A hybrid eikonal solver for accurate first-arrival traveltime computation in anisotropic media with strong contrasts

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Abstract

First-arrival traveltime computation is crucial for many applications such as traveltime tomography, Kirchhoff migration, etc. There exist two major issues in conventional eikonal solvers: the source singularity issue and insufficient numerical accuracy in complex media. Some existing eikonal solvers also exhibit the stability issue in media with strong contrasts in medium properties. We develop a stable and accurate hybrid eikonal solver for 2D and 3D transversely isotropic media with a tilted symmetry axis (TTI, or tilted transversely isotropic media). Our new eikonal solver combines the traveltime field factorization technique, the third-order Lax-Friedrichs update scheme, and a new method for computing the base traveltime field. The solver has the following three advantages. First, there is no need to assign exact traveltime values in the near-source region, and the computed traveltime field near the source location is accurate even for TTI media with strong anisotropy. Second, the computed traveltime field is high-order accurate in space. Third, the solver is numerically stable for 2D and 3D TTI media with strong anisotropy, complex structures, and strong contrasts in medium properties. We verify the stability and accuracy of our hybrid eikonal solver using several 2D
and 3D TTI medium examples. The results show that our solver is stable and accurate in 2D and 3D complex TTI media, producing first-arrival traveltime fields that are consistent with full-wavefield solutions.

**Keywords:** anisotropic media, first-arrival traveltime, eikonal equation, strong medium property contrasts

1 Introduction

Traveltime computation is important for many applications, including underwater acoustics (Martinelli, 2012), geometrical optics (Qian and Leung, 2006), quantum mechanics (Jin et al., 2005), geophysics, etc. Many geophysical applications, such as Kirchhoff migration (Gray and May, 1994; Buske, 1999) and first-arrival traveltomeography (Lin et al., 2009; Taillandier et al., 2009) for reconstructing subsurface structures and medium properties, rely on accurate and efficient traveltime computation.

There exist roughly two categories of numerical methods for traveltime computation: ray-based methods and eikonal-equation-based methods. Ray approaches are based on ray equations approximated from wave equations. A ray-tracing system is then solved in the framework of either one-point initial value problem or two-point boundary value problem using different techniques (Pereyra et al., 1980; Grechka and McMechan, 1996; Sadeghi et al., 1999; Meléndez et al., 2015). These approaches are generally very efficient for sparse source and receivers, but the computational costs may increase dramatically as the number of source-receiver pairs increases. In addition, ray tracing cannot trace rays to cover the entire model, and cannot handle complex media with strong contrasts. A more flexible ray tracing approach is the so-called the wavefront construction (WFC) method (Vinje et al., 1993; Lambaré et al., 1996; Gibson et al., 2005; Chambers and Kendall, 2008), which computes traveltome and amplitude fields in a more intuitive manner by dynamically inserting new rays where necessary. WFC generally requires more programming efforts and is more computationally demanding compared with conventional ray-based approaches.

Vidale (1988) first developed an eikonal-equation-based approach to generating first-arrival traveltimes in heterogeneous media. His method is also known as the expanding-box method. Eikonal-equation-based approaches have since gained fast development and wide applications.
Currently, the most widely used approaches include expanding wavefront methods (Podvin and Lecomte, 1991; Qin et al., 1992), fast marching methods (Sethian and Popovici, 1999; Rawlinson and Sambridge, 2004; Zhang et al., 2006), fast sweeping methods (Tsai et al., 2003; Zhao, 2004; Kao et al., 2005; Fomel et al., 2009; Luo and Qian, 2011; Waheed et al., 2015b), etc. Eikonal solvers on triangular or unstructured mesh (e.g., Qian et al., 2007; Le Bouteiller et al., 2019) can handle complex interfaces and domain boundaries. Various high-order and non-oscillatory numerical schemes (Kim and Cook, 1999; Kim, 2002; Luo and Qian, 2011; Luo et al., 2012) can improve the numerical accuracy and stability of eikonal solvers in complex media with strong contrasts. Different eikonal solvers have various computational complexities and numerical accuracy. A comparison on several popular eikonal solvers can be found in Gómez et al. (2019). One of the most distinct advantage of eikonal solvers compared with ray-based methods is that the output of an eikonal solver is a first-arrival traveltime field in the entire computational domain, as opposed to ray tracing methods that compute only traveltimes on ray paths. In addition, eikonal solvers generally allow models to be arbitrarily heterogeneous and complex, whereas ray-based methods usually require simple or smooth media.

Finite-difference eikonal solvers were first developed for isotropic media. There is an intensive need for eikonal solvers in anisotropic media. Eikonal equations for various kinds of anisotropic media are often significantly more complex than that in isotropic media. Most of the eikonal solvers in isotropic media require substantial modifications for anisotropic media if possible. In anisotropic media, the phase and group velocity directions are generally not the same (Carcione, 2015). Therefore, eikonal solvers for anisotropic media require sophisticated numerical schemes for updating traveltime fields. Eaton (1993) developed a high-order expanding-wavefront method on a hexagonal mesh to compute qP-, qSV- or qSH-wave traveltime fields for transversely isotropic (TI) media. Qian and Symes (2002) developed a paraxial eikonal equation system to compute the qP-wave traveltime field in TI media. Wang et al. (2006) developed an unconditionally-stable expanding wavefront method for the eikonal equation in TI media. Their method explicitly tracks group velocity propagation directions to ensure correct causal stencils. Waheed et al. (2015a) developed a method based on perturbation expansion to solve the TTI eikonal equation. There exist several other methods for solving eikonal equations in anisotropic media by assuming weak, elliptical anisotropy (Ettrich and Gajewski, 1998), by perturbing from elliptical reference medium
(Soukina et al., 2003), or with only low-order numerical accuracy (Lecomte, 1993).

The source singularity is a major problem in various eikonal solvers. Conventional numerical schemes for solving eikonal equations are based on local plane wave assumption for traveltime field update, and therefore cannot accurately handle large curvatures of the traveltime field around the source point. The numerical error in the near-source region can eventually deteriorate the overall numerical accuracy in the entire computational domain. The source singularity issue is generally solved with the traveltime factorization method. The total traveltime field is factorized into an addition or multiplication of a base traveltime and an additional or multiplicative traveltime field, i.e., \( t = t_0 + \tau \) or \( t = t_0 \times \tau \), where \( t \) is the total traveltime field, \( t_0 \) is the base traveltime field, and \( \tau \) is the additional or multiplicative traveltime field (Luo and Qian, 2011, 2012; Luo et al., 2012). The base traveltime field is solved in a homogeneous isotropic or elliptically anisotropic medium, while the traveltime field \( \tau \) is computed in the heterogeneous part of the medium. A notable approach based on the traveltime factorization is an iterative scheme developed by Waheed et al. (2015b) and Waheed and Alkhalifah (2017) to solve the eikonal equation in TTI media. Their method first decomposes the left-hand side of the eikonal equation into a tilted elliptically anisotropic term and an additional term, and moves the additional term to the right-hand side of the eikonal equation. During each iteration, their method updates the right-hand side term to approximate the true TTI eikonal equation.

