SOME EXAMPLES OF SINGULAR FLUID FLOWS

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Abstract. We explain the construction of some solutions of the Stokes system with a given set of singular points, in the sense of Caffarelli, Kohn and Nirenberg [1]. By means of a partial regularity theorem (proved elsewhere), it turns out that we are able to show the existence of a suitable weak solution to the Navier-Stokes equations with a singular set of positive one dimensional Hausdorff measure.

1. Introduction

There is a wide interest on the problem of regularity for the Navier-Stokes equations. One of the most interesting achievement was gained by Caffarelli, Kohn and Nirenberg in 1982. They showed that a (suitable) weak solution $u$ to the Navier-Stokes equations has a set of singular points of null one-dimensional Hausdorff measure. In their definition, a regular point for a solution is a space-time point $(t, x)$ such that the solution is essentially bounded in a small neighbourhood of $(t, x)$. A singular point is then a space-time point which is not regular. In the sequel, we will call singular set the set of all singular points for a weak solution $u$.

A few years later Scheffer in [7], [8] and [9], found some examples of solutions to the Navier-Stokes inequality (namely, vector fields that satisfy only the local energy inequality, but not necessarily the equations) having a nearly one dimensional singular set. Scheffer pointed out that the limit of the Caffarelli, Kohn and Nirenberg theorem rested in the energy method, so that a clever, complete use of the equations could give a better result.

We are indebted with some of Scheffer’s ideas. Since we deal with a linear equation, our computations are simpler. Nevertheless we ends up with a solution to the Navier-Stokes equations having a singular set whose dimension is bigger than the one of the Scheffer’s example. This is not in contrast to the Caffarelli, Kohn and Nirenberg theorem [1], since our body force is less regular (see Remark 4.2).

In view of these considerations, throughout the paper we will call thin a set having null one-dimensional Hausdorff measure, and fat a set having positive, possibly infinite, one-dimensional Hausdorff measure. Moreover, we will denote by $S(u)$ the singular set of a vector field $u$.

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2. Main results

We will present two different construction of solutions to the Stokes equation

\[
\partial_t z - \Delta z + \nabla Q = f
\]
\[
\text{div } z = 0
\]

with a given fat singular set. The main idea underlying the first construction is that the solution has a property of self-similarity, in a sense related to the one given by Leray \cite{4}: it is invariant under the scaling

\[ z(t, x) \mapsto \lambda z(\lambda^2 t, \lambda x). \]

on a fixed countable set of time intervals. In other words, we start from a solution to the Stokes equation in a small time interval. Afterwards, the solution is extended in the next interval by the self-similarity property, and so on. After a finite time the singular set of the solution contains a given fractal set.

In the second example we complete the same construction on \( \mathbb{R}^2 \), then we obtain an axially-symmetric solution on \( \mathbb{R}^3 \). The main interest of this example is that it can be used to obtain a solution to the Navier-Stokes equation with a fat singular set. For this purpose we use a partial regularity result proved in \cite{3}. The last part of the section is devoted to the explanation of this result and of its application on our example.

2.1. Solutions to the Stokes system with a fat singular set. The first construction we explain fails to be a solution with a fat singular set, if at least we want to use it for the Navier-Stokes equations (see Remark 2.2).

Anyway this example has some interest: firstly there is a more general expression of the body force (it is composed of a mean term and a fluctuation). Moreover it should be possible to use this construction in the second example below (see Proposition 2.3) to obtain in that case a even fatter singular set. Finally the second example exploits some of the idea of the first one, so that the construction should be plainer. The following proposition is proved in Section 3.

**Proposition 2.1.** Let \( \alpha \in (0, \frac{3}{2}) \), then there exist \( z_0 : \mathbb{R}^3 \to \mathbb{R}^3 \), \( f : \mathbb{R}^+ \times \mathbb{R}^3 \to \mathbb{R}^3 \) and \( g : \mathbb{R}^+ \times \mathbb{R}^3 \to \mathbb{R}^3 \) such that the solution \( z \) of the Stokes equation (2.1) has the following properties

(i) \( z \in L^\infty(0, \infty; [L^2(\mathbb{R}^3)]^3) \cap L^2(0, \infty; [H^1(\mathbb{R}^3)]^3) \),

(ii) \( z \) has compact support in \( [0, \infty) \times \mathbb{R}^3 \),

(iii) \( z(t) \in C^\infty(\mathbb{R}^3) \) for each \( t \geq 0 \),

(iv) the singular set \( S(z) \) has at least Hausdorff dimension \( \alpha \).

**Remark 2.2.** Unfortunately, the solution given by the proposition above does not satisfy the set of assumptions (B) below, so that it cannot be used to obtain solutions to the Navier-Stokes equation with a fat singular set (see Remark 3.3).

From now to the end of this section, we will consider for simplicity \( g \equiv 0 \). In order to have a solution of the Stokes system with a fat singular set satisfying the set of assumptions (B), we need to modify the previous
construction. The idea is to complete the same construction we did for the solution in Proposition 2.1 on a plane and then to consider an axially-symmetric solution in the space. Note that the equation solved by the 2D-solution is different, since at the end we want the 3D-solution to solve the Stokes equation. We obtain the following result, which will be proved in Section 4.

