Berberian Extension and its $S$-spectra in a Quaternionic Hilbert Space

B. Muraleetharan* and K. Thirulogasanthar

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Abstract. For a bounded right linear operators $A$, in a right quaternionic Hilbert space $V_{R}^{H}$, following the complex formalism, we study the Berberian extension $A^\circ$, which is an extension of $A$ in a right quaternionic Hilbert space obtained from $V_{R}^{H}$. In the complex setting, the important feature of the Berberian extension is that it converts approximate point spectrum of $A$ into point spectrum of $A^\circ$. We show that the same is true for the quaternionic S-spectrum. As in the complex case, we use the Berberian extension to study some properties of the commutator of two quaternionic bounded right linear operators.

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1. Introduction

In 1962 Berberian extended a bounded linear operator $A$ on a complex Hilbert space $X$ to an operator $A^\circ$ on a complex Hilbert space obtained from $X$. An important feature of this extension is that it converts approximate point spectrum of $A$ into point spectrum of $A^\circ$ [3]. This extension is also a useful tool in studying the spectrum of commutator of two bounded linear operators [11].

In the complex theory this extension goes as follows. Let $X$ be a complex Hilbert space. Let $l^\infty(X)$ denotes the space of all bounded sequence of elements of $X$, and let $c_0(X)$ denote the space of all null sequences in $X$. Endowed with the canonical norm, the space $\mathfrak{X} = l^\infty(X)/c_0(X)$ is a Hilbert space into which $X$ can be isometrically embedded. Every operator

*Corresponding author.
Let \( A \in B(X) \), the set of all bounded linear operators on \( X \), defines by component wise action an operator on \( l^\infty(X) \) which leaves \( c_0(X) \) invariant, and hence induces an operator \( A^\circ \in B(\mathcal{X}) \). It is immediate that \( A^\circ \) is an extension of \( A \) when \( X \) is regarded as a subspace of \( \mathcal{X} \), and that the mapping that assigns to each \( A \in B(X) \) its Berberian extension \( A^\circ \in B(\mathcal{X}) \) is an isometric algebra homomorphism.

In this note we shall study the Berberian extension of a quaternionic right linear operator \( A \) on a right quaternionic Hilbert space and show that the approximate point S-spectrum of \( A \) coincides with the point S-spectrum of the Berberian extension \( A^\circ \). Following the complex formalism given in [11], we shall also study certain S-spectral properties of the commutator of two quaternionic bounded right linear operators.

In the complex setting, in a complex Hilbert space or Banach space \( \mathfrak{H} \), for a bounded linear operator, \( A \), the spectrum is defined as the set of complex numbers \( \lambda \) for which the operator \( Q_\lambda(A) = A - \lambda I_{\mathfrak{H}} \), where \( I_{\mathfrak{H}} \) is the identity operator on \( \mathfrak{H} \), is not invertible. In the quaternionic setting, let \( \mathcal{V}_{\mathbb{H}}^R \) be a separable right quaternionic Hilbert space or Banach space, \( A \) be a bounded right linear operator, and \( R_q(A) = A^2 - 2\text{Re}(q)A + |q|^2 I_{\mathcal{V}_{\mathbb{H}}^R} \), with \( q \in \mathbb{H} \), the set of all quaternions, be the pseudo-resolvent operator. The S-spectrum is defined as the set of quaternions \( q \) for which \( R_q(A) \) is not invertible. The notion of S-spectrum was introduced in 2006 by Colombo and Sabadini. The discovery and the importance of this spectrum is well explained in [6]. Further developments on the theory of S-spectrum can be found in the book [7]. In the complex case various classes of spectra, such as approximate point spectrum, surjectivity spectrum etc. are defined by placing restrictions on the operator \( Q_\lambda(A) \). In this regard, in the quaternionic setting, these spectra are also defined by placing the same restrictions to the operator \( R_q(A) \) [12, 14].

Due to the non-commutativity, in the quaternionic case there are three types of Hilbert spaces: left, right, and two-sided, depending on how vectors are multiplied by scalars. This fact can entail several problems. For example, when a Hilbert space \( \mathcal{H} \) is one-sided (either left or right) the set of linear operators acting on it does not have a linear structure. Moreover, in a one sided quaternionic Hilbert space, given a linear operator \( A \) and a quaternion \( q \in \mathbb{H} \), in general we have that \( (qA)^\dagger \neq qA^\dagger \) (see [13] for details). These restrictions can severely prevent the generalization to the quaternionic case of results valid in the complex setting. Even though most of the linear spaces are one-sided, it is possible to introduce a notion of multiplication on both sides by fixing an arbitrary Hilbert basis of \( \mathcal{H} \). This fact allows to have a linear structure on the set of linear operators, which is a minimal requirement to develop a full theory.

2. Mathematical Preliminaries

In order to make the paper self-contained, we recall some facts about quaternions which may not be well-known. For details we refer the reader to [1, 10, 15].
2.1. Quaternions

Let $\mathbb{H}$ denote the field of all quaternions and $\mathbb{H}^*$ the group (under quaternionic multiplication) of all invertible quaternions. A general quaternion can be written as

$$q = q_0 + q_1 \mathbf{i} + q_2 \mathbf{j} + q_3 \mathbf{k}, \quad q_0, q_1, q_2, q_3 \in \mathbb{R},$$

where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the three quaternionic imaginary units, satisfying $i^2 = j^2 = k^2 = -1$ and $ij = k = -ji, \; jk = i = -kj, \; ki = j = -ik$. The quaternionic conjugate of $q$ is

$$\bar{q} = q_0 - iq_1 - jq_2 - kq_3,$$

while $|q| = (q\bar{q})^{1/2}$ denotes the usual norm of the quaternion $q$. If $q$ is non-zero element, it has inverse $q^{-1} = \frac{\bar{q}}{|q|^2}$.

2.2. Quaternionic Hilbert Spaces

In this subsection we discuss right quaternionic Hilbert spaces. For more details we refer the reader to [1,10,15].

2.2.1. Right Quaternionic Hilbert Space. Let $V^R_{\mathbb{H}}$ be a vector space under right multiplication by quaternions. For $\phi, \psi, \omega \in V^R_{\mathbb{H}}$ and $q \in \mathbb{H}$, the inner product

$$\langle \cdot \mid \cdot \rangle_{V^R_{\mathbb{H}}} : V^R_{\mathbb{H}} \times V^R_{\mathbb{H}} \longrightarrow \mathbb{H}$$

satisfies the following properties

(i) $\langle \phi \mid \psi \rangle_{V^R_{\mathbb{H}}} = \langle \psi \mid \phi \rangle_{V^R_{\mathbb{H}}}$

(ii) $\|\phi\|^2_{V^R_{\mathbb{H}}} = \langle \phi \mid \phi \rangle_{V^R_{\mathbb{H}}} > 0$ unless $\phi = 0$, a real norm

(iii) $\langle \phi \mid \psi + \omega \rangle_{V^R_{\mathbb{H}}} = \langle \phi \mid \psi \rangle_{V^R_{\mathbb{H}}} + \langle \phi \mid \omega \rangle_{V^R_{\mathbb{H}}}$

(iv) $\langle \phi \mid \psi q \rangle_{V^R_{\mathbb{H}}} = \langle \phi \mid \psi \rangle_{V^R_{\mathbb{H}}} q$

(v) $\langle \phi q \mid \psi \rangle_{V^R_{\mathbb{H}}} = \bar{q}\langle \phi \mid \psi \rangle_{V^R_{\mathbb{H}}}$

where $\bar{q}$ stands for the quaternionic conjugate. It is always assumed that the space $V^R_{\mathbb{H}}$ is complete under the norm given above and separable. Then, together with $\langle \cdot \mid \cdot \rangle_{V^R_{\mathbb{H}}}$ this defines a right quaternionic Hilbert space. Quaternionic Hilbert spaces share many of the standard properties of complex Hilbert spaces. Every separable quaternionic Hilbert space posses a basis. It should be noted that once a Hilbert basis is fixed, every left (resp. right) quaternionic Hilbert space also becomes a right (resp. left) quaternionic Hilbert space [10,15].

