Complete Minors, Independent Sets, and Chordal Graphs

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Abstract

The Hadwiger number $h(G)$ of a graph $G$ is the maximum size of a complete minor of $G$. Hadwiger’s Conjecture states that $h(G) \geq \chi(G)$. Since $\alpha(G) \chi(G) \geq |V(G)|$, Hadwiger’s Conjecture implies that $\alpha(G) h(G) \geq |V(G)|$. We show that $(2 \alpha(G) - \lceil \log_\tau(\alpha(G)/2) \rceil) h(G) \geq |V(G)|$ where $\tau \approx 6.83$. For graphs with $\alpha(G) \geq 14$, this improves on a recent result of Kawarabayashi and Song who showed $(2 \alpha(G) - 2) h(G) \geq |V(G)|$ when $\alpha(G) \geq 3$.

1 Introduction

Hadwiger’s Conjecture [8] from 1943 states the following (see [10] for a survey):

**Conjecture.** For every $k$-chromatic graph $G$, $K_k$ is a minor of $G$.

Hadwiger’s Conjecture for $k = 4$ was proved by Dirac [5], the case $k = 5$ was shown equivalent to the Four Color Theorem [11, 12, 14] by Wagner [17] and the case $k = 6$ was shown equivalent to the Four Color Theorem by Robertson et al. [15]. Hadwiger’s Conjecture for $k \geq 7$ remains open. Let $h(G)$ denote the Hadwiger number, the size of the largest complete minor of $G$. Since $\alpha(G) \chi(G) \geq |V(G)|$, Hadwiger’s Conjecture implies the following conjecture, which was observed in [7, 12, and 19].

**Conjecture.** For every graph $G$, $\alpha(G) h(G) \geq |V(G)|$.

This conjecture seems weaker than Hadwiger’s Conjecture, however Plummer et al. [13] showed that for graphs with $\alpha(G) = 2$, the two conjectures are equivalent. In 1981, Duchet and Meyniel [7] showed that $(2 \alpha(G) - 1) h(G) \geq |V(G)|$. No general improvement on this theorem has been made for the case $\alpha(G) = 2$. Seymour asked for any improvement on this result for $\alpha(G) = 2$, conjecturing that there exists an $\epsilon > 0$ such that if $\alpha(G) = 2$, then $G$ has a complete minor of size $(1/3 + \epsilon) n$. Recently, Kawarabayashi et al. [9].
showed that \((4\alpha(G) - 3)h(G) \geq 2|V(G)|\) when \(\alpha(G) \geq 3\) and Kawarabayashi and Song \cite{10} improved this to \((2\alpha(G) - 2)h(G) \geq |V(G)|\) when \(\alpha(G) \geq 3\). Wood \cite{18} proved \((2\alpha(G) - 1)(2h(G) - 5) \geq 2|V(G)| - 5\) for all graphs \(G\). Our main result is to improve the bound for graphs with \(\alpha(G) \geq 14\).

**Theorem 1.** Let \(G\) be an \(n\)-vertex graph. Then \(K_{[n/r]}\) is a minor of \(G\), where

\[
r = 2\alpha(G) - \lceil \log_r(\tau\alpha(G)/2) \rceil \quad \text{and} \quad \tau = \frac{2\sqrt{2}}{\sqrt{2} - 1} \approx 6.83.
\]

Using a more careful analysis, we are able to improve the result.

**Theorem 2.** Let \(G\) be an \(n\)-vertex graph with \(\alpha(G) = 5\). Then \(K_{5n/38}\) is a minor of \(G\).

The proof of Theorem \cite{2} appears in the appendix which is posted online \cite{3}.

A graph \(G\) is **perfect** if \(\chi(H) = \omega(H)\) for every induced subgraph \(H\) of \(G\). For two vertex sets \(T, S \subseteq V(G)\), we say \(T\) **-touches** \(S\) if \(T \cap S \neq \emptyset\) or there is an edge \(xy \in E(G)\) with \(x \in T\) and \(y \in S\). For \(T \subseteq V(G)\), we define \(\alpha(T) = \alpha(G[T])\) and \(N(T) = \{x \in V(G) : \exists y \in T, xy \in E(G)\} = \cup_{v \in T} N(v)\). If \(H\) is a subgraph of \(G\) and \(T \subseteq V(G)\), then we define \(H \cap T = G[V(H) \cap T]\). \(H\) is a **spanning subgraph** of \(G\) if \(H\) is a subgraph of \(G\) and \(V(H) = V(G)\). A graph \(G\) is **chordal** if \(G\) has no induced cycle of length at least 4. A vertex is **simplicial** if its neighborhood is a clique. A **simplicial elimination ordering** is an order \(v_n, \ldots, v_1\) in which vertices can be deleted so that each vertex \(v_i\) is a simplicial vertex of the graph induced by \(\{v_1, \ldots, v_i\}\). A **partial simplicial elimination ordering** is an ordered vertex set \(U = \{v_1, \ldots, v_k\} \subseteq V(G)\), such that for each \(v_iv_j \notin E(G)\) with \(i < j\) and \(v_i, v_j \in U\) and each component \(C\) of \(G - \{v_1, \ldots, v_j\}\) at most one of \(v_i\) or \(v_j\) touches \(C\). Dirac \cite{6} proved that a graph is chordal if and only if it has a simplicial elimination ordering, and Berge \cite{4} observed that by greedily coloring the vertices of a simplicial elimination ordering one obtains an \(\omega(G)\)-coloring of \(G\), proving that chordal graphs are perfect.

Let \(f : V(G) \rightarrow \mathbb{Q}^+\) be a weight function on \(V(G)\). For \(A \subseteq V(G)\), define \(f(A) = \sum_{v \in A} f(v)\). Then the **weighted independence number of** \(G\) relative to \(f\) is

\[
\alpha_f(G) = \max \{f(A) : A \text{ is an independent set in } G\}.
\]

We shall need the following result.

**Theorem 3.** Let \(H\) be a perfect graph and \(f\) a weight function on \(V(H)\). Then

\[
\omega(H) \geq \left\lceil \frac{f(V(H))}{\alpha_f(H)} \right\rceil.
\]

The goal of our algorithm is to find a minor \(H\) of \(G\) such that \(H\) is a chordal graph, and then to devise a weight function on the vertices of \(H\) to which we apply Theorem \cite{3}. Most of the time, the weight of a vertex \(v\) in \(H\) is the number of vertices of \(G\) which are contracted to \(v\). The algorithm builds the minor \(H\) by using two operations: extension and breaking. The key property is that at each step, the algorithm uses the operations to increase the number of vertices in a partial simplicial elimination ordering. Once all vertices are included in the partial simplicial elimination ordering, we have a simplicial elimination ordering, so that the algorithm has produced a chordal graph.

In Section 2.2, we provide an algorithm which yields an alternate proof of Kawarabayashi and Song’s \cite{10} result.

**Theorem 4.** Let \(G\) be an \(n\)-vertex graph. Then \(K_{[n/r]}\) is a minor of \(G\), where

\[
r = \begin{cases} 
  n & \alpha(G) = 1, \\
  \frac{n}{2} & \alpha(G) = 2, \\
  \frac{n}{2\alpha(G)-2} & \alpha(G) \geq 3.
\end{cases}
\]

The rest of the paper is organized as follows: in Section 2 we define the operations and define the algorithms, in Section 3 we prove some lemmas and theorems about the operations used during the algorithms, and in Section 4 we analyze the algorithm. In the appendix posted online \cite{3} we specialize the algorithm to \(\alpha(G) = 5\), and by changing the weight function we find a complete minor of size at least \(5n/38\), which is slightly larger than the \(n/8 = 5n/40\) produced by the general algorithm.
2 Definition of the algorithms

The algorithm first builds a family of disjoint vertex sets spanning connected graphs which partition \( V(G) \) and a spanning subgraph of \( G \). We start with the empty family and at each step apply an operation which either adds a new set to the family, adds vertices to an existing set in the family, or updates the spanning subgraph. To identify the spanning subgraph, we color the edges of \( G \): initially all edges are blue and during the algorithm we color some edges red. We denote the spanning subgraph induced on the blue edges by \( G_b \). When we color some edges red, we make sure that each \( T \in \mathcal{F} \) spans a connected graph in \( G_b \). Once we have obtained a partition \( \mathcal{F} \) of \( V(G) \), we define a graph \( H \) by starting with \( G_b \) and contracting each set of \( \mathcal{F} \). We need the spanning subgraph \( G_b \) because starting from \( G \) and contracting each set in \( \mathcal{F} \) might not yield a chordal graph. Throughout this paper, a subscript of \( G \) is implied on \( \alpha \) and \( N \).

2.1 Operations used in the algorithm

There are two operations that are carried out by the algorithm: extending and breaking. We are given a labeled (ordered) family of disjoint vertex sets \( \mathcal{F} \) and a red/blue coloring of the edges of \( G \). Let \( U = V(G) - \cup_{T \in \mathcal{F}} T \), and let \( G_b \) be the spanning subgraph of blue edges. We define the following operations:

Extending \( T \) into \( X \) by \( k \): Let \( T \in \mathcal{F} \), let \( X \subseteq U \) such that \( G_b[X \cup T] \) is connected, there are no red edges between \( T \) and \( X \), and let \( k \in \mathbb{Z}^+ \) such that \( k \leq \alpha(X - N(T)) \). The operation extends \( T \) into \( X \) by \( k \) by adding at most \( 2k \) vertices from \( X \) into \( T \) so that the new \( G_b[T] \) is still connected and we increase \( \alpha(T) \) by at least \( k \). When extending \( T \) into \( X \), the order of the sets in \( \mathcal{F} \) are unchanged. In the extension we always follow the algorithm described in the proof of Lemma 12. Extending \( T \) into \( X \) by \( k \) is always acceptable.

Breaking \( X \) by \( k \): Let \( k \) be a positive integer, and let \( X \subseteq U \) such that \( X \) does not touch \( U-X \) in \( G \) (i.e. \( X \) is a union of components of \( G[U] \)).

Step (a): For any \( T \in \mathcal{F} \) and any component \( D \) of \( G[X] \) with \( \alpha(D - N(T)) = \alpha(D) \), we color all edges between \( T \) and \( D \) red.

Step (b): If there exists a component \( D \) of \( G[X] \) with independence number at least \( k \), let \( I \) be an independent set in \( D \) with \( |I| \geq k \) and let \( v \) be any vertex in \( I \). Add \( T = \{v\} \) to \( \mathcal{F} \) and then extend \( T \) into \( D - T \) by \( k - 1 \). Lemma 13 shows that \( T, D - T, k \) satisfy the conditions in the extension. We then set \( X := X - T \) and continue Step (b) until every component in \( G[X] \) has independence number strictly less than \( k \). The new sets produced are added last in the ordering of \( \mathcal{F} \).

Definition. We say that breaking \( X \) by \( k \) is acceptable if both of the following conditions hold before we start breaking (before Step (a)):

- For all \( T \in \mathcal{F} \) and every component \( D \) of \( G[X] \) either the edges between \( T \) and \( D \) are already red, or \( \alpha(D - N(T)) = \alpha(D) \) (the edges will become red in Step (a)), or \( \alpha(D - N(T)) < k \).
- For every component \( D \) of \( G[X] \), \( \alpha(D) < 2k \).

In other words, an acceptable breaking means each set \( T \) in the original family and each component \( D \) of \( U \) will either have the edges between \( T \) and \( D \) colored red or touch with blue edges every set born during Step (b) in \( D \), and the new sets will touch each other as well.

Definition. We say that \( \mathcal{F} \) is formed by acceptable operations in \( G \) if \( \mathcal{F} \) is formed by starting with the empty family and then performing any sequence of acceptable operations. When we extend \( T \) into \( X \) by \( k \) we say that the amount of the extension is \( k \). For \( T \in \mathcal{F} \), define \( \text{ext}(T) \) to be one plus the total amount of extensions of \( T \), which includes the extensions in the breaking when \( T \) was born and all other extensions of \( T \).

In Theorem 5 we show that we obtain a chordal graph when we start with the graph \( G_b \) and contract each set of the partition.
Theorem 5. Let $\mathcal{F}$ be a partition of $V(G)$ formed by acceptable operations in $G$, and let $G_b$ be the spanning subgraph of blue edges. Let $H$ be the graph obtained by starting from $G_b$ and contracting each set of $\mathcal{F}$ to a single vertex. Then $H$ is a chordal graph.

Lemma 6. Consider a family $\mathcal{F}$ formed by a sequence of operations in $G$. Then for every $T \in \mathcal{F}$, $|T| \leq 2 \text{ext}(T) - 1$. Also, $\text{ext}(T) \leq \alpha(T)$.

An acceptable breaking of $X$ by $k$ requires that for each component $D$ of $G[X]$ and each $T \in \mathcal{F}$ we have $\alpha(D - N(T))$ in the correct range. The following lemma shows that we can control $\alpha(D - N(T))$ by using the extension operation.

Lemma 7. Let $T'$ be the set formed by extending $T$ into $X$ by $k$. Then $\alpha(X - N(T)) - k \geq \alpha(X - T' - N(T'))$. That is, extending $T$ into $X$ by $k$ using the procedure in Lemma 6 reduces $\alpha(X - N(T))$ by at least $k$.

2.2 The $2\alpha(G) - 2$ algorithm

Let $n = |V(G)|$. We are going to build a partition $\mathcal{F}$ of $V(G)$ using only a sequence of breaking operations. At any stage of the algorithm, let $U = V(G) - \cup_{T \in \mathcal{F}} T$.

Case $\alpha(G) = 1$. Note that this conclusion is obvious but we put a detailed argument to make the reader familiar the definitions. The algorithm is to break $V(G)$ by 1. This is an acceptable operation because before the breaking $\mathcal{F}$ is empty and every component of $G$ has independence number 1. Breaking $V(G)$ by 1 does not color any edges red because the family before the breaking is empty, and so the breaking results in a family of singleton sets $\mathcal{F} = \{\{v\} : v \in V(G)\}$ with $G_b = G$. Theorem 5 shows $G$ is chordal, and using the weight function $f(v) = 1$ we have that the total weight is $n$ and $\alpha_f(G) = \alpha(G) = 1$. Thus Theorem 5 shows that $\omega(G) \geq n$.

