}$.

(G) There exists $\beta > 0$ such that

$$\psi(b, \tau, v) \geq \frac{1}{2} \beta |v|^2 \quad \text{for all } (b, \tau, f) \in \mathcal{L} \times S^2 \times \mathbb{R}^3.$$

As remarked in the previous section, the assumptions above are broad enough to encode a wide variety of modelling assumptions made when modelling dislocation dynamics:

- The convexity assumptions (C1) and (C2) imply that dissipation potential is always positive and increases as the dislocation velocity or rate of bending increases, and no change in the energy dissipated is produced if dislocations do not deform or translate within the material. The fact that $\psi(b, \tau, v) = +\infty$ if $v \cdot \tau \neq 0$ enforces the requirement that meaningful dislocation velocity fields must be locally perpendicular to the dislocation line.

- The regularity assumption (R) ensures that energy dissipation rate varies smoothly with the velocity and orientation of the dislocation line.

- The growth assumption (G) is a technical assumption, and is not particularly restrictive, however relaxing it would make some aspects of our analysis more technical.

On the other hand, the choice to make the dissipation depend upon $\nabla_{\tau} v$ as well as $v$ appears to be a novel addition to DDD. It does not seem to be unreasonable to require that energy is dissipated by the bending or stretching of dislocation lines, and it this additional term gives us control of the bending rate, ruling out the generation of singularities similar to those shown in the figure and allowing us to prove that the evolution is well-posed. It would be of great interest to understand whether this term is indeed physically-justified via a future computational study of a realistic model, or indeed whether this additional dependence can be mathematically removed via (for example) a vanishing viscosity argument.

As an indicative example of a constitutive relation satisfying these assumptions which is closely related to a dislocation mobility which is already present in the literature, we may define

$$\Psi(S, v) = \int_S \frac{1}{2} \alpha |\nabla_{\tau} v|^2 + \psi(b, \tau, v) \, d\mathcal{H}^1(s),$$

so that $A(b, \tau) = \alpha I$, and where we set

$$\psi(b, \tau, v) = \begin{cases} \frac{1}{2} v^T B^1(b, \tau) v & v \cdot \tau = 0 \\ +\infty & v \cdot \tau \neq 0, \end{cases} \quad \text{with}$$

$$B(b, \tau) = \begin{pmatrix} \frac{|b \wedge \tau|^2}{B_{eg}} + \frac{(b \cdot \tau)^2}{B_2^2} & \frac{P(\tau)\sigma_1 \otimes P(\tau)b}{|b \wedge \tau|^2} + \sqrt{B_{ec}^2 |b \wedge \tau|^2 + B_\perp^2 (b \cdot \tau)^2 (b \wedge \tau) \otimes (b \wedge \tau)} \end{pmatrix} |b|^2 |b \wedge \tau|^2.$$

Here, $B^\dagger$ denotes the Moore–Penrose pseudo-inverse of $B$; in this case, this is simply the matrix which has the same eigenspaces as $B$ and inverts any non-zero eigenvalues. $B_{eg}, B_{ec}, B_\perp > 0$ are all mobility parameters describing dissipative timescales resulting from various different modes of motion (bending and stretching, glide of edge dislocations, climb of edge dislocations and glide of screw dislocations), and $\alpha^{-1} > 0$ is the energy dissipation rate per unit additional area swept out per unit length of dislocation.

Defining $\psi^*(b, \tau, f) = \sup \{ v \cdot f - \psi(b, \tau, v) \mid v \in \mathbb{R}^3 \}$, i.e. the Legendre–Fenchel transform of $\psi$, we find that

$$\psi^*(b, \tau, f) = \frac{1}{2} \left( f, B(b, \tau)f \right).$$
and we note that \( v = B(b,\tau)f = -\nabla \psi^*(b,\tau,-f) \) is the mobility relation for dislocations in BCC defined in equation (10.40) of [17]. We therefore see that if \( \alpha = 0 \), the model as defined above reduces to that considered in [17].

We remark that as a consequence of assumption (C2), we have the following characterisation of the subgradient of \( \psi \):

\[
\partial_v \psi(b,\tau,v) = \begin{cases} 
\emptyset & v \cdot \tau = 0, \\
\{ D^\perp_r \psi(b,\tau,v) \} & v \cdot \tau \neq 0,
\end{cases}
\]

where \( D^\perp_r \psi(b,\tau,v) \) means the gradient of \( \psi \) taken in directions perpendicular to \( \tau \).

### 2.7 Evolution problem.

When viewed as a function defined on \( H^1(S;\mathbb{R}^3) \), \( \Psi(S,) \) has a well–defined subdifferential with respect to \( v \), \( \partial_v \Psi(S,v) \subset H^1(S,\mathbb{R}^3)^* \), and recalling the definition of \( D^\perp_r \psi \) made above, \( \xi \in \partial_v \Psi(S,v) \) implies that

\[
\langle \xi, w \rangle = \int_S \nabla_r v \cdot A(b,\tau) \nabla_r w + D^\perp_r \psi(b,\tau,v) \cdot w \, dH^1
\]

as long as \( v \cdot \tau = 0 \) \( H^1 \)-almost everywhere on \( S \) (otherwise, \( \partial_v \Psi(S,v) = \emptyset \)). As shorthand for the formula above, we will write

\[
\partial_v \Psi(S,v) = -\text{div}_r [A(b,\tau) \nabla_r v] + D^\perp_r \psi(b,\tau,v).
\]

This expression corresponds to the frictional force induced on the dislocation configuration by moving according to the velocity field \( v \). In the overdamped regime where inertial effects are neglected, these frictional forces balance with the configurational forces, so that \( v \) must satisfy

\[
-\text{div}_r [A(b,\tau) \nabla_r v] + D^\perp_r \psi(b,\tau,v) = f^{PK} \quad \text{in} \quad H^1(S;\mathbb{R}^3)^*.
\]

It is straightforward to check that, since \( \Psi \) is convex on \( H^1(S;\mathbb{R}^3) \), an equivalent criterion is to require that \( v \) solves the minimisation problem

\[
\psi(S,v) = \arg\min_{v \in H^1(S;\mathbb{R}^3)} \langle D\Phi^v(S),v \rangle = \arg\min_{v \in H^1(S;\mathbb{R}^3)} \psi(S,v) - (f^{PK},v)_{L^2}.
\]

The force balance equation (22) is Eulerian in nature: it must be satisfied on the dislocations themselves as they deform. For the purpose of proving existence of solutions, we now cast an alternative Lagrangian formulation for a solution of DDD: we suppose that the position of points on the dislocation line at time \( t \) is expressed as a function of time and position on the dislocation line at the initial time. In the language of geometry, this idea is expressed as a pushforward by \( U \) (recall A.4), where \( U : [0,T] \times S^0 \to \mathbb{R}^3 \). Writing \( U(t) \) to denote the mapping at time \( t \), we require that

\[
U(0) = \text{id} \quad \text{on} \ S^0,
\]

and the ‘trajectory’ of currents is then

\[
S^t := U(0)_\# S^0 \quad \text{for all} \ t \in [0,T].
\]

In this formulation, we see that \( U \) directly identifies the family of currents \( \{S^t\} \subset \mathcal{A} \). Thanks to §4.1.14 of [35], \( S^t \) is well–defined as a current as long as \( U^t \) is a Lipschitz map on \( S^0 \).

We will find that an appropriate ‘energy space’ in which to seek to prove the existence of \( U \) is \( H^1([0,T];H^1(S^0;\mathbb{R}^3)) \), i.e. the space of \( L^2 \) Bochner–integrable functions from \([0,T]\) in \( H^1(S^0;\mathbb{R}^3) \) with \( L^2 \) Bochner–integrable weak derivatives; for further detail on the definition of such spaces, see for example Chapter 7 of [72], and we denote the weak time derivative of \( U(t) \) as \( \dot{U}(t) \).

\textit{A priori}, there is no guarantee that a generic \( U(t) \in H^1(S^0;\mathbb{R}^3) \) is Lipschitz, and therefore no guarantee that \( U(t)_\# S^0 \) is a current. In order to ensure \( S^t \in \mathcal{A} \) for all time, we additionally require that \( U(t) \in C^{0,1}(S^0;\mathbb{R}^3) \). We will say that a pair \( \{U,\{v^t\}_{t \in [0,T]}\} \) form a solution of DDD as formulated in (22) if \( U \in H^1([0,T];H^1(S^0;\mathbb{R}^3)) \), \( U(t) \in C^{0,1}(S^0;\mathbb{R}^3) \) and \( v^t \in H^1(U(t)_\# S^0;\mathbb{R}^3) \) for almost every \( t \in [0,T] \), and

\[
\partial_v \Psi(U(t)_\# S^0,v^t) \ni f^{PK}(U(t)_\# S^0) \quad \text{in} \ H^1(U(t)_\# S^0;\mathbb{R}^3)^* \\
\text{and} \quad U(t)_\# v^t = \dot{U}(t) \quad \text{in} \ H^1(S^0;\mathbb{R}^3)
\]

(24)
for almost every \( t \in [0, T] \), where the definitions of pushforward \( U(t) \# \) and pullback \( U(t) \* \) are given in \[A.2\]. Now that we have given a definition of what it means to be a solution to DDD, we have the following well-posedness result.

**Theorem 3.** If \( S^0 \in \mathcal{A} \) is a finite union of \( C^{1,\gamma} \) curves with \( \gamma \in (0, 1] \), and satisfies \( \Theta(S^0) < +\infty \), then there exists a unique solution \((U, \{\nu^t\})\) satisfying \[24\] for \( t \in [0, T]\) where

\[
T := \sup \{ t \in \mathbb{R} : \Theta(U(s) \# S^0) < +\infty \text{ for all } s \leq t \}.
\]

Moreover, \( U \in C^0([0, T]; C^1(S^0; \mathbb{R}^3)) \).

We note that a strength of this result is that it allows for dislocation collision, since there is no requirement that \( U \) is invertible, although as currently phrased in a Lagrangian form, it is not clear that dislocations satisfy the correct evolution if they merge. Indeed, in practical simulations of DDD, dislocations are remeshed exactly as dislocation segments approach separation distances of \( O(\varepsilon) \), as discussed in \S10.4 of \[17\]. It would be of great interest to understand how best to correctly incorporate this phenomenon into a mathematical theory of DDD in future.

Generically, we expect \( \Theta(U(t) \# S^0) < +\infty \) for all time, since the contrary would require a concentration of internal energy on a small set; while at present we are unable to rule out the possibility of this occurring, it would be interesting in future to confirm existence for all time for at least a large class of initial data.

2.8. **Conclusion.** We have provided a framework in which to study a regularised form of DDD in three dimensions, inspired by various ideas in both the Engineering and Mathematics literature \[9, 19, 27, 29, 61, 74\]. Computable integral formulæ for the energy of and configurational forces on a general dislocation configuration were derived, and bounds on the configurational force which depend weakly on the overall length of dislocation were obtained. A gradient flow formalism in which to study DDD was proposed, and within this framework, we obtained a well-posedness result for the evolution up until the first time an infinite density of dislocations develops.

It is hoped that the energetic framework developed here is sufficiently general to open the way to new upscaling results such as those in \[31, 38, 58, 59, 70, 75\] in a three-dimensional setting, to build upon the results of \[29\] in the case where the regularisation lengthscale \( \varepsilon \) tends to zero, and to allow the mathematical study of the numerical schemes used in practical implementations of DDD.

3. **THE ENERGY OF DISLOCATIONS**

Our aim in this section is to prove Theorem 1 providing a characterisation of the energy of dislocations.

3.1. **Minimisation problem.** Our first step towards proving Theorem 1 is to characterise the solution to the minimisation problem \[10\].

