ON THE INTEGRAL D’ALEMBERT’S AND WILSON’S FUNCTIONAL EQUATIONS

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Abstract. Let $G$ be a locally compact group, and let $K$ be a compact subgroup of $G$. Let $\mu : G \to \mathbb{C}\{0\}$ be a character of $G$. In this paper, we deal with the integral equations

\begin{align*}
W_\mu(K) : \int_K f(xkyk^{-1})dk + \mu(y) \int_K f(xky^{-1}k^{-1})dk &= 2f(x)g(y), \\
D_\mu(K) : \int_K f(xkyk^{-1})dk + \mu(y) \int_K f(xky^{-1}k^{-1})dk &= 2f(x)f(y)
\end{align*}

for all $x, y \in G$ where $f, g : G \to \mathbb{C}$, to be determined, are complex continuous functions on $G$. When $K \subset Z(G)$, the center of $G$, $D_\mu(K)$ reduces to the new version of d’Almbert’s functional equation $f(xy) + \mu(y)f(xy^{-1}) = 2f(x)f(y)$, recently studied by Davison [18] and Stetkær [35]. We derive the following link between the solutions of $W_\mu(K)$ and $D_\mu(K)$ in the following way: If $(f, g)$ is a solution of equation $W_\mu(K)$ such that $C_Kf = \int_K f(xkyk^{-1})d\omega_K(k) \neq 0$ then $g$ is a solution of $D_\mu(K)$. This result is used to establish the superstability problem of $W_\mu(K)$. In the case where $(G, K)$ is a central pair, we show that the solutions are expressed by means of $K$-spherical functions and related functions. Also we give explicit formulas of solutions of $D_\mu(K)$ in terms of irreducible representations of $G$. These formulas generalize Euler’s formula $\cos(x) = \frac{e^{ix} + e^{-ix}}{2}$ on $G = \mathbb{R}$.

1. Introduction and Preliminaries

1.1. Throughout this paper, $G$ will be a locally compact group, $K$ be a compact subgroup of $G$ and $dk$ the normalized Haar measure of the compact group $K$. The unit element of $G$ is denoted by $e$. The center of $G$ is denoted by $Z(G)$. For any function $f$ on $G$ we define the function $f(x) = f(x^{-1})$ for any $x \in G$. The space of all complex continuous functions on $G$ having compact support is designed by $C_c(G)$. We denote by $\mathcal{C}(G)$ the space of all complex continuous functions on $G$. For each fixed $x \in G$, we define the left translation operator by $(L_x f)(y) = f(x^{-1}y)$ for all $y \in G$.

For a given character $\mu : G \to \mathbb{C}\{0\}$ we consider the following integral equation

\begin{equation}
\int_K f(xkyk^{-1})dk + \mu(y) \int_K f(xky^{-1}k^{-1})dk = 2f(x)g(y), \quad x, y \in G.
\end{equation}

This equation is a generalization of the following functional equations :

\begin{equation}
\int_K f(xkyk^{-1})dk + \mu(y) \int_K f(xky^{-1}k^{-1})dk = 2f(x)f(y), \quad x, y \in G,
\end{equation}

which was studied in [7] when $\mu = 1$ and $(G, K)$ is a central pair.

If $K \subset Z(G)$ the subgroup center of $G$ and $f = g$, (1.1) becomes d’Alembert’s
functional equation

\[(1.3)\quad f(xy) + \mu(y)f(xy^{-1}) = 2f(x)f(y), \quad x, y \in G.\]

In the case where \(\mu = 1\), many authors studied the functional equation (1.3) (see [3], [15], [16], [17], [26], [29], [30], [31], [32], [33], [34], [35], [36], [37], [38], [49]).

When \(f(kxh) = f(x)\) for any \(x \in G\) and \(k, h \in K\), we obtain the functional equation

\[(1.4)\quad \int_K f(xky)dk + \int_K f(xky^{-1})dk = 2f(x)g(y), \quad x, y \in G.\]

If \(K \subset Z(G)\), (1.1) reduces to the following version of Wilson’s functional equation

\[(1.5)\quad f(xy) + \mu(y)f(xy^{-1}) = 2f(x)g(y), \quad x, y \in G.\]

If \(K \subset Z(G)\) and \(\mu = 1\), (1.1) becomes the Wilson’s functional equation

\[(1.6)\quad f(xy) + f(xy^{-1}) = 2f(x)g(y), \quad x, y \in G.\]

If \(f(xk) = \overline{\chi(k)f(x)}\), where \(x \in G\), \(k \in K\) and \(\chi\) is a unitary character of \(K\) we obtain the functional equation

\[(1.7)\quad \int_K f(xky)\overline{\chi(k)}k + \mu(y)\int_K f(xky^{-1})\overline{\chi(k)}dk = 2f(x)g(y), \quad x, y \in G.\]

If \(G\) is compact we can take \(K = G\) and consider the functional equation

\[(1.8)\quad \int_G f(xyt^{-1})dt + \mu(y)\int_G f(xty^{-1}t^{-1})dt = 2f(x)f(y), \quad x, y \in G.\]

The equations (1.4), (1.7) and (1.8) were studied in [1], [7], [9], [10] and [21]. The functional equation (1.6) appeared in several works by H. Steketee, see for example [31], [32] and [33]. For equation (1.3), we refer to the recent studies by Davison [18] and Steketee [36].

1.2. Recall on the central pairs. For a function \(f\) on \(G\), we say that the function \(f\) is \(K\)-central if \(f(kx) = f(xk)\) for all \(k \in K\) and for all \(x \in G\). We put \(\mathcal{K}_K(G) = \{f \in \mathcal{K}(G) : f(kx) = f(xk), x \in G, k \in K\}\). Under convolution, denoted \(\ast\), \(\mathcal{K}_K(G)\) is a subalgebra of the algebra \(\mathcal{K}(G)\). We recall (see [7]) that the pair \((G, K)\) is said to be a central pair if the algebra \((\mathcal{K}_K(G), \ast)\) is commutative.

A non-zero continuous function \(\varphi\) on \(G\) is called \(K\)-spherical function, if

\[(1.9)\quad \int_K \varphi(xkyk^{-1})dk = \varphi(x)\varphi(y), \quad x, y \in G.\]

for all \(x, y \in G\). We will say that a function \(f \in \mathcal{C}(G)\) satisfying

\[(1.10)\quad \int_K f(xkyk^{-1})dk = f(x)\varphi(y) + f(y)\varphi(x), \quad x, y \in G.\]

is associated with the \(K\)-spherical function \(\varphi\). Let \(C_K : \mathcal{C}(G) \rightarrow \mathcal{C}(G)\) be the operator given by

\[(C_K f)(x) = \int_K f(kxk^{-1})dk, \quad x \in G.\]

By easy computations we show that \(f\) is \(K\)-central if and only if \(C_K f = f\). For more results on the operator \(C_K\) we refer to [7, Propositions 2.1, Proposition 2.2].

