Landau level spectroscopy of Bi$_2$Te$_3$

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(Dated: April 29, 2020)

Here we report on Landau level spectroscopy in magnetic fields up to 34 T performed on a thin film of topological insulator Bi$_2$Te$_3$ epitaxially grown on a BaF$_2$ substrate. The observed response is consistent with the picture of a direct-gap semiconductor in which charge carriers closely resemble massive Dirac particles. The fundamental band gap reaches $E_g = (175\pm5)$ meV at low temperatures and it is not located on the trigonal axis, thus displaying either six or twelvefold valley degeneracy. Notably, our magneto-optical data do not indicate any band inversion. This suggests that the fundamental band gap is relatively distant from the Γ point where profound inversion exists and gives rise to relativistic-like surface states of Bi$_2$Te$_3$.

I. INTRODUCTION

Bismuth telluride (Bi$_2$Te$_3$) is nowadays a widely explored material in the condensed-matter community. Intensive investigations of Bi$_2$Te$_3$ started more than fifty years ago and they were, to a great extent, driven by its unusual thermoelectric properties. More recently, Bi$_2$Te$_3$ appeared among the very first experimentally verified three-dimensional (3D) topological insulators which host specific relativistic-like surface states.

Despite considerable experimental effort, the electronic band structure of Bi$_2$Te$_3$ is nowadays understood only partly. The current consensus implies that Bi$_2$Te$_3$ is a narrow-gap semiconductor with inverted ordering of bands around the Γ point. Nevertheless, the width, the overall character (direct versus indirect) as well as the position of the band gap in the Brillouin zone still remain under debate. The band structure model comprising six equivalent valleys – located pairwise in the mirror planes – has been invoked in the past (see Fig. 1 and used to interpret the quantum oscillations experiments in Bi$_2$Te$_3$, due to electrons in the lowest conduction band and due to holes in the topmost valence bands). In optical experiments, both direct and indirect band gaps have been reported, usually with the width below $E_g \sim 200$ meV.

Available theoretical studies neither provide us with a consistent view on the electronic bands in Bi$_2$Te$_3$. Presumably more accurate GW calculations predict multiple extrema of the highest valence band and rather indirect band gap. The deduced band gap magnitude appears to be strongly dependent on the used functional and it falls into a relatively broad range between 50 and 200 meV.

In this paper, we study the bulk magneto-optical response of a thin layer of Bi$_2$Te$_3$. We show that the relatively complex response, comprising a series of interband and intraband inter-Landau level (inter-LL) excitations, may be explained using a simple two-band model for a time-reversal-invariant direct-gap semiconductor. This implies that the charge carriers in Bi$_2$Te$_3$ behave, to a certain extent, as massive Dirac electrons. The selection rules for electric-dipole excitations observed experimentally allow us to deduce the symmetry of electronic bands around the fundamental band gap. In this way, we conclude that the band gap is located away from the trigonal axis, and therefore, it displays a multiple degeneracy ($N = 6$ or 12).

II. SAMPLE PREPARATION AND EXPERIMENTAL DETAILS

The studied Bi$_2$Te$_3$ epilayer with the thickness of 300 nm was grown using molecular beam epitaxy on a 1-mm-thick (111)-oriented cleaved BaF$_2$ substrate. The details about the growth technique and the sample characterization were presented previously. The Hall measurements at liquid-helium temperature, performed on the same sample as all our optical and magneto-optical studies, indicate $p$-type conductivity and the moderate hole density $n \sim 2 \times 10^{18}$ cm$^{-3}$.

To measure infrared magneto-transmission, nonpolarized radiation from a globar or a mercury lamp was analyzed by a commercial Bruker Vertex 80v Fourier-transform spectrometer. The radiation was then delivered via light-pipe optics to the sample kept in the helium exchange gas at the temperature $T = 2$ K and placed in the superconducting solenoid (up to 13 T) or resistive magnet (above 13 T). The light transmitted through the sample was detected by a composite bolometer, placed directly below the sample. The studied sample was probed in the Faraday configuration, with the magnetic
field applied along the trigonal axis (rhombohedral or c-axis) of Bi$_2$Te$_3$. The measured transmission spectra, $T_B$, were normalized by the zero-field transmission, $T_0$, and plotted in the form of relative magneto-absorbance, $A_B = -\ln[T_B/T_0]$. The optical response at $B = 0$ was deduced from the ellipsometric measurements realized using a commercial Woollam IR-VASE ellipsometer coupled to a closed He-cycle cryostat, for details see Ref. [19].

