Subalgebras with Converging Star Products in Deformation Quantization: An Algebraic Construction for $\mathbb{C}P^n$

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Abstract

Based on a closed formula for a star product of Wick type on $\mathbb{C}P^n$, which has been discovered in an earlier article of the authors, we explicitly construct a subalgebra of the formal star-algebra (with coefficients contained in the uniformly dense subspace of representative functions with respect to the canonical action of the unitary group) that consists of converging power series in the formal parameter, thereby giving an elementary algebraic proof of a convergence result already obtained by Cahen, Gutt, and Rawnsley. In this subalgebra the formal parameter can be substituted by a real number $\alpha$: the resulting associative algebras are infinite-dimensional except for the case $\alpha = 1/K$, $K$ a positive integer, where they turn out to be isomorphic to the finite-dimensional algebra of linear operators in the $K$th energy eigenspace of an isotropic harmonic oscillator with $n + 1$ degrees of freedom. Other examples like the $2n$-torus and the Poincaré disk are discussed.

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1 Introduction

The concept of deformation quantization as defined by F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, and D. Sternheimer in 1978 (cf. [3]) consists in a formal local deformation of the commutative algebra (a so-called star product) of all smooth complex-valued functions on a symplectic manifold such that the first order commutator equals the Poisson bracket and pointwise complex conjugation remains an antilinear involution. The existence of star products on every symplectic manifold was proved by M. DeWilde and P. B. A. Lecomte in 1983 (cf. [12]) and independently by B. Fedosov in 1985 [13], [14]. A third existence proof was given by H. Omori, Y. Maeda and A. Yoshioka [17].

One of the problems with these star products is the fact that the formal series involved is shown to never converge on the space of all complex-valued smooth functions (see e.g. [20]), i.e. for every complex number there are two functions whose star product diverges when the formal parameter is substituted for that number.

This paper is a continuation of our work [6] in which we gave a closed formula for a star product of Wick type on complex projective space by a version of quantum phase space reduction. Star products on complex projective space have already been constructed by H. Omori, Y. Maeda and A. Yoshioka [18] and by C. Moreno [16] in a less explicit way. \( \mathbb{C}P^n \) can also be regarded as a coadjoint orbit of the unitary group \( U(n+1) \) (see the work of D. Arnal, J. Ludwig and M. Masmoudi for the existence of covariant star products on more general coadjoint orbits in [2] and references therein).

The aim of the present paper is twofold: firstly, we use our formula to explicitly compute the star product for a uniformly dense subalgebra of representative functions for the \( U(n+1) \)-action on complex projective space. Secondly, we would like to use the example \( \mathbb{C}P^n \) in order to illustrate the following algebraic procedure to deal with the above convergence problem:

i.) Find a complex subalgebra \( \mathcal{U} \) of the \( \ast \)-algebra of all formal power series in \( \nu \) with smooth coefficients such that 1.) all elements of \( \mathcal{U} \) are power series with infinite radius of convergence in \( \nu \) and 2.) their coefficients may be choosen in a “sufficiently large” (e.g. uniformly dense) subspace of the space of all smooth complex-valued functions.

ii.) Verify whether the subspace \( \mathcal{I}_\alpha \) of \( \mathcal{U} \) defined by

\[
\mathcal{I}_\alpha := \{ \Phi(\nu) \in \mathcal{U} | \Phi(\alpha) = 0 \}
\]

(1)

is a star-ideal of \( \mathcal{U} \).

iii.) Identify the quotient \( \mathcal{U}/\mathcal{I}_\alpha \) with the associative algebra of quantum observables related to the “\( \hbar \)-value” \( \alpha \) and try to find a representation of this quotient in some Hilbert space.

From the physical point of view it is often required that the subalgebra \( \mathcal{U} \) contains certain “important observables” which are in some cases related to the presence of additional symmetries of the classical phase space. For a general symplectic manifold the viability of the above procedure (in particular the existence of a sufficiently large subalgebra \( \mathcal{U} \)) is –to our best knowledge– an open problem in the theory of deformation quantization.

Nevertheless, in several examples having a large symmetry group (e.g. the Moyal product on the 2-torus or the Wick product on \( \mathbb{C}^{n+1} \)) the above programme may be carried out.
For such manifolds the space of representative functions of the symmetry group plays a prominent rôle for the construction of $U$: in a remarkable article of M. Cahen, S. Gutt, and J. Rawnsley [9] the convergence of a star product for these functions has been proved for all compact Hermitean spaces (in particular for $\mathbb{C}P^n$) by analytic methods of complex differential geometry. They start from the finite-dimensional operator algebras of geometric quantization in tensor powers of a very ample regular prequantum line bundle over a compact Kähler manifold and use coherent states (see [4], [19]) to first construct star products for the Berezin-Rawnsley symbols ([4], [19]) for each tensor power separately. In a second step an asymptotic expansion of these star products in the inverse tensor power is shown to define a local star product on the manifold where the formal parameter appears as a sort of interpolation of the inverse tensor powers.

The approach of this paper is in some sense reverse to the programme of M. Cahen, S. Gutt, and J. Rawnsley (cf. [8], [9], [10], [11]) and only makes use of elementary algebraic methods: We start from the explicit star product on $\mathbb{C}P^n$ (see [8]) and define $U$ as a certain proper subspace of the space of all polynomials in the formal parameter with coefficients in the uniformly dense subspace of representative functions for the unitary group $U(n+1)$. Since all occurring star products can explicitly be computed the analysis of the ideals $I_\alpha$ and the quotient algebras $U/I_\alpha$ becomes relatively simple. The main result is that for inverse integer values of the formal parameter $\nu$ the quotient algebras turn out to be finite-dimensional full complex matrix-algebras whereas all noninteger quotients are of infinite dimension and define converging star products on the space of representative functions for these values as has been stated in [9].

From the physical point of view one can regard the finite-dimensional quantum algebras as the set of all quantum observables restricted to the eigenspace of integral energy of a harmonic oscillator of $n + 1$ degrees of freedom (where the ground state energy is zero for the Wick quantization rule).

The advantage of the above algebraic programme of prescribing a real value to the formal parameter of deformation quantization is that one may hope to transfer it to physical situations with an infinite number of degrees of freedom, i.e. field theories, where the powerful analytical methods in the theory of finite-dimensional manifolds are no longer well-defined.

The paper is organised as follows: after briefly reviewing the concepts and formulas of our last paper [8] in section 2 we compute the star product of two Berezin-Rawnsley symbols in section 3 and discuss the unitary symmetry action on the star product. Section 4 is then devoted to define the algebra $\mathcal{U}$, to compute the ideals $\mathcal{I}_\alpha$ and the quotient algebras $\mathcal{U}/\mathcal{I}_\alpha$. In section 5 we briefly consider other phase spaces already dealt with in the literature for which the above programme works: the complex vector space $\mathbb{C}^{n+1}$ with the Wick product, the $2n$-torus $T^{2n}$ with the Moyal product, and the Poincaré disk. For this last example we can recover the formula of the star product for the corresponding Berezin-Rawnsley symbols given by Cahen, Gutt, and Rawnsley in [10].

