DIFFEOTOPY GROUPS OF NON-COMPACT 4-MANIFOLDS

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Abstract. We provide information on diffeotopy groups of exotic smoothings of punctured 4-manifolds, extending previous results on diffeotopy groups of exotic \( \mathbb{R}^4 \)'s. In particular, we prove that for a smoothable 4-manifold \( M \) and for a non-empty, discrete set of points \( S \subset M \), there are uncountably many distinct smoothings of \( M \setminus S \) whose diffeotopy groups are uncountable.

We then prove that for a smoothable 4-manifold \( M \) and for a non-empty, discrete set of points \( S \subset M \), there exists a smoothing of \( M \setminus S \) whose diffeotopy groups have similar properties as \( \mathbb{R}_U \), Freedman and Taylor’s universal \( \mathbb{R}^4 \).

Moreover, we prove that if \( M \) is non-smoothable, both results still hold under the assumption that \( |S| \geq 2 \).

1. Introduction

One of the most striking results in 4-dimensional topology is the existence of uncountably many different smooth structures on \( \mathbb{R}^4 \) (see for example [Tau87; Gom93; GS99]). This wild behavior was later also detected in several other classes of non-compact topological 4-manifolds (see for example [BE98; GS99]). Up to now, it is still an open question whether all non-compact topological 4-manifolds admit uncountably many distinct smoothings. In particular, punctured 4-manifolds (i.e, manifolds with points removed) behave as wildly as \( \mathbb{R}^4 \). This is not surprising: one way to view \( \mathbb{R}^4 \) is as the 4-sphere without a point.

Although during the years a lot was proved about the exotic structures on non-compact 4-manifolds, little was known until recently on how these structures behaved in terms of self-diffeomorphism groups.

Diffeomorphism groups are normally hard to compute. A useful approximation is given by the mapping class groups. They consist of diffeomorphisms of a manifold \( M \) to itself modulo isotopy. In this paper we will refer to them as diffeotopy groups, following the terminology used in [Gom18]. Given a smooth manifold \( M \), we denote the diffeotopy group by \( D(M) \). In other words, elements of \( D(M) \) are smooth isotopy classes of self-diffeomorphisms of \( M \). If \( M \) is non-compact, we will also consider self-diffeomorphisms at infinity. Roughly speaking they are defined in terms of smooth, proper embeddings of codimension-0 submanifolds whose complement in \( M \) has compact closure. They are isotopic if they are properly isotopic as proper embeddings. We denote the diffeotopy group at infinity of \( M \) by \( D^\infty(M) \).

Each self-diffeomorphism \( f \) of \( M \) defines a diffeomorphism at infinity simply by restriction. We obtain a homomorphism \( r : D(M) \to D^\infty(M) \) by sending diffeomorphisms to their corresponding equivalence classes at infinity. If a diffeomorphism at infinity lies in the image of \( r \) we say that it extends over \( M \).

Even if mapping class groups are simpler to analyze than diffeomorphism groups, they still are quite obscure objects.

As an example, we do not know how to compute \( D^\infty(\mathbb{R}^4) \) yet (this is in fact related, as explained in [Gom18], to the smooth Schoenflies conjecture in dimension 4, which is notoriously still open).
In a recently published paper, Gompf approached the question, proving that many (in fact, uncountably many) smoothings of $\mathbb{R}^4$ have an uncountable diffeotopy group $\mathcal{D}(\mathbb{R})$ [Gom18, Theorem 4.4]. The same holds for their diffeotopy group at infinity $\mathcal{D}^\infty(\mathbb{R})$. He also proved an interesting result for $\mathbb{R}^4_U$, Freedman and Taylor’s universal smoothing of $\mathbb{R}^4$ [FT86]. Though nothing is known about $\mathcal{D}(\mathbb{R}^4_U)$ and $\mathcal{D}^\infty(\mathbb{R}^4_U)$ by themselves, we now know that the group homomorphism $r : \mathcal{D}(\mathbb{R}^4_U) \to \mathcal{D}^\infty(\mathbb{R}^4_U)$ is surjective [Gom93, Theorem 5.1]. Equivalently, every diffeomorphism at infinity of $\mathbb{R}^4_U$ extends over $\mathbb{R}^4_U$. We will refer to this extension property as universal behavior.¹

As the early results on exotic $\mathbb{R}^4$’s were used to distinguish between smooth structures on other non-compact 4-manifolds, we will use Gompf’s work as the starting point for a general discussion on diffeotopy groups of non-compact 4-manifolds. This possibility is mentioned in [Gom18, Section 4] and carried out in [Gom18, Theorem 4.10] for the interior of manifolds having a specific embedding property. However in this paper we follow a different strategy.

We will focus our attention on punctured 4-manifolds. In Section 3 and Section 4 we will prove the following two theorems:

**Theorem 3.3.** Let $M$ be a topological 4-manifold and $S \subseteq \hat{M}$ a non-empty discrete set of points. Then $M \setminus S$ admits uncountably many smoothings whose diffeotopy groups $\mathcal{D}^\infty(M \setminus S)$ and $\mathcal{D}(M \setminus S)$ are uncountable if:

1. $M$ is smoothable or
2. $M$ is non-smoothable and $|S| \geq 2$.

**Theorem 4.3.** Let $M$ be a topological 4-manifold, $S \subseteq \hat{M}$ a non-empty discrete set of points.

1. If $M$ is smoothable, then $M \setminus S$ admits a smoothing for which the map $r_{\epsilon_p}$ is surjective for each $p \in S$.
2. If $M$ is non-smoothable and $|S| \geq 2$, then $M \setminus S$ admits a smoothing for which $r_{\epsilon_p}$ is surjective for all but one $p \in S$.

Here $\hat{M}$ indicates the manifold interior, i.e $M \setminus \partial M$.

Note that in both Theorem 3.3 and Theorem 4.3 we did not address the case $M$ a topological non-smoothable 4-manifold with only one puncture. The following natural question arises:

**Question.** Do Theorems 3.3 and 4.3 apply also for $M \setminus p$, where $M$ is a topological non-smoothable 4-manifold and $p \in M$?

The proofs of the two theorems were inspired by [FO93] and [Gom93]. In the referenced papers uncountability of smooth structures was extended from $\mathbb{R}^4$ to punctured 4-manifolds exploiting the homeomorphism $M \setminus p \approx M \# \mathbb{R}^4$ for $p \in M$. We will do the same here. This is rather intuitive: in [Gom18, Theorem 4.4], the uncountability of $\mathcal{D}(\mathcal{R})$ is proven by exhibiting a group injection $G \hookrightarrow \mathcal{D}^\infty(\mathcal{R})$, where $G$ is an uncountable group with the discrete topology. The behavior at infinity was enough to detect non-isotopic diffeomorphisms of $\mathcal{R}$. It is quite natural then to try puncturing a 4-manifold $M$ in such a way to have an end diffeomorphic to that of a previously studied $\mathbb{R}^4$ homeomorph. We will then need to show how the exotic behavior at infinity can distinguish isotopy classes of self-diffeomorphisms of the punctured manifold.

