Low Complexity Sequential Search with Measurement Dependent Noise

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Abstract—This paper considers a target localization problem where at any given time an agent can choose a region to query for the presence of the target in that region. The measurement noise is assumed to be increasing with the size of the query region the agent chooses. Motivated by practical applications such as initial beam alignment in array processing, heavy hitter detection in networking, and visual search in robotics, we consider practically important complexity constraints/metrics: time complexity, computational and memory complexity, query geometry, and cardinality of possible query sets.

Two novel search strategy, dyaPM and hiePM, are proposed. In contrast to previously proposed algorithms, dyaPM and hiePM are of a connected query geometry (i.e. query set is always a connected set). We also demonstrated how they can be implemented with low computational and memory complexity. Additionally, hiePM has a hierarchical structure and has a low cardinality of possible query sets. These make hiePM suitable for applications such as beamforming in array processing where the extra computation of the query set construction dictates a codebook-based approach (the choice of query set is constrained to a pre-computed small query set collection), and the limit of memory enforces a smaller codebook size.

Through a unified analysis with Extrinsic Jensen Shannon (EJS) Divergence, dyaPM is shown to be asymptotically optimal in search time complexity (asymptotic in both resolution (rate) and error (reliability)). On the other hand, hiePM is shown to be near-optimal in rate. In addition, via numerical examples, both hiePM and dyaPM are shown to outperform prior work in the non-asymptotic regime.

Index Terms—sequential search, measurement-dependent noise, Posterior Matching, Extrinsic Jensen Shannon Divergence

I. INTRODUCTION

We consider a target search problem where at any given time, an agent can choose a query set inspected for the presence of the target. More precisely, upon querying a set, the agent receives a noisy measurement indicating the presence of the target in the set. The agent conducts multiple queries where each query set can, in general, be chosen adaptively and strategically based on previous (noisy) measurements. The main focus of this paper is to design, analyze, and compare various search strategies under a realistic model where noise statistics depends on the size of the query set. (measurement-dependent noise model).

The problem of binary noisy search for a target with measurement-independent noise [1]–[7] have been studied extensively in the literature. Relying on connections with feedback coding, the authors in [2], [3] proposed a noisy variant of the binary search algorithm. Posterior Matching strategy proposed in [8] generalizes the noisy binary search algorithm to a general DMC case. In particular, [8] established the rate-optimality where the targeting rate is defined as the asymptotic ratio of the logarithm of the search resolution over the number of queries. By allowing for a random search time, [9] characterized the reliability of the Posterior Matching algorithm, where reliability is defined to be the asymptotic ratio of the logarithm of error probability over the (expected) number of queries. Relying on a connection to hypothesis testing [6] and [7] proposed two-phase schemes that achieve the optimal rate-reliability trade-off.

In many applications of interest, such as spectrum sensing [10] in cognitive radio, Angle-of-Arrival (AoA) estimation in initial beam alignment [11], and heavy hitter detection in networking [12], the noise statistics of the observation is usually measurement-dependent. In particular, querying a larger region results in a noisier measurement than querying a smaller region. With binary measurements and Bernoulli noise, for instance, this noise behavior means that the false alarm and miss detection of each query is a non-decreasing function of the size of the query set.

The problem of noisy search with measurement-dependent noise was first introduced in [6], where the author proposed a search strategy, maxEJS, that designs the query set by maximizing the Extrinsic Jensen Shannon (EJS) divergence (a function of the posterior) exhaustively over all possible query sets. However, the prohibitive complexity of the exhaustive maximization of EJS divergence renders maxEJS impractical in many applications. Furthermore, the asymptotic analysis in [6] failed to establish rate optimality of maxEJS.

The first optimal (in terms of rate-reliability) search strategy was proposed in [13], consisting of three phases of random search strategies; in this paper, we use the shorthand 3rand to refer to this algorithm. By allowing the second and third phases of the search to adapt to the outcome of the previous phase(s), the algorithm was shown to significantly outperform all non-adaptive strategies (in terms of both rate and reliability). This is in sharp contrast to the case of noisy search with measurement-independent noise where randomized non-

1Thus we will refer to the noisy variant of binary search algorithm as median Posterior Matching, medianPM
adaptive searches are known to perform asymptotically (rate) optimal\footnote{This is nothing but a manifestation of Shannon’s original analysis establishing that feedback cannot increase the capacity of DMC.}

While 3rand strategy is optimal in asymptotic sense, it suffers from three main shortcomings due to its essential reliance on random code constructions. Firstly, the algorithms computational complexity (decode/detection complexity) is rather prohibitive. Secondly, the query geometry is not constrained where the collection of possible query sets grows exponentially (in resolution). Thirdly, by construction, 3rand does not fully utilize the profile of the noise statistics, resulting in a rather poor non-asymptotic performance with significant sensitivity to the choice of hyper parameters associated with the three phases of the algorithm.

Motivated by practical applications such as initial beam alignment in array processing, heavy hitter detection in networking, and visual search in robotics, in this paper we study the problem of measurement-dependent noisy search working, and visual search in robotics, in this paper we study the problem of measurement-dependent noisy search

Moving to the details of our work, we consider three novel fully adaptive sequential search strategies: sorted Posterior Matching (sortPM), dyadic Posterior Matching (dyaPM) and hierarchical Posterior Matching (hiePM). We analyze these algorithms by quantifying the step-by-step extrinsic Jensen-Shannon divergence.

In particular, sortPM is shown to optimize the asymptotic time complexity. dyaPM achieves similar time complexity while ensuring that the agent can only query connected sets. Lastly, we design and analyze hiePM by further limiting the query sets to be the sets restricted to those that can be represented as a decision tree, such as the bisection search set. This property of hiePM means that the collection of query sets forms a hierarchical cover of the search area hence is of small cardinality; hence hiePM is suitable for applications such as beamforming in array processing that requires pre-construction and storage of the possible query sets.

In addition to desirable time complexity, all three strategies, we show, can be implemented with low computational and memory complexities. In particular, we provide an exponential improvement in computational and memory complexities of the proposed strategies over the search strategies in the literature of measurement-dependent noisy search (maxEJS and 3rand). Furthermore, through a set of practically motivated numerical examples, we show that all the proposed search strategies have superior non-asymptotic performance compared with that of 3rand. Notably, we demonstrate superior performance of hiePM despite the lack of theoretical guarantee regarding its asymptotic optimality.

Notations: We use boldface letters to represent vectors. We write \( \pi \) to denote sorted element of the vector \( \pi \) in descending order, i.e., \( \pi_i \) represents the \( i \)th largest element of \( \pi \). For a set of indices \( S \), we write \( \pi_S \equiv \sum_{i \in S} \pi_i \). We denote the space of probability mass functions on set \( X \) as \( P(\cdot) \).

We denote the Kullback-Leibler (KL) divergence between distribution \( P \) and \( Q \) by \( D(P\|Q) = \sum_x P(x) \log \frac{P(x)}{Q(x)} \).

The mutual information between random variable \( X \) and \( Y \) is defined as \( I(X;Y) = \sum_{x,y} p(x,y) \log \frac{p(x,y)}{p(x)p(y)} \), where \( p(x,y) \) is the joint distribution, and \( p(x) \) and \( p(y) \) are the marginals of \( X \) and \( Y \). Let \( \text{Bern}(p) \) denote the Bernoulli distribution with parameter \( p \), and \( I(q,p) \) denote the mutual information of the input \( X \sim \text{Bern}(q) \) and the output \( Y \) of a BSC channel with crossover probability \( p \). Let \( C_1(p) := D(\text{Bern}(p)||\text{Bern}(1-p)) \). Let \( \mathbb{E}[\cdot] \) denote the expectation. We use \( |S_t| \) to represent the counting measure of a discrete set \( S_t \) (cardinality of \( S_t \)).

II. PROBLEM SETUP

We consider the problem of searching for a point target in a unit interval. The target is uniformly placed on the unit interval. We wish to estimate the target position to a particular resolution \( \frac{1}{2} \delta \). Given a target resolution \( \frac{1}{2} \delta \), determined and fixed in advance, without loss of generality, we can discretize the problem by quantizing the area into \( \delta \) sub-intervals before the search process begins: More precisely, let us divide the unit interval \([0, 1]\) into \( \frac{1}{2} \delta \) sub-intervals (referred to as bins), where \( \frac{1}{2} \delta \), without loss of generality, is assumed to be an integer. Let \( \theta \) be the index of the bin that contains the target.