We develop a hybrid numerical scheme based on both the monotonic Godunov scheme and the high-order weighted essentially non-oscillatory (WENO) scheme to solve the eikonal equation in 2D and 3D anisotropic media. Our hybrid eikonal solver has three advantages. First, our solver is free of source singularity issue, and there is no need to assign traveltime around the source by using multiplicative factorization of the traveltime field. Second, the computed traveltime field of our solver is high-order accurate in space by using the third-order Lax-Friedrichs WENO scheme. Third, our solver is numerically stable for 2D and 3D TTI media with strong anisotropy, highly complex structures, and strong medium property contrasts, by using the weighted non-oscillatory scheme. Our eikonal solver employs both the conventional first-order Godunov scheme and the third-order Lax-Friedrichs scheme to achieve numerical stability and high-order accuracy. Therefore, we call our solver a hybrid approach. To our knowledge, our hybrid eikonal solver is the first fast-sweeping-based method to date that simultaneously holds these three advantages.
Our paper is organized as follows. In the Methodology section, we describe the three computational steps of our anisotropic eikonal solver, including the first-order Godunov locking-sweeping step, the base traveltime field computation step in the arbitrary TTI medium, and the third-order Lax-Friedrichs fast sweeping step. We then use several numerical examples to verify the stability and accuracy of our method. In the Conclusions section, we summarize the most important features of our new anisotropic eikonal solver.

2 Methodology

We derive the formulation for our hybrid eikonal solver in 2D anisotropic media without loss of generality. We give the 3D formulation in Appendix A.

We adopt the following eikonal equation in TTI media (Waheed et al., 2015b):

\[
v_x^2 \left( \frac{\partial t}{\partial \hat{x}} \right)^2 + v_z^2 \left( \frac{\partial t}{\partial \hat{z}} \right)^2 \left[ 1 - 2(\varepsilon - \delta)v_z^2 \left( \frac{\partial t}{\partial \hat{x}} \right)^2 \right] = 1,
\]

where \( v_x(x) = V_p(x) \sqrt{1 + 2\varepsilon(x)} \) is the qP-wave velocity along the \( x \)-axis, \( v_z(x) = V_p(x) \) is the qP-wave velocity along the \( z \)-axis (i.e., the anisotropy symmetry axis), \( \varepsilon = \varepsilon(x) \) and \( \delta = \delta(x) \) are Thomsen parameters describing anisotropy properties of a VTI medium. Equation (1) is written in the rotated coordinates \( \hat{x} - \hat{z} \), and the spatial derivatives are combinations of the spatial derivatives in the unrotated coordinates \( x - z \):

\[
\frac{\partial t}{\partial \hat{x}} = a_x \frac{\partial t}{\partial x} + c_x \frac{\partial t}{\partial z},
\]

\[
\frac{\partial t}{\partial \hat{z}} = a_z \frac{\partial t}{\partial x} + c_z \frac{\partial t}{\partial z},
\]

with the coordinate transformation matrix \( R \) written as

\[
R = \begin{bmatrix} a_x & c_x \\ a_z & c_z \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix},
\]

where \( \theta = \theta(x) \) is the tilt angle of a VTI medium’s symmetry axis (i.e., the counterclockwise rotation angle of the symmetry axis with respect to the \( y \)-axis).

For notation clarity and derivation convenience, we further define

\[
\alpha_x = v_x a_x = v_x \cos \theta,
\]
\[
\gamma_x = v_x c_x = v_x \sin \theta, \quad (6)
\]
\[
\alpha_z = v_z a_z = -v_z \sin \theta, \quad (7)
\]
\[
\gamma_z = v_z c_z = v_z \cos \theta, \quad (8)
\]
and
\[
\xi = \frac{2(\varepsilon - \delta)}{1 + 2\varepsilon}, \quad (9)
\]
leading to
\[
\left(\frac{\partial t}{\partial \hat{x}}\right)^2 + \left(\frac{\partial t}{\partial \hat{z}}\right)^2 \left[1 - \xi \left(\frac{\partial t}{\partial \hat{x}}\right)^2\right] = 1, \quad (10)
\]
with
\[
\frac{\partial t}{\partial \hat{x}} = \alpha_x \frac{\partial t}{\partial x} + \gamma_x \frac{\partial t}{\partial z}, \quad (11)
\]
\[
\frac{\partial t}{\partial \hat{z}} = \alpha_z \frac{\partial t}{\partial x} + \gamma_z \frac{\partial t}{\partial z}. \quad (12)
\]
Generally, \(v_x \neq v_z\) and \(\theta \neq 0\) in TTI media.

2.1 Step I: First-order Godunov fast locking-sweeping

The first step of our hybrid eikonal solver is to solve equation (10) using an iterative first-order fast sweeping method. The iterative approach is based on rewriting equation (10) in the form of
\[
\left[1 - \frac{1}{2} \xi \left(\frac{\partial t}{\partial \hat{z}}\right)^2\right] \left(\frac{\partial t}{\partial \hat{x}}\right)^2 + \left[1 - \frac{1}{2} \xi \left(\frac{\partial t}{\partial \hat{x}}\right)^2\right] \left(\frac{\partial t}{\partial \hat{z}}\right)^2 = 1. \quad (13)
\]
By setting two coefficients
\[
c_x = 1 - \frac{1}{2} \xi \left(\frac{\partial t}{\partial \hat{z}}\right)^2, \quad (14)
\]
\[
c_z = 1 - \frac{1}{2} \xi \left(\frac{\partial t}{\partial \hat{x}}\right)^2, \quad (15)
\]
and absorbing them into \(\partial t/\partial \hat{x}\) and \(\partial t/\partial \hat{z}\), we have
\[
\left(\frac{\partial t}{\partial \hat{x}}\right)^2 + \left(\frac{\partial t}{\partial \hat{z}}\right)^2 = 1, \quad (16)
\]
with
\[
\frac{\partial t}{\partial \hat{x}} = \sqrt{|c_x|} \alpha_x \frac{\partial t}{\partial x} + \sqrt{|c_z|} \gamma_x \frac{\partial t}{\partial z}, \quad (17)
\]
\[
\frac{\partial t}{\partial z} = \sqrt{|c_z| \alpha_x \frac{\partial t}{\partial x}} + \sqrt{|c_z| \gamma_z \frac{\partial t}{\partial z}}. \tag{18}
\]

The left-hand side of equation (16) is the eikonal correspondence in an elliptically transversely isotropic medium, and can be solved using the first-order Godunov scheme.

We solve the eikonal equation (16) using the following procedure. We first set \(c_x(x) = c_z(x) = 1\), and solve equation (16) using fast sweeping; Then we update \(c_x(x)\) and \(c_z(x)\) using the computed traveltime \(t\) based on equations (14) and (15), and solve equation (16) again. The reason for such a reformatting is that a Godunov scheme for equation (10) can be fairly difficult to derive, and possibly leads to high computational complexity for local solvers. Instead, a first-order Godunov scheme for the left-hand side of equation (13) (or equation (16)) is usually simple to derive (e.g., Tsai et al., 2003).

We use a locking-sweeping procedure to reduce computational costs. At each fast sweeping iteration, we simply lock the points where \(t(x)^{(m)} = t(x)^{(m-1)}\) where the superscript \((m)\) represents the \(m\)-th round of fast sweeping, and update traveltime values only at unlocked points. The criterion \(t(x)^{(m)} = t(x)^{(m-1)}\) might not be as accurate as the one given in Bak et al. (2010) and Gómez et al. (2019) based on checking the changes of neighbor points of a certain spatial point, but is much more efficient to compute. In practice, we find that even for complex media, this criterion results in accurate traveltime fields. The locking-sweeping procedure can significantly reduce the computational costs for the first step.

