Failure of Mean Field Theory at Large N

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We study strongly coupled lattice QCD with $N$ colors of staggered fermions in $3+1$ dimensions. While mean field theory describes the low temperature behavior of this theory at large $N$, it fails in the scaling region close to the finite temperature second order chiral phase transition. The universal critical region close to the phase transition belongs to the 3d XY universality class even when $N$ becomes large. This is in contrast to Gross-Neveu models where the critical region shrinks as $N$ (the number of flavors) increases and mean field theory is expected to describe the phase transition exactly in the limit of infinite $N$. Our work demonstrates that close to second order phase transitions infrared fluctuations can sometimes be important even when $N$ is strictly infinite.

Mean field techniques provide a simple but powerful approach to gain qualitative insight of the underlying physics in a variety of field theories [1]. The Bardeen-Cooper-Schrieffer (BCS) solution to superconductivity is a well known application of such a technique. Wilson's renormalization group shows that when correlation lengths $\xi$, associated with the fluctuations of the field, become large compared to the microscopic length scale $a$ of the problem, mean field techniques become exact in dimensions greater than four. However, in lower dimensional systems infrared fluctuations can become important when $\xi \gg a$ and invalidate the mean field arguments. For this reason the mean field approach must be used with care close to second order phase transitions. The region close to the critical point where mean field theory fails is usually referred to as the Ginsburg region [2]. Using field theoretic techniques sometimes it is possible to estimate the Ginsburg region. In conventional superconductors the Ginsburg length is known to be suppressed by some power of $T_c/E_f$ where $T_c$ is the critical temperature and $E_f$ is the Fermi energy. Mean field theory is believed to be reliable outside the Ginsburg region. This is the reason why BCS is a good approach to understand the physics of superconductivity for all temperatures except very close to $T_c$.

There are certain limits in which the infrared fluctuations can be naturally suppressed even in low dimensional systems. For example consider a theory containing a field with $N$ components. When $N$ is large a saddle point approximation can be used to solve the theory and the leading term is nothing but the mean field solution. Theoretical physicists often use this feature to solve a physical theory by increasing the number of components of the field artificially. By solving the theory in the limit of large number of components and computing the leading corrections they can sometimes estimate realistic answers in the physical theory. A variety of field theories can be studied in the large $N$ limit [3].

Although large $N$ approach to field theories has been very successful, in this article we show that not all large $N$ limits lead to the suppression of infrared fluctuations. We contrast two models, the Gross-Neveu (GN) model involving $N$ species of fermions studied recently [4, 5] and strongly coupled $U(N)$ lattice QCD with staggered fermions (SCLQCD) studied here. Both these models contain a global symmetry which is spontaneously broken at low temperatures. Further, the symmetry group and the breaking pattern is not affected by $N$. These models undergo a second order phase transition to a symmetric phase at a finite critical temperature $T_c$. In the large $N$ limit they can be solved exactly using mean field techniques at low temperatures. An interesting question then is whether the mean field description is valid even close to $T_c$ when $N$ becomes infinite.

It was discovered in [4] that indeed in the GN model the critical behavior near $T_c$ belongs to the Landau-Ginsburg mean field universality class at large $N$. This implies that the Ginsburg region, where the critical behavior is governed by a non-trivial universality class that depends on the symmetry group and the breaking pattern in a dimensionally reduced theory, has a zero width at large $N$. It was later shown in [5] that indeed the Ginsburg region is suppressed by a factor $1/\sqrt{N}$. Although a $Z_2$ symmetric GN model was analyzed in [2, 7], there are reasons to believe that the arguments would hold for continuous symmetries as well [6, 7].

