On closed subgroups of the group of homeomorphisms of a manifold
Frédéric Le Roux

To cite this version:
Frédéric Le Roux. On closed subgroups of the group of homeomorphisms of a manifold. 2012. hal-00715582

HAL Id: hal-00715582
https://hal.science/hal-00715582v1
Preprint submitted on 8 Jul 2012

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
On closed subgroups of the group of homeomorphisms of a manifold

Frédéric Le Roux

July 8, 2012

Abstract

Let $M$ be a triangulable compact manifold. We prove that, among closed subgroups of Homeo$_0(M)$ (the identity component of the group of homeomorphisms of $M$), the subgroup consisting of volume preserving elements is maximal.

AMS classification. 57S05 (57M60, 37E30).

1 Introduction

The theory of groups acting on the circle is very rich (see in particular the monographs [Ghy01, Nav07]). The theory is far less developed in higher dimension, where it seems difficult to discover more than some isolated islands in a sea of chaos. In this note, we are interested in the closed subgroups of the group Homeo$_0(M)$, the identity component of the group of homeomorphisms of some compact topological $n$-dimensional manifold $M$. We will show that, when $n \geq 2$, for any good (nonatomic and with total support) probability measure $\mu$, the subgroup of elements that preserve $\mu$ is maximal among closed subgroups.

Let us recall some related results in the case when $M$ is the circle. De La Harpe conjectured that $PSL(2, \mathbb{R})$ is a maximal closed subgroup ([Bes]). Ghys proposed a list of closed groups acting transitively, asking whether, up to conjugacy, the list was complete ([Ghy01]); the list consists in the whole group, $SO(2)$, $PSL(2, \mathbb{R})$, the group Homeo$_0(S^1)$ of elements that commutes with some rotation of order $k$, and the group $PSL_k(2, \mathbb{R})$ which is defined analogously. The first conjecture was solved by Giblin and Markovic in [GM06]. These authors also answered Ghys's question affirmatively, under the additional hypothesis that the group contains some non trivial arcwise connected component. Thinking of the two-sphere with these results in mind, one is naturally led to the following questions.

**Question 1.** Let $G$ be a proper closed subgroup of Homeo$_0(S^2)$ acting transitively. Assume that $G$ is not a (finite dimensional) Lie group. Is $G$ conjugate to one of the two subgroups: (1) the centralizer of the antipodal map $x \mapsto -x$, (2) the subgroup of area-preserving elements?

Note that the centralizer of the antipodal map is the group of lifts of homeomorphisms of the projective plane; it is the spherical analog of the groups Homeo$_{k,0}(S^1)$. 

1
Question 2. Is $\text{PSL}(2,\mathbb{C})$ maximal among closed subgroups of $\text{Homeo}_0(\mathbb{S}^2)$?

On the circle the group of measure-preserving elements coincides with $\text{SO}(2)$. It is not a maximal closed subgroup since it is included in $\text{PSL}(2,\mathbb{R})$. In contrast, we propose to prove that the closed subgroup of area-preserving homeomorphisms of the two-sphere is maximal. To put this into a general context, let $M$ be a compact topological manifold whose dimension is greater or equal to $2$. We assume that $M$ is triangulable and (for simplicity) without boundary. Let us equip $M$ with a probability measure $\mu$ which is assumed to be good: this means that every finite set has measure zero, and every non-empty open set has positive measure. We consider the group $\text{Homeo}_0(M)$ of homeomorphisms of $M$ that are isotopic to the identity, and the subgroup $\text{Homeo}_0(M,\mu)$ of elements that preserve the measure $\mu$. According to the famous Oxtoby-Ulam theorem ([OU41], see also [Fat80]), if $\mu'$ is another good probability measure on $M$ then it is homeomorphic to $\mu$, meaning that there exists an element $h \in \text{Homeo}_0(M)$ such that $h_*\mu = \mu'$. In particular the subgroup $\text{Homeo}_0(M,\mu')$ is isomorphic to $\text{Homeo}_0(M,\mu)$. We equip these transformation groups with the topology of uniform convergence, which turns them into topological groups. The subgroup $\text{Homeo}_0(M,\mu)$ is easily seen to be closed in $\text{Homeo}_0(M)$. Note that according to Fathi’s theorem (first theorem in [Fat80]), $\text{Homeo}_0(M,\mu)$ coincides with the identity component in the group of measure preserving homeomorphisms. The aim of the present note is to prove the following.

