Effect of finite range interactions on roton mode softening in a multi-component BEC

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Abstract
We consider the Gross-Pitaevskii model of a Bose–Einstein Condensate (BEC) for single-component and multi-component BEC. The pseudopotential for s-wave scattering between atoms is taken to be of a width of the order of the s-wave scattering length. Such an interaction giving rise to a roton minimum in the spectrum of elementary excitations of a single-component BEC is well known. However, softening roton modes take us into the strongly interacting BEC regime where three body losses occur. We study the roton mode softening for a multi-component BEC. We show that by increasing the number of components of a multi-component BEC, the roton mode can be softened at a progressively lower value of the gas parameter ($\alpha^2 n$), thus reducing the three body losses.

Keywords: Bose-Einstein condensates, multi-component condensates, roton mode softening

(Some figures may appear in colour only in the online journal)

1. Introduction

Since the discovery of Bose–Einstein condensation (BEC) in dilute atomic gases [1, 2], there have been many experimental and theoretical investigations in this field. BEC has been shown to possess superfluidity [3], whose phenomenological theory was given by Landau [4]. An interesting extension of the idea of superfluidity is the supersolid phase, which is superfluidity with solid order [5, 6]. The search for such phases of matter has been a greatly pursued topic and there have been a lot of theoretical proposals in this regards in BEC [7, 8], along with experimental observations [9, 10]. The conventional approach to obtain such a state in BEC is through the softening of roton modes. These have been mainly proposed in dipolar interactions, Rydberg-dressed atoms and optical lattice mediated interactions in a BEC [7, 8, 11]. These approaches take into consideration long-range interactions on top of the s-wave scattering between atoms. There have also been studies taking into account the non-local nature of s-wave interactions, without taking into account additional long-range interactions on top of this [12, 13]. The occurrence and lowering of roton minimum was shown in these cases too.

Apart from changing various parameters in an atomic BEC, like the density and scattering length [14], one can also change the number of distinct atomic components. These are the multi-component BECs [15–17]. The most studied amongst multi-component BECs are two-component BECs. There have been extensive studies regarding the miscibility [18], stability [19] and varying inter-species interaction strength [20] in a two-component BEC. BEC with more than two components have also been studied [17].

The main tool for theoretical investigations in a BEC has been the mean field Gross-Pitaevskii (GP) equation [21, 22]. The local GP equation for a single-component BEC predicts the ground state of the BEC to be one with uniform density [3], with the order parameter $\psi(r, t) = \sqrt{n} e^{-i\mu t/\hbar}$. The quantity $|\psi(r, t)|^2 = n$ gives the density of the condensate. Here $\mu$ is the chemical potential of the BEC. Small amplitude oscillations on top of this uniform density state have the dispersion relation $\omega = \sqrt{(\hbar^2 k^2/(4m^2)) + (k^2 gn/m)}$ [22]. A similar ground state exists for a multi-component BEC, where all the different components have uniform densities [22]. Also, a similar dispersion relation exists for excitations on top of the different uniform densities in a multi-component BEC. The point to note here is that this dispersion is in the presence of the $\delta$-function pseudopotential, i.e. one whose range tends to zero.

For an atomic BEC with temperature of the order of few hundred nano-Kelvin, the atoms have very low momenta.
Hence, the details of the inter-atomic potential can be ignored and the real potential can be replaced by an effective pseudopotential (say $V_{gp}$) which is repulsive, for the symmetric s-wave scattering between atoms [3]. The condition for using this approximation is that $\int dr \, V_{gp} = g$ holds, where $g = 4\pi \hbar^2 a/m$ denotes the strength of the inter-atomic s-wave interactions. This pseudopotential is taken to be a $\delta$-function in a dilute BEC. However, instead of taking a $\delta$-function, one may take a repulsive rectangular barrier as the scattering pseudopotential which may be able to capture the non-local behaviour of the inter-particle interactions. This sort of pseudopotential has been considered in literature [12, 23, 24]. This sort of extended pseudopotential gives rise to the formation of a roton minimum in a single-component BEC [8, 12, 13]. This roton minimum can be further lowered by varying the interaction strength between atoms.

In this article, we study the effect of taking a non-local pseudopotential for s-wave scattering on the dispersion relation of a multi-component BEC. We consider the non-local s-wave scattering and do not consider any additional long-range interactions. We start by discussing the existing results for small amplitude excitations on top of uniform density for single-component BECs with $\delta$-function pseudopotential. We then briefly discuss the extended pseudopotential in the form of a rectangular barrier which is extensively studied in literature [8, 12, 13, 25]. This barrier pseudopotential is a simple non-local extension of the $\delta$-function pseudopotential. We use this barrier pseudopotential to study elementary excitations in single as well as multi-component BEC.

