TROPICAL GEOMETRIC COMPACTIFICATION OF MODULI, II
- $A_g$ CASE AND HOLOMORPHIC LIMITS -

YUJI ODAKA

To the memory of Kentaro Nagao

Abstract. We compactify the classical moduli variety $A_g$ of principally polarized abelian varieties of complex dimension $g$ by attaching the moduli of flat tori of real dimensions at most $g$ in an explicit manner. Equivalently, we explicitly determine the Gromov-Hausdorff limits of principally polarized abelian varieties. This work is analogous to [Od4], which compactified the moduli of curves by attaching the moduli of metrized graphs.

Then, we also explicitly specify the Gromov-Hausdorff limits along holomorphic family of abelian varieties and show that they form special non-trivial subsets of the whole boundary. We also do it for algebraic curves case and observe a crucial difference with the case of abelian varieties.

Contents

1. Introduction 2
2. Compactifying $A_g$ 4
   2.1. Gromov-Hausdorff collapse of abelian varieties 4
   2.2. Construction of $\bar{A}_g^T$ and comparison with other tropical moduli space 12
   2.3. Finite and infinite joins of $A_g$ 14
   2.4. On the (co)homology groups 15
   2.5. Gromov-Hausdorff limits with other rescaling 17
3. Along holomorphic disks 21
   3.1. Abelian varieties case 21
   3.2. Algebraic curves case 23
   3.3. Torelli maps 32
Appendix A. Morgan-Shalen type compactification 35
   A.1. Slight extensions 35
   A.2. Functoriality of MSBJ construction 44
References 45

Date: April 23, 2017.
1. Introduction

This paper is a companion paper to (or sequel of) [Od4], which gave a couple of compactifications of the moduli of hyperbolic projective curves $M_g$ and analyzed them. We work on the moduli space $A_g$ of $g$-dimensional principally polarized complex abelian varieties in this paper. What we first prove in our §2 will be roughly summarized as follows.

**Theorem 1.1** (cf., 2.1, 2.3, 2.5). The moduli space $A_g$ of principally polarized abelian varieties over $\mathbb{C}$ with the complex analytic topology admits a compactification $\bar{A}_g$ which attach (as its boundary) the moduli space of all real flat tori of dimension 1 up to $g$, the half of the original abelian varieties’ real dimension.

We interpret the real flat tori appearing here as tropical abelian varieties, as commonly defined so these days (cf., e.g., [BMV], [MZ]), so we would like to call this compactification $\bar{A}_g$ the tropical geometric compactification of $A_g$. However, in this particular case of abelian varieties, in reality the compactification is nothing but the “Gromov-Hausdorff compactification” i.e., attaching the moduli space of all possible Gromov-Hausdorff limits as metric spaces, to the original moduli space ($A_g$) as a boundary. The abstract existence of such Gromov-Hausdorff compactification of $A_g$ (without knowing what are the limits and the structure of the boundary) itself is a direct corollary of the well-known Gromov’s precompactness theorem [Grom]. Our point is to study the very explicit structures of the compactification by in particular identifying the Gromov-Hausdorff limits as specific flat tori, and also show relations with other fields.

**Remark 1.2.** The reason why we would like to allow two possible names for the identical compactification above, tropical geometric compactifications and Gromov-Hausdorff compactifications, is as follows. Basically the author sees this coincidence as a special phenomenon only for abelian varieties etc. As we also did for curves case ([Od4]), the Gromov-Hausdorff compactification has a natural uniform definition independent of class of Kähler-Einstein polarized varieties, and the boundaries parametrize only metric spaces. On the other hand, what we would like to propose and name as tropical geometric compactifications of the moduli spaces of collapsing Kähler-Einstein polarized varieties, should in general parametrize a priori more informations, such as affine structures to regard it as “polarized tropical varieties” rather than simply metric spaces at the boundary. In that way, the author believes, the compactifications should become “nicer” with more
structures. At the moment of writing this, we only have case by case “working definitions” of such compactifications for $M_g$ ([Od4]), $A_g$ and K3 surfaces ([OO]). See remark 2.6 for more explanation for our particular case.

More precisely, the point of §2 of this paper is, by using the Siegel reduction and so on, to explicitly determine the Gromov-Hausdorff limits as well as the structure of the compactification. Then, we go on to make some more basic analysis of the compactification including the relations of cohomologies and homologies. Then, in §3 in turn, we determine the holomorphic limits and compare. That is, we consider Gromov-Hausdorff limits of an arbitrary given punctured holomorphic family of either principally polarized abelian varieties of the form $(\mathcal{X}, L) \to \Delta^*$ or canonically polarized curves where $\Delta^*$ denotes a punctured smooth algebraic curve $\Delta \setminus \{0\}$. More precisely, we take a sequence $t_i \in \Delta^*$ ($i = 1, 2, \cdots$) which converges to the puncture $0 \in \Delta$ and discuss the Gromov-Hausdorff limit of $\mathcal{X}_{t_i}$. The result in particular shows that it does not depend on the sequence we take, once we fix the family.

Our results in §3 can be roughly summarized as follows. For (i), we only prove under some condition (triviality of the Raynaud extension) and fully extend in our forthcoming joint paper [OO] with Y.Oshima.

**Theorem 1.3.** (i) (cf., [3.1], also [OO])

Given a punctured algebraic family of $g$-dimensional principally polarized abelian varieties $(\mathcal{X}, L) \to \Delta^* \ni t$, the Gromov-Hausdorff limit at $0 \in \Delta$ does not depend on the choice of sequence converging to $0 \in \Delta$ and such limits form the union of $A_g$ and a dense subset (consists of “rational points”) inside the whole boundary $\partial \bar{A}_g$.

(ii) (cf., [3.2.1], [3.15])

Given a punctured algebraic family of smooth projective curves of genus $g \geq 2$ $(\mathcal{X}, L) \to \Delta^* \ni t$, the Gromov-Hausdorff limit at the puncture $0 \in \Delta$ does not depend on the choice of sequence converging to $0$ and such limits form the union of $M_g$ and a finite subset inside the whole real $3g - 4$ dimensional boundary $\partial \bar{M}_g$.

Finally, our appendix discusses the Morgan-Shalen compactification [MS], recently revisited and extended by Favre [Fav] and Boucksom-Jonsson [BJ]. We deal with slight more extensions and prove basic properties as to identify our compactifications (in e.g., [3.1], [OO]). This
appendix is used for stating Theorem 3.7 but since it is fairly independent study, we decided to put later as appendix. Hence, those interested in 3.7 could be either once skipped to the appendix and then come back to 3.7 after that or simply skip 3.7 and continue to read the main texts.

As this series of papers heavily depends on the basic theory of Gromov-Hausdorff convergence, we refer to [BB1] if needed.

**Funding.** This work was partially supported by the Japan Society for the Promotion of Science [Kakenhi, Grant-in-Aid for Young Scientists (B) 26870316] and [Kakenhi, Grant-in-Aid for Scientific Research (S), No. 16H06335].

**Acknowledgments.** The original version of this preprint was e-print arXiv:1406.7772 appeared in June 2014. §2 of this paper is a much revision of the latter half of it. The former half of the original e-print was put as another paper (v2 of arXiv:1406.7772) also with some mathematical and expository improvements. Parts of this work is done during when the author visited Chalmers university and Paris. He thanks their warm hospitality and discussions, especially R. Berman, S. Boucksom, C. Favre and M. Jonsson, A. Macpherson. Since 2014, large part of this paper (and [Od4], plus some of [OO]) has been presented in the talks at various places including Kyoto, Tokyo, Oaxaca, Gothenberg, Oxford, Kanazawa, Singapore and we appreciate the organizers.

We would like to dedicate this set of papers (with [Od4]) to the heartwarming memory of Kentaro Nagao.

We will further continue our series in a forthcoming joint paper with Y. Oshima [OO], to whom I also thank for his helpful comments to this paper as well.

### 2. Compactifying $A_g$

**2.1. Gromov-Hausdorff collapse of abelian varieties.** In this section, to each $g$-dimensional principally polarized abelian variety $(V, L)$, we associate a rescaled Kähler-Einstein metrics whose diameters are 1. That is, we consider the flat Kähler metric $g_{KE}$ whose Kähler class is $c_1(L)$ and consider the induced distance on $V$ which we denote by $d_{KE}(V)$ in this paper. Then we rescale to $\frac{d_{KE}}{\text{diam}(d_{KE})}$ of diameter 1, which will be the metric in concern. diam$(\cdot)$ means the diameter. Note that in this case, precompactness of the corresponding moduli space $A_g$ with respect to the associated Gromov-Hausdorff distance follows from the famous Gromov’s precompactness theorem [Grom] (while it also directly follows from our arguments in this section.) We sometimes
omit the principal polarization and simply write principally polarized
abelian varieties as $V$ or $V_i$ as far as it should not cause any confu-
sion. For the basics of the Gromov-Hausdorff convergence in metric
geometry, we refer to e.g., the textbook [BB].

We proceed to classification of all the possible Gromov-Hausdorff
limits of them. The author suspects it has been naturally expected by
experts and at least partially known that those collapse should be (real)
flat tori but unfortunately he could not find precise study nor results in
literatures, so we present here a precise statement as well as its proof,
and also give explicit determinations of the limits. In particular, our
arguments show that the flat tori which can appear as such Gromov-
Hausdorff limits, have its real dimension at most $g$ (which is the half
of the real dimension of the original complex abelian varieties) and is
characterized only by that condition.

For simplicity and better presentation of ideas, let us first restrict
our attention to maximally degenerating case, and establish the general
case later.

**Theorem 2.1.** Consider an arbitrary sequence of $g$-dimensional prin-
cipally polarized complex abelian varieties $\{V_i\}_{i=1,2,3,...}$ which is converg-
ing to the cusp $A_0$ of the boundary of the Satake-Baily-Borel compactifi-
cation $\bar{A}_g^{sBB}$. We denote the flat (Kähler) metrics with respect to the
polarization $d_{KE}(V_i)$ and their diameters $\text{diam}(d_{KE}(V_i))$. Then, after
passing to an appropriate subsequence, we have a Gromov-Hausdorff
limit of $\{(V_i, \frac{d_{KE}(V_i)}{\text{diam}(d_{KE}(V_i))})\}_i$ which is $(g-r)$-dimensional (flat) tori of
diameter 1 with some $0 \leq r < g$.

Conversely, any such flat $(g-r)$-dimensional torus of diameter 1
with $0 \leq r < g$ can appear as a possible Gromov-Hausdorff limit of
such sequence of $g$-dimensional principally polarised abelian varieties
with fixed diameter 1.

**Proof.** Let us first set up our notations (mainly after [Chai]) on the
Siegel upper half space and its compactification theory due to I.Satake
[Sat], as in our proof, we make essential use of the Siegel reduction
theory.

---

Footnotes:

1. The character “さ” is Hiragana type character which we pronounce “SA”, the
first syllable of Satake and the idea of using this character is after Namikawa’s book
[Nam2] which used Katakana “サ” instead (but we japaneses rarely use katakana
for writing japanese name). The corresponding Kanji character 佐 is more normal.
For a point \( Z = X + \sqrt{-1}Y \) of the Siegel upper half space \( \mathcal{H}_g \), we denote the Jacobi decomposition of \( Y \) as \( Y = tBDB \), where

\[
B = \begin{pmatrix}
1 & b_{1,2} & b_{1,3} & \cdots & b_{1,g} \\
1 & b_{2,3} & \cdots & b_{2,g} \\
0 & 1 & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1
\end{pmatrix},
\]

\[
D = \text{diag}(d_1, \ldots, d_g) = \begin{pmatrix}
d_1 & 0 & 0 & \cdots & 0 \\
d_2 & d_3 & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & d_{g-1} & d_g
\end{pmatrix}.
\]

Equivalently, writing \( Y = \sqrt{Y} \sqrt{Y} \) with a matrix \( \sqrt{Y} \in GL(g, \mathbb{R}) \),
\[
\sqrt{Y} = \text{diag}(\sqrt{d_1}, \ldots, \sqrt{d_g})B
\]
is the corresponding Iwasawa decomposition.

Using the above notation, recall that the Siegel subset \( \mathfrak{F}_g(u) \) of the Siegel upper half plane \( \mathcal{H}_g \) is defined as

\[
\{ X + \sqrt{-1}Y \in \mathcal{H}_g \mid |x_{ij}| < u, |1-b_{i,j}| < u, 1 < ud_i < ud_{i+1} \text{ for all } i, j \}.
\]

It is known to satisfy

\[
(1) \quad \text{Sp}_{2g}(\mathbb{Z}) \cdot \mathfrak{F}_g(u) = \mathcal{H}_g
\]
for \( u \gg 0 \). Let us set

\[
\mathcal{H}_g^* := \mathfrak{F}_g \sqcup \mathfrak{F}_{g-1} \sqcup \mathfrak{F}_{g-2} \sqcup \cdots \sqcup \mathfrak{F}_0.
\]

Then the Satake-Baily-Borel compactification \( \tilde{\mathcal{A}}^*_g \) can be defined as \( \mathfrak{F}_g^*/\sim \) with some equivalent relation \( \sim \) extending \( \text{Sp}_{2g}(\mathbb{Z}) \)-action on \( \mathfrak{F}_g \). Thanks to (2) we can suppose that, fixing sufficiently large \( u_0 \gg 1 \), we have \( \tilde{\mathcal{A}}^*_g = \mathfrak{F}_g^*(u_0)/\sim \) with the same equivalent relation, where

\[
\mathfrak{F}_g^*(u_0) := \mathfrak{F}_g(u_0) \sqcup \mathfrak{F}_{g-1}(u_0) \sqcup \cdots \sqcup \mathfrak{F}_0(u_0) \subset \mathcal{H}_g^*.
\]

We refer to [Chai] for the details of above discussions. Using the above reduction, we can assume that the whole sequence \( \{ V_i \}_i \) of principally polarize abelian varieties are parametrized by a sequence \( Z_i = X_i + \sqrt{-1}Y_i \) in \( \mathfrak{F}_g(u_0) \) for the fixed \( u_0 \) which is sufficiently large. Thanks to such boundedness, appropriately passing to a subsequence, we can and do assume \( X_i \) and \( B_i \) converges.
As a metric space, our principally polarized abelian variety which corresponds to $Z = X + \sqrt{-1}Y \in \mathfrak{g}$ is

$$\mathbb{C}^g/\left(\begin{array}{c}1 \\ X \\ Y \end{array}\right)\mathbb{Z}^g$$

which is isometric to $\mathbb{R}^{2g}/\mathbb{Z}^{2g}$ with metric matrix

$$(3) \begin{pmatrix} 1 & X \\ Y & Y^{-1} \end{pmatrix} \begin{pmatrix} Y^{-1} \\ Y^{-1} \end{pmatrix} = \begin{pmatrix} Y^{-1}X \\ XY^{-1} \end{pmatrix} \begin{pmatrix} Y^{-1} \\ Y^{-1} \end{pmatrix} = \begin{pmatrix} 1 \\ X \\ Y \end{pmatrix} \mathbb{Z}^g.$$

If we denote the corresponding standard basis of the lattice $\left(\begin{array}{c}1 \\ X \\ Y \end{array}\right)\mathbb{Z}^g$ as $e_1, \ldots, e_{2g}$ and the Kähler-Einstein metric as $g_{KE}$ as before, then what we meant by metric matrix is defined as

$$\{g_{KE}(e_i, e_j)\}_{1 \leq i, j \leq 2g}.$$

In particular, if $X = (0)$ (zero matrix), then the metric matrix of our torus $\mathbb{R}^{2g}/\mathbb{Z}^{2g}$ is

$$\begin{pmatrix} Y^{-1} \\ Y \end{pmatrix}.$$

Thus what we would like to classify are possible Gromov-Hausdorff limits of

$$\frac{1}{\text{diam}(V_i)} \begin{pmatrix} Y_i^{-1} & Y_i^{-1}X_i \\ X_iY_i^{-1} & X_iY_i^{-1}X_i + Y_i \end{pmatrix}.$$

We set our notation as follows. For our points $Z_i = X_i + \sqrt{-1}Y_i$ in the Siegel set $\mathcal{F}_g(u_0)$ which give our principally polarized abelian varieites $V_i$ ($i = 1, 2, \ldots$), we do the Jacobi decomposition of $\sqrt{\mathcal{Y}}$ and denote the corresponding $d_j$s as $d_j(V_i)$. Replacing $\text{diam}(V_i)$ by $d_g(V_i)$, let us first classify possible limits of

$$\frac{1}{d_g(V_i)} \begin{pmatrix} Y_i^{-1} & Y_i^{-1}X_i \\ X_iY_i^{-1} & X_iY_i^{-1}X_i + Y_i \end{pmatrix}$$

instead. Our assumption that it converges to the cusp $A_0 \in \partial \tilde{A}_g^{BB}$ (“maximally degenerating”) is equivalent to that $d_1(V_i) \to +\infty$ when $i \to +\infty$ from the definition of Satake topology (cf., [Sat], [Chai]).

