Solving Multiobjective Linear Programming Problems with Interval Parameters

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\textbf{ABSTRACT}

In the present paper, a multiobjective linear programming problem under uncertainty, particularly when parameters are given in interval forms, is investigated. In this case, it is assumed that objective coefficients and constraints parameters have arrived in interval numbers. Considering a suitable order relation for interval numbers, a solution procedure for dealing with such a problem is developed. A numerical example is provided to illustrate the efficiency of the solution procedure.

\textbf{KEYWORDS}

Multiobjective linear programming; interval number; uncertainty modelling

1. Introduction

Multiobjective programming (MOP) problems involving several, often, conflicting and incommensurate objective functions have gained the attention of many researchers in earlier decades [1]. When handling real-world MOP problems, it is often necessary to treat inexact or uncertain input data due to various measurement errors or estimations. Throughout the years, several approaches for dealing with MOP problems with imprecise data based on different sources of uncertainty have emerged.

Uncertainty may be interpreted as randomness or fuzziness [2]. To deal with randomness in MOP problems, stochastic programming approaches are applied [3,4]. On the other hand, fuzzy programming techniques are used for dealing with fuzziness in MOP problems [5–7]. In stochastic programming and fuzzy programming, the uncertain coefficients are assumed as random variables with known distributions and fuzzy numbers with known membership functions, respectively. However, determining distributions of random variables and determining membership functions of fuzzy numbers are not easy. It is due to the fact that sometimes they do not perfectly match the real situations.

Interval programming as an alternative choice could be considered to deal with uncertainty in MOP problems. In interval programming, it is assumed that the coefficients perturb independently within the given lower and upper bounds [8]. Actually, uncertain coefficients are modelled by closed intervals in this approach. Since interval programming does not require stringent applicability conditions, therefore, it has been considered a suitable tool for modelling uncertainty in many practical applications [9–20].
This paper focuses on interval programming for dealing with uncertainty in multiobjective linear programming (MOLP) problems. MOLP problems with interval coefficients have been investigated by some authors. Bitran [21] discussed MOLP problems with interval objective function coefficients and introducing two types of solutions. Urli and Nadeau [22] used an interactive method for solving MOLP problems with interval coefficients. Oliveira and Antunes [23] provided an overview of MOLP problems with interval coefficients. Also, Oliveira and Antunes [24] presented an interactive method to solve such problems. Wu [25] proposed Karush–Kuhn–Tucker optimality conditions for a multiobjective programming problem with interval objective function coefficients. Some new solution concepts and algorithms were suggested by Rivaz and Yaghoobi to MOLP problems with interval objective function coefficients in [2,19].

The current research tries to propose a solution procedure to interval MOLPs. In this sense, an order relation for interval numbers is used and a solution concept according to interval MOLPs is defined. Further, in order to solve an interval MOLP problem, a bi-objective linear programming problem is presented.

The remainder of the paper is organised as follows. In Section 2, some preliminaries are discussed. Section 3 is devoted to introduce an interval MOLP problem. In addition, a solution procedure for dealing with interval MOLP problems is discussed in the same section. In Section 4, with the aid of a numerical example, the solution procedure is illustrated. Finally, Section 5 states conclusions and proposes directions for future research.

2. Preliminaries

In this section, we recall some concepts of interval arithmetic and multiobjective optimisation which are used later [1,26]. Throughout the paper, capital letters indicate closed intervals. There are two different representations of an interval. One may represent the interval A by its left bound \(a_L\) and right bound \(a_R\) as

\[ A = [a_L, a_R] = \{x \in \mathbb{R} : a_L \leq x \leq a_R\}. \]

Another representation of interval A is by its centre point \(a_C\) and half-width length (or radius) \(a_W\) as

\[ A = \langle a_C, a_W \rangle = \{x \in \mathbb{R} : a_C - a_W \leq x \leq a_C + a_W\}, \]

where

\[ a_C = \frac{a_L + a_R}{2}, \quad a_W = \frac{a_R - a_L}{2}. \]

**Definition 2.1:** Let \(* \in \{+, -, \cdot, \div\}\) be a binary operation on \(\mathbb{R}\). If A and B are two arbitrary closed intervals, then

\[ A * B = \{a * b : a \in A, b \in B\}. \]

In the case of division, it is supposed that \(0 \not\in B\).
From Definition 2.1, it could be shown that
\[ A + B = [a_L, a_R] + [b_L, b_R] = [a_L + b_L, a_R + b_R], \]
\[ A + B = \langle a_C, a_W \rangle + \langle b_C, b_W \rangle = \langle a_C + b_C, a_W + b_W \rangle, \]
\[ kA = k[a_L, a_R] = \begin{cases} 
[ka_L, ka_R] & \text{if } k \geq 0, \\
[ka_R, ka_L] & \text{if } k < 0,
\end{cases} \]
\[ kA = k\langle a_C, a_W \rangle = \langle ka_C, |k|a_W \rangle. \]

In what follows, an order relation which represents the decision maker's preference between intervals are defined for minimisation problems.

