Hidden evidence of non-exponential nuclear decay

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Abstract

The framework to describe natural phenomena at their basics being quantum mechanics, there exist a large number of common global phenomena occurring in different branches of natural sciences. One such global phenomenon is spontaneous quantum decay. However, its long time behaviour is experimentally poorly known. Here we show, that by combining two genuine quantum mechanical results, it is possible to infer on this large time behaviour, directly from data. Specifically, we find evidence for non-exponential behaviour of alpha decay of \(^8\)Be at large times from experiments.

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The decay law in quantum mechanics is necessarily non-exponential \([1]\) and deviates from a simple exponential form at small and large times (for reviews see \([2, 3, 4]\)). Recently, some experimental facts regarding the short time behaviour have confirmed the expectations \([5]\). Till today this is not the case for the long time evolution. Although this long tail of the survival probability of an unstable quantum state is more often than not, not directly observable \([6, 7, 8]\) (indeed, we are not aware of any such successful experiment), the information about resonance time evolution, in general, and the long time behaviour, in specific, should as a matter of principle be encoded in the resonant scattering data \(A + a \rightarrow \text{resonance} \rightarrow B + b\). Since energy is a dual variable to time, we can imagine that the survival amplitude

\[
A_\Psi(t) = \langle \Psi | e^{-iHt} | \Psi \rangle = \langle \Psi | \Psi(t) \rangle
\]

(related to the survival probability, \(P_\Psi(t) = |A_\Psi(t)|^2/|A_\Psi(0)|^2\) can be written as a Fourier transform of an energy dependent quantity which can be
constructed on the premises of the scattering matrix, $S$. If this is so and the $S$-matrix itself for a resonant reaction is extractable from the experiment, the information encoded in the scattering allows us to infer on the time evolution of the resonance produced as an intermediate state in the process. To see this point clearly, let us first consider the method for the recasting of the survival amplitude as a Fourier transform of a spectral function. Ever since it was derived by Fock and Krylov [9] it has been the basis for most of the investigations on quantum unstable systems.

Consider a resonance $R^*$ formed as an intermediate unstable state in a scattering process such as $A + a \rightarrow R^* \rightarrow A + a$. Since the unstable state, $|\Psi\rangle$, cannot be an eigenstate of the (hermitian) Hamiltonian, we expand it (assuming a continuous spectrum) in terms of the energy eigenstates of the decay products $A$ and $a$ as follows:

$$|\Psi\rangle = \int dE\ a(E)|E\rangle$$

where $|E\rangle$ is the eigenstate and $E$ the total energy of the system $A + a$. Substituting now for $|\Psi\rangle$ in (1), we get,

$$A_{\Psi}(t) = \int dE' \ dE\ a^*(E') a(E) \langle E'|e^{-iHt}|E\rangle$$

$$= \int dE' \ dE\ a^*(E') a(E) e^{-iEt} \delta(E - E')$$

The proper normalization of $\Psi$ tells us that $|a(E)|^2$ should have dimension of $(1/E)$ and hence can be associated with density of states. In the Fock-Krylov method, $A_{\Psi}(t)$ then reads

$$A_{\Psi}(t) = \int_{E_{\text{th}}}^{\infty} dE\ \rho_{\Psi}(E) e^{-iEt}$$

where $E_{\text{th}}$ is the minimum sum of the masses of the decay products [10]. Since we have expanded the unstable state $|\psi\rangle$ in terms of the eigenstates, the spectral function $\rho_{\Psi}(E)$, is a probability density to find the eigenstates $|E\rangle$ in $|\Psi\rangle$, or, in other words, it is the continuum probability density of states in a resonance. Hence we can write,

$$\rho_{\Psi}(E) = \frac{d\text{Prob}_{\Psi}(E)}{dE} = |\langle E|\Psi\rangle|^2.$$
If we succeed, in a second step, to extract $\rho_\Psi(E)$ from experimental data, we can process this data by equation (4) to conclude, now from experiment, on the long tail of the survival amplitude of the resonance under consideration. This would then be the closest we can ever come to pinning down the large times in a decay directly from experiment albeit by an indirect method.

