Efficient Construction of Spanners in \( d \)-Dimensions

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Abstract

In this paper we consider the problem of efficiently constructing \( k \)-vertex fault-tolerant geometric \( t \)-spanners in \( \mathbb{R}^d \) (for \( k \geq 0 \) and \( t > 1 \)). Vertex fault-tolerant spanners were introduced by Levcopoulus et. al in 1998. For \( k = 0 \), we present an \( O(n \log n) \) method using the algebraic computation tree model to find a \( t \)-spanner with degree bound \( O(1) \) and weight \( O(\omega(MST)) \). This resolves an open problem. For \( k \geq 1 \), we present an efficient method that, given \( n \) points in \( \mathbb{R}^d \), constructs \( k \)-vertex fault-tolerant \( t \)-spanners with the maximum degree bound \( O(k) \) and weight bound \( O(k^2 \omega(MST)) \) in time \( O(n \log n) \). Our method achieves the best possible bounds on degree, total edge length, and the time complexity, and solves the open problem of efficient construction of (fault-tolerant) \( t \)-spanners in \( \mathbb{R}^d \) in time \( O(n \log n) \).

1 Introduction

In this work we consider the problem of constructing spanner graphs to approximate the complete Euclidean graph. Given an edge weighted graph \( G = (V, E, W) \), where \( w(e) \) is the weight of an edge \( e \), let \( d_G(u, v) \) denote the shortest distance from vertex \( u \) to vertex \( v \) in graph \( G \). The weight of the graph \( G, \omega(G) \), is the sum of the edge weights of edges in \( G \). A subgraph \( H = (V, E') \), where \( E' \subseteq E \), is called a \( t \)-spanner of the graph \( G \), if for any pair of vertices \( u, v \in V \), \( d_H(u, v) \leq t \cdot d_G(u, v) \). The minimum \( t \) such that \( H \) is a \( t \)-spanner of \( G \) is called the stretch factor of \( H \) with respect to \( G \). An Euclidean graph is a graph where the vertices are points in \( \mathbb{R}^d \) and the weight of every edge \( (u, v) \) is the Euclidean distance \( \|uv\| \) between its end-vertices \( u \) and \( v \). Spanner graphs in \( \mathbb{R}^d \) have been extremely well studied. We consider spanner graphs with additional properties of bounded degree, low weight and fault tolerance.

In this paper, we study \( t \)-spanners and \( k \) vertex fault-tolerant \( t \)-spanners ((\( k, t \))-VFTS for short) for a set \( V \) of \( n \) points in \( \mathbb{R}^d \). A subgraph \( H = (V, E) \) is \( k \) vertex fault-tolerant, or \( k \)-VFT for short, if for any pair of vertices \( u \) and \( v \) with \( uv \notin E \), there are \( k + 1 \) vertex disjoint paths from \( u \) to \( v \) in \( H \). Here two paths \( \Pi_1 \) and \( \Pi_2 \) from \( u \) to \( v \) are said to be vertex disjoint if the only common vertices of \( \Pi_1 \) and \( \Pi_2 \) are \( u \) and \( v \). A geometric graph \( H = (V, E) \) is termed \( (k, t) \)-VFTS if for any subset \( F \subseteq V \) of at most \( k \) vertices and any two vertices \( w_1, w_2 \in V \setminus F \), the graph \( H(V \setminus F, E_H') \), where \( E_H' = E_H \setminus \{(u, v) \mid u \in F, \text{ or } v \in F \} \), contains a path \( \Pi(w_1, w_2) \) from \( w_1 \) to \( w_2 \) with length at most \( t \|w_1w_2\| \). Given an Euclidean graph that is \( k \) vertex fault-tolerant (VFT) and a real number \( t > 1 \), the aim here is to construct a subgraph \( H \) which is a \((k, t)\)-VFTS subgraph, with a bounded vertex degree, and a bounded weight, \( i.e., \omega(H)/\omega(MST(G)) \) is bounded by a specified small constant, where \( MST(G) \) is the minimum weighted spanning tree of \( G \).

A greedy algorithm has been used to construct spanners for various graphs \[8, 15, 17, 27\]. For a graph \( G = (V, E) \) with \( |V| = n \) and an arbitrary edge weight, Peleg and Schaffer \[25\] showed that any \( t \)-spanner needs at least \( n^{1+\frac{1}{t+2}} \) edges; thus there is edge weighted graph such that any \( t \)-spanner of such a graph has weight at least \( \Omega(n^{1+\frac{1}{t+2}}\omega(MST)) \) (the bound is obtained by letting the weight of each edge be 1). Chandra et
al. [9] showed that the greedy algorithm constructs a t-spanner of weight at most \((3 + \frac{19}{7}) \frac{2t}{\ln t} \cdot \omega(\text{MST})\) for every \(t > 1\) and any \(\epsilon > 0\). Regev [28] proved that the t-spanner constructed by the greedy algorithm has weight at most \(2e^2 \ln n \cdot n^{-\frac{2}{t+1}} \cdot \omega(\text{MST})\) when \(t \in [3, 2 \log n + 1]\), and has weight at most \((1 + 4/3 + 2 \log n) \cdot \omega(\text{MST})\) when \(t > 2 \log n + 1\), by studying the girth of the constructed t-spanner. Elkin and Peleg [14] recently showed that for any constant \(\epsilon, \lambda > 0\) there exists a constant \(\beta = \beta(\epsilon, \lambda)\) such that for every \(n\)-vertex graph \(G\) there is an efficiently constructible \((1 + \epsilon, \beta(\epsilon, \lambda))\)-spanner of size \(O(n^{1+\lambda})\).

Constructing \(t\)-spanners [16, 10, 21, 23, 24, 26, 29, 30, 32] and \((k, t)\)-VFTS [13, 10, 20, 22] for Euclidean graphs has been extensively studied in the literature. For computing \(t\)-spanners of \(O(1)\) degree and \(O(\omega(\text{EMST}))\) weight, the current best result [1] using algebraic computation tree model is a method with time complexity \(O(n \log^2 n / \log \log n)\). An \(O(n \log n)\) algorithm which uses an algebraic model together with indirect addressing has been obtained in [17]. While this model is acceptable in practice, the problem of computing low weight spanners in the algebraic decision tree model in time \(O(n \log n)\) is still open. We resolve this problem and extend the techniques introduced in the first part of the paper to allow us to compute the \(k\)-fault-tolerant spanners efficiently.

In this paper we will also consider constructing \((k, t)\)-VFTS for \(k \geq 1\) for the complete Euclidean graphs on \(n\) points \(V\) in \(\mathbb{R}^d\). The problem of constructing \((k, t)\)-VFTS for Euclidean graphs was first introduced in [20].

Using the well-separated pair decomposition [7], Callahan and Kosaraju showed that a \(k\)-VFT spanner can be constructed (1) in \(O(n \log n + k^2 n)\) time with \(O(k^2 n)\) edges, or (2) in \(O(nk \log n)\) time with \(O(kn \log n)\) edges, or (3) in time \(O(n \log n + c^k n)\) with degree \(O(c^k)\) and total edge length \(O(c^k \cdot \omega(\text{EMST}))\). Here the constant \(c\) is independent of \(n\) and \(k\). Later, Lukovszki [22] presented a method to construct a \((k, t)\)-VFTS with the asymptotic optimal number of edges \(O(kn)\) in time \(O(n \log^{d-1} n + nk \log log n)\). Czumaj and Zhao [10] showed that there are Euclidean graphs such that any \((k, t)\)-VFTS has weight at least \(\Omega(k^2 \omega(\text{EMST}))\), where \(\text{EMST}\) is the Euclidean minimum spanning tree connecting \(V\). They then proved that one can construct a \((k, t)\)-VFTS using a greedy method for a set \(V\) of \(n\) nodes, that has maximum degree \(O(k)\) and total edge length \(O(k^2 \omega(\text{EMST}))\). However it is unknown, given arbitrary \(k\), an Euclidean graph and a pair of vertices \(u, v\), whether we can determine in polynomial time if there are \(k + 1\) vertex-disjoint paths connecting them and each path has a length at most a given value \(t\|uv\|\). Notice that this problem is NP-hard when we are given a graph \(G\) with arbitrary weight function. Czumaj and Zhao further presented a method to construct a \((k, t)\)-VFTS for Euclidean graphs in time \(O(nk \log d n + nk^2 \log k)\) such that it has the maximum node degree \(O(k)\) and total edge length \(O(k^2 \log n) \cdot \omega(\text{EMST})\) for \(k > 1\). Observe that there is a gap between the lower bound \(O(k^2) \cdot \omega(\text{EMST})\) and the achieved upper bound \(O(k^2 \log n) \cdot \omega(\text{EMST})\) on the total edge length.

Our Results: The contributions of this paper are as follows. In the first part of the paper, given a set \(V\) of points (such input points are called nodes hereafter) in \(\mathbb{R}^d\) and an arbitrary real number \(t > 1\), we present a method that runs in time \(O(n \log n)\) using the algebraic computation tree model and constructs a \(t\)-spanner graph whose total edge length is \(O(\omega(\text{EMST}))\). The hidden constants depend on \(d\) and \(t\), or more precisely, the number of cones used in our method, which is \(O((\frac{1}{t+1})d)\). This solves an open question of finding a method with time-complexity \(O(n \log n)\) in the algebraic computation tree model. The main techniques used in our methods are listed below.

1. We first apply a special well-separated pair decomposition, called bounded-separated pair decomposition (BSPD) which is produced using a split-tree partition of input nodes [7]. The split-tree partition uses boxes that tightly enclose a set of nodes, i.e., each side of the box contains a point from \(V\). In our decomposition, we need to ensure that every pair \((X, Y)\) of separated sets of nodes is contained in two, almost equal sized boxes, \(b(X)\) and \(b(Y)\), respectively, where \(b(X)(b(Y))\) respectively) contains only the

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\(^1\)Private communication with M. Smid.

\(^2\)Edges are processed in increasing order of length and an edge \((u, v)\) is added only if \(H\) formed by previously added edges does not have \(k + 1\) internally node-disjoint paths connecting \(u\) and \(v\), each with length at most \(t\|u - v\|\).
node set $X(Y$ respectively). These boxes are termed floating virtual boxes since they can be positioned in a number of ways. An important property of the BSPD that we construct is that for every pair of nodes sets $(X, Y)$ in the decomposition, the distance between $b(X)$ and $b(Y)$ is not only not too small (these conditions are from WSPD), but also not too large, compared with the sizes of boxes containing them respectively.

2. To facilitate the proof that the structure constructed by our method is a $t$-spanner we use neighborhood cones. At every point $x$ we use a cone partition of the space around $x$ by a set of basis vectors. To guide the addition of spanner edges we introduce the notion of General-Cone-Direction for a pair of boxes $b$ and $b'$. This notion ensures that every point $x$ contained inside the box $b$ has all the nodes inside $b'$ within a collection of cones $C$, each cone with apex $x$. The angular span of the cones $C$ is bounded from above by some constant (depending on the spanning ratio $t$), which ensures the spanner property.

3. To prove that the structure constructed by our method has low-weight, we introduce the empty-cylinder property. A set of edges $E$ is said to have the empty-cylinder property if for every edge $e \in E$, we can find an empty cylinder (that does not contain any end-nodes of $E$ inside) that uses a segment of $e$ as its axis and has radius and height at least some constants factor of the length of $e$. We prove that a set of edges $E$ with empty-cylinder property and empty-region property has a total weight proportional to the minimum spanning tree of the set of end-nodes of edge $E$.

In the second part of the paper, given $V$ in $\mathbb{R}^d$, $t > 1$ and a constant integer $k > 1$, we present a method that runs in time $O(n \log n)$ and constructs a $k$-vertex fault-tolerant $t$-spanner graph with following properties: (1) the maximum node degree is $O(k)$, and (2) the total edge length is $O(k^2 \omega(\text{EMST}))$. This achieves an optimal weight bound and degree bound of the spanner graph, which is the first such result known in the literature. The second part utilizes techniques introduced in the first part.

The paper is organized as follows. We present our method of constructing $t$-spanner in time $O(n \log n)$ in Section 2 and prove the properties of the structure and study the time complexity of our method in Section 3. In Section 4 we present and study our method of constructing $(k, t)$-VFTS. We conclude our paper in section 5.

## 2 $t$-Spanner in $\mathbb{R}^d$ Using Compressed Split-tree

In this section, we present an efficient method in the algebraic computation tree model with time complexity $O(n \log n)$ to construct a $t$-spanner for any given set of nodes $V$ in $\mathbb{R}^d$ for any $t > 1$. Our method is based on a variation of the compressed split-tree partition of $V$: We partition all pairs of nodes using a special well-separated pair decomposition based on a variant of split-tree that uses boxes with bounded aspect ratio.

### 2.1 Split Tree Partition of a set of Nodes

We use $d(x, y)$ to denote the distance between points $x$ and $y$ in $\mathbb{R}^d$ in the $L_p$-metric for $p \geq 1$. We will focus on Euclidean distance here. We define our partition of input nodes $V \in \mathbb{R}^d$ using a Compressed Split-tree, a structure first used in [31]. Let $x_i$ be the $i$th dimension in $\mathbb{R}^d$ and $x = (x_1, x_2, \cdots, x_d)$ be a point in $\mathbb{R}^d$. Then an orthogonal box $b$ in $\mathbb{R}^d$ is $\{x = (x_1, x_2, \cdots, x_d) \mid L_i \leq x_i \leq R_i\}$, where $L_i < R_i$, $i = 1, 2, \cdots, d$, are given values defining the bounding planes of the box. Given a box $b$ in $\mathbb{R}^d$ we define the following terminology:

- $|b|$ is the number of nodes from $V$ contained in the box $b$.
- $d(b_1, b_2)$ is the Euclidean distance between the boxes $b_1$ and $b_2$, i.e., $\min_{x \in b_1, y \in b_2} \|x - y\|$.
- For a box $b$, $\vartheta(b)$ denotes the size of box $b$, i.e., the length, $\max_{1 \leq i \leq d}(R_i - L_i)$, of the longest side of $b$.
- The aspect ratio of a box $b$ is defined as the ratio of the longest side-length over the smallest side-length, i.e., $\max_{1 \leq i, j \leq d}(R_i - L_i)/(R_j - L_j)$.
Given a point set, \( S \), we will refer to the smallest orthogonal box enclosing the point set \( S \) as \( b = \Box(S) \). Such a box \( b \) is called an enclosing-box of a point set \( S \) hereafter. Here the enclosing-box \( b \) does not necessarily have a good aspect ratio.

**Definition 1 (Tight-Virtual Box)** Given a box \( b = \Box(S) \) we define a box \( \Box(b) \) as a tight-virtual box if it has the following properties:

1. \( \Box(b) \supseteq b \), i.e. it contains \( b \) inside
2. longest side of \( \Box(b) \) is exactly \( \vartheta(b) \), the longest side of box \( b \).
3. \( \Box(b) \) has an aspect ratio at most a constant \( \beta \leq 2 \).

Given a set \( V \) of \( n \) \( d \)-dimensional nodes, let \( \delta(V) \) be the smallest pairwise distance between all pairs of nodes in \( V \). We next define a special split-tree similar to the structure defined in [7][8][11].

