SOLYANIK ESTIMATES AND LOCAL HÖLDER CONTINUITY OF
HALO FUNCTIONS OF GEOMETRIC MAXIMAL OPERATORS

PAUL HAGELSTEIN AND IOANNIS PARISSIS

Abstract. Let $B$ be a homothecy invariant basis consisting of convex sets in $\mathbb{R}^n$, and define the associated geometric maximal operator $M_B$ by

$$M_B f(x) := \sup_{x \in B} \frac{1}{|R|} \int_R |f|$$

and the halo function $\phi_B(\alpha)$ on $(1, \infty)$ by

$$\phi_B(\alpha) := \sup_{E \subset \mathbb{R}^n : 0 < |E| < \infty} \frac{1}{|E|} \left| \{ x \in \mathbb{R}^n : M_B \chi_E(x) > 1/\alpha \} \right|.$$

It is shown that if $\phi_B(\alpha)$ satisfies the Solyanik estimate $\phi_B(\alpha) - 1 \leq C (1 - \frac{1}{\alpha})^p$ for $\alpha \in (1, \infty)$ sufficiently close to 1 then $\phi_B$ lies in the Hölder class $C^p(1, \infty)$. As a consequence we obtain that the halo functions associated with the Hardy-Littlewood maximal operator and the strong maximal operator on $\mathbb{R}^n$ lie in the Hölder class $C^{1/(n)}(1, \infty)$.

1. Introduction

From the time of the seminal paper [13] of Hardy and Littlewood, geometric maximal functions have played a central role in analysis. For example, the Hardy-Littlewood maximal operator $M_{HL}$ has been used in a proof of the Lebesgue Differentiation Theorem as well as in proofs of the $L^p$ boundedness of a wide class of singular integral operators; [19] provides a well-known exposition of these facts.

A key property that the maximal operator $M_{HL}$ satisfies is the so-called \textit{weak type $(1,1)$ estimate}:

$$|\{ x \in \mathbb{R}^n : M_{HL} f(x) > \alpha \}| \leq \frac{C_n}{\alpha} \int_{\mathbb{R}^n} |f|, \quad \alpha \in (0, \infty).$$

It is this property that enables us to show that the collection of cubes or balls in $\mathbb{R}^n$ differentiates $L^1(\mathbb{R}^n)$. Now, the \textit{strong maximal operator} $M_S$, defined by taking maximal

2010 \textit{Mathematics Subject Classification.} Primary 42B25, Secondary: 42B35.

\textit{Key words and phrases.} maximal function, halo function, Tauberian conditions, differentiation basis.

P. H. is partially supported by a grant from the Simons Foundation (#208981 to Paul Hagelstein).

I. P. is supported by the Academy of Finland, grant 138738.
averages of a function over rectangular parallelepipeds whose sides are parallel to the coordinate axes, does not satisfy a weak type \((1,1)\) condition, although it does satisfy a weak type estimate of the form

\[
\{|\{x \in \mathbb{R}^n : M_S f(x) > \alpha\}| \leq C_n \int_{\mathbb{R}^n} \frac{|f|}{\alpha} \left(1 + \log^+ \left(\frac{|f|}{\alpha}\right)\right)^{n-1}, \quad \alpha \in (0, \infty).
\]

This weak type estimate can be used to show that the collection of rectangular parallelepipeds whose sides are parallel to the axes differentiates functions that are locally in \(L(\log L)^{n-1}(\mathbb{R}^n)\); details in this regard may be found in [7].

In order to describe more general results along these lines we introduce some terminology. A basis \(B\) is a collection of bounded open sets in \(\mathbb{R}^n\). A collection \(B\) is called a density basis if it differentiates functions in \(L^\infty(\mathbb{R}^n)\). Given a basis \(B\) we explicitly define the maximal operator \(M_B\) by

\[
M_B f(x) := \sup_{x \in \mathbb{R}^n} \frac{1}{|R|} \int_R |f|
\]

if \(x \in \bigcup_{B \in B} B\) while we set \(M_B f(x) := 0\) otherwise. We use the special notations \(M_{HL,b}\) when \(B\) is the collection of all Euclidean balls in \(\mathbb{R}^n\), \(M_{HL,c}\) when \(B\) is the basis of all cubes in \(\mathbb{R}^n\) with sides parallel to the coordinate axes, and \(M_S\) when \(B\) consists of all rectangular parallelepipeds in \(\mathbb{R}^n\) with sides parallel to the coordinate axes. If a maximal operator \(M_B\) associated with a basis \(B\) satisfies a weak type \((\Phi, \Phi)\) estimate, with \(\Phi\) being a convex non-negative, non-decreasing function in \((0, \infty)\) with \(\Phi(0) = 0\), the basis \(B\) is known to differentiate functions for which \(\Phi(f)\) is locally integrable. For this reason, given a maximal operator \(M_B\), it is highly desirable to place bounds on its distribution function \(\{|\{x \in \mathbb{R}^n : M_B f(x) > \alpha\}|\); that enables us to establish differentiation results for the basis \(B\).

A somewhat weaker estimate on a maximal operator is a so-called Tauberian condition. A maximal operator \(M_B\) associated with a basis \(B\) is said to satisfy a Tauberian condition with respect to \(\alpha \in (0,1)\) if there exists a constant \(C > 0\) such that

\[
|\{x \in \mathbb{R}^n : M_B \chi_E(x) > \alpha\}| \leq C|E|
\]

holds for all measurable sets \(E \subset \mathbb{R}^n\). This condition is quite useful. Cédula and Fefferman related Tauberian conditions of maximal operators to \(L^p\) bounds of multiplier operators in [4], and Hagelstein and Stokolos showed in [12] that if \(B\) is a homothecy invariant basis consisting of convex sets and the maximal operator \(M_B\) satisfies a Tauberian condition with respect to some \(\alpha \in (0,1)\) then \(M_B\) must be bounded on \(L^p(\mathbb{R}^n)\) for sufficiently large \(p\). Subsequent papers extending these ideas include [8] and [11].

The halo function \(\phi_B\) associated with a density basis \(B\) is defined as

\[
\phi_B(\alpha) := \sup_{E: 0 < |E| < \infty} \frac{1}{|E|} \{|x \in \mathbb{R}^n : M_B \chi_E(x) > \frac{1}{\alpha}\}.
\]
for $\alpha \in (1, \infty)$, and by convention it is defined as $\phi_B(\alpha) := \alpha$ for $\alpha \in [0, 1]$. The growth of the halo function $\phi_B(\alpha)$ as $\alpha \to \infty$ enables us to establish weak type bounds on $M_B$; in particular, if $\phi_B(\alpha) \leq C\alpha^p$ for $\alpha > 1$, then $M_B$ is of restricted weak type $(p, p)$ and accordingly $B$ differentiates all functions which are locally in $L^q(\mathbb{R}^n)$ for $q > p$. A prominent unsolved problem in differentiation theory is the halo conjecture which asserts that if $B$ is a homothecy invariant density basis, then $B$ must differentiate any measurable function $f$ for which $\phi_B(f)$ is locally integrable. Partial results regarding the halo conjecture may be found in [7], [14] and [18].

For these reasons, it is the issue of the growth properties of halo functions that has received the majority of attention in the field of differentiation theory in recent decades. A fundamental but until recently overlooked issue is that of continuity and smoothness of halo functions. Beznosova and Hagelstein proved in [1] that the halo function of a density basis must be continuous on $[0, 1]$ and $(1, \infty)$. However, they also provided an example of a density basis consisting of nonconvex sets whose halo function exhibits a jump discontinuity at 1.

