Geometry and Dynamics of Discrete Isometry Groups of Higher Rank Symmetric Spaces

Gabriele Link

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Abstract

For real hyperbolic spaces, the dynamics of individual isometries and the geometry of the limit set of nonelementary discrete isometry groups have been studied in great detail. Most of the results were generalised to discrete isometry groups of simply connected Riemannian manifolds of pinched negative curvature. For symmetric spaces of higher rank, which contain isometrically embedded Euclidean planes, the situation becomes far more complicated. This paper is devoted to the study of the geometric limit set of “nonelementary” discrete isometry groups of higher rank symmetric spaces. We obtain the natural generalisations of some well-known results from Kleinian group theory. Our main tool consists in a detailed description of the dynamics of individual isometries. As a by-product, we give a new geometric construction of free isometry groups with parabolic elements in higher rank symmetric spaces.

1 Introduction

Let $X$ be a globally symmetric space of noncompact type, $o \in X$ and $G = \text{Isom}^0(X)$ the connected component of the identity. We will denote by $\partial X$ the geometric boundary of $X$ endowed with the cone topology (see [Ba, chapter II]).

The goal of this paper is to give more insight into the dynamics of certain individual isometries of $X$ and describe geometrically the structure of the limit set $L_\Gamma := \Gamma \cdot o \cap \partial X$ of discrete isometry groups $\Gamma \subseteq G$. The main difficulties we face in the higher rank case compared to the situation in manifolds with pinched negative curvature are due to the more complicated structure of the geometric boundary. In fact, to each point $\xi \in \partial X$ we can associate a unique “direction” in a fixed Weyl chamber of $X$ (see section 2.3 for a precise definition). If the direction of $\xi$ is in the interior of the Weyl chamber, we say that $\xi$ belongs to the regular boundary $\partial X^{\text{reg}} \subseteq \partial X$. Every point in the $G$-orbit of $\xi$ possesses the same direction, in particular $G$ does not act transitively on the geometric boundary if the rank of $X$ is greater than one.

Concerning the dynamics of parabolic isometries, there are only partial results for example by P. Eberlein ([E, chapter 4.1]), A. Parreau ([P, chapter I.2]) and the author ([L, chapter 4.4]). For axial (sometimes also called loxodromic) isometries, however, we are able to describe precisely the action on the geometric boundary. The same remains true for the particular kind of “generic parabolic” isometries, which behave similarly as parabolic isometries of rank one symmetric spaces. A first application gives a new geometric construction of Schottky groups in higher rank symmetric spaces which contain parabolic isometries.
We then generalise appropriately the notion of “nonelementary” groups well-known in the context of manifolds of pinched negative curvature to higher rank symmetric spaces. Our notion is weaker than Zariski density which is normally used for discrete subgroups of real reductive linear Lie groups (see e.g. [B] and [CG]). Also, the definition is more natural and easily understandable from a geometrical point of view.

Unfortunately, the incomplete picture we have about the dynamics of parabolic isometries makes it difficult to describe the structure of the limit set of discrete isometry groups which, in general, always contain parabolics. For the large class of nonelementary groups, however, we can use a so-called “approximation argument” in order to reduce the problem to understanding the action of sequences of axial isometries. A direct consequence of this approximation argument is Theorem 4.10 which states that the set of attractive fixed points of axial isometries is dense in $L_{\Gamma}$. It is also one of the key ingredients in the proof of Theorems 4.13 and 4.15 which we state here in a simplified version. Let $P_{\Gamma}$ denote the set of directions of limit points, and $P_{\Gamma}^{reg} \subset P_{\Gamma}$ the set of directions of regular limit points.

**Theorem 1** If $\Gamma \subset G$ is a nonelementary discrete group, then the limit set $K_{\Gamma}$, considered as a subset of the Furstenberg boundary, is a minimal closed set under the action of $\Gamma$.

**Theorem 2** If $\Gamma \subset G$ is a nonelementary discrete group, then the regular limit set $L_{\Gamma}^{reg}$ splits as a product $K_{\Gamma} \times P_{\Gamma}^{reg}$.

To each axial isometry of a higher rank symmetric space we can associate a so-called “translation vector”, a notion introduced by A. Parreau ([P]) which generalises the translation length in rank one spaces. Let $\ell_{\Gamma}$ denote the set of translation vectors of axial isometries in $\Gamma$. Then we have the following

**Theorem 3** If $\Gamma \subset G$ is a nonelementary discrete group, then $P_{\Gamma}$ is equal to the closure of $\ell_{\Gamma}$.

Although these results are already known for Zariski dense subgroups of real reductive linear groups (see e.g. [B], [CG]), the advantage of our approach is its purely geometric nature which allows to easily adapt the methods to products of manifolds of pinched negative curvature (compare [DK]).

The paper is organised as follows: In section 2 we recall some basic facts about Riemannian symmetric spaces of noncompact type and decompositions of semisimple Lie groups. We describe the $G$-orbit structure of the geometric boundary $\partial X$ and introduce a family of (possibly nonsymmetric) $G$-invariant pseudo distances on $X$ which we will need later on. In section 3 we classify individual isometries and describe their dynamics and action on the geometric boundary. As a corollary we obtain Theorem 3.20 which gives a new geometric construction of free groups. Section 4 is devoted to the study of the limit set. We introduce and describe “nonelementary” groups of higher rank symmetric spaces. For these groups, we prove the “approximation argument” Proposition 4.9 and describe the dynamics of certain sequences of axial isometries. This leads directly to the proof of Theorems 4.13, 4.15 and 4.16.

2 Preliminaries

In this section we recall basic facts about symmetric spaces of noncompact type (see also [He], [BGS], [E], [W]) and introduce some notations for the sequel.
2.1 Cartan and Iwasawa decomposition

Let $X$ be a simply connected symmetric space of noncompact type with base point $o \in X$, $G = \text{Isom}^o(X)$, and $K$ the isotropy subgroup of $o$ in $G$. It is well-known that $G$ is a semisimple Lie group with trivial centre and no compact factors, and $K$ a maximal compact subgroup of $G$. Denote by $\mathfrak{g}$ and $\mathfrak{t}$ the Lie algebras of $G$ and $K$. Since $G$ acts transitively on $X$, we may identify $X$ with the homogeneous space $G/K$. The geodesic symmetry in $o$ induces a Cartan involution $\theta$ on $\mathfrak{g}$, hence $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}$, where $\mathfrak{p} \subset \mathfrak{g}$ denotes its $-1$ eigenspace. The Killing form $B$ of $\mathfrak{g}$ induces a scalar product

$$\langle X, Y \rangle := -B(X, \theta Y), \quad X, Y \in \mathfrak{g} \quad \text{(1)}$$

on $\mathfrak{g}$. The tangent space $T_oX$ of $X$ in $o$ can be identified with $\mathfrak{p}$, and the Riemannian exponential map at $o$ is a diffeomorphism of $\mathfrak{p}$ onto $X$. The scalar product $\langle \cdot, \cdot \rangle$ of $\mathfrak{g}$ restricted to $\mathfrak{p}$ therefore induces an scalar product $\langle \cdot, \cdot \rangle$ on $T_oX$ which extends to a $G$-invariant Riemannian metric on $X$ with associated distance $d$. With respect to this metric, $X$ has nonpositive sectional curvature, and, up to rescaling in each factor, this metric is the original one.

Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subspace. Its dimension $r$ is called the rank of $X$. The choice of an open Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$ determines a Cartan decomposition $G = K e^{\mathfrak{a}^+} K$, where $\mathfrak{a}^+$ denotes the closure of $\mathfrak{a}^+$. We further put $\mathfrak{a}_1 := \{ H \in \mathfrak{a} \mid \| H \| := \sqrt{\langle H, H \rangle} = 1 \}$.

If $z = ke^H o \in X$, we call $k \in K$ an angular projection and $H \in \mathfrak{a}^+$ the (unique) Cartan projection of $z$.

**Definition 2.1** For $x, y \in X$ the unique vector $H \in \mathfrak{a}^+$ with the property $x = go$ and $y = ge^H o$ for some $g \in G$ is called the Cartan vector of the ordered pair of points $(x, y) \in X \times X$ and will be denoted $H(x, y)$.

Let $\Sigma$ be the set of roots of the pair $(\mathfrak{g}, \mathfrak{a})$, and $\Sigma^+ \subset \Sigma$ the set of positive roots determined by the Weyl chamber $\mathfrak{a}^+$. We denote $\mathfrak{g}_\alpha$ the root space of $\alpha \in \Sigma$, $\mathfrak{n}^+ := \sum_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha$, and $N^+$ the Lie group exponential of the nilpotent Lie algebra $\mathfrak{n}^+$. The decomposition $G = KAN^+$ is called the Iwasawa decomposition associated to the Cartan decomposition $G = Ke^{\mathfrak{a}^+} K$.

It induces a diffeomorphism $N^+ \times \mathfrak{a} \to X$, $(n, H) \mapsto ne^H o$, and we have the formula

$$d(ne^H o, n'e^{H'} o) \geq d(e^H o, e^{H'} o) \quad \forall n, n' \in N^+ \forall H, H' \in \mathfrak{a} \quad \text{(2)}$$

Let $M$ denote the centraliser of $\mathfrak{a}$ in $K$. The Iwasawa decomposition induces a natural projection

$$\pi^I : G \to K/M$$

$$g = kan \mapsto kM,$$

which we will need in the sequel.

2.2 The Bruhat decomposition

Given an Iwasawa decomposition $G = KAN^+$, we consider the closed subgroup $P = MAN^+ \subset G$. The homogeneous space $G/P$ is called the Furstenberg boundary which is identified with $K/M$ via the bijection

$$\bar{\pi} : G/P \to K/M$$

$$gP \mapsto \pi^I(g).$$
The Furstenberg boundary hence has a natural differentiable structure arising from the Lie group structure of $K$. Geometrically it can be described as the set of equivalence classes of asymptotic Weyl chambers in $X$ (see [M]).

Let $M^*$ be the normaliser of $a$ in $K$, and $W = M^*/M$ the Weyl group of the pair $(\mathfrak{g}, \mathfrak{a})$. We denote by $w_\ast \in W$ the unique element such that $\text{Ad}(m_{w_\ast})(-a^+) = a^+$ for any representative $m_{w_\ast}$ of $w_\ast$ in $M^*$, and put $n^- := \text{Ad}(m_{w_\ast})n^+ = \sum_{\alpha \in \Sigma^+} \mathfrak{g} - \alpha$. In the sequel we will also need the opposition involution $\iota : a \mapsto -\text{Ad}(m_{w_\ast})H$.

The Bruhat decomposition of $G$ with respect to the minimal parabolic subgroup $P$ is the disjoint union

$$G = \bigcup_{w \in W} N^+m_{w_\ast}P = \bigcup_{w \in W} U_w m_{w_\ast}P,$$

where $m_{w_\ast}$ is an arbitrary representative of $w_\ast$ in $M^*$, and the sets $U_w$ are the Lie group exponentials of the subspaces

$$u_w := n^+ \cap \text{Ad}(m_{w_\ast})n^- \subset n^+.$$

Then the orbit in the decomposition (4) corresponding to $w_\ast \in W$ is parametrised by $N^+ = U_{w_\ast}$, and the restriction of the above bijection $\pi$ to $N^+m_{w_\ast}P$ defines a map

$$\kappa : N^+ \to K/M \quad n \mapsto \pi(nm_{w_\ast}P).$$

Geometrically, this map can be interpreted in the following way: If $n \in N^+$, then $\kappa(n) \in K/M$ is the unique element such that the Weyl chamber $\kappa(n)e^{a^+}o$ is asymptotic to the Weyl chamber $ne^{-a^+}o$. The following property of the map $\kappa$ is well-known:

**Proposition 2.2** ([He], chapter IX, Corollary 1.9)
The map $\kappa$ is a diffeomorphism onto an open submanifold of $K/M$ whose complement consists of finitely many disjoint manifolds of strictly lower dimension.

It follows that the orbit $N^+m_{w_\ast}P$ is a dense and open submanifold of the Furstenberg boundary $G/P$. We will call a $G$-translate of the set $N^+m_{w_\ast}P \subset G/P$ a big cell of the Furstenberg boundary.

### 2.3 Compactification of $X$

The geometric boundary $\partial X$ of $X$ is the set of equivalence classes of asymptotic geodesic rays endowed with the cone topology. This boundary is homeomorphic to the unit tangent space of an arbitrary point in $X$.