**Definition 2 (Compressed split-tree)** A compressed split-tree, termed \( CT(V) \), is a rooted tree of \( d \)-dimensional boxes defined as follows:

1. Each vertex \( u \) in the compressed split-tree is mapped to a \( d \)-dimensional box \( b = \Box(u) \), and associated with a tight-virtual box \( \Box(b) \).
2. The root vertex, termed root, of the tree \( CT(V) \) is associated with the enclosing-box \( \Box(\text{root}) = \Box(V) \) containing all the nodes in \( V \). Associated with this box is a tight-virtual box \( \Box(\Box(\text{root})) \) which has a bounded aspect ratio \( \beta \leq 2 \) enclosing the box \( \Box(\text{root}) \).
3. Each internal vertex \( u \) (associated with a box \( b = \Box(u) \) and the tight-virtual box \( \Box(b) \)) in the tree \( CT(V) \) has two children vertices, if \( b \) contains at least 2 nodes from \( V \). Consider the two boxes, \( B'_1, B'_2 \), obtained by subdividing \( b \) into 2 smaller boxes \( b'_i, 1 \leq i \leq 2 \), cutting \( b \) by a hyperplane passing through the center of \( b \) and perpendicular to the longest side of \( b \). Shrink \( B'_i, 1 \leq i \leq 2 \) to obtain minimum sized enclosing-box \( b_i \), containing the same set of nodes as \( B'_i \), i.e., each face of \( b_i \) contains a node of \( V \). Let \( \Box(b) = \{b_1, b_2\} \). With each box \( b_i \) in \( \Box(b) \), we associate a tight-virtual box \( \Box(b_i) \) with an aspect ratio at most \( \beta \leq 2 \). Then the children vertices of the vertex \( u \) are two boxes \( b_1, b_2 \). Additionally, \( \Box(b_1) \) and \( \Box(b_2) \) are disjoint and are contained inside \( b \).
4. There is a tree edge from \( b \) to every \( b_i \in \Box(b) \). Notice that neither \( b_1 \) nor \( b_2 \) is empty of nodes inside. The box \( b \) from which \( b_i \) is obtained by this procedure is referred to as the father of the \( b_i \), denoted as \( \Box(b_i) \). A box \( b \) that contains only one node is called a leaf box. For simplicity of presentation, we assume that any leaf box has a size \( c \) for sufficiently small \( 0 < c < \delta(V) \).
5. The level of a box \( b \) is the number (rounds) of subdivisions used to produce \( b \) from the root box. The level of the box \( \Box(\text{root}) \) is then 0. If a box \( b \) has level \( j \), then each box in \( \Box(b) \) has level \( j + 1 \).

In Lemma 1 we will show that the tight-virtual boxes \( \Box(b_i), i = 1, 2 \), can be constructed from \( \Box(b) \) in \( O(d) \) time.

The tree \( CT(V) \) is called a canonical partition split-tree of \( V \). One difference between our structure and the split-tree structure used in [7] is that we associate with each box \( b \) in \( CT(V) \) a tight-virtual box \( \Box(b) \), while in [7] different boxes are used. Another major difference is the floating-virtual-boxes to be introduced later.

**Lemma 1** Given a box \( b \) and its associated tight-virtual box \( \Box(b) \), we can find the tight-virtual box \( \Box(b_i) \) for each children box \( b_i \in \Box(b) \) in \( O(d) \) time.

**Proof.** Obviously, for the root box \( \Box(V) \), we can find a tight-virtual box \( \Box(\Box(V)) \) in \( O(d) \) time. We now show that given a vertex \( b = \Box(\Box(S)) \) (enclosing-box of some subset \( S \) of nodes in \( V \)) in \( CT(V) \) and a child enclosing-box \( b_1 \) obtained by subdividing \( b \) by a hyperplane \( h \), we can construct a tight-virtual box \( \Box(b_1) \) from the tight-virtual box \( \Box(b) \) efficiently as follows. Let \( \Box(b)_h \) be the box, which contains \( b_1 \), obtained by
partitioning $\square(b)$ using the hyperplane $h$. Assume w.l.o.g that $b_1$ is the one located with the same center as $\square(h)$. Box $\square(b)_h$ can now be shrunk as $b_h$ until one of the sides of the shrunk box $b_h$ meets a side of $b_1$. Other sides of $b_h$ which are larger than $\vartheta(b_1)$ can be shrunk to meet $b_1$ if the aspect ratio is not below $\beta$. This gives us $\square(b)$. The aspect ratio of $\square(b)$ is bounded by the aspect ratio of $\square(b)_h$ if $\beta \leq 2$.

For the purpose of constructing a spanner in $\mathbb{R}^d$, we introduce another box, called floating-virtual box, associated with each box in the tree $CT(V)$. For box $b$, let $\square(\varphi(b))_h$ be one of the two boxes that is produced by halving the (longest dimension of) tight-virtual box $\square(\varphi(b))$ that contains $b$ inside. Since the tight-virtual box $\square(\varphi(b))$ has an aspect ratio bounded by $\beta \leq 2$, $\square(\varphi(b))_h$ has an aspect ratio bounded by $\beta \leq 2$ also.

**Definition 3 (Floating-Virtual Box)** Consider a compressed split-tree $CT(V)$ for a set of input nodes $V$. For a box $b$, a box, denoted as $\box(b)$, is termed as a floating-virtual box associated with the box $b$ if the following properties hold:

1. it includes the tight-virtual box $\square(b)$ of the box $b$ inside,
2. it is contained inside the parent box $\varphi(b)$ of $b$,
3. it has an aspect ratio at most a constant $\beta \leq 2$, and
4. it is contained inside the box $\square(\varphi(b))_h$, halved from the tight-virtual box $\square(\varphi(b))$.

It is worth to emphasize that, for a box $b$, a floating-virtual box to be used by our method is not unique: it also depends on some other boxes to be paired with. It is also easy to show that the floating-virtual boxes of two disjoint boxes $b_1$ and $b_2$ will be always disjoint because of the property 4 in Definition 3. Table 1 summarizes some of the notations used in the paper. See Figure 1 for illustration of some concepts defined in this paper.

**Table 1: Notations and abbreviations used in this paper.**

| Symbol | Description |
|--------|-------------|
| $\square(S)$ | the enclosing-box of a set of nodes $S$ |
| $\square(b)$ | the two minimum sized boxes produced by halving $b$ and then shrinking the produced boxes to be enclosing-boxes |
| $\square(b)$ | the tight-virtual box that contains a box $b$ inside and has an aspect ratio $\leq \beta \leq 2$ |
| $\box(b)$ | a floating-virtual box containing the tight-virtual box $\square(b)$ inside ($\square(b)$ and $\box(b)$ may be same) and has an aspect ratio $\leq \beta$. Here $\box(b)$ is not unique in our algorithm: it depends on the box $b'$ to be paired with for defining edges. |
| $\vartheta(b)$ | the size of the box $b$, i.e., the length of the longest side. |
| $d(b_1, b_2)$ | the Euclidean distance between two boxes $b_1, b_2$. |
| $\ell(b_1, b_2)$ | the edge-distance between two boxes $b_1, b_2$. This is equal to $d(b'_1, b'_2)$ where $b'_1$ and $b'_2$ are the floating-virtual boxes for bounded-separated boxes $b_1$ and $b_2$. |
| $\varphi(b)$ | the parent vertex of a box $b$ in the tree $CT(V)$ |

Observe that the compressed split-tree proposed here is slightly different from the split-tree defined in [7]; we define the tight-virtual boxes and also associate a tight-virtual box with some floating-virtual boxes: these floating-virtual boxes will be determined by a procedure to be described later. Given that a split-tree can be constructed in $O(n \log n)$ steps [7], it is easy to show the following theorem (Theorem 2). The proof is similar to the proof in [7] and is thus omitted.

**Theorem 2** The compressed split-tree $CT(V)$ can be constructed in time $O(dn \log n)$ for a set of nodes $V$ in $\mathbb{R}^d$.

In the rest of the paper, we will mainly focus on the tight-virtual boxes $\square(b)$ for all enclosing-boxes $b$ produced in the compressed split-tree. For ease of description, we will refer to $b$ and $\square(b)$ by $b$ itself. Thus $\square(\square(b))$ will refer to the same box as $\square(b)$. The difference here is that $\square(b)$ has an aspect ratio $\leq \beta$ while the aspect ratio of an enclosing-box $b$ could be arbitrarily large.
Figure 1: An illustration of several concepts defined in this paper. Here for a box \( b_2' \), depending on the pairing box, the floating-virtual box for \( b_2' \) could be different. When the box \( b_2' \) is paired with the box \( b_2 \), the floating-virtual box \( \Box(b_2') \) is shaded as brown in the figure. When the box \( b_2' \) is paired with the box \( b_1' \), the floating-virtual box \( \Box(b_2') \) is shaded as green in the figure.

2.2 Well-Separated Pair Decomposition and Bounded-Separated Pair Decomposition

Our method of constructing a spanner efficiently will use some decomposition of all pairs of nodes similar to well-separated pair decomposition (WSPD) [7].

Well-Separated Pair Decomposition (WSPD): Recall that, given two sets of points \( A, B \in \mathbb{R}^d \), a set \( \mathcal{R}(A, B) = \{\{A_1, B_1\}, \{A_2, B_2\}, \ldots, \{A_p, B_p\}\} \) is called a well-separated realization of the interaction product \( A \otimes B = \{(x, y) \mid x \in A, y \in B, \ and \ x \neq y\} \) if

1. \( A_i \subseteq A \) and \( B_i \subseteq B \) for all \( i \in [1, p] \).
2. \( A_i \cap B_i = \emptyset \) for all \( i \in [1, p] \).
3. \((A_i \otimes B_i) \cap (A_j \otimes B_j) = \emptyset \) for all \( 1 \leq i < j \leq p \).
4. \( A \otimes B = \bigcup_{i=1}^{p} A_i \otimes B_i \).
5. \( A_i \) and \( B_i \) is well-separated, i.e., the distance \( d(A_i, B_i) \geq g \max(\vartheta(A_i), \vartheta(B_i)) \) for some constant \( g \), where \( \vartheta(X) \) of a point set \( X \) is the radius of the smallest disk containing \( X \), which is of the same order of the size of the smallest box \( \Box(X) \) containing \( X \).

Here \( p \) is called the size of the realization \( \mathcal{R}(A, B) \).

Given the split-tree \( CT(V) \), we say that a well-separated realization of \( V \otimes V \) is a well-separated pair decomposition (WSPD) of \( V \) based on \( CT(V) \), if for any \( i, A_i \) and \( B_i \) are the sets of nodes contained in some enclosing-boxes, \( \Box(u_i), u_i \in CT(V) \) and \( \Box(v_i), v_i \in CT(V) \), respectively. Here we overuse notations a little bit, for a box \( b \in CT(V) \), we also refer via \( b \) as the subset of nodes from \( V \) contained inside \( b \). In [7], it is shown that a well-separated pair decomposition based on a split-tree \( CT(V) \) can be constructed in linear time \( O(n) \) when the split-tree \( CT(V) \) is given.

**Theorem 3** [7] Given the split-tree \( CT(V) \), we can construct a well-separated pair decomposition based on \( CT(V) \) in linear-time \( O(n) \) and the realization of this WSPD has size \( O(n) \).

**Proof.** For completeness of presentation, we briefly review the proof here. The algorithm itself is recursive. In this proof, when we mention a box \( b \), we always mean the tight-virtual box \( \Box(b) \) corresponding to the enclosing-box used in the split-tree. Each vertex of the compressed split-tree \( CT(V) \) has 2 children enclosing-boxes, containing point sets denoted by \( A_1, A_2 \). We construct a well-separated realization for \( A_1 \otimes A_2 \).

We now focus on how to construct a realization for a pair of boxes \( b \) and \( b' \). If \( (b, b') \) is well-separated, i.e., \( d(b, b') \geq g \max(\vartheta(b), \vartheta(b')) \), then we are done for this pair. Otherwise, for simplicity, we assume that
Definition 5 (potential-edge-boxes): Two enclosing-boxes $b$ and $b'$ in the compressed split-tree $CT(V)$ (corresponding to two vertices in $CT(V)$) are said to be a pair of potential-edge-boxes, denoted as $(b, b')$, if
\[
\vartheta(b) \geq \vartheta(b').
\] In this case, let $\{c_i \mid 1 \leq i \leq 2\}$ be the 2 children enclosing-boxes of $b$ in the tree $CT(V)$. We then recursively construct a well-separated realizations of $c_i \otimes b'$, for each child box $c_i$ of $b$ and then return the union of these realizations as the well-separated realization for the pair $(b, b')$. Based on the compressed split-tree $CT(V)$, we will have a WSPD-computation tree $T$ as follows:

1. The root of $T$ is $(b_0, b_0)$, where $b_0$ is the box corresponding to the root vertex of $CT(V)$;
2. For each node $(b, b')$ in $T$, if $b = b'$, then it has the following children $(b_i, b_j)$ where $1 \leq i \leq j \leq 2$ and $b_1, b_2$ are children enclosing-boxes of $b$ in the compressed split-tree $CT(V)$. However, observe that here $(b_i, b_j) \in T$ for each children box $b_i$ of $b$.
   (a) If $d(b, b') \geq \varrho \max(\vartheta(b), \vartheta(b'))$, then it does not have any children since $(b, b')$ is a pair of well-separated boxes.
   (b) If $d(b, b') < \varrho \max(\vartheta(b), \vartheta(b'))$ and $\vartheta(b) \geq \vartheta(b')$, then it has 2 children nodes $(c_i, b')$ where $c_i$ is a tight-virtual child box of $b$ in the tree $CT(V)$. Recall that here $c_i, 1 \leq i \leq 2$ is produced by halving the longest dimension of $b$ and then shrinking the corresponding boxes into the smallest tight-virtual boxes with aspect ratio at most $\beta$.
   (c) If $d(b, b') < \varrho \max(\vartheta(b), \vartheta(b'))$ and $\vartheta(b) < \vartheta(b')$, then it has 2 children nodes $(b, c_i)$ where $c_i$ is a tight-virtual children box of $b'$ in the tree $CT(V)$.

For simplicity of analysis, for any vertex $(b, b')$ in the WSPD-computation tree $T$, we always assume that $\vartheta(b) \geq \vartheta(b')$; otherwise, we reorder them and rename them. A careful analysis in [7] show that the WSPD-computation tree $T$ has size at most $2n \cdot 2^d + 4n \cdot (4\varrho_1)^d = O(n)$ vertices. The theorem then follows.

**Bounded-Separated Pair Decomposition (BSPD):** Observe that a pair of boxes $(b, b')$ in WSPD may have a distance arbitrarily larger than $\max(\vartheta(b), \vartheta(b'))$, especially, the WSPD produced in [7]. To produce spanners with low weight, we do not want to include arbitrarily long edges in the spanner, unless it is required. To capture such a requirement, we propose a new concept, a bounded-separated pair decomposition (BSPD). A well-separated pair decomposition (WSPD) based on a compressed split-tree $CT(V)$ is called a bounded-separated pair decomposition, if it has the following additional property: each pair of tight-virtual boxes $(b, b')$ in this WSPD satisfies the property of bounded-separation of floating-virtual boxes, i.e., there is a pair of floating-virtual boxes $b_2 = \square(b)$ and $b'_2 = \square(b')$ and $(b_2, b'_2) = (\square(b), \square(b'))$ has bounded separation.

**Definition 4 (Bounded Separation)** A pair of boxes $(b_2, b'_2)$ has the property of bounded-separation if it satisfies the following properties

1. Almost Equal-size Property: $\varepsilon_1 \vartheta(b'_2) \leq \vartheta(b_2) \leq \varepsilon_2 \vartheta(b'_2)$. Here we typically choose constants $\varepsilon_2 = \varepsilon_1 = 1/2$. The two boxes $b_2$ and $b'_2$ are called almost-equal-sized.
2. Bounded-Separation Property: $g_1 \max(\vartheta(b_2), \vartheta(b'_2)) \leq d(b_2, b'_2) \leq g_2 \max(\vartheta(b_2), \vartheta(b'_2))$ for constants $1 < g_2 < g_1$ to be specified later.

