On the Estimates of the Density of Feynman-Kac Semigroups of 
$\alpha$-Stable-like Processes

Chunlin Wang
Department of Mathematics
University of Illinois
Urbana, IL 61801, USA
Email: cwang13@uiuc.edu

Abstract

Suppose that $\alpha \in (0, 2)$ and that $X$ is an $\alpha$-stable-like process on $\mathbb{R}^d$. Let $F$ be a function on $\mathbb{R}^d$ belonging to the class $J_{d,\alpha}$ (see Introduction) and $A^F_t$ be $\sum_{s\leq t} F(X_{s-}, X_s)$, $t > 0$, a discontinuous additive functional of $X$. With neither $F$ nor $X$ being symmetric, under certain conditions, we show that the Feynman-Kac semigroup $\{S_t^F : t \geq 0\}$ defined by

$$S_t^F f(x) = \mathbb{E}_x(e^{-A^F_t f(X_t)})$$

has a density $q$ and that there exist positive constants $C_1, C_2, C_3$ and $C_4$ such that

$$C_1 e^{-C_2 t^{-\frac{4}{d}} \left( 1 \wedge \frac{t^{\frac{d}{2}}}{|x-y|} \right)^{d+\alpha}} \leq q(t, x, y) \leq C_3 e^{C_4 t^{-\frac{4}{d}} \left( 1 \wedge \frac{t^{\frac{d}{2}}}{|x-y|} \right)^{d+\alpha}}$$

for all $(t, x, y) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$.

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Running Title: Density estimates of Feynman-Kac semigroups of stable processes
1 Introduction

Suppose $X = (X_t, \mathbb{P}_t)$ is a Hunt process on $\mathbb{R}^d$ with a Lévy system $(N,H)$ given by $H_t = t$ and

$$N(x, dy) = 2C(x, y)|x-y|^{-(d+\alpha)}m(dy),$$

where $m$ is a measure on $\mathbb{R}^d$ given by $m(dx) = M(x)dx$ with $M(x)$ bounded between two positive numbers. That is for any nonnegative function $f$ on $\mathbb{R}^d \times \mathbb{R}^d$ vanishing on the diagonal

$$\mathbb{E}_x \left( \sum_{s \leq t} f(X_{s-}, X_s) \right) = \mathbb{E}_x \int_0^t \int_{\mathbb{R}^d} \frac{2C(X_s, y)f(X_s, y)}{|X_s - y|^{d+\alpha}}m(dy)ds,$$

for every $x \in \mathbb{R}^d$ and $t > 0$.

We introduce \(\alpha\)-stable-like processes as follows.

**Definition 1.1** We say that $X$ is an \(\alpha\)-stable-like process if $C(x, y)$ is bounded.

In this paper, we assume that $X$ admits a transition density $p(t, x, y)$ with respect to $m$ and $p(t, x, y)$ is jointly continuously on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$ and satisfies the condition

$$M_1 t^{-\frac{d}{\alpha}} \left( 1 + \frac{1}{|x-y|} \right)^{d+\alpha} \leq p(t, x, y) \leq M_2 t^{-\frac{d}{\alpha}} \left( 1 + \frac{1}{|x-y|} \right)^{d+\alpha}, \quad \forall (t, x, z) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d,$$

(1.1)

where $M_1$ and $M_2$ are positive constants.

Here we do not assume that $X$ is symmetric. When $X$ is symmetric, it is called a symmetric \(\alpha\)-stable-like process, which was introduced in [3], where a symmetric Hunt process is associated with a regular Dirichlet form and thus Dirichlet form method can be applied. It was also shown in [3] that the transition densities of symmetric \(\alpha\)-stable-like processes satisfy (1.1).

We list some examples which are \(\alpha\)-stable-like processes and satisfy (1.1). For one dimensional strictly \(\alpha\)-stable processes with Lévy measure $\nu$ concentrated neither on $(0, \infty)$ nor on $(-\infty, 0)$, the Lévy measure $\nu(dx) = c_1x^{-1-\alpha}dx$ on $(0, \infty)$ and $\nu(dx) = c_2x^{-1-\alpha}dx$ on $(-\infty, 0)$ with $c_1 > 0$ and $c_2 > 0$, which implies $C(x, y)$ in the Lévy system as above is bounded between two positive numbers. We set $c = c_1 + c_2$ and $\beta = (c_1 - c_2)/(c_1 + c_2)$. Let $\rho = (1 + \beta)/2$ or $= (1 - \beta^2)/(2\alpha)$, according to $\alpha < 1$ or $> 1$. Without loss of generality, we can fix the parameter $c$ and assume that $c \in (0, \frac{\pi\beta}{\pi\alpha})$, $\frac{\pi}{2}$ or $\cos(\pi\beta - \alpha)$ for $\alpha < 1$, $1$, or $> 1$, respectively. [4] gave the following estimates for the continuous transition density $p(t, 0, x)$, which equals $t^{-1/\alpha}p(1, 0, t^{-1/\alpha}x)$:

1. When $x \to \infty$,

$$p(1, 0, x) \sim \frac{1}{\pi} \Gamma(\alpha + 1)(\sin(\pi\rho\alpha)x^{-\alpha-1}, \text{ if } \alpha \neq 1,$$

$$p(1, 0, x) \sim \frac{1 + \beta}{2} x^{-2}, \text{ if } \alpha = 1,$$
2. When $x \to 0$,

\[
p(1, 0, x) \to \frac{1}{\pi} \Gamma(1/\alpha + 1)(\sin \pi \alpha), \quad \text{if } \alpha \neq 1,
\]

\[
p(1, 0, x) \to \frac{1}{\pi} b_1, \quad \text{if } \alpha = 1, \beta > 0,
\]

where $b_1$ is a positive constant.

See (14.37), (14.30), (14.33) and (14.32) in [4] for details. It is clear that the dual process of the one dimensional strictly $\alpha$-stable process has the transition density $p(t, 0, -x)$. Thus applying the above estimates to $p(t, 0, -x)$, we get

3. When $x \to -\infty$,

\[
p(1, 0, x) \sim \frac{1}{\pi} \Gamma(\alpha + 1)(\sin(\pi \rho \alpha))|x|^{-\alpha - 1}, \quad \text{if } \alpha \neq 1,
\]

\[
p(1, 0, x) \sim \frac{1 + \beta}{2}|x|^{-2}, \quad \text{if } \alpha = 1.
\]

4. When $x \to 0$,

\[
p(1, 0, x) \to \frac{1}{\pi} \tilde{b}_1, \quad \text{if } \alpha = 1, \beta < 0,
\]

where $\tilde{b}_1$ is a positive constant.

One dimensional strictly $\alpha$-stable process with $\alpha = 1$ and $\beta = 0$ is a Cauchy process with drift $0$. It is easy to see that when $x \to 0$, $p(1, 0, x) \to$ a positive constant.

[4] also pointed out that $p(t, 0, x)$ is positive when the Lévy measure $\nu$ is concentrated neither on $(0, \infty)$ nor on $(-\infty, 0)$.

Therefore when the Lévy measure $\nu$ is concentrated neither on $(0, \infty)$ nor on $(-\infty, 0)$, the transition density $p$ satisfies (1.1).

For higher dimensions, [7] considered a large class of nonsymmetric strictly $\alpha$-stable processes with $0 < \alpha < 2$, which has Lévy measure $\nu$ satisfying

\[
\nu(B) = \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi) r^{-(1+\alpha)} dr
\]

for every Borel set $B$ in $\mathbb{R}^d$, where $\lambda$ is a finite measure on the unit sphere $S = \{x \in \mathbb{R}^d : |x| = 1\}$ and is called the spherical part of the Lévy measure $\nu$. $\lambda$ is assumed to have a density $\phi : S \to (0, \infty)$ such that

\[
\phi = \frac{d\lambda}{d\sigma} \text{ and } \kappa \leq \phi(\xi) \leq \kappa^{-1}, \quad \forall \xi \in S,
\]

where $\sigma$ is the surface measure on the unit sphere and $\kappa > 0$ is a positive constant. The assumption on $\phi$ implies the transition density $p(t, 0, x) > 0$ for all $t > 0$ and all $x \in \mathbb{R}^d$. It is known that
$p(1, 0, x)$ is uniformly bounded in $x \in \mathbb{R}^d$. [7] pointed out that the Lévy measure $\nu$ has a density $f(x) = \phi(x/|x|)|x|^{-(d+\alpha)}$ with respect to the $d$-dimensional Lebesgue measure, and
\[
\kappa |x|^{-(d+\alpha)} \leq f(x) \leq \kappa^{-1} |x|^{-(d+\alpha)}
\]
for every $x \in \mathbb{R}^d \setminus \{0\}$. Then the transition density $p(t, 0, x)$ of the processes satisfy
\[
p(t, 0, x) \leq C t^{-\alpha}, \quad x \in \mathbb{R}^d, \quad t > 0,
\]
and
\[
p(t, 0, x) \leq C t|x|^{-(\alpha+d)}, \quad x \in \mathbb{R}^d \setminus \{0\}, \quad t > 0,
\]
where $C$ is a positive constant. See (2.6) and (2.7) in [7] for these two inequalities. Thus we have
\[
p(t, 0, x) \leq \tilde{C} t^{-\frac{d}{\alpha}} \left( 1 \wedge \frac{t^{\frac{1}{\alpha}}}{|x-y|} \right)^{d+\alpha}, \quad x \in \mathbb{R}^d, \quad t > 0.
\]
(2.12) in [7] gave the following estimate,
\[
\tilde{c} |x|^{-(\alpha+d)} \leq p(1, 0, x) \leq \tilde{C} |x|^{-(\alpha+d)}, \quad \text{for large } x,
\]
where $\tilde{c}$ is a positive constant and $\tilde{C}$ is the same constant as above. This implies that
\[
\tilde{c} t^{-\frac{d}{\alpha}} |x|^{-(\alpha+d)} \leq p(1, 0, t^{-\frac{1}{\alpha}} x) \leq \tilde{C} t^{-\frac{d}{\alpha}} |x|^{-(\alpha+d)}, \quad \text{for large } t^{-\frac{1}{\alpha}} x.
\]
Thus
\[
\tilde{c} t^{-\frac{d}{\alpha}} |x|^{-(\alpha+d)} \leq t^{-\frac{d}{\alpha}} p(1, 0, t^{-\frac{1}{\alpha}} x) = p(t, 0, x), \quad \text{for large } t^{-\frac{1}{\alpha}} x.
\]
(1.3)

For small $t^{-\frac{1}{\alpha}} x$, since $p(1, 0, x)$ is positive and continuous in $x \in \mathbb{R}^d$, there exists a positive constant $\tilde{c}_0$ such that
\[
\tilde{c}_0 \leq p(1, 0, t^{-\frac{1}{\alpha}} x),
\]
which implies
\[
\tilde{c}_0 t^{-\frac{d}{\alpha}} \leq t^{-\frac{d}{\alpha}} p(1, 0, t^{-\frac{1}{\alpha}} x) = p(t, 0, x), \quad \text{for small } t^{-\frac{1}{\alpha}} x.
\]
(1.4)

Combining (1.2), (1.3) and (1.4), we can see that the transition density $p$ satisfies (1.1). It is clear that $C(x, y) = \phi(\frac{x-y}{|x-y|})|x-y|^{-(d+\alpha)}$ is bounded between two positive numbers.

