Accurate characterization of the stellar and orbital parameters of the exoplanetary system WASP-33 b from orbital dynamics

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1 INTRODUCTION

Steady observations of a test particle orbiting its primary over time intervals much longer than its orbital period $P_0$ can reveal peculiar cumulative features of its orbital motion which may turn out to be valuable tools to either put to the test fundamental theories or characterize the physical properties of the central body acting as source of the gravitational field. It has been just the case so far or characterize the physical properties of the central body acting to be valuable tools to either put to the test fundamental theories

ABSTRACT

By using the most recently published Doppler tomography measurements and accurate theoretical modeling of the oblateness-driven orbital precessions, we tightly constrain some of the physical and orbital parameters of the planetary system hosted by the fast rotating star WASP-33. In particular, the measurements of the orbital inclination $i_0$ to the plane of the sky and of the sky-projected spin-orbit misalignment $\lambda$ at two epochs about six years apart allowed for the determination of the longitude of the ascending node $\Omega$ and of the orbital inclination $I$ to the apparent equatorial plane at the same epochs. As a consequence, average rates of change $\dot{\Omega}$, $\dot{I}$ of this two orbital elements, accurate to a $\approx 10^{-2}$ deg yr$^{-1}$ level, were calculated as well. By comparing them to general theoretical expressions $\dot{\Omega}_g$, $\dot{I}_g$ for their precessions induced by an oblate star whose symmetry axis is arbitrarily oriented, we were able to determine the angle $i^*$ between the line of sight the star’s spin $S^*$ and its first even zonal harmonic $J_2$, obtaining $i^* = 142^{+10}_{-11}$ deg, $J_2 = (2.1_{-0.8}^{+0.5}) \times 10^{-4}$. As a by-product, the angle between $S^*$ and the orbital angular momentum $L$ is as large as about $\psi \approx 100$ deg ($\psi^{2008} = 99^{+5}_{-4}$ deg, $\psi^{2014} = 103^{+5}_{-4}$ deg), and changes at a rate $\dot{\psi} = 0.7_{-1.6}^{+1.5}$ deg yr$^{-1}$. The predicted general relativistic Lense-Thirring precessions, or the order of $\approx 10^{-3}$ deg yr$^{-1}$, are, at present, about one order of magnitude below the measurability threshold.

Key words: stars; planetary systems–gravitation–celestial mechanics

1 See, e.g., http://exoplanets.org/ on the WEB.
impact in stellar evolution, in particular towards the higher mass
(Tarafdar & Vardy 1971; Wolff, Edwards & Preston 1982; Vigneron et al. 1990; Wolff & Simon 1997; Herbst & Mundt 2005; Jackson, MacGregor & Skumanich 2005). As a na"ive measure of
the relevance of the Einsteinian theory of gravitation in a given bi-
nary system characterized by mass $M$, proper angular momentum
$S$ and extension $r$, the magnitude of the ratios of some typical
gravitational lengths to $r$ can be assumed. By taking (Bertotti, Farinella & Vokrouhlick 2003)
$$r_M = \frac{GM}{c^2},$$
$$r_S = \frac{S}{Mc},$$
where $G$ and $c$ are the Newtonian gravitational constant and the
speed of light in vacuum, respectively, it can be easily noted that,
for exoplanets hosted by Sun-like stars at, say, 0.005 au, Eqs 1
to 2 yield
$$r_M \approx 2 \times 10^{-6},$$
$$r_S \approx 4 \times 10^{-7}.$$ Such figures are substantially at the same level of, or even larger
than those of the double pulsar (Burgay et al. 2003; Lyne et al.
2004; Kramer et al. 2006), for which one has
$$r_M \approx 4 \times 10^{-6},$$
$$r_S \approx 8 \times 10^{-8}.$$ It shows that, in principle, some of the extrasolar planetary systems
may well represent important candidates to perform also tests of
relativistic orbital dynamics.