The use of the delta function pseudopotential gives good agreement with the experiments for dilute gases where $a^3n \rightarrow 1$. However, when we go into the regimes where $a^3n \rightarrow 1$, one has to consider corrections on top of the GP equation with delta function pseudopotential [26, 27] to fully characterize the properties of the system. This is mainly because as $a^3n \rightarrow 1$, the finite range effects of the BEC particle interactions come into the picture. A simple way to account for these finite range effects is to consider an extended pseudopotential in place of the $\delta$-function pseudopotential. This is exactly what we do in this paper. We would like to re-emphasize the fact that the extended pseudopotential used by us, as the name suggests, a pseudopotential and not the actual interaction potential between the BEC particles. As mentioned before, such an approach has already been considered for single-component BEC in literature [8, 12, 13].

We first revisit the roton minimum appearing in the elementary excitation spectrum of a single-component BEC with extended pseudopotential which has been studied previously [12, 13]. This roton minimum can be lowered by increasing the gas parameter $a^3n$. However, this appearance and lowering of the roton minimum lies well outside the diluteness limit, and in the strongly interacting regime, for a single-component BEC. The diluteness limit is given by $a^3n \ll 1$ for a single-component BEC. A similar diluteness limit exists for a multi-component BEC given by $\sqrt{n_i n_j a_{ij}^3} \ll 1$ [28], where $n_i$ and $n_j$ are the densities of the $i$th and $j$th components and $a_{ij}$ is the inter-species scattering length. As we continue going away from the diluteness limit and towards the strongly interacting regime $a^3n \sim 1$, the lifetime of a BEC decreases due to the appearance of three body effects. Hence, it is important to achieve roton mode softening at a lower value of gas parameter, so that we may observe it over a significant interval of time. With this motivation, we study the effect of non-locality of s-wave scattering on the appearance and lowering of the roton minimum for a BEC with multiple components. We investigate whether we can soften the roton minimum in a multi-component BEC at a lower value of the gas parameter as compared to that for a single-component BEC.

We show that for multi-component BECs, the appearance and lowering of roton minimum can be achieved at a lower value of the gas parameter as we go on increasing the number of distinct components. We observe that even for a two-component BEC, roton mode softening can be achieved at a significantly lower value of the gas parameter.

### 2. Single-component BEC

#### 2.1. Excitations with local interactions

Let us discuss briefly the dispersion relation for a single-component BEC with $\delta$-function inter-particle pseudopotential given by $V_{gp}(r - r') = g \, \delta(r - r')$. This is a standard textbook calculation [3]. In the absence of an external potential, the GP equation with such contact interactions is given by

$$i\hbar \frac{\partial \psi(r, t)}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi(r, t) + g \psi(r, t)^3 \psi(r, t). \tag{1}$$

This equation admits a solution with uniform density. This uniform density state of a BEC is given by $\psi(r, t) = \sqrt{\rho} e^{-i\omega t/\hbar}$, where $|\psi(r, t)|^2$ gives the uniform density and $\mu$ is the chemical potential. Small amplitude oscillations on top of this uniform density ground state are taken to be of the form $\theta(r, t) = \sum_i n_i(r) e^{-\int_0^t v_i(r) e^{-\int_0^t \omega_i}}$. The calculation of the small amplitude oscillations over a uniform density state is a standard procedure [3] which involves treating $n_i$ and $v_i$ as excitations with small amplitudes and hence neglecting their higher powers. Now, we drop the subscript $l$ and take $u(r) = u e^{ikr}$, $v(r) = v e^{ikr}$ and collect terms which evolve in time identically. This procedure gives us the dispersion relation for excitations as $\omega = \sqrt{(\hbar^2 k^2 / (4m^2)) + (k^2 gn/m)}$. This relation tells us that for small $k$, the dispersion is phonon-like, whereas for large $k$ it is particle-like. To keep in mind is the fact that the current approach uses the $\delta$-function approximation for the pseudopotential, which is equivalent to taking the range of a rectangular barrier pseudopotential tending to zero.

