Fundamental diagrams in traffic flow: the case of heterogeneous kinetic models

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**Abstract**

Experimental studies on vehicular traffic provide data on quantities like density, flux, and mean speed of the vehicles. However, the diagrams relating these variables (the so-called *fundamental* and *speed* diagrams) show some peculiarities not yet fully reproduced nor explained by mathematical models. In this paper, resting on the methods of the kinetic theory, we introduce a new traffic model which takes into account the heterogeneous nature of the flow of vehicles along a road. In more detail, the model considers traffic as a mixture of two populations of vehicles (e.g., cars and trucks) with different characteristics, in particular different length and maximum speed. With this approach we gain some insights into the scattering of the data in the regime of congested traffic clearly shown by actual measurements.

**Keywords:** traffic flow, kinetic models, multispecies kinetic equations, fundamental diagrams

**Mathematics Subject Classification:** 76P05, 65Z05, 90B20

1 Introduction

Prediction and control of traffic have become an important aspect in the modern world. In fact, the necessity to forecast the depletion time of a queue or to optimize traffic flows, thereby reducing the number of accidents, has arisen following the increase of circulating vehicles.

In the current mathematical literature, three different approaches are mainly used to model traffic flow phenomena. *Microscopic* models look at vehicles as single entities of traffic and predict, using a system of ordinary differential equations, the evolution of their position and speed (namely, the microscopic states characterizing their dynamics) regarded as time dependent variables. In these models, the acceleration is prescribed for each vehicle as a function of time, position, and
speed of the various entities of the system, taking also into account mutual interactions among vehicles. For example, in the well known follow-the-leader model each vehicle is assumed to adapt its speed to the one of the leading vehicle based on their instantaneous relative speed and mutual distance. On the opposite end, macroscopic models provide a large-scale aggregate point of view in which the focus is not on each single particle of the system. In this case, the motion of the vehicles along a road is described by means of partial differential equations inspired by conservation and balance laws from fluid dynamics, following the seminal works. Improvements and further evolutions of such a basic macroscopic description of traffic have been proposed over the years by several authors, from the classical mechanically consistent restatement of second order models to applications to road networks thoroughly developed in the book. In the middle, mesoscopic (or kinetic) models are based on a statistical mechanics approach, which still provides an aggregate representation of the traffic flow while linking macroscopic dynamics to pairwise interactions among vehicles at a smaller microscopic scale. These models will be the main reference background of the present paper. However, before entering the details of their discussion, some remarks about the other two types of models are in order.

Microscopic models give rise to very large systems of ordinary differential equations, when the number of vehicles is high, which makes the microscopic scale computationally not competitive. Moreover, the description of the behavior of single vehicles requires a quite detailed knowledge of several, mostly unknown or unaccessible, microscopic parameters while being often not really needed, since one usually is more interested in average quantities such as the flow rate or the mean speed. On the other hand, macroscopic models do not allow one to account for interactions among vehicles, which instead play a prominent role in triggering traffic phenomena. A further drawback is that they typically require the prescription of closure laws linking the density and the flow (or the mean speed) of vehicles in order to give rise to self-consistent solvable equations (but see also [2]). This kind of information is usually recovered from interpolation or extrapolation of experimental data, which are then plugged into the mathematical models. Nevertheless one would like these data to be reproduced by models as a consequence of more fundamental modeling procedures rather than being postulated externally.

Kinetic (mesoscopic) models, first introduced in [23, 25, 26], are based on the Boltzmann equation that describes the statistical behavior of a system of particles. From the kinetic point of view, the system is again seen as the resultant of the evolution of microscopic particles, with given microscopic position and speed, but its representation is provided in aggregate terms by a statistical distribution function, whose evolution is described by integro-differential equations. Compared to microscopic models, the kinetic approach requires a smaller number of equations and parameters. On the other hand, unlike macroscopic models, at the mesoscopic scale the evolution equations do not require an a priori closure law: the flow is provided by the statistical moments of the kinetic distribution function over the microscopic states. Kinetic models have also been extended to include multilane traffic flow [17, 18] and control problems [15], to name but just a few applications.

For an overview of vehicular traffic models at all scales, the interested reader is referred e.g., to the review papers [19, 24] and references therein.

In this paper we propose a multipopulation kinetic model for traffic flow, which draws inspiration from the ideas presented in [4] recast in the frame of the discrete-velocity kinetic models [7, 8]. The main goal is to study fundamental diagrams, computed from moments of equilibrium solutions of the kinetic equations, and in particular to show that taking into account the heterogeneous composition of the “mixture” of vehicles allows one to explain the experimentally observed scattering of such diagrams in the phase of congested traffic without invoking further elements of microscopic randomness of the system, cf. [9]. In more detail, the structure of the paper is as follows: in Section 2 we briefly review the role of fundamental diagrams in vehicular traffic practice. Next, in Section 3 we describe the discrete-velocity kinetic model developed in [7, 8] by focusing on its spatially homogeneous version, which represents the mathematical counterpart of the experimental setting in which traffic equilibria and fundamental diagrams are measured. In Section 4 we first review the multi-population macroscopic model [4] and then introduce our new two-population kinetic model, proving in particular its consistency with the original single-population model and
2 Fundamental diagrams

In this section we present a brief description of some basic tools for the analysis of traffic problems, namely the diagrams which relate the macroscopic flux and mean speed to the vehicle density in homogeneous steady conditions. The qualitative structure of such diagrams is defined by the properties of different regimes, or phases, of traffic as outlined in the following.

**Flux-density diagrams** Also called *fundamental diagrams*, they report the flow rate of vehicles as a function of the number of vehicles per unit length. At low traffic densities, the so-called *free phase* in which interactions among vehicles are rare, the flux grows nearly linearly with the density until a *critical density* value is reached, at which the flux takes its maximum (*road capacity*). Beyond such a critical value traffic switches to the *congested phase*, which in [16] is defined as complementary to the free phase. In this regime the flux decreases as the density increases. In fact interactions among vehicles are more and more frequent due to the higher packing, which causes faster vehicles to be hampered by slower ones. The formation of local slowdowns (*phantom traffic jams*) is first observed. Additional increments of the density cause a steep reduction of the flux until the so-called *traffic jam* is reached, in which the density reaches its maximum value, called *jam density*, and the flux is zero.

**Speed-density diagrams** They give the mean speed of the vehicles as a function of the local macroscopic density of traffic. In free flow conditions, vehicles travel at the maximum allowed speed, which depends on the environmental conditions (such as e.g., quality of the road, weather) infrastructure, on the mechanical characteristics of the vehicles, and on the imposed speed limits. This speed, called the *free flow speed*, can be reached when there is a large distance between vehicles on the road. Conversely, in congested flow conditions vehicles travel closer to one another at a reduced speed, until the density reaches the jam density, at which vehicles stop and have zero speed.

These diagrams play an important role in the prediction of the capacity of a road and in the control of the flow of vehicles.
Examples of fundamental diagrams provided by experimental measurements are shown in Fig. They clearly exhibit the phase transition between free and congested flow: below the critical density the flux values distribute approximately on a line with positive slope, thus the flux can be regarded as a single-valued increasing function of the density with low, though non zero, dispersion; conversely, above the critical density the flux decreases and experimental data exhibit a large scattering in the flux-density plane. In the congested phase, therefore, the flux can hardly be approximated by a single-valued function of the density.

