QED in the Exact Renormalization Group

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Abstract

The functional flow equation and the Quantum Master equation are consistently solved in perturbation for the chiral symmetric QED with and without four-fermi interactions. Due to the presence of momentum cutoff, unconventional features related to gauge symmetry are observed even in our perturbative results.

In the absence of the four-fermi couplings, one-loop calculation gives us the Ward identity, \( Z_1 = Z_2 \), and the standard results of anomalous dimensions and the beta function for the gauge coupling. It is a consequence of regularization scheme independence in one-loop computation. We also find a photon mass term.

When included, four-fermi couplings contribute to the beta function and the Ward identity is also modified, \( Z_1 \neq Z_2 \), due to a term proportional to the photon mass multiplied by the four-fermi couplings.

1 Introduction

Recently much attention has been attracted to the exact renormalization group or functional renormalization group (ERG/FRG) approach to gauge theories. The regularization scheme with a momentum cutoff \( \Lambda \) is not compatible with gauge invariance: the BRST transformation in its standard form is not a symmetry of the Wilsonian action. However it has been shown that the BRST symmetry survives in a modified form \([1, 2, 3, 4]\): the variation of a Wilsonian action \( S \) under appropriately modified BRST transformation defined at \( \Lambda \) is cancelled by the Jacobian factor of the functional measure. This cancellation mechanism, the modified Ward-Takahashi (mWT) identity, is lifted to the Quantum Master Equation (QME) \([5, 6, 7, 8]\) in the Batalin-Vilkovisky (BV) antifield formalism \([9]\). The QME and the flow equation are two basic equations to define a gauge theory in ERG/FRG. It has been a challenging problem to solve them consistently in appropriate truncation schemes.

In a previous work \([10]\), the compatibility of two equations is discussed for Yang-Mills (YM) theory in a perturbative framework (see also \([11]\)). The main results obtained there are: firstly, two equations are combined to develop BRST cohomology analysis \([12, 13, 14, 15]\) that uniquely determines the classical action of first and second orders in gauge coupling; secondly, it was shown that one-loop perturbative solution to the flow equation satisfies the QME or its Legendre transform, modified Slavnov-Taylor (mST) identity \([10]\), up to third order in the coupling; thirdly the standard results are obtained for the beta function and anomalous dimensions as a consequence of regularization scheme independent computation. It leads to the standard Slavnov-Taylor identities among renormalization constants.

In this paper, we consider the compatibility between the QME and the flow equation for a chiral invariant QED with four-fermi interactions \([17]\). This type of the model have attracted interests in connection with possible existence of a non-trivial UV-fixed point and associated chiral symmetry breaking \([18, 19, 20, 21, 22, 23, 24, 25]\). In the light of asymptotically safe scenario, there is a new interest of finding a UV completion of QED \([26]\). In this paper we will not discuss such non-perturbative structure. Instead, we take a perturbative
approach in parallel to Ref.\cite{10} and find how the higher dimensional interactions affect the realization of BRST symmetry.

Here we mainly use the Legendre transform of the QME and flow equation to avoid redundancy arising from the one-particle reducible part of the Wilsonian action. QME/mST is also best expressed in terms of $\Gamma$ since its free part $\Gamma_0$ carries no regularization that simplifies BRST cohomology analysis. Though the Legendre transform of the measure contribution $\Delta S$ in QME contains the inverse of the two-point function $\Gamma^{(2)}$, that is readily expandable perturbatively and does not cause any trouble.

We will show that, even in the presence of the four-fermi interactions, the perturbative solution to the flow equation satisfies the QME/mST up to order of $e^3$ or $eG_{S,V}$ in a general covariant gauge.

After introducing the wavefunction renormalization factors via a canonical transformation of classical fields and their antifields, the beta function of the gauge coupling and anomalous dimensions are computed by using the flow equations.

Without four-fermi couplings, the standard perturbative results are obtained as in the case of YM theory and the Ward identity $Z_1 = Z_2$ holds. It is again a consequence of regularization scheme independence. The presence of the photon mass term proportional to $e^2\Lambda^2$ is observed. Once the four-fermi interactions are taken into accounts, the beta function acquires an extra term proportional to the photon mass term and we find $Z_1 \neq Z_2$ due to the mass term.

In the next section, we give a brief summary of the 1PI formulation in ERG/FRG. In section 3, we show that one-loop perturbative solution to the flow equation satisfies the QME/mST up to the third order in couplings. The beta function and anomalous dimensions are computed in section 4. Summary and conclusions are given in section 5.

2 Legendre transform of the QME and the flow equation

The Wilsonian action consists of free and interaction parts, $S = S_0 + S_I$. In the free action

$$S_0[\phi, \phi^*] = \frac{1}{2} \phi^A K^{-1} \Delta^{-1}_{AB} \phi^B + \phi^*_A K^{-1} R^A_B \phi^B,$$

the kinetic terms $\Delta^{-1}_{AB}$ are regularized by a UV cutoff function $K(p^2/\Lambda^2)$, satisfying the requirements that $K(0) = 1$ and $K(u) \to 0$ sufficiently fast as $u \to \infty$. Also included are free BRST transformations $R^A_B \phi^B$ for fields $\phi^A$ multiplied by their antifields $\phi^*_A$ and by overall factor $K^{-1}$. By construction, the free BRST transformation satisfies the relation

$$\Delta^{-1}_{AC} R^C_B + \Delta^{-1}_{BC} R^C_A = 0$$

for fields $\phi^A$ multiplied by their antifields $\phi^*_A$ and by overall factor $K^{-1}$. By construction, the free BRST transformation satisfies the relation

$$\Delta^{-1}_{AC} R^C_B + \Delta^{-1}_{BC} R^C_A = 0$$

$S_I[\phi, \phi^*]$ consists of interaction terms and some antifield dependent terms with coupling constants. We use the condensed notation as in Refs.\cite{4,10}.

The regularized version of the antibracket and the measure operator can be defined as those in \cite{10}:

$$(X,Y)_K = \frac{\partial^r X}{\partial \phi_A^K} K \frac{\partial^l Y}{\partial \phi_A^K} K \frac{\partial^r X}{\partial \phi_A^K} K \frac{\partial^l Y}{\partial \phi_A^K},$$

and

$$\Delta_K X = (-)^{A+1} \frac{\partial^r}{\partial \phi_A^K} K \frac{\partial^r}{\partial \phi_A^K} X.$$
The Wilsonian action $S$ can be expressed as a tree-level expansion in terms of its 1PI part $\Gamma_I$ \cite{27, 28, 29}. The latter is related to $S_I[\phi, \phi^*]$ via Legendre transformation

$$\Gamma_I[\Phi, \Phi^*] = S_I[\phi, \phi^*] - \frac{1}{2} (\phi - \Phi)^A \Delta_{AB}^{-1} (\phi - \Phi)^B,$$

$$\partial^r_{\partial \phi B} S_I[\phi, \phi^*] = (\phi - \Phi)^A \bar{\Delta}_{AB}^{-1} = \partial^r_{\partial \Phi B} \Gamma_I[\Phi, \Phi^*],$$

where $\bar{\Delta}_{AB}^{-1}$ denote the inverse of the IR regulated propagators $\bar{\Delta}^{AB} = \bar{K} \Delta^{AB}$ with $\bar{K} = 1 - K$. For the aesthetic reason, we use the notation $\Phi_A^* = \phi_A^*$ for the 1PI effective action. By adding a free part, we introduce the 1PI effective action $\Gamma$ as

$$\Gamma = \frac{1}{2} \Phi^A \bar{\Delta}^{-1} \Phi^B + \Phi^*_A R^A_B \Phi^B + \Gamma_I[\Phi, \Phi^*].$$

