Parametric critical point theorems and their applications to boundary value problems on the Sierpiński Gasket

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December 1, 2017

Abstract

In this note we consider the classical variational tools like: Ekelendad’s Variational Principle, Mountain Pass Lemma and some of their corollaries subject to a parameter. Next, we investigate the behaviour of critical points obtained once a sequence of parameters is allowed to be convergent. Applications for the Dirichlet Boundary Value Problem on the Sierpiński Gasket are given in presence of assumptions which lead to fulfillment of the mountain geometry.

Keywords: Ekeland’s variational principle, Mountain Pass Theorem, problems on Sierpiński Gasket, dependence on parameters.

AMS Mathematics Subject Classification: 49J40, 35J25

1 Introduction

In this paper we aim at providing the parametric versions of two well-known variational tools, namely the Ekeland’s variational principle and the Mountain Pass Theorem. A multiplicity result based on these two tools will also be discussed. In our approach, we allow for the action functional to depend on some parameter and investigate what happens when this parameter approaches its limit. The main question is whether corresponding to this limit we obtain a critical point (or critical points) provided that for any parameter from a sequence a critical point exists. Such abstract results have not been considered in the literature yet and we believe that these will also be applied to the so called variational stability or else to the continuous dependence on parameters.
for boundary value problems which are described for example in [19]. Moreover, we underline that for the limit problem we do not impose assumptions in the same manner as for any other problem from the sequence. This means that the variational tools mentioned above cannot be directly applied.

Furthermore, we provide some applications for Laplacian problems considered on Sierpiński Gasket, namely we consider the following problem.: Let \( V \) stand for the Sierpiński gasket, \( V_0 \) be its intrinsic boundary, let \( \Delta \) denote the weak Laplacian on \( V \) and let the measure \( \mu \) denote the restriction to \( V \) of normalized \( \log N/\log 2 \)–dimensional Hausdorff measure, so that \( \mu(V) = 1 \). Let \( M > 0 \). In this paper we consider the existence of at least two nontrivial solutions to the following boundary value problem on \( V \) for \( m \in \mathbb{N} \):

\[
\begin{align*}
\Delta x(y) + a(y)x(y) + g_m(y)f(x(y))h(u_m) &= 0 \quad \text{for a.e. } y \in V \setminus V_0, \\
x|_{V_0} &= 0,
\end{align*}
\]

where \( (u_m) \subset L^2(V) \), \( (g_m) \subset C(V) \), \( |u_m(y)| \leq M \) for a.e. \( y \in V \) and \( f : \mathbb{R} \to \mathbb{R} \), \( h : [-M, M] \to \mathbb{R}_+ \) are continuous functions. Solutions to (1) are understood in the weak sense which we will describe in more detail later. Using our abstract results we will examine what happens as \( u_m \to u_0 \) and \( g_m \to g_0 \), where \( \to \) denotes norm convergence in the respective spaces.

We define \( F : \mathbb{R} \to \mathbb{R} \) by \( F(\xi) = \int_0^\xi f(x) \, dx \). Concerning the nonlinear term, we will employ the following conditions for every \( m \in \mathbb{N} \):

**A1** \( a \in L^1(V, \mu) \) and \( a \leq 0 \) almost everywhere in \( V \);

**A2** there exists positive constants \( g^1, g^2 \) such that \( g^1 \leq g_m(y) \leq g^2 \) for all \( y \in V \); there exists positive constants \( h^1, h^2 \) such that \( h^1 \leq h(v) \leq h^2 \) for all \( v \in [-M, M] \);

**A3** there exist constants \( \theta_m > 2 + \varepsilon \), where \( \varepsilon > 0 \), such that for all \( v \in \mathbb{R} \)

\[
0 < \theta_m F(v) \leq vf(v).
\]

Moreover, there is a constant \( c > 0 \) such that \( \frac{1}{2} - \frac{1}{\theta_m} \geq c \);

**A4** there are constants \( M_1 > 0 \) and \( \beta_m > 0 \) such that

\[
\max_{y \in V, |v| \leq M_1, |u| \leq M} |g_m(y)f(v)h(u)| \leq \frac{M_1}{2(\beta_m + 1)(2N + 3)^2};
\]

**A5** for each \( m \), there exists \( \eta_m > \eta > 0 \) such that for every \( v \in [-1, 1] \) we have

\[
F(v) \geq \eta_m |v|.
\]

The Sierpiński gasket has its origin in a paper by Sierpiński [23]. Our choice of this setting for the applications lies mainly in Proposition 2.24 from [12], which concerns checking the Palais-Smale condition. The Sierpiński Gasket is
can be described as a subset of the plane obtained from an equilateral triangle by removing the open middle inscribed equilateral triangle of 1/4 of the area, removing the corresponding open triangle from each of the three constituent triangles and continuing in this way.