We say that \(f \in \mathcal{C}(G)\) satisfies the Kannappan type condition if

\[
\int_K \int_K f(zkxk^{-1}hyh^{-1})dkdh = \int_K \int_K f(zkyk^{-1}hxh^{-1})dkdh, \quad x, y \in G \quad (*)
\]
When $K \subset Z(G)$, ($*$) reduces to Kannappan condition $f(xyz) = f(yxz)$ for all $x, y, z \in G$ (see [27]).

The results of the present paper are organised as follows: In section 2 we establish relationship between functional equation (1.1) and (1.2). In Theorem 2.3 we show that if $(f, g)$ is a solution of (1.1) such that $f \neq 0$ and $C_K f \neq 0$, without the assumption that $f$ satisfies ($*$), then $g$ is a solution of (1.2). In section 4 we show that if $(G, K)$ is a central pair and $f$ is a solution of (1.2), then $f$ has the form $f = \frac{\mu + \phi}{2}$ where $\phi$ is a $K$-spherical function. Furthermore we give a complete description of the solutions of equations of (1.1) and (1.2) in the case where $(G, K)$ is a central pair. The solutions are expressed by means of $K$-spherical functions and solutions of the functional equation

\begin{equation}
(1.11) \quad \int_K f(xky^{-1})dk = f(x)\varphi(y) + f(y)\varphi(x), \quad x, y \in G
\end{equation}

in which $\varphi$ is a $K$-spherical function. In Corollaries 4.3 and 4.4 we give explicit formulas of solutions of (1.2) and (1.8) in terms of irreducible representations of $G$. These formulas generalize Euler’s formula $\cos(x) = \frac{e^{ix} + e^{-ix}}{2}$ on $G = \mathbb{R}$. In the last section we study stability [48] and Baker’s superstability (see [5] and [6]) of the functional equations (1.1), (1.2), (1.3), (1.4), (1.5), (1.6) and (1.7). For more information concerning the stability problem we refer to [3], [5], [6], [11], [12], [22], [40],[41], [42], [43], [44], [45], [46], [47] and [48]. The results of the last sections generalize the ones obtained in [12] and [21].

2. General properties of equations $W_\mu(K)$

In this section we deal with the integral Wilson’s functional equation (1.1) on a locally compact group $G$. We prove, without the assumption that $f$ satisfies ($*$), that if $(f, g)$ is a solution of Wilson’s functional equation (1.1) then $g$ is a solution of d’Alembert’s functional equation (1.2).

For later use we need the following proposition

**Proposition 2.1.** Let $G$ be a locally compact group. Let $\mu : G \rightarrow \mathbb{C}\setminus\{0\}$ be a continuous character of $G$ and let $\varphi \in C(G)$ be a $K$-spherical function. Then

i) $\mu \varphi$ is a $K$-spherical function.

ii) $\frac{\mu \varphi}{\mu \varphi}$ is a solution of (1.2).

iii) Assuming $(f, g)$ is a solution of (1.1) we have: (1) the pair $(L_x f, g)$ for all $x \in G$ is a solution of (1.1), and (2) the pair $(C_K f, g)$ is a solution of (1.1).

**Proof.** We get i) and ii) by easy computations

iii) Let $x \in G$. For all $y, z \in G$ we have

\begin{align*}
\int_K C_K(L_{x^{-1}} f)(ykz^{-1})dk + \mu(z) \int_K C_K(L_{x^{-1}} f)(yk^{-1}z^{-1})dk \\
= \int_K \int_K (L_{x^{-1}} f)(hykz^{-1}h^{-1})dkdh + \mu(z) \int_K \int_K (L_{x^{-1}} f)(hykz^{-1}h^{-1})dkdh \\
= \int_K \int_K f(xhykz^{-1}h^{-1})dkdh + \mu(z) \int_K \int_K f(xhykz^{-1}h^{-1})dkdh \\
= \int_K \int_K f(xhk^{-1}yhz^{-1})dkdh + \mu(z) \int_K \int_K f(xhk^{-1}yhz^{-1})dkdh
\end{align*}
\[
\begin{align*}
&= \int_K \int_K f(xhk^{-1}ykhz^{-1})dkdh + \mu(z) \int_K \int_K f(xhk^{-1}ykhz^{-1}h^{-1})dkdh \\
&= \int_K \int_K f(xk^{-1}ykhz^{-1})dkdh + \mu(z) \int_K \int_K f(xk^{-1}ykhz^{-1}h^{-1})dkdh \\
&= \int_K \int_K f(xkyk^{-1}hzh^{-1})dkdh + \mu(z) \int_K \int_K f(xkyk^{-1}hzh^{-1}h^{-1})dkdh \\
&= 2 \int_K f(xkyk^{-1})d\omega_K(k)g(z) \\
&= 2 \int_K (L_{e^{-1}}f)(kyk^{-1})dkg(z) \\
&= 2C_K(L_{e^{-1}}f)(y)g(z).
\end{align*}
\]

\textbf{Proposition 2.2.} Let \( G \) be a locally compact group. Let \( \mu : G \rightarrow \mathbb{C}\setminus\{0\} \) be a character of \( G \) and let \( f, g \in C(G) \) such that \( f \neq 0 \) be a solution of (1.1). Then

i) \( g(x) = \mu(x)g(x^{-1}) \) for all \( x \in G \).

ii) If \( f \) is \( K \)-central with \( f(e) = 0 \), then \( f(x) = -\mu(x)f(x^{-1}) \) for all \( x \in G \).

iii) \( g(e) = 1 \).

