COLLAPSE OF UNIFORMLY ROTATING STARS TO BLACK HOLES AND THE FORMATION OF DISKS

STUART L. SHAPIRO

Received 2004 January 20; accepted 2004 March 30

ABSTRACT

Simulations in general relativity show that the outcome of collapse of a marginally unstable, uniformly rotating star spinning at the mass-shedding limit depends critically on the equation of state. For a very stiff equation of state, which is likely to characterize a neutron star, essentially all of the mass and angular momentum of the progenitor are swallowed by the Kerr black hole formed during the collapse, leaving nearly no residual gas to form a disk. For a soft equation of state with an adiabatic index \( \Gamma - 4/3 \ll 1 \), which characterizes a very massive or supermassive star supported predominantly by thermal radiation pressure, as much as 10% of the mass of the progenitor avoids capture and goes into a disk around the central hole. We present a semianalytic calculation that corroborates these numerical findings and shows how the final outcome of such a collapse may be determined from simple physical considerations. In particular, we employ a simple energy variational principle with an approximate, post-Newtonian energy functional to determine the structure of a uniformly rotating, polytropic star at the onset of collapse as a function of polytropic index \( n \), where \( \Gamma = 1 + 1/n \). We then use this data to calculate the mass and spin of the final black hole and ambient disk. We show that the fraction of the total mass that remains in the disk falls off sharply as \( 3 - n \) (equivalently, \( \Gamma - 4/3 \)) increases.

Subject headings: black hole physics — hydrodynamics — relativity — stars: rotation

1. INTRODUCTION

Determining the final state of a rotating star undergoing gravitational collapse is an important issue in relativistic astrophysics. Such a collapse is the principle route by which a rotating black hole forms in nature. Several different scenarios involving the collapse of rotating stars to black holes have been the focus of considerable attention recently. The collapse of massive stars in hypernovae explosions (“collapsars”) may be the origin of long-period gamma-ray bursts (MacFadyen & Woosley 1999; MacFadyen et al. 2001). The remnant of a binary neutron star, following inspiral and coalescence, is likely to be rapidly rotating and undergo collapse, either prompt or delayed, depending on its mass (Shibata & Uryu 2000, 2002; Baumgarte et al. 2000). Binary neutron star merger and collapse provides a plausible scenario for generating short-period gamma-ray bursts (Narayan et al. 1992; Ruffert & Janka 1999) and a good candidate for the detection of gravitational waves by Advanced LIGO and other high-frequency, gravitational wave laser interferometers. The collapse of a rotating supermassive star may be a promising route for forming the seeds of supermassive black holes (see Shapiro 2003 for a review and references), which are likely to reside at the centers of many, and perhaps most, bulge galaxies (Richstone et al. 1998; Ho 1999), including the Milky Way (Genzel et al. 1997; Ghez et al. 2000; Schödel et al. 2002), and are believed to be the engines that power active galactic nuclei (AGNs) and quasars (Rees 1998, 2001).

General relativity induces a radial instability in compact stars to catastrophic collapse. Rotation can support larger masses in stable equilibrium, but all stars become unstable to collapse when they are sufficiently compact. Uniformly rotating stars can have masses exceeding the maximum allowed mass of a nonrotating spherical star by about \( \leq 20\% \). Cook et al. (1994a, 1994b) have constructed numerical solutions of such “supramassive” stars, both for polytropes and for 14 realistic candidate nuclear equations of state. Differentially rotating stars can be “hypermassive,” with masses exceeding the maximum value for supramassive stars by factors of 2 or more (Baumgarte et al. 2000; Lyford et al. 2003). Viscosity (molecular or turbulent) and magnetic fields drive differentially rotating configurations to uniform rotation on secular timescales. (See, e.g., Shapiro 2000 and Cook et al. 2003 for recent Newtonian calculations of magnetic braking and viscous damping of differential rotation, and Liu & Shapiro 2004 for approximate, general relativistic calculations of these phenomena; see Duez et al. 2004 for detailed, general relativistic, numerical simulations of viscous damping of differential rotation.)