The field of quaternions $\mathbb{H}$ itself can be turned into a left quaternionic Hilbert space by defining the inner product $\langle q \mid q' \rangle = q\bar{q}'$ or into a right quaternionic Hilbert space with $\langle q \mid q' \rangle = \bar{q}q'$.

3. Right Quaternionic Linear Operators and Some Basic Properties

In this section we shall define right $\mathbb{H}$-linear operators and recall some basis properties. Most of them are very well known. In this manuscript, we follow
the notations in [2,10]. We shall also recall some results pertinent to the development of the paper.

**Definition 3.1.** A mapping \( A : \mathcal{D}(A) \subseteq V_H^R \rightarrow V_H^R \), where \( \mathcal{D}(A) \) stands for the domain of \( A \), is said to be right \( H \)-linear operator or, for simplicity, right linear operator, if
\[
A(\phi a + \psi b) = (A\phi)a + (A\psi)b, \quad \text{if} \quad \phi, \psi \in \mathcal{D}(A) \quad \text{and} \quad a, b \in H.
\]

The set of all right linear operators from \( V_H^R \) to \( V_H^R \) will be denoted by \( L(V_H^R) \) and the identity linear operator on \( V_H^R \) will be denoted by \( I_{V_H^R} \).

For a given \( A \in L(V_H^R) \), the range and the kernel will be
\[
\text{ran}(A) = \{ \psi \in V_H^R \mid A\phi = \psi \quad \text{for} \quad \phi \in \mathcal{D}(A) \},
\]
\[
\text{ker}(A) = \{ \phi \in \mathcal{D}(A) \mid A\phi = 0 \}.
\]

We call an operator \( A \in L(V_H^R) \) bounded if
\[
\|A\| = \sup_{\|\phi\|_{V_H^R} = 1} \|A\phi\|_{V_H^R} < \infty, \tag{3.1}
\]
or equivalently, there exist \( K \geq 0 \) such that \( \|A\phi\|_{V_H^R} \leq K\|\phi\|_{V_H^R} \) for all \( \phi \in \mathcal{D}(A) \). The set of all bounded right linear operators from \( V_H^R \) to \( V_H^R \) will be denoted by \( B(V_H^R) \).

Assume that \( V_H^R \) is a right quaternionic Hilbert space, \( A \) is a right linear operator acting on it. Then, there exists a unique linear operator \( A^\dagger \) such that
\[
\langle \psi \mid A\phi \rangle_{V_H^R} = \langle A^\dagger\psi \mid \phi \rangle_{V_H^R}; \quad \text{for all} \quad \phi \in \mathcal{D}(A), \psi \in \mathcal{D}(A^\dagger), \tag{3.2}
\]
where the domain \( \mathcal{D}(A^\dagger) \) of \( A^\dagger \) is defined by
\[
\mathcal{D}(A^\dagger) = \{ \psi \in V_H^R \mid \exists \phi \text{ such that } \langle \psi \mid A\phi \rangle_{V_H^R} = \langle \varphi \mid \phi \rangle_{V_H^R} \}.
\]

**Proposition 3.2.** [10] Let \( A, B \in B(V_H^R) \) then
\[
(a) \quad (A + B)^\dagger = A^\dagger + B^\dagger.
\]
\[
(b) \quad (AB)^\dagger = B^\dagger A^\dagger.
\]

We shall need the following results which are already appeared in [10, 12].

**Proposition 3.3.** Let \( A \in B(V_H^R) \). Then
\[
(a) \quad \text{ran}(A)^\perp = \text{ker}(A^\dagger).
\]
\[
(b) \quad \text{ker}(A) = \text{ran}(A^\dagger)^\perp.
\]
\[
(c) \quad \text{ker}(A) \text{ is closed subspace of } V_H^R.
\]

**Theorem 3.4.** [12] (Bounded inverse theorem) Let \( A \in B(V_H^R) \), then the following results are equivalent.
\[
(a) \quad A \text{ has a bounded inverse on its range.}
\]
\[
(b) \quad A \text{ is bounded below.}
\]
\[
(c) \quad A \text{ is injective and has a closed range.}
\]

**Proposition 3.5.** [12] Let \( A \in B(V_H^R) \). Then,
(a) $A$ is invertible if and only if it is injective with a closed range (i.e., $\ker(A) = \{0\}$ and $\ran(A) = \ran(A)$).
(b) $A$ is left (right) invertible if and only if $A^\dagger$ is right (left) invertible.

**Proposition 3.6.** [12] $A \in B(V^R_H)$ is surjective if and only if $A$ is right invertible.

**Proposition 3.7.** $A \in B(V^R_H)$ is injective if and only if $A$ is left invertible.

**Proof.** From point (b) of Proposition 3.5, point (b) of Proposition 3.3, and Proposition 3.6, we have, $A$ is left invertible $\iff A^\dagger$ is right invertible $\iff \ran(A^\dagger) = V^R_H \iff \ker(A) = \{0\}$. This completes the proof. □

### 3.1. Left Scalar Multiplications on $V^R_H$

We shall extract the definition and some properties of left scalar multiples of vectors on $V^R_H$ from [10] as needed for the development of the manuscript.

The left scalar multiple of vectors on a right quaternionic Hilbert space is an extremely non-canonical operation associated with a choice of preferred Hilbert basis. Since $V^R_H$ is a separable Hilbert space, $V^R_H$ has a Hilbert basis $O = \{\varphi_k \mid k \in N\}$, (3.3)

where $N$ is a countable index set. The left scalar multiplication on $V^R_H$ induced by $O$ is defined as the map $H \times V^R_H \ni (q, \phi) \mapsto q\phi \in V^R_H$ given by

$$q\phi := \sum_{k \in N} \varphi_k q \langle \varphi_k \mid \phi \rangle_{V^R_H}, \quad (3.4)$$

for all $(q, \phi) \in \mathbb{H} \times V^R_H$.

**Proposition 3.8.** [10] The left product defined in the Eq. 3.4 satisfies the following properties. For every $\phi, \psi \in V^R_H$ and $p, q \in \mathbb{H}$,

(a) $q(\phi + \psi) = q\phi + q\psi$ and $q(\phi p) = (q\phi)p$.
(b) $\|q\phi\|_{V^R_H} = |q|\|\phi\|_{V^R_H}$.
(c) $q(p\phi) = (qp)\phi$.
(d) $\langle \bar{q} \phi \mid \psi \rangle_{V^R_H} = \langle \phi \mid q \psi \rangle_{V^R_H}$.
(e) $r\phi = \phi r$, for all $r \in \mathbb{R}$.
(f) $q\varphi_k = \varphi_k q$, for all $k \in N$.