In the present work, we will deal with WASP-33 b (Collier
Cameron et al. 2010). It is a planet closely transiting a fast rotating
and oblate main sequence star along a circular, short-period ($P_b$
= 1.21 d) orbit which is highly inclined to the stellar equator. In Iorio
(2011b) it was suggested that, in view of the relatively large size
of some classical and general relativistic orbital effects, they could
be used to better characterize its parent star as long as sufficient
accurate data records were available. It has, now, become possible
in view of the latest Doppler tomography measurements processed
by Johnson et al. (2015), and of more accurate theoretical models
of the orbital precessions involved (Iorio 2011c, 2012).

The plan of the paper is as follows. In Section 2, we illustrate
our general analytical expressions for the averaged classical and
relativistic precessions of some Keplerian orbital elements in
the case of an arbitrary orientation of the stellar symmetry axis and
of an unrestricted orbital geometry. Section 3 describes the coordi-
nate system adopted in this astronomical laboratory. Our theoretical
predictions of the orbital rates of change are compared to the cor-
responding phenomenologically measured precessions in Section 4
where tight constraints on some key stellar parameters are in-
ferred, and the perspectives of measuring the Lense-Thirring effect
are discussed. Section 5 is devoted to summarizing our findings.

2 THE MATHEMATICAL MODEL OF THE ORBITAL
PRECESIONS

A particle at distance $r$ from a central rotating body of symmetry
axis direction $\hat{S} = \{\hat{S}_x, \hat{S}_y, \hat{S}_z\}$ experiences an additional non-
central acceleration (Vrbik 2005)
$$A_J = -\frac{3GMJ_R^2}{2c^4}\left[\left(1 - 5\hat{r} \cdot \hat{S}\right)^2 - 2\hat{r} \cdot \hat{S} \right], \tag{7}$$
which causes long-term orbital precessions. For a generic orienta-
tion of $\hat{S}$ in a given coordinate system, they were analytically
worked out by Iorio (2011c). Among them we have
$$\Omega_{J_2} = \frac{3n_0J_R^2}{4a^2(1 - e^2)} \left[\hat{S}_x \cos 2I \csc J \left(\hat{S}_z \sin \Omega - \hat{S}_y \cos \Omega\right) + \cos I \left[1 - 3\hat{S}_y^2 + (\hat{S}_x^2 - \hat{S}_z^2) \cos 2\Omega - 2\hat{S}_x\hat{S}_y \sin 2\Omega\right]\right], \tag{8}$$
$$I_{J_2} = -\frac{3n_0J_R^2}{2a^2(1 - e^2)} \left[(\hat{S}_y \cos \Omega + \hat{S}_x \sin \Omega) \hat{S}_z \cos I + \sin I \left(\hat{S}_x \sin \Omega - \hat{S}_y \cos \Omega\right)\right], \tag{9}$$
which will be relevant for our purposes. In Eqs 8 to 9, $a$ is the
semimajor axis, $n_0 = \sqrt{GMa^{-3}}$ is the Keplerian mean motion, $e$ is the eccentricity, $I$ is the inclination of the orbital plane with respect
to the coordinate $(x, y)$ plane adopted, and $\Omega$ is the longitude of
the ascending node counted in the $(x, y)$ plane from a reference $x$
direction to the intersection of the orbital plane with the $(x, y)$ plane
itself. Note that if the body’s equatorial plane is assumed as $(x, y)$
plane, i.e. if $\hat{S}_z = \hat{S}_x = 0$, $\hat{S}_y = 1$, Eqs 8 to 9 reduce to the well
known expressions (Bertotti, Farinella & Vokrouhlick 2003)
$$\Omega_{J_2}^{(0)} = \frac{3n_0J_R^2}{2a^2(1 - e^2)^2} \cos I, \tag{10}$$
$$I_{J_2}^{(0)} = 0; \tag{11}$$
with this particular choice, $I$ coincides with the angle $\psi$ between
$\hat{S}$ and the particle’s orbital angular momentum $L$. It is important
to stress that, in the general case, the cumbersome multiplicative
geometrical factor in Equation 8 depending on the spatial orienta-
tion of the orbit and of the spin axis does not reduce to $\cos \psi$, as
it will explicitly turn out clear in Section 3. On the other hand, it
can be easily guessed from the fact that $\cos \psi$ is linear in the com-
ponents of $\hat{S}$, while the acceleration of Equation 7 is quadratic in
them, whatever parametrization is adopted. Such an extrapolation
of a known result valid only in specific cases is rather widespread
in the literature (see, e.g., Iorio 2011b, Barnes et al. 2013, John-
son et al. 2015), and may lead to errors when accurate results are
looked for. Eqs 8 to 9 are completely general, and can be used with
any coordinate system provided that the proper identifications per-
taining the angular variables are made.