We fix a Cartan decomposition $G = Ke^{a^+}K$ and let $o \in X$ be the unique point stabilised by $K$. Then for $k \in K$ and $H \in a_+^+$, we denote by $(k, H)$ the unique class in $\partial X$ which contains the geodesic ray $\sigma(t) = ke^{Ht}o$, $t > 0$. We call $k$ an angular projection, and $H$ the Cartan projection of $(k, H)$. Again, the Cartan projection $H$ of a point $\xi \in \partial X$ is unique, whereas its angular projection $k$ is only determined up to right multiplication by an element in the centraliser of $H$ in $K$. 


If $r = \text{rank}(X) > 1$, we define the regular boundary $\partial X^{\text{reg}}$ as the set of classes with Cartan projection $H \in a_1^+$. If $\text{rank}(X) = 1$, we use the convention $\partial X^{\text{reg}} = \partial X$. Furthermore, the natural projection
\[
\pi^B : \partial X^{\text{reg}} \to K/M
\]
\[
(k, H) \mapsto kB
\]
will be important in the sequel. The following lemma relates the cone topology to the topology of $K/M$. It is a corollary of Lemma 2.9 in [L].

**Lemma 2.3** A sequence $(\xi_n) \subset \partial X^{\text{reg}}$ converges to $\xi = (k, H) \in \partial X^{\text{reg}}$ in the cone topology if and only if $\pi^B(\xi_n)$ converges to $kB$ in $K/M$ and the Cartan projections of $\xi_n$ converge to $H$ in $a_1^+$.

Hence $\pi^B$ is continuous, and $\text{rank}(X) = 1$ if and only if $\pi^B$ is a homeomorphism.

Let $\overline{X} := X \cup \partial X$. For $x \in X$ and $z \in \overline{X} \setminus \{x\}$ we denote by $\sigma_{x,z}$ the unique unit speed geodesic emanating from $x$ which contains $z$. We say that $G = Ke^{\mathfrak{a}^+}K$ is a Cartan decomposition with respect to $x \in X$ (and $\eta \in \partial X$) if $x$ is the unique point fixed by the maximal compact subgroup $K \subset G$ and $\sigma_{x,\eta}(t) \subset e^{\mathfrak{a}^+}x$ for all $t > 0$.

The isometry group of $X$ has a natural action by homeomorphisms on the geometric boundary. If $g \in G$ and $\xi = (k, H) \in \partial X$, we have $g \cdot (k, H) = (k_g, H)$, where $k_g \in K$ is an angular projection of the unique unit speed ray emanating from $o$ asymptotic to the ray $g \cdot \sigma_o.\xi$. Furthermore, the projection $\pi^B$ induces an action of $G$ by homeomorphisms on the Furstenberg boundary $K/M = \pi^B(\partial X^{\text{reg}})$. More precisely, if $G = Ke^{\mathfrak{a}^+}K$ is a Cartan decomposition with associated Iwasawa decomposition $G = KAN^+$, and $\pi^I$ the projection defined at the end of section 2.1 we have the following

**Lemma 2.4** Let $g \in G$ and $\xi = (k, H) \in \partial X$ with $k \in K$ and $H \in \mathfrak{a}_1^+$. If $k' \in K$ is such that $\pi^I(gk) = k'M$, then $g\xi = (k', H)$.

In particular, if $\xi \in \partial X^{\text{reg}}$, then $g\pi^B(\xi) = \pi^B(g\xi) = k'M$.

**Proof.** Let $o \in X$ be the fixed point of $K$ and consider the geodesic $\sigma := \sigma_{o,\xi}$, i.e. $\sigma(t) = ke^{Ht}o$ for $t \in \mathbb{R}$. We write $gk = k'an$ with $k' \in K$, $a \in A$ and $n \in N^+$. In order to prove that $g\sigma(t)$ converges to $(k', H) \in \partial X$ as $t \to \infty$, we let $R >> 1$ and $\varepsilon > 0$ arbitrary. For $t > R$ we denote $\sigma_t$ the geodesic emanating from $o$ passing through $g\sigma(t)$. If $s_t := d(o, g\sigma(t))$, then by the triangle inequality $|s_t - t| \leq d(o, go)$. Using the convexity of the distance function we estimate for $t > R$
\[
d(k'e^{HR}o, \sigma_t(R)) \leq \frac{R}{s_t} (d(k'e^{Hs_t}o, g\sigma(s_t)) + d(g\sigma(s_t), \sigma_t(s_t)))
\]
\[
= \frac{R}{s_t} (d(k'e^{Hs_t}o, ke^{Hs_t}o) + d(g\sigma(s_t), g\sigma(t)))
\]
\[
= \frac{R}{s_t} (d(k'e^{Hs_t}o, k'an e^{Hs_t}o) + d(\sigma(s_t), \sigma(t))) \leq \frac{R}{s_t} (d(o, ano) + d(o, go))
\]
since $d(e^{Hs}o, an e^{Hs}o) \leq d(o, ano)$ for all $s > 0$. From $s_t \to \infty$ as $t \to \infty$ we conclude
\[
d(k'e^{HR}o, \sigma_t(R)) < \varepsilon
\]
for $t$ sufficiently large. Hence $g\xi = (k', H)$. \hfill $\square$
Notice that $G\cdot \xi = K\cdot \xi$ for any $\xi \in \partial X$. Furthermore, $G$ acts transitively on $\partial X$ if and only if $\text{rank}(X) = 1$.

A further difficulty in the higher rank setting is the fact that, in general, a pair of boundary points can not be joined by a geodesic. The following sets will therefore play a significant role in the sequel.

**Definition 2.5** The visibility set at infinity viewed from $\xi \in \partial X$ is the set

$$\text{Vis}^\infty(\xi) := \{ \eta \in \partial X \mid \exists \text{ geodesic } \sigma \text{ such that } \sigma(-\infty) = \xi, \sigma(\infty) = \eta \}.$$  

The Bruhat visibility set viewed from $\xi \in \partial X^{\text{reg}}$ is the image of $\text{Vis}^\infty(\xi)$ under the projection $\pi^B : \partial X^{\text{reg}} \to K/M$, i.e.

$$\text{Vis}^B(\xi) = \pi^B(\text{Vis}^\infty(\xi)).$$

We remark that if $\text{rank}(X) = 1$, then $\text{Vis}^B(\xi) \cong \text{Vis}^\infty(\xi) = \partial X \setminus \{\xi\}$ for all $\xi \in \partial X$. If $\xi \in \partial X^{\text{reg}}$ is stabilised by the minimal parabolic subgroup $P \subset G$, then $\text{Vis}^B(\xi)$ is exactly the image under the map $\pi$ of the big cell $N^+ m_w P \subset G/P$ of maximal dimension (see [L, Corollary 2.15] for a more general result). In particular, $\text{Vis}^B(\xi)$ can be identified with the nilpotent Lie group $N^+$ or an arbitrary orbit $N^+ x, x \in X$. Moreover, all Bruhat visibility sets are open and dense submanifolds of $K/M$ by Proposition 2.2.

Furthermore, if $\xi \in \partial X$ is arbitrary, $x \in X$, and

$$N_{\xi} := \{ g \in G \mid \lim_{t \to \infty} d(g\sigma_{x,\xi}(t), \sigma_{x,\xi}(t)) = 0 \}$$

denotes the horospherical subgroup associated to $\xi$, then $\text{Vis}^\infty(\xi) = N_{\xi} \cdot \sigma_{x,\xi}(-\infty)$.

### 2.4 Directional distances

Let $x, y \in X$, $\xi \in \partial X$, and $\sigma$ a geodesic ray in the class of $\xi$. We put

$$B_{\xi}(x, y) := \lim_{s \to \infty} (d(x, \sigma(s)) - d(y, \sigma(s))).$$

This number is independent of the chosen ray $\sigma$, and the function

$$B_{\xi}(\cdot, y) : X \to \mathbb{R}$$

$$x \mapsto B_{\xi}(x, y)$$

is called the Busemann function centred at $\xi$ based at $y$ (see also [Ba, chapter II]). Using Buseman functions we introduce an important family of (possibly nonsymmetric) pseudo distances which we will need in the proof of Theorem 4.16.

**Definition 2.6** Let $\xi \in \partial X$. We define the directional distance of the ordered pair $(x, y) \in X \times X$ with respect to the subset $G\cdot \xi \subseteq \partial X$ by

$$B_{G\cdot \xi} : X \times X \to \mathbb{R}$$

$$(x, y) \mapsto B_{G\cdot \xi}(x, y) := \sup_{g \in G} B_{g\cdot \xi}(x, y).$$
Notice that in rank one symmetric spaces $G \cdot \xi = \partial X$, hence $B_{G,\xi}$ equals the Riemannian distance $d$ for any $\xi \in \partial X$. In general, the corresponding estimate for the Buseman functions implies

$$B_{G,\xi}(x, y) \leq d(x, y) \quad \forall \xi \in \partial X \quad \forall x, y \in X.$$ 

Furthermore, $B_{G,\xi}$ is a (possibly nonsymmetric) $G$-invariant pseudo distance on $X$ (for a proof see [L, Proposition 3.7]), and we have

$$B_{G,\xi}(x, y) = d(x, y) \sup_{g \in G} \cos \angle_x(y, g\xi).$$

In particular, if $G = K e^{a^+} K$ is a Cartan decomposition, $H_\xi \in a^+$ the Cartan projection of $\xi$, and $H(x, y) \in a^+$ the Cartan vector of the ordered pair $(x, y)$ according to Definition 2.1, then

$$B_{G,\xi}(x, y) = \langle H_\xi, H(x, y) \rangle \quad \forall x, y \in X. \quad (5)$$

3 Individual Isometries

In this section, we recall the geometric classification and an algebraic characterisation of individual isometries. We further describe their fixed point set and dynamical properties when acting on the geometric boundary of $X$.

3.1 Geometric classification of isometries

If $X$ is a Hadamard manifold, individual isometries of $X$ can be classified geometrically by means of the displacement function (compare [BGS, chapter 6])

$$d_\gamma : X \to \mathbb{R}$$

$$x \mapsto d(x, \gamma x) \quad \text{for } \gamma \in \text{Isom}(X).$$

We will denote by $l(\gamma) := \inf_{x \in X} d_\gamma(x)$ the translation length of $\gamma$.

**Definition 3.1** A nontrivial isometry $\gamma$ of $X$ is called elliptic, if $\gamma$ fixes a point in $X$, and $\gamma$ is called axial, if $d_\gamma$ assumes the infimum in $X$ and $l(\gamma) > 0$.

We call $\gamma$ parabolic, if $d_\gamma$ does not assume the infimum. If furthermore $l(\gamma) = 0$, then $\gamma$ is called strictly parabolic, if $l(\gamma) > 0$, we call $\gamma$ mixed parabolic.

For $\gamma \in \text{Isom}(X)$ we denote by $\text{Fix}(\gamma)$ the set of fixed points of $\gamma$ in $X$. The following propositions summarise a few properties of individual isometries of a Hadamard manifold. The proofs can be found in [Ba], chapter II.

**Proposition 3.2** ([Ba, Proposition II.3.2])

An isometry $\gamma \in \text{Isom}(X) \setminus \{\text{id}\}$ is elliptic if and only if $\gamma$ has a bounded orbit.

**Proposition 3.3** ([Ba, Proposition II.3.3])

An isometry $\gamma \in \text{Isom}(X) \setminus \{\text{id}\}$ is axial if and only if there exists a unit speed geodesic $\sigma$ and a number $l > 0$ such that $\gamma(\sigma(t)) = \sigma(t + l)$ for all $t \in \mathbb{R}$.

**Proposition 3.4** ([Ba, Proposition II.3.4])

If $\gamma \in \text{Isom}(X) \setminus \{\text{id}\}$ is parabolic, then there exists a point $\eta \in \text{Fix}(\gamma) \subset \partial X$ such that $B_{\eta}(x, \gamma x) = 0$ for all $x \in X$. 

7
3.2 The Jordan decomposition

From here on we restrict ourselves to the case of a globally symmetric space $X$ of noncompact type. The choice of an Iwasawa decomposition $G = KAN^+$ gives rise to a natural algebraic characterisation of certain individual isometries.

**Definition 3.5** An isometry $\gamma \in G \setminus \{id\}$ is called elliptic, if $\gamma$ is conjugate to an element in $K$, hyperbolic, if $\gamma$ is conjugate to an element in $A$, and unipotent, if $\gamma$ is conjugate to an element in $N^+$.

