A DIRECT ALGORITHM FOR CONSTRUCTING RECURSION OPERATORS AND LAX PAIRS FOR INTEGRABLE MODELS

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We suggest an algorithm for seeking recursion operators for nonlinear integrable equations. We find that the recursion operator $R$ can be represented as a ratio of the form $R = \frac{L_1 - 1}{1 L_2}$, where the linear differential operators $L_1$ and $L_2$ are chosen such that the ordinary differential equation $(L_2 - \lambda L_1)U = 0$ is consistent with the linearization of the given nonlinear integrable equation for any value of the parameter $\lambda \in \mathbb{C}$. To construct the operator $L_1$, we use the concept of an invariant manifold, which is a generalization of a symmetry. To seek $L_2$, we then take an auxiliary linear equation related to the linearized equation by a Darboux transformation. It is remarkable that the equation $L_1 U = L_2 U$ defines a Backlund transformation mapping a solution $U$ of the linearized equation to another solution $\tilde{U}$ of the same equation. We discuss the connection of the invariant manifold with the Lax pairs and the Dubrovin equations.

Keywords: Lax pair, integrable chain, higher symmetry, invariant manifold, recursion operator

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1. Introduction

In our previous works, we suggested a method for constructing recursion operators and Lax pairs for nonlinear integrable equations (see [1]–[4]). We briefly explain the core of the method. For convenience, we first take a class of evolutionary nonlinear partial differential equations (PDEs), although it is shown below that the algorithm is applicable to any integrable equations. We consider an integrable equation of the form

$$u_t = f(u, u_1, u_2, \ldots, u_k), \quad \text{where } u_j = \frac{\partial u}{\partial x_j}. \tag{1}$$

Here, $D_x$ denotes the operator of the total derivative with respect to $x$. In what follows, we use the linearization of Eq. (1) around an arbitrary solution $u(x, t)$:

$$U_t = F_* U, \quad F_* = \frac{\partial f}{\partial u} + \frac{\partial f}{\partial u_1} D_x + \frac{\partial f}{\partial u_2} D_x^2 + \cdots + \frac{\partial f}{\partial u_k} D_x^k. \tag{2}$$

We seek an ordinary differential equation

$$H(x, t; U, U_1, \ldots, U_m; u, u_1, \ldots, u_{m_1}) = 0 \tag{3}$$

of the order $m \geq 1$ compatible with linearized equation (2) for all values of the dynamical variables $u, u_1, u_2, \ldots$ regarded as parameters here. The compatibility condition for Eqs. (2) and (3) coincides with the equation

$$D_t H(x, t; U, U_1, \ldots, U_m; u, u_1, \ldots, u_{m_1}) = 0 \mod (1, 2, 3). \tag{4}$$

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We assume that all of the derivatives in (4) are expressed using Eqs. (1) and (2) and the variables $U_{s+m}$ with $s \geq 0$ are expressed using Eq. (3). In this case, we say that Eq. (3) defines an invariant manifold in the space of the variables $U, U_1, U_2, \ldots$.

By construction, Eqs. (2) and (3) are compatible if $u = u(x, t)$ is a solution of Eq. (1). It is remarkable that for an appropriate choice of Eq. (3), the converse also holds: the compatibility of Eqs. (2) and (3) implies (1). As is commonly known, just this property is crucial for the Lax pairs. Studied examples indicate that ODE (3) can be of different kinds. Here, we concentrate on two types of Eq. (3) for which $H$ is either linear or quadratic:

\begin{align}
 H &= \sum_{j=0}^{m} \alpha_j(u, u_1, \ldots)U_j, \quad (5) \\
 H &= c + \sum_{i,j=0}^{s} \alpha_{i,j}(u, u_1, \ldots)U_iU_j, \quad (6)
\end{align}

where $c$ is an arbitrary constant. We note that the first case is connected with the recursion operator $R$ for Eq. (1) and the second corresponds to the usual Lax pair for $c = 0$ and a nonlinear Lax pair for $c \neq 0$ from which the well-known Dubrovin equations were derived in the context of finite-gap integration.

We recall that the recursion operator is a solution of Eq. (16) (see below). In general, it is a pseudodifferential operator (pseudodifference for lattices) generating symmetries of Eq. (1). The Lax pair and the recursion operator are very close to each other. It follows from (16) that the operators $R$ and $d/dt - F*$ generate a Lax pair:

\begin{align}
 R\Psi &= \lambda\Psi, \\
 \frac{d}{dt}\Psi &= F_*\Psi
\end{align}

for Eq. (1) (see [5], [6]). Moreover, it was demonstrated in [3], [4] by numerous examples that Lax pair (7) is effectively converted into the classical Lax pair. A clear manifestation of this circumstance is that a linear invariant manifold is transformed into a nonlinear one by decreasing the order in the corresponding ODE (3).

We note that there is a great variety of approaches for seeking Lax pairs from the Zakharov–Shabat dressing [7], [8] and the method of pseudopotentials by Wahlquist and Estabrook [9] to the singular manifold method [10], [11] of Painlevé analysis and the three-dimensional consistency approach developed in [12]–[14]. We also mention approaches proposed in [5], [15], [16]. We stress that our method uses a similar idea but differs from the well-known method in [9], where both Lax equations are assumed to be linear and unknown. In contrast, we take the linearization of a given integrable equation as one of the Lax equations and seek the second equation, which is not assumed to be linear. In fact, we find a nonlinear Lax pair at the first stage and then linearize it by an appropriate point transformation. Because we seek only one of the two equations, our method is quite effective.

