Estimating Formation Mechanisms and Degree Distributions in Mixed Attachment Networks

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Abstract. Our work introduces an approach for estimating the contribution of attachment mechanisms to the formation of growing networks. We present a generic model in which growth is driven by the continuous attachment of new nodes according to random and preferential linkage with a fixed probability. Past approaches apply likelihood analysis to estimate the probability of occurrence of each mechanism at a particular network instance, exploiting the concavity of the likelihood function at each point in time. However, the probability of connecting to existing nodes, and consequently the likelihood function itself, varies as networks grow. We establish conditions under which applying likelihood analysis guarantees the existence of a local maximum of the time-varying likelihood function and prove that an expectation maximization algorithm provides a convergent estimate. Furthermore, the in-degree distributions of the nodes in the growing networks is analytically characterized. Simulations show that, under the proposed conditions, expectation maximization and maximum-likelihood accurately estimate the actual contribution of each mechanism, and in-degree distributions converge to a stationary distributions.

Keywords: Complex networks, Network model, Statistical inference.

1 Introduction

The aim of a wide range of network models is to provide a framework to understand how linkage mechanisms for establishing links give rise to particular topological properties, including degree distributions [4,7], clustering [20], average path lengths [1], and community partitions [20]. There has been a continuous and significant effort directed at formalizing such mechanisms and their role in the formation of networks like the Internet, the world wide web, and co-authorship associations [25,24,6,21,15,10,18,11,14,13,4].

The work in [4] explains the emergence of power law degree distributions as an outcome of the addition of new nodes to the network (a growth mechanism) and the preference of new nodes to connect to highly connected nodes (an attachment mechanism). The two mechanisms combined yield a limit degree distribution that follows a single power law. However, few empirical distributions obey a power
law for all their values (e.g., most distributions satisfy \( P(k) \sim k^{-\alpha}, \alpha > 0 \), only for large values of degree \( k \)), which suggests that multiple mechanisms of attachment underlie network formation \([24,28]\). Evidence for mixed degree distributions are found in citation networks \([21]\), opinion networks \([23]\), and protein-protein interaction networks \([12,17]\).

A number of models have been introduced to explain how attachment mechanisms give rise to topological properties of networks. An important step in developing these models entails the problem of assessing the plausibility of each mechanism. Previous research tries to estimate the contribution of a mechanism using maximum-likelihood methods. Such methods determine the optimal estimate that best describes the contribution of each mechanism based on the number of new edges established at some point in time \([27,28]\). In particular, the work in \([28]\) assigns an adjustable weight to each mechanism. The optimal estimate represents the set of weights that maximize the likelihood of all new edges. However, as the network grows, applying standard maximum-likelihood estimation does not produce a consistent estimate over time.

Our work focuses on understanding how to obtain a convergent estimate of the contribution of multiple mechanisms that influence the evolution of growing networks. In particular, it aims to provide an analytical framework to evaluate the contribution of two attachment mechanisms, namely, random and preferential attachment. We extend the method in \([24]\) and \([28]\) by presenting conditions to guarantee the existence of a realistic maximum-likelihood estimate. Moreover, an expectation maximization algorithm is applied to evaluate the contribution of these two attachment mechanisms in one simulated and two empirical citation networks.

The contributions of this paper are the following. First, we characterize the roots of the likelihood function regarding the network parameters (Lemmata \([1]\) and \([2]\)). Second, we present conditions under which the likelihood function has a maximum (Theorem \([1]\)) and an algorithm to estimate the contribution of the two attachment mechanisms (preferential and random attachment). Third, we use a discrete-time approach to characterize the in-degree distribution as a function of the parameters and contribution of each mechanisms (Theorem \([2]\)). Fourth, we show that the dynamics of the in-degree distribution converges to a stationary distribution (Corollary \([2]\)). Finally, we verify that the estimate of the contribution of random and preferential attachment yields a theoretical in-degree distribution that resembles empirical distributions of co-authorship networks.

The remainder of the paper is organized as follows. Section \([2]\) introduces the mixed attachment model. Section \([3]\) presents the proposed estimation approach. Section \([4]\) overviews the estimation based on expectation-maximization algorithm. Section \([5]\) characterizes the in-degree distribution of the model. Section \([6]\) presents simulation results. Section \([7]\) draws the concluding remarks and some future work.
2 The Network Model

The network model used in this paper is an extension of the network model in [24], which supports directed networks and includes a response mechanism. It consists of three main mechanisms, namely, growth, attachment, and response. By growth we mean that the number of nodes in the network increases by one at each time step. Attachment refers to the fact that new nodes tend to connect to existing nodes, while response refers to the fact that existing nodes tend to connect to new nodes. In mathematical terms, the network model is parametric in a probability $\alpha$ and natural numbers $m$ and $\hat{m}$ governing the attachment and response mechanisms. Internally, the attachment mechanism creates $m > 0$ outgoing edges from the new node and is characterized as a Bernoulli trial with parameter $\alpha$, where $\alpha$ represents the probability of establishing a new edge by preferential attachment and $1 - \alpha$ by random attachment. The response mechanism creates $\hat{m} \geq 0$ incoming edges from the existing nodes to the new node by random attachment.

A network is represented as a directed graph $G_t = (V_t, E_t)$ with nodes $V_t$ and edges $E_t \subseteq V_t \times V_t$. A pair $(u, v) \in E_t$ represents a directed edge from a source node $u$ to a target node $v$. The expressions $k_t(u)$ and $\hat{k}_t(u)$ denote, respectively, the in- and out-degree of node $u \in V_t$. Moreover, $n_t$ and $e_t$ denote the number of nodes and the number of edges in the network at time $t$, respectively (i.e., $n_t = |V_t|$ and $e_t = |E_t|$).

Definition 1. The algorithm used in the network model goes as follows:

1. Growth: starting from a seed network $G_0$, at each time step $t > 0$, a new node is added with $m$ outgoing edges that link the new node to $m$ different nodes already present in the network and $\hat{m}$ incoming edges that link $\hat{m}$ different nodes already present in the network to the new node.

2. Attachment: when choosing the $m$ nodes to which the new node connects, we assume that the probability $\pi_t(v \mid \alpha)$ that the new node will be connected to node $v$ is given by

$$\pi_t(v \mid \alpha) = \alpha \pi_t^{pa}(v) + (1 - \alpha) \pi_t^{ra}(v)$$

where the probability of establishing an outgoing edge from the new node to the existing node $v$ due to preferential attachment is given by

$$\pi_t^{pa}(v) = \frac{k_{t-1}(v)}{e_{t-1}}$$

and due to random attachment by

$$\pi_t^{ra}(v) = \frac{1}{n_{t-1}}.$$ 

3. Response: when choosing the $\hat{m}$ nodes that connect to the new node, we assume that the probability $\eta_t(v)$ that an existing node $v$ connects to the new node is given by

$$\eta_t(v) = \frac{1}{n_{t-1}}.$$
For the growth and response processes to be well-defined, the algorithm in Definition 1 assumes that the seed network has at least \( \max\{m, \hat{m}\} \) nodes. Although it is not required by the algorithm, it will be further assumed that the seed network does not have self-loops and each node has at least one incoming edge. In this way, the networks generated by the algorithm do not contain self-loops and each node has non-zero in-degree. As explained before, the attachment mechanism is characterized as a Bernoulli trial with parameter \( \alpha \), representing the probability of establishing a new edge via preferential attachment. For preferential attachment, the probability of establishing a new edge depends on the in-degree of the target node, meaning that the probability for an existing node of becoming a target node is directly proportional to its in-degree. For random attachment, the probability of establishing an edge to a target node follows a discrete uniform distribution. The response mechanism selects from the set of existing nodes uniformly at random which nodes connect to the new node.