**Lemma 4.** Assuming that \( C \) satisfies a Legendre–Hadamard condition, \( T \in \mathcal{P}_2(\mathbb{R}^3; \mathcal{L}) \) and \( \varepsilon_T \) is as defined in \[9\], the variational problem in \[10\] has a unique solution, which is smooth and satisfies the equation

\[
(25) \quad -C_{ijkl}u^e_{k,lj} = C_{ipqr}(\varepsilon_T)^{q}G_{ipqr}(x,y)d\mathcal{H}^2(s),
\]

in the sense of distributions. Moreover, the solution may be represented as

\[
(26) \quad u^e(x) = \int_{\mathbb{R}^3} C_{ijkl}G_{\alpha_{ij},l}(x-y)(\varepsilon_T)^{k,l}(y)dy = \int_{\mathbb{R}^3} C_{ijkl}G_{\alpha_{ij},l}(x-y)b_k(b_l)\nu_l(s)d\mathcal{H}^2(s),
\]

where \( G \) is the elastic Green’s function, which, recalling \[1.3\], is the distributional solution of \[1\], and the function \( \varepsilon_T := G * \varphi_T \) is a regularised version of \( G \), solving \[1\].

**Proof.** The existence of a minimiser follows from the fact that by Theorem 5.25 in \[30\], any quadratic function is quasiconvex if and only if it is rank-one convex, and in this case rank-one convexity of the integrand is straightforward to check, following from the assumption that \( C \) satisfies a Legendre–Hadamard condition. As a consequence, \( I \) is weakly lower semicontinuous on \( H^1(\mathbb{R}^3) \) and unique minimisers exist in this space; computing the Frechet derivative of \( I \) in the same space shows that the solution satisfies the equation \[25\] in the sense of distributions.

Convolving the distributional equation \[1\] with \( f_\beta \) and contracting the index \( \beta \), we see that

\[
-C_{ijkl}G_{\beta k} * f_{\beta,i} = f_\alpha.
\]
Setting \( f_\alpha = C_{\alpha pqr}(z_T^\varepsilon)_{qr,p} \), we find that
\[ u_\alpha^\varepsilon = C_{\beta pqr} G_{\beta \alpha} * (z_T^\varepsilon)_{qr,p}. \]
Integrating by parts to move the derivative with respect to \( x_j \) onto \( G \), then using the fact that \( G \) is a symmetric tensor, i.e. \( G_{ij} = G_{ji} \) for any \( i, j \in \{1, 2, 3\} \), we obtain the first equality in (26). The second equality follows by using Fubini’s theorem to deduce that
\[ G * (\varepsilon^\varphi * (b \otimes \nu \mathcal{H}^2 \mid \Sigma)) = (G * \varepsilon^\varphi) * (b \otimes \nu \mathcal{H}^2 \mid \Sigma). \]

We remark that while Lemma 4 establishes that the problem (10) has a unique solution \( u^\varepsilon \), it does not guarantee the positivity of the energy \( I(z^\varepsilon + Du^\varepsilon) \), since we cannot say anything about the sign of \( I(z^\varepsilon) \) unless \( z^\varepsilon \) is a gradient. We also note that under appropriate growth and quasiconvexity conditions, the existence of \( u^\varepsilon \) can be ensured if the stored energy density takes a more general nonlinear form; related ideas are discussed in [58, 79, 85] in a two-dimensional setting.

3.2. Energy. A key feature of the linear theory we consider is that it allows the derivation of a representation formula for the distortion due to a configuration of slip, which leads to the following result, allowing us to provide an explicit integral formula for the internal energy (10).

**Lemma 5.** If \( T \in \mathcal{F}_2(\mathbb{R}^3; \mathcal{L}) \), supported on \( \Sigma \), with slip vector \( b \) and normal field \( \nu \), the energy \( E^\varepsilon \) defined in (10) may be represented
\[
E^\varepsilon(T) = \int_{\mathbb{R}^3} \int_{\Sigma \times \Sigma} \frac{1}{2} C_{abcd} A_{bpl} C_{ijkl} b_k(s) G_{ai,jn}(x-s) A_{pmn} \nu_m(s) \times A_{dqh} C_{efgh} b_q(t) G_{ce,fs}(x-t) A_{qrn} \nu_r(t) d(\mathcal{H}^2 \otimes \mathcal{H}^2)(s, t) dx
\]
where \( A \) is the alternating tensor as defined in (11).

Furthermore, (27) depends only on \( \partial T \), which entails that \( \Phi^\varepsilon : \mathcal{A} \to \mathbb{R} \) with
\[ \Phi^\varepsilon(S) := E^\varepsilon(T) \quad \text{for any } S \in \mathcal{A} \text{ where } S = \partial T \]
is well-defined, and if \( S = \partial T \) is supported on \( \Gamma \), with Burgers vector \( b \) and tangent field \( \tau \), \( \Phi^\varepsilon(S) \) may be expressed as
\[
\Phi^\varepsilon(S) = \int_{\mathbb{R}^3} \int_{\Gamma \times \Gamma} \frac{1}{2} C_{abcd} A_{bpl} C_{ijkl} b_k(s) G_{ai,j}^\varepsilon(x-s) \tau_p(s) \times A_{dqh} C_{efgh} b_q(t) G_{ce,j}^\varepsilon(x-t) \tau_q(t) d(\mathcal{H}^1 \otimes \mathcal{H}^1)(s, t) dx.
\]

**Proof.** Since \( T \) is fixed during this proof, throughout, we write \( z^\varepsilon \) in place of \( z_T^\varepsilon \) to keep notation as concise as possible. Applying (26), we write the elastic distortion \( \beta^\varepsilon \) as
\[
\beta^\varepsilon_{ab} = u^\varepsilon_{ab} + z^\varepsilon_{ab} = C_{ijkl} G_{ai,j} * z^\varepsilon_{kl,b} + z^\varepsilon_{ab}.
\]
Using the definition of the elastic Green’s function given in (1) with a change of indices, integration by parts, and the major symmetry of the elasticity tensor,
\[
z^\varepsilon_{ab} = l_{ka} \delta_0 * z^\varepsilon_{kb} = -C_{ijkl} G_{ai,jl} * z^\varepsilon_{kb} = -C_{ijkl} G_{ai,jl} * z^\varepsilon_{kb} = -C_{ijkl} G_{ai,j} * z^\varepsilon_{kb}.l.
\]
Substituting this representation into (29) in place of the latter term, we obtain
\[
\beta^\varepsilon_{ab}(x) = \int_{\mathbb{R}^3} C_{ijkl} \left[ G_{ai,j}(x-y)z^\varepsilon_{kl,b}(y) - G_{ai,j}(x-y) z^\varepsilon_{kb,l}(y) \right] dy,
\]
\[
= \int_{\mathbb{R}^3} C_{ijkl} G_{ai,j}(x-y) [l_{lm} l_{bn} - l_{bm} l_{ln}] z^\varepsilon_{km,n}(y) dy,
\]
\[
= \int_{\mathbb{R}^3} A_{plb} C_{ijkl} G_{ai,j}(x-y) A_{pmn} z^\varepsilon_{km,n}(y) dy,
\]
where we have used the elementary tensor identity \( A_{pmn} A_{plb} = l_{lm} l_{bn} - l_{bm} l_{ln} \). Using the definition of \( z^\varepsilon \) given in (6), we have
\[
\beta^\varepsilon_{ab}(x) = \int_{\mathbb{R}^3} A_{plb} C_{ijkl} b_k G_{ai,j}(x-y) \int_{\Sigma} A_{pmn} \nu_m(s) \varphi^\varepsilon_n(y-s) d(\mathcal{H}^2(s)) dy.
\]
where $\tau$ is the tangent vector field on $\partial \Sigma$. If $x \notin \partial \Sigma$, Fubini’s theorem applies to the above integral representation, so using the definition of $G^\varepsilon$ given in (13) and applying Stokes’ Theorem, we find that
\[
\beta_{ab}^\varepsilon(x) = \int_{\Sigma} A_{bpl} C_{ijkl} b_k G_{ai,jn}(x-s) A_{pmm} \nu_m \nu_n dH^2(s) = \int_{\partial \Sigma} A_{bpl} C_{ijkl} b_k G_{ai,j}^\varepsilon(x-s) \tau_p(s) dH^1(s),
\]
where $\tau : \partial \Sigma \rightarrow \mathbb{S}^2$ is the tangent vector field on $\partial \Sigma$. Substituting the former expression of $\beta^\varepsilon$ into the definition of $I$ stated in (11) gives (27). Similarly, substituting the latter expression into (11) allows us to directly deduce that $\Phi^\varepsilon$ is well-defined and has the expression given in (28).

3.3. Kernel representation. Inspecting the formulae for $\mathcal{E}^\varepsilon$ and $\Phi^\varepsilon$ given in Lemma 5, we note that both expressions can be regarded as a convolution integral against an interaction kernel; this form allows us to use ideas from [9] to prove the following result, which, when combined with Lemma 4 and Lemma 5, completes the proof of Theorem 1.

**Lemma 6.** If $T \in \mathcal{F}_2(\mathbb{R}^3; \mathcal{L}')$ is supported on $\Sigma$, with slip vector $b$ and normal field $\nu$, and $S = \partial T$ is supported on $\Gamma$ with Burgers vector $b$ and tangent field $\tau$, the energy functionals $\mathcal{E}^\varepsilon(T)$ and $\Phi^\varepsilon(S)$ may be expressed as
\[
\mathcal{E}^\varepsilon(T) = \int_{\Sigma \times \Sigma} \frac{1}{2} J^\varepsilon_{abcd}(s-t) b_a(s) b_b(t) \nu_a(t) \nu_b(t) d(H^2 \otimes H^2)(s,t)
\]
\[
\Phi^\varepsilon(S) = \int_{\Gamma \times \Gamma} \frac{1}{2} K^\varepsilon_{abcd}(s-t) b_a(s) \tau_b(t) \nu_a(t) \nu_b(t) d(H^1 \otimes H^1)(s,t),
\]
where the kernels $J^\varepsilon, K^\varepsilon : \mathbb{R}^3 \rightarrow \mathbb{R}^{3 \times 3 \times 3}$ are defined to be
\[
J^\varepsilon_{kmgp}(s) := \int_{\mathbb{R}^3} C_{abcd} A_{bpl} C_{ijkl} G_{ai,jn}(x-s) A_{pmm} A_{dqn} C_{efgh} G_{ce,f}^\varepsilon(x) A_{qr} x d\nu,
\]
\[
K^\varepsilon_{pppq}(s) := \int_{\mathbb{R}^3} C_{abcd} A_{bpl} C_{ijkl} G_{ai,jn}(x-s) A_{dqn} C_{efgh} G_{ce,f}^\varepsilon(x) x d\nu,
\]
and satisfy the following properties.

1. $J^\varepsilon_{abcd}(s) = J^\varepsilon_{cdab}(s) = J^\varepsilon_{abcd}(s-t)$ and $K^\varepsilon_{abcd}(s) = K^\varepsilon_{cdab}(s) = K^\varepsilon_{abcd}(s-t)$ for any $s \in \mathbb{R}^3$.

2. $J^\varepsilon$ and $K^\varepsilon$ are smooth.

3. For any $m \in \mathbb{N}, 0 \leq j \leq m$, and vectors $v_1, \ldots, v_j \in \mathbb{S}^2$, there exists a constant $C_{m,j}$ such that for all $s \in \mathbb{R}^3$,
\[
\left| D^m K^\varepsilon(s) : \left[ v_1, \ldots, v_j, \frac{s}{|s|} \right] \right| \leq \frac{C_{m,j}}{\sqrt{\mathcal{C}^2 m + 2 + |s|^2 (m-j)!}},
\]

**Proof.** We divide the proof into a series of steps, corresponding to each of the assertions made.

**Kernel representation.** The existence of the kernels is a straightforward consequence of applying Fubini’s theorem to the expressions (27) and (28). $J^\varepsilon(s)$ and $K^\varepsilon(s)$ are finite for any $s \in \mathbb{R}^3$ since the elastic Green’s function satisfies the standard properties that $|G_{ij,k}(x)| \lesssim |x|^{-2}$ and $|G_{ij,kl}(x)| \lesssim |x|^{-3}$, and therefore there exist constants $C_\varepsilon$ such that
\[
|G_{ij,k}^\varepsilon(x)| \leq C_\varepsilon \min \{1, |x|^{-2}\} \quad \text{and} \quad |G_{ij,kl}^\varepsilon(x)| \leq C_\varepsilon \min \{1, |x|^{-3}\},
\]
and it follows that $G_{ij,k}^\varepsilon$ and $G_{ij,kl}^\varepsilon$ are in $L^2(\mathbb{R}^3)$, and hence the integrals in (13) converge for any $s \in \mathbb{R}^3$.