**Notation:** Throughout this paper we use the Einstein summation convention, i.e. summation over repeated indices is automatic. Moreover, the symbol $F(z)$ for a complex-valued function $F$ of a complex vector $z$ does not necessarily imply that $F$ is holomorphic.
2 Review of star products on complex projective space

In this section we shall give a short review of earlier work \([7]\) in which we derived an explicit formula for a star product of Wick-type on the complex projective space \(\mathbb{C}P^n\).

Let \(\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}P^n\) be the canonical projection of a complex vector \(z\) onto the complex ray through it. Let \(x := \frac{1}{z^*z}\). The usual Wick product on \(\mathbb{C}^{n+1} \setminus \{0\}\) of two complex-valued functions \(F, G \in C^\infty(\mathbb{C}^{n+1} \setminus \{0\})\) is given as the following formal power series in the parameter \(\lambda\):

\[
F \star G = \sum_{r=0}^{\infty} \frac{\lambda^r}{r!} \frac{\partial^r F}{\partial z_{i_1} \cdots \partial z_{i_r}} \frac{\partial^r G}{\partial z^{i_1} \cdots \partial z^{i_r}}.
\]

We have called a function \(F \in C^\infty(\mathbb{C}^{n+1} \setminus \{0\})\) \textit{homogeneous} iff it is invariant under the natural action of the group \(\mathbb{C} \setminus \{0\}\). These functions are precisely given by pull backs of functions \(f \in C^\infty(\mathbb{C}P^n)\), \(F = f \circ \pi = \pi^* f\). We have called a function \(R \in C^\infty(\mathbb{C}^{n+1} \setminus \{0\})\) \textit{radial} iff it is a function of \(x\), i.e. iff there is a smooth function \(\rho : \mathbb{R}^+ \to \mathbb{C}\) such that \(R = \rho \circ x\). We have defined a formal differential operator \(S : C^\infty(\mathbb{C}^{n+1} \setminus \{0\})[[\lambda]] \to C^\infty(\mathbb{C}^{n+1} \setminus \{0\})[[\lambda]]\) depending only on \(x\) and \(\partial_x := \frac{1}{2x}(z^i \partial_{z^i} + \bar{z}^i \partial_{\bar{z}^i})\) whose standard symbol \(\hat{S}(x, \alpha) := (Se_\alpha)(x)e^{-\alpha x}\) (\(e_\alpha\) denoting the exponential function \(x \mapsto e^{\alpha x}\) for \(\alpha \in \mathbb{R}\)) is given by (setting the series \(D\) in \([7]\) eqn. 9) equal to 1:

\[
\hat{S}(x, \alpha) = \exp \left( \frac{x}{\lambda} \log (1 + \lambda \alpha) - \lambda \alpha \right).
\]

\(S\) and its inverse \(S^{-1}\) act trivially on homogeneous functions, i.e. \(SF = F = S^{-1}F\), but do in general transform radial functions into radial ones, in particular (for a nonnegative integer \(r\)) \([7]\) eqn. 14]

\[
Sx^r = x^r \prod_{k=0}^{r} \left( 1 - \frac{k \lambda}{x} \right), \quad Sx^{-r} = x^{-r} \prod_{k=0}^{r} \left( 1 + \frac{k \lambda}{x} \right)^{-1}.
\]

We have used \(S\) as an equivalence transformation for a modified Wick product of two functions \(F, G \in C^\infty(\mathbb{C}^{n+1} \setminus \{0\})\):

\[
F \tilde{\star} G := S(S^{-1}F \star S^{-1}G).
\]

For two radial functions \(R_1, R_2\) and a homogeneous function \(F\) on \(\mathbb{C}^{n+1} \setminus \{0\}\) this new star product is just pointwise multiplication \([7]\) eqn. 12]:

\[
R_1 \tilde{\star} R_2 = R_1R_2 = R_2 \tilde{\star} R_1, \quad R_1 \tilde{\star} F = R_1F = F \tilde{\star} R_1,
\]

whereas for two smooth homogeneous functions \(F, G\) we get

\[
(F \tilde{\star} G)(z) = \sum_{r=0}^{\infty} \frac{1}{r!} \left( \frac{\lambda}{x} \right)^r \prod_{k=1}^{r} \left( 1 + \frac{k \lambda}{x} \right)^{-1} x^r \frac{\partial^r F}{\partial z^{i_1} \cdots \partial z^{i_r}}(z) \frac{\partial^r G}{\partial \bar{z}^{i_1} \cdots \partial \bar{z}^{i_r}}(z).
\]

The main result of \([7]\) was the fact that this formula can directly be projected to \(\mathbb{C}P^n\) by phase space reduction via the \(U(1)\)-momentum map \(J : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{R} : z \mapsto -\frac{\pi}{2}\) of the canonical \(U(1)\)-action on \(\mathbb{C}^{n+1} \setminus \{0\}\): for a negative real number \(\mu\) and a \(U(1)\) invariant
function $F$ in $C^\infty(\mathbb{C}^{n+1} \setminus \{0\})$ we write $F_\mu$ for the unique function in $C^\infty(\mathbb{C}P^n)$ obtained by first restricting $F$ to the odd sphere $J^{-1}(\mu)$ and then projecting it to $\mathbb{C}P^n$ \cite[4.3]{4}. Then the formula

$$F_\mu *^\mu G_\mu := (F * G)_\mu$$

was shown to define a star product on $\mathbb{C}P^n$. The explicit form of $*^\mu$ is obtained by replacing $\lambda/x$ by $\lambda/(-2\mu)$ in \cite{7} and noting that the bidifferential operator $\hat{M}_r(f,g)(\pi(z)) := x^r \frac{\partial^r \pi^* f(z)}{\partial z_1^{r_1} \cdots \partial z_n^{r_n}} \frac{\partial^r \pi^* g(z)}{\partial \bar{z}_1^{r_1} \cdots \partial \bar{z}_n^{r_n}}$ is well-defined on $f,g \in C^\infty(\mathbb{C}P^n)$. For simplicity we shall work with the redefined formal parameter $\nu := \lambda/(-2\mu)$ in what follows.

**Lemma 1**: The standard symbol of $S^{-1}$ is described by the formula

$$\hat{S}^{-1}(x,\alpha) = e^{\frac{i}{\hbar} e^{(\alpha \hat{L} - \alpha \Lambda)}} = e^{* \beta x} e^{-\alpha x}$$

where the last term involves the star-exponential \cite{3} of the function $x$ with respect to the usual Wick product \cite{3}.

**Proof**: Since $Se_\beta = e^{\frac{i}{\hbar} \log(1+\beta \lambda)}$ we obviously get $e_\beta = S^{-1} e^{\frac{i}{\hbar} \log(1+\beta \lambda)}$ which proves the first equation after the substitution $\alpha := \frac{1}{\hbar} \log(1+\beta \lambda)$. Secondly, note that

$$e^{* \alpha x} = S^{-1} e^{\alpha \hat{S}^{-1} x} = S^{-1} e^{* \alpha x} \bigg|_{S} S^{-1} e^{\alpha x} = \hat{S}^{-1}(x,\alpha) e^{\alpha x}$$

which proves the second equality. \qed

Remark: Note that the function $H := \frac{1}{2\hbar} x$ equals the usual Hamiltonian function of an isotropic harmonic oscillator in $n+1$ degrees of freedom. The above star-exponential of $x$ for $\alpha = -it/2\hbar$ and $\lambda = 2\hbar$ then corresponds to the quantum mechanical time development operator for this system.

