Radiation from horizons, chirality and the principle of effective theory

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Abstract. We review a modification of the gravitational anomalies method for the calculation of Hawking-like radiation from horizons of n-dimensional static or stationary spacetimes with horizon(s). This modification uses (i) the chirality near the horizon and (ii) the principle of effective theory of gravity, in order to resolve some of its defects and apply it to spacetimes with constant curvature. The new approach can be smoothly applied in the same class of spacetime as the original gravitational anomalies method.

1. Introduction
The discovery of Hawking radiation from black holes [1] is a particularly important result for theoretical physics. It implies that in the semiclassical picture of a background gravitational field on which the quantum fields propagate, matter and energy can escape from the black holes, which, when considered as objects of the classical gravitational theory, do not permit even to light to escape. Since then, it is proved that Hawking-like radiation can be produced in a series of Lorentz spacetimes with horizons, such as the Hawking-Gibbons radiation which arises at the cosmological horizon of the de Sitter spacetime [2] and the Unruh radiation arising from the horizon for an accelerating observer in the flat spacetime (Rindler spacetime) [3]. Other methods apart from the original one have also been proposed for the calculation of the radiation flux from a horizon, a conceptually clear and easily visualisable method being the tunneling [4], which pictures the radiation as the escape of virtual particles from the horizon, and the trace anomaly method [5] which is applicable only in two dimensions. The latter gave insight on the derivation of the Hawking flux from quantum gravitational anomalies near the region of the horizon and led to the gravitational anomalies method [6]. In this method, the flux is calculated from the local theory near the horizon, which is chiral and two-dimensional. The structure of the spacetime plays no other role except to impose appropriate boundary conditions so as to derive the flux. Although the gravitational anomalies method has been successfully applied in a series of spacetimes of any dimension, it fails to give a correct value for the cases of the de Sitter and the Rindler spacetimes because the quantum anomalies vanish [7]. In [8], a resolution was given by pointing out the fact that it is the chirality in the near horizon region that plays the fundamental role in the method and quantum anomalies is just a consequence of the chirality. Additionally, as postulated in the principle of effective theory [9], different observers\(^1\), who

\(^1\) Here, by observer we mean a congruence of timelike curves.
have access to different regions of the spacetime manifold will have access to different amount of information. Then, it is straightforward to define a new effective stress-energy tensor which gives the correct radiation flux for all previous studied cases and in addition for the two pathological cases (Rindler and de Sitter spacetime).

In the following sections, we briefly summarize the gravitational anomalies method in order to introduce the reader to the technical framework. Then, we explain how the principle of effective theory combined with chirality leads to the new approach and apply it to the prototype spacetimes.

2. Gravitational anomalies method

We start by describing the gravitational anomalies method in the four-dimensional Schwarzschild black hole. The metric in static (Schwarzschild) coordinates has the form

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2d\Omega^2$$

where $f(r) = 1 - 2M/r$, $M$ the mass of the black hole. The horizon is at $r_H$ where $f(r_H) = 0$ with the surface gravity given by $\kappa = \frac{f'(r_H)}{2}$. Near the horizon, the theory is reduced to a two dimensional field theory. Thus, a four dimensional quantum field acts as a two dimensional massless free field and the ingoing and outgoing modes of the field are plane waves. Since the horizon surface acts like a one-membrane surface, the ingoing modes of the field are excluded from the external region of the spacetime. Thus, the theory near the horizon becomes chiral and, as a result, quantum chiral anomalies appear in the conservation equations. These anomalies are of the form \(^{(2)}\)

$$\nabla_\mu T^\mu_{\nu(4)}(r) = 0.$$

We have thus resulted with a four dimensional theory far from the horizon and an effective two dimensional local theory in the region of the horizon. To combine the theories in the two separate regions, we ignore the transverse dimensions in the region far from the horizon, since they play no role on our discussion. Then, the energy-momentum tensor is derived by integrating over the transverse dimensions its $T^t_t$ component given by

$$T^t_t = \int d\Omega^2 r^2 T^t_t$$

and the differential equation is

$$\partial_r T^t_t = 0.$$

The next step is to integrate this differential equation and eq. \(^{(2)}\) in order to obtain the flux from the horizon. By integrating eq. \(^{(2)}\) from $r_H$ to $r$, the $\nu = t$ component of \(^{(2)}\) yields

$$T^t_t(r) = T^t_t(r_H) - N^t_t(r_H)$$

2 For simplicity, we denote as $T^\mu_\nu$ the expectation value of the energy-momentum tensor with respect to the time-asymmetric vacuum state analogue to the Unruh vacuum state in the Schwarzschild spacetime $\langle U[T^\mu_\nu]U \rangle$. \(\text{NEB 15 – Recent Developments in Gravity IOP Publishing}

