PRESENTING SCHUR ALGEBRAS

STEPHEN DOTY AND ANTHONY GIAQUINTO

Abstract. Motivated by work of R.M. Green, we obtain a presentation of Schur algebras (both the classical and quantized versions) in terms of generators and relations. The presentation is compatible with the usual presentation of the (quantized or classical) enveloping algebra of $\mathfrak{gl}_n$. As a result, we obtain a new “integral” basis for Schur algebras which is a subset of Kostant’s basis of the integral form of the enveloping algebra (or its $q$-analogue). Projection onto an appropriate component gives a new “integral” basis and a presentation for the Hecke algebra, compatible with the basis and presentation for the Schur algebra. Finally, we find a second presentation of Schur algebras which is similar to Lusztig’s modified form of the quantized enveloping algebra.

Introduction

Let $R$ be a commutative ring. The (classical) Schur algebra $S_R(n,d)$ may be defined as the algebra $\text{End}_{\Sigma_d}(V_R^\otimes d)$ of linear endomorphisms on the $d$th tensor power of an $n$-dimensional free $R$-module $V_R$ commuting with the action of the symmetric group $\Sigma_d$, acting by permutation of the tensor places (see [Gr]). The Schur algebras form an important class of quasi-hereditary algebras, and, when $R$ is an infinite field, the family of Schur algebras $\{S_R(n,d)\}_{d \geq 0}$ determines the polynomial representation theory of the general linear group $\text{GL}(V_R)$.

All these algebras, for various $R$, can be constructed from the integral form $S_Z(n,d)$ by base change, since $S_R(n,d) \cong R \otimes_Z S_Z(n,d)$. Fixing our base field at $\mathbb{Q}$ (we could use any field of characteristic zero), we henceforth write $S(n,d)$ for $S_{\mathbb{Q}}(n,d)$, $V$ for $V_{\mathbb{Q}}$.

In this paper, we give an alternative construction of Schur algebras, as follows. First we obtain a presentation of $S(n,d)$ by generators and relations (Theorem 1.1). This presentation is compatible with Serre’s presentation of the universal enveloping algebra $U = U(\mathfrak{gl}_n)$ of the Lie algebra $\mathfrak{gl}_n$ of $n \times n$ matrices. Then we construct $S_Z(n,d)$ as the precise analogue of the Kostant $Z$-form $U_Z$, and we obtain new bases for $S_Z(n,d)$ (Theorem 1.3). We also obtain a second presentation.

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of $S(n, d)$ by generators and relations (Theorem \[1.4\]) which is closely related to Lusztig’s construction (see \[Lu3\]) of the modified form $\hat{U}$ of the quantized enveloping algebra $U$.

Our approach is based on the classic “double-centralizer” theory of Schur \[Sc\] (and its quantization). The group $GL(V)$ acts on $V^{\otimes d}$ by means of the natural action in each tensor factor, and by differentiating this action on tensors we obtain an action of $U$ on $V^{\otimes d}$. These actions obviously commute with the action of the symmetric group $\Sigma_d$. So we have representations

$$U \longrightarrow \text{End}(V^{\otimes d}) \longleftrightarrow \mathbb{Q}\Sigma_d$$

induced from the commuting actions. Then Schur’s result is that the image of each representation is precisely the commuting algebra for the action of the other algebra. In particular, $S(n, d)$ is the image of the representation $U \rightarrow \text{End}(V^{\otimes d})$. It is a very natural problem to ask for an efficient generating set for the kernel of this representation. By solving this problem we obtain the presentation of Theorem \[1.1\]. We note that, in the quantum case, the analogue of the surjective map $U \rightarrow S(n, d)$ was studied by R.M. Green \[RG1, RG2\], who described a basis for the kernel.

In the quantum case one replaces in the above setup $\mathbb{Q}$ by $\mathbb{Q}(v)$ ($v$ an indeterminate), $V$ by an $n$-dimensional $\mathbb{Q}(v)$-vector space $V$, $\Sigma_d$ by the corresponding Hecke algebra $H = H(\Sigma_d)$, and $U$ by the Drinfeld-Jimbo quantized enveloping algebra $U = U(gl_n)$. Then the resulting commuting algebra, $S(n, d)$, is known as the $q$-Schur algebra, or quantized Schur algebra. It appeared first in work of Dipper and James \[DJ1, DJ2\], and, independently, Jimbo \[Ji\]. Dipper and James showed that the $q$-Schur algebras determine the representation theory of the finite general linear groups in non-describing characteristic. (Note that one should replace their parameter $q$ by $v^2$ to make the correspondence with our version of $S(n, d)$.) In \[BLM\] a geometric realization of $S(n, d)$ was given. In \[Du\] the \[BLM\] approach was reconciled with the Dipper-James approach; moreover, it was shown in that paper that $S(n, d)$ may be identified with the image of the map

$$U \longrightarrow \text{End}(V^{\otimes d}).$$

In the quantum situation, one replaces $\mathbb{Z}$ by $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$. In this case the analogue of $S_\mathcal{A}(n, d)$ is a certain $\mathcal{A}$-form $S_{\mathcal{A}}(n, d)$ in $S(n, d)$.

Our results in the quantum case are almost exact analogues of the results in the classical case, although the proofs are sometimes more difficult. First we obtain a presentation of $S(n, d)$ by generators and relations (Theorem \[2.1\]). This presentation is compatible with the usual
presentation of the quantized enveloping algebra $U$ (over $\mathbb{Q}(v)$) corresponding to $\mathfrak{gl}_n$. Then we construct $S_A(n,d)$ as the analogue of Lusztig’s $A$-form $U_A$, and we obtain new bases for $S_A(n,d)$ (Theorem 2.3). Finally, we have a second presentation of $S(n,d)$ by generators and relations (Theorem 2.4) which is closely related to the algebra $\hat{U}$. Upon specializing $v$ to 1, the presentation of Theorem 2.4 coincides with the presentation of Theorem 1.4. (This does not apply to Theorem 2.1 in relation to Theorem 1.1.)

In the final section we give some applications of our results to the Borel Schur algebras and Hecke algebras. In particular, in Proposition 11.3 we obtain a simple basis for the Borel Schur algebras which is a subset of the integral basis obtained (in Theorem 2.3) for the entire Schur algebra. We also get a new basis for the Hecke algebra (realized as a subalgebra of the Schur algebra) which is a subset of our basis in Theorem 2.3, and an integral presentation of it. We also write out some examples in the final section.

In rank 1 we have more precise results than in this paper, obtained by different arguments [DG1]. The results in the current paper were summarized in the announcement [DG3].

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1. Main results: classical case

Let $\Phi$ be the root system of type $A_{n-1}$: $\Phi = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i \neq j \leq n\}$. Here the $\varepsilon_i$ form the standard orthonormal basis of the euclidean space $\mathbb{R}^n$. Let $(\ , \ )$ denote the inner product on this space and define $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$. Then $\{\alpha_1, \ldots, \alpha_{n-1}\}$ is a base of simple roots and $\Phi^+ = \{\varepsilon_i - \varepsilon_j \mid i < j\}$ is the corresponding set of positive roots.

We now give a precise statement of our main results in the classical case. The proofs are contained in sections 3–6. The first result describes a presentation by generators and relations of the Schur algebra over the rational field $\mathbb{Q}$.

**Theorem 1.1.** Over $\mathbb{Q}$, the Schur algebra $S(n,d)$ is isomorphic with the associative algebra (with 1) on the generators $e_i$, $f_i$ ($1 \leq i \leq n-1$), and $H_i$ ($1 \leq i \leq n$) with relations

- (R1) $H_i H_j = H_j H_i$
- (R2) $e_i f_j - f_j e_i = \delta_{ij}(H_j - H_{j+1})$
- (R3) $H_i e_j - e_j H_i = (\varepsilon_i, \alpha_j)e_j$, $H_i f_j - f_j H_i = - (\varepsilon_i, \alpha_j)f_j$
\[(R4) \quad \begin{align*}
e_i^2e_j - 2e_ie_je_i + e_je_i^2 &= 0 \quad (|i - j| = 1) \\
e_i^2e_j - e_je_i &= 0 \quad (\text{otherwise})
\end{align*}\]

\[(R5) \quad \begin{align*}
f_i^2f_j - 2f_if_jf_i + f_jf_i^2 &= 0 \quad (|i - j| = 1) \\
f_i^2f_j - f_jf_i &= 0 \quad (\text{otherwise})
\end{align*}\]

\[(R6) \quad H_1 + H_2 + \cdots + H_n = d\]

\[(R7) \quad H_i(H_i - 1) \cdots (H_i - d) = 0.\]

Note that the enveloping algebra \(U = U(\mathfrak{gl}_n)\) is the algebra on the same generators but subject only to the relations \((R1)-(R5)\), and \(U(\mathfrak{sl}_n)\) is isomorphic with the subalgebra of \(U\) generated by the \(e_i, f_i, H_i - H_{i+1}\) \((1 \leq i \leq n - 1)\).

Next we introduce the root vectors \(x_\alpha \) \((\alpha \in \Phi)\), which may be defined inductively as follows. Write \(\alpha = \varepsilon_i - \varepsilon_j\) and assume that \(i < j\). If \(j - i = 1\) then \(\alpha = \alpha_i\) and we set \(x_\alpha = e_i, x_{-\alpha} = f_i\). If \(j - i > 1\) then we inductively set

\[x_\alpha = e_i x_{\alpha - \alpha_i} - x_{\alpha - \alpha_i} e_i, \quad x_{-\alpha} = x_{-\alpha + \alpha_i} f_i - f_i x_{-\alpha + \alpha_i}.\]

The linear span of the set \(\{x_\alpha\} \cup \{H_i\}\) is a subspace of \(U\) isomorphic with the Lie algebra \(\mathfrak{gl}_n\) under the Lie bracket given by \([x, y] = xy - yx\), and the \(x_\alpha\) correspond to the usual root vectors in \(\mathfrak{gl}_n\).

**Remark 1.2.** The defining relation \((R6)\) can be used to rewrite one of the \(H_i\)'s in terms of the others. Fix an integer \(i_0\) with \(1 \leq i_0 \leq n\) and set

\[G = \{x_\alpha \mid \alpha \in \Phi\} \cup \{H_i \mid i \neq i_0\}\]

and fix an arbitrary ordering for this set. We conjecture that \(S(n, d)\) has a \(\mathbb{Q}\)-basis consisting of all monomials in \(G\) (with specified order) of total degree not exceeding \(d\). This basis would be an analogue of the Poincare-Birkhoff-Witt (PBW) basis of \(U\).

Our next result constructs the integral Schur algebra \(S_Z(n, d)\) in terms of the generators given above. We need more notation. For \(B = (B_i)\) in \(\mathbb{N}^n\), we write

\[H_B = \prod_{i=1}^{n} \binom{H_i}{B_i}\]

where \(\binom{H_i}{m} = H_i(H_i - 1) \cdots (H_i - m + 1)/(m!)\) \((m \geq 1)\), \(\binom{H_i}{0} = 1\).

Let \(\Lambda(n, d)\) be the subset of \(\mathbb{N}^n\) consisting of those \(\lambda \in \mathbb{N}^n\) satisfying \(|\lambda| = d\) (here \(|\lambda| = \sum \lambda_i\); this is the set of \(n\)-part compositions of \(d\).
Given $\lambda \in \Lambda(n, d)$ we set $1_\lambda = H_\lambda$. We will show that the collection $\{1_\lambda\}$ as $\lambda$ varies over $\Lambda(n, d)$ forms a set of pairwise orthogonal idempotents in $S_\mathbb{Z}(n, d)$ which sum to the identity element.

For $m \in \mathbb{N}$ and $\alpha \in \Phi$, set $x_\alpha^{(m)} = x_\alpha^m / (m!)$. Any product of elements of the form

$$x_\alpha^{(r)}, \left(H_i \atop s\right) \quad (r, s \in \mathbb{N}, \alpha \in \Phi, 1 \leq i \leq n),$$

taken in any order, will be called a Kostant monomial. Note that the set of Kostant monomials is multiplicatively closed. We define a function $\chi$ (content function) on Kostant monomials by setting

$$\chi(x_\alpha^{(m)}) = m \varepsilon_{\max(i, j)}, \quad \chi\left(H_i \atop m\right) = 0$$

where $\alpha = \varepsilon_i - \varepsilon_j$ ($i \neq j$), and by declaring that $\chi(XY) = \chi(X) + \chi(Y)$ whenever $X, Y$ are Kostant monomials.

For $A \in \mathbb{N}^\Phi^+$ we set $|A| = \sum_{\alpha \in \Phi^+} A(\alpha)$. For $A, C \in \mathbb{N}^\Phi^+$ we write

$$e_A = \prod_{\alpha \in \Phi^+} x_\alpha^{(A(\alpha))}, \quad f_C = \prod_{\alpha \in \Phi^+} x_\alpha^{(C(-\alpha))}$$

where the products in $e_A$ and $f_C$ are taken relative to any two fixed orders on $\Phi^+$.

The first part of the next result shows that $S_\mathbb{Z}(n, d)$ is the analogue in $S(n, d)$ of Kostant’s $\mathbb{Z}$-form $U_\mathbb{Z}$ in $U$.

**Theorem 1.3.** The integral Schur algebra $S_\mathbb{Z}(n, d)$ is the subring of $S(n, d)$ generated by all divided powers $e_i^{(m)}$, $f_i^{(m)}$. Moreover, each of the disjoint unions

(a) $Y_+ = \bigcup_\lambda \{e_A1_\lambda f_C \mid \chi(e_Af_C) \preceq \lambda\}$
(b) $Y_- = \bigcup_\lambda \{f_A1_\lambda e_C \mid \chi(f_Ae_C) \preceq \lambda\}$

as $\lambda$ varies over $\Lambda(n, d)$, and where $\preceq$ denotes the componentwise partial ordering on $\mathbb{N}^n$, is a $\mathbb{Z}$-basis of $S_\mathbb{Z}(n, d)$.

Finally, we have another presentation of the Schur algebra by generators and relations. This presentation has the advantage that it possesses a quantization of the same form, in which we can specialize $v$ to 1 to recover the classical version.

