Black–Scholes Option Pricing Revisited?

Mark Mink  Frans J. de Weert

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Abstract

The hedging argument of Black and Scholes (1973) hinges on the assumption that a continuously rebalanced asset portfolio satisfies the continuous-time self-financing condition. This condition, which is a special case of the continuous-time budget equation of Merton (1971), is believed to mathematically formalize the economic concept of an asset portfolio that is rebalanced continuously without requiring an inflow or outflow of external funds. Although we sometimes find it hard to believe our results, we believe that we show with three alternative mathematical proofs that the continuous-time self-financing condition does not hold for rebalanced portfolios. In addition, we pinpoint the mistake in the derivation that Merton (1971) uses to motivate the continuous-time budget equation. Specifically, by inadvertently equating a deterministic variable to a stochastic one, Merton (1971) implicitly assumes that the portfolio rebalancing does not depend on changes in asset prices. If correct, our results invalidate the continuous-time budget equation of Merton (1971) and the hedging argument and option pricing formula of Black and Scholes (1973).

Keywords: Self-financing condition, Black–Scholes option pricing, continuous-time budget equation, binomial option pricing, probability theory.

1 Introduction

A key assumption that underlies the option pricing formula of Black and Scholes (1973) is that the continuously rebalanced asset portfolio that replicates the option satisfies the continuous-time self-financing condition. This condition is believed to mathematically
formalize the economic concept of an asset portfolio that is rebalanced continuously without requiring an inflow or outflow of external funds. The continuous-time self-financing condition has not been formally derived in continuous time, however, but is defined based on a discrete-time motivation in Merton (1971). When he derives the continuous-time budget equation of an investor, of which the continuous-time self-financing condition is a special case, Merton (1971) points out that "it is not obvious" how to derive the condition in continuous time directly. Instead, "it is necessary to examine the discrete-time formulation of the model and then to take limits carefully to obtain the continuous-time form." Merton (1973b, 1977) and subsequent studies refer back to this argument when they use the continuous-time self-financing condition, although Harrison and Pliska (1981) add that "we have no doubt that these are the right definitions, but a careful study of this issue is certainly needed." More recently, Björk (2009) emphasizes that the reasoning behind the condition "has only been of a motivating nature."

Although we sometimes find our results hard to believe, we think that our analysis mathematically proves that the continuous-time self-financing condition does not hold for rebalanced portfolios. When this condition was introduced it made techniques from continuous-time stochastic calculus available for the analysis of intertemporal consumption and portfolio optimization problems (e.g., Merton 1971, 1973a). To be able to use these techniques, however, it does not suffice mathematically to define a continuous-time self-financing condition based on a discrete-time motivation. In fact, with three alternative mathematical proofs, our analysis shows that the continuous-time self-financing condition does not hold for rebalanced portfolios. We also pinpoint the mistake in the derivation that Merton (1971) uses to motivate the continuous-time budget equation, of which the continuous-time self-financing condition is a special case. Specifically, we show that Merton (1971) inadvertently equates a deterministic variable to a stochastic one, and thereby implicitly assumes that the rebalancing does not depend on changes in asset prices. If correct, by invalidating the continuous-time self-financing condition, our analysis invalidates the continuous-time budget equation of Merton (1971) as well as the hedging argument and option pricing formula of Black and Scholes (1973).

To provide more context to the result that the option pricing formula of Black and Scholes (1973) is incorrect, we discuss the argument of Cox, Ross and Rubinstein (1979) that the continuous-time formula of Black and Scholes (1973) is equal to the limit of their discrete-time binomial option pricing formula. More specifically, we show that the binomial model of Cox, Ross and Rubinstein (1979) implicitly assumes that options and their underlying stocks are only exposed to systematic risk (i.e., to the market portfolio) and not to idiosyncratic risk. We then show that this assumption directly implies the partial differential equation of Black and Scholes and the difference equation of Cox, Ross and Rubinstein (1979), without using the (continuous-time or discrete-time) self-financing condition or a hedging argument. From a mathematical perspective, the difference equation that underlies the binomial option pricing formula therefore reflects an implicit assumption rather than a hedging argument. From an economic perspective this assumption seems difficult to rationalize, and our result that the option pricing formula of Black and Scholes is incorrect implies that it does not hold in continuous time.

If our analysis is correct, we can only speculate about the sign and magnitude of the
mispricing that results when using the formula of Black and Scholes (1973) to determine the value of an option. Moreover, there would no longer be a basis for the argument that hedging an option with the underlying stock yields a risk-free portfolio. While option traders do not use the formula of Black and Scholes (1973) directly, they typically use the underlying hedging argument. This argument contributed to the development of option markets by allowing traders to price options solely based on their view of the volatility of the underlying stock, without the need to have a view on the expected return of the stock and the expected return of the option. By disproving this hedging argument, our analysis implies that trading options cannot be reduced to trading the volatility of the underlying stock. Still, the value of an option can be expressed as its expected payoff at maturity discounted by its expected return. This expression resembles the option pricing formula of Black and Scholes (1973), but contains the percentage drift on the stock and the (time-varying) percentage drift on the option instead of the risk-free rate of return.