In order to investigate such characteristics of the traffic diagrams we will consider kinetic models of traffic flow, because they naturally allow one to obtain fundamental diagrams as stationary asymptotic solutions starting from a statistical description of microscopic interactions among vehicles. In addition, they have proved to be able to catch the transition from the free to the congested phase of traffic, see e.g., [7, 8]. However, standard kinetic models do not account for the scattered data typical of the congested regime. Usually, this characteristic of the flow is explained considering the statistical variability of driver behaviors, who may individually decide to drive at a different speed than the one resulting from the local density, see e.g., [9].

In this work we propose instead a different interpretation of the scattering of the flux in congested traffic, based on the consideration that the flow along a road is naturally heterogeneous. Namely, it is composed by different classes of vehicles with different physical and kinematic characteristics (size, maximum speed, . . .). For this we will extend the aforementioned kinetic models so as to deal with a mixture of two populations of vehicles, say cars and trucks, each described by its own statistical distribution function. The core of the model will be the statistical description of the microscopic interactions among the vehicles of the same and of different populations, which will take into account the microscopic differences of the various types of vehicles.

3 A discrete kinetic model

In this section we briefly review a kinetic traffic model recently introduced in [8], which will be the basis for our multipopulation extension. In the kinetic approach we focus on a statistical description of the microscopic states of the vehicles, therefore the evolution of their position $x$ and speed $v$ is described by means of a distribution function $f = f(t, x, v)$ such that $f(t, x, v) dx dv$ is the (infinitesimal) number of vehicles which at time $t$ are located between $x$ and $x + dx$ with a speed between $v$ and $v + dv$.

The model proposed in [8] is a discrete one in both space and speed. Hence the spatial domain, say an interval $X \subseteq \mathbb{R}$ representing the considered road, and the domain of the microscopic speeds, say $V \subseteq [0, +\infty)$, have the following structure:

$$X = \bigcup_{i=1}^{m} X_i, \quad V = \{v_1, v_2, \ldots, v_j, \ldots, v_n\},$$

where $X = \{X_i\}_{i=1}^{m}$ is a collection of pairwise disjoint space cells, each of finite length, and the $v_j$’s are speed classes such that:

$$0 \leq v_j < v_{j+1} \quad \forall j = 1, \ldots, n-1, \quad v_1 = 0, \quad v_n = V_{\text{max}},$$

$V_{\text{max}}$ being the maximum speed of a vehicle. For instance, $V_{\text{max}}$ can be chosen as a speed limit imposed by safety regulations, or by the state of the road, or by the mechanical characteristics of the vehicles.

The microscopic state of a generic vehicle is represented by the pair $(X_i, v_j)$, which belongs to the discrete state space $\mathcal{X} \times V$. Over the latter, the statistical distribution of vehicles is given by the functions:

$$f_{ij} = f_{ij}(t) : [0, T_{\text{max}}] \to [0, +\infty), \quad i = 1, \ldots, m, \quad j = 1, \ldots, n,$$

$f_{ij}(t)$ being the density of vehicles which, at time $t$, are located in the space cell $X_i$ and travel with speed $v_j$.  

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The macroscopic variables useful in the study of traffic, namely the vehicle density \( \rho_i \), flux \( q_i \), and mean speed \( u_i \) in the \( i \)-th space cell, are obtained from the \( f_{ij} \)'s as statistical moments with respect to the speed:

\[
\rho_i(t) = \sum_{j=1}^{n} f_{ij}(t), \quad q_i(t) = \sum_{j=1}^{n} v_j f_{ij}(t), \quad u_i(t) = \frac{q_i(t)}{\rho_i(t)}. \tag{1}
\]

As already mentioned in the previous Section 2, the experimental diagrams are measured under flow conditions which are as much as possible homogeneous in space and stationary. Space homogeneity can be translated mathematically by assuming that the \( f_{ij} \)'s do not depend on the index \( i \). We write therefore \( f_j = f_j(t) \) for the distribution of vehicles which, a time \( t \), travel with speed \( v_j \) and we study its evolution in time due to vehicle interactions which cause speed changes. The corresponding system of (spatially homogeneous) Boltzmann-type kinetic equations writes:

\[
\frac{df_j}{dt} = J_j[f, f], \quad j = 1, \ldots, n, \tag{2}
\]

where \( J_j \), the \( j \)-th “collisional” operator, describes the microscopic interactions among vehicles which determine the change of \( v_j \) in time. We used the vector notation \( f := \{f_j\}_{j=1}^{n} \). For mass conservation purposes, the following property is required:

\[
\sum_{j=1}^{n} J_j[f, f] = 0 \quad \forall f,
\]

in fact in this case from (2) it results:

\[
\frac{d}{dt} \sum_{j=1}^{n} f_j = \frac{d\rho}{dt} = 0, \tag{3}
\]

i.e., the macroscopic density \( \rho \) (which does not depend in turn on the index \( i \)) is constant in time.

Stationary flow conditions mean that we are actually interested in equilibrium solutions (if any) to system (4), that is constant-in-time solutions \( f^e := \{f^e_j\}_{j=1}^{n} \) such that \( J_j[f^e, f^e] = 0 \) for all \( j = 1, \ldots, n \). Due to (3), we see that in the spatially homogeneous case the density \( \rho \) plays the role of a parameter of system (4) fixed by the prescribed initial condition, hence in particular:

\[
\rho = \sum_{j=1}^{n} f_j(0) = \sum_{j=1}^{n} f^e_j.
\]

Hence equilibrium solutions are parameterized by specific values of the vehicle density, which allows one to define analytically the fundamental and speed diagrams of traffic by means of the following mappings:

\[
\rho \mapsto q(\rho) = \sum_{j=1}^{n} v_j f^e_j, \quad \rho \mapsto u(\rho) = \frac{q(\rho)}{\rho}.
\]

In particular, if for any given \( \rho \) system (4) admits a unique stable equilibrium then these mappings are actual functions of \( \rho \); otherwise, they define multivalued diagrams. We stress that, contrary to macroscopic models, the mapping \( \rho \mapsto q(\rho) \) is not based on a priori closure relations but is obtained from the large time evolution of the kinetic distribution function as a result of microscopic vehicle interactions.

### 3.1 Modeling vehicle interactions

The operator \( J_j \) models the microscopic interactions among vehicles. Following [8], the formalization of \( J_j \) is based on stochastic game theory: vehicles are the players, their pre-interaction speeds are the game strategies and the post-interaction speeds are the payoff of the game. This point
of view allows one to assign post-interaction speeds in a non-deterministic way, consistently with the intrinsic stochasticity of driver behaviors. We report here the construction of the operator $J_j$, which will be extended later to the two-population case. We consider only binary interactions among vehicles, thus the collisional operator can be written as:

$$J_j[f, f] = G_j[f, f] - f_j L_j[f]$$

where

$$G_j[f, f] := \sum_{h,k=1}^{n} \eta A_{jkh} f_h f_k$$

and

$$L_j[f] := \sum_{k=1}^{n} \eta f_k$$

are the gain and loss terms, respectively. The term $G_j$ counts statistically the number of interactions which lead, in the unit time, a so-called candidate vehicle with speed $v_h$ to switch to the test speed $v_j$ after an interaction with a field vehicle with speed $v_k$. Conversely, the term $L_j$ describes the loss of vehicles with test speed $v_j$ after interactions with any field vehicle.

