HAUSDORFF DIMENSION OF SELF-AFFINE LIMIT SETS WITH AN INVARIANT DIRECTION

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ABSTRACT. We determine the Hausdorff dimension of self-affine limit sets for some class of iterated function systems in the plane with an invariant direction. In particular, the method applies to some type of generalized non-self-similar Sierpiński triangles. This partially answers a question asked by Falconer and Lammering and extends a result by Lalley and Gatzouras.

1. Introduction. In 1997, Falconer and Lammering [7] studied generalized Sierpiński triangles, which are self-affine limit sets of an iterated function system obtained by mapping a triangle Δ in an affine way into three sub-triangles Δ₁, Δ₂, Δ₃ contained in Δ, such that each Δᵢ has one common vertex with Δ and each pair of the sub-triangles intersect at one point, which is their common vertex and lies on a side of Δ. The authors considered the case, when one of the maps is a similarity. Then, changing the coordinates in an affine way, one can assume that Δ has vertices at (0, 0), (1, 0), (0, 1) and the maps have the form

\[ f_1(x, y) = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \]
\[ f_2(x, y) = \begin{bmatrix} 1 - b & 0 \\ 0 & 1 - b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ b \end{bmatrix}, \]
\[ f_3(x, y) = \begin{bmatrix} 1 - a & 1 - a - b \\ 0 & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a \\ 0 \end{bmatrix}, \]

where \( a, b \in (0, 1) \). The system is presented in Fig. 1.

The generalized Sierpiński triangle is defined as the limit set

\[ \Lambda = \bigcap_{n=1}^{\infty} \bigcup_{i_1, \ldots, i_n \in \{1, 2, 3\}} f_{i_1} \circ \cdots \circ f_{i_n}(\Delta) \]

The case \( a = b = 1/2 \) corresponds to the standard self-similar Sierpiński triangle, where the Hausdorff and box dimension are equal to \( \log 3/\log 2 \). In all other cases at least one map is not a similarity. Such affine iterated function systems are more difficult to handle than the self-similar ones, and their dimension theory is less developed. For some results in this area refer e.g. to [1, 3, 4, 5, 6, 8, 9, 10, 11].

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In [7], the authors computed the box dimension of the generalized Sierpiński triangle in the case when $b \geq \max(a,1-a)$ or $b \leq \min(a,1-a)$ and asked several questions, e.g. on the Hausdorff dimension of the limit set. In this paper we partially answer this problem, computing the Hausdorff dimension of $\Lambda$ in the case $b \geq \max(a,1-a)$. In fact, we determine the Hausdorff and box dimension for a wider class of self-affine limit sets of iterated function systems in the plane preserving a given direction and fulfilling some separation conditions. The theorem, which extends a result by Lalley and Gatzouras from [8], is based on the paper [1].

2. Definitions and results. Consider an affine iterated function system on the plane $\mathbb{R}^2$ (i.e. a finite number of affine planar automorphisms), which preserves a direction given by a straight line $\ell \subset \mathbb{R}^2$ (i.e. a straight line parallel to $\ell$ is always mapped to another line parallel to $\ell$). By a linear change of coordinates, we can assume that $\ell$ is horizontal, which means that the matrices of linear parts of the maps are upper triangular. The system can be written as $\{f_{i,j}\}_{(i,j)\in I}$, where

$$I = \{(i,j) : i = 1, \ldots, k, \ j = 1, \ldots, m_i\}$$

for some positive integers $k, m_1, \ldots, m_k$ and

$$f_{i,j}(x,y) = \begin{bmatrix} a_{i,j} & c_{i,j} \\ 0 & b_{i} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \alpha_{i,j} \\ \beta_{i} \end{bmatrix},$$

for some real numbers $a_{i,j}, b_i, \alpha_{i,j}, \beta_i$.