**Fourier characterisation of kernels.** To prove that the kernels $J^\varepsilon$ and $K^\varepsilon$ satisfy the stated properties, we use a characterisation via the Fourier transform. Applying the Fourier transform to the definition of $G^\varepsilon$ given in (13), we obtain
\[
-C_{abcd} G_{ce,db}^\varepsilon(k) = C_{abcd} k_b k_d G_{ce}^\varepsilon(k) = I_{ac} \hat{\varphi}^\varepsilon(k).
\]
Define the 2-tensor $D(k)_{ab} := C_{abcd} k_b k_d$ and its algebraic inverse $D(k)^{-1}$, satisfying the relation $D(k)^{-1} D(k)_{ab} = I_{ac}$. $D(k)^{-1}$ is well-defined for $k \neq 0$, since the Legendre–Hadamard condition on $C$ entails that $D(k)$ is strictly positive definite in this case, and it follows that
\[
\hat{G}_{ce}^\varepsilon(k) = D(k)^{-1}_{ce} \hat{\varphi}^\varepsilon(k) \quad \text{and} \quad \hat{G}_{ab,ce}^\varepsilon(k) = -i k_c D(k)^{-1}_{ab} \hat{\varphi}^\varepsilon(k).
Since $\varphi^\rho$ was assumed to be smooth and rapidly-decreasing, the same holds for $\hat{\varphi}^\rho$. Moreover, $D(k)_{ab}^{-1} k_c$ is $-1$-homogeneous in $k$, i.e.

$D(\lambda k)_{ab}^{-1} \lambda k_c = \lambda^{-1} D(k)_{ab}^{-1} k_c$, for any $\lambda \neq 0$.

Using this observation and applying Plancherel's theorem to the definitions in (13),

$$ J_{kmgr}^c(s) = \int_{\mathbb{R}^3} C_{abcd} A_{bpl} C_{ijkl} A_{pmn} A_{dgh} C_{efgh} A_{qrs} \hat{G}^{\rho}_{\alpha,i,j,n}(k) \hat{G}^{\rho}_{\alpha,f,s,k}(k) e^{-ik \cdot s} dk $$

$$ = \int_{\mathbb{R}^3} C_{abcd} A_{bpl} C_{ijkl} A_{pmn} A_{dgh} C_{efgh} A_{qrs} k_k k_f k_s D(k)_{ai}^{-1} D(k)_{ce}^{-1} |\hat{\varphi}^\rho(k)|^2 e^{-ik \cdot s} dk, $$

$$ K_{abcd}^c(s) = \int_{\mathbb{R}^3} C_{efgh} C_{aijk} C_{clmn} A_{filb} A_{hdkm} \hat{G}^{\rho}_{\alpha,i,j,k}(k) \hat{G}^{\rho}_{\alpha,g,m,n}(k) e^{-ik \cdot s} dk $$

$$ = -\int_{\mathbb{R}^3} C_{efgh} C_{aijk} C_{clmn} A_{filb} A_{hdkm} k_k k_n D(k)_{ai}^{-1} D(k)_{gm}^{-1} |\hat{\varphi}^\rho(k)|^2 e^{-ik \cdot s} dk. $$

As $\varphi^\rho$ is assumed to be radially symmetric, it follows that $\hat{\varphi}^\rho$ is also radially-symmetric, and therefore $k_k k_n D(k)_{ai}^{-1} D(k)_{gm}^{-1} |\hat{\varphi}^\rho(k)|^2$ is even in $k$. Using the latter observation, and setting $r = |k|$ and decomposing $k = rz$ for some $z \in S^2 = \{ z \in \mathbb{R}^3 \mid |z| = 1 \}$, we transform to polar coordinates, and use (30) and the evenness of $\hat{\varphi}^\rho$ to obtain

$$ K_{abcd}^c(s) = -\int_{\mathbb{R}^3} C_{efgh} C_{aijk} C_{clmn} A_{filb} A_{hdkm} k_k k_n D(z)_{ai}^{-1} D(z)_{gm}^{-1} |\hat{\varphi}^\rho(rz)|^2 \cos(rz \cdot s) dk $$

$$ = -\int_{S^2} C_{efgh} C_{aijk} C_{clmn} A_{filb} A_{hdkm} k_k k_n D(z)_{ai}^{-1} D(z)_{gm}^{-1} \left( \frac{1}{2} \right) \int_{-\infty}^{+\infty} |\hat{\varphi}^\rho(rz)|^2 e^{irz \cdot s} dr d\mathcal{H}^2(z). $$

Standard properties of the Fourier transform imply that $\hat{\varphi}^\rho(k) = \hat{\varphi}^\rho(\xi k)$, so applying this relation and changing variable, we find

$$ \int_{-\infty}^{+\infty} |\hat{\varphi}^\rho(rz)|^2 e^{irz \cdot s} dr = \frac{1}{\varepsilon} \int_{-\infty}^{+\infty} |\hat{\varphi}^\rho(rz)|^2 e^{irz \cdot s / \varepsilon} dr. $$

Now, for any $\varepsilon > 0$, we define $\eta^\rho : \mathbb{R} \to \mathbb{R}$ to be

$$ \eta^\rho(t) := \frac{1}{\varepsilon} \int_{-\infty}^{+\infty} \left| \hat{\varphi}^\rho(\varepsilon z \cdot s) \right|^2 e^{irz \cdot s / \varepsilon} dr = \eta^\rho(t / \varepsilon) / \varepsilon. $$

It is straightforward to show that this function is rapidly-decreasing, a property it inherits from $\varphi^\rho$. In summary, we have shown that

$$ K_{abcd}^c(s) = -\int_{S^2} \frac{1}{2} C_{efgh} C_{aijk} C_{clmn} A_{filb} A_{hdkm} k_k k_n D(z)_{ai}^{-1} D(z)_{gm}^{-1} \eta^\rho(z \cdot s) d\mathcal{H}^2(z). $$

Performing a similar computation for $J^c$, we obtain

$$ J_{kmgr}^c(s) = \int_{S^2} \frac{1}{2} C_{abcd} C_{ijkl} C_{efgh} A_{bpl} A_{pmn} A_{dgh} A_{qrs} z_j z_n z_f z_s D(z)_{ai}^{-1} D(z)_{ce}^{-1} $$

$$ \times \left( \int_{-\infty}^{+\infty} r^2 |\hat{\varphi}^\rho(rz)|^2 e^{irz \cdot s} dr \right) d\mathcal{H}^2(z). $$

Considering the inner integral and performing a change of variable,

$$ \int_{-\infty}^{+\infty} r^2 |\hat{\varphi}^\rho(rz)|^2 e^{irt} dr = \frac{1}{\varepsilon^3} \int_{-\infty}^{+\infty} r^2 |\hat{\varphi}^\rho(\varepsilon z \cdot s)|^2 e^{irz \cdot s / \varepsilon} dr = -|\eta^\rho''(t)|; $$

hence

$$ J_{kmgr}^c(s) = \int_{S^2} \frac{1}{2} C_{abcd} C_{ijkl} C_{efgh} A_{bpl} A_{pmn} A_{dgh} A_{qrs} z_j z_n z_f z_s D(z)_{ai}^{-1} D(z)_{ce}^{-1} \eta^\rho(z \cdot s) d\mathcal{H}^2(z). $$

By applying a series of tensor identities, this representation can be reduced to

$$ J_{kmgr}^c(s) = \int_{S^2} \frac{1}{2} \left[ C_{kmgr} - C_{abcd} A_{bpl} A_{pmn} A_{dgh} A_{qrs} z_j z_n z_f z_s D(z)_{ai}^{-1} \right] \eta^\rho(z \cdot s) d\mathcal{H}^2(z). $$

Kernel properties. The symmetry and smoothness properties asserted in (1) and (2) follow directly from the representations (31) and (32), noting that $\eta^\rho$ is smooth by construction.
Next, we note that as $\eta^1 \in C^\infty(\mathbb{R})$ is a rapidly-decreasing function, there exist constants $C_m > 0$ for $m \in \mathbb{N}$, independent of $\varepsilon$, such that
\begin{equation}
(|\eta^r|^m)(r) \leq \frac{C_m}{\varepsilon^{m+1}} \quad \text{for all } r \in \mathbb{R}.
\end{equation}
Moreover, since $C$ satisfies a Legendre–Hadamard condition and $D(k)$ is strictly positive definite, it follows that there exist $M$ and $M'$ such that for all $z \in S^2$
\begin{equation}
|C_{efgh}C_{ijkl}C_{clmn}A_{fik}A_{hld}z_kz_nD(z)z_j^{-1}D(z)^{-1}z_m| \leq M \quad \text{and}
\end{equation}
\begin{equation}
|C_{kmgr} - C_{abgr}D(z)^{-1}C_{ijkm}z_kz_l| \leq M'.
\end{equation}
Applying the bounds (33) and (34) to the representations (31) and (32), we obtain
\begin{equation}
|D^mK^{\varepsilon}_{abcd}(s)| \leq \frac{4\pi C_m M}{\varepsilon^{m+1}} \quad \text{and} \quad |D^mJ^{\varepsilon}_{abcd}(s)| \leq \frac{4\pi C_{m+2} M}{\varepsilon^{m+3}} \quad \text{for all } s \in \mathbb{R}^3 \text{ and } m \in \mathbb{N}.
\end{equation}
Further, taking derivatives of (31), applying the resulting multilinear operator to the collection of vectors $v_1, \ldots, v_j, s, \ldots, s$, where $|v_i| = 1$, and using (34) once more, we find that
\begin{equation}
|D^mK^\varepsilon(s) : [v_1, \ldots, v_j, s, \ldots, s]| \leq M \int_{S^2} |z \cdot s|^{m-j}|(\eta^r)^{(m)}(z \cdot s)| \, dz.
\end{equation}
Expressing this upper bound using polar coordinates on $S^2$ with inclination $\theta$ measured relative to an axis parallel to $s$, and subsequently changing variable to $t = \frac{|s|}{\varepsilon} \cos \theta$, we have
\begin{equation}
|D^mK^\varepsilon(s)[v_1, \ldots, v_j, s, \ldots, s]| \leq 2\pi M \int_0^{\pi} |(s|^{m-j}|(\eta^1)^{(m)}(t)| \frac{|s|^{m-j}|\cos \theta|^{m-j}}{\varepsilon^{m+1}} | dt
= \frac{2\pi M}{\varepsilon^j|s|} \int_{-|s|/\varepsilon}^{|s|/\varepsilon} |t|^{m-j}|(\eta^1)^{(m)}(t)| \, dt
\leq \frac{2\pi M}{\varepsilon^j|s|} \int_{-\infty}^{\infty} |t|^{m-j}|(\eta^1)^{(m)}(t)| \, dt,
\end{equation}
where the integral in this upper bound is finite since $\eta^1$ is a rapidly-decreasing function. Dividing by $|s|^{m-j}$, and combining with (35) completes the proof of assertion (3). \hfill \Box

We make the following remarks concerning the Fourier representations of the kernels given in formulae (31) and (32):

- In the case where $\varphi^\varepsilon$ is a Gaussian, as in the example provided in (8), we may explicitly compute $\eta^\varepsilon$ as used in the proof above, giving
\begin{equation}
\eta^\varepsilon(t) = \frac{8\pi^{7/2}}{\varepsilon} \exp \left(-\frac{t^2}{4\varepsilon^2}\right).
\end{equation}
More generally, for the purpose of computation we may choose $\varphi^\varepsilon$ in order to obtain a convenient expression for $\eta^\varepsilon$.
- Combining a convenient choice for $\eta^\varepsilon$ with the representations (31) and (32) suggests that the kernels $J^\varepsilon$ and $K^\varepsilon$ may be efficiently computed numerically, since these expressions require integration of a smooth function over the unit sphere, and this can be accurately approximated in practice with relatively few quadrature points. Moreover, these expressions are amenable to asymptotic analysis in the case where $|s| \gg \varepsilon$, which should allow for the implementation of explicit expressions to speed-up computation.