Now, let us set

$$r := \min \left\{ (1 \leq j \leq g) \mid \lim inf_{i \to +\infty} \frac{d_j(V_i)}{d_g(V_i)} > 0 \right\} - 1.$$

Then, after appropriately passing to a subsequence again, we can assume that $\frac{d_j(V_i)}{d_g(V_i)} \to +0$ so that $\frac{d_j(V_i)}{d_g(V_i)} \to +0$ for all $j \leq r$.

By once more replacing $\{V_i\}_i$ by a subsequence if necessary, we can assume without loss of generality that for each $1 \leq j \leq g - r$ the
sequence \( \{ \frac{d\mathcal{E}(V_i)}{d_g(V_i)} \}_i \) converges. We denote that convergence values as \( a_{r+j} \) for each \( j \). We prove that then \( \{ (V_i, \frac{d\mathcal{E}(V_i)}{d_g(V_i)}) \}_i \) converges to a \((g - r)\)-dimensional torus as \( i \to +\infty \).

As \( d_g(V_i) \to +\infty \), it follows that \( Y_i^{-1}/d_g(V_i) \to +0 \). On the other hand, thanks to our preceded set of processes of replacing \( \{ V_i \}_i \) by its subsequence, the following holds

\[
\begin{pmatrix}
1 & & & \\
& d_1(V_i) & & \frac{d_2(V_i)}{d_g(V_i)} \\
& & d_3(V_i) & \\
& & & \ddots \\
0 & & & d_g(V_i)
\end{pmatrix}
\]

\[
\downarrow
\]

\[
\begin{pmatrix}
0 & & & \\
& \ddots & & \frac{0}{a_{r+1}} \\
& & 0 & \\
0 & & & a_g = 1
\end{pmatrix}
\]

when \( i \to +\infty \). Please note that the downarrow between the above big matrices “\( \downarrow \)” means convergence as \( g \times g \) real matrices, when \( i \to \infty \). From the above convergence of the matrices, it follows straightforward that \( \mathbb{R}^{2g}/\mathbb{Z}^{2g} \) whose metric matrix is

\[
\begin{pmatrix}
Y_i^{-1} & Y_i^{-1}X_i \\
X_iY_i^{-1} & X_iY_i^{-1}X_i + Y_i
\end{pmatrix}
\]

converges to a \((g-r)\)-dimensional torus in the Gromov-Hausdorff sense, when \( i \to +\infty \). From this result, we particularly deduce the following.

**Claim 2.2.** In the above setting, we have

\[
d_g(V_i) \sim d_{\mathcal{E}}(V_i)
\]

i.e., the ratio of the left hand side and the right hand side is bounded on both sides (by some positive constants) when \( i \to +\infty \).

Going back to proof of Theorem 2.1, now we would like to show the other direction i.e., to show that every \((g - r)\)-dimensional flat torus with \( 0 \leq r \leq g \) of diameter 1 can indeed appear as the above type Gromov-Hausdorff limit. Indeed, we can construct such a sequence in the following explicit manner, for instance. Fix \( (a_{r+1}, \cdots, a_g) \in \mathbb{R}^{g-r} \). Then set a sequence of \( g \times g \) diagonal real matrices \( \{ D_i \}_{i=1,2,\cdots} \) as
$D_i := \text{diag}(d_{1,i}, \ldots, d_{g,i}) := \begin{pmatrix} d_{1,i} & 0 & \cdots & 0 \\ 0 & d_{2,i} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{g,i} \end{pmatrix},$

where

$d_{j,i} := 1$

for $j \leq r$ and

$d_{j,i} := (i + 1)^r a_j$

for $j \geq r + 1$. Recall the notation at the beginning of our proof of Theorem 2.1. Let us fix “X-part and B-part”, i.e., set $Z_i = X_i + \sqrt{-1} Y_i$ with constant $X_i = X$ and $B_i = B$, where $Y_i = t^T B_0 D_i B_i$ is the Jacobi decomposition for $i = 1, 2, \cdots$ and denote the corresponding principally polarized abelian variety as $V_i$ with the associated Kähler-Einstein metric $g_{KE}(V_i)$. Then the Gromov-Hausdorff limit of $(V_i, \frac{g_{KE}(V_i)}{d_g(V_i)})$ for $i \to \infty$ is a $(g - r)$-dimensional torus whose metric matrix is

$\begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$

Letting $B$ runs over all upper triangular real $g \times g$ matrices, we get all $g \times g$ matrices of the form

$\begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$

with a positive definite $(g - r) \times (g - r)$ symmetric matrix $P$, as above limits. In such situation, the Gromov-Hausdorff limit of $(V_i, \frac{g_{KE}(V_i)}{d_g(V_i)})$ for $i \to \infty$ is $(g - r)$-dimensional flat real torus whose metric matrix is $P$. Thus we complete the proof. $\square$
Please note that $r < g$ can really happen while the conjectures [KS, Conjecture 1, p.19], [Gross, Conjecture 5.4], motivated by the Strominger-Yau-Zaslow mirror symmetry on Calabi-Yau varieties, expect the collapse only to just half dimensional affine manifolds (with singularities), i.e. $i = g$ case. This difference occurred naturally, and hence not contradicting, simply because we take an arbitrary sequence rather than dealing with proper algebraic family with maximal monodromy as they do. For a general sequence in $A_g$, we prove the following.

**Theorem 2.3.** We use the same notation as Theorem 2.1. Suppose a sequence of $g$-dimensional principally polarized complex abelian varieties $\{V_i\}_{i\geq 1}$ converges to a point of $A_c \subset \partial \bar{A}_g^{BB}$ with $0 \leq c < g$ in the Satake-Baily-Borel compactification $\bar{A}_g^{BB}$.

Then, after passing to a subsequence, $(V_i, \frac{d_{KE}(V_i)}{\text{diam}(d_{KE}(V_i))})$ converges to a $(g-r)$-dimensional (flat) tori of diameter 1 with some $(c \leq r < g)$, in the Gromov-Hausdorff sense.

Conversely, any such flat $(g-r)$-dimensional torus of diameter 1 with $c \leq i \leq g$ can appear as a possible Gromov-Hausdorff limit of such sequence of $g$-dimensional principally polarized complex abelian varieties with diameter 1 rescaled Kähler-Einstein (flat) metrics.

Before going to the proof, let us analyse what the above particularly means. Note that the set of possible limits described above is included in the corresponding limit set of the maximal degeneration case 2.1. Morally speaking, this can be seen as a special case of more general phenomenon that “degeneration / deformation” order get reversed once we pass from algebro-geometric setting to its tropical analogue. Indeed, similar phenomenon happened in curve case ([Od4]).

Another very simple fact, which is partially related to above, reflecting such general phenomenon is the following. It roughly states that Gromov-Hausdorff limit of degenerating spaces sees just “degenerating part” and ignores non-degenerating part.

**Proposition 2.4.** Suppose a sequence of compact metric spaces $\{X^{(i)}\}_{i\in \mathbb{Z}_{>0}}$ decomposes as

$$X_1^{(i)} \times \cdots \times X_m^{(i)}$$

as metric spaces with $p$-product metric for some $p > 0$. If the last component $X_m^{(i)}$ is “responsible of degeneration” in the sense that

(i) $\text{diam}(X_m^{(i)}) \to +\infty$ and

(ii) $\text{diam}(X_j^{(i)}) \leq \text{constant}$ for all $j \neq m$,
then the Gromov-Hausdorff limit “only sees $X_m^{(i)}$” in the sense that
\[ \lim_{i \to +\infty} (X^{(i)}/\text{diam}(X^{(i)})) = \lim_{i \to +\infty} (X_m^{(i)}/\text{diam}(X_m^{(i)})). \]

Here the above $\lim_{i \to +\infty}$ means the Gromov-Hausdorff limits and $(X^{(i)}/\text{diam}(X^{(i)}))$ (resp., $(X_m^{(i)}/\text{diam}(X_m^{(i)}))$) means the topological space $X^{(i)}$ (resp., $X_m^{(i)}$) with the rescaled metric of the original metric, with diameter 1.

A trivial remark is that the statement of the above proposition is just equivalent to $m = 2$ case but we stated as above just to get a better intuition for various applications.

Proof. The whole point is simply that there is a constant $c$ which satisfies the inequality
\[ \text{diam}(X_m^{(i)}) \leq \text{diam}(X^{(i)}) \leq \text{diam}(X_m^{(i)}) + c \]
for all $i$. The assertion easily follows from the above. \qed

Thus indeed if a punctured family of abelian varieties with semabelian reduction with torus rank $(g - r)$ of the central fiber, it follows that the torus part determines the Gromov-Hausdorff limit (with fixed diameters). Theorem 2.3 is reflecting that fact.

Let us now turn to the proof, i.e. the classification of our Gromov-Hausdorff limits of principally polarized abelian varieties.

Proof of Theorem 2.3 As the proof is a fairly simple extension of the proof of maximal degeneration case (2.1), without bringing essentially new ideas, here we only sketch the proof, focusing on the differences. As in (2.1), thanks to the Siegel reduction theory, we can and do fix sufficiently large $u_0 \gg 0$ so that our sequence can be parametrized by a sequence
\[ \{Z_i = X_i + \sqrt{-1}Y_i\}_{i=1, 2, \ldots} \]
in the Siegel set $\mathfrak{F}_g(u_0)$ (cf., the definition (Π)). Again in the same manner, we can and do appropriately take a subsequence so that the following conditions hold.

(i) $X_i$ converges when $i \to +\infty$,
(ii) the upper triangle matrix part $B(V_i)$ converges when $i \to +\infty$,
(iii) $d_j(V_i)$ for any $(1 \leq j(\leq c)$ converges when $i \to +\infty$,
(iv) $d_{c+j}(V_i) \to +\infty$ when $i \to +\infty$ for any $(1 \leq j(\leq (g - c))$.

Here, the notations are same as the proof of (2.1). Let us set again
\[ r := \max\{(1 \leq j(\leq g) | \lim_{i \to +\infty} \inf \frac{d_j(V_i)}{d_j^g(V_i)} = 0\}. \]
Then in our general case, we have $c \leq r \leq g$ from the definition of the Satake topology \[Sat\]. The rest of the proof that $V_i$ converges to a $(g - r)$-dimensional flat torus with diameter 1 is completely the same.

Conversely, for a given $r \geq c$, let us prove that any $(g - r)$-dimensional torus $T$ with diameter 1 can appear as the above limit. From our (2.1), we know there is a sequence of principally polarized abelian varieties $(W_i, M_i)_{i=1, 2, \ldots}$ of complex dimension $g - c$. Then, we take arbitrary $c$-dimensional principally polarized abelian variety $W'$ and set $V_i := W' \times W_i$ for each $i = 1, 2, \ldots$ which admit natural principal polarizations from the construction. Then $\{V_i\}_{i=1, 2, \ldots}$ with rescaled Kähler-Einstein metric $\frac{d_{KE}(V_i)}{\text{diam}(d_{KE}(V_i))}$ converge to $T$ in the Gromov-Hausdorff sense, as the simple combination of Proposition 2.4 and Theorem 2.1 show.

□

2.2. Construction of $\bar{A}_g^T$ and comparison with other tropical moduli space. Similarly as in the curves case, we rigorously define our tropical geometric compactification of the moduli space of principally polarized abelian varieties first set-theoretically as $\bar{A}_g^T := A_g \sqcup T_g$, where $T_g$ denotes the set (moduli) of real flat tori with diameters 1 whose dimension is $i$ with $1 \leq i \leq g$, from now on. Then we put a topology on $\bar{A}_g^T$ whose open basis can be taken as those of $A_g$ with respect to the complex analytic topology, and metric balls around point $[T]$ in $\partial \bar{A}_g^T$

$$B([T], r) := \{[X] \in \bar{A}_g^T \mid d_{GH}([X], [T]) < r\},$$

where $d_{GH}$ denotes, as in [Od4], the Gromov-Hausdorff distance with respect to the rescaled metric on each flat torus whose diameter is 1.

Then we get the following consequence of Theorem 2.3.

Corollary 2.5. $\bar{A}_g^T$ is a compact Hausdorff space containing $A_g$ as an open dense subset.

Remark 2.6. Each $k$-dimensional real flat torus $T$ ($1 \leq k \leq g$) that is parametrized at the boundary of our $\bar{A}_g^T$ carries a canonical integral affine structure (up to rescaling by positive constants) by identifying $T$ with $\mathbb{R}^k/\mathbb{Z}^k$ and consider the corresponding coordinates (or its equivalent class with respect to the positive scalar rescaling). This is indeed the integral affine structure we should put from the context of

--

Footnote: $T$ of $\bar{A}_g^T$ stands for Tropical while $T$ of $T_g$ stands for Tori.
the Strominger-Yau-Zaslow mirror symmetry picture for the principal polarization case as well (cf., e.g., [GS]).

Therefore, such affine structure does not give additional structure which enlarge the Gromov-Hausdorff compactification of $A_g$. It is the reason why we simply call the above compactification, the tropical geometric compactification. We refer to the footnote of the introduction

Note that if we forget complex structures of principally polarized abelian varieties, it gives nontrivial morphism from $A_g$ to a moduli space of certain flat tori of real dimension $2g$. The latter moduli space has the same dimension as that of $A_g$. Indeed, discreteness of the fibres of the forgetful morphism follows from the fact that, adding marking $[\pi_1(\text{complex ab. var of dim } g) \cong \mathbb{Z}^{2g}]$, which is obviously discrete data, recovers the complex structure of the abelian varieties. It easily follows from the fact that the metric matrix $[3]$ has enough information to recover $X$ and $Y$.

Let us clarify the simple relation with the moduli space $A_g^{tr}$ of tropical abelian varieties constructed in [BMV]. In their language, the boundary of our tropical geometric compactification $\bar{A}_g^T$ is

$$\partial \bar{A}_g^T \simeq A_g^{tr}/\mathbb{R}_{>0} = (\Omega^{rt} \setminus \{0\})/(GL(g, \mathbb{Z}) \cdot \mathbb{R}_{>0}),$$

where $A_g^{tr}$ is the moduli space of $g$-dimensional tropical (principally polarized) abelian varieties $\mathbb{R}^g/\Lambda$ in the sense of [BMV], $\Omega^{rt}$ (resp., $\Omega$) is the cone of positive semidefinite forms (resp., positive definite forms) on the universal covering $\mathbb{R}^g$ whose null space has a basis inside the rational vector space $\Lambda \otimes \mathbb{Q}$, following their notations. Note $\Omega \subset \Omega^{rt} \subset \bar{\Omega}$.

**Remark 2.7.** We make a simple observation on the relation between our Gromov-Hausdorff limits of principally polarized abelian varieties with the dual (intersection) complex (cf., [KS], [Gross]) of algebraic degenerations of them. Such connection is natural, after the well-known conjectures of Kontsevich-Soibelman [KS] and Gross-Siebert (cf., [Gross]) for their approach to the Strominger-Yau-Zaslow conjecture [SYZ]. In their studies, they also predict and partially establish that given a maximal degeneration of general Calabi-Yau manifolds, the dual complex of the special fiber is “close to” the Gromov- Hausdorff limit of Ricci-flat metrics with fixed diameters.

Let us think of the relative compactification of Alexeev and Nakamura ([AN], [Ale1], [Nak1], [Nak2]), of a semi-abelian reduction of a generically abelian scheme. Due to [AN (3.17)], [Ale1], [Nak1 (4.9)],

---

3For example, at the locus which parametrizes products of $g$ elliptic curves, it gives generally $2^g$ to 1 morphism.
the dual complexes are the duals of the Delaunay triangulations of \((g - r)\)-dimensional tori which are topologically of course always real torus of \((g - r)\)-dimension. This coincides with our collapsed limits, except for a slight difference that the tori can get lower dimension as we considered an arbitrary sequence there. We give a closer connection between Alexeev-Nakamura type degeneration of abelian varieties and our Gromov-Hausdorff limits later in section 3.1. Similarly, for the case of curves [Od4], the collapsed limit coincides with the dual graph of the limit stable curves and for higher dimensional semi-log-canonical models, we believe the collapsed limits along horomorphic one parameter degeneration \(X \rightarrow \Delta\) (partially analyzed in [Zha]) should be at least homeomorphic to the dual complex of lc centers of a log crepant blow up \(\tilde{X}_0\) of \(X_0\) whose normalization \(\tilde{X}_0^\nu\) with the conductor divisor \(\text{cond}(\nu)\) is a dlt pair. We also refer to [BJ] for related recent study. (The author morally sees this as a variant of the Yau-Tian-Donaldson correspondence and wishes to come back to this connection at deeper level in future.)

2.3. Finite and infinite joins of \(A_g\). Completely similarly as curve case ([Od4]), we can naturally construct joins of our tropical geometric compactifications \(\bar{A}_g^T\), thanks again to the inductive structure of the boundaries.