### 3 A star product for representative functions on complex projective space

Let $p : \mathbb{C}^{n+1} \to \mathbb{C} : z \mapsto P(z)$ be a polynomial function (in the $2n + 2$ variables $z^0, \ldots, z^n, \bar{z}^0, \ldots, \bar{z}^n$). We shall call $p$ homogeneous of degree $(k,k)$ for a nonnegative integer $k$ iff $p(\lambda z) = (\lambda \lambda)^k p(z)$ for all $\lambda \in \mathbb{C} \setminus \{0\}$. We denote by $\mathcal{E}_k$ the following subspace of $C^\infty(\mathbb{C}P^n)$:

$$\mathcal{E}_k := \left\{ \phi \mid \text{there is a homogeneous polynomial } p_k \text{ of degree } (k,k) \text{ s.t. } (\pi^* \phi)(z) = \frac{1}{x^k} p_k(z) \right\}.$$

**Lemma 2**:

i.) For each integer $k \geq 0$: $\mathcal{E}_k \subset \mathcal{E}_{k+1}$.

ii.) $\mathcal{E} := \bigcup_{k=0}^\infty \mathcal{E}_k$ is a filtered subalgebra of $C^\infty(\mathbb{C}P^n)$ with respect to the pointwise multiplication. It is closed under complex conjugation.

iii.) $\mathcal{E}$ separates points and is therefore a dense subalgebra of $C^\infty(\mathbb{C}P^n)$ with respect to the uniform topology.
Proof: i) Clearly \( \frac{p_k}{x^k} = \frac{x^k}{x^{n+1}} \in \pi^* E_{k+1} \). ii) This is obvious. iii) Consider the complex rays \( \pi(z(1)) \neq \pi(z(2)) \) for \( z(1), z(2) \in \mathbb{C}^{n+1} \setminus 0 \). This is true iff \( z(1), z(2) \) are linearly independent iff there is a \( y \in \mathbb{C}^{n+1} \) such that \( \langle y, z(1) \rangle = 1 \) and \( \langle y, z(2) \rangle = 0 \) where \( \langle y, z \rangle \) denotes the standard sesquilinear form \( \langle y, z \rangle = \bar{y}^k z^k \). Then \( \phi \in E_1 \) defined by \( \phi(z) := \frac{|< z, y >|^2}{x} \) separates \( \pi(z(1)) \) and \( \pi(z(2)) \). The density of \( E \) follows from the Stone-Weierstrass Theorem.

Consider the standard action of the unitary group \( U(n+1) \) on \( \mathbb{C}^{n+1} \setminus \{0\} \): \( (g, z) \mapsto g z =: \Phi_g(z) \) and its induced action on \( \mathbb{C}P^n \): \( (g, \pi(z)) \mapsto \pi(gz) =: \Psi_g(\pi(z)) \). A smooth complex-valued function \( f \) on \( \mathbb{C}P^n \) is called representative with respect to the \( U(n+1) \) action iff
\[
\mathbb{C}-\text{span}\{ f \circ \Psi_g | g \in U(n+1) \} \quad \text{is finite-dimensional.} \tag{11}
\]
We now get a characterization of \( E \) which should be fairly standard (see [3] sec. 3, Lemma 1):

**Lemma 3:** The algebra \( E \) is equal to the set of all representative functions on \( \mathbb{C}P^n \).

Proof: Since \( x \) and the finite-dimensional space of all homogeneous polynomials of degree \((k, k)\) are invariant under \( U(n+1) \) it follows that \( E \) consists of representative functions. In order to prove the reversed inclusion we can use a more general argument: \( \mathbb{C}P^n \) is a homogeneous space \( G/H \) for the compact Lie group \( G = U(n+1) \) with compact isotropy subgroup \( H = U(1) \times U(n) \). Now the space of all representative functions \( f \) on \( G/H \) (defined as in (11)) with \( U(n + 1) \) replaced by any compact Lie group \( G \) is clearly in one-to-one correspondence with the space of its pull-back to \( G \) under the natural projection \( G \to G/H \): this in turn is given by the space of all \( H \) right invariant representative functions on \( G \) with respect to left multiplication. Both this space and the pull-back of \( E \) to \( G \) are \( G \)-modules and are therefore closed in the space of all representative functions on \( G \) with respect to the uniform topology on \( G \) (see e.g. [3], p.126, Prop.(1.4) (iii)). Since the pull-back obviously is a continuous closed linear map with respect to the uniform topologies and \( E \) is dense in the space of all representative functions on \( G/H \) thanks to the previous Lemma it follows that \( E \) is equal to that space.

Yet another equivalent description of \( E \) is obtained in terms of the Berezin-Rawnsley symbols known from geometric quantization ([1], [19], [8]): for a fixed nonnegative integer \( k \) take the vector space \( H^{(k)} \) of complex-valued holomorphic polynomials \( \psi \) on \( \mathbb{C}^{n+1} \), which are homogeneous of degree \( k \), i.e. \( \psi(\lambda z) = \lambda^k \psi(z) \) for all complex numbers \( \lambda \). The dimension of this space is clearly \( N := \binom{n+k}{k} \) and we shall henceforth identify \( H^{(k)} \) with \( \mathbb{C}^N \) by means of the base \((z^i \cdots z^k)\). Consider the space \( B(H^{(k)}) \) of complex linear endomorphisms of \( H^{(k)} \). Any \( A \in B(H^{(k)}) \) can be viewed as a complex \( N \times N \) matrix \( A_{i_1 \cdots i_k j_1 \cdots j_k} \) where each of the indices \( i_1, \ldots, i_k, j_1, \ldots, j_k \) ranges over \( 0, 1, \ldots, n \) and the matrix elements are symmetric with respect to all permutations among the \( i_1, \ldots, i_k \) and among the \( j_1, \ldots, j_k \). To each \( A \in B(H^{(k)}) \) one can associate the polynomial function
\[
\tilde{\sigma}(A) : \mathbb{C}^{n+1} \to \mathbb{C} : z \mapsto \bar{z}^i_1 \cdots \bar{z}^i_k z^{j_1} \cdots z^{j_k} A_{i_1 \cdots i_k j_1 \cdots j_k}. \tag{12}
\]
Clearly, \( \tilde{\sigma}(A) \) is homogeneous of degree \((k, k)\), and by counting dimensions it can be seen that every homogeneous polynomial function of degree \((k, k)\) is of that form. The Berezin-
Rawnsley symbol \( \sigma(A) \) associated to \( A \) is then defined by

\[
\sigma(A) : \mathbb{C}^n \to \mathbb{C} : \pi(z) \mapsto \frac{\tilde{\sigma}(A)(z)}{x^k}.
\]  
(13)

Then the following corollary is clear:

**Corollary 1**: For each nonnegative integer \( k \) the space \( \mathcal{E}_k \) is spanned by all the Berezin-Rawnsley symbols \( \sigma(A) \) associated to \( A \in B(\mathcal{H}^{(k)}) \).

