LOW-TEMPERATURE ASYMPTOTICS OF FREE ENERGY
OF 3D ISING MODEL IN EXTERNAL MAGNETIC FIELD

Martin S. Kochmański
Institute of Physics, Pedagogical University
T.Rejtana 16 A, 35–310 Rzeszów, Poland
e-mail: mkochma@atena.univ.rzeszow.pl

Abstract

The paper presents new method for calculating the low-temperature asymptotics of free energy of the 3D Ising model in external magnetic field ($H \neq 0$). The results obtained are valid in the wide range of temperature and magnetic field values fulfilling the condition: $[1 - \tanh(h/2)] \sim \varepsilon$, for $\varepsilon \ll 1$, where $h = \beta H$, $\beta$ - the inverse temperature and $H$ - external magnetic field. For this purpose the method of transfer-matrix, and generalized Jordan-Wigner transformations, in the form introduced by the author in [1], are applied.

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I. FORMULATION OF THE PROBLEM

As is well known, till now an exact solution for the 2D Ising model in external magnetic field \((H \neq 0)\) was not found. In the case of the 3D Ising model there does not exist an exact solution for vanishing magnetic field \((H = 0)\), without even mentioning the case with magnetic field. Despite great successes in investigations of the Ising model reached using the renormalization group method \([2]\), and other approximate methods \([3–6]\), the problem of calculation various asymptotics for the 2D and 3D Ising models in external magnetic field \((H \neq 0)\) is still of great importance. In the paper \([7]\) we calculated low-temperature asymptotics for the 2D Ising model in external magnetic field \((H \neq 0)\), and free energy for this model in the limit of asymptotically vanishing magnetic field. In this paper we shortly discuss the problem of calculation of the low-temperature asymptotics for free energy in the 3D Ising model in external magnetic field \((H \neq 0)\), following the approach and the ideas we introduced in the paper \([7]\).

Let us consider a cubic lattice built of \(N\) rows, \(M\) columns and \(K\) planes, to vertices of which are assigned the numbers \(\sigma_{nmk}\) from the two-entries set \(\pm 1\). These quantities we will be calling here and everywhere below the Ising “spins.” The multiindex \((nmk)\) numbers vertices of the lattice, with \(n\) numbering rows, \(m\) numbering columns, and \(k\) numbering planes. The Ising model with the nearest neighbors interaction in external magnetic field is described by the Hamiltonian of the form:

\[
\mathcal{H} = - \sum_{(n,m,k)=1}^{N M K} (J_1 \sigma_{nmk} \sigma_{n+1,m,k} + J_2 \sigma_{nmk} \sigma_{n,m+1,k} + J_3 \sigma_{nmk} \sigma_{nm,k+1} + H \sigma_{nmk}),
\]

(1.1)

taking into account anisotropy of the interaction between the nearest neighbors \((J_{1,2,3} > 0)\), and the interaction of the spins \(\sigma_{nmk}\) with external magnetic field \(H\), directed “up” \((\sigma_{nmk} = +1)\). The main problem consists of calculation of the statistical sum for the system:

\[
Z_3(h) = \sum_{\sigma_{111}=\pm 1} \ldots \sum_{\sigma_{NMK}=\pm 1} e^{-\beta \mathcal{H}} = \sum_{\{\sigma_{nmk}=\pm 1\}} \exp \left[ \sum_{nmk} (K_1 \sigma_{nmk} \sigma_{n+1,m,k} + K_2 \sigma_{nmk} \sigma_{n,m+1,k} + K_3 \sigma_{nmk} \sigma_{nm,k+1} + h \sigma_{nmk}) \right],
\]

(1.2)

where \(K_{1,2,3} = \beta J_{1,2,3}, \quad h = \beta H, \quad \beta = 1/k_B T\). Typical boundary conditions for the variables \(\sigma_{nmk}\) are the periodic ones. We follow this standard assumption everywhere below. Let us note here that the statistical sum (1.2) is symmetric with respect to the change \((h \to -h)\).