The study of the Laplacian on fractals started in physical sciences in [2] and [21, 22]. The Laplacian on the Sierpiński gasket was first constructed in [18] and [15]. Among the contributions to the theory of nonlinear elliptic equations on fractals we mention [7, 10, 12] and [17, 24]. Concerning some recent results by variational methods and critical point theory pertaining to the existence and the multiplicity of solutions by the recently developed variational tools we must mention the following sources [5], [6], [8], [20].

2 Parametric theorems

In what follows, X stands for an arbitrary Banach space. With the advent of the Ekeland’s Variational Principle (comp. [9]), mathematicians gained an invaluable tool for the investigation of critical points. One such result can be found in [16], p. 27 (Corollary 3.4).

**Theorem 1.** Let \( \Phi : X \to \mathbb{R} \) be a \( C^1 \)-functional that is bounded from below and satisfies the Palais-Smale condition. Then there exists \( \bar{x} \in X \) such that
\[
\Phi(\bar{x}) = \inf_{x \in X} \Phi(x) \quad \text{and} \quad \partial \Phi(\bar{x}) = 0.
\]

A natural question is the following:

What can we say about the limit of a sequence of functionals in Theorem [7]?

In a sense, we expect that if the sequence of functionals \( (\Phi_n) \) behaves ‘well-enough’, then the limit functional \( \Phi \) will still have a critical point. Throughout the whole chapter, we suppose that a functional sequence \( (\Phi_n) : X \to \mathbb{R} \) is given. Moreover, we consider a functional \( \Phi : X \to \mathbb{R} \) which is in a sense (we will make it precise later) a limit of the aforementioned sequence.

At first, we discuss the boundedness from below. The following lemma provides the condition for the limit functional to be bounded from below.

**Lemma 2.** Suppose that \( (\Phi_n) : X \to \mathbb{R} \) and \( \Phi : X \to \mathbb{R} \) satisfy the following:

\((\text{BB})\) \( \inf_{x \in X} \Phi_n(x) \rightarrow \Phi(x) \), and

\((\text{C})\) the sequence \( (\Phi_n) \) converges uniformly to \( \Phi \), i.e.
\[
\forall \varepsilon > 0 \\exists N \in \mathbb{N} \forall x \in X \forall n \geq N \left| \Phi(x) - \Phi_n(x) \right| < \varepsilon.
\]

Then the functional \( \Phi \) is bounded from below.

3
Proof. Fix $\varepsilon > 0$. By (C), there exists $N \in \mathbb{N}$ such that
\[
\forall x \in X \quad \forall n \geq N \quad |\Phi(x) - \Phi_n(x)| < \varepsilon,
\]
which implies that
\[
\forall x \in X \quad \forall n \geq N \quad |\Phi_n(x)| - \varepsilon < |\Phi(x)|.
\]
Taking the infimum on both sides, we obtain
\[
-\infty \overset{(BB)}{<} \inf \limits_{x \in X, n \geq N} |\Phi_n(x)| - \varepsilon \leq \inf \limits_{x \in X} |\Phi(x)|,
\]
which ends the proof.

In order to apply Theorem 1 we need, apart from boundedness from below, some sort of Palais-Smale condition. The next result comes to our rescue.

**Lemma 3.** Suppose that:

- (Diff) $\Phi$ and $\Phi_n$ are $C^1$-functionals for every $n \in \mathbb{N}$.
- (diagPS) If $(x_n) \subset X$ is such that $\sup_{n \in \mathbb{N}} |\Phi_n(x_n)| < \infty$ and
  \[
  \lim_{n \to \infty} \|\partial_1 \Phi_n(x_n)\| = 0,
  \]
then there exists a convergent subsequence of $(x_n)$.

- (C) $\Phi$ and $\partial_1 \Phi$ satisfy (C).

Then the functional $\Phi$ satisfies Palais-Smale condition.

Proof. Fix $\varepsilon > 0$ and suppose that $(x_n) \subset X$ is a sequence such that $\Phi(x_n)$ is bounded and $\partial_1 \Phi(x_n) \to 0$. Observe that by (C), there exists $N \in \mathbb{N}$ such that
\[
\forall n \geq N \quad |\Phi(x_n) - \Phi_n(x_n)| < \varepsilon
\]
and consequently
\[
\forall n \geq N \quad |\Phi_n(x_n)| < |\Phi(x_n)| + \varepsilon.
\]
Taking the supremum on both sides, we obtain
\[
\sup_{n \geq N} |\Phi_n(x_n)| \leq \sup \limits_{n \in \mathbb{N}} |\Phi(x_n)| + \varepsilon < \infty.
\]
By (C) we know that there exists $N \in \mathbb{N}$ (possibly different than before) such that
\[
\forall n \geq N \quad \|\partial_1 \Phi_n(x_n) - \partial_1 \Phi(x_n)\| < \varepsilon.
\]
Consequently, we have
\[ \forall n \geq N \quad \| \partial_1 \Phi_n(x_n) \| \leq \varepsilon, \]
and at last:
\[ \lim_{n \to \infty} \| \partial_1 \Phi_n(x_n) \| = 0. \tag{3} \]
By (\text{diagPS}), we extract a convergent subsequence from \((x_n)\), which ends the proof.

