Suppression of blow-up in parabolic–parabolic Patlak–Keller–Segel via strictly monotone shear flows

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Abstract
In this paper we consider the parabolic–parabolic Patlak–Keller–Segel models in \( \mathbb{T} \times \mathbb{R} \) with advection by a large strictly monotone shear flow. Without the shear flow, the model is \( L^1 \) critical in two dimensions with critical mass \( 8\pi \): solutions with mass less than \( 8\pi \) are global in time and there exist solutions with mass larger than \( 8\pi \) which blow up in finite time (Schweyer 2014 (arXiv:1403.4975)). We show that the additional shear flow, if it is chosen sufficiently large, suppresses one dimension of the dynamics and hence can suppress blow-up. In contrast with the parabolic-elliptic case (Bedrossian and He 2016 *SIAM J. Math. Anal.* 49 4722–66), the strong shear flow has destabilizing effect in addition to the enhanced dissipation effect, which makes the problem more difficult.

Keywords: parabolic–parabolic Patlak–Keller Segel equation, shear flow enhanced dissipation, shear flow destabilizing effect

Mathematics Subject Classification numbers: 35, 92

1. Introduction
In this paper, we consider the two-dimensional parabolic–parabolic Patlak–Keller–Segel equations with additional effect of advection by a shear flow, which model the chemotaxis phenomena in a moving fluid:

\[ \partial_t n + \nabla \cdot (n \nabla c) + Au(y) \partial_x n = \Delta n, \]  

(1.1a)
\[ \epsilon \left( \partial_t c + Au(y) \partial_x c \right) = \Delta c + n - c, \quad (1.1b) \]
\[ n(x, y, 0) = n_m(x, y), \quad c(x, y, 0) = c_m(x, y), \quad (x, y) \in \mathbb{T} \times \mathbb{R}. \quad (1.1c) \]

Here \( n(x, y, t) \) and \( c(x, y, t) \) denote the micro-organism density and the chemo-attractant density, respectively. The divergence free vector field \( Au(y) \) represents the underlying fluid velocity. When \( Au \equiv 0 \), the system is the classical parabolic–parabolic Patlak–Keller–Segel system modeling chemotaxis in a static environment; see e.g. \([31, 40]\). In this case, the first part of \((1.1)\) describes the time evolution of the micro-organism density \( n \) subject to diffusion and chemo-attractant-triggered aggregation. The second part of \((1.1)\) models the time evolution of the chemo-attractant secreted by the micro-organism. The parameter \( \epsilon = 0, 1 \) corresponds to the parabolic-elliptic case and parabolic–parabolic case respectively. When \( Au \neq 0 \), the system \((1.1)\) takes into account the advection effect of fluid in the ambient environment.

We focus on the case where \( Au \neq 0 \) to reflect a scenario of chemotaxis taken place in moving fluid. The question we address is whether one can use a shear flow \( Au \) to prevent the micro-organism from undergoing chemotactic blow-up when \( u = 0 \). It is worth mentioning that system \((1.1)\) is one of many attempts to take into account the effect of the moving fluid. For other related models, see \([21, 26, 27, 32, 35–37, 44]\). We recall the large literature on the Patlak–Keller–Segel model in the static case \((u = 0)\), referring the interested reader to the review \([29]\) and the following works \([3, 9, 10, 12–19, 23–25, 28, 30, 34, 38, 39, 42]\).

It is well known that the Patlak–Keller–Segel equation \((1.1)\) is \( L^1 \) critical and the \( L^1 \) norm of the solution \( M : = \|n\|_1 \) is preserved. If there is no underlying moving fluid, i.e. \( Au \equiv 0 \), the existing results for the parabolic–parabolic case \((\epsilon = 1)\) can be summarized as follows. In the sub-critical case \( M < 8\pi \), the global well-posedness of the free energy solution to \((1.1)\) is known \([18, 20]\). On the other hand, if \( M > 8\pi \), it is shown in \([41]\) that there exists finite time blow-up solution on \( \mathbb{R}^2 \). In higher-dimension, there exist solutions with arbitrary mass which blow up in finite time \([45]\).

In the recent years, progress was made in proving global existence of solution to \((1.1)\) in the parabolic-elliptic regime \((\epsilon = 0)\) with total mass \( M > 8\pi \) and \( Au \neq 0 \). In \([33]\), it was shown that if the vector field \( u \) is relaxation enhancing - a generalization of weakly mixing introduced in \([22]\) - with large enough amplitude, the solution \( n \) is global in time. The authors proved that due to the mixing property of \( u \), the solution undergoes a large growth in its gradient which significantly enhances the dissipation. Once the enhanced dissipation dominates the nonlinear aggregation, suppression of chemotactic blow-up of Patlak–Keller–Segel follows. In \([8]\), it is shown that one can use a strong shear flow without degenerate critical points to suppress the blow up in \((1.1)\). The idea in the paper is to exploit the enhanced dissipation effect of shear flow using hypoenergicy \([4, 43]\) and to prove that a large shear flow can in some sense suppress one dimension in parabolic-elliptic PKS system \((1.1)\) and hence make \( 2D \) \( L^1 \) subcritical and \( 3D \) \( L^1 \) critical. It is worth mentioning that the enhanced dissipation effect of shear flow is also shown to be important for understanding the stability of the Couette flow in the \( 2D \) and \( 3D \) Navier–Stokes equations at high Reynolds number \([5–7, 11]\).

In the parabolic–parabolic setting \((\epsilon = 1)\), the situation is different. The mixing of the shear flow has both stabilizing and destabilizing effect on the system \((1.1)\). On the one hand, same as in the parabolic-elliptic case, mixing enhances the dissipation in the micro-organism evolution equation \((1.1a)\) and hence stabilizes the dynamics. On the other hand, the extra shear flow advection term \( Au(y) \partial_x c \) in the chemo-attractant evolution \((1.1b)\) creates large gradient in the chemical density \( c \). To better understand this destabilizing effect, we take a look at the passive scalar equation on a Torus \( \mathbb{T}^2 \).
\[ \partial_t \rho + Au \partial_x \rho = \Delta \rho, \quad \rho(t = 0, \cdot) = \rho_0(\cdot), \]

where \( \rho \) have average zero. We calculate the time evolution of \( ||\nabla \rho||^2 \) as follows

\[
\frac{d}{dt} ||\nabla \rho||^2 \leq -||\nabla \rho||^2 + A||u(\cdot)\partial_x \rho||2||\nabla \rho||^2.
\]

One observe that the dissipation start to take effect on the time scale \( O(1) \) but the shear flow effect takes effect on the time scale \( O(1/A) \). Therefore in the time scale between \( O(1/A) \) and \( O(1) \), shear flow effect dominates the dissipation and creates large growth in the gradient. This is called the destabilization effect of the shear flow. The large growth in the chemical gradient destabilizes the dynamics through the aggregation nonlinearity \( \nabla \cdot (\nabla cn) \) in the micro-organism evolution (1.1a). It is worth noting that this destabilizing effect of shear flow does not exist in the parabolic-elliptic regime due to the fast relaxation of chemical density to equilibrium.

As a result, it is reasonable to expect that an extra smallness assumption is needed to control the mixing destabilizing effect. In this paper, it is assumed that the \( x \)-dependent part of the initial chemical gradient is small. Since only the \( x \)-dependent part of \( \partial_x c \) is strongly forced by the shear flow, this smallness restriction is sufficient to control the growth of the chemo-attractant gradient and hence keep the aggregation nonlinearity in (1.1a) bounded independent of \( A \). Now the situation is similar to the parabolic-elliptic case, hence one can show suppression of chemotactic blow-up through shear flow.

Denote the following projections for function \( g(x, y) \):

\[
g(y) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(x, y) dx, \quad g\neq(x, y) = g(x, y) - g(y).
\]

The main theorem of this paper is as follows.

**Theorem 1.** Let the shear flow profile \( u \in C^3(\mathbb{R}) \) be a strictly monotone function whose derivative approaches nonzero numbers at \( \pm \infty \) and \( ||u'||_{W^{2,\infty}} < \infty \). Consider the equation (1.1) subject to initial condition \( n_{in} \in H^1 \cap W^{1,\infty}(T \times \mathbb{R}), \ c_{in} \in H^2 \cap W^{2,\infty}(T \times \mathbb{R}) \). Then the solution to (1.1) is global in time if the amplitude \( A \) takes values in the interval \( (\frac{A_0}{q_*}, ||\nabla (c_{in})||_2) \) where \( q_* \in (-2, 0) \) and \( A_0 = A_0(u, ||n_{in}||_{H^1} ||\nabla (c_{in})||_2) \) is independent of \( ||\nabla (c_{in})||_2 \).

We make several remarks concerning the main theorems.

**Remark 1.** For the interval \( (\frac{A_0}{q_*}, ||\nabla (c_{in})||_2) \) to be nonempty, we implicitly assume that \( ||\nabla (c_{in})||_2 \) is small compared to \( A_0/q_* \). As explained before, this smallness is applied to control the destabilizing effect of the strong shear flow. Note that if \( c_{in} \equiv 0 \), then the interval is always nonempty. This corresponds to the situation that at the initial time of the chemotaxis experiment, no chemo-attractant exists in the environment.

**Remark 2.** The difficulty is twofold. First we need to construct a hypercercivity functional adapted to the parabolic–parabolic PKS equation, which is significantly more subtle than the one in the parabolic-elliptic case [8]. Secondly, one needs to control \( ||\nabla c||_\infty \) uniformly independent of \( A \) for all time. This is delicate due to destabilizing effect of the strong shear flow.

**Remark 3.** Since the Couette flow \( (y, 0) \) is a stationary solution to the Navier–Stokes equation, this result is the first step to proving the suppression of blow-up for the parabolic–parabolic Keller–Segel–Navier–Stokes system.
1.1. Notations

1.1.1. Miscellaneous. Given quantities $X, Y$, if there exists a constant $B$ such that $X \lesssim BY$, we often write $X \lesssim Y$. We will moreover use the notation $\langle x \rangle := (1 + |x|^2)^{1/2}$.

1.1.2. Fourier analysis. For $f(x, y)$ we define the Fourier transform $\hat{f}(k, y)$ only in terms of the variable $x$, and the inverse Fourier transform as follows:

$$\hat{f}(k, y) = \frac{1}{2\pi} \int_{\mathbb{T}} e^{-ikx} f(x, y) dx, \quad \hat{g}(x, y) = \sum_{k = -\infty}^{\infty} g(k, y) e^{ikx}. $$

Define the following orthogonal projections:

$$f_0(t, y) = \frac{1}{2\pi} \int_{\mathbb{T}} f(t, x, y) dx, $$

$$f_\neq(t, x, y) = f(t, x, y) - f_0(t, y). $$

Here ‘0’ and ‘:\neq’ stand for ‘zero frequency’ and ‘non-zero frequencies’. For any measurable function $m(k)$, we define the Fourier multiplier $m(\partial_x) f := (m(k) \hat{f}(k, y))^\vee$.

1.1.3. Functional spaces. The norm for the $L^p$ space is denoted as $||\cdot||_p$ or $||\cdot||_{L^p(\cdot)}$:

$$||f||_p = ||f||_{L^p} = \left( \int |f|^p dx \right)^{1/p}, $$

with natural adjustment when $p = \infty$. If we need to emphasize the ambient space, we use the second notation, i.e. $||a||_{L^p(T \times \mathbb{R})}$. Otherwise, we use the first notation for the sake of simplicity. The Sobolev norm $|| \cdot ||_{H^s}$ is defined as follow:

$$||f||_{H^s} := ||\langle \nabla \rangle^s f||_{L^2}. $$

For a function of space and time $f = f(t, x)$, we use the following space-time norms:

$$||f||_{L^p_t L^q_x} := ||f||_{L^p_t L^q_x}, $$

$$||f||_{L^p_t H^s_x} := ||f||_{L^p_t H^s_x}. $$

The paper is organized as follows: in section 2, we set up the bootstrap argument; in section 3, we prove the enhanced dissipation of the $x$-depending part of the solution; in section 4, we prove the $L^2_t H^s_x$ estimate of $x$-dependent part the micro-organism density; in section 5, we estimate the $x$ independent part of the solution; in section 6, we prove the uniform in time $L^\infty$ estimate of the solution.

2. Preliminaries and bootstrap

2.1. Reformulation of theorem 1

In this paper, we will prove the following theorem, which implies the theorem 1.

Theorem 2. Let the shear flow profile $u$ satisfy the conditions in theorem 1. Consider the equation (1.1) subject to initial conditions $n_0 \in H^1 \cap W^{1,\infty}$, $c_0 \in H^2 \cap W^{2,\infty}$ and $||\nabla (c_0)||_{W^{1,\infty}} \leq C_0 A^{-q}$, $q > 1/2$ for any constant $C_0$ independent of $A$. Then there exists an $A_0 = A_0(u, ||n_0||_{W^{1,\infty}}, ||\nabla c_0||_{W^{1,\infty}})$ such that if $A > A_0$, the solution to (1.1) is global in time.
Proof of theorem 1. Choosing $C_{in} = 1$ in theorem 2, we have the following relation
\[
\|\nabla(c_{in})\|_{H^1/\Gamma W^{1,\infty}}^{-1/q} \geq A. \tag{2.1}
\]
Combining it with the relation $A \geq A_0$, we end up with $A \in (A_0, \|\nabla(c_{in})\|_{H^1/\Gamma W^{1,\infty}}^{-1/q}, q \in (1/2, \infty)$. Define $q_* = \frac{q}{q-1}$, we end up with the condition specified in theorem 1. \(\square\)

2.2. Bootstrap argument

Same as in the paper [8], we rescale in time and decompose the solution into $x$-independent part $n_0$, $c_0$ and $x$-dependent part $n_\neq$, $c_\neq$
\[
\partial_t n_0 + \frac{1}{A} \partial_y (\partial_y c_0 n_0) + \frac{1}{A} (\nabla \cdot (\nabla c_\neq n_\neq))_0 = \frac{1}{A} \partial_y n_0, \tag{2.2a}
\]
\[
\partial_t c_0 = \frac{1}{A} \Delta c_0 + \frac{1}{A} n_0 - \frac{1}{A} c_0; \tag{2.2b}
\]
and,
\[
\partial_t n_\neq + u(y) \partial_x n_\neq + \frac{1}{A} \nabla \cdot (\nabla c_\neq n_0) + \frac{1}{A} \partial_y (\partial_y c_\neq n_\neq) + \frac{1}{A} (\nabla \cdot (\nabla c_\neq n_\neq))_\neq = \frac{1}{A} \Delta n_\neq, \tag{2.3a}
\]
\[
\partial_t c_\neq + u(y) \partial_x c_\neq = \frac{1}{A} \Delta c_\neq + \frac{1}{A} n_\neq - \frac{1}{A} c_\neq. \tag{2.3b}
\]

To apply the machinery of the paper [4], we apply the Fourier transform \textit{only in the $x$ variable} to both sides of (2.3a) and (2.3b) to obtain
\[
\partial_t \hat{n}_k + NL_k + L_k + u(y) i k \hat{n}_k = \frac{1}{A} (\partial_y - |k|^2) \hat{n}_k, \tag{2.4a}
\]
\[
\partial_t \hat{c}_k + u(y) i k \hat{c}_k = \frac{1}{A} (\partial_y - |k|^2) \hat{c}_k + \frac{1}{A} \hat{n}_k - \frac{1}{A} \hat{c}_k, \quad k \neq 0, \tag{2.4b}
\]
where $L_k, NL_k$ are defined as follows:
\[
NL_k := \frac{1}{A} \sum_{k \neq 0 \ell} \partial_y (\partial_y \hat{c}_k \hat{c}_\ell) = \frac{1}{A} \sum_{k \neq 0 \ell} (k - \ell) \hat{c}_k \hat{c}_\ell, \tag{2.5}
\]
\[
L_k := \frac{1}{A} \partial_y (\partial_y c_0 \hat{n}_k) + \frac{1}{A} \nabla \cdot (\nabla c_0 n_0) = \frac{1}{A} \partial_y (\partial_y c_0 \hat{n}_k) - \frac{1}{A} k^2 \hat{c}_k n_0 + \frac{1}{A} \partial_y (\partial_y c_0 n_0). \tag{2.6}
\]

Here, the $L$ refers to ‘linear with respect to the nonzero frequencies’ and $NL$ refers to ‘nonlinear with respect to the nonzero frequencies’.

As is standard in the study of nonlinear mixing, we use a bootstrap argument to prove the main theorem. For constants $C_{Ed}, C_{m,L}, C_{m,Ed}, C_{n,\infty}, C_{\nabla c,\infty}$ and $A_0$ determined by the proof, define $T_*$ to be the end-point of the largest interval $[0, T_*)$ such that the following hypotheses hold for all $t \leq T_*$:

(1) Nonzero mode $L_k H_{k,y}^1$ estimate:
\[
\frac{1}{A} \int_0^{T_*} \|\nabla_t n_\neq\|_2^2 \, dt \leq 8 \|n_{in}\|_2^2. \tag{2.7a}
\]
(2) Nonzero mode enhanced dissipation estimate:
\[ ||n_{\neq}(t)||^2 + ||\nabla c_{\neq}(t)||^2 \leq 4C_{\text{ED}}(||n_{\text{in}}||^2 + 1)e^{-\frac{r}{\eta T}}, \quad \forall t < T_*, \]
where $\eta$ is a small constant depending only on $u$.

(3) Uniform in time estimates on the zero mode:
\[ ||\partial_t c_0||_{L^\infty(0,T,L^\infty)} + ||n_0||_{L^\infty(0,T,L^2)} \leq 4C_{n_0,L^2}, \quad ||\partial_t n_0||_{L^\infty(0,T,L^2)} \leq 4C_{n_0,H^1}. \]

(4) $L^\infty$ estimate of the solution $n$:
\[ ||n||_{L^\infty(0,T,L^\infty)} < 4C_{n,\infty}. \]

(5) $L^n$ estimate of the $x$-dependent part of the chemical gradient $\nabla c_{\neq}$:
\[ ||\nabla c_{\neq}||_{L^n(0,T,L^\infty)} < 4C_{\text{grad},n,L^\infty}. \]

Furthermore, we define the following constant to simplify the notation:
\[ C_{\text{grad}} := 1 + M + C_{\text{ED}}(||n_{\text{in}}||_{H^1}^2 + C_{n_0,L^2}^2 + C_{n,\infty} + C_{\text{grad},n,L^\infty} + ||\nabla(c_{\text{in}})||_{H^1 \cap W^{1,\infty}}. \]

Note that $C_{n_0,H^1}$ is not included in $C_{\text{grad}}$.

