Online Verification of Deep Neural Networks under Domain or Weight Shift

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Abstract

Although neural networks are widely used, it remains challenging to formally verify the safety and robustness of neural networks in real-world applications. Existing methods are designed to verify the network before use, which is limited to relatively simple specifications and fixed networks. These methods are not ready to be applied to real-world problems with complex and/or dynamically changing specifications and networks. To effectively handle dynamically changing specifications and networks, the verification needs to be performed online when these changes take place. However, it is still challenging to run existing verification algorithms online. Our key insight is that we can leverage the temporal dependencies of these changes to accelerate the verification process, e.g., by warm starting new online verification using previous verified results. This paper establishes a novel framework for scalable online verification to solve real-world verification problems with dynamically changing specifications and/or networks, known as domain shift and weight shift respectively. We propose three types of techniques (branch management, perturbation tolerance analysis, and incremental computation) to accelerate the online verification of deep neural networks. Experiment results show that our online verification algorithm is up to two orders of magnitude faster than existing verification algorithms, and thus can scale to real-world applications.

1 Introduction

Neural networks are widely applied to safety/security-critical applications, such as autonomous driving [1], flight control [2], facial recognition [3], social media [4], and stock trading [5]. These applications require the network to behave as expected under all circumstances. We can write these expectations in mathematical specifications, and use formal verification to check whether the network satisfies these specifications. A specification is usually encoded as an input-output property such that given arbitrary input from the input set, the output of the network falls in the safe output set. For example, for a classification network, the input set can be: images of an apple from different angles, and the safe output set is: all outputs that classify the image as apple. There are many existing works that can formally verify these input-output properties for neural networks [6].

However, deploying existing neural network verification methods to real-world applications still faces many challenges. Most existing neural network verification methods are designed for offline verification before the network is deployed. But in real-world applications, there are two cases that offline verification cannot handle in advance. The first case is when the specification is data-dependent and the data distribution is time-varying. If we want to perform offline verification in this case, the offline specification must include all possible data distributions, which can form an extremely large specification that may essentially cover the whole input space and is computationally intractable to
All these challenges motivate online verification. Instead of verifying a hard and complex problem offline as a whole, we decompose it into many online problems, which are dependent to each other temporally. We consider two common temporal dependencies: domain shift and weight shift. Then we propose three acceleration principles and derive several acceleration algorithms by applying these principles to different temporal dependencies.

verify. For example, to prove the robustness of an image classifier (e.g., a dog classifier), it is not enough to just show that the classifier can tolerate certain input perturbations on images sampled from a fixed distribution [7,8] (e.g., all dogs from ImageNet). That is because when the classifier is deployed in the real world, its input data distribution can shift (e.g., more dog images from more diverse angles). Such a shift should be considered in the verification to ensure the robustness of the classifier after deployment. However, it is challenging to foresee all possible data distributions (e.g., all possible images including dogs); and even if they can be foreseen, it would be computationally intractable to verify against all possible data distributions. The second case is when the neural network evolves after deployment. In fact, many applications require the neural network to adapt online [9]. For example, a behavior prediction network may adapt to the subject’s personality, and a meta learned robot adapts to unseen tasks [10]. If the potential online evolution is not considered, the offline verification results cannot guarantee the safety or robustness of the neural network during online execution. On the other hand, it is computationally intractable to make the offline verification cover all possible online evolution. New methods are needed to address these challenges.

All these challenges motivate online verification. Instead of verifying a hard and complex problem once for all, we can verify a sequence of time-varying problems on the fly to provide guarantees that the neural network is safe and robust to use now and in the near future. The specification for these online problems can be centered around the current data distribution instead of covering all possible data distributions, which addresses the first challenge. For example, instead of verifying that a human behavior prediction network is robust to any arbitrary human subject, we can verify that it is robust to the current subject. Moreover, in online problems, we only need to verify the current neural network under the current specification instead of verifying all possible neural networks and specifications that may be evolved online, which then addresses the second challenge.

Nevertheless, the trade-off introduced by turning offline verification into online verification is that the online problems need to be verified in real-time when either the specification or the network changes. Real-time verification is still challenging for existing verification algorithms. On the other hand, the online problems exhibit temporal dependencies (e.g., a video stream contains a sequence of correlated images that follow similar distributions, and the parameters in a network change incrementally during online adaptation). We can exploit these temporal dependencies to accelerate the verification process, e.g., by warm starting new online verification using previous verified results. In this work, we consider two common types of temporal dependency: temporal dependency in the input data under domain shift and temporal dependency in the network parameters under weight shift. We first analyze the bottleneck of existing verification algorithms in terms of computational efficiency and propose three general principles to accelerate the verification for online problems: branch management (to reuse previously verified results for certain input areas or to merely reuse previous partitions of input areas), perturbation tolerance analysis (to check if the previously verified robustness margin can tolerate the current changes in the specification or the network parameters), and incremental computation (to reuse previously verified results for certain nodes of the neural network). Then we derive various algorithms to accelerate the verification by applying these acceleration principles to leverage different temporal dependencies under domain shift and/or weight shift. Our numerical study shows that these algorithms can accelerate the verification process up to two orders of magnitude. Our analysis and solutions are based on reachability-based verification methods, but the core algorithms can also be applied to other verification methods.

In summary, our contributions are threefold: 1) introducing online verification to greatly reduce verification difficulty for problems with time-varying specifications or time-varying networks; 2) proposing three guiding principles to accelerate existing algorithms for online problems: branch
management, perturbation tolerance analysis, and incremental computation; 3) developing algorithms to efficiently verify problems with domain shift and weight shift by applying these principles.

2 Problem formulation for Online Verification

This section provides a mathematical formulation of online verification problems progressively. We consider feedforwarding ReLU neural networks with input-output specifications.

