The Hilb/Sym correspondence for $\mathbb{C}^2$: descendents and Fourier–Mukai

Rahul Pandharipande$^1$ · Hsian-Hua Tseng$^2$

Received: 2 October 2018 / Revised: 7 August 2019 / Published online: 20 August 2019
© Springer-Verlag GmbH Germany, part of Springer Nature 2019

Abstract
We study here the crepant resolution correspondence for the $\mathbb{T}$-equivariant descendent Gromov–Witten theories of $\text{Hilb}^n(\mathbb{C}^2)$ and $\text{Sym}^n(\mathbb{C}^2)$. The descendent correspondence is obtained from our previous matching of the associated CohFTs by applying Givental’s quantization formula to a specific symplectic transformation $K$. The first result of the paper is an explicit computation of $K$. Our main result then establishes a fundamental relationship between the Fourier–Mukai equivalence of the associated derived categories (by Bridgeland, King, and Reid) and the symplectic transformation $K$ via Iritani’s integral structure. The results use Haiman’s Fourier–Mukai calculations and are exactly aligned with Iritani’s point of view on crepant resolution.

Contents

0 Introduction ............................................. 510
1 Quantum differential equations ................................... 515
2 Descendent Gromov–Witten theory .................................. 519
3 Descendent correspondence ..................................... 525
4 Fourier–Mukai transformation .................................... 528
References ................................................ 539

Communicated by Vasudevan Srinivas.

$\hhtseng$ Hsian-Hua Tseng
hhtseng@math.ohio-state.edu
Rahul Pandharipande
rahul@math.ethz.ch

1 Department of Mathematics, ETH Zürich, Rämistrasse 101, 8092 Zurich, Switzerland
2 Department of Mathematics, Ohio State University, 100 Math Tower, 231 West 18th Ave., Columbus, OH 43210, USA
0 Introduction

0.1 Overview

The diagonal action on $\mathbb{C}^2$ of the torus $T = (\mathbb{C}^*)^2$ lifts canonically to the Hilbert scheme of $n$ points $\text{Hilb}^n(\mathbb{C}^2)$ and the orbifold symmetric product

$$\text{Sym}^n(\mathbb{C}^2) = \left[\left(\mathbb{C}^2\right)^n / \Sigma_n\right].$$

Both the Hilbert–Chow morphism

$$\text{Hilb}^n(\mathbb{C}^2) \to \left(\mathbb{C}^2\right)^n / \Sigma_n$$

(0.1)

and the coarsification morphism

$$\text{Sym}^n(\mathbb{C}^2) \to \left(\mathbb{C}^2\right)^n / \Sigma_n$$

(0.2)

are $T$-equivariant crepant resolutions of the singular quotient variety $\left(\mathbb{C}^2\right)^n / \Sigma_n$.

The geometries of the two crepant resolutions $\text{Hilb}^n(\mathbb{C}^2)$ and $\text{Sym}^n(\mathbb{C}^2)$ are connected in many beautiful ways. The classical McKay correspondence [19] provides an isomorphism on the level of $T$-equivariant cohomology: $T$-equivariant singular cohomology for $\text{Hilb}^n(\mathbb{C}^2)$ and $T$-equivariant Chen–Ruan orbifold cohomology for $\text{Sym}^n(\mathbb{C}^2)$. A lift of the McKay correspondence to an equivalence of $T$-equivariant derived categories was proven by Bridgeland et al. [4] using a Fourier–Mukai transformation.

Quantum cohomology provides a different enrichment of the McKay correspondence. For the crepant resolutions $\text{Hilb}^n(\mathbb{C}^2)$ and $\text{Sym}^n(\mathbb{C}^2)$, the genus 0 equivalence of the $T$-equivariant Gromov–Witten theories was proven in [5] using [6,22]. Going further, the crepant resolution correspondence in all genera was proven in [25] by matching the associated $R$-matrices and Cohomological Field Theories (CohFTs), see [24, Section 4] for a survey.

The results of [5,25] concern the $T$-equivariant Gromov–Witten theory with primary insertions. However, following a remarkable proposal of Iritani, to see the connection between the Fourier–Mukai transformation of [4] and the crepant resolution correspondence for Gromov–Witten theory, descendent insertions are required. Our first result here is a determination of the crepant resolution correspondence for the $T$-equivariant Gromov–Witten theories of $\text{Hilb}^n(\mathbb{C}^2)$ and $\text{Sym}^n(\mathbb{C}^2)$ with descendent insertions via a symplectic transformation $K$ which we compute explicitly. The main result of the paper is a proof of a fundamental relationship between the Fourier–Mukai equivalence of the associated derived categories [4] and the symplectic transformation $K$ via Iritani’s integral structure. The results use Haiman’s Fourier–Mukai calculations [12,13] and are exactly aligned with Iritani’s point of view on crepant resolutions [16,17].
0.2 Descendent correspondence

The descendent correspondence for the $T$-equivariant Gromov–Witten theories of $\text{Hilb}^n(\mathbb{C}^2)$ and $\text{Sym}^n(\mathbb{C}^2)$ is obtained from the CohFT matching of [25] together with the quantization formula of Givental [11]. Our first result is a formula for the symplectic transformation

$$K \in \text{Id} + z^{-1} \cdot \text{End}(H^*_T(\text{Hilb}^n(\mathbb{C}^2)))[[z^{-1}]]$$

defining the descendent correspondence.\footnote{1}

The formula for $K$ is best described in terms of the Fock space $\mathcal{F}$ which is freely generated over $\mathbb{C}$ by commuting creation operators $\alpha_{-k}$ for $k \in \mathbb{Z}_{>0}$ acting on the vacuum vector $v_\emptyset$. The annihilation operators $\alpha_k, k \in \mathbb{Z}_{>0}$ satisfy

$$\alpha_k \cdot v_\emptyset = 0, \quad k > 0$$

and commutation relations

$$[\alpha_k, \alpha_l] = k \delta_{k+l}.$$ 

The Fock space $\mathcal{F}$ admits an additive basis

$$|\mu\rangle = \frac{1}{\delta(\mu)} \prod_i \alpha_{-\mu_i} v_\emptyset, \quad \delta(\mu) = |\text{Aut}(\mu)| \prod_i \mu_i,$$

indexed by partitions $\mu = (\mu_1, \mu_2, \ldots)$.

An additive isomorphism

$$\mathcal{F} \otimes_{\mathbb{C}} \mathbb{C}[t_1, t_2] \cong \bigoplus_{n \geq 0} H^*_T(\text{Hilb}^n(\mathbb{C}^2)), \quad (0.3)$$

is given by identifying $|\mu\rangle$ on the left with the corresponding Nakajima basis elements on the right. The intersection pairing $(-, -)_{\text{Hilb}}$ on the $T$-equivariant cohomology of $\text{Hilb}^n(\mathbb{C}^2)$ induces a pairing on Fock space,

$$\eta(\mu, \nu) = \frac{(-1)^{|\mu|-\ell(\mu)}}{(t_1 t_2)^{\ell(\mu)}} \frac{\delta_{\mu \nu}}{\delta(\mu)}.$$ 

In the following result, we write the formula for $K$ in terms of the Fock space,

$$K \in \text{Id} + z^{-1} \cdot \text{End}(\mathcal{F} \otimes_{\mathbb{C}} \mathbb{C}[t_1, t_2])[z^{-1}],$$

using (0.3).

\footnote{1 Cohomology will always be taken here with $\mathbb{C}$-coefficients.}
Theorem 1 The descendent correspondence is determined by the symplectic transformation $K$ given by the formula

$$K(J^\lambda) = \frac{z^{2\lambda}}{(2\pi \sqrt{-1})^{2\lambda}} \prod_{w:T \text{-weights of } \text{Tan}\text{Hilb}^n(C^2)} \Gamma(w/z + 1) \bullet H^\mu_z.$$

Here, $J^\lambda$ is the Jack symmetric function defined by Eq. (1.5) of Sect. 1, and $H^\mu_z$ is the Macdonald polynomial, see [12,18,23]. The linear operator

$$\bullet : \mathcal{F} \to \mathcal{F}$$

is defined by

$$\bullet | \mu \rangle = z^{\ell(\mu)} \frac{(2\pi \sqrt{-1})^{\ell(\mu)}}{\prod_{i} \mu_i} \prod_{i} \frac{\mu^{\mu_1/z}_i \mu^{\mu_2/z}_i}{\Gamma(\mu_1/z) \Gamma(\mu_2/z)} \langle \mu \rangle.$$

The descendent correspondence in genus 0, expressed in terms of Givental’s Lagrangian cones, is explained in Theorem 10 of Sect. 3.2,

$$\mathcal{L}^{\text{Sym}} = CKq^{-D/z} \mathcal{L}^{\text{Hilb}},$$

where $D = -|(2, 1^{n-2})|$ is the T-equivariant first Chern class of the tautological vector bundle on $\text{Hilb}^n(C^2)$. The descendent correspondence for all $g$, formulated in terms of generating series,

$$e^{-F^{\text{Sym}}_i(t)} D^{\text{Sym}} = \mathcal{K} q^{-D/z} \left( e^{-F^{\text{Hilb}}_i(tD)} D^{\text{Hilb}} \right),$$

is discussed in Theorem 11 of Sect. 3.3.

For toric crepant resolutions, the symplectic transformation underlying the descendent correspondence is constructed in [9] by using explicit slices of Givental’s Lagrangian cones constructed via the Toric Mirror Theorem [7,10]. We proceed differently here. The symplectic transformation $K$ is constructed by comparing the two fundamental solutions $S^{\text{Hilb}}$ and $S^{\text{Sym}}$ of the QDE given by descendent Gromov–Witten invariants of $\text{Hilb}^n(C^2)$ and $\text{Sym}^n(C^2)$ respectively. Via the Hilb/Sym correspondence in genus 0, Theorem 1 is then simply a reformulation of the calculation of the connection matrix in [23, Theorem 4].

0.3 Fourier–Mukai

An equivalence of T-equivariant derived categories

$$\text{FM} : D^b_{\mathcal{F}}(\text{Hilb}^n(C^2)) \to D^b_{\mathcal{F}}(\text{Sym}^n(C^2))$$

\footnote{The footnote $z$ indicates a rescaling of the parameters, $H^\mu_z = H^\mu(t^1_z, t^2_z)$.}

\footnote{See for (2.5) the definition of the symplectic isomorphism $\mathbb{C}$.}
The Hilb/Sym correspondence for \( \mathbb{C}^2 \): descendents and Fourier–Mukai

is constructed by Bridgeland et al. [4] via a tautological Fourier–Mukai kernel. We also denote by \( \mathcal{FM} \) the induced isomorphism on \( T \)-equivariant \( K \)-groups,

\[
\mathcal{FM} : K_T(\text{Hilb}^n(\mathbb{C}^2)) \to K_T(\text{Sym}^n(\mathbb{C}^2)).
\]  

(0.4)

Iritani [16] has proposed a beautiful framework for the crepant resolution correspondence. In the case of \( \text{Hilb}^n(\mathbb{C}^2) \) and \( \text{Sym}^n(\mathbb{C}^2) \), the isomorphism (0.4) on \( K \)-theory should be related to a symplectic transformation

\[
\mathcal{H}_{\text{Hilb}} \to \mathcal{H}_{\text{Sym}}
\]

via Iritani’s integral structure. The Givental spaces \( \mathcal{H}_{\text{Hilb}} \) and \( \mathcal{H}_{\text{Sym}} \) will be defined below (in a multivalued form). A discussion of Iritani’s perspective can be found in [17]. Our main result is a formulation and proof of Iritani’s proposal for the crepant resolutions \( \text{Hilb}^n(\mathbb{C}^2) \) and \( \text{Sym}^n(\mathbb{C}^2) \). For the precise statement, further definitions are required.

