Duality-invariant class of two-dimensional field theories

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Abstract

We construct a new class of two-dimensional field theories with target spaces that are finite multiparameter deformations of the usual coset $G/H$-spaces. They arise naturally, when certain models, related by Poisson–Lie T-duality, develop a local gauge invariance at specific points of their classical moduli space. We show that canonical equivalences in this context can be formulated in loop space in terms of parafermionic-type algebras with a central extension. We find that the corresponding generating functionals are non-polynomial in the derivatives of the fields with respect to the space-like variable. After constructing models with three- and two-dimensional targets, we study renormalization group flows in this context. In the ultraviolet, in some cases, the target space of the theory reduces to a coset space or there is a fixed point where the theory becomes free.
1 Introduction

Cosets $G/H$ as target spaces in 2-dim field theories have been extensively studied in
the literature, as they provide examples of spaces other than group manifolds, which
give rise to integrable models. It is always of interest to find integrable deformations
of such models and if possible classify them. In the ordinary (undeformed) coset
models one starts with the usual Wess–Zumino action for a group, with Lagrangian
density proportional to $\text{Tr}(\partial_i g^{-1} \partial^i g)$, and then restricts the trace to the coset space only.
Hence, this construction, but not the corresponding models, is quite trivial. Having in
mind 2-dim field theories, with targets spaces representing continuous deformations of
the latter coset spaces, we need models with non-trivial moduli as a starting point. Such
an example was considered in [12]; we present in this paper the generalization of this to
a class of theories.

We found natural to start, in section 2, with 2-dim models related by Poisson–Lie
T-duality [13], since these have indeed a non-trivial moduli space and, moreover, their
classical equivalence has been established [14, 15]. Also, in some examples, there are
hints that point towards the classical equivalence promoted into a quantum one at 1-loop
in perturbation theory [12, 16]. We will show that in some points in this moduli space a
local (gauge-like) invariance is developing. Hence, at these points the configuration space
is lower-dimensional and we discover in a unifying manner spaces that are deformations
of the usual coset spaces. In addition, as a byproduct, we will obtain duals of these
models that are classically canonically equivalent to them as 2-dim field theories. This
equivalence is encoded in infinite-dimensional current algebras of the parafermionic type
that we construct. We derive these from the infinite-dimensional algebras with a central
extension, which were found in the proof of canonical equivalence of the Poisson–Lie
T-duality-related models in [15]. The corresponding generating functionals have the new
feature that they are not linear in the derivatives of the fields with respect to the space-like
variable. This is in contrast with the cases of Abelian duality [17], non-Abelian duality in
Principal Chiral [18, 19] and more general [20] models, as well as for Poisson–Lie T-duality
(and its possible generalizations) [15, 12]. They are, instead, non-polynomial functions
of these derivatives. Many of these aspects are explicitly demonstrated in section 3, with
a particular example. In section 4 we discuss the renormalization group (RG) flow in
this context. As in [12], we emphasize that taking the classical limit that leads to the
lower-dimensional models and then studying the RG flow does not necessarily imply that
this limit would correspond to a fixed point of the RG flow, i.e. the two procedures do not
commute. There is, however, a particular domain in parameter space, where for a wide

\[1\] Examples include the $O(N)$ [6], the principal chiral [3] and the Gross–Neveu models [8], for which
the complete $S$-matrix was found through the existence of higher-spin-conserved currents that lead to
its factorization property. Building on work in [4], comparison between the $S$-matrix results and those
obtained by perturbative techniques in the ultraviolet (UV) regime was made for the $O(N)$ $\sigma$-model [6],
the Principal Chiral models for $SU(N)$ [3], $SO(N)$ and $Sp(N)$ [7] and the $O(N)$ Gross–Neveu model
[8], finding perfect agreement.
range of energies in the UV, the description is effectively perturbative with a UV-stable
fixed point corresponding to the point where the gauge invariance develops. Then the
model becomes effectively a two-dimensional one.

We end the paper with section 5, containing concluding remarks and a discussion
on future directions of this research. We have also written an appendix, where some
mathematical aspects of our proofs are worked out explicitly.

2 General formulation

In this section we first show how new 2-dim field theories, with target spaces representing
deformed coset spaces, arise in the context of Poisson–Lie T-duality-related \( \sigma \)-models.
We then present a duality-invariant formulation and show that canonical equivalences
are encoded into algebras of the parafermionic-type in loop space.

2.1 Formulation using Poisson–Lie T-duality-related \( \sigma \)-models

The form of 2-dim \( \sigma \)-model actions related by Poisson–Lie T-duality (in the absence of
spectator fields) is [13]

\[
S = \frac{1}{2\lambda} \int E_{AB} L^A_M L^B_N \partial_+ X^M \partial_- X^N , \quad E = (E_0^{-1} + \Pi)^{-1} ,
\]

and

\[
\tilde{S} = \frac{1}{2\lambda} \int \tilde{E}^{AB} \tilde{L}_{AM} \tilde{L}_{BN} \partial_+ \tilde{X}^M \partial_- \tilde{X}^N , \quad \tilde{E} = (E_0 + \tilde{\Pi})^{-1} .
\]

The field variables in (2.1) are \( X^M, \mu = 1, 2, \ldots, d_G \) and parametrize an element \( g \) of a
group \( G \). We also introduce representation matrices \( \{ T_A \} \), with \( A = 1, 2, \ldots, d_G \) and the
components of the left-invariant Maurer–Cartan forms \( L^A_M \). The light-cone coordinates on
the 2-dim space-time are \( x^\pm = \frac{1}{2}(t \pm x) \), whereas \( \lambda \) denotes the overall coupling constant,
which is assumed to be positive. Similarly, for (2.2) the field variables are \( \tilde{X}^M, \mu = 1, 2, \ldots, d_G \),
parametrize a different group \( \tilde{G} \), whose dimension is, however, equal to
that of \( G \). Accordingly, we introduce a different set of representation matrices \( \{ \tilde{T}_A \} \), with
\( A = 1, 2, \ldots, d_G \), and the corresponding components of the left-invariant Maurer–Cartan forms \( \tilde{L}_{AM} \). In (2.1) and (2.2), \( E_0 \) is a constant \( d_G \times d_G \) matrix, whereas \( \Pi \) and \( \tilde{\Pi} \) are
antisymmetric matrices with the same dimension as \( E_0 \), but they depend on the variables
\( X^M \) and \( \tilde{X}^M \) via the corresponding group elements \( g \) and \( \tilde{g} \). They are defined as [13]

\[
\Pi^{AB} = b^{CA} a_C^B , \quad \tilde{\Pi}_{AB} = \tilde{b}_{CA} \tilde{a}_C^B ,
\]

where the matrices \( a(g), b(g) \) are constructed using

\[
g^{-1} T_A g = a_A^B T_B , \quad g^{-1} \tilde{T}^A g = b^{AB} T_B + (a^{-1})_B^A \tilde{T}_B ,
\]

and similarly for \( \tilde{a}(\tilde{g}) \) and \( \tilde{b}(\tilde{g}) \). Consistency restricts these to obey

\[
a(g^{-1}) = a^{-1}(g) , \quad b^T(g) = b(g^{-1}) , \quad \Pi^T(g) = -\Pi(g) ,
\]
and similarly for the tilded ones. There is also a bilinear invariant \( \langle \cdot | \cdot \rangle \) with the various generators obeying
\[
\langle T_A | T_B \rangle = \langle \tilde{T}^A | \tilde{T}^B \rangle = 0 \, , \quad \langle T_A | \tilde{T}^B \rangle = \delta_A^B \, .
\] (2.6)

Finally, we note that the choice of possible groups \( G \) and \( \tilde{G} \) is restricted by the fact that their corresponding Lie algebras must form a pair of maximally isotropic subalgebras into which the Lie algebra of a larger group \( D \), known as the Drinfeld double, can be decomposed \([21]\).

Let us consider two subgroups \( H \in G \) and \( \tilde{H} \in \tilde{G} \) with \( d_H = d_{\tilde{H}} \). Accordingly we split the Lie-algebra indices as \( A = (a, \alpha) \), where Latin and Greek indices refer to subgroup and coset spaces, respectively. Then we may separate the various matrices appearing in (2.1) and (2.2) into blocks as
\[
(E_0^{-1})^{AB} = \begin{pmatrix}
E_0^{ab} & E_2^{a\beta} \\
E_3^{ab} & E_1^{a\beta}
\end{pmatrix},
\]
and
\[
\Pi^{AB} = \begin{pmatrix}
\Pi_0^{ab} & \Pi_2^{a\beta} \\
-\Pi_2^{b\alpha} & \Pi_1^{\beta}
\end{pmatrix}, \quad \tilde{\Pi}_{AB} = \begin{pmatrix}
(\tilde{\Pi}_0)_{ab} & (\tilde{\Pi}_2)_{a\beta} \\
-(\tilde{\Pi}_2)_{b\alpha} & (\tilde{\Pi}_1)_{\alpha\beta}
\end{pmatrix}. \quad (2.7)
\]

We would like to take a limit in the model (2.1) and its dual (2.2) such that the number of fields \( X_M \) (and \( \tilde{X}_M \)) is reduced by \( d_H \). We would call the remaining variables by \( X^{\mu} \) (and \( \tilde{X}^{\mu} \)) with \( \mu = 1, 2, \ldots, d_{G/H} \). Consider the limit
\[
E_0^{ab} \to \infty \iff (E_0^{-1})^{ab} \to 0 \, ,
\] (2.9)
in a uniform way for all matrix elements. This means that ratios of matrix elements remain constant in this limit. Using (2.7) we find that in the limit (2.9)
\[
E_0 \approx \begin{pmatrix}
0 & 0 \\
0 & E_1^{-1}
\end{pmatrix}.
\] (2.10)