### 2.2 Step II: Base traveltime field computation

In Steps II and III, we express the traveltime field \(t = t(x)\) in heterogeneous TTI media using a multiplicative factorization (Luo and Qian, 2012; Waheed and Alkhalifah, 2017) as

\[
t(x) = t_0(x) \tau(x), \tag{19}
\]

where \(t_0(x)\) is the base traveltime field in the homogeneous TTI media, and \(\tau(x)\) is the multiplicative traveltime field that accounts for heterogeneities of the model.

This multiplicative factorization results in

\[
\frac{\partial t}{\partial x} = \alpha_x \left( t_0 \frac{\partial \tau}{\partial x} + \frac{\partial t_0}{\partial x} \tau \right) + \gamma_x \left( t_0 \frac{\partial \tau}{\partial z} + \frac{\partial t_0}{\partial z} \tau \right), \tag{20}
\]
\[
\frac{\partial t}{\partial z} = \alpha_z \left( t_0 \frac{\partial \tau}{\partial x} + \frac{\partial t_0}{\partial x} \tau \right) + \gamma_z \left( t_0 \frac{\partial \tau}{\partial z} + \frac{\partial t_0}{\partial z} \tau \right),
\]

which transform equation (10) into a factorized eikonal equation for TTI media.

For the factorized eikonal equation, the base traveltime field \( t_0 \) and its first-order spatial derivatives, say, \( t_{0x} = \partial t_0 / \partial x \) and \( t_{0z} = \partial t_0 / \partial z \), are computed analytically, and are fixed during fast-sweeping iterations (Luo and Qian, 2012; Waheed and Alkhalifah, 2017). The base traveltime field is essential to avoid the source singularity issue in eikonal-equation-based traveltime computation without specifying exact traveltime values at the points near the point source location.

Existing techniques for solving the factorized isotropic or anisotropic eikonal equation assume that the background media is either isotropic where \( \varepsilon = \delta = 0 \) (Fomel et al., 2009; Luo and Qian, 2011; Luo et al., 2012) or elliptically anisotropic where \( \varepsilon = \delta \neq 0 \) (Luo and Qian, 2011, 2012; Waheed and Alkhalifah, 2017), because there exist closed-form expressions for computing the traveltime and its spatial derivatives in the case of \( \varepsilon = \delta \).

However, the requirement of \( \varepsilon = \delta \) also limits the application of multiplicative factorization in complex or strong anisotropic media. For instance, to apply multiplicative factorization to the case where \( \varepsilon \neq \delta \), Waheed and Alkhalifah (2017) had to use an iterative scheme to update the right-hand side of the eikonal equation, and update the analytic \( t_0, t_{0x} \) and \( t_{0z} \) after several iterations. Even with such an iterative scheme, in each iteration, the background anisotropic medium is assumed to be elliptically anisotropic. Therefore, the base traveltime field \( t_0 \) never truly approximates that in the anisotropic media where \( \varepsilon \neq \delta \). For strongly anisotropic media, the group velocity curve can significantly deviate from an ellipse. The resulting base traveltime field might significantly differ from the true solution, eventually leading to suboptimal traveltime solutions, even though it helps avoid the source singularity issue.

In contrast to existing techniques where \( t_0 \) is computed for an elliptically anisotropic medium, we develop a semi-analytic approach to directly computing \( t_0 \) for anelliptically anisotropic medium where \( \varepsilon \neq \delta \). Therefore, the background anisotropic medium in our method is the non-degenerated TTI medium at the reference point. This is the most important difference between our hybrid method and existing factorized eikonal solvers.

In the following derivations, to distinguish the rotation angle \( \theta \) of the TTI symmetry axis in the following description, we use \( \vartheta \) to represent the phase velocity angle in a VTI medium, which
measures the deviation angle from the vertical symmetry axis of a VTI medium. The phase velocity of the qP-wave in a VTI medium can be written as (Tsvankin, 2012)

\[
v_{\text{phase}}(\vartheta) = V_p \sqrt{1 + \varepsilon \sin^2 \vartheta - \frac{f}{2} \left[ 1 - \sqrt{\left( 1 + 2\varepsilon \sin^2 \vartheta \right)^2 - 2(\varepsilon - \delta) \sin \vartheta \sin^2 \vartheta} \right]}, \quad (22)
\]

where \( f = 1 - V_s^2/V_p^2 \), and \( V_p \) and \( V_s \) are qP- and qS-wave velocities along the symmetry axis, respectively. By setting \( V_s = 0 \), we have

\[
v_{\text{phase}}(\vartheta) = V_p \sqrt{\frac{1}{2} + \varepsilon \sin^2 \vartheta + \sqrt{(1 + 2\varepsilon \sin^2 \vartheta)^2 - 2(\varepsilon - \delta) \sin^2 2\vartheta}}. \quad (23)
\]

Note that there is no approximation in equation (23), and therefore it is accurate even for strong anisotropy. We also have the group velocity angle \( \psi \) in terms of the phase velocity angle \( \vartheta \) as (Tsvankin, 2012)

\[
\psi(\vartheta) = \arctan \frac{\tan \vartheta + \frac{1}{v_{\text{phase}}(\vartheta)} \frac{dv_{\text{phase}}(\vartheta)}{d\vartheta}}{1 - \frac{\tan \vartheta}{v_{\text{phase}}(\vartheta)} \frac{dv_{\text{phase}}(\vartheta)}{d\vartheta}}, \quad (24)
\]

and the magnitude of the group velocity in terms of the phase velocity angle \( \vartheta \) as

\[
v_{\text{group}}(\vartheta) = v_{\text{phase}}(\vartheta) \sqrt{1 + \frac{1}{v_{\text{phase}}(\vartheta)} \frac{dv_{\text{phase}}(\vartheta)}{d\vartheta}}, \quad (25)
\]

where based on equation (23), we have

\[
\frac{dv_{\text{phase}}(\vartheta)}{d\vartheta} = \frac{V_p^2 \sin 2\vartheta}{2v_{\text{phase}}(\vartheta)} \left[ \varepsilon \left( 1 + 2\varepsilon \sin^2 \vartheta \right) - 2(\varepsilon - \delta) \cos 2\vartheta \right] \sqrt{\left( 1 + 2\varepsilon \sin^2 \vartheta \right)^2 - 2(\varepsilon - \delta) \sin \vartheta \sin^2 2\vartheta + \varepsilon}. \quad (26)
\]

For an arbitrary spatial point \( x = (x, z) \) in the computational domain, the group velocity angle at this point can be evaluated as

\[
\psi = \arctan \frac{x - x_0}{z - z_0}, \quad (27)
\]

where \( x_0 = (x_0, z_0) \) is the position of the point source.

Our goal is to compute the semi-analytic magnitude of the group velocity at this point, so that we can find the exact first-arrival traveltime at \( x \). This requires the determination of the corresponding phase velocity angle \( \vartheta \) at \( x \), by which we can evaluate the magnitude of the group velocity at \( x \) using equation (25). Unfortunately, there is no closed-form expression to compute
from $\psi$ based on equation (24), because equation (24) is a complicated transcendental equation and is extremely difficult to solve analytically, if not impossible.