Next we briefly describe the extended pseudopotential used in literature and then use it to study the dispersion relation for a single component and then multi-component BECs in the presence of non-local s-wave scattering.
represents the in
1 = \frac{c}{\sigma a} + a \sigma \varepsilon_{\text{e}}

density state of a BEC is given by
s-wave scattering. So, instead of taking the actual
are mainly concerned with the low energy scattering between
the s-wave scattering length. We would like to emphasize
spectrum of single as well as multi-component BEC. We take
barrier pseudopotential and study its effect on the excitation

Figure 1. Figure shows the dispersion relation for single-component
BEC in the presence of non-local inter-particle interaction
pseudopotential for certain values of the parameter \( a^3 n \). Here,
\( \Omega = m \omega^2 / h \) and \( \sigma = 1 \) for the sake of comparison.

2.2. Extended pseudopotential

In an atomic BEC, due to the low temperatures involved, we
are mainly concerned with the low energy scattering between
atoms. The most important contribution would then come
from s-wave scattering. So, instead of taking the actual interaction
potential between atoms, we can work with a pseudopotential(\( V_{\text{eff}} \)), with the condition that \( \int d\mathbf{r} \ V_{\text{eff}} = g \),
where \( g = 4 \pi \hbar^2 a / m \), \( a \) being the s-wave scattering length
[3]. The standard pseudopotential used is the \( \delta \)-function pseudopotential given by \( V_{\text{eff}} = g \ \delta(\mathbf{r} - \mathbf{r}') \). However, there
have been many works which use an extended pseudopotential in the form of a rectangular barrier [8, 12, 23]. This is the simplest extension of the \( \delta \)-function pseudopotential to incorporate the effects of non-locality of inter-atomic interactions. Pomeau et al [12] have taken the width of this rectangular barrier pseudopotential to be the s-wave scattering length. This is justified since, while considering low energy scattering from a rectangular barrier, the width of the barrier is of the order of the s-wave scattering length (see for example, Sakurai [29]). Considering such an extended pseudopotential leads to formation of a roton minimum in the spectrum of elementary excitations which can be further lowered by increasing the parameter \( a^3 n \). In what follows, we use this barrier pseudopotential and study its effect on the excitation spectrum of single as well as multi-component BEC. We take the width of the barrier pseudopotential to be of the order of the s-wave scattering length. We would like to emphasize here that we do not state that the width of the pseudopotential is exactly equal to the s-wave scattering length, but it is rather of the order of the s-wave scattering length.

2.3. Excitations with non-local interactions

Let us now look at the effect of non-local interactions on the small amplitude oscillation modes on top of the uniform density state of a BEC. As mentioned before, the uniform density state of a BEC is given by \( \psi(\mathbf{r}, t) = \sqrt{n} e^{-i \mathbf{k} \cdot \mathbf{r} / \hbar} \). Let us take the non-local interaction pseudopotential as
\( V_{\text{eff}}(\mathbf{r} - \mathbf{r}') = 3g / [4\pi (\sigma a)^3] \) for \( |\mathbf{r} - \mathbf{r}'| \leq \sigma a \) and zero otherwise, where \( \sigma \) is an order unity proportionality factor. Thus the width of the scattering pseudopotential is of the order of the s-wave scattering length.

The non-local GP equation in presence of the pseudopotential mentioned above can be derived from the full non-local GP equation, given by [3]
\[
\begin{align*}
\imath \hbar \frac{\partial \psi(\mathbf{r}, t)}{\partial t} &= \frac{-\hbar^2}{2m} \nabla^2 \psi(\mathbf{r}, t) \\
&\quad + \psi(\mathbf{r}, t) \int \left[ \delta_3(\mathbf{r}', t) V_{\text{eff}}(\mathbf{r} - \mathbf{r}') \psi(\mathbf{r}', t) \right] d\mathbf{r}'.
\end{align*}
\]

Inserting the pseudopotential mentioned above, we get
\[
\imath \hbar \frac{\partial \psi(\mathbf{r}, t)}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi(\mathbf{r}, t) \\
&\quad + \frac{3g}{(\sigma a)^3} \psi(\mathbf{r}, t) \int_{0}^{\sigma a} |\psi(\phi, t)|^2 d\phi,
\]
where \( \phi = \mathbf{r} - \mathbf{r}' \) and \( d\phi \) represents the infinitesimal volume element. The integral limits are so taken since we have taken the width of the scattering barrier to be of the order of the s-wave scattering length. We further set the dimensionless constant \( \sigma \) to unity for brevity. This we do while mentioning the caveat that the range of the pseudopotential is not exactly the scattering length, but is of the order of the scattering length.