Here we focus on uniformly rotating stars and consider the collapse of supramassive stars. The first investigation of the gravitational collapse of supramassive stars was performed by Shibata et al. (2000), who employed a fully relativistic hydrodynamics code in three spatial dimensions plus time. They studied the collapse of a marginally unstable, relativistic polytrope with polytropic index \( n = 1 \) spinning at the mass-shedding limit. The mass-shedding limit is the maximal spin rate for a uniformly rotating star; here matter on the equator moves in a Keplerian (geodesic) orbit about the star, supported against gravity entirely by centrifugal forces. Stars with \( n = 1 \) have stiff equations of state, are not very centrally condensed, and are fairly relativistic at the onset of collapse, with \( R_\odot/M = 5.6 \), where \( R_\odot \) is the equatorial (circumferential) radius and \( M \) is the total mass energy. Such stars provide crude models of rapidly rotating, massive neutron stars. The outcome of collapse is a Kerr black hole containing all the rest mass \( M_0 \) of the initial configuration, and essentially all of the total mass energy \( M \) and angular momentum \( J \) of the initial configuration (apart from the small amount radiated away by gravitational waves). The reason that no matter remains outside the hole is that at the equator the specific angular momentum of the gas, which is strictly conserved during axisymmetric collapse, is smaller than \( \mathcal{J}_{\text{ISCO}} \), the specific angular momentum of a particle at the innermost stable circular orbit (ISCO) about the

1 Department of Physics, Department of Astronomy, and NCSA, University of Illinois at Urbana-Champaign, Urbana, IL 61801-3080.
final hole. Hence, all the interior matter in the star is captured by the hole. This situation is similar to the collapse of a non-rotating spherical star, which forms a Schwarzschild black hole without any ambient disk. By contrast, Shibata & Shapiro (2002) performed a fully relativistic hydrodynamics simulation in axisymmetry of the collapse of a marginally unstable \( n = 3 \) polytrope at the mass-shedding limit. Such a configuration models the likely endpoint state of an evolved, radiation-dominated, very massive or supermassive star at the onset of collapse (Baumgarte & Shapiro 1999). By contrast with an dominated, very massive or supermassive star at the onset of collapse, Shibata & Shapiro (2002) performed a fully relativistic hydrodynamics simulation without any ambient disk. By contrast, Shibata & Shapiro (2002) have shown that the mass of the disk depends sensitively on the mass and spin of the final black hole and ambient disk. We calculate the mass and spin of the black hole and disk in \( \S 3 \). Henceforth we adopt geometrized units and set \( G = c = 1 \).

2. THE MARGINALLY UNSTABLE, SUPRAMASSIVE PROGENITOR STAR

In this section we analyze the equilibrium and stability of a supramassive polytrope spinning at mass shedding and identify the critical configuration at which radial instability sets in.

Our treatment is a straightforward generalization to arbitrary polytropic index \( n \) of the semianalytic analyses of Baumgarte & Shapiro (1999). We briefly review the predictions of the Roche approximation for the rotating envelope in \( \S 2.1 \). In \( \S 2.2 \) we present a post-Newtonian energy variational calculation that allows us to determine the critical configuration for each \( n \) and identify its two key parameters, \( R_p/M \) (where \( R_p \) is the polar radius) and \( J/M^2 \), needed to determine the mass and spin of the final black hole and disk. We calculate the mass and spin of the black hole and disk in \( \S 3 \).

2.1. Review of the Newtonian Roche Model

Stars with soft equations of state are extremely centrally condensed: they have an extended, low-density envelope, while the bulk of the mass is concentrated in the core. For an \( n = 3 \) Newtonian polytrope, for example, the ratio between central density to average density is \( \rho_c/\bar{\rho} = 54.2 \). The gravitational force in the envelope is therefore dominated by the massive core, and it is thus legitimate to neglect the self-gravity of the envelope. In the equation of hydrostatic equilibrium,

\[
\nabla P = -\nabla(\Phi + \Phi_c),
\]

this neglect amounts to approximating the Newtonian gravitational potential \( \Phi \) by

\[
\Phi = -\frac{M}{r}.
\]

In equation (1) we introduce the centrifugal potential \( \Phi_c \), which, for constant angular velocity \( \Omega \) about the \( z \)-axis, can be written

\[
\Phi_c = -\frac{1}{2} \Omega^2 (x^2 + y^2) = -\frac{1}{2} \Omega^2 r^2 \sin^2 \theta.
\]

