Minimal Acceleration for the Multi-dimensional Isentropic Euler Equations

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Abstract

On the set of dissipative solutions to the multi-dimensional isentropic Euler equations, we introduce a quasi-order by comparing the acceleration at all time. This quasi-order is continuous with respect to a suitable notion of convergence of dissipative solutions. We establish the existence of minimal elements. Minimizing the acceleration amounts to selecting dissipative solutions that are as close to being a weak solution as possible.

1. Introduction

The isentropic Euler equations

\[
\begin{align*}
\partial_t \varrho + \nabla \cdot (\varrho \mathbf{u}) &= 0 \quad & \text{in } [0, \infty) \times \mathbb{R}^d \\
\partial_t (\varrho \mathbf{u}) + \nabla \cdot (\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla P(\varrho) &= 0
\end{align*}
\]

model the evolution of compressible gases. The unknowns \((\varrho, \mathbf{u})\) depend on time \(t \in [0, \infty)\) and space \(x \in \mathbb{R}^d\). We assume suitable initial data is given as:

\[(\varrho, \mathbf{u})(0, \cdot) =: (\bar{\varrho}, \bar{\mathbf{u}})\).

We will think of \(\varrho\) as a map from \([0, \infty)\) into the space of non-negative, finite Borel measures, which we denote by \(\mathcal{M}_+(\mathbb{R}^d)\). The quantity \(\varrho\) is called the density; it represents the distribution of mass in time and space. The continuity equation in (1) expresses the local conservation of mass, where

\[\mathbf{u}(t, \cdot) \in \mathcal{L}^2(\mathbb{R}^d, \varrho(t, \cdot)) \quad \text{for all } t \in [0, \infty)\]

is the Eulerian velocity field taking values in \(\mathbb{R}^d\). The momentum equation in (1) expresses the local conservation of momentum \(\mathbf{m} := \varrho \mathbf{u}\). Note that \(\mathbf{m}(t, \cdot)\) is a finite \(\mathbb{R}^d\)-valued Borel measure absolutely continuous with respect to \(\varrho(t, \cdot)\) for all
$t\in [0,\infty)$, because of (2). Without loss of generality, we will assume that $\rho$ has mass one. It then follows from the continuity equation that for all times $t$ we have $\rho(t,\cdot)\in \mathcal{P}(\mathbb{R}^d)$, the space of Borel probability measures.

To obtain a closed system (1) we must prescribe an equation of state, which determines the pressure. For the isentropic Euler equations, where the thermodynamical entropy is assumed constant in time and space, the pressure is just a function of the density. We will consider polytropic gases.

**Definition 1. (Internal Energy)** Let $U(r) := \kappa r^\gamma$ for all $r \geq 0$, where $\kappa > 0$ and $\gamma > 1$ are constants. For all $\rho \in \mathcal{P}(\mathbb{R}^d)$ we define the internal energy

$$U[\rho] := \begin{cases} \int_{\mathbb{R}^d} U(r(x)) \, dx & \text{if } \rho = r \mathcal{L}^d, \\ +\infty & \text{otherwise.} \end{cases}$$

Here $\mathcal{L}^d$ is the $d$-dimensional Lebesgue measure on $\mathbb{R}^d$.

The constant $\gamma$ is called the adiabatic coefficient. Since we are interested in solutions of (1) with finite energy, the density $\rho(t,\cdot)$ must be absolutely continuous with respect to the Lebesgue measure for all $t \in [0,\infty)$. Let $r(t,\cdot)$ be its Lebesgue–Radon–Nikodým derivative. Then

$$p(t,\cdot) = P(r(t,\cdot)) \mathcal{L}^d \text{ for all } t \in [0,\infty),$$

where

$$P(r) = U'(r)r - U(r) \text{ for } r \geq 0.$$

Occasionally, we will not distinguish between $\rho$ and its Lebesgue density.

Smooth solutions $(\rho, u)$ of (1) satisfy the additional conservation law

$$\partial_t \left( \frac{1}{2} \rho |u|^2 + U(\rho) \right) + \nabla \cdot \left( \frac{1}{2} \rho |u|^2 + U'(\rho)\rho \right) u = 0,$$  \hspace{1cm} (3)

which expresses local conservation of total energy

$$E(\rho, u) := \frac{1}{2} \rho |u|^2 + U(\rho),$$

which is the sum of kinetic and internal energy. It is well-known, however, that a generic solution to the isentropic Euler equations will not remain regular, even for smooth initial data. Instead the solution will develop jump discontinuities along codimension-one submanifolds in space-time, which are called shocks. In this case, continuity and momentum equation must be considered in the sense of distributions, and the energy equation (3) does not follow automatically. A physically reasonable relaxation is to assume that no energy can be created by the fluid. Then (3) must be replaced by the inequality

$$\partial_t \left( \frac{1}{2} \rho |u|^2 + U(\rho) \right) + \nabla \cdot \left( \frac{1}{2} \rho |u|^2 + U'(\rho)\rho \right) u \leq 0$$  \hspace{1cm} (4)
distributionally. Strict inequality in (4) means that mechanical energy is transformed into a form of energy not accounted for by the model, such as heat.

A differential inequality like (4) contains some information on the regularity of solutions: The space-time divergence of a certain non-linear function of \((\varrho, u)\) is a non-positive distribution, and therefore a measure. In the one-dimensional case, one usually requires that weak solutions of (1) satisfy differential inequalities analogous to (4) for a large class of non-linear functions of \((\varrho, u)\) called entropy-entropy flux pairs. Such an assumption is called an entropy condition. Utilizing the method of compensated compactness, it is then possible to establish the global existence of weak (entropy) solutions of (1); see [1–9].

In several space dimensions the only available entropy-entropy flux pair is the total energy and the corresponding energy flux. The compensated compactness technique cannot be applied. One can, however, establish the existence of a large set of initial data for which weak solutions of (1) exist globally in time, by using non-linear iteration schemes like the ones introduced by Nash [10,11] in the context of isometric embeddings of Riemannian manifolds. We refer the reader to the groundbreaking work by De Lellis and Székelyhidi [12,13] and subsequent extensions [14–17] by various authors. These results even show that for suitable initial data there exist infinitely many weak solutions of (1), even if one requires that solutions satisfy an entropy condition in the form (4). This is related to the fact that—in addition to energy dissipation through shocks—there is an additional dissipation mechanism due to very high oscillations of the velocity field, which is reminiscent of anomalous dissipation in turbulence. Moreover, there is a precise threshold of Hölder regularity 1/3 between energy conserving and energy dissipating regimes. For incompressible flows, this has been conjectured based on physical considerations by Onsager [18]. A mathematical proof of this conjecture has been provided in a series of recent articles; see [19–21] and references therein. For related results for the compressible Euler equations, see [22]. The Cauchy problem for (1) in several space dimensions has not been solved yet: In order to apply the methodology above for given initial data, it is necessary to allow a small increase in energy initially, which violates (4).

We do not know whether weak solutions of (1) always exist. Clearly there is no uniqueness. It is conceivable that there is initial data for which no well-defined weak solution can be found, as numerical experiments in [23–25] involving particular situations such as Kelvin-Helmholtz instabilities have suggested. It has been argued that the apparent non-convergence of numerical solutions under successive mesh refinement necessitates the use of solution concepts that are weaker than weak solutions, like measure-valued or dissipative solutions; see also [25,26].

Dissipative solutions

In order to construct approximate solutions of (1), we will use the variational time discretization introduced in [27]. For given timestep \(\tau > 0\) it generates approximate solutions at discrete times \(t^k := k\tau\) with \(k \in \mathbb{N}_0\) by solving an optimization problem for each step, minimizing the sum of a suitable work functional plus the internal energy. This can be interpreted as maximizing the entropy production; see
The work functional quantifies how much mass points, which at the beginning of the time step have a well-defined location and velocity, deviate from the free transport path, i.e., it measures their acceleration. It is a second-order replacement of the Wasserstein distance, which plays a central role in the interpretation of certain parabolic equations as gradient flows on the space of probability measures; see [28].

Having determined approximate solutions of (1) at discrete times $t^k$, we then interpolate in time to obtain a curve, i.e., a map $t \mapsto (\rho^\tau, m^\tau)(t, \cdot)$ into the state space, for all $t \in [0, \infty)$. There are different ways to do this. One possibility is to interpolate piecewise constantly in time so that the approximate solution jumps at $t^k$. Another one, which was not considered in [27], utilizes a piecewise linear interpolation of the transport map and the velocity. We will need both interpolations. The second method provides a tighter control of the total energy; see Remark 8.

We then consider the limit $\tau \to 0$. Extracting subsequences as necessary, we obtain a limit density/momentum $(\rho, m)$, which is a candidate for a solution of (1). As in [27], we use Young measures to capture the weak limits of non-linear functions of $(\rho^\tau, m^\tau)$ as $\tau \to 0$. Since we consider different time interpolations, we obtain two Young measures $(\epsilon, \nu)$. Those are elements of the dual space $E^* := \mathcal{L}_w^\infty([0, \infty), \mathcal{M}(\mathbb{R}^d \times X))$, which is the space of weakly measurable, essentially bounded maps from $[0, \infty)$ into the space of finite Borel measures on $\mathbb{R}^d \times X$. Here $\mathbb{R}^d$ is the one-point compactification of the physical space, and $X$ is a suitable compactification of the state space; see Section 3.3. As $\tau \to 0$, we have the convergence

$$m^\tau \otimes m^\tau / \rho^\tau \to [m \otimes m / \rho] \quad \text{and} \quad P(\rho^\tau) \to [P(\rho)]$$

in a suitable topology, where $[\cdot]$ and $[[\cdot]]$ denote the pairing with Young measures $\nu$ and $\epsilon$, respectively. We refer the reader to Section 3.3 for details. Then $(\rho, m)$ satisfy the continuity equation and a modified momentum equation

$$\partial_t m + \nabla \cdot U = 0 \quad \text{with} \quad U := [m \otimes m / \rho + [P(\rho)]1].$$

We will call such pairs of Young measures dissipative solutions, borrowing the term from [29]. There, dissipative solutions are defined as tuples of $(\rho, m)$ and defect measures $R, \phi$ that satisfy the continuity equation and

$$\partial_t m + \nabla \cdot \left( \frac{m \otimes m}{\rho} + P(\rho)1 \right) + \nabla \cdot (R + \phi1) = 0$$

distributionally. Here $R$ and $\phi$ are measures taking values in the symmetric, positive semidefinite matrices and the non-negative numbers. Moreover

$$\frac{d}{dt} \int_{\mathbb{R}^d} \left( \frac{1}{2} \rho |u|^2 + U(\rho) + \frac{1}{2} \text{tr}(R) + \frac{1}{\gamma - 1} \phi \right) (t, dx) \leq 0$$
distributionally, which provides a bound on the sizes of \( R \) and \( \phi \). The dissipative solutions of \([29]\) have the** weak-strong uniqueness property**, i.e., dissipative solutions coincide with strong solutions of \((1)\) as long as the latter exist. We will show in Proposition 1 that our momentum flux has a decomposition

\[
\mathcal{U} = \left( \frac{m \otimes m}{\rho} + P(\rho)1 \right) + \left( R + \phi 1 \right),
\]

with defect measures \( R \) and \( \phi \) that are positive semidefinite as above. This follows from Jensen’s inequality and another Young measure representation. We prefer to think of dissipative solutions as pairs of Young measures \((\varepsilon, \nu)\) because these come with a natural topology of convergence that is useful for studying the behavior of non-linear functions of density/momentum; see Section 4. For a weak-strong uniqueness result for solutions of the isentropic Euler equations with possible occurrence of vacuum we refer the reader to \([30]\).

**Time regularity**

As is well-known, the underlying local geometry plays a central role in defining gradient flows. In the classical case of a gradient flow of some functional \( E \) defined on a Riemannian manifold \((M, g)\), for example, the Riemannian metric \( g \) is used to associate to the differential \( dE(x) \) at a point \( x \in M \), which is a cotangent vector, the gradient \( \nabla E(x) \), which is tangent. Then

\[
\dot{\gamma}(t) = -\nabla E(\gamma(t)) \quad \text{for} \quad t \in [0, \infty)
\]

makes sense as an equality of tangent vectors. For gradient flows on the space of probability measures, the Wasserstein distance induces a suitable notion of tangent vectors as the set of square-integrable velocity fields \( v \) such that

1. the continuity equation \( \partial_t \rho = -\nabla \cdot (\rho v) \) is satisfied distributionally;
2. the \( L^2(\rho_t) \)-norm of \( v_t \) is minimal, for a.e. \( t \in [0, \infty) \).

Here \( t \mapsto \rho_t \) with \( t \in [0, \infty) \) is a sufficiently smooth curve of probability measures. We use the subscript \( t \) to indicate evaluation at time \( t \). The continuity equation does not uniquely determine \( v_t \). It is always possible to add a square-integrable vector field \( w_t \) with \( \nabla \cdot (\rho_t w_t) = 0 \) without losing the equality. The minimization of the \( L^2(\rho_t) \)-norm in (2) amounts to defining tangent vectors to the space of probability measures as (limits of) gradient vector fields.

If the functional \( E \) is convex, but non-smooth, then the gradient flow equation (9) must be relaxed to \(-\dot{\gamma}(t)\) being contained in the subdifferential of \( E \) at the point \( \gamma(t) \), which is a set-valued map. The evolution typically selects the elements of minimal length (norm) in this subdifferential; see \([28,31]\). In this sense, gradients flows combine the idea of steepest descent (proceed in the direction of the subdifferential, so that the functional decreases as fast as possible) and the idea of slowest evolution (among the vectors in the subdifferential select the one with minimal norm).

When studying a variational approach for the isentropic Euler equation \((1)\), the question arises what is the “correct” metric structure for the space of momenta,
which are vector-valued Borel measures. We propose to use the bounded Lipschitz norm (also known as Monge–Kantorovich norm). It is defined by testing against bounded, Lipschitz continuous functions with norm less than or equal to one; see Definition 5. The topological dual of the space of bounded, Lipschitz continuous functions is very large, containing besides bounded measures also certain distributions of first order. However, the set of measures with total variation uniformly bounded (e.g., because of an energy bound) turns out to be closed in this dual space. This is well-known in the case of probability measures where the bounded Lipschitz norm metrizes the weak* convergence of measure; see [32]. A similar thing happens here. We refer the reader to [33–36] for further information.

The use of the bounded Lipschitz norm is very well adapted to the momentum equation $\partial_t m = -\nabla \cdot U$, which provides an expression for the time derivative of the momentum. To measure its size we test against bounded, Lipschitz continuous functions, integrate by parts, then take the supremum over all test functions with norm not bigger than 1. Since $U$ takes values in the symmetric, positive semidefinite matrices, this turns out to be very simple: We just need to integrate the trace $\text{tr}(U)$ to obtain an upper bound. Since the total energy is bounded for all times, it follows that the momentum $t \mapsto m_t$ is Lipschitz continuous with respect to the bounded Lipschitz norm. The Lipschitz continuity of $t \mapsto \rho_t$ with respect to the Wasserstein distance follows as in [28]. We refer the reader to Section 3.2 for details.

Minimal acceleration

In addition to the bounded Lipschitz norm, there is another quantity that is significant for our discussion. Recall that if $(\mathcal{S}, d)$ is a complete metric space, then a curve $t \mapsto v(t)$ into $\mathcal{S}$ is called absolutely continuous if there exists an integrable function $m \in \mathcal{L}^1_{\text{loc}}([0, \infty))$ such that

$$d(v(t), v(s)) \leq \int_s^t m(r) \, dr \quad \text{for all } 0 \leq s \leq t < \infty. \quad (10)$$

One can show that for any absolutely continuous curve $v$ in $\mathcal{S}$ the limit

$$|v'(t)| := \lim_{s \to t} \frac{d(v(s), v(t))}{|s - t|} \quad \text{exists for a.e. } t \in [0, \infty),$$

and $|v'|$ is the smallest function $m$ one can use on the right-hand side of (10); see Theorem 1.1.2 in [28]. This $|v'|$ is called the metric derivative of $v$.