The problem of finding the recursion operator for an integrable equation is not easily solved. There are several ways to study the task. Some of them use the Lax representation (see, e.g., [5], [17]–[21]). Others are based on directly studying Eq. (16). To solve this equation, most authors use the multi-Hamiltonian approach [22]–[26]. Their basic aim is to determine two Hamiltonian operators $H_1$ and $H_2$ for a given equation. The sought recursion operator is given by the simple formula

\begin{equation}
 R = H_2 H_1^{-1}. \quad (8)
\end{equation}

Here, we use an alternative representation for the recursion operator,

\begin{equation}
 R = L_1^{-1}L_2, \quad (9)
\end{equation}
where \( L_1 \) can be found as a differential operator annihilating some classical symmetries of Eq. (1) and \( L_2 \) is determined using the procedure described in Sec. 3. We note that the equation \( L_1 \tilde{U} = L_2 U \) defines a Bäcklund autotransformation for linearized equation (2). Obviously, the two representations (8) and (9) of the recursion operator differ essentially.

We briefly discuss the content of the article. In Sec. 2, we study the close connection between the symmetries of a given nonlinear equation and the invariant manifolds of its linearization, which provides an easy way to construct linear invariant manifolds. In Sec. 3, we then explain the stepwise algorithm for determining the operators \( L_1 \) and \( L_2 \) used in formula (9). In Secs. 3.1–3.4, we illustrate the application of the algorithm with the KdV equation, a Volterra-type autonomous lattice, a Volterra-type nonautonomous lattice, and a system of two lattices as examples. In Sec. 4, we demonstrate a direct method for finding nonlinear invariant manifolds and some applications of them in integrability theory.

2. Invariant manifolds and symmetries

It has been observed that there is a close connection between symmetries and invariant manifolds.

**Proposition 1.** Any set of symmetries

\[
\begin{align*}
    u_{\tau_j} &= U^{(j)}, \quad j = 1, 2, \ldots, k,
\end{align*}
\]

of Eq. (1) defines an invariant manifold of form (3) for linearized equation (2). Here, \( U^{(j)} \) is a solution of Eq. (2) depending on a finite number of dynamical variables.

**Proof.** We consider the linear space spanned by the functions \( U^{(j)} \). Any element \( U \) of the space is then represented as a linear combination

\[
\begin{align*}
    U &= c_1 U^{(1)} + c_2 U^{(2)} + \cdots + c_k U^{(k)}.
\end{align*}
\]

Here, we assume that \( U^{(j)} \) are linearly independent. We regard Eq. (11) as the general solution of a differential equation of the form

\[
\begin{align*}
    L_1 U := (D^k x + a^{(1)} D^{k-1} x + \cdots + a^{(k)}) U = 0.
\end{align*}
\]

By construction, \( U \) also solves linearized equation (2), and Eq. (12) therefore defines a linear invariant manifold. The coefficients of this linear differential equation are obviously found from the equation

\[
\begin{align*}
    L_1 U = \det \begin{pmatrix}
        U^{(1)}_k & U^{(1)}_{k-1} & \cdots & U^{(1)}_1 \\
        U^{(2)}_k & U^{(2)}_{k-1} & \cdots & U^{(2)}_1 \\
        \vdots & \vdots & \ddots & \vdots \\
        U^{(k)}_k & U^{(k)}_{k-1} & \cdots & U^{(k)}_1 \\
        U_k & U_{k-1} & \cdots & U_k
    \end{pmatrix},
\end{align*}
\]

where \( U_m = D^m x U \) and \( U^{(j)}_m = D^m x U^{(j)} \). The proof is complete.
3. Algorithm for constructing linear invariant manifolds and recursion operators for integrable equations

Invariant manifolds can be found directly by solving the defining Eq. (3), which is highly overdetermined and usually effectively solved. But this strategy becomes quite laborious when applied to systems or higher-order equations.

To seek a linear invariant manifold, we suggest a new convenient algorithm. We note that the linear invariant manifold is closely connected with the recursion operator $R$ in Eq. (1). We consider the equation

$$RU = \lambda U$$ \hspace{1cm} (14)

and then by multiplying from the left by a differential operator $L_1$ bring this equation to the form

$$L_2U = \lambda L_1U,$$ \hspace{1cm} (15)

where $L_2$ is also a differential operator. The operator $L_1$ is obviously not unique. We require that the order of $L_1$ be as small as possible. Equation (15) then defines an invariant manifold for an arbitrary value of $\lambda$ including $\lambda = 0$ and $\lambda = \infty$. Therefore, we can easily obtain a linear invariant manifold for the known recursion operator. But we apply this scheme in the opposite direction.

Here, we assume that the recursion operator admits a representation as a ratio of two differential operators, $R = L_1^{-1}L_2$, defining invariant manifolds as $L_1U = 0$ and $L_2U = 0$. It is well known that the recursion operator usually satisfies the equation [5], [27]

$$\frac{d}{dt}R = [F_*, R].$$ \hspace{1cm} (16)

We replace $R = L_1^{-1}L_2$ in (16) and after a slight transformation obtain

$$\frac{d}{dt}L_2 = \left(\frac{d}{dt}(L_1)L_1^{-1} + L_1F_*L_1^{-1}\right)L_2 - L_2F_*,$$ \hspace{1cm} (17)

where $F_*$ is linearization operator (2). The operator $L_1$ can easily be found using formula (13) through the symmetries of Eq. (1). We define a new operator $A$ as

$$A := \frac{d}{dt}(L_1)L_1^{-1} + L_1F_*L_1^{-1}.$$ \hspace{1cm} (18)

Then (17) immediately yields the equation

$$\frac{d}{dt}L_2 = AL_2 - L_2F_*,$$ \hspace{1cm} (19)

which allows finding $L_2$. As a result, we obtain the recursion operator $R = L_1^{-1}L_2$ solving Eq. (16).

**Remark 1.** Equations (18) and (19) generate two Darboux-type transformations $U^{(j)} \rightarrow V^{(j)} = L_jU^{(j)}$, $j = 1, 2$, converting solutions $U^{(j)}$ of Eq. (2) into solutions $V^{(j)}$ of some auxiliary linear differential equation $V_t = AV$. Choosing $V^{(1)} = V^{(2)}$, we obtain an equation of the form

$$L_1U^{(1)} = L_2U^{(2)}$$ \hspace{1cm} (20)

defining a Bäcklund transformation that converts a solution $U^{(2)}$ of linearized equation (2) into another solution of the same Eq. (2). To derive the invariant manifold from Bäcklund transformation (20), we set $U := U^{(2)}$ and $U^{(1)} = \lambda U$. 

