SCALING ASYMPTOTICS FOR LADDER SEQUENCES OF SPHERICAL HARMONICS AT CAUSTIC LATITUDES

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ABSTRACT. We study the concentration of ladder sequences of spherical harmonics on caustic latitude circles. We prove that they have Airy scaling asymptotics. We also determine the weak* limit of certain empirical measures of $L^2$ norms of restrictions of spherical harmonics to these latitude circles.

1. INTRODUCTION

Consider the round sphere $(S^2, g_{can})$ with standard polar coordinates $(\phi, \theta) \in (0, \pi) \times (0, 2\pi)$ where $\theta$ is the polar angle measured relative to fixed meridian geodesic. We study scaling asymptotics of certain sequences of the standard $L^2$ normalized spherical harmonics,

$$Y_N^m(\phi, \theta) = \sqrt{\frac{2N + 1}{4\pi}} \frac{(N - m)!}{(N + m)!} P_N^m(\cos \phi) e^{im\theta},$$

(1.1)

in which $m_k, N_k \to \infty$ while the ratio $c = m_k/(N_k + \frac{1}{2})$ is held fixed. These are called rational ladder sequences and have the special feature that they are semi-classical Lagrangian distributions which concentrate on Lagrangian tori $T_c \subset S^*S^2$. The torus $T_c$ projects to an annular band around the equator $\gamma_e = \{\phi = \frac{\pi}{2}\}$ and the projection has a fold singularity over the bounding latitude circles, $\gamma_c^\pm$ determined by $\sin \phi = c$. We obtain scaling asymptotics in an $(N + \frac{1}{2})^{-\frac{1}{3}}$ neighborhood around the caustic latitude circles.

Theorem 1.1. There exists an $\varepsilon > 0$ such that if $x = (\phi, \theta)$ with $c < \sin \phi < c + \varepsilon$ then, letting $h_k = (N_k + \frac{1}{2})^{-1}$, there are a sequence of smooth half densities $u_{i,j}$ on $S^2$ such that

$$Y_{N_k}^{m_k}(x) \sqrt{dV_g(x)} \sim Ai \left( -h_k^{-\frac{4}{3}} \rho(x) \right) \sum_{n=0}^{\infty} u_{0,n}(x) h_k^{-\frac{1}{6} + n}$$

$$+ Ai' \left( h_k^{-\frac{4}{3}} \rho(x) \right) \sum_{n=0}^{\infty} u_{1,n}(x) h_k^{\frac{1}{6} + n},$$

(1.2)

The argument of the Airy function and its derivative is

$$\rho(x) = \left( \frac{4}{3} \int_{\gamma_e} \alpha \right)^{\frac{2}{3}}$$

(1.3)
Here, $\gamma_x$ is the geodesic arc joining the two pre-images $\pi^{-1}(x) \in T_c$ and $\alpha$ is the canonical 1-form on $T^* S^2$. The arc is oriented so as to make the integral positive. The leading order amplitude is given by

$$u_{0,0}(x) = (2\pi)\rho(x)^{\frac{1}{4}}\pi_x d\mu_c$$

(1.4)

where $d\mu_c$ for the normalized joint flow invariant density on $T_c$.

The articles recent work of Galkowski and Toth [5],[17] obtain sharp decay estimates for joint eigenfunctions in the forbidden region $S^2 \setminus \pi(T_c)$. The scaling asymptotics we obtain are only valid on the ‘allowed region side’ of the caustic latitude, however it should be possible to extend them to two sided Airy asymptotics which agree with the decay proven by Galkowski-Toth. It should be possible to extend the Airy scaling results to joint eigenfunctions on a convex surface of revolution. One can still separate variables and conjugate the 1D Sturm-Liouville problem to a Schrödinger operator. Once a quasi-mode is constructed, the rest of the argument works the same way. Scaling asymptotics of the Legendre functions $P^m_N$ along such sequences were studied previously in [10] [12], however the ODE methods they applied did not provide explicit expressions for the quantities appearing in the expansion. We re-derive the expansion of the Legendre functions by constructing explicit quasi-modes for the Legendre operator using the well-known Maslov-WKB quantization procedure (2, 3, 19) which approximate the Legendre functions locally uniformly up to $O(h^\infty)$ error. The quasi-mode is expressible as an oscillatory integral with a degenerate critical point in the phase near the turning points. From this, the Airy expansion is obtained by putting the phase function in a cubic normal form. This idea was first developed by Chester Friedman and Ursell [1] and later by Ludwig [10],[11]. It is also discussed by Guillemin and Sternberg [9] and Hörmander [7]. The advantage of this approach is that it expresses the quantities appearing in the expansion of the Legendre functions in terms of the geometry on $S^2$.

In the previous paper [6], we studied the empirical measures

$$\mu_N(t) = \sum_{m=-\ell}^{\ell} ||\varphi^m_N||^2_{L^2(H)} \delta_0 \left( t - \frac{m}{\ell} \right)$$

(1.5)

On a convex surface of revolution $(S^2, g)$, where $H$ is the unique rotationally invariant closed geodesic. It was shown that these measures tend to a weak limit which exhibits $(1-t^2)^{-\frac{1}{2}}$ type blow-up at the end points due to the fact that the end points $t = \pm 1$ correspond to the Gaussian beam ladder sequences of joint eigenfunctions $(m/\ell = \pm 1)$ which peak on the geodesic $H$.

**Theorem 1.2.** Let $\gamma_{c_0}$ be either of the two latitude circles determined by $\sin \phi = c_0$. The normalized empirical measures

$$\mu_{N,c_0} = \frac{1}{c_0(N + \frac{1}{2})} \sum_{m=-N}^{N} ||Y^m_N||^2_{L^2(\gamma_{c_0})} \delta_0 \left( t - \frac{m}{N} \right)$$

(1.6)

Then for all $f \in C^0([-1, 1])$, 

\[
\int f(t) \, d\mu_{N,c_0}(t) = \frac{1}{c_0 \pi} \int_{-c_0}^{c_0} f(t) \left(1 - \left(\frac{t}{c_0}\right)^2\right)^{-\frac{1}{2}} \, dt.
\]  

(1.7)

This reflects the fact that the latitude \(\gamma_{c_0}\) is the caustic curve for the ladder sequence with ratio \(m/N \to c\) and these spherical harmonics are largest there, with an \((N + \frac{1}{2})^{1/6}\) peak due to the Airy bump. The measures are supported on \([-c_0, c_0]\) reflecting the fact that when \(c = m/N > c_0\), \(\gamma_c\) is outside of the projection of the Lagrangian \(T_c\) which makes the corresponding ladder sequence \(O(h^{\infty})\) on \(\gamma_c\).

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1.1. Outline. In section 2, we conjugate the associated Legendre operator to a Schrödinger operator on \(I = (0, \pi)\) and construct a global WKB quasi-mode (approximate eigenfunction) in such a way that it is a locally uniform approximation to \(P_{m,N}(\cos \phi), \phi \in (0, \pi)\) as \(m, N \to \infty\) with the ratio \(c = m/(N + \frac{1}{2})\) fixed. Section 3 contains the derivation of the Airy expansion of the quasi-mode. In section 4 we explain the relevant geometry on \(S^2\) and connect it to the expansion of the Legendre functions. Section five contains the short calculation of the weak limit of the empirical measures \((1.6)\). The rest of this section contains background on the Legendre functions and an overview of the relevant machinery of semi-classical Lagrangian distributions for those unfamiliar.

