Landau damping for the Alber equation and stability of unidirectional wave spectra

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Abstract

The Alber equation is a moment equation for the nonlinear Schrödinger equation, widely used in ocean engineering to investigate the stability of stationary and homogeneous sea states in terms of their power spectra. More specifically, a sufficient condition for instability is known. In this work we present the first well-posedness theory for the Alber equation with the help of an appropriate equivalent reformulation. Moreover, we show Landau damping, i.e. we prove that under a stability condition any perturbations of the homogeneous sea state disperse and decay in time. The idea of the proof is related to recent work in Landau damping for unconfined Vlasov equations, although there are substantial differences as well; in particular our result does not require a mean-zero assumption. Finally, the sufficient condition for stability is resolved, and compared to the known sufficient condition for instability. An algorithm for the practical checking of stability for measured spectra is also provided, and the physical implications for ocean waves are discussed.

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1 Introduction

The Alber equation \cite{2, 3, 18, 26, 30, 31, 37, 4, 32}

\[ \partial_t w + 2\pi p k \partial_x w - qi \int_{\lambda, y \in \mathbb{R}^d} e^{-2\pi i \lambda y} \left[ n(x - \frac{y}{2}, t) - n(x + \frac{y}{2}, t) \right] dy \left( P(k - \lambda) + w(x, k, t) \right) d\lambda = 0, \]

\[ n(x, t) = \int_{\xi \in \mathbb{R}^d} w(x, \xi, t) d\xi, \quad w(x, k, 0) = w_0(x, k), \]  \hspace{1cm} (1)

is a stochastic second moment of the cubic NLS equation

\[ i \partial_t u + \frac{p}{2} \Delta u + \frac{q}{2} |u|^2 u = 0, \quad u(t = 0) = u_0 \]  \hspace{1cm} (2)

governing the complex envelope of weakly nonlinear waves. It is derived by virtue of a Gaussian moment closure\textsuperscript{1}, and widely used in the context of ocean waves. The background spectrum \( P(k) \) represents the

\textsuperscript{1}The Gaussian closure involves a complex Isserlis theorem, and it is elaborated in Appendix A. Equation (1) turns out to have the same structure as the deterministic Wigner equation of (2).
distribution of energy over wavenumbers in a homogeneous sea state. The unknown function \( w(x, k, t) \) represents any non-homogeneous deviations, resolved over position \( x \), wavenumber \( k \) and time \( t \).

In this work we prove the first well-posedness results for equation (1), and show for the first time linear Landau damping (i.e. dispersion of inhomogeneities and return to the homogeneous equilibrium) under a stability condition for the spectrum \( P(k) \). We also elaborate on the stability condition, and present the first systematic, practical algorithm to determine whether it holds or not for a given spectrum.

1.1 Physical context: modulation instability and continuous spectra for ocean waves

The NLS equation (2) has been used as an envelope equation for ocean waves since the sixties [8, 38]. Considerations based on this approximation and its ramifications are to date widely used in ocean engineering, see e.g. [13, 14, 16, 21, 35] and the references therein. Eq. (2) is focusing if \( pq > 0 \); this is the physically relevant case for ocean waves. In this paper for simplicity we take \( p > 0 \), \( q > 0 \).

Moreover, ocean waves are random and in many practical applications stochastic models have to be used [21, 25, 31, 9]. It has been long established that the sea surface elevation \( \eta(x, t) \) can be approximated to leading order by a random process which is stationary in \( t \) and homogeneous in \( x \), at least in mesoscales of \( O(1 \text{ hour} - 3 \text{ days}) \) and \( O(30-100 \text{ km}) \) in the open sea [22, 34]. In this context the sea-state, i.e. the random process which generates the sea surface elevation, is characterised primarily by its power spectrum \( S(\omega) \), which measures the distribution of the wave action over different frequencies. Using a linear model for ocean waves, the frequency-resolved spectrum \( S(\omega) \) can easily be transformed to the wavenumber-resolved spectrum \( P(k) \) [25]. It is important to note that, in most practical cases, what is measured and used in the field is just the spectrum. This is a highly averaged measurement, typically extracted from hundreds (or more) of individual waves.

The approximate stationarity and homogeneity of the sea-state (at least in the mesoscales) has been well documented and widely used for long. By now it is so well-known that it is just taken for granted. But one may ask: do we have an explanation for it? Does the fundamental physics predict the existence of robust stationary and homogeneous sea-states? There is no reason to expect that, generically, a random solution of a focusing nonlinear equation with very large energy (over a spatially extended wavefield) will lead to stationary and homogeneous statistics when evolved in time. On the contrary, the Benjamin-Feir instability (also known as modulation instability) [7, 39, 38] seems to indicate that nonlinear effects will destroy the stationarity and homogeneity of an idealised plane-wave solution for the focusing NLS. In fact, the modulation instability has long been linked to localised extreme events and rogue waves [27, 9, 4, 14, 16]. How does one pass from instability for very narrow spectra to the robust preservation of somewhat broader spectra? The Alber equation [2] was introduced precisely as a systematic way to combine the NLS and the spectrum, and presents a natural way to resolve the apparent contradiction between modulation instability and stable homogeneous wavefields. A nonlinear “eigenvalue relation” was derived in [2], as a sufficient condition for instability: if solutions of the “eigenvalue relation” exist, then modulation instability is present and any inhomogeneities would grow, feeding on the energy of the background [2, 4].

1.2 Open questions and description of main results

Alber’s idea to combine NLS-type envelope equations with a stochastic approach is very natural and has attracted substantial attention in the ocean waves community [2, 3, 18, 26, 30, 31, 37, 4, 32]. However, one must not underestimate how many questions related to the Alber equation have been completely open up to now.

There exist some recent works for well-posedness and stability of nonlinear equations related to the Alber equation. In [24, 23] the authors work in operator formalism, for a defocusing problem with a regular interaction kernel \( (K(x) \ast n(x, t)) \) appears in the place of \( n(x, t) \) in the equation; in our case the interaction kernel is a \( \delta \) function). The authors proceed to exploit the defocusing character of the problem by defining a relative entropy which controls the solution in an appropriate sense; this is a key ingredient
of their proof. In [12] a similar argument is used for the defocusing problem with a \( \delta \) interaction kernel and a single background spectrum. Another related work is [15], where the stability of a fully stochastic problem (no Gaussian closure) is studied, but only in the defocusing case, and for \( d \geq 4 \) and with smooth interaction kernel. In other words, no existing results cover the focusing Alber equation (1). We show well-posedness and regularity of solutions for any dimension in Theorems 3.1, 3.2, 3.3 below.

Landau damping for the Alber equation (i.e. dispersion of inhomogeneities in the presence of a homogeneous background) has been conjectured at least since [26]. More broadly, whenever Alber’s “eigenvalue relation” has no solutions, stability was expected, but no precise results existed. Here we manage to show Landau damping for \( d = 1 \) under a Penrose-type condition, including time-decay of the force and existence of a wave operator in Theorem 3.4. Our proof makes use of ideas from [6], which are adapted to the specifics our problem, including the different dimension and function spaces. Moreover here we exploit some fine properties of the Hilbert transform to justify taking a limit of the interaction kernel. In other words, no existing results cover the focusing problem (no Gaussian closure) is studied, but only in the defocusing case, and for a single background spectrum. Another related work is [15], where the stability of a fully stochastic analysis of [6] to \( d = 1 \), where the dispersion is weaker. In [6] the authors work for \( d = 3 \) only.

Moreover here we exploit some fine properties of the Hilbert transform to justify taking a limit of the Bromwich contour’s location directly. This makes our proof quite self-contained, and we avoid the mean-zero assumption on the inhomogeneity that is made in [6]. This latter point is physically important, as the location of their proof is central to our result; they can be thought of as adaptations of Proposition 2.3 of [6] to \( d = 1 \), where the dispersion is weaker. In [6] the authors work for \( d = 3 \) only.

2 Mathematical preliminaries

2.1 Definitions and notations

The normalisation we use for the Fourier transform is

\[ \hat{u}(X) = \mathcal{F}_{x \rightarrow X}[u] = \int_{x \in \mathbb{R}^d} e^{-2\pi i x \cdot u(x)} dx, \quad \hat{u}(X) = \mathcal{F}_{x \rightarrow X}^{-1}[u] = \int_{x \in \mathbb{R}^d} e^{2\pi i x \cdot u(x)} dx \]

**Definition 2.1** (Spaces of bounded derivatives and moments). Consider a function on phase-space \( f(x, k), s \in \mathbb{N}, p \in [1, \infty] \). The \( \Sigma^{s,p} \) norm will be defined as

\[ \|f\|_{\Sigma^{s,p}} = \sum_{0 \leq |a+b+c+d| \leq s} \|x^a k^b \partial_x^c \partial_k^d f\|_{L^p(\mathbb{R}^d)}. \]

We will also use the standard Sobolev spaces \( \|f\|_{W^{s,p}} = \sum_{0 \leq |c+d| \leq s} \|\partial_x^c \partial_k^d f\|_{L^p}, \|f\|_{H^s} = \|f\|_{W^{s,2}}. \)

One readily checks the following

**Lemma 2.2** (Embeddings of the \( \Sigma^{s,p} \)). By virtue of the Sobolev embeddings,

\[ \forall q, p \in [1, \infty], s \in \mathbb{N} \quad \exists s_0 = s_0(q, p, d) \in \mathbb{N}, \quad C = C(q, p, d) > 0 \quad \|f\|_{\Sigma^{s,p}} \leq C \|f\|_{\Sigma^{s+s_0, q, p}}. \quad (3) \]

\( \text{The spacetime estimates of Lemma 6.4 are central to our result; they can be thought of as adaptations of Proposition 2.3 of [6] to } d = 1, \text{ where the dispersion is weaker. In [6] the authors work for } d = 3 \text{ only.} \)
Moreover, denoting $\mathcal{S}(\mathbb{R}^{2d})$ the Schwarz class of test-functions on phase-space, it follows that for any $q \in [1, \infty]$
\[ \bigcap_{s \in \mathbb{N}} \Sigma^s q(\mathbb{R}^{2d}) = \mathcal{S}(\mathbb{R}^{2d}). \]

Moreover, the spaces $\Sigma^{s,2}$ are closed under Fourier transforms in the sense that
\[ \mathcal{F}_{(x,k) \rightarrow (X,K)} \Sigma^{s,2} = \mathcal{F}_{k \rightarrow K} \Sigma^{s,2} = \mathcal{F}_{x \rightarrow X} \Sigma^{s,2} = \Sigma^{s,2} \]
and similarly for inverse Fourier transforms. Combined with equation (3), this means that
\[ \forall q, p \in [1, \infty], s \in \mathbb{N} \quad \exists q_0 \in \mathbb{N}, C > 0 \quad \text{such that} \quad \|f\|_{\Sigma^{s,q}} \leq C \|\mathcal{F}f\|_{\Sigma^{s+q_0,p}} \]
where $\mathcal{F}$ denotes a forward or inverse Fourier transform in the $x$, $k$, or $(x, k)$ variables.

We will also use the Laplace transform, denoted as $\mathcal{L}_t \mapsto \omega[f] = \int_{t=0}^{\infty} e^{-\omega t} u(t) dt$, and the Hilbert transform $\mathbb{H}$ and the signal transform $\mathbb{S}$
\[ \mathbb{H}[u](x) = \frac{1}{\pi} \text{p.v.} \int_{t \in \mathbb{R}} \frac{u(t)}{x-t} \, dt, \quad \mathbb{S}[u](x) = \mathbb{H}[u](x) - iu(x), \quad (4) \]
respectively.

In the context of the inverse Laplace transform we will also use an alternate “Fourier transform in time”,
\[ \tilde{\mathfrak{F}}_{t \rightarrow s}[u(t)] := \int_{t \in \mathbb{R}} e^{-i s t} u(t) dt, \quad \tilde{\mathfrak{F}}^{-1}_{s \rightarrow t}[v(s)] = \frac{1}{2\pi} \int_{s \in \mathbb{R}} e^{i s t} v(s) ds. \]

Obviously
\[ \|\tilde{\mathfrak{F}}[u]\|_{L^2} = \sqrt{2\pi} \|u\|_{L^2}, \quad \|\tilde{\mathfrak{F}}^{-1}[v]\|_{L^2} = \frac{1}{\sqrt{2\pi}} \|v\|_{L^2}, \quad \tilde{\mathfrak{F}}[tu(t)] = i\partial_s \tilde{\mathfrak{F}}[u]. \]

In the statement and proof of the main results we will also use the following

**Definition 2.3 (D_X P).** For a function $P : \mathbb{R} \rightarrow \mathbb{R}$ we will use the notation
\[ D_X P(k) = \begin{cases} \frac{P(k + \frac{X}{2}) - P(k - \frac{X}{2})}{P'(k)}, & X \neq 0 \\ \frac{X}{P'(k)}, & X = 0. \end{cases} \]

By abuse of notation all constants will be denoted by $C, C', C''$. To keep track of dependence on important parameters we will use e.g. $C = C(t, p, q)$.