2 The replacement $Q_2 \rightarrow GMJ_R^2$ must be done in the equations by Iorio
(2011c) to obtain the present ones. Other conventions exist in the literature
about dimensional quadrupole moments $Q$, mainly differing for the sign and
the inclusion of $G$.

3 Also the argument of pericenter $\omega$ and the mean anomaly $M$ are impacted
by $J_2$ with long-term precessions. We will not display them here because
they are not relevant in the present study.
3 THE COORDINATE SYSTEM ADOPTED

For consistency reasons with the conventions adopted by Johnson et al. (2015), who, in turn, followed Queloz et al. (2000), the coordinate system used in the present analysis is as follows (see Figure 1).

The line of sight, directed towards the observer, is assumed as reference y axis, while the z axis is determined by the projection of the stellar spin axis \( \hat{S}^* \) onto the plane of the sky, which is inferred from observations. The z axis is straightforwardly chosen perpendicular to both the other two axes in such a way to form a right-handed coordinate system; it generally does not point towards the Vernal Equinox \( \nu \) at a reference epoch. With the present choice, the coordinate \( \{x, y\} \) plane does not coincide with the plane of the sky which, instead, is now spanned by the \( z \) and \( x \) axes; the \( \{x, y\} \) plane is known as apparent equatorial plane (Queloz et al. 2000).

The planetary longitude of the ascending node \( \Omega \) lies in it, being counted from the \( x \) axis to the intersection of the orbital plane with the apparent equatorial plane itself; thus, in general, \( \Omega \) does not stay in the plane of the sky. Moreover, with such conventions, the angle \( \angle \) between the orbital plane and the coordinate \( \{x, y\} \) plane entering Eqs [8 to 9] and Eqs [12 to 13] is not the orbital inclination \( i_p \), which refers the plane of the sky and is one of the orbital parameters directly accessible to observations. Instead, \( \angle \), which is also the angle from the unit vector \( k \) of the \( z \) axis to the planetary orbital angular momentum \( \hat{L} \), has to be identified with the angle \( \alpha \) of Queloz et al. (2000). By considering it as a colatitude angle of \( \hat{L} \) in a spherical coordinate system, the components of the unit vector of the planetary orbital angular momentum are

\[
\hat{L}_x = \sin \theta \sin \Omega ,
\]

\[
\hat{L}_y = - \sin \theta \cos \Omega ,
\]

\[
\hat{L}_z = \cos \theta.
\]