2 Analysis

We first summarize a standard derivation of the replicating portfolio that underlies the option pricing formula of Black and Scholes (1973). Staying close to their original notation, consider a call option with value \( w_t \) at time \( t \) that is written on a stock with value \( x_t \). As we adopt a continuous-time setting, we focus on an infinitesimal \( dt \to 0 \) throughout the analysis. The value of the stock is assumed to follow a geometric Brownian motion:

\[
\frac{dx_t}{x_t} = \mu dt + \sigma dW_t \quad \text{with} \quad x_0 > 0 \text{ given,}
\]

where \( \mu \) is the percentage drift, \( \sigma > 0 \) is the percentage standard deviation, and \( \{W_t\}_{t \geq 0} \) is a Wiener process on a filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\) satisfying the usual conditions. Black and Scholes derive a pricing formula for this option by constructing a replicating portfolio. To date, their replicating portfolio is typically replaced by the one defined by Merton (1973b, 1977), who replicates the option with a portfolio of \( w_{1,t} \) stocks with value \( x_t \) and \( \beta_t \) bonds with value \( b_t = \exp(rt)b_0 \) that earn a risk-free return \( db_t = b_t r dt \). Specifically, Merton (1973, 1977) defines the option value as:

\[
w_t = w_{1,t}x_t + \beta_t b_t,
\]

where \( w_{1,t} = \partial w_t/\partial x_t \) is the partial derivative of the value of the option with respect to the value of the stock (i.e., delta). This partial derivative changes over time as it depends on the value of the stock and on the remaining maturity of the option. Black and Scholes

\[\footnote{The option pricing formula of Black and Scholes (1973) has served as a theoretical benchmark for almost fifty years. Over time, the literature documented several empirical observations that seem at odds with this benchmark. While these findings were typically regarded as evidence against the idealized capital market conditions under which the benchmark was derived, our analysis implies that even under these idealized conditions, the formula of Black and Scholes (1973) does not provide the correct value of an option.}

\[\footnote{Bergman (1981) argues that the replicating portfolio defined by Black and Scholes contains mathematical inaccuracies which can be overcome by using the replicating portfolio of Merton (1973, 1977). MacDonald (1997) therefore argues that these inaccuracies are an issue of terminology rather than methodology.}\]
thus assume that the number of stocks changes over time as well, and refer to this trading strategy as continuous rebalancing. Applying the product rule of stochastic integration to equation (2) shows that the return on the option over an infinitesimal $dt \to 0$ equals:

$$dw_t = w_{1,t} dx_t + dw_{1,t} x_t + dw_{1,t} x_t + \beta_t db_t + d\beta_t b_t,$$

which uses the fact that $db_t$ is deterministic so that $d\beta_t db_t = O(dt^2)$ and can therefore be neglected. Merton (1973a, 1977) assumes that for $dt \to 0$ the next equality also holds:

$$dw_{1,t} x_t + dw_{1,t} x_t + d\beta_t b_t = 0,$$

which he substitutes into equation (3) to obtain the return on the option:

$$dw_t = w_{1,t} dx_t + \beta_t db_t.$$

In summary, the replicating portfolio that underlies the option pricing formula of Black and Scholes is described by equations (2) and (5), and hinges on the validity of equation (4). This equation is known as the continuous-time self-financing condition (a term coined by Harrison and Kreps, 1979). While it is trivially obvious that the condition holds for an asset portfolio that is not rebalanced (i.e., for $dw_{1,t} = d\beta_t = 0$), the condition is also believed to hold for asset portfolios that are continuously rebalanced without requiring an inflow or outflow of external funds, based on a motivation that goes back to Merton (1971). The next remark shows, however, that the analysis of Merton (1971) inadvertently equates a deterministic variable to a stochastic one, and thereby implicitly assumes that the portfolio rebalancing does not depend on changes in asset prices.

**Remark 1.** Equation (8) in Merton (1971) states that an investor with a portfolio of $\alpha_t$ stocks with value $x_t$ can consume an amount $C_t \Delta t = - (\alpha_t - \alpha_{t-\Delta t}) x_t$ over a discrete time interval $\Delta t$. Merton then "increments" this equation by $\Delta t$ to obtain $C_{t+\Delta t} \Delta t = - (\alpha_{t+\Delta t} - \alpha_t) x_{t+\Delta t}$. Using this "incremented" new stochastic variable, Merton argues that taking the limit $\Delta t \to 0$ of $C_{t+\Delta t} \Delta t$ yields $C_t dt = -d\alpha_t (x_t + dx_t)$. The implications of this argument become clear when taking the limit $\Delta t \to 0$ of the original deterministic variable $C_t \Delta t$ in (his) equation (8), which yields $C_t dt = -\lim_{\Delta t \to 0} (\alpha_t - \alpha_{t-\Delta t}) x_t$. Merton therefore implicitly argues that $\lim_{\Delta t \to 0} (\alpha_t - \alpha_{t-\Delta t}) x_t = d\alpha_t (x_t + dx_t)$, which equates a deterministic variable to a stochastic one unless $d\alpha_t$ is deterministic for all $t$ (in which case $\partial \alpha_t / \partial x_t = 0$). His analysis therefore implicitly assumes that the portfolio rebalancing is deterministic and does not depend on changes in asset prices.

The implicit assumption that the portfolio rebalancing does not depend on changes in asset prices seems to enter the analysis of Merton (1971) when he takes the limit of his discrete-time model to go to continuous time. As the remark showed, before taking this limit, Merton (1971) "increments" all time subscripts by $\Delta t$ to obtain the stochastic version of his model. In the context of the self-financing condition (where the portfolio also includes $\beta_t$ bonds and consumption is equal to zero), this procedure assumes that

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Specifically, the continuous-time self-financing condition is equal to equation (9') in Merton (1971) when consumption $C_t dt = 0$, in which case the rebalanced portfolio does not exhibit inflows or outflows of funds.
because the discrete-time self-financing condition holds in realization (i.e., ex-post) it must also hold stochastically (i.e., ex-ante). While our proofs focus on the self-financing condition in continuous time, the next remark shows that this assumption raises the question of how the investor knows which Wiener process drives the stock returns, as he can only observe the ex-post realizations of this process.