**Theorem.** The group $\text{Homeo}_0(M,\mu)$ is maximal among closed subgroups of the group $\text{Homeo}_0(M)$.

In what follows we consider some element $f \in \text{Homeo}_0(M)$ that does not preserves the measure $\mu$, and we denote by $G_f$ the subgroup of $\text{Homeo}_0(M)$ generated by
\[ \{f\} \cup \text{Homeo}_0(M,\mu). \]

Our aim is to show that the group $G_f$ is dense in $\text{Homeo}_0(M)$.

## 2 Localization

In this section we show how to find some element in $G_f$ that has small support and contracts the volume of some given ball.

**Good balls** A ball is any subset of $M$ which is homeomorphic to a euclidean ball in $\mathbb{R}^n$, where $n$ is the dimension of $M$. We will need to consider balls which are locally flat and whose boundary has measure zero. More precisely, let us denote by $B_r(0)$ the euclidean ball with radius $r$ and center 0 in $\mathbb{R}^n$. A ball $B$ will be called good if $\mu(\partial B) = 0$ and if there exists a topological embedding (continuous one-to-one map) $\gamma : B_2(0) \to M$ such that $\gamma(B_1(0)) = B$. Note that, due to countable additivity, if $\gamma : B_1(0) \to M$ is any topological embedding, then for almost every $r \in (0,1)$ the ball $\gamma(B_r(0))$ is good.
Oxtoby-Ulam theorem} We will need the following consequence of the Oxtoby-Ulam theorem. Let \( B_1, B_2 \) be two good balls in the interior of some manifold \( M' \), with or without boundary (what we have in mind is either \( M' = M \) or \( M' \) is a euclidean ball). Let \( \mu' \) be a good probability measure on \( M' \) which assigns measure zero to the boundary \( \partial M' \). Denote by \( \text{Homeo}_0(M', \mu') \) the identity component of the group of homeomorphisms of \( M' \) which are supported in the interior of \( M' \) and preserve \( \mu' \). Assume \( \mu'(B_1) = \mu'(B_2) \). Then there exists \( \phi \in \text{Homeo}_0(M', \mu') \) such that \( \phi(B_1) = B_2 \). To construct \( \phi \), we first choose a good ball \( B \) in the interior of \( M' \) that contains \( B_1, B_2 \) in its interior. According to the annulus theorem ([Kir69, Qui82]), we may find a homeomorphism \( \phi' \) supported in the ball \( B \) that sends \( B_1 \) onto \( B_2 \). A first use of the Oxtoby-Ulam theorem provides a homeomorphism \( \phi_1 \) supported in \( B_2 \) and sending the measure \( (\phi'_1 \mu')|_{B_2} \) to the measure \( \mu'|_{B_2} \). A second use of the same theorem gives a homeomorphism \( \phi_2 \) supported in \( B \setminus B_2 \) and sending the measure \( (\phi'_2 \mu')|_{B \setminus B_2} \) to the measure \( \mu'|_{B \setminus B_2} \). Then \( \phi \) is obtained as \( \phi_2 \phi_1 \phi' \). Note that, since \( \phi \) is supported in the ball \( B \), Alexander’s trick ([Ale23]) provides an isotopy from the identity to \( \phi \) within the homeomorphisms of \( B \) that preserves the measure \( \mu' \), which shows that \( \phi \) belongs to \( \text{Homeo}_0(M', \mu') \).