Our starting point will be the following lemma. This is an easy case of Theorem 3.3.

¹Some care is required: the precise meaning of universal behavior depends on the complexity at infinity of the non-compact 4-manifold we are working with. We will discuss this in Section 4.
Lemma 3.1. Let $M$ be a closed, smooth 4-manifold and $p \in M$. Then there exists a smoothing of $M \setminus p$ whose diffeotopy groups $D^\infty(M \setminus p)$ and $D(M \setminus p)$ are uncountable.

The proof of this result will showcase the techniques used throughout the paper. The main idea is the following: take an uncountable group $G \in G^*$ [Gom18]. $G^*$ is the collection of all groups with discrete topology that have an action via diffeomorphisms on $\mathbb{R}^4$ that respects certain properties. Intuitively, the diffeomorphisms defined by elements of $G \in G^*$ interact equivariantly with a collection of rays properly embedded in $\mathbb{R}^4$ and keep fixed pointwise a region disjoint from these rays. The precise definition of $G^*$ is given in Section 2. We construct a $G$-action via diffeomorphisms on the punctured manifold $M \setminus p$. To do so, we will use $M \# \mathbb{R}^4 \approx M \setminus p$. Once the action is constructed, we exploit Gompf’s result to derive a contradiction. We will so prove that $G$ injects into $D^\infty(M \setminus p)$ as well as $D(M \setminus p)$.

It is worthwhile to emphasize that diffeotopy groups of compact 4-manifolds are mostly shrouded in mystery as well. This showcases once again the discrepancies between dimension 4 and other dimensions. The case of $S^4$ is for example far from understood. There are however recent breakthroughs by Watanabe, Gabai and Budney-Gabai [Wat18; BG19; Gab20] that sparked new life in this area of research, which still has a lot to offer in terms of open questions.

1.1. Outline. In Section 2 we will recall all the definitions and results we need from Gompf’s paper. Section 3 and 4 will be devoted to the proofs of Theorem 3.3 and Theorem 4.3 respectively.

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2. Group actions and Exotic $\mathbb{R}^4$’s

We recall the main definitions and results from [Gom18].

2.1. Diffeotopy groups at infinity and diffeotopy groups of the ends. Given a smooth manifold $V$, we study its diffeomorphism group modulo isotopy.

Definition 2.1. Let $V$ be a smooth manifold. The group $\mathcal{D}(V)$ of isotopy classes of self-diffeomorphisms of $V$ is called the diffeotopy group of $V$.

For a non-compact manifold $V$, we also have an analogous notion in terms of diffeomorphisms at infinity. We now give a precise definition of such objects.

Definition 2.2. Given a non-compact manifold $V$, a closed neighborhood of infinity is a codimension-0 submanifold $U \subseteq V$ that is a closed subset whose complement has compact closure (see Figure 1).

Definition 2.3. A compact exhaustion of a connected topological manifold $M$ is a sequence $\{C_i\}_{i=1}^\infty$ of compact, connected, codimension-0 submanifolds with $C_i \subseteq \text{Int}_M(C_{i+1})$ (topological interior) and $\bigcup_i C_i = M$. By letting $N_i := M \setminus C_i$, we obtain a corresponding cofinal sequence of closed neighborhoods of infinity. Each such $N_i$ has connected components $\{N^j_i\}_{j=1}^{k_i}$, where $k_i = \infty$ is also allowed. We can arrange that each $N^j_i$ is noncompact by enlarging $C_i$ to include all the compact components of $N_i$.
In the above arrangement, an end $\epsilon$ of $M$ is determined by a nested sequence $(N_i^h)_{i=1}^{\infty}$ of components of the $N_i$; each component is called a closed neighborhood of $\epsilon$ (see Figure 1).

**Remark 2.4.** Changing the sequence $\{C_i\}$ does not change the homeomorphism type of space of the ends of $M$. More information about ends of spaces can be found in [Fre31; Fre42; Pes90].

**Example 2.5.** The Euclidean space $\mathbb{R}^n$ for $n \geq 2$ has one end $\epsilon$. $\epsilon$ has a closed neighborhood that is homeomorphic to $S^{n-1} \times [0, \infty)$. The space $S^1 \times \mathbb{R}$ has two ends $\epsilon_1$ and $\epsilon_2$. They have closed neighborhoods $N_1$ and $N_2$ respectively which are both homeomorphic to $S^1 \times [0, \infty)$. A common example of an infinite-ended space is the universal cover of $S^1 \lor S^1$.

**Figure 1.** On the top, $U$ is a closed neighborhood of infinity of $S^1 \times \mathbb{R}$; $U$ is homeomorphic to the disjoint union of two copies of $S^1 \times [0, \infty)$. On the bottom, two closed neighborhoods $N_1$ and $N_2$, both homeomorphic to $S^1 \times [0, \infty)$, of the ends $\epsilon_1$ and $\epsilon_2$ respectively.

**Definition 2.6.** Given non-compact smooth manifolds $V$ and $V'$, suppose $f_i: Y_i \to Y'_i$, $i = 1, 2$, are diffeomorphisms between closed neighborhoods of infinity $Y_i \subseteq V$ and $Y'_i \subseteq V'$. We refer to $f_1$ and $f_2$ as equivalent if they agree outside some compact subset of $V$ containing the complements of $Y_1$ and $Y_2$. A diffeomorphism at infinity from $V$ to $V'$ is an equivalence class of such diffeomorphisms.

If $V$ has a single end, this is referred to as a **diffeomorphism of the end of $V$ and $V'$**. In order to simplify the notation we write $f$ instead of $[f]$ for the equivalence class.

We say that a diffeomorphism at infinity $f$ extends over $V$ if the equivalence class $f$ contains a diffeomorphism $V \to V'$. We can compose such diffeomorphisms at infinity, making the set of self-diffeomorphism of $V$ at infinity into a group. Given a group $G$, we define a $G$-action at infinity on $V$ to be a homomorphism of $G$ into this group.

**Definition 2.7.** Let $V$ be a non-compact smooth manifold. Two diffeomorphisms at infinity of $V$ will be called isotopic if they have representatives that are properly isotopic as proper embeddings into $V'$. Recall a proper map has preimage of each compact subset compact.

The group $D^\infty(V)$ of isotopy classes of self-diffeomorphisms of $V$ at infinity is called the **diffeotopy group of $V$ at infinity**.
We obtain a homomorphism \( r : \mathcal{D}(V) \to \mathcal{D}^\infty(V) \) by sending diffeomorphisms to their corresponding equivalence classes at infinity.

Focusing on closed neighborhoods of the ends instead of closed neighborhoods of infinity, we obtain an analogue of Definition 2.6.

