Existence, uniqueness and stability for a class of third order dissipative problems depending on time

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Abstract

We prove new results regarding the existence, uniqueness, (eventual) boundedness, (total) stability and attractivity of the solutions of a class of initial-boundary-value problems characterized by a quasi-linear third order equation which may contain time-dependent coefficients. The class includes equations arising in Superconductor Theory and in the Theory of Viscoelastic Materials. In the proof we use a Liapunov functional \(V\) depending on two parameters, which we adapt to the characteristics of the problem.

1 Introduction

As known, dealing with (in)stability in non-autonomous problems in general requires careful generalizations of criteria and methods valid for autonomous problems, even in linear, finite-dimensional systems (see e.g. [1, 2, 3, 4, 5]). Liapunov direct method in its general formulation applies to non-autonomous (as well as to autonomous) systems, but the construction of Liapunov functions is more complicated.

In this paper we consider a class of non-autonomous initial-boundary-value problems having a number of different physical applications and prove new results regarding the existence, uniqueness, boundedness, stability and attractivity of their solutions; the problems have the form

\[
L\varphi = h(x, t, \Phi), \quad L(t) := \partial_t + a\partial_t - C(t)\partial_x^2 - \varepsilon(t)\partial_x^2 \partial_t \quad x \in [0, \pi], \quad t > t_0,
\]

(1.1)

\[
\varphi(0, t) = \varphi_0(t), \quad \varphi(\pi, t) = \varphi_\pi(t),
\]

(1.2)

\[
\varphi(x, t_0) = \varphi_0(x), \quad \varphi_\varepsilon(x, t_0) = \varphi_1(x).
\]

(1.3)

Here \(\Phi := (\varphi, \varphi_\varepsilon, \varphi_\pi), \quad t_0 \geq 0, \quad \varepsilon \in C^2(I, I), \quad C \in C^1(I, \mathbb{R}^+)\) (with \(I := [0, \infty]\)) are functions of \(t\), with \(C(t) \geq \bar{C} = \text{const} > 0\); \(a = \text{const}, \quad \varepsilon(t) \geq 0, \quad h \in C([0, \pi] \times I \times \mathbb{R}^3); \quad \varphi_0, \varphi_\pi \in C^2(I), \quad u_0, u_1 \in C^2([0, \pi])\) are assigned and fulfill the consistency conditions

\[
\varphi_0(t_0) = \varphi_0(0), \quad \dot{\varphi}_0(t_0) = \varphi_1(0), \quad \varphi_\pi(t_0) = \varphi_\pi(0), \quad \dot{\varphi}_\pi(t_0) = \varphi_\pi(1).
\]

(1.4)

We wish to compare problem (1.1)-(1.2) to the perturbed one

\[
Lw = h(x, t, W) + k(x, t), \quad x \in [0, \pi], \quad t > t_0,
\]

(1.6)

\[
w(0, t) = \varphi_0(t) + w_0(t), \quad w(\pi, t) = \varphi_\pi(t) + w_\pi(t),
\]

(1.6)

\[
w(x, t_0) = \varphi_0(x) + w_0(x), \quad w_\varepsilon(x, t_0) = \varphi_1(x) + w_1(x).
\]

(1.6)

where \(W := (w, w_\varepsilon, w_\pi), \quad k \in C([0, \pi] \times I), \quad w_0, w_\pi \in C^2(I), \quad w_0, w_1 \in C^2([0, \pi])\) are assigned and fulfill the consistency conditions

\[
w_0(t_0) = w_0(0), \quad \dot{w}_0(t_0) = w_1(0), \quad w_\pi(t_0) = w_0(\pi), \quad \dot{w}_\pi(t_0) = w_1(\pi).
\]
Defining

\[ p(x,t) := \frac{2}{\pi}w_\pi(t) + (1 - \frac{2}{\pi})w_0(t), \quad u := w - \varphi - p, \quad u_0(x) := w_0(x) - p(x,t), \]

\[ u_1(x) := u_1(x) - (\partial_t p)(x,t_0) \quad f(x,t,U) := h(x,t,U + \Phi + P) - h(x,t,\Phi) - (Lp)(x,t) + k(x,t), \]

where \( U := (u, u_x, u_t), \ P := (p, p_x, p_t) \), we find that \( u \) fulfills the initial-boundary-value problem

\[
\left\{
\begin{array}{l}
Lu = f(x,t,U), \\
u(0,t) = 0, \\
u(\pi,t) \equiv 0, \\
u(x,t_0) = u_0(x), \\
u_t(x,t_0) = u_1(x).
\end{array}
\right.
\]

where \( u_0, u_1 \) fulfill automatically the consistency condition \( u_0(0) = u_1(0) = u_0(\pi) = u_1(\pi) = 0 \). This shows that we can reduce the questions of stability, attractivity of some \( \varphi \) and of boundedness of \( w - \varphi \) to those of the the corresponding \( u \) around the origin \( u \equiv 0 \). Note that if \( w_0 \equiv w_\pi \equiv 0 \), then \( p \equiv 0 \), \( P \equiv 0 \), \( k \equiv 0 \), \( f(x,t,0) = 0 \), and problem (1.8) admits the null solution, \( u(x,t) \equiv 0 \). In (1.1), (1.8) the \( \varepsilon \)-term is dissipative at \( t \) if \( \varepsilon(t) > 0 \), and the \( a \)-term as well if \( a > 0 \).

![Josephson Junction](image1.png)

Figure 1: Josephson Junction (left) and schematic representation of a Voigt material (right). \( W, L \) are the width and length of the JJ, \( I_s, H \) are the total superconducting current and the external magnetic field.

Physically remarkable examples of problems (1.1)-(1.2) include:

- If \( h = b \sin \varphi - \gamma \), with \( b, \gamma \) const, a modified sine-Gordon eq. describing Josephson effect [6] in the Theory of Superconductors, which is at the base (see e.g. [7]) of a large number of advanced developments both in fundamental research (e.g. macroscopic effects of quantum physics, quantum computation) and in applications to electronic devices (see e.g. Chapters 3-6 in [8]): \( \varphi(x,t) \) is the phase difference of the macroscopic wavefunctions of the Bose-Einstein condensate of Cooper pairs in two superconductors separated by a Josephson junction (JJ), i.e. a very thin and narrow dielectric strip of finite length (Fig. 1 left), the \( \gamma \) term is the (external) “bias current” providing energy to the system, the term \( a\varphi_t \) is due to the Joule effect of the residual current of single electrons across the JJ, the term \( \varepsilon\varphi_{\text{ext}} \) is due to the surface impedance of the JJ. In the simplest model adopted to describe the JJ the parameters \( \varepsilon, C \) are constant (\( \varepsilon \) is rather small), and \( a = 0 \); more accurately, \( a \) is positive but very small; even more accurately, \( h = b \sin \varphi - \gamma - \beta \varphi_t \cos \varphi \) and \( \varepsilon, C, \beta \) are positive (\( \varepsilon, \beta \) are very small), depend on the temperature and on the voltage applied to the JJ (see e.g. [9]), which can be controlled and varied with \( t \). Also \( \gamma \) can be varied with \( t \). Finally, if \( \gamma \), or the temperature \( [10] \), or the width of the junction \( [11] [12] \) are spatially dependent, then new terms linear in \( \varphi_x \) may appear in the equation; in particular if the width is exponentially shaped the system may be modelled by the choice \( h = b \sin \varphi - \gamma - \beta \varphi_t \cos \varphi - \lambda \varphi_x \) (\( \lambda = \text{const} \)).

- If \( a = 0 \), \( h = h(x,t) \), an equation (see e.g. [13] [14]) for the displacement \( \varphi(x,t) \) of the section of a rod from its rest position \( x \) in a Voigt material: \( h \) is applied density force, \( C \equiv c^2 = E/\rho, \varepsilon = 1/\eta \), where \( \rho \) is the linear density of the rod at rest, \( E, \eta \) are the elastic and viscous constants of the rod, which enter the stress-strain relation \( \sigma = E\nu + \partial_\nu/\eta \), where \( \sigma \) is the stress, \( \nu \) is the strain (as known, a discretized model of the rod is a series of elements consisting of a viscous damper and an elastic spring connected in parallel as shown in Fig. 1 right). Again, \( E, \eta \) may depend on the temperature of the rod, which can be controlled and varied with \( t \).
- Equations used to describe: heat conduction at low temperature $\varphi$ [15, 16, 17], if $\varepsilon = \varepsilon^2$, $h = 0$; sound propagation in viscous gases [18]; propagation of plane waves in perfect incompressible and electrically conducting fluids [19].

We see that the class of $f$ arising from these examples and (1.7) is rather broad.

The plan of the paper is as follows. As a preliminary step, we prove (section 2) under rather general conditions existence and uniqueness of solutions of the present problem by transforming the latter into one with constant coefficients, for which known existence and uniqueness theorems [20, 21] can be applied. In section 3 we present for the non-autonomous problem (1.8-1.9) boundedness and displaying rather different behaviours - including periodic ones, or with $\varepsilon > 2$ Existence and uniqueness of the solution

We adapt to the characteristics of the problem; their choice and the main properties of $u$ that $\varepsilon > 0$ for all $t$. To this end we note that problem (1.1-1.2) without loss of generality can be reduced in two steps to one of the same kind where $C, \varepsilon$ are constant.

