Canonical matrices of forms and pairs of forms over finite and $p$-adic fields

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Abstract

Canonical matrices of
(a) bilinear and sesquilinear forms,
(b) pairs of forms, in which every form is symmetric or skew-symmetric, and
(c) pairs of Hermitian forms
are given over finite fields of characteristic $\neq 2$ and over $p$-adic fields (i.e., finite extensions of the field $\mathbb{Q}_p$ of $p$-adic numbers) with $p \neq 2$.

These canonical matrices are special cases of the canonical matrices of (a)–(c) over a field of characteristic not 2 that were obtained by the author [Math. USSR–Izv. 31 (1988) 481–501] up to classification of quadratic or Hermitian forms over its finite extensions; we use the known classification of quadratic and Hermitian forms over finite fields and $p$-adic fields.

AMS classification: 15A21.

Keywords: Bilinear forms, Sesquilinear forms, Congruence, Canonical matrices, Finite fields, Local fields, Fields of $p$-adic numbers.

1 Introduction

We give canonical matrices of
(a) bilinear and sesquilinear forms,
(b) pairs of forms in which every form is symmetric or skew-symmetric, and
(c) pairs of Hermitian forms

over

(i) finite fields of characteristic different from 2, and
(ii) \( p \)-adic fields (i.e., finite extensions of the field \( \mathbb{Q}_p \) of \( p \)-adic numbers); for simplicity, we take \( p \neq 2 \).

Our canonical matrices are special cases of the canonical matrices of (a)–(c) over a field \( F \) of characteristic not 2 that were obtained in \([15]\) up to classification of quadratic or Hermitian forms over finite extensions of \( F \). We use the known classification of quadratic and Hermitian forms over finite extensions of (i) and (ii).

Analogous canonical matrices of (a)–(c) could be obtained over any local field (which is either a \( p \)-adic field or the field of formal power series of one variable over a finite field) since the classification of quadratic and Hermitian forms over local fields is known.

In Section 2 we recall canonical forms of (a)–(c) obtained in \([15]\). In Sections 3 and 4 we give canonical forms of (a)–(c) over (i) and (ii).

2 Canonical matrices over any field of characteristic not 2

In this section \( F \) denotes a field of characteristic different from 2 with a fixed involution \( F \to F \); that is, a bijection \( a \mapsto \bar{a} \) satisfying

\[
\overline{a + b} = \bar{a} + \bar{b}, \quad \overline{ab} = \bar{a}\bar{b}, \quad \bar{\bar{a}} = a \quad \text{for all } a, b \in F.
\]

We recall canonical forms of (a)–(c) obtained in \([15]\) by the method that was developed by Roiter and the author in \([11, 14, 15]\); it reduces the problem of classifying systems of forms and linear mappings over \( F \) to the problems of classifying

- systems of linear mappings over \( F \), and
quadratic and Hermitian forms over skew fields that are finite extensions of \( F \).

This method was applied to the problem of classifying bilinear and sesquilinear forms in [6, 7, 8] and to the problem of classifying isometric operators on vector spaces with scalar product given by a nonsingular quadratic or Hermitian form in [16].

For any matrix \( A = [a_{ij}] \) over \( F \), we write \( A^* := \overline{A}^T = [\overline{a}_{ji}] \). Square matrices \( A \) and \( B \) are said to be similar if \( S^{-1}AS = B \), congruent if \( S^TAS = B \), and *congruent if \( S^*AS = B \) for a nonsingular \( S \). Pairs of matrices \( (A_1, A_2) \) and \( (B_1, B_2) \) are congruent if \( S^T A_1 S = B_1 \) and \( S^T A_2 S = B_2 \); they are *congruent if \( S^* A_1 S = B_1 \) and \( S^* A_2 S = B_2 \) for a nonsingular \( S \). The transformations of congruence \( (A \mapsto S^T A S) \) and *congruence \( (A \mapsto S^* A S) \) are associated with the bilinear form \( x^T Ay \) and the sesquilinear form \( x^* Ay \), respectively.

The involution on \( F \) can be the identity. Thus, we consider congruence as a special case of *congruence.

Every square matrix \( A \) over \( F \) is similar to a direct sum, uniquely determined up to permutation of summands, of Frobenius blocks

\[
\Phi = \begin{bmatrix}
0 & 0 & -c_m \\
1 & \ddots & \vdots \\
\vdots & \ddots & 0 \\
0 & 1 & -c_1
\end{bmatrix},
\]

whose characteristic polynomial

\[
\chi_\Phi(x) = p_\Phi(x)^l = x^m + c_1 x^{m-1} + \cdots + c_m
\]

is an integer power of a polynomial \( p_\Phi(x) \) that is irreducible over \( F \); this direct sum is called the Frobenius canonical form or the rational canonical form of \( A \), see [2 Section 6]. If \( \chi_\Phi(x) = (x - \lambda)^m \), then \( \Phi \) is similar to the Jordan block

\[
J_m(\lambda) := \begin{bmatrix}
\lambda & 0 \\
1 & \lambda \\
\ddots & \ddots \\
0 & 1 & \lambda
\end{bmatrix} \quad (m\text{-by-}m).
\]
For each polynomial
\[ f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n \in \mathbb{F}[x], \]
we define the polynomials
\[ \bar{f}(x) := \bar{a}_0 x^n + \bar{a}_1 x^{n-1} + \cdots + \bar{a}_n, \]
\[ f^\vee(x) := \bar{a}_n^{-1}(\bar{a}_n x^n + \cdots + \bar{a}_1 x + \bar{a}_0) \quad \text{if} \quad a_n \neq 0. \]
In particular,
\[ f^\vee(x) = a_n^{-1}(a_n x^n + \cdots + a_1 x + a_0) \quad \text{(5)} \]
if the involution on \( \mathbb{F} \) is the identity.

The following lemma was proved in [15, Lemma 6] (or see [8, 16]).

**Lemma 2.1.** Let \( \mathbb{F} \) be a field with involution \( a \mapsto \bar{a} \), let \( p(x) = p^\vee(x) \) be an irreducible polynomial over \( \mathbb{F} \), and consider the field
\[ \mathbb{F}(\kappa) = \mathbb{F}[x]/p(x)\mathbb{F}[x], \quad \kappa := x + p(x)\mathbb{F}[x], \quad (6) \]
with involution
\[ f(\kappa) \circ := \bar{f}(\kappa^{-1}). \]
Then each element of \( \mathbb{F}(\kappa) \) on which the involution acts identically is uniquely representable in the form \( q(\kappa) \), in which
\[ q(x) = a_r x^r + \cdots + a_1 x + a_0 + \bar{a}_1 x^{-1} + \cdots + \bar{a}_r x^{-r}, \quad a_0 = \bar{a}_0, \quad (8) \]
r is the integer part of \((\deg p(x))/2\), \( a_0, \ldots, a_r \in \mathbb{F} \), and if \( \deg p(x) \) is even then
\[ a_r = \begin{cases} 
0 & \text{if the involution on } \mathbb{F} \text{ is the identity,} \\
\bar{a}_r & \text{if the involution on } \mathbb{F} \text{ is not the identity and } p(0) \neq 1, \\
-\bar{a}_r & \text{if the involution on } \mathbb{F} \text{ is not the identity and } p(0) = 1.
\end{cases} \]
For each square matrix \( \Phi \) and
\[ \varepsilon = \begin{cases} 
1 \text{ or } -1, & \text{if the involution on } \mathbb{F} \text{ is the identity,} \\
1, & \text{if the involution on } \mathbb{F} \text{ is nonidentity,}
\end{cases} \]
denote by $\Phi^\ast\sqrt{\Phi}$ and $\Phi^\ast\varepsilon$ fixed nonsingular matrices (if they exist) such that

\begin{align*}
\Phi^\ast\sqrt{\Phi} &= (\Phi^\ast\sqrt{\Phi})^*, \\
\Phi^\ast\varepsilon &= \Phi^\ast, \quad \Phi^\ast\varepsilon\Phi = \varepsilon(\Phi^\ast\varepsilon)^*.
\end{align*}

We use the notation $\Phi^\ast\sqrt{\Phi}$ both in the case of nonidentity involution and in the case of the identity involution on $F$, but if we know that the involution is the identity then we prefer to write $\sqrt{\Phi}$ instead of $\Phi^\ast\sqrt{\Phi}$.

It suffices to construct $\Phi^\ast\sqrt{\Phi}$ and $\Phi^\ast\varepsilon$ for canonical matrices $\Phi$ under similarity since if $\Psi = S^{-1}\Phi S$ then we can take

\begin{align*}
\Phi^\ast\sqrt{\Psi} &= S^*\Phi^\ast\sqrt{\Phi}S, \\
\Phi^\ast\varepsilon\Psi &= S^*\Phi^\ast\varepsilon\Phi S.
\end{align*}

Existence conditions and explicit forms of $\Phi^\ast\sqrt{\Phi}$ and $\Phi^\ast\varepsilon$ for all Frobenius blocks $\Phi$ will be given in Lemmas 2.6 and 2.7.

Define the skew sum of two matrices

$$[A \setminus B] := \begin{bmatrix} 0 & B \\ A & 0 \end{bmatrix}.$$ 

Theorem 2.2 ([15, Theorem 3]; see also [8, Theorem 2.2]). (a) Let $F$ be a field of characteristic different from 2 with involution (which can be the identity). Every square matrix $A$ over $F$ is *congruent to a direct sum of matrices of the following types:

(i) $J_n(0)$;

(ii) $[\Phi \setminus I_n]$, where $\Phi$ is an $n \times n$ nonsingular Frobenius block such that $\Phi^\ast\sqrt{\Phi}$ does not exist (see Lemma 2.6); and

(iii) $\Phi^\ast\sqrt{\Phi}(\Phi)$, where $\Phi$ is a nonsingular Frobenius block such that $\Phi^\ast\sqrt{\Phi}$ exists and $q(x) \neq 0$ has the form (8) from Lemma 2.1 in which $p(x) = p_\Phi(x)$ is the irreducible divisor of the characteristic polynomial of $\Phi$.

(b) The summands are determined to the following extent:

Type (i) uniquely.

Type (ii) up to replacement of $\Phi$ by the Frobenius block $\Psi$ that is similar to $\Phi^{-\ast}$ (i.e., whose characteristic polynomial is $\chi_\Psi(x)$, see [11]).
Type (iii) up to replacement of the whole group of summands

\[ \sqrt[\ast]{\Phi} q_1(\Phi) \oplus \cdots \oplus \sqrt[\ast]{\Phi} q_s(\Phi) \]

with the same \( \Phi \) by

\[ \sqrt[\ast]{\Phi} q'_1(\Phi) \oplus \cdots \oplus \sqrt[\ast]{\Phi} q'_s(\Phi) \]

in which each \( q'_i(x) \) is a nonzero function of the form (8) and the Hermitian forms

\[ q_1(\kappa)x^\circ_1y_1 + \cdots + q_s(\kappa)x^\circ_sy_s, \]
\[ q'_1(\kappa)x^\circ_1y_1 + \cdots + q'_s(\kappa)x^\circ_sy_s \]

are equivalent over the field (8) with involution (7).

(c) Frobenius blocks in (a) and (b) can be replaced by arbitrary matrices that are similar to them (for example, by Jordan blocks if \( \mathbb{F} \) is algebraically closed).

Define the \((n-1) \times n\) matrices

\[
F_n := \begin{bmatrix}
1 & 0 & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 1
\end{bmatrix}, \quad G_n := \begin{bmatrix}
0 & 1 & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0 & 1
\end{bmatrix}
\]

for each \( n = 1, 2, \ldots \), and define the direct sum of two matrix pairs:

\[ (A_1, B_1) \oplus (A_2, B_2) := (A_1 \oplus A_2, B_1 \oplus B_2). \]

**Theorem 2.3** ([15, Theorem 4]). (a) Let \( \mathbb{F} \) be a field of characteristic different from 2 with involution (which can be the identity). Let \( A \) and \( B \) be \( \epsilon \)-Hermitian and \( \delta \)-Hermitian matrices over \( \mathbb{F} \) of the same size:

\[ A^* = \epsilon A, \quad B^* = \delta B, \]

in which

\[ (\epsilon, \delta) = \begin{cases} 
(1, 1), & \text{if the involution on } \mathbb{F} \text{ is nonidentity,} \\
(1, 1) \text{ or } (1, -1) \text{ or } (-1, -1), & \text{otherwise.}
\end{cases} \]

Then \( (A, B) \) is *congruent to a direct sum of matrix pairs of the following types:
(i) \([F_n \setminus \varepsilon F_n^*], [G_n \setminus \delta G_n^*]\), in which \(F_n\) and \(G_n\) are defined in (11);

(ii) \([I_n \setminus \varepsilon I_n], [\Phi \setminus \delta \Phi^*]\), in which \(\Phi\) is an \(n \times n\) Frobenius block such that \(\Phi_\delta\) (see (10)) does not exist if \(\varepsilon = 1\);

(iii) \(A_{\Phi}^{f(x)} := (\Phi_\delta, \Phi_\delta \Phi) f(\Phi)\) only if \(\varepsilon = 1\), in which \(0 \neq f(x) = f(\delta x) \in \mathbb{F}[x]\) (see (10)) and \(\deg(f(x)) < \deg(p_\Phi(x))\);

(iv) \([J_n(0) \setminus \varepsilon J_n(0)^*], [I_n \setminus (-I_n)]\) only if \(\delta = -1\), in which \(n\) is odd if \(\varepsilon = 1\);

(v) \[
B_n^a := \begin{pmatrix}
0 & \delta & 1 & 0 \\
\delta & 1 & \cdot & \cdot \\
1 & \cdot & \delta & \cdot \\
\cdot & \cdot & \cdot & \delta \\
0 & \cdot & \cdot & 0
\end{pmatrix}, \\
\begin{pmatrix}
0 & \delta \\
\delta & 1 \\
1 & \cdot \\
\cdot & \cdot \\
0 & \cdot
\end{pmatrix}, \\
\begin{pmatrix}
0 & \delta \\
\delta & 1 \\
\cdot & \cdot \\
\cdot & \cdot \\
0 & \cdot
\end{pmatrix}, \\
\begin{pmatrix}
0 & \delta \\
\delta & 1 \\
\cdot & \cdot \\
\cdot & \cdot \\
0 & \cdot
\end{pmatrix}
\]

in which the matrices are \(n\)-by-\(n\), \(\varepsilon = 1\), \(0 \neq a = \bar{a} \in \mathbb{F}\), and \(n\) is even if \(\delta = -1\).