We also define the total 1PI effective action with regularized kinetic terms as

$$\Gamma_{\text{tot}} = \frac{1}{2} \Phi^A \bar{\Delta}^{-1} \Phi^B + \Phi^*_A R^A_B \Phi^B + \Gamma_I[\Phi, \Phi^*].$$

Note the 1PI action $\Gamma$ and $\Gamma_{\text{tot}}$ differ only in the kinetic terms and the difference vanishes as the cutoff goes to zero.

Now we rewrite the QME in \eqref{5} in terms of the 1PI action. From \eqref{6} and \eqref{7}, we find

$$\partial^r_{\partial \phi A} S_I = \phi^B \Delta_{BA}^{-1} + \phi^*_B R^B_A + \partial^r_{\partial \phi^* A} S_I = \partial^r_{\partial \Phi A} \Gamma, \quad \partial^l_{\partial \phi^* A} = \partial^r_{\partial \Phi^* A}. \quad \text{(10)}$$

Using \eqref{2}, \eqref{10} and \eqref{5}, we find

$$(S, S)_{K} = (\Gamma, \Gamma). \quad \text{(11)}$$

The antibracket on the r.h.s. is defined for arbitrary functionals of the classical fields $\Phi^A$ and their antifields $\Phi^*_A$ as

$$(Z, W) = \frac{\partial^r Z}{\partial \Phi^A} \frac{\partial^l W}{\partial \Phi^*_A} - \frac{\partial^r Z}{\partial \Phi^*_A} \frac{\partial^l W}{\partial \Phi^A}. \quad \text{(12)}$$

Note that the regulator function $K$ is absent in the above expression.

In rewriting the second term of QME, $\Delta_K S_I$, we note that only the interaction action produces field dependent contributions. Using \eqref{10}, we obtain

$$\Delta_K S_I = \frac{\partial^r K}{\partial \phi^*_A} \frac{\partial^l S_I}{\partial \Phi^*_A} = \frac{\partial^r K}{\partial \Phi^*_A} \frac{\partial^l S_I}{\partial \Phi^*_A} = \text{Tr} \left( K \Gamma^{(2)}_I \left[ 1 + \bar{\Delta}^{(2)}_I \right]^{-1} \right). \quad \text{(13)}$$

The last expression in \eqref{13} is reached by using the relation

$$\frac{\partial^r \Phi^A}{\partial \phi^*_B} = \left( \left[ 1 + \Delta^{(2)}_I \right]^{-1} \right)_B^A, \quad \text{(14)}$$

that is derived from \eqref{7}. Here we have used notations

$$\left( \Gamma^{(2)}_I \right)_{AB} = \frac{\partial^r \partial^r}{\partial \Phi^A \partial \Phi^B} \Gamma_I, \quad \text{(15)}$$

as well as

$$\left( \Gamma^{(2)}_{I*, I} \right)^A_B = \frac{\partial^r \partial^l}{\partial \Phi^*_A \partial \Phi^*_B} \Gamma_I. \quad \text{(16)}$$

\footnote{More detailed derivation will be found in \cite{10}.}
Finally, we find the modified Slavnov-Taylor (mST) identity as the Legendre transform of the QME
\[ \Sigma = \frac{1}{2} (\Gamma, \Gamma) - \text{Tr} \left( K \Gamma_I^{(2)} \left[ 1 + \Delta \Gamma_I^{(2)} \right]^{-1} \right) = 0. \]  

(17)

It is also worth pointing out that the second functional derivative of \( \Gamma_{\text{tot}} \) appeared in the second term of (17) as
\[ 1 + \Delta \Gamma_I^{(2)} = \Delta \Gamma_{\text{tot}}^{(2)}. \]  

(18)

In (17), it is interesting to find \( \Gamma \) in the first term and \( \Gamma_{\text{tot}} \) in the second term. Shortly we will find a similar trace structure in the flow equation written for the 1PI action.

The measure operator \( \Delta \) similar to (4) defined in terms of \( \Phi^A \) and \( \Phi^*_{A} \) appears as the first-order part of eq. (17)
\[ \Delta \Gamma = \text{Tr} \left( K \Gamma_I^{(2)} \right). \]  

(19)

Here an important remark is in order. The Legendre transformation (6) is not a canonical transformation from \( \{ \phi^A, \phi^*_A \} \) to \( \{ \Phi^A, \Phi^*_A \} \). Although the antibracket (12) in terms of \( \{ \Phi^A, \Phi^*_A \} \) is convenient to write the relation (11), one should not mix up two canonical structures in \( S \)-world and \( \Gamma \)-world.

Using the flow equation for \( S_I \) [30]
\[ \dot{S}_I = \Lambda \partial \lambda S_I = -\frac{1}{2} \frac{\partial r S_I}{\partial \phi^A} A_{AB} \frac{\partial l S_I}{\partial \phi^B} + \frac{1}{2} (-)^A \dot{\Delta} \frac{\partial l S_I}{\partial \phi^B} \phi^A \]  

(20)

and the Legendre transformation [6], [7] and [11], we find that
\[ \dot{\Gamma}_I = \dot{S}_I + \frac{1}{2} (-)^A \dot{\Delta} \frac{\partial l S_I}{\partial \phi^B} \phi^A. \]  

(21)

Thus, we obtain the flow equation for \( \Gamma_I \) [31] [27] [32] [33] as
\[ \dot{\Gamma}_I = -\frac{1}{2} \text{Str} \left( \dot{\Delta} \left[ 1 + \Delta \Gamma_I^{(2)} \right]^{-1} \right). \]  

(22)

The expression of the Quantum Master Functional (QMF) \( \Sigma \) in [3] and the flow equation (20) are combined to give
\[ \dot{\Sigma} = -\frac{1}{2} \frac{\partial r S_I}{\partial \phi^A} A_{AB} \frac{\partial l \Sigma}{\partial \phi^B} + \frac{1}{2} (-)^A \dot{\Delta} \frac{\partial l \Sigma}{\partial \phi^B} \phi^A. \]  

(23)

That is, the QMF satisfies the linearized flow equation as a composite operator [34] (See also [4]). The QME is stable along the RG flow once it holds at some cutoff scale.

