Self-normalized Cramér type moderate deviations for the maximum of sums

WEIDONG LIU¹, QI-MAN SHAO² and QIYING WANG³

¹Department of Mathematics and Institute of Natural Sciences, Shanghai Jiao Tong University, Shanghai, China. E-mail: liuweidong99@gmail.com
²Department of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong. E-mail: maqmshao@ust.hk
³School of Mathematics and Statistics, University of Sydney, Australia. E-mail: qiying@maths.usyd.edu.au

Let $X_1, X_2, \ldots$ be independent random variables with zero means and finite variances, and let $S_n = \sum_{i=1}^{n} X_i$ and $V_n^2 = \sum_{i=1}^{n} X_i^2$. A Cramér type moderate deviation for the maximum of the self-normalized sums $\max_{1 \leq k \leq n} S_k / V_n$ is obtained. In particular, for identically distributed $X_1, X_2, \ldots$, it is proved that

$$P(\max_{1 \leq k \leq n} S_k / V_n \geq x) = \frac{1}{1 - \Phi(x)},$$

holds uniformly for $0 < x \leq o(n^{1/6})$ under the optimal finite third moment of $X_1$.

Keywords: independent random variables; maximum of self-normalized sums

1. Introduction and main results

Let $X_1, X_2, \ldots$ be a sequence of independent non-degenerate random variables with zero means. Set

$$S_n = \sum_{j=1}^{n} X_j \quad \text{and} \quad V_n^2 = \sum_{j=1}^{n} X_j^2.$$

The past decade has brought significant developments in the limit theorems for the so-called “self-normalized” sum, $S_n / V_n$. It is now well understood that the limit theorems for $S_n / V_n$ usually require fewer moment assumptions than those for their classical standardized counterpart, and thus have much wider applicability. For examples, for identically distributed $X_1, X_2, \ldots$, a self-normalized large deviation holds without any moment assumption (Shao [11]), and a Cramér type moderate deviation (Shao [12]),

$$\lim_{n \to \infty} \frac{P(S_n \geq x V_n)}{1 - \Phi(x)} = 1,$$

holds uniformly for $x \in [0, o(n^{1/6})]$ provided that $E|X_1|^3 < \infty$, whereas a finite moment-generating condition of $\sqrt[3]{|X_1|}$ is necessary for a similar result for the standard sum $S_n / \sqrt{\text{Var}(S_n)}$ (see, e.g., Linnik [9]). For more related results, we refer to de la Peña, Lai and Shao [5] for a systematic treatment of the theory and applications of self-normalization and Wang [13] for some refined self-normalized moderate deviations.

1350-7265 © 2013 ISI/BS
As for the Cramér type moderate deviations for the maximum of self-normalized sums, namely for \( \max_{1 \leq k \leq n} S_k/V_n \), Hu, Shao and Wang [7] were the first to prove that if \( X_1, X_2, \ldots \) is a sequence of i.i.d. random variables with \( \operatorname{E} X_1^4 < \infty \), then

\[
\lim_{n \to \infty} \frac{\operatorname{P}(\max_{1 \leq k \leq n} S_k \geq x V_n)}{1 - \Phi(x)} = 2,
\]

(1.2)

uniformly for \( x \in [0, o(n^{1/6})] \). This contrasts with the moderate deviation result for the maximum of partial sums of Aleshkyavichene [1,2], where a finite moment-generating condition is required. However, in view of the result given in (1.1), it is natural to ask whether a finite third moment suffices for (1.2). The main purpose of this paper is to provide an affirmative answer to this question. Indeed, we have the following more general result for independent random variables.

**Theorem 1.** Assume that \( \max_{k \geq 1} \operatorname{E}|X_k|^{2+r} < \infty \) and \( \min_{k \geq 1} \operatorname{E}X_k^2 > 0 \), where \( 0 < r \leq 1 \). Then (1.2) holds uniformly in \( 0 \leq x \leq o(n^r/(4+2r)) \).

As in the moderate deviation result for self-normalized sum \( S_n/V_n \), Theorem 1 is sharp in both the moment condition and the range in which the result (1.2) holds true. Examples can be constructed similarly as done by Chistyakov and Götze [4] and Shao [12]. In particular, for \( r = 1 \) and identically distributed \( X_1, X_2, \ldots \), Theorem 1 establishes (1.2) under the optimal finite third moment of \( X_1 \).

Theorem 1 can be extended further; in fact, it is a direct consequence of Theorem 2 below. Set

\[
B_n^2 = \sum_{i=1}^n \operatorname{E}X_i^2, \quad L_{n,r} = \sum_{i=1}^n \operatorname{E}|X_i|^{2+r} \quad \text{and} \quad d_{n,r} = B_n/L_{n,r}^{1/(2+r)}, \quad \text{where} \quad 0 < r \leq 1.
\]

**Theorem 2.** For \( 0 < r \leq 1 \), suppose that \( d_{n,r} \to \infty \) as \( n \to \infty \), and that

\[
\frac{\sum_{j=k}^n \operatorname{E}|X_j|^{2+r}}{\sum_{j=k}^n \operatorname{E}|X_j|^2} \leq \frac{\tau L_{n,r} r/(2+r)}{d_{n,r}^{\delta}} \quad \text{for some} \ \delta, \tau > 0.
\]

(1.3)

Then (1.2) holds uniformly in \( 0 \leq x \leq \min\{B_n, o(d_{n,r})\} \).

**Remark 1.** For i.i.d. random variables with \( \operatorname{E}X_i = 0 \) and \( \operatorname{E}|X_i|^3 < \infty \), Jing, Shao and Wang [8] proved that (1.1) can be refined as

\[
\frac{\operatorname{P}(S_n \geq x V_n)}{1 - \Phi(x)} = 1 + O(1)(1 + x^3)\operatorname{E}|X_1|^3/(\operatorname{E}X_1^2)^{3/2}
\]

uniformly in \( x \in [0, n^{1/6}(\operatorname{E}X_1^2)^{1/3}](\operatorname{E}X_1^2)^{1/2}) \), where \( O(1) \) is bounded by an absolute constant. We conjecture that a similar result holds for \( \max_{1 \leq k \leq n} S_k/V_n \), that is,

\[
\frac{\operatorname{P}(\max_{1 \leq k \leq n} S_k \geq x V_n)}{1 - \Phi(x)} = 2 + O(1)(1 + x^3)\operatorname{E}|X_1|^3/(\operatorname{E}X_1^2)^{3/2}
\]

uniformly in \( x \in [0, n^{1/6}(\operatorname{E}X_1^2)^{1/2}](\operatorname{E}X_1^3)^{1/3}) \).
This paper is organized as follows. The proof of the main theorems is given in the next section. The proofs of two technical propositions are deferred to Sections 3 and 4, respectively. Throughout the paper, \( A, A_1, \ldots \) denotes absolute constants and \( C_{\delta, \tau} \) denotes a constant depending only on \( \delta \) and \( \tau \), which might be different at each appearance.

2. Proofs of theorems

**Proof of Theorem 1.** Simple calculations show that if

\[
\max_{k \geq 1} \mathbb{E}|X_k|^{2+r} < \infty \quad \text{and} \quad \min_{k \geq 1} \mathbb{E}X_k^2 > 0,
\]

then \( B_n^2 \asymp n, L_{n,r} \asymp n, d_{n,r} \asymp n^{r/(4+2r)} \) and (1.3) holds for \( \delta = 1 \) and some \( \tau > 0 \), where the notation \( a_n \asymp b_n \) denotes \( 0 < \lim_{n \to \infty} a_n/b_n < \infty \). Therefore, Theorem 1 follows immediately from Theorem 2.

