Insertion Operations on Deterministic Reversal-Bounded Counter Machines✩

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Abstract

Several insertion operations are studied applied to languages accepted by one-way and two-way deterministic reversal-bounded multicounter machines. These operations are defined by the ideals obtained from relations such as the prefix, infix, suffix, and outfix relations, as well as operations defined from inverses of a type of deterministic transducer with reversal-bounded counters attached. The question of whether the resulting languages can always be accepted by deterministic machines with the same number (or larger number) of input-turns (resp., counters, counter-reversals, etc.) is investigated.

Keywords: Automata and Logic, Counter Machines, Insertion Operations, Reversal-Bounds, Determinism, Finite Automata

1. Introduction

One-way deterministic multicounter machines are deterministic finite automata augmented by a fixed number of counters, which can each be independently increased, decreased or tested for zero. If there is a bound on the number of switches each counter makes between increasing and decreasing, then the machine is reversal-bounded [1, 2]. The family of languages accepted by one-way deterministic reversal-bounded multicounter machines is denoted by DCM (and DCM(k,l) when there are at most k counters with at most an l-reversal-bound), and the nondeterministic variant is denoted by NCM.

Reversal-bounded counter machines (both deterministic and nondeterministic) have been extensively studied. Many generalizations have been investigated, and they have found applications in areas such as verification of infinite-state systems [3, 4, 5, 6], membrane computing systems [7], biocomputing [8], Diophantine equations [6], and others. DCM in particular is an interesting family as it is more general than the family of regular languages, but still has...
decidable emptiness, infiniteness, equivalence, inclusion, universe, and disjointness problems [2]. Moreover, these problems remain decidable if the machines operate with two-way input that is finite-crossing in the sense that there is a fixed $r$ such that the number of times the boundary between any two adjacent input cells is crossed is at most $r$ [9]. In addition, for fixed $k, l$, the emptiness, membership, containment, and equivalence problems for DCM($k, l$) can be tested in polynomial time [9]. Hence, DCM has many nice decidability and complexity theoretic properties. We know of no other family more general than the regular languages that enjoy these properties. Despite this, little is known regarding the closure properties of this family, which is important for constructing other languages that remain in this family.

More recently, the DCM model has gained a resurgence of theoretical interest. It was shown that all commutative semilinear languages are in DCM, and in fact, the subfamily of DCM languages accepted by machines that cannot subtract from any counter until hitting the right end-marker was shown to be equal to the smallest family closed under inverse deterministic finite transductions, commutative closure, and right quotient with regular languages [10]. In [11], it was shown that there is a polynomial time algorithm to decide, for fixed $k, l$ whether the shuffle of two NCN($k, l$) machines is contained in a DCM($k, l$) machine. In addition, DCM was studied in [12] as part of an interesting conjecture involving holonomic functions. The authors define a family RCM that is obtained from the regular languages via so-called linear constraints on the number of occurrences of symbols, and homomorphisms. It is demonstrated that all RCM languages have generating functions which are all holonomic functions. The class of holonomic functions in one variable is an extension of the algebraic functions which contains all those functions satisfying a linear differential equation with polynomial coefficients [12]. They conjectured that DCM is contained in RCM, implying that all DCM languages have holonomic generating functions. Although this conjecture has yet to be established, it is shown to be true for a subfamily of DCM. The study of closure properties on DCM can potentially help in this regard towards establishing the conjecture. Deletion operations applied to DCM have also been recently investigated [13] using word operations such as prefix, suffix, infix, and left and right quotients. It was found that DCM is closed under right quotient with many general families defined even by nondeterministic machines such as the context-free languages, and it is shown that the left quotient of a DCM($1, 1$) language with general families such as the context-free languages always gives DCM languages. However, even the suffix closure of languages in DCM($2, 1$) or DCM($1, 3$) gives languages which are not in DCM.

Generally, various schema for insertions and deletions have been studied in automata theory, from simple concatenation [14], to more complex insertion operations [15], and they have found applications in the area of natural computing for modelling biological processes [16, 17].

In this paper, we study various insertion operations on deterministic reversal-bounded multicounter languages. The prefix, suffix, infix, and outfix deletion operations can also be used to define insertion operations. As an example, the set of all infixes of a language $L$, $\text{inf}(L) = \{w \mid xwy \in L, x, y \in \Sigma^*\}$, and then the inverse of this operation, $\text{inf}^{-1}(L) = \Sigma^* L \Sigma^*$, is the set of all words having a word in $L$ as an infix. This is the same as what is often called the two-sided ideal, or the infix ideal [18]. For the suffix operation, $\text{suffix}(L) = \{w \mid xw \in L, x \in \Sigma^*\}$, and $\text{suffix}^{-1}(L) = \Sigma^* L$,
with the latter being called the *left ideal*, or the *suffix ideal*. For prefix, \( \text{pref}(L) = \{ w \mid wy \in L, y \in \Sigma^* \} \), and \( \text{pref}^{-1}(L) = L\Sigma^* \), the *prefix ideal*, or the *right ideal*. Thus, the inverse of each operation defines a natural and simple insertion operation.

We will examine the insertion operations defined by the inverse of the prefix, suffix, infix, outfix, and embedding operations, as well as the concatenation of languages from other families. It is easy to see that all language families closed under homomorphism, inverse homomorphism and intersection with regular languages (such as the nondeterministic reversal-bounded multicounter languages, or the context-free languages) are closed under all these insertion operations. However, this is a more complex question for families accepted by deterministic machines such as \( \text{DCM} \).

In this case, if we start with a language that can be accepted with a parameterized number of counters, input tape turns, and reversals on the counters, is the result of the various insertion operations always accepted with the same type of machine? These results are summarized for all such insertion operations in column 2 of Table 1. And if they are not closed, can they always be accepted by increasing either the number of counters, or reversals on the counters (presented in column 3 of Table 1), or turns on the input tape (listed in Section 6)? Results in this paper form a complete characterization in this regard. In particular, it is rather surprising that even if we have languages accepted by deterministic 1-reversal-bounded machines with either one-way input and 2 counters, or 1 counter and 1 turn on the input, then concatenating \( \Sigma^* \) to the right can result in languages that can neither be accepted by \( \text{DCM} \) machines (any number of reversal-bounded counters), nor by two-way deterministic reversal-bounded one counter machines (2DCM(1), which have no bound on input turns). This is in contrast to deterministic pushdown languages which are closed under right concatenation with regular languages [19]. In addition, concatenating \( \Sigma^* \) to the left of a DCM(1, 1) language can create languages that are neither in \( \text{DCM} \) nor 2DCM(1).

As a consequence of the results in this paper, it is evident that the right input end-marker used for language acceptance for (one-way) \( \text{DCM} \) strictly increases the power for even one-way deterministic reversal-bounded multicontroler languages when there are at least two counters. This is usually not the case for various classes of one-way machines, such as for deterministic pushdown automata (DPDAs). Indeed, language acceptance for DPDAs is defined as being without a right end-marker, and DPDAs are closed under right quotient with a single symbol [20], meaning a right end-marker could be removed without altering the languages accepted. In contrast, language acceptance for DCMs is defined using a right end-marker and DCM is closed under right quotient with symbols (and even context-free languages) [13]. But the end-marker is necessary for this right quotient result. Moreover, if language acceptance for DCM is defined without an end-marker (defined and studied in this paper), this family of languages is not closed under right quotient with a single symbol. This demonstrates the importance of the right input end-marker.

Lastly, a type of finite transducer augmented by reversal-bounded counters is studied, and it is shown that DCM is closed under these inverse deterministic transductions. The inverses of these transductions can be used for defining many insertion operations under which DCM is closed.

Most non-closure results in this paper use techniques that simultaneously shows languages are not in \( \text{DCM} \) and not in 2DCM(1). The techniques do not rely on any pumping arguments.
| Operation | is $O_p(L) \in \text{DCM}(k, l)$? | is $O_p(L) \in \text{DCM}$? |
|-----------|---------------------------------|-------------------------------|
| $\text{pref}^{-1}(L)$ | Yes if $k = 1$, $l \geq 1$ | Cor 9 | Yes if $k = 1$, $l \geq 1$ | Cor 9 |
| | No if $k \geq 2$, $l \geq 1$ | Thm 13 | Yes if $L \in \text{DCM}_{\text{NE}}$ | Thm 7 |
| | | | No otherwise if $k \geq 2, l \geq 1$ | Thm 13 |
| $\text{suff}^{-1}(L)$ | No if $k, l \geq 1$ | Thm 16 | No if $k, l \geq 1$ | Thm 16 |
| $\text{inf}^{-1}(L)$ | No if $k, l \geq 1$ | Thm 12 | No if $k, l \geq 1$ | Thm 12 |
| $\text{outf}^{-1}(L)$ | No if $k, l \geq 1$ | Thm 19 | No if $k, l \geq 1$ | Thm 19 |
| $\text{emb}^{-1}(m, L)$ | No if $k, l, m \geq 1$ | Cor 20 | No if $k, l, m \geq 1$ | Cor 20 |
| $\text{LR}$ | Yes if $k = 1$, $l \geq 1$ | Cor 8 | Yes if $k = 1$, $l \geq 1$ | Cor 8 |
| | Yes if $L \in \text{DCM}_{\text{NE}}$ | Thm 7 | Yes if $L \in \text{DCM}_{\text{NE}}$ | Thm 7 |
| | No otherwise if $k \geq 2, l \geq 1$ | Thm 13 | No otherwise if $k \geq 2, l \geq 1$ | Thm 13 |
| $\text{RL}$ | Yes if $R$ prefix-free | Cor 6 | Yes if $R$ prefix-free | Cor 6 |
| | No otherwise if $k, l \geq 1$ | Cor 17 | No otherwise if $k, l \geq 1$ | Cor 17 |
| $\text{LD}_{\text{DCM}}L$ | No if $k, l \geq 1$ | Cor 18 | No if $k, l \geq 1$ | Cor 18 |
| $\text{LD}_{\text{DCM}}_e L$ | No if $k, l \geq 1$ | Cor 13 | Yes if $L_{\text{DCM}}_e$ prefix-free | Thm 5 |
| | | | No otherwise if $k, l \geq 1$ | Cor 13 |

Table 1: Summary of results for DCM. Assume $R \in \text{REG}$, $L_{\text{DCM}} \in \text{DCM}$, and $L_{\text{DCM}}_{\text{NE}} \in \text{DCM}_{\text{NE}}$. Then, for all $L \in \text{DCM}(k, l)$, the question in row 1 is presented for each insertion operation in column 1. When applying the operation in the first column to any $L \in \text{DCM}(k, l)$, is the result necessarily in $\text{DCM}(k, l)$ (column 2), and in $\text{DCM}$ (column 3)? This is parameterized in terms of $k$ and $l$, and the theorems showing each result is provided.