Proposition 2.3. There exist $z_0 : \mathbb{R}^3 \to \mathbb{R}^3$ and $f : \mathbb{R}_+ \times \mathbb{R}^3 \to \mathbb{R}^3$ such that for the solution $z : \mathbb{R}_+ \times \mathbb{R}^3 \to \mathbb{R}^3$ of the Stokes system (2.1) the following properties hold

(i) $z \in L^\infty(0, \infty; (L^2(\mathbb{R}^3))^3) \cap L^2(0, \infty; (H^1(\mathbb{R}^3))^3)$;
(ii) $z$ has compact support in $[0, \infty) \times \mathbb{R}^3$;
(iii) $z(t) \in C^\infty(\mathbb{R}^3)$ for all $t \geq 0$;
(iv) $z$ satisfies the assumptions (B) below;
(v) $H^1(S(z)) > 0$.

Remark 2.4. It should be possible to obtain a fatter singular set than the one obtained in the previous proposition. The idea is to apply on the 2D solution the same construction as the one given in Proposition 2.1, to obtain an axially-symmetric solution with a singular set of dimension larger than one. Indeed, the example contained in Proposition 2.1 has been given to suggest this possibility. We have not tried to show this claim because of the huge amount of annoying computations. Moreover our point is to show the existence of suitable weak solutions to the Navier-Stokes equations with a singular set fatter than the one in Caffarelli, Kohn and Nirenberg [1]. This will be given in the next section.

2.2. A suitable weak solution of Navier-Stokes equation with a fat singular set. We want to apply now the construction we have given in the previous section to the theory of singularities for the Navier-Stokes equations. In [3], the theory of singularities has been studied in the following way. Consider the Navier-Stokes equations in a bounded open domain $D \subset \mathbb{R}^3$, with regular boundary,

$$
\begin{align*}
\partial_t u - \nu \Delta u + (u \cdot \nabla)u + \nabla P &= f + \partial_t g, \\
\text{div } u &= 0, \\
u u(0) &= u_0, \\
$u(t, \cdot)$ &= 0 \quad \text{on } \partial D,
\end{align*}
$$

(2.2)

where the term $f$ represents the mean force and the term $\partial_t g$ represents the fluctuations. The solution of the equation is split

$$u = v + z, \quad P = \pi + Q,$$

where $(z, Q)$ is the solution of the Stokes problem

$$
\begin{align*}
\partial_t z - \nu \Delta z + \nabla Q &= f + \partial_t g, \\
\text{div } z &= 0, \\
z(0) &= 0, \\
z(t, \cdot) &= 0 \quad \text{on } \partial D,
\end{align*}
$$

(2.3)
and \((v, \pi)\) solves the modified Navier-Stokes equations
\[
\partial_t v - \nu \Delta v + ((v + z) \cdot \nabla) (v + z) + \nabla \pi = 0,
\]
(2.4)
\[
\text{div } v = 0,
\]
\[
v(0) = u_0,
\]
\[
v(t, \cdot) = 0 \quad \text{on } \partial D,
\]

A different notion of suitable weak solution (introduced mostly to treat the term \(\partial_t g\)) can be given.

**Definition 2.5.** Let \((z, Q)\) be the solution of the Stokes problem. A suitable weak solution \((u, P)\) to the Navier-Stokes equations (2.2) is a pair
\[
u \in L^\infty([0, \infty); [L^2(D)]^3) \cap L^2_{\text{loc}}([0, \infty); [H^1_0(D)]^3)
\]
and
\[
P \in L^{5/3}_{\text{loc}}((0, T) \times D)
\]
such that the new variables \(v = u - z\) and \(\pi = P - Q\) satisfy the modified Navier-Stokes equations (2.4) in the sense of distributions over \((0, \infty) \times D\) and moreover satisfy the local energy inequality
\[
\int_D |v(t)|^2 \varphi + 2 \int_0^t \int_D \varphi |\nabla v|^2 \leq \int_0^t \int_D \varphi |v|^2 \left( \frac{\partial \varphi}{\partial t} + \Delta \varphi \right) + \int_0^t \int_D \left( |v|^2 + 2v \cdot z \right) ((v + z) \cdot \nabla \varphi)
\]
(2.5)
\[
+ 2 \int_0^t \int_D \varphi z \cdot ((v + z) \cdot \nabla) v + \int_0^t \int_D 2\pi v \cdot \nabla \varphi
\]
for every smooth function \(\varphi : \mathbb{R} \times D \to \mathbb{R}, \varphi \geq 0\), with compact support in \((0, T] \times D\).

**Remark 2.6.** For some comments on this definition and on the connection with the suitable weak solutions, as defined by Caffarelli, Kohn and Nirenberg [1], see [3] and [5].

The forcing terms \(f, g\) are taken in the following way
\[
(A) \quad f \in L^p_{\text{loc}}((0, \infty) \times D), \quad p > 2,
\]
\[
g \in C^{4-\varepsilon}([0, \infty); H^{2\beta}(D)), \quad g(0) = 0, \quad \beta > \varepsilon > 0.
\]
The set of assumptions (A) implies (see the appendix in [3])
\[
(B) \quad z \in L^\infty_{\text{loc}}([0, \infty); L^2(D)) \cap L^2_{\text{loc}}([0, \infty); H^1_0(D)), \quad z \in L^\infty_{\text{loc}}([0, \infty); L^q_{\text{loc}}(D)) \quad \text{for } q > 6,
\]
\[
\lim_{r \to 0} \frac{1}{r} \int_{Q_r(t, x)} |\nabla z|^2 \, ds \, dy = 0 \quad \text{for all } (t, x),
\]
where \(Q_r(t, x) = \{ (s, y) | t - r^2 < s < t, \, |x - y| < r \} \). Finally, it is shown (see Theorem 5.3 in [3]) the following result.

