Abstract

A systematic method of constructing manifestly supersymmetric 1 + 1-dimensional KP Lax hierarchies is presented. Closed expressions for the Lax operators in terms of superfield eigenfunctions are obtained. All hierarchy equations being eigenfunction equations are shown to be automatically invariant under the (extended) supersymmetry. The supersymmetric Lax models existing in the literature are found to be contained (up to a gauge equivalence) in our formalism.
**Introduction.** In recent years, there has been rapidly growing interest in various $1 + 1$-dimensional soliton systems exhibiting invariance under (extended) supersymmetry and admitting Lax representations. For various contributions to this program see for instance [1, 2, 3, 4, 5, 6] and references therein.

Still, in our opinion there remains a need for a systematic approach to the problem. The purpose of this paper is to propose such a construction. The resulting formalism is manifestly supersymmetric, encompasses known supersymmetric models encountered in literature and provides a straightforward guidance for the future model building.

The starting point is the SKP$_2$ hierarchy based on the Lax operator of an even parity and correspondingly involving only even time-flows [7]. The reduction scheme we propose generalizes to $N = 1$ supersymmetry the method which has previously been applied to the standard KP Hierarchy and resulted in the constrained KP hierarchy [8, 9]. It is based on the symmetry constraints generated by pairs of conjugated eigenfunctions of the original hierarchy. It is an important feature of the formalism that the eigenfunctions remain eigenfunctions of the reduced hierarchy [9]. In the supersymmetric case the eigenfunctions are superfields and their flows provide a complete set of hierarchy equations. Due to the use of the manifestly covariant formalism the hierarchy equations are all invariant under supersymmetry. By analogy we generalize the method and obtain the $N$-extended supersymmetric Lax hierarchy models.

We also introduce a gauge transformation to the nonstandard supersymmetric KP hierarchy. As in the bosonic case the gauge transformation induces an equivalence between standard and nonstandard Lax hierarchies. This mapping is used to gauge some of the supersymmetric nonstandard Lax hierarchies present in the literature to our models.

The KP hierarchy and its reduction. The KP hierarchy consists of a set of multi-time evolution equations in the Lax form

$$ \partial_m L \equiv \frac{\partial L}{\partial t_m} = [(L^m)_+, L], \quad m = 1, 2, 3, \ldots $$

with commuting flows

$$ \frac{\partial^2 L}{\partial t_m \partial t_n} = \frac{\partial^2 L}{\partial t_n \partial t_m}. $$

The pseudo-differential Lax operator $L$ in (3) is given by

$$ L = D + \sum_{r=1}^{\infty} u_r D^{-r-1}, \quad D \equiv \frac{\partial}{\partial x} $$

In eq. (3) $(L^m)_+$ denotes the purely differential part of $L^m$. We adopt a notation that for any operator $A$ and any function $f$, the symbol $A(f)$ will indicate the action of $A$ on $f$ while the symbol $Af$ stands for the operator product of $A$ with the zero-order (multiplication) operator $f$. Accordingly, $D(f) = f'$ while $Df = f' + fD$.

Our objective is to describe the reduction of the KP hierarchy in a way which straightforwardly generalizes to the supersymmetric case. Our reduction procedure employs a set of eigenfunctions and adjoint eigenfunctions which satisfy

$$ \partial_n \Phi = B_n(\Phi) \quad ; \quad \partial_n \Psi = -B^*_n(\Psi) \quad n = 1, 2, \ldots \quad ; \quad B_n \equiv L^n_+ \quad ; \quad B^*_n \equiv (L^*)_+ $$

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for a given Lax operator $L$ obeying Sato’s flow equation (1). In (4) we have introduced an operation of conjugation, defined by the rules $D^* = -D$ and $(AB)^* = B^* A^*$.