Ultimately, the single-population model writes as:

$$\frac{df_j}{dt} = \sum_{h,k=1}^{n} \eta A_{jkh} f_h f_k - f_j \sum_{k=1}^{n} \eta f_k.$$

(4)

The coefficient $\eta > 0$ is the interaction rate, which here we assume independent of the pre-interaction speeds. We point out, however, that it may also depend on the relative speed of the interacting pairs: $\eta_{hk} = \eta(|v_k - v_h|)$ like in [6].

The matrix $A_j = \{A^j_{hk}\}_{h,k=1}^{n}, j = 1, \ldots, n$, is called the table of games. It encodes the discrete probability distribution of gaining the test speed $v_j$:

$$A^j_{hk} = \text{Prob}(v_h \rightarrow v_j|v_k, \rho), \quad h, k, j = 1, \ldots, n,$$

which in the present model is further parameterized by the macroscopic density $\rho$ so as to account for the influence of the macroscopic traffic conditions (local road congestion) on the microscopic interactions among vehicles. We stress that this is a further source of nonlinearity at the right-hand side of (4), besides the quadratic one typical of Boltzmann-like kinetic equations. The coefficients $A^j_{hk}$ fulfill the following conditions:

$$0 \leq A^j_{hk} \leq 1, \quad \sum_{j=1}^{n} A^j_{hk} = 1$$

$$\forall h, k, j = 1, \ldots, n, \quad \forall \rho \in [0, \rho_{\text{max}}],$$

(5)

$\rho_{\text{max}} > 0$ being the maximum density of vehicles that can be locally accommodated on the road in bumper-to-bumper conditions.

The table of games of model (4) is built appealing to the following assumptions:

• a candidate vehicle with speed $v_h$ can accelerate by at most one speed class at a time. However, it can possibly decelerate by an arbitrary number of speed classes when it interacts with a field vehicle with lower speed $v_k < v_h$;

• let $P$ be the probability that a candidate vehicle gets the maximum possible test speed resulting from an interaction, then we assume that $P$ is proportional to the local percentage of free room:

$$P = \alpha \left(1 - \frac{\rho}{\rho_{\text{max}}} \right).$$

The coefficient $\alpha \in [0, 1]$ can be thought of as a parameter describing the environmental conditions, for instance road or weather conditions, with $\alpha = 0, 1$ standing for worse and best conditions, respectively.
In more detail, we distinguish three types of interactions which lead us to propose the following table of games:

- **Interaction with a faster field vehicle**: in this case we have \( v_h < v_k \), or \( h < k \). Following the interaction, we assume that the candidate vehicle can either be motivated to accelerate or maintain its speed. Thus the test speed resulting from this interaction is \( v_j = v_{h+1} \) with probability \( P \) or \( v_j = v_h \) with probability \( 1 - P \), whence:

\[
A_{jh}^j = \begin{cases} 
1 - P & \text{if } j = h \\
P & \text{if } j = h + 1 \\
0 & \text{otherwise} 
\end{cases} \quad h < k = 2, \ldots, n. \quad (6)
\]

- **Interaction with a slower field vehicle**: in this case we have \( v_h > v_k \), or \( h > k \). Following the interaction, we assume that the candidate vehicle can either maintain its speed, if for instance there is enough room to overtake the leading field vehicle, or decelerate to the speed of the latter and queue up. Therefore the test speed resulting from this interaction is \( v_j = v_h \) with probability \( P \) or \( v_j = v_k \) with probability \( 1 - P \), whence:

\[
A_{jh}^j = \begin{cases} 
1 - P & \text{if } j = k \\
P & \text{if } j = h \\
0 & \text{otherwise} 
\end{cases} \quad h > k = 1, \ldots, n - 1. \quad (7)
\]

Notice that in this case \( P \) plays the role of a probability of passing as defined in [25].

- **Interaction with a field vehicle with the same speed**: in this case we have \( v_h = v_k \), or \( h = k \). Following the interaction, we assume that the candidate vehicle can either maintain its pre-interaction speed, if it is not disturbed by the field vehicle, or accelerate to overtake the latter, or decelerate to put some room between itself and the leading vehicle. Hence the test speed resulting from this interaction is either \( v_j = v_{h+1} \) with probability \( P \), or \( v_j = v_{h-1} \) with probability \( Q \), or finally \( v_j = v_h \) with probability \( 1 - (P + Q) \). In particular, it is reasonable to assume that the probability of braking \( Q \) increases with the congestion of the road \( \frac{\rho}{\rho_{\text{max}}} \) and with the badness of the environmental conditions \( 1 - \alpha \). According to this argument, we define

\[
Q = (1 - \alpha) \frac{\rho}{\rho_{\text{max}}}. 
\]

We further distinguish three cases, in fact if the candidate vehicle is either in \( v_1 = 0 \) or in \( v_n = V_{\text{max}} \) then it cannot decelerate or accelerate, respectively. Thus:

\[
A_{11}^j = \begin{cases} 
1 - P & \text{if } j = 1 \\
P & \text{if } j = 2 \\
0 & \text{otherwise} 
\end{cases} \quad (8a)
\]

\[
A_{hh}^j = \begin{cases} 
Q & \text{if } j = h - 1 \\
1 - (P + Q) & \text{if } j = h \\
P & \text{if } j = h + 1 \\
0 & \text{otherwise} 
\end{cases} \quad h = 2, \ldots, n - 1 \quad (8b)
\]

\[
A_{nn}^j = \begin{cases} 
Q & \text{if } j = n - 1 \\
1 - Q & \text{if } j = n \\
0 & \text{otherwise} 
\end{cases} \quad (8c)
\]
By studying, as previously explained, the evolution to equilibria of a given initial condition corresponding to a fixed value of $\rho$ we obtain the fundamental and speed diagrams depicted in Fig. 2 for three different values of the number $n$ of speed classes. For $\rho < \frac{1}{2}\rho_{\text{max}}$ we recognize the free phase of traffic, in which the flux is an increasing linear function of the density. Conversely, for $\rho > \frac{1}{2}\rho_{\text{max}}$ we find the congested phase, in which the specific form of the diagrams predicted by the model depends on the number $n$ of speed classes. For instance, for $n = 2$ the flux is a decreasing linear function of the density, whereas for $n > 2$ it becomes nonlinear. Here $\rho_c = \frac{1}{2}\rho_{\text{max}}$ is the critical density mentioned in Section 2, at which a bifurcation of equilibria occurs, which is the mathematical counterpart of the physical phase transition. For an analytical investigation of this problem interested readers are referred to [9].