In the next section, we consider QED with chiral invariant four-fermi interactions and show that the QME/mST (17) and the flow equation (21) can be simultaneously solved in a perturbative expansion.

3 1PI effective action in QED and the QME/mST

3.1 The classical effective action

We consider 1PI effective action for QED with a massless Dirac fermion. The free part of the covariantly gauge fixed action contains kinetic terms for the photon \( A_{\mu} \), the Dirac field \( \Psi, \overline{\Psi} \) and the FP ghost fields \( C \)
and \( \bar{C} \): the auxiliary field \( B \) and the gauge parameter \( \xi \) are introduced accordingly. In addition, here we also include antifields \( A^* \) and \( C^* \) as sources for the free BRST transformations of \( A_\mu \) and the antighost \( \bar{C} \).

\[
\Gamma_0 = \int \left[ \frac{1}{2} \left\{ (\partial_\mu A_\nu)^2 - (\partial \cdot A)^2 \right\} + \bar{\Psi} i\gamma^\mu \Psi + (A^*_\mu - i\partial_\mu \bar{C}) \partial_\mu C + \frac{1}{2} \epsilon B^2 + \left( C^* - i\partial \cdot A \right) B \right].
\] (24)

Starting from \( \Gamma_0 \) in (24), we construct a 1PI effective action that satisfies the Classical Master Equation

\[
(\Gamma_{cl}, \Gamma_{cl}) = 0
\] (25)

up to \( O(\epsilon^2) \), \( \Gamma_{cl} = \Gamma_0 + \Gamma_1 + \Gamma_2 \). The lower index is for the order of the gauge coupling. The quantum part \( \Gamma_q \) will be discussed later. In order to solve eq. (25) we utilize the BRST cohomology argument [12, 13, 14, 15] that was applied earlier to Yang-Mills theory in ERG [10].

From \((\Gamma_1, \Gamma_0) = 0\), we will uniquely determine \( \Gamma_1 \), up to some normalization factors to be discussed later. We decompose \( \Gamma_1 \) into parts of definite antighost numbers. The table 1 is the list of various gradings of (anti)fields. Looking for local field combinations with the highest antighost number, mass dimension four and of vanishing fermion and ghost numbers, we find the highest antighost number is one and \( \Gamma_1^1 = \int_x \left[ c_1 \bar{\Psi} \Psi C + c_2 \bar{\Psi} \Psi \bar{C} \right] \) with coefficients \( c_1 \) and \( c_2 \) to be determined shortly. The superscript of \( \Gamma_1^1 \) indicates the antighost number. The only candidate for \( \Gamma_0^1 \) is the minimal gauge interaction term with the coupling \( \epsilon \), \( \Gamma_0^1 = -e \int_x \bar{\Psi} / A \Psi \). Now the requirement \((\Gamma_0^1 + \Gamma_1^1, \Gamma_0) = 0\) fixes the coefficients \( c_1 \) and \( c_2 \) in \( \Gamma_1^1 \) as \( c_1 = -c_2 = -ie \). In this manner, we find

\[
\Gamma_1 = \int_x \left[ -e \bar{\Psi} A \Psi - ie \Psi^* \Psi C + ie \bar{\Psi} \Psi \bar{C} \right].
\] (26)

All contained in \( \Gamma_0 + \Gamma_1 \) are marginal terms, and \( \Lambda \) independent.

| Field | gh # | ag # | pure gh # | dimension |
|-------|------|------|-----------|-----------|
| \( A_\mu \) | 0 | 0 | 0 | 0 | 1 |
| \( C \) | 1 | 1 | 0 | 1 | 1 |
| \( \Psi, \bar{\Psi} \) | 1 | 0 | 0 | 0 | 3/2 |
| \( \Psi^*, \bar{\Psi}^* \) | 0 | -1 | 1 | 0 | 3/2 |
| \( \bar{C} \) | 1 | -1 | 1 | 0 | 1 |
| \( B \) | 0 | 0 | 1 | 1 | 2 |
| \( A^*_\mu \) | 1 | -1 | 1 | 0 | 2 |
| \( C^* \) | 0 | 0 | 0 | 0 | 2 |

Table 1: The various properties of the (anti)fields, namely, Grassmann parity, ghost number, antighost/antifield number, pure gh # = gh # + ag #, and mass dimension.

We also include chiral invariant four-fermi interactions as irrelevant terms,

\[
\Gamma_{2,cl} = \int_x \left[ \frac{G_S}{2\Lambda^2} \left\{ \left( \bar{\Psi} \gamma_\mu \Psi \right) \left( \bar{\Psi} \gamma_\nu \Psi \right) - \left( \bar{\Psi} \gamma_5 \Psi \right) \left( \bar{\Psi} \gamma_5 \Psi \right) \right\} 
+ \frac{G_V}{2\Lambda^2} \left\{ \left( \bar{\Psi} \gamma_\mu \Psi \right) \left( \bar{\Psi} \gamma_\nu \Psi \right) + \left( \bar{\Psi} \gamma_5 \gamma_\mu \Psi \right) \left( \bar{\Psi} \gamma_5 \gamma_\nu \Psi \right) \right\} \right].
\] (27)

It is easy to confirm that \( \Gamma_{cl} = \Gamma_0 + \Gamma_1 + \Gamma_2 \) satisfies the Classical Master Equation in (25). In our perturbative expansion, we regard \( G_S, G_V \) at the order of \( \epsilon^2 \).

The one-loop correction to the 1PI effective action is given as the closed-form solution to (22):

\[
\Gamma_q = \frac{1}{2} \text{Str log} \left( \Delta^{-1} + \Gamma_{I,cl}^{(2)} \right),
\] (28)

\(^2\)We take the gauge-fixed basis for antifields [10].
where $\Gamma^{(2)}_{\text{IR}}$ is the classical part of $\Sigma$, the second functional derivative of $\Gamma_1 + \Gamma_2$. $\Delta$ in (28) are the IR-regularized propagators,

$$\Delta_{\mu\nu} = (P^T_{\mu\nu} + \xi P^L_{\mu\nu}) \Delta, \quad \Delta_{\alpha\beta} = (i\phi)_{\alpha\beta} \Delta,$$

(29)

for the gauge and Dirac fields respectively. Here, $\Delta = (1 - K)/( -\partial^2)$, $P^T_{\mu\nu}$ and $P^L_{\mu\nu}$ are the transverse and longitudinal projection operators. The lowest order quantum correction is simply

$$\Gamma_{1,q} = \frac{1}{2} \text{Str} \left( \Delta \Gamma^{(2)}_{1,q} \right).$$

(30)

The r.h.s. of (30) is evaluated with first two vertices in (76) of Appendix A and turned out to be zero. The perturbative expansion of (28) starts from $O(e^2)$ term.