The reduction procedure goes as follows. First of all, we notice that for a purely differential operator $A$ and arbitrary functions $f,g$ we have the identity

$$[A, fD^{-1}g]_+ = A(f)D^{-1}g - fD^{-1}A^*(g)$$ (5)

This together with (4) gives

$$\frac{\partial}{\partial t_k} \left( \Phi D^{-1} \Psi \right) = [B_k, \Phi D^{-1} \Psi]_+ .$$ (6)

We now introduce the symmetry constraints leading to the so-called constrained KP hierarchy denoted by $cKP$ [8, 9]. Let $\partial_{\alpha_i}$ be vector fields, whose action on the standard KP Lax operator from equation (3) is induced by the (adjoint) eigenfunctions $\Phi_i, \Psi_i$ of $L$ through

$$\partial_{\alpha_i}L \equiv [L, \Phi_i D^{-1} \Psi_i] ; \quad i = 1, \ldots, m$$ (7)

As a result of (6), the vector fields $\partial_{\alpha_i}$ commute with the isospectral flows of the Lax operator:

$$[\partial_{\alpha_i} , \partial_n] L = 0 \quad n = 1, 2, \ldots$$ (8)

The reduction is then performed by imposing equality between the “ghost” flow $\sum_{i=1}^m \partial_{\alpha_i}$ and the isospectral flow $\partial_r$ of the original KP hierarchy. Comparing (7) with equation (1) we find that the negative pseudo-differential part of the Lax operator belonging to the constrained KP hierarchy is $L_r^- = \sum_{i=1}^m \Phi_i D^{-1} \Psi_i$. The positive differential part of $L'$ has a generic form $L_r^+ = D^r + \sum_{l=0}^{r-2} u_l D^l$. Especially for $r = 1$ we are led to the Lax operator:

$$L = D + \sum_{i=1}^m \Phi_i D^{-1} \Psi_i$$ (9)

still subject to the Lax equation (1). For the simplest $m = 1$ case the constrained system from (3) is equivalent to the AKNS hierarchy. Note, that the (adjoint) eigenfunctions $\Phi_i$ and $\Psi_i$ of the original Lax operator $L$ (3) used in the above construction remain (adjoint) eigenfunctions for $L$ (9).

$N = 1$ Supersymmetric KP hierarchy. Let superspace be defined in terms of the pair $(x, \theta)$ with the even variable $x$ and the odd variable $\theta$. The corresponding supercovariant derivative is

$$D_\theta = \frac{\partial}{\partial \theta} + \theta D$$ (10)

and has the property $D_\theta^2 = D$. An arbitrary pseudo-super-differential operator $\Lambda$ has the formal expression

$$\Lambda = \sum_{i=-\infty}^{N} V_i D_\theta^i$$ (11)
We denote its splitting on positive/negative parts by

\[ \Lambda_+ \equiv \sum_{i=0}^{N} V_i D_\theta^i, \quad \Lambda_- \equiv \sum_{i=-\infty}^{-1} V_i D_\theta^i. \] (12)

The multiplication of two operators \( \Lambda \) and \( \Omega \) is determined by the associativity. The basic product rule is

\[ D_\theta UV = (D_\theta U)V + (-1)^{|U|}UD_\theta V \]

for \( D_\theta \) acting on two arbitrary superfields \( U \) and \( V \). The parity of a function \( U \) is denoted by \( |U| \) and is equal to zero for \( U \) being even and one for \( U \) being odd. The above identity generalizes to the supersymmetric version of the Leibniz rule [10]:

\[ D_\theta U = \sum_{l=0}^{\infty} (-1)^{|U|} \binom{i}{l} D_\theta^l(U)D_\theta^{i-l}. \] (13)

The super-binomial coefficients \( \binom{i}{k} \) are for \( i \geq 0 \) [10]:

\[ \binom{i}{k} = \begin{cases} 
0 & \text{for } k < 0 \text{ or } k > i \text{ or } (i, k) = (0, 1) \mod 2 \\
\left( \frac{[i/2]}{[k/2]} \right) & \text{for } 0 \leq k \leq i \text{ and } (i, k) \neq (0, 1) \mod 2 
\end{cases}; \] (14)

and are expressed for \( i < 0 \) by the identity

\[ \binom{i}{k} = (-1)^{[k/2]} \binom{-i+k-1}{k}. \] (15)