Case $\alpha(G) = 2$: We first break $V(G)$ by 2. This is acceptable because before the breaking $\mathcal{F}$ is empty and every component of $G$ has independence number at most 2. No edges are colored red, and so this breaking results in a family $\mathcal{F}$ of disjoint induced $P_3$'s ($P_3$ is the unique connected graph on three vertices with independence number 2). This family $\mathcal{F}$ is maximal because the remaining vertices (the set $U$) induce a disjoint union of cliques. We next break $U$ by 1. This is acceptable because each $T \in \mathcal{F}$ dominates $U$ so $\alpha(U - N(T)) = 0$, and each component of $G[U]$ is a clique. Also, no edge is colored red because each $P_3$ in $\mathcal{F}$ dominates $U$. Thus the two breaking operations produce a partition of $V(G)$ into a maximal family of induced $P_3$'s and singleton sets of the remaining vertices, with all edges colored blue ($G_b = G$). We now contract each $P_3$ to form the graph $H$, and use the weight function $f(v) = 3$ for a vertex $v$ obtained by contracting a $P_3$, and $f(v) = 1$ otherwise. Thus $f(v)$ records the number of vertices in the set in $\mathcal{F}$ that is contracted down to $v$, and the total weight $f(V(H)) = n$. Theorem 5 shows $H$ is a chordal graph. To compute $\alpha_f(H)$, take any independent set $I$ in $H$. This independent set corresponds to a pairwise non-touching subfamily $\mathcal{I}$ of $\mathcal{F}$. Since no edges are colored red, $\mathcal{I}$ is pairwise non-touching in $G$. Then either $\mathcal{I}$ contains one $P_3$ and nothing else (in which case $f(I) = 3$) or at most two single vertices (in which case $f(I) = 2$). Thus $\alpha_f(H) \leq 3$ so Theorem 5 shows that $\omega(H) \geq \lceil n/3 \rceil$, that is we have a complete minor of $G$ of size at least $\lceil n/3 \rceil$.

Case $\alpha(G) \geq 3$: Initially, $U = V(G)$ and $\mathcal{F} = \emptyset$.

- Step 1: Let $C$ be any component of $G[U]$. If $\alpha(C)$ is 1 or 2, then we break $C$ like the above two cases. If $\alpha(C) \geq 3$, then we break $C$ by $\alpha(C) - 1$.

- Step 2: We now update $U := U - \cup_{T \in \mathcal{F}} T$ and continue Step 1 with a new $C$ until $U = \emptyset$.

First, all the breakings are acceptable. Consider a component $C$ we are about to break in Step 1. Now consider any set $T$ that has already been produced, say $T$ was born when $C'$ was broke. If $C$ is not contained in $C'$ then there are no edges between $T$ and $C$ so $\alpha(C - N(T)) = \alpha(C)$. If $C \subseteq C'$, then $\alpha(T) = \alpha(C') - 1$ so that $\alpha(C' - N(T)) \leq 1$ which implies that $\alpha(C - N(T)) \leq 1$. Thus breaking $C$ by $\alpha(C) - 1$ is acceptable.
Because $\alpha(C - N(T)) \leq 1$, the only possibility for edges to be colored red in Step 1 is when we choose a component $C$ with $\alpha(C) = 1$. Thus for each $T \in \mathcal{F}$, we have $G[T] = G_b[T]$.

Now consider the graph $H$ formed by starting with $G_b$ and contracting each set of $\mathcal{F}$. Consider the weight function $f$ on $V(H)$ where we assign to each vertex of $H$ the size of the set of $\mathcal{F}$ which it came from. Thus the total weight of $f$ on $H$ is $n$. By Theorem 6 we know that $H$ is a chordal graph.

Next, we show that $\alpha_f(H) \leq 2\alpha(G) - 2$. Consider any independent set $I$ in $H$. This corresponds to a pairwise non-touching (in $G_b$) subfamily $\mathcal{I}$ of $\mathcal{F}$. By Lemma 6, $|I| \leq 2\text{ext}(T) - 1$ so that we can bound the total weight of $I$ as follows:

$$f(I) = \sum_{T \in I} |T| \leq 2 \sum_{T \in I} \text{ext}(T) - |I|.$$ 

If $|I| = 1$ then the largest breaking we ever do is by $\alpha(G) - 1$ which produces sets with $\text{ext}(T) \leq \alpha(G) - 1$ which have size at most $2\alpha(G) - 3$. So $|I| \geq 2$.

**Claim 8.** For any pairwise non-touching family $\mathcal{I}$ in $G_b$, $\sum_{T \in \mathcal{I}} \text{ext}(T) \leq \alpha(G)$.

**Proof.** Define $\mu(\mathcal{I})$ to be the total number of red edges between sets of $\mathcal{I}$. Assume we have a counterexample minimizing $\mu(\mathcal{I})$, i.e., a pairwise non-touching family $\mathcal{I}$ in $G_b$ where $\sum_{T \in \mathcal{I}} \text{ext}(T) > \alpha(G)$ and $\mu(\mathcal{I})$ is minimized. If $\mu(\mathcal{I}) = 0$, then $\mathcal{I}$ is a pairwise non-touching family in $G$ so that $\sum_{T \in \mathcal{I}} \alpha(T) \leq \alpha(G)$. By Lemma 6, $\text{ext}(T) \leq \alpha(T)$ so $\sum_{T \in \mathcal{I}} \text{ext}(T) \leq \alpha(G)$.

Assume now that $\mu(\mathcal{I}) \geq 1$ and take some $T, R \in \mathcal{I}$ where there is a red edge between $T$ and $R$. We will produce a subfamily $\mathcal{I}'$ spanning fewer red edges. As noted above, for edges to be colored red one of $T$ or $R$ must be a single vertex. Assume $|R| = 1$, and let $C$ be the component containing $R$ (with $\alpha(C) = 1$) chosen in Step 1 which caused the edges between $T$ and $R$ to be colored red. Thus $\alpha(C - N(T)) = 1$ so there exists a vertex $v \in V(C) - N(T)$. Let $\mathcal{I}' = \mathcal{I} - R + \{v\}$. Note that since $v \in V(C)$ and $\alpha(C) = 1$, $\{v\} \in \mathcal{F}$. We now show that $v$ does not touch any other set in $\mathcal{I}'$. Say that there exists an $S \in \mathcal{I} \cap \mathcal{I}'$ where $S$ touches $v$ in $G$. Since $v \in V(C) - N(T)$, we must have $T \neq S$. First assume $\text{ext}(S) = 1$, so that $S = \{s\}$ for some vertex $s$. Then since $s$ touches $v$ we must have $s \in V(C)$. Note that when singletons are born, their component must be a clique. But then $s$ touches $R$ using a blue edge, contradicting that $S \in \mathcal{I}$. So we can assume $\text{ext}(S) \geq 2$.

First assume $T$ is indexed lower than $S$, and let $C'$ be the component chosen in Step 1 when $T$ was born. Then $\alpha(T) \geq \text{ext}(T)$ and $\text{ext}(T)$ is one plus the number of extensions during the breaking so $\text{ext}(T) = \alpha(C') - 1$. Since $S$ touches $v$ and $T$ is connected to $v$ by a path of length 2 using a vertex of $C$ we have that $S$ is contained inside $C'$. Since $\alpha(S) \geq \text{ext}(S) \geq 2$ we must have $T$ touching $S$ using blue edges, contradicting that both are in $\mathcal{I}$. Now assume that $S$ is indexed lower than $T$, and let $C'$ be the component chosen in Step 1 when $S$ was born. Then $\alpha(S) \geq \text{ext}(S) = \alpha(C') - 1$ and since $S$ touches $v$ and $T$ is connected to $v$ by a path of length 2 using a vertex of $C$ we have that $T$ is contained inside $C'$. Since $\alpha(T) \geq \text{ext}(T) \geq 2$ we have that $S$ touches $T$ using blue edges, contradicting that both are in $\mathcal{I}$. Thus $v$ does not touch any other set in $\mathcal{I}'$, so $\mathcal{I}'$ is pairwise non-touching in $G_b$ and we have reduced the number of red edges. Also, $\sum_{T \in \mathcal{I}} \text{ext}(T) = \sum_{T \in \mathcal{I}} \text{ext}(T) > \alpha(G)$ contradicting that $\mathcal{I}$ was a minimum counterexample. $\square$

Using Claim 8, we can immediately complete the proof since then $f(I) \leq 2\alpha(G) - |I| \leq 2\alpha(G) - 2$. To summarize, we can find a complete minor of $G$ of size $\lceil r \rceil$, where $r$ is defined as in Theorem 3.

### 2.3 The $2\alpha(G) - \log_2(\tau\alpha(G)/2)$ Algorithm

Given a graph $G$, we use the operations of breaking and extending to produce a partition $\mathcal{F}$ of $V(G)$ and a spanning subgraph $G_b$ of blue edges. When we start the algorithm, $\mathcal{F}$ will be the empty family and $G_b = G$. The improvement from $2\alpha(G) - 2$ to $2\alpha(G) - \log_2(\tau\alpha(G)/2)$ comes from breaking each component $C$ by $\lceil (\alpha(C) + 1)/2 \rceil$ so we produce sets of size approximately $\alpha(C)$, and then we extend the sets of $\mathcal{F}$ before future breakings only if it would prevent the breaking from being acceptable.

Given a graph $G$, with all edges are colored blue and set $\mathcal{F} = \emptyset$. 

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We pick $C$ to be any component of $G[V(G) - \cup_{T \in \mathcal{F}} T]$. If $\alpha(C) = 1$, we break $C$ by 1 which constitutes Step C. So assume $\alpha(C) \geq 2$, and run the following substeps inside $C$, which constitutes Step C.

- **Substep 1:** For each $T \in \mathcal{F}$ with $\alpha(C - N(T)) = \alpha(C)$, color all edges between $T$ and $C$ red. Then let $b = \left\lceil \frac{\alpha(C) + 1}{2} \right\rceil$, and let $A = V(C)$. Partition $\mathcal{F}$ into three classes.
  - $\mathcal{H}_0 = \{T \in \mathcal{F} : \text{all edges between } T \text{ and } C \text{ are colored red}\}$,
  - $\mathcal{H}_1 = \{T \in \mathcal{F} - \mathcal{H}_0 : \alpha(C - N(T)) < \sqrt{2}(b - 1)\}$,
  - $\mathcal{H}_2 = \mathcal{F} - \mathcal{H}_0 - \mathcal{H}_1$.

- **Substep 2:** For any $T \in \mathcal{H}_1$ and any component $D$ of $G[A]$ with $b \leq \alpha(D - N(T)) < \alpha(D)$, we extend $T$ into $V(D)$ by $\alpha(D - N(T)) - b + 1$. We then update $A := A - T$ and continue Substep 2 until no pair $T, D$ satisfies $b \leq \alpha(D - N(T)) < \alpha(D)$. Note, for the first $T$ selected during Substep 2 we will have $D = C$ so that $T$ satisfies $b \leq \alpha(C - N(T)) < \alpha(C)$ and thus is extended.

If there exists some $T \in \mathcal{H}_1$ which was not extended during Substep 2 and some component $D$ of $G[A]$ such that $b \leq \alpha(D - N(T)) = \alpha(D)$ then we do not continue to Substep 3, instead we are finished with Step C. Otherwise, continue to Substep 3.

- **Substep 3:** For any $T \in \mathcal{H}_2$ and any component $D$ of $G[A]$ with $b \leq \alpha(D - N(T)) < \alpha(D)$, we extend $T$ into $V(D)$ by $\alpha(D - N(T)) - b + 1$. We then update $A := A - T$ and continue Step 3 until no pair $T, D$ satisfies $b \leq \alpha(D - N(T)) < \alpha(D)$.

- **Substep 4:** Break $A$ by $b$.

If $\mathcal{F}$ is not yet a partition of $V(G)$, pick a new component $C$.

In Section 4, we prove that using this algorithm we can find a complete minor of $G$ of size $\lceil n/r \rceil$, where $r$ is defined in Theorem 1.

### 3 Analysis of the operations

#### 3.1 Proofs of Theorems 3 and 5

If $V(G) = \{v_1, \ldots, v_n\}$ and $H_1, \ldots, H_n$ are pairwise disjoint graphs, then the **composition** $G[H_1, \ldots, H_n]$ is the graph formed by the vertex disjoint union of $H_1, \ldots, H_n$ plus the edges $xy$ where $x \in V(H_i), y \in V(H_j)$ and $v_iv_j \in E(G)$. In 1972, Lovász [11] proved that a composition of perfect graphs is perfect.

**Proof of Theorem 3** First, we modify $f$ by multiplying each weight by their common denominator so that $f : V(H) \to \mathbb{Z}^+$. Multiplying every weight by a constant does not change $f(V(H))/\alpha_f(H)$. For $v \in V(H)$, define $H_v = f(v)K_1$ to be an independent set of size $f(v)$. Then define $H' = H[H_{v_1}, \ldots, H_{v_n}]$ as a composition of $H$. Then $f(V(H')) = |V(H')|$, $\omega(H) = \omega(H')$, and $\alpha_f(H) = \alpha(H')$. (If $I'$ is a maximal independent set in $H'$, then either $H_v \subseteq I'$ or $H_v \cap I' = \emptyset$ because $H_v$ is an independent set and every vertex in $H_v$ has the same neighborhood.) Since $H'$ is a perfect graph, we have

$$\omega(G) = \omega(H') = \chi(H') \geq \left\lceil \frac{|V(H')|}{\alpha(H')} \right\rceil = \left\lceil \frac{f(V(H))}{\alpha_f(H)} \right\rceil.$$

We say that $\mathcal{F} = \{T_1, \ldots, T_k\}$ is a **partial simplicial elimination ordering** in $G$ if for every non-touching pair $T_i, T_j$ with $i < j$ and for every component $C$ of $G - T_1 - \ldots - T_j$, at most one of $T_i$ or $T_j$ touches $C$. This corresponds exactly to a partial simplicial elimination ordering in the graph obtained by contracting each set of $\mathcal{F}$. We first prove that using acceptable operations we get a partial simplicial elimination ordering in the blue subgraph.
**Theorem 9.** Let $\mathcal{F}_0$ be a partial simplicial elimination ordering in $G$, and let $\mathcal{F}$ be any family formed by starting with $\mathcal{F}_0$ and performing any sequence of acceptable operations. Let $G_b$ be the spanning subgraph of blue edges after the operations. Then $\mathcal{F}$ is a partial simplicial elimination ordering in $G_b$.

**Proof.** Let $G_b$ be the spanning subgraph of $G$ of the blue edges at the end of all operations. We need to prove that after every acceptable operation we have a partial simplicial elimination ordering.

**Lemma 10.** Let $\mathcal{F}$ be a partial simplicial elimination ordering in $G_b$. Let $U$ be the set of vertices $V(G) - \bigcup_{T \in \mathcal{F}} T$, and let $X \subseteq U$ be a union of some components of $G[U]$. Let $k$ be an integer such that breaking $X$ by $k$ is an acceptable operation. Then the family obtained by breaking $X$ by $k$ is a partial simplicial elimination ordering in $G_b$.