We wish to estimate \( \theta \) by sequentially choosing (possible random) any query sets \( S_t \subset \{1, 2, ..., \frac{1}{2} \delta \} \). Let \( X_t = 1(\theta \in S_t) \) denote the clean binary signal indicating whether the target is in the query set \( S_t \). The agent obtains a noisy version \( X_t \), denoted by \( Y_t \):

\[
Y_t = X_t \oplus Z_t(S_t),
\]

where \( \oplus \) denotes exclusive OR operation, and \( Z_t(S_t) \) is a Bernoulli noise random variable whose statistics depends on the query set \( S_t \). In particular, we assume that \( Z_t(S_t) \sim \text{Bern}(p(\delta(S_t))) \) where \( p : (0,1) \to (0,\frac{1}{2}) \) is a continuous and non-decreasing function. We assume that the noise is conditionally (conditioned on \( S_t \)) i.i.d. across time.

After \( \tau \) queries, the agent declares the target index \( \hat{\theta} \). The search is said to have resolution \( \frac{1}{2} \delta \) and reliability \( \epsilon \) if

\[
\Pr( | \hat{\theta} - \theta | \leq \delta ) \geq 1 - \epsilon.
\]

A sequential causal strategy selects random query set \( S_t \) as a measurable random variable of the past decisions and observations \( (S_t^{t-1}, Y_t^{t-1}) \), and makes a declaration of the estimate \( \hat{\theta} \) at a stopping time \( \tau \). We say a strategy is fixed-length if stopping time \( \tau \) is a deterministic time selected independently of the observation sequences \( Y_1^{t-1} \); otherwise, we say it is variable-length. We say a strategy is non-adaptive if the selection of \( S_t \) is made independently of the realization of the past observations; this is in contrast to a strictly adaptive policy where the selection of \( S_t \) explicitly depends on the observation sequence \( Y_1^{t-1} \).

In this work, we consider the case where the decision has zero initial side information about the location of the target, hence, a uniform Bayesian prior \( \pi_i(0) := \Pr(\theta = i) = \delta \). By
this Bayesian framework, the belief vector $\pi(t)$, where its $i_{th}$ component at time $t$ is given as

$$\pi_i(t) = \mathbb{P}(\theta = i \mid Y_1^{t-1}, S_1^{t-1}),$$

is a sufficient statistics. In other words, any deterministic stationary adaptive strategy can be denoted by a function

$$\gamma : \Delta_\delta \rightarrow \{1, 2, \ldots, \frac{1}{\delta}\}$$

where $\Delta_\delta$ is the probability simplex of dimension $\frac{1}{\delta}$.

We characterize the performance of search strategies by the following:

i) The Query Time Complexity:

We are interested in search strategies that can find the target location accurately (with resolution $\frac{1}{\delta}$) and reliability (with confidence $1 - \epsilon$) as quickly as possible. We measure the asymptotic time complexity by how the (expected) number of queries, $\tau_{\epsilon, \delta}$, scales with the resolution $\frac{1}{\delta}$ and the reliability $\epsilon$. Lastly, we use the rate-reliability pair to capture the asymptotic query time complexity:

**Definition 1.** A family of search strategies $\gamma_{\epsilon, \delta}$ with resolution $\frac{1}{\delta}$, reliability $\epsilon$, and stopping time $\tau_{\epsilon, \delta}$ are said to achieve a maximum rate $R$ and a maximum reliability $E$ respectively if and only if

$$R = \lim_{\delta \rightarrow 0} \frac{\log(\frac{1}{\delta})}{\tau_{\epsilon, \delta}}, \quad E = \lim_{\epsilon \rightarrow 0} \frac{\log(\frac{1}{\epsilon})}{\tau_{\epsilon, \delta}}.$$  

ii) The Computational and Memory Complexity:

There are memory and computational requirements for computing the query set $S_t$ at every query time $t$, as well as computing the final estimate $\hat{\theta}$. Specifically, adaptive selection of $S_t$ requires updating the posterior vector. There is also the computation complexity associated with the mapping $\gamma$ from $\pi(t)$ to the next query set $S_{t+1}$.

iii) The Query Geometry and the Query Cardinality:

In many practical settings, the choice of the query set $S_t$ cannot be arbitrary. Let $A \subseteq 2^{\{1, 2, \ldots, \frac{1}{\delta}\}}$ be the set of allowable query sets, i.e. consider $S_t \in A \subseteq 2^{\{1, 2, \ldots, \frac{1}{\delta}\}}$.

We evaluate the algorithms in terms of the geometric complexity of sets in $A$. One practically relevant choice of $A$, motivated by the visual search [4] and initial beam alignment (run in parallel) applications, is when $A = \mathcal{I} := \{i \mid a \leq i \leq b \mid 1 \leq a < b < \frac{1}{\delta}\}$, i.e. when the query sets are constrained to be contiguous intervals. In such case, we say that the search strategy is with a connected/contiguous query geometry. In fact, we will see that the connected query geometry of $\mathcal{I}$ offers an immediate reduction of computational and memory complexity in tracking the posterior.

Furthermore, the cardinality of $A$ determines the memory footprint of the algorithm. A smaller query cardinality is favorable for applications where the construction of the query set itself is non-trivial and a pre-construction with a codebook-based approach is preferable (e.g. the beam alignment problem in [11]). Hence, we characterize the query cardinality of the algorithms as the cardinality of $A$.

To get an understanding of the importance of the query geometry, let us present the reduction of computational and memory complexity just by the constraint of connected query geometry. We see this through the following lemma:

**Lemma 1.** For connected query geometry $S_n \in \mathcal{I} := \{i : a \leq i \leq b \mid 1 \leq a < b < \frac{1}{\delta}\}$, $n = 1, 2, \ldots$ t with uniform prior $\pi_i(0) = \delta$ for all $i$, the posterior at time $t$ can be written as a simple function with at most $2t + 1$ intervals. Specifically, there exist a sequence of disjoint partition of $[\frac{1}{\delta}] = \bigcup_{u=0}^{2t+1} I_u^{(t)}$,

$$I_u^{(t)} \in \mathcal{I}$$

such that

$$\pi_i(t) = \sum_{u=0}^{2t+1} \frac{\pi_i^{(t)}}{I_u^{(t)}} \mathbb{1}_{I_u^{(t)}}(i), \quad t = 1, 2, \ldots$$

The proof of lemma [1] follows from Procedure [1] in particular, the complexity of tracking the complexity of tracking the posterior is of order $O(\tau)$ under the connected query geometry. In other words, restricting queries to contiguous intervals offers a logarithmic order of reduction for the computational and memory complexity from $O(\frac{1}{\delta})$ to $O(\log \frac{1}{\delta})$ (we will show that $\tau = O(\log \frac{1}{\delta})$ for all the proposed algorithms).

**Procedure 1 (Bayes’ Rule with sequential binning)**

1. **Input:** $(\pi^{(t)}(t), \mathcal{I}(t), S_{t+1}, Y_{t+1})$ where $S_{t+1} = \{i : s_1 \leq i \leq s_2\};$
2. **Output:** $(\pi^{(t+1)}(t+1), \mathcal{I}(t+1))$;
3. Find $I_{u_1}^{(t)}$, $I_{u_2}^{(t)}$ such that $s_1 \in I_{u_1}^{(t)}$ and $s_2 \in I_{u_2}^{(t)}$;
4. for $0 \leq u < t_1$ do
5. \[ I_{u_1}^{(t+1)} = \left( I_{u_1}^{(t)} \cap \pi^{(t)}(t) \right) \cap \pi^{(t+1)}(t) \];
6. \[ I_{u_2}^{(t+1)} = \left( I_{u_2}^{(t)} \cap \pi^{(t)}(t) \right) \cap \pi^{(t+1)}(t) \];
7. $I_{u_1}^{(t+1)} = \left[ s_1, \max I_{u_1} \right]$, $\pi^{(t+1)}(t+1) = \frac{\mathbb{1}_{I_{u_1}^{(t+1)}}}{\mathbb{1}_{I_{u_1}^{(t+1)}}} \pi^{(t)}(t+1)$;
8. for $t_1 + 2 \leq u < t_2 + 1$ do
9. $I_{u_1}^{(t+1)} = \left[ u_{u-1}, \pi^{(t)}(t) \right] \pi^{(t+1)}(t+1)$;
10. $I_{u_2}^{(t+1)} = \left[ \min I_{u_2}, s_{u+1} \right]$, $\pi^{(t+1)}(t+1) = \frac{\mathbb{1}_{I_{u_2}^{(t+1)}}}{\mathbb{1}_{I_{u_2}^{(t+1)}}} \pi^{(t)}(t+1)$;
11. $I_{u_1}^{(t+1)} = \left[ s_2, \max I_{u_2} \right]$, $\pi^{(t+1)}(t+1) = \frac{\mathbb{1}_{I_{u_1}^{(t+1)}}}{\mathbb{1}_{I_{u_1}^{(t+1)}}} \pi^{(t)}(t+1)$;
12. for $t_2 + 3 \leq u < 2t + 2$ do
13. $I_{u_1}^{(t+1)} = \left( u_{u-2}, \pi^{(t)}(t) \right) \pi^{(t+1)}(t+1)$;
14. # Bayes’ rule:

$$\pi^{(t+1)}(t+1) = \frac{\pi^{(t+1)}(t) \mathbb{P}(Y_{t+1} \mid X_t + 1 = \mathbb{I}(t + 1 \leq u \leq t_2 + 1 + 1))}{\sum_{u' = 0}^{2t + 2} \pi^{(t+1)}(t) \mathbb{P}(Y_{t+1} \mid X_t + 1 = \mathbb{I}(\theta \in I_{u')})}$$