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where \( N_i^r = \frac{1}{192\pi} (ff'' + f'^2) \), while integration of (5) from \( r \) to \( \infty \) gives \( T_{t(o)}^r(r) = c_0 = \Phi \) with \( c_0 \) being the constant flux at infinity. The radiation flux is obtained from the condition that the value of the energy-momentum tensor far from the horizon should be equal to the limit of the near-horizon energy-momentum tensor at infinity

\[
T_{t(o)}^r(r) = T_{t(H)}^r(r \to \infty). 
\]  

(7)

The physical meaning of this condition is the assumption that no other sources of energy are present in the spacetime. Thus we have to calculate the limit of (6) at infinity. For black hole spacetimes which are asymptotically flat, the component \( N_i^r(r \to \infty) \) vanishes. Therefore, the flux can be written

\[
T_{t(o)}^r(r) = \Phi = T_{t(H)}^r(r \to \infty) = T_{t(H)}^r(r_H) - \frac{1}{192\pi} f'^2(r_H). 
\]  

(8)

The only missing information is the value of the energy-momentum tensor on the horizon. This can be calculated by the covariant condition \( \tilde{T}_i^r(r_H) = 0 \), where \( \tilde{T}_i^r \) is the covariant tensor defined by [11]

\[
\tilde{T}_i^r(r) = T_i^r(r) + \frac{1}{192\pi} (ff'' - f'^2) \]  

(9)

while the \( T_i^r \) tensor we used in our relations till now is called consistent. The covariant condition is imposed in order to ensure that the physical quantities on the horizon do not diverge. Using (9) and the fact that \( f(r_H) = 0 \) one obtains \( T_i^r(r_H) = 2 \frac{f'^2(r_H)}{192\pi} = 2N_i^r(r_H) \). By imposing the covariant condition [12], the flux of Hawking radiation reads

\[
\Phi = N_i^r(r_H) = \frac{f'^2(r_H)}{192\pi} = \frac{\kappa^2}{48\pi} = \frac{\pi}{12} T_H^2, 
\]  

(10)

where \( \kappa \) is the surface gravity of the black hole horizon and \( T_H = \frac{\kappa}{2\pi} \) is the Hawking temperature.

Up to now, we have shown that the flux can be computed when one employs the conservation and anomaly equations of the consistent energy-momentum tensor. One can derive exactly the same results if the conservation and anomaly equations of the covariant energy-momentum tensor are considered. In particular, the anomaly equation of the covariant energy-momentum tensor reads [12]

\[
\nabla_\mu \tilde{T}_\mu^r(r) = \tilde{A}_\nu \frac{1}{96\pi} \epsilon^\mu_\nu \nabla_\mu R, 
\]  

(11)

where \( R \) is the Ricci scalar, and \( \epsilon_{\mu\nu} \) is the totally antisymmetric tensor in two dimensions. The \( \nu = t \) component of the anomalies is given by

\[
\tilde{A}_t = \frac{1}{\sqrt{-g}} \partial_r \tilde{N}_t^r \quad \tilde{N}_t^r(r) = \frac{1}{96\pi} (ff'' - f'^2) \frac{1}{2}. 
\]  

(12)

Following the same procedure and imposing the same boundary conditions as before (i.e. \( \tilde{T}_t^r(r_H) = 0 \)), one finds that the flux is

\[
\Phi = -\tilde{N}_t^t(r_H) = -\frac{1}{96\pi} \frac{f'^2(r_H)}{2} = \frac{\pi}{12} T_H^2. 
\]  

(13)

This flux is identical to the one given by the consistent equations (10). This method is successful in a large class of static and stationary n-dimensional spacetimes with horizons with metric given by the general form

\[
ds^2 = -f(r) dt^2 + h^{-1} dr^2 + P(r) d\sigma^{n-2}. 
\]  

(14)
The horizon is at $r_H$ where $f(r_H) = h(r_H) = 0$, $P(r)$ is a smooth function and $d\sigma^{n-2}_{\perp}$ is the metric of the transverse dimensions. The surface gravity of the horizon is given by

$$\kappa = \frac{1}{2}\sqrt{f'(r_H)h'(r_H)}.$$  

The dimensional reduction should be performed again in a way that the conformal factor preserves the roots of $f(r)$ and $h(r)$ in order to have physically acceptable solutions [13]. Moreover, the spacetime need not be asymptotically flat, as is the case of the black hole spacetime we studied in this section, but further modifications are needed in order to obtain the correct flux, as was shown in [14], because the $N_t^t(r)$ does not always vanish at infinity. In this category also falls the case of the spacetimes with multiple horizons such as the Schwarzschild-de Sitter spacetime. In spite of the success of this method to a broad class of metrics, there are several problems yet unresolved which raise questions whether the method can be smoothed and become more physically intuitive. First, it would appear more natural to express the law of energy-momentum conservation in a unified form instead of having two equations, one normal and one anomalous. Second, it is important to explore which form of the equations, consistent or covariant is most physically appropriate for the calculation of the radiation flux. Important explorations are also on call for the cases of time-dependent spacetimes. Although there is work done in this direction (see e.g.[15]), further research is needed. Finally, a major defect of the method is the fact that it cannot be applied in spacetimes with constant curvature. In the next section, we review the suggestion in [8] for the resolution of this problem.

