ON ELEMENTS IN ALGEBRAS HAVING
FINITE NUMBER OF CONJUGATES

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Dedicated to Professor Kálmán Győry on his 60th birthday

Abstract. Let \( R \) be a ring with unity and \( U(R) \) its group of units. Let \( \Delta U = \{ a \in U(R) \mid [U(R) : C_{U(R)}(a)] < \infty \} \) be the FC-radical of \( U(R) \) and let \( \nabla(R) = \{ a \in R \mid [U(R) : C_{U(R)}(a)] < \infty \} \) be the FC-subring of \( R \).

An infinite subgroup \( H \) of \( U(R) \) is said to be an \( \omega \)-subgroup if the left annihilator of each nonzero Lie commutator \([x, y] \) in \( R \) contains only finite number of elements of the form \( 1 - h \), where \( x, y \in R \) and \( h \in H \). In the case when \( R \) is an algebra over a field \( F \), and \( U(R) \) contains an \( \omega \)-subgroup, we describe its FC-subalgebra and the FC-radical. This paper is an extension of [1].

1. Introduction

Let \( R \) be a ring with unity and \( U(R) \) its group of units. Let

\[
\Delta U = \{ a \in U(R) \mid [U(R) : C_{U(R)}(a)] < \infty \},
\]

\[
\nabla(R) = \{ a \in R \mid [U(R) : C_{U(R)}(a)] < \infty \},
\]

which are called the FC-radical of \( U(R) \) and FC-subring of \( R \), respectively. The FC-subring \( \nabla(R) \) is invariant under the automorphisms of \( R \) and contains the center of \( R \).

The investigation of the FC-radical \( \Delta U \) and the FC-subring \( \nabla(R) \) was proposed by S. K. Sehgal and H. Zassenhaus [8]. They described the FC-subring of a \( \mathbb{Z} \)-order as a unital ring with a finite \( \mathbb{Z} \)-basis and a semisimple quotient ring.

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Definition. An infinite subgroup $H$ of $U(R)$ is said to be an $\omega$-subgroup if the left annihilator of each nonzero Lie commutator $[x, y] = xy - yx$ in $R$ contains only a finite number of elements of the form $1 - h$, where $h \in H$ and $x, y \in R$.

The groups of units of the following infinite rings $R$ contain $\omega$-subgroups of course

1. Let $A$ be an algebra over an infinite field $F$. Then the subgroup $U(F)$ is an $\omega$-subgroup.
2. Let $R = KG$ be the group ring of an infinite group $G$ over the ring $K$. It is well-known (see [6], Lemma 3.1.2, p. 68) that the left annihilator of any $z \in KG$ contains only a finite number of elements of the form $g - 1$, where $g \in G$. Thus $G$ is an $\omega$-subgroup.
3. Let $R = F\lambda G$ be an infinite twisted group algebra over the field $F$ with an $F$-basis $\{u_g \mid g \in G\}$. Then the subgroup $\overline{G} = \{\lambda u_g \mid \lambda \in U(F), g \in G\}$ is an $\omega$-subgroup.
4. If $A$ is an algebra over a field $F$, and $A$ contains a subalgebra $D$ such that $1 \in D$ and $D$ is either an infinite field or a skewfield, then every infinite subgroup of $U(D)$ is an $\omega$-subgroup.

2. Results

In this paper we study the properties of the $FC$-subring $\nabla(R)$ when $R$ is an algebra over a field $F$ and $U(R)$ contains an $\omega$-subgroup. We show that the set of algebraic elements $A$ of $\nabla(R)$ is a locally finite algebra, the Jacobson radical $J(A)$ is a central locally nilpotent ideal in $\nabla(R)$ and $A/J(A)$ is commutative. As a consequence, we describe the $FC$-radical $\Delta U$, which is a solvable group of length at most 3, and the subgroup $t(\Delta U)$ is nilpotent of class at most 2. If $F$ is an infinite field then any algebraic unit over $F$ belongs to the centralizer of $\nabla(R)$, and, as a consequence, we obtain that $t(\Delta U)$ is abelian and $\Delta U$ is nilpotent of class at most 2. These results are extensions of the results obtained by the author in [1] for groups of units of twisted group algebras.