\textbf{Proof.} i) by easy computations.

ii) let \( a \in G \) such that \( f(a) \neq 0 \), then for any \( y \in G \) we have

\begin{equation}
\int_K f(aky^{-1}k^{-1})dk + \mu(y^{-1}) \int_K f(aky^{-1}k^{-1})dk = 2f(a)g(y^{-1}).
\end{equation}

Multiplying (2.1) by \( \mu(y) \) and using the fact that \( \mu(yy^{-1}) = 1 \) we get for all \( y \in G \)

\[
2f(a)\mu(y)g(y^{-1}) = \int_K f(aky^{-1}k^{-1})k + \mu(y) \int_K f(aky^{-1}k^{-1})dk = 2f(a)g(y)
\]

which implies that \( g(y) = \mu(y)g(y^{-1}) \) for any \( y \in G \).

iii) since \( f \) is a solution of (1.1) we get by setting \( x = e \) in (1.1) that

\[
\int_K f(kyk^{-1})dk + \mu(y) \int_K f(ky^{-1}k^{-1})dk = 2f(e)g(y).
\]

Since \( f \) is \( K \)-central and \( f(e) = 0 \) we get for all \( y \in G \) that \( f(y) + \mu(y)f(y^{-1}) = 0 \).

By easy computations we get the remainder.

The next theorem is the main result of this section. We establish a relation between Wilson’s functional equation (1.1) and d’Alembert’s functional equation (1.2) on a locally compact group \( G \), without the assumption that \( f \) satisfies \((*)\).

\textbf{Theorem 2.3.} Let \( G \) be a locally compact group. Let \( f, g \in C(G) \) be a solution of Wilson’s functional equation (1.1) such that \( C_K f \neq 0 \). Then \( g \) is a solution of d’Alembert’s functional equation (1.2).
Proof. By getting ideas from [13] and [36], and [19] we discuss the following possibilities:

The first possibility is \( f(x) = -\mu(x)f(x^{-1}) \) for all \( x \in G \). We let \( x \in G \) and we put

\[
\Phi_x(y) = \int_K g(xky^{-1})dk + \mu(y) \int_K g(y^{-1}kx^{-1})dk - 2g(x)g(y), \ y \in G.
\]

According to Proposition 2.1 and the fact that \((f, g)\) is a solution of (1.1) we get for any \( x, y, z \in G \) that

\[
2f(z)\Phi_x(y) + 2f(y)\Phi_x(z) = 2f(z)\left[ \int_K g(xky^{-1})dk + \mu(y) \int_K g(y^{-1}kx^{-1})dk - 2g(x)g(y) \right] - \mu(y) \int_K f(zkx^{-1}y^{-1}h^{-1})dkdh
\]

\[
+ \mu(y) \int_K f(zkh^{-1}x^{-1}y^{-1}h^{-1})dkdh + \mu(x) \int_K f(zh^{-1}x^{-1}ky^{-1}h^{-1})dkdh + \mu(z) \int_K f(yhxz^{-1}k^{-1}h^{-1})dkdh = 0.
\]
Then for any \( C \subset C \) that

\[ \text{This completes the proof in the first possibility.} \]

If \( f \neq 0 \), then there exists \( a \in G \) such that \( f(a) = 0 \). According to (2.2) we get that

\[ f(a) \Phi_x(y) + f(y) \Phi_x(a) = 0 \]

and then \( \Phi_x(y) = 0 \) for any \( y \in G \). By setting \( y = a \) in (2.2) we get that \( c_x f(a)^2 = 0 \). This implies that \( c_x = 0 \) and then \( \Phi_x = 0 \) for any \( x \in G \). According to Proposition 2.2 we get we get by using the fact that \( \Phi_x(y) = 0 \) for any \( x, y \in G \) that

\[ 2g(x)g(y) = \int_K g(xky^{-1}k^{-1})dk + \mu(y) \int_K g(y^{-1}kxk^{-1})dk \]

\[ = \mu(x)\mu(y) \int_K g(y^{-1}kx^{-1}k^{-1})dk + \mu(y) \int_K g(y^{-1}kxk^{-1})dk \]

\[ = \mu(y) \int_K g(y^{-1}kxk^{-1})dk + \mu(x) \int_K g(y^{-1}kx^{-1}k^{-1})dk. \]

This implies that

\[ \int_K g(y^{-1}kxk^{-1})dk + \mu(x) \int_K g(y^{-1}kx^{-1}k^{-1})dk = 2g(x)\mu(y^{-1})g(y) = 2g(y^{-1})g(x). \]

This completes the proof in the first possibility.

Now we fix \( g \) and we consider

\[ W_g = \{ f \in C(G) : f \text{ is } K \text{- central, satisfies (1.1) and } f(e) = 0 \}. \]

If \( W_g \neq \{0\} \), then by using the above computations we get the desired result, so we may assume \( W_g = \{0\} \). Let \( f \in C(G) \setminus \{0\} \) be a solution of (1.1) such that \( C_K f \neq 0 \). According to Proposition 2.1 it follows that \( C_K f \) is a solution of (1.1). Since \( W_g = \{0\} \) and \( C_K(C_K f) = C_K f \) then \( C_K f(e) = f(e) \neq 0 \). Replacing \( C_K f \) by \( \frac{C_K f}{C_K f(e)} \), we may assume that \( C_K f(e) = 1 \). Let \( h \) be a solution of (1.1), then \( C_K h - (C_K h)(e)C_K f \in W_g = \{0\} \). So \( C_K h = (C_K h)(e)C_K f \).

According to Proposition 2.1 we have for any \( x \in G \) that \( C_K(L_{x^{-1}}C_K f)(y) = \int_K f(xkyk^{-1})dk \) for any \( y \in G \) is a solution of (1.1) and that \( C_K(C_K(L_{x^{-1}}C_K f)) = C_K(L_{x^{-1}}C_K f) \). So that \( C_K(L_{x^{-1}}C_K f) = C_K(L_{x^{-1}}C_K f)(e)C_K f = C_K f(x)C_K f \).

Then \( \int_K C_K f(xkyk^{-1})dk = C_K(L_{x^{-1}}C_K f)(y) = C_K f(x)C_K f(y) \), which show that
$C_K f$ is a $K$-spherical function i.e. $\int_K C_K f(xkyk^{-1}) dk = C_K f(x)C_K f(y)$ for all $x, y \in G$. Substituting this result into

$$\int_K C_K f(xkyk^{-1}) dk + \mu(y) \int_K C_K f(xky^{-1}k^{-1}) dk = 2C_K f(x)g(y), \ x, y \in G$$

we get $g(y) = \frac{C_K f(y) + \mu(y)C_K f(y^{-1})}{2}$. According to Proposition 2.1 ii) we get that $g$ satisfies equation (2.1). This finishes the proof of theorem. □