**Remark 3.9.** (1) The meaning of writing $p\phi$ is $p \cdot \phi$, because the notation from the Eq. 3.4 may be confusing, when $V^R_H = \mathbb{H}$. However, regarding the field $\mathbb{H}$ itself as a right $\mathbb{H}$-Hilbert space, an orthonormal basis $O$ should consist only of a singleton, say $\{\varphi_0\}$, with $|\varphi_0| = 1$, because we clearly have $\theta = \varphi_0\langle \varphi_0 \mid \theta \rangle$, for all $\theta \in \mathbb{H}$. The equality from (f) of Proposition 3.8 can be written as $p\varphi_0 = \varphi_0 p$, for all $p \in \mathbb{H}$. In fact, the left hand may be confusing and it should be understood as $p \cdot \varphi_0$, because the true equality $p\varphi_0 = \varphi_0 p$ would imply that $\varphi_0 = \pm 1$. For the simplicity, we are writing $p\phi$ instead of writing $p \cdot \phi$.

(2) Also one can trivially see that $(p + q)\phi = p\phi + q\phi$, for all $p, q \in \mathbb{H}$ and $\phi \in V^R_H$. 
Furthermore, the quaternionic left scalar multiplication of linear operators is also defined in \([5,10]\). For any fixed \(q \in \mathbb{H}\) and a given right linear operator \(A : V^R_H \rightarrow V^R_H\), the left scalar multiplication of \(A\) is defined as a map \(qA : V^R_H \rightarrow V^R_H\) by the setting 
\[
(qA)\phi := q(A\phi) = \sum_{k \in \mathbb{N}} \varphi_k q \langle \varphi_k \mid A\phi \rangle_{V^R_H},
\]
for all \(\phi \in V^R_H\). It is straightforward that \(qA\) is a right linear operator. We can define right scalar multiplication of the right linear operator \(A : V^R_H \rightarrow V^R_H\) as a map \(Aq : V^R_H \rightarrow V^R_H\) by the setting 
\[
(Aq)\phi := A(q\phi),
\]
for all \(\phi \in V^R_H\). It is also right linear operator. One can easily see that 
\[
(qA)^\dagger = A^\dagger \overline{q} \quad \text{and} \quad (Aq)^\dagger = \overline{q} A^\dagger.
\]

### 3.2. S-spectrum

For a given right linear operator \(A : \mathcal{D}(A) \subseteq V^R_H \rightarrow V^R_H\) and \(q \in \mathbb{H}\), we define the operator \(R_q(A) : \mathcal{D}(A^2) \rightarrow \mathbb{H}\) by
\[
R_q(A) = A^2 - 2\text{Re}(q)A + |q|^2 V^R_H,
\]
where \(q = q_0 + iq_1 + jq_2 + kq_3\) is a quaternion, \(\text{Re}(q) = q_0\) and \(|q|^2 = q_0^2 + q_1^2 + q_2^2 + q_3^2\).

In the literature, the operator is called pseudo-resolvent since it is not the resolvent operator of \(A\) but it is the one related to the notion of spectrum as we shall see in the next definition. For more information, on the notion of \(S\)-spectrum the reader may consult e.g. \([4,5,9,10]\).

**Definition 3.10.** Let \(A : \mathcal{D}(A) \subseteq V^R_H \rightarrow V^R_H\) be a right linear operator. The \(S\)-resolvent set (also called spherical resolvent set) of \(A\) is the set \(\rho_S(A) (\subset \mathbb{H})\) such that the three following conditions hold true:

(a) \(\ker(R_q(A)) = \{0\}\).
(b) \(\text{ran}(R_q(A))\) is dense in \(V^R_H\).
(c) \(R_q(A)^{-1} : \text{ran}(R_q(A)) \rightarrow \mathcal{D}(A^2)\) is bounded.

The \(S\)-spectrum (also called spherical spectrum) \(\sigma_S(A)\) of \(A\) is defined by setting \(\sigma_S(A) := \overline{\mathbb{H}} \setminus \rho_S(A)\). For a bounded linear operator \(A\) we can write the resolvent set as 
\[
\rho_S(A) = \{q \in \mathbb{H} \mid R_q(A) \text{ has an inverse in } B(V^R_H)\}
= \{q \in \mathbb{H} \mid \ker(R_q(A)) = \{0\} \quad \text{and} \quad \text{ran}(R_q(A)) = V^R_H\}
\]
and the spectrum can be written as 
\[
\sigma_S(A) = \overline{\mathbb{H}} \setminus \rho_S(A)
= \{q \in \mathbb{H} \mid R_q(A) \text{ has no inverse in } B(V^R_H)\}
= \{q \in \mathbb{H} \mid \ker(R_q(A)) \neq \{0\} \quad \text{or} \quad \text{ran}(R_q(A)) \neq V^R_H\}.
\]

The spectrum \(\sigma_S(A)\) decomposes into three major disjoint subsets as follows:
Proposition 3.11. \([8,10]\) For \(A \in B(V_H^R)\), the resolvent set \(\rho_S(A)\) is a non-empty open set and the spectrum \(\sigma_S(A)\) is a non-empty compact set.

Remark 3.12. For \(A \in B(V_H^R)\), since \(\sigma_S(A)\) is a non-empty compact set so is its boundary. That is, \(\partial \sigma_S(A) = \partial \rho_S(A) \neq \emptyset\).

Proposition 3.13. \([6]\) Let \(A \in B(V_H^R)\). Then \(\ker(R_q(A)) \neq \{0\}\) if and only if \(q\) is a right eigenvalue of \(A\). In particular every right eigenvalue belongs to \(\sigma_S(A)\).

Definition 3.14. \([12]\) Let \(A \in B(V_H^R)\). The approximate S-point spectrum of \(A\), denoted by \(\sigma_{ap}^S(A)\), is defined as

\[
\sigma_{ap}^S(A) = \{ q \in \mathbb{H} \mid \text{there is a sequence } \{\phi_n\}_{n=1}^{\infty} \text{ such that } \|\phi_n\| = 1 \text{ and } \|R_q(A)\phi_n\| \to 0 \}.
\]

Proposition 3.15. \([12]\) Let \(A \in B(V_H^R)\), then \(\sigma_{pS}(A) \subseteq \sigma_{ap}^S(A)\).

Definition 3.16. \([12,14]\) The spherical compression spectrum of an operator \(A \in B(V_H^R)\), denoted by \(\sigma_c^S(A)\), is defined as

\[
\sigma_c^S(A) = \{ q \in \mathbb{H} \mid \ker(R_q(A)) \text{ is not dense in } V_H^R \}.
\]

Definition 3.17. \([14]\) Let \(A \in B(V_H^R)\). The surjectivity S-spectrum of \(A\) is defined as

\[
\sigma_{su}^S(A) = \{ q \in \mathbb{H} \mid \ker(R_q(A)) \neq V_H^R \}.
\]

Clearly we have

\[
\sigma_c^S(A) \subseteq \sigma_{su}^S(A) \quad \text{and} \quad \sigma_S(A) = \sigma_{pS}(A) \cup \sigma_{su}^S(A). \quad (3.8)
\]

Proposition 3.18. \([12]\) Let \(A \in B(V_H^R)\). Then \(A\) has the following properties.

(a) \(\sigma_{pS}(A) \subseteq \sigma_c^S(A^\dagger)\) and \(\sigma_c^S(A) = \sigma_{pS}(A^\dagger)\).

(b) \(\sigma_{su}^S(A) = \sigma_{ap}^S(A^\dagger)\) and \(\sigma_{ap}^S(A) = \sigma_{su}^S(A^\dagger)\).

(c) \(\sigma_S(A) = \sigma_S(A^\dagger)\).