Notice that by Proposition 2.19.18 (1), (2) and (5) of [E], these definitions coincide with the definitions of “elliptic”, “hyperbolic” and “unipotent” via the image under the adjoint representation of $G$ in $GL(g)$. The following lemma further relates this algebraic characterisation to the geometric classification of the previous subsection.

**Lemma 3.6** $\gamma \in G \setminus \{id\}$ is conjugate to an element in $K$ if and only if $\gamma$ fixes a point in $X$. Hyperbolic isometries are axial, and unipotent isometries are strictly parabolic.

**Proof.** The first assertion is trivial, because the stabiliser of any point in $X$ is conjugate to $K$, the stabiliser of $o \in X$.

If $\gamma$ is hyperbolic, there exists $H \in a \setminus \{0\}$ and $g \in G$ such that $\gamma = ge^Hg^{-1}$. The unit speed geodesic $\sigma$ defined by $\sigma(t) := ge^{tH/\|H\|}o$ then satisfies

$$\gamma(\sigma(t)) = (ge^Hg^{-1})ge^{tH/\|H\|}o = ge^{H+tH/\|H\|}o = \sigma(\|H\| + t) \quad \forall t \in \mathbb{R},$$

hence the claim follows from Proposition 3.3.

If $\gamma$ is unipotent, there exists $n \in N^+ \setminus \{id\}$ and $k \in K$ such that $\gamma = knk^{-1}$. For $H \in a_1^+$ we define the geodesic $\sigma$ by $\sigma(t) := ke^{Ht}o$, $t \in \mathbb{R}$. Then

$$l(\gamma) = \inf_{x \in X} d(x, \gamma x) \leq \inf_{t > 0} d(\sigma(t), \gamma \sigma(t)) = \inf_{t > 0} d(ke^{Ht}o, knke^{Ht}o) = \inf_{t > 0} d(o, e^{-Ht}ne^{Ht}o) = 0,$$

because $e^{-Ht}ne^{Ht} \to id$ as $t \to \infty$. Furthermore, $d_\gamma$ does not assume the infimum in $X$, because otherwise $\gamma$ would be elliptic. $\square$

We have the following remarkable Jordan decomposition of elements in $G$.

**Theorem 3.7** ([E, Theorem 2.19.24])

For any element $g \in G$ there exists a unique triplet $e, h, u$ in $G$ with the following properties: $e$ is elliptic, $h$ is hyperbolic and $u$ is unipotent, $e$, $h$ and $u$ commute pairwise, and $g = ehu$. Moreover, if $g' \in G$ commutes with $g$, then $g'$ commutes with $e$, $h$ and $u$.

The triplet $e, h, u$ is called the Jordan decomposition of $g$.

Let $G = Ke^{a^+}K$ be a Cartan decomposition. Due to the rich algebraic structure of symmetric spaces, the translation length of an isometry $\gamma \in G$ can be generalised to a vector in $a^+$. If

$$C(\gamma) := \{H(x, \gamma x) \mid x \in X\} \subseteq a^+,$$

where $H(x, \gamma x)$ is the Cartan vector from Definition 2.1, then by Proposition V.2.1. in [P], the closure of $C(\gamma)$ in $a^+$ contains a unique segment $L(\gamma)$ of minimal length. This segment is called the translation vector of $\gamma$. We further have $\|L(\gamma)\| = l(\gamma)$ and $L(\gamma) = L(h)$, if $e, h, u$ is the Jordan decomposition of $\gamma$. In particular, $L(\gamma)$ is trivial if and only if $\gamma$ is elliptic or strictly parabolic.
3.3 Isometries with positive translation length

We will see in the remainder of this section that there is an essential difference between isometries \( \gamma \in G \) with \( l(\gamma) = 0 \) and \( l(\gamma) > 0 \). In the first case, we do not know a priori the accumulation points of an orbit of the cyclic group \( \langle \gamma \rangle \) in \( \overline{X} \). On the other hand, if \( l(\gamma) \) is positive, then Proposition I.2.3 (2) in [P] implies that for all \( x \in X \) the limit of the sequence \( \gamma^j x \) as \( j \to \infty \) exists and is independent of \( x \).

**Definition 3.8** Let \( \gamma \in G \) be an isometry with positive translation length. The limit \( \gamma^+ := \lim_{j \to \infty} \gamma^j o \) is called the attractive fixed point of \( \gamma \). The repulsive fixed point \( \gamma^- \) of \( \gamma \) is defined as the attractive fixed point of \( \gamma^{-1} \).

It will turn out that the fixed points \( \gamma^+ \) and \( \gamma^- \) play a significant role in the study of the dynamics of \( \gamma \). Furthermore, if \( e, h, u \) is the Jordan decomposition of \( \gamma \), then Proposition I.2.3 (2) in [P] and its proof show that \( \gamma^+ = h^+, \gamma^- = h^- \) and \( \gamma^- \in \text{Vis}^\infty(\gamma^+) \).

We fix a Cartan decomposition \( G = Ke^{\mathfrak{a}^+}K \) with respect to \( o \in X \) and the associated Iwasawa decomposition \( G = KAN^+ \). Recall that \( \pi^t \) is the natural projection \( G \to K/M \) introduced at the end of section 2.1, and \( \iota \) the opposition involution defined by (3). The following lemma describes the coordinates of the fixed points \( \gamma^+ \) and \( \gamma^- \in \text{Vis}^\infty(\gamma^+) \).

**Lemma 3.9** Let \( \gamma \in G \) be an isometry with nontrivial translation vector \( L \in \mathfrak{a}^+ \), and \( g \in G \) such that the hyperbolic component \( h \) in the Jordan decomposition of \( \gamma \) belongs to \( ge^{\mathfrak{a}^+}g^{-1} \). If \( k, k' \in K \) are such that \( \pi^t (g) = kM, \pi^t (gm^{-1}_w) = k'M, \) then \( \gamma^+ = (k, L/||L||) \) and \( \gamma^- = (k', \iota(L)/||L||) \).

**Proof.** By the remark after Definition 3.8 it suffices to prove the claim for \( \gamma \) hyperbolic. We first treat the case \( \gamma = e^L \). Then the geodesic \( \sigma \) defined by \( \sigma(t) := e^{tL/||L||}o \) for \( t \in \mathbb{R} \) is invariant under \( \gamma \), and we have \( \gamma^\pm = \sigma(\pm \infty) \).

Furthermore, \( \gamma^+ = \sigma(\infty) = (\text{id}, L/||L||) \), and for \( t > 0 \) we have
\[
\sigma(-t/||L||) = e^{-Lt}o = m^{-1}_w e^{-\Lambda d(mw, L)t}o = m^{-1}_w e^{\iota(L)t}o,
\]
hence \( \gamma^- = \sigma(-\infty) = (m^{-1}_w, \iota(L)/||L||) \).

If \( \gamma = ge^Lg^{-1} \) with \( g \in G \), then \( \gamma^\pm = g\sigma(\pm \infty) \), and the assertion follows from Lemma 2.7. \( \square \)

For \( \gamma \in G \) with \( L(\gamma) \neq 0 \) we put \( F(\gamma) := \{ x \in X \mid \sigma_{x, \gamma^+}(-\infty) = \gamma^- \} \), i.e. \( F(\gamma) \) consists of the union of all parallel geodesics joining \( \gamma^+ \) to \( \gamma^- \). By Proposition 2.11.4 of [E], \( F(\gamma) \) is a complete, totally geodesic submanifold of \( X \). Moreover, if \( e \) denotes the elliptic component in the Jordan decomposition of \( \gamma \), then \( F(\gamma) \cap \text{Fix}(e) \neq \emptyset \).

The following proposition shows that if \( \gamma^+ \) and \( \gamma^- \) are contained in the regular boundary, then \( \gamma \) is axial. Although this fact is probably well-known, we include a geometric proof for the convenience of the reader.

**Proposition 3.10** Let \( \gamma \in G \) be an isometry with \( L(\gamma) \in \mathfrak{a}^+ \setminus \{\text{id}\} \). Then \( \gamma \) is axial.

**Proof.** Let \( e, h, u \) be the Jordan decomposition, and \( \gamma^+, \gamma^- \in \partial X^{\text{reg}} \) the attractive and repulsive fixed point of \( \gamma \). We fix a Cartan decomposition \( G = Ke^{\mathfrak{a}^+}K \) with respect to \( o \in F(\gamma) \cap \text{Fix}(e) \) and \( \gamma^+ \in \partial X^{\text{reg}} \), and the associated Iwasawa decomposition \( G = KAN^+ \). By Proposition 4.1.5 of [E], \( \gamma^+ \) and \( \gamma^- \) are fixed by \( e, h \) and \( u \), in particular \( e, u \in MAN^+ = \text{Stab}_G(\gamma^+) \). Furthermore, \( h = e^{L(\gamma)} \) and \( F(\gamma) = Ao \).
We claim that \( e \in M \) and \( u = \text{id} \). Write \( e = \text{man} \) with \( m \in M \), \( a \in A \) and \( n \in N^+ \).
Since \( o \in \text{Fix}(e) \) we have 
\[
0 = d(o, eo) = d(o, mano) = d(a^{-1} o, no) \geq d(a^{-1} o, o),
\]
which implies \( a = \text{id} \) and \( n = \text{id} \). Hence \( e \in M \).

Next we write \( u = \text{man} \) with \( m \in M \), \( a \in A \) and \( n \in N^+ \), and put \( H := L(\gamma)/||L(\gamma)|| \in a^+_n \). We consider the geodesic \( \sigma = \sigma_o\gamma^+ \) and compute for \( t > 0 \)
\[
d(u\sigma(-t), \sigma(-t)) = d(\text{man}^{-H^t} o, e^{-H^t} o) \geq d(\text{man}^{-H^t} o, \text{mae}^{-H^t} o) - d(\text{mae}^{-H^t} o, e^{-H^t} o) = d(e^{H^t} e^{-H^t} o, o) = d(o, o).
\]
Now if \( n \neq \text{id} \), the righthand side is unbounded as \( t \to \infty \). Therefore \( u\sigma(-\infty) = \sigma(-\infty) \) implies \( n = \text{id} \), hence \( u = ma \). This is impossible if \( u \neq \text{id} \), because the isometry \( ma \) assumes the infimum of its displacement function, but \( u \) doesn’t. \[ \square \]

3.4 Dynamics of axial isometries

For axial isometries, we are able to describe the action on the geometric boundary more precisely. If \( \gamma \in \text{Isom}(X) \) is axial, we call the set
\[
\text{Ax}(\gamma) := \{ x \in X \mid d(x, \gamma x) = l(\gamma) \}
\]
the axis of \( \gamma \). It is invariant under the action of the cyclic group \( \langle \gamma \rangle \), closed, convex, and consists of the union of all geodesics translated by \( \gamma \) (see \( [E, \text{Proposition 1.9.2 (2)}] \)). In particular, \( \text{Ax}(\gamma) = F(\gamma) \), and the set of fixed points \( \text{Fix}(\gamma) \) of \( \gamma \) in \( X \) equals \( \partial \text{Ax}(\gamma) \).

We fix a Cartan decomposition \( G = Ke^{a^+}K \) of \( G = \text{Isom}^o(X) \) with respect to \( o \in X \), and choose \( x \in \text{Ax}(\gamma) \) arbitrary. Then the translation vector of \( \gamma \) is given by
\[
L(\gamma) = H(x, \gamma x) \in \overline{a^+},
\]
where \( H(x, \gamma x) \) denotes the Cartan vector from Definition 2.4. Furthermore, we have the following

**Proposition 3.11** ([E, Proposition 2.19.18 (3) and Corollary 2.19.19])

An isometry \( \gamma \in G \) is axial if and only if \( \gamma \) is conjugate to \( e^H m \), where \( H \in \overline{a^+} \setminus \{0\} \) and \( m \in \{ k \in K \mid \text{Ad}(k)H = H \} \). Moreover, \( H \in \overline{a^+} \) equals the translation vector \( L(\gamma) \) of \( \gamma \).

For the sake of simplicity, we will restrict ourselves here to the action of the following kind of axial isometries. For a more general treatment of arbitrary axial isometries we refer the reader to [L], chapter 5.2.

**Definition 3.12** An isometry \( \gamma \in G \) with translation vector \( L(\gamma) \in a^+ \setminus \{\text{id}\} \) is called a regular axial isometry.

