Synchronization of Fractional Reaction-Diffusion Neural Networks With Time-Varying Delays and Input Saturation

YIN WANG, SHUTANG LIU, AND XIANG WU

1Institute of Marine Science and Technology, Shandong University, Qingdao 266237, China
2School of Control Science and Technology, Shandong University, Jinan 250061, China

Corresponding author: Shutang Liu (stliu@sdu.edu.cn)

This work was supported by the National Natural Science Foundation of China under Grant U1806203 and Grant 61533011.

ABSTRACT
This study is concerned with a synchronization problem of two fractional reaction-diffusion neural networks with input saturation and time-varying delays by the Lyapunov direct method. We extend the traditional ellipsoid method by giving the novel definition of the ellipsoid and linear region of the saturated, which makes our method succinct and effective. First, we linearize the saturation terms by the properties of convex hulls. Then, by using a new Lyapunov-Krasovskii functional, we give the synchronization criteria and estimate the domain of attraction. All the results are presented in the form of linear matrix inequalities (LMIs). Finally, two numerical experiments verify the validity and reliability of our method.

INDEX TERMS Fractional reaction-diffusion, neural networks, Riemann-Liouville, input saturation.

I. INTRODUCTION
After the conception of “small world” [1] came up, the related research of complex networks has entered a rapid development stage. Complex networks is the network dynamically evolving in time whose structure is regular and complex [2]. It is an abstract description of the interaction between individuals in nature over time. Therefore, complex networks can describe not only the whole but also local behavior.

As one kind of complex networks, the neural networks has attracted many scholars’ interest because it can simulate many practical problems. Under the existing theoretical framework, the neural networks are described in two parts: the topological structure and the dynamical model. From the point of view of the dynamical model, previous studies mainly focused on the ODEs model. Still, in practice, the reaction-diffusion phenomenon cannot be ignored due to the necessity of describing the behavior of substance in space. Thus reaction-diffusion neural networks have become a research hotspot in recent years [3]–[5]. On the other hand, as an extension of the integral order reaction-diffusion equation, the fractional-order reaction-diffusion equation can model more complex phenomena due to its non-local properties. It has achieved great success in such fields as anomalous diffusion [6], image enhancement [7], and porous media seepage [8], [9]. For neural networks, the existing researches mainly focus on the problems of ODEs with Caputo and Riemann-Liouville derivative [10], utilize the Laplace transform and properties of Mittag-Leffler function to obtain stability conditions [11]–[14]. On the other hand, the adaptive control law also attracts the attention of scholars. In [15], an adaptive sliding mode control method was presented for a class of fractional-order nonlinear time-delay systems with uncertainties to solve the target output tracking problem. By employing Hermitian form Lyapunov functionals and fractional skills, [16] present some sufficient criteria for fractional complex projective synchronization. In [17], sufficient conditions for the global asymptotical stabilization of a class of fractional-order nonautonomous systems had been obtained by constructing quadratic Lyapunov functions and utilizing a new property for Caputo fractional derivative. In [18], the sliding mode control problem for a normalized singular fractional-order system with matched uncertainties was investigated. The global stabilization criteria were given in [19] for fractional memristor-based neural networks with the aid of Lyapunov functions and the comparison principle. In general, the above research mainly focuses on the
ODE system. Only the recent work [20] concern about the fractional reaction-diffusion neural networks (FRDNNs) problem with Riemann-Liouville derivative. Hence, the study of neural networks with fractional order reaction-diffusion model will further develop related fields.

Considering almost all practical applications, the time delay is unavoidable. Hence in this paper, we also take this factor into account. In many cases, researchers will construct a special Lyapunov functional to solve such problems: the Lyapunov–Krasovskii functional. The Lyapunov–Krasovskii stability theorem for fractional systems with delay had been investigated in literature [21]. And the Lyapunov–Krasovskii functional has many applications on the stability criterion and controller designing [22]–[25].

From the earlier discussion, synchronization between the nodes of neural networks is a widespread phenomenon. Usually, we need to introduce some controllers to synchronize the nodes in the neural networks. Fortunately, there many synchronization control strategies such as pinning control [26], [27], sliding mode control [28], adaptive control [29], and sampled-data control [30], which have been implemented can be applied on this topic.

On the other hand, when designing the controller for synchronization, input saturation cannot be neglected due to the maximum power or the propagation and reaction rate. Once the control signal reaches or exceeds the saturation state, the system will become hard to control or completely uncontrollable. At present, some methods such as ellipsoid method [31], anti-windup [32], [33], have been applied to solve such problems. The problem of adaptive neural control for a class of strict-feedback stochastic nonlinear systems with multiple time-varying delays subject to input saturation has been investigated in [34], neural network-based adaptive control for spacecraft under actuator failures and input saturations has been handled in [35], and [36] investigates reliable estimation problem for Markovian jump neural networks with sensor saturation. There exists extensive research on the control systems with saturation [31], [37]–[41]. The ellipsoid method [31] is simple and reliable, has been applied successfully in some discrete or ODEs model [42], [43], but no application is seen in PDEs models such as reaction-diffusion problems. In fact, some successful methods in the ODE model cannot be directly applied to the PDE model, and we must consider the evolution of the model in the whole space. Compared to other anti-windup methods, which usually introduce a dead zone, the main advantage of this method is that it is easy to linearize the saturation controller by introducing the auxiliary gain function. The estimation of the domain of attraction can be obtained by solving LMIs.

