A HOMOLOGICAL REPRESENTATION FORMULA OF COLORED
ALEXANDER INVARIANTS

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Abstract. We give a formula of the colored Alexander invariant in terms of the homological
representation of the braid groups which we call truncated Lawrence’s representation. This
formula generalizes the famous Burau representation formula of the Alexander polynomial.

1. Introduction

The Alexander polynomial is one of the most important and fundamental knot invariant having
various definitions and interpretations. Each definition brings a different prospect and often leads
to different generalizations.

In this paper we explore one of the generalizations of the Alexander polynomial, the colored
Alexander invariant introduced in [ADO]. This is a family of invariants \( \Phi^N_K \) of a link \( K \) indexed
by positive integers \( N = 2, 3, \ldots \), and \( \Phi^2_K \) coincides with the (multivariable) Alexander polynomial
[Mu1]. The first definition of the colored Alexander invariant in [ADO] uses a state-sum inspired
from solvable vertex model in physics. As is already noted in [ADO] and made it clarified in [Mu2],
the \( N \)-th colored Alexander invariant \( \Phi^N_K \) is a version of a quantum \( \mathfrak{sl}_2 \) invariant at \( 2N \)-th root of
unity.

Throughout the paper we treat the case that \( K \) is a knot. Then the colored Alexander invariant
\( \Phi^N_K(\lambda) \) is a function of one variable \( \lambda \). We give a new formulation of the colored Alexander invariant,
in the same spirit as [It2] where we gave a topological formulation of the loop expansion of the
colored Jones polynomials.

In Theorem 5.3 we show that the colored Alexander invariant is written as a sum of the traces of
homological representations which we call truncated Lawrence’s representation. These are variants
of Lawrence’s representation studied in [Law] [It1] [It2], derived from an action on the configuration
space. Our formula can be seen as a generalization of the Burau representation formula of the
Alexander polynomial and has more topological flavor.

A key result is Theorem 4.2 where we show that truncated Lawrence’s representation is identi-
ified with a certain quantum \( \mathfrak{sl}_2 \) braid group representation, defined for the case the quantum
parameter \( q \) is put as \( 2N \)-th root of unity. This generalizes a relation between Lawrence’s repre-
sentation and generic quantum \( \mathfrak{sl}_2 \)-representation [Koh] [It1], and is interesting in its own right.

Unfortunately, unlike the Burau representation formula, our formula is not completely topolog-
ical since it heavily depends on a particular presentation (closed braid representative) of knots,
and we cannot see its topological invariance directly, although it suggests a relationship to the
topology of abelian coverings of the configuration space.

We remark that the colored Alexander invariant \( \Phi^N_K(\lambda) \) at \( \lambda = (N - 1) \) is equal to the \( N \)-
colored Jones polynomial at \( N \)-th root of unity [MuMu], which in turn, is equal to Kashaev’s
invariant derived from the quantum dilogarithm function [Kas]. Thus, our formula also brings a
new prospect for Kashaev’s invariant and the volume conjecture.

Acknowledgements

The author gratefully thanks Jun Murakami who suggests a generalization of the author’s previ-
ous work on colored Jones polynomial to the colored Alexander invariants. He also wish to thank
2. Quantum $\mathfrak{sl}_2$ representation and colored Alexander invariant

2.1. Generic quantum $\mathfrak{sl}_2$ representation. For a complex parameter $q \in \mathbb{C} - \{0,1\}$, we define

$$[a]_q = \frac{q^a - q^{-a}}{q - q^{-1}}, \quad [a; i]_q = \begin{cases} 1 & (i = 0), \\ [a]_q \cdots [a + 1 - i]_q & (i > 0). \end{cases}$$

Then $[a]_q! = [a; a]_q = [a]_q[a - 1]_q \cdots [1]_q$, and

$$[a]_q! = [a; i]_q = \frac{[a]_q!}{[b]_q!}.$$

Let $U_q(\mathfrak{sl}_2)$ be the quantum $\mathfrak{sl}_2$, defined by

$$U_q(\mathfrak{sl}_2) = \left\{ K, K^{-1}, E, F \mid KK^{-1} = K^{-1}K = 1, \quad KEK^{-1} = q^{2E}, \quad KFK^{-1} = q^{-2}F \right\}.$$

For $\lambda \in \mathbb{C}^*$, let $\widehat{V}_\lambda$ be a $\mathbb{C}$-vector space spanned by $\{e_0^\lambda, e_1^\lambda, \ldots, \}$, equipped with an $U_q(\mathfrak{sl}_2)$-module structure by

$$\begin{cases} Ke_i^\lambda = q^{(\lambda - 2i)} e_i^\lambda, \\ Ee_i^\lambda = \sum_{j=0}^{i-1} e_j^\lambda, \\ Fe_i^\lambda = [i + 1]_q [\lambda - i]_q e_{i+1}^\lambda. \end{cases}$$

Here $[\lambda - i]_q = \frac{q^{\lambda - i} - q^{-(\lambda - i)}}{q - q^{-1}}$. We define the $R$-operator $R$ by

$$R = R_{\lambda, \mu} = q^{-\frac{1}{2} \mu T} \circ R : V_\lambda \otimes V_\mu \to V_\mu \otimes V_\lambda,$$

where $T$ is the transposition $T(v \otimes w) = w \otimes v$ and $R$ is a universal $R$-matrix of $U_q(\mathfrak{sl}_2)$. When $\mu = \lambda$, $R$ gives rise to a braid group representation

$$\varphi_{\widehat{V}_\lambda} : B_n \to GL(\widehat{V}_\lambda^\otimes n), \quad \varphi(\sigma_i) = \text{id}^\otimes (i-1) \otimes R \otimes \text{id}^\otimes (n-i).$$

Let $V_n, m = \widehat{V}_{n, m}(\lambda)$ be a subspace of $\widehat{V}_\lambda^\otimes n$ spanned by $\{e_0^\lambda, \ldots, e_n^\lambda \mid e_1 + \cdots + e_n = m\}$, and let $\widehat{W}_{n, m} = \widehat{W}_{n, m}(\lambda) = \text{Ker} E \cap \widehat{V}_{n, m}$. Then $V_n, m$ and $\widehat{W}_{n, m}$ are finite-dimensional braid group representations with dimension $(n + m - 1)$ and $(n + m - 2)$, respectively.

To relate $\widehat{V}_\lambda$ with irreducible finite dimensional $U_q(\mathfrak{sl}_2)$-modules, let $V_\lambda$ be subspace of $\widehat{V}_\lambda$ spanned by $\{v_0^\lambda, v_1^\lambda, \ldots, \}$, where we define $v_j^\lambda = [\lambda; j]_q e_j^\lambda$. Then $V_\lambda$ is a sub $U_q(\mathfrak{sl}_2)$-module with much familiar action

$$\begin{cases} K v_i^\lambda = q^{(\lambda - 2i)} v_i^\lambda, \\ E v_i^\lambda = \sum_{j=0}^{i-1} v_j^\lambda, \\ F v_i^\lambda = [i + 1]_q v_{i+1}^\lambda. \end{cases}$$

If $\lambda$ is equal to a positive integer $\alpha - 1$, then $v_i^\lambda = 0$ for $i \geq \alpha$, and $V_\lambda$ is nothing but the $\alpha$-dimensional irreducible $U_q(\mathfrak{sl}_2)$-module.