We say that a function $V$ on $\mathbb{R}^d$ belongs to the Kato class $K_{d, \alpha}$ if
\[
\lim_{t \downarrow 0} \sup_{x \in \mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} p(s, x, y) |V(y)| dy ds = 0,
\]
and we say that a signed measure $\mu$ on $\mathbb{R}^d$ belongs to the Kato class $K_{d, \alpha}$ if
\[
\lim_{t \downarrow 0} \sup_{x \in \mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} p(s, x, y) |\mu|(dy) ds = 0.
\]

Suppose $F$ is a function on $\mathbb{R}^d \times \mathbb{R}^d$. 4
Definition 1.2 We say $F$ belongs to $J_{d,\alpha}$ if $F$ is bounded, vanishing on the diagonal, and the function
\[ x \mapsto \int_{\mathbb{R}^d} \frac{|F(x, y)|}{|x - y|^{d+\alpha}} \, dy \]
belongs to $K_{d,\alpha}$.

For any $F \in J_{d,\alpha}$, we set
\[ A_t^F = \sum_{s \leq t} F(X_{s-}, X_s), \quad t > 0. \]

We can define a non-local Feynman-Kac semigroup as follows
\[ S_t^F f(x) = \mathbb{E}_x (e^{-A_t^F f(X_t)}), \]
where $f$ is a measurable function on $\mathbb{R}^d$. This semigroup was studied in [5] and [2].

Let $\tilde{F}(x, y) = F(y, x)$, for any $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$. In this paper, we always assume both $F$ and $\tilde{F} \in J_{d,\alpha}$.

Recently, sharp two-sided estimates of the density of the semigroup $\{S_t^F, t \geq 0\}$ were established in [6]. Under the assumption that $X$ is a symmetric $\alpha$-stable-like process, using a martingale argument and results from [2], the following result was established in [6]: Suppose that $F \in J_{d,\alpha}$ is a symmetric function, the semigroup $\{S_t^F, t \geq 0\}$ admits a density $q(t, x, y)$ with respect to $m$ and that $q$ is jointly continuous on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$. Furthermore, there exist positive constants $C_1, C_2, C_3$ and $C_4$ such that
\[ C_1 e^{-C_2 t^{\frac{1}{\alpha}} \left( 1 \wedge \frac{t^{\frac{1}{d}}}{|x - y|} \right)^{d+\alpha}} \leq q(t, x, y) \leq C_3 e^{C_4 t^{\frac{1}{\alpha}} \left( 1 \wedge \frac{t^{\frac{1}{d}}}{|x - y|} \right)^{d+\alpha}} \]
for all $(t, x, y) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$.

The question that we are going to address in this paper is the following: can we establish the same two-sided estimates for the density of the Feynman-Kac semigroup of nonsymmetric $\alpha$-stable-like process $X$ when $F \in J_{d,\alpha}$ is nonsymmetric. The proof of the above result in [6] can not be adapted to the case where neither $F$ nor $X$ is symmetric. It seems that, to answer the question, one has to use some new ideas. In this paper, we are going to tackle the question above by combining the generalization of an idea of [1], which was used to deal with the estimates of the density of continuous functionals of Brownian motion, with some results on discontinuous additive functionals.

The content of this paper is organized as follows. In section 2, we present some preliminary results on discontinuous additive functionals. In section 3, we establish the two-sided estimates on the density of Feynman-Kac semigroup under certain assumptions of $F(x, y)$. 

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2 Preliminary Results on Discontinuous Additive Functionals

For convenience, we use \( A_t \) to denote \( \sum_{s \leq t} F(X_{s-}, X_s) \) instead of \( A_t^F \). We have the following formulae for \( A_t^2 \).

\[
A_t^2 = 2 \int_0^t A_s \, dA_s - \int_0^t F(X_{s-}, X_s) \, dA_s,
\]

and

\[
A_t^2 = 2 \int_0^t (A_t - A_s) \, dA_s + \int_0^t F(X_{s-}, X_s) \, dA_s.
\]

The proof is straightforward.

In general, the formulae for \( A_t^n \) are given by the following theorem.

**Theorem 2.1**

\[
A_t^n = C_1^n \int_0^t A_s^{n-1} \, dA_s - C_2^n \int_0^t A_s^{n-2} F(X_{s-}, X_s) \, dA_s + C_3^n \int_0^t A_s^{n-3} F^2(X_{s-}, X_s) \, dA_s + \cdots
\]

\[
+ (-1)^{i-1} C_i^n \int_0^t A_s^{n-i} F^{i-1}(X_{s-}, X_s) \, dA_s + \cdots + (-1)^{n-1} C_n^n \int_0^t F^{n-1}(X_{s-}, X_s) \, dA_s,
\]

\[
A_t^n = C_1^n \int_0^t (A_t - A_s)^{n-1} \, dA_s + C_2^n \int_0^t (A_t - A_s)^{n-2} F(X_{s-}, X_s) \, dA_s
\]

\[
+ C_3^n \int_0^t (A_t - A_s)^{n-3} F^2(X_{s-}, X_s) \, dA_s + \cdots + C_i^n \int_0^t (A_t - A_s)^{n-i} F^{i-1}(X_{s-}, X_s) \, dA_s
\]

\[
+ \cdots + C_n^n \int_0^t F^{n-1}(X_{s-}, X_s) \, dA_s,
\]

where \( C_i^n = \frac{n!}{i!(n-i)!} \).

**Proof.** We use induction to show these two formulae for \( A_t^n \) hold for all \( n > 1 \). It is clear that they are true for \( n = 2 \). Suppose they hold for \( n \leq m - 1 \), we show they hold for \( n = m \).

It follows from the integration by parts formula,

\[
A_t^m = \int_0^t A_s \, dA_s^{m-1} + \int_0^t A_s^{m-1} \, dA_s
\]
where

\[
\int_0^t A_s \, dA_s^{m-1}
\]

\[
= \int_0^t (A_s - F(X_{s-}, X_s)) \, dA_s^{m-1}
\]

\[
= \int_0^t A_s \, dA_s^{m-1} - \int_0^t F(X_{s-}, X_s) \, dA_s^{m-1}
\]

\[
= \int_0^t A_s \left( \sum_{i=1}^{m-1} (-1)^{i-1} C_{m-1}^i A_s^{m-1-i} F^{i-1}(X_{s-}, X_s) \right) \, dA_s
\]

\[
- \int_0^t F(X_{s-}, X_s) \left( \sum_{j=1}^{m-1} (-1)^{j-1} C_{m-1}^j A_s^{m-1-j} F^{j-1}(X_{s-}, X_s) \right) \, dA_s
\]

(by the first formula for \(A_t^n\) when \(n = m - 1\))

\[
= \sum_{i=1}^{m-1} (-1)^{i-1} C_{m-1}^i \int_0^t A_s^{m-i} F^{i-1}(X_{s-}, X_s) \, dA_s
\]

\[
- \sum_{j=1}^{m-1} (-1)^{j-1} C_{m-1}^j \int_0^t A_s^{m-j} F^{j-1}(X_{s-}, X_s) \, dA_s
\]

( let \(j = i - 1\) )

\[
= \sum_{i=2}^{m-1} (-1)^{i-1} \left( C_{m-1}^i + C_{m-1}^{i-1} \right) \int_0^t A_s^{m-i} F^{i-1}(X_{s-}, X_s) \, dA_s
\]

\[
+ C_{m-1}^1 \int_0^t A_s^{m-1} \, dA_s - (-1)^{m-2} \int_0^t F^{m-1}(X_{s-}, X_s) \, dA_s
\]

\[
= \sum_{i=2}^{m-1} (-1)^{i-1} C_{m}^i \int_0^t A_s^{m-i} F^{i-1}(X_{s-}, X_s) \, dA_s
\]

\[
+ C_{m-1}^1 \int_0^t A_s^{m-1} \, dA_s - (-1)^m \int_0^t F^{m-1}(X_{s-}, X_s) \, dA_s
\]

( by \(C_{m-1}^i + C_{m-1}^{i-1} = C_m^i\) )

thus

\[
A_t^n = \int_0^t A_s \, dA_s^{m-1} + \int_0^t A_s^{m-1} \, dA_s = \sum_{i=1}^{m} (-1)^{i-1} C_{m}^i \int_0^t A_s^{m-i} F^{i-1}(X_{s-}, X_s) \, dA_s,
\]

i.e. the first formula for \(A_t^n\) holds for \(n = m\).
Now we go to the second formula for $A^n_t$, for $n = m$.

\[ C^1_m \int_0^t (A_t - A_s)^{m-1} dA_s \]

\[
= C^1_m \int_0^t \sum_{i=0}^{m-1} C^i_m A_t^i (-1)^{m-1-i} A_s^{m-1-i} dA_s \\
= \sum_{i=0}^{m-1} (-1)^{m-1-i} C^i_m (m-i) A_t^i \int_0^t A_s^{m-1-i} dA_s \\
= \sum_{i=0}^{m-1} (-1)^{m-1-i} C^i_m (m-i) A_t^i \int_0^t A_s^{m-1-i} dA_s \\
( \text{by } C^i_m = C^i_m (m-i) ) \\
= \sum_{i=0}^{m-1} (-1)^{m-1-i} C^i_m A_t^i \int_0^t A_s^{m-1-i} dA_s \\
( \text{by the first formula for } A_t^m \text{ when } n = m - i ) \\
= \sum_{i=0}^{m-1} (-1)^{m-1-i} C^i_m A_t^i A_t^{m-i} \\
+ \int_0^t \sum_{i=0}^{m-1} (-1)^{m-1-i} \sum_{k=2}^{m-i} (-1)^k C^i_m C^k_{m-i} A_t^i A_s^{m-i-k} F^{k-1}(X_s, X_s) dA_s,
\]

where

\[
\sum_{i=0}^{m-1} (-1)^{m-1-i} C^i_m A_t^i A_t^{m-i} = \left( \sum_{i=0}^{m-1} (-1)^{m-1-i} C^i_m A_t^i \right) A_t^m = (1) A_t^m,
\]

and

\[
\int_0^t \sum_{i=0}^{m-1} (-1)^{m-1-i} \sum_{k=2}^{m-i} (-1)^k C^i_m C^k_{m-i} A_t^i A_s^{m-i-k} F^{k-1}(X_s, X_s) dA_s \\
= \int_0^t \sum_{k=2}^{m} \sum_{i=0}^{m-k} (-1)^{m-k-i-1} C^k_{m-k} C^i_m A_t^i A_s^{m-k-i} F^{k-1}(X_s, X_s) dA_s \\
( \text{by } C^i_m C^k_{m-i} = C^k_{m-k} C^i_m \text{ and } (-1)^{m-1-i+k} = (-1)^{m-k-i-1} ) \\
= \sum_{k=2}^{m} C^k_{m} (-1)^{-1} \int_0^t (A_t - A_s)^{m-k} F^{k-1}(X_s, X_s) dA_s \\
= - \sum_{k=2}^{m} C^k_{m} \int_0^t (A_t - A_s)^{m-k} F^{k-1}(X_s, X_s) dA_s,
\]

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therefore
\[ C_1^m \int_0^t (A_t - A_s)^{m-1} dA_s = A_t^m - \sum_{k=2}^m C_k^m \int_0^t (A_t - A_s)^{m-k} F^{k-1}(X_{s-}, X_s) dA_s, \]
i.e. the second formula of \( A^n_t \) holds for \( n = m \).