Figure 1 illustrates the evolution of a seed network \( G_0 = (V_0, E_0) \) at time steps \( t = 1, 2, 3 \) in the network model with \( m = 2 \) and \( \hat{m} = 1 \) (for the sake of simplicity in the illustration, the probability \( \alpha \) is omitted). The seed network \( G_0 \) has nodes \( V_0 = \{1, 2, 3\} \) and edges \( E_0 = \{(1, 2), (2, 1), (2, 3), (3, 1)\} \). At time \( t = 1 \), the set of nodes grows by adding the new node 4, and by creating the new edges \((4, 2), (4, 3), (2, 4)\) (\( m = 2 \) outgoing edges by the attachment mechanism and \( \hat{m} = 1 \) incoming edges by the response mechanism). At time \( t = 2 \), the new node 5 establishes \( m = 2 \) outgoing edges and \( \hat{m} = 1 \) incoming edges, as it is the case for the new node 6 at time \( t = 3 \).

3 Maximum Likelihood Analysis

Consider a sequence of networks \((G_t)_{t \leq T}\) generated by the algorithm presented in Definition 1 (see Section 2), for some \( T \in \mathbb{N} \), from a given seed network \( G_0 \) and the problem of determining the values of the \( \alpha \), \( m \), and \( \hat{m} \) parameters used in the process. Inspection on such a sequence is enough for establishing the values of \( m \) and \( \hat{m} \). However, determining the value of \( \alpha \) is, in general, a tall order. The purpose of this section is two-fold. On the one hand, it argues that recently published maximum likelihood analysis approaches (e.g., in [28]) do not produce reasonable estimates in some cases. On the other hand, it presents a new approach to provide better estimates and sufficient conditions under which such an analysis is feasible.

The likelihood of creating an edge from the new node \( u \) at time \( t \) to a node \( v \in V_{t-1} \) is given by [28]:

\[
\pi_t(v | \alpha) = \alpha \pi_t(v | \alpha) + (1 - \alpha) \pi_t(v | \alpha) = \alpha \frac{k_{t-1}(v)}{e_{t-1}} + (1 - \alpha) \frac{1}{n_{t-1}} = \alpha \left( \frac{k_{t-1}(v)}{e_{t-1}} - \frac{1}{n_{t-1}} \right) + \frac{1}{n_{t-1}}. \tag{5}
\]

Since preferential attachment depends on the degree of the target node, Equation 5 can be written as a function of the in-degree of node \( v \) with \( k = k_{t-1}(v) \):
Fig. 1: Network evolution from a given seed network $G_0$ in the network model. At each time step $t \geq 1$, a new node is added with $m = 2$ outgoing edges (dashed line depicted in blue) and $\hat{m} = 1$ incoming edges (dotted line depicted in red).

$$\pi_t(k | \alpha) = \alpha \left( \frac{k}{e_{t-1}} - \frac{1}{n_{t-1}} \right) + \frac{1}{n_{t-1}}.$$  \hspace{1cm} (6)

**Definition 2.** For $t \in \mathbb{N}$, let:

- $A_t$ be the multiset of in-degrees of nodes selected at time $t$ by the attachment mechanism in Definition 1 and
- $B_t = \bigcup_{i \leq t} A_i$.

At each time $t$, the set $B_t$ is a random sample of length $mt$. Since the elements of $B_t$ are independent and identically distributed (i.i.d.), it follows from Equation 6 that the likelihood function is given by Equation 7:

$$f_t(\alpha) = \prod_{k \in B_t} \pi_t(k) = \prod_{k \in B_t} \left[ \alpha \left( \frac{k}{\pi_{t-1}} - \frac{1}{n_{t-1}} \right) + \frac{1}{n_{t-1}} \right].$$  \hspace{1cm} (7)

Note that $f_t(\alpha)$ is a polynomial in the indeterminate $\alpha$ and has order at most $mt$. If all $k \in B_t$ satisfy $kn_{t-1} - e_{t-1} \neq 0$, then the order of $f_t(\alpha)$, denoted $\deg f_t(\alpha)$, is $mt$.

**Definition 3.** For a network $G_t$, the function $f_t$ is called the $G_t$-likelihood function. The maximum likelihood estimator is defined as

$$\hat{\alpha} = \arg\max_{\alpha \in (0,1)} f_t(\alpha).$$
Consider the complete directed graph $G_0$ with 3 vertices, and a sequence $(G_t)_{t \leq 2000}$ generated by the algorithm in Definition 1 with parameters $m = 5$, $\hat{m} = 3$, and $\alpha = 0.6$. At each time $t$ the new edges are the set of links generated by the new node and likelihood analysis is applied. Let the likelihood functions $f_t^{(1)}(\alpha)$ and $f_t^{(2)}(\alpha)$ be defined as:

$$f_t^{(1)}(\alpha) = \prod_{k \in A_t} \pi_t(k) \quad \text{and} \quad f_t^{(2)}(\alpha) = \prod_{k \in B_t} \pi_t(k).$$

At each time, an estimation of $\alpha$ is provided by each likelihood function. Figure 2 shows that the standard maximum-likelihood estimation with $f_t^{(1)}$ is not capable of producing a consistent estimate over time (this witnesses a counter-example to the approach in [28] for likelihood analysis). However, the proposed approach developed in the rest of this section, which supports the maximum-likelihood estimation with $f_t^{(2)}$, yields a better estimate. The remaining of this section establishes conditions under which the estimator based on the $G_t$-likelihood function produces a consistent estimate over time for the network model in Section 2.

![Figure 2](image-url)

Fig. 2: Behavior of the parameter estimated by the likelihood functions $f_t^{(1)}$ and $f_t^{(2)}$, with $m = 5$, $\hat{m} = 3$, and $\alpha = 0.6$.

The next goal is to identify some key relationships between the number of nodes, the number of edges, and the number of roots of the $G_t$-likelihood function $f_t$, and then provide conditions under which it has a maximum.
Lemma 1. If $\deg f_t = mt$, then there exists $mt$ different elements in $B_t$. Moreover, for $k \in B_t$ such that $kn_{t-1} \neq e_{t-1}$, the roots of $f_t$ can be written as

$$\omega_k = \frac{e_{t-1}}{e_{t-1} - kn_{t-1}}.$$  

Note that the root multiplicity of $\omega_i$ corresponds to the number of times that the value of an in-degree appears in $B_t$.

**Definition 4.** Let $\Omega(p)$ denote the real roots of a polynomial $p$, and $\Omega^+(p)$ and $\Omega^-(p)$ denote the set of positive and negative roots of $p$, respectively. The expression $\mu_i$ denotes the multiplicity of the root $\omega_i$ in $\Omega(p)$.

Recall from Section 2 that the seed network $G_0$ is assumed to have at least $m$ nodes, each with an in-degree of at least 1 (and without self-loops). This assumption is key for Lemma 2 to be applied. Otherwise, if there are nodes with in-degree 0, then the minimum of the positive roots in $\Omega^+(f_t)$ would trivially be 1.