Proposition 3.19. \([12]\) If \(A \in B(V_H^R)\) and \(q \in \mathbb{H}\), then the following statements are equivalent.
(a) \( q \notin \sigma_{ap}(A) \).
(b) ker\( (R_q(A)) = \{0\} \) and ran\( (R_q(A)) \) is closed.
(c) There exists a constant \( c \in \mathbb{R}, c > 0 \) such that \( R_q(A)\phi \geq c\|\phi\| \) for all \( \phi \in D(A^2) \).

**Theorem 3.20.** [10] Let \( V^R_H \) be a right quaternionic Hilbert space equipped with a left scalar multiplication. Then the set \( B(V^R_H) \) equipped with the point-wise sum, with the left and right scalar multiplications defined in Eqs. 3.5 and 3.6, with the composition as product, with the adjunction \( A \longrightarrow A^† \), as in 3.2, as \( *- \) involution and with the norm defined in 3.1, is a quaternionic two-sided Banach \( C^*- \) algebra with unity \( \mathbb{I}_{V^R_H} \).

One can observe that in the above theorem, if the left scalar multiplication is left out on \( V^R_H \), then \( B(V^R_H) \) becomes a real Banach \( C^*- \) algebra with unity \( \mathbb{I}_{V^R_H} \).

### 4. Berberian Extension in the Quaternionic Setting

Following the definition given in [3] for complex bounded sequences, we denote by \( \text{glim} \) a Banach generalized limit defined for bounded sequences \( \{q_n\} \subseteq \mathbb{H} \) with the following properties. For \( q \in \mathbb{H} \) and \( \{q_n\}, \{p_n\} \subseteq \mathbb{H} \),

1. \( \text{glim}(q_n + p_n) = \text{glim}(q_n) + \text{glim}(p_n) \);
2. \( \text{glim}(q_n q) = \text{glim}(q_n) q \);
3. \( \text{glim}(qq_n) = q \text{glim}(q_n) \);
4. \( \text{glim}(q_n) = \lim_{n \to \infty} q_n \) whenever \( \{q_n\} \) is convergent;
5. \( \text{glim}(q_n) \geq 0 \) when \( \{q_n\} \subseteq \mathbb{R} \) and \( q_n \geq 0 \) for all \( n \).

\( \text{glim} \) defines a positive linear form on the vector space \( \mathcal{M} \) of all quaternionic bounded sequences and \( c_0 \) denotes the set of quaternionic null sequences, that is, sequences that converge to zero, and has the value 1 for the constant sequence \( \{1\} \). From properties (a) and (e) of \( \text{glim} \), \( \text{glim}(q_n) \) is real whenever \( q_n \) is real for all \( n \). Hence \( \text{glim}(q_n) = \text{glim}(q_n) \) for any bounded sequence \( \{q_n\} \subseteq \mathbb{H} \).

#### 4.1. An extension of \( V^R_H \)

Let

\[
\mathcal{B} = \left\{ s = \{\phi_n\} \mid \{\phi_n\} \subseteq V^R_H, \|\phi_n\|_{V^R_H} < \infty \forall n, \text{ that is, } \{\|\phi_n\|_{V^R_H}\} \in \mathcal{M} \right\}.
\]

If \( s = \{\phi_n\} \) and \( t = \{\psi_n\} \) write \( s = t \) whenever \( \phi_n = \psi_n \) for all \( n \). Also

\[
s + t = \{\phi_n + \psi_n\} \quad \text{and} \quad sq = \{\phi_n q\},
\]

with these operations \( \mathcal{B} \) becomes a quaternionic right linear vector space. The left scalar multiplication, on \( \mathcal{B} \), is defined as the map \( \mathbb{H} \times \mathcal{B} \ni (q, s) \longrightarrow qs \in \mathcal{B} \) given by

\[
qs := \{q\phi_n\},
\]

for all \( (q, s) = (q, \{\phi_n\}) \in \mathbb{H} \times \mathcal{B} \), where for each \( n \in \mathbb{N} \), \( q\phi_n \) is given by the Definition 3.4. Suppose that \( s = \{\phi_n\}, t = \{\psi_n\} \in \mathcal{B} \). Since

\[
|\langle \phi_n | \psi_n \rangle_{V^R_H}| \leq \|\phi_n\|_{V^R_H} \|\psi_n\|_{V^R_H}, \quad \text{for all } n,
\]

...
it is permissible to define
\[ \Phi(s, t) = \operatorname{glim}(\langle \phi_n | \psi_n \rangle)_{V_{\mathbb{R}^2}}. \]

We have the following properties for \( \Phi \).

(a) Since \( \langle \phi_n | \psi_n \rangle_{V_{\mathbb{R}^2}} = \langle \psi_n | \phi_n \rangle_{V_{\mathbb{R}^2}} \), we have \( \Phi(s, t) = \Phi(t, s) \). That is, \( \Phi \) is symmetric.

(b) Since \( \langle \phi_n | \phi_n \rangle_{V_{\mathbb{R}^2}} \geq 0 \) for all \( n \), \( \Phi(s, s) \geq 0 \) for all \( s \in B \). That is, \( \Phi \) is positive.

(c) \( \Phi \) is a bilinear functional, in the sense that \( \Phi \) is left-antilinear with respect to the first variable,
\[ \Phi(rp + sq, t) = p\Phi(r, t) + q\Phi(s, t), \quad \text{for all } p, q \in \mathbb{H} \text{ and } r, s, t \in B, \]
and \( \Phi \) is right-linear with respect to the second variable,
\[ \Phi(s, rp + tq) = \Phi(s, r)p + \Phi(s, t)q, \quad \text{for all } p, q \in \mathbb{H} \text{ and } r, s, t \in B. \]

From the Schwarz’s inequality we have
\[ |\Phi(s, t)|^2 \leq \Phi(s, s)\Phi(t, t). \]

Let
\[ \mathcal{N} = \{ s \in B | \Phi(s, s) = 0 \} = \{ s \in B | \Phi(s, t) = 0 \ \forall \ t \in B \}. \]

Clearly \( \mathcal{N} \) is a right linear subspace of \( B \). Write \( [s] = s + \mathcal{N} \) for a coset. The quotient right linear vector space \( \mathcal{P} = B/\mathcal{N} \) becomes an inner product space by defining
\[ \langle [s] | [t] \rangle_{\mathcal{P}} = \Phi(s, t). \]

If \( u = \{[\phi_n]\} = \{\phi_n\} + \mathcal{N} \) and \( v = \{[\psi_n]\} = \{\psi_n\} + \mathcal{N} \), then
\[ \langle u | v \rangle_{\mathcal{P}} = \langle [\phi_n] | [\psi_n] \rangle_{\mathcal{P}} = \operatorname{glim}(\langle \phi_n | \psi_n \rangle)_{V_{\mathbb{R}^2}}. \quad (4.2) \]

Using the left scalar multiplication defined on \( B \), by the Eq. 4.1, we can define a left scalar multiplication on \( \mathcal{P} \) by the map \( \mathbb{H} \times \mathcal{P} \ni (q, s) \mapsto q[s] \in \mathcal{P} \) given by
\[ q[s] := qs + \mathcal{N}, \quad (4.3) \]
for all \( (q, [s]) = (q, s + \mathcal{N}) \in \mathbb{H} \times \mathcal{P} \). Following proposition provides some properties of the above defined left scalar multiplication:

**Proposition 4.1.** The left product defined in the Eq. 4.3 satisfies the following properties. For every \([s], [t] \in \mathcal{P} \) and \( p, q \in \mathbb{H} \),

(a) \( q([s] + [t]) = q[s] + q[t] \) and \( q([s])p = (q[s])p \).

(b) \( \| q[s] \|_{\mathcal{P}} = |q| \| [s] \|_{\mathcal{P}} \).

