Constructing Lifshitz spaces using the Ricci flow

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Abstract. In this work we make use of the Ricci flow equations to show that, by starting from a general ansatz for the metric, we can construct two kinds of Lifshitz spaces in which: (a) the critical exponent coincides with the spatial dimension of the spacetime and therefore adopts discrete values, and (b) the critical exponent is continuous and arbitrary. These results show that Lifshitz spaces are exact solutions to the Ricci flow equations. Moreover, we found that the Ricci flow evolves towards a single fixed point for both cases which coincides with the flat spacetime.

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1 Introduction

The concept of Ricci flow was originally introduced by Richard Hamilton in 1982 [1] as a part of a developed program to resolve the well-known Poincaré conjecture or -more generally- the Thurston geometrization conjecture for closed 3-manifolds. The Ricci flow describes a mathematical treatment that allows us to continuously deform a Riemannian manifold, such deformation is realized by means of a partial differential equation that behaves like a kind of diffusion heat equation for the metric. An equivalent formulation was introduced by Dennis DeTurck [2] through a family of diffeomorphisms along the flow, giving rise to the Hamilton-DeTurck Ricci flow.

In physics, this tool has gained some attention, in part thanks to Perelman’s wonderful work [3,4] proving the geometrization conjecture almost a century after the Poincaré conjecture was proposed. The most obvious applications of Ricci’s flow in physics can be realized within the framework of general relativity, since it is by its nature a theory about the geometry of spacetime. In this context, the Ricci flow has been used to model the numerical evolution of wormholes and black holes [5]-[10]. The evolution of Morris-Thorne wormhole geometries and bubble geometries using numerical methods was studied in [5]. The question on how the Arnowitt-Deser-Misner (ADM) mass change under the Ricci flow was analyzed in [6]. Particularly, the study of the ADM mass of an asymptotic locally euclidean (ALE) space along the Ricci flow was approached. Moreover, the authors showed that the ALE property is preserved along the Ricci flow and, for a three-dimensional space, the mass is an invariant under the Ricci flow. In contrast with these results, the decay of the ADM mass of an asymptotically hyperbolic manifold of dimension $d \geq 3$ has been shown in [7]. Additionally, starting from the decay formula for the asymptotically hyperbolic mass under the curvature-normalized flow, an heuristic derivation on the invariance of the ADM mass for an asymptotically flat Ricci flow was given.

From Perelman’s gradient formulation for the Ricci flow, a possible connection between Perelman entropy and Bekenstein-Hawking entropy (geometric entropy) that arises from black hole thermodynamics was investigated in [8]. However, by studying the corresponding fixed points for the flow, no connection was found between these two entropies. Nevertheless, the authors proposed a new flow that apparently admits such connection and may have applications in black holes physics by suggesting new approaches to Penrose inequality. In [9] the ricci flow was implemented to describe the evolution of the area and Hawking mass of a 2-dimensional closed manifold. To get a physical significance of the results there are two facts from general relativity that were considered; the area of event horizons is related with the black hole entropy and, additionally, the fact that the Hawking mass of an asymptotic round 2-sphere is just the ADM mass. As a main result, an inequality which relates the evolution of the area of a closed surface and the Hawking mass was derived.

In [10] the Ricci flow was applied to the study of 4-dimensional Euclidean gravity with boundary $S^1 \times S^2$ which represents the canonical ensemble for gravity in a spherical box. It was found that at high temperatures the action has three saddle points: hot flat space, and large and small black holes. The small black hole is unstable under the Ricci flow since it has a Perry-Yaffe-type negative mode, while the other two are stable under the Ricci flow.
A review on how the Ricci flow arises in the renormalization group (RG) of non-linear sigma models was presented in [11]. Moreover, in this work a discussion of the behaviour of the mass under the Ricci flow by using a two-dimensional, rotationally symmetric, Ricci soliton was given. Additionally, the connection between the holographic renormalization group flow and the Ricci flow has been revealed in [12,13]; particularly, in [12] the authors introduced the Hamilton-Jacobi formalism to derive the RG flow equations and to show that, for the case of dual AdS/QFT theories, they are described by the Ricci flow. Another physical theory in which the Ricci flow naturally emerges is the well-known Hořava-Lifshitz gravity [14,15]; the original Ricci flow arises as a limiting case of a more general RG flow obtained by applying the Hamilton-Jacobi equation to the Hořava-Lifshitz gravity theory and setting some parameters [15]. The maximum principle was used in [16] to show that a Ricci soliton does not exist in Riemannian manifolds with boundaries such as those Lorentzian spacetimes which are asymptotically flat, Kaluza Klein, locally AdS or have extremal horizons.

