A STOCHASTIC APPROACH TO COUNTING PROBLEMS.

ADRIEN BOULANGER

Abstract. We study orbital functions associated to finitely generated Kleinian groups acting on the hyperbolic space \( \mathbb{H}^3 \), developing a new method based on the use of the Brownian motion. On the way, we give some estimates of the orbital function associated to nilpotent covers of compact hyperbolic manifolds, partially answering a question asked by M. Pollicott to the author.

1. Introduction

Historical background. Given a group \( \Gamma \) acting properly and discontinuously on a metric space \((X, d)\), we define the orbital function as follows

\[
N_\Gamma(x, y, \rho) := \sharp \{ \gamma \in \Gamma, \ d(x, \gamma \cdot y) \leq \rho \},
\]

where \(x, y \in X\) and \(\rho > 0\). The growth of orbital functions of groups acting on various types of hyperbolic spaces have been extensively studied since the 50's. Two main approaches were developed in this setting. The oldest one due to Huber [22], following Delsarte (see [2]), used Selberg's pre-trace formula in order to study orbital functions in the setting of groups acting cocompactly on the hyperbolic space \( \mathbb{H}^2 \). This line of work was generalised later on by Selberg [38, page 147], Patterson [33], Lax-Phillips [26] among many others. Another way to get estimates of the orbital function growth comes from Margulis' seminal PhD work, which exhibits a strong relation between this problem and the mixing of the geodesic flow on the quotient space \( X/\Gamma \). This second approach has the benefit to give results in the case of variable curvature as well. This idea has been widely extended by numerous authors since, dropping most of Margulis' assumptions required in his work [42] [36]. For general references, one can recommend Babillot’s survey [2] on counting problems, Section 2 of Eskin-McMullen’s article [15] where Margulis’ strategy is well explained and the author’s PhD dissertation [6, Chapitre 1]. The following theorem is, generality-wise, the most advanced of the theory.

Theorem 1.1. [36] Let \( \Gamma \) be a group acting by isometries, properly and discontinuously on a connected, simply connected complete manifold of negative sectional curvature \( X \) such that

- the length spectrum of the quotient space is non arithmetic;
- the unitary tangent bundle of the quotient space \( X/\Gamma \) admits a finite Bowen-Margulis-Sullivan measure,

then, for any points \( x, y \in X \) there exists a constant \( C \) such that

\[
N_\Gamma(x, y, \rho) \sim_{\rho \to \infty} C e^{\delta_\Gamma \rho},
\]

where \( \delta_\Gamma \) is the critical exponent of the group \( \Gamma \).

The above theorem is weaker than the one stated in [36] but the author did not want to burden this article with definitions regarding the CAT(-1) setting for which an analogous version of the above theorem holds. Let us note that the two assumptions of Theorem 1.1 are automatically satisfied when \( \Gamma \) acts convex-compactly on a hyperbolic space of any dimension, see [20, Proposition 3] for the assumption made...
on the non-arithmeticity of the length spectrum.

T. Roblin also showed, under the same assumptions of those of Theorem 1.1, that if the quotient space $X/\Gamma$ does not admit any finite Bowen-Margulis-Sullivan measure, one has

$$N_\Gamma(x, y, \rho) = o\left(e^{\delta \rho}\right).$$

for any pair of points $x, y \in X$ when $\rho \to \infty$. Nowadays, the goal is to explicit the asymptotic in this case. There is, up to the author’s knowledge, only two classes of examples where the finitness assumption had been successfully dropped. The first one is due to Pollicott and Sharp [35]. They found an asymptotic for orbital functions of groups associated to abelian covers of compact hyperbolic manifolds. The second one is due to Vidotto [41] and deals with an example in variable curvature. In both cases, the authors manage to get an asymptotic of the orbital function. Their methods rely on a finite measure property hidden somewhere: in [35] the author’s method uses a strong mixing property of the geodesic flow of the compact underlined manifold whereas in [41] the geodesic flow still admits a coding by a sub-shift of finite type.

**Description of the results.** In this article we introduce a method to study the growth of orbital functions of groups acting on the real hyperbolic space $H^3$ of dimension three, based on the use of the Brownian motion in the spirit of Sullivan’s work [39]. Our motivation for introducing the Brownian motion in this setting is to compensate the lack of known randomness of the geodesic flow - preventing to adapt Margulis’ method - when there is no finite Bowen-Margulis-Sullivan measure. Replacing the geodesic flow by the Brownian motion gives a larger range of applications, the latter being random on its own. The key facts allowing us to relate its dynamical behaviour to the orbital function is the strong homogeneity of the hyperbolic space and the so called drift property of the Brownian motion. We refer to the author’s PhD dissertation [6] for more details.

More precisely, this article aims at investigating the link between Brownian motion and orbital functions through examples. We call Kleinian group any finitely generated subgroup $\Gamma$ of the orientation preserving isometries of the hyperbolic 3-space $H^3$ acting properly and discontinuously. Regarding the counting problem for Fuchsian groups, analogous to the last but acting on $H^2$, there is not so much to be said since all the quotients $H^2/\Gamma$ turned to carry a finite Bowen-Margulis-Sullivan measure [14], so that Theorem 1.1 readily applies in this case. Therefore, one would like to picture what is the situation in dimension 3. We will call a Kleinian group $\Gamma$ degenerate when the quotient manifold $M_\Gamma := H^3/\Gamma$ does not carry any finite Bowen-Margulis-Sullivan measure. By extension, we also call degenerate the manifold $M_\Gamma$. These manifolds are central objects in some of the most famous theorems around Kleinian group theory such as the Thurston’s hyperbolisation theorem (see [28] or [31]), Ahlfors’ conjecture, corollary of [9] Marden’s tameness conjecture (see [8] and the reference therein) or the ending lamination conjecture [7], all theorems now.

Surprisingly, very little is known about the behaviour of their orbital functions. The only special case already settled being as a corollary of [35], giving an asymptotic to the orbital function for surface groups associated to $\mathbb{Z}$-covers of compact 3-manifolds, included in M. Pollicott and R. Sharp’s theorem as Abelian covers of compact manifolds.
Our two main theorems can be stated as follows, we refer to Subsection 4.1 for the description of ends of hyperbolic manifolds involved in the statements.

**Theorem 1.2.** Let $\Gamma$ be a degenerate Kleinian group such that the manifold $M_\Gamma$ has positive injectivity radius, then:

- either all ends are degenerate and the group is divergent,
- or the following upper bound for the orbital function holds: for any $x, y \in \mathbb{H}^3$ there is a constant $C > 0$ such that for $\rho$ large enough we have
  \[ N_\Gamma(x, y, \rho) \leq C \frac{e^{2\rho}}{\rho} . \]

Under some extra assumption, one can improve the above theorem into

**Theorem 1.3.** Let $\Gamma$ be a degenerate Kleinian group such that the manifold $M_\Gamma$ has positive injectivity radius, and the averaged orbital function is roughly decreasing (see Subsection 3.3), then

- either all the ends of $M_\Gamma$ are degenerate, then there is two constant $C_-, C_+$ such that for any $x, y \in \mathbb{H}^3$ there is $\rho$ big enough such that
  \[ \frac{C_- e^{2\rho}}{\sqrt{\rho}} \leq N_\Gamma(x, y, \rho) \leq \frac{C_+ e^{2\rho}}{\sqrt{\rho}} ; \]
- or, for every $x, y \in M_\Gamma$ there is two constants $C_-, C_+$ such that for $\rho$ big enough we have
  \[ \frac{C_- e^{2\rho}}{\rho^\frac{3}{2}} \leq N_\Gamma(x, y, \rho) \leq \frac{C_+ e^{2\rho}}{\rho^\frac{3}{2}} . \]

Note that the constants appearing in the first above item depend only on the group $\Gamma$. Constants involved in the second item depend on the points under consideration, which is not a flaw of the method but something expected.

To prove both Theorem 1.2 and Theorem 1.3 one needs first to link the heat kernel to the orbital function, which is the purpose of the next two theorems.

**Theorem 1.4.** There is a constant $C_+ > 0$ such that for any subgroup $\Gamma$ of the group of orientation preserving isometries of $\mathbb{H}^3$ acting properly and discontinuously and all $x \in \mathbb{H}^3$ one has for $\rho > 0$

\[ \frac{N_\Gamma(x, y, \rho)}{e^{2\rho}} \leq C_+ \sqrt{\rho} p_\Gamma \left( x, y, \frac{\rho}{2} \right) , \]

where $p_\Gamma(x, y, t)$ is the heat kernel of the manifold $\mathbb{H}^3 / \Gamma$.

**Remark 1.6.** This upper bound is relevant only when the critical exponent of the group is 2. Otherwise, it would not even have the correct exponential growth since from [39, Theorem 2.17] and [18, Theorem 10.24] we have

\[ \frac{\ln p_\Gamma(x, y, \rho)}{\rho} \to \delta_\Gamma (\delta_\Gamma - 2) . \]

One might, perhaps, get an more general upper bound in weighting the manifold $M_\Gamma$ with respect to its first eigenfunction, known to be positive [39, Theorem 2.1].
Ultimately, the proof of this theorem relies on the drift property of the Brownian motion. This upper bound is not sharp, and finer estimates - sharp upper and lower bounds up to multiplicative constants - can be proven under a polynomial control of the heat kernel in large time and some extra assumptions on the orbital functions. This assumptions and the definition of the average orbital function are stated in Subsection 3.3.

**Theorem 1.7.** Let $\Gamma$ be a Kleinian group such that

- there is $\alpha \geq 0$ such that there is two points $x,y \in M_\Gamma$ and two constants $C_-, C_+$ such that for $t$ big enough one has
  \begin{equation}
  c_- t^{-\alpha} \leq p_\Gamma(x,y,t) \leq c_+ t^{-\alpha} ;
  \end{equation}
- the averaged orbital function is roughly decreasing,

then there is two positive constants $C_-$ and $C_+$ such that
\[
\frac{C_- e^{2\rho}}{\rho^\alpha} \leq N_\Gamma(x,y,\rho) \leq \frac{C_+ e^{2\rho}}{\rho^\alpha} .
\]

Both Theorem 1.4 and Theorem 1.7 apply immediately in the context of an infinite manifold admitting the volume doubling property and a Poincaré inequality (see Section 5.1), which is the case for nilpotent covers of compact manifold. This partially answers the following question.