By the Theorem of B. H. Neumann [5], elements of finite order in $\Delta U$ form a normal subgroup, which we denote by $t(\Delta U)$, and the factor group $\Delta U/t(\Delta U)$ is a torsion free abelian group. If $x$ is a nilpotent element of the ring $R$, then the element $y = 1 + x$ is a unit in $R$, which is called the unipotent element of $U(R)$.

Let $\zeta(G)$ be the center of $G$ and $(g, h) = g^{-1}h^{-1}gh$, where $g, h \in G$.

Lemma 1. Assume that $U(R)$ has an $\omega$-subgroup. Then all nilpotent elements of the subring $\nabla(R)$ are central in $\nabla(R)$.

Proof. Let $x$ be a nilpotent element of $\nabla(R)$. Then $x^k = 0$, and by induction on $k$ we shall prove that $vx = xv$ for all $v \in \nabla(R)$.

Choose an infinite $\omega$-subgroup $H$ of $U(R)$. By Poincare’s Theorem the centralizer $S$ of the subset $\{v, x\}$ in $H$ is a subgroup of finite index in $H$. Since $H$ is infinite,
$S$ is infinite and $fx = xf$ for all $f \in S$. Then $xf$ is nilpotent and $1 + xf$ is a unit in $U(R)$. Since $v \in \nabla(R)$, the set $\{ (1 + xf)^{-1}v(1 + xf) \mid f \in S \}$ is finite. Let $v_1, \ldots, v_t$ be all the elements of this set and

$$W_i = \{ f \in S \mid (1 + xf)^{-1}v(1 + xf) = v_i \}.$$  

Obviously, $S = \cup W_i$ and there exists an index $j$ such that $W_j$ is infinite. Fix an element $f \in W_j$. Then any element $q \in W_j$ such that $q \neq f$ satisfies

$$(1 + xf)^{-1}v(1 + xf) = (1 + xq)^{-1}v(1 + xq)$$

and

$$v(1 + xf)(1 + xq)^{-1} = (1 + xf)(1 + xq)^{-1}v.$$  

Then

$$v\{(1 + xq) + (xf - xq)\}(1 + xq)^{-1} = \{(1 + xq) + (xf - xq)\}(1 + xq)^{-1}v,$$

$$v(1 + x(f - q)(1 + xq)^{-1}) = (1 + x(f - q)(1 + xq)^{-1})v$$

and

$$vx(f - q)(1 + xq)^{-1} = x(f - q)(1 + xq)^{-1}v. \quad (1)$$

Let $xv \neq vx$ and $k = 2$. Then $x^2 = 0$ and $(1 + xq)^{-1} = 1 - xq$. Since $f$ and $q$ belong to the centralizer of the subset $\{x, v\}$, from (1) we have

$$(f - q)vx(1 - xq) = (f - q)x(1 - xq)v,$$

whence $(f - q)(vx - vx^2q - xv + x^2qv) = 0$ and evidently $(f - q)(vx - xv) = 0$. Therefore, $(q^{-1}f - 1)(vx - xv) = 0$ for any $q \in W_j$. Since $q^{-1}W_j$ is an infinite subset of the $\omega$-subgroup $H$, we obtain a contradiction, and thus $vx = xv$.

Let $k > 2$. If $i \geq 1$ then $x^{i+1}$ is nilpotent of index less than $k$, thus applying an induction on $k$, first we obtain that $x^{i+1}v = vx^{i+1}$ and then

$$x(f - q)x^i q^iv = (f - q)x^{i+1}q^iv = (f - q)vx^{i+1}q^i = vx(f - q)x^iq^i.$$  

Hence

$$vx(f - q)(1 - xq + x^2q^2 + \cdots + (-1)^{k-1}x^{k-1}q^{k-1})$$

$$= x(f - q)(1 - xq)v + (x^2q^2 + \cdots + (-1)^{k-1}x^{k-1}q^{k-1}v).$$

and $(f - q)(vx - xv) = 0$. As before, we have a contradiction in the case $xv \neq xv$.