These results confirm that the kinetic approach is able to catch successfully the phase transition in traffic flow as a consequence of more elementary microscopic interaction rules. In particular, such a phase transition need not be postulated {	extit{a priori}} through heuristic closures of the flux as a given function of the density. Nevertheless, model (4) still provides a single-valued density-flux relationship. In fact, as shown in [9], for all $\rho \in [0, \rho_{\text{max}}]$ there exists a unique stable and attractive equilibrium of system (4). Therefore, given an initial condition $\mathbf{f}_0 = \{f_{0j}\}_{j=1}^n$ such that $\sum_{j=1}^n f_{0j} = \rho$, the equilibrium distribution $\mathbf{f}^e = \{f^e_j\}_{j=1}^n$ does not depend on the detailed initial one but only on the value of $\rho$ fixed by the latter. Consequently, the flux $q$ at equilibrium is uniquely determined by the initial density $\rho$, which does not explain the scattered data of the experimental diagrams.

In [9] it is suggested that the scattering of the experimental diagrams can be explained by studying the standard deviation (SD) of the equilibrium distribution $\mathbf{f}^e$. Such an SD is indeed zero for $\rho < \frac{1}{2}\rho_{\text{max}}$, hence in the free phase, and different from zero for $\rho > \frac{1}{2}\rho_{\text{max}}$, hence in the congested phase. However, it is worth pointing out that this argument actually refers to the scattering of the microscopic speeds at equilibrium rather than to that of the macroscopic flux.
4 Two-population models

Starting from the kinetic approach discussed in the previous section, we now introduce a model which treats traffic as a mixture of types of vehicles with different physical and kinematic characteristics. As far as we know, this is the first attempt to account for the heterogeneity of traffic in a kinetic model. We will see that the proposed structure allows one to understand the nature of scattered data in experimental diagrams. For the sake of simplicity we will consider a two-population model, which can be easily extended to more complex mixtures.

Multi-population models of vehicular traffic are already available in the literature. For instance, in [4] the authors describe an $n$-population generalization of the Lighthill-Whitham-Richards macroscopic traffic models [21, 28], that here we briefly illustrate in the case of $n = 2$ species as an introduction to the forthcoming kinetic approach.

The model consists of two coupled one-dimensional conservation laws:

\[
\begin{cases}
\partial_t \rho_1 + \partial_x F_1(\rho_1, \rho_2) = 0 \\
\partial_t \rho_2 + \partial_x F_2(\rho_1, \rho_2) = 0
\end{cases}
\]  

where $\rho_i = \rho_i(t, x)$ is the macroscopic density of the $i$-th species, $F_i(\rho_1, \rho_2) = \rho_i v_i(\rho_1, \rho_2)$ its flux function, and $v_i(\rho_1, \rho_2)$ is the speed-density relation, which describes the attitude of drivers of the $i$-th population to change speed on the basis of the local values of $\rho_1$, $\rho_2$. In particular, after defining the fraction of road occupancy as the dimensionless quantity

\[ s := \rho_1 l_1 + \rho_2 l_2, \]

$l_1, l_2 > 0$ being the characteristic lengths of the vehicles of either population, the following Greenshield-type speed-density relation is proposed:

\[ v_i(\rho_1, \rho_2) = (1 - s) V_i, \quad i = 1, 2 \]

where $V_i > 0$ is a constant denoting the maximum speed of population $i$. System (9) specializes then as:

\[
\begin{cases}
\partial_t \rho_1 + \partial_x (\rho_1(1 - s)V_1) = 0 \\
\partial_t \rho_2 + \partial_x (\rho_2(1 - s)V_2) = 0
\end{cases}
\]  

with fluxes $F_i(\rho_1, \rho_2) = \rho_i(1 - s)V_i$. We notice that the total flux $F_1 + F_2 = (1 - s)(\rho_1 V_1 + \rho_2 V_2)$ is not a one-to-one function of the fraction of road occupancy $s$, as there might exist different pairs $(\rho_1, \rho_2)$ giving rise to the same value of $s$ and nevertheless to different total fluxes. This is possible if $l_1 \neq l_2$ or $V_1 \neq V_2$.

4.1 A two-population kinetic model

In constructing our two-population kinetic model we confine ourselves to the spatially homogeneous case, in order to focus on the study of fundamental diagrams. To fix the ideas, we identify the two classes of vehicles with "cars" ($C$) and "trucks" ($T$), respectively. Roughly speaking, the physical and kinematic differences between them consist in that the former are shorter and quicker while the latter are longer and slower. We adopt a compact notation, which makes use of two indexes

\[ p \in \{C, T\}, \quad q \in \{C, T\} \setminus \{p\} \]

to label various quantities referred to either population of vehicles.

Entering the details of the model, we assume that the spaces of microscopic speeds for cars and trucks, $\mathcal{V}^C$, $\mathcal{V}^T$, respectively, are such that $\mathcal{V}^T \subseteq \mathcal{V}^C$. The speeds accessible to trucks are a subset of those accessible to cars, considering that the former typically reach lower maximum speeds than the latter. For instance, choosing to deal with an equispaced lattice of speeds, we define:

\[ v_j = \frac{j - 1}{n^C - 1} V_{\text{max}}, \quad 1 \leq j \leq n^C, \]
then we set $\mathcal{V}^C = \{v_j\}_{j=1}^n$, $\mathcal{V}^T = \{v_j\}_{j=1}^n$ with $n^T \leq n^C$. This way the maximum speed of cars is $V_{\text{max}}$, whereas that of trucks is a fraction of it, precisely $\frac{n^T}{n^C} V_{\text{max}}$.

Over the discrete state space $\mathcal{V}^p$ we introduce the kinetic distribution function

$$ f_j^p = f_j^p(t) : [0, T_{\text{max}}] \rightarrow [0, +\infty), $$

which gives the statistical distribution of $p$-vehicles traveling with speed $v_j$ at time $t$. The macroscopic observable quantities referred to such a class of vehicles are recovered as (cf. (11)):

$$ \rho^p(t) = \sum_{j=1}^{n^p} f_j^p(t), \quad q^p(t) = \sum_{j=1}^{n^p} v_j f_j^p(t), \quad u^p(t) = \frac{q^p(t)}{\rho^p(t)}. \quad (11) $$

Inspired by (4), we model the evolution of the $f_j^p$'s by means of the following equation:

$$ \frac{df_j^p}{dt} = J_{j}^{\rho}[\mathbf{f}^p, \mathbf{f}^q], \quad j = 1, \ldots, n^p \quad (12) $$

where the term $J_{j}^{\rho}[\mathbf{f}^p, \mathbf{f}^q]$ describes interactions for the test state $v_j$ in which $p$-vehicles play the role of candidate vehicles. Since we consider only binary interactions, we can follow an approach frequently used for mixtures of two gases in kinetic theory, see e.g., [5] [12] [13], which consists in writing the collisional operator as the sum of two terms:

$$ J_{j}^{\rho}[\mathbf{f}^p, \mathbf{f}^q] = J_{j}^{\rho \rho}[\mathbf{f}^p, \mathbf{f}^p] + J_{j}^{\rho q}[\mathbf{f}^p, \mathbf{f}^q], \quad j = 1, \ldots, n^p. \quad (13) $$