III. PROPOSED SEARCH STRATEGIES AND MAIN RESULTS

In this section, we introduce three proposed search strategies: sorted Posterior Matching (sortPM), dyadic Posterior Matching (dyadPM), and hierarchical Posterior Matching (hierPM). We give a summary of our main results and complexity in terms of rate (time complexity), computational and
memory complexity, as well as query geometry and cardinality in Table I. The three proposed algorithms are all variants of binary Posterior Matching proposed originally by Horstein [2], and later analyzed by [8], [9], [14]. As such, we will first provide a description of Posterior Matching here: Posterior Matching strategy queries the bins to the left of posterior median. In other words,
$$S_{t+1}^{PM} = \gamma_{PM}(\pi_t) = \{i : i \leq k_{PM}^t\},$$
where $k_{PM}^t$ is the bin index closet to the posterior median, i.e. $k_{PM}^t = \arg \min_k \{\pi_{[1,k]}(t) - \frac{1}{2}\}$.

**Remark 1.** By construction, $\mathbb{P}(X_{t+1} \mid Y^t_i) \approx \frac{1}{2}$. In other words, under Posterior Matching, $X_t$ has the desirable property of maximum (conditional) entropy. However, the measurement noise, whose variance increases with the size, could be excessively large. *sortPM*, described next, addresses this issue.

### A. Sorted Posterior Matching

Under Sorted Posterior Matching (*sortPM*) strategy, the posterior matching step is preceded by sorting operation on the posterior vector. In particular, consider the sorted posterior $\pi^{(t)}$ and the corresponding sorting operation $\sigma_t$: $\pi_t(t) \equiv \pi_{\sigma_t(t)}(t)$. Let $k^* = \arg \min_k \{\pi_{[1,k]}(t) - \frac{1}{2}\}$, under *sortPM*,
$$S_{t+1} = \gamma_{s}(\pi(t)) = \{i : \sigma_t(i) \in [1,k^*]\},$$
is queried.

**Theorem 1.** The expected search time of *sortPM* of achieving resolution $\delta > 0$ and reliability $0 < \epsilon < 1$ can be upper bounded by
$$E[\tau_{e,\delta}] \leq \frac{\log(1/\delta)}{I(1/2,p_\alpha)} + \frac{\log(1/\epsilon)}{C_1(p_\delta)} + o\left(\frac{1}{\delta}\right),$$
for any fixed $\alpha > (e\log \frac{1}{M})^{-K_*}$, where $K_* > 0$ a constant defined in Lemma [3].

**Remark 2.** By first taking $\delta \to 0$ and then $\alpha \to 0$, Theorem [1] together with the corresponding converse theorem [Theorem 1 in [6]] implies that *sortPM* achieves the best possible acquisition rate $I(1/2,p_{\min})$ and the best reliability exponent $C_1(p_\delta)$ (by taking $\epsilon \to 0$).

**Remark 3.** Even though *sortPM*, as well as prior works such as maxEJS [6] and 3-phase random search [15], are asymptotically optimum in time complexity under measurement-dependent noise, they, in general, do not admit any constraint on the query set they choose. In other words, the query cardinality of these algorithms are of a prohibitive order $O(2^t)$. Furthermore, the unconstrained query geometry prevents the applicability to many applications where connected query set or other specific geometry is preferred (such as visual search [6]). *HiePM*, described next, restricts the query set.

Algorithm 1: Sorted Posterior Matching

1. **Input:** resolution $\frac{1}{2^t}$, error probability $\epsilon$, fixed stopping time $n$, stopping-criterion
2. **Output:** estimate of the target location $\hat{\theta}$ after $\tau$ queries
3. **Initialization:** $\pi_t(0) = \delta$ for all $i = 1, 2, ..., 1/\delta$.
4. for $t = 0, 1, ...$
do
5. 
6. # Design the search region by sorted posterior
7. $k^* = \arg \min_k \{\pi_{[1,k]}(t) - 1/2\}$
8. $S_{t+1} = \gamma_{s}(\pi(t)) = \{i : \sigma_t(i) \in [1,k^*]\},$
9. # Take next measurement
10. $Y_{t+1} = \mathbb{I}(\theta \in S_{t+1}) \oplus Z_{t+1}$
11. # Posterior update by Bayes’ Rule
12. $\pi(t+1) \leftarrow Y_{t+1}, \pi(t)$
13. # Stopping criteria
14. case: stopping-criterion = fixed length (FL)
15. if $t+1 = n$ then
16. break;
17. case: stopping-criterion = variable length (VL)
18. if $\max_i \pi_t(t+1) > 1 - \epsilon$ then
19. break;
20. $\tau = t + 1$ (length of the search)
21. $
\hat{\theta} = \arg \max_i \pi_t(\tau)$

#### B. Hierarchical Posterior Matching

Motivated by the need of the connected query geometry, here we proposed a novel low-complexity search strategy which we call Hierarchical Posterior Matching, *HiePM*. *HiePM* utilizes the hierarchical query geometry that is used in the noiseless binary search. For the brevity of presentation, we assume that $\frac{1}{\delta} = 2^{L}$ for some $L > 0$. The hierarchical query geometry is therefore written as $\mathcal{H} = \{\mathcal{H}^m_{i}: l = 0, 1, 2, ..., m = 0, 1, 2, ..., 2^{L} - 1\}$ where $\mathcal{H}^m_{i} = \{m2^{L-l} + 1, m2^{L-l} + 2, ..., (m+1)2^{L-l}\}$. This query geometry, as shown in Fig. 1, can be represented by a binary tree recursively by
$$H^n_{1} = \mathcal{H}^{2m+1}_{i} \cup \mathcal{H}^{2m+1}_{i+1}, \quad l = 0, 1, 2, 3, ..., L.$$
Matching hierarchically along the binary tree as follows. Let
\[
\begin{align*}
I^*_t & = \arg \max_I \left\{ \max_m \pi_{H^m}(t) \geq \frac{1}{2} \right\}, \\
M^*_t & = \arg \max_m \pi_{H^m}(t),
\end{align*}
\]
and the hierarchical posterior matching with
\[
(l_{t+1}, m_{t+1}) = \arg \min_{(l', m') \in \{(l^*_t, m^*_t), (l^*_t + 1, 2m^*_t), (l^*_t + 1, 2m^*_t + 1)\}} \left| \pi_{H^{m'}}(t) - \frac{1}{2} \right| .
\]
In other words, \((l_{t+1}, m_{t+1})\) identifies the node one the decision tree \(H\) with the posterior closest to \(\frac{1}{2}\). As such, querying \(S_{t+1} = H_{l_{t+1}}^{m_{t+1}}\) ensures a high conditional entropy, while the size of the set is kept small to ensure near optimal time complexity: (See Algorithm 2 for more details on the construction of \(hiePM\)).

**Theorem 2.** The expected search time of \(hiePM\) for achieving resolution \(\delta > 0\) and reliability \(0 < \epsilon < 1\) can be upper bounded by
\[
E[\tau, \epsilon] \leq \frac{\log(1/\delta)}{I(1/3, p[2^{-l}])} + \frac{\log(1/\epsilon)}{C_1(p[\delta])} + o\left(\frac{1}{\delta \epsilon}\right),
\]
for any fixed \(l > 0\) such that \(2^{-l} > (\epsilon \log \frac{1}{\delta \epsilon})^{-K_h}\), where \(K_h > 0\) is a constant defined in Lemma 2.

**Remark 4.** As shown in Algorithm 2 both the computational and memory complexity of \(hiePM\) are dominated by the posterior representation \(\pi_{Z(t)}\), \(Z(t)\) in Procedure 1. By Theorem 2 we know that the computational and memory complexity is of order \(O(\log \frac{1}{\delta \epsilon})\).

**Remark 5.** The hierarchical query geometry \(H\) not only is connected but also is of a hierarchical structure, which is suitable for the applications such as heavy hitter detection in networking [12] (monitoring pre-fix IP addresses) and bit-wise coding [16]. Furthermore, the query cardinality is only \(|H| = O(\frac{1}{\delta})\), rendering \(hiePM\) a great candidate for beamforming applications [11].