Thus nilpotent elements of $\nabla(R)$ are central in $\nabla(R)$. \qed
Lemma 2. Let $R$ be an algebra over a field $F$ such that the group of units $U(R)$ contains an $\omega$-subgroup. Then the radical $\mathfrak{Z}(A)$ of every locally finite subalgebra $A$ of $\nabla(R)$ consists of central nilpotent elements of the subalgebra $\nabla(R)$, and $A/\mathfrak{Z}(A)$ is a commutative algebra.

Proof. Let $x \in \mathfrak{Z}(A)$, then $x \in L$ for some finite dimensional subalgebra $L$ of $A$. Since $L$ is left Artinian, Proposition 2.5.17 in [7] (p. 185) ensures that $L \cap \mathfrak{Z}(A) \subseteq \mathfrak{Z}(L)$, moreover $\mathfrak{Z}(L)$ is nilpotent. Now $x \in \mathfrak{Z}(L)$ implies that $x$ is nilpotent and the application of Lemma 1 gives that $x$ belongs to the center of $\nabla(R)$. Then Theorem 48.3 in [4] (p. 209) will enable us to verify the existence of the decomposition into the direct sum

$$L = Le_1 \oplus \cdots \oplus Le_n \oplus N,$$

where $Le_i$ is a finite dimensional local $F$-algebra (i.e. $Le_i/\mathfrak{Z}(Le_i)$ is a division ring), $N$ is a commutative artinian radical algebra, and $e_1, \ldots, e_n$ are pairwise orthogonal idempotents. Since nilpotent elements of $\nabla(R)$ belong to the center of $\nabla(R)$, by Lemma 13.2 of [4] (p. 57) any idempotent $e_i$ is central in $L$ and the subring $Le_i$ of $\nabla(R)$ is also an $FC$-ring, whence $\mathfrak{Z}(Le_i)$ is a central nilpotent ideal.

Suppose that $Le_i/\mathfrak{Z}(Le_i)$ is a noncommutative division ring. Then $1 + \mathfrak{Z}(Le_i)$ is a central subgroup and

$$U(Le_i)/(1 + \mathfrak{Z}(Le_i)) \cong U(Le_i/\mathfrak{Z}(Le_i)).$$

Applying Herstein’s Theorem [2] we establish that a noncentral unit of $Le_i/\mathfrak{Z}(Le_i)$ has an infinite number of conjugates, which is impossible. Therefore, $L/\mathfrak{Z}(L)$ is a commutative algebra and from $\mathfrak{Z}(L) \subseteq \mathfrak{Z}(A)$ and $\mathfrak{Z}(L)$ is nil (actually nilpotent) in $A$, it follows that $A/\mathfrak{Z}(A)$ is a commutative algebra. \qed

Theorem 1. Let $R$ be an algebra over a field $F$ such that the group of units $U(R)$ contains an $\omega$-subgroup, and let $\nabla(R)$ be the $FC$-subalgebra of $R$. Then the set of algebraic elements $A$ of $\nabla(R)$ is a locally finite algebra, the Jacobson radical $\mathfrak{Z}(A)$ is a central locally nilpotent ideal in $\nabla(R)$ and $A/\mathfrak{Z}(A)$ is commutative.

Proof. Since any nilpotent element of $\nabla(R)$ is central in $\nabla(R)$ by Lemma 1, one can see immediately that the set of all nilpotent elements of $\nabla(R)$ form an ideal $I$, and the factor algebra $\nabla(R)/I$ contains no nilpotent elements. Obviously, $I$ is a locally finite subalgebra in $\nabla(R)$, and all idempotents of $\nabla(R)/I$ are central in $\nabla(R)/I$.