4. Deforming dislocations and the Peach–Koehler force

Theorem 1 established a representation of the elastic energy induced in a material due to the presence of dislocations. In this section, we prove Theorem 2 computing the configurational or Peach–Koehler force induced on a dislocation configuration, and demonstrating its properties.
4.1. The Peach–Koehler force. The first step towards proving Theorem 2 is to establish the expressions (15) and (16).

**Lemma 7.** If $S \in \mathcal{A}$, the inner variation of $\Phi^\varepsilon$, defined in (28), is given by

$$
\langle D\Phi^\varepsilon(S), g \rangle = -\int_{\Gamma} f^{PK}(s) \cdot g \, d\mathcal{H}^1(s), \quad \text{where} \quad f^{PK}(s, S) := G(s, S) \wedge \tau(s)
$$

and

$$
G_k(s, S) := \int_{T} A_{klm} \varepsilon^{abcd, e}(s-t) b_a(s) b_c(t) \tau_d(t) d\mathcal{H}^1(t).
$$

Moreover, if $S = \partial T$, where $T \in \mathcal{A}(\mathbb{R}^3; \mathcal{L})$ is supported on $\Sigma$ with slip vector $b$ and normal field $\nu$, $G$ can alternatively be written

$$
G_k(s, S) = \int_{\Sigma} A_{def} A_{klm} \varepsilon^{abcd, mb}(s-t) b_a(s) b_c(t) \nu_f(t) d\mathcal{H}^2(t).
$$

**Proof.** Given $g \in C^1(\mathbb{R}^3; \mathbb{R}^3)$, we set $h^\delta := \varepsilon^d + \delta g$. Pushing forward, we find that

$$
\Phi^\varepsilon(h^\delta, S) = \int_{\Gamma \times \tau} \frac{1}{2} K^{\varepsilon\tau}_{abcd, e}(s-t) (g_e(s) - g_e(t)) b_a(s) \tau_b(s) b_c(t) \tau_d(t) + \frac{1}{2} K^{\varepsilon\tau}_{abcd, e}(s-t) b_a(s) \nabla^{\tau} g_b(s) b_c(t) \tau_d(t) d\mathcal{H}^1(s, t).
$$

Applying the definition (14), we differentiate and set $\delta = 0$, we find

$$
\langle D\Phi^\varepsilon(S), g \rangle = \int_{\Gamma \times \tau} \left[ \frac{1}{2} K^{\varepsilon\tau}_{abcd, e}(s-t) g_e(s) b_a(s) \tau_b(s) b_c(t) \tau_d(t) + \frac{1}{2} K^{\varepsilon\tau}_{abcd, e}(s-t) b_a(s) \nabla^{\tau} g_b(s) b_c(t) \tau_d(t) \right] d\mathcal{H}^1(s, t).
$$

Applying the symmetries of $K^\varepsilon$ asserted in Lemma 6, this formula reduces to

$$
\langle D\Phi^\varepsilon(S), g \rangle = \int_{\Gamma \times \tau} \left[ K^{\varepsilon\tau}_{abcd, e}(s-t) g_e(s) b_a(s) \tau_b(s) b_c(t) \tau_d(t) + K^{\varepsilon\tau}_{abcd, e}(s-t) b_a(s) \nabla^{\tau} g_b(s) b_c(t) \tau_d(t) \right] d\mathcal{H}^1(s, t).
$$

Since $S \in \mathcal{A}$ satisfies $\partial S = 0$, i.e. $S$ is formed of closed loops, we may integrate by parts in the variable $s$, passing a derivative from $g$ onto $K^\varepsilon$ in the second term, which yields

$$
\langle D\Phi^\varepsilon(S), g \rangle = \int_{\Gamma \times \tau} \left[ K^{\varepsilon\tau}_{abcd, e}(s-t) g_e(s) b_a(s) \tau_b(s) b_c(t) \tau_d(t) - K^{\varepsilon\tau}_{abcd, e}(s-t) b_a(s) g_b(s) \tau_c(t) \tau_d(t) \right] d\mathcal{H}^1(s, t).
$$

Now, applying the tensor identity $A_{ijk} A_{klm} = I_{ijkl} - I_{ilmj}$, and the definition of $G(s, S)$, we obtain the first result. To obtain the latter expression, we simply apply Stokes’ Theorem. \qed

In view of this result, we make two remarks:

- Without additional regularity assumptions on $S$, we note that $f^{PK}$ is generically only in $L^\infty(S)$, since the tangent field $\tau$ on a Lipschitz curve need not be continuous.
- More generally, the fact that $f^{PK}$ is the product of a smooth kernel with components of $b$ (which is locally constant on $S$) and the tangent field $\tau$, entails that the regularity of $f^{PK}$ at a point is dictated by the regularity of the tangent field $\tau$ at the same point. This point is one we will return to in §5 when formulating a dynamical theory.
- If $g$ and $S$ are assumed to be more regular, it is possible to compute higher–order variations of the energy in a similar way. However, it should be noted that some care is required if variations are made in different directions, since the order in which variations are taken will matter in general.
4.2. Bounds on the Peach–Koehler force. The second crucial step in proving Theorem 2 is to establish (17), which is encoded in the following result.

Lemma 8. If \( S \in \mathcal{A} \) is an admissible dislocation configuration with Burgers vector \( b \), then the Peach–Koehler force satisfies the uniform bound

\[
\left\| f_{PK}(S) \right\|_{L^\infty} \leq \frac{C}{\varepsilon} \| b \|_{L^\infty} \Theta(S) \log \left( 1 + \frac{2M(S)}{\varepsilon \Theta(S)} \right),
\]

where \( C > 0 \) is a coefficient independent of \( S \) and \( \varepsilon \), \( \| b \|_{L^\infty} \) is the maximum Burgers vector, and the mass \( M(S) \) and mass ratio \( \Theta(S) \) were respectively defined in (4) and (6).

Proof. As a consequence of assertion (3) in Lemma 6 we have that

\[
|DK^\varepsilon(s)| \leq \frac{C}{\varepsilon \sqrt{x^2 + |s|^2}},
\]

with \( C \) independent of \( \varepsilon \), and therefore the expression given for \( f_{PK} \) in (15) directly implies that

\[
|f_{PK}(s, S)| \leq \| b \|_{L^\infty} \int_\Gamma |DK^\varepsilon(s - t)||b(t)|dH^1(t) \leq \frac{C}{\varepsilon} \| b \|_{L^\infty} \int_\Gamma \frac{|b(t)|}{\sqrt{x^2 + |s - t|^2}}dH^1(t),
\]

where \( \Gamma \) is the support of \( S \).

This upper bound may now be recast in the following way: Define the function \( \mu : \mathbb{R}^+ \to \mathbb{R}^+ \) to be

\[
\mu(r) := M(S \setminus B_r(S)).
\]

This function is clearly monotonically increasing in \( r \), satisfies \( \mu(0) = 0 \), and since \( S \in \mathcal{A} \) is compactly-supported, there must exist \( R \geq 0 \) for which

\[
(38) \quad \mu(r) = M(S) \quad \text{whenever} \quad r \geq R.
\]

As a consequence of these facts, \( \mu \) is a function of bounded variation (see §3.2 of [3]), and has a weak derivative, \( \mu' \), which may in general be a measure. Using the definition of \( \mu(r) \), we have

\[
\int_\Gamma \frac{|b(t)|}{\sqrt{x^2 + |s - t|^2}}dH^1(t) = \int_0^\infty \frac{1}{\sqrt{\varepsilon^2 + r^2}}d\mu'(r).
\]

Now, to proceed, we define the inverse of \( \mu \),

\[
\rho(m) := \inf_{r > 0} \{ \mu(r) \leq m \}.
\]

We note that as a consequence of the observation made in (38), \( \rho(m) = +\infty \) for \( m > M(S) \). Changing variable by setting \( \rho(m) = r \), and since by definition, \( m = \mu(\rho(m)) \), so that \( 1 = \mu'(\rho(m))\rho'(m) \), we have

\[
\int_0^\infty \frac{1}{\sqrt{\varepsilon^2 + r^2}}d\mu'(r) = \int_0^{M(S)} \frac{1}{\sqrt{\varepsilon^2 + m^2}}dm.
\]

To estimate this integral, we use the definition of \( \Theta(S) \) given in (6) to find that

\[
\mu(r) \leq \Theta(S)r \quad \text{for all} \quad r \geq 0, \quad \text{and hence} \quad \rho(m) \geq \frac{m}{\Theta(S)} \quad \text{for all} \quad m \geq 0.
\]

It follows that the latter integral may be bounded above by

\[
\int_0^{M(S)} \frac{1}{\sqrt{\varepsilon^2 + m^2}/\Theta(S)}dm \leq \int_0^{M(S)} \frac{1}{\sqrt{\varepsilon^2 + m^2}/\Theta(S)^2}dm = \Theta(S) \log \left( \frac{M(S)}{\varepsilon \Theta(S)} + \sqrt{1 + \frac{M(S)^2}{\varepsilon^2 \Theta(S)^2}} \right) \leq \Theta(S) \log \left( 1 + \frac{2M(S)}{\varepsilon \Theta(S)} \right),
\]

which directly entails the stated result.

As discussed in [2] this estimate, while better than simply using the fact that \( DK^\varepsilon \) is globally bounded, does not take into account much detail of the geometric structure, nor the fact that the Peach–Koehler force may be cast as either a line or surface integral (see the result of Lemma 7). It may be of interest for future applications to improve this bound in order to take better account of additional geometric features of a given dislocation configuration.
As a direct consequence of (17), we obtain the following $L^2$ bound directly via Hölder’s inequality.

**Corollary 9.** We have the following bound on the Peach–Koehler force:

\[
\|f^\mathrm{PK}\|_{L^2} \leq \frac{C}{\varepsilon} \|b\|_{L^\infty} M(S)^{1/2} \Theta(S) \log \left( 1 + \frac{2 M(S)}{\varepsilon \Theta(S)} \right).
\]

Both $L^\infty$ and $L^2$ bounds, (17) and (39), will be important for the proof of Theorem 3.

**4.3. Continuity of the Peach–Koehler force.** To complete the proof of Theorem 2, we establish (18).

**Lemma 10.** If $g : S \rightarrow \mathbb{R}^3$ is a Lipschitz map and $F := \text{id} + g$, then

\[
\|F^\# (f^\mathrm{PK}(F) S) - f^\mathrm{PK}(S)\|_{L^\infty} \leq (1 + CM(S)) \|\nabla g\|_{L^\infty} + CM(S) \|g\|_{L^\infty},
\]

where $C$ is a constant independent of $g$ and $S$.

