An asymmetric mode-localized mass sensor based on the electrostatic coupling of different structural modes with distributed electrodes

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Received: 12 March 2021 / Accepted: 19 September 2021 / Published online: 18 January 2022 © The Author(s), under exclusive licence to Springer Nature B.V. 2022

Abstract Mode-localized sensor with amplitude ratio as output metric has shown excellent potential in the field of micromass detection. In this paper, an asymmetric mode-localized mass sensor with a pair of electrostatically coupled resonators of different thicknesses is proposed. Partially distributed electrodes are introduced to ensure the asymmetric mode coupling of second- and third-order modes while actuating the thinner resonator by the distributed electrode. The analytical dynamic model is established by Euler–Bernoulli theory and solved by harmonic balance method (HBM) combined with asymptotic numerical method (ANM). Detailed investigations on the linear and nonlinear behavior, critical amplitude as well as the sensitivity of the sensor are performed. The sensitivity of the proposed sensor can be enhanced by about 20 times compared to first-order mode-localized mass sensors. By exploiting the nonlinearities while driving the device beyond the critical amplitude for the in-phase mode, the sensor performs a great improvement in sensitivity up to 1.78 times compared to linear case. The influence of the coupling voltage on the sensor performance is studied, which gives a good reference to avoid mode aliasing. Moreover, the effect of the length of driving electrode on sensitivity is investigated.

Keywords Mode-localized sensor · Asymmetric mode coupling · Distributed electrodes · Mass sensing

1 Introduction

For decades, resonant MEMS sensors have attracted worldwide attentions for their potential applications in detecting biomolecular [1, 2], proteins [3] and DNA [4, 5]. Reducing the size of resonators to the scale of nanometer is an effective way to improve sensitivity and enable such sensors to molecules [6]. However, nanoscale devices bring problems about noise and instability [7]. And then, methods involving structure optimization [8], utilizing high-order vibration modes [9, 10] and nonlinear vibration [11–13] are also presented for increasing sensitivity; nevertheless,
single beam resonator still has disadvantages such as temperature drift and insufficient sensitivity.

Unlike single resonator-based sensors mentioned above, mass sensor based on mode localization of two or more resonators used the amplitude shift ratio [14, 15] or eigenstate shift [16] as the output metric of sensitivity, other than resonant frequency shift. Such sensor can simultaneously detect the variation of each resonator’s dynamic response induced by the added mass, which can attain high sensitivity and great anti-interference ability [17]. Spletzer et al. [18] used a pair of mechanically coupled cantilever beams to improve the mass sensitivity by 2 orders of magnitude and further enhanced the sensitivity by increasing the degree of freedom of the multiple resonators [19]. Besides the method of reducing geometric size, it provided an alternative way to achieve high performances like high sensitivity, anti-interference capability and low noise. Thiruvenkatanathan et al. [20] presented an electrometer composed of double-ended tuning fork (DETF) resonators with mechanical coupling elements, and the results show that the displacement of eigenstate is about three orders of magnitude larger than the resonant frequency shift at the same charge input. Ouakad et al. [21] studied the veering phenomenon and dynamic characteristics of first three vibration modes of mechanically coupled resonators for ultra-sensitive working environment. Different from mechanical coupling, Thiruvenkatanathan et al. [22] designed a mass sensor based on electrostatic coupled resonators which makes the sensitivity controllable. Ilyas et al. [23] investigated eigenfrequencies of both the electrostatically and mechanically coupled microbeam resonators, and pointed out that the two coupling methods can control their mode steering behavior by tuning the bias voltage. Meanwhile, the mode-localized sensor with electrostatic coupling has great potentials in adjusting the coupling stiffness. Pandit et al. [24] studied the nonlinear behavior of electrostatic weakly coupled resonators and concluded that the resolution gain can be increased significantly compared with the linear resonators. Lyu et al. [25] designed a novel mass sensor with two electrostatically coupled microbeams of different lengths, which explored the nonlinear behavior to improve sensitivity. Zhao et al. [26] fabricated a potential sensor using 3-DoF weakly coupled resonators, and the sensitivity of such 3-DoF sensor has been significantly increased.

Other methods to improve the sensitivity and stability of mode localization sensors have also been explored. Kasai et al. [27] used a virtual cantilever beam coupled with a real cantilever beam to complete nanogram-level mass sensing, which is 10 times more sensitive than single cantilever resonator. Li et al. [28] investigated the nonlinear bifurcation behavior of electrostatically coupled resonators. By using the bifurcation phenomenon, the detection accuracy is improved and the detection error caused by frequency drift in nonlinear vibration can be overcome. Rabenimanana et al. [29] introduced a mode-localized sensor composed of two cantilevers with different lengths to overcome the mode aliasing limitation of linear mode-localized sensors by tuning the electrostatic nonlinearity, and the sensor sensitivity has been enhanced up to 67%. It could be concluded that mode-localized sensor has advantages of ultra-high sensitivity, high resolution, low noise and high stability. In particular, the sensitivity can be further enhanced by introducing nonlinear phenomenon, increasing the degree of freedom and optimizing structure configuration. While mode veering phenomenon has been widely studied either between mode shapes of a single structure or between the same order modes of discrete systems, to our knowledge, the combination of both and its application for mass sensing have not been investigated. Actually, different order mode couplings may result in high performances for mass sensors.

In this paper, firstly, an asymmetric mode-localized mass sensor with two electrostatically coupled resonators of different thicknesses is proposed, which realized the coupling of different high-order bending modes to improve the sensitivity. The thinner resonator is actuated by a DC voltage combined with AC voltage, so that the equilibrium state can be easily realized while operating in second- and third-order mode. Secondly, the analytical dynamic model of the sensor with asymmetric structure is established and solved by harmonic balance method (HBM) combined with asymptotic numerical method (ANM) to investigate its linear and nonlinear behaviors. Moreover, the linear and nonlinear frequency–amplitude characteristics are verified via the longtime integration (LTI) and the method of multiple scales (MMS). Thirdly, the sensor specifications are analyzed such as pull-in phenomenon, critical amplitude and mode aliasing. Simulations are performed involving the sensitivity comparison between first-order mode coupling and
different mode couplings (namely second–third-order mode coupling), to emphasize the benefit of about 20 times increase in sensitivity compared to first-order mode coupling. Furthermore, the sensitivity can be enhanced by 1.78 times while introducing the nonlinearities. Finally, the effect of driving electrode parameters on sensitivity is investigated.

The rest of this paper is organized as follows. Section 2 introduces the mathematical formulation by the Euler–Bernoulli theory and Galerkin discretization. In Sect. 3, the linear and nonlinear behaviors are investigated by using the harmonic balance method (HBM) combined with asymptotic numerical method (ANM) and using the longtime integration (LTI) method and the method of multiple scales (MMS) for validation. In Sect. 4, the influences of mass position, driving voltage, coupling voltage as well as the length of driving electrode on sensitivity are studied. Finally, conclusions are presented in Sect. 5.