**Question:** (M. Pollicott) what is the asymptotic behaviour of the orbital function associated to a nilpotent cover of a compact hyperbolic manifold ?

**Corollary 1.9.** Let $\Gamma$ be a Kleinian group such that $M_\Gamma$ is roughly isometric to a nilpotent cover of a compact manifold then there exists a constant $C_+$ such that
\[
N_\Gamma(x,y,\rho) \leq \frac{C_+ \sqrt{\rho} e^{2\rho}}{\text{vol} \left( B(x, \sqrt{\rho}) \right)} .
\]

Moreover, if the averaged orbital function is roughly decreasing, then there exists two constant $C_-, C_+$ such that for every two points $x,y$ one has for $\rho$ big enough
\[
\frac{C_- e^{2\rho}}{\text{vol} \left( B(x, \sqrt{\rho}) \right)} \leq N_\Gamma(x,y,\rho) \leq \frac{C_+ e^{2\rho}}{\text{vol} \left( B(x, \sqrt{\rho}) \right)} .
\]

In view of the above theorem, it seems natural to hope that there exists a constant $C > 0$ such that for every pair of points $x, y \in \mathbb{H}^3$ one has
\[
N_\Gamma(x,y,\rho) \sim_{\rho \to \infty} \frac{C e^{2\rho}}{\text{vol} \left( B(x, \sqrt{\rho}) \right)} ,
\]
which would give a satisfying answer to Pollicott’s question.

In order to deduce our main theorems 1.2 and 1.3 from Theorems 1.4 and 1.7, we investigate the large time behaviour of the heat kernel on the manifolds under consideration, the degenerate hyperbolic manifolds.

**Theorem 1.10.** Let $\Gamma$ a degenerate Kleinian group such that $M_\Gamma$ has positive injectivity radius, then

- either all the ends of $M_\Gamma$ are degenerate, then there is two constant $C_-, C_+ > 0$ such that for every $x, y \in \mathbb{H}^3$ one has for $t$ large enough
  \[
  \frac{C_-}{\sqrt{t}} \leq p_\Gamma(x,y,t) \leq \frac{C_+}{\sqrt{t}} ;
  \]
or there are both degenerate ends and geometrically finite ends, then for every \( x, y \in M_\Gamma \) there exists a constant \( C_+ > 0 \) such that one has for \( t \) large enough

\[
pr_t(x, y, t) \leq \frac{C_+}{t^{3/2}}.
\]

Moreover, for any \( x \in M_\Gamma \) there is a constant \( C_- \) such that

\[
\frac{C_-}{t^{3/2}} \leq pr_t(x, x, t).
\]

Let us emphasise that the first part of Theorem 1.2 will follow from the fact that the above theorem implies the Brownian motion to be recurrent in the case where all ends are degenerate, see Section 5.

Outline of the paper. In Section 2 we introduce all the key notions and definitions regarding the heat kernel and its natural weighted cousin. We also state two theorems relating the large time behaviour of the heat kernel with the large scale geometry of the underlined manifold.

Section 3 is devoted to the proofs of both Theorems 1.4 and 1.7. We end up this section in proving Corollary 1.9.

In Section 4, we summarize all the geometric properties of degenerate manifolds that we will need in order to use the material introduced in Section 2. We will also recall Thurston-Sullivan’s argument, showing that there is a family of harmonic functions growing linearly on all degenerate ends, with definite limit in the geometrically finite ends. They will play a crucial role in order to apply Theorems stated in Section 3 as weights of the underlined degenerate manifolds.

Section 5 is fully dedicated to prove Theorem 1.10, which, combined with Theorem 1.4 and Theorem 1.7, readily gives Theorems 1.2 and 1.3. The intuition behind Theorem 1.10 is rather clear, and will be explained in the beginning of this section.

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2. Weighted manifolds and their heat kernels.

2.1. Weighted manifolds. We denote by \((M, g)\) a complete Riemannian manifold and by \(\mu_g\) its associated Riemannian measure. Given a positive function \(\sigma\) on \(M\), the weight, we call the metric measured space \((M, g, \sigma \mu_g)\) a weighted Riemannian manifold. We will denote the weighted Riemannian measure \(\sigma \mu_g\) by \(\mu_\sigma\) for short. The following definition aims at enlarging the Laplace operator to the setting of weighted Riemannian manifolds.

**Definition 2.1.** The **weighted Laplace operator**, denoted by \(\Delta_\sigma\), is defined as follows

\[
\Delta_\sigma f := - \sigma^{-1} \text{div}(\sigma \nabla f)
\]

where \(f\) is a smooth function on \(M\).

We refer to the non-weighted (equivalently weighted with the constant function 1) Riemannian Laplace operator as the **usual** Laplace operator.
The weighted Laplace operator may also be defined on $L^2(M, \mu_\sigma)$, the set of square integrable functions with respect to the measure $\mu_\sigma$. The resulting operator generates a semi-group, denoted by $e^{-t\Delta_\sigma}$. We also denote by $p_\sigma(x, y, t)$ the kernels of these operators, satisfying the following identity for any $f \in L^2(M, \mu_\sigma)$

$$e^{-t\Delta_\sigma}(f)(x) = \int_M p_\sigma(x, y, t) f(y) \, d\mu_\sigma(y).$$

As a comprehensive reference around heat operators and their kernels, one can refer to [18, Chapters 6 and 7].

2.2. Stochastic completeness. Provided that the weighted manifold $(M, g, \mu_\sigma)$ to satisfy a mild assumption of the volume growth of balls, one can identify the heat kernel to the fundamental solution of a Cauchy problem. We denote by $\text{vol}_\sigma(B(x, \rho))$ the - weighted - volume of the ball centred at $x$ of radius $\rho$.

**Theorem 2.2.** [18, Corollary 9.6, Theorem 11.8] Let $(M, g, \mu_\sigma)$ be a complete weighted Riemannian manifold of infinite volume. If there is a point $x_0 \in M$ such that

$$\int_1^\infty \frac{t \, dt}{\ln \left( \frac{\text{vol}_\sigma(B(x, \sqrt{t}))}{\text{vol}_\sigma(B(x, 1))} \right)} = \infty$$

then for any bounded continuous initial condition $u_0$ the following Cauchy problem

$$\begin{cases}
\Delta_\sigma u(x, t) + \partial_t u(x, t) = 0 & \\
u_0(\cdot, t) \rightarrow u_0 & \text{in the uniform topology}
\end{cases}$$

has a unique solution given by $e^{-t\Delta_\sigma}(u_0)$.

Note that the condition 2.3 is not too demanding, for example it is satisfied for all complete Riemannian manifold of Ricci curvature bounded from below - hyperbolic manifolds in particular - from the celebrated Bishop-Gromov comparison theorem.

If the Cauchy problem (2.4) has a unique solution, the weighted Riemannian manifold is said to be stochastically complete. Uniqueness gives a way to check that a given kernel is the actual heat kernel of a stochastically complete weighted Riemannian manifold. The following proposition follows from the above discussion, it is well known but we give it here a proof for the reader’s convenience.

**Proposition 2.5.** Let $(M, g)$ a stochastically complete Riemannian manifold and $h$ a positive harmonic function. If the weighted manifold $(M, g, \mu_h^2)$ is stochastically complete then the weighted heat kernel $p_{h^2}(x, y, t)$ satisfies the following identity

$$p_{h^2}(x, y, t) = \frac{p(x, y, t)}{h(x)h(y)}$$

where $p(x, y, t)$ is the usual heat kernel on $(M, g)$.

**Proof:** as emphasised above, to verify that our candidate is the weighted heat kernel one wants to check that it solves the Cauchy problem (2.4). Let us start off showing that it is solution of the weighted heat equation: we freeze the variable $x$, and we denote by $p_t(y)$ the function $(y, t) \mapsto p(x, y, t)$. One wants first to show that the function

$$\varphi_t := y \mapsto \frac{p_t(y)}{h(y)}$$

is solution of the heat equation (2.4)

$$\Delta_{h^2}\varphi_t + \partial_t \varphi_t = 0.$$
We start off the right member left term of the above equation:

\[-\Delta h^2 \varphi_t := \frac{1}{h^2} \text{div} (h^2 \nabla \varphi_t)\]

\[= \frac{1}{h^2} \text{div} \left( h^2 \nabla \left( \frac{p_t}{h} \right) \right)\]

\[= \frac{1}{h^2} \text{div} (h \nabla p_t - p_t \nabla h)\]

\[= \frac{1}{h^2} \left( \nabla h \cdot \nabla p_t - h \Delta p_t - \nabla p_t \cdot \nabla h + p_t \Delta h \right).\]

Since \(h\) is harmonic and \(p_t\) solution of \(\Delta p_t = -\partial_t p_t\), we have

\[-\Delta h^2 \varphi_t = \frac{1}{h^2} \left( h \partial_t p_t \right)\]

\[= \frac{1}{h} \partial_t \left( \frac{p_t}{h} \right) = \partial_t \varphi_t.\]

To conclude, we must show that for every continuous and bounded function \(f\) we have

\[\int_M p_{h^2}(x, y, t) f(y) \, d\mu(y) \rightarrow_{t \to 0} f(x).\]

To do it, we use the fact that such a property holds for the usual heat kernel; since the Riemannian manifold \((M, g, \mu)\) is also stochastically complete by assumption one has

\[\int_M p_h(x, y, t) f(y) \, d\mu(y) = \int_M \frac{p(x, y, t)}{h(y)} \frac{f(y)}{h(x)} h^2(y) \, d\mu(y)\]

\[= \frac{1}{h(x)} \int_M p(x, y, t) f(y) \, d\mu(y)\]

But \(p_t(x, y, t)\) is solution of the Cauchy problem 2.4 associated to the Riemannian manifold \((M, g)\), therefore:

\[\int_M p(x, y, t) f(y) h(y) \, d\mu_g(y) \rightarrow_{t \to 0} f(x) \, h(x)\]

uniformly on every compact set, which completes the proof.

Provided that the weights of two manifolds differs from one another by a positive harmonic function, the above proposition relates the two induced heat kernels by a time independent factor. In particular, the large time behaviour of these heat kernels are alike, the two points \(x\) and \(y\) being fixed. We shall make a crucial use of this fact later on.