As we will demonstrate here, in contrast to the GN model, the Ginsburg region does not shrink in SCLQCD in the large $N$ limit. While mean field theory is indeed a good approximation at low temperatures, the finite temperature phase transition is not described by mean field theory even at infinite $N$. We believe our results should be of interest to a wide range of physicists since our model can be mapped into a theory of classical dimers. Dimer models have a long history [8]. In the 1960s these models attracted a lot of attention when it was shown that the Ising model can be rewritten as a dimer model [9, 10]. In the late 1980s they gained popularity again in their quantum version [11] as a promising approach to the famous resonating-valence-bond (RVB) liquid phase [12]. More
recently, this approach has gained momentum again since it was shown that the RVB phase was indeed realized on a triangular lattice but not on a cubic lattice [12, 14]. Thus physicists attempting to use large N techniques in dimer models can benefit from our results.

The partition function of SCLQCD is given by

\[ Z(T, m) = \int [dU][d\psi d\bar{\psi}] \exp \left( -S[U, \psi, \bar{\psi}] \right), \]

where \([dU]\) is the Haar measure over \(U(N)\) matrices and \([d\psi d\bar{\psi}]\) specify Grassmann integration. The Euclidean space action \(S[U, \psi, \bar{\psi}]\) in the strong (gauge) coupling limit with staggered fermions is given by

\[ -\sum_{x,\mu} \frac{\eta_x}{2} \left[ \bar{\psi}_x U_{x,\mu} \psi_{x+\hat{\mu}} - \bar{\psi}_{x+\hat{\mu}} U_{x,\mu}^\dagger \psi_x \right] - m \sum_x \bar{\psi}_x \psi_x. \]

where \(x\) refers to the lattice site on a periodic four-dimensional hyper-cubic lattice of size \(L\) along the three spatial directions and size \(L_t\) along the euclidean time direction. The index \(\mu = 1, 2, 3, 4\) refers to the four space-time directions, \(U_{x,\mu} \in U(N)\) is the usual links matrix representing the gauge fields, and \(\psi_x, \bar{\psi}_x\) are the three-component staggered quark fields. The gauge fields satisfy periodic boundary conditions while the quark fields satisfy either periodic or anti-periodic boundary conditions. The factors \(\eta_{x,\mu}\) are the well-known staggered fermion phase factors. Using an asymmetry factor between space and time we introduce a temperature in the fermion phase factors. Using an asymmetry factor be-

By working on anisotropic lattices with \(L_t < L\) at fixed \(L_t\) and varying \(T\) continuously one can study finite temperature phase transitions [12]. In this article we fix \(L_t = 4\) for convenience.

The partition function given in eq. (1) can be rewritten as a partition function for a monomer-dimer system, which is given by

\[ Z(T, m) = \sum_{n_x} \prod_{x,\mu} (z_{x,\mu})^{b_{x,\mu}} \left( \frac{N - b_{x,\mu}}{b_{x,\mu} N!} \right) \prod_x \frac{N!}{n_x!} \prod_{x} m^{n_x}, \]

and is discussed in detail in [17, 13]. Here \(n_x = 0, 1, 2, ..., N\) refers to the number of monomers on the site \(x\), \(b_{x,\mu} = 0, 1, 2, ..., N\) represents the number of dimers on the bond connecting \(x\) and \(x + \hat{\mu}\). \(m\) is the monomer weight, \(z_{x,\mu} = \eta_{x,\mu}^2 / 4\) are the dimer weights. Note that while spatial dimers carry a weight 1/4, temporal dimers carry a weight \(T/4\). The sum is over all monomer-dimer configurations \(n_x, b_{x,\mu}\) which are constrained such that at each site, \(n_x + \sum_\mu b_{x,\mu} + b_{x,\mu} = N\).

When \(m = 0\), the action of SCLQCD, eq. (2), is invariant under \(O(2)\) chiral transformations: \(\psi_x \rightarrow \exp(i\sigma_x \theta) \psi_x\) and \(\bar{\psi}_x \rightarrow \bar{\psi}_x \exp(i\sigma_x \theta)\) where \(\sigma_x = 1\) for all even sites and \(-1\) for all odd sites. In the large \(N\) limit mean field techniques can be used to show that this chiral symmetry is spontaneously broken at low temperatures [10]. In [17] a Monte Carlo method was developed to solve the problem from first principles and it was shown that mean field theory is indeed reliable at small temperatures [13]. Unfortunately, since the algorithm was inefficient at small quark masses, the finite temperature chiral phase transition was never studied in the large \(N\) limit. Recently a very efficient cluster algorithm was discovered to solve the model at any value of \(N\) [18]. Using this algorithm it was shown with great precision that for \(N = 3\) the finite temperature phase transition belonged to the 3d XY universality class [19]. Here we extend that calculation to higher values of \(N\).