To the best of our knowledge, synchronization of FRDNNs with input saturation has not yet been fully investigated, which has theoretical and practical value to study. We hope that by putting forward such a Riemann-Liouville neural network, combined with some existing research basis, we can contribute to technology development in related fields and get some more universal results. Hence, motivated by the above reasons, the synchronization of FRDNNs with input saturation is investigated in this paper. We mainly intend to extend the ellipsoid method [44] combining with the Lyapunov–Krasovskii functional to the field of fractional partial differential model.

In this paper, we will focus on the synchronization of FRDNNs with time-varying delays and input saturation. Linearization of the saturated input is by using the properties of the convex hulls. The main contributions and innovations of this paper are as follows:

a) New definitions of the ellipsoid and linear region of the saturated are given for the FRDNNs input saturation problem.

b) A novel Lyapunov–Krasovskii functional is employed.

c) The saturation controller based on the convex hulls is extended to Riemann-Liouville FRDNNs. Meanwhile, the designed method can be easily extended to the system with Neumann boundary conditions.

d) The domain of attraction is also estimated to ensure that the initial value range does not exceed the saturation input’s control capacity.

This paper is organized as follows. Section II gives some basic concepts, symbols, assumptions, and lemmas that are needed in the later proof process. In section III–IV we give the criterion of synchronization and the estimation of the domain of attraction. In Section V, we verify the theorem given in Section III by some numerical example. In Section VI, we summarize this paper and look forward to future research.

Notation: Throughout this paper, $R^n$ denotes the n-dimensional Euclidean vector space, $I_n$ denotes the $n \times n$ identity matrix, $\otimes$ denotes the Kronecker product.

II. PRELIMINARIES

Problem Formulation: In this paper, we set the response system as the following Riemann-Liouville FRDNNs

$$
\frac{d}{\tau} u_i(t) = d_i \Delta u_i - c_i u_i(x, t) + \sum_{j=1}^{n} a_{ij} f_j(u_j(x, t)) + \sum_{j=1}^{n} g_{ij} g_j(u_j(x, t - \tau(t))) + b_j \text{sat}(J_j(x, t)), \quad t \geq 0, \quad i = 1, 2, \ldots, N,
$$

(1)

with the Dirichlet boundary conditions and initial conditions as

$$
\frac{d^{\alpha-1}}{dt^{\alpha-1}} u_i(x, s) = \phi_i(x, s), \quad (x, s) \in \Omega \times [-\tau, 0],
$$

$$
u_i(x, t) = 0, \quad (x, t) \in \partial \Omega \times [-\tau, +\infty],
$$

(2)

where $\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$ is the Laplace diffusion operator on $\Omega$; $\phi_i(x, t)$ is a bounded continuous function; $x \in R^n$ is spatial independent variable; $u_i(x, t) \in R^n$ are the n-dimensional state of the $i$-th neuron at time $t$; $c_i$ and $d_i$ are $n \times n$ dimensional constant diagonal matrix where $c_i$ represents the rate with which the $i$th neuron will reset its potential to the resting state when disconnected from the networks and external inputs in...
space \( x \), and \( d_i \) represents the transmission diffusion coefficient along the \( i \)-th neuron; \( a_{ij} \) and \( g_{ij} \) are \( n \times n \) dimensional constant matrix where \( a_{ij} \) denote the connection strength, and \( g_{ij} \) are the coupling strength between the \( i \)-th and the \( j \)-th nodes; \( f_j \) are the excitation function of the \( j \)-th node; \( \tau(i) \) is the time-varying delay satisfying \( 0 \leq \tau(t) \leq \tau \) and \( 0 \leq \bar{\tau}(t) \leq \sigma \leq 1; J_i(x, t) \) are the control input and

\[
sat(J_i(x, t)) = \text{sign}(J_i(x, t)) \min(|J_i(x, t)|, \bar{J}_i)
\]

is the saturation function with the input saturation upperbound \( \bar{J}_i \). \( R_{t_0}^t \partial_t^\alpha u_i(t) \) denotes the \( \alpha \) order Riemann-Liouville derivative which is defined as [45]

\[
R_{t_0}^t \partial_t^\alpha u_i(t, x) = \begin{cases} 
\frac{1}{\Gamma(n-\alpha)} \int_{t_0}^t \frac{(t-s)^{n-\alpha-1}}{(t-s)^{n}} u_i(s) \, ds \\
\frac{1}{\Gamma(-\alpha)} \int_{t_0}^t \frac{u_i(s)}{(t-s)^{\alpha+1}} \, ds
\end{cases} 
\]

while \( n - 1 < \alpha < n \),

where \( \Gamma(\cdot) \) denotes the Gamma function. Then, we set the drive system as

\[
R_{t_0}^t \partial_t^\alpha v_i = d_i \Delta v_i - c_i v_i(t) + \sum_{j=1}^N a_{ij} f_j(v_j(x, t)))
\]

\[+ \sum_{j=1}^N g_{ij} f_j(v_j(x, t - \tau(t))), \quad t \geq 0, \quad i = 1, 2, \ldots, N, \]

with \( R_{t_0}^t \partial_t^\alpha v_i(x, s) = \varphi_i(x, s), (x, s) \in \Omega \times [-\tau, 0], \)

\( v_i(x, t) = 0, \quad (x, t) \in \partial \Omega \times [-\tau, +\infty]. \)

Let \( e_i(x, t) = u_i(x, t) - v_i(x, t) \) as the synchronization error function, then we have the error system as

\[
R_{t_0}^t \partial_t^\alpha e_i(x, t) = d_i \Delta e_i(x, t) - c_i e_i(x, t)
\]

\[+ \sum_{j=1}^N a_{ij} \left( f_j(u_j(x, t)) - f_j(v_j(x, t)) \right)
\]

\[+ \sum_{j=1}^N g_{ij} \left( f_j(u_j(x, t - \tau(t))) - f_j(v_j(x, t - \tau(t))) \right)
\]

\[+ b_i sat(J_i(x, t)), \quad t \geq 0, \quad i = 1, 2, \ldots, N, \]

with

\[
R_{t_0}^t \partial_t^\alpha e_i(x, s) = e_0(x, s), (x, s) \in \Omega \times [-\tau, 0], \quad e_i(x, t) = 0, \quad (x, t) \in \partial \Omega \times [-\tau, +\infty]. \]

Next, some useful definitions are presented.

