Distributional Robustness and Regularization in Statistical Learning

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A central question in statistical learning is to design algorithms that not only perform well on training data, but also generalize to new and unseen data. In this paper, we tackle this question by formulating a distributionally robust stochastic optimization (DRSO) problem, which seeks a solution that minimizes the worst-case expected loss over a family of distributions that are close to the empirical distribution in Wasserstein distances. We establish a connection between such Wasserstein DRSO and regularization. More precisely, we identify a broad class of loss functions, for which the Wasserstein DRSO is asymptotically equivalent to a regularization problem with a gradient-norm penalty. Such relation provides new interpretations for problems involving regularization, including a great number of statistical learning problems and discrete choice models (e.g. multinomial logit). The connection also suggests a principled way to regularize high-dimensional, non-convex problems, which is demonstrated the training of Wasserstein generative adversarial networks (WGANs) in deep learning.

Key words: Wasserstein distance; regularization; deep learning; generative adversarial networks; choice model

1. Introduction

Statistical learning theory (Vapnik 2013) provides a framework for learning functional dependencies from past data, so as to make better predictions and decisions for the future. Typically, a statistical learning problem is written as

$$\min_{\beta \in \mathcal{D}} \mathbb{E}_{(x,y) \sim \mathbb{P}_{true}} [\ell_\beta(x, y)].$$

Here the term $\ell_\beta(x, y)$ is defined as $\ell_\beta(x, y) := \ell(f(x; \beta), y)$, where the function $f(x; \beta)$ is the hypothesis function parameterized by $\beta \in \mathcal{D}$, the function $\ell$ is the per-sample loss function, and $(x, y)$ is an input-output random vector with probability distribution $\mathbb{P}_{true}$ on the data space $\mathcal{Z} \subset \mathbb{R}^d$. 

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In practice, the true data-generating distribution $P_{\text{true}}$ might be unknown. However, a sample of observations from the distribution $P_{\text{true}}$ is often available. Thus, a common practice is to replace the expected risk under the unknown true distribution $P_{\text{true}}$ with the empirical risk under the empirical distribution $P_n$, which is constructed from $n$ data points $\{(\hat{x}^i, \hat{y}^i)\}_{i=1}^n$. Thereby we obtain the following empirical risk minimization problem:

$$\min_{\beta \in D} \mathbb{E}_{(x,y) \sim P_n} [\ell_\beta(x,y)].$$

Empirical risk minimization often yields solutions which perform well on the training data, but may perform poorly on out-of-sample data. This is known as the overfitting phenomenon. A core aim of statistical learning is to design algorithms with a better generalization ability, i.e., the ability to perform well on new, previously unseen data. To reduce the generalization error, a great number of regularization methods have been proposed. A typical regularization problem can be represented as

$$\min_{\beta \in D} \mathbb{E}_{(x,y) \sim P_n} [\ell_\beta(x,y)] + \alpha \cdot J(\ell_\beta). \quad (1)$$

This formulation not only covers commonly seen norm-penalty regularization methods, such as $\ell_1$-regularization (Tibshirani 1996) and Tikhonov regularization (Tikhonov et al. 1977), but also is (approximately) equivalent to other regularization methods, including adding noise (Bishop 1995), dropout (Wager et al. 2013, Srivastava et al. 2014), and adversarial training (Goodfellow et al. 2014b). We refer to Chapter 7 of Goodfellow et al. (2016) for a survey on regularization methods in machine learning.

Recently, the emerging field of data-driven distributionally robust stochastic optimization (DRSO) (Goh and Sim 2010, Bertsimas et al. 2013, Wiesemann et al. 2014, Ben-Tal et al. 2013, Jiang and Guan 2015, Gao and Kleywegt 2016, Esfahani and Kuhn 2017) provides another approach to robustify the learning models by considering the following minimax problem

$$\min_{\beta \in D} \sup_{Q \in \mathcal{M}(P_n)} \mathbb{E}_{(x,y) \sim Q} [\ell_\beta(x,y)],$$

where $\mathcal{M}(P_n)$ is a set of probability distributions on $\mathbb{Z}$ constructed from the empirical distribution $P_n$. By hedging against the worst possible distribution over the set $\mathcal{M}(P_n)$ of probability distributions, distributionally robust stochastic optimization seeks solutions that are robust against disturbances of data distribution. Distributionally robust stochastic
optimization is rooted in von Neumann’s game theory (Žáčková 1966), robust statistics (Huber 2011, Berger 1984), problem of moments (Shohat and Tamarkin 1943) and Fréchet class (Joe 1997, Agrawal et al. 2012). There are two typical ways to construct the set \( \mathcal{M} \). One approach considers distributions with specified properties such as symmetry, support information and moment conditions (Calafiore and El Ghaoui 2006, Popescu 2007, Goh and Sim 2010, Delage and Ye 2010, Wiesemann et al. 2014). The other approach directly works with the empirical distribution (or some other nominal distribution) and considers distributions that are close to the nominal distribution in the sense of certain statistical distance. Popular choices of the statistical distance include \( \phi \)-divergences (Ben-Tal et al. 2013, Jiang and Guan 2015, Bayraksan and Love 2015, Wang et al. 2016), Prokhorov metric (Erdoğan and Iyengar 2006), and Wasserstein distances (Wozabal 2014, Blanchet and Murthy 2016, Gao and Kleywegt 2016, Esfahani and Kuhn 2017). In addition, DRSO reduces to robust optimization (Bertsimas and Sim 2004, Ben-Tal et al. 2009) if the set is specified only by the support of the distributions.

In this paper, we establish a connection between DRSO with Wasserstein distance and regularization (1). More precisely, we consider the following Wasserstein DRSO problem

\[
\min_{\beta \in \mathcal{D}} \sup_{Q \in \mathcal{M}^\alpha_p(P_n)} \mathbb{E}_{(x,y) \sim Q}[\ell_\beta(x, y)],
\]

(Wasserstein-DRSO)

where

\[
\mathcal{M}^\alpha_p(P_n) := \{ Q \in \mathcal{P}(\mathcal{Z}) : \mathcal{W}_p(Q, P_n) \leq \alpha \},
\]

that is, \( \mathcal{M}^\alpha_p(P_n) \) contains all probability distributions whose \( p \)-Wasserstein distance \( \mathcal{W}_p(Q, P_n) \) (see Definition 1 in Section 2) to the empirical distribution \( P_n \) is no more than certain threshold \( \alpha > 0 \). We establish a connection between (Wasserstein-DRSO) and the regularization (1). Such connection has several important methodological and algorithmic implications. In particular, our contributions are as follows.

1. **Exact equivalence for the linear function class.** For many loss functions commonly used in statistical learning, there is an exact equivalence between (Wasserstein-DRSO) and norm-penalty regularization. This not only recovers several equivalence results from previous studies (see Section 1.1 below), but also develops new results (Section 3.1). The equivalence result for linear optimization provides a new interpretation for discrete choice models (e.g. multinomial logit, nested logit, and generalized extreme value...
choice models) from the perspective of distributional robustness, and offers a new economic intuition for the generalized extreme value choice models, which was introduced in the literature purely mathematically (Section 4).

2. Asymptotic equivalence for smooth function classes. For a broad class of loss functions, we show that \((\text{Wasserstein-DRSO})\) is asymptotically equivalent to the following regularization problem

\[
\min_{\beta \in \mathcal{D}} \mathbb{E}_{(x,y) \sim P_n}[\ell_\beta(x,y)] + \alpha \cdot \|\nabla_{(x,y)} \ell_\beta\|_{P_{n,p^*}},
\]

where \(p^* = \frac{p}{p-1}\) and the penalty term \(\|\nabla_{(x,y)} \ell_\beta\|_{P_{n,p^*}}\) represents the empirical \(p^*\)-norm (see Definition 2 in Section 2) of the gradient of the loss function with respect to the data (Section 3.2).

3. A principled approach to regularize statistical learning problems. The asymptotic equivalence suggests a principled way to regularize statistical learning problems, namely, by solving the regularization problem (2). This is illustrated by the training of Wasserstein generative adversarial networks.

1.1. Related Work

On the equivalence between regularization and robust optimization / DRSO. It has been shown that norm penalty regularization has a robust-optimization interpretation in some special cases, including linear/matrix regression (Xu et al. 2009a, Bertsimas and Copenhaver 2017), and support vector machine (Xu et al. 2009b). Gao and Kleywegt (2016) has shown that the problem (\text{Wasserstein-DRSO}) can be approximated by a robust optimization problem. In view of this close relationship between (\text{Wasserstein-DRSO}) and robust optimization, it is conceivable that in the above-mentioned special cases, (\text{Wasserstein-DRSO}) may also be closely related to norm penalty regularization. Indeed, such an equivalence has been established in Esfahani and Kuhn (2017) for piecewise-linear convex loss, Shafieezadeh-Abadeh et al. (2015) for logistic regression, and in Blanchet et al. (2016) for linear regression and support vector machines. Our general results not only cover these previous results, but also extend to many other learning problems. In a recent work, Shafieezadeh-Abadeh et al. (2017) studies the equivalence between regularization and DRSO with 1-Wasserstein distance \((p = 1)\) and its kernelization. As will be shown in Sections 3.2 and 5, our study on the case \(p > 1\) is of great methodological and algorithmic importance. Besides the Wasserstein distance, the equivalence between regularization and
DRSO with other distances has also been studied. For example, Gotoh et al. (2015) and Lam (2016) have pointed out that DRSO with $\phi$-divergence is first-order equivalent to variance regularization.

On interpretation of choice models. Discrete choice models are used to describe decision makers’ choices among a finite set of alternatives, and have attracted a lot of interest in economics, marketing, operations research and management science. Many choice models can be based on random utility theory (McFadden et al. 1973, McFadden 1978, McFadden and Train 2000), in which the utilities of alternatives are random, and each consumer chooses the alternative with the highest realized utility. A recent approach, called semi-parametric model (Natarajan et al. 2009, Mishra et al. 2014, Ahipasaoglu et al. 2013) combines the idea of random utility theory and distributional robustness, in which the distribution of the random utilities is given by the worst-case distribution over a set of distributions for an expected utility maximization problem. Choice models can also be based on representative agent model (Anderson et al. 1988, Hofbauer and Sandholm 2002), in which a representative agent maximizes a weighted sum of utilities of alternatives plus a regularization term that encourages diversification. We refer to Feng et al. (2017) for a study on relations between these choice models. Based on our equivalence result, the representative agent choice model can be derived from an ambiguity-averse representative agent choosing a choice probability vector that maximizes the expected utility.