2 Mathematical formulation

An electrostatically coupled MEMS resonator is designed as shown in Fig. 1a, which is composed of two double-ended clamped beams with a coupling voltage of $V_c$. The two resonators have different thicknesses, and the thinner one is excited by a combination of DC voltage $V_{dc}$ and AC voltage $V_{ac}$. To achieve second and third mode coupling, the distributed electrodes are introduced to drive the thinner resonator and the different mode of the sensor is shown in Fig. 1b. The geometric parameters of proposed mass sensor are shown in Table 1.

According to Euler–Bernoulli beam theory, the equations of motion that govern the transverse deflections $\tilde{w}_1(\tilde{x}, \tilde{t})$ and $\tilde{w}_2(\tilde{x}, \tilde{t})$ of the two resonators are written as:

$$
\begin{align*}
EI_1 \ddot{\tilde{w}}_1 + \rho A_1 \dddot{\tilde{w}}_1 + \delta_{c0}(\tilde{x} - \tilde{x}_0) m_p \dddot{\tilde{w}}_1 &= - \frac{1}{2} \frac{\epsilon_0 b V_c^2}{(g_e + \tilde{w}_1 - \tilde{w}_2)^2} H_2(\tilde{x}) \\
&+ \frac{1}{2} \frac{\epsilon_0 b [V_{dc} + V_{ac} \cos(\Omega t)]^2}{(g_e - \tilde{w}_1)^2} H_1(\tilde{x})
\end{align*}
$$

(1)

where $\tilde{x}$ represents the position along the microbeam and $\tilde{t}$ is time. $\tilde{c}_1$ and $\tilde{c}_2$ are the linear viscous damping. $h_1$ and $h_2$ are the thicknesses of two microbeams, and $h_2$ is $\delta$ times of $h_1$. $E$, $\rho$, $b$ and $l$ are the Young’s modulus, density, width and length of the beam, and $I = bh_3/12$ is the moment of inertia. $g_e$ is the gap distance between the two beams, and $g_{o}$ is the distance between the microbeam 1 and the driving electrode.

![Fig. 1](image-url)  

**Fig. 1** a Sketch of proposed mass sensor, b sketch of different mode couplings
Table 1 Parameters of the mass sensor

| Type                | Value |
|---------------------|-------|
| Coupling voltage \((V_c)\) | 10 V  |
| Height of microbeam 1 \((h_1)\) | 1 μm  |
| Microbeam width \((b)\) | 10 μm |
| Length of microbeams \((l)\) | 200 μm|
| Air gap \((g_a)\) | 1 μm  |
| Air gap \((g_c)\) | 1 μm  |
| Young’s modulus \((E)\) | 169 GPa|
| Density \((\rho)\) | 2320 kg/m³|

The dielectric constant of the gap medium is \(\varepsilon_0\). And \(V_{dc}\) is the applied bias voltage, \(V_c\) is the coupling voltage and \(m_p\) is the added mass located at \(\tilde{x}_0\) from the cantilever clamped side. \(\tilde{N}_1\) and \(\tilde{N}_2\) are mechanical axial force and \(H_1\) and \(H_2\) are Heaviside function modeling the electrostatic forces distributions, which are defined as:

\[
H_1(\tilde{x}) = |H(\tilde{x} - 0.5l)|
\]

\[
H_2(\tilde{x}) = |H(\tilde{x} - 0.2l) - H(\tilde{x} - 0.3l)|
\]

The equations of motion have the following boundary conditions:

\[
\begin{align*}
\ddot{w}_1(0, \tilde{t}) &= \ddot{\tilde{w}}_1(l, \tilde{t}) = \frac{\partial \ddot{w}_1}{\partial x} (0, \tilde{t}) = \frac{\partial \ddot{\tilde{w}}_1}{\partial x} (l, \tilde{t}) \\
\ddot{w}_2(0, \tilde{t}) &= \ddot{\tilde{w}}_2(l, \tilde{t}) = \frac{\partial \ddot{w}_2}{\partial x} (0, \tilde{t}) = \frac{\partial \ddot{\tilde{w}}_2}{\partial x} (l, \tilde{t})
\end{align*}
\]

2.1 Nondimensionalization

For simplification, the following nondimensional variables are introduced.

\[
\begin{align*}
\tilde{w}_1 &= \frac{\ddot{w}_1}{g_a}, \quad x = \frac{\tilde{x}}{l}, \quad t = \frac{\tilde{t}}{\tau}, \quad w_2 &= \frac{\ddot{w}_2}{g_a}, \quad \tau = \sqrt{\frac{\rho A_1 l^3}{E I_1}}
\end{align*}
\]

Substitute the dimensionless parameters into Eqs. (1)-(4) to remove the hats. Assuming that the device operates under moderate amplitude vibrations, the electrostatic excitation force term is expended to third-order Taylor series [30, 31]. Also, since the coupling voltage is very weak, the first-order Taylor expansion is used on coupled electrostatic force terms.

\[
\begin{align*}
\frac{\partial^4 w_1}{\partial x^4} + \frac{\partial^2 w_1}{\partial t^2} + c_1 \frac{\partial w_1}{\partial t} &+ \int_0^1 \left( \frac{\partial \tilde{w}_1}{\partial x} \right)^2 dx \frac{\partial^3 w_1}{\partial x^3} + \delta_{\tilde{y}_0}(x) \Delta m \frac{\partial^3 w_1}{\partial x^3} = -\frac{V^2_0}{R^2} H_2(x) \left( \frac{1}{R^2} \frac{2}{R^3} (w_1 - w_2) \right) \\
&+ \frac{V_0(V_{dc} + V_{ac} \cos(\Omega t))^2}{R^2} H_1(x) (1 + 2w_1 + 3w_2 + 4w_3) \\
&\delta^2 \frac{\partial^4 w_2}{\partial x^4} + \frac{\partial^2 w_2}{\partial t^2} + c_2 \frac{\partial w_2}{\partial t} &+ \frac{1}{R^2} \frac{2}{R^3} (w_1 - w_2) \\
&= \frac{V^2_0}{R^2} H_2(x) \left( \frac{1}{R^2} \frac{2}{R^3} (w_1 - w_2) \right)
\end{align*}
\]

The following boundary conditions and Heaviside functions are:

\[
\begin{align*}
w_1(0, t) &= w_1(1, t) = \frac{\partial w_1}{\partial x} (0, t) = \frac{\partial w_1}{\partial x} (1, t) \\
w_2(0, t) &= w_2(1, t) = \frac{\partial w_2}{\partial x} (0, t) = \frac{\partial w_2}{\partial x} (1, t)
\end{align*}
\]

\[
H_1(x) = |H(x - 0.5)|
\]

\[
H_2(x) = |H(x - 0.2) - H(x - 0.3)|
\]

The parameters in Eq. (6) are:

\[
\begin{align*}
h_2 &= \delta h_1, \quad c_1 = \frac{I^4_1}{E I_1}, \quad N_1 = \tilde{N}_1 \frac{I^4_1}{E I_1}, \quad \alpha_1 = 6 \left( \frac{g_a}{h_1} \right)^2, \\
\alpha_2 &= \frac{\varepsilon_0 h_1^4}{2 E I g_a}, \quad c_2 = \frac{I^4_1 \varepsilon_2}{\delta E I_1}, \quad N_2 = \tilde{N}_2 \frac{I^4_1}{\delta E I_1}, \\
\Omega &= \tilde{\Omega} \tau, \quad \Delta m = \frac{m_p}{\rho A_1 l^3} R = \frac{g_c}{g_a}
\end{align*}
\]

