THE HERMITE-HADAMARD TYPE INEQUALITIES FOR OPERATOR $s$-CONVEX FUNCTIONS

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Abstract. In this paper we introduce operator $s$-convex functions and establish some Hermite-Hadamard type inequalities in which some operator $s$-convex functions of positive operators in Hilbert spaces are involved.

Keywords: The Hermite-Hadamard inequality, $s$-convex functions, operator $s$-convex functions.

1. INTRODUCTION

The following inequality holds for any convex function $f$ defined on $\mathbb{R}$ and $a, b \in \mathbb{R}$, with $a < b$

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2} \quad (1.1)$$

Both inequalities hold in the reversed direction if $f$ is concave. The inequality (1.1) is known in the literature as the Hermite-Hadamard’s inequality. We note that the Hermite-Hadamard’s inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen’s inequality. The classical Hermite-Hadamard inequality provides estimates of the mean value of a continuous convex function $f : [a, b] \to \mathbb{R}$.

In the paper [7] Hudzik and Maligranda considered, among others, two classes of functions which are $s$-convex in the first and second senses. These classes are defined in the following way: a function $f : [0, \infty) \to \mathbb{R}$ is said to be $s$-convex in the first sense if

$$f(\alpha x + \beta y) \leq \alpha^s f(x) + \beta^s f(y)$$

holds for all $x, y \in [0, \infty)$ and $\alpha, \beta \geq 0$ with $\alpha^s + \beta^s = 1$. The class of $s$-convex functions in the first sense is usually denoted with $K_s^1$.

A function $f : \mathbb{R}^+ \to \mathbb{R}$ where $\mathbb{R}^+ = [0, +\infty)$, is said to be $s$-convex in the second sense if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda^s f(x) + (1 - \lambda)^s f(y)$$
holds for all \(x, y \in [0, \infty), \lambda \in [0, 1]\) and for some fixed \(s \in (0, 1]\). The class of \(s\)-convex functions in the second sense is usually denoted with \(K^2_s\). It can be easily seen that for \(s = 1\), \(s\)-convexity reduces to ordinary convexity of functions defined on \([0, \infty)\).

It is proved in [7] that if \(s \in (0, 1)\) then \(f \in K^2_s\) implies \(f([0, \infty)) \subseteq [0, \infty)\), i.e., they proved that all functions from \(K^2_s, s \in (0, 1)\), are nonnegative. The following example can be found in [7].

**Example 1.** Let \(s \in (0, 1)\) and \(a, b, c \in \mathbb{R}\). We define function \(f : [0, \infty) \to \mathbb{R}\) as

\[
f(t) = \begin{cases} a, & t = 0, \\ bt^s + c, & t > 0. \end{cases}
\]

It can be easily checked that

(i) If \(b \geq 0\) and \(0 \leq c \leq a\), then \(f \in K^2_s\),

(ii) If \(b > 0\) and \(c < 0\), then \(f \notin K^2_s\).

In Theorem 4 of [7] both definitions of the \(s\)-convexity have been compared as follows:

(i) Let \(0 < s \leq 1\). If \(f \in K^2_s\) and \(f(0) = 0\), then \(f \in K^1_s\),

(ii) Let \(0 < s_1 < s_2 \leq 1\). If \(f \in K^2_{s_2}\) and \(f(0) = 0\), then \(f \in K^2_{s_1}\),

(iii) Let \(0 < s_1 < s_2 \leq 1\). If \(f \in K^1_{s_2}\) and \(f(0) \leq 0\), then \(f \in K^1_{s_1}\).

In [3], Dragomir and Fitzpatrick proved the following variant of Hadamard’s inequality which holds for \(s\)-convex functions in the second sense:

**Theorem 1.** Suppose that \(f : [0, \infty) \to [0, \infty)\) is an \(s\)-convex function in the second sense, where \(s \in (0, 1)\) and let \(a, b \in [0, \infty), a < b\). If \(f \in L^1[a, b]\), then the following inequalities hold:

\[
2^{s-1} f \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{s + 1}
\]

the constant \(k = \frac{1}{s+1}\) is the best possible in the second inequality in (1.2). The above inequalities are sharp.