The rest of this paper is organized as follows. In Section 2, we define several mathematical terms and review basic results on DRSO with Wasserstein distance. Next, in Section 3, we study the connection between distributional robustness and regularization. In Section 4, we provide a new interpretation of the discrete choice models from the perspective of distributional robustness, based on the equivalence result for linear optimization. Then in Section 5, we demonstrate the algorithmic implications of the distributionally robust framework. The conclusion of the paper is made in Section 6. Auxiliary results are provided in the Appendix.

2. Preliminary

Notation. We denote by $(\mathcal{Z}, d(\cdot, \cdot))$ a metric space equipped with some metric $d(\cdot, \cdot)$. For a normed space $(\mathcal{Z}, \|\cdot\|)$, we denote by $\mathcal{Z}^*$ its dual space, and by $\|\cdot\|_*$ or $\|\cdot\|_{\mathcal{Z}^*}$ the dual norm. Let $\mathcal{P}(\mathcal{Z})$ denote the set of all Borel probability measures on $\mathcal{Z}$. For any $p \in [1, \infty]$,
we denote $p_*$ as its Hölder conjugate, i.e., $\frac{1}{p} + \frac{1}{p_*} = 1$. For any function $h : \mathcal{Z} \to \mathbb{R}$, we denote by $h^*$ its convex conjugate, i.e., $h^*(z^*) = \sup_{z \in \mathcal{Z}} \{ \langle z, z^* \rangle - h(z) \}$. We denote by $\delta_z$ the Dirac probability measure on $z$. For ease of notation, we also use the conventions $\frac{1}{\infty} = 0$, $\frac{1}{0} = \infty$, $\infty^\infty = 1$, and $\infty + c = \infty$ for any $c \in \mathbb{R}$.

We introduce several definitions and review some results on DRSO with Wasserstein distance.

**Definition 1 (Wasserstein distance).** Let $p \in [1, +\infty]$. The $p$-Wasserstein distance between distributions $P, Q \in \mathcal{P}(\mathcal{Z})$ is defined as

$$W_p(P, Q) := \begin{cases} \left( \min_{\gamma \in \Gamma(P, Q)} \left\{ \int_{\mathcal{Z} \times \mathcal{Z}} d^p(z, z') \gamma(dz, dz') \right\} \right)^{1/p}, & \text{if } 1 \leq p < \infty, \\ \inf_{\gamma \in \Gamma(P, Q)} \gamma\text{-ess sup}_{\mathcal{Z} \times \mathcal{Z}} d(z, z'), & \text{if } p = \infty, \end{cases}$$

where $\Gamma(P, Q)$ denotes the set of all Borel probability distributions on $\mathcal{Z} \times \mathcal{Z}$ with marginal distributions $P$ and $Q$, and $\gamma\text{-ess sup}_{\mathcal{Z} \times \mathcal{Z}} d(z, z')$ expresses the essential supremum of $d(\cdot, \cdot)$ with respect to the measure $\gamma$.

Given the empirical distribution $P_n := \frac{1}{n} \sum_{i=1}^n \delta_{\hat{z}_i}$, the Wasserstein ball of radius $\alpha$ is given by

$$\mathcal{M}_p(\alpha, P_n) := \{ Q \in \mathcal{P}(\mathcal{Z}) : W_p(Q, P_n) \leq \alpha \},$$

and the corresponding DRSO problem is

$$\min_{\beta \in D} \sup_{Q \in \mathcal{M}_p(\alpha, P_n)} \mathbb{E}_{z \sim Q}[\ell_\beta(z)]. \quad \text{(Wasserstein-DRSO)}$$

Note that here we do not restrict to supervised learning, in which case $z = (x, y)$. Problem (Wasserstein-DRSO) admits a strong duality reformulation, as shown by the following lemma.

**Lemma 1.** Assume $h : \mathcal{Z} \to \mathbb{R}$ satisfies $h(z) \leq C(\|z - z^0\| + 1)^q$ for some constants $q \in [1, \infty)$, $C > 0$, $z^0 \in \mathcal{Z}$, and for all $z \in \mathcal{Z}$. Then for the set $\mathcal{M}_p(\alpha, P_n) \ (p \in [q, \infty))$, it holds that

$$\sup_{Q \in \mathcal{M}_p(\alpha, P_n)} \mathbb{E}_Q[h(z)] = \min_{\lambda \geq 0} \left\{ \lambda \alpha^p + \frac{1}{n} \sum_{i=1}^n \sup_{z \in \mathcal{Z}} \left[ h(z) - \lambda \|z - \hat{z}_i\|^p \right] \right\},$$

and for $\mathcal{M}_\infty(\alpha, P_n)$, it holds that

$$\sup_{Q \in \mathcal{M}_\infty(\alpha, P_n)} \mathbb{E}_Q[h(z)] = \sup_{z^i \in \mathcal{Z}, i = 1, \ldots, n} \left\{ \frac{1}{n} \sum_{i=1}^n h(z^i) : \|z^i - \hat{z}_i\| \leq \alpha \right\}.$$
Proof of Lemma 1. If $p \in [1, \infty)$, the result follows from Corollary 2 in Gao and Kleywegt (2016). If $p = \infty$, the result follows from Theorem 3 in Gao and Kleywegt (2017) by setting $R_0 = \infty$. □

We remark that the condition $p \geq q$ in Lemma 1 is necessary, as otherwise the worst-case loss will be infinity (Gao and Kleywegt 2016).

Our last definition is on the empirical norm.

Definition 2 (Empirical norm). Let $\|\cdot\|$ be some norm on $\mathbb{R}^d$. The empirical $p$-norm $\|h\|_{P_n,p}$ of $h : \mathcal{Z} \rightarrow \mathbb{R}^d$ is defined as for $p \in [1, \infty)$,

$$\|h\|_{P_n,p} := \left( \frac{1}{n} \sum_{i=1}^{n} \|h(\hat{z}^i)\|^p \right)^{1/p},$$

and for $p = \infty$,

$$\|h\|_{P_n,\infty} := \max_{1 \leq i \leq n} \|h(\hat{z}^i)\|.$$  

3. Equivalence between Distributional Robustness and Regularization

In this section, we show that for many common statistical learning problems, the DRSO problem (Wasserstein-DRSO) and the regularization problem (1) are closely related. In Section 3.1, we show an exact equivalence between the two problems for the linear function class. In Section 3.2, we show that for some smooth function class, the two problems are asymptotically equivalent.

3.1. Exact Equivalence for the Linear Function Class

In this subsection, we consider the linear function class in any of the following cases:

(i) [Regression] $\ell_\beta(z) = \ell(\beta^\top x - y)$, where $z = (x, y) \in \mathcal{Z} = (\mathbb{R}^d, \|\cdot\|) \times (\mathbb{R}, |\cdot|)$, and

$$\|(x, y) - (x', y')\|_\mathcal{Z} = \|x - x'\| + |y - y'|.$$  

(ii) [Classification] $\ell_\beta(z) = \ell(y \cdot \beta^\top x)$, where $z = (x, y) \in \mathcal{Z} = (\mathbb{R}^d, \|\cdot\|) \times \{-1, 1\}, \mathbb{I}$, where

$$\mathbb{I}(u) = 0 \text{ if } u = 0 \text{ and } \mathbb{I}(u) = \infty \text{ otherwise, and } \|(x, y) - (x', y')\|_\mathcal{Z} = \|x - x'\| + \mathbb{I}(y - y');$$

(iii) [Unsupervised learning] $\ell_\beta(z) = \ell(\beta^\top z)$, where $z \in \mathcal{Z} = (\mathbb{R}^d, \|\cdot\|).$

Here $\ell : \mathbb{R} \rightarrow \mathbb{R}$ is some univariate Lipschitz continuous loss function (cf. Examples 1-3 below). Note that for any Lipschitz function, Rademacher’s theorem (see, e.g., Theorem 2.14 in (Ambrosio et al. 2000)) implies that the set of differentiable points of $\ell$ is dense in $\mathbb{R}$. We have the following equivalence result.
Theorem 1 (Linear predictor). Under the setup described as above, suppose $\ell$ is $L_\ell$-Lipschitz continuous. Let $\mathcal{T}$ be the set of points in $\mathbb{R}$ at which $\ell$ is differentiable. Assume $\lim_{t \in \mathcal{T}, |t| \to \infty} \ell'(t) = L_\ell$. Then for the regression (Case (i) above), it holds that
\[
\sup_{Q \in \mathcal{M}_1^\alpha (\mathbb{P}_n)} \mathbb{E}_Q[\ell_\beta(z)] = \mathbb{E}_{\mathbb{P}_n}[\ell_\beta(z)] + \alpha \cdot L_\ell \cdot \max(\|\beta\|_*, 1),
\]
and for the classification and the unsupervised learning (Cases (ii)(iii) above), it holds that
\[
\sup_{Q \in \mathcal{M}_1^\alpha (\mathbb{P}_n)} \mathbb{E}_Q[\ell_\beta(z)] = \mathbb{E}_{\mathbb{P}_n}[\ell_\beta(z)] + \alpha \cdot L_\ell \cdot \|\beta\|_*,
\]

Remark 1. Theorem 1 generalizes Theorem 3.1(ii) and Remark 3.15 in Shafieezadeh-Abadeh et al. (2017) by relaxing the convexity assumption on $\ell$. In the case of the classification, the metric structure on $\mathcal{Z}$ indicates that there is no uncertainty in the label variable $y$. Such assumption holds for many applications, including many image-related tasks (e.g., ImageNet competition (Russakovsky et al. 2015)) in which the sample images are correctly labeled. On the other hand, if there is uncertainty in the label, the equivalence no longer holds, but the DRSO can be reduced to some convex program (see, e.g., Shafieezadeh-Abadeh et al. (2015, 2017)).