3. A new approach

The gravitational anomalies appear in the area near the horizon because the horizon acts as a one way membrane and therefore the ingoing modes of any quantum field are excluded from the external part of the manifold. This makes the theory chiral and consequently the conservation equations become anomalous. However, in the case of certain spacetimes, the derivative of the anomalous flux vanishes and we get the unexpected result that there is no radiation flux from the horizon, despite the fact that the anomalous flux $N_t^t$ does not vanish. This points to the fact that the anomalies are not the fundamental quantity but rather it is chirality near the region of the horizon that one should take into account in order to calculate the radiation flux.

The second observation leads to the principle of effective theory applied in gravity. This principle states that physical theories in a given coordinate system must be formulated entirely in terms of the variables that an observer using that coordinate system can access [9]. This can be seen as a natural interpretation of general covariance [16]. Therefore, the conservation law for the energy-momentum tensor defined only in the external region of the horizon is

$$\nabla_\mu T^\mu_\nu = 0, \quad \nabla_\mu \tilde{T}^\mu_\nu = 0$$  

where now the $T^\mu_\nu \equiv T^\mu_\nu - N^\mu_\nu$ is the consistent effective energy-momentum tensor, $T^\mu_\nu$ is the consistent energy-momentum tensor it was considered in the anomalies method, while $N^\mu_\nu$ is the contribution of the modes near the horizon as discussed already. The second equation is the analogue for the covariant form of the equations. The $r-t$ component of the conservation equation is

$$\partial_r (T^r_t \mp N^r_t) = 0 \quad \text{and} \quad \partial_r (\tilde{T}^r_t \mp \tilde{N}^r_t) = 0$$  

where we have written both the consistent and covariant form of the equations. We have also included $\mp$ signs to introduce the generalization of the method in spacetimes in which the observer has access in the internal of the horizon part of the manifold as in the case that the horizon is cosmological and an observer exists in the inner to the horizon part of the manifold. Then the ingoing modes are to be taken into account and the outgoing modes are excluded. The minus represents the consideration of the outgoing modes as we considered till this point while the plus sign the consideration of the ingoing. We will follow this notation in the following.
where necessary. These effective energy-momentum tensors reproduce all the correct results for the Hawking radiation fluxes of asymptotically flat spacetimes, but in addition they work for the non-asymptotically flat, constant curvature spacetimes of de Sitter and Rindler. In the next section, we will calculate the flux for these two spacetimes. By integrating eq. (15), we find

\[ T^r_t (r \to r_0) \pm N^t_r (r \to r_0) = T^r_t (r_H) \mp N^t_r (r_H) = \Phi = \pm N^t_r (r_H) . \]  

(17)

The integration constant, \( \Phi \), is the flux defined previously in (10). We have evaluated the integrated quantity at two points \( r = r_0 \) and \( r = r_H \). Previously \( r_0 \to \infty \) since we were dealing with asymptotically flat spacetimes. We have generalized (6) to (17) since de Sitter and Rindler spacetimes are not asymptotically flat so we will need to take a limit different than \( r_0 \to \infty \).

In a similar manner we can integrate the second relationship in (16) to obtain the covariant expression

\[ \tilde{T}^r_t (r \to r_0) \mp \tilde{N}^t_r (r \to r_0) = \tilde{T}^r_t (r_H) \pm \tilde{N}^t_r (r_H) = \Phi = \pm \tilde{N}^t_r (r_H) . \]  

(18)

As previously the integration constant, \( \Phi \), is the flux defined in (13) and we have used the fact that \( N^t_r (r_H) = -\tilde{N}^t_r (r_H) \). The proper limit \( r_0 \) will be specified from the geometry of the spacetime.