2.2 Reduced-order model

Use the Galerkin method to discretize the equations and eliminate the spatial dependency, where the linear undamped mode shapes are taken as the mode basis functions. Therefore, the deflection of the microbeams can be written as:
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\[ \begin{align*}
    w_1(x, t) &= w_{s1}(x) + \sum_{i=1}^{N_n} q_{1,i}(t) \phi_{1,i}(x) \\
    w_2(x, t) &= w_{s2}(x) + \sum_{i=1}^{N_n} q_{2,i}(t) \phi_{2,i}(x)
\end{align*} \]  

(11)

where \( q_{1,i}(t) \) and \( q_{2,i}(t) \) are the \( i \)th generalized coordinate, \( \phi_{1,i}(x) \) and \( \phi_{2,i}(x) \) are the \( i \)th mode function and \( w_{s1} \) and \( w_{s2} \) are the static displacements of two microbeams. Substituting Eq. (11) into Eq. (6) and multiply by the mode shapes \( \phi_{1,i}(x) \) and \( \phi_{2,i}(x) \). Then the resulting equations are integrated from 0 to 1, the reduced-order model are performed as follows:

\[ \begin{align*}
    (1 + \Delta m \phi_{1,i}(x_0) \phi_{1,i}(x_0)) \ddot{q}_{1,i} + c_1 \dot{q}_{1,i} \\
    - \sum_{i=1}^{N_n} \sum_{k,m=1}^{N_n} z_i \int_0^1 \left( q_{1,i} \phi_{1,i}'' + w_{s1}(x) \right) \phi_{1,j} \\
    \int_0^1 \left( w_{s1}(x)^2 + q_{1,i} q_{1,m} \phi_{1,i}' \phi_{1,m}' + 2 q_{1,i} \phi_{1,i}' w_{s1}(x) \right) dx \\
    - N_1 \sum_{i=1}^{N_n} \int_0^1 \left( q_{1,i} \phi_{1,i}' + w_{s1}(x) \right) \phi_{1,i} dx \\
    + \frac{V_2^2}{R} \int \phi_{1,i} H_2(x) dx \\
    - \sum_{i=1}^{N_n} \frac{2 V_2^2}{R} \int \left( q_{1,i} \phi_{1,i} - q_{2,i} \phi_{2,i} + w_{s1}(x) - w_{s2}(x) \right) \phi_{1,i} H_2(x) dx \\
    \int (\phi_{1,i} H_1(x)(1 + 2 w_{s1}(x) + 3 w_{s1}^2(x) + 4 w_{s1}^3(x))) dx \\
    - \sum_{i=1}^{N_n} 2 x_i [V_{dc} + V_{ac} \cos(\Omega t)]^2 \\
    q_{1,i} \int_0^1 \left( \phi_{1,i} \phi_{1,i} H_1(x)(1 + 3 w_{s1}(x) + 6 w_{s1}^2(x)) \right) dx \\
    - 3 x_i [V_{dc} + V_{ac} \cos(\Omega t)]^2 \\
    \sum_{i=1}^{N_n} \sum_{k,l=1}^{N_n} q_{1,i} \int \left( \phi_{1,i} \phi_{1,k} H_1(x)(1 + 4 w_{s1}(x)) \right) dx \\
    - 4 x_i [V_{dc} + V_{ac} \cos(\Omega t)]^2 \\
    \sum_{i=1}^{N_n} \sum_{k,l=1}^{N_n} q_{1,i} q_{1,k} \int \left( \phi_{1,i} \phi_{1,k} \phi_{1,l} H_1(x) \right) dx \\
    + \int w_{s1}^{(4)}(x) \phi_{1,i} dx + \frac{1}{\lambda_1^2} q_{1,i} = 0
\end{align*} \]  

(13)

In order to investigate the validity of the Taylor series expansion, the relative error between the exact force and linearized expansion one is evaluated and plotted in Fig. 2, while assuming that they undergo moderate vibrations with up to 12% of the gap. In Fig. 2a, it can be found that the relative error is less than 0.1%, which provides a good accuracy for the driving force. For the coupling force, the relative error is below 4.7%, which is acceptable, knowing that micromachining defects may induce larger errors and that the nonlinear terms of the coupling force are negligible compared to the actuation force.

3 Numerical simulations

3.1 Static analysis

The deformation of the resonator is composed of static displacement caused by DC voltage and dynamic displacement caused by AC voltage. Excessive voltage will lead the microbeam to hit the electrode and may damage the device [32]. Therefore, it is necessary to ensure that the DC voltage applied to the resonator is less than the pull-in voltage. By neglecting the time derivative, time varying force terms in Eq. (1) and removing hats, the dimensionless static equation can be obtained.

\[ \ddot{q}_{2,i} + c_2 \dot{q}_{2,i} - \sum_{i=1}^{n} \sum_{k,m=1}^{n} z_i \int_0^1 \left( q_{2,i} \phi_{2,i}'' + w_{s2}(x) \right) \\
\phi_{2,i} \int_0^1 \left( w_{s2}''(x)^2 + q_{2,i} q_{2,m} \phi_{2,i}' \phi_{2,m}' + 2 q_{2,i} \phi_{2,i}' w_{s2}(x) \right) dx \\
- N_2 \sum_{i=1}^{n} \int_0^1 \left( q_{2,i} \phi_{2,i}'' + w_{s2}(x) \right) \phi_{2,i} dx \\
- \frac{V_2^2}{R^2} \int \phi_{2,i} H_2(x) dx \\
+ \sum_{i=1}^{n} \frac{2 V_2^2}{R^2} \int_0^1 \left( q_{1,i} \phi_{1,i} - q_{2,i} \phi_{2,i} + w_{s1}(x) - w_{s2}(x) \right) \phi_{2,i} H_2(x) dx \\
+ \delta^2 \int_0^1 w_{s2}''(x) \phi_{2,i} dx + \frac{1}{\lambda_2^2} q_{2,i} = 0
\]  

(12)
While assuming that the microbeams are not subjected to axial load, Eq. (14) can be discretized on the modal basis with three bending modes to ensure convergence [25] using the Galerkin method, transforming the coupled differential equations into a system of nonlinear algebraic equations which have been solved using the asymptotic numerical method (ANM). Also, the finite element model is implemented on COMSOL Multiphysics as shown in Fig. 3. Since the vibrations of the resonators are mainly flexural vibrations, a 2D model is used to simplify the calculation. All the solid parts are meshed with triangle elements. The air gaps between the driving electrode and resonators are also meshed to apply the electrostatic force and defined as moving mesh. The coupling voltage is applied to the 0.2 \textit{l} from the left end of resonators, and the length of coupling electrodes is 0.1 \textit{l}. After that, the clamped–clamped boundary conditions are defined on the end of resonators. Then, the displacements and eigenfrequencies of resonators with variable parameters are calculated numerically as shown in Fig. 4.