We apply this construction to the set of momentum fields $m$, which are vector-valued Borel measures. Because of the energy bound, the total variations of these momentum measures are bounded uniformly in time. Moreover, it suffices to consider measures with prescribed average: by using the constant as a test function in the momentum equation in (1), one finds that the spatial integral of $m$ (the total momentum) is preserved in time. On momentum curves $t \mapsto m_t$, we define

$$d(m_s, m_t) := \sup \left\{ \int_{\mathbb{R}^d} \zeta(x) \cdot (m_s(dx) - m_t(dx)) : \|\zeta\|_{\text{Lip}([0, \infty))} \leq 1 \right\} \quad (11)$$
for $0 \leq s \leq t$. Note that the difference between (11) and the distance induced by the bounded Lipschitz norm is that in (11) we do not require the sup-norm of the test function to be bounded by one, only its Lipschitz semi-norm. This is sufficient because $m_s$ and $m_t$ have the same total momentum; testing their difference against a constant gives zero. We refer the reader to [35] for a related discussion. We will show in Lemma 1 that, with momentum flux

$$U(t, dx) = [m \otimes m/\varrho](t, dx) + \|P(\varrho)\|1,\$$

the metric derivative of $t \mapsto m_t$ induced by the distance (11) satisfies

$$|m'(t)| = \int_{\mathbb{R}^d} \text{tr}(U(t, dx)) \quad \text{for a.e. } t \in [0, \infty).$$

We will be interested in dissipative solutions of (1) that minimize the metric derivative $|m'|$. We also say that we try to minimize the acceleration.

Given the fluid state $(\varrho_t, m_t)$ at some time $t$, there could be many admissible momentum fluxes $U$ that are consistent with the notion of dissipative solutions. This set, however, has the shape of a convex cone. Indeed we have (8) with non-negative defect measures $R$ and $\phi$ as discussed above. It is therefore not possible to make the momentum flux $U$ arbitrarily small. It is only possible to bring the momentum flux as close as possible to the vertex $m \otimes m/\varrho + P(\varrho)1$ of the convex cone. If we could reach this vertex for a.e. $(t, x)$, then the dissipative solution would actually be a weak solution. We refer to reader to [37] for a similar discussion.

Therefore our goal here is to select from the set of dissipative solutions such solutions that minimize the acceleration for almost all times.

This problem is structurally very similar to multi-objective optimization, which is an optimization problem that involves multiple objective functions $f_1(x), \ldots, f_k(x)$ defined on some set $X$. In this paper, we are interested in minimizing the metric derivative (acceleration) for almost all times, so our objective functions are indexed by $t$, and minimization is done over the set of dissipative solutions.

Since in multi-objective optimization it is typically not possible to find an $x \in X$ that minimizes all $f_i(x)$ simultaneously, different strategies have been devised to deal with such problems. One strategy consists in prioritizing the objective functions (or combinations thereof) and minimizing iteratively, meaning that one selects the minimizers of $f_{i+1}$ not from all of $X$, but from the set of minimizers of $f_i$, which has been selected in the previous step, for $i = 1 \ldots k - 1$. This procedure can also work for countably many objective functions: Since the sets of minimizers are nested it is sufficient to prove that each such set is non-empty and compact. Then the intersection of all sets is non-empty, because of the Cantor intersection theorem. Typically the procedure is sensitive to the ordering of the objective functions.

An alternative approach to multi-objective optimization is to replace the goal of finding a minimizer by the goal of finding a minimal element. That means, to find an element $m \in X$ that cannot be improved in any of the objective functions without making things worse for another objective. Such elements are called Pareto optimal. More precisely, we say that $x_1 \in X$ Pareto dominates $x_2 \in X$ if

$$f_i(x_1) \leq f_i(x_2) \quad \text{for all } i \in \{1, \ldots, k\},$$

$$f_j(x_1) < f_j(x_2) \quad \text{for at least one } j \in \{1, \ldots, k\}. \quad (12)$$
We say that $m \in X$ is Pareto optimal if there exists no $x \in X$ that Pareto dominates $m$. This can be rephrased as saying that $m$ is a minimal element with respect to the quasi-order defined by (12). Note that there may be pairs $x_1, x_2 \in X$ that cannot be compared in the sense that neither $x_1$ Pareto dominates $x_2$, nor vice versa. Moreover, (12) typically does not define a partial order because $f_i(x_1) = f_i(x_2)$ for all $i$ may not imply that $x_1 = x_2$. We refer the reader to Section 2 for more.

For suitable initial data $(\bar{\varrho}, \bar{m})$, let $S$ be the set of dissipative solutions with this initial data; see Definition 6. For any $(\varepsilon, \nu) \in S$ and all $t \in [0, \infty)$ we define

$$a(t|\varepsilon, \nu) := \int_{\mathbb{R}^d} \left( \text{tr}(m \otimes m/\varrho)(t, dx) + d[P(\varrho)](t, dx) \right).$$

(13)

The map $t \mapsto a(t|\varepsilon, \nu)$ is in $L^\infty([0, \infty))$ because of Definition 6 (ii) and (iii).

**Definition 2.** We define a quasi-order $\preceq$ on $S$: We say that $(\tilde{\varepsilon}, \tilde{\nu}) \preceq (\varepsilon, \nu)$ if

$$a(t|\tilde{\varepsilon}, \tilde{\nu}) \leq a(t|\varepsilon, \nu) \quad \text{for a.e. } t \in [0, \infty).$$

It is easy to check that $\preceq$ is reflexive and transitive (see Section 2); it may not be antisymmetric. Then we have the following result:

**Theorem 1.** For any $(\bar{\varrho}, \bar{m})$ as in (36), let $S$ be the set of dissipative solutions of the isentropic Euler equations (1) with initial data $(\bar{\varrho}, \bar{m})$. Then there exists a minimal element $(\tilde{\varepsilon}, \tilde{\nu}) \in S$ with respect to the quasi-order $\preceq$ of Definition 2: For any $(\varepsilon, \nu) \in S$, if $(\varepsilon, \nu)$ and $(\tilde{\varepsilon}, \tilde{\nu})$ are comparable at all, then

$$a(t|\tilde{\varepsilon}, \tilde{\nu}) \leq a(t|\varepsilon, \nu) \quad \text{for a.e. } t \in [0, \infty).$$

In this sense, the solution $(\tilde{\varepsilon}, \tilde{\nu})$ minimizes the acceleration.

The proof of Theorem 1 will be given in Section 4.

**Remark 1.** As the proof of Theorem 1 shows, the minimal elements are extracted from subsets of $S$ that are totally ordered by the quasi-order $\preceq$ and maximal with respect to inclusion. Such subsets do exist because of the Hausdorff maximality principle. It would be interesting to know whether there are dissipative solutions that minimize the acceleration over the whole set $S$, not only over maximal totally ordered subsets. This is an open problem (and may well be false).

**Remark 2.** Because of (8), minimizing the acceleration amounts to making the defect measures $R$ and $\phi$ as small as possible. Heuristically, one might expect that the mininization of acceleration can counteract the occurrence of highly oscillatory velocity fields, which can be the source of anomalous dissipation. By minimizing the defect measures one brings the dissipative solution as close to being a weak solution as possible; see [37] for a related discussion. It would be interesting to check whether one can reach $R = 0$ and $\phi = 0$ in subsets of $\mathbb{R}^d$ so that the dissipative solution becomes a weak solution of (1) there. This will be considered elsewhere.
To make a connection between our approach and the construction of infinitely many weak solutions of the isentropic Euler equation (1) using convex integration, we sketch the argument in [38]. The starting point is a triple \((\varrho, m, U_0)\) with
\[
\partial_t \varrho + \nabla \cdot m = 0, \quad \partial_t m + \nabla \cdot U_0 = 0 \tag{14}
\]
in some space-time domain \(Q\), where \((\varrho, m)\) are smooth density/momentum and \(U_0\) is a smooth field taking values in the trace-free, symmetric matrices. It is assumed that the following inequality holds pointwise everywhere:
\[
d\lambda_{\text{max}}\left( \left( \frac{m \otimes m}{\varrho} + P(\varrho) 1 \right) - U_0 \right) < c, \tag{15}
\]
for some positive constant \(c\). Here \(\lambda_{\text{max}}(A)\) is the largest eigenvalue of the symmetric \((d \times d)\)-matrix \(A\). The triple \((\varrho, m, U_0)\) is called a subsolution.

Since \(c\) is constant we can replace \(U_0\) in (14) by the momentum flux
\[
U := \frac{c}{d} 1 + U_0, \tag{16}
\]
and then (15) is similar to the condition \(m \otimes m/\varrho + P(\varrho) 1 \leq U\) in the sense of symmetric matrices; see (8). In order to construct a weak solution of (1) the goal is now to iteratively modify \((\varrho, m, U_0)\), while preserving (14), so that equality is achieved a.e. in (15) in the limit. This can be done by the repeated superposition of oscillatory waves and a Baire category argument, as pioneered in [12,13].

Interestingly, it is sufficient to consider the trace
\[
\left( \frac{|m|^2}{\varrho} + dP(\varrho) \right) = \text{tr} \left( \frac{m \otimes m}{\varrho} + P(\varrho) 1 - U_0 \right)
\leq d\lambda_{\text{max}}\left( \left( \frac{m \otimes m}{\varrho} + P(\varrho) 1 \right) - U_0 \right), \tag{17}
\]
which is smaller than \(c\) everywhere because of (15). Indeed the iteration in [38] is set up to achieve equality in the limit for the upper bound
\[
\int_Q \left( \frac{|m|^2}{\varrho} + dP(\varrho) \right) \leq c|Q|, \tag{18}
\]
with \(|Q|\) the Lebesgue measure of \(Q\). The integral is a lower bound of the acceleration. But instead of making (13) small, the goal in [38] is to make the left-hand side of (18) bigger. If \(|m|^2/\varrho + dP(\varrho) = c\) a.e., then equality holds in (17) and
\[
\frac{m \otimes m}{\varrho} + P(\varrho) 1 - U_0 = \frac{1}{d} \text{tr} \left( \frac{m \otimes m}{\varrho} + P(\varrho) 1 - U_0 \right) 1 = \frac{c}{d} 1 \quad \text{a.e.};
\]
see Lemma A.2.1 in [38]. Therefore \(U = m \otimes m/\varrho + P(\varrho) 1\) a.e. (recall (16)), which we can insert into (14) in place of \(U_0\) to obtain a weak solution of (1).
Remark 3. We do not know whether acceleration minimizing, dissipative solutions are unique. In order to overcome the non-uniqueness issue for hyperbolic conservation laws, it has been proposed in [39] to look for solutions that decrease the total entropy (here: the total energy) as quickly as possible. The right-hand side of (13):

$$\int_{\mathbb{R}^d} \left( \text{tr}([m \otimes m/\rho](t, dx)) + d[P(\rho)](t, dx) \right)$$

$$= \int_{\mathbb{R}^d} \left( 2\left[\frac{1}{2}|m|^2/\rho\right](t, dx) + d(\gamma - 1)[U(\rho)](t, dx) \right)$$

is a total energy, up to numerical factors. In this sense, our selection procedure searches for dissipative solutions with minimal energy. Kinetic and internal energies are somewhat independent since they involve different Young measures. Alternatively, we can apply the selection procedure above directly to

$$f(t|\varepsilon, \nu) := \int_{\mathbb{R}^d} \left[\frac{1}{2}|m|^2/\rho + U(\rho)\right](t, dx)$$

and use $f$ instead of $a$ to define the quasi-order $\preceq$. The integrand in (19) is bounded below by $\frac{1}{2}|m|^2/\rho + U(\rho)$, with density/momentum $(\rho, m)$; see Definition 66. We obtain a dissipative solution that minimizes the total energy at a.e. $t \in [0, \infty)$ in the sense of Theorem 1, i.e., along maximal totally ordered subsets. Again it would be interesting to find energy minimizers of the whole set $S$.

For a related discussion, we refer the reader to [40]. In that paper, the authors consider a countable family of objective functionals, defined as the Laplace transforms of the energy profiles $t \mapsto f(t|\varepsilon, \nu)$ for a suitable collection of Laplace parameters, and then iteratively minimize these functionals, as explained above in the context of multi-objective optimization. They get an energy minimizing dissipative solution in that sense. Analogously, instead of constructing minimal elements using a quasi-order, as done here, we could apply the iterative minimization of [40] to find acceleration minimizing dissipative solutions in the sense outlined above.

The main focus of [40], however, is not on energy minimization but the construction of a semigroup, i.e., of a solution operator that to any admissible initial data selects precisely one dissipative solution such that the following holds: If one follows a dissipative solution up to some time $t$ and then takes the density/momentum at that time as new initial data, then the solution operator will select for this new initial data precisely the original dissipative solution shifted back in time by $t$. The issue here is the restart-ability of dissipative solutions. This may be relevant to the construction of dissipative solutions with minimal acceleration profiles in the sense of the quasi-order in Definition 2 and will be considered elsewhere.

2. Quasi-Ordered Spaces

A quasi-order on a set $X$ is a binary relation $R$ with the properties

1. Reflexivity: for all $x \in X$ we have $(x, x) \in R$. 

(2) Transitivity: for all \( x, y, z \in X \) we have
\[
(x, y) \in R \text{ and } (y, z) \in R \implies (x, z) \in R.
\]

Note that we do not assume

(3) Antisymmetry: for all \( x, y \in X \) we have
\[
(x, y) \in R \text{ and } (y, x) \in R \implies x = y,
\]
which would make the quasi-order \( R \) into a partial order. To simplify the notation, we will usually write \( x \lessdot y \) instead of \( (x, y) \in R \), with \( x, y \in X \).

For any \( x \in X \) we define the set of predecessors as
\[
P(x) := \{ y \in X : y \lessdot x \}.
\]

**Theorem 2.** Suppose that \( X \) is a non-empty compact set with a quasi-order \( R \) such that \( P(x) \) is closed for every \( x \in X \). Then \( X \) has a minimal element, i.e., an element \( m \in X \) such that, if \( y \in X \) and \( m \) can be compared at all, then \( m \lessdot y \).

**Proof.** This was proved in [41]. We include the argument for the reader’s convenience. We first observe that the set \( A := \{ P(x) : x \in X \} \) is partially ordered by inclusion. By the Hausdorff maximal principle (which is equivalent to the axiom of choice), there exists a maximal totally ordered subset \( T \subseteq A \), i.e., it holds

1. For all \( P(x_1), P(x_2) \in T \) we have \( P(x_1) \subseteq P(x_2) \) or \( P(x_2) \subseteq P(x_1) \).
2. \( T \) is not properly contained in another totally ordered set.

Then the intersection \( \bigcap_{P(x) \in T} P(x) \) is non-empty. Indeed if this were false, then
\[
X = X \setminus \bigcap_{P(x) \in T} P(x) = \bigcup_{P(x) \in T} (X \setminus P(x)),
\]
by De Morgan’s laws. Since all \( P(x) \) are closed, their complements are open, and so
\[
\{X \setminus P(x)\}_{P(x) \in T} \text{ is an open covering of } X.
\]

Since \( X \) is compact, there exists a finite subcovering, i.e., there exist finitely many \( x_k \in X, k = 1 \ldots K \), for which \( P(x_k) \in T \) and the following holds:
\[
X = \bigcup_{k=1}^{K} \left( X \setminus P(x_k) \right) = X \setminus \bigcap_{k=1}^{K} P(x_k).
\]

Thus \( \bigcap_{k=1}^{K} P(x_k) \) is empty. But every finite subset of a non-empty totally ordered set has a lower bound; this can be proved using induction on the cardinality of the subset. This means there exists an index \( l \in \{ 1 \ldots K \} \) such that
\[
P(x_l) \subseteq P(x_k) \text{ for all } k = 1 \ldots K,
\]
and thus \( P(x_l) = \bigcap_{k=1}^{K} P(x_k) \). It follows that \( P(x_l) \) must be empty. But this is a contradiction because \( x_l \in P(x_l) \), by reflexivity of the quasi-order.
Therefore there exists an element \( m \in \bigcap_{P(x) \in T} P(x) \).