**Definition 2.8.** The finite join of our tropical geometric compactifications is defined inductively as

\[
\bar{A}_{\leq g}^T := \bar{A}_{\leq (g-1)}^T \cup_{T_{g-1}} \bar{A}_g^T.
\]

The union is obtained via two canonical inclusion maps \(T_{g-1} \hookrightarrow T_g\) and \(T_{g-1} \hookrightarrow \bar{A}_{\leq (g-1)}^T\). We call \(\bar{A}_{\leq g}^T\) a finite join of our tropical geometric compactifications.

From the definition, we have

\[
\cdots \bar{A}_{\leq (g-1)}^T \subset \bar{A}_{\leq g}^T \cdots.
\]

Then we set

\[
\bar{A}_\infty^T := \lim_{g \to \infty} \bar{A}_{\leq g}^T = \cup_g \bar{A}_{\leq g}^T,
\]

and call it the infinite join of our tropical geometric compactifications.

The boundary of our infinite join \(\bar{A}_\infty^T\) by which we mean the natural locus \(\cup_g (\partial \bar{A}_g^T = S^\text{wt}_g) \subset \bar{A}_\infty^T\), should be regarded as a tropical version

\footnote{also called “incidence complex” (cf., e.g. [Tyo]), “dual graph”, or “dual intersection complex” (cf., e.g., [Gross]) etc}

\footnote{However, unfortunately such existence is unknown. Also cf., [Ko2 5.22].}
of “$A_\infty$” [1] introduced and studied recently in [JJ] a while after the appearance of the first version of this paper.

Also note $\tilde{A}_\infty^T$ is connected and all our tropical geometric compactification $\tilde{A}_g^T$ is inside this infinite join. In particular, $A_g$ for all $g$ is inside this connected “big infinite dimensional moduli space”.

2.4. On the (co)homology groups. About the open dense locus $A_g$, the following has been classically known as a result of A.Borel who proved through studying the vector spaces of $Sp_{2g}(\mathbb{R})$-invariant different forms and the group cohomology interpretation that

$$H^i(A_g; \mathbb{Q}) = H^i(Sp_{2g}(\mathbb{Z}); \mathbb{Q}).$$

**Theorem 2.9** ([Bor]) $H^i(A_g; \mathbb{Q}) = \mathbb{Q}[x_2, x_6, x_{10}, \ldots]$ for $0 \leq i < g-1$, where the right hand side is a polynomial generated by $x_{4a+2}$ whose weight is $4a+2$. In particular, $H^i(A_g; \mathbb{Q}) = 0$ if $i$ is odd, less than $g-1$ and the stable cohomology is naturally $\lim_{\rightarrow g} H^*(A_g) = \mathbb{Q}[x_2, x_6, x_{10}, \ldots]$.

There are also many studies on the homology of symplectic groups such as [Char], [MV] etc. Using such topological results on $A_g$, at least partially the study of (co)homologies of the boundary $T_g$ gives some informations on those of $\tilde{A}_g^T$. For instance, a simple observation is that $dim(T_g) = 3g - 4$ combined with the long exact sequence of the Borel-Moore homology groups gives that $H_i(\tilde{A}_g^T; \mathbb{Q}) = 0$ for if $i$ is even and $i > g^2$.

Motivated partially from the above discussion, from now on, let us study the boundary $T_g$. Note that $T_g$ has the following orbifold as an open dense locus

$$\Omega/(\mathbb{R}_{>0} \cdot GL(g, \mathbb{Z})),\)$$

which we will write $T^o_g$. Then

$$T_g = T^o_g \sqcup T_{g-1},$$

so that we can partially study the (co)homology of $T_g$ inductively, once we know those of $T_g = \Omega/(\mathbb{R}_{>0} \cdot GL(g, \mathbb{Z}))$. However, the author does not know well how this cohomology behaves except for the asymptotic behaviour of the lower degree due to A.Borel [Bor], that is

$$H^i(T_g; \mathbb{Q}) = H^i(GL(g; \mathbb{Z}); \mathbb{Q}) = \mathbb{Q}[x_3, x_5, x_7, \ldots]_{\text{weight}=i},$$

for $i \leq (g - 5)/4$. Here, $x_i$ has weight $i$.

As in the discussion of the previous paper [Od4], we have a canonical chain of closed embeddings

---

6They call it “universal moduli spaces” of abelian varieties
(4) \[ T_g \hookrightarrow T_{g+1} \hookrightarrow \cdots \]

which is analogous to the boundary structure of the Satake-Baily-Borel compatification of \( A_g \). We have the following asymptotic triviality of the topologies, analogous to that of curves case \([Od4]\).

**Proposition 2.10.** The topological space \( T_\infty \) is contractible. \( \text{Im}(H_k(T_g; \mathbb{Q}) \to H_k(T_{g+1}; \mathbb{Q})) = 0 \) for any \( k \) and \( g \).

**Proof.** We imitate the idea of curve case analogue in \([Od4]\) but in this abelian varities case, it is even easier. However, the whole point is still the same, that is to construct an extension \( \psi_g : CT_g \to T_\infty \) of the identity map of \( T_g \) where \( CT_g := (T_g \times [0,1])/(T_g \times \{1\}) \), which is compatible with lower \( \psi \) i.e., \( \psi_g|_{T_g-1} = \psi_{g-1} \).

For \((X,d_X),(t) \in T_g \times [0,1] \) (\( d_X \) denotes the flat metric on \( X \)), we define

\[ \psi_g(X,t) := \text{rescale of } ((X,(1-t)d_X) \times S^1(t)) \text{ with diameter } 1. \]

The continuity of the map is obvious. Here, the product means the 2-product metric (i.e., simply the square root of the sum of squares of direction-wise distances). It is straightforward to confirm the requirements of the map. \( \square \)

Intuitively speaking, the all \( g \)-dimensional tori continuously and simultaneously change to once \((g+1)\)-dimensional tori but later collapse to a circle of circumference 1.

On the other hand, we have the following exact sequence from which high nontriviality of the topologies of \( T_g \) follows.

**Proposition 2.11.** We have the following two long exact sequences.

\( (i) \) \[ \cdots \to H_k(GL(g; \mathbb{Z}); \mathbb{Q})^* \to H^k(T_g, \mathbb{Q}) \to H^k(T_{g-1}; \mathbb{Q}) \to \cdots \]

\[ \cdots \to H_{k+1}(GL(g; \mathbb{Z}); \mathbb{Q})^* \to H^{k+1}(T_g, \mathbb{Q}) \to H^{k+1}(T_{g-1}; \mathbb{Q}) \to \cdots . \]

\( (ii) \) \[ \cdots \to H_k(T_{g-1}; \mathbb{Q}) \to H_k(T_g, \mathbb{Q}) \to H^k(GL(g; \mathbb{Z}); \mathbb{Q})^* \to \cdots \]

\[ \cdots \to H_{k-1}(T_{g-1}; \mathbb{Q}) \to H_{k-1}(T_g, \mathbb{Q}) \to H^{k-1}(GL(g; \mathbb{Z}); \mathbb{Q})^* \to \cdots . \]

**Proof.** These are simply the long exact sequences of compactly supported cohomology groups and the Borel-Moore homology groups respectively, combined with Lefschetz duality for orbifold \( T_g \setminus T_{g-1} = \Omega/GL(g, \mathbb{Z}) \). \( \square \)
2.5. **Gromov-Hausdorff limits with other rescaling.** There are of course some other ways of rescaling the metrics of abelian varieties which could produce essentially different (pointed) Gromov-Hausdorff limits. One of the nontrivial rescaling is (i) via fixing the *volume* while another is (ii) via fixing the *injectivity radius*. We discuss such two other ways of rescaling but before that, let us illustrate the differences by a simple example.

2.5.1. A *simple example.* Consider again a degenerating sequence of elliptic curves

\[ E_k := \mathbb{C}/(\mathbb{Z} + \mathbb{Z}k(a\sqrt{-1})) \]

for \( k = 1, 2, \ldots \), while \( a > 1 \) fixed. In this case, this is maximally degenerating so that the corresponding “torus rank” is \( r = 1 = g \).

The “diameter fixed” Gromov-Hausdorff limit is \( S^1(1/2\pi) \) as we observed. Instead if we fix the injectivity radius, then as the metric is standard metric of \( \mathbb{C} \) we get

\[ (\mathbb{R}/\mathbb{Z}) \times (\sqrt{-1}\mathbb{R}) \]

as the pointed Gromov-Hausdorff limit.

On the other hand, if we fix the volume of each \( E_k \), then we rescale the metric by multiplying the lengths by \( 1/\sqrt{ka} \). Then the pointed Gromov-Hausdorff limit is the imaginary axis

\[ (\sqrt{-1}\mathbb{R}) \subset \mathbb{C} \].

In our Gromov-Hausdorff interpretation of the Satake-Baily-Borel compactification \( \mathbb{C} \subset \mathbb{C}P^1 \) discussed above (2.13), this line of infinite length is corresponding to the cusp \( \{ \infty \} \) while the open part \( A_1 \simeq \mathbb{C} \) parametrizes flat 2-dimensional tori of volume 1.

2.5.2. *Fixing the injectivity radius.* In this subsection, we study pointed Gromov-Hausdorff limits of \( g \)-dimensional principally polarized abelian varieties with fixed *injectivity radius*, that is morally the “minimal” non-collapsing limits. We keep using the previous notation of this section. Recall that for our sequence \( \{ V_i \}_{i=1,2,\ldots} \) of principally polarized abelian varieties of \( g \)-dimension, the corresponding point in the Siegel set is denoted as \( Z_i = X_i + \sqrt{-1}Y_i \) with \( Y_i = iB_iD_iB_i \) (the Iwasawa decomposition of \( \sqrt{Y_i} \)).

Similarly as before, after passing to a subsequence, we can and do assume that for some \( 0 \leq r < g \),

(i) both \( X_i \) and \( B_i \) converge when \( i \) tends to infinity;
(ii) \( d_j(V_i) \) for all \( 1 \leq j \leq r \) converges to finite value while
(iii) \( d_j(V_i) \) for all \( j > r \) (strictly) diverges to infinity when \( i \) tends to infinity.
Here, what we meant by the strict divergence in the above (iii), is that all subsequences diverge. We assume the above three conditions throughout the rest of present subsection.

Let us first start with the simplest situation, i.e., those satisfying the following conditions.

(iv) $X_i = 0, B_i = I_g$ (unit matrix)
(v) $d_j(V_i) = a_j$ for all $j \leq r$ and
(vi) $d_j(V_i) = i \cdot a_j$ for all $j > r$.

The real constants $a_1, \cdots, a_g$ above satisfy that

$1 < u_0 a_0, a_i < u_0 a_{i+1}$.

Intuitively $g - r$ is the corresponding “torus rank” of limit. Then from the above assertions, it is easy to see that

**Proposition 2.12.** The pointed Gromov-Hausdorff limit of the rescaled Kähler-Einstein metrics on $V_i(i \to \infty)$ with fixed injectivity radius 1 in the above notation is isometric to

$$ \prod_{r < j \leq g} S^1\left(\frac{a_g}{2\pi a_j}\right) \times \mathbb{R}^{g+r}, $$

where $S^1(a)$ denotes a circle with radius $a$.

Note that “pointed” does not cause ambiguity in this situation, thanks to the homogeneity of abelian varieties.

**Sketch proof.** In the above simple situation, $V_i$ with the Kähler-Einstein metrics on $V_i(i \to \infty)$ is decomposed as

$$ \prod_{1 \leq j \leq g} (\mathbb{C}/\mathbb{Z} + \mathbb{Z}\sqrt{-1}(d_j(V_i))^2) $$

as Kähler manifolds and each $\mathbb{C}/\mathbb{Z} + \mathbb{Z}\sqrt{-1}(d_j(V_i))^2$ is isometric to the product metric space $S^1\left(\frac{1}{2\pi(d_j(V_i))}\right) \times S^1\left(\frac{(d_j(V_i))^2}{2\pi}\right)$. It concludes that $V_i$ with rescaled Kähler-Einstein metrics on $V_i(i \to \infty)$ with fixed injectivity radius 1 is isometric to

$$ \prod_{1 \leq j \leq g} \left(S^1\left(\frac{c_i}{2\pi d_j(V_i)}\right) \times S^1\left(\frac{c_i d_j(V_i)}{2\pi}\right)\right), $$

for positive real number $c_i$ defined as

$$ c_i := \left( \min\left\{ \frac{1}{2\pi d_j(V_i)}, \frac{d_j(V_i)}{2\pi} \mid 1 \leq j \leq g \right\} \right)^{-1}. $$

Then the assertion of Proposition 2.12 follows. \qed
Note that the limit above does not reflect any abelian part data ("$a_1, \cdots, a_r$") encoded in the boundary of the Satake-Baily-Borel compactification. We prefer the other Gromov-Hausdorff limits, hence we do not pursue the above type rescaled limits further, partially because our main intention is (still) to investigate nice moduli compactifications that occur from other rescalings.

2.5.3. Fixing the volume. We remove the assumptions (iv), (v), (vi) now while keep assuming (i), (ii), (iii) and analyse the corresponding Gromov-Hausdorff limits while fixing volumes in turn. Note that to fix the volume of $\{V_i\}$, say as 1, is simply resulting to the metric matrices

$$
\begin{pmatrix}
Y_i^{-1} & Y_i^{-1}X_i \\
X_iY_i^{-1} & X_iY_i^{-1}X_i + Y_i
\end{pmatrix}
$$

of $\mathbb{R}^{2g}/\mathbb{Z}^{2g}$ without any normalization factor. Let $B$ be $\lim_{i \to \infty} B_i$ and let $X$ be $\lim_{i \to \infty} X_i$. We extract the $(r \times r)$ upper left part $X'$ of $X$ and $Y'$ of $Y$ as

$$
X' := \begin{pmatrix}
x_{1,1} & x_{1,2} & x_{1,3} & \cdots & x_{1,r} \\
x_{2,1} & x_{2,2} & x_{2,3} & \cdots & x_{2,r} \\
\vdots & \vdots & \vdots & \cdots & \vdots \\
x_{r,1} & x_{r,2} & x_{r,3} & \cdots & x_{r,r}
\end{pmatrix},
$$

$$
Y' := \begin{pmatrix}
y_{1,1} & y_{1,2} & y_{1,3} & \cdots & y_{1,r} \\
y_{2,1} & y_{2,2} & y_{2,3} & \cdots & y_{2,r} \\
\vdots & \vdots & \vdots & \cdots & \vdots \\
y_{r,1} & y_{r,2} & y_{r,3} & \cdots & y_{r,r}
\end{pmatrix},
$$

and denote the $(r \times r)$ upper left part $B'$ of $B$ as

$$
B' := \begin{pmatrix}
1 & b_{1,2} & b_{1,3} & \cdots & b_{1,r} \\
1 & b_{2,3} & \cdots & b_{2,r} \\
0 & \ddots & \vdots \\
0 & \ddots & 1
\end{pmatrix}.
$$

Then our metric matrices converge to the following except for lower right i.e., $(\ast)$-part of $(g - r) \times (g - r)$. 
\[
\begin{pmatrix}
F & 0 & \cdots & 0 & G & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
^tG & 0 & \cdots & 0 & H & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & * & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & \cdots & * 
\end{pmatrix}.
\]

Here, the submatrices \(F, G, H\) are those defined by \(X, X', Y, Y'\) as
- \((Y')^{-1} = F\),
- \((Y')^{-1}X = G\) and
- \(X'(Y')^{-1}X' + Y' = H\).

The corresponding \((*)\)-part of our metric matrix (5) is exactly the lower right part of \(Y_i\) which is diverging due to the divergence of \(d_{r+j}(V_i)\) \((i \to +\infty)\) for any \(j > 0\). More precisely that \((g-r) \times (g-r)\) part is positive definite with all eigenvalues strictly diverge to +\(\infty\).

The diverging part \(((g+r+j)-\text{th columns for } 1 \leq j \leq (g-r))\) yields \(\mathbb{R}^{g-r}\) and the rest of part converges to the 2\(r\)-dimension real torus with the metric matrix as (6) below.

**Proposition 2.13.** In the above setting, the pointed Gromov-Hausdorff limit of our \(V_i\) with fixed volume 1 is isometric to

\[\left(\mathbb{R}^{2r}/\mathbb{Z}^{2r}\right) \times \mathbb{R}^{g-r}\]

where the corresponding metric matrix of the first factor is

\[
\begin{pmatrix}
F & G \\
^tG & H 
\end{pmatrix}.
\]

The proof follows straightforward from the discussion before the statement. Note that the metric matrix above (6) corresponds exactly to the limit of \([V_i]_{i=1,2,\ldots} \in \mathbb{A}_g\) inside the Satake-Baily-Borel compactification (cf., e.g., [Chai 4.4]). In conclusion, we have proved that

**Corollary 2.14.** The Satake-Baily-Borel compactification \(\tilde{\mathbb{A}}_g^{BB}\) parametrizes the set of pointed Gromov-Hausdorff limits of \(g\)-dimensional principally polarized abelian varieties with fixed volumes.