**Proof.** Let $\mathcal{F} = \{T_1, \ldots, T_m\}$ be the original family, and let $R_1, \ldots, R_\ell$ be the sets produced when we broke $X$ by $k$. We consider a non-touching pair in $G_b$, and show the pair satisfies the condition for a partial simplicial elimination ordering. We only need to consider pairs which contain at least one $R_i$.

Let $T_i, R_j$ be a non-touching pair in $G_b$. Let $D$ be the component of $G[X]$ containing $R_j$ (then $D$ is also a component of $G[U]$). We first show that all edges between $T_i$ and $D$ are colored red.

First assume $T_i$ does not touch $R_j$ in $G$. Then $k = \alpha(R_j) \leq \alpha(D - N(T_i))$ so by the condition in the definition of acceptable breaking we have $\alpha(D - N(T_i)) = \alpha(D)$ or all edges between $T_i$ and $D$ colored red, thus after the breaking all edges between $T_i$ and $D$ are red.

Now assume the edges between $T_i$ and $R_j$ are red. Then we either had all edges between $T_i$ and $D$ red before the breaking or we colored all edges between $T_i$ and $D$ red during the breaking. Thus we have all the edges between $T_i$ and $D$ colored in red.

Let $C$ be any component of $G_b - T_1 - \ldots - T_m - R_1 - \ldots - R_j$. We want to show that at least one of $T_i$ or $R_j$ does not touch $C$ using blue edges. $C$ is either contained inside $D$ or disjoint from $D$, because $D$ is a component of $G[U] = G - T_1 - \ldots - T_m$ and $C$ is a connected subgraph of $G[U]$. If $C$ is disjoint from $D$ then $R_j$ does not touch $C$ in $G$. If $C$ is contained inside $D$, then $T_i$ does not touch $C$ using blue edges because all edges between $T_i$ and $C$ are red.

Now consider a non-touching pair $R_i, R_j$ in $G_b$ with $i < j$. Assume $R_i$ and $R_j$ are contained in the same component $D$ of $G[X]$. Then since $2k > \alpha(D)$ we must have $R_i$ touching $R_j$ in $G$ so touching in $G_b$ (we only color red edges which have exactly one endpoint in an existing set). Thus we must have $R_i$ and $R_j$ in different components of $G[X]$. So let $C$ be any component of $G_b - T_1 - \ldots - T_m - R_1 - \ldots - R_j$ so that $V(C) \subseteq U$. Then $C[V(C) \cap X]$ is contained inside some component of $G_b[X]$ so $C[V(C) \cap X]$ cannot touch both $R_i$ and $R_j$ in $G_b$. Since there are no edges between $X$ and $U - X$ in $G$, if $R_i$ has no edges to $C[V(C) \cap X]$ then $R_j$ has no edges to $C$ and similarly if $R_j$ has no edges to $C[V(C) \cap X]$. Thus at most one of $R_i$ or $R_j$ touched $C$ using blue edges.

**Lemma 11.** Let $\mathcal{F}$ be a partial simplicial elimination ordering in $G_b$. Let $X \subseteq V(G) - \bigcup_{T \in \mathcal{F}} T$ where $G_b[X]$ is connected, and let $T_i$ be an element of $\mathcal{F}$. Then the family obtained by extending $T_i$ into $X$ is still a partial simplicial elimination ordering in $G_b$.

**Proof.** Let $\mathcal{F} = \{T_1, \ldots, T_m\}$ before the extension, and let $T'$ be the set $T_i$ plus the vertices added during the extension. Now consider a $T_j \in \mathcal{F}$ where $T_j$ does not touch $T'$ in $G_b$, let $\ell = \max\{i, j\}$ and consider any component $C$ of $G - T_1 - \ldots - T_{\ell} - T'$. Let $D$ be the component of $G - T_1 - \ldots - T_{\ell}$ which contains $C$. Because $\mathcal{F}$ is a partial simplicial elimination ordering, at least one of $T_i$ or $T_j$ does not touch $D$ using blue edges. Since $G[X]$ is connected, $X$ is either contained inside $V(D)$ or disjoint from $V(D)$. If $X$ is not contained inside $V(D)$, then $D = C$ and at least one of $T_i$ or $T_j$ does not touch $C$ using blue edges. Extension does not change this, so one of $T'$ or $T_j$ does not touch $C$ using blue edges. If $X$ is contained inside $V(D)$, then $T_j$ touches $D$ using blue edges ($T_i$ touches the new vertices in $T'$ and we only extend using blue edges) so $T_j$ does not touch $D$ using blue edges so does not touch $C$ using blue edges.

Clearly, Lemma 10 and Lemma 11 imply Theorem 9.
**Case 1:** Consider when \( \alpha \) which is trivially a partial simplicial elimination ordering. Then by Theorem 9, \( F \) is a partial simplicial elimination ordering in \( G_b \). Let \( H \) be the graph obtained from \( G_b \) by contracting each \( T_i \in F \) into a single vertex \( v_i \in V(H) \).

We show that \( H \) is chordal by giving a simplicial elimination ordering of \( H \). We order the vertices of \( H \) according to the ordering of the sets of \( F \). For each \( v_i \in V(H) \), define \( B_i = N(v_i) \cap \{v_1, \ldots, v_i\} \). Assume we had \( v_j, v_k \in B_i \) with \( j < k < i \) where \( v_jv_k \notin E(H) \). Let \( D \) be the component of \( G_b - T_1 - \ldots - T_k \) which contains \( T_i \). Then \( T_j \) does not touch \( T_k \) in \( G_b \), so by the condition on partial simplicial elimination ordering one of \( T_j \) or \( T_k \) does not touch \( D \) in \( G_b \). This contradicts that \( v_j, v_k \in B_i \), so \( B_i \) spans a clique. This happens for each \( i \), yielding that \( H \) is a chordal graph. \( \square \)

### 3.2 Some properties of the operations

In this section, let \( G \) be any graph and \( G_b \) any spanning subgraph of \( G \). Let \( T, X \subseteq V(G) \) and \( k \) any integer with \( T \cap X = \emptyset \), \( G_b[X \cup T] \) connected, no red edges between \( T \) and \( X \), and \( k \leq \alpha(X - N(T)) \). (These are the conditions when we extend \( T \) into \( X \) by \( k \) during the algorithm.)

**Lemma 12.** It is possible to extend \( T \) into \( X \) by \( k \) such that \( G_b[T] \) remains connected and \( \alpha(T) \) increases by at least \( k \) and \( |T| \) increases by at most \( 2k \).

**Proof.** Let \( T_0 = T \) so \( T_0 \) is the initial \( T \). We use the following algorithm to produce \( T_0 \subseteq T_1 \subseteq \ldots \subseteq T_k \), where \( \alpha(T_i) \geq \alpha(T_0) + i \) and \( |T_i| \leq |T_0| + 2i \). (Note that we do not define \( T_i \) for every \( i < k \).) Initially, let \( I_0 \) be any maximal independent set in \( G[X - N(T_0)] \) with \( |I_0| \geq k \).

Assume we have defined \( T_i \) and \( I_i \) with \( i < k \). We now show how to define \( T_{i+r} \) and \( I_{i+r} \) for some \( 1 \leq r \leq k - i \).

**Step 1.** Choose \( P \) to be a shortest path in \( G_b[X \cup T_0] \) between \( T_i \) and \( I_i - T_i \). The length of \( P \) is at most three because \( I_i \) is a maximal independent set in \( G_b[X - N(T_0)] \). The algorithm maintains that there are no edges between \( T_i \) and \( I_i - T_i \) when \( i < k \), so the length of \( P \) is at least two.

**Step 2.**

**Case 1:** Consider when \( P = (p_1, p_2, p_3) \) with \( p_1 \in T_i \) and \( p_3 \in I_i - T_i \). Then we add \( p_2 \) and

\[
r = \min \{k - i, |N(p_2) \cap (I_i - T_i)\} \geq 1
\]

vertices from \( N(p_2) \cap (I_i - T_i) \) to \( T_i \) to form \( T_{i+r} \). Let \( I_{i+r} = I_i \). Thus \( \alpha(T_{i+r}) = \alpha(T_i) + r \) and \( |T_{i+r}| = |T_i| + 1 + r \).

**Case 2:** Consider when \( i \leq k - 2 \) and \( P = (p_1, p_2, p_3, p_4) \) with \( p_1 \in T_i \) and \( |N(p_3) \cap (I_i - T_i)| \geq 2 \). Here, we add \( p_2, p_3 \), and

\[
r = \min \{k - i, |N(p_3) \cap (I_i - T_i)\} \geq 2
\]

vertices of \( N(p_3) \cap (I_i - T_i) \) to \( T_i \) to form \( T_{i+r} \). Let \( I_{i+r} \subseteq G[X - T_i - N(T_i)] \) be a maximal independent set containing \( I_i \). Then \( \alpha(T_{i+r}) \geq \alpha(T_i) + r \) and \( |T_{i+r}| = |T_i| + 2 + r \). Since \( r \geq 2 \), the increase in the number of vertices is at most twice the increase of \( i \).
Case 3: Consider when \( P = (p_1, p_2, p_3, p_4) \) with \( p_1 \in T_i \) and \( N(p_3) \cap I_i = \{ p_4 \} \). We set \( I_{i+1} = I_i - \{ p_4 \} + \{ p_3 \} \) and then extend \( I_{i+1} \) to be a maximal independent set in \( G[X - T_i - N(T_i)] \). Then \( I_{i+1} \) is still a maximal independent set of size at least \( k \), and we can now add \( p_2 \) and \( p_3 \) to \( T_i \) to get \( T_{i+1} \). This increases the number of vertices by two and the independence number by one.

Case 4: Consider when \( i = k - 1 \) and \( P = (p_1, p_2, p_3, p_4) \) with \( p_1 \in T_i \) and \( |N(p_3) \cap I_i| \geq 2 \). Here, we add \( p_2 \) and \( p_3 \) to \( T_i \) to get \( T_k \). Let \( I_k = I_i - \{ p_4 \} + \{ p_3 \} \). Thus \( \alpha(T_k) \geq \alpha(T_0) + k \) and \( |T_k| \leq |T_i| + 2 \).

Consider one step which did not produce \( T_k \). If this step is Case 1, then we added the entire set \( N(p_2) \cap I_i \) to \( T_i \). In Case 2, we added the entire set \( N(p_3) \cap I_i \) to \( T_i \). In Case 3, we added the entire \( N(p_2) \cap I_{i+1} \) to \( T_i \). In Case 4, we always produce \( T_k \). Note that we always maintain that there are no edges between \( T_i \) and \( I_i - T_i \) if \( i < k \). If we ever added all vertices of \( I_i \) to \( T_i \), we would have increased \( \alpha(T_i) \) to \( \alpha(T_0) + k \) because \( I_i \subseteq X - N(T_0) \) and \( |I_i| \geq k \). We continue the algorithm if \( i < k \) so we will eventually produce \( T_k \).

Lemma 13. The extension of \( T = \{ v \} \) into \( D - T \) by \( k - 1 \) during Step (b) of a breaking operation satisfies all the conditions of the extension.

Proof. Any edge colored red has at least one endpoint in a set of \( F \), so that \( G_b[D] = G[D] \) where \( G_b \) is the spanning subgraph of blue edges at the time of extension. Since \( D \) is a component of \( G[U] \), we have \( G_b[D] \) connected. Also, since \( T \subseteq V(D) \) we have no red edges between \( T \) and \( D - T \) at the time of extension. Finally, since we chose \( v \in I \) where \( I \) is an independent set of size at least \( k \), we have \( \alpha(D - N(v)) \geq |I - v| \geq k - 1 \).

Proof of Lemma 12. The extending operation does not produce new sets, so the only way to produce a new set is by breaking some set \( X \) by \( k \). In Step (b) of the breaking, we initially have \( |T| = 1 \), and then we extend \( T \) by \( \leq k - 1 \) which adds at most \( 2k - 2 \) new vertices, so \( T \) has at most \( 2k - 1 \) vertices. Since the independence number increased by at least \( k - 1 \), we have \( \alpha(T) \geq k \). Extending \( T \) by \( k \) increases the number of its vertices by at most \( 2k \) and its independence number by at least \( k \). Thus \( |T| \leq 2 \text{ext}(T) - 1 \) and \( \alpha(T) \geq \text{ext}(T) \).

Proof of Lemma 14. Let \( B = G[X - N(T)] \). Assume towards a contradiction that \( \alpha(B) - k + 1 \leq \alpha(X - T' - N(T')) \), and let \( I = I_k \) be the independent set used at the end of the proof of Lemma 12. Then \( \alpha(B) \geq \alpha(B \cap T') + \alpha(B - T' - N(T')) \). Since \( B - T' - N(T') = X - T' - N(T') \), we have \( \alpha(B) \geq \alpha(B \cap T') + \alpha(B - T') - k + 1 \), i.e. \( \alpha(B \cap T') < k \).

But \( |I \cap T'| = k \) because the algorithm added at least \( k \) vertices of \( I_k = I \) to form \( T_k = T' \). Since \( I \subseteq V(B) \), we have \( \alpha(B \cap T') \geq k \), a contradiction.

4 The \( 2\alpha(G) - \log_\tau(\tau \alpha(G)/2) \) algorithm

Definition. Let \( f : \{0\} \cup \mathbb{R}^{\geq 1} \rightarrow \mathbb{R}, \tau \in \mathbb{R} \) satisfy the following properties:

P1: \( f(0) = 0 \),
P2: \( f(4\sqrt{2}) \leq 1 \),
P3: If \( 1 \leq x \in \mathbb{R} \), then \( f(\tau x) \leq 1 + f(x) \),
P4: If \( 1 \leq x, y \in \mathbb{R} \), then \( f(2\sqrt{2}x + 2\sqrt{2}y) \leq f(x) + f(y) \),
P5: If \( 0 \leq x \leq y \in \mathbb{Z} \), then \( f(y) \leq f(x) + y - x \),
P6: If \( 1 \leq x \leq y \in \mathbb{Z} \) and \( 1 \leq r \in \mathbb{R} \), then \( f(ry) \leq f(rx) + y - x \),
P7: \( f \) is non-decreasing so by property P4, if \( x_1, \ldots, x_k \in \mathbb{R} \) then \( f(\sum_i x_i) \leq \sum_i f(x_i) \),
P8: If \( 2 \leq x \in \mathbb{Z} \), then \( \tau(\sqrt{2} - 1)[(x + 1)/2] - \tau\sqrt{2} \geq \sqrt{2}(x - 1) \),
P9: If \( 2 \leq x \in \mathbb{Z} \), then \( f(2\sqrt{2}x) \leq 1 + f(x - [(x + 1)/2]) \).
P10: If \(2 \leq x \in \mathbb{Z}\), then \(\sqrt{2x} \leq \tau \lceil (x-1)/2 \rceil\).