We claim that \( m \) is a minimal element of \( X \). Since \( m \in P(x) \) for all \( P(x) \in T \), it follows that \( m \preceq x \) for such \( x \). Consider therefore \( x \in X \) with \( P(x) \not\in T \). Suppose that \( x \preceq m \). Since \( P(x) \not\in T \) there must exist \( P(y) \in T \) to which \( P(x) \) cannot be compared, i.e., it holds neither \( P(y) \subseteq P(x) \) nor \( P(x) \subseteq P(y) \). Indeed if \( P(x) \) could be compared to every \( P(y) \in T \), then \( P(x) \in T \), by maximality of \( T \). But \( P(y) \in T \) implies that \( m \preceq y \), as we have just shown. By transitivity, our assumption \( x \preceq m \) then implies that \( x \preceq y \). Using transitivity again, we observe that for every \( z \preceq x \) we also have \( z \preceq y \), and thus \( P(x) \subseteq P(y) \). But this is a contradiction since \( P(x) \) and \( P(y) \) cannot be compared, by choice of \( y \). Therefore the case \( x \preceq m \) cannot occur, which means that either \( x \) and \( m \) cannot be compared at all, or \( m \preceq x \).

\[ \square \]

### 3. Dissipative Solutions

In this section, we introduce the class of dissipative solutions of \((1)\) from which we will select the ones with minimal acceleration. Our construction is more detailed than the one in \([42]\) since it involves Young measures to describe the defect measures \( R, \chi \) in \((6)\); see also \([37]\). We first introduce some notation.

Let \( \mathbb{R}^d \) be the space of real \((d \times d)\)-matrices and

\[
\text{Mat}_d(\mathbb{R}, \square) := \left\{ A \in \text{Mat}_d(\mathbb{R}) : v \cdot (Av) \square 0 \text{ for all } v \in \mathbb{R}^d \right\}
\]

where \( \square \) stands for either \( \geq \) or \( > \). The analogous spaces of symmetric matrices are denoted by \( \text{Sym}_d(\mathbb{R}) \) and \( \text{Sym}_d(\mathbb{R}, \square) \). We denote by \( \text{Skew}_d(\mathbb{R}) \) the space of real skew symmetric \((d \times d)\)-matrices. The Frobenius inner product is

\[
A : B := \text{tr}(A^T B) \quad \text{for } A, B \in \text{Mat}_d(\mathbb{R}).
\]

Let \( \| \cdot \| \) be the operator norm induced by the Euclidean norm \( | \cdot | \) on \( \mathbb{R}^d \).

We denote by \( \mathbb{R}^d_\infty \) the one-point compactification of \( \mathbb{R}^d \). We adjoin to \( \mathbb{R}^d \) a point \( \infty \) and define, with \( h(x) := 1/(1 + |x|) \) for all \( x \in \mathbb{R}^d \), a distance

\[
d(x, y) := \begin{cases} 
\min\{|x - y|, h(x) + h(y)\} & \text{if } x, y \in \mathbb{R}^d, \\
h(x) & \text{if } x \in \mathbb{R}^d \text{ and } y = \infty, \\
0 & \text{if } x, y = \infty;
\end{cases}
\]

see \([43]\). Then \( |x| \to \infty \) is equivalent to \( d(x, \infty) \to 0 \).

We denote by \( \mathcal{C}_s(\mathbb{R}^d; V) \) the space of continuous functions

\[
g : \mathbb{R}^d \to V \quad \text{for which } \lim_{|x| \to \infty} g(x) \in V \text{ exists},
\]

equipped with the sup-norm. Here \( V \) is some Banach space. Then

\[
\mathcal{C}_s(\mathbb{R}^d; V) = V + \mathcal{C}_0(\mathbb{R}^d; V),
\]
with \( C^0(R^d; V) \) the closure of the space of compactly supported and continuous \( V \)-valued functions in the sup-norm. Functions in \( C^\ast(R^d; V) \) can be identified with elements in \( C(R^d; V) \): To \( g \in C^\ast(R^d; V) \) we associate \( \dot{g} \in C(R^d; V) \) as

\[
\dot{g}(x) := \begin{cases} 
g(x) & \text{if } x \in R^d, \\
\lim_{|x| \to \infty} g(x) & \text{if } x = \infty.
\end{cases}
\]

For simplicity of notation, we will not distinguish between \( g \) and \( \dot{g} \).

We now define the space of test functions

\[
\mathcal{A} := \{ u \in C^1(R^d; R^D) : \nabla u \in C^\ast(R^d; \text{Mat}_{D \times d}(R)) \}, \tag{20}
\]

with \( \text{Mat}_{D \times d}(R) \) the space of \((D \times d)\)-matrices. We will not indicate the dimension \( D \) when it is clear from the context. Functions in \( \mathcal{A} \) grow at most linearly at infinity. In particular, the space \( \mathcal{A} \) contains all linear maps \( u(x) := Ax \) for all \( x \in R^d \), with \( A \in \text{Mat}_{d}(R) \), and so test functions do not have to have compact support.

Let \( C^1_c([0, \infty)) \otimes \mathcal{A} \) be the space of tensor products

\[
\eta \otimes \zeta(t, x) := \eta(t)\zeta(x) \quad \text{with } \eta \in C^1_c([0, \infty)) \text{ and } \zeta \in \mathcal{A}. \tag{21}
\]

We will assume that partial differential equations will hold in duality with \( C^1_c([0, \infty)) \otimes \mathcal{A} \), which means testing against functions of the form (21); see Section 3.4 for details. For all \( T > 0 \), the tensor product \( C([0, T]) \otimes \mathcal{A} \) is dense in \( C([0, T]; \mathcal{A}) \) with respect to the sup-norm because \( \mathcal{A} \) is a locally convex topological vector space.

### 3.1. A priori bounds

We now collect various a priori-bounds for solutions of (1). To simplify notation, we will use the subscript \( t \) to indicate the value at time \( t \), as in \( \varrho_t := \varrho(t, \cdot) \). The continuity equation in (1) implies that if \( \varrho \) has total mass one, then \( \varrho_t \in P(R^d) \) for all \( t \in [0, \infty) \). Similarly, the entropy condition (4) implies that the total energy should be non-increasing in time. Assuming that

\[
\dot{E} := \int_{R^d} \frac{1}{2} |\dot{\varrho}|^2 + U[\varrho] < +\infty, \tag{22}
\]

we will therefore be interested in solutions of (1) with finite energy, so that

\[
\int_{R^d} \frac{1}{2} \varrho_t |u_t|^2 + U[\varrho_t] \leq \dot{E} \quad \text{for all } t \in [0, \infty). \tag{23}
\]

As a consequence of Cauchy–Schwarz inequality, the momentum \( m_t := \varrho_t u_t \) is an \( R^d \)-valued Borel measure with total variation bounded by \( \dot{E} \), uniformly in time \( t \). Definition 1 and (23) provide higher integrability for \( \varrho_t \). We have

\[
\varrho \in L^\infty([0, \infty); L^\gamma(R^d)), \quad m \in L^\infty([0, \infty); L^{2\gamma}(R^d)).
\]
because of Hölder inequality. Both \( \varrho_t \) and \( m_t \) are absolutely continuous with respect to \( L^d \). We will always make assumption (22) in the following.

We will also require that the initial density \( \bar{\varrho} \) has finite second moment:

\[
\bar{M} := \left( \int_{\mathbb{R}^d} |x|^2 \bar{\varrho}(dx) \right)^{1/2} < +\infty.
\] (24)

Multiplying the continuity equation by \( \frac{1}{2} |x|^2 \) and integrating by parts, we find

\[
\frac{d}{dt} \left( \int_{\mathbb{R}^d} |x|^2 \varrho_t(dx) \right)^{1/2} \leq \left( \int_{\mathbb{R}^d} |u_t(x)|^2 \varrho_t(dx) \right)^{1/2}
\]

for all \( t \), formally. We will therefore be interested in solutions of (1), for which

\[
\left( \int_{\mathbb{R}^d} |x|^2 \varrho_t(dx) \right)^{1/2} \leq \bar{M} + t (2 \tilde{E})^{1/2} =: M(t) \quad \text{for all } t \in [0, \infty). \tag{25}
\]

This in turn implies that the momentum \( m_t \) has finite first moment for all times, which follows again from Cauchy–Scharz inequality with (23).

**Remark 4.** The products of \( (\varrho_t, m_t) \) with \( \zeta \in A \) are integrable in space since these measures have finite first moments. Moreover, the spatial derivative \( \nabla \zeta \) is bounded, therefore the integrals involving fluxes of (1) are well-defined as well.

### 3.2. Time regularity

Because of the a priori-bounds from Section 3.1, we think of solutions of the isentropic Euler equations (1) as curves \( t \mapsto (\varrho_t, m_t) \) taking values in a convex set of vector measures whose total variations are bounded uniformly in time. In order to quantify the time regularity of these curves we must choose an appropriate metric structure on the spaces of densities and momenta.

**Definition 3.** \((p\text{-Wasserstein Distance})\) For any \( \varrho^1, \varrho^2 \in \mathcal{P}(\mathbb{R}^D) \) let

\[
\text{Adm}(\varrho^1, \varrho^2) := \{ \gamma \in \mathcal{P}(\mathbb{R}^{2D}) : \mathbb{P}^k \# \gamma = \varrho^k \text{ with } k = 1 \ldots 2 \}
\]

be the space of admissible transport plans connecting \( \varrho^1 \) and \( \varrho^2 \), where

\[
\mathbb{P}^k(x^1, x^2) := x^k \text{ for all } (x^1, x^2) \in \mathbb{R}^{2D} = (\mathbb{R}^D)^2
\]

and \( k = 1 \ldots 2 \), and \( \# \) denotes the push-forward of measures. For any \( 1 \leq p < \infty \) the \( p \)-Wasserstein distance \( W_p(\varrho^1, \varrho^2) \) between \( \varrho^1, \varrho^2 \) is defined by

\[
W_p(\varrho^1, \varrho^2)^p := \inf_{\gamma \in \text{Adm}(\varrho^1, \varrho^2)} \left\{ \int_{\mathbb{R}^{2D}} |x^1 - x^2|^p \gamma(dx^1, dx^2) \right\}. \tag{26}
\]
The $p$-Wasserstein distance between the measures $\varrho^1$ and $\varrho^2$ is the minimal cost it takes to transport $\varrho^1$ into $\varrho^2$ if the cost of moving a unit mass in $\mathbb{R}^D$ from $x$ to $y$ is defined as the $p$th power of the Euclidean distance. The elements of $\text{Adm}(\varrho^1, \varrho^2)$ are called transport plans; they all transport $\varrho^1$ to $\varrho^2$. The inf in (26) is attained for a suitable $\gamma \in \text{Adm}(\varrho^1, \varrho^2)$, called an optimal transport plan. For $p = 2$ the support of an optimal transport plan is contained in the graph of a cyclically monotone map (that means, in the subdifferential of a proper, lower semicontinuous, convex function).

**Definition 4.** We denote by $\mathcal{P}_2(\mathbb{R}^d)$ the space of Borel probability measures with finite second moment, endowed with the $2$-Wasserstein distance; see Definition 3. For a function $t \mapsto \varrho_t \in \mathcal{P}_2(\mathbb{R}^d)$, $t \in [0, \infty)$, we denote by

$$
\|\varrho\|_{\text{Lip}([0,\infty); \mathcal{P}_2(\mathbb{R}^d))} := \sup_{t_1, t_2 \in [0, \infty)} \frac{W_2(\varrho_{t_1}, \varrho_{t_2})}{|t_2 - t_1|}
$$

its Lipschitz seminorm, with $W_2$ the Wasserstein distance; see (26).

For any $t \in [0, \infty)$, the momentum $m_t = \varrho_t u_t$ is an element of the set

$$
\mathcal{M}_t := \left\{ m \in \mathcal{M}(\mathbb{R}^d; \mathbb{R}^d) : \int_{\mathbb{R}^d} (1 + |x|) |m(\text{d}x)| \leq (1 + M(t))(2\bar{E})^{1/2} \right\},
$$

which is convex; see Section 3.1. The measures in $\mathcal{M}_t$ thus have bounded total variation and are uniformly tight, which means that for all $\varepsilon > 0$ there exists a compact set $K \subset \mathbb{R}^d$ with

$$
\int_{\mathbb{R}^d \setminus K} |m(\text{d}x)| < \varepsilon \quad \text{for all } m \in \mathcal{M}_t.
$$

Indeed since $m \in \mathcal{M}_t(\mathbb{R}^d)$ has finite first moment, we can estimate

$$
\int_{\mathbb{R}^d \setminus \bar{B}_R(0)} |m(\text{d}x)| \leq R^{-1} \int_{\mathbb{R}^d} |x| |m(\text{d}x)| \quad \text{for all } R > 0.
$$

The integral on the right-hand side of (28) is bounded by (27). On $\mathcal{M}_t$ the topology of narrow convergence of measures, defined in terms of testing against bounded and continuous functions, coincides with the topology induced by the bounded Lipschitz norm (also called Dudley or Monge–Kantorovich norm), which is defined as follows:

**Definition 5.** We denote by $\text{Lip}(\mathbb{R}^d; \mathbb{R}^D)$ the vector space of Lipschitz continuous maps $\zeta : \mathbb{R}^d \longrightarrow \mathbb{R}^D$. The Lipschitz constant of $\zeta \in \text{Lip}(\mathbb{R}^d; \mathbb{R}^D)$ is

$$
\|\zeta\|_{\text{Lip}(\mathbb{R}^d)} := \sup_{x_1 \neq x_2} \frac{|\zeta(x_1) - \zeta(x_2)|}{|x_1 - x_2|}.
$$

We denote by $\text{BL}(\mathbb{R}^d; \mathbb{R}^D)$ the subspace of bounded functions in $\text{Lip}(\mathbb{R}^d; \mathbb{R}^D)$. It is a Banach space when equipped with the norm

$$
\|\zeta\|_{\text{BL}(\mathbb{R}^d)} := \|\zeta\|_{\mathcal{L}^\infty(\mathbb{R}^d)} + \|\zeta\|_{\text{Lip}(\mathbb{R}^d)}.
$$
Let $\mathcal{B}L_1(\mathbb{R}^d; \mathbb{R}^D)$ be the space of all $\zeta \in \mathcal{B}L(\mathbb{R}^d; \mathbb{R}^D)$ with $\|\zeta\|_{\mathcal{B}L(\mathbb{R}^d)} \leq 1$.

We denote by $\mathcal{M}_{\mathcal{B}L}(\mathbb{R}^d; \mathbb{R}^D)$ the space of finite $\mathbb{R}^D$-valued Borel measures $m$ with finite first moment, equipped with the bounded Lipschitz norm

$$
\|m\|_{\mathcal{M}_{\mathcal{B}L}(\mathbb{R}^d)} := \sup \left\{ \int_{\mathbb{R}^d} \zeta(x) \cdot m(dx) : \zeta \in \mathcal{B}L_1(\mathbb{R}^d; \mathbb{R}^D) \right\}.
$$

The bounded Lipschitz norm is bounded above by the total variation. The integral in (30) is well-defined because $m$ has finite first moment, by assumption.

We refer the reader to [35,36] for a related discussion.

### 3.3. Young measures

We will use parameterized measures (Young measures) to describe solutions of (1). The state space for density/momentum $(\varrho, m)$ is

$$
X \cup \{(0,0)\}, \text{ where } X := \left( (0, \infty) \times \mathbb{R}^d \right).
$$

Our goal is to define a suitable compactification of the state space. Equivalently, we must specify the set of all continuous bounded functions on $X$ needed to represent the non-linearities in the isentropic Euler equations (1). In slight abuse of notation, we will use the symbols $(\varrho, m)$ for elements in $X$.

Let $h(\varrho, m) := \varrho + \left( \frac{|m|^2}{2\varrho} + U(\varrho) \right)$; see Definition 1. We then introduce the set

$$
\mathcal{W}(X) := \left\{ \Phi = \varrho + \left( c_\varrho \cdot \left( \frac{\varrho}{m} \right) + c_K : \frac{m \otimes m}{\varrho} + c_U P(\varrho) \right) / h : \varrho \in \mathcal{C}_0(X), c_\varrho \in \mathbb{R}^{d+1}, c_K \in \text{Sym}_d(\mathbb{R}), c_U \in \mathbb{R} \right\}.
$$

One can check that the functions in $\mathcal{W}(X)$ are continuous and bounded. Moreover, being a finite-dimensional augmentation of the vector space $\mathcal{C}_0(X)$, which is known to be separable, the set $\mathcal{W}(X)$ is a complete and separable vector space with respect to uniform convergence. Then there exists a compact, metrizable Hausdorff space $\mathcal{X}$ and an embedding $e : X \rightarrow \mathcal{X}$ such that $e(X)$ is dense in $\mathcal{X}$. We will call $\mathcal{X}$ a compactification of $X$. If $\mathcal{A}$ denotes the smallest closed subalgebra in $\mathcal{C}_0(X)$ that contains $\mathcal{W}(X)$, then for all maps $\Phi \in \mathcal{A}$, the composition $\Phi \circ e^{-1} : X \rightarrow \mathbb{R}$ has a continuous (hence bounded) extension to all of $\mathcal{X}$. For simplicity of notation, we will identify $X$ with its image $e(X)$, and functions $\Phi \in \mathcal{A}$ with their extensions in $\mathcal{C}(\mathcal{X})$. We refer the reader to Sections 6.4/5 in [27] for additional details.