Figure 4 shows the static displacement at the middle of microbeam 1, where the solid lines and dotted lines represents the stable and unstable solutions, respectively, and the circles represent the solution obtained by the finite element method. The results of the finite element method and the theoretical result are in good agreement. Clearly, the static displacement increases monotonously and presents a vertical slope at the pull-in voltage equal to 17.49 V. At this actuation level, the resonator touches the electrode and collapses.

3.2 Eigenvalue analysis

By neglecting the nonlinear and damping terms, the equation is simplified to a linear undamped eigenvalue problem. Then, substituting Eq. (11) into Eq. (15) and assuming \( V_{dc} \) equal to 10 V, the linear undamped eigenvalues can be obtained.

\[
\begin{align*}
\frac{\partial^4 w_{1}}{\partial x^4} + \frac{\partial^2 w_{1}}{\partial t^2} &= -\alpha_2 V_{dc}^2 H_{1}(x) \left( \frac{1}{R^2} - \frac{2}{R^3} (w_1 - w_2) \right) \\
&\quad + \alpha_2 V_{dc}^2 H_{1}(x) (1 + 2w_1) \\
\frac{\partial^4 w_2}{\partial x^4} + \frac{\partial^2 w_2}{\partial t^2} &= \frac{\alpha_2 V_{dc}^2 H_{2}(x)}{\delta} \left( \frac{1}{R^2} - \frac{2}{R^3} (w_1 - w_2) \right)
\end{align*}
\]
Figure 5a shows curves of eigenvalue frequency varying with thickness of microbeam 2, which illustrates that the thickness of microbeam 2 will significantly affect the mode veering phenomenon. The $\omega_{ij}$ in Fig. 5 are the frequencies of each mode where $i$ is the number of microbeams going from 1 to 2 and $j$ is the number of structural mode going from 1 to $n$. The variation of the thickness of microbeam 2 directly affects its stiffness; thus, the frequencies of microbeams will change accordingly. When the thicknesses of two microbeams are equal, it can be seen that veering points of same order mode coupling are generated. The increase of the thickness of microbeam 2 results in the increase of its eigenfrequency. While the thickness of microbeam 2 is about 1.956 $\mu$m, the second-order frequency of microbeam 2 is equal to the third-order frequency of microbeam 1, which achieves the different order mode couplings. Then, the influence of DC voltage on frequencies of microbeams has been studied. Figure 5b shows the frequencies of the two microbeams under different DC voltages when $h_2$ is 1.956 $\mu$m. Electrostatic force will produce electrostatic negative stiffness [12], which will affect the frequencies of microbeams. With the increase of DC voltage, the first-order frequencies of two microbeams gradually separate until the pull-in phenomenon occurs. By tuning the DC voltage, the third-order natural frequency of the microbeam 1 and the second-order natural frequency are close to each other when $V_{dc}$ equal to 10.91 V; therefore, the balance state of the mass sensor can be realized.

3.3 Dynamic response

In order to reveal the effects of nonlinearities on the dynamics of the weakly coupled resonators, the frequency responses can be obtained by solving the reduced-order model. For nonlinear ordinary differential equations, several numerical methods have been used to analyze the nonlinear dynamics of resonant MEMS, such as the shooting method [32], longtime integrating method [33] and HBM combined with ANM [31]. For instance, the stable solution in time domain response can be obtained using the Runge–Kutta scheme to integrate the ordinary differential equations until the steady state is reached. However, the appropriate initial value is required to make the longtime integrating method converge to the correct solution while the convergence region near the bifurcation point is very small. Besides, this approach is time-consuming and needs lots of computing resources. The shooting method is reliable and
powerful for obtaining the periodic solution with a low computational cost. However, it cannot capture the unstable solutions without a continuation method. Therefore, we selected harmonic balance method (HBM) combined with asymptotic numerical method (ANM) for solving the dynamic model. This technique solves nonlinear ordinary differential equations in high efficiency [34]. In addition, the longtime integration method (LTI) and the method of multiple scales (MMS) are also used to validate the results obtained by HBM and ANM. Assuming that quality factor $Q$ is 8000, $V_{ac}$ is 0.1 mV and utilizing HBM combined with ANM to solve Eq. (12) and (13), the linear behavior of the resonators can be obtained. The results of LTI and HBM combined with ANM are in good agreement.

Here we choose the maximum amplitudes when the resonators operating in antisymmetric mode and the symmetric mode to be shown in Fig. 6. Figure 6a shows the balance state of the first-order mode when $V_{dc} = 4.41$ V and $h_2 = 0.99$ μm, and Fig. 6b shows the balance state of third order of microbeam 1 coupled with the second order of microbeam 2 when $V_{dc} = 10.91$ V and $h_2 = 1.956$ μm. The characteristics of amplitude–frequency curves show that there are two peaks in the first-order mode coupling and the second–third-order mode coupling, corresponding to out-of-phase mode and in-phase mode, respectively. It can be seen that the amplitude of microbeam 2 is smaller than that of microbeam 1 when the second and third modes are coupled. In the linear case, the frequency at the resonance peak is exactly equal to the mode frequency, and the periodic solutions are totally stable.

Figure 7 shows the nonlinear dynamic response of first-order and different order mode couplings. With the increase of $V_{ac}$, the response amplitudes of microbeams increase, which show stiffness hardening behavior due to the mechanical nonlinearities [35]. With the first-order mode coupling, the dynamic responses of two resonators are equivalent and the amplitude–frequency response peaks are hysteretic. Between the two bifurcation points, the bistability occurs with up to two stable solutions for a given frequency. In Fig. 7b, when third-order mode is coupled with second-order mode, the phase–frequency response indicates that the peaks of two resonators have same hysteretic frequency while the in-phase mode of microbeam 2 has more hardening behavior than that of out-of-phase mode. It is worth to note that the amplitudes of two resonators are different in both the linear and nonlinear behavior. In order to clarify that the reason of the different amplitudes of two microbeams, the analytical solution is performed. As the linear case, the linear governing equations of microbeams can be obtained by neglecting the nonlinear terms.
\[
\begin{aligned}
\frac{\partial^4 w_1}{\partial x^4} + \frac{\partial^2 w_2}{\partial t^2} + c_1 \frac{\partial w_1}{\partial t} &= -x_2 V_c^2 \left( \frac{1}{R^2} - \frac{2}{R^3} (w_1 - w_2) \right) H_2(x) \\
\frac{\partial^2 w_2}{\partial x^2} + \frac{\partial^4 w_2}{\partial t^4} + c_2 \frac{\partial w_2}{\partial t} &= \frac{x_2 V_c^2}{\delta} \left( \frac{1}{R^2} - \frac{2}{R^3} (w_1 - w_2) \right) H_2(x)
\end{aligned}
\]

(16)

For the driving force term, the fluctuation of the driving force caused by the beam deformation can be neglected when driving displacement is small enough compared to the distance between two resonators \((x_a \gg w_1)\). And when \(V_{dc} \gg V_{ac}\), the harmonic \(\cos(2\Omega t)\) can also be neglected. Then, the effective driving force acting on the resonator is:

\[
F = 2x_2 V_{dc} V_{ac} \cos(\Omega t) H_1(x)
\]