P11: If \(2 \leq x \in \mathbb{Z}\), then \(f(\sqrt{2x}) \leq 2 \lceil (x-1)/2 \rceil\).

We can pick \(f(x) = \lceil \log_{\tau}(x/(4\sqrt{2})) \rceil \) for \(x \geq 1\) with \(\tau = 2\sqrt{\tau}/(\sqrt{2} - 1)\).

The goal of this section is to prove the following which implies our main result:

**Theorem 14.** The algorithm in Section 2.3 produces a complete minor of size \(\lceil n/(2\alpha(G) - f(2\sqrt{2}\alpha(G))) \rceil\).

To prove Theorem 14 we use Theorems 8 and 15, so we need to prove that the algorithm uses acceptable operations and give an upper bound for the weight of an independent set.

**Notation.** Let \(F\) be the partition after the algorithm terminates, and let \(G_b\) be the spanning subgraph of blue edges after the algorithm terminates. Let \(F_C\) be the family before Step \(C\) begins, and define \(A_C = V(C) - \bigcup_{R \in F_C} R\). If \(\alpha(C) > 1\), define \(F^*_i\) to be the family right before Substep \(i\) of Step \(C\), define \(F^*_i\) to be the family after all substeps of Step \(C\) are completed, and let \(A^*_C = V(C) - \bigcup_{R \in F^*_C} R\). For \(T \in F\) and 1 \(\leq i \leq 5\), define

\[
T^{C,i} = \begin{cases} 
S & S \in F^*_i \text{ and } S \subseteq T \text{ if there exists such an } S, \\
\emptyset & \text{otherwise.}
\end{cases}
\]

In other words, \(T^{C,i}\) is the set in \(F^*_i\) that gets extended to \(T\) during the rest of the algorithm.

Define \(H_0(C), H_1(C), \text{ and } H_2(C)\) to be the partition chosen in Substep 1 of Step \(C\).

**Lemma 15.** Let \(T^{C,1} \in H_1(C) \cup H_2(C)\) such that \(T^{C,1}\) was extended during Substep 2 or Substep 3 of Step \(C\). This extension satisfies all the conditions of an extension.

**Proof.** Let \(A\) be the set of vertices not in a set of \(F\) right before the extension, \(D\) the component of \(G[A]\) which \(T^{C,1}\) is extended into and \(G_b\) the spanning subgraph of blue edges at the time of the extension. Since \(T^{C,1} \notin H_0(C)\) there are no red edges between \(T^{C,1}\) and \(C\) and no red edges between \(T^{C,1}\) and \(D\). Since each red edge at the time of extension has at least one endpoint in \(F_C\), we have \(G[V(D)] = G_b[V(D)]\). Since \(D\) is a component of \(G[A]\), we have that \(G_b[V(D)]\) is connected. Since we are extending \(T^{C,1}\) we must have \(\alpha(D - N(T^{C,1})) < \alpha(D)\) implying there exist edges between \(T^{C,1}\) and \(D\). These edges must be blue so \(G_b[V(D)] \cup T^{C,1}\) is connected. Finally, we extend by \(\alpha(D - N(T^{C,1})) - b + 1\) which is smaller than \(\alpha(D - N(T^{C,1}))\) since \(b \geq 1\).

**Lemma 16.** \(F\) is formed by acceptable operations.

**Proof.** Consider Step \(C\). If \(\alpha(C) = 1\), we break \(C\) by 1. This breaking is acceptable because for each \(T \in F_C\) we either have \(\alpha(C - N(T)) = 0 < 1\) or \(\alpha(C - N(T)) = \alpha(C)\). So assume \(\alpha(C) \geq 2\).

The coloring in Substep 1 can be viewed as breaking \(T(C)\) by \(\alpha(C) + 1\). For \(T \in F\), in Step (a) of breaking \(V(C)\) by \(\alpha(C) + 1\) we colors all edges red between \(T\) and \(D\), where \(D\) is a component of \(C\) (i.e. \(D = C\)) with \(\alpha(D - N(T)) = \alpha(D)\). Since we are breaking by \(\alpha(C) + 1\), Step (b) of the breaking does not produce any new sets. This breaking is acceptable since for each \(T \in F\), \(\alpha(C - N(T)) \leq \alpha(C) < \alpha(C) + 1\).

Extensions are always acceptable, so if Step \(C\) does not continue to Substep 3 after Substep 2 then Step \(C\) uses acceptable operations. So assume that Step \(C\) continues to Substep 3 and then consider the breaking in Substep 4. Since \(b = \lceil (\alpha(C) + 1)/2 \rceil\), if any component \(D\) of \(G[A^*_C]\) we have \(\alpha(D) \leq \alpha(C) \leq 2b\). So consider some \(T^{C,1} \in F^*_C\) and some component \(D\) of \(G[A^*_C]\). If \(T^{C,1} \in H_0(C)\), then \(T^{C,1} = T^{C,4}\) and \(\alpha(D - N(T^{C,1})) \leq \alpha(C - N(T^{C,4})) < b\) or all edges between \(T^{C,1}\) and \(D\) are colored red.

If \(T^{C,1} \in H_1(C)\), then we considered extending \(T^{C,1}\) in Substep 2. If we extended \(T^{C,1}\) then by Lemma 7 we must have \(\alpha(D - N(T^{C,1})) < b\). So assume \(T^{C,1} = T^{C,4}\) and that \(b \leq \alpha(D - N(T^{C,1}))\), and let \(D'\) be the component of \(G[A^*_C]\) which contains \(D\). Then \(b \leq \alpha(D' - N(T^{C,1}))\) and since we continued to Substep 3 we must have \(\alpha(D' - N(T^{C,1})) < \alpha(D')\). In this case we should have continued Substep 2 with the pair \(T^{C,1}, D'\). Thus we must have \(\alpha(D - N(T^{C,1})) < b\).
If $T^{C,1} \in H_2(C)$ then we considered extending $T^{C,1}$ in Substep 3. If we extended $T^{C,1}$ then by Lemma \[\Box\] we must have $\alpha(D - N(T^{C,4})) < b$. So assume $T^{C,1} = T^{C,4}$. If $b \leq \alpha(D - N(T^{C,4})) < \alpha(D)$, then we should have continued Substep 3 with the pair $T^{C,1}, D$. Thus either $\alpha(D - N(T^{C,4})) < b$ or $\alpha(D - N(T^{C,4})) = \alpha(D)$, showing that the breaking in Substep 4 is acceptable.

We now need to bound the maximum weight of an independent set. Define the set of independent subfamilies of $\mathcal{F}$ in $G_b$ by

$$\text{IND}_{G_b}(\mathcal{F}) = \{I \subseteq \mathcal{F} : I \text{ is a pairwise non-touching subfamily in } G_b\}.$$ 

Independent subfamilies of $\mathcal{F}$ correspond to independent sets in $H$.

Using the weight function which weights a set with its size, the total weight is $|V(G)|$. Then the total weight of $I \in \text{IND}_{G_b}(\mathcal{F})$ is $\sum_{T \in I} |T|$. Using Lemma \[\Box\] we know that the weight of $I$ is at most $2 \sum_{T \in I} \text{ext}(T) - |I|$. We will give an upper bound of $2\alpha(G) - f(2\sqrt{2}\alpha(G))$ on the weight so that

$$f(2\sqrt{2}\alpha(G)) \leq |I| + 2\alpha(G) - 2 \sum_{T \in I} \text{ext}(T).$$

Note that when we analyzed the Section 2.2 algorithm, we showed that either $|I|$ is at least 2 or that $\alpha(G) - \sum_{T \in I} \text{ext}(T)$ is at least 1. That is, we showed that in order for the total amount of extensions of sets in $I$ to be $\alpha(G)$ we need more than one set.

Consider Figure \[\Box\] where the vertices of the tree are the steps run by the algorithm. Each step of the algorithm corresponds to a component, and the tree is the containment tree of these components. Let $I \in \text{IND}_{G_b}(\mathcal{F})$, with $T, S \in I$. Say that $T$ is born in the step labeled $C_1$ in the figure, and is extended during the steps labeled $C_2, C_3,$ and $C_4$. Assume that $S$ is born in the step labeled $C_1'$ and is extended in the step labeled $C_2'$. We would like to prove by induction on the steps of the algorithm that

$$|\{Q \in I : Q \subseteq V(C)\}| + 2\alpha(C) - 2 \sum \{\text{ext}(Q) : Q \in I, Q \subseteq V(C)\}$$

is large. We are unable to prove this directly because when the induction reaches the Step $C_1$ we must include $\text{ext}(T)$ into the sum for the first time because Step $C_1$ is the first step where $T$ is completely contained inside the component for the step. Instead, we would like our inductive bound for Step $C_2$ to include the amount of extensions of $T$ carried out in steps $C_2$ and $C_3$ which is only part of $\text{ext}(T)$, so that when we reach Step $C_1$ the inductive bounds for the smaller components contained inside $C_1$ already include most of the value $\text{ext}(T)$. So we define a notion of the gap between $\alpha(C)$ and $\sum \{\text{ext}(Q) : Q \in I, Q \subseteq V(C)\}$ which allows us to include only the amount extensions of $T$ into some subset of $V(C)$. Note that since $\mathcal{F}$ is a
partial simplicial elimination ordering, we can have at most one set $T$ which has part of its extensions inside $C$ and part outside $C$. Define for any $T \in \mathcal{I}$

$$\text{ext}(C,T) = \text{the total amount of extensions of } T \text{ into } X \text{ where } X \subseteq V(C),$$

$$\text{gap}(C,\mathcal{I},T) = \alpha(C - N(T^{C,1})) - \text{ext}(C,T) - \sum_{T \in \mathcal{I}, T \cap V(C) \neq \emptyset} \text{ext}(Q).$$

Note that if $T \neq \emptyset$ and $T \cap V(C) \neq \emptyset$ then for each $Q$ in the sum we must have $Q \subseteq V(C)$ because $\mathcal{F}$ is a partial simplicial elimination ordering. Also note that by definition,

$$\text{ext}(C,\emptyset) = 0,$$

$$\text{gap}(C,\mathcal{I},\emptyset) = \alpha(C) - \sum_{T \in \mathcal{I}, T \cap V(C) \neq \emptyset} \text{ext}(T).$$

In the next lemma, we show that $|\mathcal{I}| + 2 \text{ gap}(C,\mathcal{I},T)$ is large by induction on the steps carried out by the algorithm. For comparison with the Section 2.2 algorithm, Claim 8 proves $0 \leq \text{ gap}(G,\mathcal{I},\emptyset)$.

**Lemma 17.** Consider any component $C$ chosen as a Step during the algorithm. Let $\mathcal{I}_0 \in \text{IND}_{G_b}(\mathcal{F})$, and let $\mathcal{I} = \{Q \in \mathcal{I}_0 : Q \cap V(C) \neq \emptyset\}$.

(i) If $\alpha(C) > 1$ and there exists $T \in \mathcal{I}$ with $T^{C,1} \neq \emptyset$ and $T^{C,1} \in \mathcal{H}_1(C)$ then

$$f(\alpha(C - N(T^{C,1}))) \leq |\mathcal{I}| + 2 \text{ gap}(C,\mathcal{I},T) - 1.$$

(ii) If $\alpha(C) > 1$ and there exists $T \in \mathcal{I}$ with $T^{C,1} \neq \emptyset$ and $T^{C,1} \in \mathcal{H}_2(C)$ then

$$f(\sqrt{2}\alpha(C - N(T^{C,1}))) \leq |\mathcal{I}| + 2 \text{ gap}(C,\mathcal{I},T) - 1.$$

(iii) Otherwise,

$$f(2\sqrt{2}\alpha(C)) \leq |\mathcal{I}| + 2 \text{ gap}(C,\mathcal{I},\emptyset).$$

Using Lemma 17 we can prove Theorem 14.

**Proof of Theorem 14.** Let $H$ be the graph formed from $G_b$ by contracting each set of $\mathcal{F}$. Define $g$ to be the weight function on $V(H)$ which assigns to each $v \in V(H)$ the size of the set of $\mathcal{F}$ which contracted to $v$. By Lemma 16 and Theorem 8 $H$ is a perfect graph, so by Theorem 8 we just need to show that $\alpha_g(H) \leq 2\alpha(G) - f(2\sqrt{2}\alpha(G))$.

Let $C_1, \ldots, C_k$ be the components of $G$. Let $I$ be any independent set in $H$, which corresponds to a subfamily $\mathcal{I} \in \text{IND}_{G_b}(\mathcal{F})$. Then define $\mathcal{I}_i = \{T \in \mathcal{I} : T \cap V(C_i) \neq \emptyset\}$. Since $C_1, \ldots, C_k$ are components of $G$ we have $\mathcal{I}_i = \{T \in \mathcal{I} : T \subseteq V(C_i)\}$.

For each $T \in \mathcal{I}_i$, we have $T \subseteq V(C_i)$ which implies $T^{C_i,1} = \emptyset$. Thus we apply the bound in case (iii) of Lemma 17 for each component $C_i$ to obtain

$$\sum_{1 \leq i \leq k} f(2\sqrt{2}\alpha(C_i)) \leq |\mathcal{I}_i| + 2 \sum_{1 \leq i \leq k} \text{ gap}(C_i,\mathcal{I}_i,\emptyset). \quad (1)$$

Expanding the definition of $\text{ gap}(C_i,\mathcal{I}_i,\emptyset)$ in (1) gives,

$$\sum_{1 \leq i \leq k} f(2\sqrt{2}\alpha(C_i)) \leq |\mathcal{I}| + 2 \sum_{1 \leq i \leq k} \alpha(C_i) - 2 \sum_{1 \leq i \leq k} \sum_{T \in \mathcal{I}_i} \text{ ext}(T). \quad (2)$$

Since $\sum \alpha(C_i) = \alpha(G)$ and each $T \in \mathcal{I}$ appears in exactly one $\mathcal{I}_i$, (2) simplifies to

$$\sum_{1 \leq i \leq k} f(2\sqrt{2}\alpha(C_i)) \leq |\mathcal{I}| + 2\alpha(G) - 2 \sum_{T \in \mathcal{I}} \text{ ext}(T). \quad (3)$$
Using Lemma 6 to bound $|T|$ and rearranging 5, we have
\[
g(I) = \sum_{T \in I} |T| \leq 2 \sum_{T \in I} \text{ext}(T) - |I| \leq 2\alpha(G) - \sum_{1 \leq i \leq k} f(2\sqrt{2}\alpha(C_i)).
\]
By property P7,
\[
g(I) \leq 2\alpha(G) - f(2\sqrt{2} \sum_{1 \leq i \leq k} \alpha(C_i)) = 2\alpha(G) - f(2\sqrt{2}\alpha(G)).
\]

Before proving Lemma 17 we need some lemmas:

**Lemma 18.** Let $T^{C,1} \in H_1(C)$ and let $R$ be a set born in Substep 4 of Step $C$. Then $T^{C,4}$ touches $R$ with blue edges.