Indeed, by the change of time variable

$$ t \to \tau(t) := \frac{1}{\varepsilon} \int_{0}^{t} \varepsilon(z)dz, \quad \Rightarrow \quad \dot{\tau} = \frac{\varepsilon}{\varepsilon}, \quad \partial_{t} = \frac{\varepsilon}{\varepsilon}\partial_{\tau}, \quad \partial_{t}^{2} = \frac{\varepsilon^{2}}{\varepsilon^{2}}\partial_{\tau}^{2} + \frac{\varepsilon}{\varepsilon}\partial_{\tau} \quad (2.1) $$

(here $\varepsilon$ is a positive constant with the same dimensions as $\varepsilon$) we transform (1.1) into a new equation of the form $(\varphi_{\tau\tau} - \varepsilon \varphi_{xx}) \varepsilon^{2}[t(\tau)]/\varepsilon^{2} = ...$, where the dots stand for an expression not depending on $\varphi_{\tau\tau}, \varphi_{xx}$. By the change of the dependent variable

$$ \varphi \to \tilde{\varphi}(x, \tau) := b^{-1}(\tau) \varphi[x, t(\tau)], \quad b(\tau) := \exp \int_{0}^{\tau} dz \left\{ \frac{1}{\varepsilon} \left[ \frac{-\varepsilon C[t(z)]}{\varepsilon^{2}[t(z)]} \right] \right\} \quad (2.2) $$

the equation is further transformed into $(\tilde{\varphi}_{\tau\tau} - \tilde{\varphi}_{xx} - \varepsilon \tilde{\varphi}_{xx}) \varepsilon^{2}/\varepsilon^{2} = ...$, where the dots stand for an expression not depending on $\tilde{\varphi}_{\tau\tau}, \tilde{\varphi}_{xx}, \tilde{\varphi}_{xx}$. Multiplying by $\varepsilon^{2}/b \varepsilon^{4}$ (by the above assumptions, in any finite interval $[0, T]$ this function is bounded from above by a finite constant) the problem (1.1) & (1.2) takes the final equivalent form

$$ \begin{cases} \tilde{\varphi}_{\tau\tau} - \tilde{\varphi}_{xx} - \varepsilon \tilde{\varphi}_{xx} = \hat{f}(x, \tau, \tilde{\varphi}, \tilde{\varphi}_{x}, \tilde{\varphi}_{\tau}), & x \in ]0, \pi[; \quad T > \tau > \tau_{0}, \\ \tilde{\varphi}(0, \tau) = \tilde{\varphi}_{0}(\tau), \quad \tilde{\varphi}(\pi, \tau) = \tilde{\varphi}_{\pi}(\tau), \end{cases} \quad (2.3) $$

\footnote{The latter problem has been recently treated also in [25], by an explicit Fourier decomposition of the solution of problem (1.8-1.9).}
\[ \ddot{\varphi}(x, \tau_0) = \ddot{\varphi}_0(x), \quad \dot{\varphi}_\tau(x, \tau_0) = \dot{\varphi}_1(x), \]  

(2.4)

where

\[ \tau_0 := \frac{1}{\varepsilon} \int_0^{\tau_0} \varepsilon(z)dz, \quad \ddot{\varphi}_0(x) := \frac{\dot{\varphi}_0(x)}{b(\tau_0)}, \quad \ddot{\varphi}_1(x) := \frac{\dot{\varphi}_1(x)}{b(\tau_0)} \frac{b_x(\tau_0) - b_x(\tau_0)}{b^2(\tau_0)} \varphi_0(x), \quad \dot{\varphi}_\tau(x) := \frac{\dot{\varphi}_i(t(\tau))}{b(\tau)}, \quad i = 0, \pi, \]

\[ \tilde{f}(x, \tau, \varphi, \dot{\varphi}_x, \ddot{\varphi}_\tau) := \left[ \frac{\varepsilon^2}{bc^2} h \left( x, t, b, b_x, \frac{b_c}{\varepsilon}, \frac{b_c}{\varepsilon} \right) \right] \left( \frac{a}{\varepsilon} + \frac{\varepsilon}{b^2} \dot{\varphi}_x + \left( \frac{b_{\tau\tau}}{b} + \frac{ab_{\varepsilon\varepsilon}}{b^2} \right) \dot{\varphi}_\tau \right)_{\varepsilon = 0} \right]. \]

Under the above assumptions on \( C, \varepsilon, \tilde{f} \) is locally Lipschitz iff \( f \) is. This is now a problem already treated in \( [20] \), where a theorem of existence and uniqueness of the solution has been proved formulating it as an equivalent integral-differential equation in any time interval \( [0, T] \), and applying the fixed-point theorem. This theorem can be applied now to the present context. \[ \text{[Note that requiring } \varepsilon > 0 \text{ in all } t \text{ we are not excluding that } \varepsilon \text{ may go to zero as } t \to \infty. \text{ in any finite } [0, T] \text{ it is in any case inf}_{[0, T]} \varepsilon(t) > 0, \text{ so that the above definitions are safe.] We hope to further generalize this theorem elsewhere.} \]

### 3 Boundedness, stability, attractivity

#### 3.1 Preliminaries

The solution \( \varphi \) of problem \([1.1+1.2]\) and the solution \( w \) of the perturbed problem \([1.4+1.5]\) are ‘close’ to each other iff \( u \) is a ‘small’ solution of problem \([1.8+1.9]\) and coincide iff \( w \) is the null solution. We give a precise meaning to this introducing the distance between \( \varphi, w \) as the norm \( d(u, u_i) \) of \( u \), where the \( t \)-dependent norm \( d(\varphi, \psi) \) is defined by

\[ d^2(\varphi, \psi) \equiv d^2(\varphi, \psi) = \int_0^\pi dx \left[ \varepsilon^2(t) \varphi_{xx}^2 + \varphi_x^2 + \varphi^2 + \psi^2 \right]; \]

(3.1)

\( \varepsilon^2 \) plays the role of a weight for the second order derivative term \( \varphi_{xx}^2 \) so that for \( \varepsilon = 0 \) this automatically reduces to the proper norm needed for treating the corresponding second order problem. Using the condition \( \varphi(0) = 0 = \varphi(\pi) \) one easily derives that

\[ |\varphi(x)| \leq d(\varphi, \psi), \quad \varepsilon |\varphi(x)| \leq \pi \varepsilon d(\varphi, \psi), \quad \varepsilon |\varphi_x(x)| \leq \pi \varepsilon d(\varphi, \psi), \]

(3.2)

for any \( x \). Therefore a convergence in the norm \( d \) implies also a uniform (in \( x \)) pointwise convergence of \( \varphi \) and a uniform (in \( x \)) pointwise convergence of \( \varphi_x \) for \( \varepsilon(t) \neq 0 \).

The notions of (eventual) boundedness, stability, attractivity, etc. are formulated using this distance, i.e. the norm of \( u \), which we shall abbreviate as \( d(t) \equiv d_{<(0)} \left[ u(x, t), u_i(x, t) \right] \) whenever this is not ambiguous. Therefore, without loss of generality, to investigate these properties we consider problem \([1.8+1.9]\).

**Def. 3.1** The solutions of \([1.1+1.2]\) are **bounded** if for any \( \alpha > 0, t_0 > 0 \) there exists \( \beta(\alpha, t_0) > 0 \) such that

\[ d(u_0, u_1) \leq \alpha \Rightarrow d(u_i, u) < \beta \quad \forall t \geq t_0; \]

**eventually bounded** if \( \exists \alpha(\alpha) > 0 \) such that this holds for \( t_0 \geq s \) (eventually) **uniformly bounded** if \( \beta = \beta(\alpha) \).

\(^2\text{From } \varphi^2(x) = \int_0^x dx' dx'' dx'''(x') = \int_0^x dx' \varphi^2(x') \varphi(x') \text{ and } 2|\varphi_{xx}| \leq (|\varphi + \varphi_x|^2) \text{ it follows } \varphi^2(x) \leq \int_0^x dx' (\varphi + \varphi_x)^2 \leq \int_0^x dx' (\varphi + \varphi_x^2) \leq d^2(\varphi, \psi), \text{ as claimed. From } \varphi(0) = 0 = \varphi(\pi) \text{ it follows that there exists } \xi \in [0, \pi] \text{ such that } \varphi_x(\xi) = 0; \text{ hence, using Schwarz inequality,}

\[ \varphi_x(x) = \varphi_x(x) - \varphi_x(\xi) = \int_\xi^x \varphi_y(y)dy \Rightarrow \varepsilon |\varphi_x(\xi)| \leq \int_\xi^x |\varepsilon \varphi_y(y)|dy \leq \int_0^\pi |\varepsilon \varphi_y(y)|dy \leq \frac{\pi}{2} d \left[ \int_0^\pi |\varepsilon \varphi_y(y)|^2dy \right]^\frac{1}{2} \leq \pi \varepsilon d \]

as claimed. Finally, from \( \varphi(x) = \int_0^x \varphi_y(y)dy \) it follows \( |\varepsilon \varphi(x)| \leq \int_0^\pi |\varepsilon |\varphi_y(y)||dy \leq \frac{\pi \varepsilon}{2} d = \pi \varepsilon d = \pi \varepsilon d \).
Def. 3.2 \( u(x,t) \equiv 0 \) is eventually quasi-uniform-asymptotically stable in the large if there exists a \( \bar{t} \geq 0 \) such that for any \( \rho, \alpha > 0 \) there exist \( s(\alpha) \geq \bar{t} \) such that

\[
d(u_0,u_1) < \alpha, \quad t_0 \geq s(\alpha) \quad \Rightarrow \quad \exists T(\rho,\alpha,t_0) > 0 \quad \text{s. t.} \quad d(u,u_t) < \rho \quad \forall t \geq t_0+T.
\]

It is quasi-uniform-asymptotically stable in the large if this holds with \( s(\alpha) = \bar{t} \) and \( T = T(\rho,\alpha) > 0 \).

Suppose now that \( f(x,t,0) = 0 \), so that \( u(x,t) \equiv 0 \) is a solution of problem (1.8+1.9). If \( f \) is defined as in (1.7) this occurs for \( p \equiv 0, k \equiv 0 \).

Def. 3.3 \( u(x,t) \equiv 0 \) is stable if for any \( \sigma > 0 \) there exists a \( \delta(\sigma,t_0) > 0 \) such that

\[
d(u_0,u_1) < \delta(\sigma,t_0) \quad \Rightarrow \quad d(u,u_t) < \sigma \quad \forall t \geq t_0.
\]

It is uniformly stable if \( \delta = \delta(\sigma) \).

We introduce the non-autonomous family of Liapunov functionals

\[
V \equiv V(\varphi,\psi,t;\gamma,\theta) := \int_{0}^{\pi} \frac{1}{2} \{ \gamma \psi^2 + (\varepsilon \varphi_{xx} - \psi)^2 + |C(1+\gamma) - \varepsilon + \varepsilon(a+\theta)| \varphi_x^2 + a\theta \varphi^2 + 2\theta \varphi \psi \} dx
\] (3.3)

where \( \theta, \gamma \) are for the moment unspecified positive parameters. \( V \) reduces to the Liapunov functional of [22] for constant \( \varepsilon, C \equiv 1, \theta = a = 0 \).

