THREE DIMENSIONAL BLACK HOLES AND FOUR DIMENSIONAL
BLACK STRINGS AS NONLINEAR SIGMA MODELS

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Abstract

Two solutions of stringy gravity in three and four dimensions which admit interpretation as a black hole and a black string, respectively, are discussed. It is demonstrated that they are exact WZWN nonlinear sigma models to all orders in the inverse string tension, and hence represent exact conformal field theories on the world-sheet. Furthermore, since the dilaton for these two solutions is constant, they also solve the equations of motion of standard GR with a minimally coupled three form field strength.

* based on a poster presented in absentio at the 5th Canadian Conference on General Relativity and Relativistic Astrophysics, Waterloo, Ontario, May 13-15, 1993, and a talk presented at the Conference on Quantum Aspects of Black Holes, U. of California, Santa Barbara, California, June 21-27, 1993.
Theory of gravity has remained one of the most challenging problems of physics of our time. The present status of gravity is in many ways equivocal. Whereas in the classical domain it is described exceptionally well by Einstein’s theory of General Relativity (GR) all attempts to construct a consistent quantum theory have been foiled with grave difficulties of both technical and conceptual nature. String theory is one of those attempts, which technically looks extremely attractive, particularly for the reason of its well behaved ultraviolet regime. It still remains to be seen, however, what are the basic principles of string theory, playing the role of its cornerstone, much the same way as the Principle of Equivalence stands in GR.

So far, string theory has lead to particularly fruitful developments in the study of the gravitational sector. Perhaps one of the most important recent achievements was the suggestion how string theory might be able to avoid singularity problems which plague many GR solutions, such as black holes. The extra symmetries present in string theory provide stringent constraints on the behavior of exact solutions, and lead to a host of nonrenormalization theorems. These could be employed to construct exact nonsingular solutions to all orders in both the inverse string tension and genus expansions. In this review I will reflect on two examples, which can be viewed as a black hole in three and a black string in four dimensions. They do not change even after all higher order corrections from the inverse string tension expansion (equivalently, an expansion in powers of curvature) are included. Thence the two solutions represent exact conformal field theories on the world-sheet. Both have finite curvature everywhere, except at the origin, and hence represent structures with horizon but with controllable divergence.

The dynamics of the bosonic zero mass sector of effective string theory in $D$ dimensions is described by the effective action, in the world sheet frame and to order $O(\alpha'^0)$,

$$S = \int d^D x \sqrt{G} e^{-\sqrt{2}\kappa \Phi} \left( \frac{1}{2\kappa^2} R - H_{\mu\nu\lambda} H^{\mu\nu\lambda} + \partial_{\mu} \Phi \partial^\mu \Phi + \Lambda \right)$$

(1)
Here \( H_{\mu\nu\lambda} = \partial_{[\lambda} B_{\mu\nu]} \) is the field strength associated with the Kalb-Ramond field \( B_{\mu\nu} \) and \( \Phi \) is the dilaton field. Braces denote antisymmetrization over enclosed indices. The cosmological constant has been included to represent the central charge deficit \( \Lambda = \frac{2}{3} \delta c_T = \frac{2}{3} (c_T - D) \geq 0 \). It arises as the difference of the internal theory central charge and the total central charge for a conformally invariant theory \( c_{tot} = 26 \). Note that in the conventions adopted here positive \( \Lambda \) corresponds to a negative cosmological constant.