3 Density of Feynman-Kac Semigroups Given by Discontinuous Additive Functionals

From now on we define \( q_0(t, x, y) = p(t, x, y) \) where \( p(t, x, y) \) is the transition density of \( \alpha \)-stable-like process \( X \) and satisfies (1.1). By the second formula for \( A^n_t \), we have for any bounded measurable function \( g \)
\[
\mathbb{E}_x[A^n_t g(X_t)] = \sum_{i=1}^n C_i^n \mathbb{E}_x[\int_0^t (A_t - A_s)^{n-i} F^{i-1}(X_{s-}, X_s) g(X_t) dA_s] \\
= \sum_{i=1}^n C_i^n \mathbb{E}_x[\int_0^t \mathbb{E}_{X_s} (A^{n-i}_{t-s} g(X_{t-s})) d(\sum_{r\leq s} F^i(X_{r-}, X_r))] \\
= \sum_{i=1}^n C_i^n \mathbb{E}_x[\int_0^t \int_{\mathbb{R}^d} 2C(X_s, y) F^i(X_s, y) \mathbb{E}_{y} (A^{n-i}_{t-s} g(X_{t-s})) m(dy)ds].
\]

We define \( q_n(t, x, z) \) as follows,
\[
q_n(t, x, z) = \sum_{i=1}^n C_i^n \int_0^t \int_{\mathbb{R}^d} p(s, x, w)m(dw) \int_{\mathbb{R}^d} \frac{2C(w, y) F^i(w, y)}{|w - y|^{d+\alpha}} q_{n-i}(t - s, y, z) m(dy)ds.
\]

Then by induction, we can show that for any \( n > 0 \)
\[
\int_{\mathbb{R}^d} q_n(t, x, z) g(z) m(dz) = \mathbb{E}_x[A^n_t g(X_t)]
\]
and
\[
\mathbb{E}_x[A^n_t g(X_t)] = \sum_{i=1}^n C_i^n \mathbb{E}_x[\int_0^t \int_{\mathbb{R}^d} 2C(X_s, y) F^i(X_s, y) \frac{q_{n-i}(t - s, y, z) g(z) m(dz) m(dy)}{|X_s - y|^{d+\alpha}} ds] \\
= \sum_{i=1}^n C_i^n \int_0^t \int_{\mathbb{R}^d} p(s, x, w)m(dw) \int_{\mathbb{R}^d} \frac{2C(w, y) F^i(w, y)}{|w - y|^{d+\alpha}} \int_{\mathbb{R}^d} q_{n-i}(t - s, y, z) g(z) m(dz) m(dy)ds.
\]

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We assume that there exist positive constants $C$, $L$, $M_0$ and $M$ such that $|2C(x,y)| \leq C$, $|F(x,y)| \leq \frac{L}{2}$ and $0 < M_0 \leq M(y) \leq M$ where $m(dy) = M(y)dy$. Define $\overline{F}(w,y) = |F(w,y)| + |F(y,w)|$, which is symmetric and satisfies $|\overline{F}(w,y)| \leq L$. Define $\overline{p}(t,x,y) = p(t,x,y) + p(t,y,x)$. Then $\overline{p}(t,x,y)$ is symmetric and satisfies

$$2M_1t^{-\frac{d}{\alpha}} \left(1 \wedge \frac{t\frac{1}{\alpha}}{|x-y|}\right)^{d+\alpha} \leq \overline{p}(t,x,y) \leq 2M_2t^{-\frac{d}{\alpha}} \left(1 \wedge \frac{t\frac{1}{\alpha}}{|x-y|}\right)^{d+\alpha}, \forall (t,x,y) \in (0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d.$$

Denote $(\int_{\mathbb{R}^d} \overline{F}(w,y) \, dw)(dy)$ by $\mu(dw)$ and let $C_t = \sup_{x \in \mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} \overline{p}(s,x,w) \mu(dw)ds$. Then $C_t \downarrow 0$ as $t \downarrow 0$. It is clear that there exist two positive constants $D_1$ and $D_2$ such that $D_1 \leq \int_{\mathbb{R}^d} \overline{p}(t,x,y) \, m(dy) \leq D_2$, as $\overline{p}(t,x,y)$ is comparable to $p(t,x,y)$. Let $\overline{q}_0(t,x,z) = \overline{p}(s,x,z)$ and define $\overline{q}_n(t,x,z)$ by

$$\overline{q}_n(t,x,z) = \sum_{i=1}^n C_i^\alpha \int_0^t \int_{\mathbb{R}^d} \overline{p}(s,x,w) m(dw) \int_{\mathbb{R}^d} \frac{\overline{C}F(w,y)}{|w-y|^{d+\alpha}} \overline{q}_{n-i}(t-s,y,z) \, m(dy)ds.$$

We can see that $|\overline{q}_n(t,x,z)| \leq \overline{q}_n(t,x,z)$.

Before we move on to the main results, two lemmas are needed.

**Lemma 3.1** For any two positive constants $K < 1$ and $L$, there exists a constant $C_0(K,L)$ which depends on $K$ and $L$, such that

$$K^{n-1} + K^{n-2} \frac{L}{2!} + K^{n-3} \frac{L^2}{3!} + \cdots + K^{n-i} \frac{L^{i-1}}{i!} + \cdots + \frac{L^{n-1}}{n!} \leq C_0(K,L)K^n, \text{ for all } n > 0. \ (3.1)$$

**Proof.** Use the fact that there exists $i_0 \geq 0$, such that

$$\frac{L^{l-1}}{l!} \leq \left(\frac{K}{2}\right)^l, \text{ for } l \geq i_0.$$

\[ \square \]

**Remark 3.2** For any given $K$ and $L$, we can choose a small $t_0$ so that for a given constant $M_1$, $C_tM_1C_0(K,L) \leq 1$ for $0 \leq t \leq t_0$. Thus

$$C_t M_1 \left( K^{n-1} + K^{n-2} \frac{L}{2!} + K^{n-3} \frac{L^2}{3!} + \cdots + \frac{L^{n-1}}{n!} \right) \leq C_t M_1 C_0(K,L) K^n \leq K^n.$$

**Lemma 3.3** $\overline{q}_n(t,x,y)$ is symmetric in $x$ and $y$. 

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Proof. We know that

\[
\bar{q}_n(t, z, x) = \sum_{i_1=1}^n C_{i_1}^{t_1} \int_0^t \int_{\mathbb{R}^d} \mathcal{P}(s_1, z, w_1) m(dw_1) \int_{\mathbb{R}^d} \frac{\bar{C}F_i^{(1)}(w_1, y_1)}{|w_1 - y_1|^{d+\alpha}} \pi_{n-1}(t - s_1, y_1, x) m(dy_1) ds_1
\]

\[
= \sum_{i_1=1}^n C_{i_1}^{t_1} \sum_{i_2=1}^{n-1} C_{i_2}^{t_2} \int_0^t \int_{\mathbb{R}^d} \mathcal{P}(s_1, z, w_1) m(dw_1) \int_{\mathbb{R}^d} \frac{\bar{C}F_i^{(1)}(w_1, y_1)}{|w_1 - y_1|^{d+\alpha}} m(dy_1) ds_1 \int_0^{t-s_1} \int_{\mathbb{R}^d} \mathcal{P}(s_2, y_1, w_2) m(dw_2) \int_{\mathbb{R}^d} \frac{\bar{C}F_i^{(2)}(w_2, y_2)}{|w_2 - y_2|^{d+\alpha}} m(dy_2) ds_2 \cdots
\]

\[
= \sum_{i_1+i_2+\ldots+i_k=n} C_{i_1}^{t_1} C_{i_2}^{t_2} \cdots C_{i_k}^{t_k} \int_0^t \int_0^{t-s_1} \int_0^{t-s_1-\ldots-s_{k-1}} \int_{\mathbb{R}^d} \mathcal{P}(s_1, z, w_1) \int_{\mathbb{R}^d} \frac{\bar{C}F_i^{(1)}(w_1, y_1)}{|w_1 - y_1|^{d+\alpha}} \cdots \mathcal{P}(s_k, y_{k-1}, w_k) m(dy_1) \cdots m(dy_k)
\]

Put \(t - s_1 - \cdots - s_k = \tilde{s}_1, s_k = \tilde{s}_2, \ldots, s_l = \tilde{s}_{k+2-l}, \ldots, s_2 = \tilde{s}_l\). It is easy to see the absolute value of the Jacobian of this transformation is 1. Let \(y_k = \tilde{w}_l, \ldots, y_l = \tilde{w}_{k-l+1}, \ldots, y_1 = \tilde{w}_k, w_k = \tilde{y}_l, \ldots, w_l = \tilde{y}_{k-l+1}, \ldots, w_1 = \tilde{y}_1\) and \(j_k = i_1, \ldots, j_l = i_{k-l+1}, \ldots, j_1 = i_k\).

Thus the above equality becomes

\[
\bar{q}_n(t, z, x) = \sum_{j_1+j_2+\ldots+j_h=n} C_{j_1}^{t_1} \cdots C_{j_h}^{t_h} \int_0^{t-\tilde{s}_1-\cdots-\tilde{s}_{k-1}} \int_0^{t-\tilde{s}_1-\cdots-\tilde{s}_{k-2}} \cdots \int_0^t \int_{\mathbb{R}^d} \mathcal{P}(t - \tilde{s}_1 - \cdots - \tilde{s}_k, z, \tilde{y}_k) \int_{\mathbb{R}^d} \frac{\bar{C}F_i^{(1)}(\tilde{y}_k, \tilde{w}_k)}{|\tilde{y}_k - \tilde{w}_k|^{d+\alpha}} \mathcal{P}(\tilde{s}_k, \tilde{w}_k, \tilde{y}_{k-1})
\]

\[
= \sum_{j_1+j_2+\ldots+j_h=n} C_{j_1}^{t_1} \cdots C_{j_h}^{t_h} \int_{\mathbb{R}^d} \frac{\bar{C}F_i^{(1)}(\tilde{y}_1, \tilde{w}_1)}{|\tilde{y}_1 - \tilde{w}_1|^{d+\alpha}} \cdots \mathcal{P}(\tilde{s}_2, \tilde{w}_2, \tilde{y}_1) \int_{\mathbb{R}^d} \frac{\bar{C}F_i^{(2)}(\tilde{y}_2, \tilde{y}_1)}{|\tilde{y}_2 - \tilde{y}_1|^{d+\alpha}} \cdots \mathcal{P}(\tilde{s}_1, \tilde{w}_1, x) \int_{\mathbb{R}^d} \frac{\bar{C}F_i^{(1)}(\tilde{y}_1, \tilde{w}_1)}{|\tilde{y}_1 - \tilde{w}_1|^{d+\alpha}} \cdots \mathcal{P}(\tilde{s}_1, \tilde{w}_1, x) d\tilde{s}_1 \ldots d\tilde{s}_1 m(d\tilde{y}_1) \cdots m(d\tilde{y}_1) m(d\tilde{w}_1) \cdots m(d\tilde{w}_1).
\]

Rearranging the components of the integrand and using the fact that \(\bar{F}(x, y)\) and \(\mathcal{P}(t, x, y)\) are
symmetric in $x$ and $y$, it is easy to see that the above expression for $\overline{q}_n(t, z, x)$ is equal to $\overline{q}_n(t, x, z)$. 

In the proof of the following theorem, we use an idea similar to that used in [1] for Brownian motions.