Recently, the study of Ricci flows for maximally symmetric manifolds with constant curvature was performed in [17]. In this work it was found that spacetimes with positive constant curvature (de Sitter spaces) evolve into spacetimes with negative constant curvature (Anti-de Sitter spaces) under the Ricci flow through a singularity in the curvature, evolving further into Minkowski spacetime. There is another family of spacetimes with negative constant curvature that has a group of symmetries distinct from that of Anti-de Sitter space (which is a maximally symmetric space). The metric of these manifolds is invariant under anisotropic scalings of time and space; these spaces are called Lifshitz spacetimes and are given by the metric:

\[
ds^2 = \ell^2 \left(-r^{2z} dt^2 + r^{2z} dx_i^2 + \frac{dr^2}{r^2}\right), \quad i = 1, 2, \ldots, D,
\]

where \( z \) is a continuous parameter known as the critical exponent. This metric has a symmetry group denoted by \( \text{Lif}_D(z) \) [18,19], and is invariant under translations and spatial rotations \((H, P^i, L^{ij})\)

\[
H : t \rightarrow t' = t + \alpha; \\
P^i : x^i \rightarrow x^i' = x^i + \alpha^i; \\
L^{ij} : x^i \rightarrow x^i' = L^{ij}_{\quad k} x^k,
\]

and the non relativistic scaling symmetry \( D_2 \)

\[
D_2 : \quad r \rightarrow r' = \lambda^{\pm 1} r, \\
t \rightarrow t' = \lambda^{\mp 2} t, \\
x^i \rightarrow x^i' = \lambda^{\mp 1} x^i.
\]

The main purpose of this work is to introduce the Hamilton-DeTurck Ricci flow to study the construction of spacetime geometries with particular physical interest focused on those that allow Lifshitz symmetries. We already know that Lifshitz spaces arise as solutions to the Einstein field equations with several non-trivial stress energy tensors that depend on a given theory (see, for instance, [18,20,21]). On the other hand, the Ricci flow is a purely geometric equation that involves an external parameter that can be linked to a physical quantity; this is a very interesting fact since our results imply that one can construct Lifshitz spacetimes in a purely geometrical way, dispensing with the physical fields, for a wide family of field theories that support such metrics. Another important aspect of the study of Lifshitz spaces resides on the fact that they play a very important role in the description of the holographic quantum systems within the so-called Gravity (Lifshitz)/Condensed Matter Theory correspondence [18]. A better study of the intrinsic properties of the Lifshitz spaces will lead us to a deeper understanding of the holographic correspondence as well as to interesting applications to quantum physical systems through Lifshitz holography.

After getting exact solutions to the Ricci flow equations, we show that by means of an appropriate transformation of coordinates on the obtained metric, we get a family of Lifshitz spaces with discrete and continuous critical exponent. These metrics tend to a fixed point that corresponds to flat Minkowski spacetime as the flow evolves.

The structure of this work is as follows, in section [2] we introduced the Hamilton-DeTurck formulation of the Ricci flow. Then, we start with a quite general spacetime geometry that depends on the energy coordinate \( r \) and the Ricci parameter \( \lambda \), and evolves under the Hamilton-DeTurck Ricci flow. This evolution will render a set of nonlinear coupled partial differential equations. To solve this system, we assume an ansatz on the DeTurck vector field in such way that the system becomes consistent under the Hamilton-DeTurck Ricci flow. We further find exact solutions to the flow equations that can be recast as a Lifshitz metrics; here we also reveal that there exist one fixed point along the flow that reproduces the flat spacetime. In section [3] we present some concluding remarks and set a discussion about the fixed point of the geometry found for the Lifshitz metric constructed in this work.
2 Lifshitz space construction

