On approximate solutions of semilinear evolution equations II. Generalizations, and applications to Navier-Stokes equations.

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Abstract

In our previous paper [12], a general framework was outlined to treat the approximate solutions of semilinear evolution equations; more precisely, a scheme was presented to infer from an approximate solution the existence (local or global in time) of an exact solution, and to estimate their distance. In the first half of the present work the abstract framework of [12] is extended, so as to be applicable to evolutionary PDEs whose nonlinearities contain derivatives in the space variables. In the second half of the paper this extended framework is applied to the incompressible Navier-Stokes equations, on a torus $\mathbb{T}^d$ of any dimension. In this way a number of results are obtained in the setting of the Sobolev spaces $H^n(\mathbb{T}^d)$, choosing the approximate solutions in a number of different ways. With the simplest choices we recover local existence of the exact solution for arbitrary data and external forces, as well as global existence for small data and forces. With the supplementary assumption of exponential decay in time for the forces, the same decay law is derived for the exact solution with small (zero mean) data and forces. The interval of existence for arbitrary data, the upper bounds on data and forces for global existence, and all estimates on the exponential decay of the exact solution are derived in a fully quantitative way (i.e., giving the values of all the necessary constants; this makes a difference with most of the previous literature). Nextly, the Galerkin approximate solutions are considered and precise, still quantitative estimates are derived for their $H^n$ distance from the exact solution; these are global in time for small data and forces (with exponential time decay of the above distance, if the forces decay similarly).

Keywords: Differential equations, theoretical approximation, Navier-Stokes equations, Galerkin method.

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1 Introduction.

This is a continuation of our previous paper [12] on the approximate solutions of semilinear Cauchy problems in a Banach space $F$, and on their use to get fully quantitative estimates on: (i) the interval of existence of the exact solution; (ii) the distance at any time between the exact and the approximate solution.

In [12], we mentioned the potential interest of (i) (ii) in relation to the equations of fluid dynamics. Here we treat specifically the incompressible Navier-Stokes (NS) equations on a torus $T^d$ of any dimension $d \geq 2$, taking for $F$ a Sobolev space of vector fields over $T^d$. To be precise, we consider the Sobolev space $H^n(T^d) \equiv H^n$ of the "velocity fields" $v : T^d \to \mathbb{R}^d$ whose derivatives of order $\leq n$ are square integrable; then we choose $F := H^n_{\Sigma0}$, where the subscripts $\Sigma0$ indicate the subspace of $H^n$ formed by the divergence free, zero mean velocity fields $f : T^d \to \mathbb{R}^d$ (of course, the condition of zero divergence represents incompressibility; the mean velocity can always be supposed to vanish, passing to a convenient moving frame). We always take $n > d/2$.

The choice of $T^d$ as a space domain allows a rather simple treatment, based on Fourier analysis; we presume that the results of this paper could be extended to bounded domains of $\mathbb{R}^d$, with suitable boundary conditions. In our notations, the NS Cauchy problem is written

$$
\dot{\varphi}(t) = \Delta \varphi(t) - \mathcal{L}(\varphi(t) \cdot \partial \varphi(t)) + \xi(t), \quad \varphi(0) = f_0,
$$

where $f = \varphi(t)$ is the velocity field at time $t$, $\mathcal{L}$ the Leray projection on the divergence free vector fields and $\xi(t)$ is the external forcing at time $t$ (more precisely, what remains of the external force field after applying $\mathcal{L}$ and subtracting the mean value).

We can regard (1.1) as a realization of the abstract semilinear Cauchy problem

$$
\dot{\varphi}(t) = A\varphi(t) + \mathcal{P}(\varphi(t), t), \quad \varphi(t_0) = f_0
$$

where $A : f \mapsto Af$ is a linear operator and $\mathcal{P} : (f, t) \mapsto \mathcal{P}(f, t)$ is a nonlinear map. Of course, in the NS case $A$ is the Laplacian $\Delta$ and $\mathcal{P}(f, t) := -\mathcal{L}(f \cdot \partial f) + \xi(t)$. By a standard method, both (1.1) and its abstract version (1.2) can be reformulated as a Volterra integral equation, involving the semigroup $(e^{tA})_{t \geq 0}$.

In [12], a general setting was proposed for semilinear Volterra problems, when the nonlinearity $\mathcal{P}(\cdot, t)$ is a sufficiently smooth map of a Banach space $F$ into itself. This setting cannot be applied to Cauchy problems like (1.1). In fact, due to the presence of the derivatives $\partial f$, the map $f \mapsto \mathcal{L}(f \cdot \partial f)$ cannot be seen as a smooth map of a Sobolev space, say $H^n_{\Sigma0}$, into itself; on the contrary, the above map is smooth from $H^n_{\Sigma0}$ to $H^{n-1}_{\Sigma0}$. The external forcing $\xi : t \mapsto \xi(t)$ fits well to this situation if we require it to be a sufficiently smooth map from $[0, +\infty)$ to $H^{n-1}_{\Sigma0}$.

In view of the applications to (1.1), in the first half of the present paper (Sections 2-5) we extend the abstract framework of [12] to the case where, at each time $t$,
\( \mathcal{P}(\cdot, t) \) is a smooth map between \( \mathbf{F} \) and a larger Banach space \( \mathbf{F}_- \). A general scheme to treat approximate solutions is developed along these lines; this could be applied not only to (1.1), but also to other evolutionary PDEs (essentially, of parabolic type) with space derivatives in the nonlinear part.

In the second half of the paper (Sections 6-10) we fix the attention on the NS equations, in the framework of the above mentioned \( \mathbb{H}^n_{50} \) spaces (incidentally, we wish to point out that other function spaces could be used to analyse the same equations within our general scheme).

Some technicalities related to either the first or the second half are presented in Appendices A-H.

Of course, there is an enormous literature on NS equations, their approximation methods and the intervals of existence of the exact solutions: references [2] [3] [5] [6] [7] [8] [9] [10] [11] [14] [15] are examples including seminal works, classical treatises and recent contributions. Some differences between the present analysis and most of the published literature are the following:

(i) Our discussion of the NS approximate solutions is part of a more general framework, in the spirit of the first half of the paper.

(ii) Our analysis is fully quantitative: any function, numerical constant, etc., appearing in our estimates on the solutions is given explicitly. In the end, our approach gives bounds on the interval of existence of the exact NS solution and on its distance from the approximate solution in terms of fully computable numbers; such computations are exemplified in a number of cases.

(iii) If compared with other contributions, our approach seems to be more suitable to derive the existence of global exact solutions from suitable approximate solutions, under specific conditions (typically, of small initial data); a comment on this point appears in Remark 8.7 (iii).

Hereafter we give more details about the contents of the paper.

**First half: a general setting for the approximate solutions of (1.2).** We have just mentioned the assumption \( \mathcal{P}(\cdot, t) : \mathbf{F} \mapsto \mathbf{F}_- \). We furtherly suppose \( \mathcal{A} : \mathbf{F}_+ \to \mathbf{F}_- \) where \( \mathbf{F}_+ \) is a dense subspace of \( \mathbf{F} \), to be equipped with the graph norm of \( \mathcal{A} \); in the end, this gives a triple of spaces \( \mathbf{F}_+ \subset \mathbf{F} \subset \mathbf{F}_- \).

To go on, we require \( \mathcal{A} \) to generate a semigroup on \( \mathbf{F}_- \), with the fundamental regularizing property \( e^{t\mathcal{A}}(\mathbf{F}_-) \subset \mathbf{F} \) for all \( t > 0 \). A more precise description of all these assumptions is given in Section 2: here we suppose, amongst else, the availability of an upper bound \( u_-(t) \in (0, +\infty) \) for the operator norm of \( e^{t\mathcal{A}} \), regarding the latter as a map from \( \mathbf{F}_- \) to \( \mathbf{F} \). The bound \( u_- \) is allowed to diverge (mildly) for \( t \to 0^+ \), an indication that \( e^{t\mathcal{A}} \mathbf{F}_- \not\subset \mathbf{F} \) for \( t = 0 \): the precise assumption is \( u_-(t) = O(1/t^{1-\sigma}) \) with \( 0 < \sigma \leq 1 \). In applications to the NS system, \( \mathbf{F} = \mathbb{H}^n_{50} \) and \( \mathbf{F}_\pm = \mathbb{H}^{n+1}_{50} \); the semigroup \( (e^{t\Delta}) \) of the Laplacian has the prescribed regularizing features, with \( \sigma = 1/2. \)
In Section 3 we present a general theory of the approximate solutions, for an abstract Cauchy (or Volterra) problem of the type sketched above. The basic idea is to associate to any approximate solution \( \varphi_{ap} : [t_0, T) \to F \) of the problem an integral control inequality for an unknown function \( R : [t_0, T) \to [0, +\infty) \); this has the form

\[
\mathcal{E}(t) + \int_{t_0}^{t} ds \ u_-(t - s) \ell(R(s), s) \leq R(t)
\]  

(1.3)

where \( \mathcal{E} : [t_0, T) \to [0, +\infty) \) is an estimator for the (integral) error of \( \varphi_{ap} \), and \( \ell \) is a function describing the growth of \( \mathcal{P} \) from \( \varphi_{ap} \). The main result in this framework is the following: if the control inequality is fulfilled by some function \( R \) on \([t_0, T)\), then the semilinear Volterra problem has an exact solution \( \varphi : [t_0, T) \to F \), and

\[ \| \varphi(t) - \varphi_{ap}(t) \| \leq R(t) \]

for all \( t \) in this interval (\( \| \| \) is the norm of \( F \)).

When \( F = F \), we recover from here the framework of [12]. Similarly to the result of [12], the present theorem about \( R, \varphi_{ap} \) and \( \varphi \) can be considered as the abstract and unifying form of many statements, appearing in the literature about specific systems.

The available literature would suggest to prove the above theorem along this path: (i) derive an existence theorem for \( \varphi \) on small intervals; (ii) use some nonlinear Gronwall lemma to prove that \( \| \varphi(t) - \varphi_{ap}(t) \| \leq R(t) \) on any interval \([t_0, T') \subset [t_0, T)\) where \( \varphi \) is defined; (iii) show the existence of \( \varphi \) on the full domain \([t_0, T)\) of \( \mathcal{R} \) by the following reductio ad absurdum: if not so, \( \| \varphi(t) - \varphi_{ap}(t) \| \) would diverge before \( T \) and its upper bound via \( \mathcal{R}(t) \) would be violated.

Our proof of the theorem on \( \mathcal{R}, \varphi_{ap} \) and \( \varphi \), presented in Section 4, replaces the above strategy with a more constructive approach. The main idea is to interpret the control inequality (1.3) as individuating a tube of radius \( \mathcal{R} = \mathcal{R}(t) \) around \( \varphi_{ap} \), invariant under the action of the semilinear Volterra operator \( \mathcal{J} \) for our problem. This makes possible to construct the solution by an iteration of Peano-Picard type, starting from \( \varphi_{ap} \); the result is a Cauchy sequence of functions \( \varphi_k = \mathcal{J}^k(\varphi_{ap}) \) on \([t_0, T)\), \( k = 0, 1, 2, ... \), whose \( k \to +\infty \) limit is an exact solution of the given Volterra problem.

From this viewpoint, existence of the solution on a short time interval, with any datum \( \varphi(t_0) = f_0 \), is a very simple corollary of the previous theorem based the choice \( \varphi_{ap}(t) := \text{constant} = f_0 \).

Even though there is a basic analogy with [12], proving the main theorem on approximate solutions is technically more difficult in the present case, mainly due to the divergence of \( u_-(t) \) for \( t \to 0^+ \). Such a divergence is also relevant in applications: in fact, differently from [12], Eq. (1.3) with \( \leq \) replaced by \( = \) cannot be reduced to an ordinary differential equation. Our assumption \( u_-(t) = O(1/t^{1-\sigma}) \) relates (1.3)
to the framework of singular integral equations of fractional type (which could be interpreted in terms of the so-called ”fractional differential calculus”).

In spite of these pathologies, solving (1.3) is rather simple when the semigroup \( (e^{tA}) \) and \( \varphi_{ap} \) have suitable features, and the nonlinear function \( \mathcal{P} \) has the (affine) quadratic structure

\[
\mathcal{P}(f, t) = \mathcal{P}(f, f) + \xi(t),
\]

with \( \mathcal{P} : F_- \times F_- \to F \) a continuous bilinear form and \( \xi : [0, +\infty) \to F_- \) a (locally Lipschitz) map; this is the subject of Section 5 (where the datum \( f_0 \) of (1.2) is always specified at \( t_0 = 0 \)).

The section starts from a fairly general statement on the control inequality (1.3), which is subsequently applied with specific choices of the approximate solution.

First of all, we consider the choice \( \varphi_{ap}(t) := 0 \). In this case, for any datum \( f_0 \) and external forcing \( \xi \), we construct for the control inequality a solution \( R \) with domain \( [0, T) \); this implies the existence on \( [0, T) \) of the solution \( \varphi \) of (1.2), and gives an estimate \( \|\varphi(t)\| \leq R(t) \) on the same interval. If \( f_0 \) and \( \xi \) are sufficiently small, \( T = +\infty \) and so \( \varphi \) is global. With the stronger assumption that \( \xi(t) \) decays exponentially for \( t \to +\infty \), we derive for the control inequality a solution \( t \mapsto R(t) \) which is also exponentially decaying; so, the same can be said for \( \|\varphi(t)\| \).

Next we consider, for a small \( f_0 \) and a small, exponentially decaying \( \xi \), the approximate solution \( \varphi_{ap} \) obtained solving the linear Cauchy problem

\[
\dot{\varphi}_{ap}(t) = A\varphi_{ap}(t) + \xi(t), \quad \varphi_{ap}(0) = f_0.
\]

In this case the control equation still possesses a global, exponentially decaying solution \( R \), giving a precise estimate on the distance \( \|\varphi(t) - \varphi_{ap}(t)\| \).

**Second half: applications to the NS equations.** In Section 6 we review the Sobolev spaces of vector fields on \( T^d \), and the Leray formulation of the incompressible NS equations within this framework; furthermore, we show that the Cauchy problem with mean initial velocity \( m_0 \) can be reduced to an equivalent Cauchy problem where the initial velocity has zero mean, by a change of space-time coordinates \( (x, t) \mapsto (x - h(t), t) \), where the function \( t \mapsto h(t) \) is suitably determined.

In the same section we give explicitly the constants \( K_{nd} \equiv K_n \) such that \( \|f \cdot \partial g\|_{n-1} \leq K_n \|f\|_n \|g\|_n \) for all velocity fields \( f, g \) on \( T^d \), \( \|\|_n \) and \( \|\|_{n-1} \) denoting the Sobolev norms of orders \( n \) and \( n - 1 \). The study of these constants, inspired by our previous work [13], prepares the fully quantitative application of the methods presented in the first half of the paper.

Section 7 starts from the formulation (1.1) of the Cauchy problem, in the already mentioned Sobolev spaces \( F = H^m_{\Sigma_0}, \quad F_\Sigma = H^{m+1}_{\Sigma_0}. \) We check that (1.1) fulfills all requirements of the general theory for quadratic nonlinearities, and construct the estimator \( u_\Sigma \) for the semigroup \( (e^{t\Delta}) \).

In Section 8 we rephrase for the NS equations all the results of Section 5 on the abstract quadratic case (1.4). The estimates on the time of existence \( T \), for arbitrary data and forcing, have a fully explicit form; the same happens for the bounds on
the norms $\|f_0\|_n$, $\|\xi(t)\|_{n-1}$ which ensure global existence and, possibly, exponential decay of $\varphi(t)$ for $t \to +\infty$.

In Section 9 we discuss the approximate NS solutions provided by the Galerkin method. More precisely, for each finite set $G(\neq 0)$ of wave vectors we consider the subspace $H_{20}^G$ spanned by the exponentials $e^{ikx}$ ($k \in G$), and the projection on $H_{20}^G$ of the NS Cauchy problem; this has a solution $t \mapsto \varphi^G(t)$ (in general, on a sufficiently small interval; with special assumptions, also involving the forcing, $\varphi^G$ is global and decays exponentially for $t \to +\infty$).

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(i) For any initial datum $f_0 \in H_{\Sigma 0}^p$ of the NS Cauchy problem (1.1), and each external forcing $\xi$ with values in $H_{\Sigma 0}^{p-1}$, both $\varphi^G$ and $\varphi$ exist on a suitable interval $[0, T)$, and there is an estimate

$$\|\varphi(t) - \varphi^G(t)\|_n \leq \frac{W_{np|G|}(t)}{|G|^{p-n}} \quad \text{for } t \in [0, T);$$

$T$ can be $+\infty$, if the datum and the forcing are sufficiently small. Both $T$ and the function $t \mapsto W_{np|G|}(t)$ are given explicitly.

(ii) If $f_0 \in H_{20}^G$ is sufficiently small and there is a small, exponentially decaying forcing $t \mapsto \xi(t) \in H_{p-1}^{\Sigma 0}$, then

$$\|\varphi(t) - \varphi^G(t)\|_n \leq \frac{W_{np|G|}}{|G|^{p-n}} e^{-t} \quad \text{for } t \in [0, +\infty);$$

the upper bounds for $f_0$, $\xi$ and the coefficient $W_{np|G|}$ are also given explicitly.

The results (i) (ii) imply convergence of $\varphi^G$ to $\varphi$ as $|G| \to +\infty$, on the time interval where the previous estimates hold (which can be $[0, +\infty)$, as pointed out).

In Section 10 we exemplify our estimates giving the numerical values of $T$ and of the error estimators in (i) (ii) for certain data and forcing, with $d = 3$ and $n = 2$, $p = 4$.

## 2 Introducing the abstract setting.

**Notations.** (i) All Banach spaces considered in this paper are over the same field, which can be $\mathbb{R}$ or $\mathbb{C}$.

(ii) If $X$ and $Y$ are Banach spaces, we write

$$X \hookrightarrow Y$$

(2.1)

to indicate that $X$ is a dense vector subspace of $Y$ and that its natural inclusion into $Y$ is continuous (i.e., $\|x\|_Y \leq \text{constant} \ \|x\|_X$ for all $x \in X$).

(iii) Consider two sets $\Theta, X$ and a function $\chi : \Theta \to X$, $t \mapsto \chi(t)$. The graph of $\chi$ is

$$\text{gr} \chi := \{(\chi(t), t) \mid t \in \Theta\} \subset X \times \Theta.$$  

(2.2)
If $X = [0, +\infty]$, we define the subgraph of $\chi$ as
\[
\text{sgr}_X \chi := \{(r, t) \mid t \in \Theta, r \in [0, \chi(t))\} \subset [0, +\infty) \times \Theta .
\] (2.3)

(iv) Consider a function $\chi : \Theta \to X$, where $\Theta$ is a real interval and $X$ a Banach space. This function is locally Lipschitz if, for each compact subset $I$ of $\Theta$, there is a constant $M = M(I) \in [0, +\infty)$ such that
\[
\|\chi(t) - \chi(t')\|_X \leq M|t - t'| \quad \text{for all } t, t' \in I .
\] (2.4)

As usually, we denote with $C^{0,1}(\Theta, X)$ the set of these functions.

**General assumptions.** Throughout the section, we will consider a set
\[
\langle F_+, F_-, A, u, u_-, P \rangle
\] (2.5)
with the following properties.

(P1) $F_+$, $F$ and $F_-$ are Banach spaces with norms $\| \|_+$, $\| \|$ and $\| \|_-$, such that
\[
F_+ \hookrightarrow F \hookrightarrow F_-. \quad (2.6)
\]

Here and in the sequel, $B(f_0, r)$ will denote the open ball $\{ f \in F \mid \| f - f_0 \| < r \}$ (the radius $r$ can be $+\infty$, and in this case $B(f_0, r) = F$).

(P2) $A$ is a linear operator such that
\[
A : F_+ \to F_- , \quad f \mapsto Af . \quad (2.7)
\]

Viewing $F_+$ as a subspace of $F_-$, the norm $\| \|_+$ is equivalent to the graph norm $f \in F_+ \mapsto \| f \|_+ + \| Af \|_-$. 

(P3) Viewing $A$ as a densely defined linear operator in $F_-$, it is assumed that $A$ generates a strongly continuous semigroup $(e^{tA})_{t \in [0, +\infty)}$ on $F_-$ (of course, from the standard theory of linear semigroups, we have $e^{tA}(F_+) \subset F_+$ for all $t \geq 0$).

(P4) One has
\[
e^{tA}(F) \subset F \quad \text{for } t \in [0, +\infty) ;
\] (2.8)

the function $(f, t) \mapsto e^{tA}f$ gives a strongly continuous semigroup on $F$ (i.e., it is continuous from $F \times [0, +\infty)$ to $F$).

Furthermore, $u \in C([0, +\infty), (0, +\infty))$ is a function such that
\[
\| e^{tA}f \| \leq u(t)\| f \| \quad \text{for } t \geq 0, f \in F ;
\] (2.9)

this function will be referred to as an estimator for the semigroup $(e^{tA})$ with respect to the norm of $F$.

(P5) One has
\[
e^{tA}(F_-) \subset F \quad \text{for } t \in (0, +\infty) ;
\] (2.10)
the function \((f, t) \mapsto e^{tA}f\) is continuous from \(F_+ \times (0, +\infty)\) to \(F\) (in a few words: for all \(t > 0\), \(e^{tA}\) regularizes the vectors of \(F_+\), sending them into \(F\) continuously). Furthermore, \(u_- \in \mathcal{C}((0, +\infty), (0, +\infty))\) is a function such that

\[
\|e^{tA}f\| \leq u_-(t)\|f\|_+ \quad \text{for } t > 0, \ f \in F_+ ;
\]

\[
u_-(t) = O(\frac{1}{t^{1-\sigma}}) \quad \text{for } t \to 0^+, \ \sigma \in (0, 1]. \tag{2.12}
\]

The function \(u_-\) will be referred to as an estimator for the semigroup \(e^{tA}\) with respect to the norms of \(F\) and \(F_\); Eq. (2.12) ensures its integrability in any right neighbourhood of \(t = 0\).

(P6) One has

\[
\mathcal{P} : \text{Dom}\mathcal{P} \subset F \times R \to F_-, \quad (f, t) \mapsto \mathcal{P}(f, t), \tag{2.13}
\]

and the domain of \(\mathcal{P}\) is semi-open in \(F \times R\): by this we mean that, for any \((f_0, t_0) \in \text{Dom}\mathcal{P}\), there are \(\delta, r \in (0, +\infty]\) such that \(B(f_0, r) \times [t_0, t_0 + \delta) \subset \text{Dom}\mathcal{P}\). Furthermore, \(\mathcal{P}\) is Lipschitz on each closed, bounded subset \(C\) of \(F \times R\) such that \(C \subset \text{Dom}\mathcal{P}\); by this, we mean that there are constants \(L = L(C)\) and \(M = M(C) \in [0, +\infty)\) such that

\[
\|\mathcal{P}(f, t) - \mathcal{P}(f', t')\|_+ \leq L\|f - f'\| + M|t - t'| \quad \text{for all } (f, t), (f', t') \in C . \tag{2.14}
\]

\textbf{2.1 Remark.} As anticipated, our aim is to discuss the Cauchy problem \(\dot{\varphi}(t) = A\varphi(t) + \mathcal{P}(\varphi(t), t), \ \varphi(t_0) = f_0\) (and its equivalent formulation as a Volterra problem) for a system \((F_+, F_-, A, u, u_-, \mathcal{P})\) with the previously mentioned properties (P1),...,(P6). In comparison with the present work, the analysis of [12] corresponds to the special case

\[
F_- = F , \quad u_- = u \tag{2.15}
\]

in which, by the continuity of \(u\) at \(t = 0\), Eq. (2.12) is fulfilled with \(\sigma = 1\) \((^1)\).

\textbf{Preliminaries to the analysis of the Cauchy and Volterra problems.}

(i) In the sequel, whenever we consider an interval \([t_0, T]\), we intend \(-\infty < t_0 < T \leq +\infty\) \((^2)\).

(ii) Let us consider a function \(\omega \in \mathcal{C}([t_0, T), F_-)\) and the function

\[
\Omega : t \in [t_0, T) \mapsto \Omega(t) := \int_{t_0}^t ds e^{(t-s)A}\omega(s) . \tag{2.16}
\]

\(^1\)In [12] \(e^{tA}\) was written \(U(t)\), and \(F_-\) was simply indicated with \(\text{Dom}A\); furthermore, we assumed \(\text{Dom}\mathcal{P}\) to be open in \(R \times F\).

\(^2\)In [12], we also considered solutions of the Cauchy or Volterra problems with domain a closed, bounded interval \([t_0, T]\); the symbol \([t_0, T]\) was employed to denote an interval of either type. Here we only consider the first case (semiopen, possibly unbounded), simply to avoid tedious distinctions.
Proof

It is based on (2.17): see [1]. The derivation of (ii), which is the most

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formal definitions of the Cauchy and Volterra problems. These definitions

are similar to the ones adopted in [12], with slight changes due to the present use of
two different spaces \( F, F_\pm \).

2.2 Definition. Consider a pair \((f_0, t_0)\) \(\in\) \(\text{DomP}\), with \(f_0 \in F_+\). The Cauchy problem \(\text{CP}(f_0, t_0)\) with datum \(f_0\) at time \(t_0\) is the following one:

Find \(\varphi \in C([t_0, T), F_+) \cap C^1([t_0, T), F_-)\) such that \(\text{gr} \varphi \subset \text{DomP}\) and

\[
\varphi(t) = A\varphi(t) + P(\varphi(t), t) \quad \text{for all} \ t \in [t_0, T) , \quad \varphi(t_0) = f_0 .
\]

We note that \(C([t_0, T), F_+) \subset C([t_0, T), F)\). This fact, with the properties of \(A\) and \(P\), implies the following: if \(\varphi \in C([t_0, T), F_+)\) and \(\text{gr} \varphi \subset \text{DomP}\), the right hand side of the differential equation in (2.18) defines a function in \(C([t_0, T), F_-)\).

2.3 Definition. Consider a pair \((f_0, t_0)\) \(\in\) \(\text{DomP}\). The Volterra problem \(\text{VP}(f_0, t_0)\) with datum \(f_0\) at time \(t_0\) is the following one:

Find \(\varphi \in C([t_0, T), F)\) such that \(\text{gr} \varphi \subset \text{DomP}\) and

\[
\varphi(t) = e^{(t-t_0)A}f_0 + \int_{t_0}^t ds \ e^{(t-s)A}P(\varphi(s), s) \quad \text{for all} \ t \in [t_0, T) .
\]

2.4 Proposition. For \((f_0, t_0)\) \(\in\) \(\text{DomP}\) and \(f_0 \in F_+\), we have the following.

(i) a solution \(\varphi\) of \(\text{CP}(f_0, t_0)\) is also solution of \(\text{VP}(f_0, t_0)\);

(ii) a solution \(\varphi\) of \(\text{VP}(f_0, t_0)\) is also a solution of \(\text{CP}(f_0, t_0)\), if \(F_-\) is reflexive.

Proof It is based on (2.17): see [1]. The derivation of (ii), which is the most
technical part, uses the Lipschitz property (P6) of \(P\) and the reflexivity of \(F_-\) to
show that a solution of \(\text{VP}(f_0, t_0)\) has the necessary regularity to fulfill \(\text{CP}(f_0, t_0)\). □
2.5 Proposition. (Uniqueness theorem for the Volterra problem). Consider a pair \((f_0, t_0) \in \text{Dom}\mathcal{P}\), and assume that \(\mathcal{VP}(f_0, t_0)\) has two solutions \(\varphi \in C([t_0, T], F)\), \(\varphi' \in C([t_0, T'), F)\). Then
\[
\varphi(t) = \varphi'(t) \quad \text{for } t \in [t_0, \min(T, T')] .
\] (2.20)

Proof. We consider any \(\tau \in [t_0, \min(T, T')]\), and show that \(\varphi = \varphi'\) in \([t_0, \tau]\). To this purpose, we subtract Eq. (2.19) for \(\varphi\) from the analogous equation for \(\varphi'\); taking the norm \(|| \cdot ||\) and using Eqs. (2.11) (2.12) (2.14), for each \(t \in [t_0, \tau]\) we obtain:
\[
||\varphi(t) - \varphi'(t)|| \leq \int_{t_0}^{t} ds \, u_-(t - s)||\mathcal{P}(\varphi(s), s) - \mathcal{P}(\varphi'(s), s)||_-
\] (2.21)
\[
\leq UL \int_{t_0}^{t} ds \frac{||\varphi(s) - \varphi'(s)||}{(t - s)^{1-\sigma}} .
\]
In the above: \(L \geq 0\) is a constant fulfilling the Lipschitz condition (2.14) for \(\mathcal{P}\) on the set \(\mathcal{C} := \text{gr}(\varphi \upharpoonright [t_0, \tau]) \cup \text{gr}(\varphi' \upharpoonright [t_0, \tau])\); \(U \geq 0\) is a constant such that \(u_-(t') \leq U/t^{1-\sigma}\) for all \(t' \in (0, \tau)\) (which exists due to (2.12)).

Eq. (2.21) implies \(||\varphi(t) - \varphi'(t)|| = 0\) for all \(t \in [t_0, \tau]\); in fact, this result follows applying to the function \(z(t) := ||\varphi(t) - \varphi'(t)||\) the forthcoming Lemma. \(\square\)

2.6 Lemma. Consider a function \(z \in C([t_0, \tau], [0, +\infty))\) (with \(-\infty < t_0 < \tau < +\infty\)), and assume there are \(\Lambda \in [0, +\infty), \sigma \in (0, 1]\) such that
\[
z(t) \leq \Lambda \int_{t_0}^{t} ds \frac{z(s)}{(t - s)^{1-\sigma}} \quad \text{for } t \in [t_0, \tau] .
\] (2.22)
Then, \(z(t) = 0\) for all \(t \in [t_0, \tau]\).

Proof. See Appendix A. \(\square\)

2.7 Remark. For \(\mathcal{VP}(f_0, t_0)\) we will grant existence as well, on sufficiently small time intervals (see the forthcoming Proposition 3.10, where local existence is obtained as a simple application of the general theory of approximate solutions).

The Volterra integral operator. This is the (nonlinear) integral operator appearing in problem \(\mathcal{VP}(f_0, t_0)\). More precisely, let us state the following.

2.8 Definition. Let \((f_0, t_0) \in \text{Dom}\mathcal{P}\). The Volterra integral operator \(\mathcal{J}(f_0, t_0) \equiv \mathcal{J}\) associated to this pair is the following map:
(i) \(\text{Dom}\mathcal{J}\) is made of the functions \(\psi \in C([t_0, T], F)\) (with arbitrary \(T \in (t_0, +\infty])\) such that \(\text{gr}\psi \subset \text{Dom}\mathcal{P}\);
(ii) for each \(\psi\) in this domain, \(\mathcal{J}(\psi) \in C([t_0, T], F)\) is the function
\[
t \in [t_0, T) \mapsto \mathcal{J}(\psi)(t) := e^{(t-t_0)A}f_0 + \int_{t_0}^{t} ds \, e^{(t-s)A}\mathcal{P}(\psi(s), s) .
\] (2.23)
3 Approximate solutions of the Volterra and Cauchy problems: the main result.

Throughout the section, we consider again a set \((F_+, F, F_-, A, u, u_-, P)\), with the properties (P1)-(P6) of the previous section. The definitions that follow generalize similar notions, introduced in [12].

Approximate solutions, and their errors. We introduce them in the following way.