Once again, we take the form of excitations on top of the uniform density state as \( \psi(\mathbf{r}, t) = \sum_{n} u_n(r) e^{-i \mathbf{k} \cdot \mathbf{r}} + v_n(r) e^{i \mathbf{k} \cdot \mathbf{r}} \) and treat \( u \) and \( v \) as excitations with small amplitudes and hence neglect their higher powers. As for the local interactions case, taking \( u(\mathbf{r}) = u e^{ikr} \), \( v(\mathbf{r}) = ve^{ikr} \) and collecting terms which evolve in time identically, we get
\[
\omega = \pm \frac{\hbar^2}{m} \frac{k^4}{4} + (4\pi n \sin \sigma \lambda),
\]
which can be equivalently written as
\[
\omega = \pm \frac{\hbar^2}{a^2 m} \frac{\lambda^4}{4} + (4\pi a^3 n \sin \sigma \lambda),
\]
where \( \lambda = ka \). The gas parameter \( a^3 n \) appearing in the equation above should be small in order for the three body interactions to be noticeably small. Particularly, the well-known diluteness limit is given by \( a^3 n \ll 1 \). However, there have been recent experiments which have explored the limits of the gas parameter, trying to push it towards the strongly interacting regime \( a^3 n \sim 1 \) [26, 27].

The dispersion relation given by equation (3) exhibits a roton minimum which can be lowered as we increase \( a^3 n \) as shown in figure 1. The roton minimum touches the \( ak \) axis at a value of \( a^3 n \sim 1.5 \) which is beyond the \( a^3 n \ll 1 \) limit. This would take us well into the strongly interacting regime. Nonetheless, we see that it is possible to pull the roton minimum down in principle. Thus we get a hint that it might be possible to pull the roton minimum down, at a lower value of \( a^3 n \), by introducing additional interactions such as those in
multi-component BEC. Since the macroscopic wave-function of different components in a multi-component BEC is different, there occur cross terms between the wave-functions in the GP equation, due to inter-species scattering. Thus for multi-component BEC, there exist additional interaction terms than the ones in the GP equation for a single-component BEC. In the following section, we consider multi-component BEC and study the manifestation of inter-species interactions in the dispersion relation of small amplitude oscillations.

3. Multi-component BEC

3.1. Excitations with local interactions

BECs with more than one components, in the absence of an external potential, can be analysed by taking into account the inter-particle interaction in the GP energy functional. This can be written as [30]

\[ E = \frac{\hbar^2}{2m} \sum_j \int d\mathbf{r} \left( |\nabla \psi_j(\mathbf{r})|^2 + \sum_{\ell = 1}^{j} \int d\mathbf{r} |\psi_j(\mathbf{r})|^{-2} \right) + \int d\mathbf{r}^{\prime} V_{ij}(\mathbf{r} - \mathbf{r}^\prime) \psi_i(\mathbf{r}) \psi_j(\mathbf{r}^\prime), \]

where \( g_{ij} \) represents the inter-species s-wave scattering strength, \( g_{ij} = 4\pi\hbar^2a_{ij}(m_i + m_j)/(m_im_j) \), and \( m_i \) and \( m_j \) represent the mass of the atoms of different species whose wave-functions are \( \psi_i \) and \( \psi_j \) respectively. The inter-particle scattering length is given by \( a_{ij} \). If the scattering pseudopotential is taken to be the \( \delta \)-function as before with \( V_{ii}(\mathbf{r} - \mathbf{r}^\prime) = g_i\delta(\mathbf{r} - \mathbf{r}^\prime) \) for intra-species scattering and \( V_{ij}(\mathbf{r} - \mathbf{r}^\prime) = g_{ij}\delta(\mathbf{r} - \mathbf{r}^\prime) \), we get the local GP equation for multi-component BEC as

\[ \frac{i\hbar}{\partial \psi_j(\mathbf{r}, t)} = \frac{\hbar^2}{2m} \nabla^2 \psi_j(\mathbf{r}, t) + g_{ij} |\psi_i(\mathbf{r}, t)|^2 \psi_j(\mathbf{r}, t) + \psi_j(\mathbf{r}, t) \sum_{i=1}^{j} g_{ij} |\psi_i(\mathbf{r}, t)|, \]

for each of the several components, denoted by the subscript \( j \).

Now, as for the single-component case, these equations admit solutions with uniform density for each component given by \( \psi_j(\mathbf{r}, t) = \sqrt{n_j} e^{-i\omega_j t/\hbar} \). Such BEC might exist stably in many configurations with regards to the miscibility. The general miscibility condition for two components \( i \) and \( j \) is given by \( a_{ij} < \sqrt{a_i a_j} \) [31]. We shall consider here a uniform, homogeneous mixture state of the many components, i.e. we consider a state where the miscibility criterion holds for all components. We shall first look at the dispersion relation of small amplitude oscillations for the local GP equation given by equation (5) and then look at the non-local case in the next section.