In this letter we consider a limited version of the problem. Namely, the problem of calculation of the low-temperature asymptotics for free energy in the 3D Ising model in external magnetic field. More precisely, given the coupling constants \((J_{1,2,3} = const)\) and external magnetic field \((H = const)\), we consider the region of temperatures satisfying the condition: \(h \sim \varepsilon^{-1}, \quad \varepsilon \ll 1\). To be more exact, we introduce a small parameter in the following way:

\[
1 - \tanh(h/2) \sim \varepsilon, \quad \varepsilon \ll 1,
\]

(1.3)
Then we consider the problem of calculation of free energy per one Ising spin in the thermodynamic limit, with exactness up to quantities of the order $\varepsilon^2$ in expansions of the operators associated with interaction of spins as well among themselves as with the external field. (details of the approximation used will be presented below). In our opinion the problem formulated above is of reasonable importance, and, as far as is known to the author, it was not investigated in the existing literature.

II. PARTITION FUNCTION

Let us consider an auxiliary 4D Ising model in external magnetic field $H$ on simple 4D lattice ($N \times M \times K \times L$). We write the Hamiltonian for the 4D Ising model with the nearest neighbor interaction in the form:

$$H = - \sum_{n,m,k,l} (J_1 \sigma_{nmkl} \sigma_{n+1,mkl} + J_2 \sigma_{nmkl} \sigma_{n,m+1,kl} + J_3 \sigma_{nmkl} \sigma_{nm,k+1,l} + J_4 \sigma_{nmkl} \sigma_{nmk,l+1} + H \sigma_{nmkl}),$$

(2.1)

taking into account anisotropy of the interaction between the nearest neighbors ($J_{1,2,3,4} > 0$), and interaction of the spins $\sigma_{nmkl}$ with external magnetic field $H$, directed “up” ($\sigma_{nmkl} = +1$). Here (2.1) the multiindex $(nmkl)$ numbers the vertices of the 4D lattice, and the indices $(n, m, k, l)$ take on values from 1 to $(N, M, K, L)$, respectively. As in the case of the 3D Ising model, we introduce periodic boundary conditions for the variables $\sigma_{nmkl}$. Then we write the partition function $Z_4(h)$ in the form:

$$Z_4(h) = \sum_{\sigma_{1111}=\pm1} \ldots \sum_{\sigma_{NMKL}=\pm1} e^{-\beta H} = \sum_{\{\sigma_{nmkl}=\pm1\}} \exp \left[ \sum_{nmkl} (K_1 \sigma_{nmkl} \sigma_{n+1,mkl} + K_2 \sigma_{nmkl} \sigma_{n,m+1,kl} + K_3 \sigma_{nmkl} \sigma_{nm,k+1,l} + K_4 \sigma_{nmkl} \sigma_{nmk,l+1} + h \sigma_{nmkl}) \right],$$

(2.2)

where the quantities $K_i$ and $h$ are defined as above (1.2) [8,9]. The expression (2.2) we can write, using the well known method of transfer matrix, in the form of a trace from the $L$-th power of the operator $\hat{T}$:

$$Z_4(h) = Tr(\hat{T})^L, \quad \hat{T} = T_4 T_h^{1/2} T_3 T_2 T_1 T_h^{1/2},$$

(2.3)

where the operators $T_{1,2,3,4,h}$ are defined by the formulas:

$$T_1 = \exp \left( K_1 \sum_{nmk} \tau_{nmk}^z \tau_{n+1,mk}^z \right), \quad T_2 = \exp \left( K_2 \sum_{nmk} \tau_{nmk}^z \tau_{n,m+1,k}^z \right),$$

(2.4)

$$T_3 = \exp \left( K_3 \sum_{nmk} \tau_{nmk}^z \tau_{nm,k+1}^z \right), \quad T_4 = (2 \sinh 2K_4)^{NMK/2} \exp \left( K_4^* \sum_{nmk} \tau_{nmk}^x \right),$$

(2.5)

$$T_h = \exp \left( h \sum_{nmk} \tau_{nmk}^z \right),$$

(2.6)
and the quantities $K_4$ and $K_4^*$ are coupled by the following relations:

$$\tanh(K_4) = \exp(-2K_4^*), \quad \text{or} \quad \sinh 2K_4 \sinh 2K_4^* = 1.$$  \hspace{1cm} (2.7)