**Proof.** We recall that

\[
f^\mathrm{PK}(s, S) = \left( \int_T A_{klm} K_{atcd,m} (s - t) \beta_a(s) \beta_c(t) \tau_d(t) dH^1(t) \right) \wedge \tau(s),
\]

and so

\[
f^\mathrm{PK}(F(s), F \# S) = \left( \int_T A_{klm} K_{atcd,m} (F(s) - F(t)) \beta_a(s) \beta_c(t) DF(t) [\tau(t)] dH^1(t) \right) \wedge DF(s) [\tau(s)].
\]

Taking the difference between these formulae, applying the triangle inequality and using the fact that

\[|DF(s)[\tau(s)] - \tau(s)| \leq \|\nabla g\|_{L^\infty}\]

and since by assertion (3) of Lemma 6, $D^2K^\varepsilon$ is uniformly bounded, we may Taylor expand to obtain

\[D^2K^\varepsilon (s - t + \theta (g(s) - g(t))) - D^2K^\varepsilon (s - t) = |D^2K^\varepsilon (s - t + \theta (g(s) - g(t))) [g(s) - g(t)]| \leq C \|g\|_{L^\infty}.
\]

Taking the difference between (40) and (41), and applying the triangle inequality along with the latter estimates, we directly deduce the result. \(\square\)

### 5. Evolution Problem and Existence Results

We now prove Theorem 3. Our basic strategy for doing so follows a fairly standard scheme. We carry out the following steps:

1. Construct a family of approximate solutions.
2. Derive bounds on the approximate solutions which are independent of the approximation.
3. Use these bounds and a compactness result to extract a convergent approximating sequence.
4. Prove that the approximating sequence satisfies (24) in the limit, and verify the solution is unique.

Each of these steps is carried out in turn over the course of the following sections.

#### 5.1. Approximation scheme.** In order to prove existence of a dynamical evolution, we set up an approximation scheme, which may be viewed as an explicit Euler scheme for the gradient flow dynamics. Fixing $T > 0$ and a sequence of times

\[0 = t^0 < t^1 < \ldots < t^K = T, \quad \text{and set} \quad \delta t^i := t^{i+1} - t^i,
\]

for each $i = 0, \ldots, K - 1$, we will say that $(U, \{v^i\}_{i=0}^{K-1})$ form an approximate solution to DDD if $U(t) = U(t^i)^\# (\text{id} + (t - t^i) v^i)$ for all $t \in (t^i, t^{i+1})$ and $i = 0, \ldots, K - 1$, and $v^i$ satisfies

\[
v^i \in \text{argmin} \quad \Psi(S^i, v) - (f^\mathrm{PK}(S^i), v)_{L^2(S^i)} \quad \text{where} \quad S^{i+1} := (\text{id} + \delta t^i v^i)^\# S^i
\]

for each $i = 0, \ldots, K - 1$. 
We will prove that each of the minimisation problems \(42\) is well–posed, and given sufficient regularity of \(S^i, \psi^i\) is regular. The following lemma encodes the first of these results.

**Lemma 11.** If \(S^i \in \mathcal{A}\), then the minimisation problem in \(42\) has a unique solution \(\psi^i \in H^1(S^i; \mathbb{R}^3)\), which satisfies

\[
\partial_v \Psi(S^i, \psi^i) \ni f_{PK}(S^i) \text{ in } H^1(S^i, \mathbb{R}^3)^* 
\]

and the bound

\[
\|\psi^i\|_{H^1} \leq \frac{C \mathcal{M}(S^i)^{1/2} \Theta(S^i) \|b\|_{L^\infty}}{\varepsilon \min(\alpha, \beta)} \log \left(1 + \frac{2 \mathcal{M}(S^i)}{\varepsilon \Theta(S^i)}\right).
\]

**Proof.** First, setting \(v = 0\) demonstrates that the functional which we seek to minimise is finite for some \(v \in H^1(S^i; \mathbb{R}^3)\). Assumptions (C\(_1\)) and (G) entail that

\[
\Psi(S^i, v) \geq \int_{S^i} \frac{1}{2} \alpha |\nabla \tau v|^2 + \frac{1}{2} \beta |v|^2 \, d\mathcal{H}^1 \geq \frac{1}{2} \gamma \|v\|_{H^1}^2, \quad \text{for any } v \in H^1(S^i, \mathbb{R}^3),
\]

where \(\gamma = \min(\alpha, \beta)\). Using the bound for \(f_{PK}\) derived in Corollary 9, we also have

\[
\left| \int_{S^i} f_{PK} \cdot v \, d\mathcal{H}^1 \right| \leq \|v\|_{L^2} \frac{C}{\varepsilon \|b\|_{L^\infty} \mathcal{M}(S^i)^{1/2} \Theta(S) \log \left(1 + \frac{2 \mathcal{M}(S)}{\varepsilon \Theta(S)}\right)} \left(1 + \frac{2 \mathcal{M}(S)}{\varepsilon \Theta(S)}\right)^2.
\]

Combining (45) and (46) and using Young’s inequality in the usual way, we find that

\[
\Psi(S^i, v) - (f_{PK}(S^i), v)_{L^2} \geq \frac{1}{4} \gamma \|v\|_{H^1}^2 - \frac{C^2 \|b\|_{L^\infty} \mathcal{M}(S^i) \Theta(S) \log \left(1 + \frac{2 \mathcal{M}(S)}{\varepsilon \Theta(S)}\right)}{2\gamma \varepsilon^2} \left(1 + \frac{2 \mathcal{M}(S)}{\varepsilon \Theta(S)}\right)^2
\]

which implies that the functional which we seek to minimise is coercive. As \(\Psi\) is strictly convex, and as \(D\Phi^\varepsilon(S^i)\) is a bounded linear functional and is therefore also convex, it follows that the map \(v \mapsto \Psi(S^i, v) - (f_{PK}(S^i), v)_{L^2}\) is weakly lower semicontinuous. A standard application of the Direct Method of the Calculus of Variations therefore implies existence of \(\psi^i\), and strict convexity entails that \(\psi^i\) is unique.

To establish (43), we note that convexity of \(v \mapsto \Psi(S^i, v) + (D\Phi^\varepsilon(S^i), v)\) implies that the subdifferential at the minimum must contain 0, and therefore, by the characterisation of \(\partial_v \psi\) given in (21), it follows that

\[
\int_{S^i} \nabla \tau \psi^i \cdot A(b, \tau) \nabla \tau w + (D^+ \psi(b, \tau, \psi^i) - f_{PK}) \cdot w \, d\mathcal{H}^1 = 0 \quad \text{for any } w \in H^1(S^i; \mathbb{R}^3).
\]

Standard properties of convex functions entail that

\[
\psi(b, \tau, 0) \geq \psi(b, \tau, v) - \xi \cdot v \quad \text{for any } \xi \in \partial \psi(b, \tau, v),
\]

so since \(\psi(b, \tau, 0) = 0\) by (C\(_2\)), it follows that

\[
D^+ \psi(b, \tau, v) \cdot v \geq \psi(b, \tau, v) \geq \frac{1}{2} \beta |v|^2 \quad \text{for any } v \text{ such that } v \cdot \tau = 0,
\]

where we have applied (G). Setting \(w = \psi^i\) in (47) and bounding the left–hand side below as in (45), we thereby obtain

\[
\frac{1}{2} \gamma \|\psi^i\|_{H^1}^2 \leq \|f_{PK}(S^i)\|_{L^2} \|\psi^i\|_{L^2} \leq \|f_{PK}(S^i)\|_{L^2} \|\psi^i\|_{H^1}.
\]

The bound (44) then follows directly from the estimate established in Corollary 9.

\[\square\]

Considering the details of the proof above, we make two remarks:

- Estimate (44) hinges upon the \(L^2\) bound on \(f_{PK}\) made in Corollary 9, which in turn relies upon the \(L^\infty\) bound [17] proved in Lemma 6. Any improvement of [17] would therefore entail an improved bound on \(\psi^i\).
- The choice to assume quadratic growth of \(\psi\) in (G) allows us to directly obtain an \(H^1\) estimate on \(\psi^i\); if a weaker growth condition was assumed, we would need to apply a Poincaré–type inequality to obtain a similar estimate. Since such an inequality would inevitably depend upon \(M(S)\), this would render some aspects of the arguments which follow more technical.
As a consequence of (44), we may apply the Cauchy–Schwarz inequality to show the following corollary.

**Corollary 12.** The solution to the minimisation problem (42) satisfies

\[ \|\nabla_v v^i\|_1 \leq M(S^i)^{1/2} \|\nabla_v v^i\|_2 \leq \frac{C M(S^i) \Theta(S^i) \|b\|_L^\infty}{\epsilon \min(\alpha, \beta)} \log \left[ 1 + \frac{2M(S^i)}{\epsilon \Theta(S^i)} \right]. \]

Estimate (45) will be important later, as it provides a control on the maximal growth rate of M(S^i).

5.2. Properties of approximate solutions. Now that the existence of v^i : S^i \to \mathbb{R}^3 has been established, we wish to define S^{i+1} as the pushforward of S^i under the mapping \( \delta U^i(s) := \text{id}(s) + \delta t^i v^i(s) \).

At present, we have only established that v^i is in H^3(S^i; \mathbb{R}^3); this entails that \( \delta U^i \) is continuous as a mapping from S^i to \mathbb{R}^3, but in order to be sure that S^{i+1} \in \mathcal{A}, we must show that \( \delta U^i \) is Lipschitz, and hence we must develop a regularity theory for v^i. The following result establishes several crucial properties of v^i.

**Lemma 13.** If S^i is a finite union of C^{k,\gamma} curves with k \geq 1 and 0 < \gamma \leq 1, then the solution v^i to the minimisation problem (42) is C^{k,\gamma}, and moreover we have the bounds

\[
\|v^i\|_{L^\infty} \leq C \sqrt{1 + 2M(S^i) \Theta(S^i)} \|b\|_{L^\infty} \log \left[ 1 + \frac{2M(S^i)}{\epsilon \Theta(S^i)} \right]
\]

\[
\|\nabla_v v^i\|_{L^\infty} \leq \frac{C''}{\epsilon \alpha} \|b\|_{L^\infty} \left( 1 + \sqrt{1 + 2M(S^i)} \right) \Theta(S^i) \log \left[ 1 + \frac{2M(S^i)}{\epsilon \Theta(S^i)} \right]
\]

\[
[\nabla_v v^i] \leq \left( M(S^{1-\gamma} + [\tau], M(S)) \right) \frac{C''}{\epsilon \alpha} \|b\|_{L^\infty} \left( 1 + \sqrt{1 + 2M(S)} \right) \Theta(S) \log \left[ 1 + \frac{2M(S)}{\epsilon \Theta(S)} \right]
\]

To prove this result, we pull back to a flat domain, recast the resulting equation as an ODE system, and then use the properties assumed of A and \psi along with some elementary integral bounds.

**Proof.** We first prove regularity, then proceed to obtain the stated estimates. Since they are fixed throughout this proof, we suppress superscripts, writing v and S in place of v^i and S^i.

**Regularity.** Since S is assumed to be a union of C^{k,\gamma} curves, there exists a C^{k,\gamma} diffeomorphism g which maps the interval \((-a, a)\) to a neighbourhood of s \in S. Without loss of generality, we may assume g is an arc–length parametrisation, so g' = g'' \tau on \((-a, a)\), and therefore \|g'| = 1 since \tau = 1.

Defining V : (-a, a) \to \mathbb{R}^3 to be the pullback of v by g, i.e. V := g# v, we find it has weak derivative

\[ g# \nabla_v v = V'. \]

We recall that the equation satisfied by v on S is

\[ -\text{div}_v [A(b, \tau) \nabla_v v] + D_\tau^\perp \psi(b, \tau, v) = f^\text{PK}, \]

so defining B = g# b and F = g# f^\text{PK} and ‘pulling back’ the equation, we find that V : (-a, a) \to \mathbb{R}^3 must satisfy

\[ -[A(B(r), g'(r)) V'(r)]' + D_\tau^\perp \psi(B(r), g'(r), V(r)) = F(r) \]

almost–everywhere on \((-a, a)\). As remarked at the end of [4.1], the fact that S is assumed to by C^{k,\gamma} implies that F \in C^{k-1,\gamma}((-a, a); \mathbb{R}^3).

