Quantum group symmetry of integrable systems
with or without boundary

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Abstract

We present a construction of integrable hierarchies without or with boundary, starting from a single $R$-matrix, or equivalently from a ZF algebra. We give explicit expressions for the Hamiltonians and the integrals of motion of the hierarchy in term of the ZF algebra. In the case without boundary, the integrals of motion form a quantum group, while in the case with boundary they form a Hopf coideal subalgebra of the quantum group.

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1 Introduction

The aim of the present article is to show that, in the study of integrable systems (without or with boundary), all the relevant informations are contained in (and can be reconstructed from) a unique algebra: the ZF algebra. This fact is known in the case of the Non Linear Schrödinger equation in 2 dimensions, and we will show that this is true in general.

In the study of the Non Linear Schrödinger (NLS) equation in 2 dimensions, it has been known for a long time that a central role is played by a deformed oscillators algebra, the ZF algebra [1]. These deformed oscillators can be seen as the asymptotic states of the physical system. Thanks to this algebra, one can reconstruct the quantum Hamiltonians of the hierarchy associated to NLS, as well as its quantum canonical fields [2]. Quite recently [3], it has also been shown that the integrals of motion of this hierarchy, which form a Yangian algebra \( Y(N) \), are a part of the ZF algebra, and an explicit construction in term of basic ZF generators was given.

It appears that the same technics can be applied in the case of the Non Linear Schrödinger equation with a boundary (BNLS). Following the work of Cherednik [4] and Sklyanin [5], who studied the problem of boundaries in integrable systems in the QISM framework, Mintchev et al [6, 7] have shown that the algebraic approach used for NLS can also be applied to BNLS. In that case, the ZF algebra is replaced by a so-called boundary algebra, which contains both the asymptotic states of the system and the effect of the boundary. Indeed, the boundary algebra contains as a subalgebra the reflection algebra, which is known to be of fundamental importance in integrable systems with boundary. The Hamiltonians and the canonical fields of the hierarchy can be also reconstructed from the boundary algebra, and recently [8] it has been shown that the reflection algebra indeed correspond to the integrals of motion of the hierarchy. Thanks to this subalgebra, a classification of the boundary conditions was also given [8].

We will show that most of the above features, explicited for NLS, are valid in full generality, provided one has at its disposal a \( R \)-matrix which obeys a unitary condition. In particular, the Hamiltonians and the integrals of motion of an integrable hierarchy will be exhibited, both in the case without and with boundary. We will show that the integrals of motion form a quantum group when there is no boundary, and a Hopf coideal subalgebra of this quantum group when there is a boundary. As for NLS, the central role of the construction will be a ZF algebra, and a class of operators (contained in the ZF algebra) called well-bred operators.

The paper is organized as follows: in the first section, we treat the case without boundary; then, the next section deals with the case of boundaries; finally, we conclude in section 4. Most of the results presented here can be found in [9, 10].
and we refer to these papers for the original proofs.

2 Case without boundary

The starting point is an evaluated $R$-matrix, of size $N^2 \times N^2$, with spectral parameter, and which obeys the Yang-Baxter equation and the unitarity condition:

$$R_{12}(k_1, k_2)R_{13}(k_1, k_3)R_{23}(k_2, k_3) = R_{23}(k_2, k_3)R_{13}(k_1, k_3)R_{12}(k_1, k_2)$$

$$R_{12}(k_1, k_2)R_{21}(k_2, k_1) = \mathbb{I} \otimes \mathbb{I}$$

Note that no assumption is made on the form of the dependence in the spectral parameters $k_1$ and $k_2$.

To this $R$-matrix one can introduce two types of algebras, that we describe below.

2.1 The Z.F. algebra

The Zamolodchikov-Faddeev (ZF) algebra is an exchange algebra, that can be seen as a deformation of an oscillators algebras.