2. Preliminaries

The set of non-negative integers is represented by $\mathbb{N}_0$, and positive integers by $\mathbb{N}$. For $c \in \mathbb{N}_0$, let $\pi(c)$ be 0 if $c = 0$, and 1 otherwise.

We use standard notations for formal languages, referring the reader to [19, 14]. The empty word is denoted by $\lambda$. We use $\Sigma$ and $\Gamma$ to represent finite alphabets, with $\Sigma^*$ as the set of all words over $\Sigma$ and $\Sigma^+ = \Sigma^* \setminus \{\lambda\}$. For a word $w \in \Sigma^*$, if $w = a_1 \cdots a_n$ where $a_i \in \Sigma$, $1 \leq i \leq n$, the length of $w$ is denoted by $|w| = n$, and the reversal of $w$ is denoted by $w^R = a_n \cdots a_1$. The number of $a$’s, for $a \in \Sigma$, in $w$ is $|w|_a$. Given a language $L \subseteq \Sigma^*$, the complement of $L$, $\Sigma^* \setminus L$ is denoted by $\overline{L}$.

**Definition 1.** For a language $L \subseteq \Sigma^*$, we define the prefix, inverse prefix, suffix, inverse suffix, infix, inverse infix, outfix and inverse outfix operations, respectively:
Note that $\text{outf}(\Sigma) \subseteq \mathbb{Q}$ which is a subset of $\mathbb{Q}$. We denote by $\text{insertion operations with} \text{ pref}^ {-1}$ to prevent negative values in any counter. The symbols $S$ and $R$ indicate the direction of input tape head movement, $+k$, $−k$, respectively. The machine $M$. The language accepted by (final state in) $L$. We denote by $\text{emb}(\Sigma)$ the contents of the $m$-embedding inserts $m$ arbitrary words in every word.

A language $L$ is called prefix-free if, for all words $x, y \in L$, where $x$ is a prefix of $y$, then $x = y$.

A one-way $k$-counter machine is a tuple $M = (k, Q, \Sigma, <, δ, q_0, F)$, where $Q, \Sigma, <, q_0, F$ are respectively the finite set of states, the input alphabet, the right end-marker (not in $\Sigma$), the initial state in $Q$, and the set of final states, which is a subset of $Q$. The transition function $δ$ (defined as in [21] except with only a right end-marker since these machines only use one-way inputs) is a partial function from $Q \times (\Sigma \cup \{<\}) \times \{0, 1\}^k$ into the family of subsets of $Q \times \{S, R\} \times \{-1, 0, +1\}^k$, such that if $δ(q, a, c_1, \ldots, c_k)$ contains $(p, d, d_1, \ldots, d_k)$ and $c_i = 0$ for some $i$, then $d_i \geq 0$ to prevent negative values in any counter. The symbols $S$ and $R$ indicate the direction of input tape head movement, either stay or right respectively. The machine $M$ is deterministic if every element mapped by $δ$ is to a subset of size one. The machine $M$ is non-exiting if there are no transitions defined on final states. A configuration of $M$ is a $k + 2$-tuple $(q, w, c_1, \ldots, c_k)$ representing the fact that $M$ is in state $q$, $w$ (either in $\Sigma^*$ or $\Sigma^* <$) is the remaining input, and $c_1, \ldots, c_k \in \mathbb{N}_0$ are the contents of the $k$ counters. The relation $r_M$ is defined between configurations, where $(q, aw, c_1, \ldots, c_k) r_M (p, w', c_1 + d_1, \ldots, c_k + d_k)$, if $(p, d, d_1, \ldots, d_k) \in δ(q, a, c_1, \ldots, c_k)$ where $d \in \{S, R\}$ and $w' = aw$ if $d = S$, and $w' = w$ if $d = R$. We let $r_M$ be the reflexive, transitive closure of $r_M$. And, for $m \in \mathbb{N}_0$, let $r_M^m$ be the application of $r_M$ $m$ times. A word $w \in \Sigma^*$ is accepted by $M$ if $(q_0, w, <, 0, \ldots, 0) r_M^m (q, <, c_1, \ldots, c_k)$, for some $q \in F$, and $c_1, \ldots, c_k \in \mathbb{N}_0$. A derivation between the initial configuration and a final configuration is called an accepting computation. The language accepted by (final state in) $M$, denoted by $L(M)$, is the set of all words accepted by $M$.

The machine $M$ is $l$-reversal-bounded if, in every accepting computation, the count on each counter alternates between increasing and decreasing at most $l$ times.

We denote by $\text{NCM}(k, l)$ the family of languages accepted by one-way nondeterministic $l$-reversal-bounded $k$-counter machines. We denote by $\text{DCM}(k, l)$ the family of languages accepted by one-way deterministic $l$-reversal-bounded $k$-counter machines.
bounded \(k\)-counter machines. The union of the families of languages are denoted by \(\text{NCM} = \bigcup_{k \geq 0} \text{NCM}(k, l)\) and \(\text{DCM} = \bigcup_{k \geq 0} \text{DCM}(k, l)\).

Given a DCM machine \(M = (k, Q, \Sigma, <\!, \delta, q_0, F)\), the language accepted by \textit{final state without end-marker} is the set of words \(w\) such that \((q_0, w <\!, 0, \ldots, 0) \vdash^*_M (q', a <\!, c_1', \ldots, c_k')\), for some \(q' \in F, a \in \Sigma, c_i, c_i' \in \mathbb{N}_0, 1 \leq i \leq k\). Such a machine does not “know” when it has reached the end-marker \(<\!\). The state that the machine is in when the last letter of input from \(\Sigma\) is consumed entirely determines acceptance or rejection. It would be equivalent to require \((q_0, w, 0, \ldots, 0) \vdash^*_M (q, \lambda, c_1, \ldots, c_k), w \in \Sigma^*,\) for some \(q \in F\), but we continue to use \(<\!\) for compatibility with the end-marker definition. We use \(\text{DCM}_{\text{NE}}(k, l)\) to denote the family of languages accepted by these machines by final state without end-marker when they have \(k\) counters that are \(l\)-reversal-bounded. We define \(\text{DCM}_{\text{NE}} = \bigcup_{k \geq 0} \text{DCM}_{\text{NE}}(k, l)\).

We denote by \(2\text{DCM}(1)\) the family of languages accepted by two-way deterministic finite machines (with both a left and right input tape end-marker) augmented by one reversal-bounded counter, accepted by final state. A machine of this form is said to be \textit{finite-crossing} if there is a fixed \(k\) such that the number of times the boundary between any two adjacent input cells is crossed is at most \(k\) times, and a machine of this form is \textit{finite-turn bounded} if there is a fixed \(t\) where \(M\) makes at most \(t\) changes of direction on the input tape for every computation \cite{finite-turn}. Note a finite-turn machine is finite-crossing, but the converse does not hold in general. The family \(\text{NPCM} (\text{DPCM})\) is defined by languages accepted by one-way nondeterministic (deterministic) machines with an unrestricted pushdown augmented by reversal-bounded counters \cite{finite-turn}.

3. Closure for Insertion and Concatenation Operations

Closure under concatenation is difficult for DCM languages because of determinism. However, certain special cases are demonstrated where closure can be obtained. Towards this, a comparison of DCM to \(\text{DCM}_{\text{NE}}\) will be made. This is important as it will be shown that \(\text{DCM}_{\text{NE}}\) is closed under right concatenation with regular languages, although this will be shown not to be true generally for DCM. However, when only one reversal-bounded counter is used, the end-marker will be shown to not change the capacity. This will show that DCM languages defined by machines with one reversal-bounded counter are closed under right concatenation with regular languages. In addition, closure under left concatenation with prefix-free regular languages will be shown. These results serve to demonstrate that DCM languages are strictly more powerful with the end-marker, but add no power to DCM(1, l). This is in contrast to deterministic pushdown automata which do not need a right input end-marker.

\textbf{Lemma 3.} For any \(l \geq 1\), \(\text{DCM}(1, l) = \text{DCM}_{\text{NE}}(1, l)\).

\textbf{Proof.} Trivially, \(\text{DCM}_{\text{NE}}(1, l) \subseteq \text{DCM}(1, l)\), by removing all transitions defined on the end-marker.