**Theorem 2.7.** Assume the set of assumptions (B) for \(z\). Let \(v\) be the solution of (2.4) relative to \(z\) and satisfying the local energy inequality (2.3). Then
\[
\mathcal{H}^1(S(v)) = 0
\]
In order to give the same conclusion of the previous theorem for the true solution of Navier-Stokes $u$, we assume the following condition

\[(C) \quad f \in L^p_{\text{loc}}((0, \infty) \times D), \quad p > \frac{5}{2}, \quad g \in C^{\frac{1}{2}-\varepsilon}([0, \infty); H^{\frac{1}{2}+2\beta}(D)), \quad g(0) = 0, \quad \beta > \varepsilon > 0.\]

In fact the set of assumptions $\mathbb{A}$ implies that $z \in L^\infty_{\text{loc}}((0, \infty) \times D)$.

The idea is now to exploit the gap between the set of assumptions $\mathbb{A}$ and the set of assumptions $\mathbb{B}$, in order to find a suitable weak solution of Navier-Stokes equations having a singular set bigger than it is stated in $\mathbb{B}$.

Indeed, Proposition 2.3 above let us consider a vector field $z$ having a fat singular set and satisfying the set of assumptions $\mathbb{B}$. Consequently the solution $v$ of (2.4) has a thin singular set, so that

$$u = v + z$$

will have a fat singular set. The complete proof of the following theorem, which extends the argument above, is given in Section 4.

**Theorem 2.8.** There exist $u_0 : \mathbb{R}^3 \to \mathbb{R}^3$, $f : \mathbb{R}^+ \times \mathbb{R}^3 \to \mathbb{R}^3$, a bounded regular domain $D$ and a suitable weak solution $u$ of the Navier-Stokes system, in the sense of Definition 2.5, such that

$$\mathcal{H}^1(S(u)) > 0.$$

**Remark 2.9.** It should be interesting to find which are the minimal assumptions on the forcing terms (like $\mathbb{A}$ or $\mathbb{B}$) which give a singular set for $z$ of null one-dimensional Hausdorff measure. For example it may be guessed, with some heuristics, the critical summability for $f$. In $\mathbb{B}$ it is suggested a way to show this. We know nothing about the term $g$.

### 3. A Stokes vector field with a fractal singular set

In this section we prove Proposition 2.1. First we point out that it is possible to get rid of the divergence-free condition. In fact the following construction can be performed separately on the three components of a given divergence-free initial condition, giving a solution $z$ and the body forces $f$, $g$ that will be divergence-free vector fields. Consequently there is no loss of generality in working with a real valued $z$.

For the sake of clarity, the construction is divided into three steps. In the first step a solution which has a singular point is built, in the second step the geometry of the fractal set is explained and finally in the third step the arguments of the previous steps are glued together to obtain the singular solution.

#### 3.1. A solution with a singular point.

Consider a function $z_0 \in C^\infty_c(\mathbb{R}^3)$ such that $z_0(0) = 1$ and $z_0 \geq 0$. Fix two parameters $\sigma \in (0, 1)$ and $\lambda > 1$. Set for $N \geq 1$,

$$\sigma_0 = 0, \quad \sigma_N = \sum_{k=1}^{N} \sigma^k \quad \text{and} \quad T = \sum_{k=1}^{\infty} \sigma^k.$$
and \( I_N = [\sigma_{N-1}, \sigma_N] \). Let \( f_1, g_1 \) be functions in \( I_1 \times \mathbb{R}^3 \) (without loss of generality they can be taken smooth and with compact support), with \( g_1(0) = 0 \), such that the solution \( z_1 \) of the Stokes equation is positive, smooth, with compact support and

\[
    z_1(0, x) = z_0(x), \quad z_1(\sigma, x) = \lambda z_0\left(\frac{x}{\sqrt{\sigma}}\right)
\]

\[
    \|z_1(t)\|_{L^p(\mathbb{R}^3)} \leq C\|z_0\|_{L^p(\mathbb{R}^3)} \quad \text{for each } p \geq 2 \text{ and } t \geq 0.
\]

Then for \( N \geq 2 \), functions \( z_N, f_N \) and \( g_N \) are defined inductively in \( I_N \times \mathbb{R}^3 \) as

\[
    z_N(t, x) = \lambda z_{N-1}\left(\frac{t - \sigma}{\sigma}, \frac{x}{\sqrt{\sigma}}\right),
\]

\[
    f_N(t, x) = \frac{\lambda}{\sigma} f_{N-1}\left(\frac{t - \sigma}{\sigma}, \frac{x}{\sqrt{\sigma}}\right),
\]

\[
    g_N(t, x) = g_{N-1}(\sigma_{N-1}) + \lambda \left[ g_{N-1}\left(\frac{t - \sigma}{\sigma}, \frac{x}{\sqrt{\sigma}}\right) - g_{N-1}(\sigma_{N-2}, \frac{x}{\sqrt{\sigma}}) \right].
\]

Finally set

\[
    z(t, x) = z_N(t, x), \quad f(t, x) = f_N(t, x) \quad \text{and } g(t, x) = g_N(t, x)
\]

if \( t \in I_N \) and \( z(t, x) = f(t, x) = g(t, x) = 0 \) if \( t \geq T \).