The Hermite-Hadamard inequality has several applications in nonlinear analysis and the geometry of Banach spaces, see [8]. In recent years several extensions and generalizations have been considered for classical convexity. We would like to refer the reader to [2, 5, 13] and references therein for more information. A number of papers have been written on this inequality providing some inequalities analogous to Hadamard’s inequality given in (1.1) involving two convex functions, see [11, 1, 12]. Pachpatte in [11] has proved the following theorem for the product of two convex functions.
Theorem 2. Let $f$ and $g$ be real-valued, nonnegative and convex functions on $[a, b]$. Then

$$\frac{1}{b-a} \int_{a}^{b} f(x)g(x)dx \leq \frac{1}{3}M(a, b) + \frac{1}{6}N(a, b),$$

$$2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_{a}^{b} f(x)g(x)dx + \frac{1}{6}M(a, b) + \frac{1}{3}N(a, b),$$

where $M(a, b) = f(a)g(a) + f(b)g(b)$, $N(a, b) = f(a)g(b) + f(b)g(a)$.

Kirmaci et al. in [9] have proved the following theorem for the product of two $s$-convex functions, which is a generalization of Theorem 2.

Theorem 3. Let $f, g : [0, \infty) \to [0, \infty)$ be $s_1$-convex and $s_2$-convex functions in the second sense respectively, where $s_1, s_2 \in (0, 1)$. Let $a, b \in [0, \infty)$, $a < b$. If $f, g$ and $fg \in L^1([a, b])$ then

$$\frac{1}{b-a} \int_{0}^{1} f(x)g(x)dx \leq \frac{1}{s_1 + s_2 + 1}M(a, b) + \beta(s_1 + 1, s_2 + 1)N(a, b),$$

where $M(a, b) = f(a)g(a) + f(b)g(b)$, $N(a, b) = f(a)g(b) + f(b)g(a)$.

In this paper we show that Theorem 1 and Theorem 3 hold for operator $s$-convex functions in a convex subset $K$ of $B(H)^+$ the set of positive operators in $B(H)$. We also obtain some integral inequalities for the product of two operator $s$-convex functions.

2. OPERATOR $s$-CONVEX FUNCTIONS

First, we review the operator order in $B(H)$ and the continuous functional calculus for a bounded selfadjoint operator. For selfadjoint operators $A, B \in B(H)$ we write $A \leq B$ (or $B \geq A$) if $\langle Ax, x \rangle \leq \langle Bx, x \rangle$ for every vector $x \in H$, we call it the operator order.

Now, let $A$ be a bounded selfadjoint linear operator on a complex Hilbert space $(H; \langle ., . \rangle)$ and $C(Sp(A))$ the $C^*$-algebra of all continuous complex-valued functions on the spectrum of $A$. The Gelfand map establishes a $*$-isometrically isomorphism $\Phi$ between $C(Sp(A))$ and the $C^*$-algebra $C^*(A)$ generated by $A$ and the identity operator $1_H$ on $H$.
as follows (see for instance [6, p.3]): For \( f, g \in C(Sp(A)) \) and \( \alpha, \beta \in \mathbb{C} \)

(i) \( \Phi(\alpha f + \beta g) = \alpha \Phi(f) + \beta \Phi(g) \);
(ii) \( \Phi(fg) = \Phi(f)\Phi(g) \) and \( \Phi(f^*) = \Phi(f)^* \);
(iii) \( \|\Phi(f)\| = \|f\| := \sup_{t \in Sp(A)} |f(t)| \);
(iv) \( \Phi(f_0) = 1 \) and \( \Phi(f_1) = A \), where \( f_0(t) = 1 \) and \( f_1(t) = t \), for \( t \in Sp(A) \).

If \( f \) is a continuous complex-valued functions on \( Sp(A) \), the element \( \Phi(f) \) of \( C^*(A) \) is denoted by \( f(A) \), and we call it the continuous functional calculus for a bounded selfadjoint operator \( A \).

