AUTOMORPHISMS AND QUOTIENTS OF QUATERNIONIC FAKE QUADRICS

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Abstract. A fake quadric is a smooth surface of general type with the same invariants as the quadric in $\mathbb{P}^3$, i.e. $c_1^2 = 8$, $c_2 = 4$ and $q = p_g = 0$. We study here quaternionic fake quadrics i.e. fake quadrics constructed arithmetically by using quaternion algebras over real quadratic number fields. We provide examples of quaternionic fake quadrics $X$ with a non-trivial automorphism group and compute the invariants of the minimal desingularisation of the quotient of $X$ by this group. In that way we obtain minimal surfaces $Z$ of general type with $q = p_g = 0$ and $K^2 = 4$ or $2$ which contain the maximal number of disjoint $(-2)$-curves. We then prove that if a surface of general type has the same invariant as $Z$ and same number of $(-2)$-curves, then we can construct geometrically a surface of general type with $c_1^2 = 8$, $c_2 = 4$.

Key-Words: Surfaces of general type, Fake Quadrics, Automorphisms, Godeaux surfaces, Campedelli surfaces, Surfaces with $q = p_g = 0$.

AMS subject Classification 14J29, 14G35, 11F06, 14J50.

1. Introduction

A fake quadric is a smooth minimal surface of general type with the same numerical invariants as the quadric in $\mathbb{P}^3$, i.e. with Chern numbers $c_1^2 = 8$, $c_2 = 4$ and vanishing geometric genus $p_g = 0$. Two classes of examples of such surfaces are known and these two classes are both quotients of $\mathbb{H} \times \mathbb{H}$, where $\mathbb{H}$ is the upper-half plane, by a cocompact torsion free lattice $\Gamma \subset Aut(\mathbb{H}) \times Aut(\mathbb{H})$. In other words, their universal cover is always $\mathbb{H} \times \mathbb{H}$.

The first class of fake quadrics consists of surfaces $X = \Gamma \backslash \mathbb{H} \times \mathbb{H}$ such that the group $\Gamma$ is reducible. By reducible we mean that there exists a subgroup of finite index $\Gamma' = \Gamma_1 \times \Gamma_2$ of $\Gamma$ such that the group $\Gamma_i$ acts on $\mathbb{H}$ and $C_i = \mathbb{H}/\Gamma_i$ is a smooth algebraic curve. This case is now well understood and the full classification of these fake quadrics, called surfaces isogenous to a higher product, has been achieved in [3] by Bauer, Catanese and Grunewald. In practice, this classification and construction is done geometrically by classifying triples $(C_1, C_2, G)$ of two smooth curves $C_i$ of general type and an automorphism group $G$, such that $G$ acts freely on the surface $C_1 \times C_2$ and the quotient $(C_1 \times C_2)/G$ has the asked invariants.

In this paper we will focus on fake quadrics of the second class, that we call quaternionic fake quadrics. These fake quadrics are quotients of $\mathbb{H} \times \mathbb{H}$ by cocompact irreducible lattices $\Gamma$ in $Aut(\mathbb{H}) \times Aut(\mathbb{H})$. The lattice $\Gamma$ is then arithmetic by a theorem of Margulis and is defined by an indefinite quaternion algebra over a totally real number field.

The first quaternionic fake quadrics have been constructed by Shavel [21] in 1978. We know that these surfaces are rigid and thus that there are only a finite number of them, but at the moment we do not have a complete list of all these surfaces. We have a list of commensurability classes of fake quadrics defined by quaternion algebras over quadratic fields [7].

The situation for quaternionic fake quadrics is very similar to the case of fake projective planes which are surfaces of general type with the same numerical invariants as the projective planes.
plane. Fake projective planes are all quotients of the 2-dimensional complex unit ball $B^2$ by cocompact arithmetic lattices $\Gamma \subset PU(2, 1)$. This provides an arithmetic construction of these surfaces, but it is generally not easy to handle and construct these surfaces geometrically, e.g. as a quotient or ramified cover of some known surfaces.

In order to remedy at this situation, in [11], [13], [14], Keum studied quotients $Z$ of fake projective planes by groups of automorphisms. In this way, he obtained surfaces of general type with geometric genus $p_g = 0$ and was able to rebuild a fake projective plane by only knowing the properties of the quotient surface $Z$.

The aim of this paper is to study automorphisms of quaternionic fake quadrics and the quotients of these surfaces by groups of automorphisms. The computations in [7] leads us to the conjecture that the order of the automorphism group is less or equal $24$ (see Section 4.6).

The first main result we obtain is the following:

**Theorem A.** Let $X = \Gamma \backslash \mathbb{H} \times \mathbb{H}$ be a quaternionic fake quadric. An automorphism of $X$ has only finitely many fixed points. There exist fake quaternionic quadrics $X$ with automorphism group isomorphic to $\mathbb{Z}/2\mathbb{Z}$, $(\mathbb{Z}/2\mathbb{Z})^2$, $D_4$, $D_6$, $D_8$, or $D_{10}$, where $D_n$ is the dihedral group with order $2n$.

Let us remark that the knowledge of surfaces of general type with $p_g = 0$ and a large automorphism group can be interesting to check whether the Bloch conjecture holds (see e.g. [10]).

The second aim of this paper is to study the minimal desingularisation of the quotient of a quaternionic fake quadric by a group of automorphisms, in order to obtain new surfaces with $p_g = 0$.

**Theorem B.** Let $X$ be a quaternionic fake quadric and $G$ a finite group of automorphisms of $X$. The minimal desingularisation $Z$ of the quotient $X/G$ has the following numerical invariants:

| $G$         | $c_2(Z)$ | $c_2(Z)$ | Singularities on $X/G$ | Minimal $\kappa(Z)$ |
|-------------|----------|----------|------------------------|----------------------|
| $\mathbb{Z}/2\mathbb{Z}$ | 4        | 8        | $4A_1$                 | yes                  |
| $\mathbb{Z}/3\mathbb{Z}$ | 2        | 10       | $2A_{3,1} + 2A_2$      | -                    |
| $\mathbb{Z}/6\mathbb{Z}$ | -4       | 16       | $2A_{6,1} + 2A_5$      | no                   |
| $\mathbb{Z}/8\mathbb{Z}$ | -2       | 14       | $A_{8,3} + A_{8,5}$    | no                   |
| $\mathbb{Z}/10\mathbb{Z}$ | -12      | 24       | $2A_{10,1} + 2A_9$     | no                   |
| $(\mathbb{Z}/2\mathbb{Z})^2$ | 2        | 10       | $6A_1$                 | yes                  |
| $D_4$       | 0        | 12       | $4A_1 + A_{4,3} + A_{4,1}$ | yes                  |
| $D_8$       | -1       | 13       | $4A_1 + A_{8,3} + A_{8,5}$ | no                  |

Here, $\kappa$ indicates the Kodaira dimension of the surface $Z$.

We obtain also results and restrictions for the groups $\mathbb{Z}/4\mathbb{Z}$, $\mathbb{Z}/5\mathbb{Z}$ and $D_3$. We note that the surfaces general type we obtain have vanishing geometric genus. The classification of surfaces with $p_g = 0$ is not established and we intend to compute the fundamental groups of our examples in a forthcoming paper.

A curve $C$ on a surface is called nodal if $C \cong \mathbb{P}^1$ and $C^2 = -2$. A nodal curve is the resolution of a nodal singularity. The surfaces $Z$ we obtain as quotient of a fake quadric by an automorphism group $(\mathbb{Z}/2\mathbb{Z})^n$, $n \in \{1, 2\}$ have the maximum number of nodal curves (so-called Miyaoka bound, see [17]). If minimal, the surfaces obtained by quotient by the groups...
\( \mathbb{Z}/3\mathbb{Z} \) and \( \mathbb{D}_3 \) have also the maximum number of quotient singularities. As Keum did with fake planes, we can reverse the construction:

**Proposition C.** Let \( Z \) be a smooth minimal surface of general type with \( q = p_g = 0 \).

a) Suppose that \( c_1^2 = 4, 2, \) Pic\((Z)\) has no 2-torsion, and that there is a birational map \( Z \to Y \) onto a surface containing \( 8 - c_1^2 \) nodal singularities \( A_1 \). There exists a smooth minimal surface of general type \( S \) with invariants \( c_1^2 = 2c_2 = 8 \) and a \( (\mathbb{Z}/2\mathbb{Z})^m \)-cover \( S \to Y \) ramified over the nodes, with \( m \) such that \( 2^m = \frac{8}{c_1^2} \).

b) Suppose that \( c_1^2 = 2, \) Pic\((Z)\) has no 3-torsion, and that there is a birational map \( Z \to Y \) onto a surface with \( 2A_{3,1} + 2A_2 \) singularities. There exist a smooth surface \( S \) with invariants \( c_1^2 = 2c_2 = 8 \) and a \( (\mathbb{Z}/3\mathbb{Z}) \)-cover \( Z \to Y \) ramified over the singularities of \( Y \).

The proof of part a) of this Proposition uses mainly the results of Dolgachev, Mendes Lopes, Pardini (\[6\]) and illustrates their theory. The proof of part b) is more original because it mixes two types of singularities.

The paper is structured as follows: we begin to recall the known facts on quaternionic fake quadrics, and on quotients of surfaces. We then provide examples of fake quadrics having a large group of automorphisms, compute the quotients surfaces and then reverse the construction on the opposite direction: starting with a surface with the same invariants as the quotient, we construct a surface with \( c_1^2 = 2c_2 = 8 \).

**Acknowledgements.** The authors warmly thank Fabrizio Catanese and Miles Reid for pointing out an error in a previous version and Håkan Granath for his help to correct it. We thank also Ingrid Bauer, Margarida Mendes Lopes and Rita Pardini for many useful discussions. Part of this research was done during the second author stay in Strasbourg University and in the Instituto Superior Technico under grant FCT SFRH/BPD/72719/2010 and project Geometria Algebrica PTDC/MAT/099275/2008.

## 2. Generalities on Quaternionic Fake Quadrics

Let us give a more detailed description of quaternionic fake quadrics. First, recall that a lattice \( \Gamma < \text{PSL}_2(\mathbb{R}) \times \text{PSL}_2(\mathbb{R}) \cong \text{Aut} \mathbb{H} \times \text{Aut} \mathbb{H} \) is irreducible if it is not commensurable with a product \( \Gamma_1 \times \Gamma_2 \) of two discrete subgroups \( \Gamma_1, \Gamma_2 \subset \text{PSL}_2(\mathbb{R}) \). Equivalently, the image of \( \Gamma \) under the projection onto one of the factors \( \text{PSL}_2(\mathbb{R}) \) is a dense subgroup of \( \text{PSL}_2(\mathbb{R}) \). By a famous result of Margulis, an irreducible lattice \( \Gamma \) in \( \text{PSL}_2(\mathbb{R}) \times \text{PSL}_2(\mathbb{R}) \) is an arithmetic group, and can therefore be described in the following way:

There exists a totally real number field \( k \) of degree \( g = [k : \mathbb{Q}] \geq 2 \) and a quaternion algebra \( B = (\alpha, \beta)_k := \langle 1, i, j, ij \rangle_k \) with \( i^2 = \alpha \in k, j^2 = \beta \in k, ij = -ji \), over \( k \) such that

\[
B \otimes_{\mathbb{Q}} \mathbb{R} = \prod_{\rho \in \text{Hom}(k, \mathbb{R})} B^\rho \cong M_2(\mathbb{R}) \times M_2(\mathbb{R}) \times H_{\mathbb{R}} \times \cdots \times H_{\mathbb{R}}.
\]

Here, \( B^\rho = (\alpha^\rho, \beta^\rho)_{\mathbb{R}} \) and \( H_{\mathbb{R}} = (-1, -1)_{\mathbb{R}} \) denotes the skew field of Hamilton quaternions. Let \( \mathcal{O}_k \) be the ring of integers of \( k \) and \( \mathcal{O} \) a maximal order in \( B \), i.e. a subring of \( B \) which is a full \( \mathcal{O}_k \)-lattice in \( B \). Finally, let \( \mathcal{O}^1 \) be the subgroup of all elements in \( \mathcal{O} \) of reduced norm one. The isomorphism \( \mathcal{O} \) induces an embedding of \( \mathcal{O}^1 \) into \( \text{SL}_2(\mathbb{R}) \times \text{SL}_2(\mathbb{R}) \) by taking the element \( \gamma \in \mathcal{O}^1 \) to the pair \( (\gamma^\rho_1, \gamma^\rho_2) \in \text{SL}_2(\mathbb{R}) \times \text{SL}_2(\mathbb{R}) \), where \( \gamma^\rho_i \) is the image of \( \gamma \) in \( B^\rho_i \). The group \( \mathcal{O}^1 \) then acts on \( \mathbb{H} \times \mathbb{H} \) as a group of fractional linear transformations. Namely, if \( (z, w) \in \mathbb{H} \times \mathbb{H} \) is a point and an element \( \gamma \in \mathcal{O}^1 \) is identified with two matrices \( \gamma^\rho_1 \) and
$\gamma^{p_2} \in \text{SL}_2(\mathbb{R})$, then
\[ \gamma(z, w) = (\gamma^{p_1}z, \gamma^{p_2}w). \]
After dividing out by the ineffective kernel, one considers the group
\[ \Gamma_{\mathcal{O}} = \mathcal{O}^1/\{\pm 1\} \subset \text{PSL}_2(\mathbb{R}) \times \text{PSL}_2(\mathbb{R}) \]
and it can be proven that $\Gamma_{\mathcal{O}}$ is an irreducible lattice in $\text{PSL}_2(\mathbb{R}) \times \text{PSL}_2(\mathbb{R})$ (see [15]). In general we say that a subgroup $\Gamma \subset \text{PSL}_2(\mathbb{R}) \times \text{PSL}_2(\mathbb{R})$ is an arithmetic lattice if there exists $k, B, \rho_1, \rho_2, \mathcal{O}$ as above such that $\Gamma$ is commensurable with $\Gamma_{\mathcal{O}}$.

