Lagrangian Grassmannians, CKP Hierarchy and Hyperdeterminantal Relations

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Abstract: This work concerns the relation between the geometry of Lagrangian Grassmannians and the CKP integrable hierarchy. The Lagrange map from the Lagrangian Grassmannian of maximal isotropic (Lagrangian) subspaces of a finite dimensional symplectic vector space $V \oplus V^*$ into the projectivization of the exterior space $\Lambda_1 V$ is defined by restricting the Plücker map on the full Grassmannian to the Lagrangian sub-Grassmannian and composing it with projection to the subspace of symmetric elements under dualization $V \leftrightarrow V^*$. In terms of the affine coordinate matrix on the big cell, this reduces to the principal minors map, whose image is cut out by the $2 \times 2 \times 2$ quartic hyperdeterminantal relations. To apply this to the CKP hierarchy, the Lagrangian Grassmannian framework is extended to infinite dimensions, with $V \oplus V^*$ replaced by a polarized Hilbert space $H = H_+ \oplus H_-$, with symplectic form $\omega$. The image of the Plücker map in the fermionic Fock space $F = \Lambda^{\infty/2} H$ is identified and the infinite dimensional Lagrangian map is defined. The linear constraints defining reduction to the CKP hierarchy are expressed as a fermionic null condition and the infinite analogue of the hyperdeterminantal relations is deduced. A multiparametric family of such relations is shown to be satisfied by the evaluation of the $\tau$-function at translates of a point in the space of odd flow variables along the cubic lattices generated by power sums in the parameters.

1. Integrable Hierarchies, Grassmannians, $\tau$-Functions

It is well known, since the work of Sato [33] and his school and of Segal and Wilson [37], that solutions of the KP integrable infinite hierarchy of PDE’s are determined by abelian group flows on infinite dimensional Grassmann manifolds. The associated $\tau$-functions satisfy Hirota bilinear relations [19], which may be interpreted as infinite dimensional versions of the Plücker relations determining the embedding of the Grassmannian in an exterior product space. There are also discrete versions of the hierarchy, in which
the appropriate recursion relations appear as special Plücker-type relations [34], and solutions can be related to discrete flows on the Grassmannian [29].

For KP, and finite dimensional reductions, the systems are associated to homogeneous spaces of the A-series of Lie groups, or their infinite dimensional limit. They have generalizations, the BKP, CKP and DKP hierarchies, associated to the B, C and D-series of Lie groups, [12]. There are also discrete versions, expressed as lattice equations [3,4,15,19,23,29,30,35]. In the BKP, CKP and DKP cases, the flows are on (maximal) isotropic Grassmannians, cut out by quadratic relations on infinite exterior algebra spaces, interpreted as fermionic Fock spaces, which are the infinite analogues of spinor (or Clifford) modules for the B or D series, and of suitably restricted linear subspaces for the C series.

The link to isotropic Grassmannians is implicit in previous studies [12,13,40], of BKP or CKP, but the main focus has been solutions to the continuous hierarchies, either as vacuum expectation values in fermionic Fock space, or equivalent representation theoretic constructions, or application of some form of the Riemann–Hilbert “dressing method” [31]. Links between the continuous and discrete hierarchies have generally been based on the use of discrete symmetries, as groups of dressing transformations [30].

The aim of this paper is to derive the link between continuous and discrete hierarchies through evaluations of the \(\tau\)-function at infinite lattices of points embedded within the flow group orbits as Sato did for the case of KP [34], by suitable interpretations of the addition formulae for KP \(\tau\)-functions. In the case of the CKP hierarchy, analogous formulae are derived using flows restricted to infinite Grassmannians of Lagrangian type. The link with lattice recursion systems involves quartic relations of the hyperdeterminantal type [20,23,32]. More generally, we show how continuous KP \(\tau\) functions of the CKP type provide solutions to the hexahedron relations of Kenyon and Pemantle [25,26]. In both finite and infinite dimensions we show, over a generic set, that the “short” Plücker type relations generate the entire set of Plücker relations, and similarly, the lattice recursion relations appearing in the Lagrangian case, are in fact, just the “short” versions of the full set of quartic relations satisfied by solutions of the discrete CKP hierarchy. This discretization therefore provides families of solutions to the hyperdeterminantal relations for every continuous KP \(\tau\)-function of CKP type. (See also [4], where a link between KP tau functions, hyperdeterminantal relations and the hexahedron recurrences is made.)

1.1. The KP hierarchy, infinite Grassmannians and the Plücker map. In the study of the Kadomtsev–Petviashvili (KP) hierarchy [18,33,37], the \(\tau\)-function \(\tau^K \! P_w(t)\) is a key ingredient. It depends on an infinite sequence of commuting flow variables

\[
t = (t_1, t_2, \ldots),
\]

and is parametrized by elements \(w \in \text{Gr}_{H^+}(H)\) of an infinite Grassmannian [12–14, 33,37], consisting of subspaces \(w \subset H\) of a polarized Hilbert space \(H = H^+ + H^\perp\), commensurate with the subspace \(H^+ \subset H\). It satisfies the Hirota bilinear residue relation,

\[
\text{res}_{z = \infty} \left( e^{\sum_{i=1}^{\infty} \delta t_i z^i} \tau^K \! P_w(t - [z^{-1}]) \tau^K \! P_w(t + \delta t + [z^{-1}]) \right) d z = 0,
\]

identically in \(\delta t\), where

\[
\delta t := (\delta t_1, \delta t_2, \ldots), \quad [z^{-1}] := \left( \frac{1}{1}, \frac{1}{2 z^2}, \ldots, \frac{1}{j z^j}, \ldots \right).
\]
Expanding $\tau^K_P(w)(t)$ in a basis of Schur functions \cite{28,33}
\begin{equation}
\tau^K_P(w)(t) = \sum_{\lambda} \pi_{\lambda}(w)s_{\lambda}(t),
\end{equation}
with the flow parameters $(t_1, t_2, \ldots)$ interpreted as normalized power sums
\begin{equation}
t_i = \frac{p_i}{i}, \quad p_i := \sum_{a=1}^{\infty} x_a^i \quad i = 1, 2, \ldots,
\end{equation}
and the labels $\lambda$ denoting integer partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{l(\lambda)} > 0, \cdots)$, the coefficients $\pi_{\lambda}(w)$ may be interpreted as Plücker coordinates of the element $w \in \text{Gr}_{H^+}(H)$. These satisfy the Plücker relations \cite{17,18}, which determine the image of the infinite Grassmannian $\text{Gr}_{H^+}(H)$ under the Plücker map:
\begin{equation}
\Psi_{|H^+,H} : \text{Gr}_{H^+}(H) \to \mathbf{P}(F)
\end{equation}
embedding $\text{Gr}_{H^+}(H)$ into the projectivization of the fermionic Fock space $F$, which is the semi-infinite wedge product space
\begin{equation}F = \Lambda^\infty/2(H) = \sum_{n \in \mathbb{Z}} F_n.\end{equation}

Here $\{|\lambda; n\rangle\}$ is the standard basis \cite{18,21,33} for the fermionic charge $n$ sector $F_n$ of the Fock space, $\{w_1, w_2, \ldots\}$ is an admissible basis \cite{37} for the subspace $w \subset H$, viewed as an element of the connected component of $\text{Gr}_{H^+}(H)$, on which the Fredholm orthogonal projection operator $\Pi^+_w : w \to H^+$ has index $n$ and $\{|\phi\rangle\} \in \mathbf{P}(F)$ denotes the projective equivalence class of $|\phi\rangle \in F$. As in the finite dimensional case, the Plücker coordinates $\{\pi_{\lambda}(w)\}$ are expressible as determinants of suitably defined infinite matrices $W_{\lambda}(w)$, which are maximal minors of the homogeneous coordinate matrix $W(w)$ of the element $w$, relative to an admissible basis \cite{18,37}, and may be interpreted as holomorphic sections of the (dual) determinantal line bundle $\text{Det}^* \to \text{Gr}_{H^+}(H)$.

1.2. The CKP hierarchy, Lagrangian Grassmannians and the Lagrange map. The CKP hierarchy \cite{12,14,21,40} may similarly be parametrized by elements $w^0 \in \text{Gr}^L_{H^+}(H, \omega)$ of the sub-Grassmannian consisting of Lagrangian (i.e., maximal isotropic) subspaces of the Hilbert space $H$, with respect to a complex symplectic product $\omega$ (as defined in Sect.3.3). It only involves the odd flow variables
\begin{equation}t_o = (t_1, t_3, \ldots),\end{equation}
and the corresponding Baker function satisfies the Hirota bilinear residue equation
\begin{equation}\text{res}_{z \to \infty} \left( \Psi_{w^0}(z, t_o) \Psi_{w^0}(-z, t_o + \delta t_o) \right) dz = 0\end{equation}
identically in
\begin{equation}\delta t_o = (\delta t_1, \delta t_3, \ldots).\end{equation}
It may be expressed \([9, 10, 27]\) in terms of a CKP \(\tau\)-function \(\tau_{w^0}^{CKP}(t_o)\) as

\[
\Psi_{w^0}(z, t_o) := z^{-1/2} \left( \psi_{w^0}(z, t_o) \frac{\partial \psi_{w^0}(z, t_o)}{\partial t_1} \right)^{1/2},
\]

where

\[
\psi_{w^0}(z, t_o) := e^{\tilde{\xi}(z, t_o) \tau_{w^0}^{CKP}(t_o) - 2[z^{-1}]_o},
\]

\[
\tilde{\xi}(z, t_o) := \sum_{j=0}^{\infty} t_{2j-1} z^{2j-1}, \quad [z^{-1}]_o := \left( z^{-1}, \frac{1}{3} z^{-3}, \frac{1}{5} z^{-5}, \ldots \right).
\]

The square of \(\tau_{w^0}^{CKP}(t_o)\) is the restriction to vanishing values of the even KP flow variables \(t' := (t_1, 0, t_3, 0, \cdots)\), of a KP \(\tau\)-function \(\tau_{w^0}^{KP}(t)\)

\[
(\tau_{w^0}^{CKP}(t_o))^2 = \tau_{w^0}^{KP}(t')
\]

satisfying the auxiliary criticality condition \([9, 10, 27]\)

\[
\left. \frac{\partial \tau_{w^0}^{KP}(t)}{\partial t_2} \right|_{t=t'} = 0
\]

and, more generally,

\[
\left. \frac{\partial \tau_{w^0}^{KP}(t)}{\partial t_{2j}} \right|_{t=t'} = 0, \quad \forall \ j \in \mathbb{N}^+.
\]

It follows that we have a Schur function expansion

\[
(\tau_{w^0}^{CKP}(t_o))^2 = \sum_{\lambda} \pi_{\lambda}(w^0) s_{\lambda}(t'),
\]

in which the Plücker relations are satisfied by the coefficients \(\{\pi_{\lambda}(w^0)\}\), as well as an infinite set of linear relations which imply that \(w^0 \subset \mathcal{H}\) is a Lagrangian subspace with respect to the symplectic form \(\omega\).

### 1.3. Summary of content and results

Section 2 recalls the setting of finite dimensional Grassmannians, their Plücker embedding in a projectivized exterior space and the Lagrangian Grassmannian \(\text{Gr}^L_V(\mathcal{H}_N, \omega_N)\), consisting of subspaces \(w^0 \subset \mathcal{H}_N\) of the 2\(N\)-dimensional symplectic vector space \(\mathcal{H}_N = V \oplus V^*\) that are maximal isotropic with respect to the canonically defined symplectic form \(\omega_N\). In Sect. 2.4, the Lagrange map

\[
\mathcal{L}^N : \text{Gr}^L_V(\mathcal{H}_N, \omega_N) \to \mathbf{P}(\Lambda(V))
\]

is defined, extending the principal minors map, defined on the space of \(N \times N\) symmetric affine coordinate matrices on the big cell, to the entire Lagrangian Grassmannian.

The linear coefficients \(\mathcal{L}_J(w^0)\) of the image

\[
\mathcal{L}^N(w^0) = \left[ \sum_J \mathcal{L}_J(w^0) e_{-Jc} \right]
\]
relative to a basis $\{e_{-J^c}\}$ for $\Lambda(V)$ labelled by ordered subsets $J \subset \{1, \ldots, N\}$ of integers (where $J^c$ is the complement of $J$) coincide with the Plücker coordinates $\pi_{\lambda}(w^0)$ corresponding to symmetric partitions $\lambda = \lambda^T$. However, the map $\mathcal{L}$ is not one-to-one (cf. [39]). As explained in Sect. 2.6 its fibres are the orbits of the group $(\mathbb{Z}_2)^N$ of reflections within the symplectic 2-planes corresponding to a canonical basis and, generically, are of cardinality $2^{N-1}$.

For Lagrangian subspaces $w^0 \in \text{Gr}_V^L(\mathcal{H}_N, \omega_N)$ in the big cell, the $\mathcal{L}_J$'s are the principal minor determinants of the $N \times N$ symmetric affine coordinate matrix $A(w^0)$. As shown in [20,32], these satisfy the set of quartic relations (2.6.2), the “core” hyperdeterminantal relations, whose orbit under the symplectic subgroup

$$G_N := (\text{SL}(2))^N \times S_N \subset \text{Sp}(\mathcal{H}_N, \omega_N),$$

(1.20)
cuts out the image of the Lagrange map. Combining the quadratic Plücker relations with the linear conditions on the Plücker coordinates which assure that the element $w^0$ is in the Lagrangian Grassmannian $\text{Gr}_V^L(\mathcal{H}_N, \omega_N)$, a new proof of these relations, valid on a Zariski open subset, is provided in Sects. 2.7 and 2.8 (Propositions 2.7 and 2.8). It is also shown how a more general set of relations, the hexahedron recurrence relations, introduced in [25,26] in the study of double dimer coverings and rhombus tilings, follow from the Plücker relations and isotropy conditions for Lagrangian Grassmannians.

The realization of the KP hierarchy in terms of isospectral flows of formal pseudo-differential operators is recalled in Sect. 3.1, together with its reduction to the CKP case. The Grassmannian interpretation of this reduction consists of restricting the KP flows on the infinite Grassmannian $\text{Gr}_H^L(\mathcal{H})$ of subspaces of the underlying polarized Hilbert space $\mathcal{H} = \mathcal{H}_- \oplus \mathcal{H}_+$ of the KP hierarchy, commensurable with $\mathcal{H}_+$, to the subgroup of flows in the odd flow parameters only, acting on the Lagrangian sub-Grassmannian $\text{Gr}_H^L(\mathcal{H}, \omega) \subset \text{Gr}_H^L(\mathcal{H})$ of isotropic subspaces with respect to a suitably defined symplectic form $\omega$ on $\mathcal{H}$. The fermionic representation of the KP $\tau$-function as a vacuum expectation value (VEV) on the associated fermionic Fock space $\mathcal{F} = \Lambda^{\infty/2}(\mathcal{H})$ is recalled in Sect. 3.2.

The symplectic form $\omega$ on $\mathcal{H}$ is introduced in Sect. 3.3 and used to define the infinite symplectic group $\text{Sp}(\mathcal{H}, \omega)$ action on $\mathcal{H}$ and on $\mathcal{F}$. In Sect. 3.5, the reduction conditions from the KP to the CKP hierarchy are expressed as fermionic null conditions equivalent to the Lagrangian condition. Using the bosonization map and the Murnaghan–Nakayama rule, this is shown to imply an infinite set of linear vanishing conditions (Proposition 3.5) satisfied by the Plücker coordinates.

The infinite dimensional analog of the Lagrange map

$$\mathcal{L} : \text{Gr}_{\mathcal{H}_+}^L(\mathcal{H}, \omega) \to \mathcal{P}(\mathcal{F}^S)$$

(1.21)
is introduced in Sect. 3.6.1, mapping the Lagrangian Grassmannian $\text{Gr}_{\mathcal{H}_+}^L(\mathcal{H}, \omega)$ to the projectivization of the subspace $\mathcal{F}^S = \Lambda^{\infty/2}\mathcal{H}_+ \subset \mathcal{F}$ spanned by basis elements corresponding to symmetric partitions. Combining the Plücker relations with the Lagrangian condition, it is shown in Sect. 3.6.2 (Proposition 3.12), that the symmetric partition Plücker coordinates of an element $w^0 \in \text{Gr}_{\mathcal{H}_+}^L(\mathcal{H}, \omega)$ corresponding to a CKP type $\tau$-function satisfy the hyperdeterminantal relations. Finally, in Sect. 3.6.3 it is shown (Proposition 3.13 and Corollary 3.14), as a consequence of the addition formulae for KP $\tau$-functions (generalized Fay identities), that an $N$-parameter family of hyperdeterminantal relations is satisfied by the $\tau$-function, evaluated at the translates of a point in the space of odd flow variables by cubic lattices generated by power sums in the parameters.
2. Plücker Map, Clifford Algebra and Lagrange Map in Finite Dimensions

2.1. The Plücker map and Plücker relations. The Plücker map [17]

\[
\Psi^n_k : \text{Gr}_k(C^n) \to \mathbf{P}(\Lambda^k(C^n))
\]

where \([\phi]\) denotes the projective equivalence class of \(\phi \in \Lambda^k(C^n)\) defines an embedding of the Grassmannian \(\text{Gr}_k(C^n)\) of \(k\)-planes \(w = \text{span}\{W_1, \ldots, W_k\} \subset C^n\) in the projectivization \(\mathbf{P}(\Lambda^k(C^n))\) of the exterior space \(\Lambda^k(C^n)\). It is equivariant with respect to the natural action of the general linear group \(\text{Gl}(n, C)\) on \(\text{Gr}_k(C^n)\) and on \(\mathbf{P}(\Lambda^k(C^n))\). The image \(\Psi^n_k(\text{Gr}_k(C^n)) \subset \mathbf{P}(\Lambda^k(C^n))\) is the intersection of a number of quadrics, the Plücker quadrics, thereby realizing \(\text{Gr}_k(C^n)\) as a projective variety. The Plücker coordinates \(\pi_\lambda(w)\) are the (projectivized) linear coordinates of the image \(\Psi^n_k(w)\) in the standard basis \(\{f_L\}_{L=(L_1,\ldots,L_k)}\) for the exterior space \(\Lambda^k(C^n)\), defined by

\[
f_L := f_{L_1} \wedge \cdots \wedge f_{L_k},
\]

where the multi-index

\[
L := (L_1, \ldots, L_k) \subset \{1, \ldots, n\}
\]

is a \(k\)-element subset of \(\{1, \ldots, n\}\), written in increasing order and \(\{f_1, \ldots, f_n\}\) is the standard basis for \(C^n\). Thus

\[
\Psi^n_k(w) = \left[ \sum_\lambda \pi_\lambda(w) f_L \right],
\]

where the partition \(\lambda = (\lambda_1 \geq \cdots \geq \lambda_k \geq 0)\) associated to \(L\), which labels the Plücker coordinate \(\pi_\lambda(w)\), is given by

\[
\lambda_i = L_{k-i+1} + i - k - 1, \quad i = 1, \ldots, k
\]

and its Young diagram fits into a \(k \times (n-k)\) rectangle.

