Some remarks on sets with small quotient set

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Abstract. We prove that for any finite set $A \subset \mathbb{R}$ with $|A/A| \ll |A|$ we have $|A - A| \gg |A|^{5/3 - o(1)}$. We also show that for such sets $|A + A + A| \gg |A|^{2 - o(1)}$.

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§ 1. Introduction

Let $A, B \subset \mathbb{R}$ be finite sets. Define the sum set, the product set and the quotient set of $A$ and $B$ as

$$A + B := \{a + b: a \in A, b \in B\},$$
$$AB := \{ab: a \in A, b \in B\},$$

and

$$A/B := \{a/b: a \in A, b \in B, b \neq 0\},$$

respectively. Sometimes we write $kA$ for $k$-fold sumsets, for example, $A + A + A = 3A$. Having nonzero $\alpha \in \mathbb{R}$ we put

$$\alpha \cdot A := \{\alpha a: a \in A\}.$$

The Erdős-Szemerédi conjecture [5] says that for any $\varepsilon > 0$ one has

$$\max \{|A + A|, |AA|\} \gg |A|^{2-\varepsilon}. \quad (1.1)$$

Modern bounds concerning the conjecture can be found in [20], [9] and [10].

The first interesting case of Conjecture (1.1) was proved in [4] (see also [20]), namely that

$$|A + A| \ll |A| \text{ or } |A - A| \ll |A| \implies |AA| \gg |A|^{2-\varepsilon} \text{ and } |A/A| \gg |A|^{2-\varepsilon}.$$

The opposite case, when the operations $\times$ and $+$ interchange in the implication (it is sometimes called a weak Erdős-Szemerédi Conjecture; [13]), is wide open. Thus it is unknown whether the following statement holds:

$$|AA| \ll |A| \text{ or } |A/A| \ll |A| \implies |A + A| \gg |A|^{2-\varepsilon} \text{ or } |A - A| \gg |A|^{2-\varepsilon}. \quad (1.2)$$

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The best current lower bounds on the size of a subset of a set $A$ with small $AA$ or $A/A$ can be found in [9] and [10]. As for difference sets, it was proved in [7] (see also [19]) that

$$|AA| \ll |A| \quad \text{or} \quad |A/A| \ll |A| \implies |A - A| \gg |A|^{8/5 - \varepsilon}.$$  

The case when $A$ consists of integers was considered in [2] (where the case of lower bounds for multifold sumsets $kA$ was treated as well).

Let us formulate the first main result of our paper (see Theorem 6 below).

**Theorem 1.** Let $A \subset \mathbb{R}$ be a finite set. Then

$$|A/A| \ll |A| \implies |A - A| \gg |A|^{5/3 - \varepsilon}$$

holds for each $\varepsilon > 0$.

Our arguments use some ideas from the method of higher energies, see [14], and has some intersections with [19]. The main new ingredient is a ‘several projections’ argument (see the proof of Theorem 6).

Our second main result shows that Conjecture (1.2) holds if one considers other combinations involving $A$, for example, $A + A + A$ or $A + A - A$ (see Theorem 7).

**Theorem 2.** Let $A \subset \mathbb{R}$ be a finite set, and $|AA| \ll |A|$ or $|A/A| \ll |A|$. Then for any $\alpha, \beta \neq 0$ one has

$$|A + \alpha \cdot A + \beta \cdot A| \gg \frac{|A|^2}{\log^3 |A|}.$$

Theorem 2 is an analogue of Theorem 1 from [17] and is proved by a similar method. We also study various properties of sets with small product/quotient set; see §5.

The best results for multiple sumsets $kA$, $k \to \infty$, of sets $A$ with small product/quotient set can be found in [1] (see also our remarks in §5).

**§ 2. Notation**

Let $G$ be an abelian group. In this paper we use the same letter to denote a set $S \subseteq G$ and its characteristic function $S: G \to \{0, 1\}$. By $|S|$ we denote the cardinality of $S$.

Let $f, g: G \to \mathbb{C}$ be two functions. Put

$$(f \ast g)(x) := \sum_{y \in G} f(y)g(x - y) \quad \text{and} \quad (f \circ g)(x) := \sum_{y \in G} f(y)g(y + x).$$

Denote by $E^+(A, B)$ the **additive energy** of two sets $A, B \subseteq G$ (see, for example, [22]), that is,

$$E^+(A, B) = |\{a_1 + b_1 = a_2 + b_2: a_1, a_2 \in A, b_1, b_2 \in B\}|.$$

If $A = B$ we simply write $E^+(A)$ instead of $E^+(A, A)$. Clearly,

$$E^+(A, B) = \sum_x (A \ast B)(x)^2 = \sum_x (A \circ B)(x)^2 = \sum_x (A \circ A)(x)(B \circ B)(x).$$
Note also that
\[ E^+(A, B) \leq \min\{|A|^2|B|, |B|^2|A|, |A|^{3/2}|B|^{3/2}\}. \] (2.2)

More generally (see [14]), for \( k \geq 2 \) put
\[ E_k^+(A) = \{|a_1 - a'_1 = a_2 - a'_2 = \cdots = a_k - a'_k : a_i, a'_i \in A\}|. \]

Thus, \( E^+(A) = E_2^+(A) \).