**Definition 2.1 (ZF algebra $\mathcal{A}_R$)**

To the above $R$-matrix, one can associate a ZF algebra $\mathcal{A}_R$, with generators $a_i(k)$ and $a_i^\dagger(k)$ ($i = 1, \ldots, N$) and exchange relations:

$$a_1(k_1) a_2(k_2) = R_{21}(k_2, k_1) a_2(k_2) a_1(k_1)$$

$$a_1^\dagger(k_1) a_2^\dagger(k_2) = a_2^\dagger(k_2) a_1^\dagger(k_1) R_{21}(k_2, k_1)$$

$$a_1(k_1) a_2^\dagger(k_2) = a_2^\dagger(k_2) R_{12}(k_1, k_2) a_1(k_1) + \delta(k_1 - k_2) \delta_{12}$$

We use the notations on auxiliary spaces

$$a_1(k_1) = \sum_{i=1}^{N} a_i(k_1) e_i \otimes \mathbb{I} \ , \ a_2(k_2) = \sum_{i=1}^{N} a_i(k_2) \mathbb{I} \otimes e_i$$

$$a_1^\dagger(k_1) = \sum_{i=1}^{N} a_i^\dagger(k_1) e_i^\dagger \otimes \mathbb{I} \ , \ a_2^\dagger(k_2) = \sum_{i=1}^{N} a_i^\dagger(k_2) \mathbb{I} \otimes e_i^\dagger$$

$$\delta_{12} = \sum_{i=1}^{N} e_i \otimes e_i^\dagger \ , \ e_i^\dagger = (0, \ldots, 0, 1, 0, \ldots, 0) \ , \ e_i^\dagger \cdot e_j = \delta_{ij}$$

where $\cdot$ stands for the scalar product of vectors. Note that any central element of $\mathcal{A}_R$ is a constant, and that this algebra admits an adjoint operation:
Property 2.2 Let \( \dagger \) be defined by

\[
\begin{align*}
\dagger: & \quad \mathcal{A}_R \rightarrow \mathcal{A}_R \\
& \quad a(k) \mapsto a^\dagger(k) \\
& \quad a^\dagger(k) \mapsto a(k)
\end{align*}
\] (2.9)

together with \( (xy)^\dagger = y^\dagger x^\dagger, \forall x, y \in \mathcal{A}_R \) and \( R_{12}(k_1, k_2)^\dagger = R_{21}(k_2, k_1) \).

Then \( \dagger \) is an anti-automorphism of \( \mathcal{A}_R \).

Note that the adjoint operation is defined on \( a(k) \) and \( a^\dagger(k) \), i.e. on elements of \( \mathcal{A}_R[[k]] \otimes \mathbb{C}^N \): that is the reason why it has also to be defined on the evaluated \( R \)-matrix.

2.2 Quantum group \( \mathcal{U}_R \)

Still with the \( R \)-matrix at our disposable, we can construct a quantum group \( \mathcal{U}_R \), with Hopf structure, in the usual way.

Definition 2.3 (Quantum group \( \mathcal{U}_R \))

To the above \( R \)-matrix, one can associate a quantum group \( \mathcal{U}_R \), with generators \( T^{(n)}_{ij} \) which are gathered (using a spectral parameter \( k \)) in a \( N \times N \) matrix

\[
T(k) = \sum_{i,j=1}^{N} \sum_{n=0}^{\infty} k^{-n} T^{(n)}_{ij} E_{ij}
\] (2.10)

which is submitted to the relation

\[
R_{12}(k_1, k_2) T_1(k_1) T_2(k_2) = T_2(k_2) T_1(k_1) R_{12}(k_1, k_2)
\] (2.11)

with \( T_1(k) = T(k) \otimes \mathbb{I} \) and \( T_2(k) = \mathbb{I} \otimes T(k) \).

The coproduct in \( \mathcal{U}_R \) is given by

\[
\Delta T(k) = T(k) \hat{\otimes} T(k)
\] (2.12)

where \( \hat{\otimes} \) is the tensor product in \( \mathcal{U}_R \) and the matricial product in the auxiliary space.