In order to describe the dynamics of regular axial isometries on the geometric boundary \( \partial X \) of \( X \), we introduce an auxiliary distance for the Bruhat visibility sets \( \text{Vis}^B(\xi), \xi \in \partial X^{\text{reg}} \), defined in section 2.3.

Let \( G = Ke^{a^+}K \) be a Cartan decomposition with respect to \( x \in X \) and \( \xi \in \partial X^{\text{reg}} \) arbitrary, and \( G = KAN^+ \) the associated Iwasawa decomposition. Recall that \( N^+ \) is the Lie group exponential of \( n^+ = \sum_{\alpha \in \Sigma^+} g_\alpha \). As mentioned at the end of section 2.3 \( \text{Vis}^B(\xi) \)
can be identified with the submanifold $N^+x$ of $X$. Let $B_\alpha$ denote the scalar product on $n^+$ which equals the scalar product $\mathbf{1}$ on $g_\alpha$, and is zero on $g_\beta$ for $\beta \neq \alpha$. Then the scalar product

$$ds^2_{x,\xi} = \frac{1}{2} \sum_{\alpha \in \Sigma^+} B_\alpha$$

on $n^+$ extends to an $N^+$–invariant metric for the submanifold $N^+x$ of $X$ with associated Riemannian distance $d_{x,\xi}$ on $N^+x \cong \text{Vis}^B(\xi)$. We remark that for $y \in X$, the distance $d_{y,\xi}$ is equivalent to the distance $d_{x,\xi}$ on $\text{Vis}^\infty(\xi)$. Furthermore, since the map $\kappa$ is a diffeomorphism from $N^+$ onto a dense open subset of $K/M$, it follows that the topology induced by the distance $d_{x,\xi}$ on $\text{Vis}^B(\xi) \subset K/M$ is equivalent to the original topology on $K/M$. The following lemma describes how this distance behaves under the action of a regular axial isometry. Recall that $\iota$ is the opposition involution defined by $\mathbf{3}$.

**Lemma 3.13** Let $h$ be a regular axial isometry, $x \in Ax(h)$ and $h^+, h^-$ the attractive and repulsive fixed point of $h$. Fix a Cartan decomposition $G = Ke^{\mathfrak{a}^+}K$ with respect to $x$ and $h^+$, denote $L \in \mathfrak{a}^+$ the translation vector of $h$, and put $\alpha_+ := \min\{\alpha(L/||L||) \mid \alpha \in \Sigma^+\}$, $\alpha_- := \min\{\alpha(L/||L||) \mid \alpha \in \Sigma^+\}$. If $\eta^+ \in \partial Ax(h) \cap \partial X^{reg}$ satisfies $\pi^B(\eta^+) = \pi^B(h^+)$, and $\eta^- := \sigma_{x,\eta^+}(\iota)$, then

$$\forall \xi \in \text{Vis}^\infty(\eta^+) \quad d_{x,\eta^+}(h^{-1}\xi, \eta^-) \leq e^{-\alpha_+||L||}d_{x,\eta^+}(\xi, \eta^-),$$

$$\forall \xi \in \text{Vis}^\infty(\eta^-) \quad d_{x,\eta^-}(h \cdot \xi, \eta^+) \leq e^{\alpha_-||L||}d_{x,\eta^-}(\xi, \eta^+).$$

**Proof.** Let $G = KAn^+$ be the Iwasawa decomposition associated to the given Cartan decomposition. Then by Proposition $\mathbf{3.11}$ there exists $m \in M$ such that $h = e^Lm$. If $\sigma := \sigma_{x,\eta^+}$, then $\xi \in \text{Vis}^\infty(\eta^-)$ implies the existence of $n \in N^+$ such that $\xi = ns\sigma(\iota)$.

Let $\varepsilon > 0$ and $c : [0, 1] \to N^+x$ a curve in the submanifold $N^+x$ with $c(0) = x$, $c(1) = nx$ and

$$\int_0^1 \|\dot{c}(t)\| \, dt < d_{x,\eta^+}(\xi, \eta^-) + \varepsilon.$$ 

For $t \in [0, 1]$, we write $c(t) = n(t)x$ with $n(t) \in N^+$ and put

$$Z(t) := DL_{n(t)^{-1}} \frac{d}{ds} \bigg|_{s=t} n(s) \in n^+.$$ 

Then, by definition of the metric, $\|\dot{c}(t)\|^2 = ds^2_{x,\eta^+}(Z(t), Z(t))$.

Since $h^{-1}$ fixes $\eta^-$, and $h^{-1}\xi$ corresponds to the element $h^{-1}nhx$ in $N^+x$, the curve $c_h(t) := h^{-1}n(t)hx$ joins $x$ to $h^{-1}nhx$, hence

$$d_{x,\eta^+}(h^{-1}\xi, \eta^-) \leq \int_0^1 \|\dot{c}_h(t)\| \, dt.$$ 

Here $\|\dot{c}_h(t)\|^2 = ds^2_{x,\eta^+}(\Ad(h^{-1})Z(t), \Ad(h^{-1})Z(t))$. Since $M$ normalises $N^+$ we have $\Ad(m)Z(t) \in n^+$ for all $t \in [0, 1]$. Furthermore,

$$ds^2_{x,\eta^+}(\Ad(m^{-1})Z(t), \Ad(m^{-1})Z(t)) = ds^2_{x,\eta^+}(Z(t), Z(t))$$ 

because the scalar product $\mathbf{1}$ on $g$ and hence $B_\alpha$, $\alpha \in \Sigma^+$, is invariant by $\Ad(K)$. We conclude

$$ds^2_{x,\eta^+}(\Ad(h^{-1})Z(t), \Ad(h^{-1})Z(t)) = ds^2_{x,\eta^+}(\Ad(e^{-L})Z(t), \Ad(e^{-L})Z(t))$$

$$\leq \max_{\alpha \in \Sigma^+} e^{-2\alpha(L)} ds^2_{x,\eta^+}(Z(t), Z(t)).$$

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Putting \( \alpha_+ := \min\{\alpha(L/||L||) \mid \alpha \in \Sigma^+ \} > 0 \) we summarise

\[
d_{x,\eta^+}(h^{-1}\xi, \eta^-) \leq e^{-\alpha_+||L||} \int_0^1 \|\dot{c}(t)\| \, dt < e^{-\alpha_+||L||} \left( d_{x,\eta^+}(\xi, \eta^-) + \varepsilon \right),
\]

and the first claim follows as \( \varepsilon \) tends to zero.

Concerning the second assertion we remark that \( h^{-1} \) is regular axial with translation vector \( \nu(L) \in a^\perp \). Furthermore, \( \text{Ax}(h^{-1}) = \text{Ax}(h) \), hence the assertion follows from the first claim. \( \square \)

For the following important corollary we fix a Cartan decomposition \( G = K e^{a^\perp} K \).

**Corollary 3.14** Let \( h \) be a regular axial isometry and \( \sigma \subset \text{Ax}(h) \) a regular geodesic with \( \pi^B(\sigma(\infty)) = \pi^B(h^+), \pi^B(\sigma(-\infty)) = \pi^B(h^-) \). Then

\[
\forall \xi \in \text{Vis}^\infty(\sigma(+\infty)) \quad \lim_{j \to \infty} h^{-j} \xi = \sigma(-\infty),
\]

\[
\forall \xi \in \text{Vis}^\infty(\sigma(-\infty)) \quad \lim_{j \to \infty} h^j \xi = \sigma(+\infty).
\]

In particular, for all \( \xi \in \partial X^{reg} \) with \( \pi^B(\xi) \in \text{Vis}^B(h^-) \) the sequence \( \pi^B(h^j \xi) \) converges to \( \pi^B(h^+) \) in \( K/M \) as \( j \to \infty \).

### 3.5 Dynamics of strictly parabolic isometries

We are finally going to study nonelliptic isometries with zero translation length. If \( X \) is a rank one symmetric space, and \( \gamma \) a parabolic isometry of \( X \), then \( \gamma \) fixes a unique point \( \eta \in \partial X \). Moreover, for all \( z \in X \) we have \( \gamma^j z \to \eta \) and \( \gamma^{-j} z \to \eta \) as \( j \to \infty \). Unfortunately, this is far from being true in higher rank symmetric spaces. We only know that if \( \gamma \) is parabolic with fixed point \( \eta \) as in Proposition 3.3, then the set of accumulation points of the cyclic group \( \langle \gamma \rangle \) is contained in the boundary of every horosphere centred at \( \eta \). Furthermore, if \( \gamma \) is mixed parabolic then, as pointed out at the beginning of section 3.3, we have

\[
\lim_{j \to \infty} \gamma^j x = \gamma^+ \quad \text{and} \quad \lim_{j \to \infty} \gamma^{-j} x = \gamma^- \quad \forall x \in X.
\]

In this section we are going to describe the dynamics of strictly parabolic isometries on the geometric boundary.

**Proposition 3.15** Let \( \gamma \) be a strictly parabolic isometry, and \( \eta \in \partial X \) a fixed point of \( \gamma \). Then either \( \gamma \) fixes a point in \( \text{Vis}^\infty(\eta) \), or for any compact subset \( C \subset \text{Vis}^\infty(\eta) \) there exists an integer \( N \in \mathbb{N} \) such that

\[
\gamma^j C \cap C = \gamma^{-j} C \cap C = \emptyset \quad \forall j \geq N.
\]

**Proof.** Let \( e, h, u \) be the Jordan decomposition of \( \gamma \). Since \( \gamma \) is strictly parabolic, we have \( h = \text{id} \). Fix a Cartan decomposition \( G = K e^{a^\perp} K \) with respect to \( o \in \text{Fix}(e) \) and \( \eta \in \partial X \), and let \( G = KAN^+ \) be the associated Iwasawa decomposition. Then \( u \in N^+ \) and \( e \) fixes \( \eta \). If

\[
N^\eta := \{ g \in G \mid \lim_{t \to \infty} d(g_{\rho_0,\eta}(t), \rho_{\sigma_0,\eta}(t)) = 0 \} \subseteq N^+
\]
is the horospherical subgroup associated to \( \eta \), we have \( \text{Vis}^\infty(\eta) = N^+_\eta \cdot \sigma_{o,\eta}(-\infty) \). Furthermore, if \( H \in \mathfrak{a}^+_\eta \) is the Cartan projection of \( \eta \), its Lie algebra is given by

\[
\mathfrak{n}^+_\eta := \sum_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha \subseteq \mathfrak{n}^+,
\]

and the exponential map \( \exp : \mathfrak{n}^+_\eta \to N^+_\eta \) is a diffeomorphism. We endow \( \mathfrak{n}^+_\eta \) with the norm \( \| \cdot \| \) associated to the restriction of the scalar product \( \langle \cdot, \cdot \rangle \) to \( \mathfrak{n}^+ \), and for \( R > 0 \) we put \( \mathfrak{n}_\eta^+(R) := \{ z \in \mathfrak{n}^+_\eta \mid \| z \| < R \} \). Then, given a compact set \( C \subset \text{Vis}^\infty(\eta) \), there exists \( R > 0 \) such that

\[
C \subset \exp \left( \mathfrak{n}^+_\eta(R) \right) \cdot \sigma_{o,\eta}(-\infty).
\]

Let \( \xi \in \text{Vis}^\infty(\eta) \) arbitrary and \( Z \in \mathfrak{n}^+_\eta \) such that \( \xi = \exp(Z) \cdot \sigma_{o,\eta}(-\infty) \). We write

\[
u = \exp(\sum_{\alpha \in \Sigma^+} Y_\alpha) \quad \text{and} \quad Z = \sum_{\alpha \in \Sigma^+} Z_\alpha \quad \text{with} \quad Y_\alpha, Z_\alpha \in \mathfrak{g}_\alpha.
\]

If \( Y_\alpha = 0 \) for all \( \alpha \in \Sigma^+ \) with \( \alpha(H) > 0 \), then \( \text{ad}(H)Y_\alpha = 0 \) for all \( \alpha \in \Sigma^+ \), hence \( e^{Ht}ue^{-Ht} = u \) for all \( t \in \mathbb{R} \). Since \( e \) fixes \( \sigma_{o,\eta} \) pointwise, we conclude

\[
d(\sigma_{o,\eta}(-t), \gamma_{o,\eta}(-t)) = d(\sigma_{o,\eta}(-t), u_{\gamma_{o,\eta}(-t)}) = d(o, e^{Ht}u e^{-Ht}o) = d(o, uo)
\]

for all \( t \in \mathbb{R} \). In particular, \( \gamma \) fixes \( \sigma_{o,\eta}(-\infty) \).