1.2. Background on Legendre functions. To establish notation and collect basic facts we quote the following classical results about the Legendre functions and refer to the standard references [12],[8] for more detail. We note that these functions are called ‘Ferrer’s functions’ or ‘Legendre functions on the cut’ by some authors. For each pair of integers \((m,N)\) with \(0 \leq m \leq N\), let \(P_{m,N}(x)\) be the following function defined for \(x \in [-1,1]\):

\[
P_{m,N}(x) := \left(\frac{1}{2}\right) \frac{(N-m)!}{N!(N+m)!} \left(1-x^2\right)^{\frac{m}{2}} \partial_x^{N+m}(x^2-1)^N.
\]

(1.8)

We refer to \(P_{m,N}^0(x)\) as the normalized Legendre function of degree \(N\) and order \(m\). They are real-valued, smooth on \((-1,1)\), and satisfy

\[
(1-x^2)\partial_x^2 P_{m,N}^0(x) - 2x \partial_x P_{m,N}^0(x) + \left(\frac{1}{2} + \frac{m^2}{1-x^2} - \frac{1}{4}\right) P_{m,N}^0(x) = 0,
\]

(1.9)

\[
\int_{-1}^{1} P_{m,N}^0(x)^2 \, dx = 1.
\]

(1.10)

Proposition 1.3. For \(m \in \mathbb{Z}_{\geq 0}\), define the (positive) Legendre operator \(L_m\),

\[
L_m := -\partial_x (1-x^2)\partial_x + \frac{m^2}{1-x^2} + \frac{1}{4}.
\]

(1.11)

As an unbounded operator on \(L^2[-1,1]\) with domain \(C^\infty_c([-1,1], dx)\), \(L_m\) has only discrete spectrum consisting of simple eigenvalues.
Spec($L_m$) = \left\{ \left( N + \frac{1}{2} \right)^2 \mid N \in \mathbb{N}, N \geq m \right\}.

Each eigenspace is the complex span of $P_N^m(x)$ and the set $\{P_N^m\}_{N=m}^{\infty}$ is an orthonormal basis of $L^2([-1, 1], dx)$.

For a proof, see [15]. The formula

$$P_{n+1}'(x) = xP_n'(x) + (n+1)P_n(x)$$

together with $P_N^0(1) = 1$ implies that for all $0 \leq m \leq N$, $P_N^m(x)$ is positive near $x = 1$. Depending on the relative of parity of $m$ and $N$, $P_N^m(x)$ is either odd or even, $P_N^m(-x) = (-1)^{m+N}P_N^m(x)$. We will use these properties to match the quasi-mode with $P_N^m$.

1.3. Background on oscillatory functions associated to Lagrangian manifolds. This section contains a review of the basic theory of oscillatory integrals which we will use in the construction of the quasi-mode in section 2.

Let $(M^n, g)$ be a Riemannian manifold. The theory reviewed here depends upon working with smooth half densities rather than functions. Fix a smooth, positive density $\nu$ on $M$. We may then identify functions with half densities via the isomorphism $f(x) \sim f(x)\sqrt{\nu}$.

Let $\Lambda \subset T^*M$ be a compact Lagrangian submanifold. In order to define the space $\mathcal{O}^\mu(M, \Lambda)$ of oscillatory half densities associated to $\Lambda$, we fix a locally finite open cover $\{U_j\}$ of $\Lambda$ such that for each $j$, there exists a phase function $\psi_j(x, \theta) \in C^\infty(V_j \times \mathbb{R}^{N_j}, \mathbb{R})$ defined on some open subsets $V_j \subset M$ which are small enough so that the maps

$$i_{\psi_j} : (x, \theta) \ni C_{\psi_j} \mapsto (x, d_x \psi_j(x, \theta))$$

are embeddings onto $U_j \subset \Lambda$. Here $C_{\psi_j}$ is the zero set of $d_\theta \psi_j$ which is assumed to be an $n$ dimensional submanifold. We further fix a partition of unity $\chi_j$ subordinate to this cover.

**Definition 1.4.** The space $\mathcal{O}^\mu(M, \Lambda)$ is the space of all half densities which can be written in the form

$$u(x, h) = \left( \sum_j (2\pi h)^{-N_j} \int_{\mathbb{R}^{N_j}} a_j(x, \theta, h) e^{i\psi_j(x, \theta)} d\theta \right) \sqrt{\nu}$$

(1.12)

$$a_j(x, \theta) \sim \sum_{n=0}^{\infty} a_{j,n}(x, \theta) h^{\mu+n}$$

(1.13)

where each $a_j(x, \theta)$ is a smooth function with compact support. We write $\mathcal{O}^\infty(M, \Lambda) = \cap_{\mu \in \mathbb{R}} \mathcal{O}^\mu(M, \Lambda)$ and when $h$ is restricted to take values in a particular sequence $h_k$, we will signify this with the notation $\mathcal{O}^\mu(M, \Lambda, h_k)$. Associated to each $u(x, h) \in \mathcal{O}^\mu(M, \Lambda)$ is a geometric object $\sigma(u)$ called its principal symbol, which is a section of a certain line bundle over $\Lambda$. To define it, we first recall that the Maslov bundle
\( \mathbb{L} \to \Lambda \) is a flat complex line bundle which can be described concretely using the choice of \( \{ U_j, \psi_j \} \). On \( U_i \cap U_j \), define the locally constant functions

\[
m_{ij}(\lambda) = \frac{1}{2} (\text{sgn} \partial^2_{\theta} \psi_j - \text{sgn} \partial^2_{\theta} \psi_i)
\]

where \( \partial^2_{\theta} \psi \) is the hessian with respect to the fiber variables. The functions \( \exp i \frac{\pi}{4} m_{ij}(\lambda) \) are the transition functions of the Maslov bundle on \( U_i \cap U_j \). The choice of phase functions determines a canonical section, \( s_j \), of \( \mathbb{L} \) by

\[
s_j(\lambda) = \exp i \frac{\pi}{4} \text{sgn} \, d^2_{\theta} \psi_j(\lambda) \quad \lambda \in U_j
\]

Let \( \Psi_j \) be the lift of \( \psi_j \) to \( U_j \) via the map \( i_{\psi_j} \) and \( \Omega^s \to \Lambda \) be the half density bundle over \( \Lambda \). Fix a smooth positive density \( \rho_0 \) on \( \Lambda \) and define the space of symbols of order \( \mu \), \( S^\mu(\Lambda) \), to be the set of all smooth sections of \( \Omega^s \otimes \mathbb{L} \to \Lambda \) which may be written in the form

\[
h^\mu \left( \sum_j \exp i \frac{\Psi_j(\lambda)}{h} f_j(\lambda) s_j(\lambda) + O(h) \right) \sqrt{\rho_0}
\]

(1.14)

where \( f_j \) are smooth functions on \( \Lambda \) with \( \text{supp} f_j(\lambda) \subset U_j \). The principal symbol map \( \sigma : \mathcal{O}^\mu(M, \Lambda) \to S^\mu(\Lambda)/S^{\mu+1}(\Lambda) \) is defined so that when \( u(x, h) \) is written in the form (1.12) then

\[
[\sigma(u)](\lambda) = h^\mu \left( \sum_j \exp i \frac{\Psi_j(\lambda)}{h} a_{j,0}(\lambda) g_j(\lambda) s_j(\lambda) \right) \sqrt{\rho_0}
\]