In the same way define the multiplicative energy of two sets \( A, B \subseteq \mathbb{G} \)
\[ E^\times(A, B) = \{|a_1b_1 = a_2b_2 : a_1, a_2 \in A, b_1, b_2 \in B\}|, \]
and, similarly, \( E_k^\times(A) \). Certainly, the multiplicative energy \( E^\times(A, B) \) can be expressed in terms of multiplicative convolution as in (2.1). We often use the notation
\[ A_\lambda = A_\lambda^\times = A \cap (\lambda^{-1}A), \]
for any \( \lambda \in A/A \). Hence
\[ E^\times(A) = \sum_{\lambda \in A/A} |A_\lambda|^2. \]

For a given integer \( k \geq 2 \), a fixed vector \( \bar{\lambda} = (\lambda_1, \ldots, \lambda_{k-1}) \) and a set \( A \) put
\[ \Delta_{\bar{\lambda}}(A) = \{(\lambda_1a, \lambda_2a, \ldots, \lambda_{k-1}a, a) : a \in A\} \subseteq \mathbb{G}^k. \]

All the logarithms are base 2. The symbols ‘\( \ll \)’ and ‘\( \gg \)’ are the usual Vinogradov symbols, thus \( a \ll b \) means that \( a = O(b) \) and \( a \gg b \) means that \( b = O(a) \). Having a fixed set \( A \), we write \( a \lesssim b \) or \( b \gtrsim a \) if \( a = O(b \cdot \log^c |A|) \), with an absolute constant \( c > 0 \). For any given prime \( p \) denote by \( \mathbb{F}_p \) the finite prime field and put \( \mathbb{F}_p^* = \mathbb{F}_p \setminus \{0\} \).

§ 3. Preliminaries

Again, let \( \mathbb{G} = (\mathbb{G}, +) \) be an abelian group with group operation ‘+’. We begin with the famous Plünnecke-Ruzsa inequality (see [22], for instance).

Lemma 1. Let \( A, B \subseteq \mathbb{G} \) be two finite sets, \( |A + B| \leq K|A| \). Then for all positive integers \( n \) and \( m \) the following holds:
\[ |nB - mB| \leq K^{n+m}|A|. \] (3.1)

Further, for any \( 0 < \delta < 1 \) there is \( X \subseteq A \) such that \( |X| \geq (1 - \delta)|A| \) and for any positive integer \( k \) one has
\[ |X + kB| \leq (K/\delta)^k|X|. \] (3.2)

We need a simple lemma.

Lemma 2. Let \( A \subseteq \mathbb{R} \) be a finite set. Then there is \( z \) such that
\[ \sum_{x \in zA} |zA \cap x(zA)| \gg \frac{E^\times(A)}{|A| \log |A|}. \] (3.3)
Proof. Without loss of generality one can assume that \(0 \notin A\). We have
\[
E^x(A) = \sum_x |A \cap xA|^2 \leq 2 \sum_{x : |A \cap xA| > E^x(A)/(2|A|^2)} |A \cap xA|^2.
\]
Thus, putting \(\Delta = E^x(A)/(2|A|^2)\) and setting \(P_j\) equal to
\[
P_j = \{x : \Delta 2^{-j-1} < |A \cap xA| \leq \Delta 2^j\},
\]
for some \(j_0\) we get that
\[
\sigma_{j_0} = \sum_{x \in P_{j_0}} |A \cap xA| \gg \frac{E^x(A)}{2^{j_0} \Delta \log |A|}.
\tag{3.4}
\]
Putting \(\sigma = \sigma_{j_0}\) and \(P = P_{j_0}\), let \(A' = \{x \in A : |P \cap x^{-1}A| \geq (2|A|)^{-1} \Delta |P|\}\). Because \(P = P^{-1}\), we have
\[
\sigma = \sum_{x \in P} |A \cap xA| = \sum_{x \in A} |P \cap xA^{-1}| = \sum_{x \in A} |P \cap x^{-1}A|.
\]
It follows that there is \(x \in A\) with \(|P \cap x^{-1}A| \gg \sigma/|A|\). Put \(W = x(P \cap x^{-1}A') \subseteq A' \subseteq A\) and note that
\[
\frac{E^x(A)}{|A| \log |A|} \ll \sigma \Delta 2^{j_0} |A|^{-1} \ll |W| \Delta 2^{j_0} \ll \sum_{y \in x^{-1}W} |A \cap yA| \ll \sum_{y \in x^{-1}A} |x^{-1}A \cap y(x^{-1}A)|,
\]
as required.