So if \( \gamma \) does not fix a point in \( \text{Vis}^\infty(\eta) \), there exists \( \alpha \in \Sigma^+ \) such that \( \alpha(H) > 0 \) and \( Y_\alpha \neq 0 \). We may therefore choose \( \beta \in \Sigma^+ \) such that \( \beta(H) > 0 \), \( Y_\beta \neq 0 \) and (3) \( \beta(H) \leq \alpha(H) \) for all \( \alpha \in \Sigma^+ \) with \( \alpha(H) > 0 \) and \( Y_\alpha \neq 0 \). For \( j \in \mathbb{N} \) we write

\[
u^{\pm j} \exp(Z) = \exp(Y^{(\pm j)}) = \exp(\sum_{\alpha \in \Sigma^+} Y^{(\pm j)}_{\alpha}) \quad \text{with} \quad Y^{(\pm j)}_{\alpha} \in \mathfrak{g}_\alpha.
\]

Then \([\mathfrak{g}_\alpha, \mathfrak{g}_{\alpha'}] \subseteq \mathfrak{g}_{\alpha + \alpha'} \) for all \( \alpha, \alpha' \in \Sigma^+ \) and the Campbell Hausdorff formula imply

\[
Y^{(\pm j)}_{\beta} = \pm jY_{\beta} + Z_{\beta} + X_{\beta},
\]

where \( X_{\beta} \in \mathfrak{g}_\beta \) is a term consisting of successive Lie brackets of the \( Z_{\alpha} \). In particular, \( \| X_{\beta} \| \) is bounded, and therefore \( \| Y^{(j)}_{\beta} \| \) and \( \| Y^{(-j)}_{\beta} \| \) tend to infinity as \( j \to \infty \). Since root spaces associated to different roots in \( \Sigma^+ \) are orthogonal with respect to the scalar product \( \langle \cdot, \cdot \rangle \) (see [He, Theorem III.4.2 (iii)]), we have

\[
\| Y^{(j)} \| = \sqrt{\sum_{\alpha \in \Sigma^+} \| Y^{(j)}_{\alpha} \|^2} \geq \| Y^{(j)}_{\beta} \| \geq j\| Y_{\beta} \| - \| Z_{\beta} \| - \| X_{\beta} \|.
\]

Now by \( \text{Ad}(K) \)-invariance of the scalar product \( \langle \cdot, \cdot \rangle \) and \( e \in K \) we obtain \( \| \text{Ad}(e^{\pm j})Y^{(j)} \| = \| Y^{(j)} \| \geq \| Y^{(j)}_{\beta} \| \). Hence if \( j > (R + \| Z_{\beta} \| + \| X_{\beta} \|)/\| Y_{\beta} \| \), we deduce

\[
\gamma^{\pm j} \exp(Z)\sigma_{o,\eta}(-\infty) = \exp(\text{Ad}(e^{\pm j})Y^{(j)}_{\beta})\sigma_{o,\eta}(-\infty) \notin C,
\]

since \( e \) commutes with \( u \) and fixes \( \sigma_{o,\eta} \). By compactness of \( C \) there exists \( N \in \mathbb{N} \) such that for any \( \xi \in C \) and all \( j \geq N \) we have \( \gamma^j \xi \notin C \) and \( \gamma^{-j} \xi \notin C \). \( \Box \)

For the following kind of parabolic isometries we have similar dynamics on the geometric boundary as in rank one symmetric spaces. They will also play an important role for the construction of free groups in the following section.
**Definition 3.16** An isometry $\gamma \in G \setminus \{\text{id}\}$ is called generic parabolic, if $\gamma$ is strictly parabolic and possesses a unique fixed point in each $G$–invariant subset of the regular geometric boundary $\partial X^{reg}$.

The following lemma will be convenient in the sequel.

**Lemma 3.17** If $\gamma$ is generic parabolic and $e, h, u$ its Jordan decomposition, then $h = \text{id}$, $e$ is conjugate to an element in $M$, and $u$ is conjugate to an element $n \in N^+$ with the property $n \notin m_wN^+m_w^{-1}$ for all nontrivial Weyl group elements $w \in W \setminus \{\text{id}\}$.

**Proof.** Let $e, h, u$ be the Jordan decomposition and $\eta \in \partial X^{reg}$ a fixed point of $\gamma$. Since $\gamma$ is strictly parabolic, we have $l(\gamma) = l(h) = 0$, hence $h = \text{id}$. Furthermore, Proposition 4.1.5 of [E] implies that $\eta$ is fixed by both $e$ and $u$. Let $G = Ke^\alpha K$ be a Cartan decomposition with respect to $o \in \text{Fix}(e)$ and $\eta \in \partial X^{reg}$, and $G = KAN^+$ the associated Iwasawa decomposition. Then $u \in N^+$ and $e \in MAN^+$ = $\text{Stab}_G(\eta)$. We write $e = man$ with $m \in M$, $a \in A$ and $n \in N^+$. Since $o \in \text{Fix}(e)$ we have

$$0 = d(o, eo) = d(o, mano) = d(a^{-1}o, no) \geq d(a^{-1}o, o)$$

which implies $a = \text{id}$ and $n = \text{id}$. Hence $e \in M$.

Next suppose there exists $w \in W \setminus \{\text{id}\}$ such that $u \in m_wN^+m_w^{-1}$. Then $m_w\eta \neq \eta$ and, since $N^+$ fixes $\eta$, we have $um_w\eta = m_w\eta$. Now $e \in M$ implies $em_w\eta = m_w\eta$, i.e. $m_w\eta \in \gamma \eta$ is fixed by $\gamma = eu$, in contradiction to the fact that $\eta$ is the unique fixed point of $\gamma$ in $G \cdot \eta$. $\square$

Recall the definition of the map $\pi$ from section 2.2.

**Proposition 3.18** If $\gamma$ is generic parabolic and $\eta \in \partial X^{reg}$ a fixed point of $\gamma$, then for any $\xi \in G \cdot \eta$ we have

$$\lim_{j \to \infty} \gamma^j \xi = \lim_{j \to \infty} \gamma^{-j} \xi = \eta.$$ 

**Proof.** Let $e, h = \text{id}, u$ be the Jordan decomposition of $\gamma$. We fix a Cartan decomposition $G = Ke^aK$ with respect to $o \in \text{Fix}(e)$ and $\eta \in \partial X^{reg}$, and the associated Iwasawa decomposition $G = KAN^+$. Denote by $H \in a_1^+$ the Cartan projection of $\eta$ and let $V \subset K/M$ be an open neighbourhood of $\pi^B(\eta)$, for $w \in W$ we denote $\|\cdot\|_w$ the norm on $u_w := n^+ \cap \text{Ad}(m_w)n^-$ associated to the restriction of the scalar product $\langle \|\cdot\| \rangle$ to $u_w$. Then by the Bruhat decomposition there exists $R > 0$ such that with $U_w(R) := \{\exp Z \mid Z \in u_w \text{ with } \|Z\|_w \leq R\}$ we have

$$K/M \setminus \pi\left( \bigcup_{w \in W \setminus \{\text{id}\}} U_w(R)m_wP \right) \subset V.$$ 

Hence it suffices to prove that for all $\xi \in \partial X^{reg}$ and $j$ sufficiently large

$$\pi^B(\gamma^j \xi) \in K/M \setminus \pi\left( \bigcup_{w \in W \setminus \{\text{id}\}} U_w(R)m_wP \right).$$

Let $\xi \in \partial X^{reg}$ arbitrary. If $\pi^B(\xi) = \pi^B(\eta)$, there is nothing to prove. If $\pi^B(\xi) \neq \pi^B(\eta)$, there exists $w \in W \setminus \{\text{id}\}$ and $Z \in u_w$ such that $\pi^B(\xi) \in \pi(\exp(Z)m_wP)$. As in the proof of the previous proposition we write

$$u = \exp\left( \sum_{\alpha \in \Sigma^+} Y_\alpha \right) \quad \text{and} \quad Z = \sum_{\alpha \in \Sigma^+} Z_\alpha \quad \text{with} \quad Y_\alpha, Z_\alpha \in g_\alpha.$$
Since $u \not\in m_w N^+ m_w^{-1}$ there exists $\alpha \in \Sigma^+$ such that $Y_\alpha \neq 0$ and $\text{Ad}(m_w^{-1}) Y_\alpha \in n^-$, i.e. $Y_\alpha \in u_w$. Hence we may choose $\beta \in \Sigma^+$ such that $\beta(H) \leq \alpha(H)$ for all $\alpha \in \Sigma^+$ with $Y_\alpha \neq 0$ and $Y_\alpha \in u_w$. For $j \in \mathbb{N}$, we write

$$u^{\pm j} \exp(Z) = \exp(Y^{(\pm j)}) = \exp\left( \sum_{\alpha \in \Sigma^+} Y^{(\pm j)}_\alpha \right) \quad \text{with} \quad Y^{(\pm j)}_\alpha \in \mathfrak{g}_\alpha.$$ 

Then $[\mathfrak{g}_\alpha, \mathfrak{g}_{\alpha'}] \subseteq \mathfrak{g}_{\alpha + \alpha'}$ for all $\alpha, \alpha' \in \Sigma^+$ and the Campbell Hausdorff formula imply

$$Y^{(\pm j)}_\beta = \pm j Y_\beta + Z_\beta + X_\beta,$$

where $X_\beta \in \mathfrak{g}_\beta$ is a term consisting of successive Lie brackets of the $Z_\alpha$. In particular, $\|X_\beta\|_w$ is bounded, and therefore $\|Y^{(j)}_\beta\|_w$ and $\|Y^{(\pm j)}_\beta\|_w$ tend to infinity as $j \to \infty$. As before, we obtain by the $\text{Ad}(K)$-invariance of the scalar product $\|\text{Ad}(e^{\pm j}) Y^{(\pm j)}\|_w = \|Y^{(\pm j)}\|_w \geq 1$ for $j \to \infty$. Hence for $j \in \mathbb{N}$ sufficiently large we have $\|\text{Ad}(e^{\pm j}) Y^{(\pm j)}\|_w = \|Y^{(\pm j)}\|_w > R$ and the claim follows from $\pi^B(\gamma^{(\pm j)} \xi) = \pi^B(\exp(\text{Ad}(e^{\pm j}) Y^{(\pm j)})) m_w P)$. $\square$

### 3.6 Construction of free groups

We will now apply the results of the previous sections to produce Schottky groups, an interesting kind of free and discrete isometry groups of infinite covolume. Their construction is based on the following

**Lemma 3.19 (Klein’s Criterion) (see [Ha])**

Let $G$ be a group acting on a set $S$, $\Gamma_1, \Gamma_2$ two subgroups of $G$, where $\Gamma_1$ contains at least three elements, and let $\Gamma$ be the subgroup they generate. Assume that there exist two nonempty subsets $S_1, S_2$ in $S$ with $S_2$ not included in $S_1$ such that $\gamma(S_2) \subseteq S_1$ for all $\gamma \in \Gamma_1 \setminus \{\text{id}\}$ and $\gamma(S_1) \subseteq S_2$ for all $\gamma \in \Gamma_2 \setminus \{\text{id}\}$. Then $\Gamma$ is isomorphic to the free product $\Gamma_1 * \Gamma_2$.

For the remainder of this section we fix a Cartan decomposition $G = K\exp(n)$ with respect to the base point $o \in X$. Recall from the remark following Definition 2.5 that a finite intersection of sets $\text{Vis}^B(\xi_i) \subset K/M$, $\xi_i \in \partial X^{reg}$, is a dense and open subset of $K/M$. The following theorem describes a new geometric construction of finitely generated free groups containing parabolic isometries.