(1.15)

Here, the \( g_j \) are smooth functions on \( U_j \) defined by

\[
g_j \sqrt{\rho_0} = (i_{\psi_j}^{-1})^* \sqrt{dC_{\psi_j}} \quad \lambda \in U_j
\]

where \( dC_{\psi_j} \) is the canonical \( \delta \)-density on the critical set \( C_{\psi_j} \), determined by the density \( \nu \otimes |d\theta|^2 \) on \( V_j \times \mathbb{R}^{N_j} \). Next, we a map which takes a symbol to an oscillatory half density,

\[
\mathcal{Q} : S^\mu(\Lambda) \to \mathcal{O}^\mu(M, \Lambda).
\]

Suppose that \( \sigma \in S^\mu \) is written in the form (1.14) (which is always possible using \( \chi_j \)). Define \( \mathcal{Q}(\sigma) \) to be the smooth half density (1.12) with amplitudes \( a_j \) chosen so that \( (i_{\psi_j}^{-1})^* a_j g_j = f_j \). Then \( \mathcal{Q} \) is a right inverse for the principal symbol map. It depends on the choices \( (U_j, \psi_j, \chi_j) \) while the principal symbol map does not. Finally, suppose that \( P \) is an order zero semi-classical pseudo-differential operator with principal symbol \( \rho_0 \) and with sub-principal symbol equal to zero. If \( \rho_0 = 0 \) on \( \Lambda \) and \( \rho \) is a density on \( \Lambda \) invariant under the Hamiltonian flow \( t \to \exp tX_{p_0} \) of \( \rho_0 \), then for any \( u \in \mathcal{O}^\mu(M, \Lambda) \), \( Pu \in \mathcal{O}^{\mu+1}(M, \Lambda) \) and if \( u(x, h) \) has principal symbol

\[
\sigma(u) = \left( \sum_j \exp i \frac{\Psi_j(\lambda)}{h} f_j(\lambda) s_j(\lambda) \right) \sqrt{\rho}
\]

(1.16)

then the order \( \mu + 1 \) symbol of \( Pu \) is
\[ \sigma(Pu) = \left( \sum_j \exp i \frac{\Psi_j(\lambda)}{\hbar_k} \frac{2}{i} X_{\mu_j} f_j(\lambda) s_j(\lambda) \right) \sqrt{\rho} \]  

(1.17)

2. WKB FOR THE LEGENDRE OPERATOR

We begin by conjugating the Legendre operator on \([-1, 1]\) to a Schrödinger operator on \(I = (0, \pi)\). The following proposition is a straightforward calculation.

**Proposition 2.1.** Let \(U\) be the unitary map \(U : L^2((-1, 1), dx) \to L^2((0, \pi), d\phi)\)

\[ (Uf)(\phi) = f(\cos \phi) \sqrt{\sin \phi} \]

Let \(0 < c < 1\) and define the operator \(H_{h,c}\) for \(f \in C^\infty((0, \pi))\)

\[ H_{h,c} f(\phi) := -h^2 f''(\phi) + \left( \frac{\epsilon^2}{\sin^2 \phi} - \frac{h^2}{4 \sin^2 \phi} \right) f(\phi) \]  

(2.1)

Suppose that \(m(h)\) is an integer such that \(c = m(h)h \in (0, 1)\) for all \(h\). Then

\[ h^2 U L_{m(h)} U^* = H_{h,c} \]

For the remainder of this section we fix once and for all some \(c \in (0, 1)\) and a rational ladder sequence, that is, integers \(0 \leq m_k \leq N_k\) such that for all \(k\), \(m_k/(N_k + \frac{1}{2}) = c\). Putting \(h_k = (N_k + \frac{1}{2})^{-1}\), it follows from propositions [1.3 and 2.1] that the spectrum of \(H_{h_k,c}\) is

\[ \text{Spec}(H_{h_k,c}) = \left\{ \frac{h_k^2}{N + \frac{1}{2}} \mid N \geq m_k \right\}. \]

In particular, 1 is an eigenvalue of \(H_{h_k,c}\) for all \(k\). Moreover \(\ker(H_{h_k,c} - 1)\) is one dimensional and spanned by \(u_{h_k}(\phi) := UP^0_{N_k}\). It follows that there exists \(\delta > 0\) so that \(H_{h_k,c}\) has the spectral gap,

\[ \inf_{\lambda \in \text{Spec}(H_{h_k,c}) \setminus \{1\}} |1 - \lambda| \geq \delta h_k \]  

(2.2)

2.1. **Construction of a global \(h^\infty\) quasi-mode for \(H_{h_k,c}\).** We say that a smooth function \(v_h\) on \(I = (0, \pi)\) is a quasi-mode of order \(h^\infty\) for \(H_{h_k,c}\) with quasi-eigenvalue \(E(h)\) if

\[ \| (H_{h,c} - E(h)) v_h \|_{L^2(I)} = O(h^\infty) \]  

(2.3)

where \(E(h)\) has the semi-classical expansion \(E(h) \sim E_0 + \sum_{j=1}^{\infty} h^j E_j\). Let \((\phi, \tau)\) be coordinates for \(T^* \mathbb{R}, p : T^* \mathbb{R} \to \mathbb{R}\) the natural projection, and

\[ f(\phi, \tau) = \tau^2 + \frac{\epsilon^2}{\sin^2 \phi} \]

be the principal symbol of \(H_{h,c}\). The energy curve

\[ \Sigma = \{ f(\phi, \tau) = 1 \} \]  

(2.4)

is a smooth, closed curve symmetric about \(\tau \mapsto -\tau\), intersecting \(\{ \tau = 0 \}\) at \(\phi = \frac{\pi}{2} \pm \phi_0\) where \(\phi_{\pm}\) are the two solutions of \(\sin \phi = c, \phi \in (0, \pi)\). We follow the well-known procedure of WKB-Maslov quantization in order to construct a quasi-mode.
The existence of \( O \) solvability of (2.5) up to error

Proposition 2.3.

properties:

Let \( \phi \in \mathcal{O}(I, \Sigma, h_k) \) with \( \|v_{h_k}\|_{L^2(I)} = 1 \) and a sequence of real numbers \( E_j \) so that if \( E(h_k) \sim 1 + h_k^2 E_2 + h_k^4 E_3 + \cdots \), then

\[
\|(H_{h_k, c} - E(h_k)) v_{h_k}\|_{L^2(I)} = O(h_k^\infty). \tag{2.5}
\]

Moreover, for any fixed \( \phi \in I \),

\[
v_{h_k}(\phi) = \begin{cases} 
\sqrt{\frac{2 \sin \phi}{\pi}} \left( \cos \left( \frac{\sqrt{h_k} \int_{g_0} \alpha + \frac{\pi}{2}}{\sin^2 \alpha - c^2} \right) \right) + O(h_k) & N_k - m_k \text{ odd} \\
- \sqrt{\frac{2 \sin \phi}{\pi}} \left( \sin \left( \frac{\sqrt{h_k} \int_{g_0} \alpha + \frac{\pi}{2}}{\sin^2 \alpha - c^2} \right) \right) + O(h_k) & N_k - m_k \text{ even}
\end{cases}
\tag{2.6}
\]

and there exists an \( \varepsilon > 0 \) such that if \( \phi > \phi_+ - \varepsilon \), then

\[
v_{h_k}(\phi) = (2\pi h_k)^{-\frac{1}{2}} \int a(\tau, h) e^{\pi \sqrt{h_k}(\phi - G_4(\phi))} d\tau + O(h_k^\infty)_{L^2}, \tag{2.7}
\]

Where \( a(\tau, h) \sim \sum_j a_j(\tau) h^j \), \( a_0(\tau) = \frac{1}{\sqrt{\pi}} V'(G_4(\tau))^{-\frac{1}{2}} \), and \( G_4(\tau) \) satisfies \( G_4(0) = 0 \), \( (G_4(\tau), \tau) \in \Sigma \) on the support of \( a \).