More explicitly, the coproduct reads

\[
\Delta T^{ab}_{(n)} = \sum_{p+q=n}^{N} \sum_{c=1}^{N} T^{ac}_{(p)} \otimes T^{cb}_{(q)}
\] (2.13)

Depending on the \( R \)-matrix, one gets for \( \mathcal{U}_R \), e.g. the Yangian \( Y(N) \) based on \( gl(N) \) (for \( R(k_1, k_2) = R(k_1 - k_2) \) with \( R(u) = \frac{u^2 + g^2}{u^2} (\mathbb{I} - \frac{i}{u} P_{12}) \), where \( P_{12} \) is the permutation of the two auxiliary spaces), or the quantum group \( \mathcal{U}_q(gl_N) \), based on the affine algebra \( gl_N \) (for \( R(k_1, k_2) = R(k_1/k_2) \) as given in e.g. [14]).
2.3 Vertex operator construction

Our aim is to construct $U_R$ from $A_R$, i.e. $T(k)$ from $a(k)$ and $a^\dagger(k)$. The basic notion for such a purpose is:

**Definition 2.4 (Well-bred operators in $A_R$)**
$L(k) \in A_R \otimes M_N(\mathbb{C})[[k]]$ is said well-bred if it satisfies

$$L_1(k_1) a_2(k_2) = R_{21}(k_2, k_1) a_2(k_2) L_1(k_1)$$  \hspace{1cm} (2.14)

$$L_1(k_1) a_2^\dagger(k_2) = a_2^\dagger(k_2) R_{12}(k_1, k_2) L_1(k_1)$$  \hspace{1cm} (2.15)

In [9], it has been shown that a well-bred operator $T(k)$ can be constructed as a series in $a$’s:

$$T(k_0) = \mathbb{I} + \sum_{n=1}^{\infty} \frac{(-1)^n}{(n-1)!} a_{n-1}^\dagger T^{(n)}_{01\ldots n} a_{1\ldots n}$$ \hspace{1cm} (2.16)

with

$$a_{n-1}^\dagger = a_{\alpha_n}(k_n) \ldots a_{\alpha_1}(k_1) e_\alpha \otimes \ldots \otimes e_\alpha$$

$$a_{1\ldots n} = a_{\beta_1}(k_1) \ldots a_{\beta_n}(k_n) e_\beta \otimes \ldots \otimes e_\beta$$

$$T^{(n)}_{01\ldots n} = T^{(n)}_{\alpha_0,\beta_0,\alpha_1,\beta_1,\ldots,\alpha_n,\beta_n}(k_0, k_1, \ldots, k_n) E_{\alpha_0,\beta_0} \otimes E_{\alpha_1,\beta_1} \otimes \ldots \otimes E_{\alpha_n,\beta_n}$$

There is an implicit summation on the indices $\alpha_0, \beta_0, \ldots, \alpha_n, \beta_n = 1, \ldots, N$ and an integration over the spectral parameters $\int dk_1 \ldots dk_n$. The matrices $T^{(n)}_{01\ldots n}$ are built using only the evaluated $R$-matrix. For their exact expression, we refer to [9]. To clarify the notation, let us stress that the auxiliary spaces indices 1, 2, …, $n$ being repeated, they are ”dummy” (the corresponding indices $\alpha$’s and $\beta$’s are summed), and as such can be exchanged. As a consequence, the matrices $T^{(n)}_{01\ldots n}$ obey the following property [9]:

**Property 2.5** Without any loss of generality, the matrices $T^{(n)}_{01\ldots n}$ can be supposed to obey the following relation

$$\forall i < j \quad T^{(n)}_{01\ldots n} = B^{-1}_{ij} T^{(n)}_{01\ldots n|ij} B_{ij}$$

with

$$T^{(n)}_{01\ldots n} = T^{(n)}_{01\ldots n}(k_0, k_1, \ldots, k_n)$$

$$T^{(n)}_{01\ldots n|ij} = T^{(n)}_{01\ldots n, i-1, i, i+1, \ldots, j-1, i, j+1, \ldots, n}$$

and

$$B_{ij} = \left( \prod_{a=i+1}^{\rightarrow} R_{ia}(k_i, k_a) \right) R_{ij}(k_i, k_j) \left( \prod_{b=i+1}^{\leftarrow} R_{b,j}(k_b, k_j) \right)$$

Let us stress that in $T^{(n)}_{01\ldots n|ij}$, the spectral parameters $k_i$ and $k_j$ are also exchanged.