**Theorem 3.4** There exist two positive constants $K < 1$ and $M$, and there exists $t_0 > 0$ such that $0 < t \leq t_0$,

$$\overline{q}_n(t, x, z) \leq n!MK^n t^{-\frac{d}{\alpha}}, \text{ for all } n,$$

and

$$\int_0^t \int_{\mathbb{R}^d} \overline{q}_n(s, x, z) \mu(dz) ds \leq C t n! K^n, \text{ for all } n.$$

**Proof.** It is clear that when $n = 0$, (3.2) and (3.3) hold. We assume they hold for $n \leq m - 1$, and consider the case $n = m$. Writing $\overline{q}_m(t, x, y)$ into two terms in the following way:

$$\overline{q}_m(t, x, y) = \sum_{i=1}^m C^i_m \int_0^{\frac{t}{2}} \int_{\mathbb{R}^d} \overline{p}(s, x, w) m(dw) \int_{\mathbb{R}^d} \frac{CF^i(w, y)}{|w - y|^{d+\alpha}} \overline{q}_{m-i}(t - s, y, z) m(dy) ds$$

$$+ \sum_{i=1}^m C^i_m \int_{\frac{t}{2}}^t \int_{\mathbb{R}^d} \overline{p}(s, x, w) m(dw) \int_{\mathbb{R}^d} \frac{CF^i(w, y)}{|w - y|^{d+\alpha}} \overline{q}_{m-i}(t - s, y, z) m(dy) ds.$$

Since (3.2) and (3.3) hold for $n \leq m - 1$, we have

$$\sum_{i=1}^m C^i_m \int_0^{\frac{t}{2}} \int_{\mathbb{R}^d} \overline{p}(s, x, w) m(dw) \int_{\mathbb{R}^d} \frac{CF^i(w, y)}{|w - y|^{d+\alpha}} \overline{q}_{m-i}(t - s, y, z) m(dy) ds$$

$$\leq \sum_{i=1}^m C^i_m M^{M^2 C L_i^{-1}} (m - i)! K^{m-i} \left( \frac{t}{2} \right)^{-\frac{d}{\alpha}} \int_0^{\frac{t}{2}} \int_{\mathbb{R}^d} \overline{p}(s, x, w) \mu(dw) ds$$

$$\leq \sum_{i=1}^m C^i_m M^{M^2 C L_i^{-1}} (m - i)! K^{m-i} \left( \frac{t}{2} \right)^{-\frac{d}{\alpha}}.$$ 

Similarly,

$$\sum_{i=1}^m C^i_m \int_{\frac{t}{2}}^t \int_{\mathbb{R}^d} \overline{p}(s, x, w) m(dw) \int_{\mathbb{R}^d} \frac{CF^i(w, y)}{|w - y|^{d+\alpha}} \overline{q}_{m-i}(t - s, y, z) m(dy) ds$$

$$\leq \sum_{i=1}^m C^i_m M \left( \frac{t}{2} \right)^{-\frac{d}{\alpha}} \int_{\frac{t}{2}}^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{CF^i(w, y)}{|w - y|^{d+\alpha}} m(dw) \overline{q}_{m-i}(t - s, y, z) m(dy) ds$$

$$\leq \sum_{i=1}^m C^i_m M \left( \frac{t}{2} \right)^{-\frac{d}{\alpha}} C L_i^{-1} M^2 \int_{\frac{t}{2}}^t \int_{\mathbb{R}^d} \overline{q}_{m-i}(t - s, y, z) \mu(dy) ds.$$ 

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\[ \leq \sum_{i=1}^{m} C_m^i M \left( \frac{t}{2} \right)^{-\frac{d}{\alpha}} CL_i^{-1} M^2 C_t (m-i)! K^{m-i} \]

(by symmetry, \( \varpi_{m-i}(t-s,y,z) = \varpi_{m-i}(t-s,z,y) \) and (3.3))

\[ = \sum_{i=1}^{m} C_t C_m^i M^{1+i} CL_i^{-1} (m-i)! K^{m-i} \left( \frac{t}{2} \right)^{-\frac{d}{\alpha}}. \]

Therefore,

\[ \varpi_m(t,x,z) \leq \sum_{i=1}^{m} C_t C_m^i 2^{1+\frac{d}{\alpha}} M^{1+i} CL_i^{-1} (m-i)! K^{m-i} t^{-\frac{d}{\alpha}} \]

\[ = m! 2^{1+\frac{d}{\alpha}} M^{1+i} CL_i^{-1} (m-i)! C_t \left( \sum_{i=1}^{m} K^{m-i} \frac{L_i^{-1}}{i!} \right). \]

Let \( M_1 = 2^{1+\frac{d}{\alpha}} M^{1+i} CL_i^{-1} \). Then by Remark 3.2, we can choose a small \( t_0 \) such that for \( 0 < t \leq t_0 \),

\[ C_t M_1 \left( \sum_{i=1}^{m} K^{m-i} \frac{L_i^{-1}}{i!} \right) \leq K^m, \text{ for any } m. \]

Thus

\[ \varpi_m(t,x,y) \leq m! MK^{m-i} t^{-\frac{d}{\alpha}}, \]

i.e. (3.2) holds for \( n = m \).

Now we show (3.3) holds for \( n = m \).

\[ \int_0^t \int_{\mathbb{R}^d} \varpi_m(s,x,z) \mu(dz) ds \]

\[ = \int_0^t \int_{\mathbb{R}^d} \left( \sum_{i=1}^{m} C_m^i \int_0^s \varpi(u,x,w) m(dw) \right) \int_{\mathbb{R}^d} \frac{CF^i(w,y)}{|w-y|^{d+\alpha}} \varpi_{m-i}(s-u,y,z) m(dy) du \mu(dz) ds. \]

Let \( s-u=v \), we get

\[ \int_0^t \int_{\mathbb{R}^d} \varpi_m(s,x,z) \mu(dz) ds \]

\[ = \int_0^t \int_{\mathbb{R}^d} \left( \sum_{i=1}^{m} C_m^i \int_0^t \varpi(u,x,w) m(dw) \right) \int_{\mathbb{R}^d} \frac{CF^i(w,y)}{|w-y|^{d+\alpha}} \varpi_{m-i}(v,y,z) m(dz) dv \ m(dy) du \]

\[ = \sum_{i=1}^{m} C_m^i \int_0^t \int_{\mathbb{R}^d} \varpi(u,x,w) \left( \int_0^{t-u} \varpi_{m-i}(v,y,z) m(dz) dv \right) \int_{\mathbb{R}^d} \frac{CF^i(w,y)}{|w-y|^{d+\alpha}} m(dy) m(dw) du \]

\[ \leq \sum_{i=1}^{m} C_m^i \int_0^t \int_{\mathbb{R}^d} \varpi(u,x,w) C_t (m-i)! K^{m-i} \int_{\mathbb{R}^d} \frac{CF^i(w,y)}{|w-y|^{d+\alpha}} m(dy) m(dw) du \]

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\[
\sum_{i=1}^{m} C_m^i C_t (m-i)! K^{m-i} \int_0^t \int_{\mathbb{R}^d} \overline{p}(u, x, w) \int_{\mathbb{R}^d} \frac{CF^i(w, y)}{|w-y|^{d+\alpha}} m(dy) m(dw) du \leq \sum_{i=1}^{m} C_m^i C_t (m-i)! K^{m-i} \overline{C} L^{i-1} t^2 \int_0^t \int_{\mathbb{R}^d} \overline{p}(u, x, w) \mu(dw) du \leq C_t C_t \overline{M}^2 C_m! \left( \sum_{i=1}^{m} K^{m-i} \frac{L^{i-1}}{i!} \right).
\]

It is clear that for the previous \( t_0 \), when \( 0 < t \leq t_0 \),

\[
C_t \overline{M}^2 C \left( \sum_{i=1}^{m} K^{m-i} \frac{L^{i-1}}{i!} \right) \leq K^m.
\]

Thus

\[
C_t C_t \overline{M}^2 C_m! \left( \sum_{i=1}^{m} K^{m-i} \frac{L^{i-1}}{i!} \right) \leq C_t m! K^m,
\]
i.e.

\[
\int_0^t \int_{\mathbb{R}^d} \overline{q}_m(s, x, z) \mu(dz) ds \leq C_t m! K^m.
\]

Therefore (3.3) holds for \( n = m \).

By the above theorem, we have for \( 0 < t \leq t_0 \),

\[
\sum_{n=0}^{\infty} \frac{q_n(t, x, z)}{n!} \leq \sum_{n=0}^{\infty} MK^n t^{-\frac{d}{\alpha}} = M \frac{1}{1-K^{-\frac{d}{\alpha}}}.
\]

Since \( |q_n(t, x, z)| \leq \overline{q}_n(t, x, z) \), \( \sum_{n=0}^{\infty} \frac{q_n(t, x, z)}{n!} \) is uniformly convergent on \( [\epsilon, t_0] \times \mathbb{R}^d \times \mathbb{R}^d \), for any \( \epsilon > 0 \). Let \( q(t, x, z) = \sum_{n=0}^{\infty} (-1)^n \frac{q_n(t, x, z)}{n!} \). Then \( q(t, x, z) \) is well defined on \( (0, t_0] \times \mathbb{R}^d \times \mathbb{R}^d \).

Thus we have the following properties for \( q(t, x, z) \),

**Proposition 3.5** (i) \( \int_{\mathbb{R}^d} q(t, x, z) g(z) m(dz) = E_x[e^{-A_t} g(X_t)] \), for any \( g \) bounded measurable and any \( t > 0 \), (ii) \( \int_{\mathbb{R}^d} q(t, x, y) q(s, y, z) m(dy) = q(t+s, x, z) \), for any \( t, s > 0 \).

In the following, we estimate \( q(t, x, z) \) from above and from below.

It is clear that for any \( (t, x, y) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \), there exists a positive constant \( D_2 \) such that \( \int_{\mathbb{R}^d} \overline{p}(t, x, y) m(dy) \leq D_2 \). For this \( D_2 \), the positive constants \( M_1, L \) and \( K < 1 \) given in (1.1), Remark 3.2 and Theorem 3.4, and \( \overline{C} \) which is the upper bound of \( |2C(x, y)| \), there exists a large enough positive integer \( k \) such that \( \frac{L}{(1-\frac{1}{10} - 2^{-\frac{1}{2}})^{d+\alpha} 2^k \overline{C} D^2_2} \leq \frac{1}{8} M_1 \).

Now instead of considering

\[
\overline{q}_1(t, x, z) = \int_0^t \int_{\mathbb{R}^d} \overline{p}(s, x, w) m(dw) \int_{\mathbb{R}^d} \frac{CF(w, y)}{|w-y|^{d+\alpha}} \overline{p}(t-s, y, z) m(dy) ds,
\]

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we consider
\[
\overline{q}_{1,k}(t, x, z) = \frac{\overline{q}_1(t, x, z)}{k} = \int_0^t \int_{\mathbb{R}^d} \overline{p}(s, x, w) m(dw) \int_{\mathbb{R}^d} \frac{C\overline{T}(w, y)}{|w - y|^{d+\alpha}} \overline{p}(t - s, y, z) m(dy) ds.
\]

We have the following theorem

**Theorem 3.6** There exists a small \( t_1 \) such that when \( 0 < t \leq t_1 \),
\[
\overline{q}_{1,k}(t, x, z) \leq \frac{1}{2} p(t, x, z), \quad \forall (x, z) \in \mathbb{R}^d \times \mathbb{R}^d.
\]  

**Proof.** We write \( \overline{q}_{1,k}(t, x, z) \) into two terms
\[
\overline{q}_{1,k}(t, x, z) = \int_0^{t/2} \int_{\mathbb{R}^d} \overline{p}(s, x, w) m(dw) \int_{\mathbb{R}^d} \frac{C\overline{T}(w, y)}{|w - y|^{d+\alpha}} \overline{p}(t - s, y, z) m(dy) ds
\]
\[+ \int^t_{t/2} \int_{\mathbb{R}^d} \overline{p}(s, x, w) m(dw) \int_{\mathbb{R}^d} \frac{C\overline{T}(w, y)}{|w - y|^{d+\alpha}} \overline{p}(t - s, y, z) m(dy) ds.
\]

First we look at the first term. There are two cases.

