Numerical Study of a Jeffcott Rotor Model with a Snubber Ring

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Abstract. In this article we study a two-degrees-of-freedom model of a rotor system with a bearing clearance. During operation the rotor makes intermittent contact with an outer snubber ring, which results in complex dynamical behaviour. Specifically, the system will be analyzed numerically by a path following method, where we will use the toolbox TC-HAT, which is a module for modeling non-smooth systems by AUTO 97.

1. Introduction

Rotor systems have attracted much attention due to their rich dynamics and also because of their wide industrial applications, such as aero engines, gas turbines, compressors, generators, and others. There are different sources of nonlinearities for this type of systems. One of the most important of them are the bearing clearances, which allow rotors to make intermittent contact between their components (e.g. rotor-stator dynamics), which leads to non-smooth dynamics. This is often produced by centrifugal forces due to rotor mass imbalance. Even if the rotor is well balanced at the beginning of its operating life, the balance generally deteriorate with use, which may produce high amplitude vibrations that can lead to catastrophic failures. Therefore it is important to understand the dynamics of rotor systems in order to design suitable control methods in order to avoid unwanted behaviour.

Rotor systems with bearing clearances have been considered in the past, putting special attention on Jeffcott rotors. For instance, rub interactions in rotating machines have been studied in [1], [2–5]. Stability analysis has been performed in [6], while the spontaneous sidebanding was considered in [7]. On the other hand, subharmonic and synchronous responses of rotor systems have been investigated in [3], [8–10]. Various numerical studies have revealed the rich dynamics behind this system (see [11], [12]), which includes the existence of period-doubling bifurcations and fractal basins of attraction. Apart from this, the theoretical models have been compared with experimental data, finding good correlations with the theory, see [13], [14]. The next phase of this research is the bifurcation analysis of the periodic orbits of the rotor system, which is the main concern of this article. To this end, we will use the toolbox TC-HAT (see [15]) which encompasses a set of basic tools for the bifurcation analysis of periodic
trajectories of non-smooth dynamical systems. It functions as a driver to a modified version of AUTO 97 (cf. [16]).

2. Physical model and equations of motion

Consider a two-degrees-of-freedom model of a Jeffcott rotor system with a snubber ring, as shown in Figure 1. A rotating mass \( M \) is excited by an out-of-balance \( \rho m \), within a massless elastic snubber ring. During operation the rotor makes intermittent contact with the snubber ring, resulting in a complex dynamical behaviour. The equations of motion are obtained in the coordinate system shown in Figure 1(b), centered at the static position of the rotor \( O_{r0} \). The vector \((\varepsilon_x, \varepsilon_y)\) represents the displacement of \( O_{r0} \) from the center of the resting position of the snubber ring \( O_{s0} \), which is separated from the center of the rotor \( O_r \) by \( R \). A detailed description of the variables and parameters of the model can be found in Appendix A. Under this notation, the dimensionless equations of motion are \(^1\) (cf. [11])

\[
\frac{\ddot{x}'}{x'} + 2\nu \frac{\dot{x}'}{x'} + x' + \tilde{F}_{Nx} = \eta m \tilde{\rho} \eta^2 \cos(\eta \tau + \varphi_0),
\]

\[
\frac{\ddot{y}'}{y'} + 2\nu \frac{\dot{y}'}{y'} + y' + \tilde{F}_{Ny} = \eta m \tilde{\rho} \eta^2 \sin(\eta \tau + \varphi_0),
\]

where

\[
\begin{pmatrix}
\tilde{F}_{Nx} \\
\tilde{F}_{Ny}
\end{pmatrix} = \frac{1}{k_1 \gamma} F_N = \begin{cases}
\frac{1}{k_1 \gamma} \begin{pmatrix}
k_2 (R - \gamma) \cos(\psi) \\
k_2 (R - \gamma) \sin(\psi)
\end{pmatrix}, & R \geq \gamma, \\
0, & R < \gamma,
\end{cases}
\]

\[
= \begin{cases}
\tilde{K} (\tilde{x} - \tilde{\varepsilon}_x) \left(1 - \frac{1}{\tilde{z}}\right), & \tilde{z} \geq 1, \\
\tilde{K} (\tilde{y} - \tilde{\varepsilon}_y) \left(1 - \frac{1}{\tilde{z}}\right), & \tilde{z} < 1,
\end{cases}
\]

\(^1\) In what follows, \( \ddot{x}' \) means \( \frac{d^2x}{d\tau^2} \) and so on.
represents the dimensionless normal force produced when the rotor is in contact with the snubber ring.