(c) \( q(p[s]) = (qp)[s] \).

(d) \( \langle q[s] | [t] \rangle_{\mathcal{P}} = \langle [s] | q[t] \rangle_{\mathcal{P}} \).

(e) \( r[s] = [s]r, \text{ for all } r \in \mathbb{R} \).

**Proof.** The proof immediately follows from the Proposition 3.8 together with the Eqs. 4.1 and 4.3. \( \square \)
Let $\phi \in V^R_H$, we write \{\phi\} for the sequence all of whose terms are $\phi$ and $\phi'$ for the coset \{[\phi]\} = \{\phi\} + \mathcal{N}$. Evidently
\[
\langle [\phi] | [\psi] \rangle_{\mathcal{P}} = \langle \phi | \psi \rangle_{V^R_H},
\]
and $\phi \mapsto [\phi]$ is an isometric right linear mapping of $V^R_H$ onto a closed linear subspace $V^R_H$ of $\mathcal{P}$. Regard $\mathcal{P}$ as a linear subspace of its Hilbert space completion $\mathcal{H}$. Then $V^R_H$ is a closed linear subspace of $\mathcal{H}$ and $\mathcal{P}$ is a dense linear subspace of $\mathcal{H}$.

4.2. A Representation of $B(V^R_H)$

Every operator $A$ in $V^R_H$ determines an operator $A^\circ$ in $\mathcal{H}$ as follows.

If $s = \{\phi_n\} \in \mathcal{B}$ then the relation $\|A\phi_n\|_{V^R_H} \leq \|A\| \|\phi_n\|_{V^R_H}$ shows that $\{A\phi_n\} \in \mathcal{B}$. Define
\[
A_0 : \mathcal{B} \longrightarrow \mathcal{B} \quad \text{by} \quad A_0 s = \{A\phi_n\},
\]
then $A_0$ is a right linear mapping such that
\[
\Phi(A_0 s, A_0 s) \leq \|A\| \Phi(s, s).
\]
In particular, if $s \in \mathcal{N}$, that is $\Phi(s, s) = 0$, then $A_0 s \in \mathcal{N}$, it follows that
\[
A^\circ : \mathcal{P} \longrightarrow \mathcal{P} \quad \text{by} \quad \{[\phi_n]\} \mapsto \{[A\phi_n]\}
\]
is a well-defined right linear map. Thus
\[
A^\circ s' = (A_0 s)'
\]
and the inequality
\[
\langle A^\circ u | A^\circ u \rangle_{\mathcal{P}} \leq \|A\|^2 \langle u | u \rangle_{\mathcal{P}}
\]
is valid for all $u \in \mathcal{P}$. That is, $\|A^\circ u\|_{\mathcal{P}} \leq \|A\| \|u\|_{\mathcal{P}}$, for all $u \in \mathcal{P}$. Hence $A^\circ$ is bounded (continuous), and $\|A^\circ\| \leq \|A\|$, $\|\cdot\|$ is the norm on $B(\mathcal{H})$. The left scalar multiplication of $A^\circ$ by any $q \in \mathbb{H}$ is defined as a map $qA^\circ : \mathcal{P} \longrightarrow \mathcal{P}$ by the setting
\[
(qA^\circ)\{[\phi_n]\} := \{[q(A\phi_n)]\},
\]
for all $\{[\phi_n]\} \in \mathcal{P}$. It is straightforward that $qA^\circ$ is a right linear operator. We also have the following properties for the operators:

**Proposition 4.2.** For $A, B \in B(V^R_H)$ and $q \in \mathbb{H}$, we have

\begin{align*}
(a) \quad (A + B)^\circ &= A^\circ + B^\circ, \\
(b) \quad (qA)^\circ &= qA^\circ, \\
(c) \quad (AB)^\circ &= A^\circ B^\circ, \\
(d) \quad (A^\dagger)^\circ &= (A^\circ)^\dagger, \\
(e) \quad I_{V^R_H}^\circ &= I_{V^R_H}, \\
(f) \quad \|A^\circ\|_\circ &= \|A\|.
\end{align*}

**Proof.** Proofs of (a), (c) and (e) are straightforward from the definition of $A^\circ$. Assertion (b) immediately follows from the (definition) Eq. 4.5 as follows: for any $\{[\phi_n]\} \in \mathcal{P}$,
\[
(qA^\circ)\{[\phi_n]\} = \{[q(A\phi_n)]\} = \{[(qA)\phi_n]\} = (qA)^\circ \{[\phi_n]\}.
\]
To verify (d), let \( C = (A^\circ)^\dagger \) and \( u = \{[\phi_n]\} \) and \( v = \{[\psi_n]\} \). Then
\[
\langle A^\circ u \mid v \rangle_{\Phi} = \langle u \mid C v \rangle_{\Phi}.
\]

This implies that
\[
\langle u \mid C v \rangle_{\Phi} = \langle A^\circ u \mid v \rangle_{\Phi} = \text{glim}(\langle A\phi_n|\psi_n\rangle_{V^R}) = \text{glim}(\langle \phi_n|A^\dagger \psi_n\rangle_{V^R})
\]
\[
= \langle u \mid (A^\circ)^\dagger v \rangle_{\Phi}.
\]
Therefore \( (A^\circ)^\circ = C = (A^\circ)^\dagger \), and this completes the proof of (d). Finally let us establish the equality \( \|A^\circ\|_\circ = \|A\| \). First note that for any \( \phi \in V^R \); from the Eq. 4.2, we have \( \|\phi'\|_{\Phi} = \|\phi\|_{V^R} \). Now since \( A^\circ \phi' = (A\phi)' \), for all \( \phi \in V^R \),
\[
\|A^\circ\|_\circ = \sup_{\|\phi'\|_{\Phi} = 1} \|A^\circ \phi'\|_{\Phi} = \sup_{\|\phi\|_{V^R} = 1} \|A\phi\|_{V^R} = \|A\|.
\]
Therefore the assertion (f) follows. \( \square \)

The continuous right linear mapping \( A^\circ \) extends to a unique right linear operator in \( \mathfrak{S} \), which we also denote \( A^\circ \). Also in the Theorem 3.20, \( B(V^R) \) with left multiplication is a \( C^* \)-algebra with unity \( I_{V^R} \). In the same manner, \( B(\mathfrak{S}) \) with left multiplication is a \( C^* \)-algebra with the same unity \( I_{V^R} \). Also note that, no matter which Hilbert basis we choose to define a left multiplication the spaces \( B(V^R) \) and \( B(\mathfrak{S}) \) becomes \( C^* \)-algebras, and hence the results provided in this note are independent of the basis chosen.

**Theorem 4.3.** The mapping
\[
B(V^R) \longrightarrow B(\mathfrak{S}) \quad by \quad A \mapsto A^\circ
\]
is a faithful \( \ast \)-representation.