**Case 1.** When \(|x - z| \leq t_1^1\alpha\),
\[
\int_0^{t/2} \int_{\mathbb{R}^d} \overline{p}(s, x, w) m(dw) \int_{\mathbb{R}^d} \frac{C\overline{T}(w, y)}{|w - y|^{d+\alpha}} \overline{p}(t - s, y, z) m(dy) ds
\]
\[\leq \int_0^{t/2} \int_{\mathbb{R}^d} \overline{p}(s, x, w) m(dw) \int_{\mathbb{R}^d} \frac{C\overline{T}(w, y)}{|w - y|^{d+\alpha}} 2M_2(t - s)^{-d} m(dy) ds
\]
\[\leq 2M_2\overline{M} C^{-1} \int_0^{t/2} \int_{\mathbb{R}^d} \overline{p}(s, x, w) (t - s)^{-d} \mu(dw) ds
\]
\[\leq C t M_2\overline{M} C^{-1} 2^{1+d} t^{-d}.
\]

**Case 2.** When \(|x - z| \geq t_1^1\). Let \( B_1 = \{y \in \mathbb{R}^d | |y - z| \geq \frac{1}{10} |x - z|\}, B_2 = \{w \in \mathbb{R}^d | |w - x| \geq 2^{-\frac{d}{2}} |x - z|\} \) and \( B_3 = \{(w, y) \in \mathbb{R}^d \times \mathbb{R}^d | |y - z| < \frac{1}{10} |x - z|, |w - x| < 2^{-\frac{d}{2}} |x - z|\} \). On \( B_3 \), we have
\[ |w - y| \geq (1 - \frac{1}{10} - 2^{-\frac{1}{2}})|x - z|. \]

\[
\int_0^t \int_{\mathbb{R}^d} \overline{p}(s, x, w) m(dw) \int_{\mathbb{R}^d} \frac{C F(w, y)}{|w - y|^{d+\alpha}} \overline{p}(t - s, y, z) m(dy) ds \leq \\
\int_0^t \int_{\mathbb{R}^d} \overline{p}(s, x, w) m(dw) \int_{\mathbb{R}^d} \frac{C F(w, y)}{|w - y|^{d+\alpha}} \overline{p}(t - s, y, z) 1_{B_1}(y) m(dy) ds \\
+ \int_0^t \int_{\mathbb{R}^d} \overline{p}(s, x, w) m(dw) \int_{\mathbb{R}^d} \frac{C F(w, y)}{|w - y|^{d+\alpha}} \overline{p}(t - s, y, z) 1_{B_2}(w) m(dy) ds \\
+ \int_0^t \int_{\mathbb{R}^d} \overline{p}(s, x, w) m(dw) \int_{\mathbb{R}^d} \frac{C F(w, y)}{|w - y|^{d+\alpha}} \overline{p}(t - s, y, z) 1_{B_3}(w, y) m(dy) ds \leq \\
\int_0^t \int_{\mathbb{R}^d} \overline{p}(s, x, w) m(dw) \int_{\mathbb{R}^d} \frac{C F(w, y)}{|w - y|^{d+\alpha}} 2M_2M_10^{d+\alpha} \frac{(t - s)}{|x - z|^{d+\alpha}} 1_{B_1}(y) dy ds \\
+ \int_0^t \int_{\mathbb{R}^d} 2M_2M_1 2^{\frac{1}{2}(d+\alpha)} \frac{s}{|x - z|^{d+\alpha}} dw \int_{\mathbb{R}^d} \frac{C F(w, y)}{|w - y|^{d+\alpha}} \overline{p}(t - s, y, z) 1_{B_2}(w) dy ds \\
+ \frac{L}{(1 - \frac{1}{10} - 2^{-\frac{1}{2}})^{d+\alpha} k} \int_0^t \int_{\mathbb{R}^d} \overline{p}(s, x, w) m(dw) \int_{\mathbb{R}^d} \frac{1}{|x - z|^{d+\alpha}} \overline{p}(t - s, y, z) 1_{B_3}(w, y) m(dy) ds \leq \\
C_t 2M_2M_1 2^{-\frac{1}{2}(d+\alpha)} \frac{t}{|x - z|^{d+\alpha}} \\
+C_t 2M_2M_1^2 \frac{1}{k} 2^{\frac{3}{2}(d+\alpha)} \frac{t}{|x - z|^{d+\alpha}} \\
+ \frac{L}{(1 - \frac{1}{10} - 2^{-\frac{1}{2}})^{d+\alpha} k} C D_2^2 \frac{t}{|x - z|^{d+\alpha}} \\
( \text{by } \int_{\mathbb{R}^d} \overline{p}(s, x, w) m(dw) \leq D_2 \text{ and } \int_{\mathbb{R}^d} \overline{p}(t - s, y, z) m(dy) \leq D_2 ) \\
\leq C_t M_2M_1^2 \frac{1}{k} (2 \cdot 10^{d+\alpha} + 2^{\frac{3}{2}(d+\alpha)}) \frac{t}{|x - z|^{d+\alpha}} + \frac{1}{8} p(t, x, z). \]

Since \( C_t \downarrow 0 \) as \( t \downarrow 0 \), then for both case 1 and case 2, we can find a small \( t_{11} \) such that when \( 0 < t \leq t_{11} \),

\[
\int_0^t \int_{\mathbb{R}^d} \overline{p}(s, x, w) m(dw) \int_{\mathbb{R}^d} \frac{C F(w, y)}{|w - y|^{d+\alpha}} \overline{p}(t - s, y, z) m(dy) ds \leq \frac{1}{4} p(t, x, z). \]

For the second term of \( \overline{q}_{1,k}(t, x, z) \):

\[
\int_0^t \int_{\mathbb{R}^d} \overline{p}(s, x, w) m(dw) \int_{\mathbb{R}^d} \frac{C F(w, y)}{|w - y|^{d+\alpha}} \overline{p}(t - s, y, z) m(dy) ds. \]
Letting $t - s = \tilde{s}$, the second term becomes

$$
\int_0^{\frac{t}{2}} \int_{\mathbb{R}^d} \bar{p}(t - \tilde{s}, x, w)m(dw) \int_{\mathbb{R}^d} \frac{C F^{(w,y)}_k}{|w - y|^{d+\alpha}} \bar{p}(\tilde{s}, y, z) m(dy) d\tilde{s}.
$$

There are two cases.

Case a. When $|x - z| \leq t^\frac{1}{2}$,

$$
\int_0^{\frac{t}{2}} \int_{\mathbb{R}^d} \bar{p}(t - \tilde{s}, x, w)m(dw) \int_{\mathbb{R}^d} \frac{C F^{(w,y)}_k}{|w - y|^{d+\alpha}} \bar{p}(\tilde{s}, y, z) m(dy) d\tilde{s} \\
\leq \int_0^{\frac{t}{2}} \int_{\mathbb{R}^d} 2M_2(t - \tilde{s})^{-\frac{d}{2}} m(dw) \int_{\mathbb{R}^d} \frac{C F^{(w,y)}_k}{|w - y|^{d+\alpha}} \bar{p}(\tilde{s}, y, z) m(dy) d\tilde{s} \\
\leq M_22M_2\overline{C}_K^{(a)} \int_0^{\frac{t}{2}} \int_{\mathbb{R}^d} \bar{p}(\tilde{s}, y, z)(t - \tilde{s})^{-\frac{d}{2}} \mu(dy) ds \\
\leq C t2M_2\overline{M}^2\overline{C}_K^{(a)} \int_0^{\frac{t}{2}} \int_{\mathbb{R}^d} \bar{p}(\tilde{s}, y, z)(t - \tilde{s})^{-\frac{d}{2}} (\text{by symmetry of } \bar{p}(\tilde{s}, y, z)).
$$

Case b. When $|x - z| \geq t^\frac{1}{2}$. Let $\tilde{B}_1 = \{ y \in \mathbb{R}^d | |y - z| \geq t^\frac{1}{10} |x - z| \}$, $\tilde{B}_2 = \{ w \in \mathbb{R}^d | |w - x| \geq 2^{-\frac{1}{2}} |x - z| \}$ and $\tilde{B}_3 = \{ (w, y) \in \mathbb{R}^d \times \mathbb{R}^d | |y - z| < t^\frac{1}{10} |x - z|, |w - x| < 2^{-\frac{1}{2}} |x - z| \}$. On $\tilde{B}_3$, we have $|w - y| \geq (1 - \frac{1}{10} - 2^{-\frac{1}{2}}) |x - z|$. 

$$
\int_0^{\frac{t}{2}} \int_{\mathbb{R}^d} \bar{p}(t - \tilde{s}, x, w)m(dw) \int_{\mathbb{R}^d} \frac{C F^{(w,y)}_k}{|w - y|^{d+\alpha}} \bar{p}(\tilde{s}, y, z) m(dy) d\tilde{s} \\
\leq \int_0^{\frac{t}{2}} \int_{\mathbb{R}^d} \bar{p}(t - \tilde{s}, x, w)m(dw) \int_{\mathbb{R}^d} \frac{C F^{(w,y)}_k}{|w - y|^{d+\alpha}} \bar{p}(\tilde{s}, y, z) 1_{\tilde{B}_1}(y) m(dy) d\tilde{s} \\
+ \int_0^{\frac{t}{2}} \int_{\mathbb{R}^d} \bar{p}(t - \tilde{s}, x, w)m(dw) \int_{\mathbb{R}^d} \frac{C F^{(w,y)}_k}{|w - y|^{d+\alpha}} \bar{p}(\tilde{s}, y, z) 1_{\tilde{B}_2}(w) m(dy) d\tilde{s} \\
+ \int_0^{\frac{t}{2}} \int_{\mathbb{R}^d} \bar{p}(t - \tilde{s}, x, w)m(dw) \int_{\mathbb{R}^d} \frac{C F^{(w,y)}_k}{|w - y|^{d+\alpha}} \bar{p}(\tilde{s}, y, z) 1_{\tilde{B}_3}(w, y) m(dy) d\tilde{s} \\
\leq \int_0^{\frac{t}{2}} \int_{\mathbb{R}^d} \bar{p}(t - \tilde{s}, x, w)m(dw) \int_{\mathbb{R}^d} \frac{C F^{(w,y)}_k}{|w - y|^{d+\alpha}} 2M_2\overline{M}^210^{d+\alpha} \frac{\tilde{s}}{|x - z|^{d+\alpha}} 1_{\tilde{B}_1}(y) dy d\tilde{s} \\
+ \int_0^{\frac{t}{2}} \int_{\mathbb{R}^d} 2M_2\overline{M}^22^{\frac{1}{2}}10^{d+\alpha} \frac{(t - \tilde{s})}{|x - z|^{d+\alpha}} dw \int_{\mathbb{R}^d} \frac{C F^{(w,y)}_k}{|w - y|^{d+\alpha}} \bar{p}(\tilde{s}, y, z) 1_{\tilde{B}_2}(w) dy d\tilde{s} \\
+ \frac{L}{(1 - \frac{1}{10} - 2^{-\frac{1}{2}})^{d+\alpha}} \int_0^{\frac{t}{2}} \int_{\mathbb{R}^d} \bar{p}(t - \tilde{s}, x, w)m(dw) \int_{\mathbb{R}^d} \frac{1}{|x - z|^{d+\alpha}} \bar{p}(\tilde{s}, y, z) 1_{\tilde{B}_3}(w, y) m(dy) d\tilde{s}
$$