**Remark 6.** Taking \(\epsilon \to 0\), we see that \(hiePM\) achieves the best possible error exponent \(C_1(p_{\min})\). However, the achievable acquisition rate of \(hiePM\) by Theorem 2 is only \(I(1/3, p_{\min}) < I(1/2, p_{\min})\). This, we believe, is a byproduct of our analysis that loosely bounds the posterior distribution of \(X\). The best achievable acquisition rate when we restrict the query area \(S_{t+1}\) to \(H\) remains.

### C. Dyadic Posterior Matching

By using the hierarchical query \(H\), \(hiePM\) gives a solution that allows for constraints on the connectedness of query
geometry. To ensure the optimality in time complexity, we proposed another low-complexity search strategy which we call dyadic Posterior Matching, \textit{dyaPM}.

By the same procedure as in \textit{hiePM}, \textit{dyaPM} first finds the smallest binary interval that contains more than half posterior, i.e. \( H^{m_1}_t = \{ m_1^2 L - t^l_1 + 1, m_1^2 L - t^l - 1, \ldots, (m_1^2 + 1) 2L - t^l_1 \} \) as in equation (13). The \textit{dyaPM} algorithm then applies the Posterior Matching within \( H^{m_1}_t \) by potentially appending/excluding additional bins:

\[
S_{t+1} = [m_1^2 2^{L - t^l_1} + 1, k^*]
\]  

(17)

where \( k^* = \arg \min_k [m_1^2 2^{L - t^l_1} + 1, k)] \) (The whole procedure of \textit{dyaPM} is summarized in Algorithm 3).

**Remark 8.** As shown in Algorithm [3], both the computational and memory complexity are again dominated by tracking the posterior representation \( \pi_{T(t)}, I(t) \) in Procedure [4] By Theorem [3] we know that the computational and memory complexity of \textit{dyaPM} is of order \( O(\log \frac{1}{\varepsilon}) \).

**D. Asymptotic Results**

To illustrate the asymptotic results from Theorem [12] in Fig. 2 we illustrate the achievable rate-reliability \((R, E)\) pair for an example where we use the noise profile \( p[x] = 0.1 + 0.5x \), \( 0 \leq x \leq \frac{1}{2} \). Note that the blue line not only represents the achievable \((R, E)\) pair of the corresponding algorithms, it also illustrates the converse theorem.

![Fig. 2: Achievable rate-reliability region](image)

The noise profile is set to be \( p[x] = 0.1 + 0.5x \), \( 0 \leq x \leq \frac{1}{2} \). (this means \( p_{\min} = 0.1, p_{\max} = p[\frac{1}{2}] = 0.35 \).)

**IV. Numerical Examples**

In this section, we give numerical comparisons amongst various algorithms. We study the error probability versus the number of queries at a fixed resolution. We compare all the algorithms under measurement-dependent Bernoulli noise where \( p[\delta|S]| = 0.1+\delta|S|/2 \). For applications of our algorithm under non-Bernoulli noise profile, we refer readers to [11]. We use algorithm-VL to represent the variable length termination of the algorithm. Likewise, we use algorithm-FL to represent the fixed length termination of the algorithm.

As we see in Fig. 3 the proposed algorithms \textit{sortPM}, \textit{dyaPM}, and \textit{hiePM} all enjoy the optimal error exponent \( C_1(p[\delta]) \) with variable length (VL) operation for both measurement independent and measurement-dependent noise, as predicted by Theorem 1-3. We also note that, despite the restriction of contiguous query area, \textit{dyaPM} and \textit{hiePM} perform almost the same as \textit{sortPM} both asymptotically and non-asymptotically in reliability. As expected, the classic PM performs rather poorly. On the other hand, while 3rand is also asymptotically optimal in reliability with VL operation, we note a non-negligible non-asymptotic performance drop compared to our proposed algorithms.
strategies but also their systematic way that bridges theoretical studies of noisy search by actively querying examples for labels.

Theorem 2, we know that the (expected) number of queries of interesting extension of this paper. On the other hand, by settings such as a target localization using drone [17] is one that can be bisected. Applying hiePM applications.

V. CONCLUSION

Our formulation of the four different complexities shows a systematic way that bridges theoretical studies of noisy search problem with practical engineering problem. Not only the low time/computational/memory complexity of the proposed strategies but also their query geometry is shown to be suitable for practical applications. Particularly, restricting the query set with the hierarchical query geometry is found to be useful in the initial beam alignment problem in wireless communication [11]. Thanks to the Bayesian framework, our algorithms also adapts to different noise statistics (such as Poisson statistics in heavy hitter detection in networking), making our proposed algorithm potentially applicable in in many other target search applications.

By the hierarchical query geometry, hiePM also offers a natural generalization to a higher dimension or any structure that can be bisected. Applying hiePM to more practical settings such as a target localization using drone [17] is one of interesting extension of this paper. On the other hand, by Theorem 2 we know that the (expected) number of queries grows only linearly in the number of dimensions. This benefit also renders hiePM suitable for active learning problem where a learner tries to learn a classifier in multi-dimension by actively querying examples for labels.

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APPENDIX A

PRELIMINARIES: AVERAGE LOG-LIKELIHOOD AND THE EXTRINSIC JENSEN-SHANNON DIVERGENCE

In this subsection, we review some useful concepts in [18]. The average log-likelihood of the posterior is defined as

\[ U(t) \equiv U(\pi(t)) := \sum_{i=1}^{1/\delta} \pi_i(t) \log \frac{\pi_i(t)}{1-\pi_i(t)}, \]

with the following property:

1) \( U(t) \) is a submartingale with drift \( EJS. \)

\[ \mathbb{E}[U(t+1) \mid \pi(t)] = U(t) + EJS(\pi(t), \gamma), \]

where \( EJS \) is the Extrinsic Jensen-Shannon divergence, defined as

\[ EJS(\pi(t), \gamma) = \sum_{i=1}^{1/\delta} \pi_i(t) D \left( P_{y|S_{i+1}} \parallel P_{y|S_{i+1}\neq i,S_{i+1}} \right) \]

(21)
with
\[ P_{y_{t+1}\mid S_{t+1}} := \mathbb{P}(Y_{t+1} = y_{t+1} \mid \theta = i; S_{t+1} = \gamma(t)) \]
\[ = \mathbb{P}(Y_{t+1} = y_{t+1} \mid X_{t+1} = 1) \quad (i \in S_{t+1}) \]
and
\[ P_{y_{t+1}\neq i, S_{t+1}} := \mathbb{P}(Y_{t+1} = y_{t+1} \mid \theta \neq i; S_{t+1}) \]
\[ = \sum_{j \neq i} \frac{\pi_j(t)}{1 - \pi_i(t)} P_{y_{t+1}\mid j, S_{t+1}}, \]
\[ (22) \]
(3) Level crossing of \( U \) is directly related to the error probability, since \( \pi_i(t) < 1 - \epsilon \forall i \Rightarrow U(t) < \log \frac{1}{\epsilon} \).

Analyzing the random drift from time 0 with the initial value \( U(0) = -\log(\frac{1}{\epsilon} - 1) \) is directly related to the logarithm of resolution and hence the targeting rate.

\[ \text{EJS}(\pi(t), \gamma) \geq J_S(\pi(t), \gamma), \]
where \( J_S(\pi(t), \gamma) \) is the Jensen-Shannon (JS) divergence.

\[ \text{EJS}(\pi(t), \gamma) \geq J_S(\pi(t), \gamma), \]
\[ (32) \]
Fact 4 (Lemma 2 in [18]). The EJS divergence is lower bounded by the Jensen Shannon (JS) divergence:

\[ \text{EJS}(\pi(t), \gamma) \geq J_S(\pi(t), \gamma), \]
\[ (32) \]
with \( J_S(\pi(t), \gamma) \) defined as:

\[ J_S(\pi(t), \gamma) = \sum_{i \in I} \pi_i(t) \mathbb{D}(P_{y_{t+1}\mid i, S_{t+1}} \mid P_{y_{t+1}\mid \neq i, S_{t+1}}), \]
\[ (33) \]
Fact 5. Using the search strategy \( \text{hiePM} \) with resolution \( 1/\delta \) and reliability \( \epsilon \) on codebook \( \text{WE} \) with \( L = \log_2(1/\delta) \), we have

\[ \text{EJS}(\pi(t), \gamma_t) \geq J_S(\pi(t), \gamma_t), \quad \forall t \]
\[ (34) \]
\[ \text{EJS}(\pi(t), \gamma_t) \geq \tilde{\pi} C_1(p_{\min}), \quad \forall \max \pi_i \geq \tilde{\pi}, \]
\[ (35) \]
where \( \tilde{\pi} := 1 - \frac{1}{1 + \max\{\log(1/\delta), \log(1/\epsilon)\}}. \)