The order parameter that signals chiral symmetry breaking is the chiral condensate, defined by

\[ \Sigma = \lim_{m \rightarrow 0} \frac{1}{N L_t L^3} \frac{1}{Z} \frac{\partial}{\partial m} Z(T, m). \]

For a fixed \(T\) the large \(N\) result for \(\Sigma\) can easily be obtained by extending the calculation of [17]. One gets

\[ \Sigma = \sqrt{\frac{-9 - 17T + 18\sqrt{9 - T + T^2}}{81 - 187 + T^2}}, \]

which shows that the critical temperature, as we have defined it, is infinite. A calculation which includes the 1/N correction shows that \(T_c \sim N\). In the large \(d\) (spatial dimensions) limit one obtains \(T_c = d(N + 2) L_t / 6\) [20].

In order to determine \(\Sigma\) using Monte Carlo calculations we measure the chiral susceptibility in the chiral limit,

\[ \chi = \lim_{m \rightarrow 0} \frac{1}{L^3} \frac{\partial^2}{\partial m^2} Z(T, m), \]

The finite size scaling of this quantity is known from chiral perturbation theory [21] and one expects

\[ \chi = \frac{N^2 L_t^2 \Sigma^2 L^3}{2} \left[ 1 + \frac{0.226}{F^2 L} + \frac{\alpha}{L^2} + ... \right]. \]

The constant \(F^2\) is equal to

\[ \lim_{L \rightarrow \infty} \frac{1}{3 L^3} \left\{ \left( \sum_x J_{x,1} \right)^2 + \left( \sum_x J_{x,2} \right)^2 + \left( \sum_x J_{x,3} \right)^2 \right\}, \]

at \(m = 0\). The current \(J_{x,\mu}\) is the conserved current associated with the \(O(2)\) chiral symmetry [19]. By fitting the data for \(\chi\) to the form given in eq. (7) we can determine \(\Sigma\) accurately.

We have done extensive simulations for various values of \(L\) and \(N\) in order to extract \(\Sigma\) as discussed above. In fig. we plot \(\Sigma\) as a function of \(T\) at \(N = 6, 12, 24, 48\). For comparison we also plot the mean field result (eq. (5)). As the graph shows, at a fixed value of \(T\), our data approaches the mean field prediction quite nicely as \(N\) becomes large. However, for every value of \(N\), as \(T\) increases the order parameter approaches zero at some critical temperature \(T_c\). Close to \(T_c\), the mean field theory
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The solid line is the mean field result given in eq. (5).

is definitely not a good approximation. Using our algorithm we can determine \( T_c \) accurately for every value of \( N \) (see below). In the inset of fig. 1 we plot \( T_c \) as a function of \( N \). A fit to the form \( T_c = aN + b + c/N \) yields \( a = 1.5525(3) \), \( b = 3.126(9) \) and \( c = -0.88(4) \) with a \( \chi^2/DOF = 0.73 \) (solid line in the inset).

One might think that it is quite easy to explain why the mean field theory breaks down close to \( T_c \). Since \( T_c \) grows as \( N \), the large \( N \) theory does not know about the existence of a finite \( T_c \), unless the 1/\( N \) corrections are included. We have computed these corrections and have found that they do not improve the situation much. Perhaps one needs to develop a new mean field expansion where one holds \( T/N \) fixed as \( N \) becomes large. Let us refer to this as the finite \( T \) mean field theory (FTMFT). Although we have not yet developed this mean field theory, we will argue that it is bound to fail close to \( T_c \), since the Ginzburg region where the 3d \( XY \) universality class is observed does not shrink with \( N \). Indeed we find that \( \Sigma \sim (1 - T/T_c)^\beta \) close to \( T_c \) where \( \beta = 0.3485(2) \) independent of \( N \). A fit of the data to this form was used to determine \( T_c \) plotted in fig. 1. In fig. 2 we plot \( \Sigma \) as a function of \( T/T_c \) for \( 0.95 \leq T/T_c \leq 1 \) for \( N = 6, 12, 24, 48 \). The solid lines are fits to the form \( A(1-T/T_c)^{0.3485} \). We find \( A = 0.3668(3), 0.2630(2), 0.1862(2), 0.1315(2) \) for \( N = 6, 12, 24, 48 \) respectively all with a \( \chi^2/DOF \) less than 1.

