A NOTE ON THE CONCURRENT NORMAL CONJECTURE

ALEXANDR GREBENNIKOV AND GAIANE PANINA

Abstract. It is conjectured since long that for any convex body $K \in \mathbb{R}^n$ there exists a point in the interior of $K$ which belongs to at least $2n$ normals from different points on the boundary of $K$. The conjecture is known to be true for $n = 2, 3, 4$.

Motivated by a recent preprint of Y. Martinez-Maure, we give a short proof of his result: for dimension $n \geq 3$, under mild conditions, almost every normal through a boundary point to a smooth convex body $K \in \mathbb{R}^n$ contains an intersection point of at least $6$ normals from different points on the boundary of $K$.

1. Introduction

Given a smooth convex body $K \in \mathbb{R}^n$, its normal to a point $p \in \partial K$ is a line passing through $p$ and orthogonal to $\partial K$ at the point $p$. It is conjectured that for any convex body $K \in \mathbb{R}^n$ there exists a point in the interior of $K$ which is the intersection point of at least $2n$ normals from different points on the boundary of $K$. The concurrent normals conjecture trivially holds for $n = 2$. For $n = 3$ it was proven by Heil [2] and [3] via geometrical methods and reproved by Pardon via topological methods. The case $n = 4$ was completed also by Pardon [6].

Recently Martinez-Maure proved for $n = 3, 4$ that (under some mild conditions) almost every normal through a boundary point to a smooth convex body $K$ passes arbitrarily close to the set of points lying on normals through at least six distinct points of $\partial K$ [4].

He used Minkowski differences of smooth convex bodies, that is, the theory of hedgehogs.

The present paper is very much motivated by [4]. We give an alternative short proof of almost the same fact for all $n \geq 3$, see Theorem [2]. Our proof is based on the bifurcation theory and does not use hedgehogs.

Acknowledgments. G. Panina is supported by RFBR grant 20-01-00070A; A. Grebennikov is supported by Ministry of Science and Higher Education of the Russian Federation, agreement 075-15-2019-1619.

Key words and phrases. Bifurcation, Morse-Cerf theory, Morse points.
2. Some preliminaries and the main result

Let \( n \geq 3 \), and let \( K \) be a strictly convex \( C^\infty \)-smooth compact body in \( \mathbb{R}^n \). For a point \( x \in \partial K \) we denote by \( \mathcal{N}(x) \) the normal line to \( \partial K \) at the point \( x \). Following [4], we make use of the support function. Let \( h : S^{n-1} \to \mathbb{R} \) be the support function of \( K \). For \( y \in \mathbb{R}^n \) define
\[
h_y : S^{n-1} \to \mathbb{R} \\
h_y(v) = h_0(v) - \langle v, y \rangle.
\]
The function \( h_y \) equals the support function of \( K \) after a translation which takes the origin \( O \) to the point \( y \).

Given a point \( y \), all the normals passing through \( y \) can be read off the function \( h_y \):

**Lemma 1.** [4] A point \( y \) lies on the normal \( \mathcal{N}(x) \) for some \( x \in \partial K \) iff \( u(x) \) is a critical point of the function \( h_y \).

Here \( u(x) \in S^{n-1} \) is the outer unit normal to \( \partial K \) at the point \( x \). \[\square\]

By Morse lemma type arguments ([5], Lemma A), \( h_y \) is a Morse function almost for all \( y \). Its bifurcation diagram [1], [4] is given by the *focal surface* \( \mathcal{F}_K \) of the body \( K \).

1. The focal surface equals the locus of the centers of principal curvatures of \( \partial K \). Thus it has \( n-1 \) *sheets*. Sheet number \( k \) corresponds to the \( k \)-th curvature, assuming that the curvature radii are enumerated in the ascending order: \( r_1 \leq r_2 \leq \ldots \leq r_{n-1} \). Each sheet is an image of \( S^{n-1} \). The sheets intersect each other and may have self-intersections and singularities.

2. The focal surface cuts the ambient space \( \mathbb{R}^n \) into *cameras* (that is, connected components of the complement of \( \mathcal{F}_K \)). The type of the associated Morse functions \( h_y \) depends on the camera containing \( y \) only.

3. Transversal crossing of exactly one of the sheets (say, the sheet number \( k \)) of the focal surface at its smooth point amounts to a birth (or death) of two critical points of \( h_y \) whose indices are \( k \) and \( k-1 \).

**Theorem 2.** Let \( n \geq 3 \), and let \( K \in \mathbb{R}^n \) be a \( C^\infty \)-smooth convex body, \( x \in \partial K \). If the normal line \( \mathcal{N}(x) \) does not intersect the singular locus of the focal surface \( \mathcal{F}_K \) then \( \mathcal{N}(x) \) contains a point \( z \) such that:

1. \( z \) is an intersection point of at least 6 normals from different points on the boundary of \( K \).
2. The distance \( |xz| \) satisfies
\[
r_1(x) < |xz| < r_{n-1}(x),
\]
where \( r_1(x) \) and \( r_{n-1}(x) \) are the largest and the smallest principal curvature radii at the point \( x \).

