Photon Magnetic Moment and Vacuum Magnetization in an Asymptotically Large Magnetic Field

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We consider the effect of the photon radiative correction on the vacuum energy in a superstrong magnetic field. The notion of a photon anomalous magnetic moment is analyzed and its connection with the quasiparticle character of the electromagnetic radiation is established. In the infrared domain the magnetic moment turns out to be a vector with two orthogonal components in correspondence with the cylindrical symmetry imposed by the external field. The possibility of defining such quantity in the high energy limit is studied as well. Its existence suggests that the electromagnetic radiation is a source of magnetization to the whole vacuum and thus its electron-positron zero-point energy is slightly modified. The corresponding contribution to the vacuum magnetization density is determined by considering the individual contribution of each vacuum polarization eigenmode in the Euler-Heisenberg Lagrangian. A paramagnetic response is found in one of them, whereas the remaining ones are diamagnetic. Additional issues concerning the transverse pressures are analyzed.

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I. INTRODUCTION

Large magnetic fields $|\mathbf{B}| \gg B_c$, $B_c = m^2/e = 4.42 \cdot 10^{13}$G (m and e are the electron mass and charge, respectively) in the surface of stellar objects identified as neutron stars [1,2] might provide physical scenarios where quantum processes predicted in such a regime could become relevant for astrophysics and cosmology. According to quantum electrodynamics (QED) in strong background fields, the most important effects are likely to be pair creation [3,4], photon splitting [5,6] and photon capture [7–13]. The last two essentially depend on the drastic departure of the photon dispersion relation from the light cone curve, due to the vacuum polarization tensor $\Pi_{\mu\nu}$ which depends on both the Landau levels of virtual electron-positron pair, as well as on the external magnetic field. As a result, the issue of light propagation in empty space, in the presence of $\mathbf{B}$, is similar to the dispersion of light in an anisotropic “medium”, with the preferred direction corresponding to the external field axis. The phenomenological aspects associated with this issue has been studied for a long time and recently, the effects of the vacuum polarization tensor on the Coulomb potential in superstrong magnetic field have been considered as well [14–17]. However, the problem concerning the magnetism carried by a photon has not attracted sufficient attention, excepting [17], where an photon anomalous magnetic moment has been pointed out. The authors of this reference attempted to derive this quantity in two different regimes of the vacuum polarization tensor. On the one hand for low energies in weak fields ($|\mathbf{B}| \ll B_c$) and on the other hand (originally studied in Ref. [18]) near the first pair creation threshold and for a moderate fields ($|\mathbf{B}| \sim B_c$). However, it was ignored that the photon magnetic moment $m_\gamma$ actually is a vector defined as the coefficient of $\mathbf{B}$ when the energy is linearly approximated in term of the external magnetic field. In consequence, neither any discussion about the connection between this quantity and its angular momentum was presented nor any comment about its precession around the external field axis was made.

In the present work we reveal the asymptotic conditions where the concept of a photon anomalous magnetic moment may be adequate. We will analyze the cases of superstrong magnetic fields ($|\mathbf{B}| \gg B_c$) for both the low and the high energy limit of the vacuum polarization tensor. In these domains, one eigenvalue of $\Pi_{\mu\nu}$ depends linearly on the external field. In consequence, the Maxwell equations in the “medium” seem to describe a massless particle with a magnetic moment. In this case $m_\gamma$ may become a characteristic quantity since the external field generates a Lorentz symmetry break down which induces the nonconservation of the photon helicity. We shall see that this notion equips us with an intuitive tool to understand, in a phenomenological way, the quasiparticle behavior of a photon propagation mode in a highly magnetized vacuum.

The second purpose of this work is to show that virtual photons are a source of magnetization to the whole vacuum. We will address the question in which way the virtual electromagnetic radiation contributes to a measurement of the vacuum magnetization and therefore to increase the external field strength. This aspect might be important for astrophysics since the origin and evolution of magnetic fields in compact stellar objects remains poorly understood [19]. Some investigations in this area provide theoretical evidences that $|\mathbf{B}|$ is self-consistent due to the Bose-Einstein condensation of charged and neutral boson gases in a superstrong magnetic field [20–22]. In this context, the nonlinear QED-vacuum possesses the properties of a paramagnetic medium and seems to play an important role within the process of...
magnetization in the stars. Its properties have been studied also in [26, 29] for weak ($|B| \ll B_c$) and moderate fields ($|B| \sim B_c$) in one-loop approximation of the Euler-Heisenberg Lagrangian [30, 31]. New corrections emerge by considering the two-loop term of this effective Lagrangian [30, 31] which contains the contribution of virtual photons created and annihilated spontaneously in the vacuum and interacting with the external field through the vacuum polarization tensor.

The two-loop term of the Euler-Heisenberg Lagrangian was computed many years ago by Ritus [31, 34]. A few years latter, Dittrich and Reuter [32] obtained a simpler integral representation of this term and showed that their results agreed with those determined by Ritus in the strong magnetic field limit. In the last few years, it has been recalculated by several authors using the worldline formalism [33, 34] and it has been extended to the case of finite temperature as well [35]. Nevertheless, in all these works it is really cumbersome to discern the individual contributions given by each virtual photon propagation mode for very large magnetic fields ($|B| \gg B_c$). Whilst the complete two-loop contribution is purely paramagnetic we find a diamagnetic response coming from that $\Pi_{\mu\nu}$-eigenvalue which is used to obtain the photon anomalous magnetic moment.

In order to expose our results we have structured our paper as follows. In Sect. II we recall some basic features of photon propagation in an external magnetic field. Some aspects of the Noether currents associated with the spacetime symmetries in presence of an $B$ are discussed in Sect. III. The corresponding calculations are carried out in a basis that diagonalizes the polarization tensor. The possibility to define a photon anomalous magnetic moment in an asymptotically large magnetic field is studied in Sect. IV. We shall show that $m_\gamma$ turns out to be a vector with two orthogonal components in correspondence with the cylindrical symmetry imposed by the external field. Moreover, we will see that a nonvanishing torque exerted by $B$ might generates a precession of $m_\gamma$, which is a manifestation of the reduction of the rotation symmetry and, consequently Lorentz symmetry. In Sect. V we perform the calculation of the two-loop contributions to the Euler-Heisenberg Lagrangian given by each virtual mode. From them we obtain the modified vacuum configuration for very large magnetic fields in Sect. VI. The corresponding vacuum magnetizations and magnetic susceptibilities are analyzed as well as the transversal pressure. There is additional discussion given in the conclusions while essential steps of many calculations have been deferred to the appendices.

II. GENERAL REMARKS

A. Symmetry reduction and diagonal decomposition of vacuum polarization tensor

The standard relativistic description of a photon reflects the underlying symmetry of the Poincaré group (ISO(3,1)). Its irreducible representations are characterized by two observable degrees of freedom, corresponding to the helicity values $\lambda = J \cdot k/|k| = \pm 1$. Here, $J$ encodes the generators of the 3-dimensional rotation group and $k$ is the photon momentum. In the presence of an external electromagnetic field, space-time is no longer isotropic and Poincaré invariance breaks down. In a reference frame where the field is purely magnetic, the symmetry breaking for neutral particles has the following pattern:

$$\text{ISO}(3,1) + B \rightarrow \text{ISO}(2) \times \text{ISO}(1,1).$$

The effective physical configuration of Minkowski space manifests itself as the direct product of the 2-dimensional Euclidean group ISO(2) and the (1 + 1)-dimensional pseudoeuclidean group ISO(1,1). The two resulting symmetry groups are associated with the transversal and pseudoparallel planes with respect to $B$-direction. The conserved quantities of any uncharged relativistic quantum field within this reference frame are related to the Casimir invariants of this direct product of groups [30].

For the electromagnetic radiation this type symmetry reduction occurs as soon as radiative corrections are considered. In such a case an observable photon interacts with the external magnetic field through the virtual electron-positron ($e^\pm$) pair whose Green’s functions $\mathcal{G}(x,x'|B)$ determine the vacuum polarization tensor $\Pi_{\mu\nu}(x,x'|B) = e^2 \mathrm{Tr} \left[ \gamma_\mu \mathcal{G}(x,x'|B) \gamma_\nu \mathcal{G}(x',x|B) \right]$. Consequently, the photon behaves like a quasiparticle embodying both radiation and $e^\pm$ properties, the latter quantized by the Landau levels.

The diagonalization of $\Pi_{\mu\nu}$ is expressed as

$$\Pi_{\mu\nu} = \sum_{i=0}^{4} \kappa_i(z_1, z_2, \Delta) \frac{a_{\mu}^{(i)} a_{\nu}^{(i)}}{(a^{(i)})^2},$$

with its renormalized eigenvalues $\kappa_i$ depending on the scalars, $z_1 = k F^{+}\kappa/(2\Delta)$ and $z_2 = -k F^{+}\kappa/(\Delta\Delta)$, which together with $k^2 = z_1 + z_2$ form the Casimir invariants of the ISO(2) × ISO(1,1) Lie algebra. Here $F^{\mu\nu} = 1/2 \varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$ represents the dual of $F_{\mu\nu}$, whereas $\Delta = 1/4 F_{\mu\nu} F^{\mu\nu} = 1/2 |B|^2$ is one of the external field invariants (the remaining one, $\Phi = 1/4 F_{\mu\nu} F^{\mu\nu}$, vanishes identically).

In one-loop approximation, the eigenvalues of $\Pi_{\mu\nu}$ read

$$\kappa_1 = \frac{1}{2} k^2 \mathcal{J}_1, \quad \kappa_2 = \frac{1}{2} (z_1 \mathcal{J}_2 + z_2 \mathcal{J}_1), \quad \kappa_3 = \frac{1}{2} (z_1 \mathcal{J}_3 + z_2 \mathcal{J}_3),$$

from which the Casimir invariants $z_1, z_2, \Delta$ could be expressed 

$$z_1 = \frac{1}{2} k^2 \mathcal{J}_1, \quad z_2 = \frac{1}{2} (z_1 \mathcal{J}_2 + z_2 \mathcal{J}_1), \quad \Delta = \frac{1}{2} (z_1 \mathcal{J}_3 + z_2 \mathcal{J}_3).$$

These relations allow to obtain an explicit expression for $\Pi_{\mu\nu}$, which is given by

$$\Pi_{\mu\nu} = \sum_{i=0}^{4} \frac{1}{2} (a_{\mu}^{(i)} a_{\nu}^{(i)}) (a^{(i)})^2.$$
with
\[ S_i = \frac{4\alpha}{\pi} \int_0^\infty \int_0^{\pi} \sin \theta \cos \phi \, d\theta \cos \phi \, d\phi \, d\tau \left\{ -z_1 \frac{2}{eB} \left[ M(s, \eta) - N(s, \eta) \right] \right\}, \]

Here and in the following \( s \equiv eB\tau, \)

\[ \sigma_1(s, \eta) = \frac{1}{4} \sinh(s) \cosh(\eta s) - \eta \sinh(\eta s) \cosh(s), \]

\[ \sigma_2(s, \eta) = \frac{1}{4} - \eta^2 \cosh(s), \quad \sigma_3(s, \eta) = \frac{M(s, \eta)}{\sinh(s)}, \quad \sigma_4(s, \eta) = \frac{N(s, \eta)}{\sinh(s)}. \]

M(s, \eta) = \frac{\cosh(s) - \cosh(\eta s)}{2 \sinh(s)}, \quad N(s, \eta) = \frac{1 - \eta^2}{4} s.

The four vector \( a_{\mu}^{(i)} \) in Eq. (2) denotes the corresponding eigenvectors of \( \Pi_{\mu \nu} \):

\[ a_{\mu}^{(1)} = \frac{k^2 F^2_{\lambda \mu} k^\lambda - (k F^2) k_\mu}{k^2 - (k F^2) k_\mu}, \quad a_{\mu}^{(2)} = \frac{F_{\mu \lambda}^a k^\lambda}{(2\Delta)^{1/2}}, \]

\[ a_{\mu}^{(3)} = \frac{F_{\mu \lambda}^a k^\lambda}{(-k F^2)^{1/2}}, \quad a_{\mu}^{(4)} = \frac{k^\mu}{k^2}, \]

which fulfill both the orthogonality condition: \( a_{\mu}^{(i)} a^{* (j)}_{\nu} = \delta^{ij} (a^{(i)})^2 \) and the completeness relation: \( \delta^{\mu \nu} = \sum_{i=1}^{4} a^{\mu (i)} a^{\nu (i)} / (a^{(i)})^2 \).

Owing to the transversality property \( (k^\mu \Pi_{\mu \nu} = 0) \), the eigenvalue corresponding to the fourth eigenvector vanishes identically \( (\omega^{(4)} = 0) \). Consequently, the photon propagator can be decomposed as

\[ \mathcal{D}_{\mu \nu} = \sum_{i=1}^{3} \frac{1}{k^2 - \omega^{(i)}^2} \frac{a_{\mu}^{(i)} a_{\nu}^{(i)}}{k^2}, \]

with \( \zeta \) being the gauge parameter.