It is easy to verify that, if

\[
    \lambda \sigma^{3/4} < 1,
\]

then \( z \in L^2(0, \infty; L^2(\mathbb{R}^3)) \cap L^2(0, \infty; H^1(\mathbb{R}^3)) \) and \( z \) solves the equation in the sense of distributions. Finally it can be verified that

\[
    z(\sigma_N, 0) = z_N(\sigma_N, 0) = \lambda^N z_0(0) = \lambda^N
\]

and so the point \((T, 0)\) is a singular point for \( z \).

**Remark 3.1.** It is easy to see that \( f \in L^p(\mathbb{R}_+ \times \mathbb{R}^3) \) if

\[
    \lambda^p \sigma^{\frac{3}{2} - p} < 1,
\]

while \( g \in C^{\frac{3}{2} - \varepsilon}([0, \infty); H^{2\beta}(D)) \) if

\[
    \lambda \sigma^{\frac{3}{4} - (\beta - \varepsilon)} < 1.
\]

### 3.2. The geometrical construction.

A classical construction of self-similar fractal sets will be used and it will give a generalised Cantor set (see for example David, Semmes [4]). Fix two integers \( k \) and \( m \leq k^3 \) and consider the unit cube \( Q \) in \( \mathbb{R}^3 \). The first step of the construction is to divide the cube in \( k^3 \) equal cubes and to choose \( m \) of them.

The second step is to perform the same operation on each of these \( m \) cubes: take each of them, divide it in \( k^3 \) parts and select \( m \) of these smaller cubes in the same relative positions as in the first step. Now there are \( m^2 \) cubes, the next step is to apply this transformation to each of them to obtain \( m^3 \) cubes and so on (see the figure which shows the first three iterations with \( k = 5 \) and \( m = 8 \)).

The fractal set \( C_{k,m} \) will be the only compact subset of \( \mathbb{R}^3 \) which is invariant under the following transformation. If \( C_{k,m} \) is re-scaled with a factor \( k^{-1} \), \( m \) copies of the re-scaled set are made and each copy is put in
the place of each of the cubes chosen, then $C_{k,m}$ is obtained again. So it is easy to calculate the Hausdorff dimension $\alpha$ of $C_{k,m}$, in fact

$$\frac{m}{k^\alpha} = 1,$$

that gives $\alpha = \frac{\log m}{\log k}$

(see for example David and Semmes [2]).

3.3. A fractal set of singularities. The construction of the fractal set and the construction of the first step are mixed up together: there the starting point was a solution in the first interval $I_1$ that was $z_0$ in the beginning and a re-scaling of $z_0$ at the end. Now the basic solution will be constructed as a function which solves the equation, it is equal to $z_0$ at time $t = 0$ and is equal to $m$ re-scaling of $z_0$ placed in the points corresponding to point $x = 0$ in the construction of the fractal set. At each step the construction will be iterated, following the outline of the procedure of the construction of the fractal set.

More precisely, fix integers $k$ and $m$ as above and set $\sigma = k^{-2}$ and $\sigma_N$ and $T$ as in the previous paragraph. Consider the $m$ points $x_1, x_2, \ldots, x_m$ vertices of the $m$ chosen small cubes which are in the same position as $x = 0$ in the bigger cube and set

$$\beta_i(x) = k(x - x_i) = \frac{x - x_i}{\sqrt{\sigma}} \quad i = 1, \ldots, m$$

and for each function $u = u(t,x)$,

$$(S_i(u))(t,x) = u(\frac{t - \sigma}{\sigma}, \beta_i(x)).$$

Fix $i \in \{1, \ldots, m\}$ and define functions $Z_i$, $F_i$ and $G_i$, with $G_i(0) = 0$, in such a way that $Z_i$ is the solution of the equation with forcing term $F_i + \partial_t G_i$

and

$$Z_i(0,x) = \frac{1}{m} z_0(x) \quad Z_i(\sigma, x) = \frac{\lambda}{m} z_0(\beta_i(x)),$$

with $Z_i$ enjoying the same properties as the basic solution of the first part.

For each integer $N \geq 2$ and for all $i \in \{1, \ldots, m\}$ we define in $I_N \times \mathbb{R}^3$ the following functions

$$Z_N = \sum_{i=1}^m (Z_{N-1})_i, \quad F_N = \sum_{i=1}^m (F_{N-1})_i \quad \text{and} \quad G_N = \sum_{i=1}^m (G_{N-1})_i,$$
Proposition 3.2. The following properties hold

(i) for each $N \geq 1$, $z_N(\sigma_{N-1}) = z_{N-1}(\sigma_{N-1})$ and

$$z_{N-1}(\sigma_{N-1}, x) = \sum_{i_1, \ldots, i_{N-1}=1}^{m} \left( \frac{\lambda}{m} \right)^{N-1} z_0(\beta_{i_1} \circ \ldots \circ \beta_{i_N}(x)),$$

(ii) if $\lambda \sigma < 1$, $z$ is a solution in the sense of distributions of the equation with body force $f + \partial_t g$,

(iii) if $\lambda \sigma^{\frac{3}{4}} < 1$ then $z \in L^\infty(0, \infty; L^2(\mathbb{R}^3)) \cap L^2(0, \infty; H^1(\mathbb{R}^3))$.