For the reverse containment, consider \(M = (1, Q, \Sigma, <\!, \delta, q_0, F)\) accepting \(L\) by final state. A machine \(M'\) will be built such that the language accepted by \(M'\) by final state without end-marker is equal to \(L(M)\).
We assume without loss of generality that \( \delta \) is a total function. Let \(|Q| = n\). For each state \( q \in Q \), define the language

\[
L(q) = \{ a' \mid (q, c, i) \xrightarrow{M} (q_f, c, i'), q_f \in F, c \in \mathbb{N}_0 \},
\]

the set of counter values which lead to acceptance from the end-marker \( c \) and state \( q \). This language can be accepted by a machine in DCM with one counter, by adding the input \( i \) to the counter, then simulating \( M \) from state \( q \), and accepting if \( M \) does. Since all DCM languages are semilinear [3], \( L(q) \) is unary, all unary semilinear languages are regular \([19]\), then \( L(q) \) is regular. Thus we can accept \( L(q) \) with a DFA, say \( D(q) = (Q_{D(q)}, (a), \delta_{D(q)}, s_{D(q)}, F_{D(q)}) \).

Because these languages are unary, the structure of the DFAs are relatively simple, and well-known (see [20] for a seminal work on unary finite automata, and [23] for the informal language used here). Every unary DFA with \( m \) states is isomorphic to one with states \([0, \ldots, m - 1]\) where there exists some state \( t \), and there is a transition from \( i \) to \( i + 1 \), for all \( 0 \leq i < t \) (the “tail”), and there is a transition from \( j \) to \( j + 1 \) for all \( t \leq j < m - 2 \), plus a transition from \( m - 1 \) to \( t \) (the “loop”), and no other transitions. Furthermore, we can assume without loss of generality that each \( D(q) \), for \( q \in Q \), has the same length of tail, \( t > 0 \), equal to the maximum of the tail lengths of all of the original DFAs \( D(p) \), over all \( p \in Q \). This can be done as the tail of a unary DFA can be made longer by adding additional states to the tail and shifting the final states in the loop. Similarly, it can be assumed that the loops are all of the same length \( l \) by making it the length that is the least common multiple of the original lengths (thus making the loop length a multiple of all the originals). Thus all \( D(q) \), for \( q \in Q \) have tail length \( t > 1 \), loop length \( l \), and \( m \) states, and they all differ only in final states. Let \( \delta_D(i) \) be the state of all \( D(q) \) machines after reading \( a', i \geq 0 \). Then, \( \delta_D(i) \) is \( i \) if \( i \leq t \), and \( i - t \mod l \) otherwise.

The intuition for the construction of \( M' \) is as follows. The machine \( M' \) simulates \( M \), and after reading \( w \), if \( M \) has counter value \( c \), \( M' \) has counter value \( c - t \) if \( c > t \), with \( t \) stored in the finite control. If \( c \leq t \), then \( M' \) stores \( c \) in the finite control with zero on the counter. This allows \( M' \) to know what counter value \( M \) would have after reading a given word, but also to know when the counter value is less than \( t \) (and the specific value less than \( t \)). In the finite control, in addition to simulating \( M \), \( M' \) simulates each \( D(q) \), for all \( q \in Q \), in parallel in such a way that the (unique, for all DFAs) state \( \delta(i) \) is stored when the counter of \( M \) is \( i \). To do this, \( M' \) stores two integers, \((d, j)\), where \( 0 \leq d \leq t, 0 \leq j \leq l \), and if \( i < t \), then \((d, j) = (i, 0)\), and if \( i \geq t \), then \((d, j) = (t, i - t \mod l)\). Thus, we call the first component the “tail” counter, and the second component the “loop” counter. Then \( d + j \) is the state of \( D(q) \) after reading \( a' \). Each time \( M \) increases the counter, from \( i \) to \( i + 1 \), the state of each \( D(q) \) is determined by increasing the appropriate bounded counter (the first component of it is not yet \( t \), and the second component otherwise). Each time \( M \) decreases the counter from \( i \) to \( i - 1 \), the state of each \( D(q) \) changes deterministically by decreasing the second component \( j \) by one modulo \( l \) if \( i > t \) (going “backwards” in the loop), and if \( i \leq t \), then the counter of \( M' \) will be zero, and thus the simulation of each \( D(q) \) can tell when to switch deterministically from decreasing the loop counter to the tail counter. Then, when \( M \) is in state \( q \), \( M' \) can tell if the current counter value would be accepted using the appropriate DFA \( D(q) \).
We now provide the construction in detail:

The machine $M'$ has state set $Q_{M'} = (Q \times \{0, \ldots, l\} \times \{0\}) \cup (Q \times \{0, \ldots, l\})$. The final states of $M'$ are of the form $(q, d, j)$ if either $(d < t, j = 0, d \in F_{D(q)})$, or $(d = t, t + j \in F_{D(q)})$.

If $\delta(q, b, 0) = (p, T, a)$, for some $q, p \in Q, b \in \Sigma, T \in \{S, R\}, a \in \{0, 1\}$, we add the following transition to $\delta_{M'}$:

1. $\delta_{M'}((q, 0, 0), b, 0) = ((p, \alpha, 0), T, 0)$.

Also, if $\delta(q, b, 1) = (p, T, a)$, for some $q, p \in Q, T \in \{S, R\}, a \in \{-1, 0, 1\}$ we add the following transition in $\delta_{M'}$ for every $s = (q, d, j) \in Q_{M'}$:

2. $\delta_{M'}(s, b, 0) = ((p, d + \alpha, 0), T, 0)$ if $j = 0, 0 \leq d \leq t$, and $0 \leq d + \alpha \leq t$,

3. $\delta_{M'}(s, b, y) = ((p, t, j + \alpha \mod l), T, a)$ for $y \in \{0, 1\}$, if $d = t$ and $\alpha \in \{0, 1\}$,

4. $\delta_{M'}(s, b, 1) = ((p, t, j - 1 \mod l), T, a)$ if $d = t$ and $\alpha = -1$.

Claim 1. For all $m \in \mathbb{N}_0$, if $(q_0, w = uv, 0) \vdash^m_M (q, v, c)$ where $u, v \in \Sigma^*, q \in Q, c \in \mathbb{N}_0$, then

$$((q_0, 0, 0), uv, 0) \vdash^m_{M'} ((q, t, c - t \mod l), v, c - t),$$

when $c > t$, and

$$((q_0, 0, 0), uv, 0) \vdash^m_{M'} ((q, c, 0), v, 0),$$

when $c \leq t$.

Proof. We perform induction on $m$.

If $m = 0$ then $q = q_0, u = \lambda, c = 0, c \leq t$, thus the second condition is true.

Consider $m \geq 0$, and assume the implication holds for $m$. We will show it holds for $m + 1$.

Suppose $(q_0, uv, 0) \vdash^{m+1}_M (q, v, c)$. Then for some state $p \in Q, a \in \Sigma \cup \{\lambda\}$, and $c' \in \mathbb{N}_0$, we have $(q_0, uv, 0) \vdash^m_M (p, av, c') \vdash^1_M (q, v, c)$, with the last transition via $x$. We know that $c \in \{c' - 1, c', c' + 1\}$.

Case: $c \geq t, c' \geq t$. Notice that when $c' = t$, Equations $[1]$ and $[2]$ coincide. Then, by our hypothesis, we have

$$((q_0, 0, 0), uv, 0) \vdash^m_{M'} ((p, t, c' - t \mod l), av, c' - t).$$

If $c = c' = t$ (so $c' > t$ and $c' - t > 0$), then we know that $((p, t, c' - t \mod l), av, c' - t) \vdash^m_{M'} ((q, t, c' - t - 1 \mod l), v, c' - t - 1) = ((q, t, c - t \mod l), v, c - t)$ by the transition created by rule (4) from $x$.

If $c = c'$, then we know that $((p, t, c' - t \mod l), av, c' - t) \vdash^m_{M'} ((q, t, c' - t \mod l), v, c' - t) = ((q, t, c - t \mod l), v, c - t)$ by transition rule (3).

If $c = c' + 1$, then we know $((p, t, c' - t \mod l), av, c' - t) \vdash^m_{M'} ((q, t, c' - t + 1 \mod l), v, c' - t + 1) = ((q, t, c - t \mod l), v, c - t)$ by transition rule (3).
Case: $c \leq t, c' \leq t$. By our hypothesis we have

$$(q_0, 0, 0, u, 0) \overset{m}{\rightarrow}_M ((p, c', 0), a, 0),$$

$a \in \Sigma \cup \{\lambda\}$.

If $c = c' - 1, c = c'$ or $c = c' + 1$, then the implication holds by transition rule (2), unless $c' = 0$, in which case it holds by transition rule (1).

Thus we have shown that the implication true for $M'$ in $m + 1$ steps, and is therefore true for all $m$. \qed

Claim 2. For all $m \in \mathbb{N}_0$, let

$$(q_0, 0, 0, w = uv, 0) \overset{m}{\rightarrow}_M ((q, d, j), v, e),$$

where $u, v \in \Sigma^*$. Then the following are true:

1. $d + j = \delta_D(e + d)$,
2. $((q_0, uv, 0) \overset{m}{\rightarrow}_M (q, v, e + d)).$
3. $e > 0$ or $j > 0$ only if $d = t$.

Proof. We perform induction on $m$.

If $m = 0$ then $d = 0 < t, e = 0, j = 0$, and thus (3) is true, and conditions (1) and (2) are immediate.

Consider $m \geq 0$, and assume the implication holds for $m$.

Suppose $((q_0, 0, 0, uv, 0) \overset{m+1}{\rightarrow}_M (q, d, j), v, e)$. Then

$$(q_0, 0, 0, uv, 0) \overset{m}{\rightarrow}_M ((q', d', j'), av, e')$$

for some last transition $x$. Then, by the hypothesis, $d' + j' = \delta_D(e' + d'), (q_0, uv, 0) \overset{m}{\rightarrow}_M (q', av, e' + d')$, and $e' > 0$ or $j' > 0$ only if $d' = t$.