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**Theorem 1.** Let the operator $L_2$ satisfy Eq. (19) Then the equation

$$L_2U = 0$$  \hspace{1cm} (21)

is consistent with the linearized equation

$$\frac{d}{dt} U = F_\ast U,$$  \hspace{1cm} (22)

i.e., Eq. (21) defines an invariant manifold for (22).

**Proof.** Differentiating the left-hand side of (21), we obtain

$$\frac{d}{dt} (L_2)U + L_2 \frac{d}{dt} U.$$ 

We must now show that this expression vanishes. Indeed, from (2), (21), and (22), we obtain

$$\frac{d}{dt} (L_2)U + L_2 \frac{d}{dt} U = AL_2U + L_2 \left( \frac{d}{dt} - F_\ast \right) U = 0.$$ \hspace{1cm} (23)

This completes the proof.

**Corollary 1.** For any value of $\lambda \in \mathbb{C}$, the equation

$$(L_2 - \lambda L_1)U = 0$$ \hspace{1cm} (24)

defines an invariant manifold for Eq. (2).

Indeed, the operator $M = L_2 - \lambda L_1$ solves the equation $\dot{M} = AM - MF_\ast$ because of (18) and (19). Therefore, according to Theorem 1, the constraint $MU = 0$ is compatible with Eq. (2).

In what follows, we show the effectiveness of the algorithm in several examples.

**3.1. Example: The KdV equation.** We consider the KdV equation

$$u_t = u_{xxx} + uu_x.$$ \hspace{1cm} (25)

Its linearization has the form

$$U_t = U_{xxx} + uU_x + u_x U.$$ \hspace{1cm} (26)

In this case, $L_1$ can be defined by formula (13) with $k = 1$ and with the symmetry $U^{(1)} = u_x$. We set $L_1 = u_x D_x / u_x$ because this operator obviously annihilates $u_x$.

We define the operator $A$ by the formula

$$A = D_t (L_1) L_1^{-1} + L_1 F_\ast L_1^{-1},$$ \hspace{1cm} (27)

where $F_\ast = D_x^3 + u D_x + u_x$ is the linearization operator and obviously $L_1^{-1} = u_x D_x^{-1} / u_x$. Equation (27) implies that

$$A = D_x^3 + \left( u + 3 \left( \frac{u_{xxx}}{u_x} - \frac{u_x^2}{u_x^2} \right) \right) D_x + 3 \frac{u_{xxx} u_{xx}}{u_x u_x} - 6 \frac{u_{xx} u_{xxx}}{u_x^2} + 3 \frac{u_{xxx}^3}{u_x^3} + 2 u_x.$$ \hspace{1cm} (28)
We must now seek an operator of the order \( m > 1 \)

\[
L_2 = \alpha D_x^m + \alpha_1 D_x^{m-1} + \cdots + \alpha_m
\]

(29)

that solves the equation

\[
D_t(L_2) = AL_2 - L_2 F_\ast.
\]

(30)

It is easy to prove that for \( m = 2 \), Eq. (30) has no solution of form (29). For \( m = 3 \), we have

\[
L_2 = D^3_x - \frac{u_{xx}}{u_x} D^2_x + \frac{2u}{3} D_x + u_x - \frac{2uu_{xx}}{3u_x}.
\]

(31)

Direct computations based on equalities (27) and (30) show that the operator

\[
R = L - 1 L_2
\]

solves the equation

\[
D_t (R) = [F_\ast, R],
\]

and \( R \) is therefore a recursion operator for the KdV equation. Indeed, from the formula

\[
R = D^2_x + \frac{2u}{3} + \frac{1}{3} u_x D_x^{-1}.
\]

3.2. Constructing invariant manifolds and recursion operators for nonautonomous integrable lattices of the Volterra type. It is known from [28], [29] that integrable lattices of the form

\[
u_{n,t} = f(n, u_{n+1}, u_n, u_{n-1})
\]

(32)

satisfy a sequence of integrability conditions. We provide some of these conditions with important applications in the symmetry classification:

\[
D_t \log \left( \frac{\partial f}{\partial u_1} \right) = (D - 1) q^{(1)},
\]

(33)

\[
D_t \log \left( \frac{\partial f}{\partial u_{-1}} \right) = (D - 1) q^{(-1)},
\]

(34)

\[
r^{(k)} = (D - 1) s^{(k)}, \quad k = 1, 2,
\]

(35)

where \( D \) is the shift operator such that \( Dy(n) = y(n+1) \) and \( q^{(1)}, q^{(-1)}, s^{(1)}, \) and \( s^{(2)} \) are some functions depending on the dynamical variables \( u_n, u_{n\pm 1}, u_{n\pm 2}, \ldots \). In these formulas, we use a commonly accepted abbreviated notation, which is also used hereafter: \( u := u_n, u_1 := u_{n+1}, u_{-1} := u_{n-1}, \) and so on. In (35),

\[
r^{(1)} = \log \left( -\frac{f_{u_1}}{f_{u_{-1}}} \right), \quad r^{(2)} = s^{(1)} + 2f_u.
\]

(36)

The problem of completely classifying autonomous lattices of form (32) was solved years ago (see [15]). Recursion operators for these equations in the autonomous case were discussed in [30]. Below, we show that the method of invariant manifolds can be successfully applied to the integrable lattices in both the autonomous and nonautonomous cases. Moreover, we derive a general formula for the recursion operator applicable to a subclass of integrable lattices (32).

