Beading instability and spreading kinetics in grooves with convex curved sides

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The open channel flow problem of liquid contained in a groove with convex curved sides is considered, with particular reference to that formed between a pair of touching parallel cylinders. In contrast to a V-shaped wedge or U-shaped channel, the liquid in such a groove is self-leveling at low loading, but displays a beading instability at high loading (the behaviour is determined by the sign of the derivative of the Laplace pressure). The interesting consequences of this duality are discussed qualitatively, both for the coarsening kinetics in the case of the beading instability, and for the spreading kinetics of an isolated drop deposited in the groove.

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Open channel flow problems have attracted much interest not only because of possible applications in microfluidics [1], but also because of their relevance to liquids spreading on topographically patterned surfaces such as human skin [2,3]. The paradigmatic case of spreading in a V-shaped wedge has been analysed both when liquid is supplied by a reservoir [2,4], and in the starved (no reservoir) situation [5]. Various aspects of these predictions have been confirmed experimentally, most notably in electrowetting experiments [6]. More recently, flows in U-shaped channels (i.e. with concave sides) have been considered [7]; such channels are easily micro-machined and are thus relevant for microfluidics applications.

In the present study, I revisit the problem, in the context of a groove with convex sides, such as that formed between a pair of touching parallel cylinders (Fig. 1b)). This is motivated by possible applications for instance to oily soils spreading along fabric yarns spun from collections of fibres, and molten solder wicking in stranded copper wires and braids. This problem throws up some interesting aspects not found in the previous cases since such a groove encourages spreading at low loads but displays a beading instability at higher loading. This duality has interesting consequences, both for the coarsening kinetics in the case of the beading instability, and for the spreading kinetics of an isolated drop deposited in the groove. Moreover, as I shall argue, the behaviour is not determined by the sign of the Laplace pressure, but rather by the sign of the derivative with respect to the loading. This can easily mean that the liquid in the spreading case protrudes above the groove, and consequently is able to bridge across to adjacent channels.

To start with, consider an arbitrary open channel flow, and let $A(x,t)$ be the cross section occupied by liquid. A mass conservation law holds [3,5,8],

$$\frac{\partial A}{\partial t} + \frac{\partial (A\pi)}{\partial x} = 0, \tag{1}$$

where the mean flow rate $\pi$ satisfies a Hagen-Poiseuille (HP) law, $\pi = -(k/\eta) \partial p/\partial x$, in which $k$ is the permeability (a quantity with units of length squared, cf. Darcy’s law), $\eta$ is viscosity, and $p$ is the Laplace pressure. Combining the HP law with Eq. (1) gives what can perhaps be called the $\textit{wicking equation}$,

$$\frac{\partial A}{\partial t} = \frac{\partial}{\partial x} \left( \frac{Ak}{\eta} \frac{\partial p}{\partial A} \right). \tag{2}$$

This generally has the character of a non-linear diffusion equation. I have assumed that the cross section $A$ is weakly varying with $x$, so the contribution to the Laplace pressure from the interface curvature in the longitudinal direction can be neglected. Note also that the permeability depends on the loading, viz. $k = k(A)$.

Insight can be gained by linearising about the uniformly loaded static solution $A = A_0$ and $\pi = 0$. Let us write $A/A_0 = 1 + \epsilon(x,t)$. Then $\partial \epsilon/\partial t = D_{\text{eff}} \partial^2 \epsilon/\partial x^2$ where $D_{\text{eff}} = (Ak/\eta)p/DA$ is an effective diffusion coefficient, evaluated at $A = A_0$. If this is positive (spreading case), then perturbations will die away and the liquid will be self-leveling. If it is negative, then perturbations will grow indicative of the aforementioned beading instability, which is similar to the classic Rayleigh-Plateau instability of a liquid column. It is clear that the behavior depends,

![FIG. 1. Transverse cross sections. (a) Liquid in a V-shaped wedge: spreading case (left) and non-spreading case (right). (b) Liquid in the groove between touching parallel cylinders.](image-url)
FIG. 2. Trigonometry for calculating the radius of curvature in (a,b) negative and (c,d) positive Laplace pressure cases.

as claimed, not on the sign of Laplace pressure \( p \) but rather on the sign of \( dp/dA \). Physically, if \( dp/dA > 0 \), an overfilled region will have a higher Laplace pressure than an underfilled region, and the liquid will flow to even things out. On the other hand, if \( dp/dA < 0 \), liquid will flow from underfilled regions into overfilled regions, magnifying the initial imbalance. One of the interesting properties of a groove with convex curved sides is that both situations can occur, depending on the loading.