The vertex operator obey the fundamental property:
**Property 2.6** The vertex operator (2.14) obeys the quantum group $U_R$ relations

$$R_{12}(k_1, k_2)T_1(k_1)T_2(k_2) = T_2(k_2)T_1(k_1)R_{12}(k_1, k_2) \quad (2.17)$$

Note that the construction is unique: there is only one vertex operator (i.e. a series of the form (2.16)) which yields a well-bred operator.

Let us remark that the definition of well-bred operators provides linear equations in $L(k)$, while the quantum group relations are quadratic in $T(k)$. That is the reason why the use of well-bred operators is crucial for the construction of the quantum group. The situation is somehow analogous to the case of Drinfeld twist $\mathcal{F}$ for Hopf algebras: the cocycle condition is quadratic in $\mathcal{F}$, and it is using linear equations\cite{12, 13} in $\mathcal{F}$ that one is able to give explicit forms for $\mathcal{F}$.

Due to the relation

$$T(k) = T^{-1}(k) \quad (2.18)$$

the same expansion can be done for $T^{-1}(k)$:

$$T(k_0)^{-1} = I + \sum_{n=1}^{\infty} \frac{(-1)^n}{(n-1)!} a_{n...1}^{\dagger} T_{01...n}^{(n)} a_{1...n} \quad (2.19)$$

### 2.4 Integrable hierarchy associated to $A_R$

As in the case of undeformed oscillator algebra, one can introduce the following Hamiltonians

$$H_n = \int_{-\infty}^{\infty} dk \ k^n a^{\dagger}(k)a(k), \ n = 0, 1, 2, \ldots \quad (2.20)$$

They form an Abelian subalgebra of $A_R$, and thus define a hierarchy. The generators $a^{\dagger}(k)$ and $a(k)$ are eigenvectors:

$$[H_n, a^{\dagger}(k)] = k^n a^{\dagger}(k) \quad \text{and} \quad [H_n, a(k)] = -k^n a(k) \quad (2.21)$$

Indeed, as we will see in the case of the Non Linear Schrödinger equation, these generators correspond to asymptotic states of the physical models associated to the hierarchy.

Moreover, this integrable hierarchy is also related to the quantum groups $U_R$, thanks to the property:

**Property 2.7** $U_R$ generates integrals of motion for the hierarchy defined by the $H_n$’s.

$$[H_n, T(k)] = 0 \quad (2.22)$$
2.5 Fock space \( \mathcal{F}_R \)

We introduce the Fock space \( \mathcal{F}_R \) of \( \mathcal{A}_R \), with vacuum \( \Omega \):

\[
a(k)\Omega = 0, \quad \forall k; \quad \mathcal{F}_R = \mathcal{A}_R \Omega
\]

(2.23)

It can be decomposed into eigenspaces of \( H_n \):

\[
\mathcal{F}_R = \bigoplus_{p=0}^{\infty} \int \mathcal{F}_p(k_1, \ldots, k_p) d^p k
\]

(2.24)

with

\[
[H_n, x] = (k_1^n + k_2^n + \ldots + k_p^n)x, \quad \forall x \in \mathcal{F}_p(k_1, \ldots, k_p)
\]

(2.25)

Moreover, since \( \mathcal{U}_R \) commutes with \( H_n \), each \( \mathcal{F}_p(k_1, \ldots, k_p) \) eigenspace provides a representation of \( \mathcal{U}_R \). On this subspace, \( T(k_0) \) acts by right multiplication by \( R_{01}(k_0, k_1) \ldots R_{0p}(k_0, k_p) \), so that \( \mathcal{F}_p(k_1, \ldots, k_p) \) can be identified with the tensor product of \( p \) evaluation representations \( \mathcal{V}(k_p) \) of \( \mathcal{U}_R \).

Note that the action of \( T(k_0) \) on \( \mathcal{F}_p(k_1, \ldots, k_p) \) allows to reconstruct the coproduct for \( \mathcal{U}_R \), although \( \mathcal{A}_R \) does not possess any coproduct (see [4] for more details).