There are two types of supersymmetric extensions of the KP hierarchy. One is based on the odd pseudo-super-differential operator [10]

\[ L_{MR} = D_\theta + \sum_{i=-1}^{\infty} \bar{U}_i D_\theta^{-i-1} \] (16)

and the other on the even pseudo-super-differential operator [7]

\[ Q = D_\theta^2 + \sum_{i=-2}^{\infty} U_i D_\theta^{-i-1}. \] (17)

The latter hierarchy is called SKP\(_2\). It has been shown to be bi-Hamiltonian [7]. The coefficients \( U_i \) in (17) are functions of \( x, \theta \) and various (even) time variables \( t_m(m = 1, 2, 3, \ldots) \). The pseudo-super-differential operator \( Q \) as given by (17) has an even parity and accordingly \( |U_i| = i + 1 \mod 2 \).

The supersymmetric KP hierarchy involving the Lax operator \( Q \) is a system of infinitely many evolution equations for the functions \( U_i \) following from the Lax equations

\[ \partial_m Q \equiv \frac{\partial Q}{\partial t_m} = [(Q^m)_+, Q], \quad m = 1, 2, 3, \ldots \] (18)
From now on we will only consider the supersymmetric KP hierarchy SKP$^2$ with the Lax operator $Q$ from (17) and its evolution equations (18).

We need to extend to this setting the concept of conjugation denoted above by "$^*$". The proposed rules are as follows. On the product of two elements (including operators) $\Lambda$ and $\Omega$ the conjugation reverses the order and adds a phase depending on the grade of the elements:

$$(\Lambda \Omega)^* = (-1)^{|\Lambda||\Omega|}\Lambda^*\Omega^* .$$

Both derivatives $D$ and $D_\theta$ change sign under conjugation and therefore

$$\left(D^k\right)^* = (-1)^k D^k ; \quad \left(D_\theta^k\right)^* = (-1)^{k(k+1)/2} D_\theta^k$$

It follows for consistency that for the superfield $U$ having an arbitrary gradation we have $U^* = U$.

Defining reduction of SKP$^2$. We are looking for the reduction of the original SKP$^2$ hierarchy with the typical grade zero Lax operator given by (17) and subject to the standard Lax equation (18). For this purpose we again introduce a pair of (conjugated) eigenfunctions $\Phi(x, \theta), \Psi(x, \theta)$ of $Q$ satisfying eqs. (4) but this time with $B_n = (Q^n)_+$. In complete analogy with the pure bosonic case (see (6)) we have the following result:

$$\partial_n \left(\Phi D_\theta^{-1}\Psi\right) = \left[B_n , \Phi D_\theta^{-1}\Psi\right]_-$$(21)

Note, that as a consequence of eqs. (4) we also have

$$\left(B_n \Phi D_\theta^{-1}\Psi\right)_- = (B_n \Phi) D_\theta^{-1}\Psi = (\partial_n \Phi) D_\theta^{-1}\Psi .$$

On the other hand, a simple but tedious computation using expansions for $B_n$ and $B_n^*$

$$B_n = \sum_{i=0}^{\infty} P_i D_\theta^i ; \quad B_n^* = \sum_{k=0}^{\infty} (-1)^{k(k-1)/2} D_\theta^k P_k$$

yields

$$- \left(\Phi D_\theta^{-1}\Psi B_n\right)_- = -\Phi D_\theta^{-1} B_n^*(\Psi)$$

which along with eq. (22) verifies (21).

Let $\partial_{\alpha_i}$ be vector fields, whose action on the standard KP Lax operator $Q$ from (17) is induced by the (adjoint) eigenfunctions $\Phi_i, \Psi_i$ of $Q$ through

$$\partial_{\alpha_i} Q \equiv [Q , \Phi_i D_\theta^{-1}\Psi_i] \quad i = 1, \ldots, m$$

Note, the presence of the super-covariant derivative $D_\theta$ in the definition of the “ghosts” flows $\partial_{\alpha_i}$.

