Critical dynamics of nonconserved $N$-vector model with anisotropic nonequilibrium perturbations

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We show with dynamic renormalization group arguments that spatially anisotropic nonequilibrium perturbations can make a large class of nonconserved $N$-vector models cross over to kinetic-Ising class. We further confirm our prediction by a numerical study of a two-component $\phi^2$ model in two space-dimensions, subjecting it to dynamic perturbations that violate $Z_2$ symmetry.

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Classification of the universality exhibited by systems with macroscopic degrees of freedom, both at and away from equilibrium, is one of the main objectives that has been pursued in statistical physics ever since the advent of scaling theory and renormalization group framework. The universality classes of nonequilibrium systems are far less understood, unlike those at equilibrium, in spite of having identified many genuine nonequilibrium classes like the absorbing phase transitions \[1\], growing surfaces \[2\], self-organized criticality \[3\], driven diffusive systems \[4\], and so on. Constructing classes of infrared-stable field theories by taking a scaling limit of microscopic models is a formidable task even at equilibrium. Hence probing known field theories by various perturbations and following the induced instabilities, if any, is an alternative that can provide invaluable insights towards any classification.

Near-equilibrium critical dynamics is extensively studied and effectively captured by time-dependent Landau-Ginzburg (LG) models as categorized by Hohenberg and Halperin \[5\]. Recent studies have explored the effects of nonequilibrium perturbations on various dynamic universality classes \[6\], \[7\], \[8\], \[9\], \[10\], \[11\]. They not only include perturbations about the LG energy functionals but also genuine nonequilibrium perturbations about the critical dynamics. It is well established that the kinetic Ising systems of Model-A class (in Hohenberg-Halperin classification) are stable against local dynamic perturbations even if they violate detailed-balance condition provided the symmetries are preserved \[6\], \[12\]. Bassler and Schmittmann (BS) further found that the spatially anisotropic perturbations in spite of not respecting the $Z_2$ symmetry cannot destabilise the dynamic class of nonconserved kinetic Ising models which are described by a single scalar field order parameter \[7\]. On the other hand, the detailed-balance violating perturbations turn out to be relevant in the conserved systems \[4\], \[8\], \[13\]. This naturally brings forth the issue whether the irrelevance of such perturbations pervades throughout model-A systems or is only restricted to its subset, like those describable by a scalar order parameter. To resolve this it is fruitful to understand the role of nonequilibrium perturbations on $N$-component model-A systems. An incautious extension of the result of BS \[7\], namely the marginal irrelevance of the nonequilibrium anisotropic perturbations to model-A systems, may suggest that the $N$-component systems are also robust to such perturbations. Indeed such a misconception seems to prevail in the community \[8\], \[14\].

The Gaussian fixed-point of equilibrium $N$-vector models in $4 - \epsilon$ dimensions is unstable to both Ising and $O(N)$-symmetric perturbations and together they give rise to Ising, Heisenberg, and cubic fixed-points \[15\]. The latter two are stable for $N < N_c$ and $N > N_c$, respectively, where $N_c \approx 4$ near 4-dimensions. In either of the case the Ising fixed-point is unstable. Now if we include nonequilibrium perturbations with model-A dynamics then intuitively one may expect the dynamic class to be either model-A Heisenberg or cubic class or else some other $N$-vector nonequilibrium class depending on whether these perturbations are irrelevant or relevant. It is hard to imagine that the large-scale properties of the model could be governed by a kinetic Ising class. As we shall see, the nonequilibrium perturbations exert nontrivial effect on these models and make them cross over to a single component kinetic Ising class. We shall now describe how this crossover takes place.

Consider the nonconserved $N$-vector model that is accorded with permutation symmetry while being subjected to spatially anisotropic dynamic perturbations, and described by the Langevin equation

$$\partial_t \varphi_a(x, t) = \mathcal{F}_a(\varphi(x, t)) + \eta_a(x, t),$$

where

$$\mathcal{F}_a(\varphi) = (\nabla^2 - m^2)\varphi_a + \sum_{bc} E_{abc} \partial_\parallel(\varphi_b \varphi_c) - \sum_{bcd} G_{abcd} \varphi_b \varphi_c \varphi_d,$$