We define the auxiliary function \( \sigma := g#(A(b, \tau) \nabla_v v) \), so that V' = A(B, g')^{-1} \sigma. Solving (52) for \sigma entails that V and \sigma must satisfy the system of equations

\[ V'(r) = A(B(r), g'(r))^{-1} \sigma(r) \]

\[ \sigma'(r) = D_\xi \psi(B(r), g'(r), V(r)) - F(r). \]

Now, since v \in H^1(S; \mathbb{R}^3) \subset C^{0,\frac{1}{2}}(S; \mathbb{R}^3), and g \in C^{1,\gamma}((-a, a); S) \subset C^{0,1}((-a, a); S), it follows that V = w# g \in C^{0,\frac{1}{2}}((-a, a); \mathbb{R}^3). The regularity assumptions on \psi and the second equation therefore entail that \sigma \in C^{1,\eta}((-a, a); \mathbb{R}^3), where \eta = \min\{\frac{1}{2}, \gamma\}, and the regularity assumptions on A applied to the first equation entail that V \in C^{1,\gamma}((-a, a); \mathbb{R}^3). Bootstrapping, we ultimately find that V, \sigma \in C^{k,\gamma}((-a, a); \mathbb{R}^3).
This local argument entails \( v \in C^{k,\gamma}(S; \mathbb{R}^3) \) via a finite covering of \( S \), which is possible since \( S \in \mathcal{F} \) is a finite union of \( C^{k,\gamma} \) curves.

**Uniform bound.** Now, taking the inner product between \( V \) and \( V' \) and then integrating and applying the Cauchy–Schwarz inequality, we obtain

\[
\frac{1}{2}|v(s_1)|^2 - \frac{1}{2}|v(s_0)|^2 = \int_{\rho(s_0)}^{\rho(s_1)} V(r) \cdot V'(r) \, dr \leq \|V\|_{L^2} \|V'\|_{L^2} = \|v\|_{L^2} \|\nabla v\|_{L^2}.
\]

Integrating with respect to \( s_0 \), dividing by \( M(S) \), and using (54), we find that

\[
|v(s_1)|^2 \leq 2\|v\|_{L^2} \|\nabla v\|_{L^2} + \frac{\|v\|_{L^2}^2}{M(S)} \leq \frac{C^2(1 + 2M(S))\Theta(S)^2\|b\|_{L^\infty}^2}{\varepsilon^2 \min(\alpha, \beta)^2} \log \left(1 + \frac{2M(S)}{\varepsilon \Theta(S)}\right)^2.
\]

which leads directly to (49).

**Uniform gradient bound.** If \( V \) and \( \sigma \) solve (53) on \((-a, a)\), then by integrating the second equation, we find that

\[
|\sigma(r_1) - \sigma(r_0)| \leq C' \left(\|F\|_{L^\infty} + \|V\|_{L^\infty}\right)|r_1 - r_0|
\]

so \( \sigma \in C^{0,1}((-a, a); \mathbb{R}^3) \). Moreover, we may use (49) and the definition of \( F \) as a pullback of \( f^{PK} \) to find

\[
\|\sigma\|_{L^\infty} \leq C' \left(\|F\|_{L^\infty} + \|v\|_{L^\infty}\right)^{\frac{1}{2}} M(S)
\]

\[
\leq C'' \frac{\|b\|_{L^\infty}}{\varepsilon} \left(1 + \sqrt{\frac{1 + 2M(S)}{\min(\alpha, \beta)}}\right) \frac{M(S)\Theta(S)\log \left(1 + \frac{2M(S)}{\varepsilon \Theta(S)}\right)}{\varepsilon \Theta(S)}
\]

Using the definition of \( \sigma \), and noting that \((C_1)\) implies that \( \|A^{-1}\|_{L^\infty} \leq \alpha^{-1} \), we therefore obtain

\[
\|\nabla v\|_{L^\infty} \leq C'' \frac{\|b\|_{L^\infty}}{\varepsilon} \left(1 + \sqrt{\frac{1 + 2M(S)}{\min(\alpha, \beta)}}\right) \frac{M(S)\Theta(S)\log \left(1 + \frac{2M(S)}{\varepsilon \Theta(S)}\right)}{\varepsilon \Theta(S)}
\]

which is (50).

**Uniform bound on Hölder seminorm.** Applying the assumption (R) to deduce that \( |A^{-1}(b, \tau_1) - A^{-1}(b, \tau_2)| \leq L|\tau_1 - \tau_2| \) for some \( L \), we have

\[
|V'(r_1) - V'(r_0)| = |A(B(r_1), g'(r_1))^{-1} \sigma(r_1) - A(B(r_0), g'(r_0))^{-1} \sigma(r_0)|
\]

\[
\leq |A(B(r_1), g'(r_1))^{-1} \sigma(r_1) - A(B(r_1), g'(r_1))^{-1} \sigma(r_0)|
\]

\[
+ |A(B(r_1), g'(r_1))^{-1} \sigma(r_0) - A(B(r_1), g'(r_0))^{-1} \sigma(r_0)|
\]

\[
\leq \alpha^{-1}|\sigma(r_1) - \sigma(r_0)| + L |g'(r_1) - g'(r_0)| \|\sigma\|_{L^\infty}
\]

\[
\leq \left(\alpha^{-1}|\sigma(\gamma) + L |g'| \gamma|\|\sigma\|_{L^\infty}\right)|r_1 - r_0|
\]

Using (55), we find that

\[
[\sigma]_\gamma \leq C'' \frac{\|b\|_{L^\infty}}{\varepsilon} \left(1 + \sqrt{\frac{1 + 2M(S)}{\min(\alpha, \beta)}}\right) \frac{M(S)^{1-\gamma} \Theta(S)\log \left(1 + \frac{2M(S)}{\varepsilon \Theta(S)}\right)}{\varepsilon \Theta(S)}
\]

and estimating the other term using (56), we obtain (51).

Lemma [13] guarantees the spatial regularity of any approximate solution \( U(t) \) defined via the procedure prescribed in (42), and therefore ensures that such approximate solutions are well-defined. Our next step will be to prove that approximate solutions converge as \( \max_i \{\delta t^i\} \to 0 \), and that the limit satisfies (24).
5.3. Convergence of approximate solutions. Our approach to proving convergence is via compactness; this requires us to prove appropriate uniform a priori bounds on approximate solutions, which will subsequently allow us to employ the Arzelà–Ascoli theorem. We remark that all bounds on approximate solutions derived thus far depend upon \( M(S) \) and \( \Theta(S) \), and therefore it is these quantities we must bound; the following lemma therefore establishes a bound on the growth of \( M(U(t)\#S) \).

Lemma 14. If \( S^0 \in \mathcal{A} \) with \( \Theta(S^0) < +\infty \), then for any \( \rho > \Theta(S^0) \) and \( M > M(S^0) \), and all \( \delta > 0 \) sufficiently small, there exists \( T(M, \rho, \delta) > 0 \) such that any approximate solution of DDD (in the sense described in [11]) with \( \max_i \{ \delta t^i \} \leq \delta \) satisfies

\[
M(U(t)\#S^0) \leq M \quad \text{and} \quad \Theta(U(t)\#S^0) \leq \rho \quad \text{for all } t \in [0, T].
\]

Proof. The proof is divided into first obtaining a uniform bound on the mass growth, then using this bound to guarantee a bound on the growth of the mass ratio.

Uniform mass bound. Our first step is to establish a uniform bound on the mass. By definition, \( U(t)\#S^0 = (id + (t - t_i)v^i)\#S^i \) for \( t \in (t_i, t_{i+1}) \), and so \( (58) \) implies

\[
M(U(t)\#S^0) = M((id + (t - t_i)v^i)\#S^i) = \|t^i + (t - t_i)v^i\|_{L^1(S^i)} \leq M(S^i) + (t - t_i)^2 C \Theta(S^i) M(S^i) \log \left( 1 + \frac{2M(S^i)}{\varepsilon \min(\alpha, \beta)} \right).
\]

Estimating \( \log(1 + x) \leq x \) for \( x \geq 0 \), and employing the comparison principle for ODEs, we find that \( M(U(t)\#S^0) \) must be bounded above by the solution to

\[
m'(t) = \frac{2C}{\min(\alpha, \beta)} \frac{m(t)^2}{\varepsilon^2} \quad m(0) = M(S^0).
\]

Since this is a separable ODE, we may check that

\[
m(t) = \left( \frac{1}{M(S^0)} - \frac{2CT\|b\|_{L^\infty}}{\varepsilon^2 \min(\alpha, \beta)} \right)^{-1}
\]

for all \( t \in [0, T] \), and therefore it follows that

\[
M(U(t)\#S^0) \leq M \quad \text{for all } t \leq T(M) = C_1 \frac{M - M(S^0)}{M(S^0)},
\]

where \( C_1 \) is a constant depending on \( \|b\|_{L^\infty}, \varepsilon \) and \( \min(\alpha, \beta) \).

Uniform mass ratio bound. Suppose that \( U(t) \) is an approximate solution on \([0, T(M)]\), with \( \max_i \{ \delta t^i \} \leq \delta \leq T(M) \). The bounds \( (49) \) and \( (50) \) entail that for \( t \) small enough, the map \( \delta U^i(t) : [t^i, t_{i+1}] \times S^i \to \mathbb{R}^3 \)

\[
\delta U^i(t) : = id + (t - t_i)v^i
\]

defines a map \( \delta U^i(t) \) that satisfies

\[
\frac{[\delta U^i(t, s_1) - \delta U^i(t, s_0)]}{[s_1 - s_0]} \leq 1 + (t - t_i)^2 C_2 \Theta(S^i) \log \left( 1 + \frac{2M(S^i)}{\varepsilon \Theta(S^i)} \right)
\]

for any \( s_0, s_1 \in S^i \) for some constant \( C_2 > 0 \). Moreover, since \( \Theta(S^i) \geq 1 \) for any \( S^i \neq \emptyset \), as observed in \( (7) \), we may bound the logarithmic terms by \( \log(1 + 2M/\varepsilon) \), finding that

\[
\frac{[\delta U^i(s_1) - \delta U^i(s_0)]}{[s_1 - s_0]} \leq 1 + C_3 (t - t_i) \Theta(S^i).
\]

for some \( C_3 > 0 \).

Next, recalling an argument made in §3.2.17 of [35], supposing that \( F : S \to \mathbb{R}^3 \) satisfies

\[
[F(s_1) - F(s_0)]/[s_1 - s_0] \leq \lambda \quad \text{for all } s_0, s_1 \in B_r(s),
\]

for some \( \lambda > 1 \), it follows that

\[
F \# S \cap B_r(F(s)) \subset F \# (S \cap B_{\lambda r}(s)),
\]

and therefore

\[
\mathcal{H}^1 \left( F \# S \cap B_r(F(s)) \right) \leq \lambda \mathcal{H}^1 (S \cap B_{\lambda r}(s)).
\]
Noting the connection between the Hausdorff measure of the support of an integral current and its mass made in \( (5) \), we may take suprema over \( r \geq 0 \) and \( s \in \mathbb{R}^3 \) to obtain
\[
\Theta(F_{\#}S) \leq \lambda^2 \Theta(S).
\]
In particular, \( \delta U^i(t) \) satisfies the conditions on \( F \) with
\[
\lambda = 1 + C_3 \left( t - t^i \right) \Theta(S^i),
\]
and so we find that
\[
\frac{\Theta(\delta U^i(t)_{\#}S^i) - \Theta(S^i)}{t - t^i} \leq C_3 \left( 2 + C_3 \delta \Theta(S^i) \right) \Theta(S^i)^2.
\]
Employing the comparison principle for ODEs once more, and using the fact that, by construction,
\[
\Theta(U(t)_{\#}S^0) = \Theta(\delta U^i(t)_{\#}S^i),
\]
we find that \( \Theta(U(t)_{\#}S^0) \) is bounded above by the solution to
\[
\rho'(t) \leq C_3 \left( 2 + C_3 \delta \rho(t) \right) \rho(t)^2, \quad \text{with } \rho(0) = \Theta(S^0).
\]
Again, being a separable ODE, we may integrate to find
\[
\frac{1}{2C_3 \Theta(S^0)} - \frac{\delta}{4} \log \left| C_3 \delta + \frac{2}{\Theta(S^0)} \right| = \frac{1}{2C_3 \rho(t)} + \frac{\delta}{4} \log \left| C_3 \delta + \frac{2}{\rho(t)} \right| = t,
\]
and thereby we see that for any
\[
t \leq T(\rho, M, \delta) = \frac{\rho - \Theta(S^0)}{2C_3(M) \Theta(S^0)} + \frac{\delta}{4} \log \left| C_3(M) \delta + \frac{2}{\Theta(S^0)} \right|,
\]
the result holds.