Equivalently, let \(W\) be the \(n \times k\) homogeneous coordinate matrix of the element \(w\), whose columns are the basis vectors \((W_1, \ldots, W_k)\), viewed as column vectors, and let \(W_\lambda\) be the \(k \times k\) matrix whose \(i\)th row is the \(L_i\)th row of \(W\). Then

\[
\pi_\lambda(w) = \det(W_\lambda).
\]

The labelling by partitions \(\lambda\) or by \(k\) element subsets \(L \subset \{1, \ldots, n\}\) is equivalent, but it is sometimes more convenient to use the multi-index \(L\), in which case we write

\[
\tilde{\pi}_L(w) := \pi_\lambda(w).
\]

The fact that \(\Psi^n_k(w) \in \mathbf{P}(\Lambda^k(C^n))\) is (the projectivization of) a completely decomposable element of \(\Lambda^k(C^n)\) uniquely characterizes the image of the Plücker map. It is equivalent to \(\Psi^n_k(w)\) satisfying the quadratic Plücker relations, which are obtained by contracting it, as a projectivized \(k\)-vector, with the various possible basis elements in \(\Lambda^{k-1}(C^n)\), to obtain elements of \(w\), and noting that, due to the decomposability of \(\Psi^n_k(w)\), their exterior products with \(\Psi^n_k(w)\) must vanish. The vanishing of the components of the resulting elements of \(\Lambda^{k+1}(C^n)\), expressed in terms of Plücker coordinates, define the Plücker relations.
To express these concisely [17], let \((I, J)\) be a pair of ordered subsets of \(\{1, \ldots, n\}\) with cardinalities \(k - 1\) and \(k + 1\), respectively:

\[
I = (I_1, I_2, \ldots, I_{k-1}), \quad 1 \leq I_1 < I_2 < \cdots < I_{k-1} \leq n,
J = (J_1, J_2, \ldots, J_{k+1}), \quad 1 \leq J_1 < J_2 < \cdots < K_{k+1} \leq n.
\] (2.1.8)

For any ordered subset

\[
L = (L_1, \ldots L_r), \quad 1 \leq L_1 < \cdots < L_r \leq n
\] (2.1.9)
of cardinality \(r\), and any \(j \in \{1, \ldots, n\}, j \notin L\), denote by \(L(j)\) the ordered set with elements \((L_1, \ldots L_r, j)\) and

\[
(L_1, \ldots, \hat{L}_m, \ldots, L_r), \quad m = 1, \ldots r
\] (2.1.10)
the subset \(L \setminus \{L_m\}\) with \(L_m\) removed. The Plücker relations are then

\[
\sum_{m=1}^{k+1} (-1)^m \tilde{\pi}(I_1, I_2, \ldots, I_{k-1}; J_m) \tilde{\pi}(J_1, J_2, \ldots, J_{k+1}) = 0,
\] (2.1.11)

where the indexing has been extended to all multi-index distinct sequences, such that Plücker coordinates whose indices differ by a permutation from the increasingly ordered sequence are understood to equal the ordered one times the sign of the permutation.

The relations (2.1.11) are not independent, of course. Generically, a much smaller subset, known as the short Plücker relations, in which the intersection \(I \cap J\) is of cardinality \(k - 2\), suffices to generate them all. If we choose the first \(k - 2\) of the indices to coincide

\[
I' := (I_1 = J_1, \cdots, I_{k-2} = J_{k-2}),
\] (2.1.12)
there are only three possible distinct terms in the sum (2.1.11). Letting

\[
I_{k-1} := i \quad J_{k-1} := j_1, \quad J_k := j_2, \quad J_{k+1} := j_3,
\] (2.1.13)
these are

\[
\tilde{\pi}(I, j_1)\tilde{\pi}(I', j_2, j_3) + \tilde{\pi}(I, j_3)\tilde{\pi}(I', j_1, j_2) + \tilde{\pi}(I, j_2)\tilde{\pi}(I', j_3, j_1) = 0.
\] (2.1.14)

As shown in [18], App. D (cf. also [24]), on a Zariski open set within \(\mathbf{P}(\Lambda^k(C^n))\), these short Plücker relations are sufficient to imply the full set. This follows inductively from the Desnanot–Jacobi identity, and the generalized Giambelli identity, which expresses all Plücker coordinates as minor determinants of the matrix of hook partition Plücker coordinates. Another proof of this fact, formulated more geometrically, is provided in Sect. 2.7.
2.2. Plücker map for $\text{Gr}_V(\mathcal{H}_N)$ and the Clifford algebra. Let $V$ be a complex vector space of dimension $N$, $V^*$ its dual space, and denote by

$$\mathcal{H}_N := V \oplus V^*$$

the direct sum of the two. The Grassmannian $\text{Gr}_V(\mathcal{H}_N)$ of $N$-planes in $\mathcal{H}_N$ is the orbit of $V \subset \mathcal{H}_N$ under the action of the general linear group $\text{Gl}(\mathcal{H}_N)$. The Plücker map

$$\mathfrak{P}_V : \text{Gr}_V(\mathcal{H}_N) \rightarrow \mathbb{P}(\Lambda^N(\mathcal{H}_N))$$

for this case is the $\text{Gl}(\mathcal{H}_N)$ equivariant embedding of $\text{Gr}_V(\mathcal{H}_N)$ in the projectivization $\mathbb{P}(\Lambda^N(\mathcal{H}_N))$ of the exterior space $\Lambda^N(\mathcal{H}_N)$ defined by:

$$\mathfrak{P}_V : w \mapsto [w_1 \wedge \cdots \wedge w_N] \in \mathbb{P}(\Lambda^N(\mathcal{H}_N)),$$

where $\{w_1, \ldots, w_N\}$ is a basis for the subspace $w \in \text{Gr}_V(V \oplus V^*)$. Its image is cut out by the intersection of the Plücker quadrics (2.1.11), for $k = N$, $n = 2N$.

To anticipate the notational conventions used in the next section, we index the basis for $V$ and $V^*$ henceforth as $\{e_{-j}\}_{j=1}^N$ and $\{e_j\}_{j=0}^{N-1}$ respectively, with dualization pairing

$$e_i(e_{-j}) = (-1)^j \delta_{i+1,j}.$$  

(2.2.4)

Ordering the basis for $\mathcal{H}_N$ as $(e_{-N}, \ldots, e_{-1}, e_0, \ldots, e_{N-1})$, define the corresponding basis elements $\{|\lambda\rangle\}$ for $\Lambda^N(\mathcal{H}_N)$ by

$$|\lambda\rangle := e_{l_1} \wedge \cdots \wedge e_{l_N},$$

(2.2.5)

where $\lambda$ is any partition whose Young diagram fits in the $N \times N$ square diagram, and

$$l_j := \lambda_j - j, \quad 1 \leq j \leq N$$

(2.2.6)

are the particle positions associated to the partition (see [18], Chapt. 5, Sec. 5.1)

$$\lambda = (\lambda_1, \ldots, \lambda_{\ell(\lambda)}, 0, \ldots).$$

(2.2.7)

Thus $l_1 > \cdots > l_N$ is a strictly decreasing sequence of $N$ integers between $N - 1$ and $-N$. The “vacuum” (or highest weight) vector is defined as

$$|0\rangle := |\emptyset\rangle = e_{-1} \wedge \cdots \wedge e_{-N},$$

(2.2.8)

and its projectivization is the image $\mathfrak{P}_V(V)$ of $V$ under the Plücker map. A (complex) scalar product on $\Lambda^N(\mathcal{H}_N)$ is defined, in bra/ket notation, by requiring the $\{|\lambda\rangle\}$ basis to be orthonormal

$$\langle \lambda | \mu \rangle = \delta_{\lambda \mu}.$$  

(2.2.9)

Following Cartan [7, 8], define the natural complex scalar product $Q$ on $\mathcal{H}_N \oplus \mathcal{H}^*_N$ by

$$Q((X, \xi), (Y, \eta)) = \eta(X) + \xi(Y), \quad X, Y \in \mathcal{H}_N, \quad \xi, \eta \in \mathcal{H}^*_N,$$

(2.2.10)
and let $\text{Cl} \left( \mathcal{H}_N \oplus \mathcal{H}_N^*, Q \right)$ denote the corresponding Clifford algebra on $\mathcal{H}_N \oplus \mathcal{H}_N^*$. The standard irreducible representation

$$\Gamma : \text{Cl} \left( \mathcal{H}_N \oplus \mathcal{H}_N^*, Q \right) \to \text{End}(\Lambda(\mathcal{H}_N)),$$

$$\Gamma : \sigma \mapsto \Gamma_\sigma$$

is generated by the linear elements, which are represented by exterior and interior multiplication:

$$\Gamma_{v+i\mu} := v \wedge + i_{\mu} \in \text{End}(\Lambda(\mathcal{H}_N)),$$

$$v \in \mathcal{H}_N, \quad \mu \in \mathcal{H}_N^*.$$

The representations of the basis elements, denoted

$$\psi_i := \Gamma_{e_i} = e_i \wedge \quad \psi^\dagger_i := \Gamma_{e^*_i} = i_{e^*_i}, \quad i = -N, \ldots, N - 1,$$

are viewed as finite dimensional fermionic creation and annihilation operators, which satisfy the anticommutation relations

$$[\psi_i, \psi_j]_+ = 0, \quad [\psi^\dagger_i, \psi^\dagger_j]_+ = 0, \quad [\psi_i, \psi_j]_+ = \delta_{ij}$$

as well as the vacuum annihilation conditions

$$\psi_{-i}|0\rangle = \psi^\dagger_{i-1}|0\rangle = 0, \quad i = 1, \ldots, N.$$

2.3. Plücker coordinates on $\text{Gr}_V(\mathcal{H}_N)$. For consistency with standard notations [18,21,33] used in infinite dimensions (see Sect. 3), we index our bases as $\{e_{-N}, \ldots, e_{-1}\}$ and $\{e_0, \ldots, e_{N-1}\}$ to identify $V$ and $V^*$ with $C^N$ and $C^{N*}$, respectively, with the dualization pairing

$$e_i(e_{-j}) = (-1)^j \delta_{i,j-1}, \quad i = 0, \ldots, N - 1; \quad j = 1, \ldots, N.$$  

(2.3.1)

The dual basis $\{e^*_N, \ldots, e^*_{-1}, e^*_0, \ldots, e^*_{N-1}\}$ is thus given by

$$e^*_i = (-1)^{i+1}e_{i-1} \quad i = -N, \ldots, N - 1.$$  

(2.3.2)

Let $w \in \text{Gr}_V(\mathcal{H}_N)$ be an element of the Grassmannian of $N$-dimensional subspaces of $\mathcal{H}_N$, and let $W$ denote the $2N \times N$ dimensional homogeneous coordinate matrix whose columns $\{W_i \in \mathcal{H}_N\}_{i=1,\ldots,N}$ are a basis for $w$ expressed relative to $\{e_{-N}, \ldots, e_{-1}, e_0, \ldots, e_{N-1}\}$. The Plücker coordinates $\{\pi_\lambda(w)\}$ are thus labelled by partitions $\lambda$ whose Young diagrams fit into the $N \times N$ square $(N)^N$. Recall that any partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_{\ell(\lambda)} \geq 0, \cdots)$ of length $\ell(\lambda) \leq N$ may equivalently be labelled by its Frobenius indices [28]

$$\lambda = \lambda(\mathbf{a}|\mathbf{b}), \quad \mathbf{a}|\mathbf{b} := (a_1, \ldots, a_r|b_1, \ldots, b_r),$$  

(2.3.3)

where the Frobenius rank $r$, with $0 \leq r \leq N$, is the number of diagonal terms in the Young diagram of $\lambda$, and

$$\mathbf{a} = (a_1 \geq \cdots \geq a_r \geq 0), \quad \mathbf{b} = (b_1 \geq \cdots \geq b_r \geq 0).$$  

(2.3.4)
are two strictly decreasing sequences of nonnegative integers that represent the “arm” and “leg” lengths’ in the Young diagram (i.e., the number of squares to the right of and below the $r$ diagonal elements, respectively).

To each partition $\lambda \subset (N)^N$, we associate the $N \times N$ submatrix $W_\lambda$ of the homogeneous coordinate matrix whose rows consist of the rows of $W$ in positions $L_1 < L_2 \cdots < L_N$, where

$$L_i := l_{N-1+i}, \quad i = 1, \ldots, N. \quad (2.3.5)$$

The Plücker coordinates $\pi_\lambda(w)$ are given, up to projective equivalence, by the determinants

$$\pi_\lambda(w) = \det(W_\lambda) = \tilde{\pi}_L. \quad (2.3.6)$$

For $0 \leq r \leq N$, let

$$\{I := (I_1, \ldots, I_r) \subset (1, \ldots, N)\}, \quad \{J := (J_1, \ldots, J_r) \subset (1, \ldots, N)\}, \quad (2.3.7)$$

be a pair of (increasingly) ordered subsets of $(1, \ldots, N)$, with cardinalities $|I| = |J| = r$. Define a basis $\{e(I, J)\}$ for $\Lambda^r(\mathcal{H}_N)$, labelled by such pairs $(I, J)$, as

$$e(I, J) := e_{I-1} \wedge e_{J^c}, \quad e_{I-1} \in \Lambda^r(V^*), \quad e_{J^c} \in \Lambda^{N-r}(V), \quad r = 0, \ldots, N, \quad (2.3.8)$$

where

$$J^c = (J^c_1 < \cdots < J^c_{N-r}) \quad (2.3.9)$$

is the (increasingly ordered) complement of $J \subset (1, \ldots, N)$, and

$$e_{I-1} := e_{I_r-1} \wedge \cdots \wedge e_{I_1} \in \Lambda^r(V^*), \quad e_{J^c} := e_{J^c_1} \wedge \cdots \wedge e_{J^c_{N-r}} \in \Lambda^{N-r}(V) \quad (2.3.10)$$

are the corresponding standard basis elements for $\Lambda^r(V^*)$ and $\Lambda^{N-r}(V)$, respectively. (Note that the ordering of successive factors in the wedge products is chosen in both cases to be decreasing from left to right.) To each partition $\lambda \subset (N)^N$ of Frobenius rank $r$, we associate the unique pair $(I, J)$ such that

$$I_i = \lambda_{r-i+1} - r + i, \quad 1 \leq i \leq r, \quad J^c_i = r + i - \lambda_{r+i}, \quad 1 \leq i \leq N - r. \quad (2.3.11)$$

### Lemma 2.1

The relation between the Frobenius indices $(a|b)$ and the pairs $(I, J)$ is given by:

$$a_i = I_{r-i+1} - 1, \quad b_i = J_{r-i+1} - 1, \quad i = 1, \ldots, r, \quad r = 0, \ldots, N - 1. \quad (2.3.12)$$

and the basis elements are related by

$$e(I, J) = |\lambda\rangle = (-1)^{\sum_{i=1}^r b_i} \prod_{i=1}^r \psi_{a_i} \psi_{-b_i-1}^\dagger |0\rangle. \quad (2.3.13)$$
Proof. This follows by direct application of Eqs. (2.2.5), (2.2.6), (2.3.11) and the definition of the Frobenius indices \((a_1, \ldots, a_r | b_1, \ldots, b_r)\).

The pairs \(\{(I, J)\}\) thus provide an equivalent labelling of the partitions \(\{\lambda \subset (N)^N\}\), which we denote
\[
\lambda(I, J) := (a | b) = \lambda^T(J, I).
\] (2.3.14)

We may replace the pair \((I, J)\) by the Frobenius indices \((a | b)\), and label the basis elements equivalently as
\[
|\lambda(I, J)\rangle = |(a | b)\rangle = e(I, J).
\] (2.3.15)

The element obtained by interchanging \((I, J)\) just corresponds to the transposed partition \(e(J, I) = |(b | a)\rangle = |\lambda^T\rangle\).

(2.3.16)

The Plücker map is thus
\[
\mathfrak{P}l_V : Gr_V(\mathcal{H}_N) \to \mathbf{P}(\Lambda^N(\mathcal{H}_N))
\]
\[
\mathfrak{P}l_V : w \mapsto [W_1 \wedge \cdots \wedge W_N] = \left[ \sum_{r=0}^{N} \sum_{(I, J)} \pi_{\lambda(I, J)}(w)e_{(I, J)} \right].
\] (2.3.17)

A symmetric partition is one that equals its transpose \(\lambda = \lambda^T\), so that \(I = J, a = b\), with \((I, J)\) related to \((a, b)\) by (2.3.12).

Remark 2.1. Note that, following standard usage, \(|\lambda|\) denotes the weight of an integer partition \(\lambda\) (i.e., the sum of its parts), while \(\ell(\lambda)\) denotes its length (i.e., the number of nonzero parts). For multi-indices \(K = (K_1, \ldots, K_m)\), however, \(|K| = m\) denotes the cardinality. There should be no confusion, since we consistently use lower case Greek letters \(\lambda, \mu, \ldots\) for partitions, upper case Roman letters \(K = (K_1, \ldots, K_m)\) for increasingly ordered multi-indices and lower case Roman letters \((l_1, l_2, \ldots)\) for (finite or infinite) strictly decreasing sequences of integers.

2.4. Symplectic form and Lagrange map. Define the symplectic form \(\omega_N \in \Lambda^2(\mathcal{H}_N^*)\) on \(\mathcal{H}_N = V \oplus V^*\) as
\[
\omega_N(u + \alpha, v + \beta) = \alpha(v) - \beta(u) \quad u, v \in V, \ \alpha, \beta \in V^*.
\] (2.4.1)

In terms of the basis elements this is
\[
\omega_N := \sum_{i=1}^{N} (-1)^{i+1} e_{-i}^* \wedge e_{i-1}^* = \sum_{i=1}^{N} (-1)^{i} e_{-i} \wedge e_{i-1}.
\] (2.4.2)

where we identify \(\mathcal{H}_N \sim \mathcal{H}^*_N\) via the isomorphism (2.3.2). The symplectic group \(\text{Sp}(\mathcal{H}_N, \omega_N)\) is thus the subgroup of \(\text{Gl}(\mathcal{H}_N)\) that preserves \(\omega_N\):
\[
\text{Sp}(\mathcal{H}_N, \omega_N) = \{ g \in \text{Gl}(\mathcal{H}_N) \mid \omega_N(gX, gY) = \omega_N(X, Y), \ \forall X, Y \in \mathcal{H}_N \}.
\] (2.4.3)
The Lagrangian Grassmannian $\text{Gr}^L_V(\mathcal{H}_N, \omega_N) \subset \text{Gr}_V(\mathcal{H}_N)$ consists of those elements $\{w^0 \in \text{Gr}_V(\mathcal{H}_N)\}$ on which the restriction of $\omega_N$ is totally null:

$$\omega_N|_{w^0} = 0.$$  \hfill (2.4.4)

The “big cell” in $\text{Gr}^L_V(\mathcal{H}_N, \omega_N)$ consists of elements $w^0 \in \text{Gr}_V(\mathcal{H}_N)$ of the form

$$w^0 := \text{span}\{e_{-i} + \sum_{j=1}^N A_{ij}(w^0)(-1)^{j-1}e_{j-1}\}_{i=1,...,N},$$  \hfill (2.4.5)

where $A(w^0) = A^T(w^0)\,.$ is a symmetric $N \times N$ matrix, whose entries are the affine coordinates of $w^0$.

The exterior space $\Lambda(V) \subset \Lambda^N(\mathcal{H}_N)$ may be identified with the subspace

$$\Lambda^S(\mathcal{H}_N) \subset \Lambda^N(\mathcal{H}_N)$$  \hfill (2.4.6)

spanned by basis elements $\{e_{(J,J)} = |\lambda\rangle\}$ corresponding to symmetric partitions $\lambda = \lambda^T$ via the injection map $\iota_\Lambda(V)$:

$$\iota_\Lambda(V) : \Lambda(V) \rightarrow \Lambda^S(\mathcal{H}_N) \subset \Lambda^N(\mathcal{H}_N)$$

$$\iota_\Lambda(V) : \sum J e_{-J^c} \mapsto \sum J \mathcal{L}_J e_{(J,J)},$$  \hfill (2.4.7)

where the sum is over all increasingly ordered multi-indices $J = (J_1, \ldots, J_r) \subset (1, \ldots, N)$ of cardinality $0 \leq r \leq N$.