With $\hat{\mathbb{R}}^d$ the one-point compactification of $\mathbb{R}^d$, we now define

$$
\mathcal{E} := L^1([0, \infty), \mathcal{C}(\hat{\mathbb{R}}^d \times \mathcal{X})).
$$
as the space of measurable maps $\phi : [0, \infty) \to \mathcal{C}(\mathbb{R}^d \times \mathcal{I})$ with finite norm:

$$\|\phi\|_E := \int_0^\infty \|\phi(t, \cdot)\|_{\mathcal{C}(\mathbb{R}^d \times \mathcal{I})} \, dt < \infty.$$  

Here $\phi$ is measurable if it is the pointwise limit of a sequence of simple functions. We identify functions that differ only on a Lebesgue null set.

Since $\mathcal{I}$ is compact and metrizable it is separable. One can then show that $E$ is a separable Banach space. Its topological dual is given by

$$E^* := \mathcal{L}^\infty_w([0, \infty), \mathcal{M}(\dot{\mathbb{R}}^d \times \mathcal{I})),$$

the space of functions $\nu : [0, \infty) \to \mathcal{M}(\dot{\mathbb{R}}^d \times \mathcal{I})$ such that

$$t \mapsto \int_{\dot{\mathbb{R}}^d \times \mathcal{I}} \phi(x, \tau) \nu_t(dx, d\tau) \text{ measurable for all } \phi \in \mathcal{C}(\dot{\mathbb{R}}^d \times \mathcal{I}), \text{ and }$$

$$\|\nu\|_{E^*} := \text{ess sup}_{t \in [0, \infty)} \|\nu_t\|_{\mathcal{M}(\dot{\mathbb{R}}^d \times \mathcal{I})} < \infty.$$}

Here $\nu_t := \nu(t, \cdot)$ and $\tau := (\varrho, m) \in \mathcal{I}$. Again we identify functions that coincide almost everywhere. The duality is induced by the pairing

$$\langle \nu, \phi \rangle := \int_0^\infty \int_{\dot{\mathbb{R}}^d \times \mathcal{I}} \phi(t, x, \tau) \nu_t(dx, d\tau) \, dt \quad \text{for } \phi \in E \text{ and } \nu \in E^*. $$

Bounded closed balls in $E^*$ endowed with the weak* topology are metrizable and (sequentially) compact, by Banach-Alaoglu theorem. We write

$$[f(\varrho, m)]_\nu(t, dx) := \int_{\mathcal{I}} f(y) \nu_t(dx, d\tau) \text{ for a.e. } t \in [0, \infty)$$

and for all functions $f : X \to \mathbb{R}$ with $f/h \in \mathcal{A}$. We emphasize that the pairing (34), being an integration of $f$ with respect to $\nu$, is linear in $f$.

**Remark 5.** The compactification $\mathcal{I}$ adds points not only for the limit of large $(\varrho, m)$ but also for the case when $(\varrho, m)$ approaches vacuum. Indeed we have that

$$\lim_{\varrho \to 0} \frac{1}{h(\varrho, m)} \begin{bmatrix} \varrho \\ m \end{bmatrix} \bigg|_{m=\varrho u} = \frac{1}{1+|u|^2} \begin{bmatrix} 1 \\ u \end{bmatrix},$$

for any $u \in \mathbb{R}^d$ fixed. Notice that the right-hand side of (35) vanishes as $|u| \to \infty$. The points in $\mathcal{I}$ corresponding to vacuum therefore have the topology of $\mathbb{R}^d$. This can also be seen by using $(\varrho, m) \mapsto (\varrho, u := m/\varrho)$ to map the (non vacuum) density-momentum space $X$ to the density-velocity space $Y := [0, \infty) \times \mathbb{R}^d$, where vacuum is $\{0\} \times \mathbb{R}^d$. Let $V$ denote the set of vacuum points of the compactification $\mathcal{I}$ and $X_V := ((0, \infty) \times \mathbb{R}^d) \cup V$. The topology of $X_V$ is the one inherited from $\mathcal{I}$.  


3.4. Global existence

We can now introduce dissipative solutions of (1).

**Definition 6. (Dissipative Solutions)** Suppose that initial data

\[
\bar{\varrho} \in \mathcal{P}_2(\mathbb{R}^d) \text{ with } \mathcal{U}[\bar{\varrho}] < +\infty, \quad \bar{\mathbf{u}} \in \mathcal{L}^2(\mathbb{R}^d, \bar{\varrho}), \quad \bar{\mathbf{m}} := \bar{\varrho}\bar{\mathbf{u}}
\]  

is given. Let initial energy/moment \( \bar{\mathcal{E}}, \bar{\mathcal{M}} \) be defined by (22) and (24).

A dissipative solution of the isentropic Euler equations (1) is a pair \( \varepsilon, \nu \in \mathcal{L}_w^\infty(\mathbb{R}^d \times X) \) such that there exist \( \varrho \in \mathcal{L}_w^\infty([0, \infty) ; \mathcal{M}_+([\mathbb{R}^d \times \mathfrak{K}])) \), \( m \in \mathcal{L}_w^\infty([0, \infty) ; \mathcal{M}_w([\mathbb{R}^d])) \), with the properties (1)–(9) listed below. We denote by \( \langle \cdot, \cdot \rangle \) and \( [\cdot] \) the pairing with the Young measures \( \varepsilon \) and \( \nu \), respectively; see (34) for the definition.

**A priori bounds**

(i) The moment bound (25) holds and \( m_t \in \mathcal{M}_t \) for all \( t \in [0, \infty) \); see (27).

(ii) The map \( t \mapsto E(t) \) is non-increasing and bounded by \( \bar{\mathcal{E}} \), with

\[
E(t) := \int_{\mathbb{R}^d} \left[ \frac{1}{2} |m|^2/\varrho + U(\varrho) \right](t, \, dx) \quad \text{for all } t \in [0, \infty). \tag{38}
\]

(iii) We have \( N(t) \leq E(t) \) a.e., where

\[
N(t) := \int_{\mathbb{R}^d} \left[ \frac{1}{2} |m|^2/\varrho + U(\varrho) \right](t, \, dx) \quad \text{for all } t \in [0, \infty). \tag{39}
\]

**Time Regularity**

(iv) There exists a constant \( L \) depending only on \( d, \gamma \) such that

\[
\|\varrho\|_{\text{Lip}(0, \infty); \mathcal{P}_2(\mathbb{R}^d))} \leq (2\bar{\mathcal{E}})^{1/2},
\]

\[
\|m\|_{\text{Lip}(0, \infty); \mathcal{M}_w(\mathbb{R}^d)} \leq L\bar{\mathcal{E}}. \tag{40}
\]

**Eulerian Velocity**

(v) We have \( m =: \varrho \mathbf{u} \) with

\[
\mathbf{u}_t \in \mathcal{L}^2(\mathbb{R}^d, \varrho_t) \quad \text{for all } t \in [0, \infty).
\]

**Young measures**

(vi) The function \( \varrho, \, m \) are compatible with the Young measures \( \varepsilon, \nu \):

\[
\varrho_t(\, dx) = [\varrho](t, \, dx) = [\varrho](t, \, dx) \quad \text{for a.e. } t \in [0, \infty),
\]

\[
m_t(\, dx) = [m](t, \, dx) = [m](t, \, dx) \tag{41}
\]
(vii) There exist functions
\[ Q, R \in L^\infty_w([0, \infty); \mathcal{M}(\mathbb{R}^d; \text{Sym}_d(\mathbb{R}, \geq))], \]
\[ \phi, \chi \in L^\infty_w([0, \infty); \mathcal{M}_+(\mathbb{R}^d)), \]
such that the following decomposition holds for a.e. \( t \in [0, \infty) \):
\[ \llbracket m \otimes m / \varrho \rrbracket(t, dx) = r_t(x)u_t(x) \otimes u_t(x) dx + Q(t, dx), \]
\[ \llbracket P(\varrho) \rrbracket(t, dx) = P(r_t(x)) dx + \phi(t, dx), \] \hspace{1cm} (42)
where \( \varrho_t = r_t \mathcal{L}^d \). Formulas (42) hold analogously with \([\cdot]\) and \( R, \chi \).

**Hyperbolic conservation laws**

(viii) The initial data is attained:
\[ \varrho(0, \cdot) = \bar{\varrho}, \quad m(0, \cdot) = \bar{m}. \]

(ix) The conservation laws are satisfied:
\[ \frac{\partial}{\partial t} \varrho + \nabla \cdot m = 0, \]
\[ \frac{\partial}{\partial t} m + \nabla \cdot [m \otimes m / \varrho] + \nabla \llbracket P(\varrho) \rrbracket = 0 \] \hspace{1cm} in \( \mathcal{C}_1([0, \infty)) \otimes \mathcal{A} \). \hspace{1cm} (43)

**Remark 6.** Because of Definition 6 (vii)/(ix), the pair \( (\varrho, m = \varrho u) \) satisfies the modified momentum equation (6). We can rewrite the total energy as
\[ \int_{\mathbb{R}^d} \left( \frac{1}{2} m^2 / \varrho + U(\varrho) \right)(t, dx) \]
\[ = \int_{\mathbb{R}^d} \left( \frac{1}{2} r_t |u_t|^2 + U(r_t) \right) dx + \int_{\mathbb{R}^d} \left( \frac{1}{2} \text{tr}(Q(t, dx)) + \frac{1}{\gamma - 1} \phi(t, dx) \right), \]
where \( \varrho_t = r_t \mathcal{L}^d \) and \( t \in [0, \infty) \). Then Definition 6 (ii) implies the energy inequality (7) with \( Q \) in place of \( R \), from which (23) follows. Equality (44) also holds with \([\cdot], Q, \phi\) replaced by \([\cdot], R, \chi\), because of (41). We find that
\[ \int_{\mathbb{R}^d} \left( \frac{1}{2} \text{tr}(Q(t, dx)) + \frac{1}{\gamma - 1} \phi(t, dx) \right) \geq \int_{\mathbb{R}^d} \left( \frac{1}{2} \text{tr}(R(t, dx)) + \frac{1}{\gamma - 1} \chi(t, dx) \right) \]
for all \( t \in [0, \infty) \), as a consequence of Definition 6 (vi)/(iii).

**Remark 7.** One can prove that the boundedness of \( \varrho \) in \( L^\infty([0, \infty); L^\gamma(\mathbb{R}^d)) \) with Lipschitz continuity in a weaker topology (here: with respect to the Wasserstein distance, which metrizes the weak* convergence of measures) implies that \( t \mapsto \varrho_t \) is continuous in time with respect to the weak \( L^\gamma(\mathbb{R}^d) \)-topology. Similarly, we have that \( t \mapsto m_t \) is continuous with respect to the weak \( L^p(\mathbb{R}^d) \)-topology.

**Proposition 1.** For initial data as in (36), dissipative solutions do exist.
Proof. We utilize the time variational time discretization in [27], with minor modifications that will be pointed out below. The strategy is to generate an approximation of the solution \((\varrho, m = \varrho u)\) of the isentropic Euler equations (1) at discrete times \(t^k_\tau := k\tau\), where \(\tau > 0\) and \(k \in \mathbb{N}_0\), by solving a convex minimization problem in each timestep. By interpolating in time and passing to the limit \(\tau \to 0\), we obtain a sequence of approximate solutions that converge (up to a subsequence) towards a dissipative solution of (1) in the sense of Definition 6. We will outline the main steps below and refer the reader to [27] for additional details.

Step 1 Let us first describe the minimization problem, which is the basis for the time discretization. It was shown in [27] that for any given data

\[ \varrho \in \mathcal{P}_2(\mathbb{R}^d), \quad u \in \mathcal{L}^2(\mathbb{R}^d, \varrho) \]

with finite energy, there exists a unique minimizer \(X_\tau\) of the functional

\[ \frac{3}{4\tau^2} \int_{\mathbb{R}^d} |X(x) - (x + \tau u(x))|^2 \varrho(dx) + \int_{\mathbb{R}^d} U(r(x)) \det(\nabla X(x)^S)^{1-\gamma} dx, \]

which is defined for \(\mathbb{R}^d\)-valued functions \(X \in \mathcal{L}^2(\mathbb{R}^d, \varrho)\) that are monotone. Finiteness of the internal energy implies that \(\varrho =: r\mathcal{L}^d\), with \(r\) some Lebesgue integrable function; recall Definition 1. By slight abuse of notation, we will not always distinguish between the measure \(\varrho\) and its Lebesgue density \(r\). Monotonicity of \(X_\tau\) implies \(\text{BV}_{\text{loc}}\)-regularity, and \(\nabla X(x)^S\) is the symmetric part of the absolutely continuous part of the distributional derivative of \(X\), which is a locally finite \(\text{Mat}_d(\mathbb{R}, \geq)\)-valued measure. The optimality condition of this minimization problem takes the following form: There exists \(\lambda_\tau \in \mathcal{M}(\mathbb{R}^d; \text{Sym}_d(\mathbb{R}, >))\) such that

\[ \int_{\mathbb{R}^d} \xi(x) \cdot \frac{W_\tau(x) - u(x)}{\tau} r(x) dx = \int_{\mathbb{R}^d} \nabla \xi(x) : \left( P_\tau(x) dx + \lambda_\tau(dx) \right) \quad (45) \]

for all \(\xi \in \mathfrak{A}\) (see (20)), with pressure field

\[ P_\tau(x) := P(r(x)) \det(\nabla X_\tau(x)^S)^{1-\gamma} (\nabla X_\tau(x)^S)^{-1} \quad (46) \]

(taking values in \(\text{Sym}_d(\mathbb{R}, >)\)) and velocities

\[ W_\tau(x) := \frac{3}{2} V_\tau(x) - \frac{1}{2} u(x), \quad V_\tau(x) := \frac{X_\tau(x) - x}{\tau}, \quad (47) \]

in \(\mathcal{L}^2(\mathbb{R}^d, \varrho)\). If \(E[\varrho, u] := \int_{\mathbb{R}^d} \frac{1}{2} \varrho |u|^2 + \mathcal{U}[\varrho]\) denotes the total energy, then

\[ E[\varrho_\tau, u_\tau] + \int_{\mathbb{R}^d} \frac{1}{6} |W_\tau(x) - u(x)|^2 \varrho(dx) \]

\[ + \int_{\mathbb{R}^d} \left( P(r(x)) D_{\text{loc}}(\nabla X_\tau(x)^S - 1) dx + \text{tr}(\lambda_\tau(dx)) \right) \leq E[\varrho, u], \quad (48) \]

where \(\varrho_\tau := X_\tau \# \varrho\) and \(u_\tau := W_\tau \circ X_\tau^{-1}\) belongs to \(\mathcal{L}^2(\mathbb{R}^d, \varrho_\tau)\). Here \# indicates the push-forward of measures, defined as \(\varrho_\tau(A) := \varrho(X_\tau^{-1}(A))\) for all Borel subsets \(A \subset \mathbb{R}^d\). One can show that \(X_\tau\) can be extended by monotonicity to a \(\varrho\)-essentially injective map on all of \(\mathbb{R}^d\) so that \(u_\tau\) is indeed well-defined.
Note that $D_{\mathcal{U}}$ is the Bregman divergence of the convex function $S \mapsto \det(S)^{1-\gamma}$ with $S \in \text{Sym}_d(\mathbb{R}, >)$. For any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ with

$$\sup_{z \in \mathbb{R}^d, |z| = 1} \left| \left( z, (1 - \det(1 + S)^{1-\gamma}(1 + S)^{-1})z \right) \right| \leq \varepsilon + C_\varepsilon D_{\mathcal{U}}(S) \quad (49)$$

for all $S \in \text{Sym}_d(\mathbb{R})$ such that $1 + S$ is positive definite.

We refer the reader to [27] for motivation, proofs, and further discussion.

