Quantization as Asymptotics of Diffusion Processes in the Phase Space

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This work is an extended version of the paper [1], in which the main results were announced. We consider certain classical diffusion process for a wave function on the phase space. It is shown that at the time of order $10^{-11}$ sec this process converges to a process considered by quantum mechanics and described by the Schrodinger equation. This model studies the probability distributions in the phase space corresponding to the wave functions of quantum mechanics. We estimate the parameters of the model using the Lamb–Retherford experimental data on shift in the spectrum of hydrogen atom and the assumption on the heat reason of the considered diffusion process.

In the paper it is shown that the quantum mechanical description of the processes can arise as an approximate description of more exact models. For the model considered in this paper, this approximation arises when the Hamilton function changes slowly under deviations of coordinates, momenta, and time on intervals whose length is of order determined by the Planck constant and by the diffusion intensities.

Contents

1 Introduction
2 Description of the model
3 The main results
   3.1 The mathematical model of the process
   3.2 Analysis of the diffusion component of the equation
   3.3 Analysis of the model of the process
1 Introduction

In this paper we propose a model which, on the one hand, allows one to estimate the probability distribution of a quantum particle in the phase space in the low temperature heat field. For the first time this problem was solved by Wigner [3], but he constructed “quasi-distributions” on the phase space which can be negative and hence have no physical sense. On the other hand, the proposed model yields one more construction of quantization of mechanical systems and can be used in the new approach to foundation of the classical quantization procedure. This is an old problem. Various approaches to this problem, in particular probabilistic ones, can be found in [4, 5, 6, 7, 8, 9]. These works essentially influenced the author during the construction of the present model.

In this paper we consider the classical model of a diffusion process for a wave (complex valued) function on the phase space. The analysis of the differential equation of the model shows that the motion in the model splits into rapid and slow motions. The result of the rapid motion is that the system, starting from an arbitrary wave function on the phase space, goes to a function belonging to certain distinguished subspace. The elements of this subspace are parameterized by the wave functions depending only on the coordinates. The slow motion along the subspace is described by the Schrödinger equation.

Using the assumptions on the heat reason of the diffusions and the correspondence of the consequences of the model with the known physical experiments of Lamb–Retherford [10] (the Lamb shift in the spectrum of hydrogen atom), we estimate the diffusion coefficients and the time of the transition process from the classical description, in which the Heisenberg indeterminacy
principle does not hold, to the quantum description in which the Heisenberg principle already holds. The time of the transition process has order $1/T \cdot 10^{-11}$ sec, where $T$ is the temperature of the medium.

The results of this work have been announced in the paper [1]. Proofs of theorems 4 and 5 are instructive but rather technical, hence they are exposed in the Appendix. The estimate of the parameters of the model is also exposed in the Appendix.

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\section{Description of the model}

We consider a mathematical model of a process whose state at each moment of time is given by a wave function, which is a complex valued function $\varphi(x, p)$, where $(x, p) \in \mathbb{R}^{2n}$, and $n$ is the dimension of configuration space. In contrast to quantum mechanics, where the wave function depends only on coordinates or only on momenta, in our case the wave function depends both on coordinates and on momenta. As in quantum mechanics, it is assumed that for the wave functions the superposition principle holds, and the probability density $\rho(x, p)$ on the phase space, corresponding to the wave function $\varphi(x, p)$, is given by the standard formula

$$\rho(x, p) = \varphi^*(x, p)\varphi(x, p) = |\varphi(x, p)|^2. \quad (1)$$

In the present work we consider the classical model of diffusion process for the wave function $\varphi(x, p)$ on the phase space. It is assumed that each complex vector of the wave function is simultaneously in 4 motions:

the base point of the complex vector moves along the classical trajectory given by the Hamilton function $H(x, p)$;

the base point of the vector moves randomly with respect to coordinates and momenta, being in diffusion process with constant diffusion coefficients $a^2$ and $b^2$ with respect to coordinates and momenta, respectively;

the base point of each vector moves along a random trajectory as a result of motions described in the two preceding points, and the vector itself rotates with very large constant angular velocity $\omega = mc^2/\hbar$ in the coordinate system

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related with this point, where \( m \) is the mass of the particle, \( c \) is the light velocity, \( \hbar \) is the Planck constant;

the length of all complex vectors of the wave function at the moment \( t \) of time is multiplied by \( \exp(abnt/\hbar) \) (this is a purely technical requirement which does not affect the relative probabilities of position of the particle in the phase space).

It is assumed that the wave vector \( \varphi(x, p, t) \) at the point \((x, p)\) at the moment \( t \) of time equals, by the superposition principle, to the sum of wave vectors given by the distribution of vectors \( \varphi(x, p, 0) \) at the initial moment of time which get to the point \((x, p)\) at the moment \( t \) due to the motions described above.

3 The main results

3.1 The mathematical model of the process

Consider the diffusion process on the phase space in which the wave function \( \varphi(x, p, t) \) at the moment \( t \) satisfies the differential equation

\[
\frac{\partial \varphi}{\partial t} = \sum_{k=1}^{n} \left( \frac{\partial H}{\partial x_k} \frac{\partial \varphi}{\partial p_k} - \frac{\partial H}{\partial p_k} \frac{\partial \varphi}{\partial x_k} \right) - \frac{i}{\hbar} \left( H - \sum_{k=1}^{n} \frac{\partial H}{\partial p_k} p_k \right) \varphi + \Delta_{a,b} \varphi, \tag{2}
\]

where \( \Delta_{a,b} \varphi = a^2 \sum_{k=1}^{n} \left( \frac{\partial}{\partial x_k} - \frac{ip_k}{\hbar} \right)^2 \varphi + b^2 \sum_{k=1}^{n} \frac{\partial^2}{\partial p_k^2} \varphi + \frac{abn}{\hbar} \varphi, \tag{3}\)

where \( H(x, p) \) is the Hamilton function; \( a^2 \) and \( b^2 \) are the diffusion coefficients with respect to coordinates and momenta, respectively.

If we omit the last summand in equation (2), then we obtain a first order partial differential equation \( \partial \varphi / \partial t = A \varphi, \) where

\[
A \varphi = \sum_{k=1}^{n} \left( \frac{\partial H}{\partial x_k} \frac{\partial \varphi}{\partial p_k} - \frac{\partial H}{\partial p_k} \frac{\partial \varphi}{\partial x_k} \right) - \frac{i}{\hbar} \left( H - \sum_{k=1}^{n} \frac{\partial H}{\partial p_k} p_k \right) \varphi. \tag{4}
\]

This part of equation (2) describes the deterministic component of the motion of complex vectors \( \varphi(x, p, t) \) along the characteristics of the equation. According to the equation, in this motion the base point of each vector moves along the classical trajectory given by the Hamiltonian \( H(x, p) \), and the vector itself rotates at each point of the trajectory with the angular velocity \( \omega' = \frac{1}{\hbar} \left( H - \sum_{k=1}^{n} \frac{\partial H}{\partial p_k} p_k \right) \).
Note that in the case when the configuration space is three dimensional and \( H = E = c\sqrt{m^2c^2 + p^2} \), we have

\[
\omega' dt = \frac{1}{\hbar} \left( H - \sum_{k=1}^{n} \frac{\partial H}{\partial p_k} p_k \right) dt = \frac{mc^2}{\hbar} \frac{mc^2 dt}{H} = \frac{mc^2}{\hbar} d\tau,
\]

where \( \tau = mc^2 dt/H \), in accordance with the formulas of special relativity theory, is the proper time in the coordinate system related with the particle moving with the momentum \( p \). I.e. in this case, the vector whose base point moves along the classical trajectory, rotates with the constant angular velocity \( \omega = mc^2/\hbar \) in the coordinate system related with this point.

On the contrary, if in the right hand side of equation (2) we leave only the last summand of the form (3), then we obtain the equation

\[
\frac{\partial \varphi}{\partial t} = a^2 \sum_{k=1}^{n} \left( \frac{\partial}{\partial x_k} - \frac{ip_k}{\hbar} \right)^2 \varphi + b^2 \sum_{k=1}^{n} \frac{\partial^2}{\partial p_k^2} \varphi + \frac{ab}{\hbar} n \varphi. \tag{5}
\]

This equation describes the diffusion component of the motion of vectors \( \varphi(x, p, t) \) in the phase space. In this motion, the base points of the vectors move according to the classical homogeneous diffusion process with the diffusion coefficients with respect to coordinates and momenta equal to \( a^2 \) and \( b^2 \), respectively. And the vector itself is parallel transported during small random transports from a point \((x, p)\) to the point \((x + dx, p + dp)\), and its length at moment \( t \) is multiplied by \( \exp(abnt/\hbar) \). Note that the parallel transport of vectors on the phase space is given by a connection expressed by the following formula:

\[
L_{(dx, dp)} \varphi(x, p) - \varphi(x, p) \approx \left( \frac{i}{\hbar} \right) \varphi(x, p) p dx,
\]

where \( L_{(dx, dp)} \varphi(x, p) \) is the parallel transport of the vector \( \varphi(x, p) \) from the point \((x, p)\) along the infinitely small vector \((dx, dp)\).

In the particular case when the configuration space is three dimensional, such the connection on the phase space is related to synchronization of moving clocks at the points of the phase space. Indeed, if a particle with coordinates \( x = (x_1; x_2; x_3) \) moves with the velocity \( v = (v_1; v_2; v_3) \), then, according to formulas of special relativity theory, proper time is expressed through the observer time \( t \) by the formula

\[
\tau = \frac{t - xv/c^2}{\sqrt{1 - v^2/c^2}}, \tag{6}
\]

where \( xv = x_1v_1 + x_2v_2 + x_3v_3 \) is the scalar product of the vectors \( x \) and \( v \), and \( c \) is the velocity of light.
For a free particle with the momentum \( p = (p_1; p_2; p_3) \) and the stationary mass \( m \), the energy \( E = c\sqrt{p^2 + m^2c^2} \) and, respectively,
\[
v = \frac{pc}{\sqrt{p^2 + m^2c^2}}, \quad \sqrt{1 - \frac{v^2}{c^2}} = \frac{mc}{\sqrt{p^2 + m^2c^2}}. \quad (7)
\]
Substituting these expressions into (6), after computations we obtain:
\[
\tau = \frac{Et - xp}{mc^2}. \quad (8)
\]
Consider the distribution of complex vectors \( \varphi(x, p, t) \) on the phase space, rotating with constant angular velocity \( \omega = mc^2/\hbar \) in the proper time \( \tau \), which, at the moment \( \tau = 0 \), are equal to one and the same vector \( \varphi_0 \). We have
\[
\varphi(x, p, t) = \varphi_0 \exp\left(-i\frac{mc^2}{\hbar} \tau\right). \quad (9)
\]
Substituting formula (8) into this expression, we obtain
\[
\varphi(x, p, t) = \varphi_0 \exp\left(-i\frac{(Et - xp)}{\hbar}\right). \quad (10)
\]
Hence, if \( L_{(\Delta x, 0)}\varphi(x, p, t) \) is the shift of the vector \( \varphi(x, p, t) = \varphi_0 \exp(-imc^2\tau/\hbar) \) by the vector \( \Delta x \) along coordinates \( x \) without change of proper time, then
\[
L_{(\Delta x, 0)}\varphi(x, p, t) = \varphi(x, p, t) \exp\left(-i\frac{\Delta xp}{\hbar}\right). \quad (11)
\]
In the limit of infinitely small \( \Delta x \) we obtain the required formula for this case: \( L_{(dx, 0)}\varphi(x, p) - \varphi(x, p) = -(i/\hbar)\varphi(x, p)pdx \), where \( dx \) is the infinitely small shift of coordinates \( x \).

On the other hand, if we have a shift \( L_{(0, \Delta p)} \) of the vector \( \varphi(x, p, t) = \varphi_0 \exp(-imc^2\tau/\hbar) \) along momentum by \( \Delta p \) without change of proper time, then, since in the special relativity approximation, acceleration does not change the proper time of a particle, we have the equality \( L_{(0, \Delta p)}\varphi(x, p, t) = \varphi(x, p, t) \) and \( L_{(0, dp)}\varphi(x, p) - \varphi(x, p) = 0 \).

Hence, by linearity of \( L_{(dx, dp)} \) with respect to \( (dx, dp) \), we obtain the required equality in the general case: \( L_{(dx, dp)}\varphi(x, p) - \varphi(x, p) = -(i/\hbar)\varphi(x, p)pdx \).

Note that under these assumptions, the derivation with respect to the vector of infinitely small shift along the \( k \)-th coordinate corresponds to the
differential operator \( D_{x_k} = \partial / \partial x_k - ip_k / \hbar \), and the derivation with respect to the shift along the \( k \)-th momentum corresponds to the usual differential operator \( D_{p_k} = \partial / \partial p_k \), where \( k = 1, ..., n \).

Note also that these operators of shift along coordinates and momenta do not commute. The commutators of these differential operators read

\[
[D_{p_k}, D_{x_k}] = -i / \hbar \quad \text{and} \quad [D_{p_k}, D_{x_j}] = 0, \quad \text{where} \quad k \neq j \quad \text{and} \quad k, j = 1, ..., n.
\]

Thus, the shifts along coordinates and momenta of wave functions on the phase space realize a representation of the Heisenberg group.

### 3.2 Analysis of the diffusion component of the equation

Consider the diffusion equation (5) in more detail. This equation can be represented as follows:

\[
\frac{\partial \varphi}{\partial t} = \Delta_{a,b} \varphi = a^2 \Delta_x \varphi + b^2 \Delta_p \varphi + \frac{na b}{\hbar} \varphi, \tag{12}
\]

where

\[
\Delta_x \varphi = \sum_{k=1}^n D_{x_k}^2 \varphi = \sum_{k=1}^n \left( \frac{\partial^2 \varphi}{\partial x_k^2} - \frac{2ip_k}{\hbar} \frac{\partial \varphi}{\partial x_k} - p_k^2 \frac{\hbar}{\varphi} \right),
\]

\[
\Delta_p \varphi = \sum_{k=1}^n D_{p_k}^2 \varphi = \sum_{k=1}^n \frac{\partial^2 \varphi}{\partial p_k^2}.
\]

It is natural to call the operator \( \Delta_{a,b} \) by the diffusion operator for the representation of the Heisenberg group with the diffusion intensities \( a \) and \( b \) with respect to coordinates and momenta, respectively.

Let us look for a solution of equation (12) in the form

\[
\varphi(x, p, t) = \varphi^0(x, p, t) \exp(i x p / \hbar). \tag{13}
\]

Substituting this expression into equation (12), dividing both parts of the equality by \( \exp(i x p / \hbar) \) and transferring the summand \( (na b / \hbar) \varphi^0 \) into the left hand side, we obtain the equation

\[
\frac{\partial \varphi^0}{\partial t} - \frac{na b}{\hbar} \varphi^0 = \sum_{k=1}^n \left( a^2 \frac{\partial^2 \varphi^0}{\partial x_k^2} + b^2 \left( \frac{\partial^2 \varphi^0}{\partial p_k^2} + \frac{2ix_k}{\hbar} \frac{\partial \varphi^0}{\partial p_k} - \frac{x_k^2}{\hbar^2} \varphi^0 \right) \right), \tag{14}
\]
where \( \varphi^0 = \varphi^0(x, p, t) \) is some function.

To solve equation (14), let us decompose the function \( \varphi^0(x, p, t) \) into the Fourier integral with respect to \( p \), i.e., let us represent the function \( \varphi^0(x, p, t) \) in the form

\[
\varphi^0(x, p, t) = \mathcal{F}_h \psi^0(x, y, t) = \frac{1}{(2\pi\hbar)^{n/2}} \int_{\mathbb{R}^n} \psi^0(x, y, t) \exp\left(-\frac{iyp}{\hbar}\right) dy, \tag{15}
\]

where \( \psi^0(x, y, t) = \mathcal{F}_h^{-1} \varphi^0(x, p, t) = \frac{1}{(2\pi\hbar)^{n/2}} \int_{\mathbb{R}^n} \varphi^0(x, p, t) \exp\left(\frac{iyp}{\hbar}\right) dp. \tag{16} \]

Substituting this expression for \( \varphi^0(x, p, t) \) into the equation (14), we obtain that \( \psi^0(x, y, t) \) satisfies the equation

\[
\frac{\partial \psi^0}{\partial t} - \frac{nab}{\hbar} \psi^0 = \sum_{k=1}^{n} \left( a^2 \frac{\partial^2 \psi^0}{\partial x_k^2} - \frac{b^2(x_k - y_k)^2}{\hbar^2} \psi^0 \right), \tag{17}
\]

The right hand side of this equation is a self adjoint operator with discrete spectrum consisting of negative numbers.