**Theorem 1.4.** The $\mathbb{Q}$-algebra $S(n, d)$ is the associative algebra (with 1) given by generators $1_\lambda$ ($\lambda \in \Lambda(n, d)$), $e_i$, $f_i$ ($1 \leq i \leq n - 1$) subject
to the relations

\[(R1') \quad 1_{\lambda 1_{\mu}} = \delta_{\lambda,\mu} 1_{\lambda}, \quad \sum_{\lambda \in \Lambda(n,d)} 1_{\lambda} = 1\]

\[e_i 1_{\lambda} = \begin{cases} 1_{\lambda + \alpha_i} e_i & \text{if } \lambda + \alpha_i \in \Lambda(n,d) \\ 0 & \text{otherwise} \end{cases}\]

\[f_i 1_{\lambda} = \begin{cases} 1_{\lambda - \alpha_i} f_i & \text{if } \lambda - \alpha_i \in \Lambda(n,d) \\ 0 & \text{otherwise} \end{cases}\]

\[(R2') \quad 1_{\mu} e_i = \begin{cases} e_i 1_{\lambda - \alpha_i} & \text{if } \lambda - \alpha_i \in \Lambda(n,d) \\ 0 & \text{otherwise} \end{cases}\]

\[1_{\mu} f_i = \begin{cases} f_i 1_{\lambda + \alpha_i} & \text{if } \lambda + \alpha_i \in \Lambda(n,d) \\ 0 & \text{otherwise} \end{cases}\]

\[(R3') \quad e_i f_j - f_j e_i = \delta_{ij} \sum_{\lambda \in \Lambda(n,d)} (\lambda_j - \lambda_{j+1}) 1_{\lambda}\]

along with the Serre relations \[(R4), (R5).\]

2. Main results: quantum case

Our main results in the quantum case are similar in form to those in the classical case. Proofs are given in sections [7] [10]. The first result describes a presentation by generators and relations of the quantized Schur algebra over the rational function field \(\mathbb{Q}(v)\).

**Theorem 2.1.** Over \(\mathbb{Q}(v)\), the \(q\)-Schur algebra \(S(n,d)\) is isomorphic with the associative algebra (with 1) with generators \(E_i, F_i\) \((1 \leq i \leq n - 1)\), \(K_i, K_i^{-1}\) \((1 \leq i \leq n)\) and relations

\[(Q1) \quad K_i K_j = K_j K_i, \quad K_i K_i^{-1} = K_i^{-1} K_i = 1\]

\[(Q2) \quad E_i F_j - F_j E_i = \delta_{ij} \frac{K_i K_{i+1}^{-1} - K_i^{-1} K_{i+1}}{v - v^{-1}}\]

\[(Q3) \quad K_i E_j = v^{(\varepsilon_i, \alpha_j)} E_j K_i, \quad K_i F_j = v^{-(\varepsilon_i, \alpha_j)} F_j K_i\]

\[(Q4) \quad E_i^2 E_j - (v + v^{-1}) E_i E_j E_i + E_j E_i^2 = 0 \quad (|i - j| = 1)\]

\[E_i E_j - E_j E_i = 0 \quad (\text{otherwise})\]

\[(Q5) \quad F_i^2 F_j - (v + v^{-1}) F_i F_j F_i + F_j F_i^2 = 0 \quad (|i - j| = 1)\]

\[F_i F_j - F_j F_i = 0 \quad (\text{otherwise})\]
\( K_1 K_2 \cdots K_n = v^d \)

\( (K_i - 1)(K_i - v)(K_i - v^2) \cdots (K_i - v^d) = 0. \)

We note that the quantized enveloping algebra \( U = U(\mathfrak{gl}_n) \) is the
algebra on the same set of generators, subject only to the relations \((Q1)\)–\((Q5)\). Moreover, the quantized enveloping algebra \( U(\mathfrak{sl}_n) \) is isomorphic
with the subalgebra of \( U \) generated by all \( E_i, F_i, K_i K_{i+1}^{-1} \) \((1 \leq i \leq n-1)\).

We have \( q \)-analogues of the root vectors in \( \mathfrak{gl}_n \), which can be defined
as follows. For \( \alpha \in \Phi^+ \), write \( \alpha = \varepsilon_i - \varepsilon_j \) for \( i < j \). If \( j - i = 1 \) then
set \( X_\alpha = E_i, X_{-\alpha} = F_i \). For \( j - i > 1 \) we inductively set (following Xi \cite{Xi1}, §5.6)
\[ X_\alpha = v^{-1} E_i X_{\alpha - \varepsilon_i} - X_{\alpha - \varepsilon_i} E_i, \quad X_{-\alpha} = v X_{-\alpha + \varepsilon_i} F_i - F_i X_{-\alpha + \varepsilon_i}. \]

Our notation differs from that in \cite{Xi1} where the elements \( X_\alpha \) and \( X_{-\alpha} \)
are denoted \( E_{i,j} \) and \( F_{j,i} \), resp. Up to scalar multiplication by units
in \( \mathbb{A} \), the elements \( X_\alpha \) \((\alpha \in \Phi)\) first appeared in Jimbo’s paper \cite{Ji}.

Remark 2.2. Fix an integer \( i_0 \) with \( 1 \leq i_0 \leq n \) and write \( N_n i_0 \) for the
set of \( B \in N_n \) such that \( B_{i_0} = 0 \). We conjecture that \( S(n,d) \) has a
\( \mathbb{Q}(v) \)-basis consisting of all monomials of the form
\[ \prod_{\alpha \in \Phi^+} X_\alpha^{A(\alpha)} \prod_{i \neq i_0} K_i^{B_i} \prod_{\alpha \in \Phi^+} X_{-\alpha}^{C(\alpha)} \quad (A, C \in \mathbb{N}^{\Phi^+}, B \in \mathbb{N}_0^n) \]
of total degree not exceeding \( d \), where the products of powers of \( X_\alpha \),
\( X_{-\alpha} \) are taken with respect to arbitrary fixed orders on \( \Phi^+ \). This basis
would be the analogue of the PBW-type basis of \( U \), given in Lusztig \cite[Proposition 1.13]{Lu2}.

Our next result constructs the \( \mathcal{A} \)-form \( S_A(n, d) \) in terms of the gener-
erators given above. We write \( \left[ K_{i; t} \right] \) short for \( \left[ K_{i; 0} \right] \), where (following
Lusztig) we define
\[ \left[ K_{i; c} \right] = \prod_{s=1}^{t} \frac{K_i v^{c-s+1} - K_i^{-1} v^{-c+s-1}}{v^s - v^{-s}}, \quad \left[ K_{i; 0} \right] = 1 \]
for \( t \geq 1, c \in \mathbb{Z} \). For \( B \) in \( \mathbb{N}^n \), we write
\[ K_B = \prod_{i=1}^{n} K_i^{B_i} \].

Given \( \lambda \in \Lambda(n, d) \) we set \( 1_\lambda = K^{\lambda} \). Just as in the classical case,
the collection \( \{ 1_\lambda \} \) forms a set of pairwise orthogonal idempotents in
\( S_A(n, d) \) which sum to the identity element.
For $m \in \mathbb{Z}$ let $[m]$ denote the quantum integer $[m] = (v^m - v^{-m})/(v - v^{-1})$ and set

$$[m]! = [m][m - 1] \cdots [1], \quad [0]! = 1$$

$$\frac{c}{m} = \frac{[c][c - 1] \cdots [c - m + 1]}{[m]!}, \quad \frac{c}{0} = 1$$

for $c \in \mathbb{Z}$, $m \geq 1$. The $q$-analogues of the divided powers of root vectors are defined by $X_a^{(m)} = X_a/[m]!$. The Kostant monomials in this situation are products of elements of the form

$$X_a^{(r)}, \quad \left[ K_s \right], \quad K_i^{\pm 1} \quad (r, s \in \mathbb{N}, a \in \Phi, 1 \leq i \leq n),$$

taken in any order. As before, the set of Kostant monomials is multiplicatively closed. By analogy with the classical case, $\chi$ is defined by

$$\chi(X_a^{(m)}) = m \varepsilon_{\text{max}(i,j)}, \quad \chi\left( \left[ K_s \right] \right) = \chi(K_s) = 0$$

where $\alpha = \varepsilon_i - \varepsilon_j \in \Phi$, and by declaring that $\chi(XY) = \chi(X) + \chi(Y)$ whenever $X, Y$ are Kostant monomials. For $A, C \in \mathbb{N}^{\Phi^+}$ we write

$$E_A = \prod_{\alpha \in \Phi^+} X_{\alpha}^{(A(\alpha))}, \quad F_C = \prod_{\alpha \in \Phi^+} X_{-\alpha}^{(C(\alpha))}$$

where the products in $E_A$ and $F_C$ are taken relative to any two specified orderings on $\Phi^+$.

**Theorem 2.3.** The integral $q$-Schur algebra $S_A(n, d)$ is the subring of $S(n, d)$ generated by all quantum divided powers $E_i^{(m)}, F_i^{(m)}$, along with the elements $\left[ K_i \right]$. Moreover, each of the sets

(a) $Y_+ = \bigcup_{\lambda} \{ E_A1_{\lambda}F_C \mid \chi(E_AF_C) \preceq \lambda \}$

(b) $Y_- = \bigcup_{\lambda} \{ F_A1_{\lambda}E_C \mid \chi(F_AE_C) \preceq \lambda \}$,

as $\lambda$ ranges over $\Lambda(n, d)$, forms an $\mathcal{A}$-basis of $S_A(n, d)$.

We conjecture that the elements $\left[ K_i \right]$ lie within the subring generated by the $E_i^{(m)}, F_i^{(m)}$, in which case we would obtain the more precise analogue of Theorem 1.3.

Finally, we have another presentation of the $q$-Schur algebra by generators and relations. These relations are similar to relations that hold for the modified form $\hat{U}$ of $U$ (see [Lu3, Chap. 23]). This presentation
has the advantage that upon specializing \( v \) to 1, we recover the classical version given in Theorem 1.4.

**Theorem 2.4.** The algebra \( S(n, d) \) is the associative algebra (with 1) given by generators \( 1_\lambda (\lambda \in \Lambda(n,d)) \), \( E_i \), \( F_i \) \((1 \leq i \leq n-1)\) subject to the relations

\[
\sum_{\lambda \in \Lambda(n,d)} 1_\lambda = 1 (Q1')
\]

\[
E_i 1_\lambda = \begin{cases} 1_{\lambda+\alpha_i} E_i & \text{if } \lambda + \alpha_i \in \Lambda(n,d) \\ 0 & \text{otherwise} \end{cases} 
\]

\[
F_i 1_\lambda = \begin{cases} 1_{\lambda-\alpha_i} F_i & \text{if } \lambda - \alpha_i \in \Lambda(n,d) \\ 0 & \text{otherwise} \end{cases} 
\]

\[
1_\lambda E_i = \begin{cases} E_i 1_{\lambda-\alpha_i} & \text{if } \lambda - \alpha_i \in \Lambda(n,d) \\ 0 & \text{otherwise} \end{cases} 
\]

\[
1_\lambda F_i = \begin{cases} F_i 1_{\lambda+\alpha_i} & \text{if } \lambda + \alpha_i \in \Lambda(n,d) \\ 0 & \text{otherwise} \end{cases} 
\]

\[
E_i F_j - F_j E_i = \delta_{ij} \sum_{\lambda \in \Lambda(n,d)} [\lambda_j - \lambda_{j+1}] 1_\lambda (Q2')
\]

along with the \( q \)-Serre relations \((Q4), (Q5)\).

3. **The algebra \( T \)**

From now on we hold \( n \) and \( d \) fixed, and set \( S = S(n,d) \). We define an algebra \( T = T(n,d) \) (over \( \mathbb{Q} \)) by the generators and relations of Theorem 1.1. Since \( U \) is the algebra on the same generators but subject only to relations \((R1)-(R5)\), we have a surjective quotient map \( U \rightarrow T \) (mapping generators onto generators). Eventually we shall show that \( T \simeq S \), which will prove Theorem 1.1.

**Lemma 3.1.** Under the representation \( U \rightarrow \text{End}(V^{\otimes d}) \) the images of the \( H_i \) satisfy the relations \((R6)\) and \((R7)\). Moreover, the relation \((R7)\) is the minimal polynomial of (the image of) \( H_i \) in \( \text{End}(V^{\otimes d}) \).

**Proof.** Relation \((R6)\) is trivial in the case \( d = 1 \), from which the general case follows since each \( H_i \) acts as a derivation of \( V^{\otimes d} \). The relation \((R7)\) follows from the fact (which can be verified by induction on \( d \)) that the eigenvalues of the diagonal operators \( H_i \) are \( 0, 1, \ldots, d \). The proof is complete. \( \square \)
As we know, \( S = S(n, d) \) is the image of the representation \( U \to \text{End}(V^\otimes d) \) mentioned in the introduction. From the above lemma it follows that this surjection \( U \to S \) factors through \( T \). Because \( T, S \) are homomorphic images of \( U \), any relations between generators holding in \( U \) will automatically carry over to \( T, S \). We will not distinguish notionally between the generators or root vectors for \( U, T, \) or \( S \).

Recall the triangular decomposition of \( U \), that the multiplication map \( U^- \otimes U^0 \otimes U^+ \overset{\phi}{\to} U \) is an isomorphism of vector spaces, where \( U^+ \) (resp., \( U^- \)) is the subalgebra of \( U \) generated by the \( e_i \) (resp., \( f_i \)) and \( U^0 \) is the subalgebra of \( U \) generated by all \( H_i \). Thus \( U = U^-U^0U^+ \). From this we obtain a similar triangular decomposition of \( T \):

\[
(3.2) \quad T = T^-T^0T^+
\]

where \( T^+, T^-, T^0 \) are defined to be the images of \( U^+, U^-, U^0 \) under the quotient mapping \( U \to T \).

We also have similar factorizations over \( \mathbb{Z} \). Setting \( U_+^Z, U_-^Z, U^0_Z \) to be, respectively, the intersection of \( U^+, U^-, U^0 \) with the Kostant \( \mathbb{Z} \)-form \( U_Z \) (the subring of \( U \) generated by all \( e_i^{(m)}, f_i^{(m)}, \binom{H_i}{m} \)), we have the factorization \( U_Z = U_Z^-U_Z^0U_Z^+ \), which immediately induces similar equalities

\[
(3.3) \quad T_Z = T_Z^-T_Z^0T_Z^+
\]

where the various subalgebras are defined in the obvious way as appropriate homomorphic images of \( U_+^Z, U_-^Z, U^0_Z \).