Lemma 2 is proved.

Remark 1. An alternative way to obtain a result similar to the one contained in Lemma 2 is to use a small generalization of Exercise 1.1.8 from [22] (which can be obtained using the probabilistic method, say). This exercise asserts that for any sets \(A\) and \(B\) there exists a set \(X \subseteq A + B - B\),
\[
|X| \ll \frac{|A + B - B|}{|B|} \cdot \log |A + B|,
\]
such that \(A + B \subseteq X + B\). This approach gives the same number of logarithms in bound (3.3) but a worse dependence on the parameter \(M := |A/A|/|A|\) in the final estimate.

The method of the paper relies on the famous Szemerédi-Trotter Theorem (see [21] and also [22]). Let us recall the definitions.

We call a set \(\mathcal{L}\) of continuous plane curves a pseudo-line system if each curve in \(\mathcal{L}\) is determined by two points. Define the number of incidences \(I(\mathcal{P}, \mathcal{L})\) between points and pseudo-lines as \(I(\mathcal{P}, \mathcal{L}) = |\{(p, l) \in \mathcal{P} \times \mathcal{L} : p \in l\}|\).
Theorem 3. Let $\mathcal{P}$ be a set of points and let $\mathcal{L}$ be a pseudo-line system. Then
$$I(\mathcal{P}, \mathcal{L}) \ll |\mathcal{P}|^{2/3}|\mathcal{L}|^{2/3} + |\mathcal{P}| + |\mathcal{L}|.$$  

A simple consequence of Theorem 3 was obtained in [16]; see Lemma 7.

Lemma 3. Let $A \subset \mathbb{R}$ be a finite set. Put
$$M(A) := \min_{B \neq \emptyset} \frac{|AB|^2}{|A||B|}.$$  

Then we have
$$E_+^+(A) \ll M(A)|A|^3 \log |A|.$$  

We also need a result from [11]. Let $T(A)$ be the number of collinear triples in $A \times A$.

Theorem 4. Let $A \subset \mathbb{R}$ be a finite set. Then
$$T(A) \ll |A|^4 \log |A|.$$  

More generally, for three finite sets $A, B, C \subset \mathbb{R}$ let $T(A, B, C)$ be the number of collinear triples in $A \times A$, $B \times B$ and $C \times C$, respectively. Clearly, the quantity $T(A, B, C)$ is symmetric in all its variables. Further, it is easy to see that
$$T(A, B, C) = \left| \left\{ \frac{c_1 - a_1}{b_1 - a_1} = \frac{c_2 - a_2}{b_2 - a_2} : a_1, a_2 \in A, b_1, b_2 \in B, c_1, c_2 \in C \right\} \right| + 2|A \cap B \cap C||A||B||C|,$$
$$T(A, B, B) = \sum_{a_1, a_2 \in A} E_+^+(B - a_1, B - a_2).$$

Corollary 1. Let $A, B \subset \mathbb{R}$ be two finite sets, $|B| \leq |A|$. Then
$$T(A, B, B) \ll |A|^2|B|^2 \log |B|$$
and for any finite $A_1, A_2 \subset \mathbb{R}$, $|B| \leq |A_1|, |A_2|$, we have
$$T(A_1, A_2, B) \ll |A_1|^2|A_2|^2 \log |B|.$$  

Proof. Split $A$ into $t \ll |A|/|B|$ parts $B_j$ of size at most $|B|$. Then, using Theorem 4, we get
$$T(B_i \times B_j, B, B) \ll T(B_i \cup B_j \cup B) \ll |B|^4 \log |B|$$
and hence
$$T(A, B, B) \ll \sum_{i, j=1}^t T(B_i \times B_j, B, B) \ll t^2|B|^4 \log |B| \ll |A|^2|B|^2 \log |B|,$$
as required. The second bound follows similarly. This completes the proof of Corollary 1.

We need a result from [12], which is a consequence of the main theorem from [13].

Theorem 5. Let $A, B, C \subseteq \mathbb{F}_p$, and let $M = \max(|A|, |BC|)$. Suppose that $|A||B||BC| \ll p^2$. Then
$$E_+^+(A, C) \ll (|A||BC|)^{3/2}|B|^{-1/2} + M|A||BC||B|^{-1}.$$
§ 4. The proof of the main result

Now let us obtain a lower bound for the difference set of a set with a small quotient set. Let us explain the main idea of the proof.

Suppose that there is a family of finite sets $A_j \subset \mathbb{R}^d$, $j = 1, \ldots, n$, $d > 1$, and we want to obtain a lower bound on the size of $\bigcup_{j=1}^n A_j$ which is better than the trivial bound $\max_j |A_j|$. Let us assume the contrary and the first simple model situation, when we must show that the equality $A_1 = \cdots = A_n$ is impossible. Suppose that for any $j$ there is a map (projection) $\pi_j$ associated with the set $A_j$. We should think about the $\pi_j$ as ‘distinct’ maps somehow. If one is able to prove that $\bigcup_{j=1}^n \pi_j(A_j)$ has a strictly larger cardinal number than $\max_j |\pi_j(A_j)|$ then $A_1 = \cdots = A_n$ cannot be the case and hence $\bigcup_{j=1}^n A_j$ should be large. We will follow this scheme in our argument below and will construct appropriate sets and their projections.