3.1 Definition. Let \((f_0, t_0) \in \text{Dom} P\).
(i) An approximate solution of \(\text{VP}(f_0, t_0)\) is any function \(\varphi_{ap} \in C([t_0, T), F)\) such that \(gr\varphi_{ap} \subset \text{Dom} P\).
(ii) The integral error of \(\varphi_{ap}\) is the function
\[
E(\varphi_{ap}) := \varphi_{ap} - J(\varphi_{ap}) \in C([t_0, T), F) ,
\]
i.e., \(E(\varphi_{ap})(t) = \varphi_{ap}(t) - e^{(t-t_0)}A f_0 - \int_{t_0}^t ds e^{(t-s)}A P(\varphi_{ap}(s), s)\).
An integral error estimator for \(\varphi_{ap}\) is a function \(E \in C([t_0, T), [0, +\infty))\) such that, for all \(t\) in this interval,
\[
\|E(\varphi_{ap})(t)\| \leq E(t) . \tag{3.2}
\]

3.2 Definition. Let \((f_0, t_0) \in \text{Dom} P\), and \(f_0 \in F_+\).
(i) An approximate solution of \(\text{CP}(f_0, t_0)\) is any function \(\varphi_{ap} \in C([t_0, T), F_+) \cap C^1([t_0, T), F_-)\) such that \(gr\varphi_{ap} \subset \text{Dom} P\).
(ii) The datum error for \(\varphi_{ap}\) is the difference
\[
d(\varphi_{ap}) := \varphi_{ap}(t_0) - f_0 \in F_+ \subset F ; \tag{3.3}
\]
a datum error estimator for \(\varphi_{ap}\) is a nonnegative real number \(\delta\) such that
\[
\|d(\varphi_{ap})\| \leq \delta . \tag{3.4}
\]
(iii) The differential error of \(\varphi_{ap}\) is the function
\[
e(\varphi_{ap}) \in C([t_0, T), F_-), \quad t \mapsto e(\varphi_{ap})(t) := \dot{\varphi}_{ap}(t) - A \varphi_{ap}(t) - P(\varphi_{ap}(t), t) ; \tag{3.5}
\]
a differential error estimator for \(\varphi_{ap}\) is a function \(e \in C([t_0, T), [0, +\infty))\) such that, for \(t\) in this interval,
\[
\|e(\varphi_{ap})(t)\|_- \leq e(t) . \tag{3.6}
\]

3.3 Remarks. (i) A function \(\varphi_{ap}\) as in Definition 3.1 (resp., Definition 3.2) is a solution of \(\text{VP}(f_0, t_0)\) (resp., of \(\text{CP}(f_0, t_0)\)) if and only if \(E(\varphi_{ap}) = 0\) (resp., \(d(\varphi_{ap}) = 0\) and \(e(\varphi_{ap}) = 0\)).
(ii) Of course, the previous definitions of the error estimators can be fulfilled setting \(E(t) := \|E(\varphi_{ap})(t)\|, \delta := \|d(\varphi_{ap})\|, e(t) := \|e(\varphi_{ap})(t)\|_-\).
3.4 Lemma. Let \((f_0, t_0) \in \text{Dom}\mathcal{P}, f_0 \in F_+,\) and \(\varphi_{ap}\) be an approximate solution of \(\mathcal{CP}(f_0, t_0)\) with datum and differential errors \(d(\varphi_{ap}), e(\varphi_{ap})\). Then:

(i) \(\varphi_{ap}\) is also an approximate solution of \(\mathcal{VP}(f_0, t_0)\), with integral error

\[
E(\varphi_{ap})(t) = e(t-t_0)A d(\varphi_{ap}) + \int_{t_0}^{t} ds \ e(t-s)A e(\varphi_{ap})(s).
\]  

(ii) If \(\delta, \epsilon\) are datum and differential error estimators for \(\varphi_{ap}\), an integral error estimator for \(\varphi_{ap}\) is

\[
\mathcal{E}(t) := u(t-t_0) \delta + \int_{t_0}^{t} ds \ u_-(t-s)\epsilon(s) \quad \text{for all} \ t \in [t_0, T).
\]  

Proof. (i) To derive Eq. (3.7), use the definitions of \(E(\varphi_{ap}), d(\varphi_{ap}), e(\varphi_{ap})\) and the identity (2.17) with \(\psi := \varphi_{ap}\).

(ii) To derive the estimator (3.8), apply \(\parallel\ \parallel\) to both sides of (3.7), using Eqs. (2.9), (2.11) for \(u, u_-,\) and Eqs. (3.4), (3.6) for \(\delta, \epsilon\).

\[\square\]

Growth of \(\mathcal{P}\) from a curve. To introduce this notion, we need some notations. Let us consider a function \(\rho \in C([t_0, T), (0, +\infty])\); we recall that, according to (2.3), the subgraph of \(\rho\) is the set \(\text{sgt}\rho := \{(r, t) \mid t \in [t_0, T), r \in [0, \rho(t))\}\). Furthermore, let \(\phi \in C([t_0, T), F)\). We define the \(\rho\)-tube around \(\phi\) as the set

\[
\mathcal{T}(\phi, \rho) := \{(f, t) \mid f \in F, t \in [t_0, T), \|f - \phi(t)\| < \rho(t)\};
\]  

of course, the above tube is the whole space \(F\) if \(\rho(t) = +\infty\) for all \(t\) \(^3\).

3.5 Definition. Let \(\phi \in C([t_0, T), F)\), with \(\text{gr}\phi \subset \text{Dom}\mathcal{P}\). A growth estimator for \(\mathcal{P}\) from \(\phi\) is a function \(\ell\) with these features.

(i) The domain of \(\ell\) is the subgraph of some function \(\rho \in C([t_0, T), (0, +\infty))\), and

\[
\ell \in C(\text{sgt}\rho, [0, +\infty)) \quad (r, t) \mapsto \ell(r, t);
\]  

\(\ell\) is nondecreasing in the first variable, i.e., \(\ell(r, t) \leq \ell(r', t)\) for \(r \leq r'\) and any \(t\).

(ii) The function \(\rho\) in (i) is such that \(\mathcal{T}(\phi, \rho) \subset \text{Dom}\mathcal{P}\). For all \((f, t) \in \mathcal{T}(\phi, \rho)\), it is

\[
\|\mathcal{P}(f, t) - \mathcal{P}(\phi(t), t)\|_\perp \leq \ell(\|f - \phi(t)\|, t).
\]  

3.6 Remark. Consider any tube \(\mathcal{T}(\phi, \rho') \subset \text{Dom}\mathcal{P}\). Using the Lipschitz property (2.14) of \(\mathcal{P}\), one can easily construct a growth estimator \(\ell\) of domain \(\text{sgt}(\rho'/2)\), depending linearly on \(r\): \(\ell(r, t) = \lambda(t)r\).

\[^3\text{In [12], this notion was presented in the case }\rho = \text{constant; the present generalization is harmless, and could have been employed in our previous work as well.}\]
The main result on approximate solutions. This is contained in the following

3.7 Proposition. Let \((f_0, t_0) \in \text{Dom} \mathcal{P}\), and consider the problem \(\mathcal{V} \mathcal{P}(f_0, t_0)\). Suppose that:

(i) \(\varphi_{ap} \in C([t_0, T), F)\) is an approximate solution of \(\mathcal{V} \mathcal{P}(f_0, t_0)\), \(\mathcal{E} \in C([t_0, T), [0, +\infty))\) is an estimator for the integral error \(E(\varphi_{ap})\);

(ii) \(\ell \in C(\text{sgrp}, [0, +\infty))\) is a growth estimator for \(\mathcal{P}\) from \(\varphi_{ap}\) (for a suitable \(\rho \in C([t_0, T), (0, +\infty))\).

Consider the following problem:

Find \(\mathcal{R} \in C([t_0, T), [0, +\infty))\) such that \(\text{gr} \mathcal{R} \subset \text{sgrp}\), and

\[
\mathcal{E}(t) + \int_{t_0}^{t} ds \ u_-(t - s) \ \ell(\mathcal{R}(s), s) \leq \mathcal{R}(t) \quad \text{for } t \in [t_0, T) .
\]  

(3.12)

If (3.12) has a solution \(\mathcal{R}\) on \([t_0, T)\), then \(\mathcal{V} \mathcal{P}(f_0, t_0)\) has a solution \(\varphi\) with the same domain, and

\[
\|\varphi(t) - \varphi_{ap}(t)\| \leq \mathcal{R}(t) \quad \text{for } t \in [t_0, T) .
\]  

(3.13)

The solution \(\varphi\) is constructed by a Peano-Picard iteration of \(J\), starting from \(\varphi_{ap}\).

Proof. See the next section. \(\square\)

3.8 Definition. Eq.(3.12) will be referred to as the control inequality.

3.9 Remarks. (i) It is worthwhile stressing the following: the estimators \(u_-, \ell, \mathcal{E}\) in the control inequality (3.12) depend on \(A, \mathcal{P}, \varphi_{ap}\), and should be regarded as known when the Volterra problem and the approximate solution are specified. So, (3.12) is a problem in one unknown \(\mathcal{R}\), that one sets up using only informations about \(\varphi_{ap}\). After \(\mathcal{R}\) has been found, it is possible to draw conclusions about the (exact) solution \(\varphi\) of \(\mathcal{V} \mathcal{P}(f_0, t_0)\). In the usual language: the control inequality allows predictions on \(\varphi\) through an a posteriori analysis of \(\varphi_{ap}\).

(ii) (extending to the present framework a comment in [12]). Typically, one meets this situation: \(\varphi_{ap}, \mathcal{E}, \ell\) are defined for \(t\) in some interval \([t_0, T')\), and the control inequality (3.12) has a solution \(\mathcal{R}\) on an interval \([t_0, T) \subset [t_0, T')\); in this case one renames \(\varphi_{ap}, \mathcal{E}, \ell\) etc. the restrictions of the previous functions to \([t_0, T)\), and applies Proposition 3.7 to them (as an example, this occurs essentially in the proof of the forthcoming result).

A first implication of Proposition 3.7: local existence. The most general and simple consequence of Proposition 3.7 is the fact anticipated in Remark 2.7, i.e., the local existence for the Volterra problem. Here we formulate this statement precisely.
3.10 Proposition. Let \((f_0, t_0) \in \text{Dom} \mathcal{P}\). Then, there are \(R', T', \mathcal{E}, \ell\) such that (i)-(iii) hold:

(i) \(R' \in (0, +\infty], T' \in (t_0, +\infty]\) and \(B(f_0, R') \times [t_0, T') \subset \text{Dom} \mathcal{P}\);
(ii) \(\mathcal{E} \in C([t_0, T'), [0, +\infty))\) and, for all \(t \in [t_0, T')\),

\[
\|f_0 - e^{(t-t_0)A}f_0 - \int_{t_0}^{t} ds \, e^{(t-s)A} \mathcal{P}(f_0, s)\| \leq \mathcal{E}(t) , \quad \mathcal{E}(t_0) = 0 ; \quad (3.14)
\]

(iii) \(\ell \in C([0, R') \times [t_0, T'), [0, +\infty))\), \((r, t) \mapsto \ell(r, t)\); this function is non decreasing in the first variable and, for \((f, t) \in B(f_0, R') \times [t_0, T')\),

\[
\|\mathcal{P}(f, t) - \mathcal{P}(f_0, t)\|_{-} \leq \ell(\|f - f_0\|, t) . \quad (3.15)
\]

Given \(R', T', \mathcal{E}, \ell\) with properties (i)-(iii), we have (a)(b):

(a) there are \(R \in (0, R')\) and \(T \in (t_0, T')\) such that, for all \(t \in [t_0, T)\),

\[
\mathcal{E}(t) + \int_{t_0}^{t} ds \, u_-(t-s) \ell(R, s) \leq R ; \quad (3.16)
\]

(b) if \(T\) and \(R\) are as in item (a), \(\mathcal{V}(f_0, t_0)\) has a solution \(\varphi\) of domain \([t_0, T)\) and, for all \(t\) in this interval,

\[
\|\varphi(t) - f_0\| \leq R . \quad (3.17)
\]

Proof. Step 1. Existence of \(R', T', \mathcal{E}, \ell\). A pair \((R', T')\) as in (i) exists because \(\text{Dom} \mathcal{P}\) is semi-open (see the explanations in (P6)). A function \(\mathcal{E}\) as in (ii) can be constructed setting \(\mathcal{E}(t) := \text{the left hand side of the inequality in (3.14)}\). A function \(\ell\) as in (iii) is constructed putting \(\ell(r, t) := \sup_{f \in B(f_0, R)} \|\mathcal{P}(f, t) - \mathcal{P}(f_0, t)\|_{-}\); this sup is proved to be finite using the Lipschitz type inequality (2.14) for \(\mathcal{P}\) on each set \(\mathcal{C} := \overline{B(f_0, R)} \times \{t\}\). One checks that \(\ell(r, t)\) is continuous in \((r, t)\) and non decreasing in \(r\).

Step 2. Proof of (a). Let us pick up any \(R \in (0, R')\), and define

\[
\mathcal{G} : [t_0, T') \to [0, +\infty) , \quad t \mapsto \mathcal{G}(t) := \mathcal{E}(t) + \int_{t_0}^{t} ds \, u_-(t-s) \ell(R, s) . \quad (3.18)
\]

Then \(\mathcal{G}\) is continuous and \(\mathcal{G}(t_0) = 0\); this fact, with the positivity of \(R\), implies the existence of \(T \in [t_0, T')\) such that \(\mathcal{G}(t) \leq R\) for all \(t \in [t_0, T)\); the last inequality is just the thesis (3.16).

Step 3. Proof of (b). We apply Proposition 3.7 to the approximate solution

\[
\varphi_{ap}(t) := f_0 \quad \text{for all } t \in [t_0, T) . \quad (3.19)
\]

Due to (i)-(iii), the function \(\mathcal{E} \uparrow [t_0, T)\) is an integral error estimator for \(\varphi_{ap}\), and the function \(\ell \uparrow [0, R') \times [t_0, T)\) is a growth estimator for \(\mathcal{P}\) from \(\varphi_{ap}\) (the function
\( \rho \) in the general Definition 3.5 of growth estimator is given in this case by \( \rho(t) = \text{constant} = R' \). Eq. (3.16) tells us that the general control inequality (3.12) is fulfilled by the function \( R(t) := \text{constant} = R \) for all \( t \in [t_0, T) \). So, Proposition 3.7 implies the existence of a solution \( \varphi \) of \( \mathcal{W}(f_0, t_0) \) on \( [t_0, T) \), and also gives the inequality (3.17).

4 Proof of Proposition 3.7.

Let us make all the assumptions in the statement of the Proposition. We begin the proof introducing an appropriate topology for the space of continuous functions \([t_0, T) \to F\).

4.1 Definition. From now on, \( C([t_0, T), F) \) will be viewed as a (Hausdorff, complete) locally convex space with the topology of uniform convergence on all compact subintervals \([t_0, \tau] \subset [t_0, T)\). By this, we mean the topology induced by the seminorms \( (\| \cdot \|_\tau)_{\tau \in [t_0, T)} \), where

\[
\| \cdot \| : C([t_0, T), F) \to [0, +\infty), \quad \psi \mapsto \| \psi \|_\tau := \sup_{t \in [t_0, \tau]} \| \psi(t) \| \quad (4.1)
\]

(\( \| \cdot \| \) is the usual norm of \( F \)).

4.2 Remark. The uncountable family of seminorms \( (\| \cdot \|_\tau) \) is topologically equivalent to the countable subfamily \( (\| \cdot \|_{\tau_n}) \), where \( (\tau_n) \) is any sequence of points of \([t_0, T)\) such that \( \lim_{n \to +\infty} \tau_n = T \). Therefore, \( C([t_0, T), F) \) is a Fréchet space. \( \diamond \)

To go on we introduce a basic set, whose definition depends on \( \varphi_{ap} \) and on the function \( \mathcal{R} \) in the control inequality (3.12).

4.3 Definition. We put

\[
\mathcal{D} := \{ \psi \in C([t_0, T), F) \mid \| \psi(t) - \varphi_{ap}(t) \| \leq \mathcal{R}(t) \quad \text{for} \quad t \in [t_0, T) \} \quad (4.2)
\]

4.4 Lemma. (i) \( \mathcal{D} \) is a closed subset of \( C([t_0, T), F) \), in the topology of Definition 4.1.

(ii) For all \( \psi \in \mathcal{D} \), one has \( \text{gr} \psi \subset \text{Dom} \mathcal{P} \) (and so, \( \mathcal{J}(\psi) \) is well defined).

Proof. (i) Suppose \( \psi \in C([t_0, T), F) \) and \( \psi = \lim_{n \to -\infty} \psi_n \), where \( (\psi_n) \) is a sequence of elements of \( \mathcal{D} \). Then, for all \( t \in [t_0, T) \) we have

\[
\| \psi(t) - \varphi_{ap}(t) \| = \lim_{n \to -\infty} \| \psi_n(t) - \varphi_{ap}(t) \| \leq \mathcal{R}(t).
\]

(ii) Let us consider the function \( \rho \in C([t_0, T), (0, +\infty)) \) mentioned in the statement of Proposition 3.7. Then, \( \psi \in \mathcal{D} \implies \| \psi(t) - \varphi_{ap}(t) \| \leq \mathcal{R}(t) < \rho(t) \) for all \( t \in [t_0, T) \) \( \implies \text{gr} \psi \subset \mathcal{I}(\varphi_{ap}, \rho) \subset \text{Dom} \mathcal{P}. \) \( \square \)
From now on, our attention will be focused on the map
\[ \mathcal{D} \to C([t_0, T], \mathbb{F}), \quad \psi \mapsto \mathcal{J}(\psi). \] (4.3)

Of course, for \( \varphi \in \mathcal{D} \), we have the equivalence
\[ \varphi \text{ solves } \mathcal{VP}(f_0, t_0) \iff \mathcal{J}(\varphi) = \varphi. \] (4.4)

To clarify the sequel, let us recall that \( \sigma \in (0, 1] \) is the constant appearing in Eq. (2.12).

4.5 Lemma. (i) For each \( \tau \in [t_0, T) \) there is a constant \( \Lambda_\tau \in [0, +\infty) \) such that, for all \( \psi, \psi' \in \mathcal{D} \),
\[ \| \mathcal{J}(\psi)(t) - \mathcal{J}(\psi')(t) \| \leq \Lambda_\tau \int_{t_0}^{t} ds \frac{\|\psi(s) - \psi'(s)\|}{(t-s)^{1-\sigma}} \text{ for } t \in [t_0, \tau]. \] (4.5)

(ii) For all \( \tau \in [t_0, T) \) and \( \psi, \psi' \in \mathcal{D} \), the above equation implies a Lipschitz type inequality
\[ \| \mathcal{J}(\psi) - \mathcal{J}(\psi') \|_\tau \leq \frac{\Lambda_\tau (\tau - t_0)^{\sigma}}{\sigma} \|\psi - \psi'\|_\tau \] (4.6)
(which ensures, amongst else, the continuity of \( \mathcal{J} \) on \( \mathcal{D} \)).

Proof. (i) Let \( \psi, \psi' \in \mathcal{D} \). We consider Eq. (2.23) for \( \mathcal{J}(\psi)(t) \), and subtract from it the analogous one for \( \mathcal{J}(\psi')(t) \). After applying the norm of \( \mathcal{F} \) and using (2.11), we infer
\[ \| \mathcal{J}(\psi)(t) - \mathcal{J}(\psi')(t) \| \leq \int_{t_0}^{t} ds \ u_-(t-s) \| \mathcal{P}(\psi(s), s) - \mathcal{P}(\psi'(s), s) \|_\tau \] (4.7)
for all \( t \in [t_0, T) \). To go on, we fix \( \tau \in [t_0, T) \) and define
\[ C_\tau := \{(f, s) \in \mathcal{F} \times [t_0, \tau] \mid \|f - \varphi_{ap}(s)\| \leq \mathcal{R}(s)\}. \] (4.8)
This is a closed, bounded subset of \( \mathcal{F} \times \mathbb{R} \), and \( C_\tau \subset \bar{\mathcal{J}(\varphi_{ap}, \rho)} \subset \text{Dom} \mathcal{P} \). Therefore, by the Lipschitz property (P6) of \( \mathcal{P} \), there is a nonnegative constant \( L(C_\tau) \equiv L_\tau \) such that
\[ \| \mathcal{P}(f, s) - \mathcal{P}(f', s) \|_\tau \leq L_\tau \|f - f'\| \text{ for } (f, s), (f', s) \in C_\tau. \] (4.9)
Furthermore, recalling Eq. (2.12) for \( u_- \), we see that there is another constant \( U_\tau \) such that
\[ u_-(t') \leq \frac{U_\tau}{t'^{1-\sigma}} \text{ for } t' \in (0, \tau - t_0]. \] (4.10)
Inserting Eqs. (4.9) (4.10) into (4.7), we obtain the thesis (4.5) with \( \Lambda_\tau := U_\tau L_\tau \).
(ii) For each $t \in [t_0, \tau]$, Eq. (4.5) implies

$$
\| J(\psi)(t) - J(\psi')(t) \| \leq \Lambda_\tau \| \psi - \psi' \|_\tau \int_{t_0}^{t} ds \frac{(t - s)^{1-\sigma}}{(t - s)^{1-\sigma}} = \Lambda_\tau \| \psi - \psi' \|_\tau (t - t_0)^{\sigma}.
$$

Taking the sup over $t$, we obtain the thesis (4.6). \qed

4.6 Lemma \textit{(Main consequence of the control inequality for $\mathcal{R}$).} One has

$$J(\mathcal{D}) \subset \mathcal{D}. \quad (4.11)$$

\textbf{Proof.} Let $\psi \in \mathcal{D}$; then

$$J(\psi) - \varphi_{ap} = [J(\varphi_{ap}) - \varphi_{ap}] + [J(\psi) - J(\varphi_{ap})] = -E(\varphi_{ap}) + [J(\psi) - J(\varphi_{ap})]. \quad (4.12)$$

We write this equality at any $t \in [t_0, T)$, explicitating $J(\psi)(t) - J(\varphi_{ap})(t)$; this gives

$$J(\psi)(t) - \varphi_{ap}(t) = -E(\varphi_{ap})(t) + \int_{t_0}^{t} ds \ e^{(t-s)A} \left[ \mathcal{P}(\psi(s), s) - \mathcal{P}(\varphi_{ap}(s), s) \right]. \quad (4.13)$$

Now, we apply the norm of $\mathbf{F}$ to both sides and use Eqs. (3.2) for $E(\varphi_{ap})(t)$, (2.11) for $e^{(t-s)A}$, (3.11) for the growth of $\mathcal{P}$ from $\varphi_{ap}$; in this way we obtain

$$\| J(\psi)(t) - \varphi_{ap}(t) \| \leq \mathcal{E}(t) + \int_{t_0}^{t} ds \ u_-(t - s) \ell(\| \psi(s) - \varphi_{ap}(s) \|, s). \quad (4.14)$$

On the other hand, $\| \psi(s) - \varphi_{ap}(s) \| \leq \mathcal{R}(s)$ implies $\ell(\| \psi(s) - \varphi_{ap}(s) \|, s) \leq \ell(\mathcal{R}(s), s)$; inserting this into (4.14), and using the control inequality (3.12) for $\mathcal{R}$, we conclude

$$\| J(\psi)(t) - \varphi_{ap}(t) \| \leq \mathcal{E}(t) + \int_{t_0}^{t} ds \ u_-(t - s) \ell(\mathcal{R}(s), s) \leq \mathcal{R}(t), \quad (4.15)$$

i.e., $J(\psi) \in \mathcal{D}$. \qed

The invariance of $\mathcal{D}$ under $J$ is a central result; with the previously shown properties of $J$, it allows to set up the Peano-Picard iteration and get ultimately a fixed point of this map.

4.7 Definition. $(\varphi_k) (k \in \mathcal{N})$ is the sequence in $\mathcal{D}$ defined recursively by

$$\varphi_0 := \varphi_{ap}, \quad \varphi_k := J(\varphi_{k-1}) \ (k \geq 1). \quad (4.16)$$
4.8 Lemma. Let $\tau \in [t_0, T)$. For all $k \in \mathbb{N}$, one has

$$\|\varphi_{k+1}(t) - \varphi_k(t)\| \leq \Sigma_{\tau} \frac{\Lambda_k^k \Gamma(\sigma)^k (t - t_0)^{k\sigma}}{\Gamma(k\sigma + 1)} \quad \text{for } t \in [t_0, \tau] ,$$

where $\Lambda_{\tau}$ is the constant of Eq. (4.5) and $\Sigma_{\tau} := \max_{t \in [t_0, \tau]} \mathcal{E}(t)$. So,

$$\|\varphi_{k+1} - \varphi_k\|_{\tau} \leq \Sigma_{\tau} \frac{\Theta^k_{\tau\sigma}}{\Gamma(k\sigma + 1)} , \quad \Theta_{\tau\sigma} := \Lambda_{\tau} \Gamma(\sigma)(\tau - t_0)^{\sigma} .$$

Proof. Eq. (4.18) is an obvious consequence of (4.17). We prove (4.17) by recursion, indicating with a subscript $k$ the thesis at a specified order. We have $\varphi_1 - \varphi_0 = \mathcal{J}(\varphi_{ap}) - \varphi_{ap} = -\mathcal{E}(\varphi_{ap})$, whence $\|\varphi_1(t) - \varphi_0(t)\| \leq \mathcal{E}(t) \leq \Sigma_{\tau}$; this gives (4.17)$_0$. Now, we suppose (4.17)$_k$ to hold and infer its analogue of order $k + 1$. To this purpose, we keep in mind Eq. (4.5) and write

$$\|\varphi_{k+2}(t) - \varphi_{k+1}(t)\| = \|\mathcal{J}(\varphi_{k+1})(t) - \mathcal{J}(\varphi_k)(t)\|$$

$$\leq \Lambda_{\tau} \int_{t_0}^t ds \frac{\|\varphi_{k+1}(s) - \varphi_k(s)\|}{(t - s)^{1-\sigma}} \leq \Lambda_{\tau} \Sigma_{\tau} \frac{\Lambda_k^k \Gamma(\sigma)^k}{\Gamma(k\sigma + 1)} \int_{t_0}^t ds \frac{(s - t_0)^{k\sigma}}{(t - s)^{1-\sigma}} .$$

On the other hand, we have (4)

$$\int_{t_0}^t ds \frac{(s - t_0)^{k\sigma}}{(t - s)^{1-\sigma}} = \frac{\Gamma(k\sigma + 1)\Gamma(\sigma)}{\Gamma((k + 1)\sigma + 1)} (t - t_0)^{(k+1)\sigma} ;$$

inserting (4.20) into (4.19) we obtain the thesis (4.17)$_{k+1}$. □

The next (and final) Lemma is a generalization of the inequality (4.18), based on the Mittag-Leffler function $E_{\sigma}$ (see, e.g., [16]). For any $\sigma > 0$, this is the entire function defined by

$$E_{\sigma} : \mathbb{C} \to \mathbb{C} , \quad z \mapsto E_{\sigma}(z) := \sum_{\ell=0}^{+\infty} \frac{z^\ell}{\Gamma(\ell\sigma + 1)} .$$

In particular, $E_{\sigma}(z) \in [1, +\infty)$ for all $z \in [0, +\infty)$ and $E_1(z) = e^z$ for all $z \in \mathbb{C}$.

4To check this, make in the integral the change of variable $s = t_0 + x(t - t_0)$, with $x \in [0, 1]$, and then use the general identity

$$\int_0^1 dx x^{\alpha-1}(1-x)^{\beta-1} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)} \quad \text{for } \alpha, \beta > 0 .$$
4.9 Lemma. For all \( \tau \in [t_0, T) \) and \( k, k' \in \mathbb{N} \),
\[ \| \varphi_{k'} - \varphi_k \|_\tau \leq \sum_{j=k}^{k'-1} \frac{\Theta_j^\tau}{\Gamma(j\sigma + 1)} E_\sigma(\Theta_j^\tau), \quad h := \min(k, k') \tag{4.22} \]
(\( \Theta_j^\tau \) being defined by (4.18)); this implies that \( \varphi_k \) is a Cauchy sequence.

Proof. It suffices to consider the case \( k' > k \) (so that \( h = k \)). Writing \( \varphi_{k'} - \varphi_k = \sum_{j=k}^{k'-1} (\varphi_{j+1} - \varphi_j) \) and using Eq.(4.18) we get
\[ \| \varphi_{k'} - \varphi_k \|_\tau \leq \sum_{j=k}^{k'-1} \frac{\Theta_j^\tau}{\Gamma(j\sigma + 1)} . \tag{4.23} \]
On the other hand, for each \( z \geq 0 \),
\[ \sum_{j=k}^{k'-1} \frac{z^j}{\Gamma(j\sigma + 1)} \leq \sum_{j=k}^{+\infty} \frac{z^j}{\Gamma(j\sigma + 1)} = z^k \sum_{\ell=0}^{+\infty} \frac{z^\ell}{\Gamma((k+\ell)\sigma + 1)} \leq \frac{z^k}{\Gamma(k\sigma + 1)} \sum_{\ell=0}^{+\infty} \frac{z^\ell}{\Gamma(\ell\sigma + 1)} = \frac{z^k}{\Gamma(k\sigma + 1)} E_\sigma(z) \tag{4.24} \]
(the last inequality depends on the relation \( \Gamma(\alpha + \beta + 1) \geq \Gamma(\alpha + 1)\Gamma(\beta + 1) \) for \( \alpha, \beta \geq 0 \). With \( z = \Theta_j^\tau, \) from (4.23) (4.24) we obtain (4.22). This equation, with the obvious fact that \( z^h/\Gamma(h\sigma + 1) \to 0 \) for \( h \to \infty \) and fixed \( z \in \mathbb{C} \), implies
\[ \| \varphi_{k'} - \varphi_k \|_\tau \to 0 \quad \text{for} \ (k, k') \to \infty , \tag{4.25} \]
for each fixed \( \tau \in [t_0, T) \). In conclusion, \( \varphi_k \) is a Cauchy sequence. \( \square \)

Proof of Proposition 3.7. \( \varphi_k \) being a Cauchy sequence, \( \lim_{k \to \infty} \varphi_k := \varphi \) exists in \( C([t_0, T), \mathbb{F}) \); \( \varphi \) belongs to \( \mathcal{D} \), because this set is closed. By the continuity of \( J \), we have
\[ J(\varphi) = \lim_{k \to \infty} J(\varphi_k) = \lim_{k \to \infty} \varphi_{k+1} = \varphi . \tag{4.26} \]
Now, recalling (4.4) we get the thesis. \( \square \)

4.10 Remark. The Mittag-Leffler function \( E_\sigma \) on \( [0, +\infty) \) is strictly related to a linear integral equation. More precisely, given \( \sigma > 0 \) let us consider the following problem: find \( G \in C([0, +\infty), \mathbb{R}) \) such that
\[ G(t) = 1 + \frac{1}{\Gamma(\sigma)} \int_0^t ds \frac{G(s)}{(t-s)^{1-\sigma}} \quad \text{for all} \ t \in [0, +\infty). \tag{4.27} \]
This has a unique solution

\[ G(t) := E_\sigma(t^\sigma) \quad \text{for} \ t \in [0, +\infty) . \quad (4.28) \]

One checks directly that the above \( G \) solves (4.27) (5); uniqueness of the solution follows from the linearity of the problem and from Lemma 2.6.

Integral equations like (4.27) are related to the so-called “fractional differential equations” (see e.g. [4], also mentioning \( E_\sigma \)).

5 Applications of Proposition 3.7 to systems with quadratic nonlinearity: local and global results.

The setting. Throughout this section we consider a set \((F_+, F, F_-, A, u, u_-, \mathcal{P}, \xi)\) with the following features.

\( F_+ \), \( F \), \( F_- \) are Banach spaces, \( A \) is an operator and \( u, u_- \) are semigroup estimators fulfilling conditions (P1)-(P5). Furthermore:
(Q1) \( \mathcal{P} \) is a bilinear map such that

\[ \mathcal{P} : F \times F \to F_- , \quad (f, g) \mapsto \mathcal{P}(f, g) ; \quad (5.1) \]

we assume continuity of \( \mathcal{P} \), which is equivalent to the existence of a constant \( K \in (0, +\infty) \) such that

\[ \| \mathcal{P}(f, g) \|_- \leq K \| f \| \| g \| \quad (5.2) \]

for all \( f, g \in F \) (6).

(Q2) We have

\[ \xi \in C^{0,1}([0, +\infty), F_-) , \quad (5.3) \]

(recall that \( C^{0,1} \) stands for the locally Lipschitz maps).

Having made the above assumptions, let us fix some notations.

5.1 Definition. From now on:

(i) \( \mathcal{U} \in C([0, +\infty), [0, +\infty)) \) is a nondecreasing function such that

\[ \int_0^t ds u_-(s) \leq \mathcal{U}(t) \quad \text{for} \ t \in [0, +\infty) , \quad \mathcal{U}(0) = 0 \quad (5.4) \]

5To this purpose one inserts into (4.27) the series expansion coming from (4.21), and then uses the identity in the previous footnote.