As in the case $\widehat{V}_\lambda$, $R$ leads to the braid group representation $\varphi_{\lambda, \lambda} : B_n \to GL(V_\lambda^\otimes n)$, and $V_{n, m} = \widehat{V}_{n, m} \cap V_\lambda^\otimes n$ and $\widehat{W}_{n, m} = \widehat{W}_{n, m} \cap V_\lambda^\otimes n$ also give rise to braid group representations.

In the definition of $R$, the term $q^{-\frac{1}{2} \mu T}$ corresponds to a part of the framing correction (see also Remark 2.3). Thanks to this modification, the action of $R_{\lambda, \lambda}$ is written by

$$R(\widehat{e}_i^\lambda) = q^{-\lambda (i + j)} \sum_{n=0}^i q^{2(n - i)(n + j)} q^{\frac{n(n - 1)}{2}} \left[ \begin{array}{c} n + j \\ n \end{array} \right]_q [\lambda - i; n]_q (q - q^{-1})^n \widehat{e}_j^\lambda \otimes \widehat{e}_{i-n}^\lambda.$$

A remarkable feature is that each coefficient in (2.2) is a Laurent polynomial of $q$ and $q^\lambda$. Thus by regarding $q$ and $q^\lambda$ as abstract variables, one can define the braid group representation $V_\lambda^\otimes n$ over the Laurent polynomial ring $\mathbb{Z}[q^{\pm 1}, q^\pm \lambda]$, which we call a generic quantum $\mathfrak{sl}_2$-representation (see [JK] [II] for details).
We will consider the following two kind of genericity conditions.

**Definition 2.1.** We say that \( \lambda \) is generic with respect to \( q \) if \([\lambda - i]_q \neq 0\) for all \( i \in \mathbb{Z} \). We also say that \( q \) and \( \lambda \) are fully generic if the subset \( \{1, q, q^\lambda\} \) is algebraically independent.

**Lemma 2.2.**

(i) If \( \lambda \) is generic with respect to \( q \), then \( \widehat{V}_\lambda \) and \( V_\lambda \) are isomorphic as \( U_q(\mathfrak{sl}_2) \)-modules. In particular, the braid group representations \( \widehat{V}_\lambda \otimes^n \) and \( V_\lambda \), \( (V_{n,m} \) and \( W_{n,m} \) and \( W_{n,m} \)) are isomorphic.

(ii) If \( \lambda \) and \( q \) are fully generic, then the braid group representation \( \widehat{V}_\lambda \otimes^n \) splits as

\[
\widehat{V}_\lambda \otimes^n \cong \bigoplus_{m=0}^{\infty} V_{n,m} \cong \bigoplus_{m=0}^{\infty} \bigoplus_{k=0}^{m} F^{m-k} W_{n,k}.
\]

**Proof.** (i) follows from the definition of \( V_\lambda \). (ii) is [JK] Lemma 13. (We remark that the modules \( V_{n,m} \) and \( W_{n,m} \) in this paper correspond to \( V_{n,m} \) and \( W_{n,m} \) in [JK]). \( \square \)

From now on, we will always assume that \( \lambda \) is generic with respect to \( q \) so that we do not need to distinguish \( \widehat{V}_\lambda \) with \( V_\lambda \). We will mainly work on \( V_\lambda \).

**2.2. The case \( q \) is a root of unity.** Let us put \( q \) as the \( 2N \)-th root of unity \( \zeta = \zeta_N = \exp(\frac{2\pi i}{2N}) \). As we remarked in the previous section, we assume that \( \lambda \) is generic with respect to \( \zeta \).

By (2.1), \( \{v_0^\lambda, \ldots, v_{N-1}^\lambda\} \subset V_\lambda \) spans an irreducible \( N \)-dimensional irreducible \( U_\zeta(\mathfrak{sl}_2) \)-module, a central deformation of the \( N \)-dimensional irreducible \( U_q(\mathfrak{sl}_2) \) module [Mu2]. We denote this \( N \)-dimensional irreducible \( U_\zeta(\mathfrak{sl}_2) \)-module by \( U_N(\lambda) \), and the corresponding braid group representation by \( \varphi_{U_N} : B_n \to \text{GL}(U_N(\lambda)^{\otimes n}) \).

As an \( U_\zeta(\mathfrak{sl}_2) \)-module, the tensor product decomposes as

\[
U_N(\lambda) \otimes U_N(\mu) \cong \bigoplus_{i=0}^{N-1} U_N(\lambda + \mu - 2i)
\]

so iterated use of (2.3) gives a decomposition as an \( U_\zeta(\mathfrak{sl}_2) \) module

\[
U_N(\lambda)^{\otimes n} \cong \bigoplus_{i=0}^{N-1} T_i \otimes U_N(n\lambda - 2i).
\]

Here \( T_i \) is the intertwiner space \( \text{End}_{U_\zeta(\mathfrak{sl}_2)}(U_N(n\lambda - 2i), U_N(\lambda)^{\otimes n}) \), the set of endomorphisms equivariant with respect to the \( U_\zeta(\mathfrak{sl}_2) \)-actions. The dimension of \( T_i \) is the multiplicity of the direct summands.

The \( U_\zeta(\mathfrak{sl}_2) \)-module \( U_N(\lambda) \) is generated by a highest weight vector \( v_0^\lambda \), so we identify \( T_i \) with the subspace consisting of highest weight vectors

\[
T_i = \{ v \in U_N(\lambda)^{\otimes n} | E v = 0, K v = \zeta^{n\lambda - 2i} v \}.
\]

In particular, we may view the decomposition (2.4) as

\[
U_N(\lambda)^{\otimes n} = \bigoplus_{i=0}^{N-1} (T_i \oplus FT_i \oplus \cdots \oplus F^{N-1} T_i).
\]

Next we look at the braid group action. As in the previous section, let \( X_{n,m}^N = X_{n,m}^N(\lambda) \) be the subspace of \( U_N(\lambda)^{\otimes n} \) spanned by \( \{v_{e_1}^\lambda \otimes \cdots \otimes v_{e_n}^\lambda | e_1 + \cdots + e_n = m\} \), and let \( Y_{n,m}^N = \text{Ker} E \cap X_{n,m}^N \). Both \( X_{n,m}^N \) and \( Y_{n,m}^N \) are braid group representations, and as a braid group representation, \( U_N(\lambda)^{\otimes n} \) splits as

\[
U_N(\lambda)^{\otimes n} \cong \bigoplus_{m=0}^{n(N-1)} X_{n,m}^N.
\]
By definition, \( K v = \zeta^{n \lambda - 2m} v \) for \( v \in X^N_{n,m} \). Hence by (2.5), the intertwiner space \( T_i \) in (2.4) is identified with the direct sums of \( Y^N_{n,m} \).

\[
(2.7) \quad T_i = \left\{ v \in \bigoplus_{m=0}^{n(N-1)} X^N_{n,m} | \text{End}(V) = 0, \quad K v = \zeta^{n \lambda - 2m} v \right\} = \bigoplus_{j=0}^{n-1} Y^N_{n,i+(N-1)j}.
\]

Summarizing, the decomposition of \( U_N(\lambda)^{\otimes n} \) as \( U_\zeta(\mathfrak{sl}_2) \)-representation (2.4) is written in terms of \( Y^N_{n,m} \) as

\[
(2.8) \quad U_N(\lambda)^{\otimes n} = \bigoplus_{i=0}^{N-1} \left( \bigoplus_{k=0}^{N-1} F^k \left( \bigoplus_{j=0}^{n-1} Y^N_{n,i+(N-1)j} \right) \right).
\]

2.3. Colored Alexander invariant. The colored Alexander invariant is the quantum \( \mathfrak{sl}_2 \) invariant coming from \( U_N(\lambda) \). However, the quantum trace of the quantum representation \( \varphi_{U_N} \) is trivial so we need a trick (see [GPMT] for a general framework).