Two boxes $b$ and $b'$ present in the bounded-separated pair decomposition are called a pair of bounded-separated boxes in $CT(V)$. The choice of the constants $g_2 > g_1 \geq 2\varrho_1$ depends on the spanning ratio $t > 1$ required. The constants $g_2$ and $g_1$ will be chosen as specified later to ensure the existence of a pair of bounded-separated boxes. Observe the fact that a pair of boxes $b$ and $b'$ is in BSPD does not imply that the distance between $b$ and $b'$ is in the range $[g_1 \cdot \max(\vartheta(b), \vartheta(b')), g_2 \cdot \max(\vartheta(b), \vartheta(b'))]$; it is possible that the distance $d(b, b') > g_2 \cdot \max(\vartheta(b), \vartheta(b'))$. However, observe that $d(\mathcal{P}(b), \mathcal{P}(b')) \leq g_2 \cdot \max(\vartheta(b), \vartheta(b'))$ for the parent boxes $\mathcal{P}(b)$ and $\mathcal{P}(b')$ of $b$ and $b'$.

Given the split-tree $CT(V)$, we then briefly discuss how to connect pairs of nodes to form edges in the spanner. Our method is based on the concept of potential-edge-boxes.

**Definition 5 (potential-edge-boxes)** Two enclosing-boxes $b$ and $b'$ in the compressed split-tree $CT(V)$ (corresponding to two vertices in $CT(V)$) are said to be a pair of potential-edge-boxes, denoted as $(b, b')$, if
1. The pair of floating-virtual boxes \( b_2 = \Box(b) \) and \( b'_2 = \Box(b') \) have the property of bounded-separation; the pair of boxes \( b_2 \) and \( b'_2 \) is called the pair of bounded-separated floating-virtual boxes defining the pair of potential-edge boxes \( b \) and \( b' \).

2. None of the pairs \((b, \Box(b')), (\Box(b), b'), (\Box(b), \Box(b'))\), is a pair of potential-edge boxes in \( CT(V) \).

For a pair of potential-edge boxes \((b, b')\), we use \((\Box(b), \Box(b'))\) to denote the floating-virtual boxes of \( b \) and \( b' \) respectively that define the pair of potential-edge boxes. From the definition of floating-virtual boxes, we have the following lemma.

**Lemma 4** For a pair of potential-edge boxes \( b \) and \( b' \) and the pair of floating-virtual boxes \( b_2 \) and \( b_2' \) defining them, the floating-virtual box \( b_2 = \Box(b) \supseteq b \) cannot contain another enclosing-box that is disjoint of \( b \), i.e., \( b'' \cap b = \emptyset \implies \Box(b) \cap b'' = \emptyset \).

**Definition 6 (edge-distance of potential-edge boxes)** For a pair of potential-edge boxes \( b_1 \) and \( b'_1 \), define its edge-distance, denoted as \( \ell((b_1, b'_1)) \), as the distance between the pair of bounded-separated floating-virtual boxes \( b_2 \) and \( b'_2 \) that define the pair of potential-edge boxes \( b_1 \) and \( b'_1 \), i.e., \( \ell((b_1, b'_1)) = d(b_2, b'_2) \).

Given a compressed split-tree \( CT(V) \) and associated boxes at each node we then define the edge-neighboring boxes of an enclosing-box \( b \) as

\[
\mathcal{N}(b) = \{b' \mid \text{boxes } b \text{ and } b' \text{ are a pair of potential-edge boxes in } CT(V)\}
\]

Observe that here the distance \( d(b, b') \) could be arbitrarily larger than the maximum size of boxes \( b \) and \( b' \). However, the distance \( d(\Box(b), \Box(b')) \leq g_2 \max(\vartheta(\Box(b)), \vartheta(\Box(b'))) \) if \( b' \in \mathcal{N}(b) \). Given a fixed size \( L \), and a tight-virtual box \( b \), the number of tight-virtual boxes \( b' \) such that (1) \((\Box(b), \Box(b'))\) has the property of bounded-separation, and (2) \( \Box(b) \) has size \( L \), is clearly bounded by a constant. However, this does not mean that the number of tight-virtual boxes, the cardinality of \( \mathcal{N}(b) \), that could pair with \( b \) to form a pair of potential-edge boxes is bounded by a constant. The reason is that the floating-virtual boxes for a tight-virtual box \( b \) depend on with which box the box \( b \) will be paired (see Figure I for illustration). There is an example of nodes’ placement such that the cardinality \( |\mathcal{N}(b)| \) could be as large as \( \Theta(n) \). The example is as follows: in \( 2D \), we place a node at \( v_0 = (0, 0) \) and \( n \) nodes \( v_i \) at \((0, 2^i - \epsilon)\), for \( 1 \leq i \leq n \). An additional node \( v_{n+1} \) is placed at \((0, -2^n)\). Here \( \epsilon > 0 \) is a sufficiently small number. Then the smallest box \( b \) containing node \( v_{n+1} \) will have \( \Theta(n) \) boxes in \( \mathcal{N}(b) \) (the sizes of these boxes are about \( 2^i \), \( 0 \leq i \leq n - 1 \)).

Thus, our construction method will use another set instead

\[
\mathcal{N}_{\geq}(b) = \{b' \mid \text{boxes } b \in \mathcal{N}(b) \text{ and } \vartheta(\Box(b')) \geq \vartheta(\Box(b))\}
\]

Observe that for any pair of potential-edge boxes \( b \) and \( b' \), we either have \( b' \in \mathcal{N}_{\geq}(b) \) or \( b \in \mathcal{N}_{\geq}(b') \), or both. Consider \( \Box(b) \) and \( \Box(b') \). The distance between \( \Box(b) \) and \( \Box(b') \) is at most \( g_2 \max(\vartheta(b_2), \vartheta(b'_2)) \) since the floating-virtual box for \( b \) is always contained inside \( \Box(b) \). Thus there are at most \( \Theta((\rho_2 \cdot \epsilon_2)^d) \) boxes \( b' \).

**Lemma 5** The cardinality of \( \mathcal{N}_{\geq}(b) \) is bounded by a constant \( \Theta((\rho_2 \cdot \epsilon_2)^d) \).

We now show that we can construct a linear size BSPD based on \( CT(V) \) in linear time.

**Lemma 6** Given any pair of tight-virtual boxes \( b \) and \( b' \) in a BSPD with \( \vartheta(b') \leq \vartheta(b) \leq \vartheta(\Box(b')) \), in time \( O(d) \), we can find a floating-virtual box \( b'' \) inside \( \Box(b')_n \) of almost-equal-size with \( b \).
for some constants $\vartheta$. More specifically, for each pair of boxes $b, b'$ in the WSPD. We show that, by adjusting the sizes of pairs of boxes in the WSPD computed, we can get the exact value of $\vartheta$ inside the tight-virtual box $b$.

Projecting $P(b')$ on this dimension results in a segment, say $xy$. Let the segment $ab$ be the projection of $b'$ in the dimension of $d_1$. Note that $[a, b] \subset [x, y]$. We then align a floating-virtual box $b''$ (s.t. $\vartheta(b)/2 \leq \vartheta(b'') \leq \vartheta(b)$) such that its projection on the dimension that contains $[c, d]$ starts at $x$. If $[c, d]$ contains $[a, b]$ we are done. Otherwise, align $b''$ such that $d = b$. Since $b - x \geq d - c$ and $d - c \geq b - a$, the alignment of the box $b''$ in dimension $d_1$ is possible. This can be repeated for all dimensions. It is easy to show that the size of the floating-virtual box $b''$ is at most $\vartheta(b)$, and at least $\vartheta(b)/2$. This finishes the proof.

**Theorem 7** Given the compressed split-tree $CT(V)$, we can construct a bounded-separated pair decomposition (BSPD) (using constants $\varrho_1$ and $\varrho_2$) in linear-time $O(n)$ and the realization has size $O(n)$. The constants $\varrho_1$ and $\varrho_2$ are related to $\varrho$ of the WSPD as follows:

\[
\begin{align*}
\varrho_1 &\leq \varrho - \sqrt{d} \\
\varrho_2 &\geq 2\varrho + 4\sqrt{d}
\end{align*}
\]  

PROOF. We will prove this based on the well-separated pair decomposition computed in the proof of Theorem 3. In the proof, we will mainly focus on the tight-virtual boxes, instead of the actual enclosing-boxes. Observe that in the WSPD computation tree $T$ (defined in proof of Theorem 3) of a WSPD based on the compressed split-tree $CT(V)$, all the leaf vertices will form a WSPD. We first build a WSPD with a constant $\varrho$, where the exact value of $\varrho$ will be determined later.

Consider a pair of boxes $(b, b')$ at a leaf vertex in the WSPD-computation tree $T$, i.e. $(b, b')$ is an element of the computed WSPD. We show that, by adjusting the sizes of pairs of boxes in the WSPD computed, we can get a BSPD. More specifically, for each pair of boxes $b$ and $b'$ in WSPD, we show how to obtain two almost-equal-sized floating-virtual boxes $b_2$ (containing $b$) and $b_2'$ (containing $b'$) such that $\varrho_1 \vartheta(b_2) \leq d(b_2, b_2') \leq \varrho_2 \vartheta(b_2)$ for some constants $\varrho_1 < \varrho_2$.

Assume w.l.o.g., $\vartheta(b') \leq \vartheta(b)$. First of all, because of the properties of the WSPD computation-tree $T$, $\vartheta(b) \leq \vartheta(P(b'))$ when $(b, b')$ is a leaf node in the computation tree $T$. Notice that here, to get the vertex $(b, b')$, we could have split $P(b')$ first or we could have split $P(b)$ first in the WSPD computation-tree $T$. Thus, by Lemma 3 we can find a floating-virtual box $b'' = \mathbb{P}(b')$, which is almost-equal-sized to $b$ (i.e., $\vartheta(b)/2 \leq \vartheta(b'') \leq \vartheta(b)$), is inside $P(b')$, and contains $b'$ inside.

We now show that the distance $d(b, b'')$ is at least a constant fraction of $\vartheta(b)$. Obviously,

\[d(b, b'') > d(b, b') - \sqrt{d}\vartheta(b'') \geq \varrho \max(\vartheta(b), \vartheta(b')) - \sqrt{d}\vartheta(b) = (\varrho - \sqrt{d})\vartheta(b) \geq \varrho_1 \vartheta(b).
\]

On the relations of $d(b, b'')$ and the size $\vartheta(b)$, there are two complementary cases here:

Case 1: $d(b, b'') \leq \varrho_2 \vartheta(b)$: In this case, we have already found a pair of floating-virtual boxes $b_2 = b$ and $b_2' = b''$ for the pair of boxes $b$ and $b'$ such that the distance between the floating-virtual boxes satisfies that $\varrho_1 \vartheta(b) \leq d(b, b'') \leq \varrho_2 \vartheta(b)$. Thus, we put $(b, b')$ into BSPD.

Case 2: $d(b, b'') > \varrho_2 \vartheta(b)$: Here, from $\vartheta(b) \leq \vartheta(P(b'))$, we have $d(b, P(b')) \leq \varrho \max(\vartheta(b), \vartheta(P(b'))) = \varrho \vartheta(P(b'))$. If this is not true, clearly in the WSPD computation tree $T_1$, we will use $(b, P(b'))$ instead of $(b, b')$. Similarly, we have $d(P(b), b') \leq \varrho \max(\vartheta(P(b)), \vartheta(b'))$.

We now show how to find equal-sized floating-virtual boxes $b_2$ inside the tight-virtual box $P(b)$ and $b_2'$ inside the tight-virtual box $P(b')$. This will identify the potential edge-boxes. Let $\Delta$ be the size of the equal-sized boxes $b_2$ and $b_2'$. Then we have

\[
\begin{align*}
\{d(b_2, b_2') &\geq d(b, b'') - 2\sqrt{d}\Delta \\
d(b_2, b_2') &\leq d(b, b'')
\end{align*}
\]  

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Then the following is clearly a sufficient condition for the existence of such a pair of floating-virtual boxes $b_2$ and $b'_2$ to define the potential-edge-boxes $b$ and $b'$:

$$\begin{aligned}
\begin{cases}
\mathbf{d}(b_2, b'_2) \geq \mathbf{d}(b,b'') - 2\sqrt{\Delta} \geq \varrho_1 \Delta \\
\mathbf{d}(b_2, b'_2) \leq \mathbf{d}(b,b'') \leq \varrho_2 \Delta
\end{cases}
\end{aligned}$$

(3)

Notice that $\vartheta(b) \leq \frac{\mathbf{d}(b,b'')}{\varrho_2}$. Thus, it is equivalent to require $\Delta$ in range

$$\frac{\mathbf{d}(b,b'')}{\varrho_2} \leq \Delta \leq \frac{\mathbf{d}(b,b'')}{\varrho_1 + 2\sqrt{d}}$$

Clearly, we have a solution for $\Delta$ when

$$\varrho_2 \geq \varrho_1 + 2\sqrt{d}$$

(4)

We now show that the boxes $b_2$ and $b'_2$ will be inside the boxes $\Box(\mathbb{P}(b))_h$ and $\Box(\mathbb{P}(b'))_h$ respectively indeed. To ensure this, we only need the condition that size $\Delta$ is at most $\min(\vartheta(\mathbb{P}(b))/2, \vartheta(\mathbb{P}(b'))/2)$.

If $\vartheta(\mathbb{P}(b)) \geq \vartheta(\mathbb{P}(b'))$, then we have

$$\mathbf{d}(b, b'') < \mathbf{d}(b, \mathbb{P}(b')) + 2\sqrt{\Delta} \vartheta(\mathbb{P}(b')) \leq (\varrho + 2\sqrt{d}) \vartheta(\mathbb{P}(b')) = (2\varrho + 4\sqrt{d}) \min(\vartheta(\mathbb{P}(b))/2, \vartheta(\mathbb{P}(b'))/2)$$

(5)

If $\vartheta(\mathbb{P}(b)) \leq \vartheta(\mathbb{P}(b'))$, then we have

$$\mathbf{d}(b, b'') < \mathbf{d}(\mathbb{P}(b), b'') + 2\sqrt{d} \vartheta(\mathbb{P}(b)) \leq (\varrho + 2\sqrt{d}) \vartheta(\mathbb{P}(b)) < (2\varrho + 4\sqrt{d}) \min(\vartheta(\mathbb{P}(b))/2, \vartheta(\mathbb{P}(b'))/2)$$

(6)

Thus, when

$$2\varrho + 4\sqrt{d} \leq \varrho_2$$

(7)

we can choose $\Delta = \frac{\mathbf{d}(b,b'')}{\varrho_2} \leq \min(\vartheta(\mathbb{P}(b))/2, \vartheta(\mathbb{P}(b'))/2)$ as the final size of $b_2$ and $b'_2$ to ensure that the floating-virtual boxes $b_2$ and $b'_2$ will be inside the boxes $\Box(\mathbb{P}(b))_h$ and $\Box(\mathbb{P}(b'))_h$ respectively indeed. Observe that we do not put any condition on the locations of $b_2$ and $b'_2$ here. Thus, as in Lemma 6 we can find arbitrary locations of $b_2$ and $b'_2$ in time $O(d)$.

Then a sufficient condition for the constants $\varrho_1$ and $\varrho_2$ such that we can compute a BSPD from the WSPD with a constant $\varrho$ is

$$\begin{aligned}
\begin{cases}
\varrho_1 \leq \varrho - \sqrt{d} \\
\varrho_2 \geq 2\varrho + 4\sqrt{d}
\end{cases}
\end{aligned}$$

(8)

It is easy to show that we need $\varrho_2 \geq 2\varrho_1 + 6\sqrt{d}$. This finishes the proof.

Based on the above proof, we can find the floating-virtual boxes for each pair of boxes in the bounded-separated pair decomposition in time $O(d)$. Consequently, we also can find $\mathcal{N}_{\geq}(b)$ for all boxes $b$ in linear-time.