Then, the actions (2.1) and (2.2) take the form
\[
S = \frac{1}{2\lambda} \int \Sigma_{\alpha\beta} L^\alpha_\mu L_\nu^\beta \partial_+ X^\mu \partial_- X^\nu \, , \quad \Sigma = (E_1 + \Pi_1)^{-1},
\] (2.11)
and
\[
\tilde{S} = \frac{1}{2\lambda} \int \tilde{\Sigma}^{AB} \tilde{L}_{AB} \tilde{L}_{B\nu} \partial_+ \tilde{X}^\mu \partial_- \tilde{X}^\nu \, , \quad \tilde{\Sigma} = \left( \tilde{\Pi}_0 - \tilde{\Pi}_2 E_1^{-1} + \tilde{\Pi}_1 \right)^{-1}.
\] (2.12)

Notice that in (2.11) \( \Sigma_{\alpha\beta} \) are elements of a \( d_{G/H} \times d_{G/H} \) matrix, whereas in (2.12) \( \tilde{\Sigma}^{AB} \) are elements of a \( d_{G} \times d_{G} \) one. We have anticipated that the number of variables in (2.11) and (2.12) has been reduced to \( d_{G/H} \) upon taking the limit (2.9). However, this does not happen automatically, but depends on whether or not certain conditions, as we will next prove, are fulfilled. In order to reduce the dimensionality of (2.1) we should prove that,
after taking the limit (2.9), a local gauge invariance develops, which suffices to gauge-fix $d_H$ degrees of freedom in the actions (2.1) and similarly for (2.2). For (2.1) consider the transformation

$$g \to gh , \quad h(x^+, x^-) \in H . \quad (2.13)$$

In its infinitesimal form it reads $\delta g = i g \epsilon_a T^a$. We may show that this induces the following transformations:

$$\delta L^\alpha_\pm = f_{\gamma}^\alpha e^b L^\gamma_\pm , \quad \delta X^M = L^M_a e^a . \quad (2.14)$$

Using these and the relation (A.6) of [15], specialized for coset space indices\footnote{Our symmetrization and antisymmetrization conventions are $(ab) = ab + ba$ and $[ab] = ab - ba.$}

$$L^M_a \partial_M \Pi^\delta_1 = - \tilde{f}^\alpha_\beta a - f_{a\beta}^\gamma [\gamma \Pi^\delta_1] , \quad (2.15)$$

we may prove that (2.11) is invariant under the gauge transformation (2.13), provided that the following condition holds:

$$\tilde{f}^{\alpha\beta}_c + f_{c\gamma}^{\alpha} E_1^{\gamma} + f_{c\gamma}^{\beta} E_1^{\alpha} = 0 , \quad (2.16)$$

or equivalently

$$f_{c\gamma}^{(\alpha S^\beta)\gamma} = 0 , \quad \tilde{f}^{\alpha\beta}_c + f_{c\gamma}^{\beta} A^{\alpha\gamma} = 0 , \quad (2.17)$$

where we have denoted by $S^{\alpha\beta}$ and $A^{\alpha\beta}$ the symmetric and antisymmetric parts of the matrix $E_1^{\alpha\beta}$. When the conditions (2.16) are satisfied then we may gauge-fix $d_H$ parameters in the group element $g \in G$. The most efficient way is to parametrize the group element $g \in G$ as $g = \kappa h$, where $h \in H$ and $\kappa \in G/H$, and then set $h = I$. It can be easily seen that this completely fixes the gauge freedom.

There are $d^2_{G/H} d_H$ algebraic conditions in (2.11) for the $d^2_{G/H}$ elements of the matrix $E_1$. Hence, it is not at all obvious that they can be fulfilled for a general Drinfeld double and then for any arbitrary choice of the subgroup $H \subset G$. An obvious simplification occurs when $\tilde{G}$ is an Abelian group. Then $\Pi^{AB} = 0$, $\tilde{f}^{AB} = 0$ and eq. (2.16) is solved by $E_1^{\alpha\beta} \sim \delta^{\alpha\beta}$. Then (2.11) with $\Sigma_{\alpha\beta} \sim \delta_{\alpha\beta}$ takes the form of the usual $\sigma$-model action on the coset $G/H$ space. Accordingly (2.12) represents its usual non-Abelian dual. Hence, when both groups $G$ and $\tilde{G}$ are non-Abelian, the models (2.11) and (2.12) are deformations of the usual models on coset spaces $G/H$ and of their non-Abelian duals.

### 2.2 Duality-invariant formulation

We would like to find a duality-invariant action, from which the $\sigma$-models (2.11) and (2.12) originate. It is natural to start with the manifestly Poisson–Lie T-duality-invariant action of [22], from which the $\sigma$-models (2.1) and (2.2) originate. This action is defined in the Drinfeld double as [22],

$$S(l) = I_0(l) + \frac{1}{2 \pi} \int dx dt (l^{-1} \partial_x l |R|^{-1} \partial_x l) , \quad (2.18)$$
where \( I_0(l) \) is the WZW action for a group element \( l \in D \). The operator \( R \) is defined as

\[
R = |R_A^+\rangle \eta^{AB} \langle R_B^+| + |R_A^-\rangle \eta^{AB} \langle R_B^-| ,
\]

with

\[
R_A^\pm = T_A \pm (E_0^\pm)_{AB} \tilde{T}^B , \quad \eta_{AB} = (E_0^+)_{AB} + (E_0^-)_{AB} ,
\]

where we have used the notation \( E_0^+ = E_0 \) and \( E_0^- = E_0^T \). In the limit (2.9) we have

\[
\left( \begin{array}{c} R_A^+ \\ R_A^- \end{array} \right) \approx \left( \begin{array}{c} T_a \\ T_a \pm (E_1^+)_{\alpha\beta}^{-1} \tilde{T}^\beta \end{array} \right) ,
\]

Using this and the conditions (2.16), one can show that (2.18), in the limit (2.9), develops

the gauge invariance

\[
l \to lh , \quad h(t,x) \in H ,
\]

provided that the following constraint is obeyed

\[
\langle l^{-1} \partial_x l | T_a \rangle = 0 , \quad \forall T_a .
\]

In order to avoid introducing this constraint we may use gauge fields instead. Indeed, consider the action

\[
S(l,A_t) = I_0(l) + \frac{1}{2\pi} \int \langle l^{-1} \partial_x l | R_{g/h} | l^{-1} \partial_x l \rangle - \frac{1}{\pi} \int \langle l^{-1} \partial_x l | A_t \rangle ,
\]

where \( A_t \) takes values in the Lie algebra of \( H \), i.e. \( A_t = A_t^a T_a \). The operator \( R_{g/h} \) is defined as the restriction in \( G/H \) of the corresponding operator in (2.20)

\[
R_{g/h} = |R_A^+\rangle \eta_1^{\alpha\beta} \langle R_B^+| + |R_A^-\rangle \eta_1^{\alpha\beta} \langle R_B^-| ,
\]

where

\[
R_A^\pm = T_a \pm (E_1^+)_{\alpha\beta}^{-1} \tilde{T}^\beta , \quad \eta_1^{\alpha\beta} = (E_1^+)_{\alpha\beta} + (E_1^-)_{\alpha\beta} ,
\]

and \( \eta_1^{\alpha\beta} \) is the inverse matrix of \( (\eta_1)_{\alpha\beta} \). It can be shown that (2.24) is gauge-invariant

under (2.22) and the corresponding transformation for the gauge field

\[
A_t \to h^{-1}(A_t - \partial_t)h ,
\]

provided that \( R_{g/h} \) is invariant under the similarity transformation

\[
hR_{g/h}h^{-1} = R_{g/h} .
\]

In order to prove (2.28) we first show that

\[
hR_A^\pm h^{-1} = \Delta_{\alpha}^{\pm\beta}(h) R_B^\pm , \quad \Delta_{\gamma}^{\pm\alpha}(h) \Delta_{\delta}^{\pm\beta}(h) \eta_1^{\gamma\delta} = \eta_1^{\alpha\beta} ,
\]
for some $h$-dependent matrix $\Delta^{\pm\beta}_{\alpha}$. After repeatedly using (2.4) and a lengthy computation we find that such a matrix exists and is given by
\[
\Delta^{\pm\beta}_{\alpha}(h) = (E^\pm_1)^{-1}_{{\alpha\beta}}(E^\pm_1)^{\delta\gamma}a_{\delta\gamma}(h),
\] provided that the following condition holds:\footnote{The various algebraic manipulations are facilitated by the fact that matrix elements $a(h)_A^B$ and $b(h)^{AB}$ vanish if their indices are not both Greek or Latin.}
\[
a_{\beta}(h)a_{\delta}(h)(E^\pm_1)^{\gamma\delta} = (E^\pm_1)^{\alpha\beta} \pm \Pi^{\alpha\beta}(h),
\] or equivalently, splitting into the symmetric and antisymmetric parts,
\[
a_{\beta}(h)a_{\delta}(h)S^{\gamma\delta} = S^{\alpha\beta}, \quad a_{\beta}(h)a_{\delta}(h)A^{\gamma\delta} = A^{\alpha\beta} + \Pi^{\alpha\beta}(h).
\] At first sight it seems that (2.31) is more restrictive than the corresponding conditions in (2.16), since, unlike (2.16), they are valid for finite-gauge transformations. However, we show in the appendix that (2.16) actually implies (2.31).

