LOCAL WELL-POSEDNESS FOR HYPERBOLIC-ELLIPTIC ISHIMORI EQUATION

YUZHAO WANG

Abstract. In this paper we consider the hyperbolic-elliptic Ishimori initial-value problem with the form:

\[
\begin{aligned}
\partial_t s &= s \times \Box_x s + b(\phi_{x_1}, s_{x_2} + \phi_{x_2} s_{x_1}) \quad \text{on } \mathbb{R}^2 \times [-1, 1]; \\
\Delta \phi &= 2s \cdot (s_{x_1} \times s_{x_2}) \\
s(0) &= s_0
\end{aligned}
\]

where \(s(\cdot, t) : \mathbb{R}^2 \to S^2 \subset \mathbb{R}^3\), \(\times\) denotes the wedge product in \(\mathbb{R}^3\), \(\Box_x = \partial_{x_1}^2 - \partial_{x_2}^2\), \(b \in \mathbb{R}\). We prove that such system is locally well-posed for small data \(s_0 \in H^\sigma_Q(\mathbb{R}^2; S^2), \sigma_0 > 3/2, Q \in S^2\). The new ingredient is that we develop the methods of Ionescu and Kenig [7] and [8] to approach the problem in a perturbative way.

Contents

1. Introduction 1
2. Notations and main resolution spaces 5
3. Preliminary lemmas 6
4. Linear Estimates 15
5. Trilinear estimates 17
6. Multilinear Estimates 20
7. Proof of Theorem 1.2 25
References 29

Keywords: Hyperbolic-elliptic Ishimori equation, Local well-posedness

1. Introduction

In this paper we consider the hyperbolic-elliptic Ishimori equation, which is an integrable topological spin field model, with the form

\[
\begin{aligned}
\partial_t s &= s \times \Box_x s + b(\phi_{x_1}, s_{x_2} + \phi_{x_2} s_{x_1}); \\
\Delta \phi &= 2s \cdot (s_{x_1} \times s_{x_2}) \\
s(0) &= s_0
\end{aligned}
\]

where \(\Box_x = \partial_{x_1}^2 - \partial_{x_2}^2\), and \(s : \mathbb{R}^2 \times \mathbb{R} \to S^2 \to \mathbb{R}^3, \lim_{|x_1|, |x_2| \to \infty} s(x_1, x_2, t) = (0, 0, -1), b \in \mathbb{R}\). In [5], Ishimori introduced the system (1.1) in analogy with 2D
CCIHS chain, as a model having the same topological properties as the latter yet permitting topological vortices whose dynamics were integrable. Ishimori system (1.1) also describes the evolution of a system of static spin vortices in the plane. Furthermore, (1.1) is completely integrable when $b = 1$.

During the past decades, Ishimori system (1.1) was widely studied, see [3, 9, 11] and references therein. In [11], Soyeur proved that the Ishimori system (1.1) was well-posed in $H^m(\mathbb{R}^2)$ for $m \geq 4$. In [9], Kenig and Nahmod proved that the Ishimori system (1.1) admitted a local in time solution with large data in the Sobolev class $H^m(\mathbb{R}^2)$, $m > 3/2$, and uniqueness in $H^2(\mathbb{R}^2)$. Recently, for the completely integrable case $b = 1$, Bejenaru, Ionescu and Kenig proved in [3] that the Ishimori system (1.1) was globally well-posed with small data in the critical Sobolev space $\dot{H}^1(Q; \mathbb{S}^2)$.

In this paper, we prove a local well-posedness result for the Ishimori equation (1.1) with $b \in \mathbb{R}$, when the data is small in Sobolev space $H^\sigma_0(Q; \mathbb{S}^2)$, for $\sigma_0 > 3/2$, $Q \in \mathbb{S}^2$, and the constants in this paper will dependent on $b$. We begin with some notations. For $\sigma \geq 0$, let $J^\sigma$ denote the operator defined by Fourier multiplier $(1 + |\xi|^2)^{\sigma/2}$, and $H^\sigma$ denote the usual Sobolev space on $\mathbb{R}^2$ with the norm $\|f\|_{H^\sigma} = \|J^\sigma f\|_{L^2}$. Then for $\sigma \geq 0$ and $Q = (Q_1, Q_2, Q_3) \in \mathbb{S}^2$, we can define the metric space

$$H^\sigma_Q = H^\sigma_Q(\mathbb{R}^2; \mathbb{S}^2) = \{ f : \mathbb{R}^2 \to \mathbb{S}^2 : |f(x)| \equiv 1 \text{ and } f_l - Q_l \in H^\sigma \text{ for } l = 1, 2, 3 \}$$

with the induced distance

$$d_Q(f, g) = \left[ \sum_{l=1}^{3} \|f_l - g_l\|_{H^\sigma}^2 \right]^{1/2}.$$

For $Q \in \mathbb{S}^2$, we define the complete metric space

$$H^\infty_Q(\mathbb{R}^2) = \bigcap_{\sigma \geq 0} H^\sigma_Q(\mathbb{R}^2)$$

with the induced metric.

Let $f_Q(x) \equiv Q$, $f_Q \in H^\infty_Q$. For any metric space $X$, $x \in X$, and $r > 0$, denote $B_X(x, r) = \{ y \in X : d(x, y) < r \}$. $\mathbb{Z}_+ = \{ 0, 1, 2, \ldots \}$ is the nature number set. We now state our main result.

**Theorem 1.1.** (a) Assume $\sigma_0 > 3/2$ and $Q \in \mathbb{S}^2$. Then there is $\epsilon(\sigma_0) > 0$ such that for any $u_0 \in H^\infty_Q \cap B_{H^\sigma_0}(0, \epsilon(\sigma_0))$ there is a unique solution

$$s = S^\infty(s_0) \in C([-1, 1] : H^\infty_Q)$$

of the initial-value problem (1.1).

(b) The mapping $s_0 \to S^\infty(s_0)$ extends uniquely to a Lipschitz mapping

$$S^\sigma_0 : B_{H^\sigma_0}(0, \epsilon(\sigma_0)) \to C([-1, 1] : H^\sigma_0_Q) \quad (1.2)$$
where \( S^{\sigma_0}(s_0) \) is a weak solution of the initial-value problem (1.1) for any \( s_0 \in B_{H^Q_0}(f_Q, \epsilon(\sigma_0)) \).

(c) For any \( \sigma \in \mathbb{Z}_+ \) we have the local Lipchitz bound

\[
\sup_{t \in [-1, 1]} d_Q^{\sigma+\sigma}(S^{\sigma_0}(s_0)(t) - S^{\sigma_0}(s_0')(t)) \leq C(\sigma_0, \sigma, R)d_Q^{\sigma+\sigma}(s_0, s_0')
\]

for any \( R > 0 \) and \( s_0, s_0' \in B_{H^Q_0}(0, \epsilon(\sigma_0)) \cap B_{H^Q_0+\sigma}(0, R) \).

We use the stereographic projection to reduce (1.1) to a nonlinear non-elliptic Schrodinger equation (1.3) below. We refer the readers to [7] or [11] for the details. Finally, for Theorem 1.2 it suffices to prove Theorem 1.2 below.

**Theorem 1.2.** (a) Assume \( \sigma_0 > 3/2 \). Then there is \( \epsilon(\sigma_0) > 0 \) with the property that for any \( u_0 \in H^\infty \cap B_{H^\sigma_0}(0, \epsilon(\sigma_0)) \) there is a unique solution

\[
u = \tilde{S}^\infty(\phi) \in C([-1, 1] : H^\infty)
\]

of the initial-value problem

\[
\begin{cases}
(i\partial_t + \Box)u = \frac{2\bar{u}}{1 + u\bar{u}}[(\partial_{x_1}u)^2 - (\partial_{x_2}u)^2] + ib(\phi_{x_1}u_{x_2} + \phi_{x_2}u_{x_1}) \\
\Delta \phi = 4i\left(u_{x_1}\bar{u}_{x_2} - \bar{u}_{x_1}u_{x_2}\right) \\
u(0, x_1, x_2) = u_0(x_1, x_2)
\end{cases}
\]

(1.4)

(b) The mapping \( \phi \to \tilde{S}^\infty(\phi) \) extends uniquely to a Lipschitz mapping

\[
\tilde{S}^{\sigma_0} : B_{H^\sigma_0}(0, \epsilon(\sigma_0)) \to C([-1, 1] : H^{\sigma_0})
\]

where \( \tilde{S}^{\sigma_0}(\phi) \) is a weak solution of the initial-value problem (1.4) for any \( u_0 \in B_{H^\sigma_0}(0, \epsilon(\sigma_0)) \).

(c) For any \( \sigma' \in \mathbb{Z}_+ \) we have the local Lipchitz bound

\[
\sup_{t \in [-1, 1]} \|\tilde{S}^{\sigma_0}(\phi)(t) - \tilde{S}^{\sigma_0}(\phi')(t)\|_{H^{\sigma_0+\sigma'}} \leq C(\sigma_0, \sigma', R)\|\phi - \phi'\|_{H^{\sigma_0+\sigma'}}
\]

for any \( R > 0 \) and \( \phi, \phi' \in B_{H^\sigma_0}(0, \epsilon(\sigma_0)) \cap B_{H^\sigma_0+\sigma'}(0, R) \).