The Pauli spin matrices $\tau_{nmk}^{x,y,z}$ commute for $(nmk) \neq (n'm'k')$, and for given $(nmk)$ these matrices satisfy the usual relations $\Box$. It is easy to see that the matrices $T_{1,2,3,h}$ commute among themselves, but do not commute with the matrix $T_1$. In the case in which one of the quantities $K_i = 0$, $(i = 1, 2, 3)$, we get obviously the known expressions describing the 3D Ising model on a simple cubic lattice. Namely, the transition to the 3D Ising model with respect to the coupling constants $K_1$, $K_2$, or $K_3$ is realized by taking $(K_1 = 0)$, or $(K_2 = 0)$, or $(K_3 = 0)$, and removing summation over $n$, $(N = 1)$, or over $m$, $(M = 1)$, or over $k$, $(K = 1)$, respectively. As a result we get the standard expressions $\Box$ for the 3D Ising model in external magnetic field. In the process the operators $T_i$, $(i = 1, 2, 3)$ in every one of the cases are identically equal to the unit operator ($T_i \equiv 1$). A bit different situation appears in the case of the transition to the 3D Ising model with respect to the coupling constant $K_4$. In this case we take $(K_4 = 0, \ L = 1)$, i.e. we remove summation over $l$. In consequence we get the following expression for the operator $T_4$, (2.5):

$$T_i^* \equiv T_4(K_4 = 0) = \prod_{nmk} (1 + \tau_{nmk}^x),$$ \hspace{1cm} (2.8)

where we used the relation (2.7). Then, after transition to the limit $(K_4 = 0, \ L = 1)$ in (2.3), we can write the following expression for the partition function for the 3D Ising model:

$$Z_3(h) = Tr(T_4^* T_h^{1/2} T_3 T_2 T_1 T_h^{1/2}),$$ \hspace{1cm} (2.9)

where the matrices $T_i$ are defined as above (2.4 – 6, 8). Now we pass to the fermionic representation. For this aim one should write the matrices $T_i$ in terms of the Pauli operators $\tau_{nmk}^{\pm}$, $\Box$:

$$\tau_{nmk}^\pm = \frac{1}{2}(\tau_{nmk}^x \pm i \tau_{nmk}^y),$$ \hspace{1cm} (2.10)

which satisfy anticommutation relations for one vertex, and which commute for different vertices.

As the next step one should pass from the representation by Pauli operators (2.10) to the representation by Fermi creation and annihilation operators $\Box$. In the paper $\Box$ were introduced appropriate transformations (generalized transformations of the Jordan-Wigner type), enabling the transition to the fermionic representation:

$$\tau_{nmk}^+ = \exp \left[i\pi \left( \sum_{s=1}^{N} \sum_{p=1}^{M} \sum_{q=1}^{k-1} \alpha_{spq}^+ \alpha_{spq} + \sum_{s=1}^{N} \sum_{p=1}^{m-1} \alpha_{spk}^+ \alpha_{spk} + \sum_{s=1}^{n-1} \alpha_{smk}^+ \alpha_{smk} \right) \right] \alpha_{nmk}^+,$$

$$\tau_{nmk}^+ = \exp \left[i\pi \left( \sum_{s=1}^{N} \sum_{p=1}^{M} \sum_{q=1}^{k-1} \beta_{spq}^+ \beta_{spq} + \sum_{s=1}^{n-1} \sum_{p=1}^{M} \beta_{spk}^+ \beta_{spk} + \sum_{p=1}^{m-1} \beta_{npk}^+ \beta_{npk} \right) \right] \beta_{nmk}^+,$$

$$\tau_{nmk}^+ = \exp \left[i\pi \left( \sum_{s=1}^{N} \sum_{p=1}^{m-1} \sum_{q=1}^{K} \gamma_{spq}^+ \gamma_{spq} + \sum_{s=1}^{n-1} \sum_{p=1}^{M} \gamma_{smq}^+ \gamma_{smq} + \sum_{s=1}^{n-1} \gamma_{smk}^+ \gamma_{smk} \right) \right] \gamma_{nmk}^+,$$

\hspace{1cm} 4
\[
\tau_{nmk}^+ = \exp \left[ i\pi \left( \sum_{s=1}^{N-1} \sum_{p=1}^{m-1} \sum_{q=1}^{K} \eta_{spq}^+ \eta_{pq} + \sum_{s=1}^{n-1} \sum_{q=1}^{K} \eta_{sqm}^+ \eta_{mq} + \sum_{q=1}^{k-1} \eta_{mq}^+ \eta_{mnq} \right) \right] \eta_{nmk}^+ \\
\tau_{nmk}^- = \exp \left[ i\pi \left( \sum_{s=1}^{n-1} \sum_{p=1}^{M} \sum_{k=1}^{K} \omega_{spq}^+ \omega_{pq} + \sum_{s=1}^{m-1} \sum_{pq}^{K} \omega_{npq}^+ \omega_{npq} + \sum_{p=1}^{m} \sum_{k=1}^{K} \omega_{npk}^+ \omega_{npk} \right) \right] \omega_{nmk}^- \\
\tau_{nmk}^+ = \exp \left[ i\pi \left( \sum_{s=1}^{n-1} \sum_{p=1}^{M} \sum_{q=1}^{K} \theta_{spq}^+ \theta_{pq} + \sum_{s=1}^{m-1} \sum_{q=1}^{K} \theta_{npq}^+ \theta_{npq} + \sum_{q=1}^{k-1} \theta_{mnq}^+ \theta_{mnq} \right) \right] \theta_{nmk}^+ \tag{2.11}
\]