### 2.2 Reformulation of the problem and heuristics

To study this problem it is helpful to use equivalent reformulations. If we take the inverse Fourier transform in $x$ of the original Alber equation (1),
\[ f(X, k, t) := \mathcal{F}_{x \rightarrow X}^{-1}[w(x, k, t)] = \int_{x} e^{2\pi i x \cdot X} w(x, k, t) dx, \]
we pass to the Alber-Fourier equation
\[ \partial_t f - 4\pi^2 ipk \cdot X f + q_i \left[ P\left( k - \frac{X}{2} \right) - P\left( k + \frac{X}{2} \right) \right] \tilde{n}(X, t) + \\
+ eq_i \int_s \tilde{n}(s, t) \left[ f\left( X - s, k - \frac{s}{2} \right) - f\left( X - s, k + \frac{s}{2} \right) \right] ds = 0, \]
\[ f(X, k, 0) = f_0(X, k) = \mathcal{F}_{x \rightarrow X}^{-1}[w_0], \quad \tilde{n}(X, t) = \int_{\xi} f(X, \xi, t) d\xi = \mathcal{F}_{x \rightarrow X}^{-1}[n(x, t)]. \]
Landau damping is about the inhomogeneity dispersing as if in free-space, despite the presence of the background. One starts from the linearised problem,

\[ \partial_t f - 4\pi^2 i p k \cdot X f + q_1 \left[ P \left( k - \frac{X}{2} \right) - P \left( k + \frac{X}{2} \right) \right] \tilde{n}(X, t) = 0, \]

\[ f(X, k, 0) = f_0(X, k) = \mathcal{F}^{-1}_{x \to X} [w_0], \quad \tilde{n}(X, t) = \int_{\xi} f(X, \xi, t) d\xi = \mathcal{F}^{-1}_{x \to X} [n(x, t)]. \]  

By recasting in mild form we have

\[ f(X, k, t) - e^{4\pi^2 i p k \cdot X t} f_0(X, k) = -q_1 \int_0^t e^{4\pi^2 i p k \cdot X (t-\tau)} \left[ P \left( k - \frac{X}{2} \right) - P \left( k + \frac{X}{2} \right) \right] \tilde{n}(X, \tau) d\tau = 0, \]

and by integrating in \( k \) we obtain a closed problem in \( \tilde{n}(X, t) \),

\[ \tilde{n}(X, t) - \tilde{n}_f(X, t) = \int_{\tau=0}^t h(X, t - \tau) \tilde{n}(X, \tau) d\tau = 0, \quad h(X, t) = 2q \cdot \sin(2\pi^2 p X^2 t) \tilde{P}(2pXt), \]

where \( n_f(x, t) \) is the known “free-space position density”,

\[ n_f(x, t) := \int_k w_0(x - 2\pi p k t, k) dk \quad \Rightarrow \quad \tilde{n}_f(X, t) = \mathcal{F}^{-1}_{x \to X} [n_f(x, t)] = \int_k U(t) f_0 dk. \]

Denote

\[ \tilde{n}(X, \omega) := \mathcal{L}_{t \to \omega} [\tilde{n}(X, t)], \quad \tilde{n}_f(X, \omega) := \mathcal{L}_{t \to \omega} [\tilde{n}_f(X, t)], \quad \tilde{h}(X, \omega) := \mathcal{L}_{t \to \omega} [h(X, t)]. \]

By taking the Laplace transform of equation (8) and rearranging terms we get

\[ \tilde{n}(X, \omega) = \tilde{n}_f(X, \omega) + \tilde{h}(X, \omega) \tilde{n}(X, \omega) \quad \Rightarrow \quad X\tilde{n}(X, \omega) - X\tilde{n}_f(X, \omega) = \frac{\tilde{h}(X, \omega)}{1 - \tilde{h}(X, \omega)} X\tilde{n}_f(X, \omega). \]

This last equation will be the starting point for the proof of Theorem 3.4 in Section 6 (where the Laplace transforms will also be justified). For now it should clearly motivate the following

**Definition 2.4 (Penrose stability condition).** We will say that a spectrum \( P \in S(\mathbb{R}) \) of compact support is Penrose stable if there is some \( \kappa > 0 \) such that

\[ \inf_{\text{Re}_{\omega} > 0, \ k \in \mathbb{R}} |1 - \tilde{h}(X, \omega)| \geq \kappa > 0. \]

### 3 Main results

**Theorem 3.1 (Local well-posedness in \( L^1 \) for the Alber-Fourier equation).** Let \( f_0 \in L^1(\mathbb{R}^{2d}) \). Then there exists a maximal time \( T = T(\| f_0 \|_{L^1(\mathbb{R}^{2d})}, q, \epsilon, \| P \|_{L^1(\mathbb{R}^{2d})}) > 0 \) such that there exists a unique mild solution \( f(t) \in C([0,T], L^1(\mathbb{R}^{2d})) \) of Eq. (5).

Moreover, the blow-up alternative holds, i.e. either \( T = +\infty \) or \( \lim_{t \to T^-} \| f(t) \|_{L^1(\mathbb{R}^{2d})} = +\infty. \)

The proof can be found in Section 4.1.
Theorem 3.2 (Higher regularity for solutions of the nonlinear problem). Denote \( f(t) \) the solution of Eq. (5) with initial data \( f_0 \in \mathcal{S}(\mathbb{R}^{2d}) \), and \( T = T(|f_0|_{L^1(\mathbb{R}^{2d})}, q, r, \|P\|_{L^1(\mathbb{R}^d)}) \) as in Theorem 3.1. Assume moreover that \( P \in \mathcal{S}(\mathbb{R}^d) \). Then
\[
\forall t \in [0, T), \quad f(t) \in \mathcal{S}(\mathbb{R}^{2d}).
\]
Moreover, \( f \in C^\infty([0, T), \Sigma^{s,1}) \forall s \in \mathbb{N}. \)

Theorem 3.2 is proved in Section 4.2.

Theorem 3.3 (Global well-posedness and exponential bounds in \( \Sigma^{s,1} \) for the linearised problem). Denote \( f(t) \) the solution of the linearised Alber-Fourier equation (6) with initial data \( f_0 \in \mathcal{S}(\mathbb{R}^{2d}) \). Assume moreover \( P \in \mathcal{S}(\mathbb{R}^d) \). Then the maximal time is \( T = +\infty \) for all initial data and for each \( s \in \mathbb{N} \) there exists some \( C = C(s, d, q, P) > 0 \) so that
\[
\|f(t)\|_{\Sigma^{s,1}} \leq \|f_0\|_{\Sigma^{s,1}} Ce^{Ct}.
\]
Moreover, there exist some \( s_2 = s_2(d), C = C(s_2, d, q, P) \) so that
\[
\|\tilde{n}(t)\|_{L^\infty} + \|\tilde{c}_t\tilde{n}(t)\|_{L^\infty} + \|\partial_t f(t)\|_{L^\infty, tk} \leq \|f_0\|_{\Sigma^{s_2,1}} Ce^{Ct}.
\]

The proof can be found in Section 4.2.

Theorem 3.4 (Linear Landau damping for the Alber equation in one dimension). Let \( P \in \mathcal{S}(\mathbb{R}) \) be a background spectrum of compact support which is Penrose-stable in the sense of Definition 2.4. Consider the linearised Alber equation
\[
\partial_t w + 2\pi pk \cdot \partial_x w - q i \int_{\lambda, y \in \mathbb{R}} e^{-2\pi i y} \left[ n(x + y/2, t) - n(x - y/2, t) \right] dy P(k - \lambda) d\lambda = 0,
\]
where \( n(x, t) = \int_{\xi \in \mathbb{R}} w(x, \xi, t) d\xi \) and \( w(x, k, 0) = w_0(x, k) \in \mathcal{S}(\mathbb{R}^2) \).

Then there exists \( r \in \mathbb{N} \) large enough so that the force \( \partial_x n(x, t) \) decays in time in the sense that
\[
\|\partial_x n\|_{L^2_{x,t}} \leq \frac{C}{\kappa} \|w_0\|_{\Sigma^{r,\infty}}.
\]
Furthermore, denoting \( E(t) : w_0(x, k) \mapsto w_0(x - 2\pi pkt, k) \) the free-space propagator, there exists a wave operator \( \mathbb{W} \) so that
\[
\lim_{t \to \infty} \|w(t) - E(t)\mathbb{W}(w_0)\|_{L^\infty(\mathbb{R}^2)} = 0.
\]

The proof is given in Section 6.

Theorem 3.5 (Equivalent formulations of the Penrose-Alber stability condition). Let \( P(k) \in \mathcal{S}(\mathbb{R}) \) be the background spectrum. Assume moreover that \( P \) is of compact support. Then the following statements are equivalent:

1. \( \inf_{\text{Re} \omega > 0, \ X \in \mathbb{R}} |1 - \tilde{h}(X, \omega)| = 0 \), i.e. the spectrum is not Penrose stable in the sense of Definition 2.4.

2. \( \exists \ X_\ast \in \mathbb{R}, \ \Omega_\ast \in \mathbb{C} \setminus \mathbb{R} \) such that \( \mathbb{H}[D_{X_\ast}P](\Omega_\ast) = \mathbb{H}[D_{X_\ast}P](\Omega_\ast) = \frac{4\pi p}{q} \) or
\[
\exists \ X_\ast, \Omega_\ast \in \mathbb{R} \ such \ that \ \mathbb{H}[D_{X_\ast}P](\Omega_\ast) = \frac{4\pi p}{q} \ and \ D_{X_\ast}P(\Omega_\ast) = 0.
\]
(3). \( d(Γ, 4\pi p/q) = 0 \), where
\[
Γ_X := \{ S[D_XP(\cdot)](t), \ t ∈ \mathbb{R} \} \cup \{ 0 \}, \quad Γ_X = \{ z ∈ \mathbb{C} | z \text{ enclosed by } Γ_X \}, \quad Γ := \bigcup_{X ∈ \mathbb{R}}^∞ Γ_X. \tag{20}
\]

Moreover, we have the following sufficient condition for stability: if
\[
∀t_*$ such that $D_XP(t_*) = 0$ the condition $\mathbb{H}[D_XP](t_*) < \frac{4\pi p}{q}$ holds
then $P$ is Penrose stable in the sense of Definition 2.4.

The proof can be found in Section 7.

Remark 3.6 (Penrose stability condition and Alber’s nonlinear eigenvalue relation). In [2] a two-dimensional setup is used, but the spectrum is integrated in the transverse direction, leading to an effective one-dimensional spectrum and a condition on it. This one-dimensional “eigenvalue relation” in our notation and scalings becomes
\[
∃X_⋆ ∈ \mathbb{R}, \ \text{Re}(ω_*) > 0 \quad \text{such that} \quad qi \int k P(k + \frac{X_⋆}{2}) - P(k - \frac{X_⋆}{2}) \frac{ω_⋆}{ω_⋆ - 4\pi^2 p k X_⋆} dk = 1. \tag{21}
\]
If it is satisfied then linear instability follows. To see the relationship between this condition and (2) of Theorem 3.5 above observe that, for $X_⋆ ≠ 0$, $Ω_⋆ := ω_⋆/(4π p i X_⋆)$ equation (21) becomes
\[
∃0 ≠ X_⋆ ∈ \mathbb{R}, \ Ω_⋆ ∈ \mathbb{C} \quad \text{with} \quad \text{sign}(X_⋆) \cdot \text{Im}(Ω_⋆) < 0 \quad \text{such that} \quad \mathbb{H}[D_{X_⋆}P](Ω_⋆) = \frac{4\pi p}{q}.
\]
The form (2) in Theorem 3.5 appropriately takes into account the case $X_⋆ = 0$ as well (equation (21) by construction has no solutions for $X_⋆ = 0$, but stability may still fail due to what could be called renormalised solutions corresponding to $X_⋆ = 0$).

4 Strong solutions for the Alber equation

To simplify notations we can rewrite equation (5) as
\[
\dot{c}t f - 4\pi^2 ip k \cdot X f + \mathbb{B}[m, f] = 0, \quad m(X, t) = \int k f(X, k, t) dk, \quad f(X, k, 0) = f_0(X, k), \tag{22}
\]
where
\[
\mathbb{B}[m, f] = iq \left[ P \left( k - \frac{X}{2} \right) - P \left( k + \frac{X}{2} \right) \right] m(X, t) + \epsilon iq \int s m(s, t) \left[ f \left( X - s, k - \frac{s}{2}, t \right) - f \left( X - s, k + \frac{s}{2}, t \right) \right] ds. \tag{23}
\]

Lemma 4.1 (Bounds on $\mathbb{B}[m, f]$). Let $f, g, h ∈ L^1(\mathbb{R}^{2d}), m ∈ L^1(\mathbb{R}^{d})$ and consider $\mathbb{B}[m, f]$ as defined in equation (23). Then
\[
\| \mathbb{B}[m, f] \|_{L^1(\mathbb{R}^{2d})} ≤ 2|q| \| P \|_{L^1(\mathbb{R}^{d})} \| m \|_{L^1(\mathbb{R}^{2d})} + 2|\epsilon| \| f \|_{L^1(\mathbb{R}^{2d})} \| f \|_{L^1(\mathbb{R}^{2d})}, \tag{24}
\]
and
\[
\| \mathbb{B}[ \int_k f dk, f] \|_{L^1(\mathbb{R}^{2d})} ≤ 2|q| \| P \|_{L^1(\mathbb{R}^{d})} \| f \|_{L^1(\mathbb{R}^{2d})} + 2|\epsilon| \| f \|_{L^1(\mathbb{R}^{2d})}^2. \tag{25}
\]
Moreover,
\[
\| \mathbb{B}[ \int_k g dk, g] - \mathbb{B}[ \int_k hd k, h] \|_{L^1(\mathbb{R}^{2d})} ≤ 2|q| \left( \| P \|_{L^1(\mathbb{R}^{d})} + |\epsilon| \| g \|_{L^1(\mathbb{R}^{2d})} + |\epsilon| \| h \|_{L^1(\mathbb{R}^{2d})} \right) \| g - h \|_{L^1(\mathbb{R}^{2d})}. \tag{26}
\]
Proof: For equation (24) observe that

\[ \|B[m,f]\|_{L^1(\mathbb{R}^{2d})} \leq |q| \left\| \left[ P\left( k - \frac{X}{2} \right) - P\left( k + \frac{X}{2} \right) \right] m(X) \right\|_{L^1_{X,k}} + \\
+ |\epsilon q| \left\| \int_{s} m(s,t) \left[ f\left( X - s, k - \frac{s}{2}, t \right) - f\left( X - s, k + \frac{s}{2}, t \right) \right] ds \right\|_{L^1_{X,k}}. \]