These results immediately motivate a closer study of the behavior of halo functions near 1. The first results in this regard are due to A. A. Solyanik, who proved in [17] that the halo functions associated with the centered Hardy-Littlewood maximal operator and the strong maximal operator tend to 1 as $\alpha \to 1$. Similar estimates were shown for the uncentered Hardy-Littlewood maximal operator, defined with respect to balls, by Hagelstein and Parissis in [9], and analogues of these results were proved in the weighted case by Hagelstein and Parissis in [10].

Our previous work on Solyanik estimates was motivated primarily out of intrinsic interest. However, Michael Lacey has subsequently brought to our attention that he and Sarah Ferguson implicitly used Solyanik-type estimates for the strong maximal operator in their work establishing a commutator estimate enabling one to give a characterization of the product BMO space $\text{BMO}(\mathbb{R}^2 \times \mathbb{R}^2)$ of Chang and R. Fefferman in terms of commutators; see the appendix of their paper [5]. Two other papers where Solyanik-type estimates were implicitly used include [3] by Cabrelli, Lacey, Molter, and Pipher as well as [16] by Lacey and Terwilliger.

Given a density basis $B$ it will be convenient for us to define the sharp Tauberian constant $C_B(\alpha)$ by $\phi_B(\frac{1}{\alpha})$ for $0 < \alpha < 1$. In particular

$$C_B(\alpha) := \sup_{E: 0 < |E| < \infty} \frac{1}{|E|} \left| \left\{ x \in \mathbb{R}^n : M_B \chi_E(x) > \alpha \right\} \right|.$$  

Recall that if a basis $B$ is homothecy invariant then $C_B(\alpha)$ is always finite for $\alpha \in (0, 1)$; see [2]. A more precise quantitative version of the Solyanik estimates discussed above, collectively due to Hagelstein, Parissis, and Solyanik, is the following.
Theorem 1. [9, 17] Let $C_{HL,b}$, $C_{HL,c}$, and $C_S$ denote the sharp Tauberian constants for the Hardy-Littlewood maximal operator with respect to balls, the Hardy-Littlewood maximal operator with respect to cubes, and the strong maximal operator, respectively. Then we have the following asymptotic estimates for $\alpha \in (0, 1)$ sufficiently close to 1:

$$C_{HL,b}(\alpha) - 1 \lesssim_n (1 - \alpha)\frac{1}{n+1},$$

$$C_{HL,c}(\alpha) - 1 \sim_n (1 - \alpha)\frac{1}{n},$$

and

$$C_S(\alpha) - 1 \sim_n (1 - \alpha)\frac{1}{n}.$$

The estimates for $C_{HL,c}$ and $C_S$ are sharp in the sense that the exponent $\frac{1}{n}$ cannot be replaced by any larger exponent. Whether or not the exponent associated with $C_{HL,b}$ can be improved to $\frac{1}{n}$ or possibly $\frac{2}{n+1}$ is an open problem; see [9].

The purpose of this paper is to show that the above quantitative Solyanik estimates may be used to establish Hölder continuity results for $C_{HL,b}$, $C_{HL,c}$, and $C_S$, and accordingly yield Hölder continuity results for the corresponding halo functions. We will moreover see that, given a homothecy invariant density basis $B$ consisting of convex sets, a Solyanik estimate of the form

$$C_B(\alpha) - 1 \sim_n (1 - \alpha)^p, \quad \alpha \to 1^-,$$

implies that $\phi_B$ lies in the Hölder class $C^p(0, 1)$. A key ingredient of the proof of this result will be the careful use of the Calderón-Zygmund decomposition to show that halo sets of the form

$$\mathcal{H}_{B,\alpha}(E) := \{x \in \mathbb{R}^n : M_B \chi_E(x) > \alpha\}$$

satisfy an imbedding relation

$$\mathcal{H}_{B,\alpha}(E) \subset \mathcal{H}_{B,\alpha(1+\gamma(\alpha)\delta)}(\mathcal{H}_{B,1-c_n\delta}(E))$$

whenever $\delta \lesssim_n 1 - \alpha$. Here $\gamma(\alpha) \sim_n \min(\alpha, 1 - \alpha)^{2n}$ and $c_n > 0$ is a dimensional constant.

Notation

In this paper we will make frequent use of the following notational conventions. We write $C, c > 0$ for numerical constants that can change value even in the same line of text. The presence of a subscript as in $C_\tau$ denotes dependence on some parameter $\tau$. We write $A \lesssim B$ whenever $A \leq CB$ for some constant $C > 0$ and $A \sim B$ if $A \lesssim B$ and $B \lesssim A$. We write $A \lesssim_\tau B$ whenever the implied constant depends on the parameter $\tau$, and $A \sim_\tau B$ if $A \lesssim_\tau B$ and $B \lesssim_\tau A$. Also, given an interval $I \subset \mathbb{R}$, we say that a function $f$ lies in the Hölder class $C^p(I)$ whenever for any compact set $K \subset I$ we
have \(|f(x) - f(y)| \lesssim_K |x - y|^p\) for all \(x, y \in K\). This condition corresponds to \(f\) being locally Hölder continuous with exponent \(p\) in \(I\). We are following here the notation and terminology found, for instance, in [6]. We many times refer to a rectangular parallelepiped \(R\) as a rectangle in \(\mathbb{R}^n\). Any rectangle \(R\) gives rise to a dyadic grid consisting of homothetic copies of \(R\) simply by bisecting each side of \(R\) and iterating. Thus any \(R\) gives rise to \(2^n\) dyadic children and any dyadic descendant \(S\) of \(R\) is contained in a unique dyadic parent that we always denote by \(S^{(1)}\).

## 2. Embedding of Halo Sets associated with bases of rectangular parallelepipeds

In this section we provide the statement and proof of the following theorem regarding the embedding of halo sets associated with bases of rectangular parallelepipeds.

**Theorem 2.** Let \(B\) be a homothecy invariant collection consisting of rectangular parallelepipeds in \(\mathbb{R}^n\). Given a measurable set \(E \subset \mathbb{R}^n\) of finite measure and \(\alpha \in (0, 1)\) we define the associated halo set \(H_B,\alpha(E)\) by

\[
H_B,\alpha(E) := \{x \in \mathbb{R}^n : M_B\chi_E(x) > \alpha\}.
\]

Then for all \(\xi, \delta \in (0, 1)\) with \(\alpha < 1 - \delta < \xi\) we have

\[
H_B,\alpha(E) \subset H_B,\alpha(1 + \frac{1-\delta}{2^n})(H_B,1-\delta(E)).
\]

**Proof.** Let \(E \subset \mathbb{R}^n\) be a set of finite measure and fix \(\alpha, \delta, \xi \in (0, 1)\) with \(\alpha < 1 - \delta < \xi\). Let \(x \in H_B,\alpha(E)\). Then there exists a rectangle \(R \in B\) containing \(x\) such that

\[
\frac{1}{|R|} \int_R \chi_E > \alpha.
\]

Using the homothecy invariance of \(B\), we may assume without loss of generality that the rectangle \(R\) is sufficiently large so that \(|E \cap R|/|R| \leq \xi < 1\) holds as well.

We will now consider a modified Calderón-Zygmund decomposition of \(\chi_E\) of \(\chi_{E \cap R}\) with respect to the dyadic grid generated by \(R\) at level \(\xi\). Along the same lines as the standard Calderón-Zygmund decomposition with respect to cubes we see that there exists a collection \(\{R_j\}_j\) of dyadic subrectangles of \(R\), each \(R_j\) strictly contained in \(R\), satisfying

(i) \(E \cap R \subset \bigcup_j R_j\) a.e.,

(ii) \(\frac{1}{|R_j|} \int_{R_j} \chi_E > \xi\), and
(iii) if $S$ is any dyadic ancestor of $R_j$ contained in $R$, then \( \frac{1}{|S|} \int_S \chi_E \leq \xi \).