As in the bosonic case we find that the vector fields $\partial_{\alpha_i}$ commute with the isospectral flows of the Lax operator $Q$:

$$\left[\partial_{\alpha_i} , \frac{\partial}{\partial t_n}\right] Q = 0 \quad n = 1, 2, \ldots$$ (26)
Hence following the standard procedure we define the constrained supersymmetric KP hierarchy via identification of the “ghost” flow $\sum_{i=1}^{m} \partial_{\alpha_i}$ with the isospectral flow $\partial_r$ of the original SKP$_2$ hierarchy. We are in this way led to the Lax operator $Q_c = Q^r$ which for $r = 1$ is given by

$$Q_c = D_\theta^2 + \sum_{i=1}^{m} \Phi_i D_\theta^{-1} \Psi_i$$  \hspace{1cm} (27)$$

It can be shown that the (adjoint) eigenfunctions $\Phi_i$, $\Psi_i$ of the original Lax operator $Q$ used in the above construction remain (adjoint) eigenfunctions for $Q_c$ (27). Indeed the method of reference [9] extends to this case and one can show that as a result of plugging $Q_c$ into eq.(18) it follows that up to a $(x, \theta)$-independent phase transformation $\Phi \rightarrow e^{c_r} \Phi$ and $\Psi \rightarrow e^{-c_r} \Psi$ (which leaves the Lax operator unchanged) the superfields $\Phi$ ($\Psi$) are (adjoint) eigenfunctions. This observation carries over to the extended supersymmetric Lax operator considered below.

**Supersymmetric AKNS Model.** As an example we now consider the constrained Lax operator constructed from only one pair of (adjoint) superfield eigenfunctions $Q \equiv D_\theta^2 + \Phi D_\theta^{-1} \Psi$ \hspace{1cm} (28)

This is the supersymmetric AKNS model and we will study its corresponding AKNS (generalized NLS) equations:

$$\partial_t \Phi = (Q^2)_+ (\Phi) \quad ; \quad \partial_t \Psi = - (Q^2)_+^* (\Psi) \quad ; \quad \partial_t \equiv \partial / \partial t_2$$  \hspace{1cm} (29)$$

Although they follow automatically by construction they can also be obtained by inserting $Q$ from (28) into the eq.(18).

We find:

$$(Q^2)_+ = D^2 + 2 \Phi (D_\theta \Psi) + 2 (-1)^{|\Psi|} \Phi D_\theta$$ \hspace{1cm} (30)$$

$$(Q^2)_+^* = D^2 + +2(-1)^{|\Phi|+1}(|\Psi|+1) \Psi (D_\theta \Phi) + 2(-1)^{|\Phi|+1}(|\Psi|+1)(-1)^{|\Phi|} \Psi \Phi D_\theta$$ \hspace{1cm} (31)$$

Choose now $|\Psi| = 1$ and $|\Phi| = 0$ with parametrization:

$$\Phi = \phi_1 + \theta \psi_1 \quad ; \quad \Psi = \psi_2 + \theta \phi_2$$ \hspace{1cm} (32)$$

Plugging (30)-(31) and (32) into (29) results in

$$\partial_t \phi_1 = \phi_1'' - 2 \phi_1 \psi_2 \psi_1 + 2 \phi_1^2 \phi_2$$ \hspace{1cm} (33)$$

$$\partial_t \psi_1 = \psi_1'' + 2 \phi_1 \phi_2 \psi_2 + 2 \phi_1^2 \psi_2' + 2 \phi_1 \phi_2 \psi_1$$ \hspace{1cm} (34)$$

$$\partial_t \psi_2 = - \psi_2'' - 2 \psi_2 \phi_1 \phi_2$$ \hspace{1cm} (35)$$

$$\partial_t \phi_2 = - \phi_2'' + 2 \psi_2 \phi_1 \phi_2 - 2 \phi_1 \phi_2^2 + 2 \phi_1 \psi_2 \psi_2'$$ \hspace{1cm} (36)$$

The equations (33)- (36) contain the ordinary AKNS equations in the limit $\psi_1, \psi_2 \rightarrow 0$.