3.2 Main assumptions and preliminary estimates

We consider rather general \( t \)-dependences for \( \varepsilon, C \), like the ones depicted in figures 2. To be more precise, denote \( \dot{C}_+(t) := \begin{cases} C(t) & \text{if } \dot{C}(t) \geq 0 \\ 0 & \text{if } \dot{C}(t) < 0 \end{cases} \), \( \dot{C}_- := \dot{C} - \dot{C}_+ \). We assume that there exists a constant \( \mu > 0 \) such that

\[
C > 0, \quad \varepsilon \geq 0, \quad a + \varepsilon > 0, \quad \dot{C} - \dot{\varepsilon} \geq \mu(1+\varepsilon), \quad \mu \varepsilon^2 + \mu \varepsilon + \dot{\varepsilon} - (1+\gamma) \dot{C}_- > 0 \quad (3.4)
\]

We are not excluding the following cases: \( \varepsilon(t) = 0 \) for some \( t, \varepsilon \xrightarrow{t \to \infty} 0, \varepsilon(t) \equiv 0, \varepsilon \xrightarrow{t \to \infty} \infty \) [in view of (3.4) the latter condition requires also \( C \xrightarrow{t \to \infty} \infty \]); but, by condition (3.4) in (1.1) either the term containing \( a \) or the one containing \( \varepsilon \) in (1.1) is dissipative.

We recall Poincaré inequality, which easily follows from Fourier analysis:

\[
\phi \in C^1([0,\pi]), \quad \phi(0) = 0, \quad \phi(\pi) = 0, \quad \Rightarrow \quad \int_{0}^{\pi} dx \phi_x^2(x) \geq \int_{0}^{\pi} dx \phi^2(x).
\] (3.5)

Figure 2: Possible \( t \)-dependences for \( \varepsilon, C \)
3.3 Upper bound for $V$

From definition (3.3) and the inequalities $-2\varepsilon \varphi_{xx} \psi \leq \varepsilon^2 \varphi_{xx}^2 + \psi^2$, $2\theta \varphi \psi \leq \theta (\varphi^2 + \psi^2)$, (3.4) we easily find

$$
V(\varphi, \psi; \gamma, \theta) \leq \int_0^\pi \frac{1}{2} \left\{ (\gamma + 2 + \theta)\psi^2 + 2\varepsilon^2 \varphi_{xx}^2 + [C(1+\gamma) - \varepsilon + \varepsilon(\alpha + \theta)] \varphi_x^2 + (a + 1) \theta \varphi^2 \right\} \, dx
$$

for all $\gamma > 0, \theta \geq \theta_0 := \max\{0, -a\}$. By (3.4) there exists $\lambda \in ]0, 1[$ such that $a + \lambda \varepsilon > 0$. Choosing

$$
\theta > \theta_1 = \max \left\{ 2a, \frac{-a}{1-\lambda} \right\}, \tag{3.6}
$$

$$
\gamma \geq \gamma_1 := 2 + \max \left\{ \frac{a + \theta}{\mu}, (|\alpha|+1)\theta, \frac{4\theta}{\varepsilon\lambda + a}, 2 \frac{\theta - a}{\varepsilon + a} \right\}, \tag{3.7}
$$

and setting

$$
G(t) := C(t) - \frac{1}{2} \varepsilon(t) + 1 \tag{3.8}
$$

[note that by (3.4) it is $G(t) > 1$] we find that

$$
V(\varphi, \psi; \gamma, \theta) \leq \frac{1}{2} \int_0^\pi 2\gamma \varepsilon^2 + 2\varepsilon^2 \varphi_{xx}^2 + \gamma (2C - \varepsilon) \varphi_x^2 + 2\gamma \varphi^2 \, dx \leq \gamma G(t) \, d_2. \tag{3.9}
$$

3.4 Lower bound for $V$

We find, on one hand,

$$(\varepsilon \varphi_{xx} - \psi)^2 = \frac{\varepsilon^2}{2} \varphi_{xx}^2 + \left( \frac{\varepsilon}{\sqrt{2}} \varphi_{xx} - \sqrt{2} \psi \right)^2 \geq \frac{\varepsilon^2}{2} \varphi_{xx}^2 - \psi^2, \tag{3.10}$$

$$(\varepsilon \lambda + a) \varphi^2 + 2\theta \varphi \psi = \frac{3}{4} (\varepsilon \lambda + a) \varphi^2 + (\varepsilon \lambda + a) \theta \left[ \frac{\varphi}{\varepsilon \lambda + a} \right]^2 \left( 1 + \frac{2\psi}{\varepsilon \lambda + a} \right) - \frac{4\theta}{\varepsilon \lambda + a} \varphi^2, \tag{3.11}
$$

$$
C(1+\gamma) - \varepsilon + \varepsilon (\alpha + (1-\lambda)\theta) \geq \mu(1+\varepsilon) + C \gamma + \varepsilon (\alpha + (1-\lambda)\theta) \geq \mu(1+\varepsilon) + C \gamma + \varepsilon [a + (1-\lambda)\theta]. \tag{3.12}
$$

In the last line we have used (3.4), (3.6). On the other hand,

$$
V = \frac{1}{2} \int_0^\pi \left[ \gamma \psi^2 + (\varepsilon \varphi_{xx} - \psi)^2 + [C(1+\gamma) - \varepsilon + \varepsilon (\alpha + (1-\lambda)\theta)] \varphi_x^2 + \varepsilon \lambda \theta (\varphi_x^2 - \varphi^2) + (\varepsilon \lambda + a) \varphi^2 + 2\theta \varphi \psi \right] \, dx. \tag{3.13}
$$

From (3.10), (3.13), (3.5) and $\varepsilon \lambda + a > 0$ we find

$$
V \geq \frac{1}{2} \int_0^\pi \left[ \gamma - 1 - \frac{4\theta}{\varepsilon \lambda + a} \right] \psi^2 + \frac{\varepsilon^2}{2} \varphi_{xx}^2 + \left[ \mu(1+\varepsilon) + C \gamma + \varepsilon (\alpha + (1-\lambda)\theta) \right] \varphi_x^2 + \frac{3}{4} (\varepsilon \lambda + a) \varphi^2 \right] \, dx
$$

$$
\geq \chi_0 d^2, \quad \chi_0 := \min \left\{ \frac{1}{2} \mu(1+\varepsilon) + C \gamma + \varepsilon (\alpha + (1-\lambda)\theta), \gamma - 1 - \frac{4\theta}{\varepsilon \lambda + a}, \frac{3}{4} (\varepsilon \lambda + a) \right\}. \tag{3.14}
$$

$\chi_0$ is positive by (3.7).
3.5 Upper bound for \( \dot{V} \)

Let \( V(t; \gamma, \theta) := V(u, u_t, t; \gamma, \theta) \). Reasoning as in Ref. 23 we find

\[
\begin{align*}
\dot{V}(t; \gamma, \theta) &= \int_0^\pi \left\{ (\varepsilon u_{xx} - u_t)(\varepsilon u_{xxt} - u_{tt} + \dot{\varepsilon} u_{xx}) + [C(1+\gamma) - \dot{\varepsilon} + \dot{\varepsilon}(a+\theta)] u_t^2 \right\} dx \\
&\quad + [C(1+\gamma) - \dot{\varepsilon} + \dot{\varepsilon}(a+\theta)] u_x u_{xt} + a\theta u u_t + \theta u_t^2 + (\gamma u_t + \theta u) u_{tt} \right\} dx \\
&= \int_0^\pi \left\{ (\varepsilon u_{xx} - u_t)[u_t - C u_{xx} - f + \dot{\varepsilon} u_{xx}] + [C(1+\gamma) - \dot{\varepsilon} + \dot{\varepsilon}(a+\theta)] u_t^2 \right\} dx \\
&\quad - [C(1+\gamma) - \dot{\varepsilon} + \dot{\varepsilon}(a+\theta)] u_x u_t + a\theta u u_t + \theta u_t^2 + (\gamma u_t + \theta u)[C u_{xx} + \varepsilon u_{xxt} - a u_t + f] \right\} dx \\
&= \int_0^\pi \left\{ \varepsilon[(\dot{\varepsilon} - C) u_{xx} - f] u_x + [a(1+\gamma) - \theta] u_t^2 + [2\theta C + \dot{\varepsilon} + \dot{\varepsilon}(a+\theta) - (1+\gamma)\dot{C}] u_t^2 \right\} dx \\
&\quad + \varepsilon \gamma u_{xxt} - [(1+\gamma) u_t + \theta u] f \right\} dx. \quad (3.15)
\end{align*}
\]

We shall use the following inequalities: (3.4), (3.5) and

\[
\begin{align*}
\int_0^\pi \varepsilon[(C - \dot{\varepsilon}) u_{xx} + f u_{xx}] dx &\geq \int_0^\pi \varepsilon[\mu(1+\varepsilon) u_{xx} + f u_{xx}] dx = \int_0^\pi \left[ \varepsilon \mu u_{xx} + \frac{3}{4} \varepsilon^2 u_{xx} + \left( \frac{\sqrt{\mu}}{2} \varepsilon u_{xx} + \frac{f}{\sqrt{\mu}} \right)^2 - \frac{f^2}{\mu} \right] dx \\
&\geq \int_0^\pi \left[ \varepsilon \mu u_{xx} + \frac{3}{4} \varepsilon^2 u_{xx} - \frac{f^2}{\mu} \right] dx; \quad (3.16)
\end{align*}
\]

\[
\begin{align*}
\int_0^\pi \left[ \frac{C}{4} u_x^2 - u f \right] dx &\geq \int_0^\pi \left[ \frac{C}{4} u_x^2 - u f \right] dx = \int_0^\pi \left[ \frac{C}{4} (u_x^2 - u^2) + \left( \frac{\sqrt{C} u - 2f}{\sqrt{C}} \right)^2 - \frac{4f^2}{C} \right] dx \geq \int_0^\pi \frac{\theta f^2}{C} dx; \quad (3.17)
\end{align*}
\]

\[
\begin{align*}
\int_0^\pi \varepsilon \gamma u_{xxt} + [a(1+\gamma) - \theta] u_t^2 - (1+\gamma) f u_t \right\} dx &\geq \int_0^\pi \left[ \varepsilon \gamma (u_{xxt}^2 - u_t^2) + |\varepsilon + a(1+\gamma) - \theta| u_t^2 - (1+\gamma) f u_t \right\} dx \\
&\quad + \int_0^\pi \left[ (\varepsilon + a) \frac{\gamma - 1}{2} u_t^2 - (1+\gamma) f u_t \right\} dx \\
&\quad + \int_0^\pi \left[ (\varepsilon + a) \frac{\gamma - 1}{2} u_t^2 - (1+\gamma) f u_t \right\} dx; \quad (3.18)
\end{align*}
\]
\[
\int_{0}^{\pi} \frac{3}{2} \theta C^2 + \dot{\varepsilon} - \delta(a+\theta) - (1+\gamma) \dot{C} \frac{u_x^2}{2} dx = \int_{0}^{\pi} \left[ C \left( \frac{\theta}{2} - a \right) + \dot{\varepsilon} + (C - \dot{\varepsilon})(a+\theta) - (1+\gamma) \dot{C} \right] \frac{u_x^2}{2} dx \\
\geq \int_{0}^{\pi} \left[ C \left( \frac{\theta}{2} - a \right) + \dot{\varepsilon} + \mu(1+\varepsilon)(a+\theta) - (1+\gamma)(\dot{C}_- + \dot{C}_+) \right] \frac{u_x^2}{2} dx. \tag{3.19}
\]