**Remark 1.3.** Furthermore, we can generalize Theorem 1.2. The results in Theorem 1.2 for system (1.4) also hold for the following system

\[
\begin{cases}
(i\partial_t + \Box)u = \mathcal{N}(u) + ib(\phi_{x_1}u_{x_2} + \phi_{x_2}u_{x_1}) \\
\Delta \phi = 4i\left(u_{x_1}\bar{u}_{x_2} - \bar{u}_{x_1}u_{x_2}\right) \\
u(0, x_1, x_2) = u_0(x_1, x_2)
\end{cases}
\]

(1.7)

where \( \mathcal{N}(u) = F_1(u, \bar{u})Q_1(\nabla u, \nabla \bar{u}) + F_2(u, \bar{u})Q_2(\nabla u, \nabla \bar{u}) + F_3(u, \bar{u})Q_3(\nabla \bar{u}, \nabla \bar{u}) \), \( F_i(\cdot, \cdot), i = 1, 2, 3, \) are analytic functions, and \( Q_i(\cdot, \cdot), \) for \( i = 1, 2, 3, \) are quadratic forms. For system (1.7), there is no null structure in the nonlinear term, so we can not expect to use the method in [2, 4, 8] to get the regularity in \( H^s, s < 3/2 \).
In this paper, we use the methods of Ionescu and Kenig [7] for Schrodinger map equation
\[
\begin{aligned}
(i\partial_t + \Delta)u &= \frac{2\bar{u}}{1 + u\bar{u}}[(\partial_{x_1} u)^2 + (\partial_{x_2} u)^2] \\
u(0, x) &= \phi(x).
\end{aligned}
\] (1.8)

We sketch the proof of our main theorem here. We study (1.4) in a space with high frequency spaces that have two components: an $X^{\sigma,b}$-type component measured in the frequency space and a normalized $L^1$ component measured in the physical space. Then we set up suitable linear ($L^\infty$ smoothing estimate, $L^2$ maximal function estimate) and nonlinear (trilinear estimate, multilinear estimate) estimates in these spaces, and conclude the results by a recursive construction.

However, there are some differences between this paper and [7]. Firstly, in view of bilinear estimate, the $X^{\sigma,b}$-type spaces corresponding to non-elliptic group $e^{it\Box}$ are essentially different from the $X^{\sigma,b}$-type spaces corresponding to elliptic group $e^{it\Delta}$, see Onodera [10] for detailed argument. Secondly, the approach in [7] for the local smoothing estimate depends on the elliptic property of the group $e^{it\Delta}$. We develop another approach to get the local smoothing property, which is based on the following ingredients: Denote $P(\tau, \xi) = \tau + \xi_1^2 - \xi_2^2$, $\mathbf{e} = (\cos \theta, \sin \theta) \in S^1$, and $\mathbf{e} = (\cos \theta, -\sin \theta)$, for some $\theta \in [0, 2\pi)$, we have
\[
\eta_k(\xi \cdot \mathbf{e}) \frac{1}{P(\tau, \xi)} \approx \eta_k(\xi \cdot \mathbf{e}) \frac{1}{2k(\xi_1^2 - t^*_\epsilon(\xi_2, \tau))}.
\]
Which behaves like Hilbert transform in $\xi_1^\epsilon$ direction, see Lemma 3.4 below for further argument. Thirdly, the estimate
\[
\|\mathcal{F}^{-1}_{(2+1)}(f)\|_{L^\infty} \leq C(k + 1)\|f\|_{Z_k}
\] (1.9)
is false for non-elliptic type $Z_k$ space (see Remark 3.6 below), which is the main ingredient in the proof of algebra property (multilinear estimate) in [7]. We use bilinear estimate to overcome this problem, the key point is the identity (see (6.10) below)
\[
H(uv) = (Hu)v + u(Hv) + [\partial_{x_1} u \partial_{x_1} v - \partial_{x_2} u \partial_{x_2} v],
\] (1.10)
where $H = (i\partial_t + \Box)$. This idea has first appeared in [8] as far as we knew. Fourthly, when $b \neq 0$, the potential $\phi$ introduces a nonlocal term, we use the method in [11] to show that this nonlocal term behaves roughly like the nonlinear term $(2\bar{u}/(1 + |u|^2))(u_2^2 - u_3^2)$. Finally, the semi-group $e^{it\Delta}$ is invariant under rotation of the space, but $e^{it\Box}$ is not. For example, if we rotate the $x$-axes clockwise $\pi/4$, then $e^{it\Box}$ becomes to $e^{it\partial_{x_1} \partial_{x_2}}$. So we need to be more careful when we rotate the space to transform norm $L^q_p$ to $L^q_{p_1} L^q_{x_2,t}$.

The rest of the paper is organized as follows. In Section 2 we define some notations and the main resolution spaces. In Section 3 we establish some basic estimates. In Section 4 we prove the linear estimates. In Section 5 and 6 we prove the main
nonlinear estimates. In the last section we prove the main theorem. The key ingredients in these proofs are $L^2_c$ (maximal function) estimate in Lemma 3.5, $L^2_c$ (local smoothing) estimate in Lemma 3.7 and the algebra property in Lemma 6.3.

2. Notations and main resolution spaces

Let $\eta_0 : \mathbb{R} \to [0, 1]$ be a smooth even function supported in the set $\{\mu \in \mathbb{R} : |\mu| \leq 2\}$ and equal to 1 in the set $\{\mu \in \mathbb{R} : |\mu| \leq 1/2\}$. We define $\eta_k : \mathbb{R} \to [0, 1], k = 1, 2, \ldots$,

$$\eta_k(\mu) = \eta_0(\mu/2^k) - \eta_0(\mu/2^{k-1})$$

and $\eta_k^{(2)} : \mathbb{R}^2 \to [0, 1], k \in \mathbb{Z}_+, \eta_k^{(2)}(\mu) = \eta_k(|\mu|)$. The smooth cut-off functions $\chi_{k,l}$

$$\begin{cases}
\chi_{k,l}(r) = [1 - \eta_0(r/2^{k-l})] & \text{if } k \geq 100 \\
\chi_{k,l}(r) = 1 & \text{if } k \leq 99.
\end{cases}$$

Now we begin to define the normed spaces $X_k$ and $Y_k$. The Fourier transform of the linear part of (1.4) in the original coordinate is

$$\mathcal{F}_{(2+1)}[(i\partial_t + \Box)u](\xi, \tau) = -(\tau + \xi_1^2 - \xi_2^2)\mathcal{F}_{(2+1)}(u)(\xi, \tau).$$

We denote

$$P(\tau, \xi) = \tau + \xi_1^2 - \xi_2^2. \quad (2.1)$$

For $\xi \in \mathbb{R}^2$, $\overline{\xi}$ denotes the vector conjugate to $\xi$, say, if $\xi = (\xi_1, \xi_2)$, then $\overline{\xi} = (\xi_1, -\xi_2)$. For $k \in \mathbb{Z}_+, j \in \mathbb{Z}_+$ and $e \in \mathbb{S}^1 \subset \mathbb{R}^2$, denote

$$\begin{align*}
D_{k,j} &= \{(\xi, \tau) \in \mathbb{R}^2 \times \mathbb{R} : |\xi| \in [2^{k-1}, 2^{k+1}] \text{ and } |P(\tau, \xi)| \leq 2^{j+1}\}; \\
D_{0,j} &= \{(\xi, \tau) \in \mathbb{R}^2 \times \mathbb{R} : |\xi| \leq 2 \text{ and } |P(\tau, \xi)| \leq 2^{j+1}\}; \\
D_{k,\infty} &= \bigcup_{j \geq 0} D_{k,j}; \\
D_{k,j}^e &= \{(\xi, \tau) \in D_{k,j} : |\xi \cdot \overline{e}| \geq |\xi|/2\} \text{ for } j \in \mathbb{Z}_+ \text{ and } j = \infty.
\end{align*}$$

We define $X_k$ normed spaces by

$$\begin{cases}
X_k = \{ f \in L^2(\mathbb{R}^2 \times \mathbb{R}) : \text{supp} f \subset D_{k,\infty}; \|f\|_{X_k} < \infty \} \\
&\text{where } \|f\|_{X_k} = \sum_{j=0}^{\infty} 2^{j/2}\|\eta_j(P(\tau, \xi))f\|_{L^2}. \quad (2.2)
\end{cases}$$

For any vector $e \in \mathbb{S}^1$, we write $e = (\cos \theta, \sin \theta)$, for some $\theta \in [0, 2\pi)$, and $e^1 = (-\sin \theta, \cos \theta) \in \mathbb{S}^1$ perpendicular to $e$. For $p, q \in [1, \infty]$ the normed spaces $L^{p,q}_e = L^{p,q}_e(\mathbb{R}^2 \times \mathbb{R})$ is defined by

$$L^{p,q}_e = \left\{ f \in L^2(\mathbb{R}^2 \times \mathbb{R}) ; \|f\|_{L^{p,q}_e} = \left[ \int_\mathbb{R} \left[ \int_{\mathbb{R}^2} |f(re + ve^1)|^qdvd\tau \right]^{p/q} dr \right]^{1/p} < \infty \right\}. \quad (2.3)$$

For $k \geq 100$ and $e \in \mathbb{S}^1$ we define the $Y^e_k$ normed spaces

$$\begin{cases}
Y^e_k = \{ f \in L^2(\mathbb{R}^2 \times \mathbb{R}) : \text{supp} f \subset D^e_{k,\infty}; \|f\|_{Y^e_k} < \infty \} \\
&\text{where } \|f\|_{Y^e_k} = 2^{-k/2}\|\mathcal{F}^{-1}[(P(\tau, \xi) + i) \cdot f]\|_{L^2_{k,2}}.
\end{cases}$$

For the cases $k = 0, 1, 2, \ldots, 99$, $Y^e_k = \{0\}$. 