### 2.3 Cones and Cone Partition

We consider points in the $d$-dimensional space $\mathbb{R}^d$. Let $B = \{z_1, z_2, \ldots, z_g\}$ be a set of $g$ linearly independent vectors. The set of vectors $B$ is called a basis of $\mathbb{R}^d$ for a cone partition. We define the cone of $B$, $\mathcal{C}(B)$, as

$$\mathcal{C}(B) = \{\sum_{1 \leq i \leq g} \lambda_i z_i \mid \forall i, \lambda_i \geq 0\}$$
For a point \( x \), we define the cone with apex at \( x \) generated by a basis \( B \) as \( C(x, B) = \{ y \mid y - x \in C(B) \} \). For a given set of points \( P \in \mathbb{R}^d \), the cone region of \( P \) is defined as

\[
C(P, B) = \bigcup_{x \in P} C(x, B).
\]

A vector \( uv \) is said to be in the direction of the cone \( C(B) \) (or cone with basis \( B \)) if \( v \in C(u, B) \). A vector \( uv \) is said to be in the direction of a family of cones with a collection of bases \( B = \{ B_1, B_2, \ldots, B_m \} \) if \( v \in \bigcup_{i=1}^m C(u, B_i) \). Two vectors \( x \) and \( y \), the angle between them is denoted as \( \Phi(x, y) = \arccos\left( \frac{x \cdot y}{\|x\| \|y\|} \right) \), where \( \|x\| \) is the length of the vector. We define the angular span, based at the origin, of a set of vectors \( B \) (and its corresponding cone) as \( \Phi(B) = \max_{x, y \in B} \{ \Phi(x, y) \} \). Given a set of bases \( B \), its angular span is defined similarly. Let \( F \) be a finite family of basis of \( \mathbb{R}^d \). \( F \) is called a frame if \( \bigcup_{B \in F} C(B) = \mathbb{R}^d \). The angular span of a frame \( F \) is defined as \( \Phi(F) = \max_{B \in F} \Phi(B) \). The following lemma has been shown in [33].

**Lemma 8** For any \( 0 < \phi < \pi \), one can construct a frame \( F \) in \( \mathbb{R}^d \) with size \( \Omega\left(\frac{\epsilon}{\phi}\right)^d \) for a constant \( \epsilon \) such that \( \Phi(F) < \phi \).

The Yao graph, based on the cone partition of the space produced by a frame \( F \), contains all edges \( uv \), where \( u \) is the closest node to \( v \) in some cone \( C(u, B) \) for \( B \in F \). We use \( Y(V, F) \) to denote such graph. The following lemma was obvious.

**Lemma 9** The graph \( Y(V, F) \) is a \( t \)-spanner with \( O(n) \) edges for \( t = \frac{1}{1 - 2 \sin(\phi(F)/2)} \) when \( \phi(F) < \pi/3 \).

It is easy to construct an example of points (for example, \( n/2 \) points placed evenly on a side of a unit square and another \( n/2 \) points placed evenly on the opposite side) such that the weight of \( Y(V, F) \) is \( \Theta(n) \omega(\text{EMST}) \). The maximum node degree in \( Y(V, F) \) could also be as large as \( \Theta(n) \), e.g., when \( n - 1 \) points evenly placed in a circle and 1 node at the circle center.

**General-Cone-Direction Property:** We now define the General-Cone-Direction Property for a pair of boxes \( b \) and \( b' \). Here we require that the pair of boxes \( b \) and \( b' \) have similar sizes, i.e., \( \varepsilon_1 \cdot \vartheta(b) \leq \vartheta(b') \leq \varepsilon_2 \cdot \vartheta(b) \) for fixed constants \( \varepsilon_1, \varepsilon_2 \geq 1 \). In addition, the distance \( d(b, b') \) between the boxes \( b \) and \( b' \) is at least \( \vartheta_1 \max(\vartheta(b), \vartheta(b')) \) for a constant \( \vartheta_1 > 1 \), where \( \vartheta(b) \) is the size of a box \( b \). We also assumed that the angular span of each cone in \( F \) is at most a constant \( \alpha \) (depending on the spanning ratio \( t \)).

**Definition 7 (General-Cone-Direction Property)** Given a pair of boxes \( b \) and \( b' \), a general-cone-direction of \( b' \) with respect to the base box \( b \), denoted as \( B(b, b') \), is a set of cones \( B \subset F \) such that, for every point \( x \in b \), \( \bigcup_{B \in B} C(x, B) \) properly contains the box \( b' \). Similarly, we can define the general-cone-direction \( B'(b, b') \) of \( b \) with respect to \( b' \). Also, define \( \theta \) as the maximum angular span of \( B \) and \( B' \), i.e. \( \theta = \max\{ \Phi(B), \Phi(B') \} \).

A vector \( uv \) is said to be in the general-cone-direction of \( b' \) with respect to the box \( b \) if \( uv \) is in the direction of \( B(b, b') \).

**Lemma 10** For any pair of bounded-separated boxes \( b \) and \( b' \) in a Bounded-Separated decomposition, the maximum angular span of \( B(b, b') \) and \( B'(b, b') \) is at most \( \theta = \frac{4 \sqrt{d}}{\vartheta_1} + 3 \alpha \).

**Proof.** Figure 2 illustrates our proof that follows.

A simple computation shows that the general-cone-direction property is satisfied if we select a minimal collection \( B \) of bases from \( F \) such that all vectors in \( B_i \in B \) are within angle \( \theta_1 \) from vectors in \( B \) where

\[
\theta_1 \leq \frac{2 \sqrt{d}}{\vartheta_1} + \alpha
\]
\[ G = (V, E) \]

Given a set \( T \), we can choose \( \alpha \) and \( \theta \) such that \( \theta \) is small.

Then \( \theta \leq 2\theta_1 + \alpha \leq \frac{4\sqrt{d}}{\theta_1} + 3\alpha \).

Clearly, there is only at most a constant number of cones in \( B \) and we can find \( B \) in constant time.

2.4 Method for Low Weight \( t \)-Spanner Using Split Trees

Given a set \( V \) of \( n \) nodes in \( \mathbb{R}^d \) and any real number \( t > 1 \), we now describe our method to construct a structure \( G = (V, E) \) in time \( O(n \log n) \) such that \( G \) is (1) \( t \)-spanner, (2) each node in \( V \) has a degree \( O(1) \), and (3) the total edge length \( \omega(G) \) is \( O(\omega(MST)) \), where MST is the Euclidean minimum spanning tree over \( V \).

The basic idea of our method is as follows. We first construct the compressed split-tree \( CT(V) \) and associate geographic information with \( CT(V) \). This includes enclosing-boxes, the tight-virtual boxes, and the floating-virtual boxes. By Theorem 7, we can construct a bounded-separated pair decomposition (BSPD) in linear time using \( CT(V) \). For each box \( b \) in \( CT(V) \), we will select one node as its representative node, denoted as \( R(b) \), used as a gateway node to connect nodes inside this box to nodes at some other boxes in \( N(b) \). To ensure that a node is used as a representative node by at most a constant number of boxes (thus ensuring \( O(1) \) degrees for these nodes), we apply the following strategy in selecting the representative nodes of boxes. Each leaf vertex will have at least one node from \( V \) inside. There are at most \( n - 1 \) internal vertices in the compressed split-tree, and \( 2n - 1 \) vertices in \( CT(V) \); thus we need at most \( 2n \) representative nodes. We will assign 2 credits to each node in \( V \). For each leaf box \( b \) of \( CT(V) \), we choose a node \( R(b) \) inside \( b \) and charge 1 credit to the chosen node. Using a bottom-up approach, for each internal vertex \( b \), we will select a node \( R(b) \) from nodes contained inside \( b \) that has a non-zero credit. Since each internal vertex has at least 2 children vertices, such a representative node can always be found. Thus, we have the following lemma.

Lemma 11 Each node \( v_i \) in \( V \) is used at most 2 times as a representative node in \( CT(V) \).

Our algorithm works as follows. Given a pair of potential-edge boxes \( b_1 \) and \( b'_1 \) (defined by a pair of bounded-separated floating-virtual boxes \( b_2 \) and \( b'_2 \)), and their representative nodes \( u \) and \( v \) respectively, we add an edge \( uv \) if:

1. there is no edge \( xy \) already added, where \( x \) is inside \( b_2 \) (it is possible that \( y \notin b'_2 \)), \( xy \) crossing the boundary of \( b_2 \) (thus \( y \) is not inside \( b'_2 \)), and \( xy \) is in the general-cone-direction of the basis \( B(b_2, b'_2) \).
2. \( B \) be the basis such that the representative node \( v \) is contained inside the region \( C(b_2, B) \). Such an edge \( xy \) is called crossing edge for box \( b_2 \) in the direction of \( B \), and

Algorithm presents our method for constructing a \( t \)-spanner in \( \mathbb{R}^d \) with low-weight, and bounded degree property. In Algorithm for each enclosing-box \( b \), each basis \( B_i \in \mathcal{F} \), and each dimension, we store an edge
Algorithm 1 Constructing a $t$-spanner with low-weight

1: Define a frame $\mathcal{F}$, with a constant $c$ number of bases $B_1, B_2, \ldots, B_c$ such that the angular span of any base is at most a small angle $\alpha$. The actual value of the angle $\alpha$ will be given later in proofs.

2: Build the compressed split-tree $CT(V)$ and a BSPD. With each enclosing-box $b$ in the split-tree $CT(V)$, we associate a representative node $R(b)$. For each enclosing-box $b$, we also construct $N_\geq(b)$.

3: Sort the edge-distances (see Definition 6) between all pairs of potential-edge boxes in increasing order. (There are a total of $O(n)$ pairs of potential-edge boxes, thus, the sorting can be done in time $O(n \log n)$.)

4: for ($r = 1$ to $\sum_b N_\geq(b)$) do

5: Select the pair of potential-edge boxes $b_1$ and $b'_1$ with the $r$th smallest edge-distance. Let $b_2$ be the floating-virtual box containing $b_1$ and $b'_2$ be the floating-virtual box containing $b'_1$ such that $b_2$ and $b'_2$ are a pair of bounded-separated floating-virtual boxes. Let $u$ (and $v$ resp.) be the representative node of box $b_1$ (and $b'_1$ resp.). Let $B, B' \subset \mathcal{F}$ be the collection of bases satisfying the General-Cone-Direction Property w.r.t $b_1$ and $b'_1$, respectively. We then add an edge $uv$, only if

   1. $\forall B_i \in B, \forall h$, such that $x$ is inside the box $b_1$ and $y$ is outside of the floating-virtual box $b_2$, there is no “crossing” edge $xy$ in $CrossingEdge(b_1, B_i, h)$; and

   2. $\forall B'_i \in B', \forall h$, such that $z$ is inside the box $b'_1$ and $w$ is outside of the floating-virtual box $b'_2$, there is no “crossing” edge $zw$ in $CrossingEdge(b'_1, B_i, h)$.

6: end for

7: Let $G = (V, E)$ be the graph constructed.


\[ xy \] to array $CrossingEdge(b, B_i, h)$ (if there is any) such that (1) node $x$ is inside the enclosing-box $b$, (2) node $y$ is in the cone $C(x, B_i)$, and (3) $y$ is the node that is furthest from the box $b$ in the dimension $h$ if there are multiple edges satisfying the first two conditions. This will ensure the following lemma:

**Lemma 12** For every direction specified by the basis $B_i$, there exists an edge $w_1w_2$ with $w_1 \in b$ and $w_1w_2$ crossing a floating-virtual box $b_2$ (at some ancestor of $b$) in the direction $B_i$ if and only if there is an edge $xy \in CrossingEdge(b, B_i, h)$ for a dimension $h$ and $xy$ crosses the virtual box $b_2$ (i.e. $y \notin b_2$).

3 Properties: Low-Weight, Spanner, and Low-Degree

We next show that the constructed structure $G$ by Algorithm 1 is a $t$-spanner, has a bounded degree, and has low-weight (by choosing the angular span of the frame $\mathcal{F}$, $\theta$, and the parameters $\varrho_1, \varrho_2, \varepsilon_1, \varepsilon_2$ in bounded-separateness carefully).

3.1 Degree Property

**Theorem 13** Each node in the constructed graph $G$ by Algorithm 1, $v \in V$, has degree $\leq |F| = O((\frac{1}{\alpha})^d)$ where $\alpha$ is the angular span of the frame $\mathcal{F}$.
PROOF. Since each node will serve as a representative for at most two different enclosing-boxes, it suffices to show that for each enclosing-box \( b_1 \), we will add at most a constant number of edges for the representative node of this box. We will show that we add at most 1 edge to a node \( u \) in any cone direction when \( u \) is a representative node of a box \( b_1 \). Assume that we have already added an edge \( uv \) in a direction \( B \), where \( u \) is the representative node in box \( b_1 \) and \( v \) is the representative node of \( b'_1 \) such that \( b_1 \) and \( b'_1 \) is a pair of potential-edge boxes defined by a pair of bounded-separated virtual boxes \( b_2 \) (containing \( b_1 \)) and \( b'_2 \) (containing \( b'_1 \)). We show that we cannot add another edge \( uw \) in the same direction \( B \) later. Assume that we did add another edge \( uw \) later, because of the existence of a pair of bounded-separated virtual boxes \( b_3 \) and \( b'_3 \) that defines a pair of potential-edge boxes \( b_3 \) and \( b'_3 \) (for \( b' \neq b'_1 \)). Then there are only two complementary cases:

1. \( b_3 \) contains boxes \( b_1 \) and \( b'_1 \) inside. This violates the condition (condition 3) of the potential-edge definition: \( b_3 \) will contain the parent box (which is an enclosing-box) of \( b_1 \) and \( b'_1 \) inside. Notice that since a virtual box \( b_3 \) contains both \( b_1 \) and \( b'_1 \) inside, our compressed split-tree construction shows that the parent box of the boxes \( b_1 \) and \( b'_1 \) is inside \( b_3 \) (may be same as \( b_3 \)).

2. \( b_3 \) does not contain \( b'_1 \) inside. Then edge \( uv \) will be a crossing edge that crosses the boundary of virtual box \( b_3 \). Thus edge \( uw \) will not be added.

This finishes the proof. \( \square \)

Note that we later will show that the angular span \( \alpha \) of \( F \) depends on the spanning ratio \( t > 1 \) that is required.

3.2 The Spanner Property

We now prove that the final structure \( G \) is a \( t \)-spanner, where \( t > 1 \) is a given constant, if we choose \( \theta \), \( \varrho_1 \) and \( \varrho_2 \) carefully.

**Theorem 14** The final structure \( G \) constructed by Algorithm 1 is a \( t \)-spanner for a given constant \( t > 1 \) if we carefully choose \( \alpha = \Phi(F), \varrho_1 \) and \( \varrho_2 \) according to conditions illustrated in (10).

\[
\begin{align*}
\varrho_1 &> t\sqrt{d} \\
2\sqrt{d} \frac{\varrho_1}{\varrho_1 - 1} \leq \varrho_1 \\
\frac{2\sqrt{d}}{\varrho_1} + (1 + 2\sqrt{d})/(1 - 2\sin(\theta/2) - 2\sqrt{d}/\varrho_1) &\leq t \\
\theta &\leq \frac{4\sqrt{d} + 3\alpha}{\varrho_1} \\
\varrho_2 &\geq 2\varrho_1 + 6\sqrt{d}
\end{align*}
\]  

(10)

**PROOF.** Note that the last condition is to ensure that we can construct a BSPD. We then prove the theorem by induction on the rank of the Euclidean distance between all pairs of nodes \( u \) and \( v \) from \( V \). First, for the pair of nodes \( u \) and \( v \) with the smallest distance, edge \( uv \) clearly will be added to \( G \). Thus, we have a path in \( G \) with length at most \( td(u,v) \) to connect \( u \) and \( v \). Assume that the statement is true for all pairs of nodes with the first \( r \) smallest pairwise distance. Consider a pair of nodes \( u \) and \( v \) with \((r+1)\)th smallest distance.