III. TWO-BAND MODEL OF A DIRECT-GAP SEMICONDUCTOR

To interpret our experimental data presented and discussed below, we adopt a simple two-band model for a time-reversal-invariant direct-gap semiconductor. The corresponding Hamiltonian may be derived, e.g., using the first-order $k \cdot p$ theory applied at a particular point, $k_0$, of the Brillouin zone:

$$h = \begin{bmatrix} \Delta & h \nu_D(q_x - i q_y) \\ h \nu_D(q_x - i q_y) & -\Delta \end{bmatrix},$$

where $q = k - k_0 = (q_x, q_y, 0)$. The Hamiltonian describing electrons and holes with an opposite spin projection in the twice degenerate bands is obtained by the complex conjugation of $h$. Notably, we consider only vanishing $q_z$ momenta because the $q_z = 0$ states dominate the overall optical response when the magnetic field is applied along the z axis.

The above Hamiltonian gives rise to the conduction and valence bands with a characteristic relativistic-like hyperbolic profile: $E(k) = \pm \sqrt{\Delta^2 + h^2 \nu_D^2 k^2}$. They are separated by the band gap of $2\Delta$ and display the full particle-hole symmetry. Interestingly, there exists a non-trivial analogy between the proposed two-band model and truly relativistic systems of massive electrons described by the Dirac equation$^{32,33}$. This analogy implies that the band-edge effective masses of electrons and holes are equal to the quantity referred to as the Dirac mass, $m_D = m_e = m_h = \Delta/\nu_D^2$. The effective (cyclotron) mass of charge carriers increases linearly with the energy distance $\varepsilon$ from the band-edge: $m_{e,h}(\varepsilon) = m_D(1 + \varepsilon/\Delta) = m_D E/\Delta$. The corresponding $g$ factors are expressed as $g_e = g_h = 2m_0/m_D$, where $m_0$ stands for the bare electron mass. For large momenta, the dispersion of charge carriers approaches the ultra-relativistic limit, $E(k) \approx \pm h \nu_D |k|$. This allows us to treat the $v_D$ parameter which describes the coupling between bands, as the effective velocity of light in the explored system.

When the magnetic field is applied, the electronic band structure transforms into Landau levels with the spectrum: $E_n = \pm \sqrt{2eBn + \Delta^2}$, where $n > 0$. In addition, there exists a pair of spin-polarized zero-mode LLs ($n = 0$) with the energies $E_0 = \Delta$ and $E_0 = -\Delta$, which correspond to the $h$ and $h^*$ Hamiltonians, respectively. Our model, characterized by the full rotational symmetry around the $z$-axis, implies the standard selection rules, $n \to n \pm 1$, for electric-dipole excitations in the Faraday configuration (Fig. 2). A closer analysis$^{34,35}$ shows that the excitations within the LL spectra of the $h$ and $h^*$ Hamiltonians dominantly follow the selections rules $n \to n - 1$ and $n \to n + 1$ and they are active in $\sigma^-$.
and $\sigma^+$ polarized light, respectively. This remains valid until the energy of excitations exceeds, by an order of magnitude, the band gap of $2\Delta$.

Let us now reconcile the proposed toy model of massive Dirac electrons with the real band structure of Bi$_2$Te$_3$. We align the $z$ axis in the model with the trigonal axis of Bi$_2$Te$_3$. The bands considered in the model are associated with the lowest lying conduction and the topmost valence bands which are primarily formed from $p$-like states of bismuth ($6p$) and tellurium ($5p$). The electronic bands at higher or lower energies are completely neglected. Even though we deal with a material proven to be a topological insulator, the band inversion, putting tellurium states above bismuth ones, does not have to be present around an arbitrarily chosen point $k_0$ in the Brillouin zone, especially for the momenta far from the $\Gamma$ point. Notably, the position of the $k_0$ point in the Brillouin zone (see Fig. 1) defines the valley degeneracy $N$. When the possibility of accidental degeneracy is neglected, the valley multiplicity reaches $N = 1$ when the $k_0$ point coincides with the $\Gamma$ or $Z$ points, $N = 2$ for $k_0$ located on the trigonal axis (between $Z$ and $\Gamma$ points), $N = 6$ for the valleys located in the mirror planes or on the binary axis (perpendicular to the trigonal axis) and $N = 12$ for a general position in the Brillouin zone.