**Notation.** The symbols dist and diam denote, respectively, the Euclidean distance and diameter, while $\overline{U}$, $\text{int} U$ and $\partial U$ are, respectively, the closure, interior and boundary of a set $U$. For a set $U \subset \mathbb{R}^2$ denote by $U^{(y)}$ the horizontal section of $U$ at level $y \in \mathbb{R}$, i.e.

$$U^{(y)} = U \cap \{(x,y) : x \in \mathbb{R}\}.$$

Set also

$$y^+ = \max(y,0), \quad y^- = \min(y,0).$$

We write $\text{Leb}_1$ for 1-dimensional Lebesgue measure in $\mathbb{R} \times \{y\}$. Recall that an iterated function system $\{f_{i,j}\}_{(i,j)\in I}$ satisfies the open set condition (OSC), if there
exists a bounded open set $V$, such that $f_{i,j}(V)$ are disjoint subsets of $V$ (see e.g. [4]).

In this paper we prove the following result.

**Theorem.** Assume that

(a) $0 < |a_{i,j}| \leq |b_i| < 1$ for $(i,j) \in \mathcal{I}$, and $|a_{i,j}| < |b_i|$ for $c_{i,j} \neq 0$,
(b) $\beta_i + b_i^* \leq \beta_{i+1} + b_{i+1}^*$ for $i = 1, \ldots, k - 1$,
(c) the system $\{f_{i,j}\}_{(i,j) \in \mathcal{I}}$ satisfies OSC for an open bounded set $V \subset \mathbb{R} \times [0,1]$,
(d) there exists $\delta > 0$, such that for every $y \in \mathbb{R}$, if $V(y)$ intersects at least two different sets $f_{i,j}(V)$ for $(i,j) \in \mathcal{I}$, then $\text{Leb}_1 V(y) > \delta$.

Then the Hausdorff dimension of the limit set is equal to the maximum of the function

$$g(p) = \sum_{(i,j) \in \mathcal{I}} p_{i,j} \log \frac{\sum_{j' = 1}^{m_{i,j}} p_{i,j'}}{\sum_{(i,j) \in \mathcal{I}} p_{i,j} \log |b_i|} + \sum_{(i,j) \in \mathcal{I}} p_{i,j} \log \frac{\sum_{j' = 1}^{m_{i,j}} p_{i,j'}}{\sum_{(i,j) \in \mathcal{I}} p_{i,j} \log |a_{i,j}|}$$

(with the convention $0 \log 0 = 0$) over the simplex

$$\mathcal{S} = \left\{ p = (p_{i,j})_{(i,j) \in \mathcal{I}} \in \mathbb{R}^{m_1 + \cdots + m_k} : p_{i,j} > 0, \sum_{(i,j) \in \mathcal{I}} p_{i,j} = 1 \right\}.$$

Moreover, the box dimension of the limit set is equal to the unique real number $D$, such that

$$\sum_{(i,j) \in \mathcal{I}} |b_i|^t |a_{i,j}|^{D-t} = 1,$$

where $t$ is the unique real number such that $\sum_{i=1}^k |b_i|^t = 1$.

**Remark 1.** The condition (b) says that the horizontal strips $S_i = f_{i,j}(\mathbb{R} \times [0,1])$ have disjoint interiors.

**Remark 2.** Obviously, the condition (d) is satisfied, if all non-empty horizontal sections of $V$ have 1-dimensional Lebesgue measure greater than $\delta$ (e.g. when $V$ is a parallelogram with a pair of horizontal sides, see Fig. 2).

**Remark 3.** The limit set can be defined as

$$\Lambda = \bigcap_{n=1}^{\infty} \bigcup_{(i_{1,j_{1}}, \ldots, i_{n,j_{n}}) \in \mathcal{I}} f_{i_{1,j_{1}}} \circ \cdots \circ f_{i_{n,j_{n}}}(V).$$

Note that the maps $f_{i,j}$ need not be contractions. However, we have $\text{diam} f_{i_{1,j_{1}}} \circ \cdots \circ f_{i_{n,j_{n}}}(V) \to 0$ as $n \to \infty$ (see Lemma 3.3). This implies that $\Lambda$ is the unique non-empty compact set, for which $\Lambda = \bigcup_{(i,j) \in \mathcal{I}} f_{i,j}(\Lambda)$.