(17)

The Galerkin discretization is carried out by substituting Eq. (11) into Eq. (18):

\[
\begin{aligned}
q''_{1,i}(t) + c_1 q'_{1,i}(t) + K_{1i} q_{1,i}(t) + K_{12} q_{2,i}(t) &= F_1 \\
q''_{2,i}(t) + c_2 q'_{2,i}(t) + K_{21} q_{1,i}(t) + K_{22} q_{2,i}(t) &= F_2
\end{aligned}
\]

(18)

Parameters in Eq. (18) are:

\[
K_{ii} = \sum_{i=1}^{N_m} -\frac{2 V_{dc}^2}{R^3} \int_0^1 H_2(x) \phi_i \phi_i dx + \int_0^1 \phi_i^{(4)} \phi_i dx,
\]

\[
K_{12} = \sum_{i=1}^{N_m} -\frac{2 V_{dc}^2}{R^3} \int_0^1 H_2(x) \phi_i \phi_{i,2} dx
\]

(19)

where \(N_m\) is the number of modes. Then, Eq. (18) can be transmitted to the frequency domain by using the Laplace transformation.

\[
\begin{bmatrix}
q_{1,i} \\
q_{2,i}
\end{bmatrix} = \begin{bmatrix}
-\omega^2 + j c_1 \omega + K_{11} & K_{12} \\
\frac{1}{\delta} K_{12} & -\omega^2 + j c_2 \omega + K_{22}
\end{bmatrix}^{-1} \begin{bmatrix}
F_1 \\
F_2
\end{bmatrix}
\]

(20)

Due to \(F_2\) is equal to 0, Eq. (20) can be simplified as:

\[
\begin{bmatrix}
q_{1,i} \\
q_{2,i}
\end{bmatrix} = \det(\Delta)^{-1} \begin{bmatrix}
-\omega^2 + j c_2 \omega + K_{22}
\end{bmatrix} \begin{bmatrix}
F_1
\end{bmatrix}
\]

(21)

From Eq. (21), we found that the main reason for different amplitudes of two microbeams are the differences of thickness and also the mode function.

![Fig. 6 Linear amplitude–frequency and phase–frequency curves of the two resonators with](image-url)

Fig. 6 Linear amplitude–frequency and phase–frequency curves of the two resonators with a first-order coupling and b third–second-order mode coupling when \(V_c = 10 \text{ V}, V_{ac} = 0.1 \text{ mV}\)
3.4 Perturbation analysis

In this section, a second-order MMS [36] is used to obtain approximate analytical solutions of the system and validate the numerical results. Since the resonators are operating in third and second bending modes, respectively, only the corresponding modes are considered in the reduced-order model [third mode for Eq. (12) and second mode for Eq. (13)] and the other modes are neglected in the Galerkin procedure. By introducing a small parameter $\varepsilon$, the reduced-order model can be rewritten as follows:

\[\begin{align*}
q_{1,1} & + \varepsilon^2 c_1 q_{1,1} + k_{1,1}^2 q_{1,1} + \varepsilon k_{12} q_{1,1} \\
& + \varepsilon^2 k_{13} q_{1,1} + \varepsilon^2 k_{13} q_{1,1} + k_{2,2} q_{2,2} + \varepsilon^2 f_1 \cos(\Omega t) = 0 \\
q_{2,2} & + \varepsilon^2 c_2 q_{2,2} + k_{2,1}^2 q_{2,2} + \varepsilon k_{22} q_{2,2} \\
& + \varepsilon^2 k_{23} q_{2,2} + \varepsilon^2 k_{33} q_{2,2} + \varepsilon^2 k_{44} q_{2,2} = 0
\end{align*}\]

(22)

where the parameters in Eqs. (12)–(13) are listed in detail in “Appendix A.” To describe the response near primary resonance and the closed mode configuration, the detuning parameters $\sigma_1$ and $\sigma_2$ are introduced and the relationships between the exciting frequency $\Omega$, the frequencies of microbeam 1 of $\omega_1$ and microbeam 2 of $\omega_2$ can be considered as follows:

\[\begin{align*}
\Omega & = \omega_1 + \varepsilon^2 \sigma_1 \\
\omega_1 & = \omega_2 + \varepsilon^2 \sigma_2
\end{align*}\]

(23)

The three timescales $T_n = \varepsilon^n t$ ($n = 0, 1, 2$) are introduced, and the details about the solving procedure are also given in “Appendix B.” As a result, the following two equations can be derived.

\[a_1 = \sqrt{100a_1^4 + 60a_1^2 a_2^2 k_1^2 (3a_1^2 k_2 + 4k_4 + 8k_2 (\sigma_1 + \sigma_2)) + 9k_1^2 (9a_1^4 k_1^2 + 24a_1^2 k_1 (k_4 - 2k_2 (\sigma_1 + \sigma_2)) + 16 \left( k_4^4 - 4k_2 k_4 (\sigma_1 + \sigma_2) + k_1^2 \left( 3 + 4(\sigma_1 + \sigma_2)^2 \right) \right))}
\]

(24)

\[\left( -\frac{a_1^2 k_{11} c_1 + a_1^2 k_{21} c_2}{a_1^2 k_{13} c_1} \right)^2 + \left( \frac{10a_1^4 k_{12}^2 k_{21}}{k_1^2} + 24a_1^2 k_{11} \sigma_1 + \frac{-10a_1^4 k_{22}^2 k_{21}}{k_2^2} \right)^2 = 1
\]

(25)

where $a_1$ and $a_2$ are the amplitudes of two resonators. Recalling the relationships of frequencies in Eq. (23), the amplitude–frequency characteristic can be obtained. The nonlinear amplitude–frequency curves of the two resonators with third–second-order mode coupling using the method of multiple scales (MMS) and HBM combined with ANM are shown in Fig. 8. It can be found that the periodic solutions obtained by the two methods are in good agreement, which proves that the HBM + ANM can accurately describe the dynamic behavior of the sensor.

3.5 Critical amplitude

The critical amplitude distinguishes the linear and nonlinear range, which means that the unstable solution will appear when the amplitude of resonator exceeds the critical value. Kacem et al. [37] provided a detailed analysis of critical amplitude using a reduced-order model including the influences of mechanical and electrostatic nonlinearities. Compared with the
classical model which only considers the mechanical nonlinearity, the model is more accurate and the results show that the critical amplitude is related to the thickness of microbeam and the quality factor. At the critical driving voltage, the dynamic response curve presents an infinite slope point [35]. Figure 9a shows with the increase of $V_{ac}$, the amplitude of microbeam 1 will gradually reach the critical amplitude and enter the nonlinear region. Moreover, the variation of $V_{ac}$ will break the original balance point, and therefore, the DC voltage $V_{dc}$ needs to be tuned in order to reach a new balance state. The dynamic response at the critical point after adding 1.0 pg mass disturbance is shown in Fig. 9b. Increasing the mass will increase the nonlinearity of out-of-phase mode, making the in-phase mode linear from the critical point. For the out-of-phase mode vibration, the resonators show hardening behavior when the amplitude of microbeam 1 exceeds the critical amplitude while displaying a bifurcation topology transfer [38] between the two microbeams. Therefore, microbeam 2 also shows a hardening behavior, although its vibration is lower than the critical amplitude.

