Neutral kaon mixing from 2+1 flavor domain wall QCD

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We present the first results for neutral kaon mixing using 2+1 flavors of domain wall fermions. A new approach is used to extrapolate to the physical up and down quark masses from our numerical studies with pion masses in the range 240 – 420 MeV; only new approach is used to extrapolate to the physical up and down quark masses from our numerical

The phenomena of $CP$ violation is a central component of the Standard Model, in which $CP$ violation is only possible when all three of the quark doublets present in Nature interact. The Cabibbo-Kobayashi-Maskawa (CKM) flavor mixing matrix contains a single, physically meaningful phase which must describe all $CP$ violating phenomena. For bottom mesons, it is possible to make a direct connection between the measured $CP$ violation in $B$ decays and this CKM phase. However, for $K$ mesons, the system in which $CP$ violation was originally observed, this connection is far more challenging.

One begins with the measure of indirect $CP$ violation $\epsilon_K = 2.232 \pm 0.007 \cdot 10^{-3}$ \textsuperscript{[1]}, determined experimentally from the mixing between $K^0$ and $\bar{K}^0$ mesons. The operator product expansion relates $\epsilon_K$ to the QCD matrix element of a four quark operator $O_{VV+AA} = (\bar{s}_\gamma \gamma_\mu d)(\bar{s}_\gamma \gamma_\mu d) + (\bar{s}_\gamma \gamma_5 \gamma_\mu d)(\bar{s}_\gamma \gamma_5 \gamma_\mu d)$ between kaon states via a well known perturbative expression \textsuperscript{[2]} involving this CKM phase. This matrix element is parameterized by the renormalization scheme dependent parameter

$$ B_K = \frac{\langle K^0 | O_{VV+AA} | \bar{K}^0 \rangle}{2 f_K M_K^2}. $$

Lattice QCD offers the only first-principles determination of $B_K$, which is essential to determine if the $CP$ violations observed in the $B$ and $K$ systems have a common, Standard Model origin. We describe a lattice QCD calculation of $B_K$ in which the most important errors present in earlier lattice results have been substantially reduced. We exploit the domain wall fermion (DWF) formulation with 2+1 dynamical flavors. This suppresses $O(a)$ errors, both on- and off-shell, and also chiral symmetry breaking (measured by a (small) additive “residual mass” $m_{res}$). This allows us to renormalize $O_{VV+AA}$ multiplicatively via a non-perturbative matching \textsuperscript{[3, 4, 5]}. Thus, we simultaneously avoid the complexity of lattice operator mixing, avoid poorly convergent lattice perturbation theory and include the correct light flavor content.

Alternative lattice approaches to $B_K$ must treat either a chirality or taste mixing matrix and result in larger errors. For Wilson fermions a chirality mixing matrix can be determined using non-perturbative off-shell renormalization, but large cancelations leave results imprecise. While staggered fermion simulations successfully treat simpler quantities, current staggered results have a 10–20% error for $B_K$ due to large taste mixing \textsuperscript{[6]}.

By using large lattice volumes and meson masses as light as 243 MeV, we can determine the light quark limit with substantially improved accuracy. Instead of using a chiral perturbation theory (ChPT) which treats the $K$ meson as light compared to the chiral scale, we evaluate the chiral limit using $SU(2)_L \times SU(2)_R$ ChPT and assume that only our pions are light.

\section*{SIMULATION}

Our calculation is performed with a fixed lattice spacing and two space-time volumes, $16^3 \times 32$ and $24^3 \times 64$, using the Iwasaki gauge action \textsuperscript{[3]} with $\beta = 2.13$ and the DWF action with a fifth dimension of size 16. Each ensemble uses the same dynamical strange quark mass $am_{s,sea} = 0.04$ in lattice units. We use three $16^3$ en-
sembles with degenerate up and down quarks of mass $m_{\text{val}} \in \{0.01, 0.02, 0.03\}$ and two $24^3$ ensembles with $m_{\text{sea}} \in \{0.005, 0.01\}$. The ensembles, described in [8], were generated using the RHMC algorithm [10] with trajectories of unit length. The $16^3$ ensembles each contain 4000 trajectories, from which we omit 1000 trajectories for thermalization. We perform measurements on 150 configurations separated by 20 trajectories for each ensemble, calculating matrix elements of all possible pseudoscalar states with valence quark masses $m_{\text{val}} \in \{0.01, 0.02, 0.03, 0.04, 0.05\}$. We bin the data using up to 80 trajectories per bin to reduce the correlations between our samples. The $am_{\text{sea}} = 0.005$ and 0.01, $24^3$ ensembles are composed of 4460 and 5020 trajectories respectively with measurements performed on the final 90 configurations separated by 40 trajectories. These data are binned into blocks of 2 configurations representing 80 trajectories. All possible pseudoscalar states are studied, composed of valence quark masses $m_{\text{val}} \in \{0.001, 0.005, 0.01, 0.02, 0.03, 0.04\}$.

