Optimal spacecraft asymptotic velocity for the high inclined orbits formation using gravity assists in the planetary systems

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Abstract. The inclination changing of the celestial bodies’ orbits is one of the most energy-consuming procedures. Nevertheless, for a number of promising space projects, leaving the main plane of the planetary or satellite system is fundamentally necessary and demanded. In this case only the possibility of realizing the formation of maximally inclined spacecraft (SC) orbits using gravity assist maneuvers (GM) is remained. However, the dynamic possibilities of GM using are also limited. The maximum possible value of the SC orbital inclination (which gives the extreme point inclination pole – IncPole) depends on the modulus $V_{\text{inf}}$ of the SC asymptotic velocity on the flyby hyperbola and is directly proportional to it (GM geometric limitation). However, the magnitude of the change in inclination at one GM is inversely proportional to $V_{\text{inf}}$ (GM dynamic limitation). As a result, a compromise value of $V_{\text{inf}}$ is existing, which varies for each specific case of the flyby planet. Since in order to achieve a significant inclination a whole series of GMs may be required, in addition, each GM must ensure the "resonance" of the orbital periods of the planet and SC after the GM in order to guarantee their new meeting. According the Jacobi integral in the restricted three body problem the $V_{\text{inf}}$ is an invariant during in all interplanetary flights using GMs. It’s possible to construct the series of increasing GM in form of "jumps" along the $V_{\text{inf}}^{\text{res}}$ resonance levels from the initial inclination to the maximum possible point IncPole. This point may not belong to any isoline of one of the main resonances itself in the general case. This reduces the GMs effectiveness and necessarily increases the mission time of flight. A heuristic consideration consists in choosing (in the "resonant tuning") the such value of the design $V_{\text{inf}}$, which ensures the localization of the inclination pole in the vicinity of some resonance curves between the orbital periods of the spacecraft and the planet (resonant $V_{\text{inf}}^{\text{res}}$). Thus it is possible to get to the IncPole by jumping along the resonance curve thru the minimum amount of GM. The paper describes the $V_{\text{inf}}^{\text{res}}$ formulas and gives its estimates for the Solar system and for the planets satellite systems.

1. Introduction
Modern mission design of the interplanetary space missions required the gravity assist maneuvers using [1, 2, 3, 4, 5]. Gravity assists reduce the expense of the characteristic velocity of the spacecraft

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(SC) and can thereby solve modern comprehensive problems of space exploration. Their use is a modern highly effective trend in the interplanetary flights theory. However, the dynamic possibilities of GM using are also limited. Let’s consider the problem of inclination changing of the celestial bodies’ orbits, which is very energy-consuming procedure. The maximum possible value of the SC orbital inclination (which gives the extreme point inclination pole – IncPole) depends on the modulus \( V_{\text{inf}} \) of the SC asymptotic velocity on the flyby hyperbola and is directly proportional to it. However, the magnitude of the change in inclination at one GM is inversely proportional to \( V_{\text{inf}} \) (Dlim – GM dynamic limitation). As a result, a compromise value of \( V_{\text{inf}} \) is existing, which varies for each specific case of a planet or a massive planet' satellite, when the GM is performed near it. Since in order to achieve a significant inclination, due to the Dlim, not one GM, but a whole series of GMs may be required, in addition, each GM must ensure the "resonance" of the orbital periods of the planet and spacecraft after the GM in order to guarantee their new meeting. According the Jacobi integral existing in the restricted three body problem the \( V_{\text{inf}} \) is an invariant during in all interplanetary flights using GMs. It’s possible to construct the series of "increasing" GM in the form of cranking "jumps" along the resonance levels of the asymptotic velocity from the initial inclination to the maximum possible point IncPole. This point may not belong to any isoline of one of the main resonances itself in the general case. This reduces the GMs effectiveness and necessarily increases the mission time of flight. A heuristic consideration consists in choosing (in the "resonant tuning") the such value of the design value of \( V_{\text{inf}} \), which ensures the localization of the inclination pole in the vicinity of one of the main resonance curves between the orbital periods of the spacecraft and the planet (resonant \( V_{\text{inf}}^{\text{res}} \)).