The next section aims to relate the large time behaviour of heat kernels with the geometry of the underlined manifold trough functional identities.

2.3. Sobolev inequalities and heat kernel estimates. The following theorem, due to Varopoulos [40], relates an upper bound for the heat kernel with a Sobolev inequality.

\[\text{Theorem 2.7 (see [18] corollary 14.23 page 383). Let } (M, g, \mu_g) \text{ a weighted Riemannian manifold and } n > 2. \text{ The two following assertions are equivalent:}\]
• The Sobolev inequality: there is a constant $C > 0$ such that for all smooth compactly supported function $u$ one has

\[
\left( \int_M |u|^\frac{2n}{n-2} \, d\mu_\sigma \right)^{\frac{n-2}{2n}} \leq C \int_M |
abla u|^2 \, d\mu_\sigma .
\]

• There is a constant $C > 0$ such that for every $x, y \in M$ and all $t > 0$ one has

\[
p_\sigma(x, y, t) \leq C t^{-\frac{n}{2}} .
\]

If Equation (2.8) holds on a weighted Riemannian manifold we say the last to admit a Sobolev inequality of type $n$.

**Remark 2.10.** The condition $n > 2$ may seem artificial. There is indeed a similar statement for $n = 1, 2$, replacing the above Sobolev inequality by a Nash’s one. We refer to Saloff-Coste’s survey [37] and the references therein for a panoramic view of the results around estimates of the heat kernel.

These inequalities have the benefit of being invariant by some kind of weak isometries, the so called rough isometries, preserving the large scale geometry of the underlined manifolds.

**Definition 2.11** (Kanaï [23]). Let $(M_1, d_1, \mu_1)$ and $(M_2, d_2, \mu_2)$ two metric measured spaces. We will say of an application $f : M_1 \to M_2$ that it is a **rough isometry** if the three following conditions hold:

1. there is $\epsilon > 0$ such that the $\epsilon$-neighbourhood of $f(M_1)$ is equal to $M_2$;
2. there is $C > 0$ such that for every $x \in M_1$ we have

\[
C^{-1} \text{vol}_{\mu_1}(B(x, 1)) \leq \text{vol}_{\mu_2}(B(f(x), 1)) \leq C \text{vol}_{\mu_1}(B(x, 1));
\]
3. there is $a, b > 0$ such that for every $x, y \in M_1$ we have

\[
a^{-1} d_1(x, y) - b \leq d_2(f(x), f(y)) \leq a d_1(x, y) + b .
\]

This definition is very closed to the more familiar one of quasi-isometry. However, the latter one does not require any measured structure on the underlined metric spaces; the notion of rough isometry may therefore be considered as an extension of the one of quasi-isometry to the setting of metric measured spaces. This "measured" additional assumption on the volume of small balls is not so much to ask, for example two quasi-isometric Riemannian manifold with injectivity radius bounded and Ricci curvature bounded from below are automatically roughly isometric. The following proposition will be used to prove the Sobolev inequality (2.8) in the case of interest.

**Proposition 2.12.** [13, Proposition 4.1, page 698] The Sobolev Inequality (2.8) is invariant under rough isometries.

We conclude this section by stating another theorem which gives, on the diagonal, an lower bound for the heat kernel, provided a **doubling condition** (see (2.14) below). The theorem stated below generalises the non weighted one proved in [12].

**Theorem 2.13.** [18, Theorem 16.6 page 427] Let $(M, g, \mu_\sigma)$ be a complete weighted manifold and $x \in M$ such that

• there is a constant $C > 0$ satisfying for all $\rho > 0$

\[
\text{vol}_\sigma(B(x, 2\rho)) \leq C \text{ vol}_\sigma(B(x, \rho));
\]
the following upper bound holds for the heat kernel
\[ p_t(x, x, t) \leq \frac{C}{\text{vol}_\sigma \left( B(x, \sqrt{t}) \right)}, \]
then the heat kernel also satisfies the lower bound
\[ p_t(x, x, t) \geq \frac{C}{\text{vol}_\sigma \left( B(x, \sqrt{t}) \right)}. \]

3. The orbital function and the usual heat kernel

We start off reviewing some properties of the heat kernel on \( H^3 \) and its quotient manifolds; hyperbolic manifolds of dimension 3. We will then prove successively the upper bound of Theorem 1.4 and theorem 1.7, giving some control on the orbital function provided some knowledge of the large time behaviour of the heat kernel. On the way, we will define the averaged orbital function and the rough decreasing property required to apply Theorem 1.7. These are ones of the main two ingredients to prove Theorems 1.2 and 1.3.

3.1. Heat kernels of hyperbolic manifolds. The hyperbolic 3-space \( H^3 \) enjoys an explicit formula for its heat kernel denoted \( p_{H^3} \), denoted by \( p_3(x, y, t) \) from now on.

**Theorem 3.1** (see [19]). For any two points \( x, y \in H^3 \) at distance \( \rho \) and for all \( t > 0 \) one has
\[ p_3(x, y, t) = \frac{1}{(4\pi t)^{3/2}} \frac{\rho}{\sinh(\rho)} e^{-t - \frac{\rho^2}{4t}}. \]

For \( \rho \geq 0 \) and \( t > 0 \), we denote by \( p_3(\rho, t) \) the function giving the value of the heat kernel \( p_3(x, y, t) \) for any two points \( x, y \in H^3 \) at distance \( \rho \) from one another. Heat kernels of hyperbolic spaces of different dimensions are related by the following formula (see [19]):
\[ p_{n+2}(\rho, t) = -\frac{e^{-nt}}{2\pi \sinh(\rho)} \partial_\rho p_n(\rho, t). \]

**Remark 3.4.** There is also an explicit formula for the heat kernel of the hyperbolic space of dimension 2, so that, using the above formula one can derive a formula in any dimension. The results of this article can therefore be extended to any dimension. However, our main application being toward Kleinian groups, we will stick to the dimension three in order not to burden this article with quite a lot of computations.

**Lemma 3.5.** For any Kleinian group \( \Gamma \) and any two points \( x, y \) on \( M_\Gamma \) the heat kernel of the manifold \( M_\Gamma \) is given by the formula:
\[ p_\Gamma(x, y, t) := \sum_{\gamma \in \Gamma} p_3(\bar{x}, \gamma \cdot \bar{y}, t). \]

where \( \bar{x} \) and \( \bar{y} \) are any two lifts on \( H^3 \) of the points \( x \) and \( y \).

**Proof:** one needs first to verify that the above series makes sense in \( H^3 \). This is due to the explicit formula given by 3.1 which shows that the heat kernel decreases spatially super-exponentially whereas the orbital function \( N_\Gamma(x, y, \rho) \) as at most exponential growth in the variable \( \rho \) for any two points fixed points \( x \) and \( y \). To descend to something well defined on the quotient manifold \( M_\Gamma \), this series must not depend on the choices involved about the lifted points \( \bar{x} \) and \( \bar{y} \). Since we average this series with respect to the second variable along the group \( \Gamma \), it is automatic.
that the series does not depend on a lift of $y$. To show that the same property holds for the first factor as well, let us note that the diagonal action of an isometry on $H^3 \times H^3$ leaves the kernel $p_3(x, y, t)$ invariant since, $t$ being fixed, it depends only on the distance between $x$ and $y$. To conclude we show that $p_{\Gamma} = p_{M_{\Gamma}}$ in applying Theorem 2.2. In fact, since the volume growth of balls in hyperbolic manifolds is not faster than exponential, we know that the complete hyperbolic manifold are stochastically complete. Therefore, it remains to verify that the series $p_{\Gamma}$ satisfies the Cauchy problem (2.4), which follows readily from the fact that $p_3$ is the heat kernel of the hyperbolic space $H^3$.

3.2. The orbital function upper bound. This subsection is entirely dedicated to prove the upper bound of Theorem 1.4 announced in the introduction, that we recall here for the reader’s convenience.

**Theorem.** There is a universal constant $C_+ > 0$ such that for every Kleinian group $\Gamma$ and all $x, y \in \mathbb{H}^3$ one has for $t > 1$

$$\frac{N_\Gamma(x, y, t)}{e^{2t}} \leq C_+ \sqrt{t} p_{M_{\Gamma}} \left( x, y, \frac{t}{2} \right).$$

The above theorem relies on Lemma 3.5 and with the fact that the heat kernel $p_3(x, y, t)$ only depends on the distance between $x$ and $y$, which relates the heat kernel to the orbital function. Secondly, we use the explicit Formula given by Theorem 3.1 to conclude.

Given two points $x, y \in X$ and $\rho_2 > \rho_1$, we denote by $A_{\Gamma}(x, y, \rho_1, \rho_2)$ the following subset of $\Gamma$:

$$\{ \gamma \in \Gamma, \rho_1 \leq d(x, \gamma \cdot y) < \rho_2 \},$$

which is finite since $\Gamma$ acts discretely on $X$.