3. STUDY OF INTEGRAL WILSON’S FUNCTIONAL EQUATION $W_{\mu}(K)$ ON A CENTRAL PAIR

Let $f : G \rightarrow \mathbb{C}$. For $x \in G$ we define

$$(3.1) \quad f_x(y) = \int_K f(xkyk^{-1}) dk - f(x)f(y), \ y \in G.$$ 

When $f$ is $K$-spherical, then $f_x \equiv 0$. A generalized symmetrized sine addition law is given by

$$(3.2) \quad \int_K \omega(xkyk^{-1}) dk + \int_K \omega(ykxk^{-1}) dk = 2\omega(x)f(y) + 2\omega(y)f(x), \ x, y \in G$$

For later use we need the following results:

**Proposition 3.1.** ([7]) Let $(G, K)$ be a central pair and let $f \in C(G)$. Then we have

i) $f$ satisfies the Kannappan type condition $(\ast)$.

ii) If $f$ is $K$-central, then

$$\int_K f(xkyk^{-1}) dk = \int_K f(ykxk^{-1}) dk, \ x, y \in G.$$ 

As an immediate consequence we get the following corollary

**Corollary 3.2.** Let $(G, K)$ be a central pair and let $\omega \in C(G)$. If $\omega$ is $K$-central then (3.2) reduces to the generalized sine addition formula

$$(3.3) \quad \int_K \omega(xkyk^{-1}) dk = \omega(x)f(y) + \omega(y)f(x), \ x, y \in G.$$ 

**Proposition 3.3.** Let $f \in C \setminus \{0\}$ be a solution of the functional equation (1.2). Then

i) $f(e) = 1$,

ii) $f$ is $K$-central,

iii) $f(x) = \mu(x)f(x^{-1})$ for all $x \in G$,

iv) $\int_K f(xkyk^{-1}) dk = \int_K f(ykxk^{-1}) dk, \ x, y \in G$.

v) For any $x \in G$, $(f_x, f)$ is a solution of (3.2).

**Proof.** i) By setting $y = e$ in (1.2) and by using the fact that $f \neq 0$ we get that $f(e) = 1$. 
ii) For any $x, y \in G$ we have
\[
2f(x) \int_K f(kyk^{-1})dk = 2 \int_K f(x) f(kyk^{-1})dk = \int_K (\int_K f(xkkyk^{-1}h^{-1})dh) dk + \mu(y) \int_K f(xkkyk^{-1}h^{-1})dh = \int_K f(xhyk^{-1})dh + \mu(y) \int_K f(xhyk^{-1}h^{-1})dh
\]
So that $f(y) = \int_K f(kyk^{-1})dk$ for all $y \in G$, from which we get that $f$ is $K$-central.

iii) Since $f$ is $K$-central we get by putting $x = e$ in (1.2) that $f(y) + \mu(y)f(y^{-1}) = 2f(y)$ for all $y \in G$. Hence $f(y) = \mu(y)f(y^{-1})$ for all $y \in G$.

iv) In view of iii) we have for all $x, y \in G$ that
\[
\int_K f(xkyk^{-1})dk + \mu(y) \int_K f(xky^{-1}k^{-1}k)dk = 2f(x)f(y) = 2f(y)f(x) = \int_K f(ykzk^{-1})dk + \mu(x) \int_K f(ykzk^{-1}k^{-1})dk
\]
So that we get $\int_K f(ykzk^{-1})dk = \int_K f(xkyk^{-1})dk$ for all $x, y \in G$.

v) In the next we adapt the method used in [35]. According to (1.2) we get for any $x, y, z \in G$ that
\[
(3.4) \quad \mu(z) \int_K \int_K f(xkyk^{-1}hzh^{-1})dkdh +
\]
\[
= \mu(z) \int_K \int_K f(xkyk^{-1}hzh^{-1}1)dkdh = 2 \int_K f(xkyk^{-1})dkf(z),
\]

\[
(3.5) \quad \int_K \int_K f(xhykz^{-1}k^{-1}h^{-1})dkdh +
\]
\[
= \mu(yz^{-1}) \int_K \int_K f(xhykz^{-1}k^{-1}1h^{-1})dkdh = 2f(x) \int_K f(ykzk^{-1}1)dk,
\]

\[
(3.6) \quad \int_K \int_K f(xkzk^{-1}hy^{-1}h^{-1})dkdh +
\]
\[
= \mu(y^{-1}) \int_K \int_K f(xkzk^{-1}hgy^{-1}h^{-1})dkdh = 2f(x) \int_K f(kzk^{-1}1y)dkf(y^{-1})
\]
from which we get by multiplying (3.5) by $\mu(z)$ and (3.6) by $\mu(y)$
\[
\int_K \int_K f(xkyk^{-1}hzh^{-1})dkdh + \mu(z) \int_K \int_K f(xkyk^{-1}hzh^{-1}1)dkdh
\]
\[
= 2 \int_K f(xkyk^{-1})dkf(z),
\]
\[
\mu(z) \int_K \int_K f(xhkyz^{-1}h^{-1})dhd + \mu(y) \int_K \int_K f(xhkyz^{-1}y^{-1}h^{-1})dhd
\]
\[
= 2\mu(z)f(x) \int_K f(ykz^{-1}k^{-1})dk,
\]
\[
\mu(y) \int_K \int_K f(xkz^{-1}hy^{-1}h^{-1})dhd + \int_K \int_K f(xkz^{-1}hyh^{-1})dhd
\]
\[
= 2\mu(y) \int_K f(xkz^{-1})dkf(y^{-1}).
\]
By subtracting the middle one from the sum of the two others we get
\[
\int_K \int_K f(xkyk^{-1}hzh^{-1})dhd + \int_K \int_K f(xkz^{-1}hyh^{-1})dhd
\]
\[
= 2 \int_K f(xkyk^{-1})dkf(z) + 2\mu(y) \int_K f(xkz^{-1})dkf(y^{-1})
\]
\[
- 2\mu(z)f(x) \int_K f(ykz^{-1}k^{-1})dk.
\]
Using the fact that \( f(y) = \mu(y)f(y^{-1}) \) for any \( y \in G \) we get
\[
\int_K \int_K f(xkyk^{-1}hzh^{-1})dhd - f(x) \int_K f(ykz^{-1}k^{-1})dk
\]
\[
+ \int_K \int_K f(xkz^{-1}hyh^{-1})dhd - f(x) \int_K f(zkyk^{-1})dk
\]
\[
= 2 \int_K f(xkyk^{-1})dkf(z) - 2f(x) \int_K f(yhzk^{-1})dh
\]
\[
+ 2 \int_K f(xkz^{-1})dkf(y) - 2f(x)[f(y)f(z) - \int_K f(yhzk^{-1})dh]
\]
\[
= 2 \int_K f(xkyk^{-1})dkf(z) - 2f(x)f(y)f(z)
\]
\[
+ 2 \int_K f(xkz^{-1})dkf(y) - 2f(x)f(y)f(z).
\]
So we have for any \( x, y, z \in G \) that
\[
\int_K f_x(ykz^{-1})dk + \int_K f_x(zkyk^{-1})dk = 2f_x(y)f(z) + 2f_x(z)f(y).
\]
Hence for all \( x \in G \), the pair \((f_x, f)\) is a solution of (3.2). \( \square \)