**Proof of Theorem 2.** First note that for \( \forall \epsilon > 0 \),

\[
\frac{1}{B_n^2} \sum_{i=1}^{n} \mathbb{E}X_i^2 I(|X_i| \geq \epsilon B_n) \leq \epsilon^{1-\frac{1}{r}} - \frac{1}{\Phi(\frac{B_n}{\Delta_1 n, x})} \to 0
\]

whenever \( d_{n,r} \to 0 \). That is, the Lindeberg condition is satisfied for the sequence \( X_1, X_2, \ldots \). On the other hand, routine calculations show that, given \( d_{n,r} \to 0 \), \( V_n^2/B_n^2 \to 1 \) in probability. Given these facts, the invariance principle (see Theorem 2 of Brown [3]) and the continuous mapping theorem imply that \( \max_{1 \leq k \leq n} S_k/V_n \to D |N(0, 1)| \). This yields (1.2) uniformly for \( 0 \leq x \leq M \), where \( M \) is an arbitrary constant. Thus, Theorem 2 will follow if we can prove

\[
\lim_{M \to \infty} \lim_{n \to \infty} \sup_{M \leq x \leq \min\{B_n, o(d_{n,r})\}} \left| \frac{P(\max_{1 \leq k \leq n} S_k \geq x V_n)}{1 - \Phi(x)} - 2 \right| = 0. \tag{2.1}
\]

Toward this end, let

\[
\Delta_{n,x} = \frac{x^2}{B_n^2} \sum_{i=1}^{n} \mathbb{E}X_i^2 [|X_i| > B_n/x] + \frac{x^3}{B_n^3} \sum_{i=1}^{n} \mathbb{E}|X_i|^3 I(|X_i| \leq B_n/x),
\]

and write

\[
n_0 \equiv n_0(x) = \max \left\{ k : \sum_{j=k}^{n} \mathbb{E}X_j^2 \geq 192 B_n^2 \log(x \vee e)/x^2, 1 \leq k \leq n \right\}. \tag{2.2}
\]

It can be readily seen that the condition (1.3), together with \( 0 < x \leq \min\{B_n, o(d_{n,r})\} \) and \( d_{n,r} \to \infty \), imply the existence of an absolute constant \( A \) such that

\[
0 \leq x \leq B_n, \quad \Delta_{n,x} \leq \min(\delta^{9/2}, 1)/A. \tag{2.3}
\]
\( \Delta_{n,x} \rightarrow 0 \), and
\[
\frac{\sum_{j=n_0+1}^{n} \mathbb{E}|X_j|^3I\{|X_j| \leq B_n/x\}}{\sum_{j=n_0+1}^{n} \mathbb{E}|X_j|^2} \leq \frac{B_n}{x^{1+\delta}} \tag{2.4}
\]
for all sufficiently large \( n \), where \( \delta \) is defined as in (1.3). The result (2.1) follows immediately from the following proposition.

**Proposition 1.** For all \( x \geq 2 \) satisfying (2.3) and (2.4), we have
\[
\frac{\mathbb{P}(\max_{1 \leq k \leq n} S_k \geq x V_n)}{1 - \Phi(x)} = 2 + O(1)\left(x^{-\min[1/4, \delta/20]} + \Delta_{n,x}^{1/9}\right), \tag{2.5}
\]
where \( O(1) \) is bounded by a constant \( C_\delta \) that depends only on \( \delta \).

The main idea of the proof of Proposition 1 is to use truncation and the maximum probability inequality and then apply a moderate deviation theorem of Sakhanenko [10] to the truncated variables. A suitable truncation level is ensured by using an inequality from Jing, Shao and Wang [8], page 2181. This avoids the conjugate argument of Hu, Shao and Wang [7], and makes it possible to prove the main result under an optimal moment assumption.

It remains to prove Proposition 1. In addition to the notation in the previous section, let \( \gamma = 72^{-1} \min(\delta, 1) \),
\[
\varepsilon = \max(2\Delta_{n,x}^{2/9}, \gamma x^{-1/2}, \gamma^{-\delta/10}), \quad m = [x^2/2],
\]
\( N_0 = \emptyset \) and, for \( 1 \leq l \leq m \), \( N_l = \{j_1, j_2, \ldots, j_l\} \subseteq \{1, 2, \ldots, n\} \). Furthermore, write \( \tilde{X}_i = X_i I\{|X_i| \leq \varepsilon B_n/x\} \), and for \( 0 \leq l \leq m \) and \( 1 \leq k \leq n \),
\[
\tilde{S}_k^N = \sum_{i=1}^{k} \tilde{X}_i, \quad (\tilde{V}_n^N)^2 = \sum_{i=1}^{n} \tilde{X}_i^2, \quad (\tilde{B}_n^N)^2 = \sum_{i=1}^{n} \mathbb{E}\tilde{X}_i^2,
\]
\[
S_k^N = \sum_{i=1}^{k} X_i, \quad (V_n^N)^2 = \sum_{i=1}^{n} X_i^2, \quad (B_n^N)^2 = \sum_{i=1}^{n} \mathbb{E}X_i^2.
\]
Note that if \( s, t \in R^1 \), \( x \geq 1 \), \( c \geq 0 \) and \( s + t \geq \sqrt{x^2 + c} \), then \( s \geq (x^2 - 1)^{1/2} \sqrt{c} \). Similar to the arguments in reported by Jing, Shao and Wang [8], page 2181, we have
\[
\mathbb{P}\left(\max_{1 \leq k \leq n} S_k \geq x V_n\right) \\ \\ \leq \mathbb{P}\left(\max_{1 \leq k \leq n} \tilde{S}_k^{N_0} \geq x \tilde{V}_n^{N_0}\right) \\ \\ + \sum_{j_1=1}^{n} \mathbb{P}\left(\max_{1 \leq k \leq n} S_k \geq x V_n, |X_{j_1}| \geq \varepsilon B_n/x\right) \tag{2.6}
\]
\[ \leq P\left( \max_{1 \leq k \leq n} \tilde{S}_k^{N_0} \geq x\tilde{V}_n^{N_0} \right) \]

\[ + \sum_{j_1=1}^{n} P\left( \max_{1 \leq k \leq n} S_k^{N_1} \geq \sqrt{x^2 - 1}V_n^{N_1} \right) P(|X_{j_1}| \geq \varepsilon B_n/x) \]

and

\[ P\left( \max_{1 \leq k \leq n} S_k \geq xV_n \right) \]

\[ \geq P\left( \max_{1 \leq k \leq n} \tilde{S}_k^{N_0} \geq x\tilde{V}_n^{N_0} \right) \]

\[ - \sum_{j_1=1}^{n} P\left( \max_{1 \leq k \leq n} \tilde{S}_k^{N_1} \geq \sqrt{x^2 - 1}\tilde{V}_n^{N_1} \right) P(|X_{j_1}| \geq \varepsilon B_n/x). \]

Repeating (2.6) \( m \)–times gives

\[ P\left( \max_{1 \leq k \leq n} S_k \geq xV_n \right) \]

\[ \leq P\left( \max_{1 \leq k \leq n} \tilde{S}_k^{N_0} \geq x\tilde{V}_n^{N_0} \right) \]

\[ + \sum_{k=1}^{m} Z_k(x) + \left\{ \sum_{k=1}^{n} P(|X_k| \geq \varepsilon B_n/x) \right\}^{m+1}, \]

where

\[ Z_k(x) = \sum_{j_1=1}^{n} \cdots \sum_{j_k=1}^{n} \left[ \prod_{l=1}^{k} P(|X_{j_l}| \geq \varepsilon B_n/x) \right] \times P\left( \max_{1 \leq j \leq n} \tilde{S}_j^{N_k} \geq \sqrt{x^2 - k\tilde{V}_n^{N_k}} \right). \]

Note that

\[ \sum_{k=1}^{n} P(|X_k| \geq \varepsilon B_n/x) \]

\[ \leq \frac{x^2}{\varepsilon^2 B_n^2} \sum_{k=1}^{n} \mathbb{E}|X_k|^2 I\{|X_k| \geq \varepsilon B_n/x\} \]

\[ \leq \frac{x^2}{\varepsilon^2 B_n^2} \sum_{k=1}^{n} \mathbb{E}|X_k|^2 I\{|X_k| \geq B_n/x\} + \frac{x^3}{\varepsilon^3 B_n^3} \sum_{k=1}^{n} \mathbb{E}|X_k|^3 I\{|X_k| \leq B_n/x\} \]

\[ \leq \varepsilon^{-3} \Delta_{n,x} \leq \varepsilon^{3/2}/16 \leq 1/16. \]
Self-normalized moderate deviation for maximum

It follows from \( m = \lceil x^2 / 2 \rceil \) that

\[
\left[ \sum_{k=1}^{n} P(|X_k| \geq \varepsilon B_n / x) \right]^{m+1} \leq e^{-x^2}. \tag{2.10}
\]

This, together with (2.7) and (2.8), implies that Proposition 1 will follow if we prove the following two propositions.