4 Sensitivity analysis

4.1 Sensitivity expression considering the added mass position

As for the different mode couplings, microbeam 1 works in symmetric mode and microbeam 2 works in asymmetric mode, so its detection position has a significant influence on sensitivity. The sensitivity in this paper adopts relative shift of amplitude ratio as output signal.

$$S_a = \frac{\left( \frac{\tilde{w}_2}{w_1} - \frac{\tilde{w}^0_2}{\tilde{w}^0_1} \right)}{\frac{\tilde{w}^0_2}{\tilde{w}^0_1}} / \Delta m$$

(26)

where $\tilde{w}_1$ and $\tilde{w}_2$ are the amplitudes of two microbeams after the mass disturbance; and $\tilde{w}^0_1$ and $\tilde{w}^0_2$ are the amplitudes of equilibrium state of two microbeams. Substituting Eq. (11) into Eq. (26) and neglecting the
static displacement, the sensitivity equation with respect to generalized coordinates can be obtained.

\[ S_a = \left| \frac{\frac{q_2}{q_1} - \frac{q_0}{q_1}}{\Delta m} \right| \]

(27)

From the reduced-order model, we can conclude that only the position where the mass is added will affect the sensitivity. Figure 10 shows the relationship between detection position and sensitivity when the added mass is 0.1 pg, which is not applicable when the amplitude of microbeam 1 or microbeam 2 is 0. As theoretical analysis, the sensitivity and mode functions are basically in line with each other, and when \( \bar{x}_0 = 0.208 l \), the sensitivity reaches maximum. Hence, the mass should be added at the 0.208 times the length of microbeam 1 in order to reach the highest sensitivity.

4.2 Influence of coupling mode on sensitivity

Jaber et al. [39, 40] found that the amplitudes of high-order mode vibration can be increased by changing the electrode distribution to match up to the mode function. Our work realizes the coupling of different modes through distributed driving electrodes and microbeams with different thicknesses. Figure 11 shows the relative shift of amplitude ratio of same mode coupling and different mode couplings when the device working in the linear region. Notably, for the in-phase mode, the sensitivity of second and third mode coupling is 34.25%/pg while the first mode coupling is 1.75%/pg, which means the sensitivity is improved by about 20 times.

4.3 Influence of nonlinear behavior on sensitivity

In [25, 29], the sensitivity is improved by introducing the nonlinear behavior of mass sensor. To study the sensitivity of the proposed mass sensor in the nonlinear regime, we calculate the output of the resonator under different AC voltages while changing the mass from 0 to 1.0 pg. By comparing sub-figures in Fig. 12, we can clearly obtain that the sensitivity is prominently improved when using second and third mode coupling both out-of-phase and in-phase mode. For example, when \( V_{ac} = 5.0 \) mV and the added mass is 1.0 pg, in-phase mode sensitivity of second and third mode coupling is 23.5 times compared with that of first mode coupling. It demonstrates that the sensitivity of the sensor can be further improved by tuning the amplitude of the device beyond the critical amplitude. We also notice that the sensitivity of out-of-phase mode is much lower than that of the in-phase mode vibration. As introduced in [41] and shown Fig. 7b, in-phase mode vibration is localized in stiffer resonator and therefore more sensitive to any mass disturbances. The red dashed lines show critical sensitivity of each case, which is differentiate the linear and nonlinear region. It is worth noting that the sensitivity of the in-phase mode of second and third mode coupling is 70.21%/pg when \( V_{ac} = 5 \) mV, while the sensitivity of the critical point is 39.36%/pg, which demonstrates the possibility of tuning nonlinearities to enhance the sensitivity by a factor equal to 1.78.

\( \text{Fig. 9} \quad \text{a} \) amplitude of out-of-phase mode vibration of microbeam 1 with different \( V_{ac} \), \( \text{b} \) dynamic response of critical point with \( \Delta m = 1 \) pg

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4.4 Influence of coupling strength on sensitivity

For sensors utilizing the mode localization effect, the lower the coupling strength, the higher the sensitivity [18, 42]. For electrostatic coupling, a very weak coupling strength can be achieved by decreasing the coupling voltage, but extremely low coupling voltage will cause mode aliasing and disable the device [43]. Therefore, effective minimum coupling voltage and the influence of coupling voltage on sensitivity are studied. Figure 13a shows the relative shift of amplitude ratio with different coupling voltages $V_c$ of first-order mode coupling and Fig. 13b shows that of second and third mode coupling when $V_{ac} = 1.0$ mV. When the added mass is 1.0 pg, the sensitivity is enhanced by 6.32 times by reducing the coupling voltage from 10.0 to 4.0 V. The sensitivity of second and third mode coupling decreases due to the shift of amplitude becomes small when the coupling voltage is below 4.0 V. Although the decrease of $V_c$ will bring a great improvement of sensitivity, as $V_c$ decreases below 2.0 V, the frequencies of the two modes are closer and closer, resulting in energy loss and reducing the quality factor. Therefore, the sensor will fail due to mode aliasing.

4.5 Influence of driving electrode length on sensitivity

The length of the distributed electrodes will significantly affect the amplitude of the resonator, thus impacting the sensor sensitivity. In this work, we chose the electrode with a length equal to $l/2$ to drive the thinner beam while ensuring mode localization between different modes. Clearly, the sensor shows a highest sensitivity for mode localization between second and third bending modes. Therefore, the influence of the driving electrode length on the sensitivity is studied while assuming a constant length of the coupling electrodes. In order to maintain the characteristic of exciting the asymmetrical mode of the sensor, the length of driving electrode varies from $l/5$ up to $l/2$ for two microbeams. While imposing $V_{ac} = 0.2$ mV to ensure that the sensor is driven beyond its critical amplitude, Fig. 14 shows that, compared to the out-of-phase vibration mode, the in-
phase mode shows higher sensitivity, and hence, it is used as the output metric. As shown in Fig. 14b, the sensitivity reaches the maximum when the length of the driving electrode is equal to $l/3$, which exactly matches its operating mode (the third mode). Also, the numerical simulation results in Fig. 14d show that the sensitivity is maximum when the electrode length for driving the microbeam 2 is $l/2$, which matches well with the operating mode of second order. However, since the amplitudes of the two microbeams are different, the sensitivity is higher when the driving force is applied to microbeam 1.

