Spherically symmetric solutions in four-dimensional Poincaré gravity with non-trivial torsion

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We study a four-dimensional gauge theory of the Poincaré group with topological action which generalizes some well-known two-dimensional gravity models. We classify the spherically symmetric solutions and discuss the perturbative propagation of excitations around flat spacetime.

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1. Introduction

Two-dimensional models of gravity based on the gauge theory of the Poincaré group or one of its generalizations have been extensively studied in recent years [1]. The main reason for this interest resides in the fact that two-dimensional models are much easier to handle than four-dimensional ones, in particular in what concerns the issue of quantization and can therefore be used as toy models of four-dimensional general relativity.

However, lower-dimensional models differ in several respects from the four-dimensional gravity that they should imitate. For example, no propagating degree of freedom is present in the spectrum. This fact can be ascribed to the topological nature of the gravitational action in two dimensions. One may wonder if these peculiarities spoil the analogy with higher-dimensional gravity.

For this reason, we find interesting to investigate some aspects of the most direct generalization to four dimensions of the standard two-dimensional models of gravity and discuss its differences from general relativity. This generalization was introduced some time ago in ref. [2], where it was shown that a gauge theory of the Poincaré group with action of topological form analogous to that used in two dimensions, can be defined also in four dimensions. This action contains a multiplet of scalar fields in addition to the geometric variables and is quadratic in the curvature and the torsion.

In a previous paper [3], we have studied the riemannian sector of this model, where the torsion was set to zero. In contrast with the two-dimensional case, however, in four dimensions the vanishing of torsion is not a consequence of the field equations and hence we can extend the previous investigations to the case of non-trivial torsion. Of course, the elimination of the constraint of zero torsion may enlarge the number of degrees of freedom of the theory and change its spectrum.

In this paper, we study in the general case the spherically symmetric solutions of the field equations and the propagation of the excitations around flat space. The main results of our investigations is that, in contrast with general relativity, where the Birkhoff theorem states that there is a unique family of spherically symmetric solutions, a large class of solutions is available, which in general may depend on arbitrary functions of the radial coordinate. Moreover, we show that, in spite of the richer structure of the theory with
respect to the riemannian limit of ref. [3], also in this case no propagation of excitations takes place in Minkowski spacetime.

The paper is organized as follows: in section 2 we introduce the model and write down the field equations. In section 3 we impose a spherically symmetric ansatz and in section 4 classify all the possible solutions with this symmetry. In section 5 we discuss the perturbative propagation around the flat solution, while in the last section we make some final remarks.

2. Gauge theories of gravity in 4 dimensions

The four-dimensional Poincaré group is isomorphic to $ISO(1,3)$, with generators $M^{AB} = \{M^{ab}, M^{a4} \equiv P^a\}$, where $A,B = 0,\ldots,4$; $a,b = 0,\ldots,3$. The generators satisfy the usual commutation relations

$$[M^{AB}, M^{CD}] = h^{AC} M^{BD} - h^{AD} M^{BC} + h^{BD} M^{AC} - h^{BC} M^{AD}$$

with $h^{AB} = \text{diag} (-1,1,1,1,0)$.

As in standard Yang-Mills theory, local invariance under the Poincaré group can be enforced by introducing a gauge connection one-form $A^{AB}$ with field strength 2-form $F^{AB} = dA^{AB} + A^{AC} A^{CB}$. A gauge-invariant action of topological form can then be constructed making use of the totally antisymmetric group invariant tensor $\epsilon^{ABCDE}$. As in all even-dimensional models, one must further introduce a multiplet of scalar fields $\eta^A$, in the fundamental representation of the gauge group $ISO(1,3)$. The action can then be written as† [2]:

$$I = \int_{M_4} \epsilon^{ABCDE} \eta^A F^{BC} F^{DE} \tag{1}$$

In order to make contact with gravitation, one can then make the identifications $A^{ab} = \omega^{ab}$, $A^{a4} = e^a$, where $\omega^a_b = \omega^a_{b\mu} dx^\mu$ and $e^a = e^a_\mu dx^\mu$ are the spin connection and vierbein 1-forms of the four-dimensional manifold. These identifications imply that $F^{ab} = R^{ab}$, $F^{a4} = T^a$, where $R^{ab}$ and $T^a$ are the curvature and the torsion 2-forms of the 4-dimensional manifold, which are defined respectively as:

$$R^a_b = d\omega^a_b + \omega^b_c \omega^a_c$$

$$T^a = de^a + \omega^a_b e^b \tag{2}$$

† In the following we shall adopt the notations of [4] and omit the wedge signs.
and satisfy the Bianchi identities

\[ DT^a \equiv dT^a + \omega_b^a T^b = R^a_b e^b \]
\[ DR^a_b \equiv (DR + \omega R - R\omega)^a_b = 0 \]  \hspace{1cm} (3)

where \( D \) denotes the covariant derivative.

In terms of the geometrical quantities, the action (1) takes the form

\[ I = \int_{M_4} L = \int_{M_4} \epsilon_{abcd} \left[ \eta^4 R^{ab} R^{cd} + 4\eta^a T^b R^{cd} \right] \]  \hspace{1cm} (4)

For future reference, we remark that, alternatively, the curvature and the torsion can be written in terms of tensors in an orthonormal basis as

\[ R^{ab} = R^{ab} cde \quad T^a = T^a_b e^b e^c \]

In this formalism, the action reads

\[ I = 4 \int e d^4 x [\eta^4 (R^{abcd} R^{cdab} - 4R^{ab} R^{ba} + R^2) + 4\eta^a (T^{cd} R^{ab} - 2T^{ac} R^c_b + 2T^e R^c_a - T_a R)] \]  \hspace{1cm} (5)

where \( e = \det e^\mu_\mu \) and \( R^a_b = R^a_{bc}, \ R = R^a_a, \ T_a = T^b_{ab}. \) (Notice that the ordering of indices is essential).

In order to evaluate the field equations, we vary the action with respect to the independent fields \( e, \omega, \eta^a \) and \( \eta^4. \) From the definition of the curvature, it follows that a variation \( \delta\omega \) of the connection induces a variation of the curvature given by

\[ \delta R^{ab} = D\delta \omega^{ab} = (d\delta \omega + \omega \delta \omega + \delta \omega \omega)^{ab} \]  \hspace{1cm} (6)

Using (6), the variation of the lagrangian \( L \) can be written as

\[ \delta L = \epsilon_{abcd} \left[ R^{ab} \delta R^{cd} - \eta^4 T^{ab} \delta \eta^a + 2(\eta^4 R^{ab} + 2\eta^a T^b) D\delta \omega^{cd} \right. \\
- \left. 4\eta^a e^h R^{cd} \delta \omega^h + 4\eta^a R^{cd} D \delta e^b \right] \]  \hspace{1cm} (7)

One can rearrange the \( \omega \)-variation by noticing that, by definition of covariant derivative,

\[ D(\epsilon_{abcd} \eta^4 R^{ab} \delta \omega^{cd}) \equiv d(\epsilon_{abcd} \eta^4 R^{ab} \delta \omega^{cd}) = \epsilon_{abcd} (d\eta^4 R^{ab} \delta \omega^{cd} + \eta^4 R^{ab} D\delta \omega^{cd}) \]
\[ D(\epsilon_{abcd} \eta^a T^{b} \delta \omega^{cd}) \equiv d(\epsilon_{abcd} \eta^a T^{b} \delta \omega^{cd}) + \eta^a \omega^h \epsilon_{hbc} T^b \delta \omega^{cd} \]
\[ = \epsilon_{abcd} (d\eta^a T^{b} \delta \omega^{cd} + \eta^a R^{b} e^h \delta \omega^{cd} + \eta^a T^b D\delta \omega^{cd}) \]  \hspace{1cm} (8)
where we have made use of the Bianchi identities (3) and of the tensorial properties of \( \epsilon_{abcd} \). Similarly, for the \( e \)-variation,

\[
D(\epsilon_{abcd}\eta^a R^{cd} \delta e^b) = d(\epsilon_{abcd}\eta^a R^{cd} \delta e^b) + \eta^a \omega^h_a \epsilon_{hbcd} R^{cd} \delta e^b
\]

(9)