At this point we are able to establish a critical point for the limit functional.

\textbf{Theorem 4.} Suppose that (BB), (Diff), (\text{diagPS}) and (\partial C) are satisfied. Then there exists a point \( \bar{x} \in X \) such that
\[ \Phi(\bar{x}) = \inf_{x \in X} \Phi(x) \quad \text{and} \quad \partial \Phi(\bar{x}) = 0. \]
Moreover, if \((\bar{x}_n)\) is a sequence of critical points for \((\Phi_n)\) such that
\[ \sup_{n \in \mathbb{N}} \Phi_n(\bar{x}_n) < \infty \quad \text{and} \quad \inf_{x \in X} \Phi_n(x) = \Phi_n(\bar{x}_n), \]
then a limit of a subsequence of \((\bar{x}_n)\) is a critical point of \( \Phi \).

\textit{Proof.} Assumption (Diff) says that \( \Phi \) is a \( C^1 \)-functional. By Lemma 2 and Lemma 3 we know that \( \Phi(\cdot, \lambda) \) is bounded from below and satisfies the Palais-Smale condition. Finally, using Theorem 1 we obtain the existence of a critical point.

For the second part of theorem, by (\text{diagPS}) we know that some subsequence (we do not change the notation) of \((\bar{x}_n)\) converges. Fix \( \varepsilon > 0 \). By (C) we know that there exists \( N \in \mathbb{N} \) such that
\[ \forall x \in X \quad \forall n \geq N \quad -\varepsilon + \Phi_n(x) < \Phi(x) < \Phi_n(x) + \varepsilon. \]
Taking the infimum, for every \( n \geq N \) we have
\[ -\varepsilon + \inf_{x \in X} \Phi_n(x) < \inf_{x \in X} \Phi(x) < \inf_{x \in X} \Phi_n(x) + \varepsilon \]
\[ \iff -\varepsilon + \Phi_n(\bar{x}_n) < \inf_{x \in X} \Phi(x) < \Phi_n(\bar{x}_n) + \varepsilon \]
\[ \iff | \inf_{x \in X} \Phi(x) - \Phi_n(\bar{x}_n) | < \varepsilon \]
\[ \implies | \inf_{x \in X} \Phi(x) - \Phi(\bar{x}_n) | < \varepsilon + | \Phi(\bar{x}_n) - \Phi_n(\bar{x}_n) | < 2\varepsilon. \]
This means that
\[ \lim_{n \to \infty} \Phi(\bar{x}_n) = \Phi(\bar{x}) \]
which ends the proof.
In [3], we can find (Proposition 1) the following variation on Theorem 1:

**Theorem 5.** Let \( \Phi : X \to \mathbb{R} \) be a \( C^1 \)-functional satisfying Palais-Smale condition. If there exists an open set \( U \subset X \) such that

\[-\infty < \inf_{x \in U} \Phi(x) < \inf_{x \in \partial U} \Phi(x),\]

then there exists \( \bar{x} \in U \) such that

\[\Phi(\bar{x}) = \inf_{x \in U} \Phi(x) \quad \text{and} \quad \partial \Phi(\bar{x}) = 0.\]

In order to rewrite Theorem 5 in the parametric version, we prove the following result:

**Lemma 6.** Suppose that (C) is satisfied and

\((\text{Inf} U)\) there exists an open set \( U \subset X \) and \( \varepsilon, R > 0 \) such that

\[\forall n \in \mathbb{N} \quad -R < \inf_{x \in U} \Phi_n(x) < \inf_{x \in \partial U} \Phi_n(x) - \varepsilon.\]

Then

\[-\infty < \inf_{x \in U} \Phi(x) < \inf_{x \in \partial U} \Phi(x).\]

**Proof.** By (C) there exists \( N > 0 \) such that

\[\forall x \in X \quad \Phi_n(x) - \frac{\varepsilon}{2} < \Phi(x) < \Phi_n(x) + \frac{\varepsilon}{2}. \tag{4}\]