Since \( U_+^Z \) (resp., \( U_-^Z \)) is the \( \mathbb{Z} \)-subalgebra of \( U \) generated by the \( x_\alpha^{(m)} \) for \( \alpha \in \Phi^+ \) (resp., \( \alpha \in \Phi^- \)) and \( m \in \mathbb{N} \), the same statement applies to \( T_+^Z \) (resp., \( T_-^Z \)) in relation to \( T \). Moreover, \( U^0_Z \) is the \( \mathbb{Z} \)-subalgebra of \( U \) generated by the \( \binom{H_i}{m} \) for \( 1 \leq i \leq n \) and \( m \in \mathbb{N} \), so \( T_Z^0 \) is the \( \mathbb{Z} \)-subalgebra of \( T \) generated by the same elements.

Now we investigate the structure of the algebra \( T^0 \). We start with the algebra \( U^0 \), which is isomorphic with the polynomial ring \( \mathbb{Q}[H_1, \ldots, H_n] \) in \( n \) commuting indeterminates \( H_1, \ldots, H_n \). By the remarks following Lemma 3.3 we have surjections \( U \to T \to S \). Let \( S^0 \) be the image in \( S \) of \( U^0 \) under the map \( U \to S \). Clearly we have surjections \( U^0 \to T^0 \to S^0 \) obtained from \( U \to T \to S \) by restriction.

**Proposition 3.4.** Define an algebra \( T' = U^0/I^0 \) where \( I^0 \) is the ideal in \( U^0 \) generated by elements \( H_i(H_i - 1) \cdots (H_i - d) \ (1 \leq i \leq n) \) and \( H_1 + \cdots + H_n - d \).

(a) We have an algebra isomorphism \( T' \cong T^0 \).
(b) The set \( \{ 1_\lambda \mid \lambda \in \Lambda(n,d) \} \) is a \( \mathbb{Q} \)-basis for \( T^0 \) and a \( \mathbb{Z} \)-basis for \( T^0_\mathbb{Z} \); moreover, this set is a set of pairwise orthogonal idempotents which add up to 1.

(c) \( H_B = 0 \) for any \( B \in \mathbb{N}^n \) such that \( |B| > d \).

Proof. Consider first the algebra \( \tilde{T}' \) defined to be the quotient of \( U^0 \) by the ideal generated only by the elements \( H_i(H_i - 1) \cdots (H_i - d) \) \((1 \leq i \leq n)\). Since each of the relations in \( \tilde{T}' \) is a polynomial in just one of the variables, we have the factorization

\[
\tilde{T}' \cong \mathbb{Q}[H_1]/(p(H_1)) \otimes \cdots \otimes \mathbb{Q}[H_n]/(p(H_n))
\]

where \( p(X) = X(X - 1) \cdots (X - d) \). By the Chinese Remainder Theorem applied to each tensor factor we obtain from the above isomorphisms (products denote direct products)

\[
\tilde{T}' \cong \prod_{i=0}^{d} (\mathbb{Q}[H_1]/(H_1 - i)) \otimes \cdots \otimes \prod_{i=0}^{d} (\mathbb{Q}[H_n]/(H_n - i))
\]

and by rearranging the order of factors we obtain

\[
\cong \prod_{0 \leq \mu_1, \ldots, \mu_n \leq d} (\mathbb{Q}[H_1]/(H_1 - \mu_1) \otimes \cdots \otimes \mathbb{Q}[H_n]/(H_n - \mu_n))
\]

\[
\cong \prod_{0 \leq \mu_1, \ldots, \mu_n \leq d} \mathbb{Q}[H_1, \ldots, H_n]/(H_1 - \mu_1, \ldots, H_n - \mu_n).
\]

The isomorphism is realized by the map which sends a polynomial \( f(H_1, \ldots, H_n) \) in \( \tilde{T}' \) to the element \( (f(\mu_1, \ldots, \mu_n))_{0 \leq \mu_1, \ldots, \mu_n \leq d} \) of the direct product. Since \( T' \) is isomorphic with \( \tilde{T}'/(H_1 + \cdots + H_n - d) \), we deduce from the above that

\[
T' \cong \prod_{\mu \in \Lambda(n,d)} \mathbb{Q}[H_1, \ldots, H_n]/(H_1 - \mu_1, \ldots, H_n - \mu_n)
\]

and this isomorphism (which we denote by \( \phi \)) is realized by the map sending \( f(H_1, \ldots, H_n) \) to the element \( (f(\mu_1, \ldots, \mu_n))_{\mu \in \Lambda(n,d)} \).

Thus, given any \( \lambda \in \Lambda(n,d) \), we have

\[
\phi(1_\lambda) = \left( \binom{\mu_1}{\lambda_1} \cdots \binom{\mu_n}{\lambda_n} \right)_{\mu \in \Lambda(n,d)} = (\delta_{\lambda \mu})_{\mu \in \Lambda(n,d)}
\]

Since the vector on the right of the above equality consists of zeros and ones, with precisely one nonzero entry, and since \( \phi \) is an isomorphism, it follows that the various \( 1_\lambda \) are orthogonal idempotents which add up to the identity of \( T' \).
Now let $I$ be the ideal in $U$ generated by elements $H_i(H_i-1)\cdots(H_i-d)$ $(1 \leq i \leq n)$ and $H_1 + \cdots + H_n - d$. Then by definition $T \cong U/I$. The canonical quotient map $U \to U/I$ induces, upon restriction to $U^0$, a map $U^0 \to U/I$. The image of this map is $T^0$ and its kernel is $U^0 \cap I$, so $T^0 \cong U^0/(U^0 \cap I)$. Clearly $I^0 \subset U^0 \cap I$. Thus we obtain the following sequence of algebra surjections

\[(3.5) \quad T' = U^0/I^0 \longrightarrow U^0/(U^0 \cap I) \longrightarrow T^0 \longrightarrow S^0\]

where the last map is obtained by Lemma 3.1, and the middle map is actually an isomorphism. By the above we see that the dimension of $T'$ is the cardinality of the set $\Lambda(n,d)$. This is well-known to equal the dimension of the zero part $S^0$ of the Schur algebra $S = S(n,d)$. It follows that all the surjections above are algebra isomorphisms. This proves parts (a) and (b).

To prove part (c), suppose that $B \in \mathbb{N}^n$ such that $|B| > d$. Then for each $\mu \in \Lambda(n,d)$, there exists $i$ with $\mu_i < b_i$. Thus $\phi(H_B) = 0$ and $H_B$ must be 0 since the map $\phi$ is an isomorphism. The proof is complete.

The next result is obtained by a similar argument.

**Proposition 3.6.** Let $1 \leq i \leq n$, $b \in \mathbb{N}$, $\lambda \in \Lambda(n,d)$, and $B \in \mathbb{N}^n$. We have the following identities in the algebra $T^0$:

\[(a) \quad H_i 1_{\lambda} = \lambda_i 1_{\lambda}, \quad \left(\begin{array}{c} H_i \\ b \end{array}\right) 1_{\lambda} = \left(\begin{array}{c} \lambda_i \\ b \end{array}\right) 1_{\lambda} \]

\[(b) \quad H_B 1_{\lambda} = \lambda_B 1_{\lambda}, \quad \text{where } \lambda_B = \prod_i \left(\begin{array}{c} \lambda_i \\ B_i \end{array}\right) \]

\[(c) \quad H_B = \sum_{\lambda} \lambda_B 1_{\lambda} \]

where the sum in part (c) is carried out over all $\lambda \in \Lambda(n,d)$.

**Proof.** We apply the isomorphism $\phi$ from the proof of the preceding proposition to the product on the left-hand-side of (a), obtaining

\[\phi(\left(\begin{array}{c} H_i \\ b \end{array}\right) 1_{\lambda}) = \phi(\left(\begin{array}{c} H_i \\ b \end{array}\right) \left(\begin{array}{c} H_i \\ \lambda_1 \end{array}\right) \cdots \left(\begin{array}{c} H_n \\ \lambda_n \end{array}\right)).\]

The right-hand side of the above equality gives the vector

\[\left(\begin{array}{c} \mu_i \\ \lambda_1 \\ \cdots \\ \mu_n \\ \lambda_n \end{array}\right)_{\mu \in \Lambda(n,d)} = \left(\begin{array}{c} \mu_i \\ b \end{array}\right) \delta_{\mu,\lambda}_{\mu \in \Lambda(n,d)}\]
which is the same as \((\lambda_i \ b) \phi(1_\lambda)\) since from the preceding proof \(\phi(1_\lambda) = (\delta_{\mu_\lambda})_{\mu}\). Since \(\phi\) is an isomorphism, part (a) is proved.

Part (b) follows immediately from part (a). Then by the result of part (b) we obtain the equalities \(H_B = H_B \cdot 1 = H_B \sum_{\lambda} 1_\lambda = \sum_{\lambda} \lambda_B 1_\lambda\), proving part (c).

We write \(N_{i_0}^n\) for the set of \(B = (B_i) \in N^n\) such that \(B_{i_0} = 0\).

**Corollary 3.7.** For any fixed choice of \(i_0 (1 \leq i_0 \leq n)\) the set \(\{H_B \mid B \in N_{i_0}^n, |B| \leq d\}\) is a \(\mathbb{Q}\)-basis for \(T_0^0\) and a \(\mathbb{Z}\)-basis for \(T_0^0\).

**Proof.** The sets \(\Lambda(n,d)\) and \(\{B \in N_{i_0}^n, |B| \leq d\}\) have the same cardinality. For instance, the map \(\lambda \rightarrow \lambda - \lambda_{i_0} \varepsilon_{i_0}\) is bijective between the sets in question, with inverse map \(B \rightarrow B + (d - |B|) \varepsilon_{i_0}\).

Thus, to prove the result it is enough to show that the set \(\{H_B \mid B \in N_{i_0}^n, |B| \leq d\}\) spans \(T_0^0\). This can be deduced by considering a certain order (depending on \(i_0\)) on each of these sets.

Fixing \(1 \leq i_0 \leq n\), we order the set \(\Lambda(n,d)\) by declaring that \(\lambda\) precedes \(\lambda'\) if \(\lambda_{i_0} < \lambda'_{i_0}\), or if \(\lambda_{i_0} = \lambda'_{i_0}\) and there exists an index \(l \neq i_0\) such that \(\lambda_j \geq \lambda'_{j}\) for all \(j \in \{1, \ldots, l\} - \{i_0\}\). Similarly, we order the set \(\{B \in N_{i_0}^n, |B| \leq d\}\) by declaring that \(B\) precedes \(B'\) if there exists an index \(l \neq i_0\) such that \(b_j \geq b'_{j}\) for all \(j \in \{1, \ldots, l\} - \{i_0\}\). With these orderings, it follows from part (c) of the preceding proposition that the matrix of coefficients obtained by expressing the \(H_B\)'s in terms of the \(1_\lambda\)'s is lower unitriangular. To see this, observe that for any given \(B\), with corresponding \(\lambda = B + (d - |B|) \varepsilon_{i_0}\), any \(\mu\) which succeeds \(\lambda\) in the above order satisfies \(\mu_B = 0\), and moreover \(\lambda_B = 1\). It follows that these equations can be inverted over \(\mathbb{Z}\) and so every \(1_\lambda\) is expressible as a \(\mathbb{Z}\)-linear combination of elements from the set \(\{H_B \mid B \in N_{i_0}^n, |B| \leq d\}\). This proves that this set spans \(T_0^0\) and, by our remarks above, it must therefore be a \(\mathbb{Q}\)-basis, and thus is linearly independent over \(\mathbb{Z}\), and hence also a \(\mathbb{Z}\)-basis. The proof is complete.

The next step is to find spanning sets for the plus part \(T^+\) and minus part \(T^-\) of \(T\). For this we use the following result.
Proposition 3.8. For any $\alpha \in \Phi$, $\lambda \in \Lambda(n,d)$ we have the commutation formulas

$$x_\alpha 1_\lambda = \begin{cases} 1_{\lambda + \alpha} x_\alpha & \text{if } \lambda + \alpha \in \Lambda(n,d) \\ 0 & \text{otherwise} \end{cases}$$

and similarly

$$1_\lambda x_\alpha = \begin{cases} x_\alpha 1_{\lambda - \alpha} & \text{if } \lambda - \alpha \in \Lambda(n,d) \\ 0 & \text{otherwise} \end{cases}.$$

Proof. Write $\alpha = \varepsilon_i - \varepsilon_j$ with $i \neq j$. From the defining relation (R3) and the definition (see §1) of $x_\alpha$ we have

$$H_l x_\alpha = x_\alpha (H_l + (\varepsilon_l, \alpha)).$$

(3.9)

From this we obtain equalities

$$x_\alpha 1_\lambda = x_\alpha \left( \frac{H_1}{\lambda_1} \right) \cdots \left( \frac{H_n}{\lambda_n} \right) = \left( \left( \frac{H_i - 1}{\lambda_i} \right) \left( \frac{H_j + 1}{\lambda_j} \right) \prod_{l \neq i,j} \left( \frac{H_l}{\lambda_l} \right) \right) x_\alpha.$$

Multiplying on the left by $H_i/(\lambda_i + 1)$ and then commuting with $x_\alpha$ yields the equality

$$x_\alpha \frac{H_i + 1}{\lambda_i + 1} 1_\lambda = \frac{H_i}{\lambda_i + 1} \left( \left( \frac{H_i - 1}{\lambda_i} \right) \left( \frac{H_j + 1}{\lambda_j} \right) \prod_{l \neq i,j} \left( \frac{H_l}{\lambda_l} \right) \right) x_\alpha,$$

which by Proposition 3.6(a) simplifies to give

$$x_\alpha 1_\lambda = \left( \left( \frac{H_i}{\lambda_i + 1} \right) \left( \frac{H_j + 1}{\lambda_j} \right) \prod_{l \neq i,j} \left( \frac{H_l}{\lambda_l} \right) \right) x_\alpha,$$

which (if $\lambda_j > 0$) can be rewritten in the form

$$x_\alpha 1_\lambda = \left( \frac{H_i}{\lambda_i + 1} \right) \left( \left( \frac{H_j}{\lambda_j} \right) + \left( \frac{H_j}{\lambda_j - 1} \right) \right) \prod_{l \neq i,j} \left( \frac{H_l}{\lambda_l} \right) x_\alpha.$$

The first summand on the right-hand-side of the preceding equality vanishes, by Proposition 3.4(c). This proves the first part of the proposition in the case $\lambda_j > 0$. In case $\lambda_j = 0$ the right-hand-side is zero. This proves the first part of the proposition. The proof of the second part is similar.