We will treat each term separately. Firstly,

\[ \left\| \left[ P\left( k - \frac{X}{2} \right) - P\left( k + \frac{X}{2} \right) \right] m(X) \right\|_{L^1_{X,k}} = \int_{X,k \in \mathbb{R}^d} \left| P\left( k - \frac{X}{2} \right) - P\left( k + \frac{X}{2} \right) \right| |m(X)| dX dk \]
\[ \leq \int_{X,k \in \mathbb{R}^d} \left| P\left( k - \frac{X}{2} \right) \right| |m(X)| dX dk + \int_{X,k \in \mathbb{R}^d} \left| P\left( k + \frac{X}{2} \right) \right| |m(X)| dX dk \]
\[ = \int_{X,k \in \mathbb{R}^d} \left| P\left( k - \frac{X}{2} \right) \right| dk |m(X)| dX + \int_{X,k \in \mathbb{R}^d} \left| P\left( k + \frac{X}{2} \right) \right| dk |m(X)| dX \]
\[ = 2\|P\|_{L^1(\mathbb{R}^d)} \|m\|_{L^1(\mathbb{R}^d)}. \]

Moreover

\[ \left\| \int_{s \in \mathbb{R}^d} m(s,t) \left[ f\left( X - s, k - \frac{s}{2}, t \right) - f\left( X - s, k + \frac{s}{2}, t \right) \right] ds \right\|_{L^1_{X,k}} = \]
\[ = \int_{X,k,s \in \mathbb{R}^d} |m(s)| \left| f\left( X - s, k - \frac{s}{2} \right) - f\left( X - s, k + \frac{s}{2} \right) \right| ds dX dk \]
\[ \leq \int_{X,k,s \in \mathbb{R}^d} \left| f\left( X - s, k - \frac{s}{2} \right) \right| dX dk |m(s)| ds + \int_{X,k,s \in \mathbb{R}^d} \left| f\left( X - s, k + \frac{s}{2} \right) \right| dX dk |m(s)| ds \]
\[ = 2\|f\|_{L^1(\mathbb{R}^{2d})} \|m\|_{L^1(\mathbb{R}^d)}. \]

Equation (25) follows by virtue of the elementary observation

\[ \left\| \int_{k} fdk \right\|_{L^1(\mathbb{R}^d)} = \int_{X \in \mathbb{R}^d} \left\| \int_{k \in \mathbb{R}^d} f(X,k,t)dk \right\| dX = \int_{X,k \in \mathbb{R}^d} \left| f(X,k,t) \right| dk dX = \|f\|_{L^1(\mathbb{R}^{2d})}. \]

For equation (26) we expand

\[ \mathbb{B}\left[ \int_{k} gdk, g \right] - \mathbb{B}\left[ \int_{k} hdk, h \right] = iq \left[ P\left( k - \frac{X}{2} \right) - P\left( k + \frac{X}{2} \right) \right] \left( \int_{k} g(X,k)dk - \int_{k} h(X,k)dk \right) + \\
+ \epsilon i g \int_{s} g(s,k)dk \left[ g\left( X - s, k - \frac{s}{2} \right) - g\left( X - s, k + \frac{s}{2} \right) \right] ds + \\
+ \epsilon i \int_{s} \left( \int_{k} g(s,k)dk - \int_{k} h(s,k)dk \right) \left[ h\left( X - s, k - \frac{s}{2} \right) - h\left( X - s, k + \frac{s}{2} \right) \right] ds. \]

The result follows by treating each term as before. \[ \square \]
Moreover, consider some \( s \in \mathbb{N} \) and the multi-indices \(|\alpha + \beta + \gamma + \delta| \leq s\). Let us denote

\[
f^{\alpha,\beta,\gamma,\delta} := X^\alpha k^\beta \partial_{k}^\gamma \partial_{k}^\delta f.
\]

By direct computation one obtains

\[
\begin{align*}
\partial_t f^{\alpha,\beta,\gamma,\delta} - 4\pi^2 ipk \cdot X f^{\alpha,\beta,\gamma,\delta} &= B^{(\alpha,\beta,\gamma,\delta)}[f], \\
\alpha,\beta,\gamma,\delta}(X, k, 0) &= f_0^{\alpha,\beta,\gamma,\delta}(X, k) := X^\alpha k^\beta \partial_{k}^\gamma \partial_{k}^\delta f_0(X, k).
\end{align*}
\]

(27)

The detailed expression for \( B^{(\alpha,\beta,\gamma,\delta)}[f] \) can be found in Appendix D, and it contains terms of the form \( f^{\alpha',\beta',\gamma',\delta'} \cdot m(X, t) \), for \( \alpha' + \beta' + \gamma' + \delta' \leq \alpha + \beta + \gamma + \delta \). Furthermore, one can directly – if somewhat tediously – obtain the following

**Lemma 4.2** (Bound on the nonlinearity \( B^{(\alpha,\beta,\gamma,\delta)}[f] \)). Let \( 1 \leq s \in \mathbb{N} \), and consider multi-indices

\[ |\alpha + \beta + \gamma + \delta| \leq s, \]

and \( B^{(\alpha,\beta,\gamma,\delta)}[f] \) as in Appendix D. Assume also that \( P \in S(\mathbb{R}^d) \). Then there exists a \( C = C(s, d, q, P) > 0 \) such that

\[
\|B^{(\alpha,\beta,\gamma,\delta)}[f]\|_{L^1(\mathbb{R}^{2d})} \leq C \left[ 1 + \epsilon \|f\|_{\Sigma_{s-1,1}} \right] \|f\|_{\Sigma_s,1}.
\]

\[ \Box \]

### 4.1 Proof of Theorem 3.1

Denote

\[
U(t) : g(X, k) \mapsto e^{4\pi^2 ipk \cdot X t} g(X, k),
\]

(28)

the free-space propagator, i.e. \( g(t) = U(t)g_0 \) means that \( \partial_t g - 4\pi^2 ipk \cdot X g = 0 \) and \( g(t = 0) = g_0 \). Observe that, by construction, \( \|U(t)g\|_{L^1_{X,k}} = \|g\|_{L^1_{X,k}} \). Equation (22) can now be written in mild form

\[
f(X, k, t) = U(t)f_0 - \int_{\tau=0}^{t} U(t - \tau)B\left[\int_{k} f(\tau)dk, f(\tau)\right]d\tau.
\]

(29)

Define

\[
E := \left\{ g \in L^\infty \left([0, T_0]; L^1(\mathbb{R}^{2d})\right) \mid \|g\|_{L^\infty([0, T_0]; L^1(\mathbb{R}^{2d}))} \leq M \right\}
\]

for some \( M, T_0 > 0 \) (to be determined below), which is a complete metric space with the \( L^\infty \left([0, T_0]; L^1(\mathbb{R}^{2d})\right) \) norm. Moreover denote

\[
G : E \ni g \mapsto U(t)f_0 - \int_{0}^{t} U(t - \tau)B\left[\int_{k} g(\tau)dk, g(\tau)\right]d\tau.
\]

We will show that the operator \( G \) is a strict contraction on \( E \). First we need to show that \( GE \subseteq E \). Direct application of equation (25) from Lemma 4.1 yields

\[
\|Gg\|_{L^\infty L^1} \leq \|f_0\|_{L^1} + T_0 \|B\left[\int_{k} g(\tau)dk, g(\tau)\right]\|_{L^\infty L^1} \leq \|f_0\|_{L^1} + T_0|q| \left(2\|P\|_{L^1} \|g\|_{L^\infty L^1} + 2\epsilon \|g\|_{L^2 L^1}^2\right) \leq \|f_0\|_{L^1} + T_0|q| \left(2\|P\|_{L^1} M + 2\epsilon M^2\right).
\]

A (non-sharp) way to guarantee that \( \|Gg\|_{L^\infty L^1} \leq M \) is to set

\[
M = 2\|f_0\|_{L^1(\mathbb{R}^{2d})} \quad \text{and} \quad T_0 < \frac{1}{|q| \max\{4\|P\|_{L^1(\mathbb{R}^{2d})}, 4\epsilon\}} \left(M + 1\right).
\]

(30)
Now to proceed we will use equation (26) from Lemma 4.1; indeed, for any \( g, h \in \mathbb{E} \)

\[
\|Gg - Gh\|_{L^\infty,L^1} \leq T_0 \left[ B \left( \int k g(\tau) dk, g(\tau) \right) - B \left( \int k h(\tau) dk, h(\tau) \right) \right]_{L^\infty,L^1} \leq 2T_0 |q| \left( \|P\|_{L^1(\mathbb{R}^d)} + |\epsilon| \|g\|_{L^1(\mathbb{R}^{2d})} + |\epsilon| \|h\|_{L^1(\mathbb{R}^{2d})} \right) \leq 2T_0 |q| (\|P\|_{L^1(\mathbb{R}^d)} + 2\epsilon M) \|g - h\|_{L^1(\mathbb{R}^{2d})}.
\]

For \( T_0 \) satisfying (30) the Lipschitz constant \( L \leq T_0 |q| (\|P\|_{L^1} + 2\epsilon M) \) of the mapping is strictly smaller than 1. Therefore, by virtue of the Banach Fixed Point Theorem, there exists a unique fixed point \( f \in \mathbb{E} \), \( f = Gf \), i.e. a unique mild solution of (22) for \( t \in (0, T) \). Observe that by construction \( Gg \) is continuous in time as a mapping with values in \( L^1_{X,k} \).

Since \( \|f(T_0)\|_{L^1(\mathbb{R}^{2d})} < \infty \), we can repeat the argument and extend the solution in time. Thus the blowup alternative follows, i.e. either the solution exists for all times, or there exists a finite blow-up time \( T < \infty \) so that \( \lim_{t \to T^-} \|f(t)\|_{L^1_{X,k}} = +\infty \). Whether \( T \) is finite or infinite, it will be called the **maximal time** for which \( f(X,k,t) \) exists.

To show continuous dependence of solutions of (22) on initial data we consider \( f(X,k,t) \) as above and \( g(X,k,t) \) being a solution of (22) with initial data \( g_0(X,k) \). Take some \( T_1 \) smaller than both the maximal times of \( f \) and \( g \); then there exists some \( M_1 \) so that

\[
\|f(t)\|_{L^\infty([0,T_1],L^1)} \leq M_1. \]

Now denote \( h := f - g \); by subtracting the equations for \( f \) and \( g \) and using the same ideas as above, it follows that for all \( t \in [0, T_1] \)

\[
\|h(t)\|_{L^1} \leq \|f_0 - g_0\|_{L^1} + 2 \int_0^t \|h(\tau)\|_{L^1} (\|P\|_{L^1} + \epsilon (\|f(\tau)\|_{L^1} + \|g(\tau)\|_{L^1})) d\tau \leq \|f_0 - g_0\|_{L^1} + 2(\|P\|_{L^1} + 2\epsilon M_1) \int_0^t h(\tau) d\tau \Rightarrow \]

\[
\|h(t)\|_{L^1} \leq \|f_0 - g_0\|_{L^1} \left( 1 + t^2(\|P\|_{L^1} + 2\epsilon M_1) e^{t^2(\|P\|_{L^1} + 2\epsilon M_1)} \right) \forall t \in [0, T_1],
\]

where in the last step we used the Gronwall inequality. 

\[ \square \]

### 4.2 Propagation of regularity and Proof of Theorems 3.2, 3.3

**Theorem 4.3** (Local well-posedness for the nonlinear Alber-Fourier-I equation on \( \Sigma^{s,1} \)). *Denote \( f(X,k,t) \) the solution of (22) with initial data \( f_0(X,k) \in \Sigma^{s,1} \), \( T = T(\|f_0\|_{L^1}, q, \epsilon, \|P\|_{L^1}) \) the maximal time for which \( f(\tau) \in L^1_{X,k} \), and \( M^0(t) := \|f(\tau\|_{L^1_{X,k}} \in C([0,T]) \). Moreover, for each \( 1 \leq s \leq a_0 \) denote \( M^s(t) := \|f(t)\|_{\Sigma^{s,1}} \). Then there exist constants \( C > 0 \) depending on \( s, d, q, \epsilon, P \) and the background spectrum \( P \) such that

\[
M^s(t) \leq M^s(0) + C(s) \int_0^t M^{s-1}(\tau) M^s(\tau) d\tau \forall t \in [0,T),
\]

and therefore, for all \( s \in \mathbb{N} \),

\[
M^s(t) < \infty \forall t \in [0,T), \quad f(t) \in C([0,T), \Sigma^{s,1}).
\]
Proof: Consider multi-indices $|\alpha + \beta + \gamma + \delta| \leq s$; as was seen earlier, $f^{\alpha,\beta,\gamma,\delta} := X^\alpha k^\beta \hat{c}_X \hat{c}_k f$ satisfies equation (27). By passing to mild form we have

$$f^{\alpha,\beta,\gamma,\delta}(t) = U(t)f_0^{\alpha,\beta,\gamma,\delta} + \int_{\tau=0}^{t} U(t-\tau)\mathbb{B}(t) \mathbb{B}(\alpha,\beta,\gamma,\delta)[f(\tau)]d\tau.$$\

Taking $L^1$ norms and using Lemmata 4.1, 4.2 we have

$$\|f^{\alpha,\beta,\gamma,\delta}(t)\|_{L^1} \leq \|f_0^{\alpha,\beta,\gamma,\delta}\|_{L^1} + C \int_{\tau=0}^{t} \|f(\tau)\|_{\Sigma^{s,1}}d\tau.$$\

Equation (32) follows by summing over all $|\alpha + \beta + \gamma + \delta| \leq s$. The first part of equation (33) follows by applying recursively Gronwall’s inequality to equation (32). The second part of equation (33) follows automatically from the mild form since the time integrals now are known to exist.