The existence of \( O(h_k^\infty) \) quasi-modes is well known, see for instance [2], [3], [4]. The solvability of (2.5) up to error \( O(h_k^\infty) \) requires \( (\Sigma, h_k) \) to have the following three properties:

Proposition 2.3. Let \( \alpha = \tau d\phi|_\Sigma \), \( [m] \in H^1(\Sigma, \mathbb{Z}) \) be the Maslov class, and \( X_f \) be the Hamiltonian vector field of \( f \).

(a) For all \( k \) large enough,

\[
\frac{1}{2\pi h_k} [\alpha] - \frac{1}{4} [m] \in H^1(\Sigma, \mathbb{Z}) \tag{2.8}
\]

(b) There exists a positive density \( \rho_0 \) invariant under the flow of \( X_f \)

(c) For each smooth function \( r_0 \) on \( \Sigma \) satisfying \( \int\Sigma r_0 = 0 \), there exists a smooth function \( r_1 \) so that \( dr_1(X_f) = r_0 \).

Proof. (a) Since \( \Sigma \) is a curve, we can check this by integration. We define the Maslov class below, but to check this it suffices to know that \( \int\Sigma [m] = 2 \) when \( \Sigma \) is oriented counter-clockwise. Since the integral of \( \alpha \) is the area enclosed by \( \Sigma \) and \( \Sigma \) is symmetric across the lines \( \phi = \pi/2 \), \( \tau = 0 \), the integral is four times the area of the upper right quadrant,
\[
\int_{\Sigma} \tau \, d\phi = 4c \int_{0}^{\sqrt{1-c^2}} \frac{\tau^2 \, d\tau}{(1-\tau^2)^{1/2}} = 2\pi(1+c)
\]

Therefore

\[
\frac{1}{2\pi h_k} \int_{\Sigma} \alpha - \frac{1}{2} = N_k - m_k \in \mathbb{Z}
\]

(b) The map

\[
i : [0, \pi) \to \Sigma \quad i(t) = \exp tX_f(\phi_+, 0)
\]

is a surjective Lagrangian immersion. To see this, one only needs to note that the period of the Hamiltonian flow through \((\phi_+, 0)\) is \(\pi\). This follows from the fact the curve \(\exp \frac{t}{2}X_f(\phi_+, 0)\) can be identified with a geodesic on \(S^2\) (See section 4). The density \(\rho_0\) defined by \(i^*\rho_0 = \pi^{-1}|dt|\) is clearly positive and invariant.

(c) Pulling back under \(i\), we may assume \(r_0(t)\) is smooth on \([0, \pi)\), \(\int_{0}^{\pi} r(t) \, |dt| = 0\), and \(\lim_{t \to \pi} r_0(t) = r_0(0)\). Then the function \(r_1(t) = \int_{0}^{t} r_0(s) \, |ds|\) solves the equation.

\[
\square
\]

2.1.1. Explicit choice of phases and canonical operator. Let \(\{U_j\}_{j=1}^{4}\) be the open cover of \(\Sigma\) described as follows: pick \(0 < s < \frac{\pi}{2}\) and let \(U_1 = p^{-1}(\frac{\pi}{2} - s, \frac{\pi}{2} + s) \cap \{\tau > 0\}\) and \(U_3 = p^{-1}(\frac{\pi}{2} - s, \frac{\pi}{2} + s) \cap \{\tau < 0\}\). Then we let \(U_2 = \{(\phi, \tau) \in \Sigma \mid \phi \in (\phi_-, \frac{\pi}{2} - s + \varepsilon)\}\) and \(U_4 = \{(\phi, \tau) \in \Sigma \mid \phi \in (\frac{\pi}{2} + s - \varepsilon, \phi_+)\}\) where \(0 < \varepsilon < s\) can be arbitrary. The sets \(U_1, U_3\) are symmetric about \((\phi, \tau) \mapsto (\phi, -\tau)\) and \(U_2, U_4\) are symmetric with respect to reflection over \(\phi = \frac{\pi}{2}\).

Let \(\chi_j\) be a partition of unity subordinate to this cover so that \(\chi_1(\phi, \tau) = \chi_3(\phi, -\tau)\) and \(\chi_2(\phi, \tau) = \chi_4(\pi - \phi, \tau)\). We choose local phase functions parametrizing this open cover as follows. For \(j = 2, 4\) we put

\[
\psi_j(\phi, \tau) = \phi\tau - G_j(\tau), \quad (2.9)
\]

where \(G_j\) are chosen so that \(G_j(0) = 0\) and \((G_j'(\tau), \tau) \in \Sigma\) on the \(\tau\)-projection of \(U_j\).

For \(\phi \in p(U_1) = p(U_3)\), let

\[
\psi_1(\phi) = \int_{\gamma_0^\phi} \alpha \quad \psi_3(\phi) = \int_{-\gamma_0^\phi} \alpha = -\psi_1(\phi) \quad (2.10)
\]

Here, \(\gamma_\phi\) is the arc joining the turning point \((\phi_+, 0)\) to the point \((\phi, \tau) \in U_1\) and \(-\gamma_\phi\) is the arc joining \((\phi_+, 0)\) to \((\phi, \tau) \in U_3\). Since the lifts \(\Psi_j\) of the phases to \(\Sigma\) are primitives of \(\alpha\), they differ by a constant \(\Psi_1 - \Psi_j := C_{ij}\) on each \(U_i \cap U_j\). It is easy to see for this choice of phases that \(C_{12} = C_{23} := C\) and \(C_{34} = C_{41} = 0\). Note that this means \(\int_\Sigma \alpha = 2C\) where the integral is in the counter-clockwise direction. As described in section 1.3, this choice of phases shows that the co-cycle which defines the Maslov class \([m]\) is \(m_{21} = m_{32} = m_{43} = m_{14} = \frac{1}{2}\).
Proposition 2.4. Define constants $\beta_j$ as follows

\[
\begin{align*}
\beta_1 &= -\beta_3 = \frac{\pi}{4} & N_k - m_k \text{ odd} \\
\beta_1 &= -\beta_3 = \frac{3\pi}{4} & N_k - m_k \text{ even}
\end{align*}
\]

\[
\begin{align*}
\beta_2 &= \beta_4 = 0 & N_k - m_k \text{ odd} \\
\beta_2 &= 0, \beta_4 = \pi & N_k - m_k \text{ even}
\end{align*}
\]

Then the local expressions

\[
S_j(\lambda) = \exp i \left( \frac{\Psi_j(\lambda)}{h_k} + \beta_j \right) s_{\psi_j}(\lambda) \quad \lambda \in U_j
\]

(2.11)

define a global section of the Maslov bundle over $\Sigma$.