ISHIMORI EQUATION 5
Fix $L$ large enough, let $\{e_l\}_{l=1}^L \subset S^1$ satisfy

1. $e_l \neq e_{l'}$ if $l \neq l'$,
2. for any $e \in S^1$ there is $e_l$ such that $|e_l - e| \leq 2^{-50}$,
3. if $e \in \{e_l\}_{l=1}^L$, then $-e \in \{e_l\}_{l=1}^L$.

For $k \in \mathbb{Z}_+$, we define

$$Z_k = X_k + Y_{e_1}^e + \ldots + Y_{e_L}^e$$

(2.4)

3. Preliminary lemmas

In this section we prove some preliminary lemmas which will be used frequently in the following sections.

**Lemma 3.1** (Spaces Decomposition). For $f \in Z_k$, in view of the definitions, we can write

\[
\begin{equation}
\left\{
\begin{array}{ll}
\mathcal{Z}_k = \sum_{j \in \mathbb{Z}_+} g_j + f_{e_1} + \ldots + f_{e_L} \\
\sum_{j \in \mathbb{Z}_+} 2^{j/2} \|g_j\|_{L^2} + \|f_{e_1}\|_{Y_{e_1}^e} + \ldots + \|f_{e_L}\|_{Y_{e_L}^e} \leq 2 \|f\|_{Z_k}.
\end{array}
\right.
\end{equation}
\]

**Lemma 3.2.** Fix $\theta \in [0, 2\pi)$, $e = (\cos \theta, \sin \theta)$, $e^\perp = (-\sin \theta, \cos \theta)$, $\overline{e} = (\cos \theta, -\sin \theta)$. Let $\xi = (\xi_1, \xi_2)$ be in the original coordinate. If we write $\xi = \xi_1^e e + \xi_2^e e^\perp$, then

$$\partial_{\xi_1^e} P(\tau, \xi) = \partial_{\xi_2^e} (\xi_1 - \xi_2) = 2 \xi \cdot \overline{e},$$

(3.2)

where $P(\tau, \xi)$ is defined in (2.1).

**Proof of Lemma 3.2.** First from $\xi = (\xi_1, \xi_2)$ and $\xi = \xi_1^e e + \xi_2^e e^\perp$, we have

\[
\begin{align*}
\xi_1^e &= \xi \cdot e = \xi_1 \cos \theta + \xi_2 \sin \theta \\
\xi_2^e &= \xi \cdot e^\perp = -\xi_1 \sin \theta + \xi_2 \cos \theta
\end{align*}
\]

and so

\[
\begin{align*}
\xi_1^e &= \xi_1 \cos \theta - \xi_2 \sin \theta \\
\xi_2^e &= \xi_1 \sin \theta + \xi_2 \cos \theta
\end{align*}
\]

(3.3)

in the new coordinate $(e, e^\perp)$, we have

$$P(\tau, \xi) = \tau + \xi_1^2 - \xi_2^2$$

$$= \tau + (\xi_1^e \cos \theta - \xi_2^e \sin \theta)^2 - (\xi_1^e \sin \theta + \xi_2^e \cos \theta)^2$$

(3.4)

by a simple calculation, we have

$$\partial_{\xi_1^e} P(\tau, \xi) = 2(\xi_1^e \cos \theta - \xi_2^e \sin \theta) \cos \theta - 2(\xi_1^e \sin \theta + \xi_2^e \cos \theta) \sin \theta$$

$$= 2\xi_1 \cos \theta - 2\xi_2 \sin \theta$$

$$= 2\xi \cdot \overline{e}.\square$$

**Lemma 3.3.** (Multipliers) 1. If $m \in L^\infty(\mathbb{R}^2)$, $\mathcal{F}_2^{-1}(m) \in L^1(\mathbb{R}^2)$, and $f \in Z_k$, then $m(\xi) \cdot f \in Z_k$ and

$$\|m(\xi) \cdot f\|_{Z_k} \leq C \|\mathcal{F}_2^{-1}(m)\|_{L^1(\mathbb{R}^2)} \cdot \|f\|_{Z_k}$$

(3.5)
2. If \( f \in Z_k, k, j \in \mathbb{Z}_+ \), and \( C_1 \in \mathbb{R} \) is a constant, then
\[
\begin{align*}
\| \eta_j(P(\tau, \xi)) \cdot f \|_{X_k} & \leq C \| f \|_{Z_k} \\
\| \eta_{2k+c_1}(P(\tau, \xi)) \cdot f \|_{X_k} & \leq C \| f \|_{Z_k} \\
\| f \|_{X_k} & \leq C(k+1) \| f \|_{Z_k}.
\end{align*}
\] (3.6)

3. If \( f \in Z_k, k, j \in \mathbb{Z}_+ \), then
\[
\| \eta_{\leq j}(P(\tau, \xi)) \cdot f \|_{Z_k} \leq C \| f \|_{Z_k}
\] (3.7)

where \( P(\tau, \xi) \) is defined in (2.1).

**Proof of Lemma 3.3**  Clearly, (3.5) follows directly from the definition. Now we turn to (3.6). In view of Lemma 3.1, we can assume \( k \geq 100 \), and \( f = f_e \in Y^e_k \). Let
\[
h_e(x, t) = 2^{-k/2} F^{-1}_{(2+1)}[(P(\tau, \xi) + i) \cdot f_e](x, t),
\]
thus
\[
f_e(\xi, \tau) = \chi_{k,5}(\xi \cdot \overline{e}) \cdot \frac{2^{k/2}}{P(\tau, \xi) + i} F_{(2+1)}(h_e)(\xi, \tau)
\] (3.8)

where \( \overline{e} \) is the same as in Lemma 3.2, and \( \| f_e \|_{Y^e_k} = \| h_e \|_{L^1_{\xi, \tau}} \). By the definition, for (3.6) it suffices to prove that
\[
2^{k/2} 2^{-j/2} \| 1_{D_{k,j}}(\xi, \tau) \chi_{k,5}(\xi \cdot \overline{e}) \cdot F_{(2+1)}(h)(\xi, \tau) \|_{L^2_{\xi, \tau}} \leq C(1 + 2^{j-2k})^{-1/2} \| h \|_{L^1_{\xi, \tau}} \] (3.9)

for any \( h \in \mathcal{S}(\mathbb{R}^2 \times \mathbb{R}) \) and \( j \in \mathbb{Z}_+ \). We write \( \xi = \xi^e_1 e + \xi^e_2 e^\perp \), \( x = x^e_1 e + x^e_2 e^\perp \), and
\[
h'(x^e_1, \xi^e_2, \tau) = \int_{\mathbb{R} \times \mathbb{R}} h(x^e_1 e + x^e_2 e^\perp, t) e^{-i(\xi^e_1 x^e_2 + \tau t)} dx^e_2 dt.
\]

Thus
\[
F_{(2+1)}(h)(\xi^e_1 e + \xi^e_2 e^\perp, \tau) = \int_{\mathbb{R}} h'(x^e_1, \xi^e_2, \tau) e^{-i\xi^e_1 x^e_1} dx^e_1,
\]
and by Plancherel theorem, we get that \( \| h \|_{L^1_{\xi, \tau}} = \| h' \|_{L^2_{x^e_1 \xi^e_2 \tau}} \). Thus, for (3.9) it suffices to prove that
\[
2^{(k-j)/2} \| 1_{D_{k,j}} \cdot \chi_{k,5}(\xi \cdot \overline{e}) \cdot \int_{\mathbb{R}} h'(x^e_1, \xi^e_2, \tau) e^{-i\xi^e_1 x^e_1} dx^e_1 \|_{L^2_{\xi_1 \xi_2 \tau}} \leq C(1 + 2^{j-2k})^{-1/2} \| h' \|_{L^2_{x_1 \xi_2 \tau}}.
\]

By Hölder’s inequality, it suffices to prove that
\[
|\{\xi^e_1 : |\xi| \leq 2^k, |\xi \cdot \overline{e}| \geq 2^{k-10} \text{ and } |P(\tau, \xi)| \leq 2^{j+1}\}| \leq C \min(2^{j-k}, 2^k). \] (3.10)

This follows easily from (3.2).