Finally, the fractal set constructed in the previous paragraph is contained in the singular set of $z$. Indeed, define subsets $A_0 = \{0\} \subset \mathbb{R}^3$, $A_1 = \{x_1, \ldots, x_N\}$ and for arbitrary $N$

$$A_N = \{x \in \mathbb{R}^3 \mid \text{there exist } i_1, \ldots, i_N \text{ such that } \beta_{i_1} \circ \ldots \circ \beta_{i_N}(x) = 0 \}.$$ 

Each point of the fractal set $C_{k,m}$ is the limit of a sequence $(y_N)_{N \in \mathbb{N}}$, with $y_N \in A_N$, so it is sufficient to show that $z(\sigma_N, x)$, with $x \in A_N$, goes to infinity as $N$ goes to infinity. In fact let $x \in A_N$ and let $j_1, \ldots, j_N$ be such that $\beta_{j_1} \circ \ldots \circ \beta_{j_N}(x) = 0$, then

$$z(\sigma_N, x) = z_N(\sigma_N, x) = \sum_{i_1, \ldots, i_{N-1}=1}^{m} \left( \frac{\lambda}{m} \right)^{N} z_0(\beta_{i_1} \circ \ldots \circ \beta_{i_N}(x)) \geq \left( \frac{\lambda}{m} \right)^{N} z_0(\beta_{j_1} \circ \ldots \circ \beta_{j_N}(x)) = \left( \frac{\lambda}{m} \right)^{N},$$

since $z_0$ is positive. In conclusion if $\lambda > m$, the claim is true.

The condition $\lambda \sigma^{\frac{3}{4}} < 1$ implies that $z$ is a solution of finite energy, and this forces the Hausdorff dimension of the singular set to be any number less than $\frac{3}{2}$ since

$$\dim_H C_{k,m} = \frac{\log m}{\log k} < \frac{\log \lambda}{\log k} < 2 \frac{\log \lambda}{-\log \sigma} < 2 \frac{3}{4} = \frac{3}{2},$$

and $\lambda$ can be arbitrarily close to $m$. 

Remark 3.3. Again, we can check that if \( \lambda^p \sigma^{\frac{5}{2}} < 1 \) then \( f \in L^p((0, \infty) \times \mathbb{R}^3) \) and, if \( \lambda^{\frac{1}{2} - (\beta - \varepsilon)} \leq 1 \) then \( g \in C^1(0, \infty; H^{2\beta}(D)) \).

Moreover \( z \in L^\infty(0, \infty; L^q(D)) \) if \( \lambda^q \sigma^{\frac{3}{2}} < 1. \) Therefore, in view of assumptions (B), we can give an estimate on the Hausdorff dimension of the set \( C_{k,m} \) in dependence of \( q \):

\[
\dim_H C_{k,m} < \frac{3}{q},
\]

so that if \( q > 6 \), the dimension is smaller than \( \frac{1}{2} \).

4. The axially-symmetric construction

In this section we will prove Proposition 2.3 and Theorem 2.8. The first lemma gives the axially-symmetric construction for the solution of equation (4.1). The Proposition 2.3 will be proved once we use the lemma to obtain a divergence-free vector field enjoying the same properties. In the second part of the section, Theorem 2.8 is proved.

Lemma 4.1. There exist \( Z_0 : \mathbb{R}^3 \to \mathbb{R} \) and \( F : \mathbb{R}^+ \times \mathbb{R}^3 \to \mathbb{R} \) such that for the solution \( Z : \mathbb{R}^+ \times \mathbb{R}^3 \to \mathbb{R} \) of the equation

\[
\partial_t Z - \Delta Z = F \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^3,
\]

\[
Z(0) = Z_0
\]

the following properties hold

(i) \( Z \in L^\infty(0, \infty; L^2(\mathbb{R}^3)) \cap L^2(0, \infty; H^1(\mathbb{R}^3)) \);

(ii) \( Z \) has compact support in \( [0, \infty) \times \mathbb{R}^3 \);

(iii) \( Z(t) \in C^\infty(\mathbb{R}^3) \) for all \( t \geq 0 \);

(iv) \( Z \) satisfies the assumptions (B);

(v) \( H^1(S(Z)) > 0 \).

