

Prediction of the collapse point of overloaded materials by monitoring energy emissions

Srutarshi Pradhan
SINTEF Petroleum Research, NO-7465 Trondheim, Norway

Per C. Hemmer
Department of Physics, Norwegian University of Science and Technology, N0-7491 Trondheim, Norway

A bundle of many fibers with stochastically distributed breaking thresholds is considered as a model of composite materials. The fibers are assumed to share the load equally, and to obey Hookean elasticity up to the breaking point. The bundle is slightly overloaded, which leads to complete failure. We study the properties of emission bursts in which an amount of energy $E$ is released. The analysis shows that the size of the energy bursts has a minimum when the system is half-way from the collapse point.

PACS numbers: 02.50.-r

I. INTRODUCTION

During the failure process in composite materials under external load, bursts (avalanches) of different magnitudes are produced, where a burst consists of simultaneous rupture of several elements. At each failure, the sudden internal stress redistribution in the material is accompanied by a rapid release of mechanical energy. Therefore, with each accompanied by a burst there will be a corresponding energy emission burst. A useful experimental technique to monitor such energy bursts is to measure the acoustic emissions, the elastically radiated waves produced in the bursts [1, 2].

Fiber bundle models, with statistically distributed thresholds for the breakdown of the individual fibers, are interesting models of failure processes in materials. They are characterised by clear-cut rules for how stress caused by a failed element is redistributed on undamaged fibers. These models have been much studied since they can be analysed to an extent that is not possible for more complex materials (For reviews, see [3-8]). The statistical distribution of the magnitude of avalanches in fiber bundles is well studied [9-11], and the failure dynamics under constant load has been formulated through recursion relations which in turn explore the phase transitions and associated critical behavior in these models [12].

In this article we show that the catastrophic collapse point of an overloaded bundle can be predicted by monitoring the energy emission rate.

We consider a bundle consisting of a large number $N$ of fibers, clamped at both ends. We study equal-load-sharing models, in which the load previously carried by a failed fiber is shared equally by all the remaining undamaged fibers [13-16]. The fibers obey Hooke’s law, such that the energy stored at elongation $x$ equals $x^2/2$, where we for simplicity have set the elasticity constant equal to unity. Each fiber $i$ is associated with a breakdown threshold $x_i$ for its elongation. When its length exceeds $x_i$, the fiber breaks immediately, and does not contribute to the bundle strength thereafter. The individual thresholds $x_i$ are assumed to be independent random variables with the same cumulative distribution function $P(x)$ and a corresponding density function $p(x)$. If an external load $F$ is applied to a fiber bundle, the resulting breakdown events can be seen as a sequential process [10]. In the first step all fibers that cannot withstand the applied stress $(F/N)$ break. Then the stress is redistributed on the surviving fibers, which compels further fibers to fail, etc. This iterative process continues until all fibers fail, or an equilibrium situation with a nonzero bundle strength is reached. Since the number of fibers are finite, the total number of steps, $t_f$, in this sequential process is finite. At the stress (or elongation) $\sigma$ per surviving fiber the total force on the bundle is $\sigma$ times the number of intact fibers. The expected, or average, force at this stage is therefore

$$F(\sigma) = N \sigma \left[1 - P(\sigma)\right].$$

The maximum $F_c$ of $F(\sigma)$ corresponds to the value $\sigma_c$ for which $dF/d\sigma$ vanishes. Thus

$$1 - P(\sigma_c) - \sigma_c p(\sigma_c) = 0,$$

where the critical stress $\sigma_c$ is defined as

$$\sigma_c = F_c/N.$$ (3)

When the applied load is more than $F_c$ (or $\sigma > \sigma_c$), we call that the bundle is overloaded.

We can study the stepwise failure process in the fiber bundle when a fixed external load $F$ is applied. Then the initial external stress is $\sigma = F/N$. Let $N_t$ be the number of undamaged fibers at step no. $t$, with $N_0 = N$. 

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*Electronic address: srutarshi.pradhan@sintef.no
†Electronic address: per.hemmer@ntnu.no
[0] The Scottish Forth Road Bridge is supported by two main cables, each with 11618 fibers (wires), some of which have failed. Acoustic monitoring was installed in 2006 to detect further snapping of the wires. (See Forth Road Bridge in Wikipedia.)
We want to determine how $N_t$ decreases until the failure process stops. With $N_t$ intact fibers, an expected number

$$[N \ P(N\sigma/N_t)]$$  \hspace{1cm} (4)

of fibers will have thresholds that cannot withstand the load, and consequently these fibers break at once. Here $[X]$ denotes the largest integer not exceeding $X$. The number of intact fibers in the next step is therefore

$$N_{t+1} = N - [N \ P(N\sigma/N_t)],$$  \hspace{1cm} (5)

or, for all practical purposes,

$$n_{t+1} = 1 - P(\sigma/n_t).$$  \hspace{1cm} (6)

Here $n_t$ denotes the fraction of undamaged fibers at step $t$,

$$n_t = N_t/N$$  \hspace{1cm} (7)

In each burst a certain amount of elastic energy is released, and we consider now the energy emission process.