From (3.15, 3.19) we obtain

\[
\dot{V} \leq - \int_{0}^{\pi} \frac{3}{4} \mu u_x^2 + \frac{3}{4} \mu \varepsilon^2 u_{xx} - \frac{f^2}{\mu} + \left[ (\varepsilon + a) \frac{\gamma - 1}{2} + a - \theta \right] u_{x}^2 - \frac{1}{\varepsilon + a} \frac{f^2}{\mu} \\
+ \left[ C \left( \frac{\theta}{2} - a \right) + \dot{\varepsilon} + \mu(1+\varepsilon)(a+\theta) - (1+\gamma)(\dot{C}_- + \dot{C}_+) \right] \frac{u_x^2}{2} - \frac{f^2 \dot{\theta}}{C} \right] dx \\
= - \int_{0}^{\pi} \frac{5 \mu}{8} \varepsilon + 2 u_{xx} + \frac{5 \mu}{8} \varepsilon u_{xx} - u_{xx}^2 + \mu \varepsilon^2 u_x^2 + \mu \varepsilon^2 u_{xx} + \mu \varepsilon^2 u_{xx}^2 \right] \frac{u_x^2}{2} + \frac{1+\gamma}{2} \dot{C}_+ \int_{0}^{\pi} \frac{u_x^2}{2} dx \\
+ \left[ \frac{1}{\mu} + \frac{\gamma + 1}{2(\varepsilon + a)} + \frac{\theta}{C} \right] f^2 dx + \frac{1+\gamma}{2} \dot{C}_+ \int_{0}^{\pi} \frac{u_x^2}{2} dx \\
\leq - \int_{0}^{\pi} \frac{5 \mu}{8} \varepsilon + 2 u_{xx} + \frac{5 \mu}{8} \varepsilon u_{xx} + \left[ (\varepsilon + a) \frac{\gamma - 1}{2} + a - \theta \right] u_{x}^2 + \left[ \frac{\mu}{4} \varepsilon^2 + \frac{1}{C} \right] \frac{u_x^2}{2} + \frac{1+\gamma}{2} \dot{C}_+ \int_{0}^{\pi} \frac{u_x^2}{2} dx \\
+ \left[ \frac{1}{\mu} + \frac{\gamma + 1}{2(\varepsilon + a)} + \frac{\theta}{C} \right] f^2 dx + \frac{1+\gamma}{2} \dot{C}_+ \int_{0}^{\pi} \frac{u_x^2}{2} dx \tag{3.20}
\]

By the choices (3.6, 3.7) of \( \theta, \gamma \) we find

\[
(\varepsilon + a) \frac{\gamma - 1}{2} + a - \theta \geq (\varepsilon + a) \left[ 2 + \frac{\theta - a}{\varepsilon + a} - 1 \right] \frac{1}{2} + a - \theta = \frac{\varepsilon + a}{2} + a - \theta = \frac{\varepsilon + a}{2} > 0
\]

[the last inequality follows from (3.4)]. By (3.4) and (3.6, 3.7)

\[
\chi_1 := \frac{1}{2} \min \left\{ \frac{\mu}{4}, (\varepsilon + a) \frac{\gamma - 1}{2} + a - \theta, \frac{\mu}{\varepsilon} \varepsilon + \frac{1}{C} \right\}, \\
J(t) := \frac{1}{2} \min \left\{ \frac{\mu}{\varepsilon}, (\varepsilon + a) \frac{\gamma - 1}{2} + a - \theta, \mu \varepsilon^2 + \mu \varepsilon + \varepsilon + \mu(1+\varepsilon)(a+\theta) - (1+\gamma) \dot{C}_- \right\}, \\
B := \frac{1}{\mu} + \frac{1}{\varepsilon + a} + \frac{1}{C}, \tag{3.21}
\]

are positive, and we find

\[
\dot{V} \leq - (\chi_1 + J) d^2 + \frac{1+\gamma}{2} B \int_{0}^{\pi} f^2 dx + \frac{1+\gamma}{2} \dot{C}_+ d^2. \tag{3.22}
\]

We now assume

\[
B \int_{0}^{\pi} f^2 dx \leq \tilde{g}(t) d^2 + \tilde{g}_1(t, d^2) + \tilde{g}_2(t, d^2) \tag{3.23}
\]

where \( \tilde{g}(t), \tilde{g}_i(t, \eta) \quad (i = 1, 2 \text{ and } t \geq 0, \eta > 0) \) denote suitable non-negative continuous functions. Without loss of generality we can assume that \( \tilde{g}_i(t, \eta) \) are non-decreasing in \( \eta \); if originally this is not the case, we just need to replace \( \tilde{g}_i(t, \eta) \) by \( \max_{0 \leq s \leq \eta} \tilde{g}_i(t, u) \) to fulfill this condition.
Note that if \( f(x,t,U) = 0 \) as is the case when \( f \) is obtained as in (1.7) with \( p \equiv 0 \), then it is possible to choose \( \tilde{g} \), so that \( \tilde{g}(t,0) \equiv 0 \).

Summarizing, we have proved

**Lemma 3.1** Assume \( \varepsilon, C, \tilde{g}, \tilde{g}_1(\cdot, \eta), \tilde{g}_2(\cdot, \eta) (\eta > 0) \) are continuous nonnegative functions of \( t \in [0, \infty[ \) such that \( \varepsilon, C \) fulfill (3.4) and \( \tilde{g}, \tilde{g}_1, \tilde{g}_2 \) fulfill (3.29). Then

\[
\chi_0 d^2(t) \leq V(t) \leq \gamma G(t) d^2(t)
\]

\[
\dot{V} \leq -\psi(t) V + g_1(t, V) + g_2(t, V),
\]

where for \( i = 1, 2 \)

\[
g_i(t, V) := \frac{1+\gamma}{2} \tilde{g}(t, V), \quad \psi(t) := b(t) - \dot{g}(t), \quad b(t) := \frac{\chi_1+J(t)}{\gamma G(t)}, \quad \dot{g}(t) := \frac{1+\gamma}{2\chi_0} (\dot{C}+\dot{\tilde{g}}).
\]

By the “Comparison Principle” (a generalization of Gronwall Lemma, see e.g. [26]) \( V \) is bounded from above

\[
V(t) \leq y(t),
\]

by the solution \( y(t) \) of the Cauchy problem

\[
\dot{y} = -\psi(t) y + g_1(t, y) + g_2(t, y), \quad y(t_0) = y_0 \equiv V(t_0) \geq 0;
\]

the latter is equivalent to the integral equation

\[
y(t) = y_0 e^{-\int_{t_0}^t \psi(\tau) d\tau} + e^{-\int_{t_0}^t \psi(\tau) d\tau} \int_{t_0}^t \left[ g_1(\tau, y(\tau)) + g_2(\tau, y(\tau)) \right] e^{\int_{\tau}^t \psi(z) dz} d\tau.
\]

We therefore study the latter. If \( \tilde{C} < \infty \), then \( b(t) > \text{const}/\gamma \tilde{G} =: p > 0 \), and the theorems of [22] apply. If \( G(t) \xrightarrow{t \to \infty} \infty \) and \( \tilde{F} < \infty \), then \( b(t) \xrightarrow{t \to \infty} 0 \), and those theorems no longer apply.

**Lemma 3.2** Assume that \( \psi \in C([0, \infty[) \) and \( g_1(\cdot, \eta), g_2(\cdot, \eta) (\eta > 0) \) are continuous nonnegative functions of \( t \in [0, \infty[ \) fulfilling the following properties:

\[
\exists \tilde{t} > 0 \text{ such that } \psi(t) > 0 \quad \forall t \geq \tilde{t};
\]

\[
\lim_{t \to \infty} \frac{g_1(t, \eta)}{\psi(t)} = 0, \quad \forall \eta > 0; \quad \lim_{t \to \infty} \int_0^t g_2(\tau, \eta) d\tau =: \sigma_2(\eta) < \infty, \quad \forall \eta > 0.
\]

Then \( \forall \alpha > 0 \) there exist \( s(\alpha) \geq \tilde{t} \) such that if \( 0 \leq y_0 \leq \alpha, \ t_0 \geq s(\alpha), \) then the solution \( y(t; t_0, y_0) \) of (3.26) fulfills

\[
0 \leq y(t; t_0, y_0) < \beta(\alpha) := 3\alpha, \quad t \geq t_0 \geq s(\alpha).
\]

If \( g_1 \equiv g_2 \equiv 0, \) then \( s(\alpha) = 0. \)

**Proof** By (3.28)1, if \( 0 \leq y_0 < \alpha \) then one finds for any \( t \geq t_0 \geq \tilde{t} \)

\[
y_0 e^{-\int_{t_0}^t \psi(\tau) d\tau} \leq \alpha.
\]

On the other hand, by (3.28)2 there exists a \( s_1(\alpha) \geq \tilde{t} \) such that \( \frac{g_1(\tau, \beta)}{\psi(\tau)} < \alpha \) for all \( \tau \geq t_0 \geq s_1(\alpha) \); then

\[
e^{-\int_{t_0}^t \psi(\tau) d\tau} \int_{t_0}^t g_1(\tau, \beta) e^\int_{t_0}^t \psi(z) dz d\tau = e^{-\int_{t_0}^t \psi(\tau) d\tau} \int_{t_0}^t \psi(\tau) \frac{g_1(\tau, \beta)}{\psi(\tau)} e^{\int_{t_0}^t \psi(z) dz} d\tau \leq \alpha e^{-\int_{t_0}^t \psi(\tau) d\tau} \int_{t_0}^t \psi(\tau) e^{\int_{t_0}^t \psi(z) dz} d\tau = \alpha e^{-\int_{t_0}^t \psi(\tau) d\tau} \left[ e^{\int_{t_0}^t \psi(z) dz} \right]^{t_0} = \alpha \left[ 1 - e^{-\int_{t_0}^t \psi(\tau) d\tau} \right] < \alpha.
\]
By (3.28), there exists a $s_2(\alpha) \geq t$ such that $\int_{t_0}^{\infty} g_2(\tau, \beta) d\tau < \alpha$ for all $t_0 \geq s_2(\alpha)$; then
\[
e^{-\int_{t_0}^{t} \psi(\tau) d\tau} \int_{t_0}^{t} g_2(\tau, \beta) e^\int_{t_0}^{\tau} \psi(z) d\tau d\tau < e^{-\int_{t_0}^{t} \psi(\tau) d\tau} \int_{t_0}^{t} g_2(\tau, \beta) e^\int_{t_0}^{\tau} \psi(z) d\tau d\tau = \int_{t_0}^{t} g_2(\tau, \beta) d\tau < \int_{t_0}^{\infty} g_2(\tau, \beta) d\tau < \alpha. \tag{3.32}
\]