For a finite dimensional free \( \mathbb{C} \)-vector space \( V \), \( \text{End}(V) \cong V \otimes V^* \), where \( V^* \) is the dual of \( V \), and \( \text{trace} : \text{End}(V) \to \mathbb{C} \) is identified with the contraction. Let \( V^{\otimes n} = V_1 \otimes V_2 \otimes \cdots \otimes V_n \) be the tensor products of \( n \) copies of \( V \). The partial trace

\[
\text{trace}_{2,\ldots,n} : \text{End}(V^{\otimes n}) \to \text{End}(V) = \text{End}(V_1)
\]

is the map defined by the composition of the following natural maps,

\[
\begin{array}{ccc}
\text{End}(V^{\otimes n}) & \xrightarrow{=} & (V^{\otimes n}) \otimes (V^*)^{\otimes n} \\
\downarrow \text{trace}_{2,\ldots,n} & & \downarrow \text{id} \otimes \cdots \otimes \text{trace} \\
\text{End}(V) & \xrightarrow{=} & (V_1 \otimes V_1^*) \otimes (V_2 \otimes V_2^*) \otimes \cdots \otimes (V_n \otimes V_n^*)
\end{array}
\]

Let \( K \) be an oriented knot in \( S^3 \) represented as the closure of an \( n \)-braid \( \beta \). We cut the first strand of \( \beta \) to get an (1,1)-tangle \( T_\beta \). Let \( Q^{U_N}(T_\beta) : U_N(\lambda) \to U_N(\lambda) \) be an operator invariant of the tangle \( T_\beta \) (see [Oht, Chapter 3]), defined by

\[
(2.9) \quad Q^{U_N}(T_\beta) = \zeta^{(N-1)\lambda e(\beta)} \text{trace}_{2,\ldots,n} (\text{id} \otimes h^{(n-1)} \circ \varphi_{U_N}(\beta)) : U_N(\lambda) \to U_N(\lambda)
\]

where \( h = K^{1-N} : U_N(\lambda) \to U_N(\lambda) \) is given by \( h(v^\lambda) = \zeta^{N-1} \lambda \gamma e(\beta) v^\lambda \), and \( e : B_n \to \mathbb{Z} \) is the exponent sum homomorphism, defined by \( e(\gamma_{i+1}^\pm) = \pm 1 \).

Remark 2.3. By [Mu2] the framing correction is given by \( \zeta^{-\frac{1}{2}(\lambda + 2 - 2N)e(\beta)} \), but as we noted in the previous section, the part of the framing correction \( \zeta^{-\frac{1}{2}\lambda e(\beta)} \) is already included in the definition of \( \mathbb{R} \)-operator. This explains the framing correction term \( \zeta^{(N-1)\lambda e(\beta)} \) in (2.9).

Since \( U_N(\lambda) \) is irreducible, \( Q^{U_N}(T_\beta) \) is a scalar multiple of the identity

\[
(2.10) \quad Q^{U_N}(T_\beta) = \Phi^N_K(\lambda) \text{id}.
\]

It turns out \( \Phi^N_K(\lambda) \) is independent of a choice of a closed braid representative.

Definition 2.4 (Colored Alexander invariant [ADO, Mu2]). The colored Alexander invariant of color \( N \in \{2,3,\ldots\} \) is the scalar \( \Phi^N_K(\lambda) \) in (2.10), viewed as a function of the variable \( \lambda \).

It is convenient to put \( x = \zeta^{-2} \lambda \) and write \( CA^N_K(x) = \Phi^N_K(\lambda) \), a rational function of the variable \( x^\pm \) (Note that by (2.2) and (2.9), \( CA^N_K(x) \) is indeed a rational function of \( x^\pm \)). We call \( CA^N_K(x) \) the colored Alexander polynomial, whose name is justified by the fact \( CA^N_K(x) = \Delta_K(x) = \nabla_K(x^{\frac{1}{2}} - x^{-\frac{1}{2}}) \), where \( \nabla_K(t) \) denotes the Conway polynomial (See [Mu1] and Example 5.4).
3. Homological representations

3.1. Lawrence’s representation. We review a homological braid group representation \( L_{n,m} \) which we call Lawrence’s representation. See [Law], [11] Section 3, [H2] Appendix. In the case \( m = 2 \), it is known as Lawrence-Krammer-Bigelow representation studied in [Kra],[Big], and in the case \( m = 1 \) it is nothing but the reduced Burau representation.

We identify the braid group \( B_n \) with the mapping class group of the \( n \)-punctured disc

\[
D_n = \{ |z| \in \mathbb{C} \mid z \leq n + 1 \} - \{1, 2, \ldots, n\} = D^2 - \{p_1, \ldots, p_n\},
\]

so that the standard generator \( \sigma_i \) of \( B_n \) corresponds to the right-handed half Dehn twist along the real line segment \([i,i+1])\subset D_n\).

Let \( C_{n,m} \) be the unordered configuration space of \( m \)-points in \( D_n \),

\[
C_{n,m} = \{(z_1, \ldots, z_m) \in D^n_n \mid z_i \neq z_j \text{ for } i \neq j\}/S_m.
\]

Here \( S_m \) denotes the symmetric group acting as a permutation of the indices. Fix a sufficiently small \( \varepsilon > 0 \), and let \( d = (n+1)e^{(\frac{1}{3}+\varepsilon)s}\sqrt{\pi} \in \partial D_n \). We use \( d = \{d_1, \ldots, d_m\} \) as a base point of \( C_{n,m} \). It is known that \( H_1(C_{n,m}) = \mathbb{Z}^n \oplus \mathbb{Z} \), where the first \( n \) components are spanned by the meridians of the hyperplanes \( \{z_i = p_i\} \) \( (i = 1, \ldots, n) \), and the last component is spanned by the meridian of the discriminant \( \bigcup_{1 \leq i < j \leq n} \{z_i = z_j\} \).

Let \( \alpha \) be the homomorphism

\[
\alpha : \pi_1(C_{n,m}) \xrightarrow{\text{Hurewicz}} H_1(C_{n,m}; \mathbb{Z}) = \mathbb{Z}^n \oplus \mathbb{Z} \xrightarrow{C} \mathbb{Z} \oplus \mathbb{Z} = (x) \oplus (d)
\]

where the first map is the Hurewicz homomorphism and the second map is defined by \( C(x_1, \ldots, x_n, d) = (x_1 + \cdots + x_n, d) \). Let \( \pi : \widetilde{C}_{n,m} \to C_{n,m} \) be the covering that corresponds to \( \text{Ker} \alpha \). Take a point \( \widetilde{d} \in \pi^{-1}(d) \subset \widetilde{C}_{n,m} \) and we use \( \widetilde{d} \) as a base point of \( \widetilde{C}_{n,m} \). By identifying \( x \) and \( d \) as deck translations, \( H_m(\widetilde{C}_{n,m}; \mathbb{Z}) \) is a free \( \mathbb{Z}[x^{\pm 1}, d^{\pm 1}] \)-module of rank \( \binom{n+m-2}{m-1} \). Moreover, since \( \text{Ker} \alpha \) is invariant under the \( B_n \) action, \( B_n \) acts on \( H_m(\widetilde{C}_{n,m}; \mathbb{Z}) \) as \( \mathbb{Z}[x^{\pm 1}, d^{\pm 1}] \)-module automorphisms.