This means that the Satake-Baily-Borel compactification [Sat] can be differential geometrically naturally reconstructed, i.e., in the spirit of Gromov-Hausdorff. In our sequel with Y.Oshima [OO], we further identify \(\tilde{\mathbb{A}}_g^T\) with another Satake’s compactification.
3. Along holomorphic disks

In this section, we study the Gromov-Hausdorff limit along an arbitrarily taken meromorphic family which means (in this §3 of our paper) a flat projective family $\pi^*: (X^*, L^*) \to \Delta^*$ where $\Delta^* := \{ t \in \mathbb{C} \mid 0 < |t| < 1 \} \subset \Delta := \{ t \in \mathbb{C} \mid |t| < 1 \}$ which extends to some projective flat polarized family over whole $\Delta$. More precisely, fixing such $\pi^*$, we take a sequence of points $t(i)$ for $i = 1, 2, \cdots$ in $\Delta^*$ converging to the point $0 \in \Delta$ and consider the Gromov-Hausdorff limit of corresponding metric spaces $X_{t(i)}$ for $i = 1, 2, \cdots$. Of course, it could a priori depends on the sequence $t(i)$ we take, but as a result of the following analysis, it turned out to be not! Note that in [Od4] and our §2, we considered all sequential Gromov-Hausdorff limits and hence our task here is to show such independence of the Gromov-Hausdorff limits along a fixed family as $\pi$ above and specify the subset consists of such limits.

3.1. Abelian varieties case. In this section, we remain on the principally polarized abelian varieties case.

**Notations 1.** This section focuses on the following situation. Take an arbitrary flat projective family of $g$-dimensional principally polarized abelian varieties over $\Delta^*$ which extends to some quasi-projective family over $\Delta$. Passing to a finite base change, we can and do assume that it admits (zero-)section, i.e., is a family as algebraic groups and furthermore that we have semi-abelian reduction over $0 \in \Delta$ by the Grothendieck semiabelian reduction.

We write $(X^*, L^*) \to \Delta^* = \Delta \setminus \{ 0 \}$ for such punctured family and the extension as $(X, L) \to \Delta$. Set the completion of the local ring of holomorphic functions at 0 as $R := \mathbb{C}[[t]]^{conv}$ (the convergent series local ring) and its fraction field $K := \mathbb{C}((t))^{mero}$ (the field of meromorphic functions germs at $t = 0 \in \mathbb{C}$).

From such germ at 0 of this polarized family, one extracts the following data (“DDample”) as known to [FC90] (which also at least partially go back to Mumford, Ueno, Nakamura, Namikawa etc). See [FC90] for the details.

(i) The Raynaud extension $1 \to T \to \tilde{X} \xrightarrow{\pi} A \to 0$ over $R$
(ii) Ample line bundle $\mathcal{M}$ on $A$ and $\tilde{L} := \pi^* \mathcal{M}$,
(iii) $X := \text{Hom}(T, \mathbb{G}_m)$, $Y := \text{Hom}(\mathbb{G}_m, T)$, the polarization morphism $\phi: Y \to X$, which is isomorphic in our case.
(iv) $Y$-action on $\tilde{X}$ as follows - there is a group homomorphism $\iota: Y \to \tilde{X}(K)$, given by $\{ b(y, \chi) \in \mathcal{O}_A \}$ via a (non-unique) isomorphism $\tilde{X} \cong \text{Spec}(\oplus \chi \mathcal{O}_A)$. 
(v) Set $B'(y, \chi) := \text{val}(b(y, \chi))$ and $B(y_1, y_2) := B'(y_1, \phi(y_2))$. $B$ is known to be a symmetric positive definite quadric form.

Then we analyze the asymptotic behaviour of the metrics along this degeneration as follows.

**Theorem 3.1.** For $(\mathcal{X}, \mathcal{L}) \to \Delta$ as above, we suppose the extra assumption that Raynaud extension is the trivial extension (it is satisfied e.g. for the maximally degeneration case). Consider any sequence $t_i (i = 1, 2, \cdots) \in \Delta^*$ converging to 0 then the fiber $\mathcal{X}_{t(i)}$ with rescaled flat Kähler metric (of diameter 1) $\frac{d_{\mathcal{KE}}(\mathcal{X}_{t(i)})}{\text{diam}(\mathcal{X}_{t(i)})}$ collapses to a $r$-dimensional real torus where $r$ is the torus rank of $\mathcal{X}_0$ with metric matrix $(cB(e_i, e_j))_{i,j}$ for a basis $\{e_i\}$, $c \in \mathbb{R}_{>0}$ ($c$ is for the rescaling to make the diameter 1).

Recall that as we explained at Notation 1 any one parameter family of principally polarized abelian varieties can be reduced to the above form simply by the taking relative Picard space and then some finite base change. Hence the above result in particular confirms a conjecture by Kontsevich-Soibelman [KS, §5.1, Conjecture 1] for the abelian varieties case, and also can be regarded as abelian varieties variant as the conjecture of Gross-Wilson’s [GW, Conjecture 6.2] or [Gross, Conjecture 5.4]. The author heard A.Todorov also had similar conjecture.

**Proof.** By the triviality of the Raynaud extension, $\mathcal{X}$ is the fiber product over $R$ of a smooth projective family of $g$-dim principally polarized abelian varieties and another (degenerating) family of principally polarized abelian varieties which has maximal degeneration at $0 \in \Delta$. Then we apply simple [Od4, Proposition 3.4] and we can easily reduce to the maximally degenerating case i.e. we can assume $\mathcal{X}_0$ is an algebraic torus, without loss of generality.

In the maximally degenerating case, if we take a uniformizer $t$ of $0 \in \Delta$ and take an isomorphism $T \cong \mathbb{G}_m^{r}$ which corresponds to a basis of $Y$, $y_1, \cdots, y_r$.

From the standard way (definition) of the set of data we obtained at Notation 1 the family $\mathcal{X}_t$ with $|t| \ll 1$ is well-known to be written as $\mathcal{X}_t = \mathbb{C}^r / M \cdot \mathbb{Z}^{2r}$ where

$$M = \begin{pmatrix} 2\pi i & \cdots & \log(p_{i,j}(t)) \\ \vdots & \ddots & \vdots \\ 2\pi i & \cdots & 2\pi i \end{pmatrix}.$$ 

where $p_{i,j}(t)$ is a symmetric matrix with coefficients in the meromorphic functions field $\mathbb{C}((t))^\text{mero} \subset \mathbb{C}((t))$. Indeed, $p_{i,j}(t) = \frac{1}{b(y_i, \phi(y_j))(t)}$. Recall
$B(y_i, y_j)$'s definition from the notation, and that it is classically known to be a positive definite matrix. Taking the branch of $\log(p_{i,j}(t))$ to make its absolute value of the imaginary part at most $2\pi$, the Siegel reduction is automatically done.

From our arguments in [Od4, proofs of 3.1, 3.3], we know that the Gromov-Hausdorff limit of above is determined by the asymptotics of “Y”-(“imaginary”) part of $\log(p_{i,j}(t))$ for $t \to 0$ i.e., the orders of $p_{i,j}$. Hence, $X_t(t \neq 0)$ converges to $\mathbb{R}^r/Z$ with metric matrix $B(-,-)$, appropriately rescaled to make the diameter 1.

□

Whether the following interesting phenomenon holds was asked, by A.Macpherson to whom we appreciate.

**Corollary 3.2** (”valuative criterion of properness”). Under the assumption of triviality of the Raynaud extension, the Gromov-Hausdorff limit of degenerating abelian varieties $X_{t(i)}$ with rescaled flat Kähler metric (of diameter 1) $\frac{d_{KE}(X_{t(i)})}{\text{diam}(X_{t(i)})}$ does not depend on the converging sequences $t(i) \to 0(i \to \infty)$. 

Actually, these 3.1 and 3.2 unconditionally holds for general degeneration of principally degeneration abelian varieties i.e., without the triviality assumption of the Raynaud extension. This will be proved as a part of joint work with Y.Oshima in a forthcoming paper [OO].

**Theorem 3.3** (with Y.Oshima [OO]). Let $(\mathcal{X}, \mathcal{L}) \to C$ be as Notation 4. (We do not assume triviality of the Raynaud extension). Consider any sequence $\{t(i)\}_{i=1,2,\ldots} \in \Delta^*$ converging to 0 then the fiber $X_{t(i)}$ with rescaled flat Kähler metric (of diameter 1) $\frac{d_{KE}(X_{t(i)})}{\text{diam}(X_{t(i)})}$ collapses to a $r$-dimensional real torus where $r$ is the torus rank of $X_0$ with metric matrix $B$ appropriately rescaled (to make the diameter 1).

In particular, the Gromov-Hausdorff limit of degenerating abelian varieties $X_{t(i)}$ with rescaled flat Kähler metric (of diameter 1) $\frac{d_{KE}(X_{t(i)})}{\text{diam}(X_{t(i)})}$ does not depend on the converging sequences $t(i) \to 0(i \to \infty)$.

3.2. **Algebraic curves case.** In this subsection, we analogously study asymptotics of the rescaled Kähler-Einstein metrics of bounded diameters along punctured meromorphic families of compact Riemann surfaces. We do not logically require here the detailed construction of $M_g$ in [Od4], which is described by the language of the Teichmüller space, its Fenchel-Nielsen coordinates and the pants decompositions. Instead,

---

7In other words, the map from $\Delta^*$ sending $t$ to the underlying metric space of $\mathcal{X}_t$ with the rescaled Kähler-Einstein metric extend to $\Delta \to \{\text{compact metric spaces}\}$ as a continuous map in the sense of Gromov-Hausdorff.
the following brief review of the statement provides enough context for our purpose here.

The original analogue of Theorem 2.3 for compact Riemann surfaces case in [Od4] was as follows.

**Theorem 3.4** ([Od4, Theorem 2.4]). Let \( \{R(i)\}_{i=1,2,\cdots} \) be an arbitrary sequence of compact Riemann surfaces of fixed genus \( g \geq 2 \). Suppose \( (R(i), \frac{d_{KE}(R(i))}{\text{diam}(R(i))}) \) \( (i = 1, 2, \cdots) \) converges in the Gromov-Hausdorff sense. Here \( d_{KE} \) denotes the Kähler-Einstein metric on each \( R(i) \) and its diameter is \( \text{diam}(R(i)) \).

Then the Gromov-Hausdorff limit is either

(i) a metrized (finite) graph of diameter 1 or
(ii) a compact Riemann surface of the same genus.

Since the Deligne-Mumford compactification \( \bar{M}_g^{DM} \) with the complex analytic topology is compact, by passing to a subsequence if necessary, we can assume that \( [R(i)]_{i=1,2,\cdots} \) converges to some \( R(\infty)^{DM} \in \bar{M}_g^{DM} \) without loss of generality. Then, the case (i) happens if and only if \( R(\infty)^{DM} \) is non-smooth stable curve and in that case, the combinatorial type of the graph is a contraction of the dual graph of the corresponding stable curve \( R(\infty)^{DM} \), i.e., the limit of \( [R(i)]_{i=1,2,\cdots} \) in the Deligne-Mumford compactification of the moduli of curves \( \bar{M}_g^{DM} \), with non-negative metrics (possibly zero) on each edges.

Conversely, any metrized dual graph of the stable curve of genus \( g \) with diameter 1 can occur as the Gromov-Hausdorff limit in case (i).

**Corollary 3.5** (cf., [Od4, §2.3, §3.2]).

\[ \bar{M}_g^T := M_g \sqcup S_g^{\text{wt}} \]

with a certain natural topology is a compactification\footnote{i.e., a compact Hausdorff topological space which contains \( M_g \) as an open dense subset.} of \( M_g \) with complex analytic topology, where \( S_g^{\text{wt}} \) denotes the moduli space of metrized finite graphs, with weights \( w(v_i) \) on each vertex \( v_i \), whose underlying topological spaces satisfy purely combinatorial condition:

\[ v_1(\Gamma) + b_1(\Gamma) + \sum_i w(v_i) = g. \]

Here, we denote the number of 1-valent vertices as \( v_1(\Gamma) \) and denote the first betti number of \( \Gamma \) as \( b_1(\Gamma) \). The above condition is nothing but the characterization of finite graphs which can appear as the dual graph of some Deligne-Mumford stable curves of genus \( g(\geq 2) \) and the weights encode the genera of the components of the normalization.
In the following arguments, we specify which metrized graphs can appear as the Gromov-Hausdorff limits along meromorphic punctured family while also proving that such limits are well-defined. We start with setting up the Kuranishi space of stable curves.

3.2.1. Semi-universal deformations. Basic deformation theory of stable curve $R$ tells us that we have a semiuniversal (un-obstructed) deformation. Its tangent space $Ext^1(\Omega^1_R, \mathcal{O}_R)$ maps surjectivly to local deformation tangent space $Def^\text{loc} \cong \mathbb{C}^m$ whose $i$-th coordinate corresponds to smoothing one of $m$ nodes $x_i \in R$. We first discuss at the semi-universal deformation level in this subsection and then apply (restrict) that to one parameter deformations later at \[3.2.2\].

We anyhow need the Wolpert’s fundamental results in \[Wol\] (cf., e.g. also \[OW\]) on asymptotics of the hyperbolic metrics of compact Riemann surfaces along an arbitrary degeneration to a stable curve $R$. His constructions of smoothing and approximation of the hyperbolic metric are explained as \[\text{Step 1, Step 2}\] below respectively. We reproduce his results for the convenience of readers and to set up the stage of our later discussions.

**Step 1** ("plumbing surfaces"). Recall that there is a semiuniversal algebraic deformation $U \rightarrow Z$ on an étale cover (variety) $Z$ of $Def(R) = Ext^1(\Omega^1_R, \mathcal{O}_R)$. We re-construct its analytic germ in a differential geometric way as follows. We first take a equi-singular deformation which is a restriction of $U \rightarrow Z$ to a closed subset $Z' \subset Z$. This can be also constructed as the product of universal deformation of each components (with nodes marked). We denote this as $\{R_s\}_{s \in Z'}$.

We take the normalizations of $R_s$s which of course form a family $R^\nu_s$ again. Then around the (section formed by) preimages of $i$-th node(s) $(1 \leq i \leq m)$ $x_i(s)$ in $R^\nu_s$ which we denote by $p_i(s)$ and $q_i(s)$, we take a (holomorphic family of) local coordinates $z_i(s), w_i(s)$ around $p_i(s)$ and $q_i(s)$ respetively so that

$$z_i(s)(p_i(s)) = 0,$$
$$w_i(s)(q_i(s)) = 0.$$

Fix a small enough positive real number $c_\ast < 1$. Then we construct a small deformation of $R_s$ as

$$R_s^{t,\ast} := \left( R_s \setminus \bigcup_i \left\{ \left| z_i(s) \right| < \frac{|t_i|}{c_\ast} \right\} \sqcup \left\{ \left| w_i(s) \right| < \frac{|t_i|}{c_\ast} \right\} \right) / \sim,$$
where for $\vec{t} = \{t_i\} \in \mathbb{C}^m$ with $|t_i| < c_*^4$ and the equivalence relation $\sim$ is defined on the disjoint union of pairs of sub-annuli

$$\bigsqcup_i \left( \left\{ \frac{|t_i|}{c_*} \leq |z_i(s)| \leq c_* \right\} \sqcup \left\{ \frac{|t_i|}{c_*} \leq |w_i(s)| \leq c_* \right\} \right)$$

as

$$z_i(s) \sim w_i(s) \iff z_i(s) w_i(s) = t_i.$$  

Clearly $R_{s, \vec{t}}$ form a holomorphic flat family of compact Riemann surfaces (equisingular along $\vec{t} = 0$). We call the image of the annuli in the plumbed Riemann surface $R_{s, \vec{t}}$ as collars following [Wol] or sometimes “neck”s in literatures. In this way, we get a family $R_{s, \vec{t}}$ where $\vec{t} = (t_1, \ldots, t_m) \in \mathbb{C}^m$ with $|t_i| \ll 1$, and hence an analytic slice transversal to the equisingular locus $Z'$ inside the semi-universal deformation space $Z$. Note that the construction does depend on the local coordinates $z_i(s), w_i(s)$.