Proof of Theorem 1. Using Lemma 1 we have that
\[
\sup_{Q \in \mathcal{M}_1^\alpha (\mathbb{P}_n)} \mathbb{E}_Q[\ell_\beta(z)] - \mathbb{E}_{\mathbb{P}_n}[\ell_\beta(z)] = \min_{\lambda \geq 0} \left\{ \lambda \alpha + \frac{1}{n} \sum_{i=1}^{n} \sup_{z \in \mathcal{Z}} \left[ \ell_\beta(z) - \ell_\beta(\hat{z}_i^i) - \lambda \cdot \|z - \hat{z}_i^i\|_Z \right] \right\}.
\]
To unify the notation for different cases, we define
\[
\tilde{\beta} := \begin{cases} 
(\beta, -1), & \text{Case (i)}, \\
(\beta, 0), & \text{Case (ii)}, \\
\beta, & \text{Case (iii)}.
\end{cases}
\]
Since $\ell$ is $L_\ell$-Lipschitz, for any $\hat{z}_i^i$ and any $z \in \mathcal{Z}$ (in the case of classification, $z$ and $\hat{z}_i^i$ should have identical label assignments, see Remark 1), it holds that
\[
\ell_\beta(z) - \ell_\beta(\hat{z}_i^i) \leq L_\ell \cdot \|\tilde{\beta}\|_{Z^*} \cdot \|z - \hat{z}_i^i\|_Z,
\]
where $\|\cdot\|_{Z^*}$ represents the dual norm to the norm on $\mathcal{Z}$. Thus, for any $\lambda \geq L_\ell \cdot \|\tilde{\beta}\|_{Z^*}$,
\[
\sup_{z \in \mathcal{Z}} \left[ \ell(\beta^T z) - \ell(\beta^T \hat{z}_i^i) - \lambda \cdot \|z - \hat{z}_i^i\|_Z \right] = 0.
\]
Note that \( \lambda = L_\ell \cdot \|\beta\|_{Z^*} \) is a dual feasible solution, thereby
\[
\sup_{Q \in \mathcal{M}_1^\alpha(P_n)} E_Q[\ell_\beta(z)] - E_{P_n}[\ell_\beta(z)] \leq \alpha \cdot L_\ell \cdot \|\beta\|_{Z^*}.
\]

On the other hand, note that the Lebesgue differentiation theorem (see, e.g., Theorem 1.6.11 in Tao (2011)) implies that for any \( t_0 < t, \ell(t) - \ell(t_0) = \int_{t_0}^t \ell'(s)ds \). Since \( \lim_{t \in T, |t| \to \infty} \ell'(t) = L_\ell \), we obtain that for any \( \hat{z}^t \) and any \( \lambda < L_\ell \cdot \|\beta\|_{Z^*} \)
\[
\lim_{|t| \to \infty} \ell_\beta(\hat{z}^t + t\beta/\|\beta\|_{Z^*}) - \ell_\beta(\hat{z}^t) - \lambda t = \infty.
\]
Therefore we conclude that
\[
\min_{\lambda \geq 0} \left\{ \lambda \alpha + \frac{1}{N} \sum_{i=1}^n \sup_{z \in \mathcal{Z}} \left[ \ell_\beta(z) - \ell_\beta(\hat{z}^t_i) - \lambda \|z - \hat{z}^t_i\|_{Z^*} \right] \right\} = \min_{\lambda \geq L_\ell \cdot \|\beta\|_{Z^*}} \{ \lambda \alpha \} = \alpha \cdot L_\ell \cdot \|\beta\|_{Z^*}.
\]

Finally, note that in Case (i),
\[
\|(\beta, -1)\|_{Z^*} = \sup_{x,y} \left\{ \beta^\top x : x \in \mathcal{Z}, y \in \mathcal{Y}, \|x\| + |y| \leq 1 \right\} = \max(\|\beta\|_*, 1),
\]
which completes the proof. \( \square \)

**Example 1 (Absolute deviation regression).** Let \( \ell(t) = |t| \). Then \( \ell \) is 1-Lipschitz. By Theorem 1, we have the equivalence
\[
\sup_{Q \in \mathcal{M}_1^\alpha(P_n)} E_Q[\ell(\beta^\top z)] = E_{P_n}[\ell(\beta^\top z)] + \alpha \cdot \max(\|\beta\|_*, 1).
\]

**Example 2 (Classification).** Let \( \ell(z) = \ell(y \cdot \beta^\top x) \), where \( \ell \) is any 1-Lipschitz loss function, such as the hinge loss \( \max(1 - y \cdot \beta^\top x, 0) \), or the logistic loss \( \log(1 + \exp(-y \cdot \beta^\top x)) \). Using Theorem 1, we obtain that
\[
\sup_{Q \in \mathcal{M}_1^\alpha(P_n)} E_Q[\ell(\beta^\top z)] = E_{P_n}[\ell(\beta^\top z)] + \alpha \cdot \|\beta\|_*,
\]
which recovers Remark 3.15 in Shafieezadeh-Abadeh et al. (2017).

The result in Theorem 1 can be easily generalized to the following case, whose proof is similar to that of Theorem 1 and thus omitted.

**Corollary 1.** Let \( \mathcal{Z} = (\mathbb{R}^d, \|\cdot\|) \). Suppose there exists a positive integer \( M \) such that \( \ell_\beta(z) = \max_{1 \leq m \leq M} \ell_m(\beta^m \cdot z) \), where \( \beta^m \in \mathbb{R}^d \), \( \beta = [\beta^1; \ldots; \beta^M] \), and \( \ell_m : \mathbb{R} \to \mathbb{R} \) is \( L_m \cdot \) Lipschitz continuous. Let \( \mathcal{T}_m \) be the set of points in \( \mathbb{R} \) at which \( \ell_m \) is differentiable. Assume \( \lim_{t \in \mathcal{T}_m, |t| \to \infty} \ell'_m(t) = L_m \) for each \( 1 \leq m \leq M \). Then it holds that
\[
\sup_{Q \in \mathcal{M}_1^\alpha(P_n)} E_Q[\ell(\beta^\top z)] = E_{P_n}[\ell(\beta^\top z)] + \alpha \cdot \max_{1 \leq m \leq M} L_m \|\beta^m\|_*.
\]
Example 3 (Piecewise-linear convex loss). Let $Z = (\mathbb{R}^d, \|\cdot\|)$. Set $\ell_m$ in Corollary 1 to be the identity function, i.e., $\ell_m(z) = z$ for all $z \in Z$. Then we recover Remark 6.6 in Esfahani and Kuhn (2017):

$$\sup_{Q \in \mathcal{M}^\alpha_{\mathcal{P}_n}} \mathbb{E}_Q[\ell(\beta^T z)] = \mathbb{E}_{\mathcal{P}_n}[\ell(\beta^T z)] + \alpha \cdot \max_{1 \leq m \leq M} \|\beta^m\|_\star.$$

3.2. Asymptotic Equivalence for the Smooth Function Class

In this subsection, we consider the class of smooth functions and present an asymptotic equivalence result between Wasserstein-DRSO and regularization. Assume $\ell_\beta(\cdot)$ is differentiable. With Definition 2 in Section 2, the empirical $p$-norm of the gradient function $\nabla_z \ell_\beta$ is expressed as

$$\left\| \nabla_z \ell_\beta \right\|_{p, \mathcal{P}_n} := \begin{cases} \left( \frac{1}{n} \sum_{i=1}^n \|\nabla_z \ell_\beta(z^i)\|_\star^p \right)^{1/p}, & p \in [1, \infty), \\ \max_{1 \leq i \leq n} \|\nabla_z \ell_\beta(z^i)\|_\star, & p = \infty. \end{cases}$$

We note that the gradient is taken with respect to the data $z$, but not with respect to the learning parameter $\beta$, where the latter is seen much often in the machine learning literature.

**Theorem 2 (Asymptotic equivalence).** Let $Z = (\mathbb{R}^d, \|\cdot\|)$. Suppose either of the following conditions holds:

(i) $\ell_\beta$ is Lipschitz continuous, $p = 1$, and $\mathbb{P}_{\text{true}}$ has a continuous density on $Z$.

(ii) There exists a constant $\kappa \in (0, 1]$ and a function $h : Z \rightarrow \mathbb{R}$ such that

$$\|\nabla_z \ell_\beta(z) - \nabla_z \ell_\beta(z')\|_\star \leq h(z') \cdot \|z - z'\|^\kappa, \quad \forall z, z' \in Z.$$

Let the radius sequence $\{\alpha_n\}_{n=1}^\infty$ be a sequence of positive random variables convergent to zero almost surely. Then it holds almost surely that

$$\sup_{Q \in \mathcal{M}^\alpha_{\mathcal{P}_n}} \mathbb{E}_{z \sim Q}[\ell_\beta(z)] - \mathbb{E}_{z \sim \mathcal{P}_n}[\ell_\beta(z)] = o(\alpha_n),$$

where the “almost surely” is with respect to i.i.d. draws of samples from $\mathcal{P}_{\text{true}}$ and the randomness of $\alpha_n$.

**Remark 2.** Here we allow the radius sequence $\{\alpha_n\}_n$ to be random, since in practice it may be determined adaptively to the data-generating mechanism and converges to zero almost surely (with respect to i.i.d. draws of random data) as more and more data are collected.
Theorem 2 states that the regularization problem with penalty term \( \| \nabla \ell_\beta \|_{\mathbb{P}_n,p} \) is a first-order approximation of (Wasserstein-DRSO), and such approximation is asymptotically exact. The gradient-norm penalty has been heuristically exploited for deep learning problems, such as adversarial training \((p = \infty, \text{see Goodfellow et al. (2014b), Shaham et al. (2015)})\) and training of generative adversarial networks \((p = 2, \text{see Roth et al. (2017)})\).

The proof of Theorem 2 is based on the following two propositions, which provides upper and lower bounds of the worst-case loss \( \sup_{Q \in \mathcal{M}_p^n(P_n)} \mathbb{E}_{(x,y) \sim Q} [\ell_\beta(x,y)] \) in terms of regularization.