We therefore adopt a numerical method to compute the magnitude of group velocity $v_{\text{group}}$ at $x$. We first compute a series of group velocity values $v_{\text{group}}(\vartheta_1), v_{\text{group}}(\vartheta_2), \cdots, v_{\text{group}}(\vartheta_n)$, where $\vartheta_1, \vartheta_2, \cdots, \vartheta_n$ is an equal division of the phase angle range $[0, \pi/2]$. Meanwhile, we compute the corresponding group velocity angles $\psi(\vartheta_1), \psi(\vartheta_2), \cdots, \psi(\vartheta_n)$ based on these phase velocity angles. In any VTI medium, the group velocity angles $\psi(\vartheta_1), \psi(\vartheta_2), \cdots, \psi(\vartheta_n)$ range exactly from $[0, \pi/2]$, but are generally not equally distributed within this range. Therefore, we obtain a series of group angle-velocity pairs \{\psi(\vartheta_i), v_{\text{group}}(\vartheta_i)\} with $i = 1, 2, \cdots, n$. We then use cubic spline interpolation to obtain the group velocity for the spatial location $x$, which corresponds to a group velocity angle $\psi_x$ based on equation (27). The interpolant function of this cubic spline interpolation is built from the group angle-velocity pairs \{\psi(\vartheta_i), v_{\text{group}}(\vartheta_i)\}. In practical computations, we use a large $n$ to divide the range $[0, \pi/2]$, leading to high accuracy for cubic spline interpolation. We repeat the process until all the spatial points in the model are covered.

Therefore, there is no need to analytically compute the phase velocity angle $\vartheta$ for the spatial point $x$ in our numerical scheme. The group velocity value at any spatial point is obtained through an 1D interpolation process with smooth and continuous interpolants built from the analytic group angle-velocity pairs. As a result, the computed group velocities are practically of analytic accuracy. The computational cost associated with this part is small in the entire eikonal equation solving process.

Because for any VTI medium, the phase or group velocity is symmetric with respect to both axes, it is sufficient to build a complete group velocity profile for the entire $2\pi$ range based on the computed group velocity values in $[0, \pi/2]$. In addition, because any TTI medium is simply a coordinate rotation result of some VTI medium, the group velocity values for the TTI medium can be easily computed using the scheme described above. Assume that the tilt angle of a TTI medium is $\theta$, then for a normalized spatial location $x = (x - x_0, z - z_0)$ where $(x_0, z_0)$ is the source location, the corresponding directional vector in the unrotated coordinate is

$$x' = \begin{bmatrix} x' \\ z' \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x - x_0 \\ z - z_0 \end{bmatrix} \tag{28}$$
which indicates the group velocity angle corresponding to \( x \) should be

\[
\psi' = \arctan \frac{|x'|}{|z'|}
\]  

(29)

We take absolute value in equation (29) to ensure that the angle \( \psi' \) falls in \([0, \pi/2]\).

We then compute the group velocity value \( v_{\text{group}}|\psi' \) for \( x \) by the aforementioned interpolation procedure at \( \psi' \), and the traveltime at \( x \) is

\[
t_0(x) = \frac{\sqrt{(x-x_0)^2 + (z-z_0)^2}}{v_{\text{group}}|\psi'|}.
\]  

(30)

Finally, we use a high-order centered finite-difference scheme to compute the spatial derivatives of the base traveltime field \( t_{0x} \) and \( t_{0z} \):

\[
t_{0x}(i,j) = \frac{1}{\Delta x} \sum_{l=1}^{M} c_l [t_0(i + l, j) - t_0(i - l, j)],
\]  

(31)

\[
t_{0z}(i,j) = \frac{1}{\Delta z} \sum_{l=1}^{M} c_l [t_0(i, j + l) - t_0(i, j - l)],
\]  

(32)

where \( c_l \) are finite-difference coefficients, \( M \) is the half length of the finite-difference operator (Fornberg, 1988), and \( \Delta x \) and \( \Delta z \) are the grid sizes in the \( x \)- and \( z \)-directions, respectively. In our computation, we use \( M = 10 \), and compute the associate coefficients \( c_l \) using the procedure described in Fornberg (1988). Again, these fields are practically of analytic accuracy because \( t_0 \) is practically of analytic accuracy and \( M \) is large.

In Figure 1, we show three examples for base traveltime computation in anelliptically anisotropic TI media, including a VTI medium (Figure 1a), a HTI medium (Figure 1b), and a TTI media (Figure 1c), all containing strong anisotropies. All group velocity curves significantly deviate from an ellipse. The results indicate that our semi-analytic approach can accurately compute the base traveltime fields in complex anisotropic media. The numerical scheme to compute the base traveltime field in arbitrary TTI media is also applicable to 3D scenario as shown in Figure 2.

When a model contains multiple simultaneous sources, we need to compute the base traveltime fields for all the point sources, and compute a minimum base traveltime field by finding the minimal value from all the base traveltime field values at each point. That is,

\[
t_0(x) = \min \{ t_{01}^s(x), t_{02}^s(x), \cdots, t_{0n}^s(x) \},
\]  

(33)
where $t_{0i}$ represents the base traveltime field for the $i$-th point source. The medium properties at different point source locations can be different. Figure 1d shows a simple example of base traveltime field computation with multiple simultaneous point sources in a 2D TTI medium.

The above procedure also implies that, in homogeneous media, we can obtain the semi-analytic solution to the eikonal equation regardless of TTI anisotropy type and the number of simultaneous point sources, without the need of the aforementioned Step I or the Step III described below.

2.3 Step III: Third-order Lax-Friedrichs fast sweeping

For heterogeneous media, once the initial traveltime field $t$, the base traveltime field $t_0$ and its derivatives are computed, we solve the factorized eikonal equation (10) along with equations (20) and (21) in TTI media using the Lax-Friedrichs scheme based on a third-order WENO discretization (Zhang et al., 2006). The initial multiplicative traveltime field $\tau$ is computed as $\tau(x) = t(x)/t_0(x)$, with $\tau(x_0, z_0) \equiv 1$. The value of $\tau$ at location $(x_0, z_0)$ is kept unchanged during iterations. For multiple simultaneous point source applications, the values of $\tau$ at all the point source locations are kept unchanged during iterations. In contrast to existing high-order schemes where neighbor points of the source point should be assigned and kept unchanged during iterations, our algorithm requires only the value at the source point fixed during iterations thanks to the use of the multiplicative factorization.