Integrating equation (1) yields the Bernoulli integral

\[
h + \Phi + \Phi_c = H,
\]

where \( H \) is a constant of integration and

\[
h = \int \frac{dP}{\rho} = (n + 1) \frac{P}{\rho}
\]

is the enthalpy per unit mass. Here we have assumed a polytropic equation of state

\[
P = K \rho^\Gamma,
\]

\[
\Gamma, \quad \Gamma = 1 + 1/n = \text{const.}
\]

Evaluating equation (4) at the pole yields

\[
H = -\frac{M}{R_p},
\]

since \( h = 0 \) on the surface of the star and \( \Phi_c = 0 \) along the axis of rotation. In the following we assume that the polar radius \( R_p \) of a rotating star is always the same as in the nonrotating case. This assumption has been shown numerically to be very accurate (e.g., Papaloizou & Whelan 1973). The mass of the star is
hardly changed from its value in spherical equilibrium, which to leading Newtonian order is given by the polytopic relation

$$M = 4\pi R_p^{3-n}/(1-n) \left[ (n+1)K \right]^{n/(n-1)} \frac{r}{\xi_1^{3-n}/(n-1)} \frac{\xi_1^{2} \theta' (\xi_1)}{\xi_1}.$$  

(8)

where the Lane-Emden functions for polytropes appearing above are tabulated in Chandrasekhar (1957). A rotating star reaches mass shedding when the equator orbits with the Kepler frequency. Using equations (3) and (4), it is easy to show that at this point the ratio between equatorial and polar radius is

$$\left( \frac{R_e}{R_p} \right)_{\text{shed}} = \frac{3}{2}.$$  

(9)

The corresponding maximum angular velocity is

$$\Omega_{\text{shed}} = \left( \frac{2}{3} \right)^{3/2} \left( \frac{M}{R_p^3} \right)^{1/2}$$  

(10)

(Zel’dovich & Novikov 1971; Shapiro & Teukolsky 1983).

Since the bulk of the matter is concentrated in the core and hardly affected by the rotation, the moment of inertia of the star barely changes with rotation and is well approximated by the nonrotating value

$$I = \frac{2}{5} \kappa_n M R_p^2.$$  

(11)

where the nondimensional coefficient $\kappa_n$ is tabulated for many different polytropes $n$ in Lai et al. (1993) and for $n = 3$ and $n = 2.5$ in Table 1. The ratio between the kinetic and potential energy at mass shedding then becomes

$$\left( \frac{T}{|W|} \right)_{\text{shed}} = \left( \frac{1}{2} \right) \Omega_{\text{shed}}^2 \left( \frac{M}{R_p^3} \right) \left( \frac{8}{81} \right) \left( 1 - \frac{n}{5} \right) \kappa_n.$$  

(12)

This result predicts that $T/|W|$ of a maximally rotating polytrope of index $n$ is a universal constant, independent of mass, radius, or angular velocity.

Inserting equations (2), (3), and (5)–(10) into equation (4) yields the density throughout the extended envelope,

$$\rho = \frac{\xi_1^{3-n}}{4\pi} \left[ \frac{\xi_1^{2} \theta' (\xi_1)}{\xi_1} \right]^{n-1} \frac{M}{R_p^3} \left( \frac{r}{R_p} - 1 + \frac{4}{27} \frac{r^2}{R_p^2} \sin^2 \theta \right)^n.$$  

(13)

The stellar surface is the boundary along which $\rho = 0$ and is thus defined by the curve $r(\theta)$ given by

$$\frac{4}{27} \frac{r^3}{R_p^3} \sin^2 \theta - \frac{r}{R_p} + 1 = 0.$$  

(14)

The solution to this cubic equation is given by

$$\frac{r(\theta)}{R_p} = 3 \frac{\sin (\theta/3)}{\sin \theta}.$$  

(15)

2.2. Critical Configuration

To determine the equilibrium and stability of a rotating polytrope, we express its total energy as the sum of the (Newtonian) internal energy $U$, the potential energy $W$, the rotational energy $T$, a post-Newtonian correction $E_{\text{PN}}$, and a post–post-Newtonian correction $E_{\text{PPN}}$. Writing the terms in that order yields the energy functional

$$E(\rho_c; M, J) = k_1 KM \rho_c^{4/3} - k_2 M^{5/3} \rho_c^{1/3} + k_3 M^{7/3} \rho_c^{2/3} - k_4 M^{7/3} \rho_c^{2/3} - k_5 M^3 \rho_c.$$  