We recall that for a dyadic rectangle $S$ we denote by $S^{(1)}$ its unique dyadic parent and note that for all $j$ we have $R_j^{(1)} \subseteq R$. Let now \( \{R_j^{(1)}\}_j \) be the maximal dyadic rectangles in the collection \( \{R_j^{(1)}\}_j \). Clearly

\[
\left| \bigcup_k R_{jk} \right| \geq \frac{1}{2^n} \left| \bigcup_j R_j \right|.
\]

By continuity of the Lebesgue measure, for each $k$ there exists a homothetic copy $S_k$ of $R$ such that

\[
R_{jk} \subseteq S_k \subseteq R_j^{(1)} \quad \text{and} \quad \frac{|S_k \cap E|}{|S_k|} = \xi.
\]

Since for each $k$ we have $S_k \subseteq R_j^{(1)}$ and the $R_{jk}$’s are maximal we have that the $S_k$’s are pairwise disjoint. Furthermore note that $S_k \subset \mathcal{H}_{B,1-\delta}(E)$ since $\xi > 1 - \delta$. We conclude that

\[
|\mathcal{H}_{B,1-\delta}(E) \setminus E \cap R| \geq \sum_k \left| \mathcal{H}_{B,1-\delta}(E) \setminus E \cap S_k \right| \geq \sum_k |S_k \setminus E| = (1 - \xi) \sum_k |S_k| \geq (1 - \xi) \sum_k |R_{jk}| \geq \frac{1 - \xi}{2^n} \left| \bigcup_j R_j \right|
\]

\[
\geq \frac{1 - \xi}{2^n} |E \cap R| > \frac{1 - \xi}{2^n} \alpha |R|.
\]

The previous calculation immediately implies

\[
|\mathcal{H}_{B,1-\delta}(E) \cap R| > \alpha \left( 1 + \frac{1 - \xi}{2^n} \right) |R|
\]

so that

\[
R \subset \mathcal{H}_{B,\alpha(1 + \frac{1 - \xi}{2^n})}(\mathcal{H}_{B,1-\delta}(E)).
\]

Hence we can conclude that

\[
\mathcal{H}_{B,\alpha}(E) \subset \mathcal{H}_{B,\alpha(1 + \frac{1 - \xi}{2^n})}(\mathcal{H}_{B,1-\delta}(E))
\]

as we wanted. \( \square \)

3. Local Hölder continuity of the Halo Function of the Strong Maximal Operator

As $|\mathcal{H}_{B,\alpha}(E)| \leq C_B(\alpha)|E|$, the following corollary follows by Theorem 2 and the results in [1] regarding continuity of Tauberian constants by letting $\xi \to (1 - \delta)^+$. 
Corollary 1. Let $\mathcal{B}$ be a homothecy invariant density basis consisting of rectangular parallelepipeds in $\mathbb{R}^n$, $\alpha \in (0,1)$, and let $C_{\mathcal{B}}(\alpha)$ be the associated sharp Tauberian constant of $\mathcal{B}$ with respect to $\alpha$. Then for all $\delta \in (0,1-\alpha)$ we have

$$C_{\mathcal{B}}(\alpha) \leq C_{\mathcal{B}}(\alpha(1 + \frac{\delta}{2^m})) C_{\mathcal{B}}(1-\delta).$$

We shall now see that this corollary enables us to prove that the sharp Tauberian constant $C_S(\alpha)$, associated with the strong maximal operator $M_S$, satisfies a local Hölder continuity condition, with in fact $C_S \in C^\frac{1}{2}(0,1)$. We shall need the following technical lemma.

Lemma 1. Let $\mathcal{B}$ be a homothecy invariant density basis consisting of rectangular parallelepipeds in $\mathbb{R}^n$. Suppose that there exists $\alpha_o \in (0,1)$ such that the inequality

$$C_{\mathcal{B}}(\alpha) - 1 \lesssim_S (1-\alpha)^p$$

holds for all $\alpha \in (\alpha_o,1)$ and for some fixed $p \in (0,1)$. Then $C_{\mathcal{B}}(\alpha)$ is locally Hölder continuous with exponent $p$ on $(0,1)$, that is, $C_{\mathcal{B}} \in C^p(0,1)$.

Proof. Let us fix a compact set $K \subset (0,1)$ and let $m_K, M_K \in (0,1)$ be such that $m_K \leq x \leq M_K$ for all $x \in K$. By the results in [1] we have that $C_{\mathcal{B}}$ is continuous in $(0,1)$ thus

$$\sup_{x \in K} C_{\mathcal{B}}(x) \lesssim_B 1.$$  

We first consider the case $x, y \in K$ with $0 < y - x < \min \left( \frac{1-M_K}{2^n} m_K, \frac{1-\alpha_o}{2^n} m_K \right) =: \eta$. We write

$$C_{\mathcal{B}}(x) - C_{\mathcal{B}}(y) = C_{\mathcal{B}}(x) - C_{\mathcal{B}}(x(1 + \frac{2^n y - x}{2^n x})).$$

Since

$$\frac{2^n y - x}{x} < 2^n \frac{1-M_K}{2^n m_K} m_K = 1 - M_K \leq 1 - x,$$

we get by Corollary 1 that

$$C_{\mathcal{B}}(x) - C_{\mathcal{B}}(y) \leq C_{\mathcal{B}}(y)\left[ C_{\mathcal{B}}(1 - \frac{2^n y - x}{x}) - 1 \right] \lesssim_B \left[ C_{\mathcal{B}}(1 - \frac{2^n y - x}{x}) - 1 \right].$$

Since

$$1 - \frac{2^n y - x}{x} > 1 - \frac{1-\alpha_o}{2^n m_K} m_K = \alpha_o$$

we get by the hypothesis of the lemma that

$$C_{\mathcal{B}}(x) - C_{\mathcal{B}}(y) \lesssim_B \frac{(y-x)^p}{x^p} \lesssim_K (y-x)^p, \quad x, y \in K, \quad 0 < y - x < \eta.$$

We can now conclude that

$$\sup_{x, y \in K \atop 0 < y - x < \eta} \frac{|C_{\mathcal{B}}(x) - C_{\mathcal{B}}(y)|}{|x - y|^p} \lesssim_B 1.$$  

On the other hand, if \( x, y \in K \) with \( y - x \geq \eta \) then the Hölder bound follows trivially since \( \sup_{x,y \in K} |C_B(x) - C_B(y)| \lesssim_{B,K} 1 \) so we are done. \( \square \)

As we have that the strong maximal operator satisfies the Solyanik estimate
\[
C_S(\alpha) - 1 \sim_n (1 - \alpha)^{\frac{1}{n}}, \quad \alpha \to 1^{-},
\]
we immediately conclude the following.

**Corollary 2.** Let \( C_S(\alpha) \) denote the sharp Tauberian constant of the strong maximal operator in \( \mathbb{R}^n \) with respect to \( \alpha \in (0, 1) \). Then
\[
C_S \in C^{1/n}(0, 1).
\]

Recall that, following our previous convention, the halo function is given as \( \phi_S(x) = C_S(1/x) \) for \( x > 1 \) and \( \phi_S(x) = x \) for \( x \leq 1 \). Hence the fact that \( C_S(\alpha) - 1 \lesssim_n (1 - \alpha)^{1/n} \) enables us to immediately establish the following continuity estimate for the halo function on all of \([0, \infty)\).