**Theorem 3.4.** Let \( f, g \in C(G) \) be a solution of the functional equation

\[
\int_K f(xkyk^{-1})dk = f(x)g(y) + g(x)f(y), \quad x, y \in G.
\]

Then one of the following statements hold

i) \( f = 0 \) and \( g \) arbitrary in \( C(G) \).

ii) There exists a \( K \)-spherical function \( \varphi \) and a non-zero constant \( c \in C^* \) such that

\[
g = \frac{\varphi}{2}, \quad f = c\varphi.
\]

iii) There exist two \( K \)-spherical functions \( \varphi, \psi \) for which \( \varphi \neq \psi \) and a non-zero constant \( c \in C^* \) such that

\[
g = \frac{\varphi + \psi}{2}, \quad f = c(\varphi - \psi).
\]
iv) \( g \) is a \( K \)-spherical function and \( f \) is associated to \( g \).

**Proof.** The proof is similar to one used in [20]. \( \square \)

### 4. Solution of equation \( W_\mu(K) \) on a central pair

In this section we obtain solution of equation (1.1) in the case where \((G, K)\) is a central pair.

In the next theorem we solve the functional equation (1.2). We will adapt the method used in [35].

**Theorem 4.1.** Let \((G, K)\) be a central pair. Let \( \mu : G \rightarrow \mathbb{C}^* \) be a character on \( G \) and let \( f : G \rightarrow \mathbb{C} \) be a non-zero solution of the functional equation (2.1). Then there exists a \( K \)-spherical function \( \varphi : G \rightarrow \mathbb{C} \) such that \( f = \frac{\varphi + \mu \varphi}{2} \).

**Proof.** By using 5i) in Proposition 3.3 we get for all \( x \in G \) that the pair \((x f, f)\) is a solution of (3.2).

First case : There exists \( x \in G \) such that \( f_x \neq 0 \). According to iii) in Theorem 3.4, there exist two \( K \)-spherical functions \( \varphi \) and \( \psi \) such that \( \varphi \neq \psi \) and that \( f = \frac{\varphi + \psi}{2} \).

By substituting this in (1.2) we get for all \( x, y \in G \) that

\[
\varphi(x)[\mu(y)\varphi(y^{-1}) - \psi(y)] + \psi(x)[\mu(y)\psi(y^{-1}) - \varphi(y)] = 0.
\]

Since \( \varphi \neq \psi \), according to [19] we get that \( \varphi \) and \( \psi \) are linearly independent. Hence \( \psi(y) = \mu(y)\varphi(y^{-1}) \) for any \( y \in G \).

By iv) of Theorem 3.4 we get by a small computation the desired result.

Second case : if \( f_x = 0 \), for all \( x \in G \) then \( \int_K f(xkyk^{-1})d\omega_K(k) = f(x)f(y) \) for all \( x, y \in G \). Then by subsisting \( f \) in (1.2) we get that \( f(x) = \mu(x)f(x^{-1}) \) for all \( x \in G \). Hence \( f = \varphi = \frac{\varphi + \mu \varphi}{2} \) where \( \varphi \) is \( K \)-spherical function. \( \square \)

In the next theorem we solve the functional equation (1.1).

**Theorem 4.2.** Let \((G, K)\) be a central pair. If \((f, g)\) is a solution of (1.1) such that \( C_K f \neq 0 \), then there exists a \( K \)-spherical function \( \varphi \) such that

1) \[
g = \frac{\varphi + \mu \varphi}{2}.
\]

2) i) When \( \varphi \neq \mu \varphi \), then there exist \( \alpha, \beta \in \mathbb{C} \) such that

\[
f = \alpha \frac{\varphi + \mu \varphi}{2} + \beta \frac{\varphi - \mu \varphi}{2}.
\]

ii) When \( \varphi = \mu \varphi \), then \( f = \alpha \varphi + l \) where \( \alpha \in \mathbb{C} \) and \( l \) is a solution of the functional equation

\[
(4.1) \int_K l(xkyk^{-1})dk = l(x)\varphi(y) + \varphi(x)l(y), \ x, y \in G.
\]

**Proof.** Let \( f, g \in C(G) \setminus \{0\} \) be a solution of (1.2), then by Theorem 2.3 we get that \( g \) is a solution of (1.2). According to Theorem 4.1 there exists a \( K \)-spherical
function such that \( g(x) = \frac{\varphi(x) + \mu(x)x^{-1}}{2} \) for any \( x \in G \). By decomposing \( f \) in the following way we get for any \( x \in G \) that
\[
 f(x) = \frac{f(x) + \mu(x)f(x^{-1})}{2} + \frac{f(x) - \mu(x)f(x^{-1})}{2} = f_1(x) + f_2(x)
\]
where \( f_1(x) = \frac{f(x) + \mu(x)f(x^{-1})}{2} \) and \( f_2(x) = \frac{f(x) - \mu(x)f(x^{-1})}{2} \) for all \( x \in G \). By easy computations we get that \( f_1(x) = \mu(x)f_1(x^{-1}) \) and \( f_2(x) = -\mu(x)f_2(x^{-1}) \) for all \( x \in G \). By using the fact that \( \int_K f(xkyk^{-1})dk = \int_K f(ykxk^{-1})dk \) for all \( x,y \in G \) we get that
\[
(4.2) \quad \int_K f_1(xkyk^{-1})dk + \mu(y)\int_K f_2(xkyk^{-1})dk = 2f_1(x)g(y), \quad x,y \in G.
\]
By setting \( x = e \) in (4.3) we get that \( f_1(y) = f_1(e)g(y) = \alpha g(y) \) for any \( y \in G \). On the other hand by small computations we show that \( f_2 \) is a solution of the functional equation
\[
(4.3) \quad \int_K f_2(xkyk^{-1})dk = f_2(x)g(y) + f_2(y)g(x), \quad x,y \in G.
\]
According to iii) and iv) in Theorem 3.4 we get the remainder. \( \square \)

In the next corollary we use R. Godement’s spherical functions theory [27] to give explicit formulas in terms of irreducible representations of \( G \): Let \((\pi, \mathcal{H})\) be a completely irreducible representation of \( G \), \( \delta \) be an irreducible representation of \( K \) and \( \chi_\delta \) the normalized character of \( \delta \). The set of vectors in \( \mathcal{H} \) which under \( k \rightarrow \pi(k) \) transform according to \( \delta \) is denoted by \( \mathcal{H}_\delta \). The operator \( E(\delta) = \int_K \pi(k)\overline{\chi_\delta(k)}dk \) is a continuous projection of \( \mathcal{H} \) onto \( \mathcal{H}_\delta \).