**Definition 2.8.** Let \( V \) and \( V' \) be non-compact smooth manifolds and \( \epsilon \) and \( \epsilon' \) ends of \( V \) and \( V' \) respectively. Suppose \( f_i : N_i \to N'_i, i = 1, 2, \) are diffeomorphisms between closed neighborhoods of \( \epsilon \) and \( \epsilon' \). Assume \( N_i \) and \( N'_i \) are components of \( U_i \) and \( U'_i \), which are closed neighborhoods of infinity of \( V \) and \( V' \), respectively. We refer to \( f_1 \) and \( f_2 \) as equivalent if they agree outside some suitably large compact subset of \( V \) containing the complements of \( U_1 \) and \( U_2 \). A diffeomorphism of the end \( \epsilon \) is an equivalence class of such diffeomorphisms.

We say that a diffeomorphism \( f \) of \( \epsilon \) extends over \( V \) if the equivalence class \( f \) contains a diffeomorphism \( V \to V' \). We can compose such a diffeomorphism of the end, making the set of self-diffeomorphism of \( \epsilon \) into a group. Given a group \( G \), we define a \( G \)-action at the end to be a homomorphism of \( G \) into this group.

As before, we will denote the equivalence class with \( f \) instead of \([f]\). Note that Definition 2.8 fits with Definition 2.6 in the case that \( V \) has a single end.

**Definition 2.9.** Let \( V \) be a non-compact smooth manifold. Two diffeomorphisms of an end \( \epsilon \) of \( V \) will be called isotopic if they have representatives that are properly isotopic as proper embeddings in \( V' \).

The group \( \mathcal{D}^\epsilon(V) \) of isotopy classes of self-diffeomorphisms of \( \epsilon \) is called the diffeotopy group of \( \epsilon \).

For each end of the manifold \( V \), we also obtain an homomorphism \( r_\epsilon : \mathcal{D}(V) \to \mathcal{D}^\epsilon(V) \) by sending diffeomorphisms to their corresponding equivalence class with respect to the end \( \epsilon \).

The following two remarks assume that the ends of \( V \) are isolated, which is a consistent assumption in our setting. If we drop this condition, the constructions in remarks 2.10 and 2.11 might not be well defined.

**Remark 2.10.** We will later refer to \( \mathcal{D}^\epsilon(V) \) as a subgroup of \( \mathcal{D}^\infty(V) \). Strictly speaking, this is not quite correct. A diffeomorphism \( \phi \) of \( \epsilon \) embeds just a closed neighborhood \( U \) of the specific end \( \epsilon \). However, \( \phi \) can be turned into a diffeomorphism at infinity \( \bar{\phi} \) by extending it via the identity. Let \( U \) be a component of \( W \), a closed neighborhood of infinity of \( V \). Let \( \bar{\phi} \) be the proper embedding of \( W \) in \( V \) defined by \( \bar{\phi}|_U = \phi \) and \( \bar{\phi} = \text{identity} \) on the other components of \( W \). When we refer to \( \phi \) as a diffeomorphism at infinity we mean \( \bar{\phi} \). Under this identification we can interpret \( \mathcal{D}^\epsilon(V) \) as a subgroup of \( \mathcal{D}^\infty(V) \).

**Remark 2.11.** If \( V \) has ends \( \epsilon_i \) for \( i \in \Lambda \), then the direct product \( \prod_{i \in \Lambda} \mathcal{D}^{\epsilon_i}(V) \) can be interpreted as a subgroup of \( \mathcal{D}^\infty(V) \) as follows. Take a product \( (\phi_i)_{i \in \Lambda} \), where \( \phi_i \) embeds a closed neighborhood \( U_i \) of the end \( \epsilon_i \) inside \( V \) for each \( i \). Then \( (\phi_i)_{i \in \Lambda} \) defines a diffeomorphism \( \phi \) at infinity of \( V \); take a closed neighborhood of infinity \( U \), where \( U_i \)’s are components of \( U \), and define \( \phi \) to embed \( U_i \) via \( \phi_i \) and the remaining components via the identity. Identifying \( (\phi_i)_{i \in \Lambda} \) with \( \phi \) we turn \( \prod_{i \in \Lambda} \mathcal{D}^{\epsilon_i}(V) \) into a subgroup of \( \mathcal{D}^\infty(V) \).

2.2. **Group actions.** We follow [Gom18] for the terminology.

Let \( \mathbb{R}^4 \) denote the Euclidean space with its standard smoothing. Let \( \Sigma \) be a countable, discrete set and let \( \Gamma : \Sigma \times [0, \infty) \to \mathbb{R}^4 \) be an injection such that \( \gamma_s : = \Gamma|_{\{s\} \times [0, \infty)} \) is a ray for each \( s \in \Sigma \). Let \( G \) be a group with the discrete topology.
Definition 2.12. [Gom18, Definition 4.1] Fix $\Gamma$ and $G$ as above. A $G$-action via diffeomorphisms on $\mathbb{R}^4$ will be called $\Gamma$-compatible if $G$ acts effectively on $\Sigma$ so that for each $g \in G$ and $(s, t) \in \Sigma \times [0, \infty)$ we have $g \circ \Gamma(s, t) = \Gamma(g(s), t)$ and so that the stabilizer of each $s$ fixes pointwise a neighborhood of $\gamma_s([0, \infty))$.

Let $G$ be the set of all groups $G$ so that, after some choice of $\Gamma$, $G$ acts $\Gamma$-compatibly on $\mathbb{R}^4$. $G^+$ denotes the subset for which the action can be chosen to preserve orientation, and $G^* \subseteq G^+$ is the subset for which there is such a $\Gamma$-compatible action fixing pointwise a neighborhood of some ray $\gamma$ in $\mathbb{R}^4$ whose image is disjoint from that of $\Gamma$. We will mainly be interested in the class $G^* \subseteq G$.

Example 2.13. (1) $\mathbb{Q}^3$ lies in $G^*$. Take $\Gamma$ to be the inclusion $\mathbb{Q}^3 \times [0, \infty) \subset \mathbb{R}^3 \times \mathbb{R} = \mathbb{R}^4$ where $\mathbb{Q}^3$ is given the discrete topology. Take the obvious action on $\mathbb{R}^3 \times [0, \infty)$ tapered to be trivial on $\mathbb{R}^3 \times (-\infty, -1]$. This defines a $\Gamma$-compatible action of $\mathbb{Q}^3$ on $\mathbb{R}^4$. By [Gom18, Proposition 4.2], many uncountable groups such as $\mathbb{Q}^{2\omega}$, where $\omega$ denotes the first infinite cardinal number, lie in $G^* \subseteq G^+$.

(2) Every countable group $G$ of Euclidean or hyperbolic isometries of $\mathbb{R}^3$ lies in $G$, and if it preserves orientation, in $G^+$. Choose a point $p \in \mathbb{R}^3$ with trivial stabilizer under the elements $g \in G$. Define then $\Gamma : G \times [0, \infty) \rightarrow \mathbb{R}^3 \times \mathbb{R} = \mathbb{R}^4$ by setting $\Gamma(g, t) = (g(p), t)$.