**Lemma 3.7.** Let $X$ be a metric space, $x, y$ two points in $X$ and $\Gamma$ a discrete subgroup of $X$ such that $\Gamma \cdot y$ is a discrete set. Given $b > a \geq 0$ and $f : [a, b] \to \mathbb{R}$
a smooth function we have:

\[
\sum_{\gamma \in A_\Gamma(x,y,a,b)} f(d(x, \gamma \cdot y)) = \]

\[
N_\Gamma(x, y, b)f(b) - N_\Gamma(x, y, a)f(a) - \int_{[a, b]} N_\Gamma(x, y, \rho)f'(\rho)d\rho .
\]

(3.8)

**Proof**: the proof consists of doing a discrete integration by part. Let \( n \in \mathbb{N}^\ast \). We chop off the interval \([a, b]\) in \( n \) intervals of same length; let \( x_k := a + \frac{k(b-a)}{n} \) such that

\[
A_\Gamma(x, y, a, b) = \bigcup_{1 \leq k \leq n} A_\Gamma(x, y, x_{k-1}, x_k) .
\]

So that we also have

\[
\sum_{\gamma \in A_\Gamma(x,y,a,b)} f(d(x, \gamma \cdot y)) = \sum_{1 \leq k \leq n} \sum_{\gamma \in A_\Gamma(x,y,x_{k-1},x_k)} f(d(x, \gamma \cdot y)) .
\]

The function \( f \) being continuous one also has

\[
\sum_{\gamma \in A_\Gamma(x,y,a,b)} f(d(x, \gamma \cdot y)) = \lim_{n \to +\infty} \sum_{1 \leq k \leq n-1} \sharp A_\Gamma(x, y, x_{k-1}, x_k)f(x_k) .
\]

Now we work the right side of the above equation, \( n \) being fixed. We will perform a discrete integration by part and then we will let \( n \to +\infty \) to recognize the Riemann sum appearing in Equation (3.8). We "integrate" the sequence \( \sharp A_\Gamma(x, y, x_{k-1}, x_k) \) by \( N_\Gamma(x, y, x_k) \) and we "differentiate" \( f(x_k) \) to get

\[
\sum_{1 \leq k \leq n} \sharp A_\Gamma(x, y, x_{k-1}, x_k)f(x_k) =
\]

\[
N_\Gamma(x, y, b)f(b) - \sum_{1 \leq k \leq n-1} N_\Gamma(x, y, x_k)[f(x_k) - f(x_{k+1})] .
\]

We rewrite the right sum appearing in the above equation as

\[
\frac{b-a}{n} \sum_{1 \leq k \leq n-1} N_\Gamma(x, y, x_k) \left( \frac{f(x_k) - f(x_{k+1})}{\frac{b-a}{n}} \right) .
\]

The function \( f \) being smooth one can replace the parenthesis

\[
\left( \frac{f(x_k) - f(x_{k+1})}{\frac{b-a}{n}} \right)
\]

by \( f'(x_k) \) so that the question is now to determine whether

\[
\int_{[a, b]} N_\Gamma(x, y, \rho)f'(\rho)d\rho
\]

is actually limit of the Riemann sums associated to the function

\[
\rho \mapsto N_\Gamma(x, y, \rho)f'(\rho) .
\]

Which is clear here since the \( \Gamma \)-orbit of the point \( y \) is discrete, implying this function to be piecewise continuous. \( \blacksquare \)

We now use the above lemma 3.7 to get the following integral formula for the heat kernel.
Corollary 3.9. Let $\Gamma$ a Kleinian group, then

\begin{equation}
pr(x, y, t) = -\int_{\mathbb{R}_+} N_\Gamma(x, y, \rho) \partial_\rho p_3(\rho, t) d\rho
\end{equation}

Proof: Lemma 3.7 with the data

\[ f : \mathbb{R}_+ \rightarrow \mathbb{R}, \quad \rho \mapsto p_3(\rho, t), \]

and $b \rightarrow +\infty$ gives

\[ \sum_{\gamma \in \Gamma} p_3(x, \gamma \cdot y, t) = \lim_{b \rightarrow +\infty} \left( N_\Gamma(x, y, b)p_3(b, t) - \int_{[0,b]} N_\Gamma(x, y, \rho) \partial_\rho p_3(\rho, t) d\rho \right). \]

The heat kernel $p_3(x, y, \rho)$ being decreasing spatially super-exponentially, one also has

\[ \lim_{b \rightarrow +\infty} N_\Gamma(x, y, b)p_3(b, t) = 0, \]

and therefore

\begin{equation}
pr(x, y, t) = -\lim_{b \rightarrow +\infty} \int_{[0,b]} N_\Gamma(x, y, \rho) \partial_\rho p_3(\rho, t) d\rho.
\end{equation}

From Equation 3.3 we know that

\[-N_\Gamma(x, y, \rho)\partial_\rho p_3(\rho, t) \geq 0,\]

so that, letting $b \rightarrow \infty$, we get

\begin{equation}
pr(x, y, t) = -\int_{\mathbb{R}_+} N_\Gamma(x, y, \rho) \partial_\rho p_3(\rho, t) d\rho,
\end{equation}

which is the desired result. 

To conclude the proof of Theorem 1.4 we use the explicit formula for the heat kernel on $\mathbb{H}^3$ as announced. The key feature, from our perspective, given by Formula 3.1 is the drift of the Brownian motion. Brownian paths tend to concentrate in large time on an annulus of small radius $2t - \sqrt{t}$ and of large radius $2t + \sqrt{t}$. This $\sqrt{t}$-lack-of-concentration on the perfect sphere of radius $2t$ is responsible for the non optimality of the upper bound of Theorem 1.4, even for compact manifolds. We shall come back later on this and see how to strengthen the result adding an extra assumption on the $\Gamma$-action, cf. Subsection 3.3.

Proof of Theorem 1.4: Because the function $-\partial_\rho p_3(\rho, t)$ is positive, one has the following lower bound of the left member of (3.11):

\begin{equation}
pr(x, y, t) \geq \int_{[2t,2t+1]} N(x, y, \rho) [-\partial_\rho p_3(\rho, t)] d\rho.
\end{equation}

The orbital function $\rho \mapsto N_\Gamma(x, y, \rho)$ being increasing and non negative as well, one also has

\[ pr(x, y, t) \geq N_\Gamma(x, y, 2t) \int_{[2t,2t+1]} -\partial_\rho p_3(\rho, t) d\rho. \]

Therefore, the existence of a constant $C_1 > 0$ such that

\begin{equation}
[-\partial_\rho p_3](\rho, t) \geq \frac{C_1}{\sqrt{t}} e^{-4t},
\end{equation}

\end{document}
would give
\[ p_\Gamma(x, y, t) \geq N_\Gamma(x, y, 2t) \frac{C_1}{\sqrt{t}} e^{-4t}, \]
and thus the upper bound (1.5) of Theorem 1.4 in substituting \( t \) with \( 2t \).

To show that the lower bound (3.14) holds, we recall Formula 3.2:
\[ p_3(\rho, t) = \frac{1}{(4\pi t)^{\frac{3}{2}}} \frac{\rho}{\sinh(\rho)} e^{-\rho^2/4t}, \]
hence,
\[ \partial_\rho p_3(\rho, t) = \frac{1}{(4\pi t)^{\frac{3}{2}}} \left( \frac{1}{\sinh(\rho)} - \frac{\rho \cosh(\rho)}{\sinh(\rho)^2} - \frac{\rho^2}{2t \sinh(\rho)} \right) e^{-\rho^2/4t}, \]
and then,
\[ \partial_\rho^2 p_3(\rho, t) = \frac{1}{(4\pi t)^{\frac{3}{2}}} \left( 1 - \frac{\rho \cosh(\rho)}{\sinh(\rho)} - \frac{\rho^2}{2t} \right) e^{-\rho^2/4t}. \]

From which one gets a constant \( C_2 > 0 \) such that for every \( t > 1 \) and \( \rho \in [2t, 2t + 1] \) we have:
\[ -\partial_\rho p_3(\rho, t) \geq \frac{C_2}{(4\pi t)^{\frac{3}{2}}} e^{-2t} \left( 2t + \frac{4t^2}{2t} \right) e^{-\frac{4t^2}{4t}} \geq \frac{C_3}{t^\frac{3}{2}} e^{-4t}, \]
which concludes the proof.

As already emphasised, this theorem may be re-enforced under some extra assumptions.

3.3. Sharper estimates toward finer results. Under a "rough decreasing" assumption defined below, one can sharpen the proof of Theorem 1.4 to get a better control of the orbital function growth.
Definition 3.16. Let $I \subset \mathbb{R}$ an interval. A function $f : I \to \mathbb{R}_+$ is said to be roughly decreasing if there is a positive constant $C$ such that for $\rho$ large enough and any $s \geq t \geq \rho$, one has

$$f(s) \leq C f(t).$$

This rough decreasing property will be required for the averaged orbital function defined as

$$\tilde{N}(x, y, \rho) := \rho \mapsto \frac{N(x, y, \rho)}{e^{2\rho}},$$

where $e^{2\rho}$ is thought as the volume of an hyperbolic ball of radius $\rho$. Before entering in the proof of Theorem 1.7, we recall here its statement for the reader’s convenience.

**Theorem.** Let $\Gamma$ be any Kleinian group such that

- there is $\alpha \geq 0$ such that there is two points $x, y \in M_\Gamma$ and two constants $C_-, C_+$ such that for $t$ large enough one has
  \begin{equation}
  \begin{aligned}
  c_- t^{-\alpha} &\leq p_\Gamma(x, y, t) \leq c_+ t^{-\alpha} \\
  \end{aligned}
  \end{equation}

- the averaged orbital function $\tilde{N}(x, y, \rho)$ is roughly decreasing,

then there is two positive constants $C_-$ and $C_+$ such that

$$\frac{C_- e^{2\rho}}{\rho^\alpha} \leq N_\Gamma(x, \rho) \leq \frac{C_+ e^{2\rho}}{\rho^\alpha}.$$ 

We shall see later that the first condition made on the heat kernel is fulfilled in natural examples, as nilpotent covers of a given compact hyperbolic manifold or for thick degenerate Kleinian groups, see Sections 4 and 5. The second condition is quite natural: up to the author’s knowledge, we do not know any orbital function not satisfying it. However it doesn’t seem easy to verify it in general.

**Proof:** We prove first the upper bound appearing in the conclusion of Theorem 1.7 and then we will use it to prove the lower one. For what follows the points $x, y$ are fixed, so that all the functions appearing along the proof have to be seen as functions defined on $\mathbb{R}_+$. Given two functions $f, g : \mathbb{R}_+ \to \mathbb{R}_+$, we denote $f(t) \prec g(t)$ the transitive relation: there is a constant $C > 0$ such that $f(t) \leq C g(t)$ for $t$ large enough, omitting the value of the constants.

The proof of the upper bound is rather similar to the proof of Theorem 1.4. Recall first Equation 3.11:

$$p_\Gamma(x, y, t) = -\int_{\mathbb{R}_+} N_\Gamma(x, y, \rho) \partial_\rho p_\Gamma(\rho, t) d\rho .$$

Using the upper bound assumed on the heat kernel, one has

$$\int_{\mathbb{R}_+} N_\Gamma(x, y, \rho) [-\partial_\rho p_\Gamma(\rho, t)]d\rho \prec \frac{1}{t^{\alpha}} .$$

Instead of the rough lower bound giving by Equation(3.13) we prefer

$$\int_{2t-\sqrt{t}}^{2t+\sqrt{t}} N_\Gamma(x, y, \rho) [-\partial_\rho p_\Gamma(\rho, t)]d\rho \prec \frac{1}{t^{\alpha}},$$

that can be rewrite as

$$\int_{2t-\sqrt{t}}^{2t+\sqrt{t}} \tilde{N}_\Gamma(x, y, \rho) e^{2\rho}[-\partial_\rho p_\Gamma(\rho, t)]d\rho \prec \frac{1}{t^{\alpha}} .$$
A STOCHASTIC APPROACH TO COUNTING PROBLEMS.