4. Constant curvature spacetimes

4.1. De Sitter spacetime

The metric of de Sitter spacetime in static coordinates is written as

\[ ds^2 = -\left(1 - \frac{r^2}{a^2}\right)dt^2 + \left(1 - \frac{r^2}{a^2}\right)^{-1}dr^2 \]  

(19)

with the cosmological horizon located at \( r_H = a \). The method of gravitational anomalies can be applied to de Sitter spacetime only by using the consistent expressions of the equations and fails if one applies the covariant expressions of the equations [7]. The reason for this failure is that the covariant anomaly vanishes since the Ricci scalar is a constant, i.e. \( R = \frac{2}{a^2} \). However, the elements \( N^t_r \) and \( \tilde{N}^t_r \) do not vanish at the proper limit, \( r_0 = 0 \), namely \( N^t_r (r \to 0) = \frac{1}{192\pi} (-\frac{1}{a^2}) \) and \( \tilde{N}^t_r (r \to 0) = \frac{1}{96\pi} (-\frac{2}{a^2}) \). The proper limit is \( r_0 = 0 \) since the flux is flowing from the cosmological horizon inwards. The accessible part of the manifold inside the cosmological horizon and the flux of Hawking radiation has to be calculated at the point \( r_0 = 0 \). Using the equations for the effective consistent energy-momentum tensor using the lower signs for these two equations since the flux is coming inward from the cosmological horizon at \( r_H \), we obtain

\[ \Phi = T^r_t (r \to 0) = T^r_t (r_H) + N^t_r (r_H) = -N^t_r (r_H) . \]  

(20)

Similarly using the equations for the effective covariant energy-momentum tensor (17) and (18) (again using the lower sign in these two equations) we obtain

\[ \Phi = \tilde{T}^r_t (r \to 0) = \tilde{T}^r_t (r_H) + \tilde{N}^t_r (r_H) = -\tilde{N}^t_r (r_H) \]  

(21)

It is evident that employing either the effective consistent energy-momentum tensor or the effective covariant energy-momentum tensor, the flux of the Hawking radiation in de Sitter spacetime is equal to

\[ \Phi = -N^t_r (r_H) = +\tilde{N}^t_r (r_H) = -\frac{f^{12} (r_H)}{192\pi} = -\frac{\pi}{12} T^2_{GH} , \]  

(22)

where \( T_{GH} \) is the Gibbons-Hawking temperature of the cosmological horizon [2], i.e. \( T_{GH} = 1/2\pi a \).
4.2. Rindler spacetime

The metric of the Rindler spacetime in static coordinates is

\[ ds^2 = -(1 + 2ar)dt^2 + (1 + 2ar)^{-1}dr^2 , \]  

(23)

where \( a \) is the acceleration of an observer, a horizon at \( r_H = -1/2a \) blocks the observer from accessing the whole manifold. With respect to this observer, there exists a radiation flux, called Unruh flux [17]. Thus, it is expected that this flux can be calculated with the gravitational anomalies method. However, in [7] it was shown that both forms of the anomaly method, consistent and covariant, fail to give the correct value of the Unruh flux, since the curvature of the spacetime in (23) is zero, \( R = 0 \). However, the components \( N^t_t \) and \( \tilde{N}^t_t \) of the anomaly part of the equations, were non-zero and constant at the limit point \( r_0 = 0 \), where the static observer resides in that coordinate system. We need the quantities \( N^r_t = a^2/48 \) and \( \tilde{N}^r_t = -a^2/48 \) to calculate the flux. The limit point is at \( r_0 = 0 \) rather than infinity. However the horizon is located at \( r_H = -1/2a \) so that the flux is “outward” from negative \( r \) toward positive \( r \). Thus one should use the upper signs of the equations since the flux is “outward”. The flux then reads

\[ \Phi = T^r_t(r \to 0) - N^r_t(r_H) = N^r_t(r_H) \]  

(24)

and using the covariant expressions

\[ \Phi = \tilde{T}^r_t(r \to 0) - \tilde{N}^r_t(r_H) = -\tilde{N}^r_t(r_H) = N^r_t(r_H) \]  

(25)

In both of the above equations \( r_0 = 0 \) and \( r_H = -1/2a \). It is again evident that employing either the equations of the effective consistent energy-momentum tensor, or the equations for the effective covariant energy-momentum tensor, the flux of the Unruh radiation in Rindler spacetime is equal to

\[ \Phi = N^r_t(r_H) = -\tilde{N}^r_t(r_H) = \frac{f^2(r_H)}{192\pi} = \frac{\pi}{12} T^2_H \]  

(26)

where \( T_H = a/2\pi \) is the Unruh temperature seen by the accelerated observer.

5. Conclusions

In this article, we reviewed the method of gravitational anomalies and discussed a modification introduced in [8] for the calculation of the flux radiation and temperature of horizons. This modification resolved the problem the gravitational anomalies method had when applied in spacetimes with constant curvature. The definition of the effective stress-energy tensor allowed the equations to take a general form independent of the spacetime asymptotic behaviour. However, further investigation is needed in order to understand and clarify other aspects of this method such as why both the consistent and the covariant form of the equations give equivalent results and whether this modification can be applied in time-dependent horizons.

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