**Corollary 3.** Let \( \phi_S(\alpha) \) be the halo function associated with the strong maximal operator on \( \mathbb{R}^n \). Then
\[
\phi_S \in C^{1/n}([0, \infty)).
\]

The reasoning in this section applies not only to the strong maximal operator \( M_S \), but also the Hardy-Littlewood maximal operator \( M_{HL,c} \) with respect to cubes. Observing that the exponents in their Solyanik estimates are the same, see Theorem 1, we have the following.

**Corollary 4.** Let \( \phi_{HL,c}(\alpha) \) be the halo function associated with the Hardy-Littlewood maximal operator with respect to cubes on \( \mathbb{R}^n \). Then
\[
\phi_{HL,c} \in C^{1/n}([0, \infty)).
\]

4. **Local Hölder continuity of the Halo Function of the Hardy-Littlewood Maximal Operator averaging over balls.**

A basis consisting of rectangular parallelepipeds enjoys the advantage of enabling arguments along the lines of the Calderón-Zygmund decomposition employed in the proof of Theorem 2. To yield Hölder continuity estimates for the halo function of the Hardy-Littlewood maximal operator with respect to **balls**, however, additional arguments must be made that we provide here.
Theorem 3. Let $\phi_{\text{HL},b}(\alpha)$ be the halo function associated with the Hardy-Littlewood maximal operator with respect to balls on $\mathbb{R}^n$. Then

$$\phi_{\text{HL},b} \in C^{1/n+1}([0, \infty)).$$

Moreover, the associated Tauberian constants $C_{\text{HL},b}(\alpha)$ satisfy

$$C_{\text{HL},b} \in C^{1/n}(0, 1)$$

and consequently

$$\phi_{\text{HL},b} \in C^{1/n}(1, \infty).$$

Proof. Let $E \subset \mathbb{R}^n$ be a set of finite measure and $\alpha \in (0, 1)$. Let $x \in H_{\text{HL},b,\alpha}(E)$. By the definition of $H_{\text{HL},b,\alpha}(E)$ there exists some ball $\tilde{B} \subset \mathbb{R}^n$ with $\tilde{B} \ni x$ and

$$\frac{1}{|\tilde{B}|} \int_{\tilde{B}} \chi_E > \alpha.$$

Then there exists a ball $B \supset \tilde{B}$ such that

$$\frac{1}{|B|} \int_B \chi_E = \alpha.$$

By scaling, we can also assume that $B$ is the unit ball of $\mathbb{R}^n$. Let also $0 < \epsilon < \min(\alpha, 1 - \alpha)$ be a small parameter to be chosen later. We now denote by $\mathcal{C}_m$ the collection of all dyadic cubes of sidelength $2^{-m}$ which are strictly contained in $B$. We choose $m$ to be a large positive integer so that

$$\left| B \setminus \bigcup_{C \in \mathcal{C}_m} C \right| < \epsilon |B|.$$

Observing that the measure of the union of all dyadic cubes in $\mathcal{C}_m$ intersecting the boundary of $B$ is $\sim_n 2^{-m}$ and using elementary geometric arguments we see that we may assume $2^{-m} \sim_n \epsilon$.

Next we claim that there exists a cube $R \subseteq B$, with $|R| = 2^{-mn} \sim_n \epsilon^n$, such that

$$\frac{\alpha - \epsilon}{1 - \epsilon} \leq \frac{1}{|R|} \int_R \chi_E \leq \frac{\alpha}{1 - \epsilon}.$$  \hspace{1cm} (1)

Indeed, if for all $C \in \mathcal{C}_m$ the left hand side inequality of (1) failed, we would have

$$|B \cap E| = \left| \bigcup_{C \in \mathcal{C}_m} C \cap E \right| + \left| (B \setminus \bigcup_{C \in \mathcal{C}_m} C) \cap E \right|$$

$$< \frac{\alpha - \epsilon}{1 - \epsilon} \sum_{C \in \mathcal{C}_m} |C| + |B| - \left| \bigcup_{C \in \mathcal{C}_m} C \right| = |B| - \frac{1 - \alpha}{1 - \epsilon} \left| \bigcup_{C \in \mathcal{C}_m} C \right|$$

$$\leq |B| - \frac{1 - \alpha}{1 - \epsilon} (1 - \epsilon)|B| = \alpha |B|.$$
contradicting the choice of \( B \). On the other hand, if for all \( C \in C_m \) the right hand side inequality of (1) failed we would get
\[
|B \cap E| \geq \sum_{C \in C_m} |C \cap E| > \frac{\alpha}{1 - \epsilon} \sum_{C \in C_m} |C| = \frac{\alpha}{1 - \epsilon} \bigcup_{C \in C_m} C
\]
\[\geq \frac{\alpha}{1 - \epsilon} (1 - \epsilon)|B| = \alpha|B|
\]
which also contradicts our hypotheses on \( B \). Thus there exist dyadic cubes \( R_1, R_2 \in C_m \) such that
\[
\frac{1}{|R_1|} \int_{R_1} \chi_E \geq \frac{\alpha - \epsilon}{1 - \epsilon} \quad \text{and} \quad \frac{1}{|R_2|} \int_{R_2} \chi_E \leq \frac{\alpha}{1 - \epsilon}.
\]
As \( R_1 \) can be mapped onto \( R_2 \), inside \( B \), by a continuous rigid motion and using the intermediate value theorem we conclude that there exists a cube \( R \subseteq B \) with \(|R| = |R_1| = |R_2| = 2^{-mn}\), implying that (1) holds.

For \( \delta \in (0, 1 - \frac{\alpha}{1 - \epsilon}) \) we now perform a Calderón-Zygmund decomposition of \( \chi_{E \cap R} \) with respect to the dyadic grid generated by \( R \), at level \( 1 - \delta \), to get the Calderón-Zygmund cubes \( \{R_j\}_j \), with \( E \cap R \subseteq \cup_j R_j \) a.e., \(|R_j \cap E|/|R_j| > 1 - \delta\) and the cubes \( R_j \) being maximal with respect to this property. For every \( j \) we now have \( R_j(1) \subseteq R \). Indeed, if not, then \( R \) would be itself a Calderón-Zygmund cube. However we have
\[
|R \cap E|/|R| \leq \frac{\alpha}{1 - \epsilon} \leq 1 - \delta.
\]
Let \( \{R_j^{(1)}\}_k \) be maximal among the parents \( R_j^{(1)} \). We have
\[
|\bigcup_k R^{(1)}_k| \geq \frac{1}{2^n} |\bigcup_j R_j|.
\]
By the continuity of the Lebesgue measure we can find, for each \( k \), a cube \( S_k \) with
\[
R^{(1)}_j \subset S_k \subseteq R^{(1)}_j \quad \text{and} \quad \frac{|S_k \cap E|}{|S_k|} = 1 - \delta.
\]
Note that the \( S_k \)'s are a.e. pairwise disjoint and for each \( k \) we have \( S_k \subseteq \mathcal{H}_{S,1-\eta}(E) \) for every \( \eta \in (0, 1) \). Thus for such \( \eta \) we get
\[
|\mathcal{H}_{S,1-\eta}(E) \setminus E| \cap R| \geq \sum_k |(\mathcal{H}_{S,1-\eta}(E) \setminus E) \cap S_k| \geq \frac{\delta}{2^n} |E \cap R|.
\]
Accordingly
\[
|(\mathcal{H}_{S,1-\eta}(E) \setminus E) \cap B| \geq \frac{\delta}{2^n} |E \cap R| \geq \frac{\delta}{2^n} \frac{\alpha - \epsilon}{1 - \epsilon} |R| \geq \delta \epsilon^n \frac{\alpha - \epsilon}{1 - \epsilon} |B|.
\]
Hence
\[
B \subseteq \mathcal{H}_{HL,b,\alpha + C_n \epsilon \delta + \epsilon} \left( \mathcal{H}_{S,1-\eta}(E) \right)
\]
for some dimensional constant $C_n > 0$. We conclude that
$$C_{HL,b}(\alpha) \leq C_{HL,b}(\alpha + C_n \epsilon \frac{\alpha - \epsilon}{1 - \epsilon}) C_S(1 - \eta).$$