Since we produce a BSPD for the set \( V \) of nodes using \( CT(V) \), in the box tree \( CT(V) \), there will be a pair of floating-virtual boxes \( b_2 \) (containing \( u \)) and \( b'_2 \) (containing \( v \)) that is a bounded-separated pair. Let \( b_1 \) be the largest enclosing-box (from tree \( CT(V) \)) that is contained inside \( b_2 \) and contains \( u \); and \( b'_1 \) be the largest enclosing-box (from tree \( CT(V) \)) that is contained inside \( b'_2 \) and contains \( v \). Then the pair of boxes \( b_1 \) and \( b'_1 \) is a pair of potential-edge boxes. Depending on whether we have a crossing edge \( xy \) when processing the pair of potential-edge-boxes \( b_1 \) and \( b'_1 \), we have the following two complementary cases.

**Case 1:** We have an edge \( xy \) where \( x \) is a representative node of box \( b_1 \) and \( y \) is a representative node of box \( b'_1 \). In this case, we have \( d(u,v) \geq d(b_1,b'_1) \geq d(b_2,b'_2) \geq \varrho_1 \max(\varrho(b_2),\varrho(b'_2)) \). By choosing

\[
\varrho_1 > t\sqrt{d}
\]  

(11)
we have \( d(u, x) \leq \vartheta(b_1) \cdot \sqrt{d} \leq d(b_2, b'_2) \sqrt{d}/\vartheta_1 = \ell(b_1, b'_1) \sqrt{d}/\vartheta_1 \leq d(u, v) \sqrt{d}/\vartheta_1 < d(u, v)/t \). Then by induction, we have a path connecting \( x \) and \( u \) with length at most \( td(u, x) \); this is true because this path can only use edges with length smaller than \( d(u, v) \), and \( d(u, x) < d(u, v) \). Similarly, we have a path in \( G \) connecting \( v \) and \( y \) with length at most \( td(v, y) \). Thus, in the final structure \( G \), we have a path (with subpath from \( u \) to \( x \), subpath from \( y \) to \( v \), and edge \( xy \)) connecting \( u \) and \( v \) with length at most

\[
td(u, x) + td(v, y) + d(x, y) \leq (t + 1)d(u, x) + (t + 1)d(v, y) + d(u, v) \leq ((2(t + 1) \sqrt{d}/\vartheta_1) + 1)d(u, v)
\]

This is at most \( td(u, v) \) if

\[
2\sqrt{t^2 + 1} / (t - 1) \leq \vartheta_1
\] (12)

Figure 3: An illustration of the proof that the final structure \( G \) is a \( t \)-spanner. Here for a pair of nodes \( u \) and \( v \) we will have a path with length at most \( td(u, v) \), where \( u \) and \( v \) are representative nodes of the potential-edge boxes \( b_1 \) and \( b'_1 \).

**Case 2:** We do not have an edge \( xy \) where \( x \) is a representative node of enclosing-box \( b_1 \) and \( y \) is a representative node of box \( b'_1 \). In this case, one or both of the following conditions is true:

1. there is a crossing edge \( w_1w_2 \) such that \( w_1 \) is inside \( b_1 \), \( w_2 \) is outside of floating-virtual box \( b_2 \) and \( w_2 \) is in the general-cone-direction of \( C(w_1, B) \), or
2. there is a crossing edge \( w_1w_2 \) such that \( w_2 \) is inside \( b_1 \), \( w_1 \) is outside of floating-virtual box \( b_1 \) and \( w_1 \) is in the general-cone-direction of \( C(w_2, B) \).

W.l.o.g., we assume that the first condition is true. See Figure 3 for the illustration of the proof that follows.

Consider the general-cone-direction \( B(b_1, b'_1) \) of box \( b'_1 \) with respect to the base box \( b_1 \). The set \( B \) of cones will be called a meta-cone. The angular span of \( B \) is at most a value \( \theta \) (from Lemma 10). Observe that since the meta-cone \( C(w_1, B) \) will contain the box \( b'_1 \), it will also contain the node \( v \) inside. Recall that the edge \( w_1w_2 \) has the same direction as the meta-cone \( B \), the meta-cone \( C(w_1, B) \) also contains \( w_2 \) inside.

Then for the node \( w_2 \) and node \( v \), they must be contained in a pair of boxes in BSPD from the definition of BSPD. Consider the bounded-separated pair of floating-virtual boxes (say \( s \) and \( s' \)) containing them respectively. When the angle \( \theta < \pi/3 \), we have \( d(w_2, v) < d(w_1, v) \). Together with the fact that \( w_2 \) is outside of the floating-virtual box \( b_2 \), we can show that the edge-distance (i.e., \( d(s, s') \)) of the pair of potential-edge boxes containing
$w_2$ and $v$ respectively is less than the edge-distance $\ell(b_1, b_1')$ between $b_1$ and $b_1'$. In other words, the pair of nodes $w_2$ and $v$ has been processed before the pair of nodes $u$ and $v$. Thus, we either will have a directed edge $w_3w_4$ such that $w_3$ and $w_4$ are representative nodes of the boxes $s$ and $s'$ respectively; or we will have an edge $z_3z_4$ such that $z_3z_4$ has the same direction as the meta-cone $B(s, s')$, i.e., $z_3z_4$ is inside the meta-cone $B(s, s')$. Observe that the distance between the boxes $s$ and $s'$ is smaller than the distance between the boxes $b_1$ and $b_1'$.

We can repeat the above process and get a sequence of edges $w_1w_2, w_3w_4, w_5w_6, \ldots, w_{2k-1}w_{2k}$, by renaming the nodes and the pairs of potential-edge boxes, and the pairs of bounded-separated floating-virtual boxes, with the following properties:

1. $w_1$ is inside an enclosing-box $b_1$ and $w_2$ is outside of floating-virtual box $b_2$ containing $b_1$ (if it is not, we can pick the first one $w_{2i-1}w_{2i}$ such that this property is satisfied); node $w_2$ is inside an enclosing-box, called $b_1'$, which is inside a floating-virtual box, called $b_2'$. The pair of boxes $b_2, b_2'$ is a pair of bounded-separated floating-virtual boxes. Observe that here the boxes $b_1, b_1', b_2$ and $b_2'$ may be different from what we called at the beginning of our proof.

2. In general, for $i \geq 1$, node $w_{2i-1}$ is inside an enclosing-box $b_{2i-1}$ which is inside a floating-virtual box $b_{2i}$; node $w_{2i}$ is inside an enclosing-box $b_{2i-1}'$ which is inside a floating-virtual box $b_{2i}'$. Here, for $i \geq 1$, the pair of bounded-separated floating-virtual boxes $b_{2i}, b_{2i}'$ contain the pair of potential-edge boxes $w_{2i-1}, w_{2i}$, which is used to define the edge $w_{2i-1}w_{2i}$, i.e., $w_{2i-1}$ (resp. $w_{2i}$) is a representative node of the enclosing-box $b_{2i-1}$ (resp. $b_{2i}'$). Notice that here either the box $b_{2i-1}'$ or the box $b_{2i+1}$ could be the larger one between them, although both contain node $w_{2i+1}$. We also have that the node $w_{2i+1}$ is inside the enclosing-box $b_{2i+1}'$ for $i \geq 1$, while $w_{2i}$ is outside of the floating-virtual box $b_{2i}'$, for $i \geq 1$.

3. The angle $\angle w_{2i-1}w_{2i}w_{2i+1}$ is at most $\theta$ for a value $\theta$ in Lemma 10.

Thus, we have a path

$$ u \sim w_1w_2 \sim \ldots \sim w_{2i-1}w_{2i} \sim \ldots \sim w_{2k-1}w_{2k} \sim \ldots \sim v $$

to connect the pair of nodes $u$ and $v$. Here $p \sim q$ denotes a path constructed recursively to connect nodes $p$ and $q$. By induction, we know that the length of path $u \sim w_1$ is at most $td(u, w_1) \leq t\sqrt{d(b_1)} \leq t\sqrt{d(u, v)_1}$; similarly the length of the path $w_{2i} \sim \ldots \sim w_{2i+1}$ is at most $t\sqrt{d(b_{2i-1}', b_{2i+1})}$ since either (1) $w_{2i}$ and $w_{2i+1}$ inside $b_{2i-1}'$ or (2) $w_{2i}$ and $w_{2i+1}$ inside $b_{2i+1}'$.

Notice that $\max(\vartheta(b_{2i-1}'), \vartheta(b_{2i+1})) \leq \frac{d(w_{2i-1}, w_{2i})}{\vartheta_0}$ from the definition of potential-edge boxes. Additionally, $\max(\vartheta(b_{2i-1}'), \vartheta(b_{2i+1})) \leq \epsilon_2 \min(\vartheta(b_{2i-1}'), \vartheta(b_{2i+1}))$ since the floating-virtual boxes $b_{2i-1}'$ and $b_{2i+1}$ are required to have similar sizes (within a factor $\epsilon_1 = 1/\epsilon_2$ of each other). Then the total length of the path $w_1w_2 \sim \ldots \sim w_{2i-1}w_{2i} \sim \ldots \sim w_{2k-1}w_{2k}$ is at most

$$ \left( \sum_{i=1}^{k} d(w_{2i-1}, w_{2i}) \right) \cdot (1 + \frac{2t\sqrt{d}}{\vartheta_0}) $$

Thus, the length of the path $u \sim w_1w_2 \sim \ldots \sim w_{2i-1}w_{2i} \sim \ldots \sim w_{2k-1}w_{2k} \sim \ldots \sim v$ is at most

$$ \left( \sum_{i=1}^{k} d(w_{2i-1}, w_{2i}) \right) \cdot (1 + \frac{2t\sqrt{d}}{\vartheta_0}) + \frac{2t\sqrt{d}}{\vartheta_0} \cdot d(u, v) \tag{13} $$

We then bound the length $\sum_{i=1}^{k} d(w_{2i-1}, w_{2i})$. From the general-cone-direction property, when $\theta < \pi/3$, it is easy to show that

$$ d(v, w_{2i-1}) - d(v, w_{2i}) \geq (1 - 2\sin(\theta/2))d(w_{2i-1}, w_{2i}) $$

Since $d(v, w_{2i+1}) - d(v, w_{2i}) \leq d(w_{2i}, w_{2i+1}) \leq \sqrt{d} \max(\vartheta(b_{2i-1}), \vartheta(b_{2i+1})) \leq \frac{\sqrt{d}}{\vartheta_0} \max(d(w_{2i-1}, w_{2i}), d(w_{2i+1}, w_{2i+2}))$.

we have

$$ d(v, w_{2i}) - d(v, w_{2i+1}) \geq -\frac{\sqrt{d}}{\vartheta_0} \max(d(w_{2i-1}, w_{2i}), d(w_{2i+1}, w_{2i+2})). $$
Then, we have
\[ d(u, v) + d(w_1, u) \geq d(w_1, v) \geq \sum_{i=1}^{k} [d(v, w_{2i-1}) - d(v, w_{2i})] + \sum_{i=1}^{k-1} [d(v, w_{2i+1}) - d(v, w_{2i})] \]
\[ \geq (1 - 2 \sin(\theta/2) - 2 \sqrt{d/g_1}) \sum_{i=1}^{k} d(w_{2i-1}, w_{2i}) \]
Consequently, the ratio of the length of the path we found over \( d(u, v) \) is at most
\[ \frac{2t \sqrt{d}}{g_1} + (1 + \frac{2t \sqrt{d}}{g_1})(1 + \frac{\sqrt{d}}{g_1})/(1 - 2 \sin(\theta/2) - 2 \sqrt{d/g_1}) \leq t \] (14)
when \( \theta, \) and \( g_1 \) are chosen carefully (\( \theta \) is small enough and \( g_1 \) is large enough).
This finishes the proof of the spanner property. \( \square \)

It is easy to show that we can carefully choose \( \alpha = \Phi(F), \ g_1 \) and \( g_2 \) that satisfy the conditions in (10). Notice that these conditions are weaker than the conditions required to achieve low weight property, illustrated in (17).

### 3.3 The Weight Property

We next show that the weight \( \omega(G) \) of the graph \( G \) constructed is \( O(\omega(MST)) \). Our proof technique is based on the proofs used in \([11,13]\). Recall that an edge \( e \) is added to graph \( G \) when we process a pair of potential-edge boxes that are defined by a pair of bounded-separated floating-virtual boxes \( b \) and \( b' \in N(b) \). We then say that floating-virtual boxes \( b, b' \) and the edge \( e \) form a dumbbell (as defined in \([11]\)). For a dumbbell formed by edge \( e = (u, v) \), for both node \( u \) and node \( v \), we associate a cylinder with each node, and call it dumbbell head, of suitable size. A dumbbell head is a cylinder of radius \( \delta_1 \|uv\| \) and height \( \delta_2 \|uv\| \) with \( 0 < \delta_1 \ll \delta_2 \ll 1 \). These dumbbell heads are always contained inside the corresponding floating-virtual boxes. Similar to \([11]\), we can group edges of \( G \) into \( g = O(1) \) groups \( E_1, E_2, \ldots, E_g \) such that for edges in each group \( E_i \), we have

**Near-Parallel Property:** any pair of edges \( u_1u_2 \) and \( v_1v_2 \) in a group are nearly parallel, \( i.e. \), the angle formed by vectors \( u_2 - u_1 \) and \( v_2 - v_1 \) is bounded by a constant \( \theta_0 \). This clearly can be achieved using a partition based on cones: we first use a constant number of cones to partition the space \( \mathbb{R}^d \) (where the angular span of the cone base is at most \( \theta_0 \)). Then each cone defines a group of edges: all the (directed) edges \( uv \) contained in the direction of this cone.

**Length-Grouping Property:** In a group, any two edges have lengths that are either nearly equal or differ by more than a large constant factor. This can be achieved by first grouping edges into buckets (the \( i \)th bucket contains edges with lengths in \( [\delta^{i+1}L, \delta^iL] \) where \( L \) is the length of the longest edge and constant \( \delta \in (0,1) \)). Then form a group as the union of every \( i \)th bucket (so the edge lengths from different buckets differ by at least \( \delta^i \) factor).

**Empty-Region Property:** In a group, any two edges that have nearly equal length of value \( x \) are far apart, \( i.e. \), the distance between end-nodes of these edges are at least \( \epsilon_1x \) for some constant \( \epsilon_1 \). Here \( \epsilon_1 > 0 \) could be any constant (even larger than 1). This clearly can be done by showing that for each edge \( uv \) of length \( x \), there are at most \( O(1) \) edges that are of similar length and are not far apart (that has at least one end-node within distance \( \epsilon_1x \) of an end-node of \( uv \)). Recall that, in our method, for every added edge \( uv \), we will only add at most 1 edge for the pair of potential-edge boxes defining \( uv \) and the size of the floating-virtual boxes \( b_2 \) and \( b'_2 \) is at least a constant fraction of the edge length \( d(u, v) \). The virtual boxes \( b_2, b'_2 \) used to add an edge \( uv \) will be used to define the dumbbells of the edge \( uv \). Recall that the virtual boxes will be either disjoint or one is completely contained inside the other. This implies that, given any edge \( uv \), there is only a constant number of edges \( xy \) that are of similar length and are nearby edge \( uv \).
Consequently, we have the following lemma:

**Lemma 15** We can group edges of $G$ into $O(1)$ groups such that the edges in each group satisfy the preceding properties: near-parallel, length-grouping, and empty-region.

Here the number of groups produced depending on the values $\varrho_1$, $\varrho_2$, $\varepsilon_1$, $\varepsilon_2$, and $\beta$. Recall that the bounded aspect ratio is at most $\beta \leq 2$ for all the tight-virtual boxes. However, the aforementioned properties do not ensure that the total edge weight of edges in a group is $O(1)\omega(SMT_i)$ where $SMT_i$ is the Steiner minimum tree connecting the endpoints of edges in $E_i$. We can construct an example of edges satisfying the aforementioned properties such that the total edge weights could be as large as $O(\log n)\omega(SMT_i)$. To prove that the graph produced by our method is low-weighted, we need an additional property:

**Empty-Cylinder Property:** for every edge $uv$ and its associated dumbbells, there is a cylinder (with the height $\geq \eta_1 d(u,v)$ and radius at least $\eta_2 d(u,v)$ for some positive constants $\eta_1$ and $\eta_2$) with axis using some segment of the edge $uv$ such that the cylinder is empty of any end-node of edges in the same group. This cylinder is called a protection cylinder of the edge $uv$.