We will say that a function \( f \) is quasi-bounded if there exists a semi-norm \( \rho(x) \) such that \( \sup_{x \in G} \frac{|f(x)|}{\rho(x)} < +\infty \) (see [27]). According to [7, Theorem 4.1, Theorem 6.2] we have the following corollary

**Corollary 4.3.** Let \((G,K)\) be a central pair. Let \( f \) be a non-zero quasi-bounded continuous function on \( G \) satisfying \( \chi_\delta \ast f = f \). Then \( f \) is a solution of (1.2) if and only if there exists a completely irreducible representation \((\pi, \mathcal{H})\) of \( G \) such that
\[
f(x) = \frac{1}{2\dim(\delta)}(\text{tr}(E(\delta)\pi(x)) + \mu(x)\text{tr}(E(\delta)\pi(x^{-1}))), \quad x \in G
\]
where \( \text{tr} \) is the trace on \( \mathcal{H} \) and \( \dim(\delta) \) is the dimension of \( \delta \).

In the next corollary we assume that \( G \) is a compact. Then \((G,G)\) is a central pair (see [7] and [9]).

**Corollary 4.4.** Let \( G \) be a compact group and let \( f \) be a continuous function on \( G \). Then \( f \) is a solution of (1.8) if and only if there exists a continuous irreducible representation \((\pi, \mathcal{H})\) such that
\[
f(x) = \frac{\chi_\pi(x) + \mu(x)\chi_\pi(x^{-1})}{2d(\pi)}, \quad x \in G
\]
where \( \chi_\pi \) and \( d(\pi) \) are respectively the character and the dimension of \( \pi \).
5. Superstability of $W_\mu(K)$ on a Locally Compact Group

In this section, by using Theorem 2.4, we study the superstability problem of equations $W_\mu(K)$ and $D_\mu(K)$ on non abelian case.

**Lemma 5.1.** Let $\delta > 0$. Let $\mu : G \to \mathbb{C}$ be a unitary character of $G$. Let $f, g \in C(G)$ such that $f$ is unbounded and $(f, g)$ is a solution of the inequality

$$|\int_K f(xky^{-1}k^{-1})dk + \mu(y)\int_K f(xky^{-1}k^{-1})dk - 2f(x)g(y)| \leq \delta, \ x, y \in G. \quad (5.1)$$

Then

i) For all $x \in G$, $(C_K(L_x^{-1}f), g)$ is a solution of the inequality $(5.1)$.

ii) $g(y) = \mu(y)g(y^{-1})$ for all $y \in G$.

iii) $g$ is $K$ central.

**Proof.** i) and iii) by easy computation.

ii) Since $(f, g)$ is a solution of (5.1), then we get for any $x, y \in G$

$$|\int_K f(xky^{-1}k^{-1})dk + \mu(y)\int_K f(xky^{-1}k^{-1})dk - 2f(x)g(y)| \leq \delta$$

and

$$|\int_K f(xky^{-1}k^{-1})dk + \mu(y^{-1})\int_K f(xky^{-1}k^{-1})dk - 2f(x)g(y^{-1})| \leq \delta.$$ 

By multiplying the last inequality by $\mu(y)$ we get that

$$|\mu(y)\int_K f(xky^{-1}k^{-1})dk + \int_K f(xky^{-1}k^{-1})dk - 2f(x)\mu(y)g(y^{-1})| \leq \delta.$$ 

By triangle inequality we get

$$|2f(x)||g(y) - \mu(y)g(y^{-1})| \leq 2\delta$$

for all $y \in G$. Since $f$ is unbounded we get that $g(y) = \mu(y)g(y^{-1})$ for all $y \in G$. □

**Lemma 5.2.** Let $\delta > 0$. Let $\mu : G \to \mathbb{C}^*$ be a unitary character of $G$. Let $f, g \in C(G)$ such that $g$ is unbounded solution of inequality (5.1). Then $g$ is a solution of d’Alembert’s functional equation (1.2).

**Proof.** By using the same method as in [14, Corollary 2.7 iii] we get that $g$ is a solution of (1.2). □

**Lemma 5.3.** Let $\delta > 0$. Let $\mu : G \to \mathbb{C}^*$ be a unitary character of $G$. Let $f, g \in C(G)$ such that $f$ is unbounded solution of inequality (5.1) and that the function $x \to f(x) + \mu(x)f(x^{-1})$ is bounded and $C_K f \neq 0$. Then $g$ is a solution of d’Alembert’s functional equation (1.2).

**Proof.** First case: $g$ is bounded. Let

$$\psi(x, y) = \int_K g(xky^{-1}k^{-1})dk + \mu(y)\int_K g(y^{-1}k^{-1}z^{-1}k)dk - 2g(x)g(y), \ x, y \in G.$$ 

Then according to the proof of Theorem 2.4 we get for all $x, y, z \in G$ that

$$2f(z)\psi(x, y) + 2f(y)\psi(x, z) = 2g(x)[\int_K f(zky^{-1}k^{-1})dk + \int_K f(ykz^{-1}k^{-1})dk]$$

$$-\mu(z)\int_K \int_K f(yhxz^{-1}k^{-1}h^{-1}k^{-1}h)dkdh - \mu(x)\mu(y)\int_K \int_K f(zhx^{-1}ky^{-1}k^{-1}h^{-1}k^{-1}h)dkdh$$

$$- \mu(z)\int_K \int_K f(yhxz^{-1}k^{-1}h^{-1}k^{-1}h)dkdh - \mu(x)\mu(y)\int_K \int_K f(zhx^{-1}ky^{-1}k^{-1}h^{-1}k^{-1}h)dkdh.$$
As in the proof of Theorem 2.4, we get that $g + \psi f$.