**Step 2** We now explain in which sense the minimizer of Step 1 generates an approximate solution of (1) for one timestep. We define the interpolants

$$X_t(x) := x + tV_t(X), \quad W_t(x) := \left(1 - \frac{t}{\tau}\right)u(x) + \frac{t}{\tau} W_t(x) \quad (50)$$

for $\mathcal{Q}$-a.e. $x \in \mathbb{R}^d$ and $t \in [0, \tau]$. Then we interpolate density and velocity as

$$\mathcal{Q}_t := X_t \# \mathcal{Q}, \quad u_t := W_t \circ X_t^{-1}. \quad (51)$$

This is well-defined because $X_t$ is $\mathcal{Q}$-essentially injective.

For any $\eta \in C^1(\mathbb{R})$ and $\xi \in \mathcal{X}$, we now compute (integrating by parts)

$$-\int_0^\tau \eta'(t) \int_{\mathbb{R}^d} \xi(X_t(x)) \cdot W_t(x) \mathcal{Q}(dx) \, dt + \eta(\tau) \int_{\mathbb{R}^d} \xi(X_t(x)) \cdot W_t(x) \mathcal{Q}(dx) - \eta(0) \int_{\mathbb{R}^d} \xi(x) \cdot u(x) \mathcal{Q}(dx)$$

$$= \int_0^\tau \eta(t) \int_{\mathbb{R}^d} \nabla \xi(X_t(x)) : (W_t(x) \otimes \dot{X}_t(x)) \mathcal{Q}(dx) \, dt + \int_0^\tau \eta(t) \int_{\mathbb{R}^d} \xi(X_t(x)) \cdot \dot{W}_t(x) \mathcal{Q}(dx) \, dt. \quad (52)$$

Using definition (51), we have that

$$-\int_0^\tau \eta'(t) \int_{\mathbb{R}^d} \xi(X_t(x)) \cdot W_t(x) \mathcal{Q}(dx) \, dt = -\int_0^\tau \eta'(t) \int_{\mathbb{R}^d} \xi(z) \cdot u_t(z) \mathcal{Q}_t(dz) \, dt.$$

$$\eta(\tau) \int_{\mathbb{R}^d} \xi(X_t(x)) \cdot W_t(x) \mathcal{Q}(dx) = \eta(\tau) \int_{\mathbb{R}^d} \xi(z) \cdot u_t(z) \mathcal{Q}_t(dz).$$

The first integral on the right-hand side of (52) can be rewritten as

$$\int_0^\tau \eta(t) \int_{\mathbb{R}^d} \nabla \xi(X_t(x)) : \left( W_t(x) \otimes \dot{X}_t(x) \right) \mathcal{Q}(dx) \, dt \quad (53)$$

$$= \int_0^\tau \eta(t) \int_{\mathbb{R}^d} \nabla \xi(X_t(x)) : \left( W_t(x) \otimes (V_t(x) - W_t(x)) \right) \mathcal{Q}(dx) \, dt + \int_0^\tau \eta(t) \int_{\mathbb{R}^d} \nabla \xi(X_t(x)) : \left( W_t(x) \otimes W_t(x) \right) \mathcal{Q}(dx) \, dt.$$

Because of (47) and (50), we have that

$$V_t(x) - W_t(x) = \left(\frac{2}{3} - \frac{t}{\tau}\right)(W_t(x) - u(x)) \quad \text{for all} \ t \in [0, \tau].$$
By Young’s inequality, for any \(\varepsilon > 0\), there exists a constant \(C_\varepsilon > 0\) with
\[
\| W_t(x) \otimes (V_r(x) - W_t(x)) \|
\leq \varepsilon \left( \left(1 - \frac{t}{\tau}\right) |u(x)|^2 + \frac{t}{\tau} |W_r(x)|^2 \right) + C_\varepsilon \left| \frac{2}{3} - \frac{t}{\tau} \right| |W_r(x) - u(x)|^2.
\]

It follows that
\[
\begin{align*}
\int_0^\tau \eta(t) \int_{\mathbb{R}^d} \nabla \zeta(X_t(x)) : \left( W_t(x) \otimes (V_r(x) - W_t(x)) \right) \varrho(dx) \, dt \\
&\leq C \tau \left\{ \varepsilon \int_{\mathbb{R}^d} (|u(x)|^2 + |W_r(x)|^2) \varrho(dx) + C_\varepsilon \int_{\mathbb{R}^d} |W_r(x) - u(x)|^2 \varrho(dx) \right\}.
\end{align*}
\]
where \(C\) depends on the sup-norms of \(\eta\) and \(\nabla \zeta\). The integral multiplied by \(\varepsilon\) can be bounded by the total energy, the one multiplied by \(C_\varepsilon\) by the energy dissipation; see (48). The second integral on the right-hand side of (53) is
\[
\int_0^\tau \eta(t) \int_{\mathbb{R}^d} \nabla \zeta(X_t(x)) : \left( W_t(x) \otimes W_t(x) \right) \varrho(dx) \, dt
= \int_0^\tau \eta(t) \int_{\mathbb{R}^d} \nabla \zeta(z) : \left( u_t(z) \otimes u_t(z) \right) \varrho_t(dz) \, dt
\]
because of definition (51).

The second integral on the right-hand side of (52) can be rewritten as
\[
\begin{align*}
\int_0^\tau \eta(t) \int_{\mathbb{R}^d} \zeta(X_t(x)) & \cdot W_t(x) \varrho(dx) \, dt \\
= \int_0^\tau \eta(t) \int_{\mathbb{R}^d} \left( \zeta(X_t(x)) - \zeta(x) \right) \cdot \frac{W_r(x) - u(x)}{\tau} \varrho(dx) \, dt \\
&+ \int_0^\tau \eta(t) \int_{\mathbb{R}^d} \zeta(x) \cdot \frac{W_r(x) - u(x)}{\tau} \varrho(dx) \, dt.
\end{align*}
\]
Since \(\zeta \in \mathcal{A}\) we can estimate the first integral on the right-hand of (55) by
\[
\begin{align*}
\int_0^\tau \eta(t) \int_{\mathbb{R}^d} \left( \zeta(X_t(x)) - \zeta(x) \right) \cdot \frac{W_r(x) - u(x)}{\tau} \varrho(dx) \, dt \\
&\leq C \tau \left\{ \varepsilon \int_{\mathbb{R}^d} |V_r(x)|^2 \varrho(dx) + C_\varepsilon \int_{\mathbb{R}^d} |W_r(x) - u(x)|^2 \varrho(dx) \right\}
\end{align*}
\]
for any \(\varepsilon > 0\) and a corresponding constant \(C_\varepsilon\), by Cauchy–Schwarz and Young’s inequality. The constant \(C\) depends on the sup-norms of \(\eta\) and \(\nabla \zeta\). The integral with \(V_r\) is bounded by the total energy, while the integral with \(W_r - U\) is bounded by the energy dissipation; see (48). Notice that \(V_r\) is a convex combination of \(u\) and \(W_r\); see (47). Because of equation (45), the last integral in (55) is
\[
\int_0^\tau \eta(t) \int_{\mathbb{R}^d} \zeta(x) \cdot \frac{W_r(x) - u(x)}{\tau} \varrho(dx) \, dt
\]
We can now argue as in [27]. First, we estimate
\[
\left| \int_0^\tau \eta(t) \int_{\mathbb{R}^d} \nabla \zeta(x) : \left( P_\tau(x) - P(r(x)) \mathbf{1} \right) \, dx \, dt \right|
\leq C\tau \left\{ \varepsilon \int_{\mathbb{R}^d} U(r(x)) \, dx + C\varepsilon \int_{\mathbb{R}^d} P(r(x)) \, D_t \left( \nabla X_\tau(x)^S - \mathbf{1} \right) \, dx \right\},
\]
using (46)/(49) and the fact that \( P = (\gamma - 1)U \) for polytropic gases. Again the first integral on the right-hand side can be bounded by the total energy, the second one by the energy dissipation; see (48). Notice that since \( P_\tau, \mathbf{1} \) are symmetric, only the symmetric part of \( \nabla \zeta(x) \) is relevant for the estimate, therefore (49) can indeed be used. Finally, the last integral in (57) can be estimated as
\[
\left| \int_0^\tau \eta(t) \int_{\mathbb{R}^d} \nabla \zeta(x) : \lambda_t(dx) \, dt \right| \leq C\tau \int_{\mathbb{R}^d} \text{tr} \left( \lambda_t(dx) \right),
\]
and the integral can be bounded by the energy dissipation; see (48). We have used that \( \lambda_t \) takes values in positive semidefinite matrices, so the trace is equivalent to the Frobenius norm. The constant \( C \) in (58) and (59) depends on the sup-norms of \( \eta \) and \( \nabla \zeta \). Collecting all terms, we obtain the identity
\[
- \int_0^\tau \eta'(t) \int_{\mathbb{R}^d} \zeta(x) \cdot u_\tau(x) \, \varrho_t(dx) \, dt
+ \eta(\tau) \int_{\mathbb{R}^d} \zeta(x) \cdot u_\tau(x) \, \varrho_t(dx) - \eta(0) \int_{\mathbb{R}^d} \zeta(x) \cdot u(x) \, \varrho(dx)
\]
\[
= \int_0^\tau \eta(t) \int_{\mathbb{R}^d} \nabla \zeta(z) : \left( u_t(z) \otimes u_t(z) \right) \, \varrho_t(dz) \, dt
+ \int_0^\tau \eta(t) \int_{\mathbb{R}^d} P(r(x)) \nabla \cdot \zeta(x) \, dx \, dt + \text{ERROR}.
\]
Note that the spatial integral in (60) involving \( P(r(x)) \) does not depend on \( t \). The ERROR term collects the contributions from (54), (56), and (58). Thus
\[
|\text{ERROR}| \leq C\tau \left( \varepsilon \mathcal{E}[\varrho, u] + C\varepsilon \left( \mathcal{E}[\varrho, u] - \mathcal{E}[\varrho_t, u_t] \right) \right).
\]

We can now iterate the procedure outlined above, using the final data \((\varrho_t, u_\tau)\) of one timestep as the initial data for the next minimization problem. Notice that because of (48) the new initial data again has finite total energy and is therefore admissible. We will denote by \((\varrho^k_t, u_t^k)\) and \(X_t^k, W_t^k\) the corresponding approximate solutions and transport maps/velocities at times \( t^k_\tau := k\tau \), with \( k \in \mathbb{N}_0 \). Here we use the subscript \( \tau \) to emphasize the dependence on the timestep. Let
\[
X_{t,\tau}(x) := \frac{t^k_\tau + 1 - t}{\tau} x + \frac{t - t^k_\tau}{\tau} X_t^k(x), \quad W_{t,\tau}(x) := \frac{t^k_\tau + 1 - t}{\tau} u^k_t(x) + \frac{t - t^k_\tau}{\tau} W_t^k(x)
\]
\[
(62)
\]
for $\varrho^k_t$-a.e. $x \in \mathbb{R}^d$ be the interpolation of transport/velocity, and

$$\varrho_{\tau,t} := X_{\tau,t} \# \varrho^k_t, \quad u_{\tau,t} := W_{\tau,t} \circ X_{\tau,t}^{-1}$$

the interpolated density and Eulerian velocity, for $t \in [t^k, t^{k+1}_\tau]$ and $k \in \mathbb{N}_0$.

If now $\eta \in \mathscr{C}^{-1}_c((0, \infty))$ and $\zeta \in \mathfrak{A}$, then $(\varrho_{\tau}, u_{\tau})$ satisfies the analogue of (60), with time integration over $[0, \infty)$. When adding up the error terms, we find that the contributions from the first term on the right-hand side of (61) amounts to adding $\tau$ for every timestep because $\mathcal{E}[\varrho_{\tau}^k, u_{\tau}^k]$ is bounded by $\bar{E}$ uniformly in $k, \tau$; see (48). Since $\eta$ has compact support in some interval $[0, T]$, this contribution is bounded by $CT \epsilon \bar{E}$, which can be made arbitrarily small, uniformly in $\tau$, by choosing $\epsilon$ small. On the other hand, the contributions from the second term on the right-hand side of (61) can be summed up over all timesteps because the dissipation terms form a telescope sum. This contribution is therefore bounded by $CT \epsilon \bar{E}$, which converges to zero as $\tau \to 0$. We conclude that the error term in the momentum equation vanishes in the limit. The continuity equation is satisfied, by construction, non-increasing in time and bounded by $\bar{E}$.

Step 3 We will now pass to the limit $\tau \to 0$. The main issue here is to define suitable Young measures that can represent the weak limits of non-linear terms of the form $\varrho u \otimes u$ and $P(\varrho)$. We will use two such Young measures. First, let

$$\int_{\mathbb{R}^d \times \mathcal{X}} \phi(x, x) \epsilon_{\tau}(t, dx, dx) := \int_{\mathbb{R}^d} \phi(x, r^k_{\tau}(x), m^k_{\tau}(x)) \cdot h(r^k_{\tau}(x), m^k_{\tau}(x)) \, dx$$

for all $\phi \in \mathscr{C}(\mathbb{R}^d \times \mathcal{X})$ and $t \in [t^k, t^{k+1}_\tau]$, $k \in \mathbb{N}_0$, where

$$\epsilon^k_{\tau} := r^k_{\tau} \mathcal{L}^d, \quad m^k_{\tau} := r^k_{\tau} u^k_{\tau}.$$

Notice that $\epsilon_{\tau}$ is piecewise constant in time. Its total variation is given by

$$\|\epsilon_{\tau}(t, \cdot)\|_{\mathcal{M}(\mathbb{R}^d \times \mathcal{X})} = \int_{\mathbb{R}^d \times \mathcal{X}} \epsilon_{\tau}(t, dx, dx) = 1 + \mathcal{E}[\varrho^k_{\tau}, u^k_{\tau}]$$

for $t \in [t^k, t^{k+1}_\tau]$, which is bounded by the initial total energy $\bar{E}$ because of (48), uniformly in $k, \tau$. Indeed we have more: The function

$$t \mapsto E_{\tau}(t) := \|\epsilon_{\tau}(t, \cdot)\|_{\mathcal{M}(\mathbb{R}^d \times \mathcal{X})} - 1,$$

which equals the total energy, is non-increasing in time with $E_{\tau}(0) = \bar{E}$. For any sequence $\tau_n \to 0$ as $n \to \infty$, there now exists a subsequence (not relabeled, for simplicity) and a Young measure $\epsilon \in \mathbb{E}^*$ such that $\epsilon_{\tau_n} \rightharpoonup \epsilon$ weak* in $\mathbb{E}^*$ as $n \to \infty$ (which means testing against functions in $\mathbb{E}$), by Banach–Alaoglu theorem. Because of Helly’s selection theorem, we may extract another subsequence if necessary and obtain that $E_{\tau_n}(t) \to E(t)$ pointwise for all $t \in [0, \infty)$, with total enery

$$t \mapsto E(t) := \|\epsilon(t, \cdot)\|_{\mathcal{M}(\mathbb{R}^d \times \mathcal{X})} - 1$$

non-increasing in time and bounded by $\bar{E}$. 
For the second Young measure we use a continuous time interpolation: Let
\[
\int_{\mathbb{R}^d \times \mathcal{X}} \phi(x, \nu) \nu_t(t, dx, d\nu) := \int_{\mathbb{R}^d} \phi(x, r_{\tau,t}(x), m_{\tau,t}(x)) h(r_{\tau,t}(x), m_{\tau,t}(x)) \, dx
\]
for all \( \phi \in \mathcal{C}(\mathbb{R}^d \times \mathcal{X}) \) and \( t \in [0, \infty) \), with \((\varrho_{\tau,t}, u_{\tau,t})\) defined in (63)/(62),
\[
\varrho_{\tau,t} :=: r_{\tau,t} \mathcal{L}^d, \quad m_{\tau,t} := r_{\tau,t} u_{\tau,t}.
\]  
(64)

Again we must establish uniform boundedness. The map
\[
t \mapsto \int_{\mathbb{R}^d} \frac{1}{2} |u_{\tau,t}(z)|^2 \varrho_{\tau,t}(dx) = \int_{\mathbb{R}^d} \frac{1}{2} |W_{\tau,t}(x)|^2 \varrho_{\tau}^F(dx)
\]
is convex for \( t \in [t^k_\tau, t^{k+1}_\tau] \), \( k \in \mathbb{N}_0 \); see (62). Similarly, the map
\[
t \mapsto U[X_{\tau,t}|\varrho_{\tau,t}] := \int_{\mathbb{R}^d} U(r^k_{\tau}(x)) \det(\nabla X_{\tau,t}(x)^{\xi})^{1-\gamma} \, dx
\]
is convex in each timestep; see the proof of Proposition 5.23 in [27]. Thus
\[
t \mapsto \int_{\mathbb{R}^d} \frac{1}{2} |u_{\tau,t}(z)|^2 \varrho_{\tau,t}(dx) + U[X_{\tau,t}|\varrho_{\tau,t}] \quad \text{with} \quad t \in [t^k_\tau, t^{k+1}_\tau]
\]
is convex. In fact, it is bounded by the total energy at time \( t^k_\tau \), see Proposition 5.23 in [27]. Since \( U[\varrho_{\tau,t}] \leq U[X_{\tau,t}|\varrho_{\tau,t}] \) (see Remark 5.14 in [27]) it follows that
\[
\mathcal{E}[\varrho_{\tau,t}, u_{\tau,t}] \leq \mathcal{E}[\varrho_{\tau}^k, u_{\tau}^k] \quad \text{for} \quad t \in [t^k_\tau, t^{k+1}_\tau], \; k \in \mathbb{N}_0.
\]  
(65)

In particular, the total variation of \( \nu_{\tau,t} \) can be bounded by the total variation of \( \varrho_{\tau,t} \) for all \( t \in [0, \infty) \), which in the limit \( \tau \to 0 \) will produce Property (iii) of Definition 6. Extracting another subsequence of \( \tau_n \to 0 \) if necessary (not relabeled), we may assume that \( \nu_{\tau_n} \rightharpoonup \nu \) weak* in \( \mathbb{E}^* \) as \( n \to \infty \), for some \( \nu \in \mathbb{E}^* \).