According to Eq. (7) three nontrivial dispersion relations arise

\[ k^2 = \omega_0^2(z_2, z_1, \Delta) \quad \text{for} \quad i = 1, 2, 3. \]

We should keep in mind that the general structures of \( z_2 \) and \( z_1 \) are complicated (see Appendix A) since both of them depend on the relative \( B \)-orientation with respect to a reference frame. Substantial simplifications are achieved in reference frames which are either at rest or moving parallel to the external field. In these cases \( z_2 = k_2^2 \) and \( z_1 = k_1^2 - \omega^2 \). Here \( k_\perp \) and \( k_\parallel \) are the components of \( k \) perpendicular and along the external field respectively, with \( k^2 = k_\perp^2 + k_\parallel^2 - \omega^2 \). Note that, as a consequence, \( z_i \) depend on both \( \Delta = 1/2(B_2^2 + B_3^2 + B_2^2) \) and on the \( B \)-direction with respect to our reference frame.

Solving Eq. (9) for \( k^2 = k^2 - \omega^2 \) in terms of \( z_2 \) yields

\[ \omega_0^2 = k^2 + m_0^2(z_2, \Delta), \]

with the term \( m_0 \) arising as a sort of dynamical mass. Note that \( m_0 \) vanishes for \( z_2 = 0 \) due to the gauge invariance condition \( \omega(0, 0, \Delta) = 0 \). Obviously, the dispersion law given in Eq. (9) differs from the usual light cone equation. This difference increases near the free pair creation thresholds remarking the quasiparticle feature of a photon in an external magnetic field. For more details we refer the reader to Ref. [2].

By considering \( a_{\mu}^{(i)}(k) \) as the electromagnetic four-vector describing the eigenmodes, we obtain the corresponding electric and magnetic fields of each mode

\[ e^{(i)} = i(\omega^{(i)} a^{(i)} - k_0^{(i)}) \quad \text{and} \quad h^{(i)} = -i k \times a^{(i)}. \]

Up to a non essential proportionality factor, they are explicitly given by:

\[ e^{(1)} = -i n_\perp \omega, \quad h^{(1)} = i k_\perp \times n_\perp, \]

\[ e^{(2)} = i k_\parallel k_\perp, \quad h^{(2)} = i n_\parallel (k_\perp - \omega^2), \]

\[ e^{(3)} = i n_\parallel (n_\perp - n_\parallel), \quad e^{(4)} = i n_\parallel (k_\perp - n_\perp) \quad \text{and} \quad h^{(3)} = -i n_\perp k_\perp. \]

Here, \( n_\perp = k_\perp / |k_\perp| \) and \( n_\parallel = k_\parallel / |k_\parallel| \) are the unit vectors associated with the parallel and perpendicular direction with respect \( B \).

III. NOETHER CURRENTS OF THE RADIATION FIELD IN PRESENCE OF \( B \)

A. The Poynting vector and the physical propagation modes

Whenever the Minkowski space is translations invariant, the associated Noether current of the electromagnetic radiation \( (A_\mu(x)) \) provides a conserved stress-energy tensor:

\[ T^{\mu \nu} = \frac{\delta^{\mu \lambda} \delta^{\nu \sigma}}{4} \lambda - \frac{1}{4} \eta^{\mu \nu} \delta^{\lambda \sigma} \delta^{\sigma \lambda}. \]

Here \( \delta^{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) is the electromagnetic tensor, whereas the metric tensor \( \eta^{\mu \nu} \) has signature \( + + + - \) with \( \eta^{11} = \eta^{22} = \eta^{33} = -\eta^{00} = 1 \). In this context the momentum density is defined by \( \mathcal{P}^{\mu \nu} \equiv T^{0 \mu \nu}. \) Note that the symmetry reduction by the external magnetic field does not alter the translational group involved within ISO(3, 1). Therefore, all \( \mathcal{P}^{\mu \nu} \)-components are conserved.

The spatial part of \( \mathcal{P} \) is the Poynting vector density

\[ \mathcal{P} = E \times H. \]

For each eigenmode it reads \( \mathcal{P}^{(i)} = e^{(i)} \times h^{(i)}. \) In particular

\[ \mathcal{P}^{(1)} = \omega k_\parallel, \quad \mathcal{P}^{(2)} = \omega k, \quad \mathcal{P}^{(3)} = \omega k_\perp, \quad \mathcal{P}^{(4)} = \omega |k|^2 k_\perp. \]
Different photon degrees of freedom contribute in the presence of $B$, depending on the direction of wave propagation: for a pure longitudinal propagation $k \parallel B$, the Poynting vector $p^{(2)} = 0$. As a consequence, eigenmode 2 is a pure longitudinal non physical electric wave and does not carry energy. On the other hand, the first and third mode have well-defined Poynting vectors along the external field $p^{(1)} = p^{(3)}$. In this case, each set \{e^{(1)}, h^{(1)}, p^{(1)}\} and \{e^{(3)}, h^{(3)}, p^{(3)}\} forms an orthogonal set of vectors and represent waves polarized in the transverse plane to $B$. Consequently, for pure parallel propagation $a_\mu^{(1)}$ and $a_\mu^{(3)}$ represent physical waves.

Now, if the photon propagation involves a nonvanishing transversal momentum component $k_\perp \neq 0$, we are allowed to perform the analysis in a Lorentz frame that does not change the value $k_\parallel$, but gives $k_\perp = 0$ and does not introduce an external electric field. As a consequence, the energy flux of the first eigenmode $p^{(1)} = 0$ and becomes purely electric longitudinal and a nonphysical. In the same context each set \{e^{(2)}, h^{(2)}, p^{(2)}\} and \{e^{(3)}, h^{(3)}, p^{(3)}\} forms an orthogonal set of vectors and represent waves polarized in the transverse plane to $B$. Hence, for a photon whose three-momentum is directed at any nonzero angle with the external magnetic field, the two orthogonal polarization states $a_\mu^{(2)}$ and $a_\mu^{(3)}$ propagate. Note that mode 3 represents a physical wave independent of the direction of propagation.

B. The spin and boost of the eigenwaves

The Noether current associated with rotational invariance is usually split into two pieces: orbital angular momentum density and intrinsic spin density. For an electromagnetic field, the latter can be derived from the third rank tensor

$$S^{\nu}_{\alpha\beta}(x) = \frac{\partial L_0}{\partial (\partial^\rho A^\nu_{\alpha\beta}(x))} (J_{\alpha\beta})^\nu_\rho A^\sigma(x) \quad (15)$$

with $L_0 = -\frac{1}{4} F^\mu_{\nu\rho\sigma} F^{\mu\nu}_{\rho\sigma}$ being the free part of the QED-Lagrangian. Here $(J_{\alpha\beta})_{\nu}^\rho = -i \left( \delta^\rho_{\alpha} \eta_{\beta\nu} - \delta^\rho_{\beta} \eta_{\alpha\nu} \right)$ denotes the four-dimensional representation of the Lorentz generators. By considering the previous definition we can express Eq. (15) as

$$S^{\nu}_{\alpha\beta}(x) = -i \left[ \delta^\nu_{\alpha} A^\beta(x) - \delta^\nu_{\beta} A^\alpha(x) \right] \quad (16)$$

We fix $\mu = 0$ in Eq. (16) and take into account the spatial part of the remaining tensor. Under such a condition the intrinsic angular momentum density of the electromagnetic field is reduced to

$$S^{0}_{ij} = -i \left[ E_i(x) A_j(x) - E_j(x) A_i(x) \right], \quad (17)$$

with $E_i(x) = \tilde{h}^{0i}$ being the electric field. The spatial components defined from \frac{1}{2} e^{\nu\rho} S^{0}_{ij} with $e^{123} = 1$ define the classical spin density of the electromagnetic field

$$S(x, t) = -i E(x, t) \times A(x, t). \quad (18)$$

For $\mu = \beta = 0$ we obtain the corresponding Noether current related to the boost transformations, which is given by $j_b(x, t) = i E(x, t) A_0(x, t)$.

In order to analyze the behavior of $S$ and $j_b$ in presence of $B$ we regard $e^{(i)}$ (see Eq. (10)) as the electric field associated with the fourth potential $(A^{(i)})$. Up to a nonessential factor of proportionality we have

$$s^{(i)} = -i e^{(i)} \times a^{(i)} = -i a_0^{(i)} h^{(i)} \quad \text{and} \quad \kappa^{(i)} = i a_0^{(i)} e^{(i)}. \quad (19)$$

Manifestly Eq. (19) expresses the connection between the Noether currents and the different polarization planes associated with each eigenmode. In particular

$$s^{(1)} = \frac{\omega}{k^2} (k_\parallel \times k_{\perp}), \quad \kappa^{(1)} = \frac{\omega^2}{k^2} k_{\perp},$$

$$s^{(2)} = \omega (k_\parallel \times k_{\perp}), \quad \kappa^{(2)} = k_{\perp} k_{\parallel}, \quad k_\parallel = k_\parallel (k_{\parallel} - \omega^2),$$

$$s^{(3)} = 0, \quad \kappa^{(3)} = 0. \quad (20)$$

According to the reduced Lorentz symmetry, i.e. Eq. (11), only the parallel components $s^{(1)}$ and $\kappa^{(1)}$ are related to conserved quantities and therefore, just the component of the electric and magnetic field along $B$ can generate conserved charges. However, for a purely parallel propagation ($k_\perp = 0$), this connection is rather obscure due to the absence of $e^{(1, 3)}_\parallel$ and $h^{(1, 3)}_\parallel$. In this case the first and third mode may be combined to form a circularly polarized transversal wave which is allowed by the degeneracy property:

$$k_2(z_1, 0, \tilde{\theta}) = k_3(z_1, 0, \tilde{\theta}). \quad (21)$$

In this context the photon is labeled by the helicity $\lambda = J^\parallel \cdot n^\parallel$ which seems to be a conserved quantity. For nonvanishing transversal propagation $k_\perp \neq 0$, however, $\lambda$ stops being a well-defined quantum number due to the nonconserved rotations transversal to $B$.

C. Connection between helicity and spin for pure parallel propagation

In order to establish the connection between helicity and classical spin of the electromagnetic field we express $S^{\mu}_{\alpha\beta} = S^{\mu+}_{\alpha\beta} + S^{\mu-}_{\alpha\beta}$. Here

$$S^{\pm}_{\alpha\beta} = \frac{i}{2} \left( \tilde{S}_{\alpha\beta}^{\mu+} \pm i \tilde{S}_{\alpha\beta}^{\mu-} \right) A_\beta - \frac{i}{2} \left( \tilde{S}_{\alpha\beta}^{\mu+} \pm i \tilde{S}_{\alpha\beta}^{\mu-} \right) A_\alpha \quad (22)$$

and $\tilde{S}_{\alpha\beta}^{\mu+}$ is the dual of $\tilde{S}_{\alpha\beta}^{\mu}$. Adopting a similar procedure to those developed below Eq. (16) the vectors read

$$S^{\pm} = -\frac{i}{2} \left( E \pm i H \right) \times A. \quad (23)$$
The complex fields $\frac{1}{2}\{E \pm iH\}$ transform irreducibly under spin $(1,0)$ and $(0,1)$ representation of $SO(3,1) \sim SU(2) \times SU(2)$, respectively. These field combinations fulfill the free Maxwell equation for left- and right circularly polarized radiation

$$\nabla \times (E \pm iH) \mp \frac{i}{\partial c}(E \pm iH) = 0$$

(24)

corresponding to $\lambda = \mp 1$.

For propagation purely parallel to the external magnetic field we define the electric field $e^{(c)} = e^{(1)} + e^{(3)}$ and magnetic field $h^{(c)} = -ik \times a^{(c)} = h^{(1)} + h^{(3)}$ associated with $a^{(c)} = a^{(1)} + a^{(3)}$, respectively. At this point it is meaningful to analyze

$$s^{(c)} = -ie^{(c)} \times a^{(c)} \quad \text{and} \quad p^{(c)} = e^{(c)} \times h^{(c)}.$$

(25)

Inserting the explicit expression of $e^{(c)}$, $h^{(c)}$ and $a^{(c)}$ in Eq. (25) we get

$$s^{(c)} = s^{(1)} + \frac{\omega k_\perp}{k^2} \left(n_\parallel k_\perp - n_\perp k_\parallel\right),$$

$$p^{(c)} = p^{(1)} + p^{(3)} - \omega (k_\perp \times n_\parallel).$$

(26)

Note that

$$\lim_{k_\perp \to 0} s^{(c)} = 0 \quad \text{and} \quad \lim_{k_\perp \to 0} p^{(c)} = 2\omega k_\parallel.$$

(27)

The resulting limits are expected for a circularly polarized wave.