Hamilton's original formulation for the Ricci flow is defined by the equation \[\partial_\lambda g_{\mu\nu} = -2R_{\mu\nu}, \quad g(0) = g_0,\] (4)

where \(R_{\mu\nu}\) is the Ricci tensor constructed from the metric \(g_{\mu\nu}\), which additionally depends on the flow parameter \(\lambda\) and \(g_0\) is the initial condition on the metric when \(\lambda = 0\). This equation describes the evolution of the geometry under the flow parameter that is external to the spacetime in question. Since (4) is diffeomorphism invariant, it is a degenerate nonlinear PDE and therefore we say that it is weakly parabolic \[23\]. Nevertheless, DeTurck showed that it is possible to modify the original formulation of the Ricci flow and obtain a strongly coupled parabolic nonlinear PDE \[22\]. This new formulation is named the Hamilton-DeTurck Ricci flow and it is defined by:

\[
\partial_\lambda g_{\mu\nu} = -2R_{\mu\nu} + \nabla_\mu V_\nu + \nabla_\nu V_\mu,
\]

(5)

here \(V_\mu\) is the DeTurck vector field that generates diffeomorphisms along the flow.

We shall begin by proposing a general geometry defined by

\[
ds^2 = l^2 \left[ -f_1(\lambda, r) dt^2 + \frac{1}{r^2} dr^2 + f_3(\lambda, r) dx_i dx^i \right], \quad i = 1, 2, \ldots, D,
\]

(6)

where \((t, x_i)\) stand for time and spatial coordinates, respectively, while \(r\) denotes an extra coordinate that has the physical meaning of energy within the holographic correspondence framework, and \(f_1(\lambda, r)\) and \(f_3(\lambda, r)\) are arbitrary functions of \(r\) and the Ricci flow parameter \(\lambda\). We wish to construct solutions to the Hamilton-DeTurck Ricci flow (5). First of all, we need to compute the non-zero Christoffel symbols associated with the metric (6)

\[
\Gamma^i_{tr} = \frac{1}{2} f'_i(\lambda, r), \quad \Gamma^i_{rr} = -\frac{1}{r}, \quad \Gamma^i_{jr} = \frac{1}{2} f'_3(\lambda, r) \delta_{ij},
\]

(7)

With the aid of these symbols, we now compute the non-zero components of the Ricci tensor:

\[
R_{tt} = \frac{r^2}{2} f''_1 - \frac{r^2}{4} f^2_1 f'_1 + \frac{r}{2} f'_1 + \frac{D r^2 f'_1 f'_3}{4 f_3},
\]

\[
R_{rr} = -\frac{1}{2} f''_1 - \frac{D}{2} f'_3 - \frac{1}{4} f''_1 f'_3 + \frac{f'_3}{f_3} - \frac{1}{2 r f_1} - \frac{D f'_3}{2 r f_3}
\]

\[
R_{ij} = \left[ -\frac{r^2 f''_3}{2} + (2 - D) r^2 f^2_1 f'_3 - \frac{r^2 f'_3}{2} f'_3 + \frac{r^2 f'_1 f'_3}{4 f_3} \right] \delta_{ij},
\]

(8)

We are now in position to consider the Hamilton-DeTurck Ricci flow (5) and find a system of \(D + 2\) nonlinear coupled partial differential equations (only three of them are independent) that describes the flow. Here we shall invoke our first ansatz by assuming that the DeTurck vector field has only \(r\) component, that is \(V_\mu = (0, V_r(\lambda, r), 0, \ldots, 0)\). With this fact in mind, the independent flow equations read

\[
l^2 f_1 = r^2 f''_1 - \frac{r^2 f^2_1}{2} f'_1 + \frac{D r^2 f'_1 f'_3}{f_3} + r^2 f'_1 V_r,
\]

\[
l^2 f_3 = r^2 f''_3 - \frac{r^2 f^2_3}{2} f'_3 + \frac{r^2 f''_1 f'_3}{2 f_3} + r^2 f'_3 V_r,
\]

\[
0 = \frac{f''_1}{f_1} - \frac{1}{2} f'_1 f'_1 f'_3 f_3 f'_3 + \frac{1}{r f_3} + D \left[ \frac{f'_3}{f_3} - \frac{1}{2} f'_3 f'_3 + \frac{f'_3}{r f_3} \right] + 2 V'_r + \frac{2}{r} V_r,
\]