**Lemma 5.4.** Let $\psi(x, y) = 2f(z)\psi(x, y) + 2f(y)\psi(x, z)$, $-\infty < \beta < +\infty$. By using the inequality (5.2), it follows that there exists $c_\beta \in \mathbb{C}$ such that $\psi(x, y) = c_\beta f(y)$ for all $x, y \in G$. So that the function $(x, y, z) \rightarrow 2f(z)c_\beta f(y) + 2f(y)c_\beta f(z)$ is bounded. Since $f$ is unbounded it follows that $c_\beta = 0$ for all $x \in G$.

As in the proof of Theorem 2.4, we get that $g$ is a solution of functional equation (1.2).

Second case: $g$ is unbounded. According to lemma 5.2 we get that $g$ is a solution of (1.2).

**Lemma 5.4.** Let $\delta > 0$. Let $\mu : G \rightarrow \mathbb{C}^*$ be a unitary character of $G$. Let $f, g \in C(G)$ such that $f$ is an unbounded solution of inequality (5.1). Then $g$ is a solution of d’Alembert’s long functional equation

$$\int_K g(xky^{-1}k^{-1}h^{-1})dk + \mu(y)\int_K g(xky^{-1}k^{-1}h^{-1})dk + \int_K g(ykx^{-1}k^{-1}h^{-1})dk + \mu(y)\int_K g(ykx^{-1}k^{-1}h^{-1})dk = 4g(x)g(y)$$

for all $x, y \in G$.

**Proof.** For all $x, y, z \in G$ we have

$$2|f(z)|\int_K g(xky^{-1}k^{-1}h^{-1})dk + \mu(y)\int_K g(xky^{-1}k^{-1}h^{-1})dk + \int_K g(ykx^{-1}k^{-1}h^{-1})dk + \mu(y)\int_K g(ykx^{-1}k^{-1}h^{-1})dk = 4g(x)g(y)$$

$$+ \mu(y)\int_K g(ykx^{-1}k^{-1}h^{-1})dk - 4g(x)g(y)$$

$$\leq |\int_K f(zhxky^{-1}k^{-1}h^{-1})dkdh + \mu(xy)\int_K f(zy^{-1}kx^{-1}k^{-1}h^{-1})dkdh$$

$$- 2f(z)\int_K g(xky^{-1}k^{-1}h^{-1})dk|$$

$$+ |\mu(y)\int_K f(zhx^{-1}kky^{-1}k^{-1}h^{-1})dkdh + \mu(x)\int_K f(zyx^{-1}k^{-1}k^{-1}h^{-1})dkdh$$

$$- 2\mu(y)f(z)\int_K g(xky^{-1}k^{-1}h^{-1})dk|$$

$$+ |\int_K f(zyx^{-1}k^{-1}k^{-1}k^{-1}h^{-1})dkdh + \mu(yx)\int_K f(zhx^{-1}k^{-1}k^{-1}h^{-1})dkdh$$

$$- 2f(z)\int_K g(ykx^{-1}k^{-1}h^{-1})dk|$$

$$+ |\mu(y)\int_K f(zhyx^{-1}kxx^{-1}k^{-1}h^{-1})dkdh + \mu(x)\int_K f(zhx^{-1}kkyx^{-1}h^{-1})dkdh$$

$$- 2\mu(y)f(z)\int_K g(y^{-1}kx^{-1}k^{-1}h^{-1})dk|$$
If \( f \) is a solution of d'Alembert's functional equation (1.2). According to Lemma 5.3 we have that \( g \) is a solution of d'Alembert's functional equation (5.3).

Proof. Since \((f, g)\) is a solution of the inequality (5.1) we get according to lemma 5.1 that \((C_K f, g)\) is also a solution of (5.1). By setting \( x = e \) in (5.1) and by the fact that \( C_K f \) is \( K \)-central it follows that

\[
|C_K f(y) + \mu(y)C_K f(y^{-1}) - 2f(e)g(y)| \leq \delta, \quad y \in G.
\]

If \( f(e) = 0 \) we get that the function \( y \mapsto C_K f(y) + \mu(y)C_K f(y^{-1}) \) is bounded. According to Lemma 5.3 we have that \( g \) is a solution of d'Alembert's functional equation (1.2).

If \( f(e) \neq 0 \). Replacing \( C_K f \) by \( \frac{C_K f}{f(e)} \) we may assume that \( f(e) = 1 \). Consider the function \( C_K(L_n f)(x) = \int_K f(ax_kk^{-1})d\omega_K(k) \) for all \( x \in G \). According to lemma 5.1 we get that \((C_K(L_n f), g)\) is a solution of (5.1). Let \( a \in G \) such that

\[
h = C_K(L_n f) - C_K(L_n f)(e)C_K f.
\]

If there exists \( a \in G \) such that \( h \) is unbounded on \( G \). Since \( h(e) = 0 \) and \( C_K h = h \) and that the function \( h \) is a solution of the inequality (5.1) it follows that \( x \mapsto h(x) + \mu(x)h(x^{-1}) \) is bounded. According to lemma 5.3 we get that \( g \) is a solution of d'Alembert's functional equation (1.2).

Now assume that \( h \) is bounded, that is there exists \( M(x) > 0 \) such that

\[
|\int_K f(xk_xk^{-1})dk - f(x)C_K f(y)| \leq M(x)
\]
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for all \( x, y \in G \). Since \( f \) is \( K \)-central we get for all \( x, y \in G \) that

\[
| \int_K f(xkyk^{-1})dk - f(x)f(y) | \leq M(x).
\]

By using triangle inequality we get for all \( x, y, z \in G \) that

\[
| f(z)|| \int_K f(xkyk^{-1})dk - f(x)f(y) |
\]

\[
\leq | - \int_K \int_K f(xkyk^{-1}hz^{-1})kdh + \int_K f(xkyk^{-1})dkf(z) | \\
+ | \int_K \int_K f(xkyk^{-1}hz^{-1})dkdh - f(x) \int_K f(ykzk^{-1})dk | \\
+ | f(x)|| \int_K f(ykzk^{-1})dk - f(y)f(z) |
\]

\[
\leq \int_K M(xkyk^{-1})dk + M(x) + |f(x)|M(y). 
\]

Since \( f \) is unbounded it follows that \( \int_K f(xkyk^{-1})dk - f(x)f(y) \) for all \( x, y \in G \). Substituting this result into inequality (5.1) it follows that

\[
|f(x)||f(y) + \mu(y)f(y^{-1}) - 2g(y)| \leq \delta
\]

for all \( x, y \in G \). Since \( f \) is unbounded we get that \( g(y) = \frac{f(y) + \mu(y)f(y^{-1})}{2} \) for all \( y \in G \). According to Proposition 2.1 we get that \( g \) is a solution of d’Alembert’s functional equation (1.2). \( \square \)

The next theorem is the main result of this section

**Theorem 5.6.** Let \( \delta > 0 \) be fixed, \( \mu \) be a unitary character of \( G \) and let \( f, g : G \to \mathbb{C} \) such that \((f, g)\) satisfies (5.1) and \( f \) is \( K \)-central. Then

1) \( f, g \) are bounded or
2) \( f \) is unbounded and \( g \) satisfies d’Alembert’s functional equation (1.2) or
3) \( g \) is unbounded and \( f \) satisfies the functional equation (1.1) (if \( f \neq 0 \) such that \( C_Kf \neq 0 \), then \( g \) satisfies the d’Alembert’s functional equation (1.2)).