The goal is to prove the following improvement to the above hypotheses:

**Proposition 1.** For all $n_{\text{in}}, c_{\text{in}}$ and $u$ satisfying the assumption of theorem 2, there exists an $A_0 = A_0(u, ||n_{\text{in}}||_{H^1 \cap L^\infty}, ||\nabla c_{\text{in}}||_{H^1 \cap W^{1,\infty}})$ such that if $A > A_0$ then the following conclusions hold on the interval $[0, T_*)$:

\[ (1) \quad \frac{1}{A} \int_0^{T_*} ||\nabla_{(\mu)} n_{\neq}||_2^2 \, dt \leq 4||n_{\text{in}}||_2^2; \]

\[ (2) \quad ||n_{\neq}(t)||^2 + ||\nabla c_{\neq}(t)||^2 \leq 2C_{\text{ED}}(||n_{\text{in}}||_{H^1}^2 + 1)e^{-\frac{r}{\eta T}}, \quad \forall t < T_*; \]

\[ (3) \quad ||\partial_t c_0||_{L^\infty(0,T,L^\infty)} + ||n_0||_{L^\infty(0,T,L^2)} \leq 2C_{n_0,L^2}, \quad ||\partial_t n_0||_{L^\infty(0,T,L^2)} \leq 2C_{n_0,H^1}; \]

\[ (4) \quad ||n||_{L^\infty(0,T,L^\infty)} \leq 2C_{n,\infty}; \]

\[ (5) \quad ||\nabla c_{\neq}||_{L^n(0,T,L^\infty)} \leq 2C_{\text{grad},n,L^\infty}. \]

Proposition 1 together with the local wellposedness of the equation (1.1) implies that the time interval $[0, T_*)$ on which the estimates (2.7) hold is both open and closed on $\mathbb{R}_+$. Since the estimates are trivially satisfied at the initial time, we obtain that $[0, T_*)$ is nonempty and hence $T_*$ must be infinity, which in turn implies theorem 2.

**Remark 4.** For the sake of completeness, we prove the blow-up criterion for the system (1.1) in the appendix. The criteria implies that as long as $||n||_{\infty}$ is bounded uniformly in time, all initial bounds on higher $H^p$ norms of the solution can be propagated.
Remark 5. The constants in the proof are determined in the following order

\[ C_{\text{ED}} \Rightarrow C_{n, L^2} \Rightarrow C_{n, \infty} \Rightarrow C_{\nabla \psi \cdot \infty} \Rightarrow C_{n, bL^2} \Rightarrow A_0. \tag{2.9} \]

The magnitude of the flow \( A_0 \) will be chosen large depending on the constants in the hypotheses and the intermediate constants in the proof.

Remark 6. We need to control the destabilizing effect of the shear flow in the proof of (2.8b), (2.8d) and (2.8e).

2.3. Chemical gradient \( \partial_t c_0 \) estimate

The following estimate of the chemical gradient \( \partial_t c_0 \) is applied in the latter sections.

Lemma 2.1. Consider the solution to (2.2a) subject to initial data \((c_0)_0\). For \( s \in \mathbb{N} \) and any \((p, q)\) pair such that either \( 2 \leq p < \infty \), \( 1 \leq q \leq p \) or \( p = \infty \), \( 1 < q \leq p \) is satisfied, the following estimates hold for the solution \( c_0 \)

\[
\begin{align*}
||\partial_t c_0(t)||_p &\leq \sup_{0 \leq \tau \leq t} ||n_0(\tau)||_q + ||(\partial_t c_0)_0||_p; \\
||\partial_t^{q+1} c_0(t)||_p &\leq \sup_{0 \leq \tau \leq t} ||\partial_{\tau} n_0(\tau)||_q + ||(\partial_t^{q+1} c_0)_0||_p, \quad 2 \leq p \leq \infty. 
\end{align*}
\tag{2.10}
\]

Proof. For \( 2 \leq p \leq \infty \), using the heat mild solution representation, Minkowski’s integration inequality and Young’s inequality, we have the following

\[
\begin{align*}
||\partial_t c_0||_p &\leq \left\| \int_0^t \int_{\mathbb{R}} \frac{e^{-\frac{(x - y)^2}{2(t - s)A^{-1}}} 1}{\sqrt{4\pi(t - s)A^{-1}}} e^{-\frac{|y - \theta|^2}{4A^{-1}(t - s)}} - \frac{n_0}{A} (y, s) dy ds \right\|_p \\
&\leq \int_0^t \frac{e^{-\frac{t}{2A^{-1}}}}{2A^{-1}(t - s)} \left( \int_{\mathbb{R}} \frac{|x - y|^2}{4\pi A^{-1}(t - s)} e^{-\frac{|y|^2}{4A^{-1}(t - s)}} - \frac{n_0}{A} (y, s) dy \right) ds + ||(\partial_t c_0)_0||_p \\
&\leq \int_0^t \frac{e^{-\frac{t}{2A^{-1}}}}{2A^{-1}(t - s)} \left( \int_{\mathbb{R}} \frac{|y|^2}{4\pi A^{-1}(t - s)} e^{-\frac{|y|^2}{4A^{-1}(t - s)}} - \frac{n_0}{A} (y, s) dy \right) ds + ||(\partial_t c_0)_0||_p \\
&\leq \frac{1}{p} + 1 = \frac{1}{q} + 1 \quad \text{and} \quad 1 \leq r < \infty. \quad \text{The proof for the higher derivative case is similar, so we omit the proof. This concludes the proof of the lemma.}\]
\]

3. Enhanced dissipation estimate (2.8b)

3.1. Enhanced dissipation functional \( F \)

In this subsection, we construct the functional \( F \) to exploit the enhanced dissipation in the equation (1.1).

We start by introducing the basic ideas of Hypocoercivity [4, 43]. Consider the following passive scalar equation on \( [\mathbb{T} \times \mathbb{R}, \mathbb{T} [\mathbb{T} \times \mathbb{R}]. \]

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\[ \partial_t f + u(y) \partial_y f = \frac{1}{A} \Delta f. \quad f(t = 0, \cdot) = f_{in}(\cdot). \quad (3.1) \]

By applying Fourier transform in the \( x \) variable, we obtain the following equation:

\[ \partial_t \hat{f}_k + u(y) i k \hat{f}_k = \frac{1}{A} (\partial_y) \hat{f}_k - \frac{|k|^2}{A} \hat{f}_k. \]

The term \( u(y) i k \hat{f}_k \) is called the conservative part of the equation (3.1) because it generates a unitary semigroup which preserves the \( L^2 \) norm. The terms \( -\frac{1}{A} (\partial_y) \hat{f}_k - \frac{|k|^2}{A} \hat{f}_k \) are the dissipative part of the dynamics because they cause decay in the \( L^2 \) norm \( ||\hat{f}_k||_2^2 \). The idea of Hypocoercivity is to construct a functional \( \Phi \), which is more ‘coercive’ than the \( H^1 \) norm, to exploit the commutator structure between the conservative part and the dissipative part of the dynamics. The functional is defined as:

\[ \Phi_k[f] := ||\hat{f}_k||_2 + \alpha ||\partial_y \hat{f}_k||_2^2 + \beta \Re(\langle [\partial_y, u(y) i k \hat{f}_k, \partial_y \hat{f}_k] \rangle_{L^2}) + \gamma ||\partial_y, u(y) i k \hat{f}_k||_2^2, \]

where \([,] \) denotes the commutator of operators and \( \alpha, \beta \) and \( \gamma \) are constants chosen properly. By noting that

\[ [\partial_y, u(y) i k \hat{f}_k] = \partial_y(u(y) i k \hat{f}_k) - u(y) i k \partial_y \hat{f}_k = u'(y) i k \hat{f}_k, \]

the functional can be represented as follows

\[ \Phi_k[f(t)] = ||\hat{f}_k(t)||_2^2 + ||\sqrt{\gamma} \partial_y \hat{f}_k(t)||_2^2 + 2 |k|^2 ||\sqrt{\gamma} u'(y) \hat{f}_k(t)||_2^2; \quad (3.2) \]

\[ \Phi[f(t)] = \sum_{k \neq 0} \Phi_k[f(t)] = ||f'(t)||_2^2 + ||\sqrt{\gamma} \partial_y f'(t)||_2^2 + 2 |\beta u'| \partial_y f'(t) \partial_y f(t) + ||\sqrt{\gamma} u' \partial_y |f(t)||_2^2. \quad (3.3) \]

Here \( \alpha, \beta, \) and \( \gamma \) are \( A, k \)-dependent constants

\[ \alpha(A, k) = \epsilon_\alpha A^{2/3} |k|^{-2/3} \quad (3.4a) \]

\[ \beta(A, k) = \epsilon_\beta A^{-1/3} |k|^{-4/3} \quad (3.4b) \]

\[ \gamma(A, k) = \epsilon_\gamma |k|^{-2} \quad (3.4c) \]

where \( \epsilon_\alpha, \epsilon_\beta, \) and \( \epsilon_\gamma \) are small constants depending only on \( u \). Since we are concerned with strictly monotone shear flows instead of nondegenerate shear flows, we employ slightly different multipliers \( \alpha, \beta, \gamma \) from the ones in the paper [8]. Notice that in [4] for treating general situations one must also take \( \alpha, \beta, \) and \( \gamma \) to be \( y \)-dependent, however, as suggested by [2], this is not necessary to treat strictly monotone shear flows with \( y \in \mathbb{R} \). The parameters \( \epsilon_\alpha, \epsilon_\beta, \) and \( \epsilon_\gamma \) are tuned such that,

\[ \Phi_k[f] \approx ||\hat{f}_k||_2^2 + ||\sqrt{\gamma} \partial_y \hat{f}_k||_2^2 + |k|^2 ||\sqrt{\gamma} u' \hat{f}_k||_2^2, \quad (3.5) \]

and hence

\footnote{The constants \( \epsilon_\alpha, \epsilon_\beta, \) and \( \epsilon_\gamma \) are chosen so that all the potentially positive terms in the time derivative of \( \Phi_k \) are absorbed by the negative terms in the \( \frac{d}{dt} \Phi_k \). Since the explicit form is too complicated, we refer the interested to the paper [4].}
\[ \Phi_k[f] \approx \left\| \hat{f}_k \right\|_2^2 + |k|^{-2/3} A^{-2/3} \left\| \partial_x \hat{f}_k \right\|_2^2. \quad (3.6) \]

As a result, \( \Phi_k[f(t)] \) is equivalent to the \( H^1 \) norm of \( \hat{f}_k \) but with constants that depend on \( A \) and \( k \). The primary step in the results of \cite{4} is that for \( u(y) \) satisfying the hypotheses in theorem 2, then for the passive scalar equation (3.1), the norm \( \Phi_k[f(t)] \) satisfies the following differential inequality for some small constant \( \tilde{\epsilon} \) independent of \( k, A \) (but depending on \( u \)):

\[ \frac{d}{dt} \Phi_k[f(t)] \leq -\tilde{\epsilon} \frac{|k|^{2/3}}{A^{1/3}} \Phi_k[f(t)]. \]

Note that the decay rate of the functional \( \Phi_k[f] \left( = \frac{|k|^{2/3}}{A^{1/3}} \right) \) is much larger than the classical heat decay rate \( \left( = \frac{1}{A} \right) \) for the passive scalar equation (3.1) when \( A \) is chosen big. This is the enhanced dissipation effect of the shear flow.

Recall the estimate of the time evolution of \( \Phi_k[f(t)] \) in [4].

**Proposition 2 ([4]).** Consider the solution to the passive scalar equation (3.1). For \( \tilde{\epsilon} \) sufficiently small depending only on \( u \), there holds,

\[ \frac{d}{dt} \Phi_k[f(t)] \leq -\tilde{\epsilon} \frac{|k|^{2/3}}{A^{1/3}} \Phi_k[f(t)]. \]

\[ \Rightarrow \left| \frac{d}{dt} \Phi_k[f(t)] \right| \leq -\frac{1}{4} \frac{|k|^{2/3}}{A^{1/3}} \Phi_k[f(t)]. \]

\[ =: \mathcal{N}_k[f]. \]

**Remark 7.** The notation ‘\( \mathcal{N} \)’ stands for ‘negative terms’.

**Remark 8.** In theorem 2.1 of the paper [4], it is proved that

\[ \frac{d}{dt} \Phi_k[f(t)] \leq -\tilde{\epsilon} \lambda_{1-k} \Phi_k[f(t)], \]

where \( \lambda_{1-k} = |k|^{2/3} A^{-1/3} \) for strictly monotone shear flows. By the equivalence relation (3.6), we obtain the first three negative terms in the time evolution estimate (3.7). The other negative terms are the remnant of the negative terms in the time derivative of \( \Phi_k[f] \). We refer the interested reader to the lemma 2.2 in the paper [4] for further calculation details.

The functional we construct to exploit the enhanced dissipation effect in the equation (1.1) is the following:

**Definition 1.** Define the functional \( \mathcal{F} \) as

\[ \mathcal{F}_k := \Phi_k[n \rho] + \Phi_k[\partial_t c \rho] + \Phi_k[\partial_t c | \rho] + A |k| \Phi_k[c | \rho]. \]

\[ \mathcal{F} := \sum_{k \neq 0} \mathcal{F}_k[n \rho] + \sum_{k \neq 0} \mathcal{F}_k[\partial_t c \rho] + \sum_{k \neq 0} \mathcal{F}_k[\partial_t c | \rho] + \sum_{k \neq 0} A |k| \Phi_k[c | \rho] = \sum_{k \neq 0} \mathcal{F}_k. \]

The goal in this subsection is to show that:

**Theorem 3.** Assume the hypothesis of proposition 1. There exists a constant \( \eta > 0 \) depending only on \( u \) such that the following time decay estimate holds if \( A \) is chosen large enough.
\[
\frac{d}{dt} F \leq - \frac{\eta}{A^{1/3}} F. \tag{3.11}
\]

The theorem implies the conclusion (2.8b).

**Proof of the conclusion (2.8b).** Combining theorem 3 and the equivalence (3.6) yields the conclusion (2.8b). First by solving the differential inequality (3.11), we have that
\[
F(t) \leq F(0)e^{-\frac{\eta t}{A^{1/3}}}. \tag{3.12}
\]

Thanks to the assumption on the initial chemical gradient
\[
||\nabla (c_m) \|_{H^1} \leq C_n A^{-q}, \quad q > 1/2,
\]
the initial value \(F(0)\) is bounded
\[
F(0) \leq C(\epsilon_{\alpha}, \epsilon_{\beta}, \epsilon_{\gamma}, u, C_n) \left( \left| \|n_m\|_{H^1}^2 + \|\nabla c_m\|_{H^1}^2 \right| (1 + A) \right) \leq C(\epsilon_{\alpha}, \epsilon_{\beta}, \epsilon_{\gamma}, u, C_n) \left( ||n_m||_{H^1}^2 + 1 \right).
\]

Here we can choose the \(C_{ED} \) in (2.7b) to be much larger than the constant appeared in the estimate and obtain
\[
F(0) \leq 2C_{ED}(\epsilon_{\alpha}, \epsilon_{\beta}, \epsilon_{\gamma}, u, C_n) \left( ||n_m||_{H^1}^2 + 1 \right). \tag{3.13}
\]

The equivalence relation (3.6) yields
\[
||n_{\neq}(t)||_{H^1}^2 + ||\nabla c_{\neq}(t)||_{H^1}^2 \leq F(t).
\]

Combining this with the estimates (3.12) and (3.13), we obtain (2.8b).

In order to show the idea behind the construction of the functional \( F \), we first list all the related equations here:
\[
\partial_t \hat{n}_k = \frac{1}{A} \partial_y \hat{n}_k - \frac{|k|^2}{A} \hat{n}_k - u(y) ik \hat{n}_k - L_k - NL_k; \tag{3.14}
\]
\[
\partial_t \partial_y \hat{c}_k = \frac{1}{A} \partial_y \partial_y \hat{c}_k - \frac{|k|^2}{A} \partial_y \hat{c}_k - u(y) ik \partial_y \hat{c}_k - u'(y) ik \hat{c}_k + \frac{1}{A} \partial_y \hat{c}_k - \frac{1}{A} \partial_y \hat{c}_k; \tag{3.15}
\]
\[
\partial_t \hat{c}_k = \frac{1}{A} \partial_y \hat{c}_k - \frac{|k|^2}{A} \hat{c}_k - u(y) ik \hat{c}_k + \frac{1}{A} \hat{n}_k - \frac{1}{A} \hat{c}_k, \tag{3.16}
\]
where \( L_k, NL_k \) are defined as in (2.6) and (2.5). Our primary goal is to obtain the \( L^2 \) enhanced dissipation estimate of \( n_{\neq} \). However, we are not able to close the estimate on \( d\Phi_k[n_{\neq}]/dt \) without further information about the chemical gradient \( \partial_y c_{\neq} \). Specifically speaking, the terms in \( L_k, NL_k \) involving \( \partial_y (\partial_y c_{\neq} \hat{n}_{\neq} \phi) \) cannot be absorbed by the negative terms in \( d\Phi_k[n_{\neq}]/dt \). Therefore, in the first step, we add \( \Phi_k[\nabla c_{\neq}] \) in the functional \( F \) to make use of the extra negative terms in \( d\Phi_k[\nabla c_{\neq}]/dt \). The drawback is that it introduces destabilizing effect of the strong shear flow into the functional since problematic terms involving \( -u'(y)ik \hat{c}_k \) are created. These terms will typically involve large powers of \( A \) and \( |k| \). In the second step, we add the term \( A|k|\Phi_k[c_{\neq}] \) in \( F \) to compensate for this destabilizing effect of shear flow. Finally, we show that the negative terms in \( d\Phi_k[n_{\neq}]/dt \) absorb all terms involving \( n_{\neq} \) in \( A|k|\Phi_k[c_{\neq}] \). By
completing this loop, we have shown that all the terms are absorbed by the negative terms in the time derivative of $\mathcal{F}$ and the exponential decay ($3.11$) follows.

**Proposition 3.** For $\epsilon$ sufficiently small depending only on $u$, there holds,

$$\frac{d}{dt} \Phi_k[n_p(t)] \leq \mathcal{N}_k[n_p] + \left\{ 2\text{Re}[-L_k, \tilde{n}_k] - 2\text{Re}(\alpha \partial_{\gamma} \tilde{\zeta}_k, -L_k) - 2k\text{Re}[(\beta u^* L_k, \partial_{\gamma} \tilde{n}_k) + (\beta u \tilde{n}_k, \partial_{\gamma} L_k)] + 2|k|^2 \text{Re}(\gamma(u')^2 \tilde{n}_k, -L_k) \right\}$$

$$+ \left\{ -2\text{Re}[-N_k, \tilde{n}_k] + 2\text{Re}(\alpha \partial_{\gamma} \tilde{n}_k, N_k L_k) - 2k\text{Re}[(\beta u^* NL_k, \partial_{\gamma} \tilde{n}_k) + (\beta u \tilde{n}_k, \partial_{\gamma} NL_k)] - 2|k|^2 \text{Re}(\gamma(u')^2 \tilde{n}_k, N_k L_k) \right\}$$

$$= \mathcal{N}_{nk} + \{L_k^1 + N_k^\Beta + N_k^\alpha \} + \{NL_k^1 + NL_k^\Beta + NL_k^\alpha \}. \tag{3.17}$$

Recall that $\mathcal{N}_k$ is defined in ($3.7$) and $L_k, NL_k$ are defined in ($2.5$) and ($2.6$). The time derivative of $\Phi_k[\partial_{\epsilon} c_k \phi], \Phi_k[\partial_{\epsilon} c_k \phi]$ are bounded,

$$\frac{d}{dt} \Phi_k[\partial_{\epsilon} c_k \phi](t) \leq \mathcal{N}_k[\partial_{\epsilon} c_k \phi] + \left\{ 2\text{Re} \left( \frac{\partial_{\epsilon} \tilde{n}_k}{A}, \partial_{\epsilon} \tilde{\zeta}_k \right) - 2\text{Re} \left( \alpha \partial_{\gamma} \tilde{\zeta}_k, \frac{\partial_{\epsilon} \tilde{n}_k}{A} \right) + 2k\text{Re} \left( i\beta u^* \tilde{n}_k, \partial_{\gamma} \tilde{\zeta}_k \right) + \left\{ -2\text{Re} \left( u^* \tilde{\zeta}_k, \partial_{\gamma} \tilde{n}_k \right) + 2\text{Re} \left( \alpha \partial_{\gamma} \tilde{\zeta}_k, u^* \tilde{n}_k \right) - 2k\text{Re} \left( \gamma(u')^2 \tilde{\zeta}_k, \partial_{\gamma} \tilde{n}_k \right) \right\}$$

$$= \mathcal{N}_{nk}^\partial + \{L_{n k}^1 + L_{n k}^\alpha + L_{n k}^\Beta \} + \{NL_{n k}^1 + NL_{n k}^\Beta + NL_{n k}^\alpha \}. \tag{3.18}$$

The time derivative of $\mathcal{A}[k|\Phi_k[\partial_{\epsilon} c_k \phi]]$ is bounded,

$$\frac{d}{dt} \mathcal{A}[k|\Phi_k[\partial_{\epsilon} c_k \phi](t)] \leq \mathcal{A}[k|\mathcal{N}_k[\partial_{\epsilon} c_k \phi]] + \mathcal{A}[k] \left\{ 2\text{Re} \left( \frac{\partial_{\epsilon} \tilde{n}_k}{A}, \tilde{\zeta}_k \right) - 2\text{Re} \left( \alpha \partial_{\gamma} \tilde{\zeta}_k, \frac{\partial_{\epsilon} \tilde{n}_k}{A} \right) + 2k\text{Re} \left( i\beta u^* \tilde{n}_k, \partial_{\gamma} \tilde{\zeta}_k \right) + \left\{ -2\text{Re} \left( u^* \tilde{\zeta}_k, \partial_{\gamma} \tilde{n}_k \right) + 2\text{Re} \left( \alpha \partial_{\gamma} \tilde{\zeta}_k, u^* \tilde{n}_k \right) - 2k\text{Re} \left( \gamma(u')^2 \tilde{\zeta}_k, \partial_{\gamma} \tilde{n}_k \right) \right\}$$

$$= \mathcal{A}[k|\mathcal{N}_k] + \mathcal{A}[k|\{T_{n k}^1 + T_{n k}^\alpha + T_{n k}^\Beta \} + \mathcal{A}[k|T_{n k}^\partial]. \tag{3.19}$$

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Proof. Applying the equations (3.7), (3.14)–(3.16) and integration by parts, the estimates follow.