(9)

(10)

(11)

We could in principle, add one more arbitrary function \(f_2(\lambda, r)\) in the \(g_{rr}\) component of this metric. Here we set this function to unity for the sake of simplicity.
where the dots represent derivatives with respect to the flow parameter $\lambda$ and the primes are derivatives with respect to the coordinate $r$. For the sake of simplicity we shall perform the following transformation $f_j(\lambda, r) = e^{u_j(\lambda, r)}$, where $j = 1, 3$, rendering the following system

\[
\begin{align*}
\dot{u}_1 &= r^2 u''_1 + \frac{r^2}{2} u_1'^2 + r u_1' + \frac{D}{2} r^2 u_1' u_3' + r^2 u_1' V_r, \\
\dot{u}_3 &= r^2 u''_3 + \frac{D}{2} r^2 u_3'^2 + r u_3' + \frac{r^2}{2} u_3' u_3' + r^2 u_3' V_r, \\
0 &= u''_1 + \frac{1}{2} u_1'^2 + \frac{1}{r} u_1' + D \left[ u''_3 + \frac{1}{2} u_3'^2 + \frac{1}{r} u_3' \right] + 2 V''_r + \frac{2}{r} V_r. 
\end{align*}
\]

By adding the equations (12) and (13) we can define a new function

\[
w(\lambda, r) = u_1(\lambda, r) + D u_3(\lambda, r)
\]

such that the sum of these equations gives

\[
\dot{\bar{w}} = r^2 w'' + \frac{r^2}{2} w'^2 + r w' + r^2 w' V_r,
\]

### 2.1 Solutions to the flow with discrete critical exponent

By taking the definition of $w(\lambda, r)$ we can express (13) as follows

\[
\frac{D(D + 1)}{2} u_3'^2 - D w' u_3' + w'' + \frac{1}{2} w'^2 + \frac{1}{r} w' + 2 V''_r + \frac{2}{r} V_r = 0.
\]

Since $V_r$ is an arbitrary function of $r$ and $\lambda$, we choose our second ansatz such that

\[
w'' + \frac{1}{2} w'^2 + \frac{1}{r} w' + 2 V''_r + \frac{2}{r} V_r = -\frac{q(\lambda)^2}{r^2},
\]

where $q(\lambda)$ is an arbitrary function of $\lambda$. This gives us a system of PDE’s that couples $w(\lambda, r)$ with $V_r(\lambda, r)$:

\[
\begin{align*}
\dot{\bar{w}} &= r^2 w'' + \frac{r^2}{2} w'^2 + r w' + r^2 w' V_r, \\
0 &= r^2 w'' + \frac{r^2}{2} w'^2 + r w' + 2 r^2 V''_r + 2 r V_r + q(\lambda)^2.
\end{align*}
\]

As a simple solution for this system we choose

\[
V_r(\lambda, r) = -\frac{1}{2} q(\lambda)^2 \ln r + \frac{h(\lambda)}{r}
\]

\[
w(\lambda, r) = c = \text{const.}
\]

where $h(\lambda)$ is one more arbitrary function of $\lambda$. Then, by using (17) we can find solutions for $u_3$ and then for $u_1$, obtaining:

\[
u_1(\lambda, r) = \pm D \sqrt{\frac{2}{D(D + 1)}} q(\lambda) \ln r - D m(\lambda) + \text{const.},
\]

\[
u_3(\lambda, r) = \pm \sqrt{\frac{2}{D(D + 1)}} q(\lambda) \ln r + m(\lambda),
\]

where $m(\lambda)$ is an arbitrary function of $\lambda$. By substituting (22) and (23) into (12) and (13) we find some restrictions upon $q(\lambda)$, $h(\lambda)$ and $m(\lambda)$. By considering the solutions with either plus or minus sign before the square root of both (22) and (23) we obtain

\[
q(\lambda) = \sqrt{\frac{l^2}{\lambda - \lambda_0}},
\]

\[
h(\lambda) = \mp \sqrt{\frac{2 D l^2}{(D + 1)(\lambda - \lambda_0)}},
\]