**Proposition 1 (Upper bound on the worst-case loss).**

i) Suppose \( \ell_\beta \) is \( L_{\ell_\beta} \)-Lipschitz. Let \( p = 1 \). Then it holds that

\[
\sup_{Q \in \mathcal{M}_p^n(P_n)} \mathbb{E}_Q[\ell_\beta(z)] \leq \mathbb{E}_{P_n}[\ell_\beta(z)] + \alpha \cdot L_{\ell_\beta}.
\]

ii) Suppose \( \ell_\beta \) is differentiable, and there exists constants \( \kappa \in (0,1] \), \( q \in (1,\infty) \), \( C \geq 0 \) and a function \( h : \mathbb{Z} \to \mathbb{R} \) such that

\[
\| \nabla_z \ell_\beta(z) - \nabla_z \ell_\beta(z') \|_* \leq h(z') \cdot \| z - z' \|^{\kappa} + C \cdot \| z - z' \|^{q}, \quad \forall z,z' \in \mathbb{Z}.
\]

Let \( p \in [q+1,\infty) \) if \( C > 0 \), and \( p \in [\kappa+1,\infty) \) if \( C = 0 \). Then it holds that

\[
\sup_{Q \in \mathcal{M}_p^n(P_n)} \mathbb{E}_Q[\ell_\beta(z)] \leq \mathbb{E}_{P_n}[\ell_\beta(z)] + \alpha \cdot \| \nabla_z \ell_\beta \|_{\mathbb{P}_n,p} + \alpha^{q+1} \| h \|_{\mathbb{P}_n,p} + C \cdot \alpha^{q+1}.
\]

**Proof of Proposition 1.** (i) follows from Proposition 6.5(i) in Esfahani and Kuhn (2017). (ii) By the assumption on \( \ell_\beta \) and the mean-value theorem, it holds that

\[
\ell_\beta(z) - \ell_\beta(z') \leq \| \nabla_z \ell_\beta(z') \|_* \cdot \| z - z' \| + h(z') \cdot \| z - z' \|^{\kappa+1} + C \cdot \| z - z' \|^{q+1}, \quad \forall z,z' \in \mathbb{Z}.
\]

When \( p = \infty \), using Lemma 1 yields that

\[
\sup_{Q \in \mathcal{M}_\infty^n(P_n)} \mathbb{E}_Q[\ell_\beta(z)] - \mathbb{E}_{P_n}[\ell_\beta(z)] \leq \frac{1}{n} \sum_{i=1}^n \bigg( \| \nabla_z \ell_\beta(\tilde{z}_i) \|_* \cdot \alpha + h(\tilde{z}_i) \cdot \alpha^{q+1} + C \cdot \alpha^{q+1} \bigg)
\]

\[
\leq \alpha \cdot \| \ell_\beta \|_{\mathbb{P}_n,1} + \alpha^{q+1} \cdot \| h \|_{\mathbb{P}_n,1} + C \alpha^{q+1}.
\]

We next consider \( p \in (1,\infty) \). In view of Lemma 1, we provide an upper bound on

\[
\sup_{z \in \mathbb{Z}} \{ [\ell_\beta(z) - \ell_\beta(\tilde{z}_i)] - \lambda \cdot \| z - \tilde{z}_i \| \}
\]

We have that

\[
\sup_{z \in \mathbb{Z}} \{ [\ell_\beta(z) - \ell_\beta(\tilde{z}_i)] - \lambda \cdot \| z - \tilde{z}_i \| \}
\]

\[
\leq \sup_{z \in \mathbb{Z}} \{ \| \nabla_z \ell_\beta(\tilde{z}_i) \|_* \cdot \| z - \tilde{z}_i \| + h(\tilde{z}_i) \cdot \| z - \tilde{z}_i \|^{q+1} + C \cdot \| z - \tilde{z}_i \|^{q+1} - \lambda \cdot \| z - z_i \| \}
\]

\[
\leq \sup_{t \geq 0} \{ \| \nabla_z \ell_\beta(\tilde{z}_i) \|_* \cdot t + h(\tilde{z}_i) \cdot t^{q+1} + C \cdot t^{q+1} - \lambda \cdot t \}.
\]
Using Lemma 2 in Appendix A, for any $\delta = (\delta_1, \delta_2) > 0$, it holds that
\[
\sup_{t \geq 0} \left\{ \left\| \nabla z \ell_\beta(\hat{z}^t) \right\|_* \cdot t + h(\hat{z}^t) \cdot t^{\kappa+1} + C \cdot t^{p+1} - \lambda \cdot t^p \right\} \leq \sup_{t \geq 0} \left\{ \left\| \nabla z \ell_\beta(\hat{z}^t) \right\|_* + \frac{p - \kappa - 1}{p - 1} \cdot h(\hat{z}^t) \cdot \delta_1 + \frac{p - q - 1}{p - 1} \cdot C \cdot \delta_2 \cdot t \cdot \frac{p - q - 1}{p - 1} - \left( \lambda - \frac{\kappa}{p - 1} \cdot h(\hat{z}^t) \cdot \delta_1 \cdot \frac{p - q - 1}{p - 1} - \frac{q}{p - 1} \cdot C \cdot \delta_2 \cdot \frac{p - q - 1}{p - 1} \right) \cdot t^p \right\}
\]
\[
= \sup_{t \geq 0} \left\{ g_\delta(\hat{z}^t) \cdot t - (\lambda - C_\delta) \cdot t^p \right\}.
\]
Solving this maximization problem over $t$ gives
\[
\sup_{t \geq 0} \left\{ g_\delta(\hat{z}^t) \cdot t - (\lambda - C_\delta) \cdot t^p \right\} = \begin{cases} \frac{p - \kappa - 1}{p - 1} \cdot (g_\delta(\hat{z}^t))^{\frac{p - \kappa - 1}{p - 1}}, & \lambda > C_\delta, \\ +\infty, & \lambda \leq C_\delta. \end{cases}
\]
It then follows Lemma 1 that
\[
\sup_{\mathcal{Q} \in \mathcal{W}_p^\alpha(P_n)} \mathbb{E}_\mathcal{Q}[\ell_\beta(z)] - \mathbb{E}_{\mathcal{F}_n}[\ell_\beta(z)] \leq \inf_{\lambda \geq C_\delta} \left\{ \lambda \alpha^p + \frac{p - \kappa - 1}{p - 1} \cdot (g_\delta(\hat{z}^t))^{\frac{p - \kappa - 1}{p - 1}} \cdot \lambda \right\}. \]
Solving the right-hand side yields
\[
\sup_{\mathcal{Q} \in \mathcal{W}_p^\alpha(P_n)} \mathbb{E}_\mathcal{Q}[\ell_\beta(z)] - \mathbb{E}_{\mathcal{F}_n}[\ell_\beta(z)] \leq \alpha \cdot \|g_\delta\|_{p, p^*} + C_\delta \cdot \alpha^p.
\]
Plugging in the expressions for $g_\delta$ and $C_\delta$ on the right-hand side, we have
\[
\alpha \cdot \|g_\delta\|_{p, p^*} + C_\delta \cdot \alpha^p = \alpha \cdot \left( \left\| \nabla z \ell_\beta \right\|_{p, p^*} + \frac{p - \kappa - 1}{p - 1} \cdot \delta_1 \cdot \|h\|_{p, p^*} + \frac{p - q - 1}{p - 1} \cdot C \cdot \delta_2 \right) + \alpha^p \cdot \left( \|h\|_{p, p^*} \cdot \frac{\kappa}{p - 1} \cdot \delta_1 \cdot \frac{p - q - 1}{p - 1} + C \cdot \frac{q}{p - 1} \cdot \delta_2 \cdot \frac{p - q - 1}{p - 1} \right).
\]
Minimizing over $\delta > 0$ for the right-hand side gives the result.

**Proposition 2 (Lower bound on the worst-case loss).** Let $\mathcal{Z} = \mathbb{R}^d$. Suppose $\ell_\beta$ is differentiable, and there exists a constant $\kappa \in [0, 1]$ and a function $h : \mathcal{Z} \to \mathbb{R}$ such that
\[
\|\nabla z \ell_\beta(z) - \nabla z \ell_\beta(z')\|_* \leq h(z') \cdot \|z - z'\|_* \quad \forall z, z' \in \mathcal{Z}.
\]
Then for $p \in (\kappa + 1, \infty]$, it holds that
\[
\sup_{\mathcal{Q} \in \mathcal{W}_p^\alpha(P_n)} \mathbb{E}_\mathcal{Q}[\ell_\beta(z)] \geq \mathbb{E}_{P_n}[\ell_\beta(z)] + \alpha \cdot \|\nabla z \ell_\beta\|_{p, p^*} - \alpha^{\kappa+1} \cdot \|h\|_{p, \frac{p}{p - \kappa - 1}},
\]
else for $p \in [1, \kappa + 1)$, it holds that
\[
\sup_{\mathcal{Q} \in \mathcal{W}_p^\alpha(P_n)} \mathbb{E}_\mathcal{Q}[\ell_\beta(z)] \geq \mathbb{E}_{P_n}[\ell_\beta(z)] + \alpha \cdot \|\nabla z \ell_\beta\|_{p, p^*} - \alpha^{\kappa+1} \cdot \|h\|_{p, \infty}.
\]
Proof of Proposition 2. The proof uses the following observation: a lower bound on the worst-case loss is given by only considering distributions that are supported on \( n \) points. More specifically, \( \sup_{\beta \in \mathcal{M}_p(\mathbb{P}_n)} \mathbb{E}_Q[\ell_\beta(z)] - \mathbb{E}_{\mathbb{P}_n}[\ell_\beta(z)] \) is lower bounded by the following quantity

\[
\sup_{z_i \in Z, i = 1, \ldots, n} \left\{ \frac{1}{n} \sum_{i=1}^n \left[ \ell_\beta(z_i) - \ell_\beta(\hat{z}^i) \right] : \left( \frac{1}{n} \sum_{i=1}^n \| z^i - \hat{z}^i \|^p \right)^{1/p} \leq \alpha \right\}.
\]

(3)

Indeed, for \( p = \infty \), problem (3) is equivalent to \( \sup_{\beta \in \mathcal{M}_p(\mathbb{P}_n)} \mathbb{E}_Q[\ell_\beta(z)] - \mathbb{E}_{\mathbb{P}_n}[\ell_\beta(z)] \) by Lemma 1, and for \( p \in [1, \infty) \), the feasible set of problem (3) can be viewed as a subset of \( \mathcal{M}_p(\mathbb{P}_n) \), which contains distributions that are supported on at most \( n \) points. Then by the assumption on \( \nabla_z \ell_\beta \) and the mean-value theorem, the quantity \( \sup_{\beta \in \mathcal{M}_p(\mathbb{P}_n)} \mathbb{E}_Q[\ell_\beta(z)] - \mathbb{E}_{\mathbb{P}_n}[\ell_\beta(z)] \) is lower bounded by

\[
\sup_{z_i \in Z, i = 1, \ldots, n} \left\{ \frac{1}{n} \sum_{i=1}^n \left[ \nabla_z \ell_\beta(\hat{z}^i) \cdot \| z^i - \hat{z}^i \| - h(\hat{z}^i) \cdot \| z^i - \hat{z}^i \|^\kappa+1 \right] : \left( \frac{1}{n} \sum_{i=1}^n \| z^i - \hat{z}^i \|^p \right)^{1/p} \leq \alpha \right\},
\]

which is further lower bounded by

\[
\sup_{z_i \in Z, i = 1, \ldots, n} \left\{ \frac{1}{n} \sum_{i=1}^n \nabla_z \ell_\beta(\hat{z}^i) \cdot \| z^i - \hat{z}^i \| : \left( \frac{1}{n} \sum_{i=1}^n \| z^i - \hat{z}^i \|^p \right)^{1/p} \leq \alpha \right\}
- \sup_{z_i \in Z, i = 1, \ldots, n} \left\{ \frac{1}{n} \sum_{i=1}^n h(\hat{z}^i) \cdot \| z^i - \hat{z}^i \|^\kappa+1 : \left( \frac{1}{n} \sum_{i=1}^n \| z^i - \hat{z}^i \|^p \right)^{1/p} \leq \alpha \right\}.
\]