• Define the operators \( \deg_{\text{Hilb}}^0 \), \( \rho_{\text{Hilb}} \), and \( \mu_{\text{Hilb}} \) on \( H^*_T(\text{Hilb}^n(\mathbb{C}^2)) \) as follows. For \( \phi \in H^*_T(\text{Hilb}^n(\mathbb{C}^2)) \),

\[
\deg_{\text{Hilb}}^0(\phi) = k\phi,
\]

\[
\mu_{\text{Hilb}}(\phi) = \left( \frac{k}{2} - \frac{2n}{2} \right) \phi,
\]

\[
\rho_{\text{Hilb}}(\phi) = c^T_1(\text{Hilb}^n(\mathbb{C}^2)) \cup \phi.
\]

The multi-valued Givental space \( \tilde{\mathcal{H}}_{\text{Hilb}} \) for \( \text{Hilb}^n(\mathbb{C}^2) \) is defined by

\[
\tilde{\mathcal{H}}_{\text{Hilb}} = H^*_T(\text{Hilb}^n(\mathbb{C}^2), \mathbb{C}) \otimes \mathbb{C}[t_1, t_2] \mathbb{C}(t_1, t_2)[[\log(z)]]((z^{-1})).
\]

Definition 2 Let \( \Psi_{\text{Hilb}} : K_T(\text{Hilb}^n(\mathbb{C}^2)) \to \tilde{\mathcal{H}}_{\text{Hilb}} \) be defined by

\[
\Psi_{\text{Hilb}}(E) = z^{-\mu_{\text{Hilb}}} z^{\rho_{\text{Hilb}}} \left( \Gamma_{\text{Hilb}} \cup (2\pi \sqrt{-1})^{\deg_{\text{Hilb}}^0} \text{ch}(E) \right),
\]

where \( \text{ch}(-) \) is the \( T \)-equivariant Chern character, \( \Gamma_{\text{Hilb}} \in H^*_T(\text{Hilb}^n(\mathbb{C}^2)) \) is the \( T \)-equivariant Gamma class of \( \text{Hilb}^n(\mathbb{C}^2) \) of [9, Section 3.1], and the operators

\[
z^{-\mu_{\text{Hilb}}} : \tilde{\mathcal{H}}_{\text{Hilb}} \to \tilde{\mathcal{H}}_{\text{Hilb}}, \quad z^{\rho_{\text{Hilb}}} : \tilde{\mathcal{H}}_{\text{Hilb}} \to \tilde{\mathcal{H}}_{\text{Hilb}}
\]

are defined by

\[
z^{-\mu_{\text{Hilb}}} = \sum_{k \geq 0} \frac{(-\mu_{\text{Hilb}} \log z)^k}{k!}, \quad z^{\rho_{\text{Hilb}}} = \sum_{k \geq 0} \frac{(\rho_{\text{Hilb}} \log z)^k}{k!}.
\]
Since $|\mu|$ is identified with the corresponding Nakajima basis element, we have
\[
\deg_{\text{Hilb}}^0 |\mu\rangle = 2(n - \ell(\mu))|\mu\rangle.
\]
Also, since $t_1, t_2$ both have degree 2, we have
\[
\deg_{\text{Hilb}}^0 t_1 = 2 = \deg_{\text{Hilb}}^0 t_2.
\]

- Define the operators \(\deg^0_{\text{Sym}}, \rho^\text{Sym}, \) and \(\mu^\text{Sym}\) on \(H^*_T(\text{Sym}^n(C^2))\) as follows. For \(\phi \in H^k_T(\text{Sym}^n(C^2))\),
\[
\deg^0_{\text{Sym}}(\phi) = k\phi,
\]
\[
\mu^\text{Sym}(\phi) = \left(\frac{\deg_{\text{CR}}(\phi)}{2} - \frac{2n}{2}\right)\phi,
\]
\[
\rho^\text{Sym}(\phi) = c_1^T(\text{Sym}^n(C^2)) \cup_{\text{CR}} \phi.
\]

There are two degree operators here: \(\deg^0_{\text{Sym}}\) extracts the usual degree of a cohomology class on the inertia orbifold, and \(\deg_{\text{CR}}\) extracts the age-shifted degree. Also, we have
\[
\deg_{\text{CR}} t_1 = \deg^0_{\text{Sym}} t_1 = 2 = \deg_{\text{CR}} t_2 = \deg^0_{\text{Sym}} t_2.
\]

The multi-valued Givental space \(\tilde{H}^\text{Sym}\) for \(\text{Sym}^n(C^2)\) is defined by
\[
\tilde{H}^\text{Sym} = H^*_T(\text{Sym}^n(C^2)) \otimes_{\mathbb{C}[t_1, t_2]} \mathbb{C}(t_1, t_2)[[\log z]]((z^{-1})).
\]

**Definition 3** Let \(\Psi^\text{Sym} : K_T(\text{Sym}^n(C^2)) \rightarrow \tilde{H}^\text{Sym}\) be defined by
\[
\Psi^\text{Sym}(E) = z^{-\mu^\text{Sym}} z^{\rho^\text{Sym}} \left(\Gamma^\text{Sym} \cup (2\pi \sqrt{-1})^{\deg^0_{\text{Sym}}} \tilde{\text{ch}}(E)\right),
\]
where \(\tilde{\text{ch}}(-)\) is the T-equivariant orbifold Chern character, \(\Gamma^\text{Sym} \in H^*_T(\text{Sym}^n(C^2))\) is the T-equivariant Gamma class of \(\text{Sym}^n(C^2)\) of [9, Section 3.1], and the operators
\[
z^{-\mu^\text{Sym}} : \tilde{H}^\text{Sym} \rightarrow \tilde{H}^\text{Sym}, \quad z^{\rho^\text{Sym}} : \tilde{H}^\text{Sym} \rightarrow \tilde{H}^\text{Sym}
\]
are defined by
\[
z^{-\mu^\text{Sym}} = \sum_{k \geq 0} \frac{(-\mu^\text{Sym} \log z)^k}{k!}, \quad z^{\rho^\text{Sym}} = \sum_{k \geq 0} \frac{(\rho^\text{Sym} \log z)^k}{k!}.
\]

\(^4\) In the definition of \(\rho^\text{Sym}\) we denote by \(\cup_{\text{CR}}\) the Chen–Ruan cup product on cohomology of the inertia stack.
The precise relationship between $\mathcal{FM}$ and $\mathcal{K}$ via Iritani’s integral structure is the central result of the paper.

**Theorem 4** The following diagram is commutative$^5$:

$$
\begin{array}{ccc}
\mathcal{K}_T(\text{Hilb}^n(\mathbb{C}^2)) & \xrightarrow{\mathcal{FM}} & \mathcal{K}_T(\text{Sym}^n(\mathbb{C}^2)) \\
\Psi_{\text{Hilb}} & & \Psi_{\text{Sym}} \\
\tilde{\mathcal{H}}_{\text{Hilb}} & \xrightarrow{\mathcal{C}\tilde{\mathcal{K}}_{z \mapsto -z}} & \tilde{\mathcal{H}}_{\text{Sym}}.
\end{array}
$$

The bottom row of the diagram of Theorem 4 is determined by the analytic continuation of solutions of the quantum differential equation of $\text{Hilb}^n(\mathbb{C}^2)$ along the ray from 0 to $-1$ in the $q$-plane [23, Theorem 4]. A lifting of monodromies of the quantum differential equation of $\text{Hilb}^n(\mathbb{C}^2)$ to autoequivalences of $D^b_T(\text{Hilb}^n(\mathbb{C}^2))$ has been announced by Bezrukavnikov and Okounkov in [20, Sections 3.2.8 and 5.2.7] and [21, Section 3.2]. In their upcoming paper [2], commutative diagrams parallel to Theorem 4 are constructed in cases of flops of holomorphic symplectic manifolds.$^6$

Theorem 4 fits into the framework of [2] if the relationship between $\text{Hilb}^n(\mathbb{C}^2)$ and $\text{Sym}^n(\mathbb{C}^2)$ is viewed morally as a flop in their sense.

A special aspect of the ray from 0 to $-1$ is the identification of the end result of the analytic continuation (the right side of the diagram) with the orbifold geometry $\text{Sym}^n(\mathbb{C}^2)$. The identification of the end results of other paths from 0 to $-1$ with geometric theories is an interesting direction of study. Are there twisted orbifold theories which realize these analytic continuations?

### 1 Quantum differential equations

#### 1.1 The differential equation

We recall the quantum differential equation for $\text{Hilb}^n(\mathbb{C}^2)$ calculated in [22] and further studied in [23]. We follow here the exposition [22,23].

The quantum differential equation (QDE) for the Hilbert schemes of points on $\mathbb{C}^2$ is given by

$$
q \frac{d}{dq} \Phi = M_D \Phi, \quad \Phi \in \mathcal{F} \otimes_{\mathbb{C}} \mathbb{C}(t_1, t_2),
$$

(1.1)

---

$^5$ Our variable $z$ corresponds to $-z$ in [9] as can be seen by the difference in the quantum differential Eq. (2.2) here and the quantum differential equation [9, equation (2.5)]. After the substitution $z \mapsto -z$ in $\mathcal{K}$, Theorem 4 matches the conventions of Iritani’s framework in [9].

---

$^6$ In fact, the study of commutative diagrams connecting derived equivalences and the solutions of the quantum differential equation has old roots in the subject. See, for example, [3,14]. These papers refer to talks of Kontsevich on homological mirror symmetry in the 1990s for the first formulations.
where $M_D$ is the operator of quantum multiplication by $D = -|2, 1^{n-2} \rangle$,

$$M_D = (t_1 + t_2) \sum_{k>0} \frac{k}{2} \frac{(-q)^k + 1}{(-q)^k - 1} \alpha_{-k} \alpha_k - \frac{t_1 + t_2}{2} \frac{(-q) + 1}{(-q) - 1} | \cdot | + \frac{1}{2} \sum_{k,l>0} \left[ t_1 t_2 \alpha_{k+l} \alpha_{-k} \alpha_{-l} - \alpha_{-k-l} \alpha_k \alpha_l \right].$$

Here $| \cdot | = \sum_{k>0} \alpha_{-k} \alpha_k$ is the energy operator.

While the quantum differential equation (1.1) has a regular singular point at $q = 0$, the point $q = -1$ is regular.

The quantum differential equation considered in Givental’s theory contains a parameter $z$. In the case of the Hilbert schemes of points on $\mathbb{C}^2$, the QDE with parameter $z$ is

$$zq \frac{d}{dq} \Phi = M_D \Phi, \quad \Phi \in \mathcal{F} \otimes_{\mathbb{C}} \mathbb{C}(t_1, t_2).$$

(1.3)

For $\Phi \in \mathcal{F} \otimes_{\mathbb{C}} \mathbb{C}(t_1, t_2)$, define

$$\Phi_z = \Phi \left( \frac{t_1}{z}, \frac{t_2}{z}, q \right).$$

(1.4)

Define $\Theta \in \text{Aut}(\mathcal{F})$ by

$$\Theta | \mu \rangle = z^{\ell(\mu)} | \mu \rangle.$$

The following Proposition allows us to use the results in [23].

**Proposition 5** If $\Phi$ is a solution of (1.1), then $\Theta \Phi_z$ is a solution of (1.3).

Proposition 5 follow immediately from the following direct computation.

**Lemma 6** For $k > 0$, we have $\Theta \alpha_k = \frac{1}{z} \alpha_k \Theta$ and $\Theta \alpha_{-k} = z \alpha_{-k} \Theta$.

1.2 Solutions

We recall the solution of QDE (1.1) constructed in [23]. Let

$$J_{\lambda} \in \mathcal{F} \otimes_{\mathbb{C}} \mathbb{C}(t_1, t_2)$$

be the integral form of the Jack symmetric function depending on the parameter $\alpha = 1/\theta$ of [18,23]. Then

$$J_{\lambda} = t_2^{|\lambda|} \ell(\cdot) J_{\lambda} |_{\alpha=-t_1/t_2}$$

(1.5)
is an eigenfunction of $M_D(0)$ with eigenvalue $-c(\lambda; t_1, t_2) := -\sum_{(i, j) \in \lambda} (j - 1) t_1 + (i - 1) t_2$. The coefficient of

$$|\mu\rangle \in \mathcal{F} \otimes \mathbb{C}(t_1, t_2)$$

in the expansion of $J^\lambda$ is $(t_1 t_2)^{\ell(\mu)}$ times a polynomial in $t_1$ and $t_2$ of degree $|\lambda| - \ell(\mu)$.