Finally, using the continuity results from [1] and letting $\eta \to \delta^+$ we get that for all $\delta \leq 1 - \frac{\alpha}{1 - \epsilon}$ we have
$$C_{HL,b}(\alpha) \leq C_{HL,b}(\alpha + C_n \epsilon \frac{\alpha - \epsilon}{1 - \epsilon}) C_S(1 - \delta).$$

At this point we set $\epsilon := \frac{1}{2} \min(\alpha, 1 - \alpha)$. Then for all $\alpha \in (0, 1)$ we have $\frac{\alpha - \epsilon}{1 - \epsilon} \gtrsim \alpha$ and
$$1 - \frac{\alpha}{1 - \epsilon} = \begin{cases} \frac{2 - 3\alpha}{2 - \alpha}, & \alpha \leq \frac{1}{2}, \\ \frac{1 - \alpha}{1 + \alpha}, & \alpha > \frac{1}{2}. \end{cases}$$

From this we readily see that our estimates are valid for all $\delta \lesssim 1 - \alpha$. Using the fact that $C_{HL,b}(\alpha)$ is non-increasing in $\alpha$ we can summarize our result to the estimate
$$C_{HL,b}(\alpha) \leq C_{HL,b}(\alpha(1 + c_n \min(\alpha, 1 - \alpha)^n \delta)) C_S(1 - \delta), \quad \delta \lesssim 1 - \alpha,$$

for some dimensional constant $c_n > 0$. The proof of the statement $C_{HL,b} \in C_{1/n}(0, 1)$ is completed by using the Solyanik estimate $C_S(\alpha) - 1 \sim (1 - \alpha)^{\frac{1}{n}}$ as $\alpha \to 1^-$ as in the proof of Lemma 1.

By the definition of the halo function the argument above implies that $\phi_{HL,b} \in C^{1/n}(1, \infty)$. However, to yield Hölder continuity of $\phi_{HL,b}$ at 1 we must use the Solyanik estimates for $C_{HL,b}$ itself, not the Solyanik estimates for $C_S$ that we were able to use above in order to find smoothness estimates on $C_{HL,b}$ on compact subsets of $(0, 1)$. By Theorem 1 we have that $C_{HL,b}(\alpha) - 1 \lesssim_n (1 - \alpha)^{1/n+1}$ as $\alpha \to 1^-$. This result combined with the Solyanik estimate for $C_S$ and estimate (2) above gives
$$\phi_{HL,b} \in C^{1/n+1}([0, \infty)),$$
as we wanted.

5. Bases of Convex Sets

We now show that, if $B$ is a homothecy invariant density basis consisting of convex sets which satisfies a Solyanik estimate of the form $C_B(\alpha) - 1 \lesssim_B (1 - \alpha)^p$ for $\alpha$ sufficiently close to 1, then $\phi_B \in C^p([0, \infty))$. We need some preliminary results. A schematic for the lemmas that follow and their corresponding proofs is contained in Figure 1.

The following technical lemma uses a classical lemma of Fritz John [15] and will help us reduce the study of regularity estimates for the sharp Tauberian constants associated
with bases of convex sets to estimates concerning rectangles in $\mathbb{R}^n$. We will refer to it as the Fritz John lemma.

**Lemma 2.** Let $\Lambda \subseteq \mathbb{R}^n$ be a bounded convex set in $\mathbb{R}^n$. Then there exists a rectangular parallelepiped $R_\Lambda \subseteq \mathbb{R}^n$ such that

$$R_\Lambda \subseteq \Lambda \subseteq n^{\frac{3}{2}}R_\Lambda.$$

**Proof.** Let $\Lambda$ be a bounded convex set in $\mathbb{R}^n$. As was proven by Fritz John in [15], $\Lambda$ must contain an ellipsoid $E_\Lambda$ such that

$$E_\Lambda \subseteq \Lambda \subseteq n^{1/2}E_\Lambda.$$

Here the dilation $nE_\Lambda$ is taken with respect to the center of the ellipsoid. Let $S_\Lambda$ be a rectangular parallelepiped of minimal volume containing $E_\Lambda$. By elementary geometry, we have

$$n^{-\frac{1}{2}}S_\Lambda \subseteq E_\Lambda \subseteq \Lambda \subseteq nE_\Lambda \subseteq nS_\Lambda.$$

The desired rectangle is now given as $R_\Lambda := n^{-\frac{1}{2}}S_\Lambda$. \qed

We proceed with a simple geometric lemma that quantifies the measure theoretic approximation of a convex set by finite unions of dyadic cubes.

**Lemma 3.** Let $\Lambda \subseteq \mathbb{R}^n$ be a convex set such that, letting $Q := [0, 1]^n$ and $cQ$ the $c$-fold concentric dilate of $Q$, we have $Q \subset \Lambda \subset n^{3/2}Q$. For every $\epsilon \in (0, 1)$ there exists a collection of a.e. disjoint dyadic cubes $\{C_j\}_j$ contained in $\Lambda$, each of sidelength $\sim_n \epsilon$, so that $|\Lambda \setminus \bigcup_j C_j| < \epsilon|\Lambda|$.

**Proof.** Let $C_m$ denote the collection of dyadic cubes of sidelength $2^{-m}$, where $m$ is a non-negative integer, which are contained in $\Lambda$. Suppose $x \in S$ and $\text{dist}(x, \partial \Lambda) > \sqrt{n}2^{-m}$. Then there exists a dyadic cube $C \in C_m$ with $C \ni x$. Using the convexity of $\Lambda$ we conclude that

$$\left| \Lambda \setminus \bigcup_{C \in C_m} \right| \leq |\{x \in \Lambda : \text{dist}(x, \partial \Lambda) \leq \sqrt{n}2^{-m}\}| \lesssim_n 2^{-m} \sim_n 2^{-m}|\Lambda|.$$

Choosing $2^{-m} \sim_n \epsilon$ proves the lemma. \qed

We proceed by showing that if the average of $\chi_E$ with respect to some convex set $\Lambda$ is equal to $\alpha$ then $\chi_E$ must have average close to $\alpha$ with respect to some cube $R \subseteq \Lambda$, where we also have a control on the measure of the cube $R$. 