**Proposition 2.** For all \( 0 \leq l \leq m \), all \( x/2 \leq y \leq x \), and all \( x \geq 2 \) satisfying (2.3) and (2.4), we have

\[
P\left( \frac{\max_{1 \leq k \leq n} \tilde{S}_k^{N_l} \geq y \tilde{V}_n^{N_l}}{1 - \Phi(y)} \right) \leq 2 + C_{\delta, \tau} (\varepsilon^{-2} \Delta_{n,x} + \varepsilon). \tag{2.11}
\]

**Proposition 3.** For all \( x \geq 2 \) satisfying (2.3) and (2.4), we have

\[
P\left( \frac{\max_{1 \leq k \leq n} \tilde{S}_k^{N_0} \geq x \tilde{V}_n^{N_0}}{1 - \Phi(x)} \right) = 2 + C_{\delta, \tau} (\varepsilon^{-2} \Delta_{n,x} + \varepsilon). \tag{2.12}
\]

Indeed, noting that

\[
\frac{x}{\sqrt{2\pi(1 + x^2)}} e^{-x^2/2} \leq 1 - \Phi(x) \leq \frac{1}{\sqrt{2\pi x}} e^{-x^2/2}
\]

for \( x \geq 1 \), we have that for \( 1 \leq k \leq m = \lceil x^2 / 2 \rceil \) and \( x \geq 1 \),

\[
\frac{1 - \Phi(\sqrt{x^2 - k})}{1 - \Phi(x)} \leq 2 e^{k/2}.
\]

This, together with (2.8)–(2.11), implies that for all \( x \geq 2 \) satisfying (2.3) and (2.4),

\[
P\left( \max_{1 \leq k \leq n} S_k \geq x V_n \right) \leq e^{-x^2} + 2 \left\{ 1 - \Phi(x) + \sum_{k=1}^{m} \left[ 1 - \Phi(\sqrt{x^2 - k}) \right] \left\{ \sum_{j=1}^{n} P(|X_j| \geq \varepsilon B_n / x) \right\} \right\}^k \times \left\{ 1 + C_{\delta, \tau} (\varepsilon^{-2} \Delta_{n,x} + \varepsilon) \right\}
\]

\[
\leq 2 \left( 1 - \Phi(x) \right) \left\{ 1 + C_{\delta, \tau} (\varepsilon^{-3} \Delta_{n,x} + \varepsilon + x^{-1}) \right\}
\]

\[
\leq 2 \left( 1 - \Phi(x) \right) \left\{ 1 + C_{\delta, \tau} (x^{-\min\{1/4, \delta/20\}} + \Delta_{n,x}^{1/9}) \right\}.
\tag{2.13}
\]

Similarly, by (2.7), (2.11) and (2.12), we obtain that for all \( x \geq 2 \) satisfying (2.3) and (2.4),

\[
P\left( \max_{1 \leq k \leq n} S_k \geq x V_n \right) \geq 2 \left( 1 - \Phi(x) \right) \left\{ 1 - C_{\delta, \tau} (x^{-\min\{1/4, \delta/20\}} + \Delta_{n,x}^{1/9}) \right\}.
\tag{2.14}
\]

Combining (2.13) and (2.14), we obtain (2.5), and thus Proposition 1.
It remains to prove Propositions 2 and 3, which we give in Sections 3 and 4, respectively. The proof of Theorem 2 is now complete. □

3. Proof of Proposition 2

Let $b = y/B_n^{N_l}$. First, note that

$$
P\left(\max_{1 \leq k \leq n} \tilde{\gamma}_{k}^{N_l} \geq y\bar{V}_n^{N_l}, \left|b\bar{V}_n^{N_l} - y\right| \geq \varepsilon\right) \leq P\left(\max_{1 \leq k \leq n} \tilde{\gamma}_{k}^{N_l} \geq (b\bar{V}_n^{N_l})^2 + y^2 - \varepsilon^2\right)
$$

(3.1)

Furthermore, we have

$$
P\left(\max_{1 \leq k \leq n} \tilde{\gamma}_{k}^{N_l} \geq y\bar{V}_n^{N_l}, \left|b\bar{V}_n^{N_l} - y\right| \geq \varepsilon\right)
\leq P\left(\max_{1 \leq k \leq n} \tilde{\gamma}_{k}^{N_l} \geq y\bar{V}_n^{N_l}, b^2(\bar{V}_n^{N_l})^2 > y^2 + \varepsilon y\right)
$$

(3.2)

$$
+ P\left(\max_{1 \leq k \leq n} \tilde{\gamma}_{k}^{N_l} \geq y\bar{V}_n^{N_l}, b^2(\bar{V}_n^{N_l})^2 < y^2 - \varepsilon y\right)
=: I_1 + I_2
$$

and

$$
P\left(2b \max_{1 \leq k \leq n} \tilde{\gamma}_{k}^{N_l} \geq b^2(\bar{V}_n^{N_l})^2 + y^2 - \varepsilon^2\right)
\leq P\left(\bigcup_{k=1}^{n} \{2b\tilde{\gamma}_{k}^{N_l} \geq b^2(\bar{V}_n^{N_l})^2 + y^2 - \varepsilon^2, \ E[(\bar{V}_n^{N_l})^2 - (\bar{V}_k^{N_l})^2] - [(\bar{V}_n^{N_l})^2 - (\bar{V}_k^{N_l})^2] \geq \varepsilon^2/b^2\} \right)
$$

(3.3)

$$
+ P\left(\bigcup_{k=1}^{n} \{2b\tilde{\gamma}_{k}^{N_l} \geq b^2(\bar{V}_k^{N_l})^2 + b^2E[(\bar{V}_n^{N_l})^2 - (\bar{V}_k^{N_l})^2] + y^2 - 2\varepsilon^2\} \right)
=: I_3 + I_4.
$$

By (3.1)–(3.3), Proposition 2 follows from the following Lemma 1.

Lemma 1. Under the conditions of Proposition 2, we have

$$
I_1 \leq C_{\delta,\gamma} y^{-2} \exp(-y^2/2),
$$

(3.4)

$$
I_2 \leq C_{\delta,\gamma} y^{-2} \exp(-y^2/2),
$$

(3.5)
which, together with (3.9), implies that
\[ I_3 \leq C_{\delta, \tau} y^{-2} \exp(-y^2/2), \]  
(3.6)
\[ I_4 \leq 2[1 - \Phi(y)][1 + C_{\delta, \tau} (\varepsilon^{-2} \Delta_{n,x} + \varepsilon)]. \]  
(3.7)

To prove Lemma 1, we start with some preliminaries. Note that
\[ \gamma \max\{x^{-1/2}, x^{-\delta/10}\} \leq \varepsilon \leq \min\{1/24, \delta/72\}, \quad \Delta_{n,x} \leq (\varepsilon/2)^{9/2}. \]  
(3.8)
This fact (3.8) is repeatedly used in the proof without further explanation. Define \( k_0 = 0, k_T = n \) and \( k_i, 1 \leq i < T, \) by
\[ k_i = \max \left\{ k : \sum_{j=k_{i-1}+1}^{k} \mathbb{E}X_j^2 \leq 2^{-1} \varepsilon^3 B_n^2/x^2 \right\}. \]

By the definition of \( k_i, \)
\[ \sum_{j=k_{i-1}+1}^{k_i} \mathbb{E}X_j^2 \leq 2^{-1} \varepsilon^3 B_n^2/x^2 \quad \text{and} \quad \sum_{j=k_{i-1}+1}^{k_{i+1}} \mathbb{E}X_j^2 > 2^{-1} \varepsilon^3 B_n^2/x^2 \]  
(3.9)
for any \( 1 \leq i < T. \) By (2.3) and (3.8),
\[ x^2 \max_{1 \leq k \leq n} \mathbb{E}X_k^2 \leq x^2 \max_{1 \leq k \leq n} [\mathbb{E}X_k^2 (|X_k| > B_n/x)] + (\mathbb{E}|X_k|^3 I\{|X_k| \leq B_n/x\})^{2/3} \]
\[ \leq B_n^2 (\Delta_{n,x} + \Delta_{n,x}^{2/3}) \leq \varepsilon^3 B_n^2/4, \]
(3.10)
which, together with (3.9), implies that
\[ \sum_{j=k_{i-1}+1}^{k_i} \mathbb{E}X_j^2 \geq 4^{-1} \varepsilon^3 B_n^2/x^2. \]

Therefore,
\[ (T - 1)4^{-1} \varepsilon^3 B_n^2/x^2 \leq \sum_{i=1}^{T-1} \sum_{j=k_{i-1}+1}^{k_i} \mathbb{E}X_j^2 \leq B_n^2, \]
which yields \( T \leq 4x^2/\varepsilon^3 + 1. \) For \( k_i - 1 + 1 \leq j \leq k_i - 1, \) define events
\[ A_j = \left\{ \tilde{S}_j^{N_j} \geq y \sqrt{(B_n^{N_j})^2 (1 + \varepsilon/y)} \right\}, \quad C_j = \left\{ \sum_{k=j+1, k \notin N_j}^{k_i} (\tilde{X}_k - \mathbb{E}\tilde{X}_k) \geq -\varepsilon B_n^{N_j}/y \right\}. \]

Note that \( \sum_{k \in N_j} \mathbb{E}X_k^2 \leq \varepsilon^3 B_n^2/8 \) for all \( 0 \leq l \leq m = \lfloor x^2/2 \rfloor \) by (3.10), and thus
\[ B_n^2 \geq (B_n^{N_j})^2 = \sum_{k=1}^{n} \mathbb{E}X_k^2 - \sum_{k \in N_l} \mathbb{E}X_k^2 \geq (1 - \varepsilon^3/8)B_n^2 \geq \frac{7}{8} B_n^2. \]  
(3.11)
Applying the Chebyshev inequality, we have, for any \( k_{i-1} \leq j \leq k_i \) and \( x/2 \leq y \leq x \),

\[
P(C_j) \geq 1 - \frac{y^2 \sum_{k=j+1}^{k_i} EX^2_k}{\epsilon^2 (B_n^N)^2} \geq 1 - 4\epsilon/7 \geq 1/2. \tag{3.12}
\]

We are now ready to prove Lemma 1.