\[
m(\lambda) = 0.
\]
These restrictions ensure that the system is solved in a consistent way. Therefore, the solutions for \( u_1 \) and \( u_3 \) are given by:

\[
u_1(\lambda, r) = \pm D \sqrt{\frac{2l^2}{D(D+1)(\lambda-\lambda_0)}} \ln r + c, \quad u_3(\lambda, r) = \pm \sqrt{\frac{2l^2}{D(D+1)(\lambda-\lambda_0)}} \ln r,
\]

since \( c \) is a constant, it can be absorbed into the coordinates and it does not matter if we choose it as zero. Therefore

\[
f_1(\lambda, r) = r^{\pm D\alpha(\lambda,D)}, \quad f_3(\lambda, r) = r^{\pm \alpha(\lambda,D)},
\]

where the involved parameter \( \alpha(\lambda, D) \) reads

\[
\alpha(\lambda, D) = \sqrt{\frac{2l^2}{D(D+1)(\lambda-\lambda_0)}}.
\]

These results allow us to write the evolving geometry proposed in (6) as

\[
d^2_{\mathcal{A}} = l^2 \left[ -r^{\pm 2D\alpha(\lambda,D)} dt^2 + \frac{1}{r^2} dr^2 + r^{\pm\alpha(\lambda,D)} dx_i dx^i \right], \quad i = 1, 2, \ldots, D.
\]

We can see that (28) is well defined in the interval \( \lambda \in [\lambda_0, \infty] \). Nevertheless, here there is no analytical continuation like the one that takes place in [17] for maximally symmetric spacetimes. We can write (28) in a canonical Lifshitz form by using the following transformation of coordinates

\[
t \to \frac{2}{\alpha} \tilde{t}, \quad r \to \tilde{r}^{\frac{2}{\alpha}}, \quad x_i \to \frac{2}{\alpha} \tilde{x}_i,
\]

so that the metric adopts the form

\[
d^{2}_{\tilde{\mathcal{A}}} = \tilde{l}^2(\lambda, D) \left[ -\tilde{r}^{\pm 2D} d\tilde{t}^2 + \tilde{r}^{-2} d\tilde{r}^2 + \tilde{r}^{\pm 2} d\tilde{x}_i d\tilde{x}^i \right], \quad i = 1, 2, \ldots, D,
\]

with

\[
\tilde{l}(\lambda, D) = \frac{2l}{\alpha(\lambda,D)}.
\]

The factor that multiplies the brackets is a constant for a given specific value of the flow parameter \( \lambda \). In the limit when \( \alpha \to 2 \) we identify the spacetime generated in (30) as the Lifshitz space for the case in which the critical exponent is equal to the spatial dimension. Additionally, the curvature scalar of this geometry is given by

\[
R = -\frac{(5D+1)}{2(D+1)} \left( \frac{1}{\lambda-\lambda_0} \right).
\]

This invariant is negative definite, and when \( \lambda \to \infty \) it goes to zero. This means that in this limit we reproduce the flat spacetime. By considering the definition of fixed points

\[
\partial_\lambda g_{\mu\nu} = 0
\]

applied to the metric (28) we find the conditions

\[
\partial_\lambda g_{tt} = \pm \frac{l^4}{(D+1)(\lambda-\lambda_0)^2} r^{\pm D\alpha(\lambda,D)-1} = 0,
\]

\[
\partial_\lambda g_{rr} = 0,
\]

\[
\partial_\lambda g_{ii} = \pm \frac{l^4}{D(D+1)(\lambda-\lambda_0)^2} r^{\pm\alpha(\lambda,D)-1} = 0,
\]

so, in order to satisfy them, we find that \( \lambda \to \infty \) for every \( r \). This implies that there is a single fixed point along the flow which coincides with the flat spacetime.
2.2 Solutions to the flow with continuous critical exponent