The paper [23] also uses a Hermitian pairing $\langle - , - \rangle_H$ on the Fock space $\mathcal{F}$ defined by the three following properties

- $\langle \mu | \nu \rangle_H = \frac{1}{(t_1 t_2)^{\delta(\mu)}},$ 
- $\langle af, g \rangle_H = a \langle f, g \rangle_H,$ $a \in \mathbb{C}(t_1, t_2),$ 
- $\langle f, g \rangle_H = \overline{\langle g, f \rangle}_H,$ where $a(t_1, t_2) = a(-t_1, -t_2).$

By a direct calculation, we find

$$\langle J^\lambda, J^\mu \rangle_H = \eta(J^\lambda, J^\mu),$$

where $\eta$ is the $T$-equivariant pairing on $\text{Hilb}^n(\mathbb{C}^2)$. Since $J^\lambda$ corresponds to the $T$-equivariant class of the $T$-fixed point of $\text{Hilb}^n(\mathbb{C}^2)$ associated to $\lambda$,

$$||J^\lambda||^2 = ||J^\lambda||^2_H = \prod_{w : \text{tangent weights at } \lambda} w$$

see [23].

There are solutions to (1.1) of the form

$$Y^\lambda(q)q^{-c(\lambda; t_1, t_2)}, \quad Y^\lambda(q) \in \mathcal{F} \otimes \mathbb{C}(t_1, t_2)[[q]],$$

which converge for $|q| < 1$ and satisfy $Y^\lambda(0) = J^\lambda$. We refer to [15, Chapter XIX] for a discussion of how these solutions are constructed.

By [23, Corollary 1],

$$\langle Y^\lambda(q), Y^\mu(q) \rangle_H = \delta_{\lambda, \mu} ||J^\lambda||^2_H = \langle J^\lambda, J^\mu \rangle_H.$$

As in [23, Section 3.1.3], let $Y$ be the matrix whose column vectors are $Y^\lambda$. Fix an auxiliary basis $\{e_\lambda\}$ of $\mathcal{F}$. We then view $Y$ as the matrix representation of an operator such that $Y(e_\lambda) = Y^\lambda$.

Define the following further diagonal matrices in the basis $\{e_\lambda\}$:

| Matrix | Eigenvalues |
|--------|-------------|
| $L$    | $z^{-|\lambda|} \prod_{w : \text{tangent weights at } \lambda} w^{1/2}$ |
| $L_0$  | $q^{-c(\lambda; t_1, t_2)/\varepsilon}$ |

7 In the domain of $Y$ we use the basis $\{e_\lambda\}$, while in the range of $Y$ we use the basis $\{|\mu\rangle\}$. Springer
Define

\[ Y_z = Y \left( \frac{t_1}{z}, \frac{t_2}{z}, q \right). \]

Consider the following solution to (1.3),

\[ S = \Theta Y_z L^{-1} L_0. \] (1.9)

We may view \( S \) as the matrix representation of an operator where in the domain we use the basis \( \{ e_\lambda \} \) while in the range we use the basis \( \{ |\mu\rangle \} \).

**Proposition 7** \( \Theta Y_z L^{-1} \) can be expanded into a convergent power series in \( 1/z \) with coefficients \( \text{End}(\mathcal{F}) \)-valued analytic functions in \( q, t_1, t_2 \).

**Proof** Let \( \Phi^\lambda \) be the column of \( \Theta Y_z L^{-1} \) indexed by \( \lambda \). By construction of \( Y \),

\[
\Theta Y_z L^{-1} \bigg|_{q=0} = \Theta J_z L^{-1},
\]

hence \( \Phi^\lambda \bigg|_{q=0} = \Theta J_z \lambda \prod \text{tangent weights at } \lambda \ w^{-1/2} \). Write \( J^\lambda = \sum \epsilon J^\lambda_\epsilon (t_1, t_2) |\epsilon\rangle \).

Then we have

\[
\Theta J^\lambda_\epsilon \lambda \prod = \sum \epsilon J^\lambda_\epsilon (t_1/z, t_2/z) z^{\ell(\epsilon)} z^{\lambda |\epsilon|} |\epsilon\rangle
\]

\[ = \sum \epsilon J^\lambda_\epsilon (t_1, t_2) z^{-2\ell(\epsilon)} z^{\lambda |\epsilon|} z^{\ell(\epsilon)} z^{\lambda |\epsilon|} |\epsilon\rangle = J^\lambda. \]

Together with (1.7), we find \( \Phi^\lambda \bigg|_{q=0} = J^\lambda / ||J^\lambda||. \)

Since \( S \) is a solution to (1.3), \( \Phi^\lambda \) is a solution to the differential equation

\[ zq \frac{d}{dq} \Phi^\lambda = (M_D + c(\lambda; t_1, t_2)) \Phi^\lambda. \] (1.10)

By uniqueness of solutions to (1.10) with given initial conditions, \( \Phi^\lambda \) can also be constructed using the Peano–Baker series (see [1]) with the initial condition

\[ \Phi^\lambda \bigg|_{q=0} = J^\lambda / ||J^\lambda||. \]

As the Peano-Baker series is manifestly a power series in \( z^{-1} \) with analytic coefficients, the Proposition follows. \( \square \)
2 Descendent Gromov–Witten theory

2.1 Hilbert schemes

Let $S_{\text{Hilb}}(q, t_D)$ be the generating series of genus 0 descendent Gromov–Witten invariants of $\text{Hilb}^n(C^2)$,

$$\eta(a, S_{\text{Hilb}}(q, t_D)b) = \eta(a, b)
+ \sum_{k \geq 0} z^{-1-k} \sum_{m,d} \frac{q^d}{m!} \langle a, t_D D, \ldots, t_D D, b \psi_{m+2}^{k} \rangle_{0, d}^\text{Hilb}^n(C^2)$$  \hspace{1cm} (2.1)

By definition, $S_{\text{Hilb}}$ is a formal power series in $1/z$ whose coefficients are in $\text{End}(\mathcal{F})[t_D][[q]]$, written in the basis $\{|\mu\rangle\}$. $S_{\text{Hilb}}(q, t_D)$ satisfies the following two differential equations:

$$z \frac{\partial}{\partial t_D} S_{\text{Hilb}}(q, t_D) = (D \star t_D) S_{\text{Hilb}}(q, t_D),$$  \hspace{1cm} (2.2)

$$z q \frac{\partial}{\partial q} S_{\text{Hilb}}(q, t_D) - z \frac{\partial}{\partial t_D} S_{\text{Hilb}}(q, t_D) = - S_{\text{Hilb}}(q, t_D)(D \cdot).$$  \hspace{1cm} (2.3)

Here $(D \star t_D) = (D \star t_D D)$ is the operator of quantum multiplication by the divisor $D$ at the point $t_D D$, and $(D \cdot)$ is the operator of classical cup product by $D$. In particular,

$$(D \star t_D) \big|_{t_D = 0} = M_D(q), \quad (D \cdot) \big|_{t_D = 0} = M_D(0).$$  \hspace{1cm} (2.4)

Equation (2.2) follows from the topological recursion relations in genus 0. Equation (2.3) follows from the divisor equations for descendent Gromov–Witten invariants.

We first determine $S_{\text{Hilb}}\big|_{t_D = 0}$. Combining (2.2) and (2.3) and setting $t_D = 0$, we find

$$z q \frac{\partial}{\partial q} \left( S_{\text{Hilb}}\big|_{t_D = 0} \right) = M_D(q) \left( S_{\text{Hilb}}\big|_{t_D = 0} \right) - \left( S_{\text{Hilb}}\big|_{t_D = 0} \right) M_D(0).$$

\hspace{1cm} \footnote{We use $t_D$ to denote the coordinate of $D$.}
So, we see

\[ zq \frac{\partial}{\partial q} \left( S_{\text{Hilb}} \big|_{tD=0} J^\lambda / ||J^\lambda|| \right) \]

\[ = M_D(q) \left( S_{\text{Hilb}} \big|_{tD=0} J^\lambda / ||J^\lambda|| \right) - M_D(0) J^\lambda / ||J^\lambda|| \]

\[ = M_D(q) \left( S_{\text{Hilb}} \big|_{tD=0} J^\lambda / ||J^\lambda|| \right) + c(\lambda; t_1, t_2) \left( S_{\text{Hilb}} \big|_{tD=0} J^\lambda / ||J^\lambda|| \right). \]

Since \( S_{\text{Hilb}} \big|_{tD=0, q=0} = \text{Id} \), we have \( (S_{\text{Hilb}} \big|_{tD=0} J^\lambda / ||J^\lambda||) \big|_{q=0} = J^\lambda / ||J^\lambda|| \). Comparing the result with the proof of Proposition 7, we conclude

\[ S_{\text{Hilb}} \big|_{tD=0} J^\lambda / ||J^\lambda|| = \Phi^\lambda, \]

as \( \mathcal{F} \)-valued power series.

Let \( A : \mathcal{F} \to \mathcal{F} \) be defined by \( A(e_\lambda) = J^\lambda / ||J^\lambda|| \). The above discussion yields the following result.

**Proposition 8** As power series in \( 1/z \), we have \( S_{\text{Hilb}} \big|_{tD=0} A = SL_0^{-1} \).

By definition, \( S_{\text{Hilb}} \) is a formal power series in \( q \). By Proposition 8, \( S_{\text{Hilb}} \) is analytic in \( q \).

By the divisor equation for primary Gromov–Witten invariants, we have

\[ q \frac{\partial}{\partial q} (D\ast t_D) - \frac{\partial}{\partial t_D} (D\ast t_D) = 0. \]

A direct calculation then shows that the two differential operators

\[ z \frac{\partial}{\partial t_D} - (D\ast t_D) \] and \[ z q \frac{\partial}{\partial q} - z \frac{\partial}{\partial t_D} - (-)(D\ast) \]

commute. Therefore, Eq. (2.2) and Proposition 8 uniquely determine \( S_{\text{Hilb}}(q, t_D) \).

### 2.2 Symmetric products

We introduce another copy of the Fock space \( \mathcal{F} \) which we denote by \( \tilde{\mathcal{F}} \). An additive isomorphism

\[ \tilde{\mathcal{F}} \otimes \mathbb{C}[t_1, t_2] \simeq \bigoplus_{n \geq 0} H^*_T(I\text{Sym}^n(\mathbb{C}^2), \mathbb{C}), \]

is given by identifying \( |\mu\rangle \in \tilde{\mathcal{F}} \) with the fundamental class \( [I_\mu] \) of the component of the inertia orbifold \( I\text{Sym}^n(\mathbb{C}^2) \) indexed by \( \mu \). The orbifold Poincaré pairing \( (-, -)^{\text{Sym}} \)

\( \tilde{\mathcal{F}} \) Springer
induces via this identification a pairing on $\tilde{F}$,

$$\tilde{\eta}(\mu, \nu) = \frac{1}{(t_1 t_2)^{\ell(\mu)}} \delta_{\mu \nu} z^{\ell(\mu)}.$$ 

Following [25, Equation (1.6)], we define

$$|\tilde{\mu}\rangle = (-\sqrt{-1})^{\ell(\mu)}|\mu\rangle \in \tilde{F}.$$ 

We will use the following linear isomorphism

$$C : F \to \tilde{F}, \quad |\mu\rangle \mapsto |\tilde{\mu}\rangle,$$ 

which is compatible with the pairings $\eta$ and $\tilde{\eta}$.

We recall the definition of the ramified Gromov–Witten invariants of $\text{Sym}^n(C^2)$ following [25, Section 3.2]. Consider the moduli space $\overline{M}_{g,r+b}(\text{Sym}^n(C^2))$ of stable maps to $\text{Sym}^n(C^2)$ and let

$$\overline{M}_{g,r,b}(\text{Sym}^n(C^2)) = \left[ \left( \text{ev}_{r+1}^{-1}(I_2) \cap \cdots \cap \text{ev}_{r+b}^{-1}(I_2) \right) / \Sigma_b \right]$$

where the symmetric group $\Sigma_b$ acts by permuting the last $b$ marked points. Define ramified descendent Gromov–Witten invariants by

$$\langle \prod_{i=1}^r I_{\mu_i} \psi^{k_i} \rangle_{\text{Sym}^n(C^2)} = \int_{[\overline{M}_{g,r,b}(\text{Sym}^n(C^2))]} \prod_{i=1}^r \text{ev}_i^*([I_{\mu_i}]) \psi^{k_i}.$$ 

Let $S^\text{Sym}(u, \tilde{t})$ be the generating function of genus 0 ramified descendent Gromov–Witten invariants of $\text{Sym}^n(C^2)$,

$$\tilde{\eta}(a, S^\text{Sym}(u, \tilde{t}) b) = \tilde{\eta}(a, b) + \sum_{k \geq 0} z^{-1-k} \sum_{m,d} \frac{u^d}{m!} (a, \tilde{t} I_{2m+2,2}, \ldots, \tilde{t} I_{2m+2,2}, b \psi^{k_m}_{m+2})_{\text{Sym}^n(C^2)}.$$

By definition, $S^\text{Sym}$ is a formal power series in $1/z$ whose coefficients are in $\text{End}(\overline{F})[[u]]$, written in the basis $\{|\tilde{\mu}\rangle\}$. $S^\text{Sym}$ satisfies the following two differential equations:

$$z \frac{\partial}{\partial \tilde{t}} S^\text{Sym}(u, \tilde{t}) = (I_{2, \tilde{t}}) S^\text{Sym}(u, \tilde{t}),$$ 

$$\frac{\partial}{\partial u} S^\text{Sym}(u, \tilde{t}) = \frac{\partial}{\partial \tilde{t}} S^\text{Sym}(u, \tilde{t}).$$

\text{Springer}
Here \((I(2)\ast \tilde{t}) = (I(2)\ast \tilde{t}I(2))\) is the operator of quantum multiplication by the divisor \(I(2)\) at the point \(\tilde{t}I(2)\),

\[
\tilde{\eta}((I(2)\ast \tilde{t})a, b) = \sum_{m,d} \frac{u^d}{m!} \langle (I(2), a, \tilde{t}I(2), \ldots, \tilde{t}I(2)), b \rangle_{\text{Sym}^m(C^2)}^{0,d}.
\]

Equation (2.7) follows from the genus 0 topological recursion relations for orbifold Gromov–Witten invariants, see [26]. Equation (2.8) follows from divisor equations for ramified orbifold Gromov–Witten invariants, see [5].