Theoretically, many different forms of $\rho_\Psi(E)$ are available and many have been used in a rather ad-hoc fashion. The then sometimes specious results, especially with regard to the large times, are not always compatible with each other. Secondly, mostly these are purely theoretical constructs for which it is hard to find a bona fide connection with experiments. In all this, one important result originating from statistical mechanics, seems to have been overlooked, at least in connection with unstable states. In calculating the second virial coefficients $B$ and $C$ for the equation of states in a gas, $pV = RT[1 + B/V + C/V^2 + ...]$, Beth and Uhlenbeck [11] (the derivation of their result is reproduced in [12], see also [13, 14]) found that the difference between the density of states with interaction, $n_l$, and without, $n_l^{(0)}$, is given by the derivative of the scattering phase shift $\delta_l$ as,

$$n_l(k_{CM}) - n_l^{(0)}(k_{CM}) = \frac{2l + 1}{\pi} \frac{d\delta_l(E_{CM})}{dE_{CM}} \tag{6}$$

where $l$ is the angular momentum of the $l^{th}$ partial wave and $k_{CM}$ and $E_{CM}$ are the momentum and energy in the centre-of-mass system of the scattering particles, respectively. If a resonance is formed during the scattering process, $E_{CM}$ becomes the energy of the resonance in its rest frame. Certainly, the density of states and the probability density in (5) are connected. Note that in the absence of interaction, since no resonance can be produced, one would expect (5) to be zero. Switching off the interaction (by, say, letting the coupling constants go to zero), $n_l$ will not become zero, but tend to $n_l^{(0)}$ from above. Therefore, as long as $n_l - n_l^{(0)} \geq 0$ (this is always the case for an isolated resonance), we can write for the continuum probability density of states of the decay products in a resonance,

$$\frac{d\text{Prob}_{\Psi_l}(E_{CM})}{dE_{CM}} = \text{const.} \frac{d\delta_l(E_{CM})}{dE_{CM}} \tag{7}$$

which is the sought after connection between data, here in the form of $\delta_l$, and the survival amplitude in (4). This method is a general (i.e. without any
further restrictions with the exception of our belief in quantum mechanics) feasible tool for studying the time evolution of resonances from data, only if there are no overlapping resonances, i.e., if we handle one isolated resonance. In reality, this is difficult to realize and in most cases overlapping resonances will “distort” $d\delta t/dE_{CM}$, which can then have several maxima and minima, even negative (see Fig.1). These negative regions are bound to appear between resonances as noted by Wigner long ago in a different context [15]. The realistic situation of several overlapping resonances implies that the identification (7) is operative starting from threshold and extending over one resonance region, but often not beyond, as far as real experiments are concerned. However, one very useful feature remains when we restrict ourselves to large times. Large times correspond to small energies, which implies that in order to experimentally extract information on this region, all we need to know is $\delta t$ at threshold and in the vicinity of the resonance. The exact form of how $\rho_\Psi(E)$ falls off at large $E$, well beyond the resonance region, is not important to conclude on the behaviour of $A_\Psi(t)$ as $t \to \infty$. This is the reason why we restrict ourselves to large time behaviour in applying the method of Fock and Krylov with the inclusion of the result of Beth and Uhlenbeck.