The result of Lemma \( 14 \) entails that, given an upper limit on the mass and mass ratio, and a maximum step size, there exists an infinite family of approximate solutions on some time interval \([0, T]\) with \( T > 0 \). This is a crucial result which allows us to establish existence by compactness in the following lemma.

**Lemma 15.** Suppose \( S^0 \in \mathcal{A} \) is \( C^{1, \gamma} \) for some \( \gamma \in (0, 1] \) and fix \( \rho \) and \( M \) such that \( M(S^0) \leq M < +\infty \) and \( \Theta(S^0) < \rho < +\infty \); then there exists \( T > 0 \) such that there exists a unique \( C^0([0, T]; C^1(S^0; \mathbb{R}^3)) \) solution to \( (24) \), and moreover \( M(U(t)_{\#}S^0) \leq M \) and \( \Theta(U(t)_{\#}S^0) \leq \rho \) for all \( t \in [0, T] \).

**Proof.** In Lemma \( 14 \) we established the existence of \( T \) such that if \( U_n : [0, T] \times \mathbb{R}^3 \) is an approximation solution of DDD with \( \max \{ \delta t^i_n \} \leq \delta \), then \( M(U_n(t)_{\#}S^0) \leq M \) and \( \Theta(U_n(t)_{\#}S^0) \leq \rho \). Therefore, taking a sequence of such solutions with \( \max \{ \delta t^i_n \} \to 0 \), we wish to show that this sequence contains a convergent subsequence via an application of the Arzelà–Ascoli Theorem (or equivalently, the compactness properties of Hölder spaces).

**Compactness.** When combined with the result of Lemma \( 14 \) the bounds established in Lemma \( 13 \) entail that
\[
\left\| \tilde{U}_n(t) \right\|_{C^1,G} \leq C(M, \rho, \delta) \quad \text{for all } n \in \mathbb{N};
\]
in turn, this entails that
\[
\left\| U(t_1) - U(t_0) \right\|_{C^1,G} \leq \int_{t_0}^{t_1} \left\| \tilde{U}_n(t) \right\|_{C^1,G} dt \leq C(M, \rho, \delta) |t_1 - t_0|.
\]
Setting \( t_0 = 0 \), it follows that \( U_n : [0, T] \times S^0 \to \mathbb{R}^3 \) are uniformly bounded in \( C^{0,1}([0, T]; C^{1, \gamma}(S^0; \mathbb{R}^3)) \), from which it follows that the sequence of approximate solutions is compact in \( C^0([0, T]; C^1(S^0; \mathbb{R}^3)) \). We may therefore extract a convergent subsequence, which we do not relabel; we denote the limit \( U_\infty \).

Moreover, the same results also imply that \( U_n(t)_{\#}v^i_n \) is uniformly bounded in \( L^2([0, T]; H^1(S^0; \mathbb{R}^3)) \). It follows that we may extract a further subsequence (which again, we do not label) such that \( U_n(t)_{\#}v^i_n \) converges weakly in \( H^1(S^0; \mathbb{R}^3) \) for almost every \( t \in [0, T] \).

Now that we have demonstrated the existence of a candidate limit, we must demonstrate that it solves \( (24) \).

**Convergence of dissipation potential.** By virtue of the fact that \( U_n \to U_\infty \) in \( C^{0,1}([0, T]; C^1(S^0)) \),
\[
U_n(t)_{\#}\tau_n(t) = \nabla_\tau U_n(t) \to \nabla_\tau U_\infty(t) = U_\infty(t)_{\#}\tau_\infty(t), \quad \text{uniformly for } t \in [0, T] \text{ as } n \to \infty,
\]
where \( \tau_n(t) : U_n(t) \# S^0 \to S^2 \) and \( \tau_\infty(t) : U_\infty(t) \# S^0 \to S^2 \) are tangent fields. Recalling the definition made in\(^{(42)}\), by construction, approximate solutions satisfy

\[
\partial_t \Psi(U_n(t_n^i) \# S^0, U_n(t_n^i) \# U_n(t_n^i)) = f_{PK}(U_n(t_n^i) \# S^0)
\]

for each \( t_n^i \).

Applying assumption (R), we have

\[
A(U_n(t)^# b, U_n(t)^# \tau_n) \to A(U_\infty(t)^# b, U_\infty(t)^# \tau_\infty)
\]

in \( L^\infty(S^0; \mathbb{R}^{3 \times 3}) \) uniformly in \( t \) as \( n \to \infty \), and therefore if \( V_n \) is a sequence of functions in \( H^1(S; \mathbb{R}^3) \) such that \( \nabla \tau V_n \to \nabla \tau V_\infty \) weakly in \( L^2(S^0; \mathbb{R}^3) \), we have

\[
\liminf_{n \to \infty} \int_{S^0} \frac{1}{2} A(U_n(t)^# b, U_n(t)^# \tau_n) : [\nabla \tau V_n, \nabla \tau V_n] \, dH^1 \geq \int_{S^0} \frac{1}{2} A(U_\infty(t)^# b, U_\infty(t)^# \tau_\infty) : [\nabla \tau V_\infty, \nabla \tau V_\infty] \, dH^1.
\]

Furthermore, the convexity and regularity assumptions (C2) and (R) imply that if \( V_n \in L^2(S^0; \mathbb{R}^3) \) is a sequence of functions such that \( V_n \to V_\infty \) weakly in \( L^2(S^0; \mathbb{R}^3) \) as \( n \to \infty \), then

\[
\liminf_{n \to \infty} \int_{S^0} \psi(U_n(t)^# b, U_n(t)^# \tau_n, V_n) \, dH^1 \geq \int_{S^0} \psi(U_\infty(t)^# b, U_\infty(t)^# \tau_\infty, V_\infty) \, dH^1.
\]

Together, (61), (62) and the fact that \( U_n(t)^# v_n^t \) converges weakly in \( L^2([0, T]; H^1(S^0; \mathbb{R}^3)) \) imply that

\[
\liminf_{n \to \infty} U_n(t)^# \Psi(U_n(t)^# S^0; v_n) \geq U_\infty(t)^# \Psi(U_\infty(t)^# S^0; v_\infty)
\]

for almost every \( t \in [0, T] \).

**Convergence of Peach–Koehler force.** Applying Lemma\(^{(10)}\) and the fact that \( U_n(t) \to U_\infty(t) \) in \( C^1(S^0; \mathbb{R}^3) \) uniformly in \( t \), we find that

\[
U_n(t)^# f_{PK}(U_n(t)^# S^0) \to U_\infty(t)^# f_{PK}(U_\infty(t)^# S^0)
\]

in \( L^\infty(S^0; \mathbb{R}^3) \) as \( n \to \infty \) uniformly in \( t \). Since \( U_n(t)^# v_n^t \to U_\infty(t)^# v_\infty \) weakly in \( L^2([0, T]; H^1(S^0; \mathbb{R}^3)) \), we further obtain that

\[
(U_n(t)^# f_{PK}(U_n(t)^# S^0), U_n(t)^# v_n^t)_{L^2} \to (U_\infty(t)^# f_{PK}(U_\infty(t)^# S^0), U_\infty(t)^# v_\infty)_{L^2}
\]

as \( n \to \infty \) for almost every \( t \in [0, T] \).

**Convergence of dissipation potential.** Since \( \Psi(S, \cdot) \) is a strictly convex functional defined on \( H^1(S; \mathbb{R}^3) \), it follows that it has a convex conjugate \( \Psi^*(S, \cdot) : H^1(S; \mathbb{R}^3)^* \to \mathbb{R} \), defined to be

\[
\Psi^*(S, \xi) = \sup \left\{ \langle \xi, v \rangle \, \psi(S, v) : v \in H^1(S; \mathbb{R}^3) \right\}.
\]

Employing this definition, we obtain

\[
\lim_{n \to \infty} U_n(t)^# \Psi^*(U_n(t)^# S^0, \xi) = \lim_{n \to \infty} \sup \left\{ \langle \xi, v \rangle - U_n(t)^# \Psi(U_n(t)^# S^0, v) : v \in H^1(S; \mathbb{R}^3) \right\}
\]

\[
= \sup \left\{ \langle \xi, v \rangle - \lim_{n \to \infty} U_n(t)^# \Psi(U_n(t)^# S^0, v) : v \in H^1(S; \mathbb{R}^3) \right\}
\]

\[
= \sup \left\{ \langle \xi, v \rangle - U_\infty(t)^# \Psi(U_\infty(t)^# S^0, v) : v \in H^1(S; \mathbb{R}^3) \right\}
\]

\[
= U_\infty(t)^# \Psi^*(U_\infty(t)^# S^0, \xi).
\]

Moreover, since (64) holds, it follows that

\[
U_n(t)^# D \Psi^*(U_n(t)^# S^0) \to U_\infty(t)^# D \Psi^*(U_\infty(t)^# S^0)
\]

in \( H^1(S^0; \mathbb{R}^3)^* \), and we conclude that

\[
U_n(t)^# \Psi(U_n(t)^# S^0, f_{PK}(U_n(t)^# S^0)) \to U_\infty(t)^# \Psi(U_\infty(t)^# S, U_\infty(t)^# f_{PK}(U_\infty(t)^# S^0))
\]

as \( n \to \infty \).

**Conclusion.** Finally, by using standard properties of the Legendre–Fenchel transform\(^{(4,54,71)}\), we note that (24) is equivalent to requiring that

\[
U(t)^# \Psi(U(t)^# S^0, v^t) + U(t)^# \Psi^*(U(t)^# S^0, -D \Psi^*(U(t)^# S^0)) - \langle D \Psi^*(U(t)^# S^0), v^t \rangle = 0
\]
for almost every $t \in [0, T]$. By considering this expression with $U_n(t)$ in place of $U(t)$, and $v_n^t$ in place of $v^t$, we may combine (39), (43) and (50), to pass to the limit, demonstrating that

$$0 = \int_0^T \left[ U_\infty(t) \# \Psi(U_\infty(t) \# S^0, v_\infty^t) + \Psi(U_\infty(t) \# S^0, U_\infty(t) \# f^{PK}(U_\infty(t) \# S^0)) + \left(U_\infty(t) \# f^{PK}(U_\infty(t) \# S^0), U_\infty(t) \# v_\infty^t\right)_{L^2} \right] dt.$$ 

This entails that, for almost every $t$, we have

$$\partial_t \Psi(U_\infty(t) \# S^0, v_\infty^t) \ni f^{PK}(U_\infty(t) \# S^0),$$

and since $U_\infty(t) \# v_\infty^t = \lim_{n \to \infty} U_n(t) \# v_n^t = \lim_{n \to \infty} \dot{U}_n(t) = \dot{U}_\infty(t)$, we have proved that $U_\infty$ solves (24).

To demonstrate uniqueness of the limit, we note that assumptions (C1) and (C2) entail that the functional $\Psi(S, v)$ is strictly convex on $H^1(S; \mathbb{R}^3)$, and therefore a standard argument guarantees that the above limit procedure is independent of the subsequence chosen. \hfill \Box

5.4. Conclusion of the proof. Now that we have proved existence for a finite time, we show that we may extend the solution to a possibly infinite time by demonstrating that the total mass of the dislocation configuration is bounded as long as the mass ratio remains bounded.