Corollary 3.10. We have in $T$ the equalities $x_\alpha^{d+1} = 0$ for all $\alpha \in \Phi$. 

Proof. By iterating the result of the preceding proposition we see that $x_{\alpha}^{d+1}1_\lambda = 0$ for any $\alpha \in \Phi$ and any $\lambda \in \Lambda(n,d)$, since it is clear that $\lambda + (d+1)\alpha$ does not belong to $\Lambda(n,d)$. Thus we have equalities

$$x_{\alpha}^{d+1} = x_{\alpha}^{d+1} \cdot 1 = x_{\alpha}^{d+1} \sum_{\lambda \in \Lambda(n,d)} 1_\lambda = \sum_{\lambda \in \Lambda(n,d)} x_{\alpha}^{d+1}1_\lambda = 0$$

and this proves the claim. \qed

4. Straightening

We need the following variants of the notion of content. We define functions $\chi_L$ and $\chi_R$ (left and right content) on Kostant monomials by setting

$$\chi_R(x_{\alpha}^{(m)}) = m \varepsilon_j, \quad \chi_L(x_{\alpha}^{(m)}) = m \varepsilon_i,$$

where $\alpha = \varepsilon_i - \varepsilon_j$, and again using the rule $\chi_L(XY) = \chi_L(X) + \chi_L(Y)$ (similarly for $\chi_R$) whenever $X$ and $Y$ are Kostant monomials. Note that for $A, C \in \mathbb{N}^{\Phi^+}$ we have

$$\chi(e_A) = \chi_R(e_A), \quad \chi(f_C) = \chi_L(f_C).$$

From this it follows immediately that

$$\chi(e_Af_C) = \chi_R(e_A) + \chi_L(f_C).$$

From Proposition 3.8 it follows that for any $A, C \in \mathbb{N}^{\Phi^+}, \lambda \in \Lambda(n,d)$ we have equalities

$$e_A1_\lambda f_C = 1_\lambda e_Af_C = e_Af_C1_{\lambda''}$$

where $\lambda' = \lambda + \sum_{\alpha} A(\alpha) \alpha$, $\lambda'' = \lambda + \sum_{\alpha} C(\alpha) \alpha$ (both sums over $\Phi^+$). Moreover, we have equalities

$$\sum_{\alpha \in \Phi^+} A(\alpha) \alpha = \sum_{\alpha = \varepsilon_i - \varepsilon_j; i < j} A(\alpha)(\varepsilon_i - \varepsilon_j)$$

$$= \chi_L(e_A) - \chi_R(e_A)$$

$$= -\chi_L(f_A) + \chi_R(f_A)$$

from which it follows that

$$\lambda' = \lambda - \chi_R(e_A) + \chi_L(e_A), \quad \lambda'' = \lambda + \chi_R(f_C) - \chi_L(f_C).$$

Lemma 4.6. $\chi(e_A1_\lambda f_C) \preceq \lambda \iff \chi_L(1_{\lambda'} e_Af_C) \preceq \lambda' \iff \chi_R(e_Af_C1_{\lambda''}) \preceq \lambda''$. 


Proof. By the definitions we have $\chi(e_A1_\lambda f_C) = \chi(e_A f_C)$, with similar equalities for $\chi_R, \chi_L$. From equation (4.3) and the above we have the following equivalences

$$\chi(e_A1_\lambda f_C) \preceq \lambda \iff \chi_R(e_A) + \chi_L(f_C) \preceq \lambda$$

$$\iff \chi_L(f_C) \preceq \lambda - \chi_R(e_A)$$

$$\iff \chi_L(f_C) \preceq \lambda - \chi_R(e_A) + \chi_L(e_A)$$

$$\iff \chi_L(1_\lambda e_A f_C) \preceq \lambda'.$$

This establishes the first equivalence of the lemma. The second equivalence is established by a similar argument. \qed

By similar reasoning one can obtain similar equivalences in which the $f_A$ precede the $e_C$.

It follows from the preceding lemma that the set $Y_+$ (see Theorem [1.3]) can be rewritten in either of the forms

$$\bigcup_{\lambda'} \{1_{\lambda'} e_A f_C \mid \chi_L(e_A f_C) \preceq \lambda' \}$$

(4.7)

$$= \bigcup_{\lambda''} \{e_A f_C 1_{\lambda''} \mid \chi_R(e_A f_C) \leq \lambda'' \}$$

with a similar statement applying to the set $Y_-$.

The subspace of $U$ spanned by the $H_i$ and $x_\alpha$ is isomorphic with the Lie algebra $\mathfrak{gl}_n$ of $n \times n$ matrices. The isomorphism is determined by sending $e_i \rightarrow e_{i,i+1}, f_i \rightarrow e_{i+1,i}$. Here the notation $e_{ij}$ stands for the matrix with all entries 0, except for the $(i,j)$th entry, which is 1. It follows from the definition of $x_\alpha$ (see $(1)$) that the isomorphism carries $H_i$ to $e_{ii}$ and $x_\alpha$ to $e_{ij}$ when $\alpha = \varepsilon_i - \varepsilon_j \in \Phi$. From this one can now easily verify that (for $\alpha = \varepsilon_i - \varepsilon_j, \beta = \varepsilon_k - \varepsilon_\ell$ for $i \neq j, k \neq \ell$)

$$x_\alpha x_\beta - x_\beta x_\alpha = \begin{cases} H_\alpha & \text{if } \alpha + \beta = 0 \\ c_{\alpha,\beta} x_{\alpha+\beta} & \text{if } \alpha + \beta \in \Phi \\ 0 & \text{otherwise} \end{cases}$$

(4.8)

where $H_\alpha = H_i - H_j$ and where for $\alpha + \beta \in \Phi$ we have

$$c_{\alpha,\beta} = \begin{cases} 1 & \text{if } j = k \text{ and } i \neq \ell \\ -1 & \text{if } i = \ell \text{ and } j \neq k. \end{cases}$$

The above relations hold in the enveloping algebra $U$, since we use only the defining relations (R1)–(R5) in their verification. Hence they are valid as well as in the quotient $T$.

For $r, s \in \mathbb{N}$, $\alpha, \beta \in \Phi$, one deduces the following commutation relations from the above by induction (compare with Kostant [Ko], Lemma
\[ x^{(r)}_{\alpha} x^{(s)}_{-\alpha} = \sum_{j=0}^{\min(r,s)} x^{(s-j)}_{-\alpha} \left( \frac{H_\alpha - r - s + 2j}{j} \right) x^{(r-j)}_{\alpha}, \]  

and, for \( \alpha + \beta \neq 0, \alpha + \beta \in \Phi \)

\[ x^{(r)}_{\alpha} x^{(s)}_{\beta} = x^{(s)}_{\beta} x^{(r)}_{\alpha} + \sum_{j=1}^{\min(r,s)} c^{j}_{\alpha,\beta} x^{(s-j)}_{\beta} x^{(j)}_{\alpha+\beta} x^{(r-j)}_{\alpha}, \]

For \( \alpha + \beta \neq 0, \alpha + \beta \notin \Phi \)

\[ x^{(r)}_{\alpha} x^{(s)}_{\beta} = x^{(s)}_{\beta} x^{(r)}_{\alpha}. \]

From these formulas it follows that we can always interchange the order of a product of two divided powers of root vectors, up to a \( \mathbb{Z} \)-linear combination of terms of strictly lower degree and (right or left) content.

In the following argument we make use of the fact that \( (H_\alpha - t)^s \) \((s, t \in \mathbb{Z}, \alpha \in \Phi)\) belongs to the subring of \( U \) generated by the divided powers of all root vectors, and thus belongs to \( T^0_\mathbb{Z} \). This follows from [Ko, Lemma 1] by an inductive argument. We will also use the identity

\[ x^{(a)}_{\gamma} x^{(b)}_{\gamma} = \left( \frac{a + b}{a} \right) x^{(a+b)}_{\gamma} \ (\gamma \in \Phi), \]

which follows immediately from the definitions.

**Proposition 4.10.** The sets \( Y_+ \) and \( Y_- \) span the algebra \( T_\mathbb{Z} \).

**Proof.** We prove just the claim about \( Y_+ \), as the other case is similar. We use the second formulation of \( Y_+ \) given in (4.7). The algebra \( T_\mathbb{Z} \) is spanned by the set of all products formed from divided powers of root vectors and idempotents \( 1_\lambda \). By Proposition 3.8, in any such product, one may always commute the idempotents all the way to the right. Hence, \( T_\mathbb{Z} \) is spanned (over \( \mathbb{Z} \)) by the set of all monomials of the form

\[ M = x^{(\psi_1)}_{\gamma_1} \cdots x^{(\psi_m)}_{\gamma_m} 1_\mu, \]

for various \( m \in \mathbb{N}, \gamma \in \Phi^m, \psi \in \mathbb{N}^m, \mu \in \Lambda(n, d) \). We can assume that \( \gamma_i \neq \gamma_{i+1} \) for all \( i \leq m - 1 \), for otherwise the monomial is an integral multiple of a monomial having that property, by (4.9). Call monomials of such form special.

Let \( \chi = \chi_R(M) \). We define the deviation of \( M \) by \( \delta = \sum_{\chi_i > \mu_i} (\chi_i - \mu_i) \). Note that if \( M' = x^{(\psi_1')}_{\gamma_1'} \cdots x^{(\psi_m')}_{\gamma_m'} 1_\mu \),

\[ x^{(a)}_{\gamma} x^{(b)}_{\gamma} = \left( \frac{a + b}{a} \right) x^{(a+b)}_{\gamma}. \]
is another special monomial with the same idempotent part \(1_\mu\), then 
\(\chi_R(M) \preceq \chi_R(M') \Rightarrow \delta(M) \leq \delta(M')\) and 
\(\chi_R(M) \preceq \mu \Leftrightarrow \delta(M) = 0\).

Now we claim that all special monomials \(M\) of deviation 0 lie in the 
\(\mathbb{Z}\)-span of \(Y_+\). We argue by induction on the degree 
\(r = \sum \psi_i\) of \(M\).

If \(r = 0\) then \(M = 1_\mu\), which belongs to \(Y_+\). Suppose that \(r > 0\). If 
\(M \in Y_+\) then we are done. Otherwise, we can apply the commutation 
relations (4.8) to reorder the factors in \(M\), obtaining an equality of the 
form

\[M = x^{(\psi_1)} \cdots x^{(\psi_m)} = c e_A f_C + \text{lower terms}\]

for some \(c \in \mathbb{Z}\), \(A, C \in \mathbb{N}^{\Phi^+}\). Here, the lower terms are integral 
multiples of Kostant monomials of strictly lower degree and content. The 
constant \(c \in \mathbb{Z}\) arises not from the commutation formulas but when two 
divided powers are combined via the equality (4.9). Note that, in the 
lower order terms, whenever a factor of the form 
\(\left( H_\alpha - t \right) \) (\(s, t \in \mathbb{Z}\))

appears, we express it in terms of a \(\mathbb{Z}\)-linear combination of 1's, and 
then apply Proposition 3.8 to commute the idempotents as far to the 
right as possible. Upon right multiplication of the above equality by 
\(1_\mu\) we obtain the equality

\[M = c e_A f_C 1_\mu + \text{lower terms}\]

where the lower terms are integral multiples of terms of the same form 
as \(M\) (each having a factor \(1_\mu\) on the right), but again, of strictly lower 
degree and content than that of \(M\). Now, in the above equality the 
right content of \(e_A f_C 1_\mu\) is equal to the right content of \(M\), and thus 
\(e_A f_C 1_\mu \in Y_+\). By induction the lower terms belong to the 
\(\mathbb{Z}\)-span of \(Y_+\). The claim is proved.

Now we proceed by induction on deviation. (The above claim forms the 
base step in the induction.) Let \(M\) be a special monomial of the 
form (1.11) of deviation \(\delta = \delta(M) > 0\). Set \(\chi = \chi_R(M)\). Since \(\chi \not\preceq \mu\), 
there exists an index \(j\) for which \(\chi_j > \mu_j\). Fixing this index \(j\), we call 
\(\beta \in \Phi\) bad if \(\beta = \varepsilon_i - \varepsilon_j\) for some \(i \neq j\) and call \(\beta\) good otherwise. We 
extend this terminology to the factors \(x^{(\psi_i)}\) of \(M\). We define the defect 
\(D\) of the monomial \(M\) by the equality 
\(D(M) = \sum \psi_\alpha\) where the sum 
is taken over the set \(\{x^{(\psi_\alpha)}\}\) of good factors in \(M\) which appear to the 
right of some bad factor. Note that \(D = 0\) if and only if all the bad 
factors in \(M\) appear as far to the right as possible. From Proposition 
3.8 it follows immediately that \(M = 0\) in \(T\) whenever \(D(M) = 0\).