We first compare the operators \((D\ast tD D)\) and \((I(2)\ast \tilde{t}I(2))\). For simplicity, write \((2, 1^{n-2})\) for the partition \((2)\). By [25, Theorem 4], we have

\[
\langle D, D, \ldots, D, \lambda, \mu \rangle_{\text{Hilb}} = (-1)^{k+1} \langle (2), (2), \ldots, (2), \lambda, \mu \rangle_{\text{Hilb}}
\]

\[
= (-1)^{k+1} \langle (\tilde{2}), (\tilde{2}), \ldots, (\tilde{2}), \tilde{\lambda}, \tilde{\mu} \rangle_{\text{Sym}}
\]

\[
= \langle (-\tilde{2}), -(-\tilde{2}), \ldots, -(-\tilde{2}), \tilde{\lambda}, \tilde{\mu} \rangle_{\text{Sym}},
\]

where \((-\tilde{\cdots})\) is defined in [25, Equation (1.6)]. Therefore, under the identification \(|\mu\rangle \mapsto |\tilde{\mu}\rangle\), we have

\[
D\ast tD D = -\tilde{\langle \tilde{2} \rangle} \ast tD (-\tilde{\tilde{2}}).
\]

(2.9)

Now,

\[
\tilde{\tilde{2}} = (-i)^{n-1-n} I(2) = (-i)^{-1} I(2) = i I(2).
\]

Hence we have, after \(-q = e^{iu}\),

\[
D\ast tD D = (-i) I(2) \ast \tilde{t}I(2), \quad \tilde{t} = (-i) tD.
\]

(2.10)

Consider now \(S_{\text{Sym}}^{\tilde{t}=0}\). By (2.7) and (2.8), we have

\[
\frac{\partial}{\partial u} S_{\text{Sym}}(u, \tilde{t}) = (I(2)\ast \tilde{t}) S_{\text{Sym}}(u, \tilde{t}).
\]

Setting \(\tilde{t} = 0\) and using (2.4) and (2.10), we find

\[
\frac{\partial}{\partial u} \left( S_{\text{Sym}}^{\tilde{t}=0} \right) = i M_D (-e^{iu}) \left( S_{\text{Sym}}^{\tilde{t}=0} \right).
\]
Since \( \frac{\partial}{\partial u} = i q \frac{\partial}{\partial q} \), we find that, after \(-q = e^{iu}\),

\[
zh \frac{\partial}{\partial q} \left( S_{\text{Sym}} \bigg|_{\tilde{t}=0} \right) = M_D(q) \left( S_{\text{Sym}} \bigg|_{\tilde{t}=0} \right).
\]

(2.11)

Recall \( S = \Theta Y_z L^{-1} L_0 \) also satisfied the same equation. We may then compare \( \Theta Y_z L^{-1} L_0 \) and \( (S_{\text{Sym}} \big|_{\tilde{t}=0}) \) by comparing them at \( u = 0 \) which corresponds to \(-q = -1\). Set

\[
B = S \bigg|_{q=-1} = \Theta Y_z L^{-1} L_0 \bigg|_{q=-1}.
\]

Since \( S_{\text{Sym}} \big|_{\tilde{t}=0, q=0} = \text{Id} \), we have, after \(-q = e^{iu}\),

\[
S_{\text{Sym}} \bigg|_{\tilde{t}=0} = CSB^{-1}C^{-1}.
\]

(2.12)

By Proposition 8, we have

\[
CSB^{-1}C^{-1} = CS_{\text{Hilb}} \bigg|_{t_D=0} A L_0 B^{-1} C^{-1}.
\]

(2.13)

Since \( A L_0 A^{-1} = q^{D/z} \),

\[
A L_0 B^{-1} = A L_0 A^{-1} A B^{-1} = q^{D/z} A B^{-1}.
\]

Define \( K = B A^{-1} \). We can then rewrite (2.13) as

\[
S_{\text{Sym}} \bigg|_{\tilde{t}=0} = CS_{\text{Hilb}} \bigg|_{t_D=0} q^{D/z} K^{-1} C^{-1}.
\]

(2.14)

By the divisor equation for orbifold Gromov–Witten invariants in [5] (see also [25, Section 3.2]), we have

\[
\frac{\partial}{\partial u} (I_{(2)\star \tilde{t}}) - \frac{\partial}{\partial \tilde{t}} (I_{(2)\star \tilde{t}}) = 0.
\]

A direct calculation then shows that the two differential operators

\[
z \frac{\partial}{\partial \tilde{t}} - (I_{(2)\star \tilde{t}}) \quad \text{and} \quad \frac{\partial}{\partial u} - \frac{\partial}{\partial \tilde{t}}
\]

commute. Therefore \( S_{\text{Sym}}(u, \tilde{t}) \) is uniquely determined by Eq. (2.7) and \( S_{\text{Sym}} \bigg|_{\tilde{t}=0} \). By (2.10), we have
\[
\frac{\partial}{\partial t_D} (D \star_D) = i \left( \frac{\partial}{\partial t} - (I(2) \star t) \right),
\]

after \(-q = e^{iu}\). Then Eq. (2.14) implies the following result.

**Theorem 9** After \(-q = e^{iu}\) and \(\bar{\tilde{t}} = (-i)t_D\), we have

\[
S_{\text{Sym}}(u, \bar{\tilde{t}}) = CS_{\text{Hilb}}(q, t_D) q^{D/z} K^{-1} C^{-1}.
\]

### 2.3 Proof of Theorem 1

By the definition of \(B\) and Proposition 7, \(K\) is an \(\text{End}(\mathcal{F})\)-valued power series in \(1/z\) of the form

\[
K = \text{Id} + O(1/z).
\]

By Theorem 9 and the fact that \(S_{\text{Hilb}}\) and \(S_{\text{Sym}}\) are symplectic, it follows that \(K\) is also symplectic.

Next, we explicitly evaluate \(K\). By the definition of \(B\) and [23, Theorem 4], we have

\[
B = \left( \Theta Y_z L^{-1} L_0 \right) \bigg|_{q=-1} = \frac{1}{(2\pi \sqrt{-1})^{|\cdot|}} \Theta \Gamma_z H_z \left( G_{\Delta T}^{-1} L_0 \right) \bigg|_{q=-1} L^{-1}.
\]

Here \(|\cdot| = \sum_{k>0} \alpha_k \alpha_k^*\) is the energy operator. \(G_{\Delta T}\) is the diagonal matrix in the basis \(\{e_\lambda\}\) with eigenvalues

\[
q^{-c(\lambda; t_1, t_2)} \prod_{w: \text{tangent weights at } \lambda} \frac{1}{\Gamma(w + 1)},
\]

see [23, Section 3.1.2]. The operator \(\Gamma\) is given by

\[
\Gamma |\mu\rangle = \frac{(2\pi \sqrt{-1})^{\ell(\mu)}}{\prod_{i} \mu_i} G_{\text{GW}}(t_1, t_2) |\mu\rangle,
\]

see [23, Section 3.3], where

\[
G_{\text{GW}}(t_1, t_2) |\mu\rangle = \prod_{i} g(\mu_i, t_1) g(\mu_i, t_2) |\mu\rangle,
\]

and

\[
g(\mu_i, t_1) g(\mu_i, t_2) = \frac{\mu_i^{t_1 \mu_i} \mu_i^{t_2 \mu_i}}{\Gamma(\mu_i t_1) \Gamma(\mu_i t_2)},
\]
The Hilb/Sym correspondence for $\mathbb{C}^2$: descendents and Fourier–Mukai

see [23, Section 3.1.2]. Define

$$\Gamma_z = \Gamma \left( \frac{t_1}{z}, \frac{t_2}{z} \right).$$

Since

$$K = BA^{-1} = \frac{1}{(2\pi \sqrt{-1})^{|\lambda|}} \Theta \Gamma_z \left( G_{DT}^{-1} L_0 \right) \bigg|_{q=-1} L^{-1} A^{-1},$$

and $||J^\lambda|| = \prod_{w: \text{tangent weights at } \lambda} w^{1/2}$, we see that $K$ is the operator given by

$$K(J^\lambda) = \frac{z^{||\lambda||}}{(2\pi \sqrt{-1})^{|\lambda|}} \prod_{w: \text{tangent weights at } \lambda} \Gamma (w/z + 1) \Theta \Gamma_w H^\lambda_w.$$  \hspace{1cm} (2.16)

The proof Theorem 1 is complete. \hfill \Box

3 Descendent correspondence

3.1 Variables

We compare the descendent Gromov–Witten theories of $\text{Hilb}^n(\mathbb{C}^2)$ and $\text{Sym}^n(\mathbb{C}^2)$. The following identifications will be used throughout:

$$-q = e^{iu}, \quad \tilde{t} = (-i)t_D.$$  \hspace{1cm} (3.1)

3.2 Genus 0

Following [11], consider the Givental spaces

$$\mathcal{H}^{\text{Hilb}} = H^*_T(\text{Hilb}^n(\mathbb{C}^2)) \otimes_{\mathbb{C}[t_1, t_2]} \mathbb{C}(t_1, t_2)[[q]]((z^{-1})),$$

$$\mathcal{H}^{\text{Sym}} = H^*_T(\text{Sym}^n(\mathbb{C}^2)) \otimes_{\mathbb{C}[t_1, t_2]} \mathbb{C}(t_1, t_2)[[u]]((z^{-1})).$$

equipped with the symplectic forms

$$(f, g)^{\mathcal{H}^{\text{Hilb}}} = \text{Res}_{z=0} (f(-z), g(z))^{\text{Hilb}}, \quad f, g \in \mathcal{H}^{\text{Hilb}},$$

$$(f, g)^{\mathcal{H}^{\text{Sym}}} = \text{Res}_{z=0} (f(-z), g(z))^{\text{Sym}}, \quad f, g \in \mathcal{H}^{\text{Sym}}.$$

The choice of bases

$$\{ |\mu \rangle | \mu \in \text{Part}(n) \} \subset H^*_T(\text{Hilb}^n(\mathbb{C}^2)), \quad \{ |\tilde{\mu} \rangle | \mu \in \text{Part}(n) \} \subset H^*_T(\text{Sym}^n(\mathbb{C}^2)),$$
yields Darboux coordinate systems \( \{ p_a^{\mu}, q_b^v \}, \{ \tilde{p}_a^{\mu}, \tilde{q}_b^v \} \). General points of \( \mathcal{H}^{\text{Hilb}} \), \( \mathcal{H}^{\text{Sym}} \) can be written in the form

\[
\sum_{a \geq 0} \sum_{\mu} p_a^{\mu} |\mu| \frac{(t_1 t_2) \ell(\mu)}{(\mu|-\ell(\mu)} (-z)^{-a-1} + \sum_{b \geq 0} \sum_{v} q_b^v v z^b \in \mathcal{H}^{\text{Hilb}},
\]

\[
\sum_{a \geq 0} \sum_{\mu} \tilde{p}_a^{\mu} |\mu| \frac{(t_1 t_2) \ell(\mu)}{\mu |\mu|} \frac{3(\mu)}{1} (-z)^{-a-1} + \sum_{b \geq 0} \sum_{\nu} \tilde{q}_b^\nu \nu z^b \in \mathcal{H}^{\text{Sym}},
\]