Moreover, since the curves \( t \mapsto (\varrho_{\tau,t}, m_{\tau,t}) \) are Lipschitz continuous with respect to the Wasserstein distance and the bounded Lipschitz norm, respectively, uniformly in \( \tau \), one can use the Arzelà–Ascoli theorem to conclude the existence of another subsequence (not relabeled) and of limit density/momentum \((\varrho, m)\) such that
\[
(\varrho_{\tau_n,t}, m_{\tau_n,t}) \rightharpoonup (\varrho_t, m_t) \quad \text{weak* in the sense of measures},
\]
for all \( t \in [0, \infty) \). We refer the reader to Lemmas 6.1 and 6.2 in [27]; see also the proof of Lemma 3 for a similar argument. Then lower semicontinuity of the internal energy (see Definition 1) and the fact the the total energy stays uniformly bounded imply that the limit density \( \varrho_t \) is again absolutely continuous with respect to the Lebesgue measure. Indeed since \( U(r) = kr^\gamma \) with \( \gamma > 1 \) has superlinear growth at infinity, the internal energy would be infinite if the density had any singular parts. Similarly, lower semicontinuity of the kinetic energy forces \( m_t \) to be absolutely continuous with respect to \( \varrho_t \), and thus in turn with respect to the Lebesgue measure; see again the proof of Lemma 3 for a similar argument.
Step 4 We now establish the decomposition in Definition 6 (vii). We consider only the Young measure we define the weakly measurable map \( \nu^\tau : [0, \infty) \times \mathbb{R}^d \rightarrow \mathcal{P}(X_V) \) by
\[
\int_0^\infty \int_{\mathbb{R}^d} \phi(t, x) \int_{X_V} \psi(t) \nu^\tau_{t,x}(d\tau) \ dx \ dt := \int_0^\infty \int_{\mathbb{R}^d} \phi(t, x) \nu^\tau_{t,x}(d\tau, x) \ dx \ dt
\]
for all \( \phi \in \mathcal{C}_c([0, \infty) \times \mathbb{R}^d) \) and \( \psi \in \mathcal{C}_c(X_V) \); see (64) and Remark 5 for notation. As was explained there, the set \( X_V \) is topologically equivalent to \([0, \infty) \times \mathbb{R}^d\), where \( V \) is identified with \([0] \times \mathbb{R}^d\). Thus \( \nu^\tau_{t,x} \) is the Dirac measure at \((r^\tau_{t,x}, m^\tau_{t,x})\); in vacuum it assigns mass one to the point in \( V \) corresponding to the velocity \( u^\tau_{t,x} \). Since \( \nu^\tau_{t,x} \) is a probability measure for a.e. \((t, x)\) and all \( \tau \), we have that
\[
\int_A \nu^\tau_{t,x}(X_V) \ dx \ dt = \mathcal{L}^{d+1}(A) \text{ for all } A \subset [0, \infty) \times \mathbb{R}^d \text{ Borel.}
\]
Consider now the compact sets \( \Omega_N := [0, N] \times \tilde{B}_N(0) \) with \( N \in \mathbb{N} \). Then
\[
\sup_{\tau > 0} \int_{\Omega_N} \int_{X_V} h(t) \nu^\tau_{t,x}(d\tau) \ dx \ dt \leq \sup_{\tau > 0} \int_0^N \int_{\mathbb{R}^d} \left( \varrho_\tau + \left( \frac{|m^\tau|^2}{2\varrho_\tau} + U(\varrho_\tau) \right) \right) \ dx \ dt \leq N(1 + \tilde{E}) < +\infty,
\]
with \( h \) extended to a lower semicontinuous and convex map
\[
(\varrho, m) \mapsto \begin{cases} 
\varrho + \left( \frac{|m|^2}{2\varrho} + U(\varrho) \right) & \text{if } (\varrho, m) \in (0, \infty) \times \mathbb{R}^d \\
0 & \text{if } (\varrho, m) \in V \\
+\infty & \text{otherwise}
\end{cases}
\]
It follows that the map \((\varrho, m) \mapsto h(\varrho, m)\) is inf-compact, i.e., its sublevel sets are compact. We can then apply Theorem 4.3.2 in [44] to conclude that the family \( \{\nu^\tau\}_\tau \), when restricted to \( \Omega_N \times X_V \), is strictly tight and therefore (sequentially) relatively compact with respect to the topology of \( S \)-stable convergence; see Section 2.1 and Theorem 4.3.5 in [44]. In particular, if \( \tau_n \rightarrow 0 \) is the sequence of timesteps that generates the Young measures \( \nu, \varepsilon \) of Step 3, by a diagonal argument with \( N \rightarrow \infty \), we obtain a weakly measurable map \( \nu : [0, \infty) \times \mathbb{R}^d \rightarrow \mathcal{P}(X_V) \) such that
\[
\int_0^\infty \int_{\mathbb{R}^d} \phi(t, x) \psi(r^\tau_{t,x}(x), m^\tau_{t,x}(x)) \ dx \ dt \rightarrow \int_0^\infty \int_{\mathbb{R}^d} \phi(t, x) \nu_{t,x}(d\tau) \ dx \ dt \text{ asn } \rightarrow \infty
\]
for all \( \phi \in \mathcal{C}_c([0, \infty) \times \mathbb{R}^d) \) and \( \psi \in \mathcal{C}_c(X_V) \), along suitable subsequences that are not relabeled, for simplicity. Using a similar argument, we also obtain
\[
\varrho_t(dx) = \int_{X_V} \varrho \nu_{t,x}(d(\varrho, m)) \ dx,
\]
\[
m_t(dx) = \int_{X_V} m \nu_{t,x}(d(\varrho, m)) \ dx
\]
for a.e. \( t \). Recall that \( \rho_t, m_t \) are absolutely continuous with respect to the Lebesgue measure \( \mathcal{L}^d \). By convexity and Jensen’s inequality, it follows that
\[
P(r_t(x)) \leq \int_{X_V} P(\rho) \, v_{t,x}(d(\rho, m)),
\]
\[
\frac{m_t(x) \otimes m_t(x)}{r_t(x)} \leq \int_{X_V} \frac{m \otimes m}{\rho} \, v_{t,x}(d(\rho, m))
\]
in the sense of symmetric matrices, with \( \rho_t = r_t \mathcal{L}^d \) and \( m_t := r_t u_t \), for a.e. \((t, x)\). Because of Step 3 and (66), we observe that for all \( \varphi \in \mathcal{C}_c(X_V) \) and a.e. \( t \)
\[
\int_{X_V} \varphi(\tau) \, v_{t,x}(d\tau) \, dx = \int_{X_V} \frac{\varphi(\tau)}{h(\tau)} \, v_t(dx, d\tau).
\]
(69)

Here we have used that \( \varphi \) vanishes on \( \mathcal{X} \setminus X_V \), so it is sufficient to integrate over \( X_V \) in (69) instead of \( \mathcal{X} \). By monotone convergence (recall that \( v_t \) has finite total variation), we generalize (69) to \( \varphi \in \mathcal{C}(X_V) \) with \( |\varphi(\tau)| \leq C h(\tau) \) for all \( \tau \in X \), where \( C \) is some constant. From this, we obtain the inequalities
\[
\int_{X_V} P(\rho) \, v_{t,x}(d(\rho, m)) \, dx \leq \int_{\mathcal{X}} \frac{P(\rho)}{h(\rho, m)} \, v_t(dx, d(\rho, m)),
\]
\[
\int_{X_V} \frac{m \otimes m}{\rho} \, v_{t,x}(d(\rho, m)) \, dx \leq \int_{\mathcal{X}} \frac{m \otimes m}{h(\rho, m)} \, v_t(dx, d(\rho, m))
\]
in the sense of symmetric matrices. Notice that on the right-hand side of (71) we included the contributions that the Young measure \( v_t \) may have “at infinity”, i.e., in the set \( \mathcal{X} \setminus X_V \). Combining (68) and (71), we have
\[
P(r_t(x)) \, dx \leq [P(\rho)](t, dx),
\]
\[
r_t(x) u_t(x) \otimes u_t(x) \, dx \leq [m \otimes m](t, dx)
\]
(72)
in the sense of symmetric matrices. The differences between the left- and right-hand sides of (72) define the defect measures \( R, \chi \) in Definition 6 (vii). For the second Young measure \( \varepsilon \), we use a sum in the definition of the \( \mathcal{BL}(\mathbb{R}^d) \)-norm (29), not a max. Both forms are equivalent. Moreover, we do not assume that the initial total momentum vanishes.

\[\square\]

Remark 8. In [27] a different time interpolation was used: The velocity was updated to \( W_t \) at the beginning of the timestep, then transported constantly. As a consequence, the total energy could jump upward first, before decreasing continuously to its value at the end of the timestep. Here we use the piecewise linear interpolation (62) instead, which gives us the energy inequality (65). The necessary changes in the derivation of the momentum equation have been outlined in Step 2 above. The proof of Lipschitz continuity of the momentum \( t \mapsto m_t \) in Lemma 6.2 in [27] must also be adapted, taking into account the error estimate (56). Note that (56) is already uniform in the \( \mathcal{BL}(\mathbb{R}^d) \)-norm of the test function. Indeed, since the constant \( C \) on the right-hand side of (56) only depends on the sup-norm of \( \nabla \zeta \), one can take the supremum over all \( \zeta \in \mathcal{BL}(\mathbb{R}^d) \) to conclude that the error is small in the bounded Lipschitz norm. We omit the details. Different from [27], here we use a sum in the definition of the \( \mathcal{BL}(\mathbb{R}^d) \)-norm (29), not a max. Both forms are equivalent. Moreover, we do not assume that the initial total momentum vanishes.
3.5. Properties

The following lemma makes precise in which sense the quantity \((13)\) is in fact a measure of the acceleration of the dissipative solution.

**Lemma 1.** Suppose \((\varepsilon, \nu)\) is a dissipative solution of the isentropic Euler equations \((1)\), as introduced in Definition 6. Let \(t \mapsto m_t\) be the momentum associated to \((\varepsilon, \nu)\). With \(a(t|\varepsilon, \nu)\) defined by \((13)\), we have that

\[
a(t|\varepsilon, \nu) = |m'|(t) \quad \text{for a.e. } t \in [0, \infty),
\]

where \(|m'|\) is the metric derivative of \(m\) induced by the distance \((11)\).

**Proof.** For any \(\eta \in C^1_c((0, \infty))\) and \(\xi \in C^1_c(\mathbb{R}^d; \mathbb{R}^d)\) we have that

\[
-\int_0^\infty \eta'(t) \int_{\mathbb{R}^d} \xi(x) \cdot m_t(dx) \, dt = \int_0^\infty \eta(t) \int_{\mathbb{R}^d} \nabla \cdot (U(t, dx) : m_t) \, dt
\]

because of the momentum equation; see Property (ix) of Definition 6. Here

\[
U(t, dx) = [m \otimes m/\rho](t, dx) + [P(\rho)](t, dx)1
\]

is the momentum flux. It suffices to integrate over \(\mathbb{R}^d\) on the right-hand side of \((73)\) (instead of the compactification \(\hat{\mathbb{R}}^d\)) because \(\xi\) vanishes at infinity. Consider now a sequence of test functions \(\eta_k\) that converges pointwise a.e. to the characteristic function of some time interval \([s, t]\) where \(0 \leq s \leq t\). Since the map \(r \mapsto m_r\) is Lipschitz continuous with respect to the bounded Lipschitz norm (testing against \(\xi \in BL(\mathbb{R}^d, \mathbb{R}^d)\)), we can pass to the limit \(k \to \infty\) in \((73)\) to obtain

\[
-\int_{\mathbb{R}^d} \xi(x) \cdot (m_t(dx) - m_s(dx)) = \int_s^t \int_{\mathbb{R}^d} \nabla (U(r, dx) : m_t) \, dr.
\]

With \(\| \cdot \|\) the operator norm induced by the Euclidean norm \(| \cdot |\) on \(\mathbb{R}^d\), we have that

\[
\|\xi\|_{Lip(\mathbb{R}^d)} \leq 1 \quad \implies \quad \|\nabla \xi(x)\| \leq 1 \quad \implies \quad \xi \cdot \nabla \xi(x) \xi \leq |\xi|^2
\]

for all \(x, \xi \in \mathbb{R}^d\). Recall that \(\xi\) is differentiable, by assumption. Then

\[
\left| \int_{\mathbb{R}^d} \xi(x) \cdot (m_t(dx) - m_s(dx)) \right| \leq \int_s^t \int_{\mathbb{R}^d} tr(U(r, dx)) \, dr.
\]

Recall that \(U\) takes values in the symmetric, positive semi-definite matrices so that its trace is non-negative and we can change the domain of integration to \(\hat{\mathbb{R}}^d\).

We claim that \((75)\) remains true for all maps \(\xi : \mathbb{R}^d \to \mathbb{R}^d\) with \(\|\xi\|_{Lip(\mathbb{R}^d)} \leq 1\). Assuming for the moment that the claim is true, we take the supremum on both sides of \((75)\) over such \(\xi\) and conclude that

\[
d(m_t, m_s) \leq \int_s^t \int_{\hat{\mathbb{R}}^d} tr(U(r, dx)) \, dr \quad \text{for all } 0 \leq s \leq t;
\]
see (11). In particular, the map \( t \mapsto a(t; \varepsilon, v) \) defined in (13) is an upper bound of the metric derivative \(|\mathbf{m}'|\) for all times. On the other hand, we can use the test function \( \zeta = \text{id} \) in the momentum equation because \( \text{id} \in \mathfrak{A} \). This gives

\[
- \int_0^\infty \eta'(t) \int_{\mathbb{R}^d} x \cdot \mathbf{m}_t(\text{d}x) \, \text{d}t = \int_0^\infty \eta(t) \int_{\mathbb{R}^d} \text{tr}(U(t, \text{d}x)) \, \text{d}t \tag{77}
\]

Note that we integrate over \( \mathbb{R}^d \) on the right-hand side of (77). Considering again a sequence of test functions \( \zeta^k \) converging pointwise a.e. to the characteristic function of \( [s, t] \), we want to pass to the limit \( k \to \infty \) on either side of (77). This time we may not have Lipschitz continuity of the map \( t \mapsto \int_{\mathbb{R}^d} x \cdot \mathbf{m}_t(\text{d}x) \) but this map is still in \( \mathcal{L}^\infty_{\text{loc}}([0, \infty)) \). Then the Lebesgue differentiation theorem implies

\[
- \int_{\mathbb{R}^d} x \cdot (\mathbf{m}^t(\text{d}x) - \mathbf{m}^s(\text{d}x)) = \int_s^t \int_{\mathbb{R}^d} \text{tr}(U(r, \text{d}x)) \, \text{d}r \tag{78}
\]

for almost every \( 0 \leq s \leq t \). Estimating the left-hand side of (78) from above by taking the supremum over all Lipschitz continuous test functions with \( \| \zeta \|_{\text{Lip}(\mathbb{R}^d)} \leq 1 \) (of which \( \zeta = \text{id} \) is one), we obtain the opposite inequality to (76), and therefore equality for generic \( 0 \leq s \leq t \). It only remains to prove the claim made above.