6Of course, in the trivial case \( \mathcal{P} = 0 \) we could fulfill (5.2) with \( K = 0 \) as well. In the sequel we always assume \( K > 0 \), to avoid tedious specifications in many subsequent statements and formulas. In any case, such statements and formulas could be extended to \( K = 0 \) by elementary limiting procedures.
(e.g., \( U(t) := \int_0^t ds u_-(s) \); in any case, (5.4) and the positivity of \( u_- \) imply \( U(t) > 0 \) for all \( t > 0 \). We put \( U(+\infty) := \lim_{t \to +\infty} U(t) \in (0, +\infty] \).

(ii) \( \Xi_- \in C([0, +\infty), [0, +\infty)) \) is any nondecreasing function such that

\[
\|\xi(t)\|_\infty \leq \Xi_-(t) \quad \text{for } t \in [0, +\infty) \tag{5.5}
\]

(e.g., \( \Xi_-(t) := \sup_{s \in [0,t]} \|\xi(s)\|_- \)). We put \( \Xi_-(+\infty) := \lim_{t \to +\infty} \Xi_-(t) \in [0, +\infty] \).

(iii) \( \mathcal{P} \) is the (affine) quadratic map induced by \( \mathcal{P} \) and \( \xi \) in the following way:

\[
\mathcal{P} : \mathbf{F} \times [0, +\infty) \to \mathbf{F}_- , \quad (f, t) \mapsto \mathcal{P}(f, t) := \mathcal{P}(f, f) + \xi(t) . \tag{5.6}
\]

**Properties of \( \mathcal{P} \).** Let us analyse this map, so as to match the schemes of the previous sections. First of all we note that \( \text{Dom}\mathcal{P} \) is semiope in \( \mathbf{F} \times \mathbf{R} \), in the sense defined in (P6). Furthermore, we have the following.

### 5.2 Proposition

For all \( (f, t) \) and \( (f', t') \in \mathbf{F} \times [0, +\infty) \),

\[
\|\mathcal{P}(f, t) - \mathcal{P}(f', t')\|_- \leq 2K\|f\|_\infty \|f - f'\| + K\|f - f'\|^2 + \|\xi(t) - \xi(t')\|_- . \tag{5.7}
\]

**Proof.** For the sake of brevity, we define \( h := f - f' \). Then

\[
\mathcal{P}(f, t) = \mathcal{P}(f + h, f' + h) + \xi(t) = \mathcal{P}(f', f') + \mathcal{P}(f', h) + \mathcal{P}(h, f') + \mathcal{P}(h, h) + \xi(t) ;
\]

subtracting \( \mathcal{P}(f', t') \), we get

\[
\mathcal{P}(f, t) - \mathcal{P}(f', t') = \mathcal{P}(f', h) + \mathcal{P}(h, f') + \mathcal{P}(h, h) + \xi(t) - \xi(t') .
\]

We apply \( \| \|_- \) to both sides, taking into account Eq. (5.2); this gives \( \|\mathcal{P}(f, t) - \mathcal{P}(f', t')\|_- \leq 2K\|f\|_\infty \|h\| + K\|h\|^2 + \|\xi(t) - \xi(t')\|_- \), yielding the thesis (5.7). \( \square \)

The previous proposition has two straightforward consequences.

### 5.3 Corollary

For each bounded set \( \mathcal{C} \) of \( \mathbf{F} \times [0, +\infty) \), there are two constants \( L = L(\mathcal{C}) \), \( M = M(\mathcal{C}) \) such that

\[
\|\mathcal{P}(f, t) - \mathcal{P}(f', t')\|_- \leq L\|f - f'\| + M|t - t'| \quad \text{for } (f, t), (f', t') \in \mathcal{C} ; \tag{5.8}
\]

so, \( \mathcal{P} \) fulfills condition (P6). Let us denote with \( \mathcal{B} \) and \( \mathcal{I} \) the projections of \( \mathcal{C} \) on \( \mathbf{F} \) and \([0, +\infty) \), respectively. Then we can take \( L := 4K\|\mathcal{B}\| \), where \( \|\mathcal{B}\| := \sup_{f \in \mathcal{B} \|f\|} \); furthermore, we can take for \( M \) any constant such that \( \|\xi(t) - \xi(t')\|_- \leq M|t - t'| \) for \( t, t' \in \mathcal{I} \).

**Proof.** In Eq. (5.7), we substitute the relations \( \|f'\| \leq \|\mathcal{B}\| \), \( \|f - f'\|^2 \leq (\|f\| + \|f'\|)\|f - f'\| \leq 2\|\mathcal{B}\||f - f'\| \), and the inequality defining \( M \). \( \square \)
5.4 Corollary. Let us consider any function \( \phi \in C([0,T), F) \). Then
\[
\|P(f,t) - P(\phi(t),t)\| \leq \ell(\|f - \phi(t)\|, t) \quad \text{for all } f \in F, t \in [0,T) ,
\]
\[
\ell : [0, +\infty) \times [0, T) \to [0, +\infty) , \quad (r,t) \mapsto 2K\|\phi(t)\| r + K r^2 .
\]
The function \( \ell \) is a growth estimator for \( P \) from \( \phi \), in the sense of Definition 3.5 (with a radius of the tube \( \rho(t) := +\infty \) for all \( t \)).

Proof. Use Eq. (5.7) with \((f',t') := (\phi(t),t)\). \( \square \)

Cauchy and Volterra problems; approximate solutions. These problems will always be considered taking \( t_0 := 0 \) as the initial time; we will write
\[
VP(f_0) := VP(f_0,0) , \quad CP(f_0) := CP(f_0,0) \quad \text{for each } f_0 .
\]

For the above problems, we have the following results.
(a) If \( f_0 \in F_+ \) and \( F_- \) is reflexive, \( VP(f_0) \) is equivalent to \( CP(f_0) \) (see Proposition 2.4).
(b) For any \( f_0 \in F \), uniqueness and local existence are granted for \( VP(f_0) \); see Propositions 2.5 and 3.10.
(c) We can apply to \( VP(f_0) \) Proposition 3.7 on approximate solutions, choosing arbitrarily the approximate solution \( \varphi_{ap} \); as an error estimator, we can use at will the function \( \ell \) in Corollary 5.4 (or any upper bound for it). This yields the following statement.

5.5 Proposition. Let \( f_0 \in F \) and \( T \in (0, +\infty] \). Let us consider for \( VP(f_0) \) an approximate solution \( \varphi_{ap} \in C([0,T), F) \), and suppose there are functions \( E,D,R \in C([0,T), [0, +\infty)) \) such that (i)-(iii) hold:
(i) \( \varphi_{ap} \) has the integral error estimate
\[
\|E(\varphi_{ap}(t))\| \leq E(t) \quad \text{for } t \in [0,T) ;
\]
(ii) one has
\[
\|\varphi_{ap}(t)\| \leq D(t) \quad \text{for } t \in [0,T) ;
\]
(iii) with \( K \) as in (5.2) and \( U \) as in (5.4), \( R \) solves the control inequality
\[
E(t) + K \int_0^t ds \ u_- (t-s) (2D(s)R(s) + R^2(s)) \leq R(t) \quad \text{for } t \in [0,T) .
\]

Then, (a) and (b) hold:
(a) \( VP(f_0) \) has a solution \( \varphi : [0,T) \to F \);
(b) one has
\[
\|\varphi(t) - \varphi_{ap}(t)\| \leq R(t) \quad \text{for } t \in [0,T) .
\]
Proof. We refer to the control inequality (3.12) in Proposition 3.7 (with \( t_0 = 0 \)). Due to Corollary 5.4, the growth of \( \mathcal{P} \) from \( \varphi_{ap} \) has the quadratic estimator \( \ell(r, t) := 2K\|\varphi_{ap}(t)\| r + Kr^2 \); binding \( \|\varphi_{ap}(t)\| \) via (5.13) we get another estimator, that we call again \( \ell \), of the form

\[
\ell(r, t) = 2K\mathcal{D}(t)r + Kr^2 .
\]  

(5.16)

With this choice of \( \ell \), the control inequality (3.12) takes the form (5.14) and (a) (b) follow from Proposition 3.7. \( \square \)

How to handle the control inequality (5.14). Rephrasing Remark 3.9 in the present case, we repeat that \( \mathcal{R} \) is the only unknown in (5.14). In fact, the functions \( \mathcal{E}, \mathcal{D} \) appearing therein can be determined when \( \varphi_{ap} \) is given, and \( u_- \), \( K \) can be obtained from \( \mathcal{A}, \mathcal{P} \) (as an example, the computation of \( u_- \) and \( K \) for the NS equations will be presented in Sections 6-7). Two basic strategies to find a function \( \mathcal{R} \) solving (5.14) on an interval \([0, T]\), if it exists, can be introduced:

(a) the analytical strategy: one makes some ansatz for \( \mathcal{R} \), substitutes it into (5.14) and checks whether the inequality is fulfilled;

(b) the numerical strategy.

Let us mention that a numerical approach was presented in [12], for the simpler control inequality considered therein; in that case it was possible to transform the control equality (with \( \leq \) replaced by \( = \)) into an equivalent Cauchy problem for \( \mathcal{R} \), and then solve it by a standard package for ODEs.

The approach of [12] can not be used for (5.14) due to the singularity of \( u_-(t) \) for \( t \to 0^+ \); a different numerical attack could be used, but this is not so simple and its features suggest to treat it extensively elsewhere. For the above reasons, in the present work we only give an introductory sketch of the numerical strategy: see Appendix B. In the rest of the paper, starting from the next paragraph, we will use the analytical strategy (a).

Some special results on \( \mathcal{V}\mathcal{P}(f_0) \). All these results will be derived solving the control inequality (5.14) by analytic means, in special cases.

5.6 Proposition. Let \( f_0 \in \mathcal{F} \) and \( T \in (0, +\infty] \). Let us consider for \( \mathcal{V}\mathcal{P}(f_0) \) an approximate solution \( \varphi_{ap} \in C([0, T], \mathcal{F}) \), and suppose there are functions \( \mathcal{E}, \mathcal{D} \in C([0, T], [0, +\infty)) \) such that (i)-(iii) hold:

(i) \( \mathcal{E} \) is nondecreasing and binds the integral error as in (5.12);

(ii) \( \mathcal{D} \) is nondecreasing and binds \( \varphi_{ap} \) as in (5.13);

(iii) with \( K \) as in (5.2) and \( \mathcal{U} \) as in (5.4) \(^7\),

\[
2\sqrt{K\mathcal{U}(T)\mathcal{E}(T)} + 2K\mathcal{U}(T)\mathcal{D}(T) \leq 1 .
\]

(5.17)

\(^7\)Of course, in the case \( T = +\infty \) (5.17) implies \( \mathcal{U}(+\infty) < +\infty \)
Then $\Phi(f_0)$ has a solution $\varphi : [0, T) \to F$ and, for all $t \in [0, T)$,

$$
\|\varphi(t) - \varphi_{ap}(t)\| \leqslant \mathcal{R}(t),
$$

(5.18)

$$
\mathcal{R}(t) := \begin{cases} 
\frac{1 - 2KU(t)D(t) - \sqrt{(1 - 2KU(t)D(t))^2 - 4KU(t)E(t)}}{2KU(t)} & \text{if } t \in (0, T), \\
E(0) & \text{if } t = 0;
\end{cases}
$$

the above prescription gives a well defined, nondecreasing function $\mathcal{R} \in C([0, T), [0, +\infty)).$

**Proof.** We refer to Proposition 5.5, and try to fulfill the control inequality (5.14) with a nondecreasing $\mathcal{R} \in C([0, T), [0, +\infty))$. Noting that $\mathcal{R}(s) \leqslant \mathcal{R}(t), D(s) \leqslant D(t)$ for $s \in [0, t]$, we have

$$
\mathcal{E}(t) + K \int_0^t ds u_-(t - s) (2D(s)\mathcal{R}(s) + \mathcal{R}^2(s))
$$

(5.19)

$$
\leqslant \mathcal{E}(t) + K(2D(t)\mathcal{R}(t) + \mathcal{R}^2(t)) \int_0^t ds u_-(t - s)
$$

$$
\leqslant \mathcal{E}(t) + K(2D(t)\mathcal{R}(t) + \mathcal{R}^2(t)) \mathcal{U}(t);
$$

the last inequality follows from $\int_0^t ds u_-(t - s) = \int_0^t ds u_-(s) \leqslant \mathcal{U}(t)$. Due to (5.19), (5.14) holds if $\mathcal{E}(t) + K(2U(t)D(t)\mathcal{R}(t) + \mathcal{U}(t)\mathcal{R}^2(t)) \leqslant \mathcal{R}(t)$, i.e.,

$$
KU(t)\mathcal{R}(t)^2 - (1 - 2KU(t)D(t))\mathcal{R}(t) + \mathcal{E}(t) \leqslant 0.
$$

(5.20)

This inequality is fulfilled as an equality if we define $\mathcal{R}$ as in (5.18), provided that $2\sqrt{KU(t)\mathcal{E}(t) + 2KU(t)D(t)} \leqslant 1$; this happens for each $t \in [0, T)$ due to the assumption (5.17) (8).

The function $\mathcal{R}$ on $[0, T)$ defined by (5.18) is continuous and nonnegative; to conclude the proof, we must check it to be nondecreasing. To this purpose, we note that

$$
\mathcal{R}(t) = \Upsilon(KU(t), D(t), \mathcal{E}(t)),
$$

(5.21)

$$
\Upsilon(\mu, \delta, \epsilon) := \begin{cases} 
\frac{1 - 2\mu\delta - \sqrt{(1 - 2\mu\delta)^2 - 4\mu\epsilon}}{2\mu} & \text{if } \mu > 0, \\
\epsilon & \text{if } \mu = 0.
\end{cases}
$$

The above function $\Upsilon$ has domain

$$
\text{Dom}\Upsilon := \{(\mu, \delta, \epsilon) \mid \mu, \delta, \epsilon \geqslant 0, \ 2\mu\delta + 2\sqrt{\mu\epsilon} \leqslant 1 \}.
$$

(5.22)

8Obviously enough, we take for $\mathcal{R}(t)$ the definition (5.18) since this gives the smallest nonnegative solution of Eq. (5.20).
containing all triples \((KU(t), D(t), E(t))\) due to (5.17); one checks by elementary means (e.g., computing derivatives) that \(\Upsilon\) is a nondecreasing function of each one of the variables \(\mu, \delta, \epsilon\), when the other two are fixed. \(\square\)

5.7 Remark. The inequality (5.17) is certainly fulfilled if \((E(T), D(T))\) or \(T\) are sufficiently small (recall that \(U(T)\) vanishes for \(T \to 0^+\)).

An example: the zero approximate solution. For simplicity, we suppose

\[ u(t) \leq 1 \quad \text{for all } t \in [0, +\infty) . \quad (5.23) \]

Let \(f_0 \in \mathcal{F} , T \in (0, +\infty] \); we apply Proposition 5.6, choosing for \(\mathcal{VP}(f_0)\) the trivial approximate solution

\[ \varphi_{ap}(t) := 0 \quad \text{for } t \in [0, T) . \quad (5.24) \]

5.8 Lemma. \(\varphi_{ap} := 0\) has the integral error

\[ E(\varphi_{ap})(t) = -e^{tA}f_0 - \int_0^t ds e^{-(t-s)A}\xi(s) ; \quad (5.25) \]

with \(\Xi_\_\) as in Definition 5.5, \(\|E(\varphi_{ap})(t)\|\) has the estimate

\[ \|E(\varphi_{ap})(t)\| \leq \mathcal{F}(t) , \quad \mathcal{F}(t) := \|f_0\| + \Xi_\_(t)U(t) . \quad (5.26) \]

Proof. Eq. (5.25) follows from the general definition (3.1) of integral error, and from the fact that \(\mathcal{P}(\varphi_{ap}(s), s) = \xi(s)\).

Having Eq. (5.25), we derive (5.26) in the following way. First of all,

\[ \|E(\varphi_{ap})(t)\| \leq u(t)\|f_0\| + \int_0^t ds u_\_(t-s)\|\xi(s)\|_\_ ; \quad (5.27) \]

but \(u(t) \leq 1, \|\xi(s)\|_\_ \leq \Xi_\_(t)\) for \(s \in [0, t]\), so

\[ \|E(\varphi_{ap})(t)\| \leq \|f_0\| + \Xi_\_(t) \int_0^t ds u_\_(t-s) \leq \|f_0\| + \Xi_\_(t)U(t) . \quad \square (5.28) \]

From the previous Lemma and Proposition 5.6, we infer the following result.

5.9 Proposition. With \(u, \mathcal{F}\) as in (5.23) (5.26), assume

\[ 4KU(T)\mathcal{F}(T) \leq 1 . \quad (5.29) \]

Then \(\mathcal{VP}(f_0)\) has a solution \(\varphi : [0, T) \to \mathcal{F}\) and, for all \(t\) in this interval,

\[ \|\varphi(t)\| \leq \mathcal{F}(t) \mathcal{X}(4KU(t)\mathcal{F}(t)) . \quad (5.30) \]

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Here $\mathcal{X} \in C([0,1],[1,2])$ is the increasing function defined by

$$
\mathcal{X}(z) := \begin{cases} 
\frac{1 - \sqrt{1 - z}}{(z/2)} & \text{for } z \in (0,1), \\
1 & \text{for } z = 0.
\end{cases}
$$

\textbf{Proof.} According to the previous Lemma, we can apply Proposition 5.6 with $\varphi_{ap} = 0$, $\mathcal{E} = \mathcal{F}$; obviously, we have for $\|\varphi_{ap}(t)\|$ the estimator $D(t) := 0$. Eqs. (5.17), (5.18) yield the present relations (5.29), (5.30), (5.31); in particular, the function $\mathcal{R}$ in (5.18) is given by $\mathcal{R}(t) = (1-\sqrt{1-4K\mathcal{U}(t)\mathcal{F}(t)})/2K\mathcal{U}(t) = \mathcal{F}(t) \mathcal{X}(4K\mathcal{U}(t)\mathcal{F}(t))$.

Of course, the basic inequality (5.29) is fulfilled if $(f_0,\Xi_-(T))$ or $T$ are sufficiently small.

\textbf{Further results (global in time) for }\textbf{VP}(f_0)\textbf{, under special assumptions.}

We keep the assumptions (P1)-(P5) of Section 2 and (Q1)(Q2) at the beginning of this section, and put more specific requirements on the semigroup estimators $u,u_-$. More precisely, we add to (P4) (P5) the following conditions, involving two constants

$$
B \in [0, +\infty) , \quad N \in (0, +\infty) .
$$

(P4’) The semigroup estimator $u$ has the form

$$
u(t) = e^{-Bt} \text{ for } t \in [0, +\infty) .
$$

(P5’) The semigroup estimator $u_-$ has the form

$$
u_-(t) = \mu_-(t) e^{-Bt} \text{ for } t \in (0, +\infty) ,
$$

$$
\mu_- \in C((0, +\infty), (0, +\infty)) \quad \mu_-(t) = O\left(\frac{1}{t^{1-\sigma}}\right) \text{ for } t \to 0^+ \quad (\sigma \in [0,1)) ,
$$

$$
\int_0^t ds \mu_-(t-s) e^{-Bs} \leq N \quad \text{for } t \in [0, +\infty) .
$$

\textbf{5.10 Proposition.} Given $f_0 \in \mathcal{F}$, let us consider for $\text{VP}(f_0)$ an approximate solution $\varphi_{ap} \in C([0, +\infty), \mathcal{F})$. Suppose there are constants $E , D \in [0, +\infty)$ such that:

(i) $\varphi_{ap}$ admits the integral error estimate

$$
\|E(\varphi_{ap})(t)\| \leq E e^{-Bt} \quad \text{for } t \in [0, +\infty) ;
$$

(ii) for all $t$ as above,

$$
\|\varphi_{ap}(t)\| \leq D e^{-Bt} ;
$$

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(iii) with \( N \) as in (5.36) and \( K \) as in (5.2), one has
\[
2\sqrt{KNE} + 2KND \leq 1 .
\] (5.39)

Then \( \mathcal{VP}(f_0) \) has a global solution \( \varphi : [0, +\infty) \to \mathcal{F} \) and, for all \( t \in [0, +\infty) \),
\[
\|\varphi(t) - \varphi_{ap}(t)\| \leq Re^{-Bt} ,
\] (5.40)
\[
R := \frac{1 - 2KND - \sqrt{(1 - 2KND)^2 - 4KNE}}{2KN} .
\]

**Proof.** *Step 1: the control inequality.* We use again Proposition 5.5 and the control inequality (5.14). With the notations of the cited proposition, we have \( \mathcal{E}(t) := Ee^{-Bt} \), \( \mathcal{D}(t) := De^{-Bt} \) and \( u_- \) has the expression (5.34). So, (5.14) takes the form
\[
Ee^{-Bt} + K \int_0^t ds \mu_-(t-s)e^{-B(t-s)}(2De^{-Bs}\mathcal{R}(s) + \mathcal{R}^2(s)) \leq \mathcal{R}(t) ,
\] (5.41)
that we regard as an inequality for an unknown nonnegative function \( \mathcal{R} \).

*Step 2: searching for a global solution \( \mathcal{R} \) of (5.41).* We try to fulfill (5.41) with
\[
\mathcal{R}(t) := Re^{-Bt} \quad \text{for all } t \in [0, +\infty) ,
\] (5.42)
with \( R \geq 0 \) an unknown constant. Then, the left hand side of (5.41) is
\[
e^{-Bt} \left[ E + K(2DR + R^2) \int_0^t ds \mu_-(t-s)e^{-Bs} \right] \leq e^{-Bt} \left( E + K(2DR + R^2)N \right) ,
\] (5.43)
where the inequality depends on (5.36). The last expression is bounded by \( \mathcal{R}(t) \) if \( R \) fulfills the inequality \( E + K(2DR + R^2)N \leq R \), i.e.,
\[
KNR^2 - (1 - 2KND)R + E \leq 0 .
\] (5.44)

This condition is fulfilled as an equality if we define \( R \) as in (5.40); this \( R \) is well defined and nonnegative due to the assumption (5.39) \(^9\). Due to the above considerations, the thesis is proved. \( \square \)

**Applications to cases with exponentially decaying forcing.**

From here to the end of the section, we add to (P1)-(P5')(Q1)(Q2) the following condition:

\(^9\)And, in fact, is the smallest nonnegative solution of (5.44).
(Q3) There is a constant $J \in [0, +\infty)$ such that (with $B$ as in (P4')(P5'))
\[ \|\xi(t)\|_\ast \leq Je^{-2Bt} \quad \text{for all } t \in [0, +\infty). \] (5.45)

Two cases where (Q3) holds are: (i) the trivial case $\xi(t) = 0$ for all $t$; (ii) situations where the external forcing is switched off in the future.

Hereafter we present two applications of Proposition 5.10, corresponding to different choices for $\varphi_{ap}$. Both of them give global existence for the exact solution $\varphi$ of $\text{VP}(f_0)$ when the datum $f_0$ is sufficiently small, with suitable estimates of the form (5.40).

**Example: the zero approximate solution.** Let $f_0 \in F$; we reconsider, from the viewpoint of Proposition 5.10, the $\text{VP}(f_0)$ approximate solution
\[ \varphi_{ap}(t) := 0 \quad \text{for } t \in [0, +\infty). \] (5.46)

**5.11 Lemma.** $\varphi_{ap} := 0$ has the integral error estimator
\[ \|E(\varphi_{ap})(t)\| \leq Fe^{-Bt}, \quad F := \|f_0\| + NJ. \] (5.47)

**Proof.** The integral error $E(\varphi_{ap})$ was already computed, see Eq. (5.25). From this equation and the assumptions (5.33) (5.34) on $u$, $\mu_-$ we infer
\[ \|E(\varphi_{ap})(t)\| \leq e^{-Bt}\|f_0\| + \int_{0}^{t} ds \, e^{-B(t-s)}\mu_-(t-s)\|\xi(s)\|_\ast; \] (5.48)

inserting here the bound (5.45) for $\|\xi(s)\|_\ast$, and using Eq. (5.36) for $\mu_-$ we get
\[ \|E(\varphi_{ap})(t)\| \leq e^{-Bt}\|f_0\| + Je^{-Bt}\int_{0}^{t} ds \, \mu_-(t-s)e^{-Bs} \leq e^{-Bt}\|f_0\| + Je^{-Bt}N. \] (5.49)

From the previous Lemma and Proposition 5.10, we infer the following.

**5.12 Proposition.** With $F$ as in (5.47), let
\[ 4KNF \leq 1. \] (5.50)

Then $\text{VP}(f_0)$ has a global solution $\varphi : [0, +\infty) \to F$ and, for all $t \in [0, +\infty)$,
\[ \|\varphi(t)\| \leq F \mathcal{X}(4KNF) e^{-Bt}, \] (5.51)

with $\mathcal{X}$ as in (5.31).
Proof. We apply Proposition 5.10 with \( \varphi_{ap} := 0 \). The constants of the cited proposition are \( E = F \) and \( D = 0 \) (the first equality follows from the previous lemma, the second one is obvious). Eqs. (5.39) (5.40) yield the present relations (5.50) (5.51) (5.31); in particular, the constant in (5.40) is given by \( R = (1 - \sqrt{1 - 4KNF})/(2KN) = F \mathcal{A}(4KNF) \).

Example: the ”\( \mathcal{A} \) -flow” approximate solution. Given \( f_0 \in F \), we consider for \( \mathcal{V}(f_0) \) the approximate solution

\[
\varphi_{ap}(t) := e^{tA}f_0 + \int_0^t ds \ e^{(t-s)A}\xi(s) \quad \text{for } t \in [0, +\infty)
\]  

(5.52)

(i.e., we use the flow of the linear equation \( \dot{f} = Af + \xi(t) \)).

5.13 Lemma. Let \( F \) be as in (5.47). The above \( \varphi_{ap} \) has the integral error estimator

\[
\| E(\varphi_{ap})(t) \| \leq KNF^2 e^{-Bt}
\]

(5.53)

and fulfills for all \( t \in [0, +\infty) \) the norm estimate

\[
\| \varphi_{ap}(t) \| \leq Fe^{-Bt}.
\]

(5.54)

Proof. We first derive Eq. (5.54). To this purpose, we note that the definition (5.52) of \( \varphi_{ap} \), the assumptions (5.33) (5.34) on \( u, u_- \) and Eqs. (5.45) for \( \| \xi(\cdot) \|_\cdot \), (5.36) for \( \mu_- \) imply

\[
\| \varphi_{ap}(t) \| \leq e^{-Bt}\| f_0 \| + \int_0^t ds \ e^{-B(t-s)}\mu_-(t-s)\| \xi(s) \|_\cdot
\]

(5.55)

\[
\leq e^{-Bt}\| f_0 \| + Je^{-Bt} \int_0^t ds \ \mu_-(t-s)e^{-Bs} \leq e^{-Bt}\| f_0 \| + Je^{-Bt}N ;
\]

by comparison with the definition (5.47) of \( F \), we get the thesis (5.54).

Let us pass to the proof of (5.53). To this purpose we note that the definition (3.1) of integral error gives, in the present case,

\[
E(\varphi_{ap})(t) = -\int_0^t ds \ e^{(t-s)A}\mathcal{P}(\varphi_{ap}(s), \varphi_{ap}(s)) .
\]

(5.56)

From here and from Eqs. (5.34) for \( u_- \), (5.2) for \( \mathcal{P} \) we infer

\[
\| E(\varphi_{ap})(t) \| \leq \int_0^t ds \ \mu_-(t-s)e^{-B(t-s)}\| \mathcal{P}(\varphi_{ap}(s), \varphi_{ap}(s)) \|_\cdot
\]

(5.57)

\[
\leq K \int_0^t ds \ \mu_-(t-s)e^{-B(t-s)}\| \varphi_{ap}(s) \|^2 .
\]
In the last inequality, we insert the bound (5.54) and then recall (5.36). This gives
\[ \| E(\varphi_{ap})(t) \| \leq K F^2 e^{-Bt} \int_0^t ds \ e^{-B s} \mu_-(t-s) \leq K N F^2 e^{-Bt}. \] (5.58)

Let us return to Proposition 5.10; with the previous Lemma, this implies the following result.

5.14 Proposition. Let us keep the definition (5.47) for \( F \), and the assumption (5.50) \( 4 K N F \leq 1 \). The global solution \( \varphi : [0, +\infty) \to F \) of \( Vf(f_0) \) is such that, for all \( t \in [0, +\infty) \),
\[ \| \varphi(t) - e^{tA} f_0 \| \leq K N F^2 \mathcal{X}(4 K N F) e^{-Bt}; \] (5.59)
here \( \mathcal{X} \in C([0, 1], [1, 4]) \) is the increasing function defined by
\[ \mathcal{X}(z) := \begin{cases} \frac{1 - (z/2) - \sqrt{1-z}}{(z^2/8)} & \text{for } z \in (0, 1), \\ 1 & \text{for } z = 0. \end{cases} \] (5.60)

Proof. According to the previous Lemma, we can apply Proposition 5.10 with \( E = K N F^2 \) and \( D = F \). Eqs. (5.39) (5.40) yield the present relations (5.50) (5.59) (5.60); in particular, the constant in Eq. (5.40) is
\[ R = (1 - 2 K N F - \sqrt{1 - 4 K N F})/(2 K N) = K N F^2 \mathcal{X}(4 K N F). \] (5.61)

5.15 Remark. Most of the results presented in this section could be extended to the case \( \mathcal{P}(f, t) = \mathcal{P}(f, ..., f) + \xi(t) \), where \( \mathcal{P} : F^m \to F_\pm \) is continuous and \( m \)-linear for some integer \( m \geq 3 \). In this case, the growth of \( \mathcal{P} \) from any approximate solution admits an estimator \( \ell(r, t) \) more general than (5.10), which is polynomial of degree \( m \) in \( r \). One could extend the analysis as well to the case \( \mathcal{P}(f, t) = \mathcal{P}(f, ..., f, t) + \xi(t) \), involving a time dependent multilinear map \( \mathcal{P} : F^m \times [0, +\infty) \to F_\pm, (f_1, ..., f_m, t) \mapsto \mathcal{P}(f_1, ..., f_m, t) \). These generalizations are not written only to save space.

6 The Navier-Stokes (NS) equations on a torus.

From here to the end of the paper, we work in any space dimension
\[ d \geq 2. \] (6.1)

Preliminaries: distributions on \( \mathbb{T}^d \), Fourier series and Sobolev spaces. Throughout this section, we use \( r, s \) as indices running from 1 to \( d \) and employ for them the Einstein summation convention on repeated, upper and lower indices; \( \delta_{rs} \) or \( \delta^{rs} \) is the Kronecker symbol.
Elements $a, b, ...$ of $\mathbf{R}^d$ or $\mathbf{C}^d$ will be written with upper or lower indices, according to convenience: $(a^s)$ or $(a_s)$, $(b^s)$ or $(b_s)$. In any case, $a \bullet b$ is the sum of product of the components of $a$ and $b$, that will be written in different ways to accomplish with the Einstein convention. Two examples, corresponding to different positions for the indices of $a$, are

$$a \bullet b = a_s b^s, \quad a \bullet b = \delta_{rs} a^r b^s. \quad (6.2)$$

Let us consider the $d$-dimensional torus

$$\mathbf{T}^d := \mathbf{T} \times ... \times \mathbf{T} \quad (d \text{ times}), \quad \mathbf{T} := \mathbf{R}/(2\pi \mathbf{Z}). \quad (6.3)$$

A point of $\mathbf{T}^d$ will be generally written $x = (x^r)_{r=1,...,d}$. We also consider the “dual” lattice $\mathbf{Z}^d$ of elements $k = (k^r)_{r=1,...,d}$ and the Fourier basis $(e_k)_{k \in \mathbf{Z}^d}$, made of the functions

$$e_k : \mathbf{T}^d \to \mathbf{C}, \quad e_k(x) := \frac{1}{(2\pi)^{d/2}} e^{i k \bullet x}. \quad (6.4)$$

($k \bullet x := k^r x^r$ makes sense as an element of $\mathbf{T}^d$). We introduce the space of periodic distributions $D'(\mathbf{T}^d, \mathbf{C}) \equiv D'_\mathbf{C}$, which is the dual of $C^\infty(\mathbf{T}^d, \mathbf{C}) \equiv C^\infty_\mathbf{C}$ (equipping the latter with the topology of uniform convergence of all derivatives); we write $\langle v, f \rangle$ for the action of a distribution $v \in D'_\mathbf{C}$ on a test function $f \in C^\infty_\mathbf{C}$. The weak topology on $D'_\mathbf{C}$ is the one induced by the seminorms $p_f(v) := |\langle v, f \rangle|$.