In the remainder of this subsection, we consider the classical equations of motion for the (manifestly) duality and gauge-invariant action (2.24). Its variation with respect to all fields is
\[
\delta S(l, A_t) = -\frac{1}{\pi} \int \delta(l^{-1}\partial_x l) \left(l^{-1}\partial_t l - R_{g/h}l^{-1}\partial_x l + A_t \right) + \delta A_t l^{-1}\partial_x l.
\] Specializing to subgroup and coset space indices, we find the equations of motion
\[
\delta(l^{-1}\partial_x l) : \quad \langle l^{-1}\partial_x l | R_{\alpha}^\pm \rangle = 0, \quad \langle l^{-1}\partial_t l | T_a \rangle = 0,
\] where we have used also the fact that, because of (2.6), $\langle A_t | T_a \rangle = 0$. Hence, the constraint (2.23) follows as the equation of motion for $A_t$. Using (2.21), the equations of motion in (2.34) can be cast into the form $\langle l^{-1}\partial_x l | R_{\alpha}^\pm \rangle = 0$. These have the same form as the equations of motion for the action (2.18) [22].

We finally note that the action (2.18) is manifestly invariant under the transformation $l \to l_0(t)l$ for some $t$-dependent group element $l_0 \in D$ [22]. By introducing gauge fields this symmetry can be promoted into a gauge symmetry with $l_0$ a function of $t$ and $x$. This type of gauge invariance, though interesting enough in its own right to be further investigated, has no apparent relation to the one we have just discussed.

### 2.3 The canonical transformation

Poisson–Lie T-duality-related models are canonically equivalent under the transformation \[14, 15\]
\[
\tilde{P}^A = J^A, \quad \tilde{J}_A = P_A, \quad J^A = L^A + \Pi^{AB}P_B, \quad \tilde{J}_A = \tilde{L}_x A + \tilde{\Pi}_{AB} \tilde{P}^B.
\]
This transformation preserves the equal-time Poisson brackets of the conjugate pairs of variables \((J^A, P_A)\) and \((\tilde{J}^A, \tilde{P}^A)\) given by \[^{13}\]

\[
\{J^A, J^B\} = \tilde{f}^{AB} C J^C \delta(x-y) , \\
\{P_A, P_B\} = f_{AB} C P_C \delta(x-y) , \\
\{J^A, P_B\} = \left( f_{BC} A J^C - \tilde{f}^{AC} B P_C \right) \delta(x-y) + \frac{1}{\lambda} \delta_B^A \delta'(x-y) ,
\]

and

\[
\{\tilde{J}^A, \tilde{J}^B\} = f_{AB} C \tilde{J}^C \delta(x-y) , \\
\{\tilde{P}^A, \tilde{P}^B\} = \tilde{f}^{AB} C \tilde{P}^C \delta(x-y) , \\
\{\tilde{J}^A, \tilde{P}^B\} = \left( \tilde{f}^{BC} A \tilde{J}^C - f_{AC} B \tilde{P}^C \right) \delta(x-y) + \frac{1}{\lambda} \delta_A^B \delta'(x-y) ,
\]

where \(\epsilon(x-y)\) is the antisymmetric step function that equals \(+1(-1)\) for \(x > y\) \((x < 0)\). Notice that the above Poisson brackets are independent of the details of the \(\sigma\)-models related by Poisson–Lie T-duality. They are simply the central extensions, in loop space, of the usual Lie-(bi-)algebras defined in the Drinfeld double. One may also show that the Hamiltonians of the two dual actions \((\ref{2.1})\) and \((\ref{2.2})\) are equal \[^{14}\] as required for canonical transformation with no explicit \(t\)-dependence. After some algebraic manipulations, these Hamiltonians can be written as

\[
H = \frac{\lambda}{2} J^A (G_0 - B_0 G_0^{-1} B_0)_{AB} J^B + \frac{\lambda}{2} P_A (G_0^{-1})^{AB} P_B - \lambda J^A (B_0 G_0^{-1})_A^B P_B ,
\]

and

\[
\tilde{H} = \frac{\lambda}{2} \tilde{J}^A (\tilde{G}_0 - \tilde{B}_0 \tilde{G}_0^{-1} \tilde{B}_0)_{AB} \tilde{J}^B + \frac{\lambda}{2} \tilde{P}^A (\tilde{G}_0^{-1})_{AB} \tilde{P}^B - \lambda \tilde{J}^A (\tilde{B}_0 \tilde{G}_0^{-1})_A^B \tilde{P}^B ,
\]

where \(G_0\) and \(B_0\) are the symmetric and antisymmetric parts of \(E_0^+\) and similarly \(\tilde{G}_0\) and \(\tilde{B}_0\) are the symmetric and antisymmetric parts of \((E_0^+)\)\(^{-1}\). Notice that in the limit \((\ref{2.3})\) the conjugate momenta \(P_a\) vanish. This is consistent with the development of a local gauge invariance \((\ref{2.13})\). At the level of the Poisson brackets the vanishing of \(P_a\), together with its conjugate \(J^a\), has to be imposed as a constraint. In fact they form a set \(\varphi_a = (P_a, J^a)\) of second-class constraints. We may see that in the limit \((\ref{2.3})\) and upon using \((\ref{2.17})\), the Hamiltonians \((\ref{2.38})\) and \((\ref{2.39})\) reduce to

\[
H_{G/H} = \frac{\lambda}{2} (S^{-1})_{\alpha\beta} J^\alpha J^\beta + \frac{\lambda}{2} S_{\alpha\beta} \tilde{P}_\alpha \tilde{P}_\beta + \lambda (S^{-1} A)_{\alpha\beta} J^\alpha \tilde{P}_\beta , 
\]

\[^{4}\]We will not display explicitly the 2-dim space-time dependence of the phase-space variables involved in the various Poisson brackets. It is understood that the first one in the bracket is always evaluated at \(x\) and the second one at \(y\), whereas the \(t\)-dependence is common. Also, compared with \((\ref{15})\), we have restored in the various Poisson brackets the dependence on the scale \(\lambda\).

\[^{5}\]The proof that \(\tilde{H} = H\) uses the fact that

\[
\tilde{G}_0^{-1} = G_0 - B_0 G_0^{-1} B_0 , \quad \tilde{B}_0 \tilde{G}_0^{-1} = - G_0^{-1} B_0 ,
\]

as well as the similar expressions obtained by interchanging tilded and untilded symbols.
and
\[ \tilde{H}_{G/H} = \frac{\lambda}{2} S^{\alpha \beta} \tilde{J}_\alpha \tilde{J}_\beta + \frac{\lambda}{2} (S^{-1})_{\alpha \beta} \tilde{P}^\alpha \tilde{P}^\beta + \lambda (S^{-1})_\alpha^\beta \tilde{P}^\alpha \tilde{J}_\beta . \] (2.42)

We may show, with the help of (2.3\textsuperscript{3}) and (2.3\textsuperscript{4}), that \( \{H_{G/H}, P_a\} = \{H_{G/H}, J^a\} \cong 0 \) (weakly). Hence, no new constraints are generated by the time \( t \)-evolution.

In general (see, for instance, [23]), in the presence of a set of second-class constraints \( \{\varphi_a\} \), one computes the antisymmetric matrix associated with their Poisson brackets \( D_{ab} = \{\varphi_a, \varphi_b\} \). When \( D_{ab} \) is invertible one simply postulates that the usual Poisson brackets are replaced by Dirac brackets, defined as

\[ \{A, B\}_D = \{A, B\} - \{A, \varphi_a\}(D^{-1})^{ab}\{\varphi_b, B\} , \] (2.43)

for any two phase-space variables \( A \) and \( B \). In our case we compute the (infinite-dimensional) matrix

\[ \begin{aligned} D(x, y) & = \frac{1}{\lambda} \left( \begin{array}{cc} 0 & \delta_a^b \\ \delta_a^b & 0 \end{array} \right) \delta'(x - y) , \end{aligned} \] (2.44)

with inverse

\[ \begin{aligned} D^{-1}(x, y) & = \frac{\lambda}{2} \left( \begin{array}{cc} 0 & \delta_a^b \\ \delta_a^b & 0 \end{array} \right) \epsilon(x - y) . \end{aligned} \] (2.45)

Then the Dirac brackets can be computed using (2.43). We find (for notational convenience in the rest of the paper, we omit the subscript \( D \) from the Dirac brackets):

\[ \begin{aligned} \{J^\alpha, J^\beta\} & = \tilde{f}^{\alpha \beta} \tilde{J}^\gamma \delta(x - y) - \frac{\lambda}{2} \epsilon(x - y) F_{1}^{\alpha \beta}(x, y) , \\ F_1^{\alpha \beta}(x, y) & \equiv (f_{c\gamma}^\alpha \tilde{f}^{c\beta} - f_{\delta}^\beta \tilde{f}^{c\alpha}) J^\gamma(x) J^\beta(y) \\ & \quad - \tilde{f}^{\alpha \gamma} \tilde{f}^{c\beta} P_{\gamma}(x) J^\beta(y) - \tilde{f}^{\beta \gamma} \tilde{f}^{c\alpha} P_{\gamma}(x) J^\alpha(x) , \end{aligned} \] (2.46)

\[ \begin{aligned} \{P_a, P_b\} & = f_{\alpha \beta}^a \gamma P_{\gamma} \delta(x - y) - \frac{\lambda}{2} \epsilon(x - y) (F_2)_{a\beta}(x, y) , \\ (F_2)_{a\beta}(x, y) & \equiv (f_{c\gamma}^\alpha \tilde{f}^{c\beta} - f_{\delta}^\beta \tilde{f}^{c\alpha}) P_{\gamma}(x) P_{\beta}(y) \\ & \quad - f_{\alpha \gamma}^c f_{c\delta}^\beta J^\gamma(x) P_{\delta}(y) - f_{\beta \gamma}^c f_{c\alpha}^\delta J^\gamma(y) P_{\delta}(x) , \end{aligned} \] (2.47)

\[ \begin{aligned} \{J^\alpha, J_\beta\} & = (f_{\beta \gamma}^\alpha J^\gamma - \tilde{f}^{\gamma \beta} P_{\gamma}) \delta(x - y) + \frac{1}{\lambda} \delta^{\alpha \beta} \delta'(x - y) - \frac{\lambda}{2} \epsilon(x - y) F_{3}^{\alpha \beta}(x, y) , \\ F_{3}^{\alpha \beta}(x, y) & \equiv \left( f_{c\gamma}^\alpha J^\gamma(x) - \tilde{f}^{\gamma \beta} P_{c\gamma}(x) \right) \left( f_{\delta}^\beta J^\delta(y) - \tilde{f}^{c\beta} P_{\delta}(y) \right) + \tilde{f}^{\alpha \gamma} f_{c\beta}^\delta J^\gamma(x) P_{\delta}(y) . \end{aligned} \] (2.48)