It is straightforward to prove that the generalized AKNS equations (33)-(36) are invariant under the supersymmetry transformations:

$$\delta_\epsilon \phi_1 = \epsilon \psi_1 \quad ; \quad \delta_\epsilon \psi_1 = \epsilon \phi_1'$$ \hspace{1cm} (37)$$

$$\delta_\epsilon \phi_2 = \epsilon \psi_2' \quad ; \quad \delta_\epsilon \psi_2 = \epsilon \phi_2$$ \hspace{1cm} (38)$$
The above supersymmetry transformations can be rewritten in a following covariant form

\[ \delta_\epsilon \Phi = \epsilon D_\theta^\dagger \Phi \quad ; \quad \delta_\epsilon \Psi = \epsilon D_\theta^\dagger \Psi \quad ; \quad D_\theta^\dagger \equiv \frac{\partial}{\partial \theta} - \theta D \] (39)

In fact, the preceding results concerning supersymmetry invariance can be elegantly reassembled and generalized by applying covariant notation. As we now show the use of a covariant language will make transparent the origin of invariance of eqs. (33)-(36) as well as all the remaining hierarchy equations, due to their eigenfunction form \[ \partial_n \Phi(x, \theta) = (Q^n)_+ \Phi(x, \theta) \] and \[ \partial_n \Psi(x, \theta) = (Q^n)_+ \Psi(x, \theta) \]. Note, namely that in this setting applying the supersymmetry transformation to the hierarchy equations amounts to transforming the right hand side into

\[ \delta_\epsilon (Q^n)_+ \Phi = \epsilon D_\theta^\dagger (Q^n)_+ \Phi \] (40)

since \((Q^n)_+\) has an expansion in terms of the superfields \(\Phi, \Psi\). The advantage of using covariant notation is now obvious. We namely know that

\[ \epsilon D_\theta^\dagger (Q^n)_+ \Phi = \epsilon D_\theta^\dagger \partial_n \Phi = \partial_n \epsilon D_\theta^\dagger \Phi \] (41)

since \([\partial_n, D_\theta^\dagger] = 0\) for all \(n\). Hence applying \(\delta_\epsilon\) on both sides of \(\partial_n \Phi = (Q^n)_+ \Phi\) is equivalent to acting with the derivation \(\epsilon D_\theta^\dagger\). Consequently, all higher flows of the constrained supersymmetric KP hierarchy remain covariant.

Gauge Transformations for the Supersymmetric KP. In general the Lax operator can have “constant” terms i.e. superfields. For the non-supersymmetric case it has been shown [11, 12, 13] that the constant terms can be gauged away via a suitable transformation and that the resulting Lax operator satisfies the standard Lax eq. (1). We extend this procedure to the supersymmetric case.

Let \(\phi\) be an arbitrary superfield. Define

\[ G = \exp \left( - \int x D_\theta \phi \right) \] (42)

Then the differential operator

\[ Z = D_\theta^{2-|\phi|} + \phi \] (43)

can be gauge-transformed into a new operator

\[ \tilde{Z} \equiv G^{-1} Z G \] (44)

whose constant term is zero. We will verify this for the two possible values of \(|\phi|\).

Let first \(|\phi| = 0\). Then eq. (12) becomes \(G = \exp (- \int x \phi \phi)\). Using, that \(D_\theta^{2} = D\) and \(G_2^{-1} G_2' = -G_2^{-1} \phi G_2\), we obtain as stated \(\tilde{Z} \equiv G^{-1} (D + \phi) G = D\).

Next, consider the remaining case \(|\phi| = 1\). Then eq. (12) reads \(G = \exp (- \int x D_\theta (\phi) dx)\) and it is straightforward to verify that indeed \(\tilde{Z} \equiv G^{-1} (D_\theta + \phi) G = D_\theta\) due to \(G^{-1} D_\theta (G) = -G^{-1} \phi G\).