(16)

where $\rho_c$ is the central density and we have defined $J = J/M^2$ and neglected corrections due to deviations from sphericity. This neglect is justified, since these corrections scale with $T/|W|$, which, according to equation (12), is always very small for centrally condensed configurations. Even though the value of the post–post-Newtonian correction $E_{\text{PPN}}$ is very small, this term is crucial for determining the critical, marginally stable configuration for $n = 3$, as emphasized by Zel’dovich & Novikov (1971) and Baumgarte & Shapiro (1999). The values of the nondimensional coefficients $k_i$ are determined by quadratures over Lane-Emden functions. They are listed in Table 1 for $n = 3$ and $n = 2.5$; in our analysis below we evaluate them for arbitrary $2.5 \leq n \leq 3.0$ by linearly interpolating between these two values of $n$.

Note that for any polytrope, $K^{n/2}$ has units of length. We can therefore introduce nondimensional coordinates by setting $K = 1$ (see Cook et al. 1992). We denote values of nondimensional variables in these coordinates with a bar (e.g., $\bar{M}$). Values of these quantities for any other value of $K$ can be recovered easily by rescaling with an appropriate power of $K^{n/2}$; for example, $\bar{M} = K^{n/2} M$ and $\rho = K^{-n} \tilde{\rho}$.

Taking the first derivative of equation (16) with respect to the central density, holding $M$ and $J$ constant, yields the condition for hydrostatic equilibrium:

$$0 = \frac{\partial \tilde{E}}{\partial x} = \left( \frac{3}{n} \right) k_1 M^{3/3-n} - k_2 M^{5/3} + k_3 M^{7/3} x,$$

(17)

where $x = \tilde{\rho}^{1/3}$. For stable equilibrium, the second derivative of equation (16) has to be positive. A root of the second derivative therefore marks the onset of radial instability:

$$0 = \frac{\partial^2 \tilde{E}}{\partial x^2} = \left( \frac{3}{n} \right) \left( \frac{3}{n} - 1 \right) k_1 M^{3/3-n} + k_3 M^{7/3} - k_4 M^{7/3} - k_5 M^3 x.$$  

(18)
To find the critical configuration at mass shedding, equations (17) and (18) must be solved simultaneously for \( x \) and \( \dot{M} \) subject to the constraint

\[
\frac{\dot{T}}{|W_{\text{shed}}|} = \frac{k_3 j^2 M^{7/3} x^2}{k_2 M^{5/3} x} = \frac{k_3 j^2 M^{2/3} x}{k_2} ,
\]

where the left-hand side is given by equation (12).

Key parameters characterizing the critical configurations are given in columns (2)–(4) of Table 2. The first row for each \( n \) gives the solution found by solving equations (17) and (18) simultaneously, substituting equation (19). The other rows for selected \( n \) give the results of a careful integration of the full Einstein equations of general relativity for the critical equilibrium configuration. The key global parameters needed to calculate the final black hole mass fraction and spin are given in columns (5)–(7) of Table 2. The first row for each \( n \) gives the solution found by solving equations (17) and (18). The calculation of the escaping envelope mass and angular momentum of the envelope with specific angular momentum exceeding \( J_{\text{ISCO}} \) was demonstrated to be negligible. This criterion suggests a simple iterative scheme for calculating the final mass and spin of the hole and disk from the initial stellar density and angular momentum profile. First, guess the mass and spin of the hole, \( M_h \) and \( J_h \). For our initial guess, we take a black hole that has consumed all the mass and angular momentum of the star, so that \( M_h/M = 1 \) and \( J_h/J_\text{ISCO} = (J/M_\text{ISCO})_{\text{crit}} \). Next, use the initial stellar density and angular momentum profiles to “correct” this guess by calculating the escaping mass and angular momentum of the outermost envelope with specific angular momentum exceeding \( J_{\text{ISCO}} \). We note that the value \( J_{\text{ISCO}} \) depends on \( M_h \) and \( J_h \). We then “correct” the black hole mass and spin by deducting the values of the escaping mass and angular momentum of the envelope material from the guessed values of \( M_h \) and \( J_h \). We recompute \( J_{\text{ISCO}} \) for the “corrected” black hole mass and spin, and repeat the calculation of the escaping envelope mass and angular momentum until convergence is achieved. The calculation described exploits the theorem that for an axisymmetric dynamical system, the specific angular momentum spectrum, i.e., the integrated baryon rest mass of all fluid elements with specific angular momentum \( j \) less than a given value (e.g., \( j_{\text{ISCO}} \)), is strictly conserved in the absence of viscosity (Stark & Piran 1987). Any viscosity, if present, is expected to be unimportant on dynamical timescales, as required by the theorem.