Viewing $\{e_{(I,J)}\}_{I,J \subseteq (1,\ldots,N), |I|+|J|=N}$ as an orthonormal basis for $\Lambda^N(\mathcal{H}_N)$ and identifying $\Lambda(V)$ with its image under the injection map $\iota_\Lambda(V)$, we have the orthogonal projection

$$\text{Pr}_\Lambda(V) : \Lambda^N(\mathcal{H}_N) \rightarrow \Lambda(V)$$

$$\text{Pr}_\Lambda(V) : \sum_{I,J \subseteq (1,\ldots,N), |I|+|J|=N} \pi_{(I,J)} e_{(I,J)} \mapsto \sum_{J \subseteq (1,\ldots,N)} \pi_{(J,J)} e_{-J^c}$$  \hfill (2.4.8)

**Definition 2.1.** The Lagrange map

$$\mathcal{L}^N : \text{Gr}^L_V(\mathcal{H}_N, \omega_N) \rightarrow \text{P}(\Lambda(V))$$  \hfill (2.4.9)

is defined to be the composition of the restriction of the Plücker map $\mathcal{P}|_{\text{Gr}^L_V(\mathcal{H}_N, \omega_N)}$ to $\text{Gr}^L_V(\mathcal{H}_N, \omega_N)$ with the projection $\text{Pr}_\Lambda(V)$:

$$\mathcal{L}^N := \text{Pr}_\Lambda(V) \circ \mathcal{P}|_{\text{Gr}^L_V(\mathcal{H}_N, \omega_N)}.$$  \hfill (2.4.10)

It is therefore expressed in terms of the basis as

$$\mathcal{L}^N(w^0) = \left[ \sum_{J \subseteq (1,\ldots,N)} \mathcal{L}_J(w^0)e_{-J^c} \right],$$  \hfill (2.4.11)

where

$$\mathcal{L}_J(w^0) := \pi_{\lambda(J,J)}(w^0)$$  \hfill (2.4.12)
will be referred to as the Lagrange coefficients. It follows from the generalized Giambelli identity ([18], Appendix C) that on the big cell of Gr$_V^C(\mathcal{H}_N, \omega_N)$, the Plücker coordinates are, up to projective equivalence, the minor determinants of the affine coordinate matrix $A(w^0)$. In particular, for $w^0$ in the big cell, the $L_J(w^0)$'s are, within projective equivalence, the determinants of the principal submatrices $A_J(w^0)$ with rows and columns in $J$

$$L_J(w^0) = \det \left( A_J(w^0) \right). \quad (2.4.13)$$

Thus, for $w^0$ in the big cell, $L$ maps $w^0$ to an element of $\mathbf{P}(\Lambda^1(V))$ which may be expressed as

$$L^N(w^0) = \left\{ \sum_{I \subseteq (1, \ldots, N)} \det(A_J(w^0))e_{-J^c} \right\}. \quad (2.4.14)$$

in the standard basis $\{e_{-J^c}\}_{J \subseteq (1, \ldots, N)}$.

2.5. Decomposition of $\Lambda(\mathcal{H}_N)$ into irreducible representations of $\text{Sp}(\mathcal{H}_N, \omega_N)$. Viewed as an endomorphism of $\Lambda(\mathcal{H}_N)$, the inner product with the symplectic form will be denoted

$$\hat{\omega}_N^\dagger : \Lambda(\mathcal{H}_N) \to \Lambda(\mathcal{H}_N)$$

$$\hat{\omega}_N : \mu \mapsto i_{\omega_N} \mu, \quad (2.5.1)$$

and the (dual) exterior product as

$$\hat{\omega}_N : \Lambda(\mathcal{H}_N) \to \Lambda(\mathcal{H}_N)$$

$$\hat{\omega}_N : \mu \mapsto \omega_N \wedge \mu. \quad (2.5.2)$$

In terms of fermionic creation and annihilation operators, these can be written as

$$\hat{\omega}_N^\dagger = \sum_{i=0}^{N-1} (-1)^i \psi_{-i-1}^\dagger \psi_i^\dagger. \quad (2.5.3)$$

$$\hat{\omega}_N = - \sum_{i=0}^{N-1} (-1)^i \psi_{-i-1} \psi_i. \quad (2.5.4)$$

Definition 2.2. For every pair of integers $j, k \in \mathbb{N}$ satisfying

$$0 \leq j \leq N, \quad 2j \leq k \leq N + j, \quad (2.5.5)$$

define the subspace $P_{k-2j}^N \subset \Lambda^k(\mathcal{H}_N)$ as

$$P_{k-2j}^N = (\hat{\omega}_N)^j \left( \ker(\hat{\omega}_N^\dagger)|_{\Lambda^{k-2j}(\mathcal{H}_N)} \right) \subset \Lambda^k(\mathcal{H}_N). \quad (2.5.6)$$

In particular $P_N^N \subset \Lambda^N(\mathcal{H}_N)$ is defined by either of the equivalent linear relations

$$\phi \in P_N^N \text{ if and only if } \hat{\omega}_N^\dagger(\phi) = 0, \quad \phi \in P_N^N \text{ if and only if } \hat{\omega}_N(\mu) = 0. \quad (2.5.7)$$
The following is a standard result in the representation theory of $\text{Sp}(\mathcal{H}_N, \omega_N)$ [5, 6, 16]

**Proposition 2.2.** The subspaces $P^k_{k-2j} \subset \Lambda^k(\mathcal{H}_N)$ are invariant and irreducible under the $\text{Sp}(\mathcal{H}_N, \omega_N) \subset \text{Gl}(\mathcal{H}_N)$ action on $\Lambda^k(\mathcal{H}_N)$, which decomposes into their direct sum:

$$\Lambda^k(\mathcal{H}_N) = \bigoplus_{j=0}^{[k/2]} P^k_{k-2j}. \quad (2.5.8)$$

The exterior algebra $\Lambda(\mathcal{H}_N)$ thus decomposes into the direct sum:

$$\Lambda(\mathcal{H}_N) = \bigoplus_{k=0}^{N} \bigoplus_{j=0}^{[k/2]} P^k_{k-2j}. \quad (2.5.9)$$

The isomorphism class of $P^k_{k-2j}$ is given by the partition $(1)^{k-2j}$, and has dimension

$$\begin{aligned}
\left(\begin{array}{c}
2k \\
(2j - 2)
\end{array}\right) - \left(\begin{array}{c}
2k \\
(k - 2j)
\end{array}\right) &= \frac{2j + 1}{k + 1} \left(\begin{array}{c}
2k + 2 \\
(k - 2j)
\end{array}\right).
\end{aligned} \quad (2.5.10)$$

The fact that a subspace $w^0 \subset \mathcal{H}_N$ is Lagrangian is defined by the condition

$$\omega_N(u, v) = 0, \quad \forall u, v \in w^0, \quad (2.5.11)$$

which implies:

**Proposition 2.3.** The image of the restriction of the Plücker map

$$\mathfrak{P}V : \text{Gr}^C(\mathcal{H}_N, \omega_N)) \rightarrow \mathbf{P}(\Lambda^N(\mathcal{H}_N)) \quad (2.5.12)$$

to the Lagrangian Grassmannian $\text{Gr}^C(\mathcal{H}_N, \omega_N))$ lies in $P^N_N$, which is its linear span. The number of independent linear relations (2.5.7) that determine it is thus $\left(\begin{array}{c}2N \\
N - 2\end{array}\right)$.

**Remark 2.2.** A simple way to express these linear relations in terms of Plücker coordinates is given in [11]. For any subset $\alpha \subset \{-N, \ldots, N - 1\}$ of cardinality $N - 2$, whose negative indices are denoted $-I$ and nonnegative indices $J - 1$, let $\lambda(I(\alpha, i), J(\alpha, i))$ denote the partition obtained by adding the pair $(i, i)$ to the increasingly ordered sets $(I, J)$, where $i$ does not belong to $I \cup J$. The linear relations determining $P^N_N$ are then expressed in terms of the Plücker coordinates by

$$\sum \pi(\lambda(I(\alpha, i)), (J(\alpha, i)) = 0, \quad (2.5.13)$$

where the sum is over all $i \in \{1, \ldots, N\}$ that do not belong to $I \cup J$.

In particular all basis elements

$$|\lambda\rangle = |(a|a)) := e_{(I, I)} \quad (2.5.14)$$

corresponding to symmetric partitions $\lambda = \lambda^T$ belong to $P^N_N$. Their linear span may be viewed as a subspace of $P^N_N$ or, equivalently, as the exterior space $\Lambda(V)$, under the injection $i_{\Lambda(V)}$ defined in (2.4.7).
Corollary 2.4. Of these linear relations, it is possible to choose
\[ \sum_{j=1}^{\lfloor N/2 \rfloor} (-1)^{j-1} \binom{2N}{N-2j} = \frac{1}{2} \binom{2N}{N} - 2^{N-1} \]  \hspace{1cm} (2.5.15)

independent ones involving two terms only, consisting of the equalities
\[ \pi_{\lambda}(w^0) = \pi_{\lambda}^0(w^0) \]  \hspace{1cm} (2.5.16)

between Plücker coordinates corresponding to all pairs \((\lambda, \lambda^T)\) of distinct partitions within the square \((N)^N\).

2.6. Lagrange map, hyperdeterminantal relations and inverse.

2.6.1. Lagrange coefficients, principal minors, hyperdeterminantal relations  On the big cell, the **hyperdeterminantal relations**\[20,32\] are satisfied by the principal minor determinants of the affine coordinate matrix which, up to projectivization, coincide with the Lagrange coefficients
\[ D_J(A(w^0)) : = \det(A_J(w^0)) = L_J(w^0). \]  \hspace{1cm} (2.6.1)

To express these, we extend the definition of \(L_J\) to allow any distinct subset \(J = (J_1, \ldots, J_r) \subset (1, \ldots, N)\) of cardinality \(r\), regardless of order, with the value of \(L_J\) the same for all orderings. Now choose an additional triplet \((j_1, j_2, j_3)\) of distinct elements of \(\{1, \ldots, N\}\) which are also distinct from the elements of \(J\) (so \(r \leq N - 3\)). and denote by \((J, j_a), (J, j_a, j_b)\) and \((J, j_1, j_2, j_3)\), for \(a, b = 1, \ldots, 3\), \(a \neq b\), the subsets of \((1, \ldots, N)\) with the indicated elements. We then have the following result:

Proposition 2.5. The Lagrange coefficients satisfy the “core” hyperdeterminantal relations
\[ L_{j_1, j_2, j_3}^J L_{j_1, j_2, j_3}^J - 2 L_{j_1, j_2, j_3}^J L_{j_1, j_2, j_3}^J = 0. \]  \hspace{1cm} (2.6.2)

Remark 2.3. This result is proved in \[20,32\] for the principal minor determinants of any symmetric \(N \times N\) matrix. We give another proof in Sects. 2.7 and 2.8, based on combining the Plücker relations for any element \(w^0 \in Gr_V(H_N)\) with the linear relations that assure it belongs to the Lagrangian Grassmannian \(w^0 \in Gr_V(H_N, \omega_N)\). In \[32\], it was shown that the image of the principal minors map is cut out by the orbit of the “core” quartic hyperdeterminantal relations (2.6.2), under the subgroup
\[ G_N := (SL(2))^N \times S_N \subset Sp(H_N, \omega_N), \]  \hspace{1cm} (2.6.3)

where the \(SL(2)\)’s act within the planes \(\{e_{-i}, e_{i-1}\}_{i=1,\ldots,N}\) and \(S_N\) by permuting them.
In Sect. 2.8, we identify the eight distinct principal minors of size \((r + a) \times (r + a)\), for \(a = 0, 1, 2, 3\) that correspond to the nonzero columns and rows appearing in (2.6.2). These are all of the same form as the single quartic relation satisfied by the eight symmetric Plücker coordinates for \(\text{Gr}^{L}(\mathbb{C}^3 \oplus \mathbb{C}^5^*, \omega_3)\). By varying the choice of \((J, j_1, j_2, j_3)\) as subsets of \([1, \ldots, N]\), we obtain the core hyperdeterminantal relations (2.6.2).

### 2.6.2. Inverse of the Lagrange map

The Lagrange map (2.4.10) is constant on the orbits of the subgroup

\[
(Z_2)^N = I_e := \{\text{diag}(\epsilon_{-N}, \ldots, \epsilon_{-1}, \epsilon_0, \ldots, \epsilon_{N-1})\}, \subset \text{Sp}(\mathcal{H}_N, \omega_N), \quad (2.6.4)
\]

where

\[
\epsilon_{-i} = \epsilon_{i-1} = \pm 1, \quad i = 1, \ldots, N,
\]

consisting of any number of reflections inside the canonical coordinate 2-planes \(\{\epsilon_{-i}, \epsilon_{i-1}\}_{i=1,\ldots,N}\), since it leaves invariant the Plücker coordinates \(\pi_{\lambda}(w^0)\) for all symmetric partitions \(\lambda = \lambda^T\). In fact, the converse is also true [39]; two elements of \(\text{Gr}^{L}(\mathcal{H}_N, \omega_N)\) have the same image under the Lagrange map if and only if they lie on the same \((Z_2)^N\) orbit. Generically, \((Z_2)^N\) has the 2-element subgroup \(\{\pm I_{2N}\}\) as stability subgroup, and there is an open dense stratum in which all the orbits have \(2^{N-1}\) elements. But there are strata consisting of orbits of all cardinalities \(2^k\), for \(0 \leq k \leq N - 1\), so the quotient by this group action is not a manifold, but an orbifold.

As with the Plücker coordinates [17], the Lagrange coefficients \(\{L_J\}\) may be interpreted as holomorphic sections of a line bundle: the (dual) determinantal line bundle \(\text{Det}^* \rightarrow \text{Gr}^{L}(\mathcal{H}_N, \omega_N)\), defined as the pullback, under the Lagrange map, of the hyperplane section bundle \(O(1) \rightarrow \mathbf{P}(\Lambda(V))\)

\[
\text{Det}^* \rightarrow \text{Gr}^{L}(\mathcal{H}_N, \omega_N) := \mathcal{L}^*_N(O(1) \rightarrow \mathbf{P}(\Lambda(V))). \quad (2.6.6)
\]

Although this is equivalent to the restriction of the dual determinantal line bundle \(\text{Det}^* \rightarrow \text{Gr}(\mathcal{H}_N)\) to \(\text{Gr}^{L}(\mathcal{H}_N, \omega_N)\), the sections corresponding to symmetric partitions \(\Lambda^*(V)\), which is realized as the \(2^N\) dimensional subspace of \(\Lambda^{N*}(\mathcal{H}_N)\) defined by the injection map (2.4.7) or, equivalently, by the basis elements corresponding to symmetric partitions.

### 2.7. The geometry of Plücker relations: restriction to Lagrangian Grassmannians

Recall that the image of the Lagrangian Grassmannian \(\text{Gr}^{L}(\mathcal{H}_N, \omega_N)\) under the Plücker map is cut out in \(\mathbf{P}(\Lambda^N(\mathcal{H}_N))\) by the combination of the Plücker relations, corresponding to a decomposable \(N\)-vector defining the \(N\)-plane and the linear relations following from the fact that the \(N\)-plane is a Lagrangian subspace.

We can restate the Plücker relations as follows. They are determined by first choosing a “seed” multi-index \(I_0\) of cardinality \(i_0 < k - 1\), and then completing it with indices \(i_1, \ldots, i_{k-1-i_0}\) to a multi-index \(J\) of cardinality \(k - 1\), and with \(j_1, \ldots, j_{k+1-i_0}\) to a multi-index \(J\) of cardinality \(k + 1\), in such a way that all the added indices are distinct. The corresponding Plücker relation on \(\mathfrak{M}_k^L(w)\) is then

\[
\sum_{s=1}^{k+1-i_0} (-1)^s \tilde{\pi}_{I_0,j_1,j_2,\ldots,j_{k-1-i_0},j_s} \tilde{\pi}_{I_0,j_1,\ldots,\hat{j_s},\ldots,j_{k+1-i_0}} = 0. \quad (2.7.1)
\]
Thus, the number of terms in the sum is $k + 1 - i_0$. The “short” Plücker relations, occur when $i_0 = k - 2$, and consist of a three term sum

$$
\sum_{cycl(j,k,\ell)} \tilde{\pi} f^0_{i,j} \tilde{\pi} f^0_{k,\ell} = 0.
$$

(2.7.2)

We can show, by restriction and projection to subspaces, that a combination of suitably chosen short Plücker relations with the isotropy condition for Lagrangian Grassmannians imply the full set of Plücker relations with the isotropy condition, at least on a generic locus. We will see in the next section that the short relations and isotropy then determine the hyperdeterminantal conditions. The idea, roughly, is to intersect with a family of coordinate subspaces. To illustrate this, first consider the corresponding statement, for the ordinary Grassmannian of $k$-planes in $n$-space, that the short (three-term) Plücker relations determine the Grassmannian, on a generic locus. This is proved in [18], App. D (cf. also [24]), using determinantal identities. To see it geometrically, note that for the multi-index $I_0$, intersection with a subspace determined by the complementary multi-index $I_0^c$ is simply given on the level of the exterior algebra by contraction by $f_{I_0}$, as long as this contraction gives a non-zero result (this is the necessary genericity). The Plücker relation “survives” this operation simply by removal of the $I_0$, and becomes a relation for smaller dimensional planes in a smaller dimensional space. In particular, if $i_0 = k - 2$, so that the Plücker relation has length three, the relation becomes one for a two dimensional space in $n - k + 2$ space, if $n$ is our initial dimension. But for these the only Plücker relations are short. Doing this for all possible choices of $I_0$ of length $k - 2$, we find:

**Proposition 2.6.** Let $\phi$ be a $k$-vector in $\Lambda^k(U)$ which is generic, in the sense of belonging to the Zariski open set on which the contractions $i_{f^0_I}(\phi)$ are non-zero. If when contracted with every coordinate $k - 2$-vector it defines a 2-plane, (i.e. is a decomposable 2-vector), then $\phi$ defines a $k$-plane (i.e., is decomposable). In consequence, the three-term Plücker relations for the $k$-vector imply the full set of Plücker relations.

The proof consists in taking local coordinates, and then looking at all these two-planes, checking that they are in a suitable set compatible, and then piecing together the result into a $k$-plane.

We now consider a similar question for Lagrangian subspaces. Here, there will be a family of six dimensional spaces obtained from the coordinate subspaces. However, the procedure will not just be one of intersection, but rather intersection followed by projection. (Note that this is completely in the spirit of symplectic reduction, where one first restricts to a subvariety, then quotients by a null foliation.) Define the basis

$$
\{f_1, f_2, \ldots, f_N, f_1^*, f_2^*, \ldots, f_N^*\}
$$

(2.7.3)

of the $2N$-dimensional symplectic space $\mathcal{H}_N$ by

$$
f_i := e_{i-1}, \quad f_i^* := (-1)^{i-1} e_i, \quad 1 \leq i \leq N,
$$

(2.7.4)

In this basis, the symplectic form is

$$
\omega_N = \sum_j f_j \wedge f_j^*.
$$

(2.7.5)
Denote the components of a vector \( v \in \mathcal{H}_N \) relative to this basis as \( \{a_i, a_i^*\}_{i=1,\ldots,N} \).

\[
v = \sum_i (a_i f_i + a_i^* f_i^*). \tag{2.7.6}
\]

The corresponding basis \( \{f_K\} \) for \( \Lambda^N(\mathcal{H}_N) \) is given by

\[
f_K = f_{K_1} \wedge \cdots f_{K_N}, \tag{2.7.7}
\]

where \( K \) is a multi-index \((K_1, K_2, \ldots K_N)\) with distinct, increasingly ordered \( K_i \)'s, first of type \( j \), followed by those of type \( j^* \), \( j \in \{1, \ldots, N\} \). Relative to this basis, any \( N \)-vector \( \phi \in \Lambda^N(\mathcal{H}_N) \) may be expressed as

\[
\phi = \sum_K \tilde{\pi}_K f_K. \tag{2.7.8}
\]

Note that for \( K \) to correspond to a symmetric partition, in the notation of the preceding sections means that, for all \( j \in \{1, \ldots, N\} \), \( K = (K_1, \ldots, K_N) \) contains either \( j \) or \( j^* \), but not both. This is equivalent to the corresponding decomposable \( N \)-vector \( f_K \) being the Plücker image of a Lagrangian (i.e., maximal isotropic) subspace. We also will require basis multi-vectors for \( \Lambda^k(\mathcal{H}_N) \) of degree \( k < N \) which satisfy the symmetry condition that they contain either \( j \) or \( j^* \), but not both. The corresponding multi-indices, viewed as subsets of \( \{i, i^*\}_{i=1,\ldots,N} \) are defined as follows. Let \( I = (I_1, \ldots, I_k) \subset (1, \ldots, N) \) be an increasingly ordered subset, viewed as a multi-index of cardinality \( k \). We then choose a function \( A \) on the space of such \( k \)-indices which, to each \( I_j \) associates either \( A(I_j) = I_j \) or \( I_j^* \). Let \((A, I)\) denote the corresponding “marked” multi-index of cardinality \( k \)

\[
(A, I) = (A(I_1), A(I_2), \ldots A(I_k)) \tag{2.7.9}
\]

(written in their correct order) and denote the corresponding basis multivectors \( f_{(A, I)} \). For each \((A, I)\), there is a complementary \((B, J)\) consisting of the complementary elements; i.e. \( B(I_j) = I_j^* \) (resp. \( I_j \)) if and only if \( A(I_j) = I_j \) (resp. \( I_j^* \)). Let

\[
p_{(B, I)} : \mathcal{H}_N \to \mathcal{H}_N \tag{2.7.10}
\]

denote the projection map onto the \((N+k)\)-dimensional subspace spanned by the basis vectors complementary to \( \{f_{B(I_1)}, \ldots, f_{B(I_k)}\} \), with kernel the space span\( \{f_{B(I_1)}, \ldots, f_{B(I_k)}\} \), and let

\[
p_J^{(B, I)} : \Lambda^J(\mathcal{H}_N) \to \Lambda^J(\mathcal{H}_N), \quad j \in \{1, \ldots, N\} \tag{2.7.11}
\]

denote the lift of this map to \( \Lambda^J(\mathcal{H}_N) \).