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Since \( A \) decreases with increasing \( N \) one might argue that \( A \) will vanish in the limit \( N \to \infty \). In that case some higher order term will become dominant in the large \( N \) limit and it is not possible to rule out mean field behavior. However, we do not think this is the case and attribute the change in \( A \) to a renormalization effect as a function of \( N \). In order to justify this, we next focus on a correlation length scale which does not need renormalization. If correlation lengths in the theory indeed scale with \( N \), we would expect the correlation lengths to be the same at any fixed \( T/T_c \). In our model \( F^2 \) can be defined as one such inverse correlation length scale. This implies \( F^2 \sim B(1-T/T_c)^\nu \), where \( \nu = 0.6175(3) \) in the 3d \( XY \) model [22]. In fig. 3 we plot \( F^2 \) as a function of \( T/T_c \) for \( N = 12, 24, 48 \). When \( T/T_c > 0.8 \) all the points at a fixed \( T/T_c \) fall at different \( N \) fall on top of each other. Further, when \( 0.95 \leq T/T_c \leq 1 \), all the data points fit extremely well to the form \( B(1-T/T_c)^{0.6715} \), with \( B = 0.939 \) with a \( \chi^2/DOF = 0.65 \). We believe that this is strong evidence that even at infinite \( N \) the phase
transition belongs to the 3d XY universality class. Outside the Ginsburg region one would expect FTMFT to be valid. Interestingly, when $0.85 \leq T/T_c \leq 0.93$ we find that the data fits to the form $B(1-T/T_c)$ with $B = 1.507$ with a $\chi^2/DOF = 0.8$, suggesting $\nu = 1$. Hence, we suspect that the FTMFT in our model is similar to the mean field theory in 3d $O(N)$ models which yields $\nu = 1$. The cross over to mean field theory occurs when $T/T_c < 0.95$ so that the correlation length is $\xi \sim 1/F^2 < 8$ in lattice units.

One often hears the lore that in the large $N$ limit SCLQCD is solvable using mean field theory. While this is true in certain cases, in this article we have demonstrated that the finite temperature chiral phase transition belongs to the 3d XY universality class even in the large $N$ limit. In an earlier study we found similar results in two spatial dimensions. In two dimensions continuous symmetries cannot break at any finite temperature. However an $O(2)$ symmetry is special and a phase transition in the Berezinski-Kosterlitz-Thouless (BKT) universality class is possible. The BKT prediction for our model is that $\chi \sim L^{2-\eta}$ where $\eta$ is a function of temperature $0 \leq \eta(T) \leq 1/4$ when $T \leq T_c$. At the critical temperature $\eta = 1/4$. We found all this to be true even at large $N$. Interestingly, according to Witten a large $N$ mean field theory would find $\eta \sim 1/N$. While our results agree with this observation at a fixed temperature $T$, we find that if $T/T_c$ is held fixed then $\eta$ approaches a non-zero value in the predicted range showing that the mean field approach again breaks down in the large $N$ theory close to the phase transition.

Our study shows that the large $N$ limit may not always be able to suppress infrared fluctuations close to second order phase transitions. In a sense this is an indication that the perturbation expansion starting from a mean field solution breaks down. In other words a careful analysis of the FTMFT should be able to reveal this. This has not yet been done and is a useful project for the future. It can help us classify the types of large $N$ models where mean field theory can break down.

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