Lemma 4. Let \( \alpha \in (0, 1) \) and \( 0 < \epsilon < \min(\alpha, 1 - \alpha) \) and let \( \Lambda \subset \mathbb{R}^n \) be a convex set satisfying \( Q \subset \Lambda \subset n^{3/2}Q \). Suppose that \( E \subset \mathbb{R}^n \) is a measurable set of finite measure for which \( \frac{1}{|\Lambda|} \int_{\Lambda} \chi_E = \alpha \). Then there exist a cube \( R \subseteq \Lambda \) with \( |R| \sim n^\epsilon \) such that

\[
\frac{\alpha - \epsilon}{1 - \epsilon} \leq \frac{1}{|R|} \int_R \chi_E \leq \frac{\alpha}{1 - \epsilon}.
\]

Proof. Let \( \{C_j\}_j \) be the collection of dyadic cubes provided by Lemma 3. We have that \( C_j \subseteq \Lambda \) for every \( j \), \( |C_j| \sim n^\epsilon \) and \( |\Lambda \setminus \bigcup_j C_j| \leq \epsilon|\Lambda| \). Arguing as in the proof of Theorem 3 we see that there exist dyadic cubes \( R_1, R_2 \in \{C_j\}_j \) such that

\[
\frac{1}{|R_1|} \int_{R_1} \chi_E \geq \frac{\alpha - \epsilon}{1 - \epsilon} \quad \text{and} \quad \frac{1}{|R_2|} \int_{R_2} \chi_E \leq \frac{\alpha}{1 - \epsilon}.
\]

As the convex hull of the union of any two \( C_j \)'s lies in \( \Lambda \), by the intermediate value theorem we see there exists a cube \( R \subset \Lambda \) of measure \( \sim n^\epsilon \) that satisfies the conclusion of the lemma. \( \square \)

We will need to work with a smaller cube inside the one provided by the previous lemma. The following lemma summarizes the technical details of this construction.

Lemma 5. Let \( \alpha \in (0, 1) \), \( 0 < \epsilon < \min(\alpha, 1 - \alpha) \), and \( R \) be the cube provided by Lemma 4. There exists a cube \( R^* \subseteq R \) with \( |R| \sim |R^*| \) and

\[
\frac{1}{2} \frac{\alpha - \epsilon}{1 - \epsilon} \leq \frac{1}{|R^*|} \int_{R^*} \chi_E \leq \frac{1}{2} \left(1 + \frac{\alpha}{1 - \epsilon}\right)
\]

Proof. For \( t \in (0, 1) \) we have

\[
1 - t^{-n} \left[1 - \frac{\alpha - \epsilon}{1 - \epsilon}\right] \leq \frac{1}{|tR|} \int_{tR} \chi_E \leq t^{-n} \frac{\alpha}{1 - \epsilon}
\]

We now choose \( t_o \) by setting

\[
t^{-n}_o := \min \left( \left(1 + \frac{1}{2} \left(\frac{1 - \epsilon}{\alpha} - 1\right)\right), \frac{1 - \epsilon}{1 - \alpha} - \frac{1}{2} \frac{\alpha - \epsilon}{1 - \alpha}\right)
\]

and define \( R^* := t_o R \). We have that \( t_o > 0 \) whenever \( \alpha, \epsilon \in (0, 1) \) while the restriction \( \epsilon < \min(\alpha, 1 - \alpha) \) guarantees that \( t_o < 1 \). The fact that \( |R^* \cap E|/|R^*| \) satisfies the desired inequalities follows immediately by the definition of \( t_o \). Furthermore we can easily estimate

\[
t^{-n}_o = \frac{1}{2} \min \left(\frac{1 - \epsilon + \alpha}{\alpha}, \frac{2 - \epsilon - \alpha}{1 - \alpha}\right) \sim \frac{1}{\max(\alpha, 1 - \alpha)}
\]

and thus \( t_o \sim n^{\max(\alpha, 1 - \alpha)} \). This gives \( |R^*| \sim |R| \) as we wanted. \( \square \)
The following theorem constitutes the heart of the matter in this paper regarding the embedding of halo sets associated with convex bases.

**Theorem 4.** Let $\mathcal{B}$ be a homothecy invariant basis consisting of convex sets. For every $\alpha \in (0, 1)$ and every measurable set $E \subset \mathbb{R}^n$ of finite measure we have

$$H_{\mathcal{B},\alpha}(E) \subseteq H_{\mathcal{B},\alpha}(1 + c_n \min(\alpha, 1-\alpha) \delta) \{ H_{\mathcal{B},1-3n^{3n/2}}(E) \},$$

for all $\delta \lesssim n^{1-\alpha}$. Here $c_n > 0$ is a numerical constant that depends only upon the dimension.

---

**Figure 1.** Schematic for the proofs of §5

*Proof.* Suppose that $E \subset \mathbb{R}^n$ is of finite measure and $\alpha \in (0, 1)$. Let $x \in H_{\mathcal{B},\alpha}(E)$ and consider a convex set $\tilde{\Lambda} \in \mathcal{B}$ such that $x \in \tilde{\Lambda}$ and

$$\frac{1}{|\tilde{\Lambda}|} \int_{\tilde{\Lambda}} \chi_E > \alpha.$$

By considering a homothecy $\Lambda$ of $\tilde{\Lambda}$ satisfying $\Lambda \supseteq \tilde{\Lambda} \ni x$ we have

$$\frac{1}{|\Lambda|} \int_{\Lambda} \chi_E = \alpha.$$

Using the F. John lemma we can find a rectangular parallelepiped $Q_{\Lambda} \subset \mathbb{R}^n$ such that $Q_{\Lambda} \subseteq \Lambda \subseteq n^{3/2}Q_{\Lambda}$. Finally, by invariance under affine transformations, we can map $Q_{\Lambda}$ onto the unit cube $Q = [0,1]^n$ through a bijective linear transformation reducing the
problem to the case that $\Lambda$ satisfies (3) and $Q \subseteq \Lambda \subseteq n^{3/2}Q$, as in Lemma 4. It thus suffices to show that

$$\Lambda \subseteq \mathcal{H}_{\mathcal{B}_\Lambda, \alpha(1+c_n \min(\alpha, 1-\alpha)^2 \delta)}(\mathcal{H}_{\mathcal{B}_\Lambda, 1-3n^{3/2}\delta}(E)),$$

where $\mathcal{B}_\Lambda$ is the basis consisting of all the homothecies of $\Lambda$.

With these notations and reductions taken as understood we now set $\epsilon := \frac{1}{2} \min(\alpha, 1-\alpha)$ and apply Lemma 4 to get a cube $R \subset \Lambda$ with $|R| \sim_n \epsilon^n$ and such that

$$\frac{\alpha - \epsilon}{1-\epsilon} \leq \frac{1}{|R|} \int_R \chi_E \leq \frac{\alpha}{1-\alpha}. \quad (4)$$

For technical reasons we will have to consider the smaller cube $R^{in} = t_o R$ provided by Lemma 5, where we remember that $t_o \sim_n \max(\alpha, 1-\alpha)^{1/2}$. A schematic associated to the relationships among $R^{in}$, $R$, $Q$, and $\Lambda$ is indicated in Figure 1. Proceeding as in the proof of Theorem 3, for $0 < \delta \leq \frac{1}{2} \min(1 - \frac{\alpha}{1-\epsilon}, \frac{1}{3} n^{-\frac{3}{2}} (1-\alpha))$ we perform a Calderón-Zygmund decomposition of $\chi_{E \cap R^{in}}$ with respect to the dyadic grid generated by $R^{in}$, at level $1 - \delta$. This results in the collection of Calderón-Zygmund cubes $\{R_j\}_j$ for which $E \cap R^{in} \subseteq \bigcup_j R_j$ a.e.,

$$\frac{1}{|R_j|} \int_{R_j} \chi_E > 1 - \delta,$$

and the cubes $R_j$ are maximal with respect to this property in the dyadic grid generated by $R^{in}$. Note that

$$\frac{1}{|R^{in}|} \int_{R^{in}} \chi_E \leq \frac{1}{2} + \frac{1}{2} \frac{\alpha}{1-\epsilon} \leq 1 - \delta$$

so the cube $R^{in}$ is not itself a Calderón-Zygmund cube and thus, for all $j$, $R_j \subseteq R^{in}$ and $R_j^{(1)} \subseteq R^{in}$. We will use these properties in what follows without particular mention.