**Triangulations** We will also need triangulations which have good properties with respect to the measure \( \mu \). We begin with any triangulation \( \mathcal{T} \) of \( M \). We would like the \( (n-1) \)-skeleton of \( \mathcal{T} \) to have measure zero, but some \( (n-1) \)-dimensional simplices may have positive measure. We fix this as follows. Each \( n \)-dimensional simplex \( s \) of \( \mathcal{T} \) is homeomorphic to the standard \( n \)-dimensional simplex; let \( \mu_s \) be a probability measure on \( s \) which is the homeomorphic image of the Lebesgue measure on the standard simplex. The measure

\[
\mu' = \frac{1}{N} \sum \mu_s
\]

(where \( N \) denotes the number of \( n \)-dimensional simplices of \( \mathcal{T} \)) is a good probability measure on \( M \) for which the \( n-1 \)-dimensional simplices have measure zero. We apply the Oxtoby-Ulam theorem to get a homeomorphism \( h \) of \( M \) sending \( \mu' \) to \( \mu \). Then we consider the image triangulation \( \mathcal{T}_0 = h_*(\mathcal{T}) \), whose \( (n-1) \)-skeleton has measure zero. In addition to this, all the simplices of \( \mathcal{T}_0 \) have the same mass. Using successive barycentric subdivisions we get a sequence \( (\mathcal{T}_p)_{p \geq 0} \) of nested triangulations with both properties: the \( (n-1) \)-skeleton have no mass and all the simplices have the same mass. Denote by \( m_p \) the common mass of the simplices of \( \mathcal{T}_p \), and by \( d_p \) the supremum of the diameters of the simplices of \( \mathcal{T}_p \) (for some metric which is compatible with the topology on \( M \)). Then the sequences \( (m_p) \) and \( (d_p) \) tends to zero.

Here is a useful consequence. Let \( O \) be any open subset of \( M \). We define inductively \( \mathcal{O}_p \) as the set of all the \( n \)-dimensional open simplices of \( \mathcal{T}_p \) that are included in \( O \) but not in some \( s \in \mathcal{O}_{p-1} \). The elements of \( \mathcal{O} := \cup \mathcal{O}_p \) are pairwise

\footnote{One may probably avoid the use of the annulus theorem here, since the ball \( B \) may be constructed explicitly by gluing the two good balls \( B_1 \) and \( B_2 \) to a piecewise linear tube connecting them.}

3
disjoint and their closures cover $O$. Since the $(n-1)$-skeleton of our triangulations have no mass, we have the equality

$$\mu(O) = \sum_{U \in \mathcal{O}} \mu(U) \quad (1).$$

We call a (closed) simplex of some $T_p$ good if it is a good ball in $M$. We notice that for every $p > 0$, all the $n$-dimensional simplices that are disjoint from the $(n-1)$-skeleton of $T_0$ are good. Thus equality (1) still holds if, in the definition of the $O_p$'s, we replace the simplices by the simplices whose closure is good. As a consequence, if two probability measures $\mu, \mu'$ give the same mass to all the good simplices of $T_p$ for every $p$, then they are equal.

In the first Lemma we look for elements of the group $G_f$ that do not preserve the measure and have small support.

**Lemma 2.1.** For every positive $\varepsilon$ there exists a good ball $B$ of measure less than $\varepsilon$ and an element $g \in G_f$ which is supported in $B$ and does not preserve the measure $\mu$.

**Proof.** By hypothesis the probability measures $\mu$ and $f_*\mu$ are not equal. According to the discussion preceding the Lemma, there exists some $p > 0$ and some simplex of the triangulation $T_p$ whose closure $B_1$ is a good ball, and such that $\mu(B_1) \neq \mu(f^{-1}(B_1))$. To fix ideas let us assume that

$$\mu(f^{-1}(B_1)) > \mu(B_1).$$

This implies the same inequality for at least one of the simplices of $T_{p+1}$ that are included in $B_1$; thus, by induction, we see that we may choose $p$ to be arbitrarily large. Note that we have $\mu(f^{-1}(M \setminus B_1)) < \mu(M \setminus B_1)$. Thus the same reasoning, applied to $M \setminus B_1$, provides a (closed) simplex $B_2$ of some $T_{p'}$, disjoint from $B_1$, such that