Let $x_1, x_2, \ldots, x_s$ be algebraic elements of $\nabla(R)/I$. We shall prove that the subalgebra generated by $x_1, x_2, \ldots, x_s$ is finite dimension.

For every $x_i$ the subalgebra $\langle x_i \rangle_F$ of the factor algebra $\nabla(R)/I$ is a direct sum of fields

$$\langle x_i \rangle_F = F_{i1} \oplus F_{i2} \oplus \cdots \oplus F_{i\tau_i},$$
where $F_{ij}$ is a field and is finite dimensional over $F$. Choose $F$-basis elements $u_{ijk}$ ($i = 1, \ldots, s$, $j = 1, \ldots, r_i$, $k = 1, \ldots, [F_{ij} : F]$) in $F_{ij}$ over $F$ and denote by $w_{ijk} = 1 - e_{ij} + u_{ijk}$, where $e_{ij}$ is the unit element of $F_{ij}$. Obviously, $w_{ijk}$ is a unit in $\nabla(R)/I$. We collect in the direct summand all these units $w_{ijk}$ for each field $F_{ij}$ ($i = 1, \ldots, s$, $j = 1, \ldots, r_i$) and this finite subset in the group $U(\nabla(R)/I)$ is denoted by $W$.

Let $H$ be the subgroup of $U(\nabla(R)/I)$ generated by $W$. The subgroup $H$ of $\nabla(R)/I$ is a finitely generated $FC$-group, and as it is well-known, a natural number $m$ can be assigned to $H$ such that for any $u, v \in H$ the elements $u^m, v^m$ are in the center $\zeta(H)$, and $(uv)^m = u^m v^m$ (see [5]). Since $H$ is a finitely generated group, the subgroup $S = \{v^m \mid v \in H\}$ has a finite index in $H$ and $\{w^m \mid w \in W\}$ is a finite generated system for $S$. Let $t_1, t_2, \ldots, t_l$ be a transversal to $S$ in $H$.

Let $H_F$ be the subalgebra of $\nabla(R)/I$ spanned by the elements of $H$ over $F$. Clearly, the commutative subalgebra $S_F$ of $H_F$, generated by central algebraic elements $w^m$ ($w \in W$), is finite dimensional over $F$ and any $u \in H_F$ can be written as

$$u = u_1 t_1 + u_2 t_2 + \cdots + u_l t_l,$$

where $u_i \in S_F$. Since $t_i t_j = \alpha_{ij} t_{r(ij)}$ and $\alpha_{ij} \in S_F$, it yields that the subalgebra $H_F$ is finite dimensional over $F$. Recall that

$$x_i = \sum_{j,k} \beta_{jk} w_{ijk} - \sum_{j,k} \beta_{jk}(1 - e_{ij}),$$

where $\beta_{jk} \in F$ and $e_{ij}$ are central idempotents of $\nabla(R)/I$. The subalgebra $T$ generated by $e_{ij}$ ($i = 1, 2, \ldots, s$, $j = 1, 2, \ldots, r_i$) is finite dimensional over $F$ and $T$ is contained in the center of $\nabla(R)/I$. Therefore, $x_i$ belongs to the sum of two subspaces $H_F$ and $T$ and the subalgebra of $\nabla(R)/I$ generated by $H_F$ and $T$ is finite dimensional over $F$. Since $\langle x_1, \ldots, x_s \rangle_F$ is a subalgebra of $\langle H_F, T \rangle_F$, is also finite dimensional over $F$. We established that the set of algebraic elements of $\nabla(R)/I$ is a locally finite algebra. One can see that all the algebraic elements of $\nabla(R)$ form a locally finite algebra $A$ (see [3], Lemma 6.4.1, p. 162). Since the radical of an algebraic algebra is a nil ideal, according to Lemma 1 we have that $3(A)$ is a central locally nilpotent ideal in $\nabla(R)$, and $A/3(A)$ is commutative by Lemma 2.