Further considering Eq. (28) we obtain

$$s^\pm = -\frac{i}{2} \left(e^{(c)} \pm iH^{(c)}\right) \times a^{(c)}$$

$$= \frac{1}{2} s^{(c)} \mp \frac{1}{2} h^{(c)} \times a^{(c)}$$

(28)

Note that

$$h^{(c)} \times a^{(c)} = ik + \frac{k_\perp^2 - \omega^2}{k^2} k_\parallel + i \frac{k_\parallel^2}{k^2} k_\perp - \frac{i \omega^2}{k^2} (k_\perp \times n_\parallel).$$

The above relation was obtained by inserting the explicit form of $a^{(c)}$, $h^{(c)}$ and $a^{(c)}$. Therefore, the angular momentum associated with left and right circular polarization are represented by

$$\lim_{k_\perp \to 0} s^\pm \simeq \mp k_\parallel.$$

(29)

Regarding the normalized version of the above limit ($s^\pm \equiv \mp n_\parallel$) as the angular momentum used to define the helicity, we can write $\lambda = s^\pm \cdot p^{(c)} = \mp 1$, where $p^{(c)} = n_\parallel$ is the normalized version of the second limit computed in Eq. (27).

D. Spin density of a free electromagnetic field for perpendicular propagation

Let us consider a free photon propagating perpendicular to the external magnetic field ($\omega (2,3) = |k|$). In this case, the behavior of the photon spin density reads

$$s^{(1)} = -ie^{(1)} \times a^{(1)} = -ia_0^{(1)} h^{(1)}.$$

(30)

Here $e^{(1)} = e^{(2)} + e^{(3)}$ and $h^{(1)} = H^{(2)} + H^{(3)}$ denotes the electric and magnetic field associated with $a_0^{(1)} = a_0^{(2)}$ and $a_0^{(3)}$, respectively. Because of $a_0^{(2)} = -k_\parallel$, $a_0^{(3)} = 0$ and Eq. (11), we find

$$s^{(1)} = s^{(2)} = n \times s^{(2)}$$

(31)

with $n = k/|k|$ being the corresponding wave vector.

Additionally, taking Eq. (27) into account we introduce the vector

$$s^\gamma = \begin{cases} 0 & \text{for } k_\perp = 0 \\ s^{(2)} - n \times s^{(2)} & \text{for } k_\perp \neq 0 \end{cases}.$$

(32)

with $s^{(2)} \equiv s^{(2)} / |s^{(2)}| = (n_\parallel \times n_\perp)$. The above expression is total spin density, which itself depends on the direction of propagation.

Because of the fact that $s^{(2)} \cdot k = 0$, the total spin density of the electromagnetic field is orthogonal to the wave vector ($s_\gamma \cdot n = 0$). We remark that both $s^{(2)}$ and $n \times s^{(2)}$ are orthogonal to each other. Note that for transversal propagation the total spin density of the electromagnetic field has a parallel component to $B$ given by $s_\gamma = -\frac{k}{|k|^2} n_\parallel$.

IV. PHOTON MAGNETIC MOMENT

In this section we explore the possibility that a photon might carry a magnetic moment. We will restrict ourselves to asymptotically large magnetic field $b = |B|/c \gg 1$. In this limit the eigenvalue of the second propagation mode contains a term linearly growing with the magnetic field strength at low and high energy limits. Both cases deserve a separate study because the $\propto 2$-structures differs from one to the other energy domain. Under the same conditions $\propto 2$ and $\propto 3$ cannot create a virtual electron-positron pair in the ground state [11]. These show logarithmic dependences on $|B|$ and their corresponding physical dispersion laws are independent of the external field (for details see Ref. [13]). Therefore the cases concerning to the first and third propagation mode are not relevant in the current context. So, in this section, we will analyze the effects produced due to the second eigenmode.

A. Infrared structure of $\propto 2$: Covariant decomposition of the photon interaction energy

In the limit $m^2 b \gg m^2 \gg \omega^2 - k_\perp^2$ with $m^2 b \gg k_\parallel^2$, the second eigenvalues of $\Pi_{\mu\nu}$ shows a linear function on
the external field strength
\[ \varkappa_2^R = -\varrho(b) z_1 = -\frac{\alpha}{3\pi} \frac{e^2}{m^2} F_{\mu\nu} k^\mu a^{(2)}_{\nu}, \] (33)
with \( \varrho(b) = \frac{\alpha}{3\pi} b \) and \( \alpha = e^2/4\pi = 1/137 \) being the fine-structure constant. Note that we have used the decomposition: \( z_1 = k^\mu F_{\mu\nu}^* F_{\nu\lambda}^* k_\lambda/(2\Delta) = k^\mu F_{\mu\nu}^* a^{(2)}_{\nu}/(2\Delta)^{1/2}. \) The expression for \( \varkappa_2^R \) is analogous to that of the invariant interaction energy of the electron [24]:
\[ \varepsilon = \frac{e^2}{2m^2} F_{\mu\nu} p^\mu p^\nu S^\nu. \] (34)
Here \( p^\mu \) is the electron fourth momentum and \( S^\mu = \gamma^5 (\gamma^\mu - \frac{\Pi}{\alpha m}) \) is the electron spin with \( \Pi^\mu = p^\mu + \frac{1}{2} e F_{\mu\nu} x^\nu. \) In the electron rest frame, \( \varepsilon \) describes the interaction energy between the electron magnetic moment and the external magnetic field. In the photon case the structure of Eq. (33) shows that the electron spin \( S^\mu \) is replaced by the photon polarization \( a^{(2)}_{\mu} \), describing the intrinsic rotation of the photon in the plane perpendicular to the external field. Note, however, that the absence of a photon rest frame prevents the definition of a photon spin, unlike the electron case. To avoid this problem and for further convenience we will analyze the photon propagation by investigating its dispersion law Eq. (8), which reads
\[ \omega_2^2 = k^2 - \varrho(b) (1 + \varrho(b))^{-1} z_2. \] (35)
For magnetic field strength \( 10 < b \ll 3\pi/\alpha \) one should treat \( \varrho(b) \) as small. The expansion of \( \omega \) up to first order in \( \varrho(b) \) gives the dispersion law
\[ \omega_2 = |k| - \frac{1}{2} \varrho(b) z_2/|k|. \] (36)
Obviously, the first term in Eq. (36) corresponds to the light cone equation whereas the second arises due to the dipole moment contribution of the virtual electron-positron pair. In this approximation the dispersion law does not essentially deviate from its vacuum shape and its interacting term grows linearly with the external magnetic field. This fact attracts our attention because it seems that a magnetic moment may be ascribed to a mode-2 photon.

Hereafter we will denote the interaction energy by
\[ \mathcal{W} = \frac{1}{2} \varrho(b) z_2 = \frac{\alpha}{6\pi} \frac{e^2}{m^2} \frac{|z_2|^{1/2}}{|k|} F_{\mu\nu} k^\mu a^{(3)}_{\nu} \] (37)
where the decomposition \( z_2 = -k^\mu F_{\mu\nu}^* F_{\nu\lambda}^* k_\lambda/(2\Delta) = (z_2)^{1/2} k^\mu F_{\mu\nu}^* a^{(3)}_{\nu}/(2\Delta)^{1/2} \) has been used. Note that \( (z_2)^{1/2} = k_\perp \) in reference frames which are at rest or moving parallel to \( B \). With this in mind, we find by symmetrization of Eq. (37)
\[ \mathcal{W} = -\frac{1}{2} \mathcal{M}_{\mu\nu} F^{\mu\nu} \] with \( \mathcal{M}_{\mu\nu} = -ig \frac{e}{2m} f[k_\perp] S_{\mu\nu}. \) (38)
Here \( g = 4\pi \) is a kind of Landé factor, whereas \( f[k_\perp] \equiv k_\perp/|k_\perp| \). In addition, \( S_{\mu\nu} = \frac{3}{2\pi} f[(k_\perp)] \) is tensor with \( \frac{3}{2\pi} f[(k_\perp)] \equiv -ik_{\mu} a^{(3)}_{\nu} + ik_{\nu} a^{(3)}_{\mu} \) referring to the antisymmetric electromagnetic tensor generated by the third propagation mode. Note that \( a^{(3)}_{\nu} = e i k_{\nu} k_\mu. \)

The spatial part of \( \mathcal{M}_{\mu\nu} \) can be written as:
\[ \mathcal{M}_{ij} = \frac{g e}{2m} f[k_\perp] \frac{b^{(3)}_{i}}{|k|} F_{j}^{(k)} \] (39)
Here \( F_{j}^{(k)} = -i e \delta_{ilm} \) is the 3-dimensional representation of the SO(3) -generators, fulfilling \[ [\mathcal{J}^{(i)}, \mathcal{J}^{(j)}] = i e J^{jk} \mathcal{J}^{(k)} \] and \( (\mathcal{J}^{(i)})^2 = \mathcal{J}^{(i)} \mathcal{J}^{(i)} = 2\delta^{(i)} \).

In a Lorentz frame where \( F_{\mu\nu} \) is purely magnetic the structure of Eq. (38) is expanded to
\[ \mathcal{W} = -\frac{g e}{2m} f[k_\perp] \left[ n_i \sin \phi - n_\perp \cos \phi \right] \cdot B, \] (40)
where \( 0 \leq \phi \leq \pi \) is the polar angle between the wave vector \( n \) and the external field (\( \tan \phi = k_\perp/k_\parallel \)). The expression inside the brackets is a unit vector orthogonal to the direction of propagation. We then write the interaction energy as
\[ \mathcal{W} = -m_\gamma \cdot B \] (41)
with \( m_\gamma = \frac{g e}{2m} f[k_\perp] \left( n \times s^{(2)} \right). \) (42)

The structure of the interaction energy is similar to a potential energy of a magnetic dipole in an external magnetic field. Thus, in first approximation (with respect \( \varrho(b) \)), the second propagation mode in a superstrong magnetic field seems to behave as a quasiparticle having a magnetic moment \( m_\gamma \). This behavior occurs in some scenarios of condensate matter and has allowed to introduce the concept of polariton. The latter results from strong coupling of electromagnetic waves with a magnetic dipole-carrying excitation.

According to Eq. (40), this magnetic moment is the sum of two orthogonal components, as dictated by the cylindrical symmetry of our problem: the first one along the external field direction
\[ m_{\gamma \parallel} = \frac{g e}{2m} f[k_\perp] n_\parallel \sin \phi, \] (43)
whereas the second one is perpendicular to \( B \)
\[ m_{\gamma \perp} = -\frac{g e}{2m} f[k_\perp] n_\perp \cos \phi. \] (44)

Furthermore, both components of \( m_\gamma \) show opposite magnetic behavior: while the parallel one is essentially paramagnetic \( (m_{\gamma \parallel} > 0) \), the perpendicular one is purely diamagnetic \( (m_{\gamma \perp} < 0) \). They become nonmagnetic for \( k_\perp < 0 \) and become nonmagnetic for \( k_\perp < 0 \). This behavior occurs in some scenarios of condensate matter and has allowed to introduce the concept of polariton. The latter results from strong coupling of electromagnetic waves with a magnetic dipole-carrying excitation.

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the magnetic field. Note that for \(k_\perp \rightarrow 0\) the photon anomalous magnetic moment vanishes. This is a direct consequence of the gauge invariance of the eigenvalues of \(\Pi_{\mu\nu}\). Clearly, the effective magnetic interaction is related to \(m_\parallel\gamma\) which is invariant under rotation around the external field.

within the range of frequencies for which \(m_\parallel\) is displayed in Fig. 1 for purely perpendicular propagation with respect to \(B\). Note that in this case \(m_\gamma \cdot n = 0\). Within the range of frequencies for which \(m_\gamma\) is defined, it is 4 orders of magnitude smaller than the electron anomalous magnetic moment \(\mu' \sim \frac{e}{2m}\) \([3]\) but it is 12 orders of magnitude larger than the neutrino magnetic moment \(\mu_\nu = 10^{-10} \mu_B\) \([23]\). However, due to the astonishing experimental precision with which the anomalous magnetic moment of both the electron and muon are measured, there is some hope for an experimental -probably astronomical- measurement of the photon anomalous magnetic moment.

Note that although the interaction energy Eq. (41) seems to be linearized on the external field and \(m_\gamma\) is independent of \(|B|\) it turns out to be a vector depending on the external field direction. Therefore, if the photon anomalous magnetic moment is interpreted as \(\mu_\gamma = -\partial\omega/\partial B\), some differences with respect to \(m_\gamma\) are expected. However, in Appendix 3 we find that

\[\mu_\gamma \simeq m_\gamma.\]

We would like to bring the attention that this procedure identifies the magnetic moment only if the dispersion law depends linearly on the external magnetic field (for the electron case see Ref. 33-41). Otherwise \(\mu_\gamma\) must be understood as a sort of “photon magnetization” rather than a magnetic moment. However the latter notion equips us with an intuitive tool for qualitative analysis of the magnetization generated by a background of observable photons.