Both on \( \mathbb{C}^{n+1} \setminus \{0\} \) and \( \mathbb{C}P^n \) there are momentum maps (see [1] for general definitions) for the \( U(n+1) \)-action which can be expressed in terms of Berezin-Rawnsley symbols contained in \( \mathcal{E}_1 \):

**Lemma 4**: Let \( u(n+1) \) denote the space of complex antihermitean \((n+1) \times (n+1)\)-matrices, i.e. the Lie algebra of the unitary group \( U(n+1) \). Then:

i) The following map

\[
\tilde{P} : \mathbb{C}^{n+1} \setminus \{0\} \to u(n+1)^* : z \mapsto (A \mapsto \tilde{\sigma}(A))
\]

is a momentum map for the \( U(n+1) \)-action on \( \mathbb{C}^{n+1} \setminus \{0\} \).

ii) The following map

\[
P := \tilde{P}_\mu : \mathbb{C}P^n \to u(n+1)^* : \pi(z) \mapsto (A \mapsto -2\mu \sigma(A)(\pi(z)) = \tilde{\sigma}(A)_\mu(\pi(z)))
\]

is a momentum map for the \( U(n+1) \)-action on \( \mathbb{C}P^n \).

Recall that the index \( \mu \) refers to the \( U(1) \)-phase space reduction of \( \mathbb{C}^{n+1} \setminus \{0\} \) by the \( U(1) \) momentum map \( J(z) = -\frac{x}{2} \).

**Proof**: i) The Hamiltonian vector field of \( P(A) \) is given by

\[
X_{\tilde{\sigma}(A)}(z) = \frac{2}{i}(A_{ij}z^i\partial z^j + A_{ij}\bar{z}^i\partial \bar{z}^j),
\]

since \( A_{ij} = -A_{ji} \) which obviously equals the infinitesimal generator \( A_{\mathbb{C}^{n+1}\setminus\{0\}} \) of the \( U(n+1) \)-action. The \( Ad^* \)-equivariance of this map is obvious.

ii) Recall the projection \( \pi_\mu : J^{-1}(\mu) \to \mathbb{C}P^n \) to the reduced phase space (compare [1, 4.3]). Since \( \Phi_g \circ \pi_\mu = \pi_\mu \circ \Psi_g \) it is clear that \( T\pi_\mu A_{\mathbb{C}^{n+1}\setminus\{0\}} = A_{\mathbb{C}P^n} \pi_\mu \) where \( A_{\mathbb{C}P^n} \) is the infinitesimal generator of the \( U(n+1) \)-action on \( \mathbb{C}P^n \). Using the identity \( A_{\mathbb{C}P^n} \pi_\mu(z) = X_{\tilde{\sigma}(A)_\mu}(z) \) we get by phase-space reduction

\[
T_x\pi_\mu X_{\tilde{\sigma}(A)_\mu}(z) = T_x\pi_\mu A_{\mathbb{C}^{n+1}\setminus\{0\}}(z) = X_{\tilde{\sigma}(A)_\mu}(\pi(z)).
\]

The \( Ad^* \)-equivariance of this map follows at once from the \( Ad^* \)-equivariance of \( \tilde{P} \). \( \square \)

We shall now compute star products for elements of \( \mathcal{E} \): Set \( \nu := \frac{\lambda}{2\mu}, \nu^{(0)} := 1, \nu^{(1)} := 1 \) and

\[
\nu^{(k)} := (1 - \nu) \ldots (1 - (k - 1)\nu)
\]

(16)
Theorem 1: For \( f \in \mathcal{E}_k, g \in \mathcal{E}_l \) we get
\[
(\pi^* f) \ast (\pi^* g)(z) = \sum_{r=0}^{\min(k,l)} \frac{\lambda^r}{r!} S(x^{k+l-r}) x^{k+l-r} \frac{1}{\partial z_1 \ldots \partial z_r} (z) \partial^r (x^k \pi^* f)(z) \partial^r (x^l \pi^* g)(z)
\] (17)
and
\[
f \ast^\mu g = \sum_{r=0}^{\min(k,l)} \frac{\lambda^r}{r!} \frac{\nu^r \nu^{(k-l-r)}}{\nu(k) \nu(l)} M_r^{(k,l)}(f,g)
\] (18)
with the following bidifferential operator on \( C^\infty(\mathbb{C}P^n) \)
\[
\pi^* M_r^{(k,l)}(f,g) := \frac{1}{x^{k+l-r}} \frac{\partial^r (x^k \pi^* f)}{\partial z_1 \ldots \partial z^r} \frac{\partial^r (x^l \pi^* g)}{\partial \bar{z}_1 \ldots \partial \bar{z}^r}
\] (19)

Proof: The usual Wick product \([2]\) of the two polynomials \( \tilde{f}_k := x^k \pi^* f, \tilde{g}_l := x^l \pi^* g \) is given by
\[
\tilde{f}_k \ast \tilde{g}_l = \sum_{r=0}^{\min(k,l)} \frac{\lambda^r}{r!} \frac{\partial^r \tilde{f}_k}{\partial z_1 \ldots \partial z^r} \frac{\partial^r \tilde{g}_l}{\partial \bar{z}_1 \ldots \partial \bar{z}^r}.
\] (20)
Using definition \([3]\) and the formulas \([3]\) and \([4]\) we find
\[
(Sx^k)(Sx^l)(\pi^* f) \ast (\pi^* g) = \left( S(x^k \pi^* f) \right) \ast \left( S(x^l \pi^* g) \right) = S((x^k \pi^* f) \ast (x^l \pi^* g))
\]
\[
= S \left( \sum_{r=0}^{\min(k,l)} \frac{\lambda^r}{r!} \frac{\partial^r \tilde{f}_k}{\partial z_1 \ldots \partial z^r} \frac{\partial^r \tilde{g}_l}{\partial \bar{z}_1 \ldots \partial \bar{z}^r} \right)
\]
\[
= \sum_{r=0}^{\min(k,l)} \frac{\lambda^r}{r!} S \left( x^{k+l-r} \frac{1}{x^{k+l-r}} \frac{\partial^r \tilde{f}_k}{\partial z_1 \ldots \partial z^r} \frac{\partial^r \tilde{g}_l}{\partial \bar{z}_1 \ldots \partial \bar{z}^r} \right)
\]
\[
= \sum_{r=0}^{\min(k,l)} \frac{\lambda^r}{r!} S \left( x^{k+l-r} \frac{1}{x^{k+l-r}} \frac{\partial^r \tilde{f}_k}{\partial z_1 \ldots \partial z^r} \frac{\partial^r \tilde{g}_l}{\partial \bar{z}_1 \ldots \partial \bar{z}^r} \right)
\]
which proves the first equation. The second equation immediately follows by the reduction formula \([8]\).

Remark: Thanks to the straightforward recursion formulas \( M_r^{(k+1,l)} = M_r^{(k,l)} + r(l - (r - 1))M_r^{(k,l)} \) and \( M_r^{(k,l+1)} = M_r^{(k,l)} + r(k - (r - 1))M_r^{(k,l-1)} \) it can easily be checked that the right hand side of (17) is well-defined, i.e. if \( \tilde{f} \) is regarded as an element of \( \mathcal{E}_{k+a} \) for a positive integer \( a \).