**Remark 2.** For a portfolio of $\alpha_t$ stocks and $\beta_t$ bonds, the ex-post (i.e., in realization) discrete-time self-financing condition is:

$$[\Delta \alpha_t x_t + \Delta \alpha_t \Delta x_t + \Delta \beta_t b_t + \Delta \beta_t \Delta b_t | F_{t+\Delta t}] = 0,$$

(6)

which Merton (1971) assumes to also hold stochastically (i.e., ex-ante), so that:

$$[\Delta \alpha_t x_t + \Delta \alpha_t \Delta x_t + \Delta \beta_t b_t + \Delta \beta_t \Delta b_t | F_t] = 0.$$

(7)

The assumption that equation (7) holds implies that an investor can ex-ante choose the stochastic processes $[\Delta \alpha_t | F_t]$ and $[\Delta \beta_t | F_t]$ so that they depend on the stochastic process $[\Delta x_t | F_t]$. The latter is, however, fundamentally unknown to the investor. For example, in the context of equation (1), the investor knows that the stock returns are driven by a Wiener process $\{W_t\}_{t \geq 0}$, but he does not know which one out of the infinitely many Wiener process this is, as he can only observe the ex-post realizations of this process.

When we define $dw_{1,t} = d\alpha_t$, Merton (1971) obtains the continuous-time self-financing condition in equation (4) by taking the limit of the (stochastic) discrete-time self-financing condition in equation (7). He thereafter observes in footnote 10 that "it is not obvious from the continuous version directly" that equation (4) should contain the term $dw_{1,t} dx_t$, but he argues that this term appears when one takes the limit of the discrete-time self-financing condition to obtain its continuous-time form. The next proposition shows, however, that the continuous-time self-financing condition does not hold for the replicating portfolio of Black and Scholes (i.e., it implies that $\partial w_{1,t} / \partial x_t = 0$, which contradicts the fact that the portfolio rebalancing depends on changes in asset prices).

**Proposition 1.** The continuous-time self-financing condition does not hold for the replicating portfolio of Black and Scholes.

**Proof.** Dividing the continuous-time self-financing condition in equation (4) by $b_t$ and subtracting $dw_{1,t}$ and $d\beta_t$ from both sides of the equation yields:

$$\frac{x_t - b_t}{b_t} dw_{1,t} + \frac{dw_{1,t} dx_t}{b_t} = -dw_{1,t} - d\beta_t.$$

(8)

A Taylor expansion of $dw_{1,t}$ gives:

$$dw_{1,t} = w_{11,t} dx_t + w_{12,t} dt + \frac{1}{2} w_{111,t} \sigma^2 x_t^2 dt,$$

(9)

which can be substituted into the above equation to obtain:

$$\frac{x_t - b_t}{b_t} w_{11,t} dx_t + \frac{(x_t - b_t) (w_{12,t} + \frac{1}{2} w_{111,t} \sigma^2 x_t^2)}{b_t} + w_{11,t} \sigma^2 x_t^2 dt = -dw_{1,t} - d\beta_t,$$

(10)
where we used $dw_{1,t}dx_t = w_{11,t} \sigma^2 x_t^2 dt$. Using equation (1) and defining:

$$\kappa_t = \frac{(x_t - b_t) \left( w_{12,t} + \frac{1}{2} w_{111,t} \sigma^2 x_t^2 + w_{11,t} x_t \mu \right) + w_{11,t} \sigma^2 x_t^2}{b_t},$$

$$\lambda_t = \frac{x_t - b_t}{b_t} w_{11,t} x_t \sigma,$$

allows us to write equation (10) as:

$$\kappa_t dt + \lambda_t dW_t = -dw_{1,t} - d\beta_t.$$  (13)

If we define:

$$\theta_t = -w_{1,t} - \beta_t,$$  (14)

taking the integral over time of equation (13) while conditioning on $F_{t_0}$ yields:

$$\theta_T = \theta_{t_0} + \int_{t_0}^{T} \kappa_t dt + \int_{t_0}^{T} \lambda_t dW_t,$$  (15)

where $T > t_0$ is the maturity date of the option. The option contract implies that $\theta_T = -w_{1,T} - \beta_T = 0$ (since $w_{1,T} = 0$ and $\beta_T = 0$ if the option matures out of the money, and $w_{1,T} = 1$ and $\beta_T = -1$ if the option matures in the money), so that the left hand side of equation (15) is deterministic. The right hand side of the equation must therefore be deterministic as well, which due to the Markov property of the Wiener process implies that $\lambda_t = 0$ (as we formalize in the appendix) and therefore $w_{11,t} = 0$ for all $t_0 < t < T$. However, this implication contradicts the fact that $\partial w_{1,t}/\partial x_t = w_{11,t} > 0$.

Proposition 1 shows that the continuous-time self-financing condition does not hold for the replicating portfolio of Black and Scholes. The next proposition shows, in addition, that the option pricing formula of Black and Scholes only holds if the continuous-time self-financing condition holds (see also Harrison and Kreps, 1979). Taken together, these propositions therefore imply that the option pricing formula is incorrect.

**Proposition 2.** The option pricing formula of Black and Scholes only holds if the continuous-time self-financing condition holds.