Theorem 6. Let $A \subset \mathbb{R}$ be a finite set. Then

$$|A - A|^6 |A/A|^{13} \gtrsim |A|^{23}. \quad (4.1)$$

In particular, if $|A/A| \ll |A|$ then $|A - A| \gtrsim |A|^{5/3}$.

Proof. Let $\Pi = A/A$. Put $M$ equal to $|\Pi|/|A|$. Without loss of generality one can assume that $0 \notin A$. Let $D = A - A$. Also let $\mathcal{P} = D \times D$. Then for any $\lambda \in \Pi$ one has

$$Q_{\lambda} := A \times A - A - \Delta_{\lambda}(A_{\lambda}) \subseteq \mathcal{P}.$$ 

Further, for an arbitrary $\lambda \in \Pi$ consider a projection $\pi_{\lambda}(x, y) = x - \lambda y$. Then, it is easy to check that $\pi_{\lambda}(Q_{\lambda}) \subseteq D$. In other words, if we denote by $\mathcal{L}_{\lambda}$ the set of all lines of the form $\{(x, y) : x - \lambda y = c\}$ intersecting the set $Q_{\lambda}$, we obtain that $|\mathcal{L}_{\lambda}| \leq |D|$. Finally, take any set $\Lambda \subseteq \Pi$, $\Lambda = \Lambda^{-1}$, and put $\mathcal{L} = \bigsqcup_{\lambda \in \Lambda} \mathcal{L}_{\lambda}$. It follows that

$$|\mathcal{L}| = \sum_{\lambda \in \Lambda} |\mathcal{L}_{\lambda}| \leq |D| |\Lambda|. \quad (4.2)$$

By construction the number of incidences $\mathcal{I}(\mathcal{P}, \mathcal{L})$ between points $\mathcal{P}$ and lines $\mathcal{L}$ is at least $\mathcal{I}(\mathcal{P}, \mathcal{L}) \geq \sum_{\lambda \in \Lambda} |Q_{\lambda}|$. Applying Theorem 3, using formula (4.2), and making simple calculations we get

$$\sum_{\lambda \in \Lambda} |Q_{\lambda}| \leq \mathcal{I}(\mathcal{P}, \mathcal{L}) \ll (|D| |\mathcal{P}|)^{2/3} + |\mathcal{L}| + |\mathcal{P}| \ll |D|^2 |\Lambda|^{2/3}. \quad (4.3)$$

Hence, our task is to find a good lower bound on the sum $\sum_{\lambda \in \Lambda} |Q_{\lambda}|$. For any $\lambda \in \Pi$ we have

$$|A| |A_{\lambda}|^2 = \sum_{x, y} \sum_z A_{\lambda}(z) A(\lambda z + x) A_{\lambda}(z + y) = \sum_{(x, y) \in Q_{\lambda}} \sum_z A_{\lambda}(z) A(\lambda z + x) A_{\lambda}(z + y)$$

and, thus, by the Cauchy-Schwarz inequality

$$|A| |A_{\lambda}|^2 \leq |Q_{\lambda}|^{1/2} \cdot \left( \sum_{x, y} \left( \sum_z A_{\lambda}(z) A(\lambda z + x) A_{\lambda}(z + y) \right)^2 \right)^{1/2}. \quad (4.4)$$
Summing over $\lambda \in \Lambda$, making the change of variables $z' - z = w$ and applying the Cauchy-Schwarz inequality one more time we obtain

$$|A|^2(E^x_\Lambda(A))^2 := |A| \left( \sum_{\lambda \in \Lambda} |A_\lambda|^2 \right)^2$$

$$\leq \sum_{\lambda \in \Lambda} |Q_\lambda| \cdot \sum_{\lambda \in \Lambda} \sum_{x,y,z} \left( \sum_{\lambda} A_\lambda(z)A(\lambda z + x)A_\lambda(z + y) \right)^2$$

$$= \sum_{\lambda \in \Lambda} |Q_\lambda| \cdot \sum_{\lambda \in \Lambda} \sum_{x,y,z,z'} A_\lambda(z)A(\lambda z + x)A_\lambda(z + y)A_\lambda(z' + x)A_\lambda(z' + y)$$

$$\leq \sum_{\lambda \in \Lambda} |Q_\lambda| \cdot \sum_{\lambda \in \Lambda} \sum_{w} (A_\lambda \circ A_\lambda)^2(w)(A \circ A)(\lambda w) = \sigma_1 \cdot \sigma_2.$$