We first linearize Eq. (32):

\[
U_t = \frac{\partial f}{\partial u_1} U_1 + \frac{\partial f}{\partial u} U + \frac{\partial f}{\partial u_{-1}} U_{-1}.
\]

(37)

The notion of an invariant manifold introduced above for evolutionary PDEs is easily adapted to other classes of equations.
We say that an ODE of the form
\[ H(n, t, U_m, \ldots, U_m; u_1, \ldots, u_t) = 0, \tag{38} \]
where \( U = U(n, t) \) is regarded as an unknown function and the variables \( u_j \) are regarded as parameters, determines an invariant manifold for linearized equation (37) if Eqs. (37) and (38) are consistent for any choice of the solution \( u = u(n, t) \) of Eq. (32).

The simplest invariant manifold is found very easily. For this, we use a classical symmetry of Eq. (32), \( u_\tau = cu_t \). Obviously, the function \( U = cu_t \) determines the general solution of the linear difference equation \( (D - 1)(1/u_t)U = 0 \). Therefore, the equation \( L_1 U = 0 \) with \( L_1 = (D - 1)(1/u_t) \) defines an invariant manifold for linearized equation (37).

We first consider only those equations for which the operators \( L_1 \) and \( L_2 \) have the forms
\[ L_1 = (D - 1)\frac{1}{u_t}, \quad L_2 = \alpha D^2 + \beta D + \gamma + \delta D^{-1}, \tag{39} \]
where the coefficients \( \alpha, \beta, \gamma, \) and \( \delta \) are functions of the dynamical variables \( u_j \). We note that this class contains all the lattices in Yamilov’s list (see [28]) except the equation \( V_4 \), for which \( L_1 \) is a second-order difference operator annihilating \( u_t \) and the next symmetry of the equation \( V_4 \). We introduce notation for the linearization operator of lattice (32) by setting \( f^* = f_{u_1}D + f_u + f_{u_1}D^{-1} \). The operator \( A \) is defined by the formula \( A := d/dt \left( L_1 \right) L_1^{-1} + L_1 f^* L_1^{-1} \) (cf. (18)):
\[ A = (D - 1)\left( \frac{f_{u_1}f_1}{f} - \frac{f_{u_1}f^{-1}}{f} D^{-1} \right). \tag{40} \]

We now seek the operator \( L_2 \) using the equation
\[ \frac{d}{dt} L_2 = AL_2 - L_2 f^*. \tag{41} \]

We rewrite (41) in an explicit form:
\[ \alpha_t D^3 + \beta_t D + \gamma_t + \delta_t D^{-1} = (D - 1)\left( \frac{f_{u_1}f_1}{f} - \frac{f_{u_1}f^{-1}}{f} D^{-1} \right)(\alpha D^2 + \beta D + \gamma + \delta D^{-1}) - (\alpha D^2 + \beta D + \gamma + \delta D^{-1})(f_{u_1}D + f_u + f_{u_1}D^{-1}). \tag{42} \]

Comparing the coefficients of \( D^3 \) and \( D^{-2} \) in this equation, we find
\[ \alpha = \left( \frac{f_{u_1}}{f} \right)_1, \quad \delta = \left( \frac{f_{u_1}}{f} \right)_1. \tag{43} \]

We recall that \( h_1 \) denotes \( D(h) \). Comparing the coefficients of \( D^2 \) yields the equation
\[ \alpha_t = \left( \frac{f_{u_1}f_1}{f} \right)_1 \beta_1 - \left( \frac{f_{u_1}f}{f_1} + \frac{(f_{u_1})_1 f}{f_1} \right) \alpha - \alpha(f_u)_2 - \beta(f_{u_1})_1. \tag{44} \]

We now divide (44) by \( \alpha \) and obtain
\[ D_t \log \alpha = (D - 1)\beta f_1 - \left( \frac{f_{u_1}f_1}{f} + \frac{(f_{u_1})_1 f}{f_1} \right) - (f_u)_2. \]
We evaluate the left-hand side of this equation using (43) and obtain

\[ D_t \log \alpha = D(D_t \log f_{u_1} - D_t \log f). \]  

(45)

Combining Eqs. (44) and (45) and replacing in accordance with (33), we obtain the equation

\[(D - 1) \left( \beta f_1 - q_1^{(1)} + \frac{f_{u_1}f_1}{f} - (f_u)_1 \right) = 0,\]

which implies that

\[ \beta = \frac{1}{f_1} q_1^{(1)} - \frac{f_{u_1}}{f} - (f_u)_1. \]

(46)

Similarly, gathering the coefficients of \(D^{-1}\) and then applying conservation law (34), we can derive an explicit expression for \(\gamma\):

\[ \gamma = -\frac{1}{f} q_1^{(-1)} + \frac{f_u}{f} - \left( \frac{(f_{u_1})_1}{f_1} \right). \]

(47)

Hence, all the coefficients of the sought operator \(L_2\) are found, but to show that Eq. (42) is completely satisfied, we must check the last two equations, obtained by collecting the coefficients of \(D\) and \(D^0\):

\[ \beta_t = \left( \frac{f_{u_1}f_1}{f} \right) \gamma_1 - \left( \frac{f_{u_1}f_1}{f} + \frac{(f_{u_1})_1 f}{f_1} \right) \beta + \frac{f_{u_1}f_1}{f} \alpha_1 - \alpha(f_{u_2})_2 - \beta(f_u)_1 - \gamma f_{u_1}, \]

(48)

\[ \gamma_t = \left( \frac{f_{u_1}f_1}{f} \right) \delta_1 - \left( \frac{f_{u_1}f_1}{f} + \frac{(f_{u_1})_1 f}{f_1} \right) \gamma + \frac{f_{u_1}f_1}{f} \beta_1 - \beta(f_{u_1})_1 - \gamma(f_u) - \delta(f_{u_1})_1. \]

We assume that the coefficients \(\alpha, \beta, \delta, \) and \(\gamma\) found above satisfy Eqs. (48). The operator \(R = L_1^{-1}L_2,\)

\[ R = f(D - 1)^{-1}(\alpha D^2 + \beta D + \gamma + \delta D^{-1}), \]

(49)

is then the recursion operator for lattice (32). Indeed, it is easy to verify that the found \(R\) solves the equation

\[ R_t = [f^*, R], \]

(50)

which is just the defining equation for the recursion operator (see [28]).