Proof of Theorem 3.2: For the proof of Eq. (13) it suffices to observe that the $\bigcap_{s \in \mathbb{N}} \Sigma^{s,1}(\mathbb{R}^{2d})$ regularity is propagated in time by virtue of Theorem 4.3, and that it implies Schwarz-class regularity by virtue of Lemma 2.2.

For the proof of smoothness with respect to the time variable stated in (14), observe that upon applying the operator $\hat{c}_t$ to Eq. (22), one obtains the problem

$$\hat{c}_t(\hat{c}_t f) - 4\pi^2 i p k \cdot X(\hat{c}_t f) + \mathbb{B}[m, \hat{c}_t f] = \mathbb{B}(t)[f], \quad m(X,t) = \sum_k f(X,k,t)dk,$$

where

$$\mathbb{B}(t)[f] = -\epsilon i q \sum_{0 \leq l < l'} \left(\frac{t}{l'}\right)_{\delta} \left[\hat{c}_t^{l-l'} m(s,t) \hat{c}_t^{l'} f\left(X - s, k - \frac{s}{2}, t\right) - \hat{c}_t^{l'} f\left(X - s, k + \frac{s}{2}, t\right)\right]ds,$$

and

$$\mathbb{B}(0)[f] = 0.$$\

By working recursively in $l$ as in the proof of Theorem 4.3, the result follows.

Proof of Theorem 3.3: We start by recasting equation (27) in mild form and taking the $L^1$ norm, with the help of Lemma 4.2. Using the fact that $\epsilon = 0$, we obtain

$$\|f^{\alpha,\beta,\gamma,\delta}(t)\|_{L^1} \leq \|f_0^{\alpha,\beta,\gamma,\delta}\|_{L^1} + C \int_{\tau=0}^{t} \|f(\tau)\|_{\Sigma^{s,1}}d\tau.$$\

Summing over all $|\alpha + \beta + \gamma + \delta| \leq s$ yields

$$\|f(t)\|_{\Sigma^{s,1}} \leq \|f_0\|_{\Sigma^{s,1}} + C \int_{\tau=0}^{t} \|f(\tau)\|_{\Sigma^{s,1}}d\tau.$$\

Equation (15) follows by Gronwall’s inequality.

To proceed we will keep in mind that, by virtue of Lemma 2.2, for any $r$ and for $s'$ large enough we have $\|f(t)\|_{\Sigma^{*,r}} \leq C \|f(t)\|_{\Sigma^{s',1}} \leq Ce^{Ct} \|f_0\|_{\Sigma^{s',1}}$.

Now for the position density observe that

$$\|\hat{m}(t)\|_{L^\infty} \leq \sup_{X \in \mathbb{R}^d} \left|\int f(X,\xi,t)d\xi\right| \leq \int_{\mathbb{R}^d} \left|\sup_{X \in \mathbb{R}^d} \left|\frac{d}{d\xi}(f(X,\xi,t))\right|\right|_{\mathbb{R}^d} \sup_{X,\xi} |(1 + |\xi|^{d+1})f(X,\xi)| \leq C \|f(t)\|_{\Sigma^{s+1,\infty}} \leq C'e^{C't} \|f_0\|_{\Sigma^{s',1}}.$$ (34)
Moreover, going back to equation (6) we observe that
\[
\hat{c}_t f = 4\pi^2 ik \cdot X f - iq \left[ P\left(k - \frac{X}{2}\right) - P\left(k + \frac{X}{2}\right) \right] \int f(X, t) d\xi \Rightarrow \\
\Rightarrow \|\hat{c}_t f(t)\|_{L^2_{\xi, k}} \leq \|k \cdot X f(t)\|_{L^2_{\xi, k}} + C\|\hat{n}(t)\|_{L^2_{\xi}} \leq C\|f(t)\|_{\Sigma^{r, 1}}
\]
for some \(s_1 \in \mathbb{N}\) large enough. Similarly
\[
\hat{\tilde{n}} = \int \hat{c}_t f(X, k, t) dk = 4\pi^2 ip \left[ k \cdot X f dk - iq \left[ P\left(k - \frac{X}{2}\right) - P\left(k + \frac{X}{2}\right) \right] \right] d\xi \int f(X, t, t) d\xi \Rightarrow \\
\Rightarrow \|\hat{\tilde{n}}(t)\|_{L^2_{\xi}} \leq C\|k\|^{d+2}\|X f(t)\|_{L^2_{\xi, k}} + C\|\tilde{n}(t)\|_{L^2_{\xi}} \leq C\|f(t)\|_{\Sigma^{r', 1}}
\]
for some \(s'_1 \in \mathbb{N}\) large enough. Thus equation (16) follows by selecting \(s_2 = \max\{s', s_1, s'_1\}\).

5 The free-space position density

In this section we will establish some properties of the free-space position density that we will need later on, in the proof of Theorem 3.4. Recall \(n_f(x, t)\) as it was defined in (9).

**Lemma 5.1** (Alternative expression for \(\tilde{n}_f\)).
\[
\tilde{n}_f(X, t) := F_{x \rightarrow X}^{-1}[n_f(x, t)] = \tilde{w}_0(X, 2\pi pt X),
\]
where \(\tilde{w}_0(A, B) = F_{(x, k) \rightarrow (A, B)}^{-1}[w_0(x, k)]\).

**Proof:** Indeed, one readily checks
\[
\tilde{n}_f(X, t) = \int e^{2\pi i x \cdot X} n_f(x, t) dx = \int e^{2\pi i x \cdot X} w_0(x - 2\pi pt k, dk dx = \\
= \int e^{2\pi i x \cdot X} w_0(A, B) dk dx AdB = \\
= \int e^{2\pi i x \cdot X} w_0(A, B) dk dx AdB = \tilde{w}_0(X, 2\pi pt X).
\]

**Lemma 5.2** (Uniform bound for \(X\tilde{n}_f\)). Assume that there exists some \(D_r > 0\) such that
\[
|\tilde{w}_0(X, K)| \leq \frac{D}{1 + |X|^2 + |K|^2}
\]
and denote \(\tilde{n}_f(X, \omega)\) as in equation (10). Then, there exists a constant \(C > 0\) such that for all \(X \in \mathbb{R}\)
\[
\sup_{\text{Re} \omega > 0} |X\tilde{n}_f(X, \omega)| \leq C \cdot D.
\]

**Proof:** Using Lemma 5.1 one readily checks that
\[
\sup_{\text{Re} \omega > 0} |X\tilde{n}_f(X, \omega)| = \sup_{t = 0}^{\infty} \int e^{-\omega t} X\tilde{n}_f(X, t) dt \leq \int_{t = 0}^{\infty} |X\tilde{n}_f(X, t)| dt = \\
= \int_{t = 0}^{\infty} |X\tilde{w}_0(X, 2\pi pt X)| dt \leq \int_{t = 0}^{\infty} |\tilde{w}_0|_{\Sigma^{r, \infty}} dt \leq \frac{D}{1 + |2\pi pt|^2} dt.
\]

**Observation 5.3.** We will use assumptions of the form
\[
|\tilde{w}_0(X, K)| \leq \frac{D_r}{1 + |X|^r + |K|^r}
\]
in the sequel, which are weaker versions of \(\tilde{w}_0 \in \Sigma^{r, \infty}\). By virtue of Lemma 2.2 it follows that, for some \(r'\) large enough
\[
D_r \leq \|\tilde{w}_0\|_{\Sigma^{r, \infty}} \leq C\|w_0\|_{\Sigma^{r', \infty}}.
\]
Lemma 5.4 (Space-time $L^2$ estimates for the free-space position density). Let
\[
|\tilde{w}_0(X,K)| \leq \frac{D_r}{1+|X|^r + |K|^r}
\]
for some large enough $r$. Assume moreover $r - \frac{1}{2} > a > b \geq 0$ ($a,b,r$ don’t have to be integer.) Then
\[
\|X^a t^b \tilde{w}_f(X,t)\|_{L^2_{X,t}} \leq C(a,b) D_r.
\]

Proof:
\[
\|X^a t^b \tilde{w}_f(X,t)\|_{L^2_{X,t}}^2 = \int_{t,X} |X^a t^b \tilde{w}_0(X,2\pi ptX)|^2 dX dt = I_1 + I_2 + I_3 + I_4 + I_5 + I_6 =
\]
\[
= \int_{|X|<1,0<|t|<1} + \int_{|X|<1,1<|t|<1/|X|} + \int_{|X|>1,0<|t|<1/|X|} + \int_{|X|>1,1<|t|} + \int_{1/|X|<|t|<1} + \int_{|X|>1,|X|<|t|<1}
\]
see also Figure 1. One readily observes that

Figure 1: The domains of integration for the integrals $I_j$, $j = 1, \ldots, 6$.

\[
I_1 = \int_{|X|<1,|t|<1} |X^a t^b \tilde{w}_0(X,2\pi ptX)|^2 dX dt \leq CD_r^2,
\]
\[
I_2 = \int_{|X|<1,1<|t|<1/|X|} |X^a t^b \tilde{w}_0(X,2\pi ptX)|^2 dtdX \leq D_r^2 C \int_{|X|<1} |X|^{2a} \int_{t=1}^{1/|X|} t^{2b} dtdX \leq
\]
\[
\leq D_r^2 C \int_{|X|<1} |X|^{2a-2b-1} dX \leq D_r^2 C.
\]
\[
I_3 = \int_{|X|>1} \int_{t=0}^{1/|X|} |X^a t^b \tilde{w}_0(X,2\pi ptX)|^2 dtdX \leq D_r^2 \int_{|X|>1} \int_{t=0}^{1/|X|} |X|^{2a} \frac{2}{|X|^2r} dtdX = CD_r^2 \int_{1}^{\infty} X^{2(a-r)} dX \leq CD_r^2.
\]
where, by using the elementary observation that \( \frac{x^{2a}t^{2b}}{(1+x^r+(xt)^r)^2} \leq \frac{x^{2a}t^{2b}}{(xt)^{2r}} = x^{2(a-r)}t^{2(b-r)} \) for \( t \geq 0, x \geq 0 \), we have

\[
I_4 = \int_{t=1}^{\infty} \int_{|X| \geq 1} |X^{a+b} \tilde{w}_1n(X, 2\pi ptX)|^2 dtdX 
\leq CD_r^2 \int_{t=1}^{\infty} \int_{|X| \geq 1} |X|^{2(a-r)}t^{2(b-r)} dtdX \leq CD_r^2.
\]

\[
I_5 = \int_{t=1}^{\infty} \int_{|X| = 1/t} |X^{a+b} \tilde{w}_1n(X, 2\pi ptX)|^2 dtdX 
\leq CD_r^2 \int_{t=1}^{\infty} \int_{|X| = 1/t} |X|^{2(a-r)}t^{2(b-r)} dX dt =
\]

\[
= CD_r^2 \int_{t=1}^{\infty} t^{2(b-r)} \int_{|X| = 1/t} |X|^{2(a-r)} dX dt 
\leq C'D_r^2 \int_{t=1}^{\infty} t^{2(b-a-1)} dt + \int_{t=1}^{\infty} t^{2(b-r)} dt 
\leq CD_r^2.
\]

Finally, by using the elementary observation that \( \frac{x^{2a}t^{2b}}{(1+x^r+(xt)^r)^2} \leq \frac{x^{2a}t^{2b}}{x^{2r}} = x^{2(a-r)}t^{2b} \) we have

\[
I_6 = \int_{|X| = 1} \int_{t=1/|X|} |X^{a+b} \tilde{w}_1n(X, 2\pi ptX)|^2 dtdX 
\leq CD_r^2 \int_{|X| = 1} \int_{t=0} t^{2b} |X|^{2(a-r)} dt dX =
\]

\[
= CD_r^2 \int_{|X| = 1} |X|^{2(a-r)} dX \int_{t=0}^{\infty} t^{2b} dt 
\leq C'D_r^2.
\]

We will see that the position density for the linearised problem inherits these estimates in the stable case.

6 Proof of Theorem 3.4

6.1 The Laplace transform picture

Theorem 3.3 automatically implies that the Laplace transforms \( \tilde{n}(X, \omega) \), is well-defined and analytic for \( \text{Re}(\omega) \) large enough. Moreover we are justified in using Fubini to the effect that \( \tilde{n} = \mathcal{L}[\int f dk] = \int \mathcal{L}[f] dk \), for \( \text{Re}(\omega) \) large enough. The same follows for \( \tilde{n}_f(X, \omega) \) by setting \( P(k) = 0 \).

Thus if we first take the Laplace transform of equation (6),

\[
\omega \tilde{f} - f_0(X, k) - 4\pi^2 i pk \cdot X \tilde{f} + q \left[ P \left( k - \frac{X}{2} \right) - P \left( k + \frac{X}{2} \right) \right] \tilde{n}(X, \omega) = 0,
\]

re-arrange terms

\[
f(X, k, \omega) = \frac{f_0(X, k) - q i \left[ P \left( k - \frac{X}{2} \right) - P \left( k + \frac{X}{2} \right) \right] \tilde{n}(X, \omega)}{\omega - 4\pi^2 i pk \cdot X},
\]

and integrate in \( k \) and re-arrange terms once more, we obtain

\[
\tilde{n}(X, \omega) = \int_k \frac{f_0(X, k)}{\omega - 4\pi^2 i pk \cdot X} dk - q i \int_k \frac{P \left( k - \frac{X}{2} \right) - P \left( k + \frac{X}{2} \right)}{\omega - 4\pi^2 i pk \cdot X} dk \cdot \tilde{n}(X, \omega).
\]

This is exactly the first expression in equation (11), and several things follow from this alternative derivation: first of all, for \( X \neq 0 \) and for \( d = 1 \)

\[
\tilde{n}_f(X, \omega) = \int_k \frac{f_0(X, k)}{\omega - 4\pi^2 i pk \cdot X} dk = \frac{1}{4\pi i p X} \mathbb{H} \left[ f_0(X, \cdot) \right] \left( \frac{\omega}{4\pi^2 i p X} \right).
\]
Moreover, we have
\[
\tilde{h}(X, \omega) = q i \int_k \frac{P(k + \frac{X}{2}) - P(k - \frac{X}{2})}{\omega - 4\pi^2 ip X} dk = \frac{q}{4\pi p} \mathbb{H}[D_X P(\cdot)](\frac{\omega}{4\pi^2 p X})
\]
where the second equality again only holds for for $X \neq 0$.