Now we turn to prove (3.7). In view of (3.6), we can assume that \( k \geq 100 \). By the definition, it suffices to prove that
\[
\| F_{(2+1)}^{-1}[\eta_{\leq j}(P(\tau, \xi)) \cdot f \cdot \eta_k(\xi) \cdot \chi_{k,5}(\xi \cdot \overline{e})] \|_{L^1_{\xi, \tau}} \leq C \| F_{(2+1)}^{-1}(f) \|_{L^1_{\xi, \tau}} \] (3.11)
for $j \leq 2k - 100$. We write $\xi = \xi_1^e \mathbf{e} + \xi_2^e \mathbf{e}^\perp$, $x = x_1^e \mathbf{e} + x_2^e \mathbf{e}^\perp$. Using Plancherel theorem and Hölder’s, for (3.11) it suffices to show
\[
\left\| \int \! e^{ix_1^e \xi_1^e} \eta_{\leq j} (P(\tau, \xi)) \cdot \eta_k(\xi) \cdot \chi_{k,5}(\xi \cdot \mathbf{e}) d\xi_1^e \right\|_{L^1_{\xi_1^e} L^\infty_{\xi_2^e,\tau}} \leq C. \tag{3.12}
\]
In view of (3.2) and (3.4), we have that $|\partial_{\xi_1^e} P(\tau, \xi)| = 2 |\xi_1^e| \leq C 2^k$ and $|\partial_{\xi_1^e} P(\tau, \xi)| = |\cos^2 \theta - \sin^2 \theta| \leq 1$. From integration by parts and (3.10), we get that
\[
\left| \int \! e^{ix_1^e \xi_1^e} \eta_{\leq j} (P(\tau, \xi)) \cdot \eta_k(\xi) \cdot \chi_{k,5}(\xi \cdot \mathbf{e}) d\xi_1^e \right| \leq C \frac{2^{j-k}}{1 + (2^{-k} y_1^e)^2}, \tag{3.13}
\]
which gives (3.12).

\[\square\]

**Lemma 3.4** (Representation for $Y_k^e$ functions). If $k \geq 100$, $e \in S^1$, $e = (\cos \theta, \sin \theta)$ and $f \in Y_k^e$ then we can write
\[
f^e(\xi_1^e \mathbf{e} + \xi_2^e \mathbf{e}^\perp, \tau) = 2^{k/2} \int_{-k-100}^{k+100} (\xi_1^e - t^*_e) \chi_{k,5}(M^*_e) \frac{1}{2M^*_e} \int \! e^{-iy_1^e \xi_1^e} h(y_1^e, \xi_2^e \mathbf{e}^\perp) dy_1^e + g \tag{3.14}
\]
where $\xi_1^e, \xi_2^e, \tau \in \mathbb{R}$, and $t^*_e = t^*_e(\xi_2^e, \tau)$ satisfies $P(\tau, t^*_e \mathbf{e} + \xi_2^e \mathbf{e}^\perp) = 0$, that is
\[
\tau + (t^*_e \cos \theta - \xi_2^e \sin \theta)^2 - (t^*_e \sin \theta + \xi_2^e \cos \theta)^2 = 0 \tag{3.15}
\]
Denote
\[
M^*_e = M^*_e(\xi_2^e, \tau) = t^*_e(\cos \theta - \sin \theta) - 2 \xi_2^e \cos \theta \sin \theta \tag{3.16}
\]
Furthermore, we have
\begin{enumerate}
\item $t^*_e \in \mathbb{R}$ in the support of $f^e$.
\item $M^*_e = (t^*_e \mathbf{e} + \xi_2^e \mathbf{e}^\perp) \cdot \mathbf{e}^\perp \sim 2^k$ in the support of $f^e$.
\item For the $g$ and $h$ defined above, we have
\[
||g||_{L^1_{\xi_1^e} L^\infty_{\xi_2^e,\tau}} + ||h||_{L^1_{\xi_1^e} L^\infty_{\xi_2^e,\tau}} \leq C ||f^e||_{Y_k^e}. \tag{3.17}
\]
\item In the support of $f^e$, we have
\[
|\partial_{t^*_e}(\xi_2^e, \tau)| \geq C 2^{-k}. \tag{3.18}
\]
\end{enumerate}

**Proof of Lemma 3.4** Let
\[h'(x, t) = 2^{-k/2} \mathcal{F}^{-1}[(P(\tau, \xi) + i) \cdot f^e](x, t).\]
Thus
\[
\begin{cases}
  f^e(\xi_1^e \mathbf{e} + \xi_2^e \mathbf{e}^\perp, \tau) = \chi_{k,5}(\xi \cdot \mathbf{e}) \cdot \frac{2^{k/2}}{P(\tau, \xi) + i} \mathcal{F}(h')(\xi_1^e \mathbf{e} + \xi_2^e \mathbf{e}^\perp, \tau);
  \\
  ||h'||_{L^1_{\xi_1^e}} = C ||f^e||_{Y_k^e}. \tag{3.19}
\end{cases}
\]
Let
\[
h''(y_1^e, \xi_2^e, \tau) = \int_{P_0 \times \mathbb{R}} h'(y_1^e \mathbf{e} + y_2^e \mathbf{e}^\perp, t) e^{-i(y_2^e \xi_2^e + t\tau)} dy_2^e dt.
\]
By (3.19), we have
\[
\left\{ \begin{array}{l}
 f^e(\xi^e_1 e + \xi^e_2 e^\perp, \tau) = \chi_{k,5}(\xi) \cdot \frac{2^{k/2}}{P(\tau, \xi) + i} \int_{\mathbb{R}} h''(y^e_1, \xi^e_2, \tau) e^{-iy^e_1} dy^e_1; \\
 \|h^i\|_{L^1,2} = \|h''\|_{L^1_{y^e_1} L^2_{\xi^e_2, \tau}}.
\end{array} \right.
\] (3.20)

In view of (3.6),
\[
|\eta_{\geq 2k-101}(P(\tau, \xi)) \cdot f^e| \leq C|\|f^e\|_{\gamma^e_k}.
\]

It remains to write \(\eta_{\leq 2k-100}(P(\tau, \xi)) \cdot f^e\) as in (3.14). From (3.20), we obtain
\[
\left. \begin{array}{l}
\eta_{\leq 2k-100}(P(\tau, \xi)) \cdot f^e(\xi^e_1 e + \xi^e_2 e^\perp, \tau) \\
= 2^{k/2} \cdot \chi_{k,5}(\xi) \cdot \frac{\eta_{\leq 2k-100}(P(\tau, \xi))}{P(\tau, \xi) + i} \int_{\mathbb{R}} h''(y^e_1, \xi^e_2, \tau) e^{-iy^e_1} dy^e_1.
\end{array} \right.\] (3.21)

Let
\[
S = \{ (\xi, \tau) : |\xi| \leq 2^{k+2}, |\xi \cdot \overline{\xi}| \geq 2^{k-6}, |P(\tau, \xi)| \leq 2^{2k-80} \}. \] (3.22)

It is easy to see that the right-hand side of (3.21) is supported in \(S\).

Now assume \((\xi, \tau) \in S\), and analyze the behavior of \(P(\tau, \xi) = \tau + \xi^2 - \xi^2_2\). In view of (3.15), if we fix \(\xi, \xi^e\), then there exists a \(t^*_e(t^*_e, \xi_2, \tau)\), so that
\[
0 = \tau + (t^*_e)^2(\cos^2 \theta - \sin^2 \theta) - 4t^*_e \xi^e_2 \cos \theta \sin \theta - (\xi^e_2)^2(\cos^2 \theta - \sin^2 \theta). \] (3.23)

Thus we get
\[
P(\tau, \xi) = (\xi^e_1 + t^*_e)(\cos^2 \theta - \sin^2 \theta) - 4 \xi^e_2 \cos \theta \sin \theta, \] (3.24)

we reduce
\[
P(\tau, \xi) = (\xi^e_1 - t^*_e)K(\tau, \xi). \] (3.25)

Now we show that \(|K(\tau, \xi)| \geq 2^{k-10} for (\xi, \tau) \in S\), for proper \(t^*_e\). Notice that
\[
\xi \cdot \overline{\xi} = (\xi^e_1 e + \xi^e_2 e^\perp) \cdot \overline{\xi} = \xi^e_1(\cos^2 \theta - \sin^2 \theta) - 2 \xi^e_2 \cos \theta \sin \theta \] (3.26)

we get
\[
K(\tau, \xi) = -(\xi^e_1 - t^*_e)(\cos^2 \theta - \sin^2 \theta) + 2 \xi \cdot \overline{\xi}. \] (3.27)

If \(\cos^2 \theta - \sin^2 \theta = 0\), then \(P(\tau, \xi) = \tau \pm 2 \xi^e_1 \xi^e_2\), this case is easy. If \(\cos^2 \theta - \sin^2 \theta \neq 0\), we can assume that at least one solution satisfies
\[
|K(\tau, \xi)| = \left| (\xi^e_1 - t^*_e)(\cos^2 \theta - \sin^2 \theta) - 2 \xi \cdot \overline{\xi} \right| > 2^{k-20} \text{ for } (\xi, \tau) \in S. \] (3.28)
Otherwise, we denote $t_1, t_2$ to be the two solutions of \( (3.23) \). Since \( (3.28) \) and \( |\xi \cdot \bar{\xi}| \geq 2^{k-6} \) in $S$, we have

\[
| (\xi^e_i - t_i)(\cos^2 \theta - \sin^2 \theta) | \geq 2^{k-10} \quad \text{for } i = 1, 2, \text{ and } (\xi, \tau) \in S, \tag{3.30}
\]

and by the assumption on $t_i$, we have

\[
P(\tau, \xi) = (\cos^2 \theta - \sin^2 \theta)(\xi^e_i - t_i)(\xi^e_i - t_2).	ag{3.31}
\]

Combining \( (3.30) \) and \( (3.31) \), we conclude that \( |P(\tau, \xi)| \geq 2^{2k-20} \), which contradicts with \( (\xi, \tau) \in S \). Then we select $t^*_e$ to be the solution that satisfies \( |K(\tau, \xi)| > 2^{k-20} \) for \( (\xi, \tau) \in S \). Furthermore, we have

\[
|\xi^e_1 - t^*_e(\xi^e_2, \tau)| \leq 2^{k-80} \quad \text{for } (\xi^e_1 \pm \xi^e_2, \tau) \in S. \tag{3.32}
\]

Now we show that \( t^*_e \in \mathbb{R} \). Let \( \Re(t^*_e) \) and \( \Im(t^*_e) \) denote the real and imaginary part of \( t^*_e \), we notice that

\[
\Re M^*_e(\xi^e_2, \tau) = \Re(t^*_e)(\cos^2 \theta - \sin^2 \theta) - 2\xi^e_2 \cos \theta \sin \theta. \tag{3.33}
\]

In view of \( (3.27) \), we have

\[
\xi \cdot \bar{\xi} = (\xi^e_1 - \Re(t^*_e))(\cos^2 \theta - \sin^2 \theta) + \Re M^*_e(\xi^e_2, \tau)
\]

where \( |\xi^e_1 - \Re(t^*_e)| \leq |\xi^e_1 - t^*_e| \leq 2^{k-80} \) by \( (3.32) \).