Our operations will be: contraction \( i_{f_{(A, I)}} \phi \) of \( \phi \in \Lambda^N(\mathcal{H}_N) \) with \( f_{(A, I)} \) (so intersection with the co-isotropic plane corresponding to \( \tilde{\pi}_{(A, I)} = 0 \)), giving an element of \( \Lambda^{N-k}(\mathcal{H}_N) \), followed by projection \( p_{(B, I)}^{N-k} \) to \( \Lambda^{N-k}(p_{(B, I)}(\mathcal{H}_N)) \subset \Lambda^{N-k}(\mathcal{H}_N) \) (i.e., setting the corresponding coordinates to zero.) Note that if we define the contraction and projection in such a way that we stay in \( \mathcal{H}_N \), the two operations commute, in the sense that

\[
p_{(B, I)}^{N-k} \circ i_{f_{(A, I)}} = i_{f_{(A, I)}} \circ p_{(B, I)}^N \tag{2.7.12}
\]
Now let \((A, B, I)^c\) denote the set of \(2N - 2k\) indices in the complement of the union of the multi-indices \((A, I), (B, I)\); i.e., the indices \(\{j, j^*\}_{j \in I^c}\), and let \(H_{(A, B, I)^c}\) be the \(2N - 2k\) dimensional space spanned by the vectors with indices in \((A, B, I)^c\). Thus
\[
P_{(B, I)}^{N-k} \circ i_{f(A, I)}(\phi) = i_{f(A, I)} \circ P_{(B, I)}^N(\phi) \in \Lambda^{n-k} \in \Lambda^{N-k}(H_{(A, B, I)^c}) \subset \Lambda^{N-k}(H_N)
\]
gives us an \(N - k\)-vector in \(\Lambda^{N-k}(H_{(A, B, I)^c})\) for any \(\phi \in \Lambda^N(H_N)\).

We now fix \(k = N - 3\). The result of the contraction and projection is now a 3-vector on the 6-dimensional subspace \(H_{(A, B, I)^c} \subset H_N\). Applying this to a decomposable isotropic element \(\phi \in \Lambda^N(H_N)\), the resulting 3-vector is again decomposable and isotropic in \(\Lambda^3(H_{(A, B, I)^c})\). We also have the converse:

**Proposition 2.7.** A generic element \(\phi \in \Lambda^N(H_N)\) is the Plücker image of a Lagrangian plane \(w^0 \in \text{Gr}_{N}^{\mathbb{C}}(H_N)\) if and only if, for all \((A, I)\) with \(I\) of cardinality \(N - 3\), the elements
\[
P_{(B, I)}^3(i_{f(A, I)}(\phi)) = i_{f(A, I)}(P_{(B, I)}^N(\phi))
\]
represent null (isotropic) 3-planes in \(\Lambda^3(H_{(A, B, I)^c})\).

The proof proceeds in essence in taking all these Lagrangian three-planes, and seeing that the fact that they come from a common element \(\phi\) allows us to piece them together into a Lagrangian \(n\)-plane. Again, the genericity required is that the contractions and projections give non-zero results.

Now consider what this means in terms of the coordinates \(\tilde{\pi}_K\) of the original \(N\)-vector \(\phi \in \Lambda^N(H_N)\), and Plücker relations for the \(P_{(B, I)}^3i_{f(A, I)}(\phi)\). The relations on the 3-planes are given by taking the multi-index \((A, I)\) of cardinality \(N - 3\) as “seed”. This is completed in turn by adding to \((A, I)\) first two indices \((L_1, L_2)\), giving a multi-index \(L\) of cardinality \(N - 1\), and then four indices \((K_1, K_2, K_3, K_4)\), giving a multi-index \(K\) of cardinality \(N + 1\). We require that these extra indices \(L, K\) avoid the elements of \((B, I)\). They thus lie in the set \(\{i, i^*, j, j^*, k, k^*\}\), where \(i, j, k\) are the three indices not in \(I\), and \((L_1, L_2)\) and \((K_1, K_2, K_3, K_4)\) can overlap by at most one element. (If they overlap by one element, we add that to the seed.) The corresponding Plücker relations are then
\[
\sum_{s=1}^{4} (-1)^s \tilde{\pi}_{(A, I), L_1, L_2, K_s, \tilde{\pi}_{(A, I), K_1, \ldots, K_4} = 0
\]
for no overlap and
\[
\sum_{s=2}^{4} (-1)^s \tilde{\pi}_{(A, I), L_1, L_2, K_s, \tilde{\pi}_{(A, I), L_1, \ldots, K_s, K_4} = 0
\]
when \(L_1 = K_1\). Varying \(L, K\) gives the equations for the 3-plane in the 6-plane \(H_{(A, B, I)^c}\) corresponding to \((A, I)\), essentially by a correspondence \(\tilde{\pi}(A, I), \mu, \nu, \sigma \leftrightarrow \tilde{\pi}(A, I, \mu, \nu, \sigma)\). The Plücker relations restrict to Plücker relations on \(\text{Gr}_3(H_{(A, B, I)^c})\), for an appropriate choice of indices. Restriction of the isotropy condition is simpler; we just require that contractions with the restriction
\[
\omega_I := \omega_N|H_{(A, B, I)^c}
\]
of the symplectic form to \(H_{(A, B, I)^c}\) give zero. Thus:
Proposition 2.8. For generic \( \phi \), the Plücker relations, together with the symplectic isotropy conditions on \( \phi \), are equivalent to the relations (2.7.15), (2.7.16) for all \((A, I)\) with \( I \) of cardinality \( N - 3 \), together with the isotropy conditions
\[
i_{\alpha | f(3, I)}(p^{3}_{(B, I)}(i_{f(A, I)}(\phi))) = 0,
\]
where \( p^{3}_{(B, I)}(i_{f(A, I)}(\phi)) \) is viewed as an element of \( \Lambda^{3}(H_{(A, B, I)}^c) \).

2.8. Hyperdeterminantal relations for \( \text{Gr}_{C_3}(\mathbb{C}^{3} \oplus \mathbb{C}^{3*}, \omega_3) \). We have thus reduced the problem, at least on an open dense set, to a family of Plücker relations and isotropy conditions in dimensions \((3, 6)\); that is, for elements of \( \Lambda^{3}(\mathbb{C}^{3} \oplus \mathbb{C}^{3*}) \) corresponding to isotropic 3-planes. Our aim is to now combine these into one relation, the hyperdeterminantal relation, for each of these 3-planes.

For \( i, j, k \in \{1, 2, 3, 1^*, 2^*, 3^* \} \), let \( \tilde{\pi}_{ijk} \), in the indicated order, denote the Plücker coordinates of a 3-dimensional subspace \( w^0 \subset \mathbb{C}^{3} \oplus \mathbb{C}^{3*} \), viewed as an element of the Grassmannian \( \text{Gr}_{C_3}(\mathbb{C}^{3} \oplus \mathbb{C}^{3*}, \omega_3) \) whose Plücker image, up to projectivization, is given by
\[
\phi := \mathbb{P}_{N}^{3}(w^0) = \sum_{i,j,k \in \{1,2,3,1^*,2^*,3^*\}} \tilde{\pi}_{ijk} f_i \wedge f_j \wedge f_k \in \Lambda^{3}(\mathbb{C}^{3} \oplus \mathbb{C}^{3*}).
\]
This gives 20 projective coordinates, and so 19 parameters. Eight of these correspond to symmetric partitions:
\[
S_0 := \tilde{\pi}_{123}, \quad S_1 := \tilde{\pi}_{231}^{*}, \quad S_2 := -\tilde{\pi}_{132}^{*}, \quad S_3 := \tilde{\pi}_{123}^{*},
\]
\[
S_{0^{*}} := \tilde{\pi}_{1*2*3*}, \quad S_{1^{*}} := \tilde{\pi}_{1*2*3*}, \quad S_{2^{*}} := -\tilde{\pi}_{2*1*3*}, \quad S_{3^{*}} := \tilde{\pi}_{3*1*2*},
\]
in which the \( \tilde{\pi}_{ijk} \) are chosen such that \( i = 1 \) or \( 1^{*} \), \( j = 2 \) or \( 2^{*} \), \( k = 3 \) or \( 3^{*} \). The remaining 12 “nonsymmetric” Plücker coordinates form, by the linear Lagrange conditions, six equal pairs, which are labelled by mutually dual partitions
\[
T_1 := \tilde{\pi}_{122^{*}} = -\tilde{\pi}_{133^{*}}, \quad T_2 := \tilde{\pi}_{233^{*}} = \tilde{\pi}_{121^{*}}, \quad T_3 := -\tilde{\pi}_{232^{*}} = -\tilde{\pi}_{131^{*}},
\]
\[
T_{1^{*}} := -\tilde{\pi}_{212^{*}} = -\tilde{\pi}_{313^{*}}, \quad T_{2^{*}} := \tilde{\pi}_{323^{*}} = \tilde{\pi}_{112^{*}}, \quad T_{3^{*}} := -\tilde{\pi}_{111^{*}} = \tilde{\pi}_{222^{*}}.
\]
There are 120 three term (“short”) Plücker relations:
\[
\sum_{\nu=1}^{3} (-1)^{\nu} \tilde{\pi}_{i_{\nu}j_{\nu}j_{\nu}} \tilde{\pi}_{i_{\nu}j_{\nu}j_{\nu}j_{\nu}} = 0,
\]
with five distinct indices \((i_{1}, i_{2}, j_{1}, j_{2}, j_{3})\), and 15 four term ones
\[
\sum_{\nu=0}^{3} (-1)^{\nu} \tilde{\pi}_{i_{\nu}i_{\nu}j_{\nu}j_{\nu}} \tilde{\pi}_{j_{\nu}j_{\nu}j_{\nu}} = 0,
\]
with six distinct indices \((i_{1}, i_{2}, j_{0}, j_{1}, j_{2}, j_{3})\). (These are obviously very redundant, since the Lagrangian Grassmannian has dimension 6.)

We can eliminate the non-symmetric coordinates from some of the Plücker relations to obtain one quartic relation for the remaining 8 symmetric coordinates which, in
addition to projectivization, cuts us down to 6 dimensions, and so gives the isotropic Grassmannian, at least on an open set. The two short Plücker relations

\[ \tilde{n}_{123} \tilde{n}_{12} + \tilde{n}_{13} \tilde{n}_{132} + \tilde{n}_{12} \tilde{n}_{13} = 0, \]  
(2.8.6a)

\[ \tilde{n}_{231} \tilde{n}_{13} + \tilde{n}_{21} \tilde{n}_{12} + \tilde{n}_{21} \tilde{n}_{31} = 0, \]  
(2.8.6b)

give

\[ T_1^2 = -S_0 S_1^* + S_2 S_3, \]  
(2.8.7a)

\[ T_1^* = -S_0 S_1 - S_2 S_3^*, \]  
(2.8.7b)

Similarly, we have

\[ T_2^2 = -S_0 S_2^* + S_1 S_3, \]  
(2.8.7c)

\[ T_2^* = -S_0 S_2 - S_1 S_3^*, \]  
(2.8.7d)

\[ T_3^2 = -S_0 S_3^* + S_1 S_2, \]  
(2.8.7e)

\[ T_3^* = -S_0 S_3 - S_1 S_2^*, \]  
(2.8.7f)

The four term relations

\[ \tilde{n}_{123} \tilde{n}_{12} + \tilde{n}_{13} \tilde{n}_{132} + \tilde{n}_{12} \tilde{n}_{13} = 0, \]  
(2.8.8a)

\[ -\tilde{n}_{231} \tilde{n}_{12} - \tilde{n}_{21} \tilde{n}_{13} - \tilde{n}_{21} \tilde{n}_{31} = 0, \]  
(2.8.8b)

give

\[ T_1 T_1^* + T_2 T_2^* = S_0 S_0^* - S_3 S_3^*, \]  
(2.8.9a)

\[ T_1 T_1^* - T_2 T_2^* = S_1 S_1^* - S_2 S_2^*, \]  
(2.8.9b)

and hence

\[ 2T_1 T_1^* = S_0 S_0^* + S_1 S_1^* - S_2 S_2^* - S_3 S_3^*, \]  
(2.8.10a)

\[ 2T_2 T_2^* = S_0 S_0^* - S_1 S_1^* + S_2 S_2^* - S_3 S_3^*, \]  
(2.8.10b)

and similarly, we have

\[ 2T_3 T_3^* = S_0 S_0^* - S_1 S_1^* - S_2 S_2^* + S_3 S_3^*. \]  
(2.8.10c)

Squaring (2.8.10a) and equating this to the product of the expressions (2.8.7a), (2.8.7b) gives

\[ S_0^2 S_0^* + S_1^2 S_1^* + S_2^2 S_2^* + S_3^2 S_3^* = 2S_0 S_1 S_1^* + 2S_0 S_2 S_2^* + 2S_0 S_3 S_3^* + 2S_1 S_2 S_2^* + 2S_1 S_3 S_3^* + 2S_2 S_3 S_3^* - 4S_0 S_1 S_2 S_3 - 4S_0 S_1 S_2 S_3^* \]  
(2.8.11)

or, equivalently,

\[ \tilde{n}_{123} \tilde{n}_{123} + \tilde{n}_{12} \tilde{n}_{12} + \tilde{n}_{13} \tilde{n}_{13} + \tilde{n}_{12} \tilde{n}_{13} = 0, \]  
(2.8.12)
which is the single hyperdeterminantal relation for \( \text{Gr}^L_{\mathbf{C}^3}(\mathbf{C}^3 \oplus \mathbf{C}^3^*, \omega_3) \). (The same relation may be derived mutatis mutandis using the pairs \((T_2, T_2^*)\) or \((T_3, T_3^*)\).)

As explained in Sect. 2.4, on the big cell of the Lagrangian Grassmannian, the \( N \)-dimensional subspace \( w^0 \) is represented as the graph of a map \( A(w^0) : \mathbf{C}^N \rightarrow (\mathbf{C}^N)^* \), given by the affine coordinate matrix \( A(w^0) \), which is symmetric, and the Plücker coordinates corresponding to symmetric partitions are projectively equivalent to its principal minors. Relation (2.8.12) is an example of the “core” hyperdeterminantal relations studied in [20,32]. In [32], it was shown that these relations, orbited by the group \( G_N \) defined in (1.20) as the semi-direct product of \( SL(2, \mathbf{C})^N \) with the symmetric group on \( N \) letters, where the \( SL(2, \mathbf{C}) \)’s act within the 2-planes spanned the dual pairs \((f_i, f_i^*)\), and \( S_N \) permutes them, cut out the variety defined by the principal minors of \( A \).

Note that the symmetric partition Plücker coordinates do not quite determine the isotropic plane. As explained for general \( N \) in Sect. 2.6.2, the short Plücker relations (2.8.7a)–(2.8.7f) only determine the non-symmetric coordinates \((T_1, T_2, T_3, T_{1^*}, T_{2^*}, T_{3^*})\) up to the action of the group \((\mathbf{Z}_2)^3\) of sign changes within the canonical coordinate planes, which replaces these by \((\epsilon_1 T_1, \epsilon_2 T_2, \epsilon_3 T_3, \epsilon_1 T_{1^*}, \epsilon_2 T_{2^*}, \epsilon_3 T_{3^*})\) for \( \{\epsilon_i = \pm 1\}_{i=1,2,3} \). The hyperdeterminantal relation cuts out the variety obtained as the image of any of the points on an orbit, which, generically, is of cardinality 8.

On \( \mathcal{H}_N = V \oplus V^* \), the definitions of the multi-indices \((A, I), (B, I)\) may be adapted to the basis \((e_{-N}, \ldots, e_{N-1})\) as follows. The multinest \((A, I)\) is defined by combining \( I = (I_1, \ldots, I_{N-3}) \subset (1, 2, \ldots, N) \) as before with the “marking” function \( A \) that associates to each \( I_j \) either \( A(I_j) = -I_j \) or \( I_j - 1 \) giving

\[
(A, I) = (A(I_1), A(I_2), \ldots, A(I_{N-3})
\]

(2.8.13)

In the same way, we define the complementary assignments \( B(I_j) = -I_j \) (resp. \( I_j - 1 \)) if \( A(I_j) = -I_j \) (resp. \( -I_j \)), and the complementary marked multi-index \((B, I)\). The operators \( p_{(B, I)}^3 I_{f(A, I)} \) are given, mutatis mutandis, by the same operations of contraction and projection. With thus have:

**Proposition 2.9.** A generic element \( \phi \in \Lambda^N(\mathcal{H}_N) \) represents a Lagrangian plane if and only if the eight symmetric coordinates of all the elements \( p_{(B, I)}^3 I_{f(A, I)}(\phi) \) satisfy the “core” hyperdeterminantal relations (2.6.2).

Finally, if we let the multi-index \( J \) denote the set of indices \( i \in I \) for which \( A(i) = i - 1 \), the symmetric Plücker coordinates of \( p_{(B, I)}^3 I_{f(A, I)}(\phi) \) are precisely the Lagrangian coefficients \( L_J, L_{J, j_1}, L_{J, j_1, j_2}, L_{J, j_1, j_2, j_3} \) of the image of the Lagrange map. Therefore a generic element of \( \mathcal{H}_N \) represents a Lagrangian plane if and only if the image of the Lagrange map satisfies the hyperdeterminantal relations (2.6.2) of Proposition 2.5.

### 2.9. Hexahedron recurrence equations

The hyperdeterminantal relations (2.6.2) were introduced as integrable systems of recurrence relations on lattices by Kashaev [23], who showed that the star triangle relations satisfied by Boltzmann weights for the Ising model imply these for a suitably defined \( \tau \)-function on the \( \mathbf{Z}^3 \) integer lattice. They were studied subsequently by Schief and others [3,4,15,35], as discrete analogs of the CKP hierarchy.

Kenyon and Pemantle [25,26] extended these to a larger system, which they called the hexahedron recurrence, and applied them to the study of double dimer covers and rhombus tilings. These can either be derived directly or, if we include both symmetric
and nonsymmetric Plücker coordinates, by again combining the Plücker relations with the linear Lagrangian condition.

To see this, multiply the short Plücker relations (2.8.7b), by $S_1$ to get:

$$S_1(S_0S_1^* + T_1^2 - S_2S_3) = 0.$$  \hfill (2.9.1)

and another short Plücker relation by $T_1$ to get

$$T_1(S_0S_1^* + S_1T_1 - T_2T_3) = 0.$$  \hfill (2.9.2)

Taking the difference gives

$$S_0(T_1T_1^* - S_1S_1^*) = T_1T_2T_3 - S_1S_2S_3.$$  \hfill (2.9.3a)

which, up to some changes of notation, \(^1\) is one of the hexahedron relations. The others

$$S_0(T_2T_2^* - S_2S_2^*) = T_1T_2T_3 - S_1S_2S_3,$$  \hfill (2.9.3b)

$$S_0(T_3T_3^* - S_3S_3^*) = T_1T_2T_3 - S_1S_2S_3,$$  \hfill (2.9.3c)

$$S_0^+(T_1T_1^* - S_1S_1^*) = T_1^*T_2T_3^* - S_1^*S_2S_3^*,$$  \hfill (2.9.3d)

$$S_0^+(T_2T_2^* - S_2S_2^*) = T_1^*T_2T_3^* - S_1^*S_2S_3^*,$$  \hfill (2.9.3e)

$$S_0^+(T_3T_3^* - S_3S_3^*) = T_1^*T_2T_3^* - S_1^*S_2S_3^*,$$  \hfill (2.9.3f)

are derived similarly. The degree six relation (1.4d) in [26] follows by solving eqs. (2.9.3a)–(2.9.3c) for $T_1^*$, $T_2^*$ and $T_3^*$ and substituting either in (2.9.3d), (2.9.3e) or (2.9.3f).