For a Kerr black hole of mass \( M_h \) and spin parameter \( a = J_h/M_h \), the value of \( J_{\text{ISCO}} \) is given by

\[
J_{\text{ISCO}} = \frac{\sqrt{M_h r_{\text{ms}} (r_{\text{ms}}^2 - 2 M_h r_{\text{ms}} + a^2)} - \frac{r_{\text{ms}}^2 + 2 M_h r_{\text{ms}} + a^2}{r_{\text{ms}} (r_{\text{ms}}^2 - 3 M_h r_{\text{ms}} + 2 a^2)^{1/2}}}{2} ,
\]
where \( r_{\text{ms}} \) is the ISCO given by

\[
    r_{\text{ms}} = M_h \left\{ 3 + Z_2 - \left[ (3 - Z_1)(3 + Z_1 + 2Z_2) \right]^{1/2} \right\},
\]

where

\[
    Z_1 \equiv 1 + \left( 1 - \frac{a^2}{M_h^2} \right)^{1/3} \left[ \left( 1 + \frac{a}{M_h} \right)^{1/3} + \left( 1 - \frac{a}{M_h} \right)^{1/3} \right] \]

and

\[
    Z_2 = \left( 3 - \frac{a^2}{M_h^2} + Z_1^2 \right)^{1/2}
\]

(see, e.g., Shapiro & Teukolsky 1983). Clearly, the infalling gas co-rotates with the black hole.

The mass of the escaping matter in the envelope with \( j > j_{\text{ISCO}} \) is given by

\[
    \Delta M = \int \int 2\pi \varpi \, d\varpi \, \rho,
\]

where the density is given by equation (13) and the quadrature is performed over all cylindrical shells in the star with cylindrical radii \( \varpi > \varpi_{SCO} = (j_{SCO} / \Omega)^{1/2} \). Here we set \( \Omega = \Omega_{\text{held}} \) given by equation (10). (The quantity computed in eq. [24] is actually the escaping rest mass, but when the envelope is nearly Newtonian, as we assume here, we can neglect the small difference between rest mass and total mass energy.) Defining \( \tilde{\varpi} = \varpi / R_p \) and \( \tilde{Z} = Z / R_p \) gives the non-dimensional integral

\[
    \frac{\Delta M}{M} = \xi_1^{(3-n)} \left[ \xi_1 [\theta'(\xi_1)] \right]^{(n-1)} \times \int \int \tilde{\varpi} \, d\tilde{\varpi} \, d\tilde{Z} \left[ \frac{1}{(\tilde{\varpi}^2 + \tilde{Z}^2)^{1/2}} - 1 + \frac{4}{\tilde{\varpi}^2} \right]^n.
\]

Similarly, the angular momentum carried off by the escaping matter in the envelope is given by

\[
    \Delta J = \int \int 2\pi \varpi \, d\varpi \, \rho \varpi \Omega \, d\omega
\]

or

\[
    \frac{\Delta J}{M^2} = \xi_1^{(3-n)} \left[ \xi_1 [\theta'(\xi_1)] \right]^{(n-1)} \left( \frac{2}{3} \right)^{3/2} \left( \frac{R_p}{M} \right)^{1/2} \times \int \int \tilde{\varpi}^3 \, d\tilde{\varpi} \, d\tilde{Z} \left[ \frac{1}{(\tilde{\varpi}^2 + \tilde{Z}^2)^{1/2}} - 1 + \frac{4}{\tilde{\varpi}^2} \right]^n,
\]

where \( R_p / M \) is given in Table 2 and where once again the integral is performed over all cylindrical shells in the star with \( \varpi > \varpi_{SCO} = (j_{SCO} / \Omega)^{1/2} \). Equations (25) and (27) agree with Shapiro & Shibata (2002), equations (22) and (24), for \( n = 3 \).