From \( (3.31) \), we get that \( |\Re M^*_e| \geq 2^{k-10} \) in $S$. In view of \( (3.23) \), we have

\[
0 = 2i\Re(t^*_e)\Im(t^*_e)(\cos^2 \theta - \sin^2 \theta) - 4i\Im(t^*_e)\xi^e_2 \cos \theta \sin \theta = 2i\Re(M^*_e)\Im(t^*_e)
\]

which implies \( \Im(t^*_e) = 0 \), thus \( t^*_e \in \mathbb{R} \).

Then we can rewrite \( (3.34) \) as

\[
\xi \cdot \bar{\xi} = (\xi^e_1 - t^*_e)(\cos^2 \theta - \sin^2 \theta) + M^*_e(\xi^e_2, \tau)
\]

where \( |\xi^e_1 - t^*_e| \leq 2^{k-80} \) by \( (3.32) \).

In view of \( (3.25) \) and \( (3.16) \), we have

\[
K(\tau, \xi) = (\xi^e_1 - t^*_e)(\cos^2 \theta - \sin^2 \theta) + 2M^*_e(\xi^e_2, \tau)
\]

where \( |\xi^e_1 - t^*_e| \leq 2^{k-80} \) by \( (3.32) \).

Now we begin to prove the representation formula. In view of \( (3.26) \), we have

\[
\chi_{k,5}(\xi \cdot \bar{\xi}) \cdot \frac{\eta_{\leq 2k-100}(P(\tau, \xi))}{P(\tau, \xi) + i} = \chi_{k,5}(\xi \cdot \bar{\xi}) \cdot \frac{\eta_{\leq 2k-100}((\xi^e_1 - t^*_e)K(\tau, \xi))}{(\xi^e_1 - t^*_e)K(\tau, \xi) + i}.
\]
Combining (3.36) and (3.35), we get
\[
\chi_{k,5}(\xi \cdot \overrightarrow{e}) \cdot \frac{\eta \leq 2k-100((\xi_1^e - t_1^e)K(\tau, \xi))}{(\xi_1^e - t_1^e)K(\tau, \xi) + i} = \chi_{k,5}(M_0^e) \cdot \frac{\eta \leq 2k-100(\xi_2^e - t_2^e)}{(\xi_2^e - t_2^e) + i/2k} + E(\xi_1^e, \xi_2^e, \tau),
\]
(3.37)
where \(E(\xi_1^e, \xi_2^e, \tau)\) is the error term with the estimate
\[
|E(\xi_1^e, \xi_2^e, \tau)| \leq C\chi_{k,10}(M_0^e) \cdot \eta \leq (\xi_1^e - t_1^e)
\]
\[
\times \left[ \frac{1}{2k} + (1 + |P(\tau, \xi)|)^2 \right].
\]
(3.38)

We substitute (3.37) into (3.21) and notice that the error term corresponding to \(E(\xi_1^e, \xi_2^e, \tau)\) can be controlled in \(X_k\) (as in Lemma 3.3). The main term in the right-hand side of (3.37) leads to the representation (3.14), with \(h = h''\).

For (3.18), differentiate the equation (3.23) in \(\tau\) axis, we have
\[
1 = 2(\partial, t_1^e)\left[-t_1^e(\cos^2 \theta - \sin^2 \theta) + 2\xi_2^e \cos \theta \sin \theta\right]
\]
\[
= -2(\partial, t_1^e)M_0^e
\]
(3.39)
where \(M_0^e \sim 2^k\), thus we conclude (3.18). □

The following local-smoothing estimates is the main lemma in this paper, the corresponding lemma in Schrodinger case is Lemma 3.2 in [7].

**Lemma 3.5** (Local-smoothing estimate). If \(k \in \mathbb{Z}^+\), \(e' \in S^1\) and \(f \in Z_k\) then
\[
\|F^{-1}_{(2+1)}[f \cdot \chi_{k,30}(\xi \cdot \overrightarrow{e'})]\|_{L^\infty_{e'}} \leq C2^{-k/2}\|f\|_{Z_k}
\]
(3.40)

**Proof of Lemma 3.5** In view of the space decomposition (3.1). Assume first that \(f = g_j \in X_k\). In view of the definitions, it suffices to prove that if \(j \geq 0\) and \(g_j\) is supported in \(D_{k,j}\), then
\[
\left\| \int_{\mathbb{R}^{2+1}} e^{ix\cdot\xi} e^{it\tau} g_j(\xi, \tau) \cdot \chi_{k,30}(\xi \cdot \overrightarrow{e}) d\xi d\tau \right\|_{L^\infty_{e'}} \leq C2^{-k/2}j^{j/2}\|g_j\|_{L^2}.
\]
(3.41)
Let \(g_j^\#(\xi, \tau) = g_j(\xi, \tau - \xi_2^e + \xi_2^e)\). By Hölder’s inequality, for (3.41) it reduce to show
\[
\left\| \int_{\mathbb{R}^2} e^{ix\cdot\xi} e^{-it(\xi_2^e - \xi_2^e)} h(\xi) \cdot \chi_{k,30}(\xi \cdot \overrightarrow{e}) d\xi \right\|_{L^\infty_{e'}} \leq C2^{-k/2}\|h\|_{L^2}.
\]
(3.42)
which follows easily from Plancherel theorem and a change of variables(let \(\nu = \xi_1^e - \xi_2^e\), then \(\chi_{k,30}(\xi \cdot \overrightarrow{e})d\xi = 2^{-k}\chi_{k,30}(\xi \cdot \overrightarrow{e})dv\overrightarrow{e}^e\) in view of (3.2)).

Assume now that \(f = f_e \in Y_k\), \(k \geq 100\). We adopt the notations in Lemma (3.4). The estimates in Lemma (3.4) show that (since \(|\chi_{k,30}(\xi \cdot \overrightarrow{e}) - \chi_{k,30}((t_1^e e + \xi_2^e e^\perp) \cdot \overrightarrow{e})| \leq C2^{-k}|\xi_1^e - t_1^e|\),
\[
\|f_e \cdot [\chi_{k,30}(\xi \cdot \overrightarrow{e}) - \chi_{k,30}((t_1^e e + \xi_2^e e^\perp) \cdot \overrightarrow{e})]\|_{X_k} \leq C\|f_e\|_{Y_k}.
\]
Since (3.41) was already proved for \(f \in X_k\), it suffices to show that
\[
\left\| \int_{\mathbb{R}^2 \times \mathbb{R}} e^{ix \cdot \xi \cdot \mu} e^{ix \cdot \xi \cdot e^{it \tau}} f^e(\xi_1 \cdot \tau, \tau) \chi_{k, 5}(\xi \cdot \tau) \times \chi_{k, 30}(t'e + \xi_2 e^{1}) \cdot \overrightarrow{e} d\xi d\xi d\tau \right\|_{L^2_{\mu, \tau}} \leq C 2^{-k/2} \| f^e \|_{Y_k^e}.
\] (3.43)

By substituting (3.14) to (3.43) and then integrating the left-hand side of (3.43) according to the variable \(\xi^e\), we can reduce (3.43) to show
\[
\left\| \int_{\mathbb{R} \times \mathbb{R}} e^{ix \cdot \xi \cdot \mu} e^{ix \cdot \xi \cdot e^{it \tau}} \times \chi_{k, 30}(t'e + \xi_2 e^{1}) \cdot \overrightarrow{e} d\xi d\xi d\tau \right\|_{L^2_{\mu, \tau}} \leq C 2^{-k/2} \| h'' \|_{L^2_{\mu, \tau}},
\] (3.44)
for any \(h' \in L^2(\mathbb{R} \times \mathbb{R})\). As before, \(e = (\cos \theta, \sin \theta)\), we use the substitutions \(\tau = - (\mu \cos \theta + \xi_2 \sin \theta)^2 + (\mu \sin \theta - \xi_2 \cos \theta)^2 \leq Q(\mu, \xi^e)\) (so \(t'e = \mu\)), and
\[
h''(\xi, \mu) = 2^{-k/2} \cdot \partial_\mu(Q(\mu, \xi^e), \chi_{k, 5}((\mu e + \xi_2 e^{1}) \cdot \overrightarrow{e}) \cdot h'(\xi_2, Q(\mu, \xi^e))).
\]
We notice that \(\partial_\mu(Q(\mu, \xi^e)) = (\mu e + \xi_2 e^{1}) \cdot \overrightarrow{e} \approx 2^k\), so \(\|h''\|_{L^2} \leq C \|h''\|_{L^2}.\) For (3.44) it suffices to prove that
\[
\left\| \int_{\mathbb{R} \times \mathbb{R}} e^{ix \cdot \mu} e^{ix \cdot \xi \cdot e^{it \tau}} \times \chi_{k, 30}((\mu e + \xi_2 e^{1}) \cdot \overrightarrow{e}) \cdot h''(\xi_2, \mu) d\xi d\xi d\mu \right\|_{L^2_{\mu, \tau}} \leq C 2^{-k/2} \| h'' \|_{L^2_{\mu, \tau}},
\]
which follows from (3.42). \(\square\)

Remark 3.6. If we remove the cut-off function \(\chi_{k, 30}(\xi \cdot \tau)\), the estimate
\[
\|F^{-1}_{(2+1)}(f)\|_{L^2_{\mu, \tau}} \leq C 2^{-k/2} \| f \|_{Z_k}
\] (3.45)
is not true. Furthermore, the following inequality is false
\[
\|F^{-1}_{(2+1)}(f)\|_{L^2_{\mu, \tau}} \leq C(k + 1) \| f \|_{Z_k}
\] (3.46)
To see this, let \(e = (\cos(\pi/4), \sin(\pi/4))\), then \(P(\tau, \xi) = \tau + \xi \cdot e\), define \(f(\tau, \xi \cdot e + \xi \cdot e^{1}) = 1_Q(\tau, \xi \cdot e + \xi \cdot e^{1})\), where
\[
Q = \{N \leq |\xi| \leq 2N; |\xi_2^e| \leq 1/(4N); |\tau| \leq 1/2\}.
\]

Next we give a maximal function estimate.