Proof. We consider \( \lambda, \sigma, \sigma_N \) and \( I_N \) as in the previous section, we assume

\( \lambda^q \sigma < 1 \quad \text{for a } q > 6 \)

and we suppose also that \( \sigma < \frac{1}{2} \). Moreover we set

\[
\rho_N = \sum_{k=1}^N (\sqrt{\sigma})^k \quad \text{and} \quad \rho_\infty = \sum_{N=1}^\infty (\sqrt{\sigma})^N.
\]

Let \( z_0 = z_0(\rho, y) \in C^\infty_0(\mathbb{R}^2) \) be a function such that

\[
\supp z_0 \subset (-1, 1) \times (-M, M) \quad z_0(0, 0) = 1
\]

\[
\frac{\partial^k z_0}{\partial \rho^k}(t, 0, y) = 0, \quad \text{for } k = 1, 2,
\]

for some \( M > 0 \). Then there exist \( z_1 : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R} \) and \( f_1 : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R} \)

such that

\[
\frac{\partial z_1}{\partial t} - \Delta z_1 - \frac{1}{\rho} \frac{\partial z_1}{\partial \rho} = f_1 \quad t \in I_1, \quad (\rho, y) \in \mathbb{R}^2.
\]

Moreover

\[
z_1(0, \rho, y) = z_0(\rho, y) \quad \text{and} \quad z_1(\sigma, \rho, y) = \lambda z_0\left(\rho - \sqrt{\sigma}, \frac{y}{\sqrt{\sigma}}\right)
\]
and
\[ \int_{\mathbb{R}^2} |z_1(t)|^q \leq C(q) \quad \text{for all } t \in I_1 \text{ and each } q \geq 1. \]

If \( N \geq 2 \) and \( t \in I_N \), we set for \((\rho, y) \in \mathbb{R}^2\),

\[ z_N(t, \rho, y) = \lambda z_{N-1}(\frac{t-\sigma}{\sigma}, \frac{\rho-\sqrt{\sigma}}{\sqrt{\sigma}}, \frac{y}{\sqrt{\sigma}}) \tag{4.4} \]

and
\[ f_N(t, \rho, y) = \frac{\lambda}{\sigma} f_{N-1}(\frac{t-\sigma}{\sigma}, \frac{\rho-\sqrt{\sigma}}{\sqrt{\sigma}}, \frac{y}{\sqrt{\sigma}}) \]

Finally we set
\[ Z(t, x) = \begin{cases} z_N(t, \sqrt{x_1^2 + x_2^2}, x_3) & \text{if } t \in I_N, \\ 0 & \text{if } t \geq T, \end{cases} \]

and
\[ F(t, x) = \begin{cases} f_N(t, \sqrt{x_1^2 + x_2^2}, x_3) & \text{if } t \in I_N, \\ 0 & \text{if } t \geq T, \end{cases} \]

First, we observe that \((\sigma_N, \rho_N, 0) \to (T, \rho_\infty, 0)\) and
\[ z_N(\sigma_N, \rho_N, 0) = \ldots = \lambda^N z_0(0, 0) = \lambda^N, \]
so that
\[ \{(t, x) \in \mathbb{R}_+ \times \mathbb{R}^3 \mid t = T, \ x_1^2 + x_2^2 = \rho_\infty^2, \ x_3 = 0 \} \subset S(Z) \]
and property \((v)\) holds true.

Then we prove property \((ii)\). By assumptions \((4.2)\) we can deduce that \(\text{Supp } z_1(t) \subset (-1, 1) \times (-M, M)\), so that, using \((4.3)\), for \( t \in I_N \),
\[ \text{Supp } z_N(t) \subset (\rho_{N-1}, \rho_N + \sigma^{N/2}) \times (-\sigma^{\frac{N-1}{2}} M, \sigma^{\frac{N-1}{2}} M), \tag{4.5} \]
and property \((ii)\) holds true.

Property \((iii)\) is obvious. We prove property \((i)\). First we show that \(Z \in L^2(0, \infty; H^1(\mathbb{R}^3))\). Note that, if \( t \in I_N \),
\[ |\nabla Z(t, x)| = |\nabla z_N(t, \sqrt{x_1^2 + x_2^2}, x_3)| \]
and, using \((4.2)\), this is true for all \( x \in \mathbb{R}^3 \). Then by the change of variable \( x_1 = \rho \cos \theta, \ x_2 = \rho \sin \theta \) and \( x_3 = y \),
\[ \int_{0}^{+\infty} \int_{\mathbb{R}^3} |\nabla Z(t, x)|^2 \, dx \, dt = 2\pi \sum_{N=1}^{\infty} \int_{I_N} \int_{\mathbb{R}} \int_{0}^{\infty} \rho |\nabla z_N(t, \rho, y)|^2 \, d\rho \, dy \, dt \]
and, by formula (4.4) and iterating a change of variables $N - 1$ times,
\[
\int_{I_N} \int_{I_{N-1}} \int_{I_{N-2}} \int_{I_{N-3}} \int_{I_{N-4}} \int_{I_{N-5}} \rho |\nabla z_{N-1}(t, \rho, y)|^2 \, d\rho \, dy \, dt = 
\]
\[
= \lambda^2 \sigma \int_{I_{N-1}} \int_{I_{N-2}} \int_{I_{N-3}} \int_{I_{N-4}} \int_{I_{N-5}} \int_{I_{N-6}} (\rho_1 + \sqrt{\sigma} \rho) |\nabla z_{N-1}(t, \rho, y)|^2 \, d\rho \, dy \, dt = \ldots = 
\]
\[
= (\lambda^2 \sigma)^{N-1} \int_{I_1} \int_{I_2} \int_{I_3} \int_{I_4} \int_{I_5} (\rho_{N-1} + \sigma \frac{N-1}{\sigma} \rho) |\nabla z_1(t, \rho, y)|^2 \, d\rho \, dy \, dt 
\]
\[
\leq (\lambda^2 \sigma)^{N-1} (1 + \rho_\infty) \int_{I_1} \int_{I_2} \int_{I_3} |\nabla z_1(t, \rho, y)|^2 \, d\rho \, dy \, dt,
\]
where the last bound comes from formula (4.5).