Using the rough decreasing assumption made on the averaged orbital function we get

$$\tilde{N}_Γ(x, y, 2t + \sqrt{t}) \int_{2t - \sqrt{t}}^{2t + \sqrt{t}} \left( -e^{2ρ} \partial_ρ p_3(ρ, t) dρ \right) \prec \frac{1}{t^α}.$$  

Recalling Equation (3.15) one has

$$e^{2ρ} \partial_ρ p_3(ρ, t) = \frac{1}{(4πt)^{3/2}} \left( 1 - \frac{ρ \cosh(ρ)}{\sinh(ρ)} - \frac{ρ^2}{2t} \right) e^{-t - \frac{ρ^2}{4t} + t^2}.$$  

from which we deduce the existence of two universal constants $C_1, C_2 > 0$ such that for every $ρ \in (t, 3t)$ and $t > 1$ one has:

$$C_1 e^{-t - \frac{ρ^2}{4t} + t^2} \leq -e^{2ρ} \partial_ρ p_3(ρ, t) \leq C_2 e^{-t - \frac{ρ^2}{4t} + t^2}.$$  

In particular,

$$\int_{2t - \sqrt{t}}^{2t + \sqrt{t}} -e^{2ρ} \partial_ρ p_3(ρ, t) dρ \succ C.$$  

So that Equation (3.18) gives

$$\tilde{N}_Γ(x, y, 2t + \sqrt{t}) \prec \frac{1}{t^α}.$$  

Note that the function $g := t \mapsto 2t + \sqrt{t}$ being asymptotic to the function $t \mapsto 2t$, one knows the inverse function $g^{-1}$ to be asymptotic to $2t$ as well. Therefore, since $(g^{-1})^α \sim (2t)^α$ and one gets the following in substituting $t$ with $g^{-1}(t)$ in the above inequality:

$$\tilde{N}_Γ(x, y, t) \prec \frac{1}{t^α},$$  

which gives the desired upper bound in multiplying both terms by $e^{2t}$.

The strategy to get the lower bound is rather similar to what we have done for the upper one, but more subtle. Let us mention that we will need the already proven upper bound in order to get the lower one.

We can not bound from below as roughly as we did for the upper bound, we prefer to start off the mid-rough approximation

$$p_Γ(x, y, t) \sim \int_{t}^{3t} N_Γ(x, y, ρ) \partial_ρ p_3(ρ, t) dρ,$$  

valid since the polynomial growth assumption made on the heat kernel and because the two remaining integrals are decreasing faster than any rational function (from the rough upper bound $N_Γ(x, y, ρ) \prec e^{2ρ}$). From the polynomial lower bound assumed on the heat kernel, we also have

$$\frac{1}{t^α} \prec \int_{t}^{3t} N_Γ(x, y, ρ) \partial_ρ p_3(ρ, t) dρ.$$  

We now use the right side of Inequality (3.19) to get

$$\frac{1}{t^α} \prec \int_{t}^{3t} \frac{N_Γ(x, y, ρ)}{e^{2ρ}} \frac{e^{-t - \frac{ρ^2}{4t} + t^2}}{t^2} dρ.$$
which can be rewrite

\[(3.20) \quad \frac{1}{t^{\alpha}} \prec \int_t^{3t} \tilde{N}_T(x, y, \rho) \frac{e^{-t - \frac{\rho^2}{2t} + \rho}}{t^{\frac{1}{2}}} \, d\rho.\]

Let chop off the above integral the following way, \(k > 0\) will be chosen latter on.

\[
\int_t^{3t} \frac{N_T(x, y, \rho)}{e^{2\rho}} \frac{e^{-t - \frac{\rho^2}{2t} + \rho}}{t^{\frac{1}{2}}} \, d\rho = \int_t^{2t-k\sqrt{t}} \frac{N_T(x, y, \rho)}{e^{2\rho}} \frac{e^{-t - \frac{\rho^2}{2t} + \rho}}{t^{\frac{1}{2}}} \, d\rho \quad := I_1(t)
\]
\[
+ \int_{2t-k\sqrt{t}}^{2t+k\sqrt{t}} \frac{N_T(x, y, \rho)}{e^{2\rho}} \frac{e^{-t - \frac{\rho^2}{2t} + \rho}}{t^{\frac{1}{2}}} \, d\rho \quad := I_2(t)
\]
\[
+ \int_{2t+k\sqrt{t}}^{3t} \frac{N_T(x, y, \rho)}{e^{2\rho}} \frac{e^{-t - \frac{\rho^2}{2t} + \rho}}{t^{\frac{1}{2}}} \, d\rho \quad := I_3(t).
\]

The goal is now to show the

**Lemma 3.21.**

\[I_i(t) \prec \frac{e^{-k^2}}{t^{\alpha}}\]

for \(i = 1, 3\).

Before proving the above lemma, let us show how to conclude using almost the same argumentation than the one used to get the upper bound. Combined with Equation (3.20) the above lemma gives

\[
\frac{1}{t^{\alpha}} \prec \frac{e^{-k^2}}{t^{\alpha}} + I_2(t) + \frac{e^{-k^2}}{t^{\alpha}},
\]

taking \(k\) large enough we get \(t^{-\alpha} \prec I_2(t)\), namely:

\[
\frac{1}{t^{\alpha}} \prec \int_{2t-k\sqrt{t}}^{2t+k\sqrt{t}} \frac{N_T(x, y, \rho)}{e^{2\rho}} \frac{e^{-t - \frac{\rho^2}{2t} + \rho}}{t^{\frac{1}{2}}} \, d\rho.
\]

Using the rough decreasing assumption made one has

\[
\frac{1}{t^{\alpha}} \prec \tilde{N}_T(x, y, 2t - k\sqrt{t}) \int_{2t-k\sqrt{t}}^{2t+k\sqrt{t}} \frac{e^{-t - \frac{\rho^2}{2t} + \rho}}{t^{\frac{1}{2}}} \, d\rho.
\]

The right above integral being bounded, we also have

\[
\frac{1}{t^{\alpha}} \prec \tilde{N}_T(x, y, 2t - k\sqrt{t}).
\]

We conclude, the same way as done for the upper bound, in noticing that \(2t - k\sqrt{t} \sim 2t\) so that one can substitute \(2t - k\sqrt{t}\) with \(2t\) in the right member of the above equation.

\[\blacksquare\]

It remains to show that Lemma 3.21 holds.

**Proof:** (of Lemma 3.21) The proofs for bounding from above the integrals \(I_1(t)\) and \(I_3(t)\) being very similar, we will perform here only the computation concerning
$I_1(t)$. We start off using the already proven upper bound $N_f(x, y, \rho) \prec \frac{e^{2\rho}}{t^\alpha}$:

$$I_1(t) := \int_t^{2t-k\sqrt{t}} \frac{N_f(x, y, \rho) e^{-\frac{\rho^2}{2\sqrt{t}}}}{t^\alpha} \, d\rho \prec \int_t^{2t-k\sqrt{t}} \frac{e^{-(\frac{2t-\rho^2}{2\sqrt{t}})^2}}{\rho^\alpha \sqrt{t}} \, d\rho \prec \frac{1}{t^\alpha} \int_t^{2t-k\sqrt{t}} e^{-(\frac{2t-\rho^2}{2\sqrt{t}})^2} \, d\rho.$$

We substitute $\rho$ by $\frac{2t-\rho^2}{2\sqrt{t}}$ in the above integral to get

$$\int_t^{2t-k\sqrt{t}} e^{-(\frac{2t-\rho^2}{2\sqrt{t}})^2} \, d\rho = \int_k^{\sqrt{t}} e^{-u^2} \frac{2\sqrt{t} du}{\sqrt{t}} \leq 2e^{-k^2},$$

which gives

$$I_1(t) \prec \frac{e^{-k^2}}{t^\alpha},$$

the expected result.

### 3.4. First application: nilpotent covers of compact manifolds.

We apply Theorems 1.4 and 1.7 to nilpotent covers of compact hyperbolic manifold for which the large time behaviour of the heat kernel is well known. We refer to [37] for a comprehensive survey about the study of the large time behaviour of the heat kernel in this setting.

**Theorem 3.22.** See [37] and the references therein. Let $M$ be a manifold roughly isometric to a nilpotent cover of a compact manifold $M_0$, then there are two constant $C_-, C_+$ such that for every two points $x, y$ the heat kernel $p_M(x, y, t)$ satisfies for $t$ large enough

$$\frac{C_-}{\text{vol} \left( B(x, \sqrt{t}) \right)} \leq p_M(x, y, t) \leq \frac{C_+}{\text{vol} \left( B(x, \sqrt{t}) \right)}.$$

The proof of this theorem is a corollary of Grigor’yan’s work, stating an equivalence between the large scale geometry of the manifold and the long time behaviour of its associated heat kernel, we will come back to this aspect through the last Remark 5.7 of Subsection 5.1.

From Milnor’s work [29], improved later on by Pansu [32], one knows that the volume growth of finitely generated nilpotent groups is polynomial: there are constants $\alpha, C_1, C_+$ such that

$$C_- \rho^\alpha \leq \text{vol} \left( B(x, \rho) \right) \leq C_+ \rho^\alpha.$$

**Theorem 3.22** combined with Theorems 1.4, Theorem 1.7 and the above discussion gives Corollary 1.9.

### 4. The thick tamed degenerate hyperbolic manifolds.

This section is devoted to the description of the topology and the geometry of degenerate hyperbolic manifolds of positive injectivity radius, called thick. It basically gathers results from the literature.
4.1. **Their geometry.** In this subsection we gather results describing the geometry and the topology of a thick hyperbolic manifold with finitely generated fundamental group. The Tameness theorem, a result proved by Agol and Calegari-Wise independently (see [8]), asserts that the quotient of $\mathbb{H}^3$ by a finitely generated Kleinian group is homeomorphic to the interior of a compact manifold. On the geometric side, this theorem asserts that if $\Gamma$ is a finitely generated Kleinian group acting on $\mathbb{H}^3$ then the quotient manifold can be described as a compact submanifold $K$ on which one has attached finitely many ends $E_1, \ldots, E_p$. These ends can be put in two families: the **geometrically finite ends** and the **degenerate ends**, see Figure 3. The geometry of the geometrically finite ones is well understood and features the same behaviour than the 2-dimensional funnel.