We expand the QMF according to the order of couplings as

$$\Sigma = \Sigma_0 + \Sigma_1 + \Sigma_2 + \Sigma_3 + \cdots,$$

(31)

and we find $\Sigma_0 = (\Gamma_0, \Gamma_0)/2 - \Delta \Gamma_0 = 0$ and $\Sigma_1 = (\Gamma_1, \Gamma_0) - \Delta \Gamma_1 = 0$ with $\Delta$ defined in eq. (19).

In the following two subsections, we evaluate $\Sigma_{2,3}$, higher order terms in (31), after obtaining quantum corrections, $\Gamma_{2,q}$ and $\Gamma_{3,q}$.

### 3.2 Second order in gauge coupling

Expanding eq. (28) to the orders of $e^2$ and $G_{S,V}$, we obtain a quantum part of the action

$$\Gamma_{2,q} = \frac{1}{2} \text{Str} \left( \Delta \Gamma^{(2)}_{2,q} \right) + \text{Str} \left( -\frac{1}{4} \Delta \Gamma^{(2)}_1 \Delta \Gamma^{(2)}_1 \right),$$

(32)

which has gauge and fermion fields contributions. We write them separately as

$$\Gamma_{2,q}^{AA} = \frac{1}{2} \Delta_{\alpha\beta} \tau_{\alpha\beta}(-A) \overline{\Delta} \tau_{\alpha\beta}(-A) = \frac{1}{2} e^2 \left[ (i\phi) \overline{\Delta} A (i\phi) \overline{\Delta} A \right]$$

(33)

and

$$\Gamma_{2,q}^{\psi\psi} = \overline{\Delta}_{\alpha\beta} \tau_{\alpha\beta}(\psi\psi) + \overline{\Delta}_{\alpha\beta} \tau_{\alpha\beta}(-\gamma^\mu) \overline{\Delta}_{\beta\gamma} \tau_{\gamma\mu}( \psi\psi) = -e^2 \left[ \overline{\Delta}_{\mu\nu} \overline{\psi} (i\partial^\mu) \Delta \gamma^\nu \psi \right].$$

(34)

Here the quantities $\tau$ are the vertices obtained from $\Gamma^{(2)}$ listed in Appendix A. Note that the four-fermi interactions give vanishing contribution in eq. (28). In momentum space, it becomes

$$\overline{\Delta}_{\alpha\beta} \tau_{\alpha\beta}(\psi\psi) \rightarrow \frac{2}{A^2} \left( G_S - 4G_V \right) \int_{p,q} \overline{\psi} (p) \gamma^\mu \psi (p) \frac{1 - K(q)}{q^2} q^\mu = 0.$$ 

(35)

The integral over $q$ vanishes due to the Lorentz covariance.

Having constructed $\Gamma_{2,q}$, we may calculate the QMF at $O(e^2)$,

$$\Sigma_2 = (\Gamma_0, \Gamma_{2,q}) + \left[ K \Gamma^{(2)}_{1,2} \Delta \Gamma^{(2)}_1 \right],$$

(36)

where $\Gamma^{(2)}_{1,2}$ is $O(e)$ part of $\Gamma^{(2)}_1$. It turned out that both terms of $\Sigma_2$ are proportional to $A_\mu C$, the gauge field multiplied by the ghost. The second term in (36) becomes

$$\Sigma_2|_K = \left[ K \Gamma^{(2)}_{1,2} \Delta \Gamma^{(2)}_1 \right] = \left[ K \tau_{\alpha\beta} C \overline{\Delta} \tau_{\alpha\beta}(A^\dagger) + K \tau_{\alpha\beta} \Delta \tau_{\alpha\beta}(A^\dagger) \right]$$

$$= e^2 \left[ KC \overline{\Delta} A - A \overline{\Delta} C K \right] = 8e^2 \left[ KC \overline{\Delta} A \right].$$

(37)

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3Strictly speaking, this happens due to a cancellation of divergent contributions, because $\int_0^\infty q^2 dq^2$ is UV divergent. We may regularize the $q^2$ integral $\int_0^\infty dq^2$ as $\int_0^L dq^2$ with a UV cutoff $\Lambda_0$ or we may instead use the dimensional regularization.
On the other hand, as shown in Appendix B, the first term of eq. (36) becomes
\[ \Sigma_2 |_{(r_2,r_{2,n})} = -\frac{e^2}{2} \text{tr} (\gamma_{\mu} \gamma_{\nu} \gamma_{\sigma}) \left[ \partial_{\mu} \Delta A_{\nu} \partial_{\rho} \Delta \partial_{\sigma} C + \partial_{\nu} \Delta \partial_{\sigma} C \partial_{\rho} \Delta A_{\sigma} \right] \]
\[ = 8 e^2 \left[ (1 - K) C \partial_{\nu} \Delta A_{\nu} \right]. \quad (38) \]

Therefore,
\[ \Sigma_2 = \Sigma_2 |_{(r_2,r_2)} + \Sigma_2 |_{K} = 8 e^2 \left[ C \partial_{\nu} \Delta A_{\nu} \right] \]
\[ = -8 e^2 \int_{p,q} C(p) \frac{(1 - K(q))}{q^2} q_{\mu} A_{\nu}(-p) = 0. \quad (39) \]

We have shown that the QME and the flow equation are consistently solved at \( \mathcal{O}(e^2) \) and \( \mathcal{O}(G_{S,V}) \).

\( \Gamma_{AA} \) given in eq. (38) may be written as
\[ \Gamma_{AA} = \frac{1}{2} e^2 \int_{p} A_{\mu}(-p) \left[ P^T \mathcal{T}(p) + P^L \mathcal{L}(p) \right] A_{\nu}(p). \quad (40) \]

Its longitudinal part \( \mathcal{L} \)
\[ \mathcal{L}(p) = -8 \int_{q} K(p + q) \frac{(p \cdot q)}{p^2 q^2} \]
is necessary to satisfy the QME at \( \mathcal{O}(e^2), \Sigma_2 = 0 \). Once we put the IR cutoff \( K = 0 \) by sending \( \Lambda \to 0 \), we recover the standard WT relation \( \mathcal{L}(p) = 0 \).