**Lemma 2.** If there exist some $k \in B_t$ such that $0 < k < \left\lfloor \frac{e_{t-1}}{n_{t-1}} \right\rfloor$, then $\Omega^+(f_t)$ and $\Omega^-(f_t)$ are non-empty sets. Moreover, $\min \Omega^+(f_t) = 1 + \left( \frac{n_{t-1}}{m+m-1} \right)$ as $t \to \infty$.

**Proof.** Without loss of generality, assume that $\deg f_t = mt$. By Lemma 1, each root of $f_t$ can be written as

$$\omega_k = \frac{e_{t-1}}{e_{t-1} - kn_{t-1}}. \quad (8)$$

Since $e_{t-1} > 0$, the root $\omega_k$ is positive if and only if $e_{t-1} - kn_{t-1} > 0$. This last proposition is true for some $k$ because $0 < k < \left\lfloor \frac{e_{t-1}}{n_{t-1}} \right\rfloor$ by the hypothesis. Hence, $\Omega^+(f_t) \neq \emptyset$. Moreover, if $k = 1$ then the denominator of Equation 8 yields the maximum positive integer. The minimum element of $\Omega^+(f_t)$ is then given by

$$\min \Omega^+(f_t) = \frac{e_{t-1}}{e_{t-1} - n_{t-1}} = 1 + \frac{n_{t-1}}{e_{t-1} - n_{t-1}}.$$ 

Furthermore, note that $e_t = e_0 + (m + \hat{m})t$ and $n_t = n_0 + t$. Then

$$\lim_{t \to \infty} \left( \min \Omega^+(f_t) \right) = \lim_{t \to \infty} \left( 1 + \frac{n_{t-1}}{e_{t-1} - n_{t-1}} \right) = \lim_{t \to \infty} \left( 1 + \frac{n_0 + t - 1}{e_0 + (m + \hat{m})(t-1) - (n_0 + t-1)} \right) = 1 + \frac{1}{m + m - 1}.$$

**Theorem 1.** If $\Omega^+(f_t)$ and $\Omega^-(f_t)$ are non-empty, and the sum of the multiplicities of positive roots is even, then there exist $\omega_a \in \mathbb{R}^-$, $\omega_b \in \mathbb{R}^+$, and $c \in \mathbb{R}$ such that $c \in (\omega_a, \omega_b)$ is a local maximum of the $G_t$-likelihood function $f_t$. 
Proof. Without loss of generality, assume that \( \text{deg } f_t = mt \). Note that \( f_t \) can be written as the following product of factors

\[
f_t(\alpha) = \prod_{k \in B_t} \frac{\alpha[kn_{t-1} - e_{t-1} + e_t - 1]}{e_{t-1}n_{t-1}}.\]

Then, by Lemma 1

\[
f_t(\alpha) = \left(\frac{1}{e_{t-1}n_{t-1}}\right)^{mt} \prod_{k \in B_t} (\alpha - \omega_k).\]  

The hypothesis implies that the sets of real numbers \( \Omega^+(f_t) \) and \( \Omega^-(f_t) \) are finite and non-empty, and hence max \( \Omega^+(f_t) \) and min \( \Omega^+(f_t) \) exist. Let \( \omega_a \) represent the maximum element in \( \Omega^-(f_t) \) and \( \omega_b \) the minimum element in \( \Omega^+(f_t) \). Note that the function \( f_t \) is continuous and differentiable in \((\omega_a, \omega_b)\). Moreover, since \( \omega_a \) and \( \omega_b \) are roots of \( f_t \), \( f_t(\omega_a) = f_t(\omega_b) = 0 \). By Rolle’s Theorem, there exists a constant \( c \in (\omega_a, \omega_b) \) such that \( f'(c) = 0 \). That is, the maximum argument of \( f_t \) exists for \( \alpha \in (\omega_a, \omega_b) \).

To show that the point \((c, f_t(c))\) is a local maximum of \( f_t \), consider the following two cases.

**Case 1.** Assume that \( \mu_i = 1 \) for each root \( \omega_i \in \Omega(f_t) \). Since all the roots are simple, the function \( f_t \) can be written as

\[
f_t(\alpha) = \left(\frac{1}{e_{t-1}n_{t-1}}\right)^{mt} \prod_{k \in B_t} (\alpha - \omega_k).\]

The derivative of \( f_t \) with respect to \( \alpha \) is

\[
f'(\alpha) = \left(\frac{1}{e_{t-1}n_{t-1}}\right)^{mt} \sum_{k \in B_t \text{ } i \neq k} (\alpha - \omega_i).\]  

Each term in the summation is a polynomial in the indeterminate \( \alpha \) and has degree \( mt - 1 \). Let

\[
q_k(\alpha) = \prod_{i \in B_t \setminus k} (\alpha - \omega_i).\]

By the definition of \( q_k \), the root \( \omega_k \in \Omega(f_t) \) does not belong to \( \Omega(q_k) \), i.e., \( \Omega(q_k) \subseteq \Omega(f_t) \) for each \( k \in B_t \). Based on Equation 10, Equation 9 can be written as

\[
f'(\alpha) = \left(\frac{1}{e_{t-1}n_{t-1}}\right)^{mt} \sum_{k \in B_t} q_k(\alpha).\]
By Lemma 1, there exists an in-degree \( j \in B_t \) satisfying \( \omega_j = \omega_a \). In particular, there exists \( q_j \) such that \( \omega_a \) does not belong to \( \Omega(q_j) \). Therefore,

\[
f'(\omega_a) = \left( \frac{1}{e_{t-1} n_{t-1}} \right)^{mt} q_j(\omega_a). \tag{12}
\]

Since the first factor in Equation (12) is positive, the sign of \( f'(\omega_a) \) is the sign of \( q_j(\omega_a) \). Note that the product in Equation (11) can be split into two factors, one containing the positive roots of \( q_j \) and another one with the negative roots of \( q_j \). More precisely,

\[
q_j(\alpha) = \prod_{i \in B_t \setminus q_j} (\alpha - \omega_i) = \prod_{i \in \Omega^+(q_j)} (\alpha - \omega_i) \prod_{i \in \Omega^-(q_j)} (\alpha - \omega_i). \tag{13}
\]

The expression \( q_j(\omega_a) \) can be rewritten as:

\[
q_j(\omega_a) = \prod_{i \in \Omega^+(q_j)} (\omega_a - \omega_i) \prod_{i \in \Omega^-(q_j)} (\omega_a - \omega_i). \tag{14}
\]

In Equation (14) by the hypothesis, the first factor is a product of an even number of negative terms, which implies it is positive. The factors in the second product are positive because \( \omega_a \) is the maximum element in \( \Omega(f_t) \) and \( \omega_a < -\omega_i \), for all \( \omega_i \in \Omega^-(q_j) \).