Corollary 2: For \( A, B \in B(\mathcal{H}^{(1)}) \) we have
\[
\sigma(A)^k \ast^\mu \sigma(B)^l = \sum_{r=0}^{\min(k,l)} \frac{\lambda^r}{r!} \frac{k! \cdot l!}{(k-r)!(l-r)!} \frac{\nu^r \nu^{(k-l-r)}}{\nu(k) \nu(l)} \sigma(AB)^r \sigma(A)^{k-r} \sigma(B)^{l-r}
\] (21)

Proof: This is easily seen by setting \( f = \sigma(A)^k \) and \( g = \sigma(B)^l \) and using Theorem \([1]\). □

We shall now show that the momentum map \( P \) for the \( U(n+1) \)-action on \( \mathbb{CP}^n \) is even a quantum momentum map. More precisely:
**Lemma 5**: \( P \) is a quantum momentum mapping on \( \mathbb{C}P^n \) for the \( U(n+1) \)-action, i.e. for every smooth function \( \phi : \mathbb{C}P^n \to \mathbb{C} \) the following equation holds:

\[
P(A) \ast \mu \phi - \phi \ast \mu P(A) = \frac{i\lambda}{2} \{ P(A), \phi \}_\mu,
\]

i.e. the star product \( \ast \mu \) is strongly \( U(n+1) \)-invariant.

**Proof**: First we prove the equation \( \pi^* \sigma(A) \ast f - f \ast \pi^* \sigma(A) = \frac{i\lambda}{2} \{ \pi^* \sigma(A), f \} \) for \( f = \pi^* \phi \). Then eqn. (22) follows by eqn. (8). We have the strong invariance of the Wick product

\[
\tilde{\sigma}(A) \ast f - f \ast \tilde{\sigma}(A) = \frac{i\lambda}{2} \{ \tilde{\sigma}(A), f \}.
\]

With the equivalence transformation \( S \) we find

\[
S^{-1}\left( S \tilde{\sigma}(A) \ast S f - S f \ast S \tilde{\sigma}(A) \right) = \frac{i\lambda}{2} \{ \tilde{\sigma}(A), f \}
\]

and with \( \pi^* \sigma(A) = x \tilde{\sigma}(A) \) and \( S f = f \) this leads to

\[
S x \left( \pi^* \sigma(A) \ast f - f \ast \pi^* \sigma(A) \right) = \frac{i\lambda}{2} S \left( \{ x \pi^* \sigma(A), f \} \right).
\]

The Poisson bracket \( \{ x \pi^* \sigma(A), f \} = x \{ \pi^* \sigma(A), f \} \) is again homogeneous. Hence the right hand side of the last equation is simply \( x \{ \pi^* \sigma(A), f \} \). With \( S x = x \) the proof is complete.

Remark: The case \( k = l = 1 \) in Cor. 2 shows that the functions \( \sigma(A) \) and \( \sigma(B) \) commute with respect to \( \ast \mu \) iff the operators \( A \) and \( B \) commute. Let \( A_1, \ldots, A_n \) be a linearly independent set of commuting traceless Hermitean matrices. It follows in particular that the functions \( \sigma(A_1), \ldots, \sigma(A_n) \) are functionally independent and in involution, i.e. the Poisson bracket of \( \sigma(A_i) \) with \( \sigma(A_j) \) vanishes for all \( i, j \). In other words they form a **completely integrable system** in the sense of Liouville on \( \mathbb{C}P^n \). Note that this system is nontrivial in the sense that there is no global chart of action-angle-variables, i.e. \( \mathbb{C}P^n \) is not symplectomorphic to some \( T^r \times \mathbb{R}^{2n-r} \) for a nonnegative integer \( r \leq n \). The above Corollary now implies that these functions also commute with respect to the star product and can be viewed as a **quantum integrable system** on \( \mathbb{C}P^n \).

4 Construction of the subalgebra \( \mathcal{U} \), the ideals \( \mathcal{I}_\alpha \), and the quotients \( \mathcal{U}/\mathcal{I}_\alpha \)

If we take two functions \( \phi_k \in \mathcal{E}_k \) and \( \psi_l \in \mathcal{E}_l \) and multiply them with the \( \nu \)-polynomial \( \nu^{(k)} \) and \( \nu^{(l)} \), respectively, then formula (18) shows that their \( \ast \mu \)-product contains only **polynomials** in the parameter \( \nu \). Moreover, that particular combination is restored, i.e. the functions \( M_{t}^{(k,l)}(\phi_k, \psi_l) \in \mathcal{E}_{k+l-t} \) appear only in combination with \( \nu^{(k+l-t)} \). This motivates the definition of the following subspaces of \( C^\infty(\mathbb{C}P^n)[[\nu]] \):

\[
\mathcal{U}_0 := \mathcal{E}_0 \quad \text{and} \quad \mathcal{U}_k := \nu^{(k)} \mathcal{E}_k + \nu \nu^{(k-1)} \mathcal{E}_{k-1} + \cdots + \nu^{k-1} \nu^{(1)} \mathcal{E}_1 + \nu^k \mathcal{E}_0 \quad \forall \ k \in \mathbb{N}
\]
where each $U_k$ is a (not necessarily direct) sum of subspaces of $C^\infty(\mathbb{C}P^n)[[\nu]]$. We denote by $E[[\nu]]$ the polynomials in $\nu$ with coefficients in $E$. The following theorem describes the structure of $U$.

**Theorem 2**:

i.) For each integer $k$: $U_k \subset U_{k+1}$. Define $U := \bigcup_k^\infty U_k$. Then $U$ is a proper $\mathbb{C}[[\nu]]$-submodule of $E[[\nu]]$.

ii.) $U$ is a filtered subalgebra of $C^\infty(\mathbb{C}P^n)[[\nu]]$ with respect to $*^\mu$, i.e.

$$U_k *^\mu U_l \subset U_{k+l}$$

**Proof:** i.) Let $\Phi \in U_k$. $\Phi$ is of the form $\Phi = \sum_{r=0}^k \nu^r \nu^{(k-r)} \phi_{k-r}$ with $\phi_{k-r} \in \mathcal{E}_{k-r}$. The filtration of $E$ implies $\phi_{k-r} \in E_{k-r+1}$ so $\nu^r \nu^{(k-r+1)} \phi_{k-r} \in U_{k+1}$. The element $\nu^{r+1} \nu^{(k-r)} \phi_{k-r}$ is also contained in $U_{k+1}$. By eqn. (16) we get $\nu^{(k-r+1)} = (1 - (k-r)\nu)\nu^{(k-r)}$. Hence

$$\nu^r \nu^{(k-r)} \phi_{k-r} = \nu^r \nu^{(k-r+1)} \phi_{k-r} + (k-r) \nu^{r+1} \nu^{(k-r)} \phi_{k-r} \in U_{k+1}$$

which proves the inclusion $U_k \subset U_{k+1}$. It is clear from the definition (23) that $\nu U_k \subset U_{k+1} \subset E[[\nu]]$ whence $U$ is $\mathbb{C}[[\nu]]$-submodule of $E[[\nu]]$. Note that for fixed $y \in \mathbb{C}^{n+1} \setminus \{0\}$ the function $\pi(z) \mapsto \frac{[y, z]^t}{z^2} \in E_2$ is not contained in $U$. ii.) By Theorem (11) the star product for arbitrary $\Phi_k = \sum_{r=0}^k \nu^{r-k} \nu^r \phi_r \in U_k$ and $\Psi_l = \sum_{s=0}^l \nu^{l-s} \nu^s \psi_s \in U_l$ with $\phi_r \in \mathcal{E}_r$ and $\psi_s \in \mathcal{E}_s$ is given by

$$\Phi_k *^\mu \Psi_l = \sum_{r=0}^k \sum_{s=0}^l \sum_{t=0}^{\min(r,s)} \frac{1}{t!} \nu^{r-k-l+s+t} \nu^{(r+s-t)} M_t^{(r,s)}(\phi_r, \psi_s),$$

hence each summand is an element of $U_{k+l}$. This proves the second part.