### 3.3 Third order in gauge coupling

Expanding \( (28) \) to \( \mathcal{O}(e^3) \) and \( \mathcal{O}(G_{S,V}) \), we obtain
\[ \Gamma_{3,q} = \frac{1}{6} \text{Str} \left( \Delta \Gamma_1^{(2)} \Delta \Gamma_1^{(2)} \Delta \Gamma_2^{(2)} \right) - \frac{1}{4} \text{Str} \left( \Delta \Gamma_1^{(2)} \Delta \Gamma_2^{(2)} \right). \quad (42) \]

This gives quantum corrections to \( \bar{\Psi} A \Psi \) vertex. Two terms in \( (42) \) are proportional to \( e^3 \) and \( e G_{S,V} \) respectively,
\[ \Gamma_{3,q} = \Gamma_{3,q} |_{e^3} + \Gamma_{3,q} |_{eG}, \quad (43) \]

where
\[ \Gamma_{3,q} |_{e^3} = \frac{1}{3} \left( -\Delta_{\alpha \alpha} \tau_{\alpha \beta}(-\gamma^\nu) \Delta_{\mu \nu} \tau_{\beta \gamma} \Delta_{\beta \beta} \tau_{\beta \alpha}(-\bar{A}) \right. \]
\[ -\Delta_{\alpha \alpha} \tau_{\alpha \beta}(-\bar{A}) \Delta_{\mu \nu} \tau_{\beta \gamma}(-\bar{A}) \Delta_{\beta \beta} \tau_{\beta \alpha}(-\gamma^\nu) + \bar{\Delta}_{\alpha \alpha} \tau_{\alpha \beta}(-\bar{A}) \Delta_{\beta \beta} \tau_{\beta \alpha}(-\gamma^\nu) \right) \]
\[ = -e^3 \left[ \bar{\Delta}_{\mu \nu} \bar{\Psi} \gamma^\nu(i \bar{\theta}) \bar{A} (i \bar{\theta}) \bar{A} \gamma^\mu \Psi \right], \]
\[ \Gamma_{3,q} |_{eG} = \frac{1}{3} \left( -\Delta_{\alpha \alpha} \tau_{\alpha \beta}(-\bar{A}) \Delta_{\beta \beta} \tau_{\beta \alpha}(-\bar{\Psi}) \right) \]
\[ = 2 e G_{S,A} \left[ \bar{\Psi} (i \bar{\theta}) \bar{A} (i \bar{\theta}) \bar{A} \Psi \right] + 2 e G_{V} \frac{\Lambda^2}{\lambda^2} \left[ \bar{\Psi} \gamma_{\mu} (i \bar{\theta}) \bar{A} (i \bar{\theta}) \bar{A} \gamma_{\mu} \right] \]
\[ -4 e G_{V} \frac{\Lambda^2}{\lambda^2} \left[ i \partial_{\mu} \bar{A} \partial_{\mu} \Delta - i \partial_{\mu} \Delta A_{\mu} i \partial_{\mu} \Delta + i \partial_{\mu} \Delta A_{\mu} i \partial_{\mu} \Delta \right] \left( \bar{\Psi} \gamma_{\mu} \Psi \right). \quad (44) \]

Now we may calculate QME at \( \mathcal{O}(e^3) \) and \( \mathcal{O}(e G_{S,V}) \) as
\[ \Sigma_3 = (\Gamma_{3,q}, \Gamma_0) + (\Gamma_1, \Gamma_{2,q}) + \left[ K \Gamma_1^{(2)} \Delta \Gamma_2^{(2)} \right] - \left[ K \Gamma_1^{(2)} \Delta \Gamma_2^{(2)} \Delta \Gamma_1^{(2)} \right]. \quad (45) \]
All the terms in $\Sigma_3$ are proportional to $\bar{\Psi}\Psi C$ with coefficients of $O(e^3)$ or $O(eG)$: $\Sigma_3 = \Sigma_{3,e^3} + \Sigma_{3,eG}$. There are three $O(e^3)$ terms

$$\Sigma_{3,e^3} = \Sigma_{3,e^3}|_K + \Sigma_{3,e^3}|_{(\Gamma_0, \Gamma_3, n)} + \Sigma_{3,e^3}|_{(\Gamma_1, \Gamma_2, n)}$$

where

$$\Sigma_{3,e^3}|_K = -\left[K\Gamma^{(2)}_{1*}\Delta\Gamma^{(2)}_{1*}\right]$$

$$= -\left[K\tau^C_{\alpha \beta} \bar{\Delta}_\beta \tau^\alpha_{\bar{\alpha}}(\gamma \Psi) + K\tau^C_{\alpha \bar{\alpha}} \bar{\Delta}_\alpha \tau^{\bar{\alpha}}_{\beta}(\gamma \Psi)\right]$$

$$= e^3\left[\bar{\Psi}\gamma_\mu KC\bar{\Delta}\bar{\gamma}_\mu \Psi \bar{\Delta}_{\mu \nu}\right] - e^3\left[\bar{\Psi}\gamma_\mu \bar{\Delta} C K \gamma_\mu \Psi \bar{\Delta}_{\mu \nu}\right],$$

$$\Sigma_{3,e^3}|_{(\Gamma_1, \Gamma_2, n)} = -\frac{\partial \Gamma_1}{\partial \Psi^*} \frac{\partial \Gamma_2}{\partial \Psi} - \frac{\partial \Gamma_1}{\partial \Psi} \frac{\partial \Gamma_2}{\partial \Psi^*}$$

$$= e^3\left[\bar{\Psi}\gamma_\mu \bar{\Delta} C \bar{\gamma}_\mu \Psi \bar{\Delta}_{\mu \nu}\right] - e^3\left[\bar{\Psi}\gamma_\mu \bar{\Delta} C \bar{\gamma}_\mu \Psi \bar{\Delta}_{\mu \nu}\right],$$

$$\Sigma_{3,e^3}|_{(\Gamma_0, \Gamma_3, n)} = -e^3\left[\bar{\Psi}\gamma_\mu \bar{\Delta} \bar{\gamma}_\mu \Psi \bar{\Delta}_{\mu \nu}\right]$$

$$= e^3\left[\bar{\Psi}\gamma_\mu (1 - K) C \bar{\gamma}_\mu \Psi \bar{\Delta}_{\mu \nu}\right] - e^3\left[\bar{\Psi}\gamma_\mu \bar{\Delta} (1 - K) \gamma_\mu \Psi \bar{\Delta}_{\mu \nu}\right].$$

The above results lead to

$$\Sigma_{3,e^3} = 0.$$  

(50)

As for $O(eG)$ terms, we find two contributions

$$\Sigma_{3,eG} = \Sigma_{3,eG}|_K + \Sigma_{3,eG}|_{(\Gamma_0, \Gamma_3, n)}.$$  

(51)

We may calculate them as

$$\Sigma_{3,eG}|_K = \left[K\Gamma^{(2)}_{1*}\Delta\Gamma^{(2)}_{1*}\right]$$

$$= \left[K\tau^C_{\alpha \beta} \bar{\Delta}_\beta \tau^\alpha_{\bar{\alpha}}(\gamma \Psi) + K\tau^C_{\alpha \bar{\alpha}} \bar{\Delta}_\alpha \tau^{\bar{\alpha}}_{\beta}(\gamma \Psi)\right]$$

$$= 2e\frac{G_S - 2G_V}{\Lambda^2}\left[\bar{\Psi}\left[K\bar{\Delta} \bar{\gamma} - \bar{\phi} \bar{\Delta} C K\right] \Psi\right]$$

$$- \frac{4eG_V}{\Lambda^2}\left[\left(KC \partial_\mu \bar{\Delta} - \partial_\mu \bar{\Delta} C K\right) (\bar{\Psi}\gamma_\mu \Psi)\right],$$