### 3. The CKP Hierarchy, Infinite Lagrangian Grassmannians and Hyperdeterminantal Relations

#### 3.1. Baker function, Lax operators and CKP reduction

We recall the formulation of the

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3. The CKP Hierarchy, Infinite Lagrangian Grassmannians and Hyperdeterminantal Relations.

The CKP hierarchy as an infinite abelian group action on an infinite dimensional Grassmannian [33,37], its relation to isospectral flows of pseudo-differential operators and reduction to the CKP hierarchy [12,21].

Let $\mathcal{H}$ denote a separable Hilbert space, with orthonormal basis $\{e_i\}_{i \in \mathbb{Z}}$ labelled by the integers. Concretely, we may think of $\mathcal{H}$ as the space of square integrable functions $L^2(S^1)$ on the unit circle $S^1 = \{z := e^{i\theta}, \ 0 \leq \theta < 2\pi\}$ in the complex plane with hermitian inner product

$$\langle f, g \rangle := \frac{1}{2\pi i} \oint_{z \in S^1} \overline{f(z)}g(z) \frac{dz}{z}.$$  \hfill (3.1.1)

and (for reasons of historical conventions), choose the basis elements as the monomials

$$e_i := z^{-i-1}, \ i \in \mathbb{Z}.$$  \hfill (3.1.2)

\(^1\) To compare with the notation of [20,25,26], set

$$S_0 = A_0 = h = a_0, \ S_1 = A_1 = h_{(1)} = a_1, \ S_2 = A_2 = -h_{(2)} = -a_2, \ S_3 = A_3 = h_{(3)} = a_3;$$

$$S_0^+ = A_{123} = h_{(123)} = a_6^*, \ S_1^+ = A_{23} = h_{(23)} = a_4, \ S_2^+ = A_{13} = -h_{(13)} = -a_5, \ S_3^+ = A_{12};$$

$$h_{(12)} = a_6, \ T_1 = h^{(x)} = a_1, \ T_2 = h^{(y)} = a_2, \ T_3 = h^{(z)} = a_3, \ T_1^* = h^{(x)}_{(1)} = a_1^*, \ T_2^* = h^{(y)}_{(2)} = a_2^*; \ T_3^* = h^{(z)}_{(3)} = a_3^*. $$
Split $\mathcal{H}$ as a direct sum

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$$  \hspace{1cm} (3.1.3)

of Hardy spaces

$$\mathcal{H}_+ := \text{span}\{z^i = e^{-i} e^{1}\}_{i \in \mathbb{N}}, \quad \mathcal{H}_- := \text{span}\{z^{-i} = e^{1} e^{i}\}_{i \in \mathbb{N}^+},$$  \hspace{1cm} (3.1.4)

consisting of elements $f \in \mathcal{H}_+$ that admit analytic continuation to the interior of $S^1$ and $f \in \mathcal{H}_-$ that admit analytic continuation outside $S^1$, with $f(\infty) = 0$ (or, equivalently, the positive and negative power Fourier series). By the infinite Grassmannian $\text{Gr}_{\mathcal{H}_+}(\mathcal{H})$, we mean a suitably defined Banach manifold (see [37]) consisting of subspaces $w \subset \mathcal{H}$ that are commensurable with $\mathcal{H}_+$, in the sense that orthogonal projection $\pi_+ : w \to \mathcal{H}_+$ is a Fredholm operator (with index $n \in \mathbb{Z}$) while the projection $\pi_- : w \to \mathcal{H}_-$ is “small” (either Hilbert–Schmidt, or compact, depending on the context).

We skip the analytic details (see [37] or [18], Chapt. 3), and just require that, via a suitable choice of “admissible basis”, we may identify the spaces $w$ and $\mathcal{H}_+$ as isomorphic, so it is meaningful to define the determinant $\det(\pi_+ : w \to \mathcal{H}_+)$ of the projection map. We also define (as in [37]) the general linear group $\text{Gl}(\mathcal{H})$ of invertible endomorphisms of $\mathcal{H}$ (satisfying certain admissibility conditions), its Lie algebra $\text{gl}(\mathcal{H})$, and the abelian subgroup of shift flows $\Gamma_+ \subset \text{Gl}(\mathcal{H})$

$$\Gamma_+ := \{\gamma_+ (t) \in \text{Gl}(\mathcal{H}), \; \gamma_+ (t)\gamma_+(s) = \gamma_+(t+s)\},$$  \hspace{1cm} (3.1.5)

where $\mathbf{t} = (t_1, t_2, \ldots)$ are the KP flow variables, and the abelian group

$$\Gamma_+ := \{\gamma_+(t) := e^{z(t)}, \; \xi(z, t) := \sum_{i=1}^{\infty} i z^i\}$$  \hspace{1cm} (3.1.6)

acts on $f \in \mathcal{H} = L^2(S^1)$ by multiplication. This lifts in the standard way to an action on the Grassmannian

$$\Gamma_+ \times \text{Gr}_{\mathcal{H}_+}(\mathcal{H}) \to \text{Gr}_{\mathcal{H}_+}(\mathcal{H})$$

$$\gamma_+(t), w) \mapsto \gamma_+(t)w.$$  \hspace{1cm} (3.1.7)

The orbit of an element $w \in \text{Gr}_{\mathcal{H}_+}(\mathcal{H})$ under this action is denoted $\mathcal{O}_w = \{w(t)\}$. The KP $\tau$-function $\tau^K_w$ corresponding to the element $w$ is defined to be

$$\tau^K_w (t) := \det(\pi_+ : w(t) \to \mathcal{H}_+).$$  \hspace{1cm} (3.1.8)

This then satisfies the Hirota bilinear residue equations

$$\text{res}_{z=\infty} (e^{\xi(z, \delta t)} \tau^K_w (t - [z^{-1}]) \tau^K_w (t + \delta t + [z^{-1}])) dz$$  \hspace{1cm} (3.1.9)

identically in the parameters $\delta \mathbf{t} = (\delta t_1, \delta t_2, \ldots)$, where

$$[z^{-1}] := \left(\frac{1}{z}, \frac{1}{2z^2}, \ldots\right),$$  \hspace{1cm} (3.1.10)

and the formal residue $\text{res}_{z=\infty} (\cdots) dz$ signifies evaluation of the coefficient of the $\frac{1}{z}$ term in the formal Laurent series appearing in each coefficient of the monomials in the shift parameters $\{\delta t_i\}$. 

The formal Baker–Akhiezer function (or wave function) and its dual are given by the Sato formulae [21,33] as

\[ \Psi(z, t) := e^{\xi(z,t)} \frac{\tau(t - [z^{-1}])}{\tau(t)} = e^{\xi(z,t)}(1 + \sum_{i=1}^{\infty} a_i(t)z^{-i}), \]  
(3.1.11)

\[ \Psi^*(z, t) := e^{-\xi(z,t)} \frac{\tau(t + [z^{-1}])}{\tau(t)} = e^{-\xi(z,t)}(1 + \sum_{i=1}^{\infty} a_i^*(t)z^{-i}). \]  
(3.1.12)

The formal pseudo-differential “wave operator” and its dual are defined by

\[ \hat{W} := 1 + \sum_{i=1}^{\infty} a_i(t)\partial^{-i}, \]  
(3.1.13)

\[ \hat{W}^\dagger := 1 + \sum_{i=1}^{\infty} a_i^*(t)\partial^{-i}, \]  
(3.1.14)

where

\[ \partial := \frac{\partial}{\partial x}, \]  
(3.1.15)

so that

\[ \Psi(z, t) = \hat{W}(e^{\xi(z,t)}), \quad \Psi^*(z, t) = (\hat{W}^\dagger)^{-1}(e^{-\xi(z,t)}), \]  
(3.1.16)

and the Lax pseudo-differential operator operator is

\[ \mathcal{L} := \hat{W}\partial\hat{W}^{-1} = \partial + \sum_{i=1}^{\infty} u_i(t)\partial^{-i}. \]  
(3.1.17)

It follows (see [33,37], or [18], Chapt. 3) that \( \Psi(z, t) \) satisfies

\[ \frac{\partial \Psi}{\partial t_i} = \mathcal{D}_i \Psi, \quad \forall i \in \mathbb{N}, \]  
(3.1.18)

where

\[ \mathcal{D}_i := (\mathcal{L}_i)_+ \]  
(3.1.19)

is the differential operator part of the pseudo-differential operator \( \mathcal{L}_i \), and \( \mathcal{L} \) satisfies the Lax equations

\[ \frac{\partial \mathcal{L}}{\partial t_i} = [\mathcal{D}_i, \mathcal{L}]. \]  
(3.1.20)

The compatibility conditions

\[ \frac{\partial \mathcal{D}_i}{\partial t_j} - \frac{\partial \mathcal{D}_j}{\partial t_i} + [\mathcal{D}_i, \mathcal{D}_j] = 0 \]  
(3.1.21)

give an infinite set of constant coefficient partial differential equations for the functions \( \{u_i(t)\}_{i \in \mathbb{N}^+} \), each involving derivatives with respect to a triple \( (x, t_i, t_j)_{1 < i < j} \),
with $x$ identified, within a translation constant, with the flow variable $t_1$. The functions $\{u_i(t)\}_{i \in \mathbb{N}^+}$ are uniquely determined, through Eqs. (3.1.12), (3.1.14), (3.1.17), in terms of derivatives of the $\tau$-function, and the set of equations (3.1.21) are equivalent to the Hirota residue equation (3.1.9).

The KP hierarchy is reduced to the CKP one [12, 21] by imposing additional conditions. In terms of the Lax operator $L$, we require the formal anti-self-adjointness condition

$$L^\dagger = -L, \quad (3.1.22)$$

to be satisfied, which implies that

$$D^\dagger_{2j-1} = -D_{2j-1}, \quad j \in \mathbb{N}^+. \quad (3.1.23)$$

It follows that

$$\tau(t) = \tau(\tilde{t}), \quad (3.1.24)$$

where

$$\tilde{t} := (t_1, -t_2, t_3, -t_4, \ldots), \quad (3.1.25)$$

and

$$\Psi^*(z, t) = \Psi(-z, \tilde{t}). \quad (3.1.26)$$

The Hirota bilinear equation (1.2) therefore reduces to

$$\text{res}_{z=0} \Psi(z, t)\Psi(-z, \tilde{t} + \delta \tilde{t})dz = 0 \quad (3.1.27)$$

or, for vanishing even flow variables

$$\text{res}_{z=0} \Psi(z, t')\Psi(-z, t' + \delta t')dz = 0, \quad (3.1.28)$$

which is (1.9) with

$$\Psi(z, t') := \Psi_{w^0}(z, t_0) \quad (3.1.29)$$

3.2. Fermionic representation of KP $\tau$-functions. The fermionic Fock space is the semi-infinite wedge product space of $\mathcal{H}$ with itself

$$\mathcal{F} := \Lambda^{\infty/2} (\mathcal{H}) = \bigoplus_{n \in \mathbb{Z}} \mathcal{F}_n, \quad (3.2.1)$$

which is the orthogonal direct sum of the subspaces $\mathcal{F}_n$ with fermionic charge $n$. Orthonormal bases $\{|\lambda; n\rangle\}$, labelled by pairs $(\lambda, n)$ of integer partitions $\lambda$ of any weight and integers $n \in \mathbb{Z}$, are defined as

$$|\lambda; n\rangle := e_{l_1} \wedge e_{l_2} \wedge e_{l_3} \wedge \cdots, \quad (3.2.2)$$
where \( l_1 > l_2 > \cdots \) is a strictly decreasing sequence of integers, called \textit{particle locations} which saturates, after \( \ell(\lambda) \) terms, to become a sequence of successive decreasing integers. These are determined in terms of the parts \( \{\lambda_i\}_{i \in \mathbb{N}^*} \) of \( \lambda \) and \( n \) (where \( \lambda_i := 0 \) if \( i > \ell(\lambda) \)) by

\[
 l_i := \lambda_i - i + n, \quad i \in \mathbb{N}^*. \tag{3.2.3}
\]

The vacuum element in the sector \( \mathcal{F}_n \) is

\[
 |\emptyset; n\rangle = :|n\rangle = e_{n-1} \wedge e_{n-2} \wedge \cdots. \tag{3.2.4}
\]

As in the finite dimensional case, the Fermi creation and annihilation operators are elements of the representation

\[
 \hat{\Gamma} : \text{Cl}(H \oplus H^*, Q) \to \text{End}(\mathcal{F}) \tag{3.2.5}
\]

of the infinite dimensional Clifford algebra \( \text{Cl}(H \oplus H^*, Q) \), where \( Q \) is the canonical quadratic form on \( H \oplus H^* \).

\[
 Q(v + \mu) = 2\mu(v), \quad v \in H, \; \mu \in H^*, \tag{3.2.6}
\]

generated by exterior and interior multiplication by the basis elements and their duals:

\[
 \hat{\Gamma}_X^{+\xi} := X \wedge +i\xi \in \text{End}(\mathcal{F}), \quad X \in H, \; \xi \in H^*, \tag{3.2.7}
\]

\[
 \hat{\Gamma}_{e_i} = \psi_i := e_i \wedge, \quad \hat{\Gamma}_{e_i^*} = \psi_i^* := i e_i^*: \tag{3.2.8}
\]

These satisfy the anticommutation relations

\[
 [\psi_i, \psi_j]_+ = [\psi_i^+, \psi_j^+]_+ = 0, \quad [\psi_i, \psi_j^+]_+ = \delta_{ij}, \tag{3.2.9}
\]

and the vacuum annihilation conditions

\[
 \psi_{-i}|0\rangle = 0, \quad \psi_{j-1}^+|0\rangle = 0, \quad \forall \; i \in \mathbb{N}^+. \tag{3.2.10}
\]

An equivalent way \cite{18} of representing the basis elements is then

\[
 |\lambda; n\rangle = (-1)^{\sum_{i=1}^r b_i} \prod_{i=1}^r \psi_{a_i+n} \psi_{-b_{i-1}+n}^+ |n\rangle, \tag{3.2.11}
\]

where \((a_1, \ldots, a_r, b_1, \ldots, b_r)\) are the Frobenius indices of the partition \( \lambda \).

The Clifford representation of elements of the Lie algebra \( \mathfrak{gl}(H) \) is

\[
 \hat{A} = \sum_{i, j \in \mathbb{Z}} A_{ij} :\psi_i \psi_j^+:, \tag{3.2.12}
\]

where normal ordering \( :O:\) of bilinear elements means

\[
 :\psi_i \psi_j^+: := \psi_i \psi_j^+ - \langle 0| \psi_i \psi_j^+|0\rangle. \tag{3.2.13}
\]

The corresponding Clifford representation of an element \( g = e^A \in \text{Gl}_0(H) \) in the identity component of the general linear group \( \text{Gl}(H) \) is given by exponentiation

\[
 \hat{g} = e^{\hat{A}}. \tag{3.2.14}
\]
The current components are defined by
\[ J_i := \sum_{j \in \mathbb{Z}} \psi_j \psi_{ji}^\dagger, \quad \text{for } \pm i \in \mathbb{N}^+, \tag{3.2.15} \]
and the abelian group of KP “shift flows” is represented fermionically by
\[ \hat{\gamma}_+(t) = e^{\sum_{i=1}^{\infty} t_i J_i}. \tag{3.2.16} \]
For any \( g \in \text{Gl}_0(\mathcal{H}) \) in the identity component of the general linear group \( \text{Gl}(\mathcal{H}) \)
for which
\[ w := g(\mathcal{H}_+) \in \text{Gr}_{\mathcal{H}_+}^0(\mathcal{H}) \tag{3.2.17} \]
belongs to the virtual dimension 0 component \( \text{Gr}_{\mathcal{H}_+}^0(\mathcal{H}) \) of the Segal–Wilson Grassmannian \([18,37]\), the corresponding KP \( \tau \)-function is given by the fermionic vacuum expectation value (see \([14]\), or \([18]\), Chapt. 5)
\[ \tau^K_{w}(t) = \langle 0 | \hat{\gamma}_+(t) \hat{g} | 0 \rangle, \tag{3.2.18} \]
where \( \hat{g} \) is the fermionic representation of \( g \).
Under the bosonization isomorphism (in the zero fermionic charge sector \( \mathcal{F}_0 \))
\[ \mathcal{I}_{BF} : |v; 0 \rangle \mapsto \langle 0 | \hat{\gamma}_+(t) |v; 0 \rangle, \tag{3.2.19} \]
the element \( \hat{g} |0\rangle \) gets mapped to the KP \( \tau \)-function \( \tau^K_{w}(t) \), the basis elements \(|\lambda; 0 \rangle \) get mapped to Schur functions
\[ \mathcal{I}_{BF}(|\lambda; 0 \rangle) = s_\lambda(t) \tag{3.2.20} \]
and the current components get mapped to:
\[ \mathcal{I}_{BF} \cdot J_i \cdot \mathcal{I}_{BF}^{-1} = \frac{i}{\partial t_i}, \quad \mathcal{I}_{BF} \cdot J_{-i} \cdot \mathcal{I}_{BF}^{-1} = t_i. \tag{3.2.21} \]
The Plücker coordinates \( \{ \pi_\lambda(w) \} \) of the element \( w = g(\mathcal{H}_+) \) appearing as coefficients in the expansion over Schur functions
\[ \tau^K_{w}(t) = \sum_\lambda \pi_\lambda(w) s_\lambda(t), \tag{3.2.22} \]
of the \( \tau \)-function \( \tau^K_{w}(t) \) defined in (3.2.18) are the fermionic matrix elements
\[ \pi_\lambda(w) := \langle \lambda; 0 | \hat{g} |0 \rangle, \tag{3.2.23} \]
and the Plücker map \( \mathfrak{P}_{\mathcal{H}_+} : \text{Gr}_{\mathcal{H}_+}(\mathcal{H}) \to \text{P}(\mathcal{F}) \) applied to an element \( w \) with admissible basis \(([37]) \{ w_1, w_2, \cdots \} \) gives
\[ \mathfrak{P}_{\mathcal{H}_+}(w) := [w_1 \wedge w_2 \wedge \cdots] =: |w\rangle = \left[ \sum_\lambda \pi_\lambda(w) |\lambda; 0 \rangle \right] = \bigcap_{i \in \mathbb{N}} \ker(\hat{\Gamma}_{w_i}). \tag{3.2.24} \]
3.3. Symplectic form $\omega$ on $\mathcal{H}$. Lagrangian Grassmannians and fermionic representation of $\mathfrak{sp}(\mathcal{H}, \omega)$. We define the symplectic form $\omega$ on $\mathcal{H} = L^2(S^1)$ by

$$\omega(f, g) = \frac{1}{2\pi i} \oint_{z \in S^1} f(z)g(-z)dz.$$  \hspace{1cm} \text{(3.3.1)}$$

The subspaces $\mathcal{H}_\pm \subset \mathcal{H}$ are maximal isotropic, i.e. Lagrangian, with respect to $\omega$, and may be viewed as mutually dual $\mathcal{H}_- \sim \mathcal{H}_+^*$ under the pairing

$$f_-(g_+) := \omega(f_-, g_+), \quad \text{for } f_- \in \mathcal{H}_- \text{ and } g_+ \in \mathcal{H}_+.$$  \hspace{1cm} \text{(3.3.2)}$$