**Observation 6.1** (Case $X = 0$). For $X = 0$ we have $\tilde{h}(0, \omega) = 0$, and $\tilde{n}_f(0, \omega) = \frac{1}{2} \int_k f_0(0, k) dk$, which is of course consistent with Lemma 5.1 and its consequence $\tilde{n}_f(0, t) = \tilde{n}_0(0, 0)$. Thus it follows that $\tilde{n}(0, t) = \tilde{n}_f(0, t) = \tilde{n}_0(0, 0)$ for all $t$. Without loss of generality we will work for $X \neq 0$ from now on.

**Observation 6.2** (Domain of analyticity & Sokhotski-Plemelj). From the above explicit expressions it follows that, for each $X \in \mathbb{R}$, the Laplace transforms $\tilde{h}(X, \omega)$, $\tilde{n}(X, \omega)$, $\tilde{n}_f(X, \omega)$ are analytic in $\omega$ for all $\text{Re}(\omega) > 0$.

Moreover, for $X \neq 0$, we have
\[
\tilde{H}(X, s) := \lim_{\eta \to 0} \tilde{h}(X, \eta + is) = \lim_{\eta \to 0} \frac{q}{4\pi p} \mathbb{H}[D_X P(\cdot)](\frac{\eta + is}{4\pi^2 p X}) = \frac{q}{4\pi p} \mathbb{S}[D_X P](\frac{s}{4\pi^2 p X})
\]
and
\[
\tilde{N}_f(X, s) := \lim_{\eta \to 0} \tilde{n}_f(X, \eta + is) = \lim_{\eta \to 0} \frac{1}{4\pi ip X} \mathbb{S}[f_0(0, \cdot)](\frac{\eta + is}{4\pi^2 p X}) = \frac{1}{4\pi ip X} \mathbb{S}[f_0(0, \cdot)](\frac{s}{4\pi^2 p X})
\]
by virtue of the Sokhotski-Plemelj formula, cf. Theorem B.2. Moreover, observe that
\[
|1 - \tilde{h}(X, \omega)| \geq \kappa \quad \forall X \in \mathbb{R}, \text{Re}(\omega) > 0 \quad \Rightarrow \quad |1 - \tilde{H}(X, s)| \geq \kappa \quad \forall X, s \in \mathbb{R}.
\]

### 6.2 Inverting the Laplace Transform

Recalling equation (11), we set
\[
\tilde{I}(X, \omega) := \frac{\tilde{h}(X, \omega)}{1 - \tilde{h}(X, \omega)} X \tilde{n}_f(X, \omega);
\]
then
\[
X \tilde{n}(X, t) - X \tilde{n}_f(X, t) = \mathcal{L}_{\omega \to s}[\tilde{I}(X, \omega)].
\]
As we saw above, $\tilde{I}(X, \omega)$ is analytic on the open half-plane $\{\text{Re}(\omega) > 0\}$; we will thus use Theorem B.3 to invert the Laplace transform for each $0 \neq X \in \mathbb{R}$. To that end, we will need to check that its assumptions are satisfied. First of all, equations (39) and (40) yield the existence of
\[
I(X, s) := \lim_{\eta \to 0^+} \tilde{I}(X, \eta + is) = \frac{\tilde{H}(X, s)}{1 - \tilde{H}(X, s)} X \tilde{n}_f(X, s).
\]
By inspection it follows that $I(X, s)$ is continuous in $s$. To show that it is $L^1_s$ observe that
\[
\int_s |I(X, s)| ds \leq \frac{1}{\kappa} \cdot \sup_{X \tilde{n}_f(X, s')} |X \tilde{n}_f(X, s')| \cdot \int_s |\tilde{H}(X, s)| ds \leq C
\]
where we used equation (41), Lemma 5.2 for $X \tilde{n}_f$, and Theorem 6.7 for $\tilde{H}$ (observe in particular that, by construction, $D_X P$ is a function of compact support with integral $\int_k D_X P(k) dk = 0$ for all $0 \neq X \in \mathbb{R}$, hence Theorem 6.7 indeed applies). Moreover,
\[
\lim_{\rho \to \infty} \sup_{\text{Re}(\omega) > 0} |\tilde{I}(X, \omega)| \leq \sup_{\text{Re}(\omega') > 0} |X \tilde{n}_f(X, \omega')| \cdot \lim_{\rho \to \infty} \sup_{|\omega| > \rho} |\tilde{H}(X, \omega)| = C \cdot 0
\]
15
Lemma 6.4. Let \( \omega \) will use equation (42) to prove the following

\[
X \tilde{n}(X,t) - X \tilde{n}_f(X,t) = \int_{-\infty}^{\infty} e^{ist} \frac{\tilde{H}(X,s)}{1 - \tilde{H}(X,s)} X \tilde{N}_f(X,s) ds. \tag{42}
\]

Remark 6.3. If one tries to use Theorem B.3 directly on \( X \tilde{n}(X,\omega) \) then the only way to guarantee the \( L^1 \) requirement of equation (60) seems to be requiring \( \int_{k \in \mathbb{R}} f_0(X,k) dk = 0 \) for all \( X \in \mathbb{R} \). Here instead we only require that the difference \( X \tilde{n}(X,\omega) - X \tilde{n}_f(X,\omega) \) has an \( L^1_s \) limit for all \( X \in \mathbb{R} \) as \( \omega = s + i\eta, \eta \to 0 \), avoiding any extra assumptions on the initial data.

6.3 Establishing space-time estimates for the force

We will use equation (42) to prove the following

Lemma 6.4. Let \( a > 1, b > 0 \), and moreover recall that, since \( f_0 \in \mathcal{S}(\mathbb{R}^2) \),

\[
|\tilde{w}_0(X,K)| \leq \frac{\|w_0\|_{\Sigma',\infty}}{1 + |X|^r + |K|^r}
\]

for any \( r \), in particular for \( r \geq \max\{a + \frac{1}{2}, b + \frac{1}{2}\} \). Then there exists a \( C = C(a,b,P) \) so that

\[
\|t|X|^a \tilde{n}\|_{L^2_{X,t}} + \|X^b \tilde{n}\|_{L^2_{X,t}} \leq \frac{C(a,b,P)D_r}{\kappa}
\]

for all \( 0 \neq X \in \mathbb{R} \). Note that, by virtue of Observation 5.3, \( D_r \leq C\|w_0\|_{\Sigma',\infty} \) for some \( r' \) large enough.

Proof: First we will bound \( X^a t^b \tilde{n} \) norms from \( X^a t^b \tilde{n}_f \) norms. Using the alternate Fourier transform \( \mathfrak{F} \) introduced in Section 2.1, we have

\[
(42) \quad \Rightarrow \quad X \tilde{n}(X,t) - X \tilde{n}_f(X,t) = \mathfrak{F}^{-1}_{s \to t} \left[ \frac{X \tilde{N}_f(X,s) \tilde{H}(X,s)}{1 - \tilde{H}(X,s)} \right] \quad \Rightarrow
\]

\[
\|X^b \tilde{n} - X^b \tilde{n}_f\|_{L^2_{X,t}} = \|\mathfrak{F}^{-1}_{s \to t} \left[ \frac{X^b \tilde{N}_f(X,s) \tilde{H}(X,s)}{1 - \tilde{H}(X,s)} \right]\|_{L^2_{X,t}} = C \|X^b \tilde{N}_f(X,s) \tilde{H}(X,s)\|_{L^2_{X,t}} \leq
\]

\[
C \left( \sup_{X,s} \left| \frac{1}{1 - \tilde{H}(X,s)} \right| \right) \cdot \left( \sup_{X,s} |\tilde{H}(X,s)| \right) \cdot \|X^b \tilde{N}_f\|_{L^2_{X,t}}
\]

For the first factor we use equation (41). For the second factor observe that, by virtue of Theorem B.1, we have

\[
\sup_{X,s} |\tilde{H}(X,s)| = \frac{q}{4\pi p} \sup_{\zeta,t} |\mathbb{S}(D\zeta P)(t)| \leq C \sup_{\zeta} \|\mathbb{S}(D\zeta P)\|_{H^1} \leq C' \sup_{\zeta} \|D\zeta P\|_{H^1} \leq C''
\]

so finally

\[
\|X^b \tilde{n}\|_{L^2_{X,t}} \leq C \|X^b \tilde{N}_f\|_{L^2_{X,t}} = C' \|X^b \tilde{n}_f\|_{L^2_{X,t}} \tag{43}
\]

since \( \tilde{n}_f = \mathfrak{F}^{-1}_{s \to t} \tilde{N}_f \).
Working similarly we have

\[ (42) \Rightarrow X^a t(\tilde{n}(X, t) - \tilde{n}_f(X, t)) = i \int_{-\infty}^{\infty} e^{ist} \partial_s X^a \tilde{N}_f(X, s) \tilde{H}(X, s)\, ds \Rightarrow \]

\[ \Rightarrow \| X^a t(\tilde{n}(X, t) - \tilde{n}_f(X, t)) \|_{L^2_{X,t}} = C \| \partial_s X^a \tilde{N}_f(X, s) \tilde{H}(X, s) \|_{L^2_{X,s}} \leq \]

\[ \leq C \| \partial_s X^a \tilde{N}_f \|_{L^2_{X,s}} \left( \sup_{X,s} \| \tilde{H}(X, s) \|_{1-H(X, s)} \right) + C \left( \int_{X,s} \| X^a \tilde{N}_f(X, s) \|_{2}^2 \left( \sup_{s'} \| \partial_s \tilde{H}(X, s') \|_{1-H(X, s')} \right)^2 \, ds dX \right)^{1/2} \leq \]

\[ \leq \frac{C}{\kappa} \| t X^a \tilde{n}_f \|_{L^2_{X,t}} + \frac{C}{\kappa} \left( \int_{X,s} \| X^a \tilde{N}_f(X, s) \|_{2}^2 \left( \sup_{s'} \| \partial_s \tilde{H}(X, s') \|_{1-H(X, s')} \right)^2 \, ds dX \right)^{1/2} \leq \]

\[ = \frac{C}{\kappa} \| t X^a \tilde{n}_f \|_{L^2_{X,t}} + \frac{C}{\kappa} \left( \int_{X,s} \| X^a \tilde{N}_f(X, s) \|_{2}^2 \left( \sup_{s'} \| S[D_X P'] \left( \frac{s'}{4\kappa^2 pX} \right) \right)^2 ds dX \right)^{1/2} \leq \]

\[ \leq \frac{C}{\kappa} \| t X^a \tilde{n}_f \|_{L^2_{X,t}} + \frac{C}{\kappa} \left( \sup_{s'} \| S[D_X P'] \left( \frac{s'}{4\kappa^2 pX} \right) \right) \| X^a \tilde{n}_f \|_{L^2_{X,t}}. \]

Obviously

\[ \sup_{\zeta,t} \| S[D_X P'] \left( \frac{s'}{4\kappa^2 pX} \right) \| \leq \sup_{\zeta,t} \| S[D_X P'] \|_{H1} \leq C, \]

and therefore

\[ \| t X^a \tilde{n}_f \|_{L^2_{X,t}} \leq \frac{C}{\kappa} \left( \| t X^a \tilde{n}_f \|_{L^2_{X,t}} + \| X^a \tilde{n}_f \|_{L^2_{X,t}} \right) \]

\[ (44) \]

Combining equations (43) and (44) with Lemma 5.4 the result follows.

Equation (18) follows directly from Lemma 6.4.

### 6.4 Using the space-time estimates to construct the wave operator

Equation (7) implies

\[ e^{-4\pi^2 ipk \cdot X t} f(X, k, t) - f_0(X, k) = J(X, k, t), \]

\[ J(X, k, t) := qi \int_{\tau=0}^{t} e^{-4\pi^2 ipk \cdot X \tau} \frac{P\left(k + \frac{\tau}{2}\right) - P\left(k - \frac{\tau}{2}\right)}{X} X \tilde{n}(X, \tau) d\tau \]

(45)

Now for any \( 0 < \theta < 1/2, \gamma > 1, \)

\[ \int_{X \in \mathbb{R}} |J(k, X, t)| \, dX \leq C \sup_{\zeta,t} |D_\zeta P'(s)| \int_{R=0}^{+\infty} \int_{X=0}^{+\infty} \frac{\sqrt{\lambda + ((X^{\theta}\tau)^2 + |X|^{2\gamma})}}{\sqrt{\lambda + |X|^{2\gamma}}} X |\tilde{n}(X, \tau)|^2 d\tau dX \leq \]

\[ \leq C'' \int_{X=0}^{+\infty} \int_{\tau=0}^{t} \frac{1 + ((X^{\theta}\tau)^2 + |X|^{2\gamma})}{1 + ((X^{\theta}\tau)^2 + |X|^{2\gamma})} X |\tilde{n}(X, \tau)|^2 d\tau dX \]

(46)

by virtue of the Cauchy-Schwarz inequality.