Observe that the empty-region property does not imply the empty-cylinder property, and neither does the empty-cylinder property implies the empty-region property.

**Lemma 16** By carefully choosing $\varrho_1$, and $\alpha$ (and thus $\theta$), according to conditions illustrated in inequality (15), every added edge $uv$ by our Algorithm 1 has the empty-cylinder property.

**Proof.** Assume that $uv$ is added due to the pair of potential-edge boxes $b_1$ and $b'_1$, which is defined by a pair of bounded-separated (floating-virtual) boxes $b_2$ and $b'_2$. Thus $b_1$ and $b'_1$ are contained inside $b_2$ and $b'_2$ respectively. Let $B$ be the base such that $v \in C(b_2, B)$ and let $\mathcal{B}$ be the minimal collection of bases such that for any point $p$ inside the box $b_2$, $b'_2 \in C(p, \mathcal{B})$, i.e., bases that are in the general-cone-direction $\mathcal{B}$.

Since $uv$ is added, we know that there is no edge $xy$ crossing $b_2$ with $x \in b_2$ and $xy$ is in the general-cone-direction $\mathcal{B}$. We will show by a simple contradiction that there is a node $p$, such that the cone $C(p, \mathcal{B})$ is empty of nodes $w \notin b_2$ with distance $d(p,w) \leq d(p,b'_1)$. If this is not true, consider all the pairs of nodes $x$ and $y$ with $x \in b_2$, $y \notin b_2$, $y \in C(x, B)$ and $d(x,y) \leq d(x,b'_1)$. Let $p, q$ be the pair with the smallest distance among all such pairs of nodes $x, y$. Then edge $pq$ will exist in the graph $G$, which contradicts the existence of edge $uv$.

Since the cone $C(p, \mathcal{B})$ is empty of nodes, then by choosing a large enough $\varrho_1$, we will have a large empty-cylinder at the middle of the segment $uv$. For example, if we let $\varrho_1$ be four times of the value of $\varrho_1$ that satisfies condition (13), i.e.,

$$\theta \leq 3\alpha + \frac{16\sqrt{d}}{\varrho_1}$$

then we have an empty-cylinder near the middle of the segment $uv$ with height almost half of the length $d(u,v)$. In other words, if condition (15) is satisfied, we have $\eta_1 \approx 1/2$, and $\eta_2 = \varrho_1/(4\varrho_2)$. Recall that here $\varrho_1$ and $\varrho_2 = \Theta(\varrho_1)$ are constants used to define the bounded-separateness of two almost-equal-sized virtual boxes.

Thus, for any edge $uv$ added by our method, we know that there is a cylinder using a segment $wz$ of $uv$ as axis with radius at least $\eta_2 \|uv\|$ for a constant $\eta_2$, $wz$ has length $\|uv\|/2$ and in the center of segment $uv$.

**Definition 8 (Isolation Property)** A set of edges $E$ is said to satisfy the isolation property if

1. With every edge $e = uv \in E$ can be associated a cylinder $C(e)$ whose axis is a segment of $uv$, and the size of the cylinder is not small, i.e., the height is at least $\eta_1 d(u,v)$ and the radius of the basis is at least $\eta_2 d(u,v)$ for some positive constants $\eta_1$ and $\eta_2$.

\footnote{Place $n$ nodes evenly on a line and connect every pair of nodes. Then there is a group of edges produced by the preceding partitioning will have a total edge weights of $O(\log n)\omega(SMT_i)$.}
2. For every edge \( e \), its associated cylinder \( C(e) \) is not intersected by any other edge.

The following theorem was proved in \([13]\) by Das et al.

**Theorem 17** \([13]\) If a set of edges \( E \) in \( \mathbb{R}^d \) satisfies the isolation property, then \( \omega(E) = O(1)\omega(SMT) \), where SMT is the Steiner minimum spanning tree connecting the endpoints of \( E \).

Based on this theorem, we then show that the graph \( G \) produced by our method is also low-weighted. Observe that a group of edges from the graph \( G \), partitioned as previously to satisfy the near-parallel, length-grouping, and empty-region properties, may not satisfy the isolation property directly.

**Theorem 18** The set of edges \( E_i \) that satisfies empty-region property and empty-cylinder property has a total weight at most \( O(\omega(SMT_i)) \) where \( SMT_i \) is the Steiner minimum spanning tree that spans the vertices in \( E_i \).

**Proof.** We first use the grouping approach to partition the edges into a constant number of groups \( E_i \), \( 1 \leq i \leq g \), with each group of edges satisfying the near-parallel, length-grouping, and empty-region properties. It now suffices to study the weight of a group \( E_i \). We essentially will show that, for each group \( E_i \) of edges produced, we can remove some edges such that (1) the total length of all removed edges is bounded by a constant factor of the total length of the remaining edges, and (2) the set of the remaining edges satisfies the isolation property. If these two statements were proven to be true, the theorem then directly follows. Figure 4 illustrates the proof that will follow.

Figure 4: Illustration of the proof of the low weight property. Here (a) a long edge may intersect the cylinders (dark shaded rectangles) of many shorter edges and (b) the length of the similar sized edges that are intersected by an edge is bounded.

Recall that when we added an edge \( uv \) to the graph \( G \), edge \( uv \) satisfies the Empty-Cylinder Property. It is easy to show that the cylinder associated with \( uv \) will not intersect any edge \( xy \) (with a much shorter length) added before \( uv \) and \( xy \in E_i \). This can be done by shrinking the protection cylinder by at most a small constant factor.

On the other hand, it is possible that edge \( uv \) may intersect cylinders of many edges \( xy \in E_i \) with shorter lengths. See Figure 4(a) for an illustration of such case. Here the black shaded regions are protection cylinders. Given an edge \( uv \), let \( I(uv) \) be the set of edges \( xy \in E_i \) such that \( uv \) intersects the protection cylinder of the edge \( xy \). We then process edges \( uv \in E_i \), starting from the longest edge, to produce \( E'_i \) as follows: the longest edge \( uv \in E_i \) is added to \( E'_i \) and update \( E_i \) by removing all edges \( I(uv) \) from \( E_i \); we repeat this procedure till \( E_i \) is empty. Clearly, the set of edges \( E'_i \) satisfies the isolation property and thus has total weight at most \( \omega(SMT) = O(\omega(EMST)) \). Observe that, the preceding processing of \( E_i \) is only for the proof of the low-weight property; we will not remove these edges \( I(uv) \) from the constructed structure \( G \). Also observe that all edges \( xy \in I(uv) \), where \( xy \) has a length at most \( \delta^s d(u,v) \), \( s > 1 \), must be inside a cylinder with axis \( uv \), height almost \( d(u,v) \), and radius at most \( \eta_2 \delta^s \cdot d(u,v) \).

We then show that the length of edges in \( I(uv) \) is at most \( O(\|uv\|) \). For simplicity, we assume that all edges in \( I(uv) \) are parallel to \( uv \). The rest of the proof will still hold (with different constants) since the edges in \( E_i \) are almost parallel. Recall that the edges in \( I(uv) \) have the length grouping property and the length of every edge is at most \( \|uv\| \). Let \( I_i(uv) \) be edges from \( I(uv) \) with length in the range of \( [\delta^{s+1} d(u,v), \delta^s d(u,v)] \), where \( i \geq 0 \) is any integer. Here \( s \gg 1 \) and \( 0 < \delta \ll 1 \) are positive constants used in deriving the length
grouping property. Let $X_i$ be the total length of edges in $I_i(uv)$. We first show that $X_i$ is at most $\epsilon d(u, v)$ for a small constant $0 < \epsilon < 1$. Let $x_1 y_1, x_2 y_2, \ldots, x_n y_n$ be edges from $I_1(uv)$ such that the projection $x'_i$ of $x_i$ on edge $uv$ is at the righthand side of the projection $y'_{i−1}$ of node $y_{i−1}$ on edge $uv$. Then $\sum_{j=1}^{a} d(x_j y_j) \leq (1 − \eta_1) d(u, v) + 2\delta s d(u, v) \leq \epsilon d(u, v)$ for a constant $\epsilon = 1 − \eta_1 + 2\delta < 1$ when integer $s$ is chosen large enough. Here $\eta_1$ is the constant used to define the ratio of the height of a protection cylinder over the length of the edge.

We then show that we will not have edges $wz$ in $I_1(uv)$ such that it will have endpoints $w$ such that its projection $w'$ on $uv$ is in the interval $[x'_j, y'_j]$ for some $1 \leq j \leq a$. Figure 3 (b) illustrates the situation for this case. This can be proved by choosing a large constant $\epsilon_1 \gg 1$ in defining the empty-region property: for any edge $x_j y_j$, there is no edge $wz$ of similar length such that $w$ is within distance $\epsilon_1 d(x_j, y_j)$ of $x_j$ or $y_j$.

We can show that the protection cylinders defined by edges in $I_1(uv)$ are disjoint, and these protection cylinders are also disjoint from the protection cylinder of the edge $uv$. Obviously, any edge $wz$ from $I_i(uv)$ cannot have node $w$ or $z$ falling inside the protection cylinders of the edges in $I_j(uv)$ for $j < i$. Recall that our choices of protection cylinders (their sizes) already ensure that any edge $wz$ from $I_i(uv)$ cannot intersect the cylinders of edges from $I_i(uv)$ with $t \leq i$. Thus, the total length of edges in $I_i(uv)$, denoted as $X_i$, is at most

$$X_i \leq X- \eta_2 X_i \sum_{t=0}^{i-1} X_t.$$

Here $X = d(u, v)$, and $\eta_2 \sum_{t=0}^{i-1} X_t$ is the total height of the protection cylinders defined by edges in $I_0(uv) = \{uv\}, I_1(uv), I_2(uv), \ldots, I_{i-1}(uv)$. These protection cylinders are empty of nodes, and also empty of edges from $I_i(uv)$. Then it is easy to show by induction that, for any $i \geq 1$, $\sum_{t=0}^{i} X_t \leq \eta X$ for a constant $\eta = \frac{1}{\eta_2}$. This finishes the proof.

Thus, by choosing the parameters $\alpha$, $\varrho_1$, $\varrho_2$, and $\theta$ satisfying the following conditions

$$\begin{align*}
\varrho_1 &\geq 4\sqrt{d} &\text{from Theorem}\ [7] \\
\varrho_2 &\geq 6\sqrt{d} + 2\varrho_1 &\text{from Theorem}\ [7] \\
\theta &\geq 3\alpha + \frac{16\sqrt{d}}{\varrho_1} &\text{from Lemma}\ [16] \\
\varrho_1 &> t\sqrt{d} &\text{from Theorem}\ [14] \\
2\sqrt{\frac{d^{i+1}}{\varrho_1}} &\leq \varrho_1 &\text{from Theorem}\ [14] \\
\left(\frac{2\sqrt{d}}{\varrho_1} + (1 + \frac{2\sqrt{d}}{\varrho_1})\left(1 + \frac{\sqrt{d}}{\varrho_1}\right)/(1 − 2\sin(\theta/2) − 2\sqrt{d}/\varrho_1)\right) &\leq t &\text{from Theorem}\ [14]
\end{align*}$$

(16)

our structure is a $t$-spanner, with bounded degree, and is low-weighted. To satisfy the aforementioned conditions, it suffices to satisfy the following conditions when $t \leq 3$

$$\begin{align*}
\varrho_2 &\geq 6\sqrt{d} + 2\varrho_1 &\text{from Theorem}\ [7] \\
\theta &\geq 3\alpha + \frac{16\sqrt{d}}{\varrho_1} &\text{from Theorem}\ [14] \\
\varrho_1 &\geq 2\sqrt{\frac{d^{i+1}}{\varrho_1}} &\text{from Lemma}\ [16] \\
\left(\frac{2\sqrt{d}}{\varrho_1} + (1 + \frac{2\sqrt{d}}{\varrho_1})\left(1 + \frac{\sqrt{d}}{\varrho_1}\right)/(1 − 2\sin(\theta/2) − 2\sqrt{d}/\varrho_1)\right) &\leq t &\text{from Theorem}\ [14]
\end{align*}$$

(17)

It is easy to show that we do have solutions for $\alpha$, $\varrho_1$, $\varrho_2$ and $\theta$. For example, if we let $x = \sqrt{d}/\varrho_1$, $\alpha = x$, and $\theta = 19x$. By choosing $21x < 1$, we get $x = \frac{26t+1−\sqrt{(26t+1)^2−160(t−1)}}{80t}$ is a valid solution.\footnote{With a small additive value whose total length over all $i$ is bounded by $X$. This is for the case that we may have an edge $xy$ from $I_i(uv)$ such that $x$ is outside of the cylinder using $uv$ as axis and radius proportional to $\delta^{\alpha} d(u, v)$, and $y$ is inside this cylinder. Obviously, the total length of such edges are at most $X = d(u, v)$.}
Thus, \( \frac{t-1}{2m+1} \leq x = \frac{\sqrt{d}}{g_1} \leq \frac{2(t-1)}{2m+1} \) clearly is a solution satisfying the last condition here. This solution also satisfies the other conditions in Inequalities (17). Thus the number of cones produced is proportional to \( (\frac{1}{c})^d = O((\frac{1}{c})^d) = O((\frac{1}{c})^d) \). Then we have our main theorem:

**Theorem 19** Given a set of nodes \( V \) in \( \mathbb{R}^d \), Algorithm (1) constructs a structure that has low-weight \( O(\omega(MST)) \), has a constant bounded degree \( O((\frac{1}{c})^d) \), and is a t-spanner in time \( O((\frac{1}{c})^d n \log n) \), where the constant \( c \) depends on the spanning ratio \( t \) and \( \frac{1}{c} \) is the number of cones needed in our cone decomposition.

4 \( (k, t) \)-VFTS Spanner Using Compressed Boxtree

In this section, we show how to extend the previous approach to an efficient method for constructing a \( k \geq 1 \) fault-tolerant \( t \) spanner for any given set of nodes \( V \) in \( \mathbb{R}^d \), \( t > 1 \) and \( k \geq 1 \).

4.1 The Method

Our approach follows the construction of the \( t \)-spanners as in the previous sections. We will assume the construction of the compressed split-tree \( CT(V) \) and a bounded-separated pair decomposition (BSPD) based on \( CT(V) \). As has already been proven previously (Lemma [11], at each box of \( CT(V) \), we can choose 1 representative node so that each node \( v_i \in V \) is used at most 2 times as a representative node of some enclosing-boxes. Since we want a \( k \)-VFTS, we will choose \( k + 1 \) representative nodes for an enclosing-box, if it contains at least \( k + 1 \) nodes inside. For easy of presentation, we define various boxes.

**Definition 9** We call an enclosing-box \( b \) a \( k \)-box if it contains at least \( k + 1 \) nodes inside; otherwise it is called a non-\( k \)-box. A \( k \)-box is called a leaf-\( k \)-box if it is a \( k \)-box and none of its children in the compressed split-tree is a \( k \)-box.

For each \( k \)-box, we will choose \( k + 1 \) nodes contained inside \( b \) as the representative nodes \( R(b) \) of the box \( b \). We will discuss in detail on how to choose \( R(b) \) for a \( k \)-box later. For any box \( b \) that contains at most \( k \) nodes inside, all nodes will serve as the representative nodes \( R(b) \) of this box.