Neural network and specifications Consider an \( n \)-layer feedforward neural network that represents a function \( f \) with input \( x \in \mathcal{D}_x \subset \mathbb{R}^{k_0} \) and output \( y \in \mathcal{D}_y \subset \mathbb{R}^{k_n}, \) i.e. \( y = f(x) \), where \( k_0 \) is the input dimension and \( k_n \) is the output dimension. Each layer in \( f \) corresponds to a function \( f_i : \mathbb{R}^{k_i} \rightarrow \mathbb{R}^{k_{i-1}} \), where \( k_i \) is the dimension of the hidden variable \( z_i \) in layer \( i \). Hence, the network can be represented by \( f = f_n \circ f_{n-1} \circ \cdots \circ f_1 \). The function at layer \( i \) is \( z_i = f_i(z_{i-1}) = \sigma_i(W_i z_{i-1}; 1) \), where \( W_i \in \mathbb{R}^{k_i \times k_{i-1}} \) is the weight matrix (including the bias term), and \( \sigma_j : \mathbb{R}^{k_i} \rightarrow \mathbb{R}^{k_i} \) is the activation function. We only consider ReLU activation in this paper. And for simplicity, we use \( W \) to denote all the weights and bias of the network. This paper considers input-output specifications that for all \( x \in \mathcal{X} \), we require \( y = f(x) \in \mathcal{Y} \). For simplicity, we assume that \( \mathcal{X} \subset \mathbb{R}^{k_0} \) is a polytope defined by \( m_{x} \) linear constraints. And \( \mathcal{Y} \subset \mathbb{R}^{k_n} \) is the output safe set defined by \( m_{y} \) linear constraints. Denote \( \{ y : y = f(x), \forall x \in \mathcal{X} \} \) as \( f(\mathcal{X}) \). Then the specification can be written as \( f(\mathcal{X}) \subseteq \mathcal{Y} \).

Online verification In real applications, the input-output specification and the network weights may change with time. The verification problem for time-varying systems is defined as \( \mathcal{P}\{t_0, \mathcal{X}(t), f^t(\cdot), \mathcal{Y}(t)\} \), which is a tuple of the initial time, the time-varying input set, the time-varying network, and the time-varying safe output set. Since the rate of change in \( \mathcal{X}, f \), and \( \mathcal{Y} \) highly depends on the data received online, it is difficult to fully characterize these time-varying functions offline. Hence offline verification needs to ensure the specification is satisfied in the worst-case scenarios, i.e. the output of the network given any possible input should comply with any output constraint: \( f^t(\mathcal{X}(t)) \subseteq \bigcap_t \mathcal{Y}(t), \forall t' \). This specification is conservative and can be intractable in many cases. In contrast, if the verification is performed online, then at any time step \( t \), the online specification is simply:

\[
f^t(\mathcal{X}(t)) \subseteq \mathcal{Y}(t), \forall t. \tag{1}
\]

We need to verify that (1) holds at every time step before using the output of the network in subsequent tasks. Otherwise, we need to stop the online process and repair the network. The repairing is out of the scope of this paper and will be left for future work. In the following discussion, we assume that at any time \( t \), earlier specifications are all satisfied.

Temporal dependencies To run online verification of (1) efficiently, we need to leverage temporal dependencies of the online problems. We say an online verification problem has temporal dependency if the problem at a different time \( t \) has a bounded changing rate in a certain metric. Two common types of temporal dependency are domain shift and weight shift. Domain shift corresponds to the case that the input set changes throughout time. This appears in many systems with fixed networks, such as a vision module with video input, or a robot controller with nonstationary state input. We assume that only \( \mathcal{X}(t) \) changes with time, and \( \forall t, f^t = f^{t_0}, \mathcal{Y}(t) = \mathcal{Y}(t_0) \). To measure the changing rate of the input set, we define a set distance function (maximum nearest distance) \( \Delta : \mathbb{R}^{k_1} \times \mathbb{R}^{k_2} \rightarrow \mathbb{R}^{+} \), an input set distance function \( \Delta_{in} : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^{+} \), and the maximum input set distance \( \Delta_{in}^* \in \mathbb{R}^{+} \):

\[
\Delta(S_1, S_2) = \max_{x' \in S_1} \min_{x \in S_2} \|x' - x\|, \quad \Delta_{in}(t_1, t_2) = \Delta(\mathcal{X}(t_1), \mathcal{X}(t_2)), \quad \Delta_{in}^* = \max_t \Delta_{in}(t, t + 1). \tag{2}
\]

Weight shift corresponds to the case that the network weights change. This appears in robot learning, meta-learning, and online adaptation, such as a vehicle trajectory prediction network adapting to drivers’ personalities. We assume that only \( W(t) \) changes with time, and \( \forall t, \mathcal{X}(t) = \mathcal{X}(t_0), \mathcal{Y}(t) = \mathcal{Y}(t_0) \). A special case of weight shift is network fine-tuning, where only the last layer of the network \( W_n(t) \) changes with time. To measure the changing rate of the weights, we define a network difference function \( \Delta_{net} : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^{+} \), and a one-step maximum network difference \( \Delta_{net}^* \in \mathbb{R}^{+} \):

\[
\Delta_{net}(t_1, t_2) = \max_i \|W(t_1) - W(t_2)\|_{\infty}, \quad \Delta_{net}^* = \max_t \Delta_{net}(t, t + 1). \tag{3}
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refinement is required. The reachability plus branching method refines the approximation by splitting
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branching algorithm by $\mathcal{S}(\mathcal{X}, f, Y)$. The output of $\mathcal{S}(\mathcal{X}, f, Y)$ is a set of branches $\{\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_m\}$,
where $m$ is the total number of branches. Due to the computation time and memory limit, we can
only keep a finite number of branches. The reachability algorithm then computes the reachable
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Related work To the best knowledge of the authors, there has not been any work that performs
online verification of (1) throughout time. There exist formal verification algorithms that verify (1) at
a given time step [6]. These works can be broadly categorized into optimization-based methods and
reachability-based methods. Optimization-based methods try to falsify the input-output specification.
And reachability-based methods [11][12] perform layer-by-layer reachability analysis to compute the
reachable set from the input set. Both types can be combined with branching methods (called
search-based methods in [6]) to improve efficiency and accuracy. Branching methods partition the
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methods include MaxSens [13], ReLuVal [14], and Neurify [15].

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We develop online verification algorithms based on reachability plus branching methods. In the
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discuss the procedures that we can accelerate and propose three acceleration principles. Finally, we
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specification holds if all branches comply with the output constraints i.e. $O(\mathcal{X}_i, f) \subseteq Y$, $\forall i$. 

![Figure 2: An example of reachability plus branching method on a simple neural network with one hidden layer and ReLU activation. The reachable set is computed by interval arithmetic [3], which propagates the lower bound and upper bound of neurons layer-by-layer. The input set is $[-5, 3]$, and the safe output set is $[-2, 12]$. If we directly compute the reachable set, the approximated reachable set is $[0, 13]$, which is not a subset of the safe output set. But if we split the input set into two branches $[-5, -1]$ and $[-1, 3]$ and compute the reachable sets. Then the two reachable sets $[2, 10]$ and $[0, 5]$ are both subsets of the safe output set.](image-url)
\(O(X, f) \subseteq Y\). For online verification problems, the online specification at time \(t\) can be formulated as: \(\bigcup_i O(X(t), f^i) \subseteq Y(t)\). For convenience, we denote \(O(X(t), f^i)\) by \(O_i(t)\).