Substituting in (7) the values of \( D\delta \omega \) and \( D\delta e \) obtained from (8) and (9) and discarding the total derivatives, since the variation of the fields are independent it follows that

\[
\begin{align*}
\epsilon_{abcd} R^{ab} R^{cd} &= 0 \\
\epsilon_{mbcd} T^{b} R^{cd} &= 0 \\
\epsilon_{mbcd} D\eta^b R^{cd} &= 0 \\
\epsilon_{mncd} (d\eta^d R^{cd} + 2D\eta^c T^d + 2\eta^c R^d h) - \epsilon_{mbcd} \eta^b R^{cd} e_n + \epsilon_{mbcd} \eta^b R^{cd} e_m &= 0
\end{align*}
\]

(10)

where we have defined \( D\eta^a \equiv d\eta^a + \omega^a_h \eta^h \).

3. The field equations

In the following, we shall be interested in static, spherically symmetric solutions of the field equations. The most general spherically symmetric ansatz, which is also invariant under reflections can be written as \((i,j,k = 2,3)^\dagger\):

\[
\begin{align*}
e^0 &= f(r) \, dt \\
e^1 &= g(r) \, dr \\
e^i &= r \, d\hat{x}^i \\
\omega^{01} &= c(r) \, dt + d(r) \, dr \\
\omega^{0i} &= a(r) \, d\hat{x}^i \\
\omega^{1i} &= b(r) \, d\hat{x}^i \\
\omega^{ij} &= \frac{1}{2} (x^i d\hat{x}^j - x^j d\hat{x}^i)
\end{align*}
\]

(11)

where \( r \) is the radial coordinate and \( d\hat{x}^i = (1 + x^k x_k/4)^{-1} dx^i \). Moreover, we shall assume that \( \eta^a \) and \( \eta^4 \) depend only on \( r \).

\[^\dagger\] A more general ansatz can be obtained if one does not require reflection invariance. In this case \( \omega \) can depend on eight independent functions instead of four [5].
In terms of these variables, the curvature and torsion 2-forms are:

\[ R^{01} = c' \, dr \, dt \]
\[ R^{1i} = (b' + ad) \, dr \, d\tilde{x}^i + ac \, dt \, d\tilde{x}^i \]
\[ R^{0i} = (a' + bd) \, dr \, d\tilde{x}^i + bc \, dt \, d\tilde{x}^i \]
\[ R^{ij} = (1 + a^2 - b^2) \, d\tilde{x}^i \, d\tilde{x}^j \]
\[ T^0 = (f' - cg) \, dr \, dt \]
\[ T^1 = df \, dr \, dt \]
\[ T^i = (1 + bg) \, dr \, d\tilde{x}^i - af \, dt \, d\tilde{x}^i \]

where a prime denotes derivative with respect to \( r \).

In order to obtain the spherically symmetric solutions, one must now substitute these expressions into the field equations (10). The independent equations so obtained are listed below:

\[(1 + a^2 - b^2)c' + 2(aa' - bb')c = 0 \quad (13.a)\]
\[(1 + a^2 - b^2)df + 2ac(1 + bg) + 2a(b' + ad)f = 0 \quad (13.b)\]
\[(1 + a^2 - b^2)(f' - cg) + 2bc(1 + bg) + 2a(a' + bd)f = 0 \quad (13.c)\]
\[(1 + a^2 - b^2)(\eta^1' + d\eta^0) + 2(b' + ad)(a\eta^0 - b\eta^1 - \frac{1}{4}x_k\eta^k) = 0 \quad (13.d)\]
\[(1 + a^2 - b^2)(\eta^0' + d\eta^1) + 2(a' + bd)(a\eta^0 - b\eta^1 - \frac{1}{4}x_k\eta^k) = 0 \quad (13.e)\]
\[[(1 + a^2 - b^2)\eta^0 + 2a(a\eta^0 - b\eta^1 - \frac{1}{4}x_k\eta^k)]c = 0 \quad (13.f)\]
\[[(1 + a^2 - b^2)\eta^1 + 2b(a\eta^0 - b\eta^1 - \frac{1}{4}x_k\eta^k)]c = 0 \quad (13.g)\]
\[ac\eta^i' = bcn\eta^i' = 0 \quad (13.h, i)\]
\[(a^2 - b^2)c\eta^i = 0 \quad (13.j)\]
\[c'\eta^i = 0 \quad (13.k)\]
\[[a(b' + ad) - b(a' + bd)]\eta^i = 0 \quad (13.l)\]
\[(ac\eta^0)' = (bcn\eta^1)' \quad (13.m)\]
\[(f' - cg)\eta^i = df\eta^i = 0 \quad (13.n, o)\]
\[(a^2f + rbc)\eta^i = a(bf + rc)\eta^i = 0 \quad (13.p, q)\]
\[ [r(a' + bd) - a(1 + bg)]\eta^i = [r(b' + ad) - b(1 + bg)]\eta^i = 0 \] (13.r, s)