If we take the Volterra chain

\[ u_t = u(u_1 - u_{-1}) \]

(51)

as an illustrative example, then the operators \(L_1\) and \(L_2\) are

\[ L_1 = (D - 1) \frac{1}{u(u_1 - u_{-1})}, \]

\[ L_2 = \frac{1}{u_2 - u} D^2 + \left( \frac{u_2 + u_1}{u_1(u_2 - u)} - \frac{1}{u_1 - u_{-1}} \right) D + \frac{1}{u_2 - u} - \frac{u + u_{-1}}{u(u_1 - u_{-1})} - \frac{1}{u_1 - u_{-1}} D^{-1}. \]

(52)
Therefore, the formula \( R = L_1^{-1}L_2 \) gives the well-known recursion operator

\[
R = uD + u + u_1 + uD^{-1} + u(u_1 - u_{-1})(D - 1)^{-1} \frac{1}{u},
\]

(53)

found years ago in [25]. We note that the operators \( L_1 \) and \( L_2 \) differ from the Hamiltonian operators \( H_1 \) and \( H_2 \) used in [30] to represent recursion operator (53) as a ratio \( R = H_2H_1^{-1} \), where

\[
H_1 = u(D - D^{-1})u, \quad H_2 = u(DuD + uD + Du - uD^{-1} - D^{-1}u - D^{-1}uD^{-1})u.
\]

We studied examples of the lattices above assuming that \( L_1 \) is a first-order operator (see (39) above), but this assumption is too restrictive for some lattices, such that the needed \( L_2 \) does not exist. We then take \( L_1 \) of an order higher than one. Proposition 1 discussed above can also be applied to the lattice case.

**Proposition 2.** Any set of symmetries

\[
u_{r_j} = U^{(j)}, \quad j = 1, 2, \ldots, k,
\]

(54)

of Eq. (32) defines an invariant manifold of form (38) for linearized equation (37). Here, \( U^{(j)} \) is a solution of Eq. (37) depending on a finite number of dynamical variables.

We omit the proof of Proposition 2 because it is an almost verbatim repeat of the proof of Proposition 1. The sought invariant manifold is defined by the equation

\[
L_1U = \det \begin{pmatrix}
U_1^{(1)} & U_1^{(1)} & \cdots & U^{(1)} \\
U_2^{(1)} & U_2^{(1)} & \cdots & U^{(2)} \\
\vdots & \vdots & \ddots & \vdots \\
U_k^{(1)} & U_k^{(1)} & \cdots & U^{(k)} \\
U_1 & U_1 & \cdots & U \\
\end{pmatrix} = 0,
\]

(55)

where \( U_m = D^mU \) and \( U_m^{(j)} = D^mU^{(j)} \).

### 3.3. Example of a recursion operator for a nonautonomous lattice.

In this section, we consider the nonautonomous lattice of the form

\[
u_t = \frac{(-1)^{n+m}u(u^2 - 1)(u_1u_{n-1} + 1)}{uu_1}, \quad w = uu_{n-1} - u_{-1} + u + 1,
\]

(56)

found in [31] as a symmetry in the direction of \( n \) of the quad equation

\[
u_{n+1,m+1}u_{n,m}(u_{n+1,m} - 1)(u_{n,m+1} + 1) + (u_{n+1,m} + 1)(u_{n,m+1} - 1) = 0
\]

(57)

derived in [32]. Lattice (56) reduces to one of the equations in Yamilov’s list in [28] by a nonautonomous change of variables. Here, our goal is to find the recursion operator for lattice (56) according to the scheme above. We set \( m = 0 \) for simplicity. We first find the linearization of (56):

\[
u_t = f^*U,
\]

(58)
where \( f^* \) is the Frechét derivative of \( f \)

\[
f^* = (-1)^{n+1} \frac{u(u^2-1)}{w_1} D + (-1)^n \frac{u_1 u_{-1} + 1}{w_1 w} \times \left( 3u^2 - 1 - \frac{u(u^2-1)}{w_1} - \frac{u(u^2-1)(u_{-1} - 1)}{w_{-1}} \right) + (-1)^n \frac{u(u^2-1)}{w} D^{-1}. \tag{59}
\]

We take the operator \( L_1 \) as in (39): \( L_1 = (D - 1)(1/u) \). We then find \( A \) using formula (40):

\[
A = A^{(1)} D + A^{(0)} + A^{(-1)} D^{-1}, \tag{60}
\]

where

\[
A^{(1)} = (-1)^{n+1} \frac{u_2(u^2-1)(u_3 u_2 + 1) w_1}{(u_2 u + 1) w_2 w_3},
\]

\[
A^{(0)} = (-1)^{n+1} \frac{u(u^2-1)(u_1 u_{-1} + 1) w_2}{(u_2 u + 1) w_2^2 w} + (-1)^{n+1} \frac{u_1 (u_1^2 - 1)(u_2 u + 1) w}{(u_1 u_{-1} + 1) w_2^2 w_2}, \tag{61}
\]

\[
A^{(-1)} = (-1)^{n+1} \frac{u_{-1} (u_{-1}^2 - 1)(u u_{-2} + 1) w_1}{(u_1 u_{-1} + 1) w^2 w_{-1}}.
\]

