1.1 Orbits of $\mathbb{Z}_N^k$ under the action of $S_k$

In this Section we follow Saldarriaga [2], see also [1]. Let $G$ be the group $\mathbb{Z}_N^k$, and for each $N$-tuple of nonnegative integers $(a_0, a_1, ..., a_{N-1})$ such that $a_0 + a_1 + ... + a_{N-1} = k$

we define the subset of $G$

$$[(a_0, a_1, ..., a_{N-1})] = \{ x \in \mathbb{Z}_N^k \mid j \text{ occurs } a_j \text{ times in } x, 0 \leq j \leq N - 1 \}.$$  \hspace{1cm} (1.1)

Then $G$ is a disjoint union of these subsets.

Note that the symmetric group $S_k$ acts on $G$ by permuting $k$-tuples, the set of orbits under this action, $O = O(N, k)$, consists of the subsets (1.1) defined above, and each orbit contains a unique representative in standard form $((N-1)^{a_{N-1}}, ..., 1^{a_1}, 0^{a_0})$ \hspace{1cm} (1.2)

where the exponent indicates the number of repetitions of the base.

**Notación 1.** Given $x \in \mathbb{Z}_N^k$, we will denote the orbit of $x$ by $[x]$ and the representative in standard form of this orbit will be denoted by $\hat{x}$.

For orbits $[a], [b]$ and $[c]$, we define the set:

$$T([a], [b], [c]) = \{ (x, y, z) \in [a] \times [b] \times [c] \mid x + y = z \}.$$  

Note that $\sigma \in S_k$ acts on $(x, y, z) \in T([a], [b], [c])$ by $\sigma(x, y, z) = (\sigma x, \sigma y, \sigma z)$.

**Definición 1.1.** Denote by $M^{[k][c]}_{[a],[b]}$ the number of $S_k$-orbits of $T([a], [b], [c])$.

This suggests that we could use these numbers as the structure constants of an algebra (depending on $N$ and $k$ and over any field of characteristic 0) with basis $O$, by defining a bilinear product as follows:

$$[a] \times [b] = \sum_{c \in O} M^{[k][c]}_{[a],[b]} [c].$$ \hspace{1cm} (1.3)

The following is a description of how to compute the product of two $S_k$-orbits of $\mathbb{Z}_N^k$.

**Definición 1.2.** Let $[a], [b] \in O$, and assume that $[b] = \{y_1, ..., y_t\}$. For $1 \leq i \leq t$ set:

$$z_i = \hat{a} + y_i.$$ \hspace{1cm} (1.4)

We say that the equation $z_j = \hat{a} + y_j$ in the list (1.4) is redundant, if for some $i < j$ and some $\sigma \in S_k$ we have

$$\sigma \hat{a} = \hat{a}, \sigma y_j = y_i \text{ and } \sigma z_j = z_i,$$

that is, if the triples $(\hat{a}, y_i, z_i)$ and $(\hat{a}, y_j, z_j)$ are in the same $S_k$-orbit of $T([a], [b], [z_i])$. 
Then the product of two orbits can be computed as follows: let \([a], [b] \in \mathcal{O}\) and fix a representative from the orbit \([a]\), say the representative in standard form, \(\hat{a}\), and assume that \([b] = \{y_1, \ldots, y_t\}\). For every \(y_i \in [b]\), set

\[ z_i = \hat{a} + y_i, \quad 1 \leq i \leq t. \]

Remove all redundant equations from this list, and without loss of generality, assume that after removing all redundancies, we are left with the first \(s\) equations, for some \(s \leq t\). That is, the list:

\[ z_i = \hat{a} + y_i, \quad 1 \leq i \leq s, \]

has no redundancies. Then

\[ [a] \times [b] = [z_1] + [z_2] + \ldots + [z_s]. \]

Note that several \(z_i\)’s could be in the same orbit, and for every \([c] \in \mathcal{O}\),

\[ M^{(k)[c]}_{[a],[b]} = \text{Card}\{1 \leq i \leq s \mid z_i \in [c]\}. \tag{1.5} \]

**Observación 1.** From Equation (1.5), we also get that \(M^{(k)[c]}_{[a],[b]}\) can be computed by removing all redundancies from the list of equations

\[ z = \hat{a} + y, \]

where \(y \in [b]\) and \(z \in [c]\).

The following Theorem was proved in [2]

**Teorema 1.3.** Let \([a], [c] \in \mathcal{O}\) and \([b] = [(1^m, 0^{k-m})]\) for some \(m \leq k\). Suppose that \(M^{(k)[c]}_{[a],[b]} \neq 0\) then \(M^{(k)[c]}_{[a],[b]} = 1\).