**Proof.** The assertion (c) of Proposition 4.2 verifies that the above map is a homomorphism. To check the injectivity of this map, suppose that \( A, B \in B(V^R) \) with \( A^\circ = B^\circ \). Then for any \( \{[\phi_n]\} \in \Phi \), we have
\[
\{[A\phi_n]\} = \{[B\phi_n]\} \Rightarrow \{(A - B)\phi_n\} \in \mathcal{H}
\]
\[
\Rightarrow \text{glim}(\langle (A - B)\phi_n \mid (A - B)\phi_n\rangle_{V^R}) = 0.
\]
Let \( \phi \in V^R \), and choose \( \phi_n = \phi \), \( \forall n \in \mathbb{N} \). Then \( \|(A - B)\phi\|_{V^R} = 0 \). This concludes that \( A = B \). Therefore the above map is injective. Hence the theorem follows. \( \square \)

Suppose \( A \geq 0 \), that is \( \langle A\phi|\phi\rangle_{V^R} \geq 0 \) for all \( \phi \in V^R \). If \( u = \{\phi_n\}' \in \Phi \), then \( \langle A\phi_n|\phi_n\rangle_{V^R} \geq 0 \) for all \( n \), hence
\[
\langle A^\circ u | u \rangle_{\Phi} = \text{glim}(\langle A\phi_n|\phi_n\rangle_{V^R}) \geq 0.
\]
Hence \( \langle A^\circ v | v \rangle_{\Phi} \geq 0 \) for all \( v \in \mathfrak{S} \). Thus clearly for an operator \( A \) in \( V^R \) we have
\[
A \geq 0 \Leftrightarrow A^\circ \geq 0. \quad (4.6)
\]

**Proposition 4.4.** If \( A \in B(V^R) \), then \( \sigma_{ap}^S(A^\circ) = \sigma_{ap}^S(A) \).
Proposition 5.1. Let \( q \in \mathbb{H} \). Then, \( q \not\in \sigma_{ap}^{S}(A) \) if and only if there exists \( \epsilon > 0 \) such that \( R_q(A^\dagger)R_q(A) \geq \epsilon I_{V_{ii}^R} \). By Eq. 4.6, this condition is equivalent to \( R_q((A^\circ)^\dagger)R_q(A^\circ) \geq \epsilon I_{V_{ii}^R} \) thus \( q \not\in \sigma_{ap}^{S}(A^\circ) \). 

Theorem 4.5. For every operator \( A \in B(V_{i}^R) \), we have \( \sigma_{ap}^{S}(A) = \sigma_{ap}^{S}(A^\circ) = \sigma_{pS}(A^\circ) \).

Proof. From Propositions 3.15, 4.4 the relation \( \sigma_{ap}^{S}(A) = \sigma_{ap}^{S}(A^\circ) \supseteq \sigma_{pS}(A^\circ) \) is clear. Let \( q \in \sigma_{ap}^{S}(A) \). Then there exists a sequence \( \{ \phi_n \} \subseteq V_{i}^R \) with \( \| \phi_n \|_{V_{ii}^R} = 1 \) such that \( \| R_q(A)\phi_n\|_{V_{ii}^R} \to 0 \). Set \( u = \{ \phi_n \}' \), clearly \( \| u \|_{\mathfrak{P}} = 1 \). Also

\[
\| R_q(A^\circ)u \|_{\mathfrak{P}} = \lim_i\| R_q(A)\phi_n\|_{V_{ii}^R} \to 0.
\]

Therefore, by Proposition 3.13, \( q \) is a right eigenvalue of \( A^\circ \). Hence \( q \in \sigma_{pS}(A^\circ) \), which completes the proof.

5. Application to Commutators in the Quaternionic Setting

In the complex setting, the Berberian extension is very useful in studying spectral properties of commutators [11]. Following the complex formalism, in this section, we shall study some properties of \( S \)-spectrum of commutators in the quaternionic setting.

Proposition 5.1. Let \( A, B \in B(V_{i}^R) \) such that \( AB = BA \), then

(a) \( \sigma_{ap}^{S}(A + B) \subseteq \sigma_{ap}^{S}(A) + \sigma_{ap}^{S}(B) \),

(b) \( \sigma_{su}^{S}(A + B) \subseteq \sigma_{su}^{S}(A) + \sigma_{su}^{S}(B) \),

(c) \( \sigma_{S}(A + B) \subseteq \sigma_{S}(A) + \sigma_{S}(B) \).

Proof. (a) Since \( AB = BA \) we have \( A^\circ B^\circ = B^\circ A^\circ \). Let \( q \in \sigma_{ap}^{S}(A + B) = \sigma_{pS}(A^\circ + B^\circ) \). Let \( Z = \ker(R_q(A^\circ + B^\circ)) \). Then \( Z \neq \emptyset \). Let \( \psi \in A^\circ Z \), then \( \psi = A^\circ \phi \) for some \( \phi \in Z \) and also \( R_q(A^\circ + B^\circ)\phi = 0 \). Now

\[
R_q(A^\circ + B^\circ)\psi = R_q(A^\circ + B^\circ)A^\circ \phi = A^\circ R_q(A^\circ + B^\circ)\phi = 0.
\]

Therefore \( \psi \in Z \), hence \( A^\circ Z \subseteq Z \). That is, \( Z \) is invariant under \( A^\circ \), and therefore \( \sigma_{ap}^{S}(A^\circ |Z) \neq \emptyset \). Let \( p \in \sigma_{ap}^{S}(A^\circ |Z) = \sigma_{pS}(A^\circ |Z) \), hence \( A^\circ - p I_{S} = 0 \). Since \( q \in \sigma_{pS}(A^\circ + B^\circ) \), we have \( A^\circ + B^\circ - q I_{S} = 0 \), that is \( B^\circ = q I_{S} - A^\circ \).

Therefore,

\[
B^\circ - (q - p) I_{S} = -(A^\circ - p) I_{S} = 0 \text{ on } Z.
\]

Thus \( q - p \in \sigma_{pS}(B^\circ |Z) \). Hence, from Proposition 4.4,

\[
q = p + (q - p) \in \sigma_{pS}(A^\circ) + \sigma_{pS}(B^\circ) = \sigma_{ap}^{S}(A^\circ) + \sigma_{ap}^{S}(B^\circ)
\]

\[
\sigma_{ap}^{S}(A) + \sigma_{ap}^{S}(B).
\]

This completes the proof of (a).

(b) Since \( AB = BA \), we have \( A^\dagger B^\dagger = B^\dagger A^\dagger \), and therefore (a) holds for \( A^\dagger, B^\dagger \). Further from Proposition 3.18, part (b), \( \sigma_{su}^{S}(A) = \sigma_{ap}^{S}(A^\dagger) \). Thus (b) follows.
(c) For any $A \in B(V^R_H)$, from Eq. 3.8, Proposition 3.15, we have $\sigma_S(A) = \sigma_{ps}(A) \cup \sigma_{su}(A) \subseteq \sigma_{ap}(A) \cup \sigma_{ap}(A)$, and clearly $\sigma_{ap}(A), \sigma_{su}(A) \subseteq \sigma_S(A)$. Therefore, from (a) and (b), we have

$$\sigma_{ap}(A + B) \subseteq \sigma_{ap}(A) + \sigma_{ap}(B) \subseteq \sigma_S(A) + \sigma_S(B)$$

and

$$\sigma_{su}(A + B) \subseteq \sigma_{su}(A) + \sigma_{su}(B) \subseteq \sigma_S(A) + \sigma_S(B).$$

Thus

$$\sigma_S(A + B) \subseteq \sigma_{ap}(A + B) \cup \sigma_{su}(A + B) \subseteq \sigma_S(A) + \sigma_S(B).$$

Hence the inclusion (c) holds. \qed

**Definition 5.2.** Given $S, T \in B(V^R_H)$, the commutator $C(S, T) : B(V^R_H) \rightarrow B(V^R_H)$ is the mapping

$$C(S, T)(A) = SA - AT = L_S(A) - R_T(A),$$

for all $A \in B(V^R_H)$, where $L_S(A) = SA$ and $R_T(A) = AT$. It is clear that $A \in B(V^R_H)$ intertwines the pair $(S, T)$ precisely when $C(S, T) = 0$.