The next step is to find the operator \( L_2 \) from Eq. (41). We find the coefficients of \( L_2 \) using the explicit formulas (42)–(47) and the first canonical conservation laws (32)–(34). As a result, we obtain

\[
L_2 = -\frac{w_1}{w_2 (uw_2 + 1)} D^2 - \left( \frac{2(u + 1)}{(u_1 + 1)(uw_2 + 1)} + \frac{w_1 w_2}{2u(u^2-1)(uw_2 + 1)} + \right. \\
+ \left. \frac{u_{-1} u_1 - 1}{u_1 (u_1^2 - 1)(uw_2 + 1)} - \frac{2u_1}{u_1^2 - 1} - \frac{1}{u_1 (u_{-1} u_1 + 1)} + \frac{u_2 - 1}{w_2} \right) D + \\
+ \left( \frac{2(u_1 - 1)}{(u - 1)(u_1 u_{-1} + 1)} - \frac{2u(u^2-1)(u_1 u_{-1} + 1)}{u_1 (u_1 u_{-1} + 1)} + \frac{w w_1}{u_1 u_{-1} + 1} \right) + \\
+ \left( \frac{2u}{u^2 - 1} + \frac{1}{u(u_2 u + 1)} - \frac{w_1}{u_1 u_{-1} + 1} \right) D^{-1}. \tag{62}
\]

The important step is to verify whether the corresponding equations of form (48) are satisfied. We verified that these equations hold in this case. Hence, we have the recursion operator \( R = L_1^{-1} L_2 \):

\[
R = (-1)^{n+1} \frac{u(u^2-1)}{w_1^2} D + (-1)^{n+1} \left( \frac{u(u^2-1)(u_1 - 1)(u_1 u_{-1} + 1)}{w_1^2 w} + \frac{w}{2u_{-1} w_1} - \\
- \frac{2(u_{-1} - 1)(u_1 + 1)}{u_{-1} u w_1} + \frac{(u_{-1} - 1)(u_{-1} u_{-1} + 1)}{u_{-1} w w_1} + \\
+ \frac{2u(u - 1)(u_{-1} - 1)(u_{-1} u_{-1} + 1)}{u_{-1} w w_1} + \frac{u_{-1} - 1}{u_{-1} w w_1} - \frac{u + 1}{u_{-1} w w_1} - \\
- \frac{(u - 1)(u_{-1}^2)(u_{-1} u_{-1} + 1)}{2u_{-1} w w_1^2} \right) + (-1)^{n+1} \frac{u(u^2-1)}{w^2} D^{-1} - \\
- u(D - 1)^{-1} \left( \frac{2(u_{-1} + 1)}{w} + \frac{2(u_1 - 1)}{w_1} + \frac{3u^2 - 1}{u(u^2 - 1)} \right). \tag{63}
\]
3.4. Recursion operator for a coupled lattice. We discuss an algorithm for constructing the recursion operator in the example of a system of lattices [33] that reduces to one of the equations on Yamilov’s list (see [28]) by a point transformation:

\[ u_{n,t} = \frac{1}{v_{n+1}} + u_n^2 v_n, \]
\[ v_{n,t} = -\frac{1}{u_{n-1}} - u_n v_n^2. \]

The linearized system can be represented as

\[ \frac{d}{dt} \begin{pmatrix} U_n \\ V_n \end{pmatrix} = F_n^* \begin{pmatrix} U_n \\ V_n \end{pmatrix}, \]

where the linearization operator or the Fréchet derivative is given by

\[ F_n^* = F_n^{(1)} D_n + F_n^{(0)} + F_n^{(-1)} D_n^{-1}, \]

where

\[ F_n^{(1)} = \begin{pmatrix} 0 & -\frac{1}{u_n^2} \\ 0 & 0 \end{pmatrix}, \quad F_n^{(0)} = \begin{pmatrix} 2u_n v_n & u_n^2 \\ -v_n^2 & -2u_n v_n \end{pmatrix}, \quad F_n^{(-1)} = \begin{pmatrix} 0 & 0 \\ \frac{1}{u_n^2} & 0 \end{pmatrix}. \]

The linearized equation in the coordinate representation has the form

\[ U_t = -\frac{1}{v_1^2} V_1 + 2uvU + u^2 V, \quad V_t = \frac{1}{u_{-1}^2} U_{-1} - 2uvV - v^2 U. \]

To construct the linear operator \( L_1 \), we use the classical symmetry \( u_\tau = cu, \ v_\tau = -cv \), which is obviously connected with the solution of the linearized equation \( U = cu, \ V = -cv \). Eliminating the constant parameter \( c \), we obtain the system of equations

\[ (D-1)\frac{1}{v} V = 0, \quad U = -\frac{u}{v} V. \]

defining an invariant manifold. We hence have

\[ L_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} D + \begin{pmatrix} 1 & \frac{u}{v} \\ 0 & -\frac{1}{v} \end{pmatrix}. \]

We find the operator \( A \) from the equation \( A = D_1(L_1) L_1^{-1} + L_1 F^* L_1^{-1} \):

\[ A = \begin{pmatrix} 0 & 0 \\ -v_1 & 0 \end{pmatrix} D + \begin{pmatrix} uv & -\frac{1}{v_1} \\ v + \frac{1}{v_1u^2} & \frac{1}{wv_1} \end{pmatrix} + \begin{pmatrix} u & \frac{u}{vu_{-1}} \\ \frac{1}{vu_{-1}} & -\frac{1}{v_1u_{-1}} \end{pmatrix} D^{-1}. \]

We then seek the operator \( L_2 = aD + b + cD^{-1} \) from the equation \( D_1(L_2) = AL_2 + L_2 F^* \). The answer is

\[ L_2 = \begin{pmatrix} 0 & -\frac{1}{v_1} \\ v_1 - \frac{2}{v_1^2u} \end{pmatrix} D + \begin{pmatrix} -uv & -u^2 \\ v - \frac{1}{v_1^2u} & 2u \end{pmatrix} + \begin{pmatrix} \frac{1}{vu_{-1}} & 0 \\ \frac{u}{vu_{-1}} & 0 \end{pmatrix} D^{-1}. \]
Therefore, the recursion operator is given by $R = L^{-1}_1 L_2$. We write it in the explicit form [3]

$$R = L^+ + 2 \left( \begin{array}{c} -u \\ v \end{array} \right) (D - 1)^{-1} \left( \begin{array}{c} 1 \\ \frac{1}{u^2 v_1} - v, \frac{1}{v^2 u_1} - u \end{array} \right),$$