II. ENERGY RELEASE MINIMUM

For a burst in which the number of intact fibers is reduced from $N_{t-1}$ to $N_t$, all fibers with thresholds $x$ between the values $F/N_{t-1} = \sigma/n_{t-1}$ and $F/N_t = \sigma/n_t$ break. Since there are $Np(x) \ dx$ fibers with thresholds in $(x, x + dx)$, the energy emitted in this burst is given by

$$E_t = \int_{\sigma/n_{t-1}}^{\sigma/n_t} \frac{1}{2} x^2 N p(x) \ dx.$$  \hspace{1cm} (8)

We consider external stresses that are slightly above the critical value,

$$\sigma = \sigma_c + \epsilon,$$  \hspace{1cm} (9)

where $\epsilon$ is small and positive. Large stresses are less interesting, since the system will then break down quickly (See the Appendix). Simulations for a uniform threshold distribution, $p(x) = 1$ for $0 \leq x \leq 1$, show that the energy emission has a minimum at some value $t_E(\epsilon)$, and that for varying loads the minima all occur at value close to $\frac{1}{2}$ when plotted as function of the scaled variable $t/t_f$ (Fig. 1). Here $t_f$ is the number of iterations corresponding to complete bundle failure. We will show this analytically, and in addition demonstrate that the result is not limited to the uniform distribution. Fig. 1 shows that for large $N$ and near the minimum the energy emission $E_t$ appears to be an almost continuous function of $t$. The minimum is therefore located where the derivative of $E_t$ with respect to $t$ vanishes. From (5) we obtain

$$dE_t/dt = \frac{1}{2} N \left[-\sigma^2 n^{4-t} p(\sigma/n_t) \dot{n}_t + \sigma^2 n^{4-t} p(\sigma/n_{t-1}) \dot{n}_{t-1}\right]$$  \hspace{1cm} (10)

where the dot denotes differentiation with respect to time.

To eliminate quantities at time step $t - 1$ we use the connection (6), which gives

$$\dot{n}_t = \dot{n}_{t-1} p(\sigma/n_{t-1}) \sigma n_{t-1}^{-2}.$$  \hspace{1cm} (11)

Insertion into (10) yields

$$\frac{dE}{dt} = \dot{n}_t N \left[\frac{\sigma^3 p(\sigma/n_t)}{n_t^2} + \frac{\sigma^2}{n_{t-1}^2}\right].$$  \hspace{1cm} (12)

We also need $n_{t-1}$, Eq. (6) gives

$$\sigma/n_{t-1} = P^{-1}(1 - n_t),$$  \hspace{1cm} (13)

where $P^{-1}$ denotes the inverse function to $P$. In conclusion,
Since \( \dot{n}_t \) is always negative, the minimum of energy emission occurs when the relative number of intact fibers satisfies

\[
P^{-1}(1 - n_t) = \sqrt{\sigma^3 p(\sigma/n_t) n_t^{-4}},
\]

i.e.

\[
1 - n_t = P \left( \sqrt{\sigma^3 p(\sigma/n_t) n_t^{-4}} \right).
\]

Now we turn to specific cases.

### III. Explicit Results

We start with the simplest case, the uniform distribution, on which the simulations in Fig. 1 were based.

#### A. Uniform distribution

For the uniform threshold distribution, \( P(x) = x \) for \( 0 \leq x \leq 1 \), the condition (16) takes the form

\[
1 - n_t = \sqrt{\sigma^3 n_t^{-4}},
\]

or

\[
(1 - n_t)n_t^2 = \sigma^{3/2}.
\]

For this case the maximum of the force, Eq. (1), is \( F = N/4 \), corresponding to the critical stress \( \sigma_c = \frac{1}{4} \). For a small excess stress \( \epsilon = \sigma - \sigma_c = \sigma - \frac{1}{4} \), Eq. (15) takes the form

\[
(1 - n_t)n_t^2 = \left( \frac{1}{4} + \epsilon \right)^{3/2} = \frac{1}{5} \left( 1 + 6\epsilon + \mathcal{O}(\epsilon^2) \right).
\]

To lowest order \( n_t = \frac{1}{2} \), and one shows easily that

\[
n_t = \frac{1}{2} + 3\epsilon
\]

satisfies (19) to first order in \( \epsilon \).