**Theorem 3.20** Let $X$ be a globally symmetric space of noncompact type, and $\{\xi_1, \xi_2, \ldots, \xi_{2l+1}, \ldots, \xi_{2l+p}\} \subset \partial X^{reg}$ a set of $2l + p$ points such that

$$\pi^B(\xi_i) \in \bigcap_{n=1}^{2l+p} \text{Vis}^B(\xi_n) \quad \forall i \in \{1, 2, \ldots, 2l + p\}.$$ 

Then there exist regular axial isometries $\gamma_1, \gamma_2, \ldots, \gamma_l$ with $\gamma_m^+ = \xi_{2m}$ and $\pi^B(\gamma_{m}^-) = \pi^B(\xi_{2m-1})$ for $1 \leq m \leq l$, and generic parabolic isometries $\gamma_{l+1}, \gamma_{l+2}, \ldots, \gamma_{l+p}$ with fixed points $\xi_{2l+1}, \xi_{2l+2}, \ldots, \xi_{2l+p}$ respectively. Furthermore, there exist pairwise disjoint open
neighbourhoods \( U_i \subset K/M \) of \( \pi^B(\xi_i) \), \( 1 \leq i \leq 2l + p \), such that with \( C := \bigcup_{i=1}^{2l+p} U_i \) we have
\[
\gamma_m \left( C \setminus U_{2m-1} \right) \subset U_{2m} \quad \text{and} \quad \gamma_m \left( C \setminus U_{2m} \right) \subset U_{2m-1}
\quad \text{for} \quad 1 \leq m \leq l,
\gamma_m \left( K/M \setminus U_m \right) \subset U_{m+1} \quad \text{and} \quad \gamma_m^{-1} \left( K/M \setminus U_m \right) \subset U_{m+1}
\quad \text{for} \quad l + 1 \leq m \leq l + p.
\]

In particular, the finitely generated group \( \langle \gamma_1, \gamma_2, \ldots, \gamma_l, \gamma_{l+1}, \ldots, \gamma_{l+p} \rangle \subset \text{Isom}^0(X) \) is free and discrete.

**Proof.** For \( 1 \leq i \leq 2l + p \) we choose open neighbourhoods \( U_i \subset K/M \) of \( \pi^B(\xi_i) \) such that
\[
\bigcup_{n=1}^{2l+p} \bigcap_{n \neq i} \text{Vis}^B(\xi_n), \quad \text{and put} \quad C := \bigcup_{i=1}^{2l+p} U_i \subset K/M.
\]

Since for any \( m \in \{1, 2, \ldots, l\} \) we have \( \xi_{2m} \in \partial X^{reg} \) and \( \pi^B(\xi_{2m-1}) \in \text{Vis}^B(\xi_{2m}) \), there exist regular unit speed geodesics \( \sigma_m : \mathbb{R} \to X \) such that \( \sigma_m(\infty) = \xi_{2m} \) and \( \pi^B(\sigma_m(-\infty)) = \pi^B(\xi_{2m-1}) \). For \( 1 \leq m \leq l \) let \( h_m \) be a regular axial isometry with \( h_m \sigma_m(t) = \sigma_m(t+1) \) for all \( t \in \mathbb{R} \). In particular, \( h_m \) possesses the attractive and repulsive fixed points \( h^+_m = \sigma_m(\infty) = \xi_{2m} \) and \( h^-_m = \sigma_m(-\infty) \). By Corollary 3.14 and the fact that \( C \setminus U_{2m-1} \subset \text{Vis}^B(\xi_{2m-1}) \), \( C \setminus U_{2m} \subset \text{Vis}^B(\xi_{2m}) \) are compact, there exists \( k_m \in \mathbb{N} \) such that for all \( j \geq k_m \) we have \( h^+_m(U_{2m-1}) \subset U_{2m} \) and \( h^-_m(U_{2m}) \subset U_{2m-1} \). Putting \( \gamma_m := h^+_m \), we obtain the desired regular axial isometries and the corresponding open neighbourhoods.

Similarly, for \( 1 \leq m \leq p \) we choose a generic parabolic isometry \( u_m \) in the horospherical subgroup associated to \( \xi_{l+m} \). By Proposition 3.18 there exists \( k_m \in \mathbb{N} \) such that for all \( j \geq k_m \) we have
\[
u^+_m(U_{l+m}) \subset U_{l+m} \quad \text{and} \quad \nu^-_m(U_{l+m}) \subset U_{l+m}.
\]

Putting \( \gamma_{l+m} := u^+_{l+m} \), we obtain the desired generic parabolic isometries with corresponding open neighbourhoods \( U_{l+1}, \ldots, U_{l+p} \).

In order to apply Klein’s Criterion, we put \( S_1 := U_1 \cup U_2 \) and \( S_2 := U_3 \cup U_4 \). Since \( S_2 \subset C \setminus (U_1 \cup U_2) \) we have \( \langle \gamma_1 \rangle \cdot S_2 \subset S_1 \). Similarly, \( S_1 \subset C \setminus (1 \cup U_4) \) implies \( \langle \gamma_2 \rangle \cdot S_1 \subset S_2 \). Hence the group generated by \( \gamma_1 \) and \( \gamma_2 \) is free by Klein’s Criterion.

For \( i \in \{1, \ldots, l+p\} \) we denote \( \Gamma_i \) the group generated by the elements \( \gamma_m \) for \( m \leq i \). If \( 2 \leq i \leq l+1 \) we put \( S_i := \bigcup_{n=1}^{i} (U_{2n+1} \cup U_{2n}) \) and \( S_{i+1} := U_{2i+1} \cup U_{2i+2} \). Since
\[
S_{i+1} \subset C \setminus \bigcup_{n=1}^{2i} U_n
\]
we have \( \gamma \cdot S_{i+1} \subset S_i \) for all \( \gamma \in \Gamma_i \setminus \{\text{id}\} \). From \( S_i' \subset C \setminus (U_{2i+1} \cup U_{2i+2}) \) we further obtain \( \langle \gamma_{i+1} \rangle \cdot S_i' \subset S_{i+1} \), hence the group \( \Gamma_{i+1} \) generated by the elements \( \gamma_m \) for \( m \leq i+1 \), is free. We conclude inductively that \( \Gamma_i := \langle \gamma_1, \gamma_2, \ldots, \gamma_i \rangle \) is free.

We next consider the sets \( S_i' \) and \( U_{2i+1} \). Since \( S_i' \subset K/M \setminus U_{2i+1} \) and
\[
U_{2i+1} \subset C \setminus \bigcup_{n=1}^{2i} U_n
\]
we have \( \langle \gamma_{l+1} \rangle \cdot S'_l \subset U_{2l+1} \) and \( \gamma \cdot U_{2l+1} \subset S'_l \) for all \( \gamma \in \Gamma_l \setminus \{ \text{id} \} \), hence the group \( \Gamma_{l+1} \) is free.

For \( m \in \{2, \ldots, p-1\} \) we put \( S'_{l+m} := \bigcup_{n=1}^{2l+m} U_n \). Again from \( S'_{l+m} \subset K/M \setminus U_{2l+m+1} \) and

\[
U_{2l+m+1} \subset C \setminus \bigcup_{n=1}^{2l+m} U_n
\]

we obtain \( \langle \gamma_{l+m+1} \rangle \cdot S'_{l+m} \subset U_{2l+m+1} \) and \( \gamma \cdot U_{2l+m+1} \subset S'_{l+m} \) for all \( \gamma \in \Gamma_{l+m} \setminus \{ \text{id} \} \). We conclude inductively that \( \langle \gamma_1, \gamma_2, \ldots, \gamma_{l+p} \rangle \) is free.

Finally suppose \( \Gamma := \langle \gamma_1, \gamma_2, \ldots, \gamma_{l+p} \rangle \) is not discrete. Then there exists a sequence \( (h_j) \subset \Gamma \) converging to the identity. For \( j \in \mathbb{N} \) we write \( h_j := s_{k_j}^{(j)} s_{k_j-1}^{(j)} \ldots s_1^{(j)} \) as a reduced word, i.e. \( s_{m}^{(j)} \in \{ \gamma_1, \gamma_1^{-1}, \gamma_2, \gamma_2^{-1}, \ldots, \gamma_{l+p}, \gamma_{l+p}^{-1} \} \) for \( 1 \leq m \leq k_j \), and \( s_{m+1}^{(j)} \neq (s_{m}^{(j)})^{-1} \) for \( 1 \leq m \leq k_j - 1 \). Passing to a subsequence if necessary, we may assume that \( s_{k_j}^{(j)} = s \) and \( s_1^{(j)} = s' \) for all \( j \in \mathbb{N} \). We denote \( U, U' \in \{ U_1, U_2, \ldots, U_{2l}, \ldots, U_{2l+p} \} \) the corresponding neighborhoods of the (attractive) fixed point of \( s, s' \). Let \( \eta \in \partial X^{\text{reg}} \) such that \( \pi^B(\eta) \in C \setminus (\overline{U} \cup \overline{U'}) \). Then \( \pi^B(s_1^{(j)} \eta) \in U' \), hence by the dynamics of the generators of \( \Gamma \) the point \( \pi^B(h_j \eta) \) is contained in \( U \) for all \( j \in \mathbb{N} \). On the other hand, since \( h_j \) converges to the identity, \( \pi^B(h_j \eta) \) converges to \( \pi^B(\eta) \) in \( C \setminus (\overline{U} \cup \overline{U'}) \), a contradiction. \( \square \)

## 4 Discrete isometry groups

In this section we define the geometric limit set of a discrete isometry group of a globally symmetric space \( X \) of noncompact type. We extend the familiar notion of “nonelementary” groups from rank one to higher rank symmetric spaces. We will then describe the structure of the limit set of such nonelementary groups using Theorem 4.2 as our main tool.

In this section, \( X \) will again denote a globally symmetric space of noncompact type with base point \( o \in X \), and \( G = \text{Isom}^o(X) \) the connected component of the identity.

### 4.1 The limit set

**Definition 4.1** A subgroup \( \Gamma \subset G \) is called discrete, if it has a discrete orbit in \( X \). In this case, the geometric limit set \( L_\Gamma \) of \( \Gamma \) is defined by \( L_\Gamma := \Gamma \cdot o \cap \partial X \).

We remark that this definition can be extended to isometry groups of arbitrary Hadamard manifolds. Furthermore, the geometric limit set does not depend on the chosen base point \( o \).

Due to the rich algebraic structure of symmetric spaces, we can consider various sets describing asymptotic properties of a discrete isometry group \( \Gamma \). In order to do so, we fix a Cartan decomposition \( G = K e^{\mathfrak{a}^+} K \) with respect to \( o \in X \). Recall that the map \( \pi^B : \partial X^{\text{reg}} \to K/M \) denotes the projection introduced in section 2.3.

**Definition 4.2** We call the set of regular limit points \( L^{\text{reg}}_\Gamma := L_\Gamma \cap \partial X^{\text{reg}} \) the regular limit set, and the projection \( K_\Gamma := \pi^B(L^{\text{reg}}_\Gamma) \subset K/M \) the transversal limit set of \( \Gamma \). The set of Cartan projections of all points in the geometric limit set \( L_\Gamma \) of \( \Gamma \) is called the directional limit set \( P_\Gamma \subset \mathfrak{a}_1^{\perp} \), the limit cone \( \ell_\Gamma \subset \mathfrak{a}_1^{\perp} \) of \( \Gamma \) is defined by

\[
\ell_\Gamma := \left\{ L(\gamma)/\|L(\gamma)\| \mid \gamma \in \Gamma \text{ regular axial} \right\}.
\]
The following lemma is well-known and remains true for discrete isometry groups of Hadamard manifolds. We include the proof for the convenience of the reader.

**Lemma 4.3** Let $\Gamma \subset G$ be a discrete group and $h \in \Gamma$ axial. If $\varphi \in \Gamma$ fixes $h^+$, then $\varphi$ commutes with a power of $h$.

**Proof.** Let $x \in \text{Ax}(h)$ and $l > 0$ the translation length of $h$. Then for all $n \in \mathbb{N}$

$$d(x, h^{-n} \varphi h^n x) = d(h^n x, \varphi h^n x) = d(\sigma_{x,h^+} (nl), \varphi \sigma_{x,h^+} (nl))$$

is bounded from above by a constant $r \geq 0$ because $\varphi$ fixes $h^+$. Since $\#(\Gamma x \cap B_x(r))$ is finite by discreteness of $\Gamma$, we conclude that

$$h^{-n} \varphi h^n = h^{-m} \varphi h^m$$

for integers $m > n$, i.e. $\varphi$ commutes with $h^{m-n} \neq \text{id}$. \hfill $\square$

### 4.2 Nonelementary groups

We are now going to generalise to symmetric spaces $X$ of higher rank the notion of “nonelementary groups” familiar in the context of isometry groups of real hyperbolic spaces. Let $G = Ke^{B^+}K$ be a Cartan decomposition with respect to $o \in X$.

**Definition 4.4** A discrete subgroup $\Gamma$ of the isometry group $\text{Isom}^o(X)$ is called **nonelementary** if $L^{\text{reg}}_\Gamma \neq \emptyset$ and if for all $\xi, \eta \in L^{\text{reg}}_\Gamma$ we have

$$\pi^B(\Gamma \cdot \xi) \cap \text{Vis}^B(\eta) \neq \emptyset.$$ 

Otherwise $\Gamma$ is called **elementary**.