Importantly, the symmetry of the crystal and the position of the $k_0$ point in the Brillouin zone (see Fig. 1) may have a profound impact on the related magneto-optical response. The particular location of the $k_0$ point in the Brillouin zone implies a specific form of high-order momentum terms that result from the $k \cdot p$ expansion. Often, these terms do not impact the LL spectrum significantly, and therefore, they are not included in the Hamiltonian. Nevertheless, the lowered rotational symmetry gives rise, in principle, to additional sets of electric-dipole transitions.

For the point $k_0$ located on the trigonal axis, the angular momentum of an absorbed photon is conserved only modulo $\frac{1}{3}$. Therefore, additional sets of electric-dipole transitions may emerge in the excitations spectrum: $n \rightarrow n \pm 2, 5, 7, \ldots$. The situation is similar to $K$ point electrons in bulk graphite for which the pronounced trigonal warping does not alter significantly the LL spectrum, but it gives rise to a series of cyclotron resonance (CR) harmonics that would be strictly forbidden in the electric-dipole approximation in system with a full rotational symmetry. Further sets of inter-LL excitations may appear for the $k_0$ point located away from the trigonal axis of Bi$_2$Te$_3$ – for instance, interband transitions that conserve the LL index: $n \rightarrow n$.

IV. EXPERIMENTAL DATA AND DISCUSSION

The magneto-optical data collected on the studied Bi$_2$Te$_3$ epilayer at $T = 2$ K in the middle infrared spectral range are presented in Figs. 3a, b, as a stacked-plot and false-color plot of relative magneto-absorbance spectra, $A_B = -\ln[T_B/T_0]$, respectively. The data comprise a series of pronounced resonances that follow a sub-linear in $B$ dependence, and in the limit of the vanishing magnetic field, extrapolate to a finite (positive) energy. We interpret these resonances as inter-LL excitations that promote electrons across the band gap. Notably, the res-
FIG. 4. The false-color plot of the second derivative of relative magneto-absorbance in the middle infrared spectral range, $d^2/d\omega^2(-\ln[T_B/T_0])$. The solid/dashed lines correspond to the theoretically expected positions of interband inter-LL transitions that follow three different selection rules: $n \to n \pm 1$ (green), $n \to n \pm 0$ (red) and $n \to n \pm 2$ (cyan).

Resonances have pronounced high-energy tails that are typical of interband inter-LL excitations in bulk systems. A more detailed analysis shows the presence of additional transitions with a considerably weaker intensity. These latter transitions become clearly visible in the second derivative of relative magneto-absorbance, plotted as a false-color plot in Fig. 4. Since the observed resonances are relatively sharp, we associate the minima in $d^2A_B/d\omega^2$ curves directly with the positions of excitations. No resonances attributable to surface states were identified, in contrast to preceding studies.

Let us now compare our experimental data with expectations based on the proposed two-band model. In the first step, we assign the series of dominant transitions, which contains up to eight well-resolved lines, with the position of $n \to n \pm 1$ resonances expected in a system with the full rotational symmetry. Almost perfect agreement is found for an energy band gap $E_g = 2\Delta = (175 \pm 5)$ meV and a velocity parameter $v_D = (4.7 \pm 0.1) \times 10^5$ m/s, as shown by green solid lines in Fig. 4. In this way, all dominant lines may be explained using only one widely tunable parameter $v_D$. This is because the band gap $2\Delta$ falls into fairly narrow interval given by the zero-field extrapolation of lines.

The parameters $\Delta$ and $v_D$ deduced from the fit of dominant interband transitions allows us to predict the position and $B$-dependence of the fundamental CR mode, 1 $\to$ 0, in our p-type sample (cf. Fig. 2). The expected position matches perfectly the experimental data (red solid line in Fig. 5) without any additional adjustment of $v_D$ or $\Delta$. Notably, the cyclotron energy deviates only weakly from a linear dependence in $B$. This allows us to describe it approximately by the formula for the cyclotron energy of a parabolically dispersing particle: $h\omega_c = eB/m_D$ using the Dirac mass of $m_D = 0.07m_0$ (dashed gray line in Fig. 5). This value is in very good agreement with preceding magneto-transport studies which for holes reported the band-edge mass of 0.08$m_0$.