(b) The summands are determined to the following extent:

**Type (i)** uniquely.

**Type (ii)** up to replacement of \(\Phi\) by the Frobenius block \(\Psi\) with \(\chi_\Psi(x) = (\varepsilon \delta)^{\deg x \Phi^*} \chi_\Phi(\varepsilon \delta x)\).

**Type (iii)** up to replacement of the whole group of summands

\[
A_{\Phi}^{f_1(x)} \oplus \cdots \oplus A_{\Phi}^{f_s(x)}
\]

with the same \(\Phi\) by

\[
A_{\Phi}^{g_1(x)} \oplus \cdots \oplus A_{\Phi}^{g_s(x)}
\]

such that the Hermitian forms

\[
f_1(\omega)x_1^\omega y_1 + \cdots + f_s(\omega)x_s^\omega y_s,
\]

\[
g_1(\omega)x_1^\omega y_1 + \cdots + g_s(\omega)x_s^\omega y_s
\]

are equivalent over the field \(\mathbb{F}(\omega) = \mathbb{F}[x]/p_\Phi(x)\mathbb{F}[x]\) with involution \(f(\omega)^\circ = f(\delta \omega)\).
Type (iv) uniquely.

Type (v) up to replacement of the whole group of summands

\[ B_{n}^{a_{1}} \oplus \cdots \oplus B_{n}^{a_{s}} \]

with the same \( n \) by

\[ B_{n}^{b_{1}} \oplus \cdots \oplus B_{n}^{b_{s}} \]

such that the Hermitian forms

\[ a_{1}\bar{x}_{1}y_{1} + \cdots + a_{s}\bar{x}_{s}y_{s}, \]

\[ b_{1}\bar{x}_{1}y_{1} + \cdots + b_{s}\bar{x}_{s}y_{s} \]

are equivalent over \( \mathbb{F} \).

(c) Frobenius blocks in (a) and (b) can be replaced by arbitrary matrices that are similar to them (for example, by Jordan blocks if \( \mathbb{F} \) is algebraically closed).

Taking \( \varepsilon = \delta = -1 \) in Theorem 2.3, we obtain the following well-known canonical form of pairs skew-symmetric matrices; see, for example, [12, 17].

**Corollary 2.4.** Over any field of characteristic not 2, each pair of skew-symmetric matrices of the same size is congruent to a direct sum, uniquely determined up to permutation of summands, of pairs of the form:

(i) \( ([F_{n} \setminus -F_{n}^{T}], [G_{n} \setminus -G_{n}^{T}]) \), in which \( F_{n} \) and \( G_{n} \) are defined in (11);

(ii) \( ([I_{n} \setminus -I_{n}], [\Phi \setminus -\Phi^{T}]) \), in which \( \Phi \) is an \( n \times n \) Frobenius block;

(iii) \( ([J_{n}(0) \setminus -J_{n}(0)^{T}], [I_{n} \setminus -I_{n}]) \).

**Remark 2.5.** If \( \delta = -1 \) then the matrix pair \( B_{n}^{a} \) defined in (12) consists of \( n \times n \) matrices and \( n \) is even. In this case, the pair

\[ C_{n}^{a} := \begin{pmatrix} \alpha \begin{bmatrix} 0 & 1 & 0 \\ 1 & \ddots & \vdots \\ 0 & \vdots & 0 \end{bmatrix}, \alpha \begin{bmatrix} 0 & \cdots & 1 \\ -1 & \ddots & \vdots \\ \vdots & \ddots & 0 \end{bmatrix} \end{pmatrix} \]

(15)
of symmetric and skew-symmetric matrices of size \( n \times n \) can be used in (12)–(14) instead of \( B_n^a \). This follows from the proof of Theorem 4 in [15] since the pairs \( B_n^a \) and \( C_n^a \) are equivalent; that is, \( RB_n^a S = C_n^a \) for some nonsingular \( R \) and \( S \).

Let
\[
f(x) = \gamma_0 x^m + \gamma_1 x^{m-1} + \cdots + \gamma_m \in \mathbb{F}[x], \quad m \geq 1, \quad \gamma_0 \neq 0 \neq \gamma_m.
\]
A vector \((a_1, a_2, \ldots, a_n)\) over \( \mathbb{F} \) is called \( f \)-recurrent if either \( n \leq m \), or
\[
\gamma_0 a_l + \gamma_1 a_{l+1} + \cdots + \gamma_m a_{l+m} = 0 \quad \text{for all} \quad l = 1, 2, \ldots, n - m.
\]
Thus, this vector is completely determined by any fragment of length \( m \).

Existence conditions and explicit forms of \( \sqrt[\Phi]{\Phi} \) and \( \Phi_{\varepsilon} \) for Frobenius blocks \( \Phi \) are given in the following two lemmas.

**Lemma 2.6** ([15, Theorem 7]; a detailed proof in [8, Lemma 2.3]). Let \( \mathbb{F} \) be a field of characteristic not 2 with involution (possibly, the identity). Let \( \Phi \) be an \( n \times n \) nonsingular Frobenius block whose characteristic polynomial is a power of an irreducible polynomial \( p_\Phi(x) \).

(a) \( \sqrt[\Phi]{\Phi} \) exists if and only if
\[
p_\Phi(x) = p_{\Phi}(x) \quad \text{(see (14)), and}
\]
if the involution on \( \mathbb{F} \) is the identity, also \( p_\Phi(x) \neq x + (-1)^{n+1} \). \hspace{1cm} \text{(17)}

(b) If \((16)\) and \((17)\) are satisfied and\[
\chi_\Phi(x) = x^n + c_1 x^{n-1} + \cdots + c_n
\]
is the characteristic polynomial of \( \Phi \), then for \( \sqrt[\Phi]{\Phi} \) one can take the Toeplitz matrix\[
\sqrt[\Phi]{\Phi} := [a_{i-j}] = \begin{bmatrix}
a_0 & a_{-1} & \cdots & a_{1-n} 
a_1 & a_0 & \cdots & \cdots 
\cdots & \cdots & \cdots & a_{-1} 
a_{n-1} & \cdots & a_1 & a_0
\end{bmatrix}, \quad \text{(19)}
\]
whose vector of entries \((a_{1-n}, a_{2-n}, \ldots, a_{n-1})\) is the \(\chi_\Phi\)-recurrent extension of the vector

\[
v = (a_{1-m}, \ldots, a_m) = (a, 0, \ldots, 0, \bar{a})
\]

of length

\[
2m = \begin{cases} 
n & \text{if } n \text{ is even}, \\
n + 1 & \text{if } n \text{ is odd}, 
\end{cases}
\]

in which

\[
a := \begin{cases} 
1 & \text{if } n \text{ is even, except for the case } \quad p_\Phi(x) = x + c \text{ with } c^{n-1} = -1, \\
\chi_\Phi(-1) & \text{if } n \text{ is odd and } p_\Phi(x) \neq x + 1, \\
e - \bar{e} & \text{otherwise, with any fixed } \bar{e} \neq e \in \mathbb{F}.
\end{cases}
\]

Lemma 2.7 ([15, Theorem 8]). Let \(\mathbb{F}\) be a field of characteristic not 2 with involution (possibly, the identity). Let \(\Phi\) be an \(n \times n\) Frobenius block \((1)\) over \(\mathbb{F}\). Existence conditions for the matrix \(\Phi_\varepsilon\) are:

\[
p_\Phi(x) = \varepsilon^n \bar{p}_\Phi(\varepsilon x) \text{ (see (3))},
\]

if \(\varepsilon = -1\) then also \(\chi_\Phi(x) \notin \{x^2, x^4, x^6, \ldots\}\).

With these conditions satisfied, one can take

\[
\Phi_\varepsilon = [\varepsilon^i a_{i+j}],
\]

in which the sequence \((a_2, a_3, \ldots, a_{2n})\) is \(\chi\)-recurrent, and is defined by the fragment

\[
(a_2, \ldots, a_{n+1}) = \begin{cases} 
(1, 0, \ldots, 0) & \text{if } \Phi \text{ is nonsingular}, \\
(0, \ldots, 0, 1) & \text{if } \Phi \text{ is singular}.
\end{cases}
\]

3 Canonical forms over finite fields

In this section we give canonical matrices of bilinear and sesquilinear forms, pairs of symmetric or skew-symmetric forms, and pairs of Hermitian forms over a finite field \(\mathbb{F}\) of characteristic not 2. We use Theorems 2.2 and 2.3, in which these canonical matrices are given up to classification of quadratic and Hermitian forms over finite extensions of \(\mathbb{F}\) (that is, over finite fields of characteristic not 2), and the following lemma.
Lemma 3.1 ([3] Chap. 1, § 8]). (a) Each quadratic form of rank \( r \) over a finite field \( \mathbb{F} \) of characteristic not 2 is equivalent to

\[
either x_1^2 + x_2^2 + \cdots + x_r^2, \text{ or } \zeta x_1^2 + x_2^2 + \cdots + x_r^2,
\]

where \( \zeta \) is a fixed nonsquare in \( \mathbb{F} \).

(b) Each Hermitian form of rank \( r \) over a finite field of characteristic not 2 with nonidentity involution is equivalent to \( \bar{x}_1y_1 + \cdots + \bar{x}_ry_r \).

Utv. (b) eshe iz Scharlau ch 10, 1.6, examples (i).

3.1 Canonical matrices for congruence and *congruence

Define the \( n \)-by-\( n \) matrix

\[
\Gamma_n = \begin{pmatrix}
0 & & & \\
& -1 & & \\
& & 1 & \\
& & & 1
\end{pmatrix} \quad (\Gamma_1 = [1]). \quad (26)
\]

Theorem 3.2. Every square matrix over a finite field \( \mathbb{F} \) of characteristic different from 2 is congruent to a direct sum that is uniquely determined up to permutation of summands and consists of any number of summands of the following types:

(i) \( J_n(0) \);

(ii) \( [\Phi \setminus I_n] \), in which \( \Phi \) is an \( n \times n \) nonsingular Frobenius block such that

\[
p_\Phi(x) \neq p_\Phi^\vee(x) \text{ (see (5))} \quad \text{or} \quad p_\Phi(x) = x + (-1)^{n+1},
\]

and \( \Phi \) is determined up to replacement by the Frobenius block \( \Psi \) with \( \chi_\Psi(x) = \chi_\Phi^\vee(x) \);

(iii) \( \sqrt[\vee]{\Phi} \), in which \( \Phi \) is a nonsingular Frobenius block such that \( p_\Phi(x) = p_\Phi^\vee(x) \) and \( \deg p_\Phi(x) \geq 2 \);

(iv) for each \( n = 1, 2, \ldots \):
• \( \Gamma_n \),
• at most one summand \( \zeta \Gamma_n \), in which \( \zeta \) is a fixed nonsquare of \( \mathbb{F} \).

**Proof.** Let \( \mathbb{F} \) be a finite field of characteristic not 2 with the identity involution. By Theorem 2.2(a), every square matrix \( A \) over \( \mathbb{F} \) is congruent to a direct sum of matrices of the form

\[
(a) \ J_n(0), \hspace{1cm} (b) \ [\Phi \setminus I_n] \text{ if } \sqrt[n]{\Phi} \text{ does not exist}, \hspace{1cm} (c) \ \sqrt[n]{\Phi q(\Phi)}.
\]

Consider each of these summands.

**Summands (a).** Theorem 2.2(b) ensures that the summands of the form \( J_n(0) \) are uniquely determined by \( A \), which gives the summands (i) of the theorem.

**Summands (b).** By Lemma 2.6(a), \( \sqrt[n]{\Phi} \) does not exist if and only if (27) holds. Theorem 2.2(b) ensures that the summands of the form \([\Phi \setminus I_n]\) are uniquely determined by \( A \), up to replacement of \( \Phi \) by \( \Psi \) with \( \chi(\Psi(x)) = \chi(\sqrt[n]{\Phi}(x)) \). This gives the summands (ii).