B. Discussion

We have seen that the external magnetic field leaves only the rotational symmetry around of \(B\) invariant. Consequently, \(m_\parallel\) is an invariant under the rotation around the axis of \(B\) while \(m_\perp\) does not. Note, in addition, that \(m_\perp\) does not contribute to the interaction energy. However \(m_\perp\) turns out to be relevant in another case: the external field does exert a torque on the magnetic dipole which tends to line up \(m_\gamma\) with \(B\). Indeed,

\[\tau_\gamma = m_\gamma \times B = m_\perp \times B = \frac{g}{2m} e f[k_\perp] |B| \cos \phi (45)\]

Because of the fact that the torque is collinear to \(s^{(2)}\), the transversal components of total angular momentum in that direction is not conserved. This is a manifestation of the reduction of the rotation symmetry and, consequently Lorentz symmetry. Therefore \(m_\gamma\) is a signal of a break-down of the Lorentz symmetry. This fact corroborates and extends the result presented in Ref. 12 which claims that a magnetic dipole moment of truly elementary massive neutral particles is a signal of Lorentz symmetry violation. Note, in addition, that \(\tau_\gamma\) is orthogonal to \(n \times s^{(2)}\) which might causes a precession of \(m_\gamma\) around \(B\) in analogy to the Larmor precession for electrons.

The Lorentz symmetry breaking implies that the photon helicity is no longer a conserved quantity due to rotations transversal to \(B\) (see Subsect. 1113). As a consequence, alternative conserved elements are needed to identify a photon interacting with \(B\). Once the magnetic moment is a coefficient of the linear contribution in the magnetic field it may become a characteristic quantity since one may attribute it an “spin.” However this is only a possible point of view and only further studies can tell how far the interpretation can be stretched. A realistic treatment of this problem requires a group theoretical analysis including \(m_\gamma\) similar to those developed by Wigner for a free photon \([38]\). The latter is beyond the scope of this manuscript, but a study in this direction is in progress.

Let us consider now a system consisting of \(N\) independent photons described by the dispersion equation

\[\omega = |k| - m_\gamma \cdot B\]

where we have considered Eq. 36. Eq. 37 and Eq. 41. In addition we will suppose that they propagate purely transversal to the external magnetic field. According to these assumptions, the total interaction energy of the system is

\[\mathcal{U}_{\text{total}} = - \sum_{i=1}^{N} m_\gamma^{(i)} |B| = -N\langle m_\gamma^{(i)}\rangle |B|\]

where \(m_\gamma^{(i)}\) denotes the magnetic moment of each photon. Here the average of the photon magnetic moment
is $\langle m_\parallel \rangle = \sum_{i=1}^N m_i^{(i)} / N \sim \langle k_\perp \rangle$. For a uniform photon distribution in the transversal plane to $B$, one expects that $\langle k_\perp \rangle = 0$ which implies that $\mathcal{W}_\text{total} = 0$. As a consequence, the system does not carry a magnetization $\mathcal{M}_\gamma = -\partial \mathcal{W}_\text{total} / \partial B$. If the system is considered as a monochromatic beam $\langle k_\perp \rangle \neq 0$, $\mathcal{W}_\text{total} \neq 0$ and

$$\mathcal{W}_\gamma = -\partial \mathcal{W}_\text{total} / \partial B = \mathcal{N} \langle \bar{m}_\parallel \rangle$$

(48)

In consequence the beam carries a nonzero magnetization which alters the external field $B$:

$$H_B = B + 4\pi (\mathcal{M}_\gamma + M_\text{vac})$$

Here $M_\text{vac} > 0$ is the vacuum magnetization (for details see Sect. VI). Within the magnetic field interval for which $\mathcal{M}_\gamma$ is defined, $|M_\text{vac}| \sim 10^{10} - 10^{12} \text{erg/}(\text{cm}^3 \text{G})$. Now, in the frequency range of x-rays ($\langle k_\perp \rangle \geq 150$ eV) the averaged magnetic moment $\langle m_\gamma \rangle \sim 10^{-2} \text{erg/G}$. In order to produce a photon magnetization of the order of $\sim |M_\text{vac}|$, a photon density of order $\rho_\gamma = \mathcal{N}/V \sim 10^{36} - 10^{38} \text{cm}^{-3}$ would be necessary. Moreover, for $\rho_\gamma \gtrsim 10^{42} \text{cm}^{-3}$ the magnetization carried by the beam is $|\mathcal{M}_\gamma| \sim 10^{15} \text{erg}/(\text{cm}^3 \text{G})$ and thus larger than $|M_\text{vac}|$ even more, it is 1 order of magnitude larger than the external magnetic field $|B| \sim 10^{14} \text{G}/\text{cm}^3$.

Finally, we want to conclude this subsection pointing out that Eq. (46) allows to determine the corresponding vacuum refraction index in terms of the photon magnetic moment:

$$n = |k| / \omega \simeq 1 + m_\gamma \cdot B / |k|.$$  

(49)

The latter expression depends on the direction of the photon momentum and reaches its maximum for a purely transversal propagation ($k_\parallel = 0$) in which case $n_{\text{max}} \simeq 1 + \sqrt{2m_\gamma / \pi e |B|}$. For $b \sim 100$, $n_{\text{max}} \approx 1.038$ which exceeds the values of typical gases at atmospheric pressure in absence of $B$. We want to remark that Eq. (49) might play an important role in the evaluation of the magnetic effect on gravitational lenses in the vicinity of highly magnetized stellar objects.

### C. Ultraviolet domain

Next we consider the high-energy regime. In the limit $m^2 b \gg \omega^2 - k_\parallel^2 \to \infty$ and $m^2 b \gg k_\perp^2$, the second eigenvalue approaches

$$\nu_2 = \frac{2\alpha}{\pi e |B|}.$$  

(50)

Here $m_2$ is the photon effective mass corresponding to the topological one in the 2-dimensional Schwinger model. Its derivation is closely related to the chiral limit in which the axial current is not conserved (for more details we refer the reader to Ref. 47). As a consequence, an ultraviolet photon seems to behave like a neutral massive vector boson whose mass is quasi-confined in $1 + 1$ dimensions

$$\omega \approx \left( k_2^2 + m_2^2 \right)^{1/2}. $$

(51)

In contrast to the previous case $\nu_2^{\text{UV}}$ does not depend explicitly on the polarization mode. This fact is also manifest in the dispersion law which cannot be linearized in the external magnetic field and therefore a photon magnetic moment is not expected. However, this kind of photon carries a magnetization density given by

$$\mu_\gamma = -\frac{\partial \nu}{\partial B} \bigg|_{B_\parallel \to 0} = -\frac{\alpha e}{\pi \omega} n_\parallel. $$

(52)

Note that $\mu_\gamma < 0$ behaves diamagnetically and depends on the external field strength. In particular for $k_\parallel^2 \ll m_2^2$

$$\mu_\gamma \simeq -\frac{\alpha e}{\pi m_2} n_\parallel = -\left( \frac{2\alpha}{\pi b} \right)^{1/2} \mu_B n_\parallel, $$

(53)

which tends to vanish for $|B| \to \infty$.

For $b \sim 10^3$ corresponding to magnetic field $|B| \sim 10^{16} \text{G}$ the photon magnetization of an ultraviolet radiation reaches values of the order of $\mu_\gamma \sim -\mu_B$. For the same magnetic field strength, an infrared photon propagating perpendicular to $B$ has $\mu_\parallel \sim \sqrt{\frac{2\alpha}{2m_e}} \phi(b)^{-5/2} \sim 10^{-13} \mu_B$ (see Appendix A). As a consequence, the magnetic response of a photon background composed by an equal number of infrared an ultraviolet radiations will be dominated by the ultraviolet contribution, which for mode -2 is diamagnetic. The latter behavior is also manifest in the contribution of this eigenmode to the vacuum magnetization density (see Sect. VI).

### V. TWO LOOP TERM OF THE EULER-HEISENBERG LAGRANGIAN

Virtual photons can interact with the external magnetic field by means of the vacuum polarization tensor. As a consequence they might be a source of magnetization to the whole vacuum. In what follows we compute this contribution for very large magnetic fields ($b \gg 1$), which might exist in stellar objects like neutron stars.

#### A. The unrenormalized contribution due to the vacuum polarization eigenvalues

We start our analysis with the Euler-Heisenberg Lagrangian

$$\mathcal{L}_{\text{EH}} = \mathcal{L}_R^{(0)} + \mathcal{L}_R^{(1)} + \ldots $$

(54)
with
\[ \mathcal{L}_R^{(0)} = -\frac{1}{2}B^2 \quad \text{and} \quad B = B_0Z^{-1/2}_{3(1\text{loop})}. \] (55)

Here \( \mathcal{L}_R^{(0)} \) is the free renormalized Maxwell Lagrangian in a Lorentz frame where the electric field vanishes, \( \mathbf{E} = 0 \). Hereafter \( B_0, m_0 \) and \( e_0 \) will be referred as the “bare magnetic field strength”, “bare electron mass” and “bare charge”, respectively. Without the index 0 these quantities must be understood as renormalized. On the other hand, \( \mathcal{L}_R^{(1)} \) is the regularized one-loop contribution due to the virtual electron-positron pairs created and annihilated spontaneously in vacuum and interacting with the external field \[ \mathbf{B}: \]

\[ \mathcal{L}_R^{(1)} = -\frac{1}{8\pi^2} \int_0^\infty \frac{d\tau}{\tau^3} e^{-m_0^2\tau} \left( \frac{s}{\coth(s)} - 1 - \frac{s^2}{3} \right) \] (56)

with \( s = eB\tau \) and \( e = e_0Z^{3/2} \). In this context, the one-loop renormalization constant is given by

\[ Z^{-1}_{3(1\text{loop})} = \lim_{\tau_0 \to 0} \left\{ 1 + \frac{\alpha}{3\pi} \ln \left( \frac{1}{\gamma m_0^2\tau_0} \right) \right\}, \] (57)

where \( \ln(\gamma) = 0.577 \ldots \) is the Euler constant.

The contribution due to virtual photons interacting with the external field by means of the vacuum polarization tensor is expressed as

\[ \mathcal{L}^{(2)} = \frac{i}{2} \int \frac{d^4k}{(2\pi)^4} \Pi_{\mu\nu}(k) \mathcal{D}^{\mu\nu}(k). \] (58)

Substitution of Eq. (2) and Eq. (7) into the latter expression gives

\[ \mathcal{L}^{(2)} = \frac{i}{2} \sum_{i=1}^3 \int \frac{d^4k}{(2\pi)^4} \frac{\kappa_i}{(k^2 - \kappa_i)}. \] (59)

Manifestly, this expression shows that each photon propagation mode contributes independently. The quantity \( \Pi_{\mu\nu}(k) \mathcal{D}^{\mu\nu}(k) = \sum_i \kappa_i(k^2 - \kappa_i)^{-1} \) represents the interaction energy between the full photon propagator with the vacuum polarization tensor. In order to obtain the leading term in an expansion in powers of \( e^2 \), i. e. the two-loop contribution as shown in Fig. (2), we just neglect \( \kappa_0 \) in the denominator of the photon propagator. Hereafter we consider this approximation, in which case

\[ \mathcal{L}^{(2)} = \sum_{i=1}^3 \mathcal{L}_{i}^{(2)} \quad \text{with} \quad \mathcal{L}_{i}^{(2)} = \frac{i}{2} \int \frac{d^4k}{(2\pi)^4} \frac{\hat{\kappa}_i}{k^2}. \] (60)

\[ \mathcal{L}_{EH} = \cdots \]

**FIG. 2:** Two-loop expansion of the Euler-Heisenberg Lagrangian. The solid lines represent the electron-positron Green’s functions, whereas the wavy line refers to the photon. Here \( \mathcal{L}^{(1)} \) is represented by the one-loop graph which gives the contribution of the virtual free electron-positron pairs created and annihilated spontaneously in vacuum and interacting with the external field. The radiative corrections (involved in \( \mathcal{L}^{(2)} \)) emerge from the two-loop graph due to exchange of virtual photons.

\[ \mathcal{L}^{(2)} = \cdots \]

**FIG. 3:** Diagrammatic decomposition of \( \mathcal{L}^{(2)} \) in terms of the vacuum polarization eigenmodes.

\[ \mathcal{L}_{i}^{(2)} = \frac{1}{32\pi^2} \int_0^\infty dz dz_1 \hat{\kappa}_i(z_2, z_1) e^{-s(z_2 + z_1)}. \] (61)

Substitution of \( \hat{\kappa}_i \) in Eq. (61) yields:

\[ \mathcal{L}_{i}^{(2)} = -\frac{\alpha_0}{32\pi^2} \int_0^\infty \frac{d\tau}{\tau^3} \exp(-m_0^2\tau) \int_0^1 d\eta \mathcal{Q}_i(s, \eta) \] (62)

with \( s = e_0B_0\tau \) and

\[ \mathcal{Q}_1(s, \eta) = \frac{\sigma_1}{\sinh s} \left\{ \mathcal{V}(s, \eta) + \mathcal{W}(s, \eta) \right\}, \] (63)

\[ \mathcal{Q}_2(s, \eta) = -\frac{\sigma_2}{\sinh s} \mathcal{V}(s, \eta) + \frac{\sigma_1}{\sinh s} \mathcal{W}(s, \eta), \] (64)

\[ \mathcal{Q}_3(s, \eta) = \frac{\sigma_1}{\sinh s} \mathcal{V}(s, \eta) + \frac{\sigma_3}{\sinh s} \mathcal{W}(s, \eta). \] (65)