We already know the suitable ansatz to DeTurck vector field $V_r(\lambda, r)$ that will allow us to reproduce Lifshitz spaces, so we can obtain a more general solution taking the form of $V_{r, \lambda}$ and noting that it is possible to find a more general solution to (16) that it is not necessarily a constant. So, we first postulate $V_r(\lambda, r)$ as

$$V_r(r, \lambda) = -\rho(\lambda) \frac{\ln r}{r} + \frac{F(\lambda)}{r}$$

a new solution to (16) is given by

$$w(r, \lambda) = \pm \chi(\lambda) \ln(r) + G(\lambda)$$

provided that

$$F(\lambda) = -\chi(\lambda) G(\lambda) = \text{const.} = \alpha$$

since $w(r, \lambda) = u_1(r, \lambda) + Du_3(r, \lambda)$, we can choose $u_1(r, \lambda), u_3(r, \lambda)$ and $F(\lambda)$ as follows

$$u_1(r, \lambda) = \pm 2\beta(\lambda) \ln r, \quad u_3(r, \lambda) = \pm 2\gamma(\lambda) \ln r, \quad F(\lambda) = (\beta(\lambda) + D\gamma(\lambda))$$

by substituting these solutions and (37) in (12), (13) and (14) we obtain

$$l^2 \dot{\beta}(\lambda) + (\beta(\lambda) \rho(\lambda) = 0$$

$$l^2 \dot{\gamma}(\lambda) + (\gamma(\lambda) \rho(\lambda) = 0$$

$$\beta^2(\lambda) + D\gamma^2(\lambda) - \rho(\lambda) = 0$$

with solutions given by

$$\beta(\lambda) = k \sqrt{\frac{l^2}{2(k^2 + D)(\lambda - \lambda_0)}}, \quad \gamma(\lambda) = \sqrt{\frac{l^2}{2(k^2 + D)(\lambda - \lambda_0)}}, \quad \rho(\lambda) = \frac{l^2}{2(\lambda - \lambda_0)}$$

were $k$ is a continuous parameter that may have any real value. Then

$$f_1(r, \lambda) = r^{\pm 2k} \sqrt{\frac{l^2}{2(k^2 + D)(\lambda - \lambda_0)}}, \quad f_3(r, \lambda) = r^{\pm 2} \sqrt{\frac{l^2}{2(k^2 + D)(\lambda - \lambda_0)}}, \quad F(\lambda) = -(k + D) \sqrt{\frac{l^2}{2(k^2 + D)(\lambda - \lambda_0)}}$$

So the evolving metric has the form

$$ds_B^2 = l^2 \left[ -r^{\pm 2k} \sqrt{\frac{l^2}{2(k^2 + D)(\lambda - \lambda_0)}} dt^2 + r^2 dr^2 + r^{\pm 2} \sqrt{\frac{l^2}{2(k^2 + D)(\lambda - \lambda_0)}} dx_i dx^i \right] , \quad i = 1, 2, \ldots D.$$  

if we perform a similar transformation of coordinates like (29), i.e.

$$t \rightarrow \frac{1}{\gamma(\lambda)} \tilde{t}, \quad r \rightarrow \tilde{r}^{\pm 1}, \quad x_i \rightarrow \frac{1}{\gamma(\lambda)} \tilde{x}_i,$$

the metric adopts the form

$$ds_B^2 = \tilde{l}^2(\lambda) \left[ -\tilde{r}^{\pm 2z} dt^2 + \frac{1}{\tilde{r}^2} d\tilde{r}^2 + \tilde{r}^{\pm 2} d\tilde{x}_i d\tilde{x}^i \right], \quad i = 1, 2, \ldots D,$$

with

$$\tilde{l}^2(\lambda) = \frac{2(z^2 + D)(\lambda - \lambda_0)}{l^2},$$

is easy to note that the solution (40) is more general than (30), this follows from the fact that now the critical exponent ($z = k$) may adopt continuous values since is an arbitrary continuous constant. The curvature scalar in the continuous case is given by

$$R = -\left( \frac{D + 2zD + 2z^2}{2(D + z^2)} \right) \frac{1}{(\lambda - \lambda_0)}.$$  

Following similar arguments like that in the discrete case, is easy to check that this flow has one fixed point that also coincides with the flat spacetime.
3 Discussion and concluding remarks