Proof. For each $\lambda \in U_i \cap U_j$, recall that we have

\[
s_{\psi_j}(\lambda) = s_{\psi_i}(\lambda) \exp i \frac{\pi}{2} m_{ij}.
\]

(2.12)

Therefore, the above expression defines a global section if and only if

\[
\frac{\Psi_j - \Psi_i}{h_k} + \frac{\pi}{2} m_{ij} + \beta_j - \beta_i = 0 \mod 2\pi
\]

(2.13)

The quantization condition (2.8) implies that

\[
\frac{\Psi_j - \Psi_i}{h_k} - \frac{\pi}{2} = \pi(N_k - m_k)
\]

(2.14)

for $j = 1$, $i = 2$ and $j = 2$, $i = 3$. Using this together with $\Psi_1 = \Psi_4$ and $\Psi_3 = \Psi_4$ on the intersection of their domains, we easily verify the values of the $\beta_j$, are determined except for a $\pm$ sign ambiguity and this is removed by requiring $\beta_2 = 0$. \hfill $\Box$

2.1.2. Conclusion of the proof of proposition 2.3. Let $\rho_0$ be the positive invariant density on $\Sigma$ in proposition 2.3 Define the symbol $\sigma_0 \in S^0(\Sigma)$ by

\[
\sigma_0 = \left( \sum_j \exp i \frac{\Psi_j(\lambda)}{h_k} \chi_j(\lambda) s_j(\lambda) \right) \sqrt{\rho_0}
\]

(2.15)

We inductively find a sequence of smooth functions $r_j(\lambda)$ on $\Sigma$ and complex numbers $E_j$ so that for each $n \geq 0$,

\[
(H_{h_k,c} - (1 + h^2 E_1 + \cdots + h^{n+1} E_n)) (\mathcal{D}(\sigma_0) + h \mathcal{D}(r_1 \sigma_0) + \cdots + h^n \mathcal{D}(r_n \sigma_0)) \in \mathcal{O}^{n+2}
\]

(2.16)

With $r_0 = E_0 = 1$, the $n = 0$ case follows from formula (1.17) and $\mathcal{L} X_j, \sigma_0 = 0$. Supposing it holds for $n \geq 0$, let

\[
U_n = \mathcal{D} \left( 1 + \sum_{j=1}^n r_j \right) \sigma_0 \in \mathcal{O}^{n+2}
\]

\[
E_n = 1 + \sum_{j=1}^n h^{j+1} E_j
\]

Then with $E_{n+1}$ and $r_{n+1}$ to be determined, the function

\[
(H_{h_k,c} - E_n - h^{n+2} E_{n+1}) \left( U_n + h^{n+1} \mathcal{D}(r_{n+1} \sigma_0) \right) \in \mathcal{O}^{n+2}
\]

(2.17)

is in $\mathcal{O}^{n+2}$ and its principal symbol is the same as the principal symbol of
\[ U_n + h^{n+2} E_{n+1}(\sigma_0) + h^{n+1}(H_{h_k,c} - 1)\mathcal{Q}(r_{n+1}\sigma_0) \]

which vanishes if and only if

\[ \frac{2}{\ell} dr_{n+1}(X_f) + E_{n+1} + u_n = 0 \]  

(2.18)

where \( h^n u_n \sigma_0 = \sigma(U_n) \). If \( E_{n+1} = - \int \sigma_0 \rho_0 \), then proposition \( 2.3 \) implies that there is a smooth \( r_{n+1} \) which solves this equation. Now letting \( r \sim 1 + \sum_{j=1}^{\infty} r_j h^n \sigma_0 \), \( v_{h_k} = \mathcal{Q}(r \sigma_0) \) satisfies \( (H_{h_k,c} - E(h))v_{h_k} \in \mathcal{O}^\infty(I, \Sigma, h_k) \). Finally, to verify the pointwise asymptotics, we write \( K_j = i \psi_j \chi_j \) and observe that

\[ \mathcal{Q}(\sigma_0) = \sum_{j=1,3} K_j(\phi) a_j(\phi) e^{i \left( \frac{\psi_j}{k} + \beta_j \right)} + \sum_{j=2,4} (2\pi h_k)^{-\frac{1}{4}} \int K_j(\tau) a_j(\tau) e^{i \left( \frac{\psi_j}{k} + \beta_j \right)} d\tau \]  

(2.19)

\[ a_j(\phi) = \frac{1}{\sqrt{2\pi}} \frac{1}{(1 - V(\phi))^{\frac{1}{4}}} = \frac{1}{\sqrt{2\pi}} \frac{\sqrt{\sin \phi}}{(\sin^2 \phi - c^2)^{\frac{1}{4}}} \quad j = 1, 3 \]  

(2.20)

\[ a_j(\tau) = \frac{1}{\sqrt{\pi}} \frac{1}{|V'(G_j^\prime(\tau))|^{\frac{1}{2}}} \quad j = 2, 4 \]  

(2.21)

Notice that \( |G''_j(\tau)|^{\frac{1}{2}} = \frac{1}{\sqrt{2}} |V'(G'_j(\tau))|^{\frac{1}{2}} \), so if we apply stationary phase to \( j = 2, 4 \) terms at some fixed \( \phi \in (\phi_-, \phi_+) \), the amplitudes match those in the \( j = 1, 3 \) terms. Proposition \( 2.4 \) implies that the phases match as well, so using the fact that the \( K_j \) are a partition of unity when lifted to \( \Sigma \), we get \( (2.4) \). The statement \( (2.7) \) is obvious since the \( j = 4 \) term does not have any critical points away from the projection of \( U_2 \) and the \( j = 1, 3 \) are supported away from \( (\phi_+, 0) \). Now take the real part of \( v_{h_k} \). It satisfies the equation \( (2.5) \) with \( E(h) \) replaced by its real part. The principal symbol of \( \mathfrak{G}_{h_k} \) is

\[ \sigma(\mathfrak{G}_{h_k})(\phi, \tau) = \sigma(v_{h_k})(\phi, \tau) = \sigma_0 \]

so \( ||\text{Re}(v_{h_k})||_{L^2(I)} = 1 + O(h) \). It follows that \( L^2 \) normalizing \( \text{Re} v_{h_k} \) only multiplies the lower order terms in the full symbol by a constant. And therefore the expression for the leading part of \( ||\text{Re} v_{h_k}||_{L^2(I)} \) \( \text{Re} v_{h_k} \) is the same as \( (2.4) \).

2.2. Comparison of the quasi-mode to the mode. Here we show that the quasi-mode \( v_{h_k} \) of proposition \( 2.2 \) is locally uniformly close to the true mode \( u_{h_k} = UP^{m_k}_{N_k} \).