**Definition 1:** Define \( \varepsilon(P, \rho) = \{ \bar{e}^T(x, t) \in R^n : \bar{e}^T(t) P \bar{e}(t) \leq \rho \bar{e}^T(t), \bar{e}^T(t) = \max(e_i(x, t), x \in \Omega) \} \), where \( P \) is a positive definite matrix, \( V(\Omega) \) denotes the volume of \( \Omega \).

**Definition 2:** The range of state values in which the control input remains linear with respect to \( e_i(x, t) \) is defined as \( L(K) = \{ \bar{e}_i(t) \in R^n : k_i \bar{e}_i(t) \leq \bar{J}_i, \bar{e}_i(t) = \max(e_i(x, t), x \in \Omega), i = 1, 2, \ldots, n, i \in N \} \).

**Remark 1:** Definition 1 and 2 extends the conception of the ellipsoid and linear region of the saturated in [44]. By introducing the spatial variables, we use the maximum value of the function on the definition domain to represent the function’s properties. We can find that this definition is very convenient to deal with the FRDNNs problem in later proof.

**Definition 3 ([44]):** The convex hulls of \( e_i \) is defined as

\[
co \{ e_i : i = 1, N \} := \left\{ \sum_{i=1}^N \theta_i e_i : \sum_{i=1}^N \theta_i = 1, \theta_i \geq 0 \right\}.
\]

**Definition 4 ([44]):** For initial condition \( \phi(t_0) \), the domain of attraction for \( u \) is defined as

\[
S := \left\{ \phi(t_0) : \lim_{t \to \infty} u(t, \phi(t_0)) = 0 \right\}.
\]

The assumptions given below are essential assets to achieve the main results of this paper.

**Assumption 1 ([29]):** For any \( u(x, t), v(x, t) \in R^n \), there exist constants \( \delta_i > 0 (i = 1, 2, \ldots, N) \), such that:

\[
|f_i(u) - f_i(v)| \leq \delta_i |u - v|,
\]

and \( \delta_{\max} = \max(\delta_i) \).

The following important lemmas will be employed during the proof process in the later section.

**Lemma 1 ([20]):** Let \( u(x, t) \in C^1[\Omega \times [t_0, +\infty)] \) be a continuous function with the Riemann-Liouville fractional-order derivative existing, then the following inequality holds:

\[
\frac{1}{2} R_{t_0}^t \partial_t^\alpha u^T(x, t) P u(x, t) \leq u(x, t) P R_{t_0}^t \partial_t^\alpha u^T(x, t),
\]

\[\forall \alpha \in (0, 1), \quad t > t_0, \]

where \( P \in R^{n \times n} \) is a positive definite matrix.

Then, inspired by [43], we note that the saturation terms’ expressions can be treated independently of spatial coordinates. Thus we can give the expressions of \( sat(K \bar{e}(x, t)) \) as the following lemma:

**Lemma 2:** Let \( \Theta \) be the set of \( n \times n \) diagonal matrices whose diagonal elements are either 1 or 0. Suppose each element of \( \Theta \) is labeled as \( \Theta_i \) and denote \( \Theta_{\bar{i}} = I_n - \Theta_i \). Clearly, if \( \Theta_i \in \Theta \), then \( \Theta_{\bar{i}} \in \Theta \). Let \( K, H \in R^{n \times n} \), then, for any \( u(x, t) \in L(H) \), we have \( sat(Ku(x, t)) = \sum_{i=1}^{2^n} \theta_i (\Theta_i K + \Theta_{\bar{i}} H) u(x, t) \), where \( 0 \leq \theta_i \leq 1 (l = 1, 2, \ldots, 2^n) \) are some scalars satisfying \( \sum_{i=1}^{2^n} \theta_i = 1 \).
Lemma 3 ([46]): For any vector \( x, y \in \mathbb{R}^n \), positive definite matrix \( H \in \mathbb{R}^{n \times n} \), the following inequality holds

\[
\pm 2xy \leq x^T Hx + y^T H^{-1}y.
\]

Hence, according to Lemma 2 and Kronecker product properties, the synchronization errors (7) can be rewritten into a compact form as

\[
\dot{R} \delta^a e(x, t) = D \Delta e(x, t) - Ce(x, t)
+ A (f(u(x, t)) - f(v(x, t)))
+ G (f(u(x, t) - \tau(t)) - f(v(x, t) - \tau(t)))
+ B \sum_{l=1}^{2^n} \theta_l(I_N \otimes \Theta_l)K + I_N \otimes \Theta_l H) e(x, t),
\]

where \( D = \text{diag}[d_i], C = \text{diag}[c_i], G = \{g_{ij}\}, B = \text{diag}[b_l] \) with compatible dimension and \( f(u(x, t)) = (f_1(u_1(x, t)) \ldots f_N(u_N(x, t)))^T \).