**Summands (c).** Let \( \Phi \) be a nonsingular \( n \times n \) Frobenius block for which \( \sqrt[n]{\Phi} \) exists. Then by Lemma 2.6(a)

\[ p(\Phi(x)) = p(\sqrt[n]{\Phi}(x)), \hspace{1cm} p(\Phi(x)) \neq x + (-1)^{n+1}. \]  

Consider the whole group of summands of the form \( \sqrt[n]{\Phi q(\Phi)} \) with the same \( \Phi \):

\[ \sqrt[n]{\Phi q_1(\Phi)} \oplus \ldots \oplus \sqrt[n]{\Phi q_s(\Phi)}. \]  

Let first \( \text{deg } p(\Phi(x)) > 1 \). Then the involution \( f(\kappa)^c := f(\kappa^{-1}) \) on the field \( \mathbb{F}(\kappa) = \mathbb{F}[x]/p(x)\mathbb{F}[x] \) (see (6) and (7)) is nonidentity; otherwise \( \kappa = \kappa^c = \kappa^{-1}, \kappa^2 - 1 = 0, (x^2 - 1)p(\Phi(x)) \), and hence \( p(x) = x \pm 1 \) since it is irreducible.

By Lemma 3.1(b), the Hermitian form

\[ q_1(\kappa)x_1^c y_1 + \ldots + q_s(\kappa)x_s^c y_s \]

over \( F(\kappa) \) is equivalent to \( x_1^c y_1 + \ldots + x_s^c y_s \). By Theorem 2.2(b), the matrix \( (29) \) is congruent to \( \sqrt[n]{\Phi} \oplus \ldots \oplus \sqrt[n]{\Phi} \) and the summands of the form \( \sqrt[n]{\Phi} \) with \( \text{deg } p(\Phi(x)) > 1 \) are uniquely determined by \( A \). This gives the summands (iii).

Let now \( p(\Phi(x)) = x + c \). Then by (28) and (4) \( x + c = c^{-1}(cx + 1), c = c^{-1}, c = \pm 1 \). The inequality in (28) implies

\[ p(x) = x + (-1)^n. \]  

12
By Eq. (70),

\[
\Gamma_n^{-T} \Gamma_n = \Upsilon_n := (-1)^{n+1} \begin{bmatrix}
1 & 2 & * \\
1 & \ddots & \\
\vdots & & 2 \\
0 & & 1
\end{bmatrix}.
\]

(31)

Hence \(\Gamma_n = \sqrt[\Upsilon_n]\) and \(\Upsilon_n\) is similar to \(J_n((-1)^{n+1})\), which is similar to \(\Phi\) due to (30). By Theorem 2.2(c) we can take \(\Upsilon_n\) instead of \(\Phi\) with \(p_\Phi(x) = x + (-1)^n\) in Theorem 2.2(a,b). The field \(F(\kappa) = F[x]/p_\Phi(x)F[x]\) is \(F\) with the identity involution; all polynomials \(q_i(x)\) in (29) are some scalars \(a_i \in F\).

By Lemma 3.1(a), the quadratic form

\[
q_1(\kappa)x_1^2 + \cdots + q_s(\kappa)x_s^2 = a_1x_1^2 + \cdots + a_sx_s^2
\]

over \(F\) is equivalent to

either \(x_1^2 + \cdots + x_r^2\),

or \(\zeta x_1^2 + x_2^2 + \cdots + x_r^2\),

in which \(\zeta\) is a fixed nonsquare of \(F\). Theorem 2.2(b) ensures that (29) is congruent to

either \(\Gamma_n \oplus \cdots \oplus \Gamma_n\),

or \(\zeta \Gamma_n \oplus \Gamma_n \oplus \cdots \oplus \Gamma_n\),

and this sum is uniquely determined by \(A\). This gives the summands (iv).

Note that if \(\sqrt[\Upsilon_n]\) exists, then \(\deg p_\Phi(x)\) is even or \(p_\Phi(x) = x + (-1)^n\).

Indeed, \(p_\Phi(x) = p_\Upsilon(\Phi)\) by (16). Let \(p_\Phi(x) = x^n + c_1 x^{n-1} + \cdots + c_n\). Then

\[
x^n + c_1 x^{n-1} + \cdots + c_n = c_n^{-1}(c_n x^n + \cdots + c_1 x + 1),
\]

and so \(p_\Phi(-\theta) = 0\). Since \(p_\Phi(x)\) is irreducible, \(p_\Phi(x) = x + \theta\). By the inequality (17), \(p_\Phi(x) = x + (-1)^n\).

JaJa Example: \(F_5/(x^2 + x + 1)\).

**Theorem 3.3.** Let \(F\) be a finite field of characteristic not 2 with nonidentity involution. Every square matrix over \(F\) is *congruent to a direct sum, uniquely determined up to permutation of summands, of matrices of the following types:
(i) \( J_n(0) \);

(ii) \([\Phi \setminus I_n] \), in which \( \Phi \) is an \( n \times n \) nonsingular Frobenius block such that \( p_\Phi(x) \neq p_\Phi^\chi(x) \) (see (4)) and \( \Phi \) is determined up to replacement by the Frobenius block \( \Psi \) with \( \chi_\Psi(x) = \chi_\Phi(x) \);

(iii) \( \sqrt[\Phi]{\Phi} \), in which \( \Phi \) is a nonsingular Frobenius block such that \( p_\Phi(x) = p_\Phi^\chi(x) \).

**Proof.** Let \( \mathbb{F} \) be a finite field of characteristic not 2 with nonidentity involution. By Theorem 2.2(a), every square matrix \( A \) over \( \mathbb{F} \) is *congruent to a direct sum of matrices of the form

\[
\begin{align*}
(a) & \quad J_n(0), \\
(b) & \quad [\Phi \setminus I_n] \text{ if } \sqrt[\Phi]{\Phi} \text{ does not exist,} \\
(c) & \quad \sqrt[\Phi]{q(\Phi)}.
\end{align*}
\]

Consider each of these summands.

**Summands** (a). Theorem 2.2(b) ensures that the summands of the form \( J_n(0) \) are uniquely determined by \( A \), which gives the summands (i) of the theorem.

**Summands** (b). By Lemma 2.6(a), \( \sqrt[\Phi]{\Phi} \) does not exist if and only if \( p_\Phi(x) \neq p_\Phi^\chi(x) \). Theorem 2.2(b) ensures that the summands of the form \([\Phi \setminus I_n] \) are uniquely determined by \( A \), up to replacement of \( \Phi \) by \( \Psi \) with \( \chi_\Psi(x) = \chi_\Phi(x) \). This gives the summands (ii).

**Summands** (c). Let \( \Phi \) be a nonsingular \( n \times n \) Frobenius block for which \( \sqrt[\Phi]{\Phi} \) exists; this means that \( p_\Phi(x) = p_\Phi^\chi(x) \). Consider the whole group of summands of the form \( \sqrt[\Phi]{q(\Phi)} \) with the same \( \Phi \):

\[
\sqrt[\Phi]{q_1(\Phi)} \oplus \cdots \oplus \sqrt[\Phi]{q_s(\Phi)}. \tag{32}
\]

The involution \( f(\kappa)^\circ := f(\kappa^{-1}) \) on the field \( \mathbb{F}(\kappa) = \mathbb{F}[x]/p_\Phi(x)\mathbb{F}[x] \) (see (3) and (7)) is nonidentity since it extends the nonidentity involution on \( \mathbb{F} \). By Lemma 3.1(b), the Hermitian form

\[
q_1(\kappa)x_1^\circ y_1 + \cdots + q_s(\kappa)x_s^\circ y_s
\]

over \( F(\kappa) \) is equivalent to \( x_1^\circ y_1 + \cdots + x_s^\circ y_s \), and so by Theorem 2.2(b) the matrix (32) is *congruent to \( \sqrt[\Phi]{\Phi} \oplus \cdots \oplus \sqrt[\Phi]{\Phi} \) and this direct sum is uniquely determined by \( A \). This gives the summands (iii). \( \square \)
3.2 Canonical pairs of symmetric or skew-symmetric matrices

In this section, we give canonical matrices of pairs consisting of symmetric or skew-symmetric forms. Canonical matrices of pairs of skew-symmetric forms are given in Corollary 2.4; it remains to consider pairs, in which the first form is symmetric and the second is symmetric or skew-symmetric.

For square matrices \( A, B, C, D \) of the same size, we write
\[
(A, B) \oplus (C, D) = (A \oplus C, B \oplus D), \quad (A, B)C = (AC, BC).
\]

**Theorem 3.4.** Each pair of symmetric matrices of the same size over a finite field \( \mathbb{F} \) of characteristic not 2 is congruent to a direct sum that is uniquely determined up to permutation of summands and consists of any number of summands of the following types:

(i) \( ([F_n \setminus F_n^T], [G_n \setminus G_n^T]) \), where \( F_n \) and \( G_n \) are defined in (11);

(ii) for each nonsingular Frobenius block \( \Phi \):

- \( (\Phi_1, \Phi_1 \Phi) \), in which \( \Phi_1 \) is defined in (10),
- at most one summand \( (\Phi_1, \Phi_1 \Phi)f_\Phi(\Phi) \), in which \( f_\Phi(x) \in \mathbb{F}[x] \) is a fixed (for each \( \Phi \)) polynomial of degree \( \deg(p_\Phi(x)) \) such that
  \[
f_\Phi(\omega) \in \mathbb{F}(\omega) := \mathbb{F}[x]/p_\Phi(x)\mathbb{F}[x]
\]
is not a square;

(iii) for each \( n = 1, 2, \ldots \):

- the pair of \( n \times n \) matrices
  \[
  B_n := \begin{pmatrix}
  0 & 1 & 0 & \cdots & 0 \\
  1 & 0 & \cdots & \ddots & \vdots \\
  \vdots & \ddots & \ddots & \ddots & 0 \\
  0 & \cdots & \cdots & 1 & 0 \\
  \end{pmatrix},
  \begin{pmatrix}
  0 & 1 \\
  1 & 0 \\
  \end{pmatrix},
  \begin{pmatrix}
  1 & 0 \\
  \end{pmatrix},
  \begin{pmatrix}
  1 & 0 \\
  \end{pmatrix},
  \begin{pmatrix}
  1 & 0 \\
  \end{pmatrix}
  \end{pmatrix}, \quad (33)
  \]

- at most one summand \( \zeta B_n \), in which \( \zeta \) is a fixed nonsquare of \( \mathbb{F} \).
Proof. Let $\mathbb{F}$ be a finite field of characteristic not 2 with the identity involution. By Lemma 2.7, the matrix $\Phi_1$ exists for each nonsingular Frobenius block $\Phi$ over $\mathbb{F}$. By Theorem 2.3(a), each pair $(A, B)$ of symmetric matrices of the same size is congruent to a direct sum of pairs of the form

(a) $([F_n \setminus F_n^T], [G_n \setminus G_n^T])$,  
(b) $A_f^{(x)} := (\Phi_1, \Phi_1 \Phi)f(\Phi)$,  
(c) $B_n^a$,

in which $f(x) \in \mathbb{F}[x]$ is a nonzero polynomial of degree $< \deg(p_{\Phi}(x))$ and $0 \neq a \in \mathbb{F}$.

Consider each of these summands.

**Summands (a).** Theorem 2.3(b) ensures that the summands of the form (a) are uniquely determined by $(A, B)$, which gives the summands (i) of the theorem.

**Summands (b).** Consider the whole group of summands of the form $A_{\Phi}^{g(x)}$ with the same nonsingular Frobenius block $\Phi$:

$$A_{\Phi}^{g_1(x)} \oplus \cdots \oplus A_{\Phi}^{g_s(x)}.$$  

By Lemma 3.1(a), the quadratic form

$$q_1(\omega)x_1^2 + \cdots + q_s(\omega)x_2^2$$

over $\mathbb{F}(\omega) = \mathbb{F}[x]/p_{\Phi}(x)\mathbb{F}[x]$ is equivalent to

either $x_1^2 + \cdots + x_r^2$,  
or $f_{\Phi}(\omega)x_1^2 + x_2^2 + \cdots + x_r^2$,

in which $f_{\Phi}(\omega) \in \mathbb{F}[x]$ is a fixed nonzero polynomial of degree $< \deg(p_{\Phi}(x))$ such that $f_{\Phi}(\omega) \in \mathbb{F}(\omega)$ is not a square. Theorem 2.2(b) ensures that (34) is congruent to

either $A_\Phi^1 \oplus \cdots \oplus A_\Phi^1$,  
or $A_{\Phi}^{f_\Phi(x)} \oplus A_\Phi^1 \oplus \cdots \oplus A_\Phi^1$,

and this sum is uniquely determined by $(A, B)$. This gives the summands (ii).

**Summands (c).** Consider the whole group of summands of the form $B_n^a$ with the same $n$:

$$B_n^{a_1} \oplus \cdots \oplus B_n^{a_s}.$$  

By Lemma 3.1(a), the quadratic form $a_1x_1^2 + \cdots + a_sx_s^2$ over $\mathbb{F}$ is equivalent to either $x_1^2 + \cdots + x_r^2$, or $\zeta x_1^2 + x_2^2 + \cdots + x_r^2$, in which $\zeta$ is a fixed nonsquare of $\mathbb{F}$. Theorem 2.2(b) ensures that (35) is congruent to

either $B_n^1 \oplus \cdots \oplus B_n^1$,  
or $B_n^\zeta \oplus B_n^1 \oplus \cdots \oplus B_n^1$.  

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and this sum is uniquely determined by \((A, B)\). This gives the summands (iii).