Notice that an abelian discrete group $\Gamma \subset \text{Isom}^o(X)$ of axial isometries is elementary, because its limit set is contained in the boundary of the invariant maximal flats. Hence $\Gamma \cdot \xi = \xi$ for every $\xi \in L^{\text{reg}}_\Gamma$ which implies $\pi^B(\Gamma \cdot \xi) = \pi^B(\xi) \notin \text{Vis}^B(\xi)$. The same argument shows that a discrete group $\Gamma \subset \text{Isom}^o(X)$ is elementary, if it is contained in the stabiliser of a regular limit point.

The following lemma provides a first example of nonelementary groups:

**Lemma 4.5** If $\Gamma \subset G$ is a discrete group containing $l \geq 2$ regular axial isometries $\gamma_1, \gamma_2, \ldots, \gamma_l$ with

$$\pi^B(\gamma_i^+), \pi^B(\gamma_i^-) \in \bigcap_{\substack{m=1 \atop m \neq i}}^l \left( \text{Vis}^B(\gamma_m^+) \cap \text{Vis}^B(\gamma_m^-) \right) \quad \text{for} \quad 1 \leq i \leq l, \quad \text{and}$$

$$K_\Gamma \subseteq \bigcup_{i=1}^l \left( \text{Vis}^B(\gamma_i^+) \cup \text{Vis}^B(\gamma_i^-) \right), \quad (6)$$

then $\Gamma$ is nonelementary.

In particular, free groups generated by $l \geq 2$ regular axial isometries $\gamma_1, \ldots, \gamma_l$ as in Theorem 3.20 and the additional property (6) are nonelementary.
Proof. We denote $\Gamma' := \langle \gamma_1, \gamma_2, \ldots, \gamma_l \rangle$ and let $U_1, U_2, \ldots, U_{2l} \subset K/M$ be pairwise disjoint open sets as in Theorem 3.20. Choose $\xi, \eta \in L^{reg}$ arbitrary. Then by (6) there exist $g, h \in \{\gamma_1, \gamma_1^{-1}, \ldots, \gamma_l, \gamma_l^{-1}\}$ such that $\pi^B(\xi) \in \text{Vis}^B(g^+) \text{ and } \pi^B(\eta) \in \text{Vis}^B(h^+)$. If $g \neq h$, then $\pi^B(g^-) \in \text{Vis}^B(h^-)$. Since $\text{Vis}^B(h^-)$ is open, and $\pi^B(g^{-j}\xi)$ converges to $\pi^B(g^-)$ as $j \to \infty$ by Corollary 3.14, there exists $n \in \mathbb{N}$ such that $\pi^B(g^{-n}\xi) \in \text{Vis}^B(h^-)$. Now $\pi^B(h^+) \in \text{Vis}^B(\eta)$ and the same argument as before imply the existence of $m \in \mathbb{N}$ such that $\pi^B(h^mg^{-n}\xi) \in \text{Vis}^B(\eta)$.

If $g = h$, we choose $\gamma \in \{\gamma_1, \gamma_1^{-1}, \ldots, \gamma_l, \gamma_l^{-1}\} \setminus \{g, g^{-1}\}$. Then we have $\pi^B(\xi) \in \text{Vis}^B(g^+), \pi^B(g^-) \in \text{Vis}^B(g^-), \pi^B(\gamma^+) \in \text{Vis}^B(g^-) \text{ and } \pi^B(\gamma^+) \in \text{Vis}^B(\eta)$. Therefore Corollary 3.14 implies the existence of integers $n, m \in \mathbb{N}$ such that $\pi^B(g^{-n}\xi) \in \text{Vis}^B(\gamma^-), \pi^B(\gamma^mg^{-n}\xi) \in \text{Vis}^B(g^-)$ and finally $\pi^B(g^k\gamma^mg^{-n}\xi) \in \text{Vis}^B(\eta)$. $\square$

We remark that condition (6) is not very restrictive, because the sets $\text{Vis}^B(\gamma_i^+) \text{ are dense and open in } K/M$. Moreover, the same arguments as in the proof above yield the following lemma:

**Lemma 4.6** Let $\Gamma \subset G$ be a discrete group containing a generic parabolic isometry $u$ with fixed point $\eta \in \partial X^{reg}$. If $\Gamma$ further contains a regular axial isometry $h$ such that $\pi^B(\eta) \in \text{Vis}^B(h^+) \cap \text{Vis}^B(h^-)$, or a second generic parabolic isometry with fixed point $\zeta \in \partial X^{reg}$ and $\pi^B(\zeta) \in \text{Vis}^B(\eta)$, then $\Gamma$ is nonelementary.

In particular, free groups $\langle \gamma_1, \ldots, \gamma_l, \gamma_{l+1}, \ldots, \gamma_{l+p} \rangle$ as in Theorem 3.20 with at least two generators and $p \geq 1$ are nonelementary.

We are finally able to justify our choice of the notion “nonelementary”. The following lemmata state that for torsion free discrete isometry groups of rank one symmetric spaces, our definition coincides with the familiar one.

**Lemma 4.7** If $\text{rank}(X) = 1$, then a discrete isometry group $\Gamma \subset \text{Isom}^o(X)$ is nonelementary if it possesses infinitely many limit points.

**Proof.** Since $\text{rank}(X) = 1$, we have $\partial X^{reg} = \partial X$ by convention, $\partial X$ is homeomorphic to $\pi^B(\partial X)$, and $\partial X = \text{Vis}^\infty(\zeta) \cup \{\zeta\}$ for any point $\zeta$ in the geometric boundary.

Suppose $\Gamma \subset G = \text{Isom}^o(X)$ possesses infinitely many limit points, and assume there exist $\xi, \eta \in L_G$ such that $\Gamma \xi \cap \text{Vis}^\infty(\eta) = \emptyset$. Then $\gamma \xi = \eta$ for all $\gamma \in \Gamma$, in particular $\xi = \eta$. This implies that every element in $\Gamma$ fixes $\xi$. Let $\Gamma' \subseteq \Gamma$ be a torsion free subgroup of finite index which exists by Selberg’s Lemma. Since $\Gamma'$ does not contain elliptic elements, $\Gamma'$ contains only parabolic and axial isometries which all fix $\xi$. By Lemma 4.8 the set of axial elements in $\Gamma'$ must all have the same axis. We conclude that $\Gamma'$ possesses at most two limit points, hence $\Gamma$ possesses only finitely many limit points, a contradiction. $\square$

**Lemma 4.8** If $\text{rank}(X) = 1$, then a torsion free nonelementary discrete isometry group $\Gamma \subset \text{Isom}^o(X)$ possesses infinitely many limit points.

**Proof.** Suppose $\Gamma$ possesses only finitely many limit points. Since $L^{reg}_\Gamma \neq \emptyset$ by definition, we first treat the case $L_G = \{\xi\}$. But then the $\Gamma$-invariance of the limit set implies $\Gamma \xi = \xi$, hence in particular $\Gamma \xi \cap \text{Vis}^\infty(\xi) = \emptyset$, a contradiction.

Next suppose that $L_G = \{\xi, \eta\}$ with $\eta \neq \xi$. By $\Gamma$-invariance of the limit set we conclude that $\Gamma$ leaves invariant the unique geodesic $\sigma$ joining $\xi$ to $\eta$. Since $\Gamma$ does not contain elliptic elements, we conclude that $\Gamma$ is a cyclic group generated by an axial
element which translates \( \sigma \), hence \( \Gamma \cdot \xi = \xi \) and \( \Gamma \cdot \eta = \eta \). Again \( \Gamma \cdot \xi \cap \text{Vis}^{\infty}(\xi) = \emptyset \) gives a contradiction.

Finally, if \( L_\Gamma \) contains at least three points \( \xi, \eta, \zeta \), then \( \Gamma \) contains either three parabolics \( g, \gamma, p \) such that \( g\xi = \xi, \gamma \eta = \eta \) and \( p\zeta = \zeta \), or an axial isometry \( h \) and a second isometry \( \gamma \) which does not fix \( h^+ \) or \( h^- \). In the first case, the dynamics of parabolic isometries of rank one symmetric spaces imply that the points \( p^j \xi \) for \( j \in \mathbb{Z} \) are all disjoint, and contained in the limit set. Hence \#\( L_\Gamma = \infty \).

In the second case, if \( \xi \in L_\Gamma \) denotes the point not fixed by \( h \), the points \( h^j \xi, j \in \mathbb{Z} \), are disjoint and belong to the limit set, hence again \#\( L_\Gamma = \infty \). \( \square \)

We remark that the restriction to torsion free groups is only necessary for the following particular situation which may occur: If \( \Gamma \) consists of an axial isometry \( h \) which translates a geodesic \( \sigma \), and an elliptic isometry \( e \) which leaves invariant \( \sigma \) and permutes the extremities of \( \sigma \), then \( \langle h, e \rangle \) is nonelementary with respect to our definition, but possesses only two limit points.

The above proof and the remark before Lemma 4.3 show that a torsion free discrete isometry group of a rank one symmetric space is elementary if and only if it is contained in the stabiliser of a limit point.

### 4.3 The approximation argument

Since we do not know much about the dynamics of parabolic isometries, the description of the limit set of discrete isometry groups which, in general, contain parabolics, is difficult. Fortunately, the following “approximation argument” allows to approach every regular limit point in a nonelementary discrete group by a sequence of regular axial isometries.

The main tool in the proof is Proposition 4.5.14 in [E].

**Proposition 4.9** Let \( \Gamma \subset G = \text{Isom}^o(X) \) be a nonelementary discrete group. Then for every \( \xi \in L^\text{reg}_\Gamma \) there exists a sequence of axial isometries \( (h_j) \subset \Gamma \) such that \( h^+_j \xi \rightarrow \xi \) and \( h^-_j \xi \rightarrow \text{a point in Vis}^{\infty}(\xi) \). Furthermore, \( d(o, Ax(h_j)) \) is bounded as \( j \rightarrow \infty \).

**Proof.** Fix \( \xi \in L^\text{reg}_\Gamma \) and let \( \zeta \in L^\text{reg}_\Gamma \) arbitrary. Since \( \Gamma \) is nonelementary, there exists \( \gamma \in \Gamma \) such that \( \pi^B(\gamma \xi) \in \text{Vis}^B(\xi) \). This implies the existence of a regular unit speed geodesic \( \sigma \) with \( \sigma(\infty) = \zeta \) and \( \pi^B(\sigma(-\infty)) = \pi^B(\gamma \xi) \).

Let \( \Phi_t : \mathbb{R} \rightarrow SX \) denote the geodesic flow of \( X \) and \( p : SX \rightarrow X \) the foot point projection. By Proposition 4.5.14 in [E], there exists a sequence of unit tangent vectors \( (v_j) \subset SX \) converging to the tangent vector \( \tilde{\sigma}(0) \in SX \) of \( \sigma(0) \), and a sequence \( (h_j) \subset \Gamma \) of regular axial isometries with translation lengths \( (l_j) \subset \mathbb{R} \) such that

\[
h_j \Phi_t v_j = \Phi_{t+l_j} v_j \quad \text{for all} \quad j \in \mathbb{N}.
\]

Hence \( h^+_j \rightarrow \sigma(\infty) = \xi \), \( h^-_j \rightarrow \sigma(-\infty) \), and

\[
d(o, Ax(h_j)) \leq d(o, \sigma(0)) + d(\sigma(0), p(v_j)) \rightarrow d(o, \sigma(0)) \quad \text{as} \quad j \rightarrow \infty.
\]

Hence \( (h_j) \) is the desired sequence. \( \square \)

An immediate corollary of this proposition is the following

**Theorem 4.10** If \( \Gamma \subset G = \text{Isom}^o(X) \) is a nonelementary discrete group, then the set of attractive fixed points of regular axial isometries is a dense subset of the limit set \( L_\Gamma \).
4.4 Sequences of axial isometries

From here on \( \Gamma \subset G \) will always denote a nonelementary discrete group. The following equivalence for certain sequences of axial isometries in \( \Gamma \) will be necessary in the proof of Theorem \[4.12\].

**Lemma 4.11** Fix \( x \in X \) and let \((h_j) \subset \Gamma\) be a sequence of regular axial isometries such that \( d(x, \text{Ax}(h_j)) \) remains bounded as \( j \to \infty \). Then \((h_jx) \subset X\) converges to a boundary point \( \xi \in \partial X \) in the cone topology if and only if the sequence of attractive fixed points \((h_j^+) \subset \partial X^{reg}\) of \((h_j)\) converges to \( \xi \) in the cone topology.