By using Lemma 1 there exists an in-degree \( j' \) such that \( \omega_{j'} = \omega_b \), so the sign of \( f'(\omega_b) \) depends on the sign of \( q_{j'}(\omega_b) \). Using the same argument as above:

\[
q_{j'}(\omega_b) = \prod_{i \in \Omega^+(q_{j'})} (\omega_b - \omega_i) \prod_{i \in \Omega^-(q_{j'})} (\omega_b - \omega_i). \tag{15}
\]

Note that \( \omega_b - \omega_i < 0 \) because \( \omega_b \) is the minimum positive root of \( f_t \) and the roots are unique. Moreover, the number of terms in the first product in Equation (15) is an odd number. Thus, this product is negative. By Lemma 2 all positive roots are greater than 1, so the second product in Equation (15) is positive. Consequently, \( q_{j'}(\omega_b) \) is negative. Therefore, \( f'(\omega_a) > 0 \) and \( f'(\omega_b) < 0 \). Hence, \( (c, f(c)) \) is a local maximum of \( f_t \).

Case 2. Assume that there is at least one root \( \omega_i \) in \( \Omega(f_t) \) with \( \mu_i \geq 2 \). Let \( B_t \) denote the set of elements of \( B_t \) without repetitions. Since \( \mu_i \geq 1 \) for all roots \( \omega_i \in \Omega(f_t) \), the function \( f_t \) can be written as

\[
f_t(\alpha) = \left( \frac{1}{e_{t-1} n_{t-1}} \right)^{mt} \prod_{k \in B_t} (\alpha - \omega_k)^{\mu_i}.
\]
The derivative of \( f_t \) with respect to \( \alpha \) is

\[
f'(\alpha) = \left(\frac{1}{e_{t-1}n_{t-1}}\right)^{mt} \sum_{k \in B_t} \mu_k (\alpha - \omega_k)^{\mu_k-1} \prod_{i \neq k} (\alpha - \omega_i)^{\mu_i}.
\]  \hfill (16)

The summands in Equation (16) are polynomials, each with the same roots as \( f_t \), but with multiplicity of at most \( m \). As for case 1, the goal is to show that \( f'(\omega_a + \epsilon) > 0 \) and \( f'(\omega_b - \epsilon) < 0 \), for some \( \epsilon > 0 \).

Since its elements are unique, the set \( B_t \) can be indexed as \( B_t = \{i_1, i_2, i_3, \ldots\} \).

Expanding the second factor in Equation (16)

\[
\sum_{k \in B_t} \left[ \mu_k (\alpha - \omega_k)^{\mu_k-1} \prod_{i \neq k} (\alpha - \omega_i)^{\mu_i} \right]
= \mu_{i_1} (\alpha + \omega_{i_1})^{\mu_{i_1}-1}[(\alpha - \omega_{i_2})^{\mu_{i_2}} (\alpha - \omega_{i_3})^{\mu_{i_3}} \ldots]
+ \mu_{i_2} (\alpha + \omega_{i_2})^{\mu_{i_2}-1}[(\alpha - \omega_{i_1})^{\mu_{i_1}} (\alpha - \omega_{i_3})^{\mu_{i_3}} \ldots]
+ \mu_{i_3} (\alpha + \omega_{i_3})^{\mu_{i_3}-1}[(\alpha - \omega_{i_1})^{\mu_{i_1}} (\alpha - \omega_{i_2})^{\mu_{i_2}} \ldots]
+ \ldots
= \prod_{k \in B_t} (\alpha + \omega_k)^{\mu_k-1} \sum_{k \in B_t} \mu_k \prod_{i \neq k} (\alpha + \omega_i).
\]  \hfill (17)

Using Equation (15), Equation (16) can be written as

\[
f'(\alpha) = \left(\frac{1}{e_{t-1}n_{t-1}}\right)^{mt} \prod_{k \in B_t} (\alpha + \omega_k)^{\mu_k-1} \sum_{k \in B_t} \mu_k \prod_{i \neq k} (\alpha + \omega_i).
\]  \hfill (19)

In Equation (19) the factor \( \sum_{k \in B_t} \mu_k \prod_{i \neq k} (\alpha + \omega_i) \) is equal to \( \sum_{k \in B_t} \mu_k q_k(\alpha) \).

On the one hand, \( q_k(\omega_a) > 0 \) if and only if \( |\Omega(f_t)| \) is even. Since \( q_k \) is continuous over its domain, there exist \( M > 0 \) and \( \delta > 0 \) such that if \( (\omega_a + \epsilon) \in (\omega_a - \epsilon, \omega_b + \epsilon) \) and \( |(\omega_a + \epsilon) - \omega_a| < \delta \), then \( q_k(\omega_a + \epsilon) > 0 \). \hfill (3)

The second term in Equation (19) can be rewritten as

\[
\prod_{\omega_j \in \Omega^-(f_t)} (\alpha + \omega_j)^{\mu_j-1} \prod_{\omega_i \in \Omega^+(f_t)} (\alpha + \omega_i)^{\mu_i-1}.
\]  \hfill (20)

The first term in Equation (20) is positive evaluated at \( \omega_a + \epsilon \). The second term has the property that the sum of all exponents is an even number and, evaluated at \( \omega_a + \epsilon \), is also positive. Hence, \( f'(\omega_a + \epsilon) > 0 \).

If \( |\Omega(f_t)| \) is odd, the function \( q_k \) evaluated at \( \omega_a \) is negative. Using the same argument as above, it can be shown that \( q_k(\omega_a + \epsilon) < 0 \). There exists an odd number of even multiplicities by the hypothesis. Consequently, the second factor in Equation (20) evaluated at \( \omega_a + \epsilon \), is negative. In either case \( f'(\omega_a + \epsilon) > 0 \).
Similarly, it can be shown that $f'(\omega_b - \epsilon) < 0$. Therefore, $(c, f(c))$ is a local maximum of $f_t$.

To illustrate the application of Theorem 1 consider the complete graph $G_0$ with $N_0 = 3$ nodes, and a sequence $(G_t)_{t \leq 2000}$ generated by the algorithm in Definition 1 with parameters $m = 5$, $\hat{m} = 3$, and $\alpha = 0.6$. At each time $t$, the $G_t$-likelihood function is the one in Definition 3 and an estimate for $\alpha$ is computed by using Theorem 1. Figure 3 shows how maximizing the $G_t$-likelihood function yields the estimate $\tilde{\alpha}$ for the parameter $\alpha$.

Fig. 3: Convergence of the parameters associated to preferential (dotted line depicted in blue) and random (solid line depicted in red) attachment mechanisms for the complete graph $G_0$ with $N_0 = 3$ nodes, and a sequence $(G_t)_{t \leq 2000}$ generated by the algorithm in Definition 1 with parameters $m = 5$, $\hat{m} = 3$, and $\alpha = 0.6$.

4 Using an Expectation-Maximization Algorithm

This section presents how determining the contribution of the two attachment mechanisms used by the algorithm in Definition 1 is equivalent to computing the likelihood estimator with an Expectation-Maximization (EM) algorithm. The basic idea is to associate a complete-data problem, which is better suited for maximum likelihood estimation, to a given incomplete-data problem, for which the same estimation can become a wild-goose chase. For an EM algorithm to be effective, two key issues need to be addressed: first, it needs to be proved convergent; second, it needs to be efficient. This section addresses these issues by identifying sufficient conditions for the algorithm to converge and by presenting a recursive definition for estimating $\alpha$, which can be used for incremental computations in the EM algorithm.
Definition 5. Let $K_t$ be a random variable that characterizes the in-degree of the selected nodes due to an attachment mechanism up to time $t$ from $G_t$. Moreover, let $P(x \mid \theta)$ denote the probability mass function (pmf) as a function of the parameter vector $\theta$.