\[ bcn\eta'^0 - af\eta' \eta^0' - (bdf - acg + a'f + af')\eta^0 - (3bcg + adf + c - bf')\eta^1 = 0 \] (13.t)

\[ acn\eta'^1 - af\eta'^1 - (acg - bdf)\eta^1 - (3adf + bcg + c + b'f)\eta^0 = 0 \] (13.u)

\[ af\eta'^0 - 2(acg + bdf + a'f)\eta^i = 0 \] (13.v)

\[ [(1 + a^2 - b^2)\eta^0 + 2a(an^0 - bn^1 - \frac{1}{4}x_k\eta^k)]f = 0 \] (13.w)

\[ (1 + a^2 - b^2)(\eta'^0 - g\eta^1) - 2(1 + bg)(an^0 - bn^1 - \frac{1}{4}x_k\eta^k) = 0 \] (13.x)

4. Spherically symmetric solutions

In spite of the huge number of equations to be solved, it turns out that in general they are not sufficient to determine uniquely all the eleven variables \(a, b, c, d, f, g, \eta^a, \eta^4\), but almost all solutions depend on arbitrary functions. This is similar to what occurs in the riemannian limit [3], where in general the solutions depend on one arbitrary function.

In the following, we shall be interested only in the solutions with \(f, g \neq 0\), since they can be given a geometrical interpretation in terms of spacetime, but it must be stressed that one can also find many solutions with vanishing metric functions, which may be interesting from the point of view of topological field theory.

One can make some general considerations about the solutions of the system (13). If \(\eta^i\) vanishes, many of the field equations are automatically satisfied. The case \(\eta^i \neq 0\) is therefore quite special. Let us examine it in more detail: in the hypothesis \(f \neq 0\), eqs. (13.n, o) imply that \(f' = cg\) and \(d = 0\). Moreover, if \(\eta^i\) does not depend on \(x^k\), the terms containing \(x_k\eta^k\) in the field equations must vanish. This is possible iff \(a = 0, b = \text{const} \neq 0, g = -b^{-1}\) and \(c = 0\), which in turn implies \(f = \text{const}\). The remaining equations admit two solutions: either \(b^2 = 1\) (case A.1), or \(b^2 \neq 1\) with \(\eta^0 = \eta^1' = 0, \eta^1'' = \eta^1/b = \text{const}\) (case A.2). The first case is the most interesting since it corresponds to flat spacetime with vanishing torsion.

Let us consider now the case \(\eta^i = 0\). First, we assume that \(1 + a^2 - b^2\) does not vanish. In this case, eq. (13.a) can be integrated to yield

\[ c = \frac{A}{1 + a^2 - b^2} \] (14)
with $A$ an arbitrary constant. (The possibility of integrating this equation is a consequence of the fact that the Gauss-Bonnet term in the action is a total derivative in four dimensions). Moreover, if $c \neq 0$, eqs. (13.f,g) can be combined as

$$
(1 + a^2 - b^2)(b\eta^0 - a\eta^1) = 0
$$

(15.a)

$$
(1 + 3a^2 - 3b^2)\eta^0 = (1 + 3a^2 - 3b^2)\eta^1 = 0
$$

(15.b)

The equations (15) admit two different solutions, which we shall denote B.1 and B.2: in the first case, $\eta^0 = \eta^1 = 0$, while in the second case, $1 + 3a^2 - 3b^2 = 0$ and $b\eta^0 = a\eta^1$. In case B.1, all equations except (13.b,c) are automatically satisfied if $\eta^4 = \text{const}$; using the remaining equations one can hence determine two functions, say $b$ and $d$, in terms of the others and is left with three arbitrary functions $a$, $f$ and $g$.