If \( A \) is a bounded selfadjoint operator and \( f \) is a real-valued continuous function on \( Sp(A) \), then \( f(t) \geq 0 \) for any \( t \in Sp(A) \) implies that \( f(A) \geq 0 \), i.e., \( f(A) \) is a positive operator on \( H \). Moreover, if both \( f \) and \( g \) are real-valued functions on \( Sp(A) \) such that \( f(t) \leq g(t) \) for any \( t \in sp(A) \), then \( f(A) \leq f(B) \) in the operator order in \( B(H) \).

A real valued continuous function \( f \) on an interval \( I \) is said to be operator convex (operator concave) if

\[
f((1 - \lambda)A + \lambda B) \leq (\geq)(1 - \lambda)f(A) + \lambda f(B)
\]

in the operator order in \( B(H) \), for all \( \lambda \in [0, 1] \) and for every bounded self-adjoint operators \( A \) and \( B \) in \( B(H) \) whose spectra are contained in \( I \).

As examples of such functions, we give the following examples, another proof of them and further examples can be found in [6].

Example 2. (i) The convex function \( f(t) = \alpha t^2 + \beta t + \gamma \) \((\alpha \geq 0, \beta, \gamma \in \mathbb{R})\) is operator convex on every interval. To see it, for all self-adjoint operators \( A \) and \( B \):

\[
\frac{f(A) + f(B)}{2} - f\left(\frac{A + B}{2}\right) = \alpha \left(\frac{A^2 + B^2}{2} - \left(\frac{A + B}{2}\right)^2\right) + \beta \left(\frac{A + B}{2} - \frac{A + B}{2}\right) + (\gamma - \gamma)
\]

\[
= \frac{\alpha}{4}(A^2 + B^2 - AB - BA) = \frac{\alpha}{4}(A - B)^2 \geq 0.
\]

(ii) The convex function \( f(t) = t^3 \) on \([0, \infty)\) is not operator convex. In fact, if we put

\[
A = \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \quad \& \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},
\]
then
\[ \frac{A^3 + B^3}{2} - \left( \frac{A + B}{2} \right)^3 = \frac{1}{8} \begin{bmatrix} 67 & -34 \\ -34 & 17 \end{bmatrix} \neq 0. \]

For some fundamental results on operator convex (operator concave) and operator monotone functions, see [6] and the references therein.

We denoted by \( B(H)^+ \) the set of all positive operators in \( B(H) \) and
\[ C(H) := \{ A \in B(H)^+ : AB + BA \geq 0, \text{ for all } B \in B(H)^+ \}. \]
It is obvious that \( C(H) \) is a closed convex cone in \( B(H) \).

**Definition 1.** Let \( I \) be an interval in \( [0, \infty) \) and \( K \) be a convex subset of \( B(H)^+ \). A continuous function \( f : I \to \mathbb{R} \) is said to be operator \( s \)-convex on \( I \) for operators in \( K \) if
\[ f((1 - \lambda)A + \lambda B) \leq (1 - \lambda)^s f(A) + \lambda^s f(B) \]
in the operator order in \( B(H) \), for all \( \lambda \in [0, 1] \) and for every positive operators \( A \) and \( B \) in \( K \) whose spectra are contained in \( I \) and for some fixed \( s \in (0, 1] \). For \( K = B(H)^+ \) we say \( f \) is operator \( s \)-convex on \( I \).

First of all we state the following lemma.

**Lemma 1.** If \( f \) is operator \( s \)-convex on \( [0, \infty) \) for operators in \( K \), then \( f(A) \) is positive for every \( A \in K \).

**Proof.** For \( A \in K \), we have
\[ f(A) = f \left( \frac{A}{2} + \frac{A}{2} \right) \leq \left( \frac{1}{2} \right)^s f(A) + \left( \frac{1}{2} \right)^s f(A) = 2^{1-s} f(A). \]
This implies that \( (2^{1-s} - 1) f(A) \geq 0 \) and so \( f(A) \geq 0 \).