the indices $a, b, c$ run over from 1 to $N$, and $\eta_a(x, t)$ denotes the Gaussian noise with zero mean, and variance $\langle \eta_a(x, t) \eta_a(x', t') \rangle = 2T \delta_{ab} \delta(x - x') \delta(t - t')$. The
bias couplings $E_{abc}$ distinguish one of the $d$ dimensions, $x_j$, from its orthogonal space $x_\perp$, and introduce spatial anisotropy. Unlike the other terms in equation (2) they spoil the symmetry of the dynamics $\phi_a \rightarrow -\phi_a$ for any $a$. The permutation symmetry in this model is assured if the coupling constants are such that $G_{P_aP_bP_cP_d} = G_{abed}$ and $E_{P_aP_bP_c} = E_{abc}$ for all permutations $P$. Note that the above interaction terms are the most general marginal perturbations with permutation symmetry at $d = 4$. Terms like $\phi_c \partial_\mu \phi_c$ can be set to zero without any loss of generality for this symmetry also demands $E_{abc}$ to be symmetric in $b$ and $c$. Renormalization will of course modify the above form of $F_a(\varphi)$.

In the following, we essentially address the question: which of the interactions are consistent with a dynamic $N$-vector field theory having a single characteristic length-scale in the large-time limit? To this end it is convenient to analyse in Martin-Siggia-Rose (MSR) formalism [16]. The MSR action of the Langevin equation (1) is given by

$$S(\varphi_a, \varphi_a) = \int_{xt} \sum_a \left[ \varphi_a \left( \partial_\mu \varphi_a - F_a(\varphi) \right) - T \varphi_a \varphi_a \right],$$

where $\varphi_a$ refers to the auxiliary field, and the notation $\varphi_a = \varphi_a(x,t)$ and $\varphi_a = \varphi_a(x,t)$ is used. The form of $F_a$ as given in Eq. (2) is not the most general action in accord with the permutation symmetry, as there are various other allowed terms: $\sum_{ab} \varphi_a \varphi_b$, $\sum_{ab} \varphi_a \varphi_b$, and $\sum_{ab} \varphi_a \partial_\mu \varphi_b$. Some of these terms will appear when loop-corrections are taken into account in the effective action that describes the long-wavelength modes. Most importantly it is the off-diagonal mass term $\sum_{ab} \varphi_a \varphi_b$, if generated, that will introduce an extra length scale and turn some of the components of the $N$-vector field noncritical. Which of the interactions will avoid this scenario of dynamical breaking of permutation symmetry?

Let us first examine the near-equilibrium perturbations ($E_{abc} = 0$) that are consistent with the $N$-vector permutation symmetry. This symmetry in the dynamics will restrict the number of independent couplings to seven, which are denoted as $\{G_{1111}, G_{1112}, G_{1122}, G_{1222}, G_{1223}, G_{1232}, G_{1234}\}$. The notation $G_{1111}$ refers to those couplings $G_{abcd}$ where all the indices $b, c, d$ are same as $a$, and $G_{1112}$ is used when one of the indices $b, c, d$ is different from $a$, and so on. The one-loop correction to the quadratic part of the effective action due to these perturbations is given by

$$\delta_G S^{(1)}(\varphi_a, \varphi_a) = 3AT \int_{xt} \sum_{a,b,c} G_{abc} \varphi_a(x,t) \varphi_b(x,t),$$

where $A$ is a regularization dependent constant, and in dimensional-regularization with minimal subtraction (MS) scheme $A = m^{d-2}\Gamma(-d/2 + 1)/(4\pi)^{d/2}$. Hence, if the coupling constants do not satisfy the relation,

$$\sum_{c=1}^{N} G_{abc} = g\delta_{ab}, \quad (5)$$

for some arbitrary $g$, then the mass-matrix has off-diagonal elements. All these elements have the same value due to the permutation symmetry, and in turn render the matrix with two different eigenvalues, one of which is $N - 1$ degenerate. Therefore the symmetry gets dynamically broken at one-loop when the constraint is not satisfied. This is essentially a single constraint, and when explicitly expressed is given by

$$G_{1112} + G_{1222} + (N - 2)G_{1223} = 0, \quad (6)$$

for $N \geq 2$. For $N = 1$, the only nonzero coupling is $G_{1111}$.