In the same way, it is possible to show that $Z \in L^\infty(0, \infty; L^q(\mathbb{R}^3))$.

In order to show property (iv), it is sufficient to show that
\[
\lim_{N \to \infty} \frac{1}{r_N} \int_{Q_{r_N}} |\nabla Z|^2 = 0
\]
for $r_N = \sqrt{T \sigma^N}$ and a cylinder $Q_{r_N}$ centred in a point $(T, x^0)$ (for the other points the property is obvious). In fact, by similar computations as in the proof of property (i), we obtain
\[
\int_{\sigma_N}^{T} \int_{B_{r_N}} |\nabla Z|^2 \, dx \, dt = 2\pi \sum_{k=N+1} \int_{I_{N+1}} \int_{I_{N+2}} \int_{I_{N+3}} \int_{I_{N+4}} \int_{I_{N+5}} \rho |\nabla z_{N+1}|^2 \, d\rho \, dy \, dt
\]
and so, by $N$ changes of variables and using (4.5),
\[
\int_{I_{N+1}} \int_{I_{N+2}} \int_{I_{N+3}} \int_{I_{N+4}} \int_{I_{N+5}} \rho |\nabla z_{N+1}|^2 \, d\rho \, dy \, dt \leq C(\lambda^2 \sigma)^N,
\]
where $C$ is a constant independent of $N$. Finally, if $r \in (\sqrt{T \sigma^{N+1}}, \sqrt{T \sigma^N})$,
\[
\frac{1}{r} \int_{T-r^2}^{T} \int_{B_r(x^0)} |\nabla Z|^2 \leq \frac{1}{r_{N+1}} \int_{T-r^2}^{T} \int_{B_{r_N}(x^0)} |\nabla Z|^2 \leq C(\lambda^2 \sigma)^N \to 0.
\]

In order to conclude the proof of the proposition, we have only to show that $Z$ is a solution in the sense of distributions of equation (4.1). Let $\phi \in C_c^\infty(\mathbb{R}^+ \times \mathbb{R}^3)$, we have to show that
\[
\int_{0}^{\infty} \int_{\mathbb{R}^3} (Z \frac{\partial \phi}{\partial t} + Z \Delta \phi + \phi F) \, dx \, dt + \int_{\mathbb{R}^3} Z(0) \phi(0) \, dx = 0.
\]
Since $Z$ solves equation (4.3), it follows that
\[
\int_{0}^{\infty} \int_{\mathbb{R}^3} (Z \frac{\partial \phi}{\partial t} + Z \Delta \phi + \phi F) \, dx \, dt = \sum_{N=1}^{\infty} \int_{\mathbb{R}^3} Z(t) \phi(t) \, dx = \lim_{N \to \infty} \int_{\mathbb{R}^3} Z(\sigma_N) \phi(\sigma_N) \, dx.
\]
This limit is equal to zero since, by the usual formula (4.4) plus $N - 1$ changes of variables plus formula (4.5) argument, it follows that
\[
\int_{\mathbb{R}^3} Z(\sigma_N) \phi(\sigma_N) \, dx \leq \ldots \leq C \|\phi\|_{L^\infty(\lambda \sigma)^N},
\]
where the constant $C$ does not depend on $N$. \hfill \square

4.1. **The proof of Proposition 2.3.** So far, we have considered real valued solutions of the equation. In order to pass to divergence-free vector fields, we consider the functions $Z, F$ obtained in the previous lemma. We ask also that

\begin{equation}
\int_{\mathbb{R}} Z(0, x_1, x_2, \xi) \, d\xi = 0 \quad \text{for all } (x_1, x_2) \in \mathbb{R}^2
\end{equation}

(this reduces to a similar assumption on the function $z_0$ of the previous lemma). We define

$z(t, x) = (Z(t, x), Z(t, x), z_3(t, x))$ and $f(t, x) = (F(t, x), F(t, x), f_3(t, x))$

where $z_3$ is defined as

$z_3(t, x) = -\int_{-\infty}^{x_3} \partial Z \partial x_1(t, x_1, x_2, \xi) + \partial Z \partial x_2(t, x_1, x_2, \xi) \, d\xi,$

in such a way that $z$ is divergence free. The function $f_3$ will be defined later. Consequently the proposition is proved if we show that properties (i)-(iv) of Lemma 4.1 hold for $z_3$. Without loss of generality, we consider

$z_3(t, x) = -\int_{-\infty}^{x_3} \partial Z \partial x_1(t, x_1, x_2, \xi) \, d\xi.$

Property (ii) follows from assumption (4.6), while (iii) is obvious. We prove property (i). We start by proving that $z_3 \in L^\infty(0, \infty; H^1(\mathbb{R}^3))$. By direct computation

\[
\frac{\partial z_3}{\partial x_1} = -\int_{-\infty}^{x_3} \frac{\partial^2 Z}{\partial x_1^2} \, d\xi,
\]

\[
\frac{\partial z_3}{\partial x_2} = -\int_{-\infty}^{x_3} \frac{\partial^2 Z}{\partial x_1 \partial x_2} \, d\xi,
\]

\[
\frac{\partial z_3}{\partial x_3} = -\frac{\partial Z}{\partial x_1}.
\]