**Sketch of the proof of theorem 3.** The main idea of the proof of theorem 3 is to estimate all the terms $L_A^{(1)}$, $NL_A^{(k)}$, $\partial_t T_{\partial_k}^{(1)}$, $\partial_t T_{\partial_k}^{(\beta)}$, and $\partial_t T_{\partial_k}^{(\beta)}$ and $A[k]T_{\partial_k}^{(1)}$ in (3.17)–(3.20), and show that the sum of all these terms are smaller than $\frac{\epsilon}{4} \left( \mathcal{N}_{\partial_k} + \mathcal{N}_{\partial_k} + A[k] \mathcal{N}_{\partial_k} \right)$. Once we show this estimate, we end up with

$$
\frac{d}{dt} \mathcal{F} \leq \frac{3}{4} \sum_{k \neq 0} \left( \mathcal{N}_{\partial_k} + \mathcal{N}_{\partial_k} + A[k] \mathcal{N}_{\partial_k} \right) \leq -\frac{3}{4} \frac{\epsilon}{A^{1/3}} \mathcal{F}.
$$

(3.21)

This is the same as (3.11).

The remaining part of this section is organized as follows: in section 3.2, we estimate all the terms in (3.20); in section 3.3, we estimate (3.18) and (3.19); in section 3.4, we estimate (3.17).

### 3.2. Time evolution estimates: $A[k] \frac{d}{dt} \Phi_k [c_x]$

In this subsection, we estimate terms in (3.20). First the $A[k]T_{c,1,k}^1$ term in (3.20) can be estimated using Hölder inequality and Young’s inequality:

$$
T_{c,1,k}^1 = 2 \text{Re} \left\{ \frac{\bar{\partial}_k}{A} \tilde{c}_k \right\} \leq \frac{B}{A^{2/3}} \frac{\partial_k}{|k|^{2/3}} \left| \tilde{c}_k \right|^2 + \frac{|k|^{2/3}}{A^{1/3} \epsilon} \left| \tilde{c}_k \right|^2.
$$

(3.22)

We show that $A[k]T_{c,1,k}^1$ is consistent with (3.11) given that $B$, then $A$, are chosen large. For the second term in (3.22), it can be absorbed by the negative term $A[k] \mathcal{N}_k [c_x]$ in (3.20) given $B$ chosen large enough. For the first term, we can use the negative term $-\frac{\epsilon}{4} \frac{|k|^{2/3}}{A^{1/3} \epsilon} \left| \tilde{c}_k \right|^2$ in (3.17) to absorb it given $A$ chosen large enough compared to $B$ and $1/\epsilon$, i.e.

$$
A[k] \frac{B}{A^{2/3}} \frac{\partial_k}{|k|^{2/3}} \left| \tilde{c}_k \right|^2 \leq \frac{B}{A^{2/3}} \frac{\partial_k}{|k|^{2/3}} \left| \tilde{c}_k \right|^2 \leq -\frac{7}{16} \frac{|k|^{2/3}}{A} \left| \tilde{c}_k \right|^2.
$$

The second term $A[k]T_{c,1,k}^0$ in (3.20) is estimated using Hölder inequality, Young’s inequality and the definition of $\alpha$ (3.4):

$$
T_{c,1,k}^0 \leq \frac{1}{AB} \left| \sqrt{\alpha} \partial_k \tilde{c}_k \right|^2 + \frac{B}{A^{2/3}} \frac{\partial_k}{|k|^{2/3}} \left| \tilde{c}_k \right|^2.
$$

which by (3.7), (3.17) and (3.20) is consistent with (3.11) given $A$ large. For the $A[k]T_{c,1,k}^\beta$ term in (3.20), we estimate it using the fact that $\|u''\|_\infty \leq C$, the definition of $\beta$ (3.4), Hölder inequality and Young’s inequality as follows

$$
T_{c,1,k}^\beta = 4 \text{Re} \left\{ i \beta u' \tilde{c}_k, \partial_k \tilde{c}_k \right\} = 2 \text{Re} \left\{ i \beta u'' \tilde{c}_k, \tilde{c}_k \right\}
$$

$$
\leq \frac{B}{A^{2/3}} \frac{|k|^{2/3}}{A} \left| \sqrt{\beta u'} \tilde{c}_k \right|^2 + \frac{|\partial_k \tilde{c}_k|^2}{4} + \frac{|\tilde{c}_k|^2}{4AB} + \frac{B}{A^{1/3}} \mathcal{F}.
$$

which by (3.7), (3.17) and (3.20) is consistent with (3.11) given $B$, then $A$ large. Similarly, the $A[k]T_{c,1,k}^\beta$ term in (3.20) can be estimated using Hölder inequality and Young’s inequality.
\[ T_{c,1,k}^\gamma = 2|k|^2 \text{Re} \left( \gamma (u')^2 \hat{c}_k, \hat{n}_\alpha \right) \leq \frac{|k|^{8/3}}{A^{1/3} B} \| \sqrt{\gamma} u' \hat{c}_k \|^2_2 + \frac{B |k|^{4/3}}{A^{1/3}} \| \sqrt{\gamma} u' \hat{n}_k \|^2_2, \]

which is consistent with (3.11) given that \( B \), then \( A \), are chosen large enough thanks to (3.7), (3.17) and (3.20). The \( A|k|T_{c,2,k}^\beta \) term in (3.20) can be estimated using Hölder inequality, Young’s inequality and the definition of \( \gamma \) as follows

\[ A|k|T_{c,2,k}^\beta \leq A|k| \left( \frac{8|k|^2 \| \sqrt{\gamma} u' \hat{c}_k \|^2_2}{A} + \| \partial_t \hat{c}_k \|^2_2 \right). \] (3.23)

which can be absorbed by \( A|k|N_k[c_{\neq}] \) in (3.20) given that \( A \) is chosen large enough. This completes the estimation of all the terms in (3.20).

3.3. Time evolution estimates: \( \hat{\Phi} |\nabla c_{\neq}| \)

In this subsection, we estimate the time evolution of \( \Phi |\nabla c_{\neq}| \) (3.18) and (3.19). We start by estimating the terms in \( \hat{\Phi} |\partial_t c_{\neq}| \) since they involve destabilizing effect of strong shear flow. First we estimate the term \( T_{\partial_{c,x}^0}^1 \) in (3.18) using the definition of \( \beta \) (3.4), Hölder inequality and Young’s inequality as follows:

\[ T_{\partial_{c,x}^0}^1 \lesssim \frac{|k|^{2/3} \| \partial_x \hat{c}_k \|^2_2}{B A^{1/3}} + B \left( A^{2/3} |k|^{2/3} \right) |k|^2 \| \sqrt{\beta} u' \hat{c}_k \|^2_2. \]

Now we see that the first term is absorbed by the negative terms in (3.18) given \( B \) chosen large enough, and the second term can be absorbed by the term \( -A|k|^3 \| \sqrt{\beta} u' \hat{c}_k \|^2_2 \) in (3.20) given \( A \) chosen large enough. Now we see that this term is consistent with (3.11). Next, combining the definition of \( \alpha, \beta \) (3.4), Hölder inequality and Young’s inequality, the \( \alpha \) term in \( T_{\partial_{c,x}^0}^n \) can be estimated as follows:

\[ T_{\partial_{c,x}^0}^n \lesssim \frac{1}{AB} \| \sqrt{\alpha} \partial_{yy} \hat{c}_k \|^2_2 + B \left( A^{2/3} |k|^{2/3} \right) |k|^2 \| \sqrt{\beta} u' \hat{c}_k \|^2_2, \]

which is consistent with (3.11) given that \( B \), then \( A \), are chosen large. For the first \( \beta \) term in \( T_{\partial_{c},2,k}^\gamma \), combining the definition of \( \beta \) (3.4), the fact that \( \| u' \|_{W^{1,\infty}} \leq C \), integration by parts, Hölder inequality and Young’s inequality yields

\[ 2 \text{Re} \{ i u' (-u'^2 \hat{c}_k, \partial_x \hat{c}_k) \} = -2|k|^2 \text{Re} \{ \beta u^2 \partial_x \hat{c}_k, \partial_x \hat{c}_k \} - 2|k|^2 \text{Re} \{ \beta u'^2 \partial_x \hat{c}_k, \partial_x \hat{c}_k \} \]

\[ \lesssim \frac{|k|^{2/3} \| \partial_x \hat{c}_k \|^2_2}{B A^{1/3}} + \frac{B |k|^2 \| \sqrt{\beta} u' \hat{c}_k \|^2_2}{A^{1/3}}, \]

which can be absorbed by the negative term \( A|k|N_k[c_{\neq}] \) in (3.20) given \( A \) large enough. By applying integration by parts, we see that the second \( \beta \) term in \( T_{\partial_{c,x}^2}^\gamma \) is equivalent to the first one up to the following term, which can be estimated using the definition of \( \beta \) (3.4), \( \| u'' \|_{\infty} \leq C \), Hölder inequality and Young’s inequality

\[ 2 \text{Re} \{ i \beta u'' \partial_x \hat{c}_k, u' \partial_k \hat{c}_k \} \leq \frac{|k|^{2/3} \| \partial_x \hat{c}_k \|^2_2}{A^{1/3} B} + \frac{B |k|^2 \| \sqrt{\beta} u' \hat{c}_k \|^2_2}{A^{1/3}}. \]

Since the first terms can be absorbed by \( N_k[\partial_x c_{\neq}] \) and the second term can be absorbed by \( A|k|N_k[c_{\neq}] \), this is consistent with (3.11) given that \( B \), then \( A \), are chosen large. The \( T_{\partial_{c,x}^0}^\gamma \) for \( \partial_x \hat{c}_k \) gives

\[ 2 \text{Re} \{ i \beta u'' \partial_x \hat{c}_k, u' \partial_k \hat{c}_k \} \leq \frac{|k|^{2/3} \| \partial_x \hat{c}_k \|^2_2}{A^{1/3} B} + \frac{B |k|^2 \| \sqrt{\beta} u' \hat{c}_k \|^2_2}{A^{1/3}}. \]

which completes the estimation of all the terms in (3.20).
term in (3.18) can be estimated using \( \|u'\|_\infty \leq C \), Hölder inequality and Young’s inequality as follows:

\[
T_{\delta,c,2,k}^{\gamma} \lesssim \frac{|k|^{8/3}}{A^{1/3}} B \left\| \sqrt{|u'|} \partial_y \partial_y \partial_c \partial_k \right\|_2 + B \left( A^{2/3} |k|^{2/3} \right) \frac{\left\| \sqrt{|u'|} \partial_y \partial_y \partial_c \partial_k \right\|_5}{A^{1/3}}.
\]

Now we see that the first term is absorbed by the negative term \( N_k[\partial_c \partial_y \partial_c \partial_y] \) in (3.18) if \( B \) is chosen large, and the second term is absorbed by \( A |k| \tilde{N}_k[c \partial_y] \) in (3.20) given that \( A \) is chosen large. This finishes the estimation of the terms \( T_{\delta,c,2,k}^{(1)} \) in (3.18).

For the terms of the form \( T_{\delta,c,1,k}^{(1)} \) in (3.18), we will use the negative terms in (3.17) and (3.18) to absorb them. For the \( T_{\delta,c,1,k}^{(2)} \) in (3.18), we have that by Hölder inequality and Young’s inequality,

\[
T_{\delta,c,1,k}^{(1)} \lesssim \frac{1}{A^{1/4}} \left\| \partial_y \partial_c \partial_k \right\|_2^2 + \frac{1}{A^{1/4}} \left\| \partial_y \partial_y \partial_c \partial_k \right\|_2^2.
\]

By choosing \( A \) large, these two terms can be absorbed by the negative terms in (3.17) and (3.18). Combining the definition of \( \alpha \) (3.4), Hölder inequality and Young’s inequality, the \( T_{\delta,c,1,k} \) term in (3.18) can be estimated as follows,

\[
T_{\delta,c,1,k}^{(2)} \lesssim \frac{1}{A^{1/3}} |k|^{1/3} \left\| \sqrt{|u'|} \partial_y \partial_y \partial_c \partial_k \right\|_2 + \frac{1}{A^{1/3}} \left\| \partial_y \partial_c \partial_k \right\|_2,
\]

which is consistent with (3.11) for \( A \) large enough. For the first \( \beta \) term in \( T_{\delta,c,1,k}^{(3)} \), we can estimate it using the definition of \( \beta \) (3.4), the fact that \( ||u'||_\infty \leq C \), Hölder inequality and Young’s inequality as follows

\[
2k \text{Re} \left( i \beta u' \frac{\partial_y \partial_c \partial_k}{A} \partial_y \partial_c \partial_k \right) \lesssim \frac{1}{A^{1/3}} \left\| \partial_y \partial_c \partial_k \right\|_2^2 + \frac{1}{A^{1/3}} \left\| \partial_y \partial_y \partial_c \partial_k \right\|_2^2.
\]

This term is consistent with (3.11) given \( A \) chosen large. The second term in \( T_{\delta,c,1,k}^{(3)} \) is the same as the first one through integration by part up to a controllable term, which can be estimated using the definition of \( \beta \) (3.4), the fact that \( ||u'||_\infty \leq C \), Hölder inequality and Young’s inequality as follows

\[
-2k \text{Re} \left( i \beta u'' \partial_y \partial_c \partial_k \frac{\partial_y \partial_c \partial_k}{A} \right) \lesssim \frac{1}{A^{1/3}} \left\| \partial_y \partial_c \partial_k \right\|_2^2 + \frac{1}{A^{1/3}} \left\| \partial_y \partial_y \partial_c \partial_k \right\|_2^2.
\]

As long as \( A \) is large enough, these two terms can be absorbed by the negative terms in (3.17) and (3.18). Finally, for the \( \gamma \) term \( T_{\delta,c,1,k}^{(4)} \), we estimate it using the definition of \( \gamma \) (3.4), \( ||u'||_{H^1} \leq C \), Hölder inequality and Young’s inequality as follows

\[
T_{\delta,c,1,k}^{(4)} \lesssim \frac{\left\| \partial_y \partial_c \partial_k \right\|_2}{A^{2/3}} + \frac{\left\| \partial_y \partial_y \partial_c \partial_k \right\|_2}{A^{4/3}}.
\]

This is consistent with (3.11) given that \( A \) is chosen large enough. The treatment of the term \( T_{\delta,c,3,k}^{(3)} \) in (3.18) is similar to the treatment of (3.23), so we omit the estimate for the sake of brevity. This concludes the estimate of the time evolution \( \frac{d}{dt} \Phi_k[c \partial_y \partial_c \partial_y] \).

The estimate of the time derivative \( \frac{d}{dt} \Phi_k[c \partial_y \partial_c \partial_y] \) is similar to the estimates of the terms \( T_{\delta,c,1,k}^{(1)} \) and \( T_{\delta,c,3,k}^{(3)} \) in (3.18), hence we omit it for the sake of brevity.
3.4. Time evolution estimates: $\frac{d}{dt}\Phi[n_{\rho}]$

3.4.1. Estimate on the $L$ terms in (3.17). These terms are linear in the $k$th mode, and it accordingly makes sense to estimate these terms $k$-by-$k$. In this subsection we prove that for $A$ sufficiently large, the $L_{k}^{i}$ terms can be absorbed by the negative terms in the $\frac{d}{dt}\mathcal{F}$, i.e.

$$ L_{k}^{1} + L_{k}^{\alpha} + L_{k}^{\beta} + L_{k}^{\gamma} \leq -\frac{1}{4}N_{k}[n_{\rho}] - \frac{1}{4}N_{k}[\nabla c_{\rho}]. $$

(3.24)

We start by estimating the $L_{k}^{1}$ term in (3.17). We decompose it into two parts:

$$ L_{k}^{1} = 2\text{Re} \langle -L_{k}, \hat{n}_{k} \rangle = \frac{2}{A} \text{Re} \langle -\nabla \cdot (\nabla \hat{c}_{0} n_{0}) - \partial_{\gamma}(\partial_{\gamma} c_{0} \hat{n}_{k}), \hat{n}_{k} \rangle =: L_{k,1}^{1} + L_{k,2}^{1}. $$

(3.25)

The term $L_{k,1}^{1}$ can be estimated as follows:

$$ L_{k,1}^{1} \leq \frac{1}{AB} ||\nabla \hat{n}_{k}||_{2}^{2} + \frac{B}{A} ||\nabla \hat{c}_{k}||_{2}^{2} ||n_{0}||_{\infty}. $$

Thanks to the hypothesis (2.7d), it is consistent with (3.24) if we choose $B$ then $A$ large enough.

The term $L_{k,2}^{1}$ is estimated using Hölder inequality, Young’s inequality, Gagliardo–Nirenberg–Sobolev inequality and the chemical gradient $\partial_{\gamma} c_{0} L^{4}$ estimate (2.10) as follows

$$ L_{k,2}^{1} \leq \frac{1}{AB} ||\partial_{\gamma} \hat{n}_{k}||_{2}^{2} + \frac{B}{A} ||\hat{n}_{k}||_{2}^{2} ||\partial_{\gamma} c_{0}||_{4}^{8/3} \leq \frac{1}{AB} ||\partial_{\gamma} \hat{n}_{k}||_{2}^{2} + \frac{B}{A} ||\hat{n}_{k}||_{2}^{2} M^{8/3}, $$

which is consistent with (3.24) if $B$, then $A$, are chosen large.