Lawrence’s representation is a variant of \( H_m(\widetilde{C}_{n,m}; \mathbb{Z}) \). It is a sub-representation \( \mathcal{H}_{n,m} \) of \( H_m^f(\widetilde{C}_{n,m}; \mathbb{Z}) \), the homology of locally finite chains.

Let \( Y \) be the \( Y \)-shaped graph with four vertices \( c, r, v_1, v_2 \) and oriented edges as shown in Figure 1. A fork \( F \) with the base point \( d \in \partial D_n \) is an embedded image of \( Y \) into \( D^2 = \{ z \in \mathbb{C} \mid |z| \leq n + 1 \} \) such that:

- All points of \( Y \setminus \{r, v_1, v_2\} \) are mapped to the interior of \( D_n \).
- The vertex \( r \) is mapped to \( d \).
- The other two external vertices \( v_1 \) and \( v_2 \) are mapped to the puncture points.

The image of the edge \([r,c]\), and the image of \([v_1, v_2] = [v_1, c] \cup [c, v_2] \) viewed as a single oriented arc, are denoted by \( H(F) \) and \( T(F) \) respectively. We call \( H(F) \) and \( T(F) \) the handle and the time of \( F \).

![Figure 1](image)

**Figure 1.** Forks and multiforks: to distinguish tines and handle, we often write time of forks by a bold gray line.

A **multifork** of dimension \( m \) is an ordered tuples of \( m \) forks \( \mathbb{F} = (F_1, \ldots, F_m) \) such that:

- \( F_i \) is a fork based on \( d_i \).
• \( T(F_i) \cap T(F_j) \cap D_n = \emptyset \ (i \neq j) \).
• \( H(F_i) \cap H(F_j) = \emptyset \ (i \neq j) \).

Figure 1(2) illustrates an example of a multifork of dimension 3. We represent a multifork consisting of \( k \) parallel forks by drawing a single fork labelled by \( k \), as shown in Figure 1(3).

Let \( E_{n,m} = \{ (e_1, \ldots, e_{n-1}) \in \mathbb{Z}_{\geq 0}^{n-1} \mid e_1 + \cdots + e_{n-1} = m \} \). For each \( e \in E_{n,m} \), we assign a multifork \( F_e = \{ F_1, \ldots, F_m \} \) in Figure 2, which we call a standard multifork.

![Figure 2. Standard multifork \( F_e \) for \( e = (e_1, \ldots, e_{n-1}) \in E_{n,m} \)](image)

A multifork \( F \) of dimension \( m \) gives rise to a homology class \( H^m_{lf}(\tilde{C}_{n,m}; \mathbb{Z}) \) in the following manner. Each handle \( H(F_i) \) of \( F \) is seen as a path \( \gamma_i : [0,1] \to D_n \), so the handles of \( F \) defines a path \( H(F) = \{ \gamma_1, \ldots, \gamma_m \} : [0,1] \to C_{n,m} \). Let \( \tilde{H}(F) : [0,1] \to \tilde{C}_{n,m} \) be a lift of \( H(F) \), taken so that \( \tilde{H}(F)(0) = \tilde{d} \). Let \( \Sigma(F) = \{ (z_1, \ldots, z_m) \in C_{n,m} \mid z_i \in T(F_i) \} \) and let \( \tilde{\Sigma}(F) \) be the connected component of \( \pi^{-1}(\Sigma(F)) \) containing \( \tilde{H}(F)(1) \). We equip an orientation of \( \tilde{\Sigma}(F) \) so that a canonical homeomorphism \( T(F_1) \times \cdots \times T(F_n) \to \Sigma(F) \) is orientation preserving. Then \( \tilde{\Sigma}(F) \) represents an element of \( H^m_{lf}(\tilde{C}_{n,m}; \mathbb{Z}) \). By abuse of notation, we use \( F \) to represent both multifork and the homology class \( [\tilde{\Sigma}(F)] \).

We graphically express relations among homology classes represented by multiforks by Figure 3, which we call fork rules.

![Figure 3. Geometric rewriting formula for multiforks. Here \( \{ a \}_q = \frac{q^a - 1}{q - 1} \) is a different version of \( q \)-integer, and \( \{ a \}_b \) is the version of \( q \)-binomial coefficient.](image)

Let \( \mathcal{H}_{n,m} \) be the subspace of \( H^m_{lf}(\tilde{C}_{n,m}; \mathbb{Z}) \) spanned by all multiforks. The set of standard multiforks forms a free basis of \( \mathcal{H}_{n,m} \). \( \mathcal{H}_{n,m} \) is invariant under the \( B_n \) action, so we have the braid
group representation
\[ L_{n,m} : B_n \to \text{GL}(H_{n,m}) = \text{GL}\left(\binom{n+m-2}{m} : \mathbb{Z}[x^\pm 1, d^\pm 1]\right) \]
which we call Lawrence’s representation.

When \( x \) and \( d \) are put as fully generic complex numbers, all representations \( H_{n,m} \), \( H_{n,m}^{lf}(\widetilde{C}_n;m) \), and \( H_m(\widetilde{C}_n;m; \mathbb{Z}) \) are the same, but at non-generic parameters they might be different.

**Example 3.1.** Here we give a sample graphical calculation using the fork rules.

\[
\sigma_1 \left( \begin{array}{c} 3 \\ 2 \end{array} \right) = \begin{array}{c} 3 \\ 2 \end{array} \] \quad (F1,F2) \quad (-1)^3(-d)^3 \cdot \begin{array}{c} 3 \\ 2 \end{array}
\]

\[
= d^3x^3 \left\{ \begin{array}{c} 2 \\ 0 \end{array} \right\} - d + \left\{ \begin{array}{c} 2 \\ 1 \end{array} \right\} - d + \left\{ \begin{array}{c} 2 \\ 2 \end{array} \right\} - d
\]

3.2. **Truncated Lawrence’s representation.** We introduce a new braid group representation, defined in the case that the variable \( -d \) is put as the \( N \)-th root of unity \( \zeta_N = \exp(\frac{2\pi i}{N}) \).