Define the Lagrangian cones associated to the generating functions of genus 0 descendent and ancestor Gromov–Witten invariants as follows:

\[
\mathcal{L}^{\text{Hilb}} = \{(p, q) | p = d_q F_0^{\text{Hilb}} \} \subset \mathcal{H}^{\text{Hilb}}, \quad \mathcal{L}^{\text{Hilb}}_{\text{an}, tD} = \{(p, q) | p = d_q F^{\text{Hilb}}_{\text{an}, tD, 0} \} \subset \mathcal{H}^{\text{Hilb}},
\]

\[
\mathcal{L}^{\text{Sym}} = \{(\tilde{p}, \tilde{q}) | \tilde{p} = d_q F_0^{\text{Sym}} \} \subset \mathcal{H}^{\text{Sym}}, \quad \mathcal{L}^{\text{Sym}}_{\text{an}, \tilde{t}} = \{(\tilde{p}, \tilde{q}) | \tilde{p} = d_q F^{\text{Sym}}_{\text{an}, \tilde{t}, 0} \} \subset \mathcal{H}^{\text{Sym}},
\]

where

\[
F_0^{\text{Hilb}}(t) = \sum_{d, k \geq 0} \frac{q^d}{d!} \langle t(\psi), \ldots , t(\psi) \rangle_{0,d}^{\text{Hilb}},
\]

\[
F^{\text{Hilb}}_{\text{an}, tD, 0}(t) = \sum_{d, k, l \geq 0} \frac{q^d}{d!} \langle t(\psi), \ldots , t(\psi), tD D_1, \ldots , tD D_l \rangle_{0,d}^{\text{Hilb}},
\]

\[
F^{\text{Sym}}_0(\tilde{t}) = \sum_{b, k \geq 0} \frac{u^b}{d!} \langle \tilde{t}(\psi), \ldots , \tilde{t}(\psi) \rangle_{0,b}^{\text{Sym}},
\]

\[
F^{\text{Sym}}_{\text{an}, \tilde{t}, 0}(\tilde{t}) = \sum_{b, k, l \geq 0} \frac{u^b}{d!} \langle \tilde{t}(\psi), \ldots , \tilde{t}(\psi), tI(2), \ldots , tI(2) \rangle_{0,b}^{\text{Sym}}.
\]

Here, \( q = t - 1, \tilde{q} = \tilde{t} - 1 \) are dilaton shifts.

By the descendent/ancestor relations \([8]\), we have

\[
\mathcal{L}^{\text{Hilb}} = S^{\text{Hilb}}(q, tD)^{-1} \mathcal{L}^{\text{Hilb}}_{\text{an}, tD}, \quad \mathcal{L}^{\text{Sym}} = S^{\text{Sym}}(u, \tilde{t})^{-1} \mathcal{L}^{\text{Sym}}_{\text{an}, \tilde{t}}.
\]

By the genus 0 crepant resolution correspondence proven\(^9\) in \([5]\), we have

\[
\mathcal{C} \mathcal{L}^{\text{Hilb}}_{\text{an}, tD} = \mathcal{L}^{\text{Sym}}_{\text{an}, \tilde{t}}.
\]

**Theorem 10** We have \( \mathcal{L}^{\text{Sym}} = CK q^{-D/2} \mathcal{L}^{\text{Hilb}} \).

\(^9\) In particular, the results of \([5]\) implies that \( \mathcal{L}^{\text{Hilb}}_{\text{an}, tD} \) is analytic in \( q \).
Proof Using Theorem 9, we calculate

\[
\mathcal{L}^{\text{Sym}} = \mathcal{S}^{\text{Sym}}(u, \tilde{t})^{-1} \mathcal{L}^{\text{Sym}}_{\text{an}, i} = \mathcal{S}^{\text{Sym}}(u, \tilde{t})^{-1} \mathcal{L}^{\text{Hilb}}_{\text{an}, i, D} = \mathcal{C}Kq^{-D/z} \mathcal{S}^{\text{Hilb}}_g(q, t_D)^{-1} \mathcal{L}^{\text{Hilb}}_{\text{an}, i, D} = \mathcal{C}Kq^{-D/z} \mathcal{L}^{\text{Hilb}}.
\]

\[\square\]

3.3 Higher genus

Consider the total descendent potentials,

\[
\mathcal{D}^{\text{Hilb}} = \exp \left( \sum_{g \geq 0} h_g^{g-1} \mathcal{F}^{\text{Hilb}}_g \right), \quad \mathcal{F}^{\text{Hilb}}_g(t) = \sum_{d, k \geq 0} \frac{q^d}{k!} \langle t(\psi), \ldots, t(\psi) \rangle^{\text{Hilb}}_{g, d, k},
\]

\[
\mathcal{D}^{\text{Sym}} = \exp \left( \sum_{g \geq 0} h_g^{g-1} \mathcal{F}^{\text{Sym}}_g \right), \quad \mathcal{F}^{\text{Sym}}_g(\tilde{t}) = \sum_{b, k \geq 0} \frac{u^b}{k!} \langle \tilde{t}(\psi), \ldots, \tilde{t}(\psi) \rangle^{\text{Sym}}_{g, b, k},
\]

and the total ancestor potentials,\(^\text{10}\)

\[
\mathcal{A}^{\text{Hilb}}_{\text{an}, i, D} = \exp \left( \sum_{g \geq 0} h_g^{g-1} \mathcal{F}^{\text{Hilb}}_{\text{an}, i, D, g} \right),
\]

\[
\mathcal{F}^{\text{Hilb}}_{\text{an}, i, D, g}(t) = \sum_{d, k, l \geq 0} \frac{q^d}{k!l!} \langle t(\bar{\psi}), \ldots, t(\bar{\psi}), t_D D, \ldots, t_D D \rangle^{\text{Hilb}}_{g, d, k, l},
\]

\[
\mathcal{A}^{\text{Sym}}_{\text{an}, i} = \exp \left( \sum_{g \geq 0} h_g^{g-1} \mathcal{F}^{\text{Sym}}_{\text{an}, i, g} \right),
\]

\[
\mathcal{F}^{\text{Sym}}_{\text{an}, i, g}(\tilde{t}) = \sum_{b, k, l \geq 0} \frac{u^b}{k!l!} \langle \tilde{t}(\bar{\psi}), \ldots, \tilde{t}(\bar{\psi}), t I(2), \ldots, t I(2) \rangle^{\text{Sym}}_{g, b, k, l}.
\]

Givental’s quantization formalism \([11]\) produces differential operators by quantizing quadratic Hamiltonians associated to linear symplectic transforms by the following rules:

\(^\text{10}\) The results of \([25]\) imply that \(A^{\text{Hilb}}_{\text{an}, i, D}\) depends analytically in \(q\).
\[
\hat{q}_a q_b = \hbar \frac{\partial}{\partial q_a}, \hat{q}_a p_b = \hbar \frac{\partial}{\partial p_b}, \hat{p}_a q_b = \hbar \frac{\partial}{\partial q_b}, \hat{p}_a p_b = \hbar \frac{\partial}{\partial p_b},
\]

By the descendent/ancestor relations [8], we have

\[
D_{\text{Hilb}} = e^{F_1^{\text{Hilb}}(t_D)} S_{\text{Hilb}}(q, t_D) - 1 A_{\text{an}, t_D}^{\text{Hilb}},
\]

\[
D_{\text{Sym}} = e^{F_1^{\text{Sym}}(\tilde{t})} S_{\text{Sym}}(u, \tilde{t}) - 1 A_{\text{an}, \tilde{t}}^{\text{Sym}},
\]

where \(F_1^{\text{Hilb}}\) and \(F_1^{\text{Sym}}\) are generating functions of genus 1 primary invariants with

insertions \(D\) and \(I(2)\) respectively. \(F_1^{\text{Sym}}\) and \(F_1^{\text{Hilb}}\) can be easily matched using [25, Theorem 4].

**Theorem 11** We have \(e^{-F_1^{\text{Sym}}(\tilde{t})} D_{\text{Sym}} = \hat{C} Kq^{-D/z} (e^{-F_1^{\text{Hilb}}(t_D)} D_{\text{Hilb}})\).

**Proof** By [25, Theorem 4], we have \(\hat{C} A_{\text{an}, t_D}^{\text{Hilb}} = A_{\text{an}, \tilde{t}}^{\text{Sym}}\). Using Theorem 9, we calculate

\[
S_{\text{Sym}}(u, \tilde{t}) - 1 A_{\text{an}, \tilde{t}}^{\text{Sym}}
\]

\[
= \hat{C} Kq^{-D/z} S_{\text{Hilb}}(q, t_D) - 1 A_{\text{an}, t_D}^{\text{Hilb}}.
\]

Therefore, we conclude

\[
e^{-F_1^{\text{Sym}}(\tilde{t})} D_{\text{Sym}} = S_{\text{Sym}}(u, \tilde{t}) - 1 A_{\text{an}, \tilde{t}}^{\text{Sym}}
\]

\[
= \hat{C} Kq^{-D/z} S_{\text{Hilb}}(q, t_D) - 1 A_{\text{an}, t_D}^{\text{Hilb}}
\]

\[
= \hat{C} Kq^{-D/z} \left( e^{-F_1^{\text{Hilb}}(t_D)} D_{\text{Hilb}} \right).
\]

\[\square\]

4 Fourier–Mukai transformation

4.1 Proof of Theorem 4

We first localize the top row of the diagram of Theorem 4:

\[
K_T(\text{Hilb}''(\mathbb{C}^2))_{\text{loc}} \xrightarrow{\text{FM}} K_T(\text{Sym}''(\mathbb{C}^2))_{\text{loc}}
\]

\[
\hat{\psi}_{\text{Hilb}} \downarrow \quad \hat{\psi}_{\text{Sym}}
\]

\[
\hat{\mathcal{K}}_{\text{Hilb}} \quad \hat{\mathcal{K}}_{\text{Sym}}
\]

\[\odot\] Springer
Here, loc denotes tensoring by Frac(R(T)), the field of fractions of the representation ring R(T) of the torus T. The maps \( \Psi_{\text{Hilb}} \) and \( \Psi_{\text{Sym}} \) are still well-defined since the T-equivariant Chern character of a representation is invertible. The commutation of the above diagram immediately implies the commutation of the diagram of Theorem 4.

Let \( k_\lambda \in K_T(\text{Hilb}^n(\mathbb{C}^2)) \) be the skyscraper sheaf supported on the fixed point indexed by \( \lambda \). The set \( \{ k_\lambda \mid \lambda \in \text{Part}(n) \} \) is a basis of \( K_T(\text{Hilb}^n(\mathbb{C}^2)) \) loc as a Frac\( (R(T)) \)-vector space. The commutation of the localized diagram is then a consequence of the following equality: for all \( \lambda \in \text{Part}(n) \),

\[
\text{CK}|_{z \mapsto -z} \circ \Psi_{\text{Hilb}}(k_\lambda) = \Psi_{\text{Sym}} \circ \text{FM}(k_\lambda). \tag{4.1}
\]

To prove (4.1), we will match the two sides by explicit calculation.

### 4.2 Iritani’s Gamma class

For a vector bundle \( \mathcal{V} \) on a Deligne–Mumford stack \( \mathcal{X} \),

\[
\mathcal{V} \to \mathcal{X},
\]

Iritani has defined a characteristic class called the \textit{Gamma class}. Let

\[
I\mathcal{X} = \bigsqcup_i \mathcal{X}_i
\]

be the decomposition of the inertia stack \( I\mathcal{X} \) into connected components. By pulling back \( \mathcal{V} \) to \( I\mathcal{X} \) and restricting to \( \mathcal{X}_i \), we obtain a vector bundle \( \mathcal{V}|_{\mathcal{X}_i} \) on \( \mathcal{X}_i \). The stabilizer element \( g_i \) of \( \mathcal{X} \) associated to the component \( \mathcal{X}_i \) acts on \( \mathcal{V}|_{\mathcal{X}_i} \). The bundle \( \mathcal{V}|_{\mathcal{X}_i} \) decomposes under \( g_i \) into a direct sum of eigenbundles

\[
\mathcal{V}|_{\mathcal{X}_i} = \bigoplus_{0 \leq f < 1} \mathcal{V}_{i,f},
\]

where \( g_i \) acts on \( \mathcal{V}_{i,f} \) by multiplication by \( \exp(2\pi \sqrt{-1} f) \). The orbifold Chern character of \( \mathcal{V} \) is defined to be

\[
\widetilde{\text{ch}}(\mathcal{V}) = \bigoplus_i \sum_{0 \leq f < 1} \exp(2\pi \sqrt{-1} f) \text{ch}(\mathcal{V}_{i,f}) \in H^*(I\mathcal{X}), \tag{4.2}
\]

where \( \text{ch}(\cdot) \) is the usual Chern character.