To facilitate our description, we define two functionals associated with the multiplicative time field $\tau$ and its spatial derivatives $\tau_x$ and $\tau_z$:

\begin{align}
F_x(\tau, \tau_x, \tau_z) &= \alpha_x (t_0 \tau_x + t_{0x} \tau) + \gamma_x (t_0 \tau_z + t_{0z} \tau), \\
F_z(\tau, \tau_x, \tau_z) &= \alpha_z (t_0 \tau_x + t_{0x} \tau) + \gamma_z (t_0 \tau_z + t_{0z} \tau).
\end{align}

Then the Hamiltonian for equation (10) can be written as

\begin{align}
H(\tau, \tau_x, \tau_z) &= F^2_x(\tau, \tau_x, \tau_z) + F^2_z(\tau, \tau_x, \tau_z) - \xi F^2_x(\tau, \tau_x, \tau_z) F^2_z(t, \tau_x, \tau_z). \tag{36}
\end{align}

In the Lax-Friedrichs update scheme, it is necessary to compute a set of artificial viscosities. To improve numerical stability, we use the following artificial viscosities:

\begin{align}
\omega_x = \max_{\Omega} \left( \frac{\partial H}{\partial \tau} + \frac{\partial H}{\partial \tau_x} \right), \tag{37}
\end{align}
\[ \omega_z = \max_{\Omega} \left( \left| \frac{\partial H}{\partial \tau} \right| + \left| \frac{\partial H}{\partial \tau_z} \right| \right), \tag{38} \]

where \( \Omega \) represents the entire computational domain. Based on equation (36), we have

\[
\frac{\partial H}{\partial \tau} = 2 \left[ (1 - \xi \zeta_z^2) \frac{\partial F_x}{\partial \tau} F_x + (1 - \xi \zeta_z^2) \frac{\partial F_z}{\partial \tau} F_z \right] = 2 \left[ (1 - \xi \zeta_z^2)(\alpha_x t_0 + \gamma_z t_0) F_x + (1 - \xi \zeta_z^2)(\alpha_z t_0 + \gamma_z t_0) F_z \right], \tag{39} \\
\frac{\partial H}{\partial \tau_x} = 2 \left[ (1 - \xi \zeta_z^2) \frac{\partial F_x}{\partial \tau_x} F_x + (1 - \xi \zeta_z^2) \frac{\partial F_z}{\partial \tau_x} F_z \right] = 2 \tau_0 \left[ (1 - \xi \zeta_z^2) \alpha_x F_x + (1 - \xi \zeta_z^2) \alpha_z F_z \right], \tag{40} \\
\frac{\partial H}{\partial \tau_z} = 2 \left[ (1 - \xi \zeta_z^2) \frac{\partial F_x}{\partial \tau_z} F_x + (1 - \xi \zeta_z^2) \frac{\partial F_z}{\partial \tau_z} F_z \right] = 2 \tau_0 \left[ (1 - \xi \zeta_z^2) \gamma_x F_x + (1 - \xi \zeta_z^2) \gamma_z F_z \right]. \tag{41} \\
\]

In our algorithm, we compute the quantities \( |\partial H/\partial t|, |\partial H/\partial \tau_x| \) and \( |\partial H/\partial \tau_z| \) at every spatial points of the model, and find \( \omega_x \) and \( \omega_z \) based on the maximum values of these quantities. Note that we do not adopt the artificial viscosities defined by Luo and Qian (2012) and Luo et al. (2012), which may lead to unstable results in complex anisotropic media with large medium property contrasts.

We then obtain the following Lax-Friedrichs scheme to update the multiplicative traveltime field \( \tau \) at the spatial grid point \((i, j)\):

\[
\tau_{i,j}^{(m+1)} = \frac{1 - H \left( \tau_{i,j}^{(m)}, \tau_{0x;i,j}^{(m),\dagger}, \tau_{0z;i,j}^{(m),\dagger} \right)}{\omega_x/\Delta x + \omega_z/\Delta z} + \tau_{i,j}^{(m)}, \tag{42} \\
\]

where the superscripts \((m)\) and \((m+1)\) represent the values of \( \tau_{i,j} \) at the \( m \)th and \((m+1)\)th sweeping iterations, respectively, and according to Zhang et al. (2006),

\[
\tau_{0x;i,j}^{\dagger} = \frac{1}{2} \left( \tau_{0x;i,j}^{\dagger} + \tau_{0x;i,j}^{-} \right), \tag{43} \\
\tau_{0x;i,j}^{-} = \frac{1}{2} \left( \tau_{0x;i,j}^{\dagger} - \tau_{0x;i,j}^{-} \right), \tag{44} \\
\]

with the third-order WENO discretizations

\[
\tau_{0x;i,j}^{\dagger} = \left( 1 - w_x^+ \right) \frac{\tau_{i+1,j} - \tau_{i-1,j}}{2\Delta x} + w_x^{+} \frac{-\tau_{i+2,j} + 4\tau_{i+1,j} - 3\tau_{i,j}}{2\Delta x}, \tag{45} \\
\tau_{0x;i,j}^{-} = \left( 1 - w_x^- \right) \frac{\tau_{i+1,j} - \tau_{i-1,j}}{2\Delta x} - w_x^{-} \frac{-\tau_{i-2,j} + 4\tau_{i-1,j} - 3\tau_{i,j}}{2\Delta x}. \tag{46} \\
\]
\[ w_x^+ = \left\{ 1 + 2 \frac{\epsilon + (\tau_{i+2,j} - 2\tau_{i+1,j} + \tau_{i,j})^2}{\epsilon + (\tau_{i+1,j} - 2\tau_{i,j} + \tau_{i-1,j})^2} \right\}^{-1}, \quad (47) \]

\[ w_x^- = \left\{ 1 + 2 \frac{\epsilon + (\tau_{i-2,j} - 2\tau_{i-1,j} + \tau_{i,j})^2}{\epsilon + (\tau_{i+1,j} - 2\tau_{i,j} + \tau_{i-1,j})^2} \right\}^{-1}, \quad (48) \]

where \( \epsilon \) is a small number to avoid singularity. The expressions for \( \tau_{0z;i,j}^\dagger \) and \( \tau_{0z;i,j}^* \) can be analogously derived.

Because we actually use the first-order traveltime field computed at Step I as the initial solution for Step III, Step III in our algorithm requires much fewer iterations to achieve accurate results compared with the LF-3 method that directly solves the eikonal equation from a rough or constant initial guess.

### 2.4 Workflow of our hybrid eikonal solver

We summarize the workflow of our hybrid eikonal solver as follows:

1. Compute an initial, first-order accurate traveltime field \( t(x) \) using the first-order Godunov algorithm based on the locking-sweeping procedure in the following order:

   \[ i = 1, \ldots, N_x, \quad j = 1, \ldots, N_z, \quad (49) \]

   \[ i = N_x, \ldots, 1, \quad j = 1, \ldots, N_z, \quad (50) \]

   \[ i = 1, \ldots, N_x, \quad j = N_z, \ldots, 1, \quad (51) \]

   \[ i = N_x, \ldots, 1, \quad j = N_z, \ldots, 1, \quad (52) \]

   where \( i \) and \( j \) are indices of the finite-difference grids, and \( N_x \) and \( N_z \) are the number of grids in the model in the \( x \)- and \( z \)-directions, respectively. The order of fast-sweeping is not important.

2. Compute the base traveltime field \( t_0(x) \) and its spatial derivatives using the semi-analytic approach described in Step II.

3. Compute the multiplicative traveltime field \( \tau(x) \) using the numerical scheme described in Step III. The sweeping follows the order listed in equations (49)-(52).
3 Numerical Results

We use five numerical examples to verify the stability and accuracy of our hybrid eikonal solver for 2D and 3D TTI anisotropic media. We compare the results from four different methods:

1. Godunov: the first-order Godunov method based on the iterative scheme developed in Waheed et al. (2015b).
2. Factorized Godunov: the first-order Godunov method based on the iterative scheme for the factorized eikonal equation developed in Waheed and Alkhalifah (2017).
3. LF-3: the third-order direct Lax-Friedrichs method without traveltime factorization developed in Luo and Qian (2012).
4. Hybrid: our hybrid eikonal solver in this paper.