**Definition 2.2:** If \( A = [a_L, a_R] = \langle a_C, a_W \rangle \) and \( B = [b_L, b_R] = \langle b_C, b_W \rangle \) are two intervals, then the order relation \( \leq_{RC} \) is defined as \( A \leq_{RC} B \) iff \( a_R \leq b_R \) and \( a_C \leq b_C \), \( A <_{RC} B \) iff \( A \leq_{RC} B \) and \( A \neq B \).

A MOLP problem can be formulated as follows:
\[
\begin{align*}
\min z_k(x) &= \sum_{j=1}^{n} c_{kj}x_j, \quad k = 1, \ldots, p, \\
\text{s.t. } \sum_{j=1}^{m} a_{ij}x_j &\leq b_i, \quad i = 1, \ldots, m, \\
x_j &\geq 0, \quad j = 1, \ldots, n.
\end{align*}
\]

To present the following definitions, two order relations in \( \mathbb{R}^p \) are needed. Consider two vectors \( A = (a_1, \ldots, a_p)^t \) and \( B = (b_1, \ldots, b_p)^t \) in \( \mathbb{R}^p \). Then, \( A \preceq B \) if \( a_i \leq b_i \) for \( i = 1, \ldots, p \) and there is at least one \( 1 \leq q \leq p \) with \( a_q < b_q \). Also, \( A < B \) if \( a_i < b_i \) for \( i = 1, \ldots, p \).

**Definition 2.3:** A feasible solution \( x^0 = (x^0_1, \ldots, x^0_n)^t \) of Problem (1) is efficient if there is no another feasible solution \( x = (x_1, \ldots, x_n)^t \) such that \( (z_1(x), \ldots, z_p(x))^t \preceq (z_1(x^0), \ldots, z_p(x^0))^t \).

**Definition 2.4:** A feasible solution \( x^0 = (x^0_1, \ldots, x^0_n)^t \) of Problem (1) is weakly efficient if there is no another feasible solution \( x = (x_1, \ldots, x_n)^t \) such that \( (z_1(x), \ldots, z_p(x))^t < (z_1(x^0), \ldots, z_p(x^0))^t \).

**Definition 2.5:** Consider \((w_1, \ldots, w_p)\) such that \( w_k \geq 0, \ k = 1, \ldots, p \). The weighted sum linear programming problem (for short, weighted sum problem) associated with the MOLP Problem (1) is as follows:
\[
\begin{align*}
\min \sum_{k=1}^{p} \sum_{j=1}^{n} w_k c_{kj}x_j, \\
\text{s.t. } \sum_{j=1}^{m} a_{ij}x_j &\leq b_i, \quad i = 1, \ldots, m, \\
x_j &\geq 0, \quad j = 1, \ldots, n.
\end{align*}
\]
Theorem 2.6: A feasible solution \( x^0 = (x^0_1, \ldots, x^0_n)^t \) is a weakly efficient solution to the MOLP Problem (1) if and only if there exists a \( w = (w_1, \ldots, w_p) \) with \( w_i \geq 0, i = 1, \ldots, p \), such that \( x^0 \) is an optimal solution to Problem (2).

Theorem 2.7: A feasible solution \( x^0 = (x^0_1, \ldots, x^0_n)^t \) is an efficient solution to the MOLP Problem (1) if and only if there exists a \( (w^1, \ldots, w^p) \) with \( w^i > 0, i = 1, \ldots, p \), such that \( x^0 \) is an optimal solution to Problem (2).