If the model further possesses a $\varphi_a \rightarrow -\varphi_a$ symmetry for each $a$, then there are only two allowed coupling constants, the cubic coupling $G_{1111}$ and the $O(N)$-symmetric coupling $G_{1122}$, and the constraint (6) is automatically satisfied. This special case is the time-dependent Landau-Ginzburg model with $O(N)$ symmetry, broken by cubic anisotropies. Equation (6) does not imply that $G_{abcd}$ is symmetric in $a$ and $b$. If $G_{abcd}$ are chosen to be symmetric in all the indices, which is also a necessary condition for the system to reach equilibrium, then there are only five independent $G$-couplings as $G_{1222} = G_{1112}$ and $G_{1223} = G_{1232}$, and Eq. (5) reduces to the "trace condition" [17].

Now consider the bias interactions which are genuine nonequilibrium perturbations. The symmetry conditions, $E_{P_aP_bP_c} = E_{abc}$ for all permutations $P$, and $E_{abc}$ is symmetric in the indices $b$ and $c$, will allow only four E-couplings: $\{E_{1111}, E_{1122}, E_{1222}, E_{1223}\}$. The one-loop correction to the MSR effective action due to the bias interactions is

$$\delta_E S^{(1)}(\varphi_a, \varphi_a) = BT \int_{xt} \sum_{a,b,c_1,c_2} E_{abc_1c_2} \varphi_a(x,t) \partial_\mu^2 \varphi_b(x,t), \quad (7)$$

where $B = (2 - d)m^{-2}A$ in MS scheme. Apart from the diagonal term, $\varphi_a \partial_\mu^2 \varphi_a$, the loop-corrections also bring in the term $\sum_{b} \varphi_a \partial_\mu^2 \varphi_b$, that is uncensored by the symmetry. We denote the ratios of these two terms with respective to $\varphi_a \partial_\mu^2 \varphi_a$ term in the 1-loop effective action by $\rho$ and $\tilde{\rho}$, respectively. For a generic choice of couplings the value of $\tilde{\rho}$ will be nonzero, if not at 1-loop then at some higher order, unless the only nonvanishing couplings are $E_{1111}$ and $G_{1111}$. Thus there are nonzero off-diagonal corrections to the propagator, $\langle \varphi_a \varphi_b \rangle \neq 0$ for $a \neq b$, which at
two-loops leads to the following off-diagonal mass term,
\[
\delta_{\text{EGS}}^{(2)} = 3 \alpha T \left( \frac{1}{\sqrt{\rho + N \rho}} - \frac{1}{\sqrt{\rho}} \right) \times 
\sum_{a_1, a_2, a_3, a_4} \int x_t G_{a_1 a_2 a_3 a_4} \varphi_{a_1}(x, t) \varphi_{a_2}(x, t). 
\tag{8}
\]

At this order, the effective action also has a term proportional to \(\delta_{G}^{(1)}\) which is essentially a correction to \(g\) in equation (5). The off-diagonal corrections (8) imply that in order to avoid a crossover from a dynamic \(N\)-vector class the system has to satisfy yet another necessary constraint,
\[
\sum_{c=1}^{N} \sum_{d=1}^{N} G_{abcd} = g' \delta_{ab}, 
\tag{9}
\]
for some arbitrary \(g'\). Using equation (5) in the above expression leads to the explicit relation,
\[
2 G_{1122} + 2(N-2) \left( G_{1123} + G_{1223} \right) + (N-2) (N-3) G_{1234} = 0. 
\tag{10}
\]

Are there any consistent subsets of coupling constants satisfying the two constraints, (6) and (10), and closed under loop-corrections? Let us extend, for illustration, the earlier special case with only two G-couplings, \(G_{1111}\) and \(G_{1122}\), which has the additional \(\varphi_a \to -\varphi_a\) symmetry in the absence of bias. If we weaken this symmetry by compounding its action with \(x \parallel \to -x \parallel\) then the allowed E-couplings are \(E_{111}\) and \(E_{122}\). These E-couplings give a nonzero \(\tilde{\rho}\), and hence the constraint (10) would require \(G_{1122} = 0\). But the given set of couplings is not closed under loop-corrections without \(G_{1122}\) interaction and is incompatible with a single length-scale theory. So the \(O(N)\)-symmetric perturbations make the critical dynamics of this model to eventually cross over to BS model. In general, in the absence of any other symmetry than permutation symmetry, the only consistent sets are the trivial one \(\{G_{1111}, E_{111}\}\) and the set with all the couplings. The former is the BS model, while the latter is vulnerable to crossovers.