Let $\Gamma$ be irreducible and $X_{\Gamma} := \Gamma\backslash \mathbb{H} \times \mathbb{H}$ be the orbit space of the discontinuous action of $\Gamma$ on $\mathbb{H} \times \mathbb{H}$. Then, there is a natural structure of compact algebraic surface on $X_{\Gamma}$ and $X_{\Gamma}$ is smooth if and only if $\Gamma$ is torsion free. The numerical invariants of a smooth $X_{\Gamma}$ are computed in [15], see also [21]. It follows that $X_{\Gamma}$ is a fake quadric if and only if $c_2(X_{\Gamma}) = 4$ (see [21]).

### 3. Generalities on quotients of a surface

In this section we recall results from the theory of singularities and on the resolution of the quotient of a surface by a finite group. The main reference for these topics is [2], see also [20].

Let us denote by $G$ an automorphism group acting on $S$, by $X = S/G$ the quotient surface and by $\pi : Z \to S/G$ the minimal desingularisation map.

**Proposition 3.4.** [Topological Lefschetz formula] Let $\sigma$ be an automorphism acting on $S$ and $S^\sigma$ the fixed point set of $\sigma$. We have
\[ e(S^\sigma) = \sum_{j=0}^{j=4} (-1)^j Tr(\sigma|H^j(S, \mathbb{Z})_{mt}) \]
where $H^j(S, \mathbb{Z})_{mt}$ is the group $H^j(S, \mathbb{Z})$ modulo torsion.

Note that for a fake quadric $S$ we have $q = p_g = 0$, thus
\[ H^1(S, \mathbb{Z})_{mt} = \{0\}, \quad H^2(S, \mathbb{Z}) \otimes \mathbb{C} = H^1(S, \Omega_S). \]

**Corollary 3.5.** Let $S$ be a fake quadric and $\sigma$ an automorphism of order $n > 1$ acting on $S$. We have $e(S^\sigma) = 2$ or $4$. If $\sigma = \tau^2$ for an automorphism $\tau$ (e.g. if $n$ is prime to 2), we have $e(S^\sigma) = 4$.

**Proof.** For a fake quadric, the space $H^1(S, \Omega_S)$ is 2-dimensional and is generated by the classes of 2 curves in the Néron-Severi group. As an automorphism preserves the canonical divisor, the invariant subspace of $H^1(S, \Omega_S)$ is at least 1 dimensional. Therefore the trace of $\sigma$ on $H^1(S, \Omega_S)$ is 2 or 0. If we suppose that this action is not trivial, then 2 divides the order of $\sigma$, moreover we see that the action of $\sigma^2$ is always trivial. \[ \square \]

Let $\xi$ be a primitive $n$th-root of unity. Let us recall that for $1 \leq q \leq n - 1$ coprime to $n$, the quotient of $\mathbb{C}^2$ by the action of
\[ (x, y) \to (\xi x, \xi^q y) \]
has a unique singularity, called a $A_{n,q}$ singularity. For $n, m > 0$ two numbers, we denote $[n, m] = n - \frac{1}{m}$. A $A_{n,q}$ singularity is resolved by a chain of smooth rational curves $C_1, \ldots, C_k$ such that $C_i$ cuts $C_{i+1}$ for $2 \leq i \leq k - 1$ and $C_i^2 = -n_i$ for integers $n_i \geq 2$ determined by the relation:
\[ \frac{n}{q} = [n_1, [n_2, \ldots, [n_{k-1}, n_k] \ldots]]. \]
We denote classically $A_{n,n-1} = A_{n-1}$.

Let $S$ be a surface with $p_g = q = 0$ and let $\sigma$ be an order $n \geq 2$ automorphism such that the fixed points of the $\sigma^k$, $k = 1, \ldots, n - 1$ are isolated.

**Proposition 3.6.** (Holomorphic Lefschetz fixed point formula, [1] p. 567). Let $S^\sigma$ be the fixed point set of $\sigma$. Then

$$1 = \sum_{s \in S^\sigma} \frac{1}{\det(1 - d\sigma|_{T_{S,s}})}$$

where $d\sigma|_{T_{S,s}}$ denotes the action of $\sigma$ on the tangent space $T_{S,s}$.

Suppose moreover that the automorphism $\sigma$ has prime order $p$. Let $\xi$ be a primitive $p^{th}$ root of unity. Let $r_i$ be the number of isolated fixed points of $\sigma$ whose image in $S/\sigma$ are $A_{p,i}$ singularities.

**Proposition 3.7.** (Zhang’s formula, [2] Lemma 1.6). We have:

$$\sum_{i=1}^{i=p-1} r_i a_i(p) = 1$$

where

$$a_i(p) = \frac{1}{p-1} \sum_{j=1}^{j=p-1} \frac{1}{(1 - \xi^j)(1 - \xi^{ij})}$$

In particular, we have :

$$a_1(p) = \frac{5-p}{12}, a_2(p) = \frac{11-p}{24}, a_3(5) = \frac{1}{4}, a_4(5) = \frac{1}{2}.$$

Let $1 \leq i < p$ and $1 \leq k < p$ be such that $ik = 1 \mod p$. As $A_{p,i} = A_{p,k}$, the notations for $r_i$ and $r_k$ in Zhang’s Lemma can be confusing. However, as $a_i(p) = a_k(p)$, there should be no trouble in taking the convention that $r_i + r_k$ is the total number of $A_{p,i} = A_{p,k}$ singularities, rather that choosing a representative $i$ or $k$ for every such pair $(i, k)$.

Let us recall that an automorphism of a vector space is called a reflection if all its eigenvalues but one are equal to 1. Let $S$ be a surface and $G$ an automorphism group acting on $S$. Suppose that for every automorphism of $G$ the fixed point set is finite. Let $s$ be a fixed point of $G$; recall (see [2]):

**Lemma 3.8.** The action of the group $G$ on the tangent space $T_{S,s}$ is faithful and has no reflection.

In particular, if $G$ is cyclic of order $n$, the singularity type of the image of the fixed point $s$ in the quotient $S/G$ is always an $A_{n,q}$ with $q$ prime to $n$.

**Lemma 3.9.** The Euler number of $S/G$ is given by the formula

$$e(S/G) = \frac{1}{|G|} (e(S) + \sum_{n \geq 2} (n - 1)e(S_n)),$$

where $S_n = \{s \in S/|\text{Stab}(G,s)| = n\}$. The Euler number of the minimal resolution $Z$ is the sum of $e(S/G)$ and the number of irreducible components of the exceptional curves of the resolution $\pi: Z \rightarrow S/G$. 
Let $C_1,\ldots,C_k$ be the irreducible components of the one dimensional fibers of $\pi: Z \to X = S/G$. We have the relations $K_Z = \pi^*K_X - \sum_{i=1}^k a_iC_i$, for rational numbers $a_i$ such that $K_ZC_k = -2 - C_k^2$ and $C_k\pi^*K_X = 0$.

Moreover we have the equality $K_Z^2 = \frac{k^2}{c_1}$ where $|G|$ is the order of $G$. As $K_S$ ample, the canonical $\mathbb{Q}$-divisor $K_X$ is ample and $\pi^*K_X$ is nef. We remark also that $K_Z^2 \leq K_X^2$.

**Lemma 3.10.** Let $S$ be a surface with $q = p_g = 0$. The minimal resolution $Z$ of the quotient of $S$ by a group $G$ has always $q = p_g = 0$.

Suppose that $S$ is moreover minimal of general type and the fixed points of automorphisms in $G$ are isolated, then:

**Lemma 3.11.** If $K_Z^2 = 0$, the surface $Z$ has Kodaira dimension $\kappa \geq 1$. If $K_Z^2 > 0$, the surface $Z$ has Kodaira dimension $\kappa = 2$.

**Proof.** (We follow the ideas from [13].) The quotient surface has $q = p_g = 0$ and thus $\chi(\mathcal{O}_Z) = 1$. Let $m \geq 1$ be an integer, then $-mK_Z^{\pi}K_{S/G} = -mK_{S/G}^2 = -\frac{8}{|G|}m < 0$, therefore $H^0(Z, -mK_Z) = \{0\}$ for every $m \geq 1$. Let be $m \geq 2$, then by using Serre duality and Riemann-Roch:

$$H^0(Z, mK_Z) = \chi(\mathcal{O}_Z) + \frac{m(m-1)}{2}K_Z^2 + h^1(Z, mK_Z).$$

If $K_Z^2 > 0$, then immediately, $Z$ has general type. If $K_Z^2 = 0$, the surface has $h^0(Z, 2K_Z) \neq 0$ and cannot be rational by Castelnuovo criterion. Moreover, as $\chi = 1$ it cannot be a ruled surface. Suppose that $Z$ is an Enriques surface. As $K_Z^2 = 0$, it is a minimal surface, but this is impossible because $h^0(Z, 3K_Z) \neq 0$; therefore $\kappa > 0$. \qed

Let us now specialize to surfaces $X_\Gamma = \Gamma \backslash \mathbb{H} \times \mathbb{H}$ where $\Gamma$ is a cocompact and irreducible torsion-free lattice. Let $\mu: \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{H} \times \mathbb{H}$ be the involution exchanging the two factors. The group $\text{Aut}(\mathbb{H} \times \mathbb{H})$ is the semi-direct product of $\text{Aut}\mathbb{H} \times \text{Aut}\mathbb{H}$ by the group generated by $\mu$. Let $\bar{\Gamma} < \text{Aut}\mathbb{H} \times \text{Aut}\mathbb{H}$ be a cocompact torsion-free lattice and $X_\bar{\Gamma} = \bar{\Gamma} \backslash \mathbb{H} \times \mathbb{H}$, then, since $\mathbb{H} \times \mathbb{H}$ is the universal covering of $X_\Gamma$, every automorphism $\sigma$ of $X_\Gamma$ lifts to an automorphism $\bar{\sigma}$ of $\mathbb{H} \times \mathbb{H}$. If $\bar{\sigma}$ is in $\text{Aut}\mathbb{H} \times \text{Aut}\mathbb{H}$ it obviously normalizes $\bar{\Gamma}$. Therefore, the factor preserving automorphism group of $X_\Gamma$ is $\bar{\Gamma} \bar{\Gamma}/\Gamma$, where $\bar{\Gamma}$ is the normalizer of $\Gamma$ in $\text{Aut}\mathbb{H} \times \text{Aut}\mathbb{H}$.

Altogether, every automorphism is either represented by a coset $\gamma \Gamma$ with $\gamma \in \bar{\Gamma}$ or is of type $(\gamma \Gamma) \circ \mu$. The following result is a key for our computations:

**Theorem 3.12.** An automorphism $\sigma$ of $X_\Gamma$ has only finitely many fixed points or $\sigma$ is an involution whose fixed point set is purely one-dimensional. The latter never happens for a quaternionic fake quadric.

**Proof.** Suppose that $\gamma$ is in the subgroup $\text{Aut}\mathbb{H} \times \text{Aut}\mathbb{H}$. Then, as explained above, $\sigma$ can be represented by a coset $\gamma \Gamma$ with $\gamma \in \bar{\Gamma}$, where $\bar{\Gamma}$ is the normalizer of $\Gamma$ in $G$. It is sufficient to show that $\gamma$ as a mapping $\mathbb{H} \times \mathbb{H} \rightarrow \mathbb{H} \times \mathbb{H}$ has only finitely many fixed points in $\mathbb{H} \times \mathbb{H}$ modulo the action of $\Gamma$. Assume that $\gamma(z, w) = (\gamma^{\rho_1}z, \gamma^{\rho_2}w) = (z, w)$. Then $\gamma$ is an elliptic transformation, i.e. $4 \det(\gamma) - \text{Tr}(\gamma^{\rho_1})^2 > 0$ for $i = 1, 2$. The reason is the following: if $(z, w)$ is a fixed point, then in particular $z$ is a fixed point of $\gamma^{\rho_1}$ and $w$ is a fixed point of $\gamma^{\rho_2}$. The only automorphisms of $\mathbb{H}$ with fixed points in $\mathbb{H}$ are elliptic transformations. Then, by definition $\gamma$ is elliptic.
Every non-trivial elliptic transformation of $\mathbb{H}$ has a unique fixed point in $\mathbb{H}$ (the eigenvalue of the matrix which has the positive imaginary part). Moreover, because $\Gamma$ is irreducible, $\gamma^{\rho_1}$ is non-trivial if and only if $\gamma^{\rho_2}$ is non-trivial (observe here that if $\Gamma$ were not irreducible, there would be an automorphism of the form $(\gamma_1, 1)$ with non-isolated fixed points).

Let thus $(z, w)$ be the unique fixed point of $\gamma$ in $\mathbb{H} \times \mathbb{H}$. The $\mathbb{N}_\Gamma$-orbit of $(z, w)$ is discrete in $\mathbb{H} \times \mathbb{H}$, therefore there is only one representative of $(z, w)$ modulo the action of $\mathbb{N}_\Gamma$. Now, $\mathbb{N}_\Gamma$ is a finite index extension of $\Gamma$, therefore there are only finitely many representatives of $(z, w)$ modulo $\Gamma$.

