The role of the mean and forced eccentricities in the secular dynamic of circumbinary planets

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Abstract. Within the framework of the Restricted Three Body Problem (RTBP), we consider the orbital evolution of a circumbinary (CB) planet. We develop a simple analytical model to explain the mean behaviour of the planetary eccentricity and better identify the nature of the main contributors of eccentricity oscillations. Our theory is validated by comparing with N-body simulation. We analyse the effect of including dissipative forces in our theory and compare with a N-body simulation in which the dissipative force is the interaction with a protoplanetary disc. Finally, we use our model to explain dynamical maps in the CB region and distinguish domains where the secular dynamics dominate over the resonant perturbations.

1. Introduction
Although only nine CB systems have been discovered by the Kepler mission until today, they share some similarities. The binary components define compact systems with a wide range of eccentricities and mass ratios. On the other hand, the CB planets have a diversity of masses but they are typically very close to the binary, in coplanar and low-eccentric orbits.

It is well accepted that in situ formation so close to the binary is unlikely due to the strong perturbation of the secondary (e.g. [1, 2]). However, as we move away from the binary, the gravitational potential became much more similar to that of a single star and planetary formation appears to be easier, following usual core-accretion models. This suggest that circumbinary planets could have formed farther out, and later migrated inward due to interaction with a primordial disc and stalled near their current orbits [3].

Once the disc is dissipated, the planet evolves only by gravitational interaction with the binary, with the particularity that the mass of the perturbing body (i.e. the secondary star) is of the same order than the central mass. This background invites us to revisit the orbital dynamics of a planet in such a particular environment and the effect of including dissipative forces in its orbital evolution. Our aim is to test if the interaction of these forces is able to explain the orbital configuration of any of the CB planets observed.

As an additional motivation, in Figure 1 we show a dynamical map of the CB region around a fictitious binary with $\mu = m_B/(m_A + m_B) = 0.3$ and eccentricity $e_B = 0.1$. Although the observed CB Kepler systems are located around binaries with different mass ratios and eccentricities, their typical systems are similar to this values for a qualitative description. We
2. The model

Let us consider the orbital evolution of a circumbinary planet with symbolic mass $m$ around a binary system with stellar masses $m_A$ and $m_B$. Let us suppose that the planetary mass is negligible compared to the stellar masses in such a way that the RTBP is an accurate model to describe the planetary dynamics.

We adopt a Jacobi reference frame for the coordinates and velocities, where the position vector $\vec{r}_B$ of the secondary star $m_B$ is defined as $m_A$-centric, while the position vector $\vec{r}$ of the planet $m$ is measured with respect to the barycenter of $m_A$ and $m_B$. The planet’s orbit, considered coplanar with the binary, will be characterized by its semimajor axis $a$, eccentricity $e$, mean longitude $\lambda$ and longitude of pericenter $\varpi$, all calculated from the Jacobi state vectors. Similar notation, but with subindex $B$ will be used for the orbit of the secondary star $m_B$ around $m_A$.

The equation of motion of the planet is \[\frac{d^2 \vec{r}}{dt^2} = -G(m_A + m_B)\frac{\vec{r}}{r^3} + \nabla_r R\] (1)

where $G$ is the gravitational constant and

\[R = \left(\frac{Gm_A}{r^2 + \frac{m_B}{m_A + m_B}r_B^2}\right) + \left(\frac{Gm_B}{r^2 - \frac{m_A}{m_A + m_B}r_B^2}\right) - \frac{G(m_A + m_B)}{r}\right).\] (2)

Adopting a Legendre expansion truncated to third order in the ratio $\alpha = a_B/a$, the perturbation can be approximated by [5]

\[R = -(G/a_B^2) \sum_{l=2}^{3} \mathcal{M}_l \alpha^{l+1} \left(\frac{r_B}{a_B^2}\right)^l \left(\frac{a}{r}\right)^{l+1} P_l(\cos(\theta))\] (3)
where $P_l$ is the Legendre polynomial of degree $l$, $\theta = f + \varpi - f_B - \varpi_B$ is the angle between $\vec{r}$ and $\vec{r}_B$ and

$$M_l = m_A m_B \frac{m_{l-1}^A - (-m_B)^{l-1}}{(m_A + m_B)^l}$$  \hspace{1cm} (4)