3. Problem Statement and Main Results

Let the interval MOLP problem be given as

\[
\min z_k(x) = \sum_{j=1}^{n} [c^L_{kj}, c^R_{kj}] x_j, \quad k = 1, \ldots, p,
\]

\[
s.t. \sum_{j=1}^{m} [a^L_{ij}, a^R_{ij}] x_j \leq [b^L_i, b^R_i], \quad i = 1, \ldots, m,
\]

\[
x_j \geq 0, \quad j = 1, \ldots, n.
\]

(3)

It should be noted that if each of the intervals is a real value, then Problem (3) is a MOLP problem.

With respect to Definitions 2.2–2.4, the following solution concepts according to Problem (3) are defined.

Definition 3.1: A feasible solution \( x^0 = (x^0_1, \ldots, x^0_n)^t \) of Problem (3) is RC-efficient if there is no feasible solution \( x = (x_1, \ldots, x_n)^t \) such that \( (z_1(x), \ldots, z_p(x)) \preceq_{RC} (z_1(x^0), \ldots, z_p(x^0)) \), i.e. \( z_i(x) \leq_{RC} z_i(x^0) \) for \( i = 1, \ldots, p \) and there is at least one \( 1 \leq q \leq p \) with \( z_q(x) <_{RC} z_q(x^0) \).

Definition 3.2: A feasible solution \( x^0 = (x^0_1, \ldots, x^0_n)^t \) of Problem (3) is RC-weakly efficient if there is no feasible solution \( x = (x_1, \ldots, x_n)^t \) such that \( (z_1(x), \ldots, z_p(x)) \prec_{RC} (z_1(x^0), \ldots, z_p(x^0)) \), i.e. \( z_i(x) <_{RC} z_i(x^0) \) for \( i = 1, \ldots, p \).

In order to solve Problem (3), an attempt is being made to obtain an equivalent crisp problem. To do that, firstly by using the weighted sum approach [1], the following linear programming problem with interval parameters is achieved.

\[
\min \sum_{k=1}^{p} \sum_{j=1}^{n} [w_k c^L_{kj}, w_k c^R_{kj}] x_j,
\]

\[
s.t. \sum_{j=1}^{m} [a^L_{ij}, a^R_{ij}] x_j \leq [b^L_i, b^R_i], \quad i = 1, \ldots, m,
\]

\[
x_j \geq 0, \quad j = 1, \ldots, n.
\]

(4)

where \( w_k \geq 0, k = 1, \ldots, p \) denotes the weight of the \( k \)th interval objective function. Finally, considering the order relation \( \leq_{RC} \), the following deterministic bi-objective problem is
obtained.

\[
\begin{align*}
\min z_1(x) &= \sum_{k=1}^{p} \sum_{j=1}^{n} w_k c^R_{kj} x_j, \\
\min z_2(x) &= \sum_{k=1}^{p} \sum_{j=1}^{n} w_k c^C_{kj} x_j,
\end{align*}
\]

(5)

s.t. \[
\sum_{j=1}^{m} (a^L_{ij} + \alpha_i (a^R_{ij} - a^L_{ij})) x_j \leq b^R_i - \alpha_i (b^R_i - b^L_i), \quad i = 1, \ldots, m,
\]

\[
x_j \geq 0, \quad j = 1, \ldots, n,
\]

where \(0 \leq \alpha_i \leq 1, \quad i = 1, \ldots, m\). As an example, if \(\alpha_i = 0, \quad i = 1, \ldots, m\), then the largest feasible region is achieved.

Relations between Problems (3)–(5) are stated in the following theorems.

**Theorem 3.3:** According to the order relation \(\leq_{RC}\), any efficient solution of Problem (5) with \(\alpha_i = 0, \quad i = 1, \ldots, m\), is an optimal solution of Problem (4).

**Proof:** Suppose \(\hat{x} = (\hat{x}_1, \ldots, \hat{x}_n)^t\) is an efficient solution of Problem (5) with \(\alpha_i = 0, \quad i = 1, \ldots, m\). Further, assume that \(\hat{x}\) is not an optimal solution of Problem (4) according to the order relation \(\leq_{RC}\). Therefore, there is some feasible solution \(x = (x_1, \ldots, x_n)^t\) of Problem (4) such that

\[
\begin{align*}
\sum_{k=1}^{p} \sum_{j=1}^{n} [w_k c^L_{kj}, w_k c^R_{kj}] x_j &<_{RC} \sum_{k=1}^{p} \sum_{j=1}^{n} [w_k c^L_{kj}, w_k c^R_{kj}] \hat{x}_j,
\end{align*}
\]

which is a contradiction to the efficiency of \(\hat{x}\) for problem (5) with \(\alpha_i = 0, \quad i = 1, \ldots, m\) and the proof is completed.