Since we neglect the interface curvature in the longitudinal direction, the transverse profile of the free surface is characterised by an arc of a circle with radius \( R \). Taking \( R > 0 \) to indicate the surface is convex outwards, one has \( p = \frac{\gamma}{R} \) where \( \gamma \) is surface tension. Hence \( dp/dA \) has the opposite sign to \( dR/dA \), which is a purely geometrical problem. Often it is convenient to introduce an intermediate parameter, such as a filling depth \( h \). We can write

\[
\frac{dp}{dA} \propto \frac{dR}{dh} \times \left(\frac{dA}{dh}\right)^{-1}. \tag{3}
\]

Since we can often arrange for \( dA/dh \) to be strictly positive, this means that \(-dR/dh\) can serve as a proxy for \( dp/dA \). A judicious choice for \( h \) can then simplify the calculation.

As an example of this, consider the V-shaped wedge (Fig. 1a). In this case the geometry is strictly scale invariant and \( R \propto h \) where \( h \) is the filling depth and the constant of proportionality depends only on the wedge opening angle \( \phi \) and the contact angle \( \theta \) \[4\]. Hence, \( p \propto 1/h \), and consequentially \( p \) and \( dp/dh \) have opposite signs. The permeability and cross sectional area obey

\[
k \propto A \propto h^2, \text{ so that } dA/dh > 0 \text{ (which is obvious). Elementary considerations } [4] \text{ now show that if } \theta < \frac{\pi}{2} - \frac{1}{2} \phi, \text{ the liquid surface is concave (} R < 0 \text{; Fig. 1h left). In this situation } p < 0 \text{ and } dp/dA > 0, \text{ so the uniform loaded state is stable. On the other hand if } \theta > \frac{\pi}{2} - \frac{1}{2} \phi \text{ the liquid surface is convex (} R > 0 \text{; Fig. 1h right). In this other situation, } p > 0 \text{ and } dp/dA < 0, \text{ and the uniformly loaded state is unstable. The liquid contained in the wedge will break up into droplets.}

For the V-shaped wedge, \( p \) and \( dp/dA \) have strictly opposite signs. In the case of a groove with curved sides, this is no longer necessarily true. The specific case study I have in mind is wetting in the groove between touching parallel cylinders (Fig. 1b). Let the radius of the cylinders be \( a \), and let the angle \( \alpha \) be used to measure the degree of loading. Note that \( dA/d\alpha \) is strictly positive so the comments at the end of the initial discussion apply and we can focus our attention on \( dR/d\alpha \). The geometrical problem pertaining to this is solved using the trigonometry shown in Fig. 2. I initially consider separately the cases where the liquid surface is concave or convex. In the first case (Fig. 2a,b), we have that

\[
a \cos \alpha + R \sin \psi = a. \tag{4}
\]

Inspecting the right angled triangle in Fig. 2b whose hy-
potenue is $R$, we see that $\sin \psi = \cos(\theta + \alpha)$. Inserting this in Eq. 4 and rearranging gives

$$\frac{a}{R} = -\frac{\cos(\theta + \alpha)}{1 - \cos \alpha}.$$  \hspace{1cm} (5)

In this situation the Laplace pressure is negative so the sign has been inserted in accord with my convention for the sign of $R$. Now consider the second case (Fig. 2b). Here Eq. 4 also holds but now $\psi = \alpha + \theta - \frac{\pi}{2}$ so that $\sin \psi = -\cos(\theta + \alpha)$ (see triangles in Fig. 2a). Hence, noting that in this case $R$ should be positive, we again find that Eq. 5 holds. From this we find (after a little rearrangement)

$$\frac{dR}{d\alpha} = -\frac{R^2 \cos(\frac{\pi}{2} \alpha + \theta)}{2 \sin^2(\frac{\pi}{2} \alpha)}.$$ \hspace{1cm} (6)

To re-iterate, Eqs. 5 and 6 are valid irrespective of whether the liquid surface is convex or concave.

Since the Laplace pressure $p$ is inversely proportional to $R$, Eqs. 5 and 6 show that $p$ increases through zero at $\alpha + \theta = \frac{\pi}{2}$ to reach a weak maximum at $\frac{\pi}{4} \alpha + \theta = \frac{\pi}{2}$ (i.e. $\alpha = \pi - 2 \theta$). A specific example is shown in Fig. 3a, for a contact angle $\theta = 30^\circ$. The zero crossing is at $\alpha = 60^\circ$ and the Laplace pressure maximum is at $\alpha = 120^\circ$. Fig. 3b shows a selection of the corresponding filling states.