Lemma 16. If $U : [0, T] \times S^0 \to \mathbb{R}^3$ is a solution of DDD in the sense described in (24), such that $\Theta(U(t) \# S^0) \leq \rho$ for all $t \in [0, T]$, then we have the uniform bound

$$M(U(t) \# S^0) \leq \min \left\{ \frac{M(S^0)}{1 - C_1(b, \varepsilon) M(S^0) T}, \frac{2 M(S^0)}{\varepsilon \log \left( \frac{2 M(S^0) + 2 m(t)}{\varepsilon} \right)} \right\},$$

where $C_1(b, \varepsilon) = \frac{2 C \|b\|_{L^\infty}}{\varepsilon^2 \min(\alpha, \beta)}$ and $C_2(b, \varepsilon, \rho) = \frac{C \rho \|b\|_{L^\infty}}{\varepsilon \min(\alpha, \beta)}$.

Proof. Using the fact that $v^t$ must satisfy (48), and by assumption, $1 \leq \Theta(U(t) \# S^0) \leq \rho$, we have

$$\frac{d}{dt} M(U(t) \# S^0) \leq \frac{C \rho M(U(t) \# S^0) \|b\|_{L^\infty}}{\varepsilon \min(\alpha, \beta)} \log \left( 1 + \frac{2 M(U(t) \# S^0)}{\varepsilon} \right).$$

Employing the comparison principle for ODEs, it follows that for all $t \in [0, T]$, $M(U(t) \# S^0)$ must be bounded above by the solution to

$$m'(t) = \frac{C \rho \|b\|_{L^\infty}}{2 \min(\alpha, \beta)} \left( 1 + \frac{2 m(t)}{\varepsilon} \right) \log \left( 1 + \frac{2 m(t)}{\varepsilon} \right), \quad m(0) = M(S^0).$$

Integrating, we find that

$$\log \left( 1 + \frac{2 m(t)}{\varepsilon} \right) = \log \left( 1 + \frac{2 M(S^0)}{\varepsilon} \right) \exp \left( \frac{C t \rho \|b\|_{L^\infty}}{\varepsilon \min(\alpha, \beta)} \right)$$

for all $t \in [0, T]$, which, when combined with (57) which was used to bound the mass in the proof of Lemma 14 now immediately yields the bound stated. \hfill \Box

With this result in place, we now deduce the following result, which complete the proof of Theorem 3.

Corollary 17. If $S^0 \in \mathcal{A}$ satisfies $\Theta(S^0) < +\infty$, then there exists a unique solution $(U, \{v^t\})$ of (24) up until $T := \sup \left\{ t \in \mathbb{R} : \Theta(U(s) \# S^0) < +\infty \text{ for all } s \leq t \right\}$.

Proof. For any $\rho > \Theta(S^0)$, and $M > M(S^0)$, Lemma 15 establishes the existence of a unique solution up until the first time $T(M, \rho)$ at which either $M(U(T) \# S^0) = M$ or $\Theta(U(T) \# S^0) = \rho$, and we note that this existence time is bounded below by a function which is monotone in both $M$ and $\rho$. Lemma 16 entails that in fact $M(U(t) \# S^0)$ is finite for any $t \leq T^*(\rho)$, where $T^*$ is the first time at which $\Theta(U(T^*) \# S^0) = \rho$, and thus the maximal existence time is independent of the mass constraint. Letting $\rho \to \infty$, we obtain the result. \hfill \Box
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Appendix A. Vectors, forms and currents

This appendix recalls various definitions from the theory of currents, as described in Chapters 1 and 4 of both [35] and [57].

A.1. Vectors and covectors. Recall that the usual exterior product∧ is multilinear and alternating, i.e. it satisfies
\[(u + \lambda v) \wedge w = u \wedge w + \lambda(v \wedge w) \quad \text{and} \quad u \wedge v = -v \wedge u.\]
Suppose that \(e_1, \ldots, e_n\) form an orthonormal basis of \(\mathbb{R}^n\). The space of \(m\)-vectors \(\Lambda^m \mathbb{R}^n\) is then the span
\[\Lambda^m \mathbb{R}^n := \text{span}\{e_{i_1} \wedge \cdots \wedge e_{i_m} \mid i_j \in \{1, \ldots, n\}, i_1 < \cdots < i_m\};\]
\(m\)-vectors should be thought of as describing oriented \(m\)-dimensional subspaces of \(\mathbb{R}^n\). The space of \(m\)-covectors, denoted \(\Lambda^m \mathbb{R}^n^*\), is the space of linear functions on \(\Lambda^m \mathbb{R}^n\), i.e. \(\Lambda^m \mathbb{R}^n^* := [\Lambda^m \mathbb{R}^n]^*\). \(\Lambda^m \mathbb{R}^n\) may be identified with \(\Lambda^m (\mathbb{R}^n)^*,\) the span of \(m\)-fold wedge products of dual vectors \(e^*_i\), defined to satisfy \(\langle e^*_i, e_j \rangle = \delta_{ij}\).

We may define an inner product, \((\cdot, \cdot)\) and corresponding norm, \(|u| = (u, u)^{1/2}\) on \(\Lambda^m \mathbb{R}^n\), which makes \(e_{i_1} \wedge \cdots \wedge e_{i_m}\) with \(i_1 < \cdots < i_m\) an orthonormal basis for the space (see §1.7 of [35]). A similar inner product can be constructed on \(\Lambda^m \mathbb{R}^n^*\), which makes \(e^*_i \wedge \cdots \wedge e^*_m\) with \(i_1 < \cdots < i_m\) an orthonormal basis for this space.

While the above definitions are general, throughout this work we will exclusively consider \(n = 3,\) and \(m = 1\) or \(m = 2,\) since these are the cases of interest for the modelling of dislocations. In this case, we have the isometric isomorphisms
\[\mathbb{R}^3 \cong \Lambda_2 \mathbb{R}^3 \cong \Lambda_1 \mathbb{R}^3 \cong \Lambda^2 \mathbb{R}^3 \cong \Lambda^1 \mathbb{R}^3.\]
In particular, it should be noted that the identification of \(\Lambda_2 \mathbb{R}^3\) with \(\mathbb{R}^3\) corresponds to identifying \(u \wedge v\) with the usual vector cross product on \(\mathbb{R}^3\).

A.2. Forms. An \(m\)-form is a function \(\phi : \mathbb{R}^n \to \Lambda^m \mathbb{R}^n.\) Using the basis of \(\Lambda^m \mathbb{R}^n\) discussed in §A.1 any such function may be expressed as
\[\phi(x) = \sum_{i_1 < \cdots < i_m} f_{i_1 \cdots i_m}(x) e^*_{i_1} \wedge \cdots \wedge e^*_{i_m}.\]
The exterior derivative \(d\phi\) is the \((m + 1)\)-form defined via
\[d\phi(x) = \sum_{i_1 < \cdots < i_m} \left( \sum_{i_{m+1}} f_{i_1 \cdots i_m, i_{m+1}} e^*_{i_{m+1}} \right) \wedge (e^*_{i_1} \wedge \cdots \wedge e^*_{i_m}).\]
For any open set \(U \subseteq \mathbb{R}^n,\) we define the vector space of smooth \(m\)-forms which are compactly-supported in \(U\) to be
\[\mathcal{D}^m(U) := \{ \phi : U \to \Lambda^m \mathbb{R}^n \mid \phi \text{ is } C^\infty \text{ with compact support in } U \}.\]

A.3. Currents. A current is a generalisation of the notion of a distribution [37]; whereas distributions act on scalar-valued functions, currents instead act on the spaces \(\mathcal{D}^m(\mathbb{R}^n)\). In particular, an \(m\)-dimensional current (or \(m\)-current) \(T\) is a linear functional which acts on \(\mathcal{D}^m(\mathbb{R}^n)\), and we denote the action of a current on \(\phi \in \mathcal{D}^m(\mathbb{R}^n)\) to be \(\langle T, \phi \rangle \in \mathbb{R}\). The boundary of an \(m\)-dimensional current is the \((m - 1)\)-current \(\partial T\), defined to be
\[\langle \partial T, \phi \rangle = \langle T, d\phi \rangle \quad \text{for all } \phi \in \mathcal{D}^m(\mathbb{R}^n).\]
The support of a current is defined to be the closed set
\[
\text{supp}(T) := \mathbb{R}^n \setminus \left( \bigcup \left\{ U \subset \mathbb{R}^n \mid U \text{ is open and } \langle T, \phi \rangle = 0 \text{ for all } \phi \in \mathcal{D}^m(U) \right\} \right).
\]

We recall that \( \Sigma \subset \mathbb{R}^n \) is \( m \)-rectifiable if it may be expressed as a countable union of images of bounded subsets of \( \mathbb{R}^m \) under Lipschitz maps, and an \( m \)-current is rectifiable if there exists an \( m \)-rectifiable set \( \Sigma \subset \mathbb{R}^n \), a Borel measurable function \( \tau : \Sigma \to \Lambda_m \mathbb{R}^n \) with \( |\tau| = 1 \) on \( \Sigma \), and a Borel measurable \( \mu : \Sigma \to \mathbb{N} \) such that
\[
\langle T, \phi \rangle = \int_{\Sigma} \langle \tau, \phi \rangle \, d\mathcal{H}^m,
\]
where \( \mathcal{H}^m \) denotes the \( m \)-dimensional Hausdorff measure on \( \mathbb{R}^n \). This representation demonstrates that rectifiable \( m \)-currents generalise the elementary vector calculus notion of integrals over \( m \)-dimensional subsets of \( \mathbb{R}^n \).

An \( m \)-dimensional rectifiable current is an integral current if both \( T \) and \( \partial T \) are rectifiable currents. We will denote the space of integral \( m \)-currents as \( \mathcal{I}_m \), and we exclusively consider current in these classes.

A.4. **Pushforward and pullback.** Given a Lipschitz map \( f : \mathbb{R}^m \to \mathbb{R}^n \), we recall that \( f \) has a Frechet derivative \( Df(x) \in \mathcal{L}(\mathbb{R}^m; \mathbb{R}^n) \) for almost every \( x \in \mathbb{R}^m \), where \( \mathcal{L}(\mathbb{R}^m; \mathbb{R}^n) \) is the space of bounded linear operators mapping \( \mathbb{R}^m \) to \( \mathbb{R}^n \).

Following the definitions in §4.3A of [57] (which in turn follow the definitions given in §4.1.6 and §4.1.7 of [35]), we define the pushforward of a simple \( m \)-vector \( v_1 \wedge \cdots \wedge v_m \in \Lambda_m \mathbb{R}^n \) under a linear map \( A \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n) \) to be
\[
\Lambda_m(A)[v_1 \wedge \cdots \wedge v_m] := A[v_1] \wedge \cdots \wedge A[v_m].
\]
Noting that \( \Lambda_m \mathbb{R}^n \) is spanned by simple \( m \)-vectors, this definition can be extended linearly to apply to general \( m \)-vectors \( \xi \in \Lambda_m \mathbb{R}^n \). In particular, we recall that the key property of the pushforward is that \( \Lambda_m(Df(x)) \) gives the transformation of tangent space of a manifold under the map \( f \).

The pullback of an \( m \)-form \( \phi \in \mathcal{D}^m(\mathbb{R}^n) \) by a Lipschitz map \( f \), denoted \( f^\# \phi \), is defined to be
\[
\langle \xi, f^\# \phi(x) \rangle = \langle \Lambda_m(Df(x)) [\xi], \phi(f(x)) \rangle.
\]
Note that \( \phi \) is evaluated on the target space \( f(\mathbb{R}^m) \subseteq \mathbb{R}^n \).
This definition allows us to define the pushforward of a rectifiable current by duality as
\[
\langle f^\# T, \phi \rangle := \langle T, f^\# \phi \rangle.
\]

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