Let \( \mathcal{H}_{n,m}^N \) be the subspace of \( H_{n,m} \) spanned by \( \{ \mathbb{F}_e \mid e \in \mathcal{E}_{n,m}^N \} \), where

\[ \mathcal{E}_{n,m}^N = \{ e = (e_1, \ldots, e_{n-1}) \in \mathcal{E}_{n,m} \mid e_i \geq N \text{ for some } i \} \subset \mathcal{E}_{n,m} \]

We define \( \overline{\mathcal{H}}_{n,m}^N = H_{n,m}/\mathcal{H}_{n,m}^N \). By abuse of notation, we use the same symbol \( \mathbb{F} \) to represent the image \( \pi(\mathbb{F}) \in \overline{\mathcal{H}}_{n,m}^N \) of the projection \( \pi : H_{n,m} \to \overline{\mathcal{H}}_{n,m}^N \). By definition, the set of standard multiforks without more than \( (N-1) \) parallel tines \( \{ \mathbb{F}_e \}_{e \in \mathcal{E}_{n,m}^N} \), where

\[ \mathcal{E}_{n,m}^N = \{ e = (e_1, \ldots, e_{n-1}) \in \mathcal{E}_{n,m}^N \mid e_i \leq N-1 \text{ for all } i \} \subset \mathcal{E}_{n,m} \]

forms a basis of \( \overline{H}_{n,m}^N \). We denote the dimension of \( \overline{H}_{n,m}^N \), the cardinality of \( \mathcal{E}_{n,m}^N \) by \( d_{n,m}^N \).

**Proposition-Definition 3.2.** At \( d = -\zeta_N^2 = -\exp(\frac{2\pi i}{N}) \), \( \mathcal{H}_{n,m}^N \) is a \( B_n \)-invariant subspace of \( H_{n,m} \), hence we have a linear representation

\[ l_{n,m}^N : B_n \to \text{GL}(\overline{H}_{n,m}^N) = \text{GL}(d_{n,m}^N; \mathbb{Z}[x^\pm 1]) \]

We call \( l_{n,m}^N \) truncated Lawrence’s representation.

**Proof.** For \( \beta \in B_n \) and a standard multifork \( \mathbb{F}_e \) with \( e \in \mathcal{E}_{n,m}^N \), \( \beta(\mathbb{F}_e) \) is a multifork more than \( (N-1) \) parallel tines. A crucial point is that for \( k \geq N \), \( \{ k \}_{\{ i \}_{i \leq k}} = 0 \) unless \( i = 0, k \). This observation and an iterated use of the fork rule (F4) show \( \beta(\mathbb{F}_e) \in \mathcal{H}_{n,m}^N \), as desired. \( \square \)

4. **Truncated Lawrence’s representation and quantum representation**

In this section we explore the connection among the braid group representations we introduced in Section 2 and 3. We continue to assume that \( \lambda \) is generic with respect to \( g \).

Put \( V_{n,m} = \mathbb{C}v_0^N \otimes V_{n-1,m} \subset V_{n,m} \). There is an isomorphism as \( \mathbb{C} \)-vector spaces \( \Phi : V_{n,m} \to W_{n,m} \) (for the precise definition of \( \Phi \), which is not important here, see \[12\]). Thus, by sending a natural basis of \( V_{n,m} \) we get a basis \( \{ w_e = \Phi(v_0^N \otimes (v_{e_1}^N \otimes \cdots \otimes v_{e_{n-1}}^N)) \}_{e = (e_1, \ldots, e_{n-1})} \in \mathcal{E}_{n,m} \) of \( W_{n,m} \).
indexed by the same set $E_{n,m}$ as the standard multiforks. Using this basis, we express the braid group representation $W_{n,m}$ as

$$\varphi_{n,m}^W : B_n \to \text{GL}\left(\binom{n + m - 2}{m}; \mathbb{C}\right).$$

The next result says that $\varphi_{n,m}^W$ and $L_{n,m}$ are completely the same.

**Theorem 4.1.** [Koh, Theorem 6.1], [11], Theorem 4.5] If $\lambda$ is generic with respect to $q$, then for an $n$-braid $\beta \in B_n$ we have an identity of matrices

$$\varphi_{n,m}^W(\beta) = L_{n,m}(\beta)|_{x = q^{-2\lambda}, d = -q^2}.$$  

We extend the identity (4.1) for the quantum representation $Y_{n,m}$ and truncated Lawrence’s representation $\lambda_{n,m}$. For $e \in E_{n,m}$, we define $y_e = \Phi(v_0^\lambda \otimes (v_{e_1}^\lambda \otimes \cdots \otimes v_{e_{n-1}}^\lambda))$. Then $y_e \in Y_{n,m}$ and $\{y_e\}_{e \in E_{n,m}}$ forms a basis of $Y_{n,m}$. We express the braid group representation $Y_{n,m}$ by using this basis as

$$\varphi_{n,m}^Y : B_n \to \text{GL}(d_{n,m}^N; \mathbb{C}).$$

**Theorem 4.2.** If $\lambda$ is generic with respect to $q$, for an $n$-braid $\beta \in B_n$ we have an identity of matrices

$$\varphi_{n,m}^Y(\beta) = (\lambda_{n,m}(\beta))|_{x = \zeta^{-2\lambda}, d = -\zeta^2}.$$  

**Proof.** We view $U_N(\lambda)$ as a quotient $V_\lambda|_{q = \zeta}/\{v_i^\lambda = 0\}$ for $i \geq N$, rather as a submodule of $V_\lambda|_{q = \zeta}$, and denote the quotient map by $p : V_\lambda|_{q = \zeta} \to U_N(\lambda)$. Then $Y_{n,m}$ is also seen as a quotient, $Y_{n,m} = p_{\otimes n}(W_{n,m}|_{q = \zeta})$ where $p_{\otimes n} : (V_\lambda|_{q = \zeta})_{\otimes n} \to U_N(\lambda)^{\otimes n}$ is the quotient map from $p$. Since $p$ is an $U_\zeta(\mathfrak{sl}_2)$-module homomorphism, $p_{\otimes n}$ is a surjection as a braid group representation.

By definition for the inclusion map $\iota : Y_{n,m} \hookrightarrow W_{n,m}|_{q = \zeta}$, $p \circ \iota = \text{id}$. Thus, $y_e = p \circ \iota(y_e) = p(w_e)$ for $e \in E_{n,m}$. This observation, together with Theorem 4.1, shows that the basis $\{y_e\}_{e \in E_{n,m}}$ of $Y_{n,m}$ corresponds to the standard multifork basis $\{F_e\}_{e \in E_{n,m}}$ of $H_{n,m}$, with substitution $x = \zeta^{-2\lambda}, d = -\zeta^2$.  

5. Homological representation formula of the colored Alexander invariant

**Theorem 4.2.** Philosophically provides a formula of the colored Alexander invariant, but rewriting the definition of colored Alexander invariant in terms of $I_{n,m}^N(\beta) \cong \varphi_{n,m}^Y(\beta)$ requires several non-trivial computations. Our main task is to express the partial trace as a linear combination of the traces on intertwiner space.

To begin with, we compute the partial trace for $n = 2$. Recall that $U_N(\lambda) \otimes U_N(\mu) = \bigoplus_{i=0}^{N-1} U_N(\lambda + \mu - 2i) \otimes T_i(\lambda, \mu)$, where $T_i = T_i(\lambda, \mu)$ is the intertwiner space which is one-dimensional. Thus $f = f(\lambda, \mu) \in \text{End}_U(\lambda)(U_N(\lambda) \otimes U_N(\mu))$ acts on each $T_i$ as a scalar multiple by $f_i = f_i(\lambda, \mu)$.