3 Case with boundary

3.1 Integrable models with boundary

We remind here the results obtained by Mintchev et al. [6, 7], following [4, 5], on the quantum integrable systems with boundary. As for the case without boundary, the central role is played by an algebra which both encodes the asymptotic states of the system and the effect of the boundary.

Definition 3.1 (Boundary algebra \( \mathcal{B}_R \))

To the same \( R \)-matrix, one can associate another algebra, the boundary algebra \( \mathcal{B}_R \), with generators \( \tilde{a}_i(k) \), \( \tilde{a}_i^+(k) \) and \( b_{ij}(k) \ (i, j = 1, \ldots, N) \) and exchange relations:

\[
\tilde{a}_1(k_1) \tilde{a}_2(k_2) = R_{21}(k_2, k_1) \tilde{a}_2(k_2) \tilde{a}_1(k_1)
\]

(3.1)

\[
\tilde{a}_1^+(k_1) \tilde{a}_2^+(k_2) = \tilde{a}_2^+(k_2) \tilde{a}_1^+(k_1) R_{21}(k_2, k_1)
\]

(3.2)

\[
\tilde{a}_1(k_1) \tilde{a}_2^+(k_2) = \tilde{a}_2^+(k_2) R_{12}(k_1, k_2) \tilde{a}_1(k_1) + \frac{1}{2} \delta_{12} \delta(k_1 - k_2) + \frac{1}{2} b_{12}(k_1) \delta(k_1 + k_2)
\]

(3.3)

\[
\tilde{a}_1(k_1) b_2(k_2) = R_{21}(k_2, k_1) b_2(k_2) R_{12}(-k_1, k_2) \tilde{a}_1(k_1)
\]

(3.4)

\[
b_1(k_1) \tilde{a}_2^+(k_2) = \tilde{a}_2^+(k_2) R_{21}(k_2, k_1) b_1(k_1) R_{21}(k_2, -k_1)
\]

(3.5)

\[
b(k) b(-k) = \mathbb{I}
\]
We have completed the notations (2.6-2.8) by:

\[ b_{12}(k_1) = \sum_{i,j=1}^{N} b_{ij}(k_1) e_i \otimes e_j^{\dagger} \] (3.6)

\[ b_1(k_1) = \sum_{i,j=1}^{N} b_{ij}(k_1) E_{ij} \otimes I \] ; \[ b_2(k_1) = \sum_{i,j=1}^{N} b_{ij}(k_1) I \otimes E_{ij} \] (3.7)

The \( B_R \) algebras have been introduced in [7], where they were shown to play a fundamental role in the study of integrable systems with boundaries. They allow for instance the determination of off-shell correlation functions. As for the case without boundary, a hierarchy can be defined for \( B_R \):

\[ \tilde{H}_{2n} = \int_{0}^{\infty} dk \, k^{2n} \tilde{a}^{\dagger}(k)\tilde{a}(k) \] (3.8)

They satisfy \([H_{2n}, \tilde{a}^{\dagger}(k)] = k^{2n} \tilde{a}^{\dagger}(k)\) and \([H_{2n}, \tilde{a}(k)] = -k^{2n} \tilde{a}(k)\) for \( k > 0 \). Thus, \( \tilde{a}(k) \) and \( \tilde{a}^{\dagger}(k) \) can be regarded as asymptotic states for the hierarchy with boundary.

As for the \( A_R \) algebra, one can defined an adjoint operation on the \( B_R \) algebra:

**Property 3.2** Let \( \dagger \) be defined by

\[
\begin{align*}
\dagger &: \quad A_R \rightarrow A_R \\
\tilde{a}(k) &\mapsto \tilde{a}^{\dagger}(-k) \\
\tilde{a}^{\dagger}(k) &\mapsto \tilde{a}(-k) \\
b(k) &\mapsto b(-k)
\end{align*}
\] (3.9)

together with \((xy)^\dagger = y^{\dagger}x^{\dagger} \quad \forall \ x, y \in B_R \) and \( R_{12}(k_1, k_2)^\dagger = R_{21}(k_2, k_1) \).

Then \( \dagger \) is an anti-automorphism of \( B_R \).