To demonstrate the connection between the spectral density and phase shift derivative as in (7), we quote a simple example below. An amplitude which describes the resonant scattering process around the pole is often taken as,

$$T = \frac{\Gamma/2}{E_R - E - i\Gamma/2}, \quad (8)$$

from which one easily gets [16],

$$\frac{d\delta}{dE} = \frac{\Gamma/2}{(E_R - E)^2 + \frac{1}{4}\Gamma^2}, \quad (9)$$

i.e., the Lorentzian (Breit-Wigner) form. The right hand side of (9), up to a constant, is commonly taken as the spectral function in (4) to display the fact that a one-pole approximation as in (8) leads to the exponential decay law [17, 18, 19]. Note that we obtained this spectral function as a derivative of the scattering phase shift. If a resonance, $R^*$, is produced as an intermediate state in the scattering of two particles, $A$ and $a$ as, $A + a \to R^* \to A + a$, then the energy derivative of the scattering phase shift $\delta$ for this reaction,
gives the spectral function for the unstable state or resonance $R^*$. If we compute (9) in the rest frame of the resonance, then the energy $E$ in (9) is the energy available in the centre of mass of the scattering particles $A$ and $a$. In the present work, we use the energy derivative of experimental scattering phase shifts to obtain the spectral function corresponding to the intermediate unstable state in scattering.

Suppose now, that we have a fit to the data of $\delta_l$ with a reasonable scan of the threshold/resonance region which will determine the large time behaviour and which to know is therefore a conditio sine qua non. We said already that in calculating the survival probability for large times, the large energy behaviour of every single resonance, in the situation where several resonances are overlapping is not of importance. We now make this statement more precise gaining as a byproduct more insight into the late time domain. Since the phase shift $\delta_l$ has a threshold behaviour, so will $\rho_{\psi}$. We can parametrize it without loss of generality in the form $\rho_{\psi} \propto (E_{CM} - E_{th.})^{\gamma_l}$ to account for the threshold. Hence we have

\[ \rho_{\psi}(E_{CM}) = G(E_{CM})(E_{CM} - E_{th.})^{\gamma_l}. \]  

We impose the following condition on $G(E_{CM})$: (i) $G(E_{th.}) \neq 0$ since we have factorized the threshold already, (ii) $G(E_{CM}) \to 0$ sufficiently fast as $E_{CM} \to \infty$ (in case of several overlapping resonances this is a theoretical assumption which, however, is inherent in the Fock-Krylov method) and (iii) mathematically, we allow the function to have poles $z_{0i}$ in the complex plane, i.e. $1/G(z_{0i}) = 0$ such that $\Im(z_{0i}) < 0$ and $\Re(z_{0i}) > 0$; since the derivative of the phase shift carries the information about the poles [20, 21] we would expect on physics grounds a single pole at $E_R - i\Gamma/2$ signifying the resonance parameters. This generalizes the simple Breit-Wigner form in (9) which has the pole but no threshold behaviour. These general properties allow us to compute the survival probability by going to the complex plane (though this method of finding the survival probability is standard and known [2, 4], we describe it here as it is not exactly equivalent to that in [2, 4, 22]). We choose the closed path $C_R = C_3 + C_R + C_R^{1/4}$, starting from zero (after change of variables $y = E_{CM} - E_{th}$) along the real axis ($C_R$) attaching to it a quarter of a circle with radius $R$ ($C_R^{1/4}$) in the clockwise direction and completing the path by going upward the imaginary axis up to zero ($C_3$). In the integral we let $R$ go to infinity noting that along $C_R^{1/4}$, the property (ii) and the fact
that $e^{-izt} \propto e^{3mz}$ ($3mz \leq 0$), ensures that the integral is zero. We subtract the contribution along the imaginary axis. This gives

$$A_{\Psi_l}(t) = A_{\Psi_l}^E(t) + A_{\Psi_l}^P(t)$$

(11)

with

$$A_{\Psi_l}^E(t) = e^{-iE_{th}t} \lim_{R \to \infty} \oint_{C_R} dz e^{-izt} z^\gamma G(z + E_{th}) = C_1 e^{-iE_{th}t} e^{-\Gamma/2t}$$