III. MAIN RESULTS

In this section, we will derive sufficient conditions for synchronization of the systems with the Dirichlet boundary and control input saturation, that is:

Theorem: Suppose the assumption 1 holds, then system (1) and (5) will achieve synchronization if there exists a positive definite matrix \( Q \) and arbitrary matrix \( K, H \) such that

\[
\Phi = \begin{pmatrix}
\Phi_{1,1} & 0 & 0 \\
* & \Phi_{2,2} & 0 \\
* & * & \Phi_{3,3}
\end{pmatrix} \leq 0,
\]

and

\[
\varepsilon(I, \rho) \subseteq L(H),
\]

where

\[
\begin{align*}
\Phi_{1,1} &= \frac{1}{2}AA^T + \frac{1}{2}\delta^T \delta + \frac{1}{2}GG^T + Q - C \\
\Phi_{2,2} &= -D, \\
\Phi_{3,3} &= -(1 - \sigma)Q + \frac{1}{2} \delta^T \delta.
\end{align*}
\]

with known matrix \( A, B, C, D, G. \)

Proof: Choose the following Lyapunov functional

\[
V(t) = V_1(t) + V_2(t),
\]

where

\[
V_1(t) = \frac{1}{2} \int_{\Omega}^t \int_0^t \delta^a e(x, t) e(x, t) dx dt,
\]

\[
V_2(t) = \int_{t-\tau(t)}^t \int_{\Omega} \delta^a e(x, s) Q e(x, s) ds dt.
\]

Thus \( V(t) \geq 0 \) holds obviously. Then, according to Lemma 1, we get the derivative of \( V_1(t) \) along the trajectories of system (10) as follows:

\[
\dot{V}_1(t) \leq \int_{\Omega}^t e^T(x, t) \delta^a e(x, t) dx dt
= \int_{\Omega}^t e^T(x, t) (D \Delta e(x, t) - Ce(x, t)
+ A (f(u(x, t)) - f(v(x, t)))
+ G (f(u(x, t) - \tau(t)) - f(v(x, t) - \tau(t)))
+ B \sum_{l=1}^{2^n} \theta_l(I_N \otimes \Theta_l)K + I_N \otimes \Theta_l H) e(x, t)) dx dt
= \int_{\Omega}^t e^T(x, t) D \Delta e(x, t) dx dt
+ \int_{\Omega}^t e^T(x, t) A (f(u(x, t)) - f(v(x, t))) dx dt
+ \int_{\Omega}^t e^T(x, t) G (f(u(x, t) - \tau(t)) - f(v(x, t) - \tau(t))) dx dt
+ \int_{\Omega}^t e^T(x, t) B \sum_{l=1}^{2^n} \theta_l(I_N \otimes \Theta_l)K + I_N \otimes \Theta_l H) e(x, t) dx dt.
\]

Utilizing Green’s formula and the boundary conditions, we have

\[
\int_{\Omega}^t e^T(x, t) D \Delta e(x, t) dx dt
= e^T(x, t) De_s(x, t) \bigg|_0^t - \int_0^t e^T(x, t) De_s(x, t) dx dt
= -\int_0^t e^T(x, t) De_s(x, t) dx dt.
\]

According to assumption (A1) and lemma 3, the third and fourth term satisfy the inequalities

\[
\int_{\Omega}^t e^T(x, t) A (f(u(x, t)) - f(v(x, t))) dx dt
\leq \frac{1}{2} \int_{\Omega}^t e^T(x, t) AA^T e(x, t) dx dt
+ \frac{1}{2} \int_{\Omega}^t e^T(x, t) \delta^T \delta e(x, t) dx dt.
\]
Similarly, the derivative of \( V \) satisfies the following inequality
\[
\dot{V}(t) \leq -\int_{\Omega} e_{x}^T (x, t)D e_{x}(x, t)dx - \int_{\Omega} e_{x}^T (x, t)Ce(x, t)dx + \frac{1}{2} \int_{\Omega} e^T (x, t)AA^T e(x, t)dx + \frac{1}{2} \int_{\Omega} e^T (x, t)\delta^T \delta e(x, t - \tau(t))dx + \frac{1}{2} \int_{\Omega} e^T (x, t)GG^T e(x, t)dx + \frac{1}{2} \int e^T (x, t)B \sum_{l=1}^{2^n} \theta_l (I_N \otimes \Theta_l K + I_N \otimes \Theta_l H) e(x, t)dx \leq -\int_{\Omega} e^T (x, t)\Phi_1 e(x, t)dx - (1 - \sigma) \int_{\Omega} e^T (x, t - \tau(t))Q e(x, t - \tau(t))dx.
\]  
(21)

Substituting (21) and (22) into (14), we have
\[
\dot{V}(t) = \dot{V}_1(t) + \dot{V}_2(t) \\
\leq -\int_{\Omega} e_{x}^T (x, t)D e_{x}(x, t)dx - \int_{\Omega} e_{x}^T (x, t)Ce(x, t)dx + \frac{1}{2} \int_{\Omega} e^T (x, t)AA^T e(x, t)dx + \frac{1}{2} \int_{\Omega} e^T (x, t)\delta^T \delta e(x, t)dx + \frac{1}{2} \int_{\Omega} e^T (x, t)GG^T e(x, t)dx + \frac{1}{2} \int e^T (x, t)B \sum_{l=1}^{2^n} \theta_l (I_N \otimes \Theta_l K + I_N \otimes \Theta_l H) e(x, t)dx + \frac{1}{2} \int e^T (x, t - \tau(t))\delta^T \delta e(x, t - \tau(t))dx + \frac{1}{2} \int e^T (x, t)GG^T e(x, t)dx + \frac{1}{2} \int e^T (x, t)GG^T e(x, t)dx + \frac{1}{2} \int e^T (x, t)B \sum_{l=1}^{2^n} \theta_l (I_N \otimes \Theta_l K + I_N \otimes \Theta_l H) e(x, t)dx \leq -\int_{\Omega} e^T (x, t)\Phi e(x, t)dx - (1 - \sigma) \int_{\Omega} e^T (x, t - \tau(t))Q e(x, t - \tau(t))dx.
\]  
(22)