The basic idea of our method is as follows. Consider a pair of potential-edge boxes \( b_1 \) and \( b'_1 \), and their \( k + 1 \) representative nodes (if both boxes are \( k \)-boxes). Here \( b_1 \) and \( b'_1 \) are tight-virtual boxes. Consider the pair of bounded-separated floating-virtual boxes \( b_2 \) and \( b'_2 \), with \( b_2 \) contained inside the tight-virtual box \( P(b_1) \) and \( b'_2 \) contained inside the tight-virtual box \( P(b'_1) \). We add an edge \( uv \) where \( u \) is a representative node of \( b_1 \) and \( v \) is a representative node of \( b'_1 \), while the following is true:

1. there are less than \( k + 1 \) disjoint edges of the form \( xy \) where \( x \) is from \( b_2 \) and \( xy \) is in the general-cone-direction defined by \( C(x, B) \) such that \( B \) satisfies the General-Cone-Direction property w.r.t. \( B \). Here \( B \) is the cone basis such that the representative node \( v \) is contained inside the region \( C(b_2, B) \); and
2. there are less than \( k + 1 \) disjoint edges of the form \( zw \) where \( z \) is from \( b'_2 \) and \( zw \) is in a general-cone-direction \( B' \) w.r.t. \( B' \) where \( B' \) is the basis such that the representative node \( u \) is contained inside the region \( C(b'_2, B') \);

**Data Structures Used:** In our method, for each enclosing-box \( b \), each dimension \( h \), and each cone basis \( B_i \) of a frame \( F \), we store a set of at most \( k + 1 \) disjoint edges \( x_i y_i \) in an array DisjointCrossingEdge, denoted as DCE\((b, B_i)\), such that

1. Node \( x_i \) is inside the enclosing-box \( b \),
2. Node \( y_i \) is in the cone \( C(x_i, B_i) \).

Another set AllCrossingEdge, denoted as ACE\((b, B_i)\), stores all edges \( xy \) with \( x \in b \), \( y \not\in b \) and \( y \in C(x, B_i) \). Our method will ensure that the cardinality of ACE\((b, B_i)\) is at most \((k+1)^2\) for each cone direction.
B_i. Using AllCrossingEdge array ACE(b, B_i) for all cones B_i ∈ B, we can find the maximum number of disjoint edges DCE(b_i, B) in the general-cone-direction B, which cross the box b_i (one node inside b_i and one node outside of b_i). This can be done using a maximum matching in the bipartite graph over two sets of nodes: one set is all nodes inside b and the other set is all nodes y ⊈ b with an edge xy for some node x ⊈ b. Let X(b_i) be the set of end-nodes of edges DCE(b_i, B) that is inside b_i. Let X(b'_i) be the set of end-nodes of edges DCE(b'_i, B') that is inside b'_i. Clearly, for X(b_i) and X(b'_i), each has size at most k + 1. For any node u and a direction B, let deg(u, B) be the number of edges incident onto u in the direction of B. These edges were added because of processing some pairs of potential-edge boxes with smaller distance. At each step in the algorithm, edges are added based on the degree of the representative nodes and edges in the array, DCE. Our detailed method for constructing a (k, t)-VFTS is presented in Algorithm 2.

4.2 Properties: (k, t)-VFTS Spanner, Low-Weight, and Low-Degree

Lemma 20 The set AllCrossingEdge ACE(b, B_i) has size at most (k + 1)^2 when k ≥ 1.

Proof. Observe that, for each node v in the box b, we add at most k + 1 edges in any direction B_i. The moment the set ACE(b, B_i) has size (k + 1)^2, we must have at least k + 1 disjoint edges crossing the box b in the direction B_i. This follows from the observation that we will not add any more edges in our algorithm if DCE(b, B_i) is at least k + 1.

We show that the constructed graph is (k, t)-VFTS. Recall that a structure is called (k, t)-VFTS if for every pair of nodes u and v either (1) edge uv is presented or (2) there exist k + 1 node disjoint paths connecting u and v such that the length of each path is at most t||uv||.

Theorem 21 The graph G is a (k, t)-VFTS.

Proof. We prove that for every pair of nodes u and v, either edge uv ∈ G or there are k + 1 internally vertex disjoint paths Π_1, Π_2, · · · , Π_k+1 connecting them and each path has length at most t · ||uv||. In other words, G is a (k, t)-VFTS. We will prove this by induction on the distance between b and b', the bounded-separated floating-virtual boxes containing u and v, respectively. We will refer to this distance as the distance between u and v. It is easy to show that the pair of nodes v_1, v_2 with the shortest distance will be in G.

Assume that for any 1 ≤ j ≤ r, for the pair of nodes whose distance is the j-th smallest, our statement is true. We then consider the pair of nodes u and v with the (r + 1)-th smallest distance.

Consider the boxes b and b' such that u ∈ b and v ∈ b' and b and b' are bounded-separated floating-virtual boxes and let b' ∈ C(b, B) for some cone basis B ∈ F. Assume for simplicity that the direction defining B is horizontal and b is aligned with the coordinate axis. For notational simplicity, let |b| be the number of nodes from V that are inside the box b.

We have three complementary cases depending on the number of points in b and b': (1) |b| = p ≤ k, (2) |b'| = q ≤ k, and (3) p > k + 1 and q > k + 1.

First consider the case |b| = p ≤ k. The case |b'| = q ≤ k follows the same proof. When we processed a box b' from N_{≥}(b), let q = |b'|. If q ≤ k + 1 − g (where g is the number of edges incident on u in the direction of B), then edges are added between the node u and all nodes of b'. Thus, we have an edge uv already. Otherwise we know that node u is connected to k + 1 nodes, say w_1, w_2, · · · , w_{k+1} with the condition that the distance between u and w_i is no more than uv since we processed the boxes b' in N(b) in increasing order of distance to b. Additionally, since we only focus on edges in one specific cone direction, we know the length of w_i v is also less than that of uv. Then the general proof below applies.

In general, w.l.o.g., assume that neither u nor v are representative nodes in b and b'. In this case, we must have p > k + 1 and q > k + 1. Let the k + 1 representative nodes for b and b' be U = {u_1 . . . u_{k+1}} and U' = {v_1 . . . v_{k+1}} respectively. The following cases arise...
Algorithm 2 Constructing a \((k, t)\)-VFTS Spanner with Low-weight

1. Build the compressed split-tree \(CT(V)\). For each enclosing \(k\)-box \(b\) in the tree \(CT(V)\), associate it with \(k + 1\) representative nodes \(R(b)\). For each box \(b\), we sort the edge-distances \(\ell(b, b')\) for \(b' \in N_\geq(b)\). We also sort all the distances \(\{\ell(b, b') \mid b' \in N_\geq(b)\}\) for every box \(b\) in time \(O(n \log n)\).

   We then process pairs of potential-edge boxes \(b\) and \(b'\) in increasing order of their distance \(\ell(b, b')\).

   Initiate the array \(ACE(b, B_i)\) and \(DCE(b, B_i)\) as empty.

2. for \((r = 1 \to \sum_b N_\geq(b))\) do
3.   Consider an enclosing-box \(b_1\) with \(p\) representative nodes \(u_1, u_2, \ldots, u_p, 1 \leq p \leq k + 1\), and the enclosing-box \(b'_1 \in N(b)\) with \(q\) representative nodes \(u'_1, u'_2, \ldots, u'_q, 1 \leq q \leq k + 1\) such that the distance \(\ell(b_1, b'_1)\) has rank \(r\) among all pairs of potential-edge boxes. Let \(b_2 = \square(b_1)\) and \(b'_2 = \square(b'_1)\) be a pair of bounded-separated boxes.

   Let \(B \in \mathcal{F}\) be the base such that the majority representative nodes of \(b'_1\) is contained inside \(C(b_2, B)\).

   Let \(B' \in \mathcal{F}\) be the base such that the majority representative nodes of \(b_1\) is contained inside \(C(b_1, B')\).

   Let \(B, B' \subset \mathcal{F}\) be the collection of bases satisfying the General-Cone-Direction Property w.r.t \(B\) and \(B'\) respectively.

4. Let \(l\) be the maximum of the cardinality of \(DCE(b_1, B)\) and \(DCE(b'_1, B')\).
5. Let \(Y_k(b, B)\) be the sorted list of all nodes \(\{u \mid u\) is inside box \(b, u \notin X(b), \) and \(deg(u, B) \leq k\}\), in increasing order of \(deg(u, B)\). Thus, we update \(Y_k(b_1, B)\) and \(Y_k(b'_1, B')\).
6. if \((p \leq k)\) and \((q \leq k)\) then
7.   Add all edges \(u_iu_j\) for all pairs of \(u_i\) and \(u_j\) inside \(b_1\). Add all edges \(u'_iu'_j\) for all pairs of \(u'_i\) and \(u'_j\) inside \(b'_1\).
8.   for each node \(u_i\) in \(b_1\) with \(deg(u_i, B) \leq k\) do
9.     We add an edge \(u_iu'_j\) to a node \(u'_j\) in \(b'_1\) where \(u'_j\) has the smallest degree \(deg(u'_j, B')\) among all nodes inside \(b'_1\). Update the degree for all nodes and arrays \(ACE\) and \(DCE\).
10. end for
11. for each node \(u'_j\) in \(b'_1\) with \(deg(u'_j, B') \leq k\) do
12.     We add an edge \(u_iu'_j\) to a node \(u_i\) in \(b_1\) where \(u_i\) has the smallest degree \(deg(u_i, B)\) among all nodes inside \(b_1\) and \(u_iu'_j\) was not added before. Update the degree for all nodes and arrays \(ACE\) and \(DCE\).
13. end for
14. end if
15. if \((p \geq k + 1)\) and \((q \geq k + 1)\) then
16.   Add \(k + 1 - l\) edges of the form \(u_iu'_i\), where \(l = \max(|DCE(b_1, B)|, |DCE(b'_1, B')|)\). Here \(u_i, i \leq k + 1 - l\), are the first \(k + 1 - l\) nodes in \(Y_k(b_1, B)\) and \(u'_i, i \leq k + 1 - l\), are the first \(k + 1 - l\) nodes in \(Y_k(b'_1, B')\).
17.   Update the set \(DCE(b, B_i)\) and \(ACE(b, B_i)\) accordingly.
18. end if
19. if \((p \leq k)\) or \((q \leq k)\), but not both then
20.    Without loss of generality, we assume that \(p \leq k\) and \(q \geq k + 1\).
21.   Add all edges \(u_iu_j\) for all pairs of \(u_i\) and \(u_j\) inside \(b_1\). Here \(b_1\) contains exactly \(p\) nodes inside.
22.   for each representative \(u_i\) of \(b_1\) do
23.     Let \(g'\) be the cardinality of \(DCE(b'_1, B')\).
24.     If \(|Y_k(b'_1, B')| \geq \min(k + 1 - deg(u_i, B), k + 1 - g')\), we add \(\min(k + 1 - deg(u_i, B), k + 1 - g')\) edges from \(u_i\) to the first \(k + 1 - deg(u_i, B)\) nodes in \(Y_k(b'_1, B')\); otherwise, we add \(|Y_k(b'_1, B')|\) edges from \(u_i\) to nodes in \(Y_k(b'_1, B')\).
25.   Update the array \(Y_k(b_1, B)\) and \(Y_k(b'_1, B')\), and the set \(DCE(b'_1, B_i)\) and \(ACE(b'_1, B'_i)\) accordingly.
26. end for
27. end if
28. end for
29. Let \(G = (V, E)\) be the graph constructed.
Case 1: For all $i \leq k + 1$ there are edges $u_i v_i$ connecting $u_i \in U$ and $v_i \in U'$. Note that since $b$ and $b'$ are bounded-separated, $d(u, u_i) < d(u, v)$, for $u_i \in U$ and $d(v, v_i) < d(u, v)$ for $v_i \in U'$. Thus, by induction, there exist $k + 1$ disjoint paths between $u$ and $u_i \in U$, and there exist $k + 1$ disjoint paths between $v$ and $v_i \in U'$. Since $\forall i, u_i v_i$ is part of $G$, by using Mengers theorem it is easy to see that there are $k + 1$ vertex disjoint paths between $u$ and $v$ in $U'$. It is also easy to show that the length of each of such $k + 1$ paths is at most $td(u, v)$.

Case 2: There is an $i$, such that $u_i v_i$ does not exist. If $l$ edges of the form $u_i v_i$ are added where $u_i$ and $v_i$ are representatives of $b$ and $b'$ then one of the following subcases arises:

(a) There are $k + 1 - l$ edges of the form $(x, y)$ where $x \in b$ and $y \in C(x, B)$, $B$ in the general cone direction as $B$. Let the nodes inside $b$ be $x_1, x_2 \ldots x_m, m = k + 1 - l$ and the edges satisfying the preceding condition be $x_i y_i$. Here $y_i$ is a node inside some other box in the direction of $B$ of the box $b$. Note that distance between $y_i, 1 \leq i \leq m$ and nodes $v_j$ (by measure of the distance $\ell(b, b')$ for two edge-boxes $b$ containing $y_i$ and $b'$ containing $v_j$) is less than the distance between $u$ and $v_j$. We can thus apply induction to show that there are $k + 1$ vertex disjoint paths between $u$ and $x_j, 1 \leq j \leq m$. And by induction there are $k + 1$ node disjoint paths between $y_i$ and $v_j, 1 \leq j \leq m$. The result follows from Mengers theorem that there are $k + 1$ node disjoint paths connecting the $k + 1$ representatives $U$ to $k + 1$ representatives $U'$.

(b) A similar result is true when there are edges $zy, y \in b'$ and $z \in C(z, B')$, where $B'$ in the same general direction as $B'$, where $b \in C(b', B')$.

This finishes our proof that the structure is $k$-fault tolerant. Similar to the proof of Theorem 14 we can show that for every pair of nodes $u$ and $v$, each of the $k + 1$ disjoint paths found in the preceding constructive proof has a length at most $t\|u - v\|$. Thus, the structure we constructed is a $(k, t)$-VFTS.

**Theorem 22** In the graph $G = (V, E)$ constructed by our method, $\omega(E) = O(k^2) \cdot \omega(MST)$ when $k \geq 1$.

**Proof.** Consider the edges added at every node in $CT(V)$. We group edges into $O(k^2)$ groups and will show that the total length of edges in each group is at most $O(\omega(MST))$.

Note that for a pair of potential-edge boxes $b$ and $b'$, it is possible that $O(k^2)$ edges are added during the construction procedure. This happens when there are $p < k + 1$ (with $p = \Theta(k)$) representatives in one box during the procedure. As in Section 3 the edges added to connect representative nodes of pairs of potential-edge boxes can be partitioned into $O(k^2)$ groups such that the edges in each group satisfy the properties outlined: the near-parallel property, length-grouping property, empty-region property, and empty-cylinder property.

To partition edges into groups, we assume that the representative nodes $U$ inside an enclosing-box $b$ are numbered as $u_1, u_2, \ldots, u_p$, where $p \leq k + 1$ and the representative nodes $U'$ inside an enclosing-box $b'$ are numbered as $v_1, v_2, \ldots, v_q$, where $q \leq k + 1$. We group edges sequentially in increasing order of the distance between the pair of the potential-edge boxes $b$ and $b'$. For simplicity, we will only consider one cone base $B$ and all edges added in the direction of $B$. Since there are only a constant number of cone bases, if the edges added in the direction of any cone is at most $O(k^2)\omega(MST)$, the total weight of all edges is still $O(k^2)\omega(MST)$. For each edge $uv$ in the cone direction of $B$, we will put it into one of the $k^2$ groups: $E_{i, j}, 1 \leq i, j \leq k$.

Notice that, for a pair of potential-edge boxes $b$ and $b'$, we add edges based on rules defined for three different cases based on $|b| = p, |b'| = q$: Case 1) $|b| = p \leq k$, Case 2) $|b'| = q \leq k$, and Case 3) $p \geq k + 1$ and $q \geq k + 1$.