**Proposition 2.5.** Let \( v_{h_k} \in \mathcal{O}^0(I, \Sigma, h_k) \) be as in proposition \( 2.2 \) and \( u_{h_k} \) be the \( L^2 \) normalized, real-valued function satisfying \( H_{h_k,c} u_{h_k} = u_{h_k} \). Let \( \Pi \) denote orthogonal projection onto \( \ker H_{h_k,c} - 1 \). Then

\[ ||v_{h_k} - \Pi v_{h_k}||_{L^2(I)} = O(h_k^{\infty}) \]  

(2.22)

**Proof.** From the spectral gap \( (2.2) \), it follows that the lower bound

\[ ||(H_{h_k,c} - 1)u||_{L^2(I)} \geq \delta h_k ||u||_{L^2(I)} \]

holds for \( u \in (\ker H_{h_k,c} - 1)^\perp \). The estimate

\[ ||(H_{h_k,c} - E(h))v_{h_k}||_{L^2(I)} = O(h_k^{\infty}) \]
implies that there is an eigenvalue of $H_{h_k}$ in an $O(h_k^N)$ neighborhood of $E(h)$ for all large $N$. Since $E(h) = 1 + O(h^2)$ and the eigenvalues of $H_{h_k,c}$ are separated by $O(h)$ distances, this means that $E(h) = 1 + O(h_k^N)$. Therefore

$$||(H_{h_k,c} - 1)(v_{h_k} - \Pi v_{h_k})||_{L^2(I)} = O(h_k^\infty) \quad (2.23)$$

which proves the estimate in view of the lower bound above.

\[ \square \]

**Proposition 2.6.** For each $\delta > 0$, with $I_\delta = (\delta, \pi - \delta)$,

$$||v_{h_k} - u_{h_k}||_{L^\infty(I_\delta)} = O_\delta(h_k^\infty) \quad (2.24)$$

**Proof.** Writing $\partial^2_\phi = -h_k^{-2}(H_{h_k,c} - V)$. we have

$$||\partial^2_\phi(v_{h_k} - \Pi v_{h_k})||_{L^2(I_\delta)} \leq h_k^{-2} (||H_{h_k,c}(v_{h_k} - \Pi v_{h_k})||_{L^2(I_\delta)} + ||V(v_{h_k} - \Pi v_{h_k})||_{L^2(I_\delta)})$$

From proposition 2.5 $||H_{h_k,c}(v_{h_k} - \Pi v_{h_k})||_{L^2(I)} = O(h_k^\infty)$ and since $V$ is bounded on $I_\delta$ depending on $\delta$, the right hand side is $O_\delta(h_k^\infty)$. Applying the Sobolev estimate

$$||f||_{L^\infty} \leq C||f'||_{L^2}$$

twice on the interval $I_\delta$ together with the above inequality yields

$$||v_{h_k} - \Pi v_{h_k}||_{L^\infty(I_\delta)} = O_\delta(h_k^\infty) \quad (2.25)$$

Similarly, we have

$$||u_{h_k}||_{L^\infty(I_\delta)} \leq C||u''_{h_k}||_{L^2(I_\delta)} = h_k^{-2} (H_{h_k,c} - V) u_{h_k}||_{L^2(I_\delta)} = O_\delta(h_k^{-2}) \quad (2.26)$$

so

$$||v_{h_k} - u_{h_k}||_{L^\infty(I_\delta)} \leq ||v_{h_k} - \Pi v_{h_k}||_{L^\infty(I_\delta)} + ||(\zeta(h_k) - 1)u_{h_k}||_{L^\infty(I_\delta)} = O_\delta(h_k^\infty) \quad (2.27)$$

Where we have written $\Pi v_{h_k} = \zeta(h_k)u_{h_k}$ and $\zeta(h_k) = 1 + O(h_k^\infty)$ since $v_{h_k}$ is real valued and positive in a neighborhood of $\phi = 0$.

\[ \square \]

3. **Airy expansion of $v_{h_k}$ at the turning points**

The goal of this section is to prove the following Airy expansion for $v_{h_k}$ in a neighborhood of the turning points $\phi_{\pm}$.

**Proposition 3.1.** Let $v_{h_k}$ be the quasi-mode in proposition 2.2. There exists $\varepsilon > 0$ such that for $\phi_+ - \varepsilon < \phi < \phi_+$, $v_{h_k}(\phi)$ has the full asymptotic expansion

$$v_{h_k}(\phi) \sim \sin^{-\frac{1}{6}}(\phi) \sum_{n=0}^{\infty} u_{0,n}(\phi) h_{h_k}^\frac{1}{6} + \sin^{\frac{1}{6}}(\phi) \sum_{n=0}^{\infty} u_{1,n}(\phi) h_{h_k}^{-\frac{1}{6}} \quad (3.1)$$

The leading part of the expansion is

$$v_{h_k}(\phi) \sim \sin^{-\frac{1}{6}}(\phi) \sum_{n=0}^{\infty} u_{0,n}(\phi) h_{h_k}^\frac{1}{6} + \sin^{\frac{1}{6}}(\phi) \sum_{n=0}^{\infty} u_{1,n}(\phi) h_{h_k}^{-\frac{1}{6}} \quad (3.2)$$

Here, the argument of the Airy function is...
\[
\rho(\phi) = \left( \frac{3}{4} \int_{\gamma_\phi} \alpha \right) \frac{2^3}{3} 
\]  

(3.3)

where \( \gamma_\phi \) is the arc on \( \Sigma \) passing through \((\phi_+, 0)\) from \((\phi, \tau_-)\) to \((\phi, \tau_+)\).

To prove this we write

\[
v_{h_k}(\phi) = \left( 2\pi h_k \right)^{-\frac{1}{2}} \int a(\tau, h) \exp \left( \frac{\psi_4(\phi, \tau)}{h_k} + \beta_4 \right) + O(h_k^\infty) 
\]

(3.4)

For \( \phi \) in a neighborhood of the turning point \((\phi_+, 0)\). The expansion is a consequence of the following proposition from Hörmander:

**Proposition 3.2** (Ho1, Theorem 7.7.18). Let \( f(t, x) \) be a real-valued smooth function defined in a neighborhood \((0, 0) \in V \subset \mathbb{R}^2\). Suppose that \( \partial_t f(0, 0) = \partial^2_t f(0, 0) = 0 \) and \( \partial^3_t f(0, 0) \neq 0 \). Then there exists smooth, real-valued functions \( a(x), b(x) \) and smooth compactly supported functions \( u_{0,n}(x), u_{1,n}(x) \) such that

\[
e^{-\frac{2}{3}b(x)} \int u(t, x) e^{\frac{2}{3}f(t, x)} \, dt \sim A_i(h^{-\frac{2}{3}} a(x)) h^{\frac{1}{3}} \sum_{n=0}^\infty u_{0,n}(x) h^n + A_i'(h^{-\frac{2}{3}} a(x)) h^{\frac{2}{3}} \sum_{n=0}^\infty u_{1,n}(x) h^n
\]

(3.5)

For a smooth, compactly supported amplitude \( u(t, x) \) supported sufficiently close to \((0, 0)\).