The three-dimensional black hole solution is the extension of the recent construction of Banados, Teitelboim and Zanelli\(^1\) (BTZ) into the framework of string theory\(^2\). It is incorporated in string theory by the addition of the Kalb-Ramond axion, which in this case is completely determined by the third cohomology group probed by the three-form axion field strength \( H_{\mu\nu\lambda} \). The dilaton is, surprisingly, constant due to the contribution of the axion to the dilaton field equation which cancels the cosmological constant. This solution can be formulated as a nonlinear sigma model on the world-sheet. To show it, one needs to recall the Polyakov action for a nonlinear sigma model on the world-sheet. It is given by \((2\sqrt{2/3} \) arises from normalizations of the wedge product) \[
S_{\sigma} = \frac{1}{\pi} \int d^2 \sigma \left( G_{\mu\nu} + 2\sqrt{2/3} B_{\mu\nu} \right) \partial_+ X^\mu \partial_- X^\nu \tag{2}
\]
where the metric \( G_{\mu\nu} \) and the axion field \( B_{\mu\nu} \) play the role of the effective coupling constants of the 2D field theory defined by (2). In general, there exists plethora of various constructions which lead to a dynamical theory described by (2). One such approach is the Wess-Zumino-Witten-Novikov (WZWN) conformal field theory, which has first arisen in the study of non-abelian bosonization in two dimensions. The WZWN nonlinear sigma model action is defined by \[
S_{\sigma} = \frac{k}{4\pi} \int d^2 \sigma Tr \left( g^{-1} \partial_+ g g^{-1} \partial_- g \right) - \frac{k}{12\pi} \int_M d^3 \zeta Tr \left( g^{-1} dg \wedge g^{-1} dg \wedge g^{-1} dg \right) \tag{3}
\]
where \( g \) is an element of some group \( G \), and \( k \) is the (positive integer) level of the associated Kač-Moody algebra. The action above has a very big global invariance, the continuous part of which is \( G \times G \).
One way to construct the string solutions of this theory, which can be put in form (2), is choosing a group $G$, the parameter space of which represents the target manifold, and maintaining conformal invariance. The other may be to identify a part of the parameter manifold by locally factoring out a subgroup of the global invariance group $G \times G^3$. This is accomplished with choosing an anomaly-free subgroup $H \subset G \times G$ and gauging it with stationary gauge fields. Either way, after the group has been parametrized, (3) can be rewritten in terms of the parameters in the form (2) and the metric and the axion are just simply read off from the resulting expressions. The dilaton then can be computed from the effective action (1), as has been mentioned above. Its appearance owes to the requirement of conformal invariance.

The stringy version of the BTZ black hole can be constructed either as a sigma model on the group $SL(2, R)/P$ or on an extremally axially gauged coset $(SL(2, R) \times R)/(R \times P)$. The group $P$ is a discrete subgroup of $SL(2, R)$ generated by the angular Killing vector of the metric, and is isomorphic with $Z$. It appears as means of identification of the angle $\phi + 2n\pi \rightarrow \phi$. The group $SL(2, R) \times R$ can be parametrized as (with $ab + uv = 1$)

$$\begin{pmatrix} a & u \\ -v & b \end{pmatrix} \exp \left( \frac{q}{\sqrt{k}} \theta' \right)$$

The central charge of this target for the level $k$ is $c_T = 3k/(k - 2) + 1 - 1$, where $\pm 1$ correspond to the free boson and the gauging, respectively. Hence, $c_T = 3k/(k - 2)$ and the cosmological constant is $\Lambda = 4/k$. The gauge transformations from the axial subgroup of $SL(2, R) \times SL(2, R)$ mixed with translations along the free boson are

$$\delta a = 2\epsilon a \quad \delta b = -2\epsilon b \quad \delta u = \delta v = 0 \quad \delta \theta' = \frac{2\sqrt{2}}{q} \epsilon c \quad \delta A_j = -\partial_j \epsilon$$