Let us now suppose that the automorphism $\sigma$ is represented by $\gamma \mu \in \text{Aut}(\mathbb{H} \times \mathbb{H})$. Then $(\sigma \gamma \mu)^2 = (\gamma^{\rho_1} \gamma^{\rho_2}, \gamma^{\rho_2} \gamma^{\rho_1})$ is an element of $\text{Aut}\mathbb{H} \times \text{Aut}\mathbb{H}$ that acts on the surface. Suppose that $\sigma$ has an infinite number of fixed points, then $\sigma^2$ must be the identity and $\gamma^{\rho_1} \gamma^{\rho_2}$ must be in $\Gamma$. A fixed point $(z, w)$ satisfies

$$(\gamma^{\rho_1} w, \gamma^{\rho_2} z) = \lambda(z, w)$$

for a $\lambda \in \Gamma$. After the change of $\gamma$ by $\lambda^{-1} \gamma$, we can assume that $\lambda = 1$, thus $w = \gamma^{\rho_2} z$ and $\gamma^{\rho_1} \gamma^{\rho_2} = 1$ because $\gamma^{\rho_1} \gamma^{\rho_2}$ is in $\Gamma$ that is torsion free. Reciprocally, let be $t \in \mathbb{H}$; as $\gamma^{\rho_1} \gamma^{\rho_2} = 1$ the point $(t, \gamma^{\rho_2} t)$ satisfy

$$\gamma \mu(t, \gamma^{\rho_2} t) = (t, \gamma^{\rho_2} t).$$

Therefore there are no isolated fixed points for $\sigma$.

Assume now that $X_\Gamma$ is a quaternionic fake quadric. The fixed locus $C$ of $\sigma$ is a smooth curve. The topological Lefschetz formula (see Corollary 3.5) implies that the genus of the irreducible components of $C$ is negative, thus the automorphism has only a finite number of fixed points. \hfill $\square$

4. Quaternionic fake quadrics with non-trivial automorphism groups.

As already mentioned, a series of examples of quaternionic fake quadrics has been constructed by I. Shavel in [21]. There, the author concentrates on arithmetic lattices $\Gamma \supseteq \Gamma^1_\mathcal{O}$ which are defined by quaternion algebras over real quadratic fields of class number one. More recently, in [7], more examples of quaternionic quadrics associated with quaternion algebras over quadratic fields have been found. In this section we will list all known examples of quaternionic fake quadrics together with their automorphism groups. We refer the reader to [21] for generalities on quaternion algebras.

Let us first make a few general observations, before we discuss the examples in detail. For technical reasons it is more practical to consider the group $\text{PGL}_2^+(\mathbb{R}) \times \text{PGL}_2^+(\mathbb{R})$, where $\text{PGL}_2^+(\mathbb{R}) = \text{GL}_2^+(\mathbb{R})/\mathbb{R}^*$ and $\text{GL}_2^+(\mathbb{R})$ is the group of all $2 \times 2$ matrices with positive determinant, instead of $\text{PSL}_2(\mathbb{R}) \times \text{PSL}_2(\mathbb{R})$. We identify $\text{PGL}_2^+(\mathbb{R}) \times \text{PGL}_2^+(\mathbb{R})$ with the group $\text{Aut}\mathbb{H} \times \text{Aut}\mathbb{H}$ of holomorphic automorphisms which preserve the two factors.

From the point of view of Theorem 3.12 it is more interesting to consider the automorphism subgroups $G \leq \mathbb{N}_\Gamma/\Gamma =: \text{Aut}(X_\Gamma)$ of factor preserving automorphisms, which we will do in the following. Since fake quadrics $X_\Gamma$ have relatively small covolume, they tend to be large groups and therefore the order $|\text{Aut}(X_\Gamma)| = |\mathbb{N}_\Gamma/\Gamma|$ is not too big. The normalizers $\mathbb{N}_\Gamma$ will be maximal and all such lattices can be described arithmetically as follows (see [5]). If $X_\Gamma$ is a quaternionic fake quadric, there is an associated tuple $(k, \rho_1, \rho_2, B, \mathcal{O})$ as described in Section 2. The quaternion algebra $B$ is for fixed $\rho_1, \rho_2$ uniquely determined (up to isomorphism) by the reduced discriminant $d_B = v_1 \cdot \ldots \cdot v_r$, the formal product over finite places $v_i$ of $k$ where $B$ is ramified, i.e. $B \otimes_k k_{v_i} \not\cong M_2(k_{v_i})$, hence $(k, \rho_1, \rho_2, B, \mathcal{O}) = (k, \rho_1, \rho_2, d_B, \mathcal{O})$. In
the following we will often abbreviate such a datum which determines the quaternion algebra $B$ with $B(k, d_B)$ or $B(k, v_1 \ldots v_r)$. Let us fix such a datum $B(k, v_1 \ldots v_r)$ and let $B^+$ be the group of all $x \in B^*$ such that the reduced norm $\text{Nrd}(x)$ is totally positive. It is known that

\begin{equation}
\text{NT}^+_\mathcal{O} = \{ x \in B^* \mid x\mathcal{O}x^{-1} = \mathcal{O}\}/k^*
\end{equation}

is a maximal lattice. $\text{NT}^+_\mathcal{O}$ contains $\Gamma_\mathcal{O}$ and it is known that $\Gamma_\mathcal{O}$ is normal in $\text{NT}^+_\mathcal{O}$ with $\text{NT}^+_\mathcal{O}/\Gamma_\mathcal{O} \cong (\mathbb{Z}/2\mathbb{Z})^l$ an elementary abelian $2$-group with $l \geq r$ and $r$ is the number of ramified places in $B$ (see [21] for instance). If the class number of $k$ is one (as will be the case in all the consideres examples) there is an alternative description of $\text{NT}^+_\mathcal{O}$ as

\begin{equation}
\text{NT}^+_\mathcal{O} = \{ \alpha = \rho_1^{\epsilon_1} \cdots \rho_r^{\epsilon_r} \lambda \tau \in B^* \mid \text{Nrd}(\alpha) \text{ totally positive, } \tau \in k^*, \lambda \in \mathcal{O}^*, \epsilon_i \in \{0,1\}, \text{Nrd}(\rho_i) \text{ divides } d_B\}/k^*
\end{equation}

(see [21], p. 223). It follows that a quaternionic fake quadric $X_\Gamma$ with $\Gamma \supseteq \Gamma_\mathcal{O}$ will have an elementary abelian $2$-group as the automorphism group $\text{Aut}(X_\Gamma)$. All Shavel’s examples will provide such automorphism groups.

4.1. A fake quadric with automorphism group $\mathbb{Z}/2\mathbb{Z}$. There are examples of quaternionic fake quadrics $X_\Gamma$ whose automorphism group is $\mathbb{Z}/2\mathbb{Z}$ and, as mentioned, they already appear in [21].

For example, let $k = \mathbb{Q}(\sqrt{2})$ and let $B = B(k, p_3p_7)$ be the (unique) quaternion algebra over $k$ which is ramified exactly at the two finite primes $p_3$ and $p_7$ of $k$ lying over the rational primes $3$ and $7$ respectively. Since $k$ has the class number one, there is the unique (up to conjugation) maximal order $\mathcal{O}$ in $B$. Consider the group $\Gamma_\mathcal{O}$. By [21], Proposition 4.7, $X_{\Gamma_\mathcal{O}}$ is smooth. By the already mentioned general result of Matsushima and Shimura [15], $q(X_{\Gamma_\mathcal{O}}) = 0$. The Euler number $c_2(X_{\Gamma_\mathcal{O}})$ is computed via the volume formula of Shimizu (see [21], Theorem 3.1). Since the prime $3$ is inert and $7$ is decomposed in $k$, this formula gives $c_2(X_{\Gamma_\mathcal{O}}) = 8$. The normalizer of $\Gamma_\mathcal{O}$ is $\text{NT}^+_\mathcal{O}$ and by [21], Proposition 1.3 and 1.4 we have

\[ \text{Aut}(X_{\Gamma_\mathcal{O}}) \cong L_1/L_2 = ([p_3],[p_7]) \cong (\mathbb{Z}/2\mathbb{Z})^2, \]

where $L_1$ is the group of principal fractional ideals of type $(p_3)(p_7)I^2$ (I a principal fractional ideal) for which one can find a totally positive generator and $L_2$ consists of all principal ideals of type $(a^2)$ with $a \in k$ (See also [22], 3.12). Let $\Gamma_{p_3}$ be the kernel of the canonical homomorphism

\[ \text{NT}^+_\mathcal{O} \longrightarrow L_1/L_2 \longrightarrow ([p_7]). \]

By Shavel’s criterion (see [21], Theorem 4.11) $\Gamma_{p_3}$ is torsion free and as $[\Gamma_{p_3} : \Gamma_\mathcal{O}] = 2$, $X_{\Gamma_{p_3}}$ is a fake quadric with $\text{Aut}(X_{\Gamma_{p_3}}) \cong \mathbb{Z}/2\mathbb{Z}$.

4.2. A fake quadric with automorphism group $(\mathbb{Z}/2\mathbb{Z})^2$. Consider again $k = \mathbb{Q}(\sqrt{2})$ and now the quaternion algebra $B = B(k, p_2p_5)$ over $k$ which is ramified exactly at the two finite places $p_2$ and $p_5$. Again there is the unique maximal order $\mathcal{O}$ in $B$ and as in the previous example, Shavel’s results show that $X_{\Gamma_\mathcal{O}}$ is smooth. The prime $2$ is ramified and $5$ is inert in $k$ and therefore Shimizoo’s volume formula gives $c_2(X_{\Gamma_\mathcal{O}}) = 4$. Hence $X_{\Gamma_\mathcal{O}}$ is a fake quadric. With the same arguments as in the previous example $\text{Aut}(X_{\Gamma_{p_3}})$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$. 
4.3. A fake quadric with automorphism group of order 20. Consider $k = \mathbb{Q}(\sqrt{5})$ and the quaternion algebra $B = B(k, p_2p_5)$ over $k$ which is ramified exactly at the primes $p_2$ and $p_5$. In this case the group $\Gamma^1_O$ (where $O$ is again a maximal order in $B$), contains torsion elements of order 5 and no other torsions (see [21], Proposition 4.7 and Theorem 4.8)\(^1\). Volume formula of Shimizu gives in this case $c_2(X^1_{\Gamma^1_O}) = 4/5$. Let us now give a torsion-free subgroup $\Gamma < \Gamma^1_O$ of index 5. The corresponding surface $X_{\Gamma}$ will be a fake quadric. Since $p_2$ is ramified in $B$, there is a prime ideal $\mathfrak{p}_2$ in $O$ lying over $p_2$ and satisfying $\mathfrak{p}_2^2 = p_2O$. Let

$$\Gamma = \Gamma^1_O(\mathfrak{p}_2) = \{x \in \Gamma^1_O \mid x \equiv 1 \mod \mathfrak{p}_2^2\}.$$ 

$\Gamma^1_O(\mathfrak{p}_2)$ is a normal subgroup in $\Gamma^1_O$ and the index can be computed via the localisation of $B$ at $\mathfrak{p}_2$. Namely, observe first that $\Gamma^1_O/\Gamma^1_4(\mathfrak{p}_2)$ is isomorphic to the factor group $O^1/O^1(\mathfrak{p}_2)$, where

$$O^1(\mathfrak{p}_2) = \{x \in O^1 \mid x \equiv 1 \mod \mathfrak{p}_2^2\}.$$ 

This is because $-1$ is in $O^1(\mathfrak{p}_2)$. Let $O_{\mathfrak{p}_2}$ be the maximal order in $B_{\mathfrak{p}_2}$, i.e. $O_{\mathfrak{p}_2} = O \otimes_{O_k} O_{\mathfrak{p}_2}$, where $O_{\mathfrak{p}_2}$ is the ring of integers in $k_{\mathfrak{p}_2}$. Its maximal ideal $\mathfrak{p}_2$ is the topological closure of $\mathfrak{p}_2$. By the strong approximation property $O^1/O^1(\mathfrak{p}_2) \cong O^1_{\mathfrak{p}_2}/O^1_{\mathfrak{p}_2}(\mathfrak{p}_2)$ Note that $\mathfrak{p}_2 = O_{\mathfrak{p}_2}$, since $O_{\mathfrak{p}_2}$ is the subring of $B_{\mathfrak{p}_2}$ consisting of elements whose reduced norm is less or equal 1. We use a theorem of C. Riehm (see [19], Theorem 7) by which

$$O^1_{\mathfrak{p}_2}/O^1_{\mathfrak{p}_2}(\mathfrak{p}_2) \cong \ker((O_{\mathfrak{p}_2}/\mathfrak{p}_2)^* \xrightarrow{Nr} (O_{\mathfrak{p}_2}/\mathfrak{p}_2)^*) \cong \ker(F_4^* \xrightarrow{F_4} \mathbb{F}_4^*) \cong \mathbb{Z}/5\mathbb{Z}.$$ 

(Note here that the norm map induces a surjective homomorphism of multiplicative groups).