In summary, the verification process with reachability plus branching methods can be concluded as the following three steps:

1. The branching algorithm \(S(X(t), f^t, Y(t))\) divides the input set \(X\) into many subsets \(X_i\), which we call branches.
2. We compute the over-approximated reachable set \(O_i(t)\) for each branch.
3. We check whether all over-approximated reachable sets comply with the output constraints \(Y(t)\).

From our experiment in section 4.2.1, without any acceleration algorithms, these three steps take 50.32%, 49.07%, and 0.61% time separately. Steps 1 and 2 are the bottleneck to achieving efficient online verification. That is because every time the setting changes, we need to construct the branches and compute the corresponding over-approximated reachable set for every branch from scratch. Our goal is to leverage temporal dependency to accelerate these steps. To accelerate step 1, we want to reuse previous branches and reduce the frequency to re-construct the branches. We propose a principle named branch management, which keeps the branches untouched as long as possible while maintaining the verification accuracy. Two examples are shown in Fig.3(a) and Fig.4. There is a trade-off between accuracy and efficiency: when the setting changes, reconstructing all branches gives the most accurate results, but is time-consuming. Branch management aims to find a balance between accuracy and speed. To accelerate step 2, we need to consider the status of a branch:

- When a branch is unchanged, that is \(X_i(t) = X_i(t-1), f^i = f^{i-1}\), and \(Y(t) = Y(t-1)\). We can directly reuse the previous result.
- When a branch is changed, either \(X_i(t) \neq X_i(t-1)\) or \(f^i \neq f^{i-1}\), we wish to directly infer the verification results by checking whether the previously verified robustness margin can tolerate the current changes, especially when the current changes are bounded by \(\Delta_{in}(t_1, t_2)\) in (2) and \(\Delta_{net}(t_1, t_2)\) in (3). An example is shown in Fig.3. This leads to the perturbation tolerance analysis principle. Formally, we want to establish several \(\delta, \epsilon\) conditions. For domain shift, we want to find \(\delta\) and \(\epsilon\) such that: \(\Delta(O_i(t), O_i(t_0)) < \epsilon\), when \(\Delta_{in}(t_1, t_2) < \delta\). And for weight shift, we want to find \(\delta\) and \(\epsilon\) such that: \(\Delta(O_i(t), O_i(t_0)) < \epsilon\), when \(\Delta_{net}(t_1, t_2) < \delta\). If \(\epsilon\) is smaller than the gap between \(Y(t_0)\) and \(O_i(t_0)\), i.e. \(\max_{y \in Y(t_0)} \min_{y' \in O_i(t_0)} \|y - y'\|\), then we know \(O_i(t) \subset Y(t_0)\) and no additional computation is needed.
- If the direct inference fails, we have to re-compute \(O_i(t)\). We wish to accelerate the re-computation by using the previous result. This leads to the incremental computation principle. For reachability algorithms, we can reuse the reachable sets of unchanged layers. An example is shown in Fig.5.

These principles combine with reachability plus branching method form a framework of online verification algorithm, as shown in appendix A.3.2. In the following discussion, we derive detailed acceleration algorithms by applying these principles to the two kinds of temporal dependencies: domain shift and weight shift. The pseudocode of acceleration algorithms is shown in appendix A.3.

### 3.2 Domain shift

We derive three acceleration algorithms for domain shift (where only input changes with time): branch management for input, reachable set relaxation, and Lipschitz bound. Fig.5 shows how these algorithms accelerate the computation. We only consider the case that the input set expands throughout time. Because the specification will remain satisfied if the set is retracting or unchanged.

**Branch Management for Input (BMI)** Following the branch management principle, when the input changes, we only recompute the reachable sets for affected branches. Each branch is defined by a set of constraints. A branch is considered affected only if non-redundant constraints change. E.g. \(x < 2\) is a redundant constraint when there is a constraint \(x < 1\). When only redundant constraints change, the input set can only be unchanged or retracting, so a verified branch remains verified. Thus
When the input set changes from $[-5, 3]$ to $[-6, 3]$, only the first branch is affected. Therefore, we can skip the computation of the second branch. (b) Reachable Set Relaxation. When we compute the reachable set of $[-5, \delta]$ to $[-6, 0]$, we relax the input set from $[-5, \delta]$ to $[-6, 0]$ to tolerate potential input perturbations. The corresponding reachable set is $[0, 12]$. When the input set changes to $[-6, \delta]$, we first check that it is a subset of the relaxed input set $[-6, 0]$. Then we can directly assert that the reachable set must be a subset of the $[0, 12]$. (c) Lipschitz Bound. The Lipschitz constant of the network is 2. The distance between the new input set and the original input set is 1. With the original reachable set as $[2, 10]$, we can assert that the new reachable set must be a subset of $[2, 10] + 2 \cdot 1 = [0, 12]$.

We can skip the computation. Redundant constraints identification can be solved efficiently [16]. A detailed description is in appendix A.3.3. Besides, if we know which constraint changes, the process can be further accelerated. This algorithm does not increase over-approximation.

**Reachable set relaxation (RSR)** RSR follows the perturbation tolerance analysis principle. We can compute the reachable set of an enlarged input set. In this way, if an input set of time $t$ is a subset of the enlarged input set at time $t - 1$, we can assert the output set at time $t$ must be a subset of time $t - 1$. Then we can stop the computation earlier and directly use the previous results. Otherwise, we compute the reachable set of the enlarged input set of current steps for future use. The relaxation is done by adding an offset to the input set. A larger offset enables larger tolerance to perturbations, thus the computation time is reduced more. But since the input set is enlarged, the over-approximation is increased. The over-approximation and speed of RSR also depend on the number of branches. When there are more branches, the measure of each branch is smaller, more ReLU nodes have determined activation status, the nonlinearity of the network is reduced, therefore less over-approximation. Then we may be able to use a larger offset to achieve the same over-approximation. We draw a trade-off curve to guide the user to choose the appropriate offset in section 4.3.