Let us estimate the sum $\sigma_2$. Putting $\tilde{A_\lambda} = A \cap \lambda A$ we see that by the Hölder inequality

$$\sigma_2 = \sum_{\lambda \in \Lambda} \sum_{w} (A_\lambda \circ A_\lambda)^2 \left( \frac{w}{\lambda} \right)(A \circ A)(w) = \sum_{\lambda \in \Lambda} \sum_{w} (\bar{A_\lambda} \circ \tilde{A_\lambda})^2(w)(A \circ A)(w)$$

$$\leq (E^+_3(A))^{1/3} \cdot \sum_{\lambda \in \Lambda} (E^+_3(\tilde{A_\lambda}))^{2/3}.$$

We need to estimate the last sum. Although $\tilde{A_\lambda} \subseteq A$ yields the rough bound $E^+_3(\tilde{A_\lambda}) \leq E^+_3(A)$, it is not enough for our purposes. Take $\Lambda \subseteq \Pi$ such that for some $1 \leq j \lesssim 1$ one has

$$\Lambda = \Lambda_j = \{\lambda : 2^j < |\tilde{A_\lambda}| \leq 2^{j+1}\}$$

and

$$\frac{|A|^3}{M} \leq E^x(A) \lesssim E^x_\Lambda(A).$$

The first bound in (4.5) is just the Cauchy-Schwarz inequality (2.2) and the existence of the set $\Lambda$ follows from simple pigeonholing and the formula

$$\frac{|A|^3}{M} \leq E^x(A) \leq 2 \sum_{\lambda : |\tilde{A_\lambda}| \geq |A|/M} |\tilde{A_\lambda}|^2 = 2 \sum_{j=1}^{[\log M]} \sum_{\lambda \in \Lambda_j} |\tilde{A_\lambda}|^2.$$
Since $\tilde{A}_\lambda \subseteq A$ it is easy to see that

$$M(\tilde{A}_\lambda) \leq \frac{|A\tilde{A}_\lambda|^2}{|A||\tilde{A}_\lambda|} = \frac{|AA|^2}{|A||\tilde{A}_\lambda|} \leq \frac{M^2|A|}{|\tilde{A}_\lambda|} \leq M^3$$  \hspace{1cm} (4.6)

and hence

$$\sigma_2 \leq \frac{M^8/3}{|A|} \cdot \sum_{\lambda \in \Lambda} |\tilde{A}_\lambda|^2 = \frac{M^8/3}{|A|} \cdot \sum_{\lambda \in \Lambda} |A_\lambda|^2 = \frac{M^8/3}{|A|} \cdot E^\wedge_8(A).$$

Returning to (4.4) and using the Cauchy-Schwarz inequality, we get

$$\sum_{\lambda \in \Lambda} |Q_\lambda| \gtrsim \frac{|A|^4}{M^{11/3}}.$$  

Combining the last bound with (4.3) we obtain

$$\frac{|A|^{12}}{M^{11}} \lesssim |D|^6|A|^2 \leq M^2 |A|^2 |D|^6,$$

as required.

Theorem 6 is proved.

**Remark 2.** A careful analysis of the proof (for example, one should use the estimate $M(\tilde{A}_\lambda) \leq M^2|A|/|\tilde{A}_\lambda|$ from (4.6)) shows that we have obtained an upper bound on the higher energy $E^\wedge_8(A)$. Namely,

$$|A|^7 E^\wedge_8(A) \lesssim |A/A|^6 |A - A|^6.$$  \hspace{1cm} (4.7)

The last bound is always better than Elekes’ inequality for quotient sets [3]

$$|A|^5 \ll |A/A|^2 |A \pm A|^2.$$  

Now let us prove our second main result, which corresponds to the main theorem in [17].

**Theorem 7.** Let $A \subseteq \mathbb{R}$ be a finite set, and $|AA| \leq M|A|$ or $|A/A| \leq M|A|$. Then for any $\alpha \neq 0$,

$$E^\wedge(A + \alpha) \ll M^4 |A|^2 \log |A|.$$ \hspace{1cm} (4.7)

In particular,

$$|AA + A + A| \geq |(A + 1)(A + 1)| \gg \frac{|A|^2}{M^4 \log |A|}.$$  \hspace{1cm} (4.8)

Finally, for any $\alpha, \beta \neq 0$ the following holds,

$$|A + \alpha \cdot A + \beta \cdot A| \gg \frac{|A|^2}{M^6 \log^3 |A|}.$$ \hspace{1cm} (4.9)
Proof. Without loss of generality one can assume that \(0 \notin A\). Let \(\Pi = AA\) and \(Q = A/A\). Applying the second estimate of Corollary 1 with \(B = -\alpha \cdot A\) and \(A_1 = A_2 = \Pi\) as well as (3.7), we get

\[
\sum_{a, a' \in A} E^\times (\Pi + \alpha a, \Pi + \alpha a') \ll M^4 |A|^4 \log |A|.
\]