Since $\Gamma^1_O(\mathfrak{p}_2)$ is embedded in $O^1_{\mathfrak{p}_2}/\pm 1$ and the latter group is a pro-$2$-group (again by [19]) it can not contain elements of order 5. Therefore, $\Gamma^1_O(\mathfrak{p}_2)$ is a torsion-free group and $X_{\Gamma^1_O(\mathfrak{p}_2)}$ is a fake quadric. Since $\Gamma^1_O$ contains a 5-torsion and $\Gamma^1_O$ normalizes $\Gamma^1_O(\mathfrak{p}_2)$, $X_{\Gamma^1_O(\mathfrak{p}_2)}$ contains an automorphism of order 5. In order to determine the full automorphism group $\text{Aut}(X_{\Gamma^1_O(\mathfrak{p}_2)})$ we first need to find the normaliser of $\Gamma^1_O(\mathfrak{p}_2)$. By definition elements of $N_{\Gamma^1_O}(\mathfrak{p}_2)$ normalise $\mathfrak{p}_2$, i.e. $xOx^{-1} = O$. Let $\gamma \in \Gamma^1_O(\mathfrak{p}_2)$. Since the class number of $k$ is one, every two-sided $O$-ideal is principal and we can choose $\Pi_2 \in O$ such that $\Pi_2O = \mathfrak{p}_2$. Moreover, as $\mathfrak{p}_2$ is uniquely determined by the property that the $\mathfrak{p}_2$-ideal $\text{Nrd}(\mathfrak{p}_2)$ is $p_2$, we can choose $\Pi_2$ such that $\text{Nrd}(\Pi_2) = 2$. Then $\gamma = \pm(1 + m\Pi_2)$ with $m \in O$. For $x \in N_{\Gamma^1_O}$ we have

$$x^2x^{-1} = 1 + xmx^{-1} = 1 + mx^m \Pi_2^{-1} \text{ with some } m' \in O.$$ 

The element $x\Pi_2x^{-1}$ lies in $O$ and $\text{Nrd}(x\Pi_2x^{-1}) = \text{Nrd}(\Pi_2) = 2$. Since $\mathfrak{p}_2 = (\Pi_2)$ is the unique prime ideal over 2, $x\Pi_2^{-1} \in \mathfrak{p}_2$ and $x\gamma x^{-1} \in \Gamma^1_O(\mathfrak{p}_2)$. It follows that the normaliser of $\Gamma^1_O(\mathfrak{p}_2)$ is $N^+_{\Gamma^1_O}(\mathfrak{p}_2)$. This leads to an exact sequence

\begin{equation} 
\begin{align*}
1 \longrightarrow \Gamma^1_O/\Gamma^1_O(\mathfrak{p}_2) \longrightarrow N^+_{\Gamma^1_O}(\mathfrak{p}_2) \longrightarrow N^+_{\Gamma^1_O}/\Gamma^1_O \longrightarrow 1,
\end{align*}
\end{equation}

which we can write abstractly as

$$1 \longrightarrow \mathbb{Z}/5\mathbb{Z} \longrightarrow \text{Aut}(X_{\Gamma^1_O}(\Pi_2)) \longrightarrow \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \longrightarrow 1.$$ 

Let $\lambda \in O^1$ satisfy $\lambda^5 = -1$, i.e. $\lambda$ gives rise to a 5-torsion in $\Gamma^1_O$. Then $\lambda$ satisfies the equation $\lambda^2 - \frac{1+\sqrt{5}}{2}\lambda + 1 = 0$ over $k$. We can assume that $\lambda$ generates $\Gamma^1_O/\Gamma^1_O(\mathfrak{p}_2)$. Let $g = \lambda + 1$. The

\[^1\] Note that the symbol $\left(\frac{p}{\mathfrak{p}}\right)$ in Theorem 4.8 of [21] for $p = 2$ should be read as the Kronecker symbol, i.e. $\left(\frac{p}{\mathfrak{p}}\right) = 1 \iff d \equiv \pm 1 \mod 8$ and $= -1 \iff d \equiv \pm 3 \mod 8$.\]
reduced norm of $g$ is $\text{Nrd}(g) = (\lambda + 1)(\overline{\lambda} + 1) = \text{Nrd}(\lambda) + \text{Trd}(\lambda) + 1 = 2 + \frac{1 + \sqrt{5}}{2} = \frac{5 + \sqrt{5}}{2}$, where $\text{Trd}$ is the reduced trace. Since $\frac{5 + \sqrt{5}}{2}$ is a totally positive generator of the prime ideal over 5, $g$ defines an element of $\mathcal{N}\mathcal{G}^{+}_1$ (see (1.2)). On the other hand $g^2 = (\lambda + 1)^2 = \lambda^2 + 2\lambda + 1 = (\frac{1 + \sqrt{5}}{2}\lambda - 1) + 2\lambda + 1 = (\frac{5 + \sqrt{5}}{2})\lambda$. This shows that $g$ has order 10 in $\mathcal{N}\mathcal{G}^{+}_1$ and hence gives an element of order 10 in $\mathcal{N}\mathcal{G}^{+}_1/\mathcal{G}^{1}_1\mathcal{Q}_2$. Moreover the image of $g$ in $\mathcal{N}\mathcal{G}^{+}_1/\mathcal{G}^{1}_1\mathcal{O}$ is not trivial. Using the computer algebra system PARI, we can check that both ramified primes $p_2$ and $p_5$ are not split in $k$ and $B = k\Pi_2$, $\mathcal{N}\mathcal{G}^{+}_1\mathcal{O} = \mathcal{G}^{1}_1\mathcal{Q}_2$. Hence, $\Pi_2$, considered as an element of $\mathcal{N}\mathcal{G}^{+}_1$, is of order 2 and the images of $g$ and $\Pi_2$ in $\mathcal{N}\mathcal{G}^{+}_1/\mathcal{G}^{1}_1\mathcal{O}$ generate this group.

**Lemma 4.13.** Let $g$ and $\Pi_2$ be elements constructed above Then in $\mathcal{N}\mathcal{G}^{+}_1$ we have the relation $\Pi_2g\Pi_2 = g^{-1}$ modulo $\mathcal{G}^{1}_1\mathcal{Q}_2$.

**Proof.** The element $\Pi_2$ generates $\mathcal{Q}_2$. Consider $g$ and $\Pi_2$ as the elements of the localisation $B_{p_2}$ of $B$ at $p_2$. This is a division quaternion algebra over $k_{p_2}$ and has a representation

$$B_{p_2} = L_{p_2} \oplus \Pi_2 L_{p_2},$$

where $L_{p_2}$ is the unique unramified quadratic extension of $k_{p_2}$ (see [21], p.34). For every $t \in L_{p_2}$ we have $t\Pi_2 = \Pi_2 \overline{t}$, where $\overline{t}$ is the Galois-conjugate of $t$ in $L_{p_2}$. The element $g$ lies in $k_{p_2}(\lambda) = k_{p_2}(\xi_5)$ which is an unramified quadratic extension of $k_{p_2}$, so $L_{p_2} = k_{p_2}(\lambda)$. Therefore $g \in L_{p_2}$ and $g\Pi_2 = \Pi_2 g$. Because $g\mathcal{Q}$ is in $k^*$ we have that $\mathcal{Q} = g^{-1}$ considered as an element of $\mathcal{N}\mathcal{G}^{+}_1 \subset B_{p_2}^*/k_{p_2}^*$. This gives a relation $\Pi_2 g \Pi_2 = g^{-1}$ in $\mathcal{N}\mathcal{G}^{+}_1$, since $\Pi_2^2 = 1$ in $\mathcal{N}\mathcal{G}^{+}_1$. Also, $g$ and $\Pi_2 g \Pi_2 = g^{-1}$ are not equal modulo $\mathcal{G}^{1}_1\mathcal{Q}_2$ because this would imply that $g^2 \in \mathcal{G}^{1}_1\mathcal{Q}_2$. But as $\mathcal{G}^{1}_1\mathcal{Q}_2$ is torsion-free and $g^2$ is of finite order, this is impossible. □

**Proposition 4.14.** With above notations we have

$$\text{Aut}(X_{\mathcal{G}^{1}_1\mathcal{Q}_2}) \cong \mathbb{D}_{10}.$$ 

**Proof.** By the above discussion, $\text{Aut}(X_{\mathcal{G}^{1}_1\mathcal{Q}_2})$ is of order 20 and is generated by elements $g$ of order 10 and $\Pi_2$ of order 2 satisfying $\Pi_2 g \Pi_2 = g^{-1}$. The only group of order 20 with these relations is $\mathbb{D}_{10}$. □

4.4. A fake quadric with automorphism group of order 8. We consider $k = \mathbb{Q}(\sqrt{5})$ and $B = B(k, p_2 p_{11})$, the unique quaternion algebra ramified exactly at the primes $p_2$ and $p_{11}$. Since 2 is inert and 11 is decomposed in $k$, Shimizu’s volume formula gives $c_2(X_{\mathcal{G}^{1}_1}) = \frac{4}{\sqrt{12}}(4 - 1)(11 - 1) = 2$ as the value of the second Chern number of the quotient $X_{\mathcal{G}^{1}_1}$, where again $\mathcal{G}^{1}_1$ is the norm-1 group of a maximal order in $B$. As before, results of [21] show that $\mathcal{G}^{1}_1 \mathcal{O}$ contains only torsion elements of order 2 and no other torsions (Here, observe that 2 is split in $\mathbb{Q}(\sqrt{-25})$, hence, by [21], Theorem 4.8 there are no elements of order 3 in $\mathcal{G}^{1}_1\mathcal{O}$, and note that there are no elements of order 5 because $11 \equiv 1 \mod 5$ which implies that $p_{11}$ is split in $k(\xi_5)$). Since $p_{11}$ is ramified in $B$, there is the unique prime ideal $\mathcal{P}_{11}$ in $\mathcal{O}$ such that $\mathcal{P}_{11}^2 = p_{11}\mathcal{O}$. Consider the principal congruence subgroup

$$\Gamma^1_{\mathcal{O}}(\mathcal{P}_{11}) = \{x \in \Gamma^1_{\mathcal{O}} \mid x \equiv 1 \mod p_{11}\}. $$

It is a normal subgroup in $\Gamma^1_{\mathcal{O}}$. The quotient $\Gamma^1_{\mathcal{O}}/\Gamma^1_{\mathcal{O}}(\mathcal{P}_{11})$ is isomorphic to $\mathcal{O}^1 / \pm \mathcal{O}^1(\mathcal{P}_{11})$ because $-1 \notin \mathcal{O}^1(\mathcal{P}_{11})$. In order to compute the latter quotient we change over to the
localisation at the prime $p_{11}$. Let
\[ B_{p_{11}} = B \otimes_k k_{p_{11}} = B \otimes_k \mathbb{Q}_{11}. \]
It is the unique division quaternion algebra over $\mathbb{Q}_{11}$. Let us write $\mathcal{O}_{p_{11}} = \mathcal{O} \otimes_k \mathbb{Z}_{11}$ for its maximal order. As in the previous example let $\hat{\mathcal{P}}_{11}$ denote the prime ideal of $\mathcal{O}_{p_{11}}$. We have
\[ \mathcal{O}^1/\mathcal{O}^1(\hat{\mathcal{P}}_{11}) \cong \mathcal{O}_{p_{11}}^1/\mathcal{O}_{p_{11}}^1(\hat{\mathcal{P}}_{11}) \]
by the strong approximation theorem. By C. Rieml’s result, \[12\], Theorem 7,
\[ \mathcal{O}_{p_{11}}^1/\mathcal{O}_{p_{11}}^1(\hat{\mathcal{P}}_{11}) \cong \ker((\mathcal{O}_{p_{11}}/\hat{\mathcal{P}}_{11})^\ast \xrightarrow{N_r} (\mathcal{O}_{k_{p_{11}}}/p_{11})^\ast) \cong \ker(F_{121}^\ast \rightarrow F_{11}^\ast). \]
Since $F_{121} = F_{11}(\xi_{12})$, where $\xi_{12}$ denotes a primitive twelfth root of unity we conclude that $\mathcal{O}_{p_{11}}^1/\mathcal{O}_{p_{11}}^1(\hat{\mathcal{P}}_{11})$ is isomorphic to $\mu_{12} = \langle \xi_{12} \rangle$. Hence
\[ \Gamma_1^O/\Gamma_1^B(\mathcal{P}_{11}) \cong \mathcal{O}_{p_{11}}^1/\pm \mathcal{O}_{p_{11}}^1(\hat{\mathcal{P}}_{11}) \cong \mu_6 = \langle \xi_6 \rangle. \]
Let us now define an intermediate group
\[ \Gamma = \{ x \in \Gamma_1^O | x \mod \mathcal{P}_{11} \in \langle \xi_6^2 \rangle \subset \mu_6 \}. \]
$\Gamma < \Gamma_1^O$ is a subgroup of index 2, hence $e_2(X_\Gamma) = 4$. Moreover, $\Gamma$ is torsion-free since it can not contain elements of order 2. For if an order-two element $x$ is in $\Gamma$, then its image $x$ mod $\mathcal{P}_{11}$ in $\Gamma_1^O/\Gamma_1^B(\mathcal{P}_{11})$ lies in a cyclic group $\langle \xi_6^2 \rangle$ of order three, hence it must be the identity. But this means that $x$ is in $\Gamma_1^B(\mathcal{P}_{11})$. On the other hand $\Gamma_1^O(\mathcal{P}_{11})$ is torsion-free because it embeds in a pro-11 group $\mathcal{O}_{p_{11}}^1(\hat{\mathcal{P}}_{11})/\pm 1$. This contradicts the assumption on $x$. All this shows that $X_\Gamma$ is a fake quadric.

**Proposition 4.15.** Let $\mathcal{N}_O^+$ be defined as in \[4.3\]. Then $\mathcal{N}_O^+$ is the normaliser of $\Gamma$ and $\mathcal{N}_O^+\Gamma$ is isomorphic to $D_4$.