By similar trigonometric arguments the filling depth is

$$h = R + a \sin \alpha - R \sin(\alpha + \theta).$$ \hspace{1cm} (7)

An interesting corollary is that the maximum stable filled state may protrude above the cylinders (cf. the heavy filled line in Fig. 3b). This occurs if $h \geq a$, which can be shown to correspond to $\theta \leq \frac{\pi}{2}$ (at the maximum filling condition $\alpha = \pi - 2 \theta$). Thus the liquid column will protrude if the contact angle is smaller than $60^\circ$, and moreover will protrude further as the contact angle is reduced. The behavior is completely opposite to what one finds for a droplet sitting as a spherical cap on a flat surface. This consideration potentially influences the spreading of liquids in networks of aligned fibres, bearing in mind the applications mentioned in the introduction.

Beyond the maximum in the Laplace pressure curve, $dp/dA < 0$, and therefore one expects to see a beading instability in which the uniformly loaded state is unstable towards the growth of perturbations. First, what could be the final state in such a situation? (From here on my discussion will focus largely on just the qualitative aspects.) In principle one can have two different states of loading at same Laplace pressure, which can therefore be in coexistence, for example points P and Q in Fig. 3. However the higher loaded state is always in the unstable region. The only logical conclusion is that the excess liquid is expelled into a large droplet that sits somewhere on the two cylinders, coexisting with a stable column of liquid in the groove with $\alpha < \pi - 2\theta$. Since the Laplace pressure has to be positive to coexist with a droplet, the loading would also have to satisfy $\alpha > \pi - \theta$. An interesting consequence of this is that as more liquid is added, the amount contained in the groove should paradoxically go down: adding liquid must increase the size of the large droplet and reduce its Laplace pressure (towards zero), thereby in parallel diminishing the amount contained in the groove to maintain an equal Laplace pressure.

Coexistence between loading states has interesting consequences for the coarsening kinetics in the beading instability. A uniformly overloaded state will certainly break up into droplets, but these are always connected by liquid columns with a finite filling depth according to the above argument. Therefore the larger droplets can easily eat the smaller ones, via transport of liquid along the connecting liquid columns. This stands in contrast to the V-shaped wedge, where spatially separated droplets are disconnected [9] and the droplet population has to coarsen by some other mechanism, such as via a prewetting film [10], or transport through the vapor phase in the case of a volatile liquid. Thus one expects beading instability coarsening to occur much more rapidly in a groove with convex curved sides, compared to a V-shaped wedge or U-shaped channel.

The remaining point of discussion concerns the fate of a droplet of liquid deposited onto an initially empty groove. The droplet will start to empty into the groove, presumably driving a Bell-Cameron-Lucas-Washburn (BLCS) type flow from what is in effect a shrinking droplet reservoir [2][4][8][11]. This should persist all the way until the loading falls below the Laplace pressure maximum. Past this point, by analogy to the V-shaped wedge [5], one expects the spreading rate to slow down since the reservoir has been exhausted. However, the geometry suggests a clean similarity solution will not be permitted until the very late stages where everywhere $\alpha \ll 1$.

In these final stages, the wetted portion of the groove has shrunk to a narrow channel with a width of the order $R \sim \alpha^2 a$ (see Eq. 5 in the limit $\alpha \to 0$) and a depth of the order $aa$. One therefore expects $A \sim \alpha^3 a^2$, and presumably $k \sim \alpha^4 a^2$ since the permeability should largely be determined by the width. The scaling analogue of the HP law is $dL/dt \sim (k/\eta) \times \Delta p/L$, where $L$ is the length of the liquid column and $\Delta p \sim \gamma/R$ is the Laplace pressure. Substituting the above scaling expressions gives $dL/dt \sim \gamma \alpha^2 a/(\eta L)$. An additional constraint is that the droplet volume $\Omega \sim \alpha^3 a^2 L$ should be conserved. Eliminating $\alpha$ between this volume constraint and the HP scaling law yields $dL/dt \sim \gamma L^{8/3}/(\eta a^{1/3} L^{5/3})$. This integrates to the final rather esoteric result

$$L \sim (\gamma t/\eta)^{3/8} \Omega^{1/4} a^{-1/8}.$$ \hspace{1cm} (8)

In other words, the initial $L \sim t^{1/2}$ spreading law (BLCS) is expected to have diminished to an $L \sim t^{3/8}$ power law in the final starved state.
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