Recall that by Neumann’s Theorem [5] the set $t(\Delta U)$ of $\Delta U$ containing all elements of finite order of $\Delta U$ is a subgroup.

**Theorem 2.** Let $R$ be an algebra over a field $F$ such that the group of units $U(R)$ contains an $\omega$-subgroup. Then

1. the elements of the commutator subgroup of $t(\Delta U)$ are unipotent and central in $\Delta U$;
2. if all elements of $\nabla(R)$ are algebraic then $\Delta U$ is nilpotent of class 2;
3. $\Delta U$ is a solvable group of length at most 3, and the subgroup $t(\Delta U)$ is nilpotent of class at most 2.

Proof. It is easy to see that $\Delta U \subseteq \nabla(R)$, and any element of $t(\Delta U)$ is algebraic. According to Theorem 1 the set $A$ of algebraic elements of $\nabla(R)$ is a subalgebra, the Jacobson radical $\mathfrak{J}(A)$ is a central locally nilpotent ideal in $\nabla(R)$, and $A/\mathfrak{J}(A)$ is commutative. The isomorphism

$$U(A)/(1 + \mathfrak{J}(A)) \cong U(A/\mathfrak{J}(A)),$$

implies that $(t(\Delta U)(1 + \mathfrak{J}(A)))/(1 + \mathfrak{J}(A))$ is abelian, the commutator subgroup of $t(\Delta U)$ is contained in $1 + \mathfrak{J}(A)$ and consists of unipotent elements.

By Neumann’s Theorem $\Delta U/t(\Delta U)$ is abelian, therefore $\Delta U$ is a solvable group of length at most 3. □

Let $R$ be an algebra over a field $F$. Let $m$ be the order of the element $g \in U(R)$ and assume that the element $1 - \alpha^m$ is a unit in $F$ for some $\alpha \in F$. It is well-known that $g - \alpha \in U(R)$ and

$$(g - \alpha)^{-1} = (1 - \alpha^m)^{-1} \sum_{i=0}^{m-1} \alpha^{m-1-i}g^i.$$

We know that the number of solutions of the equation $x^m - 1 = 0$ in $F$ does not exceed $m$. If $F$ is an infinite field, then it follows that, there exists an infinite set of elements $\alpha \in F$ such that $g - \alpha$ is a unit. We will show that this is true for any algebraic unit.

Lemma 3. Let $g \in U(R)$ be an algebraic element over the field $F$. Then there are infinitely many elements $\alpha$ of the field $F$ such that $g - \alpha$ is a unit.

Proof. Since $g$ is an algebraic element over $F$, $F[g]$ is a finite dimensional subalgebra over $F$. Let $T$ be the radical of $F[g]$. There exists an orthogonal system of idempotents $e_1, e_2, \ldots, e_s$ such that

$$F[g] = F[g]e_1 \oplus F[g]e_2 \oplus \cdots \oplus F[g]e_s,$$

and $Te_i$ is a nilpotent ideal such that $F[g]e_i/Te_i$ is a field. It is well-known that $F[g]e_i$ is a local ring, and all elements of $F[g]e_i$, which do not belong to $Te_i$, are units. Moreover, if $\alpha \in F$, then

$$g - \alpha = (ge_1 - \alpha e_1) + (ge_2 - \alpha e_2) + \cdots + (ge_s - \alpha e_s).$$

Clearly, $ge_i$ is a unit and $ge_i \notin Te_i$ for every $i$. Put

$$L_i = \{ge_i - \alpha e_i \mid \alpha \in F\}.$$
Suppose that $ge_i - \beta e_i$ and $ge_i - \gamma e_i$ belong to $Te_i$ for some $\alpha, \beta \in F$. Then

$$(ge_i - \beta e_i) - (ge_i - \gamma e_i) = (\gamma - \beta)e_i \in Te_i,$$

which is impossible for $\beta \neq \gamma$. Therefore, $Te_i$ contains at most one element from $L_i$. Since $F[g]e_i$ is a local ring, all elements of the form $ge_i - \alpha e_i$ with $ge_i - \alpha e_i \notin Te_i$ are units, and there are infinitely many units of the form (2). □