Indeed, the equation for eigenvalues of this operator reads

\[
\sum_{k=1}^{n} \left( a^2 \frac{\partial^2 \chi}{\partial x_k^2} - \frac{b^2(x_k - y_k)^2}{\hbar^2} \chi \right) = \lambda \cdot \chi, \tag{18}
\]

where \( \chi = \chi(x, y) \) is a function of \( x \) and \( y \). This equation is the stationary Schrodinger equation for harmonic oscillator, and it is well studied (see, for instance [13], p. 94).

In particular, it is known that on the set of functions which tend to zero as \( x \) tends to infinity, equation (18) has discrete spectrum consisting of negative eigenvalues \( \lambda_1 > \lambda_2 \geq \ldots \). The greatest eigenvalue \( \lambda_1 = -nab/\hbar \) corresponds to the eigenfunction \( \chi_1(x, y) = (b/a\pi\hbar)^{n/4} \exp(-b(x - y)^2/2ah) \). The next eigenvalues are less than \( \lambda_1 \), and the difference is greater than or equal to \( ab/\hbar \).

Since the eigenfunctions \( \chi_k(x, y) \) of the operator (18) form a complete system of functions in the class of functions tending to zero as \( x \) tends to infinity, an arbitrary function \( \psi^0(x, y, t) \) from this class can be represented as a series

\[
\psi^0(x, y, t) = \sum_{k=1}^{\infty} c_k(y, t) \chi_k(x, y),
\]
where \( c_k(y, t) = \int_{\mathbb{R}^n} \psi^0(x, y, t) \chi_k(x, y)dx \) (19)

are the coefficients of the decomposition of the function \( \psi^0(x, y, t) \) with respect to eigenfunctions \( \chi_k(x, y) \).

Substituting the expression of the function \( \psi^0(x, y, t) \) in the form of this series into equation (17), we obtain that this equation in the orthonormal basis of eigenfunctions \( \chi_k(x, y) \), \( k = 1, 2, \ldots \), splits into an infinite system of equations:

\[
\frac{\partial c_k(y, t)}{\partial t} = \left( \lambda_k + \frac{nab}{\hbar} \right) c_k(y, t) \quad k = 1, 2, \ldots
\]

where \( \lambda_1 + nab/\hbar = 0 \) and \( \lambda_k + nab/\hbar \leq -ab/\hbar \) for \( k > 1 \).

Hence \( c_1(y, t) = c_1(y, 0) \), and \( c_k(y, t) = c_k(y, 0) \exp((\lambda_k + nab/\hbar) t) \) exponentially decay with time for \( k > 1 \). Hence the summand in \( \psi^0 \) corresponding to the first eigenvalue will give the main contribution into the function \( \psi^0 \) after time of order \( (\hbar/ab) \).

Thus, we have obtained that a solution of equation (17), after time \( t \) of order \( \hbar/ab \), becomes exponentially close to the function \( \psi^0(x, y) \), where

\[
\psi^0(x, y) = \lim_{t \to \infty} \psi^0(x, y, t) = c_1(y, 0) \chi_1(x, y). \quad (20)
\]

Respectively, since by definition \( \varphi^0(x, p, t) = \mathcal{F}_h \psi^0(x, y, t) \) and Fourier transform is continuous, we have

\[
\varphi^0(x, p) \overset{\text{def}}{=} \mathcal{F}_h \psi^0(x, y) = \lim_{t \to \infty} \mathcal{F}_h \psi^0(x, y, t) = \mathcal{F}_h (c_1(y, 0) \chi_1(x, y)). \quad (21)
\]

Since we will not use other eigenfunctions, introduce the notation

\[
\chi(x - y) \overset{\text{def}}{=} \chi_1(x, y) = \left( \frac{b}{a^2 \pi \hbar} \right)^{n/4} \exp \left( -\frac{b(x - y)^2}{2a \hbar} \right). \quad (22)
\]

To make the formulas shorter, let us also denote \( \psi(y) \overset{\text{def}}{=} c_1(y, 0) \), where \( c_1(y, 0) \) is expressed by the formula (19) with \( k = 1 \) and \( t = 0 \). That is,

\[
\psi(y) = \int_{\mathbb{R}^n} \psi^0(x, y, 0) \chi_1(x, y)dx = \int_{\mathbb{R}^n} \psi^0(x, y, 0) \chi(x - y)dx. \quad (23)
\]

Since by formula (13) \( \varphi(x, p, t) = \varphi^0(x, p, t) \exp(ixp/\hbar) \), then by formulas (21), (22) and using the notation \( \psi(y) \overset{\text{def}}{=} c_1(y, 0) \) above and also the equality (23) and notation (16), we obtain the following theorem.
Theorem 1. Let \( \varphi(x,p,0) \) be an arbitrary function such that Fourier transform of the function \( \varphi(x,p,0) \exp(-ixp/\hbar) \) with respect to \( p \) tends to zero as \( x \to \infty \). Then the solution \( \varphi(x,p,t) \) of the diffusion equation (23) exponentially with time (with the number in the exponent equal to \(-abt/\hbar\)) tends to a stationary solution of the form

\[
\varphi_0(x,p) = \lim_{t \to \infty} \varphi(x,p,t) = \frac{1}{(2\pi\hbar)^{n/2}} \int_{R^n} \psi(y) \chi(x-y) e^{-iyyp/\hbar} dy,
\]

where

\[
\psi(y) = \frac{1}{(2\pi\hbar)^{n/2}} \int_{R^{2n}} \varphi(x,p,0) e^{iyyp/\hbar} \chi(x-y) dpdx,
\]

and

\[
\chi(x-y) = \left(\frac{b}{a\pi\hbar}\right)^{n/4} e^{-b(x-y)^2/(2ah)}.
\]

Note that \( \chi^2(x-y) \) is the probability density of the normal distribution with respect to \( x \) with the mathematical expectation \( y \) and dispersion \( ah/(2b) \). If the quantity \( ah/(2b) \) is small, then the function \( \chi^2(x-y) \) is close to the delta function of \( x - y \).

The composition of expressions (25) and (24) yields a projector \( P_0 \) form the space of wave functions defined on the phase space onto certain subspace. The elements of this subspace are parameterized by functions of the form \( \psi(y) \), where \( y \in R^n \), i.e., by wave functions on the configuration space.

Theorem 2. The projection operator \( P_0 \) given by composition of expressions (25) and (24), has the form

\[
P_0 \varphi = \frac{1}{(2\pi\hbar)^n} \int_{R^{2n}} \varphi(x', p') e^{-b(x'-x)^2/4ah} e^{-a(pp')^2/2b\hbar} e^{i(x-x')(p+p')/2\hbar} dx' dp'.
\]

The operator \( P_0 \) is self-adjoint and commutes with the operator \( \Delta_{a,b} \).

Proof of this theorem is obtained by substitution of formulas (25) and (26) into (24), by an algebraic transformation of the number in the exponent, and by computation of the integral over \( y \). The integral over \( y \) is the Fourier transform of the exponent of a quadratic polynomial, whose analytical expression is known. The author performed the computations using the system Mathematica \[14\], supporting symbolic mathematical computations.
The commutativity of the operators $P_0$ and $\Delta_{a,b}$ follows from the fact that the orthogonal projector $P_0$ distinguishes the subspace of eigenvectors of the self-adjoint operator $\Delta_{a,b}$ with the zero eigenvalue.

Formulas (24) and (1) imply the following statement.

**Theorem 3.** If $\psi(x)$ is a wave function on the configuration space and $\varphi(x,p)$ is the wave function on the phase space corresponding to it by formula (27), then the probability density in the phase space $\rho(x,p) = |\varphi(x,p)|^2$ is expressed by the formula

$$\rho(x,p) = \frac{1}{(2\pi \hbar)^n} \left( \frac{b}{4a\hbar} \right)^{n/2} \int_{\mathbb{R}^{2n}} \psi \left( x + \frac{x'' - x'}{2} \right) \psi^* \left( x + \frac{x'' + x'}{2} \right) \exp \left( -\frac{b(x'')^2}{4ah} \right) \exp \left( -\frac{b(x')^2}{4ah} \right) \exp \left( \frac{ix'p}{\hbar} \right) dx'' dx'. \quad (28)$$

In contrast to quasi-distributions

$$W(x,p) = \frac{1}{(2\pi \hbar)^n} \int_{\mathbb{R}^n} \psi \left( x - \frac{x'}{2} \right) \psi^* \left( x + \frac{x'}{2} \right) \exp \left( \frac{ix'p}{\hbar} \right) dx', \quad (29)$$

defined by Wigner [3], the density $\rho(x,p)$ in the phase space, given by expression (28), is always nonnegative. Its expression differs from the expression for the Wigner function by exponents under integral, which give the densities of distributions close to delta functions.

To prove Theorem 3, one should substitute into formula (11) expression (24). We obtain

$$\rho(x,p) = \varphi \varphi^* = \frac{1}{(2\pi \hbar)^n} \int_{\mathbb{R}^{2n}} \psi(y) \psi^*(y') \chi(x-y) \chi(x-y') e^{-i(y-y')p/\hbar} dy dy', \quad (29)$$

where $\chi(x-y)$ is given by relation (20). After substitution of (26) into (29), the change of coordinates $y = x + (x'' - x')/2$ and $y' = x + (x'' + x')/2$ under integral and a transformation of the numbers in the exponents, we obtain formula (28).

The algebra of observables given by real functions on the phase space but averaged over the probability densities of the form (28), has been studied in [15].

The function $\rho(x)$ of density of probability distribution in the configuration space is expressed through the density $\rho(x,p)$ in the phase space by the
formulas $\rho(x) = \int_{\mathbb{R}^n} \rho(x,p) dp$. Hence, integrating expression (29) over $p$, we obtain the following statement.

**Corollary 1.** If $\psi(x)$ is a wave function on the configuration space, then the corresponding probability density $\rho(x)$ in the configuration space is given by the formula

$$\rho(x) = \int_{\mathbb{R}^n} |\psi(y)|^2 \chi^2(x - y) dy,$$

(30)

where $\chi(x - y)$ is given by (27). That is, $\rho(x)$ is obtained from $|\psi(x)|^2$ by smoothing (convolution) with the density of normal distribution with dispersion $\hbar/(2\pi)$, and the exactness of defining coordinate is bounded by the quantity $\sim (\hbar/(2\pi))$. 

As it is known [16], in quantum electrodynamics the minimal error of measuring coordinates of an electron in the stationary system is bounded by the quantity $\hbar/(mc)$, where $m$ is the mass of the electron, and $c$ is the light velocity. Hence the statement of Corollary 1, although not corresponding to non-relativistic quantum mechanics (in which it is assumed that coordinates can be measured with any degree of exactness), does not contradict with a more exact theory, quantum electrodynamics.

If one assumes that the diffusion is induced by heat action on the electron, then the diffusion coefficients with respect to coordinates and momenta are expressed in statistical physics (see, for example, [17], Ch.7, §4 and §9) through the temperature $T$ by the formulas $a^2 = kT/(m\gamma)$ and $b^2 = \gamma kT m$, where $k$ is the Boltzmann constant, $m$ is the mass of the electron, $\gamma$ is the friction coefficient of the medium per unit of mass. Hence $a/b = (\gamma m)^{-1}$ and $ab = kT$. That is, in this case, the quantity $a/b$, which enters expression (26) and determines the dispersion of smoothing in Corollary 1, does not depend on the temperature. On the other hand, the time $t$ of the transformation process determined in Theorem 1, has the form $t \sim \hbar/(ab) = \hbar/(kT) = T^{-1} \cdot 7.638 \cdot 10^{-12}$. More detailed formulas for the estimate of the quantity $\hbar/(ab)$ are given in Appendix 3.

### 3.3 Analysis of the model of the process

Let us return to the study of the main equation (2). Taking into account the estimate made at the end of the previous subsection, let us consider the quantity $\hbar/(ab)$ in equation (2) as a small parameter, and let us assume that coordinates and momenta change a little at this time in the classical motion.
defined by the Hamiltonian $H(x, p)$, and also let us assume that the function $H(x, p)$ and all its derivatives grow at infinity no faster than a polynomial.

**Theorem 4.** The motion described by equation (2) asymptotically splits as $\hbar/(ab) \to 0$ into a rapid motion and a slow one.

1) As a result of rapid motion, an arbitrary wave function $\varphi(x, p, 0)$ turns at the time of order $\hbar/(ab)$ into a function of the form (24):

$$\varphi(x, p) = \frac{1}{(2\pi\hbar)^{n/2}} \int_{\mathbb{R}^n} \psi(y) \chi(x - y) e^{-i(y - x)p/\hbar} dy,$$  

where

$$\chi(x - y) = \left( \frac{b}{a\pi\hbar} \right)^{n/4} e^{-b(x - y)^2/(2a\hbar)}.$$  

The wave functions of the form (31) form a linear subspace. Elements of this subspace are parameterized by wave functions $\psi(y)$ depending only on coordinates $y \in \mathbb{R}^n$.

2) The slow motion starting from a nonzero wave function $\varphi(x, p, 0)$ of the form (31) from this subspace, goes inside this subspace, and is parameterized by the wave function $\psi(y, t)$ depending on time. The function $\psi(y, t)$ satisfies the Schrodinger equation of the form $i\hbar \partial \psi / \partial t = \hat{H} \psi$, where

$$\hat{H} \psi = \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^n} \left( H(x, p) - \sum_{k=1}^{n} \left( \frac{\partial H}{\partial x_k} + \frac{ib}{a} \frac{\partial H}{\partial p_k} \right) (x_k - y_k') \right) \chi(x - y) \chi(x - y') e^{i\hbar(y - y')p/\hbar} \psi(y', t) dy' dx dp,$$

and $\chi(x - y)$ is given by formula (32).

Proof of Part 1 of Theorem 4 is postponed till Appendix, due to its large volume and technicalities.

**Proof of Part 2 of Theorem 4.** In the first Part of Theorem 4 it is stated that after the rapid motion, the initial distribution $\varphi(x, p, 0)$ turns to the form with just a small difference with (31). After the slow motion the distribution remains in the class of functions of the form (31), but changes in time.

To study the slow motion, let us look for a solution of equation (2) in the form (31) in which $\psi = \psi(y', t)$ is considered as dependent on time.

Let us substitute expression (31) into equation (2). Since by construction, this expression for $\varphi$ satisfies equation $\Delta_{a,b} \varphi = 0$, then after substitution of expression (31) into equation (2) and after dividing both parts of the equation
by \( \exp (ixp/\hbar) \), we obtain the following equation:

\[
\frac{1}{(2\pi\hbar)^{n/2}} \int_{R^n} \frac{\partial \psi(y', t)}{\partial t} \chi(x - y') \exp \left(-\frac{iyp}{\hbar}\right) dy'
\]

\[
= \frac{1}{(2\pi\hbar)^{n/2}} \int_{R^n} \left[ \sum_{k=1}^{n} \left( \frac{\partial H}{\partial x_k} \left( -\frac{i(y_k' - x_k)}{\hbar} \right) - \frac{\partial H}{\partial p_k} \left( -\frac{b}{a\hbar}(x_k - y_k') + \frac{i}{\hbar}p_k \right) \right) \right.
\]

\[
- \frac{i}{\hbar} \left( H - \sum_{k=1}^{n} \frac{\partial H}{\partial p_k} p_k \right) \psi(y', t) \chi(x - y') \exp \left(-\frac{iyp}{\hbar}\right) dy'.
\]  

(34)

If in the obtained equation one opens the brackets, reduces the similar terms, multiplies both parts of the equation by \( \frac{1}{(2\pi\hbar)^{n/2}} \chi(x - y) \exp(iyp/\hbar) \) and integrates by \( p \) and by \( x \), then, taking into account the equality \( \int_{R^n} \chi^2(x - y) dx = 1 \), we obtain the following equation:

\[
\frac{\partial \psi}{\partial t} = -\frac{i}{\hbar} \hat{H} \psi \quad \text{or} \quad i\hbar \frac{\partial \psi}{\partial t} = \hat{H} \psi,
\]  

(35)

where the operator \( \hat{H} \) is obtained from the function \( H \) by formula (33), required in Part 2 of Theorem 4.