The rest of the proof is divided into two complementary cases:

Case I: for some $j_o$ we have $|4n^{3/2} R_{j_o} \setminus R| > 0$.

In this case, letting $\text{side}(S)$ denote the sidelength of a cube $S$, we have that

$$4n^{3/2} \text{side}(R_{j_o}) > \frac{1}{t_o} \text{side}(R^{in}) - \text{side}(R^{in}) \frac{1}{2},$$

implying

$$|R_{j_o}| \gtrsim_n |R^{in}|(t_o^{-1} - 1)^n.$$
Assuming that $R_{j_0} = T_{j_0}(Q)$ for some homothecy $T_{j_0}$ we set $\Lambda_{j_0}^{\text{small}} := n^{-\frac{3}{2}}T_{j_0}(\Lambda)$, the dilation taken here with respect to the center of $R_{j_0}$. Then

\[ n^{-\frac{3}{4}}R_{j_0} \subseteq \Lambda_{j_0}^{\text{small}} \subseteq R_{j_0} \quad \text{and} \quad |R_{j_0}| \leq n^{\frac{3}{4}}|\Lambda_{j_0}^{\text{small}}|. \]

Remembering that $|\Lambda \cap E|/|\Lambda| = \alpha < 1 - 6n^{3n/2}\delta$, we can conclude that there exists a homothety $\Lambda_{j_0}$ of $\Lambda$ such that

\[ \Lambda_{j_0}^{\text{small}} \subseteq \Lambda_{j_0} \subseteq \Lambda \quad \text{and} \quad \int_{\Lambda_{j_0}} \chi_E = 1 - 2n^{3n/2}\delta. \]

The measure of $\Lambda_{j_0}$ can be estimated from below as follows:

\[ |\Lambda_{j_0}| \geq |\Lambda_{j_0}^{\text{small}}| \geq n^{-\frac{3}{2}}|R_{j_0}| \gtrsim_n (t_o^{-1} - 1)^n|R^{\text{in}}| = t_o^n(t_o^{-1} - 1)^n|R|. \]

Note that $\Lambda_{j_0} \subset \mathcal{H}_{B_{1-3n^{3n/2}\delta}}(E)$. We conclude that

\[ |(\mathcal{H}_{B_{1-3n^{3n/2}\delta}}(E) \setminus E) \cap \Lambda| \geq |(\mathcal{H}_{B_{1-3n^{3n/2}\delta}}(E) \setminus E) \cap \Lambda_{j_0}| \]

\[ = |\Lambda_{j_0} \setminus E| \gtrsim_n \delta|\Lambda_{j_0}| \gtrsim_n t_o^n(t_o^{-1} - 1)^n \epsilon^n|\Lambda|. \]

Remembering the definitions of $\epsilon$ and $t_o$ and using Lemma 5 it is not hard to obtain the estimate $t_o^n(t_o^{-1} - 1)^n \sim_n \min(\alpha, 1 - \alpha)^n$ for $\alpha \in (0, 1)$.

In this case we have thus proved

\[ \Lambda \subseteq \mathcal{H}_{B_{1-2n^{3n/2}\delta}}(\mathcal{H}_{B_{1-3n^{3n/2}\delta}}(E)). \]

Case II: for every $j$ we have $4n^{3/2}R_j \subset R$.

In this case we apply the Vitali covering lemma to the collection $\{4n^{3/2}R_j\}_j$ resulting in a subcollection $\{R_{j_k}\}_{k} \subset \{R_j\}_j$ such that the cubes in $\{4n^{3/2}R_j\}_k$ are a.e. pairwise disjoint and $| \bigcup_k 4n^{3/2}R_{j_k} | \gtrsim_n | \bigcup_j 4n^{3/2}R_j |$. Thus

\[ \left| \bigcup_k R_{j_k} \right| \sim_n \left| \bigcup_k 4n^{3/2}R_{j_k} \right| \gtrsim_n \left| \bigcup_j 4n^{3/2}R_j \right| \geq \left| \bigcup_j R_j \right| \geq |E \cap R^{\text{in}}|. \]

By Lemma 5 the measure $|E \cap R^{\text{in}}|$ can be estimated from below by

\[ |E \cap R^{\text{in}}| \geq \frac{1}{2} \frac{\alpha - \epsilon}{1 - \epsilon} |R^{\text{in}}| \gtrsim_n \frac{\alpha - \epsilon}{1 - \epsilon} |R| \gtrsim_n \frac{\alpha - \epsilon}{1 - \epsilon} \epsilon^n |\Lambda|. \]

We thus have showed that

\[ \left| \bigcup_k R_{j_k} \right| \gtrsim_n \frac{\alpha - \epsilon}{1 - \epsilon} \epsilon^n |\Lambda|. \]
Now if \( R_{jk} =: T_k(Q) \) for some homothecy \( T_k \) we define \( \Lambda^\text{small}_k := n^{-\frac{3}{2}} T_k(\Lambda) \); the dilation is considered here with respect to the center of the cube \( R_{jk} \). Furthermore, if \( R^{(1)}_{jk} =: T^{(1)}_k(Q) \) for some homothecy \( T^{(1)}_k \) we define \( \Lambda^\text{big}_k := T^{(1)}_k(\Lambda) \). These definitions imply

\[
n^{-\frac{3}{2}} R_{jk} \subseteq \Lambda^\text{small}_k \subseteq R_{jk} \quad \text{and} \quad |\Lambda^\text{small}_k| \geq n^{-\frac{3n}{2}} |R_{jk}|,
\]

\[
R^{(1)}_{jk} \subseteq \Lambda^\text{big}_k \subseteq n^{-\frac{3}{2}} R^{(1)}_{jk} \subseteq 4n^{-\frac{3}{2}} R_{jk} \subseteq R \quad \text{and} \quad |R^{(1)}_{jk}| \geq n^{-\frac{3n}{2}} |\Lambda^\text{big}_k|.
\]

Using the fact that \(|R_j \cap E|/|R_j| > 1 - \delta \) and the estimates resulting from the definitions above we can estimate

\[
\frac{1}{|\Lambda^\text{small}_k|} \int_{\Lambda^\text{small}_k} \chi_E = \frac{|\Lambda^\text{small}_k| - |\Lambda^\text{small}_k \setminus E|}{|\Lambda^\text{small}_k|} > \frac{|\Lambda^\text{small}_k| - |R_{jk} \setminus E|}{|\Lambda^\text{small}_k|} \\
\quad \geq \frac{|\Lambda^\text{small}_k| - \delta |R_{jk}|}{|\Lambda^\text{small}_k|} \geq 1 - n^{3n/2} \delta.
\]

For the parent cubes \( R^{(1)}_{jk} \) we have \(|R^{(1)}_{jk} \cap E|/|R^{(1)}_{jk}| \leq 1 - \delta \) by the Calderón-Zygmund decomposition. Thus for \( \Lambda^\text{big}_k \) we can estimate

\[
\frac{1}{|\Lambda^\text{big}_k|} \int_{\Lambda^\text{big}_k} \chi_E = \frac{|\Lambda^\text{big}_k| - |\Lambda^\text{big}_k \setminus E|}{|\Lambda^\text{big}_k|} \leq \frac{|\Lambda^\text{big}_k| - |R^{(1)}_{jk} \setminus E|}{|\Lambda^\text{big}_k|} \\
\quad \leq \frac{|\Lambda^\text{big}_k| - \delta |R^{(1)}_{jk}|}{|\Lambda^\text{big}_k|} \leq 1 - n^{-3n/2} \delta.
\]