In view of Eqs [16 to 18]

\[
\frac{d\hat{L}}{dt} = \frac{\Omega^{(0)}}{\cos \theta} (\hat{L} \cdot \hat{S}) \hat{S} \times \hat{L},
\]

\[
\frac{d\Omega}{dt} = \Omega^{(0)} \hat{S} \times \hat{L},
\]

concisely summarize Eqs [8 to 9] and Eqs [12 to 13]. Another angle which is measurable is the projected spin-orbit misalignment \( \lambda \). It lies in the plane of the sky, and is delimited by the projections of both the stellar spin axis and of the planetary orbital angular momentum. In our coordinate system, \( \lambda, i_p \) are the longitude and the colatitude spherical angles, respectively, with \( \lambda \) reckoned from the \( z \) axis to the projection of \( \hat{L} \) onto the plane of the sky. As such, the components of the planetary orbital angular momentum versor can also be written as

\[
\hat{L}_x = \sin i_p \sin \lambda ,
\]

\[
\hat{L}_y = \cos i_p ,
\]

\[
\hat{L}_z = \sin i_p \cos \lambda .
\]

In general, both \( \lambda \) and \( \Omega \), which explicitly enter Eqs [8 to 9] and Eqs [12 to 13] are not directly measurable; they must be expressed in terms of the observable angles \( i_p, \lambda \). To this aim, it is useful to use the unit vector \( \hat{N} \) directed along the line of the nodes towards the ascending node, which is defined as

\[
\hat{N} = \frac{k \times \hat{L}}{|k \times \hat{L}|}.
\]

From Eqs [21 to 23], its components are

\[
\hat{N}_x = - \frac{\cos i_p}{\sqrt{\cos^2 i_p + \sin^2 i_p \sin^2 \lambda}} ,
\]

\[
\hat{N}_y = \frac{\sin i_p \sin \lambda}{\sqrt{\cos^2 i_p + \sin^2 i_p \sin^2 \lambda}},
\]

\[
\hat{N}_z = 0.
\]

Eqs [16 to 18] and the definition of Equation [24] allow to express the components of \( \hat{N} \) in terms of \( I, \Omega \) as

\[
\hat{N}_x = \cos \Omega ,
\]

\[
\hat{N}_y = \sin \Omega ,
\]

\[
\hat{N}_z = 0.
\]
By adopting the convention \( 0 \leq \Omega \leq 2\pi \), Equation (31) yields \[ \Omega = \arccos \hat{N}_s \text{ for } \hat{N}_s \geq 0, \] and \[ \Omega = 2\pi - \arccos \hat{N}_s \text{ for } \hat{N}_s < 0, \]
where \( \hat{N}_s, \hat{N}_t \) are expressed in terms of \( i_p, \lambda \) by means of Eqs (25) to (26). The inclination \( I \), defined in the range \( 0 \leq I \leq \pi \), is obtained in terms of \( i_p, \lambda \) from
\[ I = \arccos (k \cdot \hat{L}) \] (33)
and Eqs (27) to (29).

If \( i^* \) is the angle between from the line of sight to \( \hat{S}^* \), the components of the star’s spin axis in our coordinate system are
\[ \hat{S}_x = 0, \] (34)
\[ \hat{S}_y = \cos i^*, \] (35)
\[ \hat{S}_z = \sin i^*. \] (36)

The angle \( \psi \) between the stellar angular momentum \( \hat{S}^* \) and the planetary orbital angular momentum \( \hat{L} \) can be computed from Eqs (34) to (36) as
\[ \hat{S} \cdot \hat{L} = \cos \psi = \cos i_p \cos i^* + \sin i_p \sin i^* \cos \lambda, \]
in agreement with, e.g., Fabrycky & Winn (2009), Iorio (2011b).

Incidentally, Equation (37) along with an analogous one could be straightforwardly obtained from Eqs (16) to (18) and Eqs (34) to (36) in terms of \( I, \Omega \), explicitly shows that the node precession cannot be generally proportional to \( \cos \psi \), as previously remarked in Section 2.