Proof of (3.4). It follows from (3.12) and the independence between \( C_j \) and \( \{A_l, l \leq j\} \) that

\[
I_1 \leq \sum_{i=1}^{T} \left[ P(A_{k_{i-1}+1}) + \sum_{j=k_{i-1}+2}^{k_i} P(A_{k_{i-1}}^c, \ldots, A_{j-1}^c, A_j) \right] \leq 2 \sum_{i=1}^{T} P(\bar{S}_{k_i}^N - E \bar{S}_{k_i}^N \geq y\sqrt{(B_n^N)^2(1 + \epsilon/y) - \epsilon B_n^N/y - D_{k_i}}),
\]

where \( D_{k_i} = \sum_{j=1}^{k_i} E|X_j| I\{X_j| > \epsilon B_n/x\} \). Taking \( t = y\sqrt{1 + \epsilon/y}/B_n^N \) and noting

\[
t(\epsilon B_n/y + D_{k_i}) \leq 2\epsilon + 1,
\]

we have

\[
P(\bar{S}_{k_i}^N - E \bar{S}_{k_i}^N \geq y\sqrt{(B_n^N)^2(1 + \epsilon/y) - \epsilon B_n^N/y - D_{k_i}}) \leq 9 \exp(-y^2 - \epsilon y) \prod_{j=1, j \notin N_i}^{k_i} \exp(t(\bar{X}_j - E\bar{X}_j)) \leq 9 \exp(-y^2 - \epsilon y) \prod_{j=1, j \notin N_i}^{k_i} \left(1 + \frac{EX^2_j}{2}t^2 + 8t^3E|\bar{X}_j|^3 e^{2\epsilon B_n/x}\right) \leq 9 \exp(-y^2/2 - \epsilon y/2 + A\Delta_{n,x}).
\]

Submitting this estimate into (3.13) and recalling \( T \leq 4x^2/\epsilon^3 + 1, x/2 \leq y \leq x \) and \( \epsilon \geq \gamma x^{-1/2} \), we obtain

\[
I_1 \leq (4\epsilon^{-3}x^2 + 1) \exp(-y^2/2 - \epsilon y/2 + A\Delta_{n,x}) \leq C_{\delta, \tau} y^{-2} e^{-y^2/2}. \tag{3.14}
\]
This proves (3.4). □

**Proof of (3.5).** For this part, let $Y_{k_i} = \sum_{j=k_i-1+1, j \notin N_l} \bar{X}_j^2$, and define

$$\tilde{A}_j = \{ \bar{S}_{N_l}^{N_j} \geq y \sqrt{(\bar{V}_{N_l}^{N_j})^2 - Y_{k_i}}, b^2[(\bar{V}_{N_l}^{N_j})^2 - Y_{k_i}] < y^2 - \varepsilon y \}, \quad 1 \leq j \leq n.$$  

From (3.12) and the independence between $C_j$ and $\{\tilde{A}_l, l \leq j\}$, it follows that

$$I_2 \leq \sum_{i=1}^{T} \left[ \mathbb{P}(A_{k_i-1+1}) + \sum_{j=k_i-1+2}^{k_i} \mathbb{P}(A_{k_i-1}, \ldots, A_{j-1}, A_j) \right]$$

$$\leq 2 \sum_{i=1}^{T} \left[ \mathbb{P}(A_{k_i-1+1}, C_{k_i-1+1}) + \sum_{j=k_i-1+2}^{k_i} \mathbb{P}(A_{k_i-1}, \ldots, A_{j-1}, A_j, C_j) \right]$$

$$\leq 2 \sum_{i=1}^{T} \mathbb{P}(\bar{S}_{k_i}^{N_j} - \mathbb{E}\bar{S}_{k_i}^{N_j} \geq y \sqrt{(\bar{V}_{N_l}^{N_j})^2 - Y_{k_i} - \varepsilon B_{N_l}^{N_i}/y - D_{k_i}}, b^2[(\bar{V}_{N_l}^{N_j})^2 - Y_{k_i}] < y^2 - \varepsilon y)$$

$$=: 2 \sum_{i=1}^{T} I_{2,i},$$

where, as before, $D_{k_i} = \sum_{j=1}^{k_i} \mathbb{E}|X_j| I(|X_j| > \varepsilon B_n/x)$. Furthermore, for $i = 1, \ldots, T$,

$$I_{2,i} \leq \mathbb{P}((\bar{V}_{N_l}^{N_j})^2 - Y_{k_i} < (1 - \varepsilon)(B_{N_l}^{N_i})^2)$$

$$+ \sum_{k=1}^{[y]} \mathbb{P}(\bar{S}_{k_i}^{N_j} - \mathbb{E}\bar{S}_{k_i}^{N_j} \geq y \sqrt{(B_{N_l}^{N_i})^2[1 - (k + 1)\varepsilon/y] - \varepsilon B_{N_l}^{N_i}/y - D_{k_i}},$$

$$(B_{N_l}^{N_i})^2[1 - (k + 1)\varepsilon/y] < (\bar{V}_{N_l}^{N_j})^2 - Y_{k_i} < (B_{N_l}^{N_i})^2[1 - k\varepsilon/y])$$

$$=: I_{2,i,0} + \sum_{k=1}^{[y]} I_{2,i,k}.$$

Note that, for any $t_1 \geq 0$ and $t_2 \geq 0$,

$$\mathbb{E}\exp(t_1(\bar{X}_k - \mathbb{E}\bar{X}_k) + t_2(\mathbb{E}\bar{X}_k^2 - \bar{X}_k^2))$$

$$\leq 1 + \frac{1}{2}\mathbb{E}(t_1(\bar{X}_k - \mathbb{E}\bar{X}_k) + t_2(\mathbb{E}\bar{X}_k^2 - \bar{X}_k^2))^2$$

$$+ (8t_1^2\mathbb{E}|\bar{X}_k|^3 + 8t_2^2\mathbb{E}|\bar{X}_k|^6)\mathbb{E}2t_1\varepsilon B_n/x + t_2\mathbb{E}X_k^2$$

$$\leq \exp\left(\frac{1}{2}t_1^2\mathbb{E}\bar{X}_k^2 + \frac{1}{2}(4t_1t_2 + t_2^2\varepsilon B_n/x)\mathbb{E}|\bar{X}_k|^3\right)$$

$$+ (8t_1^3 + 8t_2^3B_n^3/x^3)\mathbb{E}|\bar{X}_k|^3\mathbb{E}2t_1\varepsilon B_n/x + t_2\mathbb{E}X_k^2.$$  