We should now like to substitute the formal parameter $\nu$ for a fixed nonzero real number $\alpha$. In the subalgebra $U$ this is well-defined because $U$ only contains polynomials in $\nu$. The kernel of the substitution homomorphism

$$I_\alpha := \{ \Phi(\nu) \in U | \Phi(\alpha) = 0 \}$$

will turn out to be a $*^\mu$-ideal.

**Lemma 6** $I_\alpha$ is a $*^\mu$-ideal and the general form of an element $\Phi \in I_\alpha \cap U_k$ is

i.) for $\alpha \not\in \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$

$$\Phi(\nu) = (\nu - \alpha) u_{k-1}(\nu) \quad \text{with } u_{k-1}(\nu) \in U_{k-1}$$

ii.) for $\alpha = \frac{1}{K}$ with $K \in \mathbb{N} \setminus \{0\}$ for $k \leq K$

$$\Phi(\nu) = (\nu - \frac{1}{K}) u_{k-1}(\nu)$$

and for $k > K$

$$\Phi(\nu) = \nu^k \phi_k + \cdots + \nu^{k-K-1} \nu^{(k-K+1)} \phi_{k-K+1} + (\nu - \frac{1}{K}) u_{k-1}(\nu)$$

with some $u_{k-1}(\nu) \in U_{k-1}$ and $\phi_r \in \mathcal{E}_r$. 

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Proof: It is clear that $\mathcal{I}_\alpha$ is a $\ast^\mu$-ideal if the eqn. (25) resp. (26, 27) are valid because of the form of $\nu^{(k)}$ in eqn. (16) and eqn. (18) for the $\ast^\mu$-product. So we only have to prove eqn. (25) and eqs. (26, 27).

Case i.) $\alpha \not\in \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ We prove (25) by induction on $k$. With $U_1 = \nu \mathcal{E}_0 + \mathcal{E}_1$ we get $\Phi(\nu) = c + \nu \sigma(A)$ for some $c \in \mathbb{C}$ and $\sigma(A) \in \mathcal{E}_1$. $\Phi(\alpha)$ is $0$ implies $A = -\frac{a}{\alpha} \mathbf{1}$ where $\mathbf{1}$ is the unit matrix. So we have $\Phi(\nu) = c(1 - \frac{\nu}{\alpha})$. This proves the case $k = 1$.

Consider now the induction step $k-1 \to k$. Since $U_k = \nu^{(k)} \mathcal{E}_k + \nu U_{k-1}$ we write for $\Phi \in U_k$

$$\Phi(\nu) = \nu^{(k)} \phi_k + \nu u_{k-1}(\nu)$$

with $\phi_k \in \mathcal{E}_k$ and $u_{k-1}(\nu) \in U_{k-1}$. Then $\Phi(\alpha) = 0$ implies $\phi_k = -\frac{\alpha}{\alpha^{(k)}} u_{k-1}(\alpha) \in \mathcal{E}_{k-1}$ because $\alpha^{(k)} \neq 0$. Writing $\nu^{(k)} = (1 - (k - 1)\nu) \nu^{(k-1)}$ we get

$$\Phi(\nu) = \nu^{(k-1)} \phi_k + \nu \left( - (k - 1) \nu^{(k-1)} \phi_k + u_{k-1}(\nu) \right) =: \nu^{(k-1)} \phi_k + \nu u'_{k-1}(\nu)$$

with $u'_{k-1}(\nu) \in U_{k-1}$. In $\Phi(\nu) = (\nu - \alpha) u'_{k-1}(\nu) + \nu^{(k-1)} \phi_k + \alpha u''_{k-1}(\nu)$ the first term vanishes at $\nu = \alpha$ so we have

$$\left. \nu^{(k-1)} \phi_k + \alpha u'_{k-1}(\nu) \right|_{\nu = \alpha} = 0.$$

But $\nu^{(k-1)} \phi_k + \alpha u'_{k-1}(\nu)$ is an element in $U_{k-1}$ so we can apply the induction and conclude

$$\Phi(\nu) = (\nu - \alpha) u'_{k-1}(\nu) + (\nu - \alpha) u''_{k-2}(\nu)$$

with some $u''_{k-2}(\nu) \in U_{k-2}$. This proves the first part.

Case ii.) Let $\alpha = \frac{1}{K}$ with $K \in \mathbb{N} \setminus \{0\}$ and $\Phi(\nu) \in U_k$. We have to consider the two cases $k \leq K$ and $k > K$ separately:

a.) For $k \leq K$ we have $\left(\frac{1}{K}\right)^{(k)} \neq 0$ and we can apply the same arguments as in the first part, hence $\Phi(\nu) = (\nu - \frac{1}{K}) u_{k-1}(\nu)$ with $u_{k-1}(\nu) \in U_{k-1}$.

b.) For $k > K$ we have for $r \geq 1$ $\nu^{(K+r)} \big|_{\nu = \frac{1}{K}} = 0$ according to the definition (16) of $\nu^{(k)}$. Hence in every $\Phi(\nu) \in U_k$

$$\Phi(\nu) = \nu^{(k)} \phi_k + \cdots + \nu^{k-K-1} \nu^{(K+1)} \phi_{k+1} + \nu^{k-K} u_K(\nu)$$

the first terms are automatically elements of $\mathcal{I}_{\frac{1}{K}}$ and $\Phi(\frac{1}{K}) = 0$ implies $u_K(\frac{1}{K}) = 0$. But this is an element of $U_K$ and we can apply the case a.) Hence $u_K(\nu) = (\nu - \frac{1}{K}) u'_{K-1}(\nu)$ which proves the second part. $$\square$$

The quotient algebras $U/\mathcal{I}_\alpha$ can now easily be described:

**Theorem 3** The quotient algebra $A_\alpha := U/\mathcal{I}_\alpha$ is isomorphic to one of the following algebras:

i.) For $\alpha$ not equal to one of the rational numbers $1, \frac{1}{2}, \frac{1}{3}, \ldots$ the algebra $A_\alpha$ is isomorphic to the vector space of representative functions $\mathcal{E}$ equipped with the multiplication $\ast_\alpha$ defined by ($f \in \mathcal{E}_k, g \in \mathcal{E}_l; k, l \in \mathbb{N}$):

$$f \ast_\alpha g := \sum_{r=0}^{\min(k,l)} \frac{\alpha^r \alpha^{(k+l-r)}}{r! \alpha^{(k+l)}} M_r^{(k,l)}(f, g)$$

(28)
where the real number $\alpha^{(k)}$ is defined by the formula (14) and $M_{r}^{(K)}(f, g)$ is given in (13).

ii.) Let $\alpha$ be equal to $\frac{1}{K}$ with $K$ a positive integer. Then $A_\alpha$ is isomorphic to the finite-dimensional complex algebra of linear endomorphisms of $\mathbb{C}^N$ with $N := \binom{n+K}{K}$. The isomorphism is given by the map

$$A \mapsto \frac{\nu^{(K)}}{(1/K)^{(K)}} \sigma(A) \mod \mathcal{I}_{1/K} \quad (29)$$

where $\sigma(A)$ is the Berezin-Rawnsley symbol of the complex $N \times N$-matrix $A$ (see (13)). The matrix product $(AB)_{i_1 \cdots i_K,j_1 \cdots j_K}$ is given by $A_{i_1 \cdots i_K,a_1 \cdots a_K} B_{a_1 \cdots a_K,j_1 \cdots j_K}$.