To find \( t_E \), the value of \( t \) corresponding to (20) we use the previously derived solution of the iteration (6)

\[
n_t = \frac{1}{2} - \sqrt{\epsilon} \tan(At - B),
\]

where \( A = \tan^{-1}(2\sqrt{\epsilon}) \approx 2\sqrt{\epsilon} \) and \( B = \tan^{-1}(1/2\sqrt{\epsilon}) \).

We see that (20) requires \( \tan(At_E - B) = -3\sqrt{\epsilon} \), i.e.

\[
t_E - B/A = -3\sqrt{\epsilon}/A.
\]

To lowest order we therefore obtain

\[
t_E = B/A - \frac{3}{2}.
\]

Moreover we see from (21) that at for

\[
t = t_f = 2B/A
\]

we have

\[
n_{t_f} = \frac{1}{2} - \sqrt{\epsilon} \tan(B) = 0,
\]

signifying complete collapse of the fiber bundle. By comparison between (22) and (23) we obtain

\[
t_E = \frac{1}{2} t_f - \frac{3}{2}.
\]

Thus the minimum energy emission occurs almost halfway to complete bundle collapse, in agreement with the simulations shown in Fig. 1.

#### B. Weibull distribution

To illustrate the generality of the connection between the energy emission minimum and the bundle collapse, we now turn to a completely different threshold distribution, viz. a Weibull distribution of index 5,

\[
P(x) = 1 - e^{-x^5}.
\]

The critical stress for this distribution is \( \sigma_c = (5\epsilon)^{-1/5} = 0.5933994 \). Simulations reveal a qualitatively similar behavior as for the uniform distribution (Fig. 2).

The condition (10) for minimal energy emission takes in this case the form

\[
\ln n_t = 5^{5/2}\sigma^{35/2} n_t^{-20} \exp(-5\sigma^5/2n_t).
\]

It is straightforward to verify that for \( \sigma = \sigma_c = (5\epsilon)^{-1/5} \) (27) has the solution \( n_t = e^{-1/5} \). We are interested in a slightly overloaded bundle,

\[
\sigma = \sigma_c + \epsilon = (5\epsilon)^{-1/5} + \epsilon,
\]
with $\epsilon$ small and positive, and we seek the corresponding value of $n_t$. Putting $28$ and $n_t = e^{-1/5} + \delta$ into the condition $27$, and expanding in $\epsilon$ and $\delta$, we obtain to first order in the small quantities

$$e^{1/5} \delta = 3(5\epsilon)^{1/5} \epsilon - \frac{7}{9} e^{1/5} \delta. \tag{29}$$

Thus $\delta = \frac{7}{9} 5^{1/5} \epsilon$. Hence

$$n_t = e^{-1/5} + \frac{2}{9} 5^{1/5} \epsilon \tag{30}$$

is the relative number of undamaged fibers when the energy emission is minimal.

To find $t_E$, the value of $t$ corresponding to $30$, we take advantage of the ground work already done in $18$ for the Weibull distribution. Eq. (29) in $18$ shows that for small $\epsilon$ the iteration is of the form

$$n_t = e^{-1/5} - 5^{1/5} \sqrt{\epsilon/C} \tan(t\sqrt{\epsilon/C} - c), \tag{31}$$

with $C = \frac{5}{9} (5\epsilon)^{1/5}$ and the constant $c = \tan^{-1}[(1 - e^{-1/5}) 5^{-1/5} \sqrt{C/\epsilon}]$ ensures that the initial condition $n_0 = 1$ is satisfied.

Comparison between $30$ and $31$ gives

$$\tan(t_E\sqrt{\epsilon/C} - c) = -\frac{2}{9} \sqrt{\epsilon C}. \tag{32}$$

To dominating order, then,

$$t_E \sqrt{\epsilon C} - c = -\tan^{-1}(\frac{2}{9} \sqrt{\epsilon C}) \simeq -\frac{2}{9} \sqrt{\epsilon C}. \tag{33}$$

For small $\epsilon$ the constant $c$ is very close to $\pi/2$, so that in good approximation we have

$$t_E = \frac{\pi}{2\sqrt{\epsilon C}} - \frac{2}{9}. \tag{34}$$

The collapse time $t_f$ was also evaluated in $18$ to be

$$t_f = \frac{\pi}{\sqrt{\epsilon C}}. \tag{35}$$

Consequently we have

$$t_E = \frac{1}{5} t_f - \frac{2}{9}. \tag{36}$$

To excellent approximation the minimum of energy emission occur halfway to the final collapse also for the Weibull distribution. There is every reason to believe that this feature is general. Note that since $t_E$ and $t_f$ are large numbers, the constants $\frac{2}{9}$ and $\frac{7}{9}$ in $28$ and $30$ are of no significance.