Note that \( Z = \mathbb{R}^d \), then it follows from Hölder’s inequality that

\[
\sup_{z_i \in Z, i = 1, \ldots, n} \left\{ \frac{1}{n} \sum_{i=1}^n \nabla_z \ell_\beta(\hat{z}^i) \cdot \| z^i - \hat{z}^i \| : \left( \frac{1}{n} \sum_{i=1}^n \| z^i - \hat{z}^i \|^p \right)^{1/p} \leq \alpha \right\}
= \sup_{t_i \in \mathbb{R}, i = 1, \ldots, n} \left\{ \frac{1}{n} \sum_{i=1}^n \nabla_z \ell_{\beta}(\hat{z}^i) \cdot t_i : \left( \frac{1}{n} \sum_{i=1}^n \| t_i \|^p \right)^{1/p} \leq \alpha \right\}
= \alpha \cdot \| \nabla_z \ell_\beta \|_{\mathbb{P}_n, p, \kappa},
\]

and that

\[
\sup_{z_i \in Z, i = 1, \ldots, n} \left\{ \frac{1}{n} \sum_{i=1}^n h(\hat{z}^i) \cdot \| z^i - \hat{z}^i \|^\kappa+1 : \left( \frac{1}{n} \sum_{i=1}^n \| z^i - \hat{z}^i \|^p \right)^{1/p} \leq \alpha \right\}
= \sup_{t_i \in \mathbb{R}, i = 1, \ldots, n} \left\{ \frac{1}{n} \sum_{i=1}^n h(\hat{z}^i) \cdot t_i^{\kappa+1} : \left( \frac{1}{n} \sum_{i=1}^n \| t_i \|^p \right)^{1/p} \leq \alpha \right\}.
\]

Therefore, the results follow by observing that when \( p > \kappa + 1 \),

\[
\sup_{t_i \in \mathbb{R}, i = 1, \ldots, n} \left\{ \frac{1}{n} \sum_{i=1}^n h(\hat{z}^i) \cdot t_i^{\kappa+1} : \left( \frac{1}{n} \sum_{i=1}^n \| t_i \|^p \right)^{1/p} \leq \alpha \right\} = \alpha^{\kappa+1} \cdot \| h \|_{\mathbb{P}_n, p, \kappa+1},
\]
and when $p \leq \kappa + 1$,
\[
\sup_{t_i \in \mathbb{R}, i = 1, \ldots, n} \left\{ \frac{1}{n} \sum_{i=1}^{n} h(\hat{z}^i) \cdot t_i^{\kappa+1} : \left( \frac{1}{n} \sum_{i=1}^{n} t_i^p \right)^{1/p} \leq \alpha \right\} \leq \alpha^{\kappa+1} \cdot \|h\|_{\mathbb{P}_n, \infty}.
\]

With Propositions 1 and 2, we are ready to prove Theorem 2.

**Proof of Theorem 2.** When $p > 1$, Propositions 1 and 2 imply that
\[
\left| \sup_{Q \in \mathcal{M}_p} \mathbb{E}_{z \sim Q}[\ell_\beta(z)] - \mathbb{E}_{z \sim \mathbb{P}_n}[\ell_\beta(z)] - \alpha_n \cdot \|\nabla_z \ell_\beta\|_{\mathbb{P}_n,p} \right| \leq \alpha_n^{\kappa+1} \cdot \|h\|_{\mathbb{P}_n, p}.
\]

Then the integrability assumption on $h$ and the Law of Large Numbers ensure that the remainder on the right-hand side is $o(\alpha_n)$ almost surely.

When $p = 1$ and $\ell_\beta$ is Lipschitz continuous, we denote by $L_{\ell_\beta}$ the smallest Lipschitz constant of $\ell_\beta$. In view of Propositions 1 and 2, it suffices to prove
\[
\mathbb{P}\left\{ \lim_{n \to \infty} \|\nabla_z \ell_\beta\|_{\mathbb{P}_n, \infty} = L_{\ell_\beta} \right\} = 1,
\]
where $\mathbb{P}$ denotes that the probability is taken with respect to i.i.d. draws of samples from $\mathbb{P}_{\text{true}}$. To this end, observe from (2) that, for any $\epsilon > 0$,
\[
\delta := \mathbb{P}_{\text{true}}\{ z \in \mathcal{Z} : \|\nabla_z \ell_\beta(z)\|_\alpha > L_{\ell_\beta} - \epsilon \} > 0.
\]

It then follows that
\[
\mathbb{P}\{ \|\nabla_z \ell_\beta\|_{\mathbb{P}_n, \infty} < L_{\ell_\beta} - \epsilon \} < (1 - \delta)^n,
\]
Thus,
\[
\sum_{n=1}^{\infty} \mathbb{P}\{ \|\nabla_z \ell_\beta\|_{\mathbb{P}_n, \infty} < L_{\ell_\beta} - \epsilon \} < \sum_{n=1}^{\infty} (1 - \delta)^n < \infty.
\]
Then by Borel-Cantelli lemma (see, e.g. Feller (2008)), $\|\nabla_z \ell_\beta\|_{\mathbb{P}_n, \infty}$ converges to $L_{\ell_\beta}$ almost surely, which completes the proof. □

**4. Application of the Equivalence in Discrete Choice Modeling**

In this subsection, we consider linear optimization $\ell_\beta(z) = \beta^T z$, and apply the equivalence result to discrete choice modeling. This is a special case of the linear function class with Lipschitz loss considered in Section 3.1, but here we allow a more general metric structure on $\mathcal{Z}$ other than the norm $\|\cdot\|$, and allow arbitrary nominal distribution $\mathbb{P}_{\text{nom}} \in \mathcal{P}(\mathcal{Z})$ rather than the empirical distribution $\mathbb{P}_n$. 


For ease of exposition, we consider a soft-constrained version of \((\text{Wasserstein-DRSO})\):

\[
\max_{\beta \in D} \inf_{Q \in P(Z)} \left\{ \mathbb{E}_Q[\beta^\top z] + \eta \cdot \mathcal{W}_1(Q, P_{\text{nom}}) \right\},
\]

where \(\eta > 0\). Problem (4) can be interpreted as follows. Suppose \(z\) represents a vector of random utilities of \(d\) products. Let \(D = \{\beta \in \mathbb{R}^d : \sum_{k=1}^d \beta_k = 1\}\) be the set of choice probability vectors for these products. Consider a consumer who wants to maximize her total expected utility, but is ambiguous about the true distribution of the random utilities. We can model such ambiguity through a Wasserstein ball of distributions centered at some reference distribution \(P_{\text{nom}}\). Thus, problem (4) is interpreted as an ambiguity-averse consumer choosing a choice probability vector \(\beta \in D\) that maximizes the worst-case total expected utility.

We make the following assumptions on the data space \((Z, d)\) in the Definition 1 of Wasserstein distance. Let \(Z\) be a linear subspace of \(\mathbb{R}^d\). Assume that the distance function \(d\) is translation-invariant, i.e., it can be expressed as \(d(\cdot, \cdot) = D(\cdot - \cdot)\) for some function \(D : Z \to \mathbb{R}\). We also assume that \(D(u)\) is strictly convex and even in the sense that \(D(u) = D(|u|)\) for all \(u \in Z\), where \(|u|\) represents component-wise absolute value function. Furthermore, we assume that \(D(u)\) is non-decreasing on positive orthant, i.e., \(D(u_1, \ldots, u_K)\) is non-decreasing in each component on \(\{u \in Z : u \geq 0\}\).

All the above assumptions seems to be natural if we view \(D\) as a function that describes consumer’s attitude towards ambiguity. The faster \(D\) grows, the larger penalty is imposed on the deviation from the nominal distribution, and the larger penalty, the less likely of perturbations of utilities from the nominal distribution, and thus the more certain about the nominal distribution the consumer is. An extreme case is when \(D(0) = 0\) and \(D(u) = +\infty\) for all \(u \neq 0\), where \(u\) represents the deviation, then the consumer is completely certain about the nominal distribution. Indeed, in this case the only distribution that makes the inner minimization problem of (4) finite is the nominal distribution.

**Theorem 3 (Choice model).** Set \(\bar{z} = \mathbb{E}_{P_{\text{nom}}}[z]\). Then under the above setup, problem (4) is equivalent to

\[
\max_{\beta \in D} \left\{ \beta^\top \bar{z} - \eta \cdot D^\star \left( \frac{\beta}{\eta} \right) \right\},
\]

where \(D^\star\) is the convex conjugate of \(D\). The optimal solution \(\beta^0\) satisfies

\[
\beta^0 = \eta \cdot \nabla D(\bar{z} + \alpha^0 1),
\]

for some \(\alpha^0 \in \mathbb{R}\) and an all-one vector \(1 := (1, 1, \ldots, 1)^\top\).
Remark 3. The regularization problem (5) is equivalent to the formulation of representative agent choice model (Anderson et al. 1988, Hofbauer and Sandholm 2002), which states that the choice probability vector is given by the solution of the regularization problem (5). Thus, the equivalence result provides a new interpretation of the representative agent choice model from the perspective of distributional robustness, i.e., the choice probability equals the optimal solution to the distributionally robust utility maximization problem (4).

Theorem 3 states that the vector of the choice probabilities is proportional to the gradient of the distance function $D$ at $\bar{z} + \alpha^0 \mathbf{1}$. The following examples show that with proper choices of $D$, the vector of the choice probabilities is also proportional to the gradient of the distance function $D$ at $\bar{z}$. As a multivariate function, the distance function $D$ summarizes the consumer’s cross-ambiguity attitude towards multiple products.

Example 4 (Multinomial logit). Let $D_{MNL}(u) = \sum_{k=1}^{d} \exp(u_k)$. We have
\[
\nabla_k D_{MNL}(\bar{z} + \alpha^0 \mathbf{1}) = \exp(\bar{z}_k + \alpha^0) = \exp(\bar{z}_k) \cdot \exp(\alpha^0).
\]
Thus $\beta^0_k$ is proportional to $\exp(\bar{z}_k)$. Using the normalization condition $\beta \in \Delta$, we recover the multinomial logit choice model
\[
\beta^0_k = \frac{\exp(\bar{z}_k)}{\sum_{j=1}^{K} \exp(\bar{z}_j)}.
\]
For the multinomial logit model, $D_{MNL}$ is separable and is additive in each deviation $u_k$, which suggests that the consumer is cross-ambiguity neutral for all products, i.e., the consumer’s judgment on the likely of perturbations of utilities are independent across products.