Note that there is an automorphism on \( B_R \) given by [7]:

\[
\begin{align*}
\rho &\left\{ \begin{array}{c}
B_R \rightarrow B_R \\
\tilde{a}(k) &\mapsto b(k)\tilde{a}(-k) \\
\tilde{a}^{\dagger}(k) &\mapsto \tilde{a}^{\dagger}(-k)b(-k) \\
b(k) &\mapsto b(k)
\end{array} \right.
\] (3.10)

This automorphism encodes in algebraic terms of the effects of the boundary in the physical system. It is compatible with the adjoint operation:

\[ \rho(x^{\dagger}) = \rho(x)^{\dagger}, \quad \forall \ x \in B_R \] (3.11)
\textbf{Definition 3.3 (Reflection algebra $S_R$)}

The reflection algebra $S_R$ is the subalgebra of the boundary algebra, with generators $b_{ij}(k)$ ($i, j = 1, \ldots, N$). It has exchange relations:

\begin{align*}
R_{12}(k_1, k_2) b_1(k_1) R_{21}(k_2, -k_1) b_2(k_2) &= b_2(k_2) R_{12}(k_1, -k_2) b_1(k_1) R_{21}(-k_2, -k_1) \\
b(k)b(-k) &= \mathbb{I}
\end{align*}

($3.12$)

$S_R$ algebras enter into the class of $ABCD$-algebras introduced in $[14]$. In the case of the nonlinear Schrödinger equation with boundary, this algebra have been introduced by Cherednik $[4]$. They correspond, in the boundary algebra approach, to the symmetries of the underlying model (with boundary). Indeed, we have the property (proved by direct calculation):

\textbf{Property 3.4} The reflection algebra generates integrals of motion of the hierarchy associated to the $\tilde{H}_2n$'s:

$$[S_R, \tilde{H}_2n] = 0 \quad (3.13)$$

We will study in more details the reflection algebra in the following.

\section*{3.2 Construction of $B_R$ from $A_R$}

\textbf{Theorem 3.5} Let $A_R$ be a ZF algebra, and $T(k)$ its corresponding well-bred vertex operator. Let $B(k)$ be a $N \times N$ matrix such that

\begin{align*}
R_{12}(k_1, k_2) B_1(k_1) R_{21}(k_2, -k_1) B_2(k_2) &= B_2(k_2) R_{12}(k_1, -k_2) B_1(k_1) R_{21}(-k_2, -k_1) \\
B(k)B(-k) &= \mathbb{I}_N
\end{align*}

($3.14$)

Then, the following generators obey a boundary algebra $B^B_R$:

\begin{align*}
\tilde{a}(k) &= \frac{1}{2} \left( a(k) + b(k)a(-k) \right) \\
\tilde{a}^\dagger(k) &= \frac{1}{2} \left( a^\dagger(k) + a^\dagger(-k)b(-k) \right) \\
b(k) &= T(k)B(k)T(-k)^{-1}
\end{align*}

($3.15$) ($3.16$) ($3.17$)

$B(k)$ is called the reflection matrix.

From the form of $b(k)$ in term of $T(k)$, one can immediately deduce:

\textbf{Corollary 3.6} The reflection algebra $S_R$ is a subalgebra of the quantum group $U_R$. It is also a coideal of $U_R$:

$$\Delta t^{ab}(k) = T^{ae}(k)T^{-1}(-k)^{gb} \otimes b^{eg}(k) \quad \text{i.e.} \quad \Delta S_R \subset U_R \otimes S_R \quad (3.18)$$
**Proof:** From (3.17), one has obviously a morphism from $B$ into $U_R$. It remains to show that the kernel of this homomorphism is reduced to \{0\}. The construction has been done in [13] for the case of the Yangian $Y(N)$ and the corresponding reflection algebra. The proof in the general case just follows the same lines. One introduces a gradation $gr$ on $U_R$ and $S_R$

$$gr b_{(n)}^{ab} = n \quad \text{and} \quad gr T_{(n)}^{ab} = n \quad (3.19)$$

and shows that morphism between the filtered algebras $gr S_R$ and $gr U_R$ has trivial kernel. The proof relies on a PBW theorem for the quantum group $U_R$. For more details, see [15].