Recall that, for a fixed time $t$, Equation 6 defines the likelihood of forming an edge $(u, v)$ from the new node $u$ to node $v \in V_{t-1}$ through an attachment mechanism. Such an equation can be rewritten as:

$$
\pi_t(k) = \alpha \left( \frac{k}{\epsilon_{t-1}} - \frac{1}{n_{t-1}} \right) + \frac{1}{n_{t-1}} = \alpha \frac{k}{\epsilon_{t-1}} + (1 - \alpha) \frac{1}{n_{t-1}}.
$$

(21)

The set $B_t = \{k_1, \ldots, k_\ell\}$ can be considered as a sample of length $\ell$ generated by the random variable $K_t$ in Definition 5.

The EM algorithm computes iteratively a maximum likelihood estimator for data with unobserved variables [8]. In this case, the input to the EM algorithm comes from a mixed distribution in which the mixture weight is unknown. Based on Equation 21, the incomplete likelihood function can be modeled as follows:

$$
\log P(\alpha \mid B_t) = \sum_{i=1}^\ell \log \pi_t(k_i)
= \sum_{i=1}^\ell \log \left( \alpha \frac{k_i}{\epsilon_{t-1}} + (1 - \alpha) \frac{1}{n_{t-1}} \right).
$$

(22)

The attachment mechanism operates on two groups of nodes. One group consists of nodes that attach to the network using preferential attachment and the other of the nodes that attach using random attachment. Let $Y = (y_{ij}) \in \mathbb{R}^{n \times 2}$ be an $n$ by $2$ matrix where $y_{ij} = 1$ if the $i$-th observation is in group $j$ ($j \in \{1, 2\}$) and 0 otherwise. The expression $y_i$ denotes the $i$-row of $Y$. The variable $y$ is an indicator variable that defines the component from which an observation arises; it is known as a latent (hidden) variable [19].

Notice that $P(y_{i1} = 1 \mid \alpha) = \alpha$ and $P(y_{i2} = 1 \mid \alpha) = 1 - \alpha$. For a given $i$, the value of $y_{ij}$ is 0 except for one component, since each observation belongs to only one group. The conditional probabilities of the data and the unobservable data are:

$$
P(k_i \mid y_{i}, \alpha) = \left( \frac{k_i}{\epsilon_{t-1}} \right)^{y_{i1}} \left( \frac{1}{n_{t-1}} \right)^{y_{i2}}
$$

$$
P(y_i \mid \alpha) = \alpha^{y_{i1}} (1 - \alpha)^{y_{i2}}.
$$

By defining $Z = (B_t, Y)$ as the complete-data, the complete log-likelihood function is given by

$$
\log P(Z \mid \alpha) = \sum_{i=1}^\ell \log (P(k_i \mid y_{i}, \alpha)P(y_{i} \mid \alpha))
= \sum_{i=1}^\ell \log \left( \alpha \frac{k_i}{\epsilon_{t-1}} \right)^{y_{i1}} \left[ (1 - \alpha) \frac{1}{n_{t-1}} \right]^{y_{i2}}
= \sum_{i=1}^\ell y_{i1} \log \left( \alpha \frac{k_i}{\epsilon_{t-1}} \right) + \sum_{i=1}^\ell y_{i2} \log \left( (1 - \alpha) \frac{1}{n_{t-1}} \right).
$$

(23)
Equation 23 is linear in the unobservable data \( y_{ij} \). Consider a function \( Q(\alpha \mid \alpha^{(d)}) \) that represents the conditional expectation given the observed data using the \( d \)-th fit for the unknown parameter \( \alpha \). In particular,

\[
Q(\alpha \mid \alpha^{(d)}) = \mathbb{E} [\log \mathbb{P}(Z)]
\]

\[
= \mathbb{E} \left[ \sum_{i=1}^{\ell} y_{i1} \log \left( \alpha \frac{k_i}{e_{t-1}} \right) + \sum_{i=1}^{\ell} y_{i2} \log \left( (1 - \alpha) \frac{1}{n_{t-1}} \right) \right]
\]

\[
= \sum_{i=1}^{\ell} \mathbb{E} [y_{i1} \mid B_t, \alpha] \log \left( \alpha \frac{k_i}{e_{t-1}} \right)
\]

\[
+ \sum_{i=1}^{\ell} \mathbb{E} [y_{i2} \mid B_t, \alpha] \log \left( (1 - \alpha) \frac{1}{n_{t-1}} \right).
\]  \hspace{1cm} (24)

Let \( \mathbb{E} [y_{ij} \mid B_t, \alpha] = y_{ij}^{(d)} \). By Bayes' Theorem,

\[
y_{ij}^{(d)} = \frac{\mathbb{P} (y_{ij} = 1 \mid k_i, \alpha^{(d)})}{\mathbb{P} (k_i \mid \alpha^{(d)})} = \frac{\mathbb{P} (y_{ij} = 1, \alpha^{(d)} \mid k_i)}{\mathbb{P} (k_i \mid \alpha^{(d)})}.
\]

In particular, for \( j = 1 \)

\[
y_{i1}^{(d)} = \frac{\mathbb{P} (y_{i1} = 1, \alpha^{(d)} \mid k_i, \alpha^{(d)})}{\mathbb{P} (k_i \mid \alpha^{(d)})} = \frac{k_i}{n_{t-1} \alpha^{(d)} + (1 - \alpha^{(d)})}
\] \hspace{1cm} (25)

and for \( j = 2 \)

\[
y_{i2}^{(d)} = \frac{\mathbb{P} (y_{i2} = 1, \alpha^{(d)} \mid k_i, \alpha^{(d)})}{\mathbb{P} (k_i \mid \alpha^{(d)})} = \frac{1}{n_{t-1} (1 - \alpha^{(d)}) + (1 - \alpha^{(d)})}.
\] \hspace{1cm} (26)

By substituting equations (25) and (26) in Equation (24) the following equation is obtained:

\[
Q(\alpha \mid \alpha^{(d)}) = \sum_{i=1}^{\ell} y_{i1}^{(d)} \log \left( \alpha \frac{k_i}{e_{t-1}} \right) + \sum_{i=1}^{\ell} y_{i2}^{(d)} \log \left( (1 - \alpha) \frac{1}{n_{t-1}} \right).
\]

Consider \( x_1 = \alpha \) and \( x_2 = 1 - \alpha \). In order to maximize \( Q(\alpha \mid \alpha^{(d)}) \), the Lagrange multiplier \( \lambda \) with the constrain \( x_1 + x_2 = 1 \) is introduced with the goal of solving the following equation:

\[
\frac{\partial}{\partial x_1} \left[ \sum_{i=1}^{\ell} y_{i1}^{(d)} \log \left( x_1 \frac{k_i}{e_{t-1}} \right) + \sum_{i=1}^{\ell} y_{i2}^{(d)} \log \left( x_2 \frac{1}{n_{t-1}} \right) + \lambda (x_1 + x_2 - 1) \right] = 0 \tag{27}
\]
Equation 27 has the solution:
\[
\alpha^{(d+1)} = \frac{1}{\ell} \sum_{i=1}^{\ell} \frac{k_i}{e_{i-1}} \alpha^{(d)} + \frac{1}{n_{t-1}} (1 - \alpha)^{(d)}.
\]  
(28)

Equation 28 provides a recursive formulation of \(\alpha\) that can be used for estimation purposes in Algorithm 1, namely, in the EM algorithm. This algorithm computes an estimate for \(\alpha\) on an input matrix \((a_{ij}) \in \mathbb{Z}^{\ell \times 3}\) where, for a fixed time \(t\), the values \(a_{i1}, a_{i2}, \) and \(a_{i3}\) denote the in-degree of the target node, the number of edges, and the number of nodes in the network, respectively. Lines 1-5 compute the probability of the new edge forming by preferential attachment or random attachment mechanism (they implement the E-step of the EM algorithm). Lines 6-13 approximate the value for the parameter \(\alpha\) until a sufficiently accurate value is reached (they implement the M-step of the EM algorithm). Note that the recursive definition of \(\alpha\) presented in Equation 28 is used in line 9 to increase the value of the variable \(S\) at each (internal) iteration. Finally, the convergence of Algorithm 1 is obtained as a corollary of Lemma 2.1 in [9].