**Proof.** The option pricing formula of Black and Scholes states that the value of a call option with maturity date $T > t$ and strike price $k > 0$ is equal to:

$$w_t = N(d_{1,t}) x_t - \exp \left( -r (T-t) \right) N(d_{2,t}) k,$$  (16)

where $d_{1,t} = \frac{1}{\sigma \sqrt{T-t}} (\ln \left( \frac{x_t}{k} \right) + (r + \frac{1}{2} \sigma^2) (T-t))$ and $d_{2,t} = d_{1,t} - \sigma \sqrt{T-t}$, and where $N(\cdot)$ is the standard normal cumulative distribution function. Since $w_{1,t} = \partial w_t/\partial x_t = N(d_{1,t})$, equation (16) can also be written as:

$$w_t = w_{1,t} x_t + \beta_t b_t,$$  (17)

where $\beta_t = -N(d_{2,t})$ and $b_t = \exp \left( -r (T-t) \right) k$. Moreover, following Black and Scholes,
using Itô’s lemma implies that the option return over an infinitesimal \( dt \to 0 \) equals:

\[
dw_t = w_{1,t} dx_t + w_{2,t} dt + \frac{1}{2} w_{11,t} \sigma_t^2 x_t^2 dt,
\]

where \( w_{2,t} = \partial w_t / \partial t \) is the partial derivative of the value of the option with respect to the remaining maturity of the option (i.e., theta), and where \( w_{11,t} = \partial^2 w_t / \partial x_t^2 \) is the partial derivative of \( w_{1,t} \) with respect to the value of the stock (i.e., gamma). As is well-known, equation (16) implies that these partial derivatives are equal to

\[
w_{2,t} = -\frac{x_t N'(d_1,t)}{\sqrt{T-t}} + \beta_t r
\]

and

\[
w_{11,t} = \frac{N'(d_1,t)}{x_t \sigma_t \sqrt{T-t}},\]

where \( N'(\cdot) \) is the standard normal probability density function. Substituting these partial derivatives in equation (18) and using \( b_t r dt = db_t \) yields:

\[
dw_t = w_{1,t} dx_t + \beta_t db_t.
\]

Equations (17) and (19) are the same as equations (2) and (5), and therefore only hold if the continuous-time self-financing condition in equation (4) holds.

While Proposition 1 focused on the continuous-time self-financing condition in the context of the replicating portfolio of Black and Scholes, two alternative proofs in the appendix show that the continuous-time self-financing condition does not hold for any rebalanced portfolio (the second proof applies regardless of whether the rebalancing depends on changes in asset prices). As the discussion around Remarks 1 and 2 illustrated, Merton (1971) obtains this condition by taking the limit of the discrete-time self-financing condition, and both proofs use this limiting procedure as well. In fact, to be more precise, Merton (1971) does not derive the continuous-time self-financing condition directly, but this condition is a special case of his continuous-time budget equation. More specifically, for an investor with wealth \( w_t \) held in a portfolio of \( \alpha_t \) stocks and \( \beta_t \) bonds, Merton (1971) models the value of the portfolio as

\[
w_t = \alpha_t x_t + \beta_t b_t
\]

and models the return as

\[
dw_t = \alpha_t dx_t + \beta_t db_t.
\]

When defining \( \alpha_t = w_{1,t} \), these equations are the same as equations (2) and (5), while equation (4) describes the special case where consumption is equal to zero (see footnote 4). Our analysis therefore implies that the continuous-time budget equation of Merton (1971) is incorrect as well.

As an alternative to the option pricing formula of Black and Scholes, the value of an option can be expressed as its expected payoff at maturity discounted by its expected return. The next remark illustrates that when the value of the stock evolves according to equation (1), this expression resembles the option pricing formula of Black and Scholes, but contains the percentage drift on the stock and the (time-varying) percentage drift on the option instead of the risk-free rate of return.

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5 When reading the paper of Merton (1971) with this result in mind, we realized that his notation does not reflect his economic narrative on the sequencing between changes in asset prices and changes in portfolio weights. More specifically, Merton (1971) states that “prices are known at the beginning of the period” (italics are his) and that thereafter the new portfolio weights are chosen so that “trades are made at (known) current prices.” His notation does not reflect this sequencing, since discrete-time equation (8) in his paper gives the new portfolio weights the same time subscript as the asset prices. In hindsight, if our analysis is correct, this simplification does not seem innocuous.
Remark 3. At time $t$, the expected payoff on an option at maturity date $T$ is:

$$
E[w_T|F_t] = \int_k^\infty (x - k)dF_T(x) = \exp(\mu (T - t)) N (d^*_1,t) x_t - N (d^*_2,t) k,
$$

(20)

where $F_T(x) = \Pr(x_T \leq x | x_t)$ is the cumulative distribution function of $x_T$ conditional on $F_t$. Note that $d^*_1,t = \frac{1}{\sigma \sqrt{T-t}} (\ln \left( \frac{x_t}{k} \right) + (\mu + \frac{1}{2} \sigma^2) (T - t))$ and $d^*_2,t = d^*_1,t - \sigma \sqrt{T-t}$ differ from $d_{1,t}$ and $d_{2,t}$ in the formula of Black and Scholes unless $\mu = r$. The option value $w_t$ is equal to the expected payoff at maturity times the discount factor $\exp(-\mu w, t (T - t))$, where $\mu_{w,t}$ is the (time-varying) percentage drift of the option. If $\mu_{w,t} = \mu = r$ the resulting expression is the same as the option pricing formula of Black and Scholes.