Lemma 3.7 (Maximal Function Estimate). If \(k \in \mathbb{Z}_+, e \in S^1\) and \(f \in Z_k\) then
\[
\|1_{[-2,2]}(t) \cdot F^{-1}_{(2+1)}(f)\|_{L^2_{\mu, \tau}} \leq C 2^{k/2}(k + 1)^2 \| f \|_{Z_k}
\] (3.47)
Proof of Lemma 3.7. In view of Lemma 3.3, it suffices to prove that
\[
\|1_{[-2,2]}(t) \cdot \mathcal{F}^{-1}_{(2+1)}(g_j)\|_{L^2_x} \leq C2^{k/2}(k+1)2^{j/2}\|g_j\|_{L^2_x}, \tag{3.48}
\]
for any function \(g_j\) supported in \(D_{k,j}\). Denote \(g_j^\#(\xi, \tau) = g_j(\xi, \tau - \xi_1^2 + \xi_2^2)\), the left-hand side of (3.48) is dominated by
\[
\int_{[-2^{j+1},2^{j+1}]} \left\|1_{[-2,2]}(t) \cdot \int_{\mathbb{R}^2} g_j^\#(\xi, \mu) e^{ix \cdot \xi} e^{-it(\xi_1^2 - \xi_2^2)} d\xi \right\|_{L^2_x} \, d\mu.
\]

Thus for (3.48) it suffices to prove that
\[
\left\|1_{[-2,2]}(t) \cdot \int_{\mathbb{R}^2} h(\xi) e^{ix \cdot \xi} e^{-it(\xi_1^2 - \xi_2^2)} d\xi \right\|_{L^2_x} \leq C2^{k/2}(k+1)\|h\|_{L^2_x} \tag{3.49}
\]
for any function \(h\) supported in the set \(\{\xi \in \mathbb{R}^2 : |\xi| \leq 2^{k+1}\}\).

Let \(e = (\cos \theta, \sin \theta)\), rotate the x-axes to the e direction, then (3.49) changes to
\[
\left\|1_{[-2,2]}(t) \int_{\mathbb{R}^2} e^{ix_1 \xi_1} e^{ix_2 \xi_2} e^{-itQ(\xi,e)} \eta_{\leq k+2}(\xi) h(\xi) d\xi \right\|_{L^2_{x,t}} \lesssim 2^{k/2}(k+1)\|h\|_{L^2_x} \tag{3.50}
\]
where \(Q(\xi,e) = (\xi_1 \cos \theta + \xi_2 \sin \theta)^2 - (\xi_1 \sin \theta - \xi_2 \cos \theta)^2\). We use standard TT* argument to prove (3.50), it means to show that
\[
\left\|1_{[-4,4]}(t) \int_{\mathbb{R}^2} e^{ix_1 \xi_1} e^{ix_2 \xi_2} e^{-itQ(\xi,e)} \eta_{\leq k+2}(\xi) d\xi \right\|_{L^2_{x,t}} \lesssim 2^k(k+1)^2. \tag{3.51}
\]
Notice that
\[
\left|1_{[-4,4]}(t) \int_{\mathbb{R}^2} e^{ix_1 \xi_1} e^{ix_2 \xi_2} e^{-itQ(\xi,e)} \eta_{\leq k+2}(\xi) d\xi \right| \lesssim 2^k,
\]
rotate again, we get
\[
\left|1_{[-4,4]}(t) \int_{\mathbb{R}^2} e^{ix_1 \xi_1} e^{ix_2 \xi_2} e^{-itQ(\xi,e)} \eta_{\leq k+2}(\xi) d\xi \right| = \left|1_{[-4,4]}(t) \int_{\mathbb{R}^2} e^{ix_1 \xi_1} e^{ix_2 \xi_2} e^{-itQ(\xi,e)} \eta_{\leq k+2}(\xi) d\xi \right| \lesssim |t|^{-1}.
\]
Integration by parts when \(|x_1| > 2^{k+10}|t|\), then
\[
\left|1_{[-4,4]}(t) \int_{\mathbb{R}^2} e^{ix_1 \xi_1} e^{ix_2 \xi_2} e^{-itQ(\xi,e)} \eta_{\leq k+2}(\xi) d\xi \right| \lesssim \frac{2^k}{(1 + 2^k|x_1|)^2}.
\]
We collect all the estimates above and let \(K(x_1,x_2,t)\) denote the function in the left-hand side of (3.51), then
\[
\sup_{x_2,|t|<4} |K(x_1,x_2,t)| \lesssim 2^k|x_1|^{-1}1_{\{2^{-k} \leq |x_1| < 2^k\}} + \frac{2^k}{(1 + 2^k|x_1|)^2}.
\]
The bound (3.51) follows. \(\square\)
Lemma 3.8. If \( k \in \mathbb{Z}_+ \), \( t \in \mathbb{R} \) and \( f \in Z_k \), then
\[
\sup_{t \in \mathbb{R}} \| \mathcal{F}^{-1}_{(2+1)}(f)(\cdot, t) \|_{L^2_L} \leq C \| f \|_{Z_k}.
\]  
(3.52)

Thus
\[
\| \mathcal{F}^{-1}_{(2+1)}(f)(\cdot, t) \|_{L^\infty_{t, x}} \leq C 2^k \| f \|_{Z_k}.
\]  
(3.53)

**Proof of Lemma 3.8.** By Plancherel theorem, it suffices to prove that
\[
\left\| \int_{\mathbb{R}} f(\xi, \tau) e^{it\tau} d\tau \right\|_{L^2_{x}} \leq C \| f \|_{Z_k}.
\]  
(3.54)

By Lemma 3.1, we assume that \( f = g_j \in X_k \).
\[
\left\| \int_{\mathbb{R}} g_j(\xi, \tau) e^{it\tau} d\tau \right\|_{L^2_{x}} \leq C \| g_j(\xi, \tau) \|_{L^2_{\xi, t}} \leq C 2^{j/2} \| g_j \|_{L^2_{\xi, t}}
\]  
(3.55)

which gives (3.54) in this case.

Turn to the case \( k \geq 100 \) and \( f = f_e \in \mathcal{Y}_{k}^e, \ e \in \{e_1, \ldots, e_L\} \), we need to prove that
\[
\left\| \int_{\mathbb{R}} f_e(\xi, \tau) e^{it\tau} d\tau \right\|_{L^2_{x}} \leq C \| f_e \|_{\mathcal{Y}_{k}^e}.
\]  
(3.56)

By writing
\[
h_e(x, t) = 2^{-k/2} \mathcal{F}^{-1}_{(2+1)}[(P(\tau, \xi) + i) \cdot f_e](x, t)
\]
we get
\[
f_e(\xi, \tau) = \chi_{k, 10}(\xi \cdot \overline{e}) \cdot \frac{2^{k/2}}{P(\tau, \xi) + i} \mathcal{F}_{(2+1)}(h_e)(\xi, \tau)
\]

Let \( \xi = \xi_1^e + \xi_2^e e^{\pm} \), \( x = x_1^e + x_2^e e^{\pm} \). For (3.56) it suffices to prove that
\[
2^{k/2} \left\| \chi_{k, 10}(\xi \cdot \overline{e}) \int_{\mathbb{R}} \frac{1}{P(\tau, \xi) + i} \mathcal{F}_{(2+1)}(h)(\xi_1^e + \xi_2^e e^{\pm}, \tau) e^{it\tau} d\tau \right\|_{L^2_{x}} \leq C \| h \|_{L^{1, 2}_{k} t}
\]  
(3.57)

for any \( h \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}) \) and \( t \in \mathbb{R} \). As in the proof of Lemma 3.5 we define
\[
h'(x_1^e, \xi_2^e, \tau) = \int_{\mathbb{R} \times \mathbb{R}} h(x_1^e, x_2^e, t) e^{-i(x_2^e \cdot \xi_2^e + \tau t)} dx_2^e dt
\]
So
\[
\mathcal{F}_{(2+1)}(h)(\xi_1^e + \xi_2^e e^{\pm}, \tau) = \int_{\mathbb{R}} h'(x_1^e, \xi_2^e, \tau) e^{-i x_1^e \cdot \xi_1^e} dx_1^e,
\]
and \( \| h \|_{L^{1, 2}_{k} t} = C \| h' \|_{L^{1, 2}_{t} \mathcal{Y}_{k}^e} \). Let
\[
h^*_e(x_1^e, \xi_2^e, \mu) = \int_{\mathbb{R}} \frac{1}{\tau + \mu + i} h'(x_1^e, \xi_2^e, \tau) e^{it\tau} d\tau.
\]
Lemma 4.1. If \( \psi \) assume 
\[
\text{Proof of Lemma 4.1.}
\]
Just notice \( \partial_\psi^\tau (\xi_1^2 - \xi_2^2) = 2\xi_2 \cdot e \), (3.58) follows from changes of variables. \( \Box \)