We take the small amplitude excitations over the state \( \psi_j(\mathbf{r}, t) = \sqrt{n_j} e^{-i\omega_j t/\hbar} \) of the form \( \theta_j(\mathbf{r}, t) = \sum_l [u_{lj}(\mathbf{r}) e^{-i\omega_j t/\hbar} + v_{lj}(\mathbf{r}) e^{i\omega_j t/\hbar}] \). Further, as for the single-component case, we take the form of \( u_j \) and \( v_j \) as \( u_{lj}(\mathbf{r}) = u_{lj} e^{i\mathbf{k}_l \cdot \mathbf{r}} \) and \( v_{lj}(\mathbf{r}) = v_{lj} e^{-i\mathbf{k}_l \cdot \mathbf{r}} \). For a two-component BEC, this dispersion relation has two distinct branches [32]. In general for a multi-component BEC we may get a complicated dispersion relation for excitations. To simplify the system, we assume that all the intra-species scattering lengths are equal \( (a_{ij}, \text{for example}) \). This simplification enables us to look particularly into the effects of the presence of multiple components in a BEC on the roton mode softening. This simplification also brings out the role of inter-species interaction on roton mode softening, which is our core interest here.

Taking species with equal densities \( (n) \) and comparable masses \( (m) \) we get the following matrix structure for excitations:

\[
\begin{pmatrix}
U_- & X & c & c & ... \\
X & U_+ & c & c & ... \\
c & c & U_+ & X & c & c & ... \\
... & ... & ... & ... & ... & ... & ...
\end{pmatrix}
\begin{pmatrix}
u_l \\
v_i \\
u_j \\
... \\
... \\
v_l \\
0
\end{pmatrix} = 0, \quad (6)
\]

where

\[ U_\pm = \pm \omega \mp \frac{\hbar^2 k^2}{2m} - g_in \]
\[ X = -g_in \]
\[ c = -g_jn, \]

with \( s \) as the total number of distinct components in the multi-component BEC.

To get a non-trivial solution for excitations \( u \) and \( v \) on top of the uniform density ground state, the determinant of the \( 2s \times 2s \) square matrix in equation (6) must be zero. This condition gives us

\[
\left| w^2 - \frac{\hbar^2}{m^2} \left( \frac{k^4}{4} + 4\pi na_1 k^2 + (s-1)4\pi na_2 k^2 \right) \right| \left| w^2 - \frac{\hbar^2}{m^2} \left( \frac{k^4}{4} + 4\pi na_1 k^2 - 4\pi na_2 k^2 \right) \right|^{(s-1)} = 0.
\]

This equation gives us two branches of the dispersion relation as

\[
w = \frac{\hbar}{m} \left[ \frac{k^4}{4} + 4\pi na_1 k^2 + (s-1)4\pi na_2 k^2 \right]^{1/2},
\]

or/and

\[
w = \frac{\hbar}{m} \left[ \frac{k^4}{4} + 4\pi na_1 k^2 - 4\pi na_2 k^2 \right]^{1/2}. \quad (7)
\]
3.2. Excitations with non-local interactions

Let us now consider non-local pseudopotential for both intraspecies and inter-species interactions in a multi-component BEC. Let the interaction pseudopotential be given by \( V_i^{ij}(r-r') = 3g_{ij}/4\pi \sigma_{ij} \) for \( |r-r'| \leq \sigma_{ij} \) and zero otherwise for the intra-species interaction. For the inter-species interaction, we take the interaction pseudopotential as \( V_i^{ij}(r-r') = 3g_{ij}/4\pi \sigma_{ij} \) for \( |r-r'| \leq \sigma_{ij} \) and zero otherwise. We would like to re-emphasize the fact that we do not claim the range of the pseudopotential to be exactly equal to that of the scattering length, but it is of the order of the scattering length, which is what is considered here to simplify the model.