Next, we decompose the $L_{k}^{\alpha}$ term into two parts:

$$ L_{k}^{\alpha} = -2\text{Re} \langle \alpha \partial_{\gamma} \hat{n}_{k}, -L_{k} \rangle = \frac{2}{A} \text{Re} \langle \alpha \partial_{\gamma} \hat{n}_{k}, \nabla \cdot (\nabla \hat{c}_{0} n_{0}) + \partial_{\gamma}(\partial_{\gamma} c_{0} \hat{n}_{k}) \rangle =: L_{k,1}^{\alpha} + L_{k,2}^{\alpha}. $$

(3.26)

Combining the definition of $\alpha$ (3.4), the hypothesis (2.7c) and (2.7d), Hölder inequality, Gagliardo–Nirenberg–Sobolev inequality and Young’s inequality, the $L_{k,1}^{\alpha}$ term can be estimated as follows:

$$ L_{k,1}^{\alpha} = \frac{2}{A} \text{Re} \langle \alpha \partial_{\gamma} \hat{n}_{k}, (\partial_{\gamma} - k^{2}) \hat{c}_{k} n_{0} \rangle + \frac{2}{A} \text{Re} \langle \alpha \partial_{\gamma} \hat{n}_{k}, (\partial_{\gamma} c_{0} \hat{n}_{k}) \partial_{\gamma} \hat{c}_{k} - \partial_{\gamma} n_{0} \rangle $$

$$ \leq \frac{1}{AB} ||\alpha \partial_{\gamma} \hat{n}_{k}||_{2}^{2} + \frac{B}{A^{5/3}} ||\partial_{\gamma} \hat{c}_{k}||_{2}^{2} ||n_{0}||_{\infty}^{2} + \frac{B_{k}^{4}}{A^{5/3}} ||\hat{c}_{k}||_{2}^{2} ||n_{0}||_{\infty}^{2} $$

$$ + \frac{1}{BA} ||\alpha \partial_{\gamma} \hat{n}_{k}||_{2}^{2} + \frac{B_{k}^{4}}{A} ||\partial_{\gamma} \hat{c}_{k}||_{2}^{2} ||\partial_{\gamma} n_{0}||_{2}^{2} $$

$$ \leq \frac{1}{AB} ||\alpha \partial_{\gamma} \hat{n}_{k}||_{2}^{2} + \frac{B}{A^{5/3}} ||\partial_{\gamma} \hat{c}_{k}||_{2}^{2} C_{n,\infty}^{2} + \frac{B_{k}^{4}}{A^{5/3}} ||\hat{c}_{k}||_{2}^{2} C_{n,\infty}^{2} $$

$$ + \frac{||\partial_{\gamma} \hat{c}_{k}||_{2}^{2}}{A^{5/3}} + \frac{B_{k}^{4}}{A^{5/3}} ||\partial_{\gamma} \hat{c}_{k}||_{2}^{2} C_{n,\infty}^{4}, $$

which is consistent with (3.11) if we choose $B$ then $A$ to be large enough. For the $L_{k,2}^{\alpha}$ term in (3.26), we can estimate them using the definition of $\alpha$ (3.4), hypothesis (2.7c), lemma 2.10, Gagliardo–Nirenberg–Sobolev inequality, Hölder inequality and Young’s inequality as follows:
\( L_{k,3}^\beta = 2 \sqrt{\frac{\alpha}{A^{3/2}}} (\sqrt{\kappa} \partial_\gamma \tilde{\eta}, \Delta_\gamma \tilde{\eta} + \partial_\gamma c_0 \tilde{\eta}) \)

\[
\leq \frac{1}{AB} \left\| \sqrt{\kappa} \partial_\gamma \tilde{\eta} \right\|^2 + \frac{B}{A^{3/2}} \left\| \Delta_\gamma c_0 \right\| \left\| \tilde{\eta} \right\|^2 + \frac{B}{A^{3/2}} \left\| \partial_\gamma c_0 \right\| \left\| \tilde{\eta} \right\|^2 + \frac{B}{A^{3/2}} \sup_{0 < s < t} \left( n_0(s) \right) \left\| \tilde{\eta} \right\|^2
\]

\[
\leq \frac{1}{AB} \left\| \sqrt{\kappa} \partial_\gamma \tilde{\eta} \right\|^2 + \frac{B}{A^{3/2}} \left\| \Delta_\gamma c_0 \right\| \left\| \tilde{\eta} \right\|^2 + \frac{B}{A^{3/2}} \left\| \partial_\gamma c_0 \right\| \left\| \tilde{\eta} \right\|^2
\]

\[
+ \frac{BC_{2,\infty}^2}{A^{3/2}} \left\| \partial_\gamma \tilde{\eta} \right\|^2
\]

\[
\leq \frac{1}{AB} \left\| \sqrt{\kappa} \partial_\gamma \tilde{\eta} \right\|^2 + \frac{B}{A^{3/2}} \left( C_{2,\infty}^2 + C_{m,\infty}^2 \right) \left\| \tilde{\eta} \right\|^2 + \frac{B}{A^{3/2}} \left( C_{2,\infty}^2 + C_{m,\infty}^2 \right) \left\| \partial_\gamma \tilde{\eta} \right\|^2,
\]

which is consistent with (3.11) if we choose \( B \) then \( A \) large.

For the \( L_{k,4}^\beta \) terms in (3.17), we decompose it into four parts

\[
L_{k,4}^\beta = -\frac{2\kappa}{A} \text{Re} \left[ i \beta u^\prime (\nabla \cdot (\nabla \tilde{c}_0 n_0) + \partial_\gamma (\partial_\gamma c_0 \tilde{\eta})), \partial_\gamma \tilde{\eta} \right] + \left( i \beta u^\prime \tilde{\eta}, \partial_\gamma (\nabla \tilde{c}_0 n_0) + \partial_\gamma (\partial_\gamma c_0 \tilde{\eta}) \right]
\]

\[
= L_{k,4,1}^\beta + L_{k,4,2}^\beta + L_{k,4,3}^\beta + L_{k,4,4}^\beta.
\]

The term \( L_{k,4,1}^\beta \) can be estimated using the definition of \( \alpha, \beta \) (3.4), hypothesis (2.7d), the fact that \( \|u\|_{W_{1,\infty}} \leq C \), integration by parts, Hölder inequality and Young’s inequality as follows:

\[
L_{k,4,1}^\beta \leq \frac{2\kappa}{A} \text{Re} \left[ i \beta u^\prime \nabla \tilde{c}_0 n_0, \partial_\gamma \tilde{\eta} \right] + \frac{2\kappa}{A} \text{Re} \left[ i \beta u^\prime \nabla c_0 n_0, \partial_\gamma \tilde{\eta} \right]
\]

\[
\leq \frac{\left\| \partial_\gamma \tilde{\eta} \right\|^2}{A^{3/2}} + \frac{1}{A^{3/2}} \left\| \partial_\gamma \tilde{\eta} \right\|^2 \left\| n_0 \right\|^2 + \frac{1}{AB} \left\| \sqrt{\kappa} \partial_\gamma \tilde{\eta} \right\|^2 + \frac{B}{A} \left\| \partial_\gamma \tilde{\eta} \right\|^2 \left\| n_0 \right\|^2
\]

\[
+ \frac{1}{A^{3/2}} \left\| \partial_\gamma \tilde{\eta} \right\|^2 + \frac{B}{A^{3/2}} \left\| \partial_\gamma \tilde{\eta} \right\|^2 \left\| n_0 \right\|^2
\]

\[
\leq \frac{\left\| \partial_\gamma \tilde{\eta} \right\|^2}{A^{3/2}} + \frac{1}{A^{3/2}} \left\| \partial_\gamma \tilde{\eta} \right\|^2 C_{n,\infty}^2 + \frac{1}{AB} \left\| \sqrt{\kappa} \partial_\gamma \tilde{\eta} \right\|^2 + \frac{B}{A} \left\| \partial_\gamma \tilde{\eta} \right\|^2 C_{n,\infty}^2
\]

\[
+ \frac{1}{A^{3/2}} \left\| \partial_\gamma \tilde{\eta} \right\|^2 + \frac{B}{A^{3/2}} \left\| \partial_\gamma \tilde{\eta} \right\|^2 C_{n,\infty}^2,
\]

which is consistent with (3.24) if we choose \( B \) then \( A \) large. The term \( L_{k,4,3}^\beta \) is the same as the \( L_{k,4,1} \) term up to the following controllable term, which can be estimated using hypothesis (2.7d), the fact that \( \|u''\|_{\infty} \leq C \), Hölder inequality and Young’s inequality as follows

\[
\frac{2\kappa}{A} \text{Re} \left[ i \beta u'' \tilde{c}_0 \nabla \cdot (\nabla \tilde{c}_0 n_0) \right] = -\frac{2\kappa}{A} \text{Re} \left[ i \beta u'' \tilde{c}_0 \nabla \tilde{c}_0 n_0 \right] - \frac{2\kappa}{A} \text{Re} \left[ i \beta u'' \nabla \tilde{c}_0 \nabla \tilde{c}_0 n_0 \right]
\]

\[
\leq \frac{1}{A^{3/2}} \left\| \tilde{\eta} \right\|^2 + \frac{1}{A^{3/2}} \left\| \partial_\gamma \tilde{\eta} \right\|^2 \left\| n_0 \right\|^2 + \frac{1}{A^{3/2}} \left\| \nabla \tilde{\eta} \right\|^2 + \frac{1}{A^{3/2}} \left\| \nabla \tilde{c}_0 \right\| \left\| n_0 \right\|^2
\]

\[
\leq \frac{1}{A^{3/2}} \left\| \tilde{\eta} \right\|^2 + \frac{1}{A^{3/2}} \left\| \partial_\gamma \tilde{\eta} \right\|^2 C_{n,\infty}^2 + \frac{1}{A^{3/2}} \left\| \nabla \tilde{\eta} \right\|^2 + \frac{1}{A^{3/2}} \left\| \nabla \tilde{c}_0 \right\| C_{n,\infty}^2,
\]

which is consistent with (3.11) if we choose \( B \) then \( A \) large. For the \( L_{k,4,2}^\beta \) term in (3.27), it can be estimated using integration by parts, hypothesis (2.7c), the chemical gradient \( \partial_\gamma c_0 L^4 \) estimate (2.10), the fact that \( \|u''\|_{\infty} \leq C \), definition of \( \alpha, \beta \) (3.4), Hölder inequality and Young’s inequality as follows:

\[ \text{3666} \]
\[ L_{k,3}^\beta = \frac{2k}{A} \text{Re}(i\beta u'\partial_x c_0 \hat{n}_k, \partial_x \hat{n}_k) + \frac{2k^2}{A} \text{Re}(i\beta u'\partial_x c_0 \hat{n}_k, \partial_x \partial_x \hat{n}_k) \]
\[ \leq \frac{1}{A^{1/3} B} \| \partial_x \hat{n}_k \|_2^2 + \frac{B}{A^{1/3}} \| \partial_x c_0 \|_2 \| \hat{n}_k \|_2^2 \| \partial_x \hat{n}_k \|_2^2 / 2 \]
\[ + \frac{1}{AB} \| \sqrt{\alpha} \partial_x \hat{n}_k \|_2^2 + \frac{Bk^{4/3}}{A^{2/3}} \| \sqrt{\beta} u' \hat{n}_k \|_2^2 \| \partial_x c_0 \|_2^2 \]
\[ \leq \frac{1}{A^{1/3} B} \| \partial_x \hat{n}_k \|_2^2 + \frac{B}{A^{1/3}} \| \partial_x c_0 \|_2^2 + \frac{Bk^{4/3}}{A^{2/3}} \| \sqrt{\beta} u' \hat{n}_k \|_2^2 \| C_{A,B} \|, \]

which is consistent with (3.11) if we choose \( B \) then \( A \) large. The \( L_{k,3}^\beta \) term is similar to the \( L_{k,1}^\gamma \) up to the following controllable term, which can be estimated using hypothesis (2.7c), the chemical gradient \( \alpha, \beta \) estimate (2.10), Hölder inequality, the boundedness of \( \| u' \|_{w^{1,\infty}} \) and the definition of \( \alpha, \beta (3.4) \) as follows

\[ 2k \text{Re}(i\beta u' \partial_x c_0 \hat{n}_k, \partial_x \partial_x \hat{n}_k) = -\frac{2k}{A} \text{Re}(i\beta u' \partial_x c_0 \hat{n}_k, \partial_x \partial_x \hat{n}_k) \]
\[ \leq \frac{1}{A^{1/3}} \| \partial_x \hat{n}_k \|_2 \| \partial_x c_0 \|_\infty + \frac{1}{A^{1/3}} \| \partial_x \hat{n}_k \|_2^2 + \frac{1}{AB} \| \partial_x c_0 \|_4^2 \| \hat{n}_k \|_2^2 \]
\[ \leq \frac{1}{A^{1/3}} \| \partial_x \hat{n}_k \|_2 \| \partial_x c_0 \|_\infty + \frac{1}{A^{1/3}} \| \partial_x \hat{n}_k \|_2^2 + \frac{1}{AB} \| \partial_x \hat{n}_k \|_2^2 \| C_{A,B} \|, \]

It is consistent with (3.11) if we choose \( B \) then \( A \) large.

Finally, we decompose the \( L_{k,3}^\gamma \) term in (3.17) as follows:

\[ L_{k,3}^\gamma = -\frac{2k^2}{A} \text{Re}(\gamma u' \partial_x \hat{n}_k, (\nabla \cdot \nabla c_0) + \partial_x (\partial_x c_0 \partial_x \hat{n}_k)) =: L_{k,3}^{\gamma 1} + L_{k,3}^{\gamma 2}. \]  

The term \( L_{k,3}^{\gamma 1} \) can be estimated using integration by parts, hypothesis (2.7c), the fact that \( \| u' \|_{w^{1,\infty}} \leq C \), definition of \( \gamma (3.4) \), Hölder inequality and Young’s inequality as follows:

\[ L_{k,3}^{\gamma 1} \leq \frac{2k^2}{A} \text{Re}(\gamma u' u'' \partial_x \hat{n}_k + \gamma u' \partial_x \partial_x \hat{n}_k, \partial_x \partial_x \hat{n}_k) \]
\[ + \frac{2k^2}{A} \| \partial_x \hat{n}_k \|_2 \| \partial_x \partial_x \hat{n}_k \|_2 \| c_0 \|_\infty \]
\[ \leq \frac{2}{A} \| \partial_x \hat{n}_k \|_2 \| \partial_x \partial_x \hat{n}_k \|_2 \| c_0 \|_\infty + \frac{1}{AB} \| \partial_x \hat{n}_k \|_2 \| \partial_x \partial_x \hat{n}_k \|_2 \| c_0 \|_\infty \]
\[ + \frac{2}{A} \| \partial_x \hat{n}_k \|_2 \| \partial_x \partial_x \hat{n}_k \|_2 \| c_0 \|_\infty + \frac{B}{A} \| \partial_x \partial_x \hat{n}_k \|_2 \| c_0 \|_\infty \]

which is consistent with (3.11) if we choose \( B \) then \( A \) large. The term \( L_{k,3}^{\gamma 2} \) in (3.28) can be estimated using integration by parts, hypothesis (2.7c), the fact that \( \| u' \|_{w^{1,\infty}} \leq C \), definition of \( \gamma (3.4) \), Hölder inequality and Young’s inequality as follows:

\[ L_{k,3}^{\gamma 2} = \frac{2k^2}{A} \text{Re}(\gamma u' u'' \partial_x \hat{n}_k + \gamma u' \partial_x \partial_x \hat{n}_k, \partial_x \partial_x \hat{n}_k) \]
\[ \leq \frac{2}{A} \| \partial_x \hat{n}_k \|_2 \| \partial_x \partial_x \hat{n}_k \|_2 \| c_0 \|_\infty + \frac{1}{AB} \| \partial_x \partial_x \hat{n}_k \|_2 \| \partial_x \partial_x \hat{n}_k \|_2 \| c_0 \|_\infty \]
\[ \leq \frac{2}{A} \| \partial_x \hat{n}_k \|_2 \| \partial_x \partial_x \hat{n}_k \|_2 \| c_0 \|_\infty + \frac{B}{A} \| \partial_x \partial_x \hat{n}_k \|_2 \| c_0 \|_\infty \]

which is consistent with (3.11) if we choose \( B \) then \( A \) large.

\[ 3.4.2. \text{Estimate on NL terms.} \] As these terms are nonlinear in non-zero frequencies, it is more natural to consider all of the frequencies at once. For the \( N_{k}^\beta \) term in (3.17), we estimate it as follows:

\[ - \sum_{k \neq 0} 2 \text{Re}(N_{k}, \hat{n}_k) = -2 \left( \frac{1}{A} \nabla \cdot (n_\rho \nabla c_\rho), n_\rho \right) + \frac{2}{A} \langle n_\rho \nabla c_\rho, \nabla n_\rho \rangle \leq \frac{2}{A} \| \nabla c_\rho \|_\infty \| \nabla n_\rho \|_2 \| n_\rho \|_2. \]
By hypothesis (2.7e), for some constant $B > 0$,

$$- \sum_{k \neq 0} 2 \text{Re} \langle NL_k \hat{n}_k \rangle \lesssim \frac{1}{AB} \| \nabla n_{\phi} \|_2^2 + \frac{B}{A} C^2 \| n_{\phi} \|_2^2.$$ 

By first choosing $B$ large relative to the implicit constant, and then choosing $A$ large (relative to constants and $B$), these terms are absorbed by the negative terms in (3.17).

For the $NL_{k}^\alpha$ term in (3.17), we use the bootstrap hypotheses to deduce (using the definition of $\alpha$; recall that $\alpha$ is a Fourier multiplier in $x$),

$$2 \text{Re} \sum_{k \neq 0} \langle \alpha(k) \partial_x \hat{n}_k, NL_4 \rangle = \frac{2}{A} \langle \alpha(k) \partial_x n_{\phi}, \nabla \cdot (n_{\phi} \nabla c_{\phi}) \rangle$$

$$\lesssim \frac{1}{A^{\beta_k}} \| \sqrt{\alpha} \partial_x n_{\phi} \|_2 \| \nabla n_{\phi} \|_2 \| \nabla c_{\phi} \|_\infty + \| n_{\phi} \|_\infty \| \nabla^2 c_{\phi} \|_2$$

$$\lesssim \frac{1}{A^{\beta_k}} \| \sqrt{\alpha} \partial_x n_{\phi} \|_2^2 + \frac{C^2}{A^\beta} \| \nabla n_{\phi} \|_2 + \frac{C^2}{A^\beta} \| \nabla^2 c_{\phi} \|_2^2,$$

and choosing $A$ large, these terms are absorbed by the negative terms in (3.17), (3.19) and (3.18).