Now let us suppose ad absurdum that, even if $0 \leq y_0 < \alpha$, there exists $t_1 > t_0 \geq s(\alpha) := \max\{s_1(\alpha), s_2(\alpha)\}$ such that
\[
0 \leq y(t; t_0, y_0) < \beta \quad \text{for} \quad t_0 \leq t < t_1 \tag{3.33}
\]
y(t_1; t_0, y_0) = \beta. \tag{3.34}
Because of (3.33) and the monotonicity of $g_i(t, \cdot)$ w.r.t. $\eta$, for $t \in [t_0, t_1]$ the right-hand side (rhs) of (3.27) is bounded from above by the sum of the rhs’s of (3.30-3.32); this implies
\[
y(t_1; t_0, y_0) < \beta, \]
against the assumption (3.34). Hence, (3.29) is proved.

By the previous lemma and (3.25), (3.24), for any $\alpha > 0$ not only the solution $y(t)$ of the Cauchy problem (3.26), but also $V(t)$ and $d^2(u, \bar{u}_t)$, remain eventually uniformly bounded by $\beta(\alpha)$ if $0 \leq y_0 \leq \alpha$. By the monotonicity of $g_i(t, \eta)$ w.r.t. $\eta$ and the comparison principle [26] we find that $y(t)$ is also bounded
\[
y(t) \leq z(t), \quad t \geq t_0, \tag{3.35}
\]
by the solution $z(t)$ of the Cauchy problem
\[
\dot{z} = -\psi(t) z + g_1(t, \beta) + g_2(t, \beta), \quad z(t_0) = z_0 \tag{3.36}
\]
[which differs from (3.26) in that the second argument of $g_i$ is now a fixed constant $\beta > 0$], provided that $z_0 = y_0$, and $t_0 \geq s(\alpha)$. We therefore study the Cauchy problem (3.36), keeping in mind that for our final purposes we will choose $\beta = \beta(\alpha) = 3\alpha$, $t_0 = t_0(\alpha) \geq s(\alpha)$.

**Lemma 3.3** Assume $\psi \in C([0, \infty])$ and $g_1(\cdot, \eta)$, $g_2(\cdot, \eta)$ ($\eta > 0$) are continuous nonnegative functions of $t \in [0, \infty]$ fulfilling (3.28) and
\[
\int_{0}^{\infty} \psi(t) dt = \infty. \tag{3.37}
\]
Then for any $\rho > 0$, $t_0 \geq t$, $\alpha > 0$ there exists $T(\rho, \alpha, \beta, t_0) > 0$ such that, if $0 \leq z_0 < \alpha$, the solution $z(t; t_0, z_0)$ of (3.36) is bounded as follows:
\[
0 \leq z(t; t_0, z_0) < \rho, \quad \text{if} \quad t \geq t_0 + T. \tag{3.38}
\]

**Proof** The solution $z(t) = z(t; t_0, z_0)$ is of the form
\[
z(t) = z_0 e^{-\int_{t_0}^{t} \psi(\tau) d\tau} + e^{-\int_{t_0}^{t} \psi(\tau) d\tau} \int_{t_0}^{t} [g_1(\tau, \beta) + g_2(\tau, \beta)] e^{\int_{t_0}^{\tau} \psi(\lambda) d\lambda} d\tau. \tag{3.39}
\]
We now consider each of the three terms at the rhs of (3.39) separately. By (3.37), for all $t_0 \geq 0$ $\int_{t_0}^{t} \psi(\tau) d\tau \xrightarrow{t \to \infty} \infty$; hence, there exists a $T_0(\rho, \alpha, t_0) > t_0$ such that
\[
z_0 e^{-\int_{t_0}^{t} \psi(\tau) d\tau} < \frac{\rho}{3}, \quad \text{if} \quad t \geq T_0, \quad z_0 \in [0, \alpha]. \tag{3.40}
\]
By (3.28)$_2$, there exist $\sigma_1(\beta) > 0$ and $T_1(\beta, \rho) \geq \bar{t}$ such that
\[
\frac{g_1(t, \beta)}{\psi(t)} \leq \sigma_1 \quad \text{if } t \geq \bar{t}, \\
\frac{g_1(t, \beta)}{\psi(t)} \leq \frac{1}{6} \rho \quad \text{if } t \geq T_1.
\]
(3.41)
Moreover, by (3.37) there exists a $T_2(\beta, \rho) \geq T_1$ such that for $t \geq T_2$
\[
e^{-\int_0^t \psi(\tau) d\tau} \sigma_1(\beta) e^{-\int_0^t \psi(\tau) d\tau} < \frac{\rho}{6}.
\]
(3.42)
Therefore, for $t_0 \geq \bar{t}$ and $t \geq T_2 + t_0$,
\[
e^{-\int_0^t \psi(\tau) d\tau} \int_{t_0}^t g_1(\tau, \beta) e^{\int_0^\tau \psi(\lambda) d\lambda} d\tau
\]
\[
\leq e^{-\int_0^t \psi(\tau) d\tau} \int_{t}^{T_2} g_1(\tau, \beta) e^{\int_0^\tau \psi(\lambda) d\lambda} d\tau + e^{-\int_0^t \psi(\tau) d\tau} \int_{T_1}^t g_1(\tau, \beta) e^{\int_0^\tau \psi(\lambda) d\lambda} d\tau
\]
\[
\leq e^{-\int_0^t \psi(\tau) d\tau} \int_{t}^{T_2} \psi(\tau) \sigma_1(\beta) e^{\int_0^\tau \psi(\lambda) d\lambda} d\tau + e^{-\int_0^t \psi(\tau) d\tau} \int_{T_1}^t \psi(\tau) \sigma_1(\beta) e^{\int_0^\tau \psi(\lambda) d\lambda} d\tau
\]
\[
= e^{-\int_0^t \psi(\tau) d\tau} \sigma_1(\beta) \left[ e^{\int_0^T \psi(\lambda) d\lambda} + e^{-\int_0^t \psi(\tau) d\tau} \rho \left( e^{\int_0^T \psi(\lambda) d\lambda} - e^{\int_0^{T_1} \psi(\lambda) d\lambda} \right) \right]
\]
\[
< e^{-\int_0^t \psi(\tau) d\tau} \sigma_1(\beta) e^{\int_0^T \psi(\lambda) d\lambda} + e^{-\int_0^t \psi(\tau) d\tau} \rho \left( e^{\int_0^T \psi(\lambda) d\lambda} - e^{\int_0^{T_1} \psi(\lambda) d\lambda} \right)
\]
\[
< \frac{\rho}{6} (1 + 1) = \frac{\rho}{3},
\]
(3.43)
where we have used (3.41), the nonnegativity of $g_1(t, \psi(t)$ for $t \geq \bar{t}$, and (3.42). As for the third term at the rhs of (3.39), by (3.28)$_3$ there exists $T_3(\beta, \rho) > t_0$ such that
\[
\int_{T_3}^{\infty} g_2(\tau, \beta) d\tau < \frac{\rho}{6}.
\]
(3.44)
and by (3.37) there exists $T_4(\beta, \rho) > T_3$ such that for $t \geq T_4$
\[
e^{-\int_{T_3}^t \psi(\tau) d\tau} \int_0^{T_3} g_2(\tau, \beta) d\tau < \frac{\rho}{6}.
\]
(3.45)
Therefore for $t \geq T_4$
\[
e^{-\int_0^t \psi(\tau) d\tau} \int_{t_0}^t g_2(\tau, \beta) e^{\int_0^\tau \psi(\lambda) d\lambda} d\tau
\]
\[
< e^{-\int_0^t \psi(\tau) d\tau} \int_{T_3}^{\infty} g_2(\tau, \beta) d\tau + e^{-\int_0^t \psi(\tau) d\tau} \int_{t_0}^{T_3} g_2(\tau, \beta) e^{\int_0^\tau \psi(\lambda) d\lambda} d\tau + e^{-\int_0^t \psi(\tau) d\tau} \int_{T_3}^{t} g_2(\tau, \beta) d\tau
\]
\[
< e^{-\int_0^t \psi(\tau) d\tau} \sigma_1(\beta) e^{\int_0^t \psi(\lambda) d\lambda} + e^{-\int_0^t \psi(\tau) d\tau} \rho \left( e^{\int_0^t \psi(\lambda) d\lambda} - e^{\int_0^{T_3} \psi(\lambda) d\lambda} \right)
\]
\[
< \frac{\rho}{6} (1 + 1) = \frac{\rho}{3},
\]
(3.46)
\[
\leq e^{-t_3 \psi(t)dt} \int_0^{T_3} g_2(\tau, \beta) d\tau + \int_{T_3}^{\infty} g_2(\tau, \beta) d\tau
\]

where we have used the nonnegativity of \(g_2(t, \psi(t))\) for \(t \geq t_1\), (3.44) and (3.46).