### IV. CONCLUDING REMARKS

In summary, we have considered energy emission burst from slightly overloaded fiber bundles. During the degradation process there is a stage $t_E$ at which the energy emission is minimal, and we have demonstrated that the total bundle collapse occurs near $2t_E$. The demonstration has been performed merely for two very different distributions of the fiber breaking thresholds, but the result is doubtlessly universal. Thus the minimal energy emission gives an excellent estimate of when the bundle failure will take place. In our earlier work $19$ we found that the fiber breaking rate has a minimum at half way to complete collapse. However, for practical purposes energy emission burst is a better entity to measure than the fiber breaking rate.

### Acknowledgement

S. P. acknowledges financial support from Research Council of Norway (NFR) through project number 199970/S60.

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For a given stress \( \sigma \), the relative number of unbroken fibers at the emission minimum is given by Eq. (18),

\[
(1-n_t)n_t^2 = \sigma^{3/2}. \tag{37}
\]

Since the left-hand side of (37) does not exceed 4/27, the maximum stress when an emission minimum is present, equals

\[
\sigma_m = 2^{4/3}/9 = 0.279982. \tag{38}
\]

For a given stress, \( \sigma_c \leq \sigma \leq \sigma_m \), Eq. (37) determines \( n_t \), and the corresponding value of \( t_E/t_f \) can be calculated using

\[
n_t = \frac{1}{2} + \sqrt{\frac{\epsilon \tan[B(1-2t_E/t_f)]}{}} \tag{39}
\]

with \( B = \tan^{-1}(1/2\sqrt{\epsilon}) \). We have combined Eqs. (21) and (23) in the main text. Fig. 3 shows the result. The position of the emission minimum, relative to \( t_f \), decreases slowly with increasing stress. In this derivation the integer \( t \) is considered as a continuous variable. This treatment is satisfactory for slightly overloaded bundles, but less so when the \( t_E \) is smaller. To find the smallest stress that produces a minimum, we consider the first few energy emissions. Eq. (8) for the uniform threshold distribution,

\[
E_t = \frac{1}{6}N\sigma^3\left(n_t^{-3} - n_{t-1}^{-3}\right), \tag{40}
\]

and the iteration (6), \( n_0 = 1, n_1 = 1-\sigma, n_2 = \frac{1-2\sigma}{1-\sigma} \), etc., gives

\[
E_0 = \sigma^3 \tag{41}
\]
\[
E_1 = \left(\frac{\sigma}{1-\sigma}\right)^3 - \sigma^3 \tag{42}
\]
\[
E_2 = \left(\frac{\sigma(1-\sigma)}{1-2\sigma}\right)^3 - \left(\frac{\sigma}{1-\sigma}\right)^3 \tag{43}
\]
\[
E_3 = \left(\frac{\sigma(1-2\sigma)}{1-3\sigma+\sigma^2}\right)^3 - \left(\frac{\sigma(1-\sigma)}{1-2\sigma}\right)^3 \tag{44}
\]

where the common factor \( N/6 \) is omitted. For all subcritical stresses \( E_1 > E_0 \), but a local minimum may occur already at \( t = 2 \). It is easy to show that

\[
E_1 = E_2 \text{ for } \sigma = \sigma_1 = 0.278764 \tag{45}
\]
\[
E_2 = E_3 \text{ for } \sigma = \sigma_2 = 0.273054. \tag{46}
\]

For \( \sigma \geq \sigma_1 \) the energy emission bursts increase monotonically in size. For \( \sigma_2 < \sigma < \sigma_1 \) we have \( E_2 < E_1 \) and also \( E_2 < E_3 \). Hence there is a local emission minimum at \( t_E = 2 \) in this range.

Thus a local emission minimum is no longer present when the stress value exceeds \( \sigma_1 \). For the limiting value \( \sigma = \sigma_1 \), the bundle collapse occurs at \( t_f = 8 \), signified by the first non-positive value of \( n_t \). This gives \( t_E/t_f = 0.25 \) at \( \sigma = \sigma_1 \).

This limiting value is a little lower than the value \( \sigma_m \), the end point of the graph in Fig. 3, obtained by the continuous procedure above. However, at the maximum stress \( \sigma_1 \) that produces a local emission minimum, the value of \( t_E/t_f \) equals 0.230, not far from the exact value 0.25. We conclude that the graph in Fig. 3 is very precise for small supercritical stresses and fairly accurate up to the stress \( \sigma_1 \).