Algorithm 1 EM Algorithm

| Input: | A \(\ell \times 3\) matrix \(A\) of integers with \(\ell > 0\), an initial value for \(\alpha\) and, an error bound \(\epsilon\). |
|--------|----------------------------------------------------------------------------------------------------------------------------------|
| Output:| An estimate for \(\alpha\). |
| 1:     | \(\tau = 1\) |
| 2:     | \(S = 0\) |
| 3:     | for \(i = 1\) to \(\ell\) do |
| 4:     | \(\pi_{i1} = \frac{a_{i1}}{a_{i2}}; \pi_{i2} = \frac{a_{i1}}{a_{i3}}\) |
| 5:     | end for |
| 6:     | while \(\tau \geq \epsilon\) do |
| 7:     | \(\beta = \alpha\) |
| 8:     | for \(i = 1\) to \(\ell\) do |
| 9:     | \(S = S + \frac{\pi_{i1} \alpha}{\alpha \pi_{i1} + (1 - \alpha) \pi_{i2}}\) |
| 10:    | end for |
| 11:    | \(\alpha = \frac{\beta}{\tau}\) |
| 12:    | \(\tau = |\beta - \alpha|\) |
| 13:    | end while |
| 14:    | return \(\alpha\) |

Corollary 1. Algorithm 1 converges.

The main observation in the proof of Corollary 1 is that the function \(\log P(\alpha | B_t)\) in Equation 22 is concave.

5 In-degree Distribution

This section characterizes the in-degree distribution of the nodes in a sequence of networks \((G_t)_{t \leq T}\) generated by the algorithm presented in Definition 1 for
some $T \in \mathbb{N}$, from a given seed network $G_0$. It also presents how the dynamics of the in-degree distribution converges to a stationary distribution and illustrates the approach with experiments on sequences of networks $(G_t)_{t \leq T}$.

**Definition 6.** Let $K_t$ be a random variable that characterizes the in-degree of a node selected uniformly at random at time $t$ from $G_t$. Moreover, let $P_t(k) = \mathbb{P}(K_t = k)$ denote the probability that $K_t$ is equal to $k$ at time $t$.

The notion of asymptotic equivalence between two real sequences is used to prove the existence of $\lim_{t \to \infty} P_t(k)$. Theorem 2 ensures that the probability of the in-degree distribution of the model converges.

**Theorem 2.** For $t \in \mathbb{N}$, $\lim_{t \to \infty} P_t(k)$ exists for all $k \geq \hat{m}$.

**Proof.** First, note that the expected number of nodes of in-degree $k \geq \hat{m}$ is

$$n_t P_t(k) = n_{t-1} P_{t-1}(k) - P_{t-1} k - m(1 - \alpha) n_{t-1} P_{t-1}(k)$$

$$+ m(1 - \alpha) n_{t-1} P_{t-1}(k - 1). \tag{29}$$

According to Equation 29

$$n_t P_t(k) = \left( n_{t-1} - \frac{m \hat{m} n_{t-1}}{e_{t-1}} - m(1 - \alpha) \right) P_{t-1}(k)$$

$$+ \left( \frac{m \alpha n_{t-1}}{e_{t-1}} + m(1 - \alpha) \right) P_{t-1}(k - 1). \tag{30}$$

Since $\hat{m}$ nodes establish an edge to the new node, the expected number of nodes of in-degree $k = \hat{m}$ is

$$n_t P_t(\hat{m}) = \left( n_{t-1} - \frac{m \hat{m} n_{t-1}}{e_{t-1}} - m(1 - \alpha) \right) P_{t-1}(\hat{m}) + 1. \tag{31}$$

The proof proceeds by induction over $k$.

**Base case.** When $k = \hat{m}$, by using Equation 31 $P_t(\hat{m})$ can be expressed using the recurrence

$$P_t(\hat{m}) = \frac{1}{n_t} \left( n_{t-1} - \frac{m \hat{m} n_{t-1}}{e_{t-1}} - m(1 - \alpha) \right) P_{t-1}(\hat{m}) + \frac{1}{n_t}. \tag{32}$$

This is a non-autonomous, first-order difference equation. It can be shown by induction over $t$ that the solution of Equation 32 is given by

$$P_t(\hat{m}) = \prod_{i=1}^{t} a_i P_0(\hat{m}) + \sum_{i=1}^{t} \left[ \prod_{j=i+1}^{t} a_i \right] b_i, \tag{33}$$

where $a_t = \frac{1}{n_t} \left( n_{t-1} - \frac{m \hat{m} n_{t-1}}{e_{t-1}} - m(1 - \alpha) \right)$ and $b_t = \frac{1}{n_t}$. The first term in Equation 33 can be written in terms of the Gamma functions as

$$\prod_{i=1}^{t} a_i P_0(\hat{m}) = \frac{\Gamma(i - \xi_1) \Gamma(i - \xi_2) P_0(\hat{m})}{\Gamma(\frac{\xi_0}{m + \hat{m}} + i) \Gamma(n_0 + i)},$$
where \( \xi_1 \) and \( \xi_2 \) are constant real numbers that do not depend on time. Moreover,

\[
\lim_{t \to \infty} \frac{\Gamma(t - \xi_1)\Gamma(t - \xi_2)P_0(\hat{m})}{\Gamma\left(\frac{e_0}{m + \hat{m}} + t\right)\Gamma(a_0 + t)} = 0.
\]

It can further be shown that the second term in Equation 33 is the convergent series

\[
\sum_{i=1}^{\infty} \prod_{j=i+1}^{\infty} a_i b_i = \frac{m + \hat{m}}{m + \hat{m} + m^2 + m\hat{m} - m^2}\alpha.
\]

Therefore,

\[
\lim_{t \to \infty} P_t(\hat{m}) = \frac{m + \hat{m}}{m + \hat{m} + m^2 + m\hat{m} - m^2}\alpha.
\]

**Inductive step.** Let \( k > \hat{m} \) and assume that \( \lim_{t \to \infty} P_t(k) \) exists for all \( k > \hat{m} \). For a large enough \( t \), \( P_{t-1}(k) \sim P_t(k) \) and \( P_{t-1}(k + 1) \sim P_t(k + 1) \).