In particular, the term $J_{j}^{\rho \rho}[\mathbf{f}^p, \mathbf{f}^p]$ accounts for self-interactions within the population $p$, i.e., interactions in which $p$-vehicles play also the role of field vehicles. Conversely, the term $J_{j}^{\rho q}[\mathbf{f}^p, \mathbf{f}^q]$ accounts for cross-interactions between the two populations, namely interactions in which the role of field vehicles is played by $q$-vehicles. Following the same logic underlying the single population model, cf. Section 3.3, each term is written as a balance of gain and loss contributions:

$$ J_{j}^{\rho \rho}[\mathbf{f}^p, \mathbf{f}^p] = \sum_{h,k=1}^{n^p} \eta A_{hk}^{\rho,j} f_h^p f_k^p - \sum_{k=1}^{n^p} \eta f_k^p, \quad j = 1, \ldots, n^p, \quad (14) $$

$$ J_{j}^{\rho q}[\mathbf{f}^p, \mathbf{f}^q] = \sum_{h=1}^{n^p} \sum_{k=1}^{n^q} \eta B_{hk}^{\rho q,j} f_h^p f_k^q - \sum_{k=1}^{n^q} \eta f_k^q, \quad j = 1, \ldots, n^p, \quad (15) $$

where $A_{hk}^{\rho,j}$, $B_{hk}^{\rho q,j}$, $j = 1, \ldots, n^p$, are the self-interaction and cross-interaction tables of games, respectively. By further requiring:

$$ 0 \leq A_{hk}^{\rho,j}, B_{hk}^{\rho q,j} \leq 1, \quad \forall h, k, j, p, q, \quad (16) $$

$$ \sum_{j=1}^{n^p} A_{hk}^{\rho,j} = \sum_{j=1}^{n^p} B_{hk}^{\rho q,j} = 1, \quad \forall h, k, p, q \quad (17) $$

we obtain that the coefficients of the two tables of games are indeed probabilities and furthermore that

$$ \sum_{j=1}^{n^p} J_{j}^{\rho}[\mathbf{f}^p, \mathbf{f}^q] = \sum_{j=1}^{n^p} J_{j}^{\rho \rho}[\mathbf{f}^p, \mathbf{f}^p] + \sum_{j=1}^{n^p} J_{j}^{\rho q}[\mathbf{f}^p, \mathbf{f}^q] = 0, \quad (18) $$

whence from (12) the mass conservation for each species:

$$ \frac{df_j^p}{dt} \sum_{j=1}^{n^p} f_j^p = \frac{df^p}{dt} = 0. \quad (19) $$

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The two-population model resulting from [12]–[14] writes finally as:

$$\frac{df_j}{dt} = \sum_{h,k=1}^{n^p} A_{h,k;j} f_h^p f_k^p + \sum_{h=1}^{n^p} \sum_{k=1}^{n^q} B_{h,k;j}^{pq} f_h^p f_k^q - f_j^p \left( \sum_{k=1}^{n^p} f_k^p + \sum_{k=1}^{n^q} f_k^q \right), \quad j = 1, \ldots, n^p, \quad (15)$$

where, using the assumption of constant interaction rate, we have hidden $\eta$ in the time scale.

Notice that the loss term can be possibly rewritten in the analytically equivalent form $-f_j^p (\rho^p + \rho^q)$, cf. [11]. Nevertheless, from the point of view of the numerical solution of (15), such an equivalence is more delicate and has to be handled with care, see the next Section 4.2.

4.1.1 Modeling self- and cross-interactions

The total number of $p$-vehicles present in a stretch of road of length $L > 0$ is $N^p = L \sum_{j=1}^{n^p} f_j^p$. If $l^p > 0$ is the characteristic length of any of such vehicles, the total space occupied by population $p$ along the road is then $N^p l^p$, while the total space occupied by all vehicles is $S = \sum_{p \in \{C,T\}} N^p l^p$. Ultimately, the fraction of road occupancy over the length $L$ is

$$s := \frac{S}{L} = \sum_{p \in \{C,T\}} N^p l^p = \sum_{p \in \{C,T\}} \left( \sum_{j=1}^{n^p} f_j^p \right) l^p = \sum_{p \in \{C,T\}} \rho^p l^p. \quad (16)$$

Obviously, given $l^C, l^T$, the admissible pairs of densities $(\rho^C, \rho^T) \in [0, \rho^C_{\text{max}}] \times [0, \rho^T_{\text{max}}]$ are those such that $0 \leq s \leq 1$.

Notice that $\rho^p_{\text{max}} = \frac{1}{l^p}$, therefore $s$ can be rewritten as

$$s = \sum_{p \in \{C,T\}} \frac{\rho^p}{\rho^p_{\text{max}}}.$$ 

From this expression it is clear that $s$ is the natural generalization of the term $\frac{\rho}{\rho_{\text{max}}}$ appearing in the probabilities $P, Q$ of the single-population model, cf. Section 3.1. Therefore it is reasonable that in the two-population model the transition probabilities depend on $s$ and on the quality of the environment $\alpha \in [0, 1]$:

$$P = \alpha(1 - s), \quad Q = (1 - \alpha)s.$$ 

For the tables of games $A_{p,j}$ we use the same construction as in [6]–[8c], because they express self-interactions within either population of vehicles regardless of the presence of the other population. The only difference is that they are $n^p \times n^q$ matrices, hence their dimensions change depending on the specific population.

The tables of games $B_{p,q,j}$ are instead $n^p \times n^q$ rectangular matrices, therefore we need to slightly revise the basic interaction rules of the single-population model in order to take into account the different maximum speeds of the two populations in the description of the speed transitions.

The table $B_{CT,j}$ gives the probability distribution that candidate cars switch to the test speed $v_j$ upon interacting field trucks. The coefficients $B_{h,k;j}^{CT}$ are constructed like in [6]–[8c], considering however that the case (8b) applies only for $h = 2, \ldots, n^T \leq n^C$ and that the case (8c) applies only if $n^T = n^C$.

Conversely, the table $B_{TC,j}$ gives the probability distribution that candidate trucks switch to the test speed $v_j$ upon interacting with field cars. If the candidate truck is faster than the field car then the coefficient $B_{h,k;j}^{TC}$ is constructed like in [6]. Instead, when interactions involve also accelerations it is necessary to consider that the candidate truck might not be able to further increase its speed if it is already traveling at its maximum possible one $v_{n^T}$, which, apart from the very special case $n^T = n^C$, is in general smaller than $V_{\text{max}}$. In other words, in the case (6) the option of accelerating may not apply. Hence for candidate trucks traveling at speed $v_{n^T}$ which encounter faster field cars, i.e., cars traveling at speed $v_k$ with $k = n^T + 1, \ldots, n^C$, we modify the
transition probabilities (6) as:

$$B_{n^T k}^{TC, j} = \begin{cases} 1 & \text{if } j = n^T, \\ 0 & \text{otherwise}, \end{cases} \quad k > n^T.$$ 

Notice instead that the other cases which include an acceleration, namely (8a) and (8b), can be borrowed from the single-population model without modifications.

Interestingly, model (15) along with the tables of games discussed above satisfies an indifferentiability principle similar to the one valid for kinetic models of gas mixtures, see e.g., [1]: when all the species composing the gas are identical one recovers the equations of a single-component gas. In the present case, the indifferentiability principle can be stated as follows:

**Theorem 4.1 (Indifferentiability principle).** Assume that the two types of vehicles are actually the same type, i.e., they have the same physical and kinematic characteristics (length and space of discrete speeds, respectively). Then the total distribution function

$$f_j := \sum_{p \in \{C, T\}} f^p_j$$

obeys the evolution equations of the single-population model (4).