(3.16)
Let $t_1 = y \sqrt{1 - (k + 1) \varepsilon / y} / B_{N_j}$ and $t_2 = \varepsilon^{-1} y^2 / (B_{N_j}^2)$. Noting that
$$t_1 (y B_{N_j} y + D_k) \leq \varepsilon + 1,$$
we have for $1 \leq k \leq [x]$,
$$I_{2,i,k} \leq P(t_1 (\tilde{S}_{ki}^{N_j} - E \tilde{S}_{ki}^{N_j}) + t_2 [E(\tilde{V}_n^{N_j})^2 - Y_{ki}] - [(\tilde{V}_n^{N_j})^2 - Y_{ki}])$$
$$\geq y^2 - (k + 1) \varepsilon / y + t_2 k \varepsilon (B_{N_j}^2) / y - 2$$
$$\leq \exp \left( -y^2 + (k + 1) \varepsilon / y - t_2 k \varepsilon (B_{N_j}^2) / y + 2 \right)$$
$$\times \prod_{k=1, k \notin N_i}^{k_{i-1}} \mathbb{E} \exp \left( t_1 (\tilde{X}_k - E \tilde{X}_k) + t_2 (E \tilde{X}_k^2 - \tilde{X}_k^2) \right)$$
$$\times \prod_{k=k_{i-1}+1, k \notin N_i}^{k_i} \mathbb{E} \exp \left( t_1 (\tilde{X}_k - E \tilde{X}_k) \right) \times \prod_{k=k_{i+1}, k \notin N_i}^{n} \mathbb{E} \exp \left( t_2 (E \tilde{X}_k^2 - \tilde{X}_k^2) \right)$$
$$\leq \exp \left( -y^2 / 2 + 2^{-1} (k + 1) \varepsilon / y - t_2 k \varepsilon (B_{N_j}^2) / y + 2 \right)$$
$$+ A(t_1 t_2 + t_2^2 \varepsilon B_n / x + t_1^3 + t_2^3 \varepsilon^3 B_n^3 / x^3) \sum_{k=1}^{n} \mathbb{E} |\tilde{X}_k|^3$$
$$\leq \exp \left( -y^2 / 2 + 2^{-1} (k + 1) \varepsilon / y - k y + A \varepsilon^{-1} \Delta_{n,x} + 2 \right)$$
$$\leq A \exp(-y^2 / 2 - y / 2).$$
Similarly, by (3.16) with $t_1 = 0$, we have
$$I_{2,i,0} \leq P(t_2 [E((\tilde{V}_n^{N_j})^2 - Y_{ki}) - [(\tilde{V}_n^{N_j})^2 - Y_{ki}]) \geq t_2 \varepsilon (B_{N_j}^2) - \varepsilon^2)$$
$$\leq A \exp(-y^2 + A \varepsilon^{-1} \Delta_{n,x}) \leq A_1 \exp(-y^2 / 2 - y).$$
Combining above inequalities yields
$$I_2 \leq A(4 x^2 / \varepsilon^3 + 1) e^{-y^2 / 2 - y} \leq A_1 y^{-2} e^{-y^2 / 2}. \quad (3.17)$$
The proof of (3.5) is now complete. \hfill \Box

**Proof of (3.6).** Following the arguments in the estimates of $I_1$ and $I_2$, we have
$$I_3 \leq \sum_{i=1}^{T} P \left( \bigcup_{j=k_{i-1}+1}^{k_i} \{ 2b \tilde{S}_{j}^{N_j} \geq b^2 (\tilde{V}_n^{N_j})^2 + y^2 - \varepsilon^2, \right.$$}
$$\left. E[(\tilde{V}_n^{N_j})^2 - (\tilde{V}_j^{N_j})^2] - [(\tilde{V}_n^{N_j})^2 - (\tilde{V}_j^{N_j})^2] \geq \varepsilon^2 (B_{N_j}^2 / y^2) \right) \quad (3.18)$$
Self-normalized moderate deviation for maximum

\[
\leq \sum_{i=1}^{T} P \left( \bigcup_{j=k_i - 1 + 1}^{k_i} \{ 2b \tilde{S}_j^{N_i} \geq b^2 \left[ (\tilde{V}_n^{N_i})^2 - Y_{k_i} \right] + y^2 - \varepsilon^2 \} \right),
\]

\[
\sum_{k=k_i+1, k \notin N}^n (E \bar{X}_k^2 - \bar{X}_k^2) \geq 2^{-1} \varepsilon^2 (B_n^{N_i})^2 / y^2 \]

\[
\leq 2 \sum_{i=1}^{T} P \left( 2b \tilde{S}_{k_i}^{N_i} \geq b^2 \left[ (\tilde{V}_n^{N_i})^2 - Y_{k_i} \right] + y^2 - 2\varepsilon, \right.
\]

\[
\sum_{k=k_j+1, k \notin N}^n (E \bar{X}_k^2 - \bar{X}_k^2) \geq 2^{-1} \varepsilon^2 (B_n^{N_i})^2 / y^2 \]

\[
=: 2 \sum_{i=1}^{T} I_{3i}.
\]

As in the proof of (3.16), it can be easily shown that for \( \alpha \geq 0 \),

\[
E e^{b \bar{X}_j - \alpha b^2 \bar{X}_j^2} \leq \exp \left\{ (1/2 - \alpha) b^2 E \bar{X}_j^2 + A \Delta_{n,x}^{(j)} \right\}, \tag{3.19}
\]

where

\[
\Delta_{n,x}^{(j)} = \frac{x^2}{B_n^2} E X_j^2 I \{ |X_j| \geq \varepsilon B_n / x \} + \frac{x^3}{B_n^3} E |X_j|^3 I \{ |X_j| \leq \varepsilon B_n / x \}
\]

\[
\leq \varepsilon^{-1} \left( \frac{x^2}{B_n^2} E X_j^2 I \{ |X_j| \geq B_n / x \} + \frac{x^3}{B_n^3} E |X_j|^3 I \{ |X_j| \leq B_n / x \} \right)
\]

and

\[
E e^{\alpha (E \bar{X}_j^2 - \bar{X}_j^2) - b^2 \bar{X}_j^2 / 2} \leq \exp \left\{ -\frac{1}{2} b^2 E \bar{X}_j^2 + (2\alpha^2 B_n / x + x^3 / B_n^3) \varepsilon E |\bar{X}_j|^3 e^{\alpha \max_{1 \leq k \leq n} E X_k^2} \right\}. \tag{3.20}
\]

Next, let \( t \) satisfy

\[
t e^{t \max_{1 \leq k \leq n} E X_k^2} = \frac{\varepsilon B_n}{24x \sum_{j=k_i+1}^{n} E |\bar{X}_j|^3}.
\]

Clearly \( t \) exists. Furthermore, we have \( t \geq x^2 / B_n^2 \). Indeed, if \( t \max_{1 \leq k \leq n} E X_k^2 \geq \varepsilon \), then by (3.10) and recalling \( \varepsilon \leq 1/24 \),

\[
t \geq \varepsilon / \max_{1 \leq k \leq n} E X_k^2 \geq 4\varepsilon^{-2} x^2 / B_n^2 \geq x^2 / B_n^2.
\]
If \( t \max_{1 \leq k \leq n} \mathbb{E}X_k^2 \leq \varepsilon \), then
\[
\begin{align*}
t &\geq \frac{\varepsilon B_n}{24 e^x \sum_{j=k+1}^n \mathbb{E}|\tilde{X}_j|^3} \geq \frac{\varepsilon x^2}{30 B_n^3 \Delta_n x} \geq \frac{1}{15} \Delta_n^{-7/9} x^2 / B_n^2 \geq x^2 / B_n^2.
\end{align*}
\]
Now it follows from (3.19) and (3.20) with \( \alpha = t \) that
\[
I_{3i} \leq P \left( b \tilde{S}_{ki} - 2^{-1} b^2 [(V_n^{N_i})^2 - t] \sum_{k=k+1}^n (\mathbb{E}X_k^2 - \bar{X}_k^2) \right)
\geq y^2 / 2 - 2 \varepsilon + 2^{-1} \varepsilon / y^2
\leq \exp \left[ 2 \varepsilon - y^2 / 2 - \frac{\varepsilon^2 (B_n^N)^2}{2 y^2} \right] \prod_{j=1, j \notin N_i}^{k_i-1} \exp b \tilde{X}_j - 2^{-1} b^2 \bar{X}_j^2
\times \prod_{j=k_i+1, j \notin N_i}^{k_i} \exp b \tilde{X}_j \times \prod_{j=k_i+1, j \notin N_i}^n \exp -2^{-1} b^2 \bar{X}_j^2 + t (\mathbb{E}X_j^2 - \bar{X}_j^2)
\leq A \exp (-y^2 / 2) \exp \left( \frac{e^{-1} \Delta_n x - \varepsilon^2 B_n^2 t}{3 x^2} - \frac{y^2 \sum_{j=k_i+1}^n \mathbb{E}X_j^2}{2 (B_n^N)^2} + \frac{y^2 \sum_{j=k_i+1}^n \mathbb{E}X_j^2}{2 (B_n^N)^2} \right)
\leq A \exp (-y^2 / 2) \exp \left( -\frac{e^{-1} B_n^2}{4 x^2} - \frac{y^2 \sum_{j=k_i+1}^n \mathbb{E}X_j^2}{4 B_n^2} \right).
\]
Note that when \( t \leq \frac{2 \varepsilon x^2 \log x}{B_n^3 \varepsilon^3} \), \( t \max_{1 \leq k \leq n} \mathbb{E}X_k^2 < \frac{\delta}{2} \log x \) by (3.10). Thus, by the definition of \( t \),
\[
\begin{align*}
t &\geq \frac{\varepsilon B_n}{24 x^{1+\delta / 2} \sum_{j=k+1}^n \mathbb{E}|\tilde{X}_j|^3}.
\end{align*}
\]
Now considering \( t \leq \frac{2 \varepsilon x^2 \log x}{B_n^3 \varepsilon^3} \) and \( t \geq \frac{2 \varepsilon x^2 \log x}{B_n^3 \varepsilon^3} \), we have, by (3.21),
\[
I_{3i} \leq A e^{-y^2 / 2} \exp (-y^2 / 2)
+ A e^{-y^2 / 2} \exp \left( -\frac{y^2 \sum_{j=k_i+1}^n \mathbb{E}X_j^2}{4 B_n^2} - \frac{e \varepsilon^3 B_n^3}{144 x^{3+\delta / 2} \sum_{j=k_i+1}^n \mathbb{E}|\tilde{X}_j|^3}. \right)
\]
From the definition of $n_0$, $\sum_{j=n_0+1}^{n} E X_j^2 \leq 192 B_n^2 x^{-2} \log x$ and thus by (2.4)

$$\sum_{j=n_0+1}^{n} E |\bar{X}_j|^3 \leq \frac{192 \tau B_n^3 \log x}{x^{3+\delta}}.$$ 