Rephrasing the analysis, if we consider a permutation symmetric dynamic \(N\)-vector model, (right at the outset) with
\[
\mathcal{F}_a(\varphi) = \left( \nabla \cdot + \rho \partial_t \varphi - m^2 \right) \varphi_a + \sum_b \tilde{\rho} \partial_t \varphi_b 
+ \sum_{bc} E_{abc} \partial_t \left( \varphi_b \varphi_c \right) - \sum_{bcd} G_{abcd} \varphi_b \varphi_c \varphi_d, \tag{11}
\]
where the quartic couplings \(G_{abcd}\) violate any of the two constraints, (6) and (10), then the 1-loop corrections will make the system cross over from a dynamic N-vector class in \(4 - \epsilon\) dimensions. For, when these constraints are not satisfied, one of the eigenvalues of the mass-matrix is different from the rest. Therefore we can either have a one component critical dynamics, which is the BS model, or a dynamic \((N - 1)\)-vector model with bias. If we decide to tune the system to \((N - 1)\)-vector model then it will again cross over when any of the two constraints, with \(N\) replaced by \((N - 1)\), fail to hold. This result is likely to hold even in lower dimensions since the arguments leading to the crossovers solely rely on the loop-corrections to a most relevant operator, namely, the off-diagonal mass-matrix.

To be specific, if a bias is introduced in any system at \(O(N)\)-symmetric critical point in \(d < 4\), even for \(N < 4\), then we expect that it will cross over to the kinetic Ising class to which BS model belongs to. We have confirmed this prediction by the following numerical study.

Consider a \(O(2)\)-symmetric model on a 2-dimensional lattice described by the Hamiltonian
\[
\mathcal{H} = - \sum_{\langle n,m \rangle} \tilde{\phi}_n \cdot \tilde{\phi}_m + \sum_n \left( \frac{r + 2d}{2} \phi_n^2 + \frac{u}{4} (\phi_n^2)^2 \right), \tag{12}
\]
where \(n\) is the site index, \(\langle n,m \rangle\) denotes sum over all nearest neighbor pairs, \(\tilde{\phi}_n = (\phi_{n,1}, \phi_{n,2})\) is a real 2-component vector field, and \(\tilde{\phi}^2 := \phi_1^2 + \phi_2^2\). The dynamics of the field \(\phi_{n,a}\) in the presence of a bias is governed by the following Langevin equation
\[
\frac{\partial}{\partial t} \phi_{n,a} = - \frac{\partial \mathcal{H}}{\partial \phi_{n,a}} + E \partial_t \left( \phi_{n,a}^2 \right) + \eta_{n,a}(t), \tag{13}
\]
where \(\eta_{n,a}\) is the white noise with correlation \(\langle \eta_{n,a}(t) \eta_{n',a'}(t') \rangle = \delta_{nn'} \delta_{aa'} \delta(t - t')\), and \(\partial_t (\phi_{n,a}^2) := \phi_{n+1,a}^2 - \phi_{n-1,a}^2\), where \(n + 1\) and \(n - 1\) refer to the two nearest neighbors of \(n\) along a specified direction. In the unbiased case, \(E = 0\), the steady state of Eq. (13) is the equilibrium state with partition function
\[
Z = \int_{-\infty}^{\infty} \prod_n \prod_{a=1}^2 d\phi_{n,a} e^{-2\mathcal{H}}. \tag{14}
\]

We have taken the two-dimensional square lattice to be of size \(L \times L\) with periodic boundary conditions. The values of \(u\) and the bias \(E\) are set to unity, \(u = E = 1\). Equation (13) is then integrated numerically by employing Euler method with \(\Delta t = 0.0025\). The initial condition is taken to be \(\phi_{n,a} = \delta_{n,1}\) for all realizations. The system sizes of \(L = 2^6, 2^7,\) and \(2^8\) are considered, and the equilibration time is set to 20 000. After equilibration we measured the magnetization \(\bar{M} = \sum_n (\phi_{n,1}, \phi_{n,2}) / L^2\) as well as \(M_2 := |\bar{M}|^2\) and \(M_4 := M_2^2\) at every 5 unit times, namely after every 2000 iterations with the above mentioned \(\Delta t\), and then obtained the averages for all these quantities. The critical point \(r_c\) is located using the Binder cumulant
\[
U_L = 1 - \frac{(M_4)}{3(M_2)^2}. \tag{15}
\]
The critical exponents, $\beta$ and $\nu$, are found from finite size scaling by taking the scaling form for $\sqrt{\langle M^2 \rangle}$ to be

$$\sqrt{\langle M^2 \rangle} = L^{-\beta/\nu} f(r_c - r) L^{1/\nu}. \quad (16)$$

The asymptotic behavior of the universal scaling function $f$ is given by

$$f(y) \rightarrow \begin{cases} y^\beta & \text{as } y \rightarrow \infty, \\ (-y)^{\beta-\nu} & \text{as } y \rightarrow -\infty. \end{cases} \quad (17)$$