**Proof.** If the two types of vehicles are the same we have $l^C = l^T = l$, $\rho^C_{\text{max}} = \rho^T_{\text{max}} = \frac{1}{l} =: \rho_{\text{max}}$.

It follows

$$s = \rho^C + \rho^T = \frac{\rho}{\rho_{\text{max}}} ,$$

which, together with $n^C = n^T = n$, implies ultimately $A^p, j = B^{pq}, j = A^j$, the latter being the table of games of the single-population model, cf. Section 3.1. Taking these facts into account and summing (15) over $p$ yields:

$$\frac{d}{dt} \sum_{p \in \{C, T\}} f^p_j = \sum_{h,k=1}^n A^j_{hk} \sum_{p \in \{C, T\}} f^p_h (f^p_k + f^q_k) - \sum_{p \in \{C, T\}} f^p_j \sum_{k=1}^n (f^p_k + f^q_k) ,$$

whence, using the definition of $f_j$ given in the statement of the theorem and considering that, by definition of $p$, $q$, it results also $f^p_j + f^q_j = f_j$, we discover

$$\frac{df_j}{dt} = \sum_{h,k=1}^n A^j_{hk} f_h f_k - f_j \sum_{k=1}^n f_k ,$$

which concludes the proof (recall that the interaction rate $\eta$ is hidden in the time scale). \qed

**Remark 4.2.** In [1] the indifferentiability principle is proved for a model featuring a single collision operator, which hinders the description of cross-interactions among particles of different species in the mixture. In more standard models for gas mixtures the collision terms are separate, like in our case, but the indifferentiability principle holds only at equilibrium. Here, instead, Theorem 4.1 holds at all times, moreover without having to merge the two collision terms into one.

### 4.2 A well-balanced formulation for computing equilibria

Our numerical evidence suggests that, for any pair of densities $(\rho^C, \rho^T) \in [0, \rho^C_{\text{max}}] \times [0, \rho^T_{\text{max}}]$, all initial distributions $(f^C (0), f^T (0))$ such that $\sum_{j=1}^n f^p_j (0) = \rho^p$, $p \in \{C, T\}$, converge in time to the same pair of equilibrium distributions $(f^C e, f^T e)$, which is therefore uniquely determined by $\rho^C$ and $\rho^T$. To get the correct equilibrium, however, it is important to devise a well balanced numerical scheme. As we will see, round off error can drive the solution to spurious equilibrium states, if the model is not integrated properly.
For illustrative purposes, we will henceforth assume $\alpha = 1$ in the expressions of the probabilities $P$, $Q$, which become

$$P = 1 - s, \quad Q = 0.$$ 

As a result, the structure of the tables of games is considerably simplified, nevertheless the wealth of information which can be extracted from the model is still surprising. In order to appreciate the sparsity pattern of the matrices $A^{p,j}$, $B^{pq,j}$, which permits a fast evaluation of the collision terms in (12), we write out explicitly their expressions. Let $r := 1 - s$. We report only the non zero elements, drawing a circle around those of the $j$-th row and column.

Concerning the self-interaction table of games we have:

$$A^{p,1} = \begin{bmatrix} s & s & s & \cdots & s \\ s & s & s & \cdots & s \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s \\ \end{bmatrix},$$

$$A^{p,n_p} = \begin{bmatrix} 0 & 0 & \cdots & r & r & \cdots & r \\ r & 0 & \cdots & r & r & \cdots & r \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ r \\ \end{bmatrix},$$

while the general expression for $1 < j < n_p$ is

$$A^{p,j} = \begin{bmatrix} r & \cdots & r \\ \vdots & \ddots & \vdots \\ r & \cdots & r \\ \end{bmatrix}.$$ 

These matrices are all $n_p \times n_p$.

The cross-interaction matrices $B^{CT,j}$ between cars (candidates) and trucks (fields) have the same structure as the $A^{p,j}$'s, apart from being rectangular of dimensions $n_C \times n_T$. Differences however arise for $j \geq n_T$, for example:

$$B^{CT,n_T} = \begin{bmatrix} 0 & \cdots & r \\ \vdots & \ddots & \vdots \\ r & \cdots & r \\ \end{bmatrix},$$

$$B^{CT,n_T+1} = \begin{bmatrix} 0 & \cdots & r \\ \vdots & \ddots & \vdots \\ r & \cdots & r \\ \end{bmatrix},$$

and in general

$$B^{CT,j} = \begin{bmatrix} \circ \cdots \circ \\ \vdots & \ddots & \vdots \\ \circ \cdots \circ \\ \end{bmatrix}.$$ 

Finally, the cross-interaction matrices $B^{TC,j}$ between trucks (candidates) and cars (fields) are $n_T \times n_C$. They can in turn be easily derived from the $A^{p,j}$'s, the only different case being the one for $j = n_T$:

$$B^{TC,n_T} = \begin{bmatrix} 0 & \cdots & r \\ \vdots & \ddots & \vdots \\ r & \cdots & r \\ \end{bmatrix}.$$
Still under the simplifying assumption $\alpha = 1$, in [9] existence and uniqueness of stable equilibria for the single-population model (4) are established and their analytic expressions are computed, thereby shedding light on the structure of the fundamental diagrams. Here instead we will proceed by integrating numerically in time the system of ODEs (15) until steady states are reached.

It is worth pointing out that in [9] equilibria are studied by rewriting the loss term $-f_j \sum_{k=1}^{n} f_k$ of (4) in the analytically equivalent form $-\rho f_j$. This allows one to take advantage of the fact that $\rho$ is indeed a parameter of system (4) fixed by the initial condition, since owing to (3) it is constant in time. However, such a simplification cannot be carried out when the system is integrated numerically, because of instabilities triggered by round-off errors. For the sake of simplicity, we illustrate this phenomenon for the single-population model (4) but our considerations apply to the two-population model (15) as well.

In practice, we need to discuss the numerical approximation of the following two analytically equivalent formulations of the single-population model:

$$\frac{df_j}{dt} = \sum_{h,k=1}^{n} A_{hk} f_h f_k - f_j \sum_{k=1}^{n} f_k \quad \text{(17a)}$$

$$\frac{df_j}{dt} = \sum_{h,k=1}^{n} A_{hk} f_h f_k - \rho f_j \quad \text{(17b)}$$

whose the first, which we will call well-balanced, leads to the computation of the correct equilibria while the second does not preserve stationary solutions and possibly leads to a violation of the mass conservation. The context is similar to the construction of well-balanced numerical schemes for balance laws, where particular care is needed in order to preserve stationary solutions at the discrete level, see e.g., [20, 22] and references therein.

Call $y = \sum_{j=1}^{n} f_j$. Summing over $j$ both sides of (17a) yields $\frac{dy}{dt} = 0$ as expected, while the same operation performed on (17b) gives

$$\frac{dy}{dt} = (y - \rho)y.$$

In this case $y$ is in general not constant in time and moreover two equilibria exist, $y_1^e = 0$ and $y_2^e = \rho$, whose the first is stable and attractive, whereas the second is unstable. This means that mass conservation $y(t) = \rho$ holds for all $t$ if and only if $y$ is computed without round-off. Otherwise, for every small perturbation of $y$ mass conservation fails, since the numerical solution is invariably driven toward the spurious equilibrium $y_1^e$.