We use the mass of the $\Omega^-$ baryon, linearly extrapolated to $m_l = (m_u + m_d)/2$, and $m_K$ and $m_\pi$ treated in $SU_L(2) \times SU_R(2)$ ChPT to determine $1/a = 1.73(3)$ GeV, and unrenormalised masses $am_l + am_{\text{res}} = 0.0375(16)$ and $am_l + am_{\text{res}} = 0.00130(6)$. Since in our simulation, $am_{s} = 0.04$ and $am_{\text{res}} = 0.00315(2)$, we must take into account our 15% too large input value of $m_s$. We obtained the pseudoscalar decay constant $f_\pi = 0.0718(18)$ using $Z_A(P)A_0(0) = -im_{\text{res}}f_\pi$ and the axial current renormalization constant $Z_A = 0.7162(2)$. (Note, the above errors are all statistical.)

We use an established method [4, 11, 12] for evaluating the lattice matrix elements. Zero-momentum kaon states are created and annihilated using Coulomb-gauge-fixed wall sources at times 5 and 27 for the $16^3$ volume and 5 and 59 for $24^3$, with $O_{VV+AA}$ inserted at each intervening point. Use of a combination of periodic and anti-periodic boundary conditions in time removes unwanted propagation around the boundary, resulting in a good plateau for the ratio of matrix elements

$$B^0_{\pi K} = \frac{(K^0(t_1)\bar{O}_{VV+AA}(t)\bar{K}^0(t_2))}{\frac{5}{3}(K^0(t_1)\bar{A}_0(t)\bar{A}_0(t)\bar{K}^0(t_2))}. \quad (2)$$

A sample fit is displayed in Fig. 1. For each ensemble the pseudoscalar mass, $M_P$, decay constant, $f_P$, and $B$-parameter, $B_P$, are computed for each combination ($m_{\text{val}}, m_{\text{val}}$). We tabulate a portion of the $16^3$ and $24^3$ results for $B_P$ and $M_P$ in Tables I and II.

Our lightest dynamical pion masses are 331 MeV. We must extrapolate our result for $B_P$ to the physical value of $m_l = (m_u + m_d)/2$, and we treat only the up and down quarks as light by using the $SU(2)_L \times SU(2)_R$ partially quenched ChPT (PQChPT) formula [13]

$$B_P(m_x, m_y) = B_P(0) \left\{ 1 + c_1 m_x + c_2 m_x - k \ln \frac{2B_0 m_x}{\Lambda_\text{ch}} \right\}.$$

Here $m_x$ is the light valence quark mass, $m_l$ the light sea quark mass, $\Lambda_\text{ch}$ the chiral scale, $B_P(0)$, $c_0$ and $c_1$ are $m_x$-dependent low energy constants, $k = B_0 m_l/(4\pi f_\pi)^2$, $B_0$ is the constant in the expression $m_P^2 = 2B_0 m_x$, and we include $m_{\text{res}}$ in all masses entering these formulae.

$M_P$ is now the lowest scale that can dimensionally balance higher order terms in the chiral expansion, and $m_P^2/\Lambda_\text{ch}^2$ will determine the suppression of successive orders. This is a better expansion parameter than the $m_P^2/\Lambda_\text{ch}^2$ of $SU(3)$ ChPT. We should note that the pseudoscalar masses and decay constants are also well described by a $SU(2)_L \times SU(2)_R$ PQChPT analysis.