Obviously the third term is in $L^2$, so we consider only the first derivative (the second can be treated in the same way). Again a direct computation gives

\[
\frac{\partial z_3}{\partial x_1} = -\int_{-\infty}^{x_3} x_2^2 \frac{\partial z_3}{\partial x_1} \frac{\partial Z}{\partial x_1}(t, \sqrt{x_1^2 + x_2^2}, \xi) \, d\xi
\]

\[
-\int_{-\infty}^{x_3} \frac{x_2^2}{x_1^2 + x_2^2} \frac{\partial^2 z_3}{\partial x_1^2}(t, \sqrt{x_1^2 + x_2^2}, \xi) \, d\xi
\]

for $t \in I_N$. Note that this formula holds true also when $x_1 = x_2 = 0$, by virtue of (4.2). For the sake of simplicity, we examine only the second term. We proceed as in the proof of Lemma 4.1: we divide the integral in time in
an infinite sum and we estimate each term.

\[
\int_{I_N} \int_{\mathbb{R}^3} \left| \int_{-\infty}^{x_2} \frac{x_1^2}{x_1^2 + x_2^2} \frac{\partial^2 z_N}{\partial \rho^2} (t, \sqrt{x_1^2 + x_2^2}, \xi) \right|^2 \, dx \, dt
= \int_{I_N} \int_{\mathbb{R}} \int_{0}^{2\pi} \int_{-\infty}^{y} \cos^2 \theta \left| \int_{-\infty}^{y} \frac{\partial^2 z_N}{\partial \rho^2} (t, \rho, \xi) \, d\xi \right|^2 \, dy \, d\theta \, d\rho \, dt
= C \lambda^2 \sigma \int_{I_N} \int_{\mathbb{R}} \int_{-\infty}^{\rho \frac{N-1}{N}} \left( \frac{\rho \frac{N-1}{N-1} \rho + \rho_{N-1}}{\frac{N-1}{N-1}} \right)^2 \left( \frac{\rho \frac{N-1}{N-1} \rho + \rho_{N-1}}{\frac{N-1}{N-1}} \right)^2 \, d\xi \, d\rho \, dt
\leq C (\lambda^2 \sigma)^{N-1},
\]

where \( C \) does not depend on \( N \). In the same way it is possible to prove that \( z_3 \in L^\infty (0, \infty; L^2 (\mathbb{R}^3)) \).

Moreover a modification of the previous computation can be used to show that also property \((iv)\) holds: it is sufficient to proceed as in the proof of the previous lemma.

Finally we show that \( z_3 \) solves equation (4.1). We set

\[
f_3(t, x) = -\int_{-\infty}^{x_2} \left( \frac{\partial F}{\partial x_1} + \frac{\partial F}{\partial x_2} \right) (t, x_1, x_2, \xi) \, d\xi.
\]

It is easy to see then that in each interval \( I_N \), \( z_3 \) solves the equation

\[
\frac{\partial z_3}{\partial t} - \Delta z_3 = f_3.
\]

Consequently it is sufficient, as in the proof of the previous lemma, to show that for each test function \( \phi \),

\[
\lim_{N \to \infty} \int_{\mathbb{R}^3} z_3 (\sigma_N) \phi (\sigma_N) = 0,
\]

and this can be done in the same way as before. The proof of the proposition is completed.

**Remark 4.2.** It is easy to see that \( f \not\in L^2 ((0, \infty) \times \mathbb{R}^3) \). In fact,

\[
\int_0^\infty \int_{\mathbb{R}^3} |f_1(t, x)|^p \, dx \, dt = \sum_{N=1}^{\infty} \int_{I_N} \int_{\mathbb{R}^3} |f_1(t, x)|^p \, dx \, dt
\]

and

\[
\int_{I_N} \int_{\mathbb{R}^3} |f_1(t, x)|^p \, dx \, dt = \ldots = C (\lambda^p \sigma^{2-p})^N,
\]

so that \( p < 2 \). More precisely

\[
p \leq \frac{2q}{1 + q}.
\]
4.2. **The proof of Theorem 2.8.** Now we can conclude the proof of Theorem 2.8. From the previous proposition we have a vector field $z$ solution of the Stokes system, with initial condition $z(0) \in C^\infty_c$. Let $\overline{z}$ be the solution of the Stokes system with initial condition $-z(0)$ and a suitable regular body force in such a way that $\overline{z}$ has compact support in space. We set $\zeta = z + \overline{z}$. The vector field $\overline{z}$ is regular, so that

$$S(\zeta) = S(z).$$

Finally, $\zeta$ is a solution of the Stokes system with zero initial condition, so that if $v$ is the solution of the modified Navier-Stokes equation (2.4) associated to $\zeta$ in a suitable regular bounded domain which contains the support of $\zeta$ (the existence of such solution is stated in [5]), the vector field

$$u = v + \zeta$$

is a suitable weak solution of Navier-Stokes equation and

$$\mathcal{H}^1(S(u)) > 0.$$

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