Thus there are \(a, a' \in A\) such that

\[
E^{\times} (\Pi + \alpha a, \Pi + \alpha a') \ll M^4 |A|^2 \log |A|.
\]

In other words,

\[
E^\times (\Pi/a + \alpha, \Pi/a' + \alpha) = E^\times (\Pi + \alpha a, \Pi + \alpha a') \ll M^4 |A|^2 \log |A|.
\]  (4.10)

Clearly, \(A \subseteq \Pi/a\) and \(A \subseteq \Pi/a'\) and hence \(E^\times (A + \alpha) \ll M^4 |A|^2 \log |A|\). To obtain the same estimate with \(Q\) just note that for any \(a \in A\) one has \(A \subseteq Qa\) and apply the same arguments with \(B = -\alpha \cdot A^{-1}\). Further, by estimate (4.7) with \(\alpha = 1\) and bound (2.2), we have

\[
|AA + A + A| = |AA + A + A + 1| \geq (A + 1)(A + 1) \gg \frac{|A|^2}{M^4 \log |A|}
\]

and (4.8) follows.

It remains to prove (4.9). Using Lemma 2 we find \(z\) such that

\[
\frac{|A|^2}{M \log |A|} \ll \sum_{\lambda \in zA} |(zA) \cap \lambda^{-1}(zA)|.
\]

With some abuse of notation redefine \(A\) to be \(zA\); thus we have

\[
\frac{|A|^2}{M \log |A|} \ll \sum_{\lambda \in A} |A \cap \lambda^{-1}A| = \sum_{\lambda \in A} |A_\lambda|.
\]  (4.11)

Further, we want to use the previous arguments but in the asymmetric situation. To do this apply the second estimate of Corollary 1 with \(A_1 = \alpha \cdot A^{-1}\), \(A_2 = \beta \cdot A^{-1}\) and \(B = Q\). We get

\[
\sum_{a, a' \in A} E^\times (Q + \alpha/a, Q + \beta/a') \ll M^4 |A|^4 \log |A|
\]  (4.12)

and, similarly,

\[
\sum_{a, a' \in A} E^\times (\Pi + \alpha a, \Pi + \beta a') \ll M^4 |A|^4 \log |A|.
\]  (4.13)

We consider the case of the set \(Q\); the second situation is similar. From (4.12) we see that there are \(a, a' \in A\) such that

\[
\sigma := \left\{|(q_1 a + \alpha)(q_1 a' + \beta) = (q_2 a + \alpha)(q_2 a' + \beta) : q_1, q_1', q_2, q_2' \in Q\right\}|
\]

\[
= E^\times (Q + \alpha/a, Q + \beta/a') \ll M^4 |A|^2 \log |A|.
\]
Using the inclusion $A \subseteq Qa$, $a \in A$, once more time it is easy to check that

$$
\sigma \geq \left| \{(a_1 + \alpha)(a'_1 + \beta) = (a_2 + \alpha)(a'_2 + \beta) : a_1, a_2 \in A, a'_1 \in A_{a_1}, a'_2 \in A_{a_2} \} \right|
$$

$$
= \sum_x n^2(x),
$$

where

$$
n(x) = \left| \{(a_1 + \alpha)(a'_1 + \beta) = x : a_1, a'_1 \in A_{a_1} \} \right|.
$$

Clearly, the support of the function $n(x)$ is contained in $A + \alpha \cdot A + \beta \cdot A + \alpha \beta$ because by the definition of the set $A_{a_1}$ we have $a_1 A_{a_1} \subseteq A$. Using the Cauchy-Schwarz inequality and bound (4.11) we obtain

$$
\frac{|A|^4}{M^2 \log^2 |A|} \ll \left( \sum_{\lambda \in A} |A_{\lambda}| \right)^2 = \left( \sum_x n(x) \right)^2 \leq |A + \alpha \cdot A + \beta \cdot A| \cdot \sum_x n^2(x)
$$

$$
\leq |A + \alpha \cdot A + \beta \cdot A| \cdot \sigma \ll |A + \alpha \cdot A + \beta \cdot A| \cdot M^4 |A|^2 \log |A|,
$$

as required.

Theorem 7 is proved.

§ 5. Appendix

Now let us make some further remarks on sets with small quotient/product set. First of all we say a few words about multiple sumsets $kA$ of sets $A$ with small multiplicative doubling. As was noted before, when $k$ tends to infinity the best results in this direction were obtained in [1]. For small $k > 3$ other methods work. We follow the arguments from [8] with some modifications.