2. Gravity assists: dynamic and geometric limitations

As a result of the analysis of the spatial GM geometry (Golubev et al., 2016), the following statement is obtained (Golubev et al., 2017b; 2016). Let the asymptotic value of the velocity of the spacecraft (invariant before and after GM) be less than the modulus of the average planet orbital velocity \( V_p \). In this case, during execution of any GM sequence near planet, the following assessment holds for the maximum orbit inclination \( i_{\text{max}} \) of the spacecraft (Golubev et al., 2017b; 2016; Labunsky et al., 1998; Kawakatsu, 2009):

\[
i_{\text{max}} \leq \arcsin(V_{\text{inf}}/V_p).
\]

It is the exact upper bound of the set of possible inclinations of the spacecraft’s orbit. From this estimate, it follows that in order to increase the orbital inclination of the spacecraft relative to the ecliptic by more than it is necessary before performing the GM to increase the asymptotic speed of a spacecraft relative to Venus to at least 17.5 km/s. Such acceleration of the spacecraft can be accomplished using a low thrust engine and/or a GAM sequence near other planets. A single GM around the planet to increase the inclination of the orbit to the desired value may not be enough due to the presence of restrictions. In this case, it is necessary to synthesize a sequence of “resonance” GMs increasing in inclination (that is, such that the few orbital periods of the spacecraft after each GM are commensurate with the orbital period of the planet, providing a new encounter with it). The execution of resonant GMs is a basic prerequisite for such missions.

The maximum rotation angle \( \varphi \) of the asymptotic velocity vector of the spacecraft during a single GM is explained, according [1] as

\[
\sin(\varphi/2) = \frac{\mu}{\mu + r_x V_{\text{inf}}^2}^{-1}
\]

where \( \mu \) is the gravity parameter of a planet and \( r_x \) is the distance of pericenter of the flyby hyperbola of the spacecraft that cannot be less than the mean radius of the planet \( R_p \).
3. Asymptotic velocity resonant lines
By setting the input asymptotic velocity vector in a tube of permissible SC trajectories with the same $V_{\text{inf}}$, we consider the region formed by the ends of the possible vectors of asymptotic velocity after performing the gravity maneuver. This area is the intersection of the $V_{\text{inf}}$-sphere [3] and the solid angle, the solution of which is twice the maximum rotation angle of the asymptotic velocity vector for a single GM (a spherical cap). We confine ourselves to considering the case of tangential GMs, when the GM is performed on the apsis line of the spacecraft’s orbit. Let’s obtain the SC velocity corresponding to one or another resonance heliocentric trajectory. We plot on the $V_{\text{inf}}$-sphere the indicated spherical caps and the ends of the asymptotic velocity vectors, which correspond to the main resonances {3:4}, {1:1}, {4:3} between the orbital periods of the spacecraft and the planet around the Sun. The opportunity for a new encounter with the flyby planet is provided by one of two options of the GM (Fig. 1): the GM is performed along a fixed resonance line on the $V_{\text{inf}}$-sphere; the spherical cap on the $V_{\text{inf}}$-sphere, corresponding to the GM, should cover the nearest resonance lines (Figure 2).

Figure 1. $V_{\text{inf}}$-sphere, corresponding to the GM, should cover the nearest resonance lines.

Figure 2. It’s possible to construct the series of increasing GM in form of “jumps” along the resonance levels from the initial inclination to the maximum possible point IncPole.

In [2, 5, 6] a formula for the final orbital inclination of the spacecraft when performing a GM around a planet was presented (it’s analogue consisted in [3]). In spherical coordinates $(\varphi, \psi)$ (latitude and longitude at the $V_{\text{inf}}$-sphere) it has the form:

$$i = \arctg[V_{\text{inf}} \sin \varphi(V_p + V_{\text{inf}} \cos \varphi \cos \psi)^{-1}].$$

(3)

According coordinates of the inclination pole (IncPole) — the inclination extremum on the $V_{\text{inf}}$-sphere after performing any GM sequence — can be found:

$$T_{\text{pole}} \{\psi_{\text{pole}} = \pi; \varphi_{\text{pole}} = \pi - \arccos(V_{\text{inf}} / V_p)\} i_{\text{max}} \leq \arcsin(V_{\text{inf}} / V_p).$$

(4)

Table 1 shows the coordinates of for various values of IncPole. Coordinates of the point for the main resonances in a model case $V_{\text{inf}}/V_p = 0.5$ are given in Table 1. By definition, this point corresponds to the extremum of inclination angle on the line of fixed resonance.
Table 1. Coordinates of the point of maximum inclination on a fixed line of resonance in the case $V_{\text{inf}} = V_p / 2$

| Resonance | $\varphi^*, \text{grad}$ | $\psi^*, \text{grad}$ |
|-----------|--------------------------|--------------------------|
| 1:1       | 75.5                     | 180                      |
| 3:4       | 62.5                     | 180                      |
| 4:3       | 85.7                     | 180                      |
| 5:4       | 83.6                     | 180                      |
| 3:2       | 89.3                     | 180                      |
| 1:2       | 34.0                     | 180                      |
| 2:1       | 83.0                     | 0                        |
| 3:1       | 75.5                     | 0                        |