Now suppose that \(D(M) > 0\). So there exists at least one good 
factor \(x^{(a)}_\alpha\) in \(M\) appearing to the right of some bad factor \(x^{(b)}_\beta\). We
may assume that $x_\alpha^{(a)}$ is the leftmost such good factor in $M$. It has one or more bad factors immediately to its left. We successively commute $x_\alpha^{(a)}$ with each bad factor to its left, using relations (4.8). As before, we express factors of the form \( \left( \frac{H_\alpha - t}{s} \right) \) in terms of idempotents $1_\lambda$, and commute these all the way to the right using Proposition 3.8. By Proposition 3.6 we know that such factors act as integral scalars on $1_\mu$. The result of all this is thus, up to a $\mathbb{Z}$-linear combination $L$ of Kostant monomials (all involving the same idempotent $1_\mu$ on the right) of strictly lower right content, an integral multiple of a monomial $M'$ of the same right content as $M$ but of strictly lower defect. By induction (on defect) $M'$ lies in the $\mathbb{Z}$-span of $Y_+$, and, since the monomials in $L$ have lower deviation than $M$ does, $L$ must also lie within the $\mathbb{Z}$-span of $Y_+$. This proves that $M$ lies in the $\mathbb{Z}$-span of $Y_+$, and hence that $Y_+$ spans $T_\mathbb{Z}$. The argument for $Y_-$ is similar (interchange right and left, $+$ and $-$ in the above argument).

\[ \square \]

5. Proof of Theorems 1.1 and 1.3

Since $S = S(n,d)$ is the image of the map $U \to \text{End}(V^\otimes d)$, it follows immediately from Lemma 3.1 that the images of the $H_i$ in the Schur algebra $S$ satisfy the defining relations for the algebra $T$, so the surjection $U \to S$ factors through the algebra $T$. In particular, this gives a surjection $T \to S$. It follows that $\dim T \geq \dim S$. In order to produce the opposite inequality, which will complete the proof of Theorem 1.1, it is enough to produce a spanning set in $T$ of cardinality equal to the dimension of $S$. We know from Proposition 4.10 that the sets $Y_+$, $Y_-$ span $T$, so the proof of Theorem 1.1 is completed by the following lemma.

**Lemma 5.1.** The cardinality of $Y_+$ and $Y_-$ is equal to the dimension of $S = S(n,d)$.

**Proof.** By symmetry, it is enough to prove this for $Y_+$. It is well known (see [Gr]) that the dimension of $S(n,d)$ is given by the number of monomials of total degree $d$ in $n^2$ variables. This is the same as the number of monomials in $n^2 - 1$ variables of total degree not exceeding $d$; in other words, the dimension of $S(n,d)$ is the same as the cardinality of the set

\[ P = \{ e_A H_B f_C \mid B \in \mathbb{N}_1^n, A, C \in \mathbb{N}^{\Phi^+}, |A| + |B| + |C| \leq d \}. \]
Thus, to prove the result it suffices to give a bijective correspondence between $P$ and $Y_+$. One such is given by the map

$$e_A H_B f_C \rightarrow e_A 1_\lambda f_C$$

where $\lambda = (d - |A| - |B| - |C|) \varepsilon_1 + B + \chi(e_A f_C)$. The inverse map is given by

$$e_A 1_\lambda f_C \rightarrow e_A H_B f_C$$

where $B = \lambda - \chi(e_A f_C) - \lambda_1 \varepsilon_1$. The lemma is proved.

It remains to prove Theorem 1.3. It follows from the preceding arguments that the quotient map $T \rightarrow S$ is an isomorphism of algebras, and from Proposition 4.10 we conclude that $Y_+$ and $Y_-$ are bases for $T$ (over $\mathbb{Q}$). Hence these sets are linearly independent over $\mathbb{Q}$, and thus also over $\mathbb{Z}$. Thus they are $\mathbb{Z}$-bases for $T_\mathbb{Z}$. Carter and Lusztig [CL Thm. 3.1] showed that the restriction to $U_\mathbb{Z}$ of the map $U \rightarrow S$ gives a surjection $U_\mathbb{Z} \rightarrow S_\mathbb{Z} = S_{\mathbb{Z}}(n, d)$. It follows that the restriction map $T_\mathbb{Z} \rightarrow S_\mathbb{Z}$ is an isomorphism, and that the sets $Y_+, Y_-$ are integral bases for the Schur algebra $S_\mathbb{Z}$. Moreover, the restriction of the map $U_\mathbb{Z} \rightarrow S_\mathbb{Z}$ to $U_{\mathbb{Z}}(\mathfrak{s}_n)$ is still surjective, according to [DG, p. 44], and thus the image is generated by all $e_i^{(m)}, f_i^{(m)}$. This proves Theorem 1.3.

Remark 5.2. We conjecture that the set $P$ appearing in the proof of Lemma 5.1 is actually another integral basis for $S_{\mathbb{Z}}(n, d)$. More generally, for any fixed $i_0$, $(1 \leq i_0 \leq n)$, either of the sets

$$\{e_A H_B f_C\}, \quad \{f_A H_B e_C\}$$

$(B \in \mathbb{N}_{i_0}^n, A, C \in \mathbb{N}^{\Phi}_+, |A| + |B| + |C| \leq d)$ should be an integral basis of $S_{\mathbb{Z}}(n, d)$. This would be a truncated form of Kostant’s well-known basis for $U_\mathbb{Z}$. The conjecture is true when $n = 2$; see [DG1].

6. Proof of Theorem 1.4

Let $B$ be the $\mathbb{Q}$-algebra given by the generators and relations of Theorem 1.4. In $B$ we define elements $H_j$ for $j = 1, \ldots, n$ by setting

$$H_j = \sum_{\lambda} \lambda_j 1_\lambda,$$

where the sum is carried over all $\lambda \in \Lambda(n, d)$.

Since the various $1_\lambda$’s commute it follows that the $H_j$’s must also commute, so relation (R1) holds for the elements $H_j$. Relation (R2) follows immediately from the defining relations (R3) and the definition of the $H_j$. 
From the defining relations (R1) and the definition of the $H_j$ it follows that

$$
\sum_{j=1}^{n} H_j = \sum_{j=1}^{n} \sum_{\lambda} \lambda_j 1_{\lambda} = \left( \sum_{j=1}^{n} \lambda_j \right) 1_{\lambda} = \sum_{\lambda} d 1_{\lambda} = d \cdot 1 = d.
$$

This proves that the $H_j$ satisfy relation (R6). Moreover, we also have equalities

$$
1_{\lambda} H_j = H_j 1_{\lambda} = \sum_{\mu} \mu_j 1_{\mu} 1_{\lambda} = \lambda_j 1_{\lambda}
$$

for each $\lambda, j$, from which we obtain the equalities

$$
H_j (H_j - 1) \cdots (H_j - b) = \left( \sum_{\lambda} \lambda_j 1_{\lambda} \right) (H_j - 1) \cdots (H_j - b) = \sum_{\lambda} \lambda_j (\lambda_j - 1) \cdots (\lambda_j - b) 1_{\lambda}
$$

for any $b \in \mathbb{N}$. This is 0 when $b = d$, since $\lambda$ is a composition of $d$, and thus each part $\lambda_j$ of $\lambda$ is an integer in the interval $0, \ldots, d$. This proves that the $H_j$ satisfy relation (R7).

We now want to show that the $H_j$ also satisfy relation (R3). For this we will use the defining relations (R2). For convenience, we extend the definition of the symbol $1_{\lambda}$ to all $\lambda \in \mathbb{Z}^n$ such that $|\lambda| = \sum \lambda_i = d$, defining it to have the value 0 if any part of $\lambda$ is negative. With this convention we have

$$
H_j e_i = \sum_{\lambda} \lambda_j 1_{\lambda} e_i = \sum_{\lambda} \lambda_j e_i 1_{\lambda-\alpha_i}
$$

where the sums are taken over all $\lambda \in \mathbb{Z}^n$ such that $|\lambda| = d$. Replacing $\lambda - \alpha_i$ by $\mu$ and noting that $\lambda_j = \mu_j$ if $j \neq i, i+1$, $\lambda_j = \mu_j + 1$ if $j = i$, and $\lambda_j = \mu_j - 1$ if $j = i + 1$ we obtain

$$
H_j e_i = \sum_{\mu} \lambda_j e_i 1_{\mu} = e_i H_j + (\delta_{ij} - \delta_{i+1,j}) e_i.
$$

where again the sum is over all $\mu \in \mathbb{Z}^n$ satisfying $|\mu| = d$. This proves that the $H_j, e_i$ satisfy relation (R3); a similar argument shows the same for the $H_j, f_i$.

From (6.1) it follows (upon replacing $b$ by $b - 1$ and dividing by $b!$) that

$$
\binom{H_j}{b} = \sum_{\lambda} \binom{\lambda_j}{b} 1_{\lambda}
$$
where the sum is over $\Lambda(n, d)$. It then follows from relations (R1) and the above, by a simple calculation, that for any $\mu \in \Lambda(n, d)$ we have
\[
\prod_{j=1}^{n} \left( \frac{H_j}{\mu_j} \right) = 1
\]
and thus the $H_j$, $(1 \leq j \leq n)$ together with the $e_i, f_i$ $(1 \leq i \leq n - 1)$ generate the algebra $B$. Since we have proved that these generators satisfy the defining relations for the algebra $T$, it follows that $B$ is a homomorphic image of $T$.

On the other hand, by Proposition 3.8 we know that the elements $e_i, f_i, 1_\lambda$ of $T$ satisfy relations (R2). They also satisfy relation (R1), clearly, and relation (R3), by Proposition 3.6. Moreover, $T$ is generated by the $1_\lambda, e_i, f_i$ since by Proposition 3.6(c) we know that $H_j$ is expressible as a linear combination of the $1_\lambda$. Thus $T$ is a homomorphic image of $B$. Combining this with the conclusion of the preceding paragraph, we see that $B \cong T \cong S(n, d)$. Theorem 1.4 is proved.

7. The algebra $T$.

We turn now to the quantum case. Fix $n$ and $d$, and set $S = S(n, d)$. We define an algebra $T = T(n, d)$ (over $\mathbb{Q}(v)$) by the generators and relations of Theorem 2.1. Since $U$ is the algebra on the same generators but subject only to relations (Q1)–(Q5), we have a surjective quotient map $U \to T$. Eventually we shall show that $T \cong S$, which will prove Theorem 1.4.

The $q$-analogue of Lemma 3.1 is the following. The proof is similar to the proof in the classical case, except that it is multiplicative where the classical argument is additive.

**Lemma 7.1.** Under the representation $U \to \text{End}(V^{\otimes d})$ the images of the $K_i$ satisfy the relations (Q6) and (Q7). Moreover, the relation (Q7) is the minimal polynomial of $K_i$ in $\text{End}(V^{\otimes d})$.

Since $S = S(n, d)$ is the image of the representation $U \to \text{End}(V^{\otimes d})$, we have a surjection $U \to S$. From the lemma it follows that the surjection $U \to S$ factors through $T$. Because $T, S$ are homomorphic images of $U$, any relations between generators in $U$ will carry over to $T, S$. We do not distinguish notationally between the generators or root vectors for $U, T, S$.

Rosso [Ro] has shown that multiplication defines a $\mathbb{Q}(v)$-linear isomorphism $U^- \otimes U^0 \otimes U^+ \cong U$, where $U^+$ (resp., $U^-$) is the subalgebra of $U$ generated by the $E_i$ (resp., $F_i$), and $U^0$ is the subalgebra of $U$
generated by all $K_i, K_i^{-1}$. It follows that $U = U^- U^0 U^+$. From this we obtain a similar triangular decomposition of $T$:

\[(7.2) \quad T = T^- T^0 T^+\]

where $T^+, T^-, T^0$ are defined to be the images of $U^+, U^-, U^0$ under the quotient mapping $U \to T$.

There are similar factorizations over $A$. Setting $U^+_A, U^-_A, U^0_A$ to be, respectively, the intersection of $U^+, U^-, U^0$ with Lusztig’s $A$-form $U_A$ (the $A$-subalgebra of $U$ generated by the $E_i^{(m)}, F_i^{(m)}, K_i^{\pm 1}, \left[ \frac{K_i}{m} \right]$), Du [Du, §2] has shown (using results of Lusztig) that $U_A = U^-_A U^0_A U^+_A$; thus

\[(7.3) \quad T_A = T^-_A T^0_A T^+_A\]

where the various subalgebras are defined in the obvious way as appropriate homomorphic images of $U^+_A, U^-_A, U^0_A$.

Since $U^+_A$ (resp., $U^-_A$) is the $A$-subalgebra of $U$ generated by the $E_i^{(m)}$ (resp., $F_i^{(m)}$) for $\alpha \in \Phi^+$ and $m \in \mathbb{N}$, the same statement applies to $T^+_A$ (resp., $T^-_A$) in relation to $T$. Moreover, $U^0_A$ is the $A$-subalgebra of $U$ generated by the $K_i^{\pm 1}, \left[ \frac{K_i}{m} \right]$ for $1 \leq i \leq n, m \in \mathbb{N}$, so $T^0_A$ is the $A$-subalgebra of $T$ generated by the same elements.

Now we determine the structure of the algebra $T^0$. As we shall see, the structure turns out to be essentially the same as that in the classical case. Consider first the algebra $U^0$, which may be identified with the commutative polynomial algebra $Q(v)[K_1^{\pm 1}, \ldots, K_n^{\pm 1}]$. We define $S^0$ to be the image of $U^0$ under the quotient map $U \to S$. As in the classical case, this map factors through the algebra $T^0$.

**Proposition 7.4.** Define an algebra $T' = U/I^0$ where $I^0$ is the ideal in $U^0$ generated by the elements $(K_i - 1)(K_i - v) \cdots (K_i - v^d)$ ($1 \leq i \leq n$) and $K_1 K_2 \cdots K_n - v^d$.

(a) We have an algebra isomorphism $T' \cong T^0$.

(b) The set $\{1_\lambda \mid \lambda \in \Lambda(n, d)\}$ is a $Q(v)$-basis for $T^0$ and an $A$-basis for $T^0_A$; moreover, this set is a set of pairwise orthogonal idempotents which add up to 1.

(c) $K_B = 0$ for any $B \in \mathbb{N}^n$ such that $|B| > d$.