4. Linear Estimates

In this section, we prove two linear estimates for the smi-group \( e^{it\Box_x} \) by following some ideas in [7]. For \( \sigma \geq 0 \) we define the normed spaces
\[
F^\sigma = \{ u \in C(\mathbb{R} : H^\infty) : \| u \|_{F^\sigma}^2 = \sum_{k=0}^{\infty} 2^{2\sigma k} \| \eta_k^{(d)}(\xi) \cdot F_{(d+1)} u \|_{L^2_k}^2 < \infty \},
\]
and
\[
N^\sigma = \{ u \in C(\mathbb{R} : H^\infty) : \| u \|_{N^\sigma}^2 = \sum_{k=0}^{\infty} 2^{2\sigma k} \| \eta_k^{(d)}(\xi) \cdot F_{(d+1)} u \|_{L^2_k}^2 < \infty \}.
\]

For \( \phi \in H^\infty \) let \( W(t)\phi \in C(\mathbb{R} : H^\infty) \) denote the solution of the free Schrödinger evolution
\[
[W(t)\phi](x,t) = c_0 \int_{\mathbb{R}^2} e^{ix \cdot \xi} e^{-it(\xi_1^2 - \xi_2^2)} F_{(2)}(\phi)(\xi) d\xi.
\]
Assume \( \psi : \mathbb{R} \to [0,1] \) is an even smooth function supported in the interval \([-8/5, 8/5]\) and equal to 1 in the interval \([-5/4, 5/4]\).

**Lemma 4.1.** If \( \sigma \geq 0 \) and \( \phi \in H^\infty \) then \( \psi(t) \cdot [W(t)\phi] \in F^\sigma \) and
\[
\| \psi(t) \cdot [W(t)\phi] \|_{F^\sigma} \leq C_\sigma \| \phi \|_{H^\sigma}.
\]

**Proof of Lemma 4.1** A straightforward computation shows that
\[
F[\psi(t) \cdot [W(t)\phi]](\xi, \tau) = F_{(2)}(\phi)(\xi) \cdot F_{(1)}(\psi)(P(\tau, \xi)).
\]
Then, directly from the definitions,
\[
\| \psi(t) \cdot [W(t)\phi] \|_{F^\sigma}^2 = \sum_{k \in \mathbb{Z}^+} 2^{2\sigma k} \| \eta_k^{(2)}(\xi) F_{(2)}(\phi)(\xi) F_{(1)}(\psi)(P(\tau, \xi)) \|_{L^2_k}^2
\]
\[
\leq \sum_{k \in \mathbb{Z}^+} 2^{2\sigma k} \| \eta_k^{(2)}(\xi) F_{(2)}(\phi)(\xi) F_{(1)}(\psi)(P(\tau, \xi)) \|_{L^2_k}^2
\]
\[
\leq C \sum_{k \in \mathbb{Z}^+} 2^{2\sigma k} \| \eta_k^{(2)}(\xi) \cdot F_{(2)}(\phi)(\xi) \|_{L^2}^2
\]
\[
\leq C \| \phi \|_{H^\sigma}^2.
\]
Lemma 4.2. If $\sigma \geq 0$ and $u \in N^\sigma$ then $\psi(t) \cdot \int_0^t W(t-s)(u(s)) \, ds \in F^\sigma$ and
\[
\| \psi(t) \cdot \int_0^t W(t-s)(u(s)) \, ds \|_{F^\sigma} \leq C \|u\|_{N^\sigma}.
\]

Proof of Lemma 4.2. A straightforward computation shows that
\[
\mathcal{F}\left[ \psi(t) \cdot \int_0^t W(t-s)(u(s)) \, ds \right](\xi, \tau) = c \int_{\mathbb{R}} \mathcal{F}(u)(\xi, \tau') \frac{\hat{\psi}(\tau - \tau') - \hat{\psi}(P(\tau, \xi))}{P(\tau, \xi)} \, d\tau',
\]
where, for simplicity of notation, $\hat{\psi} = \mathcal{F}(\psi)$. For $k \in \mathbb{Z}$ let
\[
f_k(\xi, \tau') = \mathcal{F}(u)(\xi, \tau') \cdot \eta_k(2)(\xi) \cdot (P(\tau', \xi) + i)^{-1}.
\]
For $f \in \mathcal{Z}_k$ let
\[
T(f)(\xi, \tau) = \int_{\mathbb{R}} f(\xi, \tau') \frac{\hat{\psi}(\tau - \tau') - \hat{\psi}(P(\tau, \xi))}{P(\tau, \xi)} (P(\tau', \xi) + i) \, d\tau'.
\]
where $P(\tau, \xi) = \tau + \xi_1^2 - \xi_2^2$. In view of the definitions, it suffices to prove that
\[
\|T\|_{\mathcal{Z}_k \to \mathcal{Z}_k} \leq C \text{ uniformly in } k \in \mathbb{Z}. \tag{4.5}
\]

To prove (4.5) we use the representation (3.1). Assume first that $f = g_j$ is supported in $D_{k,j}$. Let $g_j^\#(\xi, \mu') = g_j(\xi, \mu' - \xi_1^2 + \xi_2^2)$ and $[T(g)]^\#(\xi, \mu) = T(g)(\xi, \mu - \xi_1^2 + \xi_2^2)$. Then,
\[
[T(g)]^\#(\xi, \mu) = \int_{\mathbb{R}} g_j^\#(\xi, \mu') \frac{\hat{\psi}(\mu - \mu') - \hat{\psi}(\mu)}{\mu'} (\mu' + i) \, d\mu'.
\]
We use the elementary bound
\[
\left| \frac{\hat{\psi}(\mu - \mu') - \hat{\psi}(\mu)}{\mu'} (\mu' + i) \right| \leq C[(1 + |\mu|)^{-4} + (1 + |\mu - \mu'|)^{-4}].
\]
Then, using (4.6),
\[
|T(g)^\#(\xi, \mu)| \leq C(1 + |\mu|)^{-4} \cdot 2^{j/2} \left[ \int_{\mathbb{R}} |g_j^\#(\xi, \mu')|^2 \, d\mu' \right]^{1/2}
+ C1_{[-2^{j+10}, 2^{j+10}]}(\mu) \int_{\mathbb{R}} |g_j^\#(\xi, \mu')|(1 + |\mu - \mu'|)^{-3} \, d\mu'.
\]
It follows from the definition of the spaces $X_k$ that
\[
\|T\|_{X_k \to X_k} \leq C \text{ uniformly in } k \in \mathbb{Z}_+, \tag{4.7}
\]
as desired.
Assume now that \( f = f^e \in Y_k^e,\ k \geq 100,\ e \in \{e_1, \ldots, e_L\} \). We write
\[
f^e(\xi, \tau') = \frac{\tau' + \xi_1^2 - \xi_2^2}{\tau' + \xi_1^2 - \xi_2^2 + i} f^e(\xi, \tau') + \frac{i}{\tau' + \xi_1^2 - \xi_2^2 + i} f^e(\xi, \tau').
\]
Using Lemma 3.3 [\( ||(i(\tau' + \xi_1^2 - \xi_2^2 + i)^{-1} f^e(\xi, \tau'))||_{X_k} \leq C||f^e||_{Y_k^e} \). In view of (1.4) and (4.1), for (4.5) it suffices to prove that
\[
|| \int f^e(\xi, \tau') \hat{\psi}(\tau - \tau') d\tau' ||_{Z_k} + || \hat{\psi}(P(\tau, \xi)) \int f^e(\xi, \tau') d\tau' ||_{X_k} \leq C||f^e||_{Y_k^e}. \quad (4.8)
\]
The bound for the first term in the left-hand side of (4.8) follows easily from the definition. The bound for the second term in the left-hand side of (4.8) follows from (3.56) with \( t = 0 \). \( \square \)