The functions \( \mathcal{V}(s, \eta) \) and \( \mathcal{W}(s, \eta) \) are given by the following integral representations

\[ \mathcal{V}(s, \eta) = \int_0^\infty dz dz_1 z_1 e^{-s_2(\frac{\eta}{\sinh s} + \eta)} z_1(\frac{\eta}{\sinh s} + \eta), \] (66)

\[ \mathcal{W}(s, \eta) = \int_0^\infty dz dz_1 z_2 e^{-s_2(\frac{\eta}{\sinh s} + \eta)} z_1(\frac{\eta}{\sinh s} + \eta). \] (67)
These integrals can be explicitly calculated:

$$V(s, \eta) = \frac{2e_0^2 B_0^3}{(M - N) N} - \frac{2e_0^2 B_0^3}{(M - N)^2} \ln \left( \frac{M}{N} \right),$$

(68)

$$W(s, \eta) = \frac{2e_0^2 B_0^3}{(N - M) M} + \frac{2e_0^2 B_0^3}{(M - N)^2} \ln \left( \frac{M}{N} \right),$$

(69)

such that

$$V(s, \eta) + W(s, \eta) = 2e_0^2 B_0^3 M^{-1}(s, \eta) N^{-1}(s, \eta).$$

(70)

**B. Renormalized contributions due to the photon polarization modes**

While the $\tau$-integral in Eq. (62) does not diverge at $\tau \to \infty$, the integrand is singular for $\tau \to 0$. In order to regularize $\mathcal{L}_1^{(2)}$ we introduce a finite lower limit $\tau_0 > 0$ for the proper time integral of the electron propagators contained in the polarization tensor. As a consequence, we can write

$$\mathcal{L}_1^{(2)} = -\frac{\alpha_0}{32\pi^3} \int_{2\tau_0}^{\infty} d\tau e^{-m_0^2 \tau} \int_0^{\eta_0} d\eta \{ Q_1(s, \eta) - Q_{20}(\tau, \eta) \}$$

(71)

with $\eta_0 = 1 - 2\tau_0/\tau$ and

$$Q_{20}(s, \eta) = \frac{8(e_0 B_0)^3}{(1 - \eta^2)s^3}.$$  

(72)

The substraction of this term guarantees that $\mathcal{L}_1^{(2)} = 0$ for vanishing magnetic fields. Obviously, Eq. (71) differs from the original two-loop contributions in a term which is magnetic field independent and therefore a constant.

We proceed by adding and subtracting the functions $Q_{12}(s, \eta)$ in the integrand Eq. (71), such that

$$\mathcal{L}_1^{(2)} = -\frac{\alpha_0}{32\pi^3} \int_{2\tau_0}^{\infty} d\tau e^{-m_0^2 \tau} \int_0^{\eta_0} d\eta Q_{12}(s, \eta)$$

(73)

$$- \frac{\alpha_0}{32\pi^3} \int_{2\tau_0}^{\infty} d\tau e^{-m_0^2 \tau} \int_0^{\eta_0} d\eta \{ Q_1(s, \eta) - Q_2(s, \eta) \}$$

with

$$Q_{12}(s, \eta) = -\frac{2(e_0 B_0)^3}{3s}, \quad Q_{22}(s, \eta) = \frac{4(e_0 B_0)^3}{3s} \frac{1}{1 - \eta^2},$$

$$Q_{32}(s, \eta) = -\frac{(e_0 B_0)^3}{3s} - \frac{4(e_0 B_0)^3}{3s} \frac{1}{1 - \eta^2}.$$  

Note that the function $Q_1(s, \eta) = Q_{20}(s, \eta) + Q_{12}(s, \eta)$ is the expansion of $Q_1(s, \eta)$ up to quadratic terms in the external field. Obviously, for $\tau_0 \to 0$ the first term in Eq. (73) is logarithmically divergent ($\sim \ln \tau_0^{-1}$). This divergence will be “reabsorbed” in the course of charge renormalization.

To regularize the remaining integration over $\eta$ we express Eq. (73) in the following way

$$\mathcal{L}_1^{(2)} = -\frac{\alpha_0}{32\pi^3} \int_{2\tau_0}^{\infty} d\tau e^{-m_0^2 \tau} \int_0^{\eta_0} d\eta Q_{12}(s, \eta)$$

(74)

$$- \frac{\alpha_0}{32\pi^3} \int_{2\tau_0}^{\infty} d\tau e^{-m_0^2 \tau} \int_0^{\eta_0} d\eta \{ Q_1(s, \eta) - Q_2(s, \eta) \}$$

$$- \frac{\alpha_0}{32\pi^3} \int_{2\tau_0}^{\infty} d\tau e^{-m_0^2 \tau} \int_0^{\eta_0} d\eta \{ Q_1 - Q_4 - \mathcal{J}_1 \}$$

where $\mathcal{J}_1(s, \eta)$ is a function determined by the singular part of the Laurent series of $Q_1(s, \eta) - Q_2(s, \eta)$ at $\eta = 1$.

As a consequence, the integrations in the third line of Eq. (74) converges for $\tau_0 \to 0$. In particular,

$$\mathcal{J}_1(s, \eta) = \frac{4(e_0 B_0)^3}{s^3(1 - \eta^2)} \left( s \coth(s) + \frac{s^2}{\sinh^2(s)} - 2 \right),$$

(75)

$$\mathcal{J}_2(s, \eta) = \frac{2(e_0 B_0)^3}{s^3(1 - \eta^2)} \left( 3s \coth(s) + \frac{s^2}{\sinh^2(s)} - \frac{2s^2}{3} - 4 \right),$$

$$\mathcal{J}_3(s, \eta) = \frac{2(e_0 B_0)^3}{s^3(1 - \eta^2)} \left( s \coth(s) + \frac{3s^2}{\sinh^2(s)} + \frac{2s^2}{3} - 4 \right).$$

We consider the integration over $\eta$ in the first and second line of Eq. (74) to express (after some algebraical manipulations)

$$\mathcal{L}_1^{(2)} = -\frac{\alpha_0^2}{6\pi^2} \mathcal{G}^{(0)} \int_{2\tau_0}^{\infty} d\tau e^{-m_0^2 \tau} + \frac{3}{4} \mathcal{L}_1^{(1)} \frac{\partial^2}{\partial m_0^2} \mathcal{L}_1^{(2)} + \mathcal{L}_1^{(1)} \mathcal{L}_1^{(2)}$$

(76)

$$\mathcal{L}_2^{(2)} = -\frac{\alpha_0^2}{6\pi^2} \mathcal{G}^{(0)} \left[ \ln \left( \gamma m_0^2 \tau_0 \right) \int_{2\tau_0}^{\infty} d\tau e^{-m_0^2 \tau} \right.$$  

$$\left. - \int_{2\tau_0}^{\infty} d\tau e^{-m_0^2 \tau} \ln \left( \gamma m_0^2 \tau \right) - \frac{\alpha_0^2}{2\pi} \ln \left( \gamma m_0^2 \tau_0 \right) \right]$$

$$+ \frac{1}{6} \delta m^2 \frac{\partial^2}{\partial m_0^2} \mathcal{L}_1^{(2)} + \mathcal{L}_3^{(2)}$$

(77)

$$\mathcal{L}_3^{(2)} = -\frac{\alpha_0^2}{12\pi^2} \mathcal{G}^{(0)} \int_{2\tau_0}^{\infty} d\tau e^{-m_0^2 \tau} + \frac{\alpha_0^2}{6\pi^2} \mathcal{G}^{(0)} \left[ \ln \left( \gamma m_0^2 \tau_0 \right) \right.$$  

$$\times \int_{2\tau_0}^{\infty} d\tau e^{-m_0^2 \tau} - \int_{2\tau_0}^{\infty} d\tau e^{-m_0^2 \tau} \ln \left( \gamma m_0^2 \tau \right)$$

$$+ \frac{\alpha_0}{2\pi} \ln \left( \gamma m_0^2 \tau_0 \right) \mathcal{L}_1^{(1)} + \frac{1}{2} \delta m^2 \frac{\partial^2}{\partial m_0^2} \mathcal{L}_1^{(1)} + \mathcal{L}_3^{(2)}$$

(78)

with $\mathcal{G}^{(0)} = -1/2B_0^2$. Here

$$\delta m^2 = \frac{3\alpha_0^2 e_0^2}{2\pi} \left[ \ln \left( \frac{1}{\gamma m_0^2 \tau_0} \right) + \frac{5}{6} \right]$$

(79)

is the correction to the square of the “bare” electron mass ($m^2 = m_0^2 + \delta m^2$) [34, 35, 32], whereas the renormalized two loop term $\mathcal{L}_2^{(2)}$ corresponding to each eigenmode
reads
\[
\varpi^{(2)}_{1R} = -\frac{5\alpha_0 m_0^2}{12\pi} \frac{\partial \varpi^{(1)}_{1R}}{\partial m_0^2} - \frac{\alpha_0}{2\pi^2} \int_0^\infty \frac{d\tau}{\tau} e^{-m_0^2 \tau} \ln \left( \gamma m_0^2 \right) \\
\times \left[ \text{scot}(s) + \frac{s^2}{\sinh^2(s)} - 2 \right] - \frac{\alpha_0}{32\pi^3} \int_0^\infty d\tau e^{-m_0^2 \tau} \\
\times \int_0^1 dq \left\{ \varpi_1(s, \eta) - \varpi_2(s, \eta) - \varpi_3(s, \eta) \right\}, \tag{80}
\]

\[
\varpi^{(2)}_{2R} = -\frac{5\alpha_0 m_0^2}{24\pi} \frac{\partial \varpi^{(1)}_{2R}}{\partial m_0^2} - \frac{\alpha_0}{32\pi^3} \int_0^\infty \frac{d\tau}{\tau} e^{-m_0^2 \tau} \ln \left( \gamma m_0^2 \right) \\
\times \left[ 3s \text{cosh}(s) + \frac{s^2}{\sinh^2(s)} - 4 \right] - \frac{\alpha_0}{32\pi^3} \int_0^\infty d\tau \\
\times e^{-m_0^2 \tau} \int_0^1 dq \left\{ \varpi_2(s, \eta) - \varpi_2(s, \eta) - \varpi_3(s, \eta) \right\}, \tag{81}
\]

\[
\varpi^{(2)}_{3R} = -\frac{15\alpha_0 m_0^2}{24\pi} \frac{\partial \varpi^{(1)}_{3R}}{\partial m_0^2} - \frac{\alpha_0}{32\pi^3} \int_0^\infty \frac{d\tau}{\tau} e^{-m_0^2 \tau} \ln \left( \gamma m_0^2 \right) \\
\times \left[ \text{cosh}(s) + \frac{3s^2}{\sinh^2(s)} - 4 + \frac{2s^2}{3} \right] - \frac{\alpha_0}{32\pi^3} \int_0^\infty d\tau \times e^{-m_0^2 \tau} \int_0^1 dq \left\{ \varpi_3(s, \eta) - \varpi_3(s, \eta) - \varpi_3(s, \eta) \right\}. \tag{82}
\]

Inserting Eqs. (76) (78) into Eq. (60), the two-loop correction to the Euler-Heisenberg Lagrangian is given by
\[
\mathcal{L}^{(2)} = -\frac{\alpha_0^2}{4\pi^2} \varpi^{(0)}_{\text{2-loop}} = \varpi^{(1)}_{\text{2-loop}} + \varpi^{(2)}_{\text{2-loop}} + \varpi^{(2)}_{\text{2-loop}} \tag{83}
\]

with \( \varpi^{(2)} = \sum_{i=1}^3 \varpi^{(2)}_{R(i)} \).

Considering Eq. (64) and Eq. (83), we obtain
\[
\mathcal{L}_{\text{EH}} = \mathcal{L}^{(0)} Z^{-1}_{3(2\text{loop})} + \mathcal{L}^{(1)} + \varpi^{(2)}_{\text{2-loop}} + \varpi^{(2)}_{\text{2-loop}} \tag{84}
\]

where
\[
Z^{-1}_{3(2\text{loop})} = Z^{-1}_{3(1\text{loop})} - \lim_{\tau_0 \to 0} \frac{\alpha_0^2}{4\pi^2} \int_{2\tau_0}^\infty \frac{d\tau}{\tau} e^{-m_0^2 \tau}. \tag{85}
\]

For \( \tau_0 \to 0 \) the integral \( \int_{2\tau_0}^\infty \frac{d\tau}{\tau} e^{-m_0^2 \tau} = \ln(2\gamma m_0^2 \tau_0)^{-1} \), and therefore
\[
Z^{-1}_{3(2\text{loop})} \bigg|_{\tau_0 \to 0} = 1 + \frac{\alpha_0}{3\pi} \ln \left( \frac{1}{\gamma m_0^2 \tau_0} \right) - \frac{\alpha_0^2}{4\pi^2} \ln \left( \frac{1}{2\gamma m_0^2 \tau_0} \right) \tag{86}
\]

Now, we are able to identify
\[
\varpi^{(2)}_{\text{vR}}(m^2) = \varpi^{(1)}_{\text{vR}}(m_0^2) + \delta m^2 \frac{\partial \varpi^{(1)}_{\text{vR}}}{\partial m_0^2} \bigg|_{\delta m^2 = 0} \tag{86}
\]

where \( m \) is the renormalized electron mass. The renormalized charge and field strength are introduced by means of the relations \( e = e_0 Z_{3(2\text{loop})}^{3/2} \) and \( B = B_0 Z_{3(2\text{loop})}^{-1/2} \). Under this condition \( \varpi^{(0)}_{\text{vR}} = \mathcal{L}^{(0)} Z^{-1}_{3(2\text{loop})} \).