Figure 2 shows $B_P$ versus $m_x$ together with the $SU(2)_L \times SU(2)_R$ partially quenched ChPT fit to the $24^3$
MS is a 2% effect at NLO, consistent with the quenched result in relation to those quenched\[11, 14, 15\]. The valence strange quark mass is fixed at its unitary value \(m_{\upsilon} = 0.03\) and \(m_s\) respectively. The valence strange quark mass is increased from 0.05 to 0.01 by ChPT. Some \(m_s\) values are slightly shifted for clarity.

FIG. 2: Results for \(B_K\) together with the NLO partially quenched \(SU(2)_L \times SU(2)_R\) ChPT fit to the 24\(^3\) data plotted versus the light valence quark lattice mass \(am_{\upsilon}\). From top to bottom on the left-hand-side, the three curves are \(am_{\upsilon} = 0.01, 0.005\) and \(am_s\) respectively. The valence strange quark mass is fixed at its unitary value \(am_{\upsilon} = 0.03\) by ChPT. Some \(m_s\) values are slightly shifted for clarity.

The fit does not include correlations so the resulting \(\chi^2/\text{dof} = 0.14\) is not a meaningful indication of goodness of fit. The 16\(^3\) unitary data, also shown in Fig. 2, are well described by a straight line which, if simply extrapolated to the physical limit gives a result about 6% larger than the more accurate chiral extrapolation that is possible if the smaller masses in the 24\(^3\) simulation are used\[22\]. This \(SU(2)_L \times SU(2)_R\) chiral extrapolation gives \(B_K^{\text{lat}} = 0.565(10)\) for physical \(m_{\upsilon}\). \(B_K\) is determined at the correct valence value of \(am_{\upsilon} = 0.0343\) by linearly interpolating between \(am_{\upsilon} = 0.03\) and 0.04.)

We have previously demonstrated\[11, 14, 15\] that the wrong-chirality mixing in our simulation is sufficiently suppressed that we may ignore it. Thus, we consider only multiplicative renormalization of \(O_{V,V+AA}\), and use the RI-MOM non-perturbative renormalization technique to match our lattice scheme to the MS scheme via \(B_K^{\text{MS}} = (Z_{\text{RI-MOM}}^{V,V+AA}/Z_{\text{MS}}^{V})B_K^{\text{lat}} = Z_{B_K}^{\text{lat}}\).

The technique performs well since domain wall fermions are off-shell improved. We evaluate the amputated, four-leg and two-leg vertex functions \(\Lambda_{QV,V+AA}\), \(\Lambda_A\), and \(\Lambda_V\) in Landau gauge. Because of the relatively low lattice cut-off, \(\Lambda_A\) and \(\Lambda_V\) differ by 2%. We use their average and add their difference to the systematic error.

We both quote lattice results in the RI-MOM scheme without perturbative error and convert to other schemes using the continuum NLO result\[16, 17\]. We obtain \(Z_{B_K}^{\text{RI-MOM}}(2\text{ GeV}) = 0.910(5)(13)\). The first error in parenthesis is statistical and the second systematic. As shown in Figure 3 we use the RGI scheme as a scale invariant intermediate step to reveal and remove possible \((ap)^2\) errors. Only weak scale dependence is seen in the window 1.0 \(\leq (ap)^2 \leq 2.5\), implying artefact-free perturbative behavior. We obtain \(\gamma_{B_K}^{\text{RGI}} = 1.275(10)(25)\) by linearly extrapolating to \((ap)^2 = 0\) to remove \(O(a^2)\) effects.

Conversion to MS is a 2% effect at NLO, consistent with \(O(1) \times \alpha_s/4\pi\). While the error estimate could be as low as \(O(\text{few}) \times (\alpha_s/4\pi)^2\), we add in quadrature the size of the NLO correction itself as a perturbative systematic, giving \(Z_{B_K}^{\text{MS}}(2\text{ GeV}) = 0.928(5)(23)\). For comparison, 1-loop lattice perturbation theory\[18\] gives \(Z_{B_K}^{\text{MS}}(2\text{ GeV}) = 1.007\) with the difference likely due to slow convergence of lattice perturbation theory.

FIG. 3: A plot of \(Z_{B_K}^{\text{RGI}}(p^2)\) showing that the perturbative running, removed from \(Z_{B_K}^{\text{RGI}}\), accounts for most \(p^2\) dependence.