There are two terms in $NL_{k}^\beta$ in (3.17); we estimate the first by using that $\beta(k) \lesssim A^{-1/3} |k|^{-4/3}$ and defines a self-adjoint operator, chemical gradient $\partial_x c_{\phi}$ $L^p$ estimate (2.10), the fact that $|u''|_{\infty} \leq C$ and that $u$ does not depend on $\alpha$:

$$-2k \sum_{k \neq 0} \text{Re} \langle \beta(k) u' NL_4, \partial_x \hat{n}_k \rangle = \frac{2}{A} \langle \beta(k) u' \partial_x \nabla \cdot (n_{\phi} \nabla c_{\phi}), \partial_x n_{\phi} \rangle$$

$$\lesssim \frac{1}{A^{\beta_k}} \| \partial_x n_{\phi} \|_2 \| \nabla n_{\phi} \|_\infty \| \nabla^2 c_{\phi} \|_2$$

$$\lesssim \frac{1}{A^4} \| \partial_x n_{\phi} \|_2^2 + \frac{B}{A^{\beta_k}} \| \nabla n_{\phi} \|_2 \| \nabla c_{\phi} \|_\infty \| \nabla^2 c_{\phi} \|_2.$$

Recalling the bootstrap hypothesis, these terms are absorbed by the negative terms in (3.17)–(3.19) given $B$ then $A$ large enough. For the second term in $NL_{k}^\beta$ we use integration by parts to decompose it as follows

$$2 \sum_{k \neq 0} \langle \beta(k) u' k \hat{n}_k, \partial_x NL_4 \rangle = \frac{2}{A} \langle \beta(k) u' \partial_k n_{\phi}, \partial_x (n_{\phi} \nabla c_{\phi}) \rangle$$

$$= \frac{2}{A} \langle \beta(k) u' \partial_k n_{\phi}, \nabla \cdot (n_{\phi} \nabla c_{\phi}) \rangle$$

$$= NL_{k,1}^\beta + NL_{k,2}^\beta.$$

Using the definition of $\beta(k)$, the fact that $|u''|_{\infty} \leq C$ and that $u$ does not depend on $x$, we have,

$$\left| NL_{k,1}^\beta \right| \lesssim \frac{1}{A} \| n_{\phi} \|_2^2 + \frac{1}{A^{\beta_k}} \| \nabla n_{\phi} \|_2 \| \nabla c_{\phi} \|_\infty^2 + \frac{1}{A^{\beta_k}} \| n_{\phi} \|_\infty \| \nabla^2 c_{\phi} \|_2^2,$$

yielding terms which are absorbed by the negative terms in (3.17), (3.19) and (3.18) for $A$ sufficiently large. The treatment of $NL_{k,2}^\beta$ is similar to (3.29), hence it is omitted for the sake of brevity.

Turn finally to term $NL_{k}^\gamma$ in (3.17) associated with $\gamma$: 3668
\[ NL_{k,1}^\gamma = -\frac{2}{A} \langle \gamma(\partial_t)u' \partial_t n_{\gamma}, u' \partial_t \nabla \cdot (n_{\gamma} \nabla c_{\gamma}) \rangle \\
= \frac{2}{A} \langle \gamma(\partial_t)u' \partial_t \nabla n_{\gamma}, u' \partial_t (n_{\gamma} \nabla c_{\gamma}) \rangle + \frac{4}{A} \langle \gamma(\partial_t)u'' \partial_t n_{\gamma}, \partial_t (n_{\gamma} \partial_t c_{\gamma}) \rangle \\
=: NL_{k,1}^\gamma + NL_{k,2}^\gamma. \quad (3.30) \]

Then we use \( \gamma(\partial_t) = c_\gamma |\partial_t|^{-2} \) and interpolation to deduce the following bound for \( NL_{k,1}^\gamma \):

\[ NL_{k,1}^\gamma \lesssim \frac{1}{A} \left\| \sqrt{\gamma} u' \partial_t \nabla n_{\gamma} \right\|_2 \left\| \sqrt{\gamma} \partial_t (u' n_{\gamma} \nabla c_{\gamma}) \right\|_2 \]
\[ \lesssim \frac{1}{A} \left\| \sqrt{\gamma} u' \partial_t \nabla n_{\gamma} \right\|_2 \left\| u' n_{\gamma} \nabla c_{\gamma} \right\|_2 \]
\[ \lesssim \frac{1}{AB} \left\| \sqrt{\gamma} u' \partial_t \nabla n_{\gamma} \right\|_2^2 + \frac{B}{A} \left\| n_{\gamma} \right\|_\infty^2 \left\| \nabla c_{\gamma} \right\|_2^2. \]

For \( B \), then \( A \), chosen large, we may absorb these contributions in the negative terms in (3.17), (3.19) and (3.18). Next we estimate the \( NL_{k,2}^\gamma \) term in (3.30) using the definition of \( \gamma \) (3.4), the fact that \( ||u''||_\infty \leq C \) and the hypothesis (2.7e) as follows

\[ NL_{k,2}^\gamma \lesssim \frac{1}{A} \left\| \sqrt{\gamma} u' |\partial_t n_{\gamma}| \right\|_2 \left\| n_{\gamma} \partial_t c_{\gamma} \right\|_2 \lesssim \frac{1}{A} \left\| \sqrt{\gamma} u' |\partial_t n_{\gamma}| \right\|_2^2 + \frac{C_{2,\infty}^2}{A} \left\| n_{\gamma} \right\|_2^2. \]

Hence, for \( A \) chosen large, we may absorb these contributions in the negative terms in (3.17). This finishes the estimate of the \( NL \) terms.

4. Nonzero mode \( L_1^2 H_1^\gamma \) estimate (2.8a)

The nonzero mode \( L_1^2 H_1^\gamma \) estimate (2.8a) comes from an estimate on the \( \frac{d}{dt} ||n_{\gamma}||_2^2 \) and the knowledge that \( ||n_{\gamma}||_2^2 \) is bounded by \( 4C_{ED} (||n_{\gamma}||_H^2 + 1) \) from Hypothesis (2.7b). Indeed, from the nonzero mode equation (2.3a) and lemma 2.1, there holds for some universal constant \( B \),

\[ \frac{1}{2} \left\{ ||n_{\gamma}||_2^2 \right\} + \frac{1}{A} \Delta n_{\gamma} - \frac{1}{A} \partial_t (\partial_t c_0 \cdot n_{\gamma}) - \frac{1}{A} \nabla \cdot (\nabla c_{\gamma} n_{\gamma}) \right\}_0^T, \]
\[ \lesssim \frac{1}{2A} \left\{ ||\nabla n_{\gamma}||_2^2 \right\} + \frac{4}{A} \left\{ ||\partial_t c_0||_H^2 ||n_{\gamma}||_2^2 \right\} + \frac{4}{A} \left\{ ||\nabla c_{\gamma}||_2^2 ||n_{\gamma}||_\infty^2 \right\} \]
\[ \lesssim \frac{1}{2A} \left\{ ||\nabla n_{\gamma}||_2^2 \right\} + \frac{BM^2}{A} \left\{ ||n_{\gamma}||_2^2 \right\} + \frac{BC_{2,\infty}^2}{A} \left\{ ||\nabla c_{\gamma}||_2^2 \right\}. \quad (4.1) \]

Here \( M \) is the total mass \( M := ||n_{\gamma}||_{L^1(\mathbb{T} \times \mathbb{R})} \). Recalling hypothesis (2.7b),

\[ \frac{1}{A} \int_0^T \left( ||n_{\gamma}||_2^2 + ||\nabla c_{\gamma}||_2^2 \right) \, dt \leq \frac{1}{A} \int_0^T C_{ED} (||n_{\gamma}||_2^2 + 1) e^{-rt/4} \, dt \leq \frac{C}{A \gamma^3}, \quad (4.2) \]

which implies the following by inequality (4.1) given \( A \) large,

\[ \frac{1}{A} \int_0^T ||\nabla n_{\gamma}||_2^2 \, dt \leq \frac{1}{A \gamma^3} + 2 ||n_{\gamma}||_2^2 \leq 4 ||n_{\gamma}||_2^2. \quad (4.3) \]

As a result, we have proved (2.8a).
5. Zero mode estimate (2.8c)

Before estimating the $L^2$ norm of the solution, we note that by non-negativity and the divergence structure of the equation (1.1), the $L^1$ norm of $n_0(y)$ is constant in time

$$\|n_0\|_{L^1(\mathbb{R})} = \frac{1}{2\pi} \int |n_0(x)| dx = \frac{M}{2\pi}. \quad \text{(5.5)}$$

We first estimate $\|n_0\|_2^2$, then estimate $\|\partial_0 n_0\|_2^2$. From equation (2.2a) we have, by Minkowski’s inequality and lemma 2.1,

$$\frac{1}{2} \frac{d}{dt} \|n_0\|_2^2 = \left\langle n_0, \frac{1}{A} \partial_4 n_0 - \frac{1}{A} \partial_4 (\partial_4 c_0 n_0) - \frac{1}{A} (\nabla \cdot (\nabla c_0 n_0))_0 \right\rangle$$

$$= - \frac{1}{2A} \|\partial_0 n_0\|_2^2 + \frac{1}{A} \langle \partial_0 n_0, \partial_4 c_0 n_0 \rangle + \frac{1}{A} (\partial_0 c_0, (\partial_4 c_0 n_0)_0)$$

$$\leq - \frac{1}{2A} \|\partial_0 n_0\|_2^2 + \frac{1}{A} \|\partial_0 c_0\|_2^2 \|n_0\|_2^2 + \frac{1}{A} \|\partial_4 c_0 n_0\|_2^2$$

$$\leq - \frac{1}{4A} \|\partial_0 n_0\|_2^2 + \frac{M_{0/3}}{A} \|n_0\|_2^2 + \frac{1}{A} \|\partial_4 c_0 n_0\|_2^2 \|n_0\|_2^2 \|L^\infty(\mathbb{R})\|.$$

Recall the following Nash inequality on $\mathbb{R}$:

$$\|\rho\|_{L^2(\mathbb{R})} \leq \|\rho\|_{L^2(\mathbb{R})} \|\partial_0 \rho\|_{L^{2/3}(\mathbb{R})}. \quad \text{(5.1)}$$

Hence, by setting $\rho = n_0$, we have

$$-\|\partial_0 n_0\|_2^2 \leq - \frac{\|n_0\|_2^2}{C} \frac{1}{\|n_1\|_2^2} \leq - \frac{\|n_0\|_2^2}{CM^2}$$

Combining this with the time evolution of $\|n_0\|_2^2$, we obtain

$$\frac{d}{dt} \|n_0\|_2^2 \lesssim - \frac{\|n_0\|_2^2}{A} \left( \frac{\|n_0\|_2^2}{C} - M_{0/3} \right) + \frac{1}{A} \|\partial_4 c_0\|_2^2 \|n_0\|_2^2 \|L^\infty(\mathbb{R})\|.$$

Define the following quantity $G$ to be

$$G(t) := \int_0^t \frac{B}{A} \|\partial_4 c_0\|_2^2 \|n_0\|_\infty^2 d\tau, \quad \forall t \geq 0. \quad \text{(5.3)}$$

By the bootstrap hypotheses,

$$G(t) \leq \int_0^t \frac{B}{A} C_{ED} (\|n_m\|_m^2 + 1) e^{-\eta T/3} C_{n,\infty} d\tau \leq \frac{C}{A^{2/3}}. \quad \text{(5.4)}$$

hence $G \lesssim 1$ given $A$ large. Applying this in (5.2), we have

$$\frac{d}{dt} (\|n_0\|_2^2 - G(t)) \lesssim - \frac{1}{A} \|n_0\|_2^2 \left( \frac{\|n_0\|_2^2}{C} - M_{0/3} \right)$$

$$\lesssim - \frac{1}{A} \|n_0\|_2^2 \left( \frac{\|n_0\|_2^2}{C} - G(t) \right) \left( \frac{\|n_0\|_2^2}{C} + M_{0/3} \right).$$

Choosing $A$ large relative to $\|n_m\|_m^2$, $C_{n,\infty}$ and universal constants, we have

$$\|n_0\|_2^2 \lesssim G(t) + M_{0/3} + \|n_{m}\|_2^2$$

$$\leq C \left( 1 + \|n_m\|_2^2 + M_{0/3} \right)$$

$$=: C_{m,L} (\|n_m\|_2^2, M). \quad \text{(5.5)}$$

This completes the estimate on $\|n_0\|_2$. Combining with lemma 2.10 and adjusting the constant $C_{m,L}$ defined in (5.5) yield the first estimate in conclusion (2.8c).
Before estimating \( \|\partial_t n_0\|_2 \), we first note the following estimate on \( \int_0^{T_*} \|\nabla^2 c_{\rho}\|_2^2 \) from hypothesis (2.7b) and (3.18), (3.19),

\[
\frac{1}{8A} \int_0^{T_*} \|\nabla^2 c_{\rho}\|_2^2 \, dt \leq F(0) - F(T_*) \leq C_{BD} \left( \|n_0\|_{H^1}^2 + 1 \right). \tag{5.6}
\]

Now we estimate the \( \|\partial_t n_0\|_2 \). Estimating the time evolution of \( \|\partial_t n_0\|_2 \) using Gagliardo–Nirenberg–Sobolev inequality, Young's inequality, Minkowski inequality, lemma 2.1 and the time integral estimate of \( \|\nabla^2 c_{\rho}\|_2^2 \) (5.6), we have that

\[
\frac{1}{2} \frac{d}{dt} \|\partial_t n_0\|_2^2 \leq -\frac{1}{2A} \|\partial_t^2 n_0\|_2^2 + \frac{4}{A} \|\partial_t c_{\rho}\|_2 \|n_0\|_{L^\infty}^2 + \frac{4}{A} \|\partial_t (\partial_t c_{\rho} n_0)\|_2^2 + \frac{4}{A} \|\partial_t (\partial_t c_{\rho} n_0)\|_2^2 \|n_0\|_{L^\infty}^2 + \frac{4}{A} \|\partial_t c_{\rho}\|_2 \|n_0\|_{L^\infty}^2 \|\partial_t n_0\|_2^2 \|n_0\|_{L^\infty}^2 \frac{C}{A} \sup_{0 \leq t \leq T} \|\partial_t n_0(t)\|_2^2 - G(t) + \left( M^{1/3} + 1 \right) C_{C_{\infty},L^2}^2.
\]

We define

\[
G(t) := \frac{4}{A} \int_0^t \left( \|\partial_t^2 c_{\rho}\|_2^2 \|n_0\|_{L^\infty}^2 + \|\partial_t c_{\rho}\|_2 \|\partial_t n_0\|_2^2 \right) ds. \tag{5.7}
\]

Note that by the hypothesis (2.7a) and time integral control (5.6), we have that

\[
G(t) \lesssim C_{C_{\infty}}^2 \tag{5.9}
\]

for all \( t \leq T_* \). Now we apply Gagliardo–Nirenberg–Sobolev inequality, lemma 2.1 and definition of \( G(t) \) (5.8) to rewrite the inequality (5.7) as follows:

\[
\frac{d}{dt} \left( \|\partial_t n_0\|_2^2 - G(t) \right) \leq \frac{1}{A} \left( -\|\partial_t n_0\|_2^2 \right) \|\partial_t n_0\|_2^2 - G(t) + \frac{C_{C_{\infty},L^2}}{4} \|\partial_t n_0\|_2^2 \|n_0\|_{L^\infty}^2 + \sup_{0 \leq t \leq T} \|\partial_t n_0(t)\|_2^2 - G(t) + \left( M^{1/3} + 1 \right) C_{C_{\infty},L^2}^2.
\]

Now because \( G(t) \lesssim C_{C_{\infty}}^2 \), by a comparison principle, we can prove that

\[
\|\partial_t n_0\|_2^2 \leq 2C_{C_{\infty},L^2}, \tag{5.10}
\]

where \( C_{C_{\infty},L^2} \) is chosen properly. The reasoning is as follows. Since we can set the \( C_{C_{\infty},L^2} \) large such that \( \|\partial_t n_0(t_*)\|_2^2 \leq C_{C_{\infty},L^2} \), \( \|\partial_t n_0\|_2^2 - G(t) \) will reach the value \( C_{C_{\infty},L^2}^2 \) at the first time \( t_* > 0 \). At time \( t_* \), we have that \( \sup_{0 \leq t \leq t_*} \|\partial_t n_0(t)\|_2^2 = \|\partial_t n_0(t_*)\|_2^2 - G(t_*) = \|\partial_t n_0(t_*)\|_2^2 - G(t_*) \leq 2C_{C_{\infty},L^2}^2 \). Combining this fact and the differential inequality (5.10) yields

\[
\frac{d}{dt} \left( \|\partial_t n_0\|_2^2 - G(t) \right) \bigg|_{t=t_*} \leq \frac{1}{A} \left( -\|\partial_t n_0(t_*)\|_2^2 \right) \|\partial_t n_0(t_*)\|_2^2 + \frac{C_{C_{\infty},L^2}^2}{4} \|\partial_t n_0(t_*)\|_2^2 + \|\partial_t n_0(t_*)\|_2^2 - G(t_*) + \left( M^{1/3} + 1 \right) C_{C_{\infty},L^2}^2 \lesssim \frac{1}{A} \left( -\frac{C_{C_{\infty},L^2}^4}{4} + \frac{C_{C_{\infty},L^2}^2}{4} + \frac{C_{C_{\infty},L^2}^2}{4} \right) < 0.
\]
The last inequality < 0 is true if we pick $C_H'$ large enough. On the other hand, $\frac{d}{dt} (||\partial_t n_0||^2 - G)_{\mu=0} \geq 0$ at the first break through time $t_\ast$. As a result, we reach a contradiction. Therefore, we have that

$$||\partial_t n_0||^2_{L^\infty(0,T_\ast; L^2)} \lesssim \sup_{0 \leq t \leq T_\ast} G(t) + C_H'^2 \lesssim BC_{2,\infty}^4.$$

Now we just need to choose $C_H'^2$ much larger than the right hand side to conclude the proof of (2.8e).

### 6. Uniform $L^\infty$ control (2.8d) and (2.8e)

In this section we prove the uniform $L^\infty$ control (2.8d) and (2.8e). We separate the proof into two different time regimes, namely, the initial time $t \leq A^{1/3+\epsilon}$ and the long time $t \geq A^{1/3+\epsilon}$. Here $\epsilon > 0$ is a small constant determined by the proof. For the sake of clarity, we use $C^\infty_{n,\infty}$, $C^\infty_{V(\epsilon),\infty}$ to denote bounds in the initial time and $C^\long_{n,\infty}$, $C^\long_{V(\epsilon),\infty}$ to denote bounds in the long time. At the end of the proof, we will take the $C^\infty_{n,\infty}$ to be large compared to $C^\long_{n,\infty}$ and $C^\long_{V(\epsilon),\infty}$ and take the $C^\infty_{V(\epsilon),\infty}$ large compared to $C^\long_{n,\infty}$ and $C^\long_{V(\epsilon),\infty}$.

#### 6.1. Initial time layer estimate

In this subsection, we would like to prove the following lemma:

**Lemma 6.1.** Under the assumptions of proposition 1, there exist a constant $0 < \epsilon < \frac{1}{17}$ independent of the solution and constants $C^\infty_{n,\infty}, C^\infty_{V(\epsilon),\infty}, C^\long_{n,\infty}$ depending on $C_{ED}, n_\infty, M$ such that the following estimates hold on the time interval $0 \leq t \leq A^{1/3+\epsilon}$ when $A$ is chosen large enough:

$$||n(t)||_{L^\infty} \leq C^\infty_{n,\infty}(n_\infty, C_{ED}, M); \quad (6.1a)$$

$$||\nabla c_x(t)||_{L^\infty} \leq C^\infty_{V(\epsilon),\infty}(n_\infty, C_{ED}, M); \quad (6.1b)$$

$$||\partial_t n(t)||_{L^\infty} \leq C^\long_{n,\infty}(||n_\infty||_{H^1}), \quad \forall t \in [0, A^{1/3+\epsilon}]. \quad (6.1c)$$

**Remark 9.** In the proof of the lemma, the destabilizing effect of shear flow has to be treated carefully because the enhanced dissipation effect of the shear flow is too weak at the initial time. We will propagate the estimates (6.1) till $t = A^{1/3+\epsilon}$. After this time threshold, the enhanced dissipation kicks in to stabilize the dynamics.