and analogously for the operators \(\tau_{nmk}^-\). In the paper [4] we obtained formulas for relations between various Fermi operators, and commutation relations for them. Further in this paper we will use the fact that the following equality of local occupation numbers is valid:

\[
\tau_{nmk}^+ \tau_{nmk}^- = \alpha_{nmk}^+ \alpha_{nmk}^- = \beta_{nmk}^+ \beta_{nmk}^- = \gamma_{nmk}^+ \gamma_{nmk}^- = \eta_{nmk}^+ \eta_{nmk}^- = \omega_{nmk}^+ \omega_{nmk}^- = \theta_{nmk}^+ \theta_{nmk}^- \tag{2.12}
\]

Then, applying the expressions (2.10 – 12) and considerations from the paper [4], we can write the partition function (2.9) in the form:

\[
Z_3(h) = (2 \cosh^2 h/2)^{NMK} < 0|T^*|0 >= A < 0|U + \mu^2CU D|0 >, \quad U \equiv T_h^l T_h T_l T_h^r \tag{2.13}
\]

where \(A = (2 \cosh^2 h/2)^{NMK} \) and \(\mu = \tanh(h/2)\), and the operators \(T_{1,2,3}, T_h^r \) and \(C, D\) are of the form:

\[
T_1 = \exp \left[ K_1 \sum_{n,m,k=1}^{N,M,K} (\alpha_{nmk}^+ - \alpha_{nmk})(\alpha_{n+1,m,k}^+ + \alpha_{n+1,m,k}) \right], \\
T_2 = \exp \left[ K_2 \sum_{n,m,k=1}^{N,M,K} (\beta_{nmk}^+ - \beta_{nmk})(\beta_{n+1,m,k}^+ + \beta_{n+1,m,k}) \right], \\
T_3 = \exp \left[ K_3 \sum_{n,m,k=1}^{N,M,K} (\theta_{nmk}^+ - \theta_{nmk})(\theta_{nm,k+1}^+ + \theta_{nm,k+1}) \right], \tag{2.14}
\]

and

\[
T_h^r = \exp \left\{ \mu^2 \left[ \sum_{nmk}^{N-n} \alpha_{nmk}^+ \alpha_{n+s,m,k}^+ + \sum_{nn\prime mk}^{M-m} \alpha_{nmk}^+ \alpha_{n,m+t,k}^+ + \sum_{nn\prime mn'k}^{K-k} \alpha_{nmk}^+ \alpha_{n'm,k+l}^+ \right] \right\}, \\
T_h^l = \exp \left\{ \mu^2 \left[ \sum_{nmk}^{K-k} \theta_{nm,k+t}\theta_{nmk}^+ + \sum_{nmkk'}^{M-m} \theta_{n,m,t,k}\theta_{nmk'}^+ + \sum_{nn\prime mn'k'}^{N-n} \theta_{n+s,m,k}\theta_{nmk'}^+ \right] \right\}, \tag{2.15}
\]

\[
C = \sum_{nmk} \theta_{nmk}, \quad D = \sum_{nmk} \alpha_{nmk}^+ 
\]

Here and below \(\sum_{n,m,...}\) means summation over the complete set of indices \((n = 1,...N; \quad m = 1,...M; \quad etc.)\). It is obvious that the operator \(\hat{G}\):

\[
\hat{G} = (-1)^\hat{S}, \quad \hat{S} = \sum_{nmk} \alpha_{nmk}^+ \alpha_{nmk} 
\]
where  $\hat{S}$ is the operator of the total number of particles, commutes with the operator $T^*$, (2.13). Therefore, we can divide all states of the operator $T^*$ into states with even ($\lambda_G = +1$) or odd number of particles ($\lambda_G = -1$) with respect to the operator $G$, (2.16). The form of the operators $T_{1,2,3}$ does not change during the course, only the boundary conditions for the operators ($\alpha_{nmk}, \ldots$) do. In the case of even states ($\lambda_G = +1$) antiperiodic boundary conditions, and in the case of odd states periodic ones, are chosen.