**Theorem 4.1.** [34] Let $E$ be a geometrically finite end. Then there is a Riemannian surface $(S, g)$ such that $E$ is roughly-isometric to the twisted metric space 

\[ ((S \times \mathbb{R}_+), e^{2t}g + dt^2) . \]

On the other hand, degenerate ends have been described more recently, thanks to the work of Minsky (see [30]). The picture is now quite simple and provided the manifold $M_\Gamma$ to be thick, the degenerate ends can not be too wild: they are quasi-cylinders.

**Definition 4.2.** A **quasi-cylinder** is a manifold with boundary which is roughly isometric to $\mathbb{R}_+$.

**Theorem 4.3.** [4, Theorem 16.3] Let $\Gamma$ a finitely generated Kleinian group such that $M_\Gamma$ is a thick tamed manifold, then all its degenerate ends are quasi-cylinders.

![Figure 3](image)

**Figure 3.** Here the manifold carries 3 ends, $E_1$, $E_2$ and $E_3$. The fist one is geometrically finite and the two last ones are degenerate. The chosen compact core, not unique, is represented in orange.

In [4], the authors actually prove, under the thickness assumption, that any product region - a region which is topologically given by the product on an interval and some surface - in the convex core is $(1, C)$-quasi-isometric to an interval, the constant $C$ depending only on the rank of the underlined surface group and the injectivity radius. In particular, their result apply for degenerate ends. This quasi-isometry to $\mathbb{R}_+$ is automatically upgraded to a rough isometry using again the positivity of the injectivity radius.
4.2. **Thurston-Sullivan’s harmonic function.** In this subsection we introduce the Thurston-Sullivan’s harmonic function, which will ultimately plays the role of the weight we will use in order to estimate the large time behaviour of the heat kernel.

**Theorem 4.4.** [27] [5] [4] Let $\Gamma$ be a finitely generated Kleinian group such that the manifold $M_\Gamma$ is thick and carries both degenerate and geometrically finite ends, then there exists an harmonic function $h$ such that

1. the function $h_0$ has linear growth in degenerate ends: given an marked point $x_0 \in M_\Gamma$, there is two constants $a, b > 0$ satisfying that for all $x$ in the union of all the degenerate ends then
   \[ a^{-1}d(x_0, x) - b \leq h_0(x) \leq ad(x_0, x) + b; \]

2. the harmonic function $h$ converges to 0 in the geometrically finite ends.

The story of this theorem is more intricate than the above statement might let think. In [27, Theorem 3 page 135], carrying further Thurston’s ideas, Sullivan showed that a non constant harmonic function has to have linear growth on quasi-cylinder. Latter on, Bishop and Jones [5, lemma 1.7] constructed a positive harmonic function in the full generality of tamed thick hyperbolic manifolds without requiring the ends to be quasi-cylinder. Moreover they classify them according to the number of degenerate ends. Roughly, there showed that the set of the harmonic functions described above is the convex cone generated by the finitely many harmonic function;

\[ \sum_{1 \leq i \leq d} \lambda_i h_i(x), \]

where $d$ is the number of degenerate ends, $h_i$ is a definite harmonic function and the vector $(\lambda_i)_{1 \leq i \leq d}$ has positive coordinates. The function $h_0$ appearing in Theorem 4.4 can be taken randomly in the previous family. We will come back on this aspect in the next section. We will denote by $h_c$ the harmonic function $h_0 + c$, which enjoys the same properties than $h_0$, except that it converges to $c$ in the geometrically finite ends.

Lacking a precise description of the degenerate ends, they could not manage to get the linear growth. Theorem 4.3 comes here to fill this lack in showing that the ends are actual quasi-cylinders, so that Thurston-Sullivan’s argument was enough in the end, to get existence at least.

### 5. Proof of the main theorems 1.2 and 1.3

The two counting Theorems 1.2 and 1.3 announced in our introduction are corollaries of Theorems 1.4, 1.7 and 1.10. Namely, combined with Theorems 1.4 and 1.7 one readily gets Theorem 1.2 and the first part of Theorem 1.3. Second part of Theorem 1.3 follows from Theorem 1.7 combined with second part of Theorem 1.10 which gives that for every point $x \in M_\Gamma$ there is two constants $C_-$ and $C_+$ such that for $\rho$ large enough one has

\[ \frac{C_- e^{2\rho}}{\rho^2} \leq N_\Gamma(x, x, \rho) \leq \frac{C_+ e^{2\rho}}{\rho^2}. \]

This inequalities self improves immediately to the statement proposed in Theorem 1.3 since one has

\[ N_\Gamma(x, x, \rho - d(x, y)) \leq N_\Gamma(x, y, \rho) \leq N_\Gamma(x, x, \rho + d(x, y)), \]

which gives the desired result, up to enlarging the constants.
It remains then to prove Theorem 1.10; estimating the large time behaviour of the heat kernel on thick degenerate hyperbolic manifolds.

The proof splits depending on whether or not $M_\Gamma$ carries geometrically finite ends:

- if all the ends of $M_\Gamma$ are degenerate, we will say that $\Gamma$ is **fully degenerate**;
- if $M_\Gamma$ carries both degenerate ends and geometrically finite ones, we will say that $\Gamma$ is of **mixed type**.

We shall deal with the fully degenerate case first.

5.1. **The fully degenerate case.** We have to prove the following

**Theorem 5.1.** Let $M_\Gamma$ be fully degenerate, then there is two constants $C_-, C_+$ such that for any two points $x, y \in M_\Gamma$, for $t$ big enough we have

$$
\frac{C_-}{\sqrt{t}} \leq p_\Gamma(x, y, t) \leq \frac{C_+}{\sqrt{t}}.
$$

One can deduce the above theorem from [16, Theorem 2.3 (i) page 10]. However, we prefer to give it here a proof in order to bypass all the refined technology involved in [16] which was developed for harder-to-get estimates.

In the case where the manifold $M_\Gamma$ carries only two degenerate ends, as it is for the historical examples given by degeneration of quasi-Fuchsian manifolds [28], it follows from Theorem 4.3 that $M_\Gamma$ is roughly isometric to $\mathbb{Z}$. If one would be able to compare the Brownian motion on $M_\Gamma$ and the random walk on $\mathbb{Z}$, the intuition behind Theorem 5.1 becomes clear since the local limit theorem for a random walk on $\mathbb{Z}$ is well known to behave as in the above theorem.

Theorem 5.1 implies in particular that the Brownian motion is recurrent.

**Definition 5.2.** Let $M$ be a complete Riemannian manifold. The Brownian motion on $M$ is said to be **recurrent** if there is a $x, y \in M$ such that

$$
\int_1^{+\infty} p_M(x, y, t) \, dt = +\infty.
$$

As a corollary, we get first part of Theorem 1.2.

**Corollary 5.3.** Let $\Gamma$ a thick fully degenerate Kleinian group, then the group $\Gamma$ is divergent. Equivalently, the geodesic flow is recurrent with respect to the Liouville measure.

The fact that the group is divergent straightforward follows from Sullivan’s work [39, Proposition 23]. The divergence of the group is equivalent to the recurrence of the geodesic flow with respect to the unique Bowen-Margulis-Sullivan measure [36, Theorem 1.7]. From uniqueness, since the critical exponent is 2 in our setting, this measure has to be the Liouville measure.

**Proof of Theorem 5.1:** The proof relies on the discretisation of a Riemannian manifold $M$. A discretisation of $M$ is a graph $G$ built as follows: given an $\epsilon > 0$ we define the vertex of $G$ as any $\epsilon$-separated maximal family of points of $M$. An edge relates two vertexes if and only if the two points in consideration are at distance at most $2\epsilon$. See [3, Section 1]. A discretisation, as a graph, comes with a structure of metric measured space in taking the classical distance given by the edges and the counting measure.
A STOCHASTIC APPROACH TO COUNTING PROBLEMS.

From the work performed in the previous section, a discretisation of a thick manifold $M_\Gamma$ associated to a fully degenerate Kleinian group would be given by the graph built from a rooted point - the convex core - on which we attach $d$ copies of $\mathbb{N}$ - corresponding to the number of degenerate ends - as in Figure 4. We denote such a graph by $G_d$. A very first example of two metric measured spaces roughly isometric is actually given by a Riemannian manifold with Ricci curvature bounded from below and of positive injectivity radius and one of its discretisation [24, Section 6].

In [13] the authors proved that many large scale properties of a metric measured space are invariant under rough isometries, as the doubling volume property (2.14) and the Poincaré inequality (see below). The following theorem is therefore "invariant" under rough isometries.

**Theorem 5.4.** [17], see also [37] and the references therein. Let $(M,g,\sigma)$ be a weighted complete manifold, the two following proposition are equivalent:

- The two sided heat kernel estimates: there is four constants $c_\pm$ and $C_\pm$ such that
  \[
  C_- \frac{e^{-c_- \frac{d(x,y)^2}{4t}}}{\text{vol}_\sigma(B(x,\sqrt{t}))} \leq p_\sigma(x,y,t) \leq C_+ \frac{e^{-c_+ \frac{d(x,y)^2}{4t}}}{\text{vol}_\sigma(B(x,\sqrt{t}))}.
  \]

- the conjunction of the doubling volume property and the following Poincaré inequality: There is $k,P > 0$ such that for any $x \in M$, $r > 0$ and Lipschitz function $f$ we have
  \[
  \int_{B(x,r)} |f - \tilde{f}|^2 d\mu \leq P r^2 \int_{B(x,kr)} |\nabla f|^2 d\mu,
  \]

where $\tilde{f}$ is the mean of $f$ on the ball $B(x,r)$.