(52)

and

$$\Sigma_{3,eG}|_{(\Gamma_0, \Gamma_3, n)} = \frac{2e\frac{G_S - 2G_V}{\Lambda^2}}{\Lambda^2}\left[\bar{\Psi}\left[(1 - K)C \bar{\gamma} \bar{\Delta} - \bar{\delta} \bar{\Delta} C (1 - K)\right] \Psi\right]$$

$$- \frac{4eG_V}{\Lambda^2}\left[\left((1 - K)C \partial_\mu \bar{\Delta} - \partial_\mu \bar{\Delta} C (1 - K)\right) (\bar{\Psi}\gamma_\mu \Psi)\right].$$

(53)

Eqs. (52) and (53) sum up to give a vanishing result,

$$\Sigma_{3,eG} = \frac{2e\frac{G_S - 2G_V}{\Lambda^2}}{\Lambda^2}\left[\bar{\Psi}\left(C \bar{\gamma} \bar{\Delta} - \bar{\phi} \bar{\Delta} C\right) \Psi\right] - \frac{4eG_V}{\Lambda^2}\left[\left(C \partial_\mu \bar{\Delta} - \partial_\mu \bar{\Delta} C\right) (\bar{\Psi}\gamma_\mu \Psi)\right] = 0.$$  

(54)

From (50) and (51), we finally obtain the result

$$\Sigma_3 = 0.$$  

(55)

We have confirmed that the QME/mST can be solved consistently with the flow equation up to the orders of $e^3$ and $eG_{S,V}$. We have seen that the four-fermi interactions generate quantum corrections to $\bar{\Psi}A\Psi$ vertex function and, in $\Sigma_3$, their free BRST transformation and the measure factor cancel each other.
4 Wavefunction renormalization constants and $\beta$ functions

In order to take account of $\Lambda$ evolution, we introduce renormalization constants for fields and couplings. The corresponding $Z$ factors are defined as

$$A_\mu \to Z_3^{1/2} A_\mu, \quad \Psi \to Z_2^{1/2} \Psi, \quad \bar{\Psi} \to Z_2^{1/2} \bar{\Psi}, \quad C \to Z_3^{1/2} C,$$

$$\bar{C} \to Z_3^{-1/2} \bar{C}, \quad B \to Z_3^{-1/2} B, \quad A_\mu^* \to Z_3^{-1/2} A_\mu^*,$$

$$\Psi^* \to Z_2^{-1/2} \Psi^*, \quad \bar{\Psi}^* \to Z_2^{-1/2} \bar{\Psi}^*, \quad C^* \to Z_3^{1/2} C^*. \tag{56}$$

For the gauge coupling, four-fermi couplings and gauge parameter, we set $e \to e_\Lambda = Z_{e} e, G_{S,V} \to Z_{S,V} G_{S,V}$ and $\xi \to Z_3 \xi$. There is some ambiguity in introducing wavefunction renormalization factors. Here, we require that the $Z$ factors in eq. $^{56}$ to be wavefunction rescalings due to canonical transformations so that the fields and antifields are rescaled in opposite directions $^{[9]}$.

The 1PI effective action is expressed as

$$\Gamma_0 = \int \left[ \frac{Z_3}{2} A_\mu (-\partial^2 P_{\mu \nu}^T) A_\nu + Z_2 \bar{\Psi} i \bar{\gamma} \Psi + (A_\mu^* - i \partial_\mu C) \partial_\mu C + \frac{\xi}{2} B^2 + (C^* + i \partial \cdot A) B \right],$$

$$\Gamma_1 = \int \left[ -e (Z_e Z_3^{1/2} Z_2) \bar{\Psi} A \Psi - ie (Z_e Z_3^{1/2}) \bar{\Psi} \Psi C + ie (Z_e Z_3^{1/2}) \bar{\Psi}^* \bar{\Psi} C \right],$$

$$\Gamma_2 = \int \left[ (Z_S^{1/2} Z_2^2 G_{S,V}) \frac{2A^2}{2\Lambda^2} \left\{ (\bar{\Psi} \Psi) (\bar{\Psi} \Psi) - (\bar{\Psi} \gamma_5 \Psi) (\bar{\Psi} \gamma_5 \Psi) \right\} \right.$$

$$\left. + (Z_V^{1/2} Z_2^2)^{-1} \right] \frac{G_{S,V}}{2\Lambda^2} \left\{ (\bar{\Psi} \gamma_{\mu} \Psi) (\bar{\Psi} \gamma_{\mu} \Psi) + (\bar{\Psi} \gamma_{5 \mu} \Psi) (\bar{\Psi} \gamma_{5 \mu} \Psi) \right\}. \tag{57}$$

At one loop level anomalous dimensions of the photon and Dirac fields are expressed as

$$Z_{2,3} = 1 - \eta A \log(\Lambda / \mu). \tag{58}$$

For the gauge coupling, its beta function is expressed as $\beta_e = \eta e$ where

$$Z_e = 1 + \eta e \log(\Lambda / \mu). \tag{59}$$

We first compute $\eta_A, \eta_e$ in the absence of the four-fermi interactions, $G_S = G_V = 0$. For the photon two-point functions, it follows from $^{38}$ in momentum space

$$\hat{\Gamma}_2^{AA} = \frac{e^2}{\Lambda} \int \frac{dp}{2\pi} \text{Tr} \left\{ \hat{K}(q) \hat{K}(p+q) \frac{q^2 (p+q)^2}{\hat{q} \hat{A}(-p)(\hat{p} + \hat{q}) \hat{A}(p)} \right\},$$

$$= \frac{e^2}{\Lambda} \int \frac{dp}{2} A_\mu(-p) \hat{A}_\mu(p) A_\nu(p). \tag{60}$$

We expand $A_{\mu \nu}$ in external momentum up to $p^2$. $\hat{A}_{\mu \nu}(0)$ gives a photon mass term

$$M_A^2 = \frac{e^2}{4 \pi^2} \Lambda^2 \int_0^\infty du \ u \hat{K}'(u) \hat{K}(u), \tag{61}$$

while $O(p^2) \propto P_{\mu \nu} p^2$ part yields

$$\eta_A = -\frac{e^2}{6 \pi^2} \int_0^\infty du \left\{ \left[ (u \hat{K}(u))' \right]' - (\hat{K}^2(u))' \right\} = \frac{e^2}{6 \pi^2}. \tag{62}$$

In momentum space $\Lambda \partial_A = \partial_t$ derivative of the fermion two-point function $^{53}$ takes the form

$$\hat{\Gamma}_2^{\Psi \Psi} = -\frac{e^2}{\Lambda} \int \frac{dp}{2\pi} \Psi(-p) \hat{\Gamma}_2^{\Psi \Psi}(p) \Psi(p). \tag{63}$$
\[ \dot{\Gamma}_2^\Psi(p) = \int_q \gamma_\nu (\gamma^\mu K(q)(p + q) + \gamma^\nu \dot{K}(q)(p + q)) \left( \delta_{\mu\nu} + (\xi - 1) \frac{q_\mu q_\nu}{q^2} \right). \]  