**Lipschitz Bound (LB)** Another way to apply the perturbation tolerance analysis principle is to use the Lipschitz constant of the neural network to build a relationship between the change of the input set and the change of the reachable set. As mentioned earlier, we only need to consider expanding branches. For an expanding branch $X_i^{\prime}$, suppose the output constraints are linear, i.e., $Y := \{y \mid A_y < b\}$. We have the following lemma:

**Lemma 3.1.** (Lipschitz Tolerance) Suppose $f(X_i(t_0)) \subseteq Y$, then $f(X_i(t)) \subseteq Y$ if

$$\Delta_{i, t}(t_0, t) \leq \min_j \min_{y \in C(X_i(t_0), x)} \frac{b_j - a_j^T y}{\|a_j^T\| L},$$

where $\Delta_{i, t}(t_0, t)$ is defined in (2), $a_j$ is the $j^{th}$ row of $A$, and $b_j$ is the $j^{th}$ value of $b$, and $L$ is the Lipschitz constant of the neural network. Proof can be found in appendix A.1
Figure 4: Online verification algorithms to address weight shift. Orange denotes network weights change and blue denotes algorithm operation. The right figure shows an interval network that is constructed at $t_0$. Each weight is an interval. The left figure shows the network at $t_1$. Assume one weight changes from 2 to 2.1 at $t_1$. With branch management, we skip the reconstruction and use the same branches before change. And with the interval network, because $-2.1 \in [-2.1, -1.9]$, we can assert that the reachable set at $t_1$ must be a subset of the reachable set of the interval network. Therefore no additional computation is needed.

The RHS of (4) can be computed at time step $t_0$ when we check whether the reachable set violates the output constraints without additional time cost. And if $\Delta(t_0, t)$ is unknown, computing $\Delta$ can be formulated as a linear programming problem, which is efficient comparing to computing reachable set. Because the network does not change, the Lipschitz constant $L$ can be computed offline in advance by semidefinite programming [17]. Therefore, this algorithm can reduce the computation time without increasing over-approximation.

### 3.3 Weight shift

We introduce two algorithms to address weight shift: branch management for weight and interval neural network. An example is shown in Fig. 4. We also propose an algorithm, incremental computation, specifically for the fine-tuning case where only the last layer of the network changes.

**Branch management for weight (BMW)** Following the branch management principle, when the network changes, we first leave the branch division unchanged and see whether the over-approximation increases. If it increases, we re-construct the branches. Otherwise, we keep the division. The over-approximation can be measured in multiple ways, one way is to use the coverage rate we are going to define in section 4.1.

**Interval neural network (INN)** We convert the neural network into an interval neural network [18] following the perturbation tolerance analysis principle. In an interval network, any weight becomes a range rather than a certain value. As long as the weights change within the interval range, we do not need to re-compute the output set. When the network change exceeds the interval range, we re-construct an interval network based on the current network. Computing the reachable set of an interval neural network does not introduce extra time costs than ordinary networks. But the over-approximation becomes larger. The over-approximation magnitude depends on the interval range. When the interval range is larger, the verification result can tolerate larger perturbations. Although it saves more computation, the over-approximation makes the verification less accurate. Similar to RSR, the over-approximation and speed of INN also depend on the number of branches. When the number of branches is larger, we may be able to use a larger interval range to achieve the same over-approximation. Our experiments show that $\Delta_{\text{net}} \times 3$ is a good choice to balance the speed and over-approximation, which can greatly boost the speed with an only insignificant increase of over-approximation.

**Incremental computation (IC)** Following the incremental computation principle, we derive this algorithm specifically for network fine-tuning scenarios. An example is shown in Fig. 5. Since only the last layer of the network changes, we can compute the reachable sets from the last unchanged layer. This method requires storing the reachable sets of the last unchanged layer.
The experiments are designed to answer the following questions: 1) How much can the online verification algorithm accelerate under domain shift or weight shift? 2) Do online verification algorithms generalize to different network architectures? 3) For those methods trade accuracy for speed, what is the best trade-off point? To answer the first two questions, we did ablation studies under three different network architectures. We implement all the algorithms on top of NeuralVerification.jl [6], an MIT licensed verification library. As for the third question, only INN and RSR trade accuracy for speed. We plot their trade-off curves to find the best balanced point.

4 Experiments

4.1 Experiment Setting

Task. Most deep RL tasks use fully connected networks with low dimensional inputs and outputs [19, 20]. Here we consider a trajectory prediction task as a representation [21]. In the trajectory prediction task, the neural network receives three historical velocity vectors $x = [v_{t-2}, v_{t-1}]$ as input, and is supposed to give prediction of the next 3 velocity vectors, $\hat{y} = [\hat{v}_t, \hat{v}_{t+1}, \hat{v}_{t+2}]$, where $v, \hat{v} \in \mathbb{R}^3$. We denote the ground truth by $y$. And the task last for $T$ time steps. We want to verify whether the model conforms with physics, whether the predicted velocity and acceleration are less than a threshold when the input velocity and acceleration are less than a threshold. The input set and output safe set are defined as:

$$X = \{ [v_{t-j}, v_{t-j-1}] \mid \|v_{t-i} - v_{t-i-1}\| < v_x, i = 1, 2, 3; \|v_{t-j} - v_{t-j-1}\| < a_x, j = 1, 2 \}, \quad (5)$$

$$Y = \{ [\hat{v}_t, \hat{v}_{t+1}, \hat{v}_{t+2}] \mid \|\hat{v}_{t+i} - \hat{v}_t\| < v_y, i = 0, 1, 2; \|\hat{v}_{t+j+1} - \hat{v}_{t+j}\| < a_y, j = 0, 1 \} \quad (6)$$

where $\| \cdot \|$ is $l_\infty$ norm, $v_x, v_y, a_x$ and $a_y$ are constants.

Network architectures. To demonstrate the generalizability of our algorithms to different architectures. We consider three network architectures with the same total number of hidden neurons: 1) 2 layers, 60 hidden neurons; 2) 3 layers, 40 hidden neurons; 3) 4 layers, 30 hidden neurons. All the layers are fully connected layer with ReLU activation.

Coverage rate. Some methods trade accuracy for speed, we need to define a metric of verification accuracy for a fair comparison. It’s hard to compute the over-approximation precisely. In alternative, we use the verified input set coverage rate (coverage rate for short), which is the ratio of the volume of verified input set to the volume of the total input set. Because when an algorithm over-approximates more, more branches are likely to be unknown. The coverage rate reduces. We use a sampling-based method to compute the coverage rate because high dimensional volumes are hard to compute. We randomly sample $N$ points from the input set $X$. A point is considered verified if it belongs to a verified branch $X_i$. We define the coverage rate as: $N_{\text{verified}}/N$. The higher the coverage rate is, the less over-approximation an algorithm has under the assumption that the specification should hold.

4.2 Online verification

The following experiments are tested on $T = 100$ continuous time steps. The total number of branches is $m = 100$, and the constants are $v_x = 1, a_x = 0.1, v_y = 8, a_y = 8.6$.