Theorem 5. If \( ah/b \) is a small quantity and \( H(x, p) = \frac{p^2}{2m} + V(x) \), then the operator \( \hat{H} \), up to terms of order \( ah/b \), has the form

\[
\hat{H} \approx -\frac{\hbar^2}{2m} \left( \sum_{k=1}^{n} \frac{\partial^2}{\partial y_k^2} \right) + V(y) - \frac{ah}{4b} \sum_{k=1}^{n} \frac{\partial^2 V}{\partial y_k^2} + \frac{3nb\hbar}{4ma}.
\]  

(36)

Proof of Theorem 5 is given in Appendix 2.

The first two summands in formula (36) give the standard Hamilton operator. The last summand is a constant and can be neglected. The previous summand before the last one will be considered (due to the smallness of \( ah/b \)) as a perturbation of the Hamilton operator.

Assuming that the deviations in the spectrum of the hydrogen atom (the Lamb shift) observed in the Lamb–Retherford experiments [10] are induced by the previous to the last summand in formula (36), one can estimate the quantity \( a/b \). The computations by the standard method of perturbation theory analogous to the computations of [18], give the following estimate: \( a/b = 3.41 \cdot 10^4 \sec/g \) (see the computations in Appendix 3). Hence, the standard deviation of the normal distribution \( \chi^2 \), with which we make smoothing in formula (30), has the form \( \sqrt{ah/(2b)} = 4.24 \cdot 10^{-12} \text{cm} \). This quantity is much less than the radius of the hydrogen atom, and it is close to the Compton wave length of the electron \( \hbar/(mc) = 3.86 \cdot 10^{-11} \text{cm} \).
4 Conclusion

In this paper it is shown that the standard quantum mechanical description of a process can arise as an approximation of certain classical model for a diffusion process for a wave function in the phase space. The computations show that the proposed model in the form of differential equation \( \frac{d}{dt} \) describes physical processes in a rather adequate manner in the standard cases for standard Hamiltonian. But this model can be applied as well for computations of processes with a nonstandard Hamiltonian or a Hamiltonian rapidly changing in time, as in sudden perturbations \([19]\) or for periodically changing potential with frequency of order \( \frac{ab}{\hbar} \), and it can be compared with experimental data.
APPENDICES

Appendix 1

Proof of Part 1 of Theorem 4

Let \( \varphi(x,p,t) \) be a solution of equation (2). In the notations (3) and (4) equation (2) takes the form

\[
\frac{\partial \varphi}{\partial t} = A\varphi + \Delta_{a,b}\varphi,
\]

where \( A \) is a skew Hermitian operator, and \( \Delta_{a,b} \) is the self-adjoint diffusion operator.

Consider the derivative with respect to time of \( ||\varphi||^2 \) (the square of the norm of the function \( \varphi \)). We have

\[
\frac{d}{dt} ||\varphi||^2 = \frac{d}{dt} \langle \varphi; \varphi \rangle = \langle \frac{\partial \varphi}{\partial t}; \varphi \rangle + \langle \varphi; \frac{\partial \varphi}{\partial t} \rangle = \langle A\varphi + \Delta_{a,b}\varphi; \varphi \rangle + \langle \varphi; A\varphi + \Delta_{a,b}\varphi \rangle.
\]

Hence, by linearity of the scalar product \( \langle ; \rangle \) and by the equalities \( \langle A\varphi; \varphi \rangle = -\langle \varphi; A\varphi \rangle \) and \( \langle \Delta_{a,b}\varphi; \varphi \rangle = \langle \varphi; \Delta_{a,b}\varphi \rangle \), we obtain the equality

\[
2||\varphi|| \frac{d}{dt} ||\varphi|| = \frac{d}{dt} ||\varphi||^2 = 2\langle \Delta_{a,b}\varphi; \varphi \rangle \quad \text{or} \quad \frac{d}{dt} ||\varphi|| = \frac{1}{||\varphi||} \langle \Delta_{a,b}\varphi; \varphi \rangle.
\]

Denote by \( \bar{\varphi} \equiv \varphi/||\varphi|| \) the normalized function \( \varphi \). If the function \( \varphi \) satisfies equation (37), then, taking into account formula (38), we have

\[
\frac{\partial \bar{\varphi}}{\partial t} = \frac{\partial}{\partial t} \left( \frac{\varphi}{||\varphi||} \right) = \frac{1}{||\varphi||} \frac{\partial \varphi}{\partial t} - \frac{\varphi}{||\varphi||^2} \frac{d}{dt} ||\varphi|| = A\bar{\varphi} + \Delta_{a,b}\bar{\varphi} - \langle \Delta_{a,b}\bar{\varphi}; \bar{\varphi} \rangle \bar{\varphi}.
\]

By definition of the function \( \bar{\varphi} \) its norm \( ||\bar{\varphi}|| \equiv 1 \), and it is natural to call equation (39) the equation for the normalized wave function.

Consider now the projection of the function \( \bar{\varphi}(x,p,t) \) on the subspace of stationary solutions of the diffusion equation (5).

Let us represent the function \( \bar{\varphi}(x,p,t) \) in the form \( \bar{\varphi} = \varphi_0 + \varphi_1 \), where \( \varphi_0 = P_0\bar{\varphi} \), \( \varphi_1 = (1-P_0)\bar{\varphi} = P_1\bar{\varphi} \), and \( P_0 \) is the projector from Theorem 2. The self-adjointness of the projection operator \( P_0 \) implies the equalities \( \langle \varphi_0; \varphi_1 \rangle = 0 \) and \( ||\varphi_0||^2 + ||\varphi_1||^2 = ||\bar{\varphi}||^2 = 1 \).
For the proof of the first Part of Theorem 4 one needs to show that the quantity \(||\varphi_1||^2 = 1 - ||\varphi_0||^2\) becomes small after time of order \(\hbar/(ab)\).

To make formulas shorter, introduce the notation

\[
\eta(t) \overset{\text{def}}{=} ||\varphi_0(t)||^2, \quad \text{where} \quad \varphi_0 = P_0\varphi. \tag{40}
\]

**Statement 1.** If \(\varphi\) satisfies equation (37), then the quantity \(\eta(t) = ||\varphi_0||^2\) satisfies the differential equation

\[
\dot{\eta} = \alpha(t)\sqrt{1 - \eta} + \beta(t)(1 - \eta)\eta. \tag{41}
\]

where \(\beta(t) \geq \beta_{\text{min}} = 2ab/h\) and \(|\alpha(t)| \leq \alpha_{\text{max}} = 2 \cdot \max_{\varphi_0} ||A\varphi_0 - P_0A\varphi_0||\).

The maximum in the latter expression is taken over all normalized functions \(\varphi_0\) from the subspace of stationary solutions of the diffusion equation (5), for which the function \(||\varphi_0(x,p)||^2\) gives a probability distribution in the physical region of the phase space for the given process.

To prove Statement 1, let us consider how the quantity \(||\varphi_0||^2\) changes in time, where \(\varphi_0 = P_0\varphi\) and \(\varphi\) satisfies equation (37).

We have the following equalities which follow from the definition of the function \(\varphi_0\), from independence of the operator \(P_0\) of time, from equality (39), from linearity of the scalar product \(\langle \ , \ \rangle\) with respect to each argument, from commutativity of \(P_0\) with \(\Delta_{a,b}\), from self-adjointness of the operators \(P_0\) and \(\Delta_{a,b}\), from skew Hermitian property of the operator \(A\) and from the projection property \(P_0 = P_0^2\):

\[
\dot{\eta} = \frac{d\langle \varphi_0; \varphi_0 \rangle}{dt} = \frac{d\langle P_0\varphi; P_0\varphi \rangle}{dt} = \langle P_0 \frac{\partial \varphi}{\partial t}; P_0\varphi \rangle + \langle P_0\varphi; P_0 \frac{\partial \varphi}{\partial t} \rangle
\]

\[
= \langle \frac{\partial \varphi}{\partial t}; P_0^2\varphi \rangle + \langle P_0^2\varphi; \frac{\partial \varphi}{\partial t} \rangle = \frac{\partial \varphi}{\partial t}; \varphi_0 \rangle + \langle \varphi_0; \frac{\partial \varphi}{\partial t} \rangle
\]

\[
= \langle A\varphi + \Delta_{a,b}\varphi - \langle \Delta_{a,b}\varphi; \varphi \rangle\varphi_0 \rangle + \langle \varphi_0; A\varphi + \Delta_{a,b}\varphi - \langle \Delta_{a,b}\varphi; \varphi \rangle\varphi \rangle
\]

(substitute expression for \(\varphi = \varphi_0 + \varphi_1\), use the linearity of operators and equalities \(A^* = -A, \Delta_{a,b}\varphi_0 = 0\))

\[
= -\langle \varphi_1; A\varphi_0 \rangle - \langle A\varphi_0; \varphi_1 \rangle - 2\langle \Delta_{a,b}\varphi_1; \varphi_1 \rangle \cdot \langle \varphi_0; \varphi_1 \rangle
\]

\[
= -\left( \frac{\varphi_1}{||\varphi_1||} ; \frac{\varphi_0}{||\varphi_0||} \right) \cdot ||\varphi_1|| \cdot ||\varphi_0|| - \left( \frac{A\varphi_0}{||\varphi_0||} ; \frac{\varphi_1}{||\varphi_1||} \right) \cdot ||\varphi_1|| \cdot ||\varphi_0||
\]

\[
-2\left( \frac{\Delta_{a,b}\varphi_1}{||\varphi_1||} ; \frac{\varphi_1}{||\varphi_1||} \right) \cdot ||\varphi_1||^2 \cdot ||\varphi_0||^2
\]

\[
= -\langle \varphi_1; A\varphi_0 \rangle - \langle A\varphi_0; \varphi_1 \rangle ||\varphi_1|| \cdot ||\varphi_0|| - 2\langle \Delta_{a,b}\varphi_1; \varphi_1 \rangle ||\varphi_1||^2 \cdot ||\varphi_0||^2. \tag{42}
\]
where $\varphi_1$ and $\varphi_0$ are the normalized functions $\varphi_1$ and $\varphi_0$, respectively.

Since $\eta = ||\varphi_0||^2$ and $||\varphi_0||^2 + ||\varphi_1||^2 = ||\varphi||^2 = 1$, then $||\varphi_1|| = \sqrt{1-\eta}$.

Substituting these equalities into equation (42) and introducing notations

$$
\alpha(t) \overset{\text{def}}{=} -\langle \varphi_1; A\varphi_0 \rangle + \langle A\varphi_0; \varphi_1 \rangle \quad \text{and} \quad \beta(t) \overset{\text{def}}{=} -2\langle \Delta_{a,b}\varphi_1; \varphi_1 \rangle,
$$

we obtain the equation from Statement 1: $\dot{\eta} = \alpha(t)\sqrt{1-\eta}\sqrt{\eta} + \beta(t)(1-\eta)\eta$.

For the estimate of $|\alpha(t)|$, note that since $\langle \varphi_1; A\varphi_0 \rangle = \langle A\varphi_0; \varphi_1 \rangle^*$, where at the end of the equality the star * denotes complex conjugation, then $\alpha(t) = 2 : Re(\langle -A\varphi_0; \varphi_1 \rangle)$, where $Re$ denotes the real part of a complex number. Hence we obtain the following estimates:

$$
|\alpha(t)| = 2|\text{Re}(\langle -A\varphi_0; \varphi_1 \rangle)| \leq 2|\langle A\varphi_0; \varphi_1 \rangle| = 2|\langle A\varphi_0 - P_0 A\varphi_0; \varphi_1 \rangle| = 2|\langle (A - P_0 A)\varphi_0; \varphi_1 \rangle| \leq 2||A - P_0 A|| ||\varphi_1|| = 2||A\varphi_0 - P_0 A\varphi_0||.
$$

The latter inequality in the previous formula is obtained using the Cauchy–Schwartz inequality estimating the absolute value of the scalar product. The function $-P_0 A\varphi_0$, which is orthogonal to the function $\varphi_1$, has been added to $A\varphi_0$, in order to obtain the projection of the function $A\varphi_0$ onto the subspace orthogonal to the space of stationary solutions of the diffusion equation (5).

Let us now estimate the quantity $\beta(t) = -2\langle \Delta_{a,b}\varphi_1; \varphi_1 \rangle$. To this end, decompose the function $\varphi_1$ with respect to the orthonormal eigenfunctions $\chi_i$ of the self-adjoint operator $\Delta_{a,b}$. We have $\varphi_1 = \sum_{i=0}^{\infty} c_i \chi_i$. Since the vector $\varphi_1$ is orthogonal to the vectors with eigenvalue equal to 0, we have $\Delta_{a,b}\varphi_1 = \sum_{i: \lambda_i \neq 0} \lambda_i c_i \chi_i$ and $\sum_{i: \lambda_i \neq 0} c_i^2 = 1$. Hence

$$
\langle \Delta_{a,b}\varphi_1; \varphi_1 \rangle = \langle \sum_{i: \lambda_i \neq 0} \lambda_i c_i \chi_i; \sum_{i=1}^{\infty} c_i \chi_i \rangle = \sum_{i: \lambda_i \neq 0} \lambda_i c_i^2 \leq \sum_{i: \lambda_i \neq 0} \lambda_{\text{max}} c_i^2 = \lambda_{\text{max}},
$$

where $\lambda_{\text{max}}$ is the maximal nonzero eigenvalue of the self-adjoint operator $\Delta_{a,b}$.

Using this inequality and also knowing that $\lambda_{\text{max}} = -ab/\hbar$ (see Proof of Theorem 2), we obtain the estimate required in Statement 1:

$$
\beta(t) = -2\langle \Delta_{a,b}\varphi_1; \varphi_1 \rangle \geq \beta_{\text{min}} = -2\lambda_{\text{max}} = \frac{2ab}{\hbar}.
$$

Statement 1 is proved.

**Statement 2.** As $\hbar \to 0$ the estimate of the quantity $\alpha_{\text{max}}$ has the form $\alpha_{\text{max}} = (C + o_h(1))/\sqrt{\hbar}$, where $o_h(1)$ is an infinitely small quantity with
respect to \( \hbar \), and the constant \( C \) is determined by the maximal value of the absolute values of the first and second derivatives of the Hamilton function \( H(x,p) \), when the coordinates and momenta belong to the physical region of values for the given process.

Proof of Statement 2 will be postponed until the end of this section.

For the estimate of the time of the transformation process described in Theorem 4, let us rewrite the equation from Statement 1 in the following form:

\[
dt = \frac{d\eta}{\alpha(t)\sqrt{1 - \eta\sqrt{\eta} + \beta(t)(1 - \eta)\eta}}
\]

or

\[
t = \int_{\varepsilon}^{1-\varepsilon} \frac{d\eta}{\alpha(t)\sqrt{1 - \eta\sqrt{\eta} + \beta(t)(1 - \eta)\eta}} \leq \int_{\varepsilon}^{1-\varepsilon} \frac{d\eta}{-\alpha_{\text{max}}\sqrt{1 - \eta\sqrt{\eta} + \beta_{\text{min}}(1 - \eta)\eta}}.
\]

(45)

where the integral is taken over the interval \((\varepsilon, 1 - \varepsilon)\), on which the denominator of the expression under the latter integral is positive. Simple computations show that for this the following inequalities should hold:

\[
\frac{1}{2} > \varepsilon > \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{\alpha_{\text{max}}^2}{\beta_{\text{min}}^2}}.
\]

(46)

If one takes into account that \( \beta_{\text{min}} = 2ab/\hbar \) (see Statement 1) and \( \alpha_{\text{max}} = (C + o_h(1))/\sqrt{\hbar} \) (see Statement 2), then, as \( \hbar \to 0 \), the right hand side of the previous inequality can be represented in the following form by decomposing into the Taylor series:

\[
\frac{1}{2} - \sqrt{\frac{1}{4} - \frac{\alpha_{\text{max}}^2}{\beta_{\text{min}}^2}} = \frac{\alpha_{\text{max}}^2}{\beta_{\text{min}}^2} + O\left(\frac{\alpha_{\text{max}}}{\beta_{\text{min}}}\right)^4 = \frac{C^2\hbar}{4a^2b^2} + o\left(\frac{\hbar}{ab}\right).
\]

This and inequality (46) imply that the number \( \varepsilon \) can be chosen to be arbitrary (small, as \( \hbar \to 0 \)), satisfying the inequality

\[
\frac{1}{2} > \varepsilon > \frac{C^2\hbar}{4a^2b^2} + o\left(\frac{\hbar}{ab}\right).
\]

(47)

The last integral in inequality (45) of the form

\[
\int_{\varepsilon}^{1-\varepsilon} \frac{d\eta}{-\alpha_{\text{max}}\sqrt{1 - \eta\sqrt{\eta} + \beta_{\text{min}}(1 - \eta)\eta}}.
\]

(48)
has been computed using the system Mathematica 5.0 (see [14]), and the result was decomposed into the series with respect to $\beta_{\min}$ as $\beta_{\min} \to \infty$.