**Remark 5.3.** It is worth noting the following results: for any $q \in H$ and $S, T \in B(V^R_H)$,

1. $L_S R_T = R_T L_S$,
2. $R_q(L_S) = R_q(S)$,
3. $R_q(R_T)A = AR_q(T)$; for all $A \in B(V^R_H)$,
4. $L_{R_q(S)} = R_q(L_S)$,
5. $R_{R_q(T)} = R_q(R_T)$.

The verifications of these results are elementary.

The next proposition gathers some useful identities to prove the S-spectral properties of commutators which are provided in Theorem 5.5.

**Proposition 5.4.** For arbitrary operators $S, T \in B(V^R_H)$ the following assertions hold true:

(a) $\sigma_{ap}^S(L_S) = \sigma_{ap}^S(S)$,
(b) $\sigma_{ap}^S(R_T) = \sigma_{ap}^S(T)$,
(c) $\sigma_{ap}^S(L_S) = \sigma_{ap}^S(S)$,
(d) $\sigma_{ap}^S(R_T) = \sigma_{ap}^S(T)$.

**Proof.** (a) To prove (a), let $q \in \sigma_{ap}^S(L_S)$, then there exists a sequence $\{A_n\} \subseteq B(V^R_H)$ with $\|A_n\| = 1$ such that $\|R_q(L_S)A_n\| \rightarrow 0$. That is, $\|R_q(S)A_n\| \rightarrow 0$.

Set $\psi_n = \phi$, for all $n$ and for some $0 \neq \phi \in V^R_H$. Then $\|A_n \psi_n\|_{V^R_H} = 1$, for all $n$ and $\|R_q(S)A_n \psi_n\|_{V^R_H} \rightarrow 0$. Thus $q \in \sigma_{ap}^S(S)$ and $\sigma_{ap}^S(L_S) \subseteq \sigma_{ap}^S(S)$.

To see the other inclusion, let $q \in \sigma_{ap}^S(S)$, then there exists a sequence $\{\psi_n\} \subseteq V^R_H$ with $\|\psi_n\|_{V^R_H} = 1$ and $\|R_q(S)\psi_n\|_{V^R_H} \rightarrow 0$. Pick a linear functional $\Phi$ in $(V^R_H)^*$ which is the dual of $V^R_H$ with $\|\Phi\| = 1$, where $\| \cdot \|^*$ is a norm on the dual of $V^R_H$. Define, for each $n$, an operator $A_n \in B(V^R_H)$ by

$$A_n \phi = \psi_n \Phi(\phi), \quad \text{for all } \phi \in V^R_H.$$
Then \( \| A_n \| = 1 \) for all \( n \), and
\[
\| R_q(L_S)A_n \| = \| R_q(S)A_n \| = \| R_q(S)\psi_n \|_{V^R_{\mathbb{H}}} \to 0.
\]
Thus \( q \in \sigma^S_{ap}(L_S) \) and therefore \( \sigma^S_{ap}(L_S) = \sigma^S_{ap}(S) \).

(b) To establish the equality \( \sigma^S_{ap}(R_T) = \sigma^S_{su}(T) \), let \( q \not\in \sigma^S_{su}(T) \), that is \( R_q(T) \) is surjective. Now for each \( A \in B(V^R_{\mathbb{H}}) \),
\[
\| R_q(R_T)A \| = \| AR_q(T) \| \geq \| AR_q(T)\phi \|_{V^R_{\mathbb{H}}}, \quad \text{for all } \phi \in V^R_{\mathbb{H}}.
\]
That is, as \( R_q(T) \) is surjective, \( \| R_q(R_T)A \| \geq \| A\psi \|_{V^R_{\mathbb{H}}} \), for all \( \psi = R_q(T)\phi \in V^R_{\mathbb{H}} \). Hence
\[
\| R_q(R_T)A \| \geq \sup_{\| \psi \| = 1} \| A\psi \|_{V^R_{\mathbb{H}}} = \| A \|, \quad \text{for all } A \in B(V^R_{\mathbb{H}}).
\]
Therefore \( R_q(R_T) \) is bounded below, and hence by Proposition 3.19, \( q \not\in \sigma^S_{ap}(R_T) \). Conversely suppose that \( R_q(R_T) \) is bounded below. Then there exists \( c > 0 \) such that \( c\| A \| \leq \| R_q(R_T)A \| = \| AR_q(T)\phi \|_{V^R_{\mathbb{H}}} \), for all \( A \in B(V^R_{\mathbb{H}}) \); where \( \| \cdot \| \) is the norm on \( B(B(V^R_{\mathbb{H}})) \). Choose a unit vector \( \psi \in V^R_{\mathbb{H}} \). For arbitrary linear functional \( \Phi \in (V^R_{\mathbb{H}})^* \), let \( A_{\Phi} \in B(V^R_{\mathbb{H}}) \) given by
\[
A_{\Phi}(\phi) = \psi\Phi(\phi), \quad \text{for all } \phi \in V^R_{\mathbb{H}}.
\]
Then
\[
c\| \Phi \| = c\| A_{\Phi} \| \leq \| A_{\Phi}R_q(T) \| = \| \Phi \circ R_q(T) \|, \quad \text{for all } \Phi \in (V^R_{\mathbb{H}})^*.
\]
Hence \( R_q(T) \) is bounded below. That is, by Proposition 3.4, \( R_q(T)^{\dagger} \) is injective. Therefore by Propositions 3.3, \( \text{ran}(R_q(T)^{\dagger}) = \ker(R_q(T)^{\dagger}) = \{ 0 \} \), and so \( R_q(T) \) is surjective. Thus we have \( \sigma^S_{ap}(R_T) = \sigma^S_{su}(T) \).

(c) Now
\[
q \not\in \sigma^S_{su}(L_S) \iff R_q(L_S) \text{is surjective} \iff R_q(S) \text{is surjective}
\]
\[
\iff q \not\in \sigma^S_{su}(S).
\]
Therefore \( \sigma^S_{ap}(L_S) = \sigma^S_{su}(S) \).

(d) In order to verify the inclusion \( \sigma^S_{ap}(R_T) \subseteq \sigma^S_{ap}(T) \), let \( q \not\in \sigma^S_{ap}(T) \), then by Proposition 3.19, \( R_q(T) \) is bounded below on \( V^R_{\mathbb{H}} \). Therefore, by Proposition 3.4, \( R_q(T) \) is injective. Thus by Proposition 3.7, \( R_q(T) \) is left invertible on \( V^R_{\mathbb{H}} \). Therefore, there exist \( P \in B(V^R_{\mathbb{H}}) \) such that \( PR_q(T) = I_{V^R_{\mathbb{H}}} \). This implies that
\[
R_q(R_T)R_PA = R_P(A)R_q(T) = APR_q(T) = A, \quad \text{for all } A \in B(V^R_{\mathbb{H}}).
\]
That is, \( R_q(R_T) \) is right invertible on \( B(V^R_{\mathbb{H}}) \), thus by Proposition 3.6, \( R_q(R_T) \) is surjective. Hence \( q \not\in \sigma^S_{su}(R_T) \), and we get \( \sigma^S_{su}(R_T) \subseteq \sigma^S_{ap}(T) \). To verify the other inclusion \( \sigma^S_{ap}(T) \subseteq \sigma^S_{su}(R_T) \), suppose that \( q \not\in \sigma^S_{su}(R_T) \), then \( R_q(R_T) \) is surjective. This implies that for each \( A \in B(V^R_{\mathbb{H}}) \), there exists \( B \in B(V^R_{\mathbb{H}}) \) such that \( R_q(R_T)B = A \). That is, \( BR_q(T) = A \). Assuming \( A, B \neq 0 \) without loss of generality, we get
\[
\| R_q(T) \| \geq \frac{\| A \|}{\| B \|}.
\]
This gives that \( R_q(T) \) is bounded below, and \( q \not\in \sigma_{ap}(T) \). Therefore the equality \( \sigma_{su}(R_T) = \sigma_{ap}(T) \) holds. \( \square \)

The following theorem is the main result about the S-spectral properties of commutators which we provide in this note.