RESULTS AND CONCLUSIONS

We combine the bare \(B_K\) given above with these \(Z\) factors to obtain physically normalized results from 2+1 flavor DWF at \(a^{-1} = 1.73(3)\) GeV on a 24\(^3\) × 64 volume, where the second error is the renormalisation systematic: \(B_K^{\text{RI-MOM}}(2\text{ GeV}) = 0.514(10)(7)\), and \(B_K^{\text{MS}}(2\text{ GeV}) = 0.524(10)(13)\). The RI result involves no use of perturbation theory and has a reduced systematic error.

We plot the MS result in relation to those quenched\[11, 19\] and two flavor\[12\] DWF results that could allow dependence on flavor content to be seen. (Of course differences arising from different lattice spacing and gauge action are also to be expected.) In addition, we include the 2+1 flavor staggered fermion result\[6\] in Fig. 4.

Finite volume chiral perturbation theory\[20\] suggests that finite volume effects are negligible for all masses and volumes in our simulation except for \(am_{s} = 0.001\) where the effect may be 2%. However, since our fit is insensitive to this point, we adopt our 1%, 16\(^3\) - 24\(^3\), \(am_{l}\) difference as an estimate of the finite volume error.

Finite lattice spacing errors are likely larger, but also difficult for us to estimate. We can make use of the quenched, perturbatively renormalized results of CP-PACS\[19\] also obtained for the Iwasaki and DWF lattice action which suggest a (poorly determined) scaling violation of size 3.5% at our coarser lattice spacing. We choose a 4% systematic error as the most likely estimate for \(O(a^2)\) effects in our 2+1 flavor result. A second approach to estimating discretization errors is to compare a
variety of presumably reliable quantities computed from our \(1/a = 1.73(3)\) GeV ensembles with experiment. For example, we find \(f_a\) and \(f_K\) about 4\% below experiment, a discrepancy consistent with our 4\% error estimate.

While we have interpolated to the physical valence strange quark mass, we have results for only one strange sea quark mass. We estimate the error resulting from our 15\% too large \(m_s\) by observing that for fixed valence masses \((0.01,0.04)\) \(B_P\) increases by 3\% when the light sea quark mass is changed from 0.03 to 0.02. Scaling this to the 0.0057 change needed for a single flavor of sea quark implies a 1\% error. Finally we add a 2\% chiral extrapolation error by estimating the effects of NNLO ChPT as the 6\% difference between the linear and NLO chiral limits in Fig. 2 scaled by \(m_s/m_a = 0.4\) for \(am_t = 0.01\).

Thus, we take our central value, which removes all chiral scaling errors, and add the 1\% finite volume, 4\% scaling, 1\% \(m_s\) extrapolation, 2\% ChPT and renormalization error estimates in quadrature and obtain:

\[
B^{\text{RI}}_K(2\text{ GeV}) = 0.514(10)(25),
\]
\[
B^{\text{RS}}_K(2\text{ GeV}) = 0.524(10)(28),
\]
\[
\hat{B}_K = 0.720(13)(37),
\]

where the first error is statistical and the second is the estimated systematic. A recent review, including all lattice data then available, quoted a continuum limit value of \(B^{\text{RS}}_K(2\text{ GeV}) = 0.583(6)[21]\). Our result is consistent with this and reduces both types of error substantially. This improvement arises because, using QCDOC computers, we have for the first time combined the correct dynamical flavor content with a lattice formulation with good chiral symmetry, \(O(a)\) improvement, control over operator mixing, non-perturbative renormalization and some use of \(SU(2)_L \times SU(2)_R\) ChPT for kaons.

In order to reduce the significant discretisation systematic error in our result for \(B_K\), the RBC and UKQCD collaborations are now doing simulations at a smaller lattice spacing, which will give quantitative control of this effect. A continuum NNLO perturbative calculation is required to convert lattice results to MS with better precision.

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![Graph](image)

**FIG. 4:** We compare our 2+1 flavor results with earlier quenched [10] and 2 flavor DWF [12] as well as 2+1 flavor staggered calculations [8]. The quenched Iwasaki points show statistical errors only while our point and the staggered point include renormalisation systematics. While our point lies below these Iwasaki results due to our improved chiral limit and flavor content, we expect similar \(a^2\) dependence.

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