---

1Equivalently, one can work with the squared distance function to the boundary.
3. Proof of Theorem 2

Definition 1. A one-parametric family $f_t \in C^\infty(S^{n-1}, \mathbb{R})$, $t \in \mathbb{R}_+$ is nice if the following properties hold:

1. $f_t$ depends smoothly on $t$;
2. $f_t$ is a Morse function for each $t$ except for finitely many bifurcation points $t_1, \ldots, t_m$;
3. each of the bifurcation points is of one of the two types:
   a. A birth/death point. Some two critical points with some neighbor indices $k$ and $k-1$ collide and disappear, or, vice versa, two critical points with neighbor indices appear.
   b. An index exchange point. Two critical points with neighbor indices $k$ and $k-1$ collide at $t = t_i$. There arises a degenerate critical point which splits afterwards into two critical points with the same indices $k$ and $k-1$.

For a nice family, denote $T = \mathbb{R}_+ \setminus \{t_1, \ldots, t_m\}$. For each $t \in T$ and $0 \leq k \leq n-1$ let $C_{t,k}$ be the number of critical points of $f_t$ of index $k$, and let $N_t$ be the total number of critical points of the function $f_t$.

Lemma 3. Let $f_t$ be a nice family of functions. Let also $C_{t,0} \geq 2$, $C_{t,n-1} \geq 2$ and

$$\sum_{k=1}^{n-2} C_{t,k} \geq 1 \text{ for all } t \in T \cap [t_1, t_2].$$

Then there exists $t \in T \cap [t_1, t_2]$ such that $N_t \geq 6$.

Proof. Assume the contrary, that is, for every $t$ we have $N_t < 6$. By assumption of the lemma, there are other critical points of $f_t$ than max and min, so $N_t$ is at least 3 for all $t$. Besides, $N_t$ is even, so $N_t$ necessarily equals 4 for all $t \in T \cap [t_1, t_2]$.

Therefore there are no birth/death bifurcations on $[t_1, t_2]$. Together with the conditions on $C_{t,0}$ and $C_{t,n-1}$, this implies that $f_t : S^n \to \mathbb{R}$ has two maxima and two minima for any $t \in T \cap [t_1, t_2]$ and no other critical points. A contradiction. $\Box$

Now we are ready to prove Theorem 2. Assume that the point $x$ is such that the normal $\mathcal{N}(x)$ does not meet the singularity locus of the focal surface. Denote by $u = u(x)$ the outer unit normal to $\partial K$ at the point $x$. Then the principal curvature radii $r_1 < r_2 < \ldots < r_{n-1}$ of $\partial K$ at the point $x$ are all different, and the family of functions $\{h_{x-tu}\}_{t \in \mathbb{R}_+}$ is nice. The set $T$ of bifurcation points includes $\{r_1, \ldots, r_{n-1}\}$. Let us prove that there exists $r \in (r_1, r_{n-1}),$ such that $x - ru$ lies on $\geq 6$ normals.

By Lemma 1 $u$ is a critical point of $h_{x-tu}$ for all $t$. It is easy to check that its Morse index equals 0 when $t \in (0, r_1)$, equals 1 when $t \in (r_1, r_2)$, $\ldots$, equals $n-1$ when $t \in (r_{n-1}, +\infty)$.

We shall apply Lemma 3 for the family $\{h_{x-tu}\}$ on the segment $[t_1, t_2] := [r_1 + \varepsilon, r_{n-1} - \varepsilon]$ for sufficiently small $\varepsilon > 0$. Since the Morse index of the
point $u$ is neither 0, nor $n-1$ on this segment, the condition $\sum_{k=1}^{n-2} C_{t,k} \geq 1$ is satisfied. So, we only need to check that $C_{t,0} \geq 2, C_{t, n-1} \geq 2$.

We will check the first inequality, the second may be verified in a similar way. The point $u$ at $t = r_1$ is an index exchange point. This means that as $t$ equals $r_1 + \varepsilon$ and tends to $r_1$, there is a (local) minimum point $u_{\varepsilon}$ of $h_{x-(r_1+\varepsilon)u}$ which tends to the critical point $u$ of index 1. We conclude that for small $\varepsilon$, the point $u_{\varepsilon}$ is not the global minimum of $h_{x-(r_1+\varepsilon)u}$. Therefore there are two local minima, that is, two distinct critical points of index 0.

Application of Lemma 3 completes the proof.

\[ \square \]

References

[1] V.Arnol’d, A.Varchenko, S.Gusein-Zade, Singularities of differentiable maps I and II, Birkhäuser, 1985 and 1988.
[2] E. Heil, Concurrent normals and critical points under weak smoothness assumptions. In Discrete geometry and convexity (New York, 1982), volume 440 of Ann. New York Acad. Sci., 170–178. New York Acad. Sci., New York, 1985.
[3] E. Heil, Existenz eines 6-Normalenpunktes in einem konvexen Körper. Arch. Math. (Basel), 32(4):412–416, 1979.
[4] Y. Martinez-Maure, On the concurrent normals conjecture for convex bodies. 2021 hal-03292275v3
[5] J. Milnor, Lectures on the h-cobordism theorem. Princeton University Press, Princeton, NJ, 1965.
[6] J. Pardon, Concurrent normals to convex bodies and spaces of Morse functions. Math. Ann. 352 (2012), no. 1, 55–71.

(A. Grebennikov) SAINT-PETERSBURG STATE UNIVERSITY, DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCES
Email address: sagresash@yandex.ru

(G. Panina) ST. PETERSBURG DEPARTMENT OF STEKLOV MATHEMATICAL INSTITUTE
Email address: gaiane-panina@rambler.ru