The mass and angular momentum of the black hole can then be determined from equations (25) and (27) according to

\[
    M_h / M = 1 - \Delta M / M
\]

and

\[
    J_h / M_h^2 = \frac{(J / M^2 - \Delta J / M^2)}{(1 - \Delta M / M)}.
\]

Once the iteration of equations (20)–(23) with equations (25), (28), and (29) converges, the mass of the ambient disk can be found from

\[
    M_{\text{disk}} / M = \Delta M / M.
\]

Typically, convergence to better than 1% is achieved after only four iterations.

The calculated parameters for the final black hole and disk are given in the last three columns of Table 2 for different values of \( n \). The values for \( n = 3 \) are the most reliable, since the calculation of the density and angular momentum distributions in the outermost layers by means of a Newtonian Roche model is most accurate for such a soft equation of state. For \( n = 3 \) we are able to compare the semianalytic calculations with the numerical collapse simulation of Shibata & Shapiro (2002). Allowing for numerical error inherent in the simulation (\( \lesssim \) a few percent), the agreement is quite good for the final masses of the black hole and disk and reasonable, but less accurate, for the final black hole spin. The numerical simulation (Table 2, third row) used an initial model obtained by a numerical integration of the full general relativistic, equilibrium equations for the critical configuration at the onset of collapse. When the parameters for this model are also used in the analytic derivation of the final black hole and disk (Table 2, second row) in place of the values determined by the variational principle (Table 2, first row), the agreement with the spin is much improved.

The dependence of the black hole spin and disk mass on the polytropic index is summarized in Figures 1 and 2. The spin parameter increases moderately with increasing \( n \) and increasing central concentration in the critical star. The disk mass is far more sensitive to the equation of state and increases very rapidly with increasing \( n \). There is hardly any mass in the disk unless \( n \) is very close to 3, a result that is consistent with the
The curves and points are labeled as in Fig. 1. 

4. APPLICATION: VERY MASSIVE STAR COLLAPSE

A very massive, uniformly rotating star \( \left(M/M_\odot \gg 100\right) \) is characterized by the following properties:

1. It is dominated by thermal radiation pressure.
2. It is fully convective.
3. It is governed by nearly Newtonian gravitation.
4. It is described by the Roche model in the outer envelope.

Such stars behave as polytropes with adiabatic index \( \Gamma = 1 + 1/n \) given by

\[
\Gamma \approx 4/3 + \beta/6 + \mathcal{O}(\beta^2), \quad \beta = \frac{P_g}{P_r},
\]

where \( P_r = \frac{1}{3}aT^4 \) is the radiation pressure and \( P_g = n k T / \mu \) is the gas pressure (see, e.g., Shapiro & Teukolsky 1983, chap. 17). The ratio \( \beta \ll 1 \) is related to the radiation entropy per baryon \( s_r \) (approximately constant throughout the star) according to \( \beta = (4/\mu)(s_r/k) \), where \( \mu \) is the mean molecular weight, while the mass is related to the entropy according to \( s_r/k \approx 0.942(M/M_\odot)^{1/2} \).

Combining the above equations and evaluating \( \mu \approx 2/(1 + 3X + 0.5Y) \) for a zero-metallicity, primordial composition \( X = 0.75 \) and \( Y = 0.25 \); Cyburt et al. 2003) likely to characterize massive Population III objects yields, to lowest order,

\[
M/M_\odot \approx \frac{116}{(3 - n)^2}.
\]