### 3.1. Proof of Proposition 3.1

As explained in [9], page 234 the functions \( a(x) \) and \( b(x) \) can be calculated by putting the phase function into the following cubic normal form

**Proposition 3.3** (Ho1, Theorem 7.5.13). Let \( f(t, x) \) be a real valued smooth defined in a neighborhood \((0, 0) \in V \subset \mathbb{R}^2\) such that \( \partial_t f(0, 0) = \partial^2_t f(0, 0) = 0 \) and \( \partial^3_t f(0, 0) \neq 0 \). Then there exists a real valued smooth function \( T(t, x) \) in a neighborhood of \((0, 0)\) with \( T(0, 0) = 0 \), \( \partial_t T(0, 0) > 0 \) and smooth functions \( a(x), b(x) \) such that

\[
f(t, x) = \frac{T^3(t, x)}{3} + a(x) T(t, x) + b(x)
\]

(3.6)

We apply this theorem to the phase

\[
\psi_4(\phi, \tau) = \phi \tau - G_4(\tau)
\]

It has a degenerate critical point at the turning point \((\phi_+, 0)\). Indeed, by differentiating the Eikonal equation,

\[
\tau^2 + \frac{c^2}{\sin^2 G_4'(\tau)} = 1
\]

(3.7)

We see that \( \partial^2_\tau \psi_4(\phi_+, 0) = -G''_4(0) = 0 \) and \( \partial^3_\tau \psi_4(\phi_+, 0) = -G'''_4(0) = \frac{4c}{\sqrt{1-c^2}} \neq 0 \). The functions \( a(x) \) and \( b(x) \) are calculated in the next proposition.
Proposition 3.4. There exists a smooth function \( T(\phi, \tau) \) in a neighborhood of \((\phi_+, 0)\) as in proposition 3.3 such that

\[
\psi_2(\phi, \tau) = \frac{T^3(\phi, \tau)}{3} + a(\phi)T(\phi, \tau) + b(\phi)
\]  

(3.8)

If \( \phi_+ - \varepsilon < \phi < \phi_+ \), then

\[
a(\phi) = -\left(\frac{3}{4} \int_{\gamma_0} \alpha\right)^{2/3} 
\]

(3.9)

\[
b(\phi) = \beta_4 
\]

(3.10)

Where \( \gamma_0 \) is the arc on \( \Sigma \) defined in proposition 3.3.

Proof. Existence follows from proposition 3.3. Put \( \rho(\phi) = -a(\phi) \). Take the \( \tau \)-derivative of (3.8) and observe that \( \partial_\tau \psi_2(\phi, \tau) = \phi - G'_4(\tau) = 0 \) if and only if \( T^2(\phi, \tau) = \rho(\phi) \). For a fixed \( \phi \in (\phi_+ - \varepsilon, \phi_+) \) let \( \tau_\pm \) be the two \( \tau \)-critical points of \( \psi_4 \), the \( \tau \)-coordinates of the two points \((\phi, \tau_\pm) \in \Sigma \) lying over \( \phi \),

\[
\tau_\pm(\phi) = \pm \sqrt{1 - \frac{c^2}{m^2}} \phi 
\]

(3.11)

Since \( T^2(\phi, \tau_\pm(\phi)) = \rho(\phi) \), we can write \( T(\phi, \tau_+(\phi)) = -\sqrt{\rho(\phi)} \) and \( T(\phi, \tau_-(\phi)) = \sqrt{\rho(\phi)} \). These imply that \( \psi_2(\phi, \tau_\pm) = \mp \rho(\phi)^{3/2}/3 - \pm \rho^{3/2}(\phi) + b(\phi) \) which means

\[
\frac{4}{3} \rho^{3/2}(\phi) = \psi_4(\phi, \tau_+) - \psi_4(\phi, \tau_-) 
\]

(3.12)

\[
2b(\phi) = \psi_4(\phi, \tau_+) + \psi_4(\phi, \tau_-) 
\]

The formulas then follow since \( \psi_4 \) is odd in \( \tau \) and \( \Psi_4(\tau) = \psi_4(G'_4(\tau), \tau) \) is a primitive for \( a|\psi_4| \).

Now let \( \chi(\phi) \) be a bump function equal to one on \((\phi_+ - \frac{\varepsilon}{2}, \phi_+ + \frac{\varepsilon}{2})\) and supported in \((\phi_+ - \varepsilon, \phi_+ + \varepsilon)\). The amplitude \( \chi(\phi)a_4(\tau, h) \) appearing in (3.4) will then have no critical points outside of a \( \tau \) neighborhood \( B_\varepsilon(0) \) of \( \tau = 0 \), \( r = o(\varepsilon) \). Split up the integral by inserting a \( \tau \) cutoff \( \eta(\tau) \), \( \chi(\phi)a_4(\tau, h) = \chi(\phi)\eta(\tau)a(\tau, h) + \chi(\phi)(1 - \eta(\tau))a_4(\tau, h) \) supported on \( B_\varepsilon(0) \), equal to 1 on \( B_{\varepsilon/2}(0) \). If \( \varepsilon \) is small enough, the first term is supported close enough to \( \tau = 0 \) to apply proposition 3.2.2, and the second term is \( O(h_k^{-\infty}) \). Finally, we calculate the leading order amplitude \( u_{0,0}(\phi) \) appearing in the expansion. The leading term is

\[
v_{h_k}(\phi) \sim (-1)^{m_k+N_k}(2\pi)^{\frac{1}{4}} h_k^{-\frac{7}{6}} u_{0,0}(\phi)Ai(-h_k^{-\frac{7}{4}}\phi) 
\]

(3.13)

Using the standard expansion of the Airy function for large \( t > 0 \) (see [9] pg. 215)

\[
Ai(-t) \sim \frac{1}{\sqrt{\pi}t^{1/4}} \cos\left(\frac{2}{3}t^{\frac{3}{2}} - \frac{\pi}{4}\right) 
\]

(3.14)

we see that when \( h_k^{-\frac{7}{4}}\phi \gg 0 \),

\[
v_{h_k}(\phi) \sim u_{0,0}(\phi) \left(\pi^{-\frac{1}{4}}\rho(\phi)^{-\frac{1}{4}} \sin\left(h_k^{-1}\int_{\gamma_0} \alpha + \frac{\pi}{4}\right)\right)
\]

(3.15)

But this must match the leading term in proposition 2.2 which forces...
\[ u_{0,0}(\phi) = \sqrt{\sin \phi} \left( \frac{4\rho(\phi)}{\sin^2 \phi - c^2} \right)^{\frac{1}{2}} \]  

(3.16)

4. Geometry of ladder sequences of spherical harmonics

Recall that the generator of rotations, \( D_\theta = -i \partial_\theta \) commutes with the Laplacian on \( S^2 \). The Clairaut integral, \( p_\theta(x, \xi) = \langle \xi, \partial_\theta \rangle \) is the symbol of the \( D_\theta \) so \( \{p_\theta, q\} = 0 \) where \( q(x, \xi) = |\xi| \). Together they generate a homogeneous Hamiltonian torus action, \( \Phi(t, s) \) on \( T^*S^2 \),

\[ \Phi(t, s, x, \xi) = \exp sX_{p_\theta} \circ \exp tX_q(x, \xi) \]  

(4.1)

which acts transitively on the level sets of the moment map,

\[ \mu : T^*S^2 \to \Gamma \subset \mathbb{R}^2 \]  

\[ \mu(x, \xi) = (q(x, \xi), p_\theta(x, \xi)) \]  