Hence, we obtain

\[\forall n > N \quad -\infty < \inf_{x \in U} \Phi_n(x) - \frac{\varepsilon}{2} \leq \inf_{x \in U} \Phi(x) \leq \inf_{x \in U} \Phi_n(x) + \frac{\varepsilon}{2} \]

\[< \inf_{x \in \partial U} \Phi_n(x) - \frac{\varepsilon}{2} \leq \inf_{x \in \partial U} \Phi(x), \]

which ends the proof. \( \square \)

**Theorem 7.** (parametric critical point theorem on open set)

Let \((\text{Diff}), (\text{diagPS}), (\partial C)\) and \((\text{Inf} U)\) be satisfied. Then there exists a point \( \bar{x} \in U \) such that

\[\Phi(\bar{x}) = \inf_{x \in U} \Phi(x) \quad \text{and} \quad \partial \Phi(\bar{x}) = 0.\]
Proof. Assumption (Diff) says that $\Phi$ is a $C^1$-functional. By Lemma 3 and Lemma 6 we know that $\Phi$ satisfies the Palais-Smale condition and

$$-\infty < \inf_{x \in U} \Phi(x) < \inf_{x \in \partial U} \Phi(x).$$

Finally, by Theorem 5, we obtain the desired critical point. \qed

Another result, alongside the Ekeland’s Variational Principle, which is extremely useful in critical point theory, is the celebrated Mountain Pass Theorem (comp. [1]). We recall the version which comes from [16], p. 66:

**Theorem 8.** Let $\Phi : X \to \mathbb{R}$ be a $C^1$-functional satisfying Palais-Smale condition. Suppose that there is a nonzero element $x_* \in X$ and $r > 0$ with $r < \|x_*\|$ such that

$$\max(\Phi(0), \Phi(x_*)) < \inf_{x \in \mathcal{S}(0, r)} \Phi(x).$$

Then the functional $\Phi$ has a critical value $\bar{y} \geq \inf_{x \in \mathcal{S}(0, r)} \Phi(x)$ characterized by

$$\bar{y} = \inf_{\gamma \in \Gamma} \sup_{x \in \gamma([0, 1])} \Phi(x),$$

where

$$\Gamma = \left\{ \gamma \in C([0, 1], X) : \gamma(0) = 0, \gamma(1) = x_* \right\}.$$

Here is our version of the above classical result:

**Theorem 9.** (Parametric Mountain Pass Theorem)
Suppose that (Diff), (diagPS), (C) are satisfied. Moreover, assume that:

(PMPT1) $\Phi_n(0) = 0$ for every $n \in \mathbb{N},$

(PMPT2) there exists $r > 0$ such that

$$\inf_{x \in \mathcal{S}(0, r)} \Phi_n(x) > 0,$$

(PMPT3) there exists $x_*$ such that $\|x_*\| > r$ and $\sup_{n \in \mathbb{N}} \Phi_n(x_*) < 0.$

Then there exists a point $\bar{x} \in X$ such that

$$\Phi(\bar{x}) \geq \inf_{x \in \mathcal{S}(0, r)} \Phi(x) > 0 \quad \text{and} \quad \partial_1 \Phi(\bar{x}) = 0.$$
Proof. By (PMPT1) and (C), it is easy to see that that $\Phi(0) = 0$. If we put

$$\varepsilon := \frac{1}{2} \inf_{x \in S(0,r)} \inf_{n \in \mathbb{N}} \Phi_n(x),$$

then by (C) there exists $N > 0$ such that

$$\forall x \in X \quad \forall n > N \quad \Phi_n(x) - \varepsilon < \Phi(x).$$

As a result, we have $\varepsilon < \inf_{x \in S(0,r)} \Phi(x)$.

We now put

$$\varepsilon := -\sup_{n \in \mathbb{N}} \Phi_n(x_*),$$

which is positive by (PMPT3). By (C) there exists $N > 0$ (possibly different than before) such that

$$\forall n > N \quad \Phi(x_*) < \Phi_n(x_*) + \varepsilon.$$

Taking the supremum, we conclude that $\Phi(x_*) \leq 0$.

Finally, we observe that $\Phi$ satisfies the assumptions of the classical Mountain Pass Theorem. As a consequence, we obtain the desired critical point. \qed

Finally, we are able to rewrite Proposition 2 from [3], which reads as follows:

**Theorem 10.** Let $\Phi : X \to \mathbb{R}$ be a $C^1$-functional satisfying Palais-Smale condition. Suppose that there is nonzero element $x_* \in X$ and $r > 0$ such that

1. $\Phi(0) = 0$ and

$$-\infty < \inf_{x \in B(0,r)} \Phi(x) < 0 < \inf_{x \in S(0,r)} \Phi(x),$$

2. $\Phi(x_*) \leq 0$ and $\|x_*\| > r$.

Then $\Phi$ has at least two nontrivial critical points.