Notice the parafermionic character of this algebra\footnote{This is reminiscent of the parafermionic algebras that appeared [24] in the study of classical aspects of exact conformal field theories corresponding to gauged WZW models.} which is encoded in the terms containing \( \epsilon(x - y) \). The Dirac brackets for the pair \( (\tilde{J}_\alpha, \tilde{P}^\alpha) \) are obtained from (2.46)–(2.48).
by replacing untilded symbols by tilded ones and vice versa. It is instructive to write down the Dirac brackets for the case that the group $\tilde{G}$ is Abelian, i.e. $\tilde{f}^{AB}C = 0$. We find

$$\{J^\alpha, J^\beta\} = 0,$$

$$\{P^\alpha, P^\beta\} = f_{\alpha\beta}^{\gamma\delta}(x-y)$$

$$+ \frac{\lambda}{2} \epsilon^{(x-y)} \left(f_{\alpha\gamma}^{\epsilon\delta} f_{\epsilon\delta}^{\gamma\delta}(x) P^\beta(y) + f_{\beta\gamma}^{\epsilon\delta} f_{\epsilon\delta}^{\gamma\delta}(y) P^\alpha(x)\right),$$

(2.49)

$$\{J^\alpha, P^\beta\} = f_{\beta\gamma}^{\alpha\delta}(x-y) + \frac{1}{\lambda} \delta^{\alpha\beta}(x-y) - \frac{\lambda}{2} \epsilon^{(x-y)} f_{\epsilon\delta}^{\alpha\gamma} f_{\epsilon\gamma}^{\beta\delta} J^\gamma(x) J^\delta(y).$$

The above Dirac brackets can also be obtained from the ones in (2.46)–(2.48) via a contraction that Abelianizes the group $\tilde{G}$, i.e. $J^\alpha \rightarrow \frac{1}{\epsilon} J^\alpha$, $\lambda \rightarrow \epsilon \lambda$, $\epsilon \rightarrow 0$.

### 3 An explicit example

In this section we explicitly demonstrate many of the general aspects developed in section 2, using 3- and 2-dim models related by Poisson–Lie T-duality. That includes the explicit construction of the metric and antisymmetric tensor fields, of the Dirac-bracket algebra for canonical equivalence, and also of the corresponding generating functional.

#### 3.1 The Drinfeld double

Our example will be based on the 6-dim Drinfeld double considered in [12, 14, 25], which we first review by following [12]. It is just the non-compact group $SO(3,1)$ with $G = SU(2)$ and dual $\tilde{G} = E_3 = \text{solv}(SO(3,1))$ given by the Iwasawa decomposition of $SO(3,1)$ [20]. The associated 3-dim algebras $su(2)$ and $e_3$ have generators denoted by $\{T_A\}$ and $\{T^A\}$, where $A = 1, 2, 3$. Leaving aside the details we only present the elements that are necessary in this paper. It is convenient to split the index $A = (3, \alpha), \alpha = 1, 2$. The non-vanishing structure constants for the algebras $su(2)$ and $e_3$ are

$$f_{\alpha\beta}^{3\gamma} = f^{3\alpha\beta} = \epsilon_{\alpha\beta}, \quad \tilde{f}^{3\alpha\beta} = \delta_{\alpha\beta},$$

(3.1)

where our normalization is such that $\epsilon_{12} = \delta_{11} = 1$. We parametrize the $SU(2)$ group element in terms of the three Euler angles $\phi, \psi$ and $\theta$. It is represented by the $4 \times 4$ block-diagonal matrix

$$g_{SU(2)} = \text{diag}(g, g),$$

(3.2)

where

$$g = e^{\frac{i}{2} \phi \sigma_3} e^{\frac{i}{2} \theta \sigma_2} e^{\frac{i}{2} \psi \sigma_3} = \begin{pmatrix} \cos \frac{\theta}{2} e^{\frac{i}{2}(\phi+\psi)} & \sin \frac{\theta}{2} e^{\frac{i}{2}(-\phi+\psi)} \\ -\sin \frac{\theta}{2} e^{-\frac{i}{2}(\phi-\psi)} & \cos \frac{\theta}{2} e^{-\frac{i}{2}(\phi+\psi)} \end{pmatrix}. \quad (3.3)$$

7Recently, a classification was made of all possible Drinfeld doubles based on the 3-dim real Lie algebras (Bianchi algebras) [25]. It will be interesting to use them for the construction of more examples that could be useful for the investigation of various issues presented in this and the following section.
Also the group element of $E_3$ is parametrized in terms of three variables $y_1$, $y_2$ and $\chi$ and represented by the following $4 \times 4$ block-diagonal matrix

$$
\tilde{g}_{E_3} = \text{diag}(\tilde{g}_+, \tilde{g}_-),
$$

where

$$
\tilde{g}_+ = \begin{pmatrix} e^{\frac{\chi}{2}} & \chi_+ \\ 0 & e^{-\frac{\chi}{2}} \end{pmatrix}, \quad \tilde{g}_- = \begin{pmatrix} e^{-\frac{\chi}{2}} & 0 \\ \chi_- & e^{\frac{\chi}{2}} \end{pmatrix},
$$

$$
\chi_\pm = \pm e^{-\frac{\chi}{2}}(y_1 \mp iy_2).
$$

The Maurer–Cartan forms in the parametrization of the $SU(2)$ group element (3.3) are

$$
L_1 = \cos \psi \sin \theta d\phi - \sin \psi d\theta,
$$

$$
L_2 = \sin \psi \sin \theta d\phi + \cos \psi d\theta,
$$

$$
L_3 = d\psi + \cos \theta d\phi.
$$

Similarly, using the parametrization (3.4) for the $E_3$ group element we find

$$
\tilde{L}_1 = e^{-\chi} dy_1, \quad \tilde{L}_2 = e^{-\chi} dy_2, \quad \tilde{L}_3 = d\chi.
$$

The antisymmetric matrices $\Pi$ and $\tilde{\Pi}$ are

$$
\Pi = \begin{pmatrix} 0 & -\sin \psi \sin \theta & \cos \psi \sin \theta \\ \sin \psi \sin \theta & 0 & 1 - \cos \theta \\ -\cos \psi \sin \theta & \cos \theta - 1 & 0 \end{pmatrix},
$$

$$
\tilde{\Pi} = \begin{pmatrix} 0 & -y_2 e^{-\chi} & y_1 e^{-\chi} \\ y_2 e^{-\chi} & 0 & -\frac{1}{2} (1 - (1 + y_1^2 + y_2^2)e^{-2\chi}) \\ -y_1 e^{-\chi} & \frac{1}{2} (1 - (1 + y_1^2 + y_2^2)e^{-2\chi}) & 0 \end{pmatrix}.
$$

### 3.2 Explicit three- and two-dimensional models

Consider the $\sigma$-model action (2.1) for the case of our double based on $SO(3,1)$. Let us single-out the 1-dim subgroup $H \simeq U(1)$ that is generated by $T_3$. For our purposes it will be sufficient to use the following form for the $3 \times 3$ matrix $E_0^{-1}$

$$
E_0^{-1} = \begin{pmatrix} (1 + g)^{-1}a & 0 & 0 \\ 0 & a & b - 1 \\ 0 & 1 - b & a \end{pmatrix},
$$

where we have kept the conventions of (2.7) for the enumeration of the matrix elements. Using (3.6), (3.8) and (3.10), it is the easy to compute the metric and antisymmetric tensor fields corresponding to (2.1). We find a metric given by

$$
\frac{ds^2}{V} = a \left( (L_1)^2 + (L_2)^2 + (g + 1)(L_3)^2 + \frac{1 + g}{a^2} ((b \cos \theta - 1) d\phi + (b - \cos \theta) d\psi)^2 \right),
$$
and an antisymmetric tensor given by

\[
B = 2 \frac{\sin \theta}{V} \, d\theta \wedge ( (g+1)d\psi + (b+g\cos\theta)d\phi ) ,
\]

where

\[
V \equiv a^2 + (b - \cos \theta)^2 + (1 + g) \sin^2 \theta .
\]

Notice that the antisymmetric tensor can be (locally) gauged away since the corresponding 3-form field strength is zero. Also, for our purposes, we will not need the explicit expressions for the metric and antisymmetric tensor corresponding to the dual σ-model \((2.2)\). For \(b = 1\), but general \(a\) and \(g\), the above example (with its dual) was considered in \([12]\) (also in \([16]\) for \(a = b = 1\) and \(g = 0\)).