**Step 2** (“grafting metric”). Next, we set a negative real number $a_0 < 0$ with $|a_0| \ll 1$ (depending on the construction of Step 1) fixed and a $C^\infty$ (“bump”) function $\eta: \mathbb{R} \to \mathbb{R}$ so that

$$\eta(a) = \begin{cases} 1 & \text{if } a \leq a_0 \\ \in [0, 1] & \text{if } a_0 < a < 0 \\ 0 & \text{if } a > 0. \end{cases}$$

We call the annuli $\{ \frac{|t_i|}{e^{a_0 c_*}} < |z_i(s)| < e^{a_0 c_*} \}$ the collar cores and the complement in the collars i.e., the set of pairs of annuli $B_{z_i} := \{ e^{a_0 c_*} \leq |z_i(s)| \leq c_* \}$, $B_{w_i} := \{ e^{a_0 c_*} \leq |w_i(s)| \leq c_* \}$ the collar bands. Then we set a function $\eta_{z_i}$ (resp., $\eta_{w_i}$) at an open neighborhood of $B_{z_i}$ (resp., $B_{w_i}$) as

$$\eta_{z_i} := \eta \left( \log \frac{|z_i(s)|}{c^*} \right) \quad \text{(resp., } \eta_{w_i} := \eta \left( \log \frac{|w_i(s)|}{c^*} \right).$$

Now we “glue” the complete hyperbolic metric on $R_s \setminus \{x_1, \ldots, x_m\}$ (i.e. smooth locus) which we denote as $dg_2^s$ and the local model metric (“with long neck”) $dg_{loc, t_i}^2$ around $x_i$, defined as below for $|t_i| \ll 1$. The gluing uses the above bump functions $\eta_{z_i}$ and $\eta_{w_i}$. Here the local model metric $dg_{loc, t_i}^2$ is defined as the restriction of

$$\left( \frac{\pi}{\log |t_i|} \log |z_i(s)| |dz_i(s)| \right)^2.$$

In particular, it does not essentially depend on $s$. 
Then the actual definition of the smooth (hermitian) metrics family $dg^2_{s,t}$ on $R_{s,t}$ by Wolpert, which he calls grafting, is as follows.

$$
\begin{align*}
\text{(7) } dg^2_{s,t} :=
\begin{cases}
(dg^2_s)^{1-\eta_z} \cdot (dg^2_{loc,t_i})^{\eta_z} & \text{around } B_{z_i} \\
(dg^2_{loc,t_i}) & \text{at the collar core of } x_i \\
(dg^2_s)^{1-\eta_w} \cdot (dg^2_{loc,t_i})^{\eta_w} & \text{around } B_{w_i} \\
dg^2_s & \text{otherwise}
\end{cases}
\end{align*}
$$

The above definition by patching is well-defined since at the collar core we have $\eta_z = \eta_w = 1$.

Again, we do the above procedure (Step 2) of grafting the metrics for all nodes $x_i (i=1, \ldots, m)$ simultaneously. See [Wol, §3], [OW, §2] for more details if needed. As a result of the above two steps construction, we get a smoothing family of $R_{s,t}$ over a $(t_1, \ldots, t_m)$-polydisc on $\mathbb{C}^m$ which we denote as $R_{s,t}$ and $C^\infty$–hermitian metrics family $dg^2_{s,t}$ on $R_{s,t}$.

**Step 3.** The crucial result of [Wol] we use compares the grafted metrics $dg^2_{s,t}$ with the hyperbolic metrics on $R_{s,t}$. In conclusion, he proved that they have the “same asymptotic behaviour”.

From the construction of Step 1 and 2, it is obvious that the grafted metric $dg^2_{s,t}$ is the same as local model hyperbolic metric $ds^2_{loc,t_i}$ on the collar core with respect to $x_i$ i.e.,

$$
\left\{ \left| \frac{t_i}{e^{a_0 c_s}} \right| < |z_i(s)| < e^{a_0 c_s} \right\}
$$

and coincides with the restriction of the original hyperbolic metric $ds^2_s$ on $R(s)$. On the collar bands i.e.,

$$
\{ e^{a_0 c_s} \leq |z_i(s)| \leq c_s \} \sqcup \{ e^{a_0 c_s} \leq |w_i(s)| \leq c_s \},
$$

the grafted metric is a mixture of $dg^2_s$ and $dg^2_{loc,t_i}$. The crucial result we will use is the following.

**Fact 3.6** ([Wol, Lemma3.5, §3.4, 4.2] cf., also [OW, p690]). The grafted metric $dg^2_{s,t}$ is asymptotically equivalent to hyperbolic metric $dg^2_{hyp,s,t}$ in the sense that

$$
dg_{s,t}^2 = dg_{hyp,s,t}^2 \cdot (1 + O(\sum_i (\log|t_i|)^{-2}),
$$

when $\vec{t} \to 0$.
From the above gluing construction of the metrics by Wolpert, we straightforwardly see that
\[
diam(i\text{-th collar core of } (R_{s,t_i} \cdot dq^2_{s,t_i})) = \int_{\tau_1}^{\tau_2} \pi \frac{\csc\left(\frac{\pi\log|z_i|}{\log|t_i|}\right)}{z_i} \frac{dz_i}{z_i} + O(1),
\]
for \(t_i \to 0\), where the last \(O(1)\) contribution comes from the \(arg(z(i))(\in \mathbb{R}/2\pi\mathbb{Z})\)-direction of the collars and the existence of the collarbands. By putting \(|t_i| = e^{T_i} z = e^{T_i x_i}\) with \(T_i, x_i \in \mathbb{R}\), the above can be re-expressed as
\[
\int_{T_i x_i = \log(c^*)}^{T_i x_i = \log(|t_i|)} \csc(\pi x_i) d(T_i x_i) + O(1).
\]
Then a simple calculation shows
\[
\int_{T_i x_i = \log(|t_i|)}^{T_i x_i = \log(c^*)} \frac{\pi}{T_i} \csc(\pi x_i) d(T_i x_i) = \int_{T_i x_i = \log(|t_i|)}^{T_i x_i = \log(c^*)} \csc(\pi x_i) dx_i
\]
\[
= \int_{T_i x_i = \log(|t_i|)}^{T_i x_i = \log(c^*)} \frac{1}{\sin(\pi x_i)} dx_i
\]
\[
= \int_{T_i x_i = \log(|t_i|)}^{T_i x_i = \log(c^*)} \frac{dx_i}{\pi x_i} + O(1)
\]
\[
= 2 \log(- \log(t_i)) + O(1),
\]
for \(t_i \to 0\). The above calculation is reflecting that, for fixed \(i\), the metrics \(dq^2_{loc,t_i}\) uniformly converge to \(\left(\frac{dz_i}{|z_i|\log|z_i|}\right)^2\) on any compact subsets of \(\{0 < |z(s)| < c_s\}\) when \(|t_i| \to 0\). This gives the proof of the Lemma 3.12 when combined with 3.9.

If we are allowed to use our slight extension of Morgan-Shalen-Boucksom-Jonsson type compactification for more general gluing function (Appendix A.1.3) plus the result of Abramovich-Caporaso-Payne [ACP], we can rephrase and summarize the above outcome into the following statement 3.7. We refer the readers to Appendix A.1.3 and [ACP] for the details of such preparation and we simply use it without recalling it. However, if one would not have the background and not much interested, then one could possibly skip the following as it essentially just rephrases the above discussions. For those interested, please read Appendix (especially §A.1.3) and [ACP] as a preparation. Let
me only roughly mention that the Morgan-Shalen-Boucksom-Jonsson partial compactification and our extension attach “dual intersection complex” to a given space.

**Theorem 3.7.** There is a natural homeomorphism between our compactification $\tilde{M}_g^T$ (in [Od4]) and a variant of Morgan-Shalen-Boucksom-Jonsson compactification of $M_g$ introduced at our §A.1.3. That is,

$$\tilde{M}_g^T \cong \tilde{M}_g^{hyb,(\log(-\log|\cdot|))},$$

in the notation of §A.1.3 in our appendix. More precisely, the above homeomorphism extends the identity of $M_g$ and preserves the corresponding weighted metrized graphs or compact Riemann surfaces’ isomorphism classes when we see the boundary $\partial \tilde{M}_g^{hyb,(\log(-\log|\cdot|))}$ as the moduli of such metrized graphs by [ACP].

**Proof of Theorem 3.7.** We use the above analysis of (approximation of) hyperbolic metrics based on [Wol], for the proof. From our construction of §A.1.3 the boundary of the right hand side compactification is the dual complex of the boundary of the Deligne-Mumford compactification stack $\partial \tilde{M}_g^{DM}$, i.e., the quotient of the dual intersection complex of the boundary divisors in charts, divided by the natural equivalence relation induced by the stack structure. We denote such topological space (dual complex) as $\Delta_g$. On the other hand, Abramovich-Caporaso-Paybe [ACP] identified $\Delta_g$ with $S_{wt}^g$ in our notation, preserving the real affine structures on the both sides, thus we have a canonical bijection between the above two spaces extending the identity on $M_g$. We refer to [ACP] for more details, if needed.

To show that the two topologies, the (generalized) hybrid topology ([BJ], A.1.3) and the (weighted) Gromov-Hausdorff topology (cf., [Od4] and our subsection 1.1) coincide, it is enough to see that for any given net (in the sense of Bourbaki) \( \{x_i \in (M_g \cup \Delta_g)\}_{i \in I} \) converging in both topologies converge to the same point. Since both topologies extend the usual analytic topologies on $M_g$ and also the natural euclidean topology on the boundary $\Delta_g$, we can further suppose that all $x_i$ are in $M_g$ and it converges to $y \in \Delta_g$ for the hybrid topology (resp., $z \in \Delta_g$ for the (weighted) Gromov-Hausdorff topology). What we want to show is $y = z$.

Passing to its sub-net if necessary, we can suppose that $\{x_i\}$ converges to a stable curve $R$ in the Deligne-Mumford compactification $\tilde{M}_g$ with analytic topology. Then we can lift the net, again by passing to subnet if necessary, to the level of $Def^{hyb}(R)$ (recall our notation from previous section) which we denote by $\{\tilde{x}_i\}_{i \in I}$. Then we want to
see the equivalence of convergence of $\{\tilde{x}_i\}_{i \in I}$ to some lift $\tilde{y}$ in $\Delta^{loc}(R)$ in the hybrid topology and in the (weighted) Gromov-Hausdorff topology. On the other hand, it actually straightforward follows from our proof of Proposition 3.10 since the convergence in the hybrid topology is the convergence of ratio of logarithms of the absolute values of the local coordinates corresponding to the nodes, while it is the same for the weighted Gromov-Hausdorff topology as we analyzed by using [Wol]. We complete the proof.

Remark 3.8. Theorem 3.7 somewhat refines the slogan after Abramovich-Caporaso-Payne [ACP] that “the moduli of skeleton is skeleton of the moduli”.

For the case of $A_g$, in our joint work with Y.Oshima [OO], we also proved that $\tilde{A}_g^T$ is identical to Morgan-Shalen-Boucksom-Jonsson compactification of toroidal compactifications of $A_g$ in a slightly extended sense (§A.14). By definition, it in particular proved that the dual complex (in the sense of [ACP] §6, Appendix) of toroidal compactification of $A_g$ does not depend on the choice of cone decompositions and is canonically homeomorphic to our tropical moduli space $T_g = \partial \tilde{A}_g^T$. This is the abelian variety version of the Abramovich-Caporaso-Payne theorem [ACP Theorem 1.2.1].

3.2.2. One parameter families of curves. Now we discuss one parameter family of stable curves. Let us set the notation first.

**Notations 2.** Let $\mathcal{X} \to \Delta$ be a flat projective family of curves over the disk $\Delta = \{|t| < 1\}$, whose central fiber $\mathcal{X}_0 := \tilde{R}$ is a stable curve. We denote its restriction over the punctured disk $\Delta^*$, the smooth projective family as $\mathcal{X}^* \to \Delta^*$. Suppose $\mathcal{X}_0$ has nodes $x_1, \ldots, x_m$ and around $x_i \in \mathcal{X}$, we have local equation $zw = t^{m_i}$ with $A_{m_i-1}$-singularity. The following is also a well-known fact and easy to confirm.

**Fact 3.9.** Consider the natural morphism $f: \Delta \to \text{Def}^{loc}$ associated to our family $\mathcal{X} \to \Delta$. Then $\text{ord}_i(f^*t_i) = m_i$, where $t_i$ is a coordinate corresponding to the direction of smoothing $x_i$ out.

**Proposition 3.10.** Consider any sequence $t(i)(i = 1, 2, \ldots) \in \Delta^*$ converging to 0 and suppose that the fiber $\mathcal{X}_{t(i)}$ with rescaled hyperbolic metric (of diameter 1) $d_{KE}(\mathcal{X}_{t(i)})$ collapses to a metrized finite graph (cf., [Od4 2.4]). Then the metrized graph is nothing but the dual graph of $\mathcal{X}_0$ whose edges have all the same lengths.

\textsuperscript{9}as addressed in A.Macpherson’s talk
Remark 3.11. We note that the above identification of Gromov-Hausdorff limits for meromorphic family is not sufficient for constructing the compactification itself.

Benefiting from the above fact, our proof of 3.10 is reduced to the following.

**Lemma 3.12.** The Gromov-Hausdorff limit of the rescaled grafted metrics on $X_{t(i)}$ with the diameter 1 when $t(i) \to 0$ is nothing but the dual graph of $X_0$ whose edges have all the same lengths.

**proof of Proposition 3.10.** From our discussion above, it is enough to analyze the diameter contribution of the collars in $R_{s,t}$. Recall that the grafted metric is nothing but the local model ("long neck") metric on the collar core while, on the collar bands, the grafted metric lies between the hyperbolic metric on $R$ and the local model, thus its contribution to the diameter is bounded above. Then the assertion follows straightforward from our previous discussions.

The above claim especially shows that the Gromov-Hausdorff limit one can get as above in $\partial \bar{M}_g^T$ along holomorphic one parameter deformations consist of only finite points! In particular, we also have an analogue of Corollary 3.2 for curve case as well.

**Corollary 3.13.** Once we fix the family $(X^*, \mathcal{L}^*) \to \Delta^*$, the Gromov-Hausdorff limit of $X_{t(i)}$ with rescaled hyperbolic metric (of diameter 1) $d_{KE}(X_{t(i)}) / \text{diam}(X_{t(i)})$ for converging sequences $t(i) \to 0 (i \to \infty)$ do not depend on the choice of the sequence we take and the set of such limits form only a finite subset inside $\partial \bar{M}_g^T$.

Remark 3.14. As we briefly introduced at our previous paper [Od4], L.Lang [LL] also introduced the following notion of convergence of compact Riemann surfaces to a metrized finite graph (which he calls tropical curve in the paper), which we analyze and compare with our compactification (cf., also his discussion at [LL, v2, §1.3]).

**Definition 3.15 (LL Definition 1.1).** A sequence of compact Riemann surfaces of fixed genus $g(\geq 2)$ $R_{i, i=1,2,...}$ converges in the sense of "tropical convergence" ([LL]) to a metrized finite graph $C$ if the following holds:

\[\text{(The expression is somewhat different from the original but the equivalence with it is just a matter of unwinding his definition.)}\]
$R_i$ converges to a stable curve $R_\infty$ in $\bar{M}_g^{DM}$ while shrinking a set of simple geodesics $l_a(R_i)$ and $C$ underlies a dual graph $\Gamma$ of $R_\infty$ such that

$$\text{length}(l_a(R_i)) = c_i(1 + o(1)) \frac{1}{\text{length}(l_a(\Gamma))}$$

for $i \to \infty$, for some constants $c_i$ which are independent of $i$.

Here, $l_a(\Gamma)$ means the edge of $\Gamma$ corresponding to the node comes from the shrinking of $l_a(R_i)$ for $i \to \infty$.

It directly follows from the known fact [3.6] (cf., the whole §3.2.1 of our review), that his notion of convergence [3.15] is different from our (weighted) Gromov-Hausdorff convergence and actually coincides with the Morgan-Shalen-Boucksom-Jonsson compactification in the slightly extended sense for stacks in the sense of Appendix A.1.2. We only sketch the proof as it is quite simple: the approximating grafted metric (7) has the simple closed geodesic of length proportional to $\frac{1}{|\log(t)|}$ and we can apply such approximation result [3.6] to a finite set of the semi-universal local deformations of stable curves, covering the whole (compact) boundary of the Deligne-Mumford compactification $\partial \bar{M}_g^{DM}$.

**Remark 3.16.** From the above results, we get a morphism

$$R: M_{g}^{an} \to \bar{M}_g^T,$$

where $M_{g}^{an}$ denotes the Berkovich analytification of $M_g$ over the complete discrete valuation field $\mathbb{C}((t))$, simply by considering reduction (compare with [ACP]). The map $R$ is neither continuous nor anti-continuous, in fact it is rather a combination of anti-continuous reduction map over the inner part $M_g$ and continuous tropicalization map over the boundary $\partial \bar{M}_g^T$. Again, from our results in §1 we also have a completely analogous map $A_{g}^{an} \to \bar{A}_g^T$ which is neither continuous nor anti-continuous.

### 3.3. Torelli maps

It is natural to think how or whether the classical period map

$$t^\text{alg}: M_g \to A_g$$

extends between our two compactifications $\bar{M}_g^T$ and $\bar{A}_g^T$.

Let us briefly recall the recent study of Torelli problem in tropical setting by other mathematicians before. For a unweighted (or weighted with zeroes) metrised graph $\Gamma$, the tropical Jacobian [MZ], [BMV] is
TROPICAL GEOMETRIC MODULI COMPACTIFICATION OF $A_g$

simply $H_1(\Gamma, \mathbb{R}/\mathbb{Z})$ with the following positive definite quadratic form $Q$. It is defined as

$$Q(\sum_{\text{edge}} \alpha_e \cdot e) := \sum_e \alpha_e^2 \cdot l(e)$$

for each 1-cycle $\sum_{\text{edge}} \alpha_e \cdot e$ where $l(\cdot)$ denotes the length function. Later, this turned out to be equivalent as the skeleton of (generalized) Jacobian in non-archimedean sense by [Viv, BR].