Suppose $e' > 0$. Then $x$ must be of type (3) or (4) in the construction, and $d' = t = d$, therefore the third condition is true. Then the transition $x$ that changes the counter by $a$ is created from a transition that changes the counter of $M$ similarly. Thus, the second condition holds. For the first condition, $\delta_D(e + d)$ must be in the “loop” of $D(q_i)$ since $t = d$, and $\delta_D(e + d) = \delta_D(e' + d' + \alpha) = d + (j' + \alpha \mod l) = d + j$.

Suppose $e' = 0$. Then $x$ must be of type (1), (2), or (3). If it is type (3), then $d' = t = d$ (here $\alpha \in \{0, 1\}$), then the conditions hold just like the case above. For both types (1) and (2), then $e' = e = j = j' = 0$, and so condition 3 is true. For both, $d'$ changes to $d$ in the same way as the counter of $M$. Then the second condition holds. For the first condition, $\delta_D(e + d) = d = e + d$. \qed

Then, $w$ is accepted by final state in $M$, if and only if $(q_0, w, c, 0) \overset{*}{\rightarrow}_M (q, c, c)$, for some $q \in Q, q_f \in F$, and $(q, c, c)$ is the first configuration to reach $\epsilon$, if and only if $(q_0, w, c, 0) \overset{*}{\rightarrow}_M (q, c, c)$, for some $q \in Q, (q, c, c)$
is the first configuration to reach $\prec$, and $a' \in L(q)$ (from the definition of $L(q)$), if and only if $(q_0, w, 0) \vdash_{M}^* (q, \lambda, c)$ for some $q \in Q$ such that $a' \in L(q)$. We will show that this is true if and only if $M'$ accepts $w$ by final state without end-marker.

Assume $(q_0, w, 0) \vdash_{M}^* (q, \lambda, c)$ for some $q \in Q$ such that $a' \in L(q)$. Then $\delta_D(c)$ is final in $D(q)$, and

$$(q_0, 0, 0, w, 0) \vdash_{M'}^* (q, f, j, \lambda, e),$$

for some $f, e, j$ by Claim[1] where $c > t$ implies $f = t, e = c-t$, and $j = c-t \mod l$, and $c \leq t$ implies $f = c, j = e = 0$.

In the second case, it is immediate that $(q, c, 0)$ is final since $a' \in L(q)$. In the first case, it follows since $j = c-t \mod l$ and by the structure of $D(q)$ that $\delta_D(c) = t + j$. Then $(q, f, j)$ is final in $M'$ and $M'$ accepts $w$ by final state without end-marker.

Conversely, assume $M'$ accepts $w$ by final state without end-marker. Then $((q_0, 0, 0, w, 0) \vdash_{M'}^* ((q, f, j), \lambda, e))$, either $f = t$ and $t + j \in F_{D(q)}$, or $f < t, j = 0$ and $f \in F_{D(q)}$. Then $(q_0, w, 0) \vdash_{M}^* (q, \lambda, e + f)$ with $\delta_D(e + f) = f + j$ by Claim[2] if $f = t$, then $a'^+f \in L(q)$, and we are done. If $f < t$, then $e = f = 0$ by Claim[2] $\delta_D(e + f) = \delta_D(f) = f \in F_{D(q)}$, and $a'^+f \in L(q)$. Thus, $(q_0, w, 0) \vdash_{M}^* (q, \lambda, e + f)$ for some $1 \leq i \leq n$ such that $a'^+f \in L(q)$.

Hence, $w$ is accepted by final state in $M$ if and only if $w$ is accepted by final state without end-marker in $M'$.}$

We will extend these closure results with a lemma about prefix-free $DCM_{NE}$ languages. It is known that a regular language is prefix-free if and only if there is a non-exiting DFA accepting the language $[24]$.

**Lemma 4.** Let $L \in DCM_{NE}$. Then $L$ is prefix-free if and only if there exists a DCM-machine $M$ accepting $L$ by final state without end-marker which is non-exiting.

**Proof.** Let $L \in DCM_{NE}$, with $M$ a machine accepting $L$ by final state without end-marker. Then $\{w \mid (q_0, w, 0) \vdash_{M}^* (q_f, \lambda, c), q_f \in F\} = L$. Assume without loss of generality that no stay transitions can switch to a final state, because a word can only be accepted by final state without end-marker after a transition moving right. Thus, any stay transition switching to a final state $q_f$ can switch to a non-final state $q'_f$ that operates just like $q_f$.

( $\Rightarrow$ ) Suppose $L$ is prefix-free. Construct $M'$ from $M$ such that all transitions out of final states are removed, and so $M'$ is not non-exiting. Then $L(M') \subseteq L(M)$ since all transitions of $M'$ are in $M$. For the reverse containment, consider $w \in L(M)$ such that $(q_0, w_0, i_{0,1}, \ldots, i_{0,k}) \vdash_{M}^* \vdash_{M} (q_n, w_n, i_{n,1}, \ldots, i_{n,k})$, $n \geq 0$, $q_n \in F, w_n = \lambda, w_0 = w, i_{0,1} = \cdots i_{0,k} = 0$, via transitions $\alpha_1, \ldots, \alpha_n$ respectively. Assume that there exists $j < n$ such that $q_j \in F$. Thus, $n > 0$. If $j > 1$, then $\alpha_{j-1}$ must be a right transition instead of a stay transition, as no stay transition switches to a final state. But then the sequence $\alpha_1, \ldots, \alpha_{j-1}$ (or the empty sequence if $j = 1$), is the computation accepting the right quotient of $w$ with $w_j$, which is a proper prefix of $w$ since $j < n$ and since $\alpha_n$ must be a right transition. But $L$ is prefix-free, a contradiction. Thus, all of $q_0, \ldots, q_{n-1}$ are non-final, and $\alpha_1, \ldots, \alpha_n$ are in $M'$. Thus $L(M) \subseteq L(M')$ as well.

( $\Leftarrow$ ) Suppose $M$ is non-exiting. Consider $w \in L$. Then after reading $w$ deterministically, there are no transitions to follow, so $wx$ is not accepted for any $x \neq \lambda$. Thus $L$ is prefix-free.
From this, we obtain a special case where DCM is closed under concatenation, if the first language can be both accepted by final state without end-marker, and is prefix-free. The construction considers a non-exiting machine accepting \( L_1 \) by final state without end-marker, where transitions into its final state are replaced by transitions into the initial state of the machine accepting \( L_2 \).

**Theorem 5.** Let \( L_1 \in \text{DCM}_{\text{NE}}(k,l), L_2 \in \text{DCM}(k',l'), \) with \( L_1 \) prefix-free. Then \( L_1L_2 \in \text{DCM}(k+k', \max(l,l')) \).

*Proof.* Our construction is simple. Consider non-exiting \( M_1 \) accepting \( L_1 \) by final state without end-marker, and \( M_2 \) accepting \( L_2 \). Assume without loss of generality that only transitions that move right in \( M_1 \) switch to a final state. We form \( M' \) where \( L(M') = L_1L_2 \). Indeed, \( M' \) has the states and transitions from \( M_1, M_2 \) combined, with the start state of \( M_1 \) as its start state. Any transition into an accepting state of \( M_1 \) is replaced by an equivalent transition into the starting state of \( M_2 \). The accepting states are the accepting states of \( M_2 \). The machine has separate counters for the counters of \( M_1 \) and \( M_2 \), each of which performs the same reversals they would in their original machine.

Let \( w \in L_1 \) and \( x \in L_2 \). Since \( M_1 \) accepts without end-marker, and since no proper prefix of \( w \) leads to an accepting state of \( M_1 \), we know reading \( w \) in \( M_1 \) leads to an accepting state in \( M_1 \), even without reading \( \langle \cdot \rangle \). So, in \( M' \), we know that reading \( w \) will lead to the start state of \( M_2 \). Reading \( x \) from the start of \( M_2 \) leads to an accepting state, since \( x \in L_2 \). Thus reading \( x \) from the start of \( M_2 \) in \( M' \) leads to acceptance.

Let \( y \in L(M') \). Then \( M' \) starts in the start of \( M_1 \), so the only path to an accepting state is through the start of \( M_2 \). Thus there is some division of \( y \) into \( w, x \) where reading \( w \) in \( M_1 \) leads to acceptance (because it leads to the start state of \( M_2 \) in \( M' \)), and reading \( x \) in \( M_2 \) leads to acceptance, because we get to an accepting state in \( M' \). Thus \( y \in L_1L_2 \). \( \square \)

Notice that it is also possible to make \( L_1L_2 \in \text{DCM}(\max(k,k'), 1+1') \) by resetting and reusing the same counters for \( M_1 \) and \( M_2 \).

If we remove the condition that \( L_1 \) is prefix-free however, the theorem is no longer true, as we will see in the next section that even the regular language \( \Sigma^* \) (which is in \( \text{DCM}_{\text{NE}}(0,0) \)) concatenated with a DCM language produces a language outside DCM.

**Corollary 6.** Let \( L \in \text{DCM}(k,l), R \in \text{REG}, \) where \( R \) is prefix-free. Then \( RL \in \text{DCM}(k,l) \).

In contrast to left concatenation of a regular language with a DCM language (Corollary 5), where it is required that \( R \) be prefix-free (the regular language is always in \( \text{DCM}_{\text{NE}} \)), for right concatenation, it is only required that it be a \( \text{DCM}_{\text{NE}} \) language. We will see in the next section that this is not true if the restriction that \( L \) accepts by final state without end-marker is removed.