We would like to take the analogue of the limit \((2.9)\). It is clear that in our case this corresponds to letting \(g \to -1\). Comparing \((3.10)\) to \((2.7)\) we see that the 2 × 2 matrix \(E_1\) is

\[
E_1 = \begin{pmatrix} a & b-1 \\ 1-b & a \end{pmatrix} .
\]

It is easily seen that this is the most general 2 × 2 matrix that solves \((2.16)\), with structure constants given by \((3.1)\). In agreement with our general discussion, the σ-model action with metric \((3.11)\) and antisymmetric tensor \((3.12)\) develops a local invariance under the transformation

\[
\delta \psi = \epsilon(t, x) .
\]

This allows to gauge-fix the variable \(\psi = 0\). Explicitly computing \((2.11)\) we find that the metric and antisymmetric tensors are given by

\[
ds^2 = \frac{a}{a^2 + (b - \cos \theta)^2} \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) ,
\]

\[
B = 2 \frac{\sin \theta (b - \cos \theta)}{a^2 + (b - \cos \theta)^2} \, d\theta \wedge d\phi .
\]

Equivalently, the same result follows if we set \(g = -1\) directly into the expressions for the metric \((3.11)\) and antisymmetric \((3.12)\) tensors. Similarly, the dual model action \((2.12)\) is invariant under the local transformation

\[
\delta y_\alpha = \epsilon(t, x) \epsilon_{\alpha\beta} y_\beta .
\]

Hence, we may evaluate \((2.12)\) in the gauge \(y_1 = 0\). The corresponding metric (the antisymmetric tensor turns out to be zero) is found to be

\[
ds^2 = \frac{a_1/2}{1+a_1z} \left( \frac{dz^2}{\rho^2} + \left( d\rho + \left[ \frac{b-1}{a} + \frac{z-a_1\rho^2/4}{1+a_1z} \right] \frac{dz}{\rho} \right)^2 \right) ,
\]

\[
a_1 \equiv \frac{2a}{a^2 + (b-1)^2} ,
\]

(3.18)
where we have changed variables as $y_2^2 = \frac{1}{4} \rho^2 a_1^2$ and $e^{2x} = 1 + a_1 z$. The metric in (3.16) is free of singularities (since (3.15) has no fixed point) and represents a deformed 2-sphere. In contrast, (3.18) is singular for $r = 0$. This is related to the fact that $y_1 = y_2 = 0$ is a fixed point of the gauge transformation (3.17). The singularity at $1 + a_1 z = 0$ is only a coordinate singularity and can be removed by an appropriate change of variables.

It is worth while to consider some analytic continuations of the models (3.16) and its dual (3.18). If we let $\theta \to i r$, where $r \in [0, \infty)$, and also we change the sign of the overall coupling constant $\lambda$, then (3.16) becomes

$$ds^2 = \frac{a}{a^2 + (b - \cosh r)^2} \left( dr^2 + \sinh^2 r d\phi^2 \right),$$

$$B = \frac{2 \sinh r (b - \cosh r)}{a^2 + (b - \cosh r)^2} dr \wedge d\phi.$$ 

(3.19)

The corresponding analytic continuation in the dual metric (3.18) should be $\rho \to i \rho$, with a parallel change of sign in the overall coupling constant. The metric in (3.19) is reduced to the Euclidean $AdS_2$ metric if we rescale the coupling constant $\lambda \to \lambda/a$ and then take the limit $a \to \infty$ (keeping the new coupling finite). However, for generic values of the constant $a$, it represents a space that is topologically a cigar. Indeed, for $r \to 0$ we get the 2-dim Euclidean space $E^2$ in polar coordinates, whereas for $r \to \infty$ we get, after an appropriate change of variables, $R^1 \times S^1$. For $b > 0$, the cigar-shaped space develops a “pump” corresponding to the maximum of the metric components $G_{\phi\phi}$ at

$$\cosh r = \sqrt{(1 + a^2 + b^2)^2 - 4b^2 + 1 + a^2 + b^2} \quad \text{(3.20)}$$

We note that the cigar-shape topology is also a characteristic of the Euclidean black hole corresponding to the coset $SL(2, \mathbb{R})/U(1)$ exact conformal field theory [28]. However, in our case the model (3.19) is not conformal. The Drinfeld double for (3.19) and its dual model is $SO(2, 2)$, with $G = SL(2, \mathbb{R})$, instead of $SU(2)$.

### 3.3 The Dirac brackets and the generating functional

The Dirac brackets for the conjugate variables in our example are most easily written down in the basis $J^\pm = J_1 \pm i J_2$ and $P_\pm = P_1 \pm i P_2$, where the non-zero structure constants are $f_{3 \pm \pm} = \pm$, $f_{+ - 3} = 2$ and $f_{3 \pm - \pm} = 1$. Using (2.46)–(2.48) we obtain

$$\{J^\pm, J^\pm\} = \mp \lambda \epsilon(x - y) J^\pm(x) J^\pm(y),$$

$$\{J^+, J^-\} = 0.$$ 

(3.21)

If we set $b = 1$ and redefine $a \to 2/a$ and $\lambda \to \lambda a/2$ the metric (3.16) and its dual (3.18) become those derived in [12] using a limiting procedure, equivalent to (2.9). The deeper reason that validates such a procedure is, as we have shown in the present paper, the development of a local gauge invariance in this limit.
\[
\{P_\pm, P_\pm\} = +\lambda(x-y)\left(\mp P_\pm(x)P_\pm(y) + J^\mp(x)P_\pm(y) + P_\pm(x)J^\mp(y)\right),
\]
\[
\{P_+, P_-\} = -\lambda(x-y)\left(J^-(x)P_-(y) + P_+(x)J^+(y)\right),
\]
\[
\{J^\pm, P_\pm\} = \frac{1}{\chi}\delta'(x-y) - \lambda(x-y)\left(J^\pm(x)J^\mp(y) \mp J^\pm(x)P_\pm(y)\right),
\]
\[
\{J^\pm, P_\mp\} = \lambda(x-y)\left(J^\pm(x)J^\mp(y) \pm J^\pm(x)P_\pm(y)\right),
\]

where the underlined terms should be omitted in the Abelian limit of the dual group \(\hat{G} = E_3\). In this case the above algebra provides a canonical equivalence between the \(\sigma\)-model for \(S^2\) and its non-Abelian dual with respect to the left (or right) action of \(SU(2)\). Note also that the generators \(J^\pm\) form a subalgebra \([12]\).

The generating functional that demonstrates the classical equivalence between \(\sigma\)-models related by Poisson–Lie T-duality based on our Drinfeld double was explicitly constructed in \([12]\). In a slightly different form than that in \([12]\), it reads\footnote{We also correct a misprint in eq. (26) of \([12]\). In the expression for \(B_\psi\) and in the argument for \(\cot^{-1}(y_1 \cos \psi + y_2 \sin \psi)\) should be replaced by \((y_1 \cos \psi + y_2 \sin \psi) \tan \frac{\theta}{2}\).}

\[
F = \int dx \left( A\partial_x \phi + (\psi + \alpha - \phi)\partial_x \chi - \frac{2\rho \tan^{-1} B}{\sqrt{1 + \rho^2 \cos^2(\psi + \alpha)}}\partial_x (\rho \cos(\psi + \alpha))\right),
\]
\[
A \equiv -\ln \left(e^{2x} \cos^2 \frac{\theta}{2} + e^x \rho \sin \theta \sin(\psi + \alpha) + (1 + \rho^2) \sin^2 \frac{\theta}{2}\right),
\]
\[
B \equiv \frac{e^x \cot \frac{\theta}{2} + \rho \sin(\psi + \alpha)}{\sqrt{1 + \rho^2 \cos^2(\psi + \alpha)}}, \quad (y_1, y_2) = \rho(\sin \alpha, \cos \alpha) .
\]

Notice that the above generating functional depends only on the combination \(\psi + \alpha\); it is therefore invariant under the \(U(1)\) gauge transformation \(\delta \psi = \epsilon\) and \(\delta \alpha = -\epsilon\). The generating functional for the deformed coset models \([3,10]\) is obtained by solving the equation \(\frac{\delta F}{\delta \psi} = 0\) (equivalently \(\frac{\delta F}{\delta \alpha} = 0\)) for \(\psi + \alpha\) and inserting the result back into \((3.24)\). The result is a generating functional, which is non-polynomial in derivatives with respect to \(x\). The obtained expressions are quite complicated and not very illustrative, so that we decided to present the corresponding result for the \(\sigma\)-model for \(S^2\) and its non-Abelian dual. We start with the generating functional corresponding to the 2-dim \(\sigma\)-models for \(S^3\) and its non-Abelian dual with respect to the left (or right) action of \(SU(2)\) that was obtained in \([18]\). In our notation it is given by \(F = -\int dx(y_1L_1 + y_2L_2 + zL_3)\). This is easily modified to depend on the angles \(\psi\) and \(\alpha\) only through the combination \(\psi + \alpha\), by adding the term \(-\int dx \alpha \partial_x z\). Such a term, being dependent on the variables of only
one of the dual models, can be absorbed as total derivative into the corresponding action and hence it does not affect the classical dynamics. Explicitly, the resulting generating functional is

$$F = -\int dx \left( (z \cos \theta + \rho \sin \theta \sin(\psi + \alpha)) \partial_x \phi + \rho \cos(\psi + \alpha) \partial_x \theta - (\psi + \alpha) \partial_x z \right).$$ \hspace{1cm} (3.25)

The variation of $F$ with respect to $\psi + \alpha$ gives

$$\tan(\psi + \alpha) = \frac{\partial_x z \sqrt{\rho^2 \left((\partial_x \theta)^2 + \sin^2 \theta (\partial_x \phi)^2\right) - (\partial_x z)^2} - \rho^2 \sin \theta \partial_x \theta \partial_x \phi}{(\partial_x z)^2 - \rho^2 (\partial_x \theta)^2}.$$ \hspace{1cm} (3.26)

Substituting back into (3.25) we obtain

$$F = -\int dx \left( \sqrt{\rho^2 \left((\partial_x \theta)^2 + \sin^2 \theta (\partial_x \phi)^2\right) - (\partial_x z)^2} + z \cos \theta \partial_x \phi - (\psi + \alpha) \partial_x z \right),$$ \hspace{1cm} (3.27)

where $\psi + \alpha$ is given by (3.26). The generating functional (3.27) is non-polynomial in the derivatives of the fields with respect to $x$. In that sense it belongs to a new class of generating functionals, which depend not only on the fields of the two dual $\sigma$-models, but also on their first derivatives with respect to the space-like variable in a non-trivial way. For comparison, up to now, either in the case of non-Abelian duality [18, 19, 20] or for Poisson–Lie T-duality (and its possible generalizations) [14, 15], there was no dependence of the generating functional on more than the first power of these derivatives (see, for example (3.24)). Finally, we note that, according to the work in [31], generating functionals of the form (3.27), being non-linear, are expected to receive quantum corrections. Consequently, the corresponding duality rules relating the 2-dim field theories, as well as the algebra (2.46)–(2.48), are expected to be quantum-corrected.