To this end, consider \( \varphi \in \mathcal{C}^1_c([\mathbb{R}^d]) \) with \( \varphi(\mathbb{R}^d) \subset [0, 1] \) and

\[\varphi(x) = 1 \text{ if } |x| \leq 1, \quad \varphi(x) = 0 \text{ if } |x| \geq 2.\]

For any \( R, \varepsilon > 0 \) we define the rescaled cut-off function/mollifier

\[\eta_R(x) := \varphi(x/R), \quad \varphi_\varepsilon(x) := \varepsilon^{-d} \varphi(x/\varepsilon)\]

where \( x \in \mathbb{R}^d \). For given \( \zeta : \mathbb{R}^d \to \mathbb{R}^d \) with \( \| \zeta \|_{\text{Lip}(\mathbb{R}^d)} \leq 1 \) let

\[\zeta_\varepsilon := \zeta \ast \varphi_\varepsilon \quad \text{and} \quad \zeta_{R, \varepsilon} := \eta_R \zeta_\varepsilon,\]

with the latter being an element of \( \mathcal{C}^1_c([\mathbb{R}^d]) \) for every \( R, \varepsilon > 0 \). We still have (74) with \( \zeta_{R, \varepsilon} \) in place of \( \zeta \). We decompose the right-hand side in the form

\[
\int_s^t \int_{\mathbb{R}^d} \nabla \zeta_{R, \varepsilon}(x) : U(r, \text{d}x) \, \text{d}r = \int_s^t \int_{\mathbb{R}^d} \frac{1}{R} \nabla \eta\left(\frac{x}{R}\right) \otimes \zeta_\varepsilon(x) : U(r, \text{d}x) \, \text{d}r \\
+ \int_s^t \int_{\mathbb{R}^d} \eta_R(x) \nabla \zeta_\varepsilon(x) : U(r, \text{d}x) \, \text{d}r. \tag{79}
\]

By the properties of mollifications, the assumption on \( \zeta \) implies \( \| \nabla \zeta_\varepsilon \|_{\mathcal{L}^\infty(\mathbb{R}^d)} \leq 1 \) so that \( |\zeta_\varepsilon(x)| \leq A + |x| \) for all \( x \in \mathbb{R}^d \), with \( A \geq 0 \) some constant. Then

\[
\left| \int_{\mathbb{R}^d} \frac{1}{R} \nabla \eta\left(\frac{x}{R}\right) \otimes \zeta_\varepsilon(x) : U(r, \text{d}x) \right| \leq C \frac{A + 2R}{R} \int_{\mathbb{R}^d} 1_{\mathbb{C}B_R(0)}(x) \text{tr}(U(r, \text{d}x)),
\]

with \( C \) a constant depending only on \( \nabla \eta \), for any fixed \( r \). Note that this estimate is uniform in \( \varepsilon \). As the integrand converges pointwise to zero as \( R \to \infty \), the integral vanishes in the limit, by dominated convergence. Moreover, since the \( x \)-integral over \( \text{tr}(U) \) is bounded uniformly in time, we can pass to the limit \( R \to \infty \) also in
the corresponding time integral in (79), using dominated convergence once again. In a similar fashion, we argue that the second integral on the right-hand side of (79) is bounded by \( \int_{t}^{s} \int_{\mathbb{R}^d} \text{tr}(U(r, dx)) \, dr \), uniformly in \( \varepsilon, R \). This gives the right-hand side of (75). To pass to the limit on the left-hand side of (75), we first let \( \varepsilon \to 0 \) with \( R \) fixed. Since \( \dot{m}_t \) is absolutely continuous with respect to the Lebesgue measure and, in fact, in \( L^p(\mathbb{R}^d) \) for some \( p > 1 \) [see (37)], we conclude that

\[
\int_{\mathbb{R}^d} \eta_R(x) \zeta(x) \cdot \dot{m}_t(dx) \to \int_{\mathbb{R}^d} \eta_R(x) \zeta(x) \cdot \dot{m}_t(dx) \quad \text{as} \quad \varepsilon \to 0,
\]

using the properties of mollifications. Moreover, since \( |\zeta(x)| \leq A + |x| \) for all \( x \in \mathbb{R}^d \) and some constant \( A \geq 0 \), and since \( m_t \) has finite first moment, we can again use the dominated convergence theorem to obtain

\[
\int_{\mathbb{R}^d} \eta_R(x) \zeta(x) \cdot m_t(dx) \to \int_{\mathbb{R}^d} \zeta(x) \cdot m_t(dx) \quad \text{as} \quad R \to \infty.
\]

For the integral involving \( m_s \) we argue analogously. \( \square \)

**4. Minimal Acceleration**

For initial data \((\bar{\rho}, \bar{m})\) as in (36), we define the set

\[
S := \{ (\varepsilon, \nu) \text{ dissipative solution of (1) with initial data } (\bar{\rho}, \bar{m}) \}, \tag{80}
\]

which is non-empty; see Proposition 1. We now introduce a topology such that (80) becomes a *compact metric space*. This requires a suitable notion of convergence of Young measures. Recall that Young measures are elements of the dual space \( \mathbb{E}^* \) [see (32)], which comes equipped with the weak* topology. By Banach–Alaoglu theorem, bounded sets in \( \mathbb{E}^* \) are weak* precompact. Moreover, since the Banach space \( \mathbb{E} \) [see (31)] is separable, the weak* topology is metrizable on such bounded sets, so that *compactness and sequential compactness are equivalent*. We will say that a sequence of dissipative solutions \((\varepsilon^k, \nu^k) \in S\) converges to \((\varepsilon, \nu)\) if

\[
\varepsilon^k \rightharpoonup \varepsilon, \quad \nu^k \rightharpoonup \nu \quad \text{weak* in } \mathbb{E}^* \text{ as } k \to \infty, \tag{81}
\]

which is defined in terms of testing against functions in \( \mathbb{E} \).

**Lemma 2.** The set \( S \) is a (sequentially) compact metric space under (81).

**Proof.** It suffices to prove that every sequence of dissipative solutions \((\varepsilon^k, \nu^k) \in S\) admits a subsequence converging as in (81) to a pair of Young measures \((\varepsilon, \nu)\) that is again a dissipative solution in \( S \). We establish uniform boundedness of \( \varepsilon^k \) and \( \nu^k \) in the \( \mathbb{E}^* \)-norm and apply Banach–Alaoglu theorem. Let \( \bar{E}, \bar{M} \) be the total energy and second moment determined by \((\bar{\rho}, \bar{m})\); see (22) and (24). Then

\[
\left\| \varepsilon^k(t, \cdot) \right\|_{\mathbb{E}(\mathbb{R}^d \times \mathcal{X})} = \int_{\mathbb{R}^d \times \mathcal{X}} \varepsilon^k(t, dx, dv) = \int_{\mathbb{R}^d} \left[ \bar{\rho} + \frac{1}{2} |m|^2 / \bar{\rho} + U(\bar{\rho}) \right] \varepsilon^k(t, dx),
\]
which is bounded by $1 + \bar{E}$ for every $t \in [0, \infty)$ uniformly in $k$, by Definition 6 (ii)/(vi). Recall that the pairing (34) is linear in $f$. It follows that
\[
\sup_{k \in \mathbb{N}} \|\varepsilon^k\|_{\mathbb{E}^*} \leq 1 + \bar{E};
\]
see (33). Extracting a subsequence (not relabeled, for simplicity), we obtain (81), for some $\varepsilon \in \mathbb{E}^*$. The same argument applies to the Young measures $\nu^k$.

We can now use Lemma 3 to finish the proof. 

Lemma 3. Suppose that a sequence of dissipative solutions $(\varepsilon^k, \nu^k) \in S$ converges to a pair of Young measures $(\varepsilon, \nu)$ in the sense of (81). Then $(\varepsilon, \nu) \in S$. 

Proof. We split the proof into two steps.

Step 1 We will first establish the existence of curves $t \mapsto (\varrho_t, m_t)$, for which we will check the properties of Definition 6. Consider densities/momenta $(\varrho^k, m^k)$ that are associated with the Young measures $(\varepsilon^k, \nu^k)$ as in Property (vi) of Definition 6. We will show that for a suitable subsequence (not relabeled) we have
\[
\varrho^k_t \rightharpoonup \varrho_t \quad \text{and} \quad m^k_t \rightharpoonup m_t \quad \text{weak* in the sense of measures (82)}
\]
(testing against functions in $\mathcal{C}_b(\mathbb{R}^d; \mathbb{R}^D)$, pointwise for all $t \in [0, T]$).

In order to prove (82), we use the refined version of the Arzelà–Ascoli theorem in Proposition 3.3.1 of [28]. For any $T > 0$ fixed, the curves $t \mapsto \varrho^k_t$ have second moments bounded by (25) for all $t \in [0, T]$, uniformly in $k$. Therefore, for each such $t$, the set $\{\varrho^k_t\}_{k \in \mathbb{N}}$ is tight, hence narrowly precompact in $\mathcal{P}(\mathbb{R}^d)$, by Prokhorov’s theorem. This implies in particular the weak* precompactness in the sense of measures on $\mathbb{R}^d$. The curves are Lipschitz continuous with respect to the Wasserstein distance, with Lipschitz constant bounded uniformly in $k$, because of (40). Then Proposition 3.3.1 in [28] establishes the convergence of a subsequence (not relabeled) of the $\varrho^k$ and a limit density $\varrho$ with (82) for every $t \in [0, T]$. The curve $t \mapsto \varrho_t$ is again Lipschitz continuous with respect to the Wasserstein distance, and estimates (25) and (40) still hold. Repeating this argument for $T \to \infty$, extracting subsequences as necessary, we obtain (82) for all $t \in [0, \infty)$. Notice that narrow convergence of measures in $\mathcal{P}(\mathbb{R}^d)$ is metrizable (see Remark 5.1.1 of [28]) and that the Wasserstein distance $W$ is lower semicontinuous with respect to narrow convergence; see (2.1.1) in [28].

We will now show that the limit density $\varrho_t$ is in fact uniquely determined by the Young measure $\nu_t$ for a.e. $t \in [0, \infty)$. In particular, the limit is independent of the particular choice of subsequence used in the Arzelà–Ascoli compactness argument, and so the whole sequence converges for a.e. $t$. Indeed, since the map
\[
(t, x, (\varrho, m)) \mapsto \eta(t)\varphi(x) \varrho / h(\varrho, m)
\]
with $\eta \in L^1([0, \infty))$ and $\varphi \in \mathcal{C}(\mathbb{R}^d)$ is an element in $\mathbb{E}$, we have
\[
\int_0^\infty \eta(t) \int_{\mathbb{R}^d} \varphi(x) \varrho(t, dx) \, dt = \lim_{k \to \infty} \int_0^\infty \eta(t) \int_{\mathbb{R}^d} \varphi(x) [\varrho]_{\nu^k}(t, dx) \, dt
\]
\[
= \lim_{k \to \infty} \int_0^\infty \eta(t) \int_{\mathbb{R}^d} \varphi(x) \varrho_t^k(\mathrm{d}x) \, \mathrm{d}t = \int_0^\infty \eta(t) \int_{\mathbb{R}^d} \varphi(x) \varrho_t(\mathrm{d}x) \, \mathrm{d}t.
\]

For the second equality we have used that (41) holds for \(v^k\); the third identity follows from (82) and dominated convergence. Recall that \(\varrho_t^k, \varrho_t\) have finite second moments, uniformly in \(k\) and \(t\) bounded. Therefore the domain of integration can be restricted to \(\mathbb{R}^d\). Since \(\eta, \varphi\) are arbitrary, we conclude that \([\varrho](t, \mathrm{d}x) = \varrho_t(\mathrm{d}x)\) for a.e. \(t \in [0, \infty)\). The argument works the same for \(e^k\) and \(\varepsilon\).

For the moment we argue similarly: On compact time intervals \([0, T]\), the \(m_t^k\) are bounded in \(\mathcal{M}_t\) [see (27)] uniformly in \(k\), which implies uniformly bounded first moments and therefore tightness/narrow precompactness. On \(\mathcal{M}_t\), the topology of narrow convergence coincides with the one induced by the bounded Lipschitz norm; see Corollary 3.2 and Remark 3.2 in [36]. The curves \(t \mapsto m_t\) are Lipschitz continuous with respect to the bounded Lipschitz norm, with Lipschitz constant bounded by (40) uniformly in \(k\). Then Proposition 3.3.1 in [28] establishes convergence of a subsequence of the \(m_t^k\) (not relabeled) towards a limit momentum that is again Lipschitz continuous. Also \(m_t \in \mathcal{M}_t\) for all \(t \in [0, T]\) and estimate (40) still holds. We let \(T \to \infty\) and extract further subsequence as needed to obtain (82).

To show that we also have convergence when testing against functions in \(\mathcal{A}\), which may grow linearly at infinity, we can adapt the truncation argument of Lemma 1.

We can now argue as before to show that the limit momentum \(m_t\) is uniquely determined by the Young measures \((\varrho, \varepsilon)\) for a.e. \(t \in [0, \infty)\). In particular, density and momentum \((\varrho, m)\) satisfy the compatibility condition in Property (vi) of Definition 6. Since all \((\varrho^k, m^k)\) have the same initial data \((\bar{\varrho}, \bar{m})\), the convergence (82) with \(t = 0\) implies that \((\varrho, m)\) also has initial data \((\bar{\varrho}, \bar{m})\), which proves Property (viii) of Definition 6. Lipschitz continuity in (iv) has already been discussed, and the a priori bounds (i) and (37) can be derived from weak* precompactness and lower semicontinuity of the internal and kinetic energies; see also Remark 6.

Step 2 Property (v) of Definition 6 follows from lower semicontinuity of

\[
(\varrho_t, m_t) \mapsto \sup_{\xi \in \mathcal{H}_b(\mathbb{R}^d, \mathbb{R}^d)} \int_{\mathbb{R}^d} \left( \xi(x) \cdot m_t(\mathrm{d}x) - \frac{1}{2} |\xi(x)|^2 \varrho_t(\mathrm{d}x) \right), \tag{83}
\]

which represents the kinetic energy, under weak* convergence of measures. Notice that the functional (83) gives \(+\infty\) if \(m_t\) is not absolutely continuous with respect to \(\varrho_t\). Then we use the weak* convergence (82) for every \(t \in [0, \infty)\).

Property (ii) of Definition 6 follows from (81) and Helly’s selection theorem for sequences of monotone functions; see also Step 3 in the proof of Proposition 1. To prove Property (iii) we observe that the weak* convergence of the Young measures in \(\mathcal{E}^*\) implies weak* convergence in \(\mathcal{L}^{\infty}(\mathbb{R}^d)\) of \(E^k\) and \(N^k\), which are defined as in (38) and (39) with \(e^k, v^k\) in place of \(\varepsilon, \nu\). Indeed the map

\[
(t, x, (\varrho, m)) \mapsto \eta(t) \left( \frac{1}{2} |m|^2 / \varrho + U(\varrho) \right) / h(\varrho, m)
\]

with \(\eta \in \mathcal{L}^1([0, \infty))\) is an element in \(\mathcal{E}\) (no \(x\)-dependence). Choosing test functions \(\eta_T := 1_{\{N > E\} \cap [0, T]}\), which are in \(L^1([0, \infty))\) for any \(T \in \mathbb{N}\), we find that

\[
0 \leq \int_0^\infty \eta_T(t) (N(t) - E(t)) \, \mathrm{d}t
\]
Recall that the energy bounds imply the function space inclusions (37).