**Proof.** As a subgroup of index two in $\Gamma_1^O$ the group $\Gamma$ is normal in $\Gamma_1^B$. On the other hand, for the same reason as in the previous example, $\Gamma_1^O(\mathcal{P}_{11})$ as well as $\Gamma_1^O$ is normal subgroup in $\mathcal{N}_O^+$. This already implies that $\Gamma$ is normal in $\mathcal{N}_O^+$ because any conjugate of $\Gamma$ will be a subgroup between $\Gamma_1^O(\mathcal{P}_{11})$ and $\Gamma^O$ of index 2 in $\Gamma_1^B$. There is only one such group, namely $\Gamma$, since $\Gamma_1^B/\Gamma_1^O(\mathcal{P}_{11}) \cong \mathbb{Z}/6\mathbb{Z}$. Similar exact sequence as \[4.3\] now shows that $\Aut(X_\Gamma)$ is an extension of $\mathbb{Z}/2\mathbb{Z}$ by the Klein’s four group. Since the $2$-torsions in $\Gamma_1^B$ come from embeddings of fourth root of unity into $\mathcal{O}$ there is $\lambda \in \mathcal{O}^1$ such that $\lambda^2 = -1$. Let $g = \lambda + 1$. Then, as $\text{Trd}(\lambda) = 0$, we have $\text{Nrd}(g) = (\lambda + 1)(\lambda + 1) = 2$ and also $g^2 = (\lambda + 1)^2 = 2\lambda$ which implies that $g$ defines an element of order 4 in $\mathcal{N}_O^+$ and hence an element of order 4 in $\mathcal{N}_O^+\Gamma$. Moreover, the image of $g$ in $\mathcal{N}_O^+/\Gamma_1^O$ is not trivial. Since both prime divisors 2 and $\pi_{11}$ of the reduced discriminant do not split in $k(\sqrt{-\pi_{11}})$ (as can be checked using PARI for instance), the element $\Pi_{11} = \sqrt{-\pi_{11}}$ is in $B$ and moreover $\Pi_{11}$ defines an element of $\mathcal{N}_O^+$ of order 2 such that the images of $\Pi_{11}$ and $g$ in $\mathcal{N}_O^+/\Gamma_1^B$ generate this group. Same argument as in Lemma \[4.14\] gives a relation between $\Pi_{11}$ and $g$: Consider $\Pi_{11}$ as the generator of the prime ideal $\mathcal{P}_{11}$. Locally, $B_{p_{11}}$ can be written as $B_{p_{11}} = L_{p_{11}} \oplus \Pi_{11} L_{p_{11}}$, where $L_{p_{11}} = k_{p_{11}}(\xi_{12})$ is the unique unramified quadratic extension of $k_{p_{11}} \cong \mathbb{Q}_{11}$ with the multiplication rule $t\Pi_{11} = \Pi_{11} t^{-1}$ for all $t \in L_{p_{11}}$. The element $g$ is in $L_{p_{11}}$, namely $g = 1 + \xi_{12}^3$. Then $g\Pi_{11} = \Pi_{11} g = h(1 + \xi_{12}) = (1 + \xi_{12}^2)$. In $\mathcal{N}_O^+$ the relations $g = g^{-1}$ and $\Pi_{11}^2 = 1$ hold, hence $\Pi_{11} g \Pi_{11} = g^{-1}$ in $\mathcal{N}_O^+$, $\mu_6 = \langle \xi_6 \rangle$. Also $g \neq g^{-1}$ modulo $\Gamma$, since otherwise $g^2$ would be in $\Gamma$ which is not possible because $g^2$ is torsion and $\Gamma$.
Proposition 4.17. We have $\Gamma/\Gamma$ isomorphic to $\mathbb{D}_4$ which is the only group of order 8 generated by two elements $\Pi_{11}$ of order 2 and $g$ of order 4 with $h \neq g^2$ and $\Pi_{11}g\Pi_{11} = g^{-1}$.

**Remark 4.16.** Considering $k = \mathbb{Q}(\sqrt{13})$, the quaternion algebra $B = B(k, \mathbb{Z})$ and $\Gamma = \Gamma_1^1(\mathbb{Q}_3)$, the arguments as in the examples before will show that $X_{\Gamma_0^1}(\mathbb{Q}_3)$ is a fake quadric whose automorphism group is isomorphic to $\mathbb{D}_4$.

4.5. A fake quadric with an automorphism group $\mathbb{D}_8$. This time we consider the quadratic field $k = \mathbb{Q}(\sqrt{2})$ and the quaternion algebra $B = B(k, \mathbb{Z})$. The norm-1 group $\Gamma_0^1$ of a maximal order in $B$ contains torsion elements of order 3, but no elements of order 2, because $p_3$ is decomposed in $k(\sqrt{-1})$. The second Chern number of the quotient $X_{\Gamma_0^1}$ is $c_2(X_{\Gamma_0^1}) = (9 - 1)/6 = 4/3$. Let $\Gamma_1^1(\mathbb{Z}_2)$ be the principal congruence subgroup corresponding to the prime ideal $\mathbb{Z}_2 \subset \mathbb{O}$, defined by the relation $\mathbb{Z}_2^2 = p_3\mathbb{O}$. Again by Riehm’s theorem and with arguments as in Section 4.3, $\Gamma_1^1(\mathbb{Z}_2)$ is torsion free normal subgroup in $\Gamma_0^1$ of order 3, hence $X_{\Gamma_0^1}(\mathbb{Z}_2)$ is a fake quadric. The automorphism group $\text{Aut}(X_{\Gamma_0^1}(\mathbb{Z}_2))$ is isomorphic to the factor group $\mathbb{N}\Gamma_0^1/\Gamma_0^1(\mathbb{Z}_2)$.

which is an extension of $\Gamma_0^1/\Gamma_0^1(\mathbb{Z}_2) \cong \mathbb{Z}/3\mathbb{Z}$ by $\mathbb{N}\Gamma_0^1/\Gamma_0^1 \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

**Proposition 4.17.** We have $\text{Aut}(X_{\Gamma_0^1}(\mathbb{Z}_2)) \cong \mathbb{D}_6$.

**Proof.** Let $\lambda \in \mathcal{O}_1$ be an element with $\lambda^3 = -1$ and $g = \lambda + 1$. Such $\lambda$ exists since $\Gamma_0^1$ contains 3-torsions. We can take $\pm \lambda$ to be the generator of $\Gamma_0^1/\Gamma_0^1(\mathbb{Z}_2)$. Since $\text{Trd}(\lambda) = 1$, we have $\text{Nrd}(g) = 3$ which implies that $g$ defines an element in $\mathbb{N}\Gamma_0^1$. Additionally $g^2 = \lambda^2 + 2\lambda + 1 = 3\lambda$ which means that $g$ has order 6 considered as an element of $\mathbb{N}\Gamma_0^1$. The totally positive element $\pi_2 = 2 + \sqrt{2} \in k$ generates $\mathbb{Z}_2$ and since neither $\pi_3 = 3$ nor $\pi_2$ are split in $k(\sqrt{-1})$, $\pi_2 = \sqrt{-\pi_2}$ lies in $B$ and defines an element in $\mathbb{N}\Gamma_0^1$ of order 2 such that the classes of $g$ and $\Pi_2$ in $\mathbb{N}\Gamma_0^1/\Gamma_0^1$ generate this group. In particular, $\Pi_2$ is a generator of $\mathbb{Z}_2$. Locally $B_{\mathbb{F}_2} = L_{\mathbb{F}_2} \oplus \Pi_2L_{\mathbb{F}_2}$, where $L_{\mathbb{F}_2} = \mathbb{F}_2(\xi_6)$ is the unramified quadratic extension of $k_{\mathbb{F}_2} \cong \mathbb{F}_2$. As in previous examples, $g$ lies in $L_{\mathbb{F}_2}$ and $\Pi_{2g} = \pi_2 = g^{-1}$ in $\mathbb{N}\Gamma_0^1$. This gives a relation $\Pi_2g\Pi_2 = h^{-1}$ in $\mathbb{N}\Gamma_0^1/\Gamma_0^1(\mathbb{Z}_2)$. As $\Pi_2$ is not a power of $g$, the finite group generated by $g$ and $\Pi_2$ is isomorphic to $\mathbb{D}_6$. □

4.6. Automorphism groups of order 16 and 24. There are more examples of quaternionic fake quadrics with a non-trivial automorphism group. For instance, all examples in Shavel’s paper have $\mathbb{Z}/2\mathbb{Z}$ or $(\mathbb{Z}/2\mathbb{Z})^2$ as the full group of automorphisms. As in previous examples we show

**Proposition 4.18.** Let $B(\mathbb{Q}(\sqrt{2}), \mathbb{Z}_2)$ be the indefinite quaternion algebra over $k = \mathbb{Q}(\sqrt{2})$ with reduced discriminant $d_B = \mathbb{Z}_2$ and $\Gamma_0^1(\mathbb{Z}_2)$ the congruence subgroup in $\Gamma_0^1$ corresponding to a maximal order $\mathcal{O}$ in $B$ with respect to the prime ideal $\mathbb{Z}_2$ of $\mathcal{O}$ lying over the ramified prime $\mathbb{Z}_2$. Then $X_{\Gamma_0^1}(\mathbb{Z}_2)$ is a fake quadric with the automorphism group $\text{Aut}(X_{\Gamma_0^1}(\mathbb{Z}_2)) \cong \mathbb{D}_8$.

**Proof.** The proof goes along the same lines as in the examples before. By Riehm’s Theorem, $\Gamma_0^1/\Gamma_0^1(\mathbb{Z}_2) \cong \mathbb{Z}/4\mathbb{Z}$ and we obtain $c_2(X_{\Gamma_0^1}(\mathbb{Z}_2)) = 4$ by Shimizu’s formula. By Shavel’s criterion for the existence of torsions, we find that the maximal order $\mathcal{O}$ contains a primitive eighth root of unity $\lambda$ which leads to an element of order 4 in $\Gamma_0^1$. We can take $\lambda$ as a generator of this quotient. As in the examples before take $g = 1 + \lambda$. Then, as $\lambda$ satisfies $\lambda^2 - \sqrt{2}\lambda + 1 = 0$ over $k$, $\text{Nrd}(g) = \text{Nrd}(\lambda + 1) = 2 + \sqrt{2}$, hence $g$ defines an element in $\mathbb{N}\Gamma_0^1$. We have $g^2 = \lambda^2 + 2\lambda + 1 = \sqrt{2}\lambda + 2\lambda = (2 + \sqrt{2})\lambda$. Hence, $g$ is an element of order 8 in
\( NT_\mathcal{O}^+ \) and its image in \( NT_\mathcal{O}^+/\Gamma_\mathcal{O}^+ \) is not trivial. The rational prime 7 is split in \( k \), hence, there are two possible choices of \( \mathfrak{p}_7 \). Fix a prime \( \mathfrak{p}_7 = (\pi_7) \) \((\pi_7 = 3 + \sqrt{7} \text{ say})\). Both \( \pi_7 \) as well as \( \pi_2 \) are ramified in \( k(\sqrt{-\pi_7}) \), hence \( \sqrt{-\pi_7} \in B \) defines an element \( \Pi_7 \in B \) which defines an order-2 element in \( NT_\mathcal{O}^+ \). As in the previous examples we have \( \Pi_7 g \Pi_7 = \overline{g} \) because locally in \( B_{\mathfrak{p}_7} \), \( \Pi_7 = \sqrt{-\pi_7} \) generates the unique prime ideal of the maximal order \( \mathcal{O}_{p_7} \) and \( g \) lies in the unramified quadratic extension \( L_{\mathfrak{p}_7} = \mathcal{O}_7(\xi_8) \). This gives a relation \( \Pi_7 g \Pi_7 = g^{-1} \) in \( NT_\mathcal{O}^+/\Gamma_\mathcal{O}^+(\mathfrak{p}_7) \). Also \( \Pi_7 \) is not a power of \( g \) modulo \( \Gamma_\mathcal{O}^+(\mathfrak{p}_7) \) since the reduced norms of \( \Pi_7 \) and \( g \) are different primes. The only group of order 16 with these relations is \( \mathbb{D}_8 \). □

Let us finally sketch the construction of a fake quadric with an automorphism group of order 24.

**Proposition 4.19.** Let \( B(\mathbb{Q}(\sqrt{3}), \mathfrak{p}_2, \mathfrak{p}_3) \) be the indefinite quaternion algebra over \( k = \mathbb{Q}(\sqrt{3}) \) ramified over the prime ideals \( \mathfrak{p}_2 \) and \( \mathfrak{p}_3 \) and let \( \Gamma_\mathcal{O}^1(\mathfrak{p}_2 \mathfrak{p}_3) \triangleleft \Gamma_\mathcal{O}^1 \) be the principal congruence subgroup with respect to the principal ideal \( \mathfrak{p}_2 \mathfrak{p}_3 \) of a maximal order \( \mathcal{O} \subset B \) lying over \( \mathfrak{p}_2 \mathfrak{p}_3 \). Then \( X_{\Gamma_\mathcal{O}_2(\mathfrak{p}_2 \mathfrak{p}_3)} \) is a fake quadric with \( \lvert \text{Aut}(X_{\Gamma_\mathcal{O}_2(\mathfrak{p}_2 \mathfrak{p}_3)}) \rvert = 24 \). \( \text{Aut}(X_{\Gamma_\mathcal{O}_2(\mathfrak{p}_2 \mathfrak{p}_3)}) \) contains a cyclic subgroup of order 12.