Proof. The proof of Fact 5 is a modification of proof of Proposition 3 in [18] by using Fact 4. We first prove equation (35). By the selection rule of \( \text{hiePM} \), the last level codebook \( S_{t+1} = D(t_{t+1}|\log_2(\frac{1}{\delta})) \) is used whenever \( \max \pi_i(t) \geq \tilde{\pi} > 1/2. \) Therefore,

\[ \text{EJS}(\pi(t), \gamma_t) \geq \frac{1/\delta}{1} \sum_{i \in I} \pi_i(t) \mathbb{D}(P_{y_{t+1}\mid i, S_{t+1}} \mid P_{y_{t+1}\mid \neq i, \gamma_t}), \]
\[ (36) \]
It remains to show equation (34). For notational simplicity, let

\[ \rho \equiv \pi_{D^t+1}(t) := \sum_{i \in D^t+1} \pi_i(t) \]
\[ (37) \]
and $B_0 = \text{Bern}(p[l_{t+1}])$, $B_1 = \text{Bern}(1-p[l_{t+1}])$. We separate the proof into two cases:

If $\rho > 2/3$, we know that $l_{t+1} = \log_2(\frac{1}{\alpha})$ by the selection rule of $\text{hiePM}$. Therefore, the set $D^{l_{t+1}}$ is of the smallest size 1. Let $D^{l_{t+1}} = \{l_{t+1}\}$, we have

$$EJS(\pi(t), \gamma_h) = \sum_{i=1}^{1/\delta} \pi_i(t) D \left( P_{\gamma_{t+1}[i, \gamma_h]} \| P_{\gamma_{t+1}[i \neq l_{t+1}, \gamma_h]} \right)$$

$$= \rho D_B(B^1 \| B^0)$$

$$+ \sum_{i \neq l_{t+1}} \pi_i(t) D \left( B^0 \| 1 - \pi_i(t) B^1 + \frac{1 - \rho - \pi_i(t)}{1 - \pi_i(t)} B^0 \right)$$

$$\geq D \left( B^0 \| \frac{1}{2} B^1 + \frac{1}{2} B^0 \right)$$

$$= I(1/2, p[l_{t+1}]) \geq I(1/3, p[l_{t+1}])$$

where (a) is by the fact that $D(B^1 \| B^0) = D(B^0 \| B^1)$ and that $D(B^0 \| \rho B^1 + (1 - \rho) B^0)$ is increasing in $\alpha$ for $0 \leq \alpha \leq 1$, together with $\frac{\rho}{\pi_i(t)} > 2/3 > 1/2$.

For the other case where $\rho \leq 2/3$, again by the selection rule of $\text{hiePM}$, we have $1/3 \leq \rho \leq 2/3$. Now we can lower bound the $EJS$ as

$$EJS(\pi(t), \gamma_h) \geq J_S(\pi(t), \gamma_h)$$

$$= \rho D_B(B^1 \| \rho B^0 + (1 - \rho) B^1)$$

$$+ (1 - \rho) D_B(B^0 \| \rho B^0 + (1 - \rho) B^1)$$

$$= I(\rho, p[l_{t+1}]) \geq I(1/3, p[l_{t+1}])$$

where (a) is by Fact 4 and (b) is by the concavity of the mutual information with respect to the input distribution, the symmetric of $I(\rho, p[l_{t+1}])$ around $\rho = 1/2$ for symmetric channels, and together with $1/3 \leq \rho \leq 2/3$. This concludes the assertion.

A. Upper-bounding the Expected Search Time with Measurement-Dependent Noise

From the expected search time upper bound via the use of EJS (Fact 1) and the search size $\delta[S_{t+1}]$ dependent lower bound of EJS given in Fact 2 and Fact 3 we see that intuitively we need $I(1/2, p[\delta[S_{t+1}]]$ or $I(1/3, p[\delta[S_{t+1}]]$ to be large, or equivalently the size of the search region $|S_{t+1}|$ to be small, in a certain sense. In particular, we can handle the search size shrinkage in a probabilistic manner by providing an exponentially decay tail. Indeed, we have the following proposition:

**Lemma 2.** Given any search strategy $\gamma$ with $\delta|S_{t+1}| \leq 1/2$ and

$$EJS(\pi(t), \gamma) \geq R(\delta|S_{t+1}|), \ \forall \ t$$

$$EJS(\pi(t), \gamma) \geq \tilde{\pi} E, \ \forall \ max \ \pi_i \geq \tilde{\pi},$$

for some $R(\delta|S_{t+1}|) > 0$ increasing in $\delta|S_{t+1}|$ and $E > 0$. If further

$$P(\delta \ | \ S_{t+1} \ | \ > \alpha) \leq k_0 e^{-t E_0}, \ \forall \ t > T_0$$

for some $1/2 > \alpha > \delta$, $k_0 > 0$, $E_0 > 0$, and $T_0 > [\log(\frac{1}{\delta})]$, the expected time of the strategy $\gamma$ achieving resolution $1/\delta$ and reliability $\epsilon$ can be upper bounded by

$$E[\tau_{\epsilon, \delta}] \leq \log(1/\delta) + \log(1/\epsilon) R(\alpha) + g_{R, \epsilon}(\epsilon, \delta),$$

where

$$g_{R, \epsilon}(\epsilon, \delta) := \frac{k_0 e^{2E_0}}{(1 - e^{-\epsilon}) (\log \frac{1}{\delta}) E_0} \times$$

$$\left( \left[ \log \frac{1}{\delta} \right]_e + \log \frac{1}{\delta} \frac{1}{R(1/2)} + \frac{1}{E} + f_{R(1/2), \epsilon}(\epsilon, \delta) \right)$$

$$+ \left( \frac{k_0 e^{-2E_0}}{(1 - e^{-\epsilon})^2 (\log \frac{1}{\delta}) E_0} + \log \frac{1}{\delta} \right) + f_{R(\alpha), \epsilon}(\epsilon, \delta)$$

is of $\alpha(\frac{1}{\delta})$ as $\delta \to 0$ or $\epsilon \to 0$.

**Proof.** We prove this proposition via the total probability theorem and the re-start of the time homogeneous Markov chain $\pi(t)$. Specifically, let us define the “bad” event $E_t = \{\delta|S_{t+1}| > \alpha\}$ and the “good” event $F_n = \bigcup_{t=n}^{\infty} E_t$. For every $n$, by total probability theorem and the union bound, we have

$$E[\tau_{\epsilon, \delta}] = \sum_{t=n}^{\infty} P(E_t) \left[ \tau_{\epsilon, \delta} \right] + \sum_{t=n}^{\infty} E[\tau_{\epsilon, \delta} \ | \ \pi(t)] \left[ t + \log \frac{1}{\delta} \frac{1}{R(1/2)} + \log \frac{1}{E} + f_{R(1/2), \epsilon}(\epsilon, \delta) \right]$$

$$+ \sum_{t=n}^{\infty} F_n \left[ t + \log \frac{1}{\delta} \frac{1}{R(1/2)} + \log \frac{1}{E} + f_{R(1/2), \epsilon}(\epsilon, \delta) \right]$$

$$\leq \sum_{t=n}^{\infty} P(E_t) \left[ t + \log \frac{1}{\delta} \frac{1}{R(1/2)} + \log \frac{1}{E} + f_{R(1/2), \epsilon}(\epsilon, \delta) \right]$$

$$+ \sum_{t=n}^{\infty} P(E_t) \left[ t + \log \frac{1}{\delta} \frac{1}{R(1/2)} + \log \frac{1}{E} + f_{R(1/2), \epsilon}(\epsilon, \delta) \right]$$

$$+ n + \log \frac{1}{\delta} \frac{1}{R(1/2)} + \log \frac{1}{E} + f_{R(\alpha), \epsilon}(\epsilon, \delta),$$

where $f_{R, \epsilon}(\epsilon, \delta)$ is as defined in Fact 4. Here (a) follows from the time homogeneity of the Markov chain $\pi(t)$ re-starting at time $t$, together with Fact 1 and $\delta|S_{t+1}| \leq 1/2$, written as

$$E[\tau_{\epsilon, \delta} \ | \ \pi(t)] \leq t + \log \frac{1}{\delta} \frac{1}{R(1/2)} + \log \frac{1}{E} + f_{R(1/2), \epsilon}(\epsilon, \delta).$$

(46)