2.3. Main results from [Gom18]. We next state the results of [Gom18] that we will generalize.

Lemma 2.14 ([Gom18, Theorem 4.4, Case (1)]). There are uncountably many diffeomorphism types of $\mathbb{R}^4$ homeomorphs $\mathcal{R}^2$ satisfying the following characterization. $\mathcal{R}$ embeds in $\mathbb{R}^4$ and every $G \in \mathcal{G}$ has an action on $\mathcal{R}$ that injects into the diffeotopy groups $\mathcal{D}(\mathcal{R})$ and $\mathcal{D}^{\infty}(\mathcal{R})$. The elements of $G$ define orientation preserving diffeomorphisms of $\mathcal{R}$ if and only if the defining $G$-action on $\mathbb{R}^4$ preserves orientation.

We briefly discuss the construction in the proof of [Gom18, Theorem 4.4]; this will be useful in order to better understand our own construction in Section 3.

Construction 2.15. This is the construction of an $\mathbb{R}^4$ homeomorph $\mathcal{R}$ satisfying the characterization in Lemma 2.14. We do not provide the proof for the injection property; it can be found in [Gom18].

We will build $\mathcal{R}$ as an end sum.\footnote{An $\mathbb{R}^4$ homeomorph is a manifold homeomorphic to $\mathbb{R}^4$ but not necessarily diffeomorphic to it. We follow [Gom18] for the terminology.} For a fixed $G \in \mathcal{G}$, $\Sigma$ and $\Gamma$ (as in Definition 2.12), where $G$ acts $\Gamma$-compatibly on $\mathbb{R}^4$, identify $\Sigma$ with $\mathbb{Z}^+$. We will find an increasing sequence $(t_n)$ such that, when each ray $\gamma_n := \Gamma|_{t_n}$ is restricted to the interval $[t_n, \infty)$, we get a multiray $\Gamma$ with an equivariant\footnote{This is the non-compact analogue of the boundary sum. Given two smoothings $\mathcal{R}_1, \mathcal{R}_2$ of $\mathbb{R}^4$, the end sum $\mathcal{R}_1 \sharp \mathcal{R}_2$ is defined as follows: let $\gamma_1 : [0, \infty) \rightarrow \mathcal{R}_1$ and $\gamma_2 : [0, \infty) \rightarrow \mathcal{R}_2$ be two smooth properly embedded rays with tubular neighborhoods $\nu_1, \nu_2$ respectively. Then $\mathcal{R}_1 \sharp \mathcal{R}_2 := \mathcal{R}_1 \cup_{\phi_1} ([0, 1] \times \mathbb{R}^3) \cup_{\phi_2} \mathcal{R}_2$, where $\phi_1 : [0, \frac{1}{2}] \times \mathbb{R}^3 \rightarrow \nu_1$ and $\phi_2 : (\frac{1}{2}, 1] \times \mathbb{R}^3 \rightarrow \nu_2$ are orientation preserving diffeomorphisms which respect the $\mathbb{R}^3$ bundle structures. For a more comprehensive treatment of the end sum operation, see [CG19].} tubular neighborhood map, whose restriction over each truncated ray we denote by $\varphi_n : [t_n, \infty) \times \mathbb{R}^3 \rightarrow \mathbb{R}^4$.

We truncate the rays $\gamma_n$ for each $n$ so that they can be tubed without getting undesired intersections.

To do so, construct the sequence by induction on $n$, starting with $t_1 = 0$. Given $n \geq 1$, suppose that for each $i \leq n - 1$ we have already equivariantly defined each
and from each $\gamma$ we say that a diffeotopy group at infinity. In order to simplify the notation, we will omit the quotient map. When composing this with the quotient map, we obtain a homomorphism into the homomorphism we defined in Section 2.

See Figure 2 for a visual representation of $\varphi$. Since in Construction 2.15 we used end-sums along the (truncated) rays $\gamma_n$ to construct $\mathcal{R}$, $\gamma$ defines a proper embedding of $[0, \infty)$ in $\mathcal{R}$ as well. Then the $G$-action inherited by $\mathcal{R}$ fixes pointwise $\gamma$ and its neighborhood.

3. Proof of Theorem 3.3

The goal of this section is to extend Lemma 2.14; how far can we extend this result to general punctured 4-manifolds? The starting point will be the case of a once punctured smooth, closed 4-manifold $M$ (Lemma 3.1). This first step will give us the construction we will use to prove Theorem 3.3 in full generality.

**Lemma 3.1.** Let $M$ be a closed smooth 4-manifold and $p \in M$. Then there exists a smoothing of $M \setminus p$ whose diffeotopy groups $\mathcal{D}_\infty(M \setminus p)$ and $\mathcal{D}(M \setminus p)$ are uncountable.

**Proof.** We start by choosing an $\mathbb{R}^4$ homeomorph $\mathcal{R}$. For every uncountable group $G \in G^*$ there exists (Lemma 2.14) an $\mathbb{R}^4$ homeomorph, call it $\mathcal{R}$, with a $G$-action fixing pointwise a neighborhood of some ray $\gamma$ (see Remark 2.16). $G$ injects in the diffeotopy groups $\mathcal{D}(\mathcal{R})$ and $\mathcal{D}_\infty(\mathcal{R})$ by Lemma 2.14.

$M \# \mathbb{R}^4$ is homeomorphic to $M \setminus p$; smooth it as the smooth connected sum between $M$ and $\mathcal{R}$. This gives a smoothing of $M \setminus p$.

As stated before $G$ acts on $\mathcal{D}(\mathcal{R})$ and $\mathcal{D}_\infty(\mathcal{R})$. Call the group homomorphism $\theta$, i.e. each $g \in G$ defines a self-diffeomorphism $\theta(g)$ of $\mathcal{R}$.

We extend the $G$-action to an action on $\mathcal{D}(M \setminus p)$ and $\mathcal{D}_\infty(M \setminus p)$. Each $g$ will define a self-diffeomorphism of the punctured manifold. To do so, take a disk $D$ inside the $G$-fixed neighborhood of $\gamma$; use this disk to perform the smooth connected sum with $M$. Let $g \in G$. Since the $G$-action is trivial in $D$, the diffeomorphism $\theta(g)$ is the identity on $D$. We can then extend it via the identity to $M \setminus D'$, where $D'$ is a disk in $M$ around the point $p$. Hence for each $g \in G$ we get a diffeomorphism of $M \setminus p$ which is the identity on $M \setminus D'$.