**Theorem 5.5.** For arbitrary operators \( S, T \in B(V^R) \) the following assertions hold true for their commutator \( C(S, T) \).

(a) \( \sigma_S(C(S, T)) = \sigma_S(S) - \sigma_S(T) \),
(b) \( \sigma_{ap}(C(S, T)) = \sigma_{ap}(S) - \sigma_{su}(T) \),
(c) \( \sigma_{ap}(C(S, T)) = \sigma_{ap}(S) - \sigma_{ap}(T) \).

**Proof.** To prove (b), in order to establish \( \sigma_{ap}(S) - \sigma_{su}(T) \subseteq \sigma_{ap}(C(S, T)) \), let \( q \in \sigma_{ap}(S) \) and \( p \in \sigma_{su}(T) \). It follows, from (a) and (b) in the Proposition 5.4, that \( q \in \sigma_{ap}(L_S) \) and \( p \in \sigma_{ap}(R_T) \). By Proposition 4.5 we have

\[ q \in \sigma_{ap}(L_S) = \sigma_{ap}(L_S^\circ) = \sigma_{ps}(L_S^\circ). \]

Therefore, there exists \( A \in B(V^R) \) such that \( A \neq 0 \) and \( (L_S^\circ - q)A = 0 \). That is, \( (S^\circ - q)A = 0 \). Again by Proposition 4.5,

\[ p \in \sigma_{ap}(R_T) = \sigma_{ap}(R_T^\circ) = \sigma_{ps}(R_T^\circ). \]

Therefore there exists \( B \in B(V^R) \) such that \( B \neq 0 \) and \( (R_T^\circ - p)B = 0 \). That is \( B(T^\circ - p) = 0 \). Consider

\[ (C(S, T)^\circ - q + p)AB = (L_S^\circ - R_T^\circ - q + p)AB = (L_S^\circ - q)AB - (R_T^\circ - p)AB = (S^\circ - q)AB - AB(T^\circ - p) = 0. \]

Thus

\[ q - p \in \sigma_{ps}(C(S, T)^\circ) = \sigma_{ap}(C(S, T)^\circ) = \sigma_{ap}(C(S, T)). \]

Therefore \( \sigma_{ap}(S) - \sigma_{su}(T) \subseteq \sigma_{ap}(C(S, T)) \). Now since \( \sigma_{ap}(L_S) = \sigma_{ap}(S) \) and \( \sigma_{ap}(R_T) = \sigma_{ap}(T) \), we get from part (a) of Proposition 5.1,

\[ \sigma_{ap}(C(S, T)) = \sigma_{ap}(L_S - R_T) \subseteq \sigma_{ap}(L_S) - \sigma_{ap}(R_T) \subseteq \sigma_{ap}(S) - \sigma_{ap}(T). \]

This concludes the proof for (b).
(c) Applying \( \sigma_{su}(L_S) = \sigma_{su}(S) \), and \( \sigma_{su}(R_T) = \sigma_{su}(T) \) in part (b) of Proposition 5.1, the inclusion \( \subseteq \) in assertion (c) is established. Next to show that \( \sigma_{su}(S) - \sigma_{ap}(T) \subseteq \sigma_{su}(C(S, T)) \), let \( q \in \sigma_{su}(S) \) and \( p \in \sigma_{ap}(T) \). It follows from (a) and (b) in Proposition 5.4, and Proposition 4.5 that

\[ q \in \sigma_{ap}(R_S) = \sigma_{ap}(R_S^\circ) = \sigma_{ps}(R_S^\circ) \quad \text{and} \quad p \in \sigma_{ap}(L_T) = \sigma_{ap}(L_T^\circ) = \sigma_{ps}(L_T^\circ). \]

Thus by the definition of point spectrum \( \overline{q} \in \sigma_{ps}(R_S^\circ) \) and \( \overline{p} \in \sigma_{ps}(L_T^\circ) \). Therefore, there exists \( A, B \in B(V^R) \) with \( A \neq 0 \) and \( B \neq 0 \) such that \( A(S^\circ - \overline{q}) = 0 \) and \( (T^\circ - \overline{p})B = 0 \). Hence, by Proposition 3.2, we have

\[ ((S^\circ)^\dagger - \overline{q})A^\dagger = 0 \quad \text{and} \quad B^\dagger((T^\circ)^\dagger - \overline{p}) = 0. \]

Hence, by Proposition 3.2,
Therefore \( q - p \in \sigma_{ps}(C(S, T)\circ \top) \). By proposition 4.2, \( q - p \in \sigma_{ps}(C(S, T)\circ \top) \). Now by propositions 4.5 and 3.18, we have

\[
\text{ker}(\sigma) = \sigma_{ap}(C(S, T)\circ \top) = \sigma_{su}(C(S, T)).
\]

Hence \( \sigma_{su}(S) - \sigma_{ap}(T) \subseteq \sigma_{su}(C(S, T)) \), which completes the proof of (c).

To establish (a), let \( S, T \). We have

\[
\text{ker}(\sigma) = \sigma_{ap}(C(S, T)\circ \top) = \sigma_{su}(C(S, T)).
\]

Hence \( \sigma_{su}(S) - \sigma_{ap}(T) \subseteq \sigma_{su}(C(S, T)) \), which completes the proof of (c).

On the other hand, using Eq. 5.1, from part (c),

\[
\sigma_{su}(C(S, T)) = \sigma_{su}(C(R_L)) \supseteq \sigma_{ap}(C(R_L)) = \sigma_{su}(S) - \sigma_{su}(T) = \sigma_{su}(C(S, T)).
\]

(5.2)

as \( \sigma_{ap}(R_L) = \sigma_{su}(T) \) and \( \sigma_{su}(C(R_L)) = \sigma_{su}(C(S, T)). \) Similarly from part (b), we get

\[
\sigma_{ap}(C(S, T)) = \sigma_{ap}(C(R_L)) \supseteq \sigma_{ap}(C(R_L)) = \sigma_{ap}(S) - \sigma_{ap}(T) = \sigma_{ap}(C(S, T)).
\]

(5.3)

as \( \sigma_{ap}(R_L) = \sigma_{ap}(T) \) and \( \sigma_{ap}(C(R_L)) = \sigma_{ap}(C(S, T)). \) Now the inclusions (5.2) and (5.3) guarantee that the other inclusion in assertion (a) holds,
\[ \sigma_S(C(S,T)) \supseteq \sigma_S(L_S) - \sigma_S(R_T) \subseteq \sigma_S(S) - \sigma_S(T). \]

Therefore the assertion (a) follows. Hence the theorem holds. \(\square\)

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B. Muraleetharan  
Department of Mathematics and Statistics  
University of Jaffna  
Thirunelveli  
Sri Lanka  
e-mail: muralee@univ.jfn.ac.lk

K. Thirulogasanthar  
Department of Computer Science and Software Engineering  
Concordia University  
1455 De Maisonneuve Blvd. West  
Montreal QCH3G 1M8  
Canada  
e-mail: santhar@gmail.com

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