Each $v \in D'_\mathbf{C}$ has a unique (weakly convergent) series expansion

$$v = \sum_{k \in \mathbf{Z}^d} v_k e_k, \quad (6.5)$$

with coefficients $v_k \in \mathbf{C}$ for all $k$, given by

$$v_k = \langle v, e_{-k} \rangle. \quad (6.6)$$

The ”Fourier series transformation” $v \mapsto (v_k)$ is one-to-one between $D'_\mathbf{C}$ and the space of sequences $s'(\mathbf{Z}^d, \mathbf{C}) \equiv s'_\mathbf{C}$, where

$$s'_\mathbf{C} := \{ c = (c_k)_{k \in \mathbf{Z}^d} \mid c_k \in \mathbf{C}, \quad |c_k| = O(|k|^p) \text{ as } k \to \infty, \quad \text{for some } p \in \mathbf{R} \}. \quad (6.7)$$

In the sequel, we often use the mean of a distribution $v \in D'_\mathbf{C}$, which is

$$\langle v \rangle := \frac{1}{(2\pi)^d} \langle v, 1 \rangle = \frac{1}{(2\pi)^{d/2}} v_0 \quad (6.8)$$

(in the first passage, $\langle v, 1 \rangle$ means the action of $v$ on the test function $1$; the second relation follows from (6.6) with $k = 0$, noting that $e_0 = 1/(2\pi)^{d/2}$. Of course, $\langle v, 1 \rangle = \int_{\mathbf{T}^d} v(x) dx$ if $v$ is an ordinary, integrable function).
The complex conjugate of a distribution \( v \in D'_C \) is the unique distribution \( \overline{v} \) such that \( \langle v, f \rangle = \langle \overline{v}, f \rangle \) for each \( f \in C^\infty_C \); one has \( \overline{v} = \sum_{k \in \mathbb{Z}^d} \overline{v}_k e^{-ik} \).

From now on we will be mainly interested in the space of real distributions \( D'(T^d, \mathbb{R}) \equiv D' \), defined as follows:

\[
D' := \{ v \in D'_C \mid \overline{v} = v \} = \{ v \in D'_C \mid \overline{v}_k = v_{-k} \text{ for all } k \in \mathbb{Z}^d \} ;
\]

we note that \( v \in D' \) implies \( \langle v \rangle \in \mathbb{R} \). The weak topology on \( D' \) is the one inherited from \( D'_C \).

Let us write \( \partial_s (s = 1, \ldots, d) \) for the distributional derivative with respect to the coordinate \( x^s \); from these derivatives, we define the distributional Laplacian \( \Delta := \delta^{rs} \partial_r \partial_s : D'_C \rightarrow D'_C \). Of course \( \partial_s e_k = i k_s e_k, \Delta e_k = -|k|^2 e_k \) for each \( k \). For any \( v \in D'_C \), this implies

\[
\partial_s v = i \sum_{k \in T^d} k_s v_k e_k , \quad \Delta v = - \sum_{k \in T^d} |k|^2 v_k e_k ;
\]

\[
(1 - \Delta)^m v = \sum_{k \in T^d} (1 + |k|^2)^m v_k e_k \tag{6.11}
\]

for \( m \in 0, 1, 2, 3, \ldots \). For any \( m \in \mathbb{R} \), we will regard (6.11) as the definition of \( (1 - \Delta)^m \) as a linear operator from \( D'_C \) into itself. Comparing the previous Fourier expansions with (6.8), we find

\[
\langle \partial_s v \rangle = 0 , \quad \langle \Delta v \rangle = 0 . \tag{6.12}
\]

All the above differential operators leave invariant the space of real distributions, more interesting for us; in the sequel we will fix the attention on the maps \( \partial_s, \Delta, (1 - \Delta)^m : D' \rightarrow D' \).

Let us consider the real Hilbert space \( L^2(T^d, \mathbb{R}, dx) \equiv L^2, \) i.e.,

\[
L^2 := \{ v : T^d \rightarrow \mathbb{R} \mid \int_{T^d} v^2(x)dx < +\infty \} = \{ v \in D' \mid \sum_{k \in \mathbb{Z}^d} |v_k|^2 < +\infty \} ; \tag{6.13}
\]

this has the inner product and the associated norm

\[
\langle v | w \rangle_{L^2} := \int_{T^d} v(x)w(x)dx = \sum_{k \in \mathbb{Z}^d} v_k w_k ,
\]

\[
\| v \|_{L^2} := \sqrt{\int_{T^d} v^2(x)dx} = \sqrt{\sum_{k \in \mathbb{Z}^d} |v_k|^2} . \tag{6.15}
\]
To go on, we introduce the Sobolev spaces $H^n(T^d, \mathbb{R}) \equiv H^n$. For each $n \in \mathbb{R}$,

$$H^n := \{ v \in D' \mid (1-\Delta)^{n/2} v \in L^2 \} = \{ v \in D' \mid \sum_{k \in T^d} (1+|k|^2)^n |v_k|^2 < +\infty \} ; \quad (6.16)$$

this is also a real Hilbert space with the inner product

$$\langle v | w \rangle_n := \langle (1-\Delta)^{n/2} v | (1-\Delta)^{n/2} w \rangle_{L^2} = \sum_{k \in T^d} (1+|k|^2)^n v_k w_k \quad (6.17)$$

and the corresponding norm

$$\| v \|_n := \| (1-\Delta)^{n/2} v \|_{L^2} = \sqrt{\sum_{k \in T^d} (1+|k|^2)^n |v_k|^2} \quad (6.18)$$

One proves that

$$n \geq n' \implies H^n \hookrightarrow H^{n'} \quad \| \|_{n'} \leq \| \|_n \quad (6.19)$$

In particular, $H^0$ is the space $L^2$ and contains $H^n$ for each $n \geq 0$; for any real $n$, $\Delta$ is a continuous map of $H^n$ into $H^{n-2}$. Finally, let us recall that

$$H^n \hookrightarrow D' \quad \text{for each } n \in \mathbb{R} \quad (6.20)$$

$$H^n \hookrightarrow C^q \quad \text{if } q \in \mathbb{N}, \quad n \in (q + d/2, +\infty) \quad (6.21)$$

In the above $H^n$ carries its Hilbertian topology, and $D'$ the weak topology; $C^q$ stands for the space $C^q(T^d, \mathbb{R})$, with the topology of uniform convergence of all derivatives up to order $q$.

Obviously enough, we could define as well the complex Hilbert spaces $L^2_C$ and $H^n_C$; however, these are never needed in the sequel.

**Spaces of vector valued functions on $T^d$.** To deal with the NS equations, we need vector extensions of all the above spaces and mappings. Let us stipulate the following: if $V(T^d, \mathbb{R}) \equiv V$ is any vector space of real functions or distributions on $T^d$, then

$$\mathbb{V} := V^d = \{ v = (v^1, \ldots, v^d) \mid v^r \in V \quad \text{for all } r \} \quad (6.22)$$

This notation allows to define the spaces $\mathbb{D}'$, $\mathbb{L}^2$, $\mathbb{H}^n$. Any $v = (v^r) \in \mathbb{D}'$ will be referred to as a vector field on $T^d$. We note that $v$ has a unique Fourier series expansion (6.5) with coefficients

$$v_k = (v_{k}^r)_{r=1,\ldots,d} \in \mathbb{C}^d, \quad v_k^r := \langle v^r, e_k \rangle \quad (6.23)$$

again, the reality of $v$ ensures $\overline{v_k} = v_{-k}$. We define componentwisely the mean $\langle v \rangle \in \mathbb{R}^d$ of any $v \in \mathbb{D}'$ (see Eq. (6.8)), the derivative operators $\partial_s : \mathbb{D}' \to \mathbb{D}'$, their
iterates and, consequently, the Laplacian $\Delta$. The prescription (6.11) gives a map $(1-\Delta)^m : \mathcal{D}' \to \mathcal{D}'$ for all real $m$. Whenever $V$ is made of ordinary functions, a $d$-uple $v \in V$ can be identified with a function $v : T^d \to \mathbb{R}^d$, $x \mapsto v(x) = (v^r(x))_{r=1,\ldots,d}$.

$L^2$ is a real Hilbert space. Its inner product is as in (6.14), with $v(x)w(x)$ and $\overline{v_k}w_k$ replaced by

$$v(x) \cdot w(x) = \delta_{rs} v^r(x) w^s(x) , \quad \overline{v_k} \cdot w_k = \delta_{rs} \overline{v^r} w^s_k ;$$

the corresponding norm is as in (6.15), replacing $v^2(x)$ with $|v(x)|^2 = \sum_{r=1}^d v^r(x)^2$ and intending $|v_k|^2 = \sum_{r=1}^d |v^r_k|^2$.

For any real $n$, the Sobolev space $\mathbb{H}^n$ is made of all $d$-uples $v$ with components $v^r \in H^n$; an equivalent definition can be given via Eq.(6.16), replacing therein $L^2$ with $L^2$. $\mathbb{H}^n$ is a real Hilbert space with the inner product

$$\langle v | w \rangle_n := \langle (1-\Delta)^{n/2} v | (1-\Delta)^{n/2} w \rangle_{L^2} = \sum_{k \in T^d} (1+|k|^2)^n \overline{v_k} \cdot w_k .$$

The corresponding norm $\| \|_n$ is given, verbatim, by Eq. (6.18); Eq. (6.19) holds as well for $\mathbb{H}^n, \mathbb{H}^n'$ and their norms. Let us consider the Laplacian operator $\Delta : \mathcal{D} \to \mathcal{D}$; for any real $n$

$$\Delta \mathbb{H}^n \subset \mathbb{H}^{n-2} ,$$

and $\Delta$ is continuous with respect to the norms $\| \|_n, \| \|_{n-2}$. The embeddings (6.19) (6.20) (6.21) have obvious vector analogues.

**Zero mean vector fields.** The space of these vector fields is

$$\mathcal{D}'_0 := \{ v \in \mathcal{D}' \mid \langle v \rangle = 0 \}$$

(of course, $\langle v \rangle = 0$ is equivalent to the vanishing of the Fourier coefficient $v_0$).

**Divergence free vector fields.** Let us consider the divergence operator (linear, weakly continuous)

$$\text{div} : \mathcal{D}' \to \mathcal{D}' , \quad v \mapsto \text{div } v := \partial_r v^r ;$$

we put

$$\mathcal{D}'_{\Sigma} := \{ v \in \mathcal{D}' \mid \text{div } v = 0 \}$$

and refer to this as to the space of divergence free (or solenoidal) vector fields. The description of these objects in terms of Fourier transform is obvious, namely:

$$\text{div } v = i \sum_{k \in \mathbb{Z}^d} (k \cdot v_k) e_k \quad \text{for all } v = \sum_{k \in \mathbb{Z}^d} v_k e_k \in \mathcal{D}' ;$$

34
\[ \mathcal{D}' = \{ v \in \mathcal{D}' \mid k \cdot v_k = 0 \text{ for all } k \in \mathbb{Z}^d \} \quad (6.31) \]

\[ = \{ v \in \mathcal{D}' \mid v_k \in \prec k \succ \perp \text{ for all } k \in \mathbb{Z}^d \}, \]

where \( \prec k \succ \) is the subspace of \( \mathbb{C}^d \) spanned by \( k \), and \( \perp \) is the orthogonal complement with respect to the inner product \( (b, c) \in \mathbb{C}^d \times \mathbb{C}^d \rightarrow \mathbb{C} \).

In the sequel, we will consider as well the subspace

\[ \mathcal{D}'_\Sigma_0 := \mathcal{D}'_\Sigma \cap \mathcal{D}'_0. \quad (6.32) \]

**Gradient vector fields.** Let us consider the gradient operator (again linear, weakly continuous)

\[ \partial : \mathcal{D}' \rightarrow \mathcal{D}' \; \; ; \; \; p \mapsto \partial p := (\partial_s p)_{s=1,\ldots,d}; \quad (6.33) \]

if \( p = \sum_{k \in \mathbb{Z}^d} p_k e_k \), then

\[ \partial p = i \sum_{k \in \mathbb{Z}^d} kp_k e_k. \quad (6.34) \]

The image

\[ \mathcal{D}'_\Gamma = \{ \partial p \mid p \in \mathcal{D}' \} \quad (6.35) \]

is a linear subspace of \( \mathcal{D}' \), hereafter referred to as the space of gradient vector fields; for any vector field \( w \), comparison with (6.34) gives

\[ \mathcal{D}'_\Gamma = \{ w \in \mathcal{D}' \mid w_k \in \prec k \succ \; \text{ for all } k \in \mathbb{Z}^d \}. \quad (6.36) \]

Of course, if \( w \) is in this subspace, the distribution \( p \) such that \( w = \partial p \) is defined up to an additive constant. We put

\[ \partial^{-1} w := \text{unique } p \in \mathcal{D}' \text{ such that } w = \partial p \text{ and } p_0 = 0. \quad (6.37) \]

This gives a linear map

\[ \partial^{-1} : \mathcal{D}'_\Gamma \rightarrow \mathcal{D}'. \quad (6.38) \]

**The Leray projection.** Using the Fourier representations (6.31) (6.36), one easily proves the following facts.

(i) One has

\[ \mathcal{D}' = \mathcal{D}'_\Sigma \oplus \mathcal{D}'_\Gamma \]

in algebraic sense, i.e., any \( v \in \mathcal{D}' \) has a unique decomposition as the sum of a divergence free and a gradient vector field.

(ii) The projection

\[ \mathcal{L} : \mathcal{D}' \rightarrow \mathcal{D}'_\Sigma, \quad v \mapsto \mathcal{L}v \]

(6.40)

corresponding to the decomposition (6.39) is given by
\[ \mathcal{L}v = \sum_{k \in \mathbb{Z}^d} (\mathcal{L}_k v_k) e_k \quad \text{for all } v = \sum v_k e_k \in \mathbb{D}, \]

(6.41)

\[ \mathcal{L}_k := \text{orthogonal projection of } \mathcal{C}^d \text{ onto } \prec k \succ \perp; \]

more explicitly, for all \( c \in \mathcal{C}^d \),

\[ \mathcal{L}_0 c = c, \quad \mathcal{L}_k c = c - \frac{(k \cdot c) k}{|k|^2} \quad \text{for } k \in \mathbb{Z}^d, k \neq 0. \]

(6.42)

As usually, we refer to \( \mathcal{L} \) as to the Leray projection; this operator is weakly continuous. From the Fourier representations of \( \mathcal{L} \), of the mean and of the derivatives one easily infers, for all \( v \in \mathbb{D}' \),

\[ \langle \mathcal{L}v \rangle = \langle v \rangle, \quad \mathcal{L}(\partial_s v) = \partial_s (\mathcal{L}v), \quad \mathcal{L}(\Delta v) = \Delta (\mathcal{L}v). \]

(6.43)

A Sobolev framework for the previous decomposition. For \( n \in \mathbb{R} \), let us define

\[ H^n_\Gamma := H^n \cap \mathbb{D}'_\Gamma = \{ v \in H^n \mid \text{div } v = 0 \}; \]

(6.44)

\[ H^n_\Gamma := H^n \cap \mathbb{D}'_\Gamma = \{ w \in H^n \mid w = \partial p, \ p \in D' \}; \]

(6.45)

\[ \partial H^n := \{ \partial p \mid p \in H^n \}. \]

(6.46)

Then the following holds for each \( n \):

(i) \( H^n_\Gamma \) is a closed subspace of the Hilbert space \( (H^n, \langle \cdot | \cdot \rangle_n) \) (because div is continuous between this Hilbert space and \( D' \) with the weak topology).

(ii) One has

\[ H^n_\Gamma = \partial H^{n+1}. \]

(6.47)

and \( H^n_\Gamma \) is also a closed subspace of \( H^n \). The map \( \partial^{-1} \) of Eq. (6.37) is continuous between \( H^n_\Gamma \) and \( H^{n+1} \).

(iii) Denoting with \( \perp_n \) the orthogonal complement in \( (H^n, \langle \cdot | \cdot \rangle_n) \), we have

\[ H^n_\Sigma \perp_n = H^n_\Gamma \]

(6.48)

and \( \mathcal{L} \upharpoonright H^n \) is the orthogonal projection of \( H^n \) onto \( H^n_\Sigma \); so, as usual for Hilbertian orthogonal projections,

\[ \| \mathcal{L}v \|_n \leq \| v \|_n \quad \text{for all } v \in H^n. \]

(6.49)

Other Sobolev spaces of vector fields. For \( n \in \mathbb{R} \), we put

\[ H^n_0 := H^n \cap \mathbb{D}'_0 = \{ f \in H^n \mid \langle f \rangle = 0 \}; \]

(6.50)

\[ H^n_{\Sigma 0} := H^n_\Sigma \cap H^n_0 := \{ f \in H^n \mid \text{div } f = 0, \langle f \rangle = 0 \}. \]

(6.51)
Then, $\mathbb{H}_0^n$ is a closed subspace of $\mathbb{H}^n$ (by the continuity of $\langle \cdot \rangle : \mathbb{H}^n \to \mathbb{C}$); the same holds for $\mathbb{H}_{\Sigma 0}^n$, since this is the intersection of two closed subspaces. The space (6.51) plays an important role in the sequel; we will often use the Fourier representation

$$\mathbb{H}^n_{\Sigma 0} = \{ f \in D | \sum_{k \in \mathbb{Z}^d} (1 + |k|^2)^n |f_k|^2 < +\infty , k \cdot f_k = 0 \text{ for all } k \in \mathbb{Z}^d, f_0 = 0 \} .$$

### Some inclusions

We note that the relations $\Delta \mathbb{H}^n \subset \mathbb{H}^{n-2}$ and $\text{div}(\Delta v) = \Delta(\text{div} v), \langle \Delta v \rangle = 0$ for all $v \in D$ imply the following, for each real $n$:

$$\Delta \mathbb{H}^n \subset \mathbb{H}^{n-2}_{0}, \quad \Delta \mathbb{H}^n_{\Sigma} \subset \mathbb{H}^{n-2}_{\Sigma 0} .$$

### A digression: estimates on certain series

Let us define

$$Z^d_0 := Z^d \setminus \{0\} ;$$

throughout the section, $n$ is a real number such that

$$n > \frac{d}{2} .$$

The series considered hereafter are used shortly afterwards to derive quantitative estimates on the fundamental bilinear map appearing in the NS equations: by this we mean the map sending two vector fields $v, w$ on $T^d$ into the vector field $v \cdot \partial w$ (see the next paragraph). The estimates we give are also useful for the numerical computation of those series. In both Lemmas hereafter,

$$Z^d := Z^d \text{ or } Z^d_0 .$$

### 6.1 Lemma

One has

$$\Sigma_n := \frac{1}{(2\pi)^d} \sum_{h \in \mathbb{Z}^d} \frac{1}{(1 + |h|^2)^n} < +\infty .$$

For any real "cutoff" $\lambda \geq 2\sqrt{d}$, one has

$$S_n(\lambda) < \Sigma_n \leq S_n(\lambda) + \delta S_n(\lambda)$$

where

$$S_n(\lambda) := \frac{1}{(2\pi)^d} \sum_{h \in \mathbb{Z}^d, |h| < \lambda} \frac{1}{(1 + |h|^2)^n} ,$$

$$\delta S_n(\lambda) := \frac{(1 + d)^n}{2^{d-1} \pi^{d/2} \Gamma(d/2)(2n - d)(\lambda - \sqrt{d})^{2n-d}} .$$
Proof. See Appendix C.

6.2 Lemma. For \( k \in \mathbb{Z}^d \), define

\[
\mathcal{K}_n(k) \equiv \mathcal{K}_{nd}(k) := \frac{(1 + |k|^2)^{n-1}}{(2\pi)^d} \sum_{h \in \mathbb{Z}^d} \frac{|k-h|^2}{(1 + |h|^2)^n(1 + |k-h|^2)^n} ;
\]

then, (i) (ii) hold.

(i) One has \( \mathcal{K}_n(k) < +\infty \) for all \( k \in \mathbb{Z}^d \); furthermore, with \( \Sigma_n \) as in (6.57),

\[
\mathcal{K}_n(k) \to \Sigma_n \quad \text{for } k \to \infty .
\]

Thus,

\[
\sup_{k \in \mathbb{Z}^d} \mathcal{K}_n(k) < +\infty .
\]

(ii) Let us choose any "cutoff function" \( \Lambda : \mathbb{Z}^d \to [2\sqrt{d}, +\infty) \) and define

\[
\lambda : \mathbb{Z}^d \to (0, +\infty) , \quad k \mapsto \lambda(k) := \begin{cases} 
1 + |k|^2 & \text{if } \Lambda(k) < |k| , \\
\frac{1 + |k|^2}{1 + (\Lambda(k) - |k|)^2} & \text{if } \Lambda(k) \geq |k| .
\end{cases}
\]

Then, for all \( k \in \mathbb{Z}^d \),

\[
\mathcal{K}_n(k) < \mathcal{K}_n(k) \leq \mathcal{K}_n(k) + \delta\mathcal{K}_n(k) ,
\]

where

\[
\mathcal{K}_n(k) := \frac{(1 + |k|^2)^{n-1}}{(2\pi)^d} \sum_{h \in \mathbb{Z}^d, |h| < \Lambda(k)} \frac{|k-h|^2}{(1 + |h|^2)^n(1 + |k-h|^2)^n} ,
\]

\[
\delta\mathcal{K}_n(k) := \frac{(1 + d)^n\lambda(k)^{n-1}}{2^{d-1}\pi^{d/2}\Gamma(d/2)(2n-d) \left( \Lambda(k) - \sqrt{d} \right)^{2n-d}} .
\]

Finally, suppose the cutoff \( \Lambda \) has the property

\[
\alpha |k| \leq \Lambda(k) \leq \beta |k| \quad \text{for all } k \in \mathbb{Z}^d \text{ such that } |k| \geq \chi
\]

\[
(\chi > 0, \quad 1 < \alpha \leq \beta) ;
\]

then

\[
\mathcal{K}_n(k) \to \Sigma_n , \quad \delta\mathcal{K}_n(k) = O\left( \frac{1}{|k|^{2n-d}} \right) \to 0 \quad \text{for } k \to \infty .
\]
Proof. See Appendix C. □

The fundamental bilinear map. Let \( v, w \in \mathbb{H}^n \). For each \( r, s \in \{1, 2, ..., d\} \), \( v^r \in H^n \) and \( \partial_r w^s \in H^{n-1} \) are ordinary real functions: note that \( v^r \in C^0 \) by the embedding (6.21), and \( \partial_r w^s \in L^2 \) since \( n - 1 > d/2 - 1 \geq 0 \). These functions can be multiplied pointwisely, and this allows to define

\[
  v \cdot \partial w := (v \cdot \partial w^s)_{s=1,...,d} \quad v \cdot \partial w^s := v^r \partial_r w^s : T^d \to \mathbb{C} .
\]  

6.3 Proposition. (i) Consider \( v, w \in \mathbb{H}^n \). The vector field \( v \cdot \partial w \) has Fourier coefficients

\[
  (v \cdot \partial w)_k = \frac{i}{(2\pi)^{d/2}} \sum_{h \in \mathbb{Z}^d} [(v_h \cdot (k - h)) w_{k-h}] .
\]

Furthermore, \( v \cdot \partial w \in \mathbb{H}^{n-1} \).

(ii) The bilinear map

\[
  \mathbb{H}^n \times \mathbb{H}^n \to \mathbb{H}^{n-1} , \quad (v, w) \mapsto v \cdot \partial w
\]

admits an estimate (indicating continuity)

\[
  \| v \cdot \partial w \|_{n-1} \leq K_n \| v \|_n \| w \|_n
\]

for all \( v, w \) as above, with a suitable constant \( K_n \equiv K_{nd} \in (0, +\infty) \). For the latter one can take any constant such that

\[
  \sqrt{\sup_{k \in \mathbb{Z}^d} K_n(k)} \leq K_n \; ,
\]

\[
  K_n(k) := \frac{(1 + |k|^2)^{n-1}}{(2\pi)^d} \sum_{h \in \mathbb{Z}^d} \frac{|k - h|^2}{(1 + |h|^2)^n(1 + |k - h|^2)^n}
\]

(as in (6.61), with \( \mathbb{Z}^d = \mathbb{Z}^d \)).

Proof. See Appendix D. □

6.4 Lemma. Let \( v \in \mathbb{H}_\Sigma^n , w \in \mathbb{H}^n \). Then

\[
  \langle v \cdot \partial w \rangle = 0 ;
\]

combined with Proposition 6.3, this gives \( v \cdot \partial w \in \mathbb{H}_o^{n-1} \).

Proof. For \( s = 1, ..., d \), integration by parts and the assumption \( 0 = \text{div} v = \partial_r v^r \) give

\[
  \langle v \cdot \partial w^s \rangle = \frac{1}{(2\pi)^d} \int_{T^d} dx v^r \partial_r w^s = -\frac{1}{(2\pi)^d} \int_{T^d} dx (\partial_r v^r) w^s = 0 . \quad \Box
\]
6.5 Proposition. The bilinear map
\[ H^n_{\Sigma_0} \times H^n_{\Sigma_0} \to H^{n-1}_0, \quad (f, g) \mapsto f \bullet \partial g \] (6.77)
admits an estimate
\[ \|f \bullet \partial g\|_{n-1} \leq K_n \|f\|_n \|g\|_n \] (6.78)
for all \( f, g \in H^n_{\Sigma_0} \), with a suitable constant \( K_n \equiv K_{nd} \in (0, +\infty) \). One can take for the latter any constant such that
\[ \sqrt{\sup_{k \in \mathbb{Z}^d} K_n(k)} \leq K_n, \] (6.79)
\[ K_n(k) := \frac{(1 + |k|^2)^{n-1}}{(2\pi)^d} \sum_{h \in \mathbb{Z}^d_0} \frac{|k - h|^2}{(1 + |h|^2)^n(1 + |k - h|^2)^n} \] (6.80)
(as in (6.61), with \( \mathbb{Z}^d = \mathbb{Z}^d_0 \)).

Proof. See Appendix D. \( \square \)

To conclude, we report another integral identity frequently used in the sequel.

6.6 Lemma. For any \( v \in H^n_{\Sigma} \), one has
\[ \langle v|v \bullet \partial v\rangle_{L^2} = 0 \] (6.81)

Proof. We have (with \( q, s, r \in \{1, \ldots, d\} \)),
\[ \langle v|v \bullet \partial v\rangle_{L^2} = \int_{\mathbb{T}^d} dx \delta_{qs} v^q (v^r \partial_r v^s) . \] (6.82)
From here we infer, integrating by parts,
\[ \langle v|v \bullet \partial v\rangle_{L^2} = - \int_{\mathbb{T}^d} dx \delta_{qs} \partial_r (v^q v^r) v^s \]
\[ = - \int_{\mathbb{T}^d} dx \delta_{qs} (\partial_r v^q) v^r v^s - \int_{\mathbb{T}^d} dx \delta_{qs} v^q (\partial_r v^r) v^s = - \int_{\mathbb{T}^d} dx \delta_{qs} (\partial_r v^q) v^r v^s \]
(6.83)
since \( \text{div} v = 0 \); by comparison with (6.82) we obtain
\[ \langle v|v \bullet \partial v\rangle_{L^2} = - \langle v|v \bullet \partial v\rangle_{L^2} , \] (6.84)
whence the thesis (6.81). \( \square \)
6.7 Remark. The inequality (6.73) (or (6.78)) is known from the literature, in this form or in some variant: see, for example, one of the standard references on NS equations cited in the Introduction. To our knowledge, the novelty of Proposition 6.3 (or 6.5) with respect to the already published material is the rule (6.74) (or (6.79)) to determine $K_n$, that can be used with Lemmas 6.1, 6.2 to provide a numerical value for this constant. Examples of this computation appear in Section 10 and Appendix H.

The method employed in Appendix D to prove Propositions 6.3, 6.5 is very similar to one employed in [13] to estimate the product of two scalar functions in $H^n(R^d)$. In that paper, already mentioned in the Introduction, we have given a rule similar to (6.74) (or (6.79)) to find a constant $C_n \equiv C_n$ such that
$$
\|pq\|_{H^n(R^d)} \leq C_n \|p\|_{H^n(R^d)} \|q\|_{H^n(R^d)};
$$
Furthermore, using convenient trial functions $p, q$ we have shown this constant to be very close to the smallest one fulfilling the inequality.

Due to the similarities with [13], the constant $K_n$ provided by (6.74) (or (6.79)) is hopefully close to the smallest one for the inequality (6.73) (or (6.78)).

Functions on $T^d$, depending on time. Suppose we have a function
$$
\rho : [t_0, T) \subset R \rightarrow V \text{ or } V , \quad t \mapsto \rho(t) \quad (6.85)
$$
where $V, \mathbb{V}$ stand for some spaces of $R$ or $R^d$ valued functions on $T^d$. At each "time" $t \in [t_0, T)$, this gives a function $x \in T^d \mapsto \rho(t)(x)$; of course, we will use the more common notation
$$
\rho(x, t) := \rho(t)(x) \quad (6.86)
$$

The incompressible NS equations, in the Leray formulation.

Let us recall that $C^{0,1}$ indicates the locally Lipschitz maps.

6.8 Definition. The incompressible NS Cauchy problem with initial datum $v_0 \in \mathbb{H}^{n+1}_\Sigma$ and forcing term $\eta \in C^{0,1}([0, +\infty), \mathbb{H}^{n-1}_\Sigma)$, in the Leray formulation, is the following.

Find $\nu \in C([0, T), \mathbb{H}^{n+1}_\Sigma) \cap C^1([0, T), \mathbb{H}^{n-1}_\Sigma)$, such that
$$
\dot{\nu}(t) = \Delta \nu(t) - \nabla(\nu(t) \cdot \partial \nu(t)) + \eta(t) \quad \text{for } t \in [0, T) , \quad \nu(0) = v_0 \quad (6.87)
$$
(for some $T \in (0, +\infty]$).

6.9 Remarks. (i) The requirement $\nu \in C([0, T), \mathbb{H}^{n+1}_\Sigma)$ ensures by itself that the right hand side of the above differential equation is in $C([0, T), \mathbb{H}^{n-1}_\Sigma)$.
(ii) The differential equation in (6.87) can be interpreted as the usual NS equation for an incompressible fluid, in a convenient adimensional formulation. For each $t \in [0, T)$, $\nu(t) : x \in T^d \mapsto \nu(x, t)$ is the velocity field of the fluid at time $t$; $\eta(t) : x \mapsto \eta(x, t)$ is the Leray projection of the density of external forces. Of course,
ν(t) is taken in the divergence free space $H_{\mathbb{E}^1}^{n+1}$ to fulfill the condition of incompressibility; the pressure gradient does not appear in (6.87), having been eliminated by application of $\mathcal{L}$. For completeness, all these facts are surveyed in Appendix E. 

The statement that follows refers to the time evolution of the functions $t \mapsto \langle \nu(t) \rangle = (2\pi)^{-d} \int_{T^d} \nu(x,t)dx$ and $t \mapsto (1/2)\|\nu(t)\|_{L^2}^2$. Up to dimensional factors, the first one gives the mean value of the velocity field or, equivalently, the total momentum; the second one gives the total kinetic energy.

6.10 Proposition. Suppose $\nu$ fulfills (6.87) on an interval $[0,T)$. Then, for all $t$ in this interval, we have the following relations:

(i) Balance of momentum:

\[
\frac{d}{dt} \langle \nu(t) \rangle = \langle \eta(t) \rangle . \tag{6.88}
\]

(ii) Balance of energy:

\[
\frac{1}{2} \frac{d}{dt} \|\nu(t)\|_{L^2}^2 = \langle \nu(t) | \Delta \nu(t) \rangle_{L^2} + \langle \nu(t) | \eta(t) \rangle_{L^2} \leq \langle \nu(t) | \eta(t) \rangle_{L^2} . \tag{6.89}
\]

Proof. (i) Let us observe that

\[
\frac{d}{dt} \langle \nu \rangle = \langle \dot{\nu} \rangle = \langle \Delta \nu \rangle - \langle \mathcal{L}(\nu \cdot \partial \nu) \rangle + \langle \eta \rangle ; \tag{6.90}
\]

the means of $\Delta \nu$ and $\mathcal{L}(\nu \cdot \partial \nu)$ vanish due to Eqs. (6.12) (6.43) (6.76), so we get the thesis (6.88).