**Proof.** The argument is similar to the proof of Proposition 3.4. Consider first the algebra $\tilde{T}'$ defined to be the quotient of $U^0$ by the ideal generated only by the $p(K_i) = (K_i - 1)(K_i - v) \cdots (K_i - v^d)$
for $i = 1, \ldots, n$. Since $p(K_i)$ is a non-constant polynomial with non-zero constant term, each $K_i^{-1}$ already lies in the ring $\tilde{T}'$. Thus it follows that

$$\tilde{T}' \cong \mathbb{Q}(v)[K_1]/(p(K_1)) \otimes \cdots \otimes \mathbb{Q}(v)[K_n]/(p(K_n)).$$

From the Chinese Remainder Theorem we obtain, as in the proof of 3.4, an isomorphism (products denote direct products)

$$\tilde{T}' \cong \prod_{0 \leq \mu_1, \ldots, \mu_n \leq d} \mathbb{Q}(v)[K_1, \ldots, K_n]/(K_1 - v^\mu_1, \ldots, K_n - v^\mu_n).$$

Since $T' = \tilde{T}'/(K_1 \cdots K_n - v^d)$, we deduce that

$$T' \cong \prod_{\mu \in \Lambda(n,d)} \mathbb{Q}(v)[K_1, \ldots, K_n]/(K_1 - v^\mu_1, \ldots, K_n - v^\mu_n).$$

The preceding isomorphism, which we denote by $\phi$, is given by the map sending $f(K_1, \ldots, K_n)$ to the vector $(f(v^{\mu_1}, \ldots, v^{\mu_n}))_{\mu \in \Lambda(n,d)}$. In particular, if $\lambda \in \Lambda(n,d)$, then

$$\phi(1_\lambda) = \left( \begin{bmatrix} \mu_1 \\ \lambda_1 \\ \vdots \\ \mu_n \\ \lambda_n \end{bmatrix} \right)_{\mu \in \Lambda(n,d)} = (\delta_{\lambda \mu})_{\mu \in \Lambda(n,d)}.$$

Thus the various $1_\lambda$ are orthogonal idempotents whose sum is the identity.

Now let $I$ be the ideal in $U$ generated by elements $(K_i - 1)(K_i - v) \cdots (K_i - v^d) \ (1 \leq i \leq n)$ and $K_1 \cdots K_n - v^d$. Then by definition $T \cong U/I$. The canonical quotient map $U \to U/I$ induces, upon restriction to $U^0$, a map $U^0 \to U/I$. The image of this map is $T^0$ and its kernel is $U^0 \cap I$, so $T^0 \cong U^0/(U^0 \cap I)$. Clearly $I^0 \subset U^0 \cap I$. Thus we obtain the following sequence of algebra surjections

$$T' = U^0/I^0 \longrightarrow U^0/(U^0 \cap I) \longrightarrow T^0 \longrightarrow S^0$$

where the last map is obtained by Lemma 7.1 and the middle map is actually an isomorphism. By the above we see that the dimension of $T'$ is the cardinality of the set $\Lambda(n,d)$. This is the same as $\dim S^0$, so all the surjections above are algebra isomorphisms. This proves parts (a) and (b).

Part (c) is proved in exactly the same way as part (c) of Proposition 3.4.

Proposition 7.6. Suppose $1 \leq i \leq n$, $c \in \mathbb{Z}$, $t \in \mathbb{N}$, $\lambda \in \Lambda(n,d)$, and $B \in \mathbb{N}^0$. Then we have the following identities in the algebra $T^0$:

(a) $K_i^{\pm 1}1_\lambda = v^{\pm \lambda_i}1_\lambda$; $$\left[ \begin{bmatrix} K_i; c \\ t \end{bmatrix} \right] 1_\lambda = \left[ \begin{bmatrix} \lambda_i + c \\ t \end{bmatrix} \right] 1_\lambda.$$
(b) \[ K_B 1_\lambda = \lambda_B 1_\lambda, \quad \text{where} \quad \lambda_B = \prod_i \left[ \frac{\lambda_i}{B_i} \right] \]

(c) \[ K_B = \sum_\lambda \lambda_B 1_\lambda, \]

where the sum in part (c) is carried out over all \( \lambda \in \Lambda(n,d) \).

**Proof.** We prove part (a). Apply the isomorphism \( \phi \) from the proof of Proposition 7.4. We have

\[ \phi(K_i^{\pm 1} 1_\lambda) = (v^{\pm \mu_i} \delta_{\lambda\mu}) \mu = v^{\pm \lambda_i} \phi(1_\lambda). \]

It follows that \( K_i^{\pm 1} 1_\lambda = v^{\pm \lambda_i} 1_\lambda \). Similarly, we have

\[ \phi\left( [K_i; c] 1_\lambda \right) = \phi\left( \prod_{s=1}^t \left( \frac{K_i v^{c-s+1} - K_i^{-1} v^{-c+s-1}}{v^s - v^{-s}} \right) 1_\lambda \right) \]

\[ = \left( \prod_{s=1}^t \left( \frac{v^{\mu_i} v^{c-s+1} - v^{-\mu_i} v^{-c+s-1}}{v^s - v^{-s}} \delta_{\lambda\mu} \right) \right)_\mu \]

\[ = \left[ \begin{array}{c} \mu_i + c \\ t \end{array} \right] \delta_{\lambda\mu} \]

\[ = \left[ \begin{array}{c} \lambda_i + c \\ t \end{array} \phi(1_\lambda) \right] \]

which proves the second equality in (a). Note that the equality

\[ (7.7) \quad [K_i; c] 1_\lambda = [\lambda_i; t] 1_\lambda \]

is the case \( c = 0 \) of the above.

The rest of the proof is similar to the proof of Proposition 3.6. \( \square \)

By essentially the same argument as in the classical case we obtain

**Corollary 7.8.** For any fixed choice of \( i_0 \) \((1 \leq i_0 \leq n)\) the set \( \{K_B \mid B \in \mathbb{N}_{i_0} \mid |B| \leq d\} \) is a \( \mathbb{Q}(v) \)-basis for \( T^0 \) and an \( \mathcal{A} \)-basis for \( T^0_{\mathcal{A}} \).

We also have the following exact analogue of Proposition 3.8.

**Proposition 7.9.** For any \( \alpha \in \Phi \) and any \( \lambda \in \Lambda(n,d) \) we have the commutation formulas

\[ X_\alpha 1_\lambda = \begin{cases} 1_{\lambda + \alpha} X_\alpha & \text{if } \lambda + \alpha \in \Lambda(n,d) \\ 0 & \text{otherwise} \end{cases} \]

and similarly

\[ 1_\lambda X_\alpha = \begin{cases} X_\alpha 1_{\lambda - \alpha} & \text{if } \lambda - \alpha \in \Lambda(n,d) \\ 0 & \text{otherwise} \end{cases}. \]
Proof. We will need the following identities (see \([\text{Lu2}, \S 2.3 \text{ (g3), (g4)}]\)):

\[
\begin{bmatrix} K_i ; 0 \\ t \end{bmatrix} \begin{bmatrix} K_i ; -t \\ t' \end{bmatrix} = \begin{bmatrix} t + t' \\ t \end{bmatrix} \begin{bmatrix} K_i ; 0 \\ t + t' \end{bmatrix} \quad (t, t' \in \mathbb{N})
\]

\[
\begin{bmatrix} K_i ; c + 1 \\ t \end{bmatrix} = v^t \begin{bmatrix} K_i ; c \\ t \end{bmatrix} + v^{t-c-1} K_i^{-1} \begin{bmatrix} K_i ; c \\ t - 1 \end{bmatrix} \quad (t \geq 1).
\]

From the defining relation (Q3) and the definition (see \([\text{Z}]\)) of the root vector \(X_\alpha\) one proves that

\[
K_i X_\alpha = v^{(\varepsilon_i, \alpha)} X_\alpha K_i \quad (\alpha \in \Phi).
\]

From this it follows by a simple calculation that

\[
\begin{bmatrix} K_i ; c \\ t \end{bmatrix} X_\alpha = X_\alpha \begin{bmatrix} K_i ; c + (\varepsilon_i, \alpha) \\ t \end{bmatrix} \quad (\alpha \in \Phi, c \in \mathbb{Z}, t \in \mathbb{N}).
\]

From this last equality it follows that for \(\lambda \in \Lambda(n, d), \alpha = \varepsilon_i - \varepsilon_j \in \Phi \quad (i \neq j)\) we have

\[
X_\alpha 1_{\lambda} = \begin{bmatrix} K_i ; -1 \\ \lambda_i \end{bmatrix} \begin{bmatrix} K_j ; 1 \\ \lambda_j \end{bmatrix} \left( \prod_{l \neq i, j} [K_l] \right) X_\alpha.
\]

Multiply both sides of the preceding equality by \(\begin{bmatrix} K_i \\ 1 \end{bmatrix} = \begin{bmatrix} K_i ; 0 \\ 1 \end{bmatrix}\) and use \((7.10)\) to simplify the right-hand-side and use \((7.13)\) to simplify the left-hand-side. The result is the equality

\[
X_\alpha \begin{bmatrix} K_i ; 1 \\ 1 \end{bmatrix} 1_{\lambda} = \begin{bmatrix} \lambda_i + 1 \\ 1 \end{bmatrix} \begin{bmatrix} K_i \\ \lambda_i + 1 \end{bmatrix} \begin{bmatrix} K_j ; 1 \\ \lambda_j \end{bmatrix} \left( \prod_{l \neq i, j} [K_l] \right) X_\alpha.
\]

By Proposition \((7.6)\) (a) the left-hand-side of this equality simplifies to give the equality

\[
\begin{bmatrix} \lambda_i + 1 \\ 1 \end{bmatrix} X_\alpha 1_{\lambda} = \begin{bmatrix} \lambda_i + 1 \\ 1 \end{bmatrix} \begin{bmatrix} K_i \\ \lambda_i + 1 \end{bmatrix} \begin{bmatrix} K_j ; 1 \\ \lambda_j \end{bmatrix} \left( \prod_{l \neq i, j} [K_l] \right) X_\alpha.
\]

and after cancelling the common scalar factor and expanding by means of \((7.11)\) (assuming \(\lambda_j > 0\)) this becomes

\[
X_\alpha 1_{\lambda} = \begin{bmatrix} K_i \\ \lambda_i + 1 \end{bmatrix} \left( v^{\lambda_j} \begin{bmatrix} K_j \\ \lambda_j \end{bmatrix} + v^{\lambda_j-1} K_j^{-1} \begin{bmatrix} K_j \\ \lambda_j - 1 \end{bmatrix} \right) \left( \prod_{l \neq i, j} [K_l] \right) X_\alpha.
\]

In case \(\lambda_j > 0\) after multiplying through in the above expression and applying Proposition \((7.4)\) (c) we see that the first summand must be zero, so the above equality in that case simplifies to \(X_\alpha 1_{\lambda} = v^{\lambda_j - 1} K_j^{-1} 1_{\lambda + \alpha} X_\alpha\).
Now by Proposition 7.6(a) $K_j^{-1}$ acts on $1_{\lambda+\alpha}$ as $v^{-(\lambda_j-1)}$. Thus we obtain the equality in the first part of the proposition in the case $\lambda_j > 0$.

If $\lambda_j = 0$ then one sees easily by Proposition 7.4(c) that the right-hand-side vanishes. The first part of the proposition is proved. The proof of the other part is similar.

\begin{corollary}
We have in $\mathbf{T}$ the equalities $X^{d+1}_\alpha = 0$ for all $\alpha \in \Phi$.
\end{corollary}

Again, the proof is the same as in the classical case.

8. Quantum straightening

Similar to the classical case, we define left and right content $\chi_L$, $\chi_R$ on Kostant monomials by

\begin{align}
\chi_R\left( m \right) &= m \varepsilon_j, \quad \chi_L\left( m \right) = m \varepsilon_i, \\
\chi_R\left( K_i \right) &= \chi_L\left( K_i \right) = \chi_R\left( K_i \right) = 0
\end{align}

(8.1)

where $\alpha = \varepsilon_i - \varepsilon_j$, and again using the rule $\chi_L(XY) = \chi_L(X) + \chi_L(Y)$ (similarly for $\chi_R$) whenever $X$ and $Y$ are Kostant monomials.

The $q$-analogues of (4.2)–(4.7) hold, with the same argument as in the classical case. In particular, we can work with the description of $Y_+$ in which the idempotent appears on the right.

Now we write $X_{ij}$ for the root vector $X_\alpha$ when $\alpha = \varepsilon_i - \varepsilon_j$ and write $K_\alpha = K_{ij} = K_i K_j^{-1}$ $(1 \leq i \neq j \leq n)$. Note that $E_i = X_{i,i+1}$, $F_i = X_{i+1,i}$. We assume that $i < j$ and $k < l$. By Xi [Xi, §5.6] we have the following root vector commutation formulas, listed below in formulas (8.2) and (8.3) (some of which appeared already in [Lu2] and [Ro]). Note that our notation differs from Xi’s: he writes $E_{ij}$ (resp., $F_{ij}$, $K_{ij}$) where we write $X_{ij}$ (resp., $X_{ji}$, $K_{ij}$).

\begin{align}
X_{ij} X_{kl} = & \begin{cases} 
X_{kl} X_{ij} & (j < k \text{ or } k < i < j < l) \\
v^{-1} X_{kl} X_{ij} & (i = k < j < l \text{ or } i < k < j = l) \\
vX_{il} + vX_{kl} X_{ij} & (j = k) \\
X_{kl} X_{ij} + (v^{-1} - v) X_{il} X_{kj} & (i < k < j < l)
\end{cases}
\end{align}

(8.2)

which, as Xi (loc. cit.) proves, lead to the following

\begin{align}
X^{(M)}_{ij} X^{(N)}_{kl} = & X^{(N)}_{kl} X^{(M)}_{ij} & (j < k \text{ or } k < i < j < l) \\
X^{(M)}_{ij} X^{(N)}_{kl} = & v^{-MN} X^{(N)}_{kl} X^{(M)}_{ij} & (i = k < j < l \text{ or } i < k < j = l)
\end{align}

(8.2a)

(8.2b)
\begin{align*}
X_{ij}^{(M)} X_{kl}^{(N)} &= \sum_{t=0}^{\min(M,N)} v^{(M-t)(N-t)+t} X_{kl}^{(N-t)} X_{il}^{(M-t)} X_{ij}^{(t)} \\
&= \min(M,N) \sum_{t=0}^{\min(M,N)} v^{(M-t)(N-t)+t} X_{kl}^{(N-t)} X_{il}^{(M-t)} X_{ij}^{(t)} \\
&= (j = k) \tag{8.2c}
\end{align*}

\begin{align*}
X_{ij}^{(M)} X_{kl}^{(N)} &= \sum_{t=0}^{\min(M,N)} v^{-t(t-1)/2} (v^{-1} - v)^{t} X_{kj}^{(t)} X_{kl}^{(N-t)} X_{ij}^{(M-t)} X_{il}^{(t)} \\
&= (i < k < j < l) \tag{8.2d}
\end{align*}

We note that these formulas lead to others, of the same form as those given above, except for signs and (in some cases) a scalar factor of some integral power of \(v\). The new formulas are obtained from the ones listed above by solving for the term \(X_{kl}^{(N)} X_{ij}^{(M)}\) and then interchanging \((i,j)\) and \((k,l)\).