3. Bifurcation analysis with TC-HAT

For the numerical implementation of the rotor system in TC-HAT, we need first to write the equations (1)–(2) as a set of first-order ODEs with discontinuous right-hand side:

\[
v' = \begin{cases} 
    f_{\text{imp}}(v, \alpha, \eta), & \tilde{z} \geq 1 \text{ (i.e. the rotor is in contact with the snubber ring)}, \\
    f_{\text{ins}}(v, \alpha, \eta), & \tilde{z} < 1 \text{ (i.e. no contact with the snubber ring)},
\end{cases}
\]

where \( v = (v_1, v_2, v_3, v_4, s) \) represents the state variables of the model with \((v_1, v_2) = (\tilde{x}, \tilde{y})\), \( s = \eta \tau \mod 2\pi \) and \( \alpha = (\eta_m, \tilde{\rho}, \nu, K, \varphi_0, \bar{\varepsilon}_x, \bar{\varepsilon}_y) \). Furthermore, the vector fields are defined as follows:

\[
f_{\text{ins}}(v, \alpha, \eta) = \begin{pmatrix} v_3 \\ v_4 \\ \eta \nu v_3 - 2v_4 - v_2 \\ \eta \nu v_3 - 2v_4 - v_2 \end{pmatrix},
\]

\[
f_{\text{imp}}(v, \alpha, \eta) = \begin{pmatrix} v_3 \\ v_4 \\ \eta \nu (v_3 - 2v_4 - v_1 - K(v_1 - \bar{\varepsilon}_x) \left(1 - \frac{1}{2}\right)) \\ \eta \nu (v_3 - 2v_4 - v_1 - K(v_1 - \bar{\varepsilon}_x) \left(1 - \frac{1}{2}\right)) \end{pmatrix}.
\]

Moreover, in order to describe the trajectories of the piecewise-smooth system (5), we have introduced several segments and event functions as shown in Table 1 (cf. [15]). In this table

| Index | Segment      | Vector Field | Event Function | Jump Function |
|-------|--------------|--------------|----------------|---------------|
| I_1   | Inside       | \( f_{\text{ins}} \) | \( h_{\text{imp}} \) | \( g_{\text{id}} \) |
| I_2   | Impact       | \( f_{\text{imp}} \) | \( h_{\text{imp}} \) | \( g_{\text{id}} \) |
| I_3   | Phase-inside | \( f_{\text{ins}} \) | \( h_{\text{ph}} \) | \( g_{\text{ph}} \) |
| I_4   | Phase-impact | \( f_{\text{imp}} \) | \( h_{\text{ph}} \) | \( g_{\text{ph}} \) |
| I_5   | Turning      | \( f_{\text{ins}} \) | \( h_{\text{tur}} \) | \( g_{\text{id}} \) |

Table 1. Segments and event functions for the numerical analysis.

the event and jump functions are defined as follows: \( h_{\text{imp}}(v, \alpha) = (v_1 - \bar{\varepsilon}_x)^2 + (v_2 - \bar{\varepsilon}_y)^2 - 1 \), \( h_{\text{ph}}(v, \alpha) = s - 2\pi \), \( h_{\text{tur}}(v, \alpha) = v_3 \), \( g_{\text{id}}(v) = v \) and \( g_{\text{ph}}(v) = (v_1, v_2, v_3, v_4, s - 2\pi) \). The first row of Table 1 indicates that the segment \( I_1 \) corresponds to the situation when the rotor is not in contact with the snubber ring, so the dynamics of the system is governed by the vector field \( f_{\text{ins}} \), and during this segment we monitor the condition \( h_{\text{imp}}(v, \alpha) = 0 \) in order to detect an impact with the snubber ring, in which case the jump function \( g_{\text{id}} \) gives the initial point for the next segment, and so on.

To begin with the numerical analysis we choose the initial non-impacting periodic orbit shown in Figure 2(a), for the parameter values \( \eta = 0.85 \), \( \eta_m = 0.0017 \), \( \tilde{\rho} = 70 \), \( \nu = 0.125 \), \( K = 30 \), \( \varphi_0 = 0 \), \( \bar{\varepsilon}_x = 0.75 \) and \( \bar{\varepsilon}_y = 0 \). This orbit has the cyclic signature \( \{I_5, I_3\} \), which is marked by the colors black \( (I_5) \) and green \( (I_3) \) in Figure 2(a). The red curve represents the impact condition
\( h_{\text{imp}}(v, \alpha) = 0 \), which means that in this regime the rotor is close to impacting the snubber ring. Indeed, when performing the continuation of this orbit with TC-HAT by increasing the frequency ratio \( \eta \), we find a grazing bifurcation at \( \eta \approx 0.85281 \). With this value we can find a numerical approximation of the relation between the mass imbalance of the rotor (\( \eta_m \)) and the frequency of rotation (\( \eta \)) that produces the impact with the snubber ring (grazing curve). This two-parameter continuation is shown in Figure 2(b), for different values of viscous damping \( \nu \). Here we can identify a region between the horizontal axis and the grazing curves, where the rotor does not impact the snubber ring no matter how high the frequency is. This information is important for the analysis of forced vibrations in rotor systems, where the main concern is to determine the relation between the mass imbalance of the rotor and the frequency of rotation in order to avoid impacts with the outer snubber ring, as this phenomenon may cause serious malfunctions and damage in the rotor system.