Figure 3 shows the results of the numerical integration of the two equations (17a), (17b) in the simple case with $n = 2$ speed classes and optimal environmental conditions ($\alpha = 1$), starting from initial conditions for which the vehicle density is either $\rho = 0.3$, $\rho = 0.5$, or $\rho = 0.7$. As proved in [9], the correct equilibrium distribution is $f^e = (0, \rho)$ for $\rho \leq 0.5$, but from the first two panels.
of Fig. 3 it can be seen that only the solution of (17a) converges to such an equilibrium, that of (17b) being instead attracted toward the state (0, 0). For $0.5 < \rho < 1$ the correct equilibrium distribution $f^*$ consists instead of two strictly positive values, which once again are reached only by the numerical solution to the well-balanced formulation (17a) as it is evident from the third panel of Fig. 3.

The phase portrait of system (17b), see Fig. 4, shows that any initial condition can either be driven to the spurious equilibrium (0, 0) or become unbounded for any slight perturbation.

5 Fundamental diagrams of the two-population model

In this section we investigate numerically the fundamental diagrams resulting from the two-population kinetic model (15). As we will see, they do not only capture the main qualitative features of the experimental diagrams of Fig. 1, including especially the data dispersion in the congested flow regime, but they also provide tools to better understand the behavior of traffic at the macroscopic scale.

In all of the addressed case studies, system (15) is integrated numerically up to equilibrium, using the well balanced formulation (17a). Once the equilibrium distributions have been obtained, the flux and the mean speed are obtained as moments of the kinetic distributions as indicated in (11). Since in the space homogeneous case the total density $\rho = \sum_p \rho^p$ is constant in time, it acts as a parameter, fixed by the initial condition, indexing the aforesaid macroscopic quantities.

As a matter of fact, also each $\rho^p$ is constant in time, therefore so is the fraction of road occupancy $s$ defined in (16). It is then possible to study the flux and mean speed at equilibrium also as functions of $s$. Summarizing, we will study two types of equilibrium diagrams:

- **Flux-density diagrams**, that is diagrams relating the total flux at equilibrium $\sum_p \sum_{j=1}^{n^p} t_j f^*_{j, p}$ to the total density $\rho = \sum_p \rho^p$, irrespective of the size of the different vehicles. Experimental diagrams are indeed expected to represent such a relationship.

- **Flux-space diagrams**, that is diagrams relating the total flux at equilibrium to the fraction of road occupancy $s$.

Except when otherwise stated, all simulations are performed with the parameters indicated in Table 1. Initially we consider $n^C = 3$ speed classes for cars and $n^T = 2$ speed classes for trucks, hence the corresponding spaces of microscopic speeds are

$$\mathcal{V}^C = \{0, 50 \text{ km/h}, 100 \text{ km/h}\}, \quad \mathcal{V}^T = \{0, 50 \text{ km/h}\},$$
Table 1: Parameters of model \(^{(15)}\) common to all simulations.

| Parameter | Description                  | Value               |
|-----------|------------------------------|---------------------|
| \(\alpha\) | Quality of the environment   | 1                   |
| \(l_C\)   | Typical length of a car      | 4 m                 |
| \(l_T\)   | Typical length of a truck    | 12 m                |
| \(\rho_C^{\text{max}}\) | Maximum car density | 250 vehicles/km |
| \(\rho_T^{\text{max}}\) | Maximum truck density       | 83.3 vehicles/km   |
| \(V_{\text{max}}\) | Maximum speed               | 100 km/h            |

Table 2: Deterministic pairs \((\rho_C, \rho_T)\) used in the fundamental diagrams of Figs. 5–7 for given values of the fraction of road occupancy \(s\).

| Combination type                  | Marker   | Expression                                      | \(\rho_C\) | \(\rho_T\) |
|-----------------------------------|----------|-------------------------------------------------|------------|------------|
| Space occupied mostly by cars     | Crosses  | \(\rho_T^T = \frac{1}{2} \rho_C^T l_C\)        | \(\frac{1}{2}\) | \(\frac{1}{2}\) |
| Space evenly occupied by cars and trucks | Circles  | \(\rho_T^T = \rho_C^T l_C\)                  | \(\frac{1}{2}\) | \(\frac{1}{2}\) |
| Space occupied mostly by trucks   | Dots     | \(\rho_T^T = 2 \rho_C^T l_C\)                  | \(\frac{1}{2}\) | \(\frac{1}{2}\) |

Figure 5: Flux-space diagrams for the three conditions of road occupancy listed in Table 2.

cf. Section 4.1.

In Fig. 5 we show the flux-space diagrams of either class of vehicles obtained using deterministic initial conditions: for each \(s \in [0, 1]\) we select three prototypical pairs \((\rho_C, \rho_T) \in [0, \rho_C^{\text{max}}] \times [0, \rho_T^{\text{max}}]\) such that \(\rho_C^T l_C + \rho_T^T l_T = s\), corresponding to different conditions of road occupancy, cf. Table 2. The obtained fundamental diagrams are qualitatively similar to those resulting from the single-population model \((2)\) with analogous numbers of speed classes. For instance, the fundamental diagram of the sole trucks compares well with the one shown in Fig. 2 with \(n = 2\).
speed classes.

The diagrams show clearly that there is a critical threshold of occupied space beyond which the flux decreases. In particular, the graphs in the top row of Fig. 5, which display the flux of cars and trucks as a function of the sole space $\rho^p$ occupied by the corresponding class of vehicles, show that such a threshold changes depending on the considered component of the traffic mixture. This implies, in other words, that the dynamics of a single species still depend on those of the entire mixture. Conversely, the graphs in the bottom row of Fig. 5, which display the flux of both cars and trucks as a function of the total fraction of occupied space $s$, show that there is a single value for the critical threshold, specifically $s = 0.5$, common to all of the three combinations of space occupancy listed in Table 2. This result, which is consistent with the one found in [9] for the single-population model, suggests that phase transition in traffic does not depend on how in detail, but rather on how much on the whole, the road is occupied.

In Fig. 6 we compare the fundamental diagrams obtained by using either the three deterministic pairs $(\rho^C, \rho^T)$ given in Table 2 or three pairs chosen randomly for each $s$. In practice, the latter case shadows forth the appearance of the fundamental diagram as a filled region in the $q - s$ plane which would result if, for each $s \in [0, 1]$, all possible pairs $(\rho^C, \rho^T) \in [0, \rho^C_{\text{max}}] \times [0, \rho^T_{\text{max}}]$ such that $\rho^C l^C + \rho^T l^T = s$ were used. In spite of the apparent data dispersion, such a diagram does not however reproduce the experimental data, because the information brought by the fraction of road occupancy $s$ is too synthetic as far as taking into account the heterogeneity of traffic is concerned.

Motivated by this argument, we turn now to considering flux-density diagrams, which catch better the microscopic composition of traffic because they are based on the number of vehicles per kilometer of road. As previously anticipated, experimental fundamental diagrams are indeed expected to result out of this type of observations.