**Remark 4.** The number $g(p)$ is equal to the Hausdorff dimension of the Bernoulli measure on $\Lambda$ defined by associating probabilities $p_{i,j}$ to the maps $f_{i,j}$. Hence, the Hausdorff dimension of $\Lambda$ is the supremum of Hausdorff dimension of the Bernoulli measures supported on $\Lambda$. The function $g$ is continuous on $\mathcal{S}$, so the supremum is attained. In most cases, the supremum cannot be computed explicitly. See [1] for details.
Some examples of fractal limit sets satisfying the assumptions of the above theorem are presented in Fig. 3.

It is easy to check that the system generating the generalized Sierpiński triangle described in Section 1 satisfies the conditions of the above theorem for \( V = \text{int} \Delta \) in the case \( b \geq \max(a, 1-a) \). Hence, we have the following.

**Corollary 1.** The Hausdorff dimension of the generalized Sierpiński triangle in the case \( b \geq \max(a, 1-a) \) is equal to the maximum of the function

\[
g(p_1, p_2, p_3) = \frac{(p_1 + p_3) \log(p_1 + p_3) + p_2 \log p_2}{(p_1 + p_3) \log b + p_2 \log(1-b)}
+ \frac{p_1 \log p_1 + p_3 \log p_3 - (p_1 + p_3) \log(p_1 + p_3)}{p_1 \log a + p_2 \log(1-b) + p_3 \log(1-a)}
\]

over the simplex \( S = \{(p_1, p_2, p_3) : p_1, p_2, p_3 \geq 0, \ p_1 + p_2 + p_3 = 1\} \).

Note that in the case \( b = 1-a \) the corollary follows directly from [1] (Example 3.3 and Proposition 3.6). Essentially, it is also a consequence of a result from the earlier paper [8] (formally, in [8] the considered maps are not similarities, but the arguments
remain the same). In [2], the Hausdorff and box dimension were computed for a non-linear modification of the Sierpiński triangle, where the triangles $\Delta$ and $\Delta_1, \Delta_2, \Delta_3$ are isosceles instead of equilateral.

3. Proof of the result. It is easy to check that

$$f_{i_1, j_1} \circ \cdots \circ f_{i_n, j_n}(x, y) = \begin{bmatrix} A_{i_1, j_1, \ldots, i_n, j_n} & C_{i_1, j_1, \ldots, i_n, j_n} \\ 0 & B_{i_1, \ldots, i_n} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} A_{i_1, j_1, \ldots, i_n, j_n} \\ B_{i_1, \ldots, i_n} \end{bmatrix} \tag{1}$$

for

$$A_{i_1, j_1, \ldots, i_n, j_n} = a_{i_1, j_1} \cdots a_{i_n, j_n}, \quad B_{i_1, \ldots, i_n} = b_{i_1} \cdots b_{i_n}$$

and some $C_{i_1, j_1, \ldots, i_n, j_n}, A_{i_1, j_1, \ldots, i_n, j_n}, B_{i_1, \ldots, i_n} \in \mathbb{R}$. By the condition (a),

$$|A_{i_1, j_1, \ldots, i_n, j_n}| \leq |B_{i_1, \ldots, i_n}|. \tag{2}$$

Moreover, the following holds (cf. Lemma 1 in [7]).