Example 5 (Nested logit). Let $G$ be a partition of $\{1, \ldots, d\}$. Each element of the partition is called a nest. Denote by $u_g$ the sub-vector of $u$ whose indices belong to $g \in G$, and denote by $g(k)$ the nest that $k$ belongs to. Let
\[
D^g_{NL}(u) = \sum_{g \in \hat{G}} \|\exp(u_g)\|_{1/\tau_g} = \sum_{g \in \hat{G}} \left( \sum_{k \in g} \exp(|u_k|/\tau_g) \right)^{\tau_g},
\]
where $\tau_g > 0$ are parameters. Then
\[
\beta^0_k \propto \nabla_k D^g_{NL}(\bar{z} + \alpha^0 \mathbf{1})
\]
\[
= \left( \sum_{k' : g(k') = g(k)} \exp((\bar{z}_{k'} + \alpha^0)/\tau_{g(k')}) \right)^{\tau_{g(k)}} \cdot \frac{\exp((\bar{z}_k + \alpha^0)/\tau_{g(k)})}{\sum_{k' : g(k') = g(k)} \exp((\bar{z}_{k'} + \alpha^0)/\tau_{g(k')})}
\]
\[
= \exp(\bar{z}_k/\tau_{g(k)}) \left( \sum_{k' : g(k') = g(k)} \exp(\bar{z}_{k'}/\tau_{g(k')}) \right)^{\tau_{g(k)}-1},
\]
where $\propto$ represents “proportional to”. With proper scaling, we recover the nested logit model (see, e.g., (4.2) in (Train 2009)). Unlike the multinomial logit model, the distance function $D_{NL}$ for the nested logit model is not additively separable in products, but only additively separable in nests. This indicates that the consumer’s judgment on the likely of perturbations of utilities are interrelated across the products within the same nest, and are independent across different nests.

**Example 6 (GEV).** Let $D_{GEV}(u) = D_0(\exp(u_1), \ldots, \exp(u_d))$ for some strictly convex differentiable function $D_0 : \mathbb{R}_+^d \to \mathbb{R}$. Assume that $D_0$ is homogeneous, i.e., $D_0(tY_1, \ldots, tY_d) = tD_0(Y_1, \ldots, Y_d)$ for all $t > 0$ and some $s > 0$. It follows that

$$\beta_k^0 \propto \nabla_k D_{GEV}(\bar{z} + \alpha^0 \mathbf{1})$$

$$= \exp(\alpha^0) \cdot \exp(\bar{z}_k) \cdot \nabla_k D_0((\exp(\alpha^0) \cdot \exp(\bar{z}_1), \ldots, \exp(\alpha^0) \cdot \exp(\bar{z}_d)))$$

$$\propto \exp(\bar{z}_k) \cdot \nabla_k D_0(\exp(\bar{z}_1), \ldots, \exp(\bar{z}_d)).$$

This exactly corresponds to the expression of Generalized Extreme Value (GEV) choice models proposed by McFadden (1978). We note that as pointed out in the original GEV framework, the function $D_0$ has little economic intuition. Our equivalence result Theorem 3 endows the function $D_0$ with an economic interpretation – it reflects the consumer’s ambiguity attitude.

**Proof of Theorem 3.** Using Definition 1 of Wasserstein distance, we have that

$$\inf_{Q \in P(Z)} \left\{ E_Q[\beta^T z] + \eta \cdot W_1(Q, P_{nom}) \right\} = \inf_{\gamma \in \Gamma(P, Q)} E_{\gamma}[\beta^T z + \eta \cdot D(z - z')] = \inf_{\gamma \in \Gamma(P, Q)} \{ \int_Z \int_Z [\beta^T z + \eta \cdot D(z - z')] \gamma_{z'}(dz) P_{nom}(dz') \} = \inf_{\gamma \in \Gamma(P, Q)} \{ \int_Z \int_Z [\beta^T z + \eta \cdot D(z - z')] \gamma_{z'}(dz) P_{nom}(dz') \}$$

For a random vector $(z, z')$ with joint distribution $\gamma$, we denote by $\gamma_{z'}$ the condition distribution of $z$ given $z'$. It then follows from the disintegration theorem (see, e.g., Theorem 5.3.1 in Ambrosio et al. (2008)) that

$$\inf_{\gamma \in \Gamma(P, Q)} \{ \int_Z \int_Z [\beta^T z + \eta \cdot D(z - z')] \gamma_{z'}(dz) P_{nom}(dz') \} = \inf_{\gamma \in \Gamma(P, Q)} \{ \int_Z \int_Z [\beta^T z + \eta \cdot D(z - z')] \gamma_{z'}(dz) P_{nom}(dz') \}$$

Using interchangeability principle (see, e.g, Theorem 7.80 in Shapiro et al. (2014)), we exchange the infimum and integration:

$$\inf_{\gamma \in \Gamma(P, Q)} \{ \int_Z \int_Z [\beta^T z + \eta \cdot D(z - z')] \gamma_{z'}(dz) P_{nom}(dz') \} = \int_Z \left\{ \inf_{\gamma \in \Gamma(P, Q)} \{ \int_Z [\beta^T z + \eta \cdot D(z - z')] \gamma_{z'}(dz) \} P_{nom}(dz') \right\}$$

$$= \int_Z \left\{ \inf_{\gamma \in \Gamma(P, Q)} \{ \int_Z [\beta^T z + \eta \cdot D(z - z')] \gamma_{z'}(dz) \} P_{nom}(dz') \right\}.$$
Observe that \( \inf_{\gamma \in \mathcal{P}(\mathcal{Z})} \left\{ \int_{\mathcal{Z}} [\beta^T z + \eta \cdot D(z - z')] \gamma_{z'}(dz) \right\} \) is attained on some Dirac measure, thereby

\[
\inf_{\gamma z' \in \mathcal{P}(\mathcal{Z})} \left\{ \int_{\mathcal{Z}} [\beta^T z + \eta \cdot D(z - z')] \gamma_{z'}(dz) \right\} = \inf_{z \in \mathcal{Z}} [\beta^T z + \eta \cdot D(z - z')]
= \beta^T z' + \inf_{z \in \mathcal{Z}} [\beta^T (z - z') + \eta \cdot D(z - z')].
\]

Combining the equations above yields

\[
\inf_{Q \in \mathcal{P}(\mathcal{Z})} \left\{ \mathbb{E}_Q[\beta^T z] + \eta \cdot W_1(Q, \mathbb{P}_{\text{nom}}) \right\} = \mathbb{E}_{\text{nom}}[\beta^T z'] + \int_{\mathcal{Z}} \inf_{z \in \mathcal{Z}} [\beta^T (z - z') - \eta \cdot D(z - z')] \mathbb{P}_{\text{nom}}(dz').
\]

By the assumption that \( \mathcal{Z} \) is a linear space, we have that for any \( z' \in \mathcal{Z} \),

\[
\inf_{z \in \mathcal{Z}} [\beta^T (z - z') - \eta \cdot D(z - z')] = \inf_{u \in \mathcal{Z}} [\beta^T u - \eta \cdot D(u)].
\]

Thus it follows that

\[
\inf_{Q \in \mathcal{P}(\mathcal{Z})} \left\{ \mathbb{E}_Q[\beta^T z] + \eta \cdot W_1(Q, \mathbb{P}_{\text{nom}}) \right\} = \beta^T z' - \eta \cdot \sup_{u \in \mathcal{Z}} [u^T (\beta/\eta) - D(-u)].
\]

Observe that the strict convexity of \( D \) implies that the optimal solution of \( \sup_{u \in \mathcal{Z}} [u^T (\beta/\eta) - D(-u)] \) is unique. Moreover, the non-negativity of \( \beta \) and \( D(u) = D(|u|) \) for all \( u \) imply that the optimal solution is non-negative, and

\[
\sup_{u \in \mathcal{Z}} [u^T (\beta/\eta) - D(-u)] = \sup_{u \in \mathcal{Z}} [u^T (\beta/\eta) - D(|u|)] = \sup_{u \in \mathcal{Z}} [u^T (\beta/\eta) - D(u)].
\]

Hence problem (4) is equivalent to

\[
\max_{\beta \in \mathcal{D}} \left\{ \beta^T \bar{z} - \eta \cdot \sup_{u \in \mathcal{Z}} [u^T (\beta/\eta) - D(u)] \right\} = \max_{\beta \in \mathcal{D}} \{ \beta^T \bar{z} - \eta \cdot D^*(\beta/\eta) \}. \tag{6}
\]

We next drive the optimality condition of (6). Consider the following relaxation of (6):

\[
\max_{\beta \in \mathbb{R}^d} \left\{ \beta^T \bar{z} - \eta \cdot D^*(\beta/\eta) : \sum_{k=1}^d \beta_k = 1 \right\}.
\]

Let \( \beta^0 \) be its optimal solution, which is unique since \( D^* \) is also strictly convex. Its first-order optimality condition yields that the optimal solution \( \beta^0 \) should satisfy

\[
\bar{z} + \alpha^0 1 = \eta \cdot \nabla D^*(\beta^0/\eta)/\eta = \nabla D^*(\beta^0/\eta),
\]
where $\nabla D^*$ represents the gradient of $D^*$, $\alpha^0$ is the Lagrangian multiplier of the constraint $\beta_k = 1$, and $1 = (1, \ldots, 1)$. Since $\nabla D^* = (\nabla D)^{-1}$ (see, e.g., Exercise 3.40 in Boyd and Vandenberghe (2004)), it follows that

$$\beta^0 = \eta \cdot \nabla D(\bar{z} + \alpha^0 1).$$

Observe that $\nabla D$ is non-negative by our assumption, therefore $\beta^0$ is also an optimal solution of (6), which completes the proof. \hfill \Box

5. A Principled Way to Regularize Learning Problems

In this section, we discuss the algorithmic implications of the equivalence between distributionally robust framework (\textbf{Wasserstein-DRSO}) and regularization. We have the following two observations.

1) The worst-case loss, which is obtained by solving a minimax problem (\textbf{Wasserstein-DRSO}), admits an asymptotically exact approximation that can be obtained by solving a single minimization problem

$$\min_{\beta \in D} \mathbb{E}_{P_n}(\ell(z)) + \alpha_n \cdot \|\nabla z \ell(z)\|_{\mathbb{P}_n, p^*},$$

which is often much easier to solve.