**Theorem 11.** (Double Critical Point Theorem)
Suppose that (Diff), (diagPS), (C) are satisfied. Moreover, assume that:

(DCPT1) $\Phi_n(0) = 0$ for every $n \in \mathbb{N}$,

(DCPT2) there exists $r, R > 0$ such that

$$\forall n \in \mathbb{N} \quad -R < \inf_{x \in B(0,r)} \Phi_n(x) < 0 < \inf_{x \in S(0,r)} \Phi_n(x),$$

(DCPT3) there exists $x_*$ such that $\|x_*\| > r$ and $\sup_{n \in \mathbb{N}} \Phi_n(x_*) < 0$.

Then there exist two nontrivial critical points.
Proof. Looking at Lemma 6 and the Parametric Mountain Pass Theorem, (DCPT2) implies that

\[-\infty < \inf_{x \in B(0,r)} \Phi(x) \text{ and } 0 < \inf_{x \in S(0,r)} \Phi(x).\]

It remains to prove that \(\inf_{x \in B(0,r)} \Phi(x) < 0\). Then, applying Theorem 10 for \(\Phi\), we obtain two nontrivial critical points.

We put

\[\varepsilon := -\frac{1}{2} \inf_{n \in \mathbb{N}} \Phi_n(x),\]

which is positive by (DCPT2). By (C) we know that there exists \(N \in \mathbb{N}\) such that

\[\forall x \in V_n, \quad \Phi(x) < \Phi_n(x) + \varepsilon.\]

This implies that

\[\inf_{x \in B(0,r)} \Phi(x) < -\varepsilon < 0,\]

which ends the proof. \(\square\)

3 Remarks on the abstract fractal setting

Concerning the Sierpiński gasket we follow remarks collected in [5]. Let \(N \geq 2\) be a natural number and let \(p_1, \ldots, p_N \in \mathbb{R}^{N-1}\) be so that \(|p_i - p_j| = 1\) for \(i \neq j\). Define, for every \(i \in \{1, \ldots, N\}\), the map \(S_i : \mathbb{R}^{N-1} \to \mathbb{R}^{N-1}\) by

\[S_i(x) = \frac{1}{2}x + \frac{1}{2} p_i.\]

Let \(S := \{S_1, \ldots, S_N\}\) and denote by \(G : P(\mathbb{R}^{N-1}) \to P(\mathbb{R}^{N-1})\) the map assigning to a subset \(A\) of \(\mathbb{R}^{N-1}\) the set

\[G(A) = \bigcup_{i=1}^{N} S_i(A).\]

It is known that there is a unique non-empty compact subset \(V\) of \(\mathbb{R}^{N-1}\), called the attractor of the family \(S\), such that \(G(V) = V\); see, Theorem 9.1 in [11].

The set \(V\) is called the Sierpiński gasket in \(\mathbb{R}^{N-1}\). It can be constructed inductively as follows: Put \(V_0 := \{p_1, \ldots, p_N\}\) which is called the intrinsic boundary of \(V\) and define \(V_m := G(V_{m-1})\) for \(m \geq 1\). Put \(V_* := \bigcup_{m \geq 0} V_m\). Since \(p_i = S_i(p_i)\) for \(i \in \{1, \ldots, N\}\), we have \(V_0 \subseteq V_1\), hence \(G(V_*) = V_*\). Taking into account that the maps \(S_i, \ i \in \{1, \ldots, N\}\), are homeomorphisms, we conclude that \(V_*\) is a fixed point of \(G\). On the other hand, denoting by \(C\) the convex hull of the set \(\{p_1, \ldots, p_N\}\), we observe that \(S_i(C) \subseteq C\) for \(i = 1, \ldots, N\). Thus \(V_m \subseteq C\) for every \(m \in \mathbb{N}\), so \(V_* \subseteq C\). It follows that \(V_*\) is non-empty and compact, hence \(V = V_*\).
We endow $V$ with the relative topology induced from the Euclidean topology on $\mathbb{R}^{N-1}$. By $C(V)$, we denote the space of real-valued continuous functions on $V$ and by 

$$C_0(V) := \left\{ u \in C(V) : u|_{V_0} = 0 \right\}.$$ 

The spaces $L^2(V, \mu)$, $C(V)$ and $C_0(V)$ are endowed with the usual norms, i.e. the norm induced by the inner product 

$$\langle v | h \rangle = \int_V v(y)h(y) \, d\mu$$

and supremum norm $\| \cdot \|_\infty$, respectively.