**Theorem 3.5.** Each pair consisting of a symmetric matrix and a skew-symmetric matrix of the same size over a finite field \(\mathbb{F}\) of characteristic not 2 is congruent to a direct sum that is uniquely determined up to permutation of summands and consists of any number of summands of the following types:

(i) \(([F_n \setminus F_n^T], [G_n \setminus -G_n^T])\), in which \(F_n\) and \(G_n\) are defined in (11);

(ii) \(([I_n \setminus I_n], [\Phi \setminus -\Phi^T])\), in which \(\Phi\) is an \(n \times n\) Frobenius block such that

\[
p_\Phi(x) \not\in \mathbb{F}[x^2], \quad \Phi \neq J_1(0), J_3(0), J_5(0), \ldots \tag{36}
\]

(see (2)), and \(\Phi\) is determined up to replacement by the Frobenius block \(\Psi\) with \(\chi_\Psi(x) = (-1)^{\det x} \chi_\Phi(-x)\);

(iii) \((\Phi_-, \Phi_- \Phi)\), in which \(\Phi\) is a Frobenius block such that \(p_\Phi(x) \in \mathbb{F}[x^2]\);

(iv) \(([J_n(0) \setminus J_n(0)^T], [I_n \setminus -I_n])\), in which \(n\) is odd;

(v) for each \(n = 1, 2, 3, \ldots\):

- the pair of \(n\)-by-\(n\) symmetric and skew-symmetric matrices defined as follows:

\[
C_n := \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots \\ 1 & \cdots & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots \\ -1 & 0 & \cdots & 0 \\ 0 & \cdots & -1 & 0 \end{pmatrix} \tag{37}
\]

if \(n\) is odd, and

\[
C_n := \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots \\ 1 & \cdots & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots \\ -1 & 0 & \cdots & 0 \\ 0 & \cdots & -1 & 0 \end{pmatrix} \tag{38}
\]

if \(n\) is even,
• at most one summand of the form \( \zeta C_n \), in which \( \zeta \) is a fixed non-square of \( \mathbb{F} \).

Proof. Let \( \mathbb{F} \) be a finite field of characteristic not 2 with the identity involution. By Theorem 2.3(a) and Remark 2.5, each pair \((A, B)\) consisting of a symmetric matrix \( A \) and a skew-symmetric matrix \( B \) of the same size is congruent to a direct sum of pairs of the form

\[
\begin{align*}
\text{(a)} & \quad ([F_n \setminus F_n^T], [G_n \setminus -G_n^T]), \\
\text{(b)} & \quad ([I_n \setminus I_n], [\Phi \setminus -\Phi^T]) \text{ if } \Phi^{-1} \text{ does not exist,} \\
\text{(c)} & \quad A_{\Phi}^{f(x)} := (\Phi_{-1}, \Phi_{-1} \Phi) f(\Phi), \text{ in which } 0 \neq f(x) = f(-x) \in \mathbb{F}[x] \text{ and } \deg(f(x)) < \deg(p_{\Phi}(x)), \\
\text{(d)} & \quad ([J_n(0) \setminus J_n(0)^T], [I_n \setminus -I_n]), \text{ in which } n \text{ is odd,} \\
\text{(e)} & \quad C_n^a \text{ (defined in (15))}, \text{ in which } n \text{ is even and } 0 \neq a \in \mathbb{F}.
\end{align*}
\]

Consider each of these summands.

**Summands (a).** Theorem 2.3(b) ensures that the summands (a) are uniquely determined by \((A, B)\), which gives the summands (i) of the theorem.

**Summands (b).** By Lemma 2.7, \( \Phi_{-1} \) does not exist if and only if (36) is satisfied. Theorem 2.3(b) ensures that the summands (b) are uniquely determined by \((A, B)\), up to replacement of \( \Phi \) by \( \Psi \) with \( \chi_{\Psi}(x) = (-1)^{\det_{\chi_{\Phi}} \chi_{\Phi}(-x)} \), which gives the summands (ii).

**Summands (c).** By Lemma 2.7, \( \Phi_{-1} \) exists if and only if (36) is not satisfied; that is,

\[ p_{\Phi}(x) \in \mathbb{F}[x^2] \quad \text{or} \quad \Phi = J_1(0), J_3(0), J_5(0), \ldots \quad (39) \]

Consider the whole group of summands of the form \( A_{\Phi}^{f(x)} \) with the same nonsingular Frobenius block \( \Phi \):

\[ A_{\Phi}^{f_1(x)} \oplus \cdots \oplus A_{\Phi}^{f_s(x)}. \quad (40) \]

Let first \( p_{\Phi}(x) \in \mathbb{F}[x^2] \). Then the involution \( f(\omega) = f(-\omega) \) on \( \mathbb{F}(\omega) = \mathbb{F}[x]/p_{\Phi}(x) \mathbb{F}[x] \) is nonidentity (since \( \omega^2 = -\omega \neq \omega \)). By Lemma 3.1(b), the Hermitian form

\[ f_1(\omega)x_1^2y_1 + \cdots + f_s(\omega)x_s^2y_s \]
over $\mathbb{F}(\omega)$ is equivalent to $x_1^2y_1 + \cdots + x_r^2y_r$. Theorem 2.2(b) ensures that (40) is congruent to $A^1_\Phi \oplus \cdots \oplus A^1_\Phi$, which gives the summands (iii).

Let now $\Phi = J_n(0)$ with $n = 2m + 1$ and $m = 1, 2, \ldots$. The equalities (10) hold for the $n \times n$ matrices

$$
\Phi' := \begin{bmatrix}
0 & 0 & \cdots & 0 \\
-1 & 0 & 0 & \cdots \\
\cdots & -1 & 0 & 0 \\
0 & 0 & \cdots & 1 \\
0 & 1 & 0 & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
0 & 1 & 0 & 1
\end{bmatrix}, \quad \Phi'_{-1} := \begin{bmatrix}
0 & \cdots & 1 \\
1 & \cdots & 0
\end{bmatrix}
$$

(41)

instead of $\Phi$ and $\Phi_{-1}$. Since $\Phi$ and $\Phi'$ are similar, by Theorem 2.3(c) we can take $C^f_n(x) := (\Phi'_{-1}, \Phi'_{-1}\Phi')f(\Phi')$ instead of $A^f_n(x)$ and

$$
C^f_n(x) \oplus \cdots \oplus C^f_s(x)
$$

(42)

instead of (40).

Since $p_\Phi(x) = x$, the field $\mathbb{F}(\omega) = \mathbb{F}[x]/p_\Phi(x)\mathbb{F}[x]$ is $\mathbb{F}$ with the identity involution and all polynomials $f_i(x)$ in (42) are some scalars $a_i \in \mathbb{F}$. By Lemma 3.1(a), the quadratic form $a_1x_1^2 + \cdots + a_sx_s^2$ over $\mathbb{F}$ is equivalent to either $x_1^2 + \cdots + x_r^2$, or $\zeta x_1^2 + x_2^2 + \cdots + x_r^2$, in which $\zeta$ is a fixed nonsquare of $\mathbb{F}$. Theorem 2.2(b) ensures that (40) is congruent to

- either $C^1_n \oplus \cdots \oplus C^1_n$, or $C^\zeta_n \oplus C^1_n \oplus \cdots \oplus C^1_n$, (43)

and this sum is uniquely determined by $(A, B)$. This gives the summands (v) with odd $n$.

**Summands** (d). Theorem 2.3(b) ensures that the summands of the form (d) are uniquely determined by $(A, B)$, which gives the summands (iv).

**Summands** (e). Consider the whole group of summands of the form $C^a_n$ with the same $n$:

$$
C^{a_1}_n \oplus \cdots \oplus C^{a_s}_n.
$$

(44)

By Lemma 3.1(a), the quadratic form $a_1x_1^2 + \cdots + a_sx_s^2$ over $\mathbb{F}$ is equivalent to either $x_1^2 + \cdots + x_r^2$, or $\zeta x_1^2 + x_2^2 + \cdots + x_r^2$, in which $\zeta$ is a fixed nonsquare of $\mathbb{F}$. Theorem 2.2(b) and Remark 2.5 ensure that (44) is congruent to

- either $C^1_n \oplus \cdots \oplus C^1_n$, or $C^\zeta_n \oplus C^1_n \oplus \cdots \oplus C^1_n$,
and this sum is uniquely determined by \((A, B)\), which gives the summands (v) with even \(n\).

\[ \square \]

### 3.3 Canonical pairs of Hermitian matrices

**Theorem 3.6.** Let \( F \) be a finite field of characteristic not 2 with nonidentity involution. Let \( F_0 \) be the fixed field of \( F \). Each pair of Hermitian matrices of the same size over \( F \) is *congruent to a direct sum, uniquely determined up to permutation of summands, of pairs of the following types:

(i) \([F_n \backslash F_n^*, [G_n \backslash G_n^*]]\), in which \( F_n \) and \( G_n \) are defined in (11);

(ii) \([I_n \backslash I_n, [\Phi \backslash \Phi^*]]\), in which \( \Phi \) is an \( n \times n \) Frobenius block over \( F \) such that \( p_\Phi(x) \notin F_0[x] \), and \( \Phi \) is determined up to replacement by the Frobenius block \( \Psi \) with \( \chi_\Psi(x) = \overline{\chi_\Phi(x)} \) (see (3));

(iii) \((\Phi_1, \Phi_1 \Phi)\), in which \( \Phi \) is a Frobenius block over \( F_0 \);

(iv) the pair of \( n \times n \) matrices

\[
B_n := \begin{pmatrix}
0 & \cdots & 1 & 0 \\
1 & \cdots & 0 & 0 \\
0 & 1 & 0 & 0 \\
\end{pmatrix}, \quad n = 1, 2, \ldots
\]

**Proof.** Let \( F \) be a finite field of characteristic not 2 with nonidentity involution. By Theorem 2.3(a), each pair \((A, B)\) of Hermitian matrices over \( F \) of the same size is *congruent to a direct sum of pairs of the form

(a) \([F_n \backslash F_n^*, [G_n \backslash G_n^*]]\),

(b) \([I_n \backslash I_n, [\Phi \backslash \Phi^*]]\) if \( \Phi_1 \) does not exist,

(c) \(A_\Phi^f(x) := (\Phi_\delta, \Phi_1 \Phi)f(\Phi)\), in which \( 0 \neq f(x) = \bar{f}(x) \in F[x] \) and \( \deg(f(x)) < \deg(p_\Phi(x)) \),

(d) \(B_a^n\) (defined in (12)), in which \( 0 \neq a = \bar{a} \in F \).

Consider each of these summands.

**Summands (a).** Theorem 2.3(b) ensures that the summands of the form (a) are uniquely determined by \((A, B)\), which gives the summands (i) of the theorem.
Summands (b). By Lemma 2.7(a), $\Phi_1$ does not exist if and only if $p_{\Phi}(x) \not\equiv \bar{p}_{\Phi}(x)$; that is, $p_{\Phi}(x) \not\in \mathbb{F}_0[x]$. Theorem 2.3(b) ensures that the summands of the form (b) are uniquely determined, up to replacement of $\Phi$ by $\Psi$ with $\chi_{\Psi}(x) = \bar{\chi}_{\Phi}(x)$. This gives the summands (ii).

Summands (c). Consider the whole group of summands of the form $A_f(x)^{\Phi}$ with the same nonsingular Frobenius block $\Phi$:

$$A_{f_1}^{\Phi} \oplus \cdots \oplus A_{f_s}^{\Phi}. \quad (46)$$

By Lemma 3.1(b), the Hermitian form

$$f_1(\omega)x_1^2y_1 + \cdots + f_s(\omega)x_s^2y_s$$

over $\mathbb{F}(\omega) = \mathbb{F}[x]/p_{\Phi}(x)\mathbb{F}[x]$ with involution $f(\omega)^{\circ} = \bar{f}(\omega)$ is equivalent to $x_1^2x_1 + \cdots + x_s^2x_s$. Theorem 2.3(b) ensures that (46) is *congruent to $A_{f_1}^{\Phi} \oplus \cdots \oplus A_{f_s}^{\Phi}$ and this sum is uniquely determined by $(A, B)$. This gives the summands (iii).

Summands (d). Consider the whole group of summands of the form $B_n^a$ with the same $n$:

$$B_1^{a_1} \oplus \cdots \oplus B_1^{a_s}. \quad (47)$$

By Lemma 3.1(b), the Hermitian form $a_1x_1^2y_1 + \cdots + a_sx_s^2y_s$ over $\mathbb{F}$ is equivalent to $\bar{x}_1y_1 + \cdots + \bar{x}_s y_s$. Theorem 2.2(b) ensures that (47) is *congruent to $B_1^{a_1} \oplus \cdots \oplus B_1^{a_s}$ and this sum is uniquely determined by $(A, B)$. This gives the summands (iv).

$$\square$$

4 Canonical forms over $p$-adic fields

In this section $\mathbb{K}$ is a finite extension of $\mathbb{Q}_p$ with $p \neq 2$.