5 Conclusion

In this paper, the coupling of the second and third modes is introduced to improve the sensitivity of mode-localized mass sensor with distributed electrodes. By using distributed electrodes and resonators of different thicknesses, the asymmetric mode-localized mass sensor can be easily operated in different modes. The Euler–Bernoulli theory is used to establish the mathematical model, and the reduced-order model is obtained according to the Galerkin method, solved by HBM combined with ANM and validated by the LTI method and the method of multiple scales. The critical amplitude is studied to distinguish the linear and nonlinear region of resonators. In particular, the in-phase mode of the asymmetric mode-localized sensor shows obvious hardening behavior as well as higher sensitivity than that of out-of-phase mode in nonlinear case. By comparing with the sensors of first-order mode coupling, the sensitivity of the proposed sensor is enhanced by about 20 times in the linear range. Furthermore, by tuning the driving voltage beyond the critical point to engage in the nonlinear range, the sensitivity is improved by 1.78 times. The influence of coupling strength on sensitivity are investigated in order to avoid mode aliasing. In

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In addition, the effect of the partial driving electrode parameters on the output of sensitivity is analyzed to optimal performance. Notably, by adjusting both the distribution of electrodes and the thickness of the coupling beams, the resonator can be coupled between any adjacent modes. In this paper, only the second and third-order mode coupling is studied. Future work will aim to investigate the coupling of multiple modes of different orders.

**Fig. 13** Relative shift of amplitude ratio with different coupling voltages $V_c$. (a) first mode coupling, (b) second and third mode coupling.

**Fig. 14** Relative shift of amplitude ratio with driving electrode length from $l/5$ to $l/2$ with $V_{ac} = 0.2$ mV.
The parameters in Eq. (22) are defined as:

\[
\begin{align*}
    k_{11} &= \sqrt{x^2} - N_1\int \frac{w_0^2(x)}{\phi_{11} dx - 2\varepsilon^2 V_2^2} \left( \phi_{12}^2 H_1(x) 1 + 3w_2^1(x) + 6w_2^2(x) \right) dx - z_1 \int \left( q_{11} \phi_{11}^2 \right) \phi_{11} \left( w_0^2(x) \right) dx - z_1 \int \left( 2q_{11} \phi_{11}^2 \right) \phi_{11} \left( w_0^2(x) \right) dx \\
    k_{12} &= -z_1 \int \left( q_{11} \phi_{11}^2 \right) \phi_{11} \left( w_0^2(x) \right) dx - \int \left( 2q_{11} \phi_{11}^2 \right) \phi_{11} \left( w_0^2(x) \right) dx - z_1 \int \left( q_{12} \phi_{12}^2 \right) \phi_{12} \left( w_0^2(x) \right) dx - 3z_2 V_2^1 q_{11}^2 \\
    k_{13} &= -z_1 \int \left( q_{11} \phi_{11}^2 \right) \phi_{11} \left( w_0^2(x) \right) dx - 2z_2 V_2^1 \phi_{11} H_1(x) dx \\
    k_{14} &= -\frac{2V_2^2 z_2}{R^2} \int q_{11} \phi_{11} \phi_{11} H_1(x) dx \\
    k_{22} &= \frac{2V_2^2 z_2}{R^2} \int q_{12}^2 \phi_{12}^2 \phi_{12} H_1(x) dx \\
    f_1 &= -2z_2 V_2 V_4 \cos(\Omega t) \left( \phi_{11} H_1(x) \left( 1 + 2w_1(x) + 3w_2^1(x) + 4w_2^2(x) \right) \right) dx \\
    k_{23} &= \sqrt{x^2} \left( 1 + z_1 \int \left( q_{12} \phi_{12}^2 \right) \phi_{12} \left( w_0^2(x) \right) dx - z_1 \int \left( q_{12} \phi_{12}^2 \right) \phi_{12} \left( w_0^2(x) \right) dx - \int \left( 2q_{12} \phi_{12}^2 \right) \phi_{12} \left( w_0^2(x) \right) dx - z_1 \int \left( q_{12} \phi_{12}^2 \right) \phi_{12} \left( w_0^2(x) \right) dx \right) dx \\
    k_{24} &= -\frac{2V_2^2 z_2}{R^2} \int \left( q_{12} \phi_{12}^2 \right) \phi_{12} \left( w_0^2(x) \right) dx \\
    k_{33} &= -\frac{2V_2^2 z_2}{R^2} \int \left( q_{11} \phi_{11}^2 \right) \phi_{11} \left( w_0^2(x) \right) dx \\
    k_{44} &= -\frac{2V_2^2 z_2}{R^2} \int \left( q_{12} \phi_{12}^2 \right) \phi_{12} \left( w_0^2(x) \right) dx \\
\end{align*}
\]

\[ T_n = \varepsilon^n t \quad (\varepsilon < 1) \quad (29) \]

The partial differential of time \( t \) becomes:

\[
\frac{d}{dt} + \frac{\partial}{\partial T_0} + \frac{\partial}{\partial T_1} + \frac{\partial}{\partial T_2} + \cdots = D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \cdots \quad (30)
\]

where \( \sigma_1 \) is the detuning parameter in Eq. (23) and

Appendix B: Analytical solution using the method of multiple scales

The timescales \( T_n \) are introduced depending on the parameter \( \varepsilon \) and defined by:

\[ T_n = \varepsilon^n t \quad (\varepsilon < 1) \]

The partial differential of time \( t \) becomes:

\[
\frac{d}{dt} = \frac{\partial}{\partial T_0} + \frac{\partial}{\partial T_1} + \frac{\partial}{\partial T_2} + \cdots
\]

where \( \sigma_1 \) is the detuning parameter in Eq. (23) and
An asymmetric mode-localized mass sensor

\[ \sigma_2 = \frac{\omega_1 - \omega_2}{\varepsilon^2} \tag{31} \]

The solution of Eq. (22) is assumed by a second-order timescales form as follows:

\[ q_1 = q_{10}(T_0, T_1, T_2) + \varepsilon q_{11}(T_0, T_1, T_2) + \varepsilon^2 q_{12}(T_0, T_1, T_2) \tag{32} \]

\[ q_2 = q_{20}(T_0, T_1, T_2) + \varepsilon q_{21}(T_0, T_1, T_2) + \varepsilon^2 q_{22}(T_0, T_1, T_2) \tag{33} \]

By substituting Eqs. (32) and (33) into Eq. (22), the equations of each order of \( \varepsilon \) can be obtained:

**Order \( \varepsilon^0 \):**

\[ D_0^2 q_{10} + k_{11}^2 q_{10} = 0 \tag{34} \]

\[ D_0^2 q_{20} + k_{21}^2 q_{20} = 0 \tag{35} \]

**Order \( \varepsilon^1 \):**

\[ D_0(D_0 q_{11} + D_1 q_{10}) + D_1 D_0 q_{10} + k_{11}^2 q_{11} = -k_{12} q_{10}^2 \tag{36} \]

\[ D_0(D_0 q_{21} + D_1 q_{20}) + D_1 D_0 q_{20} + k_{21}^2 q_{21} = -k_{22} q_{20}^2 \tag{37} \]

**Order \( \varepsilon^2 \):**

\[ D_0(D_0 q_{12} + D_1 q_{11} + D_2 q_{10}) + D_1(D_0 q_{11} + D_1 q_{10}) + D_2 D_0 q_{10} + k_{11}^2 q_{12} = -c_1 D_0 q_{10} - (k_{12} q_{10} + k_{13} q_{10}) - k_{12} q_{10} q_{11} - f_1 \cos(k_{11} T_0 + \sigma_1 T_2) \tag{38} \]

\[ D_0(D_0 q_{22} + D_1 q_{21} + D_2 q_{20}) + D_1(D_0 q_{21} + D_1 q_{20}) + D_2 D_0 q_{20} + k_{21}^2 q_{22} = -c_2 D_0 q_{20} - (k_{23} q_{20} + k_{24} q_{20}) - k_{23} q_{20} q_{21} \tag{39} \]