Suppose that $A \subset \mathbf{G}$ is a finite set, where $\mathbf{G}$ is an abelian group with group operation ‘$\times$’. Let $\|A\|_{\mathcal{U}^k}$ be the Gowers non-normalized $k$th-norm [6] of the characteristic function of $A$ (in multiplicative form); see, for instance, [15]:

$$
\|A\|_{\mathcal{U}^k} = \sum_{x_0, x_1, \ldots, x_k} \prod_{\omega \in \{0,1\}^k} A \left( x_0 \prod_{j=1}^k x_j^{\omega_j} \right),
$$

where $\omega = (\omega_1, \ldots, \omega_k)$. For example,

$$
\|A\|_{\mathcal{U}^2} = \sum_{x_0, x_1, x_2} A(x_0)A(x_0 x_1)A(x_0 x_2)A(x_0 x_1 x_2) = E^\times(A)
$$

is the multiplicative energy of $A$ and

$$
\|A\|_{\mathcal{U}^3} = \sum_{\lambda \in A/A} E^\times(A_{\lambda}).
$$

Moreover, the induction property holds for Gowers norms (see [6]):

$$
\|A\|_{\mathcal{U}^{k+1}} = \sum_{\lambda \in A/A} \|A_{\lambda}\|_{\mathcal{U}^k}. \quad (5.1)
$$
It was proved in [6] that the $k$th-norms of the characteristic function of any set are connected with each other. In [15] the author shows that for the non-normalized norms the connection does not depend on the size of $G$. Here we formulate a particular case of Proposition 35 from [15], which relates $\|A\|_{\mathcal{U}^k}$ and $\|A\|_{\mathcal{U}^2}$ (see Remark 36 in [15]).

Lemma 4. Let $A$ be a finite subset of an abelian group $G$ with group operation ‘$\times$’. Then for any integer $k \geq 1$

$$\|A\|_{\mathcal{U}^k} \geq \mathbb{E}^\times(A)^{2^{k-1}-k-1}|A|^{-(3 \cdot 2^{k-4} - 4)}.$$ 

Now let us prove a lower bound for $|kA|$, where $A$ has a small product/quotient set. The obtained estimate gives us a connection between the size of sumsets of a set and the Gowers norms of its characteristic function.

Proposition 1. Let $A \subset \mathbb{R}$ be a finite set and $k$ be a positive integer. Then

$$|2^k A|^2 \gg_k \|A\|_{\mathcal{U}^{k+1}} \cdot \log^{-k} |A|.$$ 

Proof. We follow the arguments from [8] and use induction. The case $k = 1$ was obtained in [20], so assume that $k > 1$. Put $L = \log |A|$.

Without loss of generality one can assume that $0 \notin A$. Taking any subset $S = \{s_1 < s_2 < \cdots < s_r\}$ of $A/A$, we have by the main argument of [8]

$$|2^k A|^2 \geq \sum_{j=1}^{r-1} |2^{k-1} A_{s_j} | |2^{k-1} A_{s_{j+1}}|.$$ 

(5.3)

Now let $S$ be a subset of $A/A$ such that

$$\sum_{s \in S} |2^{k-1} A_s|^2 \gg_k L^{-1} \sum_s |2^{k-1} A_s|^2$$

and for any two numbers $s$ and $s'$ the quantities $|2^{k-1} A_s|$ and $|2^{k-1} A_{s'}|$ differ by a multiplicative factor of at most 2 on $S$. Clearly, such $S$ exists by the pigeonhole principle. Further, put $\Delta = \min_{s \in S} |2^{k-1} A_s|$. Then plugging the set $S$ into (5.3), we get

$$|2^k A|^2 \gg_k \Delta \sum_{s \in S} |2^{k-1} A_s| \gg_k L^{-1} \sum_s |2^{k-1} A_s|^2.$$ 

Now from the induction hypothesis and formula (5.1) we see that

$$|2^k A|^2 \gg_k L^{-k} \sum_s \|A_s\|_{\mathcal{U}^k} = L^{-k} \|A\|_{\mathcal{U}^{k+1}}.$$ 

This completes the proof of Proposition 1.

Proposition 1 above has an immediate consequence.

Corollary 2. Let $A \subset \mathbb{R}$ be a finite set and $k$ be a positive integer. Also let $M \geq 1$ and

$$|AA| \leq M|A| \quad \text{or} \quad |A/A| \leq M|A|.$$ 

(5.4)
Then
\[ |2^k A| \gg_k |A|^{1+k/2} M^{-u_k} \cdot \log^{-k/2} |A|, \]  
where
\[ u_k = 2^k - k/2 - 1. \]

**Proof.** Combining Proposition 1 and Lemma 4 we obtain
\[ |2^k A|^2 \gg_k \log^{-k} |A| \cdot E^\times(A)^{2^{k+1} - k/2} |A|^{-(3 \cdot 2^{k+1} - 4k - 8)}. \]  
By assumption (5.4) and the Cauchy-Schwarz inequality (2.2), we get
\[ E^\times(A) \gg |A|^3/M. \]  
Substituting the last bound into (5.6), we have
\[ |2^k A|^2 \gg_k \log^{-k} |A| \cdot |A|^{k+2} M^{-(2^{k+1} - k/2)}, \]  
as required.