**Lemma 5.1.** The partial trace of $f = f(\lambda, \mu) \in \text{End}_U(\lambda)(U_N(\lambda) \otimes U_N(\mu))$ is given by

$$\text{trace}_2(\text{id} \otimes h)f = \left(\sum_{i=0}^{N-1} C_i^N(\lambda, \mu) f_i \right) \text{id}.$$  

Here $C_i^N(\lambda, \mu)$ satisfies the symmetry

$$C_i^N(\lambda, \mu - 2j) = C_{i+j}^N(\lambda, \mu),$$

and is given by the formula

$$C_i^N(\lambda, \mu) = \frac{[\lambda; N - 1]}{[\lambda + \mu - 2i; N - 1]}.$$
**Proof.** Throughout the proof we adopt the convention that all indices are considered modulo \( N \), unless otherwise specified.

First we prove the symmetry \((5.2)\). It is sufficient to check \((5.2)\) for the case \( j = 1 \). A key observation is that

\[
U_N(\lambda) \otimes U_N(\mu - 2) = \bigoplus_{i=0}^{N-1} U_N(\lambda + \mu - 2 - 2i) \cong \bigoplus_{i=0}^{N-1} U_N(\lambda + \mu - 2i) = U_N(\lambda) \otimes U_N(\mu).
\]

This implies an isomorphism of intertwiner spaces \( T_i(\lambda, \mu - 2) \cong T_{i+1}(\lambda, \mu) \), hence \( f_i(\lambda, \mu - 2) = f_{i+1}(\lambda, \mu) \). Also by definition of \( h \),

\[
h(v_i^{\mu-2}) = \zeta^{(N-1)(-\mu+2i+2)} v_i^{\mu-2}, \quad h(v_i^{\mu+1}) = \zeta^{(N-1)(-\mu+2i)} v_i^{\mu+1}.
\]

That is, the actions of \( h \) on \( T_i(\lambda, \mu - 2) \) and \( T_{i+1}(\lambda, \mu) \) are the same. Therefore we conclude

\[
0 = \text{trace}_2(\text{id} \otimes h)f(\lambda, \mu - 2) - \text{trace}_2(\text{id} \otimes h)f(\lambda, \mu)
\]

\[
= \sum_{i=0}^{N-1} (C_i^N(\lambda, \mu - 2)f_i(\lambda, \mu - 2) - C_i^N(\lambda, \mu)f_i(\lambda, \mu))
\]

\[
= \sum_{i=0}^{N-1} (C_{i+1}^N(\lambda, \mu - 2) - C_i^N(\lambda, \mu))f_i(\lambda, \mu)
\]

This proves \( C_{i+1}^N(\lambda, \mu - 2) = C_i^N(\lambda, \mu) \).

Then we proceed to determine \( C_i^N(\lambda, \mu) \). Thanks to \((5.2)\), it is sufficient to determine \( C_0^N(\lambda, \mu) \).

As a subspace, \( T_i \) is identified with

\[
T_i = \ker E \cap \text{span}\{v_i^\lambda \otimes v_i^\mu \mid k + l = i\}.
\]

Here the condition on the indices \( k + l = i \) is strict one, not regarded as modulo \( N \). Take \( t_i \in T_i \) so that

\[
t_i = v_i^\lambda \otimes v_i^\mu + \text{(terms of } v_{i+1}^\lambda \otimes v_{i+1}^\mu\text{)}. Then } \{F^{i-j}t_j\}_{j=0,\ldots,i} \text{ is a basis of the subspace spanned by } \{v_k^\lambda \otimes v_l^\mu \mid k + l = i\} \text{ (again, the condition on the indices } k + l = i \text{ is strict one, not regarded as modulo } N). Hence we can write

\[
v_i^\lambda \otimes v_i^\mu = \sum_{j=0}^{i} \alpha_{i,j} F^{i-j}t_j.
\]

for some \( \alpha_{i,j} \). By applying \( E \) to the both sides we get

\[
[\mu + 1 - i]_\zeta v_i^\lambda \otimes v_{i-1}^\mu = \sum_{j=0}^{N-1} \alpha_{i,j} E F^{i-j}t_j.
\]

Since \([E, F^i] = [i]_\zeta F^{i-1} \frac{\zeta^{(i-1)\zeta^{-1}K^{-1}} - \zeta^{(i-1)\zeta^{-1}K^{-1}}}{\zeta^{-1}}\) and \( Et_i = 0 \) for \( i \geq 1 \),

\[
[\mu + 1 - i]_\zeta v_i^\lambda \otimes v_{i-1}^\mu = \sum_{j=0}^{N-1} \alpha_{i,j} [i - j]_\zeta F^{(i-j)-1} \frac{\zeta^{-(i-j)\zeta^{-1}K^{-1}} - \zeta^{-(i-j)\zeta^{-1}K^{-1}}}{\zeta^{-1}} t_j
\]

\[
= \sum_{j=0}^{N-1} \alpha_{i,j} [i - j]_\zeta [\lambda + \mu + 1 - (i + j)]_\zeta F^{(i-1)-j}t_j.
\]

This shows

\[
\alpha_{i,j} = \frac{[\mu + 1 - i]_\zeta}{[i - j]_\zeta [\lambda + \mu + 1 - (i + j)]_\zeta} \alpha_{i-1,j}.
\]

By definition, \( y_0 = v_0^\lambda \otimes v_0^\mu \) so \( \alpha_{0,0} = 1 \). Therefore

\[
(5.4) \quad \alpha_{i,0} = \frac{[\mu; i]_\zeta}{[i]_\zeta [\lambda + \mu; i]_\zeta}.
\]

On the other hand, by using the basis \( \{v_k^\lambda \otimes v_l^\mu\}_{k+l=i} \), \( F^{i-j}t_j \) is expressed as

\[
F^{i-j}t_j = F^{i-j}(v_0^\lambda \otimes v_j^\mu + \text{(other terms)}) = \zeta^{-(i-j)\lambda}[i; j]_\zeta v_0^\lambda \otimes v_j^\mu + \text{(other terms)}.
\]
Thus, the action of \((\text{id} \otimes h) f\) is given by

\[
(id \otimes h) f(v_0^\lambda \otimes v_i^\mu) = (id \otimes h) \left( \sum_{j=0}^{i} \alpha_{i,j} F^{i-j} t_j \right) = (id \otimes h) \left( \sum_{j=0}^{i} \alpha_{i,j} f_j F^{i-j} t_j \right)
\]

\[
= (id \otimes h) \left\{ \sum_{j=0}^{i} \alpha_{i,j} f_j \zeta^{-(i-j)} \zeta^{-(N-1)(\mu-2i)[i; i-j]} \right\} v_0^\lambda \otimes v_i^\mu + \text{(other terms)}
\]

Therefore

\[
\text{trace}_2(id \otimes h) f = \left\{ \sum_{j=0}^{N-1} \sum_{i=0}^{i} \alpha_{i,j} f_j \zeta^{-(i-j)} \zeta^{-(N-1)(\mu-2i)[i; i-j]} \right\} \text{id}
\]

\[
= \left\{ \sum_{j=0}^{N-1} \left( \sum_{i=j}^{N-1} \alpha_{i,j} f_j \zeta^{-(i-j)} \zeta^{-(N-1)(\mu-2i)[i; i-j]} \right) \right\} \text{id}
\]