In case 1, $|b| \leq k$, for each $u_i$, we add $\min(k + 1, q)$ edges $u_iv_j$ for some nodes $v_j \in U'$. Then the edge $u_iv_j$ is added to group $E_{i, j}$ for $i \leq k + 1$ and $j \leq k + 1$. Notice that we also added $p(p - 1)/2 < k^2$ edges inside the box $b$. Observe that each such added edge (inside the box $b$) has length at most a small constant fraction of the
edge \( u_i v_j \) (added to connect representative nodes of \( b \) and \( b' \)). We will not add these edges to any group \( E_{i,j} \). A simple charging (charge the total edge length to one such crossing edge \( u_i v_j \)) method shows that these omitted edges have total edge length at most \( O(k^2) \) times the total edge length of edges in a group \( E_{i,j} \). We will show that \( \omega(E_{i,j}) \) is at most \( \omega(MST) \).

For case 2, for each \( v_i \) we will add \( \min(k + 1, p) \) edges \( u_j v_i \) for some nodes \( u_j \in U \) and then we similarly add the edge \( u_j v_i \) to group \( E_{j,i} \). Thus, all edges added in these cases are put into different groups, i.e., for any group \( E_{i,j} \), and any pair of potential-edge boxes \( b \) and \( b' \), we have at most one edge \( uv \) with \( u \) from \( b \) and \( v \) from \( b' \).

The third case is that both boxes \( b \) and \( b' \) have at least \( k + 1 \) nodes inside. Let \( b_2 \) and \( b'_2 \) be the pair of bounded-separated boxes that contain the boxes \( b \) and \( b' \) respectively. In this case, we will add \( k + 1 - l \) edges, \( u_i v_i \), where \( u_i \in U \) and \( v_i \in V \). Recall that we added these \( l \) edges because

1. there are at most \( l \leq k + 1 \) disjoint edges already leaving the floating-virtual box \( b_2 \) in the direction of the cone \( B \), and
2. there are at most \( l \leq k + 1 \) disjoint edges already leaving the floating-virtual box \( b'_2 \) in the direction of the cone \( B' \).

Notice that those edges were added when processing a pair of potential-edge boxes with shorter distance. Thus they have already been put into some groups (at most \( 2l \) different groups, since there are at most \( 2l \) such edges). Then a newly added edge \( u_i v_i \) is put into a group that is different from those \( 2l \) groups. Notice that this is always possible since \( 2l + k + 1 - l \leq 2(k + 1) \leq (k + 1)^2 \) when \( k \geq 1 \). Thus, when we put an edge \( u_i v_i \) into some group, we know that there is no edge \( xy \) in the same group such that \( x \) is inside \( b \) (resp. \( b' \)) and edge \( xy \) crossing the boundary of floating-virtual box \( b_2 \) (resp. \( b'_2 \)) in the direction of \( B \) (resp. \( B' \)). Then similar to Theorem 18 we can prove that each group \( E_{i,j} \) of edges can be further partitioned into a constant number of subgroups such that each subgroup of edges satisfies all properties outlined previously, and thus the total weight of all edges in each group \( E_{i,j} \) is at most \( O(\omega(MST)) \). This finishes the proof of the theorem.

\[ \square \]

**Theorem 23** Each node \( v \in V \) has a degree \( O(k) \) in the graph \( G \) when \( k \geq 1 \).

**Proof.** Notice that for each node \( u \in V \), we add edges incident to it only when (1) it is a node contained inside some non-\( k \)-box, or (2) when it is a representative node in some \( k \)-box. For simplicity, we will only concentrate on edges added in the direction of one cone.

1. We first study how many edges will be added to a node \( u \) when \( u \) is inside some non-\( k \)-box. When node \( u \) is inside a non-\( k \)-box, let \( b_u \) be the largest non-\( k \)-box that contains \( u \) inside. Assume that \( b_u \) contains \( p < k + 1 \) nodes inside. According to our method, we will add \( p - 1 \) edges to other \( p - 1 \) nodes inside the box \( b_u \), and add \( \min(k + 1 - g_u, q) \) edges for a potential-edge box \( b'_i \) with \( q \) nodes inside, where \( g_u \) is the number of edges incident on \( u \), in the direction of \( B \), and with shorter distances. We have thus added at most \( p - 1 + k + 1 = O(k) \) edges in the direction of \( B \) when we have processed all enclosing-boxes \( b'_i \) in \( N(b) \), when \( u \) is a node inside some non-\( k \)-boxes.

2. We then study how many edges \( uv \) will be added to \( u \) when we process a pair of potential-edge boxes \( b \) and \( b' \) such that \( b \) is a \( k \)-box, \( b' \) is a non-\( k \)-box, such that \( u \) is inside the \( k \)-box \( b \) and \( v \) is inside the non-\( k \)-box \( b' \).

Let \( b^*_u = \mathbb{P}(b_u) \) be the smallest enclosing-box that contains \( u \) inside and is a \( k \)-box. In other words, box \( b^*_u \) is a leaf-\( k \)-box. We adopt a charging argument where we assign credits to nodes inside \( k \)-boxes to account for edges added to those nodes. With each node inside the \( k \)-box is assigned \( (k + 1) \) TYPE-1-credits for each cone direction (another set of TYPE-2-credits is assigned in the next case).

Now let us see what will happen when we process the enclosing-box \( b_1 = b^*_u \). The total free TYPE-1-credits of this box \( b^*_u \) required to charge of edges in a certain cone direction is \( |b^*_u|((k + 1) \) We will prove by induction that
Lemma 24  Every k-box b will have at least \((k + 1)(k + 1) − e_2\) free TYPE-1-credits where \(e_2\) is the number of edges that

1. are added during the processing of a pair of potential-edge boxes \(c\) and \(c'\), where \(c\) is a descendant k-box of \(b\) and \(c'\) is a non-k-box in \(\mathcal{N}(c)\), and

2. cross the boundary of \(b\) in the given cone direction when we process this box \(b\) and its potential-edge boxes to add some edge after its children boxes have been processed.

Proof of Lemma 24  This is clearly true for all leaf-k-boxes since it has been assigned \((k + 1)\) TYPE-1-credits for each node and a given direction, and it has at least \(k + 1\) nodes inside. In the rest of the proofs, when we count the number of edges crossing the boundary of \(b\), we will only count the edges that are added during the processing of a pair of potential-edge boxes \(b\) and \(b'\) for some non-k-box \(b'\). The edges added when \(b'\) is a k-box will be studied in the subsequent case.

Observe that when we process \(b_1\) and all boxes \(b'_1 \in \mathcal{N}(b_1)\) where \(b'_1\) is a non-k-box, the total number of edges added to the node \(u \in b_1\) in any cone \(B\) direction is at most \(k + 1\). Consider the case \(b_1 = b^*_1\). Let \(b'_1, b'_2, b'_3, \cdots, b'_i, b'_{i+1}, \cdots, b'_t\) be all potential-edge boxes that are non-k-boxes (with respect to \(b_1\)) in the direction of a given cone \(B\), i.e., \(b'_j \in \mathcal{C}(b_1, B)\) and \(|b'_j| \leq k\). We further assume that \(d(b'_j, b) < d(b'_{j+1}, b)\) for \(1 \leq j \leq t - 1\). Notice that some of these enclosing-boxes may be inside the box \(\mathbb{P}(b_1)\), while some of them may be outside of \(\mathbb{P}(b_1)\). Assume that the first \(i\) non-virtual potential-edge boxes \(b'_1, b'_2, b'_3, \cdots, b'_i\) are inside \(\mathbb{P}(b_1)\), while the rest of potential-edge boxes \(b'_{i+1}, \cdots, b'_t\) are outside of the box \(\mathbb{P}(b_1)\). There are two cases here:

(a) First consider the case \(x_1 = \sum_{j=1}^{i} |b'_j| \geq k + 1\). Assume that \(f \leq i\) is the smallest index such that \(\sum_{j=1}^{i} |b'_j| \geq k + 1\). We consider the number of total free TYPE-1-credits we will have for the enclosing-box \(\mathbb{P}(b_1)\) in this direction. The total TYPE-1-credits charged to nodes inside \(b_1\) for adding edges in this direction when processing enclosing-boxes \(b'_1, b'_2, b'_3, \cdots, b'_i\) is at most \((k + 1)(k + 1)\) since we add at most \(k + 1\) edges for the “closest” \(k + 1\) nodes inside \(b'_j\), \(j \leq f\) and we will not add any edges from nodes inside \(b'_j\), \(j > f\), to nodes inside \(b_1\). Observe that these new enclosing-boxes \(b'_1, b'_2, b'_3, \cdots, b'_f\) will also contribute at least \((k + 1)\) TYPE-1-credits to the enclosing-box \(\mathbb{P}(b_1)\) in this direction. Thus, the box \(\mathbb{P}(b_1)\) has at least \(x_1 \geq k + 1\) nodes, each with \((k + 1)\) free TYPE-1-credits left. In other words, our claim holds. Recall that a node gets a free TYPE-1-credit when it first becomes a node in some k-box and is thus not charged here. Thus, any node \(v\) inside boxes \(b'_1, b'_2, b'_3, \cdots, b'_f\) will not be charged any of its TYPE-1-credits when we add edges \(vu\) between \(b'_j\) and \(b_1\), \(j \leq i\).

(b) We then consider the case that \(x_1 = \sum_{j=1}^{i} |b'_j| < k + 1\). Let \(x_2 = \sum_{j=1}^{t} |b'_j|\) be the number of nodes inside non-k-boxes outside of \(b_1\). In this case, every node inside \(b'_1, b'_2, b'_3, \cdots, b'_t\) will connect \(k + 1\) edges to some nodes inside \(b_1\). So the total TYPE-1-credits charged to nodes inside \(b_1\) is \((k + 1)\). The nodes inside \(\bigcup_{j=1}^{t} b'_j\) will contribute at most \(\min(x_2, k + 1 - x_1) \cdot (k + 1)\) edges since only each of the closest \(\min(x_2, k + 1 - x_1)\) nodes will connect up to \(k + 1\) edges to nodes inside \(b_1\). Thus, the TYPE-1-credits left by all nodes inside \(\mathbb{P}(b_1)\) is at least \(|b_1| \cdot (k + 1) - \min(k + 1, x_1 + x_2) \cdot (k + 1)\), while the TYPE-1-credits contributed by nodes inside boxes \(b'_1, b'_2, b'_3, \cdots, b'_t\) is at least \(x_1(k + 1)\). Thus, the total free TYPE-1-credits in box \(\mathbb{P}(b_1)\) is at least \(|b_1| \cdot (k + 1) - \min(k + 1, x_1, x_2) \cdot (k + 1)\). Observe that, when we process the box \(\mathbb{P}(b_1)\), we already have \(e_2 = \min(k + 1, x_1, x_2)(k + 1)\) edges crossing the box \(\mathbb{P}(b)\). These edges are added from processing pairs of box \(b_1\) and non-k-boxes \(b'_j\), \(j > i\). Thus, our claim holds.

We can show by induction that Lemma 24 holds for all k-boxes. This finishes the proof of lemma.
define variable $x_1$ (as number of nodes from $b'_j$ that are inside $\mathbb{P}(b)$) and $x_2$ (as number of nodes from $b'_j$ that are outside $\mathbb{P}(b)$) similarly. We only have to add at most $\min(k+1, x_1 + x_2)(k+1) - e_2$ edges to nodes inside $b$. Similarly, we can show that our statement (Lemma 24) holds for the parent $k$-box $\mathbb{P}(b)$ also.

Observe that in our algorithm, when we decide to add edges $uv$ for a pair of potential-edge boxes, we choose a pair of nodes each has the smallest degree in the corresponding direction. Thus, the degree difference among all nodes is at most 1. Since we assigned $k + 1$ TYPE-1-credits to every node for adding edges in the case $uv$ when $u$ is from a $k$-box, and $v$ is from a non-$k$-box, the maximum number of edges added to a node $u$ in this case will be at most $k + 1$ in any cone direction.

3. We then study the total number of edges $uv$ that are added to a node $u$ when $u$ is inside some $k$-box $b_1$ and $v$ is inside some $k$-box $b'_1$ and the boxes $b_1$ and $b'_1$ are a pair of potential-edge boxes.

Notice that we add at most $k + 1$ edges when we processing each pair of potential-edge boxes $b$ and $b'$ that are $k$-boxes. In this case, we will assign 2 TYPE-2-credits for each node inside a $k$-box. Recall that the TYPE-2-credits are different from the TYPE-1-credits in the previous case. We charge a node $u$ a TYPE-2-credit if an edge $uv$ is added where $u$ and $v$ are from $b$ and $b'$ respectively. Similar to Lemma 24 we can prove that

Lemma 25 Every $k$-box $b$ will have at least $2(k + 1) - e_2$ free TYPE-2-credits where $e_2$ is the number of edges crossing the boundary of $b$ in the given cone direction when we start processing this box $b$ and its potential-edge boxes to add some edge after its children boxes have been processed.

We only added at most 1 edge when a node $u$ is served as a representative node of a $k$-box $b_1$. The statement clearly is true for all leaf-$k$-boxes. Then consider a non leaf-$k$-box $b$. If we have a $k$-box $b'$ such that $b'$ is inside $\mathbb{P}(b)$, then after processed $b$ and $b'$, $\mathbb{P}(b)$ will have at least $2(k + 1)$ free TYPE-2-credits, where $b$ and $b'$ contributed $k + 1$ TYPE-2-credits each. Similarly we can show that the lemma is true when $b'$ is outside of $\mathbb{P}(b)$.

Thus the theorem follows.

It is also not difficult to show the following theorem.

Theorem 26 Algorithm can be implemented to run in time $O(kc_2 n + c_2 n \log n)$, where $c_2 = \Theta((\frac{1}{(1-\varepsilon)})^d)$ is the number of cones partitioned (which in turn is dependent on the spanning ratio $\varepsilon$).

Proof. For each box $b$ and each direction $B_i$, we store $k + 1$ disjoint edges in an array $DCE(b, B_i)$ such that the end-nodes are furthest from the box in this direction. These edges $uv$ will be sorted based on the distances from $v$ to the box $b$, where $u$ is a node inside $b$. For a box $b$, given the array $DCE(b_1, B_i)$, and $DCE(b_2, B_i)$ where $b_1$ and $b_2$ are two children boxes of $b$, we clearly can update the list for box $b$ in $O(k)$ time from 2 sorted lists from children boxes $b_1$ and $b_2$ as follows. We greedily compare the top elements (the link with the furthest node) of two children boxes and get the link $uv$ with $v$ being furthest from $b$ in direction $B_i$; the process is repeated till we find $k + 1$ links. Notice that these newly found $k + 1$ links surely will be disjoint also. Here if a node $v$ is connected to multiple nodes $u_1$ and $u_2$ for DisjointCrossingEdge arrays $DCE(b_1, B)$ and $DCE(b_2, B)$ for both children boxes $b_1$ and $b_2$, we will only pick one link $u_i v$ for one of the children boxes and discard the other. The total time of such processing is $2 \cdot (k + 1)$. Since there are $O(n)$ boxes in total, the total time complexity of updating the furthest links list $DCE$ can be done in time $O(kn)$ for a single cone direction. The time complexity then follows from the fact that there are $c_2$ cones.

5 Conclusion

In this paper, we studied the spanner construction for a set of $n$ points in $\mathbb{R}^d$ and also fault-tolerant spanners for a set of points in $\mathbb{R}^d$. Our main contribution is an algorithm that runs in time $O(n \log n)$ to construct a $(k, t)$-
VFTS for Euclidean graph with maximum node degree $O(1 + k)$, and weight at most $O((1 + k)^2)\omega(MST)$ for $k \geq 0$. All bounds are asymptotically optimum. It remains an interesting future work to extend the method to geodesic distance when we are given $n$ nodes on a surface.

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