In case B.2, the solutions are harder to find. Combining (13.d,e) with (13.f,g) one can show that $\eta^0 = Ba$, $\eta^1 = Bb$, with $B$ an integration constant. Substituting this result in (13.x) one gets $\eta^4 = -B$. With these values for the scalar fields, making use of the condition $1 + 3a^2 - 3b^2 = 0$, eqs. (13.b,c,t,u) reduce to three independent equations, which can be written as

$$
a f' - acg - bdf = 0 \hspace{1cm} b f' - bcg - adf = 0 \hspace{1cm} (16.a,b)
$$

$$
b f' + b' f + c = 0 \hspace{1cm} (16.c)
$$

Taking into account the relation between $a$ and $b$, the first two equations are solved by $df = f' - cg = 0$, while integrating the third after noticing that, due to (14), $c = \text{const} = 3A/2$, one gets

$$
f = -\frac{1}{b} \left( \frac{3A}{2} r + C \right)
$$

(17)

with $C$ an integration constant.

We pass now to consider the case in which $1 + a^2 - b^2 = 0$, which we call B.3. In this case eq. (13.a) is satisfied for arbitrary $c$. Moreover, if $c \neq 0$, eqs. (13.f,g) imply $a\eta^0 = b\eta^1$. Making use of these conditions, one is left with only two independent equations, which can be written as

$$
d = -\frac{c(1 + bg)}{af} - \frac{a'}{b} \hspace{1cm} bcn^4' = af\eta^0' + \left( \frac{c(b + g)}{a} - a^2 a'f \right) \eta^0
$$

(18)
Therefore it results that in this case the five functions $a, c, f, g, \eta^0$ can be chosen arbitrarily.

There are some further special solutions to the field equations which correspond to the vanishing of some of the functions $c, d, \ldots$, but we shall not discuss them here, since they do not exhibit any particularly relevant feature.

In table 1 we have listed the solutions discussed above. The entry "any" means that the corresponding function can be chosen freely, while we have denoted by $\mathcal{F}(\ldots)$ the functional dependence on other variables implied by the field equation (of course there is some arbitrariness in choosing which variables are independent).

A common characteristic of all the solutions except B.1 is that some of the components of the curvature or the torsion vanish. In table 2 we have listed the vanishing components for the given solutions. Some special cases have physical relevance. For example, if one imposes vanishing torsion, one can recover the solutions of ref. [3]. In fact, vanishing torsion implies $a = d = 0, c = f'/g, b = -1/g$. Substituting in the field equations one obtains four possible classes of solutions (here we disregard the scalar fields):

1) arbitrary $f, g^2 = 1$;
2) arbitrary $g, f = \text{const}$;
3) $f = Ar + B, g^2 = 3$;
4) arbitrary $g, f = A(g^2 - 1)/g$;

These are exactly the solutions found in ref. [3].

Also interesting is the existence of solutions with vanishing curvature, but non-trivial torsion. These are obtained whenever $b^2 = 1 + a^2, c = 0, d = -a'/b$ and are therefore a special case of the solutions B.3. No further restrictions are imposed on $f$ and $g$ by the field equations.

5. Perturbative degrees of freedom

The study of the propagation of excitations around a given ground state is useful in order to investigate the physical content of the theory, and in particular the number of degrees of freedom. Due to the topological nature of the model we are considering, we expect no propagating degrees of freedom to appear in the spectrum.

This question can be investigated by evaluating the part of the action quadratic in the
perturbations of the fields around the ground state. For our purposes, the most suitable ground state is of course Minkowski spacetime, which we have shown to be a solution of the field equations corresponding to $f = g = 1, a = c = d = 0, b = -1$ and vanishing background values of the curvature and torsion. From the general form of our solutions, we are also induced to choose $\bar{\eta}^a = 0, \bar{\eta}^4 = \text{const}$ as background values for the scalars.