2) The distributionally robust formulation (\textbf{Wasserstein-DRSO}) suggests a principled way to regularize statistical learning problem — by solving the gradient-norm regularization problem (7).

To illustrate the second point, we apply the regularization scheme (1) to an important problem in deep learning – the training of Wasserstein Generative Adversarial Networks (WGANs).

We start with a brief introduction to Generative Adversarial Networks (GANs) (Goodfellow et al. 2014a, Goodfellow 2016). GANs are a powerful class of \textit{generative models}, which aim to answer the following central question in unsupervised learning:

\begin{quote}
How to learn a probability distribution from data?
\end{quote}

The primary motivations to study this question includes: (i) learning conditional distributions that are used in reinforcement learning and semi-supervised learning, (ii) generating realistic samples for real-world tasks such as single image super-resolution (Ledig et al. 2016), art creation (Zhu et al. 2016), and image-to-image translation (Isola et al. 2016),
and (iii) testing our ability to represent and manipulate multi-modal high-dimensional probability distributions.

To answer the question above, the classical approach is to perform density estimation. This is often done by considering a parametric family of distributions and find one that maximizes the likelihood function on the data. However, this approach does not work well for high-dimensional data-generating distributions in many applications (Arjovsky and Bottou 2017), such as natural images, symbols in natural language corpora, and audio waveform containing speech. Instead of estimating the explicit density function, an implicit approach works as follows. Let $Z$ be a random variable with a simple and fixed distribution $P_0$ (such as a Gaussian distribution). Passing the random variable through a parametric function $g_\theta$ (called a generator and usually described by a neural network), we denote the distribution of the random variable $g_\theta(Z)$ by $P_\theta$, which is called the model distribution. By varying $\theta$ in the parameter space $\Theta$, we can find one model distribution that is “closest” to the empirical distribution. GANs are well known examples of this approach. Among lots of variants of GANs that exploit different notion of closeness between distributions, Wasserstein GAN (WGAN) (Arjovsky et al. 2017) has recently attracts great attention in deep learning, as its training requires few parameter tuning, which is ideal for many deep learning problems. In WGAN, the closeness between the model distribution $P_\theta$ and the empirical distribution $P_n$ is measured by the 1-Wasserstein distance:

$$\min_{\theta \in \Theta} \mathcal{W}_1(P_\theta, P_n).$$

Estimation of the Wasserstein distance between high-dimensional distributions is hard. In fact, the sample complexity exponentially depends on the dimension (Sriperumbudur et al. 2010). In WGAN, the following method is used to estimate $\mathcal{W}_1(P_\theta, P_n)$. By Kantorovich-Rubinstein duality (Villani 2003), the 1-Wasserstein distance can be written as

$$\sup_{f} \mathbb{E}_{x \sim P_n}[f(x)] - \mathbb{E}_{z \sim P_\theta}[f(z)],$$

where the inner supremum is taken over the class of 1-Lipschitz functions (or $L$-Lipschitz functions with any $L > 0$):

$$|f(x) - f(z)| \leq \|x - z\|, \quad \forall x \in \text{supp} P_n, z \in \text{supp} P_\theta.$$
To compute (9), the set of Lipschitz functions is often parametrized through a critic neural network \( \{ f_w \}_{w \in W} \), as the gradient computation for a neural network is efficient. The conceptual diagram of WGAN is shown in Figure 1. Samples from the standard Gaussian distribution are fed into the generator network \( g_\theta \), whose outputs (fake images with distribution \( P_\theta \)) are compared with the true samples (real images with empirical distribution \( \mathbb{P}_n \)). The comparison is done by approximately computing \( W_1(\mathbb{P}_\theta, \mathbb{P}_n) \) using another neural network \( f_w \).

![Conceptual model of WGAN](image)

Figure 1  Conceptual model of WGAN

To enforce the Lipschitz condition (10) on the neural network \( f_w \), Arjovsky et al. (2017) proposes to clip the weight vector \( w \) within a rectangle

\[
\{ w : -c \leq w \leq c \},
\]

and formulating a minimax problem

\[
\min_{\theta \in \Theta} \max_{-c \leq w \leq c} \mathbb{E}_{x \sim \mathbb{P}_n} [f_w(x)] - \mathbb{E}_{z \sim \mathbb{P}_\theta} [f_w(z)].
\]

However, the weight clipping does not describe the set of 1-Lipschitz functions, but only a subset of \( L \)-Lipschitz functions, where \( L \) depends on \( c \). Yet another natural way to enforce the Lipschitz condition is considering a soft-constraint penalty term

\[
\min_{\theta \in \Theta} \max_{w \in W} \mathbb{E}_{x \sim \mathbb{P}_n} [f_w(x)] - \mathbb{E}_{z \sim \mathbb{P}_\theta} [f_w(z)] - \lambda \cdot \mathbb{E}_{x \sim \mathbb{P}_n, z \sim \mathbb{P}_\theta} \left( \frac{|f(x) - f(z)|}{\|x - z\|} - 1 \right)_{+}^2.
\]

We here propose a new training method using ideas from Section 3.2. We consider a distributional robust formulation of problem (8) for computing \( W_1(\mathbb{P}_n, \mathbb{P}_\theta) \) by regularizing the inner maximization of problem (11):

\[
\max_{w \in W} \min_{Q \in \mathcal{M}_2(\mathbb{P}_n)} \mathbb{E}_{x \sim Q} [f_w(x)] - \mathbb{E}_{z \sim \mathbb{P}_\theta} [f_w(z)].
\]
Using Theorem 2, this can be approximated by the regularization problem
\[
\max_{w \in W} \mathbb{E}_{x \sim P_n}[f_w(x)] - \mathbb{E}_{z \sim P_0}[f_w(z)] - \alpha \cdot \|\nabla_x f_w(x)\|_{P_n,2}.
\]

Then we add a soft-constraint penalty term and as a result, we formulate a distributionally robust WGAN (DR-WGAN) objective:
\[
\min_{\theta \in \Theta} \max_{w \in W} \mathbb{E}_{x \sim P_n}[f_w(x)] - \mathbb{E}_{z \sim P_0}[f_w(z)] - \lambda \cdot \mathbb{E}_{x \sim P_n, z \sim P_0} \left[ \left( \frac{|f(x) - f(z)|}{\|x - z\|} - 1 \right)_+ \right] - \alpha \cdot \|\nabla f_w(x)\|_{P_n,2}.
\]

We compare our proposed approach with two above-mentioned benchmarks: the weight clipping approach (Arjovsky et al. 2017) and the soft-constraint without regularization (12). The neural networks architecture is similar to the setup in Arjovsky et al. (2017), in which both the generator and critic are 4-layer ReLU-MLP with 512 hidden units. Each approach is trained using stochastic gradient descent, in which the learning rate is adjusted using Adam algorithm (Kingma and Ba 2015) with default choice of parameters. The detailed training algorithm for DR-WGAN is presented in Algorithm 1, which is modification of the algorithm in Arjovsky et al. (2017).

**Algorithm 1** The proposed DR-WGAN.

1: \textbf{while} \( \theta \) has not converged \textbf{do}
2: \hspace{1em} \textbf{for} \( t = 0, \ldots, n_{\text{critic}} \) \textbf{do}
3: \hspace{2em} \textbf{for} \( i = 1, \ldots, m \) \textbf{do}
4: \hspace{3em} Sample \( x \sim P_n, \ z \sim P_0 \)
5: \hspace{3em} \( L^{(i)} \leftarrow f_w(x) - f_w(g_\theta(z)) + \lambda \cdot \left( \frac{|f(x) - f(z)|}{\|x - z\|} - 1 \right)_+ + \alpha \cdot \|\nabla f_w(x)\|_2 \)
6: \hspace{2em} \textbf{end for}
7: \hspace{1em} \( w \leftarrow \text{Adam}(\nabla_w (\frac{1}{m} \sum_{i=1}^m L^{(i)}), w) \)
8: \hspace{1em} \textbf{end for}
9: \hspace{1em} Sample \( \{z^{(i)}\}_{i=1}^m \sim P_0 \) a batch of prior samples.
10: \hspace{1em} \( \theta \leftarrow \text{Adam}(\nabla_\theta (\frac{1}{m} \sum_{i=1}^m f_w(g_\theta(z^{(i)}))), \theta) \)
11: \textbf{end while}

We test the performance on two standard datasets: CIFAR-10 data set (Krizhevsky and Hinton 2009) and the CAT dataset (Zhang et al. 2008). The CIFAR-10 dataset includes 60,000 32 \times 32 color images in 10 classes, with 6000 images per class. The performance
is measured by the inception score (Salimans et al. 2016) which mimics the human eye’s judgment on the similarity between the generated images and the real images, and the higher inception score, the better the performance is. Figure 3 plots the inception scores over the course of training with WGAN and our proposed DR-WGAN. We observe that our method converges much faster and achieves a higher inspection score. We do not plot the training curve for the soft-constraint approach (12), as it does not even converge and the inception score remains at a relatively low value.

The CAT dataset consists of 10,000 cat images, which are preprocessed such that cat faces are aligned, and scaled to $64 \times 64$ (Jolicoeur-Martineau 2017). Figure 3a plots the real images sampled from the dataset. For each of the three approaches, we generate images from the learned parametric distribution with different random seeds, i.e., we input the generator network $g_\theta$ with i.i.d Gaussian samples, and the generated images are shown in Figure 3b-3d. We observe that the image generated by WGAN exhibits mode collapse, i.e., a lack of variety, and comparing to the other two benchmarks, DR-WGAN generates images that are much more close to reality.

6. Concluding Remarks

In this paper, we propose the Wasserstein distributionally robust formulation for solving statistical learning problems with guaranteed small generalization error. We show that it is asymptotic equivalent to a specific gradient-norm regularization problem. Such connection provides new interpretations for regularization, and offers new algorithmic insights. For the
future work, it is interesting to provide generalization error bounds for statistical learning problems based on distributional robustness, and apply the regularization scheme to other machine learning problems.