(72)

where

$$L^{(+)} = \left( \begin{array}{cc} 0 & -\frac{1}{v_1^2} \\ 0 & 0 \end{array} \right) D + \left( \begin{array}{cc} 2u v & u^2 - \frac{2u}{u_1 v^2} \\ -v^2 & \frac{2}{u_1 v} \end{array} \right) + \left( \begin{array}{cc} 0 & 0 \\ -\frac{1}{v_1^2} & 0 \end{array} \right) D^{-1}. \quad (73)$$

4. Quadratic invariant manifolds, Lax pairs, and Dubrovin equations

We illustrate the use of the nonlinear invariant manifolds to construct Lax pairs for integrable models with the example of the KdV equation

$$u_t = u_{xxx} + u u_x. \quad (74)$$

Substituting $U = W_x$, we change the linearized equation

$$U_t = U_{xxx} + u U_x + u U \quad \Rightarrow \quad W_t = W_{xxx} + u W_x \quad (75)$$

and find an ODE compatible with these equations (also see [4])

$$W_{xx} = F(W_x, W, u), \quad (76)$$

i.e., we require that

$$\left. \frac{d}{dt}(W_{xx}) - \frac{d^2}{dx^2}(W_t) \right|_{(74), (75), (76)} = 0. \quad (77)$$

Equation (77) must be satisfied identically for all values of $W$, $W_x$, $u$, $u_x$, $u_{xx}$, etc. The consistency condition reduces to a huge equation, which splits into seven equations and is effectively solved,

$$3F_{uu} u_x u_{xx} + (3W_x F_{Wu} + W_x + 3F_{Wxu} F) u_{xx} + (3F_{Wuu} W_x + 3F_{Wux} u F +$$
$$+ 3F_{Wxu} F) u_x^2 + F_{uuu} u_x^3 + (3F_{Wx} F_{Wx} u F + 2F + 3F_{WWu} W_x^2 - F_{W} W_x +$$
$$+ 3W_x F_{Wx} W + 6W_x F_{Wx} W u F + 3W_x F_{Wx} W F u + 3F_{Wu} F +$$
$$+ 3F_{Wx} W F^2 + 3W_x F_{W} W F + 3W_x F_{Wx} W F + 3F_{Wx} W^2 F + 3F_{Wx} W^2 F^2 +$$
$$+ 3W_x^2 F_{Wx} W F + 3W_x^2 F_{Wx} W F + 3W_x F_{Wx} W F + 3W_x F_{Wx} W F + 3W_x F_{Wx} W F^2 = 0. \quad (78)$$

We compare the coefficients of the independent variables $u_{xx}$ and $u_x$:

1. for $u_{xx} u_x$,

$$\frac{d^2}{du^2} F(W_x, W, u) = 0 \quad \Rightarrow \quad F(W_x, W, u) = F_1(W_x, W) u + F_2(W_x, W),$$

2. for $u_{xx} u$,

$$\frac{d}{dW_x} F_1(W_x, W) = 0 \quad \Rightarrow \quad F_1(W_x, W) = F_1(W),$$

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3. for $u_{xx}$,

$$3 \frac{d}{dW} F_1(W) + 1 = 0 \quad \Rightarrow \quad F_1(W) = -\frac{1}{3} W + c_1,$$

4. for $u_x u$,

$$\frac{d^2}{dW^2} F_2(W_x, W) - \frac{1}{W - 3c_1} = 0 \quad \Rightarrow \quad F_2(W_x, W) = \frac{W_x^2}{2(W - 3c_1)} + F_4(W) W_x + F_3(W),$$

5. for $u_x$,

$$(W - 3c_1) \frac{d}{dW} F_3(W) + F_4(W) = 0 \quad \Rightarrow \quad F_3(W) = \frac{c_2}{W - 3c_1},$$

6. for $uW_x$, $c_2 = 0$, and

7. for $u$,

$$(W - 3c_1)^2 \frac{d^2}{dW^2} F_3(W) + (W - 3c_1) \frac{d}{dW} F_3(W) - F_3(W) = 0 \quad \Rightarrow \quad F_3(W) = \frac{c_3}{W - 3c_1} + c_4(W - 3c_1).$$

Finally, we obtain the ODE $W_{xx} = F(W_x, W, u)$, where

$$F(W_x, W, u) = \left( -\frac{1}{3} W + c_1 \right) u + \frac{W_x^2}{2(W - 3c_1)} + \frac{c_3}{W - 3c_1} + c_4(W - 3c_1).$$

The parameter $c_1$ is easily eliminated by the shift $\overline{W} = W - 3c_1$, and we therefore set $c_1 = 0$. Let $c_4 = \lambda$. Then

$$W_{xx} W - \left( \lambda - \frac{1}{3} u \right) W^2 - \frac{1}{2} W_x^2 - c_3 = 0.$$  \((78)\)

We have two choices: $c_3 = 0$ and $c_3 \neq 0$.

1. If $c_3 = 0$, then we obtain a nonlinear Lax pair:

$$W_{xx} = -\frac{1}{3} W u + \frac{W_x^2}{2W} + \lambda W,$$

$$W_t = \left( 2\lambda - \frac{2}{3} u \right) W_x - \frac{1}{3} u_x W.$$  \((79)\)

It is linearized by setting $W = \varphi^2$ and reduces to the usual pair

$$\varphi_{xx} = \left( -\frac{1}{6} u + \frac{1}{2} \lambda \right) \varphi,$$

$$\varphi_t = \left( 2\lambda - \frac{2}{3} u \right) \varphi_x - \frac{1}{6} u_x \varphi.$$  \((80)\)