These computations yield the following equality:

$$t_\varepsilon = \frac{4 \text{arctanh}(1 - 2\varepsilon)}{\beta_{\min}} + \frac{4(1 - 2\varepsilon)}{\sqrt{\varepsilon(1 - \varepsilon)} \beta_{\min}^2} + O\left(\frac{1}{\beta_{\min}^3}\right),$$  \hspace{1cm} (49)$$

where $\text{arctanh}(x) = (\ln(1 + x) - \ln(1 - x))/2$ is the hyperbolic arctangens.

If one assumes that $\varepsilon$ and $\bar{h}$ are small quantities, then, substituting into this equality the expressions for $\beta_{\min} = 2ab/\bar{h}$ and $\alpha_{\max} = (C + o_h(1))/\sqrt{\bar{h}}$ from Statements 1 and 2, and decomposing the obtained expression into a series with respect to $\varepsilon$, one can obtain the following estimate:

$$t_\varepsilon = (-\ln \varepsilon + o(\varepsilon)) \frac{h}{ab} + \frac{1 + o(\varepsilon)}{\sqrt{\varepsilon}} \frac{C}{\sqrt{\bar{h}}} \left(\frac{h}{ab}\right)^2 + O\left(\frac{h^2}{a^2b^2}\right).$$  \hspace{1cm} (50)$$

Inequality (45), formulas (47), (48), (50), and the definition of the function $\eta(t)$ (40) immediately imply the following statement.

**Statement 3.** Let $\bar{\varphi}(x,p,t)$ be a normalized solution of equation (2), and, at the initial moment for $t = 0$, let the following inequality hold: $\eta(0) \overset{\text{def}}{=} ||P_0\bar{\varphi}(x,p,0)||^2 \geq \varepsilon$, where $P_0$ is the projection operator from Theorem 2 and $\varepsilon$ is an arbitrary number satisfying the inequalities

$$\frac{1}{2} > \varepsilon > \frac{C^2h}{4a^2b^2} + o\left(\frac{h}{ab}\right).$$

Then, for any $t > t_\varepsilon$, where

$$t_\varepsilon = (-\ln \varepsilon + o(\varepsilon)) \frac{h}{ab} + \frac{1 + o(\varepsilon)}{\sqrt{\varepsilon}} \frac{C}{\sqrt{\bar{h}}} \left(\frac{h}{ab}\right)^2 + O\left(\frac{h^2}{a^2b^2}\right),$$  \hspace{1cm} (51)$$

the quantity $\eta(t) \overset{\text{def}}{=} ||P_0\bar{\varphi}(x,p,t)||^2 \geq 1 - \varepsilon$, i. e., the square of the distance from the function $\bar{\varphi}(x,p,t)$ to the subspace of stationary solutions of the diffusion equation, described by Theorem 1, will be less than or equal to $\varepsilon$.

Part 1) of Theorem 4, being proved at this section, immediately follows from Statement 3. For conclusion of the proof it remains to prove Statement 2.

**Proof of Statement 2.** In the proof of Statement 2 we shall need computations of some integrals containing the function $\chi(y)$ given by expression (26) from Theorem 1. The results of computations of these integrals are listed in the following two Lemmas.
Lemma 1. Let $\tilde{\chi}$ be the Fourier transform of the function $\chi$, where $\chi(y) = (b/a\pi h)^{n/4} \exp(-by^2/(2ah))$. Then the following equality holds:

$$\tilde{\chi}(k) = \frac{1}{(2\pi h)^{n/2}} \int_{R^n} \chi(y)e^{i\frac{y}{h}k}dy = \left(\frac{a}{b\pi h}\right)^{n/4} e^{-\frac{ak^2}{h}}. \quad (52)$$

On the contrary, the Fourier transform of $\tilde{\chi}$ yields $\chi$. I. e.,

$$\chi(y) = \frac{1}{(2\pi h)^{n/2}} \int_{R^n} \tilde{\chi}(k)e^{-\frac{ik}{h}y}dk. \quad (53)$$

One has more general integrals:

$$\frac{1}{(2\pi h)^{n/2}} \int_{R^n} \chi(x - y)e^{i\frac{y}{h}(p - k)}dy = \tilde{\chi}(p - k)e^{i\frac{p}{h}k}, \quad (54)$$

$$\frac{1}{(2\pi h)^{n/2}} \int_{R^n} \tilde{\chi}(p - k)e^{i\frac{k}{h}(y - x)}dk = \chi(x - y)e^{-\frac{ip}{h}y}. \quad (55)$$

Functions $\chi$ and $\tilde{\chi}$ satisfy the following relations:

$$\chi(\alpha)\chi(\beta) = \chi\left(\frac{\alpha + \beta}{\sqrt{2}}\right)\chi\left(\frac{\alpha - \beta}{\sqrt{2}}\right) \quad \text{and} \quad \tilde{\chi}(\alpha)\tilde{\chi}(\beta) = \tilde{\chi}\left(\frac{\alpha + \beta}{\sqrt{2}}\right)\tilde{\chi}\left(\frac{\alpha - \beta}{\sqrt{2}}\right). \quad (56)$$

The derivatives of the functions $\chi(y)$ and $\tilde{\chi}(k)$ have the form

$$\frac{\partial \chi(y)}{\partial y_j} = -b/(ah) \; y_j\chi(y); \quad \frac{\partial \tilde{\chi}'(k)}{\partial k_j} = -a/(bh) \; k_j\tilde{\chi}(k). \quad (57)$$

The functions $\chi^2$ and $\tilde{\chi}^2$ are the densities of the normal distribution in the configuration space and in the space of momenta, respectively, with the zero mathematical expectations and the dispersions equal to $ah/(2b)$ and $bh/(2a)$. I. e., the following equalities for the function $\chi$ hold:

$$\int_{R^n} \chi^2(\eta)d\eta = 1; \quad \int_{R^n} \eta_i\chi^2(\eta)d\eta = 0, \; i = 1, \ldots, n;$$
$$\int_{R^n} \eta_i\eta_j\chi^2(\eta)d\eta = 0, \; \text{for} \; i \neq j; \quad \int_{R^n} \eta_i^2\chi^2(\eta)d\eta = ah/(2b), \; i = 1, \ldots, n. \quad (58)$$

The other moments have order $o(h)$. Analogous equalities hold for the function $\tilde{\chi}$:

$$\int_{R^n} \tilde{\chi}^2(\xi)d\xi = 1; \quad \int_{R^n} \xi_i\tilde{\chi}^2(\xi)d\xi = 0, \; i = 1, \ldots, n;$$
$$\int_{R^n} \xi_i\xi_j\tilde{\chi}^2(\xi)d\xi = 0, \; \text{for} \; i \neq j; \quad \int_{R^n} \xi_i^2\tilde{\chi}^2(\xi)d\xi = bh/(2a), \; i = 1, \ldots, n. \quad (59)$$
Besides that, the following equalities hold:

\[ \frac{1}{(2\pi\hbar)^{n/2}} \int_{R^n} \eta_j \xi_j \chi(\eta) \bar{\chi}(\xi) e^{i\eta\xi / \hbar} d\eta d\xi = 0, \text{ for } i \neq j; \]  
\[ \frac{1}{(2\pi\hbar)^{n/2}} \int_{R^n} \eta_j \xi_j \chi(\eta) \bar{\chi}(\xi) e^{i\eta\xi / \hbar} d\eta d\xi = \frac{i\hbar}{2}, \quad j = 1, \ldots, n. \]  
(60)

(61)

All integrals from Lemma 1, except for the latter one (61), are well known. The latter integral has been computed by the author using the system Mathematica 5.0 [14].

**Lemma 2.** Let

\[ D = \frac{1}{(2\pi\hbar)^{n/2}} \chi(\eta) \chi(\eta'/\sqrt{2}) \bar{\chi}(\xi) \bar{\chi}(\xi') e^{i(\eta\xi + \eta\xi' + \eta'\xi + \eta'\xi')/\hbar}. \]  
(62)

Then the following equalities hold:

1) \[ \int_{R^{4n}} D\eta \ d\xi \ d\eta' d\xi' = 1; \]
2) \[ \int_{R^{4n}} \eta_j D\eta \ d\xi \ d\eta' d\xi' = 0, \quad j = 1, \ldots, n; \]
3) \[ \int_{R^{4n}} (\eta_j + \eta_j') D\eta \ d\xi \ d\eta' d\xi' = 0, \quad j = 1, \ldots, n; \]
4) \[ \int_{R^{4n}} \xi_j D\eta \ d\xi \ d\eta' d\xi' = 0, \quad j = 1, \ldots, n; \]
5) \[ \int_{R^{4n}} (\xi_j + \xi_j') D\eta \ d\xi \ d\eta' d\xi' = 0, \quad j = 1, \ldots, n; \]
6) \[ \int_{R^{4n}} \eta_i (\eta_j + \eta_j') D\eta \ d\xi \ d\eta' d\xi' = 0, \quad i, j = 1, \ldots, n; \]
7) \[ \int_{R^{4n}} (\xi_i + \xi_i') \xi_j D\eta \ d\xi \ d\eta' d\xi' = 0, \quad i, j = 1, \ldots, n; \]
8) \[ \int_{R^{4n}} \eta_i \eta_j D\eta \ d\xi \ d\eta' d\xi' = 0, \text{ for } i \neq j; \]
\[ \int_{R^{4n}} \eta_j^2 D\eta \ d\xi \ d\eta' d\xi' = a\hbar/(2b), \quad j = 1, 2, \ldots, n; \]
9) \[ \int_{R^{4n}} (\eta_i + \eta_i') (\eta_j + \eta_j') D\eta \ d\xi \ d\eta' d\xi' = 0, \text{ for } i \neq j; \]
\[ \int_{R^{4n}} (\eta_j + \eta_j')^2 D\eta \ d\xi \ d\eta' d\xi' = a\hbar/(2b), \quad j = 1, 2, \ldots, n; \]
10) \[ \int_{R^{4n}} \xi_i \xi_j D\eta \ d\xi \ d\eta' d\xi' = 0, \text{ for } i \neq j; \]
\[ \int_{R^{4n}} \xi_j^2 D\eta \ d\xi \ d\eta' d\xi' = b\hbar/(2a), \quad j = 1, 2, \ldots, n; \]
11) \[ \int_{R^{4n}} (\xi_i + \xi'_i)(\xi_j + \xi'_j) \, d\eta \, d\xi \, d\eta' \, d\xi' = 0, \text{ for } i \neq j; \]
\[ \int_{R^{4n}} (\xi_j + \xi'_j)^2 \, d\eta \, d\xi \, d\eta' \, d\xi' = \frac{bh}{(2a)}, \text{ for } j = 1, 2, \ldots, n; \]

12) \[ \int_{R^{4n}} \eta_i \xi_j \, d\eta \, d\xi \, d\eta' \, d\xi' = 0, \text{ for } i, j = 1, \ldots, n; \]

13) \[ \int_{R^{4n}} \eta'_i \xi'_j \, d\eta \, d\xi \, d\eta' \, d\xi' = 0, \text{ for } i, j = 1, \ldots, n; \]

Proof of Lemma 2 is based on the use of formulas from Lemma 1. The other integrals are computed in a similar way.

For the first integral, after substitution into it, instead of \( D \), of expression (52), we obtain
\[
\frac{1}{(2\pi h)^{n/2}} \int_{R^{4n}} \chi(\eta) \chi(\eta'/\sqrt{2}) \tilde{\chi}(\xi'/\sqrt{2}) \chi(\xi) e^{i(\eta \xi + \eta' \xi' + \eta \xi + \eta' \xi'/2)\hbar} \, d\eta \, d\eta' \, d\xi \, d\xi'
= \{ \text{integrating over } \xi \text{ by formula (53)} \}
= \frac{1}{(2\pi h)^{n/2}} \int_{R^{4n}} \chi(\eta) \chi(\eta'/\sqrt{2}) \tilde{\chi}(\xi'/\sqrt{2}) \chi(\eta + \eta') e^{i(\eta + \eta' \xi'/2)\hbar} \, d\eta \, d\eta' \, d\xi \, d\xi'
= \{ \text{integrating over } \xi'/\sqrt{2} \text{ by formula (53)} \}
= \int_{R^{2n}} \chi(\eta) \chi(\eta'/\sqrt{2}) \chi(\sqrt{2}\eta + \eta'/\sqrt{2}) \chi(\eta + \eta') \, d\eta \, d\eta'
= \{ \text{apply formula (56) to the two middle factors } \chi \}
= \int_{R^{2n}} \chi^2(\eta) \chi^2(\eta + \eta') \, d\eta \, d\eta' = 1 \{ \text{by formula (58)} \}.

In the proof of Statement 2 we shall also need the averaging of functions \( F(x, p) \) on the phase space \( (x, p) \in R^{2n} \) with respect to the density of the
distribution

\[ W_\psi' = \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^n} \psi(x)\psi^*(y)e^{\frac{i(y-x)p}{\hbar}}dy. \] (63)

This density looks similar to Wigner’s quasidistribution, but does not coincide with it.

Denote by \( \bar{F}_{W_\psi'} \) the average of the function \( F(x,p) \) with respect to the density \( W_\psi' \). That is,

\[ \bar{F}_{W_\psi'} = \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^{3n}} F(x,p)\psi(x)\psi^*(y)e^{\frac{i(y-x)p}{\hbar}}dydxdp. \] (64)

Lemma 3. If \( F(x,p) \) is a smooth function which, together with all its derivatives, grows at infinity no faster than a polynomial, and \( \psi(x) \) is an arbitrary smooth complex valued function rapidly decreasing at infinity, then the following equality holds:

\[ \lim_{\hbar \to 0} \bar{F}_{W_\psi'} = \int_{\mathbb{R}^n} F(x,0)\psi(x)\psi^*(x)dx. \]

Proof of Lemma 3. Let us make the change of variables \( k = p/\hbar \) under the integral (64), and let us integrate over \( y \). We obtain

\[ \bar{F}_{W_\psi'} = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{3n}} F(x,\hbar k)\psi(x)\psi^*(y)e^{i(y-x)k}dydxdk \]

\[ = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^{2n}} F(x,\hbar k)\psi(x)\bar{\psi}^*(k)e^{-ikx}dkdx, \] (65)

where \( \bar{\psi}^*(k) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \psi^*(y)e^{iyk}dy \) (66)

is the Fourier transform of the function \( \psi^*(y) \). Since the function \( \psi^*(y) \) is rapidly decreasing, its Fourier transform \( \bar{\psi}^*(k) \) is also a function rapidly decreasing at infinity.