**Theorem 3.4:** According to the order relation \(\leq_{RC}\), any optimal solution of Problem (4) with \(w_k > 0, \quad k = 1, \ldots, p\) is an RC-efficient solution of interval MOLP problem (3).

**Proof:** Assume that \(\hat{x} = (\hat{x}_1, \ldots, \hat{x}_n)^t\) is an optimal solution of Problem (4) according to the order relation \(\leq_{RC}\) and it is not an RC-efficient solution to Problem (3). Therefore, by Definition 3.1, there is some feasible solution \(x = (x_1, \ldots, x_n)^t\) of problem (3) such that

\[
(z_1(x), \ldots, z_p(x)) \not\leq_{RC} (z_1(\hat{x}), \ldots, z_p(\hat{x})),
\]

which means \(z_i(x) <_{RC} z_i(\hat{x})\) for \(i = 1, \ldots, p\) and \(z_q(x) <_{RC} z_q(\hat{x})\) for some \(1 \leq q \leq p\). This is a contradiction to the optimality of \(\hat{x}\) for Problem (4) and the proof is completed.

**Theorem 3.5:** According to the order relation \(\leq_{RC}\), a unique optimal solution of Problem (4) with \(w_k \geq 0, \quad k = 1, \ldots, p\) is an RC-efficient solution of interval MOLP problem (3).

**Proof:** The proof can be written in the same manner as the proof of Theorem 3.4.
Theorem 3.6: According to the order relation $\leq_{RC}$, any optimal solution of Problem (4) with $w_k \geq 0, k = 1, \ldots, p$ is an RC-weakly efficient solution of interval MOLP problem (3).

Proof: Considering Definition 3.2, the proof is similar to that of Theorem 3.4.  

4. Numerical Example

Suppose that a factory can produce three products, namely, $P_1, P_2$ and $P_3$. According to the past experience, for selling each unit of $P_1, P_2$ and $P_3$, the factory can earn income that are intervals $[7, 8], [2, 3]$ and $[4, 6]$ (in $\$$), respectively. With respect to the workers’ experience, producing each unit of $P_1, P_2$ and $P_3$, consume the quantity of a rare resource that are $[6, 9], [2, 4]$ and $[4, 5]$ (in kg), respectively. The production cost of each unit of $P_1, P_2$ and $P_3$ are $[2, 2.5], [2.5, 3]$ and $[1, 2]$ where the total financial resources is $[100, 108]$ (in $\$$). The manager of the factory aims to maximise the income and minimise the consumption of the rare resource subject to the financial limitation and two other constraints on amounts of products.

The problem is formulated as an interval MOLP problem:

$$\begin{align*}
\text{max } z_1(x) &= [7, 8]x_1 + [2, 3]x_2 + [4, 6]x_3, \\
\text{min } z_2(x) &= [6, 9]x_1 + [2, 4]x_2 + [4, 5]x_3, \\
\text{s.t. } 2.5x_1 + 3x_2 + 2x_3 &\leq 100, \\
x_1 + x_2 + x_3 &\geq 40, \quad x_3 \leq 25, \\
x_i &\geq 0, \quad i = 1, 2, 3,
\end{align*}$$

(6)

where $x_i, i = 1, 2, 3$, denotes the amount of product $P_i, i = 1, 2, 3$, that should be produced. By applying the proposed method with $w_1 = w_2 = 0.5$, the bi-objective linear program (7) is yielded.

$$\begin{align*}
\text{min } z_R(x) &= x_1 + x_2 + 0.5x_3, \\
\text{min } z_C(x) &= 0.25x_2 - 0.25x_3, \\
\text{s.t. } 2.5x_1 + 3x_2 + 2x_3 &\leq 100, \\
x_1 + x_2 + x_3 &\geq 45, \\
x_3 &\leq 25, \\
x_i &\geq 0, \quad i = 1, 2, 3.
\end{align*}$$

(7)

It should be noted that the smallest feasible region is used in Problem (7) which could be changed according to the decision maker’s idea.