Now, we write the non-local GP equation which can be derived from the energy functional in equation (4) by considering the interaction pseudopotential to be non-local, as given above. It is of the form

\[
i\hbar \frac{\partial \psi_j(r,t)}{\partial t} = -\hbar^2 \frac{\nabla^2}{2m} \psi_j(r,t) + \frac{3g_i}{(\sigma_{ij})^3} \psi_j(r,t) \\
\times \int_0^{\sigma_{ij}} |\psi_i(\phi,t)|^2 d\phi + \sum_i \left[ \frac{3g_i}{(\sigma_{ij})^3} \psi_j(r,t) \right] \\
\times \int_0^{\sigma_{ij}} \psi_i^*(\phi,t) \cdot \psi_j(\phi,t) d\phi.
\]

As for the single-component case, \( \phi = r-r' \) and \( d\phi \) represents the infinitesimal volume element. The above equation admits a solution where each of the species has a uniform density, same as that for the local case. This can be verified by plugging in the form for different \( \psi_j \) as \( \psi_j(r,t) = \sqrt{N_j} e^{-i\theta_j/\hbar} \). As before, considering small amplitude oscillations \( \theta_j(r,t) \) on top of this uniform density state, we get the following matrix by mode matching:

\[
\begin{pmatrix}
A_{-} & B & D & D & \ldots \ldots \\
B & A_{-} & D & D & \ldots \ldots \\
D & D & A_{-} & B & D & \ldots \ldots \\
D & D & B & A_{-} & D & \ldots \ldots \\
\ldots \ldots & D & D & A_{-} & B & \ldots \ldots \\
\ldots \ldots & D & D & B & A_{-} & \ldots \ldots \\
\end{pmatrix}
\begin{pmatrix}
u_1 \\
u_1 \\
u_1 \\
u_1 \\
u_1 \\
u_1 \\
\end{pmatrix}
= \begin{pmatrix}
0 \\
0 \\
\ldots \ldots \\
\ldots \ldots \\
0 \\
0 \\
\end{pmatrix},
\]

\( A_{\pm} = \pm \omega - \frac{\hbar^2 k^2}{2m} - g_i \frac{\sin \sigma_{ij}}{a_k} \)

\( B = -g_i \frac{\sin \sigma_{ij}}{a_k} \)

\( D = -g_i \frac{\sin \sigma_{ij}}{a_k} \).

Here, too, we have considered the simplification that all intraspecies scattering lengths are equal \( (a_1) \) and so are the interspecies scattering lengths \( (a_2) \).

As before, to get a non-trivial solution for excitations \( u \) and \( v \) on top of the uniform density ground state, the determinant of the \( 2 \times 2 \) square matrix in equation (9) must be zero. Since the matrix structure is similar to the one for the local multi-component case, we get the following equation:

\[
\omega^2 - \frac{\hbar^2 k^4}{4m^2} + 4\pi n a_k^2 \left[ \frac{\sin \sigma a_k}{a_k} \right] \\
+ (s-1)4\pi n a_k^2 \left[ \frac{\sin \sigma a_k}{a_k} \right] \\
\times \left[ \omega^2 - \frac{\hbar^2 k^4}{4m^2} + 4\pi n a_k^2 \left[ \frac{\sin \sigma a_k}{a_k} \right] \\
- 4\pi n a_k^2 \left[ \frac{\sin \sigma a_k}{a_k} \right] \right]^{(s-1)} = 0.
\]

This condition gives us two branches of the dispersion relation for excitations as

\[
\omega = \frac{\hbar}{m} \left[ \frac{k^4}{4} + 4\pi n a_k^2 \left[ \frac{\sin \sigma a_k}{a_k} \right] \right] \\
+ (s-1)4\pi n a_k^2 \left[ \frac{\sin \sigma a_k}{a_k} \right]^{\frac{2}{s}} \\
or/and \\
\omega = \frac{\hbar}{m} \left[ \frac{k^4}{4} + 4\pi n a_k^2 \left[ \frac{\sin \sigma a_k}{a_k} \right] \right]^{-\frac{2}{s}}.
\]

The above equations exhibit a roton minimum, similar to the one for a single-component BEC with non-local pseudopotential. However, there is a crucial difference, since now the factor \( s \) in the above equation means that the dispersion relation would depend on the number of distinct species in the multi-component BEC. Additionally, for a single-component BEC, there was just one length scale for the scattering length. However, for a multi-component BEC, on top of the intra-species scattering length there exists inter-species scattering length as well.