In [10], Moslehian and Najafi proved the following theorem for positive operators as follows:

**Theorem 4.** Let \( A, B \in B(H)^+ \). Then \( AB + BA \) is positive if and only if \( f(A+B) \leq f(A) + f(B) \) for all non-negative operator monotone functions \( f \) on \( [0, \infty) \).

As an example of operator \( s \)-convex function, we give the following example.

**Example 3.** Since for every positive operators \( A, B \in C(H) \), \( AB + BA \geq 0 \), utilizing Theorem 4 we get
\[ ((1-t)A + tB)^s \leq (1-t)^s A^s + t^s B^s. \]
Therefore the continuous function \( f(t) = t^s \) \((0 < s \leq 1)\) is operator \( s \)-convex on \([0, \infty)\) for operators in \( C(H) \).
Dragomir in [4] has proved a Hermite-Hadamard type inequality for operator convex function as follows:

**Theorem 5.** Let $f : I \rightarrow \mathbb{R}$ be an operator convex function on the interval $I$. Then for all selfadjoint operators $A$ and $B$ with spectra in $I$ we have the inequality

$$\left(f\left(\frac{A+B}{2}\right) \leq \frac{1}{2} \left[f\left(\frac{3A+B}{4}\right) + f\left(\frac{A+3B}{4}\right)\right]\right)$$

$$\leq \int_0^1 f((1-t)A+tB)dt$$

$$\leq \frac{1}{2} \left[f\left(\frac{A+B}{2}\right) + \frac{f(A) + f(B)}{2}\right] \left(\leq \frac{f(A) + f(B)}{2}\right).$$

Let $X$ be a vector space, $x, y \in X$, $x \neq y$. Define the segment $[x, y] := (1-t)x + ty; t \in [0, 1]$.

We consider the function $f : [x, y] \rightarrow \mathbb{R}$ and the associated function

$$g(x, y) : [0, 1] \rightarrow \mathbb{R},$$

$$g(x, y)(t) := f((1-t)x + ty), t \in [0, 1].$$

Note that $f$ is convex on $[x, y]$ if and only if $g(x, y)$ is convex on $[0, 1]$. For any convex function defined on a segment $[x, y] \in X$, we have the Hermite-Hadamard integral inequality

$$f\left(\frac{x+y}{2}\right) \leq \int_0^1 f((1-t)x + ty)dt \leq \frac{f(x) + f(y)}{2},$$

which can be derived from the classical Hermite-Hadamard inequality (1.1) for the convex function $g(x, y) : [0, 1] \rightarrow \mathbb{R}$.

**Lemma 2.** Let $f : I \subseteq [0, \infty) \rightarrow \mathbb{R}$ be a continuous function on the interval $I$. Then for every two positive operators $A, B \in K \subseteq B(H)^+$ with spectra in $I$ the function $f$ is operator s-convex for operators in $[A, B] := \{(1-t)A + tB : t \in [0, 1]\}$ if and only if the function $\varphi_{x,A,B} : [0, 1] \rightarrow \mathbb{R}$ defined by

$$\varphi_{x,A,B} = \langle f((1-t)A+tB)x, x \rangle$$

is s-convex on $[0, 1]$ for every $x \in H$ with $\|x\| = 1$. 
Proof. Let $f$ be operator $s$-convex for operators in $[A, B]$ then for any $t_1, t_2 \in [0, 1]$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ we have

$$\varphi_{x,A,B}(\alpha t_1 + \beta t_2) = \langle f((1 - (\alpha t_1 + \beta t_2))A + (\alpha t_1 + \beta t_2)B)x, x \rangle$$

$$= \langle f(\alpha[(1 - t_1)A + t_1B] + \beta[(1 - t_2)A + t_2B])x, x \rangle$$

$$\leq \alpha^s\langle f((1 - t_1)A + t_1B)x, x \rangle + \beta^s\langle f((1 - t_2)A + t_2B)x, x \rangle$$

$$= \alpha^s\varphi_{x,A,B}(t_1) + \beta^s\varphi_{x,A,B}(t_2).$$

showing that $\varphi_{x,A,B}$ is a $s$-convex function on $[0, 1]$.