By mapping any tropical curve to its tropical Jacobian, [CaV] and [BMV] essentially established the existence of a natural map

$$t_{\text{Trop}}: (S_g^{\text{wt}} \setminus S_g^{\text{wt.tree}}) \to T_g$$

which is not only continuous but also compatible with their “stacky fan” structure [BMV] over the cones of these. Here, $S_g^{\text{wt.tree}}$ denotes the closed locus of $S_g^{\text{wt}}$ which parametrizes those which underly trees, that is disjoint from $S_g^{\text{wt.o}}$. (If $\Gamma$ is a tree, then the tropical Jacobian of [MZ], [BMV] is just a point so that we cannot rescale to make the diameter 1. ) The Torelli property i.e., the injectivity of the above does not literally hold even in $g = 2$ case as pointed out in [MZ]. Indeed, the closure of only one of the 2-cells of $S_2$ which parametrizes those without connecting edge maps onto $T_2$. Nevertheless, they proved that it is “generically one to one” [CV], [BMV].

Now, it is natural to ask the following question of Namikawa-Mumford-Alexeev type (cf., [Nam1], [Ale2]) i.e., about the extension of $t_{\text{alg}}: M_g \to A_g$ to compactifications. To be more precise, by combining the above two “period maps”

$$t_{\text{alg}}: M_g \to A_g$$

and

$$t_{\text{Trop}}: (S_g^{\text{wt}} \setminus S_g^{\text{wt.tree}}) \to T_g$$

we get a map

$$t_g(:= t_{\text{alg}} \sqcup t_{\text{Trop}}): M^{T}_g \setminus S_g^{\text{wt.tree}} \to A^{T}_g.$$ 

Now it is natural to ask the questions of continuity of the map $t_g$. It essentially asks the compactibility of Jacobians and tropical Jacobians. Although we have reviewed this theory of (tropical) Jacobians for the sake of expository completeness, we are afraid that the answer is no!

**Proposition 3.17.** The above map $t_g$ is not continuous for any $g > 1$.

*Proof.* Although we see this failure of continuity in a more systematic way later, we give explicit examples with $g = 2$ here. We use the fact that usual $\bar{t}: \bar{M}^{DM}_g \to \bar{A}^{Vor}_g$ of Mumford-Namikawa [Nam1]
§18 is isomorphism, which seems to be well-known to experts (cf., e.g., [Nam1, Example 18.14]). From more modern perspective, it can be re-explained a little more simpler as follows. First, the pairs of their degenerate abelian varieties and their theta divisors form semi-log-canonical pairs [Ale0, 3.10], [Ten]. Hence by adjunction, we conclude that such theta divisors are connected nodal curves with ample canonical classes, i.e., stable curves. (Note that, from our modular interpretation, the above isomorphism can be ascended to stacky level \( \bar{M}_g \cong \bar{A}_g^{Vor} \).)

Take a stable curve \( C_0 := C_1 \cup C_2 \) which can be described as follows. Two irreducible components are isomorphic \( C_1 \cong C_2 \) and are rational curves with one self-intersecting nodal singularity \( p_i(i = 1, 2) \) each. Furthermore, \( C_1 \) and \( C_2 \) intersect transversally at one nodal point \( q \).

Take the semi-universal deformation of \( C_0 \), which is three dimensional smooth germ with normal crossing discriminant divisors \( D = D_1 \cup D_2 \cup D' \) components of which are corresponding to local smoothing of \( p_i \) and \( q \) respectively. We take a complex analytic coordinates \((z_1, z_2, z_3)\) corresponding to the divisor \( D \), i.e. which satisfy \([z_i = 0] = D_i\) for \( i = 1, 2 \) and \([z_3 = 0] = D\). Consider analytic family of curves \( \{[C_t] = \varphi(t) \in \bar{M}_g \} \), over the unit disk \( \Delta \) described as \( z_i = t^{a_i} \) with \( a_i \in \mathbb{Z}_{>0} \).

Now, suppose that the map \( t_g \) is continuous. Then the family \( \varphi(t) \) in \( M_2 \) converging to a point in \( \partial \bar{M}_g^T \) corresponding to the graph consists of two circles joined by an edge, which looks like a handcuff. All the three edges have same lengths by Theorem 3.7 for \( t \to 0 \) (independent of a convergent sequence of \( t \) we take). We denote a point corresponding to this metric graph by \( h \). Its tropical Jacboian \( t_g(h) \) is a 2-dimensional real flat tori \( S^1(c) \times S^1(c) \) appropriately rescaled by a positive constant \( c \) with the diameter 1. (The exact value of \( c \) is \( \sqrt{2} \) after all). Here, the value inside the parathesis denote the length of the circumferences of the metrized circle \( S^1 \).

On the other hand, by Theorem 3.1 \( t_g(p_i) \) converges in the Gromov-Hausdorff sense to \( S^1(ca_1) \times S^1(ca_3) \) with appropriate positive constant \( c \) to make the diameter 1. Clearly that metric space depends on the parameters \( a_i \) which contradict to the above. For any \( g > 2 \), we can create a counterexample to the continuity of \( m_g \) as the above counterexample of \( g = 2 \) attached with \( g - 2 \) elliptic tails.

\[ \square \]

Remark 3.18. If we think of the fact that our compactification \( \bar{M}_g^{hyb} \), to be introduced at our Appendix A.1.2, coincides with [LL]'s compactification, then the existence of continuous map \( \bar{M}_g^{hyb} \to \bar{A}_g^T \) also follows from our later discussion, as a special case of A.15. For further details, we refer to Appendix A.2 but...
I hope this to also serve as an introductory motivation for the following appendix.

**Appendix A. Morgan-Shalen type compactification**

In this appendix, we discuss Morgan-Shalen compactifications \([MS]\), in particular, its variants and extensions. They exist for fairly general “spaces” which do not necessarily have good known modular interpretations. The original work \([MS]\) was later revisited by DeMarco-McMullen \([DMM]\), Kiwi \([Kiwi]\), Favre \([Fav]\) to relate to the Berkovich geometric context and then was partially extended by Boucksom-Jonsson \([BJ]\) more recently. Our intension of this appendix is to give natural further extensions of \([BJ]\) (hence \([MS]\) partially) to algebraic stacks with some mild singularities allowed, and then establish some basic properties. This appendix could be read for independent interest.

The main purpose of our extension is that (later) we use such extensions for our studies of tropical geometric compactifications (cf., e.g., Theorem 3.7 and the remark at the end of § Introduction) by comparison which will also continue in \([OO]\). In particular, our extensions provide a language to describe our tropical geometric compactifications of \(M_g, A_g\) (cf., also \([OO]\)).

Some part of this appendix requires birational geometric jargon but interested readers who were not accustomed to such language, could assume the whole space \(X\) to be an smooth orbifold and the boundary divisor \(X \setminus U = D\) to be its simple normal crossing divisor. Indeed, it is the most important special case of both our dlt stacky pairs (to be introduced) and toroidal stacks. Indeed, for our practical applications, such case will be enough.

**A.1. Slight extensions.**

**A.1.1. Brief review of \([MS], [BJ]\).** We start with briefly recalling the original constructions of Morgan-Shalen and Boucksom-Jonsson \([MS]\), \([BJ] \S 2\) in this section. In 1980s, Morgan-Shalen \([MS]\) constructed the compactifications of affine complex varieties \(U = \text{Spec}(R)\) in terms of ring theory and valuations, depending on finite generators (as \(\mathbb{C}\)-algebra) of \(R\). We refer to \([MS] \S 1.3\) for the details. They were motivated by studying the character varieties.

Recently, Boucksom-Jonsson \([BJ]\) partially extended the construction to give compactifications of smooth complex varieties \(U\) as follows. Starting from algebraic compactification \(U \subseteq X\), i.e., \(X\) is a smooth proper variety with \(D := X \setminus U\) simple normal crossing divisor, they
constructed a “hybrid compactification” of $U$ as follows. We often denote it as $\bar{U}^{hyb}$ instead of $U^{hyb}(X)$ \footnote{Boucksom-Jonsson \cite{BJ} wrote this as $X^{hyb}$ in our notation.} although it depends on $X$, in the case if $X$ is obvious from the context. Set theoretically, the compactification is simply

$$\bar{U}^{hyb} := U(C) \sqcup \Delta(D),$$

where $\Delta(D)$ denotes the so-called “dual (intersection) complex” (also called the “incidence complex”). See \cite{Kul, HPKX} for example. As in \cite{MS}, they used the logarithmic function to provide compact “hybrid” topology to the above. The topology is characterised by the following: if $x \in D$ and local coordinates $f_i(i=1, \ldots, \dim(X))$ at an open neighborhood satisfying $|f_i| < 1$ for all $i$, then a sequence $x_j(j=1, 2, \ldots)$ of $U$ converging to $x$, in turn converges in $\bar{U}^{hyb}$ to a point in $\Delta(D)$ with coordinates given by $(\ldots, \lim_{j \to \infty} \frac{\log |f_i(x_j)|}{\log \prod|f_i(x_j)|}, \ldots)$. We refer to \cite{BJ} §2 for the details.

Remark A.1. It is easy to see that the above Boucksom-Jonsson hybrid compactification \cite{BJ} for a log pair $U \subset X$ where $X$ is smooth and $X \setminus U$ a simple normal crossing divisor, satisfies the same property as our compactifications of $A_g$ and $M_g$ as proved in Corollaries 3.2 and 3.13. That is, if $(C \setminus \{p\}) \to U$ is a holomorphic morphism from a punctured disk which extends holomorphically from $C$ to $X$, the original morphism also extends to a continuous map $C \to \bar{U}^{hyb}(X)$.

A.1.2. Extending to algebraic stack. All the discussions in this subsection works over general algebraically closed field $k$. We aim at extending the story to the category of stacks, and for that, we first give a natural set of stacky definitions as a preparation. We start with some obviously natural stacky extension, which was also discussed in the literatures for other purposes (cf., e.g., \cite{Yas}, §4).

In this paper, étale chart of a Deligne-Mumford stack of finite type (over $k$) means an étale surjective morphism from a locally finite type scheme over $k$.

Definition A.2. A prime divisor of a normal separated Deligne-Mumford stack $\mathcal{X}$ of finite type over $k = \mathbb{C}$ (DM stack, for short from now on) is a reduced closed substack of $\mathcal{X}$ of pure codimension 1 which does not decompose as a union of proper closed substacks again of pure codimensions 1. Such a divisor $\mathcal{D}$ is $\mathbb{Q}$-Cartier if its pull back to any étale chart is a (algebraically) $\mathbb{Q}$-Cartier divisor. It is easy to see that this condition does not depend on the charts. A $\mathbb{Q}$-divisor on $\mathcal{X}$ is a
formal $\mathbb{Q}$-linear combination of prime divisors $D_1, \cdots, D_s$ in the form $\sum_{1 \leq i \leq s} a_i D_i$ where all $a_i \in \mathbb{Q}$.

Discussions from the next definition A.3 until A.7 or A.10 need to assume some acquaintance of the readers with the basic theory of the Minimal Model Program but for interested readers without it, one might be able to assume that being dlt is only slight extension of simple normal crossings, although very useful, invented by V. Shokurov.

**Definition A.3.** We succeed the above notation. The pair $(\mathcal{X}, \sum_{1 \leq i \leq s} a_i D_i)$ is said to be a *stacky log pair* if, for any étale chart $p: V \to \mathcal{X}$, the pair $(V, \sum_i a_i p^* D_i)$ is a log pair in the sense that $K_V + \sum_i a_i p^* D_i$ is $\mathbb{Q}$-Cartier. The above pullback $p^* D_i$ makes sense since $p$ is étale and it is straightforward to see that this condition does not depend on the presentation $p$.

We remark that by the Keel-Mori theorem [KeM], we always have a coarse algebraic space $\mathcal{X}$ and its primes divisors $D_i$ as coarse subspaces of $\mathcal{X}$. If we take an étale cover $V \to \mathcal{X}$ and suppose the natural map $V \to X$ branches at prime divisors $B_j \subset X$ with order $m_j$, we call the pair $(X, D_X := \sum_i a_i D_i + \sum_j \frac{m_j - 1}{m_j} B_j)$ the “coarse pair” of the stacky log pair $(\mathcal{X}, \sum_i a_i D_i)$. By [KoM, 5.20] for instance, this $(X, D_X)$ is also a log pair in the sense the log canonical divisor is $\mathbb{Q}$-Cartier.

**Definition A.4.** We succeed the above notation. The stacky log pair $(\mathcal{X}, \sum_{1 \leq i \leq s} a_i D_i)$ is said to be

(i) *kawamata-log-terminal* if $(X, D_X)$ is so.

(ii) *log canonical* if $(X, D_X)$ is so.

**Definition A.5.** We succeed the above notation. The stacky log pair $(\mathcal{X}, \sum_{1 \leq i \leq s} a_i D_i)$ is said to be *locally divisorially-log-terminal* or simply *dlt stacky pair* for bravity, if there is an étale chart $p: V \to \mathcal{X}$ with $(V, \sum_i a_i p^* D_i)$ dlt with $\mathbb{Q}$-Cartier $p^* D_i$s.

Here the above $\mathbb{Q}$-Cartierness again means the algebraic $\mathbb{Q}$-Cartierness on given normal variety $V$. The above notion, extending the (schematic) $\mathbb{Q}$-Cartier dlt pair, plays a central role in this appendix. Recall that an useful point of the concept of dlt comes from that all the lc centers inside the boundary of dlt pair are generically normal crossings as [Fjn, §3.9] shows (cf., also [Kol2, 4.16]). However there is a subtlety that a log pair (in the category of varieties) being dlt stacky pair is not quite the same as dlt pair nor $\mathbb{Q}$-factorial dlt pair, first as the condition is only required étale locally and second for the
Q-Cartierliness assumption of the boundaries. The coarse pair of dlt stacky pair around 0-dimensional lc center with Q-Cartier boundary components is called “qdlt” (quotient-dlt) in [dFKX]. For our purposes, dual complex of the boundary at the coarse moduli space is not enough and essentially need stack structures as the following simple example shows (cf., also [ACP, 6.1.7]).

**Example A.6.** Think of the quotient stack \([\mathbb{A}^2_{x,y}, (xy = 0)]/G\], where \(G := \mathbb{Z}/2\mathbb{Z}\) acts on the affine plane by switching the coordinates i.e., \(x \mapsto y, y \mapsto x\). Then the dual complex of the quotient is just one point while that of the stack is a segment divided by \(\mathbb{Z}/2\mathbb{Z}\). (If one would like a compact example, then replace \(\mathbb{A}^2\) simply by its projective compactification \(\mathbb{A}^2 \subset \mathbb{P}^2\).)

**Definition A.7.** We succeed the above notation. A line bundle on a DM stack \(\mathcal{X}\) is said to be nef (resp., ample) if it descends to a nef (resp., ample) Q-line bundle on \(X\).

**Definition A.8.** We succeed the above notation. The stacky log pair \((\mathcal{X}, \sum_{1 \leq i \leq s} a_i D_i)\) is said to be

(i) stacky klt model if it is stacky klt and \(K_X + \sum_{1 \leq i \leq s} a_i D_i\) is nef.

(ii) stacky lc model if it is stacky lc and \(K_X + \sum_{1 \leq i \leq s} a_i D_i\) is ample.

(iii) stacky dlt model if it is dlt and \(K_X + \sum_{1 \leq i \leq s} a_i D_i\) is nef.

Please do not confuse stacky dlt pair and stacky dlt model (only the latter, which is the special cases of the former, requires the log-minimality condition). We can define the dual complex of any stacky dlt pairs as follows.

**Definition-Proposition A.9 (Skeleta).** For an arbitrary separated stacky dlt pair \((\mathcal{X}, D_X = \sum_i D_i)\), we take an étale cover \(p: V \to \mathcal{X}\) and set \(W := V \times_{\mathcal{X}} V\) with naturally induced morphisms \(q_i(i = 1, 2): W \to V\) (so that \(\mathcal{X} = [W \rightrightarrows V]\)). Now we consider the colimit of topological spaces \(\Delta(\{(p \circ q_i)^* D_X\}) \rightrightarrows \Delta(\{p^* D_X\})\), where \(i = 1, 2\), \(\Delta(-)\) denotes the dual complex (as in [dFKX]) and the morphisms are affine linear at each simplex which extends the maps of vertices. We denote the colimit as topological space by \(\Delta(D_X)\) and call the dual (intersection) complex or the skeleton of the stacky dlt pair \((\mathcal{X}, D_X)\).

Then, \(\Delta(D_X)\) does not depend on the choice of \(p\) and hence well-defined which we call dual complex of the stacky dlt pair \((\mathcal{X}, D_X)\).