The following proof takes a DCM machine \( M_1 \) accepting by final state without end-marker, and \( M_2 \) a DFA accepting \( R \), and builds a DCM machine \( M' \) accepting \( LR \) by final state without end-marker.
**Theorem 7.** Let $L \in \text{DCM}_{\text{NE}}(k, l)$, $R \in \text{REG}$. Then $LR \in \text{DCM}_{\text{NE}}(k, l)$. Hence, $\text{pref}^{-1}(L) = L^* \in \text{DCM}_{\text{NE}}(k, l)$.

**Proof.** Let $M_1 = (k, Q_1, \Sigma, \cdot, \delta_1, q_1, F_1)$ be a DCM machine accepting $L$ by final state without end-marker where, without loss of generality, final states are only reached after transitions that move right. Let $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ be a DFA accepting $R$. A DCM machine $M' = (k, Q', \Sigma, \cdot, \delta', q', F')$ will be built that will accept $LR$ by final state without end-marker. Assume without loss of generality that $M_1$ reads all the way to the end of every input. This can be assumed, similar to the proof of closure of DCM under complement [2] by removing the ability of $M_1$ to enter an infinite loop on the input which can be detected using the finite control, and instead switching to a “dead state” while reading all of the input.

Intuitively, $M'$ will simulate $M_1$ while also storing a subset of $Q_2$ in the second component of the state. Every time it reaches a final state of $M_1$, it places the initial state of $M_2$ in the second component. And, then it continues to simulate $M_1$, while in parallel simulating the DFA $M_2$ on every state in the second component in parallel.

Formally, $Q' = Q_1 \times 2^{Q_2}$; $q'_1 = (q_1, \emptyset)$ if $q_1 \notin F_1$ and $q'_1 = (q_1, \{q_2\})$ otherwise, $F' = \{(q, X) \mid q \in Q_1, X \cap F_2 \neq \emptyset\}$ and $\delta'$ is defined as follows: for every transition, $\delta_1(q, a, x) = (p, T, i)$, $p, q \in Q_1, a \in \Sigma, x \in \{0, 1\}^k, T \in \{S, R\}, i \in \{-1, 0, 1\}^k$, introduce $\delta'(q, Y, a, x) = ((p, Z), T, i)$, for all $Y \in 2^{Q_2}$, where

- $Z = Y$ if $T = S$ (and hence $p \notin F_1$)
- $Z = \delta_2(Y, a)$ if $T = R$ and $p \notin F_1$
- $Z = \delta_2(Y, a) \cup \{q_2\}$ if $T = R$ and $p \in F_1$

**Claim 3.** $L(M_1)L(M_2) \subseteq L(M')$.

**Proof.** Let $uv \in \Sigma^*$, where $u \in L(M_1), v \in L(M_2)$. Then there is a computation $(p_1, u_1, i_1(1), \ldots, i_k(1)) \vdash_M \cdots \vdash_M (p_n, u_n, i_1(n), \ldots, i_k(n))$ where $p_1 = q_1, u_1 = uv, i_1(1) = \cdots = i_k(1) = 0, p_n \in F_1, u_n = v$. Furthermore, since $M_1$ reads every input, $(p_n, u_n, i_1(n), \ldots, i_k(n)) \vdash_{M_1} (p', \lambda, i_1(1), \ldots, i_k(1))$, for some $p', i_1, \ldots, i_k$. Then, by the construction, $((p_1, Y_1), uv, 0, \ldots, 0) \vdash_{M'} ((p_n, Y_n), v, i_1(n), \ldots, i_k(n))$, where $Y_1 = \emptyset$ and $q_2 \in Y_n$ since $p_n \in F_1$. Furthermore, it must be the case that $((p_n, Y_n), v, i_1(n), \ldots, i_k(n)) \vdash_{M'} ((p', Y'), \lambda, i_1, \ldots, i_k)$ and that $\delta(q_2, v) \in Y'$ since every transition applied to $M_1$ while reading $v$ that consumes an input letter, also changes state via that letter according to the DFA $M_2$. Thus, there is a final state from $F_2$ in $Y'$ causing $M'$ to also accept. 

**Claim 4.** $L(M') \subseteq L(M_1)L(M_2)$

**Proof.** Let $w \in L(M')$. Then,

$$((p_1, Y_1), u_1, i_1(1), \ldots, i_k(1)) \vdash_{M'} \cdots \vdash_{M'} ((p_n, Y_n), u_n, i_1(n), \ldots, i_k(n)),$$

where $u_1 = w, p_1 = q_1, Y_1 = \emptyset, u_n = \lambda, i_1(1) = \cdots = i_k(1) = 0, Y_n \cap F_2 \neq \emptyset$. Let $q_f$ be some state in $F_2 \cap Y_n$. Then, by the construction, there exists some $j, 1 \leq j \leq n$ such that $p_j \in F_1, q_2 \in Y_j$, and for every transition from the $j$th
configuration to the last one, while reading $u_j$, the sets $Y_j, \ldots, Y_n$ consecutively stay the same on a stay transition, and on a right transition that consumes the next input letter of $u_j$, puts the state $\hat{\delta}_2(q_2, u'_j)$ for each consecutive prefix $u'_j$ of $u_j$ in the sets $Y_j, \ldots, Y_n$. Hence, $u_j \in R$, and since $p_j \in F_1$, it must be that $wx_j^{-1} \in L(M_1)$. □

Hence, $LR \in \text{DCM}_{\text{NE}}(k, l)$. □

As a corollary, we get that DCM$(1, l)$ is closed under right concatenation with regular languages. This corollary could also be inferred from the proof that deterministic context-free languages are closed under concatenation with regular languages [19].

**Corollary 8.** Let $L \in \text{DCM}(1, l)$ and $R \in \text{REG}$. Then $LR \in \text{DCM}(1, l)$.

**Corollary 9.** If $L \in \text{DCM}(1, l)$, then $\text{pref}^{-1}(L) \in \text{DCM}(1, l)$.

### 4. Relating (Un)Decidable Properties to Non-closure Properties

In this section, we use a technique that proves non-closure properties using (un)decidable properties. A similar technique was used in [25] for showing that there is a language accepted by a 1-reversal DPDA that cannot be accepted by any NCM. In particular, we use this technique to prove that some languages are not in both DCM and 2DCM$(1)$ (i.e., accepted by two-way DFAs with one reversal-bounded counter). Since 2DCM$(1)$s have two-way input and a reversal-bounded counter, it does not seem easy to derive “pumping” lemmas for these machines. 2DCM$(1)$s are quite powerful, e.g., although the Parikh map of the language accepted by any finite-crossing 2NCM (hence by any NCM) is semilinear [2], 2DCM$(1)$s can accept non-semilinear languages. For example, $L_1 = \{a^ib^k \mid i, k \geq 2, i \text{ divides } k\}$ can be accepted by a 2DCM$(1)$ whose counter makes only one reversal. This technique is used to establish that the inverse infix, inverse suffix, and inverse outfix closure of a language in DCM$(1, 1)$ can be outside of both DCM and 2DCM$(1)$. It is also used to show that the inverse prefix closure of a DCM$(2, 1)$ language can be outside of both DCM and 2DCM$(1)$.

We will need the following result (the proof for DCM is in [2]; the proof for 2DCM$(1)$ is in [26]):

**Proposition 10.**

1. The class of languages DCM is closed under Boolean operations. Moreover, the emptiness problem is decidable.

2. The class of languages 2DCM$(1)$ is closed under Boolean operations. Moreover, the emptiness problem is decidable.

We note that the emptiness problem for 2DCM$(2)$, even when restricted to machines accepting only letter-bounded languages (i.e., subsets of $a_1^* \cdot \cdots \cdot a_k^*$ for some $k \geq 1$ and distinct symbols $a_1, \ldots, a_k$) is undecidable [2].

We will show that there is a language $L \in \text{DCM}(1, 1)$ such that $\text{inf}^{-1}(L)$ is not in $\text{DCM} \cup \text{2DCM}(1)$. 13
The proof uses the fact that there is a recursively enumerable language $L_{re} \subseteq \mathbb{N}_0$ that is not recursive (i.e., not decidable) which is accepted by a deterministic 2-counter machine \cite{27}. Thus, the machine when started with $n \in \mathbb{N}_0$ in the first counter and zero in the second counter, eventually halts (i.e., accepts $n \in L_{re}$).

A close look at the constructions in \cite{27} of the 2-counter machine, where initially one counter has some value $d_1$ and the other counter is zero, reveals that the counters behave in a regular pattern. The 2-counter machine operates in phases in the following way. The machine’s operation can be divided into phases, where each phase starts with one of the counters equal to some positive integer $d_i$ and the other counter equal to 0. During the phase, the positive counter decreases, while the other counter increases. The phase ends with the first counter having value 0 and the other counter having value $d_{i+1}$. Then in the next phase the modes of the counters are interchanged. Thus, a sequence of configurations corresponding to the phases will be of the form:

$$(q_1, d_1, 0), (q_2, 0, d_2), (q_3, d_3, 0), (q_4, 0, d_4), (q_5, d_5, 0), (q_6, 0, d_6), \ldots$$

where the $q_i$’s are states, with $q_1 = q_e$ (the initial state), and $d_1, d_2, d_3, \ldots$ are positive integers. Note that in going from state $q_i$ in phase $i$ to state $q_{i+1}$ in phase $i + 1$, the 2-counter machine goes through intermediate states. Note that the second component of the configuration refers to the value of $c_1$ (first counter), while the third component refers to the value of $c_2$ (second counter).