**Proof.** We get 1) by easy computations.
2) Assume that \( f \) is unbounded. According to Lemmas 5.1, 5.3 and 5.5 we get the proof.
3) Assume that \( g \) is unbounded, then for \( f = 0 \) the pair \((f, g)\) is a solution of equation (5.1). Afterward we suppose that \( f \neq 0 \). By using (5.1) and the following
decomposition

\[
2|g(z)||f(xkyk^{-1})dk + \mu(y)f(xky^{-1}k^{-1})dk - 2f(x)g(y)k| \\
= | - 2g(z)f(xkyk^{-1})dk - 2g(z)\mu(y)f(xky^{-1}k^{-1})dk - 4g(z)f(x)g(y)k| \\
\leq | \int_{\mathbb{K}} f(xkyk^{-1}hz^{-1})dkdh + \mu(z)\int_{\mathbb{K}} f(xkyk^{-1}hz^{-1})dkdh | \\
- 2f(x) \int_{\mathbb{K}} g(yzk^{-1}k^{-1})dk \\
+ | \mu(z)\int_{\mathbb{K}} f(xkyk^{-1}z^{-1}h^{-1})dkdh + \mu(z)\mu(yz^{-1})\int_{\mathbb{K}} f(xzk^{-1}hy^{-1}h^{-1})dkdh | \\
- 2\mu(z) \int_{\mathbb{K}} f(x)g(yzk^{-1}k^{-1})dk \\
+ | \mu(z)\int_{\mathbb{K}} f(xhz^{-1}kyk^{-1}z^{-1}h^{-1})dkdh + \mu(z)\mu(z^{-1})\int_{\mathbb{K}} f(xkyk^{-1}z^{-1}hy^{-1}h^{-1})dkdh | \\
- 2\mu(z) \int_{\mathbb{K}} f(x)g(z^{-1}kyk^{-1})dk \\
+ | \int_{\mathbb{K}} f(xh^zg^{-1}h^{-1}z^{-1}h^{-1})dkdh + \mu(z)\int_{\mathbb{K}} f(xkyk^{-1}z^{-1}h^{-1})dkdh | \\
- 2f(x) \int_{\mathbb{K}} g(zkyk^{-1})dk \\
+ | \mu(z)\int_{\mathbb{K}} f(xh^{-1}kyk^{-1}z^{-1}h^{-1})dkdh + \mu(z)\mu(y)\int_{\mathbb{K}} f(xzk^{-1}hy^{-1}h^{-1})dkdh | \\
- 2\mu(z) \int_{\mathbb{K}} f(xzk^{-1}g(y)dk \\
+ | \int_{\mathbb{K}} f(xzk^{-1}hyk^{-1}z^{-1}h^{-1}h^{-1})dkdh + \mu(y)\int_{\mathbb{K}} f(xzk^{-1}hy^{-1}h^{-1})dkdh | \\
- 2f(x) \int_{\mathbb{K}} g(xzk^{-1})g(y)dk \\
+ 2|f(x)|| \int_{\mathbb{K}} f(yzk^{-1}k^{-1})dk + \mu(z)\int_{\mathbb{K}} g(yzk^{-1}k^{-1})dk + \int_{\mathbb{K}} g(yzk^{-1})dk \\
+ \mu(z) \int_{\mathbb{K}} g(yzk^{-1}k^{-1})dk - 4g(yg(z)) | \\
+ 2|g(y)|| \int_{\mathbb{K}} f(xzk^{-1}k^{-1})dk + \mu(z)\int_{\mathbb{K}} f(xzk^{-1}k^{-1})dk - 2f(x)g(z) | \\
\leq \delta + |\mu(y)\delta + \delta + 2|\mu(z)|\delta + \delta + |\mu(z)|\delta + \delta + 2|f(x)| \times 0 + 2|g(y)|\delta \\
= \delta(8 + 2|g(y)|).
\]

Since \( g \) is unbounded it follows that \( f, g \) satisfy the functional equation (1.1). According to Theorem 2.4 we get the remainder \( \square \).
As a consequence we get the superstability of the functional equations (1.2), (1.3), (1.4) and (1.7)

**Corollary 5.7.** Let $\delta > 0$ be fixed, $\mu$ be a unitary character of $G$. Let $f : G \rightarrow \mathbb{C}$ such that

\[(5.4) \quad \left| \int_K f(xkyk^{-1})dk + \mu(y) \int_K f(xky^{-1}k^{-1})dk - 2f(x)f(y) \right| \leq \delta, \quad x, y \in G. \]

Then either $f$ is bounded or $f$ is a solution of the functional equation (1.2).

Let $f(kzh) = \chi(k)f(x)\chi(h), \quad k, h \in K \text{ and } x \in G$. Then we have the following corollary

**Corollary 5.8.** Let $\delta > 0$ be fixed, $\mu$ be a bounded character of $G$. Let $f : G \rightarrow \mathbb{C}$ such that

\[(5.5) \quad \left| \int_K f(xky)\overline{\chi(k)}dk + \mu(y) \int_K f(xky^{-1})\overline{\chi(k)}dk - 2f(x)f(y) \right| \leq \delta, \quad x, y \in G. \]

Then either $f$ is bounded or $f$ is a solution of the functional equation (1.7).

In the next corollary we assume that $K \subset Z(G)$. Then we get

**Corollary 5.9.** Let $\delta > 0$ be fixed, $\mu$ be a unitary character of $G$. Let $f : G \rightarrow \mathbb{C}$ such that

\[(5.6) \quad \left| f(xy) + \mu(y)f(xy^{-1}) - 2f(x)f(y) \right| \leq \delta, \quad x, y \in G. \]

Then either $f$ is bounded or $f$ is a solution of the functional equation (1.3).

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