The first factor is estimated by

\[ \left( \int_{X=0}^{+\infty} \int_{\tau=0}^{t} \left[ 1 + ((X^{\theta}\tau)^2 + |X|^{2\gamma}) \right] X^2 |\tilde{n}(X, \tau)|^2 d\tau dX \leq \|(1 + |X|^{\theta}\tau + |X|^{\gamma}) X \tilde{n}\|_{L^2_{X,t}} \leq \| X \tilde{n}\|_{L^2_{X,t}} + \|t\|_{X}^{1+\theta} \tilde{n}\|_{L^2_{X,t}} + \|X\|^{1+\gamma} \tilde{n}\|_{L^2_{X,t}} \leq C(\theta, \gamma, P) \|w_0\|_{Y_\gamma'.X} \]

(47)
for some \( r' \) large enough by virtue of Lemma 6.4.

For the other factor we break the integral up over the contributions from different regions,

\[
\int_0^{+\infty} \int_0^t \frac{1}{1 + (|X|^\theta \tau)^2 + |X|^{2\gamma}} d\tau dX = I_1 + I_2 + I_3 + I_4 + I_5 + I_6.
\]

For simplicity we use the same breakdown as in the proof of Lemma 5.4, cf. Figure 1 in the Appendix. Without loss of generality we assume \( t > 1 \).

\[
I_1 \leq \int_0^1 \int_0^1 d\tau dX = 1.
\]

Moreover,

\[
I_2 \leq \int_0^{\infty} \int_0^{1/t} \frac{1}{x^2 \tau^2} d\tau dx = \int_0^{\infty} \tau^{-2} \int_0^{1/t} x^{-2\theta} d\tau dx = C \int_0^{\infty} \tau^{-2+2\theta} d\tau = C(1 + t^{-2+2\theta}) \leq C'.
\]

Here we used \(-2\theta > -1 \iff \theta < 1/2\) for the \( x \)--integral to exist; and \(-3+2\theta < -1 \iff \theta < 1\) for the \( \tau \)--integral to exist. Moreover

\[
I_3 \leq \int_1^{\infty} \int_0^{1/x} x^{-2\gamma} dx dt = \int_1^{\infty} x^{-2\gamma-1} dx = C,
\]

where we used \(-2\gamma - 1 < -1 \iff \gamma > 0\). For \( I_4 \) we use Lemma 6.5 with \( \zeta = 3/4 \) to the effect that

\[
\frac{1}{(x^\theta \tau)^2 + x^{2\gamma}} \leq \frac{1}{(x^\theta \tau)^{2/3} x^{2/3}} = \tau^{-\frac{2}{3} x^{-\frac{2}{3} - \frac{2}{3} \theta}}.
\]

Thus

\[
I_4 \leq C \int_1^{\infty} \int_1^{t} \tau^{-\frac{2}{3} x^{-\frac{2}{3} - \frac{2}{3} \theta}} d\tau dx = C \left( \tau^{-\frac{1}{2}} \right) \left( x^{1-\frac{2}{3} - \frac{2}{3} \theta} \right)^{\infty} = C'(1 + t^{-\frac{1}{2}})
\]

where we used the fact that, by assumption, \( \gamma/2 + 3\theta/2 > 5/4 > 1 \). Moving on,

\[
I_5 \leq C \int_0^{\infty} \int_0^{1/x} x^{2\theta} dx dt = C \int_0^{\infty} x^{-2\theta} \int_0^{1/x} \tau^{-2} d\tau dx = C \int_0^{\infty} x^{-2\theta} \left( \tau^{-1} \right)^{\infty} dx = C \int_0^{\infty} x^{1-2\theta} dx = C'
\]

since \( 1 - 2\theta > -1 \iff \theta < 1/2 \). Finally,

\[
I_6 \leq C \int_1^{\infty} \int_1^{1/x} x^{-2\gamma} dx d\tau \leq \int_1^{\infty} x^{-2\gamma-1} dx \leq C.
\]

So we showed that

\[
\int_{X \in \mathbb{R}} |J(k, X, t)| dX \leq C \|w_0\|_{\Sigma', \infty}.
\]

Since \( J(X, k, t) \) is an integral in \( t \), the uniform-in-\( t \) bound automatically implies the existence of

\[
J^\infty (X, k) := \lim_{t \to \infty} J(X, k, t) = qi \int_0^{\infty} e^{-4\pi^2 ipk \cdot X} \frac{P(k + X/2) - P(k - X/2)}{X} Xn(X, \tau) d\tau \in L^1_{\Sigma} L^1_X.
\]
Now equation (45) can be recast as
\[ U(-t)f(t) - f_0 = J(t) \quad \Rightarrow \quad \lim_{t \to \infty} (U(-t)f(t) - f_0) = J^\infty. \]
By setting \( \mathcal{W}(w_0) := \mathcal{F}_{X \to \mathbb{C}}[f_0 + J^\infty] \), we have
\[ \|w(t) - E(t)\mathcal{W}(w_0)\|_{L^p_{x,k}} \leq \|f(t) - U(t)(f_0 + J^\infty)\|_{L^p_{x,k}} = \|U(-t)f(t) - f_0 - J^\infty\|_{L^p_{x,k}}, \]
hence equation (19) follows.

6.5 Auxiliary lemmata

Lemma 6.5. Let \( A, B > 0, \zeta \in (0,1) \). Then
\[ \frac{1}{A+B} \leq \frac{1}{A^\zeta B^{1-\zeta}}. \]

Proof: The well-known Young’s inequality for products implies that, for \( a, b > 0, p \in (1, \infty), \frac{1}{p} + \frac{1}{q} = 1, \)
\[ ab \leq \frac{a^p}{p} + \frac{b^q}{q} \leq a^p + b^q. \]
Now setting \( A = a^p, B = b^q \) we have
\[ A^{1/p}B^{1/q} \leq A + B \quad \Rightarrow \quad \frac{1}{A+B} \leq \frac{1}{A^{1/p}B^{1/q}}. \]
By setting \( \zeta = 1/p \) and observing that \( 1/q = 1 - 1/p = 1 - \zeta \) the conclusion follows. \( \square \)

Lemma 6.6. Let \( \tilde{h}(X,s) \) be as in equation (38). Then
\[ \lim_{\rho \to \infty} \sup_{\text{Re}(\omega) > 0} |\tilde{h}(X,\omega)| = 0. \]

Proof: Recall that \( P \in \mathcal{S}(\mathbb{R}) \) is of compact support. Hence by construction \( XD_X P(k) = P(k + X/2) - P(k - X/2) \) is also of compact support for each \( X \in \mathbb{R} \). Let \( M = M(X) \) be such that \( \text{supp} XD_X P \subseteq [-M,M] \).
Then for all \( \rho \) large enough we have
\[ G(\rho) := \sup_{\text{Re}(\omega) > 0} |\tilde{h}(X,\omega)| \leq \sup_{\text{Re}(\omega) > 0} |q| \int_{k \in \mathbb{R}} \frac{|XD_X P(k)|}{|X^2 - 4\pi^2 q X k|} dk = |q| \sup_{\text{Re}(\omega) > 0} \int_{k = -M}^{M} \frac{|XD_X P(k)|}{|X^2 - 4\pi^2 q X k|} dk \leq \]
\[ |q| \int_{k \in \mathbb{R}} |XD_X P(k)| dk \sup_{\text{Re}(\omega) > 0} \int_{|\omega| > 0, |k| < M} \frac{1}{|X^2 - 4\pi^2 q X k|}. \]
Clearly \( \lim_{\rho \to \infty} G(\rho) = 0. \) \( \square \)

Theorem 6.7 (Conditional integrability of the Hilbert transform). Let \( f \in \mathcal{S}(\mathbb{R}) \) be a function of compact support with \( \int f(t)dt = 0 \). Then
\[ \|\mathbb{H}[f]\|_{L^1(\mathbb{R})} < \infty. \]
Proof: Choose an \( M > 0 \) so that the support of \( f \) is contained in \( [-M,M], f(x) = 0 \forall |x| \geq M \). We will also use the “double” interval, \( J := [-2M,2M] \) and its complement \( J^c = \mathbb{R} \setminus J \). By an elementary estimate we have
\[ \|\mathbb{H}[f]\|_{L^1(\mathbb{R})} \leq 4M \|\mathbb{H}[f]\|_{L^\infty(\mathbb{R})} + \int_{J^c} |H[f](x)| dx \leq 4CM\|\mathbb{H}[f]\|_{L^1(\mathbb{R})} + \int_{J^c} |H[f](x)| dx \]
where $C$ is the constant of the Sobolev embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$. Moreover, using the fact that $\int f(t)dt = 0$ we have

$$I := \int_{J^c} |H[f](x)|dx = \frac{1}{\pi} \int_{x \in J^c} |x| \sum_{t \in \mathbb{R}} \frac{f(t)}{x-t}dt|dx = \frac{1}{\pi} \int_{x \in J^c} \sum_{t \in \mathbb{R}} \left( \frac{f(t)}{x-t} - \frac{f(t)}{x} \right) dt|dx = \frac{1}{\pi} \sum_{t \in J^c} \int_{x \in \mathbb{R}} f(t) \frac{t}{x(x-t)}dt|dx,$$

where in the last step we also used the fact that $f$ is supported inside $[-M, M]$. Now observe that for any $x \notin [-2M, 2M]$, $t \in [-M, M]$

$$|t| \leq |x-t| \quad \Rightarrow \quad |x| = |x+t-t| \leq 2|x-t| \quad \Rightarrow \quad \frac{1}{|x-t|} \leq \frac{2}{|x|}$$

hence

$$\left| \frac{t}{x(x-t)} \right| \leq \frac{2M}{x^2} \quad \Rightarrow \quad I \leq \frac{2M}{\pi} \int_{x \in J^c} \frac{1}{x^2} dx \int_{t=-M}^{M} |f(t)|dt < \infty.$$

\[ \square \]

7 Proof of Theorem 3.5

7.1 Elaboration and symmetry of (1).

Assuming condition (1) holds, there exists a sequence $(X_n, \omega_n) = (X_n, a_n + ib_n) \in \mathbb{R} \times \{ \text{Re}(z) > 0 \}$ such that $\lim_{n \to \infty} \tilde{h}(X_n, \omega_n) = 1$.

**Symmetry:** Recalling equation (38), $\tilde{h}(X, \omega) = qi \int_{k} \frac{P(k+X)-P(k-X)}{\omega-4\pi^{2}pkX} dk$, and therefore one readily checks that, for $X_n, a_n, b_n \in \mathbb{R}$,

$$\tilde{h}(X_n, a_n + ib_n) \to 1 \iff \tilde{h}(X_n, -a_n + ib_n) \to 1,$$

i.e.

$$\exists X_n \in \mathbb{R}, \omega_n \in \mathbb{C} : \tilde{h}(X_n, \omega_n) \to 1 \iff \exists X_n \in \mathbb{R}, \text{Re}(\omega_n) \geq 0 : \tilde{h}(X_n, \omega_n) \to 1.$$ 

Indeed all the equivalent conditions have this symmetry.

**Claim I:** The sequence $(X_n, \omega_n)$ is bounded.

**Proof of the claim:** If $|X_n| + |\omega_n| \to \infty$, then $\lim_{n \to \infty} \tilde{h}(X_n, \omega_n) = qi \lim_{n \to \infty} \int_{k} \frac{P(k+X_n)-P(k-X_n)}{\omega_n-4\pi^{2}pkX_n} dk = 0 \neq 1$.

Thus $(X_n, \omega_n)$ has accumulation points in $\mathbb{R} \times \{ \text{Re}(z) \geq 0 \}$; from now on we will denote by

$$(X_a, a + ib) = (X_a, \omega) := \lim_{n \to \infty} (X_n, \omega_n)$$

(48)

(up to extraction of a subsequence).

**Claim II:** Denote

$$\Omega_n := \frac{\omega_n}{4\pi pX_n} = \frac{b_n - ia_n}{4\pi pX_n}.$$  \hspace{1cm} (49)

Then $\Omega_n$ is bounded.

**Proof of the claim:** Without loss of generality we can assume $X_n \neq 0$ for all $n \in \mathbb{N}$. (It suffices to observe that $\tilde{h}(0, \omega) = 0$ for all $\omega$. $X_n$ can still be zero of course).

Then $\Omega_n$ is well defined and, by virtue of equation (38),

$$\tilde{h}(X_n, \omega_n) = \frac{q}{4\pi p} \mathbb{H}[D_{X_n}P](\Omega_n).$$ \hspace{1cm} (50)

Clearly, if $|\Omega_n| \to \infty$ then $(q/4\pi p)\mathbb{H}[D_{X_n}P](\Omega_n) \to 0 \neq 1$. Thus, by extracting yet another subsequence if necessary, we have $(X_n, \Omega_n) \to (X_a, \omega) \in \mathbb{R} \times \mathbb{C}.$
7.2 (1) \iff (2).

Case 1: If \( \text{Im}(\Omega_\ast) \neq 0 \) then, by continuity,
\[
\mathcal{h}(X_n, \omega_n) \to 1 \iff \frac{q}{4\pi p} \mathbb{H}[D_{X\ast}P](\Omega_\ast) = 1.
\]

Case 2: If \( \text{Im}(\Omega_\ast) = 0 \) then, by the Sokhotski-Plemelj formula (cf. Theorem B.2), for \( X_\ast > 0 \) we have
\[
\mathcal{h}(X_n, \omega_n) \to 1 \iff \frac{q}{4\pi p} \mathbb{S}[D_{X\ast}P](\Omega_\ast) = 1 \iff \left\{ \begin{array}{l}
\frac{q}{4\pi p} \mathbb{H}[D_{X\ast}P](\Omega_\ast) = 1, \\
i \frac{q}{4\pi p} D_{X\ast}P(\Omega_\ast) = 0.
\end{array} \right.
\]
while for \( X_\ast < 0 \) we have \( \mathbb{S}[D_{X\ast}P](\Omega_\ast) = 1 \), leading to the same end result. For \( X_\ast = 0 \) observe that both one-sided limits \( \text{Im}(\omega_n) \to 0^\pm \), yield the same result as well.