Let now $\varphi_{x,A,B}$ be $s$-convex on $[0, 1]$, we show that $f$ is operator $s$-convex for operators in $[A, B]$. For every $C = (1 - t_1)A + t_1B$ and $D = (1 - t_2)A + t_2B$ in $[A, B]$ we have

$$\langle f(\lambda C + (1 - \lambda)D)x, x \rangle$$

$$= \langle f[\lambda((1 - t_1)A + t_1B) + (1 - \lambda)((1 - t_2)A + t_2B)]x, x \rangle$$

$$= \langle f[(1 - (\lambda t_1 + (1 - \lambda)t_2))A + (\lambda t_1 + (1 - \lambda)t_2)B]x, x \rangle$$

$$= \varphi_{x,A,B}(\lambda t_1 + (1 - \lambda)t_2)$$

$$\leq \lambda^s\varphi_{x,A,B}(t_1) + (1 - \lambda)^s\varphi_{x,A,B}(t_2)$$

$$= \lambda^s\langle f((1 - t_1)A + t_1B)x, x \rangle + (1 - \lambda)^s\langle f((1 - t_2)A + t_2B)x, x \rangle$$

$$\leq \lambda^s\langle f(C)x, x \rangle + (1 - \lambda)^s\langle f(D)x, x \rangle.$$

The following theorem is a generalization of Theorem 1 for operator $s$-convex functions.

**Theorem 6.** Let $f : I \to \mathbb{R}$ be an operator $s$-convex function on the interval $I \subseteq [0, \infty)$ for operators in $K \subseteq B(H)^+$. Then for all positive operators $A$ and $B$ in $K$ with spectra in $I$ we have the inequality

$$2^{s-1}f\left(\frac{A + B}{2}\right) \leq \int_0^1 f((1 - t)A + tB)dt \leq \frac{f(A) + f(B)}{s + 1}. \quad (2.1)$$

**Proof.** For $x \in H$ with $\|x\| = 1$ and $t \in [0, 1]$, we have

$$\langle ((1 - t)A + tB)x, x \rangle = (1 - t)\langle Ax, x \rangle + t\langle Bx, x \rangle \in I, \quad (2.2)$$

since $\langle Ax, x \rangle \in Sp(A) \subseteq I$ and $\langle Bx, x \rangle \in Sp(B) \subseteq I$.

Continuity of $f$ and $(2.2)$ imply that the operator-valued integral

$$\int_0^1 f((1 - t)A + tB)dt$$

exists.

Since $f$ is operator $s$-convex, therefore for $t$ in $[0, 1]$ and $A, B \in K$ we have

$$f((1 - t)A + tB) \leq (1 - t)^sf(A) + t^sf(B). \quad (2.3)$$
Integrating both sides of (2.3) over $[0, 1]$ we get the following inequality
\[
\int_0^1 f((1 - t)A + tB)dt \leq \frac{f(A) + f(B)}{s + 1}.
\]
To prove the first inequality in (2.1) we observe that
\[
f \left(\frac{A + B}{2}\right) \leq \frac{f(tA + (1 - t)B) + f((1 - t)A + tB)}{2}.
\] (2.4)
Integrating the inequality (2.4) over $t \in [0, 1]$ and taking into account that
\[
\int_0^1 f((1 - t)A + tB)dt = \int_0^1 f(((1 - t)A + tB)dt
\]
then we deduce the first part of (2.1). □

Let $f : I \to \mathbb{R}$ be operator $s_1$-convex and $g : I \to \mathbb{R}$ operator $s_2$-convex function on the interval $I$. Then for all positive operators $A$ and $B$ on a Hilbert space $H$ with spectra in $I$, we define real functions $M(A, B)$ and $N(A, B)$ on $H$ by
\[
M(A, B)(x) = \langle f(A)x, x \rangle \langle g(A)x, x \rangle + \langle f(B)x, x \rangle \langle g(B)x, x \rangle \quad (x \in H),
\]
\[
N(A, B)(x) = \langle f(A)x, x \rangle \langle g(B)x, x \rangle + \langle f(B)x, x \rangle \langle g(A)x, x \rangle \quad (x \in H).
\]