Baumgarte & Shapiro (1999) have shown that, following cooling and contraction, such a massive star will likely settle into rigid rotation and evolve to the mass-shedding limit, assuming the viscous or magnetic braking timescale for angular momentum transfer is shorter than the evolution timescale in such an object (Bisnovatyi-Kogan et al. 1967; Zel’dovich & Novikov 1971; Shapiro 2000; but see New & Shapiro 2001 for an alternative scenario). Moreover, Newtonian simulations suggest that zero-metallicity rotating stars with \( M/M_\odot \approx 260 \) do not encounter the pair-production instability, so they collapse to black holes without exploding (Fryer et al. 2001). The relation derived above between the mass \( M \) and polytropic index \( n \) for very massive stars then permits us to apply the results of the previous section to determine the final fate of these objects after they reach the onset of radial instability and collapse. For example, if the star has a mass \( M/M_\odot \approx 10^5 \), then \( 3 - n \ll 0.1 \). So according to Table 2 and Figures 1 and 2, the spin of the resulting hole will be moderate, \( J_h/M_h^2 \approx 0.5 \), and the mass of the disk leftover from the collapse will be non-negligible, \( M_D/M \approx 10^{-2} \). The relativistic simulations of Shibata & Shapiro 2002 for pure \( n = 3 \) polytropes serve to confirm these predictions in the limiting regime \( 3 - n \ll 1 \).

5. SUMMARY AND CONCLUSIONS

We have employed a simple analysis to determine the effect of the stiffness of the equation of state on the fate of the collapse of a marginally unstable, relativistic polytrope spinning uniformly at the mass-shedding limit. We have used a variational principle and an approximate, post-Newtonian energy functional to determine the key parameters defining the structure of the critical progenitor star. We compared these parameters with the results of detailed numerical model calculations of stationary configurations for select cases. We then employed a Roche model to obtain analytic expressions for the density and angular momentum profiles in the envelope of the marginally stable star. We substituted these profiles into quadratures that determine the fractions of the stellar mass and angular momentum that escape capture by the central black hole assumed to form during the collapse. The fraction of the progenitor mass and spin that go into the hole versus the ambient disk is then iterated until convergence is achieved.

We find that for the stars treated here, the mass fraction in the disk is about 10% for an \( n = 3 \) polytrope and decreases rapidly as \( n \) decreases and the equation of state stiffens. The results are in agreement with the numerical simulations in 3 + 1 by Shibata et al. (2000) for \( n = 1 \) and simulations by Shibata (2003) in axisymmetry for \( 2/3 \leq n \leq 2 \), which show that the mass fraction in a disk, if present at all, is less than 0.1% of the total mass. For the special case of \( n = 3 \), the results are also in agreement with the simulation performed by Shibata & Shapiro (2002) in axisymmetry, as has been discussed previously by Shapiro & Shibata (2002). The spin parameter of the black hole is less sensitive to the stiffness of the equation of state, decreasing from \( J_h/M_h^2 \approx 0.75 \) for \( n = 3 \) to \( J_h/M_h^2 \approx 0.39 \) for \( n = 2.5 \).

The approach outlined here is more reliable for soft equations of state. In this limit, the marginally unstable configuration is nearly Newtonian, and the post-Newtonian energy functional describing the bulk of the mass, the Newtonian Roche model describing the envelope, and the quadrature determining the escaping mass-energy fraction (assumed to equal the escaping rest-mass fraction) all become better approximations. Interestingly, this is precisely the limit in which fully relativistic numerical simulations become more taxing, because of the large dynamic range and very long time integrations required for a hydrodynamic calculation.
While the semianalytic approach presented here to predict the final black hole and disk masses and spins was applied to treat the collapse of stars with uniform rotation, the method can be used equally well to treat the collapse of unstable stars with differential rotation. Recently, Duez et al. (2004) performed fully relativistic numerical simulations of hypermassive stars with appreciable differential rotation. Hypermassive stars may form from the merger of binary neutron stars or from rotating core collapse in supernovae (Baumgarte et al. 2000). Magnetic braking or viscous damping of differential rotation in such stars can drive them unstable to collapse on secular timescales. Duez et al. (2004) performed simulations to demonstrate this effect in the case of viscosity and showed that the final black hole and disk parameters are in agreement with the values predicted by the method outlined in § 3. In particular, they showed that for rapid differential rotation characteristic of hypermassive stars, the disk mass fraction is typically large (10%–20% of the total initial mass) for stiff equations of state with $n = 1$. We therefore conclude that, in general, the parameters of the final black hole and the disk formed during the collapse of an unstable star depend both on the equation of state and on the degree of differential rotation.

It is a pleasure to thank T. W. Baumgarte, C. F. Gammie, and M. Shibata for valuable discussions. This work was supported in part by NSF grants PHY 00-90310 and PHY 02-05155 and NASA grant NAG5-10781 at the University of Illinois at Urbana-Champaign.