For each $i$, there are 5 cases for the value of $d_{i+1}$ in terms of $d_i$: $d_{i+1} = d_i, 2d_i, 3d_i, d_i/2, d_i/3$. (The division operation is done only if the number is divisible by 2 or 3, respectively.) The case is determined by $q_i$. Thus, we can define a mapping $h$ such if $q_i$ is the state at the start of phase $i$, $d_{i+1} = h(q_i)d_i$ (where $h(q_i)$ is either 1, 2, 3, 1/2, 1/3).

Let $T$ be a 2-counter machine accepting a recursively enumerable set $L_{re}$ that is not recursive. We assume that $q_1 = q_e$ is the initial state, which is never re-entered, and if $T$ halts, it does so in a unique state $q_0$. Let $T$’s state set be $Q$, and 1 be a new symbol.

In what follows, $\alpha$ is any sequence of the form $#I_1#I_2# \cdots #I_{2m}#$ (thus we assume that the length is even), where $I_q = q^k$ for some $q \in Q$ and $k \geq 1$, represents a possible configuration of $T$ at the beginning of phase $i$, where $q$ is the state and $k$ is the value of counter $c_1$ (resp., $c_2$) if $i$ is odd (resp., even).

Define $L_0$ to be the set of all strings $\alpha$ such that

1. $\alpha = #I_1#I_2# \cdots #I_{2m}#$;
2. $m \geq 1$;
3. for $1 \leq j \leq 2m - 1$, $I_j \Rightarrow I_{j+1}$, i.e., if $T$ begins in configuration $I_j$, then after one phase, $T$ is in configuration $I_{j+1}$ (i.e., $I_{j+1}$ is a valid successor of $I_j$);

**Lemma 11.** $L_0$ is not in DCM $\cup$ 2DCM(1).

**Proof.** Suppose $L_0$ is accepted by a DCM (resp., 2DCM(1)). The following is an algorithm to decide, given any $n$, whether $n$ is in $L_{re}$.
1. Let $R = \#q_01^n((\#Q_1^1\#Q_1^1))^{m}\#q_01^n\#$. Then $R$ is regular.

2. Then $L' = L_0 \cap R$ is also in DCM (resp., $2\text{DCM}(1)$) by Proposition 10.

3. Check if $L'$ is empty. This is possible, since emptiness of DCM (respectively, $2\text{DCM}(1)$) is decidable by Proposition 10.

The claim follows, since $L'$ is empty if and only if $n$ is not in $L_{\text{re}}$. □

4.1. Non-closure Under Inverse Infix

**Theorem 12.** There is a language $L \in \text{DCM}(1, 1)$ such that $\inf^{-1}(L) = \Sigma^*\Sigma^*$ is not in $\text{DCM} \cup 2\text{DCM}(1)$.

**Proof.** Let $T$ be a 2-counter machine. Let $L = \{\#q_1^m\#p_1^n\# \mid \text{in } T, q_1^m \Rightarrow p_1^n\}$. That is, $L$ contains all pairs of configurations of $T$ where, when starting in state $q$ with $m$ and zero on one counter and $n$ on the other, at the next phase, $T$ does not reach state $p$ with the first counter empty, and $n$ in the second counter. Thus, $L = \{\#I\#I'\# \mid I$ and $I'$ are configurations of $T$, and $I'$ is not a valid successor of $I\}$. Since $T$ is a deterministic counter machine, that, within one phase, only decreases one counter while increasing another, $L \in \text{DCM}(1, 1)$ since the input tape of the $\text{DCM}(1, 1)$ machine can be used to simulate the decreasing counter (by reading the first configuration) while using the counter to simulate the increasing counter, then verifying that the configuration reached does not match the second input configuration.

We claim that $L_1 = \inf^{-1}(L)$ is not in $\text{DCM} \cup 2\text{DCM}(1)$. Otherwise, by Proposition 10 $L_1$ (the complement of $L_1$) is also in $\text{DCM} \cup 2\text{DCM}(1)$, and $L_1 \cap (\#Q_1^1\#Q_1^1)^+\# = L_0$ would be in $\text{DCM} \cup 2\text{DCM}(1)$. This contradicts Lemma 11. □

4.2. Non-closure Under Inverse Prefix

**Theorem 13.** There exists a language $L$ such that $L \in \text{DCM}(2, 1)$ and $L \in 2\text{DCM}(1)$ (accepted by a two-way machine that makes one turn on the input tape and the counter is 1-reversal-bounded) such that $\text{pref}^{-1}(L) = \Sigma^* \notin \text{DCM} \cup 2\text{DCM}(1)$.

**Proof.** Consider $L = \{\#w\# \mid w \in \{a, b, \#\}^*, |w|_a \neq |w|_b\}$. Then $L \in \text{DCM}(2, 1)$, as a machine can be built that records the number of $a$’s and $b$’s in two counters, and then once it hits the end-marker, subtracts both in parallel to verify that they are different (it can also be accepted by a $2\text{DCM}(1)$ machine that records the number of $a$’s, then makes a turn on the input and verifies that the number of $b$’s is different).

Suppose to the contrary that $\text{pref}^{-1}(L) \in \text{DCM} \cup 2\text{DCM}(1)$. Then, $L' \in \text{DCM} \cup 2\text{DCM}(1)$, where $L' = \text{pref}^{-1}(L) \cap (\#(a, b, \#)^*\#) = \{\#w_1 \cdots w_n\# \mid \exists i. |w_1 \cdots w_i|_a \neq |w_1 \cdots w_i|_b\}$.

Let $L'' = L_1 \cap (\#a^m b^n\#)^+\#$. It follows that $L''$ is in $\text{DCM}$ and $2\text{DCM}(1)$ since both are closed under complement and intersection with regular languages. Then $L'' \in \text{DCM} \cup 2\text{DCM}(1)$. Further, $L'' = \{\#a^m b^n\# \cdots a^k b^n\# \mid m > 0\}$.

We will show that $L''$ is not in $\text{DCM} \cup 2\text{DCM}(1)$, which will lead to a contradiction. Define two languages:
• \( L_1 = \{ \# ^{1^k} \#^{1^k} \# \cdots \#^{1^k} \#^{1^k} \mid m \geq 1, k_i \geq 1 \} \).

• \( L_2 = \{ \# ^{1^k} \#^{1^k} \# \cdots \#^{1^k} \#^{1^k} \mid m \geq 1, k_i \geq 1 \} \).

Note that \( L_1 \) and \( L_2 \) are similar. In \( L_1 \), the odd-even pairs of blocks 1’s are the same, but in \( L_2 \), the even-odd pairs of blocks of 1’s are the same. If \( M'' \) accepts \( L'' \) in \( \text{DCM} \cup \text{2DCM}(1) \), then it is possible to construct (from \( M'' \)) \( M_1 \) and \( M_2 \) in \( \text{DCM} \cup \text{2DCM}(1) \) to accept \( L_1 \) and \( L_2 \), respectively.

We now refer to the language \( L_0 \) that was shown not to be in \( \text{DCM} \cup \text{2DCM}(1) \) in Lemma 11. We will construct a DCM (resp., \( \text{2DCM}(1) \)) to accept \( L_0 \), which would be a contradiction. Define the languages:

• \( L_{\text{odd}} = \{ \# ^{1^i} \#^{1^i} \# \cdots \#^{1^i} \#^{1^i} \mid m \geq 1, I_1, \cdots, I_{2m} \text{ are configurations of the 2-counter machine } T, \text{ for odd } i, I_{i+1} \text{ is a valid successor of } I_i \} \).

• \( L_{\text{even}} = \{ \# ^{1^i} \#^{1^i} \# \cdots \#^{1^i} \#^{1^i} \mid m \geq 1, I_1, \cdots, I_{2m} \text{ are configurations of the 2-counter machine } T, \text{ for even } i, I_{i+1} \text{ is a valid successor of } I_i \} \).

Then \( L_0 = L_{\text{odd}} \cap L_{\text{even}} \). Since \( \text{DCM} \) (resp., \( \text{2DCM}(1) \)) is closed under intersection, we need only to construct two DCMs (resp., \( \text{2DCM}(1) \)’s) \( M_{\text{odd}} \) and \( M_{\text{even}} \) accepting \( L_{\text{odd}} \) and \( L_{\text{even}} \), respectively. We will only describe the construction of \( M_{\text{odd}} \), the construction of \( M_{\text{even}} \) being similar.

**Case:** Suppose \( L'' \in \text{DCM} \):

First consider the case of DCM. We will construct two machines: a DCM \( A \) and a DFA \( B \) such that \( L(M_{\text{odd}}) = L(A) \cap L(B) \).

Let \( L_A = \{ \# ^{1^i} \#^{1^i} \# \cdots \#^{1^i} \#^{1^i} \mid m \geq 1, I_1, \cdots, I_{2m} \text{ are configurations of the 2-counter machine } T, \text{ for odd } i, I_{i+1} = h(q_i)d_i \} \). We can construct a DCM \( A \) to accept \( L_A \) by simulating the DCM \( M_1 \). For example, suppose \( h(q_i) = 3 \). Then \( A \) simulates \( M_1 \) but whenever \( M_1 \) moves its input head one cell, \( A \) moves its input head 3 cells. If \( h(q_i) = 1/2 \), then when \( M_1 \) moves its head 2 cells, \( A \) moves its input head 1 cell. (Note that \( A \) does not use the 2-counter machine \( T \).)