2. If $c_3 \neq 0$, then we obtain a well-known equation connected with finite-gap solutions (see [34]):

$$W_{xx} W - \left( \lambda - \frac{1}{3} u \right) W^2 - \frac{1}{2} W_x^2 - c_3(\lambda) = 0.$$  \((81)\)
We assume that $W$ and $c_3 = c_3(\lambda)$ are polynomials of the forms

$$W = \prod_{j=1}^{n} (\lambda - r_j), \quad c_3(\lambda) = - \prod_{j=1}^{2n+1} (\lambda - e_j). \quad (82)$$

We recall that the functions $r_j$ satisfy the Dubrovin equations

$$r_{j,x} \prod_{i \neq j} (r_j - r_i) = \sqrt{2} \prod_{s=1}^{2n+1} (r_j - e_s), \quad j = 1, \ldots, n. \quad (83)$$

The potential $u$ is found as

$$u = 3 \sum_{i=1}^{2n+1} e_i - 6 \sum_{i=1}^{n} r_i.$$

We consider the Volterra lattice

$$u_{n,t} = u_n(u_{n+1} - u_{n-1}). \quad (84)$$

The linearization of (83),

$$U_{n,t} = U_n(u_{n+1} - u_{n-1}) + u_n(U_{n+1} - U_{n-1}), \quad (85)$$

is rather complicated: it contains $u_n, u_{n+1},$ and $u_{n-1}$. We simplify it by substituting $U_n = u_n(P_{n+1} - P_{n-1})$, which is the linearization of the substitution $\log u_n = p_{n+1} - p_{n-1}$ relating the Volterra lattice to the equation $p_{nt} = e^{p_{n+1} - p_{n-1}}$. From (84), we thus obtain the equation

$$P_{n,t} = u_n(P_{n+1} - P_{n-1}), \quad (86)$$

which contains only a single parameter $u_n$. We seek the invariant manifold in the form

$$P_{n+1} = F(P_n, P_{n-1}, u_n),$$

i.e., an ODE with the parameter $u_n$.

The answer is

$$u_n(P_{n+1} + P_n)(P_n + P_{n-1}) = \lambda P_n^2 + c, \quad c = \text{const.} \quad (87)$$

We assume that the functions $P_n$ and $c$ are polynomials in $\lambda$:

$$P_n = \prod_{j=1}^{m} (\lambda - \gamma_j(n)), \quad c(\lambda) = - \prod_{j=1}^{2m+1} (\lambda - e_j). \quad (88)$$

For $\gamma_j(n)$ and $u_n$, we then obtain difference equations from (87),

$$u_n \prod_{i=1}^{m} (\gamma_j(n) - \gamma_i(n + 1))(\gamma_j(n) - \gamma_i(n - 1)) = -2 \prod_{i=1}^{2m+1} (\gamma_j(n) - e_i),$$

$$u_n = \frac{1}{4} \sum_{i=1}^{2m+1} e_i - \frac{1}{2} \sum_{i=1}^{m} \gamma_i(n), \quad j = 1, \ldots, m.$$
which are the discrete versions of the Dubrovin equations (see [34]).

We assume that \( c = 0 \). The invariant surface then becomes

\[
u_n(P_{n+1} + P_n)(P_n + P_{n-1}) = \lambda P_n^2.
\]

(89)

From linearized equation (89), we now obtain the equation

\[
P_{n,t} = u_n \left( \frac{\lambda P_n^2}{P_n + P_{n-1}} - P_n - P_{n-1} \right)
\]

(90)

describing the time evolution of \( P \). We derive the Lax pair for the Volterra lattice. Because Eq. (89) is homogeneous, it is reasonable to set \( Z = P/P_{n-1} \) and reduce (89) to the discrete Riccati equation

\[
Z_{n+1} = \frac{Z_n(\lambda/u_n - 1) - 1}{Z_n + 1}.
\]

(91)

We standardly linearize (91) by introducing \( Z_{n+1} = \varphi_n/\varphi_{n-1} \):

\[
\varphi_{n+1} = \frac{\lambda}{u_n}(\varphi_n - \varphi_{n-1}).
\]

(92)

Differentiating the equality \( P_n/P_{n-1} = (\varphi_n/\varphi_{n-1}) - 1 \), after some transformations, we obtain

\[
\frac{\varphi_{n,t}}{\varphi_n} - \frac{\varphi_{n-1,t}}{\varphi_{n-1}} = \lambda - u_n + u_{n-1} - \lambda \frac{\varphi_{n-1}}{\varphi_n} - u_{n-1} \frac{\varphi_n}{\varphi_{n-1}}.
\]

(93)

We now assume that the time evolution of the eigenfunctions is given by a linear system of the form

\[
\varphi_{n,t} = a_n \varphi_n + b_n \varphi_{n-1}, \quad \varphi_{n-1,t} = c_n \varphi_n + d_n \varphi_{n-1}
\]

and substitute this expression in (93). Comparing the coefficients of powers of the independent variables \( \varphi_n \) and \( \varphi_{n-1} \), we obtain equations for the sought functions \( a_n, b_n, c_n, \) and \( d_n \):

\[
b_n = -\lambda, \quad c_n = u_{n-1}, \quad a_n - d_n = \lambda - u_n + u_{n-1}.
\]

(94)

In addition, we have \( D(\varphi_{n-1,t}) = \varphi_{n,t} \), which implies that \( a_n = \lambda + d_{n+1} \). Summarizing the above reasoning, we have

\[
\varphi_{n,t} = (\lambda - u_n) \varphi_n - \lambda \varphi_{n-1},
\]

(95)

\[
\varphi_{n-1,t} = u_{n-1} \varphi_n - u_{n-1} \varphi_{n-1}.
\]

(96)

The change of the variables \( \varphi_n = \lambda^{n/2} \prod_{j=n}^{\infty} u_j \psi_{n-1} \) reduces Eqs. (92) and (95) to the usual Lax pair for the Volterra equation:

\[
\psi_{n+1} = \xi \psi_n - u_n \psi_{n-1}, \quad \xi = \sqrt{\lambda},
\]

\[
\psi_{n,t} = (\xi^2 + u_n) \psi_n - \xi u_n \psi_{n-1}.
\]
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