Appendix A: Auxiliary Results

Lemma 2. Let $\kappa > 0$ and $p \geq \kappa + 1$. Then for any $\delta, t > 0$, it holds that

$$t^{n+1} \leq \frac{p-1-\kappa}{p-1} \cdot \delta \cdot t + \frac{\kappa}{p-1} \cdot \delta \cdot t^{n-1} \cdot t^n.$$
Proof of Lemma 2. When \( p = \kappa + 1 \), the inequality holds as equality. When \( p > \kappa + 1 \), set \( u = \frac{p - 1}{p - 1 - \kappa} \), \( v = \frac{1}{\kappa} - \frac{1}{v} = 1 \). Then the result is a consequence of the following Young’s inequality
\[
\frac{(\delta^{1/u} t^{1/u})^u}{u} + \frac{(\delta^{-1/u} p^{p/v})^v}{v} \geq t^{u+1}.
\]

References
Agrawal S, Ding Y, Saberi A, Ye Y (2012) Price of correlations in stochastic optimization. Operations Research 60(1):150–162.
Ahipasaoglu SD, Li X, Natarajan K (2013) A convex optimization approach for computing correlated choice probabilities with many alternatives. Available on Optimization Online 4034.
Ambrosio L, Fusco N, Pallara D (2000) Functions of bounded variation and free discontinuity problems (Oxford university press).
Ambrosio L, Gigli N, Savaré G (2008) Gradient flows: in metric spaces and in the space of probability measures (Springer Science & Business Media).
Anderson SP, De Palma A, Thisse JF (1988) A representative consumer theory of the logit model. International Economic Review 461–466.
Arjovsky M, Bottou L (2017) Towards principled methods for training generative adversarial networks. arXiv preprint arXiv:1701.04862.
Arjovsky M, Chintala S, Bottou L (2017) Wasserstein gan. arXiv preprint arXiv:1701.07875.
Bayraksan G, Love DK (2015) Data-driven stochastic programming using phi-divergences. The Operations Research Revolution, 1–19 (INFORMS).
Ben-Tal A, Den Hertog D, De Waegenaere A, Melenberg B, Rennen G (2013) Robust solutions of optimization problems affected by uncertain probabilities. Management Science 59(2):341–357.
Ben-Tal A, El Ghaoui L, Nemirovski A (2009) Robust optimization (Princeton University Press).
Berger JO (1984) The robust bayesian viewpoint. Studies in Bayesian Econometrics and Statistics: In Honor of Leonard J. Savage., Edited by Joseph B. Kadane 4(2):63–124.
Bertsimas D, Copenhagen MS (2017) Characterization of the equivalence of robustification and regularization in linear and matrix regression. European Journal of Operational Research.
Bertsimas D, Gupta V, Kallus N (2013) Data-driven robust optimization. Mathematical Programming 1–58.
Bertsimas D, Sim M (2004) The price of robustness. Operations research 52(1):35–53.
Bishop CM (1995) Training with noise is equivalent to tikhonov regularization. Neural computation 7(1):108–116.
Blanchet J, Kang Y, Murthy K (2016) Robust wasserstein profile inference and applications to machine learning. arXiv preprint arXiv:1610.05627.
Blanchet J, Murthy KR (2016) Quantifying distributional model risk via optimal transport. arXiv preprint arXiv:1604.01446.
Boyd S, Vandenberghe L (2004) Convex optimization (Cambridge university press).
Calafiore GC, El Ghaoui L (2006) On distributionally robust chance-constrained linear programs. Journal of Optimization Theory and Applications 130(1):1–22.
Delage E, Ye Y (2010) Distributionally robust optimization under moment uncertainty with application to data-driven problems. Operations research 58(3):595–612.
Erdogdu E, Iyengar G (2006) Ambiguous chance constrained problems and robust optimization. Mathematical Programming 107(1-2):37–61.
Esfahani PM, Kuhn D (2017) Data-driven distributionally robust optimization using the wasserstein metric: performance guarantees and tractable reformulations. Mathematical Programming ISSN 1436-4646.
Feller W (2008) An introduction to probability theory and its applications, volume 1 (John Wiley & Sons).
Feng G, Li X, Wang Z (2017) On the relation between several discrete choice models. Operations Research.
Gao R, Kleywegt AJ (2016) Distributionally robust stochastic optimization with wasserstein distance. arXiv preprint arXiv:1604.02199.
Gao R, Kleywegt AJ (2017) Distributionally robust stochastic optimization with dependence structure. arXiv preprint arXiv:1701.04200.
Goh J, Sim M (2010) Distributionally robust optimization and its tractable approximations. Operations research 58(4-part-1):902–917.
Goodfellow I (2016) Nips 2016 tutorial: Generative adversarial networks. arXiv preprint arXiv:1701.00160.
Goodfellow I, Pouget-Abadie J, Mirza M, Xu B, Warde-Farley D, Ozair S, Courville A, Bengio Y (2014a) Generative adversarial nets. Advances in neural information processing systems, 2672–2680.
Goodfellow IJ, Bengio Y, Courville A (2016) Deep learning (MIT Press).
Goodfellow IJ, Shlens J, Szegedy C (2014b) Explaining and harnessing adversarial examples. arXiv preprint arXiv:1412.6572.
Gotoh Jy, Kim MJ, Lim A (2015) Robust empirical optimization is almost the same as mean-variance optimization.
Hofbauer J, Sandholm WH (2002) On the global convergence of stochastic fictitious play. Econometrica 70(6):2265–2294.
Huber PJ (2011) Robust statistics (Springer).
Isola P, Zhu JY, Zhou T, Efros AA (2016) Image-to-image translation with conditional adversarial networks. arXiv preprint arXiv:1611.07004.
Jiang R, Guan Y (2015) Data-driven chance constrained stochastic program. Mathematical Programming 1–37.
Joe H (1997) Multivariate models and multivariate dependence concepts (CRC Press).
Jolicoeur-Martineau A (2017) Deep learning with cats. GitHub repository.
Kingma D, Ba J (2015) Adam: A method for stochastic optimization. Proceedings of the 3rd International Conference on Learning Representations (ICLR)
Krizhevsky A, Hinton G (2009) Learning multiple layers of features from tiny images. Technical report, University of Toronto.
Lam H (2016) Robust sensitivity analysis for stochastic systems. Mathematics of Operations Research 41(4):1248–1275.
Ledig C, Theis L, Huszár F, Caballero J, Cunningham A, Acosta A, Aitken A, Tejani A, Totz J, Wang Z, et al. (2016) Photo-realistic single image super-resolution using a generative adversarial network. arXiv preprint arXiv:1609.04802.
McFadden D (1978) Modeling the choice of residential location. Transportation Research Record (673).
McFadden D, Train K (2000) Mixed mnl models for discrete response. Journal of applied Econometrics 447–470.
McFadden D, et al. (1973) Conditional logit analysis of qualitative choice behavior. Institute of Urban and Regional Development, University of California.
Mishra VK, Natarajan K, Padmanabhan D, Teo CP, Li X (2014) On theoretical and empirical aspects of marginal distribution choice models. Management Science 60(6):1511–1531.
Natarajan K, Song M, Teo CP (2009) Persistency model and its applications in choice modeling. Management Science 55(3):453–469.
Popescu I (2007) Robust mean-covariance solutions for stochastic optimization. Operations Research 55(1):98–112.
Roth K, Lucchi A, Nowozin S, Hofmann T (2017) Stabilizing training of generative adversarial networks through regularization. *arXiv preprint arXiv:1705.09367*.

Russakovsky O, Deng J, Su H, Krause J, Satheesh S, Ma S, Huang Z, Karpathy A, Khosla A, Bernstein M, Berg AC, Fei-Fei L (2015) ImageNet Large Scale Visual Recognition Challenge. *International Journal of Computer Vision (IJCV)* 115(3):211–252.

Salimans T, Goodfellow I, Zaremba W, Cheung V, Radford A, Chen X (2016) Improved techniques for training gans. *Advances in Neural Information Processing Systems*, 2234–2242.

Shafieezadeh-Abadeh S, Esfahani PM, Kuhn D (2015) Distributionally robust logistic regression. *Advances in Neural Information Processing Systems*, 1576–1584.

Shafieezadeh-Abadeh S, Kuhn D, Esfahani PM (2017) Regularization via mass transportation. *arXiv preprint arXiv:1710.0016*.

Shaham U, Yamada Y, Negahban S (2015) Understanding adversarial training: Increasing local stability of neural nets through robust optimization. *arXiv preprint arXiv:1511.05432*.

Shapiro A, Dentcheva D, et al. (2014) *Lectures on stochastic programming: modeling and theory*, volume 16 (SIAM).

Shohat JA, Tamarkin JD (1943) *The problem of moments*. Number 1 (American Mathematical Soc.).

Sriperumbudur BK, Fukumizu K, Gretton A, Schölkopf B, Lanckriet GR (2010) Non-parametric estimation of integral probability metrics. *Information Theory Proceedings (ISIT), 2010 IEEE International Symposium on*, 1428–1432 (IEEE).

Srivastava N, Hinton G, Krizhevsky A, Sutskever I, Salakhutdinov R (2014) Dropout: A simple way to prevent neural networks from overfitting. *The Journal of Machine Learning Research* 15(1):1929–1958.

Tao T (2011) *An introduction to measure theory*, volume 126 (American Mathematical Soc.).

Tibshirani R (1996) Regression shrinkage and selection via the lasso. *Journal of the Royal Statistical Society. Series B (Methodological)* 267–288.

Tikhonov AN, Arsenin VY, John F (1977) *Solutions of ill-posed problems*, volume 14 (Winston Washington, DC).

Train KE (2009) *Discrete choice methods with simulation* (Cambridge university press).

Vapnik V (2013) *The nature of statistical learning theory* (Springer science & business media).

Villani C (2003) *Topics in optimal transportation*. Number 58 (American Mathematical Soc.).

Wager S, Wang S, Liang PS (2013) Dropout training as adaptive regularization. *Advances in neural information processing systems*, 351–359.

Wang Z, Glynn PW, Ye Y (2016) Likelihood robust optimization for data-driven problems. *Computational Management Science* 13(2):241–261.

Wiesemann W, Kuhn D, Sim M (2014) Distributionally robust convex optimization. *Operations Research* 62(6):1358–1376.

Wozabal D (2014) Robustifying convex risk measures for linear portfolios: A nonparametric approach. *Operations Research* 62(6):1302–1315.

Xu H, Caramanis C, Mannor S (2009a) Robust regression and lasso. *Advances in Neural Information Processing Systems*, 1801–1808.

Xu H, Caramanis C, Mannor S (2009b) Robustness and regularization of support vector machines. *Journal of Machine Learning Research* 10(Jul):1485–1510.

Žáčková J (1966) On minimax solutions of stochastic linear programming problems. *Časopis pro pěstování matematiky* 91(4):423–430.

Zhang W, Sun J, Tang X (2008) Cat head detection-how to effectively exploit shape and texture features. *Computer Vision–ECCV 2008* 802–816.

Zhu JY, Krähenbühl P, Shechtman E, Efros AA (2016) Generative visual manipulation on the natural image manifold. *European Conference on Computer Vision*, 597–613 (Springer).