Now Let \( L_B = \{ \# ^{1^i} \#^{1^i} \# \cdots \#^{1^i} \#^{1^i} \mid m \geq 1, I_1, \cdots, I_{2m} \text{ are configurations of the 2-counter machine, for odd } i, I_i = h_i, \text{ then } T \text{ in configuration } I_i \text{ ends phase } i \text{ in state } q_{i+1} \} \). Then, a DFA \( B \) can accept \( L_B \) by simulating \( T \) for each odd \( i \) starting in state \( q_i \) on \( 1^d \) without using a counter, and checking that the phase ends in state \( q_{i+1} \). (Note that the DCM \( A \) already checks the “correctness” of \( d_{i+1} \).)

We can then construct from \( A \) and \( B \) a DFA \( M_{\text{odd}} \) such that \( L(M_{\text{odd}}) = L(A) \cap L(B) \). In a similar way, we can construct \( M_{\text{even}} \).

**Case:** Suppose \( L'' \in 2\text{DCM}(1) \):

The case \( 2\text{DCM}(1) \) can be shown similarly. For this case, the machines \( M_{\text{odd}} \) and \( M_{\text{even}} \) are \( 2\text{DCM}(1) \)’s, and machine \( A \) is a \( 2\text{DCM}(1) \), but machine \( B \) is still a DFA.

\[ \square \]
The language $L$ in the proof above can be accepted by a DCM(2, 1) machine that uses the end-marker. However, we see next that this language $L$ cannot be accepted by any DCM NE machine.

**Corollary 14.** There are languages in DCM(2, 1) that are not in DCM NE.

**Proof.** Consider the language $L$ from the proof of Theorem 13. This theorem shows $L \in DCM(2, 1)$, but that $L \Sigma^* \notin DCM$, which therefore implies $L \Sigma^* \notin DCM_{NE}$. Suppose, by contradiction that $L \in DCM_{NE}$. But, DCM NE is closed under concatenation with $\Sigma^*$ by Theorem 7, and therefore $L \Sigma^* \in DCM_{NE}$, a contradiction. $\square$

Hence, the right end-marker is necessary for deterministic counter machines when there are at least two 1-reversal-bounded counters. In fact, without it, no amount of reversal-bounded counters with a deterministic machine could accept even some languages that can be accepted with two 1-reversal-bounded counters could with the end-marker.

Furthermore, if $L$ is a DCM language, then $LS$ ($S$ a new symbol) is in DCM NE. Therefore, if DCM NE were closed under right quotient with a single symbol, then DCM would be equal to DCM NE which is not true. Thus, the following result is obtained.

**Corollary 15.** DCM NE is not closed under right quotient with a single symbol.

This is in contrast to DCM which is closed under right quotient with context-free languages [13], but requires the end-marker for this proof, and therefore the end-marker cannot be removed.

4.3. Non-closure for Inverse Suffix, Outfix and Embedding

**Theorem 16.** There exists a language $L \in DCM(1, 1)$ such that $\text{suffix}^{-1}(L) \notin DCM$ and $\text{suffix}^{-1}(L) \notin 2DCM(1)$.

**Proof.** Let $L$ be as in Theorem 12. We know DCM(1, 1) is closed under $\text{pref}^{-1}$ by Corollary 9 so $\text{pref}^{-1}(L) \in DCM(1, 1)$. Suppose $\text{suffix}^{-1}(\text{pref}^{-1}(L)) \in DCM$. This implies that $\text{inf}^{-1}(L) \in DCM$, but we showed this language was not in DCM. Thus we have a contradiction. A similar contradiction can be reached if we assume $\text{suffix}^{-1}(\text{pref}^{-1}(L)) \in 2DCM(1)$. $\square$

**Corollary 17.** There exists $L \in DCM(1, 1)$ and regular language $R$ such that $RL \notin DCM$ and $RL \notin 2DCM(1)$.

This implies that without the prefix-free condition on $L_1$ in Theorem 5, concatenation closure does not follow.

**Corollary 18.** There exists $L_1 \in DCM_{NE}(0, 0)$ (regular), and $L_2 \in DCM(1, 1)$, where $L_1L_2 \notin DCM$ and $L_1L_2 \notin 2DCM(1)$.

The result also holds for inverse outfix.

**Theorem 19.** There exists a language $L \in DCM(1, 1)$, $L \subseteq \Sigma^*$ such that $\text{outf}^{-1}(L) \notin DCM$ and $\text{outf}^{-1}(L) \notin 2DCM(1)$, where $\text{outf}^{-1}(L) \subseteq (\Sigma \cup \{\$\})^*$.
Proof. Consider \( L \subseteq \Sigma^* \) where \( L \in \text{DCM}(1, 1) \), and \( \text{suff}^{-1}(L) \notin \text{DCM} \) and \( \text{suff}^{-1}(L) \notin 2\text{DCM}(1) \). The existence of such a language is guaranteed by Theorem 16. Let \( \Gamma = \Sigma \cup \{\%\} \).

Suppose \( \text{outf}^{-1}(L) \in \text{DCM} \) over \( \Gamma^* \). Then \( L' \in \text{DCM} \), where \( L' = \text{outf}^{-1}(L) \cap \%\Sigma^* \). We can see \( L' = \{xy \mid x \in L, y \in \Sigma^*\} \), since the language we intersected with ensures that the section is always added to the beginning of a word in \( L \).

However, we also have \( \%^{-1}L' \in \text{DCM} \) because \( \text{DCM} \) is closed under left quotient with a fixed word (this can be seen by simulating a machine on that fixed word before reading any input letter). We can see \( \%^{-1}L' = \{yx \mid x \in L, y \in \Sigma^*\} \). This is just \( \%^{-1}(L) \), so \( \%^{-1}(L) \notin \text{DCM} \), a contradiction.

The result is the same for \( 2\text{DCM}(1) \), relying on the closure of the family under left quotient with a fixed word, which can be shown by simulating the symbol to be removed on the left input end-marker. \( \square \)

Corollary 20. Let \( m \in \mathbb{N} \). There exists a language \( L \in \text{DCM}(1, 1), L \subseteq \Sigma^* \) such that \( \text{emb}^{-1}(m, L) \notin \text{DCM} \) and \( \text{emb}^{-1}(m, L) \notin 2\text{DCM}(1) \), where \( \text{emb}^{-1}(m, L) \subseteq (\Sigma \cup \{\#, \%\})^* \).

Proof. Consider \( L \) as in Theorem 19 above, and let \( \Gamma = \Sigma \cup \{\#, \%\} \). Let \( \text{emb}^{-1}(m, L) \) over \( \Gamma^* \). Then

\[
\text{emb}^{-1}(m, \#^m L) \cap (\#^m \% \Sigma^*) = \{(\#^m \% y)x \mid x \in L, y \in \Sigma^*\},
\]

since this enforces that all \( m \)-embedded words are of the form \( \%\# \) except the \( m' \)th, which may also insert an arbitrary \( y \in \Sigma^* \) before \( x \in L \). The rest proceeds just like Theorem 19. \( \square \)

5. Inverse Transducers

This section studies transducers with reversal-bounded counters and other stores attached. Using the inverse of such transducers allows for creating elaborate methods of insertion (such as in Example 1 below). It is shown that \( \text{DCM} \) is closed under inverse deterministic reversal-bounded multicont ACTer transductions, and \( \text{NCM} \) is closed under inverse nondeterministic reversal-bounded multicont ACTer transductions, and they are both the smallest family of languages where this holds. Hence, this demonstrates a method of defining insertion operations under which \( \text{DCM} \) is closed (in contrast to the insertion methods of Section 4).

Definition 21. A \( k \)-counter transducer \( A = (k, Q, \Sigma, \Gamma, \prec, \delta, q_0, F) \) where \( Q, \Sigma, \Gamma, \prec, q_0, F \) are respectively the sets of states, input alphabet, output alphabet, right end-marker (not in \( \Sigma \cup \Gamma \)), initial state \( q_0 \in Q \), and set of final states \( F \subseteq Q \). The transition function is a partial function from \( Q \times (\Sigma \cup \{\prec\}) \times \{0, 1\}^k \times \Gamma^* \) into the family of subsets of \( Q \times (R, S) \times \{-1, 0, +1\}^k \times \Gamma^* \). \( M \) is deterministic if every element mapped by \( \delta \) is to a subset with one element in it, and if \( \delta(F \times \{\prec\}) \times \{0, 1\}^k = \emptyset \) to prevent multiple outputs from the same input on deterministic transducers. A configuration of \( A \) is of the form \( (q, w, c_1, \ldots, c_k, z) \), where \( q \in Q \) is the current state, \( w \in \Sigma^* \) is the remaining input, \( c_1, \ldots, c_k \in \mathbb{N}_0 \) are the counter contents, and \( z \in \Gamma^* \) is the accumulated output. Then, \( (q, aw, c_1, \ldots, c_k, z) \) \( r_A \) \( (p, w', c_1 + d_1, \ldots, c_k + d_k, z') \), \( a \in \Sigma \cup \{\prec\}, aw, w' \in \Sigma^* \), where \( (p, d, d_1, \ldots, d_k, x) \in \delta(q, a, (\pi(c_1), \ldots, \pi(c_k))), z' = zx, (d = S \Rightarrow aw = w') \).
and \((d = R \Rightarrow w = w')\). Then \(\vdash_A^*\) is the reflexive-transitive closure of \(\vdash_A\). In the definition above, if there are no counters, then \(k\) and the counter contents are left off of the definitions.