Our results are shown in Fig. 1. The data collapse with the asymptotic behavior Eq. (17) is in good agreement with the critical exponents of two dimensional Ising model, $\beta = 1.17$ and $\nu = 1.08$. The critical point is located at $r_c = -0.9545 \pm 0.0010$ as shown in the inset of Fig. 1. The value of the critical cumulant is consistent with that of the Ising model on a square lattice ($\approx 0.6107$) \cite{18}. Thus the model with dynamics Eq. (13) clearly shows the order-disorder phase transition and exhibits critical behavior unlike its equilibrium counterpart which can not undergo such a transition in two dimensions \cite{19}. The kinetic Ising behaviour is actually what is expected from our analytic study. The fact that the model shows equilibrium behaviour further suggests that the bias is irrelevant, and thus also reconfirms BS result \cite{7} in two dimensions.

In summary, we have studied the effect of spatially anisotropic nonequilibrium perturbations on nonconserved $N$-vector models, and obtained the necessary "trace conditions" their viable field theories are required to satisfy. The dynamic renormalization group analysis suggests, contrary to the widespread belief, that model-A systems can succumb to such perturbations. In general these perturbations are highly relevant and will make a dynamic $N$-vector model whether in Heisenberg or in cubic universality class to cross over to kinetic Ising class. We confirmed this scenario by a numerical study on a two-component $\phi^4$ model in two dimensions.

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\begin{thebibliography}{99}
\bibitem{1} J. Marro and R. Dickman, Nonequilibrium Phase Transitions in Lattice Models (Cambridge University Press, Cambridge, 1999).
\bibitem{2} A.-L. Barabasi and H. E. Stanley, Fractal Concepts in Surface Growth ((Cambridge University Press, Cambridge, 1995).
\bibitem{3} P. Bak, How Nature Works: The Science of Self-Organized Criticality (Springer, Berlin, 1996)
\bibitem{4} B. Schmittmann and R. K. P. Zia, in Phase Transitions and Critical Phenomena, edited by C. Domb and J. L. Lebowitz (Academic Press, London, 1995), Vol. 17.
\bibitem{5} P. C. Hohenberg and B. I. Halperin, Rev. Mod. Phys. 49, 435 (1977).
\bibitem{6} G. Grinstein, C. Jayaprakash, and Y. He, Phys. Rev. Lett 55, 2527 (1985).
\bibitem{7} K. E. Bassler and B. Schmittmann, Phys. Rev. Lett 73, 3345 (1994).
\bibitem{8} U. C. Täuber, V. K. Akkineni, and J. E. Santos, Phys. Rev. Lett 88, 045702 (2002).
\bibitem{9} V. K. Akkineni and U. C. Täuber, Phys. Rev. E 69, 036113 (2004).
\bibitem{10} U. C Täuber and E Frey, Europhysics Letters 59, 655 (2002).
\bibitem{11} J. Das, M. Rao, and S Ramaswamy, Europhysics Letters 60, 418 (2002).
\bibitem{12} F. Haake, M. Lewenstein, and M. Wilkens, Z. Phys. B 55, 211 (1984).
\bibitem{13} K. E. Bassler and Z. Rácz, Phys. Rev. E 52, R9 (1995).
\bibitem{14} Géza Ódor, Rev. Mod. Phys. 76, 663 (2004).
\bibitem{15} E. Brézin, J. C. Le Guillou and J. Zinn-Justin, Phys. Rev. B 10, 892 (1974).
\bibitem{16} P. C. Martin, E. D. Sigia, and H. H. Rose, Phys. Rev. A 8, 423 (1973).
\bibitem{17} See, e.g., M. Plischke and B. Berghersen, Equilibrium Statistical Physics (World Scientific, Singapore, 2006), 3rd ed.
\bibitem{18} G. Kamieniarz and H. W. J. Blöte, J. Phys. A: Math. Gen. 26, 201 (1993); W. Selke, Eur. Phys. J. B 51, 223 (2006).
\bibitem{19} N. D. Mermin and H. Wagner, Phys. Rev. Lett. 17, 1133 (1966).
\end{thebibliography}