PROOF: i) According to the preceding Lemma the ideal $\mathcal{I}_\alpha$ is equal to $(\nu - \alpha)\mathcal{U}$ which amounts to simply substituting $\nu = \alpha$ in (13) which is obviously well-defined.

ii) According to the second part of the preceding Lemma we get $\mathcal{U}_K/\mathcal{I}_{1/K} = \mathcal{U}_K/\mathcal{I}_{1/K}$ for $k \geq K$. For $\mathcal{U}_K$ we may substitute $\nu$ for $1/K$ since $(1/K)^{(k)} \neq 0$ for $k \leq K$. Hence $\dim \mathcal{A}_{1/K} = \dim \mathcal{E}_K = \binom{n+K}{K}^2$. Moreover,

$$\frac{\nu^{(K)}}{(1/K)^{(K)}} \sigma(A) \star \frac{\nu^{(K)}}{(1/K)^{(K)}} \sigma(B) \mod \mathcal{I}_{1/K}$$

$$= \sum_{r=0}^{K} \frac{\nu^r}{r!} \frac{\nu^{(2K-r)}}{(1/K)^{(K)}(1/K)^{(K)}} M_{r}^{(K,K)}(\sigma(A), \sigma(B)) \mod \mathcal{I}_{1/K}$$

$$= \frac{(1/K)^K}{K!} \frac{\nu^{(K)}}{(1/K)^{(K)}(1/K)^{(K)}} M_{r}^{(K,K)}(\sigma(A), \sigma(B)) \mod \mathcal{I}_{1/K}$$

$$= \frac{\nu^{(K)}}{(1/K)^{(K)}} \sigma(AB) \mod \mathcal{I}_{1/K},$$

since a simple calculation shows that

$$\pi^*(M_{K}^{(K,K)}(\sigma(A), \sigma(B)))(z) = K!K! \frac{\tilde{\sigma}(AB)(z)}{x^K}.$$ 

This proves the Theorem. $\square$

Remarks: Note that for each $A \in \mathfrak{u}(n+1)$ the Berezin-Rawnsley symbol $\sigma(A)$ is contained in $\mathcal{E}_K$ and thus uniquely corresponds to a linear operator in $\mathcal{B}(\mathcal{H}^{(K)})$ which is mapped via the linear map (23) to $\mathcal{A}_{1/K}$. By Lemma 4 it follows that this defines a representation of $\mathfrak{u}(n+1)$ in $\mathbb{C}^N$, which is irreducible: in fact, by Lemma 3 we know that the momentum map $P(A)$ star-commutes with some function iff it Poisson commutes with that function which is only possible iff that function is constant since the unitary group acts transitively on $\mathbb{C}P^n$. Since the Poisson bracket with $P(A)$ obviously preserves each $\mathcal{E}_K$, it follows that this reasoning carries over to the quotient by $\mathcal{I}_{1/K}$.

From the physical point of view the second part of the preceding Theorem can be viewed as follows: The classical phase space reduction of $\mathbb{C}^{n+1} \setminus \{0\}$ to $\mathbb{C}P^n$ was motivated by the $U(1)$-action on $\mathbb{C}^{n+1} \setminus \{0\}$ which is just the flow of the classical isotropic $(n+1)$-dimensional
harmonic oscillator with Hamiltonian $H = \frac{1}{2}x$ (The frequency and mass are normalised to 1). Passing from $\mathbb{C}^{n+1} \setminus \{0\}$ to $\mathbb{C}P^n$ for a fixed value $\mu \in \mathbb{R}^-$ of the momentum mapping $J = -\frac{1}{2}x$ means classically that the harmonic oscillator is considered at a fixed energy $E = -\mu$. This is also true in the quantum mechanical case but now the energy is quantised. Only for the discrete values $\frac{1}{n}$, $K$ a positive integer, of the formal parameter $\nu = \frac{\lambda}{2\pi}$ one obtains finite dimensional algebras of observables $U/\mathcal{I}_{1/K}$ as one would physically expect for the compact phase space $\mathbb{C}P^n$ because the phase volume is finite and each state ‘occupies a phase cell of volume not smaller than $\hbar^n$ which results in a finite-dimensional Hilbert space for the quantum mechanical states. With $\lambda = 2\hbar$ the quantised energy is given by

$$E_K = \hbar K$$

(where the usual ground state energy $\frac{1}{2}\hbar(n + 1)$ is absent because of the Wick ordering). Note that in this interpretation the formal parameter $\lambda = 2\hbar$ is not quantised but the energy $E = -\mu$ is. The dimensions of the operator algebras for a fixed $K \in \mathbb{N}$ correspond to the well-known degeneracy of the $K$th energy eigenvalue of the isotropic harmonic oscillator (see e.g. [15 eqn XII.64]).

## 5 Other Examples

In this section we briefly sketch how the programme mentioned in the introduction applies to the deformation quantization of other phase spaces which have been dealt with in the literature:

1. Consider complex $n + 1$-space $\mathbb{C}^{n+1}$ as a symplectic manifold in the usual manner, i.e. with symplectic form $\omega = \frac{1}{2} \sum_{i=0}^{n} dz^i \wedge d\bar{z}^i$. The Wick product (4) then defines a star product on this space. It is natural to consider the action of $\mathbb{C}^{n+1}$ on itself by translations. Suppose that the smooth complex-valued function $F$ on $\mathbb{C}^{n+1}$ is representative with respect to this group action, i.e. there is a finite number $L$ of linearly independent smooth complex-valued functions $F_1, \ldots, F_L$ on $\mathbb{C}^{n+1}$ such that $F(z + v) = \sum_{a=1}^{L} \beta_a(v) F_a(z)$ with smooth complex-valued coefficients $\beta_a$. The same equation holds for each $F_a$ thus giving rise to a coefficient matrix $\beta_{ab}(v)$. Since $\mathbb{C}^{n+1}$ is abelian, $\beta_{ab}(v)$ commutes with each $\beta_{ab}(w)$ whence all these matrices can simultaneously be transformed to Jordan normal form. It can easily be seen that the generalized eigenvectors of $\beta_{ab}(v)$ are of the form $p(z) e^{b_i z^i + c_i \bar{z}^i}$ where $p$ is a complex-valued polynomial function of $(z, \bar{z})$ and $b_i, c_j$ are complex numbers. Conversely, it is easy to see that each linear combination of the functions of this form is indeed representative. Since this space of functions is a subalgebra of the algebra of all smooth complex-valued functions on $\mathbb{C}^{n+1}$ (with respect to pointwise multiplication) which clearly is closed under complex conjugation, contains a unit element, and separates points it is dense in $C^\infty(\mathbb{C}^{n+1})$ with respect to the uniform topology on compacta thanks to the Stone-Weierstrass Theorem. Note that the space of polynomials is a dense proper $\mathbb{C}^{n+1}$ submodule of representative functions which would be impossible for compact Lie groups (compare the proof of Lemma [3]).