(12)

by Cauchy’s theorem and

$$A_{\Psi_l}^P(t) = C_2 e^{-iE_{th}t} \int_0^\infty dx e^{-xt} x^\gamma G(-ix + E_{th})$$

$$\simeq C_2 e^{-iE_{th}t} G(E_{th}) \Gamma(\gamma + 1) \frac{1}{t^{\gamma+1}}$$

(13)

for the integral along $C_3$, where the approximation is valid for large times $t$. In the above, $\Gamma(x)$ is the Euler’s gamma function and $C_1$ is a constant easily calculable in terms of the parameters $E_R$, $\Gamma$ and $E_{th}$, and $C_2$ is $(-i)^{\gamma+1}$. Equations (12) and (13) show the general features which we would expect: an exponential decay law followed by inverse power law corrections. The latter is independent of the details of $G(E_{CM})$ as claimed, displaying also nicely the dual nature of time and energy. Formulae (12) and (13) substantiate our previous claims about the method to extract information on large time behaviour of resonances even if the data have several overlapping structures. To be mathematically precise, we note that the way we derived (13) is valid only for integer values of the exponent $\gamma$. We refer the reader to [4] for the general case, where, however, a similar result with the survival amplitude being proportional to $1/t^{\gamma+1}$ is obtained for large times.

To make a direct contact with experiment we need experimental data on $\delta_l$ which, parametrized, can be used to extract the survival amplitude. However, the data have to start very close to the threshold. This is not always easy to find in literature, as threshold regions are not the most interesting regions to focus on in an experiment.

We opted for an experiment with many data points at threshold and relatively small error bars. It is the $\alpha-\alpha$ D-wave resonant scattering in nuclear physics [17-21]

$$\alpha + \alpha \to {}^8\text{Be}(2^+) \to \alpha + \alpha.$$
In Fig. 1 we display the phase shift and its derivative over a wide region, using a simple polynomial fit to the phase shift. We find the established $^8$Be levels, shown in the figure. Motivated by a Lorentzian form [28] with an energy dependent width $\Gamma(E_{CM})$, we parametrized the data in the region of the first $2^+$ resonance, in the following form:

$$\delta_l(E_{CM}) = \tan^{-1}\left[\frac{\Gamma(E_{CM})}{E_0 - E_{CM}}\right] e^{-\beta E_{CM}}$$

(14)

with

$$\Gamma(E_{CM}) = \Gamma_0 \left(\frac{E_{CM}^2 - E_{th}^2}{E_0^2 - E_{th}^2}\right)^{\kappa/2}$$

(15)

which is valid for the elastic case. The derivative of this parametrized

![Graph showing D-wave phase shifts and their derivatives](image)

Figure 1: D-wave phase shifts (upper half) in $\alpha$-$\alpha$ elastic scattering from refs [17-21], polynomial fit to these data (solid line) and the derivative of phase shift (lower half) calculated from the fit showing all established $^8$Be($2^+$) levels, as a function of the excitation energy $E_{ex} = E_{CM} - E_{^8Be(groundstate)}$ and plotted here in arbitrary units (arb. units). The negative region in derivative of phase shift (lower half) between 5 – 15 MeV, due to the slowly falling phase shift is not obvious in the plot due to the scale of the vertical axis.
Figure 2: D-wave phase shift (upper half) and its derivative (lower half) in $\alpha$-$\alpha$ elastic scattering as a function of $E_{\text{CM}} - E_{\text{threshold}}$, in the region of the first $2^+$ level of $^8\text{Be}$. The dashed line shows the fit mentioned in the text. The inset displays the accuracy of the fit near the threshold energy region which is crucial for the large time behaviour of the decay law.