The calculations are most easily performed in an orthonormal frame in cartesian coordinates. Expanding $\omega_{ab}^\mu = \bar{\omega}_{ab}^\mu + \chi_{ab}^\mu, e^a_\mu = \bar{e}^a_\mu + h^a_\mu,$ where $\bar{\omega}_{ab}^\mu = 0, \bar{e}^a_\mu = \delta^a_\mu$ are the background values of the connection and vierbein corresponding to Minkowski spacetime, and $\chi$ and $h$ are small perturbations, one has at first order

\begin{align*}
R^{(1)}_{abcd} &= \partial_d \chi_{abc} - \partial_c \chi_{abd} \\
T^{(1)}_{abc} &= \partial_a h_{bc} - \partial_c h_{ba}
\end{align*}

with $\chi_{abc} \equiv \delta^\mu_c \chi_{ab}^\mu$ and $h_{ab} \equiv \delta^\mu_b h^a_\mu$. We also expand the scalar multiplet as $\eta^A = \bar{\eta}^A + \psi^A$.

Substituting the expansion into the action (5), one sees that the linear part of both $e(\mathcal{R}_{abcd} \mathcal{R}^{cdab} - 4 \mathcal{R}_{ab} \mathcal{R}^{ba} + \mathcal{R}^2)$ and $e(\mathcal{T}^b_{cd} \mathcal{R}^{cd}_{ab} - 2 \mathcal{T}^b_{ac} \mathcal{R}^c_b + 2 \mathcal{T}^c_c \mathcal{R}^{cb} - \mathcal{T}^a_c)$ vanishes, since these terms are quadratic in the torsion and the curvature, whose background values are null. Moreover, using (19), one can check explicitly that the quadratic part of both terms is a total derivative. From these results is easy to see that since $\bar{\eta}^4$ and $\bar{\eta}^a$ are constant, the full quadratic action is a total derivative and hence no propagation arises around flat space.

The absence of propagating degrees of freedom confirms the topological nature of the theory. Of course, choosing a different ground state, in principle one could find different properties for the propagation. A determination of the true degrees of freedom can be better performed in a hamiltonian framework.

6. Final remarks

It would be straightforward to extend our investigations to the case of de Sitter or anti-de Sitter groups. This has been done in [3] for the riemannian limit. However, we do not expect any qualitatively new feature to emerge from this generalization, even if in these cases the action contains also terms of the Einstein-Hilbert form.
A more promising development would be the investigation of the theory in the hamiltonian formalism. As in two dimensions, this should give interesting informations on the theory and its quantization and permit a rigorous determination of its degrees of freedom.

We finally remark that different versions of four-dimensional gravity with actions of topological type have been proposed [6]. These do not contain scalar fields and hence are in some sense closer to general relativity than the model considered here.

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\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|c|c|}
\hline
   & a & b & c & d & f & g & \(\eta^0\) & \(\eta^1\) & \(\eta^i\) & \(\eta^4\) \\
\hline
A.1 & 0 & \(\pm 1\) & 0 & 0 & \text{const} & \(\mp 1\) & any & any & any & any \\
A.2 & 0 & \text{const} & 0 & 0 & \text{const} & \(-b^{-1}\) & 0 & \text{const} & any & \(Ar + B\) \\
B.1 & any & \(\mathcal{F}(a, f, g)\) & \(A(1 + a^2 - b^2)^{-1}\) & \(\mathcal{F}(a, f, g)\) & any & any & 0 & 0 & 0 & \text{const} \\
B.2 & any & \(\pm \sqrt{a^2 + 1/3}\) & \text{const} & 0 & \(\mathcal{F}(a, c)\) & \(\mathcal{F}(a, c)\) & \(Ba\) & \(Bb\) & 0 & \(-Br + D\) \\
B.3 & any & \(\pm \sqrt{a^2 + 1}\) & any & \(\mathcal{F}(a, c, f, g)\) & any & any & any & \(a\eta^0/b\) & 0 & \(\mathcal{F}(a, c, f, g, \eta^0)\) \\
\hline
\end{tabular}
\caption{Classification of the solutions of the field equations.}
\end{table}

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
   & \(R^{00}\) & \(R^{01}\) & \(R^{11}\) & \(T^0\) & \(T^1\) & \(T^i\) \\
\hline
A.1 & 0 & 0 & 0 & 0 & 0 & 0 \\
A.2 & 0 & 0 & 0 & 0 & 0 & 0 \\
B.1 & & & & & & \\
B.2 & 0 & & & 0 & 0 & \\
B.3 & & & 0 & & & \\
\hline
\end{tabular}
\caption{The vanishing components of \(R\) and \(T\) for the different classes of solutions.}
\end{table}