Thus, according to the condition (11), we have
\[
\dot{V}(t) = \int_{\Omega} e^T (x, t)\Phi e(x, t)dx \leq 0.
\]  
(23)

Since (12) is equivalent to
\[
\min \{ e^T (t)\tilde{e}(t) : h\tilde{e}(t) = \tilde{J}_i \} \geq \rho,
\]  
(24)

we can transform it as
\[
\min \{ e^T (t)\tilde{e}(t) : h\tilde{e}(t) = \tilde{J}_i \} = \int \tilde{J}_i^2 (h\tilde{e}(t))^2 dx = M\tilde{J}_i^2 (h\tilde{e}(t))^2 dx = M\tilde{J}_i^2 (h\tilde{e}(t))^2 dx = \frac{M\tilde{J}_i^2}{\rho}.
\]  
(25)

According to the Schur complement, (28) can be expressed as the following LMIs form
\[
\begin{bmatrix}
I \\
h_i^T \frac{M\tilde{J}_i^2}{\rho}
\end{bmatrix} \succeq 0.
\]  
(26)

Thus, system (1) and (5) can achieve synchronization under the saturation input control.

Meanwhile, according to (14) and we have
\[
V(0) = \frac{1}{2} \int_{\Omega} e^T (x, 0)e(x, 0)dx + \int_{\Omega} e^T (x, s)Q e(x, s)dxds = \theta,
\]  
(27)

where \( \theta \) is a constant. Accordingly, since \( \dot{V}(t) \leq 0 \), it concludes that
\[
\int_{\Omega} e^T (x, t)e(x, t) \leq V(t) \leq V(0) = \theta.
\]  
(28)
In other words, for any initial value \( e(x, 0) \in \varepsilon(I, \rho) \), \( e(x, t) \) will not leave \( \varepsilon(I, \rho) \) indicating that for all \( t > 0 \), \( e(x, t) \in \varepsilon(I, \rho) \subseteq L(H) \) holds. The proof is completed.

**Remark 1:** In [47] when dealing with the Lyapunov functional \( V \), the fractional derivation is directly carried out and get the Mittag-Leffler stability. For the time-delay problem, fractional derivation on the functional \( V \) cannot work, so to use Lyapunov-Krasovskii functional and derive the functional \( V \) with respect to \( t \) is a more convenient way.

**Remark 2:** The Lyapunov-Krasovskii functional presented in our paper is a traditional and mature approach for related works of ODEs. Still, research seldom handles the Riemann-Liouville derivative with reaction-diffusion and saturation comprehensively, so we have made an original exploration of this issue.

**Corollary:** Assume that \( \tau(t) \equiv 0 \), then system (1) and (5) can reach synchronization with the feedback control

\[
J_i(x, t) = k_i e_i(x, t),
\]

if the following conditions

\[
\bar{\Phi} = \begin{pmatrix}
\bar{\Phi}_{1,1} & 0 \\
* & \bar{\Phi}_{2,2}
\end{pmatrix} \leq 0,
\]

and

\[
\varepsilon(I, \rho) \subseteq L(H),
\]

hold, where

\[
\begin{aligned}
\bar{\Phi}_{1,1} &= \frac{1}{2} A A^T + \delta^T \delta + \frac{1}{2} G G^T + Q - C - Q \\
+ &B \sum_{l=1}^{2^n} \Theta_l(I_N \otimes \Theta_l K + I_N \otimes \Theta_l H), \\
\bar{\Phi}_{2,2} &= -D.
\end{aligned}
\]

with known matrix \( A, B, C, D, G \), positive definite matrix \( Q \) and arbitrary matrix \( K, H \).

**IV. ESTIMATE THE DOMAIN OF ATTRACTION**

Due to the nonlinear influence of saturation, the stability region is often local. In this section, we will give sufficient conditions for the initial conditions which can ensure the two system reach synchronization during a finite time.

It is difficult to deal with spatial variables, so we simplify the problem. Consider the set of the maximum values of the initial value of the system in its domain

\[
\chi = \{ \| e_1(x, t) \|_{\max}, \| e_2(x, t) \|_{\max}, \ldots, \| e_N(x, t) \|_{\max} \}
\]

which conform to some certain shape reference set \( \chi_R \), then we hope that the shape reference set can fill the attraction region of the system as fully as possible. That is to solve the problem

\[
\sup_{Q > 0, D_A G, C_B K H} \gamma
\]

s.t.

\[
\begin{aligned}
& a) \gamma \chi \subseteq \varepsilon(I, \rho), \\
& b) \bar{\Phi} \leq 0, \\
& c) \varepsilon(I, \rho) \subseteq L(H).
\end{aligned}
\]

If \( \chi_R \) is a polygon, i.e.

\[
\chi_R = \text{co} \{ e_1, e_2, \ldots, e_N \}
\]

thus the constraint \( a) \) is equivalent to \( \gamma^2 M e_{\max}^T e_{\max} \leq \rho, i = 1, \ldots, N \). According to the Schur component, we have

\[
\frac{M e_{\max}^T e_{\max}}{p} \leq 1 \Leftrightarrow \left( \frac{1}{\gamma^2} \frac{e_{\max}^T e_{\max}}{p} \right) \geq 0, \quad i = 1, \ldots, N.
\]

Hence, we get (38) as the sufficient condition for the domain of initial conditions that can ensure the two systems can achieve synchronization under the above theorem conditions. Also, we can solve the following optimization problem to get the maximal volume of \( \varepsilon(I, \rho) \),

\[
\min_{Q > 0, D_A G, C_B K H} \xi
\]

s.t.

\[
\begin{aligned}
& a) \bar{\Phi} \leq 0, \\
& b) \varepsilon(I, \chi, \gamma^{-1}) \subseteq L(H).
\end{aligned}
\]

where \( \xi = \frac{1}{\gamma^2} \).