For arbitrary values latitude of point of the maximum resonance can be calculated analytically [2, 6]:

$$\cos \varphi^* = V_p / (2V_{\text{ven}}) = V_{\text{inf}} / 2$$

(5)

then, for example, for cosmic missions with the working velocity $V_{\text{inf}} = V_p / 2$ one obtains:

$$\varphi^* = \arccos \frac{1}{4} \approx 75.52^\circ.$$  

(6)

Table 2 displays the resources of rotation angles for one GM near the Earth group planets, corresponding to the required orbit inclination of the spacecraft. The analysis of the tables shows that the transitions between the principal resonances when performing a GM near Venus (according to Table 1, they require changing the current value of $\varphi^*$ by 9.8–13.3 degrees) are effective up to $i_{\text{max}} = 40^\circ$. Otherwise, it is necessary design a GAM sequence in accordance with a monoresonance. As we can see from Table 2, any GM near Mercury is ineffective for performing the transitions between resonances (the cells are marked dark). The GMs near Venus have the same property for $i_{\text{max}} > 40^\circ$ and near Mars for $i_{\text{max}} \geq 30^\circ$. Practically, the magnitude of inclination that ensures the execution of the main flight mission lies in a certain range of values. Using this property and the tables presented, the design value $V_{\text{inf}}$ can be refined by varying the inclination pole on the $V_{\text{inf}}$-sphere in order to reach the point with the maximum inclination on the line of the resonance chosen.

Table 2. Maximum possible rotation angles of the asymptotic velocity vector of a spacecraft during a single transit of terrestrial planets for the missions that differ in the orbital inclination required

| Planet     | First cosmic velocity km/s | $V_p$, km/s | $\varphi_{\text{max}}$ for $i_{\text{max}} = 20^\circ$ | $\varphi_{\text{max}}$ for $i_{\text{max}} = 30^\circ$ | $\varphi_{\text{max}}$ for $i_{\text{max}} = 45^\circ$ |
|------------|----------------------------|-------------|-----------------------------------------------------|-----------------------------------------------------|-----------------------------------------------------|
| Mercury    | 3.1                        | 47.3        | 4.0                                                  | 1.9                                                  | 1.1                                                  |
| Venus      | 7.2                        | 35.0        | 31.0                                                 | 16.7                                                 | 9.0                                                  |
| Earth      | 7.9                        | 29.7        | 44.1                                                 | 25.3                                                 | 14.1                                                 |
| Mars       | 3.5                        | 24.1        | 17.9                                                 | 9.1                                                  | 5.7                                                  |
4. Conclusion
The maximum possible value of the SC orbital inclination IncPole depends on the modulus \( V_{inf} \) of the SC asymptotic velocity on the flyby hyperbola and is directly proportional to it (the GM geometric limitation). However, the magnitude of the change in inclination at one GM is inversely proportional to \( V_{inf} \) (GM dynamic limitation). As a result, a compromise value of \( V_{inf} \) is existing, which varies for each specific case of a planet or a massive planet's satellite, when the GM is performed near it. Since in order to achieve a significant inclination, due to the GM dynamic limitation, not one GM, but a whole series of GMs may be required, in addition, each GM must ensure the "resonance" of the orbital periods of the planet and spacecraft after the GM in order to guarantee their new meeting. According the Jacobi integral existing in the restricted three-body problem the \( V_{inf} \) is an invariant during in all interplanetary flights using GMs. With help of cranking "jumps" along the resonance levels of the asymptotic velocity we can "scramble" from the initial inclination to the maximum possible point IncPole. This point may not belong to any isoline of one of the main resonances itself in the general case. A heuristic consideration consists in choosing (in the "resonant tuning") the such value of the design value of \( V_{inf} \), which ensures the localization of the inclination pole in the vicinity of one of the main resonance curves between the orbital periods of the spacecraft and the planet (resonant \( V_{res}^{inf} \)). Thus, with the help of the GM, it is possible to get to the IncPole by jumping along the resonance curve thru the minimum amount of GM. The paper describes the \( V_{res}^{inf} \) calculating formulas and gives its estimates for the Solar system and for the planets satellite systems.

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