So for every \( k \) there exists a homothecy \( \Lambda_k \) of \( \Lambda \) such that

\[
\Lambda^\text{small}_k \subseteq \Lambda_k \subseteq \Lambda^\text{big}_k \subseteq R \subseteq \Lambda
\]

and

\[
1 - \delta n^{3n/2} < \frac{1}{|\Lambda_k|} \int_{\Lambda_k} \chi_E \leq 1 - n^{-3n/2} \delta.
\]

We get

\[
\left| (\mathcal{H}_{\mathcal{B}_A,1-\delta n^{3n/2}}(E) \setminus E) \cap \Lambda_k \right| \gtrsim_n \delta |\Lambda_k|.
\]

Note that for each \( k \) we have \( n^{-\frac{3}{2}} R_{jk} \subseteq \Lambda_k \subseteq 4n^{-\frac{3}{2}} R_{jk} \) so the \( \Lambda_k \)'s are a.e. pairwise disjoint in \( R \) and \(|\Lambda_k| \sim_n |R_{jk}| \). We can therefore estimate

\[
\left| (\mathcal{H}_{\mathcal{B}_A,1-\delta n^{3n/2}}(E) \setminus E) \cap \Lambda \right| \gtrsim_n \delta \sum_k |\Lambda_k| \sim_n \delta \sum_k |R_{jk}| \gtrsim_n \delta e^{n} \frac{\alpha - \epsilon}{1 - \epsilon} |\Lambda|
\]

where in the last estimate we used (5).

Since \( \frac{\alpha - \epsilon}{1 - \epsilon} \gtrsim \alpha \) for \( \alpha \in (0,1) \) we thus have

\[
\Lambda \subseteq \mathcal{H}_{\mathcal{B}_A,\alpha + c_n \min(\alpha,1-\alpha) \alpha \delta}(\mathcal{H}_{\mathcal{B}_A,1-\delta n^{3n/2}}(E)).
\]
for a dimensional constant $c_n > 0$.

The two complementary cases studied above imply that for all $\delta \lesssim n 1 - \alpha$ we have

$$\mathcal{H}_{B, \Lambda, \alpha}(E) \subseteq \mathcal{H}_{B, \Lambda, \alpha(1 + c_n \min(\alpha, 1 - \alpha)^{2n}\delta)}(\mathcal{H}_{B, \Lambda, 1 - 3n^{3n/2}\delta}(E)),$$

where $c_n > 0$ is a numerical constant depending only on dimension. □

By an argument along the lines of the proof of Lemma 1 we then have the following.

**Theorem 5.** Let $B$ be a homothecy invariant density basis of convex sets in $\mathbb{R}^n$. Suppose $C_B(\alpha) \lesssim_B (1 - \alpha)^p$ holds for $\alpha$ sufficiently close to 1 and for some fixed $0 < p < 1$. Then

$$C_B \in C^p(0, 1)$$

and consequently

$$\phi_B \in C^p([0, \infty)).$$

**Remark.** It is quite possible that we should be able to have improved smoothness results for $C_B(\alpha)$ for $0 < \alpha < 1$, especially in the cases that the maximal operator $M_B$ is the Hardy-Littlewood or strong maximal operator. Arguments along these lines would have to be substantially more sophisticated than what we have provided here, however, as the known Solyanik estimates for the strong maximal operator are known to be sharp; see [9]. This is a subject of ongoing research.

**References**

[1] O. Beznosova and P. A. Hagelstein, *Continuity of halo functions associated to homothecy invariant density bases*, Colloq. Math. 134 (2014), no. 2, 235–243. MR3194408

[2] H. Busemann and W. Feller, *Zur Differentiation der Lebesgueschen Integrale*, Fundamenta Mathematicae 22 (1934), no. 1, 226-256 (ger). ↑3

[3] C. Cabrelli, M. T. Lacey, U. Molter, and J. C. Pipher, *Variations on the theme of Journé’s lemma*, Houston J. Math. 32 (2006), no. 3, 833–861. MR2247912 (2007e:42011)

[4] A. Córdoba and R. Fefferman, *On the equivalence between the boundedness of certain classes of maximal and multiplier operators in Fourier analysis*, Proc. Nat. Acad. Sci. U.S.A. 74 (1977), no. 2, 423–425. MR0433117 (55 #6096)

[5] S. H. Ferguson and M. T. Lacey, *A characterization of product BMO by commutators*, Acta Math. 189 (2002), no. 2, 143–160. MR1961195 (2004e:42026)

[6] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, 2nd ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 224, Springer-Verlag, Berlin, 1983. MR737190 (86c:35035)

[7] M. de Guzmán, *Differentiation of integrals in $\mathbb{R}^n$*, Measure theory (Proc. Conf., Oberwolfach, 1975), Springer, Berlin, 1976, pp. 181–185. Lecture Notes in Math., Vol. 541. MR0476978 (57 #16523)

[8] P. A. Hagelstein, T. Luque, and I. Parissis, *Tauberian conditions, Muckenhoupt weights, and differentiation properties of weighted bases*, Trans. Amer. Math. Soc. (to appear), available at 1304.1015. ↑2
P. A. Hagelstein and I. Parissis, Solyanik estimates in harmonic analysis, to appear in Springer Proceedings in Mathematics & Statistics (2014), available at 1310.3771.

P. A. Hagelstein and I. Parissis, Weighted Solyanik Estimates for the Hardy-Littlewood maximal operator and embedding of $A_\infty$ into $A_p$ (2014), available at 1405.6631.

P. A. Hagelstein and A. Stokolos, An extension of the Córdoba-Fefferman theorem on the equivalence between the boundedness of certain classes of maximal and multiplier operators, C. R. Math. Acad. Sci. Paris 346 (2008), no. 19-20, 1063–1065 (English, with English and French summaries). MR2462049 (2010a:42064)

P. A. Hagelstein and A. Stokolos, Tauberian conditions for geometric maximal operators, Trans. Amer. Math. Soc. 361 (2009), no. 6, 3031–3040. MR2485416 (2010b:42023)

G. H. Hardy and J. E. Littlewood, A maximal theorem with function-theoretic applications, Acta Math. 54 (1930), no. 1, 81–116. MR1555303

C. A. Hayes Jr., A condition of halo type for the differentiation of classes of integrals, Canad. J. Math. 18 (1966), 1015–1023. MR0199318 (33 #7466)

F. John, Extremum problems with inequalities as subsidiary conditions, Studies and Essays Presented to R. Courant on his 60th Birthday, January 8, 1948, Interscience Publishers, Inc., New York, N. Y., 1948, pp. 187–204. MR0030135 (10,719b)

M. Lacey and E. Terwilleger, Hankel operators in several complex variables and product BMO, Houston J. Math. 35 (2009), no. 1, 159–183. MR2491875 (2010c:42071)

A. A. Solyanik, On halo functions for differentiation bases, Mat. Zametki 54 (1993), no. 6, 82–89, 160 (Russian, with Russian summary); English transl., Math. Notes 54 (1993), no. 5-6, 1241–1245 (1994). MR1268374 (95g:42033)

F. Soria, Note on differentiation of integrals and the halo conjecture, Studia Math. 81 (1985), no. 1, 29–36. MR818168 (87j:42058)

E. M. Stein, Singular integrals and differentiability properties of functions, Princeton Mathematical Series, No. 30, Princeton University Press, Princeton, N. J., 1970. MR0290095 (44 #7280)

Department of Mathematics, Baylor University, Waco, Texas 76798

E-mail address: paul_hagelstein@baylor.edu

Department of Mathematics, Aalto University, P. O. Box 11100, FI-00076 Aalto, Finland

E-mail address: ioannis.parissis@gmail.com