The next step is transition to the momentum representation:

$$\alpha_{nmk}^+ = \frac{\exp(i\pi/4)}{(NMK)^{1/2}} \sum_{qpu} e^{-i(nq+mq+kr)} \xi_{qpu}^+$$

and introduction for fixed ($qpu$) corresponding bases for $\xi^-, \eta^-$ and $\zeta^-$ Fermi creation and annihilation operators in the representation in terms of occupation numbers in the finite-dimensional Fock space of dimension $2^6 = 256$). Then, after a series of transformations and calculations we arrive at the following formula for the partition function (2.13):

$$Z_{3D}^+(h) = A \left( \prod_{0<q,p,\nu<\pi} A_1^4(q) \right) \left( \prod_{0<q,p,\nu<\pi} A_3^3(\nu) \right) <0|T_1^+(h)T_2^+(h)|0>, \quad (2.17)$$

where the operators $T_1^+(h)$, $T_2$, $T_3^+(h)$ are of the form

$$T_1^+(h) = \exp \left[ \sum_{0<q,p,\nu<\pi} B_1(q)(\xi_{q-p-\nu}^+ \xi_{q-p-\nu}^+ + \xi_{q-p-\nu}^+ \xi_{q-p-\nu}^+ + \xi_{q-p-\nu}^+ \xi_{q-p-\nu}^+) \right],$$

$$T_2 = \exp \left\{ 2K_2 \sum_{0<q,p,\nu<\pi} [\cos p(\eta_{qpu} \eta_{qpu} + \ldots) + \sin p(\eta_{q-p-\nu} \eta_{q-p-\nu} + \ldots)] \right\},$$

$$T_3^+(h) = \exp \left[ \sum_{0<q,p,\nu<\pi} B_3(\nu)(\xi_{q-p-\nu} \xi_{q-p-\nu} + \xi_{q-p-\nu} \xi_{q-p-\nu} + \xi_{q-p-\nu} \xi_{q-p-\nu}) \right], \quad (2.18)$$

and $A_1(q, h), \ldots$ are defined by the expressions:

$$A_1(q, h) = \cosh 2K_1 - \sinh 2K_1 \cos q + \alpha(h, q) \sinh 2K_1 \sin q,$$

$$A_3(\nu, h) = \cosh 2K_3 + \sinh 2K_3 \cos \nu + \alpha(h, \nu) \sinh 2K_3 \sin \nu,$$

$$B_1(q, h) = \alpha(h, q) [\cosh 2K_1 + \sinh 2K_1 \cos q] \sinh 2K_1 \sin q,$$

$$B_3(\nu, h) = \alpha(h, \nu) [\cosh 2K_3 + \sinh 2K_3 \cos \nu] \sinh 2K_3 \sin \nu,$$

$$\alpha(h, q) = \tanh^2(h/2) \frac{1 + \cos q}{\sin q}, \quad \alpha(h, \nu) = \tanh^2(h/2) \frac{1 + \cos \nu}{\sin \nu}. \quad (2.19)$$

In the formula for $Z_{3D}^+(h)$ the sign (+) means that we consider the case of even states ($\lambda_G = +1$) with respect to the operator $G$, (2.16). It is obvious that for $h = 0$ we arrive at the 3D Ising model in vanishing magnetic field. Then, for $K_1 = 0$ (or $K_2 = 0, \ \text{or} \ K_3 = 0$) the expression (2.17) for the statistical sum describes the 2D Ising model in external magnetic field.
III. SOLUTION

Let us consider calculation of free energy per one Ising spin in external magnetoc field in the approximation described shortly in the introduction. For this aim let us consider the operators $T_1^*(h)$ and $T_3^*(h)$ in the ”coordinate” representation:

$$T_1^*(h) = \exp \left[ \sum_{nmk} \sum_{s=1}^{N-n} a(s) \alpha^+_{nmk} \alpha^+_{n+s,nm} \right],$$

$$T_3^*(h) = \exp \left[ \sum_{nmk} \sum_{l=1}^{K-n} c(l) \theta_{nm,k+l} \theta_{nmk} \right],$$  