**Remark 4.20.** The full automorphism group in this case has order 24. To our knowledge, this is the largest known automorphism group of a fake quadric. The precise abstract group structure of \( \text{Aut}(X_{\Gamma_\mathcal{O}_2(\mathfrak{p}_2 \mathfrak{p}_3)}) \) is not known to us, since the local method, used in previous examples does not apply directly in this case.

**Proof.** That \( X_{\Gamma_\mathcal{O}_2(\mathfrak{p}_2 \mathfrak{p}_3)} \) has the correct numerical invariants follows again from Riehm’s Theorem, Shimizu’s formula and the observation that for the index we have \( [\Gamma_\mathcal{O}^1 : \Gamma_\mathcal{O}^1(\mathfrak{p}_2 \mathfrak{p}_3)] = [\Gamma_\mathcal{O}^1 : \Gamma_\mathcal{O}^1(\mathfrak{p}_2)][\Gamma_\mathcal{O}^1 : \Gamma_\mathcal{O}^1(\mathfrak{p}_3)] \). By Shavel’s criterion, \( B \) contains \( k(\xi_{12}) \) where \( \xi_{12} \) is a primitive twelfth root of unity, hence there is an element \( \lambda \in \mathcal{O} \) with \( \lambda^6 = -1 \). So to show that \( \Gamma_\mathcal{O}^1(\mathfrak{p}_2 \mathfrak{p}_3) \) is torsion free we have to exclude the existence of 6-torsions in \( \Gamma_\mathcal{O}^1(\mathfrak{p}_2 \mathfrak{p}_3) \). But since the reduced trace of \( \lambda \) is \( \pm \sqrt{3} \) which is not congruent 2 modulo \( \mathfrak{p}_2 \mathfrak{p}_3 \), \( \lambda \) is not contained in \( \Gamma_\mathcal{O}^1(\mathfrak{p}_2 \mathfrak{p}_3) \). The element \( g = \lambda + 1 \) has norm \( \text{Nrd}(g) = 2 + \sqrt{3} \) which is a totally positive unit of \( \mathcal{O}_k \) unit, hence \( g \) lies in \( \Gamma_\mathcal{O}^1 = \mathcal{O}^+ / \mathcal{O}_k^+ \), where \( \mathcal{O}^+ \) denotes the group of all units whose reduced norm is totally positive. The group \( \Gamma_\mathcal{O}^+ \) which is an index-2-extension of \( \Gamma_\mathcal{O}^1 \) since the fundamental unit \( 2 + \sqrt{3} \) is totally positive. Also \( g^2 = (2 + \sqrt{3}) \lambda \) which shows that \( g \) has order 12 in \( \Gamma_\mathcal{O}^1 \triangleleft NT_\mathcal{O}^+ \). The image of \( g \) in \( NT_\mathcal{O}^+ / \Gamma_\mathcal{O}^+ \) is not trivial and the discussion in [21], p. 223-224 shows that \( NT_\mathcal{O}^+ / \Gamma_\mathcal{O}^+ \) is generated by the class of an element \( \Pi \in NT_\mathcal{O}^+ \) with \( \text{Nrd}(\Pi) = \pi_2 \pi_3 \) where \( \mathfrak{p}_2 = \langle \pi_2 \rangle, \mathfrak{p}_3 = \langle \pi_3 \rangle \) (note that the generators \( \pi_2 \) and \( \pi_3 \) can not be chosen totally positive). Therefore, \( \text{Aut}(X_{\Gamma_\mathcal{O}_2(\mathfrak{p}_2 \mathfrak{p}_3)}) \) is of order 24 and is an extension of \( \mathbb{Z}/6\mathbb{Z} \) by the Klein’s four group. □

## 5. Computations of the quotient surfaces

Let \( S \) be a quaternionic fake quadric, \( G \) a group of automorphisms of \( S \), \( X = S/G \) the quotient surface and let \( \pi : Z \to S/G \) be the minimal desingularisation map.

Let us first study the case where \( G \) is generated by an involution \( \sigma \).

**Proposition 5.21.** An involution \( \sigma \) has 4 fixed points. The invariants of \( Z \) are:

\[
K_Z^2 = 4, \quad c_2 = 8, \quad q = p_g = 0, \quad h^{1,1} = 5.
\]

The surface \( Z \) is minimal of general type.
Proof. By Lefschetz formula (Proposition 3.6), $1 = \sum_{s=\sigma(s)} 1$, therefore $\sigma$ has 4 fixed points. Their images in $S/\sigma$ are 4 $A_1$ singularities, resolved by 4 $(-2)$-curves on $Z$. The invariants of $Z$ are easy to compute. The surface $Z$ is of general type and is minimal because $K_Z$ is the pullback of the nef divisor $K_X$. \hfill \Box

Proposition 5.22. Let $G = \mathbb{Z}/3\mathbb{Z}$. The singularities of quotient surface $X$ are $2A_{3,1} + 2A_{3,2}$. The surface $Z$ has general type and:

$$K_Z^2 = 2, \; c_2 = 10, \; q = p_g = 0.$$  

Proof. We use the notations of Zhang’s formula (Proposition 3.7). In this case this formula gives $r_1 + r_2 = 4$. A $A_{3,1}$ singularity is resolved by a $(-3)$-curve and we have

$$K_Z^2 = \frac{8}{3} - \frac{r_1}{3}.$$  

Therefore $r_1 = 2$ and $r_2 = 2$. The singularities of $X = S/\sigma$ are $2A_{3,1} + 2A_{3,2}$. Moreover, as $q = p_g = 0$, we have $c_2 = 10$. $Z$ is of general type by Lemma 3.11. \hfill \Box

Proposition 5.23. There is no quaternionic fake quadric with $G = (\mathbb{Z}/3\mathbb{Z})^2 \subset \text{Aut}X$. 

Proof. Let $\sigma_1, \sigma_2$ be the two generating elements of $G$. Let $p$ be a fixed point of $\sigma_1$. Then as $\sigma_1$ and $\sigma_2$ commute, $\sigma_2p, \sigma_2^2p$ are also fixed points of $\sigma_1$ and there are one or four points fixed by the whole group. Let $p$ be such a fixed point. Then there must be a faithful action of $G$ on $T_{S,p}$. But such a faithful action has elements which have eigenvalue 1. This is impossible as an automorphism has only isolated fixed points. \hfill \Box

Proposition 5.24. Let $G = \mathbb{Z}/4\mathbb{Z}$. The singularities of the quotient $X$ are $2A_{4,1} + 2A_{4,3}$ or $A_1 + 2A_{4,3}$. The invariants of the resolution $Z$ are

$$K_Z^2 = 0, \; c_2 = 12, \; q = p_g = 0$$  
in the first case, and in the second case $Z$ is minimal and satisfies

$$K_Z^2 = 2, \; c_2 = 10, \; q = p_g = 0.$$  

Remark 5.25. Proposition 5.33 gives an example of the first case. 

Proof. Let $s$ be a fixed point of an order 4 automorphism $\sigma$ acting on $S$. As the involution $\sigma^2$ has only isolated fixed points, the eigenvalues of $\sigma$ acting on $T_{S,s}$ cannot be $(i, -1)$ or $(-i, -1)$. Let $a, b, c$ be the number of fixed points such that the eigenvalues of $\sigma$ are $(i, i), (-i, -i)$ and $(i, -i)$ respectively. The Lefschetz holomorphic fixed point formula implies

$$-\frac{a}{2i} + \frac{b}{2i} + \frac{c}{2} = 1 \text{ and } a + b + c = 4 \text{ or } 2,$$

thus there are two cases:

1) $a = b = 1$ and $c = 2$. The singularities of $S/G$ are $2A_{4,1} + 2A_{4,3}$.
2) $a = b = 0$ and $c = 2$. In this case, the singularities of $S/G$ are $A_1 + 2A_{4,3}$ because $\sigma^2$ has 4 fixed points.

A $A_{4,1}$ singularity is resolved by a $(-4)$-curve $C_k$. A $A_{4,3}$ singularity is resolved by a chain of three $(-2)$-curve and we have

$$K_Z = \pi^*K_{S/\sigma} - \sum_{k=1}^{k=2} \frac{1}{2} C_k.$$
thus $K_Z^2 = \frac{8}{3} - 2 = 0$ in the first case. Additionally,

$$e(S/\sigma) = \frac{1}{4}(4 + (4 - 1)4) = 4,$$

thus $c_2(Z) = 4 + 8 = 12$. The invariants in the second case are computed in a similar way. \(\square\)

**Proposition 5.26.** Let $G = \mathbb{Z}/5\mathbb{Z}$. The singularities of $S/G$ are $4A_{5,2}$, or $A_{5,1} + 2A_{5,2} + A_{5,4}$ or $2A_{5,1} + 2A_{5,4}$. The invariants of the surface $Z$ are respectively:

$$
\begin{align*}
K_Z^2 &= 0, c_2 = 12, \\
K_Z^2 &= -1, c_2 = 13, \\
K_Z^2 &= -2, c_2 = 14,
\end{align*}
$$

and in any case $q = p_g = 0$.

**Remark 5.27.** 1) In Proposition 5.30 below, we give an example of a surface such that the quotient by an order 5 automorphism has $2A_{5,1} + 2A_{5,4}$ singularities.

2) For the same reasons as for $(\mathbb{Z}/3\mathbb{Z})^2$ (see Proposition 5.23), there is no fake quadric $X$ with $(\mathbb{Z}/5\mathbb{Z})^2 \subset \text{Aut}X$.

**Proof.** Using the notations of Proposition 5.31, the number of fixed points $r_1 + r_2 + r_3 + r_4$ equals 4. As $e(S/\sigma) = \frac{1}{5}(4 + (5 - 1)4) = 4$ Zhang’s formula yields

$$(a_1, ..., a_4) = \left(0, \frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right),$$

with

$$\sum 4a_ir_i = r_2 + r_3 + 2r_4 = 4.$$

Thus $r_1 = r_4$. Therefore the possibilities for $(r_1, r_2, r_3, r_4)$ are $(0, i, j, 0)$ with $i + j = 4$, or $(1, i, j, 1)$ with $i + j = 2$ or $(2, 0, 0, 2)$. The singularities on the quotient are respectively:

$$4A_{5,2},$$

$$A_{5,1} + 2A_{5,2} + A_{5,4},$$

$$2A_{5,1} + 2A_4.$$

A singularity $A_{5,i}$ ($i = 1, .., 4$) contributes (respectively)

$$-\frac{9}{5}, -\frac{2}{5}, -\frac{2}{5}, 0$$

to $K_Z^2$. Thus the self-intersection number is

$$K_Z^2 = \frac{1}{5}(8 - 9r_1 - 2(r_2 + r_3)),
$$

and according to the possible tuples $(r_1, \ldots, r_4)$ as above: $K_Z^2 = 0$, $K_Z^2 = -1$ and $K_Z^2 = -2$. As $e(S/G) = 4$, we get $c_2 = 12, 13$ or 14 according to the three possible singular loci.

Let us justify our computation of $K_Z^2$. A $A_{5,1}$-singularity is resolved by a $(-5)$-curve $C_5$, thus we have to add $-\frac{2}{5}C_5$ to the canonical divisor. This contributes $(-\frac{2}{5}C_5)^2 = -\frac{2}{5}$ to $K_Z^2$.

On the other hand, a $A_{5,2}$-singularity is resolved by a chain of two curves $C_2, C_3$ with $C_k^2 = -k$.

We have to add $-\frac{2}{5}C_3 - \frac{1}{5}C_2$ to $\pi^*K_X$, and the contribution to $K_Z^2$ is

$$\left(\frac{2}{5}C_3 + \frac{1}{5}C_2\right)^2 = -\frac{2}{5}.$$ 

Finally, note that $A_{5,3} = A_{5,2}$ and that the $A_{5,4}$-singularity doesn’t contribute to $K_Z^2$. \(\square\)
Proposition 5.28. If $G = \mathbb{Z}/6\mathbb{Z}$, then $S/G$ has singularities $2A_{6,1} + 2A_{6,5}$. The minimal resolution $Z$ has invariants:

$$K^2_Z = -4, c_2 = 16, q = p_g = 0.$$

Proof. Let $s$ be a fixed point of an order 6 automorphism $\sigma$. Let $\alpha$ be a primitive third root of unity. By Lemma 3.8, the action of $\sigma$ on $T_{S,s}$ has eigenvalues $(-\alpha, (-\alpha)^a)$ or $(-\alpha^2, (-\alpha^2)^a)$ with $a = 1$ or 5. Let $r_1$, $r_2$ and $r_3$ be respectively the number of fixed points of $\sigma$ with eigenvalues $(-\alpha, -\alpha)$, $(-\alpha^2, -\alpha^2)$ and $(-\alpha, -\alpha^5)$. Lefschetz fixed point formula (Proposition 3.6) implies the relation

$$r_1 \frac{(1 + \alpha)^2}{(1 + \alpha^2)^2} + r_2 \frac{(1 + \alpha^2)^2}{(1 + \alpha^4)^2} + r_3 = 1,$$

therefore $r_1 = r_2$ and $r_1 + r_3 = 1$. By Corollary 3.5, $\sigma$ has 2 or 4 fixed points. The only possibility for $(r_1, r_3)$ is therefore $(1, 2)$. The singularities are $2A_{6,1} + 2A_{6,5}$ and the minimal resolution $Z$ of $S/\sigma$ has $K^2_Z = \frac{8}{3} - 2 \frac{4}{3} = -4$. Moreover $e(Z) = \frac{1}{8}(4 + 5 \cdot 4) + 2 + 2 \cdot 5 = 16$. 