This defines a $G$-action on $\mathcal{D}(M \setminus p)$ (and, by post-composing it with $r$, also on $\mathcal{D}_\infty(M \setminus p)$). Call this action $\tilde{\theta}$; the diffeomorphism defined by $g \in G$ is then $\tilde{\theta}(g)$. We give the above construction, which assigns for each $\theta(g)$ diffeomorphism of $\mathcal{R}$ the diffeomorphism $\tilde{\theta}(g)$ of $M \setminus p$, the name $\#$, reminding us that the connected sum is used to define it. So $\#(\theta(g)) = \tilde{\theta}(g)$ is defined as $\tilde{\theta}(g)|_{\mathcal{R} \setminus D} = \theta(g)|_{\mathcal{R} \setminus D}$ and $\tilde{\theta}(g)|_{M \setminus D'} = id$. See Figure 2 for a visual representation of $\#$.

Define $\phi$ (resp. $\tilde{\phi}$) to be the composition $r \circ \theta$ (resp. $r \circ \tilde{\theta}$) where $r : \mathcal{D}(\mathcal{R}) \rightarrow \mathcal{D}_\infty(\mathcal{R})$ is the homomorphism we defined in Section 2.6.

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5In Definition 2.6 we defined a $G$-action at infinity as a homomorphism of $G$ into the diffeomorphism group at infinity. Composing this with the quotient map, we obtain a homomorphism into the diffeotopy group at infinity. In order to simplify the notation, we will omit the quotient map. When we say that a $G$-action at infinity defines a diffeomorphism at infinity we usually mean an isotopy class of diffeomorphisms at infinity. We do the same for $G$-actions via diffeomorphisms.

6By $r|$ we mean the restriction of $r$ on $\theta(G)$ (resp. $\tilde{\theta}(G)$).
Figure 2. The construction of the diffeomorphism $\tilde{\theta}(g)$. The $G$-fixed ray $\gamma$ going to infinity is in violet. In light gray, the two 4-disks $D'$ and $D$ used to perform the connected sum. In blue we indicate the definition of $\theta(g)$ on $M\# \mathbb{R}$: $\theta(g)$ is extended via the identity on the boundaries of the disks.

Finally define $\#^\infty$ to be the homomorphism sending an equivalence class $\phi(g) \in \phi(G) \subseteq D^\infty(\mathbb{R})$ to the corresponding equivalence class $\tilde{\phi}(g)$ in $\tilde{\phi}(G) \subseteq D^\infty(M\setminus p)$. A representative of $\phi(g)$ is given by the embedding of some closed neighborhood of infinity $U$ defined as $\theta(g)|_U$. Shrink $U$ if necessary to make it disjoint from $D$. Then $\theta(g)|_U$ embeds $U$ in $M\setminus p$ as well. In particular, the restriction of $\theta(g)$ on $U$ is the same as the restriction of $\tilde{\theta}(g)$ on $U$ by our connected sum construction. So $\#^\infty(\phi(g))$ is the equivalence class of the diffeomorphism at infinity given by the embedding $\tilde{\theta}(g)|_U$ of $U$ in $M\setminus p$, i.e the equivalence class $\tilde{\phi}(g)$.

We summarize everything we have obtained so far in the following diagram:

$$
\begin{array}{ccc}
D(\mathbb{R}) & \xrightarrow{r} & D^\infty(\mathbb{R}) \\
\cup & & \cup \\
G & \xrightarrow{\theta} & \theta(G) \\
\downarrow & \downarrow & r| \\
\tilde{\theta}(G) & \xrightarrow{\#} & \tilde{\phi}(G) \\
\tilde{\phi}(G) & \xrightarrow{r|} & D(M\setminus p) \\
\setminus & \setminus & \setminus \\
D^\infty(M\setminus p)
\end{array}
$$

Gompf proved [Gom18, Theorem 4.4] that the map $\phi$ (hence also $\theta$) in the upper row is injective, i.e the $G$-action defines an injection of $G$ in $D^\infty(\mathbb{R})$ (hence and also in $D(\mathbb{R})$) (see Lemma 2.14 above). By our definitions of $\tilde{\theta}, \tilde{\phi}, \#$ and $\#^\infty$, both the left triangle and the right square in the diagram commute.

We want to prove that $\tilde{\phi} = r \circ \tilde{\theta}$ is an injection. To do so, it suffices to show that $\#^\infty$ is injective, since by commutativity of the diagram $\tilde{\phi} = \#^\infty \circ \phi$, and we already know that $\phi$ is injective.

**Claim.** The group homomorphism $\#^\infty$ constructed as above is injective.
Proof of the claim. For ease of notation, throughout the proof we call \( \phi(g) =: g \), meaning that this is the class of diffeomorphisms of infinity that behave like the diffeomorphism \( \theta(g) \) on some closed neighborhood of infinity \( U \).

If \( \#^\infty(g) =: \text{id} \in \mathcal{D}^\infty(M \setminus p) \), then there are two representatives of their equivalence classes that are properly isotopic as embeddings of \( U \), a closed neighborhood of infinity in \( M \setminus p \). By our definition of \( \#^\infty(g) \), a representative is given by the restriction of \( \tilde{\theta}(g) \) on some closed neighborhood of infinity of \( M \# \mathcal{R} \) entirely contained in \( \mathcal{R} \setminus \partial \mathcal{D} \). Then shrinking \( U \) if necessary, we can assume that the two representatives of \( \tilde{\phi}(g) \) and \( \text{id} \) which are properly isotopic embed \( U \) in \( M \setminus p \) as \( \theta(g)|_U \) and as \( \text{id} \). So we have \( H : U \times [0, 1] \to M \setminus p \) a proper isotopy with \( H_0 = \theta(g)|_U \) and \( H_1 = \text{id} \).

We wish to show that there is such a proper isotopy between representatives of \( g \) and \( \text{id} \in \mathcal{D}^\infty(\mathcal{R}) \), i.e a proper isotopy \( G : U' \times [0, 1] \to \mathcal{R} \) between embeddings \( \theta(g)|_{U'} \) and \( \text{id} \) of \( U' \) a closed neighborhood of infinity of \( \mathcal{R} \).

We can consider the track \( H \) of the proper isotopy, i.e the (level preserving) proper embedding \( H : U \times [0, 1] \to (M \setminus p) \times [0, 1] \). Let \( K := \partial D' \) (the boundary of the disk used to perform the gluing) and \( C := H(U \times [0, 1]) \cap (K \times [0, 1]) \) (see Figure 3). \( U \times [0, 1] \) is closed; \( H \) is proper and hence a closed map, which means \( H(U \times [0, 1]) \) is closed as well. \( K \times [0, 1] \) is compact, so \( C \) is compact as it is the intersection between a closed and a compact set.

Take the preimage \( H^{-1}(C) \), which is a compact subset of \( U \times [0, 1] \). Exhaust \( U \times [0, 1] \) by compact sets \( K_N \times [0, 1] \); since \( H^{-1}(C) \) is compact, it is contained in some \( K_N \times [0, 1] \), \( N \in \mathbb{N} \). Henceforth, we can find a closed (connected) neighborhood of infinity \( U' \subseteq U \) so that \( U' \times [0, 1] \) is disjoint from \( H^{-1}(C) \) (see Figure 3). Then the image of \( U' \times [0, 1] \) under \( H \) is entirely contained in \( \mathcal{R} \setminus \partial \mathcal{D} \), since the intersection with \( \partial D' \) is trivial and \( H(U' \times [0, 1]) \) is connected. So the restriction of \( H \) to \( U' \) defines a proper isotopy between \( \theta(g)|_{U'} \) and \( \text{id} \) as embeddings of \( U' \) in \( \mathcal{R} \). This means that \( g = \text{id} \) as equivalence classes in \( \mathcal{D}^\infty(\mathcal{R}) \).