Finally, we observe that for \( \eta \in C^1_c([0, \infty)) \) and \( \zeta \in \mathcal{A} \)

\[
\int_0^\infty \eta(t) \int_{\mathbb{R}^d} \nabla \zeta(x) : \left( [m \otimes m/\varrho]_{\nu_k}(t, \, dx) + \|P(\varrho)\|_{\nu_k}(t, \, dx) \right) \, dt
\]

\[
\longrightarrow \int_0^\infty \eta(t) \int_{\mathbb{R}^d} \nabla \zeta(x) : \left( [m \otimes m/\varrho]_{\nu(t, \, dx)} + \|P(\varrho)\|_{\nu(t, \, dx)} \right) \, dt
\]

as \( k \to \infty \) because the map

\[
(t, x, (\varrho, m)) \mapsto \eta(t) \nabla \zeta(x) : (m \otimes m/\varrho)/h(\varrho, m)
\]

belongs to the space \( \mathcal{E} \); see (31). Then we use (81) again. The same argument works for the pressure term. On the other hand, we have for \( \zeta \in C^1_c([0, \infty)) \) that

\[
\int_0^\infty \eta'(t) \int_{\mathbb{R}^d} \zeta(x) \cdot m_t^k(dx) \, dt \longrightarrow \int_0^\infty \eta'(t) \int_{\mathbb{R}^d} \zeta(x) \cdot m_t(dx) \, dt \tag{84}
\]

as \( k \to \infty \). Indeed (82) establishes convergence of the spatial integrals in (84) for a.e. \( t \in [0, \infty) \); recall that the momentum \( m_t \) is in \( \mathcal{M} \), which is defined in (27). Since the spatial integrals are bounded uniformly in \( t, k \), by Cauchy–Schwarz inequality and (23)/(25), we can then pass to the limit in (84) using dominated convergence. Recall that \( \eta \) has compact support in \([0, \infty)\). This proves that \((\epsilon, \nu)\) and \( m \) satisfy the momentum equation in (43) when testing with \( \eta \in C^1_c([0, \infty)) \) and \( \zeta \in C^1_c([0, \infty)) \) and \( \zeta \in C^1_c([0, \infty)) \). A truncation argument like the one in the proof of Lemma 1 can then be used to show that the momentum equation holds for all \( \zeta \in \mathcal{A} \). The continuity equation can be handled in a similar way, which proves Property (ix) of Definition 6.

Finally, to prove Property (vii) we argue as in Step 4 of the proof of Proposition 1:

We introduce weakly measurable maps \( \nu^k : [0, \infty) \times \mathbb{R}^d \longrightarrow \mathcal{P}(X_V) \) with

\[
\int_0^\infty \int_{\mathbb{R}^d} \phi(t, x) \int_{X_V} \varphi(x) \nu^k_{t,x}(dx) \, dx \, dt := \int_0^\infty \int_{\mathbb{R}^d} \phi(t, x) \varphi(r^k_t(x), m^k_t(x)) \, dx \, dt
\]

for all \( \phi \in C_c([0, \infty) \times \mathbb{R}^d) \) and \( \varphi \in C_c(X_V) \). Here \( (r^k, m^k) \) is the Lebesgue density of density/momentum \((\varrho^k, m^k)\). As before, one can show that the sequence of \( \nu^k \) admits a weak* converging subsequence (not relabeled) such that

\[
\int_0^\infty \int_{\mathbb{R}^d} \phi(t, x) \varphi(r^k_t(x), m^k_t(x)) \, dx \, dt
\]

\[
\longrightarrow \int_0^\infty \int_{\mathbb{R}^d} \phi(t, x) \int_{X_V} \varphi(x) \nu^k_{t,x}(dx) \, dx \, dt \quad \text{as } k \to \infty
\]
for all \( \phi \in \mathcal{C}_c([0, \infty) \times \mathbb{R}^d) \) and \( \varphi \in \mathcal{C}_c(X_V) \), where \( \nu: [0, \infty) \times \mathbb{R}^d \rightarrow \mathcal{P}(X_V) \) is some weakly measurable map. This limit \( \nu \) is uniquely determined by identity (69), which proves in particular that the whole sequence converges. Again we have (67) and (68), by Jensen’s inequality. Combining this with (71), we get (72) in the sense of symmetric matrices. The difference between the left- and right-hand sides of (72) define the defect measures \( R, \chi \) in Definition 6 (vii). For the second Young measure \( \epsilon \) and \( \left\lfloor \cdot \right\rfloor \) the argument works the same. □

For \( (\epsilon, \nu) \in S \) we define the set of predecessors

\[
P(\epsilon, \nu) := \{ (\tilde{\epsilon}, \tilde{\nu}) \in S : (\tilde{\epsilon}, \tilde{\nu}) \preceq (\epsilon, \nu) \},
\]

where \( \preceq \) is the quasi-order introduced in Definition 2.

**Lemma 4.** For every \( (\epsilon, \nu) \in S \) the set \( P(\epsilon, \nu) \) is closed under (81).

**Proof.** Because of Lemma 3, we know that if \( (\epsilon^k, \nu^k) \in S \) converges to \( (\tilde{\epsilon}, \tilde{\nu}) \) in the sense of (81), then \( (\tilde{\epsilon}, \tilde{\nu}) \in S \). Suppose that \( (\epsilon^k, \nu^k) \preceq (\epsilon, \nu) \), i.e., that

\[
a(t|\epsilon^k, \nu^k) \leq a(t|\epsilon, \nu) \quad \text{for a.e. } t \in [0, \infty)
\]

and all \( k \in \mathbb{N} \). We argue as in the proof of Lemma 3: Since the maps

\[
(t, x, (\rho, m)) \mapsto \eta(t) \left\{ \frac{m \otimes m/\rho}{P(\rho)} \right\} / h(\rho, m)
\]

with \( \eta \in L^1([0, \infty)) \) belong to \( \mathbb{E} \) (no \( x \)-dependence), (81) implies that

\[
a(\cdot|\epsilon^k, \nu^k) \rightharpoonup a(\cdot|\tilde{\epsilon}, \tilde{\nu}) \quad \text{weak* in } L^\infty([0, \infty)).
\]

Choosing test functions \( \eta_T := \mathbf{1}_{a(\cdot|\tilde{\epsilon}, \tilde{\nu}) > a(\cdot|\epsilon, \nu)|[0,T]} \), which are in \( L^1([0, \infty)) \) for any \( T \in \mathbb{N} \), we use inequality (85) and (86) to obtain the estimate

\[
0 \leq \int_0^\infty \eta_T(t)(a(t|\tilde{\epsilon}, \tilde{\nu}) - a(t|\epsilon, \nu)) \, dt \\
\leq \lim_{k \to \infty} \int_0^\infty \eta_T(t)(a(t|\tilde{\epsilon}, \tilde{\nu}) - a(t|\epsilon^k, \nu^k)) \, dt = 0.
\]

It follows that \( a(t|\tilde{\epsilon}, \tilde{\nu}) \leq a(t|\epsilon, \nu) \) for a.e. \( t \in [0, \infty) \), thus \( (\tilde{\epsilon}, \tilde{\nu}) \in P(\epsilon, \nu) \). □

Applying Theorem 2, we then obtain our main result Theorem 1.

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**References**

1. CHEN, G.-Q.: Compactness methods and nonlinear hyperbolic conservation laws. In: Some Current Topics on Nonlinear Conservation Laws. AMS/IP Stud. Adv. Math., vol. 15, pp. 33–75 (2000). https://doi.org/10.1090/amsip/015/02
2. DING, X.X., CHEN, G.Q., Luo, P.Z.: Convergence of the Lax–Friedrichs scheme for the system of equations of isentropic gas dynamics. I. Acta Math. Sci. (Chin.) 7(4), 467–480, 1987
3. DING, X.X., CHEN, G.Q., Luo, P.Z.: Convergence of the Lax–Friedrichs scheme for the system of equations of isentropic gas dynamics. II. Acta Math. Sci. (Chin.) 8(1), 61–94, 1988
4. CHEN, G.-Q., LEFLOCH, P.G.: Compressible Euler equations with general pressure law. Arch. Ration. Mech. Anal. 153(3), 221–259, 2000. https://doi.org/10.1007/s002050000091
5. CHEN, G.-Q., PEREPELITSA, M.: Vanishing viscosity limit of the Navier–Stokes equations to the Euler equations for compressible fluid flow. Commun. Pure Appl. Math. 63(11), 1469–1504, 2010. https://doi.org/10.1002/cpa.20332
6. DiPERNA, R.J.: Convergence of the viscosity method for isentropic gas dynamics. Commun. Math. Phys. 91(1), 1–30, 1983
7. LEFLOCH, P.G., WESTDICKENBERG, M.: Finite energy solutions to the isentropic Euler equations with geometric effects. J. Math. Pures Appl. (9) 88(5), 389–429, 2007. https://doi.org/10.1016/j.matpur.2007.07.004
8. LIONS, P.-L., PERTHAME, B., SOUGANIDIS, P.E.: Existence and stability of entropy solutions for the hyperbolic systems of isentropic gas dynamics in Eulerian and Lagrangian coordinates. Commun. Pure Appl. Math. 49(6), 599–638, 1996. https://doi.org/10.1002/(SICI)1097-0312(199606)49:6h599::AID-CPA213.0.CO;2-5
9. LIONS, P.-L., PERTHAME, B., TADMOR, E.: Kinetic formulation of the isentropic gas dynamics and p-systems. Commun. Math. Phys. 163(2), 415–431, 1994
10. NASH, J.: C¹ isometric imbeddings. Ann. Math. 60(2), 383–396, 1954. https://doi.org/10.2307/1969840
11. NASH, J.: The imbedding problem for Riemannian manifolds. Ann. Math. 63(2), 20–63, 1956. https://doi.org/10.2307/1969989
12. DE LELLIS, C., SZÉKELYHIDI, L., JR.: The Euler equations as a differential inclusion. Ann. Math. (2) 170(3), 1417–1436, 2009. https://doi.org/10.4007/annals.2009.170.1417
13. De Lellis, C., Székelyhidi, L., Jr.: On admissibility criteria for weak solutions of the Euler equations. *Arch. Ration. Mech. Anal.* 195(1), 225–260, 2010. https://doi.org/10.1007/s00205-008-0201-x
14. Chiodaroli, E.: A counterexample to well-posedness of entropy solutions to the compressible Euler system. *J. Hyperbolic Differ. Equ.* 11(3), 493–519, 2014. https://doi.org/10.1142/S0219891614500143
15. Chiodaroli, E., Kreml, O.: On the energy dissipation rate of solutions to the compressible isentropic Euler system. *Arch. Ration. Mech. Anal.* 214(3), 1019–1049, 2014. https://doi.org/10.1007/s00205-014-0771-8
16. Chiodaroli, E., De Lellis, C., Kreml, O.: Global ill-posedness of the isentropic system of gas dynamics. *Commun. Pure Appl. Math.* 68(7), 1157–1190, 2015. https://doi.org/10.1002/cpa.21537
17. Feireisl, E.: Maximal dissipation and well-posedness for the compressible Euler system. *J. Math. Fluid Mech.* 16(3), 447–461, 2014. https://doi.org/10.1007/s00021-014-0163-8
18. Onsager, L.: Statistical hydrodynamics. *Nuovo Cimento (9)* 6(Supplemento, 2 (Convegno Internazionale di Meccanica Statistica)), 279–287, 1949
19. Buckmaster, T.: Onsager's conjecture almost everywhere in time. *Commun. Math. Phys.* 333(3), 1175–1198, 2015. https://doi.org/10.1007/s00220-014-2262-z
20. Buckmaster, T., De Lellis, C., Székelyhidi, L., Jr.: Dissipative Euler flows with Onsager-critical spatial regularity. *Commun. Pure Appl. Math.* 69(9), 1613–1670, 2016. https://doi.org/10.1002/cpa.21586
21. Isett, P.: A proof of Onsager's conjecture. *Ann. Math. (2)* 188(3), 871–963, 2018. https://doi.org/10.4007/annals.2018.188.3.4
22. Feireisl, E., Gwiazda, P., Świerczewska-Gwiazda, A., Wiedemann, E.: Regularity and energy conservation for the compressible Euler equations. *Arch. Ration. Mech. Anal.* 223(3), 1375–1395, 2017. https://doi.org/10.1007/s00205-016-1060-5
23. Lim, H., Yu, Y., Glimm, J., Li, X.L., Sharp, D.H.: Chaos, transport and mesh convergence for fluid mixing. *Acta Math. Appl. Sin. Engl. Ser.* 24(3), 355–368, 2008. https://doi.org/10.1007/s10255-008-8019-8
24. Lim, H., Iwerks, J., Glimm, J., Sharp, D.H.: Nonideal Rayleigh–Taylor mixing. *Proc. Natl. Acad. Sci. USA* 107(29), 12786–12792, 2010. https://doi.org/10.1073/pnas.1002410107
25. Fiordholm, U.S., Mishra, S., Tadmor, E.: On the computation of measure-valued solutions. *Acta Numer.* 25, 567–679, 2016. https://doi.org/10.1017/S0962492916000088
26. Feireisl, E., Lukáčová-Medvid’ová, M., Mizerová, H.: A finite volume scheme for the Euler system inspired by the two velocities approach. *Numer. Math.* 144(1), 89–132, 2020. https://doi.org/10.1007/s00211-019-01078-y
27. Cavalletti, F., Sedjro, M., Westdickenberg, M.: A variational time discretization for compressible Euler equations. *Trans. Am. Math. Soc.* 371(7), 5083–5155, 2019. https://doi.org/10.1090/tran/7747
28. Ambrosio, L., Gigli, N., Savaré, G.: Gradient Flows in Metric Spaces and in the Space of Probability Measures, 2nd edn. Lectures in Mathematics ETH Zürich (2008)
29. Breit, D., Feireisl, E., Hofmanová, M.: Solution semiflow to the isentropic Euler system. *Arch. Ration. Mech. Anal.* 235(1), 167–194, 2020. https://doi.org/10.1007/s00205-019-01420-6
30. Ghoshal, S.S., Jana, A., Wiedemann, E.: Weak-strong uniqueness for the isentropic Euler equations with possible vacuum. *Partial Differ. Equ. Appl.* 3(4), 54–21, 2022. https://doi.org/10.1007/s42985-022-00191-2
31. Brezis, H.: Opérateurs Maximaux Monotones et Semi-groupes de Contractions dans les Espaces de Hilbert. North-Holland Mathematics Studies, No. 5 (1973)
32. Dudley, R.M.: Convergence of Baire measures. *Studia Math.* 26, 251–268, 1966. https://doi.org/10.4064/sm-27-3-251-268
33. Pachl, J.K.: Measures as functionals on uniformly continuous functions. *Proc. Am. Math. Soc.* 82(2), 515–521, 1979
34. Pachl, J.: Uniform Spaces and Measures. Fields Institute Monographs, vol. 30 (2013). https://doi.org/10.1007/978-1-4614-5058-0
35. Chitescu, I., Ioana, L., Miculescu, R., Nita, L.: Monge–Kantorovich norms on spaces of vector measures. Results Math. 70(3–4), 349–371, 2016. https://doi.org/10.1007/s00025-016-0531-1
36. Hille, S.C., Szarek, T., Worm, D.T.H., Ziemiańska, M.A.: Equivalence of equicontinuity concepts for Markov operators derived from a Schur-like property for spaces of measures. Stat. Probab. Lett. 169, 108964–7, 2021. https://doi.org/10.1016/j.spl.2020.108964
37. Feireisl, E., Hofmanová, M.: On convergence of approximate solutions to the compressible Euler system. Ann. PDE 6(2), 1–24, 2020. https://doi.org/10.1007/s40818-020-00086-8
38. Markfelder, S.: Convex Integration Applied to the Multi-dimensional Compressible Euler Equations. Lecture Notes in Mathematics, vol. 2294 (2021). https://doi.org/10.1007/978-3-030-83785-3
39. Dafermos, C.M.: The entropy rate admissibility criterion for solutions of hyperbolic conservation laws. J. Differ. Equ. 14, 202–212, 1973. https://doi.org/10.1016/0022-0396(73)90043-0
40. Breit, D., Feireisl, E., Hofmanová, M.: Dissipative solutions and semiflow selection for the complete Euler system. Commun. Math. Phys. 376(2), 1471–1497, 2020. https://doi.org/10.1007/s00220-019-03662-7
41. Wallace, A.D.: A fixed-point theorem. Bull. Am. Math. Soc. 51, 413–416, 1945. https://doi.org/10.1090/S0002-9904-1945-08367-0
42. Feireisl, E., Ghoshal, S.S., Jana, A.: On uniqueness of dissipative solutions to the isentropic Euler system. Commun. Partial Differ. Equ. 44(12), 1285–1298, 2019. https://doi.org/10.1080/03605302.2019.1629958
43. Mandelkern, M.: Metrization of the one-point compactification. Proc. Am. Math. Soc. 107(4), 1111–1115, 1989. https://doi.org/10.2307/2047675
44. Castaing, C., Raynaud de Fitte, P., Valadier, M.: Young measures on topological spaces. Math. Appl. (2004). https://doi.org/10.1007/1-4020-1964-5

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