REFERENCES

Baumgarte, T. W., & Shapiro, S. L. 1999, ApJ, 526, 941
Baumgarte, T. W., Shapiro, S. L., & Shibata, M. 2000, ApJ, 528, L29
Bisnovatyi-Kogan, G. S., Zel’dovich, Ya. B., & Novikov, I. D. 1967, Soviet Astron., 11, 419
Chandrasekhar, S. 1957, An Introduction to the Study of Stellar Structure (2nd ed.; New York: Dover)
Cook, G. B., Shapiro, S. L., & Teukolsky, S. A. 1992, ApJ, 398, 203
———. 1994a, ApJ, 422, 227
———. 1994b, ApJ, 424, 823
Cook, J. N., Shapiro, S. L., & Stephens, B. C. 2003, ApJ, 599, 1272
Cyburt, R. H., Fields, B. D., & Olive, K. A. 2003, Phys. Lett. B, 567, 227
Duez, M. D., Liu, Y. T., Shapiro, S. L., & Stephens, B. C. 2004, Phys. Rev. D, in press (astro-ph/0402502)
Fryer, C. L., Woosley, S. E., & Heger, A. 2001, ApJ, 550, 372
Genzel, R., Eckart, A., Ott, T., & Eisenhauer, F. 1997, MNRAS, 291, 219
Ghez, A. M., Morris, M., Becklin, E. E., Tanner, A., & Kremenek, T. 2000, Nature, 407, 349
Ho, L. 1999, in Observational Evidence for Black Holes in the Universe, ed. S. K. Chakrabarti (Dordrecht: Kluwer), 157
Lai, D., Rasio, F. A., & Shapiro, S. L. 1993, ApJS, 88, 205
Liu, Y. T., & Shapiro, S. L. 2004, Phys. Rev. D, 69, 044009
Lyford, N. D., Baumgarte, T. W., & Shapiro, S. L. 2003, ApJ, 583, 410
MacFadyen, A. I., & Woosley, S. E. 1999, ApJ, 524, 262
MacFadyen, A. I., Woosley, S. E., & Heger, A. 2001, ApJ, 550, 410
New, K. C. B., & Shapiro, S. L. 2001, ApJ, 548, 439
Narayan, R., Paczynski, B., & Piran, T. 1992, ApJ, 395, L83
Papaloizou, J. C. B., & Whelan, F. A. J. 1973, MNRAS, 164, 1
Rees, M. J. 1998, in Black Holes and Relativistic Stars, ed. R. M. Wald (Chicago: Univ. Chicago Press), 79
———. 2001, in Black Holes in Binaries and Galactic Nuclei, ed. L. Kaper, E. P. J. van den Heuvel, & P. A. Woudt (New York: Springer), 351
Richstone, D., et al. 1998, Nature, 395, 14
Ruffert, M., & Janka, H.-Th. 1999, A&A, 344, 573
Schödel, R., et al. 2002, Nature, 419, 694
Shapiro, S. L. 2000, ApJ, 544, 397
———. 2003, in AIP Conf. Proc. 686, The Astrophysics of Gravitational Wave Sources, ed. J. M. Centrella (Melville: AIP), 50
Shapiro, S. L., & Shibata, M. 2002, ApJ, 577, 904
Shapiro, S. L., & Teukolsky, S. A. 1983, Black Holes, White Dwarfs, and Neutron Stars (New York: Wiley-Interscience)
Shibata, M. 2003, ApJ, 595, 992
———. 2004, preprint (astro-ph/0403172)
Shibata, M., Baumgarte, T. W., & Shapiro, S. L. 2000, Phys. Rev. D, 61, 044012
Shibata, M., & Shapiro, S. L. 2002, ApJ, 572, L39
Shibata, M., & Uryu, K. 2000, Phys. Rev. D, 61, 064001
———. 2002, Prog. Theor. Phys., 107, 265
Stark, R. F., & Piran, T. 1987, Comput. Phys. Rep., 5, 221
Zel’dovich, Ya. B., & Novikov, I. D. 1971, Relativistic Astrophysics, Vol. 1 (Chicago: Univ. Chicago Press)