Let us recall the classification of quadratic and Hermitian forms over $\mathbb{K}$. Each nonzero element of $\mathbb{Q}_p$ can be represented in the form

$$a = \alpha_zp^z + \alpha_{z+1}p^{z+1} + \cdots, \quad z \in \mathbb{Z}, \quad \alpha_z \neq 0, \quad (48)$$

in which all $\alpha_i \in \{0, 1, \ldots, p - 1\}$. The *exponential variation* on $\mathbb{K}$ is the following mapping $\nu : \mathbb{K} \to \mathbb{R} \cup \{+\infty\}$:

$$\nu(a) := \begin{cases} 
+\infty, & \text{if } a = 0, \\
\nu(\alpha_m)/m, & \text{if } a \not\in \mathbb{Q}_p \text{ is represented in the form (48),} \\
\nu(\alpha_m)/m, & \text{if } a \not\in \mathbb{Q}_p \text{ and its minimum polynomial of over } \mathbb{Q}_p \text{ is } x^m + \alpha_1x^{m-1} + \cdots + \alpha_m.
\end{cases}$$

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The ring
\[ \mathcal{O}(K) := \{ a \in K \mid \nu(a) \geq 0 \} \] (49)
is called the ring of integers of \( K \); it is a principal ideal ring, whose unique maximal ideal is
\[ m := \{ a \in K \mid \nu(a) > 0 \} = \pi \mathcal{O}(K). \] (50)
Each generator \( \pi \) of \( m \) is called a prime element. The set
\[ \mathcal{O}(K)^\times := \{ a \in K \mid \nu(a) = 0 \} \] (51)
is the group of all invertible elements of \( \mathcal{O}(K) \); they are called the units of \( K \).

The factor ring
\[ \mathcal{O}(K)/m \] (52)
is a field, which is called the residue field of \( K \); it is a finite extension of the residue field \( \mathbb{F}_p = \mathbb{Q}_p/p\mathbb{Q}_p \) of \( \mathbb{Q}_p \).

**Lemma 4.1.** Let \( K \) be a finite extension of \( \mathbb{Q}_p \) with \( p \neq 2 \). Let its residue field \( \mathcal{O}(K)/m \) consist of \( p^m \) elements. Let \( u \in \mathcal{O}(K)^\times \setminus K^\times \) be a unit that is not a square, and \( \pi \) be a prime element. Then each quadratic form of rank \( r \geq 1 \) over \( K \) is equivalent to exactly one form
\[ c_1x_1^2 + c_2x_2^2 + \cdots + c_tx_t^2 + x_{t+1}^2 + \cdots + x_r^2, \] (53)
in which \( (c_1, \ldots, c_t) \) is one of the sequences:
(1), \( (u) \), \( (\pi) \), \( (u\pi) \), \( (u, \pi) \), \( (\pi, u\pi) \), \( (u, \pi, u\pi) \), \( (\pi, \pi, \pi) \), \( (u, \pi, \pi, u\pi) \), \( (\pi, \pi, \pi, \pi) \), (54)
where
\[ r := \begin{cases} u & \text{if } p^m \equiv 1 \mod 4, \\ 1 & \text{if } p^m \equiv 3 \mod 4. \end{cases} \] (55)

**Lemma 4.2.** Let a field \( K \) with nonidentity involution be a finite extension of \( \mathbb{Q}_p \), \( p \neq 2 \). Let \( K_0 \) be the fixed field with respect to this involution. Let \( u \in \mathcal{O}(K_0)^\times \setminus K_0^\times \) be a unit that is not a square, and \( \pi \) be a prime element of \( K_0 \). Then each Hermitian form with nonzero determinant over \( K \) is classified by dimension and determinant; moreover, it is equivalent to
\[ \text{either } \bar{x}_1y_1 + \cdots + \bar{x}_ny_n, \quad \text{or } t\bar{x}_1y_1 + \bar{x}_2y_2 + \cdots + \bar{x}_ny_n, \]
in which
\[ t := \begin{cases} 
\pi & \text{if } K = K_\circ(\sqrt{u}), \\
u & \text{if } K = K_\circ(\sqrt{\pi}) \text{ or } K_\circ(\sqrt{u\pi}).
\end{cases} \tag{56} \]

Proof. By [13, Ch. 10, Example 1.6(ii)], regular Hermitian forms over $K$ are classified by dimension and determinant. \hfill \Box

4.1 Canonical matrices for congruence and *congruence

Theorem 4.3. Let a field $F$ be a finite extension of $\mathbb{Q}_p$ with $p \neq 2$. Every square matrix over $F$ is congruent to a direct sum that is uniquely determined up to permutation of summands and consists of any number of summands of the following types:

(i) $J_n(0)$;

(ii) $[\Phi \setminus I_n]$, in which $\Phi$ is an $n \times n$ nonsingular Frobenius block over $F$ such that
\[ p_\Phi(x) \neq p_\Phi^\text{\#}(x) \text{ (see (51))} \quad \text{or} \quad p_\Phi(x) = x + (-1)^{n+1} \tag{57} \]
and $\Phi$ is determined up to replacement by the Frobenius block $\Psi$ with $\chi_\Phi(x) = \chi_\Psi(x)$;

(iii) for each nonsingular Frobenius block $\Phi$ over $F$ such that $p_\Phi(x) = p_\Phi^\text{\#}(x)$ and $\deg p_\Phi(x) \geq 2$:

- $\sqrt[p]{\Phi}$,
- at most one summand of the form
\[ \begin{cases} 
\sqrt[p]{\Phi}(\pi) & \text{if } K = K_\circ(\sqrt{u}), \\
\sqrt[p]{\Phi}(\bar{u}(\Phi)) & \text{if } K = K_\circ(\sqrt{\pi}) \text{ or } K = K_\circ(\sqrt{u\pi}),
\end{cases} \]
in which $K$ is the following field with involution:
\[ K := F(\kappa) = F[x]/p_\Phi(x)F[x], \quad f(\kappa)^\circ := f(\kappa^{-1}) \]
(see (6) and (7)), \(K_\circ\) is its fixed field, $\pi$ is a prime element of $K_\circ$, $u \in \mathcal{O}(K_\circ)^\times \setminus K_\circ^\times$ is a unit that is not a square, $\pi(x), \bar{u}(\kappa) \in F[x, x^{-1}]$ are the functions of the form (8) such that $\bar{\pi}(\kappa) = \pi$ and $\bar{\bar{u}}(\kappa) = u$;
(iv) for each \( n = 1, 2, \ldots \):

- \( \Gamma_n \) (defined in (26)),
- at most one summand from the list

\[
\begin{align*}
&u\Gamma_n, \quad \pi\Gamma_n, \quad u\pi\Gamma_n, \quad u\Gamma_n \oplus \pi\Gamma_n, \quad u\pi\Gamma_n, \\
&\pi\Gamma_n \oplus r\pi\Gamma_n, \quad u\Gamma_n \oplus \pi\Gamma_n \oplus r\pi\Gamma_n,
\end{align*}
\]

in which

\[
\begin{align*}
r &:= \begin{cases} 
  u & \text{if } p^m \equiv 1 \mod 4, \\
  1 & \text{if } p^m \equiv 3 \mod 4,
\end{cases}
\end{align*}
\]

\( p^m \) is the number of elements of the residue field \( \mathcal{O}(F)/\mathfrak{m} \) of \( F \), \( u \in \mathcal{O}(F)^\times \setminus F^\times 2 \) is a unit that is not a square, and \( \pi \) is a prime element of \( \mathcal{O} \).

**Proof.** Let a field \( F \) be a finite extension of \( \mathbb{Q}_p \) with \( p \neq 2 \). By Theorem 2.2(a), every square matrix \( A \) over \( F \) is congruent to a direct sum of matrices of the form

- (a) \( J_n(0) \),
- (b) \( [\Phi \setminus I_n] \) if \( \sqrt[p]{\Phi} \) does not exist,
- (c) \( \sqrt[p]{\Phi q}(\Phi) \).

Consider each of these summands.

**Summands (a).** By Lemma 2.6(a), \( \sqrt[p]{\Phi} \) does not exist if and only if (57) holds. Theorem 2.2(b) ensures that the summands of the form \( [\Phi \setminus I_n] \) are uniquely determined by \( A \), up to replacement of \( \Phi \) by \( \Psi \) with \( \chi_\Phi(x) = \chi_\Psi(x) \).

**Summands (b).** Let \( \Phi \) be a nonsingular \( n \times n \) Frobenius block for which \( \sqrt[p]{\Phi} \) exists. Then by Lemma 2.6(a)

\[
p_\Phi(x) = p_\Phi^\chi(x), \quad p_\Phi(x) \neq x + (-1)^{n+1}.
\]

Consider the whole group of summands of the form \( \sqrt[p]{\Phi q}(\Phi) \) with the same \( \Phi \):

\[
\sqrt[p]{\Phi q_1}(\Phi) \oplus \cdots \oplus \sqrt[p]{\Phi q_k}(\Phi).
\]

Let first \( \deg p_\Phi(x) > 1 \). Then the involution \( f(\kappa)^\circ := f(\kappa^{-1}) \) on the field \( \mathbb{F}(\kappa) = \mathbb{F}[x]/p_\Phi(x)\mathbb{F}[x] \) (see (6) and (7)) is nonidentity; otherwise \( \kappa = \kappa^\circ =
\[ \kappa^{-1}, \kappa^2 - 1 = 0, \ k' \text{ divides } p_\Phi(x), \text{ and hence } p_\Phi(x) = x \pm 1 \] since it is irreducible. By Lemma 4.2 the Hermitian form

\[ q_1(k)x_1^0y_1 + \cdots + q_s(k)x_s^0y_s \]

over \( \mathbb{F}(k) \) is equivalent to either \( x_1^0y_1 + \cdots + x_s^0y_s \) or \( tx_1^0y_1 + x_2^0y_2 + \cdots + x_s^0y_s \), in which \( t \) is defined in (56). Theorem 2.2(b) ensures that the matrix (60) is congruent to either \( \sqrt{\Phi} \oplus \cdots \oplus \sqrt{\Phi} \) or \( \sqrt{\Phi}(\Phi) \oplus \sqrt{\Phi} \oplus \cdots \oplus \sqrt{\Phi} \),

in which \( \sqrt{\Phi}(\Phi) \) is the function of the form (8) such that \( \sqrt{\Phi}(\kappa) = \sqrt{\Phi} \).

This sum is uniquely determined by \( A \), which gives the summands (iii).

Let now \( p_\Phi(x) = x + c \). Then by (59) and (4), \( x + c = c^{-1}(cx + 1), \ c = c^{-1}, \ c = \pm 1 \). The inequality in (59) implies

\[ p_\Phi(x) = x + (-1)^n. \] (61)

By (31), \( \Gamma_n \equiv \sqrt{\Upsilon_n} \) and \( \Upsilon_n \) is similar to \( J_n((-1)^{n+1}) \), which is similar to \( \Phi \) due to (30). By Theorem 2.2(c) we can take \( \Upsilon_n \) instead of \( \Phi \) with \( p_\Phi(x) = x + (-1)^n \) in Theorem 2.2(a,b). The field \( \mathbb{F}(k) = \mathbb{F}[x]/p_\Phi(x)\mathbb{F}[x] \) is \( \mathbb{F} \) with the identity involution; all polynomials \( q_i(x) \) in (60) are some scalars \( a_i \in \mathbb{F} \), by Lemma 4.1 with \( K = \mathbb{F} \), the quadratic form

\[ q_1(k)x_1^2 + \cdots + q_s(k)x_s^2 = a_1x_1^2 + \cdots + a_sx_s^2 \]

over \( \mathbb{F} \) is equivalent to exactly one form (53), in which \( (c_1, \ldots, c_t) \) is one of the sequences (54). Theorem 2.2(b) ensures that (60) is congruent to a direct sum of matrices of the form (iv), and this sum is uniquely determined by \( A \). This gives the summands (iv).

Theorem 4.4. Let a field \( \mathbb{F} \) with nonidentity involution be a finite extension of \( \mathbb{Q}_p \) with \( p \neq 2 \). Every square matrix \( A \) over \( \mathbb{F} \) is *congruent to a direct sum that is uniquely determined up to permutation of summands and consists of any number of summands of the following types:

(i) \( J_n(0) \);

(ii) \( [\Phi \setminus I_n] \), in which \( \Phi \) is an \( n \times n \) nonsingular Frobenius block over \( \mathbb{F} \) such that \( p_\Phi(x) \neq p_\Psi(x) \) (see (11)) and \( \Phi \) is determined up to replacement by the Frobenius block \( \Psi \) with \( \chi_\Psi(x) = \chi_\Phi(x) \);
(iii) for each nonsingular Frobenius block $\Phi$ over $\mathbb{F}$ such that $p_\Phi(x) = p_\Phi^\vee(x)$:

- $\sqrt[\Phi]{\Phi}$,
- at most one summand of the form

$$\begin{cases} 
\sqrt[\Phi]{\tilde{\pi}(\Phi)} & \text{if } K = K_\circ(\sqrt{u}), \\
\sqrt[\Phi]{\tilde{u}(\Phi)} & \text{if } K = K_\circ(\sqrt{\pi}) \text{ or } K = K_\circ(\sqrt{u\pi}), 
\end{cases}$$

in which $K$ is the following field with involution:

$$K := \mathbb{F}(\kappa) = \mathbb{F}[x]/p_\Phi(x)\mathbb{F}[x], \quad f(\kappa)^\circ := \bar{f}(\kappa^{-1})$$

(see (3), (6) and (7)), $K_\circ$ is its fixed field, $\pi$ is a prime element of $K_\circ$, $u \in \mathcal{O}(K_\circ)^\times \setminus K_\circ^{\times 2}$ is a unit that is not a square, $\tilde{\pi}(x), \tilde{u}(x) \in \mathbb{F}[x, x^{-1}]$ are the functions of the form (3) such that $\tilde{\pi}(\kappa) = \pi$ and $\tilde{u}(\kappa) = u$.