**Notación 2.** If \(a = (a_1, \ldots, a_k) \in \mathbb{Z}^k_N\), we will denote for \((a, 0)\) the \(k+1\)-tuple \((a_1, \ldots, a_k, 0) \in \mathbb{Z}^{k+1}_N\) and for \([a, 0]\) its corresponding orbit.

**Teorema 1.4.** Let \(a, b, c \in \mathbb{Z}^k_N\) with \(b = (1^m, 0^{k-m})\) then \(M^{(k)[c]}_{[a],[b]} \leq M^{(k+1)[c,0]}_{[a,0],[b,0]}\)

**Proof.** It is clear that \(M^{(k)[c]}_{[a],[b]}\) is a non-negative number, hence if \(M^{(k)[c]}_{[a],[b]} = 0\) the result follows.

Now, assume that \(M^{(k)[c]}_{[a],[b]} \neq 0\), then from Theorem 1.3 we get that \(M^{(k)[c]}_{[a],[b]} = 1\). Hence from Remark 1 we get that there exists \(y \in [b]\) and \(z \in [c]\) so that \(\hat{a} + y = z\), therefore we get that

\[ (\hat{a}, 0) + (y, 0) = (z, 0). \tag{1.6} \]

It is clear that \((\hat{a}, 0) = (\hat{a}(a, 0), (\hat{a}), 0) \in [a,0], (y, 0) \in [y,0]\) and \((z, 0) \in [z,0]\). Hence, from Equation (1.6) and Remark 1 we get that \(M^{(k+1)[c,0]}_{[a,0],[b,0]} \neq 0\) and then the result follows. \(\blacksquare\)
1.2 Particular case of the level increasing conjecture

Let \( \lambda = a_1 \lambda_1 + \cdots + a_{N-1} \lambda_{N-1} \) a weight for \( A_{N-1} \) of level \( k \), hence \( a_1 + \cdots + a_{N-1} \leq k \); then obviously \( a_1 + \cdots + a_{N-1} \leq k+1 \) and \( \lambda \) can also be considered as a weight of level \( k+1 \). Notice that the corresponding orbit \([\lambda]\) in \( \mathbb{Z}_N^k \) associated to the weight \( \lambda \) is given by

\[
[\lambda] = \left[ \left( (N - 1)^{a_{N-1}}, (N - 2)^{a_{N-2}}, \ldots, 1^{a_1}, 0^{k-\sum_{i=1}^{N-1} a_i} \right) \right]
\]

and if we see \( \lambda \) as a weight of level \( k+1 \) then its corresponding orbit is

\[
[\left( (N - 1)^{a_{N-1}}, (N - 2)^{a_{N-2}}, \ldots, 1^{a_1}, 0^{k+1-\sum_{i=1}^{N-1} a_i} \right)] = [\lambda, 0]
\]

In other words, if \([a]\) is the corresponding orbit to \( \lambda \) in \( \mathbb{Z}_N^k \), then \([a, 0]\) is the corresponding orbit in \( \mathbb{Z}_N^{k+1} \) of \( \lambda \) seen as a weight of level \( k+1 \).

The following theorem was proved in [2]

**Theorem 1.5.** Let \( \lambda = m \lambda_1 \) be a multiple of the first fundamental weight for \( A_{N-1} \), \( m \leq k \), \( \mu = a_1 \lambda_1 + \cdots + a_{N-1} \lambda_{N-1} \) any other weight of level \( k \) and \([\lambda]\) and \([\mu]\) their corresponding orbits in \( \mathbb{Z}_N^k \). Then \( N_{\mu, \lambda}^{(k)} [\nu] = M_{[\mu],[\lambda]}^{(k)} \) for any weight \( \nu \) of level \( k \).

So we get the following special case of the Level Increasing Conjecture

**Corollary 1.6.** (Special case of the Level Increasing Conjecture) Let \( \lambda, \mu \) and \( \nu \) weights for \( A_{N-1} \) of level \( k \) with \( \lambda = m \lambda_1 \) a multiple of the first fundamental weight, then we get

\[
N_{\mu, \lambda}^{(k)} [\nu] \leq N_{\mu, \lambda}^{(k+1)} [\nu].
\]

**Proof.** From Theorem 1.5 we get \( N_{\mu, \lambda}^{(k)} [\nu] = M_{[\mu],[\lambda]}^{(k)} [\nu] \) and from Theorem 1.4 we get that \( M_{[\mu],[\lambda]}^{(k)} [\nu] \leq M_{[\mu,0],[\lambda,0]}^{(k+1)} [\nu] \) and since \( M_{[\mu,0],[\lambda,0]}^{(k+1)} [\nu] = N_{\mu, \lambda}^{(k+1)} [\nu] \) we get

\[
N_{\mu, \lambda}^{(k)} [\nu] \leq N_{\mu, \lambda}^{(k+1)} [\nu].
\]
Bibliography

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