**Proof.** We use a bootstrap argument to prove the lemma. Assume that for constants $C^\infty_{n,\infty}$, $C^\infty_{V(\epsilon),\infty}$, $C^\long_{n,\infty}$ depending on the proof, $T_{**} \in [0, A^{1/3+\epsilon}]$ is the maximal time on which the following hypothesis is satisfied:

$$||n(t)||_{L^\infty} \leq 2C^\infty_{n,\infty}; \quad (6.2a)$$

$$||\nabla c_x(t)||_{L^\infty} \leq 2C^\infty_{V(\epsilon),\infty}; \quad (6.2b)$$

$$||\partial_t n(t)||_{L^\infty} \leq 2C^\long_{n,\infty}, \quad \forall t \in [0, T_{**}), T_{**} \leq \min\{A^{1/3+\epsilon}, T_\ast\}. \quad (6.2c)$$
We will show that all the estimates (6.2) hold on the same time interval $[0, T_*]$ with ‘1’ instead of ‘2’ if we choose $A_0$ large. These improvements combined with the local well-posedness of the equation (1.1) yield (6.1).

We split the proof into three steps. In the first step, we obtain the improvement to (6.2a) together with a suboptimal estimate of $\| \nabla c_{\phi} \|_p$, $\forall p < \infty$. Here the estimate in $\| \nabla c_{\phi} \|_p$, $\forall p < \infty$ is suboptimal in the sense that on the interval $[0, T_*]$, the estimate loses a small power of $A$, i.e. $\| \nabla c_{\phi} \|_p \lesssim A^d$, $d > 0$. In order to compensate for the loss in powers of $A$, we need information about the higher regularity of $n_{\phi}$. This is why we propagate another estimate (6.1c) in the initial time layer $[0, T_{**}]$. In the second step, we complete the proof of (6.1c). In the last step, we use the extra regularity information to get the optimal $L^\infty$ bound of $\nabla c_{\phi}$.

**First step:** we prove the improvement to (6.2a) on $[0, T_*)$. We start with the estimate on $\| \partial_t c_{\phi} \|_4$. Direct energy estimate yields

$$
\frac{d}{dt} \| \partial_t c_{\phi} \|_4^4 \leq -\frac{3}{2A} \| \nabla (\partial_t c_{\phi})^2 \|_2^2 + \frac{6}{A} \| \partial_t c_{\phi} \|_3^2 \| n_{\phi} \|_4^2 - \frac{4}{A} \| \partial_t c_{\phi} \|_4^4. 
$$

(6.3)

Integration in time yields

$$
\| \partial_t c_{\phi}(t) \|_4 \leq \sqrt{3} \frac{\sqrt{t}}{A^{1/2}} \sup_{0 \leq t \leq T} \| n_{\phi}(s) \|_4 + \| \partial_t (c_{in})_{\phi} \|_4.
$$

(6.4)

With the equation (2.3b), we estimate the time evolution of the $L^4$ norm of $\partial_t c_{\phi}$:

$$
\frac{d}{dt} \| \partial_t c_{\phi} \|_4^4 \leq -\frac{3}{2A} \| \nabla (\partial_t c_{\phi})^2 \|_2^2 + \frac{6}{A} \| n_{\phi} \|_3^2 \| \partial_t c_{\phi} \|_3^2 + 4 \| \partial_t c_{\phi} \|_4 \| u' \partial_t c_{\phi} \|_4 - \frac{4}{A} \| \partial_t c_{\phi} \|_4^4.
$$

As in the $\partial_t c_{\phi}$ case, we drop the negative term at the moment, and end up with the following inequality

$$
\frac{d}{dt} \| \partial_t c_{\phi} \|_4^4 \leq \frac{6}{A} \| n_{\phi} \|_3^2 \| \partial_t c_{\phi} \|_3^2 + 4 \| \partial_t c_{\phi} \|_4 \| u' \partial_t c_{\phi} \|_4.
$$

(6.5)

Recall from the statement of the main theorem 2 that $\| (c_{in})_{\phi} \|_{H^1 \cap W^{1, \infty}} \leq C_{in} A^{-q}$, $q > 1/2$. Now the idea is to compare $\| \partial_t c_{\phi} \|_4$ with the solution to the following differential equation,

$$
\frac{d}{dt} f^4 = 10 q^3 \left( \| u' \partial_t c_{\phi} \|_4 + \| n_{\phi} \|_3^2 + \frac{1}{A} \right),
$$

(6.6)

$$
f(0) = 1 > C_{in} A^{-q} \geq \| \partial_t (c_{in})_{\phi} \|_4
$$

(6.7)

and show that $\| \partial_t c_{\phi}(t) \|_4 \leq f(t)$ for $t \leq T_*$. The function $f$ is estimated using (6.4) and the fact $q > 1/2$ as follows:

$$
f(t) \leq 1 + \frac{1}{A^{1/2}} + \int_0^t \| u' \partial_t c_{\phi}(s) \|_4 + \| n_{\phi}(s) \|_3^2 \frac{1}{A} ds
$$

$$
\leq 1 + A^{1/2-q} + \frac{1}{A^{1/2}} \sup_{0 \leq s \leq T} \| n_{\phi}(s) \|_4 + \frac{1}{A^{1/2}} \sup_{0 \leq s \leq T} \| n_{\phi}(s) \|_3^2, \quad \forall t \leq A^{1/3+\epsilon}.
$$

(6.8)

Next we show that $\| \partial_t c_{\phi} \|_4 \leq f$ for $\forall t \in [0, T_*)$. Since $f$ is strictly increasing in time, $f \geq 1$. Assume that there exists a first time $t^* \leq T_*$ such that $\| \partial_t c_{\phi}(t^*) \|_4$ is equal to the
function \( f^4(t^*) \). At time \( t^* \), we have \( ||\partial_t c_{\rho}(t^*)||_4 = f(t^*) \geq 1 \), which yields the following relation

\[
||\partial_t c_{\rho}(t^*)||_4^2 \geq ||\partial_t c_{\rho}(t^*)||_4^2.
\]  

(6.9)

Combining this with (6.5) and (6.6) yields that at time \( t^* \),

\[
\frac{d}{dt} ||\partial_t c_{\rho}||_4^2 \bigg|_{t=t^*} \leq \left( \frac{6||\partial_t c_{\rho}||_4^2}{A} + 4||n'\partial_t c_{\rho}||_4 \right) ||\partial_t c_{\rho}||_4^2 \bigg|_{t=t^*} \leq \frac{d}{dt} \bigg|_{t=t^*}.
\]  

(6.10)

On the other hand, \( \frac{d}{dt} ||\partial_t c_{\rho}||_4^2 \bigg|_{t=t^*} \geq \frac{d}{dt} f^4 \bigg|_{t=t^*} \) at the first break-through time \( t^* \), which is a contradiction. As a result, we have that \( ||\partial_t c_{\rho}(t)||_4 \leq f(t), \ \forall t \leq T_{**} \), which together with (6.8) yields the following estimate

\[
||\partial_t c_{\rho}(t)||_4 \leq 1 + \frac{3}{A} \sup_{0 \leq s \leq T} ||n(x)||_4 + \frac{1}{A} \sup_{0 \leq s \leq T} ||n(s)||_4^2, \ \forall t \leq T_{**}.
\]  

(6.11)

Recall the Gagliardo–Nirenberg–Sobolev inequality on \( \mathbb{R} \times \mathbb{R} \),

\[
||f||_4 \leq \frac{1}{2} ||\nabla f||_2^{1/2} ||f||_2^{1/2} + ||f||_2.
\]

Combining this with lemma 2.10, \( ||\nabla c_{\rho}||_4 \) estimates (6.4) and (6.11), we estimate the time evolution of \( ||n||_4^2 \) as follows

\[
\frac{d}{dt} ||n||_4^2 \leq -\frac{3}{A} ||\nabla(n^2)||_4^2 + \frac{1}{A} \sup_{0 \leq s \leq T} ||n(x)||_4 + \frac{1}{A} \sup_{0 \leq s \leq T} ||n(s)||_4^2.
\]

Thanks to the hypothesis (2.7d), conservation of mass and Hölder inequality, we can take \( A \) large enough such that the above estimate can be simplified as follows:

\[
\frac{d}{dt} ||n||_4^2 \leq \frac{C}{A} ||n||_4^2 (M^4 + 1 + ||\partial_t (c_\infty)||_4^4 + A^6 \sup_{0 \leq s \leq T} ||n(s)||_4^4),
\]

where the constant \( C \) is the implicit constant in the estimate above. Now we can compare the \( ||n||_4^2 \) to the solution to the following differential equation:

\[
\frac{d}{dt} f = \frac{2Cf}{A}(M^4 + ||\partial_t (c_\infty)||_4^4 + 1 + A^6f), \ \ f(0) > \max\{1, ||n_0||_4^4\}.
\]

The strictly increasing solution \( f \) is bounded \( f \leq C(n_\infty) \) on the interval \([0, A^{1/3+\varepsilon}] \) if \( \varepsilon \) is chosen small enough and \( A \) is chosen large enough compared to \( M \). Assume that there exists a first time \( 0 < t_\star \leq A^{1/3+\varepsilon} \) such that \( ||n(t_\star)||_4^2 \) is equal to the function \( f(t_\star) \). Since \( f \) is strictly increasing, at the first break-through time \( t_\star \), we have \( ||n(t_\star)||_4 = \sup_{0 \leq s \leq t_\star} ||n(s)||_4 \), which yields the following relation

\[
\frac{d}{dt} ||n||_4^2 \bigg|_{t=t_\star} \leq \frac{C}{A} ||n||_4^2 (M^4 + ||\partial_t (c_\infty)||_4^4 + 1 + A^6||n||_4^4) \bigg|_{t=t_\star} \leq \frac{d}{dt} f \bigg|_{t=t_\star}.
\]  

(6.12)

On the other hand, \( \frac{d}{dt} ||n||_4^2 \bigg|_{t=t_\star} \geq \frac{d}{dt} f \bigg|_{t=t_\star} \) at the first break-through time \( t_\star > 0 \), which is a contradiction. As a result, we have that
Next we start the iteration process. Assume that \( ||n||_p \) is bounded, we estimate the \( ||n||_{2p} \) in terms of \( ||n||_p \). We start with estimating the \( ||\partial_x c_p||_{2p} \). By calculating the time derivative, we see that

\[
\frac{1}{2p} \frac{d}{dt} ||\partial_x c_p||_{2p}^2 = \frac{2p-1}{Ap^2} ||\nabla(\partial_x c_p)\partial_x||_2^2 + \frac{2p-1}{Ap} ||\nabla(\partial_x c_p)\partial_x||_2 ||\partial_x c_p\partial_x\partial_x||_2 - \frac{1}{A} ||\partial_x c_p||_{2p}^2
\]

As a result, we have that

\[
\frac{d}{dt} ||\partial_x c_p||_{2p}^2 \leq \frac{2p}{A} ||n||_{2p}^2,
\]

which yields

\[
||\partial_x c_p(t)||_{2p} \leq \sqrt{p} \sup_{0 \leq s \leq t} ||n(s)||_{2p}^{1/2} + ||\partial_x (c_m)\partial_x||_{2p}, \quad \forall t \in [0, A^{1/3+\epsilon}].
\]

Next we estimate the time evolution of \( ||\partial_t c_p||_{2p} \)

\[
\frac{d}{dt} ||\partial_t c_p||_{2p}^2 \leq 2p ||u'||_{2p} ||\partial_t c_p||_{2p} ||\partial_x c_p||_{2p}^{2p-1} + \frac{2p^2}{A} ||\partial_t c_p||_{2p}^{2p-2} ||n||_{2p}^2 - \frac{2p}{A} ||\partial_t c_p||_{2p}^2.
\]

By comparing the solution with the following strictly increasing function \( f \)

\[
f(t) = 1 + \frac{1}{A^{1/2}} \sqrt{p} \sup_{0 \leq s \leq t} ||n(s)||_{2p} + A^{-1/3+\epsilon} + \frac{p}{A^{1/2}} \sup_{0 \leq s \leq t} ||n||_{2p}^2,
\]

and applying a similar argument to prove (6.13), we have that

\[
||\partial_t c_p(t)||_{2p} \leq f(t) \leq 1 + \frac{1}{A^{1/2}} \sqrt{p} \sup_{0 \leq s \leq t} ||n||_{2p} + A^{-1/3+\epsilon} + \frac{p}{A^{1/2}} \sup_{0 \leq s \leq t} ||n||_{2p}^2, \quad \forall t \in [0, A^{1/3+\epsilon}].
\]

Next we estimate the time evolution of \( ||n||_{2p} \). Applying the hypothesis, \( ||\nabla c_p||_{2p} \) estimates (6.18), (6.15), lemma 2.10 and the Gagliardo–Nirenberg–Sobolev inequality on \( \mathbb{T} \times \mathbb{R} \)

\[
||f||_2 \leq ||\nabla f||_{2p}^{1/2} ||f||_1^{1/2} + ||f||_1,
\]

we have the following estimate by picking \( A \) large

\[
\frac{1}{2p} \frac{d}{dt} ||n||_{2p}^2 = \frac{2p-1}{Ap^2} ||\nabla (n^n)\nabla||_2^2 + \frac{2p-1}{Ap} ||\nabla (n^n)\nabla||_2 ||n^n\nabla c||_2
\]

\[
\leq \frac{2p-1}{Ap^2} ||\nabla (n^n)\nabla||_2^2 + \frac{2p-1}{Ap} ||\nabla (n^n)\nabla||_2 ||n^n||_{L^{1/2}} ||n||_{L^{3/2}} ||\nabla c||_2^2 + \frac{Cp^2}{A} ||n^n||_2^{3/2} ||n||_{L^{3/2}} ||\nabla c||_2^2
\]

\[
\leq \frac{Cp^2}{A} ||n||_2^{3/2} ||n^n||_1^{3/2} \sup_{0 \leq s \leq t} ||n^n||_{2p} + A^{3/2} \sup_{0 \leq s \leq t} ||n||_{2p}^2,
\]
where the constant $C$ is a universal constant depending on the constant in the Gagliardo–
Nirenberg–Sobolev inequality. Time integrating on both side of the estimate and applying
the hypothesis (2.7d), conservation of mass and Hölder inequality, we have

$$\sup_{0 \leq t < T_{\ast}} |n(s)|^{\frac{2}{p}} \leq \frac{p^8}{A^{\frac{1}{2} - \frac{N}{4}}} C(C_{2, \infty}, \partial_t (c_{in})) \left( \sup_{0 \leq t < T_{\ast}} |n(s)|^{\frac{2}{p} - \frac{p}{p-1}} + \sup_{0 \leq t < T_{\ast}} |n(s)|^{\frac{2}{p} - 1} \right) + |n_{in}|^{\frac{2}{p}}.
$$

Finally, we use the (6.19) together with (6.13) to prove the $\|n\|_{L^\infty(0, T_{\ast}, L^\infty)} \leq C_{n, \infty}^m$. Note
that if for $j \in \mathbb{N}$, $\sup_{0 \leq t < T_{\ast}} |n(s)|_{2^j} \leq 1$, we have that $\sup_{0 \leq t < T_{\ast}} |n|_{\infty} \leq 1$, and the result follows. Therefore, we define $4 < p_\ast = 2^{4j} \in 2^\mathbb{Z}$ to be the first integer such that

$$\sup_{0 \leq t < T_{\ast}} |n|_{p} \geq 1. \text{ Note that for } p = p_\ast / 2,$$

$$\|n\|_{L^\infty(0, T_{\ast}, L^{2^j/p})} \leq \max \{ C_{n, j}^m, 1 \}. \quad (6.20)$$

In the following argument, we will only care about $p > p_\ast$ since we want to find the limit
of $\|n\|_{L^\infty(0, T_{\ast}, L^{2^j/p})}$ as $p \to \infty$.

By the Hölder’s inequality,

$$1 \leq \sup_{0 \leq t < T_{\ast}} |n|_p \leq |n(t)|_p \sup_{0 \leq t < T_{\ast}} |n(s)|^{1-\theta}. \quad \forall p > p_\ast.$$

Combining this estimate with the conservation of mass, we can get a lower bound for

$$\sup_{0 \leq t < T_{\ast}} |n|_p \geq (1 + M) \frac{\theta}{2^{4j}} \geq (1 + M)^{-2j/3}. \quad \forall p \in 2^j j < j_\ast > 2. \quad (6.21)$$

Combining this with (6.19), we have that

$$\sup_{0 \leq t < T_{\ast}} |n|_p \geq \frac{p^8}{A^{\frac{1}{2} - \frac{N}{4}}} C(C_{2, \infty}) \sup_{0 \leq t < T_{\ast}} |n(s)|^{\frac{2}{p}} + |n_{in}|^{\frac{2}{p}}. \quad (6.22)$$

Now we can pick the $A$ big such that

$$\sup_{0 \leq t < T_{\ast}} |n|_p \geq \frac{p^8}{A^{\frac{1}{2} - \frac{N}{4}}} \sup_{0 \leq t < T_{\ast}} |n(s)|^{\frac{2}{p}} + |n_{in}|^{\frac{2}{p}}, \quad \forall p = 2^j \geq p_\ast, j \in \mathbb{N}. \quad (6.23)$$

Now by the $L^2$ bound of $n$ (6.13), the $L^{p_\ast/2}$ bound of $n$ (6.20) and the standard Moser–
Alikakos iteration [1], we have that

$$\sup_{0 \leq t < T_{\ast}} |n|_{\infty} \leq C_{n, \infty}^m (n_{in}). \quad \text{(6.24)}$$

**Second step:** we prove the improvement to (6.2c). First we estimate the time evolution of

$$\frac{1}{2p} \left( |\partial_t c_{\neq} |^2_{2p} \right) = - \frac{2p-1}{2p} |\nabla (\partial_t c_{\neq})|^2_{2p} + \frac{p}{A} |\partial_t n|_{2p} |\partial_t c_{\neq}|^2_{2p} - \frac{1}{A} |\partial_t c_{\neq}|^2_{2p}.$$

Here we use the fact that $\partial_t n = \partial_t n_{\neq}$. As a result, we see that

$$|\partial_x c_{\neq}(t)|_{2p} \leq \sqrt{\frac{2p}{A}} \sup_{0 \leq t < T_{\ast}} |\partial_t n(s)|_{2p} + \frac{p}{A} |\partial_x^2 (c_{in})_{\neq}|_{2p}, \quad \forall t \in [0, T_{\ast}). \quad (6.25)$$

By a similar argument as in the estimate of the term $|\partial_t c_{\neq}|_{2p}$ in (6.18), we have that
\[ \|\partial_y c_y(t)\|_{2p} \leq 1 + \frac{\sqrt{m}^{1/2}}{A^{1/2}} \sup_{0 \leq s \leq t} \|\partial_y n(s)\|_{2p} + p \frac{\sup_{0 \leq s \leq t} \|\partial_y n(s)\|_{2p}^2}{A^{1/2}}, \quad t \in [0, T_\ast). \]  

(6.26)

Now we can calculate the time evolution of \( \|\partial_y n\|_{2p}^2 \):

\[ \frac{1}{2p} \frac{d}{dt} \|\partial_y n\|_{2p}^2 \leq -\frac{2p - 1}{Ap^2} \|\nabla (\partial_y n)^p\|_{2p}^2 + \frac{2p - 1}{Ap} \left( \|\nabla (\partial_y n)^p\|_2 \|\partial_y n\|_{2p} \right) \]

\[ + \|\nabla (\partial_y n)^p\|_2 \|\partial_y n\|_{2p} \nabla c_y \|_2 \]

\[ =: -\frac{2p - 1}{Ap^2} \|\nabla (\partial_y n)^p\|_2^2 + T_1 + T_2. \]  

(6.27)

In the first line, we have used the fact that \( \partial_y \nabla c = \partial_y \nabla c_y \). Now we need to separate the estimate into two cases, \( p = 1 \) and \( p \neq 1 \). First we discuss the \( p = 1 \) case. The \( T_1 \) term in (6.27) can be estimated using the \( \|\nabla \partial_y c_y\|_{2p} \) estimates (6.25) and (6.26) as follows:

\[ T_1 \leq \frac{1}{4A} \|\nabla (\partial_y n)\|_2^2 + \frac{1}{A} \|\partial_y \nabla c_y\|_{2p}^2 \|n\|_{2p}^2 \]

\[ \leq \frac{1}{4A} \|\nabla (\partial_y n)\|_2^2 + \left( \frac{2}{A^{2/3-p}} \sup_{0 \leq s \leq t} \|\partial_y n(s)\|_2^2 + 1 + \frac{M^2}{A^2} \sup_{0 \leq s \leq t} \|\partial_y n(s)\|_2^2 \right) \|n\|_{2p}^2. \]  

(6.28)

The \( T_2 \) in (6.27) can be estimated using \( \nabla c_y \) \( L^4 \) estimates (6.4), (6.11), lemma 2.10, Gagliardo–Nirenberg–Sobolev inequality on \( \mathbb{T} \times \mathbb{R} \) and Hölder inequality as follows:

\[ T_2 \leq \frac{1}{4A} \|\nabla (\partial_y n)\|_2^2 + \frac{1}{A} \|\partial_y \nabla c_y\|_{2p}^2 \left( \frac{1}{\sqrt{A}} + \frac{M^4}{A^2} \sup_{0 \leq s \leq t} \|\partial_y n(s)\|_2^2 \right) \]

\[ + \frac{p^2}{A^2} \sup_{0 \leq s \leq t} \|n(s)\|_2^4 + \frac{1}{A^2} \sup_{0 \leq s \leq t} \|n(s)\|_2^8. \]  

(6.29)

Now combining (6.24), (6.27)–(6.29), lemma 2.1, and, we obtain that

\[ \frac{d}{dt} \|\partial_y n\|_2^2 \leq \frac{C(C_{2\infty}, \partial_y (c_{\infty})_0)}{A^{1-\delta c}} \left( 1 + \sup_{0 \leq s \leq t} \|\partial_y n(s)\|_2^2 \right). \]

Now use a comparison argument similar to the one used to prove (6.13), we end up with the following estimate given \( A \) chosen large enough

\[ \|\partial_y n(t)\|_2 \leq C_{0, n, L^2}(n_m), \quad \forall t \in [0, T_\ast). \]  

(6.30)

This finishes the treatment of the case \( p = 1 \).