Since \( \hbar \) is a small quantity, let us decompose the smooth function \( F(x,\hbar k) \) over \( \hbar k \) by the Taylor formula up to the terms of first order. We have

\[ F(x,\hbar k) = F(x,0) + \hbar \sum_{i=1}^n k_i \frac{\partial F}{\partial p_i}(x,\theta \hbar k), \]

where \( \theta = (\theta_1, \ldots, \theta_n) \) and \( 0 \leq \theta_i \leq 1 \) for \( i = 1, \ldots, n \).
Let us substitute this expression of the function $F(x, \hbar k)$ into (65). We obtain

$$
\tilde{F}_\psi = \frac{1}{(2\pi)^n/2} \int_{R^{2n}} \left( F(x, 0) + \hbar \sum_{i=1}^{n} k_i \frac{\partial F}{\partial p_i} (x, \theta h k) \right) \psi(x) \tilde{\psi}^*(k) e^{-ixk} dx dk
$$

$$
= \frac{1}{(2\pi)^n/2} \int_{R^{2n}} F(x, 0) \psi(x) \tilde{\psi}^*(k) e^{-ixk} dx dk
$$

$$
+ \frac{\hbar}{(2\pi)^n/2} \int_{R^{2n}} \sum_{i=1}^{n} k_i \frac{\partial F}{\partial p_i} (x, \theta h k) \psi(x) \tilde{\psi}^*(k) e^{-ixk} dx dk.
$$

(67)

Let us estimate the coefficient before $\hbar$ in the second summand of the obtained equality. We have

$$
\left| \frac{1}{(2\pi)^n/2} \int_{R^{2n}} \sum_{i=1}^{n} k_i \frac{\partial F}{\partial p_i} (x, \theta h k) \psi(x) \tilde{\psi}^*(k) e^{-ixk} dx dk \right|
$$

$$
\leq \frac{1}{(2\pi)^n/2} \int_{R^{2n}} \left| \sum_{i=1}^{n} k_i \frac{\partial F}{\partial p_i} (x, \theta h k) \right| |\psi(x)||\tilde{\psi}^*(k)| dx dk
$$

$$
= \frac{1}{(2\pi)^n/2} \lim_{r \to \infty} \int_{D_r} \left| \sum_{i=1}^{n} k_i \frac{\partial F}{\partial p_i} (x, \theta h k) \right| |\psi(x)||\tilde{\psi}^*(k)| dx dk
$$

$$
\leq \frac{1}{(2\pi)^n/2} \lim_{r \to \infty} \max_{D_r} \left( \left| \sum_{i=1}^{n} k_i \frac{\partial F}{\partial p_i} (x, \theta h k) \right| \right) |\psi(x)||\tilde{\psi}^*(k)| dx dk
$$

$$
\leq \frac{1}{(2\pi)^n/2} \lim_{r \to \infty} \max_{D_r} \left( \left| \sum_{i=1}^{n} k_i \frac{\partial F}{\partial p_i} (x, \theta h k) \right| \right) |\psi(x)||\tilde{\psi}^*(k)| dx dk
$$

$$
= \lim_{r \to \infty} \int_{D_r} O(x^n, k^n)|\psi(x)||\tilde{\psi}^*(k)| dx dk = M.
$$

The latter equalities follow from the fact that by the statement of Lemma 3, the expression in the integral, standing under the operation max, grows no faster than a polynomial of certain degree $N$ with respect to each variable, and $|\psi(x)|$ and $|\tilde{\psi}^*(k)|$ are rapidly decreasing functions (decreasing at infinity faster than any power), and the limit as $r \to \infty$ of the integral of a positive rapidly decreasing function exists and is equal to some $M$.

Formula (67) and the boundedness of the coefficient before $\hbar \to 0$ in the second summand of this formula imply that

$$
\tilde{F}_\psi = \frac{1}{(2\pi)^n/2} \int_{R^{2n}} F(x, 0) \psi(x) \tilde{\psi}^*(k) e^{-ixk} dx dk + O(h).
$$

(68)
Hence
\[
\lim_{\hbar \to 0} \bar{F}_{W'} = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^{2n}} F(x, 0) \psi(x) \tilde{\psi}^*(k) e^{-i x k} dx dk = \int_{\mathbb{R}^n} F(x, 0) \psi(x) \psi^*(x) dx.
\]

The latter equality is obtained by computing the integral over \( k \). This integral over \( k \) is the inverse Fourier transform of the function \( \tilde{\psi}^*(k) \), and it yields the function \( \psi^*(x) \). Lemma 3 is proved.

The distribution \( W' \psi \), as Wigner’s distribution, is not nonnegative, and the distribution \( \rho \psi \), given by expression (29) for the wave function \( \psi \), is nonnegative.

Denote by \( \bar{F}_{\rho} \) the average of the function \( F(x, p) \) with respect to the distribution \( \rho \). That is,
\[
\bar{F}_{\rho} = \int_{\mathbb{R}^{2n}} F(x, p) \rho(x, p) dx dp = \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^{2n}} F(x, p) \psi(x) \psi^*(y) \chi(x - y) \chi(x - y') e^{i(x - y') p / \hbar} dy dp dx.
\]

Lemma 4. If \( F(x, p) \) is a smooth function which, together with its derivatives, grows at infinity no faster than a polynomial, and \( \psi(x) \) is an arbitrary smooth complex valued function rapidly decreasing at infinity, then the following equality holds:
\[
\bar{F}_{W'} = \bar{F}_{\rho} + O(\hbar) = \bar{F}_{\rho} + o_{\hbar}(1),
\]
where \( o_{\hbar}(1) \) is an infinitely small quantity with respect to \( \hbar \).

Proof of Lemma 4. Consider \( \bar{F}_{\rho} \) given by expression (69). Let us represent the function \( \tilde{\psi}^*(y') \), using composition of the direct and inverse Fourier transform (15), in the following form:
\[
\tilde{\psi}^*(y') = \mathcal{F}^{-1}[\mathcal{F}[\tilde{\psi}^*(y''), y'' \to k], k \to y'],
\]
i.e., in the following form:
\[
\tilde{\psi}^*(y') = \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^{2n}} \tilde{\psi}^*(y'') e^{i(y'' - y') k / \hbar} dk dy''.
\]
Let us substitute this expression into expression (69). After simple transformations under integral we obtain

\[
\bar{F}_{\rho \psi} = \frac{1}{(2\pi \bar{\hbar})^{2n}} \int_{R^{2n}} F(x, p) \psi(y) \psi(y') \chi(x - y) \chi(x - y') \times e^{i(y''k - yp + y'k')/\hbar} dydy'dy'y'dkydkdp. \tag{71}
\]

In this expression, let us integrate over \( y' \) using formula (54) from Lemma 1. We obtain

\[
\bar{F}_{\rho \psi} = \frac{1}{(2\pi \bar{\hbar})^{3n/2}} \int_{R^{2n}} F(x, p) \psi(y) \psi(y') \chi(x - y) \tilde{\chi}(p - k) \times e^{i(y''k - yp + xk')/\hbar} dydy'dy'y'dkydkdp. \tag{72}
\]

In this expression, let us make change of variables, introducing new variables \( \xi = p - k \) and \( \eta = x - y \). Then, \( p = k + \xi, \ x = y + \eta, \) and

\[
\bar{F}_{\rho \psi} = \frac{1}{(2\pi \bar{\hbar})^{3n/2}} \int_{R^{2n}} F(y + \eta, k + \xi) \psi(y) \psi(y') \chi(\eta) \tilde{\chi}(\xi) \times e^{i(y''k - yk + \eta\xi)/\hbar} d\eta d\xi dydy'dy'y'dk. \tag{73}
\]

Assuming \( \eta \) and \( \xi \) to be small quantities, let us decompose the function \( F(y + \eta, k + \xi) \) by the Taylor formula at the point \( (y, k) \) up to the terms of second order. We obtain the following expression, in which the values of the function \( F \) and its derivatives are taken at the point \( (y, k) \):

\[
F(y + \eta, k + \xi) = F(y, k) + \sum_{i=1}^{n} (F'_{x_i} \eta_i + F'_{p_i} \xi_i) + \frac{1}{2} \sum_{i,j=1}^{n} (F''_{x_i,x_j} \eta_i \eta_j + 2F''_{x_i,p_j} \eta_i \xi_j + F''_{p_i,p_j} \xi_i \xi_j) + o(\eta^2, \xi^2, \eta \xi). \tag{74}
\]

Let us substitute this decomposition instead of the function \( F(y + \eta, k + \xi) \) into the latter integral, and let us integrate over the variables \( \eta \) and \( \xi \) using the formulas written out in Lemma 1. We obtain

\[
\bar{F}_{\rho \psi} = \frac{1}{(2\pi \bar{\hbar})^{2n}} \int_{R^{2n}} \left[ F(y, k) + \frac{1}{2} \sum_{j=1}^{n} \left( F''_{x_j,x_j} \frac{\alpha h}{2b} + F''_{x_j,p_j} (ih) + F''_{p_j,p_j} \frac{bh}{2a} \right) + o(h) \right] \times \psi(y) \psi(y') e^{i(y''k - yk)/\hbar} dydy'dk
\]

\[
= \bar{F}_{W_\psi} + \hbar \bar{S}_{W_\psi} + o(\hbar), \tag{75}
\]
where $S(x, p) = \frac{1}{2} \sum_{j=1}^{n} \left( F''_{x_j, x_j} (a/2b) + i F''_{x_j, p_j} + F''_{p_j, p_j} (b/2a) \right)$ is the second summand under the integral in expression (75) divided by $\hbar$, and the average values $\bar{F}_{W_\psi}$ and $\bar{S}_{W_\psi}$ of the functions $F$ and $S$ with respect to the distribution $W_\psi$ are given by expression (64).

Since by statement of Lemma 4, the function $F(x, p)$ grows at infinity, together with its derivatives, no faster than a polynomial, the same property holds for the function $S(x, p)$. Let us apply Lemma 3 to the function $S(x, p)$. We obtain that $\bar{S}_{W_\psi}$ is bounded as $\hbar \to 0$. This and equality (75) imply the equality $\bar{F}_{\rho_\psi} = \bar{F}_{W_\psi} + O(\hbar)$, which is equivalent to the equality required in Lemma 4.

Thus, all the Lemmas necessary for the proof of Statement 2, are proved. Let us now proceed to the proof of Statement 2 itself.

By Statement 1, $\alpha_{\max} \overset{\text{def}}{=} 2 \cdot \max \bar{\phi}_0 || A_2 \bar{\phi}_0 - P_0 A_s \bar{\phi}_0 ||$, where $A$ is given in our case by expression (4), the projection operator $P_0$ is given by expression (27), and normalized functions $\bar{\phi}_0$ are given by expression (24), and the function $\bar{\phi}_0 \bar{\phi}_0^*$ is the probability distribution in the physical region of the phase space for the given process.

If the operator $A$ is represented as a sum $A = \sum_{j=1}^{2n+1} A_j$, then, by the property of the norm stating that the norm of the sum of vectors is no greater than the sum of norms of these vectors, we have

$$||A_2 \bar{\phi}_0 - P_0 A_2 \bar{\phi}_0|| = || \sum_{s=1}^{2n+1} (A_s \bar{\phi}_0 - P_0 A_s \bar{\phi}_0)|| \leq \sum_{s=1}^{2n+1} ||A_s \bar{\phi}_0 - P_0 A_s \bar{\phi}_0||. \quad (76)$$

Hence, for the estimate of the quantity $||A_2 \bar{\phi}_0 - P_0 A_2 \bar{\phi}_0||$ it suffices to estimate the quantities $||A_s \bar{\phi}_0 - P_0 A_s \bar{\phi}_0||$, for $s = 1, \ldots, 2n + 1$.

In our case the operator $A$ is given by expression (4):

$$A_\varphi = \sum_{j=1}^{n} \left( \frac{\partial H}{\partial x_j} \frac{\partial \varphi}{\partial p_j} - \frac{\partial H}{\partial p_j} \frac{\partial \varphi}{\partial x_j} \right) - \frac{i}{\hbar} \left( H - \sum_{j=1}^{n} \frac{\partial H}{\partial p_j} p_j \right) \varphi; \quad (77)$$

and $A_{2j-1} = \frac{\partial H}{\partial x_j} \frac{\partial}{\partial p_j}$, $A_{2j} = -\frac{\partial H}{\partial p_j} \frac{\partial}{\partial x_j}$, for $j = 1, \ldots, n$;

and $A_{2n+1} = -\frac{i}{\hbar} f(x, p) = -\frac{i}{\hbar} \left( H - \sum_{j=1}^{n} \frac{\partial H}{\partial p_j} p_j \right). \quad (78)$
Note also that since $P_0$ is a self-adjoint projection operator, the vectors $P_0 A_s \bar{\varphi}_0$ and $A_s \bar{\varphi}_0 - P_0 A_s \bar{\varphi}_0$ are orthogonal. The sum of these vectors equals $A_s \bar{\varphi}_0$. Hence the following equality holds:

$$||A_s \bar{\varphi}_0 - P_0 A_s \bar{\varphi}_0||^2 = ||A_s \bar{\varphi}_0||^2 - ||P_0 A_s \bar{\varphi}_0||^2.$$  \hspace{1cm} (79)

Let us start estimating these quantities, starting with $s = 2n + 1.$

1. The case $A_{2n+1} = -i/\bar{h}f(x,p)$. We estimate $||A_{2n+1}\bar{\varphi}_0 - P_0 A_{2n+1}\bar{\varphi}_0||^2$ when the operator $A_{2n+1}$ is the operator of multiplication by the function $-i/\bar{h}f(x,p)$ and $f(x,p) = H - \sum_{j=1}^{n} \partial H / \partial p_j p_j. \hspace{1cm} (80)$

1.1. An estimate of $||A_{2n+1}\bar{\varphi}_0||^2$. In this case, for the quantity $||A_{2n+1}\bar{\varphi}_0||^2 = \langle A_{2n+1}\bar{\varphi}_0, A_{2n+1}\bar{\varphi}_0 \rangle$, after substitution of $i/\bar{h}f(x,p)$ instead of $A_{2n+1}$, substitution of expressions (24) for $\bar{\varphi}_0$, and multiplication of both parts of the equality by $\bar{h}^2$, we have

$$\bar{h}^2 ||A_{2n+1}\bar{\varphi}_0||^2 = \frac{1}{(2\pi \bar{h})^n} \int_{R^{2n}} dx dp f^2(x,p) \int_{R^n} dy \bar{\psi}(y) \chi(x-y)e^{i(x-y)p}$$
$$\times \int_{R^n} \bar{\psi}^*(y') \chi(x-y')e^{i(x-y)p} dy'$$
$$= \frac{1}{(2\pi \bar{h})^{n/2}} \int_{R^{2n}} dx dp f^2(x,p) \int_{R^n} dy \bar{\psi}(y) \chi(x-y)e^{i(x-y)p} I_{int}. \hspace{1cm} (81)$$

where $I_{int} = \frac{1}{(2\pi \bar{h})^{n/2}} \int_{R^n} \bar{\psi}^*(y') \chi(x-y')e^{i(x-y)p} dy'. \hspace{1cm} (82)$

To transform the integral $I_{int}$ in this expression, let us represent the function $\bar{\psi}^*(y')$, using composition of the direct and inverse Fourier transform (15), in the form

$$\bar{\psi}^*(y') = \mathcal{F}^{-1} \{ \mathcal{F}[\bar{\psi}^*(y'')], y'' \rightarrow k, k \rightarrow y' \},$$

i. e., in the form

$$\bar{\psi}^*(y') = \frac{1}{(2\pi \bar{h})^n} \int_{R^{2n}} \bar{\psi}^*(y'') e^{i(y''-y')k} dk dy''.$$

29
Let us substitute this expression into expression \([82]\) for \(I_{\text{int}}\). We obtain

\[
I_{\text{int}} = \frac{1}{(2\pi\hbar)^{3n/2}} \int_{R^{3n}} \tilde{\psi}^*(y'')e^{\frac{i(y''-y')}{\hbar}} \chi(x-y')e^{\frac{i(x-y)}{\hbar}} dy'dkdy''
\]

\[
= \frac{1}{(2\pi\hbar)^{3n/2}} \int_{R^{3n}} \tilde{\psi}^*(y'') \int_{R^n} \chi(x-y')e^{\frac{i(x-y)}{\hbar}} dy'dkdy''. \tag{83}
\]

Let us compute the latter integral over \(y'\) by formula \([54]\) from Lemma 1. After substitution we obtain

\[
I_{\text{int}} = \frac{1}{(2\pi\hbar)^n} \int_{R^{2n}} \tilde{\psi}^*(y'')e^{\frac{i(y''-y')}{\hbar}} \chi(p-k)e^{\frac{i(x-p)}{\hbar}} dk dy''
\]

\[
= \frac{1}{(2\pi\hbar)^n} \int_{R^{2n}} \tilde{\psi}^*(y'') \tilde{\chi}(p-k)e^{\frac{i(y''-xy)}{\hbar}} dk dy''. \tag{84}
\]