**Proof.** First we untangle the abstract definition of \(\Delta(D_X)\) as a cell complex in more concrete terms. Our topological colimit of \(\Delta(\{(p \circ \Delta \mathcal{X})\} \rightrightarrows \Delta(\{p^* D_X\})\),
the 0-skeleton $\Delta$ has an inductive “skeleton”\footnote{In the context of cell complexes, rather than that of Berkovich geometry.} structure, as being a cell complex, as follows. It is simply because both $\Delta(\{(p \circ q_i) \ast \mathcal{D}_X\})$ and $\Delta(\{p \ast \mathcal{D}_X\})$ have stratifications by the (inner parts of) $k$-skeleta and the two maps between them preserve the stratifications. Now, the 0-skeleton $\Delta(\{\mathcal{D}_X\}) \subset \Delta(\{\mathcal{D}_X\})$ is simply the colimit set (with discrete topology) of two maps $\Delta(\{(p \circ q_i) \ast \mathcal{D}_X\}) \Rightarrow \Delta(\{p \ast \mathcal{D}_X\})$. Above $\Delta(\{\ast \})$ simply denotes the sets of irreducible components (of each divisor $\ast$). We then proceed inductively as follows. Suppose we have constructed up to $(k - 1)$-skeleton part ($k \in \mathbb{Z}_{\geq 0}$) of $\Delta(\{\mathcal{D}_X\})$ which we denote as $\Delta(k - 1)(\{\mathcal{D}_X\})$. We write the set of codimension $k$ log-canonical centers of $\{p \ast \mathcal{D}_X\}$ (resp., $\{(p \circ q_i) \ast \mathcal{D}_X\}$ as $C(k)(\{p \ast \mathcal{D}_X\})$ (resp., $C(k)(\{(p \circ q_i) \ast \mathcal{D}_X\})$) and let $\tilde{\Delta}^k(\{p \ast \mathcal{D}_X\})$ (resp., $\tilde{\Delta}^k(\{(p \circ q_i) \ast \mathcal{D}_X\})$) be defined as

$$\bigcup_{S \in C(k)\{(p \ast \mathcal{D}_X)\}} k \text{ - simplex } \Delta_S$$

(resp.,

$$\bigcup_{S \in C(k)\{(p \circ q_i) \ast \mathcal{D}_X\}} k \text{ - simplex } \Delta_S$$

Then, as the next step, we glue the topological colimit of the induced diagram

$$\tilde{\Delta}^k(\{p \ast \mathcal{D}_X\}) \Rightarrow \tilde{\Delta}^k(\{(p \circ q_i) \ast \mathcal{D}_X\})$$

along the natural boundary map $\partial\tilde{\Delta}^k(\{(p \circ q_i) \ast \mathcal{D}_X\}) \to \Delta(k - 1)(\mathcal{D}_X)$. Note that the above maps are all cellular. Then we continue up to $k = \dim(\mathcal{X})$ so that the final outcome is nothing but our colimit $\Delta(\{\mathcal{D}_X\})$ using the cover $V \to \mathcal{X}$.

What we want to show is that the above $\Delta(\{\mathcal{D}_X\})$ constructed via the chart $V$ does not depend on the choice of $V$. Such independence assertion amounts to show the following: if $[W' \rightrightarrows V']$ is another presentation of $\mathcal{X}$ with an étale morphism $f : V' \to V$, then

$$\Delta(\{(p \circ r) \ast \mathcal{D}_X\}) \Rightarrow \Delta(\{(p \circ f) \ast \mathcal{D}_X\})^2 \times \Delta(\{p \ast \mathcal{D}_X\}) \times \Delta(\{q_i \ast \mathcal{D}_X\})$$

i.e., the above natural morphism is surjective, where $r : (V' \times V') \times (V \times V) \rightrightarrows V' \to V$ denotes the naturally induced morphism. Also note that since $q_1 \ast \mathcal{D}_X = q_2 \ast \mathcal{D}_X$ as $\mathcal{D}_X$ is a stacky divisor, the right

\footnote{It is not injective in general which makes an obstacle to define the dual complex of algebraic stacks at topological stack level for our general setting. See [ACP], 6.1.9, 6.1.10} for related discussions.
hand side of (8) is independent of \(i\). To prove the above required surjectivity (8) at the level of \(k\)-skleta by induction on \(k\) is fairly straightforward as follows. First, such assertion for the \(k = 0\) case is surjectivity of the natural map

\[
C^{(0)}((p \circ r)^* D_X) 
\rightarrow C^{(0)}((p \circ f)^* D_X)^2 \times_{((\Delta^{(0)}((p \circ f)^* D_X))^2} C^{(0)}((p \circ q_i)^* D_X).
\]

(9)

This holds immediately as \((q_1 \times q_2): W \to V \times V\) is étale to its image and \(f\) is also étale. Suppose that we know (8) up to \((k - 1)\)-skleta level. Then we want to show that the \(k\)-dimensional cells canonically coincides between the both hand sides of (8). This is nothing but the same claim as (9) above also holds when we replace 0 by \(k\) but, by definition, it is straightforward by the same reason that \(q_1 \times q_2\) and \(f\) are étale at open neighborhoods of generic points of codimension \(k\) log-canonical centers.

The above obviously extends the construction in schematic case (cf., e.g., [dFKX, NX] and also coincides with [ACP, §6] when overlaps. In particular, note that the above defined dual complex is not the same as the dual complex of the coarse pair. Indeed, the previous Ex. A.6 [ACP 6.1.7] provide simple counterexamples.

It is natural to expect that roughly speaking the dual complex of “minimal model” does not depend on the choice. More precisely, we conjecture the following after [dFKX] which establish its some versions for schematic case.

**Conjecture A.10** (Minimal skeleton). Once we fix a (kawamata-)log terminal DM stack \(U\), then the homeomorphic type of the dual complex of stacky dlt model \((X, \sum_{1 \leq i \leq s} D_i)\) with \(X \setminus \sum_i D_i = U\), does not actually depend on the choice of such compactifications.

**Definition-Proposition A.11** (Compactifications). We keep the notation of A.9. Then for the open substack \(U := X \setminus \text{Supp}(D_X)\) and its coarse moduli space \(U\), we can construct a Morgan-Shalen-Boucksom-Jonsson partial compactification \(U^{hyb}(X) := U \sqcup \Delta([D_X])\) with a Hausdorff topology extending the complex analytic topology of \(U(\mathbb{C})\). If \(X\) is proper and \(D_X\) is a (effective) \(\mathbb{Z}\)-divisor, then \(U^{hyb}(X)\) is also compact.

**Proof.** We simply imitate the construction of Boucksom-Jonsson [BJ] which we reviewed at §A.1.1 We write for the preimage of \(U\) to \(V\) (resp., \(W\)) as \(U_V\) (resp., \(U_W\)).

Then we first construct \(U^{hyb}_V(V)\) essentially following the method of [BJ] §2 as follows. For each point \(x \in V \setminus U_V\), consider the log-canonical center \(Z\) which includes \(x\). By the definition of stacky dlt
pair, if we replace $V$ by sufficiently small open subset $V_x$ of $V$, we can and do assume such log canonical center is the intersection of $m$ \mathbb{Q}\text{-Cartier boundary divisor} \, D_{V,i}(i = 1, \cdots, m)$ and for sufficiently divisible $l_i \gg 0$, $l_i D_{V,i}$ is Cartier so that they can be written as $f_i = 0$ by some holomorphic $f_i$ on $V_x$. We shrink $V_x$ small enough to the locus $|f_i| < 1$ if necessary. By running all such $x$ and using these $f_i$s on each $V_x$, we can construct the partial compactification of $V_x$ as $\bar{U}^{hyb}(V_x)$ completely similarly as $\mathbb{B}\mathbb{J}$, §2.2. Then, from our constructions, $\{(V_x \setminus (\cup_{x,i} D_{V,x,i}))^{hyb}(V_x)\}_x$ naturally glue together to form $U^{hyb}(V)$. In the same way, we also get $\bar{U}^{hyb}(W)$. Now, we construct $\bar{U}^{hyb}(X)$ as the (topological) colimit of the natural diagram $\bar{U}^{hyb}(X') \Rightarrow \bar{U}^{hyb}(V)$. Independence of such colimit from the cover $V$ is proved completely similarly as the above proof of A.9, thus we omit the details.

\[ \square \]

\textbf{Example A.12 (Torus embedding case description).} We give a description for toric case. What we means is the following. Starting from arbitrarily proper torus embedding $T \subset X$, we can do toric log resolution of $(X, X \setminus T)$; $f : X \to X$. Then with trivial stack structure, we can talk about the dual complex of $\hat{X} \setminus T$ and corresponding Morgan-Shalen-Boucksom-Jonsson compactification of $T(\mathbb{C})$. Then we can concretely see the resulting compactifications indeed do not depend on the (complete) fan structure. See [KKMS] for the basics of the toric (or toroidal) geometry. Let $N \cong \mathbb{Z}^n$ be a lattice and $T := T_N := N \otimes_{\mathbb{Z}} \mathbb{G}_m$ be the associated algebraic torus. We take a basis of $N$ so that we sometimes identify $N$ as $\mathbb{Z}^n$ and $T$ as $(\mathbb{G}_m)^n$. We consider the tropicalization map

\[ m : T(\mathbb{C}) \to N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}, \]

which is, via the basis of $N$, written as

\[ (z_1, \cdots, z_n) \mapsto (-\log|z_1|, -\log|z_2|, \cdots, -\log|z_n|). \]

It is easy to verify that this definition does not depend on the choice of basis of $N$. This logarithmic mapping is a key in the construction of $\mathbb{B}\mathbb{J}$.

It is natural to attach the infinite hyperplane to form the natural projective compactification $N_{\mathbb{R}} \subset \mathbb{P}_{N_{\mathbb{R}}} := \mathbb{P} = N_{\mathbb{R}} \cup ((N_{\mathbb{R}} \setminus \{0\})/\mathbb{R}^*)$. 

\[^{14}\text{It can be also seen as the moment map with respect to the } (S^1)^n(\subset T)\text{-action and the Kähler form } \prod \frac{dz_i \wedge \bar{dz}_i}{|z_i|^2}. \]
We also denote the boundary \(((N_\mathbb{R} \setminus \{0\})/\mathbb{R}^*)\) as \(\partial \mathbb{P}\). Note this compactification is different from \(N_\mathbb{R} \cong \mathbb{R}^n \subset (\mathbb{R} \cup \{+\infty\})^n\) relative to the interior.

From the compactness of the real projective space \(\mathbb{P}\), for any sequence \(x_i(i = 1, 2, \cdots) \in T\), after passing to subsequence if necessary, \(m(x_i)\) converge to some point in \(\mathbb{P}\). Then, the natural hybrid compactification for a torus embedding \(T \subset X\) with simple normal crossing toric boundary \(X \setminus T\) by [BJ] can be reconstructed as

\[ T \sqcup (((N_\mathbb{R} \setminus \{0\})/\mathbb{R}^*)). \]

The natural topology we put on the above space as an extension of the complex analytic topology on \(T\) is defined by the convergence of sequences (or nets) of points \(x_i\) of \(T\), which does not converge inside \(T\), to a point of boundary \(\partial \mathbb{P}\) as the convergence of the sequence \(m(x_i)\).

It is easy to see that this is equivalent to the definition of [BJ] (cf., also [A.1.1]), thus independent of the choice of above toric compactification.

Here is the local description. For an affine toric variety \((T \subset) X = U_\sigma = \text{Spec}(\mathcal{S}_\sigma)\), we simultaneously imitate and extend the above. Each \(m \in \mathcal{S}_\sigma\) corresponds to a regular function \(e(m)\) on \(U_\sigma\) of monomial type and we take a finite subset \(S \subset \mathcal{S}_\sigma\) which generates the function ring \(\mathcal{S}_\sigma\). Then we define the moment map

\[ m_S: X \to \mathbb{R}^S \]

as \(x \mapsto (-\log |e(m)(x)|)_{m \in S}\). It is standard to see that this is nothing but the combination of surjection \(X \to X/CT\), where \(CT := N \otimes U(1)\) is the natural compact form of \(T\), followed by the well-known topological embedding of \(X/CT\) into a manifold with corners (cf., [Oda]).

Then finally we consider

\[ \partial^\sigma X := \{ \lim_i m_S(x_i) \in ((N_\mathbb{R} \setminus \{0\})/\mathbb{R}^*) \mid x_i(i = 1, 2, \cdots) \text{ converges in } U_\sigma \}, \]

and set \(\bar{X}^\sigma := X \sqcup \partial^\sigma X\) with natural topology defined by the above convergence. It follows straightforward from the construction that \(X\) is open dense inside \(\bar{X}^\sigma\). It is also easy to see that the partial compactification \(X \subset \bar{X}^\sigma\) does not depend on the choice of \(S\), and furthermore that as far as \(S\) with the origin spans the maximal \((n-)\) dimensional space, the above construction works and gives the same outcome \(\bar{X}^\sigma\). Our construction of this \(\bar{X}^\sigma\) is a special case of [MS, §I.3].

**Example A.13 (For toroidal stacks [ACP]).** Here we follow [ACP] and [Thu]. Although not all toroidal DM stack (cf., [ACP 6.1.1]) form dlt
TROPICAL GEOMETRIC MODULI COMPACTIFICATION OF $A_g$

stacky pairs as not all toric singularities are dlt, [Thu] [ACP] nevertheless gives a natural partial generalization of the dual complex construction to toroidal embedding stack $U \subset X$ to form $\Delta(X \setminus U)$ (especially [ACP, §6]) extending the skeleton of Thuillier [Thu]. Note that this construction [Thu, ACP] essentially uses the natural log structure and indeed it is further extended to general fine saturated log schemes by [Uli]. Let us briefly review the construction and give a corresponding Morgan-Shalen type compactification from a perspective of the previous discussion \[A.12\]. The resulting compactification will be denoted as $\bar{U}_{hyb} := U \sqcup (\Delta(X \setminus U))$, where $U$ is the coarse moduli space of $U$.

Let us take an étale cover $p: V \to X$, denote the preimage of $U$ as $U_V$ and set $W := V \times_X V$ as before. Then we put $D_V := V \setminus U_V$. For each point $x \in D_V$, we can take an euclidean open neighborhood of $x \in U_V$ so that all the boundary components intersect the open subset $U_{V,x}$ and denote the isomorphism as $i_x: U_{V,x} \cong V_x$. Denote the corresponding cone of $V_x$ by $\sigma_x$ and set the dual

$$S_{\sigma_x} := \{ m \in M := Hom_{\mathbb{Z}}(N, \mathbb{Z}) \mid (m, n) \geq 0 \text{ for all } n \in \sigma_x \}$$

so that $V_x = \text{Spec}(\mathbb{C}[S_{\sigma_x}])$. Each $m \in M$ corresponds to a function $e(m)$ on $X$, so $i_x^*(e(m))$ on $U_{V,x}$. Take a finite generating system $\{m_i\}_{i \in S}$ of the semigroup $S_{\sigma_x}$ and exploits the partial compactification in the previous section i.e. we consider $\partial^{V_x} X_x$ and correspondingly we take partial compactification of $V_x$ which we denote by $\bar{V}^{(x)}$. It is straightforward to see that $\bar{V}^{(x)}$ glues together to form a partial compactification $\bar{V}$. Indeed, if we take another isomorphism $i'_p: U_x \cong V_x \subset X_x$, by shrinking $U_x$ if necessary, $\left( \frac{(i'_p)^*(e(m))}{i_x^*(e(m))} \right)$ are non-vanishing well-defined function on $U_x$ for any $m \in M$ (we can check this by restricting to finite generators). Similarly we can do the same construction to form a partial compactification $\bar{W}$ of $W$. As in the previous \[A.11\] we define the desired generalized hybrid compactification of $U$ as the colimit of $\bar{W} \rightrightarrows \bar{U}$. Independence of the construction from $V$ is proved completely similarly as \[A.11\] so we avoid to repeat the details of its proof.

**Remark A.14 ([OO]).** For toroidal compactifications of locally Hermitian symmetric space [AMRT], we can also naturally assign hybrid compactification as either special case of Definition-Proposition \[A.11\] (when it is smooth stack with normal crossing boundary) or Example \[A.13\] in general. It is straightforward from the constructions that it does not depend on the admissible cone decompositions. Then, in a forthcoming paper [OO], we showed that first it does not depend on
the admissible cone decompositions and such compactification for $A_g$ case coincides with our $\bar{A}_g^T$.

A.1.3. About “gluing function”. This subsection means to be a simple remark that the logarithmic function used in [BJ] for the hybrid compactification can be replaced by more general diverging function $f$, which we would call glueing function. Here is the condition for such functions to be used:

$$f: D^*(\epsilon) \to \mathbb{R}_{>0}$$ is a continuous function from

$$D^*(\epsilon) := \{z \in \mathbb{C} \mid 0 < |z| < \epsilon\}$$ for $0 < \epsilon \ll 1$

such that

(i) $f(z) \to +\infty$ when $z \to 0$,
(ii) for any $c \in \mathbb{C}^*$ $f(cz) - f(z) = O(1)$ when $z \to 0$.