Note that \( eB = e_0 B_0 \). So, the variable \( s \) is an invariant under the renormalization. Clearly, \( m_0 \) must be replaced by \( m \) wherever it appears as well as \( \alpha_0 \to \alpha \). Keeping this in mind,
\[
\mathcal{L}_{\text{EH}} = \mathcal{L}^{(0)} + \mathcal{L}^{(1)} + \sum_{i=1}^3 \varpi^{(2)}_{R(i)} + \ldots \tag{87}
\]

with
\[
\varpi^{(2)}_{1R} = -\frac{5\alpha_0 m_0^2}{12\pi} \frac{\partial \varpi^{(1)}_{1R}}{\partial m_0^2} - \frac{\alpha_0}{16\pi^3} \int_0^\infty \frac{d\tau}{\tau} e^{-m_0^2 \tau} \ln \left( \gamma m_0^2 \right) \\
\times \left[ \text{scot}(s) + \frac{s^2}{\sinh^2(s)} - 2 \right] - \frac{\alpha_0}{32\pi^3} \int_0^\infty d\tau e^{-m_0^2 \tau} \times \int_0^1 dq \left\{ \varpi_1(s, \eta) - \varpi_2(s, \eta) - \varpi_3(s, \eta) \right\}, \tag{88}
\]

\[
\varpi^{(2)}_{2R} = -\frac{5\alpha_0 m_0^2}{24\pi} \frac{\partial \varpi^{(1)}_{2R}}{\partial m_0^2} - \frac{\alpha_0}{32\pi^3} \int_0^\infty \frac{d\tau}{\tau} e^{-m_0^2 \tau} \ln \left( \gamma m_0^2 \right) \\
\times \left[ 3s \text{cosh}(s) + \frac{s^2}{\sinh^2(s)} - 4 \right] - \frac{\alpha_0}{32\pi^3} \int_0^\infty d\tau \times e^{-m_0^2 \tau} \int_0^1 dq \left\{ \varpi_2(s, \eta) - \varpi_2(s, \eta) - \varpi_3(s, \eta) \right\}, \tag{89}
\]

\[
\varpi^{(2)}_{3R} = -\frac{15\alpha_0 m_0^2}{24\pi} \frac{\partial \varpi^{(1)}_{3R}}{\partial m_0^2} - \frac{\alpha_0}{32\pi^3} \int_0^\infty \frac{d\tau}{\tau} e^{-m_0^2 \tau} \ln \left( \gamma m_0^2 \right) \\
\times \left[ \text{cosh}(s) + \frac{3s^2}{\sinh^2(s)} - 4 + \frac{2s^2}{3} \right] - \frac{\alpha_0}{32\pi^3} \int_0^\infty d\tau \times e^{-m_0^2 \tau} \int_0^1 dq \left\{ \varpi_3(s, \eta) - \varpi_3(s, \eta) - \varpi_3(s, \eta) \right\}. \tag{90}
\]

where we have replaced the integration variable \( \tau \) by \( s \) and \( b = B/B_c \). Here we have used the function \( f_i(s, \eta) = \{ \tilde{Q}_i(s, \eta) - \tilde{Q}_i(s, \eta) - \tilde{H}_i(s, \eta) \} \) with \( \tilde{Q}_i(s, \eta) = Q_i(s, \eta)/(eB)^3 \), \( \tilde{Q}_i(s, \eta) = Q_i(s, \eta)/(eB)^3 \) and \( \tilde{H}_i(s, \eta) = H_i(s, \eta)/(eB)^3 \).

C. Asymptotic behavior at large magnetic field strength

In the asymptotic region of superstrong magnetic field \( b \gg 1 \), Eq. (64) behaves like \( \tilde{Q}_i(s, \eta) = \frac{m_0^4 \delta}{24\pi} \left( \frac{b}{\gamma \pi} \right) + \frac{6}{\pi^2} \zeta(2) \), \tag{91}

where \( 6\pi^2 \zeta(2) = -0.5699610 \ldots \) with \( \zeta(x) \) the Riemann zeta-function.
For the same magnetic field regime the leading term of $\sigma^{(2)}_{ir}$ is derived in Appendix [B]

$$\sigma^{(2)}_{ir} \approx -\frac{\alpha m^4 b^3}{16 \pi^3} N_1,$$
$$\sigma^{(2)}_{2ir} \approx \frac{\alpha m^4 b^2}{32 \pi^3} \left[ N_2 \ln \left( \frac{b}{\gamma \pi} \right) - \frac{1}{3} \ln^2 \left( \frac{b}{\gamma \pi} \right) + A \right],$$
$$\sigma^{(2)}_{3ir} \approx \frac{\alpha m^4 b^2}{32 \pi^3} \left[ N_3 \ln \left( \frac{b}{\gamma \pi} \right) + \frac{1}{3} \ln^2 \left( \frac{b}{\gamma \pi} \right) + B \right].$$

These expressions are calculated with accuracy of terms decreasing with $b$ like $\sim b^{-1} \ln(b)$ and faster. Here the numerical constants are $N_1 = 1.25$, $N_2 = \frac{1}{3} - \frac{4\zeta(2)}{\pi^2}$, $N_3 = \frac{2}{3} + \frac{4\zeta(2)}{\pi^2}$, $A = 4.21$ and $B = 0.69$. Note $N_2 + N_3 = 1$.

Taking all this into account, the asymptotic behavior of the full two-loop term is

$$\sigma^{(2)}_R = \sum_{i=1}^{3} \sigma^{(2)}_{iri} \approx \frac{\alpha m^4 b^2}{32 \pi^3} \left[ \ln \left( \frac{b}{\gamma \pi} \right) + 2.4 \right],$$

which coincides with the results reported in references [30, 32].

VI. VACUUM MAGNETIC PROPERTIES IN A SUPERSTRONG MAGNETIC FIELD

A. Role of the photon polarization modes on the vacuum magnetization

In presence of an external magnetic field, the zero-point vacuum energy $\mathcal{E}_{\text{vac}}$ is modified by the interaction between $B$ and the virtual QED-particles. The latter is determined by the effective potential coming from the quantum-corrections to the Maxwell Lagrangian which is also contained within the finite temperature formalism. According to Eq. (87) it is expressed as

$$\mathcal{E}_{\text{vac}} = -\sigma^{(1)}_R - \sum_{i=1}^{3} \sigma^{(2)}_{iri} + \ldots$$

Consequently the vacuum acquires a non trivial magnetization $M_{\text{vac}} = -\partial \mathcal{E}_{\text{vac}} / \partial B$ induced by the external magnetic field. In what follows we will write

$$M_{\text{vac}} = M^{(1)}_{\text{vac}} + M^{(2)}_{\text{vac}} + \ldots$$

in correspondence with the loop-term $\sigma^{(1)}_R$. In this sense, the one-loop contribution at very large magnetic field $b \gg 1$ can be computed by means of Eq. (11) and gives:

$$M^{(1)} = \partial \mathcal{E}^{(1)}_R / \partial B \approx \frac{m^4 b}{24 \pi^3 B_c} \left[ 2 \ln \left( \frac{b}{\gamma \pi} \right) + 1 + \frac{12\zeta(2)}{\pi^2} \right].$$

The dependence on the external field is shown in FIG. 4. The data depicted is obtained in the field interval $10 \leq b \leq 10^2 \cdot 3\pi / \alpha$. Within this approximation, we find that the vacuum reacts paramagnetically and has a nonlinear dependence on the external field (see Eq. (96)).

The two-loop correction is given by $M^{(2)} = \sum_{i=1}^{3} M^{(2)}_{iri}$ where $M^{(2)}_{iri} = \partial \mathcal{E}^{(2)}_{iri} / \partial B$ is the contribution corresponding to a photon propagation mode. Making use of Eqs. (92) we find

$$M^{(2)}_1 \approx -\frac{\alpha m^4 b}{8 \pi^3 B_c} N_1,$$
$$M^{(2)}_2 \approx -\frac{\alpha m^4 b}{32 \pi^3 B_c} \left[ \frac{2}{3} \ln^2 \left( \frac{b}{\gamma \pi} \right) + \frac{8\zeta(2)}{\pi^2} \ln \left( \frac{b}{\gamma \pi} \right) \right. - N_2 - 2A \bigg],$$
$$M^{(2)}_3 \approx \frac{\alpha m^4 b}{32 \pi^3 B_c} \left[ \frac{2}{3} \ln^2 \left( \frac{b}{\gamma \pi} \right) + \left( 2 + \frac{8\zeta(2)}{\pi^2} \right) \ln \left( \frac{b}{\gamma \pi} \right) \right.$$
$$+ N_3 + 2B \bigg].$$

According to these results, in a superstrong magnetic field limit, $M^{(2)}_1 < 0$ and $M^{(2)}_2 < 0$ behave diamagnetically whereas $M^{(2)}_3 > 0$ is purely paramagnetic (see Fig. 5). Within the range of magnetic field values for which the photon anomalous magnetic moment is defined ($10^{14} G \lesssim |B| \lesssim 10^{15} G$) the vacuum magnetization density is $M^{(2)}_3 \sim -10^9 \text{erg}/(\text{cm}^3 \text{G})$. Moreover, while $M^{(2)}_1$ depends linearly on $b$, the contributions of the second and third propagation mode depend logarithmically on the external field. Note that the leading behavior of the complete two-loop contribution is

$$M^{(2)} \approx \frac{\alpha m^4 b}{32 \pi^3 B_c} \left[ 2 \ln \left( \frac{b}{\gamma \pi} \right) + 5.8 \right] > 0$$

which points out a dominance of the third mode.

FIG. 4: One-loop contribution to the vacuum magnetization density with regard to the external field.

As it was expected $M^{(1)} / M^{(2)} \sim \alpha^{-1}$. This ratio is also manifested between the corresponding magnetic suscep-
FIG. 5: Contribution of the vacuum polarization eigenvalues to the vacuum magnetization density with regard to the external field. Here $10 < b < 10^2 \cdot 3\pi/\alpha$. The dashed line represents the complete two-loop contribution to the vacuum.

FIG. 6: Contribution of the vacuum polarization eigenvalues to the vacuum transverse pressure density with regard to the external field. Here $10 < b < 10^2 \cdot 3\pi/\alpha$. The dashed line represents the complete two-loop contribution.

According to Eqs. (102) and Eqs. (99) they read:

\[
P_{\perp,1}^{(2)} \approx \frac{\alpha m^4 b^2}{16\pi^3} N_1 > 0, \tag{104}
\]

\[
P_{\perp,2}^{(2)} \approx \frac{\alpha m^4 b^2}{32\pi^3} \left[ \frac{1}{3} \ln^2 \left( \frac{b}{\gamma\pi} \right) + \left( N_2 + \frac{2\zeta'(2)}{\pi^2} \right) \ln \left( \frac{b}{\gamma\pi} \right) - N_2 - A \right] > 0, \tag{105}
\]

\[
P_{\perp,3}^{(2)} \approx -\frac{\alpha m^4 b^2}{32\pi^3} \frac{1}{3} \ln^2 \left( \frac{b}{\gamma\pi} \right) + \left( 2 - N_3 + \frac{2\zeta'(2)}{\pi^2} \right) \times \ln \left( \frac{b}{\gamma\pi} \right) + N_3 + B < 0, \tag{106}
\]

with the complete two-loop term given by

\[
P_{\perp}^{(2)} \approx -\frac{\alpha m^4 b^2}{32\pi^3} \left[ \ln \left( \frac{b}{\gamma\pi} \right) + 3.4 \right] < 0. \tag{107}
\]

For $b \sim 10^5$, corresponding to magnetic fields $B \sim 10^{18} G$, the transverse pressure generated by the first and second polarization mode is positive and reaches values of the order $\sim 10^{30} \text{dyn/cm}^2$ and $\sim 10^{31} \text{dyn/cm}^2$, respectively (see Fig. 6). In contrast, the contribution given by the third mode is negative with $P_{\perp,3}^{(2)} \sim -10^{31} \text{dyn/cm}^2$. In the same context $P_{\perp}^{(2)} \sim -10^{31} \text{dyn/cm}^2$. The one-loop contribution is correspondingly even of the order of $P_{\perp}^{(1)} \sim -10^{31} \text{dyn/cm}^2$.