2.1 Binomial option pricing

The previous analysis implies that the continuous-time option pricing formula of Black and Scholes is incorrect. Cox, Ross and Rubinstein (1979) argue, however, that the formula of Black and Scholes is a special limiting case of their discrete-time option pricing formula. Together, these two observations raise the question of how the mathematical commonality between both formulas can be explained, which we examine in this section. Specifically, we show that the binomial model of Cox, Ross and Rubinstein (1979) implicitly assumes that the stock and the option are only exposed to systematic risk and not to idiosyncratic risk. We then show that this assumption directly implies the partial differential equation of Black and Scholes and the difference equation of Cox, Ross and Rubinstein (1979), without using the (continuous-time or discrete-time) self-financing condition or a hedging argument.

To derive their discrete-time option pricing formula, Cox, Ross and Rubinstein (1979) assume that the stock price and the option price each follow a binomial process, and point out that “three-state or trinomial stock price movements will not lead to an option pricing formula based solely on arbitrage considerations.” An implicit restriction of the binomial process is that the asset is only exposed to systematic risk and not to idiosyncratic risk. Specifically, assuming binomial processes for the stock and the option implies that each of them is exposed to just a single risk factor. Otherwise, if the stock would be exposed to two risk factors, the number of potential outcomes for its price would at least be equal to four instead of two, and likewise for the option. Since Cox, Ross and Rubinstein (1979) assume that the expected returns on the stock and the option exceed the risk-free return, their single risk factor is priced and is therefore systematic. Propositions 3 and 4 show that this assumption directly implies the partial differential equation of Black and Scholes and the difference equation of Cox, Ross and Rubinstein (1979).

Proposition 3. The continuous-time partial differential equation of Black and Scholes is directly implied (without using the self-financing condition or a hedging argument) when assuming that the stock and the option are only exposed to systematic risk.

Proof. If a stock with value $x_t$ and an option with value $w_t$ are exposed to systematic risk
only, their excess rates of return are equal to:

\[ \frac{dx_t}{x_t} - rdt = \beta_x \left( \frac{dm_t}{m_t} - rdt \right), \]  
\[ \frac{dw_t}{w_t} - rdt = \beta_w \left( \frac{dm_t}{m_t} - rdt \right), \]  

(21)

(22)

where \( \frac{dm_t}{m_t} - rdt \) is the rate of return on the market portfolio in excess of the risk-free rate of return, and where \( \beta_x \) and \( \beta_w \) indicate the exposure of the stock and of the option to the market portfolio.\(^6\)

Defining \( \alpha_t = w_t \beta_w / x_t \beta_x \) and combining both equations yields:

\[ dw_t = w_t rdt + \alpha_t dx_t - \alpha_t x_t rdt, \]  

(23)

Furthermore, restating equation (18) for the return on the option gives:

\[ dw_t = w_{1,t} dx_t + w_{2,t} dt + \frac{1}{2} w_{11,t} \sigma^2 x^2_t dt, \]  

(24)

Subtracting equation (24) from equation (23) yields:

\[ (w_{1,t} - \alpha_t) dx_t = \left( w_t r - \alpha_t x_t r - w_{2,t} - \frac{1}{2} w_{11,t} \sigma^2 x^2_t \right) dt. \]  

(25)

Since the right-hand side of this equation is deterministic the left-hand side must be deterministic as well, which implies that:

\[ \alpha_t = w_{1,t}. \]  

(26)

Substituting this result in equation (25) and dropping \( dt \) yields:

\[ w_{2,t} = rw_{t} - rw_{1,t} x_t - \frac{1}{2} w_{11,t} \sigma^2 x^2_t, \]  

(27)

which is the partial differential equation in equation (7) of Black and Scholes. \( \square \)

**Proposition 4.** The discrete-time difference equation of Cox, Ross and Rubinstein (1979) is directly implied (without using the self-financing condition or a hedging argument) by their assumption that the stock and the option are only exposed to systematic risk.

**Proof.** Cox, Ross and Rubinstein (1979) start with a one-period binomial tree in discrete time. Using their notation, the stock price at the start of the period is \( S \) and the option price at this time is \( C \). At the end of the period, the stock price is either equal to \( uS \) or to \( dS \), where \( u > r^* > d > 0 \) and \( r^* = 1 + r \) (we added an * to distinguish their notation from our symbol \( r \)). Likewise the option price at the end of the period is either \( C^u \) or \( C^d \). As this binomial tree implies that the stock and the option are only exposed to systematic

\(^6\)In the capital asset pricing model (CAPM) these parameters are referred to as ‘beta’, and are defined as the covariance of the return on the asset with the return on the market portfolio, divided by the variance of the return on the market portfolio.
risk, their excess rates of return are equal to:

\[
(uS - S)/(S - (r^* - 1)) = \beta_S \left( (m^u - m) / m - (r^* - 1) \right),
\]

(28)

\[
(dS - S)/(S - (r^* - 1)) = \beta_S \left( (m^d - m) / m - (r^* - 1) \right),
\]

(29)

\[
(C^u - C)/C - (r^* - 1) = \beta_C \left( (m^u - m) / m - (r^* - 1) \right),
\]

(30)

\[
\left( C^d - C \right)/C - (r^* - 1) = \beta_C \left( (m^d - m) / m - (r^* - 1) \right),
\]

(31)

where \((m^u - m)/m\) is the rate of return on the market portfolio when the market goes up, \((m^d - m)/m\) is the rate of return when it goes down, and \(\beta_S\) and \(\beta_C\) indicate the exposure of the stock and of the option to the market portfolio. Combining these equations yields:

\[
C^u - Cr^* = \frac{C\beta_C}{S\beta_S} (uS - Sr^*),
\]

(32)

\[
C^d - Cr^* = \frac{C\beta_C}{S\beta_S} (dS - Sr^*).
\]

(33)

Using one of these equations to substitute for \(\beta_C/\beta_S\) in the other gives:

\[
C = \left[ \left( \frac{r^* - d}{u - d} \right) C^u + \left( \frac{u - r^*}{u - d} \right) C^d \right] / r^*,
\]

(34)

which is the difference equation in equation (2) of Cox, Ross and Rubinstein (1979).