4.2.1 Domain shift

In this experiment, the network and the output safe set are fixed. But the input set is constantly changing. We add an additional constraint $|v_{t-1}| < [v_x, v_x, v_x - 10^{-3} \cdot (T - t)]^T$ to the input set $X$.

Figure 5: Network fine-tuning and incremental verification. The left figure shows only the last layer of the network changes in fine-tuning. The right figure shows that we can accelerate the computation by computing reachable sets from the last unchanged layer.

4 Experiments
to make it expand with time. We use the following online verification algorithms: branch management for input (BMI), Lipschitz bound (LB), and reachable set relaxation (RSR). RSR uses a relaxation offset $1 \times 10^{-3}$. The ablation study results are shown in Table 1. BMI reduces the verification time to about 1/3, and LB further reduces the time to 1/10 without loss of accuracy. But when combined with RSR, although the time reduces slightly, the accuracy gets worse. All these algorithms achieve a similar acceleration effect on the three networks.

### Table 1: Ablation study for different temporal dependencies and network architectures.

| Network Layer          | Time (s)       | Average coverage rate |
|------------------------|----------------|-----------------------|
|                        | 2              | 3                     | 4                     |
|                        | 2              | 3                     | 4                     |
| Domain Shift           |                |                       |                       |
| None                   | 26.74          | 379.07                | 387.40                | 100.0% | 91.2% | 18.6% |
| BMI                    | 10.02          | 99.04                 | 50.75                 | 100.0% | 91.3% | 18.8% |
| BMI + LB               | 2.99           | 38.08                 | 36.09                 | 100.0% | 91.3% | 18.8% |
| BMI + RSR              | 5.37           | 56.33                 | 36.55                 | 100.0% | 85.5% | 17.2% |
| BMI + LB + RSR         | 2.63           | 30.68                 | 32.07                 | 100.0% | 85.5% | 17.2% |
| Weight Shift           |                |                       |                       |
| None                   | 35.35          | 385.08                | 391.75                | 100.0% | 67.5% | 20.0% |
| BMW                    | 13.43          | 131.95                | 136.89                | 100.0% | 67.5% | 20.0% |
| BMW + INN              | 2.26           | 17.77                 | 14.48                 | 100.0% | 67.3% | 20.0% |
| Fine-tuning            |                |                       |                       |
| None                   | 48.08          | 399.35                | 403.69                | 100.0% | 78.3% | 23.6% |
| BMW                    | 10.08          | 136.20                | 141.32                | 100.0% | 78.3% | 23.6% |
| BMW + INN              | 2.85           | 30.54                 | 22.12                 | 100.0% | 78.3% | 23.2% |
| BMW + IC               | 0.93           | 10.15                 | 10.54                 | 100.0% | 78.3% | 23.6% |
| BMW + INN + IC         | 0.46           | 5.20                  | 4.88                  | 100.0% | 78.3% | 23.2% |

### 4.2.2 Weight shift

In this experiment, the input set and the output constraints are fixed, only the network weights change. We define an $l_2$ norm loss function $l = \|y - \hat{y}\|_2$. At each time step, the network weights are updated by backpropagation from this loss function with a learning rate $= 10^{-3}$. $\Delta_{net}$ is at the level of $10^{-4}$. We use branch management for weight (BMW) and interval neural network (INN) to accelerate this scenario. INN uses an interval range of $3 \times 10^{-3}$. The ablation study results are shown in Table 1. BMW reduces the verification time to about 1/3, and INN further reduces the time to about 1/20 without significant loss of accuracy. INN trades accuracy for speed, but because the interval range is pretty small in this experiment, we only see a slight accuracy drop in the 3-layer architecture case. All these algorithms achieve a similar acceleration effect on 3 architectures.

We also did an additional experiment for the fine-tuning case. The update rule and learning rate are the same as above. But we only update the last layer. On top of weight change algorithms, we also use incremental computation (IC). The ablation study results are shown in Table 1. BMW reduces the verification time to about 1/3. When BMW is combined with INN, the time is reduced to about 1/15 without significant loss of accuracy. And when BM is combined with IC, the time is reduced to about 1/40 without loss of accuracy. With all these algorithms, the time is reduced up to two orders of magnitude. All these algorithms achieve a similar acceleration effect on 3 architectures.

### 4.3 Trade-off analysis

Interval Neural Network (INN) and Reachable Set Relaxation (RSR) trade accuracy for speed. To help the user find proper hyper-parameters, we plot the trade-off curves between accuracy and speed. The experiment results can be found in the appendix.

### 5 Discussion and Limitation

In this work, we proposed online verification as a potential solution to verify neural networks in real-world applications. Online verification requires real-time verification, which can not be achieved by existing methods. Therefore, We studied how to leverage temporal dependencies to accelerate existing verification algorithms to real-time level. We proposed a framework to design acceleration algorithms and derived several ones based on it. In the future, we will extend our methods to more types of temporal dependencies, study how to design the best acceleration algorithms, and how to automatically select optimal acceleration algorithms.
Another challenge we will address is, with online verification algorithms, some time steps take more computation than others. For example, when we build an Interval Neural Network (INN) at time step \( t_1 \), we can save the computation for several future time steps, until we have to rebuild it at \( t_2 \) when the change exceeds the interval range. But \( t_1 \) and \( t_2 \) take significantly more computation than other time steps, which may stall the system in practice. We can address this issue by asynchronous parallel computing. Following the previous example, we can use a second process to compute an INN at time step \( (t_1 + t_2)/2 \). Then at time step \( t_2 \), we can switch the second process as the main process. In this way, the INN is constructed in the background and will not stall the system. The total computation time is also reduced.

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Checklist

1. For all authors...
   (a) Do the main claims made in the abstract and introduction accurately reflect the paper’s contributions and scope? [Yes]
   (b) Did you describe the limitations of your work? [Yes] See section 5.
   (c) Did you discuss any potential negative societal impacts of your work? [No] This work proposes a tool to against potential negative societal impacts of neural networks by providing formal guarantees.
   (d) Have you read the ethics review guidelines and ensured that your paper conforms to them? [Yes]

2. If you are including theoretical results...
   (a) Did you state the full set of assumptions of all theoretical results? [Yes] See section 2 and appendix A.1
   (b) Did you include complete proofs of all theoretical results? [Yes] See appendix A.1

3. If you ran experiments...
   (a) Did you include the code, data, and instructions needed to reproduce the main experimental results (either in the supplementary material or as a URL)? [Yes] They are in the supplementary material.
   (b) Did you specify all the training details (e.g., data splits, hyperparameters, how they were chosen)? [N/A] Our methods do not require training.
   (c) Did you report error bars (e.g., with respect to the random seed after running experiments multiple times)? [No] Our methods do not depend on random seeds.
   (d) Did you include the total amount of compute and the type of resources used (e.g., type of GPUs, internal cluster, or cloud provider)? [No] The effectiveness of our method is demonstrated by the acceleration ratio, not the computation time.