It is possible to write the perturbation as [6]

$$R = \left(\frac{G}{a_B}\right) \sum_{j_1=-\infty}^{\infty} \sum_{j_2=-\infty}^{\infty} \sum_{k=0}^{\infty} D_{j_1,j_2,k}(\alpha, e; e_B) \cos(j_1 M + j_2 M_B + k \Delta \varpi)$$  \hspace{1cm} (5)

where $M$ and $M_B$ are the mean anomalies of the planet and the binary respectively, $\Delta \varpi = \varpi - \varpi_B$ and the coefficients of the armonics $D_{j_1,j_2,k}$ are functions of $\alpha$ and $e$, and $e_B$ is a fixed parameter.

Since we will be working within the Hamiltonian formalism, we introduce the modified Delaunay canonical variables of the problem

$$L = \sqrt{\mu a} ; \hspace{1cm} \lambda$$
$$S = \sqrt{\mu a} (1 - \sqrt{1 - e^2}) ; \hspace{1cm} -\varpi$$
$$\Lambda ; \hspace{1cm} \lambda_B$$  \hspace{1cm} (6)

where $\mu = G(m_A + m_B)$. We remember that, as we are considering the RTBP, the third degree of freedom associated to $(\Lambda, \lambda_B)$ appears when passing to the extended phase space to eliminate the non-autonomous character of the perturbation, and its value is not known a priori.

The Hamiltonian of the system can be easily found from the Equation (1) and (5) as:

$$F = -\frac{\mu^2}{2L^2} + n_B \Lambda + \epsilon \sum_{j_1,j_2=-\infty}^{\infty} \sum_{k=0}^{\infty} D_{j_1,j_2,k}(L, S) \cos \left( j_1 (\lambda - \varpi) + j_2 (\lambda_B - \varpi_B) + k \Delta \varpi \right)$$  \hspace{1cm} (7)

where $n_B$ the mean motion of the binary and we define $\epsilon = \frac{G}{a_B}$ as a parameter that serves as a guide of the relative magnitudes between the perturbation $F_1$ and the unperturbed integrable Hamiltonian $F_0$.

Having an explicit expression for $F_1 = F_1(L, S, \Lambda, \lambda, -\varpi, \lambda_B)$, we may now apply Hori’s method to study the secular dynamics. The idea is to search for a Lie-type canonical transformation, with associated generating function $B = \epsilon B_1 + \epsilon^2 B_2 + \ldots$, to a new set of variables $(L^*, S^*, \Lambda^*, \lambda^*, -\varpi^*, \lambda_B^*)$ such that the transformed Hamiltonian $F^*$ is independent of the fast angles $\lambda^*$ and $\lambda_B^*$. Up to first order in the small parameter, the new Hamiltonian may be written as

$$F^*(S^*, -\varpi^*; L^*, \Lambda^*) = F^*_0 + \epsilon F^*_1$$  \hspace{1cm} (8)

where

$$F^*_0 = F_0(L^*, \Lambda^*) = -\frac{\mu^2}{2L^{*2}} + n_B \Lambda^*$$
$$F^*_1 = \left< F_1 \right>_{\Lambda, \lambda_B}(S^*, -\varpi^*; L^*)$$  \hspace{1cm} (9)

where $\left< \right>_{\Lambda, \lambda_B}$ denotes the averaging with respect to both mean longitudes. Only the terms with $j_1 = j_2 = 0$ survive when we average $F_1$ over the mean longitudes, and a simple and close expression in the eccentricities can be obtained

$$F^*_1 = \sum_{k=0}^{\infty} D_{0,0,k}(L^*, S^*) \cos(k \Delta \varpi^*)$$  \hspace{1cm} (10)
The first-order generating function $B_1$ can be easily obtained in terms of the adopted expansion of the disturbing function (Equation 5) as

$$B_1 = - \sum_{\ell = -\infty}^{\infty} \sum_{\vec{l} \neq \vec{0}}^{\infty} D_{l_1,l_2,k}(L^*, S^*) \sin(l_1(\lambda^* - \varpi^*) + l_2(\lambda_B^* - \varpi_B^*) + k\Delta \varpi^*)$$

and the new variables are related to the older ones by means of the partial derivatives of the first-order generating function as follows.