$$\mu(f^{-1}(B_2)) < \mu(B_2).$$

Again, by induction, we may assume that $p' = p$ and this is an arbitrarily large integer. In particular $B_1$ and $B_2$ are good balls with the same mass. Let $B'$ be a ball whose interior contains $B_1$ and $B_2$. Since $B_1$ and $B_2$ have the same measure, by the above mentioned version of the Oxtoby-Ulam theorem there exists $\phi \in \text{Homeo}_0(M, \mu)$ supported in $B'$ and sending $B_1$ onto $B_2$. Now we consider the element

$$g = f^{-1}\phi f$$

of the group $G_f$. It has support in the ball $B = f^{-1}(B')$. It sends the ball $f^{-1}(B_1)$ to the ball $f^{-1}(B_2)$, and we have

$$\mu(f^{-1}(B_1)) > \mu(B_1) = \mu(B_2) > \mu(f^{-1}(B_2))$$

so that $g$ does not preserve the measure $\mu$, as required by the Lemma.

It remains to see that in the above construction we may have chosen $B$ to be a good ball of arbitrarily small measure. Since $\mu$ has no atom, for every $\varepsilon > 0$

---

2Note that there may be simplices in $T_0$ that fail to be good balls if $T_0$ is a triangulation but not a PL-triangulation.
Lemma 2.2. 

where there exists some $\eta > 0$ such that 

$$\mu(f^{-1}(B_1)) + \mu(f^{-1}(B_2)) \leq \epsilon.$$ 

Then we choose $B$ as a ball whose interior contains $f^{-1}(B_1)$ and $f^{-1}(B_2)$ and which still has measure less than $\epsilon$. Finally we shrink $B$ a little bit to turn it into a good ball. This completes the proof of the Lemma. 

We subdivide the euclidean unit ball $B_1(0)$ of $\mathbb{R}^n$ into the half-balls $B_1^- = B_1(0) \cap \{x \leq 0\}$ and $B_1^+ = B_1(0) \cap \{x \geq 0\}$. Let $\Sigma$ be the disk $B_1^- \cap B_1^+$ that separates the half-balls. We consider a given ball $B$ and some homeomorphism $g$ supported in $B$. For every homeomorphism $\gamma : B_1(0) \to B$ we let $\gamma^\pm = \gamma(B_1^\pm)$; we say that $\gamma$ is thin if $\gamma(\Sigma)$ has measure zero. We now consider the set $\mathcal{I}(\gamma, g)$ of all the numbers of the type 

$$\mu(g(\gamma^+)) - \mu(\gamma^+)$$

where $\gamma$ is thin.

Lemma 2.2. If $g$ does not preserve the measure $\mu$ then $\mathcal{I}(\gamma, g)$ contains an interval $[a^-, a^+]$ with $a^- < 0 < a^+$.

Proof. First we want to prove that there exists some $\gamma : B_1(0) \to B$ which is thin and such that $\mu(g(\gamma^+)) \neq \mu(\gamma^+)$. Since $g$ does not preserve the measure $\mu$, we may find some good ball $b$ in the interior of $B$ such that $\mu(b) \neq \mu(f^{-1}(b))$. To fix ideas we assume that $\mu(b) < \mu(f^{-1}(b))$. Thanks to the Oxtoby-Ulam theorem we may identify $B$ with a euclidean ball in $\mathbb{R}^n$, $b$ with another euclidean ball inside $B$, and $\mu$ with the restriction of the Lebesgue measure on $\mathbb{R}^n$. All our balls are centered at the origin. Let $b'$ be a ball slightly greater than $b$, and $T$ be a thin tube in $B \setminus b'$ connecting the boundary of $B$ and that of $b'$. There exists a homeomorphism $\gamma : B_1(0) \to B$ such that $\gamma^+ = T \cup b'$. The construction may be done so that the (Lebesgue) measure of $\gamma^+$ is arbitrarily close to that of $b$, and then we have $\mu(\gamma^+) < \mu(g^{-1}(\gamma^+))$, as wanted.