Writing $e_{(a,b)}$ for the exponential function $z \mapsto e^{a_i z^i + b_i \bar{z}^i}$ parametrized by two complex vectors $a, b \in \mathbb{C}^{n+1}$ we easily obtain the following formula:

$$e_{(a,\rho, b, \sigma)} \ast e_{(a', \rho', b', \sigma')} = e^{\lambda(a_i + \rho_i)(b'_i + \sigma'_i)} e_{(a, \rho, b + \sigma)} e_{(a' + \rho', b', \sigma')},$$

(30)
where $a, a', b, b', \rho, \rho', \sigma, \sigma' \in \mathbb{C}^{n+1}$. After differentiating this formula a finite number of times with respect to the components of $\rho, \rho', \sigma, \sigma'$ at $\rho = \rho' = \sigma = \sigma' = 0$ (which generates polynomial prefactors) we see that the Wick-product of two representative functions is again representative and entire analytic in the formal parameter $\lambda$. Therefore the algebra $\mathcal{U}$ can be chosen to be the subalgebra of $C^\infty(\mathbb{C}^{n+1})[[\lambda]]$ which consists of polynomials in $\lambda$ with coefficients in the space of representative functions. Substituting $\lambda$ for a real number $\alpha$ then is straight-forward.

2. Consider now the $2n$-torus $T^{2n} := S^1 \times \cdots \times S^1$ ($2n$ factors). Let $(\varphi^1, \ldots, \varphi^{2n}) =: \varphi$, $0 \leq \varphi^i < 1$, denote the standard angle co-ordinates on $T^{2n}$ and take any nondegenerate complex $2n \times 2n$-matrix $(\Lambda^i_j)$. The Moyal product may then be defined by for two smooth, complex-valued functions $f, g$ on $T^{2n}$ as follows:

$$f * g := \sum_{r=0}^{\infty} \frac{(\lambda/2\pi i)^r}{r!} \Lambda^{i_1j_1} \cdots \Lambda^{i_rj_r} \frac{\partial^r f}{\partial \varphi^{i_1} \cdots \partial \varphi^{i_r}} \frac{\partial^r g}{\partial \varphi^{j_1} \cdots \partial \varphi^{j_r}}$$

It is known that this formula defines an associative deformation for the pointwise multiplication in $C^\infty(T^{2n})$ (see e.g. [3]). The space of representative functions for the torus action on itself is spanned by the Fourier modes $\partial^{r} \varphi^{i_1} \cdots \partial^{r} \varphi^{i_r}$ which again is a holomorphic sheaf in the formal parameter $\lambda$. This space is spanned by the Fourier modes consists of representative functions. Since it is a subalgebra of $C^\infty(T^{2n})$ which is closed under complex conjugation, contains a unit element, and separates points it is a dense $T^{2n}$-submodule of the space of all representative functions (by the Stone-Weierstrass Theorem) which has to be equal to that space since $T^{2n}$ is compact (compare again the proof of Lemma (3)). The Moyal product of two Fourier modes is then simply be computed by

$$T_k * T_{k'} = e^{2\pi i k K} T_k T_{k'}$$

which again is an entire function in the formal parameter $\lambda$, whence the choice of $\mathcal{U}$ and the substitution of the formal parameter is completely analogous to the previous example.

Suppose now that the matrix $(\Lambda^i_j)$ is integral and the greatest common divisor of the matrix elements is equal to 1. Choosing $\lambda = \frac{1}{K}$, $K$ a positive integer, it is easily seen from the above formula (32) that the subspace $\mathcal{J}_{1/K}$ spanned by all elements of the form $T_k - T_{k+Kk'}$ with $k, k' \in \mathbb{Z}^{2n}$ is an $*$-ideal in the $*$-algebra of representative functions. It can be shown (see [4]) that the quotient algebra is a simple complex algebra of dimension $K^{2n}$, which is related to the geometric quantization on the $2n$-torus in the theta-bundle and its tensor powers.

3. Let $\mathbb{D} := \{ v \in \mathbb{C} \mid |v|^2 < 1 \}$ be the Poincaré disk in the complex plane. As we have indicated in the last section of [4] we can use the projective representation of $\mathbb{D}$: in $\mathbb{C}^2 \setminus \{0\}$ consider the open subset defined by the inequality $0 < y := |z^0|^2 - |z^1|^2$ (the function $y$ was defined with an erroneous sign in [3]). The image of this open set under the projection $\pi$ is an open set of $\mathbb{C}P^1$ which is holomorphically diffeomorphic to $\mathbb{D}$ via $\pi(z) \mapsto v := \frac{z^1}{z^0}$. In their article [5] Cahen, Gutt, and Rawnsley have considered the following functions $f_{p,q}(v)$ ($p, q \in \mathbb{N}$) on $\mathbb{D}$

$$f_{p,q}(v) := v^p(\overline{v}^{q,\frac{q}{2}})$$

Their pull-back to $\mathbb{C}^2 \setminus \{0\}$ is simply given by $(\pi^*f_{p,q})(z) = (z^1/z^0)^p (z^0/z^1)^q$. In [4] we have defined a star product on $\mathbb{D}$ by essentially replacing $x$ by $y$ and the operators $M_r$ by

$$\tilde{M}_r(G, H) := y^r g^{i_1j_1} \cdots g^{i_rj_r} \frac{\partial^r G}{\partial z^{i_1} \cdots \partial z^{i_r}} \frac{\partial^r G}{\partial \overline{z}^{j_1} \cdots \partial \overline{z}^{j_r}}$$

(34)
with \( g^{ij} := \text{diag}(1, -1) \). Observing that eqs (17) and (18) remain valid for arbitrary smooth complex-valued functions when the upper bound of the sum is \( \infty \) and that these formulas pass to the noncompact case with the above replacements and adapting the sign and ordering conventions of [10] we obtain the following star product:

\[
(f_{p,q} \ast f_{r,s})(\pi(z)) = \sum_{m=0}^{\infty} \frac{(-\nu)^m}{m!} \frac{(-\nu)^{(q+s-m)}}{(-\nu)^q (-\nu)^s} g^{m-q-s} g^{i_1j_1} \cdots g^{i_kj_k}
\]

\[
\times \frac{\partial^r (y^0 \pi^* f_{p,q})}{\partial z_1 \cdots \partial z_m}(z) \frac{\partial^r (y^s \pi^* f_{r,s})}{\partial z_1 \cdots \partial z_m}(z)
\]

\[
= \sum_{m=0}^{\infty} \frac{(-\nu)^m}{m!} \frac{(-\nu)^{(q+s-m)}}{(-\nu)^q (-\nu)^s} g^{m-q-s} (-1)^m
\]

\[
\times \frac{\partial^m ((z^1/z^0)^p (z^0 z^1)^q)}{\partial (z^1)^m}(z) \frac{\partial^m ((z^1/z^0)^r (z^0 z^1)^s)}{\partial (z^1)^m}(z)
\]

\[
= \sum_{m=0}^{\min(q,r)} \frac{\nu^m}{m!} \frac{(-\nu)^{(q+s-m)}}{(-\nu)^q (-\nu)^s} \frac{q!}{(q-m)!} \frac{r!}{(r-m)!} f_{p-r-m,q+s-m}(\pi(z))
\]

which reproduces the result of [10]. In [11] star products were computed on more general bounded symmetric domains.

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