An efficient solution to Problem (7) by using the weighted sum method with the weight vector $(0.5, 0.5)$ is

$$x_1 = 20, \quad x_2 = 0, \quad x_3 = 25.$$

The corresponding interval objectives are $z_1 = [240, 310]$ and $z_2 = [220, 305]$. 
5. Conclusion

In this paper, a multiobjective linear programming problem with interval parameters was considered. To deal with such a problem, a new solution procedure based on a suitable order relation of intervals, was presented. A numerical example was provided to illustrate the efficiency of the proposed method. Try to propose new solution methods with suitable properties for dealing with MOLP problems could be considered as a general topic for further research.

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References

[1] Ehrgott M. Multicriteria optimization. Berlin: Springer; 2005.
[2] Rivaz S, Yaghoobi MA. Minimax regret solution to multiobjective linear programming problems with interval objective function coefficients. Central Eur J Oper Res. 2013;21:625–649.
[3] Birge JR, Louveaux F. Introduction to stochastic programming. New York (NY): Physica-Verlag; 1993.
[4] Prokopa A. Stochastic programming. Boston: Kluwer; 1995.
[5] Jimenez M, Billbao A. Pareto-optimal solutions in fuzzy multi-objective linear programming. Fuzzy Sets Syst. 2009;160:2714–2721.
[6] Luhandjula MK, Rangoaga MJ. An approach for solving a fuzzy multiobjective programming problem. Eur J Oper Res. 2014;232:249–255.
[7] Nasseri SH, Baghban Al. A new approach for solving fuzzy multi-objective quadratic programming of water resource allocation problem. J Ind Eng Manag Stud. 2019;6:78–102.
[8] Garajova E, Hladik M, Rada M. Interval linear programming under transformations: optimal solutions and optimal value range. Central Eur J Oper Res. 2019;27:601–614.
[9] Batamiz A, Allahdadi M, Hladik M. Obtaining efficient solutions of interval multi-objective linear programming problems. Int J Fuzzy Syst. 2020;22:873–890.
[10] Bhurjee AK, Kumar P, Padhan SK. Solid transportation problem with budget constraints under interval uncertain environments. Int J Process Manag Benchmark. 2017;7:172–182.
[11] Ferdowski F, Maleki HR, Rivaz S. Air refueling tanker allocation based on a multi-objective zero-one integer programming model. Oper Res. 2020;20:1913–1938.
[12] Fortin J, Zielinski P, Dubois D, Fargier H. Criticality analysis of activity networks under interval uncertainty. J Sched. 2010;13:609–627.
[13] Garajova E, Hladik M. On the optimal solution set in interval linear programming. Comput Optim Appl. 2019;72:269–292.
[14] Giove S, Funari S, Nardelli C. An interval portfolio selection problem based on regret function. Eur J Oper Res. 2006;170:253–264.
[15] Ida M. Portfolio selection problem with interval coefficients. Appl Math Lett. 2003;16:709–713.
[16] Inuiguchi M, Kume Y. Goal programming problems with interval coefficients and target intervals. Eur J Oper Res. 1991;52:345–360.
[17] Kumar P, Panda G, Gupta U. An interval linear programming approach for portfolio selection model. Int J Oper Res. 2016;27:149–164.
[18] Lai KK, Wang SY, Xu JP, et al. A class of linear interval programming problems and its application to portfolio selection. IEEE Trans Fuzzy Syst. 2002;10:698–704.
[19] Rivaz S, Yaghoobi MA. Weighted sum of maximum regrets in an interval MOLP problem. Int Trans Oper Res. 2018;25:1659–1676.
[20] Yu VF, Hu KJ, Chang AY, An interactive approach for the multi-objective transportation problem with interval parameters. Int J Prod Res. 2015;53:1051–1064.
[21] Bitran GR. Linear multiobjective problems with interval coefficients. Manage Sci. 1980;26:694–706.
[22] Urli B, Nadeau R. An interactive method to multiobjective linear programming problems with interval coefficients. INFOR. 1992;30:127–137.
[23] Oliveira C, Antunes CH. Multiple objective linear programming models with interval coefficients—an illustrative overview. Eur J Oper Res. 2007;181:1434–1463.
[24] Oliveira C, Antunes CH. An interactive method of tackling uncertainty in interval multiple objective linear programming. J Math Sci. 2009;161:854–866.
[25] Wu HC. The Karush–Kuhn–Tucker optimality conditions in multiobjective programming problems with interval-valued objective functions. Eur J Oper Res. 2009;196:49–60.
[26] Ishibuchi H, Tanaka H. Multiobjective programming in optimization of the interval objective function. Eur J Oper Res. 1990;48:219–225.