Similar argument can be made for (b) with $\delta|S_{t+1}| \leq \alpha$ for $t \geq n$ under event $F_n^C$. Now, plugging the assumption $P(E_t) = P(\delta \ | \ S_{t+1} \ | \ > \alpha) \leq k_0 e^{-t E_0}$ into (45) with some
algebra, we have
\[ E[\tau_{e, \delta}] \leq \frac{k_\delta e^{-n E_0}}{1 - e^{-E_0}} \times \left( n + \frac{e^{-n E_0}}{1 - e^{-E_0}} + \frac{\log \frac{1}{\delta}}{R(1/2)} + \frac{\log \frac{1}{E}}{E} + f_{R(1/2), E}(\epsilon, \delta) \right) + n + \frac{\log \frac{1}{R(\alpha)}}{E} + f_{R(\alpha), E}(\epsilon, \delta). \]
Letting \( n = \lceil \log \log \frac{1}{\delta} \rceil \), we have the assertion of the proposition. \( \square \)

By proposition 2, we can see that for proving Theorem 1, it is sufficient to provide exponential decay tail probability of a large search size \( \mathbb{P}(\delta \mid S_{t+1} > \alpha) \) for each of the proposed algorithm \( S_{t+1} = \gamma(\pi(t)) \). The main idea of studying the event \( \{\delta[S_{t+1} > \alpha] \} \) is to group the region into coarse bins of size \( \alpha \) according to each of the search algorithm. By the nature of each algorithm the event \( \{\delta[S_{t+1} > \alpha] \} \) is equivalent to the event that one coarse bin has posterior larger than half. By further considering a similar submartingale of an average log-likelihood as in (19) but over the coarse bin posterior, the problem is then transformed to be the tail probability of a level of strictly positively drifted submartingale, where we can bound it by the Azuma’s Inequality (Lemma 6). Next, let us provide the details:

1) Proof of Theorem 7

Along with the operation of sortPM, we first sort the posterior, and then group into bins with size \( \delta[\text{bin}(q)] = \alpha \), written as
\[ \pi^q_\alpha(t) := \sum_{i \in \text{bin}(q)} \pi^i_\alpha(t), \quad q = 1, 2, ..., 1/\alpha, \tag{48} \]
where \( \pi^i_\alpha \) is the sorted posterior, \( \text{bin}(q) := \{ \frac{q}{\alpha}(q - 1) + 1, \frac{q}{\alpha}(q - 1) + 2, ..., \frac{q}{\alpha}q \} \). For notational simplicity, we deal with the case where \( 1/\alpha \) and \( \alpha/\delta \) are both integer (the proof follows similarly for non-integer case). Let us further define the average log-likelihood of the binned sorted posterior
\[ U_\alpha(t) := U(\pi^\alpha_\alpha(t)) \]
\[ = \frac{1}{\alpha} \sum_{q=1}^{1/\alpha} \pi^q_\alpha(t) \log \frac{\pi^q_\alpha(t)}{1 - \pi^q_\alpha(t)}. \tag{49} \]

By the search set selection rule in Algorithm 1, together with the definition of \( U_\alpha(t) \), under sortPM strategy we have
\[ \mathbb{P}(\delta \mid S_{t+1} > \alpha) \leq \mathbb{P} \left( \pi^\alpha_\alpha(t) < \frac{1}{2} \right) = \mathbb{P}(U_\alpha(t) < 0). \tag{50} \]

Now, by fact 3 and Lemma 3, \( U_\alpha(t) \) is a submartingale with bound difference
\[ |U_\alpha(t + 1) - U_\alpha(t)| \leq B_\alpha := \log(1/\alpha) + \log \frac{1 - p_{\min}}{p_{\min}} + e. \tag{51} \]
Further note that \( U_\alpha(0) = -\log(1/\alpha - 1) < -\log(1/\alpha) \) and together with Lemma 6 we have
\[ \mathbb{P}(\delta \mid S_{t+1} > \alpha) \leq \mathbb{P}(U_\alpha(t) < 0) \leq k_\epsilon e^{-\alpha S_{t+1} + \frac{K_S^2}{\alpha}} \tag{52} \]
where \( k_\epsilon = e^{-\gamma/\epsilon^2} \). Since \( \alpha > (e \log \frac{1}{\delta})^{1/K_S} \), and furthermore \( \log(1/\alpha) < |\log \frac{1}{\delta}| \), by proposition 2, we can conclude the assertion.

2) Proof of Theorem 2 and 3

Without loss of generality, we may assume that the resolution \( \delta \) is such that \( L = \log_2(1/\delta) \) is an integer. If otherwise, we can choose a smaller \( \delta' \) such that \( \log_2(1/\delta') \) is an integer and the analysis will follow similarly without affecting the asymptotic conclusions. One of the key attributes of dyapM and hiemP is the nested resolution due to the natural bisection. To analyze it, we introduce the posterior vector \( \pi^l_1(t) \) of a nested resolution level \( l < L \) with length \( 2^l \) where its elements are defined as
\[ \pi^q_1(t) := \sum_{i \in \text{bin}(q)} \pi^i_1(t), \quad q = 1, 2, ..., 2^l, \tag{53} \]
where \( \text{bin}(q) := \{ (q - 1)2^{L-l} + 1, (q - 1)2^{L-l} + 2, ..., q2^{L-l} \} \). Further, we can also define the average log-likelihood on \( \pi^l_1 \) as
\[ U^l_1(t) := \sum_{q=1}^{2^l} \pi^q_1(t) \log \frac{\pi^q_1(t)}{1 - \pi^q_1(t)}. \tag{54} \]
We have
\[ \mathbb{P}(\delta \mid S_{t+1} > 2^{-l}) \leq \mathbb{P}(\max_q \pi^q_1(t) < \frac{1}{2}) \leq \mathbb{P}(U^l_1(t) < 0). \tag{55} \]

The proof then follows similarly as in the proof of Theorem 1. Applying proposition 2 with \( \alpha = 2^{-l} \), where the corresponding submartingale properties of \( U^l_1(t) \) is by Lemma 4 and Lemma 3 for dyapM and hiemP, respectively, hence we omitted the rest.

APPENDIX B

TECHNICAL LEMMANS

Lemma 3. Using sortPM with resolution \( \delta \), the coarse binned sorted log-likelihood \( U_\alpha(t) \) defined by (48) and (49) is a submartingale with respect to \( \pi(t) \). In particular, we have
\[ \mathbb{E}[U_\alpha(t + 1) - U_\alpha(t)] \geq K_\alpha := \max \left\{ \frac{1}{2} D \left( \frac{1}{4} B_1 + \frac{3}{4} B_0 B_0 \right), \frac{1}{8} D \left( B_1 \parallel \frac{3}{4} B_1 + \frac{1}{4} B_0 \right) \right\} \tag{56} \]
for all \( t > 0 \) where \( B_1 = \text{Bern}(1 - p(1/2)) \) and \( B_0 = \text{Bern}(p(1/2)) \).

Proof. Let \( \sigma_t \) be the permutation such that \( \sigma_t(\pi(t)) = \pi^\alpha(t) \). To emphasize the effect of the different permutations at different time \( t \), for a given permutation \( \sigma \) we define
\[ \pi^\sigma(t) := \]
\[ \sigma(\pi(t)) \quad \text{and} \quad U^\sigma_\alpha(t) := U(\pi^{\alpha, \sigma}) \]
\[ = \sum_{q=1}^{1/\alpha} \pi_q^{\alpha, \sigma}(t) \log \frac{\pi_q^{\alpha, \sigma}(t)}{1 - \pi_q^{\alpha, \sigma}(t)}, \quad (57) \]
where
\[ \pi_q^{\alpha, \sigma}(t) := \sum_{i \in \text{bin}(q)} \pi_i^q(t), \quad q = 1, 2, \ldots, 1/\alpha. \quad (58) \]

By definition, we have \( U_\alpha(t) \equiv U^\sigma_\alpha(t) \). Now, we can lower bound the expected drift as
\[ \text{E}[U_\alpha(t + 1) \mid \pi(t)] - U_\alpha(t) \]
\[ = \text{E}[U^\sigma_{\alpha+1}(t + 1) \mid \pi(t)] - U^\sigma_\alpha(t) \]
\[ \geq \text{E}[U^\sigma_{\alpha+1}(t + 1) \mid \pi(t)] - U^\sigma_\alpha(t) \]
\[ \geq \frac{1}{\alpha} \sum_{q=1} \pi_{q, \alpha}^{\sigma, \alpha}(t) D \left( P^\sigma_{\gamma+1|\text{bin}(q), S_{\gamma+1}} \mid P^\sigma_{\gamma+1|\text{bin}(q), S_{\gamma+1}} \right), \quad (59) \]
where
\[ P^\sigma_{\gamma+1|\text{bin}(q), S_{\gamma+1}} := \frac{1}{\pi_{\alpha, \gamma}^q(t)} \sum_{i \in \text{bin}(q)} \pi_i^q(t) P_{\gamma+1|\text{i}, S_{\gamma+1}} \]
\[ P_{\gamma+1|\text{bin}(q), S_{\gamma+1}} := \sum_{q' \neq q} \pi_{q', \alpha}^{\sigma, \alpha}(t) P^\sigma_{\gamma+1|\text{bin}(q), S_{\gamma+1}} \quad (60) \]
and \( P_{\gamma+1|\text{bin}(q), S_{\gamma+1}} \) is as defined in (22). Here the inequality (a) follows from \( \pi_{q, \alpha}^{\sigma, \alpha}(t + 1) \geq \pi_{q, \alpha}^{\sigma, \alpha}(t + 1) \) and that \( U(\pi) \) is Schur-convex with respect to \( \pi \). And (b) is a similar manipulation using Bayes’ rule as was done in the proof of [Theorem 1 in [18]].