(64)

It gives

\[ \frac{\dot{\rho}}{p^2} \dot{\Gamma}_2^\Psi(p)|_{p^2 = 0} = \frac{\xi}{8\pi^2} \int_0^\infty du [\dot{K}(u)]' = \frac{\xi}{8\pi^2}. \]

(65)

Therefore, the anomalous dimension for the Dirac fields is

\[ \eta_\Psi = \frac{e^2}{8\pi^2}. \]

(66)

For the gauge interaction vertex, we have

\[ \dot{\Gamma}_3^{\Psi A}(p, q, r) = -e^3 \int_{p, q, r} \dot{\Gamma}_3^{\Psi A}(p, q, r) \bar{\Psi}(p) A^\rho(q) \Psi(r) \delta(p + q + r), \]

(67)

where

\[ \dot{\Gamma}_3^{\Psi A}(0, 0, 0) = \frac{\partial}{\partial t} \int_q \dot{K}^3(q) \frac{\gamma_\rho \gamma_\mu \gamma_\nu}{q^6} \left( \delta_{\mu\nu} + (\xi - 1) \frac{q_\mu q_\nu}{q^2} \right) = -\frac{2\xi}{(4\pi)^2} \int_0^\infty du (\dot{K}^3(u))' = -\frac{\xi \gamma_\rho}{8\pi^2}. \]

(68)

Therefore,

\[ \dot{\Gamma}_3^{\Psi A} = \frac{e^3}{8\pi^2} \bar{\Psi} A \Psi. \]

(69)

The flow equation for the gauge interaction vertex takes the form

\[ e^2 \left( \frac{1}{2} \eta_A + \eta_\Psi - \eta_c \right) = \frac{e^2}{8\pi^2}, \]

(70)

which leads to

\[ \eta_c = \frac{1}{2} \eta_A = \frac{e^2}{12\pi^2}. \]

(71)

This is equivalent to the well-known Ward identity

\[ Z_1 = Z^{1/2}_S Z_c Z_2 = Z_2, \]

(72)

in our one-loop computation.

The beta function for the gauge coupling is given by

\[ \dot{\beta}_A = \beta_c = \eta_c e^2 = \frac{e^2}{12\pi^2}. \]

(73)

We stress that the anomalous dimensions \( \eta_A, \Psi \) and the beta function \( \beta_c \) are universal, being independent of the choice of cutoff function \( K \). They are the same as those obtained by using gauge invariant regularization as the dimensional regularization in the standard perturbation theory.

We now include the four-fermi interactions, \( G_S \) and \( G_V \). The presence of these couplings leave the anomalous dimensions \( \eta_A, \Psi \) unchanged, while there arises an additional contribution in the r.h.s. of the flow eq. (70). Instead of (71), we have

\[ \eta_c = \frac{1}{2} \eta_A - \frac{1}{4\pi^2} (G_S - 4G_V) \int_0^\infty du \bar{K}(u) \dot{K}(u). \]

(74)

The coefficient of \( G_S - 4G_V \) depends on choice of the cutoff function, but is related to the photon mass term. Therefore, we obtain

\[ \eta_c = \frac{1}{2} \eta_A - \bar{M}_A^2 (G_S - 4G_V), \]

(75)

where \( \bar{M}_A^2 = M_A^2/(e^2 A^2) \).
5 Summary and discussion

In the ERG with a momentum cutoff, the flow equation generates gauge non-invariant quantum corrections such as a photon mass term. BRST transformation of these symmetry breaking corrections are systematically cancelled, if the QME is fulfilled. Using 1PI formulation, we have shown that the perturbative solutions to flow equation also solve the QME/mST for QED with four-fermi interactions in a general covariant gauge.

As for Λ evolution, we obtain the standard anomalous dimensions and the standard beta function of the gauge coupling together with $Z_1 = Z_2$ relation when removing four-fermi interactions. This reflects regularization-scheme independence in the one-loop computation for these objects. When included, the four-fermi terms yields a new contribution to the beta function. Its coefficient depends on the choice of cutoff function, and expressed in terms of the photon mass term. Even for $Z_1 \neq Z_2$, BRST symmetry is unbroken because the QME/mST remains intact for the rescaled 1PI action \[\beta_e\]. It is a consequence of invariance of QME/mST under a canonical transformation used in introducing wavefunction renormalization factors.

The use of such a canonical transformation will induce an undesirable Λ evolution in $eZ_e Z_1^{1/2} \Psi^* \Psi C$ and $eZ_e Z_3^{1/2} \bar{\Psi}^* \bar{\Psi} C$ vertices for $Z_1 \neq Z_2$, i.e. $Z_e Z_3^{1/2} \neq 1$. Hence, RG flows should be computed suppressing the antifields at the final stage. Then, \[eZ_3\] is satisfied, and RG flows stay in BRST invariant submanifold of the theory space.

Let us consider the cutoff removing limit $K \to 0$ ($\Lambda \to 0$). In the one-loop formula \[23\], the IR-regulated propagators $\Delta^{AB}$ are replaced with unregularized ones $\Delta^{AB}$. The quantum actions $\Gamma_2,q$ and $\Gamma_3,q$ whose UV divergences are removed using the dimensional regularization satisfy the Zinn-Justin equations: $(\Gamma_0, \Gamma_2,q) = 0$, $(\Gamma_0, \Gamma_3,q) + (\Gamma_1, \Gamma_2,q) = 0$. The first equation leads to vanishing quantum corrections in the longitudinal part of the photon two-point functions, $\mathcal{L} = 0$. The second equation gives the standard relation between the mass operator of the fermion two-point functions and the gauge interaction vertices. Since the four-fermi interactions yield no contribution to $\beta_e$ as seen from \[75\] with $M_A = 0$, they do not affect the WT relation $Z_1 = Z_2$. Therefore, we observe that the classical BRST symmetry persists at quantum level, irrespective of the presence of higher dimensional operators such as the four-fermi interactions. These results should be compared with those for $\Lambda \neq 0$.

Our perturbative results imply that a non-perturbative study of the chiral invariant QED will certainly observe a similar modification of the Ward identity if we include the four fermi interactions.