## 4 Renormalization group flow

In this section we study the 1-loop RG equations corresponding to the three-dimensional model (3.11), (3.12). We will show that there are no fixed points in the flow and also that the correct description of the models is a non-perturbative one. However, for large domains in parameter space and for a wide range of energies in the UV, the description is effectively perturbative and the model becomes a two-dimensional one. Finally, by performing some analytic continuations we will find three- and two-dimensional models with fixed points under the RG flow, where the theory becomes free.

We begin this section with a short review of RG flow in 2-dim field theories with curved target spaces. and (3.14). The 2-dim $\sigma$-model corresponding to the metric (3.11)

\[11\] A generating functional of the type (3.27), containing first derivatives of the fields in a non-polynomial way, has appeared in a study on the canonical equivalence between Liouville and free field theories [32] (also [33] as quoted in [34]).
and antisymmetric tensor (3.12) is of the form

\[
S = \frac{1}{2\lambda} \int Q^+_{\mu\nu} \partial_+ X^\mu \partial_- X^\nu, \quad Q^+_{\mu\nu} \equiv G_{\mu\nu} + B_{\mu\nu}.
\]

It will be renormalizable if the corresponding counter-terms, at a given order in a loop expansion, can be absorbed into a renormalization of the coupling constant \(\lambda\) and (or) of some parameters labelled collectively by \(a_i\), \(i = 1, 2, \ldots\). In addition, we allow for general field redefinitions of the \(X^\mu\)’s, which are coordinate reparametrizations in the target space. This definition of renormalizability of \(\sigma\)-models is quite strict and similar to that for ordinary field theories. A natural extension of this is to allow for the manifold to vary with the mass scale and the RG to act in the infinite-dimensional space of all metrics and torsions [34]. Further discussion of this generalized renormalizability will not be needed for our purposes. Perturbatively, in powers of \(\lambda\), we express the bare quantities, denoted by a zero as a subscript, as

\[
\lambda_0 = \mu^\epsilon \lambda \left( 1 + \frac{J_1(a)}{\pi \epsilon} \lambda + \cdots \right),
\]

\[
a^i_0 = a^i + \frac{a^i_1(a)}{\pi \epsilon} \lambda + \cdots \equiv a^i \left( 1 + \frac{y a^i}{\epsilon} + \cdots \right),
\]

\[
X^\mu_0 = X^\mu + \frac{X^\mu(X, a)}{\pi \epsilon} \lambda + \cdots.
\]

The ellipses stand for higher-order loop- and pole-terms in \(\lambda\) and \(\epsilon\) respectively. Then, the beta-functions up to one loop are given by \(\beta_{\lambda} = \lambda^2 \frac{\partial \lambda}{\partial \lambda} = \frac{\lambda^2}{\pi} J_1\) and \(\beta_{a^i} = \lambda a^i \frac{\partial a^i}{\partial \lambda} = \frac{2}{\pi} a^i_1\), where, as usual, \(\beta_{\lambda} = \frac{da^i}{dt}\), \(\beta_{a^i} = \frac{da^i}{dt}\) and \(t = \ln \mu\). The equations to be satisfied by appropriately choosing \(J_1, a^i_1\) and \(X^\mu_1\) are given by

\[
\frac{1}{2} R^-_{\mu\nu} = -J_1 Q^+_{\mu\nu} + \partial_+ Q^+_{\mu\nu} a^i_1 + \partial_\lambda Q^+_{\mu\nu} X^\lambda_1 + Q^+_{\lambda\nu} \partial_\mu X^\lambda_1 + Q^+_{\mu\lambda} \partial_\nu X^\lambda_1,
\]

where \(R^-_{\mu\nu}\) are the components of the “generalized” Ricci tensor defined with a connection that includes the torsion, i.e. with \(\Gamma^\mu_{\nu\rho} - \frac{1}{2} H^\mu_{\nu\rho}\). The corresponding counter-terms were computed in the dimensional regularization scheme (see, for instance, [35]).

### 4.1 Models with no fixed points

#### 4.1.1 Three-dimensional models

In the metric (3.11) there are three parameters \(a, b\) and \(g\) and the three Euler angles \(\theta, \psi\) and \(\phi\) will be denoted by \(X^\mu\). Also for the antisymmetric tensor in (3.12) we have \(H^\mu_{\nu\rho} = 0\). Examining (4.3) we find that the coupling \(\lambda\) and the coordinates \((\theta, \psi, \phi)\) do not renormalize and therefore the corresponding beta-functions are zero\(^{12}\). In contrast,

\(^{12}\)We believe that the non-renormalization of the overall coupling constant \(\lambda\) will persist in general for all Poisson–Lie T-duality-related models with actions (2.1) and (2.2). On the other hand, models...
for the parameters $a, b$ and $g$ we find

\[
\begin{align*}
\beta_a &= \frac{\lambda}{4\pi} \frac{1 + a^2 - b^2}{a^2} \left( (g - 1)a^2 + (g + 1)(b^2 - 1) \right), \\
\beta_b &= \frac{\lambda}{2\pi} \frac{b}{a} \left( (g - 1)a^2 + (g + 1)(b^2 - 1) \right), \\
\beta_g &= \frac{\lambda}{2\pi} \frac{1 + g}{a} \left( g(1 + a^2) + (g + 2)b^2 \right).
\end{align*}
\]

(4.4)

This system of coupled non-linear equations can be considerably simplified. First, using (4.4), we may easily show that there is a RG-flow-invariant defined as

\[
\frac{a^2 + b^2 + 1}{b} \equiv 2\nu = \text{const.},
\]

(4.5)

which implies that

\[
a = \sqrt{(b_+ - b)(b - b_-)} \geq 0, \quad b_\pm \equiv \nu \pm \sqrt{\nu^2 - 1}, \quad |\nu| \geq 1.
\]

(4.6)

Without loss of generality we may assume that $\nu > 0$ since (4.5) remains invariant under $\nu \rightarrow -\nu$ and $b \rightarrow -b$. Then, using the last two equations in (4.4) we may derive an equation for $b$ as a function of $g$ whose solution is

\[
b = -g \left( \nu \pm \sqrt{\nu^2 - 1 + e^{-2C(1 + 1/\nu)^2}} \right),
\]

(4.7)

where $C$ is a real constant, which is determined by the initial conditions for $b$ and $g$. The sign in front of the square root in (4.7) is changed when $g = 0$, in order to ensure the continuity of $b$ as a function of the energy scale $t = \ln \mu$. Hence, the only differential equation we still have to solve is the one for $g$, which, after using (4.5), takes the form

\[
\beta_g = \frac{\lambda}{\pi a} (g + 1)(b + \nu g),
\]

(4.8)

where $a$ and $b$ are determined by (4.6) and (4.7). Since the RG equations are real, $a^2$ will stay strictly non-negative and therefore $b$ will oscillate with $t = \ln \mu$ between its minimum and maximum values $b_-$ and $b_+$, where $a = 0$. When $a \simeq 0$, for finite values of corresponding to a limit of (3.11) and with target space $S^3$ or its deformation along a direction in the Cartan subalgebra of $SU(2)$ (see also the comments after (4.9) below), have an overall coupling constant that gets renormalized. The reason for this apparent paradox is that, in these models, the overall coupling constant is related to our $\lambda$ by rescalings, such as those described in footnote 8, with parameters that get renormalized.

13Presumably, the dual to the (3.11), (3.12) model will also have the same beta-functions (4.4). We also note that it is highly non-trivial that the change of the matrix (3.10) under the RG eqs. (4.4) preserves its form. For example, had we started, as in [16], with a matrix $E_0^{-1}$ proportional to the identity, then eqs. (4.4) would have generated off-diagonal elements. This is clearly seen by computing the right-hand sides of eqs. (4.4) for $a = b = 1$ and $g = 0$. By doing so we obtain the infinitesimal change of $E_0^{-1} = I$ (to lowest order) and eq. (102) of [16] (with $c = 1$).
the overall coupling constant $\lambda$, the curvature for the metric (3.11) approaches infinity and the perturbative expansion of the RG equations becomes meaningless.