In terms of this pairing, the symplectic form is

$$\omega(f_+ + f_-, g_+ + g_-) = f_-(g_+) - g_-(f_+), \quad \text{for } f_\pm, g_\pm \in \mathcal{H}_\pm$$  \hspace{1cm} \text{(3.3.3)}$$

or, in terms of basis elements

$$\omega(e_i, e_j) = -\omega(e_j, e_i) = (-1)^i \delta_i, -j - 1, \quad e_i(e_{j-1}) = (-1)^i \delta_{ij}, \quad i, j \in \mathbb{Z}.$$  \hspace{1cm} \text{(3.3.4)}$$

Following [21], the fermionic representation of the $C_\infty$ Lie algebra is realized as the subalgebra of $\mathfrak{gl}(\mathcal{H}) \sim A_\infty$ consisting of the fixed points

$$\sigma_{-1}(\hat{A}) = \hat{A}$$  \hspace{1cm} \text{(3.3.5)}$$

under the Clifford algebra automorphism generated by

$$\sigma_{-1}(\psi_i) := (-1)^{i+1} \psi^\dagger_{-i-1}, \quad \sigma_{-1}(\psi^\dagger_i) := (-1)^{i+1} \psi_{-i-1}.$$  \hspace{1cm} \text{(3.3.6)}$$

The entire algebra is generated by forming successive commutators from the Chevalley basis elements:

$$\hat{E}_0 = \psi_{-1}\psi^\dagger_0, \quad \hat{F}_0 = \psi_0\psi^\dagger_{-1}, \quad \hat{H}_0 = \psi_{-1}\psi^\dagger_{-1} - \psi_0\psi^\dagger_0,$$

$$\hat{E}_j = \psi_{j-1}\psi^\dagger_j + \psi_{j-1}\psi^\dagger_{j-1}, \quad \hat{F}_j = \psi_j\psi^\dagger_{j-1} + \psi_j\psi^\dagger_{j-1}, \quad \text{for } j \geq 1,$$

$$\hat{H}_j = \psi_{j-1}\psi^\dagger_{j-1} - \psi_j\psi^\dagger_{j-1} - \psi_{j-1}\psi^\dagger_{j-1} - \psi_{j-1}\psi^\dagger_{j-1}, \quad \text{for } j \geq 1.$$  \hspace{1cm} \text{(3.3.7)}$$

This corresponds to the following representation on $\mathcal{H}$ as generators of a subalgebra of $A_\infty \sim \mathfrak{gl}(\mathcal{H})$:

$$E_0e_i = \delta_{i,0} e_{-1}, \quad F_0e_i = \delta_{i,-1} e_0, \quad H_0e_i = \delta_{i,-1} e_{-1} - \delta_{i,0} e_0,$$

$$E_je_i = \delta_{i,j} e_{j-1} + \delta_{i,-j} e_{j-1}, \quad F_je_i = \delta_{i,j-1} e_j + \delta_{i,-j} e_j, \quad \text{for } j \geq 1,$$

$$H_je_i = (\delta_{i,j-1} - \delta_{i,j} + \delta_{i,-j-1} - \delta_{i,-j}) e_i, \quad \text{for } j \geq 1.$$  \hspace{1cm} \text{(3.3.8)}$$

It follows that all these elements $X$ satisfy

$$\omega(Xe_i, e_j) + \omega(e_i, Xe_j) = 0,$$  \hspace{1cm} \text{(3.3.9)}$$

and so do the commutators $[X, Y]$ of any two such elements, and all successive commutators, and hence any element $X \in C_\infty$. The symplectic form $\omega$ is therefore invariant under this $C_\infty$ action, and we may identify $C_\infty \sim \mathfrak{sp}(\mathcal{H}, \omega) \subset \mathfrak{gl}(\mathcal{H})$. 

---

**Notes:**

1. The subspaces $\mathcal{H}_\pm \subset \mathcal{H}$ are maximal isotropic, i.e. Lagrangian, with respect to $\omega$, and may be viewed as mutually dual $\mathcal{H}_- \sim \mathcal{H}_+^*$ under the pairing $f_-(g_+) := \omega(f_-, g_+)$, for $f_- \in \mathcal{H}_- \text{ and } g_+ \in \mathcal{H}_+$.

2. The symplectic form $\omega$ is defined as $\omega(f, g) = \frac{1}{2\pi i} \oint_{z \in S^1} f(z)g(-z)dz$.

3. The fermionic representation of the $C_\infty$ Lie algebra is realized as the subalgebra of $\mathfrak{gl}(\mathcal{H})$ consisting of the fixed points $\sigma_{-1}(\hat{A}) = \hat{A}$.

4. The Clifford algebra automorphism generated by $\sigma_{-1}(\psi_i) := (-1)^{i+1} \psi^\dagger_{-i-1}$.

5. The entire algebra is generated by forming successive commutators from the Chevalley basis elements.

6. The symplectic form $\omega$ is invariant under this $C_\infty$ action.
The orbit of \( \mathcal{H}_+ \subset \mathcal{H} \) under the subgroup \( \text{Sp}(\mathcal{H}, \omega) \subset \text{Gl}(\infty) \) preserving the symplectic form \( \omega \) is the Lagrangian Grassmannian \( \text{Gr}^L_{\mathcal{H}_+}(\mathcal{H}, \omega) \subset \text{Gr}_{\mathcal{H}_+}(\mathcal{H}) \) consisting of maximal isotropic subspaces \( w^0 \subset \mathcal{H} \), on which the restriction of \( \omega \) to any \( w^0 \in \text{Gr}^L_{\mathcal{H}_+}(\mathcal{H}, \omega) \) vanishes
\[
\omega|_{w^0} = 0. \tag{3.3.10}
\]

Bose–Fermi equivalence identifies the basis state \(| \lambda \rangle \) in the zero fermionic charge sector \( \mathcal{F}_0 \) with the Schur function \( s_\lambda(t) \). The operators that are the fermionization of the (Murnaghan–Nakayama) operator \([38]\) of multiplication by \(jt_j\) and its dual \(\frac{\partial}{\partial t_j}\) when acting on the basis of Schur functions are the current components
\[
J_{-j} := \sum_{i \in \mathbb{Z}} \psi_i \psi_{i-j}^\dagger = I_{\mathcal{F}_B}^{-1} \cdot j t_j \cdot I_{\mathcal{F}_B}, \tag{3.3.11}
\]
\[
J_j := \sum_{i \in \mathbb{Z}} \psi_i \psi_{i+j}^\dagger = I_{\mathcal{F}_B}^{-1} \cdot \frac{\partial}{\partial t_j} \cdot I_{\mathcal{F}_B}, \quad r \in \mathbb{N}^+.
\]

### 3.4. Decomposition of \( \mathcal{F} \) into \( \text{Sp}(\mathcal{H}, \omega) \) invariant submodules.

The operator \( i_{\omega_N} \) defined in (2.5.1), may be identified in the infinite dimensional setting as the fermionic operator
\[
\hat{\omega}^\dagger := \sum_{i=0}^{\infty} (-1)^i \psi_{-i-1}^\dagger \psi_i^\dagger, \tag{3.4.1}
\]
which lowers the fermionic charge by 2. Denote the kernel of \( \hat{\omega}^\dagger \), restricted to \( \mathcal{F}_0 \), as
\[
\mathcal{F}_0^{(0)} := \{ |v \rangle \in \mathcal{F}_0 | \hat{\omega}^\dagger |v \rangle = 0 \}. \tag{3.4.2}
\]
This is the infinite dimensional counterpart of the \( \text{Sp}(W, \omega_N) \) invariant submodule \( P_N \subset \Lambda^N(W) \) defined in Sect.2.5. We also define the dual fermionic operator
\[
\hat{\omega} := - \sum_{i=0}^{\infty} (-1)^i \psi_{-i-1}^\dagger \psi_i, \tag{3.4.3}
\]
which raises the fermionic charge by 2, and has the same kernel
\[
\mathcal{F}_0^{(0)} := \{ |v \rangle \in \mathcal{F}_0 | \hat{\omega} |v \rangle = 0 \}. \tag{3.4.4}
\]
It follows that both \( \hat{\omega} \) and \( \hat{\omega}^\dagger \) commute with all elements of \( \text{sp}(\mathcal{H}, \omega) \).

**Lemma 3.1.**
\[
[\hat{\omega}, \hat{X}] = 0 \quad \text{and} \quad [\hat{\omega}^\dagger, \hat{X}] = 0 \quad \forall \hat{X} \in \text{sp}(\mathcal{H}, \omega). \tag{3.4.5}
\]

**Proof.** This is a direct computation for the case of the Chevalley elements (3.3.7). By the Jacobi identity, it also holds for all commutators of such elements, and hence for all elements \( \hat{X} \in \text{sp}(\mathcal{H}, \omega) \). \( \square \)
Remark 3.1. Note that the automorphism $\sigma_{-1}$ in (3.3.6) may be expressed as

$$\sigma_{-1}(\psi_i) = [\hat{\omega}, \psi_i], \quad \sigma_{-1}(\psi_i^+) = [\hat{\omega}^+, \psi_i^+].$$

(3.4.6)

Definition 3.1. For all $j, n \in \mathbb{N}$, define the subspaces

$$\mathcal{F}_n^{(j)} := (\hat{\omega})^j \left( \ker \left( \hat{\omega}^+|_{\mathcal{F}_{n-2j}} \right) \right) \subset \mathcal{F}_n.$$  

(3.4.7)

As in the finite dimensional case, we have a direct sum decomposition:

Proposition 3.2. (cf. [14,21,22]) The fermionic Fock space decomposes into a direct sum of $\mathfrak{sp}(\mathcal{H}, \omega)$ submodules:

$$\mathcal{F} = \bigoplus_{n \in \mathbb{Z}} \bigoplus_{j \in \mathbb{N}^+} \mathcal{F}_n^{(j)}.$$  

(3.4.8)

For any pair of integers $n \in \mathbb{Z}$ and $j \in \mathbb{N}$ satisfying $n \leq j$, $\mathcal{F}_n^{(j)}$ is irreducible.

In what follows, we only consider the submodules $\mathcal{F}_n^{(j)}$ that lie within the zero fermionic charge sector $\mathcal{F}_0$. As in the finite dimensional case, these are all highest weight modules. To see this, define, for all $j \in \mathbb{N}^+$, the element

$$|v(j)\rangle := (\hat{\omega})^j |-2j\rangle,$$

(3.4.9)

i.e., the image of the vacuum element in the fermionic charge sector $\mathcal{F}_{-2j}$ under the $j$th power of the symplectic “raising” map $\hat{\omega}$. Since, as is easily verified, the charged vacuum vector $|-2j\rangle \in \mathcal{F}_{-2j}$ is in the kernel of $\hat{\omega}^+|_{\mathcal{F}_{-2j}}$

$$\hat{\omega}^+|-2j\rangle = 0,$$

(3.4.10)

it follows that $|v(j)\rangle \in \mathcal{F}_0^{(j)}$. Lemma 3.1 then implies that the $\mathfrak{sp}(\mathcal{H}, \omega)$ action on $|v(j)\rangle$ is the same as its action on the vacuum vector $|-2j\rangle \in \mathcal{F}_{-2j}$ in each sector. We have

$$|-2j\rangle = e_{-2j-1} \wedge e_{-2j-2} \wedge \cdots,$$

(3.4.11)

so the action of the raising operators $\{\hat{E}_m|_{m \in \mathbb{N}}\}$, and the Cartan elements $\{\hat{H}_m|_{m \in \mathbb{N}}\}$ on the vacuum vector $|-2j\rangle$ and hence also on $|v(j)\rangle$ are easily computed.

Lemma 3.3.

$$\hat{E}_m |v(j)\rangle = 0, \quad \hat{H}_m |v(j)\rangle = \delta_{j,m} |v(j)\rangle, \quad \forall m \in \mathbb{N}.$$  

(3.4.12)

Proof. The corresponding relations on $|-2j\rangle$, are verified directly from the definition of the operators $\{\psi_i, \psi_i^+\}$. They therefore also hold on $|v(j)\rangle$ by the equivariance of the maps $(\hat{\omega})^j$ and $(\hat{\omega}^+)^j$ implied by Lemma 3.1. \hfill \Box

Since the $|v(j)\rangle$’s are annihilated by the raising operators $\{\hat{E}_m|_{m \in \mathbb{N}}\}$ and are eigenvectors of the Cartan elements $\{\hat{H}_m\}$, with weights given by the eigenvalues $\{\delta_{m,j}\}$, they are highest weight vectors in the various submodules $\mathcal{F}_n^{(j)}$. The submodules $\mathcal{F}_0^{(j)}$ can therefore be viewed as the linear span of the elements $\{\hat{F}_\alpha|v(j)\rangle\}$, where $\alpha = (\alpha_1 > \cdots > \alpha_r = 2j)$ is any strict partition ending with $\alpha_r = 2j$, for all $r \in \mathbb{N}$ and

$$\hat{F}_\alpha := \hat{F}_{\alpha_1} \cdots \hat{F}_{\alpha_r}.$$  

(3.4.13)
Remark 3.2. The lowering operators \( \{ \hat{F}_m \} \) for \( m \neq 2j \) also annihilate the highest weight vectors \( |v(j)\rangle \):

\[
\hat{F}_m |v(j)\rangle = 0 \quad \text{if} \quad m \neq 2j. \tag{3.4.14}
\]

The elements \( \hat{F}_\alpha \) may be viewed as spanning the universal enveloping algebra \( U(\mathcal{N}_-) \) of the subalgebra \( \mathcal{N}_- \subset \mathfrak{sp}(\mathcal{H}, \omega) \) generated by the Chevalley elements \( \{ \hat{F}_m \}_{m \in \mathbb{N}} \). The submodule \( \mathcal{F}^{(j)}_0 \) may be viewed as a quotient of the Verma module corresponding to this universal enveloping algebra, with highest weight the same as \( |v(j)\rangle \), for which we choose a basis \( \{ \mathcal{F}_\alpha \} \) labelled by the strict partitions corresponding to the elements \( \{ \hat{F}_\alpha \} \), and quotient by the span of all those elements \( \mathcal{F}_\alpha \) for which \( |v(j)\rangle \) is in the kernel of \( \hat{F}_\alpha \).

It follows, as in the finite dimensional case (Proposition 2.3), that the image of the Lagrangian Grassmannian \( \text{Gr}^{\mathcal{L}}_{\mathcal{H}^+}(\mathcal{H}, \omega) \subset \text{Gr}_{\mathcal{H}^+}(\mathcal{H}) \) under the Plücker map (3.2.24) is contained within the kernel \( \mathcal{F}^{(0)}_0 \) of \( \hat{\omega} \) (or \( \hat{\omega}^\dagger \)) acting on \( \mathcal{F}_0 \). For \( w^0 \in \text{Gr}^{\mathcal{L}}_{\mathcal{H}^+}(\mathcal{H}, \omega) \), let \( [|w^0\rangle] := \mathfrak{P}^{\mathcal{L}}_{\mathcal{H}^+}(w^0) \) denote its image under the Plücker map. Then

\[
\hat{\omega} |w^0\rangle = 0, \quad \hat{\omega}^\dagger |w^0\rangle = 0, \quad \forall \ w^0 \in \text{Gr}^{\mathcal{L}}_{\mathcal{H}^+}(\mathcal{H}, \omega), \tag{3.4.16}
\]

and that these kernels are equal to the entire submodule \( \mathcal{F}^{(0)}_0 \)

\[
\ker(\hat{\omega})|_{\mathcal{F}_0} = \ker(\hat{\omega}^\dagger)|_{\mathcal{F}_0} = \mathcal{F}^{(0)}_0. \tag{3.4.17}
\]

Proposition 3.4. The images \( \{ [|w^0\rangle] \} \) of the elements \( w^0 \in \text{Gr}^{\mathcal{L}}(\mathcal{H}^+(\mathcal{H}, \omega)) \) of the Lagrangian Grassmannian under the Plücker map span the \( \mathfrak{sp}(\mathcal{H}, \omega) \)-submodule \( \mathcal{F}^{(0)}_0 \).

Proof. By construction, the image of the Lagrangian Grassmannian must span a non-trivial \( \mathfrak{sp}(\mathcal{H}, \omega) \) submodule of \( \mathcal{F}^{(0)}_0 \). By Proposition 3.2 we know that \( \mathcal{F}^{(0)}_0 \) is irreducible, and hence must coincide with the span of the image of the Lagrangian Grassmannian. \( \square \)

3.5. The CKP reduction condition. Combining these results, it follows that a KP \( \tau \)-function admitting a Schur function expansion

\[
\tau^KP(t) = \sum_{\lambda} \pi_{\lambda} s_\lambda(t) \tag{3.5.1}
\]

is of CKP type if and only if its fermionic counterpart \( \sum_{\lambda} \pi_{\lambda} |\lambda\rangle \) is in the submodule \( \mathcal{F}^{(0)}_0 \subset \mathcal{F}_0 \); i.e. if, in addition to the Plücker relations, the linear constraint

\[
\hat{\omega}(\sum_{\lambda} \pi_{\lambda} |\lambda\rangle) = 0 \tag{3.5.2a}
\]

or, equivalently,

\[
\hat{\omega}^\dagger(\sum_{\lambda} \pi_{\lambda} |\lambda\rangle) = 0 \tag{3.5.2b}
\]
is satisfied. This may be expressed equivalently as a set of linear relations for the Plücker coefficients.

Another way to express the fact that a KP \( \tau \)-function

\[
\tau_{w^0}^{KP}, \quad w^0 = h(H_+) \quad (3.5.3)
\]

is the bosonization of an element in \( F^{(0)}_0 \) is to note that in the fermionic VEV representation

\[
\tau_{KP}^{t}(\mathbf{t}) = \langle 0 | \hat{\gamma}^+ (\mathbf{t}) \hat{h} | 0 \rangle, \quad (3.5.4)
\]

the group element \( h \) belongs to \( \text{Sp}(H, \omega) \), so that

\[
\sigma_{-1}(\hat{h}) = \hat{h}. \quad (3.5.5)
\]

From its definition \((3.3.6)\), \(\sigma_{-1}\) acts on the shift flow current component generators as

\[
\sigma_{-1}(J_j) = (-1)^{j+1} J_j, \quad (3.5.6)
\]

and therefore

\[
\sigma_{-1} (\hat{\gamma}^+ (\mathbf{t})) = \hat{\gamma}^+ (\mathbf{i}), \quad (3.5.7)
\]

where

\[
\mathbf{i} := (t_1, -t_2, t_3, -t_4, \ldots). \quad (3.5.8)
\]

Since the (right) ideal of the fermionic Clifford algebra \( \text{Cl}(H \oplus H^*) \), \( Q \) generated by

the annihilators \( \{ \psi_{-i} , \psi_i^+ \}_{i \in \mathbb{N}} \) of the vacuum \( |0\rangle \) is invariant under \( \sigma_{-1} \), the action of \( \sigma_{-1} \) passes to the quotient by this ideal, and hence projects to the Fock space, so that

\[
\langle 0 | \sigma_{-1} (O) | 0 \rangle = \langle 0 | O | 0 \rangle \quad (3.5.9)
\]

for any element \( O \in \text{Cl}(H + H^*, Q) \) of the Clifford algebra. Therefore

\[
\langle 0 | \sigma_{-1} (\hat{\gamma}^+ (\mathbf{t}) \hat{h}) | 0 \rangle = \langle 0 | (\hat{\gamma}^+ (\mathbf{i}) \hat{h}) | 0 \rangle, \quad (3.5.10)
\]

and hence

\[
\tau_{w^0}^{KP} (\mathbf{t}) = \tau_{w^0}^{KP} (\mathbf{i}), \quad \forall \mathbf{t} = (t_1, t_2, \ldots). \quad (3.5.11)
\]

In particular, this implies the conditions

\[
\left. \frac{\partial \tau_{w^0}^{KP} (\mathbf{t})}{\partial t_{2j}} \right|_{t_i = 0} = 0, \quad \forall j \in \mathbb{N}^+. \quad (3.5.12)
\]

As explained in the introduction, the square of any CKP \( \tau \)-function \( \tau_{w^0}^{CKP} (\mathbf{t}_o) \) can be expressed as the restriction to \( \mathbf{t}^\prime := (t_1, 0, t_3, 0, \ldots) \) of a KP \( \tau \)-function \( \tau_{w^0}^{KP} (\mathbf{t}) \),

\[
(\tau_{w^0}^{CKP} (\mathbf{t}_o))^2 = \tau_{w^0}^{KP} (\mathbf{t}^\prime), \quad (3.5.13)
\]
satisfying the auxiliary criticality conditions [27]. It follows that we have a Schur function expansion

$$\tau_{w^0}^{KP}(t') = \sum_{\lambda} \pi_{\lambda}(w^0)s_{\lambda}(t'),$$

(3.5.14)

in which the Plücker coordinates \{\pi_{\lambda}(w^0)\} are subject to the linear constraints (3.5.12).