We write the dispersion relation in equation (10) in a form similar to that of equation (3) as

\[
\omega = \frac{\hbar}{a_1^2 m} \left[ \frac{\lambda^4}{4} + 4\pi a_1^3 n \lambda \sin \sigma \lambda + (s-1)4\pi a_1^3 n \lambda \sin \sigma \beta \lambda \right] \\
or/and \\
\omega = \frac{\hbar}{a_2^2 m} \left[ \frac{\lambda^4}{4} + 4\pi a_2^3 n \lambda \sin \sigma - 4\pi a_2^3 n \lambda \sin \sigma \beta \lambda \right]^{\frac{2}{s}}
\]

where \( \lambda = a_1 k \) and \( \beta = a_2/a_1 \). As the miscibility criteria states, we should stay in the limit \( a_2 < a_1 \), implying \( \beta < 1 \) [31]. In addition, as we have taken the densities of each component to be the same, the gas parameter takes the form \( na_1^2 \) and \( na_2^2 \) for the inter-species and intra-species scattering length respectively [28]. Since \( a_2 < a_1 \), \( na_2^2 < na_1^2 \).

As we had seen in the single-component case with non-local interactions, roton lowering was possible due to the
The $\sin ak$ term, since it attains negative values. Here, too, the roton minimum is obtained due to $\sin ak$ and $\sin \beta ak$ terms. However, the behaviour of the upper and lower branches of equation (11) is different due to the opposite sign of $\sin ak$ and $\sin \beta ak$, and the absence of $s$ in the lower branch. Also, notice that the lower branch is common to all the multi-component BEC with $s \geq 2$. In figure 2 we plot this common branch for various values of the gas parameter. The lowest value of the gas parameter for which the roton mode of this branch may be pulled down to the $\omega = 0$ axis is $a_1^3 n \sim 0.616$. Notice here that since $\sin \lambda$ and $\sin \beta \lambda$ have opposite signs, the roton minimum would be lowered when $\sin \beta \lambda$ is positive and $\sin \lambda$ is negative. This is in contrast to the upper branch where both the terms being negative will enhance the roton lowering. Hence, for the common branch, the value of $\beta = a_2/a_1$ for which the roton mode softening is enhanced is $\beta \sim 0.49$. For the upper branch, as we shall see, the roton lowering would be enhanced for $\beta$ very close to 1. Having discussed the common branch, we next discuss the behaviour of the upper branch of equation (11).

In figure 3 we plot the dispersion relation of elementary excitations for a two-, three- and four-component BEC for the upper branch in equation (11). This figure shows that, for a fixed value of the parameter $a_1^3 n$, the roton minimum is lowered as we increase the number of distinct components in a BEC. From the figure, we also see that the lower branch of $\omega$ in equation (11) does not show any instability in the regime of parameters considered. The values taken in the figure are such that $a_2 < a_1$, such that the miscibility limit is satisfied. For the single-component BEC (figure 1) the roton lowering was achieved at $a_1^3 n \sim 1.5$. However, here the roton lowering is achieved at a value of $a_1^3 n \sim a_3^3 n \sim 0.43$ for a four-component BEC and $a_1^3 n \sim a_3^3 n \sim 0.57$ for a three-component BEC. Upon further varying the value of $a_1^3 n$ we may observe roton lowering for a two-component BEC in the upper branch. The value of gas parameter for this to happen is $a_1^3 n \sim 0.85$. However, we have already seen that the common branch of excitations for a multi-component BEC gives us roton mode softening at a lower value of gas parameter, viz., $a_1^3 n \sim 0.616$. Since this branch is present for $s \geq 2$, it is present for a two-component BEC as well. Hence the common (lower)branch would give roton lowering for a two-component BEC at a smaller value of the gas parameter than the upper branch.

The condition for three body losses to be negligible in a BEC is given by the diluteness condition $a_1^3 n \ll 1$. As the parameter $a_1^3 n \to 1$ or beyond one, loss of atoms from the condensate occurs. Nonetheless, BEC in such strongly interacting regimes have been obtained and studied [27]. This is achieved by tuning the s-wave scattering length [33]. The lifetime of such a strongly interacting BEC would depend on the gas parameter and hence it is important to obtain the roton lowering at a lower value of the gas parameter.

The loss rate for a BEC is given by the equation $dN(t)/dt = -K_3^{\text{tot}}(n^3)N(t)/6$, where $N(t)$ is the number of atoms at time $t$, $(n^3)$ is the mean square density of the condensate and $K_3^{\text{tot}}$ is the total three body loss rate constant [34]. The factor $K_3^{\text{tot}}$ is dependent on the gas parameter $a_1^3 n$. From Feshbach resonance experiments for commonly used atomic BEC components like rubidium, caesium, potassium, lithium and sodium [34–38], one can see that obtaining values of $a_1^3 n$ near 1 requires working very close to the resonance. These experiments have also measured value of $K_3^{\text{tot}}$ for a range of values of the scattering lengths. One can see from these experiments that $K_3^{\text{tot}}$ increases very sharply near the resonance. Hence, an increase in the value of $a_1^3 n$ from 0.6 to 1.5 would increase the loss rate $K_3^{\text{tot}}$ by two to three orders of magnitude. Thus, it can be seen that the reduction in the value of $a_1^3 n$ for roton mode softening that we have shown is quite significant.