The previous two propositions show that the partial differential equation of Black and Scholes and the difference equation of Cox, Ross and Rubinstein (1979) are directly implied when one assumes that the stock and the option are only exposed to systematic risk. While Black and Scholes do not make this assumption, Cox, Ross and Rubinstein (1979) do so implicitly by modelling the stock price and the option price as binomial processes. From a mathematical perspective, their difference equation results from this assumption rather than from a hedging argument. In continuous time, this same assumption implies the partial differential equation of Black and Scholes, which explains why both option pricing formulas are mathematically related. Hence, this relationship between both formulas does not indicate that the continuous-time hedging argument of Black and Scholes can be interpreted as the limit of a discrete-time hedging argument by Cox, Ross and Rubinstein (1979). Moreover, because Propositions 1 and 2 in the previous section show that the option pricing formula of Black and Scholes is incorrect, the assumption that the stock and the option are only exposed to systematic risk does not hold in continuous time. Hence, in continuous time, changes in the value of an option are partially idiosyncratic even if the underlying stock is only exposed to systematic risk.

3 Conclusion

If our analysis is correct, the continuous-time self-financing condition does not hold for rebalanced asset portfolios. This result implies that the continuous-time budget equation of Merton (1971) and the hedging argument and option pricing formula of Black and
Scholes (1973) are incorrect. Moreover, there would no longer be a basis for the argu-
ment that hedging an option with the underlying stock yields a risk-free portfolio. Still,
the value of an option can be expressed as its expected payoff at maturity discounted
by its expected return. This expression resembles the option pricing formula of Black
and Scholes (1973), but contains the percentage drift on the stock and the (time-varying)
percentage drift on the option instead of the risk-free rate of return.

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Appendix

Proposition A.1. If $\theta_T = \theta_0 + \int_{t_0}^{T} \kappa_t dt + \int_{t_0}^{T} \lambda_t dW_t$ and $\theta_T = 0$ then $\lambda_t = 0$ for all $t_0 < t < T$.

Proof. As $\theta_T = \theta_0 + \int_{t_0}^{T} \kappa_t dt + \int_{t_0}^{T} \lambda_t dW_t + \int_{t_1}^{T} \lambda_t dW_t$, the following equality holds:

\[
\text{Cov} \left( \int_{t_0}^{t_1} \kappa_t dt, \int_{t_0}^{t_1} \kappa_t dt + \int_{t_0}^{t_1} \lambda_t dW_t \right) = - \text{Cov} \left( \int_{t_0}^{t_1} \kappa_t dt, \theta_0 + \int_{t_1}^{T} \kappa_t dt + \int_{t_1}^{T} \lambda_t dW_t \right),
\]

where the first line uses $\theta_T = 0$, and the second line uses the facts that $\theta_0$ is deterministic and that the Markov property of the Wiener process implies $\text{Cov} \left( \int_{t_0}^{t_1} \kappa_t dt, \int_{t_1}^{T} \lambda dW_t \right) = 0$ (i.e., the Wiener process does not depend on past events). Moreover, $\theta_T = \theta_0 + \int_{t_0}^{T} \kappa_t dt + \int_{t_1}^{T} \lambda_t dW_t$ for $\theta_T = 0$ implies that $\text{Cov} \left( \int_{t_1}^{T} \kappa_t dt, \int_{t_1}^{T} \lambda_t dW_t \right) = -1$. Because this perfect correlation is not stochastic, it is equal to $-1$ irrespective of whether one conditions on $\mathcal{F}_{t_1}$ or on $\mathcal{F}_{t_0}$. Combining this perfect correlation with the above result that $\text{Cov} \left( \int_{t_0}^{t_1} \kappa_t dt, \int_{t_1}^{T} \lambda dW_t \right) = 0$ implies that $\text{Cov} \left( \int_{t_0}^{t_1} \kappa_t dt, \int_{t_1}^{T} \lambda dW_t \right) = 0$, so that equation (35) becomes:

\[
\text{Cov} \left( \int_{t_0}^{t_1} \kappa_t dt, \int_{t_0}^{t_1} \kappa_t dt + \int_{t_0}^{t_1} \lambda_t dW_t \right) = 0.
\]

Furthermore, we can also use $\text{Cov} \left( \int_{t_0}^{t_1} \kappa_t dt, \int_{t_1}^{T} \kappa_t dt \right) = 0$ to write:

\[
\text{Var} \left( \int_{t_0}^{T} \kappa_t dt \right) = \text{Var} \left( \int_{t_0}^{t_1} \kappa_t dt \right) + \text{Var} \left( \int_{t_1}^{T} \kappa_t dt \right),
\]

and since the Markov property implies that $\text{Cov} \left( \int_{t_0}^{t_1} \lambda dW_t, \int_{t_1}^{T} \lambda dW_t \right) = 0$, we also have:

\[
\text{Var} \left( \int_{t_0}^{T} \lambda_t dW_t \right) = \text{Var} \left( \int_{t_0}^{t_1} \lambda_t dW_t \right) + \text{Var} \left( \int_{t_1}^{T} \lambda_t dW_t \right).
\]