In the same way, still following [13], one proves the coideal property. Using the coproduct (2.12), one first gets (with implicit summation on repeated indices):

$$\Delta T^{-1}(k)^{gb} = T^{-1}(k)^{gb} \otimes T^{-1}(k)^{ac} \quad (3.20)$$

which leads to

$$\Delta b^{gb}(k) = \Delta \left( T^{ae}(k) T^{-1}(-k)^{db} \right) B^{cd}(k)$$

$$= \left( T^{ae}(k) \otimes T^{ec}(k) \right) \left( T^{-1}(-k)^{gb} \otimes T^{-1}(-k)^{dg} \right) B^{cd}(k)$$

$$= T^{ae}(k) T^{-1}(-k)^{gb} \otimes T^{ec}(k) B^{cd}(k) T^{-1}(-k)^{dg}$$

$$= T^{ae}(k) T^{-1}(k)^{gb} \otimes b^{eg}(k)$$

The difference between the algebras $B^B_R$ and $B_R$ can be seen in the following lemma:

**Lemma 3.7** In $B^B_R$, the automorphism $\rho$ given in (3.10) is the identity:

$$\tilde{a}(k) = b(k) \tilde{a}(-k) \quad \text{and} \quad \tilde{a}^\dagger(k) = \tilde{a}^\dagger(-k) b(-k) \quad (3.21)$$

**Remark:** Strictly speaking, $B^B_R$ corresponds to the coset (noted $B^B_R$) of the abstract boundary algebra $B_R$ (given by definition 3.1) by the relation $\rho + id = 0$, so that the construction of $B^B_R$ in theorem 3.5 defines inclusions of $B^B_R$ into $A_R$ (see also theorem 3.11).

**Property 3.8** The Fock space $F_R$ of $A_R$ provides a Fock space representation for $B^B_R$, defined by

$$\tilde{a}(k) \Omega = 0 \quad \text{and} \quad b(k) \Omega = B(k) \Omega \quad (3.22)$$
From the definition of the Fock space for $A_R$, one has $a(k)\Omega = 0$, and thus $\tilde{a}(k)\Omega = 0$. The well-bred vertex operator $T(p)$ satisfies $T(p)\Omega = \Omega$, which implies $b(k)\Omega = B(k)\Omega$.

**Remark:** In [7, 8], the boundary algebra $B_R$ has several Fock spaces, depending on the value of $b(k)$ on $\Omega$. In the present article, the algebra $B_R^B$ has only one Fock space, but $B$ is given within the construction of $B_R^B$, and there are as much $B_R^B$-algebra constructions in the present approach, as there are Fock spaces in the approach of [4, 5].

### 3.3 Vertex operator construction

One can do the same construction for the $b$ operator:

**Property 3.9 (Reflection operators as vertex operators)**

In term of the $A_R$ generators, the reflection operators read

$$b_0(k_0) = \mathbb{I} + \sum_{n=1}^{\infty} \frac{(-1)^n}{(n-1)!} a_{n-1}^{(n)} b_{01\ldots n}$$

$$\beta_{01\ldots n}^{(n)} = T_{01\ldots n} B_0 + B_0 T_{01\ldots n}^{(n)} + (n-1) \sum_{p=1}^{n-1} \binom{n-2}{p-1} T_{0p+1\ldots n} B_0 T_{01\ldots p}^{(n-p)} R_p'$$

where the prime ' indicates that one has to consider $-k_0$ instead of $k_0$, $B_0 = \lim_{k \to \infty} B(k)$, and

$$R_p' = R_{p+10}^{-1} \cdots R_{n0}^{-1} = \prod_{s=p+1}^{n} R_s (-k_0, k_s)$$

Let us stress that the expansion is done in term of the $A_R$ generators $a$ and $a^\dagger$, not in term of $\tilde{a}$ and $\tilde{a}^\dagger$, generators of $B_R$. It is possible that such an expansion would lead to a more simple expression for $\beta^{(n)}$.