**Proof.** Since $T^{C,1} \in H_1(C)$, all edges between $T^{C,1}$ and $C$ are colored blue at the start of Step $C$. We produced an $R$ in Substep 4 so $\alpha(C) \geq 2$, so we consider extending $T^{C,1}$ in Substep 2. Since we continued to Substep 3 after Substep 2, we have for each component $D$ of $G[A_{C,1}^{2}]$, $\alpha(D - N(T^{C,3})) < b$. Now consider the component $D'$ of $G[A_{C,1}^{2}]$ which contains $R$. Then there exists a component $D''$ of $G[A_{C,1}^{2}]$ which contains $D'$ so $\alpha(D' - N(T^{C,4})) \leq \alpha(D - N(T^{C,4})) < b$ so no edges incident to $T^{C,4}$ are colored red during Substep 4. Also, since $b \leq \alpha(R)$ by Lemma 6, we must have an edge of $G$ between $T^{C,4}$ and $R$. This edge is colored blue at the end of the algorithm because no edges incident to $T^{C,4}$ are colored red during Substep 4 and all future edge colorings only color edges between an existing set in $\mathcal{F}$ and a vertex of $A$.

**Lemma 19.** Let $T^{C,1} \in H_2(C)$, and assume that $T^{C,1}$ was extended in Substep 3 of Step $C$. Let $R$ be a set born during Substep 4 of Step $C$. Then $T^{C,4}$ touches $R$ with blue edges.

**Proof.** Since $T^{C,1} \in H_2(C)$ and we extended $T^{C,1}$, let $A$ be the subset of vertices of $C$ not yet in $C$ at the time we extend $T^{C,1}$, and let $D$ be the component of $G[A]$ where $b \leq \alpha(D - N(T^{C,1})) < \alpha(D)$. By Lemma 7, $\alpha(D - T^{C,4}) - N(T^{C,4}) < b$. Also, consider any other component $D'$ of $G[A]$ besides $D$. Since $\alpha(C)/2 < b < \alpha(D)$, we must have $\alpha(D') < b$ since $\alpha(D) + \alpha(D') \leq \alpha(C)$. Thus for all components $D'$ of $G[A - T^{C,4}]$ we have $\alpha(D' - N(T^{C,4})) < b$. Now $R$ is connected so $R$ must be contained inside some component $D''$ of $G[A_{C,1}^{2}]$ which is contained inside some component $D'$ of $G[A - T^{C,4}]$. If $T^{C,4}$ does not touch $R$ in $G$, then $\alpha(D - N(T^{C,4})) \geq \alpha(R) \geq b$ which gives a contradiction. If $T^{C,4}$ touches $R$ with red edges, then we must have colored the edges between $T^{C,4}$ and $D''$ red during Substep 4 so $\alpha(D'' - N(T^{C,4})) = \alpha(D'') \geq \alpha(R) \geq b$, again giving a contradiction.

**Lemma 20.** Let $T^{C,1} \in H_3(C) \cup H_2(C)$. Then $\alpha(C - N(T^{C,1})) < \alpha(C)$.

**Proof.** Assume $T^{C,1} \in H_1(C) \cup H_2(C)$. If $\alpha(C - N(T^{C,1})) = \alpha(C)$, then we color all edges between $T$ and $C$ red in Substep 1 of Step $C$. Since all edges are red, $T^{C,1} \in H_0(C)$ which contradicts $T^{C,1} \in H_1(C) \cup H_2(C)$.

We now prove Lemma 17. The proof works by induction on $|V(C)|$, where Step $C$ is a step carried out by the algorithm. Fix an $I_0 \in \text{IND}_{G_0}(\mathcal{F})$ and a component $C$ chosen by the algorithm and consider Step $C$. For the rest of this section, let $D_1, \ldots, D_k$ be the components of $G[A_{C,1}^{2}]$. We can then apply induction into each of the components $D_i$ because at some future time in the algorithm $D_i$ will be selected as a Step. Let $I = \{T \in I_0 : T \cap V(C) \neq \emptyset\}$ and $I_T = \{T \in I : T \cap V(D_i) \neq \emptyset\}$. We can apply induction into $D_i$ with the independent subfamily $I_T$.

If $\alpha(C) = 1$, we need to show $f(2\sqrt{2}) \leq |I| + 2\text{gap}(C, I, \emptyset)$. If $\text{gap}(C, I, T) = 0$ then $|I| = 1$. As $f$ is non-decreasing, property P2 shows $f(2\sqrt{2}) \leq 1$.

Now assume $\alpha(C) > 1$ and consider the possibilities for $T \in I$ with $T^{C,1} \neq \emptyset$. We cannot have two sets $T, R \in I$ with $T^{C,1} \neq \emptyset$ and $R^{C,1} \neq \emptyset$, because this would contradict that $\mathcal{F}$ forms a partial simplicial
elimination ordering since $T$ and $R$ touch $C$ with blue edges ($G_b[T]$ is connected and $T \cap V(C) \neq \emptyset$) but all edges between $T$ and $R$ are red. Also, we cannot have two sets $T, R \in \mathcal{I}$ which were born in Substep 4 of Step $C$ because the sets born in Substep 4 are pairwise touching using blue edges. Thus define $T$ to be the set in $\mathcal{I}$ with $T^{C,1} \neq \emptyset$ if it exists and otherwise define $T = \emptyset$, and define $R$ to be the set in $\mathcal{I}$ which was born in Substep 4 of Step $C$, otherwise $R = \emptyset$. Note that $T \neq \emptyset$ implies that $T \in \mathcal{I}$ so that $T \cap V(C) \neq \emptyset$ which implies there are blue edges between $T$ and $C$ which implies $T^{C,1} \notin \mathcal{H}_0(C)$. Thus if $T \neq \emptyset$ we need to prove the inequality in either case (i) or (ii) of Lemma 17. If $T = \emptyset$ we need to prove the inequality in case (iii) of Lemma 17.

For each $D_i$, at most one of $T$ or $R$ can touch $D_i$ using blue edges. (If both touch $D_i$ using blue edges, then we contradict the partial simplicial elimination ordering.) Define $Q_i$ to be $T$ or $R$ depending on which is contained in $\mathcal{I}_i$, and define $Q_i = \emptyset$ if neither is in $\mathcal{I}_i$. Define

$$
\gamma_i = \begin{cases} 1 & Q_i^{D_i,1} \in \mathcal{H}_1(D_i), \\ \sqrt{2} & Q_i^{D_i,1} \in \mathcal{H}_2(D_i), \\ 2\sqrt{2} & Q_i = \emptyset. \end{cases}
$$

Claim 21.

$$
\sum_{1 \leq i \leq k} f(\gamma_i \alpha(D_i - N(Q_i^{C,5}))) - \sum_{1 \leq i \leq k} 2 \text{gap}(D_i, \mathcal{I}_i, Q_i) \leq |\mathcal{I} - \{T, R\}|.
$$

We actually use Claim 21 in the following form:

$$
\sum_i f(\gamma_i \alpha(C - N(Q_i^{C,5}))) + 2 \text{gap}(C, \mathcal{I}, \emptyset) - 2 \sum_i \text{gap}(D_i, \mathcal{I}_i, \emptyset) \leq |\mathcal{I}| + 2 \text{gap}(C, \mathcal{I}, \emptyset). \quad (4)
$$

Proof. Assume we have indexed the components so that for $1 \leq i \leq h_1$, $Q_i = T$ and for $h_1 < i \leq h_2$, $Q_i = R$ and for $h_2 < i \leq k$, $Q_i = \emptyset$. We consider Step $D_i$. Then $Q_i^{D_i,1}$ is the set in $\mathcal{I}_i$ which will touch $D_i$ in blue and be considered in the statement of Lemma 17 and $\gamma_i$ is the coefficient inside the function $f$. Thus for $1 \leq i \leq h_2$, we obtain

$$
f(\gamma_i \alpha(D_i - N(Q_i^{D_i,1}))) \leq |\mathcal{I}_i| - 1 + 2 \text{gap}(D_i, \mathcal{I}_i, Q_i).
$$

Note that $D_i \cap N(Q_i^{D_i,1}) = D_i \cap N(Q_i^{C,5})$ so we have

$$
f(\gamma_i \alpha(D_i - N(Q_i^{C,5}))) \leq |\mathcal{I}_i| - 1 + 2 \text{gap}(D_i, \mathcal{I}_i, Q_i).
$$

Here, we can think of $|\mathcal{I}_i| - 1$ as counting the number of sets in $\mathcal{I}_i$ besides $Q_i$. For $h_2 < i \leq k$ we obtain

$$f(\gamma_i \alpha(D_i)) \leq |\mathcal{I}_i| + 2 \text{gap}(D_i, \mathcal{I}_i, \emptyset).
$$

Again $|\mathcal{I}_i|$ is counting the sets of $\mathcal{I}_i$ besides $Q_i$. Thus

$$
\sum_{i=1}^{h_2}(|\mathcal{I}_i| - 1) + \sum_{i=h_2+1}^{k} |\mathcal{I}_i| = |\mathcal{I} - \{T, R\}|.
$$

Thus summing the inductive bounds over all $i$ we obtain (for $h_2 < i \leq k$, $Q_i = \emptyset$ so that $\alpha(D_i - N(Q_i^{C,5})) = \alpha(D_i)$)

$$
\sum_{1 \leq i \leq k} f(\gamma_i \alpha(D_i - N(Q_i^{C,5}))) \leq |\mathcal{I} - \{T, R\}| + \sum_{1 \leq i \leq k} 2 \text{gap}(D_i, \mathcal{I}_i, Q_i).
$$

$$\square$$
We finish the proof of Lemma 17 by showing that the inequality in Claim 21 simplifies in all cases to the inequalities in Lemma 17. For the simplification, we add \( \text{gap}(C, I, T) \) to both sides of Claim 21 and then use lower bounds on \( \text{gap}(C, I, T) - \sum_i \text{gap}(D_i, I_i, Q_i) \) and lower bounds on \( \sum_i f(\gamma_i \alpha(D_i - N(Q_i^{C,5})) \)). Define
\[
\theta = \text{ext}(C, T) - \sum_{1 \leq i \leq k} \text{ext}(D_i, T) = \text{the amount of extensions of } T \text{ during Step C},
\]
\[
\lambda = \sum_{1 \leq i \leq k} \alpha(D_i - N(Q_i^{C,5})),
\]
\[
J = \left\{ i : \alpha(D_i - N(Q_i^{C,5})) > 0 \right\}.
\]

We claim the following bounds.

**Bound 1:** If \( T = R = \emptyset \), then \( f(2\sqrt{2} \lambda) \leq \sum_i f(\gamma_i \alpha(D_i - N(Q_i^{C,5}))) \).

**Bound 2:** If \( |J| \geq 2 \), then \( f(2\sqrt{2} \lambda) \leq \sum_i f(\gamma_i \alpha(D_i - N(Q_i^{C,5}))) \).

**Bound 3:** If \( J = \{i\} \), then \( f(\gamma_i \lambda) = \sum_i f(\gamma_i \alpha(D_i - N(Q_i^{C,5}))) \).

**Bound 4:** \( f(\lambda) \leq \sum_i f(\gamma_i \alpha(D_i - N(Q_i^{C,5}))) \).

**Bound 5:** If \( R = \emptyset \), then \( \text{gap}(C, I, T) - \sum_i \text{gap}(D_i, I_i, Q_i) = \alpha(C - N(T^{C,1})) - \theta - \lambda \).

**Bound 6:** If \( R \neq \emptyset \), then \( \text{gap}(C, I, T) - \sum_i \text{gap}(D_i, I_i, Q_i) = \alpha(C - N(T^{C,1})) - b - \lambda \).

For Bound 1, \( T = R = \emptyset \) implies that \( Q_i = \emptyset \) and \( \gamma_i = 2\sqrt{2} \) for all \( i \) so the inequality follows by property P7. Bound 2 follows from property P4 since \( |J| \geq 2 \). Bound 3 is an equality by definition. For Bound 4, property P7 and \( \gamma_i \geq 1 \) imply
\[
f(\lambda) \leq \sum_{1 \leq i \leq k} f(\alpha(D_i - N(Q_i^{C,5}))) \leq \sum_{1 \leq i \leq k} f(\gamma_i \alpha(D_i - N(Q_i^{C,5}))).
\]

Now consider Bound 5 and assume \( R = \emptyset \). First, using the definition of \( Q_i \) we have
\[
\sum_{1 \leq i \leq k} \text{ext}(D_i, Q_i) = \sum_{1 \leq i \leq k} \text{ext}(D_i, T) + \sum_{h_i < i \leq k} \text{ext}(D_i, \emptyset) = \sum_{1 \leq i \leq h_1} \text{ext}(D_i, T).
\]
For \( h_1 < i \leq k \), we have \( Q_i = \emptyset \) implying that \( T \cap V(D_i) = \emptyset \) which implies \( \text{ext}(D_i, T) = 0 \). Thus the equality in (5) expands to
\[
\sum_{1 \leq i \leq k} \text{ext}(D_i, Q_i) = \sum_{1 \leq i \leq k} \text{ext}(D_i, T).
\]

Then expanding the definition of gap,
\[
\sum_{1 \leq i \leq k} \text{gap}(D_i, I_i, Q_i) = \sum_{1 \leq i \leq k} \alpha(D_i - N(Q_i^{C,5})) - \sum_{1 \leq i \leq k} \text{ext}(D_i, Q_i) - \sum_{1 \leq i \leq k} \sum_{W \in I_i, W \neq Q_i} \text{ext}(W)
\]
\[
= \lambda - \sum_{1 \leq i \leq k} \text{ext}(D_i, T) - \sum_{W \in I, W \neq T} \text{ext}(W).
\]

Then expanding the definition of \( \text{gap}(C, I, T) \) and combining with the equality in (7) gives
\[
\text{gap}(C, I, T) - \sum_{1 \leq i \leq k} \text{gap}(D_i, I_i, Q_i) = \alpha(C - N(T^{C,1})) - \text{ext}(T, C) - \sum_{W \in I, W \neq T} \text{ext}(W) - \sum_{1 \leq i \leq k} \text{gap}(D_i, I_i, Q_i)
\]
\[
= \alpha(C - N(T^{C,1})) - \text{ext}(T, C) - \lambda + \sum_{1 \leq i \leq k} \text{ext}(D_i, T)
\]
\[
= \alpha(C - N(T^{C,1})) - \lambda - \theta.
\]
This completes the proof of Bound 5.