For $\theta_T = 0$, the equality $\theta_T = \theta_0 + \int_{t_1}^{T} \kappa_t dt + \int_{t_1}^{T} \lambda_t dW_t$ implies that $\text{Var} \left( \int_{t_1}^{T} \kappa_t dt \right) = \text{Var} \left( \int_{t_1}^{T} \lambda_t dW_t \right)$, and likewise the equality $\theta_T = \theta_0 + \int_{t_0}^{T} \kappa_t dt + \int_{t_0}^{T} \lambda_t dW_t$ implies that $\text{Var} \left( \int_{t_0}^{T} \kappa_t dt \right) = \text{Var} \left( \int_{t_0}^{T} \lambda_t dW_t \right)$. Combining equations (37) and (38) therefore yields:

\[
\text{Var} \left( \int_{t_0}^{t_1} \kappa_t dt \right) = \text{Var} \left( \int_{t_0}^{t_1} \lambda_t dW_t \right).
\]
Using this result the following equality holds:

\[
\text{Var} \left( \int_{t_0}^{t_1} \kappa_t \, dt + \int_{t_0}^{t_1} \lambda_t \, dW_t \right) = 2 \text{Var} \left( \int_{t_0}^{t_1} \kappa_t \, dt \right) + 2 \text{Cov} \left( \int_{t_0}^{t_1} \kappa_t \, dt, \int_{t_0}^{t_1} \lambda_t \, dW_t \right),
\]

\[
= 2 \text{Cov} \left( \int_{t_0}^{t_1} \kappa_t \, dt, \int_{t_0}^{t_1} \kappa_t \, dt + \int_{t_0}^{t_1} \lambda_t \, dW_t \right),
\]

\[
= 0,
\]

(40)

where we used equation (36) in the last step. This equality states that the integral \( \int_{t_0}^{t_1} \kappa_t \, dt + \lambda_t \, dW_t \) has zero variance and is therefore deterministic for all \( t_0 < t_1 < T \). The integrand must therefore be deterministic as well, which implies that \( \lambda_t = 0 \).

\[\square\]

**Proposition A.2.** The continuous-time self-financing condition does not hold if the portfolio rebalancing depends on changes in asset prices.

**Proof.** According to Merton (1971), the continuous-time self-financing condition can be derived by starting with the ex-post discrete-time self-financing condition, which for a portfolio of \( \alpha_t \) stocks and \( \beta_t \) bonds is equal to:

\[
[\Delta \alpha_t x_t + \Delta \alpha_t \Delta x_t + \Delta \beta_t b_t + \Delta \beta_t \Delta b_t | \mathcal{F}_T] = 0.
\]

(41)

While this condition holds ex-post when trades have taken place (i.e., conditional on \( \mathcal{F}_T \), where \( T \geq t + \Delta t \) is the maturity date of the option), Merton argues that equation (41) also holds ex-ante (i.e., conditional on \( \mathcal{F}_t \)). He then takes the limit \( \Delta t \to 0 \) to obtain:

\[
\lim_{\Delta t \to 0} [\Delta \alpha_t x_t + \Delta \alpha_t \Delta x_t + \Delta \beta_t b_t | \mathcal{F}_t] = 0,
\]

(42)

where we used the fact that \( \Delta b_t \) is deterministic because \( b_t \) is a risk-free bond, so that \( \lim_{\Delta t \to 0} [\Delta \beta_t \Delta b_t | \mathcal{F}_t] \) is of higher order than the other terms and can therefore be neglected. Likewise, taking the limit \( \Delta t \to 0 \) of equation (41) yields:

\[
\lim_{\Delta t \to 0} [\Delta \alpha_t x_t + \Delta \alpha_t \Delta x_t + \Delta \beta_t b_t + \Delta \beta_t \Delta b_t | \mathcal{F}_T] = 0,
\]

(43)

which is conditioned on \( \mathcal{F}_T \) so that all terms are deterministic (i.e., they are numbers) and not stochastic. Because of this conditioning, \( \lim_{\Delta t \to 0} [\Delta \alpha_t \Delta x_t | \mathcal{F}_T] \) and \( \lim_{\Delta t \to 0} [\Delta \beta_t \Delta b_t | \mathcal{F}_T] \) go to zero faster than \( \lim_{\Delta t \to 0} [\Delta \alpha_t x_t | \mathcal{F}_T] \) and \( \lim_{\Delta t \to 0} [\Delta \beta_t b_t | \mathcal{F}_T] \)\(^7\) Neglecting these higher order terms yields:

\[
\lim_{\Delta t \to 0} [\Delta \alpha_t x_t + \Delta \beta_t b_t | \mathcal{F}_T] = 0.
\]

(44)

Because equation (44) holds for all realized paths of \( \Delta \alpha_t \) and \( \Delta \beta_t \), it also holds stochastically (i.e., conditional on \( \mathcal{F}_t \)) so that:

\[
\lim_{\Delta t \to 0} [\Delta \alpha_t x_t + \Delta \beta_t b_t | \mathcal{F}_t] = 0,
\]

(45)