(ii) Let us write

\[
\frac{1}{2} \frac{d}{dt} \|\nu\|_{L^2}^2 = \langle \nu | \dot{\nu} \rangle_{L^2} = \langle \nu | \Delta \nu \rangle_{L^2} - \langle \nu | \mathcal{L}(\nu \cdot \partial \nu) \rangle_{L^2} + \langle \nu | \eta \rangle_{L^2} ; \tag{6.91}
\]

on the other hand by the symmetry of $\mathcal{L}$, the equality $\mathcal{L}\nu = \nu$ and Lemma 6.6,

\[
\langle \nu | \mathcal{L}(\nu \cdot \partial \nu) \rangle_{L^2} = \langle \mathcal{L}\nu | \nu \cdot \partial \nu \rangle_{L^2} = \langle \nu | \nu \cdot \partial \nu \rangle_{L^2} = 0 ; \tag{6.92}
\]

these facts yield the equality in (6.89). The subsequent inequality in (6.89) follows via elementary integration by parts:

\[
\langle \nu | \Delta \nu \rangle_{L^2} = \delta_{rs} \int_{T^d} \nu^r \Delta \nu^s = -\delta_{rs} \int_{T^d} \partial \nu^r \cdot \partial \nu^s \leq 0 . \tag{6.93}
\]

Reducing the NS equations to the case of a zero mean velocity field. Let us recall the notation $H_{\mathbb{E}^0}^n$ for the space of divergence free, zero mean vector fields (see Eq. (6.51) and subsequent comments); we regard this as a Hilbert space, with the inner product $\langle \ | \ \rangle_n$ and the norm $\| \ |_n$ inherited from $H^n$.

The purpose of this paragraph is to show that the general Cauchy problem (6.87) can be reduced to a Cauchy problem for zero mean vector fields; let us define the latter precisely.
6.11 **Definition.** Let \( f_0 \in \mathbb{H}^{n+1}_{\infty,0} \) and \( \xi \in C^{0,1}([t_0, +\infty), \mathbb{H}^{n-1}_\Sigma) \). The incompressible, zero mean NS Cauchy problem with initial datum \( v_0 \) and forcing term \( \xi \) is the following.

\[
\begin{align*}
\text{Find } \varphi & \in C([0,T), \mathbb{H}^{n+1}_{\infty,0}) \cap C^1([0,T), \mathbb{H}^{n-1}_\Sigma) \text{ such that } \\
\dot{\varphi}(t) &= \Delta \varphi(t) - \mathcal{L}(\varphi(t) \cdot \partial \varphi(t)) + \xi(t) \quad \text{for } t \in [0,T), \quad \varphi(0) = f_0 \quad (6.94)
\end{align*}
\]

(for some \( T \in (0, +\infty) \)).

Let us connect this problem with the previous one (6.87), for given \( v_0 \) and \( \eta \). To this purpose, we need a bit more regularity on the forcing \( \eta \); to be precise, we assume

\[
v_0 \in \mathbb{H}^{n+1}_\Sigma, \quad \eta \in C^{0,1}([0, +\infty), \mathbb{H}^{n-1}_\Sigma) \cap C([0, +\infty), \mathbb{H}^{n}_\Sigma) .
\quad (6.95)
\]

Let us define from \( v_0 \) and \( \eta \) the following objects:

\[
m_0 := \langle v_0 \rangle \in \mathbb{R}^d ; \quad m \in C([0, +\infty), \mathbb{R}^d), \quad t \mapsto m(t) := m_0 + \int_0^t \langle \eta(s) \rangle ; \quad (6.96)
\]

\[
h \in C^1([0, +\infty), \mathbb{R}^d) , \quad t \mapsto h(t) := \int_0^t ds \langle \eta(s) \rangle ; \quad (6.97)
\]

\[
f_0 := v_0 - m_0 \in \mathbb{H}^{n+1}_{\infty,0} ; \quad (6.98)
\]

\[
\xi : [0, T) \to \mathbb{H}^{n}_{\Sigma,0} , \quad t \mapsto \xi(t) \text{ such that } \xi(x,t) := \eta(x + h(t), t) - \langle \eta(t) \rangle . \quad (6.99)
\]

The statement \( \xi(t) \in \mathbb{H}^{n}_{\Sigma,0} \) for each \( t \) is evident from the definition. In Appendix F we will prove that

\[
\xi \in C^{0,1}([0, +\infty), \mathbb{H}^{n-1}_{\Sigma,0}) . \quad (6.100)
\]

6.12 **Proposition.** Let us consider:

(i) the Cauchy problem (6.87), with any datum \( v_0 \) and forcing \( \eta \) as in (6.95);

(ii) the above definitions of \( m_0, m, h, f_0, \xi \) and the Cauchy problem (6.94).

A function \( \nu \) of domain \([0,T)\) fulfills (6.87) if and only if there is a function \( \varphi \) on \([0,T)\) fulfilling (6.94) such that, for all \( x \in T^d \) and \( t \in [0,T) \),

\[
\nu(x,t) = m(t) + \varphi(x - h(t), t) . \quad (6.101)
\]

**Proof.** Step 1. We suppose (6.94) with the above datum \( f_0 \) to have a solution \( \varphi \) on \([0,T)\); we define \( \nu \) as in (6.101) and prove that it solves problem (6.87). It is clear that \( \nu \) is in the functional space prescribed by (6.87), and that the following holds:

\[
\nu(x,0) = m_0 + \varphi(x,0) = m_0 + f_0(x) = v_0(x) ; \quad (6.102)
\]

\[
\nu(t,x) = \langle \eta(t) \rangle + \dot{\varphi}(x - h(t), t) - h(t) \cdot \partial \varphi(x - h(t), t) \quad (6.103)
\]

\[
= \langle \eta(t) \rangle + \dot{\varphi}(x - h(t), t) - m(t) \cdot \partial \varphi(x - h(t), t) .
\]

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\[ \Delta \nu(x, t) = \Delta \varphi(x - h(t), t) ; \]  
(6.104) 

\[ (\nu \cdot \partial \nu)(x, t) = (m(t) \cdot \partial \varphi)(x - h(t), t) + (\varphi \cdot \partial \varphi)(x - h(t), t) . \]  
(6.105) 

The Leray projection \( \mathcal{L} \) commutes with space translations, so the last equation implies

\[ \mathcal{L}(\nu \cdot \partial \nu)(x, t) = \mathcal{L}(m(t) \cdot \partial \varphi)(x - h(t), t) + \mathcal{L}(\varphi \cdot \partial \varphi)(x - h(t), t) . \]  
(6.106)

To conclude, we note that

\[ \mathcal{L}(m(t) \cdot \partial \varphi) = m(t) \cdot \partial \varphi ; \]  
(6.107)

to prove this, it suffices to check that \( m(t) \cdot \partial \varphi \) is divergence free. In fact,

\[ \text{div}(m(t) \cdot \partial \varphi) = \partial_s(m^r(t) \partial_r \varphi^s) = m^r(t) \partial_r(\partial_s \varphi^s) = m^r(t) \partial_r(\text{div} \varphi) = 0 , \]  
(6.108)

since \( \varphi \) is divergence free at all times. From Eq. (6.102) we see that \( \nu \) fulfills the initial condition in (6.87); from Eqs. (6.103) (6.104) (6.106) (6.107) and (6.94), we see that \( \nu \) fulfills the evolution equation in (6.87).

**Step 2.** Let us consider function \( \nu \) on \([0, T)\), fulfilling (6.87); we will prove the existence of a function \( \varphi \) on \([0, T)\) fulfilling (6.94), such that \( \nu \) and \( \varphi \) are related by (6.101). To this purpose, let us define a function \( \varphi \) by

\[ \varphi(x, t) := \nu(x + h(t), t) - m(t) ; \]  
(6.109)

then, at each time \( t \in [0, T) \),

\[ \langle \varphi(t) \rangle = \langle \nu(t) \rangle - m(t) = 0 \]  
(6.110)

on account of Eq. (6.88) for \( \langle \nu \rangle \) and of the definition (6.96) of \( m \). Besides having zero mean, \( \varphi \) belongs to the function spaces prescribed by (6.87) due to the properties of \( \nu \). Now, computations very similar to the ones of Step 1 prove that \( \varphi \) fulfills the initial condition and the evolution equation in (6.94). \( \square \)

### 6.13 Remark.

Let us regard Eqs. (6.98) (6.99) as defining a transformation

\[ \mathcal{T} : \mathbb{H}_{\Sigma}^{n+1} \times C^{0,1}([0, +\infty), \mathbb{H}_{\Sigma}^{n-1}) \to \mathbb{H}_{\Sigma_0}^{n+1} \times C^{0,1}([0, +\infty), \mathbb{H}_{\Sigma_0}^{n-1}) , \]  
(6.111)

\[ (v_0, \eta) \mapsto (f_0, \xi) = \mathcal{T}(v_0, \eta) . \]

The map \( \mathcal{T} \) is onto, due to the trivial equality \( (f_0, \xi) = \mathcal{T}(f_0, \xi) \).
7 The NS equations in the general framework for evolution equations with quadratic nonlinearity.

Basic notations. The zero mean version (6.94) is the final form for the NS Cauchy problem, to which we stick from now on. Let us recall that $d \in \{2, 3, \ldots\}$; throughout the section, we fix $n > \frac{d}{2}$, $\xi \in C^0([0, +\infty), \mathbb{H}^{n-1}_{20})$.

(7.1) (the function $\xi$ is regarded to be given by itself, independently of any function $\eta$ as in the previous section). Our aim is to apply the formalism of Section 5 to the Cauchy problem (6.94) (and to the equivalent Volterra problem); in this case

$$F_\pm \equiv (F_\pm, \|\|_\pm) := (\mathbb{H}^{n+1}_{20}, \|\|_{n+1}), \quad F \equiv (F, \|\|) := (\mathbb{H}^n_{20}, \|\|_n),$$

(7.2)

$$A := \Delta : \mathbb{H}^{n+1}_{20} \to \mathbb{H}^{n-1}_{20}, \quad f \mapsto \Delta f ;$$

(7.3)

$$\mathcal{P} : \mathbb{H}^n_{20} \times \mathbb{H}^n_{20} \to \mathbb{H}^{n-1}_{20}, \quad \mathcal{P}(f, g) := -\mathcal{L}(f \cdot \partial g) ; \quad \xi \text{ as in (7.1)} .$$

(7.4)

From $\mathcal{P}$ and $\xi$, we construct the function

$$\mathcal{P} : \mathbb{H}^n_{20} \times [0, +\infty) \to \mathbb{H}^{n-1}_{20} , \quad \mathcal{P}(f, t) := \mathcal{P}(f, f) + \xi(t) = -\mathcal{L}(f \cdot \partial f) + \xi(t) ,$$

(7.5)

which appears in (6.94).

For subsequent reference, we record the Fourier representations

$$\|f\|_{n \pm 1} = \sqrt{\sum_{k \in \mathbb{Z}_0^d} (1 + |k|^2)^{n \pm 1}|f_k|^2}, \quad \|f\|_n = \sqrt{\sum_{k \in \mathbb{Z}_0^d} (1 + |k|^2)^n|f_k|^2},$$

(7.6)

$$(\Delta f)_k = -|k|^2 f_k .$$

(7.6)

Verification of properties (P1)-(P5'). The above set $(F_-, F, F_+, A)$, completed with suitable semigroup estimators $u, u_-$, has the properties (P1)-(P5) prescribed in Section 2, and (P4')(P5') of Section 5. We will indicate which parts of the proof are obvious, and give details on the nontrivial parts.

7.1 Proposition. $F = \mathbb{H}^n_{20}, F_+ = \mathbb{H}^{n+1}_{20}$ and $A = \Delta$ fulfill conditions (P1)-(P2).

Proof. Everything follows easily from the Fourier representations. □

7.2 Proposition. (i) $\Delta$ generates a strongly continuous semigroup on $\mathbb{H}^{n-1}_{20}$, given by

$$e^{t\Delta} f = \sum_{k \in \mathbb{Z}_0^d} e^{-|k|^2 t} f_k e_k , \quad \text{for } f \in \mathbb{H}^{n-1}_{20}, \ t \in [0, +\infty) .$$

(7.7)
So, (P3) holds.

(ii) For $f \in \mathbb{H}^n_{\Sigma_0}$ and $t \in [0, +\infty)$ one has

$$e^{t\Delta} f \in \mathbb{H}^n_{\Sigma_0}, \quad \|e^{t\Delta} f\|_n \leq u(t)\|f\|_n, \quad u(t) := e^{-t}; \quad (7.8)$$

the function $(f, t) \mapsto e^{t\Delta} f$ gives a strongly continuous semigroup on $\mathbb{H}^n_{\Sigma_0}$.

(iii) For $f \in \mathbb{H}^{n-1}_{\Sigma_0}$ and $t \in (0, +\infty)$ one has

$$e^{t\Delta} f \in \mathbb{H}^{n-1}_{\Sigma_0}, \quad \|e^{t\Delta} f\|_{n-1} \leq u_-(t)\|f\|_{n-1}, \quad (7.9)$$

$$u_-(t) := e^{-t} \mu_-(t), \quad \mu_-(t) := \begin{cases} 
\frac{e^{2t}}{\sqrt{2ct}} & \text{for } 0 < t \leq \frac{1}{4}, \\
\sqrt{2} & \text{for } t > \frac{1}{4}; 
\end{cases} \quad (7.10)$$

Note that $u_-(t), \mu_-(t) = O(1/\sqrt{t})$ for $t \to 0^+$. The function $(f, t) \mapsto e^{t\Delta} f$ is continuous from $\mathbb{H}^{n-1}_{\Sigma_0} \times (0, +\infty)$ to $\mathbb{H}^n_{\Sigma_0}$.

With $u_-$ as in (7.10), the function $t \mapsto U(t) := \int_0^t ds u_-(s)$ is given by

$$U(t) := \begin{cases} 
\frac{\gamma(t)}{\sqrt{2}} & \text{for } 0 < t \leq \frac{1}{4}, \\
\frac{\gamma(1/4)}{\sqrt{2}} + \sqrt{2}(e^{-1/4} - e^{-t}) & \text{for } \frac{1}{4} < t \leq +\infty, 
\end{cases} \quad (7.11)$$

$$\gamma(t) := \int_0^t ds \frac{e^s}{\sqrt{s}} \quad \text{for } 0 \leq t \leq \frac{1}{4}. \quad (7.12)$$

(In particular, $U(+\infty) = \gamma(1/4)/\sqrt{2} + \sqrt{2}e^{-1/4} \in (1.872, 1.873)$).

(iv) With $\mu_-$ as in (7.10), one has

$$\sup_{t \in [0, +\infty)} \int_0^t ds \mu_-(t-s)e^{-s} = \sqrt{2}. \quad (7.13)$$

(v) In conclusion, (P4) and (P5) hold with $B = 1$, $\sigma = \frac{1}{2}$, $N = \sqrt{2}$.

Proof. (i) This follows basically from Eq. (7.6) for $\Delta$.

(ii) We only give details on the derivation of Eqs. (7.8)-(7.11). Let $f \in \mathbb{H}^n_{\Sigma_0}$, $t \in [0, +\infty)$. Then,

$$\sum_{k \in \mathbb{Z}_0^d} (1 + |k|^2)^n |(e^{t\Delta} f)_k|^2 = \sum_{k \in \mathbb{Z}_0^d} (1 + |k|^2)^n e^{-2t|k|^2} |f_k|^2 \quad (7.15)$$

$$\leq e^{-2t} \sum_{k \in \mathbb{Z}_0^d} (1 + |k|^2)^n |f_k|^2, \quad$$

since $|k| \geq 1$ for $k \in \mathbb{Z}_0^d$; this yields Eq. (7.8). Now, let $f \in \mathbb{H}^{n-1}_{\Sigma_0}$ and $t \in (0, +\infty)$;
then,
\[
\sum_{k \in \mathbb{Z}_d^d} (1 + |k|^2)^n |(e^{t\Delta} f)_k|^2 = \sum_{k \in \mathbb{Z}_d^d} (1 + |k|^2) e^{-2t|k|^2} (1 + |k|^2)^{n-1} |f_k|^2
\]  
(7.16)

\[
\leq \left( \sup_{\vartheta \in [1, +\infty)} U_t(\vartheta) \right) \left( \sum_{k \in \mathbb{Z}_d^d} (1 + |k|^2)^{n-1} |f_k|^2 \right),
\]

\[
U_t(\vartheta) := (1 + \vartheta) e^{-2t\vartheta},
\]

and an elementary computation gives
\[
\sup_{\vartheta \in [1, +\infty)} U_t(\vartheta) = \begin{cases} 
U_t\left(\frac{1}{2t} - 1\right) = \frac{e^{2t}}{2et} & \text{for } 0 < t \leq \frac{1}{4}, \\
U_t(1) = 2e^{-2t} & \text{for } t > \frac{1}{4}.
\end{cases}
\]  
(7.17)

From (7.17) we infer Eq. (7.9) with \(u_-(t) := \sqrt{\sup_{\vartheta \in [1, +\infty)} U_t(\vartheta)}\), i.e.,
\[
u_-(t) = \begin{cases} 
\frac{e^t}{\sqrt{2et}} & \text{for } 0 < t \leq \frac{1}{4}, \\
\sqrt{2e^{-t}} & \text{for } t > \frac{1}{4}.
\end{cases}
\]  
(7.18)

this definition of \(u_-(t)\) agrees with Eq. (7.10), and (7.11) follows trivially.
(iv) See Appendix G.
(v) Obvious consequence of items (i-iv).

\[\square\]

**Analysis of \(\mathcal{P}\) and \(\mathcal{P}'\).** We turn the attention to the functions in Eqs. (7.4) (7.5).

**7.3 Proposition.** (i) \(\mathcal{P}\) is a bilinear map and admits the estimate (implying continuity)
\[
\|\mathcal{P}(f, g)\|_{n-1} \leq K_n \|f\|_n \|g\|_n
\]  
(7.19)

for all \(f, g \in \mathbb{H}_\Sigma^0\), with \(K_n \equiv K_{nd}\) any constant fulfilling (6.78) (so, condition (Q1) holds for this map).
(ii) As a consequence of (i), \(\mathcal{P}\) fulfills the Lipschitz condition (P6). Furthermore, for each function \(\phi \in C([0,T), \mathbb{H}_\Sigma^0)\), the growth of \(\mathcal{P}\) from \(\phi\) admits the estimate
\[
\|\mathcal{P}(f, t) - \mathcal{P}(\phi(t), t)\|_{n-1} \leq \ell_n(t, \|f - \phi(t)\|_n)
\]  
(7.20)

for \(t \in [0,T)\) and \(f \in \mathbb{H}_\Sigma^0\), where
\[
\ell_n : [0, +\infty) \times [0, T) \rightarrow [0, +\infty), \ (r, t) \mapsto \ell_n(r, t) := 2K_n\|\phi(t)\|_n r + K_n r^2.
\]  
(7.21)
Proof. (i) The bilinearity is obvious, the estimate follows from Eqs. (6.49) for \( \mathcal{L} \) and (6.78) for the map \((f, g) \mapsto f \cdot \partial g\).

(ii) Use Corollaries 5.3, 5.4 on quadratic maps. \(\square\)

The function \( \Xi_{n-1} \). From here to the end of the paper, we denote in this way any function in \( C([0, +\infty), [0, +\infty)) \) such that

\[
\Xi_{n-1} \in C([0, +\infty), [0, +\infty)) \text{ nondecreasing ,} \tag{7.22}
\]

(e.g., \( \Xi_{n-1}(t) := \sup_{s \in [0,t]} \| \xi(s) \|_{n-1} \)).

Cauchy and Volterra problems.

7.4 Definition. For any \( f_0 \in \mathbb{H}^{n+1}_{\Sigma_0} \), \( CP_n(f_0) \) is the Cauchy problem (6.94), i.e.,

Find \( \varphi \in C([0, T), \mathbb{H}^{n+1}_{\Sigma_0}) \cap C^1([0, T), \mathbb{H}^{n-1}_{\Sigma_0}) \) such that

\[
\varphi(t) = \Delta \varphi(t) + \mathcal{P}(\varphi(t), t) \quad \text{for all } t \in [0, T) , \quad \varphi(0) = f_0 . \tag{7.23}
\]

For any \( f_0 \in \mathbb{H}^{n}_{\Sigma_0} \), \( VP_n(f_0) \) is the Volterra problem

Find \( \varphi \in C([0, T), \mathbb{H}^{n}_{\Sigma_0}) \) such that

\[
\varphi(t) = e^{t\Delta} f_0 + \int_0^t ds \, e^{(t-s)\Delta} \mathcal{P}(\varphi(s), s) \quad \text{for all } t \in [0, T) . \tag{7.24}
\]

7.5 Remarks. (i) If \( f_0 \in \mathbb{H}^{n+1}_{\Sigma_0} \), \( VP_n(f_0) \) is equivalent to \( CP_n(f_0) \) by the reflexivity of the Hilbert space \( \mathbb{H}^{n-1}_{\Sigma_0} \) (see once more Proposition 2.4).

(ii) For any \( f_0 \in \mathbb{H}^{n}_{\Sigma_0} \), uniqueness and local existence are granted for \( VP_n(f_0) \) (Propositions 2.5 and 3.10).

8 Results for the NS equations arising from the previous framework.

We keep the assumption (7.1) and all notations of the previous section; furthermore, we fix an initial datum

\[
f_0 \in \mathbb{H}^{n}_{\Sigma_0} . \tag{8.1}
\]

The analysis of the previous section allows us to identify \( VP_n(f_0) \) with a Volterra problem of the general type discussed in Section 5, with semigroup estimators \( u, u_- \) of the form considered therein and a quadratic nonlinearity \( \mathcal{P} \). Due to Propositions
7.2, 7.3, the constant $K$ and the functions $u_-, u_-, \mu_-, U, \Xi_-$ of Section 5 can be taken as follows:

$$K = a constant \ K_n \ fulfills \ (6.78), \quad u(t) := e^{-t}, \quad u_-(t) = \mu_-(t)e^{-t}, \quad (8.2)$$

$\mu_-$ as in (7.10), $U$ as in (7.11), $\Xi_- = \Xi_{n-1}$ as in (7.22).

Hereafter, we rephrase Propositions 5.5 and 5.6 with the above specifications.

8.1 Proposition. Let us consider for $\mathbb{VP}_n(f_0)$ an approximate solution $\varphi_{ap} \in C([0, T], \mathbb{H}^m_{\Sigma_0})$, where $T \in (0, +\infty]$. Suppose there are functions $\mathcal{E}_n, \mathcal{D}_n, \mathcal{R}_n \in C([0, T], [0, +\infty))$ such that (i)-(iii) hold:

(i) $\varphi_{ap}$ has the integral error estimate

$$\|E(\varphi_{ap}(t))\|_n \leq \mathcal{E}_n(t) \quad for \ t \in [0, T); \quad (8.3)$$

(ii) one has

$$\|\varphi_{ap}(t)\|_n \leq \mathcal{D}_n(t) \quad for \ t \in [0, T); \quad (8.4)$$

(iii) $\mathcal{R}_n$ solves the control inequality

$$\mathcal{E}_n(t) + K_n \int_0^t ds u_-(t-s)(2\mathcal{D}_n(s)\mathcal{R}_n(s) + \mathcal{R}_n^2(s)) \leq \mathcal{R}_n(t) \quad for \ t \in [0, T). \quad (8.5)$$

Then, (a) and (b) hold:

(a) $\mathbb{VP}(f_0)$ has a solution $\varphi : [0, T) \to \mathbb{H}^m_{\Sigma_0}$;

(b) one has

$$\|\varphi(t) - \varphi_{ap}(t)\|_n \leq \mathcal{R}_n(t) \quad for \ t \in [0, T). \quad (8.6)$$

8.2 Proposition. Let us consider for $\mathbb{VP}_n(f_0)$ an approximate solution $\varphi_{ap} \in C([0, T], \mathbb{H}^m_{\Sigma_0})$, where $T \in (0, +\infty]$. Suppose there are functions $\mathcal{E}_n, \mathcal{D}_n \in C([0, T], [0, +\infty))$ such that (i)-(iii) hold:

(i) $\mathcal{E}_n$ is nondecreasing, and binds the integral error as in (8.3);

(ii) $\mathcal{D}_n$ is nondecreasing and binds $\varphi_{ap}$ as in (8.4);

(iii) one has

$$2 \sqrt{K_n U(T)\mathcal{E}_n(T)} + 2K_n U(T)\mathcal{D}_n(T) \leq 1. \quad (8.7)$$

Then $\mathbb{VP}_n(f_0)$ has a solution $\varphi : [0, T) \to \mathbb{H}^m_{\Sigma_0}$ and, for all $t \in [0, T)$,

$$\|\varphi(t) - \varphi_{ap}(t)\|_n \leq \mathcal{R}_n(t), \quad (8.8)$$

$$\mathcal{R}_n(t) := \left\{ \begin{array}{ll}
1 - 2K_n U(t)\mathcal{D}_n(t) - \sqrt{(1 - 2K_n U(t)\mathcal{D}_n(t))^2 - 4K_n U(t)\mathcal{E}_n(t)}}\mathcal{E}_n(t) & \text{if } t \in (0, T), \\
\mathcal{E}_n(0) & \text{if } t = 0;
\end{array} \right.$$

the above prescription gives a well defined, nondecreasing function $\mathcal{R}_n \in C([0, T], [0, +\infty)). \quad (8.9)$
In Section 5, from Proposition 5.6 we have inferred Proposition 5.9, corresponding to the approximate solution \( \varphi_{ap} := 0 \); in the present situation, this reads as follows.

### 8.3 Proposition

Let

\[
\mathcal{F}_n(t) := \|f_0\|_n + \Xi_{n-1}(t)U(t) .
\]

Suppose \( T \in [0, +\infty] \), and

\[
4K_n U(T) \mathcal{F}_n(T) \leq 1 .
\]

Then \( \mathcal{V}_p_n(f_0) \) has a solution \( \varphi : [0, T) \rightarrow \mathbb{H}_{\Sigma_0}^n \) and, for all \( t \in [0, T) \),

\[
\|\varphi(t)\|_n \leq \mathcal{F}_n(t) \mathcal{X}(4K_n U(t) \mathcal{F}_n(t))
\]  \hspace{1cm} (8.11)

(where, as in (5.31): \( \mathcal{X}(z) := \frac{1 - \sqrt{1 - z}}{(z/2)} \) for \( z \in (0, 1] \), \( \mathcal{X}(0) := 1 \)).

The other results of Section 5 were about global existence and exponential decay, under specific assumption. In the present framework the constants \( B, \sigma, N \) of the cited section are given by Eq. (7.14); this allows to rephrase Proposition 5.10 in this way.

### 8.4 Proposition

Let us consider for \( \mathcal{V}_p_n(f_0) \) an approximate solution \( \varphi_{ap} \in C([0, +\infty), \mathbb{H}_{\Sigma_0}^n) \). Suppose there are constants \( E_n, D_n \in [0, +\infty) \) such that:

(i) \( \varphi_{ap} \) admits the integral error estimate

\[
\|E(\varphi_{ap})(t)\|_n \leq E_n e^{-t} \quad \text{for} \quad t \in [0, +\infty) ;
\]  \hspace{1cm} (8.12)

(ii) for all \( t \) as above,

\[
\|\varphi_{ap}(t)\|_n \leq D_n e^{-t} ;
\]  \hspace{1cm} (8.13)

(iii) one has

\[
2 \sqrt{2} \sqrt{K_n E_n} + 2 \sqrt{2} K_n D_n \leq 1 .
\]  \hspace{1cm} (8.14)

Then \( \mathcal{V}_p_n(f_0) \) has a global solution \( \varphi : [0, +\infty) \rightarrow \mathbb{H}_{\Sigma_0}^n \) and, for all \( t \in [0, +\infty) \),

\[
\|\varphi(t) - \varphi_{ap}(t)\|_n \leq R_n e^{-t} ,
\]  \hspace{1cm} (8.15)

\[
R_n := \frac{1 - 2 \sqrt{2} K_n D_n - \sqrt{(1 - 2 \sqrt{2} K_n D_n)^2 - 4 \sqrt{2} K_n E_n}}{2 \sqrt{2} K_n} .
\]

The applications of Proposition 5.10 considered in Section 5 were based on the assumption of exponential decay for the external forcing, that in the present framework must be formulated in this way:
There is a constant $J_{n-1} \in [0, +\infty)$ such that
\[ \|\xi(t)\|_{n-1} \leq J_{n-1} e^{-2t} \quad \text{for all } t \in [0, +\infty). \] (8.16)

The above mentioned applications in Section 5 were Proposition 5.12 (corresponding to the choice $\varphi_{ap} := 0$) and Proposition 5.14 (with $\varphi_{ap}$ the $A$-flow approximate solution). These can be restated, respectively, in the following way:

8.5 Proposition. Assume (Q3), and define
\[ F_n := \|f_0\|_n + \sqrt{2}J_{n-1}; \] (8.17)
furthermore, assume
\[ 4\sqrt{2}K_n F_n \leq 1. \] (8.18)
Then $\mathcal{V}P_n(f_0)$ has a global solution $\varphi : [0, +\infty) \to \mathbb{H}^n_{x_0}$ and, for all $t \in [0, +\infty),$
\[ \|\varphi(t)\|_n \leq F_n \mathcal{X}(4\sqrt{2}K_n F_n) e^{-t} \] (8.19)
(with $\mathcal{X}$ as in (5.31)).

8.6 Proposition. Define
\[ \varphi_{ap} \in C([0, +\infty), \mathbb{H}^n_{x_0}), \quad \varphi_{ap}(t) := e^{t\Delta}f_0 + \int_0^t ds \ e^{(t-s)\Delta}\xi(s). \] (8.20)

Furthermore, let us keep the assumptions and definitions (Q3) (8.17) (8.18). Then the global solution $\varphi : [0, +\infty) \to \mathbb{H}^n_{x_0}$ of $\mathcal{V}P_n(f_0)$ is such that, for all $t \in [0, +\infty),$
\[ \|\varphi(t) - \varphi_{ap}(t)\|_n \leq \sqrt{2}K_n F_n^2 \mathcal{X}(4\sqrt{2}K_n F_n) e^{-t} \] (8.21)
(where, as in (5.60): $\mathcal{X}(z) := \frac{1 - (z/2) - \sqrt{1 - z}}{(z^2/8)}$ for $z \in (0, 1], \mathcal{X}(0) := 1$).

8.7 Remarks. (i) Condition (8.10) can be fulfilled with either $f_0, \xi$ small, $T$ large or $f_0, \xi$ large, $T$ small. Condition (8.18) is fulfilled if $f_0$ and $\xi$ are sufficiently small.

(ii) As a special case, suppose the external forcing $\xi$ to be identically zero; then we can take $\Xi_{n-1} = 0$ and $J_{n-1} = 0$. Eqs. (8.10) and (8.18) ensure global existence if the datum fulfills the conditions $4K_n \mathcal{U}(+\infty)\|f_0\|_n \leq 1$ and $4\sqrt{2}K_n\|f_0\|_n \leq 1$, respectively. The less restrictive condition on $f_0$ is the second one, since $\sqrt{2} < \mathcal{U}(+\infty)$.