**Proof.** Let a field $\mathbb{F}$ with nonidentity involution be a finite extension of $\mathbb{Q}_p$ with $p \neq 2$. By Theorem 2.2(a), every square matrix $A$ over $\mathbb{F}$ is $*$congruent to a direct sum of matrices of the form

- (a) $J_n(0)$,
- (b) $[\Phi \setminus I_n]$ if $\sqrt[\Phi]{\Phi}$ does not exist,
- (c) $\sqrt[\Phi]{\Phi}q_\Phi(\Phi)$.

Consider each of these summands.

**Summands (a).** Theorem 2.2(b) ensures that the summands of the form $J_n(0)$ are uniquely determined by $A$, which gives the summands (i) of the theorem.

**Summands (b).** By Lemma 2.6(a), $\sqrt[\Phi]{\Phi}$ does not exist if and only if $p_\Phi(x) \neq p_\Phi^\vee(x)$. Theorem 2.2(b) ensures that the summands of the form $[\Phi \setminus I_n]$ are uniquely determined by $A$, up to replacement of $\Phi$ by $\Psi$ with $\chi_\Phi(x) = \chi_\Psi(x)$. This gives the summands (ii).

**Summands (c).** Let $\Phi$ be a nonsingular $n \times n$ Frobenius block for which $\sqrt[\Phi]{\Phi}$ exists; this means that $p_\Phi(x) = p_\Phi^\vee(x)$. Consider the whole group of summands of the form $\sqrt[\Phi]{\Phi}q_\Phi(\Phi)$ with the same $\Phi$:

$$\sqrt[\Phi]{\Phi}q_1(\Phi) \oplus \cdots \oplus \sqrt[\Phi]{\Phi}q_s(\Phi). \quad (62)$$

The involution $f(\kappa)^\circ := f(\kappa^{-1})$ on the field $\mathbb{F}(\kappa) = \mathbb{F}[x]/p_\Phi(x)\mathbb{F}[x]$ (see (6) and (7)) is nonidentity since it extends the nonidentity involution on $\mathbb{F}$. By Lemma 4.2 with $K := \mathbb{F}(\kappa)$, the Hermitian form

$$q_1(\kappa)x_1^\circ y_1 \oplus \cdots \oplus q_s(\kappa)x_s^\circ y_s$$
over $\mathbb{F}(\kappa)$ is equivalent to either $x_1^\tau y_1 + \cdots + x_n^\tau y_n$, or $r x_1^\tau y_1 + x_2^\tau y_2 + \cdots + x_n^\tau y_n$, in which $t$ is defined in (66). By Theorem 2.2(b), the matrix (62) is congruent to

\[ \sqrt[n]{\Phi} \oplus \cdots \oplus \sqrt[n]{\Phi}, \text{ or } \sqrt[n]{\Phi} \tilde{t}(\Phi) \oplus \sqrt[n]{\Phi} \oplus \cdots \oplus \sqrt[n]{\Phi}, \]

where $\tilde{t}(x) \in \mathbb{F}[x, x^{-1}]$ is the function of the form [8] such that $\tilde{t}(\kappa) = t$. This sum is uniquely determined by $A$, which gives the summands (iii).

4.2 Canonical pairs of symmetric or skew-symmetric matrices

For each Frobenius block $\Phi$, denote by $\sqrt[n]{\Phi}$ and $\Phi_\varepsilon$ ($\varepsilon = \pm 1$) fixed nonsingular matrices satisfying, respectively, the conditions

\[ \sqrt[n]{\Phi} = (\sqrt[n]{\Phi})^T A, \]
\[ \Phi_\varepsilon = \Phi_\varepsilon^T, \quad \Phi_\varepsilon A = \varepsilon(\Phi_\varepsilon A)^T, \]

in which $A$ is similar to $\Phi$. Each of these matrices may not exist for some $\Phi$; existence conditions and explicit forms of these matrices were established in [15].

**Theorem 4.5.** Let a field $\mathbb{F}$ be a finite extension of $\mathbb{Q}_p$ with $p \neq 2$. Each pair of symmetric matrices of the same size over $\mathbb{F}$ is congruent to a direct sum that is uniquely determined up to permutation of summands and consists of any number of summands of the following types:

(i) $([F_n \setminus F_n^T], [G_n \setminus G_n^T])$, in which $F_n$ and $G_n$ are defined in [11];

(ii) for each nonsingular Frobenius block $\Phi$ over $\mathbb{F}$:

- $(\Phi_1, \Phi_1 \Phi)$, in which $\Phi_1$ is defined in [10],
- at most one summand of the form

\[ (\Phi_1, \Phi_1 \Phi) f_1(\Phi) \oplus \cdots \oplus (\Phi_1, \Phi_1 \Phi) f_t(\Phi), \]

in which $(f_1(x), \ldots, f_t(x))$ is a sequence of polynomials over $\mathbb{F}$ of degree $< \deg(p_\Phi(x))$ such that the sequence $(f_1(\omega), \ldots, f_t(\omega))$ of elements of the field

\[ \mathbb{K} := \mathbb{F}(\omega) = \mathbb{F}[x]/p_\Phi(x) \mathbb{F}[x] \]

27
is one of the sequences
\begin{align}
(1), \ (u), \ (\pi), \ (u\pi), \ (u, \pi), \ (u, u\pi), \ (\pi, u\pi), \ (u, \pi, r\pi),
\end{align}
where
\begin{align}
r := \begin{cases}
    u & \text{if } p^m \equiv 1 \mod 4, \\
    1 & \text{if } p^m \equiv 3 \mod 4,
\end{cases}
\end{align}
p^m is the number of elements of the residue field \( \mathcal{O}(\mathbb{K})/m \) of \( \mathbb{K} \),
u \in \mathcal{O}(\mathbb{K})^\times \setminus \mathbb{K}^\times \) is a unit that is not a square, and \( \pi \) is a prime element of \( \mathbb{K} \);

(iii) for each \( n = 1, 2, \ldots \):
- the pair of \( n \)-by-\( n \) matrices
  \begin{align}
  B_n := \begin{pmatrix}
  0 & \cdots & 1 & 0 \\
  \cdots & \cdots & \cdots & \cdots \\
  1 & 0 & 0 \\
  0 & 0 & 1
  \end{pmatrix} \quad \begin{bmatrix}
  0 \\
  \vdots \\
  1
  \end{bmatrix},
\end{align}
- at most one summand of the form
  \begin{align}
  B_{n1} \oplus \cdots \oplus B_{nt},
\end{align}
in which \( (c_1, \ldots, c_t) \) is one of the sequences
\begin{align}
(1), \ (u), \ (\pi), \ (u\pi), \ (u, \pi), \ (u, u\pi), \ (\pi, u\pi), \ (u, \pi, r\pi),
\end{align}
where
\begin{align}
r := \begin{cases}
    u & \text{if } p^m \equiv 1 \mod 4, \\
    1 & \text{if } p^m \equiv 3 \mod 4,
\end{cases}
\end{align}
p^m is the number of elements of the residue field \( \mathcal{O}(\mathbb{F})/m \) of \( \mathbb{F} \),
u \in \mathcal{O}(\mathbb{F})^\times \setminus \mathbb{F}^\times \) is a unit that is not a square, and \( \pi \) is a prime element of \( \mathbb{F} \).

Proof. Let a field \( \mathbb{F} \) with the identity involution be a finite extension of \( \mathbb{Q}_p \), \( p \neq 2 \). By Lemma 2.7, the matrix \( \Phi_1 \) exists for each nonsingular Frobenius block \( \Phi \) over \( \mathbb{F} \). By Theorem 2.3(a), each pair \((A, B)\) of symmetric matrices of the same size is congruent to a direct sum of pairs of the form
\begin{align}
(a) \ (\{F_n \setminus F_n^T\}, \ {G_n \setminus G_n^T})], \quad (b) \ A_{\Phi}^{f(x)} := (\Phi_1, \Phi) f(\Phi), \quad (c) \ B_n\alpha,
\end{align}
in which \( f(x) \in \mathbb{F}[x] \) is a nonzero polynomial of degree < \( \deg(p_{\Phi}(x)) \) and \( 0 \neq a \in \mathbb{F} \).

Consider each of these summands.

**Summands (a).** Theorem 2.3(b) ensures that the summands of the form (a) are uniquely determined by \((A, B)\), which gives the summands (i) of the theorem.

**Summands (b).** Consider the whole group of summands of the form \( A_{\Phi}^{g(x)} \) with the same nonsingular Frobenius block \( \Phi \):

\[
A_{\Phi}^{g_1(x)} \oplus \cdots \oplus A_{\Phi}^{g_s(x)}.
\]  

(71)

By Lemma 4.1, the quadratic form

\[
q_1(\omega)x_1^2 + \cdots + q_s(\omega)x_s^2
\]

over \( \mathbb{F}(\omega) = \mathbb{F}[x]/p_{\Phi}(x)\mathbb{F}[x] \) is equivalent to exactly one form (53), in which \((a_1, \ldots, a_s)\) is one of the sequences (54). Theorem 2.3(b) ensures that (71) is congruent to a direct sum of pairs of the form (ii) and this sum is uniquely determined by \((A, B)\), which gives the summands (ii).

**Summands (c).** For each \( n \), consider the whole group of summands of the form \( B_{n}^a \) with the same \( n \):

\[
B_{n}^{a_1} \oplus \cdots \oplus B_{n}^{a_s}.
\]  

(72)

By Lemma 4.1, the quadratic form \( a_1 x_1^2 + \cdots + a_s x_s^2 \) over \( \mathbb{F} \) is equivalent to to exactly one form (53), in which \((c_1, \ldots, c_t)\) is one of the sequences (54). Theorem 2.3(b) ensures that (72) is congruent to a direct sum of pairs of the form (iii) and this sum is uniquely determined by \((A, B)\), which gives the summands (iii).

\[\Box\]

**Theorem 4.6.** Let a field \( \mathbb{F} \) be a finite extension of \( \mathbb{Q}_p \) with \( p \neq 2 \). Each pair consisting of a symmetric matrix and a skew-symmetric matrix of the same size over \( \mathbb{F} \) is congruent to a direct sum that is uniquely determined up to permutation of summands and consists of any number of summands of the following types:

(i) \(([F_n \setminus F_n^T], [G_n \setminus - G_n^T])\), in which \( F_n \) and \( G_n \) are defined in (11);

(ii) \(([I_n \setminus J_n], [\Phi \setminus - \Phi^T])\), in which \( \Phi \) is an \( n \times n \) Frobenius block over \( \mathbb{F} \) such that

\[
p_{\Phi}(x) \notin \mathbb{F}[x^2], \quad \Phi \neq J_1(0), J_3(0), J_5(0), \ldots
\]  

(73)

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(see [2]), and Φ is determined up to replacement by the Frobenius block \( \Psi \) with \( \chi_\Phi(x) = (-1)^{\det \chi_\Phi} \chi_\Phi(-x) \):

(iii) for each Frobenius block \( \Phi \) over \( \mathbb{F} \) such that \( p_\Phi(x) \in \mathbb{F}[x^2] \):

- \( (\Phi_{-1}, \Phi_{-1}) \)
- at most one summand of the form

\[
\begin{cases}
(\Phi_{-1}, \Phi_{-1})\tilde{\pi}(\Phi) & \text{if } K = K_\circ(\sqrt{u}) , \\
(\Phi_{-1}, \Phi_{-1})\tilde{u}(\Phi) & \text{if } K = K_\circ(\sqrt{\pi}) \text{ or } K = K_\circ(\sqrt{u\pi}) ,
\end{cases}
\]

in which \( K \) is the following field with involution:

\[
K := \mathbb{F}(\omega) = \mathbb{F}[x]/p_\Phi(x)\mathbb{F}[x] , \\
f(\omega)^0 = f(-\omega) ,
\]

\( K_\circ \) is its fixed field, \( \pi \) is any prime element of \( K_\circ \), and \( u \in \mathcal{O}(K_\circ)^\times \setminus K_\circ^{\times 2} \) is any unit that is not a square; \( \tilde{\pi}(x) \) and \( \tilde{u}(x) \) are polynomials over \( \mathbb{F} \) of degree \( < \deg(p_\Phi(x)) \) such that

\[
\tilde{\pi}(x) = \tilde{\pi}(-x) , \\
\tilde{\pi}(\omega) = \pi , \\
\tilde{u}(x) = \tilde{u}(-x) , \\
\tilde{u}(\omega) = u ;
\]

(iv) \( ([J_n(0) \setminus J_n(0)^T], [I_n \setminus -I_n]) \), in which \( n \) is odd;

(v) for each \( n = 1, 2, 3, \ldots \):

- the pair of \( n \)-by-\( n \) symmetric and skew-symmetric matrices defined as follows:

\[
C_n := \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
1 & 0 & \cdots & 0 & \cdots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & \cdots & 1 \\
\end{pmatrix}
\]

(74)

if \( n \) is odd, and

\[
C_n := \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
1 & 0 & \cdots & 0 & \cdots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & \cdots & 1 \\
\end{pmatrix}
\]

(75)
if \( n \) is even, and

- at most one summand of the form

\[
C_n c_1 \oplus \cdots \oplus C_n c_t, \tag{76}
\]

in which \((c_1, \ldots, c_t)\) is one of the sequences

\[
(1), \ (u), \ \pi, \ (u\pi), \ (u, \pi), \ (u, ur\pi), \ (u, \pi, r\pi), \tag{77}
\]

where

\[
r := \begin{cases} 
  u & \text{if } p^m \equiv 1 \pmod{4}, \\
  1 & \text{if } p^m \equiv 3 \pmod{4},
\end{cases} \tag{78}
\]

\( p^m \) is the number of elements of the residue field \( \mathcal{O}(F)/\mathfrak{m} \) of \( F \),
\( u \in \mathcal{O}(F)^* \setminus F^* \) is a unit that is not a square, and \( \pi \) is a prime element of \( F \).