Finally, the configurations \( \{ i_p, \lambda, i^* \} \) and \( \{ \pi - i_p, -\lambda, \pi - i^* \} \) are physically equivalent since they correspond to looking at the planetary system from the opposite sides of the plane of the sky (Masuda 2015). In both case, the angle \( \psi \) remains the same, as explicitly shown by Equation (37). According to Eqs (31) to (33) the node precession remains unaltered, while the rate of \( I \) changes by the amount
\[ \Delta I_j \pm I_j (\theta, \perp, \lambda, \pi, \perp, \pi, i^*) \]
\[ = \pm \frac{3n_0 J_2 R^2 |\sin \lambda| \cos i^* \sin i_p \cos \psi}{a^2 (1 - e^2)^2 \sqrt{\cos^2 i_p + \sin^2 i_p \sin^2 \lambda}}. \] (38)

In calculating Equation (38) we used both Equation (31) and Equation (32) since the transformation \( \{ i_p, \lambda \} \rightarrow \{ \pi - i_p, -\lambda \} \) changes the sign of \( \hat{N}_s \), as shown by Equation (26). The + sign in Equation (38) corresponds to using Equation (31) in \( J_2 (\theta, \perp, \lambda, \pi, \perp, \pi, i^*) \) and Equation (32) in \( J_2 (\theta, \perp, i^*, \perp, \lambda, \pi, \perp) \) while the − sign is for Equation (31) in \( J_2 (\theta, \perp, \lambda, \pi, \perp, \pi, i^*) \) and Equation (32) in \( J_2 (\theta, \perp, i^*, \perp, \lambda, \pi, \perp) \).

4 CONSTRAINTING THE STELLAR SPIN AXIS AND OBLATENESS

4.1 Using the precessions of \( I \) and \( \Omega \)

Generally speaking, while the magnitude of the classical precessions driven by the star’s oblateness is at the \( \approx 1 \) deg yr\(^{-1} \) level, the relativistic gravitomagnetic ones about three orders of magnitude smaller. Despite this discrepancy, if, on the one hand, the current
state-of-the-art in the orbital determination of WASP-33 b (Johnson et al. 2015), based on data records 5.89 years long (from Nov 12, 2008 to Oct 4, 2014), does not yet allow for a measurement of the relativistic effects, on the other hand, they might exceed the measurability threshold in a not so distant future. Indeed, they are just ≈ 4–8 times smaller than the present-day errors, which amount to ≈ 2–8 × 10^−2 deg yr^−1 (Johnson et al. 2015) for the node.

In the following, we will reasonably assume that the measured orbital precessions of WASP-33 b are entirely due to the star’s oblateness. This will allow us to put much tighter constraints on either \( i^* \) and \( J_{2}^* \). Our approach is as follows.

The lucky availability of the measurements of both \( i_0 \) and \( \lambda \) at two different epochs some years apart leads to the calculation of the unobservable orbital parameters \( \Omega, I \), from Eqs 21 to 23 at the same epochs. According to the measured values of \( i_0, \lambda \) by Johnson et al. (2015), it is \( N_1 < 0 \), so that Equation 32 must be used yielding

\[
\Omega_{2008}^{2014} = 266.4^{+0.2}_{-0.3} \text{ deg}.
\]

The values by Johnson et al. (2015) differ from Eqs 39 to 40 by \( \pi \), likely due to the different convention adopted for the node. Since Equation 33 returns

\[
I_{2008}^{2014} = 110.0^{+0.5}_{-0.2} \text{ deg},
\]

Eqs 16 to 18 and Eqs 21 to 23 agree both in magnitude and in sign. It is straightforward to compute the average rates of change of \( \Omega_{exp}, I_{exp} \) by simply taking the ratios of the differences \( \Delta \Omega, \Delta I \) of their values at the measurement’s epochs to the time span, which in our case is \( \Delta t = 5.89 \text{ yr} \). Our results are in Table 1.