Consider the nonstandard Lax \(Q = D + U_{-1} + \sum_{i=0}^{\infty} U_i D_\theta^{-i-1}\) satisfying

\[ \partial_n Q = \left[ (Q^n)_{\geq 1}, Q \right] . \] (45)
The next step is to investigate whether the gauge transformed of an nonstandard Lax operator satisfies the standard SKP^2 flow equations.

The claim is that if Q satisfies eq. (45) then \( \tilde{Q} = G^{-1}QG \) is a solution of the standard Lax equations (18).

Using eq. (45) we obtain by direct computation

\[
\partial_n \tilde{Q} = \left[ G^{-1}(Q^n)_{\geq 1}G - G^{-1}\partial_nG, \tilde{Q} \right].
\] (46)

In general for a pseudo-differential operator \( M = \sum_{j \leq m} a_j D^j \), set \( (M)_j = a_j D^j \). Then \( (Q^n)_{\geq 1} = Q^n - \sum_{j \leq 0} (Q^n)_j \) and consequently \( \partial_nQ = \left[ (Q^n)_{\geq 1}, Q \right] = -\left[ \sum_{j \leq 0} (Q^n)_j, Q \right] \), which implies that

\[
\partial_nU_{-1} = (\partial_nQ)_0 = -\left( \sum_{j \leq 0} (Q^n)_j, Q \right)_0 = ([Q, (Q^n)_0])_0 = \frac{\partial}{\partial x}(Q^n)_0.
\]

Therefore

\[
G^{-1}\partial_nG = -G^{-1} \left( \int_x^x \partial_nU_{-1}dx \right)G = -G^{-1}(Q^n)_0 G,
\]

and eq.(46) becomes:

\[
\partial_n \tilde{Q} = \left[ G^{-1}(Q^n)_{\geq 1}G + G^{-1}(Q^n)_0 G, \tilde{Q} \right] = \left[ G^{-1}(Q^n)_{\geq 0} G, \tilde{Q} \right] = \left[ \tilde{Q}^+, \tilde{Q} \right].
\]

In the last equality, the cancellation of the constant term in \( \tilde{Q} \) by the gauge transformation was taken into account.

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Gauge connection to the Supersymmetric two-boson Hierarchy. As an example of gauge equivalence let us take the nonstandard Lax operator of the supersymmetric two-boson hierarchy considered in [1]:

\[
L_{NS} = D^2_\theta - D_\theta (\Phi_0) + D^{-1}_\theta \Phi_1
\] (47)

where \( \Phi_0 = f_0 + \theta J_0 \) and \( \Phi_1 = f_1 + \theta J_1 \) are fermionic superfields. Observe that the above nonstandard Lax operator can be transformed to the standard Lax operator from eq.(28) via the gauge transformation \( Q = G^{-1}L_{NS}G \), where \( G = \exp (\int x D_\theta (\Phi_0) dx) \) and \( Q \) is

\[
Q = D + G^{-1}D^{-1}_\theta \Phi_1 G.
\] (48)

This clearly suggests to make an identification \( \Phi \equiv G^{-1} \), \( \Psi \equiv \Phi_1G \), which in components reads:

\[
\phi_1 = e^{-\int^x J_0 dx} \quad ; \quad \psi_1 = e^{-\int^x J_0 dx} f_0
\] (49)

\[
\phi_2 = e^{-\int^x J_0 dx} (f_1 f_0 + J_1) \quad ; \quad \psi_2 = e^{-\int^x J_0 dx} J_1
\] (50)

Thus the Lax operator (17) is brought to the standard form \( Q = D + \Phi D^{-1}_\theta \Psi \). It is straightforward to show directly that indeed under substitutions (49)-(50) the component equations of reference [1] go into the AKNS eqs. (33)-(36) and vice versa. This also follows from the general statement proved above.
Let us consider the extended superspace described in terms of \((x, \theta_1, \theta_2, \ldots, \theta_N)\) where \(x\) is even and \(\theta_i\)s are odd variables. Inspired by our discussion of \(N = 1\) case we propose a particular form of the Lax operator as:

\[
Q = D + \Phi D^i D_{\theta_1}^{-1} \cdots D_{\theta_N}^{-1} \Psi \quad l < N
\]  

where \(\Phi(x, \theta_1, \theta_2, \ldots, \theta_N)\) and \(\Psi(x, \theta_1, \theta_2, \ldots, \theta_N)\) are superfields on the \(N\)-extended superspace with parity chosen such that \(Q\) is even. Denote \(B_n = (Q^n)_+\). Then eq. (52) generalizes to:

\[
\partial_n \left( \Phi D^i D_{\theta_1}^{-1} \cdots D_{\theta_N}^{-1} \Psi \right) = \left[ B_n, \Phi D^i D_{\theta_1}^{-1} \cdots D_{\theta_N}^{-1} \Psi \right]_{l,N}
\]

where \([\cdot, \cdot]_{l,N}\) denotes the commutator projected on the pseudo-differential operator present on the l.h.s. space. Indeed, assuming the relation (4) for \(\Phi\) and \(\Psi\) and using the expansions of \(B_n\) and \(B_n^*\)

\[
B_n = \sum_{i_1,...,i_N=0}^{\infty} A_{i_1,...,i_N} D_{\theta_1}^{i_1} \cdots D_{\theta_N}^{i_N}
\]

\[
B_n^* = \sum_{j_1,...,j_N=0}^{\infty} D_{\theta_1}^{j_1} \cdots D_{\theta_N}^{j_N} A_{j_1,...,j_N} (-1)^{\frac{N}{2} (\sum_{a=1}^{N} j_a)^2 + \frac{1}{2} (\sum_{a=1}^{N} j_a)}
\]

we can verify directly eq. (52). The expansion (54) takes into account that the parity of \(A_{i_1,...,i_N}\) is \(|A_{i_1,...,i_N}| = i_1 + \cdots + i_N \mod 2\), in order to assure the even character of \(B_n\).

If, like before, the Lax operator \(Q\) from (51) satisfies the Sato’s Lax hierarchy equations we find that \(\Phi\) and \(\Psi\) are (adjoint) eigenfunctions. Accordingly, the ir flow equations will be manifestly invariant under supersymmetry transformations. The above result can naturally be extended to the more general Lax operator:

\[
Q_{l,K,N}^{(m)} = D + \sum_{a=1}^{m} \sum_{i_1,...,i_N=0}^{i_0=1} \sum_{i_1 + \cdots + i_N = K} C_{i_1,...,i_N}^a \Phi_a D^i D_{\theta_1}^{-i_1} \cdots D_{\theta_N}^{-i_N} \Psi_a, \quad j = 1, \ldots, N; \quad 0 \leq K \leq N
\]

where \(C_{i_1,...,i_N}^a\) are constants and \(l < K\). By analogy with \(N = 1\) case we call the Lax operator written in the form (55) the “standard” Lax operator. The “nonstandard” operator will contain constant terms. The constant term denoted by \(\phi(x, \theta_1, \ldots, \theta_N)\) can be gauged away via the gauge transformation \(G = \exp (- f^2 \phi(x, \theta_1, \ldots, \theta_N) dx)\) as in the \(N = 1\) case. Note, that there are two parameters \(l\) and \(K\) that specify a “class” of Lax operators (55). We cannot mix in one Lax operator terms having different values of \(l\) or \(K\).

As an example, let us take the \(N = 2\) case. Then eq. (53) becomes:

\[
Q_{l,K,2}^{(m)} = D + \sum_{a=1}^{m} \sum_{i_1+1+\cdots+i_2=K} C_{i_1,i_2}^a \Phi_a D^i D_{\theta_1}^{-i_1} D_{\theta_2}^{-i_2} \Psi_a, \quad j = 1, 2; \quad 0 \leq K \leq 2
\]

Consider now the nonstandard Lax operator of supersymmetric GNLS hierarchies proposed in [2]

\[
L_{NS} = D - \frac{1}{2} \sum_{A=1}^{M} F_{A}(Z) \bar{F}_{A}(Z) - \frac{1}{2} \sum_{A=1}^{M} F_{A}(Z) \bar{D} D^{-1} \bar{D} \left( \bar{F}_{A}(Z) \right)
\]  