So that Theorem 5.1 will readily follow if one would be able to make sense for graphs of the Poincaré inequality above and prove it for the graphs $G_d$, since the volume doubling property is immediate here. The only symbol which does not make sense is the gradient of a function in the setting of a graph, that should be replace (see [3, Theorem 2.1]) for the discrete analogue

\[
\delta f(x) := \sqrt{\sum_{y \sim x} |f(y) - f(x)|^2},
\]

where $x \sim y$ stands for the relation "$x$ and $y$ are related by an edge of the graph".
The proof of the Poincaré inequality for the graphs $G_d$ is essentially the same than the short - less than a page - one proposed by Kleiner and Saloff-Costes in [25, Theorem 2.2] at the very beginning of the paper. The main idea being that it follows whenever one has "few canonical short paths" relating any two points.

**Proof of the Poincaré inequality:** here $\mu$ stands for the counting measure on the graph $G_d$. Let $f$ be a function on $G_d$, we start off the left member of Equation (5.5):

$$
\int_{B(x,r)} |f - \bar{f}|^2 \, d\mu = \int_{B(x,r)} \left( \frac{1}{\text{vol}(B(x,r))} \int_{B(x,r)} f(z) - f(y) \, d\mu(y) \right)^2 \, d\mu(z).
$$

Since we have the lower bound $\text{vol}(B(x,r)) \geq r$ we get

$$
\int_{B(x,r)} |f - \bar{f}|^2 \, d\mu \leq \frac{1}{r^2} \int_{B(x,r)} \left( \int_{B(x,r)} f(z) - f(y) \, d\mu(y) \right)^2 \, d\mu(z).
$$

Using Cauchy-Schwartz identity one has

$$
\left( \int_{B(x,r)} f(z) - f(y) \, d\mu(y) \right)^2 \, d\mu(z) \leq \text{vol}(B(x,r)) \int_{B(x,r)} (f(z) - f(y))^2 \, d\mu(y).
$$

We now use the upper bound $\text{vol}(B(x,r)) \leq d \cdot r$ to get

(5.6) \quad \int_{B(x,r)} |f - \bar{f}|^2 \, d\mu \leq \frac{d}{r} \int_{B(x,r)} \int_{B(x,r)} (f(z) - f(y))^2 \, d\mu(y) \, d\mu(z).

Let us denote by $\gamma_{y,z} = (\omega_0 = y, \omega_1, ..., \omega_n = z)$ the (unique) geodesic of $G_d$ relating the points $x$ and $y$, where $n = d(x, y)$. Writing

$$
f(z) - f(y) \leq \sum_{1 \leq k \leq n} (f(\omega_i) - f(\omega_{i-1})) ,
$$

and using again the Cauchy Schwartz identity one gets

$$(f(z) - f(y))^2 \leq n \sum_{1 \leq i \leq n} (f(\omega_i) - f(\omega_{i-1}))^2 .
$$

Note that $n \leq 2r$ since $x, y \in B(x, r)$. Note also that

$$
\sum_{1 \leq k \leq n} (f(\omega_i) - f(\omega_{i-1}))^2 \leq \sum_{\omega \in \gamma_{y,z}} (\delta f)^2(\omega) ,
$$

which gives

$$(f(z) - f(y))^2 \leq 2r \sum_{\omega \in \gamma_{y,z}} (\delta f)^2(\omega) .
$$

Looking backward at 5.6, we have

$$
\int_{B(x,r)} |f - \bar{f}|^2 \, d\mu \leq 2d \int_{B(x,r)} \int_{B(x,r)B(x,r)} (\delta f)^2(\omega) \, d\mu(y) \, d\mu(z).
$$

Any geodesic $\gamma_{z,y}$ of $G_d$ is exactly given by its endpoints, and, again because $\gamma_{z,y}$ is a geodesic, a point $p \in B(x, r)$ belongs to $\gamma_{z,y}$ at most once. Since we have prescribed in the above summation the endpoints of the geodesics to lie in $B(x, r)$,
there is at most \((d r)^2\) different such geodesics and thus at most \((d r)^2\) occurrences of the term \(\langle \delta f \rangle^2(p)\). Therefore,

\[
\int_{B(x,r)} |f - \tilde{f}|^2 \, d\mu \leq 2d^3 r^2 \int_{B(x,r)} (\delta f)^2(p) \, d\mu(p),
\]

which concludes the proof setting \(P = 2d^3\).

\textbf{Remark 5.7.} Note that, given a finitely generated nilpotent group \(N\), Theorem [25, Theorem 2.2] implies that the Poincaré inequality holds for all Cayley graphs of \(N\) since we know them to have polynomial growth. Therefore, Theorem 3.22 follows as well from the invariance of under rough isometries of the doubling volume and the Poincaré inequality.

5.2. The mixed type case. To deal with the mixed type case, we shall use the weighted version of theorem 2.7 to prove the second part of theorem 1.10. We recall here its statement for the reader’s convenience.

\textbf{Theorem 5.8.} Let \(\Gamma\) be a mixed type hyperbolic manifold then,

- for every \(x, y \in M_\Gamma\) there is a constant \(C_+\) such that for \(t\) large enough

\[
p_\Gamma(x, y, t) \leq \frac{C_+}{t^2}.
\]

- Moreover, for any \(x \in M_\Gamma\) there is a constant \(C_-\) such that

\[
\frac{C_-}{t^2} \leq p_\Gamma(x, x, t)
\]

The end of this paper is devoted to the proof of this theorem. The intuition behind 5.8 is that whenever the Brownian motion enters the geometrically finite end, which looks roughly like a hyperbolic space, it has to choose a point at infinity and goes toward it. Therefore, it tends to escape every compact set; it seems then natural to compare, in large time, the behaviour of the heat kernel to the one of a Brownian motion defined on the degenerate ends stopped whenever it enters some definite compact core region, see Figure 5. An discrete analogue would be a random walk defined on \(N\) - the degenerate end - and stopped at 0 - the compact core - known to have a probability of first return of the order of \(n^{-\frac{3}{2}}\).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure5.png}
\caption{The red Brownian path is expected to behave like a random walk on \(N^*\) when it belongs to the end \(E_2\) and to be stopped whenever it enters the geometrically finite end, analogous to 0.}
\end{figure}
To get the lower bound, we will use Theorem 2.13 which actually needs an upper bound, so that we will start in proving the upper bound using Theorem 2.7.

One would like then to show that a mixed type hyperbolic manifold admits a type 3 Sobolev inequality. Unfortunately, such a direct approach will surely fail since Theorem 2.7 gives a uniform upper bound on the heat kernel, regardless of where are the points $x$ and $y$, see Figure 6 and comments below.

We will use successively Theorem 2.7 and Theorem 2.13. We will weight the manifold $M_\Gamma$ in two different ways, depending on which theorem we use. Recall that we denoted by $h_c$ the Thurston-Sullivan’s harmonic function $h + c$.

**Proposition 5.9.** Let $M_\Gamma$ a thick tamed hyperbolic manifold and $h_1$ be the Thurston-Sullivan’s harmonic function. The weighted Riemannian manifold $(M, g, \mu_{h_2^2})$ satisfies a type 3 Sobolev inequality.

For the lower bound, we will need the following

**Proposition 5.10.** For any $x$ there is two constant $C_+, C_-$ such that

$$C_- t^3 \leq \text{vol}_{h_2^2} \left( B(x, t) \right) \leq C_+ t^3 .$$

In particular the manifold $M_\Gamma$ enjoys the volume doubling property around $x$.

Before proving these two propositions, let us show how to deduce Theorem 5.8.

**Proof:** [5.9 + 5.10 => 5.8] Proposition 5.9 combined with Theorem 2.7 give readily that there is a constant $C > 0$ such that for every $x, y \in M_\Gamma$ one has

$$p_{h_2^2}(x, y, t) \leq C \ t^{-\frac{3}{2}} .$$

On the other hand, Proposition 2.5 shows that for every $x, y \in M_\Gamma$ one has

$$p_{h_2^2}(x, y, t) = \frac{p_{h_1^1}(x, y, t)}{h_1(x) h_1(y)} .$$

Therefore, for any points $x, y \in M_\Gamma$ there is a constant $C_2$ such that

$$p_{h_1^1}(x, y, t) \leq C_2 \ t^{-\frac{3}{2}} ,$$

which is the desired upper bound. For the lower one, as announced, we weight $M_\Gamma$ with respect to $h_2^2$. By Proposition 2.5, the large time behaviour on the heat kernel at some definite point does not depend on a harmonic weight, we readily get that for every point $x \in M_\Gamma$ there is a constant $C_3$ such that

$$p_{h_2^2}(x, x, t) \leq C_3 \ t^{-\frac{3}{2}} .$$

\[\text{Figure 6. One expects the heat kernel } p_T(x, y, t), \text{ taken at } x = y \text{ far away in the degenerate end, to behave like } \frac{1}{\sqrt{t}}, \text{ as it would be on a cylinder.}\]
Proposition 5.10 gives that the volume growth of a ball is of order $t^3$ allowing one to apply Theorem 2.13, giving the lower bound for the $h_0^2$-weighted heat kernel. We use again that large values (at definite points) of the heat kernel do not depend of the harmonic weight to conclude. ■

5.3. Proof of Proposition 5.9. We will prove the Sobolev inequality of type 3 on degenerate ends and geometrically finite ones independently. We will deduce the global Sobolev inequality on $M_\Gamma$ in using the

Theorem. [10, Theorem 2.5] Let $(M, g, \mu_\sigma)$ be a Riemannian weighted manifold of infinite volume. If there is a compact $K$ such that the Sobolev inequality 2.8 holds on $M \setminus K$, then the same inequality holds for the whole manifold $(M, g, \mu_\sigma)$.

This theorem is not stated in the generality of weighted manifolds in Carron’s article, but the proof is identical in this setting.

For our purpose, we will use the above proposition with $K$ a compact core of the degenerate manifold. It remains therefore to show the Sobolev inequality for all the ends independently.

The degenerate ends. The Sobolev inequality (2.8) for a degenerate thick end $E$ comes from that we now such an inequality of type $n$ to hold for the Euclidean space of dimension $n$ [21, Theorem 5.3.1 page 204]. In particular, the euclidean space $\mathbb{R}^3$ satisfies it letting $n = 3$, and so every radial function of $\mathbb{R}^3$ satisfies it too.