The phase shift satisfies our requirements. In particular, $\kappa = 2\gamma + 2$ and the pole is highlighted by a peak structure (see Fig. 2) which is expected [20]. To be unbiased, we fitted $\kappa$, $\beta$, $\Gamma_0$ and $E_0$ simultaneously taking the error bars into consideration. Our best fit gives $\kappa = 6.36$, $\beta = 0.00359\text{ GeV}^{-1}$, $\Gamma_0 = 0.0009\text{ GeV}$ and $E_0 = 7.45838\text{ GeV}$ and is shown also in Fig. 2. In the fitting procedure, special attention was paid to the mandatory threshold. We can take the parametrization, the derivative of the same and perform numerically the integration to obtain the survival amplitude. The numerical result for the survival probability is depicted in Fig. 3 and is our main result. However, since we see our theoretical conditions (i)-(iii) on the spectral function confirmed, it suffices to use the fit together with the mathematics developed in formulae (12) and (13). We then conclude that at large times, the survival probability of the unstable $^8\text{Be}(2^+)$ state at 3.04 MeV excitation
energy, behaves as,

\[ P_{8\text{Be}}(t) \sim \frac{1}{t^{6.36}}. \]  

(16)

Theoretically we would expect that near threshold \([28, 29]\), \(\tan \delta_l(E_{CM}) \sim \kappa_{CM}^{2l+1} \), which implies, \(d\delta_l(E_{CM})/dE_{CM} \sim (E_{CM} - E_{th})^{l-1/2} \) also near threshold. This amounts to saying that \(\kappa\) is expected to be \(2l + 1\). For the \(^8\text{Be}(2^+)\) resonance, one then gets an inverse power law \(t^{-5}\) for the survival probability. The data on the phase shift do not seem to follow the standard threshold behaviour and hence we get (16). The discrepancy, however, does not look serious. Indeed, re-calculated the “\(l\)”-value of the fitted \(\kappa\) is 2.68. Interestingly, the exponent 6.36 is close to the theoretical prediction of 7 for the case of \(l = 2\) made in [2]. The deviation from the exponential decay law starts around 30 lifetimes after the onset of the decay. By this time, one could say

![Figure 3: Survival probability \(P(t)\) of the decay \(^8\text{Be}(2^+) \rightarrow \alpha + \alpha\), as a function of number of lifetimes after decay. \(P(t) = |A_{\Phi l}(t)|^2/|A_{\Phi l}(0)|^2\) is evaluated numerically using \(d\delta_l/dE_{CM}\) of Fig. 2 (which has been obtained from the fit to phase shift data) as the spectral density \(\rho_{\Phi}(E)\) in (4). The dashed line \((e^{-\Gamma t})\) shows that the decay law for the \(^8\text{Be}(2^+)\) state (solid line) is exponential up to about 30 lifetimes after which it proceeds as \(t^{-6.36}\). \(\epsilon_R\) and \(\Gamma\) are the resonance mass and width respectively.

![Diagram showing survival probability and decay law](image-url)
that the sample with which one started has depleted by about 13 orders of magnitude ($\sim e^{-30}$), making a direct measurement of such a phenomenon not feasible. The above is the case of a strong decay with short lifetime. In the case of weak decays, the onset of the non-exponential law at large times is expected to be much later [2, 3] making the direct measurement even less feasible.

In summary, we combined the Fock-Krylov method to calculate the survival amplitude of an unstable state in terms of a Fourier transform of a spectral function with a result in statistical mechanics of Beth-Uhlenbeck which identifies the continuum density of states with the energy derivative of the two body scattering phase shift (6) being proportional to the continuum probability density of states (5). Hence, using experimentally determined phase shifts, the method allowed us to compute the non-exponential long time behaviour of unstable quantum systems directly from data. An explicit example was given and the inverse power law behaviour at large times determined from data. Asked as to why the merger of the two quantum mechanical results, which when combined give insight into the quantum decay, has been overlooked so far, we can only speculate by answering that even the old results by Eisenbud [30], Wigner [15] and Smith [31] concerning the phase shift derivative have been neglected for a long time and came back into vogue only recently [30-34].

We close by quoting from [37]; “Thus it seems unlikely that nuclear decays will show deviations from the exponential decay law which they made famous.” We have shown that it is possible, as the information on the time evolutions is encoded in the scattering data.

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