We also note that the formulas one obtains in this way, together with the ones already listed, exhaust the possibilities for a commutation of two root vectors labelled by positive roots. Indeed, given two finite intervals \([i,j], [k,l]\) one of the following mutually exclusive possibilities must apply: 1) the intervals are disjoint; 2) one interval is properly included in the other without shared endpoints; 3) one interval is properly included in the other with the two intervals sharing a common endpoint; 4) the intervals coincide; 5) the intervals meet at a single point; 6) the intervals properly overlap. Looking at the formulas listed above, we see that case (a) covers possibilities 1) and 2), case (b) covers possibility 3), case (c) is possibility 5), and case (d) is possibility 6). We do not need a formula for possibility 4).

There are five cases listed in \([\Xi], \S 5.6 (d0)\] and only four cases listed above, but the second of the five cases in \((\text{loc. cit.})\) is superfluous, as we have just proved.

There are similar commutation formulas for products of negative root vectors. One way to get them is by applying the isomorphism \(\Omega : U \to U^{\text{opp}}\) of \([\Xi], 1.2\], defined by \(\Omega E_i = F_i, \Omega F_i = E_i, \Omega K_i = K_i^{-1}, \Omega v = v^{-1}\), to the positive root vector commutation formulas discussed above.

\(\Xi (\text{loc. cit.})\) also gives the following, which express the commutation between positive and negative root vectors. Still under the assumption
\[ X_{ij}X_{lk} = \begin{cases} 
X_{lk}X_{ij} & (j \leq k \text{ or } k < i < j < l) \\
X_{lk}X_{ij} + v^{-1}K_{kj}^{-1}X_{ik} & (i < k < j = l) \\
X_{lk}X_{ij} - X_{ij}K_{ij}^{-1} & (i = k < j < l) \\
X_{lk}X_{ij} + \begin{bmatrix} K_{ij}; 0 \\
1 \end{bmatrix} & (i = k, j = l) \\
X_{lk}X_{ij} + v^{-1}(v^{-1})X_{ij}K_{kj}^{-1}X_{ik} & (i < k < j < l) 
\end{cases} \]

which, as Xi proves, lead to the following

\[ X^{(M)}_{ij}X^{(N)}_{lk} = X^{(N)}_{lk}X^{(M)}_{ij} \] (\( j \leq k \text{ or } k < i < j < l \))

\[ X^{(M)}_{ij}X^{(N)}_{lk} = \sum_{t=0}^{\min(M,N)} t^t(N-t-1)X^{(N-t)}_{jk}X^{(M-t)}_{ij}X^{(t)}_{ik} \] (\( i < k < j = l \))

\[ X^{(M)}_{ij}X^{(N)}_{jk} = \sum_{t=0}^{\min(M,N)} (-1)^t X^{(M-t)}_{ij}X^{(t)}_{ik}X^{(N-t)}_{jk}X^{(t)}_{ij} \] (\( i = k < j < l \))

\[ X^{(M)}_{ij}X^{(N)}_{ji} = \sum_{t=0}^{\min(M,N)} X^{(N-t)}_{ji}X^{(M-t)}_{ij}K_{ij}^{2t-M-N} \] (\( i = k < j < l \))

\[ X^{(M)}_{ij}X^{(N)}_{ik} = \sum_{t=0}^{\min(M,N)} \xi(v, t)X^{(N-t)}_{ik}X^{(t)}_{ij}K_{kj}X^{(M-t)}_{ij}X^{(t)}_{ik} \] (\( i < k < j < l \))

where \( \xi(v, t) = v^{-t(2N+t-1)/2}(v^{-1})^t[t]! \)

As before, these formulas lead to others, of a similar form as those given above. The new formulas in this case are obtained by first applying \( \Omega \) to get a formula in \( U^{\text{opp}} \), then switching order of factors to get a formula in \( U \), and finally interchanging \((i, j)\) with \((k, l)\).

The formulas one obtains in this way, together with the ones already listed, exhaust the possibilities for a commutation involving a positive root vector followed by a negative root vector. For the possibilities 1) – 6) listed earlier for two finite intervals \([i, j], [k, l]\), we see that case (a) covers possibilities 1), 5), and 2), cases (b), (c) cover possibility 3), case (d) is possibility 4), and case (e) is possibility 6).

There are similar commutation formulas for the case of a negative root vector followed by a positive root vector. They are obtained from...
the ones already described, by simply solving for the term $X_{ik}^{(N)} X_{ij}^{(M)}$. The new formulas will be of the same form, except for signs.

Thus we see that the formulas listed in (8.2) and (8.3), together with formulas easily derivable from them, give the commutation in $U$ between $q$-divided powers of any two root vectors.

The most important feature of these commutation formulas, for our purposes, is that every product of the form $X_{\alpha}^{(r)} X_{\beta}^{(s)}$ may be expressed as a scalar multiple (by an element of $A$) of the product in the opposite order, modulo an $A$-linear combination of monomials of degree and (right or left) content which is no greater than that of the original product. This differs from the classical case, where the extra terms in any such commutation always have strictly lower degree and content.

A central result of [Lu2] is a $q$-analogue of Kostant’s basis for $U_z$. The description of this basis uses the “box ordering” on $\Phi^+$ defined as follows: if $\alpha = \varepsilon_i - \varepsilon_j$ and $\beta = \varepsilon_r - \varepsilon_s$, then $\alpha \succ \beta$ if either $s > j$ or $s = j$ and $r > i$. For $B \in \mathbb{N}^{n-1}$ and $\delta \in \{0, 1\}^{n-1}$ set

$$K_{\delta, B} = K_{\alpha_1}^{(\delta_1)} \cdots K_{\alpha_{n-1}}^{(\delta_{n-1})} \left[ \begin{array}{c} K_{\alpha_1}^{(0)} \\ B_1 \\ \vdots \\ K_{\alpha_{n-1}}^{(0)} \\ B_{n-1} \end{array} \right].$$

Then Lusztig [Lu2, Thm. 4.5] proves that the set of all elements of the form

$$F_A K_{\delta, B} E_C \quad (A, C \in \mathbb{N}^{\Phi^+}, B \in \mathbb{N}^{n-1}, \delta \in \{0, 1\}^{n-1})$$

is an $A$-basis of $U_A(\mathfrak{sl}_n)$, provided the products in $E_C$ are taken in the box order on $\Phi^+$ and the products in $F_A$ are taken in the reverse box order. In [Xi2, Theorem 2.4] it is shown that the box order is not necessary as one may take any ordering on $\Phi^+$ when forming $F_A$ and $E_C$. By applying the involution $\omega$ (see [Lu3, 3.1.3]) which interchanges the $E_i$ and $F_i$ we obtain another such basis, consisting of all elements of the form

$$E_A K_{\delta, B} F_C \quad (A, C \in \mathbb{N}^{\Phi^+}, B \in \mathbb{N}^{n-1}, \delta \in \{0, 1\}^{n-1})$$

We note that elements $\left[ K_{\alpha}^{(c)} \right]_t$ appear in some of Xi’s commutation formulas. In the following argument we use the fact that any $\left[ K_{\alpha}^{(c)} \right]_t$ ($\alpha \in \Phi, c \in \mathbb{Z}, t \in \mathbb{N}$) belongs to $U_A^0$ and thus belongs to $T_A^0$. This was proved by Lusztig [Lu1, 4.5] in case $\alpha$ is simple. Lusztig’s argument extends immediately to general $\alpha \in \Phi$ since one has a version of [Lu1, (4.3.1)] for any $\alpha$. 

We shall also need the relation
\[(8.6)\]
\[X_\gamma^{(a)}X_\gamma^{(b)} = \left[\begin{array}{c} a + b \\ a \end{array}\right] X_\gamma^{(a+b)} \quad (\gamma \in \Phi),\]
which follows immediately from the definitions.

**Proposition 8.7.** The sets \(Y_+\) and \(Y_-\) span the algebra \(T_A\).

**Proof.** We just prove the statement for \(Y_+\). As in the classical case, the algebra \(T_A\) is spanned by the set of all “special” monomials of the form
\[(8.8)\]
\[M = X_{\gamma_1}^{(\psi_1)} \cdots X_{\gamma_m}^{(\psi_m)} 1_\mu\]
for \(m \in \mathbb{N}, \gamma \in \Phi^m, \psi \in \mathbb{N}^m\). By (8.6) we may assume that \(\gamma_i \neq \gamma_{i+1}\) for all \(i \leq m - 1\). Set \(\chi = \chi_R(M)\) and define the deviation of \(M\) by \(\delta = \sum_{\chi_i > \mu_i} (\chi_i - \mu_i)\). We claim that the set of all special monomials of deviation 0 lie within the \(A\)-span of \(Y_+\). However, the proof of this claim is different in the present case. By (8.5) we can express the product \(X_{\gamma_1}^{(\psi_1)} \cdots X_{\gamma_m}^{(\psi_m)}\) (which lies in the subalgebra \(U(\mathfrak{sl}_n)\)) in terms of an \(A\)-linear combination of terms of the form \(E_A K_{\delta,B} F_C\). In this expression, each of the terms has right content not exceeding that of \(M\), because commutation does not increase degree or right content. We express each \(K_{\delta,B}\) in terms of an \(A\)-linear combination of idempotents \(1_\lambda\), and use Proposition 7.9 to commute the idempotents all the way to the right. Upon right multiplication of the resulting expression by \(1_\mu\) we obtain an equality of the form
\[M = \sum_{A,C} a_{A,C} E_A F_C 1_\mu\quad (a_{A,C} \in A)\]
in which the right content of each term on the right-hand-side is no greater than that of \(M\). It follows that each term in the right-hand-side of the above equality has deviation 0, and thus lies in \(Y_+\). This proves the claim.

Now we proceed by induction on deviation, with the above claim forming the base step in the induction. Let \(M\) be a special monomial of the form (8.8) of deviation \(\delta = \delta(M) > 0\). Set \(\chi = \chi_R(M)\). Since \(\chi \not\leq \mu\), there is an index \(j\) for which \(\chi_j > \mu_j\). Fixing this index \(j\), we call \(\beta \in \Phi\) **bad** if \(\beta = \varepsilon_i - \varepsilon_j\) for some \(i \neq j\) and call \(\beta\) **good** otherwise. We extend this terminology to the factors \(X_{\gamma_1}^{(\psi_1)}\) of \(M\). We define the **defect** \(D\) of the monomial \(M\) by the equality \(D(M) = \sum \psi_\alpha\) where the sum is over the set \(\{X_{\gamma_1}^{(\psi_1)}\}\) of good factors in \(M\) appearing to the right of some bad factor. The defect of \(M\) is 0 if and only if all the bad
factors in $M$ appear as far to the right as possible. As in the classical case we have by Proposition 7.3 that $M = 0$ whenever $D(M) = 0$.

Now suppose that $D(M) > 0$. Then there exists at least one good factor $X^{(a)}_\alpha$ appearing to the right of some bad factor. We may assume that $X^{(a)}_\alpha$ is the leftmost such good factor in $M$. It has one or more bad factors immediately to its left. If there is just one bad factor $X^{(b)}_\beta$ to the left of $X^{(a)}_\alpha$ we apply relations (8.2), (8.3) to the product

$$X^{(b)}_\beta X^{(a)}_\alpha.$$  

In all cases except (8.2d), (8.3e) the argument is similar to the classical case, and we obtain a multiple (by an element of $A$) of a Kostant monomial of the same deviation as $M$ but of strictly lower defect, modulo an $A$-linear combination of monomials of strictly lower deviation. In cases (8.2d), (8.3e) (as written) we obtain an $A$-linear combination of monomials of the same defect and deviation as $M$. But in relation (8.2d) the factors $X^{(M-t)}_{ij}$, $X^{(t)}_{il}$ commute (up to a power of $v$), and so do the factors $X^{(t)}_{kj}$, $X^{(N-t)}_{kl}$. After interchanging those pairs of factors, the factors $X^{(t)}_{kj}$, $X^{(t)}_{il}$ will be adjacent, and they commute. Thus we see that the right-hand-side of formula (8.2d) can be rewritten in such a way that the bad factors $X^{(M-t)}_{kj}$, $X^{(t)}_{ij}$ appear on the right, and thus all the monomials we obtain after applying (8.2d) to $M$ are of strictly lower defect than that of $M$. In case (8.3e) similar commutation applies to obtain the same result: here $X^{(M-t)}_{ij}$, $X^{(t)}_{ik}$ commute up to a power of $v$, then $X^{(t)}_{ij}$, $X^{(t)}_{ik}$ commute by case (8.3a) since $k - 1 < j$. (One needs also in this case to express the factor $K^{-1}_{kj}$ as an $A$-linear combination of idempotents, and then commute those all the way to the right.)

Now suppose there is more than one bad factor to the left of $X^{(a)}_\alpha$. Let $X^{(b)}_\beta$ be the rightmost such bad factor. We again apply relations (8.2), (8.3) to the product $X^{(b)}_\beta X^{(a)}_\alpha$ and repeat the argument given above. The result in the cases (8.2d), (8.3e) is an $A$-linear combination of Kostant monomials of the same defect and deviation as that of $M$, but with fewer bad factors to the left of the good factor $X^{(a)}_\alpha$. By induction on the number of such factors, we are done.

This proves that $M$ lies in the $A$-span of $Y_+$, and hence that $Y_+$ spans $T_A$. The argument for $Y_-$ is similar.

9. Proof of Theorems 2.1 and 2.3

Since $S$ is the image of the representation $U \to \text{End}(V^d)$, it follows from Lemma 7.1 that the images of the $K_i$ in the quantum Schur
algebra $S$ satisfy the defining relations for the algebra $T$, so the surjection $U \to S$ factors through the algebra $T$. In particular, this gives a surjection $T \to S$. Thus, as in the classical case, to complete the proof of Theorem 2.1 it is enough to produce a spanning set in $T$ of cardinality equal to the dimension of $S$. We know from Proposition 8.7 that $Y_+$ and $Y_-$ span the algebra $T$, and it is clear that these sets have cardinality equal to the dimension of $S$, since they are in bijective correspondence with the set $Y_+$. This proves that $T \simeq S$ and Theorem 2.1 follows. It also follows that the sets $Y_+$ and $Y_-$ are $Q(v)$-bases for $S$. Hence these sets are linearly independent over $Q(v)$, and thus also over $A$. Thus they are $A$-bases for $T_A$. By [Du], the restriction to $U_A$ of the map $U \to S$ gives a surjection $U_A \to S_A$. It follows that the restriction map $T_A \to S_A$ is an isomorphism (of $A$-algebras), and that the sets $Y_+, Y_-$ are $A$-bases for the $q$-Schur algebra $S_A$. Moreover, by Proposition 7.4, $K_i$ and $K_i^{-1}$ lie in the subalgebra of $T_A$ generated by the $\left[ K_i^a b \right]$. This proves Theorem 2.3.