The remaining steps of the procedure for obtaining the solution are to fix the gauge of the group choosing $b = \pm a$ so that the anomaly cancels, integrate out the gauge fields, rescale $\theta' \rightarrow (2c/\sqrt{k}) \theta'$ and take the limit $c \rightarrow \infty$ which effectively decouples the $SL(2, R)$
part from the gauge fields. The gauged form of the sigma model (3) is
\[
S_\sigma(g, A) = S_\sigma(g) + \frac{k}{2\pi} \int d^2\sigma A_+ \left( b\partial_- a - a\partial_- b - u\partial_- v + v\partial_- u + \frac{4qc}{\sqrt{2k}} \partial_- \theta' \right) \\
+ \frac{k}{2\pi} \int d^2\sigma A_- \left( b\partial_+ a - a\partial_+ b - v\partial_+ u + u\partial_+ v + \frac{4qc}{\sqrt{2k}} \partial_+ \theta' \right) \\
+ \frac{k}{2\pi} \int d^2\sigma 4A_+ A_- \left( 1 + \frac{2c^2}{k} - uv \right)
\]

(6)

where (the Wess-Zumino term vanishes by gauge fixing)
\[
S_\sigma = -\frac{k}{4\pi} \int d^2\sigma \left( \partial_+ u\partial_- v + \partial_- u\partial_+ v + \partial_+ a\partial_- b + \partial_- a\partial_+ b \right) + \frac{q^2}{2\pi} \int d^2\sigma \partial_+ \theta' \partial_- \theta
\]

(7)
The resulting Polyakov sigma model action can be rewritten as
\[
S_{\sigma_{\text{eff}}} = -\frac{k}{8\pi} \int d^2\sigma \frac{v^2 \partial_+ u\partial_- u + u^2 \partial_- v\partial_+ v + (2 - uv) (\partial_+ u\partial_- v + \partial_- u\partial_+ v)}{(1 - uv)} \\
+ \frac{q^2}{2\pi} \int d^2\sigma \left( 2(1 - uv) \partial_+ \theta' \partial_- \theta \right) \\
+ \frac{q\sqrt{k}}{2\sqrt{2\pi}} \int d^2\sigma \left( (u\partial_- v - v\partial_- u) \partial_+ \theta' + (v\partial_+ u - u\partial_+ v) \partial_- \theta \right)
\]

(8)

To extract the solution from (8), one needs to introduce a set of coordinate transformations, which recast (8) into the form of the 3D black hole. The first transformation is \( u = \exp \left( \sqrt{\frac{2}{k}} q t' \right) \sqrt{(R/q)^2 - 1}, \ v = -\exp \left( -\sqrt{\frac{2}{k}} q t' \right) \sqrt{(R/q)^2 - 1} \). To introduce the angular momentum, one can further ”boost” the \( t', \theta' \) coordinates to the new frame \( t, \theta \) and identify along the orbits of \( \zeta = \partial/\partial \theta \). This determines the structure of the group \( P \) introduced above: \( P = \exp(2n\pi \zeta) \), with \( n \) integers. The boost is performed according to \( x^k = \tilde{O}^{kj} x'^j \) where \( \tilde{O} \) is an \( SO(1,1) \) Lorentz transformation, and its parameter is defined by
\[
\sinh \beta = \text{sign}(J) \frac{1}{\sqrt{2}} \left( \frac{1 - \sqrt{1 - (J/M)^2}}{1 - (J/M)^2} \right)^{1/2}
\]

(9)

With more definitions of the parameters, \( \rho_+^2 = q^2 \sqrt{2\Lambda} = M(1 - (J/M)^2)^{1/2}, \ R^2 = (\sqrt{2\Lambda}/2) (\rho^2 + M - \rho_+^2), \ N^\theta = -J/2R^2 \) and \( \Lambda = 4/k \), the final solution is
\[
d s^2 = \frac{d\rho^2}{2\Lambda(\rho^2 - \rho_+^2)} + R^2 (d\theta + N^\theta dt)^2 - \frac{\rho^2}{R^2} \frac{\rho^2 - \rho_+^2}{2\Lambda} dt^2
\]
\[
B_{t\theta} = \frac{\rho^2}{\sqrt{6\Lambda}}
\]