**Lemma 3.1.** There exists $c > 0$, such that for every $n$ and every $i_1, j_1, \ldots, i_n, j_n$ we have

$$|C_{i_1, j_1, \ldots, i_n, j_n}| < c|B_{i_1, \ldots, i_n}|.$$

**Proof.** Let

$$\alpha = \max_{(i, j): c_{i, j} \neq 0} \left| \frac{a_{i, j}}{b_{i}} \right|, \quad \beta = \max_{i, j} \left| \frac{c_{i, j}}{b_{i}} \right|.$$

By the condition (a), we have $\alpha < 1$. Take a constant $c > \beta/(1 - \alpha)$ and let

$$Q_k = \frac{|C_{i_n-k, j_n-k, \ldots, i_n, j_n}|}{|B_{i_n-k, \ldots, i_n}|}$$

for $k = 0, \ldots, n - 1$. To prove the lemma, we show $|Q_{n-1}| < c$ using induction on $k$. We have

$$Q_0 = \left| \frac{c_{i_1, j_1}}{b_{i_1}} \right| \leq \beta < c$$

and by (1),

$$Q_k = \frac{|a_{i_n-k, j_n-k}C_{i_n-k+1, j_n-k+1, \ldots, i_n, j_n} + c_{i_n-k, j_n-k}B_{i_n-k+1, \ldots, i_n}|}{|b_{i_n-k}B_{i_n-k+1, \ldots, i_n}|} \leq \frac{|a_{i_n-k, j_n-k}|}{|b_{i_n-k}|}Q_{k-1} + \frac{|c_{i_n-k, j_n-k}|}{|b_{i_n-k}|}$$

for $k > 0$. Hence, if $c_{i_n-k, j_n-k} = 0$, then $Q_k \leq Q_{k-1} < c$ by (a) and induction, and if $c_{i_n-k, j_n-k} \neq 0$, then $Q_k \leq \alpha Q_{k-1} + \beta < \alpha c + \beta < c$ by induction and the definition of $c$. This completes the induction step and shows $|C_{i_1, j_1, \ldots, i_n, j_n}|/|B_{i_1, \ldots, i_n}| = |Q_{n-1}| < c$. \qed

To prove our theorem, we will use results from [1] (Theorems A and B and Remark 3.8), which give formulae for the Hausdorff and box dimension of the limit sets for so-called rectangle-like geometric constructions in the plane. For convenience, we present the definition of a rectangle-like construction suited to our case (note that the notation from [1] is changed).
Definition 3.2 (cf. Definitions 3.1 and 3.4 in [1]). Let $\Delta_{i_1,j_1,\ldots,i_n,j_n}$ for $n > 0$ and $(i_1,j_1), \ldots, (i_n,j_n) \in I$ (where $I$ is defined as in Section 2) be compact sets in the plane. Let $a_{i,j}$ and $b_i$ for $(i,j) \in I$ be real numbers, such that $0 < |a_{i,j}| \leq |b_i| < 1$. Set $A_{i_1,j_1,\ldots,i_n,j_n} = a_{i_1,j_1} \cdots a_{i_n,j_n}, B_{i_1,\ldots,i_n} = b_i \cdots b_{i_n}$. We say that $\Delta_{i_1,j_1,\ldots,i_n,j_n}$ and $\Delta_{i'_1,j'_1,\ldots,i'_n,j'_n}$ have the same type, if $i_m = i'_m$ for $m = 1, \ldots, n$. Otherwise, we say that the sets have different types.

We say that the family of sets $\Delta_{i_1,j_1,\ldots,i_n,j_n}$ defines a rectangle-like geometric construction in the plane, if there exist constants $c_1, c_2, c_3, c_4, K > 0$ and points $z_{i_1,j_1,\ldots,i_n,j_n} \in \Delta_{i_1,j_1,\ldots,i_n,j_n}$, such that for every $n$ and every $i_1,j_1,\ldots,i_n,j_n$, the following conditions are satisfied:

(i) $\Delta_{i_1,j_1,\ldots,i_n+1,j_n+1} \subset \Delta_{i_1,j_1,\ldots,i_n,j_n}$,
(ii) $\mathrm{diam} \Delta_{i_1,j_1,\ldots,i_n,j_n} < c_1|B_{i_1,\ldots,i_n}|$,
(iii) for every $z \in \Delta_{i_1,j_1,\ldots,i_n,j_n}$ there are at most $K$ different sets $\Delta_{i'_1,j'_1,\ldots,i'_n,j'_n}$, such that $1/c_0 < |A_{i'_1,j'_1,\ldots,i'_n,j'_n}|/|A_{i_1,j_1,\ldots,i_n,j_n}| < c_0$ (where $c_0 = \max_{i,j} |a_{i,j}|^{-1}$) and $\mathrm{dist}(z, \Delta_{i'_1,j'_1,\ldots,i'_n,j'_n}) < c_2|A_{i_1,j_1,\ldots,i_n,j_n}|$,
(iv) if $\Delta_{i_1,j_1,\ldots,i_n,j_n}$ and $\Delta_{i'_1,j'_1,\ldots,i'_n,j'_n}$ have the same type and do not coincide, then
\[
\mathrm{dist}(z_{i_1,j_1,\ldots,i_n,j_n}, z'_{i'_1,j'_1,\ldots,i'_n,j'_n}) < c_3|A_{i_1,j_1,\ldots,i_n,j_n}|,
\]
where $n_0 = \min\{m \leq n : (i_m,j_m) \neq (i'_m,j'_m)\}$,
(v) if $\Delta_{i_1,j_1,\ldots,i_n,j_n}$ and $\Delta_{i'_1,j'_1,\ldots,i'_n,j'_n}$ have different types, then
\[
\mathrm{dist}(z_{i_1,j_1,\ldots,i_n,j_n}, z'_{i'_1,j'_1,\ldots,i'_n,j'_n}) > c_4 \min(|B_{i_1,\ldots,i_n}|, |B_{i'_1,\ldots,i'_n}|).
\]

The following result was proved in [1].

**Dimension Formula** (cf. Theorems A and B and Remark 3.8 in [1]). Let a family of sets $\Delta_{i_1,j_1,\ldots,i_n,j_n}$ defines a rectangle-like geometric construction in the plane in the sense of Definition 3.2 and let
\[
\Lambda = \bigcap_{n=1}^{\infty} \bigcup_{(i_1,j_1),\ldots,(i_n,j_n) \in I} \Delta_{i_1,j_1,\ldots,i_n,j_n}
\]
be the limit set of the construction. Then the Hausdorff dimension of $\Lambda$ is equal to $\max S$, where
\[
g(p) = \frac{\sum_{(i,j) \in I} p_{i,j} \log \sum_{j'=1}^{m_i} p_{i,j'}}{\sum_{(i,j) \in I} p_{i,j} \log |b_i|} + \frac{\sum_{(i,j) \in I} p_{i,j} \log p_{i,j} - \sum_{(i,j) \in I} p_{i,j} \log \sum_{j'=1}^{m_i} p_{i,j'}}{\sum_{(i,j) \in I} p_{i,j} \log |a_{i,j}|}
\]
(3)
(with the convention $0 \log 0 = 0$) and
\[
S = \left\{ p = (p_{i,j})_{(i,j) \in I} : p_{i,j} \geq 0, \sum_{(i,j) \in I} p_{i,j} = 1 \right\}.
\]

Moreover, the box dimension of $\Lambda$ is equal to the unique real number $D$, such that
\[
\sum_{(i,j) \in I} |b_i|^t |a_{i,j}|^{D-t} = 1, \quad \text{where } t \text{ is the unique real number such that } \sum_{i=1}^{K} |b_i|^t = 1.
\]

To prove our theorem, we will show that the family of sets
\[
\Delta_{i_1,j_1,\ldots,i_n,j_n} = f_{i_1,j_1} \circ \cdots \circ f_{i_n,j_n}(\overline{V}),
\]
where $V$ is the set from the condition (c), defines a rectangle-like construction in the plane. This will be done in the following lemma. Then the formulae for the Hausdorff and box dimension of $\Lambda$ will follow directly from the Dimension Formula.