(iii) Let us return to Proposition 8.1; this states, amongst else, that the solution $\varphi$ of $\mathcal{V}P_n(f_0)$ exists on $[0, T)$ if the inequality (8.5) has a solution $\mathcal{R} : [0, T) \to [0, +\infty)$. Let us compare this statement with a result presented in the recent work [2], that we rephrase here in our notations.
Let us consider the (incompressible, zero mean) NS equations with external forcing $\xi$ and initial datum $f_0$; when we refer to [2] a solution of this Cauchy problem means a strong solution, as defined therein. Now suppose $\varphi_{ap}$ to be an approximate solution on an interval $[0, T)$, and

$$\|\varphi_{ap}(0) - f_0\|_{n-1} \leq \delta_{n-1}, \quad \|\varphi_{ap}(t) - \Delta \varphi_{ap}(t) - P(\varphi_{ap}(s), s)\|_{n-1} \leq \epsilon_{n-1}(t),$$  

(8.22)

$$\|\varphi_{ap}(t)\|_{n-1} \leq D_{n-1}(t), \quad \|\varphi_{ap}(t)\|_n \leq D_n(t)$$

for all $t \in [0, T)$, for suitable estimators $\delta_{n-1} \geq 0$, $\epsilon_{n-1}, D_{n-1}, D_n : [0, T) \to [0, +\infty)$. According to [2] (page 065204-10), the NS Cauchy problem has a solution $\varphi : [0, T) \to \mathbb{H}^n_{\Sigma_0}$ if

$$\delta_{n-1} + \int_0^T ds \epsilon_{n-1}(s) < \frac{1}{C_n T} e^{-C_n \int_0^T ds (D_{n-1}(s) + D_n(s))}$$

(8.23)

(with $C_n > 0$ a constant not computed explicitly, whose role is analogous to the one of $K_n$); (8.23) is an inequality involving only the approximate solution, and plays a role similar to our (8.5) to grant the existence of an exact solution on $[0, T)$.

Seemingly, Eq.(8.23) is not suited to obtain results of global existence for the exact solution. To explain this statement, suppose $\varphi_{ap}$ and its estimators to be defined on $[0, +\infty)$, with $\delta_{n-1} \neq 0$ or $\epsilon_{n-1}$ non identically zero; then (8.23) surely fails for large $T$, even in the most favourable situation where all integrals therein converge for $T \to +\infty$. In fact,

$$\begin{align*}
\text{l.h.s. of (8.23)} & \quad \xrightarrow{T \to +\infty} \quad \delta_{n-1} + \int_0^{+\infty} ds \epsilon_{n-1}(s) \in (0, +\infty], \\
0 & \leq \text{r.h.s. of (8.23)} \leq \frac{1}{C_n T} \quad \xrightarrow{T \to +\infty} \quad 0.
\end{align*}$$

On the contrary, our control inequality (8.5) can be used in certain cases to derive the existence of $\varphi$ (and bind its distance from $\varphi_{ap}$) up to $T = +\infty$; some applications of this type have appeared in the present section, further examples will be given in the next one on the Galerkin approximations (10).

---

\textsuperscript{10}To conclude this remark we wish to point out that, under special assumptions, some global existence results could perhaps be derived from the approach of [2], with a different analysis of the differential inequalities proposed by the authors to infer Eq. (8.23). A discussion of this point, and of other interesting features of [2], would occupy too much space here.
9 Galerkin approximate solutions of the NS equations.

Throughout this section, we consider a set $G$ with the following features:

$$G \subset \mathbb{Z}_0^d, \quad G \text{ finite, } \quad k \in G \Leftrightarrow -k \in G . \quad (9.1)$$

Hereafter we write $e_k \succ k \in G$ for the linear subspace of $D$ spanned by the functions $e_k$ for $k \in G$.

Galerkin subspaces and projections. We define them as follows.

9.1 Definition. The Galerkin subspace and projection corresponding to $G$ are

$$H^G_{\Sigma_0} := D_\Sigma_0' \cap \langle e_k \rangle_{k \in G} = \left\{ \sum_{k \in G} v_k e_k \mid v_k \in C^d, \overline{v_k} = v_{-k}, k \cdot v_k = 0 \text{ for all } k \right\} . \quad (9.2)$$

$$\mathcal{P}^G : D_\Sigma_0' \to H^G_{\Sigma_0}, \quad v = \sum_{k \in \mathbb{Z}_0^d} v_k e_k \mapsto \mathcal{P}^G v := \sum_{k \in G} v_k e_k . \quad (9.3)$$

It is clear that

$$H^G_{\Sigma_0} \subset C^\infty \cap D_\Sigma_0', \quad \Delta(H^G_{\Sigma_0}) \subset H^G_{\Sigma_0} ; \quad (9.4)$$

$$H^G_{\Sigma_0} \subset H^m_{\Sigma_0}, \quad \mathcal{P}^G(H^m_{\Sigma_0}) = H^G_{\Sigma_0} \text{ for all } m \in \mathbb{R} . \quad (9.5)$$

The following result will be useful in the sequel.

9.2 Lemma. Let $n, p \in \mathbb{R}, n \leq p$ and $v \in H^p_{\Sigma_0}$. Then,

$$\| (1 - \mathcal{P}^G) v \|_n \leq \frac{\| v \|_p}{|G|^{p-n}}, \quad |G| := \inf_{k \in \mathbb{Z}_0^d \setminus G} \sqrt{1 + |k|^2} . \quad (9.6)$$

Proof. We have $(1 - \mathcal{P}^G) v = \sum_{k \in \mathbb{Z}_0^d \setminus G} v_k e_k$, implying

$$\| (1 - \mathcal{P}^G) v \|_n^2 = \sum_{k \in \mathbb{Z}_0^d \setminus G} (1 + |k|^2)^n |v_k|^2 = \sum_{k \in \mathbb{Z}_0^d \setminus G} \frac{(1 + |k|^2)^p}{(1 + |k|^2)^{p-n}} |v_k|^2 \quad (9.7)$$

$$\leq \left( \sup_{k \in \mathbb{Z}_0^d \setminus G} \frac{1}{(1 + |k|^2)^{p-n}} \right) \left( \sum_{k \in \mathbb{Z}_0^d \setminus G} (1 + |k|^2)^p |v_k|^2 \right) \leq \frac{1}{|G|^{2(p-n)}} \| v \|_p^2 ,$$

whence the thesis. \qed

Galerkin approximate solutions. Let

$$\xi \in C([0, +\infty), D_\Sigma_0'), \quad f_0 \in D_\Sigma_0' \quad (9.8)$$

(of course, in the sequel $\mathcal{P}(f, t) := -\mathcal{L}(f \ast df) + \xi(t)$ whenever this makes sense).
9.3 **Definition.** The Galerkin approximate solution of NS corresponding to \( G \), with external forcing \( \xi \) and datum \( f_0 \), is the maximal solution \( \phi^G \equiv \phi^G \) of the following Cauchy problem, in the finite dimensional space \( \mathbb{H}^{G\Sigma_0} \):

\[
\text{Find } \phi^G \in C^1([0, T_G), \mathbb{H}^{G\Sigma_0}) \text{ such that } \quad (9.9)
\]

\[
\dot{\phi}^G(t) = \Delta \phi^G(t) + \mathcal{P}^G(\phi^G(t)) \quad \text{for all } t , \quad \phi^G(0) = \mathcal{P}^G f_0 .
\]

Of course, "maximal" means that \([0, T_G)\) is the largest interval of existence. In certain cases, one can prove that \( T_G = +\infty \) and derive estimates of \( \phi^G \) (we return on this in the sequel). In this section we use the functions \( \mathcal{U} \) as in Eq. (7.11), \( \mathcal{X} \) as in Eq. (5.31).

Let us consider any real number \( m \), and assume \( \xi \in C^{0,1}([0, +\infty), \mathbb{H}^{m-1}_{\Sigma_0}) \). We will use the notation \( \Xi_{m-1} \) to indicate any function in \( C([0, +\infty), [0, +\infty)) \) fulfilling Eq. (7.22) with \( n \) replaced by \( m \); in the sequel that equation will be referred to as (7.22)\(_m\). When necessary we will make the assumption (Q3)\(_m\) of exponential decay for the external forcing; this is like (Q3)\(_n\) with \( n \to m \), thus involving a constant \( J_{m-1} \in [0, +\infty) \).

The forthcoming proposition gives estimates on the interval of existence of the Galerkin solution and on its norm \( \| \|_m \), which are in fact independent of \( G \).

9.4 **Proposition.** Let \( m > d/2 \), \( \xi \in C^{0,1}([0, +\infty), \mathbb{H}^{m-1}_{\Sigma_0}) \) and \( f_0 \in \mathbb{H}^{m}_{\Sigma_0} \); then, (i)(ii) hold.

(i) Define (similarly to (8.9))

\[
\mathcal{F}_m(t) := \| f_0 \|_m + \Xi_{m-1}(t) \mathcal{U}(t) ; \quad (9.11)
\]

furthermore, let \( T \in (0, +\infty] \), and assume the inequality

\[
4K_m \mathcal{U}(T) \mathcal{F}_m(T) \leq 1 . \quad (9.12)
\]

Then the Galerkin solution \( \phi^G \) with this datum exists on \([0, T]\) and fulfills

\[
\| \phi^G(t) \|_m \leq \mathcal{D}_m(t) \quad \text{for } t \in [0, T) , \quad (9.13)
\]

\[
\mathcal{D}_m(t) := \mathcal{F}_m(t) \mathcal{X}(4K_m \mathcal{U}(t) \mathcal{F}_m(t)) . \quad (9.14)
\]

(ii) Alternatively, assume (Q3)\(_m\); define (similarly to (8.17))

\[
F_m := \| f_0 \|_m + \sqrt{2} J_{m-1} , \quad (9.15)
\]

and suppose

\[
4\sqrt{2} K_m F_m \leq 1 . \quad (9.16)
\]
Then the Galerkin solution \( \varphi^G \) with this datum is global, and fulfills
\[
\| \varphi^G(t) \|_m \leq D_m e^{-t} \quad \text{for } t \in [0, +\infty),
\]
\[
D_m := F_m \mathcal{X}(4\sqrt{2}K_m F_m)
\]
(the above equations will be referred to in the sequel as (9.11)_m, (9.12)_m, etc.).

**Proof.** We refer to the framework of Section 5 on systems with quadratic nonlinearities. In the present case \((\mathcal{F}, \| \cdot \|):=(\mathcal{H}^G_{\Sigma_0}, \| \cdot \|_m), (\mathcal{F}_\mp, \| \cdot \| := (\mathcal{H}^G_{\Sigma_0}, \| \cdot \|_{m\mp})\) (we have three copies of the same finite dimensional space, but equipped with different, though equivalent, norms); the operator \(\mathcal{A} = \Delta \restriction_{\mathcal{H}^G_{\Sigma_0}}\) and the bilinear map \(\mathcal{B} = \mathcal{A} \cdot\mathcal{P}\); the function \(\mathcal{F} \in C^0_0([0, +\infty), \mathcal{H}^G_{\Sigma_0})\); the initial datum is \(\mathcal{P}^G f_0 \in \mathcal{H}^G_{\Sigma_0}\).

To estimate \(\mathcal{P}^G \mathcal{P}\), we use the inequalities on \(\mathcal{P}\) in Proposition 5.2 with \(n \rightarrow m\), and the obvious relation \(\| \mathcal{P} \cdot \|_m \leq \| \cdot \|_{m-1}\); this gives \(\| \mathcal{P}^G \mathcal{P}(f,g) \|_{m-1} \leq K_m \| f \|_m \| g \|_m\), and so the constant \(K\) of Section 5 is, in this case, \(K_m\).

For the initial datum \(\mathcal{P}^G f_0\) and for \(\mathcal{P}^G \mathcal{F}\) we use the estimates
\[
\| \mathcal{P}^G f_0 \|_m \leq \| f_0 \|_m ,
\]
\[
\| \mathcal{P}^G \mathcal{F}(t) \|_{m-1} \leq \| \mathcal{F}(t) \|_{m-1} \leq \Xi_{m-1}(t) \text{ or } J_{m-1} e^{-2t} ;
\]
of course, the bound via \(\Xi_{m-1}\) refers to case (i) and the bound via \(J_m\) is for case (ii). Applying to this framework Propositions 5.9 and 5.12 we get the statements in (i) and (ii), respectively. \(\Box\)

**9.5 Remark.** Global existence of \(\varphi^G\) could be proved under much weaker conditions than the ones in item (ii) of the above proposition. In fact, using for \(\varphi^G\) an energy balance relation similar to (6.89), one can derive global existence and boundedness of \(\| \varphi^G(t) \|_{L^2}\) when \(f_0\) is arbitrary and the external forcing makes finite both integrals \(\int_0^\infty dt \| \mathcal{F}(t) \|_{L^2}, \int_0^\infty dt \| \mathcal{F}(t) \|_{L^2}^2\); see, e.g., [15]. However, the energetic approach does not allow to derive estimates of the specific type appearing in Proposition 9.4. \(\Diamond\)

The distance between the exact NS solution and the Galerkin approximations. From here to the end of the paragraph, we fix two real numbers
\[
p \geq n > \frac{d}{2} ;
\]
55
we also fix
\[ \xi \in C^{0,1}([0, +\infty), \mathbb{H}^{p-1}) , \quad f_0 \in \mathbb{H}_{2\alpha}^{p} \] (9.22)
and denote with \( \varphi^G \) the Galerkin approximate solution with such forcing and datum, for any \( G \) as before. This will be compared with the solution \( \varphi \) of the NS equations with the same forcing and datum.

9.6 Lemma. Let us regard \( \varphi^G \) as an approximate solution of \( \mathcal{V} \varphi_n(f_0) \); then the following holds.

(0) The integral error of \( \varphi^G \) is
\[ E(\varphi^G)(t) = -(1 - \mathfrak{P}^G) \left[ e^{t \Delta} f_0 + \int_0^t ds \, e^{(t-s) \Delta} \mathcal{P}(\varphi^G(s)) \right] . \] (9.23)

(i) Let us introduce the definitions or assumptions (9.11), (9.12), for some \( T \in (0, +\infty) \) (implying existence of \( \varphi^G \) on \([0, T])\). Then, for all \( t \in [0, T) \) we have
\[ \| E(\varphi^G)(t) \|_n \leq \frac{\mathcal{Y}_p(t)}{|G|^{p-n}} , \] (9.24)
\[ \mathcal{Y}_p(t) := \mathfrak{F}_p(t) \left[ 1 + K_p \mathcal{U}(t) \mathfrak{F}_p(t) \mathcal{X}^2(4K_p \mathcal{U}(t) \mathfrak{F}_p(t)) \right] . \] (9.25)
The function \( \mathcal{Y}_p \) is nondecreasing.

(ii) Alternatively, introduce the definitions or assumptions (Q3), (9.15), (9.16), (9.17) (implying that \( \varphi^G \) is global). Then, for all \( t \in [0, +\infty) \) we have
\[ \| E(\varphi^G)(t) \|_n \leq \frac{\mathcal{Y}_p}{|G|^{p-n}} e^{-t} , \] (9.26)
\[ \mathcal{Y}_p := \Phi_p \left[ 1 + \sqrt{2} K_p \mathfrak{F}_p \mathcal{X}^2(4\sqrt{2} K_p \mathfrak{F}_p) \right] . \] (9.27)

Proof. Derivation of (9.23). By definition
\[ E(\varphi^G)(t) = \varphi^G(t) - e^{t \Delta} f_0 - \int_0^t ds \, e^{(t-s) \Delta} \mathcal{P}(\varphi^G(s)) ; \] (9.28)

on the other hand, the Cauchy problem (9.9) defining \( \varphi^G \) has the integral reformulation
\[ \varphi^G(t) = e^{t \Delta} \mathfrak{P}^G f_0 + \int_0^t ds \, e^{(t-s) \Delta} \mathfrak{P}^G \mathcal{P}(\varphi^G(s)) ; \] (9.29)
inserting this into (9.28) we get
\[ E(\varphi^G)(t) = -e^{t \Delta} (1 - \mathfrak{P}^G) f_0 - \int_0^t ds \, e^{(t-s) \Delta} (1 - \mathfrak{P}^G) \mathcal{P}(\varphi^G(s)) . \] (9.30)
Finally, the operator \(1 - \mathcal{P}^G\) commutes with \(\Delta\) and its semigroup (as made evident by the Fourier representations); so, \(1 - \mathcal{P}^G\) can be factored out and we obtain the thesis (9.23).

Some preliminaries to the proof of (i) and (ii). From (9.23), the estimates (9.6) on \(1 - \mathcal{P}^G\) and (7.8)–(7.10) on the semigroup of \(\Delta\) we get

\[
\|D(\varphi^G(t))\|_n \leq \frac{1}{|G|^{p-n}} \left[ \|e^{\Delta} f_0\|_p + \int_0^t ds \|e^{(t-s)\Delta} \mathcal{P}(\varphi^G(s))\|_p \right] \tag{9.31}
\]

\[
\leq \frac{1}{|G|^{p-n}} \left[ e^{-t}\|f_0\|_p + \int_0^t ds e^{-(t-s)}\mu_\pm(t - s)\|\mathcal{P}(\varphi^G(s))\|_{p-1} \right].
\]

Proof of (i). From \(\mathcal{P}(\varphi^G(s)) = \mathcal{P}(\varphi^G(s), \varphi^G(s)) + \xi(s)\) we infer the following, for \(s \in (0, t)\):

\[
\|\mathcal{P}(\varphi^G(s))\|_{p-1} \leq K_p\|\varphi^G(s)\|^2_p + \|\xi(s)\|_{p-1} \tag{9.32}
\]

\[
\leq K_pD_p^2(s) + \Xi_{p-1}(s) \leq K_pD_p^2(t) + \Xi_{p-1}(t)
\]

(in the above, we have used (9.13)\_p (7.22)\_p and the relation \(D_p(s) \leq D_p(t)\)).

We insert the result (9.32) into Eq. (9.31); in this way we are left with an integral \(\int_0^t ds e^{-(t-s)}\mu_\pm(t - s) = \int_0^t ds e^{-s}\mu_\pm(s) \leq U(t)\). From this bound and \(e^{-t} \leq 1\) we obtain

\[
\|D(\varphi^G(t))\|_n \leq \frac{1}{|G|^{p-n}} \left[ \|f_0\|_p + U(t)\left(K_pD_p^2(t) + \Xi_{p-1}(t)\right) \right] \tag{9.33}
\]

\[
= \frac{1}{|G|^{p-n}} \left[ \mathcal{F}_p(t) + U(t)K_pD_p^2(t) \right],
\]

where the last passage follows from definition (9.11)\_p; now, explicitating \(D_p(t)\) we get the thesis (9.24) (9.25). Finally, \(\mathcal{Y}_p\) is nondecreasing because \(\mathcal{F}_p, U\) and \(X\) are so.

Proof of (ii). In this case, from the inequality \(\|\mathcal{P}(\varphi^G(s))\|_{p-1} \leq K_p\|\varphi^G(s)\|^2_p + \|\xi(s)\|_{p-1}\) we infer, by means of Eqs. (9.17)\_p and (Q3)\_p ,

\[
\|\mathcal{P}(\varphi^G(s))\|_{p-1} \leq K_pD_p^2 e^{-2s} + J_{p-1}e^{-2s} \tag{9.34}
\]

Inserting this result into (9.31) we are left with a term \(e^{-t} \int_0^t ds e^{-s}\mu_\pm(t - s)\), which is bounded by \(\sqrt{2} e^{-t}\) due to (7.13); the conclusion is

\[
\|D(\varphi^G(t))\|_n \leq \frac{e^{-t}}{|G|^{p-n}} \left[ \|f_0\|_p + \sqrt{2}(K_pD_p^2 + J_{p-1}) \right] \tag{9.35}
\]

\[
= \frac{e^{-t}}{|G|^{p-n}} \left[ F_p + \sqrt{2}K_pD_p^2 \right]
\]
(the last equality following from (9.15)\(_p\)). Now, explicitating \(D_p\) we get the thesis (9.26) (9.27).

The following proposition contains the main result of the section.

**9.7 Proposition.** (i) Let \(T \in (0, +\infty]\); make the assumptions and definitions (9.11)\(_n\) (9.12)\(_n\) and (9.11)\(_p\) (9.12)\(_p\) (implying the existence of \(\varphi^G\) on \([0, T]\)). Finally, with \(D_n\) and \(Y_p\) defined by (9.14)\(_n\) (9.25), assume

\[
2\sqrt{\frac{K_n U(T) Y_p(T)}{|G|^{p-n}}} + 2K_n U(T) D_n(T) \leq 1 .
\]  

(9.36)

Then \(\mathcal{V}_n(f_0)\) has a solution \(\varphi\) of domain \([0, T]\) and, for all \(t\) in this interval,

\[
\|\varphi(t) - \varphi^G(t)\|_n \leq \frac{W_{np} |G| (t)}{|G|^{p-n}} ,
\]  

(9.37)

\[
W_{np} |G| (t) := \frac{Y_p(t)}{1 - 2K_n U(t) D_n(t)} X \left( \frac{4K_n U(t) Y_p(t)}{(1 - 2K_n U(t) D_n(t))^2 |G|^{p-n}} \right) .
\]  

(9.38)

The function \(t \mapsto W_{np} |G| (t)\) is nondecreasing; a rough, \(|G|\)-independent bound for it is

\[
W_{np} |G| (t) \leq \frac{2Y_p(T)}{1 - 2K_n U(T) D_n(T)}
\]  

(9.39)

for all \(t \in [0, T]\).

(ii) Alternatively, make the assumptions and definitions (Q3)\(_n\) (9.15)\(_n\) (9.16)\(_n\) and (Q3)\(_p\) (9.15)\(_p\) (9.16)\(_p\) (implying global existence of \(\varphi^G\)). Finally, with \(D_n\) and \(Y_p\) defined by (9.18)\(_n\) (9.27), assume

\[
2 \sqrt[4]{2} \sqrt{\frac{K_n Y_p}{|G|^{p-n}}} + 2\sqrt{2}K_n D_n \leq 1 .
\]  

(9.40)

Then \(\mathcal{V}_n(f_0)\) has a solution \(\varphi\) of domain \([0, +\infty)\) and, for all \(t\) in this interval,

\[
\|\varphi(t) - \varphi^G(t)\|_n \leq \frac{W_{np} |G| e^{-t}}{|G|^{p-n}} ,
\]  

(9.41)

\[
W_{np} |G| := \frac{Y_p}{1 - 2\sqrt{2}K_n D_n} X \left( \frac{4\sqrt{2}K_n Y_p}{(1 - 2\sqrt{2}K_n D_n)^2 |G|^{p-n}} \right) .
\]  

(9.42)

The above constant has the rough, \(|G|\)-independent bound

\[
W_{np} |G| \leq \frac{2Y_p}{1 - 2\sqrt{2}K_n D_n} .
\]  

(9.43)
Proof. (i) A simple application of Proposition 8.2, with
\[ \varphi_{ap} = \varphi^G; \quad D_n \text{ as in } (9.14); \quad E_n(t) = \frac{Y_p(t)}{|G|^{p-n}}. \] (9.44)
The condition (8.7) in the cited proposition takes the form (9.36). The proposition ensures that \( \varphi \) is defined on \([0, T)\), and gives the estimate
\[ \|\varphi(t) - \varphi^G(t)\|_n \leq R_n(t), \] (9.45)
involving the nondecreasing, continuous function
\[ R_n(t) := 1 - \frac{4K_nU(t)Y_p(t)}{|G|^{p-n}}, \] (9.46)
we note that we can write
\[ R_n(t) = \frac{Y_p(t)}{(1 - 2K_nU(t)D_n(t))|G|^{p-n}}X \left( \frac{4K_nU(t)Y_p(t)}{(1 - 2K_nU(t)D_n(t))|G|^{p-n}} \right); \] (9.47)
this yields the thesis (9.37) (9.38).
The fact that \( W_{np|G|} \) is a nondecreasing function of time is apparent from its definition. The bound (9.39) for it follows from the nondecreasing nature of the function \( t \mapsto Y_p(t)/(1 - 2K_nU(t)D_n(t)) \) and from the inequality \( X(z) \leq 2 \) for all \( z \in [0, 1] \).

(ii) A simple application of Proposition 8.4, with
\[ \varphi_{ap} = \varphi^G, \quad D_n \text{ as in } (9.18), \quad E_n = \frac{Y_p}{|G|^{p-n}}. \] (9.48)
The condition (8.14) in the cited proposition takes the form (9.40). The proposition ensures that \( \varphi \) is defined on \([0, +\infty)\), and gives the estimate
\[ \|\varphi(t) - \varphi^G(t)\|_n \leq R_n e^{-t}, \] (9.49)
\[ R_n := \frac{1 - 2\sqrt{2}K_nD_n - \sqrt{(1 - 2\sqrt{2}K_nD_n)^2 - 4\sqrt{2}K_nY_p/|G|^{p-n}}}{2\sqrt{2}K_n}. \] (9.50)
We note that we can write
\[ R_n = \frac{Y_p}{(1 - 2\sqrt{2}K_nD_n)|G|^{p-n}}X \left( \frac{4\sqrt{2}K_nY_p}{(1 - 2\sqrt{2}K_nD_n)^2|G|^{p-n}} \right), \] (9.51)
yielding the thesis (9.41) (9.42); the rough bound (9.43) follows again from the inequality \( X(z) \leq 2 \). \( \square \)
9.8 Remark. Of course, if \( p > n \) the previous proposition implies convergence of the Galerkin solution \( \varphi^G \) to the exact solution \( \varphi \) of \( \mathcal{V}_n(f_0) \). More precisely, with the assumptions in (i) we infer from (9.36) (9.39) that

\[
\sup_{t \in [0,T]} \| \varphi(t) - \varphi^G(t) \|_n = O\left( \frac{1}{|G|^{p-n}} \right) \rightarrow 0 \quad \text{for } |G| \rightarrow +\infty ;
\]

(9.52)

in case (ii), we infer from (9.41) (9.43) that

\[
\sup_{t \in [0, +\infty)} e^t \| \varphi(t) - \varphi^G(t) \|_n = O\left( \frac{1}{|G|^{p-n}} \right) \rightarrow 0 \quad \text{for } |G| \rightarrow +\infty .
\]

(9.53)

10 Numerical examples.

Given the necessary constants \( K_n \), the datum norms and some bounds on the external forcing, the framework of Sections 8 and 9 yields informations on the time of existence of the solution \( \varphi \) of \( \mathcal{V}_n(f_0) \), and on its \( \mathcal{H}^n_{\Sigma_0} \) distance from an approximate solution. In the sequel we exemplify such estimates referring to Section 9, i.e., to the Galerkin approximations.

Throughout the section, we take

\[
d = 3 ; \quad n = 2 , \quad p = 4 .
\]

(10.1)

The constants \( K_2 \) and \( K_4 \) involved in calculations can be obtained from Lemmas 6.1 6.2 and Proposition 6.3; a MATLAB computation illustrated in Appendix H yields the values

\[
K_2 = 0.20 , \quad K_4 = 0.067 .
\]

(10.2)

The other calculations mentioned hereafter have been performed using MATHEMATICA.

An application of Proposition 9.7, item (i). We suppose the external forcing has bounds (7.22) with \( \Xi_1(t) = \text{const.} \equiv \Xi_1 \) and \( \Xi_3(t) = \text{const.} \equiv \Xi_3 \) for all \( t \in [0, +\infty) \). Conditions (9.12) and (9.12) are satisfied with \( T = +\infty \) if

\[
\| f_0 \|_2 + 1.88 \Xi_1 < 0.667 , \quad \| f_0 \|_4 + 1.88 \Xi_3 < 1.99 ;
\]

(10.3)

under the above inequalities for \( f_0 \) and the forcing, the Galerkin solution \( \varphi^G \) exists on \([0, +\infty)\) for each \( G \). As an example, conditions (10.3) are satisfied in the case

\[
\| f_0 \|_2 = 0.15 , \quad \| f_0 \|_4 = 1.50 , \quad \Xi_1 = 0.025 , \quad \Xi_3 = 0.25 ,
\]

(10.4)

to which we stick hereafter. In the above case, condition (9.36) with \( n = 2 , \ p = 4 \) and \( T = +\infty \) becomes \( 0.161 + 2.31/|G| \leq 1 \), which is fulfilled if

\[
|G| \geq 2.76 .
\]

(10.5)
Assuming (10.4) (10.5) the solution $\varphi$ of $VP_2(f_0)$ is also global, and

$$
\|\varphi(t) - \varphi^G(t)\|_2 \leq \frac{W_{24|G|}(t)}{|G|^2} \leq 8.71 \quad \text{for all } G \text{ as above, } t \in [0, +\infty) .
$$

(10.6)

The numerical value of $W_{24|G|}(t)$ can be computed at will from definition (9.38); here we have used the rough bound $W_{24|G|}(t) \leq 8.71$, coming from (9.39).

**Another application of Proposition 9.7, item (i).** We maintain the assumptions $\Xi_1(t) = \text{const.} \equiv \Xi_1$ and $\Xi_3(t) = \text{const.} \equiv \Xi_3$ for all $t \in [0, +\infty)$. We take

$$
\|f_0\|_2 = 0.20 , \quad \|f_0\|_4 = 2.00 , \quad \Xi_1, \Xi_3 \text{ as in (10.4)}.
$$

(10.7)

Now conditions (10.3) are not fulfilled, indicating that (9.12)$_2$ and (9.12)$_4$ are not satisfied with $T = +\infty$. On the contrary, (9.12)$_2$ and (9.12)$_4$ are found to hold with

$$
T = 1.51 ,
$$

(10.8)
i.e., the Galerkin solution $\varphi^G$ exists for any $G$ on the time interval $[0, 1.51)$. To go on, we note that condition (9.36) with $n = 2$, $p = 4$ and $T$ as above becomes $0.163 + 2.41/|G| \leq 1$, which is fulfilled if

$$
|G| \geq 2.88 .
$$

(10.9)

Under the assumption (10.9) the solution $\varphi$ of $VP_2(f_0)$ exists on the same interval, and

$$
\|\varphi(t) - \varphi^G(t)\|_2 \leq \frac{W_{24|G|}(t)}{|G|^2} \leq 11.1 \quad \text{for all } G \text{ as above, } t \in [0, 1.51) .
$$

(10.10)

Again, we can compute the numerical value of $W_{24|G|}(t)$ from the definition (9.38); here we have used the bound $W_{24|G|}(t) \leq 11.1$, coming from (9.39).

**An application of Proposition 9.7, item (ii).** Let us recall that this case refers to exponentially decaying forcing. From the datum norms $\|f_0\|_m$ and the constants $J_{m-1}$ in the forcing bounds, as in (9.15) we define the coefficients $F_m := \|f_0\|_m + \sqrt{2}J_{m-1}$ for $m = 2, 4$. Conditions (9.16) for $m = 2, 4$ become, respectively,

$$
1.14F_2 \leq 1 \text{ and } 0.380F_4 \leq 1 ; \text{ these are fulfilled if }
$$

$$
F_2 \leq 0.877 , \quad F_4 \leq 2.63 ,
$$

(10.11)

and in this case the Galerkin solution $\varphi^G$ is global for each $G$. As an example, let us suppose

$$
F_2 = 0.20 , \quad F_4 = 2.00 ;
$$

(10.12)

then, condition (9.40) becomes $0.121 + 1.75/|G| \leq 1$, which is fulfilled if

$$
|G| \geq 2.00 .
$$

(10.13)
With these assumptions the exact solution \( \varphi \) of \( VP_2(f_0) \) is global, and
\[
\| \varphi(t) - \varphi^G(t) \|_2 \leq \frac{W_{24|G|}}{|G|^2} e^{-t} \leq \frac{6.10}{|G|^2} e^{-t} \quad \text{for all } G \text{ as above, } t \in [0, +\infty). \quad (10.14)
\]
The expression of \( W_{24|G|} \) is provided by (9.42); here we have used the rough bound \( W_{24|G|} \leq 6.10 \), coming from (9.43).

### A Appendix. Proof of Lemma 2.6.

First of all, we put
\[
Z := \sup_{t \in [t_0, \tau]} z(t) ; \quad (A.1)
\]
we continue in two steps.

**Step 1.** For all \( k \in \mathbb{N} \), one has
\[
z(t) \leq Z \frac{\Lambda^k \Gamma(\sigma)^k (t - t_0)^{k\sigma}}{\Gamma(k\sigma + 1)} \quad \text{for } t \in [t_0, \tau] . \quad (A.2)
\]
To prove this, we write \((A.2)_k\) for the above equation at order \( k \), and proceed by recursion. Eq. \((A.2)_0\) is just the inequality \( z(t) \leq Z \). Now, we suppose that \((A.2)_k\) holds and infer from it Eq. \((A.2)_{k+1}\). To this purpose, we substitute \((A.2)_k\) into the basic inequality (2.22), which gives
\[
z(t) \leq Z \Lambda^{k+1} \frac{\Gamma(\sigma)^k}{\Gamma(k\sigma + 1)} \int_{t_0}^{t} ds \frac{(s - t_0)^{k\sigma}}{(t - s)^{1 - \sigma}} ;
\]
expressing the integral via the known identity (4.20), we get the thesis \((A.2)_{k+1}\).