To find these explicitly, we first recall the Murnaghan–Nakayama rule [38], which gives the product of any Schur function \(s_{\lambda}\) with the power sum symmetric functions \(p_r = r t_r, r \in \mathbb{N}^+\). To express this concisely, let \(\Xi(f)\) be the space of formal linear combinations, with complex coefficients, of symbols \(f_{\lambda}\) indexed by elements of the Young lattice of integer partitions \(\lambda\). Define

$$M_r : \Xi(f) \to \Xi(f)$$

(3.5.15)

to be the linear map generated by

$$M_r(f_{\lambda}) := \sum_{\mu} (-1)^{h(\mu/\lambda)+1} f_{\mu},$$

(3.5.16)

where the sum is over partitions \(\mu\) of weight \(|\lambda| + r\) obtained by augmenting the Young diagram for \(\lambda\) by adding \(r\) squares, such that the skew partition \(\mu/\lambda\) is a continuous border strip (i.e. of width = 1 and height \(h(\mu/\lambda)\)), and let \(M_r^* : \Xi(f) \to \Xi(f)\) be the dual map generated by

$$M_r^*(f_{\lambda}) := \sum_{\mu} (-1)^{h(\lambda/\mu)+1} f_{\mu},$$

(3.5.17)

where the sum is over partitions \(\mu\) of weight \(|\lambda| - r\) obtained by reducing the Young diagram for \(\lambda\) by removing \(r\) squares, such that the skew partition \(\lambda/\mu\) is a continuous border strip of height \(h(\mu/\lambda)\).

Viewing the Schur functions \(s_{\lambda}(t)\) as weighted homogeneous polynomials in the normalized power sums

$$t_i := \frac{1}{i} p_i, \quad i \in \mathbb{N},$$

(3.5.18)

the Murnaghan–Nakayama rule may be expressed as

$$p_r s_{\lambda} = r t_r s_{\lambda} = M_r(s_{\lambda}), \quad r \in \mathbb{N}^+$$

(3.5.19)

and the dual Murnaghan–Nakayama rule as:

$$\frac{\partial s_{\lambda}}{\partial t_r} = M_r^*(s_{\lambda}).$$

(3.5.20)

Identifying the linear space \(\Xi(f)\) with \(\mathcal{F}_0\) such that

$$f_{\lambda} \sim |\lambda\rangle,$$

(3.5.21)

it follows from the bosonization map that Eqs. (3.5.19) and (3.5.20) may equivalently be expressed fermionically as

$$J_{-r}|\lambda\rangle = M_r(|\lambda\rangle), \quad J_r|\lambda\rangle = M_r^* (|\lambda\rangle).$$

(3.5.22)
Therefore, if
\[
[w^0] = \Psi[\mathcal{H}_+](w^0) = \left[ \sum_{\lambda} \pi_{\lambda}(w^0) |\lambda\rangle \right]
\] (3.5.23)
is the image under the Plücker map of an element \( w^0 \in \text{Gr}^L_{\mathcal{H}_+}(\mathcal{H}, \omega) \), we have
\[
J_{-\tau}|w^0\rangle = M_\tau(|w^0\rangle) \quad J_{\tau}|w^0\rangle = M^*_\tau(|w^0\rangle). \quad (3.5.24)
\]
Also note that if all the even flow variables are set equal to 0,
\[
t_e := (t_2, t_4, \ldots) = (0, 0 \ldots), \quad t = t' := (t_1, 0, t_3, 0, \ldots),
\] (3.5.25)
the value of the Schur function \( s_\lambda(t') \) equals that for the transposed partition
\[
s_\lambda(t') = s_{\lambda^T}(t'). \quad (3.5.26)
\]
Defining the orthogonal projector
\[
\Pi_{S}|\lambda\rangle \mapsto \frac{1}{2} \left( |\lambda\rangle + |\lambda^T\rangle \right),
\] (3.5.27)
(extended linearly), whose image is the subspace consisting of elements that are invariant under the transpose involution \( |\lambda\rangle \rightarrow |\lambda^T\rangle \), the fermionic expression of the linear constraint (3.5.12) is therefore
\[
J_{2j} \circ \Pi_{S}|w^0\rangle = 0, \quad \forall \ j \in \mathbb{N}^+. \quad (3.5.28)
\]
Dualizing, Eq. (3.5.28) can equivalently be written in terms of the Plücker coefficients in the expansion (3.5.23).

**Proposition 3.5.** The reduction conditions (3.5.12) are equivalent to the following set of linear relations satisfied by the Plücker coefficients in the expansion (3.5.14):
\[
M_{2j}(\pi_{\lambda}(w^0)) + M_{2j}(\pi_{\lambda^T}(w^0)) = 0, \quad \forall \ j \in \mathbb{N}^+. \quad (3.5.29)
\]

3.6. Lagrange map and hyperdeterminantal relations.

3.6.1. Lagrange map As in the finite dimensional setting, we define the subspace \( \mathcal{F}_{0}^{S} \subset \mathcal{F}_{0}^{(0)} \subset \mathcal{F}_{0} \) as the span of the basis elements corresponding to symmetric partitions
\[
\mathcal{F}_{0}^{S} := \text{span}\{|\lambda\rangle\} \subset \mathcal{F}_{0}^{(0)}, \quad \lambda = \lambda^T. \quad (3.6.1)
\]
Equivalently, we may identify \( \mathcal{F}_{0}^{S} \) with the semi-infinite wedge product space
\[
\mathcal{F}^{S} := \Lambda^{\infty/2} (\mathcal{H}_+). \quad (3.6.2)
\]
spanned by basis vectors
\[
e_{-J^c} := (-1)^r \prod_{i=1}^r \psi_{-J_i}^\dagger |0\rangle = e_{\ell_1}^\wedge e_{\ell_2}^\wedge \cdots, \quad (3.6.3)
\]
in which \( J \subset \mathbb{N}^+ \) is a subset \( \{ J_1, \ldots, J_r \} \) of the positive integers of cardinality \( r \), ordered increasingly, so the sequence of indices \( (l_1 > l_2 \cdots) \) are decreasing negative integers that eventually saturate to a sequence of consecutive negative integers. The basis elements of \( \mathcal{F}_0^S \) are those, in the fermionic sectors \( \{ \mathcal{F}_- \} \), that correspond to symmetric partitions
\[
e_{-J} \leftrightarrow |\lambda; -r),
\]
where, in Frobenius notation
\[
\lambda = (J_r - 1, \cdots, J_1 - 1|J_r - 1, \cdots, J_1 - 1).
\]

The Lagrange map
\[
\mathcal{L} : \text{Gr}^L_{\mathcal{H}_s}(\mathcal{H}, \omega) \rightarrow \mathbf{P}(\mathcal{F}_S^0)
\]
is then defined, as in the finite dimensional case \((2.4.11)\), by
\[
\mathcal{L}(w^0) := \left[ \sum_j \mathcal{L}_J(w^0)e_{-J} \right],
\]
where
\[
\mathcal{L}_J(w^0) := \pi_{(J_r - 1, \cdots, J_1 - 1|J_r - 1, \cdots, J_1 - 1)}.
\]

3.6.2. Hyperdeterminantal relations in infinite dimensions. We again extend our definition of the Lagrange coefficients \( \mathcal{L}_J \) to allow the multi-index \( J = (J_1, \ldots, J_r) \) to appear in arbitrary order without changing the value of \( \mathcal{L}_J \). Choose a triplet \( (j_1, j_2, j_3) \) of distinct positive integers, and an \( r \)-tuple \( J \) of positive integers that does not contain any of these. As in Sect. 2.6.1, for \( a, b = 1, 2, 3, \ a \neq b \), we mean by \( (J, j_a), (J, j_a, j_b) \) and \( (J, j_1, j_2, j_3) \) the distinct \( r + 1, r + 2 \) and \( r + 3 \)-tuples consisting of the indicated sets of indices. As in the finite dimensional case (Proposition 2.5), we have

**Proposition 3.6.** The coefficients \( \mathcal{L}_J \) in \((3.6.7)\) satisfy the hyperdeterminantal relations
\[
\mathcal{L}_J^2 \mathcal{L}^2_j_{j_1j_2j_3} + \mathcal{L}_J^2 \mathcal{L}^2_j_{j_1j_2j_3} + \mathcal{L}_J^2 \mathcal{L}^2_j_{j_2j_1j_3} - 4 \mathcal{L}_J \mathcal{L}^2_j_{j_1j_2j_3} - 2 \mathcal{L}_J \mathcal{L}^2_j_{j_1j_2j_3} - 2 \mathcal{L}_J \mathcal{L}^2_j_{j_1j_2j_3} = 0.
\]

These determine the image of the Lagrange map \((3.6.7)\) on a Zariski open set of \( \mathbf{P}(\mathcal{F}_0^S) \). The inverse image \( \mathcal{L}^{-1}(\mathcal{L}(w^0)) \) of any element \( \mathcal{L}(w^0) \) in the variety cut out by these relations is the orbit of \( w_0 \) under the group \( (\mathbf{Z}_2)_{\infty} = \{ \epsilon := \{ \epsilon_i = \pm 1 \}_{i \in \mathbf{Z}} \subset \text{Sp}(\mathcal{H}) \} \), acting by reflections:
\[
\epsilon : (e_{-i}, e_i) \mapsto (e_i e_{-i-1}, e_i e_i)
\]
within the coordinate planes \( \{ e_{-i-1}, e_i \}_{i \in \mathbf{N}} \).
To prove these relations in the infinite dimensional setting recall that, in finite dimensions, the result was obtained by first contracting and then projecting down in a family of different ways to $\Lambda^3(C^6)$, showing that a generic element satisfied the Plücker conditions and the linear isotropy conditions if and only if these reductions to $(3, 6)$ dimensions also satisfied the Plücker conditions and linear isotropy conditions. From this, it was possible to manipulate the quadratic and linear constraints, eliminating all but the symmetric partition Plücker coordinates, to obtain the quartic hyperdeterminantal relations for the various cases of $(3, 6)$ dimensions. Tracing back, the remaining variables are exactly the coefficients of the image of the Lagrange map, and the relations are those given in Eq. (3.6.9).

In infinite dimensions we proceed essentially the same way, but first reduce to a nested sequence of elements $\phi^N \in \Lambda^3(\mathcal{H}_N)$ of the $N$-th exterior power of a nested sequence of finite dimensional symplectic subspaces $\mathcal{H}_N$, showing that if both the Plücker relations and the isotropy condition are satisfied for $N > N_0$ for some $N_0$ depending on the subspace, the result in infinite dimensions follows by taking the direct limit.

Thus, consider the image of the Lagrangian Grassmannian $\text{Gr}_{\mathcal{H}^+(\mathcal{H}, \omega)} \subset \text{Gr}_{\mathcal{H}^+(\mathcal{H})}$ under the Plücker map (2.2.3), intersected with the kernel $\mathcal{F}_0^{(0)}$ of $\hat{\omega}$ (or $\hat{\omega}^\dagger$) acting on $\mathcal{F}_0$, where it consists of the set of decomposable elements in $\mathcal{F}_0^{(0)}$. The finite dimensional criteria for decomposability can be extended to this context. The Grassmannian $\text{Gr}_{\mathcal{H}^+(\mathcal{H})}$ is an orbit space under the general linear group $\text{Gl}(\mathcal{H})$ restricted (as in [37]), such that the orthogonal projection maps from elements of $\text{Gr}_{\mathcal{H}^+(\mathcal{H})}$ to $\mathcal{H}^+$ are Fredholm, and of index zero. The corresponding Fock subspace $\mathcal{F}_0 \subset \mathcal{F}$ is such that a generic decomposable element projects to a non zero multiple of the vacuum, which is the decomposable element corresponding to $\mathcal{H}^+$. The same then holds over the whole Fock space.

Turning to criteria of decomposability, in finite dimensions decomposable elements $\phi \in \Lambda^k(C^N)$ are those whose annihilators under the exterior product

$$\text{Ann}(\phi) := \{ \alpha \in C^N \mid \alpha \wedge \phi = 0 \} \quad (3.6.11)$$

have maximal dimension $k$. In the infinite dimensional setting, we have virtual dimensions for subspaces, given by the Fredholm index of the projections onto $\mathcal{H}^+$ along $\mathcal{H}^-$, so we can define an element $|\phi\rangle \in \mathcal{F}^{(0)}$ to be virtually decomposable if its annihilator

$$\text{Ann}(|\phi\rangle) := \{ v \in \mathcal{H} \mid \hat{\Gamma}_v|\phi\rangle = 0 \} \quad (3.6.12)$$

is in $\text{Gr}_{\mathcal{H}^+(\mathcal{H})}$. We now truncate to finite dimensions. For $N \in \mathbb{N}^+$, let $\mathcal{H}_N \subset \mathcal{H}$ denote the $2N$ dimensional subspace spanned by $\{e_{-N}, e_{-N+1}, ..., e_{N-1}\}$. We then have the decomposition

$$\mathcal{H} = \mathcal{H}_{N^+} \oplus \mathcal{H}_N \oplus \mathcal{H}_{N^-} \quad (3.6.13)$$

where $\mathcal{H}_{N^+}$ is the span of $\{e_{-N-i}\}_{i \in \mathbb{N}^+}$, and $\mathcal{H}_{N^-}$ is the span of $\{e_{N+i}\}_{i \in \mathbb{N}}$. Thus $\mathcal{H}$ of Eq. (2.2.1) may be identified with $\mathcal{H}_N$. Let

$$\pi_N : \mathcal{H} \to \mathcal{H}_{N^+} \oplus \mathcal{H}_N \quad (3.6.14)$$

denote projection to $\mathcal{H}_{N^+} \oplus \mathcal{H}_N$ along $\mathcal{H}_{N^-}$, and

$$\hat{\pi}_N : \mathcal{F}_0 \to \mathcal{F}_{(0,N)} \quad (3.6.15)$$
the corresponding projection from $\mathcal{F}_0$ to the subspace $\mathcal{F}_{(0,N)} \subset \mathcal{F}_0$ spanned by those basis elements that have no factors in $\{e_{N+i}\}_{i \in \mathbb{N}}$. Decomposable elements of this space are the Plücker image of subspaces of codimension $N$ in $\mathcal{H}_{N^*} \oplus \mathcal{H}_N$.

Consider the vacuum element

$$|−N\rangle = e_{−N−1} \wedge e_{−N−2} \wedge \cdots$$  \hspace{1cm} (3.6.16)

in the fermionic charge sector $\mathcal{F}_{−N}$. In analogy with finite dimensions, the inner product map

$$i_{−N} : \mathcal{F}_{(0,N)} \to \Lambda^N(\mathcal{H}_N)$$ \hspace{1cm} (3.6.17)

is defined on basis elements by

$$i_{−N}(e_{l_1} \wedge e_{l_2} \wedge \cdots e_{l_N}) = \begin{cases} e_{l_1} \wedge e_{l_2} \wedge \cdots \wedge e_{l_N} & \text{if } l_j = −j, \forall j > N \\ 0 & \text{otherwise.} \end{cases}$$ \hspace{1cm} (3.6.18)

Now define

$$\phi^N_N := i_{−N}(\hat{\pi}_N(|\phi\rangle)) \in \Lambda^N(\mathcal{H}_N) \subset \Lambda(\mathcal{H}_N),$$ \hspace{1cm} (3.6.19)

and

$$|\phi_N\rangle := \hat{\Gamma}_{\phi^N_N} |−N\rangle \in \mathcal{F}_0,$$ \hspace{1cm} (3.6.20)

where the Grassmann algebra $\Lambda(\mathcal{H}_N)$ is identified with the finite dimensional subalgebra of the fermionic Clifford algebra $\text{Cl}(\mathcal{H} \oplus \mathcal{H}^*, Q)$ generated by $\{\psi_{−N}, \cdots, \psi_{N−1}\}$. We then have

**Lemma 3.7.** The elements $|\phi_N\rangle$ converge to $|\phi\rangle$ as $N \to \infty$.

Let $|\phi\rangle$ be virtually decomposable. The annihilator $\text{Ann}(|\phi\rangle)$ of $|\phi\rangle$ is in $\text{Gr}_0^0(\mathcal{H})$, so it has a Fredholm projection onto $\mathcal{H}_+$, and a small (compact, or Hilbert Schmidt [37]) projection to $\mathcal{H}_−$. Since its virtual dimension is zero, this means that its intersection

$$\text{Ann}(|\phi\rangle) \cap (\mathcal{H}_{N−} \oplus \mathcal{H}_N)$$ \hspace{1cm} (3.6.21)

has dimension $N$, for large $N$, and the projection

$$\text{Ann}(|\phi\rangle)^N_N \subset \mathcal{H}_N$$ \hspace{1cm} (3.6.22)

of $\text{Ann}(|\phi\rangle) \cap (\mathcal{H}_{N−} \oplus \mathcal{H}_N)$ to $\mathcal{H}_N$ has dimension $N$ for large $N$.

**Lemma 3.8.** Let $|\phi\rangle$ be virtually decomposable. Then, for large $N$, $\text{Ann}(|\phi\rangle)^N_N$ is the annihilator of $\phi^N_N$, which implies that $\phi^N_N$ is decomposable in $\Lambda^N(\mathcal{H}_N)$ and $|\phi_N\rangle$ is decomposable in $\mathcal{F}_0$.

The annihilator $\text{Ann}(|\phi_N\rangle)$ of $|\phi_N\rangle$ is obtained by adding to $\text{Ann}(\phi^N_N)$ the vectors $e_{−N−1}, e_{−N−2}, e_{−N−3}, \ldots$, to obtain an infinite dimensional space of virtual dimension 0.

**Lemma 3.9.** The annihilators $\text{Ann}(|\phi_N\rangle)$ converge to $\text{Ann}(|\phi\rangle)$.
Let \( I \) be an infinite multi-index \( I_1 < I_2 < \cdots \) such that for \( j \) beyond a certain \( N_I \), \( I_j = j + \ell_0 \) for a fixed \( \ell_0 \). The Fredholm property tells us that there is a coordinate plane \( w_I \) corresponding to such an \( I \), spanned by vectors \( e_{-I_j} \), such that the projection of \(|\phi\rangle\) to \( e_{-I_1} \wedge e_{-I_2} \wedge e_{-I_3} \cdots \) is nonzero. Going now to our decomposable \(|\phi_N\rangle\), for \( N > N_0 \), the \(|\phi_N\rangle\)’s also map non-trivially, and since they correspond to subspaces \( w_N \), they have bases \( w_{N,1}, w_{N,2}, \ldots \), of the form

\[
w_{N,j} = e_{-I_j} + \sum_{i \notin I} a_{N,i} e_{-i}.
\]  

(3.6.23)

These \( w_{N,j} \) converge individually as \( N \to \infty \), for each \( j \), and the limits \( w_{\infty,j} \) then give a decomposition

\[
|\phi\rangle = w_{\infty,1} \wedge w_{\infty,2} \wedge \cdots .
\]  

(3.6.24)

**Proposition 3.10.** Suppose that for \( N > N_0 \), the elements \( \phi^N_N \) are nonzero and decomposable; then \(|\phi\rangle\) is also. If \(|\phi\rangle\) is virtually decomposable, then it is decomposable.

An element \(|\phi\rangle \in \mathcal{F}_0^{(0)}\) will be the Plücker image of an element of the Lagrangian Grassmannian if and only if it is virtually decomposable, since the Lagrangian condition is guaranteed by its belonging to \( \mathcal{F}_0^{(0)} \). Furthermore, the finite dimensional elements \( \phi^N_N \) must also be isotropic with respect to the (finite-dimensional) symplectic form \( \omega_N \). Thus,

**Proposition 3.11.** The element \(|\phi\rangle\) corresponds to a Lagrangian subspace if and only if the elements \( \phi^N_N \) correspond to Lagrangian subspaces in finite dimensions for all \( N > N_0 \).