We have seen that the correct description of the theory is a genuine non-perturbative one. Nevertheless, for $\nu \gg 1$ we will show that there exists a wide range of energies in the UV, where the description is effectively perturbative. Moreover, there exists a fixed point at $g = -1$ where the theory has effectively a 2-dim target space. Indeed, using (4.9), we have that $a^2 \simeq 2\nu b \gg 1$ when $\nu \gg 1$. Hence, in that limit and after redefining $\lambda \to \lambda/a$ we may simplify the RG eqs. (4.4) as

$$
\beta_\lambda \simeq -\frac{\lambda^2}{4\pi}(1 - g) ,
$$

$$
\beta_g \simeq \frac{\lambda}{2\pi} g(1 + g) ,
$$

$$
\beta_b \simeq -\frac{\lambda}{2\pi} (1 - g)b .
$$

Then the metric (3.11) becomes

$$
ds^2 = (L^1)^2 + (L^2)^2 + (1 + g)(L^3)^2
= d\theta^2 + \sin^2 \theta d\phi^2 + (1 + g)(d\psi + \cos \theta d\phi)^2 ,
$$

which is the deformed $SU(2)$ Principal Chiral model considered in [36]. Also the first two of the above equations are those derived in [36] for the corresponding coupling $\lambda$ and deformation parameter $g$. In the UV the solution of (4.9) is

$$
\lambda \simeq \frac{2\pi}{t} , \quad g \simeq -1 + \text{const.} \frac{t}{2\nu} , \quad b \simeq \text{const.} \frac{t}{2\nu} , \quad \text{as } t \to \infty .
$$

Hence, in the UV $a^2 \simeq 2\nu b \sim 2\nu/t^2$. Therefore if the condition

$$
1 \ll t \ll \nu^{1/2} ,
$$

is fulfilled, then $a \gg 1$ and the model is indeed described perturbatively by (4.10). The point $g = -1$ is a UV-fixed point, where the metric (4.10) becomes $S^2$. However, outside the validity of (4.12) the correct description is non-perturbative.

### 4.1.2 Two-dimensional models

Let us now return to the 2-dim models (3.16) and (3.18). As before, there is no wave-function renormalization for $\theta$ and $\phi$, and the beta-function for the coupling $\lambda$ is zero. For the couplings $a$ and $b$ the corresponding beta-functions can be obtained by simply setting $g = -1$ into (4.4). The reason why such a procedure is consistent seems to be intimately related to the local invariance that reduces the 3-dim models into 2-dim ones. Hence, we have

$$
\beta_a = -\frac{\lambda}{2\pi} (1 + a^2 - b^2) ,
$$

$$
\beta_b = -\frac{\lambda}{\pi} ab ,
$$

(4.13)
which are nothing but the beta-functions for the 2-dim model (3.16) as well as for its
dual (3.18).

This is a strong hint that their classical equivalence can be promoted into
a quantum one as well. Having said that we note, once again, that \( g = -1 \) is not a fixed
point of the (4.8) in the UV. Since (4.5) is still a RG invariant of (4.13), it is clear that
one variable between \( a \) and \( b \) is an independent one. Eliminating \( a \) from (4.13) using
(4.5), we obtain

\[
\beta_b = -\frac{\lambda}{\pi} b \sqrt{(b_+ - b)(b - b_-)}.
\]

(4.14)

Hence, the solution for \( b \) as a function of the energy scale \( t = \ln \mu \) oscillates between \( b_+ \)
and \( b_- \) as

\[
\frac{1}{b(t)} = \nu + \sqrt{\nu^2 - 1} \sin \frac{\lambda}{\pi} (t - t_0),
\]

(4.15)

where \( t_0 \) is an arbitrary reference scale. This means that the corresponding \( \sigma \)-model
actions do not define local field theories and can be considered at most as effective
actions for scales such that \( b \) stays away from \( b_\pm \).

The usual \( S^2 \) metric and its non-Abelian dual with respect to the right (or left) action
of \( SU(2) \) are obtained from (3.16) and (3.18) if we rescale the coupling constant \( \lambda \to \lambda/a \)
and then take the limit \( a \to \infty \) (keeping the new coupling finite). However, this limit
is problematic at the quantum level since the corresponding \( \beta \)-functions do not tend to
the beta-function obtained by studying the 2-dim field theories based on \( S^2 \) (and its
non-Abelian dual) by themselves [12]. The latter is, at one-loop, just \( \beta_\lambda = -\frac{\lambda^2}{2\pi} \)
and is consistent with the fact that these models are asymptotically free. It is formally obtained
by the first of (4.13) in the limit \( a \to \infty \) after we rescale \( \lambda \to \lambda/a \) as described above.
This limit does not correspond to any fixed point of (4.13). It is easily seen that, from
a RG theory view point, these models offer an effective description of the more general
models (3.16) and (3.18) in the case of \( \alpha \simeq b \simeq \nu \gg 1 \), which, according to (4.15), occurs
at scales \( \frac{\lambda}{\pi} (t - t_0) \simeq -\frac{\pi}{2} + \frac{1}{\nu} \mod(2\pi) \).

4.2 Models with fixed points

4.2.1 Three-dimensional models

We have seen that our model (3.11), (3.12) does not have a true fixed point under the
1-loop RG eqs. (4.8). Consider, however, the analytic continuation

\[
\lambda \to -i\lambda \quad \text{and} \quad a \to ia.
\]

\[\text{In order to compare with } \beta_a \text{ and } \beta_\lambda \text{ as given by eq. (47) of [12], one should remember that these correspond to the model (3.19) with } b = 1. \text{ Imposing that } b = 1 \text{ and further requiring that } \beta_b = 0 \text{ enforces a wave-function renormalization of the } \theta, \text{i.e. in (4.2) we have } \theta_1 = -\frac{1}{a} \sin \theta, \text{in order for the model to be 1-loop-renormalizable. Then it turns out that } \beta_a = -\frac{1}{2\pi} (4 + a^2). \text{ After taking into account the redefinitions of the various parameters, as described in footnote 8 of the present paper, this implies eq. (47) of [12].}\]
Then the metric and antisymmetric tensors become
\[
\begin{align*}
    ds^2 &= \frac{a}{V} \left( (L^1)^2 + (L^2)^2 + (g+1)(L^3)^2 - \frac{1+g}{a^2}((b \cos \theta - 1) \, d\phi + (b - \cos \theta) \, d\psi)^2 \right), \\
    B &= 2i \frac{\sin \theta}{V} \, d\theta \wedge \left( (g+1) \wedge d\psi + (b + g \cos \theta) \wedge d\phi \right),
\end{align*}
\]

(4.16)
and
\[
    B = 2i \frac{\sin \theta}{V} \, d\theta \wedge \left( (g+1) \wedge d\psi + (b + g \cos \theta) \wedge d\phi \right),
\]

(4.17)
where instead of (3.13) the function \(V\) is given by
\[
    V \equiv a^2 - (b - \cos \theta)^2 - (1 + g) \sin^2 \theta.
\]
(4.18)
The fact that the antisymmetric tensor is imaginary is bothersome if we want to describe models in 2-dim Minkowskian space-times. However, for Euclidean ones, the (locally) exact 2-form measures the charge of non-trivial instanton-like configurations. The perturbative expansion is completely independent of the antisymmetric tensor, but this will definitely play a rôle in a, yet lacking, non-perturbative formulation of the model. The 1-loop RG equations for the metric (4.16) are obtained from (4.4) by the analytic continuation we have described above. Then the analogue of (4.8) is given by
\[
    \beta_g = -\frac{\lambda b}{\pi a} \, (g+1)(b + \nu g),
\]
(4.19)
where now
\[
    a = \sqrt{(b - b_+)(b - b_-)} \geq 0, \quad b_\pm \equiv \nu \pm \sqrt{\nu^2 - 1},
\]
(4.20)
and \(b\) is still given by (4.7). As before, we will assume that \(\nu > 0\) with no loss of generality. However, now \(\nu\) does not have to be larger than or equal to 1, as in (4.8), in order to ensure reality for \(a\). If \(\nu < 1\) then \(b_\pm\) are complex conjugate of each other and, unlike the case when they are real, \(b\) can take any real value without spoiling the reality of the parameter \(a\). However, now the condition \(|1 + 1/g| \geq e^C \sqrt{1 - \nu^2}\) has to be fulfilled in order for \(b\) to remain real. If on the other hand \(\nu > 1\), then \(b_\pm\) are both real and the reality condition for \(a\) requires that \(b \geq b_+ > b_-\) or \(b \leq b_- < b_+\). Since \(0 < b_- < b_+\), it turns out that there are fixed points for initial conditions where \(b\) is less than \(b_-\). Consider first the RG eq. (4.19) near the point with \(g = 1/(e^C - 1)\), \(b = 0\) and \(a = 1\). It can be written as (we take the lower sign in (4.7)):
\[
    \beta_g \simeq \frac{\lambda}{\pi} \bar{g}^*(g - g^*), \quad \bar{g}^* \equiv 1/(e^C - 1).
\]
(4.21)
The same equation near the different point with \(g = -1/(e^C + 1)\), \(b = 0\) and \(a = 1\) takes the form
\[
    \beta_g \simeq \frac{\lambda}{\pi} \bar{g}^*(g - \bar{g}^*), \quad \bar{g}^* \equiv -1/(e^C + 1).
\]
(4.22)
For \(e^C > 1\) we have \(-\frac{1}{2} < \bar{g}^* < 0 < g^*\). Hence, for \(e^C > 1\) we have an IR-stable point at \(g = g^*\) as well as a UV-stable point at \(g = \bar{g}^*\). For \(0 < e^C < 1\), we have that \(g^* < -1 < \bar{g}^* < -\frac{1}{2}\). Therefore, for \(0 < e^C < 1\) there are two UV-stable points at \(g = g^*\) and at \(g = \bar{g}^*\).
In all cases the background (4.16), (4.17) flows, either in the IR or in the UV, towards the background with

\[ ds^2 = \frac{1}{g_0} \left( \frac{d\theta^2}{\sin^2 \theta} - g_0 d\phi^2 + (g_0 + 1) d\psi^2 \right), \]

\[ B = -\frac{2i}{g_0 \sin \theta} d\theta \wedge \left( (g_0 + 1) d\psi + g_0 \cos \theta d\phi \right), \]  

(4.23)

where \( g_0 \) represents any of the two fixed points \( g^* \) or \( \tilde{g}^* \). This represents a free theory, as can be seen by changing variables as \( \sin \theta = \frac{1}{\cosh y} \). It is interesting to note that in the case \( e^C > 1 \) the signature of the metric in (4.23) is \((-+-)\) in the IR fixed point \( g_0 = g^* \) and \((+++)\) in the UV fixed point \( g_0 = \tilde{g}^* \). Also in the case of \( 0 < e^C < 1 \) the signature at the \( g_0 = g^* \) UV-stable point is \((+-+)\), but in the other UV-stable point at \( g_0 = \tilde{g}^* \) it is \((+++)\). Hence, only at \( g = \tilde{g}^* \) the metric has Euclidean signature and we expect a well-defined field-theoretical description.