Using the claim, we get that \( \tilde{\phi} \) is indeed an injection. Clearly, since \( \tilde{\phi} = r \circ \tilde{\theta} \), the same holds for \( \tilde{\theta} \). This proves that \( G \) injects in \( \mathcal{D}(M \setminus p) \) an \( \mathcal{D}^\infty(M \setminus p) \), showing that both are uncountable.

We started with once punctured closed, smooth 4-manifolds. The following corollary extends the result of Lemma 3.1 to give uncountably many smoothings having the desired property.

Corollary 3.2. Let \( M \) be a closed, smooth 4-manifold and \( p \in M \). Then there exist uncountably many smoothings of \( M \setminus p \) whose diffeotopy groups \( \mathcal{D}^\infty(M \setminus p) \) and \( \mathcal{D}(M \setminus p) \) are uncountable.

Proof. Given an uncountable group \( G \in \mathcal{G}^* \) (see Example 2.13 (1)) we can choose \( \mathcal{R} \) from an uncountable family of \( \mathbb{R}^4 \) homeomorphs \( \{ \mathcal{R}_t \} \) by equivariantly varying the model summand in the construction (see [Gom18, Lemma 3.2]).

Consider the family of smoothings of \( M \setminus p \) given by \( \{ M \# \mathcal{R}_t \} \). Since only countably many \( \mathbb{R}^4 \) homeomorphs can realize a given end [Gom93], this uncountable family contains an uncountable subfamily of smoothings of \( M \setminus p \) with pairwise non-diffeomorphic ends. Hence, this represents an uncountable family of distinct smoothings of \( M \setminus p \) so that each one satisfies the properties of the above Lemma 3.1.

We proved the basic case. We can now state the result in full generality; the techniques we will use in the proof are analogous to the ones used in the proofs of Lemma 3.1 and Corollary 3.2.
Theorem 3.3. Let $M$ be a topological 4-manifold, $S \subset \hat{M}$ a non-empty discrete set of points. Then $M \setminus S$ admits uncountably many smoothings whose diffeotopy groups $\mathcal{D}^\infty(M \setminus S)$ and $\mathcal{D}(M \setminus S)$ are uncountable if:

1. $M$ is smoothable or
2. $M$ is non-smoothable and $|S| \geq 2$.

Remark 3.4. We impose the assumption $S \subset \hat{M}$ in order to avoid removing points on the (possibly non-empty) boundary $\partial M$. Going through the proof of Lemma 3.1, we see that we made almost no use of the empty boundary assumption. However if $p \in \partial M$, then the initial idea, connect-summing with $\mathcal{R}$, needs to be modified. We have to take the boundary sum with $\mathbb{H}^4$, the upper half 4-space. We do not know how the diffeotopy groups of $\mathbb{H}^4$ behave (note that this is $D^3 \setminus \text{pt}$ where the point lies in the
Proof of Theorem 3.3. We already covered case (1) when \(|S| = 1\) and \(M\) is closed (by Remark 3.4 also compact with possibly non-empty boundary) in Lemma 3.1. To prove Theorem 3.3, it suffices to prove case (1) for non-compact 4-manifolds; if \(M\) is a smooth, compact 4-manifold, \(|S| \geq 2\) then apply the result to \(M' := M \setminus p\), in which case \(M'\) is non-compact, and \(S' := S \setminus p\), where \(p \in S\). Similarly, if \(M\) is a non-smoothable 4-manifold (2) follows from (1) applied to \(M' := M \setminus p\) (by Quinn’s smoothability theorem [Qui82] \(M \setminus p\) has a smooth structure) and \(S' := S \setminus p\), where \(p \in S\).

We first prove (1) for \(M\) a non-compact 4-manifold and \(|S| = 1\). Once we have this, we will show how to obtain the general result.

Claim. Let \(M\) be a non-compact 4-manifold and \(p \in M\). Then there exist uncountably many smoothings of \(M \setminus p\) whose diffeotopy groups \(\mathcal{D}^\infty(M \setminus p)\) and \(\mathcal{D}(M \setminus p)\) are uncountable.

Proof of the claim. The proof follows that of Lemma 3.1 with the natural modifications. Note that \(K := \partial D'\) separates \(e_p\) from all the other ends of \(M\). To show there are uncountably many distinct smoothings of \(M \setminus p\) satisfying this property we work as in Corollary 3.2. Consider the uncountable family of \(\mathbb{R}^4\) homeomorphs \(\{R_t\}_{t \in \mathbb{R}}\) with uncountable diffeotopy groups. Take the uncountable subfamily \(\{R_u\}_{u \in \mathbb{R}}\) having pairwise non-diffeomorphic ends. As in [Gom93], \(M\) can only have countably many \(S^3 \times \mathbb{R}\) collared ends. So there is another subfamily \(\{R_u\}_{u \in \mathbb{R}}\), with pairwise non-diffeomorphic ends, so that no end has the same diffeomorphism type of an end of \(M\). Consider then the family of smoothings \(\{M \# R_u\}_{u \in \mathbb{R}}\). This realizes an uncountable family of distinct smoothings of \(M \setminus p\) each one satisfying the desired property for \(\mathcal{D}(M \setminus p)\) and \(\mathcal{D}^\infty(M \setminus p)\). This completes the proof of the claim. \(\square\)

We have proved (1) when \(M\) is non-compact and \(|S| = 1\); if \(S\) is an arbitrary discrete subset, the result follows from the claim. More precisely, let \(S\) be discrete and \(M\) non-compact. Take \(p \in S\), define \(S'\) to be \(S \setminus p\) and \(M' := M \setminus S'\). Then \(M'\) is non-compact. By the claim, \(M' \setminus p = M \setminus S\) has uncountable \(\mathcal{D}(M' \setminus p)\) and \(\mathcal{D}^\infty(M' \setminus p)\) (since the set \(S\) is discrete, we can find \(K\) as in the proof of the claim separating the end \(e_p\) from the other punctured ends) and this completes the proof of Theorem 3.3. \(\square\)

Example 3.5. Theorem 3.3 implies that \(S^4 \setminus \{p, q\}\) with \(p \neq q \in S^4\), which is homeomorphic to \(S^3 \times \mathbb{R}\), has uncountably many smoothings whose diffeotopy groups \(\mathcal{D}(S^4 \setminus \{p, q\})\) and \(\mathcal{D}^\infty(S^4 \setminus \{p, q\})\) are uncountable.