Recall the definition (2.7.11) of the projection map

\[
p_{(B,I)}^j : \Lambda^j(\mathcal{H}_N) \to \Lambda^j(\mathcal{H}_N).
\]  

(3.6.25)

By Proposition (2.7), a generic \( \phi^N_N \) is decomposable and represents a Lagrangian subspace if and only if, for all multi-indices \((A,I)\) of cardinality \( N - 3 \) and complementary \((B,I)\) (as defined in (2.7.9)), the elements \( p^3_{(B,I)}(i_{f(A,I)}(\phi^N_N)) = i_{f(A,I)}(p^N_{(B,I)}(\phi^N_N)) \) represent a Lagrangian 3-space in 6 dimensions. Since in this section we have different conventions for the numbering of elements of the basis, we redefine, for a fixed \( N \), multi-indices \((A,I)\), \((B,I)\), where

\[
I = \{I_1, \ldots, I_{N-3}\} \subset \{1, 2, \ldots, N\} \text{ is a subset of cardinality } N - 3,
\]  

(3.6.26a)

\(A\) associates to each \( I_j \in I\) an integer \( A(I_j)\) which is either \( I_j - 1\), or \(-I_j\), \(A(I_j)\) (3.6.26b)

\(B\) associates to each \( I_j \in I\) an integer \( B(I_j)\) which is complementary to \( A(I_j)\),

so that either \( B(I_j) = I_j - 1\), if \( A(I_j) = -I_j\), or \( B(I_j) = -I_j\) if \( A(I_j) = I_j - 1\).

(3.6.26c)

We can define multivectors \( f_{(A,I)N} \) on the spaces \( \mathcal{H}_N \), as wedge products of the elements \( e_{A(i_j)} \), ordered so that the \( A(i_j)\) are increasing. This gives corresponding contractions \( i_{f(A,I)N} \). As above, we can project out all the basis elements \( e_{B(i_j)} \), and obtain a projection
Thus, the elements in dimensions \( p(B,I)_N \). And, as above, \( \{ p(B,I)^3, i_{f(A,I)}(\phi_N^N) \} \) give us 3-vectors in 6-space that correspond to Lagrangian subspaces for \( \phi_N^N \) to correspond to one.

There is a natural extension \( (A^+, I^+)_{N+1}, (B^+, I^+)_{N+1} \) of \( (A, I)_N, (B, I)_N \) from \( N \) to \( N+1 \), given by:

\[
I^+ = \{ I_1, \ldots, I_{N-3}, N+1 \},
\]
\[
A^+(I_j) = A(I_j), \quad j \leq N - 3, \quad \text{and} \quad A^+(N + 1) = -N - 1,
\]
\[
B^+(I_j) = B(I_j), \quad j \leq N - 3, \quad \text{and} \quad B^+(N + 1) = N.
\]

We can stabilise to infinite dimensions, and define semi-infinite multi-indices \( (A, I)_\infty \), \( (B, I)_\infty \) as follows.

\[
I = \{ I_1, I_2, \ldots \} \text{ is a subset of the positive integers, omitting only 3 integers },
\]
\[
A \text{ associates to each } I_j \text{ an integer } A(I_j) \text{ which is either } A(I_j) = I_j - 1 \text{ or } -I_j.
\]
\[
\text{For } j \text{ greater than some } j_0, \quad A(I_j) = -I_j.
\]
\[
B \text{ associates to each } I_j \text{ the integer } B(I_j) \text{ “complementary” to } A(I_j),
\]
\[
\text{that is } B(I_j) = I_j - 1 \text{ if } A(I_j) = -I_j, \quad \text{or } B(I_j) = -I_j \text{ if } A(I_j) = I_j - 1.
\]
\[
\text{For } j \text{ greater than some } j_0, \quad B(I_j) = I_j - 1.
\]

Fix \( (A, I)_\infty \), \( (B, I)_\infty \), and let \( \mathcal{H}_{(A,B,I)^c} \) be the six dimensional space defined in the same way as the finite dimensional case; i.e., spanned by the basis elements indexed by integers not in \( (A, I) \) or \( (B, I) \). Now pick an \( N > j_0 \), and set \( (A, I_{\leq N})_\infty, (B, I_{\leq N})_\infty \) to be the multi-index formed by the indices of \( (A, I)_\infty \), \( (B, I)_\infty \) with \( I_j \leq N \). Since \( N > j_0 \), they are of cardinality \( N - 3 \), and are formed by the removal of the infinite tails \( A(I_j) = -I_j, B(I_j) = I_j - 1 \) for \( I_j > N \). From this we have contractions \( i_{f(A,I)}_\infty \) and projections \( p(B,I)_\infty \) given by

\[
i_{f(A,I)}_\infty = i_{f(A,I_{\leq N})_\infty} \circ \hat{i}_{\leq N}
\]
\[
p_3(B,I)_\infty = p_3(B,I_{\leq N})_\infty \circ \hat{\pi}_N
\]

The composition

\[
i_{f(A,I)}_\infty p_3(B,I)_\infty = i_{f(A,I_{\leq N})_\infty} \circ p_3(B,I_{\leq N})_\infty \circ \hat{i}_{\leq N} \circ \hat{\pi}_N
\]

then maps us to \( \Lambda^3(\mathcal{H}_{(A,B,I)^c}) \), as in the finite dimensional case, passing through \( \mathcal{H}_N \) as an intermediary step. The result is invariant under the stabilization from \( N \) to \( N + 1 \). Thus, the elements in dimensions \( (3, 6) \) that we must test, for \( \phi_N^N \) to correspond to a Lagrangian plane, are obtainable directly from \( |\phi\rangle \) as \( i_{f(A,I)}_\infty p_3(B,I)_\infty (|\phi\rangle) \), and belong to \( \Lambda^3(\mathcal{H}_{(A,B,I)^c}) \).

**Proposition 3.12.** A generic \( |\phi\rangle \in \mathcal{F}_0 \) corresponds to an element of the Lagrangian Grassmannian if all of its \( (3, 6) \) dimensional reductions \( p_3(B,I)_\infty i_{f(A,I)}_\infty (|\phi\rangle) \) correspond to Lagrangian planes. This in turn is equivalent, for generic elements, to \( p_3(B,I)_\infty i_{f(A,I)}_\infty (|\phi\rangle) \) satisfying the hyperdeterminantal relations (3.6.9).
3.6.3. Parametric families of hyperdeterminantal relations in terms of \( \tau_{u0}^{KP} \)

Choose three arbitrary parameters \((x_1, x_2, x_3)\), such that \(x_i + x_j \neq 0\) for any distinct pair \(i, j \in \{1, 2, 3\}\), and define the \(3 \times 3\) matrix valued function \(A(t', x_1, x_2, x_3)\) of the parameters \((x_1, x_2, x_3)\) and the odd KP flow parameters \(t' = (t_1, 0, t_3, 0, \ldots)\) with matrix elements

\[
A_{ij}(t', x_1, x_2, x_3) := \frac{\tau_{u0}^{KP}(t' + [x_i] - [-x_j])}{(x_i + x_j)\tau_{u0}^{KP}(t')}, \quad i, j \in \{1, 2, 3\},
\]

(3.6.32)

where \(\tau_{u0}^{KP}(\tau)\) is a KP \(\tau\)-function satisfying the condition (3.5.11) assuring that it generates solutions to the CKP hierarchy. It follows that \(A(t', x_1, x_2, x_3)\) is a symmetric matrix

\[
A(t', x_1, x_2, x_3) = A^T(t', x_1, x_2, x_3).
\]

(3.6.33)

Define the following evaluations of \(\tau_{u0}^{KP}(t')\)

\[
\sigma_0(t', x_1, x_2, x_3) := \tau_{u0}^{KP}(t'),
\]

(3.6.34a)

\[
\sigma_i(t', x_1, x_2, x_3) := \frac{1}{2x_i} \tau_{u0}^{KP}(t' + [x_i] - [-x_i]), \quad i = 1, 2, 3
\]

(3.6.34b)

\[
\sigma_0^+(t', x_1, x_2, x_3) := \prod_{1 \leq i < j} (x_i - x_j)^2 \prod_{i,j=1}^3 (x_i + x_j) \tau_{u0}^{KP}(t' + \sum_{i=1}^3 ([x_i] - [-x_i])),
\]

(3.6.34c)

\[
\sigma_i^+(t', x_1, x_2, x_3) := \frac{(x_j - x_k)^2}{4x_jx_k(x_j + x_k)^2} \tau_{u0}^{KP}(t' + [x_j] + [x_k] - [-x_j] - [-x_k]),
\]

where \((i, j, k)\) is a cyclic permutation of \((1, 2, 3)\).

(3.6.34d)

**Proposition 3.13.** These satisfy the parametric family of hyperdeterminantal relations

\[
\sigma_0^2 \sigma_0^2 + \sigma_1^2 \sigma_1^2 + \sigma_2^2 \sigma_2^2 + \sigma_3^2 \sigma_3^2 = 2\sigma_0 \sigma_0 \sigma_1 \sigma_1 + 2\sigma_0 \sigma_0 \sigma_2 \sigma_2 + 2\sigma_0 \sigma_0 \sigma_3 \sigma_3 + 2\sigma_1 \sigma_2 \sigma_2 + 2\sigma_1 \sigma_3 \sigma_3 + 2\sigma_2 \sigma_2 \sigma_3 \sigma_3 - 4\sigma_0 \sigma_1 \sigma_2 \sigma_3 - 4\sigma_0 \sigma_1 \sigma_2 \sigma_3
\]

(3.6.35)

for all \((t', x_1, x_2, x_3)\).

**Proof.** Denote the eight principal minors of \(A(t', x_1, x_2, x_3)\),

\[
\Sigma_0(t', x_1, x_2, x_3) := 1,
\]

(3.6.36a)

\[
\Sigma_i(t', x_1, x_2, x_3) := A_{ij}(t', x_1, x_2, x_3) \quad i \in \{1, 2, 3\},
\]

(3.6.36b)

\[
\Sigma_0^+(t', x_1, x_2, x_3) := \det(A(t', x_1, x_2, x_3)),
\]

(3.6.36c)

\[
\Sigma_i^+(t', x_1, x_2, x_3) := \det \left( A_{jj}(t', x_1, x_2, x_3) A_{jk}(t', x_1, x_2, x_3) \right),
\]

(3.6.36d)

where \((i, j, k)\) is a cyclic permutation of \((1, 2, 3)\).
Since $A(t', x_1, x_2, x_3)$ is symmetric, these satisfy the hyperdeterminantal relation
\[
\sum_{0}^{3} \Sigma_{0}^{2} + \sum_{1}^{2} \Sigma_{1}^{2} + \sum_{2}^{3} \Sigma_{2}^{2} + \sum_{3}^{3} \Sigma_{3}^{2} = 2 \Sigma_{0} \Sigma_{0} + 2 \Sigma_{0} \Sigma_{2} + 2 \Sigma_{0} \Sigma_{3} + 2 \Sigma_{2} \Sigma_{3} + 4 \Sigma_{3} \Sigma_{3}.
\]
(3.6.37)

Now recall the following consequence of the addition formula for KP $\tau$-functions ([36], and [18], Chapt. 3, Prop. 3.10.4):
\[
\frac{\tau_{KP}(t + \sum_{i=1}^{k}[x_i] - \sum_{i=1}^{k}[y_i])}{\tau_{KP}(t)} \prod_{i<j}(x_i - x_j)(y_j - y_i) = \det \left( \frac{\tau_{KP}(t + [x_i] - [y_j])}{(x_i - y_j)\tau_{KP}(t)} \right)_{1 \leq i, j \leq k}.
\]
(3.6.38)

Setting
\[
\tau_{KP} = \tau_{w,0}^{KP}, \quad t = t', \quad y_i := -x_i, \quad i = 1, 2, 3,
\]
(3.6.39)

and choosing $k = 0, 1, 2$ or 3, this gives
\[
\sigma_{0}(t', x_1, x_2, x_3) = \tau_{w,0}^{KP}(t') \Sigma_{0}(t', x_1, x_2, x_3),
\]
(3.6.40a)
\[
\sigma_{i}(t', x_1, x_2, x_3) = \tau_{w,0}^{KP}(t') \Sigma_{i}(t', x_1, x_2, x_3), \quad i = 1, 2, 3
\]
(3.6.40b)
\[
\sigma_{0*}(t', x_1, x_2, x_3) = \tau_{w,0}^{KP}(t') \Sigma_{0*}(t', x_1, x_2, x_3),
\]
(3.6.40c)
\[
\sigma_{i*}(t', x_1, x_2, x_3) = \tau_{w,0}^{KP}(t') \Sigma_{i*}(t', x_1, x_2, x_3),
\]
(3.6.40d)

where $(i, j, k)$ is a cyclic permutation of $(1, 2, 3)$.

The hyperdeterminantal relation (3.6.37) may therefore be written equivalently as (3.6.35).

In fact, there is no reason to limit the number of parameters to just 3. For any $\tau$-function $\tau_{w,0}^{KP}(t)$ of Lagrangian type, choose a set of $N$ parameters $\{x_i\}_{i=1,...,N}$ satisfying
\[
x_i + x_j \neq 0, \quad \forall \ i, j \in (1, \ldots, N)
\]
(3.6.41)

(where, in principle, we could allow $N \to \infty$, provided suitable convergence conditions are satisfied), and an arbitrary point $t'$ in the space of (odd) flow parameters. Then define the map
\[
\tau : \mathbb{Z}^{N} \to \mathbb{C}
\]
\[
\tau : \mathbf{n} : \mapsto \tau_{n} := \tau_{w,0}^{KP}(t' + \sum_{i=1}^{N} n_i([x_i] - [-x_i]))
\]
(3.6.42)

\[
\mathbf{n} = (n_1, \ldots, n_N) \in \mathbb{Z}^{N}
\]
and, for each triple of integers \((i, j, k)\), \(1 \leq i < j < k \leq N\), the eight quantities

\[
\begin{align*}
\sigma^n &:= \tau^n, \\
\sigma_a^n &:= \frac{1}{2x_a} \tau^{(n_1, \ldots, n_{a+1}, \ldots, n_N)}, \quad a = i, j, k \\
\sigma_{ijk}^n &:= \frac{(x_i - x_j)^2(x_j - x_k)^2(x_k - x_i)^2}{(x_i + x_j)^2(x_j + x_k)^2(x_k + x_i)^2} \tau^{(n_1, \ldots, n_i+1, \ldots, n_{j+1}, \ldots, n_{k+1}, \ldots, n_N)}, \\
\sigma_{ab}^n &:= \frac{(x_a - x_b)^2}{4x_ax_b(x_a + x_b)^2} \tau^{(n_1, \ldots, n_{a+1}, \ldots, n_{b+1}, \ldots, n_N)}, \quad (a, b) = (i, j), (j, k), (i, k).
\end{align*}
\] (3.6.43a–d)

We then have:

**Corollary 3.14.** For all \(i < j < k\), the following \(N\)-parameter family of hyperdeterminantal relations hold:

\[
\begin{align*}
& (\sigma_i^n \sigma_{jk}^n)^2 + (\sigma_j^n \sigma_{ik}^n)^2 + (\sigma_j^n \sigma_{ik}^n)^2 + (\sigma_k^n \sigma_{ij}^n)^2 \\
& = 2\sigma_i^n \sigma_{jk}^n \sigma_j^n \sigma_{ik}^n + 2\sigma_j^n \sigma_{ik}^n \sigma_k^n \sigma_{ij}^n + 2\sigma_k^n \sigma_{ij}^n \sigma_i^n \sigma_{jk}^n \\
& + 2\sigma_i^n \sigma_{jk}^n \sigma_j^n \sigma_{ki}^n + 2\sigma_j^n \sigma_{ki}^n \sigma_k^n \sigma_{ij}^n + 2\sigma_k^n \sigma_{ij}^n \sigma_i^n \sigma_{jk}^n - 4\sigma_i^n \sigma_j^n \sigma_k^n \sigma_{ij}^n - 4\sigma_i^n \sigma_j^n \sigma_{jk}^n \sigma_{ki}^n.
\end{align*}
\] (3.6.44)

The proof is the same as for Proposition 3.13, with the replacements

\[
(x_1, x_2, x_3) \to (x_i, x_j, x_k), \quad t' \to t' + \sum_{i=1}^N n_i([x_i] - [-x_i]).
\] (3.6.45)

To obtain (2.6.2) from (3.6.44), set \((i, j, k) = (j_1, j_2, j_3)\) and \(n = n_J\), the binary vector with 1’s in positions \((J_1, \ldots, J_f)\) and 0’s elsewhere:

\[
\begin{align*}
\mathcal{L}_J := \sigma^n_J, \quad \mathcal{L}_{J Ja} := \sigma^n_{Ja}, \quad \mathcal{L}_{J,Ja/Jb} := \sigma^n_{Ja/Jb}, \quad \mathcal{L}_{J,j_1,j_2,j_3} := \sigma^n_{j_1,j_2,j_3},
\end{align*}
\] (3.6.46)

for \(a, b \in \{1, 2, 3\}, \ a < b\). Defining

\[
T_{nj}(\tau^n) := \tau^{(n_1, \ldots, n_{j+1}, \ldots, n_N)}, \quad i \in \{1, \ldots, N\}
\] (3.6.47)

and substituting Eqs. (3.6.43a–d) into (3.6.44) gives the form of the discrete CKP relations studied in [15].

### 3.7. Summary of results and further developments

We have shown that any KP \(\tau\)-function \(\tau_{w^0}\) satisfying the CKP reduction conditions (Eq. (3.3.10), or Eqs. (3.5.2a), (3.5.5) or (3.5.11)), corresponds to an element \(w^0\) belonging to the subgrassmannian \(\text{Gr}^\mathbb{C}_{\mathcal{H}_+}(\mathcal{H}, \omega) \subset \text{Gr}_{\mathcal{H}_+}(\mathcal{H})\) of Lagrangian subspaces of the symplectic Hilbert space \((\mathcal{H}, \omega)\), acted on by the subgroup consisting of those abelian flow group \(\Gamma_\ast\) that preserve \(\text{Gr}^\mathbb{C}_{\mathcal{H}_+}(\mathcal{H}, \omega)\) (i.e., the odd parameter flows only). It was proved, as a consequence of the addition formulae for KP \(\tau\)-functions, that any such \(\tau\)-function of CKP type, when evaluated at the finite or infinite lattice points and normalized as in Eqs. (3.6.43a–d), provides infinite parametric families of solutions to the hyperdeterminantal
relations, depending on the choice of parameters \( \{x_i\}_{i \in \mathbb{Z}} \) and the origin \( \gamma(t) w_0 \) in the group orbit in which the lattice is embedded. These relations may be interpreted as defining solutions of the discretized lattice form of the CKP hierarchy \([15,35]\). Moreover, as noted in Sect. 2.9 for the finite dimensional case, these may also be extended to a more general system, by adding further Plücker coordinates of the Lagrangian Grassmannian element, besides those entering in the Lagrange map, so as to provide solutions to the hexahedron recurrence relations of Kenyon and Pemantle \([25,26]\).

This suggests that, by making suitable lattice evaluations, corresponding both to symmetric partitions and some further “almost” symmetric ones, we may derive, for any CKP type \( \tau \)-function, infinite families of solutions, both of the discrete CKP hierarchy, and the hexahedron recursion relations. The detailed development of these results is done in a subsequent paper \([1]\), in which the addition formulae for KP \( \tau \)-functions are shown to imply that such normalized lattice evaluations of \( \tau \)-functions of the continuous hierarchies, both KP and CKP, provide infinite parametric families of solutions to the discrete ones. A similar result holds for the BKP hierarchy, in which the Plücker relations, which in the KP case are equivalent to the Hirota bilinear residue relations, are replaced by the corresponding Cartan relations (cf. \([7]\), Sec. 7.2 and Appendix E of \([18]\), and \([2]\)), which play a similar role in the embedding of maximal isotropic Grassmannians with respect to a complex scalar product into the projectivization of the Fock space of neutral fermions \([12,13]\).

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