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\[\leq C_t 2M_2 \mathbb{M}^2 C \frac{1}{k} 10^{d + \alpha} \frac{t}{|x - z|^{d + \alpha}} + C_t 2M_2 \mathbb{M}^2 C \frac{1}{k} 2^{\frac{1}{2} (d + \alpha)} \frac{t}{|x - z|^{d + \alpha}} + \frac{L}{(1 - \frac{1}{10} - 2^{-\frac{1}{2}})^{d + \alpha} k} \frac{1}{\mathcal{C} D_2^2 |x - z|^{d + \alpha}} \]

(by \(\int_{\mathbb{R}^d} \mathbb{P}(t - \bar{s}, x, w) m(dw) \leq D_2\) and \(\int_{\mathbb{R}^d} \mathbb{P}(\bar{s}, y, z) m(dy) \leq D_2\))

\[\leq C_t 2M_2 \mathbb{M}^2 C \frac{1}{k} (10^{d + \alpha} + 2^{\frac{1}{2} (d + \alpha)}) \frac{t}{|x - z|^{d + \alpha}} + \frac{1}{8} p(t, x, z).\]

Since \(C_t \downarrow 0\) as \(t \downarrow 0\), then for both case a and case b, we can find a small \(t_{12}\) such that when \(0 < t \leq t_{12}\),

\[\int_0^t \int_{\mathbb{R}^d} \mathbb{P}(t - \bar{s}, x, w) m(dw) \int_{\mathbb{R}^d} \frac{C \mathbb{E}[(w, y)]}{|w - y|^{d + \alpha}} \mathbb{P}(\bar{s}, y, z) m(dy) d\bar{s} \leq \frac{1}{4} p(t, x, z),\]

i.e.

\[\int_0^t \int_{\mathbb{R}^d} \mathbb{P}(s, x, w) m(dw) \int_{\mathbb{R}^d} \frac{C \mathbb{E}[(w, y)]}{|w - y|^{d + \alpha}} \mathbb{P}(t - s, y, z) m(dy) ds \leq \frac{1}{4} p(t, x, z).\]

Let \(t_1 = \min(t_{11}, t_{12})\). Then when \(0 < t \leq t_1\),

\[\bar{q}_{1,k}(t, x, z) \leq \frac{1}{2} p(t, x, z), \text{ } \forall (x, z) \in \mathbb{R}^d \times \mathbb{R}^d.\]

\[\square\]

Define \(q_{1,k}(t, x, z) = \frac{q_1(t, x, z)}{k}\). It is clear that \(|q_{1,k}(t, x, z)| \leq \bar{q}_{1,k}(t, x, z)\). Theorem 3.6 implies that when \(0 < t \leq t_1\),

\[p(t, x, z) - q_{1,k}(t, x, z) \geq p(t, x, z) - \bar{q}_{1,k}(t, x, z) \geq \frac{1}{2} p(t, x, z).\]

We know \(\int_{\mathbb{R}^d} q_{1,k}(t, x, z) g(z) m(dz) = \mathbb{E}_x [\frac{A_t}{k} g(X_t)]\), for any \(g\) measurable. Since \(1 - \frac{A_t}{k} \leq e^{-\frac{A_t}{k}}\), we have

\[\frac{1}{|B_r|} \mathbb{E}_x [(1 - \frac{A_t}{k}) 1_{B_r}(X_t)] \leq \frac{1}{|B_r|} \mathbb{E}_x [e^{-\frac{A_t}{k}} 1_{B_r}(X_t)].\]

Thus

\[\frac{1}{2} \frac{1}{|B_r|} \mathbb{E}_x [1_{B_r}(X_t)] \leq \frac{1}{|B_r|} \mathbb{E}_x [e^{-\frac{A_t}{k}} 1_{B_r}(X_t)] \leq \left( \frac{1}{|B_r|} \mathbb{E}_x [e^{-A_t} 1_{B_r}(X_t)] \right)^{\frac{1}{2}} \left( \frac{1}{|B_r|} \mathbb{E}_x [1_{B_r}(X_t)] \right)^{1 - \frac{1}{2}}\]

(by Hölder inequality).
Therefore

\[
\frac {1} {2} \frac {1} {|B_r|} \mathbb{E}_x [1_{B_r}(X_t)] \leq \frac {1} {|B_r|} \mathbb{E}_x [e^{-\lambda t} 1_{B_r}(X_t)]^{1/k},
\]

i.e.

\[
\frac {1} {2} \left( \frac {1} {|B_r|} \mathbb{E}_x [1_{B_r}(X_t)] \right)^{1/k} \leq \left( \frac {1} {|B_r|} \mathbb{E}_x [e^{-\lambda t} 1_{B_r}(X_t)] \right)^{1/k},
\]

i.e.

\[
\frac {1} {2^k} \frac {1} {|B_r|} \mathbb{E}_x [1_{B_r}(X_t)] \leq \frac {1} {|B_r|} \mathbb{E}_x [e^{-\lambda t} 1_{B_r}(X_t)].
\]

Let \( r \downarrow 0 \), we have

\[
\frac {1} {2^k} p(t, x, z) \leq q(t, x, z).
\]

Therefore when \( 0 < t \leq t_0 \),

\[
\frac {1} {2^k} M_1 t^{-d/\alpha} \left( 1 \wedge \frac {t^{1/\alpha}}{|x - z|} \right)^{d + \alpha} \leq q(t, x, z).
\]

Applying (iv) of Proposition 3.5, we have

\[
q(t, x, y) \geq C_3 e^{-C_4 t} t^{-d/\alpha} \left( 1 \wedge \frac {t^{1/\alpha}}{|x - y|} \right)^{d + \alpha}, \quad \forall (t, x, y) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d,
\]

where \( C_3 \) and \( C_4 \) are positive constants.

Next we establish the upper bound.

It is clear that for the positive constants \( L \) and \( K < 1 \) given in Remark 3.2 and Theorem 3.4, and \( \overline{M}, \overline{C} \), which are the upper bounds for \( M(y) \) and \( |2C(x, y)| \) respectively, there exists a constant \( \tilde{C} \geq 1 \) such that

\[
L^{n-1} t^{2} \overline{M} \overline{C} \leq \tilde{C}^{1/2} n! K^n, \quad \forall n \geq 1.
\]

Suppose that \( g \geq 0 \) is a measurable function and \( g \leq C_g \min \left( \frac {1} {D_2}, 1 \right) \), where \( C_g \geq 1 \) is a constant, then we have the following

**Proposition 3.7** There exists \( t_2 \geq 0 \) such that when \( 0 < t \leq t_2 \),

\[
\int_{\mathbb{R}^d} \varphi_n(t, x, z) g(z) m(dz) \leq \tilde{C} C_g C_t n! K^n, \quad \forall n \geq 1.
\]

**Proof.** When \( n = 1 \),

\[
\int_{\mathbb{R}^d} \varphi_1(t, x, z) g(z) m(dz) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_0^t \overline{F}(w, y) m(dw) \int_{\mathbb{R}^d} \frac {CF(w, y)}{|w - y|^{d + \alpha}} F(t - s, y, z) m(dy) ds g(z) m(dz)
\]

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Thus (3.5) holds for $n$.

Suppose it holds for $n \leq m - 1$, we show that it holds for $n = m$.

\[
\begin{align*}
\int_{\mathbb{R}^d} \varphi_m(t, x, z) g(z) &\ m(dz) \\
= \int_{\mathbb{R}^d} \sum_{i=1}^m C_i m \int_0^t \int_{\mathbb{R}^d} \varphi(s, x, w) m(dw) \int_{\mathbb{R}^d} \frac{CF(w, y)}{|w - y|^{d+\alpha}} \varphi_{m-i}(t - s, y, z) m(dy) ds g(z) m(dz) \\
= \sum_{i=1}^m C_i m \int_0^t \int_{\mathbb{R}^d} \varphi(s, x, w) m(dw) \int_{\mathbb{R}^d} \frac{CF(w, y)}{|w - y|^{d+\alpha}} \varphi_{m-i}(t - s, y, z) g(z) m(dz) m(dy) ds \\
\leq \sum_{i=1}^{m-1} C_i m \int_0^t \int_{\mathbb{R}^d} \varphi(s, x, w) m(dw) \int_{\mathbb{R}^d} \frac{CF(w, y)}{|w - y|^{d+\alpha}} m(dy) ds \tilde{C} C_g C_t (m-i)! K^{m-i} \\
+ \int_0^t \int_{\mathbb{R}^d} \varphi(s, x, w) m(dw) \int_{\mathbb{R}^d} \frac{CF(w, y)}{|w - y|^{d+\alpha}} m(dy) ds C_g \\
= \sum_{i=1}^{m-1} C_i m \int_0^t \int_{\mathbb{R}^d} \varphi(s, x, w) m(dw) \int_{\mathbb{R}^d} \frac{CF(w, y)}{|w - y|^{d+\alpha}} m(dy) ds C_g m! \\
= m! \sum_{i=1}^{m-1} \frac{(L^{i-1} K^{m-i})}{i!} C_t \tilde{M}^2 \tilde{C} C_g C_t + L^{m-1} \tilde{M}^2 \tilde{C} C_g C_t.
\end{align*}
\]

Since $C_t \downarrow 0$ as $t \downarrow 0$, $\exists t_2 \geq 0$ such that when $0 < t \leq t_2$

\[
\sum_{i=1}^{n-1} \frac{(L^{i-1} K^{n-i})}{i!} C_t \tilde{M}^2 \tilde{C} C_g C_t \leq \frac{1}{2} K^n, \forall n \geq 2,
\]

by the choice of $\tilde{C}$,

\[
L^{n-1} \tilde{M} \tilde{C} \leq \tilde{C} \frac{1}{2} n! K^n, \forall n \geq 1,
\]

Thus

\[
\int_{\mathbb{R}^d} \varphi_m(t, x, z) g(z) dz \leq \frac{1}{2} m! K^m \tilde{C} C_g C_t + \frac{1}{2} m! K^m \tilde{C} C_g C_t \\
= \tilde{C} C_g C_t m! K^m,
\]

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i.e. the statement holds for $n = m$. \hfill \Box

For the $L$, $K$, $C$ and $D_2$ given above, it is clear that there exists $\tilde{C}_2 \geq 1$ such that

$$\frac{L^n}{(1 - \frac{1}{m} - 2 - \frac{1}{d})^{d+\alpha}} \leq \frac{1}{8} \tilde{C}_2 n! K^n, \ \forall n \geq 0.$$ 

We claim that

**Theorem 3.8** There exist $t_3 > 0$ and $\tilde{C}_1 \geq 1$ such that when $0 < t \leq t_3$,

$$\mathcal{F}_n(t, x, z) \leq \tilde{C}_1 n! K^n t^{-\frac{d}{\alpha}} \left(1 \wedge \frac{t^{\frac{1}{\alpha}}}{|x - z|}\right)^{d+\alpha}, \ \forall n \geq 0. \ (3.6)$$

**Proof.** Since $\mathcal{F}_0(t, x, z) = \mathcal{F}(t, x, z)$, there exist $t_{13} > 0$ and $\tilde{C}_1 \geq \tilde{C}_2$ such that when $0 < t \leq t_{13}$

$$\mathcal{F}_0(t, x, z) \leq \tilde{C}_1 t^{-\frac{d}{\alpha}} \left(1 \wedge \frac{t^{\frac{1}{\alpha}}}{|x - z|}\right)^{d+\alpha},$$

i.e. the statement holds for $n = 0$. Suppose it is true for $n \leq m - 1$. We show that it holds for $n = m$. We write $\mathcal{F}_m(t, x, z)$ into two terms

$$\mathcal{F}_m(t, x, z) = \sum_{i=1}^{m} C_i^m \int_0^{\frac{t}{2}} \int_{\mathbb{R}^d} \mathcal{F}(s, x, w)m(dw) \int_{\mathbb{R}^d} \frac{CF^i(w, y)}{|w - y|^{d+\alpha}} \mathcal{F}_{m-i}(t - s, y, z) m(dy) ds$$

$$+ \sum_{i=1}^{m} C_i^m \int_{\frac{t}{2}}^t \int_{\mathbb{R}^d} \mathcal{F}(s, x, w)m(dw) \int_{\mathbb{R}^d} \frac{CF^i(w, y)}{|w - y|^{d+\alpha}} \mathcal{F}_{m-i}(t - s, y, z) m(dy) ds.$$