The above condition morally tells that the function grows not (asymptotically) faster than the logarithmic function. Indeed, it is straightforward to see that all our constructions of Morgan-Shalen-Boucksom-Jonsson partial compactifications only use the above properties. Thus, our generalized hybrid compactification is similarly defined as

$$\bar{U}_{hyb}^{(f)} = \bar{U}_{hyb}^{(\log|z|)} := U(\mathbb{C}) \sqcup \Delta(D),$$

i.e., set-theorially same as Boucksom-Jonsson hybrid space, with modified hybrid topology depending on $f$, but defined just by imitating [BJ] which was the case $f(z) = -\log|z|$. Hence, $\bar{U}_{hyb}^{(\log|z|)} = \bar{U}_{hyb}^{(f)}$ and so we tend to omit the subscript $(f)$ when $f$ is the usual logarithmic function as above. Note that in our proof of Proposition A.11, the logarithmic function can be replaced by any function satisfying above so that we can define $(U \subset)\bar{U}_{hyb}^{(f)}$ for stacky dlt pair $(X, \sum_i D_i)$.

A.2. Functoriality of MSBJ construction.

**Theorem A.15** (Functoriality). The skeleta and the Morgan-Shalen-Boucksom-Jonsson partial compactifications (MS) [BJ] A.9 A.11 A.13 A.1.3 are both functorial in the following sense.

Suppose $(X, D_X)$ and $(Y, D_Y)$ are dlt stacky pairs (resp., $(X \setminus D_X) \subset X$, $(Y \setminus D_Y) \subset Y$ are toroidal DM stacks). If $f: X \to Y$ is a representable morphism (resp., toroidal morphism) such that $f^*D_Y = D_X$. Denote the coarse moduli of $(X \setminus D_X)$ (resp., $(Y \setminus D_Y)$) by $U_X$ (resp., $U_Y$). Then the induced map $U_X \to U_Y$ with complex analytic topologies continuously extends in a unique way to $\bar{U}_{hyb}^{(f)}_X \to \bar{U}_{hyb}^{(f)}_Y$. Moreover, if $(X, D_X)$ and $(Y, D_Y)$ are both dlt and toroidal, with $f$ toroidal morphism, then the two constructions coincide. For this theorem A.15, the glueing function needs to be the usual logarithmic function.
Proof. First, we see that dual complexes are functorial. Passing to an étale cover, we can and do assume $X$ and $Y$ are varieties. (The necessary arguments for such reduction is again in the same way as our § 4.6, so we omit the details.) We create a natural map at the $k$-skeleta level of $\Delta(D_X)$ and $\Delta(D_Y)$ on induction on $k$. The $k$-simplices forming $\Delta(D_X)$ corresponds to codimension $k$ lc centers of $(X, D_X)$. If we take a general point $x$ of an arbitrary lc center $Z$ and take a sufficiently small euclidean open neighborhood $O_x$ of $x$, $Z \subset O_x$ can be written as $(z_1 = \cdots = z_k = 0)$ where $z_i(i = 1, \cdots, \dim(X))$ are local holomorphic functions of $O_x$. If we suppose that the lc center of $(Y, D_Y)$ is $l$-dimensional $Z'$, we can take local holomorphic functions around $f(x)$ as $w_1, \cdots, w_{\dim(Y)}$ with $(\prod_{1 \leq i \leq l} w_i = 0) \cap U_Y = Z'$. From the assumptions, we can write $f$ as $w_i = \prod_{1 \leq j \leq k} z_j^{m_{i,j}} g_{i,j}(\vec{z})$ for all $i = 1, \cdots, l$ with some invertible functions $g_{i,j}$. (It morally says that $f$ is not so far from monomial maps.) Furthermore, as $f^*D_Y = D_X$, $\sum_i m_{i,j} > 0$ for any $j$ and $\sum_j m_{i,j} > 0$ for any $i$, hence $k \geq l$ in particular. The matrix $(m_{i,j})_{i,j}$ induces a morphism from the $k$-simplex corresponding to $Z$ to a $l$-simplex corresponding to $Z'$. It naturally glues to form a continuous map $\Delta(D_X) \to \Delta(D_Y)$, which does not depend on the choice of $(U_X, z_j)$ and $w_i$s. From the above construction, it is also obvious that the map $U_X \sqcup \Delta(D_X) \to U_Y \sqcup \Delta(D_Y)$, which is simply obtained as a disjoint union of the two maps, is continuous. □

By applying the above Theorem A.15 to the extended Torelli maps $\mathcal{M}_g \to \mathcal{A}_g$ (Nam2, Ale2 etc) we will get more counterexamples systematically to the continuity of “glued” Torelli map $t_g$ we discussed around Theorem 3.17.

References

[ACP] D. Abramovich, L. Caporaso, S. Payne, The tropicalization of the moduli space of curves, Annales scientifiques de l’ENS (2015).

[Ale0] V. Alexeev, Log canonical singularities and complete moduli of stable pairs, arXiv:9608013

[Ale1] V. Alexeev, Complete moduli in the presence of semiabelian group action, Ann. of Math. pp.611-708, (2002)

[Ale2] V. Alexeev, Compactified Jacobians and Torelli map, Publ. R. I. M. S., Kyoto Univ. vol. 40, pp. 1241-1265 (2004)

[AN] V. Alexeev, I. Nakamura, On Mumford’s construction of degenerating abelian varieties, Tohoku Math. J. vol. 51, pp.399–420 (1999).

[AMRT] A. Ash, D. Mumford, M. Rapoport, T-s. Tai, Smooth compactifications of locally symmetric varieties, Cambridge Mathematical Library 2nd edition (2010).

[BCS] L. Borisov, L. Chen, G. Smith, The orbifold Chow ring of toric Deligne-Mumford stacks, J. A. M. S. (2005).
[BBI] D. Burago, Y. Burago, S. Ivanov, A course in metric geometry, Graduate
Studies in Mathematics, AMS, Volume: 33; (2001).
[Berk1] V. Berkovich, Spectral theory and analytic geometry over non-archimedean
fields, Mathematical surveys and monographs, no. 33, A. M. S. (1990).
[Berk2] V. Berkovich, Smooth p-adic analytic spaces are locally contractible, Invent. Math. vol. 137, no. 1, pp. 1-84 (1999).
[Berk09] V. Berkovich, A non-Archimedean interpretation of the weight zero sub-
nspaces of limit mixed Hodge structures. Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. I, 49-67, Progr. Math., 269, Birkhäuser
Boston, Inc., Boston, MA, (2009).
[Bers] L. Bers, An inequality for Riemann surfaces, Differential geometry and com-
plex analysis, Springer, Berlin, pp. 87-93 (1985).
[Bor] A. Borel, Stable real cohomology of arithmetic groups, Ann. Sci. de L’É. N.
S., vol. 4 pp. 235-272 (1974).
[BJ] S. Bouckson, M. Jonsson, Tropical and non-Archimedean limits of degener-
ating families of volume forms, to appear in Jour. de l’Ecole polytechnique
[BMV] S. Brannetti, M. Melo, F. Viviani, On the tropical Torelli map, Adv. in Math. vol. 226 pp. 2546-2586 (2011).
[BPR] M. Baker, S. Payne, and J. Rabinoff, Nonarchimedean geometry, tropical-
ization, and metrics on curves, arXiv:1104.0320.
[BR] M. Baker, J. Rabinoff, The skeleton of the Jacobian, the Jacobian of the
skeloton, and lifting meromorphic functions from tropical to algebraic
curves,
[BMV] S.Branetti, M. Melo, F. Viviani, On the tropical Torelli map, Adv. in Math. (2011)
[Cap] L. Caporaso, Algebraic and tropical curves: comparing their moduli spaces,
arXiv:1101.4821 (2011).
[CaV] L. Caporaso, F. Viviani, Torelli theorem for graphs and tropical curves Duke
Math. J. Volume 153, Number 1 (2010), 129-171.
[Chai] C.-L. Chai, Siegel Moduli schemes and their compactification over C, Chapter IX of “Arithmetic geometry” edited by G. Cornell, Joseph H. Silverman, Springer-Verlag (1986).
[Char] R. Charney, A generalization of a theorem of Vogtmann, J. Pure Appl. Alg.
vol. 44 pp. 107-125 (1987).
[CV] M.Culler, K.Vogtmann, Moduli of graphs and automorphisms of free groups,
Invent. Math. vol. 84 no.1, pp. 91–119. (1986)
[dFKX] T. de Fernex, J. Kollár, C. Xu, The dual complex of singularities, Adv.
Stud. Pure Math., Professor Kawamata’s 60th birthday volume.
[DM] P. Deligne, D. Mumford, The irreducibility of moduli of curves, Publ. I. H.
E. S. (1969).
[DMM] L.G.DeMarco, C.T.McMullen, Trees and the dynamics of polynomials, Ann. Sci. Éc. Norm. Supér. (2008).
[DS] S. Donaldson, S. Sun, Gromov-Hausdorff limits of Kahler manifolds and
algebraic geometry, arXiv:1206.2609 (2012).
[EVHS] P. Elbaz-Vincent, G. Herbert, C. Soulé, Quelques calculs de la cohomologie
de $GL_N(Z)$ et de la $K$-théorie de $Z$, C. R. Math. Acad. Sci. Paris vol. 335,
o. 4, pp.321-324 (2002).
[FMN] B. Fantechi, E. Mann, F. Nironi Smooth toric DM stacks, Crelle J (2009).
TROPICAL GEOMETRIC MODULI COMPACTIFICATION OF $A_g$

[FC90] G. Faltings, C-L. Chai, Degeneration of abelian varieties, Springer-Verlag (1990).

[Fav] C. Favre, unpublished notes, dated December, 2012.

Cf., also his recent arXiv:1611.08490.

[Fjn] O. Fujino, What is log terminal? a chapter in the book “Flips for 3-folds and 4-folds” Oxford university press (2007).

[Grom] M. Gromov, Structures métriques pour les variétés riemanniennes, Textes Mathématiques, Paris, no. 1, pp. 1-120 (1981).

[Gross] M. Gross, Mirror Symmetry and the Strominger-Yau-Zaslow conjecture, arXiv:1212.4220 (2012).

[GS] M. Gross, B. Siebert, Theta functions and Mirror symmetry, JDG conference proceeding, arXiv:1204.1991.

[GW] M. Gross, P. M. H. Wilson, Large complex structure limits of K3 surfaces, J. Differential Geom. vol. 55, No. 3, pp. 475–546 (2000).

[GTZ] M. Gross, V. Tosatti, Y. Zhang, Gromov-Hausdorff collapsing of Calabi-Yau manifolds, Comm. Anal. Geom. 24 (2016).

[IT] Y. Imayoshi, M. Taniguchi, An introduction to Teichmüller spaces, Springer-Verlag (1992)

[Iwa] I. Iwanari, The category of toric stacks, Compositio Math. (2009).

[JJ] L. Ji, J. Jost, Universal moduli spaces of Riemann surfaces, arXiv:1611.08732.

[Ke] L. Keen, Collars on Riemann surfaces, Discontinuous Groups and Riemann Surfaces, Princeton University Press, pp. 263-268 (1974).

[Kiwi] J. Kiwi, Puiseux series polynomial dynamics and iteration of complex cubic polynomials, Ann. Inst. Fourier (2006).

[KeM] S. Keel, S. Mori, Quotients by groupoids, Ann. of Math. (1997).

[KoM] J. Kollár, S. Mori, Birational Geometry of Algebraic Varieties, Cambridge Tracts of Mathematics, Cambridge University Press (1998).

[Kol1] J. Kollár, Moduli of varieties of general type, arXiv:1008.0621 (2010).

[Kol2] J. Kollár, Singularities of the minimal model program, Cambridge Tracts in Mathematics (2013).

[KKMS] G. Kempf, F. Knudsen, D. Mumford, B. Saint-Donat, Toroidal embeddings. I, Lecture Notes in Mathematics, Vol. 339. Springer-Verlag (1973).

[KS] M. Kontsevich, Y. Soibelman, Affine structures and non-archimedian geometry, The Unity of Mathematics Progress in Mathematics vol. 244, pp 321-385 (2006).

[KSu] M. Kotani, T. Sunada, Jacobian tori associated with a finite graph and its abelian covering graphs.

[Kul] V. Kulikov, Degenerations of K3 surfaces and Enriques surfaces, Izv. Akad. Nauk S.S.S.R Ser. Mat. (1977).

[LL] L. Lang, Harmonic tropical curves, arXiv:1501.07121v2.

[MS] J. Morgan, P.B. Shalen, Valuations, trees and degeneration of hyperbolic structures, Ann. Math., 122 (1985)

[MV] B. Mirzaii, W. Van der Kallen, Homology stability for symplectic groups, arXiv:0110163 (2001).

[MZ] G. Mikhalkin, I. Zharkov, Tropical curves, their Jacobians and Theta functions, Contemporary Mathematics vol. 465, Proceedings of the Interna-
tional Conference on Curves and Abelian Varieties in honor of Roy Smith’s 65th birthday, pp. 203-231 (2007).
[MS84] J. Morgan, P. B. Shalen, Valuations, trees, and degenerations of hyperbolic structures, Ann. of Math. (1984).
[Nak1] I. Nakamura, Stability of degenerate abelian varieties, Invent. Math. vol. 136, pp.659–715 (1999).
[Nak2] I. Nakamura, Another canonical compactification of the moduli space of abelian varieties, Algebraic and arithmetic structures of moduli spaces (Sapporo, 2007), Adv. Studies Pure Math. vol. 58, pp.69-135 (2010).
[Nam1] Y. Namikawa, A new compactification of the Siegel space and degenerations of abelian varieties, I, II, Math. Ann. vol. 221, pp. 97-141, pp. 201-241 (1976).
[Nam2] Y. Namikawa, Toroidal Compactification of Siegel Spaces, Lecture Notes in Mathematics, vol. 812 (1980).
[NX] J. Nicaise, C. Xu, The essential skeleton of a degeneration of algebraic varieties, Am. J. Math. (2016).
[OW] K. Obitsu, S. Wolpert, Grafting hyperbolic metrics and Eisenstein series, Math. Annalen, 341 (2008), 685-706
[Oda] T. Oda, Convex bodies and algebraic geometry, Ergeb. der Math. und ihrer Grenz. Springer (1988).
[Od1] Y. Odaka, A generalization of Ross-Thomas slope theory, Osaka J. Math. (2013)
[Od2] Y. Odaka, The Calabi conjecture and K-stability, I. M. R. N. vol. 2012, No. 10, pp. 2272-2288 (2012).
[Od3] Y. Odaka, On the moduli of Kähler-Einstein Fano manifolds, Proceeding of Kinosaki algebraic geometry symposium 2013. (arXiv:1211.4833 v4)
[OSS] Y. Odaka, C. Spotti, S. Sun, Compact moduli of Del Pezzo surfaces and Kähler-Einstein metrics. J. Diff. Geom. (2016). arXiv:1210.0858
[Od4] Y. Odaka, Tropical Geometric Compactification of Moduli, I - $M_g$ case -, arXiv:1406.7772v2. (meaning to upload simultaneously)
[OO] Y. Odaka, Y. Oshima, in preparation.
[Sat] I. Satake, On the compactification of the Siegel space, J. Indian Math. Soc., pp. 259-281 (1956).
[SYZ] A. Strominger, S. T. Yau, E. Zaslow, Mirror symmetry is T-duality, Nuclear Physics B vol. 479 pp.243-259 (1996).
[Ten] J. Tenini, On the singularities of degenerate abelian varieties, arXiv:1401.0516 (2014).
[Thu] A. Thuillier, Géométrie toroïdale et géométrie analytique non archimédienne. Application au type d’homotopie de certains schémas formels, Manuscripta Math(2007).
[Tev] J. Tevelev, Compactifications of subvarieties of tori, Amer. J. of Math. vol. 129, pp. 1087-1104 (2007).
[Tyo] I. Tyomkin, Tropical geometry and correspondence theorems via toric stacks, Math. Ann. 353 (2012), no. 3, 945-995.
[Uli] M. Ulirsch, Functorial tropicalization of logarithmic schemes: The case of constant coefficients, arXiv:1310.6269 (2013).
[Vak] R. Vakil, Murphy’s law in algebraic geometry: Badly-behaved deformation spaces, Invent. Math. 164 (2006), no. 3, 569-590.
[Viv] F. Viviani, Tropicalizing vs Compactifying the Torelli morphism, Contemp. Math. 605 (2013), 181–210. arXiv:1204.3875

[Wol] S. Wolpert, The hyperbolic metric and the geometry of the universal curve, J. Diff. Geom. vol. 31 no. 2, pp. 417-472 (1990).

[Yas] T. Yasuda, Motivic integration over Deligne-Mumford stacks, Adv. in Math. (2006) arXiv:0312115

[Zha] Y. Zhang, Collapsing of negative Kähler-Einstein metrics, arXiv:1505.04728

Contact: yodaka@math.kyoto-u.ac.jp
Department of Mathematics, Kyoto University, Kyoto 606-8285. JAPAN