Eqs 8 to 9 provide us with an accurate mathematical model of the oblateness-driven precessions which, in view of its generality, can be straightforward applied to the present case. Eqs 8 to 9 can be viewed as two functions of the two independent variables \( i^*, J_{2}^* \); By allowing them to vary within their physically admissible ranges (Lório 2011b), it is possible to evaluate \( \Omega_{12}, I_{12}, I_{exp} \) by obtaining certain stripes in the \( [i^*, J_{2}^*] \) plane whose widths are fixed by the experimental ranges of the observationally determined precessions quoted in Table 1. If our model is correct and if it describes adequately the empirical results, the two stripes must overlap somewhere in the considered portion of the \( [i^*, J_{2}^*] \) plane by determining an allowed region of admissible values for the inclination of the stellar spin axis to the line of sight and the star’s dimensionless quadrupole mass moment. It is just the case, as depicted in the upper row of Figure 2. From it, it turns that

\[
i^* = 142^{+16}_{-11} \text{ deg}.
\]

\[
J_{2}^* = (2.1^{+0.2}_{-0.3}) \times 10^{-4}.
\]

As a consequence, the angle between the orbital plane and the stellar equator and its precession is as reported in Table 1.

The lower row of Figure 2 depicts the physically equivalent case with \( \pi - i_0 \neq \lambda \). Now, \( N_1 > 0 \), and Equation 31 must be used yielding

\[
\Omega_{2008}^{2014} = 86.4^{+0.2}_{-0.3} \text{ deg},
\]

\[
\Omega_{2014}^{2008} = 88.58^{+0.04}_{-0.03} \text{ deg}.
\]

While the stripe for \( \Omega \) is the same, it is not so for \( I \), as expected from Equation 38, the intersection between the \( I, \Omega \) curves corresponds to

\[
\pi - i^* = 38.1^{+0.10}_{-0.07} \text{ deg}.
\]

It must be noted that \( J_{2}^* \) is unchanged.

### 4.2 Constraining the oblateness of Kepler-13 Ab

An opportunity to apply the present method to another exoplanet is offered by Kepler-13 Ab, also known as KOI-13.01 (Szabó et al. 2012; Shporer et al. 2014; Johnson et al. 2014; Masuda 2015). By using the values of its physical and orbital parameters determined with the gravity darkened transit light curves and other observations (Masuda 2015), it is possible to compute analytically the rate of change of \( \cos i_0 \) in terms of \( \Omega, I \) by means of Equation 17 and Equation 22 and compare it to its accurately measured value (Masuda 2015) in order to infer \( J_{2}^* \). We obtain

\[
J_{2}^* = (6.0 \pm 0.6) \times 10^{-5},
\]

in agreement with Masuda (2015), who seemingly used a different dynamical modelization. We calculated our uncertainty with a straightforward error propagation in our analytical expression of \( J_{2}^* \) thought as a function of the parameters \( d \cos i_0/dt, \cos i_0, i^*, \dot{R}_0, a/R_0, \lambda \) affected by experimental uncertainties (Shporer et al. 2014; Masuda 2015).

The definition of the impact parameter

\[
b = \left( \frac{a}{R_0} \right) \cos i_0,
\]

valid for a circular orbit, along with Equation 17 and Equation 22 allows us to use also the value of \( b_{exp} \) independently measured by Szabó et al. (2012) with the transit duration variation, although it is accurate only to 27%. We get

\[
J_{2}^* = (8.6 \pm 2.4) \times 10^{-5},
\]

which is not in disagreement with Equation 45.

From Equation 49, it turns out that the analytical expressions of \( b \) and \( d \cos i_0/dt \) are not independent, so that the availability of independently measured values for both of them do not allow to determine/constrain any further dynamical effect with respect to \( J_{2}^* \). Luckily, it seems that other precessions, independent of \( b, d \cos i_0/dt \), should be measurable via Doppler tomography in the next years or so (Johnson et al. 2015; Masuda 2015). Depending on the final accuracy reached, such an important measurement will allow, at least in principle, to dynamically measure or, at least, constrain also the stellar spin by means of the Lense-Thirring effect through, e.g., \( \lambda \) calculated with Eqs 12 to 13.