**Step 2.** \( z(t) = 0 \) for all \( t \in [t_0, \tau] \). In Eq. \((A.2)_k\), let us fix \( t \) and send \( k \) to \( \infty \); the right hand side of this inequality vanishes in this limit, yielding the thesis. \( \square \)

### B Appendix. A scheme to solve numerically the control inequality (5.14).

**Notations.** In this Appendix we often write \( \{0, ..., M\} \) where \( M \) is an integer or \(+\infty\). If \( M \) is a nonnegative integer this will mean, as usually, the set of integers \( 0, 1, 2, ..., M \). If \( M \) is a negative integer, we will intend \( \{0, ..., M\} := \emptyset \). If \( M = +\infty \), \( \{0, ..., M\} \) will mean the set \( \mathbb{N} \) of all natural numbers.
We often consider finite or infinite sequences of real numbers of the form \((t_m)_{m \in \{0, \ldots, M\}}\); if \(M = +\infty\), we intend \(t_M := \lim_{m \to +\infty} t_m\) whenever the limit exists.

The numerical scheme. Let us be given an approximate solution \(\varphi_{ap} \in C([0, T'), \mathbf{F})\) of \(\mathcal{VP}(f_0)\), where \(T' \in (0, +\infty]\); in the sequel we always intend

\[
E(t) := \|E(\varphi_{ap}(t))\|, \quad D(t) := \|\varphi_{ap}(t)\| \quad \text{for } t \in [0, T').
\]  

(B.1)

Hereafter we outline a numerically implementable algorithm to construct a solution \(\mathcal{R}\) of the integral inequality (5.14) on some interval \([0, T)\subset[0, T')\); this solution \(\mathcal{R}\) will be piecewise linear.

In order to construct the algorithm, we choose a sequence of instants \((t_m)_{m=0, \ldots, M'}\), where \(M'\) is a positive integer or +\(\infty\). We assume

\[
0 = t_0 < t_1 < t_2 < \ldots < t_{M'} = T'.
\]  

(B.2)

Furthermore, we denote with \(\mathcal{E}_m, D_m, H_{mk}, I_{mk}, N_{mk}\) some constants such that

\[
\sup_{t \in [t_m, t_{m+1})} \mathcal{E}(t) \leq \mathcal{E}_m, \quad \sup_{t \in [t_m, t_{m+1})} D(t) \leq D_m;
\]  

(B.3)

\[
\sup_{t \in [t_m, t_{m+1})} \int_{t_k}^{t_{k+1}} ds \ u_-(t-s) \left(\frac{s-t_k}{t_{k+1}-t_k}\right)^2 \leq H_{mk}, \quad \sup_{t \in [t_m, t_{m+1})} \int_{t_k}^{t_{k+1}} ds \ u_-(t-s) \frac{s-t_k}{t_{k+1}-t_k} \leq I_{mk},
\]

\[
\sup_{t \in [t_m, t_{m+1})} \int_{t_k}^{t_{k+1}} ds \ u_-(t-s) \leq N_{mk} \quad \text{for } m \in \{1, \ldots, M'-1\}, k \in \{0, \ldots, m-1\} ;
\]

\[
\sup_{t \in [t_m, t_{m+1})} \int_{t_m}^{t} ds \ u_-(t-s) \left(\frac{s-t_m}{t_{m+1}-t_m}\right)^2 \leq H_{mm}, \quad \sup_{t \in [t_m, t_{m+1})} \int_{t_m}^{t} ds \ u_-(t-s) \frac{s-t_m}{t_{m+1}-t_m} \leq I_{mm},
\]

\[
\sup_{t \in [t_m, t_{m+1})} \int_{t_m}^{t} ds \ u_-(t-s) \leq N_{mm} \quad \text{for } m \in \{0, \ldots, M'-1\}.
\]  

(B.4)

Finally, for \(m, k\) as above and all \(a, x \in \mathbf{R}\), we define

\[
\Phi_{mk}(a, x) := (H_{mk} + N_{mk} - 2I_{mk})a^2 + 2(I_{mk} - H_{mk})ax + H_{mk}x^2
\]  

\[
+ 2(N_{mk} - I_{mk})D_ka + 2I_{mk}D_kx .
\]  

(B.5)

B.1 Proposition. Suppose there is a finite or infinite sequence of nonnegative reals \((\mathcal{R}_m)_{m \in \{0, \ldots, M\}}\) (with \(1 \leq M \leq M'\)) such that

\[
\mathcal{E}_m + K \sum_{k=0}^{m} \Phi_{mk}(\mathcal{R}_k, \mathcal{R}_{k+1}) \leq \min(\mathcal{R}_m, \mathcal{R}_{m+1}) \quad \text{for } m \in \{0, \ldots, M-1\}.
\]  

(B.6)
Let \( \mathcal{R} \in C([0, t_M), [0, +\infty)) \) be the unique piecewise linear map with values \( \mathcal{R}_m \) at the times \( t_m, \) i.e.,

\[
\mathcal{R}(t) = \mathcal{R}_m + (\mathcal{R}_{m+1} - \mathcal{R}_m) \frac{t - t_m}{t_{m+1} - t_m} \quad \text{for } t \in [t_m, t_{m+1}), \ m \in \{0, \ldots, M - 1\}. \tag{B.7}
\]

Then, \( \mathcal{R} \) solves the integral inequality (5.14) on \([0, t_M)\).

**Proof.** Let \( \mathcal{R} \) be defined as above, and \( t \) in some subinterval \([t_m, t_{m+1})\) \( (m \in \{0, \ldots, M - 1\}) \). Then

\[
\text{l.h.s. of (5.14) } \leq \mathcal{E}_m + K \int_0^t ds \ u_-(t - s)(2\mathcal{D}(s) + \mathcal{R}(s))\mathcal{R}(s) \tag{B.8}
\]

\[
\leq \mathcal{E}_m + K \left( \sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} + \int_{t_m}^t \right) ds \ u_-(t - s)(2\mathcal{D}_k + \mathcal{R}(s))\mathcal{R}(s)
\]

\[
= \mathcal{E}_m + K \left( \sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} + \int_{t_m}^t \right) ds \ u_-(t - s) \left( 2\mathcal{D}_k + \mathcal{R}_k + (\mathcal{R}_{k+1} - \mathcal{R}_k) \frac{s - t_k}{t_{k+1} - t_k} \right) \times
\]

\[
\times \left( \mathcal{R}_k + (\mathcal{R}_{k+1} - \mathcal{R}_k) \frac{s - t_k}{t_{k+1} - t_k} \right) \leq \mathcal{E}_m + K \sum_{k=0}^{m} \Phi_{mk}(\mathcal{R}_k, \mathcal{R}_{k+1}) ,
\]

the last passage following from the inequalities (B.4) and the definition (B.5) of \( \Phi_{mk} \).

From here and from (B.6) we infer, for \( t \) in the same interval,

\[
\text{l.h.s. of (5.14) } \leq \min(\mathcal{R}_m, \mathcal{R}_{m+1}) \leq \mathcal{R}(t) = \text{r.h.s. of (5.14)} \tag{B.9}
\]

In conclusion, (B.6) ensures \( \mathcal{R} \) to fulfill (5.14) on \([0, t_M)\). \( \square \)

**B.2 Remarks.** (i) A sequence of constants \( (\mathcal{E}_k) \) fulfilling the first inequality (B.3) is easily obtained if \( \varphi_{ap} \in C([0, T'), \mathcal{F}_+) \cap C^1([0, T'), \mathcal{F}_-) \), and there are suitable estimators for (the semigroup and) for the datum and differential errors \( d(\varphi_{ap}) \), \( e(\varphi_{ap}). \) More precisely suppose that

\[
\|d(\varphi_{ap})\| \leq \delta , \tag{B.10}
\]

and that, for \( m \in \{0, \ldots, M' - 1\},

\[
\sup_{t \in [t_m, t_{m+1})} u(t) \leq u_m , \quad \sup_{t \in [t_m, t_{m+1})} \|e(\varphi_{ap}(t))\|_1 \leq \epsilon_m . \tag{B.11}
\]

\((u_m)_{m=0,\ldots,M'-1} \) and \((\epsilon_m)_{m=0,\ldots,M'-1} \) being sequences of nonnegative reals.
From Lemma 3.4 on the integral error we obtain, for \( t \in [t_m, t_{m+1}) \),

\[
E(t) \leq u(t) \delta + \left( \sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} + \int_{t_m}^{t} \right) ds u_-(t-s) \| e(\varphi_{ap}(s)) \|_\ast ;
\]

(B.12)

now, from (B.11) and (B.4) we infer, for \( m \in \{0, 1, ..., M' - 1\} \),

\[
E(t) \leq E_m \text{ if } t \in [t_m, t_{m+1}) ,
E_m := u_m \delta + \sum_{k=0}^{m} N_{mk} \epsilon_k .
\]

(B.13)

(In fact, one could extend this result to the case where \( \varphi_{ap} \) is continuous from \([t_0, T')\) to \( F_+ \) and piecewise \( C^1 \) from \([t_0, T')\) to \( F_- \): this typically occurs for the approximate solutions defined by finite difference schemes in time).

(ii) For any \( m \), Eq. (B.6) holds if and only if

\[
\text{either } R_{m+1} \in [0, R_m) , \quad E_m + K \sum_{k=0}^{m} \Phi_{mk}(R_k, R_{k+1}) \leq R_{m+1} ,
\]

(B.14)

or \( R_{m+1} \in [R_m, +\infty) , \quad E_m + K \sum_{k=0}^{m} \Phi_{mk}(R_k, R_{k+1}) \leq R_m \),

(B.15)

(note that each \( \Phi_{mk} \) in these inequalities is a quadratic polynomial. For \( m = 0 \), Eq.s (B.14) (B.15) define a problem for two unknowns \((R_0, R_1)\); for \( m > 0 \), we can see (B.14) (B.15) as a problem to determine recursively \( R_{m+1} \) from \( R_0, ..., R_m \).

For each \( m \), if the problem has solutions it seems convenient to choose for \( R_{m+1} \) the smallest admissible value. This criterion could be applied for \( m = 0 \) as well, choosing among all solutions \((R_0, R_1)\) the one with the smallest \( R_1 \).

(iii) In practical computations, the determination of \( R_{m+1} \) from \( R_0, ..., R_m \) goes on until problem (B.14) or (B.15) have solutions. The iteration ends if, for some finite \( M \), both the above inequalities for \( R_{M+1} \) have no solutions. In this case, we have a function \( R \) solving (5.14) on the interval \([0, t_M)\). Alternatively, the iteration might go on indefinitely.

(iv) The recursive scheme (B.6) has a typical feature of the iterative methods to solve integral equations or inequalities of the Volterra type: to find \( R_{m+1} \) one must compute a "memory term" involving \( R_0, ..., R_m \). The memory term depends non-trivially on \( m \) (through the coefficients \( l_{mk} \), etc.), so it must be fully redetermined at each step; this makes the computation more and more expensive while \( m \) grows.

An exception to this framework occurs if the semigroup estimator \( t \mapsto u_-(t) \) is (a constant \( \times \) an exponential, at least for \( t \) greater than some fixed time \( \vartheta \); this is just the case of the NS equations, see the forthcoming Remark B.4 (ii). In this special situation, Eq. (B.6) can be rephrased as a pair of recursion relations for
two real sequences \((R_m), (S_m)\); at each step, computation of \(R_{m+1}\) and \(S_{m+1}\) does not involve the whole previous history, but only the values of \(S_m\) and \(R_k\) for \(t_m - \vartheta < t_{k+1} \leq t_{m+1}\). The forthcoming Proposition explains all this.

**B.3 Proposition.** Let us suppose there are \(\vartheta \geq 0\) and \(A, B > 0\) such that

\[
u_-(t) = Ae^{-Bt} \quad \text{for } t \in (\vartheta, +\infty); \quad (B.16)
\]

furthermore, let us intend that \(k\) always means an integer in \(\{0, ..., M' - 1\}\). Then, i) ii) hold.

i) Let

\[
m \in \{0, ..., M'\}, \quad t_{k+1} \leq t_m - \vartheta; \quad (B.17)
\]

then, conditions \((B.4)\) are fulfilled with

\[
H_{mk} := Ae^{-Bt_m} H_k, \quad H_k := \frac{2(e^{Bt_{k+1}} - e^{Bt_k}) - 2Be^{Bt_{k+1}}(t_{k+1} - t_k) + B^2e^{Bt_{k+1}}(t_{k+1} - t_k)^2}{B^3(t_{k+1} - t_k)^2};
\]

\[
I_{mk} := Ae^{-Bt_m} I_k, \quad I_k := \frac{e^{Bt_k} - e^{Bt_{k+1}} + Be^{Bt_{k+1}}(t_{k+1} - t_k)}{B^2(t_{k+1} - t_k)};
\]

\[
N_{mk} := Ae^{-Bt_m} N_k, \quad N_k := \frac{e^{Bt_{k+1}} - e^{Bt_k}}{B}. \quad (B.18)
\]

Consequently, for all real \(a, x\) one has

\[
\Phi_{mk}(a, x) = Ae^{-Bt_m}\Phi_k(a, x), \quad (B.19)
\]

\[
\Phi_k(a, x) := (H_k + N_k - 2I_k)a^2 + 2(I_k - H_k)ax + H_kx^2 + 2(N_k - I_k)D_ka + 2I_kD_kx. \quad (B.20)
\]

ii) Consider a sequence \((R_m)_{m \in \{0, ..., M\}}\) of nonnegative reals. Then, \((R_m)\) fulfills Eq. \((B.6)\) if and only if there is a sequence of reals \((S_m)_{m \in \{0, ..., M - 1\}}\) such that

\[
S_m + \sum_{\{k|t_m - \vartheta < t_{k+1} \leq t_{m+1} - \vartheta\}} \Phi_k(R_k, R_{k+1}) \leq S_{m+1} \quad \text{for } m \in \{0, ..., M - 2\}; \quad (B.21)
\]

\[
E_m + K \sum_{\{k|t_m - \vartheta < t_{k+1} \leq t_{m+1}\}} \Phi_{mk}(R_k, R_{k+1}) + KAe^{-Bt_m}S_m \leq \min(R_m, R_{m+1}) \quad (B.22)
\]

for \(m \in \{0, ..., M - 1\}\).

iii) In particular, suppose the instants \(t_m\) and \(\vartheta\) to be integer multiples of a basic spacing \(\tau > 0\):

\[
t_m = m\tau \quad \text{for } m \in \{0, ..., M'\}; \quad \vartheta = L\tau \quad \text{for some } L \in \mathbb{N}. \quad (B.23)
\]
Then, for \( m \in \{0, ..., M - 1\} \),
\[
\{k | \tau_m - \vartheta < \tau_{k+1} \leq \tau_{m+1}\} = \begin{cases} 
\{m - L, ..., m\} & \text{if } m \geq L , \\
\{0, ..., m\} & \text{if } m < L , 
\end{cases}
\] (B.23)
\[
\{k | \tau_m - \vartheta < \tau_{k+1} \leq \tau_{m+1} - \vartheta\} = \begin{cases} 
\{m - L\} & \text{if } m \geq L , \\
\emptyset & \text{if } m < L . 
\end{cases}
\] (B.24)

**Proof.** i) Let \( t \in [\tau_m, \tau_{m+1}] \). For \( s \in [\tau_k, \tau_{k+1}] \) one has \( t - s > \tau_m - \tau_{k+1} + 1 \geq \vartheta \), implying \( u_-(t - s) = Ae^{-B(t-s)}, \) so,
\[
\int_{\tau_k}^{\tau_{k+1}} ds \ u_-(t - s) \left( \frac{s - \tau_k}{\tau_{k+1} - \tau_k} \right)^2 = Ae^{-Bt} \int_{\tau_k}^{\tau_{k+1}} ds \ e^{Bs} \left( \frac{s - \tau_k}{\tau_{k+1} - \tau_k} \right)^2 = Ae^{-Bt} \Phi_k (R_k, R_{k+1}) \] (B.25)
\[
\leq Ae^{-Bt} \Phi_k (R_k, R_{k+1}) = H_{mk} \] (B.26)
In conclusion, defining \( H_{mk} \) as in (B.18) we fulfill the first inequality in (B.4) (note that \( k \leq m - 1 \) due to \( \tau_{k+1} \leq \tau_m \)). Similarly, the other inequalities (B.4) are fulfilled with \( I_{mk}, N_{mk} \) as in (B.18).

Finally, inserting Eq. (B.18) into Eq. (B.5) for \( \Phi_{mk} \) we obtain the thesis (B.19).

ii) Let us rephrase Eq. (B.6) in the case under examination. To this purpose, we reexpress the sum therein writing
\[
\sum_{k=0}^{m} = \sum_{k | \tau_m - \vartheta < \tau_{k+1} \leq \tau_{m+1}} + \sum_{k | \tau_{k+1} \leq \tau_m - \vartheta} ,
\] (B.26)
and then use Eq. (B.19) for the summands with \( \tau_{k+1} \leq \tau_m - \vartheta \); in this way, Eq. (B.6) becomes
\[
\mathcal{E}_m + K \Phi_{mk} (\mathcal{R}_k, \mathcal{R}_{k+1}) + KAe^{-Bt} \sum_{k | \tau_{k+1} \leq \tau_m - \vartheta} \Phi_k (\mathcal{R}_k, \mathcal{R}_{k+1}) \leq \min(\mathcal{R}_m, \mathcal{R}_{m+1}) \] (B.27)
Let us consider any sequence \( (\mathcal{R}_m)_{m \in \{0, ..., M - 1\}} \) of nonnegative reals. If \( (\mathcal{R}_m) \) fulfills (B.27), define
\[
\mathcal{S}_m := \sum_{k | \tau_{k+1} \leq \tau_m - \vartheta} \Phi_k (\mathcal{R}_k, \mathcal{R}_{k+1}) \quad \text{for } m \in \{0, ..., M - 1\} ;
\] (B.28)
then (B.20) (B.21) follow immediately (in fact, with \( \leq \) replaced by = in (B.20)).

Conversely, suppose there is a sequence of reals \( (\mathcal{S}_m)_{m \in \{0, ..., M - 1\}} \) fulfilling Eqs (B.20) and (B.21) with \( (\mathcal{R}_m) \); then \( \sum_{k | \tau_{k+1} \leq \tau_m - \vartheta} \Phi_k (\mathcal{R}_k, \mathcal{R}_{k+1}) \leq \mathcal{S}_m \), and it is easy to infer Eq. (B.27) for \( (\mathcal{R}_m) \).

iii) Obvious. \( \square \)
B.4 Remark. (i) Eq. (B.20) does not prescribe $S_0$. It is convenient to choose $S_0 := 0$; with this position Eq. (B.21) with $m = 0$ is a problem for two unknowns ($R_0, R_1$), for which we could repeat the comments of Remark B.2 (ii). After these initial steps, we can use Eq.s (B.20) (B.21) as recursion relations to obtain $S_1, R_2, S_2, R_3$, and so on.

(ii) As anticipated, Proposition B.3 can be applied to the NS equations, in the framework of Section 7. Eq. (7.10) of the cited section gives the semigroup estimator $u_\ast(t) := e^t / \sqrt{2e^t}$ for $t \leq 1/4$, and $u_\ast(t) := \sqrt{2e^{-t}}$ for all $t > 1/4$. So, the conditions of the previous proposition are fulfilled with $\vartheta = 1/4$, $A = \sqrt{2}$, $B = 1$. These values must be substituted into Eq. (B.18) for $H_{mk}, I_{mk}$ and $N_{mk}$ giving, for example,

$$N_{mk} := \sqrt{2e^{-tm}} N_k, \quad N_k := e^{t_{k+1}} - e^{t_k} \quad \text{for } t_{k+1} \leq t_m - 1/4.$$  \hspace{1cm} (B.29)

Explicit expressions could be derived as well for $H_{mk}, I_{mk}$ and $N_{mk}$ when $t_m - 1/4 < t_{k+1} \leq t_m$, using elementary bounds on $u_\ast$ derived from the expression (7.10). However, this requires a tedious analysis of a number of cases, since the parameter $t - s$ in Eq. (B.4) can be smaller or greater than $1/4$; details will be given elsewhere, when we will treat systematically the approach outlined in this Appendix.

C Appendix. Proof of Lemmas 6.1 and 6.2.

We begin with two auxiliary Lemmas.

C.1 Lemma. Let us consider two radii $\rho, \rho_1$ such that $2\sqrt{d} \leq \rho < \rho_1 < +\infty$, and a nonincreasing function $\chi \in C([\rho - 2\sqrt{d}, \rho_1], [0, +\infty))$. Then,

$$\sum_{h \in \mathbb{Z}^d, \rho \leq |h| < \rho_1} \chi(|h|) \leq \frac{2\pi^{d/2}}{\Gamma(d/2)} \int_{\rho - 2\sqrt{d}}^{\rho_1} dt \left( t + \sqrt{d} \right)^{d-1} \chi(t).$$ \hspace{1cm} (C.1)

Proof. Let us introduce the cubes

$$h + [0, 1]^d \quad (h \in \mathbb{Z}^d) \hspace{1cm} (C.2)$$

and the annulus

$$A(\rho - \sqrt{d}, \rho_1 + \sqrt{d}) := \{ q \in \mathbb{R}^d \mid \rho - \sqrt{d} \leq |q| < \rho_1 + \sqrt{d} \}.$$ \hspace{1cm} (C.3)

we claim that

$$\bigcup_{h \in \mathbb{Z}^d, \rho \leq |h| < \rho_1} (h + [0, 1]^d) \subset A(\rho - \sqrt{d}, \rho_1 + \sqrt{d}).$$ \hspace{1cm} (C.4)

In fact, $q \in h + [0, 1]^d$ implies $|h| - \sqrt{d} \leq |q| \leq |h| + \sqrt{d}$; now, if $\rho \leq |h| < \rho_1$ we conclude $\rho - \sqrt{d} \leq |q| < \rho_1 + \sqrt{d}$.
The inclusion (C.4) implies
\[ \sum_{h \in \mathbb{Z}^d, \rho \leq |h| < \rho_1} dq \chi(|q| - \sqrt{d}) \leq \int_{A(\rho - \sqrt{d}, \rho_1 + \sqrt{d})} dq \chi(|q| - \sqrt{d}) . \] (C.5)

On the other hand, for \( h \) as in the above sum and \( q \in h + [0, 1]^d \), we have \( |q| - \sqrt{d} \leq |h| \), whence \( \chi(|q| - \sqrt{d}) \geq \chi(|h|) \); this implies
\[ \int_{h+[0,1]^d} dq \chi(|q| - \sqrt{d}) \geq \chi(|h|) \int_{h+[0,1]^d} dq = \chi(|h|) . \] (C.6)

From here and from (C.5) we obtain
\[ \sum_{h \in \mathbb{Z}^d, \rho \leq |h| < \rho_1} \chi(|h|) \leq \int_{A(\rho - \sqrt{d}, \rho_1 + \sqrt{d})} dq \chi(|q| - \sqrt{d}) . \] (C.7)

The right hand side of (C.7) can be expressed in terms of the one-dimensional variable \( r = |q| \); as well known, \( dq = (2\pi^{d/2}/\Gamma(d/2)) r^{d-1} dr \), so
\[ \sum_{h \in \mathbb{Z}^d, \rho \leq |h| < \rho_1} \chi(|h|) \leq \frac{2\pi^{d/2}}{\Gamma(d/2)} \int_{\rho - \sqrt{d}}^{\rho_1 + \sqrt{d}} dr r^{d-1} \chi(r - \sqrt{d}) ; \] (C.8)

now, a change of variables \( t = r - \sqrt{d} \) gives the thesis (C.1).

To go on, let us recall the convention (6.56)
\[ \mathbb{Z}^d := \mathbb{Z}^d \text{ or } \mathbb{Z}_0^d . \]

**C.2 Lemma.** (i) Generalizing (6.59), let
\[ S_\nu(\lambda) := \frac{1}{(2\pi)^d} \sum_{h \in \mathbb{Z}^d, |h| < \lambda} \frac{1}{(1 + |h|^2)\nu} \quad \text{for } \nu, \lambda \in (0, +\infty) . \] (C.9)

Then,
\[ S_\nu(\lambda) \leq S_\nu + \frac{(1 + d)^\nu}{2^{d-1}\pi^{d/2}\Gamma(d/2)} F_\nu(\lambda) \quad \text{for } \nu > 0, \lambda > 2\sqrt{d} , \] (C.10)

\[ S_\nu := \frac{1}{(2\pi)^d} \sum_{h \in \mathbb{Z}^d, |h| < 2\sqrt{d}} \frac{1}{(1 + |h|^2)\nu} , \]

\[ F_\nu(\lambda) := \begin{cases} \frac{1}{2\nu - d} \left( \frac{1}{\sqrt{d}^{2\nu-d}} - \frac{1}{(\lambda + \sqrt{d})^{2\nu-d}} \right) & \text{if } \nu \neq d/2 , \\ \log(\frac{\lambda + \sqrt{d}}{\sqrt{d}}) & \text{if } \nu = d/2 . \end{cases} \]

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For fixed $\nu$ and $\lambda \to +\infty$, this implies

\[
S_{\nu}(\lambda) = \begin{cases} 
O(1) & \text{if } \nu > d/2, \\
O(\log \lambda) & \text{if } \nu = d/2, \\
O(\lambda^{d-2\nu}) & \text{if } 0 < \nu < d/2.
\end{cases}
\] (C.11)

(ii) Let

\[
\Delta S_{\nu}(\lambda) := \frac{1}{(2\pi)^d} \sum_{h \in \mathbb{Z}^d, |h| \geq \lambda} \frac{1}{(1 + |h|^2)^\nu} \quad \text{for } \nu > d/2, \lambda > 0.
\] (C.12)

Then

\[
\Delta S_{\nu}(\lambda) \leq \delta S_{\nu}(\lambda) \quad \text{for } \nu > d/2, \lambda > 2\sqrt{d},
\] (C.13)

where (generalizing (6.60))

\[
\delta S_{\nu}(\lambda) := \frac{(1 + d)^\nu}{2^{d-1} \pi^{d/2} \Gamma(d/2) (2\nu - d) (\lambda - \sqrt{d})^{2\nu-d}}.
\] (C.14)

**Proof.** (i) Let $\nu > 0$, $\lambda > 2\sqrt{d}$. Dividing in two parts the sum defining $S_{\nu}(\lambda)$, we get

\[
S_{\nu}(\lambda) = S_{\nu} + \frac{1}{(2\pi)^d} \sum_{h \in \mathbb{Z}^d, 2\sqrt{d} \leq |h| < \lambda} \frac{1}{(1 + |h|^2)^\nu}.
\] (C.15)

Now, we bind the sum using (C.1) with $\chi(t) := 1/(1 + t^2)^\nu$, $\rho := 2\sqrt{d}$ and $\rho_1 := \lambda$, yielding

\[
S_{\nu}(\lambda) \leq S_{\nu} + \frac{1}{2^{d-1} \pi^{d/2} \Gamma(d/2)} \int_0^\lambda \frac{dt}{(t + \sqrt{d})^{d-1}}.
\] (C.16)

On the other hand, one establishes by elementary means the inequality

\[
\frac{1}{1 + t^2} \leq \frac{1 + d}{(t + \sqrt{d})^2} \quad \text{for } t \in [0, +\infty)
\] (C.17)

(holding as an equality when $t = 1/\sqrt{d}$). Inserting this into (C.16), we get

\[
S_{\nu}(\lambda) \leq S_{\nu} + \frac{(1 + d)^\nu}{2^{d-1} \pi^{d/2} \Gamma(d/2)} \int_0^\lambda \frac{dt}{(t + \sqrt{d})^{2\nu-d}}.
\] (C.18)

The last integral equals $F_{\nu}(\lambda)$, so (C.10) is proved. Having this result, the statement (C.11) on the limit $\lambda \to +\infty$ is obvious.

(ii) Let $\nu > d/2$, $\lambda > 2\sqrt{d}$. To bind $\Delta S_{\nu}(\lambda)$ we use (C.1) with $\chi(t) := 1/(1 + t^2)^\nu$, $\rho := \lambda$ and $\rho_1 := +\infty$, and subsequently employ the inequality (C.17). This gives
\[
\Delta S_\nu(\lambda) \leq \frac{1}{2^{d-1}\pi^{d/2}\Gamma(d/2)} \int_{\lambda-2\sqrt{d}}^{+\infty} dt \frac{(t + \sqrt{d})^{d-1}}{(1 + t^2)\nu} \\
\leq \frac{(1 + d)^\nu}{2^{d-1}\pi^{d/2}\Gamma(d/2)} \int_{\lambda-2\sqrt{d}}^{+\infty} dt \frac{1}{(t + \sqrt{d})^{2\nu-d+1}},
\]
and computing the last integral we justify Eqs. (C.13) (C.14).

From here to the end of the Appendix we fix a real number \( n \), fulfilling the relation (6.55)
\[
n > \frac{d}{2};
\]
here are the proofs of Lemmas 6.1, 6.2.

**Proof of Lemma 6.1.** We take any \( \lambda \geq 2\sqrt{d} \); with the notations of the previous Lemma, we have
\[
\Sigma_n = S_n(\lambda) + \Delta S_n(\lambda).
\]
Both terms in the right hand side are finite; the term \( \Delta S_n(\lambda) \) has the upper bounds (C.13) and the obvious lower bound \( \Delta S_n(\lambda) > 0 \). From these facts we infer the finiteness of \( \Sigma_n \), and the bounds (6.58) for it. □

**Proof of Lemma 6.2.** We consider any cutoff function \( \Lambda : \mathbb{Z}^d \to [2\sqrt{d}, +\infty) \). For \( k \in \mathbb{Z}^d \), we introduce the decomposition
\[
\mathcal{K}_n(k) = \mathcal{K}_n(k) + \Delta \mathcal{K}_n(k);
\]
here \( \mathcal{K}_n(k) \) is defined by (6.66), and
\[
\Delta \mathcal{K}_n(k) = \frac{(1 + |k|^2)^n-1}{(2\pi)^d} \sum_{h \in \mathbb{Z}^d, |h| > \Lambda(k)} \frac{|k-h|^2}{(1 + |h|^2)^n(1 + |k-h|^2)^n}.
\]
For the term \( \mathcal{K}_n(k) \), we furtherly introduce a decomposition
\[
\mathcal{K}_n(k) = \mathcal{K}_n'(k) + \mathcal{K}_n''(k),
\]
\[
\mathcal{K}_n'(k) := \frac{(1 + |k|^2)^n-1}{(2\pi)^d} \sum_{h \in \mathbb{Z}^d, |h| < |k|/2} \frac{|k-h|^2}{(1 + |h|^2)^n(1 + |k-h|^2)^n},
\]
\[
\mathcal{K}_n''(k) := \frac{(1 + |k|^2)^n-1}{(2\pi)^d} \sum_{h \in \mathbb{Z}^d, |h|/2 \leq |h| < \Lambda(k)} \frac{|k-h|^2}{(1 + |h|^2)^n(1 + |k-h|^2)^n},
\]
(the sum defining \( \mathcal{K}_n''(k) \) is meant to be zero if \( \Lambda(k) \leq |k|/2 \)). In the sequel we analyse separately \( \Delta \mathcal{K}_n \), \( \mathcal{K}_n' \) and \( \mathcal{K}_n'' \); we will frequently use the inequalities
\[
\frac{|k-h|^2}{(1 + |k-h|^2)^n} \leq \frac{1}{(1 + |k-h|^2)^{n-1}} \quad \text{for all} \ k, h \in \mathbb{Z}^d;
\]
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\[
\frac{1+z}{1+\eta z} \leq \max(1, \frac{1}{\eta}) \quad \text{for all } \eta > 0, \, z \geq 0. \tag{C.27}
\]

Step 1. For all \( k \in \mathbb{Z}^d \), one has
\[
0 < \Delta \mathcal{K}_n(k) \leq \delta \mathcal{K}_n(k); \tag{C.28}
\]

here, as in (6.67),
\[
\delta \mathcal{K}_n(k) := \frac{(1 + d)^n \lambda(|k|)^{n-1}}{2^{d-1} \pi^{d/2} \Gamma(d/2)(2n - d) (\Lambda(k) - \sqrt{d})^{2n-d}},
\]
\( \lambda \) being defined by (6.64). If \( \Lambda \) fulfills (6.68), the above upper bound is such that
\[
\delta \mathcal{K}_n(k) = O\left( \frac{1}{|k|^{2n-d}} \right) \rightarrow 0 \quad \text{for } k \rightarrow \infty. \tag{C.29}
\]

The inequality \( \Delta \mathcal{K}_n(k) > 0 \) is obvious. To prove the rest we start from Eq. (C.26), implying
\[
\Delta \mathcal{K}_n(k) \leq \frac{(1 + |k|^2)^{n-1}}{(2\pi)^d} \sum_{h \in \mathbb{Z}^d, |h| \geq \Lambda(k)} \frac{1}{(1 + |h|^2)^n(1 + |k - h|^2)^{n-1}}; \tag{C.30}
\]
setting
\[
\mu(k) := \inf_{h \in \mathbb{Z}^d, |h| \geq \Lambda(k)} |k - h|, \tag{C.31}
\]
we infer from (C.30) that
\[
\Delta \mathcal{K}_n(k) \leq \frac{1}{(2\pi)^d} \left( \frac{1 + |k|^2}{1 + \mu^2(k)} \right)^{n-1} \sum_{h \in \mathbb{Z}^d, |h| \geq \Lambda(k)} \frac{1}{(1 + |h|^2)^n}. \tag{C.32}
\]

\[
= \left( \frac{1 + |k|^2}{1 + \mu(k)^2} \right)^{n-1} \Delta \mathcal{S}_n(\Lambda(k)),
\]
where \( \Delta \mathcal{S}_n \) is defined following Eq. (C.12). To go on we claim that, for all \( k \in \mathbb{Z}^d \),
\[
\mu(k) \geq \begin{cases} 
0 & \text{if } \Lambda(k) < |k|, \\
\Lambda(k) - |k| & \text{if } \Lambda(k) \geq |k|. \end{cases} \tag{C.33}
\]

The above inequality is trivial if \( \Lambda(k) < |k| \); if \( \Lambda(k) \geq |k| \), it follows noting that \( |h| \geq \Lambda(k) \) implies \( |k - h| \geq |h| - |k| \geq \Lambda(k) - |k| \).