First we look at the first term. There are two cases:

**Case 1.** When $|x - z| \leq t^{\frac{1}{\alpha}}$,

$$\sum_{i=1}^{m} C_i^m \int_0^{\frac{t}{2}} \int_{\mathbb{R}^d} \mathcal{F}(s, x, w)m(dw) \int_{\mathbb{R}^d} \frac{CF^i(w, y)}{|w - y|^{d+\alpha}} \mathcal{F}_{m-i}(t - s, y, z) m(dy) ds$$

$$\leq \sum_{i=1}^{m} C_i^m \int_0^{\frac{t}{2}} \int_{\mathbb{R}^d} \mathcal{F}(s, x, w)m(dw) \int_{\mathbb{R}^d} \frac{CF^i(w, y)}{|w - y|^{d+\alpha}} dy ds L^{-1} \tilde{M} \tilde{C}_1 (m - i)! K^{m-i} \left(\frac{t}{2}\right)^{-\frac{d}{\alpha}}$$

$$\leq \sum_{i=1}^{m} C_i^m C_l M^2 C_l^{-1} \tilde{C}_1 (m - i)! K^{m-i} \left(\frac{t}{2}\right)^{-\frac{d}{\alpha}}$$

$$= m! \sum_{i=1}^{m} \left(\frac{L^{i-1} K^{m-i}}{i!}\right) M^2 C_l \tilde{C}_1 \tilde{C}_1 t^{-\frac{d}{\alpha}}.$$

Since there exists $t_{23} > 0$ and $t_{23} \leq t_{13}$ such that when $0 < t \leq t_{23}$,

$$\sum_{i=1}^{n} \left(\frac{L^{i-1} K^{m-i}}{i!}\right) M^2 C_l \tilde{C}_1 \tilde{C}_1 t^{-\frac{d}{\alpha}} \leq \frac{1}{2} K^n, \ \forall n \geq 1,$$

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thus in case 1, when $0 < t \leq t_{23}$,

$$
\sum_{i=1}^{m} C_{m}^{i} \int_{0}^{\frac{t}{2}} \int_{\mathbb{R}^{d}} p(s, x, w) m(dw) \int_{\mathbb{R}^{d}} \frac{CF(t, y)}{|w-y|^{d+\alpha}} \eta_{m-i}(t-s, y, z) m(dy) ds \leq \frac{1}{2} C_{1} m! K^{m-i} \frac{t}{|x-z|^{d+\alpha}}.
$$

Case 2. When $|x-z| \geq \frac{t}{10}$. Let $B_{1} = \{ y \in \mathbb{R}^{d} \mid |y-z| \geq \frac{1}{10} |x-z| \}$, $B_{2} = \{ w \in \mathbb{R}^{d} \mid |w-x| \geq 2^{-\frac{1}{2}} |x-z| \}$ and $B_{3} = \{ (w, y) \in \mathbb{R}^{d} \times \mathbb{R}^{d} \mid |y-z| < \frac{1}{10} |x-z|, |w-x| < 2^{-\frac{1}{2}} |x-z| \}$. On $B_{3}$, we have $|w-y| \geq (1 - \frac{1}{10} - 2^{-\frac{1}{2}})|x-z|$.

$$
\sum_{i=1}^{m} C_{m}^{i} \int_{0}^{\frac{t}{2}} \int_{\mathbb{R}^{d}} p(s, x, w) m(dw) \int_{\mathbb{R}^{d}} \frac{CF(t, y)}{|w-y|^{d+\alpha}} \eta_{m-i}(t-s, y, z) m(dy) ds \\
\leq \sum_{i=1}^{m} C_{m}^{i} \int_{0}^{\frac{t}{2}} \int_{\mathbb{R}^{d}} p(s, x, w) m(dw) \int_{\mathbb{R}^{d}} \frac{CF(t, y)}{|w-y|^{d+\alpha}} \eta_{m-i}(t-s, y, z) 1_{B_{1}}(y) m(dy) ds \\
+ \sum_{i=1}^{m} C_{m}^{i} \int_{0}^{\frac{t}{2}} \int_{\mathbb{R}^{d}} p(s, x, w) m(dw) \int_{\mathbb{R}^{d}} \frac{CF(t, y)}{|w-y|^{d+\alpha}} \eta_{m-i}(t-s, y, z) 1_{B_{2}}(w) m(dy) ds \\
+ \sum_{i=1}^{m} C_{m}^{i} \int_{0}^{\frac{t}{2}} \int_{\mathbb{R}^{d}} p(s, x, w) m(dw) \int_{\mathbb{R}^{d}} \frac{CF(t, y)}{|w-y|^{d+\alpha}} \eta_{m-i}(t-s, y, z) 1_{B_{3}}(w, y) m(dy) ds \\
\leq \sum_{i=1}^{m} C_{m}^{i} \int_{0}^{\frac{t}{2}} \int_{\mathbb{R}^{d}} p(s, x, w) m(dw) \int_{\mathbb{R}^{d}} \frac{CF(t, y)}{|w-y|^{d+\alpha}} ML^{i-1} \hat{C}_{1}(m-i)! K^{m-i} 10^{d+\alpha} (t-s) |x-z|^{d+\alpha} dy ds \\
+ \sum_{i=1}^{m} C_{m}^{i} \int_{0}^{\frac{t}{2}} \int_{\mathbb{R}^{d}} m(dw) \hat{C}_{1} M^{2} 2^{(d+\alpha)} s |x-z|^{d+\alpha} \int_{\mathbb{R}^{d}} \frac{CF(t, y)}{|w-y|^{d+\alpha}} \eta_{m-i}(t-s, y, z) dy ds L^{i-1} \\
+ \sum_{i=1}^{m-1} C_{m}^{i} \frac{L^{i}}{(1 - \frac{1}{10} - 2^{-\frac{1}{2}})^{d+\alpha}} \hat{C} \int_{0}^{\frac{t}{2}} \int_{\mathbb{R}^{d}} p(s, x, w) m(dw) \int_{\mathbb{R}^{d}} \frac{1}{|x-z|^{d+\alpha}} \eta_{m-i}(t-s, y, z) m(dy) ds + \frac{L^{m}}{(1 - \frac{1}{10} - 2^{-\frac{1}{2}})^{d+\alpha}} \hat{C} D_{2}^{2} \frac{t}{|x-z|^{d+\alpha}}
$$

$$
\leq \sum_{i=1}^{m} C_{m}^{i} \hat{C}_{1} M^{2} CL^{i-1} \hat{C}_{1}(m-i)! K^{m-i} 10^{d+\alpha} |x-z|^{d+\alpha} \\
+ \sum_{i=1}^{m} C_{m}^{i} \hat{C}_{1} M^{2} C_{2} 2^{(d+\alpha)} |x-z|^{d+\alpha} C_{t}(m-i)! K^{m-i} L^{i-1} \\
\quad \text{(by symmetry of $\eta_{m-i}(t-s, y, z)$ and Theorem 3.4)} \\
+ \sum_{i=1}^{m-1} C_{m}^{i} \frac{L^{i}}{(1 - \frac{1}{10} - 2^{-\frac{1}{2}})^{d+\alpha}} \hat{C} D_{2}^{2} |x-z|^{d+\alpha} \hat{C}_{2} C_{t}(m-i)! K^{m-i} \\
\quad \text{(by symmetry of $\eta_{m-i}(t-s, y, z)$, Proposition 3.7 and $\int_{\mathbb{R}^{d}} p(s, x, w) m(dw) \leq D_{2}$)} \\
+ \frac{L^{m}}{(1 - \frac{1}{10} - 2^{-\frac{1}{2}})^{d+\alpha}} \hat{C} D_{2}^{2} \frac{t}{|x-z|^{d+\alpha}}.
$$
It is easy to see that there exists $t_{33} > 0$ and $t_{33} \leq \min(t_0, t_2)$ such that when $0 < t \leq t_{33}$, the first three terms in above inequality $\leq \frac{1}{8} \tilde{C}_1 m! K^m \frac{t}{|x-z|^{d+\alpha}}$, for all $m > 0$. We can also have the fourth term in the above inequality $\leq \frac{1}{8} \tilde{C}_1 m! K^m \frac{t}{|x-z|^{d+\alpha}}$, for all $m > 0$, by the choice of $\tilde{C}_2$ and $\tilde{C}_1 \geq \tilde{C}_2$.

Thus in case 2 when $0 < t \leq t_{33}$,

$$\sum_{i=1}^{m} C^i_m \int_0^t \int_{\mathbb{R}^d} \varphi(s, x, w) m(dw) \int_{\mathbb{R}^d} \frac{\mathcal{F}^i(w, y)}{|w-y|^{d+\alpha}} \overline{\eta}_{m-i}(t-s, y, z) m(dy) ds \leq \frac{1}{2} \tilde{C}_1 m! K^m \frac{t}{|x-z|^{d+\alpha}}.$$

Combining case 1 and case 2, when $0 < t \leq \min(t_{23}, t_{33})$,

$$\sum_{i=1}^{m} C^i_m \int_0^t \int_{\mathbb{R}^d} \varphi(s, x, w) m(dw) \int_{\mathbb{R}^d} \frac{\mathcal{F}^i(w, y)}{|w-y|^{d+\alpha}} \overline{\eta}_{m-i}(t-s, y, z) m(dy) ds \leq \frac{1}{2} \tilde{C}_1 m! K^m t - \frac{d\alpha}{m} \left(1 \wedge \frac{t^{\frac{1}{\alpha}}}{|x-z|}\right)^{d+\alpha}.$$

For the second term in the expression of $\overline{\eta}_m(t, x, z)$:

$$\sum_{i=1}^{m} C^i_m \int_0^t \int_{\mathbb{R}^d} \varphi(t-s, x, w) m(dw) \int_{\mathbb{R}^d} \frac{\mathcal{F}^i(w, y)}{|w-y|^{d+\alpha}} \overline{\eta}_{m-i}(t-s, y, z) m(dy) ds.$$

Letting $t-s = \tilde{s}$, the second term becomes

$$\sum_{i=1}^{m} C^i_m \int_0^\frac{t}{2} \int_{\mathbb{R}^d} \varphi(t-\tilde{s}, x, w) m(dw) \int_{\mathbb{R}^d} \frac{\mathcal{F}^i(w, y)}{|w-y|^{d+\alpha}} \overline{\eta}_{m-i}(\tilde{s}, y, z) m(dy) d\tilde{s}.$$

There are two cases.