The solutions that we have obtained from the evolution of the Hamilton-De Turck Ricci flow (4) give us a family of Lifshitz spaces with both discrete and continuous critical exponents. The latter solution is more general than the former one, since in it the critical exponent may adopt any real value including those corresponding to the discrete case. Thus we have shown that Lifshitz spacetimes are exact solutions of the Hamilton-DeTurck Ricci flow equations when starting with a general setup given by (6). Therefore, Lifshitz geometries can be constructed in a purely geometric way with the aid of the Ricci flow, in contrast to the RG flow formalism that makes use of several $\sigma$-models defined in terms of diverse fields. In this sense, we are generating Lifshitz spaces in an universal way since these geometries can be supported by several actions where gravity is coupled to a wide range of matter fields [18,21]. Additionally, we see that the spaces that we have behave in a similar way to those discussed in [21, i. e., they preserve the property that the flow is well defined for the interval $0 \leq \lambda < \infty$, and in the limit when $\lambda \to \infty$, we recover the flat spacetime geometry; we see this fact in an invariant way from the form of the curvature scalar. With this in mind, we can say that, analogously to what happens with the diffusion of heat, for large enough values of the flow parameter, the Ricci flow equations homogenize the metric in such a way that it reaches the equilibrium state given by the flat spacetime.

We would like to contrast the result regarding the fixed point for both the obtained metric (28) and the transformed interval (30). For the former, the fixed point corresponds in a clear way to flat spacetime as $\lambda \to \infty$, which is in agreement with the behavior of the curvature scalar when the Ricci flow parameter adopts large values. By following the definition of a fixed point (33) for the transformed metric (30), it is clear that we also need to transform the flow parameter

$$\lambda \to \tilde{\lambda}(\lambda)$$

in such a way that the partial derivative respect to the Ricci flow parameter now reads

$$\partial_\lambda \to \frac{\partial \tilde{\lambda}}{\partial \lambda} \partial_{\tilde{\lambda}}$$

(50)

where $\tilde{\lambda}$ is a function of $\lambda$, that we shall call $P(\lambda)$. With this transformation we can find the fixed point equations for the metric in question, leading to the following relations

$$\partial_{\tilde{\lambda}} \tilde{g}_{tt} = 0 \Rightarrow -\frac{2D(D + 1)}{\partial_{\tilde{\lambda}} P} \tilde{t}^{2D} = 0,$$

(51)

$$\partial_{\tilde{\lambda}} \tilde{g}_{rr} = 0 \Rightarrow \frac{2D(D + 1)}{\partial_{\tilde{\lambda}} P} \tilde{r}^{-2} = 0,$$

(52)

$$\partial_{\tilde{\lambda}} \tilde{g}_{ii} = 0 \Rightarrow \frac{2D(D + 1)}{\partial_{\tilde{\lambda}} P} \tilde{r}^{2} = 0,$$

(53)

therefore, in order to recover the fixed point already computed for the original metric (28), we have to impose the condition that $\partial_{\lambda} P(\lambda) \to \infty$ as $\lambda \to \infty$: a good choice of the function $P$ would be, for example, $P(\lambda) \sim \lambda^n$ with $n > 1$. On the other hand, with this choice we also recover the behavior of the curvature scalar (32) as $\tilde{\lambda} \to \infty$, i. e. we obtain the flat spacetime as a fixed point along the flow.

A similar analysis of the metric with a continuous critical exponent (46) leads us to the same results regarding the fixed points of the Ricci flow: the family of geometries evolves towards flat spacetime.

It would be interesting as well to consider the evolution of spatially anisotropic but homogeneous Lifshitz geometries under the Hamilton-DeTurck Ricci flow like the spaces recently constructed in [18,19], i. e. when the Lifshitz geometry admits a non relativistic scaling symmetry given by

$$D_z: \ r \to r' = \lambda^{z_1} r, \ t \to t' = \lambda^{z_2} t, \ x' = \lambda^{z_3} x',$$

(54)

where $z_i$ are critical exponents. As a final remark, we would like to point out that the above constructed family of Lifshitz spaces includes two relevant cases for $D = z = 2$ and $D = z = 3$ since within the Gravity/Condensed Matter correspondence they describe physically relevant quantum systems when the critical exponent is $z = 2$ for two-dimensional materials, and $z = 3$ for three-dimensional solid state systems [21].

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2 Those spaces with translational symmetry in which each direction scales differently.
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