### 3.4 Hierarchy for $B_R^B$

**Property 3.10 (Hierarchy for $B_R^B$)**

Let

$$\hat{H}_n = \frac{1}{2} \int_{-\infty}^{\infty} dk k^n \tilde{a}\dagger(k)\tilde{a}(k)$$

Then $\hat{H}_{2n+1}$ vanish identically in $B_R^B$ and

$$\hat{H}_{2n} = \hat{\tilde{H}}_{2n} = \int_{0}^{\infty} dk k^n \tilde{a}\dagger(k)\tilde{a}(k)$$
Let \( a(k), a^\dagger(k) \) be the generators of the \( A_R \) algebra, and \( H_n \) the Hamiltonian of the \( A_R \)-hierarchy. Then, we have

\[
\tilde{H}_{2n} = H_{2n} + \int_{-\infty}^{\infty} dk \ a^\dagger(k) b(k) a(k) \tag{3.26}
\]

In other terms, one can see the Hamiltonians with boundary as the Hamiltonians without boundary (bulk term) plus a boundary term.

Proof: The different equalities follows from the identities (3.21) and (3.14).

Remark: The hierarchy defines integrable systems with boundary defined by \( B(k) \). In the framework we have adopted, the definition of the boundary is given by the data of the reflection matrix \( B(k) \), as it is presented in [4], but the boundary algebra is naturally recovered here, contrarily to [4], where it is lacking for the calculation of off-shell correlation functions. On the other hand, in [7, 6, 8], the boundary algebra is the basic data (whence the possibility of computation of correlation functions), but the data of the boundary condition (i.e. the reflection matrix) is given with the choice of a Fock space \( \mathcal{F}_B \). Thus, the present framework can be viewed as a bridge between the approaches [4] and [7, 8].

This remark is confirmed in the following theorem (proved in [10]):

**Theorem 3.11** Let \( B(k) \) be a reflection matrix of \( A_R \), and \( b(k) = T(k)B(k)T^{-1}(-k) \) the corresponding reflection operator. Let \( \rho_B \) be defined by

\[
\rho_B(a(k)) = b(k) a(-k) \quad \text{and} \quad \rho_B(a^\dagger(k)) = a^\dagger(-k) b(-k) \tag{3.27}
\]

Then:

(i) \( \rho_B \) is an automorphism of \( A_R \)

(ii) \( \mathcal{B}_R^B \) is the coset of \( A_R \) by the ideal \( \text{Ker}(\rho_B - \text{id}) \)

We present now a property which was already proved in [8] for the case of additive \( R \)-matrix \( R_{12}(k_1 - k_2) \), but the reasoning is valid in full generality:

**Property 3.12 (Integrals of motions of the hierarchy)**

The reflection algebra \( \mathcal{S}_B^B \) generates integrals of motion for the \( \mathcal{B}_R^B \)-hierarchy.

Still following the lines given in [8], one gets:

**Property 3.13 (Spontaneous symmetry breaking)**

In the Fock space representation, there is a spontaneous symmetry breaking of the symmetry algebra through

\[
b(k)\Omega = B(k)\Omega \tag{3.28}
\]
4 Conclusion

We have shown how to construct an integral hierarchy, together with is integrals of motion, starting from a single $R$-matrix which obeys to a unitary condition. It remains to construct the canonical fields associated to such hierarchy, as well as the physical systems which underlies the construction. This have been already done for the NLS hierarchy, and there is no doubt that the technics should apply in its full generality.

As far as the case with boundary is concerned, the present technics has to be related to the construction of boundary states (see e.g. [16, 17]). These later relies on the existence of a reflection matrix. Since this matrix is also used here for the construction of the boundary algebra staring from the ZF algebra, a link between these two approaches should be exhibited.

An obvious generalization of this technics is the case of elliptic algebras, where the $R$-matrix depends also of extra parameters: if such a thing could be done, one would get a insight in models such as the XYZ model and show, as a by-product, that the elliptic algebras are integrals of motions of that type of models.

Another point which need to be clarified is the correspondence with the studies done in [18] for Toda systems with boundaries, where it has been shown that the integrals of motions are coideals subalgebras.

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