By (5.4), we conclude

\[
C_0^N(\lambda, \mu) = \sum_{i=0}^{N-1} \alpha_{i,0} \zeta^{-i \lambda} \zeta^{-(N-1)(\mu-2i)[i]} = \sum_{i=0}^{N-1} \frac{[\mu; i]_{\zeta} \zeta^{-i \lambda} \zeta^{-(N-1)(\mu-2i)}}{[\lambda + \mu; i]_{\zeta}}
\]

The above formula of \(C_0^N(\lambda, \mu)\) is simplified as follows. Let us put \(A_i = q^{-\lambda}[\mu - i]_q, B_i = q^{\mu}[\lambda - i]_q,\) and \(C_i = [\lambda + \mu - i]_q.\) Then \(C_i = q^i A_{i-j} + q^{-(i-j)} B_j.\)

By induction on \(N,\) we show the identity

\[
\sum_{i=0}^{N} A_0 A_1 \cdots A_{i-1} q^{-2i} = \frac{1}{C_0 \cdots C_{N-2}} \sum_{i=1}^{N} q^{-(N-i)} \left[ \frac{N}{i} \right]_{q} A_0 \cdots A_{N-1-i} B_0 \cdots B_{i-2}.
\]

Here we use a convention that for \(X \in \{A, B, C\}, X_0X_1 \cdots X_j = 1\) if \(j < 0). By using the inductive hypothesis for \(N,\) one computes

\[
\sum_{i=0}^{N} A_0 A_1 \cdots A_{i-1} q^{-2i}
\]

\[
= \frac{1}{C_0 \cdots C_{N-1}} \left( \sum_{i=1}^{N} q^{-(N-i)} \left[ \frac{N}{i} \right]_{q} A_0 \cdots A_{N-1-i} B_0 \cdots B_{i-2} C_{N-1} + q^{-2N} A_0 \cdots A_{N-1} \right)
\]

\[
= \frac{1}{C_0 \cdots C_{N-1}} \left( \sum_{i=1}^{N} q^{-(N-i)} \left[ \frac{N}{i} \right]_{q} A_0 \cdots A_{N-1-i} B_0 \cdots B_{i-2} (q^{-1} A_{N-i} + q^{-N+i} B_{i-1}) \right)
\]

\[
+ q^{-2N} \left[ \frac{N}{0} \right]_{q} A_0 \cdots A_{N-1}
\]

\[
= \frac{1}{C_0 \cdots C_{N-1}} \left( \sum_{i=1}^{N+1} q^{-(N+1-i)} \left( q^{i} \left[ \frac{N}{i} \right]_{q} + q^{-N+i-1} \left[ \frac{N}{i-1} \right]_{q} \right) A_0 \cdots A_{N-i} B_0 \cdots B_{i-2} \right)
\]

\[
= \frac{1}{C_0 \cdots C_{N-1}} \sum_{i=1}^{N+1} q^{-(N+1-i)} \left[ \frac{N+1}{i} \right]_{q} A_0 \cdots A_{N-i} B_0 \cdots B_{i-2},
\]
as desired. Since \([N]_q|q=\zeta = 0\) for \(i = 1, \ldots, N - 1\), we conclude
\[
C^N_0(\lambda, \mu) = \sum_{i=0}^{N-1} \frac{A_0A_1 \cdots A_{i-1}}{C_0C_1 \cdots C_{i-1}} q^{-2i} = \zeta^{-(N-1)\mu} \frac{B_0B_1 \cdots B_{N-2}}{C_0C_1 \cdots C_{N-2}} |q=\zeta = \frac{[\lambda; N-1\zeta]}{[\lambda + \mu; N-1\zeta]}.
\]

Next we compute the partial trace for general \(n\). Recall that we have a splitting
\[
(5.5) \quad U_N(\lambda)^{\otimes n} = \bigoplus_{k=0}^{N-1} U_N(n\lambda - 2k) \otimes T_k
\]
where \(T_k\) is the intertwiner space which \(B_n\) acts on. We denote the braid group action on \(T_k\) by \(\varphi_{T_k} : B_n \to \text{End}(T_k)\).

**Proposition 5.2.**

\[
\text{trace}_{2, \ldots, n}(\text{id} \otimes h^{\otimes (n-1)}) f = \left( \sum_{i=0}^{N-1} [\lambda; N-1\zeta]_{q=N}^{n\lambda - 2i; N-1\zeta} \text{trace}\varphi_{T_i}(\beta) \right) \text{id}
\]

**Proof.** As in the proof of Lemma 5.1, we regard all indices are considered modulo \(N\). Let us view \(U_N(\lambda^{\otimes n}) = U_N(\lambda) \otimes (U_N(\lambda)^{\otimes (n-1)})\). By (2.3), as \(U_2(\mathfrak{g}_2)\)-module we have a splitting
\[
U_N(\lambda)^{\otimes (n-1)} \cong \bigoplus_{i=0}^{N-1} U_N((n-1)\lambda - 2i) \otimes T'_i
\]
where \(T'_i\) denotes the intertwiner space (the multiplicity of the direct summands). Hence
\[
(5.6) \quad U_N(\lambda) \otimes U_N(\lambda)^{\otimes (n-1)} = \bigoplus_{i=0}^{N-1} \left( U_N(\lambda) \otimes U_N((n-1)\lambda - 2i) \otimes T'_i \right)
\]
\[
(5.7) \quad \quad = \bigoplus_{i=0}^{N-1} \left\{ \bigoplus_{j=0}^{N-1} U_N(n\lambda - 2(i + j)) \otimes T'_i \right\}
\]
\[
\quad \quad \quad \quad \quad \text{(put } k \equiv i + j \text{ mod } N) \quad = \bigoplus_{i=0}^{N-1} \left\{ \bigoplus_{k=0}^{N-1} U_N(n\lambda - 2k) \otimes T'_i \right\}
\]
\[
(5.8) \quad \quad \quad \quad \quad = \bigoplus_{k=0}^{N-1} U_N(n\lambda - 2k) \otimes \left( \bigoplus_{i=0}^{N-1} T'_i \right)
\]

Let \(\beta_{i,j} \in \text{End}(U_N(n\lambda - 2(i + j)) \otimes T'_i)\) be the restriction of \(\varphi_{U_N}(:\beta:\) on \(U_N(n\lambda - 2(i + j)) \otimes T'_i\) in the splitting (5.7). Comparing (5.8) with (5.5) we get \(T_i \cong \bigoplus_{i=0}^{N-1} T'_i\) and
\[
(5.9) \quad \sum_{i=0}^{N-1} \text{trace} \beta_{i,k-i} = \text{trace} \varphi_{T_k}(\beta).
\]

By (5.6), \(\text{trace}_{2, \ldots, n}\) in the left hand side of (5.6) is written in terms of the right hand side as
\[
(5.10) \quad \text{trace}_{2, \ldots, n} = \sum_{i=0}^{N-1} \text{trace}_{2} \otimes \text{trace}_{T'_i}
\]
where the first term is \(\text{trace}_{2} : \text{End}(U_N(\lambda) \otimes U_N((n-1)\lambda - 2i)) \to \text{End}(U_N(\lambda))\) and second term is \(\text{trace}_{T'_i} : \text{End}(T'_i) \to \mathbb{C}\).
Example 5.4. Let us consider the case hence Proposition 5.2 gives the desired formula.