**Remark 3:** It should note that (38) is “sufficient” enough, which means that the estimation of the domain of attraction is often smaller than its theoretical one. In other words, the initial conditions obtained by the above methods are usually safe enough, but we still hope to find more conservative laws in our future work.

**V. NUMERICAL EXAMPLES**

In this section, we will give two examples. In Example 1, We take the parameters satisfying all the Theorem conditions to test the feedback control capability. Then, we will test the tolerance upbound of the initial errors in Example 2.

**Example 1:** Consider two four-nodes FDRNNs defined on \( \Omega \times [-\tau, +\infty) = [-1, 1] \times [-\tau, +\infty) \) with the following parameters:

\[
\begin{aligned}
\alpha &= 0.75, n = 1, N = 4, \\
\bar{T} &= 50, B = I_4, \\
D &= \text{diag}(0.3, 0.2, 0.35, 0.4), \\
C &= \text{diag}(-6, -3, -4, -3.6), \\
A &= G = \begin{pmatrix}
2 & 1 & 1 & 1 \\
1 & 2 & 0.4 & 1 \\
1 & 0.4 & 0.7 & 0.2 \\
1 & 1 & 0.2 & 2
\end{pmatrix}, \\
f(u(x, t)) &= \tanh(u(x, t)), \\
\tau(t) &= \frac{0.1e^t}{1 + e^t}.
\end{aligned}
\]
and the initial value are given as \( u_0(x) = 0.41 \sin(2\pi ix) \) and 
\( v_0(x) = 0, \ i = 1, 2, 3, 4 \). Then according to the definition (4), it can be transformed as 
\( u_i(x, \bar{t}) = 0.66 \sin(2\pi ix) \) and 
\( v_i(x, \bar{t}) = 0 \) for \( \bar{t} \in [-\tau, 0] \) approximately. Thus, we can get the maximal \( \rho \approx 1.4625 \) by solving (39).

**FIGURE 1.** System errors without control of example 1.

**FIGURE 2.** System errors with control of example 1.

**FIGURE 3.** System errors with control at \( x = 0.18 \) of example 1.

**FIGURE 4.** Control input of example 1.

**FIGURE 5.** Control input at \( x = 0.18 \) of example 1.

With the above control gain, FIGURE 2 illustrates that the errors between the two neural networks achieve the neighborhood of 0 on the entire domain. Looking at it from another angle, as FIGURE 3 depicts, the errors between two systems decay very quickly under the proposed control input.

Then FIGURE 4 and 5 shows the input control signal of each node. In this situation, they didn’t trigger saturation. Next, Example 2 will test the robustness of the designed control law.

**Example 2:** Consider the parameters in Example 1, and we will replace them with some “sick” initial conditions to test the maximal tolerance of the initial errors. From (38),
naturally, except for the two boundaries, we can take the initial errors as the maximum value on the interval value on the whole domain, that is $u_0(x) = e_{\text{imax}}$, $v_0(x) = 0$. Let $\gamma = 1$, thus we have $e_{\text{imax}} \approx 0.8426$, the numerical experiment indicate that two system can reach synchronization as FIGURE 6 illustrate.

Increasing $e_{\text{imax}}$ to 20, we found that although the control input has reached the saturation state, the error system can still approach the neighborhood of zero in finite time according to FIGURE 10-12.

Continue increase $e_{\text{imax}}$ to 25, we found that the errors of node 1 increase rapidly as FIGURE 13 and 14 illustrate which
indicate that under the saturation bound $J_f = 50$ the systems cannot synchronize.

VI. CONCLUSION

In this work, firstly, the definitions of the ellipsoid and the linear region of the saturated are extended to PDEs case. Under this framework, we construct a suitable Lyapunov-Krasovskii functional for synchronizing two fractional reaction-diffusion neural networks and obtain sufficient conditions under saturated control inputs by using convex hulls and some Riemann-Liouville fractional integral properties. Besides, we estimate the domain of attraction. All the conditions are presented in the form of LMIs thus can easily be solved by the MATLAB toolbox. At last, two numerical experiments show that the proposed control laws are reliable when trigger saturation state. Meanwhile, the designed control law is safe enough with our estimation of the domain of attraction. As we can see, our method is simple and sufficient, but the estimation of the domain of attraction is too small. In our future work, we can find some more suitable inequalities to achieve more conservative conditions and apply our approach on network consensus, fault-tolerant, adaptive fuzzy control, etc.