4. If you are using existing assets (e.g., code, data, models) or curating/releasing new assets...
   (a) If your work uses existing assets, did you cite the creators? [Yes] 6
   (b) Did you mention the license of the assets? [Yes] MIT License.
   (c) Did you include any new assets either in the supplementary material or as a URL? [Yes] They are in the supplementary material.
(d) Did you discuss whether and how consent was obtained from people whose data you’re using/curating? [N/A] We use a generated dataset.

(e) Did you discuss whether the data you are using/curating contains personally identifiable information or offensive content? [N/A] We use a generated dataset.

5. If you used crowdsourcing or conducted research with human subjects...

(a) Did you include the full text of instructions given to participants and screenshots, if applicable? [N/A]

(b) Did you describe any potential participant risks, with links to Institutional Review Board (IRB) approvals, if applicable? [N/A]

(c) Did you include the estimated hourly wage paid to participants and the total amount spent on participant compensation? [N/A]
A Appendix

A.1 Lipschitz Bound lemma and proof

Consider an expanding branch $\mathcal{X}_i$, we define $\Delta_{in}(t_0, t)$ as the maximum distance from $\mathcal{X}_i(t)$ to $\mathcal{X}_i(t_0)$:

\[
\Delta_{in}(t_0, t) = \max_{\mathbf{x} \in \mathcal{X}_i(t)} \min_{\mathbf{x}' \in \mathcal{X}_i(t_0)} \|\mathbf{x}' - \mathbf{x}\|.  
\] (7)

With the Lipschitz constant $L$ of the network and eq. (2), we have $\forall x' \in \mathcal{X}_i(t), x \in \mathcal{X}_i(t_0)$.

\[
\|\mathbf{f}(\mathbf{x}') - \mathbf{f}(\mathbf{x})\| \leq L \|\mathbf{x}' - \mathbf{x}\| \leq L \cdot \Delta_{in}(t_0, t).  
\] (8)

Suppose the output constraint is a linear function of the network output:

\[
\mathcal{Y} := \{y \mid A\mathbf{y} < \mathbf{b}\}.  
\] (9)

**Lemma A.1.** (Lipschitz Tolerance) $\mathbf{f}(\mathcal{X}_i(t_0)) \subseteq \mathcal{Y}$ if

\[
\Delta_{in}(t_0, t) \leq \min_j \min_{\mathbf{y} \in \mathcal{O}(\mathcal{X}_i(t_0), \mathbf{f})} \frac{b_j - \mathbf{a}_j^T \mathbf{y}}{\|\mathbf{a}_j\| L}.  
\] (10)

where $a_j$ is the $j^{th}$ row of $A$, and $b_j$ is the $j^{th}$ value of $b$.

**Proof.** $\mathbf{f}(\mathcal{X}_i(t_0)) \subseteq \mathcal{Y}$, that is

\[
A\mathbf{f}(\mathbf{x}) < \mathbf{b}, \forall \mathbf{x} \in \mathcal{X}_i(t_0).  
\] (11)

We want to verify that $\mathbf{f}(\mathcal{X}_i(t)) \subseteq \mathcal{Y}$, that is

\[
A\mathbf{f}(\mathbf{x}') < \mathbf{b}, \forall \mathbf{x}' \in \mathcal{X}_i(t).  
\] (12)

According to eq. (2), $\forall \mathbf{x}'$, we can find a $\mathbf{x}$ such that $\|\mathbf{x}' - \mathbf{x}\| < \Delta_{in}(t_0, t)$, for a row $a_j$ in $A$, $b_j$ in $b$.

\[
a_j^T \mathbf{f}(\mathbf{x}') - b_j = a_j^T \mathbf{f}(\mathbf{x}') - a_j^T \mathbf{f}(\mathbf{x}) + a_j^T \mathbf{f}(\mathbf{x}) - b_j  
\] (13)

\[
\leq a_j^T [\mathbf{f}(\mathbf{x}') - \mathbf{f}(\mathbf{x})] + a_j^T \mathbf{f}(\mathbf{x}) - b_j  
\] (14)

\[
\leq \|a_j\| \|\mathbf{f}(\mathbf{x}') - \mathbf{f}(\mathbf{x})\| + a_j^T \mathbf{f}(\mathbf{x}) - b_j  
\] (15)

\[
\leq \|a_j\| L \cdot \Delta_{in}(t_0, t) + a_j^T \mathbf{f}(\mathbf{x}) - b_j.  
\] (16)

Therefore, a sufficient condition for eq. (12) is

\[
\max_{\mathbf{x} \in \mathcal{X}_i(t_0)} \|a_j^T\| L \cdot \Delta_{in}(t_0, t) + a_j^T \mathbf{f}(\mathbf{x}) - b_j \leq 0, \forall j  
\] (17)

\[
\iff \Delta_{in}(t_0, t) \leq \min_{\mathbf{x} \in \mathcal{X}_i(t_0)} \min_j \frac{b_j - a_j^T \mathbf{f}(\mathbf{x})}{\|a_j\| L}  
\] (18)

It’s hard to compute $\min_{\mathbf{x} \in \mathcal{X}_i(t_0)} b_j - a_j^T \mathbf{f}(\mathbf{x})$, so we relax the inequality to

\[
\Delta_{in}(t_0, t) \leq \min_j \min_{\mathbf{y} \in \mathcal{O}(\mathcal{X}_i(t_0), \mathbf{f})} \frac{b_j - a_j^T \mathbf{y}}{\|a_j\| L} \leq \min_j \min_{\mathbf{x} \in \mathcal{X}_i(t_0)} \frac{b_j - a_j^T \mathbf{f}(\mathbf{x})}{\|a_j\| L}  
\] (19)

Then

\[
\Delta_{in}(t_0, t) \leq \min_j \min_{\mathbf{y} \in \mathcal{O}(\mathcal{X}_i(t_0), \mathbf{f})} \frac{b_j - a_j^T \mathbf{y}}{\|a_j\| L} \implies \mathbf{f}(\mathcal{X}_i(t)) \subseteq \mathcal{Y}  
\] (20)

\[\square\]
A.2 Trade off analysis

A.2.1 INN analysis

As discussed in section 3.3, the accuracy and speed of INN depend both on the interval range and the number of branches. Therefore, to find the best-balanced point, we test how the coverage rate changes with the interval range under three different numbers of branches. We choose the interval range as \([0, \Delta^*_\text{net} \times 20]\). And we choose the number of branches 400, 500, 600, correspond to different levels of verification accuracy. The coverage rate without INN is 70%, 90%, 100%.