We now further lower bound \((59)\) by positivity and convexity of the KL divergence. We separate the discussion into two cases:

1) If \( q^* = 1 \):
   By the selection rule of \( k^* \) in \( \text{sortPM} \), we have \( \pi_1^q(t) \geq 1/2 \) and \( \pi_{[1, k^*]}(t) \geq 1/4 \). Therefore,
   \[ \geq \frac{1}{\alpha} \sum_{q=1} \pi_{q, 1}^{\sigma, \alpha}(t) D \left( P^\sigma_{\gamma+1|\text{bin}(q), S_{\gamma+1}} \mid P^\sigma_{\gamma+1|\text{bin}(q), S_{\gamma+1}} \right), \quad (59) \]
   \[ \geq \frac{1}{2} \left( \frac{1}{4} B_1 + \frac{3}{4} B_0 \right) \quad (61) \]
   where (c) is by positivity of KL divergence and (d) is by \( \pi_i^q(t, \sigma) \geq 1/2 \) and \( \pi_{[1, k^*]}(t) \geq 1/4 \).

2) If \( q^* > 1 \):
   By the selection rule of \( k^* \) in \( \text{sortPM} \), we have \( \pi_{[1, k^*]}(t) \leq 3/4 \).
   WLOG, we assume that \( k^* < \max \text{bin}(q^*) \) otherwise it reduces to the case of Fact 2a. Together with the selection rule of \( k^* \), we have \( \pi_{[1, q^*]}(t) \geq 1/2 \). By sorting we aslo have \( \pi_{[1, q^*]}^{\sigma, \alpha}(t) \geq \pi_{q^*}^{\sigma, \alpha}(t) \). Therefore \( \pi_{[1, q^*]}^{\sigma, \alpha}(t) \geq \frac{1}{4} \).

Now can proceed the lower bound as
\[ \geq \sum_{q=1} \pi_{q, 1}^{\sigma, \alpha}(t) D \left( P^\sigma_{\gamma+1|\text{bin}(q), S_{\gamma+1}} \mid P^\sigma_{\gamma+1|\text{bin}(q), S_{\gamma+1}} \right), \quad (59) \]
\[ \geq \frac{1}{2} \left( \frac{1}{4} B_1 + \frac{3}{4} B_0 \right) \quad (61) \]

where (c) is by Fact 2a and positivity of the KL divergence, and (f) is from \( \pi_{[1, q^*]}^{\sigma, \alpha}(t) \geq \frac{1}{4} \) and \( \pi_{[1, k^*]}(t) \leq 3/4 \).

Now let
\[ K_d := \max \left\{ \frac{1}{4} D \left( \frac{1}{4} B_1 + \frac{3}{4} B_0 \right) \right\} \quad (63) \]
and by (61) and (60), we conclude the assertion of this lemma.

**□**

**Lemma 4.** Using dy\( A \)PM with resolution \( \delta \), the nested log-likelihood \( U^{(1)}(t) \) of lower resolution level \( l < \log_2(1/\delta) \) defined in (54) is a submartingale. In particular, we have
\[ \text{E}[U^{(1)}(t + 1) \mid \pi(t)] - U^{(1)}(t) \geq K_d := \min \left\{ \min_{\rho \in [0, 1]} \max \{ f(\rho), g(\rho) \}, \min_{\rho \in [1/4, 1]} f(\rho) \right\}, \quad (64) \]
\[ \geq \frac{1}{2} D \left( \frac{1}{4} B_1 + \frac{3}{4} B_0 \right) \quad (65) \]
\[ \geq \frac{1}{2} D \left( \frac{1}{4} B_1 + \frac{3}{4} B_0 \right) \quad (66) \]
\[ f(\rho) = (1/2 - \rho) D \left( \frac{1}{4} B_1 + 4 \rho B_0 \mid (1/2 + \rho) B_1 + (1/2 - \rho) B_0 \right) \]
\[ g(\rho) = (1/2 - \rho) D \left( \frac{1}{4} B_1 + 4 \rho B_0 \mid (1/2 + \rho) B_1 + (1/2 - \rho) B_0 \right) \]
\[ \text{Proof.} \quad \text{By similar algebraic effort as in [Theorem 1 in [18]], the expected drift can be written as} \]
\[ \text{E}[U^{(1)}(t + 1) \mid \pi(t)] - U^{(1)}(t) \]
\[ = \sum_{q=1} \pi_{q, 1}^{(1)}(t) D \left( P_{\gamma+1|\text{bin}(q), \gamma} \mid P_{\gamma+1|\text{bin}(q), \gamma} \right), \quad (67) \]
where
\[ P_{\gamma+1|\text{bin}(q), \gamma} := \frac{1}{\pi_{q, 1}^{(1)}(t)} \]
\[ \times \sum_{i \in \text{bin}(q)} \pi_i(t) p(y_{\gamma+1}| \theta = i, S_{\gamma+1} = \gamma(\pi(t))) \quad (68) \]
and
\[
\begin{align*}
P_{y_{t+1} \mid \text{bin}(q), \gamma} := & \frac{1}{\sum_{b \in \text{bin}(q)} \pi_b(t)} \\
& \times \sum_{b \in \text{bin}(q)} \pi_b(t) p(y_{t+1} = i, S_{t+1} = \gamma(\pi(t))) \quad (69)
\end{align*}
\]

We drop \((t)\) and write \(\pi = \pi(\pi(t))\) in the proof frequently for notational simplification. We write the starting index of \(H^{nL}_\gamma\) as \(d \equiv m^L L^{-L}\). Furthermore, let the bin of level \(l\) that contains \(k^*\) be \(q^*\), i.e. \(k^* \in \text{bin}(q^*)\) and \(b_m = \min(\text{bin}(q^*))\) and \(b_M = \max(\text{bin}(q^*))\).

The case of \(l = \log_2(1/\delta)\) is done by Fact 2. For any given \(l < \log_2(1/\delta)\), we separate into two cases:

1) \(S_{t+1} = \gamma_d(\pi(t))\) contains at least one bin of level \(l\), i.e. \(\text{bin}(q^*) \subseteq S_{t+1}\) for some \(q^*\):

\[
\mathbb{E}[U^{(1)}(t + 1) \mid \pi(t)] - U^{(1)}(t)
\]

\[
\begin{align*}
= \sum_{q=1}^q \pi_q^{(t)}(t) D \left( P_{y_{t+1} \mid \text{bin}(q), \gamma} \bigg\| P_{y_{t+1} \mid \text{bin}(q), \gamma} \right) \\
\geq \max \left\{ \pi_{[d,b_m-1]} D \left( B_1 \bigg\| \pi_{[d,k^*]} B_1 + (1 - \pi_{[d,k^*]}) B_0 \right) \right\},
\end{align*}
\]

where we used
\[
\begin{align*}
D \left( P_{y_{t+1} \mid \text{bin}(q), \gamma} \bigg\| P_{y_{t+1} \mid \text{bin}(q), \gamma} \right) & \geq D \left( P_{y_{t+1} \mid \text{bin}(q), \gamma} \bigg\| P_{y_{t+1} \mid \text{bin}(q), \gamma} \right) \\
D \left( P_{y_{t+1} \mid \text{bin}(q), \gamma} \bigg\| P_{y_{t+1} \mid \text{bin}(q), \gamma} \right) & \geq 0
\end{align*}
\]

in (a). Note that by the binary tree construction of \(H^{nL}_\gamma\), we have \([b_m,k^*] \subseteq \text{bin}(q^*) \subseteq H^{2nL}_{2\gamma t+1}\). Therefore,

\[
\pi_{[b_m,k^*]} \leq \pi_{[d,k^*]} \leq \pi_{H^{2nL}_{2\gamma t+1}} \leq \frac{1}{2} \quad (72)
\]

By the selection rule of \(k^*\) and that \(\pi_k \leq 1/2\), we also know that \(\pi_{[d,k^*]} \leq 3/4\). Together with \((72)\) we can lower bound the first part in \((70)\) as

\[
\begin{align*}
\pi_{[d,b_m-1]} D \left( B_1 \bigg\| \pi_{[d,k^*]} B_1 + (1 - \pi_{[d,k^*]}) B_0 \right) \\
\geq \rho D \left( B_1 \bigg\| (3/4) B_1 + (1/4) B_0 \right) := f(\rho)
\end{align*}
\]

where we used \(\rho \equiv \pi_{[d,b_m-1]}\) for further simplification of the notation.