The solutions of Eqs. (34) and (35) are given by:

\[ q_{10} = X_1 e^{ik_{11} T_0} + \overline{X_1} e^{-ik_{11} T_0} \tag{40} \]

\[ q_{20} = X_2 e^{ik_{21} T_0} + \overline{X_2} e^{-ik_{21} T_0} \tag{41} \]

where \( X_1 \) and \( X_2 \) are complex functions and \( \overline{X_1} \) and \( \overline{X_2} \) are the complex conjugates. Then, substituting \( q_{10} \) and \( q_{20} \) into Eqs. (36) and (37) we obtain:

\[ D_0^2 q_{11} + k_{11}^2 q_{11} = -e^{-iT k_{11} k_{12} X_1^2} - 2k_{12} \overline{X_1} X_1 \]

\[ -e^{-iT k_{11} k_{12} X_1^2} \]

\[ + 2ie^{-iT k_{11} k_{11} D_2^2 X_1 X_1} \tag{42} \]

\[ D_0^2 q_{21} + k_{21}^2 q_{21} = -e^{-2iT k_{21} k_{22} X_2^2} - 2k_{22} \overline{X_2} X_2 \]

\[ -e^{-2iT k_{21} k_{22} X_2^2} \]

\[ + 2ie^{-2iT k_{21} k_{11} D_2^2 X_2 X_2} \tag{43} \]

Eliminating the secular terms in Eqs. (42) and (43) results in:

\[ D_1 X_1 = D_1 X_2 = 0 \tag{44} \]

And the solutions of Eqs. (38) and (39) are:

\[ q_{11} = \frac{e^{-2iT k_{12} \overline{X_1} X_1} k_{12}}{3k_{11}^2} - \frac{2k_{12} \overline{X_1} X_1}{k_{11}} + \frac{e^{2iT k_{11} k_{12} X_1^2}}{3k_{11}} \tag{45} \]

\[ q_{21} = \frac{e^{-2iT k_{22} \overline{X_2} X_2} k_{22}}{3k_{21}^2} - \frac{2k_{22} \overline{X_2} X_2}{k_{21}} + \frac{e^{2iT k_{21} k_{22} X_2^2}}{3k_{21}} \tag{46} \]

By substituting the solutions of \( q_{11} \) and \( q_{21} \) into Eqs. (38) and (39), the secular terms can be obtained as:

\[ \frac{1}{2} e^{-iT k_{12} \overline{X_1} X_1} f_1 + k_{11} X_1 + ik_{11} c_1 X_1 - \frac{10k_{12}^2 \overline{X_1} X_1^2}{3k_{11}^2} \]

\[ + 3k_{13} \overline{X_1} X_1^2 + k_{22} e^{-iT k_{22} \overline{X_2} X_2} X_2 + 2ik_{11} D_2 X_1 + D_2^2 X_1 = 0 \tag{47} \]

\[ k_{11} c_1 e^{-iT k_{12} \overline{X_1} X_1} + k c_1 X_2 + ik_{21} c_2 X_2 - \frac{10k_{22}^2 \overline{X_2} X_2^2}{3k_{21}^2} \]

\[ + 3k_{23} \overline{X_2} X_2^2 + 2ik_{21} D_2 X_2 + D_2^2 X_2 = 0 \tag{48} \]

Let \( X_1 \) and \( X_2 \) be as follows:

\[ X_1 = \frac{a_1}{2} e^{i \theta_1} \]

\[ X_2 = \frac{a_2}{2} e^{i \theta_2} \tag{49} \]

Substituting Eq. (49) into Eqs. (47) and (48) and multiplying Eq. (47) and Eq. (48) by \( e^{-i \theta_1} \) and \( e^{-i \theta_2} \), respectively, then separating the real part and the imaginary part, the following relationships can be obtained:
\[ \frac{1}{2} \cos(\phi_2) f_1 - \frac{5a_1^2 k_{12}^2}{12k_{11}^2} + \frac{3}{8} a_1^2 k_{13} + \frac{1}{2} a_1 k_{c1} \\
+ \frac{1}{2} a_2 \cos(\phi_1) k_{c2} - a_1 k_{11} \sigma_1 + a_1 k_{11} \phi_2' = 0 \]

(50)

\[ \frac{1}{2} \sin(\phi_2) f_1 - \frac{1}{2} a_2 \sin(\phi_1) k_{c2} \\
+ \frac{1}{2} a_1 k_{11} c_1 + k_{11} a_1' = 0 \]

(51)

\[ -\frac{5a_1^2 k_{22}^2}{12k_{31}^2} + \frac{3}{8} a_1^2 k_{23} + \frac{1}{2} a_1 \cos(\phi_1) k_{c3} \\
+ \frac{1}{2} a_2 k_{c4} - a_2 k_{21} \sigma_1 - a_2 k_{21} \sigma_2 + a_2 k_{21} \phi_1' + a_2 k_{21} \phi_2' = 0 \]

(52)

\[ \frac{1}{2} a_1 \sin(\phi_1) k_{c3} + \frac{1}{2} a_2 k_{21} c_2 + k_{21} a_2' = 0 \]

(53)

where \( \phi_1 \) and \( \phi_2 \) are:

\[ \phi_1 = T_0 e^2 \sigma_2 + \gamma_1 - \gamma_2 \]

\[ \phi_2 = T_0 e^2 \sigma_1 - \gamma_1 \]

(54)

In the steady state, we have the conditions

\[ a_1' = a_2' = \phi_1' = \phi_2' = 0 \]; then, we get

\[ \sin(\phi_1) = -\frac{a_2 k_{21} c_2}{a_1 k_{c3}} \]

(55)

\[ \cos(\phi_1) = \frac{10a_1^2 k_{22} + 3a_1 k_{21}^2 (-3a_2^2 k_{23} - 4k_{c4} + 8k_{21} (\sigma_1 + \sigma_2))}{12a_1 k_{21} k_{c3}} \]

(56)

\[ \cos(\phi_2) = \left( \frac{10a_1^2 k_{22} + 24a_1^2 k_{11} \sigma_1 + \frac{1}{k_{c3}}}{k_{c1}} \right) \left( -\frac{10a_1^2 k_{22} k_{c2}}{k_{21}^2} + 3(3a_2^4 k_{23} k_{c2} - 3a_1^4 k_{13} k_{c3} - 4a_1^2 k_{c1} k_{c3} + 4a_1^2 k_{c2} k_{c4}) \right) \]

\[ -24a_1^2 k_{21} k_{c2} (\sigma_1 + \sigma_2) \]

(58)

From \( \sin(\phi_1)^2 + \cos(\phi_1)^2 = 1 \) and \( \sin(\phi_2)^2 + \cos(\phi_2)^2 = 1 \), Eqs. (24) and (25) can be obtained.

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