Again, for each $s \in [0, 1]$ the graphs in Fig. 7 are obtained by taking three pairs $(\rho^C, \rho^T)$ corresponding to the combinations reported in Table 2. This time, however, the equilibrium fluxes $q^p$, $\sum_p q^p$ are plotted vs. the densities $\rho^p$, $\rho = \sum_p \rho^p$, $p = C, T$. It can be noticed that each combination features now a different critical density value for the phase transition, which depends on the percentage of the different species within the mixture. Nevertheless, from the previous case study we know that such critical values, say $\rho^*_p$, satisfy the relationship

$$\rho^C l^C + 3 \rho^T l^T = 0.5,$$

whence, considering further that in the present context we are assuming $l^T = 3 l^C$ with $l^C = 4$ m, cf. Table 1,

$$\rho^*_C + 3 \rho^*_T = 125 \text{ vehicles/km.}$$
By sampling three random pairs \((\rho^C, \rho^T)\) for any given \(s \in [0, 1]\) we obtain the fundamental diagram illustrated in Fig. 8, which clearly captures the main characteristics of the experimental diagrams discussed in Section 2. In particular, at low densities the total flux grows nearly linearly with small dispersion, while at higher densities it decreases with larger dispersion due to the frequent interactions between fast and slow vehicles. In the graph, cyan circles indicate the total density-total flux pairs obtained for \(s \in [0, 0.8]\), whereas blue crosses indicate those obtained for \(s \in [0, 0.8]\). As a matter of fact, the latter are the most likely to occur in practice, since even in traffic jams vehicles attain seldom a state of maximum density and complete stop (see e.g., Fig. 1, where a residual movement always appears).

The examples discussed so far have put in evidence that the bulk characteristics of traffic at equilibrium can be predicted deterministically once the composition of traffic, i.e., the pair \((\rho^C, \rho^T)\), is known. This induces to interpret the scattering of data in the congested phase as a consequence of the possible heterogeneity of traffic for a given level of road occupancy rather than as an effect of the unpredictability of driver behaviors.

Such a statement can be articulated more precisely by considering the two-population model (15) with vehicles differing by only one microscopic characteristic. In particular, we consider the case...
in which vehicles have the same length: $l^T = l^C = 4$ m but different spaces of microscopic speeds:

\[ V^C = \{0, 50 \text{ km/h}, 80 \text{ km/h}, 100 \text{ km/h} \}, \quad V^T = \{0, 50 \text{ km/h}, 80 \text{ km/h} \}, \]

and the case in which they have different length as reported in Table 1 but the same spaces of microscopic speeds. In Fig. 9 the traffic diagrams vs. the total density on the left and on the right correspond to these two cases, respectively. By inspecting them we infer that differences in the speeds (in particular, the maximum ones) of the vehicles composing the traffic mixture are responsible for the small scattering of the data in the free flow phase, whereas differences in the length determine the larger scattering of the data in the congested flow phase. This conclusion is indeed consistent with daily experience: in free flow drivers keep different maximum speeds according to their driving style, while in congested flow they tend to travel all at the same speed, which steadily decreases as the traffic congestion increases.

The same two cases are further investigated in Fig. 10 by focusing on the speed diagram vs. the fraction of road occupancy $s$. In particular, when vehicles have different microscopic speeds but same length (diagram on the left) we deduce that, in free flow conditions, the slower population is not affected by the faster one, while the latter interacts with the former as the small dispersion in its mean speed demonstrates. In congested flow conditions, instead, both types of vehicles are forced to slow down reaching finally the same mean speed. Conversely, when vehicles have the same microscopic speeds but different lengths (diagram on the right) we discover that the mean...
speed is the same for both populations in both traffic regimes, i.e., in other words, it is a one-to-one function of the fraction of occupied space.

Finally, we stress that the kinetic approach, thanks to its closer connections with the microscopic scale of the system, seems to be essential for catching the sharp phase transition in traffic flow. Figure 11 shows the fundamental diagram for the traffic mixture modeled by the two-population macroscopic model [4] summarized in Section 4. It is immediate to notice that there is no trace of the sharp phase transition predicted by the kinetic model and that the scattering of the data is very high also at low densities.

6 Conclusions and perspectives

In this paper we have introduced a kinetic model for vehicular traffic with a new structure which accounts for the heterogeneous composition of the flow of vehicles. Our approach differs from standard kinetic models in that we consider two distribution functions describing two classes of particles with different physical features, in this case the typical length of a vehicle and its
maximum speed.

Like in [8], the model is built by assuming a discrete space of microscopic speeds and by expressing vehicle interactions in terms of transition probabilities among the admissible speed classes. We have shown that our two-population model satisfies an indifferentiability principle, which makes it consistent with the original single-population model when the particles composing the mixture share the same physical characteristics (in our case the vehicle length and maximum speed). This property, enforced in [11], is not trivial, and several kinetic models for gas mixtures possess it only at equilibrium [12, 14].

We have then used our two-population kinetic model to perform a computational analysis of the equilibria of the system and to derive in this way the fundamental diagrams predicted by the simulated dynamics. Even with a small number of microscopic speeds, such diagrams feature a structure closely resembling experimental data. In particular, they are characterized by a marked phase transition: at low vehicle densities (free flow) the flux increases almost linearly with small standard deviation, while beyond a critical density value it decreases taking widely scattered values (congested flow).

Several authors have dealt with this problem, cf. e.g., [9, 13] where this phenomenon is explained by invoking the uncertainty of driver behaviors in terms of standard deviation of the statistical distribution of speeds at equilibrium. However, such an approach predicts a zero standard deviation in the free phase of traffic and furthermore interprets the scattered distribution of the data in the congested phase as a consequence of the variability of the microscopic speeds at equilibrium. In our case, instead, we do not only recover the sharp phase transition, which seems to result naturally from a kinetic approach, but we also obtain the scattered behavior at a genuinely macroscopic level as a consequence of the fact that a given road occupancy can be obtained with different compositions of the mixture. In other words, if the flux is given as a function of the number of vehicles crossing a section of road in the unit time then our model indicates that the scattering may be due to the simultaneous presence of different types of vehicles. On the other hand, in the congested phase the mean speed of the vehicles seems to depend only on the degree of congestion of the road.

From the point of view of applications, our results can be used to direct the strategies of data collection in experimental research on traffic flow. In fact, other types of fundamental diagrams can be studied, besides the standard ones discussed so far. For example, with a multi-population model it is possible to study the volume of goods carried by trucks in a current of cars or the number of passengers traveling in a flow composed of cars and buses.

Finally, we also wish to note that the model is very simple: the complexity of the real flow is clustered in the characteristics of only two distinct populations, with a very small number of microscopic velocities. Thus, from a computational point of view, this construction is not significantly more demanding than a macroscopic model.

As far as the analytical properties of the model are concerned, we can prove the well-posedness of the Cauchy problem associated with [15], in the sense that the solution exists, is unique, depends continuously on the initial data, and moreover remains nonnegative and bounded by the initial mass. Furthermore, we can also prove that equilibria, which define the fundamental diagrams, are uniquely determined by the initial mass of the two classes of vehicles, and, in some simplified cases, they can be computed explicitly. These results will be gathered in a forthcoming paper [27]. Additional study will be dedicated to the extension of the present model to road networks and multilane highways.

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