The trade-off curves is shown in Fig. 6. We can see that, in all three sub-figures, when the interval range is at around \(3 \times \Delta^*_\text{net}\), the coverage rate has only an unnoticeable drop while the verification time reduces a lot. We consider \(3 \times \Delta^*_\text{net}\) is a well-balanced point.

![Figure 6: INN trade off curves under different number of branches. The curve shows how time and coverage rate drops in percentage with the interval range increase. The results show that \(\Delta^*_\text{net} \times 3\) is a well balanced point, where the time reduces more than 60% when the coverage rate barely changes.](image)

A.2.2 RSR analysis

Similar to INN, as we discussed in section 3.2, the accuracy and speed of RSR depend both on the relaxation offset and the number of branches. We set the relaxation magnitude range as \([0, \Delta^*_\text{in} \times 5]\) because the time does not show significant change for larger relaxation. And we choose the branches 120, 140, 160, corresponds coverage rate without set relaxation is 50%, 66%, 86%.

We draw the trade-off curves to show how time and coverage rate drop in percentage as the set relaxation magnitude increases, as shown in Fig. 7. Time drops like a log function, and the coverage rate drops about linearly. The user can choose the relaxation magnitude based on the conditions.

![Figure 7: RSR trade off curves under different number of branches. The results show that the coverage rate drops about linearly, and the time drops like a log function.](image)

A.3 Algorithm details and pseudocode

A.3.1 Reachability plus branching algorithm

The Pseudocode of reachability plus branching methods are shown in algorithm 1

A.3.2 Online verification algorithm

We give a framework of online verification algorithms based on reachability plus branching methods. We define \(tag_i\) to denote whether a branch is changed or not, and \(r_i\) to denote the verification result of
Algorithm 1: Reachability plus branching algorithm

1: function REACHABILITY PLUS BRANCH(T, X(t), f^i, Y(t))
2:   for t = 1 to T do
3:     \{X_i(t)\}_{i=1:m} \leftarrow \text{SPLIT}(X(t), f^i, Y(t)) \triangleright \text{Construct branches}
4:   for i = 1 to m do
5:     R_i(t) \leftarrow \text{REACH}(X_i(t), f^i) \triangleright \text{Compute the reachable set}
6:     r_i \leftarrow \text{CHECK}(R_i(t), Y(t)) \triangleright \text{Check whether the constraints are satisfied}
7:   end for
8: end for
9: end function

Algorithm 2: Online verification algorithm

1: function ONLINE VERIFICATION(T, X(t), f^i, Y(t))
2:   for t = 1 to T do
3:     \{X_i(t), \text{tag}_i\}_{i=1:m} \leftarrow \text{BRANCH MANAGEMENT}(X(t), f^i, Y(t)) \triangleright \text{Construct branches}
4:   for i = 1 to m do
5:     if \text{tag}_i = \text{unchanged} then
6:       r_i \leftarrow \text{CHECK}(R_i(t-1), Y(t)) \triangleright \text{Branch is not changed.}
7:     continue
8:   end if
9:   if PERTURBATION ANALYSIS(X_i(t), f^i) then
10:      r_i \leftarrow \text{CHECK}(R_i(t_0), Y(t)) \triangleright \text{Check whether change can be tolerated}
11:     continue
12:   else
13:     t_0 \leftarrow t
14:     R_i(t) \leftarrow \text{REACH}(X_i(t), f^i) \triangleright \text{Update the robust margin, incremental computation}
15:     r_i \leftarrow \text{CHECK}(R_i(t), Y(t)) \triangleright \text{Check whether the constraints are satisfied}
16:   end if
17: end for
18: end function

A.3.3 BMI

We assume the input set is composed by \(m_x\) linear constraints \(a_jx < b_j\) and constraints change gradually, that is, \(a_j(t)\) and \(b_j(t)\) change gradually.

The split algorithm divides the input set \(X(t)\) by adding more constraints. We denote the total number of branches by \(m\), the number of input set constraints by \(m_x\), and the number of additional constraints of branch \(i\) by \(m_i\).

\[ X(t) = \{ x \mid a_j(t)x \leq b_j(t), j = 1 \ldots m_x \} \]  \hspace{1cm} (21)
\[ X_i(t) = \{ x \mid a_j(t)x \leq b_j(t), j = 1 \ldots m_x + m_i \} \]  \hspace{1cm} (22)

A constraint is either a boundary constraint or a redundant constraint. Boundary constraints are:

\[ \{ j \mid \exists x \in X_i(t), a_j(t)x = b_j(t) \} \]  \hspace{1cm} (23)

Redundant constraints are:

\[ \{ j \mid \forall x \in X_i(t), a_j(t)x < b_j(t) \} \]  \hspace{1cm} (24)

BMI skips the computation of \(X_i(t)\) if boundary constraints are not changed. The pseudo-code is shown in algorithm 2.
Algorithm 3 BMI

1: function BRANCH MANAGEMENT($X(t), f^t, Y(t)$)
2: ret ← {} 
3: for i = 1 to m do 
4: tag_i ← unchanged 
5: B ← FIND BOUNDARY CONSTRAINTS($X_i(t-1)$) 
6: for j = 1 to $m_i + m$ do 
7: if $j \in B$ and $[a_j(t) \neq a_j(t_1) \text{ or } b_j(t) \neq b_j(t_1)]$ then 
8: tag_i ← changed 
9: break 
10: end if 
11: end for 
12: PUSH(ret, ($X_i(t), tag_i$)) 
13: end for 
14: return ret 
15: end function

A.3.4 RSR

For a branch $X_i(t_0)$, we add an offset $c_j$ to each $b_j(t_0)$, forms an enlarged input set: 
\[ X_i'(t_0) = \{x | a_j(t_0)x \leq b_j(t_0) + c_j, j = 1 \ldots m_x + m_i \} \]  
(25)

We compute the relaxed reachable set for this enlarged branch (assuming the relaxed reachable set is verified). Then, if $X_i(t) \subseteq X_i'(t_0)$, we can skip the computation for $X_i(t)$. $X_i(t) \subseteq X_i'(t_0)$ if 
\[ \forall j, \max_{x \in X_i(t)} a_j(t_0)x - b_j(t_0) - c_j \leq 0, \]  
(26)

which is a linear programming problem. The pseudo code is shown in algorithm 4 and algorithm 5.