Finally, instead of the Delaunay modified variables, it is convenient to use the regular variables, defined as

$$K = \sqrt{2S} \cos(-\Delta \varpi) \quad ; \quad H = \sqrt{2S} \sin(-\Delta \varpi)$$

Thus, the transformation between the old and the new variables is given by

$$L = L^* + \epsilon \left( \frac{\partial B_1}{\partial \lambda^*} \right)_{ini} ; \quad \lambda = \lambda^* - \epsilon \left( \frac{\partial B_1}{\partial L^*} \right)_{ini}$$

$$K = K^* + \epsilon \left( \frac{\partial B_1}{\partial H^*} \right)_{ini} ; \quad H = H^* - \epsilon \left( \frac{\partial B_1}{\partial K^*} \right)_{ini}$$

$$\Lambda = \Lambda^* + \epsilon \left( \frac{\partial B_1}{\partial \lambda_B^*} \right)_{ini} ; \quad \lambda_B = \lambda_B^*$$

for where the partial derivatives respect to the regular variables can be calculated from the partial derivatives respect to the modified Delaunay variables, by using the chain rule.

2.1. From osculating to mean initial conditions

Let us consider the particular case in which all initial values for the osculating angles of the binary and the planet are zero: $\lambda(0) = \lambda_B(0) = \varpi(0) = \varpi_B(0) = 0$. It can be easily shown that the new angles are also zero ($\lambda^*(0) = \lambda_B^*(0) = \varpi^*(0) = \varpi_B^*(0) = 0$) and the transformation for this instant reduces to

$$L_{ini} = L^*_{ini} + \epsilon \left( \frac{\partial B_1}{\partial \lambda^*} \right)_{ini} ; \quad \lambda_{ini} = \lambda^*_{ini} - \epsilon \left( \frac{\partial B_1}{\partial L^*} \right)_{ini} = 0$$

$$K_{ini} = K^*_{ini} \left( 1 + \frac{\epsilon}{2S^*_{ini}} \left( \frac{\partial B_1}{\partial (-\varpi^*)} \right)_{ini} \right) ; \quad H_{ini} = H^*_{ini} \left( 1 - \frac{\epsilon}{2S^*_{ini}} \left( \frac{\partial B_1}{\partial (-\varpi^*)} \right)_{ini} \right) = 0$$

$$\Lambda_{ini} = \Lambda^*_{ini} + \epsilon \left( \frac{\partial B_1}{\partial \lambda_B^*} \right)_{ini} ; \quad \lambda_{B,ini} = \lambda_{B,ini}^* = 0$$

where the subindex $ini$ indicates that the variable should be evaluated in $t = 0$.

Preliminary simulations show that it is possible to dismiss the correction for the semimajor axis. For this reason, in what follows we can approximate $L \sim L^*$ and mainly focus on the eccentricity.

In the low planetary eccentricity regime, we have $2S \sim L e^2$ and we can define the non-canonical regular variables as

$$k = \frac{1}{\sqrt{L}} K = e \cos(-\Delta \varpi) \quad ; \quad h = \frac{1}{\sqrt{L}} H = e \sin(-\Delta \varpi)$$

Due to the choice of zero initial osculating angles we have

$$k_{ini} = k^*_{ini} \left( 1 + \frac{\epsilon}{L^*_{ini} k^*_{ini}} \left( \frac{\partial B_1}{\partial (-\varpi^*)} \right)_{ini} \right)$$
Thus, for the particular case of initial zero-osculating angles and given the osculating eccentricity, we transform to the associated regular variable and numerically solve the Equation (17) to obtain the mean variable. These will be the initial condition of our secular Hamiltonian $F^*$. 

2.2. The mean eccentricity

The secular Hamiltonian of a massless planet in a CB orbit can be calculated from Equation (10). Up to third order in $e$ and up to second order in $e_f$, can be written as [4]

$$F^*(k^*, h^*; L^*) = -A(k^{*2} + h^{*2}) - Bk^*$$

where

$$A = \frac{3}{8}\alpha^3M_2(1 + \frac{3}{2}e_B^2) \quad ; \quad B = \frac{3}{8}\alpha^4M_3(-\frac{5}{2}e_B - \frac{15}{8}e_B^3)$$

The secular solution can be expressed by [4]

$$k^*(t) = e_p\cos(gt) + e_f \quad ; \quad h^*(t) = e_p\sin(gt)$$

where $e_f = \frac{-B}{2A}$ is what is commonly called forced eccentricity, determined by the binary configuration and the position of the planet, $e_p$ is the commonly called free eccentricity, which depends on the initial conditions for the planet and $g = \frac{2\alpha^4}{M_3}$ the secular frequency.