Let us study the case $G = \mathbb{Z}/8\mathbb{Z}$.

Proposition 5.29. Let $\sigma$ be an order 8 element acting on $S$. The singularities of $S/\sigma$ are $2A_{8,3} + 2A_{8,5}$. The resolution $Z$ of the quotient surface is a surface with

$$K^2_Z = -2, c_2(Z) = 14, q = p_g = 0.$$

Proof. Let $p$ be a fixed point of $\sigma$ and let $\xi_p$ be a primitive 8-th root of unity such that $\sigma$ acts on $T_{S,p}$ with eigenvalues $\xi_p, \xi_p^q$ for $q_p \in \{0, \ldots, 7\}$. Since there are no reflexions, we must have $\xi_p^i \neq 1$ and $\xi_p^{q_p} \neq 1$ for $j = 1, \ldots, 7$, thus $q_p$ is prime to 2 and $q_p \in \{1, 3, 5, 7\}$. Let $a_1, a_3, a_5$ and $a_7$ be the number of fixed points with $q_p = 1, 3, 5$ or 7 respectively. We have $\sum a_i = 2$ or 4. By summing over the powers $\sigma^k$ for $k = 1 \ldots 7$ in the formula of the Holomorphic Lefschetz Theorem, we get

$$7 = \sum_{p \in S^p} \frac{1}{\det(1 - d\sigma^k|T_{S,p})},$$

and thus

$$7 = \sum_{u=0}^{3} \sum_{k=1}^{7} \frac{a_{2u+1}}{(1 - \xi^k)(1 - \xi^{k(2u+1)})} = \frac{7}{4}a_1 + \frac{5}{4}a_3 + \frac{9}{4}a_5 + \frac{21}{4}a_7.$$

The possibilities for $(a_1, \ldots, a_4)$ are $(4, 0, 0, 0), (2, 1, 1, 0), (1, 0, 0, 1)$ and $(0, 2, 2, 0)$. For $t^2$ of order 4, we have seen that the singularities of $S/\sigma^2$ are $2A_{4,1} + 2A_{4,3}$ or $A_1 + 2A_{4,3}$. Thus the only possibility for $(a_1, \ldots, a_4)$ is $(0, 2, 2, 0)$, and the singularities of $S/\sigma$ are $2A_{8,3} + 2A_{8,5}$. The Euler number of $S/\sigma$ is

$$e(S/\sigma) = \frac{1}{8}(4 + 7 \cdot 4) = 4.$$

Since $\frac{8}{3} = 3 - \frac{1}{3}$ and $\frac{8}{3} = 2 - \frac{1}{3}$ we get

$$e(Z) = 4 + 2 \cdot 2 + 2 \cdot 3 = 14.$$

It is easy to chek that a singularity $A_{8,3}$ decreases $K^2_Z$ by 1 and a singularity $A_{8,5}$ decreases $K^2$ by $\frac{1}{2}$, thus we obtain: $K^2_Z = \frac{8}{3} - 2 \cdot 1 - 2 \cdot \frac{1}{2} = -2$. \qed
Proposition 5.30. Let S be a fake quadric with $G = \mathbb{Z}/10\mathbb{Z} \subset \text{Aut}(S)$. The singularities of the quotient surface $X = S/G$ are $2A_{10,1} + 2A_{10,9}$. The resolution $Z$ has the invariants:

$$K^2 = -12, c_2 = 24, q = p_g = 0.$$ 

Proof. Let $\sigma$ be an automorphism of order 10 acting on $S$. It has 2 or 4 fixed points. As the involution $\sigma^5$ has 4 fixed points, $\sigma$ cannot have 2 fixed points. Therefore:

$$e(S/G) = \frac{1}{10}(4 + (10 - 1)4) = 4.$$ 

Let $\xi$ be a primitive $5^{th}$-root of unity and $p$ a fixed point. There exist $a = a(p)$ and $b = b(p)$ integers invertible mod 5 such that the action of $\sigma$ on $T_{S,p}$ has eigenvalues $(-\xi^a, -\xi^{ba})$. The Lefschetz holomorphic fixed point formula yields

$$1 = \sum_{p \in S^\sigma} \frac{1}{(1 + \xi^a)(1 + \xi^{ab})}.$$ 

For $b = 1, 2, 3, 4$, the sum $c(b) = \sum_{a=1}^{a=4} \frac{1}{(1+\xi^a)(1+\xi^{ab})}$ is equal to $-4, 1, 1, 6$, respectively. Recall again that $A_{10,3} = A_{10,7}$. For $k \in \{1, 3, 9\}$, let $r_k$ be the number of points in $S^\sigma$ giving a $A_{10,k}$ singularity. By summing the Lefschetz fixed point

$$4 = -4r_1 + r_3 + 6r_9.$$ 

Taking care of the relation $r_1 + r_3 + r_9 = 4$, we have the following possibilities for $(r_1, r_3, r_9)$: $(0, 4, 0), (1, 2, 1)$ and $(2, 0, 2)$. The resolution of a $A_{10,3}$-singularity is a chain of 3 curves $C_2, C_2', C_4$ with intersection numbers $(-2) - (-2) - (-4)$. We have to add $-\frac{1}{9}(C_2 + C_2' + C_4)$ to $\pi^*K_{S/G}$. Each singularity contributes $-\frac{1}{9}(C_2 + C_2' + C_4)^2 = -\frac{6}{9}$ to $K^2_Z$.

Similarly, the resolution of a $A_{10,1}$-singularity is $(-10)$-curve $C_{10}$. A $A_{10,1}$-singularity decreases $K^2_{S/G}$ by $(-\frac{8}{10}C_{10})^2 = -\frac{32}{9}$. When the singularities of $S/G$ are respectively $4A_{10,3}, A_{10,1} + 2A_{10,3} + A_{10,9}$ and $2A_{10,1} + 2A_{10,9}$, we have: $K^2_Z = \frac{8}{10} - \frac{4}{9} = -4, K^2_Z = \frac{8}{10} - \frac{22}{9} - 2 \frac{4}{9} - 0 = -8$ and $K^2_Z = \frac{8}{10} - \frac{22}{9} = -12$. The Euler number of $Z$ is respectively $4 + 4 \cdot 2 = 12, 4 + 1 + 2 + 2 + 9 = 18$ and $4 + 2 + 2 + 9 = 24$. Only the last case is possible because 12 has to divide $K^2_Z + e(Z)$. \hfill $\square$

Proposition 5.31. Let be $G = (\mathbb{Z}/2\mathbb{Z})^2$. The quotient surface $X = S/G$ contains 6 $A_1$ singularities. The surface $Z$ is minimal of general type and has the invariants:

$$K^2_Z = 2, c_2 = 10, q = p_g = 0.$$ 

Proof. A faithful representation of $G$ on a 2-dimensional space contains reflections, therefore by Lemma 3.8 there are no points fixed by the whole $G$. The group $G$ contains 3 involutions. Each of these involutions has 4 isolated fixed points whose image in $X$ are $2A_1$ singularities. Thus there are $6A_1$ singularities on $X = S/G$ and we have

$$e(Z) = e(S/G) + 6 = \frac{1}{4}(4 + 12) + 6 = 10.$$ 

Moreover, $K_Z = \pi^*K_{S/G}$ is nef and $K^2_{S/G} = K^2_S/4 = 2$. By Lemma 3.10 we have $q = p_g = 0$. \hfill $\square$
Remark 5.32. a) As Miles Reid pointed out to us, a minimal surface of general type with $c_1^2 = 2c_2 = 8, p_g = 0$ and automorphism group containing $G = (\mathbb{Z}/2\mathbb{Z})^3$ such that each involution has only isolated points must deform, therefore $(\mathbb{Z}/2\mathbb{Z})^3$ cannot be a subgroup of the automorphism group of a quaternionic fake quadric which are rigid surfaces.

b) For $G = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, the quotient surface $S/G$ has singularities $2A_1 + 2A_3$ and the desingularisation $Z$ has invariants $K_Z^2 = 1, c_2 = 11, q = p_g = 0$. We do not know if a fake quadric $S$ with such automorphism subgroup exists.

Proposition 5.33. Let be $G = \mathbb{D}_4$ acting on the fake quadric $S$. The singularities of $S/G$ are $4A_1 + A_{4,3} + A_{4,1}$. The resolution $Z$ of the quotient surface has invariants:

\[ K_Z^2 = 0, \quad c_2(Z) = 12, \quad q = p_g = 0. \]

The elements of order 4 in $\mathbb{D}_4$ have 4 fixed points.

Proof. Let be $t, a$ be generators of $\mathbb{D}_4$ such that $t^4 = 1, a^2 = 1$ and $at = t^3a$. The elements of order 4 are $t$ and $t^3$. The elements of order 2 are $a, ta, t^2a, t^3a$ and $t^2$.

There cannot be a point of $S$ that is fixed by the whole group $G$ because any faithful 2-dimensional representation of $G$ contains a reflection $(x, y) \to (x, -y)$ and thus such a point would lie on a curve fixed by an involution. But an automorphism of $S$ has only isolated fixed points.

First case: let us suppose that $t$ has 4 fixed points: $\text{Fix}(t) = \{p_1, ap_1, p_2, ap_2\}$. The Euler number of $S/G$ is

\[ e(S/G) = \frac{1}{8} (4 + (2 - 1)(4 \cdot 4) + (4 - 1)4) = 4. \]

The singularities on $S/G$ are $4A_1 + A_{4,3} + A_{4,1}$ thus

\[ e(Z) = 4 + 4 + 3 + 1 = 12. \]

Moreover $K_Z^2 = \frac{8}{8} + (-\frac{1}{2})^2(-4) = 0$.

Second case: suppose that $t$ has 2 fixed points: $\text{Fix}(t) = \{p_1, ap_1\}$. The Euler number of $e(S/G)$ would be

\[ \frac{1}{8} (4 + (2 - 1)(18) + (4 - 1)2) = \frac{7}{2} \]

but this is not an integer. \qed

Proposition 5.34. Suppose that the dihedral group $\mathbb{D}_8$ of order 16 acts on fake quadric $S$. The singularities of $S/\mathbb{D}_8$ are $4A_1 + A_{8,3} + A_{8,5}$. The resolution $Z$ of the quotient surface has invariants:

\[ K_Z^2 = -1, \quad c_2(Z) = 13, \quad q = p_g = 0. \]

Proof. Let be $t, a$ be generators of $\mathbb{D}_8$ such that $t^8 = a^2 = 1$ and $at = t^7a$. Order 8 elements in $G$ are $t, t^3, t^5, t^7$, order 4 elements are $t^2, t^6$, order 2 elements are $a, ta, t^2a, t^3a, t^4a, t^5a, t^6a, t^7a$ and $t^4$.

By the discussion on order 8 elements, $t$ has 4 fixed points $\text{Fix}(t) = \{p_1, ap_1, p_2, ap_2\}$. Let $p$ be a fixed point of an involution $\sigma \neq t^4$. The orbit of $p$ under $G$ has 8 elements, each is a fixed point of an involution $\neq t^4$. The quotient surface has $\frac{1}{8} \cdot 8 \cdot 4A_1 + A_{8,3} + A_{8,5}$ singularities.

We have

\[ e(S/G) = \frac{1}{16} (4 + 1 \cdot (8 \cdot 4) + 7 \cdot 4) = 4 \]

and $e(Z) = 4 + 4 + 2 + 3 = 13$. Moreover $K_Z^2 = \frac{8}{16} - 1 - \frac{1}{2} = -1$. \qed
6. RECONSTRUCTION OF A SURFACE KNOWING ITS QUOTIENT.

In [17], Miyaoka gives a bound on the number of disjoint \((-2)\)-curves on a minimal smooth surface \(Y\). This implies in particular that if \(c_1^2 = 4\) or \(2\) and \(\chi(O_Y) = 1\), there are at most \(4\) or \(6\) such curves respectively. The surfaces we obtained as quotient of quaternionic fake quadrics reach that bound. For the cases \(c_1^2 = 2\) these surfaces seems to be the first known ones with this property.

In [6] Dolgachev, Mendes Lopes, Pardini study rational surfaces with the maximal number of \((-2)\)-curves. For that aim they use and developpe the theory of \((\mathbb{Z}/2\mathbb{Z})^n\)-covers ramified over \(A_1\) singularities. Using their results, we obtain:

**Proposition 6.35.** Let \(Y\) be a smooth minimal surface of general type with \(q = p_g = 0\) and \(2\text{Pic}(Y) = 0\).

a) If \(c_1(Y)^2 = 4\), \(c_2(Y) = 8\) and \(Y\) contains \(4\) disjoint \((-2)\)-curves \(C_1, \ldots, C_4\), then there exist a double cover of \(Y\) ramified over the curves \(C_i\). The minimal model of this covering has invariants \(c_1^2 = 2c_2 = 8\) and \(q \leq 1\).

b) If \(c_1(Y)^2 = 2\), \(c_2(Y) = 10\) and \(Y\) contains \(6\) disjoint \((-2)\)-curves \(C_1, \ldots, C_6\), then there exist a bi-double cover of \(Y\) ramified over the curves \(C_i\). The minimal model of this covering has invariants \(c_1^2 = 2c_2 = 8\).