Using a similar argument to the one the proof of Theorem 1 in [22] and by Equation 32,

\[
\left(1 + \frac{ma(k + 1)}{m + \hat{m}} + m(1 - \alpha)\right) P_t(k + 1) \sim \left(\frac{mak}{m + \hat{m}} + m(1 - \alpha)\right) P_t(k).
\]

By inductive hypothesis

\[
\lim_{t \to \infty} P_t(k + 1) = \frac{mak}{m + \hat{m} + m^2 + m\hat{m} - m^2}\alpha \lim_{t \to \infty} P_t(k).
\]

Therefore, \( \lim_{t \to \infty} P_t(k) \) exists for all \( k \geq \hat{m} \)

Equation 30 indicates that the expected number of nodes of in-degree \( k \geq \hat{m} \) is equal to the difference between the expected number of nodes of in-degree \( k \) selected at time \( t - 1 \) by the attachment process and the expected number of nodes of in-degree \( k - 1 \) that establish an edge with the new node.

**Corollary 2.** If \( k \geq \hat{m} \), then the asymptotic behavior of the expected complementary cumulative in-degree distribution satisfies

\[
\tilde{F}_{\infty}(k) = \begin{cases} 
\left(\frac{m}{1 + m}\right)^{k - \hat{m}}, & \alpha = 0 \\
\frac{\Gamma(k + \hat{m} + m + m^2)\Gamma(k)}{\Gamma(\hat{m})\Gamma(k + \frac{e_0 + \hat{m}}{m + \hat{m}})}, & \alpha = 1 \land \hat{m} \geq 1 \\
\frac{\Gamma(k + \hat{m} + m + m^2 + m\hat{m} - m^2)\Gamma(k + \frac{e_0 + \hat{m}(1 - \alpha)}{m + \hat{m}})}{\Gamma(\frac{e_0 + \hat{m}(1 - \alpha)}{m + \hat{m}})\Gamma(\hat{m} + m + \hat{m} + m^2)}\left(\frac{m + \hat{m}}{m + \hat{m} + m^2 + m\hat{m} - m^2}\alpha\right), & 0 < \alpha < 1
\end{cases}
\]
Proof. Let \( P_\infty(k) \) denote the limit of \( P_t(k) \) as \( t \) tends to infinity. According to Theorem 2, \( P_\infty(k) \) can be written as
\[
P_\infty(k) = \begin{cases} 
\frac{m+\hat{m}}{m^2+mr+\hat{m}^2-m-\alpha \hat{m}} & , \ k = \hat{m} \\
\frac{\alpha(km-m^2-mr-m)+m^2+mr}{\alpha(km-m^2-mr)+m^2+mr+\alpha \hat{m}^2+\hat{m}^2+m} P_\infty(k-1), & k > \hat{m}
\end{cases}
\]

Equation 34 defines a recurrence relation that varies as a function of the value of \( \alpha \). For \( \alpha = 0 \), the solution of the recurrence is
\[
P_\infty(k) = \frac{1}{m+1} \left( \frac{m}{m+1} \right)^{k-\hat{m}}.
\]

For \( \alpha = 1 \) and \( \hat{m} \geq 1 \), the solution is given by
\[
P_\infty(k) = \frac{(m+\hat{m}) \Gamma(\hat{m}+m\hat{m}+m) \Gamma(k)}{m \Gamma(\hat{m}) \Gamma(k+\frac{\hat{m}+2m}{m})}.
\]

Furthermore, for \( 0 < \alpha < 1 \), the recurrence has solution
\[
P_\infty(k) = \frac{(m+\hat{m}) \Gamma \left( \frac{r+m(1+m+r-m\alpha)}{m\alpha} \right) \Gamma \left( k + \frac{(m+r)(1-\alpha)}{\alpha} \right)}{m\alpha \Gamma \left( \frac{m+r-m\alpha}{m} \right) \Gamma \left( \frac{r+m(m+r+k\alpha-(m+r)(\alpha+1))}{m\alpha} \right)}.
\]

Since \( \bar{F}_\infty(k) = \mathbb{P}[K \geq k] = 1 - \sum_{j=\hat{m}}^{k-1} P_\infty(j) \), by using Equation 35, Equation 36, and Equation 37, the desired result is obtained.

Consider the plots in Figure 4. They summarize experiments performed on three sequences of networks generated by the algorithm in Definition 4 from the complete graph with 3 nodes. In the three sequences the parameters \( m = 5 \) and \( \hat{m} = 3 \) are fixed. However, each sequence uses a different parameter \( \alpha \): \( \alpha = 0.0 \), \( \alpha = 0.6 \), and \( \alpha = 1.0 \), respectively. The plots in Figure 4 summarize the degree distribution and the complementary cumulative degree distribution for each one of the three sequences of networks. The main observation is that the simulated distributions approach the theoretical limits, a result that follows from Theorem 2 and Corollary 2.

6 Results

This section shows an application of the EM algorithm presented in Section 4 and validates the proposed approach using the results in Section 5 to estimate the contribution of the attachment mechanisms in empirical co-authorship networks. In particular, the High Energy Physics Theory (HepTh) and Phenomenology (HepPh) co-authorship networks [16] are analyzed for specific periods of time. The nodes of each network represent authors and edges between two authors represent the existence of at least one co-authored publication. The timestamps denote dates of publication.
6.1 Case Study Analysis: HepTh Network

This case study considers authors publishing papers between September 1993 and December 1999 in the HepTh network. The seed network $G_0$ is built based on the publications from September to December, 1993. The sequence $(G_t)_{t \leq 57}$ consists of networks $G_t$ built by adding edges meeting the following criteria:

1. the target nodes must belong to $G_{t-1}$, and
2. if the target node does not belong to $G_{t-1}$, it is added to the network.

Applying Algorithm 1 to maximize the $G_t$-likelihood function at each time $t$, the incidence proportion of preferential attachment mechanism is estimated (see Figure 5).

Note that, because of the form the network is built, the parameter $\hat{m}$ is 0 (i.e., the network does not respond to incoming nodes). Based on the estimated parameter $\hat{\alpha}$, Theorem 2 and Corollary 2 are applied to specify the theoretical indegree distribution. The value of the parameter $m$ is estimated using the empirical degree distribution and the estimated parameter. Figure 6 illustrates the relationship between the theoretical and empirical degree distribution: solid lines represent the values for the theoretical degree and complementary cumulative degree distributions; the dots represent the values for the empirical degree and complementary degree distributions.

6.2 Case Study Analysis: HepPh Network

This case study considers authors publishing papers between March 1992 and December 1999 in the HepPh network. The seed network $G_0$ is built based on the publications from March to September, 1992. In the sequence $(G_t)_{t \leq 88}$, each $G_t$ is build under the criteria above for the HepTh network. Again, by using Algorithm 1 the contribution of the preferential attachment mechanism is estimated and plotted in Figure 7.

Consider the last value estimate $\hat{\alpha} = 0.62$. Using Theorem 2 and Corollary 2, the theoretical degree and complementary cumulative degree is validated. Figure 8 shows the relationship between the theoretical and empirical distributions. As before, solid lines represent the values for the theoretical degree and complementary cumulative degree distributions. The dots represent the empirical complementary degree distributions.