We can find a continuous family $(R_t)_{t \in [0,1]}$ of rotations of $B_1(0)$ such that $R_0$ is the identity and $R_1$ is a rotation that exchanges $B_1^-$ and $B_1^+$. Setting $\gamma_t := \gamma \circ R_t$, we have $\gamma_1^- = \gamma_0^- = \gamma^-$. Note that it may happen that $\gamma_t(\Sigma)$ has positive measure for some $t$. To remedy for this we consider $\gamma' = \phi \circ \gamma$, where $\phi : B \to B$ is a homeomorphism that fixes $\gamma(\Sigma)$, such that the image under $\gamma'$ of the Lebesgue measure on $B_1(0)$ is equivalent to the restriction of $\mu$ to the ball $B$, in the sense that both measures share the same measure zero sets; such a $\phi$ is provided by the Oxtoby-Ulam theorem. This ensures that $\gamma'_t := \gamma' \circ R_t$ is thin for every $t$. Note that $\gamma'_t^\pm = \gamma^\pm_t$ and $\gamma_1^\pm = \gamma_1^\pm$. We have 

$$\mu(g(\gamma_1^+)) - \mu(\gamma_1^+) = \mu(g(\gamma_0^+)) - \mu(\gamma_0^+),$$

$$= (1 - \mu(g(\gamma_0^+))) - (1 - \mu(\gamma_0^+)),$$

$$= -(\mu(g(\gamma_0^+)) - \mu(\gamma_0^+)) \neq 0.$$

Thus the set $\mathcal{I}(\gamma, g)$ contains the interval

$$\{\mu(g(\gamma_t^+)) - \mu(\gamma_t^+), t \in [0,1]\}$$

5
which contains both a positive and a negative number, as required by the lemma.

\[ \text{Corollary 2.3.} \quad \text{Let } \gamma_0 : B_1(0) \to M \text{ be a topological embedding in } M \text{ with } \mu(\gamma_0(\Sigma)) = 0, \text{ let } B_0 = \gamma_0(B_1(0)), \text{ and let } \varepsilon > 0 \text{ be less than the measure of } \gamma_0^+. \text{ Then there exists some element } g \in G_f \text{, supported in } B_0, \text{ such that } \\
\mu(g(\gamma_0^+)) = \mu(\gamma_0^+) - \varepsilon. \]

In the situation of the corollary we will say that \( g \) transfers a mass \( \varepsilon \) from \( \gamma_0^+ \) to \( \gamma_0^- \).

\[ \text{Proof.} \quad \text{Lemma 2.1 provides some element } g' \in G_f \text{ that does not preserve the measure } \mu, \text{ and which is supported on a good ball } B \text{ whose measure is less than the minimum of } \mu(\gamma_0^+) - \varepsilon \text{ and } \mu(\gamma_0^-). \text{ Then Lemma 2.2 provides some homeomorphism } \gamma : B_1(0) \to B \text{ which is thin and such that } g' \text{ transfers some mass } a \text{ from } \gamma^+ \text{ to } \gamma^-: \\
\mu(g'(\gamma^+)) - \mu(\gamma^+) = a. \]

Since such a number \( a \) may be chosen freely in an open interval containing 0, we may assume that \( a = \frac{\varepsilon}{N} \) for some positive integer \( N \). Choose some homeomorphism \( \Phi_1 \in \text{Homeo}_0(M, \mu) \) that sends \( B \) inside \( B_0, \gamma^+ \) inside \( \gamma_0^+ \) and \( \gamma^- \) inside \( \gamma_0^- \). Such a \( \Phi_1 \) is provided by Oxtoby-Ulam theorem, thanks to the fact that we have chosen the measure of \( B \) to be small enough and that \( \mu(\gamma(\Sigma)) = \mu(\gamma_0(\Sigma)) = 0. \)