VII. CONCLUSION

In the first part of this work we showed that an infrared photon propagating in a strongly magnetized vacuum ($10 < b < 10^2 \cdot 3\pi/\alpha$), seems to exhibit a nonzero vector anomalous magnetic moment [see Eq. (102)]. We have pointed out that this quantity is a signal of the Lorentz symmetry breaking due to the presence of an external magnetic field. In addition, we have shown that $m_\gamma$...
arises due to the interaction between virtual electron-positron quantum pairs with the external magnetic field and its existence is closely related to the gauge invariance. In this context, we showed that \( m_1 \) can be decomposed into two orthogonal components in correspondence with the cylindrical symmetry imposed by the external field. These components are opposite in sign, and only the one along \( B \) is conserved. We remarked that the photon paramagnetism (analyzed in \([17]\)) is only associated with a kind of photon magnetization rather than a magnetic moment.

In the last sections of this work we discussed the effect of the vacuum polarization tensor, which modifies the zero-point energy of the vacuum even in the absence of electromagnetic waves. In the limit of a superstrong magnetic field, the two-loop contribution of the magnetization density corresponding to the second and third propagation mode depends nonlinearly on the external magnetic field and their behavior is diamagnetic and paramagnetic, respectively. On the other hand, the contribution coming from the first mode is diamagnetic and depends linearly on \( B \). We have seen that for very large magnetic field the contribution due to the third mode strongly dominate the analyzed quantities. In this magnetic field regime the latter exerts a negative transverse pressures to the external field. On the contrary those contributions coming from the first and second virtual mode are positive.

We want to point out that, although the decomposition of the two-loop term in the Euler-Heisenberg Lagrangian was considered in a magnetic background it remains valid also for the electric case. This fact should allow to study the role of the vacuum polarization modes in electron-positron production in a superstrong electric field. A detailed analysis of this issue will be presented in a forthcoming work.

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Appendix A:

The term \( z_2 \) can be written as

\[
z_2 = \frac{1}{2\Delta} |B \times k|^2 = k^2 - \frac{(B \cdot k)^2}{2\Delta} \quad (A1)
\]

and its substitution into Eq. 35 leads to:

\[
\omega^2 = \frac{1}{1 + \varrho(b)} \left( k^2 - \varrho(b) \frac{(B \cdot k)^2}{2\Delta} \right). \quad (A2)
\]

Defining the photon magnetization by means of the relation \( \mu_\gamma = -\partial \omega / \partial B \) we find

\[
\mu_\gamma = -\frac{\partial \omega}{\partial B} = \frac{1}{2\omega(1 + \varrho(b))^2} \left[ k^2 + (1 + 2\varrho(b)) \frac{(B \cdot k)^2}{2\Delta} \right] \times \frac{B}{2\Delta} \frac{\varrho(b)}{\omega(1 + \varrho(b))} \frac{(B \cdot k)}{\Delta}.
\]

For \( B_\perp \to 0 \) we obtain

\[
\mu_\perp = -\varrho \frac{e}{m_0} n_\perp \cos \phi \quad \text{and} \quad \mu_\parallel = \varrho \frac{e}{m_0} n_\parallel \sin \phi. \quad (A5)
\]

1. Relation between \( \mu_\gamma \) and \( m_\gamma \)

According to these expressions we can write

\[
\mu_\gamma = m_\gamma + \frac{\partial m_\gamma}{\partial B} \cdot B. \quad (A6)
\]

The second term in Eq. (A6) is a vector orthogonal to the external field. It can be expressed as

\[
\frac{\partial m_\gamma}{\partial B} \cdot B = \frac{e}{2m} n_\perp \cos \phi = \frac{e}{2m} (n_\parallel \times s_\gamma) \cdot (n_\parallel \times s_\gamma) \quad (A7)
\]

By substituting Eq. (A2) and Eq. (A7) in Eq. (A6) we obtain

\[
\mu_\gamma = \frac{e}{2m} \left[ n + (n \cdot n_\parallel) n_\parallel \right] \times s_\gamma, \quad (A8)
\]

Substituting Eq. (A6) in Eq. (A1) gives

\[
\mathcal{V} = -m_\gamma \cdot B = -\mu_\gamma \cdot B + \frac{e}{2m} \cos \phi n_\parallel \cdot B \quad (A9)
\]

Because of the fact that the second term on the right-hand side is projected out by the scalar product (\( n_\parallel \cdot B = 0 \)), the physical consequences of \( \mu_\gamma \) might be analyzed by means of \( m_\gamma \). So, in a good approximation

\[
\mu_\gamma \approx m_\gamma. \quad (A10)
\]

the remaining transversal component involved in \( m_\gamma \) is not relevant because is projected out by the scalar product \( m_\gamma \cdot B \).
2. $\mu^0_l$ for $b \geq 3\pi/\alpha$

Note, in addition, that for purely perpendicular propagation, Eq. (A4) reduces to:

$$\mu^0_l = g^2 e^{2m} f[k_L](1 + g(b))^{-5/2}. \quad (A11)$$

In particular for $b \to \infty$

$$\mu^0_l \approx g^2 e^{2m} f[k_L](g(b))^{-5/2}.$$ 

---

**Appendix B:**

The aim of this appendix is to find the asymptotic behavior of $\mathcal{L}^{(2)}_i$ for $b \gg 1$. For this purpose, it is convenient to write

$$\mathcal{L}^{(2)}_{iR} = \mathcal{L}^{(2)}_{iR} + \mathcal{L}^{(2)}_{iR}, \quad (B1)$$

with the first term being given by

$$\mathcal{L}^{(2)}_{iR} = \mathcal{L}^{(2)}_{iR} + \mathcal{L}^{(2)}_{iR}, \quad (B2)$$

with

$$\Sigma_1 = \int_0^\infty \frac{ds}{s^3} e^{-s/b} \ln\left(\frac{s}{\pi}\right) \left( s \coth(s) + \frac{s^2}{\sinh^2(s)} - 2 \right), \quad \Sigma_2 = \int_0^\infty \frac{ds}{s^3} e^{-s/b} \ln\left(\frac{s}{\pi}\right) \left( 3s \coth(s) + \frac{s^2}{\sinh^2(s)} - 4 \right),$$

$$\Sigma_3 = \int_0^\infty \frac{ds}{s^3} e^{-s/b} \ln\left(\frac{s}{\pi}\right) \left( s \coth(s) + \frac{3s^2}{\sinh^2(s)} - 4 + \frac{2s^2}{3} \right). \quad (B3)$$

The second term in Eq. (B1) is defined as:

$$\mathcal{L}^{(2)}_{iR} = -\frac{\alpha m^4 b^2}{32\pi^3} G_i(b) \text{ with } G_i(b) = \int_0^1 d\eta \int_0^\infty dse^{-s/b} f_i(s, \eta) \quad (B4)$$

with $f_i(s, \eta) \equiv \left\{ \tilde{\Phi}_i(s, \eta) - \tilde{\Phi}_i(s, \eta) - \tilde{\delta}_i(s, \eta) \right\}$.

1. **Leading behavior of $\mathcal{L}^{(2)}_{iR}$ in an asymptotically large magnetic field**

In order to determine the leading asymptotic-magnetic field term of $\mathcal{L}^{(2)}_{iR}$, we substitute Eq. (91) into Eqs. (B2), which gives:

$$\mathcal{L}^{(2)}_{iR} \approx \frac{\alpha m^4 b^2}{16\pi^3} \left[ \frac{1}{3} \ln\left(\frac{b}{\gamma \pi}\right) + \frac{5}{18} - \Sigma_1 \right], \quad \mathcal{L}^{(2)}_{iR} \approx \frac{\alpha m^4 b^2}{32\pi^3} \left[ \left( \frac{1}{3} - \frac{4\zeta(2)}{\pi^2} \right) \ln\left(\frac{b}{\gamma \pi}\right) - \frac{2}{3} \ln^2\left(\frac{b}{\gamma \pi}\right) + \frac{5}{18} - \Sigma_2 \right],$$

$$\mathcal{L}^{(2)}_{iR} \approx \frac{\alpha m^4 b^2}{32\pi^3} \left[ \left( 1 + \frac{4\zeta(2)}{\pi^2} \right) \ln\left(\frac{b}{\gamma \pi}\right) + \frac{2}{3} \ln^2\left(\frac{b}{\gamma \pi}\right) + \frac{5}{6} + \Sigma_3 \right]. \quad (B5)$$

Note that $\Sigma_2$ and $\Sigma_3$ can be expressed as

$$\Sigma_2 = \Sigma_1 + 2\Sigma, \quad \Sigma_3 = \Sigma_1 - 2\Sigma - 2b \frac{d\Sigma}{db}, \quad (B6)$$

$$\Sigma = \int_0^\infty \frac{ds}{s^3} \ln\left(\frac{s}{\pi}\right) e^{-s/b} \left[ s \coth(s) - 1 - \frac{s^2}{3} \right]. \quad (B7)$$
To derive the second expression in Eq. (B6) we have used the identity
\[ s \coth(s) + \frac{s^2}{\sinh^2(s)} - 2 = \]
\[ - s^2 \frac{d}{ds} \left[ \frac{1}{s^2} \left( s \coth(s) - 1 - \frac{s^2}{3} \right) \right] \]
and an integration by parts.

Note that \( \Sigma_1 \) converges even without the exponential factor which approaches to 1 for \( b \rightarrow \infty \). By using MATHEMATICA code we find \( \Sigma_1 \approx 0.19 \).

\( \Sigma \) does not involve singularities in the integrands at \( s = 0 \), but would diverge at \( s \rightarrow \infty \) if one sets the limiting value \( \exp(-s/b) = 1 \). For that reason we divide the integration domain into two parts:
\[ \Sigma = \Sigma^{(L)} + \Sigma^{(H)} \]
with \( \Sigma_1 \approx 0.19 \\
\]
\[ \Sigma^{(L)} = \int_0^T \frac{ds}{s^3} \ln \left( \frac{s}{\pi} \right) e^{-s/b} \left[ s \coth(s) - 1 - \frac{s^2}{3} \right], \]
\[ \Sigma^{(H)} = \int_T^\infty \frac{ds}{s^3} \ln \left( \frac{s}{\pi} \right) e^{-s/b} \left[ s \coth(s) - 1 - \frac{s^2}{3} \right] \]
and \( T \) an arbitrary positive number.

Now, we can omit the exponential in \( \Sigma_L \) since the resulting integral converges anyway:
\[ \Sigma^{(L)} \approx \int_0^T \frac{ds}{s^3} \ln \left( \frac{s}{\pi} \right) \left[ s \coth(s) - 1 - \frac{s^2}{3} \right]. \]

Substantial simplification is achieved by splitting the integrand of \( \Sigma_H \) into its parts and neglecting the exponential factor \( e^{-s/b} \) whenever is possible
\[ \Sigma^{(H)} \approx \int_T^\infty \frac{ds}{s^3} \ln \left( \frac{s}{\pi} \right) \coth(s) - \int_T^\infty \frac{ds}{s^3} \ln \left( \frac{s}{\pi} \right) \]
\[ \left. - \frac{1}{3} \int_T^\infty \frac{ds}{s} \ln \left( \frac{s}{\pi} \right) \exp(-s/b). \right] \]
For \( s \rightarrow \infty \) the leading term of \( \coth(s) \approx 1 \). Having this in mind, we compute the integrals:
\[ \int_T^\infty \frac{ds}{s^3} \ln \left( \frac{s}{\pi} \right) \cosh(s)|_{s \rightarrow \infty} = \frac{1 + \ln \left( \frac{T}{\pi} \right)}{T}, \]
\[ \int_T^\infty \frac{ds}{s^3} \ln \left( \frac{s}{\pi} \right) = \frac{1 - 2 \ln \left( \frac{T}{\pi} \right)}{4T^2}, \]
and
\[ \int_T^\infty \frac{ds}{s} \ln \left( \frac{s}{\pi} \right) e^{-s/b} \approx \pi^2 + 2 \ln \left( \frac{b}{\pi \gamma} \right) - \frac{1}{2} \ln^2 \left( \frac{T}{\pi \gamma} \right) - \ln \left( \frac{T}{\gamma} \right) \ln (\gamma) - \frac{1}{2} \ln^2 \left( \frac{\pi}{\gamma} \right) \]
In the latter we have neglected terms decreasing as \( T/b \). Numerical calculation using MATHEMATICA code gives \( T \approx \pi \) by minimizing \( \Sigma (d\Sigma/dT = 0) \). Therefore,
\[ \Sigma \approx - \frac{1}{6} \ln \left( \frac{b}{\pi \gamma} \right) + 0.28 \]
and according to Eq. (B9)
\[ \Sigma_1 = 0.19, \quad \Sigma_2 \approx - \frac{1}{3} \ln \left( \frac{b}{\pi \gamma} \right) + 0.75, \]
\[ \Sigma_3 \approx \frac{1}{3} \ln \left( \frac{b}{\pi \gamma} \right) - 0.04. \]