Importantly, our model does not include any electron-hole asymmetry. The experimentally deduced Dirac mass $m_D$ thus represents a good estimate of the band-edge mass for both, holes and electrons. Indeed, the extracted Dirac mass of $m_D = 0.07m_0$ is close to the electron band-edge mass of 0.06$m_0$ deduced in the past in quantum oscillation experiments. Importantly, this latter agreement indicates that the conduction-band minimum, hosting the final states of the observed interband inter-LL excitations, is not just a local extremum but the global one. The parameter $2\Delta$ thus corresponds to the fundamental band gap in Bi$_2$Te$_3$. Hypothetically, one can imagine that there exist other extrema of the conduction and valence bands, which are characterized by the band-edge masses identical to valleys probed in our magneto-optical experiments. In such a case, the real energy band gap might still be smaller and indirect. However, we do not find such a coincidence probable.

The nearly perfect agreement between experimental data and our simple two-band model suggests that Bi$_2$Te$_3$ is a direct-gap semiconductor. Strictly speaking, such a conclusion is not correct. One can never exclude a small displacement $\delta k_{c-v}$ between the extrema of the conduction and valence bands. Thanks to our low-field magneto-optical data, we may find the upper limit of such a displacement. It is approximately given by the reciprocal value of the magnetic length $\delta k_{c-v} \sim 1/l_B = \sqrt{eB/\hbar} \approx 0.07$ nm, taken at the onset of LL quantization in our sample ($B \approx 3$ T). Such a distance represents only a small fraction of the whole Brillouin zone size. This allows us to speak about the direct character of the band gap in Bi$_2$Te$_3$ with reasonable justification.

The validity of the proposed model also implies that the Zeeman splitting in Bi$_2$Te$_3$ should be equal to cyclotron energy, $E_Z = E_C$, that is typical of massive Dirac electron and the $g$ factors of electrons and holes should reach $g_e = g_h = 2m_0/m_D \approx 30$. In fact, large values of $g$ factors are expected in systems composed of heavy elements with strong spin-orbit coupling. To the best of our knowledge, no results from spin-resonance experiments on Bi$_2$Te$_3$ have been reported so far. Relatively large values for $g$ factors were estimated from quantum oscillation experiments but the concluded ratio was somewhat lower than unity: $E_Z/E_C \approx 0.5 - 0.7$. The difference may be attributed to the influence of more distant bands.
possibly existing inter-LL excitations. As a matter of fact, all additionally observed lines fit perfectly to two series: \( n \rightarrow n \pm 2 \) and \( n \rightarrow n \) marked by dashed lines in Fig. 5. These two series overly overlap for higher indices \( n \), nevertheless, at low photon energies, one may clearly distinguish two separate 2 \( \rightarrow \) 0 and 1 \( \rightarrow \) 1 lines. Let us notice that the 0 \( \rightarrow \) 2 transition should not appear due to the occupation effect in our \( p \)-type sample. The proposed two-band model, implying not more than two tunable parameters, is thus capable of explaining all intra-band and interband inter-LL excitations resolved in our magneto-optical experiment, thus cannot be located on the trigonal axis of Bi\(_2\)Se\(_3\). The valley degeneracy thus must reach either \( N = 6 \) or \( N = 12 \), unless some accidental degeneracy occurs. Both possibilities have been discussed in the literature. The former one, corresponding to the energy gap within the mirror planes of Bi\(_2\)Te\(_3\) or at the binary axes (see Fig. 1), has been concluded as more probable based on quantum oscillations experiment.\(^\text{8–10,13,48}\)

Let us confront our conclusions – about the size, nature and multiplicity of the fundamental band gap in Bi\(_2\)Te\(_3\) – with the zero-field optical response obtained using ellipsometry. In line with expectations, the deduced optical conductivity (Fig. 6) shows rather steep increase at photon energies slightly above the band gap \( E_g = 175 \text{ meV} \) estimated from our magneto-optical experiments. The optical response indicated the presence of several critical points in the explored part of the infrared spectral range, which are discussed in the appendix. The lowest one, located around \( E_g^2 \sim 188 \text{ meV} \), corresponds to the onset of interband absorption, so-called optical band gap. The difference \( E_g^2 - E_g \) is usually referred to as the Moss-Burstein (MB) shift, which is characteristic of all degenerate semiconductors and which allows us to estimate the Fermi energy in our sample. Assuming the full particle-hole symmetry, we obtain \( E_F \) below 10 meV.