### 5 SUMMARY AND CONCLUSIONS

The use of a general model of the orbital precessions caused by the primary’s oblateness, applied to recent phenomenological measurements of some planetary orbital parameters of WASP-33 b taken at different epochs 5.89 years apart, allowed us to tightly constrain the inclination \( i^* \) of the spin \( S^* \) of WASP-33 to the line of sight

7 Contrary to WASP-33 b, in the case of Kepler-13 Ab also \( i^* \) is available (Masuda 2015).
and its dimensionless quadrupole mass moment $J_2^\star$. Our analytical expressions are valid for arbitrary orbital geometries and spatial orientations of the body’s symmetry axis.

By comparing our theoretical orbital rates of change of the longitude of the ascending node $\Omega$ and of the inclination $i$ of the orbital plane with respect to the apparent equatorial plane with the observationally determined ones, we obtained $i^\star = 142_{-11}^{+10}$ deg, $J_2^\star = (2.1_{-0.5}^{+0.8}) \times 10^{-4}$. Furthermore, the angle between the stellar and orbital angular momenta at different epochs is $\psi_{\text{2008}} = 99_{-4}^{+5}$ deg, $\psi_{\text{2014}} = 103_{-4}^{+5}$ deg. Thus, it varies at a rate $\dot{\psi} = 0.7_{-1.5}^{+1.5}$ deg yr$^{-1}$.

In view of the fact that WASP-33 b should transit its host star until 2062 or so and of the likely improvements in the measurement accuracy over the years, such an extrasolar planet will prove a very useful tool for an increasingly accurate characterization of the key physical and geometrical parameters of its parent star via its orbital dynamics. Moreover, also the determination of the general relativistic Lense-Thirring effect, whose predicted size is currently just one order of magnitude smaller than the present-day accuracy level in determining the planetary orbital precessions, may become a realistic target to be pursued over the next decades.

Furthermore, in view of its generality, our approach can be straightforwardly applied to any other exoplanetary system, already known or still to be discovered, for which at least the same parameters as of WASP-33 b are or will become accessible to the observation. A promising candidate, whose orbital precessions should be measurable via Doppler tomography in the next years, is Kepler-13 Ab. For the moment, we applied our method to it by exploiting its currently known parameters, and we were able to constrain its oblateness in agreement with the bounds existing in the literature.

Finally, in principle, also the periastron, if phenomenologically measurable at different epochs as in the present case, can become a further mean to investigate the characteristics of highly eccentric exoplanetary systems-and to test general relativity as well-along the guidelines illustrated here.

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WAPS-33 b parameters from orbital dynamics

Ω_{J^2} = \hat{\Omega}_{\text{exp}}

\hat{I}_{J^2} = \hat{I}_{\text{exp}}

Figure 2. Upper row: the darkest region in the plot is the experimentally allowed area in the \( \{ i^*, J^2 \} \) plane, which is enlarged in the right panel. It is determined by the overlapping of the permitted shaded stripes set by the precessions of the node \( \Omega \) and the orbital inclination to the apparent equatorial plane \( I \). We assumed that the experimental precessions \( \hat{\Omega}_{\text{exp}}, \hat{I}_{\text{exp}} \) are entirely due to the stellar oblateness \( J^2 \), within the experimental errors. For \( \Omega_{J^2}, I_{J^2} \), we used the mathematical model of Eqs 8 to 9 calculated with the values quoted in Table 1; the values for \( R^*, M^*, a \) were taken from Collier Cameron et al (2010). The curves inside the shaded areas correspond to the best estimates for \( \hat{\Omega}_{\text{exp}}, \hat{I}_{\text{exp}} \); their intersection is given by \( i^* = 142 \text{ deg}, J^2 = 2.1 \times 10^{-4} \). Lower row: same as in the upper row, but with \( \pi - i_p, \lambda \). Note that the stripe for \( I \) is different, in agreement with Equation 38. The solution for the stellar spin axis inclination corresponds to \( \pi - i^* \).

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