For each \( i \) and \( f \), let \( \delta_{i,f,j} \), for \( 1 \leq j \leq \text{rank}\mathcal{V}_{i,f} \), be the Chern roots of \( \mathcal{V}_{i,f} \). Iritani’s Gamma class\(^{11}\) is defined to be

\[
\Gamma(\mathcal{V}) = \bigoplus_i \prod_{0 \leq f < 1} \prod_{j=1}^{\text{rank}\mathcal{V}_{i,f}} \Gamma(1 - f + \delta_{i,f,j}). \tag{4.3}
\]

\(^{11}\) The substitution of cohomology classes into Gamma function makes sense because the Gamma function \( \Gamma(1 + x) \) has a power series expansion at \( x = 0 \).
As usual, $\Gamma_\chi = \Gamma(T_\chi)$.

If the vector bundle $V$ is equivariant with respect to a $T$-action, the Chern character and Chern roots above should be replaced by their equivariant counterparts to define a $T$-equivariant Gamma class.

If $X$ is a scheme, then the Gamma class simplifies considerably since there are no stabilizers. Directly from the definition, the restriction of $\Gamma_{\text{Hilb}}$ to the fixed point indexed by $\lambda$ is

$$\Gamma_{\text{Hilb}}|_\lambda = \prod_{w: \text{tangent weights at } \lambda} \Gamma(w + 1).$$

Recall that the inertia stack $I\text{Sym}^n(\mathbb{C}^2)$ is a disjoint union indexed by conjugacy classes of $S_n$. For a partition $\mu$ of $n$, the component $I_\mu \subset I\text{Sym}^n(\mathbb{C}^2)$ indexed by the conjugacy class of cycle type $\mu$ is the stack quotient

$$[\mathbb{C}^n_\sigma / C(\sigma)],$$

where $\sigma \in S_n$ has cycle type $\mu$, $\mathbb{C}^n_\sigma \subset \mathbb{C}^n$ is the $\sigma$-invariant part, and $C(\sigma) \subset S_n$ is the centralizer of $\sigma$.

**Lemma 12** The restriction of $\Gamma_{\text{Sym}}$ to the component $I_\mu$ is given by

$$\Gamma_{\text{Sym}}|_\mu = (t_1 t_2)^{\xi(\mu)} (2\pi)^{n - \ell(\mu)} \left( \prod_i \mu_i \right) \left( \prod_i \mu_i^{1 - \mu_i t_1} \mu_i^{1 - \mu_i t_2} \right) \left( \prod_i \Gamma(\mu_i t_1) \Gamma(\mu_i t_2) \right).$$

**Proof** Using the description of eigenspaces of $T_{\text{Sym}^n(\mathbb{C}^2)}$ on the component of $I\text{Sym}^n(\mathbb{C}^2)$ indexed by $\mu$ (see [25, Section 6.2]), we find that

$$\Gamma_{\text{Sym}}|_\mu = \prod_i \prod_{l=0}^{\mu_i - 1} \Gamma \left( 1 - \frac{l}{\mu_i} + t_1 \right) \Gamma \left( 1 - \frac{l}{\mu_i} + t_2 \right).$$

Using the formula

$$\prod_{k=0}^{m-1} \Gamma \left( z + \frac{k}{m} \right) = (2\pi)^{-1} \frac{m^{-1}}{z} \frac{1}{z} m^{-1} \Gamma(mz),$$

we find

$$\prod_{l=0}^{\mu_i - 1} \Gamma \left( 1 - \frac{l}{\mu_i} + t_1 \right) = t_1 (2\pi)^{\mu_i - 1} \frac{1}{\mu_i} \Gamma(\mu_i t_1).$$

\(\square\) Springer
and similarly for the other factor. Therefore,
\[
\Gamma_{\text{Sym}} \bigg|_\mu = (t_1 t_2)^{\ell(\mu)} (2\pi)^{n-\ell(\mu)} \left( \prod_i \mu_i \right) \left( \prod_i \mu_i^{-\mu_i t_1} \mu_i^{\mu_i t_2} \right) \left( \prod_i \Gamma(\mu_i t_1) \Gamma(\mu_i t_2) \right),
\]
which is the desired formula. \qed

4.3 Calculation of $C_{K \circ \Psi}^\text{Hilb}$

Since $k_\lambda$ is supported at the $T$-fixed point of $\text{Hilb}^n (\mathbb{C}^2)$ indexed by $\lambda$, the $T$-equivariant Chern character $\text{ch}(k_\lambda)$ is also supported there. Using the Koszul resolution (or Grothendieck–Riemann–Roch), we calculate

\[
\text{ch}(k_\lambda) = J^\lambda \prod_w \frac{1 - e^{-w}}{w}, \quad (4.4)
\]

We have used the fact that the class of the $T$-fixed point of $\text{Hilb}^n (\mathbb{C}^2)$ indexed by $\lambda$ corresponds to the factor

\[
\frac{J^\lambda}{\prod_w w}.
\]

By the definition of $\text{deg}_0^\text{Hilb}$, we have

\[
(2\pi \sqrt{-1})^{\frac{\text{deg}_0^\text{Hilb}}{2}} \text{ch}(k_\lambda) = (2\pi \sqrt{-1})^{\frac{\text{deg}_0^\text{Hilb}}{2}} \frac{J^\lambda}{\prod_w 2\pi \sqrt{-1}w} \prod_w \left(1 - e^{-2\pi \sqrt{-1}w}\right).
\]

Write $J^\lambda = \sum_\epsilon J^\lambda_\epsilon (t_1, t_2) |\epsilon\rangle$. Since $J^\lambda_\epsilon$ is $(t_1 t_2)^{\ell(\epsilon)}$ times a homogeneous polynomial in $t_1, t_2$ of degree $n - \ell(\epsilon)$, we have\(^{12}\)

\[
(2\pi \sqrt{-1})^{\frac{\text{deg}_0^\text{Hilb}}{2}} J^\lambda = \sum_\epsilon (2\pi \sqrt{-1})^{\frac{\text{deg}_0^\text{Hilb}}{2}} J^\lambda_\epsilon (t_1, t_2) |\epsilon\rangle
\]

\[
= \sum_\epsilon J^\lambda_\epsilon (2\pi \sqrt{-1} t_1, 2\pi \sqrt{-1} t_2) (2\pi \sqrt{-1})^{n-\ell(\epsilon)} |\epsilon\rangle
\]

\[
= \sum_\epsilon J^\lambda_\epsilon (t_1, t_2) (2\pi \sqrt{-1})^{n+\ell(\epsilon)} (2\pi \sqrt{-1})^{n-\ell(\epsilon)} |\epsilon\rangle
\]

\[
= (2\pi \sqrt{-1})^{2n} \sum_\epsilon J^\lambda_\epsilon (t_1, t_2) |\epsilon\rangle
\]

\[
= (2\pi \sqrt{-1})^{2n} J^\lambda.
\]

\(^{12}\) The calculation also follows from the fact that $J^\lambda$ is the class a $T$-fixed point (of real degree $4n$).
After putting the above formulas together, we obtain

\[
\Gamma_{\text{Hilb}} \cup (2\pi \sqrt{-1})^{\deg_{\text{Hilb}}/2} \text{ch}(k_\lambda) \\
= \left(2\pi \sqrt{-1}\right)^{2n_J^\lambda} \prod_{w: \text{tangent weights at } \lambda} \Gamma(w + 1)(1 - e^{-2\pi \sqrt{-1}w}).
\]

Recall the following identity for the Gamma function:

\[
\Gamma(1 + t)\Gamma(1 - t) = \frac{2\pi \sqrt{-1}t}{e^{\pi \sqrt{-1}t} - e^{-\pi \sqrt{-1}t}}. \tag{4.5}
\]

We have

\[
\Gamma(w + 1)(1 - e^{-2\pi \sqrt{-1}w}) = \Gamma(w + 1)(e^{\pi \sqrt{-1}w} - e^{-\pi \sqrt{-1}w})(e^{-\pi \sqrt{-1}w}) \\
= \frac{2\pi \sqrt{-1}w}{\Gamma(1 - w)}(e^{-\pi \sqrt{-1}w}).
\]

Hence

\[
\Gamma_{\text{Hilb}} \cup (2\pi \sqrt{-1})^{\deg_{\text{Hilb}}/2} \text{ch}(k_\lambda) \\
= \left(2\pi \sqrt{-1}\right)^{2n_J^\lambda} \prod_{w: \text{tangent weights at } \lambda} \frac{1}{\Gamma(1 - w)} e^{-\pi \sqrt{-1}w}.
\]

Since the operator $z^\rho_{\text{Hilb}}$ is the operator of multiplication by $z^{c_1^T(\text{Hilb}^n(\mathbb{C}^2))}$, we have

\[
z^\rho_{\text{Hilb}} \left(\Gamma_{\text{Hilb}} \cup (2\pi \sqrt{-1})^{\deg_{\text{Hilb}}/2} \text{ch}(k_\lambda)\right) \\
= z^{\rho(t_1 + t_2)} \left(2\pi \sqrt{-1}\right)^{2n_J^\lambda} \prod_{w: \text{tangent weights at } \lambda} \frac{1}{\Gamma(1 - w)} e^{-\pi \sqrt{-1}w} \\
= z^{\rho(t_1 + t_2)} e^{-\pi \sqrt{-1}n(t_1 + t_2)} \left(2\pi \sqrt{-1}\right)^{2n_J^\lambda} \prod_{w: \text{tangent weights at } \lambda} \frac{1}{\Gamma(1 - w)},
\]

where we use

\[
c_1^T(\text{Hilb}^n(\mathbb{C}^2))|_{\lambda} = \sum_{w: \text{tangent weights at } \lambda} w = n(t_1 + t_2).
\]

By the definition of $\mu_{\text{Hilb}}$, we have

\[
z^{-\mu_{\text{Hilb}}} (\phi) = z^n z^{-\deg_{\text{Hilb}}/2} (\phi) = z^n \left(\frac{\phi}{z^{k/2}}\right)
\]
for \( \phi \in H^k(\text{Hilb}^n(\mathbb{C}^2), \mathbb{C}) \), we have
\[
\left( z^{-\mu} \text{Hilb} z^{\rho} \text{Hilb} \left( \Gamma_{\text{Hilb}} \cup (2\pi \sqrt{-1})^{-\frac{\text{deg}_{\text{Hilb}}}{2}} \text{ch}(k_\lambda) \right) \right) = z^n \left( z^{n(t_1 + t_2)/z} e^{-\pi \sqrt{-1} n(t_1 + t_2)/z} \left( \frac{2\pi \sqrt{-1}}{z} \right)^{2n} \right)^{2n} \prod_{w: \text{tangent weights at } \lambda} \frac{1}{\Gamma(1 - w/z)}.
\]

Here, the operator \( z^{-\text{deg}_{\text{Hilb}}/2} \) acts on \( z^n(t_1 + t_2) \) as follows:
\[
z^{-\text{deg}_{\text{Hilb}}/2}(z^n(t_1 + t_2)) = z^{-\text{deg}_{\text{Hilb}}/2} \left( \sum_{k \geq 0} \frac{(n(t_1 + t_2) \log z)^k}{k!} \right) = \sum_{k \geq 0} \frac{(n \log z)^k (t_1 + t_2)^k}{k!} = z^n(t_1 + t_2)/z.
\]

The actions of \( z^{-\text{deg}_{\text{Hilb}}/2} \) on \( e^{-\pi \sqrt{-1} n(t_1 + t_2)} \) and \( \Gamma(1 + w) \) are similarly determined.