Now the conjugate \( g_1 = \Phi_1 g' \Phi_1^{-1} \) transfers a mass \( a \) from \( \gamma_0^+ \) to \( \gamma_0^- \):

\[ \mu(g_1(\gamma_0^+)) = \mu(\gamma_0^+) - a. \]

We repeat the process with \( \gamma_1 = g_1 \circ \gamma_0 \) instead of \( \gamma_0 \), getting an element \( g_2 \in G_f \) that transfers a mass \( a \) from \( \gamma_1^+ \) to \( \gamma_1^- \):

\[ \mu(g_2 g_1(\gamma_0^+)) = \mu(g_2(\gamma_1^+)) = \mu(\gamma_1^+) - a = \mu(g_1(\gamma_0^+)) - a = \mu(\gamma_0^+) - 2a. \]

We repeat the process \( N \) times, and get the final homeomorphism \( g \) as a composition of the \( N \) homeomorphisms \( g_N, \ldots, g_1. \)

\[ \Box \]

\[ \text{3 Proof of the theorem} \]

We consider as before some element \( f \in \text{Homeo}_0(M) \setminus \text{Homeo}_0(M, \mu) \). Let \( g \) be some other element in \( \text{Homeo}_0(M) \). In order to prove the theorem we want to approximate \( g \) with some element in the group \( G_f \) generated by \( f \) and \( \text{Homeo}_0(M, \mu) \). We fix a triangulation \( T_0 \) for which the \((n - 1)\)-skeleton has zero measure. The first step of the proof consists in finding an element \( g' \in G_f \) satisfying the following property: \( \text{for every simplex } s \text{ of } T_0, \text{ the measure of } g'(s) \text{ coincides with the measure of } g^{-1}(s). \) To achieve this, the (very natural) idea
is to use corollary 2.3 to progressively transfer some mass from the simplices $s$ whose mass is larger than the mass of their image under $g^{-1}$, to those for which the opposite holds.

Here are some details. Given a triangulation $T$ for which the $(n-1)$-skeleton has zero measure, we choose two $n$-dimensional simplices $s, s'$ of $T$, and some positive $\varepsilon$ less than $\mu(s)$; let us explain how to transfer a mass $\varepsilon$ from $s$ to $s'$. First assume that $s$ and $s'$ are adjacent. Then we may choose an embedding $\gamma : B_1(0) \to s \cup s'$ with $\gamma(\Sigma) \subset s \cap s'$, and we apply corollary 2.3. Thus we get an element $h \in G_f$, supported in $s \cup s'$, such that $\mu(h(s)) = \mu(s) - \varepsilon$, and consequently $\mu(h(s')) = \mu(s') + \varepsilon$. Now consider the general case, when $s$ and $s'$ are not adjacent. Since $M$ is connected, there exists a sequence $s_0 = s, \ldots, s_\ell = s'$ of simplices of $T$ in which two successive elements are adjacent. As described before we may transfer mass $\varepsilon$ from $s_0$ to $s_1$, then from $s_1$ to $s_2$, and so on. Thus by successive adjacent transfers of mass we get some element in $h \in G_f$ that transfers mass $\varepsilon$ from $s$ to $s'$. Note that the masses of all the other elements do not change, that is, $\mu(h(\sigma)) = \mu(\sigma)$ for every simplex $\sigma$ of $T$ different from $s$ and $s'$.

Now we go back to our triangulation $T_0$, and we construct $g'$ the following way. If each simplex $s$ has the same measure as its inverse image $g^{-1}(s)$ then there is nothing to do. In the opposite case there exists some simplex $s$ of $T_0$ such that $\mu(s) > \mu(g^{-1}(s))$. We also select some other simplex $s'$ such that $\mu(s') \neq \mu(g^{-1}(s'))$, and we use the previously described construction of a homeomorphism $g_1 \in G_f$ that transfers the mass $\mu(s) - \mu(g^{-1}(s))$ from the simplex $s$ to the simplex $s'$. After doing so the number of simplices $g_1(s) \in g_1 \times T_0$ whose mass differs from the mass of $g^{-1}(s)$ has decreased by at least one compared to $T_0$. We proceed recursively until we get an element $g' \in G_f$ such that $\mu(g'(s)) = \mu(g^{-1}(s))$ for every simplex $s$ in $T_0$, as wanted for this first step.