REFERENCES

[1] D. J. Watts and S. H. Strogatz, “Collective dynamics of ‘small-world’ networks,” Nature, vol. 393, no. 6684, pp. 440–442, 1998.
[2] M. E. J. Newman, “The structure and function of complex networks,” SIAM Rev., vol. 45, no. 2, pp. 167–256, Jan. 2003.
[3] Q. Gan, “Global exponential synchronization of generalized stochastic neural networks with mixed time-varying delays and reaction-diffusion terms,” Neurocomputing, vol. 89, pp. 96–105, Jul. 2012.
[4] L. Wang and H. Zhao, “Synchronized stability in a reaction–diffusion neural network model,” Phys. Lett. A, vol. 378, no. 48, pp. 3586–3599, Nov. 2014.
[5] B. Ambrosio, M. A. Aziz-Alaoui, and V. L. E. Phan, “Global attractor of complex networks of reaction-diffusion systems of Fitzhugh-Nagumo type,” Discrete Continuous Dyn. Syst.-B, vol. 23, no. 9, pp. 3787–3797, 2018.
[6] R. Metzler and J. Klafter, “The random walk’s guide to anomalous diffusion: A fractional dynamics approach,” Phys. Rep., vol. 339, no. 1, pp. 1–77, Dec. 2000.
[7] S. Shourya, S. Kumar, and R. K. Jha, “Adaptive fractional differential approach to enhance underwater images,” in Proc. 6th Int. Symp. Embedded Comput. Syst. Design (ISED), Dec. 2016, pp. 56–60.
[8] N. Su, P. N. Nelson, and S. Connor, “The distributed-order fractional diffusion-wave equation of groundwater flow: Theory and application to pumping and slug tests,” J. Hydrol., vol. 529, pp. 1262–1273, Oct. 2015.
[9] J.-H. He, “Approximate analytical solution for seepage flow with fractional derivatives in porous media,” Comput. Methods Appl. Mech. Eng., vol. 167, nos. 1–2, pp. 57–68, Dec. 1998.
[10] H. Zhang, R. Ye, J. Cao, and A. Alsaedi, “Delay-independent stability of Riemann–Liouville fractional neutral-type delayed neural networks,” Neural Process. Lett., vol. 47, no. 2, pp. 427–442, Apr. 2018.
[11] Y.-H. Lim, K.-K. Oh, and H.-S. Ahn, “Stability and stabilization of fractional-order linear systems subject to input saturation,” IEEE Trans. Autom. Control, vol. 58, no. 4, pp. 1062–1067, Apr. 2013.
[12] E. S. A. Shahri, A. Alfi, and J. Machado, “Stability analysis of a class of nonlinear fractional-order systems under control input saturation,” Int. J. Robust Nonlinear Control, vol. 28, no. 7, pp. 2887–2905, May 2018.
[13] L. Chuang and L. Junguo, “On the ellipsoidal invariant set of fractional order systems subject to actuator saturation,” in Proc. 34th Chin. Control Conf. (CCC), Jul. 2015, pp. 800–805.
[14] J. Luo, “State-feedback control for fractional-order nonlinear systems subject to input saturation,” Math. Problems Eng., vol. 2014, Jan. 2014, Art. no. 891639.
[15] Z. Wang, X. Wang, J. Xia, H. Shen, and B. Meng, “Adaptive sliding mode output tracking control based-FODOB for a class of uncertain fractional-order nonlinear time-delayed systems,” Sci. China Technol. Sci., vol. 63, no. 9, pp. 1854–1862, May 2020.
[16] Q. Xu, X. Xu, S. Zhuang, J. Xiao, C. Song, and C. Che, “New complex projective synchronization strategies for drive-response networks with fractional complex-variable dynamics,” Appl. Math. Comput., vol. 338, pp. 552–566, Dec. 2018.
[17] Q. Xu, S. Zhuang, X. Xu, C. Che, and Y. Xia, “Stabilization of a class of fractional-order nonautonomous systems using quadratic Lyapunov functions,” *Adv. Difference Equ.*, vol. 2018, no. 1, pp. 1–15, Jan. 2018.

[18] B. Meng, X. Wang, Z. Zhang, and Z. Wang, “Necessary and sufficient conditions for normality and slide mode control of singular fractional-order systems with uncertainties,” *Sci. China Inf. Sci.*, vol. 63, no. 5, Mar. 2020, Art. no. 152202.

[19] J. Jia, X. Huang, Y. Li, J. Cao, and A. Alsaedi, “Global stabilization of fractional-order memristor neural networks with time delay,” *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 31, no. 3, pp. 997–1009, Mar. 2020.

[20] X. Wu, S. Liu, and Y. Wang, “Stability analysis of Riemann–Liouville fractional-order neural networks with reaction-diffusion terms and mixed time-varying delays,” *Neurocomputing*, vol. 431, pp. 169–178, Mar. 2021.

[21] D. Baleanu, A. Ranjbar, S. J. Sadati, H. Delavari, and V. Gejji, “Lyapunov–Krasovskii stability theorem for fractional systems with delay,” *Romanian J. Phys.*, vol. 56, nos. 5–6, pp. 636–643, 2011.

[22] M. Di Ferdinando, P. Pepe, and S. D. Gennaro, “A converse Lyapunov–Krasovskii theorem for the global asymptotic local exponential stability of nonlinear time-delay systems,” *IEEE Control Syst. Lett.*, vol. 5, no. 1, pp. 7–12, Jan. 2021.

[23] A. Taghieh and M. H. Shafiee, “Observer-based robust model predictive control of switched nonlinear systems with time delay and parametric uncertainties,” *J. Vib. Control*, Sep. 2020, doi: 10.1177/1077546320935652.

[24] X.-M. Zhang and Q.-L. Han, “New Lyapunov–Krasovskii functionals for global asymptotic stability of delayed neural networks,” *IEEE Trans. Neural Netw. *, vol. 20, no. 3, pp. 533–539, Mar. 2009.

[25] E. Fridman, “New Lyapunov–Krasovskii functionals for stability of linear retarded and neutral type systems,” *Syst. Control Lett.*, vol. 43, no. 4, pp. 309–319, Jul. 2001.

[26] X. Yang, J. Cao, and Z. Yang, “Synchronization of coupled reaction-diffusion neural networks with time-varying delays via pinning-impulsive controller,” *SIAM J. Control Optim.*, vol. 51, no. 5, pp. 3486–3510, Jan. 2013.