Having proved (C.33), we insert it into (C.32); this gives the inequality
\[
\Delta \mathcal{K}_n(k) \leq \lambda(k)^{n-1} \Delta \mathcal{S}_n(\Lambda(k)) \tag{C.34}
\]

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and substituting therein the upper bound \((C.13)\) for \(\Delta S_n\) we get the upper bound in \((C.28)\).

To go on, suppose \(\Lambda\) fulfills \((6.68)\); then

\[
\Lambda(k) \geq \alpha |k| \geq |k|, \quad \lambda(k) \leq \frac{1 + |k|^2}{1 + (\alpha - 1)^2 |k|^2} \leq \max(1, \frac{1}{(\alpha - 1)^2}) \text{ for } |k| \geq \chi \quad (C.35)
\]

(the last inequality follows from \((C.27)\), with \(z = |k|^2\) and \(\eta = (\alpha - 1)^2\)). So,

\[
\frac{1}{\Lambda(k) - \sqrt{d}} = O\left(\frac{1}{|k|}\right), \quad \lambda(k) = O(1) \quad \text{for } k \to \infty \; ; \quad (C.36)
\]

inserting this in the definition of \(\delta K_n\), we infer Eq. \((C.29)\).

**Step 2.** With \(\Sigma_n\) as in \((6.57)\), one has

\[
K_n'(k) \to \Sigma_n \quad \text{for } k \to \infty \; . \quad (C.37)
\]

To prove this, we write

\[
K_n'(k) = \sum_{h \in \mathbb{Z}^d} c_{nk}(h) \; , \quad (C.38)
\]

\[
c_{nk}(h) := \frac{\theta(|k|/2 - |h|)|k - h|^2(1 + |k|^2)^{n-1}}{(2\pi)^d(1 + |h|^2)^n(1 + |k - h|^2)^n} , \quad \theta(z) := \begin{cases} 1 & \text{if } z \in (0, +\infty) , \\ 0 & \text{if } z \in (-\infty, 0] . \end{cases}
\]

For any fixed \(h \in \mathbb{Z}^d\), we have

\[
c_{nk}(h) \to \frac{1}{(2\pi)^d} \frac{1}{(1 + |h|^2)^n} \quad \text{for } k \to +\infty \; ; \quad (C.39)
\]

this implies

\[
K_n'(k) \to \frac{1}{(2\pi)^d} \sum_{h \in \mathbb{Z}^d} \frac{1}{(1 + |h|^2)^n} = \Sigma_n \; , \quad (C.40)
\]

if the limit \(k \to \infty\) can be exchanged with the sum over \(h\). By the Lebesgue theorem of dominated convergence, the exchange is possible if \(c_{nk}(h)\) is bounded from above by a summable function of \(h\), uniformly in \(k\); indeed this occurs, since

\[
c_{nk}(h) \leq \frac{4^{n-1}}{(2\pi)^d} \frac{1}{(1 + |h|^2)^n} \quad \text{for all } h, k \in \mathbb{Z}^d \; . \quad (C.41)
\]

Let us prove \((C.41)\). The thesis is obvious if \(|h| \geq |k|/2\), since in this case \(c_{nd}(k) = 0\); hereafter we assume \(|h| < |k|/2\). First of all, we note that
Step 3. Suppose \( \Lambda \) for \( h \) (the last inequality follows from (C.27), with \( \eta \)).

To prove this, let us take any \( k \) and \( \Lambda \) (\( k \) variable). From (C.27), with \( \eta = 1/4 \) and \( z = |k|^2 \).

\[ c_{nk}(h) = \frac{|k - h|^2(1 + |k|^2)^{n-1}}{(2\pi)^d(1 + |h|^2)^n(1 + |k - h|^2)^n} \]

\[ \leq \frac{(1 + |k|^2)^{n-1}}{(2\pi)^d(1 + |h|^2)^n(1 + |k - h|^2)^n-1}; \]

secondly, from \( |h| < |k|/2 \) we infer \( |k - h| \geq |k| - |h| \geq |k|/2 \), whence

\[ c_{nk}(h) \leq \frac{1}{(2\pi)^d} \left( \frac{1 + |k|^2}{1 + |h|^2/4} \right)^{n-1} \frac{1}{(1 + |h|^2)^n} \leq \frac{4^{n-1}}{(2\pi)^d} \frac{1}{(1 + |h|^2)^n} \]

(10.27) the last passage use again (C.27), with \( \eta = 1/4 \) and \( z = |k|^2 \).

Step 3. Suppose \( \Lambda \) fulfills (6.68); then, for \( k \to \infty \),

\[ \mathcal{K}_n''(k) = \begin{cases} O\left( \frac{1}{|k|^2} \right) & \text{if } n > d/2 + 1, \\ O\left( \frac{\log |k|}{|k|^2} \right) & \text{if } n = d/2 + 1, \\ O\left( \frac{1}{|k|^{2n-d}} \right) & \text{if } d/2 < n < d/2 + 1. \end{cases} \]

To prove this, let us take any \( k \in \mathbb{Z}^d \) such that \( |k| \geq \chi \). First of all, from (C.26) and \( \Lambda(k) \leq \beta |k| \) we infer

\[ \mathcal{K}_n''(k) \leq \frac{(1 + |k|^2)^{n-1}}{(2\pi)^d} \sum_{h \in \mathbb{Z}^d, |k|/2 \leq |h| < \Lambda(k)} \frac{1}{(1 + |h|^2)^n(1 + |k - h|^2)^n-1} \]

\[ \leq \frac{(1 + |k|^2)^{n-1}}{(2\pi)^d} \sum_{h \in \mathbb{Z}^d, |k|/2 \leq |h| < \beta |k|} \frac{1}{(1 + |h|^2)^n(1 + |k - h|^2)^n-1}. \]

For \( |h| \geq |k|/2 \) we have \( 1/(1 + |h|^2) \leq 1/(1 + |k|^2/4) \), whence

\[ \mathcal{K}_n''(k) \leq \frac{1}{(2\pi)^d(1 + |h|^2)^n} \frac{1}{1 + |h|^2/4} \sum_{h \in \mathbb{Z}^d, |k|/2 \leq |h| < \beta |k|} \frac{1}{(1 + |k - h|^2)^n-1} \]

\[ = \frac{1}{(2\pi)^d} \left( \frac{1 + |k|^2}{1 + |h|^2/4} \right)^{n-1} \frac{1}{1 + |h|^2/4} \sum_{h \in \mathbb{Z}^d, |k|/2 \leq |h| < \beta |k|} \frac{1}{(1 + |k - h|^2)^n-1} \]

\[ \leq \frac{4^{n-1}}{(2\pi)^d} \frac{1}{1 + |h|^2/4} \sum_{h \in \mathbb{Z}^d, |k|/2 \leq |h| < \beta |k|} \frac{1}{(1 + |k - h|^2)^n-1}. \]

(10.27) the last passage use again (C.27), with \( \eta = 1/4 \) and \( z = |k|^2 \). Now, a change of variable \( h = q - k \) in the last sum gives

\[ \mathcal{K}_n''(k) \leq \frac{4^{n-1}}{(2\pi)^d} \frac{1}{1 + |k|^2/4} \sum_{q \in \mathbb{Z}^d, |k|/2 \leq |q - k| < \beta |k|} \frac{1}{(1 + |q|^2)^{n-1}}. \]

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To go on we note that, for all \( q \in \mathbb{Z}^d \),
\[
|q - k| < \beta |k| \Rightarrow |q| \leq |q - k| + |k| < (\beta + 1)|k| .
\] (C.48)

Thus,
\[
\mathcal{K}''_n(k) \leq \frac{4^{n-1}}{(2\pi)^d} \frac{1}{1 + |k|^2/4} \sum_{q \in \mathbb{Z}^d, |q| < (\beta + 1)|k|} \frac{1}{(1 + |q|^2)^{n-1}} \quad (C.49)
\]
\[
= \frac{4^{n-1}}{1 + |k|^2/4} S_{n-1}((\beta + 1)|k|) ,
\]
where \( S_{n-1} \) is defined following Eq. (C.9). The \( k \to \infty \) behaviour of \( S_{n-1}((\beta + 1)|k|) \) is inferred from Eq. (C.11); this function is multiplied by \( 4^{n-1}/(1 + |k|^2/4) = O(1/|k|^2) \), and the conclusion is Eq. (C.44).

Step 4. Conclusion of the proof. For any cutoff \( \Lambda \), the decomposition (C.21) and the bounds (C.28) give the inequalities (6.65) for \( \mathcal{K}_n(k) \), also implying the finiteness of this quantity for arbitrary \( k \).

From now on \( \Lambda \) is supposed to fulfill (6.68), and we consider the limit \( k \to \infty \). Then, the decomposition \( \mathcal{K}_n = \mathcal{K}_n' + \mathcal{K}_n'' \) and Steps 2, 3 give
\[
\mathcal{K}_n(k) \to \Sigma_n ,
\]
as claimed in (6.69); the other statement in (6.69), namely, \( \delta \mathcal{K}_n(k) = O(1/|k|^{2n-d}) \to 0 \), is known from (C.29). The decomposition \( \mathcal{K}_n = \mathcal{K}_n + \Delta \mathcal{K}_n \), and Eqs. (C.28) (6.69) also give
\[
\mathcal{K}_n(k) \to \Sigma_n . \quad (C.50)
\]

Summing up, all statements in items (i)(ii) of this lemma are proved. \( \square \)

## D Appendix. Proof of Propositions 6.3 and 6.5.

**Proof of Proposition 6.3.** We consider two vector fields \( v, w \in \mathbb{H}^n \) with \( n > d/2 \), and proceed in two steps.

**Step 1. Proof of Eq. (6.71).** We have
\[
v^r = \sum_{h \in \mathbb{Z}^d} v^r_h e_h , \quad \partial_r w^s = i \sum_{\ell \in \mathbb{Z}^d} \ell_r w^s_\ell e_\ell ; \quad (D.1)
\]
this implies \( v \cdot \partial w^s = v^r \partial_r w^s = i \sum_{h, \ell \in \mathbb{Z}^d} (v^r_h \ell_r w^s_\ell) (e_h e_\ell) \) or, in vector form,
\[
v \cdot \partial w = i \sum_{h, \ell \in \mathbb{Z}^d} (v^r_h \ell) w_\ell (e_h e_\ell) . \quad (D.2)
\]
On the other hand \( e_h e_\ell = e_{h+\ell}/(2\pi)^d/2 \), so

\[
v \cdot \partial w = \frac{i}{(2\pi)^d/2} \sum_{k \in \mathbb{Z}^d} \left( \sum_{h, \ell \in \mathbb{Z}^d, h + \ell = k} (v_h \cdot \ell) w_\ell \right) e_k \tag{D.3}
\]

\[
= \frac{i}{(2\pi)^d/2} \sum_{k \in \mathbb{Z}^d} \left( \sum_{h \in \mathbb{Z}^d} [v_h \cdot (k - h)] w_{k-h} \right) e_k .
\]

The term multiplying \( e_k \) in the last expression is the Fourier coefficient \( (v \cdot \partial w)_k \); so, Eq. (6.71) is proved.

**Step 2. Proof that \( v \cdot \partial w \) is in \( \mathbb{H}^{n-1} \) and fulfills (6.73).** For any \( k \in \mathbb{Z}^d \), Eq. (6.71) implies

\[
| (v \cdot \partial w)_k | \leq \frac{1}{(2\pi)^d/2} \sum_{h \in \mathbb{Z}^d} |v_h||k-h||w_{k-h}| \tag{D.4}
\]

\[
= \frac{1}{(2\pi)^d/2} \sum_{h \in \mathbb{Z}^d} \frac{|k-h|}{\sqrt{1+|h|^2} \sqrt{1+|k-h|^2}} \sqrt{1+|h|^2} |v_h| \sqrt{1+|k-h|^2} |w_{k-h}| .
\]

Now, Hölder’s inequality \( |\sum_h a_h b_h|^2 \leq \left( \sum_h |a_h|^2 \right) \left( \sum_h |b_h|^2 \right) \) gives

\[
| (v \cdot \partial w)_k |^2 \leq c_k p_k , \tag{D.5}
\]

where

\[
c_k := \frac{1}{(2\pi)^d} \sum_{h \in \mathbb{Z}^d} \frac{|k-h|^2}{(1+|h|^2)^n(1+|k-h|^2)^n} ,
\]

\[
p_k := \sum_{h \in \mathbb{Z}^d} (1+|h|^2)^n |v_h|^2 (1+|k-h|^2)^n |w_{k-h}|^2 .
\]

The last inequality implies

\[
\sum_{k \in \mathbb{Z}^d} (1+|k|^2)^{n-1} |(v \cdot \partial w)_k |^2 \leq \sum_{k \in \mathbb{Z}^d} (1+|k|^2)^{n-1} c_k p_k \tag{D.6}
\]

\[
\leq \left( \sup_{k \in \mathbb{Z}^d} (1+|k|^2)^{n-1} c_k \right) \sum_{k \in \mathbb{Z}^d} p_k .
\]

On the other hand,

\[
(1+|k|^2)^{n-1} c_k = \mathcal{K}_n(k) , \quad \sup_{k \in \mathbb{Z}^d} (1+|k|^2)^{n-1} c_k \leq K_n^2 \tag{D.7}
\]

with \( \mathcal{K}_n \), \( K_n \) as in Eqs. (6.75) (6.74); the finiteness of \( \mathcal{K}_n(k) \) for any \( k \), and of its sup over \( k \) are known from Lemma 6.2. To go on, we note that

\[
\sum_{k \in \mathbb{Z}^d} p_k = \left( \sum_{h \in \mathbb{Z}^d} (1+|h|^2)^n |v_h|^2 \right) \left( \sum_{h \in \mathbb{Z}^d} (1+|h|^2)^n |w_h|^2 \right) = \|v\|^2_n \|w\|^2_n . \tag{D.8}
\]
Inserting Eqs. (D.7), (D.8) into (D.6), we see that 
\[ \sum_{k \in \mathbb{Z}^d} (1 + |k|^2)^{n-1} |(v \cdot \partial w)_k|^2 < +\infty, \]
implies \( v \cdot \partial w \in H^{n-1} \). Furthermore,
\[ \|v \cdot \partial w\|_{n-1}^2 = \sum_{k \in \mathbb{Z}^d} (1 + |k|^2)^{n-1} |(v \cdot \partial w)_k|^2 \leq K_n^2 \|v\|_n^2 \|w\|_n^2, \quad (D.9) \]
yielding Eq. (6.73).

\[ \square \]

Proof of Proposition 6.5. This is a simple variant of the proof given for Proposition 6.3. One takes into account the following facts: if \( f, g \in H^n_{\Sigma 0} \), then their zero order Fourier coefficients are \( f_0 = 0, g_0 = 0 \); furthermore, \( (f \cdot \partial g)_0 = 0 \), since this function has mean zero by Lemma 6.4.

\[ \square \]

E Appendix. Derivation of the NS equations (6.87).

The NS equations in physical and adimensional units.

In the space \( L \) of (oriented) lengths we fix some positive length \( \lambda \), determining the size of the system in consideration. The ”space domain” of the system is modelled as a torus \( L^d/(2\pi \lambda)^d \); we write \( x \) for any point in this domain, and \( t \) for the ”physical” time. The NS equations are
\[ \rho(\dot{v}(t) + v(t) \cdot \partial v(t)) = -\partial p(t) + \eta \Delta v(t) + k(x,t) \quad (E.1) \]
where \( v(t) : x \mapsto v(x,t), p(t) : x \mapsto p(x,t) \) are the velocity and the pressure fields, while \( k(t) : x \mapsto k(x,t) \) is the density of external forces; in the above, \( \cdot \) indicates the partial derivative with respect to time \( t \). The coefficients \( \rho > 0, \eta > 0 \) are the density and viscosity of the fluid, respectively. The velocity field \( v(t) \) is required to be divergence free, to fulfill the condition of incompressibility.

One passes to the adimensional form introducing three functions \( \nu(x,t), \pi(x,t), \kappa(x,t) \quad (x \in T^d, t \in [0,T] \subset \mathbb{R}) \) via the equations
\[ v(x,t) = \frac{\eta}{\rho \lambda} \nu(x,t), \quad p(x,t) = \frac{\eta^2}{\rho \lambda^2} \pi(x,t), \quad k(x,t) = \frac{\eta^2}{\rho \lambda^3} \kappa(x,t) \quad (E.2) \]
for \( x = \frac{x}{\lambda}, t = \frac{\eta t}{\rho \lambda^2} \).

With these positions, Eq. (E.1) is equivalent to
\[ \dot{\nu}(t) + \nu(t) \cdot \partial \nu(t) = -\partial \pi(t) + \Delta \nu(t) + \kappa(t). \]
Hereafter we formalise this adimensional version, specifying the necessary functional
spaces. First of all, we suppose
\[ \kappa \in C^{0,1}([0, +\infty), \mathbb{H}^{n-1}) ; \]  
(secondly, we stipulate the following.

**E.1 Definition.** The incompressible NS Cauchy problem with initial datum \( v_0 \in \mathbb{H}_\Sigma^{n+1} \), in the pressure formulation, is the following.

Find \( \nu \in C([0, T), \mathbb{H}_\Sigma^{n+1}) \cap C^1([0, T), \mathbb{H}_\Sigma^{n-1}) \), \( \pi \in C([0, T), H^n) \) such that
\[ \dot{\nu}(t) + \nu(t) \cdot \partial \nu(t) = -\partial \pi(t) + \Delta \nu(t) + \kappa(t) \quad \text{for} \ t \in [0, T) , \quad \nu(0) = v_0 \]  
(E.4)
(for some \( T \in (0, +\infty] \)).

We note the following: the requirements \( \nu \in C([0, T), \mathbb{H}_\Sigma^{n+1}) \) and \( \pi \in C([0, T), H^n) \) are sufficient for the right hand side of the above differential equation to be in \( C([0, T), \mathbb{H}_\Sigma^{n-1}) \).

The equivalence between the pressure formulation (E.4) and the Leray formulation (6.87) of the Cauchy problem.

Let us introduce the function
\[ \eta \in C^{0,1}([0, +\infty), \mathbb{H}_\Sigma^{n-1}) , \quad t \mapsto \eta(t) := \mathcal{L}\kappa(t) . \]  
(E.5)

For convenience, we report here the Cauchy problem in the form (6.87):

Find \( \nu \in C([0, T), \mathbb{H}_\Sigma^{n+1}) \cap C^1([0, T), \mathbb{H}_\Sigma^{n-1}) \), such that
\[ \dot{\nu}(t) = \Delta \nu(t) - \mathcal{L}(\nu(t) \cdot \partial \nu(t)) + \eta(t) \quad \text{for} \ t \in [0, T) , \quad \nu(0) = v_0 \]  
(for some \( T \in (0, +\infty] \)). The above mentioned equivalence can be stated as follows.

**E.2 Proposition.** A function \( \nu \) of domain \([0, T)\) fulfills (6.87) if and only if there is a function \( \pi \) with the same domain, such that \((\nu, \pi)\) fulfills (E.4).

**Proof.** Suppose a pair \((\nu, \pi)\) fulfills (E.4), and apply the projector \( \mathcal{L} \) to both sides of the differential equation. We have \( \mathcal{L}\nu(t) = \nu(t) \), and \( \mathcal{L} \) commutes with both the time derivative \( \cdot \) and the Laplacian \( \Delta \); finally, \( \mathcal{L}\partial \pi(t) = 0 \). These facts yield Eq. (6.87). Conversely, suppose a function \( \nu \) fulfills (6.87) and define
\[ \gamma \in C([0, T), \mathbb{H}^{n-1}) , \quad t \mapsto \gamma(t) := \Delta \nu(t) - \nu(t) \cdot \partial \nu(t) + \kappa(t) - \dot{\nu}(t) ; \]  
(E.6)
then from (6.87) one infers \( \mathcal{L}\gamma(t) = 0 \), i.e., \( \gamma(t) \in \mathbb{H}_T^{n-1} \) for all \( t \). Let
\[ \pi : t \in [0, T) \mapsto \pi(t) := \partial^{-1}\gamma(t) \]  
(E.7)
with \( \partial^{-1} \) as in (6.37); then \( \pi \in C([0, T)H^n) \) (since \( \partial^{-1} \) maps continuously \( \mathbb{H}_T^{n-1} \) into \( H^n) \). One easily checks that \((\nu, \pi)\) fulfills (E.4). \( \Box \)
Appendix. Proof of Eq. (6.100).

Let us rephrase the definition (6.99) of $\xi$ as

$$\xi(t) = \zeta(t) - \langle \eta(t) \rangle, \quad (F.1)$$

having put

$$\zeta : [0, T) \to \mathbb{H}_\Sigma^n, \quad t \mapsto \zeta(t)$$

such that $\zeta(x, t) := \eta(x + h(t), t). \quad (F.2)$

Clearly, the thesis (6.100) follows if we prove that

$$\zeta \in C^{0,1}([0, +\infty), \mathbb{H}_\Sigma^{n-1}) \quad (F.3)$$

To this purpose, we consider the Fourier coefficients $\eta_k(t), \zeta_k(t)$ of $\eta(t)$, $\zeta(t)$ and note that (F.2) implies

$$\zeta_k(t) = \eta_k(t) e^{ik \cdot h(t)} \quad (k \in \mathbb{Z}^d, t \in [0, T)). \quad (F.4)$$

Let $t, t' \in [0, T)$. We have

$$\zeta_k(t) - \zeta_k(t') = \alpha_k(t, t') + \beta_k(t, t') \quad (F.5)$$

with

$$\alpha_k(t, t') := \eta_k(t) e^{ik \cdot h(t')} \left( e^{ik \cdot (h(t) - h(t'))} - 1 \right), \quad \beta_k(t, t') := \eta_k(t') - \eta_k(t').$$

Therefore $\|\zeta(t) - \zeta(t')\|_{n-1} = \sqrt{\sum_{k \in \mathbb{Z}^d} (1 + |k|^2)^{n-1} \|\zeta_k(t) - \zeta_k(t')\|^2}$ has the bound

$$\|\zeta(t) - \zeta(t')\|_{n-1} \leq A(t, t') + B(t, t'),$$

$$A(t, t') := \sqrt{\sum_{k \in \mathbb{Z}^d} (1 + |k|^2)^{n-1} \|\alpha_k(t, t')\|^2}, \quad B(t, t') := \sqrt{\sum_{k \in \mathbb{Z}^d} (1 + |k|^2)^{n-1} \|\beta_k(t, t')\|^2}. \quad (F.6)$$

On the other hand, Eq. (F.5) and the elementary inequality $|e^{ik \cdot y} - 1| \leq |k||y|$ (for all $y \in \mathbb{R}^d$) give

$$|\alpha_k(t, t')| \leq |k| |\eta_k(t)||h(t) - h(t')|, \quad |\beta_k(t, t')| = |\eta_k(t) - \eta_k(t')|. \quad (F.7)$$

Inserting these bounds into the expressions (F.6) of $A(t, t'), B(t, t')$ (and using $1 + |k|^2)^{n-1} |k|^2 \leq (1 + |k|^2)^n$) we get

$$\|\zeta(t) - \zeta(t')\|_{n-1} \leq \|\eta(t)\|_n |h(t) - h(t')| + \|\eta(t) - \eta(t')\|_{n-1}. \quad (F.8)$$

Now, let us consider any compact subset $I$ of $[0, +\infty)$. Then, the assumptions (6.95) on $\eta$ and the $C^1$ nature of $h$ ensure the existence of constants $Q, M_1, M_2$ such that

$$\|\eta(t)\|_n \leq Q, |h(t) - h(t')| \leq M_1 |t - t'| \quad \text{and} \quad \|\eta(t) - \eta(t')\|_{n-1} \leq M_2 |t - t'| \quad \text{for} \ t, t' \in I.$$ 

This implies

$$\|\zeta(t) - \zeta(t')\|_{n-1} \leq (QM_1 + M_2) |t - t'|, \quad (F.9)$$

and (F.3) is proved.
G  Appendix. Proof of Proposition 7.2, item (iv).

Our aim is to derive Eq. (7.13); we write this as

\[ \sup_{t \in [0, +\infty)} N(t) = \sqrt{2}, \quad (G.1) \]

\[ N : [0, +\infty) \to [0, +\infty), \quad t \mapsto N(t) := \int_0^t ds \mu_-(t-s)e^{-s}. \quad (G.2) \]

Eq. (G.1) will follow from Steps 1 and 2, giving separate estimates on \( N(t) \) for \( 0 \leq t \leq 1/4 \) and \( t > 1/4 \).

Step 1. For \( 0 \leq t \leq 1/4 \) one has \( N(t) < \sqrt{2} \). In fact, with this range for \( t \) Eq. (7.10) for \( \mu_\perp \) implies

\[ N(t) = \frac{e^{2t}}{\sqrt{2}e} \int_0^t ds \frac{e^{-3s}}{\sqrt{t-s}} \leq \frac{e^{2t}}{\sqrt{2}e} \int_0^t ds \frac{1}{\sqrt{t-s}} = \frac{e^{2t}}{\sqrt{2}e} 2\sqrt{t} \leq \frac{1}{\sqrt{2}} < \sqrt{2}. \]

Step 2. One has \( \sup_{t>1/4} N(t) = \sqrt{2} \). In fact, for all \( t > 1/4 \), using again Eq. (7.10) for \( \mu_\perp \) we get

\[ N(t) = (\int_0^{t-1/4} ds + \int_{t-1/4}^t ds) \mu_- (t-s)e^{-s} = \sqrt{2} (1-e^{1/4-t}) + Ce^{-t} = \sqrt{2} - (\sqrt{2} e^{1/4} - C)e^{-t}, \]

\[ C := \int_0^{1/4} ds' \frac{e^{3s'}}{\sqrt{2}es'}. \]

One has the estimate \( C \leq 0.6 < \sqrt{2} e^{1/4} \), implying \( N(t) < \sqrt{2} \). From the above expression for \( N(t) \), we also get \( \lim_{t \to +\infty} N(t) = \sqrt{2} \); these facts yield the thesis. □

H  Appendix. The constants \( K_2 \) and \( K_4 \) in dimension \( d = 3 \).

The above constants are needed for the numerical examples in Section 10; the route to compute them is outlined in Lemmas 6.1, 6.2 and Proposition 6.3.

**Computing \( K_2 \).** We can take for it any constant such that

\[ \sqrt{\sup_{k \in \mathbb{Z}_0^d} K_2(k)} \leq K_2. \quad (H.1) \]

We have the bounds (6.65): \( K_2(k) < K_2(k) \leq K_2(k) + \delta K_2(k) \), with the explicit expressions (6.66) for \( K_2(k) \) and (6.67) for \( \delta K_2(k) \) (and \( \mathbb{Z}^d = \mathbb{Z}_0^d \) therein). Both \( K_2 \)
and $\delta K_2$ are defined in terms of some cutoff function $\Lambda_2$, that we choose in this way: 
$\Lambda_2(k) := 24$ if $|k| < 4$, and $\Lambda_2(k) := 6|k|$ if $|k| \geq 4$.

The above setting can be employed to evaluate $K_2(k)$ for some set of values of $k$; computations performed for all $k$'s with $|k_i| \leq 10$ ($i = 1, 2, 3$) seem to indicate that 

$$\sup_{k \in \mathbb{Z}_0^d} K_2(k) = \lim_{k \to \infty} K_2(k) . \quad (H.2)$$

On the other hand, according to (6.62),

$$\lim_{k \to \infty} K_2(k) = \Sigma_2 , \quad (H.3)$$

where $\Sigma_2$ is the series (6.57) with $n = 2$ and $\mathbb{Z}^d = \mathbb{Z}_0^d$. To estimate this series, we use the bounds (6.58): $S_2(\lambda) < \Sigma_2 \leq S_2(\lambda) + \delta S_2(\lambda)$ with $S_2(\lambda)$ and $\delta S_2(\lambda)$ as in Eqs. (6.59) (6.60); these depend on a cutoff $\lambda$, to be chosen as large as possible to reach a good precision. Taking $\lambda = 250$, we get

$$0.03607 \leq \Sigma_2 \leq 0.03934 , \quad (H.4)$$

which implies, taking square roots,

$$0.1899 \leq \sqrt{\sup_{k \in \mathbb{Z}_0^d} K_2(k)} \leq 0.1984 . \quad (H.5)$$

Retaining only two meaningful digits, we take as a final upper bound for $\sqrt{\sup_{k \in \mathbb{Z}_0^d} K_2}$ the quantity

$$K_2 := 0.20 . \quad (H.6)$$

**Computing $K_4$.** We can take for it any constant such that $\sqrt{\sup_{k \in \mathbb{Z}_0^d} K_4(k)} \leq K_4$.

We use again the bounds (6.65) $K_4(k) \leq K_4(k) + \delta K_4(k)$; the cutoff $\Lambda_4$ defining $K_4$ and $\delta K_4$ is chosen setting $\Lambda_4(k) := 10$ if $|k| < 10/3$ and $\Lambda_4(k) := 3|k|$ if $|k| \geq 10/3$.

Computing $(K_4 + \delta K_4)(k)$ for $|k_i| \leq 6$ ($i = 1, 2, 3$) we obtain numerical evidence that $\sup_{k \in \mathbb{Z}_0^d} (K_4 + \delta K_4)(k)$ is attained at $k = (3, 0, 0)$. So,

$$\sup_{k \in \mathbb{Z}_0^d} K_4(k) \leq (K_4 + \delta K_4)(3, 0, 0) \leq 0.004383 . \quad (H.7)$$

We also have

$$\sup_{k \in \mathbb{Z}_0^d} K_4(k) \geq K_4(3, 0, 0) \geq 0.004382 , \quad (H.8)$$

and in conclusion, taking square roots,

$$0.06619 \leq \sqrt{\sup_{k \in \mathbb{Z}_0^d} K_4(k)} \leq 0.06621 . \quad (H.9)$$
Retaining only two meaningful digits, we take as a final upper bound for \(\sqrt{\sup K_4}\) the quantity
\[
K_4 := 0.067.
\] (H.10)

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