This result is consistent with the Fermi energies found in quantum oscillations experiments performed on Bi\(_2\)Te\(_3\) crystals with similar hole densities.\(^\text{13}\) Similar to conventional semiconductors, the position of the absorption onset shifts to lower energies with increasing \( T \) (inset of
In this way, we may estimate that, at higher temperatures, the energy band gap shrinks roughly linearly with $T$: $E_g(\text{meV}) \approx 175 - 0.07 \times T[\text{K}]$. Interestingly, the optical conductivity remains non-zero, nearly temperature independent and flat at photon energies below $E_g$, which we assign to excitations from/to localized states in the band gap.

Above the optical band gap, $\sigma_1(\omega)$ in Bi$_2$Te$_3$ reaches significantly larger values as compared to the sister compound Bi$_2$Se$_3$ (see Fig. 6 and Ref. 19). The latter has similar (inverted) direct band gap located at the $\Gamma$ point and the electronic bands are also fairly well described by the Dirac Hamiltonian for massive electrons. The difference in $\sigma_1(\omega)$ thus must lie in specific band structure parameters, in the reduced mass $\mu = m_e m_h / (m_e + m_h)$ and/or valley degeneracy $N$, in particular. Assuming strictly parabolic profiles of bands, optical conductivity can be approximated using the textbook expression for a direct-gap semiconductor: $\sigma_1(\omega) \propto N \mu^{3/2} \sqrt{\hbar \omega - E_g}$. Comparing the reduced masses only, $\mu_{\text{Bi}_2\text{Se}_3} = 0.08 m_0$ and $\mu_{\text{Bi}_2\text{Te}_3} = 0.035 m_0$, one expects greater $\sigma_1$ for Bi$_2$Se$_3$, roughly by a factor of 3. In experimental data, however, opposite behavior is observed (Fig. 6). Above the corresponding optical band gaps, $\sigma_1(\omega)$ for Bi$_2$Se$_3$ is roughly by a factor of 5 smaller as compared to Bi$_2$Te$_3$. The valley multiplicity $N = 12$ thus appears to be more consistent with our optical data on Bi$_2$Te$_3$ as compared to $N = 6$, concluded in the above cited transport studies. Nevertheless, this rule of thumb analysis provides us with only very rough guide, especially when the pronounced anisotropy along the $c$-axis ($\mu^{3/2} \rightarrow \mu^{1/2}$) was completely neglected. The anisotropy in Bi$_2$Te$_3$ may considerably differ from that in Bi$_2$Se$_3$ and it cannot be deduced from the presented magneto-optical data, which provide us only with the in-plane (i.e., perpendicular to the trigonal axis) estimate of the velocity parameter.

The applicability of the simple massive Dirac model to the magneto-optical response of Bi$_2$Te$_3$ may be somewhat surprising. In fact, the generic two-band models for 2D or 3D topological insulators, always comprise dispersive diagonal elements, $\Delta \rightarrow \Delta + M k^2$ which account for the band inversion. Such dispersive diagonal elements are responsible for the appearance of the surface states, but they also profoundly impact bulk properties. For instance, when the magnetic field is applied, they lead to characteristic (anti)crossing of zero-mode ($n = 0$) LLs in all inverted systems.

Even though Bi$_2$Te$_3$ is a topological insulator – with inverted bands and experimentally confirmed surface states – such diagonal dispersive terms are not included in our model, which thus keeps the simplest possible massive Dirac form. This is, for instance, seen from the deduced electron and hole masses that approach very well the Dirac mass $m_D$. This contrasts with Bi$_2$Se$_3$, where the massive Dirac picture is also valid, nevertheless, the presence of the dispersive diagonal elements in the Hamiltonian enhances twice the mass of electrons and holes as compared to the Dirac mass.