Let us substitute the obtained expression for \(I_{\text{int}}\) into expression \([81]\) for \(h^2|A_{2n+1}\varphi_0|^2\). After simple transformations we obtain

\[
h^2|A_{2n+1}\varphi_0|^2 = \frac{1}{(2\pi\hbar)^{3n/2}} \int_{R^{3n}} f^2(x, p) \tilde{\psi}(y) \tilde{\psi}^*(y'') \chi(x-y) \tilde{\chi}(p-k) \times e^{\frac{i(y''-xy)}{\hbar}} dk dy'' dy dx dp. \tag{85}
\]

In the obtained expression, let us make change of variables, introducing new variables \(\xi = p-k\) and \(\eta = x-y\). Then \(p = k+\xi, x = y+\eta,\) and

\[
h^2|A_{2n+1}\varphi_0|^2 = \frac{1}{(2\pi\hbar)^{3n/2}} \int_{R^{3n}} f^2(y+\eta, k+\xi) \tilde{\psi}(y) \tilde{\psi}^*(y'') \chi(\eta)\tilde{\chi}(\xi) \times e^{\frac{i(y''-xy+\eta\xi)}{\hbar}} dk dy'' dy d\eta d\xi. \tag{86}
\]

Assuming that \(\eta\) and \(\xi\) are small quantities, let us decompose the function \(f^2(y+\eta, k+\xi)\) into the Taylor series at the point \((y, k)\) up to terms of the second order. We obtain the following expression, in which the values of the function \(f\) and its derivatives are taken at the point \((y, k)\),

\[
f^2(y+\eta, k+\xi) = f^2 + \sum_{i=1}^{n} (2f f_{x_i} \eta_i + 2f f'_{x_i} \xi_i) + \sum_{i,j=1}^{n} [(f'_{x_i} f'_{x_j} + f f''_{x_i, x_j}) \eta_i \eta_j + (f_{x_i} f'_{x_j} + f f''_{x_i, x_j}) \eta_i \xi_j + (f_{x_i} f'_{x_j} + f f''_{x_i, x_j}) \xi_i \eta_j + o(\eta^2, \xi^2, \eta \xi)]. \tag{87}
\]
Let us substitute this expression, instead of function \(f^2(y + \eta, k + \xi)\), into the latter integral and let us perform integration over the variables \(\eta\) and \(\xi\) using the formulas written out in Lemma 1. We obtain

\[
\hbar^2 \|A_{2n+1} \tilde{\varphi}_0\|^2 = \frac{1}{2\pi h} \int_{R^n} \left[ f^2 + \sum_{j=1}^{n} \left( f_{x_j}' f_{x_j}' + f f_{x_j,x_j}' \right) \frac{a h}{2b} \right.
\]
\[
+ \left. (f_{x_j}' f_{p_j}' + f f_{p_j,x_j}') i h + \left( f_{p_j}' f_{p_j}' + f f_{p_j,p_j}' \right) \frac{b h}{2a} \right] + o(h) \]
\[
\times \tilde{\psi}(y) \tilde{\psi}^*(y') e^{i (y' - y) / \hbar} \text{d}k \text{d}y' \text{d}y.
\]

**1.2. An estimate of \(\|P_0 A_{2n+1} \tilde{\varphi}_0\|^2\).** Let us now estimate the expression subtracted in (79), i.e., \(\|P_0 A_{2n+1} \tilde{\varphi}_0\|^2 = \langle P_0 A_{2n+1} \tilde{\varphi}_0 ; A_{2n+1} \tilde{\varphi}_0 \rangle\), where the operator \(A_{2n+1}\) is the multiplication by the smooth function \(-i/h f(x, p)\). Expanding this expression with the scalar product and substituting into it the expression for the operator \(A_{2n+1}\), expression (24) for \(\tilde{\varphi}_0\) and expression (27) for the operator \(P_0\), represented, in the notations of Lemma 1 in the form

\[
P_0 \tilde{\varphi} = \frac{1}{(2\pi h)^{n/2} 2^{n/2}} \int_{R^n} \varphi(x', p') \chi \left( \frac{x - x'}{\sqrt{2}} \right) \tilde{\chi} \left( \frac{p' - p}{\sqrt{2}} \right) e^{i (x' - x)(p + p') / 2h} \text{d}x' \text{d}p',
\]

and multiplying both parts of the equality by \(h^2\), we obtain

\[
h^2 \|P_0 A_{2n+1} \tilde{\varphi}_0\|^2 = \frac{1}{(2\pi h)^{n/2} 2^{n/2}} \int_{R^{2n}} \tilde{\psi}(y) \chi(x' - y) e^{i (y' - y) / \hbar} f(x', p')
\]
\[
\times \chi \left( \frac{x - x'}{\sqrt{2}} \right) \tilde{\chi} \left( \frac{p' - p}{\sqrt{2}} \right) e^{i (x' - x)(p + p') / 2h} f(x, p)
\]
\[
\times \tilde{\psi}^*(y') \chi(x - y') e^{i (y' - y) / \hbar} \text{d}y' \text{d}x' \text{d}p' \text{d}x \text{d}p
\]
\[
= \frac{1}{(2\pi h)^{n/2} 2^{n/2}} \int_{R^{2n}} \tilde{\psi}(y) \chi(x' - y) e^{i (y' - y) / \hbar} f(x', p')
\]
\[
\times \chi \left( \frac{x - x'}{\sqrt{2}} \right) \tilde{\chi} \left( \frac{p' - p}{\sqrt{2}} \right) e^{i (x' - x)(p + p') / 2h} f(x, p)
\]
\[
\times \text{d}f(x, p) I_{\text{int}} \text{d}x' \text{d}p' \text{d}x \text{d}p,
\]

where \(I_{\text{int}} = \frac{1}{(2\pi h)^{n/2} 2^{n/2}} \int_{R^n} \tilde{\psi}^*(y') \chi(x - y') e^{i (y' - y) / \hbar} \text{d}y'\).

The integral \(I_{\text{int}}\) is the same as in (82) in the computation of \(h^2 \|A_{2n+1} \tilde{\varphi}_0\|^2\). Let us substitute into expression (88) the representation of the integral \(I_{\text{int}}\).
in the form \( [84] \). After simple transformations we obtain,

\[
\hbar^2 |P_0 A_{2n+1} \bar{\varphi}_0|^2 = \frac{1}{(2\pi \hbar)^{2n+2}} \int_{R^n} f(x', p') f(x, p) \bar{\psi}(y) \bar{\psi}^*(y'') \\
\times \chi(x' - y) \chi \left( \frac{x' - x}{\sqrt{2}} \right) \bar{\chi} \left( \frac{p' - p}{\sqrt{2}} \right) \bar{\chi}(p - k) \\
\times e^{i(x-x')(y+y') + i(\eta''k + \eta'y'') - \bar{\eta}'x - \bar{\eta}'y - \bar{\eta}''k)} dk \ dy'' dx' dp' dx \ dp. \tag{90}
\]

In the obtained integral, let us make a change of variables, introducing the new variables \( \eta = x' - y, \ \xi = p - k, \ \eta' = x - x', \ \xi' = p' - p. \)

Then, \( x' = y + \eta, \ p = k + \xi, \ x = y + \eta' + \eta', \ p' = k + \xi + \xi'. \)

After substitution of \( x', x, p, p', \) expressed through the new variables, and after simple transformations, we obtain

\[
\hbar^2 |P_0 A_{2n+1} \bar{\varphi}_0|^2 = \frac{1}{(2\pi \hbar)^{2n+2}} \int_{R^n} f(y + \eta, k + \xi + \xi') f(y + \eta + \eta', k + \xi) \\
\times \bar{\psi}(y) \bar{\psi}^*(y'') \chi(\eta) \chi \left( \eta''/\sqrt{2} \right) \bar{\chi} \left( \xi'/\sqrt{2} \right) \bar{\chi}(\xi) \\
\times e^{i(\eta''y'' + \eta'y'') + i(\eta''k + \eta'y'' - \bar{\eta}'x - \bar{\eta}'y - \bar{\eta}''k)} dk \ dy'' dy dx' dp' dx \ dp. \tag{91}
\]

Since the functions \( \chi^2 \) and \( \bar{\chi}^2 \) yield the densities of normal distributions with small dispersions, let us decompose the function \( f(y + \eta, k + \xi + \xi') f(y + \eta + \eta', k + \xi) \) in the previous expression into the Taylor series at the point \((y, k)\) up to terms of the second order, assuming the quantities \( \eta, \eta', \xi, \xi' \) to be small. We have,

\[
f(y + \eta, k + \xi + \xi') f(y + \eta + \eta', k + \xi) \\
= f^2(y, k) + \sum_{j=1}^n [f f_{\eta j} (\xi_j + \xi'_j) + f f'_{\eta j} (\eta_j + \eta'_j)] + f f'_{\eta j} (\eta_j + \eta'_j) + f f'_{\eta j} (\xi_j + \xi'_j) \\
+ \sum_{i,j=1}^n [f f_{\eta j} (\eta_j + \eta'_j) + f f'_{\eta j} (\eta_j + \eta'_j) + f f'_{\eta j} (\xi_j + \xi'_j)] \\
+ \frac{1}{2} \sum_{i,j=1}^n [f f''_{\eta j} (\eta_j + \eta'_j) + f f''_{\eta j} (\eta_j + \eta'_j) + f f''_{\eta j} (\xi_j + \xi'_j) + f f''_{\eta j} (\xi_j + \xi'_j)] \\
+ \frac{1}{2} \sum_{i,j=1}^n [f f''_{\eta j} (\eta_j + \eta'_j) + f f''_{\eta j} (\eta_j + \eta'_j) + f f''_{\eta j} (\xi_j + \xi'_j) + f f''_{\eta j} (\xi_j + \xi'_j)] \\
+ o \left( \eta^2, (\eta + \eta')^2, \xi^2, (\xi + \xi')^2, \eta (\xi + \xi'), (\eta + \eta') \xi \right). \tag{92}
\]
After substitution of the decomposition of the function \( f(y+\eta,k+\xi,\xi') \phi(y+\eta,k+\xi) \) in the form (92) into expression (91) and computing integrals over \( \eta,\eta',\xi,\xi' \) using the integrals of Lemma 2, we obtain

\[
\hbar^2 ||P_0 A_{2n+1} \bar{\varphi}_0||^2 = \frac{1}{(2\pi \hbar)^n} \int_{R^{3n}} \left[ f^2 + \sum_{j=1}^{n} [f f''_{x_j} a \hbar/(2b) + (f'_{x_j})^2 + f f''_{p_j} b \hbar/(2a)] + o(\hbar) \right] \times \bar{\psi}(y) \bar{\psi}^*(y'') e^{-i(y'' - y)k/\hbar} dk dy'' dy.
\]  

(93)

Thus, we have estimated the expression \( \hbar^2 ||A_{2n+1} \bar{\varphi}_0||^2 \) by formula (88) and expression \( \hbar^2 ||P_0 A_{2n+1} \bar{\varphi}_0||^2 \) by formula (93). Let us substitute these formulas into expression (79), let us reduce similar terms, and let us divide both parts of the equality by \( \hbar^2 \). We obtain

\[
||A_{2n+1} \bar{\varphi}_0 - P_0 A_{2n+1} \bar{\varphi}_0||^2 = ||A_{2n+1} \bar{\varphi}_0||^2 - ||P_0 A_{2n+1} \bar{\varphi}_0||^2 = \frac{1}{h(2\pi h)^n} \int_{R^{3n}} \left[ \sum_{j=1}^{n} \left( (f'_{x_j})^2 \frac{a}{2b} + (f'_{p_j})^2 \frac{b}{2a} \right) + o(h(1)) \right] \times \bar{\psi}(y) \bar{\psi}^*(y'') e^{-i(y'' - y)k/\hbar} dk dy'' dy.
\]  

(94)

The last row of expression (94) is the density of the distribution \( W'_{\bar{\psi}} \) given by expression (63). Let us apply Lemma 4 to the right hand side of the equality (94). We obtain

\[
||A_{2n+1} \bar{\varphi}_0 - P_0 A_{2n+1} \bar{\varphi}_0||^2 = \frac{1}{h} \left[ \int_{R^{2n}} \sum_{j=1}^{n} \left( (f'_{x_j})^2 \frac{a}{2b} + (f'_{p_j})^2 \frac{b}{2a} \right) \rho_{\bar{\psi}} dk + o(n(1)) \right].
\]  

(95)

where \( \rho_{\bar{\psi}}(k) \) is the nonnegative density of distribution given by expression (29) for the function \( \bar{\psi}() \).

To estimate the required expression \( \max_{\bar{\varphi}_0} ||A_{2n+1} \bar{\varphi}_0 - P_0 A_{2n+1} \bar{\varphi}_0|| \), introduce the constant \( C_{2n+1} \) by the following equality:

\[
C_{2n+1}^2 \overset{\text{def}}{=} \max_{(x,p) \in U} \sum_{j=1}^{n} \left( (f'_{x_j})^2 \frac{a}{2b} + (f'_{p_j})^2 \frac{b}{2a} \right),
\]  

(96)

where the maximum is taken over the physical domain \( U \) of values of the coordinates and momenta for the given process, which contains the support.
of the density function of the probability distribution $\rho_\psi$. Then, equality (93) implies that

$$\|A_{2n+1}\varphi_0 - P_0A_{2n+1}\varphi_0\|^2 \leq \frac{1}{h} \left( \int_{R^n} C_{2n+1}^2 \rho_\psi dydk + o_h(1) \right)$$

$$= \left( C_{2n+1}^2 + o_h(1) \right) \frac{1}{h}. \quad (97)$$

The latter equality is obtained from the normalization condition for the density $\rho_\psi$, i.e., from the equality $\int_{R^n} \rho_\psi dydk = 1$.

Hence, taking the square root from both parts of the inequality, we finally obtain

$$\|A_{2n+1}\varphi_0 - P_0A_{2n+1}\varphi_0\| \leq \left( C_{2n+1} + o_h(1) \right) \frac{1}{\sqrt{h}}. \quad (98)$$

This finishes examining Case 1.

2. Case $A_{2j-1} = \frac{\partial H}{\partial x_j} \frac{\partial}{\partial p_j}$. Let us estimate the quantity $\|A_{2j-1}\varphi_0 - P_0A_{2j-1}\varphi_0\|^2 = \|A_{2j-1}\varphi_0\|^2 - \|P_0A_{2j-1}\varphi_0\|^2$ by the same scheme as in the previous case: we separately estimate $\|A_{2j-1}\varphi_0\|^2$ and $\|P_0A_{2j-1}\varphi_0\|^2$.

2.1. An estimate of $\|A_{2j-1}\varphi_0\|^2$. Let us substitute into $\|A_{2j-1}\varphi_0\|^2 = \langle A_{2j-1}\varphi_0, A_{2j-1}\varphi_0 \rangle$ expression (24) for $\varphi_0$, and put $A_{2j-1} = \frac{\partial H}{\partial x_j} \frac{\partial}{\partial p_j}$. We have

$$\|A_{2j-1}\varphi_0\|^2 = \int_{R^{2n}} \frac{\partial H}{\partial x_j} \frac{\partial}{\partial p_j} \left( \frac{1}{(2\pi h)^{n/2}} \int_{R^n} \bar{\psi}(y) \chi(x-y) e^{i(x-y)\varphi_h} dy \right)$$

$$\times \frac{\partial H}{\partial x_j} \frac{\partial}{\partial p_j} \left( \frac{1}{(2\pi h)^{n/2}} \int_{R^n} \bar{\psi}^*(y') \chi(x-y') e^{i(y'-y)\varphi_h} dy' \right) dx dp$$

$$= \int_{R^{2n}} \frac{\partial H}{\partial x_j} \left( \frac{1}{(2\pi h)^{n/2}} \int_{R^n} \bar{\psi}(y) \chi(x-y) \frac{i(x_j - y_j)}{h} e^{i(x-y)\varphi_h} dy \right)$$

$$\times \frac{\partial H}{\partial x_j} \left( \frac{1}{(2\pi h)^{n/2}} \int_{R^n} \bar{\psi}(y) \chi(x-y) e^{i(x-y)\varphi_h} dy \right) dx dp$$

$$= \int_{R^{2n}} \frac{\partial H}{\partial x_j} \left( \frac{1}{(2\pi h)^{n/2}} \int_{R^n} \bar{\psi}(y) \chi(x-y) \frac{i(x_j - y_j)}{h} e^{i(x-y)\varphi_h} dy \right)$$

$$\times \frac{\partial H}{\partial x_j} \left( \frac{1}{(2\pi h)^{n/2}} \int_{R^n} \bar{\psi}(y) \chi(x-y) e^{i(x-y)\varphi_h} dy \right) dx dp, \quad (99)$$

where $I_{int}$ is given by expression (52). Let us substitute here, instead of $I_{int}$, its expression in the form (84). After simple transformations and after substitution of the derivative of the function $\tilde{\chi}$ in the form (57), we obtain

$$\|A_{2j-1}\varphi_0\|^2 = \frac{1}{(2\pi h)^{3n/2}} \int_{R^{2n}} \frac{\partial H}{\partial x_j} \left( \bar{\psi}(y) \chi(x-y) \frac{i(x_j - y_j)}{h} e^{i(x-y)\varphi_h} \right)$$

\[34\]
\[
\times (p_j - k_j)\hat{\chi}(p - k) e^{\frac{i\phi''(y'' - \eta p + \eta k)}{\hbar}} dk \, dy' \, dy' \, dx \, dp
\]

In the obtained expression, let us make the change of variables \( \eta = x - y \) and \( \xi = p - k \). Then, \( x = y + \eta, \, p = k + \xi, \) and

\[
||A_{2j-1}\tilde{\varphi}_0||^2 = -\frac{ia}{bh^2 (2\pi \hbar)^{3n/2}} \int_{R^{3n}} \left( \frac{\partial H}{\partial x_j} \right)^2 \tilde{\psi}(y) \overline{\psi}^{*}(y'') \eta_j \chi(\eta) \\
\times (\xi_j) \hat{\chi}(\xi) e^{\frac{i\phi''(y'' - \eta k + \eta \xi)}{\hbar}} dk \, dy' \, dy' \, d\eta \, d\xi,
\]

where in the function \( \left( \frac{\partial H}{\partial x_j} \right)^2, \) instead of variables \( x \) and \( p, \) we have substituted \( y + \eta \) and \( k + \xi, \) respectively.