(4.2)

whose image is the cone \( \Gamma = \{(x, y) \mid x \geq 0, |y| \leq x \} \). Since \( \mu \) is homogeneous, we need only consider level sets with \( q = 1 \). For \( c \in [-1, 1] \) set \( T_c = \mu^{-1}(1, c) = S^*S^2 \cap \{p_\theta = c\} \). For \( c \in (-1, 1) \), \( T_c \) is Lagrangian torus inside of \( S^*S^2 \). When \( c = \pm 1 \), \( T_c \) is just the lift of the standard equator \( \gamma_c \) to \( S^*S^2 \). In terms of the polar coordinates \( (\phi, \tau, \theta, \eta) \) on \( T^*S^2 \),

\[ T_c = \{ (\phi, \tau, \theta, \eta) \mid \tau^2 + \frac{c^2}{\sin^2 \phi} = 1; \eta = c \} \]  

(4.3)

Therefore its projection to \( S^2 \) is \( \pi(T_c) = \{ (\phi, \theta) \mid \sin \phi \geq c \} \). The projection is an annular band consisting of all geodesics which make the fixed angle \( \arccos c \) with the equator. The energy curve associated with the associated Legendre functions is just the \( (\phi, \tau) \) cross-section of \( T_c \). For \( x \) in the interior of \( \pi(T_c) \), let \( \gamma_x \) be the geodesic arc connecting the two points lying above \( x \) in \( T_c \), from \( (x, \xi_-) \) to \( (x, \xi_+) \), where the sign corresponds to the sign of \( \tau \). It is clear that \( \int_{\gamma_x} d\theta = 0 \) since there is no change in the \( \theta \) coordinate across the arc. But the canonical 1-form is \( \alpha = \tau d\phi + \eta d\theta \). Since \( \eta = c \) is constant on \( T_c \) the second term contributes nothing to the integral over \( \gamma_x \), and the first term is clearly equal to the integral \([4.3]\) in the Legendre function expansion. The density

\[ d\mu_c = \frac{|d\phi| \otimes |d\theta|}{(2\pi)^2 |\tau|} \]  

(4.4)

is invariant under the joint flow on \( T_c \) and

\[ \pi_*d\mu_c = \frac{1}{(2\pi)^2} \frac{\sqrt{\sin \phi} |d\phi| \otimes |d\theta|}{\sin^2 \phi - c^2} \]

which verifies the formula \([4.4]\). The reason for the Airy bump at the caustic latitude circles is the presence of a fold singularity for the projection \( \pi|_{T_c} : T_c \to S^2 \). Recall that a smooth map \( f : X^n \to Y^n \) between \( n \)-dimensional manifolds is said to have a fold singularity with fold locus \( S \) if there exists a codimension one submanifold \( S \subset X \) such that

1. \( S \) is equal to the set of critical points of the map \( f \), i.e. \( S = \{ x \in X \mid df_x \) is not surjective \}

2. For each \( s \in S \), the kernel of \( df_s \) is transverse to the tangent space \( T_xS \).
Proposition 4.1. The projection \( \pi|_{T_c} \to S^2 \) is a folding map with fold locus \( S = S_+ \cup S_- \),

\[ S_\pm = \{(\phi_\pm, \theta, 0, c) \mid \theta \in [0, 2\pi)\} \]

where \( \phi_\pm \) are the two solutions of \( \sin \phi = c \). The images of \( S_\pm \) are the latitude circles which bound \( \pi(T_c) \).

Proof. Writing \((\rho, \eta)\) as dual coordinates to \((\phi, \theta)\), \(T_c\) is cut out by the equations \( \eta = c \) and \( \rho^2 + \frac{c^2}{\sin^2 \phi} = 1 \). Differentiating the second equation gives

\[ \rho d\rho - c^2 \cos \phi \sin^3 \phi d\phi = 0 \]

so writing \( x = (\phi, \theta) \), \( \xi = (\rho, \eta) \),

\[ T(x, \xi)_{T_c} = \{\alpha \partial_\phi + \beta \partial_\theta + \gamma \partial_\rho \mid \rho \gamma = c^2 \cos \phi \sin^3 \phi \alpha\} \]

So for \( v \in T(x, \xi)T_c \), \( d\pi v = 0 \) if and only if \( \alpha = \beta = 0 \). But then \( \rho \gamma = 0 \). If \( \rho = 0 \), then \( v = 0 \), so the kernel of \( d\pi \) is non-trivial only when \( \rho = 0 \), and this means that \((x, \xi) \in S\). At such points, the kernel of \( d\pi \) is the span of \( \partial_\rho \), which is transverse to \( T(x, \xi)S = \mathbb{R}\partial_\theta \).

\[ \square \]

5. Calculation of the weak limit of the empirical measures, proof of theorem 1.2

In this section we determine the weak limit of the measures (1.6). Noting that \( |Y_N^m(x)|^2 \) is constant on latitude circles, choose any \( x \in \gamma_{c_0} \) and we may write

\[ \mu_{N,c_0}(t) = \frac{2\pi}{N + \frac{1}{2}} \sum_{m=-N}^{N} |Y_N^m(x)|^2 \delta_0 \left(t - \frac{m}{N}\right). \]

Thus \( \int \mu_{N,c_0} = \frac{2\pi}{N + \frac{1}{2}} \Pi_N(x, x) = 1 \). By the Lévy continuity theorem, it suffices to show that the characteristic functions

\[ \Phi_{N,c_0}(s) = \frac{1}{2\pi} \int e^{ist} d\mu_{N,c_0}(t) \]

converge pointwise to a limit \( \Phi(s) \) which is continuous at \( s = 0 \). In this case, \( \mu_{N,c_0} \) converges weakly to the Fourier transform of \( \Phi(s) \).

Proposition 5.1. The characteristic functions \( \Phi_{N,c_0} \) converge pointwise to \( (2\pi)^{-1} J_0(c_0 s) \)

where

\[ J_0(s) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp (-is \sin \theta) \, d\theta \]

is the order zero Bessel function.

Proof. Write

\[ \Phi_{N,c_0} = \frac{1}{N + \frac{1}{2}} \sum_{m=-N}^{N} \exp \left(i \frac{sm}{N}\right) |Y_N^m(x)|^2 \]
Let \( r_s : S^2 \to S^2 \) be rotation by \( s \) in the polar angle, i.e. pullback under the flow of \( \partial_\theta \). Then the right hand side can be rewritten as

\[
\Phi_{N,c_0}(s) = \frac{1}{N} + \frac{1}{2} \Pi_N(r_{s/N}(x), x) = \frac{1}{2\pi} P_N \left( \cos d(r_{s/N}(x), x) \right)
\]

where \( d(x, y) \) is the Riemannian distance. Recall \( \cos d(x, y) = x \cdot y \) so \( d(r_s(x), x) = 1 - c_0^2(1 - \cos s) \) and in particular \( d(r_s(x), x) \leq c_0 |s| \), hence

\[
\frac{d(r_{s/N}(x), x)^2}{2} = 1 - \cos d(r_{s/N}(x), x) + O \left( \frac{s^4}{N^4} \right)
\]

By the Mehler-Heine asymptotics we have, locally uniformly in \( z \),

\[
\lim_{N \to \infty} P_N(\cos \frac{z}{N}) = J_0(z)
\]

which, together with (5.5) implies the limit.

Now the weak limit calculation is finished in light of the fact that

\[
\int_{-\infty}^{\infty} e^{-ist} J_0(s) \, ds = 1|_{[-1,1]} \frac{2}{\sqrt{1-t^2}}
\]

which is easily verified directly.

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