Let us also note that for \( \nu \gg 1 \) the RG flow is described, as before, by (4.9), (4.11) and the corresponding \( \sigma \)-model is again (4.10), provided (4.12) is satisfied.

### 4.2.2 Two-dimensional models

Now we turn to the 2-dim model (3.16) after the same analytic continuation as before, \( a \to ia \) and \( \lambda \to -i\lambda \):

\[ ds^2 = \frac{a}{a^2 - (b - \cos \theta)^2} \left( d\theta^2 + \sin^2 \theta d\phi^2 \right), \]

\[ B = 2i \frac{\sin \theta (b - \cos \theta)}{a^2 - (b - \cos \theta)^2} \; d\theta \wedge d\phi. \]  

(4.24)

The 1-loop RG equation corresponding to (4.14) is

\[ \beta_b = -\frac{\lambda}{\pi} b \sqrt{(b - b_+)(b - b_-)}. \]  

(4.25)

The form of the solution for \( b \) as a function of the energy scale \( t = \ln \mu \) depends on whether or not \( \nu \) is smaller or larger than 1. We find

\[ \frac{1}{b(t)} = \nu + \sqrt{1 - \nu^2} \sinh \frac{\lambda}{\pi} (t - t_0), \; \text{if} \; \nu < 1, \; \frac{\pi}{2\lambda} \ln \left( \frac{1 + \nu}{1 - \nu} \right) \leq t - t_0 < \infty, \]  

(4.26)

where \( t_0 \) denotes again an arbitrary reference scale. We see that in the UV there is a fixed point at \( b = 0 \) (and \( a = 1 \)). The lower bound for \( t \) above is needed for \( b \) to stay positive, since only then is (4.26) a solution of (4.24). For the case of \( \nu > 1 \), we have to distinguish the solutions between those with \( b \leq b_- \) and those with \( b \geq b_+ \). In the former case we obtain

\[ \frac{1}{b(t)} = \nu + \sqrt{\nu^2 - 1} \cosh \frac{\lambda}{\pi} (t - t_0), \; \text{if} \; \nu > 1, \; t \geq t_0, \]  

(4.27)
and
\[
\frac{1}{b(t)} = \nu - \sqrt{\nu^2 - 1} \cosh \frac{\lambda}{\pi} (t - \tilde{t}_0), \quad \text{if} \quad \nu > 1, \quad \frac{\pi}{2\lambda} \ln \left(\frac{\nu + 1}{\nu - 1}\right) \leq t - \tilde{t}_0 < \infty, \quad (4.28)
\]

where, as before, \( t_0 \) and \( \tilde{t}_0 \) are arbitrary reference scales. For the trajectory given by (4.27), \( b \) stays positive. It starts at \( b = b_- \) for \( t = t_0 \), and ends at \( b = 0 \) for \( t \to \infty \). For the trajectory given by (4.28), \( b \) is always negative and starts at \( b = -\infty \) for \( t - \tilde{t}_0 = \frac{\pi}{2\lambda} \ln \left(\frac{\nu + 1}{\nu - 1}\right) \) and ends at \( b = 0 \) for \( t \to \infty \). Hence, we see that \( b = 0 \) is a UV fixed point.

Also as we lower the scale \( t \) towards the IR, the solution becomes singular in both cases. In any case, we then run into non-perturbative regimes. For trajectories in the region \( b \geq b_+ \), the solution is still given by (4.28), but with \(-\frac{\pi}{2\lambda} \ln \left(\frac{\nu + 1}{\nu - 1}\right) \leq t - \tilde{t}_0 \leq 0 \). In the lower limit \( b \to \infty \) and in the upper limit \( b = b_+ \). Hence in that case we have a singular behaviour of the 1-loop RG equations towards the IR as well as the UV. As we have mentioned, in those cases the corresponding 2-dim field theory is not well defined at the quantum level and can be considered only as an effective field theory at scales away from the singularities.

The 2-dim model corresponding to the fixed point at \( b = 0 \) is obtained by setting \( g_0 = -1 \) in (4.23)
\[
ds^2 = \frac{1}{\sin^2 \theta} \left( d\theta^2 + \sin^2 \theta d\phi^2 \right),
\]
\[
B = -2i \cot \theta \, d\theta \wedge d\phi.
\]

The fact that (3.16) approaches a free-field conformal field theory at the fixed point is similar to the case of an integrable model (different from (3.16)), representing also a 1-parameter deformation of \( S^2 \), that was considered in [11]. It is interesting to investigate whether or not (3.16) represents also an integrable perturbation of \( S^2 \).

## 5 Concluding remarks

We have constructed a new class of 2-dim field theories with target spaces corresponding to deformations of coset spaces \( G/H \). Our models correspond to special points of the classical moduli space of models related by Poisson–Lie T-duality, where a local invariance develops. A classification of all possible models that arise with such a procedure is an interesting open problem and can be done by analyzing the general conditions (2.16), or equivalently (2.31). By construction these models come in dual pairs. The corresponding generating functionals depend non-polynomially on the derivatives of the fields with respect to the space-like variable. The latter feature is also manifested in an underlying infinite-dimensional algebra with a central extension of the parafermionic type. It would also be interesting to uncover the relation of our models to those in [37].

We have also performed a quite general RG flow analysis using specific models with 3- and 2-dim target spaces. As in [12], we conclude that quantum aspects of the lower
dimensional models do not necessarily follow by taking the same classical limit as that used to relate the corresponding 2-dim field-theoretical classical actions. Concretely, the beta-function equations for the lower-dimensional models follow from those of the original models by just setting some parameters to their prescribed values (see (4.4) and (4.13)). However, these values do not necessarily correspond to any fixed points of the solutions of these equations. Using our 3-dim example we saw that in a large domain in parameter space, and for a wide range of energies in the UV, the description is effectively perturbative with a UV-fixed point exactly where the local gauge invariance develops. We believe that this feature will persist for more general models related by Poisson–Lie T-duality. In that respect it would be very interesting to study the RG flow in general using (2.1) and (2.2) and possibly to formulate this flow in a duality-invariant way.

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Note added

I would like to thank C. Klimcik and the referee for informing me that the models constructed here have been studied before, at a purely classical level, in [37]. The present paper provides an alternative, more natural to physicists, complementary viewpoint on the classical origin of these models. Moreover, we have further elucidated their structure by providing explicit examples and exploring quantum aspects of the renormalization group flow.

A Proof of (2.31)

In this appendix we prove that (2.32) (or equivalently (2.31)) follows from the conditions (2.17) (or equivalently (2.16)). First we rewrite (2.32), using an obvious matrix notation, as

\[ S\alpha^{-1} - \alpha^T S = 0, \quad A\alpha^{-1} - \alpha^T A = -b^T. \]

(A.1)

In [15] explicit expressions for the matrices \(a, b, \Pi\) were found in terms of normal coordinates parametrizing the group manifolds.\(^15\) In our case we have \(h = e^{ix^c Ta} \in H \subset G\). Then defining two matrices \(f\) and \(\tilde{f}\) with matrix elements

\[ f_{\alpha \beta} = f_{\alpha c}^{\beta} x^c, \quad \tilde{f}^{\alpha \beta} = \tilde{f}^{\alpha c} e^c x^c, \]

(A.2)

\(^15\)We also correct a misprint in the expression for the matrix \(\tilde{b}\) as it appeared in [15]. In eq. (41), \(n!\) should be replaced by \((n + 1)!\).
we obtain
\[ \alpha = (e^{-f})_\alpha^\beta , \quad b = \sum_{n=0}^\infty \sum_{m=0}^n \frac{(-1)^m}{(n+1)!} (f^T)^{n-m} \tilde{f} f^m . \] (A.3)

Using these expressions it is easy to show that proving (A.1) is equivalent to proving
\[ S f^{n+1} + (-1)^n (f^T)^{n+1} S = 0 , \quad n \geq 0 , \]
\[ A f^{n+1} + (-1)^n (f^T)^{n+1} A = \sum_{m=0}^n (-1)^m (f^T)^m \tilde{f} f^{n-m} , \quad n \geq 0 . \] (A.4)

Their proof proceeds by induction. For \( n = 0 \), the above conditions reduce to
\[ S f + f^T S = 0 , \quad A f + f^T A = \tilde{f} . \] (A.5)

These are nothing but the conditions (2.17) (in a matrix notation after we contract by \( x^c \) appropriately) and by assumption they are satisfied. Assuming that (A.4) are valid for \( n = m \) for some \( m \geq 1 \), we may easily show, with the aid of (A.5), that they also hold for \( n = m + 1 \). That proves (A.4) for all \( n \geq 0 \).

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