To explain the absence of the dispersive terms on the diagonal of the Hamiltonian, let us note that the region with inverted ordering of bands represents only a small part of the Brillouin zone of Bi$_2$Te$_3$. It is located around the $\Gamma$ point and it does not coincide with the location of the fundamental band gap. This also implies that the velocity parameter $v_D$ deduced from our experiments on Bi$_2$Te$_3$ is not connected with the slope of the linear bands on the surface, which emerge due to the band inversion at the $\Gamma$ point. Again, this is in contrast to other topological insulators such as Bi$_2$Se$_3$ or Bi$_{1-x}$Sb$_x$ where the regions of the fundamental band gap and of the band inversion overlap, and where, the slope of the surface conical band provide us with a good estimate for the velocity parameter in the Hamiltonian describing bulk states.

V. CONCLUSIONS

We conclude that bismuth telluride is a direct-gap semiconductor with the band gap of $E_g = (175 \pm 5)$ meV at low temperatures. The performed analysis of the magneto-optical response implies that the fundamental band gap is not located at the trigonal axis, which implies its multiple degeneracy $N = 6$ or 12. We also conclude that the low-energy electronic excitations in Bi$_2$Te$_3$ are fairly well described within the model of massive Dirac electrons, which comprises only two material parameters.

ACKNOWLEDGMENTS

The authors acknowledge helpful discussions with I. Aguilera and D. M. Basko. This work was financially supported by the European Regional Development Fund Project CEITEC Nano+ (No. CZ.021.01/0.0/0.0/16_013/0001728). CzechNanoLab project LM20181110 funded by MEYS CR is also gratefully acknowledged for the financial support of the measurements at CEITEC Nano Research Infrastructure. This work was supported by the ANR DIRAC3D project (ANR-17-CE30-0023). A.D. acknowledges support by the Czech Science Foundation (GAČR) under Project No. GA20-10377S. The authors also acknowledge the support of LNCMI-CNRS, a member of the European Magnetic Field Laboratory (EMFL).

APPENDIX

To complement the magneto-optical experiments, we have also performed the ellipsometric measurements in the infrared spectral range and analyzed the obtained data using the approach adopted in our preceding study. The modeling of the optical response enabled us to determine the thickness of the film (300 nm) and the thickness of the surface roughness effective layer. As
FIG. 7. Zero-field optical data obtained by ellipsometry. Panel (a): the real and imaginary part of the dielectric function. Panel (b): the real part of the optical conductivity at 7 K. The arrows denotes the critical points. Panel (c): displays the second derivative of the real (red squares) and imaginary (blue triangles) part of the dielectric function and the model spectrum (black line). The numbers are the energies (in meV) of the related critical points.

a second step, the point by point dielectric function was obtained. The deduced real and imaginary part of the dielectric function \( \varepsilon = \varepsilon_1 + i\varepsilon_2 \) is shown in Fig. 7a. The related real part of the low-temperature optical conductivity \( \sigma_1 = \sigma_0 \omega \varepsilon_2 \) is shown in Fig. 7b.

We have analyzed the interband transitions using the critical point (CP) model \(^{20-22}\) that is widely applied to the second (or third) derivative of the dielectric function in order to enhance the CPs with respect to the background. The contribution of a parabolic CP to the second derivative is modelled as

\[
\frac{d^2 \varepsilon}{d\varepsilon^2} = A e^{i\phi}(E - E_{\text{CP}} + i\Gamma)^{n-2},
\]

where \( A \) is the amplitude, \( E_{\text{CP}} \) is the energy, \( \Gamma \) is the broadening and \( \phi \) is the phase factor. The exponent \( n \) has the values 1/2, 0, -1/2 for three-, two- and one-dimensional CP, respectively. In the simplest case of uncorrelated one-electron bands, the phase \( \phi \) takes values of the integer multiples of \( \pi/2 \). For \( A > 0 \) and 3D critical point \( (n = 1/2) \), the phases \( \phi = 0, 90, 180 \) and 270 deg correspond to \( M_1, M_2, M_3 \) and \( M_0 \) critical points, respectively. For \( A > 0 \) and 2D critical point \( (n = 0) \), \( \phi = 0, 90 \) and 180 deg correspond to a minimum \( (M_0) \), saddle point \( (M_1) \) and maximum \( (M_2) \), respectively.\(^{23}\) However, note that the phase can depart from these integer values for various reasons, e.g., when excitonic effects take place.\(^{24}\)

The model spectrum fitted to the second derivative of the real and imaginary part of the dielectric function is displayed in Fig. 7b and obtained values of parameters are shown in Table I. The main features of the second derivative are the same as the one reported in Ref.\(^{19}\) however, here the sample and data of a higher quality enabled us to resolve five critical points.