Let \(2\) be the field with 2 elements. Let be \(C_1, \ldots, C_k\) be \(k\) \((-2)\)-curves on a smooth surface \(Y\). Let

\[\psi : \mathbb{F}_2^k \to \text{Pic}(Y) \otimes \mathbb{F}_2\]

be the homomorphism sending \(v = (v_1, \ldots, v_k)\) to \(\sum v_iC_i\). We say that the curve \(C_j\) appears in the kernel \(\ker\psi\) if there is a vector \(v = (v_1, \ldots, v_k)\) in \(\ker\psi\) such that \(v_j = 1\). For \(v\) in \(\ker\psi\) we denote by \(L_v\) an element of \(\text{Pic}(Y)\) such that \(2L_v = \sum v_iC_i\) (we sometimes identify elements of \(\mathbb{F}_2\) to \(0, 1\) in \(\mathbb{Z}\)). We have:

**Proposition 6.36.** ([6], Proposition 2.3). Suppose that \(2\text{Pic}(Y) = 0\). There exists a unique smooth connected Galois cover \(\pi : Z \to Y\) such that the Galois group of \(\pi\) is \(G = \text{Hom}(\ker\psi, \mathbb{G}_m)\), the branch locus of \(\pi\) is the union of the \(C_i\) appearing in \(\ker\psi\) and the surface \(Z\) obtained by contracting the \((-1)\)-curves over the \((-2)\)-curves in \(Y\) has invariants:

\[K_Z^2 = 2^rK_Y^2 c_2(\mathcal{O}_Z) = \chi(\mathcal{O}_Z) = 2^r\chi(O_Y) - k2^{r-3}, \kappa(Z) = \kappa(Y)\]

where \(r = \dim V\).

**Proof.** (Of Proposition 6.35). We have to prove that for our surface \(Y\), \(\ker\psi\) has the required dimension and that all the curves appear in \(\ker\psi\). For \(c_1^2(Y) = 4\) and \(2\), we have \(b_2(Y) = h^{1,1}(Y) = 6\) and \(8\) respectively. As we supposed that \(2\text{Pic}(Y) = 0\), the space \(\text{Pic}(Y) \otimes \mathbb{F}_2\) is \(h^{1,1}\) dimensional. As \(p_g = 0\), it has moreover a non-degenerate intersection pairing and therefore the dimension of a totally isotropic space in \(\text{Pic}(Y) \otimes \mathbb{F}_2\) is at most \(\left\lfloor \frac{h^{1,1}}{2} \right\rfloor = 3\), or \(4\) dimensional respectively. The image of \(\psi\) is the totally isotropic space generated by the curves \(C_i\), therefore the dimension \(r\) of \(\ker\psi\) is at least \(1\) and \(2\) respectively.

A smooth double cover of a surface with \(n\) nodes can exist only if \(n\) is divisible by \(4\) (see [6]). Therefore the vectors \(v = (v_1, \ldots, v_k)\) in \(\ker\psi\) (of dimension \(\leq 7\)) have weight \(4\) i.e. the number of indices \(j\) such that \(v_j = 1\) is \(4\).

In case a), \(\ker\psi\) is one dimensional, generated by \(w_1 = (1, 1, 1, 1)\). For b), as every vector in \(\ker\psi\) has weight \(4\), by [4] Lemme 1, we have \(k \geq 2^r - 1\) and thus \(r \leq 2\) and \(r \leq 3\) respectively.
Moreover, it is easy to check that in the case b), the space \( \ker \psi \) is (up to permutation of the basis vectors) generated by \( w_1 = (1,1,1,0,0) \) and \( w_2 = (1,1,0,0,1,1) \).

Let us give a bound on the irregularity:

**Lemma 6.37.** Let \( Y \) be a surface of general type with \( \chi = 1 \) and \( q = 0 \) containing a 2-divisible set of 4 \((-2)\)-curves. Let \( Y' \to Y \) be the double cover. Then \( q(Y') \leq 1 \).

**Proof.** As \( q(Y) = 0 \), the involution \( \sigma \) on \( Y' \) given by the cover \( Y' \to Y \) acts as multiplication by \(-1\) on \( H^0(Y', \Omega_Y) \). Therefore, \( \sigma \) acts trivially on \( \wedge^2 H^0(Y', \Omega_Y) \). As \( p_g(Y) = 0 \), the map \( \wedge^2 H^0(Y', \Omega_Y) \to H^0(Y', \wedge^2 \Omega_Y) \) must be 0. Let \( Y' \to Y'' \) be the blow-down map of the 4 \((-1)\)-curves over the 4 nodal curves of \( Y \). If \( q(Y'') \geq 1 \), Castelnuovo-De Franchis Theorem implies that there is a fibration onto a curve \( B \) of genus \( q(Y'') \). By [20], we get that \( q(Y'') \leq 2 \) and if \( q(Y'') = 2 \), then \( Y'' \) is an étale bundle of genus 2 fibers onto a genus 2 curve \( B \) and \( K_{Y''}^2 = 8 \). In that case, there is a commutative diagram

\[
\begin{array}{ccc}
Y'' & \to & X \\
\downarrow & & \downarrow \\
B & \to & \mathbb{P}^1
\end{array}
\]

where the vertical maps are genus 2 fibrations and \( X \) is the surface obtained by contracting the 4\((-2)\)-curves on \( Y \). This diagram is obtained from \( B \to \mathbb{P}^1 \) by taking base change and normalizing. Since \( Y'' \to X \) is unramified in codimension 1, the 6 fibers of \( X \to \mathbb{P}^1 \) occurring at the 6 branch points of \( B \to \mathbb{P}^1 \) are double. Since \( X \) has only 4 singular points, \( X \to \mathbb{P}^1 \) has at least two double fibers contained in the smooth locus of \( X \), but a multiple fiber on a genus 2 fibration cannot exists (because of the adjunction formula). Thus \( q \leq 1 \).

Let us now consider a smooth minimal surface of general type \( Z \) with \( K^2 = 2 \), \( c_2 = 10 \), \( q = p_g = 0 \) such that there is a birational map onto a surface \( Y \) with singularities \( 2A_{3,1} + 2A_{3,2} \).

**Proposition 6.38.** Suppose that \( 3 \text{Pic}(Z) = 0 \). There exists a smooth triple cover \( X \) of \( Y \) ramified precisely over the singularities of \( Y \). The surface \( X \) is of general type and has invariants \( c_1^2 = 2c_2 = 8 \).

**Proof.** Let \( D_1, D_2 \) be the \((-3)\)-curves over the 2 singularities \( A_{3,1} \) and let \( D_3, \ldots, D_6 \) be the \((-2)\)-curves over the singularities \( A_{3,2} \), with indices satisfying: \( D_3D_4 = D_5D_6 = 1 \). Let \( W \to Y \) be the blow up at the intersection points of \( D_3, D_4 \) and of \( D_5, D_6 \). Let \( C_1, \ldots, C_6 \) be the strict transforms of the \( D_i \) in \( W \). Let

\[
\psi : \mathbb{F}_3^6 \to \text{Pic}(W) \otimes \mathbb{F}_3 = H^2(W, \mathbb{F}_3)
\]

be the homomorphism sending \( v = (v_1, \ldots, v_6) \) to \( \sum v_iC_i \). The image of \( \psi \) is a totally isotropic subspace in \( H^2(W, \mathbb{F}_3) \). As \( b_2(W) = 10 \), this image is at most 5 dimensional and therefore \( \text{dim ker} \psi \geq 1 \). Let \( v = (v_1, \ldots, v_6) \in \ker \psi, v \neq 0 \). We choose the representatives of \( \mathbb{F}_3 \) in \( \{0,1,2\} \). There exist a unique invertible sheaf \( L \) such that

\[
3L = \sum v_iC_i.
\]

Let \( T \) be the triple cover of \( W \) ramified over the \( r \) curves \( C_i \) such that \( v_i \neq 0 \). The surface \( T \) is smooth outside the curves \( C_i \) with \( v_i = 2 \). Let \( R \) be the minimal resolution of \( T \) and let \( f : R \to W \) be the composite map. By [23], Propositions 2.2, 4.1 and 4.3, the invariants of \( R \) are:

\[
K_R =_{\text{num}} f^*(K_W + \frac{2}{3}\Sigma), \quad c_2(R) = 3c_2(W) - 4r, \quad \chi(O_R) = 3\chi(O_W) - \frac{r}{3}.
\]
where $\Sigma$ is the sum of the $r$ curves $C_i$ such that $v_i \neq 0$. Therefore $r = 3$ or 6 and

$$K_R^2 = 0, \ c_2(R) = 36 - 4r, \ \chi(O_W) = 3 - \frac{r}{3}.$$ 

As there are at least 3 curves $C_i$ in the branch locus, one of the curves $C_3, \ldots, C_6$ is in that branch locus. Say it is $C_3$. Let $E$ be the exceptional curve going through $C_3$. As $C_3E = C_4E = 1$ and $E \sum v_i C_i$ is divisible by 3, it forces $C_4$ to be also in the branch locus and thus $r = 6$ (and $\dim \ker \psi = 1$). The inverse image of the 6 $(-3)$-curves are $(-1)$-curves. By the formula giving $K_R$, the inverse image of the two exceptional curves are $(-3)$-curves meeting two $(-1)$-curves. We can therefore effectuate 8 blow-downs and we obtain a fake quadric. It has general type because $Y$ has general type, it is minimal because the quotient of a fake plane by an order 3 automorphism with 4 isolated fixed points has $4A_2$ singularities. □

**References**

[1] Atiyah, M. F., Singer, I. M. The index of elliptic operators. I, II, III. *Ann. of Math.* (2) 87 (1968), 484–530, 531–545, 546–604.

[2] Barth, W. P., Hulek, K., Peters, C. A. M., Van de Ven, A. Compact complex surfaces, 2nd ed. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, Bd. 4. Springer-Verlag, Berlin, 2004.

[3] Bauer, I.C., Catanese, F., Grunewald, F. The classification of surfaces with $p_g = q = 0$ isogenous to a product of curves. *Pure Appl. Math. Q.* 4, 2, part 1 (2008), 547–586.

[4] Beauville A. Sur le nombre maximum de points doubles d’une surface dans $\mathbb{P}^3$ ($\mu(5) = 31$). In: *Journées de Géométrie Algébrique d’Angers*, pp. 207–215, Sijthoff Noordhoff, Alphen aan den Rijn, Germany, 1980.

[5] Borel, A. Commensurability classes and volumes of hyperbolic 3-manifolds. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) 8, 1 (1981), 1–33.

[6] Dolgachev, I., Mendes Lopes, M., Pardini, R. Rational surfaces with many nodes. *Compositio Math.*, 132 (2002), no. 3, 349–363.

[7] Džambić, A. Fake quadrics from irreducible lattices acting on the product of upper half planes. In preparation.

[8] Catanese F., Di Scala A. Characterization of Varieties whose Universal Cover is the Polydisk or a Tube Domain. [arXiv:1011.6544](https://arxiv.org/abs/1011.6544)

[9] Granath, H. On quaternionic Shimura surfaces. *Chalmers Tekniska Högskola (Sweden)*, 2002, Thesis (PhD).

[10] Inose, H., Mizukami, M. Rational equivalence of 0-cycles on some surfaces of general type with $p_g=0$. *Math. Ann.* 244 (1979), no. 3, 205–217.

[11] Keum, J. Toward a geometric construction of fake projective plane, *Rend. Lincei Mat. Appl.* 22 (2011), 1–19.

[12] Keum, J. Projective surfaces with many nodes. Algebraic geometry in East Asia Seoul 2008, 245–257, *Adv. Stud. Pure Math.* 60, Math. Soc. Japan, Tokyo, 2010.

[13] Keum, J. Quotients of fake projective planes. *Geom. Topol.* 12, (2008), no. 4, 2497–2515.

[14] Keum, J. A fake projective plane with an order 7 automorphism. *Topology* 45 (2006), no. 5, 919–927

[15] Matsushita, Y., and Shimura, G. On the cohomology groups attached to certain vector-to-value differential forms on the product of the upper half planes. *Ann. of Math.* (2) 78 (1963), 417–449.

[16] Mendes Lopes M., Pardini R. The bicanonical map of surfaces with $p_g = 0$ and $K^2 \geq 7$. *Bull. London Math. Soc.* 33 (3) (2001), 265–274.

[17] Miyake, Y. The maximal number of quotient singularities on surfaces with given numerical invariants. to *Math. Ann.* 268, (1984), 159–171.

[18] Pardini The classification of double planes of general type with $p_g = 0$ and $K^2 = 8$. *J. of Algebra* 259 (2003), 95–118. toto

[19] Riemann, C. Norm-1 group of a $p$-adic division algebra. *Am. Journ. of Math.* 92, 2 (1970), 499–523.

[20] Rouleau X. Quotients of Fano surfaces. *Rend. Lincei Mat. Appl.* 23 (2012), 1–25.

[21] Shavel, I.H. A class of algebraic surfaces of general type constructed from quaternion algebras. *Pacific J. Math.* 76, 1 (1978), 221–245.
[22] Shimura, G. Construction of class fields and zeta functions of algebraic curves. *Ann. of Math.* (2) 85 (1967), 58–159.

[23] Urzua, G. Arrangements of curves and algebraic surfaces. *J. Algebraic Geom.* 19, 2 (2010), 335–365.

[24] Vignéras, M.-F. Arithmétique des algèbres de quaternions. *Lecture Notes in Mathematics*, 800, Springer, Berlin, 1980.

[25] Zhang, D.-Q. Automorphisms of finite order on rational surfaces. With an appendix by I. Dolgachev. *J. Algebra* 238, 2 (2001), 560–589.

[26] Zucconi, F., Surfaces with $p_g = q = 2$ and an irrational pencil. *Canad. J. Math.* 55 (2003), no. 3, 649–672.

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