In the low planetary eccentricity regime $2S \sim L^*e^2$ and $2S^* \sim L^*e^2$, we obtain from Equation (12) the temporal evolution of the eccentricity of the planet as

$$e(t)^2 = e_f^2 + e_p^2 + 2e_f e_p \cos(gt) + \frac{2e}{L^*} \frac{\partial B_1}{\partial(-\varpi^*)} + e^2 \frac{T}{L^*}$$

where we have used the secular solution for $e^*2(t) = k^{*2}(t) + h^{*2}(t)$ given by Equation (20) and

$$T = \left(\frac{\partial B_1}{\partial H^*}\right)^2 + \left(\frac{\partial B_1}{\partial K^*}\right)^2 = \frac{1}{2S^*} \left(\frac{\partial B_1}{\partial(-\varpi^*)}\right)^2 + 2S^* \left(\frac{\partial B_1}{\partial S^*}\right)^2.$$ (22)

This term is associated to the amplitude of the short-term eccentricity oscillations due to the perturbation of the secondary star.

Notice that due to the form of $B_1$ (see Equation (11)), the partial derivative $\partial B_1/\partial(-\varpi^*)$ has no secular terms. The mean squared eccentricity can be calculated as

$$\langle e(t)^2 \rangle = e_f^2 + e_p^2 + e^2 \left\langle \frac{T}{L^*} \right\rangle.$$ (23)

To obtain an expression for the mean eccentricity, we propose a Fourier solution to Equation (23) as

$$e(t) = \langle e \rangle + e_1 \cos(\phi(t))$$ (24)

where $\langle e \rangle$ and $e_1$ should be calculated to also satisfy Equation (21).

The solution with physical sense for the mean eccentricity is

$$\langle e \rangle = \frac{1}{\sqrt{2}} \left\langle \sqrt{e_f^2 + e_p^2 + e^2 \left\langle \frac{T}{L^*} \right\rangle} \right\rangle + \left(\sqrt{e_f^2 + e_p^2 + e^2 \left\langle \frac{T}{L^*} \right\rangle} \right)^2 - 2e_f^2 e_p^2.$$ (25)

In [7], the authors found a simple analytical expression for $\langle T/L^* \rangle$, by averaging the disturbing function over the mean longitude of the binary only and retaining terms up to in both $e_1$ and $e$. It is explicitly given by

$$e^2 \left\langle \frac{T}{L^*} \right\rangle = e_{2\text{par}}^2 = \frac{9}{16} \frac{m_A^2 m_B^2}{(m_A + m_B)^4} \left(\frac{a_1}{a}\right)^4 \left(1 + \frac{34}{3} e_1^2\right).$$ (26)
Figure 2. Mean eccentricities $\langle e \rangle$ as a function of the mean motion ratio $n_B/n$ of massless CB planets in binaries with different $\mu$ and different initial eccentricities $e_0$. The mean eccentricities were calculated by two different methods: by filtering exact numerical simulations (black dots) and using our analytical theory (red curves). For illustrative purposes, when is possible, we show the forced eccentricity (magenta curves).

3. Comparison with N-body simulations

In a N-body simulation, the mean eccentricity of a CB planet can be calculated by performing long-term integration of the exact equations of motion and applying a low-pass FIR (finite impulse response) filter (e.g. [8]) to remove any temporal variation. In a dynamical system displaying regular motion, the application of the filter is equivalent to an averaging of the Hamiltonian.