For $i < i_0$, where $i_0 = \max\{i: k_i + 1 \leq n_0\}$, we have

$$y^2 \sum_{j=k_i}^{n} E X_j^2 \geq x^2 \sum_{j=n_0}^{n} E X_j^2 / 4 \geq 24 B_n^2 \log x.$$ 

It now follows from (3.22), (3.8) and the fact $T \leq 4 x^2 / \varepsilon^3 + 1$ that

$$I_3 \leq 2 \sum_{i=1}^{T} I_{3i} \leq 2 A T y^{-\delta/(3\varepsilon)} e^{-y^2/2} + 2 A e^{-y^2/2} i_0 \varepsilon \log x$$

$$+ 2 A e^{-y^2/2} \sum_{i=i_0+1}^{T} \exp \left( - \frac{\varepsilon^3 B_n^3}{144 x^{3+\delta/2} \sum_{j=k_i+1}^{n} E |\bar{X}_j|^3} \right)$$

$$\leq A_1 \left( 4 x^2 / \varepsilon^3 + 1 \right) e^{-y^2/2} (y^{-6} + e^{-4x^2 \varepsilon^3 / \log x})$$

$$\leq C_{\delta, \tau} y^{-2} e^{-y^2/2}.$$ 

This completes the proof of (3.6). \(\square\)

**Proof of (3.7).** For this result, we need the following moderate deviation theorem for the standardized sum due to Sakhanenko [10] (also see Heinrich [6]).

**Lemma 2.** Suppose that $\eta_1, \ldots, \eta_n$ are independent random variables such that $E \eta_j = 0$ and $|\eta_j| \leq 1$ for $j \geq 1$. Write $\sigma_n^2 = \sum_{j=1}^{n} E \eta_j^2$ and $L_n = \sum_{j=1}^{n} E |\eta_j|^3 / \sigma_n^3$. Then there exists an absolute constant $A > 0$ such that for all $1 \leq x \leq \min\{\sigma_n, L_n^{-1/3}\} / A$,

$$\frac{P(\sum_{j=1}^{n} \eta_j \geq x \sigma_n)}{1 - \Phi(x)} = 1 + O(1) x^3 L_n,$$ 

where $|O(1)|$ is bounded by an absolute constant.

To prove (3.7), write

$$\xi_j = 2b \bar{X}_j - b^2 \bar{X}_j^2 + b^2 E \bar{X}_j^2,$$

$$E_j = \left\{ \sum_{k=1, k \notin N_l}^{j} \xi_k \geq z \right\}, \quad \text{where} \ z = 2(y^2 - \varepsilon^2).$$
Note that $|\xi_j - E\xi_j| \leq 4\varepsilon + 2\varepsilon^2 \leq 5\varepsilon$, and by the non-uniform Berry–Esseen bound, there exists an absolute constant $A_0$ such that for any $1 \leq k \leq n$ and $c > 0$,

$$
P \left( \sum_{j = k, j \notin N_l}^n (\xi_j - E\xi_j) \leq -c\varepsilon \right)
\leq 1 - \Phi(t) + \frac{A_0 \sum_{j = k, j \notin N_l}^n E|\xi_j - E\xi_j|^3}{(1 + t)^3 s_{n,k}^3}
\leq \frac{1}{2} \left( 1 - \frac{1}{\sqrt{2\pi}} \int_0^t e^{-s^2/2} ds + \frac{5A_0}{c} (1 + t)^{-3} t, \right)
$$

where $s_{n,k}^2 = \sum_{j = k, j \notin N_l}^n \text{Var}(\xi_j)$ and $t = c\varepsilon / s_{n,k}$. Because $\int_0^t e^{-s^2/2} ds \geq t(1 + t)^{-3}/2$ for any $t \geq 0$, we may choose $c_0 \geq 10A_0\sqrt{2\pi}$ such that for all $1 \leq k \leq n$,

$$
P \left( \sum_{j = k, j \notin N_l}^n (\xi_j - E\xi_j) \leq -c_0\varepsilon \right) \leq 1/2. \quad (3.25)
$$

By virtue of (3.25), we obtain that

$$
I_4 = P(E_1) + \sum_{k = 2}^n P(E_1^c, \ldots, E_{k-1}^c, E_k)
\leq 2P \left( E_1, \sum_{j = 2, j \notin N_l}^n (\xi_j - E\xi_j) \geq -c_0\varepsilon \right)
\leq 2 \sum_{k = 2}^n P \left( E_1^c, \ldots, E_{k-1}^c, E_k, \sum_{j = k+1, j \notin N_l}^n (\xi_j - E\xi_j) \geq -c_0\varepsilon \right)
\leq 2P \left( \sum_{k = 1, k \notin N_l}^n (\xi_k - E\xi_k) \geq z - c_0\varepsilon - D_n \right),
\quad (3.26)
$$

where $D_n = \sum_{j = 1, j \notin N_l}^n |E\xi_j|$. Write $z' = z - c_0\varepsilon - D_n$. It is not difficult to show that

$$
D_n \leq 2b \sum_{j = 1, j \notin N_l}^n E|X_j|I \{ |X_j| \geq \varepsilon B_n / x \} \leq 4\varepsilon^{-2} \Delta_{n,x},
$$

$$
s_{n,1}^2 = \sum_{j = 1, j \notin N_l}^n \text{Var}(\xi_j) = 4b^2 \sum_{j = 1, j \notin N_l}^n E X_j^2 + O(1)\varepsilon^{-1}\Delta_{n,x}
= 4y^2 + O(1)\varepsilon^{-2}\Delta_{n,x},
$$
where $|O(1)| \leq 30$. This yields that

$$\frac{z'}{sn,1} = y + O(1)[(\varepsilon + \varepsilon^{-2}\Delta_{n,x})/y],$$

where $|O(1)| \leq 40$. Therefore, by Lemma 2 with $\eta_j = \xi_j - E\xi_j$

$$I_4 \leq 2[1 - \Phi(z'/sn,1)] \left[ 1 + A(z'/sn,1)^3s_{n,1}^{-3} \sum_{j=1}^{n} E|\xi_j|^3 \right]$$

$$\leq 2[1 - \Phi(y)][1 + A(\varepsilon + \varepsilon^{-2}\Delta_{n,x})],$$

(3.27)

where we have used the fact that whenever $x\theta_n \to 0$,

$$\frac{1 - \Phi(x + \theta_n)}{1 - \Phi(x)} = 1 + O(1)x\theta_n.$$

This proves (3.7), and also completes the proof of Proposition 2. □

4. Proof of Proposition 3

By Proposition 2, it suffices to show that

$$P\left( \max_{1 \leq k \leq n} \bar{S}_k \geq x\bar{V}_n \right) \geq 2(1 - \Phi(x))(1 - C_{\delta,\tau}(\varepsilon^{-2}\Delta_{n,x} + \varepsilon)).$$

(4.1)