Concerning the critical point A at 188 meV, the best fit to the data was obtained using a 2D CP profile that is expected for the absorption edge influenced by the MB effect, see, e.g., the case of the Bi\(_2\)Se\(_3\) thin film in Ref.\(^{19}\). Alternatively, this CP can be modelled with a 3D critical point (albeit with a minor increase in the mean square error) yielding \( A = 30 \, \text{eV}^{-1/2}, \, E_{\text{CP}} = 188 \, \text{meV}, \, \Gamma = 17 \, \text{meV} \) and the phase \( \phi = 285 \) deg with a relatively minor departure from 270 deg of a simple \( M_0 \) CP. Obviously, for a small MB shift in the same range as broadening \( \Gamma \), a mixture of both types of critical points can be expected, which we believe, is the case of the present data. Regardless of the dimensionality, the center energy of the lowest critical point is 188 meV, which is somewhat smaller than 202 meV reported on a similar sample in Ref.\(^{19}\). This is caused presumably by a lower doping and correspondingly smaller MB shift. Indeed, the square plasma frequency \( 4.4 \times 10^6 \, \text{cm}^{-2} \) (as determined from the far-infrared reflectivity at 300 K, not shown) is

![Table I](image-url)

**TABLE I.** The values of parameters of the CP model obtained from the fit to the data shown in Fig. 7.

| label | \( A \)  | \( E_{\text{CP}} \)  | \( \Gamma \)  | \( \phi \)  | Line shape |
|-------|---------|-----------------|-------------|-----------|------------|
| A     | 7.8     | 188             | 24          | -29       | 2D         |
| B     | 21 eV\(^{-1/2}\) | 363             | 16          | 23        | 3D         |
| C     | 8 eV\(^{-1/2}\)  | 408             | 11          | 76        | 3D         |
| D     | 6 eV\(^{-1/2}\)  | 472             | 13          | 300       | 3D         |
| E     | 16 eV\(^{-1/2}\) | 575             | 15          | 60        | 3D         |
smaller compared to the value of $5.4 \times 10^6$ cm$^{-2}$ obtained on the sample studied in Ref$^{19}$. The values of $\Gamma$ in the range of tens of meV are likely related to sample inhomogeneities, e.g., fluctuations of the local hole density and related fluctuations of the magnitude of the MB effect.

The CPs at higher energies were modelled with the 3D CP profiles. The phase values of CPs labelled as B and C (see Tab. I) suggest that they correspond to the $M_1$ and $M_2$ CPs, respectively, as expected following the $M_0$ CP. The phase of the critical point D is 300 deg, which is close to 270 deg and which suggests that it corresponds to $M_0$ CP. The phase of the critical point E is 60 deg, which is closest to the $M_2$ critical point. However, it is not excluded that $M_1$ critical point is near below this energy which effectively reduces the phase from 90 deg.

The presence of another CP is manifested in the magneto-optical data by appearance of a set of additional weak inter-LL transitions at higher energies (see Fig. 8). These correspond to excitations to/from a more distant band which is not included in the simplified model in the main text. The positions of these additional lines can be reproduced using an ad-hoc invoked massive Dirac Hamiltonian with the gap of 388 meV and the velocity parameter of $6 \times 10^7$ m/s. This additional CP thus corresponds to another onset of interband excitations, and therefore, it should be of the 3D-$M_0$ type. Nevertheless, it is too weak and to be identified directly in the zero-field optical response.

FIG. 8. The false-color plot of relative magneto-absorbance in the spectral interval well above the $E_g$. The observed series of inter-LL resonance promote electrons from/to a lower/higher lying band which is not included in the two-band model presented in the main text.

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