\(^7\)To avoid confusion, we point out that \( \lim_{\Delta t \to 0} [\Delta x_t | \mathcal{F}_T] \neq dx_t \) since \( \lim_{\Delta t \to 0} [\Delta x_t | \mathcal{F}_T] \) is conditioned on \( \mathcal{F}_T \) and is therefore deterministic, while \( dx_t \) as defined in equation (1) is conditioned on \( \mathcal{F}_t \) and is therefore stochastic.
which can be subtracted from equation (42) to obtain:

$$\lim_{\Delta t \to 0} [\Delta \alpha_t \Delta x_t | F_t] = 0. \tag{46}$$

Merton (1971) argues that equation (42) yields the continuous-time self-financing condition in equation (4) when defining $\alpha_t = w_{1,t}$. This argument among others implies that:

$$\lim_{\Delta t \to 0} [\Delta \alpha_t \Delta x_t | F_t] = d\alpha_t dx_t, \tag{47}$$

which requires that $\lim_{\Delta t \to 0} [\Delta x_t | F_t]$ converges to the geometric Brownian motion $dx_t$ as defined in equation (1). Moreover, using the Taylor expansion $d\alpha_t = \left( \frac{\partial \alpha_t}{\partial x_t} \right) dx_t + \left( \frac{\partial \alpha_t}{\partial t} \right) dt + 0.5 \left( \frac{\partial^2 \alpha_t}{\partial x_t^2} \right) dx_t^2$ and neglecting higher order terms gives:

$$d\alpha_t dx_t = \frac{\partial \alpha_t}{\partial x_t} \sigma^2 x_t^2 dt, \tag{48}$$

where we used the fact that $dx_t^2 = \sigma^2 x_t^2 dt > 0$ is the quadratic variation of the geometric Brownian motion. Combining equations (46) to (48) yields:

$$\frac{\partial \alpha_t}{\partial x_t} \sigma^2 x_t^2 dt = 0. \tag{49}$$

When the portfolio rebalancing depends on changes in the stock price, however, we have $\partial \alpha_t / \partial x_t \neq 0$, in which case equation (49) contradicts the fact that the geometric Brownian motion has positive quadratic variation.

**Proposition A.3.** The continuous-time self-financing condition does not hold if the portfolio is continuously rebalanced.

**Proof.** Merton (1971) argues that the continuous-time self-financing condition in equation (4) is obtained when conditioning on $F_t$ and taking the limit $\Delta t \to 0$ of his discrete-time self-financing condition:

$$\lim_{\Delta t \to 0} [\Delta \alpha_t x_t + \Delta \alpha_t \Delta x_t + \Delta \beta_t b_t + \Delta \beta_t \Delta b_t | F_t] = 0, \tag{50}$$

where $\alpha_t$ and $\beta_t$ are the numbers of stocks and bonds in the portfolio. The discrete-time self-financing condition implies that the change in the value of the portfolio at $t$ equals:

$$\Delta w_t = \alpha_t \Delta x_t + \beta_t \Delta b_t. \tag{51}$$

Likewise, since the discrete-time self-financing condition also holds ex-post, we have:

$$[\Delta \alpha_{t-\Delta t} x_{t-\Delta t} + \Delta \alpha_{t-\Delta t} \Delta x_{t-\Delta t} + \Delta \beta_{t-\Delta t} b_{t-\Delta t} + \Delta \beta_{t-\Delta t} \Delta b_{t-\Delta t} | F_t] = 0, \tag{52}$$

so that conditional on $F_t$, the change in the value of the portfolio over the period between $t - \Delta t$ and $t$ equals:

$$\Delta w_{t-\Delta t} = \alpha_{t-\Delta t} \Delta x_{t-\Delta t} + \beta_{t-\Delta t} \Delta b_{t-\Delta t}. \tag{53}$$
Furthermore, the value of the portfolio at $t + \Delta t$ equals:

$$w_{t+\Delta t} = w_t + \Delta w_t,$$

$$= w_{t-\Delta t} + \Delta w_{t-\Delta t} + \Delta w_t, \quad (54)$$

and the value at $t - \Delta t$ equals:

$$w_{t-\Delta t} = \alpha_{t-\Delta t}x_{t-\Delta t} + \beta_{t-\Delta t}b_{t-\Delta t}. \quad (55)$$

Substituting equations (51), (53) and (55) into equation (54) gives:

$$w_{t+\Delta t} = \alpha_t - \Delta_t x_t - \Delta_t + \beta_t - \Delta_t b_t - \Delta_t,$$

$$= \alpha_t - \Delta_t x_t + \Delta_t + \beta_t - \Delta_t b_t + \Delta_t + \Delta_t \alpha_t - \Delta_t \Delta x_t + \Delta_t \beta_t - \Delta_t \Delta b_t, \quad (56)$$

where we simplified the second line by using $x_{t+\Delta t} = x_{t-\Delta t} + \Delta x_{t-\Delta t} + \Delta x_t$, and likewise for $b_{t+\Delta t}$. Taking the limit $\Delta t \to 0$ of equation (56) conditional on $F_t$ gives:

$$\lim_{\Delta t \to 0} [w_{t+\Delta t}|F_t] = \lim_{\Delta t \to 0} [\alpha_{t-\Delta t} x_t + \Delta t + \beta_{t-\Delta t} b_t + \Delta t |F_t], \quad (57)$$

which uses the fact that $\lim_{\Delta t \to 0} [\Delta \alpha_t - \Delta_t x_t + \Delta \beta_t - \Delta_t b_t |F_t]$ is of higher order than the other terms and can therefore be neglected. Equation (57) implies, however, that the portfolio weights of a point in time smaller than $t$ can be used to replicate the portfolio value at a point in time larger than $t$, which contradicts the fact that the portfolio is continuously rebalanced (including at time $t$).