Case a. When $|x-z| \leq t^{\frac{1}{\alpha}}$,

$$\sum_{i=1}^{m} C^i_m \int_0^\frac{t}{2} \int_{\mathbb{R}^d} \varphi(t-\tilde{s}, x, w) m(dw) \int_{\mathbb{R}^d} \frac{\mathcal{F}^i(w, y)}{|w-y|^{d+\alpha}} \overline{\eta}_{m-i}(\tilde{s}, y, z) m(dy) d\tilde{s} \leq \sum_{i=1}^{m} C^i_m \int_0^\frac{t}{2} \int_{\mathbb{R}^d} \tilde{C}_1 \left(\frac{t}{2}\right)^{-\frac{d}{m}} m(dw) \int_{\mathbb{R}^d} \frac{\mathcal{F}^i(w, y)}{|w-y|^{d+\alpha}} \overline{\eta}_{m-i}(\tilde{s}, y, z) m(dy) d\tilde{s} \leq \sum_{i=1}^{m} C^i_m \tilde{C}_1 \left(\frac{t}{2}\right)^{-\frac{d}{m}} L^{i-1} M^2 q C \int_0^\frac{t}{2} \int_{\mathbb{R}^d} \overline{\eta}_{m-i}(\tilde{s}, y, z) \mu(dy) d\tilde{s} \leq \sum_{i=1}^{m} C^i_m \tilde{C}_1 \left(\frac{t}{2}\right)^{-\frac{d}{m}} L^{i-1} M^2 q C_i (m-i)! K^{m-i} \left(\text{by symmetry of } \overline{\eta}_{m-i}(\tilde{s}, y, z) \text{ and Theorem 3.4}\right) \leq m! \sum_{i=1}^{m} \left(\frac{L^{i-1} K^{m-i}}{i!}\right) M^2 C \left(\frac{1}{2}\right)^{-\frac{d}{m}} C_i t^{-\frac{d}{m}} \tilde{C}_1 \left(\frac{t}{2}\right)^{-\frac{d}{m}}.$$


Since there exists $\tilde{t}_{23} > 0$ and $\tilde{t}_{23} \leq \min(t_0, t_{13})$ such that when $0 < t \leq \tilde{t}_{23}$,

$$\sum_{i=1}^{n} \left( \frac{L^{i-1} K^{n-i}}{i!} \right) M^{2} C \left( \frac{1}{2} \right)^{-\frac{d}{2}} C_{i} \leq \frac{1}{2} K^{n}, \quad \forall n \geq 1,$$

we have in case a when $0 < t \leq \tilde{t}_{23}$,

$$\sum_{i=1}^{m} C_{i}^{m} \int_{0}^{t} \int_{\mathbb{R}^{d}} \rho(t - \tilde{s}, x, w) m(dw) \int_{\mathbb{R}^{d}} \frac{CF^{i}(w, y)}{|w - y|^{d+\alpha}} \eta_{m-i}(\tilde{s}, y, z) m(dy) d\tilde{s}$$

$$\leq \frac{1}{2} \tilde{C}_{1} m! K^{m} t^{-\frac{d}{2}},$$

i.e.

$$\sum_{i=1}^{m} C_{i}^{m} \int_{0}^{t} \int_{\mathbb{R}^{d}} \rho(s, x, w) m(dw) \int_{\mathbb{R}^{d}} \frac{CF^{i}(w, y)}{|w - y|^{d+\alpha}} \eta_{m-i}(t - s, y, z) m(dy) ds$$

$$\leq \frac{1}{2} \tilde{C}_{1} m! K^{m} t^{-\frac{d}{2}}.$$

Case b. When $|x - z| > t^{\frac{1}{2}}$. Let $\tilde{B}_{1} = \{ y \in \mathbb{R}^{d} | |y - z| \geq 1 \} \{ x - z \}$, $\tilde{B}_{2} = \{ w \in \mathbb{R}^{d} | |w - x| \geq 2^{-\frac{1}{2}} |x - z| \}$ and $\tilde{B}_{3} = \{ (w, y) \in \mathbb{R}^{d} \times \mathbb{R}^{d} | |y - z| < \frac{1}{10} |x - z|, |w - x| < 2^{-\frac{1}{2}} |x - z| \}$. On $\tilde{B}_{3}$, we have $|w - y| \geq (1 - \frac{1}{10} - 2^{-\frac{1}{2}}) |x - z|$.
\[
\leq \sum_{i=1}^{m} C_m^i C_l M^2 CL^{i-1} \tilde{C}_1 (m - i)! K^{m-i} |t|^{d+\alpha} |x - z|^{d+\alpha} \\
+ \sum_{i=1}^{m} C_m^i \tilde{C}_1 M^2 C_l t^{d+\alpha} |x - z|^{d+\alpha} C_l (m - i)! K^{m-i} L^{i-1} \\
(\text{by symmetry of } \overline{q}_{m-i}(\tilde{s}, y, z) \text{ and Theorem 3.4}) \\
+ \sum_{i=1}^{m-1} \frac{L^i}{(1 - \frac{1}{10} - 2 - \frac{1}{2})^{d+\alpha}} C_l D |x - z|^{d+\alpha} \tilde{C} \tilde{C}_1 (m - i)! \tilde{K}^{m-i} \\
(\text{by symmetry of } \overline{q}_{m-i}(\tilde{s}, y, z), \text{ Proposition 3.7 and } \int_{\mathbb{R}^d} \overline{p}(t - \tilde{s}, x, w) m(dw) \leq D_2) \\
+ \frac{L^m}{(1 - \frac{1}{10} - 2 - \frac{1}{2})^{d+\alpha}} C_l D^2 |x - z|^{d+\alpha}.
\]

It is easy to see that there exists \( \tilde{t}_{33} > 0 \) and \( \tilde{t}_{33} \leq \min(t_0, t_2) \) such that when \( 0 < t \leq \tilde{t}_{33} \), the first three terms in above inequality \( \leq \frac{1}{4} \tilde{C}_1 m! K^m |t|^{d+\alpha} |x - z|^{d+\alpha} \), for any \( m > 0 \). We can also have the fourth term in the above inequality \( \leq \frac{1}{8} \tilde{C}_1 m! K^m |t|^{d+\alpha} |x - z|^{d+\alpha} \), for any \( m > 0 \), by the choice of \( \tilde{C}_2 \) and \( \tilde{C}_1 \geq \tilde{C}_2 \). Thus in case b when \( 0 < t \leq \tilde{t}_{33} \),

\[
\sum_{i=1}^{m} C_m^i \int_0^{\tilde{t}_{33}} \int_{\mathbb{R}^d} \overline{p}(t - \tilde{s}, x, w) m(dw) \int_{\mathbb{R}^d} \overline{C} \overline{F}(w, y) \overline{q}_{m-i}(\tilde{s}, y, z) m(dy) d\tilde{s} \\
\leq \frac{1}{2} \tilde{C}_1 m! K^m |x - z|^{d+\alpha},
\]
i.e.

\[
\sum_{i=1}^{m} C_m^i \int_0^{\tilde{t}_{33}} \int_{\mathbb{R}^d} \overline{p}(s, x, w) m(dw) \int_{\mathbb{R}^d} \overline{C} \overline{F}(w, y) \overline{q}_{m-i}(t - s, y, z) m(dy) ds \\
\leq \frac{1}{2} \tilde{C}_1 m! K^m |x - z|^{d+\alpha}.
\]

Combining case a and case b, when \( 0 < t \leq \min(\tilde{t}_{23}, \tilde{t}_{33}) \),

\[
\sum_{i=1}^{m} C_m^i \int_0^{\min(\tilde{t}_{23}, \tilde{t}_{33})} \int_{\mathbb{R}^d} \overline{p}(s, x, w) m(dw) \int_{\mathbb{R}^d} \overline{C} \overline{F}(w, y) \overline{q}_{m-i}(t - s, y, z) m(dy) ds \\
\leq \frac{1}{2} \tilde{C}_1 m! K^m t^{-\frac{d}{\alpha}} \left( 1 \wedge \frac{t^{1/\alpha}}{|x - z|} \right)^{d+\alpha}.
\]

Therefore when \( 0 < t < t_3 = \min(t_{23}, t_{33}, \tilde{t}_{23}, \tilde{t}_{33}) \),

\[
\overline{q}_m(t, x, z) \leq \tilde{C}_1 m! K^m t^{-\frac{d}{\alpha}} \left( 1 \wedge \frac{t^{1/\alpha}}{|x - z|} \right)^{d+\alpha},
\]

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i.e. the statement holds for \( n = m \).

By Theorem 3.8, we have for \( 0 < t \leq t_3 \),

\[
\sum_{n=0}^{\infty} \frac{q_n(t,x,z)}{n!} \leq \sum_{n=0}^{\infty} \tilde{C}_1 K^n t^{-\frac{d}{\alpha}} \left( 1 - \frac{\frac{1}{\alpha}}{|x - z|} \right)^{d+\alpha} = \tilde{C}_1 \frac{1}{1 - K} t^{-\frac{d}{\alpha}} \left( 1 - \frac{\frac{1}{\alpha}}{|x - z|} \right)^{d+\alpha}.
\]

Since \( |q_n(t,x,z)| \leq \overline{q}_n(t,x,z) \) and \( q(t,x,z) = \sum_{n=0}^{\infty} (-1)^n \frac{q_n(t,x,z)}{n!} \),

\[
q(t,x,z) \leq \tilde{C}_1 \frac{1}{1 - K} t^{-\frac{d}{\alpha}} \left( 1 - \frac{\frac{1}{\alpha}}{|x - z|} \right)^{d+\alpha}.
\]

Applying (ii) of Proposition 3.5, we have

\[
q(t,x,z) \leq C_5 e^{C_6 t} \left( 1 - \frac{\frac{1}{\alpha}}{|x - z|} \right)^{d+\alpha}, \forall (t,x,y) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d,
\]

where \( C_5 \) and \( C_6 \) are positive constants. Thus we have established the lower and upper estimates of \( q(t,x,y) \) as follows,

**Theorem 3.9** There exist positive constants \( C_3, C_4, C_5 \) and \( C_6 \) such that

\[
C_3 e^{-C_4 t} t^{-\frac{d}{\alpha}} \left( 1 - \frac{\frac{1}{\alpha}}{|x - z|} \right)^{d+\alpha} \leq q(t,x,z) \leq C_5 e^{C_6 t} t^{-\frac{d}{\alpha}} \left( 1 - \frac{\frac{1}{\alpha}}{|x - z|} \right)^{d+\alpha} \quad (3.7)
\]

for all \( (t,x,z) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \).

From Theorem 3.9 and (ii) of Proposition 3.5, it is easy to obtain the following property for \( q(t,x,z) \),

**Proposition 3.10** \( q(t,x,z) \) is joint continuous on \((0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \).

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References

[1] Ph. Blanchard and Z. M. Ma, Semigroup of Schrödinger operators with potentials given by Radon measures, *Stochastic processes, physics and geometry*, 160-195, Word Sci. Publishing, Teaneck, NJ, 1990.

[2] Z.-Q. Chen and R. Song, Conditional gauge theorem for non-local Feynman-Kac transforms, *Probab. Theory Relat. Fields*, **125**(2003), 45-72.

[3] Z.-Q. Chen and T. Kumagai, Heat kernel estimates for stable-like processes for stable-like processes on $d$-sets, *Stochastic Process. Appl.*, **108**(2003), 27-62.

[4] K. Sato, *Lévy processes and infinitely divisible distributions*, Cambridge University Press, Cambridge, 1999.

[5] R. Song, Feynman-Kac Semigroup with discontinuous additive functionals, *J. Theoret. Probab.*, **8**(1995), 727-762.

[6] R. Song, Two-sides estimates on the density of the Feynman-Kac semigroups of stable-like processes, *Elect. J. Prob.*, **11**(2006), 146-161.

[7] Z. Vondraček, Basic potential theory of certain nonsymmetric strictly $\alpha$-stable processes, *Glasnik Matematicki.*, **37**(2002), 211-233.