Finally, consider Bound 6 and assume \( R \neq \emptyset \). Using the definition of \( Q_i \) we have

\[
\sum_{1 \leq i \leq k} \text{ext}(D_i, Q_i) = \sum_{1 \leq i \leq h_1} \text{ext}(D_i, T) + \sum_{h_1 < i \leq h_2} \text{ext}(D_i, R). \tag{8}
\]

By Lemma 18 and Lemma 19 we did not extend \( T \) during Step C so that

\[
\sum_{1 \leq i \leq h_1} \text{ext}(D_i, T) = \text{ext}(T, C). \tag{9}
\]

Also, if we have an index \( i \) with \( Q_i \neq R \), this implies \( R \cap V(D_i) = \emptyset \) so \( \text{ext}(D_i, R) = 0 \). Combining (8) and (9) gives

\[
\sum_{1 \leq i \leq k} \text{ext}(D_i, Q_i) = \text{ext}(C, T) + \sum_{1 \leq i \leq k} \text{ext}(D_i, R).
\]

Then expanding the definition of gap, we have

\[
\sum_{1 \leq i \leq k} \text{gap}(D_i, \mathcal{I}, Q_i) = \sum_{1 \leq i \leq k} \alpha(D_i - N(Q_i^{C,5})) - \sum_{1 \leq i \leq k} \text{ext}(D_i, Q_i) - \sum_{1 \leq i \leq k} \sum_{W \in \mathcal{I}_i, W \neq Q_i} \text{ext}(W)
= \lambda - \text{ext}(C, T) - \sum_{1 \leq i \leq k} \text{ext}(D_i, R) - \sum_{W \in \mathcal{I}, W \neq R} \text{ext}(W)
= \lambda - \text{ext}(C, T) + \text{ext}(R) - \sum_{1 \leq i \leq k} \text{ext}(D_i, R) - \sum_{W \in \mathcal{I}, W \neq T} \text{ext}(W)
= \lambda - \text{ext}(C, T) + b - \sum_{W \in \mathcal{I}, W \neq T} \text{ext}(W). \tag{10}
\]

The last inequality holds because \( \text{ext}(R) - \sum_i \text{ext}(D_i, R) \) is one plus the number of extensions of \( R \) during Substep 4 of Step C which is \( b \). Then expanding the definition of \( \text{gap}(C, \mathcal{I}, T) \) and combining with the equality in (10) gives

\[
\text{gap}(C, \mathcal{I}, T) - \sum_{1 \leq i \leq k} \text{gap}(D_i, \mathcal{I}_i, Q_i) = \alpha(C - N(T^{C,1})) - \text{ext}(C, T) - \sum_{W \in \mathcal{I}, W \neq T} \text{ext}(W) - \sum_{1 \leq i \leq k} \text{gap}(D_i, \mathcal{I}_i, Q_i)
= \alpha(C - N(T^{C,1})) - \lambda - b.
\]

This finishes the proof of all the bounds.

We now just need to show that in all the different cases, the inequality in Claim 21 simplifies to the inequalities in Lemma 17.

**Case 1.** \( R = T = \emptyset \).

We apply Bounds 1 and 5 to simplify (4)

\[
f(2\sqrt{2}\lambda) + 2\alpha(C) - 2\lambda \leq |\mathcal{I}| + 2\text{gap}(C, \mathcal{I}, \emptyset). \tag{11}
\]

Since \( \lambda \leq \alpha(C) \) we use property P6 with \( r = 2\sqrt{2} \) to obtain

\[
f(2\sqrt{2}\alpha(C)) \leq f(2\sqrt{2}\alpha(C)) + \alpha(C) - \lambda \leq f(2\sqrt{2}\lambda) + 2\alpha(C) - 2\lambda. \tag{12}
\]

Combining (11) with (12) proves the inequality in case (iii) of Lemma 17.

**Case 2.** \( R = \emptyset, T \neq \emptyset, T \) was extended during Step C, and \( |J| \geq 2 \).
We apply Bounds 2 and 5 to simplify (4):
\[ f(2\sqrt{2}\lambda + 2\alpha(C - N(T^{C,1})) - 2\theta - 2\lambda) \leq |I| - 1 + 2\text{gap}(C,I,T). \]
We have \(\alpha(C - N(T^{C,1})) \geq \theta + \lambda\) since \(\lambda = \sum_i \alpha(D_i - N(T^{C,5}))\) and \(\alpha(T)\) increased by at least \(\theta\) during Step C by adding vertices from \(C - N(T^{C,1})\). Also, \(\lambda \geq 1\) since \(J \neq \emptyset\). Thus we can apply property P6 with \(r = 2\sqrt{2}\) to get
\[ f(2\sqrt{2}\lambda + 2\alpha(C - N(T^{C,1})) - 2\theta - 2\lambda) \geq f(2\sqrt{2}\alpha(C - N(T^{C,1})) - 2\sqrt{2}\theta) \geq f(2\sqrt{2}(b - 1)). \]
The last inequality holds because \(\theta \leq \alpha(C - N(T^{C,1})) - b + 1\) and \(f\) is non-decreasing. Now we combine (13) with (14) to obtain
\[ f(2\sqrt{2}(b - 1)) \leq |I| - 1 + 2\text{gap}(C,I,T). \]
Then by definition of \(b\) we have for \(\alpha(C) \geq 2\)
\[ 2\sqrt{2}(b - 1) = 2\sqrt{2} \left( \frac{\alpha(C) - 1}{2} \right) \geq \sqrt{2}(\alpha(C) - 1). \]
Since we extended \(T^{C,1} \in H_1(C) \cup H_2(C)\) so by Lemma 20 \(\alpha(C - N(T^{C,1})) \leq \alpha(C) - 1\). Then since \(f\) is non-decreasing, (13) and (15) simplify to
\[ f(\sqrt{2}\alpha(C - N(T^{C,1}))) \leq f(\sqrt{2}(\alpha(C) - 1)) \leq |I| - 1 + 2\text{gap}(C,I,T). \]
This proves the inequality in case (i) and (ii) of Lemma 17

**Case 3.** \(R = \emptyset, T \neq \emptyset, T\) was extended during Step C, \(J = \{i\}\), and we continued to Substep 3.

Then we use Bounds 3 and 5 to simplify (4):
\[ f(\gamma_i\lambda) + 2\alpha(C - N(T^{C,1})) - 2\theta - 2\lambda \leq |I| - 1 + 2\text{gap}(C,I,T). \]
Since \(J \neq \emptyset\), we have \(\lambda \geq 1\). Also, \(\lambda + \theta \leq \alpha(C - N(T^{C,1}))\) since \(\alpha(T)\) increased by at least \(\theta\) during Step C and \(\lambda = \sum_i \alpha(C - N(T^{C,5}))\). Thus we can apply property P6 with \(r = \gamma_i\) to obtain
\[ f(\gamma_i\lambda) + 2\alpha(C - N(T^{C,1})) - 2\theta - 2\lambda \geq f(\gamma_i\alpha(C - N(T^{C,1})) - \gamma_i\theta) + \alpha(C - N(T^{C,1})) - \theta - \lambda \]
\[ \geq f(\gamma_i(b - 1)) + b - 1 - \lambda. \]
The last inequality holds because \(\theta \leq \alpha(C - N(T^{C,1})) - b + 1\) and \(f\) is non-decreasing. If \(\lambda < b - 1\), then using properties P10 and P3 and \(\gamma_i \geq 1\) we obtain
\[ f(\gamma_i(b - 1)) + b - 1 - \lambda \geq f(b - 1) + 1 \geq f(\tau(b - 1)) \geq f(\sqrt{2}\alpha(C)). \]
Since \(\alpha(C - N(T^{C,1})) \leq \alpha(C)\), we combine (17), (18), and (19) to prove the inequality in case (i) and (ii) of Lemma 17.

Now assume \(\lambda \geq b - 1\). Since \(|J| = 1\) we have \(\lambda = \alpha(D_i - N(Q_i^{C,5})) \leq \alpha(D_i)\) and since we ran Substep 4 we have \(\alpha(D_i) \leq b - 1\) so \(\lambda = b - 1\). We are also forced to have \(\alpha(D_i - N(Q_i^{C,5})) = \alpha(D_i)\). Assume \(Q_i \neq \emptyset\).

Since \(D_i \cap N(Q_i^{C,5}) = D_i \cap N(Q_i^{D,5,1})\) we will color all edges between \(Q_i^{D,5,1}\) and \(D_i\) red in Substep 1 of Step \(D_i\), which implies \(Q_i^{C,5} \cap V(D_i) = \emptyset\). This contradicts that \(Q_i \in I\). Thus \(Q_i = \emptyset\) and so by definition of \(\gamma_i\), we have \(\gamma_i = 2\sqrt{2}\). Then we combine (17) with (18) to obtain (\(\lambda = b - 1\))
\[ f(2\sqrt{2}(b - 1)) \leq |I| - 1 + 2\text{gap}(C,I,T). \]
By the inequality in (16) we have \(2\sqrt{2}(b - 1) \geq \sqrt{2}(\alpha(C) - 1)\). Since \(f\) is non-decreasing, (20) simplifies to
\[ f(\sqrt{2}(\alpha(C) - 1)) \leq |I| - 1 + 2\text{gap}(C,I,T). \]
Since \(T^{C,1} \in H_1(C) \cup H_2(C)\) we have by Lemma 20 that \(\alpha(C - N(T^{C,1})) < \alpha(C)\). Thus we have proved the inequality in case (i) and (ii) of Lemma 17.
\textbf{Case 4.} \( R = \emptyset, \ T \neq \emptyset, \ T \) was extended during Step C, \( J = \{i\}, \) and we did not continue to Substep 3 after Substep 2.

In this case, we must have \( T^{C,1} \in \mathcal{H}_1(C) \) because we did not run Substep 3. We apply Bounds 3 and 5 to simplify (4):

\[
f(\gamma_i \lambda) + 2\alpha(C - N(T^{C,1})) - 2\lambda - 2\theta \leq |I| - 1 + 2 \operatorname{gap}(C, I, T). \tag{21}
\]

First assume that \( \lambda < b - 1 \). We use \( \gamma_i \geq 1 \) and that \( f \) is non-decreasing to simplify (21) to

\[
f(\lambda) + 2\alpha(C - N(T^{C,1})) - 2\lambda - 2\theta \leq |I| - 1 + 2 \operatorname{gap}(C, I, T). \tag{22}
\]

Because \( \alpha(T) \) increased by at least \( \theta \) during Step C, we have \( \lambda + \theta \leq \alpha(C - N(T^{C,1})) \). We use properties P6 (\( J \neq \emptyset \) so \( \lambda \geq 1 \)), P3, and P10 to obtain

\[
f(\lambda) + 2\alpha(C - N(T^{C,1})) - 2\lambda - 2\theta \geq f(\alpha(C - N(T^{C,1})) - \theta) + \alpha(C - N(T^{C,1})) - \lambda - \theta \\
\geq f(b - 1) + b - 1 - \lambda \\
\geq f(b - 1) + 1 \\
\geq f(\tau(b - 1)) \\
\geq f(\sqrt{2}\alpha(C)). \tag{23}
\]

Since \( \alpha(C - N(T^{C,1})) \leq \alpha(C) \), combining (22) with (23) proves the inequality in both case (i) and (ii) of Lemma [17]

Now assume \( \lambda \geq b - 1 \) so that \( \alpha(D_i - N(Q^{C,5}_i)) \geq b - 1 \). Since we did not continue to Substep 3 after Substep 2 we must have some component \( D_j \), and some \( S^{C,1} \in \mathcal{H}_1(C) \) with \( b \leq \alpha(D_j) = \alpha(D_j - N(S^{C,1})) \leq \sqrt{2}(b - 1) \). If \( i = j \) then we have \( \alpha(D_i) \leq \sqrt{2}(b - 1) \) and if \( i \neq j \) then \( \alpha(D_i) \leq \alpha(C) = \alpha(D_j) \leq \alpha(C) - b \leq \sqrt{2}(b - 1) \). Thus

\[
\alpha(D_i - N(Q^{C,5}_i)) \geq b - 1 \geq \frac{1}{\sqrt{2}}\sqrt{2}(b - 1) \geq \frac{\alpha(D_j)}{\sqrt{2}}. \tag{24}
\]

Since

\[
\alpha(D_j) \geq 2 \left\lceil \frac{\alpha(D_j) - 1}{2} \right\rceil,
\]

(21) implies that

\[
\alpha(D_i - N(Q^{D,1}_i)) \geq \frac{\alpha(D_j)}{\sqrt{2}} \geq \sqrt{2} \left( \left\lceil \frac{\alpha(D_i) + 1}{2} \right\rceil - 1 \right).
\]

This shows that either \( Q_i = \emptyset \) or \( Q^{D,1}_i \in \mathcal{H}_0(D_i) \cup \mathcal{H}_2(D_i) \). If \( Q_i \neq \emptyset \) and \( Q^{D,1}_i \in \mathcal{H}_0(D_i) \) then we must have colored all edges between \( Q^{D,1}_i \) and \( D_i \) red in Substep 1 of Step D, which contradicts \( Q_i \cap V(D_i) \neq \emptyset \). Thus either \( Q_i = \emptyset \) so \( \gamma_i = 2\sqrt{2} \) or \( Q_i \neq \emptyset \) and \( Q^{D,1}_i \in \mathcal{H}_2(D_i) \) so that \( \gamma_i = \sqrt{2} \). Then (24) simplifies to

\[
f(\sqrt{2}\lambda) + 2\alpha(C - N(T^{C,1})) - 2\lambda - 2\theta \leq |I| - 1 + 2 \operatorname{gap}(C, I, T). \tag{25}
\]

Because \( \alpha(T) \) increased by at least \( \theta \) during Step C, we have \( \lambda + \theta \leq \alpha(C - N(T^{C,1})) \). Using property P6 (\( J \neq \emptyset \) so \( \lambda \geq 1 \)) we obtain

\[
f(\sqrt{2}\lambda) + 2\alpha(C - N(T^{C,1})) - 2\lambda - 2\theta \geq f(\sqrt{2}\alpha(C - N(T^{C,1})) - \sqrt{2}\theta) \geq f(\sqrt{2}(b - 1)). \tag{26}
\]

Combining (25) with (20) gives

\[
f(\sqrt{2}(b - 1)) \leq |I| - 1 + 2 \operatorname{gap}(C, I, T).
\]

Since \( T^{C,1} \in \mathcal{H}_1(C) \) we have \( \sqrt{2}(b - 1) \geq \alpha(C - N(T^{C,1})) \) so we have proved the bound in case (i) of Lemma [17].