By Eq. (2.16), we have
\[
K \bigg|_{z \mapsto -z} (J^\lambda) = \frac{(-z)^{|\mu|}}{(2\pi \sqrt{-1})^{|\mu|}} \left( \prod_{w: \text{tangent weights at } \lambda} \Gamma(-w/z + 1) \right) \Theta' \Gamma_{-z} H_{-z}^\lambda,
\]
where we define \( \Theta' |_{\mu} = (-z)^{(\mu)} |_{\mu} \). Hence,
\[
K \bigg|_{z \mapsto -z} \left( z^{-\mu} \text{Hilb} z^{\rho} \text{Hilb} \left( \Gamma_{\text{Hilb}} \cup (2\pi \sqrt{-1})^{-\frac{\text{deg}_{\text{Hilb}}}{2}} \text{ch}(k_\lambda) \right) \right) = z^n \left( z^{n(t_1 + t_2)/z} e^{-\pi \sqrt{-1} n(t_1 + t_2)/z} \left( \frac{2\pi \sqrt{-1}}{z} \right)^{2n} \right)^{2n} K \bigg|_{z \mapsto -z} (J^\lambda) \prod_{w: \text{tangent weights at } \lambda} \frac{1}{\Gamma(1 - w/z)} = z^n \left( z^{n(t_1 + t_2)/z} e^{-\pi \sqrt{-1} n(t_1 + t_2)/z} \left( \frac{2\pi \sqrt{-1}}{z} \right)^{2n} \right)^{2n} \frac{(-z)^{|\mu|}}{(2\pi \sqrt{-1})^{|\mu|}} \Theta' \Gamma_{-z} H_{-z}^\lambda.
\]
\[ \times \prod_{w: \text{tangent weights at } \lambda} \frac{\Gamma(-w/z + 1)}{\Gamma(1 - w/z)} \]

\[ = (-1)^n z^n z^{n(t_1 + t_2)/z} e^{-\pi \sqrt{-1} \text{Im}(t_1 + t_2)/z} \left( \frac{2\pi \sqrt{-1}}{z} \right)^n \Theta' \Gamma_{-z} H^\lambda_{-z}. \]

By the definition of \( \Gamma_{-z} \), we have

\[ \Gamma_{-z}|\mu\rangle = \frac{(2\pi \sqrt{-1})^{\ell(\mu)}}{\prod_i \Gamma(-\mu_i t_1/z) \Gamma(-\mu_i t_2/z)} |\mu\rangle. \]

Also, \( C|\mu\rangle = |\tilde{\mu}\rangle \), we thus obtain

\[ \text{CK}_{|z\mapsto-z} \left( z^{-\mu} \text{Hilb} z^{\rho} \text{Hilb} \left( \Gamma_{\text{Hilb}} \cup \left( \frac{2\pi \sqrt{-1}}{2} \text{ch}(k_\lambda) \right) \right) \right) = \Delta_{\text{Hilb}}(H^\lambda_{-z}), \quad (4.6) \]

where \( \Delta_{\text{Hilb}} : \mathcal{F} \to \tilde{\mathcal{F}} \) is the operator defined as follows:

\[ \Delta_{\text{Hilb}}|\mu\rangle = (-1)^n z^n z^{n(t_1 + t_2)/z} e^{-\pi \sqrt{-1} \text{Im}(t_1 + t_2)/z} \left( \frac{2\pi \sqrt{-1}}{z} \right)^n (-z)^{\ell(\mu)} \frac{(2\pi \sqrt{-1})^{\ell(\mu)}}{\prod_i \Gamma(-\mu_i t_1/z) \Gamma(-\mu_i t_2/z)} |\mu\rangle. \]

\[ (4.7) \]

### 4.4 Haiman’s result

The homomorphism \( FM \) has been calculated by Haiman [12,13]. Denote by \( F \) the operator of taking Frobenius series of bigraded \( S_n \)-modules, as defined in [12, Definition 3.2.3]. Note that \( T \times S_n \)-equivariant sheaves on \( \text{Sym}^n(\mathbb{C}^2) = [(\mathbb{C}^2)^n / S_n] \)

are \( T \times S_n \)-equivariant sheaves on \( \mathbb{C}^2 \), and hence can be identified with bigraded \( S_n \)-equivariant \( \mathbb{C}[x, y] \)-modules.\(^{13}\) Therefore, the composition

\[ \Phi = F \circ FM \]

\(^{13}\) Here, \( x = \{x_1, \ldots, x_n\} \) and \( y = \{y_1, \ldots, y_n\} \).
makes sense and takes values in a certain algebra of symmetric functions, see [12, Proposition 5.4.6]. For the analysis of the diagram of Theorem 4, we will need the following result of Haiman.

**Theorem 13** ([12], Equation (95)) Let \( k_\lambda \in K_T(Hilb^n(C^2)) \) be the skyscraper sheaf supported on the \( T \)-fixed point indexed by \( \lambda \). Then

\[
\Phi(k_\lambda) = \tilde{H}_\lambda(z; q, t).
\]

The Macdonald polynomial \( \tilde{H}_\lambda(z; q, t) \) is a symmetric function in an infinite set of variables

\[
z = \{z_1, z_2, z_3, \ldots\}
\]

and depends on two parameters \( q, t \). As explained in [25, Section 9.1], \( \tilde{H}_\lambda(z; q, t) \) of [12] is the same as \( H_\lambda \) after the following identification: the parameters \( (q, t) \) and \( (t_1, t_2) \) are related by

\[
(q, t) = (e^{2\pi \sqrt{-1}t_1}, e^{2\pi \sqrt{-1}t_2}).
\]

Symmetric functions in \( z \) are viewed as elements of \( \tilde{F} \) via the following convention. For a partition \( \mu \), the power-sum symmetric function

\[
p_\mu = \prod_k \left( \sum_{i \geq 1} z_i^{\mu_k} \right)
\]

is identified with \( z(\mu)|\mu \).

To make use of Haiman’s result, we must compare the operator \( F \) taking Frobenius series with the orbifold Chern character \( \tilde{c}H \). Let \( V^\lambda \) be the irreducible \( S_n \)-representation indexed by \( \lambda \in \text{Part}(n) \). We construct the bigraded \( S_n \)-equivariant \( \mathbb{C}[x, y] \)-module \( V^\lambda \otimes \mathbb{C}[x, y] \), which is equivalent to a \( T \)-equivariant sheaf \( V^\lambda \) on \( \text{Sym}^n(C^2) \). Define the operator \( \delta : \tilde{F} \to \tilde{F} \) by

\[
\delta|\mu = \prod_i (1 - q^{\mu_i})(1 - t^{\mu_i})|\mu \).
\]

By [12, Section 5.4.3], we have

\[
F_{V^\lambda \otimes \mathbb{C}[x,y]} = s_\lambda \left[ \frac{Z}{(1 - q)(1 - t)} \right],
\]

where \( s_\lambda \) is the Schur function. Here \( Z \) denotes the collection of variables \( z_1, z_2, \ldots \) that the functions are symmetric with respect to, according to the convention of [12].
Using the definition of plethystic substitution \( Z \mapsto Z/(1-q)(1-t) \), see [12, Section 3.3], we obtain

\[
\delta(F_{V^\lambda \otimes \mathbb{C}[x,y]}) = s_\lambda.
\]

On the other hand, by the definition of orbifold Chern character \(^{14}\) recalled in Eq. (4.2), we have

\[
\tilde{\text{ch}}(V^\lambda) = s_\lambda.
\]

Since \( K_T(\text{Sym}^n(\mathbb{C}^2)) \) is freely spanned as a \( R(T) \)-module by \( V^\lambda \otimes \mathbb{C}[x,y] \), we find

\[
\delta \circ F = \tilde{\text{ch}},
\]
after identifying \(^{15}\) \( q = e^{-t_1}, t = e^{-t_2} \). Therefore,

\[
\tilde{\text{ch}}(FM(k_\lambda)) = \delta(F(FM(k_\lambda))) = \delta(\Phi(k_\lambda)) = \delta(\tilde{H}_\lambda), \quad q = e^{-t_1}, \quad t = e^{-t_2}.
\]

### 4.5 Calculation of \( \Psi^{\text{Sym}} \circ FM \)

We have

\[
(2\pi \sqrt{-1})^{\deg_{\text{Sym}}^0} \tilde{\text{ch}}(FM(k_\lambda)) = \delta(\tilde{H}_\lambda), \quad q = e^{-2\pi \sqrt{-1}t_1}, \quad t = e^{-2\pi \sqrt{-1}t_2}.
\]

We have used the definition of \( \deg_{\text{Sym}}^0 \) and the fact that \( |\mu\rangle \in \tilde{\mathcal{F}} \) as a class in \( H_T^*(I\text{Sym}^n(\mathbb{C}^2)) \) has degree 0.

By Lemma 12, we have

\[
\Gamma_{\text{Sym}} \cup (2\pi \sqrt{-1})^{\deg_{\text{Sym}}^0} \tilde{\text{ch}}(FM(k_\lambda)) = \delta_2(\tilde{H}_\lambda), \quad q = e^{-2\pi \sqrt{-1}t_1}, \quad t = e^{-2\pi \sqrt{-1}t_2},
\]

where \( \delta_2 : \tilde{\mathcal{F}} \to \tilde{\mathcal{F}} \) is defined by

\[
\delta_2|\mu\rangle = (t_1t_2)^{\ell(\mu)} (2\pi)^{n-\ell(\mu)} \left( \prod_i \mu_i \right) \left( \prod_i \mu_i^{-\mu_i t_1} \mu_i^{-\mu_i t_2} \right) \times \left( \prod_i \Gamma(\mu_i t_1) \Gamma(\mu_i t_2) \right) \left( \prod_i (1 - e^{-2\pi \sqrt{-1} \mu_i t_1})(1 - e^{-2\pi \sqrt{-1} \mu_i t_2}) \right) |\mu\rangle.
\]

\(^{14}\) The natural basis of \( H_T^*(I\text{Sym}^n(\mathbb{C}^2)) \) is identified with \( \{ |\mu\rangle | \mu \in \text{Part}(n) \} \subset \tilde{\mathcal{F}} \).

\(^{15}\) The choice of \( T = (\mathbb{C}^*)^2 \)-action on \( \mathbb{C}^2 \) in [12, Section 5.1.1] is dual to ours.
Since $c_1^r(\text{Sym}^n(\mathbb{C}^2))|_{\mu} = n(t_1 + t_2)$, we have

$$z^{\rho_{\text{Sym}}} \left( \Gamma_{\text{Sym}} \cup (2\pi \sqrt{-1})^{\deg_{\text{Sym}}} \tilde{\text{ch}}(\mathcal{F}(k_\lambda)) \right) = z^{n(t_1+t_2)} \delta_2(\tilde{H}_\lambda),$$

$$q = e^{-2\pi \sqrt{-1}t_1}, \quad t = e^{-2\pi \sqrt{-1}t_2}.$$

Next, we write

$$z^{-\mu_{\text{Sym}}} z^{\rho_{\text{Sym}}} \left( \Gamma_{\text{Sym}} \cup (2\pi \sqrt{-1})^{\deg_{\text{Sym}}} \tilde{\text{ch}}(\mathcal{F}(k_\lambda)) \right) = \delta_3(H^\lambda_{-z}),$$

where $\delta_3 : \tilde{\mathcal{F}} \to \tilde{\mathcal{F}}$ is defined by

$$\delta_3|\mu) = z^{n(t_1+t_2)/z} (t_1 t_2/z^2)^{\ell(\mu)} (2\pi)^{n-\ell(\mu)} \left( \prod_i \mu_i \right) \left( \prod_i \mu_i^{\frac{-\mu_i t_1}{z}} \mu_i^{\frac{-\mu_i t_2}{z}} \right) \times \left( \prod_i \Gamma(\mu_i t_1/z) \Gamma(\mu_i t_2/z) \right) \left( \prod_i (1 - e^{-2\pi \sqrt{-1} \mu_i t_1/z}) (1 - e^{-2\pi \sqrt{-1} \mu_i t_2/z}) \right) \times z^{-(n-\ell(\mu))} |\mu).$$

We have used the definition of $\mu^\text{Sym}$ and the fact that $|\mu) \in \tilde{\mathcal{F}}$ as a class in $H^r_T(\text{ISym}^n(\mathbb{C}^2))$ has age-shifted degree $2(n - \ell(\mu))$. We have also used

$$z^{\deg_{\text{CR}}/2} \left( \tilde{H}_\lambda |_{q=e^{-2\pi \sqrt{-1}t_1}, \ t=e^{-2\pi \sqrt{-1}t_2}} = \tilde{H}_\lambda |_{q=e^{-2\pi \sqrt{-1}t_1/z}, \ t=e^{-2\pi \sqrt{-1}t_2/z}},$$

which is equal to $H^\lambda_{-z}$.