Let \(T(\rho, \alpha, \beta, t_0) := \max\{T_0, T_2, T_3\}\). Collecting the results (3.39), (3.40), (3.43), (3.46) we find that the solution \(z(t)\) of (3.36) fulfills (3.38), as claimed. \(\square\)

**Remark 3.1** By (3.28), (3.37) the function \(\Psi(t) := \int_0^t \psi(t) dt\) is nonnegative and increasing for \(t \geq t_1\) and diverges as \(t \to \infty\). If there exists a nonnegative, strictly increasing function \(h : [0, \infty] \to [0, \infty]\) such that \(h(0) = 0\) and

\[
\int_{t_0}^{t_0 + \Delta} \psi(t) dt \equiv \Psi(t_0 + \Delta) - \Psi(t_0) \geq h(\Delta) \quad \forall t_0 \geq t_1, \Delta \geq 0
\]

(with \(h\) not depending on \(t_0\)), then one can choose in (3.40) \(T_0 = h^{-1}\left(\log\left(\frac{3n}{\rho}\right)\right)\); then \(T\) becomes independent of \(t_0\). A sufficient condition for (3.47) is that there exists a \(\psi_0 > 0\) such that \(\psi(t) \geq \psi_0 > 0\) for all \(t \geq t_0\), whence \(h(\Delta) = \psi_0 \Delta\) and \(T_0 = \frac{1}{\psi_0} \log\left(\frac{3n}{\rho}\right)\). However it is not necessary: there are examples that satisfy (3.47) but not the latter. A class of such examples is obtained choosing \(f = r(t)\sin \varphi\), with a function \(r(t)\) such that the integral \(\int_0^t r^2(\tau) d\tau\) grows in the average as some power \(t^\chi\), where \(\chi \leq 1\) and in the case \(\chi = 1\) is smaller than \(pt\), but may vanish somewhere; e.g. we could take \(r^2\) a continuous function that vanishes everywhere except in intervals centered, say, at equally spaced points, where it takes maxima increasing with some power law \(\sim t^\rho\), but keeps the integral bounded, e.g.

\[
r^2(t) = r_0^2 \begin{cases} 
4n^{\alpha+\beta}(t - n + \frac{1}{2\pi}) & \text{if } t \in [n - \frac{1}{2\pi}, n], \\
4n^\beta - 4n^{\alpha+\beta}(t - n) & \text{if } t \in [n, n + \frac{1}{2\pi}], \\
0 & \text{otherwise},
\end{cases}
\]

with \(r_0^2 < p\), \(\alpha \geq 1\), \(\beta \in [\alpha - 1, \alpha]\) and \(n \in \mathbb{N}\) (see [22]). The graph of \((r(t)/r_0)^2\) consists of a sequence of isosceles triangles enumerated by \(n\), having bases of length \(1/n^\gamma\) and upper vertices with coordinates \((x, y) = (n, 2n^\gamma)\). Their areas are \(A_n = 1/n^\gamma\), where \(\gamma := \alpha - \beta \in [0, 1]\). Then we can set \(\tilde{g}(t) = \pi Br^2(t), \tilde{g}_1(t) \equiv \tilde{g}_2(t) \equiv 0\).

We are now in the conditions to prove the following

**Theorem 3.1** Assume that \(\varepsilon \in \mathcal{C}^2(I), C \in \mathcal{C}^1(I)\) fulfill (3.4), the function \(f\) of lhs (1.8) belongs to \(C([0, \pi] \times I \times \mathbb{R}^3)\) and is bounded as in (3.23), where \(g(t), g_1(t, \eta), g_2(t, \eta)\) \((t \geq 0, \eta > 0)\) are continuous functions fulfilling the conditions (3.28), (3.37). Then the solutions of the problem (1.8) are eventually bounded (uniformly if \(G\) is upper bounded). Moreover, \(u \equiv 0\) is eventually quasi-uniform-asymptotically stable in the large. It is quasi-uniform-asymptotically stable in the large if in addition (3.47) is fulfilled.

**Proof** Under the assumption \(d(u_0, u_1) \leq \alpha',\) by (3.24) we find \(y_0 = V(t_0) \leq \alpha,\) where \(\alpha := \alpha'^2 \gamma G(t_0)\). By (3.25) and the application of lemma 3.2 we find that \(y(t)\) [and therefore \(V(t)\)] is bounded by \(\beta(\alpha)\), and again by (3.24) we find \(d(t) \leq \beta'(\alpha) := \sqrt{\beta(\alpha)}/\chi_0\) for \(t \geq s'(\alpha') := s(\alpha)\). If \(G\) is bounded the same works with \(G(t_0)\) replaced by \(\overline{G}\), yielding the (eventual) uniform boundedness, as claimed. On the other hand, we can now apply the comparison principle (3.35, 3.36) and lemma 3.3, chosen \(\rho' > 0,\) we set \(\rho := \chi_0 \rho'^2, T'(\rho', \alpha', \alpha) := T(\rho, \alpha, \beta(\alpha), t_0(\alpha))\). As a consequence of (3.35), (3.36), (3.24) we thus find that for \(t_0 \geq s'(\alpha')\) and \(t \geq t_0 + T'(\rho', \alpha')\),

\[
\frac{d^2(t)}{\chi_0} \leq \frac{V(t)}{\chi_0} \leq \frac{y(t)}{\chi_0} \leq \frac{2(t, \beta(\alpha'))}{\chi_0} \leq \frac{\rho}{\chi_0} = \rho^2,
\]

namely \(u \equiv 0\) is eventually quasi-uniform-asymptotically stable in the large, as claimed. \(\square\)
Remark 3.2. The two lemmas and the theorem are generalizations respectively of Lemmas 1.2 and Theorem 1. in [22], in that we allow here \( t \)-dependent \( \varepsilon \) and \( C \).

We shall use also the following modified version of Lemma 3.2 where \( s(\alpha) : = \bar{t} \):

**Lemma 3.2** Assume that \( \psi \in C([0, \infty]) \) and \( g, g, \; (i = 1, 2) \) are continuous nonnegative functions depending only on \( t \in [0, \infty] \) and fulfilling the following properties

\[
\exists \bar{t} > 0 \; \text{s. t.} \; \psi(t) > 0 \; \forall t \geq \bar{t}, \quad \text{(3.28)}
\]

Then \( \forall \alpha > 0, \; t_0 \geq \bar{t}, \; \text{if} \; 0 \leq y_0 \leq \alpha \) then the solution \( y(t; t_0, y_0) \) of (3.20) fulfills

\[
0 \leq y(t; t_0, y_0) < \tilde{\beta}(\alpha), \quad t \geq t_0, \quad \text{(3.49)}
\]

where

\[
\tilde{\beta} : = \alpha + M_1 + M_2, \quad M_1 : = \sup_{t \geq \bar{t}} \left\{ \frac{g_1(t)}{\psi(t)} \right\}. \quad \text{(3.50)}
\]

The lemma is proved using again (3.30) and replacing (3.31), (3.32) respectively by

\[
e^{-f_{t_0}^t \psi(\tau)d\tau} \int_{t_0}^t g_1(\tau)e^{\int_{t_0}^\tau \psi(\xi)d\xi}d\tau < M_1 e^{-f_{t_0}^t \psi(\tau)d\tau} \int_{t_0}^t \psi(\tau)e^{\int_{t_0}^\tau \psi(\xi)d\xi}d\tau = M_1 \left[ 1 - e^{-f_{t_0}^t \psi(\tau)d\tau} \right] < M_1, \quad \text{(3.29)}
\]

\[
e^{-f_{t_0}^t \psi(\tau)d\tau} \int_{t_0}^t g_2(\tau)e^{\int_{t_0}^\tau \psi(\xi)d\xi}d\tau \leq \int_{t_0}^t \psi(\tau)e^{\int_{t_0}^\tau \psi(\xi)d\xi}d\tau \leq \int_{t_0}^\infty g_2(\tau)d\tau < M_2. \quad \text{(3.30)}
\]

Lemma 3.3 holds unmodified. Theorem 3.1 applies with \( s(\alpha) = \bar{t} \), i.e. all the properties of the solutions become no more eventual. In particular this holds in the case \( g_1 = g_2 \equiv 0 \) with \( M_1 = M_2 = 0 \); the latter situation may occur only if \( f(x, t, U = 0) \equiv 0 \), by (3.23).

**Theorem 3.2** Assume that \( \varepsilon \in C^2(\bar{t}), \; C \in C^1(\bar{t}) \) fulfill (3.34), the function \( f \) of lhs (1.8) belongs to \( C([0, \pi] \times I \times \mathbb{R}^3) \) and is bounded as in (3.23), where \( g(t), \; g_1(t, \eta), \; g_2(t, \eta) \; (\; t \geq 0, \; \eta > 0 \) ) are continuous functions fulfilling the conditions (3.28) and in addition \( f(x, t, 0) \equiv 0 \). Then the solution \( u \equiv 0 \) is stable [uniformly if \( G \) is upper bounded and (3.47) is fulfilled].

**Proof** As \( f(x, t, 0) \equiv 0 \) then \( u \equiv 0 \) is a solution of (1.8-1.9). We prove that Lemma 3.2 implies its stability. In fact, we can choose \( \beta > 0 \) as the independent parameter and \( \alpha : = \beta/3 \) as a dependent one. Under the assumptions of Lemma 3.2 for any \( \beta > 0 \)

\[
0 \leq y_0 < \alpha(\beta), \quad t_0 \geq s''(\beta) : = \bar{s}(\alpha(\beta)) \quad \Rightarrow \quad 0 \leq y(t; t_0, y_0) < \beta, \quad \text{if} \; t \geq t_0.
\]

In particular, choosing \( t_0 = s''(\beta) \) one finds

\[
y[s''(\beta)] \equiv y_0 < \alpha(\beta) \quad \Rightarrow \quad y(t; s''(\beta), y_0) < \beta, \quad \text{if} \; t \geq s''(\beta).
\]

On the other hand, by the continuity of \( y(t; t_0, y_0) \), there exists a \( \delta''(\beta, t_0) \in ]0, \alpha(\beta)[ \) such that

\[
y_0 < \delta'', \quad t_0 \in \bar{t}, \; s''(\beta), \quad t \in [t_0, s''(\beta)] \quad \Rightarrow \quad y(t; t_0, y_0) < \alpha(\beta).
\]

As \( \delta''(\beta, t_0) \) is a continuous function of \( t_0 \) in the compact domain \( \bar{t}, s''(\beta) \), this inequality holds also with \( \delta'' \) replaced by the positive function \( \delta(\beta) : = \min_{t_0 \in \bar{t}, s''(\beta)} \{ \delta''(\beta, t_0) \} \). It holds in particular for \( t = s''(\beta) \), hence, setting \( y''_0 = y[s''(\beta); t_0, y_0] \), together with the previous two relations and \( \delta(\beta) < \alpha(\beta) \) it implies

\[
y_0 < \delta, \quad t_0 \geq \bar{t}, \quad t \geq t_0 \quad \Rightarrow \quad 0 \leq y(t; t_0, y_0) < \beta. \quad \text{(3.51)}
\]
Now for any $\sigma > 0$ let $\beta \equiv \beta(\sigma) := \chi_0 \sigma^2$, $\delta'(\sigma, t_0) := \sqrt{\delta(\beta)} / \gamma G(t_0)$. By (3.24) we find that $d(u_0, u_1) \leq \delta'$ implies $y_0 < \delta$ and therefore rhs (3.51); by (3.25) and again by (3.24) we find
\[
d(t) \leq \sigma, \quad t \geq t_0 \geq \bar{t}.
\] This amounts to the stability of the null solution. Finally $\bar{G} < \infty$ and (3.47) imply that $\delta''$, $\delta$ are independent of $t_0$, so is $\delta := \sqrt{\delta(\beta)} / \gamma \bar{G}$, and the null solution is uniformly stable. \hfill \Box

## 4 Total stability

Consider the special case that one can choose $g_1, g_2$ so that
\[
g_1(\cdot, 0) \equiv 0, \quad g_2(t, y) = g_{21}(t, y) + g_{22}(t), \quad \text{with} \quad g_{21}(\cdot, 0) \equiv 0 \tag{4.1}
\]
($g_{22}$ is necessarily nonnegative). Problem (3.26) becomes
\[
\dot{y} = -\psi(t) y + g_1(t, y) + g_{21}(t, y) + g_{22}(t), \quad y(t_0) = y_0 \equiv V(t_0) \geq 0. \tag{4.2}
\]
We shall denote its solution as $y(t; t_0, y_0; g_{22})$ when we wish to emphasize the dependence on $g_{22}$.