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References

[1] Y. Igarashi, K. Itoh, and H. So, Phys. Lett. B479, 336 (2000).
  https://dx.doi.org/10.1016/S0370-2693(00)00305-1

[2] H. Sonoda, J. Phys. A40, 9675 (2007).
  https://dx.doi.org/10.1088/1751-8113/40/31/034

[3] Y. Igarashi, K. Itoh, and H. Sonoda, Prog. Theor. Phys. 120, 1017 (2008).
  https://dx.doi.org/10.1143/PTP.120.1017

[4] Y. Igarashi, K. Itoh, and H. Sonoda, Prog. Theor. Phys. Suppl. 181, 1 (2010).
  https://dx.doi.org/10.1143/PTPS.181.1

[5] Y. Igarashi, K. Itoh, and H. So, Prog. Theor. Phys. 104, 1053 (2000).
  http://dx.doi.org/10.1143/PTP.104.1053

[6] Y. Igarashi, K. Itoh, and Hiroto So, Prog. Theor. Phys. 106, 149 (2001).
  https://dx.doi.org/10.1143/PTP.106.149
Appendix A

To calculate r.h.s. of (17) and (22), we need to find field dependent parts of $\Gamma_{I}^{(2)}$ and $\Gamma_{I^*}^{(2)}$, the vertices denoted as $\tau$.

From $\Gamma_{I}$ we find the following vertices,

$$
\begin{align*}
\tau_{\alpha\beta}^{(-A)}(x,y) &= \frac{\partial^3 \partial^\gamma \Gamma_1}{\partial \Psi_\alpha(x) \partial \Psi_\beta(y)} = -c(A)_{\alpha\beta}(x) \delta(x-y), \\
\tau_{\alpha\beta}^{(A^T)}(x,y) &= \frac{\partial^3 \partial^\gamma \Gamma_1}{\partial \Psi_\alpha(x) \partial \Psi_\beta(y)} = +c(A^T)_{\alpha\beta}(x) \delta(x-y), \\
\tau_{\alpha^*\gamma}^{(-\gamma \Psi)}(x,y) &= \frac{\partial^3 \partial^\gamma \Gamma_1}{\partial A_\alpha(x) \partial \Psi_\gamma(y)} = -c(\gamma \Psi)_{\alpha}(x) \delta(x-y) = -\tau_{\alpha^*\gamma}^{(-\gamma \Psi)}(x,y), \\
\tau_{\mu^*\gamma}^{(-\bar{\Psi} \gamma)}(x,y) &= \frac{\partial^3 \partial^\gamma \Gamma_1}{\partial A_\mu(x) \partial \Psi_\gamma(y)} = -c(\bar{\Psi} \gamma)_\mu(x) \delta(x-y) = -\tau_{\mu^*\gamma}^{(-\bar{\Psi} \gamma)}(x,y).
\end{align*}
$$

(76)

Here the superscripts of $\tau$ indicate structures of vertices. Similarly from $\Gamma_{2,cl}^{(2)}$, we have

$$
\begin{align*}
\tau_{\alpha\beta}^{(\bar{\Psi} \Psi)}(x,y) &= \left[ \frac{\partial^3 \partial^\gamma \Gamma_2^{2,cl}[\bar{\Psi} \Psi]}{\partial \Psi_\alpha(x) \partial \Psi_\beta(y) \partial \bar{\Psi}(x) \partial \Psi(y)} \right]_{A=0} \\
&= G_{A}\delta(x-y) \left\{ \left[ \delta_{\alpha\beta} (\bar{\Psi}(x) \Psi(x)) - (\gamma_5)_{\alpha\beta} (\bar{\Psi}(x) \gamma_5 \Psi(x)) \right] \\
&- \left[ \bar{\Psi}_\beta(x) \Psi_\alpha(x) - (\bar{\Psi}(x) \gamma_5 \Psi_\alpha(x))_{\beta} \right] \right\} \\
&\quad + G_{IV} \delta(x-y) \left\{ \left[ (\gamma_\mu)_{\beta\gamma} (\bar{\Psi}(x) \gamma_\mu \Psi(x)) + (\gamma_5 \gamma_\mu)_{\beta\gamma} (\bar{\Psi}(x) \gamma_5 \gamma_\mu \Psi(x)) \right] \\
&- \left[ (\bar{\Psi}(x) \gamma_\mu)_{\gamma\mu} (\gamma_\mu \Psi(x)) + (\bar{\Psi}(x) \gamma_5 \gamma_\mu)_{\gamma\mu} (\gamma_5 \gamma_\mu \Psi(x)) \right]_{\alpha} \right\} \\
&= -\left(\tau^{(\bar{\Psi} \Psi)}\right)_{\beta\gamma}(x,y). \\
\end{align*}
$$

(77)

As for $\Gamma_{I^*}^{(2)}$, we need vertices out of $\Gamma_{1}$,

$$
\begin{align*}
\tau_{\alpha^*\beta}^{C} &= \frac{\partial}{\partial \Psi_\alpha^*(x)} \frac{\partial}{\partial \Psi_\beta(y)} \Gamma_1 = +ie\delta_{\alpha\beta} C(x) \delta(x-y), \\
\tau_{\alpha\beta}^{C} &= \frac{\partial}{\partial \Psi_\alpha(x)} \frac{\partial}{\partial \Psi_\beta(y)} \Gamma_1 = -ie\delta_{\alpha\beta} C(x) \delta(x-y).
\end{align*}
$$

(78)
Appendix B

The following example shows our notation for computing the QMF.

\[
\text{Tr} \left[ KC \bar{\Delta}A \right] = \int_{x,y} K(x-y)C(y)\text{tr} \left[ \partial_y \bar{\Delta}(y-x)A(x) \right] \\
= \int_{x,y} \text{tr} \left[ A^T(x)\partial_y^T \bar{\Delta}(y-x) \right] C(y)K(x-y) \\
= -\int_{x,y} \text{tr} \left[ A(x)\partial_x \bar{\Delta}(x-y) \right] C(y)K(y-x) \\
= -\left[ A\partial \bar{\Delta} C K \right], \tag{79}
\]

where the trace is taken for \(\gamma\) matrices. We have also used the charge conjugation relation \(C\gamma^T \gamma^{-1} = -\gamma\mu\), and symmetry properties \(\bar{\Delta}(x-y) = \bar{\Delta}(y-x)\) and \(K(x-y) = K(y-x)\).

In this notation, we obtain eq. (38) by making integration by parts as

\[
\Sigma_{2|\{\Gamma_0, \Gamma_2, \eta\}} = -e^2 \left[ \partial \bar{\Delta} A \partial \bar{\Delta} C + \partial \bar{\Delta} \partial C \bar{\Delta} A \right] = -e^2 \left[ \partial \bar{\Delta} \partial C \partial \bar{\Delta} A \right] \\
= e^2 \int_{x,y} \text{tr} \left[ \partial_x \partial_y \bar{\Delta}(x-y)C(y)\partial_y \bar{\Delta}(y-x)A(x) \right] + \partial_x \bar{\Delta}(x-y)C(y)\partial_y^2 \bar{\Delta}(y-x)A(x) \right] \\
= e^2 \left[ (1-K)C \partial \bar{\Delta} A - \partial \bar{\Delta} C(1-K)A \right] = 2e^2 \left[ (1-K)C \partial \bar{\Delta} A \right] \tag{80}
\]