In our tests, we adapt and program all methods to solve the eikonal equation (10). The meaning of “iterative” in the first-two approaches is that we need to update the right-hand side of the degenerated TTI eikonal equation (13) during sweeping iterations. The meaning of “factorized” in the second approach is that we use the traveltime field factorization scheme. The meaning of “direct” in the Lax-Friedrichs approach is that we directly discretize equation (10) based on the Lax-Friedrichs update scheme and the third-order WENO finite-difference scheme, without any degeneration or right-hand-side iteration as in the first two approaches.

In all the implementations, we do not assign exact values for the points around the source to study the efficacy of these methods in realistic computational tasks. In practical applications, the media around the source can be heterogeneous, where assigning exact traveltime values can be very difficult, if not impossible, particularly for heterogeneous, anelliptically anisotropic media. In all the implementations, we only fix the traveltime value at the source point (i.e., where $t = 0$) over sweeping iterations. Traveltime field values at all other spatial points can change during iterations.

Note that without accurate and fixed values around the point source, it can be very difficult to achieve convergence for LF-3. Therefore, we use the result from the Godunov method as the initial guess for LF-3.
3.1 Homogeneous model

Since we are able to obtain traveltime field with analytic accuracy solely using the algorithm in Step II, traveltime computation in homogeneous TTI media is trivial for our hybrid method. However, it can still be challenging for conventional eikonal solvers. In the first numerical example, we first compare the traveltime field computed using our method with those obtained using three conventional methods.

The model parameters for a homogeneous TTI medium are $V_p = 2000$ m/s, Thomsen parameters $\varepsilon = 0.25$ and $\delta = 0.05$, and TTI symmetry axis tilt angle $\theta = \pi/4$. Figures 3a-d show the traveltime fields in the homogeneous TTI model computed using Godunov, factorized Godunov, LF-3 and our hybrid method, respectively. The solution computed using our method is taken as the reference solution.

Visually the solutions from different methods are close to one another, except the near-source traveltime field contours in Figure 3a (Godunov method), which clearly deviate from the reference traveltime contours shown in Figure 3d (our hybrid method). We compute the differences between the traveltime fields obtained using the three conventional methods and the reference solution (Figures 4a-c). Different conventional methods have different error levels. The result of the LF-3 method is the least accurate, partially because we do not assign exact values in the near-source region for this method. The two Godunov methods have higher accuracy compared with the LF-3 method. Nevertheless, obvious errors occur at the near-source region in both solutions, and the errors become larger with the increased distance away from the source position.

3.2 Gradient model

In the second example, we study the convergence of our hybrid method using an isotropic constant gradient model. We choose this model because we can analytically compute the first-arrival traveltime in such a medium.

The model is 4 km in both spatial directions, with a constant gradient of velocity

$$\frac{1}{s(x)} = \frac{1}{s_0} + G_0 \cdot (x - x_0),$$

where $s(x)$ is the spatially variant slowness, $s_0$ is the slowness at the source point $x_0$, and $G_0$ is the constant gradient of the model. The analytical first-arrival traveltime for this medium is (Fomel
et al., 2009):

\[ t(x) = \frac{1}{|G_0|} \arccosh \left( 1 + \frac{1}{2} s(x)s_0|G_0|^2|x - x_0|^2 \right), \]  

(54)

with

\[ \arccosh(x) = \ln \left( x + \sqrt{x^2 - 1} \right). \]  

(55)

We set a point source at \( x_0 = (2.5, 2.5) \) km in the model, with a background constant slowness \( s_0 = 1/3000 \) s/m. The constant gradient is \( G_0 = (G_x, G_z) = (0.1, 0.5) \) s\(^{-1}\). The velocity in the model varies from 1500 m/s to 4500 m/s as shown in Figure 5a. Figure 5b depicts the corresponding analytical traveltime field computed using equation (54).

We compute the traveltime fields with different model grid spacing using the Godunov and our hybrid method in this isotropic model. We compare the relative \( L_2 \)-norm misfit between the numerical solutions and the analytical solution in Figure 6, demonstrating that our method is more accurate than the Godunov method. The convergence order of our method is approximately 3.13 while that of the Godunov method is approximately 0.79. Even at the largest grid size where the number of grids along each direction is 10, our hybrid method is almost two orders of magnitude more accurate than the Godunov method.

### 3.3 Ball model

We use the third numerical example to verify the numerical stability of our method in an anisotropic medium with a strong contrast as depicted in Figure 7. The background medium is an isotropic homogeneous medium with \( V_p = 5,000 \) m/s and \( \varepsilon = \delta = \theta = 0 \). The blue region at the center of the model indicates the location of a low-velocity strong TTI anisotropic anomaly with \( V_p = 1,800 \) m/s, \( \varepsilon = 0.3, \delta = -0.3 \) and \( \theta = \pi/4 \). The model size is 3.2 km in both dimensions. The grid size is 10 m in both directions.

Figures 7a-d show the traveltime fields computed using Godunov, factorized Godunov, LF-3 and our hybrid method, respectively. The Godunov solution exhibits weak instabilities around the boundaries of the TTI ball. The factorized Godunov solution in Figure 7b shows evident numerical instabilities. The computed traveltime field inside the TTI ball indicates that the factorized Godunov scheme becomes unstable for this TTI medium, and the traveltime field outside of the TTI ball is therefore mostly wrong with error propagating from the inside of the ball.
Figure 7c is the LF-3 solution. The solution is more stable than those computed using the Godunov methods. Figure 7d displays our hybrid solution. Similar with the LF-3 solution, our hybrid method produces a stable solution with the help of the weighted non-oscillatory scheme.

It is important to compare the eikonal equation solution with the full-wavefield solution. We compute the full-wavefield solution using the fully staggered-grid finite-difference method (Lisitsa and Vishnevskiy, 2010) with a high-order stencil, plus an optimal multi-axial perfectly matched layers (Gao and Huang, 2018). Figures 8a-d show the full-wavefield solution at 0.55 s after the source excitation and the corresponding eikonal equation solutions in black curves computed using the Godunov method, factorized Godunov method, LF-3 method and our hybrid method, respectively. We find that the Godunov and the LF-3 methods are stable, yet are not consistent with the full-wavefield solution. The two solutions have an obvious delay compared with the full-wavefield solution wavefront. The factorized Godunov solution is completely inconsistent with the full-wavefield solution because it is not numerically stable. Only our hybrid method produces a stable and accurate solution that is highly consistent with the full-wavefield solution shown in Figure 8d.

3.4 Block model

The fourth example in Figure 9 is a five-block anisotropic heterogeneous model. The model is 20 km in the X direction and 5 km in the Z direction, with a uniform grid sampling of 25 m in both directions. The model has uniform Thomsen parameters of $\varepsilon = 0.4$ and $\delta = -0.2$, but has a strongly contrasted $V_p$ and TTI symmetry axis tilt angle $\theta$. The $V_p$ contrast at each interface is 2000 m/s, and that of the tilt angle $\theta$ is at least $\pi/6$.

Figures 10a-d show the traveltime filed solutions computed using the Godunov method, factorized Godunov method, LF-3 method and our hybrid method, respectively. The Godunov, factorized Godunov and LF-3 solutions have obvious spurious modes starting from the X position of 10 km. This artifact is generated at the interface between the second and the third block and propagates to the far end of the model. In the factorized Godunov solution in Figure 10b, we also observe some instabilities near the interface between the first and the second block. These instabilities propagates from the first interface to the positive direction of X, eventually deteriorating the traveltime field in the entire computational domain. By contrast, the solution computed with our hybrid method
shown in Figure 10d is the only one of the four solutions that is stable in all the five TTI blocks.