For a function $u : V \to \mathbb{R}$ and for $m \in \mathbb{N}$ let 

$$W_m(u) := \left( \frac{N+2}{N} \right)^m \sum_{x,y \in V_m, |x-y|=2^{-m}} (u(x) - u(y))^2.$$  \hspace{1cm} (5)

Since $W_m(u) \leq W_{m+1}(u)$ for every natural $m$, we can put 

$$W(u) = \lim_{m \to \infty} W_m(u).$$

Define now 

$$H^1_0(V) := \left\{ u \in C_0(V) : W(u) < \infty \right\}.$$ 

This space is a dense linear subset of $L^2(V, \mu)$ equipped with the $\| \cdot \|_2$-norm. We endow $H^1_0(V)$ with the norm 

$$\| u \| = \sqrt{W(u)}.$$ 

There is an inner product defining this norm: for $u, v \in H^1_0(V)$ and $m \in \mathbb{N}$ let 

$$W_m(u, v) = \left( \frac{N+2}{N} \right)^m \sum_{x,y \in V_m, |x-y|=2^{-m}} (u(x) - u(y))(v(x) - v(y)).$$

Put 

$$W(u, v) = \lim_{m \to \infty} W_m(u, v).$$

The space $H^1_0(V)$, equipped with the inner product $W$ inducing the norm $\| \cdot \|$, becomes a real Hilbert spaces. Moreover, 

$$\| u \|_\infty \leq (2N+3)\| u \|, \quad \text{for every } u \in H^1_0(V),$$  \hspace{1cm} (6)

and the embedding 

$$(H^1_0(V), \| \cdot \|) \hookrightarrow (C_0(V), \| \cdot \|_\infty)$$
is compact, see also, [14] for further details.

Note that \((H^1_0(V), \| \cdot \|)\) is a Hilbert space which is dense in \(L^2(V, \mu)\). Furthermore, \(W\) is a Dirichlet form on \(L^2(V, \mu)\). Let \(Z\) be a linear subset of \(H^1_0(V)\), which is dense in \(L^2(V, \mu)\). Then, in [12], we find a linear self-adjoint operator \(\Delta : Z \to L^2(V, \mu)\), the (weak) Laplacian on \(V\), given by

\[-W(u, v) = \int_V \Delta u \cdot v \, d\mu, \quad \text{for every} \quad (u, v) \in Z \times H^1_0(V).\]

Let \(H^{-1}(V)\) be the closure of \(L^2(V, \mu)\) with respect to the pre-norm

\[\|u\|_{-1} = \sup_{\|h\|_{-1}=1} |\langle u|h \rangle|,\]

where \(v \in L^2(V, \mu)\) and \(h \in H^1_0(V)\). Then \(H^{-1}(V)\) is a Hilbert space and the relation

\[\forall v \in H^1_0(V) - W(u, v) = \langle \Delta u|v \rangle\]

uniquely defines a function \(\Delta u \in H^{-1}(V)\) for every \(u \in H^1_0(V)\).

4 Applications

We observe that by (6), for every \(y \in V\)

\[|x(y)| \leq \|x\|_{\infty} \leq (2N + 3)\|x\|_{H^1_0(V)}, \quad (7)\]

Using the first inequality in (7) and the fact that \(\mu(V) = 1\) we get

\[\|x\|_{L^2(V, \mu)} \leq \|x\|_{\infty} \leq (2N + 3)\|x\|_{H^1_0(V)}\]

for any \(x \in H^1_0(V)\).

In what follows, we fix \(m\) and \(u_m\). We say that a function \(x \in H^1_0(V)\) is called a weak solution of (1), if

\[W(x, v) - \int_V a(y) x(y)v(y) \, d\mu - \int_V g_m(y) f(x(y)) h(u_m(y)) v(x) \, d\mu = 0,\]

for every \(v \in H^1_0(V)\). Consequently, whenever we write that we obtain a solution to (1), we mean the weak one. The functional \(J_m : H^1_0(V) \to \mathbb{R}\), given by

\[J_m(x) = \frac{1}{2}\|x\|^2 - \frac{1}{2} \int_V a(y)x^2(y) \, d\mu - \int_V g_m(y) F(x(y)) h(u_m(y)) \, d\mu \quad (8)\]

for every \(x \in H^1_0(V)\), is the Euler action functional attached to the problem (1).
Lemma 12. Assume that A1, A2 holds. Then, the functional $J_m : H_0^1(V) \to \mathbb{R}$ defined by the relation (8) is a $C^1(H_0^1(V), \mathbb{R})$-functional. Moreover,

$$\forall w \in H_0^1(V), J'_m(x)(w) = W(u, w) - \int_V a(y)x(y)w(x) \, dy - \int_V g_m(y)f(x(y))h(u_m(y)) \, dy,$$

for each point $x \in H_0^1(V)$. In particular, $x \in H_0^1(V)$ is a weak solution of problem (1) if and only if $x$ is a critical point of $J_m$; $J_m$ is also weakly l.s.c.