**Remark 9.1.** We conjecture that for any fixed $i_0$ ($1 \leq i_0 \leq n$) either of the sets

$$\{ E_A K_B F_C \}, \quad \{ F_A K_B E_C \}$$

($B \in \mathbb{N}_{i_0}^n, A, C \in \mathbb{N}^+, |A| + |B| + |C| \leq d$) is an $A$-basis of $S_A(n,d)$. These are a truncated form of Lusztig’s analogue for $U_A$ of Kostant’s basis for $U_Z$. The conjecture is true when $n = 2$; see [DG2].

**10. Proof of Theorem 2.4**

To prove Theorem 2.4, one sets $K_j = \sum_\lambda v^{\lambda_j} 1_\lambda, K_i^{-1} = \sum_\lambda v^{-\lambda_j} 1_\lambda$ and verifies that these elements, along with the $E_i, F_i$, satisfy the relations (Q1)–(Q7). The argument is similar to that given in the proof of Theorem 1.4. The details are left to the reader.

**11. Applications**

In this section we apply our main results to study some subalgebras of $S$. In what follows, we will focus entirely on the quantum case as the corresponding results for the classical case are essentially the same.

Recall the triangular decomposition (7.3), which we now write in the form $S_A = S_A^+ S_A^0 S_A^-$ in light of the identification $S_A \cong T_A$. We consider the plus part $S_A^+$. We give a new proof for the following result of R.M. Green [RG2, Prop. 2.3].
Proposition 11.1. Let $E_A \in S_A^+$ and suppose the products are taken in the box order. Then $E_A = 0$ if $|A| > d$. Similarly if $F_C \in S_A^-$ and the products are taken in the reverse box order, then $F_C = 0$ if $|C| > d$.

Proof. Let $E_A \in S_A^+$ be given in the box order with $|A| > d$. For each $j = 2, \ldots, n$ set

$$E_{A_j} = \prod_{i=1}^{j-1} X_{j-i,j}^{(A_{j-i,j})},$$

where we write $X_{j-i,j}$ (resp., $A_{j-i,j}$) short for $X_{\varepsilon_j-\varepsilon_i}$ (resp., $A_{\varepsilon_j-\varepsilon_i}$). Then $E_A = E_{A_n} \cdots E_{A_2}$. According to Proposition 7.9, if $E_{A_j1\lambda} \neq 0$, then $E_{A_j1\lambda} = \mu E_{A_j}$, where $\mu_j = \lambda_j - \sum_{r=1}^{j-1} A_{j-r,j}$ and $\mu_s = \lambda_s$ if $s > j$. Therefore for $E_{A_11\lambda} \neq 0$ it is necessary that $\lambda_j \geq \sum_{r=1}^{j-1} A_{j-r,j}$ for all $j = 2, \ldots, n$, and thus

$$\sum_{j=2}^{n} \lambda_j \geq \sum_{j=2}^{n} \sum_{r=1}^{j-1} A_{j-r,j} = |A|.$$

This inequality cannot be satisfied since $\sum_{j=2}^{n} \lambda_j \leq d$ and $|A| > d$. Therefore $E_{A_11\lambda} = 0$ for all $\lambda \in \Lambda(n,d)$. This forces $E_A = 0$, since $1 = \sum 1\lambda$. This completes the proof for the claim about the plus part. The proof for the other claim is similar.

Proposition 11.2. Fix an order on $\Phi^+$. The set of $E_A$ (resp., $F_A$) such that $|A| \leq d$, with products of factors taken in the designated order, is an $A$-basis of $S_A^+$ (resp., $S_A^-$).

Proof. First consider the box order on $\Phi^+$. From Lusztig [Lu2, §4.7] we know that the set of all $E_A$ (products taken in the box order) is an $A$-basis of $U^+$. Thus, by Proposition 11.1, the set $\Gamma$ consisting of those $E_A$ satisfying the condition $|A| \leq d$ is an $A$-spanning set of $S_A^+$. The cardinality of this spanning set is equal to $\dim_{Q(v)} S^+$, so it follows that it forms a basis (over $Q(v)$) of $S^+$. Hence the elements in the set are linearly independent over $Q(v)$, and thus also over $A$. Hence they form an $A$-basis of $S_A^+$.

Now we form products $E_A$ with respect to an arbitrary (but fixed) order on $\Phi^+$. By [Xi2, Theorem 2.4] the set of such products spans $S_A^+$. But the commutation formulas (8.2) do not increase degree; hence when we express an element of $\Gamma$ as an $A$-linear combination of $E_A$’s (in the given fixed order on factors) the degree cannot increase. It follows that the set of $E_A$ $(|A| \leq d)$ must span $S_A^+$ (over $A$). It follows that this set is also an $A$-basis of $S_A^+$.
This proves the statement for the plus part. The proof for the minus part is similar.

Next we consider the Borel Schur algebras $S_\lambda^\geq A = S_\lambda^0 A S_\lambda^+$ and $S_\lambda^\leq A = S_\lambda^0 A S_\lambda^-.$

**Proposition 11.3.** Fix an order on $\Phi^+$. With products taken in the specified order, the set \( \{E_A1_\lambda \mid \chi(E_A) \leq \lambda \} \) (resp., \( \{1_\lambda F_C \mid \chi(F_C) \leq \lambda \} \)) is an $A$-basis of $S_\lambda^\geq A$ (resp., $S_\lambda^\leq A$).

**Proof.** From the preceding result and the decomposition $S_\lambda^\geq A = S_\lambda^0 A S_\lambda^+$ we see that the set of all $E_A 1_\lambda$ spans $S_\lambda^\geq A$. We can argue as in the proof of Proposition 8.7 that with the restriction $\chi(E_A) \leq \lambda$ the set still spans. Clearly this set is linearly independent over $A$ since it is a subset of an $A$-basis of $S_\lambda A$. This proves the first statement. The proof of the other case is similar.

**Remark 11.4.** We conjecture that the set of all elements of the form $E_A I_B \ (A \in N^{\Phi^+}, \ B \in N^0)$ satisfying $|A| + |B| \leq d$ is an $A$-basis of $S_\lambda^\geq A$, with a similar statement applying to $S_\lambda^\leq A$.

Now we consider an application to Hecke algebras. Suppose that $n \geq d$. Let $\omega = (1^d)$. Then the subalgebra $1_\omega S(n, d) 1_\omega$ is isomorphic with the Hecke algebra $H = H(\Sigma(d))$. If $E_A 1_\lambda F_C$ ($\chi(E_A F_C) \leq \lambda$) is any basis element of $S$ then by Propositions 7.4 and 7.9 we see that $1_\omega E_A 1_\lambda F_C 1_\omega = 0$ unless

\[
\lambda + \sum_{\alpha \in \Phi^+} A(\alpha) \alpha = \omega = \lambda + \sum_{\alpha \in \Phi^+} C(\alpha) \alpha,
\]

in which case $1_\omega E_A 1_\lambda F_C 1_\omega = E_A 1_\lambda F_C 1_\omega = 1_\omega E_A 1_\lambda F_C 1_\omega = 1_\omega E_A F_C 1_\omega = E_A F_C 1_\omega$. Therefore the nonzero elements $E_A 1_\lambda F_C$ of $Y_+$ satisfying condition (11.3) comprise an $A$-basis of $H$. There is a similar basis for $H$ as a subset of the basis $Y_-$. Taking $d = n$, we can see that $H$ is generated by the elements $t_i = 1_\omega E_i F_i 1_\omega \ (1 \leq i \leq n - 1)$. One can check directly from relations (Q1)–(Q7) and the propositions in §7 that these generators satisfy the following relations:

- \((H1)\) \hspace{1cm} $t_i^2 = [2] t_i$
- \((H2)\) \hspace{1cm} $t_i t_j = t_j t_i \ (|i - j| > 1)$
- \((H3)\) \hspace{1cm} $t_i t_{i+1} t_i - t_{i+1} t_i t_{i+1} = t_i - t_{i+1}.$
Setting $e_i = t_i/[2]$ and putting $q = v^2$ establishes the equivalence of the above presentation with the presentation in terms of generators $e_i$ given in Wenzl [Wz, §2].

Note that (with $q = v^2$) the elements $T_i = v^2 - vt_i$ satisfy the relations

\[(H1') \quad T_i^2 = (q - 1)T_i + q\]
\[(H2') \quad T_iT_j = T_jT_i \quad (|i - j| > 1)\]
\[(H3') \quad T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}\]

which is the “usual” presentation of $H$.

Note also that by (Q2) and Proposition 7.6(a) we have $1_\omega E_i F_i 1_\omega = 1_\omega F_i E_i 1_\omega$, so the alternative ordering of the basis elements of the Schur algebra does not yield another presentation of $H$.

**Example 11.6.** The easiest way to write down the basis elements in the basis $Y_+ (Y_- \text{ is similar})$ uses the alternate description in the $q$-analogue of (4.7), with the idempotent on the right. Once one has the basis elements, it is a simple matter to rewrite them with the idempotent anywhere one pleases using 7.9 repeatedly.

Given $n, d$ and $\lambda \in \Lambda(n, d)$ set $Y_+ (\lambda) = \{E_A F_C 1_\lambda \mid \chi_R(E_A F_C) \preceq \lambda\}$, so that $Y_+ = \bigcup \lambda Y_+ (\lambda)$ (disjointly). The partition pieces $Y_+ (\lambda)$ are obtainable as follows. One must choose orders on the factors in $E_A F_C$ (the two parts can be ordered independently). Once that is done, then for each $j = 1 \ldots, n$ one writes out the set of monomials in variables $X_{ij}$ ($i \neq j$) of total degree not exceeding $\lambda_j$. (As before, we write $X_{ij}$ short for $X_{\alpha}$ with $\alpha = \varepsilon_i - \varepsilon_j$.) Then one takes the ordered Cartesian product of these sets over $j$, respecting the given order, with the factor $1_\lambda$ at the right. (From this it is easy to write a formula for $|Y_+ (\lambda)|$ as a product of binomial coefficients.)

Note that it suffices to describe $Y_+ (\lambda)$ just for dominant $\lambda \in \Lambda(n, d)$, since the sets indexed by non-dominant $\lambda$ can be obtained from the dominant one in its orbit by applying the appropriate permutation to the indices (and then reordering the product to conform to the specified orders on factors of $E_A, F_C$, if necessary). We list below the elements $E_A F_C$ such that $E_A F_C 1_\lambda \in Y_+ (\lambda)$, for dominant $\lambda$. The elements corresponding with basis elements of the Hecke algebra are underlined.

For $S_A(2, 2)$ the elements in question are

\[1_{(2,0)} : \{1, X_{21}, X_{21}^{(2)}\}\]
\[1_{(1,1)} : \{1, X_{12}, X_{21}, X_{12}X_{21}\}\]
Thus $\dim S(2, 2) = 2 \cdot 3 + 4 = 10$. (There are two sets in the $(2, 0)$ orbit, each of cardinality 3.)

For $S_A(3, 3)$ we fix the order $(12) < (13) < (23) < (21) < (31) < (32)$. Then the sets are determined by the elements

$1_{(3,0,0)} : \{1, X_{21}, X_{31}, X_{21}^{(2)}, X_{31}^{(2)}, X_{21}X_{31}, X_{31}^{(3)}, X_{21}^{(2)}X_{31}, X_{21}X_{31}^{(2)}\}$

$1_{(2,1,0)} : \{1, X_{12}, X_{21}, X_{31}, X_{32}, X_{12}X_{21}, X_{12}X_{31}, X_{31}^{(2)}, X_{21}X_{31},$

$X_{21}X_{32}, X_{31}X_{32}, X_{12}X_{21}^{(2)}, X_{12}X_{31}^{(2)}, X_{12}X_{21}X_{31}, X_{21}X_{32},$

$X_{31}^{(2)}X_{32}, X_{21}X_{31}X_{32}\}$

$1_{(1,1,1)} : \{1, X_{12}, X_{13}, X_{23}, X_{21}, X_{31}, X_{32}, X_{12}X_{13}, X_{12}X_{23}, X_{12}X_{21},$

$X_{12}X_{31}, X_{13}X_{21}, X_{13}X_{31}, X_{13}X_{32}, X_{23}X_{21}, X_{23}X_{31}, X_{23}X_{32},$

$X_{21}X_{32}, X_{31}X_{32}, X_{12}X_{13}X_{21}, X_{12}X_{13}X_{31}, X_{12}X_{23}X_{21},$

$X_{12}X_{23}X_{31}, X_{13}X_{21}X_{32}, X_{13}X_{31}X_{32}, X_{23}X_{21}X_{32}, X_{23}X_{31}X_{32}\}$

Note that for the $\lambda = (2, 1, 0)$ case we took the ordered Cartesian product of $\{1, X_{12}, X_{32}\}$ with $\{1, X_{21}, X_{31}, X_{21}^{(2)}, X_{31}^{(2)}, X_{21}X_{31}\}$ and for the $\lambda = (1, 1, 1)$ case we computed the ordered Cartesian product of the sets $\{1, X_{13}, X_{23}\}$, $\{1, X_{12}, X_{32}\}$, and $\{1, X_{21}, X_{31}\}$. There are 10 elements in the $(3, 0, 0)$ piece, 18 in the $(2, 1, 0)$ piece, and 27 in the $(1, 1, 1)$ piece. There are 3 pieces in the $(3, 0, 0)$-orbit, 6 in the $(2, 1, 0)$-orbit, and 1 in the $(1, 1, 1)$-orbit. Thus $\dim S(3, 3) = 3 \cdot 10 + 6 \cdot 18 + 27 = 165.$

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