For the \( p \neq 1 \) case, there exists a large \( B \) such that the \( T_1 \) term in (6.27) can be estimated as follows:

\[ T_1 \leq \frac{2p - 1}{ABp^2} \|\nabla (\partial_y n)^p\|_2^2 + \frac{BCp^3}{A} \|\partial_y n\|_{2p} \|\nabla (\partial_y c_y)^p\|_2 \|n\|_{2p}^{4p/3} \]

\[ + \frac{BCp^2}{A} \|\partial_y n\|_{2p}^{2(1-p)} \|\nabla (\partial_y c_y)^p\|_2 \|n\|_{2p}^{2p}. \]

which combined with \( \nabla \partial_y c_y \) \( L^{2p} \) estimates (6.25), (6.26), hypothesis (6.2c) and \( L^2 \) estimate of \( \partial_y n \) in the initial time layer (6.30) yields
\[
T_1 \leq \frac{2p - 1}{BP^2} \left( \left| \nabla (\partial_n) \right|_2^2 + \frac{BP^2}{A^{1-\delta}} \left( \left| \nabla (\partial_n) \right|_1^{2(1-\delta)} + \left| \nabla (\partial_n) \right|_1^{2(1-\delta)} \right) \right) C(C_{2,\infty}, C_{\partial_n,\infty}, n_m).
\]

For the \( T_2 \) in (6.27), we can estimate it using lemma (2.10), \( L^\infty \) estimate of \( n \) (6.24), \( \nabla c \), \( L^p \) estimates (6.15), (6.18) and the Gagliardo–Nirenberg–Sobolev inequality on \( T \times \mathbb{R} \) as follows:

\[
T_2 \leq \frac{2p - 1}{A} \left( \left| \nabla (\partial_n) \right|_2^2 \left| \nabla (\partial_n) \right|_6 \left| \nabla c \right|_6 \right) \\
\leq \frac{2p - 1}{BP^2} \left( \left| \nabla (\partial_n) \right|_2^2 + \frac{BCP^4}{A} \left| \nabla (\partial_n) \right|_6^2 \left( \left| \nabla c \right|_6^2 + \left| \nabla c \right|_6 \right) \right) \\
\leq \frac{2p - 1}{BP^2} \left( \left| \nabla (\partial_n) \right|_2^2 + \frac{BCP^4}{A} \left| \nabla (\partial_n) \right|_6^2 \left( \left| \nabla c \right|_6 + \left| \nabla c \right|_6 \right) \right) \left| \nabla (\partial_n) \right|_p^2 C(C_{2,\infty}).
\]

Combining (6.27), (6.31) and (6.32) and integrating in time, we have that

\[
\frac{1}{2p} \left| \partial_n (t) \right|_{\frac{2p}{p}}^p \left( \frac{1}{2p} \left| \partial_n (t) \right|_{\frac{2p}{p}}^p + \frac{BP^2}{A^{1-\delta}} \sup_{0 \leq t \leq T} \left( \left| \partial_n (s) \right|_{\frac{2p(1-\delta)}{p}}^p + \left| \partial_n (s) \right|_{\frac{2p(1-\delta)}{p}}^p \right) C(C_{2,\infty}, C_{\partial_n,\infty}, n_m) \right) \\
+ \frac{BP^4}{A^{1-\delta}} \sup_{0 \leq t \leq T} \left| \partial_n (s) \right|_{\frac{p}{2}}^p C(C_{2,\infty}). \quad \forall t \in [0, T_*].
\]

Finally, we use the (6.33) together with (6.30) to get the \( \left| \partial_n \right|_{L^2 \infty (0, T, L^{p/2})} \leq 2C_{\partial_n,\infty}. \) Note that if for \( \forall j \in \mathbb{N}, \sup_{0 \leq t \leq T} \left| \partial_n (s) \right|_p \leq 1, \) we have that \( \sup_{0 \leq t \leq T} \left| \partial_n (s) \right|_p \leq 1, \) and the result follows. Therefore, we assume that there exists \( 4 \leq p_\ast = 2^{\gamma} \in 2^n \) such that it is the first integer that satisfies \( \left| \partial_n \right|_p \geq 1. \) For \( p = p_\ast \),

\[
\left| \partial_n \right|_{L^2 \infty (0, T, L^{p/2})} \leq \max \left( C_{\partial_n, L^{p/2}}, 1 \right).
\]

We will only care about \( p = 2^{\gamma} > p_\ast, \) \( j \in \mathbb{N}. \) By the Hölder’s inequality,

\[
1 \leq \sup_{0 \leq t \leq T} \left| \partial_n (s) \right|_p \leq \sup_{0 \leq t \leq T} \left| \partial_n (s) \right|_p \sup_{0 \leq t \leq T} \left| \partial_n (s) \right|_p^{-\theta}, \quad p > p_\ast, p \in 2^n.
\]

Now combining this with (6.30), we have a lower bound for \( \sup_{0 \leq t \leq T} \left| \partial_n (s) \right|_p :=

\[
\sup_{0 \leq t \leq T} \left| \partial_n (s) \right|_p \geq (1 + C_{\partial_n, L^{p/2}})^{-3}, \quad \forall p \geq p_\ast, p \in 2^n.
\]

Now combining this with (6.33), we have that

\[
\sup_{0 \leq t \leq T} \left| \partial_n (s) \right|_{\frac{p}{2}}^p \leq \left| \partial_n \right|_{\frac{p}{2}}^p \left( \frac{1}{2p} \left| \partial_n \right|_{\frac{p}{2}}^p + \frac{BP^2}{A^{1-\delta}} \sup_{0 \leq t \leq T} \left| \partial_n (s) \right|_{\frac{p}{2}}^p C(C_{2,\infty}, C_{\partial_n,\infty}, n_m). \right.
\]

Now we can take the \( A \) large such that

\[
\sup_{0 \leq t \leq T} \left| \partial_n (s) \right|_{\frac{p}{2}}^p \leq \left| \partial_n \right|_{\frac{p}{2}}^p \left( \frac{1}{2p} \left| \partial_n \right|_{\frac{p}{2}}^p + \frac{BP^2}{A^{1-\delta}} \sup_{0 \leq t \leq T} \left| \partial_n (s) \right|_{\frac{p}{2}}^p \right).
\]

Combining \( L^2 \) estimate of \( \partial_n (6.30), L^{p/2} \) estimate of \( \partial_n (6.34) \) and the standard Moser–Alkakos iteration yields

\[
\sup_{0 \leq t \leq T} \left| \partial_n (s) \right|_\infty \leq C(n_m).
\]

Now by picking \( 2C_{\partial_n,\infty} \geq C(n_m) \), we finishes the proof of the improvement to (6.2c).
**Third step:** we prove (6.1b). First we calculate the time evolution of $||\partial_t c_p||_{2p}$ using (6.30) and (6.38):

$$\frac{1}{2p} \frac{d}{dt} ||\partial_t c_p||_{2p}^p \leq \frac{1}{A} ||\partial_t n||_{2p} ||\partial_t c_p||_{2p}^{2p-1} \leq \frac{1}{A} (C_{0, \infty} (n_a) + C_{0, \infty} (n_w)) ||\partial_t c_p||_{2p}^{2p-1}.$$

This implies that

$$||\partial_t c_p(t)||_{2p} \leq \frac{t}{A} (C_{0, \infty} + C_{0, \infty} (n_a) + ||\partial_t (c_m^p)||_{2p}, \ \forall p \in [2, \infty). \ (6.39)$$

Therefore, by the assumption that $||\nabla (c_m^p)||_{||p||^1_{||y||^{q \infty}}} \leq CA^{-q}$, $q > 1/2$, we have that

$$||\partial_t c_p(t)||_{2p} \leq \frac{t}{A} (C_{0, \infty} + C_{0, \infty}) + CA^{-q}. \ (6.40)$$

For $t \leq T^*$, $A^{1/3 + \epsilon}$, we have the following estimate for $A$ chosen large enough

$$||\partial_t c_p(t)||_{\infty} \leq 1, \ \forall t \in [0, T^*). \ (6.41)$$

In order to estimate the norm $||\partial_t c_p||_{2p}$, we need to introduce a time weighted norm. To define it, we first consider the following simpler equation only taking into account the destabilizing effect of strong shear flow

$$\frac{d}{dt} f = -u'(y) \partial_y c_p - u(y) \partial_y f, \ \ f_m = \partial_y (c_m^p).$$

We can estimate the time evolution of the $L^{2p}$ norm of the solution using (6.39) as follows:

$$\frac{1}{2p} \frac{d}{dt} ||f||_{2p} \leq ||u'(\partial_y c_p)||_{2p} ||f||_{2p}^{2p-1} \leq \left( \frac{t}{A} (C_{0, \infty} + C_{0, \infty}) + CA^{-q} \right) ||u'||_{\infty} ||f||_{2p}^{2p-1}. \ (6.42)$$

Time integration yields

$$||f(t)||_{2p} \leq \frac{t}{A} ||u'||_{\infty} (C_{0, \infty} + C_{0, \infty}) + C ||u'||_{\infty} A^{-q} t + CA^{-q} =: G_{\infty}(t), 0 \leq t < T^*, \forall p \geq 2. \ (6.43)$$

Note the following relation:

$$G'_\infty(t) = 2 \frac{t}{A} (C_{0, \infty} + C_{0, \infty}) ||u'||_{\infty} + C ||u'||_{\infty} A^{-q} \geq ||u' \partial_y c_p||_{2p}. \ (6.44)$$

Next we consider the following time weighted norm:

$$F_p^{1/p}(t) := \frac{||\partial_t c_p(t)||_{p}}{e^{G_{\infty}(t)}}. \ (6.45)$$

Since $G_{\infty}$ is bounded by a universal constant if we choose $A$ large enough, the norm $F_p^{1/p}$ is equivalent to the $L^p$ norm. However, the quantity $F_p$ has better property than the usual $L^p$ norm. When we take the time derivative of $F_p$, the weight $\frac{1}{e^{G_{\infty}(t)}}$ will contribute extra negative term to compensate for the destabilizing effect of strong shear flow.

The time derivative of the $F_p$ can be estimated with the $L^\infty$ bound of $n$ in the initial time layer (6.24) and Gagliardo–Nirenberg–Sobolev inequality on $\mathbb{T} \times \mathbb{R}$ as follows.
\[
\frac{d}{dt} F_{2p} \leq \frac{2p}{c^2G^2} \left( -2p - 1 \frac{\|\partial_t \phi\|_{G^2}^{2p}}{C_p A} + \frac{2p - 1}{p^2} \frac{\|\partial_t \phi\|_{G^2}^{2p} + 2p - 1}{A} \frac{\|\partial_t \phi\|_{G^2}^{2p - 2}}{2} (M + C_{n, \infty}^n) \right)
\]

\[
\quad + \frac{2p - 1}{p^2} \frac{\|\partial_t \phi\|_{G^2}^{2p - 1} |u'| \|\partial_t \phi\|_{G^2}^{2p} (M + C_{n, \infty}^n) \frac{2p - 1}{A}. \]
\]

If \( \sup_{0 \leq s \leq T_*} \frac{\|\partial_t \phi\|_{G^2}}{c^2G^2} \leq 1 \), we have
\[
F_{2p}(t) \leq 1, \quad \forall t \in [0, T_*]. \tag{6.46}
\]

Otherwise if \( \sup_{0 \leq s \leq T_*} \frac{\|\partial_t \phi\|_{G^2}}{c^2G^2} \geq 1 \), we have that at the maximum point \( t_* \) of \( F_{2p} \)
\[
\|\partial_t \phi\|_{G^2}(t_*) \|u'| \|\partial_t \phi\|_{G^2} \geq 1. \quad \text{Combining this fact, Hölder inequality and the hypothesis e(2.7b), we obtain the following:}
\]
\[
\frac{d}{dt} F_{2p} \bigg|_{t = t_*} \leq \frac{2p}{c^2G^2} \left( \frac{2p - 1}{p^2} \frac{\|\partial_t \phi\|_{G^2}^{2p - 1} |u'| \|\partial_t \phi\|_{G^2}^{2p}}{C_p A} \left( \frac{\|\partial_t \phi\|_{G^2}^{2p - 1} |u'| \|\partial_t \phi\|_{G^2}^{2p}}{C_p A} \right)^2 + \frac{32}{pA} \left( C_{G^2}^n \left( \|n_m\|_{H^1} + 1 \right) + \frac{2p - 1}{A} \right) \right).
\]

Now we have that
\[
\sup_{0 \leq s \leq T_*} F_{2p}(s) \leq C(M, C_{n, \infty}, C_{G^2}, \|n_m\|_{H^1}) p^2 \sup_{0 \leq s \leq T_*} F_{2p}(s)^2 + \frac{\|\partial_t \phi\|_{G^2}^{2p}}{c^2G^2}. \tag{6.47}
\]

Combining (6.46) and (6.47), and noting that \( C_{n, \infty}^n \) only depends on \( n_m \) (6.24), we have
\[
\sup_{0 \leq s \leq T_*} F_{2p}(s) \leq \max \left\{ C(M, C_{G^2}, n_m) p^2 \sup_{0 \leq s \leq T_*} F_{2p}(s), 1 \right\}. \tag{6.48}
\]

for \( A \) large enough. Combining this with the fact that \( \|\partial_t \phi\|_{2} \leq \sqrt{C_{G^2}} (\|n_m\|_{H^1} + 1) < \infty \) from the hypothesis (2.7b) and using similar Moser–Alkakos iteration argument as before, we end up with
\[
\sup_{0 \leq s \leq T_*} \|\partial_t \phi\|_{H} \leq C(M, C_{G^2}, n_m). \tag{6.49}
\]

Combining this with (6.41), we have proven that
\[
\|\nabla \phi(t)\|_{H} \leq C_{G^2, \infty} (M, C_{G^2}, n_m), \quad \forall t \in [0, T_*]. \tag{6.50}
\]

Now since we have proven the bootstrap conclusion (6.38), \( T_* \) can be extended all the way to \( A^{1/3+\varepsilon} \), \( \varepsilon < \frac{1}{12} \). Therefore all the estimates we got above can be extended to \([0, A^{1/3+\varepsilon}] \). This completes the proof of the lemma.

**Remark 10.** One might slightly improve the value of \( q \) in theorem 2. However, if \( q \) is too small, we are not able to prove theorem 2. For example, we could not prove the theorem with \( \|\nabla \phi\|_{H} \leq A^{-1/4} \). The main obstacle is the estimate of the chemical gradient near the initial time. From the \( \|\partial_t \phi\|_{A} \) estimate (6.4), we can only obtain that
\[
\|\partial_t \phi\|_{A} \leq \sqrt{3} \sqrt{1/2} \sup_{0 \leq s \leq t} \|n_m\|_{A} + A^{-1/4}, \quad t \in [0, T_*].
\]
Combining this and the estimate (6.8) and the fact that \( \| \partial_t c_\varphi(t) \|_4 \leq f(t) \), we obtain that
\[
\| \partial_t c_\varphi(t) \|_4 \lesssim A^{1/12+\epsilon} + \frac{t^{1/2}}{A^{1/2}} \sup_{0 \leq s \leq t} \| n_\varphi(s) \|_4 + \frac{1}{A^{1/2}} \sup_{0 \leq s \leq t} \| n_\varphi(s) \|_4^2,
\]
which depends badly on \( A \). Since we do not have a bound which is independent of \( A \), we could not continue the proof.

6.2. Long time estimate

In this subsection, we prove (2.8d) and (2.8e) in the time interval \([A^{1/3+\epsilon}, T_*]\). The battle plan is as follows:

**First step:** we estimate the \( \| \nabla c_\varphi \|_4 \). First we need to get an estimate of the norm at the starting time \( t_0 := A^{1/3+\epsilon} \). By standard energy estimate combining with (2.7b), (6.39), (6.1a) and (6.1b), we have that
\[
\| \nabla c_\varphi(t_0) \|_4^4 \leq 2, \tag{6.51}
\]
if \( A \) is large enough.