Assuming \( \eta \) and \( \xi \) to be small quantities, let us decompose the function \( \left( \frac{\partial H}{\partial x_j} \right)^2 (y + \eta, k + \xi) \) into the Taylor series up to the zero order. We have

\[
\left( \frac{\partial H}{\partial x_j} \right)^2 (y + \eta, k + \xi) = \left( \frac{\partial H}{\partial x_j} \right)^2 (y, k) + O(\eta, \xi).
\]

Let us substitute this expression into the previous one, and integrate it over \( \eta \) and \( \xi, \) using the integral (61) from Lemma 1. Finally we obtain

\[
||A_{2j-1}\tilde{\varphi}_0||^2 = -\frac{a}{2bh (2\pi \hbar)^n} \int_{R^{3n}} \left( \left( \frac{\partial H}{\partial x_j} \right)^2 + o_h(1) \right) \tilde{\psi}(y) \overline{\psi}^{*}(y'') \\
\times e^{\frac{i\phi''(y'' - \eta p + \eta k)}{\hbar}} dk \, dy' \, dy. \quad (100)
\]

### 2.2. An estimate of \( ||P_0 A_{2j-1}\tilde{\varphi}_0||^2. \)

By construction, this expression satisfies the inequalities

\[
0 \leq ||P_0 A_{2j-1}\tilde{\varphi}_0||^2 \leq ||A_{2j-1}\tilde{\varphi}_0||^2.
\]

Hence and from relations (79) and (100) we obtain

\[
||A_{2j-1}\tilde{\varphi}_0 - P_0 A_{2j-1}\tilde{\varphi}_0||^2 = ||A_{2j-1}\tilde{\varphi}_0||^2 - ||P_0 A_{2j-1}\tilde{\varphi}_0||^2 \leq ||A_{2j-1}\tilde{\varphi}_0||^2
\]

\[
= \frac{a}{2bh (2\pi \hbar)^n} \int_{R^{3n}} \left( \left( \frac{\partial H}{\partial x_j} \right)^2 + o_h(1) \right) \tilde{\psi}(y) \overline{\psi}^{*}(y'') \\
\times e^{\frac{i\phi''(y'' - \eta p + \eta k)}{\hbar}} dk \, dy' \, dy. \quad (101)
\]

35
The last row of expression (101) is the density of the distribution $W'_\psi$ given by expression (63). Let us apply Lemma 4 to the right hand side of equality (101). We obtain

$$||A_{2j-1}\varphi_0 - P_0A_{2j-1}\varphi_0||^2 = \frac{a}{2b\hbar} \left[ \int_{R^{2n}} \left( \frac{\partial H}{\partial x_j} \right)^2 \rho_{\bar{\psi}} \rho_{\bar{\psi}} \, dk + o_\hbar(1) \right],$$

where $\rho_{\bar{\psi}}(x,k)$ is the nonnegative density of distribution given by expression (29) for the function $\bar{\psi}(\cdot)$.

Introduce the constant $C_{2j-1}$ by the following equality:

$$C_{2j-1}^2 \overset{\text{def}}{=} \frac{a}{2b} \max_{(x,p) \in U} \left( \frac{\partial H}{\partial x_j} \right)^2,$$

where the maximum is taken over the physical region $U$ of values of coordinates and momenta for the given process.

Taking into account this notation, inequality (102) implies that

$$||A_{2j-1}\varphi_0 - P_0A_{2j-1}\varphi_0|| \leq \frac{1}{\hbar} \left( \int_{R^{2n}} C_{2j-1}^2 \rho_{\bar{\psi}} \rho_{\bar{\psi}} \, dy \, dk + o_\hbar(1) \right) \leq \left( C_{2j-1}^2 + o_\hbar(1) \right) \frac{1}{\sqrt{\hbar}}.$$

The latter equality is obtained, as in the first case, from the normalization condition for the density of the probability distribution $\rho_{\bar{\psi}}$.

Hence, taking the square root from both parts of the latter inequality, we finally obtain

$$||A_{2j-1}\varphi_0 - P_0A_{2j-1}\varphi_0|| \leq (C_{2j-1} + o_\hbar(1)) \frac{1}{\sqrt{\hbar}}.$$

This finishes examining Case 2.

3. Case $A_{2j} = -\frac{\partial H}{\partial p_j} \frac{\partial}{\partial x_j}$. Let us estimate the quantity $||A_{2j}\varphi_0 - P_0A_{2j}\varphi_0||$.

If one applies the operator $A_{2j} = -\frac{\partial H}{\partial p_j} \frac{\partial}{\partial x_j}$ to the function $\varphi_0$ of the form (24), i. e., to the function

$$\varphi_0(x,p) = \frac{1}{(2\pi\hbar)^{n/2}} \int_{R^n} \bar{\psi}(y) \chi(x-y) e^{i(x-y)p/\hbar} \, dy,$$
then, taking into account formula (57) for the derivative of the function \( \chi \), we obtain the following equalities:

\[
A_{2j}\bar{\varphi}_0 = \frac{\partial H}{\partial p_j} \bar{\varphi}_0 = -\frac{\partial H}{\partial p_j} \frac{1}{(2\pi \hbar)^n/2} \int_\mathbb{R}^n \tilde{\psi}(y) \times \left( -\frac{b}{ah}(x_j - y_j) + \frac{ip_j}{\hbar} \right) \chi(x - y) e^{i(x - y)p/\hbar} dy
\]

\[
\times \left( -\frac{b}{ah}(x_j - y_j) + \frac{ip_j}{\hbar} \right) \chi(x - y) e^{i(x - y)p/\hbar} dy
\]

\[
= \frac{b}{ah} \frac{\partial H}{\partial p_j} \frac{1}{(2\pi \hbar)^n/2} \int_\mathbb{R}^n \tilde{\psi}(y)(x_j - y_j) \chi(x - y) e^{i(x - y)p/\hbar} dy
\]

\[
- \frac{ip_j}{\hbar} \frac{\partial H}{\partial p_j} \frac{1}{(2\pi \hbar)^n/2} \int_\mathbb{R}^n \tilde{\psi}(y) \chi(x - y) e^{i(x - y)p/\hbar} dy
\]

\[
= -ib \frac{\partial H}{\partial x_j} \bar{\varphi}_0 - ip_j \frac{\partial H}{\partial p_j} \bar{\varphi}_0.
\]

That is, \( A_{2j}\bar{\varphi}_0 = A'_{2j}\bar{\varphi}_0 + A''_{2j}\bar{\varphi}_0 \), where

\[
A'_{2j}\bar{\varphi}_0 \overset{\text{def}}{=} -\frac{ib}{a} \frac{\partial H}{\partial x_j} \bar{\varphi}_0, \quad A''_{2j}\bar{\varphi}_0 \overset{\text{def}}{=} -\frac{ip_j}{\hbar} \frac{\partial H}{\partial p_j} \bar{\varphi}_0.
\]

Hence, using the property of the norm that the norm of a sum of vectors is no greater than the sum of norms of these vectors, we obtain the inequality

\[
||A_{2j}\bar{\varphi}_0 - P_0 A_{2j}\bar{\varphi}_0|| \leq ||A'_{2j}\bar{\varphi}_0 - P_0 A'_{2j}\bar{\varphi}_0|| + ||A''_{2j}\bar{\varphi}_0 - P_0 A''_{2j}\bar{\varphi}_0||. \quad (105)
\]

Note that the operator \( A'_{2j} \) coincides with the operator \( A_{2j-1} \) (see the previous Case) if one replaces the function \( \frac{\partial H}{\partial x_j} \) in it to \( -ib \frac{\partial H}{\partial x_j} \). Therefore, using formulas (104) and (103), we obtain

\[
||A'_{2j}\bar{\varphi}_0 - P_0 A'_{2j}\bar{\varphi}_0|| \leq \left( C'_{2j} + o_1(1) \right) / \sqrt{\hbar}, \quad (106)
\]

where \( (C'_{2j})^2 \overset{\text{def}}{=} \frac{b}{2a} \max_{(x,p) \in \mathcal{U}} \left( \frac{\partial H}{\partial p_j} \right)^2 \).

On the other hand, the operator \( A''_{2j} \) coincides with the operator \( A_{2n+1} \) (see Case 1), in which \( f(x, p) = p_j \frac{\partial H}{\partial p_j} \). Hence, using formulas (98) and (96), we obtain

\[
||A''_{2j}\bar{\varphi}_0 - P_0 A''_{2j}\bar{\varphi}_0|| \leq \left( C''_{2j} + o_1(1) \right) / \sqrt{\hbar}, \quad (107)
\]

37
where \((C''_{2j})^2 \overset{\text{def}}{=} \max_{(x,p) \in U} \sum_{k=1}^{n} \left[ \left( \frac{\partial}{\partial x_k} \left( p_j \frac{\partial H}{\partial p_j} \right) \right)^2 \frac{a^2}{2b} + \left( \frac{\partial}{\partial p_k} \left( p_j \frac{\partial H}{\partial p_j} \right) \right)^2 \frac{b}{2a} \right]. \)

Put \(C_{2j}'' \overset{\text{def}}{=} C_{2j}'' + C_{2j}'''. \) Then using inequalities (105), (106), and (107), and using the introduced notation \(C_{2j}, \) we finally obtain

\[\|A_{2j}\overline{\varphi}_0 - P_0 A_{2j}\overline{\varphi}_0\| \leq \left( C_{2j}'' + o_h(1) \right) \frac{1}{\sqrt{h}} = (C_{2j} + o_h(1)) \frac{1}{\sqrt{h}} \] (108)

This finishes examining Case 3.

Now we are ready to finish the proof of Statement 2.

By Statement 1, \(\alpha_{\text{max}} \overset{\text{def}}{=} 2 \cdot \max_{\overline{\varphi}_0} |A\overline{\varphi}_0 - P_0 A\overline{\varphi}_0|\). Hence, applying to inequality (76) relations (98), (104), and (108), we obtain

\[\alpha_{\text{max}} = 2 \cdot \max_{\overline{\varphi}_0} |A\overline{\varphi}_0 - P_0 A\overline{\varphi}_0| \leq 2 \cdot \max_{\overline{\varphi}_0} \sum_{s=1}^{2n+1} |A_s\overline{\varphi}_0 - P_0 A_s\overline{\varphi}_0| \]

\[= 2 \sum_{s=1}^{2n+1} (C_s + o_h(1)) \frac{1}{\sqrt{h}} = (C + o_h(1)) \frac{1}{\sqrt{h}},\]

where \(C \overset{\text{def}}{=} 2 \sum_{s=1}^{2n+1} C_s. \)

Q. E. D. Statement 2 is proved.
Appendix 2

Proof of Theorem 5

For proof of Theorem 5, consider formula (33) from Theorem 4 for the case when $H(x,p) = \frac{x^2}{2m} + V(x)$. We have

$$
\hat{H}\psi = \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^n} \left( \frac{p^2}{2m} + V(x) - \sum_{k=1}^{n} \left( \frac{\partial V}{\partial x_k} + \frac{ib}{am} p_k \right) (x_k - y'_k) \right)
\times \chi(x - y) \chi(x - y') e^{\frac{ib}{\hbar} (y - y')p} \psi(y', t) dy' dp. \tag{109}
$$

Thus, in this case $\hat{H}\psi$ is represented as the sum of three integrals

$$I_1 = \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^n} \frac{p^2}{2m} \chi(x - y) \chi(x - y') e^{\frac{ib}{\hbar} (y - y')p} \psi(y', t) dy' dp, \tag{110}
$$

$$I_2 = -\frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^n} \frac{ib}{am} \sum_{k=1}^{n} p_k (x_k - y'_k)
\times \chi(x - y) \chi(x - y') e^{\frac{ib}{\hbar} (y - y')p} \psi(y', t) dy' dp, \tag{111}
$$

$$I_3 = \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^n} \left( V(x) - \sum_{k=1}^{n} \frac{\partial V}{\partial x_k} (x_k - y'_k) \right)
\times \chi(x - y) \chi(x - y') e^{\frac{ib}{\hbar} (y - y')p} \psi(y', t) dy' dp. \tag{112}
$$

Note that the expression $\int_{\mathbb{R}^n} \chi(x - y) \chi(x - y') dx$ from the first integral can be transformed to the following form:

$$
\int_{\mathbb{R}^n} \chi(x - y) \chi(x - y') dx = \left( \frac{b}{a \pi \hbar} \right)^{n/2} \int_{\mathbb{R}^n} e^{-\frac{b(x-y)^2}{4a\hbar}} e^{-\frac{b(y-y')^2}{4a\hbar}} dx
= \left( \frac{b}{a \pi \hbar} \right)^{n/2} \int_{\mathbb{R}^n} e^{-\frac{b(x-(y+y')/2)^2}{4a\hbar}} e^{-\frac{b(y-y')^2}{4a\hbar}} d\left( x - \frac{y + y'}{2} \right) = e^{-\frac{b(y-y')^2}{4a\hbar}}. \tag{113}
$$

Hence the integral $I_1$ is transformed to the form

$$I_1 = \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^n} \frac{p^2}{2m} e^{-\frac{b(y-y')^2}{4a\hbar}} e^{\frac{ib}{\hbar} (y - y')p} \psi(y', t) dy' dp. \tag{114}
$$
As known from the formulas for Fourier transform, for any smooth function \( f(y') \) the following equality holds:

\[
f(y) = \frac{1}{(2\pi \hbar)^n} \int_{\mathbb{R}^n} e^{\frac{i}{\hbar}(y-y')^p} f(y') dy' dp. \tag{115}
\]

Below we shall often use this equality. In particular, one has the equality

\[
\psi(y, t) = \frac{1}{(2\pi \hbar)^n} \int_{\mathbb{R}^n} e^{-\frac{b(y-y')^2}{4a\hbar}} e^{\frac{i}{\hbar}(y-y')^p} \psi(y', t) dy' dp. \tag{116}
\]

Let us differentiate both parts of the latter equality with respect to \( y_k \). We obtain, taking into account relation (115),

\[
\frac{\partial \psi(y, t)}{\partial y_k} = \frac{1}{(2\pi \hbar)^n} \int_{\mathbb{R}^n} \left( \frac{ip_k}{\hbar} - \frac{b(y_k - y'_k)}{2a\hbar} \right) e^{-\frac{b(y-y')^2}{4a\hbar}} e^{\frac{i}{\hbar}(y-y')^p} \psi(y', t) dy' dp
\]

\[
= \frac{1}{(2\pi \hbar)^n} \int_{\mathbb{R}^n} \frac{ip_k}{\hbar} e^{-\frac{b(y-y')^2}{4a\hbar}} e^{\frac{i}{\hbar}(y-y')^p} \psi(y', t) dy' dp. \tag{117}
\]