**Proof.** Let \((h_j)\) be a sequence of axial isometries with attractive fixed points \((h_j^+) \subset \partial X^{reg}\), and \( c \geq 0 \) such that \( d(x, \text{Ax}(h_j)) \leq c \). For all \( j \in \mathbb{N} \) we let \( x_j \in \text{Ax}(h_j) \) be the orthogonal projection of \( x \) to \( \text{Ax}(h_j) \), and \( l_j := d(x_j, h_jx_j) \) the translation length of \( h_j \).

It suffices to prove that for all \( R >> 1 \)

\[
d(\sigma_{x,h_jx}(R), \sigma_{x,h_j^+(R)}) \to 0 \quad \text{as} \quad j \to \infty.
\]

For \( j \in \mathbb{N} \) we put \( d_j := d(x, h_jx) \). Using the convexity of the distance function, the triangle inequality and \( |d_j - l_j| \leq 2c \), we compute

\[
d(\sigma_{x,h_jx}(R), \sigma_{x,h_j^+(R)}) \\
\leq \frac{R}{l_j} (d(\sigma_{x,h_jx}(l_j), \sigma_{x,h_jx}(d_j)) + d(\sigma_{x,h_jx}(d_j), h_jx_j) + d(h_jx_j, \sigma_{x,h_j^+}(l_j))) \\
= \frac{R}{l_j} (|l_j - d_j| + d(h_jx_j, h_jx_j) + d(\sigma_{x,h_j^+}(l_j), \sigma_{x,h_j^+}(l_j))) \leq 4c \frac{R}{l_j}.
\]

From \( l_j \to \infty \) as \( j \to \infty \) we conclude that \( d(\sigma_{x,h_jx}(R), \sigma_{x,h_j^+(R)}) \) tends to zero. \( \square \)

For the remainder of this section we fix a Cartan decomposition \( G = Ke^{G \cdot \rho} K \) with respect to \( o \in X \). The following result generalises Corollary \[3.14\] to sequences of axial isometries \((h_j) \subset \Gamma\). In combination with the approximation argument Proposition \[4.9\] this will be our main tool for the description of the structure of the limit set in the following section.

**Theorem 4.12** Let \((h_j) \subset \Gamma\) be a sequence of regular axial isometries such that \( h_j o \) converges to \( \xi^+ \in \partial X^{reg} \), \( h_j^{-1} o \) converges to \( \xi^- \in \text{Vis}^\infty(\xi^+) \), and \( d(o, \text{Ax}(h_j)) \) is bounded as \( j \to \infty \). Then for any \( \zeta \in \partial X^{reg} \) with \( \pi^B(\zeta) \in \text{Vis}^B(\xi^-) \) there exist integers \( n_j, j \in \mathbb{N} \), such that the sequence \((h_j^{n_j} \zeta)\) converges to the unique point \( \eta^+ \in G \cdot \zeta \) with \( \pi^B(\eta^+) = \pi^B(\xi^+) \). In particular, if \( \zeta \in \text{Vis}^\infty(\xi^-) \), then there exist integers \( n_j, j \in \mathbb{N} \), such that \( h_j^{n_j} \zeta \) converges to \( \xi^+ \).

**Proof.** Let \((h_j)\) be a sequence of regular axial isometries with the properties stated in the theorem. We denote by \( h_j^+ \) the attractive fixed point of \( h_j \) and by \( H_j \in a_1^+ \) its Cartan projection. Let \( \zeta \in \partial X^{reg} \) with \( \pi^B(\zeta) \in \text{Vis}^B(\xi^-) \) arbitrary, and \( \eta^+ \in G \cdot \zeta \) the unique point such that \( \pi^B(\eta^+) = \pi^B(\xi^+) \). By the previous lemma, \( h_j^+ \) converges to \( \xi^+ \) in the cone topology, hence Lemma \[2.23\] implies that \( \pi^B(h_j^+) \) converges to \( \pi^B(\xi^+) \) in \( K/M \), and the Cartan projections \((H_j) \subset a_1^+ \) converge to the Cartan projection \( H_{\xi^+} \) of \( \xi^+ \).

Let \( Y \subset G \cdot \zeta \) be an open neighbourhood of \( \eta^+ \). Then for \( j > N \) sufficiently large, we have \( \pi^B(h_j^+) \in \pi^B(Y) \) and \( \pi^B(\zeta) \in \text{Vis}^B(h_j^-) \), hence by Corollary \[3.4\] there exists
$n_j \in \mathbb{N}$ such that $h_j^{n_j} \zeta \in Y$. We conclude that the sequence $(h_j^{n_j} \zeta)$ converges to $\eta^+$ in the cone topology. \qed
s \cdot s_{k_j-1}^{(j)} \cdots s_{i_j}^{(j)} \xi$, and from the dynamics of the generators of $\Gamma$ we have $\pi^B(h_j \xi) \in U$ for all $j \in \mathbb{N}$. We conclude that $\pi^B(\Gamma \cdot \xi) \subset \bigcup_{i=1}^{l+p} U_i$. \hfill \square

**Theorem 4.15** Let $\Gamma \subset G = \text{Isom}^0(X)$ be a nonelementary discrete group of isometries. Then the regular geometric limit set is isomorphic to the product $K_{\Gamma} \times (P_{\Gamma} \cap \alpha^+_1)$.

**Proof.** If $\xi \in L_{\Gamma}^{reg}$, then $\pi^B(\xi) \in K_{\Gamma}$ and the Cartan projection of $\xi$ belongs to $P_{\Gamma} \cap \alpha^+_1$.

Conversely, let $kM \in K_{\Gamma}$ and $H \in P_{\Gamma} \cap \alpha^+_1$. By definition of $P_{\Gamma}$, there exists a sequence $(\gamma_j) \subset \Gamma$ such that the Cartan projections $(H_j) \subset \alpha^+_1$ of $(\gamma_j \circ o)$ satisfy $\angle(H_j, H) \to 0$ as $j \to \infty$. Furthermore, a subsequence of $(\gamma_j \circ o)$ converges to a point $\xi_0 = (k_0, H) \in L_{\Gamma}^{reg}$ where $k_0 \in K$.

By Theorem 4.14 $K_{\Gamma} = \overline{\Gamma(K_0M)}$ is a minimal closed set under the action of $\Gamma$, hence

$kM \in \overline{\Gamma(K_0M)} = \pi^B(\Gamma \cdot \xi_0)$. Since the action of $\text{Isom}^0(X)$ on the geometric boundary does not change Cartan projections, we conclude that the closure of $\Gamma \cdot \xi_0$ contains $(k, H)$. In particular, $(k, H) \in \overline{\Gamma \cdot \xi_0} \subset L_{\Gamma}^{reg}$. \hfill \square

The limit cone and the directional limit set of a discrete isometry group are related as follows:

**Theorem 4.16** If $\Gamma \subset G = \text{Isom}^0(X)$ is a nonelementary discrete group, then $P_{\Gamma} = \overline{\ell_{\Gamma}}$.

**Proof.** In order to prove $P_{\Gamma} \supset \overline{\ell_{\Gamma}}$, let $H \in \overline{\ell_{\Gamma}}$ arbitrary. Then there exists a sequence $(h_j) \subset \Gamma$ of axial isometries with translation directions $L_j := L(h_j)$ satisfying $\angle(L_j, H) \to 0$ as $j \to \infty$.

Suppose $H \notin P_{\Gamma}$, and, for $\gamma \in \Gamma$, let $H_\gamma$ denote the Cartan projection of $\gamma \circ o$. Then there exists $\varepsilon > 0$ such that $\angle(H_\gamma, H) > \varepsilon$ for all but finitely many $\gamma \in \Gamma$. Put $\xi = (id, H) \in \partial X$. For $j \in \mathbb{N}$, we let $g_j \in G$ such that $h_j = g_j e^{L_j} g_j^{-1}$, put $x_j := g_j \circ o \in \text{Ax}(h_j)$, and let $n_j$ be an integer greater than $2d(o, x_j)/\|L_j\|$. We abbreviate $\gamma_j := h_j^{n_j} = g_j e^{L_j(n_j)} g_j^{-1}$ and $H_j := H_{\gamma_j}$. By $G$-invariance of the directional distance and equation (5) we have

$$B_{G, \xi}(x_j, \gamma_j x_j) = B_{G, \xi}(g_j^{-1} x_j, g_j^{-1} \gamma_j x_j) = B_{G, \xi}(o, e^{L_j(n_j)} o) = \langle H_j, L_j n_j \rangle = n_j \langle L_j, H \rangle,$$

and $\angle(L_j, H) \to 0$ implies $\langle L_j/\|L_j\|, H \rangle \to 1$ as $j \to \infty$. Using again equation (5), the $G$-invariance and the triangle inequality for the Riemannian and the directional distance, and $2d(o, x_j) \leq n_j \|L_j\|/j$ we conclude

$$\cos \angle(H_j, H) = \frac{\langle H_j, H \rangle}{\|H_j\|} = \frac{B_{G, \xi}(o, \gamma_j o)}{d(o, \gamma_j o)} \geq \frac{B_{G, \xi}(x_j, \gamma_j x_j) - 2d(o, x_j)}{d(x_j, \gamma_j x_j) + 2d(o, x_j)} \geq \frac{n_j \langle L_j, H \rangle - n_j \|L_j\|/j}{n_j \|L_j\| + n_j \|L_j\|/j} = \frac{\langle L_j/\|L_j\|, H \rangle - 1/j}{1 + 1/j} \to 1$$

as $j \to \infty$, a contradiction to our assumption.

Conversely, we first prove $P_{\Gamma} \cap \alpha^+_1 \subset \overline{\ell_{\Gamma}}$. Given $H \in P_{\Gamma} \cap \alpha^+_1$, there exists $\xi \in L_{\Gamma}^{reg}$ with Cartan projection $H$. Let $(h_j) \subset \Gamma$ be a sequence of regular axial isometries as in Proposition 4.19 with the properties $h_j^+ \to \xi$, $h_j^- \to \eta \in \text{Vis}^\infty(\xi)$ and $d(o, \text{Ax}(h_j)) \leq c$ for
some constant \( c > 0 \). Then Lemma 4.11 implies \( h_j o \to \xi \), hence by Lemma 2.3 the Cartan projections \((H_j) \subset a^+\) of \( h_j o \) satisfy
\[
\langle H_j/\|H_j\|, H \rangle \to 1 \quad \text{as } j \to \infty.
\]
For \( j \in \mathbb{N} \), we let \( x_j \in \text{Ax}(h_j) \) be the orthogonal projection of \( o \) to \( \text{Ax}(h_j) \). If \( L_j := L(h_j) \in a^+ \) denotes the translation vector of \( h_j \), we estimate
\[
\frac{\langle L_j \rangle}{\|L_j\|}, H \rangle = \frac{B_{G,\xi}(x_j, h_j x_j)}{d(x_j, h_j x_j)} \leq \frac{B_{G,\xi}(o, h_j o) + 2d(o, x_j)}{d(o, h_j o) - 2d(o, x_j)} \leq \frac{\langle H_j/\|H_j\|, H \rangle + 2c/\|H_j\|}{1 - 2c/\|H_j\|} \to 1 \quad \text{and}
\]
\[
\frac{\langle L_j \rangle}{\|L_j\|}, H \rangle \geq \frac{B_{G,\xi}(o, h_j o) - 2d(o, x_j)}{d(o, h_j o)} \geq \frac{\langle H_j \rangle}{\|H_j\|}, H \rangle - \frac{2c}{\|H_j\|} \to 1
\]
as \( j \to \infty \). This gives \( \angle(L_j, H) \to 0 \) as \( j \to \infty \), hence \( H \in \overline{\ell_\Gamma} \).

Since the closure of \( P_\Gamma \cap a^+_1 \) contains \( P_\Gamma \), and \( \overline{\ell_\Gamma} \) is a closed set in \( \overline{a^+_1} \), we conclude that \( P_\Gamma \subseteq \overline{P_\Gamma \cap a^+_1} \subseteq \overline{\ell_\Gamma} \). \( \square \)

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Gabriele Link  
Mathematisches Institut II  
Universität Karlsruhe  
Englerstr. 2  
76 128 Karlsruhe  
e-mail: gabriele.link@math.uni-karlsruhe.de