Checking that (2) implies (1) is obvious.

7.3 (2) \iff (3).

Denote \( \mathbb{F}_X(\Omega) := \mathbb{H}[D_XP](\Omega) \). Like before, if \( \pm X_\ast > 0 \) we have \( \pm \text{Im}(\Omega_\ast) < 0 \), and for \( X_\ast = 0 \) we should take each one-sided limit separately. All these cases follow the same steps, so without loss of generality we only present the case \( X_\ast > 0 \).

Assume Case 1 of (2) above holds, i.e. \( \exists X_\ast > 0, \text{Im}(\Omega_\ast) \neq 0 \) such that \( \mathbb{H}[D_{X\ast}P](\Omega_\ast) = 4\pi p/q \).

Then by virtue of the argument principle [4, 28], for any contour \( \gamma \) within the lower half-plane containing \( \Omega_\ast \), its image \( \mathbb{F}_X(\gamma) := \{ z \in \gamma : z = \mathbb{F}_X(w) \} \) is enclosing \( 4\pi p/q \). Let us select \( \gamma_\eta \) the closed contour comprised by parts of the horizontal line \( \mathbb{R} - i\eta \) and the semicircle \( \{ \frac{e^{i\theta}}{\eta}, \ \theta \in (-\pi, 0) \} \). Clearly, \( \Omega_\ast \) will eventually be enclosed by \( \gamma_\eta \) for \( \eta \) small enough, thus \( \mathbb{F}_X(\gamma_\eta) \) is enclosing \( 4\pi p/q \) for \( \eta \) small enough. Using the decay properties of \( \mathbb{F}_X(\omega) \) as \( |\omega| \to \infty \) (cf. Lemma 6.6) and the Sokhotski-Plemelj formula, it follows that \( \lim_{\eta \to 0} \mathbb{F}_X(\gamma_\eta) = \Gamma_X \) as defined in equation (20), i.e. \( 4\pi p/q \in \bar{\Gamma}_{X\ast} \).

If Case 2 of (2) above holds, denote \( \Omega_n \) a sequence of points on \( \Gamma_{\eta_n} \) such that \( \Omega_n \to \Omega_\ast \); then by construction \( \lim_{n \to \infty} \mathbb{F}_X(\Omega_n) = 4\pi p/q \) and therefore \( 4\pi p/q \in \lim_{\eta \to 0} \mathbb{F}_X(\gamma_\eta) = \Gamma_{X\ast} \).

To prove that (3) \implies (2), first we need to observe that, since \( \lim_{|X| \to \infty} \| D_XP \|_{H^1} = 0 \), there exists \( M > 0 \) such that for \( |X| > M \) all points of \( \Gamma_X \) are inside \( \{ z \in \mathbb{C} : |z| < 2\pi p/q \} \). Thus \( 4\pi p/q \in \bar{\Gamma} \) implies \( \exists X_\ast \in [-M, M] \) such that \( d(4\pi p/q, \bar{\Gamma}_{X\ast}) \). One now readily checks that there exists \( \Omega_\ast \) with \( \text{Im}(\Omega_\ast) \leq 0 \) such that \( \lim_{\Omega \to \Omega_\ast, \text{Im}(\Omega) < 0} \mathbb{F}_X(\Omega) = 4\pi p/q \).

7.4 Sufficient condition for stability

This follows from the elementary observation that, for the curve \( \Gamma_X \) on the complex plane, which starts and ends at 0, to be winding around the real number \( 4\pi p/q \), it is necessary to intersect the real axis somewhere on the right of \( 4\pi p/q \). The argument can be adapted for limiting case \( 4\pi p/q \in \Gamma_X \) obviously. (See also Figures 3, 4 for a visualisation of this point.)

The proof is completed by observing that, according to equation (20), \( \Gamma_X = \{ \mathbb{F}[D_XP](t), t \in \mathbb{R} \} \) intersects the real axis only for those \( t_\ast \) that are quasi-critical points, \( D_XP(t_\ast) = 0 \).

8 Summary and Conclusions

We presented the first solvability and regularity theory for the linearised and for the fully nonlinear Alber equation. Our approach is flexible, and promises to be applicable to different variants of the equation.
appearing in different contexts. We also showed linear Landau damping for \( d = 1 \). Finally, we showed the relationship between Alber’s “eigenvalue relation” (a sufficient condition for instability), and the Penrose-type conditions which are sufficient conditions for stability. We also presented a constructive way to check whether a given spectrum satisfies the Penrose stability condition.

### 8.1 Stable and unstable spectra: Physical interpretation

As is broadly understood [4, 18, 26, 31, 1, 33], narrow enough spectra will be unstable (similar to what happens with a plane wave background), while broad enough spectra will be stable. At the same time, for any spectral shape \( P(k) \), for \( a \) large enough \( aP(k) \) will be modulationally unstable, while for \( a \) small enough it will be stable. The numerical investigation presented in Figure 2 confirms this behaviour, and refines existing approximations of the stable region for JONSWAP spectra. More details can be found in Appendix C in the Supplement.

Our results seem to indicate that the vast majority of measured spectra are in fact stable, in agreement with [18, 31]. On the other hand, the unstable region is not very far removed from realistic spectra; indeed it seems possible that for \( \gamma = 5 \) there exist measured spectra that would fall in the unstable region by a small margin. This reinforces the idea that Landau damping does take place in the ocean, and it plays a role in determining the “natural minimal modulational bandwidth”. If sea states corresponding to unstable spectra do form in the ocean, then these would be the prime breeding ground for rogue waves [4, 27], even if their significant wave height is not very large.

We discussed above the bifurcation from stability to instability as \( \alpha \) or \( \gamma \) increases. This has been thought of as a violent change in behavior once a borderline stable spectrum became unstable. Such a change in behavior is the object of numerical experiments in [20], where it is noted that instead only a gradual transition is found. In fact, the lack of a dramatic bifurcation was seen as a challenge for the validity Alber equation in the aforementioned works. However our proof here (and the heuristic results of [4] for the unstable case) show that indeed the Alber equation only predicts a gradual transition. As can be clearly seen in equation (18), as \( \kappa \to 0 \) the force decays more and more slowly, until it ceases to have any time decay at all, and even grows in time in the unstable case. Moreover, for a barely unstable spectrum, the growth is very slow, taking too long to be significant\(^3\). Thus a barely stable and a barely unstable spectrum would lead to very similar behaviour over relevant timescales, reconciling the findings of [20] with the analysis of the Alber equation.

### 8.2 Further work

A natural extension of this work is nonlinear Landau damping. The results of [6] per se apply only to \( d = 3 \) and for Vlasov equations with interaction kernels \( W(x) \) satisfying \( |\hat{W}(k)| \leq C/(1 + |k|^2) \). In our case, the Alber equation can be recast as a Vlasov equation with effective interaction kernel \( \hat{W}(k) = C' \sin(|k|^2t)/(C|k|^2t) \), but the interesting spatial dimensions are \( d = 1, 2 \). So one difference is that adaptations are needed due to the different dimensionality of the problem. Moreover, in our case the effective kernel is decaying in time as well, which means that one could expect genuinely stronger decay. An important point will be whether nonlinear Landau damping can still be shown without the mean-zero restriction on the initial perturbation.

In terms of the physical applicability of this analysis, it is vital to include some account of wind input and wave breaking. Indeed, in storm conditions strong wind and breaking can affect the effective stability condition, and recent experiments have offered the first quantitative glimpses of that effect [5]. Capturing analytically this effect even crudely seems possible. One could start from modified NLS equations that account for wind input and wave breaking [10, 11], and derive a modified Alber equation and modified stability condition. There are substantial challenges even for the linear stability analysis (as the starting

\(^3\)It was shown in [4] that even a strong modulation instability would take \( O(30) \) wave periods to lead to significant growth.
Figure 2: A number of points on the $(\gamma, \alpha)$ plane are tested for Penrose stability of the corresponding JONSWAP spectrum, cf. Eq. (61) in the Supplement. \(\alpha\) controls the power of the sea state (larger \(\alpha\) means larger significant wave height) and \(\gamma\) controls the effective bandwidth (larger \(\gamma\) means more narrowly peaked spectrum). The carrier wavenumber \(k_0\) can easily be seen not to affect the (in)stability of the spectrum. \((\gamma, \alpha)\) points found to be stable are marked with a full square, while points found to be unstable are marked with an empty square. For reference the proposed separatrices of [31] and [18] are shown (they are of the form \(\alpha \cdot \gamma/\beta = C\), where \(\beta\) is the mean wave steepness and \(C = 0.77\) [18] or \(C = 0.974\) [31]). More details can be found in Appendix C in the Supplement. **Top:** Linear scaling in both axes. **Bottom:** Log scaling in the \(\alpha\) (vertical) axis.
point is now a derivative-NLS equation), but conceptually the steps are the same as the ones we went through in this work.

It must also be mentioned that combining NLS-type equations with stochastic modelling is natural in many different contexts, not only ocean waves. It is thus natural that variants of the Alber equation are being independently rederived in different branches of physics, including optics [19] and many-particle systems [17]. Thus the main results of this paper are, in principle, applicable and/or generalisable to other problems as well.

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A Derivation of the Alber equation

To explain the derivation of the Alber equation (1) as a second moment of the NLS, (2), let us first look at the algebraic (deterministic) second moment: denoting

\[ R_1(\alpha, \beta, t) := u(\alpha, t)\overline{u}(\beta, t), \]

a straightforward computation leads to

\[ i\partial_t R_1 + \frac{p}{2} (\Delta_\alpha - \Delta_\beta) R_1 + \frac{q}{2} R_1(\alpha, \beta, t) [R_1(\alpha, \alpha) - R_1(\beta, \beta)] = 0 \]

(51)

for the evolution in time of \( R_1 \). That is, despite taking a second moment of a nonlinear equation, the exact algebraic moment closure

\[ |u(\alpha, t)|^2 u(\alpha, t)\overline{u}(\beta, t) = R_1(\alpha, \alpha)R_1(\alpha, \beta, t) \]

allows one to have a closed, exact second moment equation. (the same equation is called the “infinite system of fermions” in statistical physics [12].) Now consider the stochastic second moment,

\[ R(\alpha, \beta, t) := \mathbb{E}[u(\alpha, t)\overline{u}(\beta, t)]. \]

Obviously now the algebraic closure is not enough, as \( \mathbb{E}[|u(\alpha, t)|^2 u(\alpha, t)\overline{u}(\beta, t)] \) is a fourth order stochastic moment, and not exactly expressible in terms of second order moments. However, for Gaussian processes (under additional assumptions described shortly) it can be seen that

\[ \mathbb{E}[|u(\alpha, t)|^2 u(\alpha, t)\overline{u}(\beta, t)] = 2R(\alpha, \alpha)R(\alpha, \beta, t). \]

(52)

This is reminiscent of the well known real-valued Isserlis Theorem; the difference is that here \( u \) is complex valued (and the factor 2 is an artifact of the complex-valuedness of \( u \)). So the Alber equation (1) and the deterministic Wigner transform of the Schrödinger equation (2) differ only in terms of this factor of 2.

The precise result we invoke here can be summarised thus:
Observation A.1 (A complex Isserlis theorem). In [29] a moment closure result is proved; a special case of it (appearing also explicitly on the last page of [29]) is the following:

Let $z(x)$ be a Gaussian, zero-mean, stationary process with the additional property that

$$E[u(x)u(x')] = 0 \quad \forall x, x' \in \mathbb{R}. \quad (53)$$

Then

$$E[z(x_1)z(x_2)z(x_3)z(x_4)] = E[z(x_1)z(x_3)]E[z(x_2)z(x_4)] + E[z(x_2)z(x_3)]E[z(x_1)z(x_4)].$$

This result directly implies the closure relation

$$E[u(\alpha, t)u(\beta, t)u(\alpha, t)u(\alpha, t)] = 2E[u(\alpha, t)u(\alpha, t)]E[u(\beta, t)u(\alpha, t)], \quad (54)$$

which is exactly Eq. (52).

Moreover, the condition (53) is equivalent to circular symmetry, i.e. to the condition that

$$\{e^{i\theta}u(x)\}_{\theta \in [0, 2\pi]}$$

are identically distributed for all $\theta \in [0, 2\pi)$

by virtue of a result by Grettenberg [36].

Remark A.2 (Physical meaning of the Gaussian closure). Assuming that, for each $t_0$ the wave envelope $u(x, t_0)$ is a Gaussian process, with mean zero, stationary in $x$ (i.e. spatially homogeneous) and gauge invariant, $e^{i\theta}u(x, t_0) \sim u(x, t_0)$, is in line with standard modelling assumptions for linearised ocean waves [25]. In other words, the Gaussian moment closure of equation (52) can be thought of as a linearisation of the probability structure of the wave envelope.