Case 5. \( R = \emptyset , T \neq \emptyset , T \) was extended during Step \( C \), and \( J = \emptyset \).

We apply Bounds 4 and 5 to (31) and then use \( \lambda = 0 \) and property P1 to obtain
\[
2\alpha(C - N(T^{C,1})) - 2\theta \leq |I| - 1 + 2 \text{ gap}(C, I, T).
\]
Then \( \theta \leq \alpha(C - N(T^{C,1})) - b + 1 \) so
\[
2(b - 1) \leq |I| - 1 + 2 \text{ gap}(C, I, T).
\] (27)
By property P11, \( f(\sqrt{2\alpha}(C)) \leq 2(b - 1) \). Since \( \alpha(C - N(T^{C,1})) \leq \alpha(C) \) we have that (27) simplifies to
\[
f(\sqrt{2\alpha}(C - N(T^{C,1}))) \leq |I| - 1 + 2 \text{ gap}(C, I, T).
\]
This proves the inequality in cases (i) and (ii) in Lemma 17.

Case 6. \( R = \emptyset , T \neq \emptyset \) and \( T^{C,1} = T^{C,5} \).

Since \( T^{C,1} = T^{C,5} \), we have \( \theta = 0 \).

We apply Bounds 4 and 5 to simplify (31):
\[
f(\lambda) + 2\alpha(C - N(T^{C,1})) - 2\lambda \leq |I| - 1 + 2 \text{ gap}(C, I, T).
\] (28)
Since \( \lambda \leq \alpha(C - N(T^{C,1})) \), we use property P5 to obtain
\[
f(\lambda) + 2\alpha(C - N(T^{C,1})) - 2\lambda \geq f(\alpha(C - N(T^{C,1}))).
\] (29)
If \( T^{C,1} \in \mathcal{H}_1(C) \), then (20) and (28) prove the inequality in case (i) of Lemma 17.

So assume \( T^{C,1} \in \mathcal{H}_2(C) \). If \( \lambda < \alpha(C - N(T^{C,1})) \) we can apply property P5 and P3 to obtain
\[
f(\lambda) + 2\alpha(C - N(T^{C,1})) - 2\lambda \geq f(\alpha(C - N(T^{C,1}))) + 1 \geq f(\tau\alpha(C - N(T^{C,1}))).
\] (30)
Because \( \tau \geq \sqrt{2} \), we can combine (30) with (29) to prove the inequality in case (ii) of Lemma 17.

So assume \( \lambda = \alpha(C - N(T^{C,1})) \). Since \( T^{C,1} \in \mathcal{H}_2(C) \) we have \( \lambda = \alpha(C - N(T^{C,1})) \geq \sqrt{2}(b - 1) > \alpha(C)/2 \). Then \( |J| \geq 2 \) since each component of \( G[A_2^5] \) has independence number at most \( \alpha(C)/2 \) and \( \lambda > \alpha(C)/2 \).

Using \( |J| \geq 2 \) we can apply Bounds 2 and 5 to Claim 21 to get
\[
f(2\sqrt{2}\lambda) \leq |I| - 1 + 2 \text{ gap}(C, I, T).
\]
Since \( \lambda = \alpha(C - N(T^{C,1})) \), we have proved the bound in case (ii) of Lemma 17.

Case 7. \( R \neq \emptyset \) and \( T \neq \emptyset \).

First, \( T^{C,1} \in \mathcal{H}_2(C) \) by Lemma 18 and \( T^{C,4} = T^{C,1} \) by Lemma 19.

Then we apply Bounds 4 and 6 to simplify (31):
\[
f(\lambda) + 2\alpha(C - N(T^{C,1})) - 2\lambda - 2b \leq |I| - 2 + 2 \text{ gap}(C, I, T).
\] (31)
Because \( b \leq \alpha(R^{C,5}) \) and \( Q_i \) is \( T \) or \( R \) we have \( \cup_i(D_i - N(Q_i^{C,5})) \cup R^{C,5} \subseteq \alpha(C - N(T^{C,1})) \) so that \( \lambda + b \leq \alpha(C - N(T^{C,1})) \). (Note that \( T^{C,1} \notin \mathcal{H}_0(C) \) so there are no red edges between \( T^{C,1} \) and \( R^{C,5} \).) Thus we use property P5 to obtain
\[
f(\lambda) + 2\alpha(C - N(T^{C,1})) - 2\lambda - 2b \geq f(\alpha(C - N(T^{C,1})) - b).
\] (32)
Combining (31) with (32) and adding 1 to both sides we obtain
\[
f(\alpha(C - N(T^{C,1})) - b) + 1 \leq |I| - 1 + 2 \text{ gap}(C, I, T).
\] (33)
Since \( T^{C,1} \in H_2(C) \), \( \alpha(C-N(T^{C,1})) \geq \sqrt{2}(b-1) \) so \( \alpha(C-N(T^{C,1})) - b \geq \sqrt{2}(b-1) - b = (\sqrt{2} - 1)b - \sqrt{2} \). Thus using that \( f \) is non-decreasing and property P3 and P8, we obtain
\[
f(\alpha(C - N(T^{C,1}))) - b + 1 \geq f((\sqrt{2} - 1)b - \sqrt{2}) + 1 \geq f(\tau(\sqrt{2} - 1)b - \tau \sqrt{2})) \geq f(\sqrt{2}(\alpha(C) - 1)). \tag{34}
\]
Then combining (33) with (34) we obtain
\[
f(\sqrt{2}(\alpha(C) - 1)) \leq |I| - 1 + 2 \text{ gap}(C,I,T). \tag{35}
\]
Since \( T^{C,1} \in H_2(C) \), by Lemma 20 we have \( \alpha(C - N(T^{C,1})) < \alpha(C) \) so we have proved the inequality in case (ii) of Lemma 17.

Case 8. \( R \neq \emptyset \) and \( T = \emptyset \).

Using \( T = \emptyset \) and \( \theta = 0 \), we apply Bounds 4 and 6 to simplify (31)
\[
f(\lambda) + 2\alpha(C) - 2\lambda - 2b \leq |I| - 1 + 2 \text{ gap}(C,I,\emptyset). \tag{36}
\]
Since \( \lambda = \sum_i \alpha(D_i - N(R^{C,5})) \) and \( b \leq \alpha(R^{C,5}) \) we have \( \lambda + b \leq \alpha(C) \). We use property P5 to obtain
\[
f(\alpha(C) - b) \leq f(\alpha(C) - b) + \alpha(C) - b - \lambda \leq f(\lambda) + 2\alpha(C) - 2\lambda - 2b. \tag{37}
\]
We then combine (35) with (36) and add 1 to both sides to obtain
\[
f(\alpha(C) - b) + 1 \leq |I| + 2 \text{ gap}(C,I,T). \tag{38}
\]
Then property P9 shows \( f(2\sqrt{2}\alpha(C)) \leq f(\alpha(C) - b) + 1 \) so we have proved the inequality in case (iii) of Lemma 17.

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A The $\alpha(G) = 5$ algorithm

Let $G$ be a graph with $\alpha(G) = 5$. At any stage of the algorithm, let $U$ be the set of vertices of $G$ not yet added to any set in $\mathcal{F}$. Initially, $U = |V(G)|$.

- Step 1: Let $\mathcal{F}$ be a maximal family of pairwise touching connected sets, with $\alpha(T) \leq 2$ and $|T| \leq 2\alpha(T) - 1$ for each $T \in \mathcal{F}$. We consider such a family with the maximum size, that is the maximum value $|\mathcal{F}|$. Set $U = V(G) - \cup_{T \in \mathcal{F}} T$.

- Step 2: For each $T \in \mathcal{F}$ with $|T| = 1$, we extend $T$ into $U$ by 1 if $T$ touches $U$ and $T$ does not dominate $U$. We then repeat Step 2 until we have tried to extend every set.

There are now three cases. Case I is when $G[U]$ has no component with independence number 5 but has a component with independence number 4. Case II is when $G[U]$ has no component with independence number 4 or 5. Case III is when $G[U]$ is connected and has independence number 5. We run different steps in the three cases.

Here are the steps in Case I:

- Step I.3: For any $T \in \mathcal{F}$ and any component $C$ of $G[U]$ with $\alpha(C) = 4$ and $\alpha(C - N(T)) = 3$, we extend $T$ into $C$ by 1. We then update $U$ and continue Step I.3 until no pair $T, C$ satisfies the condition.

- Step I.4: Break $U$ by 3.

- Step I.5: Break $U$ by 2.

- Step I.6: Break $U$ by 1.

Here are the steps in Case II:
• Step II.3: For any $T \in \mathcal{F}$ and any component $C$ of $G[U]$ with $\alpha(C) = 3$ and $\alpha(C - N(T)) = 2$, we extend $T$ into $C$ by 1. We then update $U$ and continue Step II.3 until no pair $T, C$ satisfies the condition.

• Step II.4: Break $U$ by 2.

• Step II.5: Break $U$ by 1.

Here are the steps for Case III:

• Step III.3: Break $U$ by 4.

• Step III.4: For any $T \in \mathcal{F}$ with $\alpha(T) = 2$ and any component $C$ of $G[U]$ with $\alpha(C) = 3$ and $\alpha(C - N(T)) = 2$, we extend $T$ into $C$ by 1. We then update $U$ and continue Step III.4 until no pair $T, C$ satisfies the condition.

• Step III.5: Break $U$ by 2.

• Step III.6: Break $U$ by 1.

We claim that using this algorithm, we can find a complete minor of $G$ of size $\frac{5n}{3}$.

We set up some notation for the sets in the family at different stages of the algorithm. Let $\mathcal{F}$ be the family at the end of the algorithm, and let $G_b$ be the spanning subgraph of blue edges at the end of the algorithm. Let $H$ be the graph obtained from $G_b$ by contracting each set in $\mathcal{F}$. For each $T \in \mathcal{F}$ we use $T$ to denote both the set in $V(G)$ and the vertex of $H$ obtained by contracting $T$.

Let

$$\mathcal{F}(s,a) = \{T : T \text{ is a set in the final family, } T \text{ was first added during step } s, \text{ext}(T) = a\},$$

where $s \in \{1, I.3, \ldots, I.6, II.3, \ldots, II.5, III.3, \ldots, III.6\}$.

Define

$$\mathcal{F}(s) = \cup_a \mathcal{F}(s,a).$$

Note that we do not include the original sets which were added to the family in step $s$, but include the final configuration of the set which includes the original plus any extensions that were made. Define $\mathcal{F}_s$ to be the family right after Step $s$. Let $U_s$ be the set of vertices not yet added into any set in $\mathcal{F}$ at the end of Step $s$. Set $n = |V(G)|$ and let $\lambda n$ be the size of the largest complete minor of $G$.

**Claim 1.** $\mathcal{F}(1)$ is a pairwise touching family in $G_b$.

**Proof.** $\mathcal{F}(1)$ is pairwise touching so trivially is a partial simplicial elimination ordering. Then all breakings are acceptable so that by Theorem 9 $\mathcal{F}$ is a partial simplicial elimination ordering. Then we form a simplicial elimination ordering of the vertices of $H$, similarly to the proof of Theorem 5. □

**Claim 2.** $\mathcal{F}$ is a partial simplicial elimination ordering in $G_b$, so $H$ is a chordal graph.

**Proof.** $\mathcal{F}$ is pairwise touching so trivially is a partial simplicial elimination ordering. Then all breakings are acceptable so that by Theorem 9 $\mathcal{F}$ is a partial simplicial elimination ordering. Then we form a simplicial elimination ordering of the vertices of $H$, similarly to the proof of Theorem 5. □

**Claim 3.** Let $\mathcal{I} \in \text{IND}_{G_b}(\mathcal{F})$. Then $\sum_{T \in \mathcal{I}} \text{ext}(T) \leq 5$.

**Proof.** Can be checked by case analysis. □

**Claim 4.** In the Case III algorithm (even if $U_2$ is not connected or $U_3$ has independence number less than 5), each $T \in \mathcal{F}(1,3)$ touches every set with extension number at least 2 using blue edges.

**Proof.** Consider a $T \in \mathcal{F}(1,3)$. Then $T$ touches every set in $\mathcal{F}(1)$ by Claim 1 and by Claim 3 every set in $\mathcal{F}(III.3)$. So we only need to show that $T$ touches every set in $\mathcal{F}(III.5)$. Since ext($T$) = 3, we must have extended $T$ in step III.4. By Lemma 7 we know $T$ touches each set in $\mathcal{F}(III.5)$ using edges of $G$. It is impossible for these edges to be colored red because $\alpha(C - N(T))$ has been reduced to 1. □
**Claim 5.** In the Case III algorithm (even if $U_2$ is not connected or $U_2$ has independence number less than 5), $|\mathcal{F}(1,2)| + |\mathcal{F}(1,3)| \leq (8\lambda - 1)n$.

**Proof.** Let $f : V(H) \to \mathbb{Z}^+$ be defined by

$$f(T) = \begin{cases} |T| + 1 & T \in \mathcal{F}(1,2) \cup \mathcal{F}(1,3), \\ |T| & \text{otherwise}. \end{cases}$$

Let $\mathcal{I} \in \text{IND}_{G_b}(\mathcal{F})$. If $\mathcal{I}$ contains just a single set, the largest extension number of a set is 4 which has 7 vertices so we have $f(\mathcal{I}) \leq 8$. Assume $\mathcal{I}$ has at least two sets, and assume $\mathcal{I}$ does not contain any set in $\mathcal{F}(1,2) \cup \mathcal{F}(1,3)$. Then $f(\mathcal{I}) \leq 2 \sum_{T \in \mathcal{I}} \text{ext}(T) - |\mathcal{I}| \leq 8$. Now assume $\mathcal{I}$ contains some sets in $\mathcal{F}(1,2) \cup \mathcal{F}(1,3)$. Since $\mathcal{F}(1,2) \cup \mathcal{F}(1,3) \subseteq \mathcal{F}(1)$ are all pairwise touching, $\mathcal{I}$ can contain at most one of these sets.