On the other hand, without loss of generality we assume that \(k^* < b_M\) (otherwise if \(k^* = b_M\), it reduces to the case of Fact 2). By the selection rule of \(k^*\) and that \(k^* < b_M\), we have

\[
0 \leq \frac{1}{2} - \pi_{[d,k^*]} \leq \pi_{[d,b_M]} - \frac{1}{2} \quad (74)
\]

which can be re-written as

\[
0 \leq \frac{1}{2} - \rho - \pi_{[b_m,k^*]} \leq \rho + \pi_{q^*} - \frac{1}{2} \quad (75)
\]

Therefore,

\[
\begin{align*}
\pi_{[b_m,k^*]} \geq & \frac{1}{2} - \pi_{q^*} - 2 \rho \quad (b) \quad (66) \quad (76) \\
& \pi_{q^*} \geq \frac{1}{2} \quad (c)
\end{align*}
\]

where \((b)\) is by \((75)\) and \((c)\) by \((72)\). And again by \((72)\) we also have

\[
\pi_{[d,b_M]} \leq \pi_{[d,b_M]} = \rho + \pi_{q^*} \leq \rho + \frac{1}{2} \quad (77)
\]

With \((75), (76)\) and \((77)\), the second part in equation \((70)\) can then be lower bounded as

\[
\begin{align*}
\pi_{q^*} D \left( \frac{\pi_{[b_m,k^*]} + \pi_{[k^*+1,b_M]}}{\pi_{q^*}} B_1 \right) & \geq \frac{1}{2} - \rho D \left( \frac{(1 - \pi_{[d,k^*]}) B_1 + (1 - \pi_{[d,k^*]}) B_0}{\pi_{[d,k^*]} B_1 + (1 - \pi_{[d,k^*]}) B_0} \right) \\
& = g(\rho).
\end{align*}
\]

Therefore \((70)\) is lower-bounded by \(K_d\) defined in \((64)\). It remains to show that \(K_d > 0\). Now, since \(f(\rho) > 0\) is increasing for \(\rho \in (0, 1/2)\) and \(g(0) > 0\), we have

\[
\min_{\rho \in (0,1/4]} \max \{ f(\rho), g(\rho) \} > 0, \min_{\rho \in (1/4,1/2]} f(\rho) > 0 \quad (79)
\]

concluding case 1.

2) \(S_{t+1} = \gamma_d(\pi(t))\) is within a bin of level \(l\), i.e. \(S_{t+1} \subseteq \text{bin}(q^*)\):

\[
\mathbb{E}[U^{(1)}(t + 1) \mid \pi(t)] - U^{(1)}(t)
\]

\[
\begin{align*}
= \sum_{q=1}^q \pi_q^{(t)}(t) D \left( P_{y_{t+1} \mid \text{bin}(q), \gamma} \bigg\| P_{y_{t+1} \mid \text{bin}(q), \gamma} \right) \\
\geq \pi_q^{(t)} D \left( \frac{\pi_{[b_m,k^*]} + \pi_{[k^*+1,b_M]}}{\pi_{q^*}} B_1 \right) \\
\geq \pi_q^{(t)} D \left( \frac{\pi_{[b_m,k^*]} + \pi_{[k^*+1,b_M]}}{\pi_{q^*}} B_0 \right) \quad (80)
\end{align*}
\]

By the selection rule of \(k^*\) and that \(S_{t+1} \subseteq \text{bin}(q^*)\), we know that \(\pi_{q^*} \geq \pi_{S_{t+1}} \geq 1/4\) and that \(\pi_{[b_m,k^*]} \geq \pi_{[b_m,k^*]} \geq \pi_{S_{t+1}} \geq 1/4\). Therefore,

\[
\frac{1}{4} \leq \pi_{[b_m,k^*]} \leq \pi_{S_{t+1}} \geq 1/4 \quad (81)
\]

The result is concluded by combining the two cases from \((79)\) and \((81)\). \(\square\)

**Lemma 5.** Using hiePM with resolution \(\frac{1}{2}\), the nested log-likelihood \(U^{(1)}(t)\) of lower resolution level \(l < \log_2(1/\delta)\)
defined in \((54)\) is a submartingale. In particular, we have
\[
\mathbb{E}[U(t+1) | \pi(t)] - U(t) \\
\geq K_h := \min \left\{ I \left( \frac{1}{3}, p \left( \frac{1}{2} \right) \right), \right. \\
\left. \frac{2}{3} D \left( \frac{1}{3} \text{Bern}(1-p \left( \frac{1}{2} \right)) + \frac{2}{3} \text{Bern}(p \left( \frac{1}{2} \right)) \right) \right\}
\]
for all \(t > 0\), for any \(l < S\).

**Proof.** Given any \(l < S\), if the selected codeword \(D^{(t+1)}\) is such that \(l_{t+1} \leq l\), by Fact 5 we conclude the results. Otherwise \(l_{t+1} > l\), then we have \(D^{(t+1)} \subseteq \text{bin}(q_t)\) for some \(q_t\). For notational simplicity, let \(\rho \equiv \pi_{D^{(t+1)}}(t) := \sum_{i \in D^{(t+1)}} \pi_i(t)\) and \(B^0 = \text{Bern}(p[2^{-l_{t+1}}]), B^1 = \text{Bern}(1-p[2^{-l_{t+1}}])\). We have
\[
\mathbb{E}[U(t+1) | \pi(t)] - U(t) \\
= \sum_{q=1}^{2^l} \pi_q(t) D_{P_{y_{t+1}q\gamma}} \left( P_{y_{t+1} \neq q, \gamma} \right) \\
\geq (a) \frac{2}{3} D(\rho B^1 + (1-\rho) B^0 || B^0) \geq (b) \frac{2}{3} D(\frac{1}{3} B^1 + \frac{2}{3} B^0 || B^0) \\
\geq \frac{2}{3} D \left( \frac{1}{3} \text{Bern}(1-p \left( \frac{1}{2} \right)) + \frac{2}{3} \text{Bern}(p \left( \frac{1}{2} \right)) \right) \text{Bern}(p \left( \frac{1}{2} \right)) \right\}
\]
where (a) and (b) are by the selection rule of hiePM that \(\pi_q(t) > 2/3\) whenever \(l_t > l\) and that \(1/3 \leq \rho \leq 2/3\). This concludes the assertion. \(\square\)

**Lemma 6** (Azuma’s Inequality). Given a submartingale \(U(t)\) with \(U(0) < 0\) with respect to another random process \(\pi(t)\). If \(U(t)\) has bounded difference, i.e. \(|U(t+1) - U(t)| < B\) for some \(B \in \mathbb{R}^+\), and that the expected difference is strictly positive, i.e.
\[
\mathbb{E}[U(t+1) - U(t) | \pi(t)] \geq K > 0,
\]
then we have
\[
\mathbb{P}(U(t) < 0) < ke^{-t \frac{K^2}{2(B+K)^2}} \quad \forall t > \frac{-U(0)}{K}
\]
where \(k = e^{-\frac{KU(0)}{2(B+K)^2}}\).

**Proof.** By the positive drift, \(U(t) - tK\) is also a submartingale with bounded difference
\[
|U(t+1) - (t+1)K - (U(t) - tK)| \leq B + K,
\]
for all \(t \geq 0\). Applying Azuma’s inequality \([19]\) on \(U(t) - tK\), we have
\[
\mathbb{P}(U(t) < 0) = \mathbb{P}(U(t) - tK - U(0) < -U(0)) \\
\leq \mathbb{P}(U(t) - tK < U(0)) \\
= \exp \left( - \frac{(U(0) + tK)^2}{2t(B + K)^2} \right) \\
\leq \exp \left( - \frac{K^2 t}{2(B + K)^2} \right) \exp \left( - \frac{KU(0)}{2(B + K)^2} \right) \\
\leq e^{-\frac{K^2 t}{2(B+K)^2}} \frac{KU(0)}{2(B+K)^2}
\]
for \(t > \frac{-U(0)}{K}\), concluding the results. \(\square\)