We note that assumptions A1-A4 lead to the existence of a solution to (1) for any value of the parameter $m$ by the Mountain Pass Theorem as suggested in [12].

4.1 Application of the Parametric Mountain Pass Theorem

Now, we apply our parametric results to problem (1). We start with the applications of the Parametric Mountain Pass Theorem.

Theorem 13. Assume that A1-A4 are satisfied. Let $g_m \to g_0$ in $C(V)$ and let $u_m \to u_0$ in $L^2(V)$. Then the problem

$$\left\{ \begin{array}{l} \Delta x(y) + a(y)x(y) + g_0(y)f(x(y))h(u_0(y)) = 0 \quad \text{for a.e. } y \in V \setminus V_0, \\ x|_{V_0} = 0, \end{array} \right.$$  \tag{9}

has a nontrivial solution.

Proof. We need to demonstrate that the assumptions of Theorem 9 are satisfied. Some calculations are taken from [12] and repeated here for the Reader’s convenience. We believe, it makes our proof more clear.

From [12] p. 563, we see that (PS$\lambda$) holds. Indeed, suppose that $|J_m(x_k)| \leq b$ for all $m, k$ and that

$$\lim_{m \to \infty} \lim_{k \to \infty} \|J'_m(x_k)\|_{H_0^{-1}(V)} = 0.$$  

We see that

$$b + \frac{1}{2 + \varepsilon} \|x_k\| \geq b + \frac{1}{\theta_m} \|x_k\| \geq \left( \frac{1}{2} - \frac{1}{\theta_m} \right) \|x_k\|^2 \geq c \|x_k\|^2.$$

By Proposition 2.24 from [12] which says that if a Palais Smale sequence is bounded then it is convergent, possibly up to a subsequence, we have our assertion.

Conditions (Diff) and ($\partial C$) are obviously satisfied by continuity. Condition (PMPT1) is satisfied by definition of $J_m$. We fix a ball $B$ such that $|v(y)| \leq M_1$ for any $v \in B$. Concerning (PMPT2), we see that it follows from formula (3.8)
Indeed, a function \( x \mapsto \frac{1}{1 + x} \) is bounded from below by some \( \frac{\beta}{1 + \beta} \) on \( [\beta, +\infty) \). From the formula we mentioned earlier, we have

\[
J_m(u) \geq \frac{\beta}{1 + \beta} \frac{M_0^2}{2(2N + 3)}.
\]

As for (PMPT3) note that there exists positive constants \( b_1, b_2 \) such that for all \( v \):

\[
F(v) \geq b_1|v|^{6_m} - b_2.
\]

Let us fix \( x \in H^1_0(V) \) such that \( |x(y)| \geq 1 \). Then, for \( s > 0 \) we have

\[
\int_V g_m(y) F(sx(y)) h(u_m(y)) \, d\mu \geq b_1 g^1 h s^{6_m} - b_2.
\]

Therefore, for \( s > 1 \)

\[
J_m(sx) \leq \frac{1}{2} \left( ||x||^2 - \int_V a(y) x^2(y) \, d\mu \right) s^2 - \alpha s^{2+\varepsilon} + b_2,
\]

where \( \alpha = b_1 g^1 h^1 \). Since \( \varepsilon > 0, a \leq 0 \), then there exists \( s^* \) such that

\[
\frac{1}{2} \left( ||x||^2 - \int_V a(y) x^2(y) \, d\mu \right) (s^*)^2 - \alpha (s^*)^{2+\varepsilon} + b_2 < 0.
\]

Consequently, this implies the existence of \( x^* \), independent of \( m \) and outside of \( B \), which satisfies

\[
J_m(x^*) < 0.
\]

Now our assertion follows by Theorem 9.

4.2 Results for the double critical point theorem

**Theorem 14.** Assume that A1-A5 are satisfied. Let \( g_m \to g_0 \) in \( C(V) \) and let \( u_m \to u_0 \) in \( L^2(V) \). Then problem (9) has at least two nontrivial solutions.