By using the Gaussian closure (52) we see that $R(\alpha, \beta, t)$ satisfies the equation

$$i\partial_t R + \frac{p}{2} (\Delta_\alpha - \Delta_\beta) R + q R(\alpha, \beta, t) [R(\alpha, \alpha) - R(\beta, \beta)] = 0, \quad (56)$$

which is structurally the same as the infinite system of fermions, the only difference being an effective doubling of the coupling constant $q$. Now introducing the assumption

$$R(\alpha, \beta, t) = \Gamma(\alpha - \beta) + \epsilon \rho(\alpha, \beta, t),$$

we postulate that $R$ is in leading order homogeneous in space, and we set up an initial value problem for the inhomogeneity $\rho(\alpha, \beta, t)$,

$$i\partial_t \rho + \frac{q}{2} (\Delta_\alpha - \Delta_\beta) \rho + q [\Gamma(\alpha - \beta) + \epsilon \rho(\alpha, \beta)] [\rho(\alpha, \alpha) - \rho(\beta, \beta)] = 0. \quad (57)$$

Now denote $\mathcal{R}$ be the rotation operator on phase-space

$$\mathcal{R}[f(x, y)] := f(x + \frac{y}{2}, x - \frac{y}{2}), \quad (58)$$

and consider the average Wigner transform of the wave envelope

$$W(x, k, t) = \int_{y \in \mathbb{R}^d} e^{-2\pi iky} E[u(x + \frac{y}{2}, t)u(x - \frac{y}{2}, t)]dy = \mathcal{F}_{y \to -k} \mathcal{R}[R(x, y, t)] = \mathcal{F}_{y \to -k} [\Gamma(y) + \epsilon \rho(x + \frac{y}{2}, x - \frac{y}{2}, t)] = P(k) + \epsilon w(x, k, t). \quad (59)$$

Now the Alber equation (1) is the equation for $w(x, k, t)$, i.e. it results by applying $\mathcal{F}_{y \to -k} \mathcal{R}$ to equation (57).

So finally the relation between the unknown of the Alber equation, $w(x, k, t)$, and the wave envelope, $u(x, t)$, is

$$\mathcal{F}_{y \to -k} E[u(x + \frac{y}{2}, t)\overline{u(x - \frac{y}{2}, t)}] \approx P(k) + \epsilon w(x, k, t).$$

In particular, if $\int_{x,k} w_0(x,k)dxdk = 0$ we have just an inhomogeneous redistribution of the energy of the homogeneous sea state, while if $\int_{x,k} w_0(x,k)dxdk > 0$ we have a wavetrain of finite energy interacting with a homogeneous sea state of infinite energy.
Combining this with the Sobolev embedding $H^1 \hookrightarrow L^2$, we have

$$\|\mathbb{H}[u]\|_{L^p(\mathbb{R})} \leq C \|u\|_{L^p(\mathbb{R})}, \quad \|\mathbb{S}[u]\|_{L^p(\mathbb{R})} \leq (1 + C) \|u\|_{L^p(\mathbb{R})}.$$ 

Moreover, $C(2) = 1$ and for any $s \in \mathbb{N}$,

$$\|\mathbb{H}[u]\|_{H^s(\mathbb{R})} = \|u\|_{H^s(\mathbb{R})}, \quad \|\mathbb{S}[u]\|_{H^s(\mathbb{R})} \leq 2 \|u\|_{H^s(\mathbb{R})}.$$ 

Combining this with the Sobolev embedding $H^1(\mathbb{R}) \hookrightarrow C^0(\mathbb{R})$ it follows that

$$u \in H^1(\mathbb{R}) \quad \Rightarrow \quad \mathbb{H}[u], \mathbb{S}[u] \in C^0(\mathbb{R}).$$

**Theorem B.2** (Sokhotski-Plemelj formula). For $u \in C(\mathbb{R}) \cap L^1(\mathbb{R})$ and for any $s, c \in \mathbb{R}$

$$\lim_{\eta \to 0^+} \mathbb{H}[u]\left(\frac{s - i\eta}{c}\right) = \mathbb{S}[u]\left(\frac{s}{c}\right).$$

**Theorem B.3** (Inverse Laplace transform, open half-plane). Let $F(\omega)$ be a bounded analytic function on an open right half-plane, $\omega \in \Pi(M) := \{\text{Re } z > M\}$. Assume moreover that the limit $F_{M^+}(b) := \lim_{\varepsilon \to 0^+} F(M + \varepsilon + ib)$ exists for all $b \in \mathbb{R}$ and is a continuous function in $b$. Moreover assume that

$$\lim_{\rho \to +\infty} \sup_{\omega \in \Pi(M)} \frac{|F(\omega)|}{\omega > \rho} = 0 \quad \text{and} \quad \int_{-\infty}^{+\infty} |F_{M^+}(s)|ds < \infty. \quad (60)$$

Then

$$F(\omega) = \mathcal{L}_{\omega \to t}[f(t)] \quad \text{where} \quad f(t) = \frac{e^{Mt}}{2\pi} \int_{-\infty}^{+\infty} e^{ist} F_{M^+}(s)ds,$$

i.e.

$$\mathcal{L}_{\omega \to t}^{-1}[F] = \frac{e^{Mt}}{2\pi} \int_{-\infty}^{+\infty} e^{ist} F_{M^+}(s)ds.$$ 

**C** On the numerical investigation of stability of JONSWAP spectra

In most of the ocean waves literature radians per second, as opposed to Hertz, are used when creating the frequency spectrum, $\mathcal{F}[R](\omega) = \int e^{-i\omega t} R(t)dt$. When the spectrum is converted from being frequency-resolved, $S(\omega)$, to being wavenumber-resolved, $P(k)$ [25], this normalisation carries over. In this work, in keeping with a lot of the Wigner transform literature we use Hertz, i.e. we normalise the Fourier transform as $\mathcal{F}[R](\omega) = \int e^{-2\pi i\omega t} R(t)dt$. To make sure there are no normalisation issues affecting our results, we will apply our technique directly to Alber’s eigenvalue relation as formulate in equation (2) of [18],

$$\exists \Omega \in \mathbb{C}, \ X \in \mathbb{R} \quad 1 + \omega_0 k_0^2 \int k \in \mathbb{R} \frac{S(k + \frac{\alpha}{2}) - S(k - \frac{\alpha}{2})}{\Omega + \frac{\omega_0 k X}{4k_0}} dk = 0$$

where $S(k)$ denotes the JONSWAP spectrum with parameters $\alpha, \gamma, k_0$, namely

$$S(k) = \frac{\alpha}{2k^3} e^{-\frac{\alpha}{4} (\frac{km}{\gamma})^2} \exp[-(1 - \sqrt{k/k_0})^2/2\delta^2], \quad \delta = \delta(k) = \begin{cases} 0.07, & k \leq k_0, \\ 0.09, & k > k_0. \end{cases} \quad (61)$$
By straightforward manipulations this is equivalent to

\[ 3 \Omega \in \mathbb{C}, \; X \in \mathbb{R} \quad \mathbb{H}[D_X P](\Omega) = \frac{1}{4\pi} \]  \tag{62} \]

where

\[ P(k) = \frac{a}{2k^3} e^{-\frac{1}{2}k^2} \exp[-(1-\sqrt{k})^2/2\delta^2], \quad \delta = \delta(k) = \begin{cases} 0.07, & k \leq 1, \\ 0.09, & k > 1, \end{cases} \]

is the JONSWAP spectrum with \( k_0 = 1 \).

Now reworking (3) of Theorem 3.5 we can directly check that \( 1/4\pi \) is on, or enclosed by, the curve \( \Gamma_X := \lim_{\eta \to 0^+} \mathbb{H}[D_X P](t - i\eta) \) for some \( X \in \mathbb{R} \). \tag{63} \]

\[ \frac{1}{4\pi} \]

is on, or enclosed by, the curve \( \Gamma_X := \lim_{\eta \to 0^+} \mathbb{H}[D_X P](t - i\eta) \) for some \( X \in \mathbb{R} \).

Figure 3: Numerical investigation of the Penrose condition for a stable JONSWAP spectrum. **Left:** Plots of the curve \( \Gamma_X \) on the complex plane for different values of \( X \). Since \( 1/4\pi \) is always outside the \( \Gamma_X \), this spectrum is stable. **Right:** The span of the real parts of \( \Gamma_X \) for different values of \( X \).

After some numerical testing, it is found sufficient to approximate

\[ \Gamma_X(t) = \lim_{\eta \to 0} \mathbb{H}[D_X P(\cdot)](t - i\eta) \approx \mathbb{H}[D_X P(\cdot)](t - \text{itol}), \quad \text{itol}=1\text{e}-4. \]

In all relevant cases here we observe that condition (63) is satisfied if and only if it is satisfied for \( X = 0 \). Once we generate an approximation to \( \Gamma_X \), the built-in MATLAB function \texttt{inpolygon} is then used to determine if the target \( 4\pi p/q \) is contained in \( \Gamma_X \cup \{0\} \). Our numerical results are summarised in Figure 2.

### D Moments and Derivatives of the Alber-Fourier equation

Denote

\[ L[P_1 - P_2; m] := \left[ P_1 \left( k - \frac{X}{2} \right) - P_2 \left( k + \frac{X}{2} \right) \right]m(X, t) \]

\[ N[m; f_1 - f_2] := \int_s m(s, t) \left[ f_1 \left( X - s, k - \frac{s}{2}, t \right) - f_2 \left( X - s, k + \frac{s}{2}, t \right) \right] ds. \] \tag{64} \]

The nonlinearity \( \mathbb{B}[m, f] \) defined in equation (23) is comprised of

\[ \mathbb{B}[m, f] = iqL[P - P; m] + eiqN[m; f - f]. \]
Figure 4: Numerical investigation of the Penrose condition for an unstable JONSWAP spectrum. **Left:** Plots of the curve $\Gamma_X$ on the complex plane for different values of $X$. Since $1/4\pi$ is contained in some curves $\Gamma_X$, the spectrum is unstable. **Right:** The span of the real parts of $\Gamma_X$ for different values of $X$. In this case it highlights clearly the bandwidth of unstable wavenumbers $X$.

**Lemma D.1.** For any multi-indices $\alpha, \beta, \gamma, \delta \in (\mathbb{N} \cup \{0\})^d$ we have the following relations

$$X^\alpha L[P_1-P_2; m] = L[P_1-P_2; X^\alpha m],$$

$$k^\beta L[P_1-P_2; m] = \sum_{0 \leq \beta' \leq \beta} \binom{\beta}{\beta'} L[k^{\beta-\beta'}P_1 - (-1)^{\beta'}k^{\beta-\beta'}P_2; (\frac{X}{2})^{\beta'} m],$$

$$\partial^\gamma_X L[P_1-P_2; m] = \sum_{0 \leq \gamma' \leq \gamma} \binom{\gamma}{\gamma'} L[(-1/2)^{\gamma-\gamma'}\nabla^{\gamma-\gamma'}P_1 - (1/2)^{\gamma-\gamma'}\nabla^{\gamma-\gamma'}P_2; \partial^{\gamma'}_X m],$$

$$\partial^\delta_k L[P_1-P_2; m] = L[\nabla^{\delta}P_1 - \nabla^{\delta}P_2; m],$$

and

$$X^\alpha N[m; f_1-f_2] = \sum_{0 \leq \alpha' \leq \alpha} \binom{\alpha}{\alpha'} N[X^{\alpha-\alpha'}m; X^{\alpha'} f_1 - X^{\alpha'} f_2],$$

$$k^\beta N[m; f_1-f_2] = \sum_{0 \leq \beta' \leq \beta} \binom{\beta}{\beta'} N[(\frac{X}{2})^{\beta-\beta'} m; k^{\beta'} f_1 - (-1)^{\beta-\beta'}k^{\beta'} f_2],$$

$$\partial^\gamma_X N[m; f_1-f_2] = N[m; \partial^\gamma_X f_1 - \partial^\gamma_X f_2],$$

$$\partial^\delta_k N[m; f_1-f_2] = N[m; \partial^\delta_k f_1 - \partial^\delta_k f_2].$$

Moreover,

$$X^\alpha k^\beta \partial^\gamma_X \partial^\delta_k (k \cdot X f) = k \cdot X (X^\alpha k^\beta \partial^\gamma_X \partial^\delta_k f) + X^\alpha k^\beta \sum_{0 \leq \gamma' < \gamma \atop 0 \leq \delta' < \delta} \binom{\gamma}{\gamma'} \binom{\delta}{\delta'} (\partial^\gamma_X \partial^\delta_k f) (\partial^{\gamma-\gamma'} \partial^{\delta-\delta'} X \cdot k).$$

The proof follows from direct computations using the definition of $L[P_1-P_2; m]$ and $N[m; f_1-f_2]$. By applying the operator $X^\alpha k^\beta \partial^\gamma_X \partial^\delta_k$ to Eq. (22) and commuting according to Lemma D.1 one obtains
the equation (27) with right hand side

\[
\mathcal{B}^{(\alpha, \beta, \gamma, \delta)}[f] = - \sum_{0 \leq \gamma' \leq \gamma \atop 0 \leq \beta' \leq \delta \atop |\gamma' + \delta'| < |\gamma + \delta|} \binom{\gamma}{\gamma'} \binom{\delta}{\delta'} (\partial^{\gamma - \gamma'}_{X} \partial^{\delta - \delta'}_{k} X \cdot k) f^{\alpha, \beta, \gamma', \delta'}

- q_{i} \sum_{0 \leq \beta' \leq \beta \atop 0 \leq \gamma' \leq \gamma} \frac{1}{2} |\gamma - \gamma'| + |\beta'| \binom{\beta}{\beta'} \binom{\gamma}{\gamma'} L[k^{\beta - \beta'}(-1)^{\gamma - \gamma'} |\nabla^\gamma - \gamma' + \delta P - k^{\beta - \beta'} \nabla^{\gamma - \gamma'} + \delta P ; \int_{k} f^{\alpha + \beta', 0, 0, 0} dk]

- e_{qi} \sum_{0 \leq \alpha' \leq \alpha \atop 0 \leq \beta' \leq \beta} \frac{1}{2} |\beta - \beta'| \binom{\alpha}{\alpha'} \binom{\beta}{\beta'} N[\int_{k} f^{\alpha' - \alpha - \beta - \beta', 0, 0, 0} dk ; f^{\alpha', \beta', \gamma, \delta} - (-1)^{\beta - \beta'} f^{\alpha', \beta', \gamma, \delta}].
\]