Assume that $\psi, g_1(\cdot, \eta)$ and $g_2(t, \eta) = g_{21}(t, \eta) + \hat{g}_{22}(t)$ fulfill (4.1) and the conditions of Lemma 3.2 for any $\alpha > 0$ we apply the lemma and denote as $\hat{s}(\alpha)$ the corresponding value of $s(\alpha)$.

First, we define the set $\hat{G}$ of admissible perturbations related to $\hat{g}_{22}$ by setting
\[
\hat{G} := \left\{ r \in C([\bar{t}, \infty]) \left| \right. \begin{array}{l}
r \geq 0, \quad \int_0^\infty r(\tau) \, d\tau \leq \int_0^\infty \hat{g}_{22} \, d\tau \quad \forall t_0 \geq \hat{t}
\end{array} \right\} \tag{4.3}
\]
and note that for all $g_{22} \in \hat{G}$ inequality (3.32), and therefore also the claim (3.29), hold again for $t_0 \geq \hat{s}(\alpha)$, because
\[
\int_0^\infty g_2(\tau, \beta) \, d\tau \leq \int_0^\infty \hat{g}_2(\tau, \beta) \, d\tau \quad \forall t_0 \geq \hat{t}, \quad \beta > 0. \tag{4.4}
\]
Choosing $\beta > 0$ and setting $\alpha(\beta) := \beta / 3$, from Lemma 3.2 we find that for any $t_0 \geq \hat{s}(\beta) := \hat{s}[\alpha(\beta)] = \hat{s}(\beta)/3$
\[
0 \leq y_0 < \alpha, \quad g_{22} \in \hat{G} \quad \Rightarrow \quad 0 \leq y(t; t_0, y_0; g_{22}) < \beta, \quad \forall t_0 \geq \hat{t}. \tag{4.5}
\]
Second, we ask what we can say if $\bar{t} \leq t_0 \leq \hat{s}(\beta)$. Eq. (4.2) can be seen as a perturbation of the equation
\[
\dot{y}^* = -\psi(t) y^* + g_1(t, y^*) + g_{21}(t, y^*), \tag{4.6}
\]
which admits the solution $y^*(t) \equiv 0$; we show that to $\alpha(\beta) < \beta$ and $\bar{t} \leq \hat{s}(\beta)$ there corresponds a $\hat{s}(\beta) < \alpha(\beta)$ such that
\[
\bar{t} \leq t_0 < \hat{s}(\beta), \quad 0 \leq y_0 < \hat{s}(\beta), \quad g_{22} \in \hat{G}, \quad \text{and} \quad \begin{cases}
\int_0^\infty g_{22}(\tau) \, d\tau < \hat{s}(\beta) \\
or \sup_{\tau \geq \hat{t}} g_{22}(\tau) < \hat{s}(\beta)
\end{cases} \Rightarrow
\]
\[
0 \leq y(t; t_0, y_0; g_{22}) < \alpha \quad \forall t \in [t_0, \hat{s}(\beta)],
\]
in particular $y_0^\beta := y[\hat{s}(\beta); t_0, y_0; g_{22}] < \alpha$.

In fact, by Theorem 5.2 in [23] on the continuous dependence of the solution of (4.6), both on the initial data and on the perturbation term, to $\alpha(\beta) < \beta$ and $\bar{t} \leq \hat{s}(\beta)$ there corresponds a $\hat{s}(\beta) < \alpha(\beta)$ such that
\[
\bar{t} \leq t_0 < \hat{s}(\beta), \quad 0 \leq y_0 < \hat{s}_1(\beta), \quad g_{22} \in \hat{G}, \quad \begin{cases}
\int_0^\infty g_{22}(\tau) \, d\tau < \hat{s}_1(\beta) \\
or \sup_{\tau \geq \hat{t}} g_{22}(\tau) < \hat{s}_1(\beta)
\end{cases} \Rightarrow
\]
\[
y(t; t_0, y_0; g_{22}) \equiv |y(t; t_0, y_0; g_{22}) - 0| < \alpha \quad \forall t \in [t_0, \hat{s}(\beta)].
\]

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First, if we choose \( g_{22} \in \hat{G} \) such that \( \int_{t}^{\infty} g_{22}(\tau) d\tau < \hat{\delta}_1(\beta) \) then of course the previous inequality is fulfilled. Second, if we choose \( g_{22} \in \hat{G} \) such that \( g_{22}(\tau) < \delta_1(\beta) := \hat{\delta}_1(\beta)/\hat{s}(\beta) \) for all \( \tau \geq t \) then

\[
\int_{t}^{\infty} g_{22}(\tau) d\tau < \delta_2(\beta) \hat{s}(\beta) - t_0 \leq \hat{\delta}_1(\beta)
\]

and again the same inequality is fulfilled. These two arguments lead to (4.7) provided we set \( \hat{s}(\beta) \) at the lhs (4.7) imply more generally

Assume that Lemma 4.1

This relation and (4.5) show that we have proved (3.28), (4.1). Then, given a class of perturbations \( \hat{G} \), for any \( \beta > 0 \) there exists a \( \delta(\beta) \in ]0, \beta[ \) such that the solution of (4.2) fulfills

\[
\text{Eq. (4.9) and Proof } \Rightarrow \text{ (4.10)}
\]

We also introduce a notion of total stability, of the solution \( u(x,t) \equiv 0 \). Suppose that \( f(x,t,U) = \bar{f}(x,t,U) + j(x,t) \) with \( \bar{f}(x,t,0) \equiv 0 \), so that problem (1.8 + 1.9) admits the solution \( u(x,t) \equiv 0 \) when \( j \equiv 0 \).

**Def. 4.1** \( u(x,t) \equiv 0 \) is totally stable if for any \( \sigma > 0 \) there exist \( \delta(\sigma), \nu(\sigma) > 0 \) such that

\[
\text{Def. 4.1} \quad d(u_0, u_1) < \delta, \quad \text{and} \quad \left\{ \begin{array}{c}
\int_{t_0}^{\infty} dt \int_{0}^{\pi} j^2(x,t) dx < \nu \\
\text{or} \sup_{t \geq t_0} \int_{0}^{\pi} j^2(x,t) dx < \nu
\end{array} \right\} \Rightarrow d(u_0, u_1) < \sigma, \quad \forall t \geq t_0.
\]

We can now formulate the following total stability theorem:

**Theorem 4.1** Assume that \( \varepsilon \in C^2(I), C \in C^1(I) \) fulfill (3.4), the function \( f \) of lhs (1.8) is continuous and has the form \( f(x,t,U) = \bar{f}(x,t,U) + j(x,t) \), with \( \bar{f}(x,t,0) \equiv 0 \) and

\[
B \int_{0}^{\pi} 2j^2 dx \leq g(t) d^2 + g_1(t, d^2) + g_{21}(t, d^2),
\]

where (with notation as in Lemma 3.1) \( g(t), g_1(t,\eta), g_{21}(t,\eta) \) are continuous functions fulfilling the conditions (3.28), \( g_1(t,0) \equiv 0, g_{21}(t,0) \equiv 0 \), and

\[
\int_{t}^{\infty} dt \int_{0}^{\pi} j^2(x,t) dx < \infty.
\]

Then \( u \equiv 0 \) is totally stable.

**Proof** Eq. (4.9) and \( f^2 \leq 2(\bar{f}^2 + j^2) \) imply

\[
B \int_{0}^{\pi} f^2 dx \leq B \int_{0}^{\pi} 2(\bar{f}^2 + j^2) dx \leq g(t) d^2 + g_1(t, d^2) + g_{21}(t, d^2) + \bar{g}_{22}(t),
\]

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where $g_{22}(t) := 2B \int_0^t \int_0^x \rho^2(x,t) dx$; hence the assumptions of Lemma 4.1 are fulfilled. Therefore, recalling (3.14), for any $\sigma > 0$ we choose $\beta \equiv \beta(\sigma) := \chi_0 \sigma^2$ and $\delta'(\sigma, t_0) := \sqrt{\delta(\beta(\sigma)) / \gamma \xi(t_0)}$, where $\delta(\beta)$ is determined by the Lemma. By (3.24), (3.26) we find that $d(u_0,u_1) \leq \delta'$ implies $y_0 < \delta$ and, by the Lemma, rhs (4.3); by (3.25) and again by (3.24) we find

$$d(t) \leq \sigma, \quad t \geq t_0 \geq \bar{t}.$$

\[ \square \]

**Remark 4.1** It is easy to realize that if $j$ is bounded w.r.t. $u$, namely there exists a constant $K > 0$ and a continuous function $j_0(x,t)$ such that $|j(x,t,U)| \leq K j_0(x,t)$, then the previous theorem applies as well.

### 4.1 Application to the problem with initial and boundary perturbations

Theorem 4.1 can be applied to discuss the following two special situations.