**Lemma 4.** Let $g \in U(R)$ and $a \in R$. If $g - \alpha, g - \beta$ are units for some $\alpha, \beta \in F$ and $ag \neq ga$, then

$$(g - \alpha)^{-1}a(g - \alpha) \neq (g - \beta)^{-1}a(g - \beta).$$

**Proof.** Suppose that $(g - \alpha)^{-1}a(g - \alpha) = (g - \beta)^{-1}a(g - \beta)$. Then

$$(g - \alpha - (\beta - \alpha))(g - \alpha)^{-1}a = a(g - \alpha - (\beta - \alpha))(g - \alpha)^{-1}$$

and $(1 - (\beta - \alpha)(g - \alpha)^{-1})a = a(1 - (\beta - \alpha)(g - \alpha)^{-1})$.

Hence

$$(\beta - \alpha)(g - \alpha)^{-1}a = a(\beta - \alpha)(g - \alpha)^{-1}$$

and $(g - \alpha)^{-1}a = a(g - \alpha)^{-1}$, which provides the contradiction $ag = ga$. □

**Theorem 3.** Let $R$ be an algebra over an infinite field $F$. Then

1. any algebraic unit over $F$ belongs to the centralizer of $\nabla(R);$  
2. if $R$ is generated by algebraic units over $F$, then $\nabla(R)$ belongs to the center of $R$.  

**Proof.** Let $a \in \nabla(R)$, and $g \in U(R)$ be an algebraic element over $F$. Then by Lemma 3 there are infinitely many elements $\alpha \in F$ such that $g - \alpha$ is a unit for every $\alpha$. If $[a, g] \neq 0$, then by Lemma 4 the elements of the form $(g - \alpha)^{-1}a(g - \alpha)$ are different, and $a$ has an infinite number of conjugates, which is impossible. Therefore, $g$ belongs to the centralizer of $\nabla(R)$.

Now, suppose that $R$ is generated by algebraic units $\{a_j\}$ over $F$. Since every $w \in U(R)$ can be written as a sum of elements of the form $\alpha_{i_j}a_{i_1}^{\gamma_{i_1}} \ldots a_{i_s}^{\gamma_{i_s}}$, where $\alpha_j \in F$, $\gamma_{i_j} \in \mathbb{Z}$, by the first part of this Theorem $w$ commute with elements of $\nabla(R)$. Hence $\nabla(R)$ is central in $R$. □

**Corollary.** Let $R$ be an algebra over an infinite field $F$, and let $t(\Delta U)$ be the torsion subgroup of $\Delta U$. Then

1. $t(\Delta U)$ is abelian and $\Delta U$ is a nilpotent group of class at most 2;
2. if every unit of \( R \) is an algebraic element over \( F \), then \( \Delta U \) is central in \( U(R) \).

Proof. Clearly, all elements from \( t(\Delta U) \) are algebraic and by Theorem 3 every algebraic unit belongs to the centralizer of \( \nabla(R) \). Since \( t(\Delta U) \subseteq \nabla(R) \), it follows that \( t(\Delta U) \) is central in \( \nabla(R) \). Since \( \Delta U/t(\Delta U) \) is abelian, by Neumann’s Theorem \( \Delta U \) is a nilpotent group of class at most 2.

Let \( a \in \Delta U \) and \( g \in U(R) \) be an algebraic element over \( F \). Then by Theorem 3 we get \([a, g] = 0\). Hence, if every unit of \( R \) is an algebraic element over \( F \), then \( \Delta U \) is central in \( U(R) \). \( \square \)

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