We note that, the Beta and Gamma functions are defined respectively, as follows:
\[
\beta(x, y) = \int_0^1 t^{x-1}(1 - t)^{y-1}dt \quad x > 0, \ y > 0
\]
and
\[
\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt \quad x > 0.
\]
The following theorem is a generalization of Theorem 3 for operator $s$-convex functions.

**Theorem 7.** Let $f : I \to \mathbb{R}$ be operator $s_1$-convex and $g : I \to \mathbb{R}$ operator $s_2$-convex function on the interval $I$ for operators in $K \subseteq B(H)^+$. Then for all positive operators $A$ and $B$ in $K$ with spectra in $I$, the inequality

\[
\int_0^1 \langle f(tA + (1 - t)B)x, x \rangle \langle g(tA + (1 - t)B)x, x \rangle dt
\]
\[
\leq \frac{1}{s_1 + s_2 + 1}M(A, B)(x) + \beta(s_1 + 1, s_2 + 1)N(A, B)(x). \quad (2.5)
\]
holds for any $x \in H$ with $\|x\| = 1$. 
Proof. For $x \in H$ with $\|x\| = 1$ and $t \in [0, 1]$, we have

$$\langle (tA + (1 - t)B)x, x \rangle = t\langle Ax, x \rangle + (1 - t)\langle Bx, x \rangle \in I, \quad (2.6)$$

since $\langle Ax, x \rangle \in Sp(A) \subseteq I$ and $\langle Bx, x \rangle \in Sp(B) \subseteq I$.

Continuity of $f, g$ and (2.6) imply that the operator valued integrals \(\int_0^1 f(tA + (1 - t)B)dt\), \(\int_0^1 g(tA + (1 - t)B)dt\) and \(\int_0^1 (fg)(tA + (1 - t)B)dt\) exist.

Since $f$ is operator $s_1$-convex and $g$ is operator $s_2$-convex, therefore for $t$ in $[0, 1]$ and $x \in H$ we have

$$\langle f(tA + (1 - t)B)x, x \rangle \leq \langle (t^{s_1}f(A) + (1 - t)^{s_1}f(B))x, x \rangle, \quad (2.7)$$

$$\langle g(tA + (1 - t)B)x, x \rangle \leq \langle (t^{s_2}g(A) + (1 - t)^{s_2}g(B))x, x \rangle. \quad (2.8)$$

From (2.7) and (2.8) we obtain

$$\langle f(tA + (1 - t)B)x, x \rangle \langle g(tA + (1 - t)B)x, x \rangle \leq \langle (t^{s_1+s_2}f(A)x, x) \langle g(A)x, x \rangle + (1 - t)^{s_1+s_2} \langle f(B)x, x \rangle \langle g(B)x, x \rangle \rangle$$

$$+ t^{s_1} (1 - t)^{s_2} [(f(A)x, x) \langle g(B)x, x \rangle]$$

$$+ t^{s_2} (1 - t)^{s_1} [(f(B)x, x) \langle g(A)x, x \rangle]. \quad (2.9)$$

Integrating both sides of (2.9) over $[0, 1]$ we get the required inequality (2.5). \(\square\)

The following theorem is a generalization of Theorem 7 in [9] for operator $s$-convex functions.