In the proof of Theorem 3.3 we constructed a \(G\)-action on \(\mathcal{D}^p(M \setminus p)\). By carefully modifying the construction, we obtain the following corollary.

Corollary 3.6. Let \(G \in \mathcal{G}^*\) be uncountable. Let \(M\) be a topological 4-manifold, \(S \subseteq M\) a non-empty discrete set of points. Call \(\Lambda\) the index set of \(S\). Let \(p1\) and \(p2\) be the following properties:

- \(p1\): The direct product \(\prod_{i \in \Lambda} G\) injects into the diffeotopy groups \(\mathcal{D}(M \setminus S)\) and \(\mathcal{D}^\infty(M \setminus S)\).
- \(p2\): The direct product \(G \times G \times \cdots \times G = G^{|S|-1}\), where \(S\) is finite, injects into the diffeotopy groups \(\mathcal{D}(M \setminus S)\) and \(\mathcal{D}^\infty(M \setminus S)\).

Then there are uncountably many smoothings of \(M \setminus S\) for which property \(p1\) holds if \(M\) is a smoothable 4-manifold.
Moreover, there are uncountable many smoothings of \( M \setminus S \) for which property \( p2 \) holds when \( M \) is a non-smoothable compact 4-manifold and \( |S| \geq 2 \).

Proof. The cases \( |S| = 1 \) for \( p1 \) and \( |S| = 2 \) for \( p2 \) follow from Theorem 3.3. We just need to prove the \( p1 \) case for \( |S| \geq 2 \); the \( p2 \) case descends from it.

We start with a non-compact 4-manifold \( M \). Fix an uncountable \( G \in G^* \). Let \( M \) be non-compact, \( S \) discrete, \( |S| \geq 2 \). The index set \( \Lambda \) must be countable otherwise \( S \) would not be discrete, so we can identify it with a subset of \( \mathbb{Z}^+ \). Start with \( p1 \in S \).

In the proof of Theorem 3.3 we constructed a smoothing of \( M \setminus p1 \) with an injection \( G \hookrightarrow D^{p_1}(M \setminus p_1) \). Fix this smoothing and call the resulting smooth manifold \( M' \).

Now take \( p2 \in S \) and smooth \( M' \setminus p2 \) as in the proof of Theorem 3.3 so that \( G \) injects into \( D^{p_2}(M \setminus \{p_1, p_2\}) \). Since \( S \) is discrete we can choose the disk \( D' \) away from \( p1 \) to perform the connected sum with \( R \). The smooth structure is not changed near the puncture \( p1 \). Hence \( G \) injects in \( D^{p_1}(M \setminus \{p_1, p_2\}) \) as well.

Now continue the procedure for each index \( i \in \Lambda \). In the end we get a smoothing of \( M \setminus S \) so that \( G \) injects into \( D^{p_i}(M \setminus S) \) for each \( i \in \Lambda \). The direct product \( \prod_{i \in \Lambda} D^{p_i}(M \setminus S) \) is a subgroup of \( D^\infty(M \setminus S) \) (see Remark 2.11). Hence we get that the direct product \( \prod_{i \in \Lambda} G \) injects into \( D^\infty(M \setminus S) \). If \( M \) is compact and smoothable, the proof is analogous.

As for \( p2 \), start with \( M \) a non-smoothable compact 4-manifold, \( |S| \geq 3 \). Take \( p \in S \) and define \( M' := M \setminus p \). Then \( M' \) is non-compact and \( |S'| \geq 2 \). Apply the case of \( p1 \) to \( M' \) and \( S' \) to get that \( G^{[S']} \) injects into \( D(M' \setminus S') \) and \( D^\infty(M' \setminus S') \). Note that \( |S'| = |S| - 1 \). Then use the equality \( M' \setminus S' = M \setminus S \). To get uncountability in both cases, choose different \( R \) summands in the construction as we did in the proof of Theorem 3.3. Thus we can conclude.

Remark 3.7. If we take in consideration different uncountable groups \( G_i \in G^* \) for \( i \in \Lambda \), where \( \Lambda \) is the index set of \( S \), then we can use the proof of Corollary 3.6 combining more group actions together. This would show that for \( \{G_i\}_i \) a family in \( G^* \), the same results as in Corollary 3.6 apply for the product \( \prod_{i \in \Lambda} G_i \).

4. Proof of Theorem 4.3

The goal of this section is to extend the following result to general punctured 4-manifolds.

Lemma 4.1 ([Gom18, Theorem 5.1]). Every diffeomorphism of the end of \( R_U \) extends over \( R_U \). That is, the homomorphism \( r: D(R_U) \to D^\infty(R_U) \) is surjective.

We will be able to achieve a similar statement: in the case of non-compact 4-manifolds, however, we shall restrict only to diffeomorphisms of the punctured ends since the other ends might have a wild behavior that we cannot detect with the tools we have developed so far. As an example, there might be an end that is not topologically collared by any \( M \times \mathbb{R} \), where \( M \) is a compact 3-manifold.

As we did for Lemma 3.1, we begin with the case of smooth, compact 4-manifolds. The starting point of the proof will be again a connected sum argument. As we previously saw (see Remark 3.4), as long as the point is assumed to be taken out from the interior of the manifold, we can allow the manifold \( M \) to have non-empty boundary.

In the following discussion, if \( M \) is orientable, then we have to restrict to the orientation-preserving diffeomorphism of the ends, otherwise the Cerf-Palais disk theorem [Cer61; Pal59] (which we use in the proof of Lemma 4.2) does not hold. We will however denote the groups by the usual \( D \), suppressing the \( _4 \) subscript, to ease the notation.
Lemma 4.2. Let $M$ be a smooth compact 4-manifold and $p \in M$. Then there is a smoothing of $M \setminus p$ so that every diffeomorphism at infinity of $M \setminus p$ extends over $M \setminus p$. That is, the homomorphism $r : D(M \setminus p) \to D^\infty(M \setminus p)$ is surjective.

Proof. Smooth $M \setminus p$ as the smooth connected sum $M \# \mathcal{R}_U$, where $\mathcal{R}_U$ is Freedman and Taylor’s universal $\mathbb{R}^4$ [FT86]. We want to show that $r : D(M \setminus p) \to D^\infty(M \setminus p)$ is surjective.

Consider $\phi \in D^\infty(M \setminus p)$ and pick $\phi$ as a representative of the class. Then $\phi$ is a proper embedding of $V$, a closed neighborhood of infinity of $M \setminus p$. By shrinking $V$ if necessary, we can assume that it lies entirely in the $\mathcal{R}_U \setminus \hat{D}$ part. Call $\phi(V) =: V'$.

Our goal is to restrict $\phi$ to a diffeomorphism at infinity of $\mathcal{R}_U$. At present, $V'$ is a closed neighborhood of infinity of $M \setminus p$, but it might intersect the $M^4 \setminus \hat{D}'$ part non-trivially.