For the second and last step we consider the triangulations $(g^{-1})_*(T_0)$ and $g'_*(T_0)$. The homeomorphism $g'g$ sends the first one to the second one, and each simplex $g^{-1}(s) \in (g^{-1})_*(T_0)$ has the same measure as its image $g'(s) \in g'_*(T_0)$. We apply Oxtoby-Ulam theorem independently on each $g'(s)$ to get a homeomorphism $\Phi_s : g'(s) \to g'(s)$, which is the identity on $\partial g'(s)$, and which sends the measure $(g'g)_*(\mu_{g^{-1}(s)})$ to the measure $\mu|g'(s)$. The homeomorphism

$$\Phi := \prod_s \Phi_s$$

preserves the measure $\mu$. Furthermore by Alexander’s trick each $\Phi_s$ is isotopic to the identity, thus $\Phi$ is isotopic to the identity, and belongs to the group $\text{Homeo}_0(M, \mu)$. Now the homeomorphism $g'' = g'^{-1}\Phi$ belongs to the group $G_f$ and for each simplex $s$ of the triangulation $T_0$ we have $g''^{-1}(s) = g^{-1}(s)$. We may have chosen the triangulation $T_0$ so that each simplex has diameter less than some given $\varepsilon$. Every point $x$ in $M$ belongs to some $n$-dimensional closed simplex $g^{-1}(s)$ of the triangulation $(g^{-1})_*(T_0)$, and since both $g(x)$ and $g''(x)$ belong to $s$ they are a distance less than $\varepsilon$ apart. In other words the uniform distance from $g$ to $g''$ is less than $\varepsilon$. This proves that $g$ belongs to the closure of $G_f$, and completes the proof of the theorem.
References

[Ale23] J. W. Alexander. On the deformation of an n-cell. *Nat. Acad. Proc.*, 9:406–407, 1923.

[Bes] M. Bestvina. Questions in geometric group theory, collected by m bestvina. [http://www.math.utah.edu/~bestvina](http://www.math.utah.edu/~bestvina).

[Fat80] A. Fathi. Structure of the group of homeomorphisms preserving a good measure on a compact manifold. *Ann. Sci. École Norm. Sup. (4)*, 13(1):45–93, 1980.

[Ghy01] Étienne Ghys. Groups acting on the circle. *Enseign. Math. (2)*, 47(3-4):329–407, 2001.

[GM06] James Giblin and Vladimir Markovic. Classification of continuously transitive circle groups. *Geom. Topol.*, 10:1319–1346 (electronic), 2006.

[GP75] Casper Goffman and George Pedrick. A proof of the homeomorphism of Lebesgue-Stieltjes measure with Lebesgue measure. *Proc. Amer. Math. Soc.*, 52:196–198, 1975.

[Kir69] Robion C. Kirby. Stable homeomorphisms and the annulus conjecture. *Ann. of Math. (2)*, 89:575–582, 1969.

[Nav07] Andrés Navas. *Grupos de difeomorfismos del círculo*, volume 13 of *Ensaios Matemáticos [Mathematical Surveys]*. Sociedade Brasileira de Matemática, Rio de Janeiro, 2007.

[OU41] J. C. Oxtoby and S. M. Ulam. Measure-preserving homeomorphisms and metrical transitivity. *Ann. of Math. (2)*, 42:874–920, 1941.

[Qui82] Frank Quinn. Ends of maps. III. Dimensions 4 and 5. *J. Differential Geom.*, 17(3):503–521, 1982.