7 Conclusion and Future Work

Preferential attachment models explain the formation of power-laws in the tail of degree distribution. Such models capture the evolution of the number of connections of a small – yet significant – number of nodes with extremely large degrees. However, preferential attachment alone falls short in describing the behavior of the large majority of nodes with smaller degrees.
To overcome this limitation, mixed attachment models contemplate how degree distributions may result from a combination of multiple mechanisms. Our work is novel for it presented conditions guaranteeing that the prevalence estimate of preferential and random attachment mechanisms represents a local maximum of the likelihood function. We used the expectation maximization algorithm to find the maximum-likelihood estimate of the contribution of the two mechanisms. Our results showed that if the algorithm is applied without satisfying the proposed conditions, then the estimate fails to converge to a stationary value.

Finally, we applied the proposed approach to estimate the prevalence of random and preferential attachment mechanism in citation networks of academic papers. The estimate is evaluated by comparing the empirical degree distribution to the theoretical distribution evaluated at the estimated parameter. The results showed that mixed attachment models are able to recreate the behavior of nodes with both small and large degrees.

Future work on extending the proposed model to include new attachment and response mechanisms that can update edges and even generate clustering should be pursued. Furthermore, the analysis of the likelihood functions, and the in- and out-degree distributions of the extended models should also be investigated. Further applications to other empirical networks should be considered, taking into account the rich experience already available on using mixed attachment networks.

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References

1. R. Albert and A.-L. Barabási. Statistical mechanics of complex networks. Reviews of Modern Physics, 74(1):47–97, 2002.
2. L. A. N. Amaral, A. Scala, M. Barthelemy, and H. E. Stanley. Classes of small-world networks. Proceedings of the National Academy of Sciences, 97(21):11149–11152, 2000.
3. T. M. Apostol. Mathematical analysis. Addison Wesley Publishing Company, 1974.
4. A. L. Barabási and R. Albert. Emergence of scaling in random networks. Science, 286:509–512, 1999.
5. G. Bianconi and A.-L. Barabási. Competition and multiscaling in evolving networks. Europhysics Letters (EPL), 54(4):436–442, 2001.
6. D. S. Callaway, J. E. Hopcroft, J. M. Kleinberg, M. E. J. Newman, and S. H. Strogatz. Are randomly grown graphs really random? Physical Review E, 64(4):041902, 2001.
7. A. Clauset, C. R. Shalizi, and M. E. J. Newman. Power-law distributions in empirical data. SIAM Review, 51(4):661–703, 2009.
8. A. P. Dempster, N. M. Laird, and D. B. Rubin. Maximum likelihood from incomplete data via the EM algorithm. *Journal of the Royal Statistical Society. Series B (Methodological)*, 39(1):1–38, 1977.
9. J. Diebolt and G. Celeux. Asymptotic properties of a stochastic EM algorithm for estimating mixing proportions. *Communications in Statistics. Stochastic Models*, 9(4):599–613, 1993.
10. S. N. Dorogovtsev and J. F. F. Mendes. Evolution of networks with aging of sites. *Physical Review E*, 62(2):1842–1845, 2000.
11. S. N. Dorogovtsev, J. F. F. Mendes, and A. N. Samukhin. Structure of growing networks with preferential linking. *Physical Review Letters*, 85(21):4633–4636, 2000.
12. L. Giot, J. S. Bader, C. Brouwer, A. Chaudhuri, B. Kuang, Y. Li, Y. Hao, C. Ooi, B. Godwin, E. Vitols, et al. A protein interaction map of Drosophila melanogaster. *Science*, 302(5651):1727–1736, 2003.
13. M. O. Jackson and B. W. Rogers. Meeting strangers and friends of friends: How random are social networks? *American Economic Review*, 97(3):890–915, 2007.
14. D. Ke and T. Yi. Growing networks based on the mechanism of addition and deletion. *Chinese Physics Letters*, 21(9):1858–1860, 2004.
15. P. L. Krapivsky, S. Redner, and F. Leyvraz. Connectivity of growing random networks. *Physical Review Letters*, 85(21):4629–4632, 2000.
16. J. Leskovec, J. Kleinberg, and C. Faloutsos. Graph evolution: Densification and shrinking diameters. *ACM Trans. Knowledge Discovery from Data*, 1(1):1–40, 2007.
17. S. Li, C. M. Armstrong, N. Bertin, H. Ge, S. Milstein, M. Boxem, P.-O. Vidalain, J.-D. J. Han, A. Cheneau, T. Hao, et al. A map of the interactome network of the Metazoan C. elegans. *Science*, 303(5657):540–543, 2004.
18. Z. Liu, Y.-C. Lai, N. Ye, and P. Dasgupta. Connectivity distribution and attack tolerance of general networks with both preferential and random attachments. *Physics Letters A*, 303(5-6):337–344, 2002.
19. G. McLachlan and T. Krishnan. *The EM Algorithm and Extensions: Second Edition*. John Wiley & Sons, 2007.
20. M. E. Newman. Modularity and community structure in networks. *Proceedings of the National Academy of Sciences*, 103(23):8577–8582, 2006.
21. S. Redner. How popular is your paper? An empirical study of the citation distribution. *The European Physical Journal B-Condensed Matter and Complex Systems*, 4(2):131–134, 1998.
22. D. Ruiz and J. Finke. Stability of the Jackson-Rogers model. In *2017 IEEE 56th Annual Conference on Decision and Control (CDC)*, pages 1803–1808. IEEE, 2017.
23. A. Said, E. De Luca, and S. Albayrak. How social relationships affect user similarities. In *Proceeding of the 2010 Workshop on Social Recommender Systems*, pages 1–4, 2010.
24. Z.-G. Shao, X.-W. Zou, Z.-J. Tan, and Z.-Z. Jin. Growing networks with mixed attachment mechanisms. *Journal of Physics A: Mathematical and General*, 39(9):2035–2042, 2006.
25. P. Sheridan, Y. Yagahara, and H. Shimodaira. A preferential attachment model with Poisson growth for scale-free networks. *Annals of the Institute of Statistical Mathematics*, 60(4):747–761, 2008.
26. G. Szabó, M. Alava, and J. Kertész. Clustering in complex networks. In *Complex Networks*, pages 139–162. Springer, 2004.
27. W. Wang, Q. Zhang, and T. Zhou. Evaluating network models: A likelihood analysis. *EPL (Europhysics Letters)*, 98(2):28004, 2012.
28. Q. Zhang, X. Xu, Y. Zhu, and T. Zhou. Measuring multiple evolution mechanisms of complex networks. *Scientific Reports*, 5(1):10350, 2015.
Fig. 4: Degree distributions and complementary cumulative degree distributions for three sequences of networks. The solid lines represent the average of the ccdf of 100 runs of the model; dashed lines represent the predictions for $m = 5$, $\hat{m} = 3$ and $\alpha = 0$ in (a) and (b); $\alpha = 0.6$ in (c) and (d); $\alpha = 1.0$ in (e) and (f).
Fig. 5: Evolution of the estimated parameter for the HepTh network.

Fig. 6: Relationship between the theoretical and empirical degree distributions for HepTh network with parameters $m = 8$ and $\hat{\alpha} = 0.34$. 
Fig. 7: Evolution of the estimated parameter for the HepPh network.

Fig. 8: Relationship between the theoretical and empirical degree distributions HepPh network with parameters $m = 8$ and $\hat{\alpha} = 0.62$. 