As in Lemma 3.1, let $K := \partial D'$ be the boundary of the gluing disk. $V' \cap K$ is compact, call it $C$. $\phi^{-1}(C)$ is then a compact subset of $V$ since $\phi$ is proper. Exhaust $V$ by compact subsets $K_n$; then there exists an $N \in \mathbb{N}$ so that $\phi^{-1}(C) \subseteq K_N$. We can find $T \subseteq V$, a closed neighborhood of infinity of $M \setminus p$, and, since $V \subseteq \mathcal{R}_U \setminus \hat{D}$, also of $\mathcal{R}_U$, so that the image under $\phi$ lies entirely in $\mathcal{R}_U \setminus D^4$ (see Figure 4).

![Diagram](image_url)

**Figure 4.** Restricting $\phi$ to a diffeomorphism at infinity of $\mathcal{R}_U$. The intersection $C$ between $V' = \phi(V)$ and $\partial D'$ is colored in blue. The closed neighborhood of infinity $T$ whose image under $\phi$ is contained in $\mathcal{R}_U \setminus \hat{D}$ is colored in violet.

Here $\phi$ defines a diffeomorphism at infinity of $\mathcal{R}_U \setminus \hat{D}$, hence also of $\mathcal{R}_U$. By Lemma 4.1, $\phi$ can be extended to a diffeomorphism of $\mathcal{R}_U$. For ease of notation, call it $\phi$ as well.

The map $\phi$ defines a smooth embedding of $D$, the disk used to perform the connected sum, inside $\mathcal{R}_U$. By the *Cerf-Palais disk theorem* [Cer61; Pal59], there is an ambient isotopy $H$ of $\mathcal{R}_U$ between the two embeddings $\phi|_D : D \hookrightarrow \mathcal{R}_U$ and $\nu : D \hookrightarrow \mathcal{R}_U$, where $\nu$ denotes the inclusion. Moreover, $H$ can be assumed to be trivial on the complement of a compact subset $K$ of $\mathcal{R}_U$. 
Then $H_1 : R_U \to R_U$ is a self-diffeomorphism of $R_U$ satisfying $H_1 \circ \phi|_D = \nu$ and $H_1|_{R_U \setminus K} = id$. The composition $H_1 \circ \phi$ defines a self-diffeomorphism of $R_U$ which extends the diffeomorphism at infinity $\phi$ and restricts to the identity on the disk $D$ used to perform the connected sum. Call $H_1 \circ \phi =: \Phi$. By construction, $\Phi|_{\partial D} =$ id; hence we can extend $\Phi$ via the identity over $M \setminus \hat{D}'$, obtaining a self-diffeomorphism of $M \setminus p$, call it $\Phi\#id$, that extends $\phi$, the diffeomorphism at infinity of $M \setminus p$ (see Figure 5). This concludes the proof.

Now we can prove the general version of this result. As in the proof of Theorem 3.3, most of the techniques we will use to prove Theorem 4.3 come from the proof of Lemma 4.2.

In the statement we use the term universal behavior. This indicates that the smoothings of $M \setminus S$ satisfy a property similar to the one of $R_U$ with respect to diffeotopy groups. Lemma 4.2 is analogous to Lemma 4.1. This is because both $M \setminus p$ and $R^4$ have a single end. Under the assumptions of Lemma 4.2, the diffeotopy group at infinity $\mathcal{D}^{\infty}(M \setminus p)$ coincides with the diffeotopy group $\mathcal{D}^{\epsilon}(M \setminus p)$ of the end at $p$. Moreover, the map $r$ coincides with $r_{\epsilon_p}$. When dealing with multiple punctures, and especially non-compact manifolds, universal behavior has a slightly different meaning which we now specify.

**Theorem 4.3.** Let $M$ be a topological 4-manifold, $S \subset \hat{M}$ a non-empty discrete set of points.

1. If $M$ is smoothable, then $M \setminus S$ admits a smoothing for which the map $r_{\epsilon_p}$ is surjective for each $p \in S$.  

![Figure 5](image-url)
(2) If $M$ is non-smoothable and $|S| \geq 2$, then $M \setminus S$ admits a smoothing for which $r_{\epsilon p}$ is surjective for all but one $p \in S$.

In both cases we say that the smoothing has universal behaviour (this does implicitly depend on the pair $(M, S)$, but we suppress this from the notation). Note that this does not impose any condition on other ends of $M \setminus S$.

Proof. We already covered case (1) when $|S| = 1$ and $M$ is compact in Lemma 4.2. To prove Theorem 4.3, it suffices to prove case (1) for $M$ non-compact. Case (2) follows from (1) applied to $M' := M \setminus p$ and $S' := S \setminus p$, where $p \in S$, again using Quinn’s result [Qui82].

As in the proof of Theorem 3.3, to prove (1) it suffices to show the result is true for a non-compact 4-manifold $M$ and $|S| = 1$. Once we have this, we will show how to obtain the general result. The compact case is analogous.

Claim. Let $M$ be a non-compact 4-manifold and $p \in \check{M}$. Then there is a smoothing of $M \setminus p$ with universal behavior.

Proof of the claim. The proof is analogous to that of Lemma 4.2. Note that $\partial D'$ separates $\epsilon_p$ from all the other ends of the non-compact $M$. □

Using the claim, we can prove (1). Suppose $S$ is discrete and let $\Lambda$ denote the index set. The set $\Lambda$ must be countable otherwise $S$ would not be discrete, so we can identify it with a subset of $\mathbb{Z}^+$. Pick $p_1 \in S$. Using the claim, we obtain a smoothing of $M \setminus p_1$ for which the map $r_{\epsilon p_1}$ is surjective. Fix the smoothing and call $M'$ the resulting smooth manifold. Call $S' := S \setminus p_1$. Take $p_2 \in S$. Applying the claim to $M'$ and $p_2$, we get a smoothing of $M' \setminus p_2$ for which the map $r_{\epsilon p_2}$ is surjective. Since $S$ is discrete, we can choose the disk $\check{D}'$ away from $p_1$ to perform the connected sum with $U$. Then the smooth structure is not changed near the puncture $p_1$. Hence $r_{\epsilon p_1}$ remains surjective.

Repeat the procedure for each index $i \in \Lambda$. In the end we get a smoothing of $M \setminus S$ for which the maps $r_{\epsilon p_i}$ are surjective for each $i \in \Lambda$. Thus, we have a smoothing of $M \setminus S$ with universal behavior. As mentioned, we can prove the case $M$ compact, smoothable and $|S| \geq 2$ analogously.

To prove case (2), apply (1) to $M' := M \setminus p$ and $S' := S \setminus p$ using Quinn’s smoothability theorem [Qui82]. Keep in mind that this step causes loss of information about the end $\epsilon_p$. We obtain a smoothing of $M \setminus S$ for which the maps $r_{\epsilon p_i}$ are surjective for $p_i \neq p$. However, we do not know whether the same holds for the map $r_{\epsilon p}$. □
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