In Figure 2 we compare the mean eccentricity $\langle e \rangle$ calculated by filtering the N-body simulation (black dots) with the one predicted by our model (Equation 25) in red curves, as a function of the mean motion ratio $n_B/n$ ($n$ is constant in the secular dynamics and $n_B$ is constant in the RTBP). We consider in different columns three type of binaries characterized by different values of reduced-mass: $\mu = 0.1$ (left column), $\mu = 0.3$ (middle column) and $\mu = 0.5$ (right column). The binary eccentricity was fixed $e_B = 0.1$, as a standard case. As Equation (25) shows, the mean eccentricity depends on the initial condition through $e_0$. For this reason, we consider in each row three initial planet eccentricity: $e_{ini} = 0$ (top panels), $e_{ini} = 0.05$ (central
Figure 3. Eccentricity evolution (full in light grey and filtered in dark grey) of a CB planet migrating inwards due to interaction with a CB disc. The red curves represent the capture mean eccentricity $\langle e \rangle_{\text{cap}}$ while the magenta curves represent the classical forced eccentricity.

panels) and $e_{\text{ini}} = 0.1$ (bottom panels). With illustrative purposes, we included when possible the forced eccentricity $e_f$ of [4]. In a non-resonant domain we notice that our analytical models fits very well the mean behavior of the simulations, even in regions very close to the binary. The mean eccentricity is far from the forced eccentricity, in particular for high values of $\mu$ (where the amplitudes of the short terms oscillations became large) and for high initial eccentricity (where the proper eccentricity increases).

3.1. Dissipative dynamics

The case in which some non-conservative force is acting, in addition to the gravitational dynamic, can be incorporated in our model. If we assume that the dissipation is not strong enough to remove the short-period terms, the only effect of such force is to cancel the proper eccentricity $e_p = 0$. The ‘capture’ mean eccentricity $\langle e \rangle_{\text{cap}}$ is simple given by

$$\langle e \rangle_{\text{cap}} = \sqrt{e_f^2 + \epsilon^2 \frac{T}{L^2}}$$

(27)

In Figure 3 we consider the interaction with the CB disc as the dissipative force. The interaction is modelled including an ad-hoc external force to our N-body code as applied in [9]. We show the full eccentricity evolution (light grey curves) of a CB planet migrating towards the binary due to the dissipation of the disc and compare with $\langle e \rangle_{\text{cap}}$ (Equation 27) with the Paardekooper-model (Equation 26) in red curves and also with the classical forced eccentricity in magenta curves. We applied a low-pass filter to the full eccentricity evolution to remove the periodic oscillations (dark grey curves). As can be observed, $\langle e \rangle_{\text{cap}}$ coincides pretty well with the mean eccentricity evolution of a migrating CB planet for all the $\mu$-values considered. The limit of low-mass secondary star ($\mu = 0.1$) is mainly dominated by the secular mode while the case of equal-stars is only dominated by the secular contribution of the short-term oscillations. In the intermediate case, the interaction of both modes should be taken into account.

In [9] we construct $\Delta e$-dynamical maps for a CB planet close to the 5/1 MMR with the binary, using Kepler-38 ([10]) as a working example. In addition to the resonant structure, we observed a low planetary eccentricity stripe with more regular motion that we associated to the forced mode. In Figure 4 we built similar dynamical maps but now considering binaries with $\mu = 0.1$ (left figure) and $\mu = 0.5$ (right figure) and $e_B = 0.1$. In white curves we plot the mean eccentricity $\langle e \rangle_{\text{cap}}$ (Equation 27) and used Equation 17 to transform and be able to compare
with our osculating dynamical map. As can be observed, this curves exactly coincides with the centre of the low-excitation blue region.

4. Summary and discussion
In this work we present a simple model for the secular orbital evolution of circumbinary planets in the framework of the Restricted Three Body Problem. We applied a first-order averaging method but retained the secular contribution of the short-term periodic contribution due to the perturbation of the binary, in this particular environment in which the mass of the perturbing body is of the same order than the mass of the central one. As a result of this, the secular evolution of the eccentricity in the CB planets is the result of the interaction of this terms with the secular mode. We validate our model by comparing with filtered N-body simulation and find a very good agreement, as long as no mean motion resonance exists between the planet and the binary. We consider the case in which dissipative forces are present. This condition is easy to include in our model and we were able to reproduce very accurately the eccentricity evolution of a CB planet migrating due to interaction with the circumbinary disc. Finally, by applying our model, we were able to distinguish the secular domain from the resonant region in dynamical maps of the CB region.

Acknowledgements
This research was funded by CONICET, SECYT/UNC and FONCYT.

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