As we have seen in (2.7), the intertwiner space (truncated Lawrence’s presentations) of the braid groups.

Finally we obtain a formula of colored Alexander invariant in terms of homological representations (truncated Lawrence’s presentations) of the braid groups.

Theorem 5.3. Let $K$ be a knot represented as a closure of an $n$-braid $\beta$. Then

$$\Phi^N_K(\lambda) = \zeta^{(N-1)\lambda e(\beta)} \sum_{i=0}^{N-1} \frac{[\lambda; N-1]_\zeta}{[n\lambda - 2i; N-1]_\zeta} \left( \sum_{j=0}^{n-1} \text{trace} l_{n,i+(N-1)j}(\beta) \right)_{x=\zeta^{-2\lambda}, d=-\zeta^2}$$

Proof. As we have seen in (2.7), the intertwiner space $T_i$ is isomorphic to $\bigoplus_{j=0}^{n-1} \mathcal{Y}_{n,i+(N-1)j}$ as a braid group representation. By Theorem 4.2, $\phi^N_{n,i+(N-1)j}(\beta)$ is equal to $l_{n,i+(N-1)j}(\beta)_{x=\zeta^{-2\lambda}, d=-\zeta^2}$, hence Proposition 5.2 gives the desired formula.

Theorem 5.3 recovers the classical formula of the Alexander polynomial.

Example 5.4. Let us consider the case $N = 2$ so we put $q = \zeta = \exp(\frac{2\pi \sqrt{-1}}{4})$. By Theorem 5.3 for a knot $K$ represented as the closure of an $n$-braid $\beta$, we have

$$\Phi^2_K(\lambda) = \zeta^{\lambda e(\beta)} \left( \frac{[\lambda]}{[\lambda]_\zeta} \sum_{j=0}^{n-1} \text{trace} l_{n,2j}^2(\beta) - \frac{[\lambda]}{[\lambda]_\zeta} \sum_{j=0}^{n-1} \text{trace} l_{n,2j+1}^2(\beta) \right).$$

A crucial observation is that we have an isomorphism of the braid group representation

$$H^2_{n,k} = \bigwedge \mathcal{H}^2_{n,1,k} = \bigwedge \mathcal{H}_{n,1} = \bigwedge \bigoplus_{k=0}^{\infty} \mathcal{H}^2_{n,k} = \bigwedge \mathcal{H}^2_{n,1} = \bigwedge \mathcal{H}_{n,1}$$

That is, truncated Lawrence’s representation is identified with the exterior powers of the reduced Burau representation. This shows that $l^2_{n,k}(\beta) = \bigwedge l^2_{n,1}(\beta) = \bigwedge L_{n,1}(\beta)$ so

$$\sum_{j=0}^{n-1} \text{trace} l_{n,2j}^2(\beta) = \text{trace} \bigwedge L_{n,1}(\beta), \quad \sum_{j=0}^{n-1} \text{trace} l_{n,2j+1}^2(\beta) = \text{trace} \bigwedge L_{n,1}(\beta).$$

Here $\bigwedge^{\text{even}}$ and $\bigwedge^{\text{odd}}$ denote the even and the odd degree part of the exterior powers. Hence

$$\Phi^2_K(\lambda) = \zeta^{\lambda e(\beta)} \frac{\zeta^{\lambda} - \zeta^{-\lambda}}{\zeta^{n\lambda} - \zeta^{-n\lambda}} \left( \text{trace} \bigwedge L_{n,1}(\beta)_{x=\zeta^{-2\lambda}} - \text{trace} \bigwedge L_{n,1}(\beta)_{x=\zeta^{-2\lambda}} \right)$$

$$= \zeta^{\lambda e(\beta)} \frac{\zeta^{\lambda} - \zeta^{-\lambda}}{\zeta^{n\lambda} - \zeta^{-n\lambda}} \det(I - L_{n,1}(\beta)_{x=\zeta^{-2\lambda}}),$$

and by rewriting in terms of $x = \zeta^{-2\lambda}$, we get

$$CA^2_K(x) = \Phi^2_K(\lambda)_{x=\zeta^{-2\lambda}} = x^{-\frac{1}{2} e(\beta)} \frac{x^{\frac{1}{2}} - x^{-\frac{1}{2}}}{x^{\frac{1}{2}} - x^{-\frac{1}{2}}} \det(I - L_{n,1}(\beta)).$$
This is nothing but the well-known formula of the Alexander polynomial [Bir].

Unfortunately, for $N > 2$, we have no clear topological interpretations or simplifications of the formula (5.11). It is an interesting and important problem to understand the formula (5.11) in terms of the topology of the knot complements. In particular, it is desirable to give an independent and direct proof of the fact that (5.11) yields a knot invariant. One of a difficulty toward the better understanding of the formula (5.11) is a lack of our understanding of $l_{n,m}^N$. In a light of (5.12), we expect that $l_{n,m}^N$ is deduced from

$$
\bigoplus_{j=1}^{N-1} H_{n,j}^N = \bigoplus_{j=1}^{N-1} H_{n,j}.
$$

**References**

[ADO] Y, Akustu, T. Deguchi and T. Ohstuki, *Invariants of colored links*, J. Knot Theory Ramifications 1 (1992), 161–184.

[Big] S. Bigelow, *Braid groups are linear*, J. Amer. Math. Soc. 14 (2000), 471–486.

[Bir] J. Birman, *Braids, Links, and Mapping Class Groups*, Annals of Math. Studies 82, Princeton Univ. Press (1974).

[GPMT] N. Geer, B. Patureau-Mirand and V. Turaev, *Modified quantum dimensions and re-normalized link invariants*, Compos. Math. 145 (2009) 196–212.

[It1] T. Ito, Reading the dual Garside length of braids from homological and quantum representations, Comm. Math. Phys. 335 (2015) 345–367.

[It2] T. Ito, Topological formula of the loop expansion of the colored Jones polynomials, arXiv:1411.5418v1

[JK] C. Jackson and T. Kerler, The Lawrence-Krammer-Bigelow representations of the braid groups via $U_q(sl_2)$, Adv. Math. 228, (2011), 1689–1717.

[Koh] T. Kohno, Quantum and homological representations of braid groups, Configuration Spaces - Geometry, Combinatorics and Topology, Edizioni della Normale (2012), 355–372.

[Kra] D. Krammer, *Braid groups are linear*, Ann. Math. 155, (2002), 131–156.

[Kas] R. Kashaev, *The hyperbolic volume of knots from the quantum dilogarithm*, Lett. Math. Phys. 39 (1997) 269–275.

[Law] R. Lawrence, Homological representations of the Hecke algebra, Comm. Math. Phys. 135, (1990), 141–191.

[Mu1] J. Murakami, A state model for the multivariable Alexander polynomial, Pacific J. Math. 157 (1993), 109–135.

[Mu2] J. Murakami, Colored Alexander invariants and cone-manifolds, Osaka J. Math. 45 (2008), 541–564.

[MuMu] H. Murakami, and J. Murakami, The colored Jones polynomials and the simplicial volume of a knot, Acta Math. 186 (2001), 85–104.

[Oht] T. Ohtsuki, Quantum invariants, Series on Knots and Everything, 29. World Scientific Publishing Co., Inc., River Edge, NJ, 2002.

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