The volume of sphere in $\mathbb{R}^3$ being growing quadratically, the Euclidean Sobolev inequality implies that any radial function from $\mathbb{R}^3$ to $\mathbb{R}$, $\varphi : \mathbb{R}_+ \to \mathbb{R}$, satisfies

$$\left( \int_{\mathbb{R}_+} |\varphi|^6(x) x^2 dx \right)^{\frac{1}{6}} \leq C \int_{\mathbb{R}_+} |\nabla \varphi|^2 x^2 dx ,$$

which is equivalent to say that the space $(\mathbb{R}_+, dx^2, x^2 dx)$ satisfies a type 3 Sobolev inequality.

Recall that $h_1$ has linear growth by Theorem 4.4 in a degenerate end. Therefore the weighted degenerate end $(E, g, h_1^2)$ is roughly isometric to $(\mathbb{R}_+, dx^2, x^2 dx)$, the map being explicit in taking the harmonic function itself as the rough isometry. We conclude in recalling Proposition 2.12 which states that satisfying a Sobolev inequality of type $n$ is invariant under rough isometry.

The geometrically finite ends. Let $\mathcal{E}$ be a geometrically finite ends. Recall that the harmonic function $h_1$ values asymptotically 1 in this ends, so that the weighted Riemannian manifold $(\mathcal{E}, g, h_1^2)$ is roughly isometric to the hyperbolic manifold $(\mathcal{E}, g)$. One is therefore able to work with the non weighted Riemannian manifold, calling back again Proposition 2.12 asserting that the sobolev inequality (2.8) is invariant under rough isometries.

To prove that such an inequality occurs for the non weighted degenerate ends, we shall use the following

Proposition 5.12. [21, page 210] Let $(M, g)$ be an hyperbolic manifold of dimension $n > 2$ with injectivity radius bounded from bellow then for all square integrable
of square integrable derivative function $u$ we have

$$\left( \int_M |u|^\frac{2n}{n-2} \, d\mu_g \right)^{\frac{n-2}{2n}} \leq C \left( \int_M |\nabla u|^2 \, d\mu + \int_M |u|^2 d\mu_g \right).$$

The above proposition would allow one to conclude provided a constant $C_2$ such that the following is satisfied for all smooth function $u$

$$\int_M |u|^2 d\mu_g \leq C_2 \int_M |\nabla u|^2 d\mu_g.$$

This inequality is satisfied provided that the bottom of the $L^2$-spectrum of $E$ is positive. In order to show that it is the case here, we will use the

Lemma 5.13. [39, Theorem 2.2] Let $(M, g)$ be a Riemannian manifold such that there is a smooth positive function $f$ and a constant $c \geq 0$ such that

$$\Delta f = cf$$

then the bottom of the spectrum is greater than $c$.

The property of having positive bottom of the spectrum is invariant under quasi-isometry since it is given the lowest value of the Rayleigh quotient. Therefore, it is sufficient to show, cf. Theorem 4.1, that $((S \times \mathbb{R}^+, e^{2t}g + dt^2)$ satisfies the assumptions of Lemma 5.13, which follows from the

Lemma 5.14. the function

$$f : S \times \mathbb{R}^+ \rightarrow \mathbb{R}_{>0}$$

$$(p, t) \mapsto e^{-t}$$

satisfies $\Delta f = f$.

Proof: We denote by $M = S \times \mathbb{R}^+$. Let $x \in S$ and $(\partial_1, \partial_2)$ an orthonormal frame of $TS$ on a neighbourhood of $x$. Now, let $p = (x, t) \in M$ so that the metric product structure guarantees the family $(\partial_1, \partial_2, \partial_t)$ to be an orthogonal frame of $T_pM$. We finally denote by $\nu_1$ and $\nu_2$ the two - locally defined as well - dual differential forms of the vector fields $\partial_1, \partial_2$.

Recall that $\Delta = -*d*d$ on functions in dimension 3. We start off computing $*df = -*e^{-t}dt$. From the hodge star definition itself, we have

$$*dt = e^{2t} \nu_1 \wedge \nu_2$$

and thus

$$*df = -e^t \nu_1 \wedge \nu_2$$

$$= -d*df = -e^t dt \wedge \nu_1 \wedge \nu_2,$$

since $d\nu_1 \wedge \nu_2 = \nu_1 \wedge d\nu_2 = 0$. Therefore

$$\Delta f = e^t * (dt \wedge \nu_1 \wedge \nu_2)$$

$$= e^t \cdot e^{-2t}$$

$$= f,$$

which concludes the proof.
5.4. **Proof of Proposition 5.10.** To prove Proposition 5.10, we will show that geometrically finite ends have bounded volume with respect to the measure \( \mu_{h^2_0} := h^2_0 \mu_g \).

**Lemma 5.15.** Let \( E \) be a geometrically finite end, then \( \text{vol}_{h^2_0}(E) \) is bounded.

This implies 5.10 since the volume of a ball of centre \( x \) and of large enough radius will therefore be of the order of the volume the ball intersected with the degenerate ends: we saw this quantity to grow like \( \rho^3 \), since the the function \( h_1 \) has linear growth in the degenerate ends.

**Proof:** We start in fixing

- a point \( x_0 \in M_\Gamma \) and one of its lifts \( \tilde{x}_0 \in \mathbb{H}^3 \);
- a fundamental domain \( D \subset \mathbb{H}^3 \) for the \( \Gamma \)-action containing \( \tilde{x}_0 \), the Dirichlet one for example.

For any point \( y \in E \), we denote by \( \tilde{y}_D \) its lift in \( D \). So that for any function \( f \) defined on \( M_\Gamma \) one has \( \tilde{f}(\tilde{y}_D) = f(y) \), where \( \tilde{f} \) denotes the lift of the function \( f \).

One has a canonical geometric compactification of \( \mathbb{H}^3 \) by the sphere at infinity \( S^2 \). It is well known [39, Theorem 2.11] [1], that the lift \( \tilde{h} \) of the Thurston-Sullivan’s harmonic function \( h \) may be read from the geometric boundary at infinity since it is positive. Namely, there is a unique probability measure \( \nu_{h_0} \) on \( S^2 \) supported on the limit set of the group \( \Gamma \) such that

\[
\tilde{h}(\tilde{y}) = h(\tilde{x}_0) \int_{S^2} e^{2\beta_{\zeta}(\tilde{y}_0, \zeta)} d\nu_{h_0}(\zeta),
\]

where \( \beta_{\zeta}(x, y) \) are the Busemann functions, see Figure 7, defined as follows

\[
\beta_{\zeta}(x, y) = \lim_{t \to \infty} \left( d(g_t(y, \zeta), \tilde{x}_0) - d(g_t(x, \zeta), \tilde{x}_0) \right).
\]

Note that the above definition does not depend on the based point \( \tilde{x}_0 \).

\[\text{Figure 7. All the three geodesic rays head the same point } \zeta \text{ at infinity. They get exponentially closer, and the value of the Busemann function can therefore be read as the (oriented) distance represented in black.}\]
Using the Cauchy Schwarz inequality one gets:

\[
h_0^2(y) = \tilde{h}_0^2(\tilde{y}_D) = h_0^2(\tilde{x}_0) \left( \int_{\mathbb{S}^2} e^2 \beta_\zeta(\tilde{x}_0, \tilde{y}_D) d\nu_\zeta(\zeta) \right)^2 \leq h_0^2(\tilde{x}_0) \int_{\mathbb{S}^2} e^{4 \beta_\zeta(\tilde{x}_0, \tilde{y}_D)} d\nu_\zeta(\zeta) .
\]

We will bound from above this integral in using the following well known lemma.

**Lemma 5.16.** [11] For every \( \theta_0 \) there is a constant \( L > 0 \) such that for any triangle \( x, y, z \in \mathbb{H}^3 \) with angle at \( x \) bigger than \( \theta_0 \) one has

\[
d(x, y) + d(x, z) \leq d(y, z) + L .
\]

Since we fix the fundamental domain \( D \), it follows from the above lemma that there is a constant \( C > 0 \) such that for any point \( \zeta \) in the limit set of \( \Gamma \) and for any point \( y \in \tilde{E} \cap D \) (see Figure 8) one has:

\[
\beta_\zeta(\tilde{x}_0, \tilde{y}_D) \leq -d_{\mathbb{H}^3}(\tilde{x}_0, \tilde{y}_D) + C .
\]

**Figure 8.** The point \( \tilde{y}_D \) being in a geometrically finite end \( E \), there is a positive lower bound for the angle \( \theta \) made "between \( \tilde{y}_D \) and \( \zeta \). This follows from the fact that we now the limit set, pictured in blue, not to intersect some definite neighborhood of the trace at infinity of the fundamental domain \( D \). This neighborhood is represented in orange, and can be taken as a finite union of half hyperbolic spaces \( \mathcal{H} \).

Therefore,

\[
h_0^2(y) \leq h_0^2(\tilde{x}_0) e^C \int_{\mathbb{S}^2} e^{-4d(\tilde{x}_0, \tilde{y}_D)} d\nu_\zeta(\zeta) \leq C_2 e^{-4d(\tilde{x}_0, \tilde{y}_D)} .
\]
We conclude with a computation:

\[
\text{vol}_{\mathbb{H}_3}(E \cap B(x_0, \rho)) := \int_{E \cap B(x_0, \rho)} h_0^2(y) d\mu_g(y)
\]

\[
\leq C_2 \int_{E \cap B(x_0, \rho)} e^{-4d_{\mathbb{H}_3}(\tilde{x}_0, \tilde{y})} d\mu_g(y)
\]

\[
\leq C_2 \int_{D \cap \tilde{E} \cap B(\tilde{x}_0, \rho)} e^{-4d_{\mathbb{H}_3}(\tilde{x}_0, \tilde{y})} d\mu_g(\tilde{y})
\]

\[
\leq C_3 \int_{B(\tilde{x}_0, \rho)} e^{-4d_{\mathbb{H}_3}(\tilde{x}_0, \tilde{y})} d\mu_g(\tilde{y})
\]

since the volume of a hyperbolic sphere of radius \( \rho \) is of the order of \( e^{2\rho} \) in dimension 3.

\[\blacksquare\]

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