Toward this end, let $b = x/B_n^{N_0}$ throughout this section. Recall (3.8), which we use repeatedly in the proof without further explanation. Let $n_0$ be defined as in (2.2). It can be readily seen that

$$P\left( \max_{1 \leq k \leq n} \bar{S}_k \geq x\bar{V}_n \right)$$

$$\geq P\left( 2b \max_{n_0 \leq k \leq n} \bar{S}_k \geq b^2\bar{V}_n^2 + x^2 \right)$$

$$\geq P\left( \bigcup_{k=n_0}^{n} \{2b\bar{S}_k \geq b^2\bar{V}_k^2 + b^2E(\bar{V}_n^2 - \bar{V}_k^2) + x^2 + \varepsilon \} \right)$$

$$- P\left( \bigcup_{k=n_0}^{n} \{2b\bar{S}_k \geq b^2\bar{V}_k^2 + x^2 + \varepsilon, (\bar{V}_n^2 - \bar{V}_k^2) - E(\bar{V}_n^2 - \bar{V}_k^2) \geq \varepsilon B_n^2/x^2 \} \right)$$

$$=: I_5 - I_6.$$

To complete the proof of Proposition 3, we only need to show the following lemma.
Lemma 3. Under the conditions of Proposition 3, we have

\[ I_5 \geq 2(1 - \Phi(x))(1 - C_{\delta, \tau}(\varepsilon^{-2}\Delta_{n,x} + \varepsilon)), \quad (4.3) \]

\[ I_6 \leq C_{\tau, \delta}x^{-2}e^{-x^2/2}. \quad (4.4) \]

Proof of (4.3). We have

\[ I_5 \geq \mathbb{P}\left( \bigcup_{k=1}^{n} \{ 2b\bar{S}_k \geq b^2\bar{V}_k^2 + b^2\mathbb{E}(\bar{V}_n^2 - \bar{V}_k^2) + x^2 + \varepsilon \} \right) \]

\[ - \mathbb{P}\left( \bigcup_{k=1}^{n_0} \{ 2b\bar{S}_k \geq b^2\bar{V}_k^2 + b^2\mathbb{E}(\bar{V}_n^2 - \bar{V}_k^2) + x^2 + \varepsilon \} \right) \]

\[ =: I_{5,1} - I_{5,2}. \]

Write

\[ \xi_j = 2b\bar{X}_j - b^2\bar{X}_j^2 + b^2\mathbb{E}\bar{X}_j^2, \]

\[ \mathbf{F}_j = \left\{ \sum_{k=1}^{j} \xi_k \geq y \right\}, \quad \text{where } y = 2x^2 + \varepsilon. \]

As in the proof of (3.25), there exists a constant \( c_0 \) such that for all \( 0 \leq k \leq n - 1 \),

\[ \mathbb{P}\left( \sum_{j=k+1}^{n} (\xi_j - \mathbb{E}\xi_j) \geq c_0\varepsilon \right) \leq 1/2. \]

This, together with the independence of \( \xi_j \), yields that

\[ I_{5,1} = \mathbb{P}(\mathbf{F}_1) + \sum_{k=2}^{n} \mathbb{P}(\mathbf{F}_1^{-}, \ldots, \mathbf{F}_{k-1}^{-}, \mathbf{F}_k) \]

\[ \geq \mathbb{P}(\mathbf{F}_1, y \leq \xi_1 \leq y + 4\varepsilon) + \sum_{k=2}^{n} \mathbb{P}\left( \mathbf{F}_1^{-}, \ldots, \mathbf{F}_{k-1}^{-}, \mathbf{F}_k, y \leq \sum_{j=1}^{k} \xi_j \leq y + 4\varepsilon \right) \]

\[ \geq 2\mathbb{P}\left( \mathbf{F}_1, y \leq \xi_1 \leq y + 4\varepsilon, \sum_{j=2}^{n} (\xi_j - \mathbb{E}\xi_j) \geq c_0\varepsilon \right) \]

\[ + 2 \sum_{k=2}^{n} \mathbb{P}\left( \mathbf{F}_1^{-}, \ldots, \mathbf{F}_{k-1}^{-}, \mathbf{F}_k, y \leq \sum_{j=1}^{k} \xi_j \leq y + 4\varepsilon, \sum_{j=k+1}^{n} (\xi_j - \mathbb{E}\xi_j) \geq c_0\varepsilon \right) \]

\[ \geq 2\mathbb{P}\left( \sum_{k=1}^{n} (\xi_k - \mathbb{E}\xi_k) \geq y + (c_0 + 4)\varepsilon + D_n \right). \]
where $D_n = \sum_{j=1}^n |E\xi_j|$. Similarly to the proofs of (3.26)–(3.27), it follows from Lemma 2 with $\eta_j = \xi_j - E\xi_j$ that
\[
I_{5,1} \geq 2P\left( \sum_{k=1}^n (\xi_k - E\xi_k) \geq y + (c_0 + 4)\varepsilon + D_n \right)
\geq 2\left( 1 - \Phi(x) \right) (1 - A(\varepsilon + \varepsilon^{-2}A_{n,x})).
\] (4.6)

On the other hand, similar to the proofs of (3.26) and (3.27), we have
\[
I_{5,2} \leq 2P\left( \sum_{j=1}^{n_0} \xi_j \geq 2x^2 + (1 - c_0)\varepsilon - D_n \right)
\leq Cx^{-1} \exp\left( -\frac{x^2}{2} - \frac{x^2 \sum_{j=n_0+1}^n E\xi_j^2}{2B_n} \right)
\leq Cx^{-2}e^{-x^2/2}.
\] (4.7)

This, together with (4.6), implies (4.3). □

**Proof of (4.4).** Define $k'_0 = 1$, and $k'_i = k'_{i-1} + 1$ if $E\xi_{k_i}^2 > \varepsilon^2 B_n^2/x^6$, and otherwise
\[
k'_i = \max\left\{ k \leq n: \sum_{j=k'_{i-1}+1}^k E\xi_j^2 \leq \frac{\varepsilon^2 B_n^2}{x^6} \right\} + 1.
\]

Let $m$ satisfy $k'_{m-1} < n \leq k'_m$ and define
\[
k_i = k'_i \quad \text{for } i < m, \quad \text{and} \quad k_m = n.
\]

Because $\sum_{j=k_{i-1}+1}^{k_i} E\xi_j^2 > \varepsilon^2 B_n^2/x^6$ for $i < m$, we have
\[
B_n^2 \geq \sum_{i=1}^{m-1} \sum_{j=k_{i-1}+1}^{k_i} E\xi_j^2 > (m-1)\varepsilon^2 B_n^2/x^6,
\]
which implies that $m \leq \varepsilon^{-2}x^6 + 1$. Furthermore, suppose that $i_0$ satisfies $k_{i_0-1} < n_0 \leq k_{i_0}$, where $n_0$ is defined as in (2.2). Set
\[
\hat{X}_k = X_k I_{\{|X_k| \leq 16^{-1}\varepsilon B_n/x^3\}}, \\
\hat{X}_k = X_k I_{\{16^{-1}\varepsilon B_n/x^3 < |X_k| \leq \varepsilon B_n/x\}}, \quad \hat{Z}_{k_i} = \sum_{k=k_{i-1}+1}^{k_i-1} |\hat{X}_k|.
\]
Note that $2b|\bar{X}_k| \leq 2\varepsilon$. Simple calculations show that

$$I_6 \leq \sum_{i=10}^{m} \mathbb{P}\left( \bigcup_{k=k_{i-1}+1}^{k_i} \{2b\bar{S}_k \geq b^2\bar{V}_k^2 + x^2 + \varepsilon, \right.$$ \begin{align*}
(\bar{V}_n^2 - \bar{V}_k^2) - \mathbb{E}(\bar{V}_n^2 - \bar{V}_k^2) \geq \varepsilon B_n^2 / x^2 \left. \right\}
\right)
\end{align*}

$$\leq \sum_{i=10}^{m} \mathbb{P}\left( \bigcup_{k=k_{i-1}+1}^{k_i-1} \{2b\bar{S}_k \geq b^2\bar{V}_k^2 + x^2 - \varepsilon, \right.$$ \begin{align*}
(\bar{V}_n^2 - \bar{V}_{k_{i-1}+1}^2) - \mathbb{E}(\bar{V}_n^2 - \bar{V}_{k_{i-1}+1}^2) \geq 2^{-1} \varepsilon B_n^2 / x^2 \left. \right\}
\right) \tag{4.8}
$$

$$\leq \sum_{i=10}^{m} \mathbb{P}(2b(\bar{S}_{k_{i-1}} + \hat{Z}_{k_i} + \mathbb{E}\hat{Z}_{k_i}) \geq b^2\bar{V}_{k_i}^2 + x^2 - 2\varepsilon, \right.$$ \begin{align*}
(\bar{V}_n^2 - \bar{V}_{k_{i-1}+1}^2) - \mathbb{E}(\bar{V}_n^2 - \bar{V}_{k_{i-1}+1}^2) \geq 2^{-1} \varepsilon B_n^2 / x^2 \left. \right\}
\right)
+ \sum_{i=10}^{m} \mathbb{P}\left( \max_{k_{i-1}+1 \leq j \leq k_i-1} 2b \sum_{k=k_{i-1}+1}^{k_i} (\hat{X}_k - \mathbb{E}\hat{X}_k) \geq \varepsilon \right)
=: I_{6,1} + I_{6,2}.
$$