(3.1)

where the ”weights” $a(s)$ and $c(l)$ are defined by the formulas:

$$a(s) = \frac{1}{N} \sum_{0 < q < \pi} 2B_1(q) \sin(qs) = z_1^{s^2} + \tanh^2 h_1^* \frac{1 - z_1^{s^2}}{(1 - z_1^{s^2})^2}, \quad s = 1, 2, 3, \ldots$$

$$c(l) = \frac{1}{K} \sum_{0 < \nu < \pi} 2B_3(\nu) \sin(\nu l) = z_3^{l^2} + \tanh^2 h_3^* \frac{1 - z_3^{l^2}}{(1 - z_3^{l^2})^2}, \quad l = 1, 2, 3, \ldots$$  

(3.2)

We introduced renormalized quantities $(K_{1,3}^*, h_{1,3}^*)$ defined as follows :

$$\sinh 2K_{1,3}^* = \beta_{1,3} \sinh 2K_{1,3}(1 - \tanh^2 (h/2)],$$

$$\cosh 2K_{1,3}^* = \beta_{1,3} \cosh 2K_{1,3} + \tanh^2 (h/2) \sinh 2K_{1,3},$$

$$\beta_{1,3} = [1 + 2 \tanh^2 (h/2) \sinh 2K_{1,3} e^{2K_{1,3}^*}]^{-1/2}, \quad \tanh^2 h_{1,3}^* = \tanh^2 (h/2) \frac{\beta_{1,3} \exp(2K_{1,3})}{\cosh 2K_{1,3}^*}. \quad (3.3)$$

These formulas are valid for $(K_{1,3} \geq 0)$. As in the case of the 2D Ising model [11], [13], also in this case one can introduce a diagrammatic representation for the vacuum matrix element $S \equiv \langle 0 | T_3^*(h)T_2^rT_1^*(h)|0 \rangle$. Computation of the vacuum matrix element $S$, which enters the formula (2.17) for $Z_{3D}^+(h)$ in general case, where the ”weights” (3.2) are arbitrary is, at least at present, impossible. Nevertheless, there exists a special case in which we can calculate the quantity $S$ in the 3D case. Namely, this is the case where the ”weights” (3.2) are independent of $l$ and $s$. In this case one should, as in the 2D case [13], put the parameters $K_{1,3}$ equal zero $(K_{1,3} = 0)$ in the formula (2.13), and then express the operators $T_h^l$ in terms of the Fermi $\beta$-operators (2.11) of creation and annihilation, with the goal to calculate $S$. After transition to the momentum representation, one should calculate the vacuum matrix element $S^*(y_1, y_3, z_2)$:

$$S^*(y_1, y_3, z_2) \equiv \langle 0 | T^l(y_3)T_2^r(y_1)|0 \rangle, \quad y_{1,3} \equiv \tanh^2 h_{1,3},$$

where $z_2 = \tanh K_2$ becomes trivial.(Here we introduced the following change of notation: $h/2 \rightarrow h_1$ - for the operator $T_h^1$, and $h/2 \rightarrow h_3$ - for the operator $T_h^3$). We can write the result for $S^*(y_1, y_3, z_2)$ in the following form:

$$S^*(y_1, y_3, z_2) = (2 \cosh^2 K_2)^{\frac{N_{MK}}{2}} \prod_{0 < q < \nu < \pi} \left[ (1 - 2z_2 \cos p + z_2^2)(1 - \cos p) + 2z_2(y_1 + y_3) \sin^2 p + y_1y_3(1 + 2z_2 \cos p + z_2^2)(1 + \cos p) \right]^4. \quad (3.4)$$
This result can be used further to calculate free energy in the approximation discussed above (1.3). For this aim let us note that the conditions $[\tanh^2 h_{1,3}^*(1 - z_{1,3}^*)^2] \to 1$ are equivalent, accordingly to (3.3), to the conditions $(\exp(-2K_{1,3}))(1 - \tanh^2 h/2) \to 0)$. It follows from this equation that for fixed $(J_{1,3} = \text{const}, \ H = \text{const})$ these conditions are satisfied in the region of temperatures $T$, in which $(h/2) \sim \varepsilon^{-1}$, \( \varepsilon \ll 1 \). In this case we can use the result (3.4). Namely, let us consider the formulas (2.19) for $B_{1,3}$, written in terms of the renormalized parameters $(h_{1,3}^*, \ K_{1,3}^*)$:

$$B_{1,3} = \frac{\tanh^2 h_{1,3}^* \sin q(\nu) + 2z_{1,3}^* \sin q(\nu)}{1 - 2z_{1,3}^* \cos q(\nu) + z_{1,3}^*^2},$$

where $z_{1,3}^* = \tanh K_{1,3}^*$. Next, since the following equalities are satisfied:

$$\frac{z_{1,3}^*}{1 + z_{1,3}^*} = \frac{z_{1,3}(1 - \tanh^2 h/2)}{1 + 2z_{1,3} \tanh^2 h/2 + z_{1,3}^2},$$

then, if we introduce a small parameter $[1 - \tanh(h/2)] \sim \varepsilon$, \( \varepsilon \ll 1 \), and expand $B_{1,3}$ into a series in powers of $\varepsilon$ ($z_{1,3}^* \sim \varepsilon$), we obtain

$$B_{1,3} = \frac{(\tanh^2 h_{1,3}^* + 2z_{1,3}^*) \sin q(\nu)}{1 - \cos q(\nu)} \sim \varepsilon^2$$

This formula gives the following expressions for the "weights" $a(s)$ and $c(l)$, (3.2) in this approximation:

$$a(s) = \tanh^2 h_1^* + 2z_1^*, \quad c(l) = \tanh^2 h_3^* + 2z_3^*,$$

with exactness of the order of smallness $\sim \varepsilon^2$. As a result in this approximation the "weights" $a(s), \ c(l)$ do not depend on $(s, l)$. Finally, if we substitute to the expression (3.4) for $S^*(y_1, y_3, z_2)$ the parameters $y_1 \to a(s)$ and $y_3 \to c(l)$, (3.6), we arrive at the following formula for free energy on one Ising spin $F_{3D}(h)$ in the thermodynamic limit:

$$-\beta F_{3D}(h) \asymp \ln(2^{3/2} \cosh K_1^* \cosh K_2 \cosh K_3^* \cosh^2 h/2) + \frac{1}{2\pi} \int_0^\pi \ln \left[(1 - 2z_2 \cos p + z_2^2)\right.\left.\times(1 - \cos p) + 2z_2(\tanh^2 h_1^* + \tanh^2 h_3^* + 2z_1^* + 2z_3^*) \sin^2 p + (\tanh^2 h_1^* + 2z_1^*) (\tanh^2 h_3^* + 2z_3^*) (1 + 2z_2 \cos p + z_2^2)(1 + \cos p)\right] \, dp,$$

where $\beta = 1/k_B T$, and $z_2 = \tanh K_2$, and $h_{1,3}^*$ and $K_{1,3}^*$ are coupled with $h$ and $K_{1,3}$ by the relations (3.3). One can show that, as it was done for the $1D$ and $2D$ Ising models \([1, 12]\), in the case of the states odd ($\lambda\mathcal{G} = -1$) with respect to the operator $\mathcal{G}$, (2.16), the formula for $F_{3D}(h)$ is described in the thermodynamic limit by (3.7). Let us note that the asymptotics (3.7) obtained above can be applied also in the case of rather strong magnetic fields $(H)$, as far as it satisfies the condition $(1 - \tanh h) \sim \varepsilon$, \( \varepsilon \ll 1 \), $(T = \text{const})$.  

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IV. FINAL REMARKS

The result derived above (3.7) can be applied to the analysis of equilibrium thermodynamics of the three dimensional Ising magnetic, lattice gas, and also three dimensional models of binary alloys [13, 14] in the region of temperatures and magnetic fields (1.3) derived above. Such analysis, as well as construction of appropriate phase diagrams for the models mentioned above is, in our opinion, of great interest. They deserve presentation in a separate publication. Therefore we deliberately do not compare here our result (3.7) with the existing papers devoted to this problem. The other important feature of the presented method is the possibility of deriving expressions for the free energy of the 3D Ising model in the limiting case of the magnetic field tending to zero ($H \rightarrow 0, \ N, M, K \rightarrow \infty$) if we know exact solution for the 3D Ising model in the absence of external magnetic field ($H = 0$). This possibility results from equations (3.2 - 3.3) describing renormalised interaction constants $K^*_1,3$, and corresponds, as was presented in the paper [7], to the results obtained by C.N.Yang [15] for the 2D Ising model.

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