Proof. Let a field \( F \) with the identity involution be a finite extension of \( \mathbb{Q}_p \), \( p \neq 2 \). By Theorem 2.3(a) and Remark 2.5 each pair \((A, B)\) consisting of a symmetric matrix \( A \) and a skew-symmetric matrix \( B \) of the same size is congruent to a direct sum of pairs of the form

(a) \([(F_n \setminus F_n^T], [G_n \setminus -G_n^T])\),

(b) \([(I_n \setminus I_n], [\Phi \setminus -\Phi^T])\) if \( \Phi^{-1} \) does not exist,

(c) \( A_{\Phi}^{f(x)} := (\Phi^{-1}, \Phi^{-1}\Phi)f(\Phi) \), in which \( 0 \neq f(x) = f(-x) \in F[x] \) and \( \deg(f(x)) < \deg(p_{\Phi}(x)) \),

(d) \([(J_n(0) \setminus J_n(0)^T], [I_n \setminus -I_n])\), in which \( n \) is odd,

(e) \( C_n^a \) (defined in (15)), in which \( n \) is even and \( 0 \neq a \in F \).

Consider each of these summands.

**Summands** (a). Theorem 2.3(b) ensures that the summands of the form (a) are uniquely determined by \((A, B)\), which gives the summands (i) of the theorem.

**Summands** (b). By Lemma 2.7, \( \Phi^{-1} \) does not exist if and only if (73) is satisfied. Theorem 2.3(b) ensures that the summands of the form (b) are uniquely determined by \((A, B)\), up to replacement of \( \Phi \) by \( \Psi \) with \( \chi_{\Psi}(x) = (-1)^{\det x} \chi_{\Phi}(-x) \), which gives the summands (ii).
Summands (c). By Lemma 2.7, $\Phi^{-1}$ exists if and only if (73) is not satisfied; that is,

$$p_\Phi(x) \in \mathbb{F}[x^2] \quad \text{or} \quad \Phi = J_1(0), J_3(0), J_5(0), \ldots$$  \hspace{1em} (79)

Consider the whole group of summands of the form $A_{\Phi}^{f(x)}$ with the same nonsingular Frobenius block $\Phi$:

$$A_{\Phi}^{f_1(x)} \oplus \cdots \oplus A_{\Phi}^{f_s(x)}. \hspace{1em} (80)$$

Let first $p_\Phi(x) \in \mathbb{F}[x^2]$. Then the involution $f(\omega)^\circ = f(-\omega)$ on $\mathbb{F}(\omega) = \mathbb{F}[x]/p_\Phi(x)\mathbb{F}[x]$ is nonidentity (since $\omega^\circ = -\omega \neq \omega$). By Lemma 4.2, the Hermitian form

$$f_1(\omega)x_1^\circ y_1 + \cdots + f_s(\omega)x_s^\circ y_s$$

over $\mathbb{F}(\omega)$ is equivalent to either $x_1^\circ y_1 + \cdots + x_s^\circ y_s$, or $tx_1^\circ y_1 + x_2^\circ y_2 + \cdots + x_s^\circ y_s$, in which $t$ is defined in (56). Theorem 2.2(b) ensures that (80) is congruent to

either $A_\Phi^1 \oplus \cdots \oplus A_\Phi^1$, or $A_\Phi^1\tilde{t}(\Phi) \oplus A_\Phi^1 \oplus \cdots \oplus A_\Phi^1$, where $\tilde{t}(x) \in \mathbb{F}[x, x^{-1}]$ is the function of the form (8) such that $\tilde{t}(\kappa) = t$. This sum is uniquely determined by $(A, B)$, which gives the summands (iii).

Let now $\Phi = J_n(0)$ with $n = 2m + 1$ and $m = 1, 2, \ldots$. The equalities (10) hold for the $n \times n$ matrices $\Phi'$ and $\Phi'_{-1}$ defined in (11) instead of $\Phi$ and $\Phi_{-1}$. Since $\Phi$ and $\Phi'$ are similar, by Theorem 2.3(c) we can take $C_{n}^{f(x)} := (\Phi'_{-1}, \Phi'_{-1}\Phi')f(\Phi')$ instead of $A_{\Phi}^{f(x)}$ and

$$C_{n}^{f_1(x)} \oplus \cdots \oplus C_{n}^{f_s(x)} \hspace{1em} (81)$$

instead of (40).

Since $p_\Phi(x) = x$, the field $\mathbb{F}(\omega) = \mathbb{F}[x]/p_\Phi(x)\mathbb{F}[x]$ is $\mathbb{F}$ with the identity involution and all polynomials $f_i(x)$ in (42) are some scalars $a_i \in \mathbb{F}$. By Lemma 4.1, the quadratic form $a_1x_1^2 + \cdots + a_sx_s^2$ over $\mathbb{F}$ is equivalent to exactly one form (53), in which $(c_1, \ldots, c_t)$ is one of the sequences (54). Theorem 2.3(b) ensures that (81) is congruent to a direct sum of pairs of the form (iii), and this sum is uniquely determined by $(A, B)$. This gives the summands (v) with odd $n$.

Summands (d). Theorem 2.3(b) ensures that the summands of the form (d) are uniquely determined by $(A, B)$, which gives the summands (iv).
Summands (e). Consider the whole group of summands of the form \( C_n^a \) with the same \( n \):
\[
C_n^{a_1} \oplus \cdots \oplus C_n^{a_s}.
\]
(82)
By Lemma 4.1, the quadratic form \( a_1 x_1^2 + \cdots + a_s x_s^2 \) over \( F \) is equivalent to exactly one form (53), in which \((c_1, \ldots, c_t)\) is one of the sequences (54). Theorem 2.3(b) ensures that (82) is congruent to a direct sum of pairs of the form (iii), and this sum is uniquely determined by \((A, B)\). This gives the summands (v) with even \( n \).

4.3 Canonical pairs of Hermitian matrices

Theorem 4.7. Let a field \( F \) with nonidentity involution be a finite extension of \( \mathbb{Q}_p \), \( p \neq 2 \). Let \( F_0 \) be the fixed field of \( F \). Each pair of Hermitian matrices of the same size over \( F \) is *congruent to a direct sum that is uniquely determined up to permutation of summands and consists of any number of summands of the following types:

(i) \([F_n \setminus F_n^*], [G_n \setminus G_n^*]\), in which \( F_n \) and \( G_n \) are defined in (11);

(ii) \([I_n \setminus I_n], [\Phi \setminus \Phi^*]\), in which \( \Phi \) is an \( n \times n \) Frobenius block such that \( p_\Phi(x) \notin F_0[x] \), and \( \Phi \) is determined up to replacement by the Frobenius block \( \Psi \) with \( \chi_\Psi(x) = \bar{\chi}_\Phi(x) \) (see (3));

(iii) for each Frobenius block \( \Phi \) over \( F \) such that \( p_\Phi(x) \in F_0[x] \):

- \((\Phi_1, \Phi_1 \Phi)\), in which \( \Phi_1 \) is defined in Lemma 2.7
- at most one summand of the form

\[
\begin{cases}
(\Phi_1, \Phi_1 \Phi) f_\pi(\Phi) & \text{if } K = K_0(\sqrt{u}), \\
(\Phi_1, \Phi_1 \Phi) f_u(\Phi) & \text{if } K = K_0(\sqrt{\pi}) \text{ or } K = K_0(\sqrt{u\pi}),
\end{cases}
\]

in which \( K \) is the following field with involution:
\[
K := F(\omega) = F[x]/p_\Phi(x)F[x], \quad f(\omega)^o = \bar{f}(\omega),
\]
\( K_0 \) is its fixed field, \( \pi \) is any prime element of \( K_0 \), and \( u \in \mathcal{O}(K_0)^\times \setminus K_0^2 \) is any unit that is not a square, \( \pi(x) \) and \( \bar{u}(x) \) are polynomials over \( F_0 \) of degree < \( \deg(p_\Phi(x)) \) such that
\[
\pi(\omega) = \pi, \quad \bar{u}(\omega) = u;
\]
(iv) for each \( n = 1, 2, \ldots \):

- the pair of \( n \)-by-\( n \) matrices

\[
B_n := \begin{pmatrix}
0 & 1 & 0 \\
\ddots & \ddots & \ddots \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

- at most one summand of the form

\[
\begin{cases}
B_n \pi & \text{if } F = F_\circ(\sqrt{u}), \\
B_n u & \text{if } F = F_\circ(\sqrt{\pi}) \text{ or } F = F_\circ(\sqrt{u\pi}),
\end{cases}
\]

in which \( \pi \) is a prime element of \( F_\circ \) and \( u \in O(F_\circ)^\times \setminus F_\circ^{\times 2} \) is a unit that is not a square.

Proof. Let a field \( F \) with nonidentity involution be a finite extension of \( \mathbb{Q}_p \), \( p \neq 2 \). By Theorem 2.3(a), each pair \((A, B)\) of Hermitian matrices of the same size over \( F \) is *congruent to a direct sum of pairs of the form

(a) \(([F_n \setminus F_n^*], [G_n \setminus G_n^*])\),

(b) \(([I_n \setminus I_n], [\Phi \setminus \Phi^*])\) if \( \Phi_1 \) does not exist,

(c) \(A_{\Phi}^{f(x)} := (\Phi_2, \Phi_1 \Phi)f(\Phi)\), in which \( 0 \neq f(x) = \bar{f}(x) \in F[x] \) and \( \deg(f(x)) < \deg(p_\Phi(x))\).

(d) \(B_n^a\) (defined in (12)), in which \( 0 \neq a = \bar{a} \in F\).

Consider each of these summands.

Summands (a). Theorem 2.3(b) ensures that the summands of the form (a) are uniquely determined by \((A, B)\), which gives the summands (i) of the theorem.

Summands (b). By Lemma 2.7(a), \( \Phi_1 \) does not exist if and only if \( p_\Phi(x) \neq \bar{p}_\Phi(x) \); that is, \( p_\Phi(x) \notin F_\circ[x] \). Theorem 2.3(b) ensures that the summands of the form (b) are uniquely determined, up to replacement of \( \Phi \) by \( \Psi \) with \( \chi_\Psi(x) = \bar{\chi}_\Phi(x) \). This gives the summands (ii).

Summands (c). Consider the whole group of summands of the form \(A_{\Phi}^{f(x)}\) with the same nonsingular Frobenius block \( \Phi \):

\[
A_{\Phi}^{f_1(x)} \oplus \cdots \oplus A_{\Phi}^{f_s(x)}.
\]  (83)
By Lemma 4.2 the Hermitian form
\[ f_1(\omega)x_1^0y_1 + \cdots + f_s(\omega)x_s^0y_s \]
over \( F(\omega) = F[x]/p_\Phi(x)F[x] \) with involution \( f(\omega)^o = \bar{f}(\omega) \) is equivalent to either \( x_1^0y_1 + \cdots + x_s^0y_s \), or \( tx_1^0y_1 + x_2^0y_2 + \cdots + x_s^0y_s \), in which \( t \) is defined in (56). Theorem 2.2(b) ensures that (80) is *congruent to

either \( A_1^1 \oplus \cdots \oplus A_s^1 \), or \( A_1^1 \bar{t}(\Phi) \oplus A_1^1 \oplus \cdots \oplus A_s^1 \),

in which \( \bar{t}(x) \in F[x, x^{-1}] \) is the function of the form (8) such that \( \bar{t}(\kappa) = t \). This sum is uniquely determined by \((A, B)\), which gives the summands (iii).

Summands (d). Consider the whole group of summands of the form \( B_n^a \) with the same \( n \):
\[ B_n^{a_1} \oplus \cdots \oplus B_n^{a_s}. \] (84)

By Lemma 4.2 the Hermitian form
\[ a_1x_1^0y_1 + \cdots + a_sx_s^0y_s \]
over \( F \) is equivalent either \( x_1^0y_1 + \cdots + x_s^0y_s \), or \( tx_1^0y_1 + x_2^0y_2 + \cdots + x_s^0y_s \), in which \( t \) is defined in (56). Theorem 2.2(b) ensures that (84) is *congruent to

either \( B_n \oplus \cdots \oplus B_n \), or \( B_n \bar{t}(\Phi) \oplus B_n \oplus \cdots \oplus B_n \),

in which \( \bar{t}(x) \in F[x, x^{-1}] \) is the function of the form (8) such that \( \bar{t}(\kappa) = t \). This sum is uniquely determined by \((A, B)\), which gives the summands (iv). \( \square \)

5 Appendix: Quadratic forms over finite extensions of \( p \)-adic fields

In this section, we recall some known results on quadratic forms over finite extensions of \( p \)-adic fields that are used in the paper.

Let \( F \) be a field and let \( \nu \) be an exponential variation on \( F \); that is, a map \( \nu : F \to \mathbb{R} \cup \{ +\infty \} \) with the properties
\[
\nu(x) = +\infty \iff x = 0, \hspace{1cm} (85)
\]
\[
\min \{ \nu(x), \nu(y) \} \leq \nu(x + y), \hspace{1cm} (86)
\]
\[
\nu(x) + \nu(y) = \nu(xy) \hspace{1cm} (87)
\]
for all $x, y \in \mathbb{F}$.