After grazing, the dynamics of the rotor becomes nonlinear, which gives rise to a more complex behaviour. When we perform the one-parameter continuation for frequency values above the grazing point \( \eta \approx 0.85281 \), the following bifurcations are found: two period-doubling bifurcations (PD) at \( \eta \approx 0.85395 \) and \( \eta \approx 0.877789 \); and two fold bifurcations at \( \eta \approx 1.52393 \) and \( \eta \approx 1.3331 \). In Figure 3 we present a phase diagram of the rotor system for \( \eta = 0.878 \) (after the PD bifurcations) where the black and brown curves mark the co-existing period-1 and -2 orbits of (5), respectively.

4. Conclusions
We have studied numerically the dynamics of a two-degrees-of-freedom rotor model with a bearing clearance described by a non-smooth system of ODEs. During operation the rotor makes intermittent contact with an outer snubber ring, which produces a complex dynamical behaviour. In the past, similar systems have been investigated by brute force numerical methods. Here we implemented a path following method by using the toolbox TC-HAT, a module allowing to model non-smooth systems by AUTO 97. First, by varying the rotational frequency we determined period-1 orbits which make zero velocity impact with the snubber ring (grazing) for different values of the viscous damping, keeping the other system parameters fixed. Next, by applying the continuation method, we have found two period-doubling and two fold bifurcations. This reveals some similarity to grazing induced bifurcations (e.g. [17]), which are typical smooth...
periodic orbits of the rotor system after grazing.

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Appendix A. Variables and parameters of the model

| Notation | Description |
|----------|-------------|
| $c$      | viscous damping coefficient |
| $k_1$    | rotor stiffness |
| $k_2$    | snubber ring stiffness |
| $M$      | mass of the rotor |
| $\rho m$ | out-of-balance |
| $t$      | time |
| $x$      | displacement of the rotor in the horizontal direction |
| $y$      | displacement of the rotor in the vertical direction |
| $R$      | radial displacement of the rotor relative to the equilibrium position of the snubber ring |
| $\varepsilon_x$ | eccentricity of the rotor in the $x$-direction |
| $\varepsilon_y$ | eccentricity of the rotor in the $y$-direction |
| $F_N$    | normal force produced when the rotor hits the snubber ring |
| $\gamma$ | radial clearance between the rotor and the snubber ring |
| $\omega_n$ | natural frequency of the rotor $\sqrt{\frac{k_1}{M}}$ |
| $\varphi_0$ | initial phase shift |
| $\omega$ | shaft rotational velocity |
| $\tilde{K}$ | stiffness ratio $\frac{k_2}{k_1}$ |
| $\tilde{x}$ | displacement ratio of the rotor in the $x$-direction $\frac{x}{\gamma}$ |
| $\tilde{y}$ | displacement ratio of the rotor in the $y$-direction $\frac{y}{\gamma}$ |
| $\tilde{R}$ | radial displacement ratio $\frac{R}{\gamma}$ |
| $\tilde{\varepsilon}_x$ | dimensionless eccentricity of the rotor in the $x$-direction $\frac{\varepsilon_x}{\gamma}$ |
| $\tilde{\varepsilon}_y$ | dimensionless eccentricity of the rotor in the $y$-direction $\frac{\varepsilon_y}{\gamma}$ |
| $\eta$ | frequency ratio $\frac{\omega}{\omega_n}$ |
| $\eta_m$ | mass ratio $\frac{m}{M}$ |
| $\nu$ | damping ratio of the rotor $\frac{c}{2\sqrt{k_1M}}$ |
| $\tau$ | dimensionless time $\omega_n t$ |
| $\psi$ | angle between the radial displacement of the rotor and the horizontal axis |
| $\tilde{\rho}$ | dimensionless radius $\frac{\rho}{\gamma}$ |