If we differentiate both parts of the obtained equality with respect to \( y_k \) once more, then we obtain the equality

\[
\frac{\partial^2 \psi(y, t)}{\partial y_k^2} = \frac{1}{(2\pi \hbar)^n} \int_{\mathbb{R}^n} \frac{ip_k}{\hbar} \left( \frac{ip_k}{\hbar} - \frac{b(y_k - y'_k)}{2a\hbar} \right)
\]

\[
\times e^{-\frac{b(y-y')^2}{4a\hbar}} e^{\frac{i}{\hbar}(y-y')^p} \psi(y', t) dy' dp = -\frac{2m}{\hbar^2} I_1^k - I_4^k, \tag{118}
\]

where

\[
I_1^k = \frac{1}{(2\pi \hbar)^n} \int_{\mathbb{R}^n} \frac{p_k^2}{2m} e^{-\frac{b(y-y')^2}{4a\hbar}} e^{\frac{i}{\hbar}(y-y')^p} \psi(y', t) dy' dp, \tag{119}
\]

and

\[
I_4^k = \frac{1}{(2\pi \hbar)^n} \int_{\mathbb{R}^n} \frac{ip_k b(y_k - y'_k)}{2a\hbar} e^{-\frac{b(y-y')^2}{4a\hbar}} e^{\frac{i}{\hbar}(y-y')^p} \psi(y', t) dy' dp. \tag{120}
\]

To compute the integrals \( I_4^k \), consider the following equality which is a particular case of equality (115):

\[
0 = \frac{1}{(2\pi \hbar)^n} \int_{\mathbb{R}^n} \frac{b(y_k - y'_k)}{2a\hbar} e^{-\frac{b(y-y')^2}{4a\hbar}} e^{\frac{i}{\hbar}(y-y')^p} \psi(y', t) dy' dp. \tag{121}
\]
Let us differentiate both parts of this equality with respect to $y_k$. We obtain

$$0 = \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^n} \left( \frac{b}{2ah} + \frac{b(y_k - y_k')}{2ah} \left( \frac{ip_k}{\hbar} - \frac{b(y_k - y_k')}{2ah} \right) \right)$$

$$\times e^{-\frac{(y-y')^2}{4a^2}} e^{i\pi(y-y')p\psi(y', t)} dy dp. \tag{122}$$

Hence, taking into account notation (120) for the integral $I_k^4$, and relation (115), we obtain the equalities

$$0 = \frac{b}{2ah} \psi(y, t) + I_k^4 \quad \text{or} \quad I_k^4 = -\frac{b}{2ah} \psi(y, t). \tag{123}$$

Substituting this equality into relation (118), we obtain

$$\frac{\partial^2 \psi}{\partial y_k^2} = -\frac{2m}{\hbar^2} I_k^1 + \frac{b}{2ah} \psi(y, t).$$

Let us express $I_k^1$ from this equality:

$$I_k^1 = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial y_k^2} + \frac{bh}{4ma} \psi(y, t).$$

Taking the sum of the obtained equality over all $k$ from 1 to $n$, we obtain an expression for the integral $I_1$:

$$I_1 = \sum_{k=1}^{n} I_k^1 = -\sum_{k=1}^{n} \frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial y_k^2} + \frac{nbh}{4ma} \psi(y, t). \tag{124}$$

Let us pass to computing the integral $I_2$ given by expression (111). Consider equality (113) of the form

$$\int_{\mathbb{R}^n} \chi(x-y) \chi(x-y') dx = e^{-\frac{b(x-y')^2}{4ah}}, \quad \text{where} \quad \chi(x-y') = \left( \frac{b}{a\pi\hbar} \right)^{n/4} e^{-\frac{b(x-y')^2}{2a^2}}.$$

Let us differentiate both parts of this equality with respect to $y'_k$. We obtain

$$\int_{\mathbb{R}^n} \frac{b(x_k - y'_k)}{ah} \chi(x-y) \chi(x-y') dx = \frac{b(y_k - y'_k)}{2ah} e^{-\frac{b(y_k - y'_k)^2}{4ah}}.$$
or, after omitting common factors,

\[ \int_{\mathbb{R}^n} (x_k - y'_k) \chi(x - y) \chi(x - y') dx = \frac{y_k - y'_k}{2} e^{-\frac{(y-y')^2}{4\hbar}}. \quad (125) \]

Substituting this equality into expression (111) for the integral \( I_2 \), we obtain

\[ I_2 = -\frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^{2n}} \frac{ib}{2am} \sum_{k=1}^{n} p_k (y_k - y'_k) e^{-\frac{(y-y')^2}{4\hbar}} e^{i(y-y')p} \psi(y', t) dy' dp. \]

Comparing the latter expression with expression (120) for the integrals \( I^k_4 \), we obtain

\[ I_2 = -\frac{nh}{2m} \psi(y, t). \quad (126) \]

Now consider the integral \( I_3 \) given by expression (112). In accordance with formula (115), this integral can be transformed to the form

\[ I_3 = \psi(y, t) \int_{\mathbb{R}^n} \left( V(x) - \sum_{k=1}^{n} \frac{\partial V}{\partial x_k}(x_k - y_k) \right) \chi^2(x - y) dx, \]

where \( \chi^2(x - y) = \left( b/(a\pi\hbar) \right)^n \exp\left(-b(x - y)^2/(a\hbar)\right) \) is the density of the normal distribution, or, after the change of variables \( x'_k = x_k - y_k \), to the form

\[ I_3 = \psi(y, t) \int_{\mathbb{R}^n} \left( V(y + x') - \sum_{k=1}^{n} \frac{\partial V(y + x')}{\partial y_k} x'_k \right) \chi^2(x') dx'. \]

Whereas the previous integrals have been computed exactly, let us compute this integral approximately, assuming the dispersion \( a\hbar/2b \) of the normal distribution \( \chi^2(x') \) to be a small quantity, decomposing the function \( V(y + x') \) into the Taylor series at the point \( y \) with respect to \( x' \) up to the second order, and decomposing \( \partial V(y + x')/\partial y_k \) up to the first order. We have

\[ I_3 \approx \psi(y, t) \int_{\mathbb{R}^n} \left( V(y) + \sum_{k=1}^{n} \frac{\partial V(y)}{\partial y_k} x'_k + \frac{1}{2} \sum_{k=1}^{n} \sum_{k'=1}^{n} \frac{\partial^2 V(y)}{\partial y_k \partial y_{k'}} x'_k x'_{k'} \right) \chi^2(x') dx' - \psi(y, t) \int_{\mathbb{R}^n} \left( \sum_{k=1}^{n} \frac{\partial V(y)}{\partial y_k} x'_k + \sum_{k=1}^{n} \sum_{k'=1}^{n} \frac{\partial^2 V(y)}{\partial y_k \partial y_{k'}} x'_k x'_{k'} \right) \chi^2(x') dx'. \]

42
Hence, since for the density of the normal distribution $\chi^2(x')$ the following relations hold: $\int_{\mathbb{R}^n} \chi^2(x') \, dx' = 1$, $\int_{\mathbb{R}^n} x'_k \chi^2(x') \, dx' = 0$, $\int_{\mathbb{R}^n} x'_k x'_l \chi^2(x') \, dx' = 0$ for $k \neq k'$, and $\int_{\mathbb{R}^n} x'_k x'_k \chi^2(x') \, dx' = ah/(2b)$, we finally obtain

$$I_3 \approx \psi(y, t)V(y) - \psi(y, t) \frac{ah}{4b} \sum_{k=1}^{n} \frac{\partial^2 V(y)}{\partial y_k^2}. \quad (127)$$

Thus, since $\hat{H}\psi = I_1 + I_2 + I_3$, where the integrals $I_1, I_2, I_3$ are given respectively by expressions (110), (111), (112), then, substituting here their computed values as expressions (124), (126), (127) and reducing similar summands, we obtain the expression for $\hat{H}$ required in Theorem 5:

$$\hat{H} \approx -\frac{\hbar^2}{2m} \left( \sum_{k=1}^{n} \frac{\partial^2}{\partial y_k^2} \right) + V(y) - \frac{ah}{4b} \sum_{k=1}^{n} \frac{\partial^2 V}{\partial y_k^2} + \frac{3nbh}{4ma}. \quad (128)$$
Appendix 3

An estimate of parameters of the model

1. An estimate of the parameter $a/b$ of the model

The difference between the operator $\hat{H}$ given by expression (36) of Theorem 5 and the standard Hamilton operator in the Schrodinger equation, can cause difference between the spectra of the energy operators (in particular, for the potential energy function of hydrogen atom) for the model considered above and for the standard model of quantum mechanics. The difference between these operators is in the third summand with the factor $\frac{a\hbar}{4b}$. If one assumes that behavior of particles is described by the Schrodinger equation with the operator $\hat{H}$ of the form (36) more exactly than with the Hamilton operator, then experiments should show the non-exactness of spectra computed by means of the Hamilton operator, i. e., non-exactness of non-relativistic quantum mechanics.

The discrepancy between theoretical and experimental data in non-relativistic quantum mechanics is well known. It has been discovered in 40s in [10], and has been called the Lamb shift of levels of hydrogen atom. Later on, this effect has been explained in quantum electrodynamics by interaction of the electron with fluctuating electromagnetic field (see, for example, [16], p. 593, where one can find references to original works).

If one assumes that the data in the Lamb experiment are related with the perturbing summand in the operator $\hat{H}$, then these data allow one to estimate the quantity $\frac{a\hbar}{4b}$.

In this section the computations of the estimate of the quantity $a\hbar/4b$ have been carried out following the computations given in [15] for substantiation of the size of the Lamb shift in the spectrum of hydrogen atom.

Let

$$V(y) = -\frac{e^2}{r} = -\frac{e^2}{\sqrt{y_1^2 + y_2^2 + y_3^2}}$$

be the potential function of hydrogen atom, and $\hat{H} = \hat{E} + \hat{V}$ be the operator
from Theorem 5, where
\[ \hat{E} = -\frac{\hbar^2}{2m} \left( \sum_{k=1}^{3} \frac{\partial^2}{\partial y_k^2} \right) \quad \text{and} \quad \hat{V}(y) = V(y) - \frac{a\hbar}{4b} \sum_{k=1}^{3} \frac{\partial^2 V}{\partial y_k^2}. \]

Consider the operator \( \hat{H} \) as a perturbation of the Hamilton operator \( \tilde{H} = \hat{E} + V \) of hydrogen atom. Let us estimate eigenvalues of the operator \( \hat{H} \).

The standard perturbation theory implies that at the first approximation, the correction \( \delta E_n \) to the eigenvalue \( E_n \) of the Hamilton operator has the form
\[ \delta E_n = \int_{\mathbb{R}^3} \rho_n(y)(\hat{V}(y) - V(y))dy, \]
where \( \rho_n(y) = |\psi_n(y)|^2 \) and \( \psi_n(y) \) is the eigenfunction of the Hamilton operator with the eigenvalue \( E_n \).

Substituting the expression for \( \hat{V}(y) \) into the expression for \( \delta E_n \), we obtain
\[ \delta E_n = -\frac{a\hbar}{4b} \int_{\mathbb{R}^3} \rho_n(y) \sum_{k=1}^{3} \frac{\partial^2 V}{\partial y_k^2} dy. \]

Since the integral of the Laplace operator of the function \( V(y) = -e^2/r \) equals \( 4\pi e^2 \delta_0(y) \), where \( 4\pi e^2 \delta_0(y) \) is the delta function at zero, this implies that
\[ \delta E_n = -\frac{a\hbar \pi e^2}{b} \rho_n(0). \]

For hydrogen atom it is known (see, for example, [20], p. 342) that
\[ \rho_n(0) = |\psi_n(0)|^2 = \frac{1}{\pi n^3} \left( \frac{me^2}{\hbar^2} \right)^3. \]

Hence
\[ \delta E_n = -\frac{a\hbar \pi e^2}{b} \frac{1}{\pi n^3} \left( \frac{me^2}{\hbar^2} \right)^3 = -\frac{a m^3 \alpha^4 \epsilon^4}{b n^3 \hbar} = -\frac{a m^3 \alpha^4 \epsilon^4}{b n^3 \hbar}, \]
where \( \alpha = e^2/(\hbar c) = 1/137 \) is the fine structure constant, \( c \) is the velocity of light. Hence, \( a/b \) is expressed as follows:
\[ \frac{a}{b} = |\delta E_n| \frac{n^3 \hbar}{m^3 \alpha^4 \epsilon^4}. \]
In the Lamb–Retherford experiments for hydrogen atom it has been established that $\delta E_2 = 1058 MHz = 1058 \cdot 10^6 h$ erg, where $h = 2\pi \hbar$. Comparing this value with the obtained value of $\delta E_2$, we obtain by simple calculations the estimate of the quantity $a/b = 3.41 \cdot 10^4 sec/g$. Hence the standard deviation for the density of normal probability distribution $\chi_1^2(x')$, with which the smoothing of the potential $V$ is made, equals $\sqrt{ah/2b} = 4.24 \cdot 10^{-12} cm$. This quantity is much less than the radius of hydrogen atom.

2. An estimate of the diffusion coefficients and of the time of the transformation process

In this section we shall assume that the diffusion coefficients $a$ and $b$ are defined by the standard heat action of the surrounding medium on the moving electron. One can assume that the Brownian particle (the electron) is acted on by a fluctuating force from the surrounding medium, and also by stochastic resistance proportional to the velocity of the particle. For modelling of the motion in this situation, one usually uses (see, for example, [17], p. 196) the Langevne equation

$$\dot{p} = -\gamma p + F(t),$$

where $\gamma$ is the friction coefficient for the unit mass, and $F(t)$ represents the fluctuating force, which is assumed to be independent of the velocity, with the mean value equal to zero.

This equation is equivalent (see [17], p. 212) to the Fokker–Planck equation for the density of probability distribution $f(p, q, t)$ in the phase space, in its standard form:

$$\frac{\partial f}{\partial t} + \frac{p_i}{m} \frac{\partial f}{\partial x_i} = \gamma \frac{\partial (fp_i)}{\partial p_i} + \gamma kTm \frac{\partial^2 f}{\partial p_i^2},$$

where $m$ is the mass of the particle (in the case of electron, $m = 9.10939 \cdot 10^{-31} kg$), $k$ is the Boltzmann constant ($k = 1.38066 \cdot 10^{-23} J/K$), $T$ is the temperature of the medium.

The solution of these equations for time intervals $t$ much greater than $\gamma^{-1}$, is well known (see [21], p. 215) to yield the diffusion process with respect to coordinates $x$ with the diffusion coefficient $kT/(m\gamma)$. This relation is also known as the Einstein relation for the diffusion coefficient ([17], p. 198).
Thus, under these assumptions one can suppose that the diffusion coefficient with respect to coordinates \( a^2 = kT/(m\gamma) \), and the diffusion coefficient with respect to momenta (as in the Fokker–Planck equation) \( b^2 = \gamma kTm \).

Hence, \( a/b = (\gamma m)^{-1} \) and \( ab = kT \).

By the estimate obtained in the previous section, we have

\[
\frac{a}{b} = \frac{1}{\gamma m} = 3.41 \cdot 10^7 \text{sec/kg},
\]

and the quantity

\[
\gamma = \frac{1}{3.41 \cdot 10^7 m} = \frac{1}{3.41 \cdot 10^7 \cdot 9.10939 \cdot 10^{-31}} = 3.22 \cdot 10^{22} \text{sec}^{-1}.
\]

On the other hand, the time of the transformation process to the process described by the Schrodinger equation, is expressed, by Theorem 4, by the quantity \( h/(ab) = h/(kT) \). This time for \( T = 1^\circ K \) equals \( 7.638 \cdot 10^{-12} \text{sec} \).

For the same temperature, the diffusion coefficients are estimated in the following way:

\[
a^2 = kT/(m\gamma) = 4.708 \cdot 10^{-16} m^2 \cdot \text{sec}^{-1};
\]

\[
b^2 = \gamma kTm = 4.049 \cdot 10^{-31} (kg \cdot m \cdot \text{sec}^{-1})^2 \cdot \text{sec}^{-1}.
\]
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