Assume $T \in \mathcal{I} \cap \mathcal{F}(1,2)$. Using Claim 5 there are two possibilities. One possibility is $\mathcal{I} = \{T, Q, R\}$ with $Q \in \mathcal{F}(III.5)$ and $R \in \mathcal{F}(III.6)$. For this $\mathcal{I}$, we have $f(\mathcal{I}) = |T| + 1 + |Q| + |R| \leq 8$. The other possibility is $\mathcal{I} = \{T, P, Q, R\}$ where $P, Q, R \in \mathcal{F}(III.6)$. For this $\mathcal{I}$, we have $f(\mathcal{I}) = |T| + 1 + 3 \leq 7$. Assume $T \in \mathcal{I} \cap \mathcal{F}(1,3)$. By Claim 6 the only possibility is $\mathcal{I} = \{T, R\}$ with $R \in \mathcal{F}(III.6)$. But for this $A$, $f(\mathcal{I}) \leq |T| + 1 + |R| \leq 7$.

Thus $f(\mathcal{I}) \leq 8$, so by Theorem 3 we have

$$\lambda n \geq \frac{f(H)}{8} \geq \frac{n + |\mathcal{F}(1,2) \cup \mathcal{F}(1,3)|}{8}.$$ 

**Claim 6.** In Cases I and II, $|\mathcal{F}(1,2)| + |\mathcal{F}(1,3)| \leq (8\lambda - 1)n$.

**Proof.** Consider that instead of running the algorithm with Case I or II, we ran the Case III algorithm. Let $\mathcal{F}'$ be the family produced by the Case III algorithm. Then by Claim 5 $|\mathcal{F}'(1,2)| + |\mathcal{F}'(1,3)| \leq (8\lambda - 1)n$.

We have $\mathcal{F}(1) = \mathcal{F}'(1)$ and also have $\mathcal{F}(1,1) = \mathcal{F}'(1,1)$. No possible extension of sets in $\mathcal{F}(1,1)$ or $\mathcal{F}'(1,1)$ can happen in steps I.3, II.3, or III.4 because after Step 2, for every $T$ in $\mathcal{F}(1,1)$ or $\mathcal{F}'(1,1)$ and each component $C$ of $U_2$ either $T$ dominates $C$ or $T$ does not touch $C$.

**Claim 7.** Assume that the algorithm selected Case I or Case II. Then let $\mathcal{I} \in \text{IND}_{G_b}(\mathcal{F})$. Then $\mathcal{I}$ satisfies one of the following conditions.

- $|\mathcal{I}| = 1$,
- $|\mathcal{I}| \geq 3$,
- $\mathcal{I} = \{T, R\}$ with $\text{ext}(T) + \text{ext}(R) < 5$,
- $\mathcal{I} = \{T, R\}$ with $T \in \mathcal{F}(1,2)$ and $R \in \mathcal{F}(I.A)$ (in Case I),
- $\mathcal{I} = \{T, R\}$ with $T \in \mathcal{F}(1,3)$ and $R \in \mathcal{F}(I.5)$ (in Case I),
- $\mathcal{I} = \{T, R\}$ with $T \in \mathcal{F}(1,3)$ and $R \in \mathcal{F}(II.A)$ (in Case II).

**Proof.** Say $\mathcal{I} = \{T, R\}$ with $\text{ext}(T) + \text{ext}(R) \geq 5$. We want to show that we must be in the last three options. Since $\text{ext}(T) + \text{ext}(R) \geq 5$ and the algorithms in Case I or II never produce a set with extension number 4, we must have $\text{ext}(T) = 3$ and $\text{ext}(R) = 2$ or $\text{ext}(T) = 2$ and $\text{ext}(R) = 3$. Since $\mathcal{F}(1)$ is pairwise touching in $G_b$, at most one of them can be in $\mathcal{F}(1)$. Now consider cases separately. In Case I, say $T \in \mathcal{F}(1,2)$. Then we must have $R \in \mathcal{F}(I.4)$ since $\text{ext}(T) = 2$ and $\text{ext}(R) = 3$. Consider $T \in \mathcal{F}(1,3)$. Then the only possibility of a set with extension number 2 for $R$ is $\mathcal{F}(I.5)$. In Case II, the only place sets with extension number 3 are created is by extending a set in $\mathcal{F}(1)$. Thus $T \in \mathcal{F}(1,3)$ and the only possibility for $R$ is $\mathcal{F}(II.4)$.

**Claim 8.** Assume that the algorithm selected Case I or II. Then $\lambda \geq \frac{2}{16}$.
Proof. Use the following weight function \( f : V(H) \to \mathbb{Q}^+ \) by
\[
  f(T) = \begin{cases} 
  |T| - 1 & T \in \mathcal{F}(1,3), \\
  |T| & \text{otherwise.}
\end{cases}
\]
Using Claim 4 \( \alpha_f(H) \leq 7 \) so by Theorem 3 we have
\[
n - |\mathcal{F}(1,3)| \leq 7 \lambda n.
\]
Combining with Claim 6 we have
\[
n - (8 \lambda n - n) \leq n - |\mathcal{F}(1,3)| \leq 7 \lambda n
\]
implying the claim.

**Claim 9.** Assume the algorithm selected Case III. Then every \( T \in \mathcal{F}(1,1) \) dominates \( U_2 \) in \( G_b \).

**Proof.** Consider \( T = \{x\} \in \mathcal{F}(1,1) \). In Step 2 we tried to extend \( T \) but failed. Since we are in Case III, \( U_2 \) is connected and \( \alpha(U_2) = 5 \). So because \( T \) was not extended in Step 2, we have \( T \) dominating \( U_2 \) or \( T \) does not touch \( U_2 \) in \( G \). If \( T \) does not touch \( U_2 \) in \( G \), we can form an independent set of size 6 by combining \( T \) with an independent set in \( U_2 \) of size 5. This contradicts \( \alpha(C - N(T)) = \alpha(C) \). Since \( T \) dominates \( U_2 \), we always have \( \alpha(C - N(T)) = 0 < \alpha(C) \) so no edges incident to \( T \) are ever colored red. Thus \( T \) dominates \( U_2 \) in \( G_b \).

**Claim 10.** Assume that the algorithm selected Case III. The possible subfamilies in \( \text{IND}_{G_b}(\mathcal{F}) \) are a subset of one of the following cases:

1. One set from \( \mathcal{F}(1,1) \),
2. One set from \( \mathcal{F}(1,2) \), one set from \( \mathcal{F}(III.5) \), and one set from \( \mathcal{F}(III.6) \),
3. One set from \( \mathcal{F}(1,2) \) and three sets from \( \mathcal{F}(III.6) \),
4. One set from \( \mathcal{F}(1,3) \) and two sets from \( \mathcal{F}(III.6) \),
5. One set from \( \mathcal{F}(III.3) \) and one set from \( \mathcal{F}(III.6) \),
6. Two sets from \( \mathcal{F}(III.5) \) and one set from \( \mathcal{F}(III.6) \),
7. One set from \( \mathcal{F}(III.5) \) and three sets from \( \mathcal{F}(III.6) \),
8. Five sets from \( \mathcal{F}(III.6) \).

**Proof.** Let \( \mathcal{I} \in \text{IND}_{G_b}(\mathcal{F}) \). Assume \( \mathcal{I} \cap \mathcal{F}(1) = \emptyset \). Then using Claim 4 conditions 5 - 8 list all possibilities. So assume \( \mathcal{I} \cap \mathcal{F}(1) \neq \emptyset \). By Claim 11 we can include at most one set from \( \mathcal{F}(1) \). Say \( \mathcal{I} \cap \mathcal{F}(1) = \{T\} \). Consider \( T \in \mathcal{I} \cap \mathcal{F}(1,1) \). Then by Claim 5 \( T \) dominates \( U_2 \) so \( |\mathcal{I}| = 1 \). Consider \( T \in \mathcal{I} \cap \mathcal{F}(1,2) \). Then conditions 2 and 3 cover the two possibilities. Consider \( T \in \mathcal{I} \cap \mathcal{F}(1,3) \). Then \( T \) touches every set in \( \mathcal{F}(III.5) \) by Claim 4. Thus the maximal non-touching family extending \( \{T\} \) is adding two sets from \( \mathcal{F}(III.6) \).

**Claim 11.** Assume that the algorithm selected Case III. Then \( \lambda \geq \frac{5}{18} \).

**Proof.** First,
\[
n \leq |\mathcal{F}(1,1)| + 3|\mathcal{F}(1,2)| + 5|\mathcal{F}(1,3)| + 7|\mathcal{F}(III.3)| + 3|\mathcal{F}(III.5)| + |\mathcal{F}(III.6)|.
\]
Now define \( f : V(H) \to \mathbb{Z}^+ \) to be:

\[
f(T) = \begin{cases} 
38 & T \in \mathcal{F}(1,1), \\
21 & T \in \mathcal{F}(1,2), \\
32 & T \in \mathcal{F}(1,3), \\
35 & T \in \mathcal{F}(III.3), \\
14 & T \in \mathcal{F}(III.5), \\
3 & T \in \mathcal{F}(III.6).
\end{cases}
\]

Then considering Claim 10, we know \( \alpha_f(H) \leq 38 \). Thus using Theorem \( \mathcal{B} \) we have

\[
38\lambda n \geq 38|\mathcal{F}(1,1)| + 21|\mathcal{F}(1,2)| + 32|\mathcal{F}(1,3)| + 35|\mathcal{F}(III.3)| + 14|\mathcal{F}(III.5)| + 3|\mathcal{F}(III.6)|
\]

giving

\[
38\lambda n \geq 5n + 33|\mathcal{F}(1,1)| + 6|\mathcal{F}(1,2)| + 7|\mathcal{F}(1,3)| - |\mathcal{F}(III.5)| - 2|\mathcal{F}(III.6)|.
\]

Thus if we can show that

\[
|\mathcal{F}(III.5)| + 2|\mathcal{F}(III.6)| \leq 6|\mathcal{F}(1,2)| + 7|\mathcal{F}(1,3)|
\]

we will have \( 38\lambda n \geq 5n \) proving the claim.

Define \( \mathcal{F}' = \mathcal{F}_2 \cup \mathcal{F}(III.6) \). Define \( L \) to be the graph formed by starting with \( G_b[\cup_{T \in \mathcal{F}'}T] \) and contracting each set of \( \mathcal{F}' \). Then \( \mathcal{F}' \) is a partial simplicial elimination ordering, so that \( L \) is a chordal graph.

Define \( f : V(L) \to \mathbb{Z}^+ \) by

\[
f(T) = \begin{cases} 
5 & T \in \mathcal{F}_2 \text{ and } \text{ext}(T) = 1, \\
2 & T \in \mathcal{F}_2 \text{ and } \text{ext}(T) = 2, \\
1 & T \in \mathcal{F}(III.6).
\end{cases}
\]

Consider \( \mathcal{I} \in \text{IND}_L(\mathcal{F}') \). Say \( T \in \mathcal{I} \cap \mathcal{F}_2 \). If \( \text{ext}(T) = 1 \) then we did not extend \( T \) in Step III.4 so \( T \in \mathcal{F}(1,1) \) so by Claim \( \mathfrak{B} \) \( T \) dominates \( U_2 \) so \( |T| = 1 \) so \( f(\mathcal{I}) = 5 \). If \( \text{ext}(T) = 2 \) then we could have \( \mathcal{I} = \{ T, P, Q, R \} \) with \( P, Q, R \in \mathcal{F}(III.6) \). Then \( f(\mathcal{I}) = 5 \). If \( \mathcal{I} \cap \mathcal{F}_2 = \emptyset \) then \( \mathcal{I} \) can have at most five sets from \( \mathcal{F}(III.6) \) (by Lemma \( \mathcal{B} \)) so \( f(\mathcal{I}) \leq 5 \).

\( L \) is chordal and \( \alpha_f(L) \leq 5 \) so by Theorem \( \mathcal{B} \) \( \omega(L) \geq \frac{f(L)}{5} \). This clique in \( L \) is a pairwise touching subfamily of \( \mathcal{F}' \) with size at least \( \frac{f(L)}{5} \). This pairwise touching subfamily is a candidate for the choice of a family in Step 1. By the maximum choice in Step 1, we know

\[
|\mathcal{F}(1)| \geq \frac{f(L)}{5} \geq \frac{2|\mathcal{F}(1)| + 3|\mathcal{F}(1,1)| + |\mathcal{F}(III.6)|}{5}.
\]

Thus

\[
3|\mathcal{F}(1,2)| + 3|\mathcal{F}(1,3)| \geq |\mathcal{F}(III.6)|. \tag{37}
\]

Now define \( \mathcal{F}' = \mathcal{F}(1,1) \cup \mathcal{F}(1,2) \cup \mathcal{F}(III.5) \cup \mathcal{F}(III.6) \). Define \( L = G_b[\cup_{T \in \mathcal{F}'}T] \). Again, \( \mathcal{F}' \) is a partial simplicial elimination ordering in \( G_b[\cup_{T \in \mathcal{F}'}T] \) so that \( L \) is a chordal graph.

We then consider the weight function \( f : V(L) \to \mathbb{Z}^+ \)

\[
f(T) = \begin{cases} 
5 & T \in \mathcal{F}(1,1), \\
2 & T \in \mathcal{F}(1,2), \\
2 & T \in \mathcal{F}(III.5), \\
1 & T \in \mathcal{F}(III.6).
\end{cases}
\]
Then using Claim 10, we have \( f(I) \leq 5 \) for each \( I \in \text{IND}_{G_b}(\mathcal{F}') \). \( L \) is chordal and \( \alpha_f(L) \leq 5 \) so by Theorem 3 we have a clique in \( L \) of size \( \frac{f(L)}{5} \). This clique corresponds to a pairwise touching subfamily of \( \mathcal{F}' \) of size \( \frac{f(L)}{5} \). Again, this subfamily is a possibility for the family in Step 1. By the maximum choice in Step 1,

\[
|\mathcal{F}(1)| \geq \frac{5|\mathcal{F}(1,1)| + 2|\mathcal{F}(1,2)| + 2|\mathcal{F}(III.5)| + |\mathcal{F}(III.6)|}{5},
\]

implying

\[
3|\mathcal{F}(1,2)| + 5|\mathcal{F}(1,3)| \geq 2|\mathcal{F}(III.5)| + |\mathcal{F}(III.6)|. \tag{38}
\]

Using 1.5 (37) + 0.5 (38) we obtain

\[
6|\mathcal{F}(1,2)| + 7|\mathcal{F}(1,3)| \geq |\mathcal{F}(III.5)| + 2|\mathcal{F}(III.6)|.
\]