We further compare the accuracy of different solutions against the full-wavefield solution in Figures 11-14. Figure 11 shows the computed traveltime field overlying on the full-wavefield solution at 1 s after source excitation. Figures 11a-d are the solutions computed using the Godunov method, the factorized Godunov method, the LF-3 method, and our hybrid method, respectively. The factorized Godunov solution contains some weak instabilities near the 5 km interface. The Godunov and LF-3 solutions give a slight delay along the direction perpendicular to the TTI symmetry axis compared with the full-wavefield solution. In comparison, our hybrid method produces a solution in Figure 11d that is both stable and accurate, and is highly consistent with the full-wavefield solution wavefront along all propagation directions.

At snapshot time 2 s depicted in Figure 12, the inaccuracy caused by the instability of the factorized Godunov solution (Figure 12b) becomes fairly apparent, while the Godunov (Figure 12a) and LF-3 solutions (Figure 12c) start to show inconsistency with the full-wavefield solution at the Z position of approximately 3 km. This inconsistency is in fact the artifact in Figures 10a and c. Only our hybrid method produces a stable and accurate solution (Figure 12d) that is highly consistent with the full-wavefield solution.

The consistency check between the full-wavefield solution and the eikonal equation solution at two other time steps shown in Figures 13 and 14 further verifies that our hybrid method is able to produce stable and accurate solutions to the anisotropic eikonal equation with strong medium property contrasts where conventional methods fail.

3.5 Salt model

We verify the accuracy and stability of our hybrid method using a 3D anisotropic model in Figure 15 modified from the SEG/EAGE salt model. The model dimension is 6.76 km in both the X and Y directions and 2 km in the Z direction. The P-wave velocity model shown in Figure 15a has a value range from 1500 m/s to 4500 m/s. We create the models of Thomsen parameters $\varepsilon$ and $\delta$, with values varying from 0 to 0.4 and -0.3 to 0.3, respectively, from the original velocity model. We create the TTI symmetry axis tilt angle $\theta$ and $\phi$ models with a value range from 0 to 90° and 0 to 180°, respectively. The $\phi$ model has the same spatial pattern as the $\theta$ model shown in
Figures 15d.

Figures 16a and b compares the full-wavefield solution at 0.2 s with the Godunov and our hybrid method solutions, respectively. There exist obvious inconsistency between the Godunov solution and the full-wavefield solution in the X-Z slice of Figure 16a at a depth of approximately 0.3 km. The Godunov solution is faster than the wavefront of the full-wavefield solution around this depth. At a depth of approximately 1.7 km, the Godunov solution is slower than the wavefront of the full-wavefield solution. By contrast, our hybrid method solution in Figure 16b shows good consistency with the full-wavefield solution in both shallow and deep regions.

The full-wavefield and Godunov solution consistency check in Figure 17a for the snapshot of 0.3 s shows that the Godunov solution is faster than the wavefront of the full-wavefield solution at the depth around 1.1 km in the X-Z slice. There exist obvious inconsistency between the two solutions in the X-Y slice in Figure 17a. By contrast, our hybrid method produces a solution in Figure 17b that is consistent with the full-wavefield solution in all three slices.

Figure 18 depicts a full-wavefield snapshot at 0.4 s superimposed with the corresponding traveltime contours obtained using the Godunov method and our hybrid method. The results further verify that that our hybrid method is stable and accurate for 3D heterogeneous anisotropic media with strong contrasts.

4 Conclusions

We have developed a hybrid eikonal solver for computing first-arrival traveltime in 2D and 3D anisotropic media. The numerical scheme of our hybrid eikonal solver consists of three steps: the Godunov fast locking-sweeping step, the base traveltime computation step in anelliptically anisotropic media, and the third-order Lax-Friedrichs fast sweeping step. There are three advantages in our hybrid eikonal solver compared with existing eikonal solvers for anisotropic media. (1) The solver avoids the source singularity issue by multiplicative traveltime factorization and requires no specification of near-source traveltime values. (2) It is high-order accurate in space. (3) It can produces stable and accurate solution in models with strong anelliptically anisotropy, strong medium property contrasts, and complex structures. We have used five numerical examples, including four 2D examples and one 3D example, to verify the stability and high-order accuracy.
of our hybrid eikonal solver. The results show that our new method is advantageous in terms of stability and accuracy compared with conventional approaches. Future work aims at extending our method to address media with more complex anisotropies such as orthorhombic and rotated orthorhombic anisotropies.

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### Appendix A: Hybrid eikonal solver for 3D TTI media

In 3D, our hybrid eikonal solver is based on the following eikonal equation in TTI media:

\[
\sqrt{v_x^2 \left( \frac{\partial t}{\partial \hat{x}} \right)^2 + v_y^2 \left( \frac{\partial t}{\partial \hat{y}} \right)^2 + v_z^2 \left( \frac{\partial t}{\partial \hat{z}} \right)^2} \left\{ 1 - 2(\varepsilon - \delta)v_z \left[ \left( \frac{\partial t}{\partial \hat{x}} \right)^2 + \left( \frac{\partial t}{\partial \hat{y}} \right)^2 \right] \right\} = 1, \tag{56}
\]

where \(v_x(x) = v_y(x) = V_p(x)\sqrt{1 + 2\varepsilon(x)}\) is the qP-wave velocity along the \(x\)- and \(y\)-axes, \(v_z(x) = V_p(x)\) is the qP-wave velocity along the \(z\)-axis (i.e., the anisotropy symmetry axis), \(\varepsilon = \varepsilon(x)\) and \(\delta = \delta(x)\) are Thomsen parameters describing a VTI medium’s anisotropy properties.

Equation (56) is written in the rotated coordinates, and the spatial derivatives are combinations...
of the spatial derivatives in the original coordinates:

\[ \frac{\partial t}{\partial x} = \alpha_x \frac{\partial t}{\partial x} + \beta_x \frac{\partial t}{\partial y} + \gamma_x \frac{\partial t}{\partial z}, \]
\[ \frac{\partial t}{\partial y} = \alpha_y \frac{\partial t}{\partial x} + \beta_y \frac{\partial t}{\partial y} + \gamma_y \frac{\partial t}{\partial z}, \]
\[ \frac{\partial t}{\partial z} = \alpha_z \frac{\partial t}{\partial x} + \beta_z \frac{\partial t}{\partial y} + \gamma_z \frac{\partial t}{\partial z}, \]

with the coordinate transformation matrix \( R \) written as

\[
R = \begin{bmatrix}
\alpha_x & \beta_x & \gamma_x \\
\alpha_y & \beta_y & \gamma_y \\
\alpha_z & \beta_z & \gamma_z \\
\end{bmatrix} = \begin{bmatrix}
\cos \theta \cos \phi & \cos \theta \sin \phi & \sin \theta \\
-\sin \phi & \cos \phi & 0 \\
-\sin \theta \cos \phi & -\sin \theta \sin \phi & \cos \theta \\
\end{bmatrix},
\]

where \( \theta = \theta(x) \) is the tilt angle of a VTI medium’s symmetry axis (i.e., the counterclockwise rotation angle of the symmetry axis w.r.t. the \( y \)-axis), \( \phi = \phi(x) \) is the azimuth angle of a VTI medium’s symmetry axis (i.e., the counterclockwise rotation angle of the symmetry axis w.r.t. the \( z \)-axis).