For \( t \geq A^{1/3+\epsilon} \), applying the estimate (6.3) and the bootstrap hypothesis (2.7b), (2.7d) and (2.7e) and the Hölder’s inequality, we have that
\[
\frac{d}{dt} \| \partial_t c_\varphi \|_4^4 \leq \frac{1}{A} \| \partial_t c_\varphi \|_4^2 \| n_\varphi \|_4^2 \leq \frac{1}{A} \| \partial_t c_\varphi \|_4^2 \| n_\varphi \|_4^2 \| \partial_t c_\varphi \|_4 \| n_\varphi \|_4 \| \partial_t n_\varphi \|_4 \leq \frac{1}{A} e^{-\eta \pi \varphi} C_{ED}(\| n_\varphi \|_{L_t^4}^2 + 1) C_{2,\infty}^2.
\]

Time integrating the above inequality and combining it with (6.51), we obtain the following estimate by taking \( A \) large:
\[
\| \partial_t c_\varphi(t) \|_4^4 \leq 4, \quad \forall t \in [A^{1/3+\epsilon}, T_*]. \tag{6.52}
\]

Applying the time evolution estimate of \( \| \partial_t c_\varphi \|_4^4 \) (6.5), the fact that \( \| u' \|_\infty \leq C \), the bootstrap hypothesis (2.7b), (2.7d) and (2.7e) and the Hölder’s inequality, we can estimate the time evolution of \( \| \partial_t c_\varphi \|_4^4 \) as follows:
\[
\frac{d}{dt} \| \partial_t c_\varphi \|_4^4 \leq \frac{1}{A} \| \partial_t c_\varphi \|_4^2 \| n_\varphi \|_4^2 + \frac{1}{A} \| \partial_t c_\varphi \|_4^2 (\| u' \partial_t c_\varphi \|_4 A) \leq \frac{1}{A} \| \partial_t c_\varphi \|_4^2 \| \partial_t c_\varphi \|_4 \| n_\varphi \|_4 \| n_\varphi \|_4 + \frac{1}{A} \| \partial_t c_\varphi \|_4^2 \leq 1/2 \| \partial_t c_\varphi \|_4^2 \| \partial_t c_\varphi \|_4^2 \| \partial_t n_\varphi \|_4 \| n_\varphi \|_4 \| n_\varphi \|_4 \| \partial_t n_\varphi \|_4 \| \partial_t n_\varphi \|_4 \leq \frac{1}{A} e^{-\eta \pi \varphi} C_{ED}(\| n_\varphi \|_{L_t^4}^2 + 1) C_{2,\infty}^2 + \frac{1}{A} C_{ED}(\| n_\varphi \|_{L_t^4}^2 + 1) e^{-\eta \pi \varphi} C_{2,\infty}^2 \left( e^{-\frac{1}{2} A} \right).
\]

Note that \( e^{-\eta \pi A} \leq C(\epsilon, \eta) \). Now integrate this in time and use the initial condition (6.51), we have the following estimate by picking \( A \) large:
\[
\| \partial_t c_\varphi(t) \|_4^4 \leq 4, \quad \forall t \in [A^{1/3+\epsilon}, T_*]. \tag{6.53}
\]
This concludes the first step.
**Second step:** we estimate the time evolution of $||n||^2_A$: 
\[
\frac{d}{dt} \int |n|^2 \, dx \, dy = 4 \int n^2 \left( \frac{\Delta n - \nabla \cdot (\nabla c n)}{A} \right) \, dx \, dy \\
\lesssim - \int \frac{|\nabla (n)^2|^2}{A} \, dx \, dy + \frac{||n^2||^2 v ||\nabla v||^2}{A}.
\]
Applying the following Gagliardo–Nirenberg–Sobolev inequalities 
\[
||n^2||_2 \lesssim ||\nabla (n)^2||_2^{1/2} ||n^2||_1^{1/2} + ||n^2||_1, \\
||n^2||_4 \lesssim ||\nabla (n)^2||_2^{1/2} ||n^2||_2^{1/2} + ||n^2||_2
\]
and Young’s inequality in the above differential inequality yields 
\[
\frac{d}{dt} \int |n|^2 \, dx \, dy \lesssim - \frac{||n^2||^2}{CA ||n^2||^{q_2} 1} + \frac{1}{A} ||n^2||_1 + \frac{||n^2||^2 (||\nabla v||^2 + 1)}{A}.
\]
Combining this with the $L^4$ estimates of $\nabla c_\not{n}$ (6.52) and (6.53), initial time estimate (6.1) lemma 2.10 and hypothesis (2.7b), we have 
\[
\sup_{t_0 \leq t \leq T} ||n(t)||_4^2 \lesssim ||n(t_0)||_4^2 + \sup_{t_0 \leq t \leq T} ||n^2(t)||^2_1 + \sup_{t_0 \leq t \leq T} (1 + ||\nabla t||_4^2) \\
\lesssim \left( C_{nA}^\text{long}(n_m, \partial_j c_m 0, C_{m,E}, A) \right)^4.
\]
Since $||n_0||_{L^4(\mathbb{T} \times \mathbb{R})} \leq ||n||_{L^4(\mathbb{T} \times \mathbb{R})}$, we have $||n||_4 \leq 2 ||n||_4 \leq 2 C_{nA}^\text{long}$. 

**Third step:** now we can start to do the Moser–Alikakos iteration on $||\partial_j c_\not{n}||_{2p}$, $p \in 2\mathbb{N}$ to get 
$||\partial_j c_\not{n}||_\infty$ bound on $[A^{1/3+}, T_*]$. The time evolution of $||\partial_j c_\not{n}||_{2p}^2$ can be estimated as follows 
\[
\frac{d}{dt} \int |\partial_j c_\not{n}|^{2p} \, dx \, dy = 2p \int (\partial_j c_\not{n})^{2p-1} \left( \frac{\Delta \partial_j c_\not{n} + \partial_j n_\not{n}}{A} - u'(y) \partial_j c_\not{n} \right) \, dx \, dy \\
\lesssim - \left( 2 - \frac{2}{p} \right) \frac{1}{A} \int |\nabla (\partial_j c_\not{n})^p|^2 \, dx \, dy + \frac{2p^2}{A} ||\partial_j c_\not{n}||_{2p-2}^2 ||n_\not{n}||_4^2 \\
+ \frac{2p}{A} ||\partial_j c_\not{n}||_{2p-2}^2 ||u' \partial_j c_\not{n}||_2 A. \quad (6.55)
\]
Note that from hypothesis (2.7b), for $A$ chosen large enough, we have that 
\[
||u' \partial_j c_\not{n}||_2 A \leq \sqrt{C_{ED}(||n_m||_\infty + 1) e^{-\eta/2 A^{1/3}}} ||u'||_\infty A \leq C(\eta, \epsilon), \quad t \geq A^{1/3+}. \quad (6.56)
\]
We also have the following type of H"older’s inequality: 
\[
||\partial_j c_\not{n}||_{4p-4} \leq ||\partial_j c_\not{n}||_{4p-4}^{1/2p} ||\partial_j c_\not{n}||_2^{1/2p}; \\
||\partial_j c_\not{n}||_{4p-2} \leq ||\partial_j c_\not{n}||_{4p-2}^{1/2p} ||\partial_j c_\not{n}||_2^{1/2p}.
\]
Applying all these estimates together with Gagliardo–Nirenberg–Sobolev inequality on $\mathbb{T} \times \mathbb{R}$ and hypothesis (2.7b) in the above differential inequality (6.55), we have that
Now apply the Moser–Alikakos iteration and (6.1a), we have that
\[ |n(t)|_\infty \leq C_{n,\infty}^{\text{long}}(C_{ED}, n_{m, C_{m, L}}, \partial_j (c_{in})_0), \quad \forall t \in [t_0, T_*]. \tag{6.59} \]
By picking the \( C_{n,\infty} \gg \max\{ C_{n,\infty}^{\text{in}}, C_{n,\infty}^{\text{long}} \} \) in (2.7d), we prove (2.8d). By picking the \( C_{\nabla c_{\phi},\infty} \gg \max\{ C_{\nabla c_{\phi},\infty}^{\text{in}}, C_{\nabla c_{\phi},\infty}^{\text{long}} \} \) in (2.7e), we prove (2.8e). This completes the proof of proposition 1.

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Appendix

First we prove the blow-up criterion for (1.1).

**Lemma A.1 (Blow-up criteria).** Consider the parabolic-parabolic PKS system (1.1) subject to initial data \( n_{n0} \in H^1 \cap W^{1,\infty}, \nabla c_{in} \in H^1 \cap W^{1,\infty}, \forall s \geq 1 \). Assume that the shear flow profile satisfies \( u' \in W^{s+1,\infty} \). Assume that the solutions blow up at the first time \( T < \infty \), then the following condition must be satisfied on \([0, T]\):

\[
sup_{0 \leq t \leq T} \|n(t)\|_{\infty} = \infty. \tag{A.1}
\]

**Proof.** We prove the lemma by contradiction. Assume that

\[
sup_{0 \leq t \leq T} \|n(t)\|_{\infty} \leq C_{n,\infty} < \infty. \tag{A.2}
\]

Since the magnitude \( A \) does not play a crucial role here, we drop the magnitude \( A \) in the equation (1.1) throughout the proof. First we estimate the time evolution of the \( L^2 \) norm of the chemical gradient \( \|\nabla c\|_2 \) using integration by parts:

\[
\frac{1}{2} \frac{d}{dt} \|\nabla c\|_2^2 = \int \nabla c \cdot \nabla (\Delta c + n - c - u(y)\partial_y c) \, dxdy \tag{A.3}
\]

\[
\leq - \frac{1}{2} \|\nabla^2 c\|_2^2 + \|n\|_2^2 + (|u'|_{\infty} - 1) \|\nabla c\|_2^2
\]

\[
\leq |n|_2^2 + (|u'|_{\infty} - 1) \|\nabla c\|_2^2. \tag{A.4}
\]

By solving the differential inequality, we have that

\[
\|\nabla c(t)\|_2^2 \leq \|\nabla c_{in}\|_2^2 e^{2(|u'|_{\infty} - 1)t} + \int_0^t 2|n(s)|_2^2 e^{2(|u'|_{\infty} - 1)(t-s)} \, ds. \tag{A.5}
\]

Combining it with the fact that \( |n|_2 \leq |n|_1^{1/2} |n|_\infty^{1/2} \leq M^{1/2} C_{n,\infty}^{1/2} \), we have that

\[
\|\nabla c(t)\|_2 \leq C_{\nabla, u', C_{n,\infty}, M, T} < \infty. \tag{A.6}
\]

Similarly we estimate the time evolution of the \( L^4 \) norm of \( \nabla c \) using integration by parts, Hölder’s inequality and Young’s inequality,

\[
\frac{1}{4} \frac{d}{dt} (|\partial_x c|_4^4 + |\partial_y c|_4^4)
\]

\[
= \int (\partial_x c)^3 \partial_x (\Delta c + n - c - u(y)\partial_y c) \, dxdy + \int (\partial_y c)^3 \partial_y (\Delta c + n - c - u(y)\partial_y c) \, dxdy
\]

\[
\leq - \frac{1}{4} \|\nabla (\partial_x c)^2\|_2^2 - \frac{1}{4} \|\nabla (\partial_y c)^2\|_2^2 + \|n\|_2^2 + (|u'|_{\infty} + 3) \|\nabla c\|_2^2.
\]
By the $L^\infty$ bound of $n$ (A.2) and Hölder’s inequality, we have that
\[
\|\nabla c(t)\|_4 \leq \|\nabla c\|_2 \|e^{4(|\omega'|\infty + 3)t}\| + \int_0^T 4\|n(s)\|_4 \|e^{4(|\omega'|\infty + 3)(t-s)}\|ds \\
\leq \|\nabla c\|_2 \|e^{4(|\omega'|\infty + 3)T}\| + e^{4(|\omega'|\infty + 3)T} \int_0^T 4MC^3 \|ds,
\]
which in term yields that
\[
\|\nabla c(t)\|_4 \leq C_{\nabla, L}(\nabla c, C_{n, \infty}, M, T) < \infty, \quad \forall t \in [0, T]. \tag{A.7}
\]
Next we estimate the time evolution of $\|\nabla^2 c\|_2$:
\[
\frac{1}{2} \frac{d}{dt} \|\nabla^2 c\|_2^2 \leq -\frac{1}{2} \|\nabla^3 c\|_2 + \|\nabla n\|_2 + (\|\nabla'|\infty - 1)\|\nabla^2 c\|_2 + \|\nabla'\|_\infty \|\nabla c\|_2. \tag{A.8}
\]
Applying the $L^2$ estimate of $\nabla c$ (A.6), we have that
\[
\|\nabla^2 c(t)\|_2^2 \leq C_{\nabla, L} (\nabla', \nabla c, C_{n, \infty}, T) \left( 1 + \int_0^t \|\nabla n(s)\|_2^2 ds \right), \quad \forall t \in [0, T]. \tag{A.9}
\]
Now we can estimate the time evolution of $\|\nabla n\|_2$ using the $L^\infty$ bound (A.2), (A.9) and the Gagliardo–Nirenberg–Sobolev inequality on $T \times \mathbb{R}$ as follows
\[
\frac{d}{dt} \|\nabla n\|_2^2 \leq -\|\nabla^2 n\|_2^2 + \|\nabla^2 c\|_2^2 \|n\|_\infty^2 + \|\nabla c\|_2^2 \|\nabla n\|_2^2 + \|\nabla'|\infty \|\nabla n\|_2^2 \\
\leq -\frac{3}{4} \|\nabla^2 n\|_2^2 + \left( 1 + \int_0^t \|\nabla n(s)\|_2^2 ds \right) C_{n, \infty}^2 \\
+ C_{\nabla, L}^2 (\|\nabla n\|_2^2 + \|n\|_\infty^2) + \|\nabla'|\infty \|\nabla n\|_2^2 \\
\leq -\frac{1}{2} \|\nabla^2 n\|_2^2 + C_{n, \infty}^2 \left( 1 + T \sup_{0 \leq s \leq t} \|\nabla n(s)\|_2^2 \right) \\
+ (C_{\nabla, L}^2 + \|\nabla'|\infty \|) \|\nabla n\|_2^2 \\
\leq 1 + \sup_{0 \leq s \leq t} \|\nabla n(s)\|_2^2 + \|\nabla n\|_2^2. \tag{A.10}
\]
Now consider the following differential equation on $[0, T]$:
\[
\frac{d}{dt} f = C_*(2f + 1), \quad \|\nabla n\|_2^2 \leq f(0) \leq 2 \|\nabla n\|_2^2 + 1, \tag{A.11}
\]
where the constant $C_*$ is chosen such that it is strictly larger than the implicit constant appeared in (A.10). Direct calculation yields that $f \leq C_*(T, C_{n, \infty}, \nabla c, n_m) < \infty$ on the interval $[0, T]$. Now we apply a comparison argument to prove that $\|\nabla n(t)\|_2^2$ is bounded on the time interval $[0, T]$. Assume that the function $\|\nabla n(t)\|_2^2$ reaches the value of $f(t)$ at the first time $t_* \in (0, T]$. Since $f$ is strictly increasing in time, we must have that $\sup_{0 \leq s \leq t_*} \|\nabla n(s)\|_2^2 = \|\nabla n(t_*)\|_2^2$. Otherwise, one can use a continuity argument to show that $t_*$ is not the first time when the function $\|\nabla n(t)\|_2^2 - f(t)$ reach zero. At time $t_*$, combining (A.10) and (A.11) yields that
\[
\frac{d}{dt} \|\nabla n\|_2^2 \Big|_{t=t_*} \leq \frac{d}{dt} f \Big|_{t=t_*}. \quad \text{On the other hand, at the first break-through time } t_*, \text{ we have}
\]

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\[ \frac{d}{dt} \| \nabla n(t) \|^2 \leq \frac{d}{dt} f \bigg|_{t=t_0}, \] which is a contradiction. As a result, we have that \( \| \nabla n(t) \|_2 \leq f(t) \) for \( \forall t \in [0, T] \). Combining (A.9) and (A.12), we end up proving
\[ \| n(t) \|_{H^r} + \| \nabla c(t) \|_{H^r} \leq C_{H^r} < \infty , \quad \forall t \in [0, T]. \tag{A.12} \]

The proof of higher \( H' \) norm estimate is similar. Assume that we have already proven
\[ \| n(t) \|_r^2 + \| \nabla c(t) \|_r^2 \leq C_{H'-1}(T, C_{n, \infty}, n_m, \nabla c_m, u') , \quad t \in [0, T] \tag{A.13} \]
for some \( 2 \leq r \in \mathbb{N} \). We prove \( \| n(t) \|_{H^r} + \| \nabla c(t) \|_{H^r} \leq C_{H^r} < \infty \). First we estimate the time evolution of \( \| \nabla^{r+1} c(t) \|_{H^r}^2 \),
\[ \frac{d}{dt} \| \nabla^{r+1} c(t) \|_{H^r}^2 \leq - \frac{1}{2} \| \nabla^{r+2} c(t) \|_{H^r}^2 + \| \nabla' n(t) \|_{H^r}^2 + \| \nabla' c(t) \|_{H^r}^2 + C(u', C_{H-r-1}). \tag{A.14} \]

As a result, we have that
\[ \| \nabla^{r+1} c(t) \|_{H^r}^2 \leq C_{n, \infty, r, C_{H-r-1}, n_m, \nabla c_m, u'} \left( 1 + \int_0^t \| \nabla' n(s) \|_{H^r}^2 ds \right) , \quad \forall t \in [0, T]. \tag{A.15} \]

Next we estimate the time evolution of \( \| \nabla' n(t) \|_{H^r}^2 \) using the estimate (A.15), induction hypothesis (A.13) and Gagliardo–Nirenberg–Sobolev inequality on \( T \times \mathbb{R} \),
\[ \frac{d}{dt} \| \nabla' n(t) \|_{H^r}^2 \]
\[ \leq - \frac{1}{2} \| \nabla^{r+1} n(t) \|_{H^r}^2 + \| \nabla^{r+1} c(t) \|_{H^r}^2 \| \nabla' n(t) \|_{H^r}^2 + \sum_{j=1}^r \| \nabla^j c(t) \|_{H^r}^2 \| \nabla^{r-j} n(t) \|_{H^r}^2 \]
\[ \leq - \frac{1}{2} \| \nabla^{r+1} n(t) \|_{H^r}^2 + \| \nabla^{r+1} c(t) \|_{H^r}^2 C_{n, \infty}^2 + \left( \| \nabla^{r+1} c(t) \|_{H^r}^2 + \| \nabla' c(t) \|_{H^r}^2 \right) \left( \| \nabla' n(t) \|_{H^r}^2 + \| \nabla' c(t) \|_{H^r}^2 \right) \]
\[ + \sum_{j=2}^{r-1} \left( \| \nabla^{r+1} c(t) \|_{H^r}^2 + \| \nabla' c(t) \|_{H^r}^2 \right) \left( \| \nabla^j n(t) \|_{H^r}^2 + \| \nabla' n(t) \|_{H^r}^2 \right) + \| \nabla' n(t) \|_{H^r}^2 C_{H-r-1} \]
\[ \leq - \frac{1}{4} \| \nabla^{r+1} n(t) \|_{H^r}^2 + \| \nabla^{r+1} c(t) \|_{H^r}^2 + \| \nabla' n(t) \|_{H^r}^2 + 1 \]
\[ \leq \sup_{0 \leq s \leq t} \| \nabla' n(s) \|_{H^r}^2 + \| \nabla' n(t) \|_{H^r}^2 + 1. \]

Now by a comparison argument we used to prove (A.12), we have that
\[ \| \nabla' n(t) \|_{H^r}^2 + \| \nabla^{r+1} c(t) \|_{H^r}^2 \leq C_{H^r}(M, C_{n, \infty, r, C_{H-r-1}, n_m, \nabla c_m, u', T) < \infty , \quad \forall t \in [0, T]. \]

Since the \( H' \) norms of \( n(t) \) and \( \nabla c(t) \) are bounded at the time \( T \), by the local existence theory of parabolic equations, we can extend the existence time to \( [0, T + \epsilon) \) for some \( \epsilon > 0 \), which contradicts the fact that \( [0, T] \) is the maximal existence time interval. This concludes the proof of the lemma. \( \Box \)
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