The substitution of the latter into Eqs. (B6) yields
\[ \mathcal{Q}^{(2)}_{1R} \approx \frac{\alpha m b^2}{16\pi} \left[ \frac{1}{3} \ln \left( \frac{b}{\pi \gamma} \right) + 0.09 \right], \]
\[ \mathcal{Q}^{(2)}_{2R} \approx \frac{\alpha m b^2}{32\pi^2} \left[ \left( 1 + \frac{4\zeta(2)}{\pi^2} \right) \ln \left( \frac{b}{\pi \gamma} \right) + \frac{1}{3} \ln^2 \left( \frac{b}{\pi \gamma} \right) + 0.87 \right]. \]

2. Leading behavior of \( \mathcal{Q}^{(2)}_{iR} \) in an asymptotically large magnetic field

The asymptotic behavior of \( \mathcal{Q}^{(2)}_{iR} \) is obtained from the integral \( G_i \). We start our analysis by dividing the integration domain into two regions:
\[ G_i = \int_0^T ds \int_0^1 d\eta_1 \ldots + \int_T^\infty ds \int_0^1 d\eta_1 \ldots \] (B17)
with \( T > 0 \). We denote the corresponding integrals by \( G^{(L)}_i \) and \( G^{(H)}_i \), respectively. In the latter we replace the upper integration limit over \( \eta \) variable by a parameter \( \eta_0 = 1 - T/s > 0 \). In order to find the behavior of \( G^{(L)}_i \) we write
\[ G^{(L)}_i = \frac{1}{b} \int_0^T ds \int_0^1 d\eta_1 f_i (s, \eta) \exp(-s/b) \] (B18)
\[ \approx \int_0^T ds \int_0^1 d\eta_1 f_i (s, \eta) \] (B19)
where we have set \( \exp(s/b) \approx 1 \) since \( f_i \) converges within the corresponding integration domain. For \( T \rightarrow 0 \) the behavior of the remaining integrand is
\[ f_1 (s, \eta) \approx \frac{1}{180} (\eta^2 - 13)s, \quad f_2 (s, \eta) \approx \frac{1}{270} (\eta^2 - 3)s, \]
\[ f_3 (s, \eta) \approx - \frac{1}{540} (\eta^2 + 27)s. \] (B20)
The integrals over \( s \) and \( \eta \) are trivial to perform and give:
\[ G_1^{(L)} \approx - \frac{19}{540} T^2, \quad G_2^{(L)} \approx - \frac{2}{405} T^2, \]
\[ G_3^{(L)} \approx - \frac{41}{1620} T^2. \] (B21)
Let us consider now, the contributions coming from the second integration domain which concerns to \( G^{(H)}_4 \). In order to this we consider the asymptotic expression of \( f_1(s, \eta) \) for \( s \to \infty \). First of all, the asymptotic expansion of Eqs. (3) in powers of \( \exp(-s) \) and \( \exp(s) \) produces an expansion of Eq. (1) in a sum of contributions coming from the thresholds, the singular behavior in the threshold points originating from the divergences of the \( s \)-integration in Eq. (4) near \( s = \infty \) as it was developed in [3]. The leading terms in the expansion of Eqs. (3) at \( s \to \infty \) are

\[
\left. \frac{\sigma_1(s, \eta)}{\sinh s} \right|_{s \to \infty} \approx \frac{1 + \eta}{4} e^{-(1+\eta)s} + \frac{1 - \eta}{4} e^{-(1-\eta)s},
\]

\[
\left. \frac{\sigma_2(s, \eta)}{\sinh s} \right|_{s \to \infty} \approx \frac{1 - \eta^2}{4},
\]

\[
\left. \frac{\sigma_3(s, \eta)}{\sinh s} \right|_{s \to \infty} \approx 2 \exp(-2s).
\]

These expressions are in correspondence with the lowest threshold \((n = 0, n' = 1) \) or viceversa for \( i = 1, n = n' = 1 \) for \( i = 3 \), and \( n = n' = 0 \) for \( i = 2 \). The fact that \( M(\infty, \eta) \approx 1/2 \) and \( N(\infty, \eta) \geq M(\infty, \eta) \) allow us to write

\[
\mathcal{V}(s, \eta) \approx -\frac{2}{N^2(s, \eta)} + \frac{2 \ln[2N(s, \eta)]}{N^2(s, \eta)} \approx \frac{2 \ln[2N]}{N^2},
\]

\[
\mathcal{W}(s, \eta) \approx \frac{4}{N(s, \eta)} - \frac{2 \ln[2N(s, \eta)]}{N^2(s, \eta)} \approx \frac{4}{N(s, \eta)},
\]

with \( \mathcal{V}(s, \eta) + \mathcal{W}(s, \eta) \approx 4N^{-1}(s, \eta) \). Considering only the terms which decrease most slowly as a function of \( s \), we find

\[
\left. \tilde{Q}_1(s, \eta) \right|_{s \to \infty} \approx \frac{4e^{-(1-\eta)s}}{(1 + \eta)s} + \frac{4e^{-(1+\eta)s}}{(1 - \eta)s},
\]

\[
\left. \tilde{Q}_2(s, \eta) \right|_{s \to \infty} \approx Q_1(s, \eta) \left. \right|_{s \to \infty},
\]

\[
\left. \tilde{Q}_3(s, \eta) \right|_{s \to \infty} \approx \frac{8 \exp(-2s)}{(1 - \eta^2)s}.
\]

Additionally, the most significant terms arising from \( \tilde{Q}_i + \tilde{\delta}_i \) for \( s \to \infty \) are

\[
\left[ \tilde{Q}_1(s, \eta) + \tilde{\delta}_1(s, \eta) \right] \left. \right|_{s \to \infty} \approx -\frac{2}{3s} + \frac{16e^{-2s}}{(1 - \eta^2)s},
\]

\[
\left[ \tilde{Q}_2(s, \eta) + \tilde{\delta}_2(s, \eta) \right] \left. \right|_{s \to \infty} \approx \frac{8e^{-2s}}{(1 - \eta^2)s},
\]

\[
\left[ \tilde{Q}_3(s, \eta) + \tilde{\delta}_3(s, \eta) \right] \left. \right|_{s \to \infty} \approx -\frac{1}{3s} + \frac{24e^{-2s}}{(1 - \eta^2)s}.
\]

The behavior of \( f_i = \exp(-b/s)(\tilde{Q}_i - \tilde{\delta}_i - \tilde{\delta}_i) \) for \( s \to \infty \) is found out by considering Eqs. (B24-B25). In fact

\[
f_1(s, \eta) \approx \frac{4e^{-(1-\eta)s}}{(1 + \eta)s} + \frac{4e^{-(1+\eta)s}}{(1 - \eta)s} - \frac{16e^{-2s}}{(1 - \eta^2)s} + \frac{2}{3s}.
\]

\[
f_2(s, \eta) \approx \frac{4e^{-(1-\eta)s}}{(1 + \eta)s} + \frac{4e^{-(1+\eta)s}}{(1 - \eta)s} - \frac{8e^{-2s}}{(1 - \eta^2)s}.
\]

\[
f_3(s, \eta) \approx -\frac{16 \exp(-2s)}{(1 - \eta^2)s} + \frac{1}{3s}.
\]

Except for the last term in \( f_1(s, \eta) \) and \( f_3(s, \eta) \), we may use the \( \exp(-s/b) \sim 1 \) so that

\[
G_1^{(H)} \approx \int_T^\infty ds \int_{-\infty}^{\eta_0} d\eta \left\{ \frac{4e^{-(1-\eta)s}}{(1 - \eta)s} - \frac{8e^{-2s}}{(1 - \eta^2)s} + \frac{e^{-s/b}}{3s} \right\},
\]

\[
G_2^{(H)} \approx 4 \int_T^\infty ds \int_{-\infty}^{\eta_0} d\eta \left\{ \frac{e^{-(1+\eta)s}}{(1 + \eta)s} - \frac{e^{-2s}}{(1 - \eta^2)s} \right\},
\]

\[
G_3^{(H)} \approx \int_T^\infty ds \int_{0}^{\eta_0} d\eta \left\{ -\frac{16e^{-2s}}{(1 - \eta^2)s} + \frac{e^{-s/b}}{3s} \right\}.
\]

Performing the integration over \( \eta \), \( G_i^{(H)} \) are given by

\[
G_1^{(H)} \approx 4 \int_T^\infty ds \frac{e^{-2s}}{s} \left[ \text{Ei}(2s - T) - \text{Ei}(T) \right],
\]

\[
-8 \int_T^\infty ds \ln \left( \frac{2s}{T} - 1 \right) e^{-2s} + \frac{2}{3} \text{Ei} \left( -\frac{T}{b} \right) - \frac{2T}{3} \int_T^\infty ds \frac{e^{-s/b}}{s^2},
\]

\[
G_2^{(H)} \approx 4 \int_T^\infty ds \frac{e^{-2s}}{s} \left[ \text{Ei}(2s - T) - \text{Ei}(T) \right],
\]

\[-4 \int_T^\infty ds \ln \left( \frac{2s}{T} - 1 \right) \exp(-2s),
\]

\[
G_3^{(H)} \approx -8 \int_T^\infty ds \frac{\ln \left( \frac{2s}{T} - 1 \right) e^{-2s}}{s} + \frac{1}{3} \text{Ei} \left( -\frac{T}{b} \right) - \frac{T}{3} \int_T^\infty ds \frac{e^{-s/b}}{s^2},
\]

where \( \text{Ei}(-T/b) \) is the exponential-integral function whose asymptotic expansion for very large magnetic field \( b \to \infty \) is

\[
\int_T^\infty \frac{ds}{s} \exp \left( -\frac{s}{b} \right) \approx \ln \left( \frac{b}{\gamma \pi} \right) - \ln \left( \frac{T}{\pi} \right).
\]

This expression is calculated with accuracy of terms that decrease with \( b \). To be consistent the last integral of Eq. (B28) and Eq. (B30) can be neglected as well, since the terms decrease as fast as \( \sim b^{-1} \ln b \) and \( \sim b^{-1} \). The remaining integrals present in Eq. (B29) and Eq. (B30) depend on the parameter \( T \). Taking all this into account,
the leading asymptotic behavior of $G_i$ for $b \to \infty$ reads

$$G_1 \approx \frac{2}{3} \ln \left( \frac{b}{\gamma} \right) + C_1, \quad G_2 \approx C_2,$$

$$G_3 \approx \frac{1}{3} \ln \left( \frac{b}{\gamma} \right) + C_3. \quad \text{(B32)}$$

where we have used Eq. (B31). Here, the numerical constants $C_{1,3}$ are determined by imposing the condition $dC_{1,3}/dT = 0$ with

$$C_1 = \int_0^\infty dT \frac{e^{-2T}}{T} \left[ Ei(2s - T) - Ei(T) \right] - \frac{19T^2}{540}$$

$$- 8 \int_0^\infty dT \ln \left( \frac{2s}{T} - 1 \right) e^{-2s/T} - \frac{2}{3} \ln \left( \frac{T}{\pi} \right), \quad \text{(B33)}$$

$$C_3 = -8 \int_0^\infty dT \frac{e^{-2s/T}}{s} \ln \left( \frac{2s}{T} - 1 \right) e^{-2s/T} - \frac{1}{3} \ln \left( \frac{T}{\pi} \right)$$

$$- \frac{41T^2}{1620}. \quad \text{(B34)}$$

This yields

$$C_1 \approx 2.67, \quad \text{and} \quad C_3 \approx 0.18. \quad \text{(B35)}$$

Note that there is not value fulfilling the condition $dC_2/dT = 0$. However, in order to compute it, we first set $\frac{d}{dT} \sum_{i=1}^{3} G_i = 0$. This condition leads to $T \approx 0.46$ and

$$G = \sum_{i=1}^{3} G_i \approx \ln \left( \frac{b}{\gamma} \right) - 1.82. \quad \text{(B36)}$$

Obviously, $G_2 = G - G_1 - G_3$. Thus, by taking into account Eqs. (B32) and Eq. (B33) we obtain $C_2 \approx -4.68$. Such that

$$G_1 \approx \frac{2}{3} \ln \left( \frac{b}{\gamma} \right) + 2.67, \quad G_2 \approx -4.68,$$

$$G_3 \approx \frac{1}{3} \ln \left( \frac{b}{\gamma} \right) + 0.18. \quad \text{(B37)}$$

Substitution of Eqs. (B37) in Eq. (B1) allows to obtain

$$Q_{1R}^{(2)} \approx - \frac{a}{16\pi^3} \left[ \frac{1}{3} \ln \left( \frac{b}{\gamma} \right) + 1.34 \right], \quad \text{(B38)}$$

$$Q_{2R}^{(2)} \approx - \frac{a}{32\pi^3} b^2 C_2 \quad \text{with} \quad C_2 = -4.68, \quad \text{(B39)}$$

$$Q_{dR}^{(2)} \approx - \frac{b}{32\pi^3} \left[ \frac{1}{3} \ln \left( \frac{b}{\gamma} \right) + 0.18 \right]. \quad \text{(B40)}$$

Finally, the behavior of $Q_{1R}^{(2)}$ presented in Eq. (92) is derived by inserting the expression below Eqs. (B10) and Eqs. (B38-B40) into Eq. (B11).

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