We can see that for finite $k$, setting the width of the pseudopotential interaction tending to zero, as is done for local interactions, would mean setting the scattering length tending to zero. One can set $a \to 0$ in equation (10). This would essentially give us back the multi-component dispersion relation in equation (7), which we had got by neglecting the non-locality of the pseudopotential in the multi-component BEC.

The phenomenon seen in figure 3 of the roton mode softening at a lower value of gas parameter arises mainly because of inter-species interaction. The lower value of gas parameter ensures that the three body losses reduce as we increase the number of components. As the strongly interacting regime $a_1^3 n \to 1$ has been achieved experimentally, such a roton lowering is realizable for a multi-component
BEC by tuning the scattering lengths involved at lower loss rates.

4. Discussion

In this paper, we have studied elementary excitations in a BEC in the presence of non-local s-wave interactions for multi-component BECs. We started by studying the effect of non-local interactions on the spectrum of elementary excitations in a single-component BEC. We considered these elementary excitations on top of the uniform density ground state. The consideration of finite range of inter-boson interactions gives us a roton mode in the spectrum of elementary excitations. This roton mode can be lowered by increasing the gas parameter $(a^3n)$. The roton energy becomes zero for a value of $a^3n > 1$ for a single-component BEC. We then have studied elementary excitations on top of a ground state of multi-component BEC, where each of the components has a uniform density. We work in the limit where the various components are miscible. Consideration of finite range of inter-boson interactions gives us a roton mode for multi-component BEC as well. We investigated to find the lowest value of the gas parameter $a^3n$ for which we get zero energy roton modes. We have shown that for BEC with two components and above, zero energy roton modes can be obtained for $a^3n < 1$. Furthermore, we have shown that as we go on increasing the distinct components of a multi-component BEC, zero energy roton modes are obtained for lower and lower values of $a^3n$.

Note that we have just considered s-wave scattering and not any other form of long-range interactions. The non-local s-wave scattering pseudopotential is taken to be a rectangular barrier with width of the order of the s-wave scattering length. Previous works have shown that in the presence of non-local interaction pseudopotential, there appears a roton minimum in the dispersion relation for a single-component BEC. This roton minimum can be lowered by modifying the width of interactions, which can be taken to be of the order of the scattering length. However, for achieving this roton mode softening, one has to go beyond the diluteness limit $(\sim a^3n)$ and into the strongly interacting regime.

The experimental progress in obtaining strongly interacting BEC is substantial with the gas parameter being pushed very close to $a^3n \sim 1$. This is achieved by tuning the s-wave scattering length. However, as we go near the strongly interacting BEC regime $a^3n \sim 1$ value, three body losses become significant. The magnitude of such losses would increase with the increase of the gas parameter. Hence it is necessary to obtain zero energy roton modes at a lower value of the gas parameter and this is where multi-component BEC helps. We have shown that even using just a two-component BEC, we are able to push the value of gas parameters for obtaining zero energy roton modes from 1.5 to 0.616. As we increase the number of components further, we get zero energy roton modes for even lower values of gas parameters.

As the roton mode exhibits itself in a single-component BEC, it is but natural to expect such a roton minimum in a multi-component BEC as well. What is peculiar to the multi-component BEC is the softening of the roton mode, at a certain value of gas parameter, just by increasing the number.

Figure 3. Figures show the dispersion relation for small amplitude excitations of a BEC, in the presence of non-local interaction pseudopotential between atoms, with various number of distinct components as labelled in the graphs. The plots are for different values of the intra-species gas parameter $a^3n$ and $\beta = a_2/a_1$. Here, $\Omega = m_w a^2 / h$ and $\sigma = 1$ for the sake of comparison.
of distinct components. This roton mode softening comes about mainly due to the fact that every species in a multi-component BEC is represented by a particular wave-function. This brings about the presence of cross-coupling terms in the GP equation used to describe the multi-component system. Due to such terms, the small amplitude oscillation modes of one component couple to the ground state of other component at the expense of energy which helps lower the roton energy. This is seen as roton mode softening in the dispersion relation. Since the strongly interacting regime of a BEC is achieved experimentally, the technique suggested by us would help realize zero energy roton modes at a lower value of gas parameter, thus reducing losses.

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