Noting that

$$\sigma_{ni}^2 := \sum_{k=k_{i-1}+1}^{k_i-1} \mathbb{E}\hat{X}_k^2 \leq \frac{\varepsilon^2 B_n^2}{x^6} \quad \text{and} \quad |\hat{X}_k| \leq 16^{-1} \varepsilon B_n / x^3,$$

it follows from $m \leq \varepsilon^{-2}x^6 + 1$ and Lévy’s inequality that with $t = 2bx^2 / \varepsilon$

$$I_{6,2} \leq \sum_{i=10}^{m} \mathbb{P}\left( \sum_{k=k_{i-1}+1}^{k_i-1} (\hat{X}_k - \mathbb{E}\hat{X}_k) \geq \varepsilon / (2b) - \sqrt{2}\sigma_{ni} \right)$$

$$\leq \sum_{i=10}^{m} e^{-(\varepsilon / (2b) - \sqrt{2}\sigma_{ni})} \prod_{k=k_{i-1}+1}^{k_i-1} e^{t(\hat{X}_k - \mathbb{E}\hat{X}_k)}$$

$$\leq Ae^{-x^2} \sum_{i=10}^{m} \exp \left[ A_1^2 \sigma_{ni}^2 \right]$$

$$\leq 2A_1(\varepsilon^{-2}x^6 + 1)e^{-x^2} \leq C \gamma x^{-2} e^{-x^2 / 2},$$

where we used the fact that $\varepsilon \geq \gamma x^{-1 / 2}.$
Recall, by the definition of $k_i$, Similar to the estimate for $I_3$ in (3.4), we obtain

$$I_{6,1} \leq \sum_{i=i_0}^m P(2b(\tilde{S}_{k_i-1} + \hat{Z}_{k_i} - E\hat{Z}_{k_i}) \geq b^2 \tilde{V}_{k_i-1}^2 + x^2 - 18\varepsilon)$$

$$\leq \sum_{i=i_0}^m P(b(\tilde{S}_{k_i-1} - E\tilde{S}_{k_i-1} + \hat{Z}_{k_i} - E\hat{Z}_{k_i}) - b^2 \tilde{V}_{k_i-1}^2/2 + t(\tilde{V}_{n,i-1}^2 - \tilde{V}_{k_i-1}^2) - tE(\tilde{V}_{n,i-1}^2 - \tilde{V}_{k_i-1}^2) \geq x^2/2 + 12\log x - 9\varepsilon)$$

$$\leq Ax^{-12}e^{-x^2/2} \sum_{i=i_0}^m \left\{ \prod_{j=1}^{k_i-1} Ee^{b(\tilde{X}_j - E\tilde{X}_j) - b^2 \tilde{X}_j^2} \right\}$$

$$\leq Ax^{-12}e^{-x^2/2} \sum_{i=i_0}^m \exp \left( A\Delta_{n,x} + A \frac{x^2 \sum_{j=k_i-1+1}^{k_i-1} E\tilde{X}_j^2}{B_n^2} e^{24\varepsilon \log x} \right)$$

$$+ A \frac{x^3 \sum_{j=k_i-1+1}^{k_i-1} E|\tilde{X}_j|^3}{B_n^3} e^{24\varepsilon \log x} e^{-1} \log x$$

$$+ A \frac{x^4 \sum_{j=k_i-1+1}^{k_i-1} E\tilde{X}_j^4}{B_n^4} e^{24\varepsilon \log x} (e^{-1} \log x)^2$$

$$+ A \frac{x^4 \sum_{j=k_i}^{k_i} E\tilde{X}_j^4}{B_n^4} e^{24\varepsilon \log x} (e^{-1} \log x)^2$$.

Recall, by the definition of $k_i$,

$$\frac{x^2}{B_n^2} \sum_{j=k_i-1+1}^{k_i-1} E\tilde{X}_j^2 \leq e^2 x^{-4},$$

$$\frac{x^4}{B_n^4} \sum_{j=k_i-1+1}^{k_i-1} E\tilde{X}_j^4 \leq e^3 \frac{x^3}{B_n^3} \sum_{j=k_i-1+1}^{k_i-1} E|\tilde{X}_j|^3 \leq e^2 x^2 \frac{k_i-1}{B_n^2} \sum_{j=k_i-1+1}^{k_i-1} E\tilde{X}_j^2 \leq e^4 x^{-4}.$$
On the other hand, we have
\[ \varepsilon^{-1} x^{-24} \gamma^{-1} \min \{ x^{\delta/10 + \delta/3}, x^{3/2} \} \leq \gamma^{-1} x^{\min \{ \delta/2, 3/2 \}}, \]
and by (2.2)–(2.4) and the inequality \( \sum_{j=n_0+1}^{n} \mathbb{E} X_j^2 \leq 192 x^{-2} B_n^2 \log x \), for all \( i \geq i_0 \),
\[ \frac{x^4}{B_n^4} \sum_{j=k_i}^{n} \mathbb{E} \bar{X}_j^4 \leq (\varepsilon B_n/x) \sum_{j=n_0+1}^{n} \mathbb{E} |\bar{X}_j|^3 \]
\[ \leq (\varepsilon \tau B_n^2/x^{2+\delta}) \sum_{j=n_0+1}^{n} \mathbb{E} X_j^2 \leq C_{\tau, \delta} \varepsilon x^{-\delta} \log x. \]
Substituting these estimates into (4.10) gives
\[ I_{6,1} \leq C_{\delta, \tau} (\varepsilon^{-2} x^6 + 1)x^{-12} e^{-x^2/2} \exp \left( C_{\tau, \delta} x^{-1/2} \log^2 x + C_{\tau, \delta} x^{-\delta/2} \log^3 x \right) \]
\[ \leq C_{\delta, \tau} x^{-2} e^{-x^2/2}. \] (4.11)
This proves (4.4), which also completes the proof of Lemma 3.

Acknowledgements

Weidong Liu’s research partially supported by a foundation of national excellent doctoral dissertation of PR China. Qi-Man Shao’s research partially supported by Hong Kong RGC CERG 602608 and 603710. Qiying Wang’s research partially supported by an ARC discovery grant. We thank the Associate Editor and the referee for their helpful comments, which led to a significant improvement of the presentation of the paper.

References

[1] Aleshkyavichene, A.K. (1979). Probabilities of large deviations for the maximum of sums of independent random variables. Theory Probab. Appl. 24 16–33.
[2] Aleshkyavichene, A.K. (1979). Probabilities of large deviations for the maximum of sums of independent random variables. II. Theory Probab. Appl. 24 318–331.
[3] Brown, B.M. (1971). Martingale central limit theorems. Ann. Math. Statist. 42 59–66. MR0290428
[4] Chistyakov, G.P. and Götze, F. (2003). Moderate deviations for Student’s statistic. Theory Probab. Appl. 47 415–428.
[5] de la Peña, V.H., Lai, T.L. and Shao, Q.M. (2009). Self-normalized Processes: Limit Theory and Statistical Applications. Probability and Its Applications (New York). Berlin: Springer. MR2488094
[6] Heinrich, L. (1985). Nonuniform estimates, moderate and large deviations in the central limit theorem for \( m \)-dependent random variables. Math. Nachr. 121 107–121. MR0809317
[7] Hu, Z., Shao, Q.M. and Wang, Q. (2009). Cramér type moderate deviations for the maximum of self-normalized sums. Electron. J. Probab. 14 1181–1197. MR2511281
[8] Jing, B.Y., Shao, Q.M. and Wang, Q. (2003). Self-normalized Cramér-type large deviations for independent random variables. *Ann. Probab.* **31** 2167–2215. MR2016616

[9] Linnik, Yu.V. (1962). Limit theorems for sums of independent random variables, taking account of large deviations. *Theory Probab. Appl.* **7** 175–129.

[10] Sakhanenko, A.I. (1991). Berry–Esseen type estimates for large deviation probabilities. *Sib. Math. J.* **32** 647–656.

[11] Shao, Q.M. (1997). Self-normalized large deviations. *Ann. Probab.* **25** 285–328. MR1428510

[12] Shao, Q.M. (1999). A Cramér type large deviation result for Student’s *t*-statistic. *J. Theoret. Probab.* **12** 385–398. MR1684750

[13] Wang, Q. (2011). Refined self-normalized large deviations for independent random variables. *J. Theoret. Probab.* **24** 307–329. MR2795041

Received October 2010 and revised October 2011