**Lemma 3.3.** The family of sets

$$\Delta_{i_1,j_1,\ldots,i_n,j_n} = f_{i_1,j_1} \circ \cdots \circ f_{i_n,j_n}(V)$$

satisfies the conditions (i)--(v) from Definition 3.2.

**Proof.** The assertion (i) follows from the inclusion $f_{i,j}(V) \subseteq V$. To show (ii), note that $f_{i_1,j_1} \circ \cdots \circ f_{i_n,j_n}$ maps the unit square onto a parallelogram of height $|B_{i_1,\ldots,i_n}|$, horizontal sides of length $|A_{i_1,j_1,\ldots,i_n,j_n}|$ and other sides of length

$$\sqrt{|B_{i_1,\ldots,i_n}|^2 + |C_{i_1,j_1,\ldots,i_n,j_n}|^2}.$$ 

Take $M > 0$, such that $V \subseteq [-M,M]^2$. Then by (2) and Lemma 3.1 we have

$$\text{diam } \Delta_{i_1,j_1,\ldots,i_n,j_n} < 2M(1 + \sqrt{1 + c^2})|B_{i_1,\ldots,i_n}|,$$

which shows (ii).

Choose a point $v \in V$ and let

$$z_{i_1,j_1,\ldots,i_n,j_n} = (x_{i_1,j_1,\ldots,i_n,j_n}, y_{i_1,j_1,\ldots,i_n,j_n}) = f_{i_1,j_1} \circ \cdots \circ f_{i_n,j_n}(v).$$

To show (iv), take different $\Delta_{i_1,j_1,\ldots,i_n,j_n}$ and $\Delta_{i'_1,j'_1,\ldots,i'_n,j'_n}$ of the same type. Then by (1), $y_{i_1,\ldots,i_n} = y_{i'_1,\ldots,i'_n}$, so dist$(z_{i_1,j_1,\ldots,i_n,j_n}, z_{i'_1,j'_1,\ldots,i'_n,j'_n}) = |x_{i_1,j_1,\ldots,i_n,j_n} - x_{i'_1,j'_1,\ldots,i'_n,j'_n}|$. Take $n_0$ as in (iv). If $n_0 = 1$, then (iv) is obvious, so suppose $n_0 > 1$. Then

$$z_{i_1,j_1,\ldots,i_n,j_n}, z_{i'_1,j'_1,\ldots,i'_n,j'_n} \in \Delta_{i_1,j_1,\ldots,i_{n_0-1},j_{n_0-1}}.$$ 

Since all horizontal sections of $\Delta_{i_1,j_1,\ldots,i_{n_0-1},j_{n_0-1}}$ have diameter at most

$$2M|A_{i_1,j_1,\ldots,i_{n_0-1},j_{n_0-1}}|,$$

we have

$$|x_{i_1,j_1,\ldots,i_n,j_n} - x_{i'_1,j'_1,\ldots,i'_n,j'_n}| \leq 2M|A_{i_1,j_1,\ldots,i_{n_0-1},j_{n_0-1}}|/|a_{i_{n_0},j_{n_0}}| \leq 2c_0M|A_{i_1,j_1,\ldots,i_{n_0},j_{n_0}}|,$$

which gives (iv).

By the conditions (b) and (c), the sets $\Delta_{i,j}$ are contained in the horizontal strips $S_i = f_{i,j}(\mathbb{R} \times [0,1])$ of height $|b_i|$ with disjoint interiors for different $i$. Hence, $\Delta_{i_1,j_1,\ldots,i_n,j_n}$ are in the horizontal strips

$$S_{i_1,\ldots,i_n} = f_{i_1,j_1} \circ \cdots \circ f_{i_n,j_n} (\mathbb{R} \times [0,1])$$