For example, the field $\mathbb{Q}_p$ of $p$-adic numbers possesses an exponential variation that is defined on each nonzero $p$-adic number as follows:

$$v(a_z p^z + a_{z+1} p^{z+1} + \ldots) = z,$$

where $a_i \in \{0, 1, \ldots, p - 1\}$, $a_z \neq 0$, and $z \in \mathbb{Z}$.

In this section $\mathbb{F}$ denotes a finite extension of $\mathbb{Q}_p$, $p \neq 2$. In this case the exponential variation (20) can be extended to an exponential variation of $\mathbb{F}$. This variation is unique and is given by the formula:

$$\nu(a) = \frac{1}{n} v(N(a)) \quad \text{for all } a \in \mathbb{F},$$

in which $n := (\mathbb{F} : \mathbb{Q}_p) = \dim_{\mathbb{Q}_p} \mathbb{F}$ is the degree of $\mathbb{F}$ over $\mathbb{Q}_p$ and $N(a)$ is the norm of $a$ in $\mathbb{F}$ over $\mathbb{Q}_p$; that is, the determinant of the linear mapping $x \mapsto xa$ on $\mathbb{F}$ as a vector space over $\mathbb{Q}_p$. If $x^m + \alpha_1 x^{m-1} + \cdots + \alpha_m$ is the minimum polynomial of $a \in \mathbb{F}$ over $\mathbb{Q}_p$ then the variation (89) can be also given by the formula:

$$\nu(a) = \frac{1}{m} v(\alpha_m) \quad \text{for all } a \in \mathbb{F}.\quad (90)$$

Note that there exists a natural number $e$ such that $e\nu(\mathbb{F}^\times) = \mathbb{Z}$. The ring

$$\mathcal{O} := \{x \in \mathbb{F} | \nu(x) \geq 0\}$$

is called the ring of integers (with respect to $\nu$);

$$\mathfrak{m} := \{x \in \mathbb{F} | \nu(x) > 0\} = \pi \mathcal{R}$$

is the unique maximal ideal of $\mathcal{R}$ and its generator $\pi$ is called a prime element (it is any element of $\mathbb{F}$ with the smallest positive $\nu(\pi)$; that is, $\nu(\pi) = 1/e$); the factor ring

$$\mathcal{O}/\mathfrak{m}$$

is a field, which is called the residue field; and the set

$$\mathcal{O}^\times := \{x \in \mathbb{F} | \nu(x) = 0\}$$

is the group of all invertible elements of $\mathcal{O}$ (which are called units).
The residue field $\mathcal{O}/m$ is an extension of the residue field $\mathbb{F}_p$ of $\mathbb{Q}_p$ and
\[ e(\mathcal{O}/m : \mathbb{F}_p) = n = (\mathbb{F} : \mathbb{Q}_p) \tag{95} \]
By [9, Section VI, Theorem 2.2] or [13, Ch. 6, Facts 4.1] $\mathbb{F}^\times/\mathbb{F}^{\times 2}$ consists of 4 cosets, represented by 1, $u$, $\pi$, $u\pi$, where $u \in \mathcal{O}^\times$ is a unit with $u \notin \mathbb{F}^{\times 2}$ (or, which is equivalent, with $u + m \notin (\mathcal{O}/m)^{\times 2}$; see [4, Example 3.11], recall that $p \neq 2$).

The Hilbert symbol is defined for $a, b \in \mathbb{F}^\times$ by
\[ (a, b)_{\mathbb{F}} := \begin{cases} 
1 & \text{if } ax^2 + by^2 \text{ represents 1}, \\
-1 & \text{otherwise}. 
\end{cases} \tag{96} \]

The Hasse invariant of a form $q \sim a_1x_1^2 + a_2x_2^2 + \cdots + a_rx_r^2$ with $a_1, \ldots, a_r \in \mathbb{F}^\times$ is
\[ c(q) := \prod_{i<j}(a_i, a_j)_{\mathbb{F}} \tag{97} \]
(see [10, Ch. VIII, p. 210]).

By [10, Ch. VIII, Theorem 4.10], two quadratic forms over $\mathbb{F}$ are equivalent if and only if they have the same rank $n$, the same discriminant $d$ (in $\mathbb{F}^\times/\mathbb{F}^{\times 2}$), and the same Hasse invariant. By [10, Ch. VIII, Proposition 4.11], if $q$ is a quadratic form of rank $r$, then
- If $r = 1$ then $c(q) = 1$.
- If $r = 2$ and $c(q) = -1$ then $d(q) \not\equiv -1 \pmod{\text{squares}}$.

Apart from these constraints, every triple $r \geq 1$, $d \in \{1, u, \pi, u\pi\} \pmod{\text{squares}}$, $c = \pm 1$ occurs as the set of invariants of a quadratic form over $\mathbb{F}$.

**Theorem 5.1.** Let $\mathbb{F}$ be a finite extension of $\mathbb{Q}_p$ with $p \neq 2$. Let its residue field $\mathcal{O}/m$ consist of $p^m$ elements. Let $u \in \mathcal{O}^\times \setminus \mathbb{F}^{\times 2}$ be a unit that is not a square, and $\pi$ be a prime element. Then each quadratic form of rank $r \geq 1$ over $\mathbb{F}$ is equivalent to exactly one form
\[ a_1x_1^2 + a_2x_2^2 + \cdots + a_t x_t^2 + x_{t+1}^2 + \cdots + x_r^2, \tag{98} \]
in which $(a_1, \ldots, a_t)$ is one of the sequences:
\[ (1), \ (u), \ (\pi), \ (u\pi), \ (u, \pi), \ (u, u\pi), \tag{99} \]
\[ \begin{cases} 
(\pi, u\pi), \ (u, \pi, u\pi) & \text{if } p^m \equiv 1 \pmod{4}, \\
(\pi, \pi), \ (u, \pi, \pi) & \text{if } p^m \equiv 3 \pmod{4}. \tag{100} \end{cases} \]
Proof. Let us show that the forms (98) give all possible invariant triples \((r, d, c)\).

The forms (98) with \(t = 1\) and \(a_1 \in \{1, u, \pi, u\pi\}\) give all possible triples \((r, d, c)\) with \(c = 1\); in particular, all possible triples with \(r = 1\).

The remaining forms (98) have the Hasse invariant \(c = -1\) since:

- \((u, \pi)_{F} = -1\) by [4, p. 53, Case 2].
- \((\pi, \pi)_{F} = -1\) by \([4, p. 53, Case 3]\). Thus, \((\pi, \pi)_{F} = -1\) if \(p^m \equiv 3 \mod 4\).
- \((\pi, u\pi)_{F} = -1\) if \(p^m \equiv 1 \mod 4\).

In particular, we have 3 invariant triples with \(r = 2, c = -1,\) and

\[
d \in \begin{cases} 
{u\pi, \pi, u} \pmod{\text{squares}} & \text{if } p^m \equiv 1 \pmod{4}, \\
{u\pi, \pi, 1} \pmod{\text{squares}} & \text{if } p^m \equiv 3 \pmod{4}.
\end{cases}
\]

But if \(r = 2\) and \(c = -1\) then \(d\) can have only 3 values (mod squares) since \(d \neq -1\) (mod squares); thus, we have all possible invariant triples with \(r = 2\).

We have all possible invariant triples with \(r \geq 3\) and \(c = -1\), since then \(d \in \{1, u, \pi, u\pi\} \pmod{\text{squares}}\).

5.1 Irreducible polynomials over \(\mathbb{Q}_p\)

Let \(f(x) \in \mathbb{Z}_p[x]\) be a monic polynomial whose reduction modulo \(p\) is irreducible in \(\mathbb{F}_p[x]\). Then \(f(x)\) is irreducible over \(\mathbb{Q}_p\). [5, Corollary 5.3.8]

The Eisenstein criterion. Suppose that the polynomial \(f(x) = x^n + a_1 x^{n-1} + \cdots + a_n \in \mathbb{Z}_p[x]\) satisfies the conditions \(p|a_i\) for all \(i\) and \(p^2 \nmid a_n\). Then \(f(x)\) is irreducible over \(\mathbb{Q}_p\). [1, Theorem 5.5].

Let \(n\) and \(m\) be coprime natural numbers. Then the polynomial \(x^n - p^m\) is irreducible over \(\mathbb{Q}_p\). [1, Theorem 5.3].

5.2 Hermitian forms over local rings

Theorem 5.2. Let \(\mathbb{F}\) be a finite extension of \(\mathbb{Q}_p\), \(p \neq 2\), with a fixed non-identity involution. Let \(\mathbb{F}_0\) be the fixed field with respect to this involution. Let \(u \in \mathcal{O}(\mathbb{F}_{0})^x \setminus \mathbb{F}_{0}^{\times 2}\) be a unit that is not a square, and \(\pi\) be a prime element.
of $\mathbb{F}_o$. Then each regular (= with nonzero determinant) Hermitian form over $\mathbb{F}$ is equivalent to either

$$\bar{x}_1y_1 + \cdots + \bar{x}_ny_n,$$

or

$$\begin{cases} 
\pi \bar{x}_1y_1 + \bar{x}_2y_2 + \cdots + \bar{x}_ny_n & \text{if } \mathbb{F} = \mathbb{F}_o(\sqrt{u}), \\
\bar{x}_1y_1 + \bar{x}_2y_2 + \cdots + \bar{x}_ny_n & \text{if } \mathbb{F} = \mathbb{F}_o(\sqrt{\pi}) \text{ or } \mathbb{F}_o(\sqrt{u\pi}).
\end{cases}$$

Proof. Since $(\mathbb{F} : \mathbb{F}_o) = 2$, we have $\mathbb{F} = \mathbb{F}_o(\alpha)$, where $\alpha$ is a root of $f(x) = x^2 + 2ax + b \in \mathbb{F}_o[x]$. Write $\lambda := \alpha + a$, then $(\lambda - a)^2 + 2a(\lambda - a) + b = \lambda^2 - a^2 + 2 = 0$.

Therefore, we can take $\alpha$ such that $\alpha^2 = \beta \in \mathbb{F}_o$. Moreover, $(\alpha a)^2 = \beta a^2$ for each $a \in \mathbb{F}_o$. But $\mathbb{F}_o^\times /\mathbb{F}_o^{\times 2}$ consists of 4 cosets, represented by $1, u, \pi, u\pi$. Hence, we can take $\alpha$ such that

$$\alpha^2 = \beta \in \{1, u, \pi, u\pi\},$$

then $\mathbb{F}$ is $\mathbb{F}_o(\sqrt{u})$, or $\mathbb{F}_o(\sqrt{\pi})$, or $\mathbb{F}_o(\sqrt{u\pi})$. Since $\bar{\alpha}^2 = \beta$, the involution on $\mathbb{F}$ is $c + d\alpha \mapsto a - d\alpha$, $c, d \in \mathbb{F}_o$. The element

$$N(c + d\alpha) = (c + d\alpha)(c - d\alpha) = c^2 - d^2\beta \in \mathbb{F}_o$$

is the norm of $c + d\alpha$. The set $N(\mathbb{F}_o^\times)$ of norms of all nonzero elements is a group. By [13, Ch. 6, Fact 4.3], the norm residue group $\mathbb{F}_o^\times /N(\mathbb{F}_o^\times)$ consists of 2 elements.

- Let $\mathbb{F} = \mathbb{F}_o(\sqrt{u})$. Then $\alpha^2 = u$, $N(c + d\alpha) = c^2 - d^2u$. If $\pi \in N(\mathbb{F}_o^\times)$ then there is $c + d\alpha$ such that $N(c + d\alpha) = c^2 - d^2u = \pi$. Then $c^2 - d^2u = 0 \mod \pi$; i.e. $u = (c/d)^2 \mod \pi$. A contradiction. Therefore, the cosets of $\mathbb{F}_o^\times /N(\mathbb{F}_o^\times)$ are represented by $1, \pi$.

- Let $\mathbb{F} = \mathbb{F}_o(\sqrt{\pi})$. Then $\alpha^2 = \pi$, $N(\alpha) = -\pi$. But $-\pi$ is a prime element too, so $1, u, -\pi, -u\pi$ represent 4 cosets of $\mathbb{F}_o^\times /\mathbb{F}_o^{\times 2}$. Thus, the cosets of $\mathbb{F}_o^\times /N(\mathbb{F}_o^\times)$ are represented by $1, u$.

- Let $\mathbb{F} = \mathbb{F}_o(\sqrt{u\pi})$. Then $\alpha^2 = u\pi$, $N(\alpha) = -u\pi$. But $-u\pi$ is a prime element too. Thus, the cosets of $\mathbb{F}_o^\times /N(\mathbb{F}_o^\times)$ are represented by $1, u$.

Let $\phi(x, y) = \alpha_1\bar{x}_1y_1 + \cdots + \alpha_n\bar{x}_ny_n$ be a regular (all $\alpha_i$ are nonzero) Hermitian form over $\mathbb{F}$. Then the determinant $\det(\phi) := a_1 \cdots a_nN(\mathbb{F}_o^\times) \in \mathbb{F}_o^\times /N(\mathbb{F}_o^\times)$ is an invariant of $\phi(x)$. By [13, Ch. 10, Example 1.6(ii)], regular Hermitian forms over $\mathbb{F}$ are classified by dimension and determinant. \(\square\)
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