(57)
where $F_A(Z)$ and $\bar{F}_A(Z)$ is a pair of chiral and anti-chiral superfields, $Z = (x, \theta, \bar{\theta})$ represents a coordinate point in $N = 2$ superspace, and

$$
\mathcal{D} = \frac{\partial}{\partial \theta} - \frac{1}{2} \bar{\theta} D \quad ; \quad \bar{\mathcal{D}} = \frac{\partial}{\partial \bar{\theta}} - \frac{1}{2} \theta \bar{D}
$$

are the supersymmetric covariant derivatives. It is easy to see that under the change of coordinates in superspace defined by

$$
\theta_1 = \frac{i}{2}(\theta + \bar{\theta}) \quad ; \quad \theta_2 = \frac{1}{2}(-\theta + \bar{\theta})
$$

the supercovariant derivatives $D_{\theta i} = \frac{\partial}{\partial \theta_i} + \theta_i D_{\theta}$ ($i=1,2$) can be expressed as:

$$
D_{\theta 1} = -i \left( \mathcal{D} + \bar{\mathcal{D}} \right) \quad ; \quad D_{\theta 2} = -\mathcal{D} + \bar{\mathcal{D}}.
$$

Correspondingly the Lax (57) becomes:

$$
L_{NS} = D - \frac{1}{2} \sum_{A=1}^{M} F_A \bar{F}_A + \frac{1}{8} \sum_{A=1}^{M} \left[ F_A D_{\theta 1}^{-1} D_{\theta 1} \left( \bar{F}_A \right) - iF_A D_{\theta 1}^{-1} D_{\theta 2} \left( \bar{F}_A \right) + F_A D_{\theta 2}^{-1} D_{\theta 2} \left( \bar{F}_A \right) + iF_A D_{\theta 2}^{-1} D_{\theta 1} \left( \bar{F}_A \right) \right]
$$

Applying the gauge transformation induced by $G = \exp \left( \frac{1}{2} \sum_{A=1}^{M} \int x F_A \bar{F}_A \, dx \right)$ the constant term in eq.(51) is removed. Via the identifications

$$
\Phi_a = \Phi_b = G^{-1} F_A \quad ; \quad \Psi_a = D_{\theta 1} \left( \bar{F}_A \right) G \quad ; \quad \Psi_b = D_{\theta 2} \left( \bar{F}_A \right) \quad a, b = 1, \ldots, M
$$

the transformed Lax $Q = G^{-1} L_{NS} G$ can be cast into the form (56) where $l = 0, K = 1$. Note that this form falls into the same equivalence class ($l = 0, K = 1$) with the Lax operator proposed in [3].

Comments. It is important to observe that although the Lax operator proposed in eq.(53) contains $K$ supercovariant derivatives $D_{\theta i}$, the corresponding hierarchy equations possess invariance under the complete $N$-extended supersymmetry even for $K < N$. For example, the Lax operator $Q^{(1)}_{l=-1, K=0, N} = D + \Phi(x, \theta_1, \ldots, \theta_N) D^{-1} \Psi(x, \theta_1, \ldots, \theta_N)$ will still be invariant under $N$ supersymmetry transformations $\delta_\epsilon \Phi = \epsilon_i D_{\theta i} \Phi$ and $\delta_\epsilon \Psi = \epsilon_i D_{\theta i} \Psi$ with $i = 1, \ldots, N$ although the supercovariant derivatives are missing altogether in its expression.

We would like to stress that the proposal of the Lax operator of the $N$ extended supersymmetry was dictated by its symmetry structure and analogy with the $N = 1$ case. It remains to supplement this proposal by investigation of the Hamiltonian structure which we expect to underlie the Lax hierarchy equations. Also of interest is to investigate the possible additional symmetry structure of the constrained KP hierarchies we proposed here along the lines of what was done in the bosonic case in [14]. We plan to address these issues in the near future.
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