Algorithm 4 RSR

1: function PERTURBATION ANALYSIS($X(t), f^t$)
2: for j = 1 to $m_x$ do
3: if $\max_{x \in X(t)} a_j(t_0)x - b_j(t_0) - c_j > 0$ then 
4: $t_0 \leftarrow t$
5: return False
6: end if
7: end for
8: return True
9: end function

Algorithm 5 RSR

1: function REACH($X(t), f^t$)
2: $R_0(t) \leftarrow X(t)$
3: for j=1 to $m_x$ do
4: $R_0(t), b_j + c_j \quad \triangleright$ Add an offset to the constant term.
5: end for
6: for k=1 to n do
7: $R_k(t) \leftarrow REACH((R_{k-1}(t), W_k(t))$
8: end for
9: return $R_n(t)$
10: end function

A.3.5 LB

Suppose the output constraints are 
\[ Y = \{y | a_jy \leq b_j, j = 1 \ldots m_y \} \]  
(27)

The pseudo code of Lipschitz bound method is shown in algorithm 6.
Algorithm 6 LB

1: function PERTURBATION ANALYSIS($\mathcal{X}(t), f^t$)  
2: $d \leftarrow \Delta_{m}(t_0, t)$ \quad \triangleright \text{Linear programming.}$  
3: $\delta \leftarrow \min_{j} \min_{y \in \mathcal{O}(\mathcal{X}_i(t_0), f)} \frac{b_j - a_j^T y}{\|a_j\|_2}$. \quad \triangleright \text{Can be computed in last round constraints check}$  
4: return $d \leq \delta$  
5: end function

A.3.6 BMW

BMW modifies the framework a little, the modified framework is shown in algorithm 8. We mentioned in appendix A.3.3 that the split algorithm divides the input set $\mathcal{X}(t)$ by adding additional constraints. We denote the set defined by these additional constraints as $\mathcal{X}^a_i(t) = \{x \mid a_j(t)x \leq b_j(t), j = m_x + 1, \ldots, m_x + m_i\}$. The pseudo code for BMW is shown in algorithm 7.

Algorithm 7 BMW

1: function BRANCH MANAGEMENT(reuse, $\mathcal{X}(t), f^t, \mathcal{Y}(t)$)  
2: if reuse = True then  
3: return SPLIT($\mathcal{X}(t), f^t, \mathcal{Y}(t)$)  
4: end if  
5: $\{\mathcal{X}_i(t)\}_{i=1:m} = \{\}$  
6: for $i = 1$ to $m$ do  
7: $\mathcal{X}_i(t) \leftarrow \mathcal{X}(t) \cap \mathcal{X}^a_i(t)$ \quad \triangleright \text{Reuse split of time step } t - 1  
8: end for  
9: return $\{\mathcal{X}_i(t)\}_{i=1:m}$  
10: end function

Algorithm 8 Online verification with BMW

1: function ONLINE VERIFICATION($T, \mathcal{X}(t), f^t, \mathcal{Y}(t)$)  
2: for $t = 1$ to $T$ do  
3: for reuse in [True, False] do \quad \triangleright \text{First try reuse previous split}$  
4: $\{\mathcal{X}_i(t), \text{tag}_i\}_{i=1:m} \leftarrow \text{BRANCH MANAGEMENT}(reuse, \mathcal{X}(t), f^t, \mathcal{Y}(t))$  
5: for $i = 1$ to $m$ do  
6: if tag$_i$ = unchanged then  
7: $r_i \leftarrow \text{CHECK}(R_i(t - 1), \mathcal{Y}(t))$  
8: continue  
9: end if  
10: if PERTURBATION ANALYSIS($\mathcal{X}_i(t), f^t$) then  
11: $r_i \leftarrow \text{CHECK}(R_i(t_0), \mathcal{Y}(t))$  
12: continue  
13: else  
14: $t_0 \leftarrow t$  
15: $R_i(t) \leftarrow \text{REACH}(\mathcal{X}_i(t), f^t)$  
16: $r_i \leftarrow \text{CHECK}(R_i(t), \mathcal{Y}(t))$  
17: end if  
18: end for  
19: $cr(t) \leftarrow \text{COVERAGE RATE}(\{\mathcal{X}_i(t)\}, \{r_i\})$ \quad \triangleright \text{Check split accuracy}$  
20: if $cr(t) \geq cr(t - 1)$ then  
21: Break \quad \triangleright \text{Previous split leads to accurate results, no need to split again.}$  
22: end if  
23: end for  
24: end for  
25: end function
A.3.7 INN

We denote the weight ranges of the interval network by \( W_{\text{int}} \). You can find how to build the interval network in [18]. The pseudo code is shown in algorithm 9 and algorithm 10.

Algorithm 9 INN

1: function \textsc{Perturbation Analysis}(\( \mathcal{X}(t), f^t \))
2: \quad return \( W(t) \in W_{\text{int}}(t_0) \)
3: end function

Algorithm 10 INN

1: function \textsc{Reach}(\( \mathcal{X}(t), f^t \))
2: \quad \( R_0(t) \leftarrow \mathcal{X}(t) \)
3: \quad \( W_{\text{int}}(t) \leftarrow \text{Build Interval Net}(W(t)) \)
4: \quad for k=1 to n do
5: \quad \quad \( R_k(t) \leftarrow \text{Reach}((R_{k-1}(t), W_{\text{int}}(t))) \)
6: \quad end for
7: \quad return \( R_n(t) \)
8: end function

A.3.8 IC

We denote the \( k^{th} \) layer of the reachable set at time step \( t \) by \( R_k(t) \). The pseudo code of IC is shown in algorithm 11.

Algorithm 11 IC

1: function \textsc{Reach}(\( \mathcal{X}(t), f^t \))
2: \quad if \( \mathcal{X}(t - 1) = \mathcal{X}(t) \) and only the last layer weights \( W_n(t) \) changes then
3: \quad \quad \( R_{n-1}(t) \leftarrow R_{n-1}(t - 1) \)
4: \quad \quad return \text{Reach}((R_{n-1}(t), W_n(t)))
5: \quad else
6: \quad \quad \( R_0(t) \leftarrow \mathcal{X}(t) \)
7: \quad \quad for k=1 to n do
8: \quad \quad \quad \( R_k(t) \leftarrow \text{Reach}(R_{k-1}(t), W_k(t)) \)
9: \quad \quad end for
10: \quad return \( R_n(t) \)
11: \quad end if
12: end function