Corollary 2 is proved.

Thus, for \( |AA| \ll |A| \) or \( |A/A| \ll |A| \) we have, in particular, that \( |4A| \gg |A|^2 \). Actually, a stronger bound holds. We thank S. V. Konyagin for pointing out this fact to us.

**Corollary 3.** Let \( A \subset \mathbb{R} \) be a finite set with \( |A/A| \ll |A| \). Then
\[ |4A| \gg |A|^{2+c}, \]  
where \( c > 0 \) is an absolute constant.

**Proof.** Without loss of generality one can assume that \( 0 \notin A \). We use the arguments and the notation of the proof of Proposition 1. By formula (5.3) we have
\[ |4A|^2 \geq \sum_{j=1}^{r-1} |A_{s_j} + A_{s_j}| |A_{s_j+1} + A_{s_j+1}|. \]  
By Theorem 11 from [16], for any finite \( B \subset \mathbb{R} \) one has \( |B + B| \gg_{M(B)} |B|^{3/2+c} \), where \( c > 0 \) is an absolute constant. Choose our set \( S \) such that \( \sum_{s \in S} |A_s|^{3+2c} \gg \sum_s |A_s|^{3+2c} \) and for any two numbers \( s \) and \( s' \) the quantities \( |A_s| \) and \( |A_{s'}| \) differ at most by a factor of 2 on \( S \). Clearly, such \( S \) exists by the pigeonhole principle. Further, put \( \Delta = \min_{s \in S} |A_s| \). By the Hölder inequality and our assumption \( |A/A| \ll |A| \) one has \( \sum_s |A_s|^{3+2c} \gg |A|^{4+2c} \) and hence \( \Delta \gg |A| \). It follows that \( M(A_s) \ll 1 \) for any \( s \in S \) (see the definition of the quantity \( M(A_s) \) in (3.5)). Applying Theorem 11 from [16] to the sets \( A_{s_j} \) and combining with (5.7) and the previous calculations we obtain
\[ |4A|^2 \gg \sum_{s \in S} |A_s|^{3+2c} \gg \sum_s |A_s|^{3+2c} \gg |A|^{4+2c}. \]  
This completes the proof of Corollary 3.
The proof of our last proposition of this paper uses the same idea as the arguments of Theorem 7 and improves the symmetric case of Lemma 33 from [18] for small $M$. The result is simple but it shows that for any set with small $|AA|$ or $|A/A|$ there is a ‘coset’ splitting, similar to multiplicative subgroups in $\mathbb{F}_p^*$.

**Proposition 2.** Let $p$ be a prime number and $A \subseteq \mathbb{F}_p$ be a set, $|AA| \ll p^{2/3}$. Put $|AA| = M|A|$. Then

$$\max_{x \neq 0} |A \cap (A + x)| \ll M^{9/4}|A|^{3/4}. \quad (5.8)$$

If $|A/A| = M|A|$ and $M^4|A|^3 \ll p^2$ then

$$\max_{x \neq 0} |A \cap (A + x)| \ll M^3|A|^{3/4}. \quad (5.9)$$

**Proof.** Without loss of generality one can assume that $0 \not\in A$. Let $\Pi = AA$ and $Q = A/A$. First of all, let us prove (5.8). It is easy to see that for any $x \in \mathbb{F}_p$ the following holds:

$$(A \circ A)(x) \leq (\Pi \circ \Pi)(ax) \quad \text{for all } a \in A. \quad (5.10)$$

Hence

$$(A \circ A)^2(x) \leq |A|^{-1} \sum_{a \in A} (\Pi \circ \Pi)^2(x/a) \leq |A|^{-1} \sum_{a} (\Pi \circ \Pi)^2(a) = |A|^{-1} E^+(\Pi). \quad (5.11)$$

By Lemma 1 there is $A' \subseteq A$, $|A'| \geq |A|/2$, such that $|A'\Pi| \ll M^2|A|$. In particular, $|\Pi| |A'| |A'\Pi| \ll M^3|A|^3 \ll p^2$. Using Theorem 5 with $A = C = \Pi$ and $B = A'$ we get

$$E^+(\Pi) \ll M^{9/2}|A|^{5/2}. \quad$$

Combining the last bound with (5.11) we obtain (5.8).

To prove (5.9) note that the following analogue of formula (5.10) holds:

$$(A \circ A)(x) \leq (Q \circ Q)(x/a) \quad \text{for all } a \in A, \quad (5.12)$$

and hence we can apply the previous arguments. In this situation, by formula (3.1) of Lemma 1 one has $|QA| \leq M^3|A|$ and thus Theorem 5 with $A = C = Q$ and $B = A$ gives us

$$|A| \cdot \max_{x \neq 0} (A \circ A)^2(x) \leq E^+(Q) \ll M^6|A|^{5/2}. \quad$$

This completes the proof of Proposition 2.

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