of height $|B_{i_1,\ldots,i_n}|$ with disjoint interiors for different sequences $i_1,\ldots,i_n$. Since $v \in \mathbb{R} \times (0,1)$, we have $z_{i_1,\ldots,i_n} \in \text{int } S_{i_1,\ldots,i_n}$ and

$$\text{dist}(z_{i_1,\ldots,i_n}, \partial S_{i_1,\ldots,i_n}) \geq a|B_{i_1,\ldots,i_n}|,$$

where $a = \text{dist}(v, \mathbb{R} \times \{0,1\}) > 0$. Hence, if $\Delta_{i_1,j_1,\ldots,i_n,j_n}$ and $\Delta_{i'_1,j'_1,\ldots,i'_n,j'_n}$ have different types, then $S_{i_1,j_1,\ldots,i_n,j_n}, S_{i'_1,j'_1,\ldots,i'_n,j'_n}$ have disjoint interiors and

$$\text{dist}(z_{i_1,j_1,\ldots,i_n,j_n}, z_{i'_1,j'_1,\ldots,i'_n,j'_n}) \geq |y_{i_1,\ldots,i_n} - y_{i'_1,\ldots,i'_n}| \geq a(|B_{i_1,\ldots,i_n}| + |B_{i'_1,\ldots,i'_n}|),$$

which shows (v).

Finally, we show (iii). Fix $i_1, j_1, \ldots, i_n, j_n$ and a point

$$z = (x, y) \in \Delta_{i_1,j_1,\ldots,i_n,j_n}.$$
Let $\mathcal{D}$ be the collection of all sets $\Delta_{i',j_1',\ldots,i_n',j_n'}$, such that

$$\frac{1}{c_0} < |A_{i_1,j_1',\ldots,i_n,j_n'}|/|A_{i_1,j_1,\ldots,i_n,j_n}| < c_0$$

and $\text{dist}(z, \Delta_{i',j_1',\ldots,i_n',j_n'}) < c_2|A_{i_1,j_1,\ldots,i_n,j_n}|$. To prove (iii), we need to show that $\mathcal{D}$ has at most $K$ elements.

First, note that the sets $\Delta_{i',j_1',\ldots,i_n',j_n'} \in \mathcal{D}$ of different types lie in different strips $S_{i',\ldots,i_n'} = \Delta_{i',j_1',\ldots,i_n',j_n'}$. This easily implies that to show (iii), it is enough to check that $\mathcal{D}'$ has at most $K'$ elements for a constant $K'$, where $\mathcal{D}'$ is the collection of all sets from $\mathcal{D}$ of the same type as $\Delta_{i_1,j_1,\ldots,i_n,j_n}$. Recall that for every $\Delta_{i',j_1',\ldots,i_n',j_n'} \in \mathcal{D}'$ we have $i_m' = i_m$ for $m = 1, \ldots, n$ and $\Delta_{i_1,j_1',\ldots,i_n,j_n'} = \Delta_{i_1,j_1,\ldots,i_n,j_n} \subset S_{i_1,\ldots,i_n}$.

Now we will show that for every $\Delta_{i',j_1',\ldots,i_n',j_n'} \in \mathcal{D}'$ there exists a point $z' = (x',y) \in \Delta_{i_1,j_1',\ldots,i_n,j_n'}$, such that

$$|x' - x| < c_0 |A_{i_1,j_1,\ldots,i_n,j_n}|$$

for a constant $c_0 > 0$ independent of $n, i_1, j_1, \ldots, i_n, j_n$. Now we prove (iii). Obviously, we can assume $\pi_Y$ to be the projection from the plane onto the $Y$-axis. For every $\Delta_{i_1,j_1',\ldots,i_n,j_n'} \in \mathcal{D}'$ and $(x,y) \in \Delta_{i_1,j_1',\ldots,i_n,j_n'}(y)$ define

$$I_0 = V'(\psi), \quad I_m = V'(f_{i_m,j_m'}^{-1} \circ \cdots \circ f_{i_1,j_1'}^{-1}(x,y))$$