(10)
The dilaton can be found from the associated effective action, or from a careful computation of the Jacobian determinant arising from integrating out the gauge fields. Inspection of the Jacobian before the limit $c \to \infty$ is taken gives $J \propto 1/\left(1 + (2c^2/k) - uv\right) = (k/2c^2)/(1 + (k/2c^2)(1 - uv))$. As $c \to \infty$ the non-constant terms decouple and do not contribute to the dilaton. Thus $\Phi = \Phi_0 = \text{const.}$

The metric part is almost precisely the BTZ solution. The only difference is, the cosmological constant in (10) is half that of what one obtains in ordinary GR in three dimensions. The reason for this discrepancy is, that the presence of the axion introduces an extra contribution to the cosmological constant, which just cancels one half of it, since the dilaton is constant. This property of the solution (10) is interesting, since the absence of the dilaton dynamics guarantees that the solution is also a solution of standard GR with a minimally coupled two form, as can be immediately verified from action (1), after $\Phi = \text{const.}$ is substituted.

The solution (10) can be immediately extended to four dimensions, by tensoring it by a flat direction. The only change in the solution (10) is that in four dimensions there is an extra additive $dz^2$ term in the metric. This solution can be understood as a black string in four dimensions. Its conformal field theory representation is an extremally axially gauged coset $(SL(2, R) \times R^2)/(R \times P)$ on the level $k$. Interpretation of this extension of (10) as a rotating black string is best seen if one replaces the three form $H_{\mu\nu\lambda}$ by its dual. The dual axion field strength $V = \sqrt{6\Lambda}dz = da(z)$ can be integrated between any two spacelike ($t = \text{const}$) hypersurfaces $z_{1, 2} = \text{const}$ to give $a(z_2) - a(z_1) = \sqrt{6\Lambda}\Delta z$. Therefore, the axion solution can be understood as a constant gradient of the pseudoscalar axion field. As $z_{1, 2} \to \infty$, the axion diverges. This is easy to explain: it is a consequence of the assumption that the string is infinitely long. In reality, one should expect some cut-off sufficiently far away along the string. The situation is analogous to that of the electrostatic potential between the plates of a parallel plate capacitor in ordinary electromagnetism. The cut-off occurs on the plates of the capacitor, where the potential is constant. The gradient is just
$\tilde{\nabla}V = (\Delta V/\Delta L)z$. This analogy shows that the black string solution should be viewed as a gravitational configuration which arose inside a transitory region separating two domains within which the axion is constant, $a_1$ and $a_2$ respectively. The axion gradient inside this region corresponds to the adiabatic change in the axion vacuum, where the adiabatic approximation is better if the transitory region (and hence the string) is bigger. The string evidently needs the domain of axionic gradient for its existence (because the axion gradient stops the dilaton from rolling), and thence can be labelled primordial. It should be noted that in four dimensions, the axion also plays role of a Higgs field. The axion condensate $6\Lambda$ in (1) breaks the normal general covariance group $GL(3,1)$ of (1) down to $GL(2,1)$.

Higher order corrections could now be investigated following the recently established resummation procedure$^5)$. It turns out, that both configurations actually survive the corrections, and appear to be exact solutions of string theory to all orders in $\alpha'$. The only effect of the higher order $\alpha'$ corrections is finite renormalization of the parameters in (10), and in particular, renormalization of the semiclassical expression for the cosmological constant.

In summary, in this review it was shown how the complex structure of string theory can be employed for the construction of exact solutions which are consistent to all orders in the inverse string tension expansion. A clear advantage of this program is that the solutions constructed as nonlinear sigma models can be analyzed in relationship to the exact effective action involving higher powers of curvature in a rather elegant way. Furthermore, the two examples exhibited here are found to be exact solutions of the exact effective action, and hence may be good candidates for consistent quantum gravitational configurations.
Acknowledgements

Thanks are due to B. Campbell, G. Hayward, V. Husain, W. Israel and D. Page for useful conversations.

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