#### 4.1.1 Application to a non-analytic forcing term

Let us consider the special case that the function $h$ of (1.14) be of the form

$$h(x,t,\Phi) \equiv h_0(t)|\varphi(x,t)|^\omega(\varphi(x,t), \quad \omega \in \mathbb{R}^+.$$

(4.11)

By (1.7) the corresponding $f$ is given by $f(x,t,U) = h_0|u+\varphi+p|^\omega(u+\varphi+p) - h_0|\varphi|^\omega \varphi - L \omega + k$, whence

$$f^2 \leq h_0(\omega + |\varphi| + p)^\omega + h_0(\omega)|\varphi|^\omega \varphi - L \omega + k] \leq h_0(\omega + |\varphi| + p)^\omega + h_0(\omega)|\varphi|^\omega \varphi - L \omega + k]^2

= h_0(\omega + |\varphi| + p)^\omega + h_0(\omega)|\varphi|^\omega \varphi - L \omega + k]^2 \leq 2h_0(\omega + |\varphi| + p)^\omega + h_0(\omega)|\varphi|^\omega \varphi - L \omega + k]^2

In the second and third inequality we have used the known one

$$\left[ \sum_{k=1}^{n} a_k \right]^s \leq \sum_{k=1}^{n} (a_k)^s \quad \text{if } s > 1, \quad n \in \mathbb{N}, \quad a_k \geq 0. \quad (4.12)$$

But applying (3.2) to $u$ one finds $|u(x,t)| \leq d(u, u_1), |u(x,t)| \leq \frac{\pi^{3/2}}{\sqrt{\eta}} d(u, u_1)(t)$ for all $x \in [0, \pi]$, whence

$$\int_0^\pi f^2 dx \leq 2h_0^2 h_0^2 |u|^{2\omega + 2} dx + 2 \int_0^\pi [h_0(\omega + |\varphi| + p)^\omega + h_0^2(\omega)|\varphi|^\omega \varphi - L \omega + k]^2 dx

\leq 2 \int_0^\pi [h_0(\omega + |\varphi| + p)^\omega + h_0^2(\omega)|\varphi|^\omega \varphi - L \omega + k]^2 dx + h_0^2 2^{2\omega + 1} 2^{2\omega + 2} \times \left( \frac{\pi^{3\omega + 4}}{2^{3\omega + 2}} \right).

If $\varphi \equiv 0$, then (3.23) holds with

$$\tilde{g} \equiv 0, \quad \frac{\tilde{g}_1}{B} := \frac{1}{2} h_0^2 (2d)^{2k+2} \times \left( \frac{\pi^{3\omega + 4}}{2^{3\omega + 2}} \right), \quad \frac{\tilde{g}_2}{2B} := \frac{\tilde{g}_2}{B} \equiv \frac{\tilde{g}_2}{2B} := \int_0^\pi [h_0(\omega + |\varphi| + p)^\omega + h_0^2(\omega)|\varphi|^\omega \varphi - L \omega + k]^2 dx,

namely (4.1) is fulfilled (with $g_{21} \equiv 0$); if the conditions (3.28) are fulfilled, then we can apply Lemma 4.1 and Theorem 4.1.

If $\varphi \neq 0$, then (3.23) holds with the same $\tilde{g}, \tilde{g}_1$ and $\tilde{g}_2(t) = 2B \int_0^\pi [h_0(\omega + |\varphi| + p)^\omega + h_0^2(\omega)|\varphi|^\omega \varphi - L \omega + k]^2 dx$; if the conditions (3.28) are fulfilled, then we can apply Lemma 3.2 and Theorem 3.1 with the conclusions of the latter holding non-eventually.
4.1.2 Application to Lipschitz forcing terms

We can apply Lemma 4.1 also to the case that \( h(x,t,W) \) be a Lipschitz function w.r.t. the \( W \) variables with a ‘constant’ (i.e. a maximal Lipschitz function) \( h_0 \) depending only on \( t \):

\[
|h(x,t,W + \Phi) - h(x,t,\Phi)| \leq h_0(t) ||w| + |w_x| + |w_t||;
\]

in fact, by (4.7)

\[
|f(x,t,U)| = |h(x,t, U + P + \Phi) - h(x,t, \Phi) - (Lp)(x,t) + k(x,t)|
\leq h_0(||u + p|| + |u_x + p_x| + |u_t + p_t|) + |Lp - k|
\leq h_0(||u|| + |u_x| + |u_t||p_x| + |p_x| + |p_t|) + |Lp - k|,
\]

what implies

\[
f^2 \leq 7 \{ h_0^2 [u^2 + u_x^2 + u_t^2 + p^2 + p_x^2 + p_t^2] + (Lp-k)^2 \} \Rightarrow \\
\int_0^\pi f^2 \, dx \leq \int_0^\pi \{ 7 \{ h_0^2 [u^2 + u_x^2 + u_t^2 + p^2 + p_x^2 + p_t^2] + 3(p_t)^2 + 3(\rho)^2 + 3k^2 \} \} \, dx
\leq \hat{g}(t)d^2 + g_{22}(t), \tag{4.14}
\]

\[
\hat{g}(t) := 7h_0^2, \quad g_{22}(t) := \int_0^\pi \{ h_0^2 [p^2 + p_x^2 + p_t^2] + 3(p_t)^2 + 3(\rho)^2 + 3k^2 \} \, dx;
\]

here we have used again (4.12) in the first line and the relation \( Lp = \rho + \rho \) in the second line. This shows that (4.1) is fulfilled with \( \hat{g}_1 \equiv 0, \hat{g}_{22} \equiv 0; \) if the conditions (3.28) are fulfilled, then we can apply Lemma 4.1 and Theorem 4.1. As an example we may choose \( h(x,t,W) = h_0(t) \sin \omega, \) what would make (4.1) a modified sine-Gordon equation.

5 Examples

Many \( \varepsilon(t), C(t), a, f \) fulfill (3.4), (3.23), and (3.28), but not the hypotheses of older theorems:

**Example 1:** We assume that \( a > 0, \varepsilon(t) = \varepsilon_0 (1 + t)^{-\nu}, \) with \( \nu \geq 0, \varepsilon_0 \geq 0, C > 0, \) are constant (see Fig. 3-left, where we have chosen \( p = \varepsilon_0 = 1, C = 1.2 \)) and condition (3.23) is fulfilled with

\[
\hat{g}_i(t) \equiv 0, \quad \hat{g}(t) \leq g_0, \quad \text{with some constant } g_0 < \frac{2\chi_0 \chi_1}{\gamma(C+1)},
\]

Then conditions (3.4) are fulfilled defining \( \mu = \min \{ C/2, (\nu + 1)(C/2\varepsilon_0)^{1/(1+\nu)} \}. \) Note that \( \varepsilon, \hat{\varepsilon}, \tilde{\varepsilon} \to 0 \) as \( t \to \infty. \) We find \( G(t) = C + \nu \varepsilon_0 [1 + t]^{-\nu - 1} + 1 \to C+1 \) and,

\[
G(t) \to C+1, \quad J(t) \to 0, \quad b(t) \to \frac{\chi_1}{\gamma(C+1)}, \quad \psi(t) \to \frac{\chi_1}{\gamma(C+1)} - \frac{g_0(\gamma + 1)}{2\chi_0} > 0,
\]

as \( t \to \infty, \) so that (3.28) and (3.47) are fulfilled. Theorem 3.1 applies: the solutions of (1.1-1.2) are eventually quasi-uniform asymptotically stable in the large. If in addition \( f(x,t,0) \equiv 0, \) then Theorem 3.2 applies and the solution \( u \equiv 0 \) is uniformly stable.

**Example 2:** We assume that \( \varepsilon(t) = \varepsilon_0 (1 + t)^q, \) \( C(t) = C_0 (1 + t)^q, \) condition (3.23) is fulfilled with \( \hat{g}_i(t) \equiv 0, \hat{g}(t) \leq K(1 + t)^q, \) where \( K, p, q, r, C_0, \varepsilon_0, a \) are some constants such that

\[
K > 0, \quad 1 > p > 0, \quad 2p - q > r \geq 0, \quad \varepsilon_0 \geq 0, \quad C_0 > p\varepsilon_0, \quad a > -\varepsilon_0;
\]

this is illustrated in Fig. 3-right, where more specifically we have chosen \( \varepsilon_0 = 1, C_0 = 2, p = q = 1/2. \) Then conditions (3.4) are fulfilled defining \( \mu = (C_0 - \nu \varepsilon_0)/(1 + \varepsilon_0). \) Note that \( C(t), \varepsilon(t) \to \infty, \) but \( \varepsilon(t), C(t) \to 0 \)
Figure 3: $t$-dependences for $\varepsilon, C$ in Examples 1,2

as $t \to \infty$. $G(t) \sim t^q$, $b(t) \sim t^{2q}$, whence $\psi(t) \to \infty$ as $t \to \infty$: therefore (3.28) and (3.47) are fulfilled. Theorem 3.1 applies: the solutions of (1.1-1.2) are eventually uniformly bounded, and $u(x,t) \equiv 0$ is eventually quasi-uniform-asymptotically stable in the large. If in addition $f(x,t,0) \equiv 0$, then Theorem 3.2 applies and the solution $u \equiv 0$ is uniformly stable.

Example 3: We assume that $a > 0$; that $\varepsilon, \dot{\varepsilon}, \ddot{\varepsilon}$ are bounded (in particular may be periodic); that $C(t) = C_0 + C_1 (1+t)^{-q}$ with $C_1 > 0, q \geq 0, C_0 > \max \left\{ 0, \bar{\varepsilon} \right\}$; that (3.23) is fulfilled with

$$\dot{g}(t) = 0, \quad \tilde{g}(t) \leq g_0, \quad \text{with some constant } g_0 < \frac{2\chi_0 \chi_1}{\gamma(\gamma+1)(C_0+C_1+1-\bar{\varepsilon}/2)}.$$

This is illustrated in Fig. 4 where more specifically we have chosen $\varepsilon(t) = 0.7 + 0.3 \cos t$, $C_0 = 2$, $C_1 = 1$, $q = -1/2$. We find $G(t) \leq C_0+C_1 - \bar{\varepsilon}+1 < \infty$. Conditions (3.4) are fulfilled with $\mu = (C_0-\bar{\varepsilon})/(1+\bar{\varepsilon})$. We find $\dot{C}_+ = 0$, $G(t) \leq C_0+C_1 - \bar{\varepsilon}/2+1 < \infty$, $\psi(t) \geq \frac{\chi_1}{\gamma(\gamma+1)} - \frac{2\chi_0 \chi_1}{4\chi_0} > 0$: (3.28) and (3.47) are fulfilled. Theorem 3.1 applies: the solutions of (1.1-1.2) are eventually uniformly bounded, and $u(x,t) \equiv 0$ is eventually quasi-uniform-asymptotically stable in the large. If in addition $f(x,t,0) \equiv 0$, then Theorem 3.2 applies and the solution $u \equiv 0$ is stable.

Figure 4: $t$-dependences for $\varepsilon, C$ in Examples 3

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