[27] H. Chen, P. Shi, and C.-C. Lim, “Pinning impulsive synchronization for stochastic reaction–diffusion dynamical networks with delay,” *Neural Netw.*, vol. 106, pp. 281–293, Oct. 2018.

[28] H. Dinis, J. J. Winkin, and A. V. Wouwer, “A sliding mode observer for a linear reaction–convective–diffusion equation with disturbances,” *Syst. Control Lett.*, vol. 124, pp. 40–48, Feb. 2019.

[29] J.-L. Wang and H.-N. Wu, “Synchronization and adaptive control of an array of linearly coupled reaction-diffusion neural networks with hybrid coupling,” *IEEE Trans. Cybern.*, vol. 44, no. 8, pp. 1350–1361, Aug. 2014.

[30] R. Li and H. Wei, “Synchronization of delayed Markov jump memristive neural networks with reaction-diffusion terms via sampled data control,” *Int. J. Mach. Learn. Cybern.*, vol. 7, no. 1, pp. 157–169, Feb. 2016.

[31] T. Hu, Z. Lin, and B. M. Chen, “An analysis and design method for linear systems subject to actuator saturation and disturbance,” *Automatica*, vol. 38, no. 2, pp. 351–359, Feb. 2002.

[32] J. M. G. da Silva and S. Tarbouriech, “Anti-windup design with guaranteed regions of stability: An LMI-based approach,” *IEEE Trans. Autom. Contr.*, vol. 50, no. 1, pp. 106–111, Jan. 2005.

[33] N. El Fezazi, F. El Haoussi, E. H. Tissir, and F. Tadeo, “Delay dependent stability conditions for normalization and sliding mode control of singular fractional-order nonautonomous systems using quadratic Lyapunov,” *IEEE Access*, vol. 6, pp. 50066–50076, Sep. 2018.

[34] Z. Li and S. Ahreza, “Low-and-high gain design technique for linear systems subject to input saturation—A direct method,” *Int. J. Robust Nonlinear Control*, vol. 7, no. 12, pp. 1071–1101, Dec. 1997.

[35] F. Zhou, K. Liu, Y. Li, and G. Liu, “Distributed fault-tolerant control of modular and reconfigurable robots with consideration of actuator saturation,” *Neural Comput. Appl.*, vol. 32, no. 17, pp. 13591–13604, Sep. 2020.

[36] N. Gu, D. Wang, Z. Peng, and L. Liu, “Adaptive bounded neural network control for coordinated path-following of networked underactuated autonomous surface vehicles under time-varying state-dependent cyber-attack,” *ISA Trans.*, vol. 104, pp. 212–221, Sep. 2020.

[37] E. S. A. Shahri, A. Alfi, and J. A. T. Machado, “Lyapunov method for the stability analysis of uncertain fractional-order systems under input saturation,” *Appl. Math. Model.*, vol. 81, pp. 663–672, May 2020.

[38] E. S. A. Shahri, A. Alfi, and J. A. T. Machado, “An extension of estimation of domain of attraction for fractional order linear system subject to saturation control,” *Appl. Math. Lett.*, vol. 47, pp. 26–34, Sep. 2015.

[39] C. Yang, L. Ma, X. Ma, and X. Wang, “Stability analysis of singularly perturbed control systems with actuator saturation,” *J. Franklin Inst.*, vol. 353, no. 6, pp. 1294–1296, Apr. 2016.

[40] P. Selvaraj, R. Sakthivel, and O. M. Kwon, “Synchronization of fractional-order complex dynamical network with random coupling delay, actuator faults and saturation,” *Nonlinear Dyn.*, vol. 94, no. 4, pp. 3101–3116, Dec. 2018.

[41] T. Hu and Z. Lin, *Control Systems With Actuator Saturation: Analysis and Design*. Boston, MA, USA: Birkhäuser, 2001.

[42] S. Liu, X. Wu, X. Zhou, and W. Jiang, “Asymptotical stability of Riemann–Liouville fractional nonlinear systems,” *Nonlinear Dyn.*, vol. 86, pp. 65–71, Oct. 2016.

[43] X. Song, X. Li, Z. Ning, M. Wang, and J. Man, “Synchronization for hybrid coupled reaction-diffusion neural networks with stochastic disturbances via spatial sampled-data control strategy,” *Proc. Inst. Mech. Eng. I, J. Syst. Control Eng.*, Jul. 2020, doi: 10.1177/0959651820935668.

[44] H. Lv and X. Zhang, “Finite-time neural network backstepping control of an uncertain fractional-order duffing system with input saturation,” *Frontiers Phys.*, vol. 8, pp. 1–8, May 2020.

**YIN WANG** received the M.S. degree from Shandong University, China, in 2015, where he is currently pursuing the Ph.D. degree. His research interest includes the control and synchronization of complex networks.

**SHUTANG LIU** received the Ph.D. degree in control theory and control engineering from the South China University of Technology and the City University of Hong Kong, in 2002. From 2003 to 2005, he did postdoctoral research at the Academy of Mathematics and Systems Science, Chinese Academy of Sciences, China. He is currently a Professor and a Doctoral Supervisor with the College of Control Science and Engineering, Shandong University, China. His research interests include spatial chaotic theory of nonlinear dynamical systems and its application, qualitative theory and qualitative control of complex systems, control and applications of fractuals, and so on.

**XIANG WU** received the M.S. degree from Anhui University, Hefei, China, in 2015. He is currently pursuing the Ph.D. degree with the School of Control Science and Engineering, Shandong University, Jinan, China. His current research interests include the stability of fractional differential equations and singular systems with delays, and synchronization and control of fractional complex networks and neural networks.