**Theorem 8.** Let $f : I \to \mathbb{R}$ be operator $s_1$-convex and $g : I \to \mathbb{R}$ be $s_2$-convex function on the interval $I$ for operators in $K \subseteq B(H)^+$. Then for all positive operators $A$ and $B$ in $K$ with spectra in $I$, the inequality

$$2^{s_1+s_2-1} \left\langle f \left( \frac{A + B}{2} \right)x, x \right\rangle \left\langle g \left( \frac{A + B}{2} \right)x, x \right\rangle$$

$$\leq \int_0^1 \langle f(tA + (1 - t)B)x, x \rangle \langle g(tA + (1 - t)B)x, x \rangle \, dt$$

$$+ \beta(s_1 + 1, s_2 + 1)M(A, B)(x) + \frac{1}{s_1 + s_2 + 1}N(A, B)(x), \quad (2.10)$$

holds for any $x \in H$ with $\|x\| = 1$. 
Proof. Since \( f \) is operator \( s_1 \)-convex and \( g \) operator \( s_2 \)-convex, therefore for any \( t \in I \) and any \( x \in H \) with \( \|x\| = 1 \) we observe that

\[
\left\langle f\left(\frac{A + B}{2}\right) x, x \right\rangle \left\langle g\left(\frac{A + B}{2}\right) x, x \right\rangle \\
= \left\langle f\left(\frac{tA + (1-t)B}{2} + \frac{(1-t)A + tB}{2}\right) x, x \right\rangle \\
\quad \times \left\langle g\left(\frac{tA + (1-t)B}{2} + \frac{(1-t)A + tB}{2}\right) x, x \right\rangle \\
\leq \frac{1}{2^{s_1+s_2}} \left\{ \left[ \langle f(tA + (1-t)B)x, x \rangle + \langle f((1-t)A + tB)x, x \rangle \right] \right. \\
\quad \times \left[ \langle g(tA + (1-t)B)x, x \rangle + \langle g((1-t)A + tB)x, x \rangle \right] \\
\leq \frac{1}{2^{s_1+s_2}} \left\{ \left[ \langle f(tA + (1-t)B)x, x \rangle \langle g(tA + (1-t)B)x, x \rangle \\
\quad + \langle f((1-t)A + tB)x, x \rangle \langle g((1-t)A + tB)x, x \rangle \right] \\
\quad + \left( t^{s_1} \langle f(A)x, x \rangle + (1-t)^{s_1} \langle f(B)x, x \rangle \right) 2 + \left( t^{s_2} \langle g(A)x, x \rangle + (1-t)^{s_2} \langle g(B)x, x \rangle \right) \right\} \\
\quad \times \left( \langle f(A)x, x \rangle + \langle f(B)x, x \rangle \right) \left( \langle g(A)x, x \rangle + \langle g(B)x, x \rangle \right) \right\} \\
= \frac{1}{2^{s_1+s_2}} \left\{ \left[ \langle f(tA + (1-t)B)x, x \rangle \langle g(tA + (1-t)B)x, x \rangle \\
\quad + \langle f((1-t)A + tB)x, x \rangle \langle g((1-t)A + tB)x, x \rangle \right] \\
\quad + \left( t^{s_1}(1-t)^{s_2} + t^{s_2}(1-t)^{s_1} \right) \left[ \langle f(A)x, x \rangle \langle g(A)x, x \rangle + \langle f(B)x, x \rangle \langle g(B)x, x \rangle \right] \\
\quad + \left( (1-t)^{s_1+s_2} + t^{s_1+s_2} \right) \left[ \langle f(A)x, x \rangle \langle g(B)x, x \rangle + \langle f(B)x, x \rangle \langle g(A)x, x \rangle \right] \right\}.
\]

By integration over \([0,1]\), we obtain

\[
\left\langle f\left(\frac{A + B}{2}\right) x, x \right\rangle \left\langle g\left(\frac{A + B}{2}\right) x, x \right\rangle \\
\leq \frac{1}{2^{s_1+s_2}} \left( \int_0^1 \left[ \langle f(tA + (1-t)B)x, x \rangle \langle g(tA + (1-t)B)x, x \rangle \\
\quad + \langle f((1-t)A + tB)x, x \rangle \langle g((1-t)A + tB)x, x \rangle \right] dt \\
\quad + 2\beta(s_1 + 1, s_2 + 1)M(A, B)(x) + \frac{2}{s_1 + s_2 + 1} N(A, B)(x) \right) .
\]

This implies the required inequality (2.10). \(\square\)
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