for $m = 1, \ldots, n-1$. By the horizontal invariance of the maps $f_{i,j}$, the sets $I_m$ do not depend on $j_1', \ldots, j_n'$ and $x$. Note that $I_m$ intersects $\Delta_{i_1,j_1',\ldots,i_n,j_n'} \subset f_{i_1,j_1'}(V)$ for every $\Delta_{i_1,j_1',\ldots,i_n,j_n'} \in \mathcal{D}'$, in particular $I_m$ intersects $f_{i_1,j_1',j_1'}(V)$. Suppose first that for every $m = 0, \ldots, n-1$ the set $I_m$ intersects only one set $f_{i,j}(V)$ (i.e. $f_{i_1,j_1',j_1'}(V)$). Then we have $j_m' = j_m$ for $m = 1, \ldots, n$, which means that $\mathcal{D}'$ contains only one set (i.e. $\Delta_{i_1,j_1,\ldots,i_n,j_n}$). This obviously proves (iii) in that case. Hence, we can assume that there exists a maximal number $0 \leq l \leq n-1$,
such that \( I_l \) intersects at least two sets \( f_{i,j}(V) \). Take \( \Delta_{i_1,j'_1,\ldots,i_n,j'_n} \in D' \). By the definition of \( I_l \), we have

\[
j'_m = j_m \quad \text{for every } l + 1 < m \leq n. \tag{7}
\]

In particular, this implies

\[
\frac{1}{c_0 c'} < \frac{|A_{i_1,j'_1,\ldots,i_l,j'_l}|}{|A_{i_1,j_1,\ldots,i_l,j_l}|} < c_0 c'
\]

for \( c' = \max_{i,j,j'} |a_{i,j}|/|a_{i,j}| \). By the condition (d), we have \( \text{Leb}_1 I_l > \delta \), so using (8) we get

\[
\text{Leb}_1(\text{int} \; \Delta_{i_1,j'_1,\ldots,i_l,j'_l}^{(y)}) = \text{Leb}_1(f_{i_1,j'_1} \circ \ldots \circ f_{i_l,j'_l}(I_l)) \geq \delta |A_{i_1,j'_1,\ldots,i_l,j'_l}| > \frac{\delta}{c_0 c'} |A_{i_1,j_1,\ldots,i_l,j_l}|. \tag{9}
\]

Moreover, by (8),

\[
\text{diam \; int} \; \Delta_{i_1,j'_1,\ldots,i_l,j'_l}^{(y)} \leq 2Mc'|A_{i_1,j_1,\ldots,i_l,j_l}|,
\]

so by (5) we have

\[
\text{int} \; \Delta_{i_1,j'_1,\ldots,i_l,j'_l}^{(y)} \subset \{ (\tilde{x}, y) : |\tilde{x} - x| < (c_0 + 2Mc')|A_{i_1,j_1,\ldots,i_l,j_l}| \}. \tag{10}
\]

By the condition (c), the sets \( \text{int} \; \Delta_{i_1,j'_1,\ldots,i_l,j'_l}^{(y)} \) for different sequences \( j'_1,\ldots,j'_l \) are pairwise disjoint subsets of \( \mathbb{R} \times \{ y \} \) containing \( \text{int} \; \Delta_{i_1,j'_1,\ldots,i_l,j'_l}^{(y)} \), so using (9) and (10) we conclude that the number of sequences \( j'_1,\ldots,j'_l \) such that \( \Delta_{i_1,j'_1,\ldots,i_l,j'_l} \in D' \), is less than \( K_0 = 2(c_0 + 2Mc')c_0 c'/\delta \). By (7), this means that the number of sequences \( j'_1,\ldots,j'_n \) such that \( \Delta_{i_1,j'_1,\ldots,i_n,j'_n} \in D' \) is less than \( K' = K_0 \max_i m_i \), which proves (iii). \( \square \)

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