We need only verify condition (DCPT2). Let us fix \( x \in H^1_0(V) \) such that \( ||x|| \leq (2N + 3)^{-1} \) and \( \frac{1}{2} \leq |x(y)| \leq 1 \) for all \( y \in V \). By A5 we see that for \( s > 0 \)

\[
\int_V g_m(y) F(sx(y)) h(u_m(y)) \, d\mu \geq \eta_m s \int_V |x(y)| g_m(y) h(u_m(y)) \, d\mu \geq \alpha s,
\]

where \( \alpha = \frac{1}{2} \eta g^1 h^1 \). Put

\[
\tau = \left( ||x||^2 - \int_V a(y) x^2(y) \, d\mu \right) > 0.
\]

Thus

\[
J_m(sx) = \frac{1}{2} s^2 \tau - \alpha s.
\]

Therefore for \( s \) small enough \( J_m(sx) < \gamma < 0 \), where \( \gamma \) does not depend on \( m \).
Acknowledgement 15. This research of the first author was supported by grant no. 2014/15/B/ST8/02854 “Multiscale, fractal, chemo-hygro-thermo-mechanical models for analysis and prediction the durability of cement based composites”

References

[1] Ambrosetti A., Rabinowitz P. H.: Dual Variational Methods in Critical Point Theory and Applications, Journal of Functional Analysis, 14 (1973), 349-381

[2] S. Alexander: Some properties of the spectrum of the Sierpiński gasket in a magnetic field, Phys. Rev. B 29 (1984), 5504–5508.

[3] C. Bereanu, P. Jebelean, J. Mawhin: Multiple solutions for Neumann and periodic problems with singular φ-Laplacian, J. Funct. Anal. 261 (2011), 3226–3246.

[4] G. Bonanno: Relations between the mountain pass theorem and local minima, Adv. Nonlinear Anal. 1, No. 3, 205-220 (2012).

[5] G. Bonanno, G. Molica Bisci, V. Rădulescu: Qualitative analysis of gradient-type systems with oscillatory nonlinearities on the Sierpiński gasket, Chin. Ann. Math. Ser. B 34 (2013), no. 3, 381–398.

[6] G. Bonanno, G. Molica Bisci, V. Rădulescu: Variational analysis for a nonlinear elliptic problem on the Sierpiński gasket, ESAIM Control Optim. Calc. Var. 18 (2012), no. 4, 941–953.

[7] B.E. Breckner, D. Repovš and Cs. Varga: On the existence of three solutions for the Dirichlet problem on the Sierpiński gasket, Nonlinear Anal. 73 (2010), 2980–2990.

[8] B.E. Breckner, V. Rădulescu and Cs. Varga: Infinitely many solutions for the Dirichlet problem on the Sierpiński gasket, Analysis and Applications 9 (2011), 235–248.

[9] I. Ekeland: On the Variational Principle, Journal of Mathematical Analysis and Applications 47, 324-353 (1974)

[10] K.J. Falconer: Semilinear PDEs on self-similar fractals, Commun. Math. Phys. 206 (1999), 235–245.

[11] K.J. Falconer: Fractal Geometry: Mathematical Foundations and Applications, 2nd edition, John Wiley & Sons, 2003.

[12] K.J. Falconer and J. Hu: Nonlinear elliptical equations on the Sierpiński gasket, J. Math. Anal. Appl. 240 (1999), 552–573.
[13] M. Ferrara, G. Molica Bisci, D. Repovš: Existence results for nonlinear elliptic problems on fractal domains, Adv. Nonlinear Anal. 5, No. 1, 75-84 (2016).

[14] M. Fukushima and T. Shima: On a spectral analysis for the Sierpiński gasket, Potential Anal. 1 (1992), 1–35.

[15] S. Goldstein: Random walks and diffusions on fractals, in “Percolation Theory and Ergodic Theory of Infinite Particle Systems” (H. Kesten, ed.), 121-129, IMA Math. Appl., Vol. 8, Springer, New York, 1987.

[16] Y. Jabri: The mountain pass theorem. Variants, generalizations and some applications, Encyclopedia of Mathematics and its Applications, 95.

[17] J. Kigami: Analysis on Fractals, Cambridge University Press, Cambridge, 2001.

[18] S. Kusuoka: A diffusion process on a fractal, Probabilistic Methods in Mathematical Physics (Katata/Kyoto, 1985), 251-274. Academic Press, Boston, MA, 1987.

[19] U. Ledzewicz, H. Schättler, S. Walczak: Optimal control systems governed by second-order ODEs with Dirichlet boundary data and variable parameters, Ill. J. Math. 47 (2003), No. 4, 1189-1206.

[20] G. Molica Bisci, V. Rădulescu: A characterization for elliptic problems on fractal sets, Proc. Amer. Math. Soc. 143 (2015), no. 7, 2959–2968.

[21] R. Rammal: A spectrum of harmonic excitations on fractals, J. Physique Lett. 45 (1984), 191–206.

[22] R. Rammal and G. Toulouse: Random walks on fractal structures and percolation clusters, J. Physique Lett. 44 (1983), L13–L22.

[23] W. Sierpiński: Sur une courbe dont tout point est un point de ramification, Comptes Rendus (Paris) 160 (1915), 302–305.

[24] R.S. Strichartz: Differential Equations on Fractals. A Tutorial, Princeton University Press, Princeton, NJ, 2006.

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