By (4.5), we have

$$\Gamma(t) \Gamma(-t) = \frac{\Gamma(1+t)}{t} \frac{\Gamma(1-t)}{-t} = \frac{1}{e^{2\pi \sqrt{-1}t} - e^{-2\pi \sqrt{-1}t}} = \frac{2\pi \sqrt{-1}}{e^{-2\pi \sqrt{-1}t} - e^{2\pi \sqrt{-1}t}}.$$

Hence

$$\Gamma(t)(1 - e^{-2\pi \sqrt{-1}t}) = (-1)e^{-\pi \sqrt{-1}t} 2\pi \sqrt{-1} \frac{1}{t} \Gamma(-t).$$
We then obtain
\[
\left( \prod_i \Gamma(\mu_i t_1/z) \Gamma(\mu_i t_2/z) \right) \left( \prod_i \left( 1 - e^{-2\pi \sqrt{-1} \mu_i t_1/z} \right) \left( 1 - e^{-2\pi \sqrt{-1} \mu_i t_2/z} \right) \right)
\]
\[
= (-1)^{2\ell(\mu)} e^{-\pi \sqrt{-1} \ln(t_1 + t_2)/z} (2\pi \sqrt{-1})^{2\ell(\mu)} \left( \prod_i \frac{z}{\mu_i t_1/\mu_i t_2} \right)
\times \left( \prod_i \frac{1}{\Gamma(-\mu_i t_1/z) \Gamma(-\mu_i t_2/z)} \right)
\]
\[
= (-1)^{2\ell(\mu)} e^{-\pi \sqrt{-1} \ln(t_1 + t_2)/z} (2\pi \sqrt{-1})^{2\ell(\mu)} \left( \frac{z^2}{t_1 t_2} \right)^{\ell(\mu)} \left( \prod_i \frac{1}{\mu_i} \right)^2
\times \left( \prod_i \frac{1}{\Gamma(-\mu_i t_1/z) \Gamma(-\mu_i t_2/z)} \right).
\]

Therefore, we can write \( \delta_3 |\mu\rangle \) as
\[
z^n z^{n(t_1 + t_2)/z} (t_1 t_2/z)^2 \ell(\mu) (2\pi)^{n-2\ell(\mu)} \left( \prod_i \mu_i \right) \left( \prod_i \mu_i^{-\mu_i t_1/z} \mu_i^{-\mu_i t_2/z} \right)
\times (-1)^{2\ell(\mu)} e^{-\pi \sqrt{-1} \ln(t_1 + t_2)/z} (2\pi \sqrt{-1})^{2\ell(\mu)} \left( \frac{z^2}{t_1 t_2} \right)^{\ell(\mu)} \left( \prod_i \frac{1}{\mu_i} \right)^2
\times \left( \prod_i \frac{1}{\Gamma(-\mu_i t_1/z) \Gamma(-\mu_i t_2/z)} \right) z^{-(n-2\ell(\mu))} |\mu\rangle
\]
\[
= z^{\ell(\mu)} z^n (t_1 + t_2)/z e^{-\pi \sqrt{-1} \ln(t_1 + t_2)/z} \frac{1}{\prod_i \mu_i} \frac{1}{\prod_i \mu_i^{-\mu_i t_1/z} \mu_i^{-\mu_i t_2/z} \Gamma(-\mu_i t_1/z) \Gamma(-\mu_i t_2/z)}
\times (2\pi)^{n-2\ell(\mu)} (2\pi \sqrt{-1})^{2\ell(\mu)} (-1)^{2\ell(\mu)} |\mu\rangle.
\]

4.6 Proof of Theorem 4

The last step of the proof is the matching
\[
\delta_3 |\mu\rangle = \Delta^{\text{Hilb}} |\mu\rangle. \tag{4.8}
\]

By comparing the expression above for \( \delta_3 |\mu\rangle \) with Eq. (4.7), we see the matching (4.8) follows from the following equality in \( \tilde{\mathcal{F}} \):
\[
(-1)^{n+\ell(\mu)} (2\pi \sqrt{-1})^{n+\ell(\mu)} |\tilde{\mu}\rangle = (2\pi)^{n-\ell(\mu)} (2\pi \sqrt{-1})^{2\ell(\mu)} (-1)^{2\ell(\mu)} |\mu\rangle. \tag{4.9}
\]
We verify (4.9) as follows. By definition, $|\tilde{\mu}\rangle = (-\sqrt{-1})^{\ell(\mu)-n}|\mu\rangle$. Thus,
\[
(-1)^{n+\ell(\mu)}(2\pi \sqrt{-1})^{n+\ell(\mu)}|\tilde{\mu}\rangle = (-1)^{n+\ell(\mu)}(2\pi \sqrt{-1})^{n+\ell(\mu)}(-\sqrt{-1})^{\ell(\mu)-n}|\mu\rangle.
\]
We calculate
\[
(-1)^{n+\ell(\mu)}(2\pi \sqrt{-1})^{n+\ell(\mu)}(-\sqrt{-1})^{\ell(\mu)-n} = (2\pi)^{n+\ell(\mu)}(-1)^{2\ell(\mu)} \sqrt{-1}^{2\ell(\mu)},
\]
\[
(2\pi)^{n-\ell(\mu)}(2\pi \sqrt{-1})^{2\ell(\mu)}(-1)^{2\ell(\mu)} = (2\pi)^{n+\ell(\mu)}(-1)^{2\ell(\mu)} \sqrt{-1}^{2\ell(\mu)}.
\]
This proves (4.9), hence (4.8).

In summary, our calculations establish the equation
\[
z^{-\mu}_{\text{Sym}} z^{\rho}_{\text{Sym}} \left( \Gamma_{\text{Sym}} \cup (2\pi \sqrt{-1})^{-\frac{\deg_{\text{Sym}}}{2}} \sim \text{ch}(\mathbb{P}M(k)) \right)
= \text{CK}|_{z \to -z} \left( z^{-\mu}_{\text{Hilb}} z^{\rho}_{\text{Hilb}} \left( \Gamma_{\text{Hilb}} \cup (2\pi \sqrt{-1})^{-\frac{\deg_{\text{Hilb}}}{2}} \text{ch}(k) \right) \right),
\]
which completes the proof of Theorem 4. \qed

Acknowledgements We thank J. Bryan, T. Graber, Y.-P. Lee, A. Okounkov, and Y. Ruan for many conversations about the crepant resolution correspondence for $\text{Hilb}^n(C^2)$ and $\text{Sym}^n(C^2)$. The paper began with Y. Jiang asking us about the role of the Fourier–Mukai transformation in the results of [25]. We are very grateful to H. Iritani for detailed discussions about his integral structure and crepant resolution framework. R. P. was partially supported by SNF-200020162928, ERC-2012-AdG-320368-MCSK, ERC-2017-AdG-786580-MACI, SwissMAP, and the Einstein Stiftung. H.-H. T. was partially supported by NSF Grant DMS-1506551. The research presented here was furthered during a visit of the authors to Humboldt University in Berlin in June 2018. The project has received funding from the European Research Council (ERC) under the European Union Horizon 2020 Research and Innovation Program (Grant no. 786580).

References

1. Baake, M., Schlägel, U.: The Peano-Baker series, Tr. Mat. Inst. Steklova 275 (2011) (Klassicheskaya i Sovremennaya Matematika v Pole Deyatelnosti Borisa Nikolaevicha Delone, 167–171; reprinted in Proc. Steklov Inst. Math. 275 (2011), 155–159)
2. Bezrukavnikov, R., Okounkov, A.: Monodromy and Derived Equivalences (in preparation)
3. Borisov, L., Horja, R.P.: Mellin–Barnes integrals as Fourier–Mukai transforms. Adv. Math. 207, 876–927 (2006)
4. Bridgeland, T., King, A., Reid, M.: The McKay correspondence as an equivalence of derived categories. J. Am. Math. Soc. 14, 535–554 (2001)
5. Bryan, J., Graber, T.: The crepant resolution conjecture. Algebraic Geometry–Seattle 2005, Part 1, Proceedings of Symposia in Pure Mathematics, vol. 80. American Mathematical Society, Providence, RI, pp. 23–42 (2009)
6. Bryan, J., Pandharipande, R.: The local Gromov–Witten theory of curves. J. Am. Math. Soc. 21, 101–136 (2008)
7. Coates, T., Corti, A., Iritani, H., Tseng, H.-H.: A mirror theorem for toric stacks. Compos. Math. 151, 1878–1912 (2015)
8. Coates, T., Givental, A.: Quantum Riemann–Roch, Lefschetz and Serre. Ann. Math. 165, 15–53 (2007)
9. Coates, T., Iritani, H., Jiang, Y.: The crepant transformation conjecture for toric complete intersections. Adv. Math. 329, 1002–1087 (2018). arXiv:1410.0024
10. Givental, A.: A mirror theorem for toric complete intersections. Topological Field Theory, Primitive Forms and Related Topics (Kyoto, 1996), Progress in Mathematics, vol. 160, pp. 141–175. Birkhäuser, Boston (1998)

11. Givental, A.: Gromov–Witten invariants and quantization of quadratic Hamiltonians, Dedicated to the memory of I. G. Petrovskii on the occasion of his 100th anniversary. Mosc. Math. J. 1, 551–568 (2001).

12. Haiman, M.: Combinatorics, symmetric functions and Hilbert schemes. Current Developments in Mathematics 2002, vol. 1, pp. 39–111. International Press of Boston, Somerville (2002)

13. Haiman, M.: Notes on Macdonald polynomials and the geometry of Hilbert schemes. Symmetric Functions 2001: Surveys of Developments and Perspectives, Proceedings of the NATO Advanced Study Institute held in Cambridge, June 25–July 6, 2001, Sergey Fomin, editor. Kluwer, Dordrecht, pp. 1–64 (2002)

14. Horja, R.P.: Hypergeometric functions and mirror symmetry in toric varieties. Ph.D. Thesis, Duke University (1999). arXiv:math/9912109

15. Ince, E.L.: Ordinary Differential Equations. Dover Publications, New York (1944)

16. Iritani, H.: An integral structure in quantum cohomology and mirror symmetry for toric orbifolds. Adv. Math. 222, 1016–1079 (2009)

17. Iritani, H.: Ruan’s conjecture and integral structures in quantum cohomology. New Developments in Algebraic Geometry, Integrable Systems and Mirror Symmetry (RIMS, Kyoto, 2008). Advanced Studies in Pure Mathematics, vol. 59, pp. 111–166. Mathematical Society, Tokyo (2010)

18. Macdonald, I.: Symmetric Functions and Hall Polynomials, Second Edition. With Contributions by A. Zelevinsky. Oxford Mathematical Monographs. Oxford Science Publications. Oxford University Press, New York (1995)

19. McKay, J.: Graphs, singularities and finite groups. The Santa Cruz Conference on Finite Groups (Univ. California, Santa Cruz, Calif., 1979), Proceedings of Symposia in Pure Mathematics, vol. 37. American Mathematical Society, Providence, pp. 183–186 (1980)

20. Okounkov, A.: Enumerative geometry and geometric representation theory. Algebraic Geometry–Salt Lake City 2015, Part 1, Proceedings of Symposia in Pure Mathematics, vol. 97, Part 1, American Mathematical Society, Providence, pp. 419–458 (2018). arXiv:1701.00713

21. Okounkov, A.: On the crossroads of enumerative geometry and geometric representation theory. Proceedings of ICM 2018. arXiv:1801.09818

22. Okounkov, A., Pandharipande, R.: Quantum cohomology of the Hilbert scheme of points in the plane. Invent. Math. 179, 523–557 (2010)

23. Okounkov, A., Pandharipande, R.: The quantum differential equation of the Hilbert scheme of points in the plane. Transform. Groups 15, 965–982 (2010)

24. Pandharipande, R.: Cohomological Field Theory Calculations. Proceedings of ICM 2018, vol. 1, pp. 869–898. World Scientific, Singapore (2018). arXiv:1712.02528

25. Pandharipande, R., Tseng, H.-H.: Higher Genus Gromov–Witten Theory of Hilb^6(C^2) and CohFTs associated to local curves, Forum of Mathematics, Pi (2019), Vol. 7, e4, p. 63. arXiv:1707.01406

26. Tseng, H.-H.: Orbifold quantum Riemann–Roch, Lefschetz and Serre. Geom. Topol. 14, 1–81 (2010)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.