We then solve the factorized eikonal equation using the Lax-Friedrichs scheme based on a third-order WENO discretization. To facilitate our description, we define

\[
F_x(\tau, \tau_x, \tau_y, \tau_z) = \alpha_x (t_0 \tau_x + t_{0x} \tau) + \beta_x (t_0 \tau_y + t_{0y} \tau) + \gamma_x (t_0 \tau_z + t_{0z} \tau),
\]
\[
F_y(\tau, \tau_x, \tau_y, \tau_z) = \alpha_y (t_0 \tau_x + t_{0x} \tau) + \beta_y (t_0 \tau_y + t_{0y} \tau) + \gamma_y (t_0 \tau_z + t_{0z} \tau),
\]
\[
F_z(\tau, \tau_x, \tau_y, \tau_z) = \alpha_z (t_0 \tau_x + t_{0x} \tau) + \beta_z (t_0 \tau_y + t_{0y} \tau) + \gamma_z (t_0 \tau_z + t_{0z} \tau).
\]

Then the Hamiltonian for equation (1) can be written as

\[
H(\tau, \tau_x, \tau_y, \tau_z) = F_x^2(\tau, \tau_x, \tau_y, \tau_z) + F_y^2(\tau, \tau_x, \tau_y, \tau_z) + F_z^2(\tau, \tau_x, \tau_y, \tau_z)
- \xi F_x^2(\tau, \tau_x, \tau_y, \tau_z) F_z^2(t, \tau_x, \tau_y, \tau_z) - \xi F_y^2(\tau, \tau_x, \tau_y, \tau_z) F_z^2(t, \tau_x, \tau_y, \tau_z),
\]

which leads to

\[
\frac{\partial H}{\partial \tau} = 2 \left[ (1 - \xi F_x^2) \frac{\partial F_x}{\partial \tau} F_x + (1 - \xi F_y^2) \frac{\partial F_y}{\partial \tau} F_y + (1 - \xi F_z^2 - \xi F_x^2) \frac{\partial F_z}{\partial \tau} F_z \right]
= 2 \left[ (1 - \xi F_x^2) (\alpha_x t_{0x} + \beta_x t_{0y} + \gamma_x t_{0z}) F_x + (1 - \xi F_y^2) (\alpha_y t_{0x} + \beta_y t_{0y} + \gamma_y t_{0z}) F_y 
+ (1 - \xi F_z^2 - \xi F_x^2) (\alpha_z t_{0x} + \beta_z t_{0y} + \gamma_z t_{0z}) F_z \right],
\]
\[
\frac{\partial H}{\partial \tau_i} = 2 \left[ (1 - \xi F^2) \frac{\partial F_x}{\partial \tau_i} F_x + (1 - \xi F^2) \frac{\partial F_y}{\partial \tau_i} F_y + (1 - \xi F^2 - \xi F^2) \frac{\partial F_z}{\partial \tau_i} F_z \right] \\
= 2t_0 \left[ (1 - \xi F^2) \alpha_x F_x + (1 - \xi F^2) \alpha_y F_y + (1 - \xi F^2 - \xi F^2) \alpha_z F_z \right], \\
\frac{\partial H}{\partial \tau_j} = 2 \left[ (1 - \xi F^2) \frac{\partial F_x}{\partial \tau_j} F_x + (1 - \xi F^2) \frac{\partial F_y}{\partial \tau_j} F_y + (1 - \xi F^2 - \xi F^2) \frac{\partial F_z}{\partial \tau_j} F_z \right] \\
= 2t_0 \left[ (1 - \xi F^2) \beta_x F_x + (1 - \xi F^2) \beta_y F_y + (1 - \xi F^2 - \xi F^2) \beta_z F_z \right], \\
\frac{\partial H}{\partial \tau_k} = 2 \left[ (1 - \xi F^2) \frac{\partial F_x}{\partial \tau_k} F_x + (1 - \xi F^2) \frac{\partial F_y}{\partial \tau_k} F_y + (1 - \xi F^2 - \xi F^2) \frac{\partial F_z}{\partial \tau_k} F_z \right] \\
= 2t_0 \left[ (1 - \xi F^2) \gamma_x F_x + (1 - \xi F^2) \gamma_y F_y + (1 - \xi F^2 - \xi F^2) \gamma_z F_z \right].
\]

We then compute the following artificial viscosities along the three spatial axes:

\[
\omega_x = \max_\Omega \left( \left| \frac{\partial H}{\partial \tau} \right| + \left| \frac{\partial H}{\partial \tau_i} \right| \right), \tag{69}
\]

\[
\omega_y = \max_\Omega \left( \left| \frac{\partial H}{\partial \tau} \right| + \left| \frac{\partial H}{\partial \tau_j} \right| \right), \tag{70}
\]

\[
\omega_z = \max_\Omega \left( \left| \frac{\partial H}{\partial \tau} \right| + \left| \frac{\partial H}{\partial \tau_k} \right| \right), \tag{71}
\]

where \( \Omega \) is the entire computational domain.

This results in the following Lax-Friedrichs update scheme for the traveltime field \( \tau \):

\[
\tau_{i,j,k}^{(m+1)} = \frac{1 - H \left( \tau_{i,j,k}^{(m)}, \tau_{i,j,k}^{(m), \uparrow}, \tau_{i,j,k}^{(m), \downarrow}, \tau_{i,j,k}^{(m), \rightarrow}, \tau_{i,j,k}^{(m), \leftarrow} \right) + \omega_x \tau_{0x;i,j,k}^{(m), \ast} + \omega_y \tau_{0y;i,j,k}^{(m), \ast} + \omega_z \tau_{0z;i,j,k}^{(m), \ast}}{\omega_x / \Delta x + \omega_y / \Delta y + \omega_z / \Delta z} + \tau_{i,j,k}^{(m)}, \tag{72}
\]

where \( \Delta x, \Delta y \) and \( \Delta z \) are the regular grid sample intervals along the \( x \)-, \( y \)- and \( z \)-axis, respectively.

The fast sweepings in 3D consist of the following sweepings:

\[
i = 1, \ldots, N_x, \quad j = 1, \ldots, N_y, \quad k = 1, \ldots, N_z, \tag{73}
\]

\[
i = N_x, \ldots, 1, \quad j = 1, \ldots, N_y, \quad k = 1, \ldots, N_z, \tag{74}
\]

\[
i = 1, \ldots, 1, \quad j = N_y, \ldots, 1, \quad k = 1, \ldots, N_z, \tag{75}
\]

\[
i = N_x, \ldots, 1, \quad j = N_y, \ldots, 1, \quad k = 1, \ldots, N_z, \tag{76}
\]

\[
i = 1, \ldots, N_x, \quad j = 1, \ldots, N_y, \quad k = N_z, \ldots, 1, \tag{77}
\]

\[
i = N_x, \ldots, 1, \quad j = 1, \ldots, N_y, \quad k = N_z, \ldots, 1, \tag{78}
\]

\[
i = 1, \ldots, N_x, \quad j = N_y, \ldots, 1, \quad k = N_z, \ldots, 1, \tag{79}
\]
\[ i = N_x, \cdots, 1, \quad j = N_y, \cdots, 1, \quad k = N_z, \cdots, 1. \] (80)

where, again, the sweeping order is not important as long as all the listed sweepings are implemented.
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