Let \(L \subseteq \Sigma^*\), and let \(A = (k, Q, \Sigma, \Gamma, s, \delta, q_0, F)\) be a \(k\)-counter transducer. Then

\[
A(L) = \{x \mid (q_0, w\epsilon, 0, \ldots, 0, A) \vdash_A^* (q_f, \epsilon, c_1, \ldots, c_k, x), w \in L, q_f \in F\}.
\]

Let \(L \subseteq \Gamma^*\). Then

\[
A^{-1}(L) = \{w \mid (q_0, w\epsilon, 0, \ldots, 0, A) \vdash_A^* (q_f, \epsilon, c_1, \ldots, c_k, x), x \in L, q_f \in F\}.
\]

Also, \(A\) is \(l\)-reversal-bounded if all counters are \(l\)-reversal-bounded on input \(\Sigma^*\).

From this definition, the following closure property can be obtained.

**Lemma 22.** DCM is closed under inverse deterministic reversal-bounded counter transductions, and NCM is closed under inverse reversal-bounded counter transductions.

**Proof.** Let \(M = (k, Q, \Gamma, s, q_0, F)\) be a \(k\)-counter \(l\)-reversal-bounded DCM. Let \(A = (k_A, Q_A, \Sigma, s, \epsilon, \delta_A, q_{A0}, F_A)\) be a deterministic \(l_A\)-reversal-bounded \(k_A\)-counter transducer.

Then we construct a \(\max\{l, l_A\}\)-reversal-bounded DCM machine \(M' = (k + k_A, Q', \Sigma, \epsilon, \delta', q_{A0}', F')\) accepting \(A^{-1}(L(M))\) as follows: \(M'\) takes as input a word \(a_1 \cdots a_n \in \Sigma^*\), \(a_i \in \Sigma\) followed by the end marker \(\epsilon\). In the states of \(Q'\), \(M'\) keeps a buffer of at most length \(a = \max\{|x| \mid (p, d, d_1, \ldots, d_k, x) \in \delta_A(q, a, i_1, \ldots, i_k)\} + 1\). Then on each input letter, \(a_i\), \(M'\) simulates one transition of \(A\) on \(a_i\), and stores the (deterministically calculated) output in the buffer, while using the first \(k_A\) counters. If the buffer becomes non-empty, \(M'\) simulates \(M\) on the buffer and the remaining \(k\) counters. Once the buffer becomes empty again, \(M'\) continues the simulation of \(A\) (on \(a_i\) if the transition of \(A\) applied last was a stay transition, and on \(a_{i+1}\) if it was a right transition). If \(M'\) reaches the end-marker of \(A\), and \(A\) is in a final state, then \(M'\) puts the end-marker \(\epsilon\) at the end of the output buffer. If this occurs, then \(M'\) continues simulating \(M\) on the buffer, accepting if it reaches a state of \(F\) with only \(\epsilon\) in the buffer.

The proof is similar for NCM.

This same proof technique can be generalized to other models where stores can be combined without increasing the capacity. But even when, for example, combining two arbitrary (non-reversal-bounded counters) counters, such machines already have the full power of Turing machines.

From this, we can immediately get a relatively simple characterization of DCM and NCM languages.

**Theorem 23.** \(L\) is in DCM (NCM respectively) if and only if there is a deterministic (nondeterministic) reversal-bounded counter transducer \(A\) such that \(L = A^{-1}(\{A\})\). Hence, DCM (NCM respectively) is the smallest family of languages containing \(\{A\}\) that is closed under inverse deterministic (nondeterministic) reversal-bounded counter transductions.
Prove. Let \( M \) be a DCM machine which, without loss of generality, does not have any transitions defined on a final state and the end-marker (these can be removed without changing the language accepted). Let \( A \) be the reversal-bounded multicontroller transducer that is obtained from \( M \) (same states, transitions, and final states), but outputs \( \lambda \) on every transition. Then \( A \) is deterministic and \( A^{-1}(\{\lambda\}) = \{w \mid w \in L(M)\} \). Similarly for NCM.

A brief example will be given next showing how such a transducer can define an insertion into a DCM language.

**Example 1.** Consider \( L = \{a^nb^n \mid n \geq 0\} \in \text{DCM}(1, 1) \). Then define a transducer \( A \) with one counter that on input \( a \), on input \( b \) outputs \( b \), and on inputs \( c \) and \( d \) outputs \( \lambda \), while verifying that all \( c \)'s occur before any \( d \)'s and that they have the same number of occurrences. Then \( A^{-1}(L) = \{w \mid w \text{ consists of } a^n b^n \text{ shuffled with } c^m d^m, n, m \geq 0\} \). Thus, \( A^{-1} \) can “shuffle” words with the same number of \( c \)'s and \( d \)'s. Alternatively, the same language could be obtained from \( \{\lambda\} \) using the inverse of a transducer \( A \) with two counters that checks that the number of \( a \)'s is the same as the number of \( b \)'s and that all \( a \)'s occur before any \( b \)'s, and similarly with \( c \)'s and \( d \)'s.

In the same way that we attached reversal-bounded counters to transducers, we will briefly consider attaching a single (unrestricted) counter, and also pushdowns. The following shows that Lemma 22 and Theorem 23 do not generalize for acceptors and transducers with an unrestricted counter or with a 1-reversal pushdown.

**Theorem 24.**  
1. There is a language \( L \) accepted by a deterministic one counter automaton (i.e., a DFA with one unrestricted counter) and a deterministic one-counter transducer (i.e. a deterministic one-counter automaton with outputs) \( A \) such that \( A^{-1}(L) \) is not in NPCM.

2. There is a language \( L \) accepted by a 1-reversal deterministic pushdown automata and a deterministic 1-reversal pushdown transducer (i.e., a 1-reversal deterministic pushdown with outputs) \( A \) such that \( A^{-1}(L) \) is not in NPCM.

**Proof.** For Part 1, let \( L = \{a^i # a^i # a^i # \cdots a^i # \mid k \geq 2 \text{ is even}, i_1 = 1, i_{j+1} = i_j + 1 \text{ for odd } j\} \). This language can be accepted by a deterministic one-counter automaton.

Construct a deterministic counter transducer \( A \) which, on input \( w \), outputs \( w \), and accepts if the following holds:

1. \( w \) is of the form \( (a^i #)^k \) for some even \( k \geq 2 \). (The finite-state control can check this.)

2. In \( w \), \( i_{j+1} = i_j + 1 \) for even \( j \). (This needs an unrestricted counter.)

Then \( A^{-1}(L) = \{a^i # a^i # a^i # \cdots a^i # \mid k \geq 2 \text{ is even}, i_1 = 1, i_{j+1} = i_j + 1 \text{ for all } j, 1 \leq j < k\} \). However, the Parikh map of \( A^{-1}(L) \) is not semilinear [1].

For Part 2, let \( L = \{a^i # a^i # a^i # \cdots a^{i_{2^j-1}} # a^{i_{2^j-1}} # a^{i_{2^j-1}} # \cdots a^{i_k} # \mid k \geq 1, i_1 = 1, i_{j+1} = i_j + 1 \text{ for odd } j\} \). Then \( L \) can be accepted by a 1-reversal deterministic pushdown automaton.

We construct a deterministic 1-reversal pushdown transducer \( A \) which, on input \( w \), outputs \( w \), and accepts if the following holds:
1. $w$ is of the form $(a^+)^m$S$(a^+)^n$ for some even $m, n \geq 1$ (The finite-state control can check this.)

2. In $w$, $i_{j+1} = i_j + 1$ for even $j$. (This needs a 1-reversal stack).

Then $A^{-1}(L) = \{a_i^i#a_i^i#\cdots a_i^i#\mid k \geq 1, i_1 = 1, i_{j+1} = i_j + 1 \text{ for all } j\}$, which is not semilinear.

However, we have:

**Theorem 25.** If $L$ is in $\text{DCM (NMc)}$ and $A$ is a deterministic (nondeterministic) transducer with a pushdown and reversal-bounded counters, or $L$ is in $\text{DPCM (NPCM)}$ and $A$ is a deterministic (nondeterministic) reversal-bounded counter transducer, then $A^{-1}(L)$ is in $\text{DPCM (NPCM)}$.

**Proof.** Similar to the proof of Lemma 22.

### 6. Summary of Results

This section summarizes insertion closure properties demonstrated in this paper. For one-way machines, all closure properties, both for $\text{DCM}(k, l)$ and $\text{DCM}$ are summarized in Table 1. Also, for two-way machines with one reversal-bounded counter, $\text{2DCM}(1)$, the results are summarized as follows:

- There exists $L \in \text{DCM}(1, 1)$ (one-way), s.t. $\text{suff}^{-1}(L) \notin \text{2DCM}(1)$ (Theorem 16).
- There exists $L \in \text{DCM}(1, 1)$ (one-way), $R$ regular, s.t. $RL \notin \text{2DCM}(1)$ (Corollary 17).
- There exists $L \in \text{DCM}(1, 1)$ (one-way), s.t. $\text{out}^{-1}(L) \notin \text{2DCM}(1)$ (Theorem 19).
- There exists $L \in \text{DCM}(1, 1)$ (one-way), s.t. $\text{inf}^{-1}(L) \notin \text{2DCM}(1)$ (Theorem 12).
- There exists $L \in \text{2DCM}(1)$, 1 input turn, 1 counter reversal, s.t. $\text{pref}^{-1}(L) \notin \text{2DCM}(1)$ (Theorem 13).
- There exists $L \in \text{2DCM}(1)$, 1 input turn, 1 counter reversal, $R$ regular, s.t. $LR \notin \text{2DCM}(1)$ (Theorem 13).

This resolves every open question summarized above, optimally, in terms of the number of counters, reversals on counters, and reversals on the input tape. Also, it was shown that the right input end-marker is necessary for $\text{DCM}$, and that $\text{DCM}$ is closed under inverse deterministic reversal-bounded multicontroller transducers that can define natural insertion operations.

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