5. Trilinear estimates

In this section, we set up a trilinear estimate. First we reduce the nonlinear term of (1.4). Let \( u(t) \in C(\mathbb{R} : H^\infty) \), and write the nonlinear term of (1.4) as
\[
\mathfrak{F}(u) = \frac{2\bar{u}}{1 + uu} [(\partial_{x_1} u)^2 - (\partial_{x_2} u)^2] + ib(\phi_{x_1} u_{x_2} + \phi_{x_2} u_{x_1})
\]
\[
\Delta \phi = 4i \left[ \left( \frac{u_{x_1} \bar{u}}{1 + |u|^2} \right)_{x_2} - \left( \frac{u_{x_2} \bar{u}}{1 + |u|^2} \right)_{x_1} \right], \quad (5.1)
\]
thus we have
\[
\phi_{x_1} u_{x_2} + \phi_{x_2} u_{x_1} = 4i \left[ \frac{\partial_{x_1} \partial_{x_2}}{\Delta} \left( \frac{u_{x_1} \bar{u}}{1 + |u|^2} \right) - \frac{\partial_{x_1}^2}{\Delta} \left( \frac{u_{x_2} \bar{u}}{1 + |u|^2} \right) \right] u_{x_2}
+ 4i \left[ \frac{\partial_{x_2}^2}{\Delta} \left( \frac{u_{x_2} \bar{u}}{1 + |u|^2} \right) - \frac{\partial_{x_1} \partial_{x_2}}{\Delta} \left( \frac{u_{x_1} \bar{u}}{1 + |u|^2} \right) \right] u_{x_1}. \quad (5.2)
\]
Here \( \frac{\partial_{x_1}}{\Delta}, \frac{\partial_{x_2}}{\Delta}, \frac{\partial_{x_2}^2}{\Delta} \) are defined by Fourier multipliers \( \frac{\xi_1 \xi_2}{\xi_1^2 + \xi_2^2}, \frac{\xi_1^2}{\xi_1^2 + \xi_2^2}, \frac{\xi_2^2}{\xi_1^2 + \xi_2^2} \). And we use \( \mathcal{R}_i (i = 1, 2, 3) \) to denote these three \( L^2 \)-bounded operators accordingly. Thus the nonlinear term of (1.4) can be written as
\[
\mathfrak{F}(u) = \mathcal{N}_0(u) [(\partial_{x_1} u)^2 - (\partial_{x_2} u)^2] - 4b \mathcal{R}_1(\mathcal{N}_0(u) \partial_{x_1} u) \partial_{x_2} u
+ 4b \mathcal{R}_2(\mathcal{N}_0(u) \partial_{x_2} u) \partial_{x_1} u - 4b \mathcal{R}_3(\mathcal{N}_0(u) \partial_{x_1} u) \partial_{x_2} u
+ 4b \mathcal{R}_1(\mathcal{N}_0(u) \partial_{x_2} u) \partial_{x_1} u, \quad (5.3)
\]
where \( \mathcal{N}_0(u) = 2\bar{u}/(1 + |u|^2) \).

We consider here the nonlinear term
\[
\mathcal{N}(u) = \psi(t) \mathfrak{F}(u) \in C(\mathbb{R} : H^\infty), \quad (5.4)
\]
and are looking for the control of
\[
||\mathcal{N}(u) - \mathcal{N}(v)||_{\mathcal{N}_\sigma} \quad \sigma > 3/2,
\]
where \( u, v \in F^\sigma \).
For \( k \in \mathbb{Z}_+ \) we define the normed spaces
\[
\tilde{Z}_k = \{ f \in L^2(\mathbb{R}^2 \times \mathbb{R}) : \text{supp} f \in I_k \times \mathbb{R}, \| f \|_{\tilde{Z}_k} < \infty \}
\]
where
\[
\| f \|_{\tilde{Z}_k} = 2^{-k/2}(k + 1)^{-2} \sup_{t \in S^1} \| 1_{[-2,2]}(t) \mathcal{F}^{-1}_{(2+1)}(f) \|_{L^2_{\sigma}} \nonumber \]
\[
+ 2^{k/2} \sup_{t \in S^1} \| \mathcal{F}^{-1}_{(2+1)}[f \cdot \chi_{20}(\xi \cdot \overline{\tau})] \|_{L^\infty_{\sigma}}. \tag{5.5}
\]
For \( \sigma \geq 0 \) we define the normed spaces
\[
\tilde{F}^\sigma = \left\{ u \in C(\mathbb{R} : H^\infty) : \| u \|_{\tilde{F}^\sigma} = \sum_{k=0}^{\infty} 2^{2\sigma k} \| \eta_k(\xi) \mathcal{F}^{-1}_{(2+1)}(u) \|_{\tilde{Z}_k} < \infty \right\},
\]
and
\[
F^\sigma = \{ u \in C(\mathbb{R} : H^\infty) : \| u \|_{F^\sigma} = \| u \|_{\tilde{F}^\sigma} \}. \tag{5.6}
\]
It is easy to see that
\[
\| \tilde{u} \|_{F^\sigma + F^\sigma} = \| u \|_{F^\sigma + F^\sigma}.
\]
By Lemma 3.5 and Lemma 3.7, we obtain
\[
\left\{ \begin{array}{l}
\| f \|_{\tilde{Z}_k} \leq C \| f \|_{Z_k} \text{ for any } k \in \mathbb{Z}_+ \text{ and } f \in Z_k \\
\| u \|_{\tilde{F}^\sigma} \leq C \| u \|_{F^\sigma + F^\sigma} \text{ for any } \sigma \geq 0 \text{ and } u \in F^\sigma + F^\sigma. \end{array} \right. \tag{5.6}
\]
For \( \sigma \in \mathbb{R} \) let \( J^\sigma \) denote the operator defined by the Fourier multiplier \((\xi, \tau) \to (1 + |\xi|^2)^{\sigma/2}\).

**Lemma 5.1.** For \( \sigma > 3/2 \) we have
\[
\| \mathcal{R}(J^1(u_1)J^1(u_2))J^1(u_3) \|_{N^\sigma} \leq C_\sigma \| u_1 \|_{\tilde{F}^\sigma} \| u_2 \|_{\tilde{F}^\sigma} \| u_3 \|_{\tilde{F}^\sigma}, \tag{5.7}
\]
where \( \mathcal{R} \) denotes \( I, \frac{\partial}{\partial x}, \frac{\partial}{\partial \xi}, \) or \( \frac{\partial^2}{\partial x^2} \), and \( J^\sigma \) denotes the operator defined by the Fourier multiplier \((\xi, \tau) \to (1 + |\xi|^2)^{\sigma/2}\).

**Proof of Lemma 5.1** Let \( k_{\text{max}} = \max\{k_1, k_2, k_3\} \), and similarly \( k_{\text{med}} \) and \( k_{\text{min}} \). In view of the definition, for (5.7), it suffices to prove a dyadic trilinear estimate
\[
2^{k_1+k_2+k_3} \| (P_3, \tau, \xi) - i \mathcal{F}^{-1}_{(2+1)}[P_k(\mathcal{R}P_k(u_1P_ku_2P_ku_3))] \|_{Z_k} \nonumber \]
\[
\leq C 2^{k_{\text{max}}-\frac{1}{2}} 2^{3k_{\text{med}}+3k_{\text{min}}} (k_{\text{med}} + 1)^3 (k_{\text{min}} + 1)^3 \times \| P_ku_1 \|_{\tilde{Z}_{k_1}} \| P_ku_2 \|_{\tilde{Z}_{k_2}} \| P_ku_3 \|_{\tilde{Z}_{k_3}}. \tag{5.8}
\]
For \( e \in \{ e_1, \ldots, e_L \} \), let
\[
\eta_{k,e}(\xi) = \begin{cases} \eta_{k,1}(\xi) \cdot \eta_{k-5,k,5}^+(\xi \cdot \overline{\tau}) & \text{if } k \geq 100; \\
\eta_{k,2}(\xi) & \text{if } k < 100. 
\end{cases}
\]
For (5.8) it suffices to prove that for any \(e \in \{e_1, \ldots, e_L\},\)

\[
2^{k_1+k_2+k_3} \| \eta_{k,e}(\xi)(P(\tau, \xi) + i)^{-1} \mathcal{F}_{(2+1)} [P_k(\mathcal{R}(P_{k_1} u_1 P_{k_2} u_2) P_{k_3} u_3)] \|_{Z_k} \\
\leq C 2^{k_{\max-3}} 2^{\frac{3k_{\med}+3k_{\min}}{2}} (k_{\med} + 1)^3 (k_{\min} + 1)^3 \\
\times \| P_{k_1} u_1 \|_{\bar{Z}_{k_1}} \| P_{k_2} u_2 \|_{\bar{Z}_{k_2}} \| P_{k_3} u_3 \|_{\bar{Z}_{k_3}}.
\]

(5.9)

We first consider \(k_1 \geq k_{\max} - 20\). So \(k_1 \geq k - 25\). By an angular partition of unity in frequency, we can assume \(\mathcal{F}(P_{k_1} u_1)\) is supported in the set

\[
\{ (\xi, \tau) : |\xi| \in [2^{k_1-1}, 2^{k_1+1}] \text{ and } \xi \cdot \mathbf{e}_0 \geq 2^{k_1-5} \}
\]

for some vector \(\mathbf{e}_0 \in S^1\). Thus we have

\[
\| P_{k_1} u_1 \|_{L_{200}^\infty} \lesssim 2^{-k_1/2} \| P_{k_1} u_1 \|_{\bar{Z}_{k_1}}.
\]

(5.10)

By Hölder’s inequality and (5.10) we have

\[
2^{k_1+k_2+k_3} \| \eta_{k,e}(\xi)(P(\tau, \xi) + i)^{-1} \mathcal{F}_{(2+1)} [P_k(\mathcal{R}(P_{k_1} u_1 P_{k_2} u_2) P_{k_3} u_3)] \|_{Z_k} \\
\leq C 2^{-\frac{3}{2}} 2^{k_1+k_2+k_3} \| P_k(\mathcal{R}(P_{k_1} u_1 P_{k_2} u_2) P_{k_3} u_3) \|_{L^1_{\bar{Z}}} \\
\leq C 2^{-\frac{3}{2}} 2^{k_1+k_2+k_3} \| P_{k_1} u_1 \|_{L^\infty_{200}} \| P_{k_2} u_2 \|_{L^2_{200}} \\
\leq C 2^{-\frac{3}{2}} 2^{k_1+k_2+k_3} \| P_{k_1} u_1 \|_{L^\infty_{200}} \| P_{k_2} u_2 \|_{L^2_{200}} \\
\leq C 2^{-\frac{3}{2}} 2^{k_1+k_2+k_3} \| P_{k_1} u_1 \|_{L^\infty_{200}} \| P_{k_2} u_2 \|_{L^2_{200}} \| P_{k_3} u_3 \|_{\bar{Z}_{k_3}}.
\]

(5.11)

Which is enough for (5.8). The proof for \(k_2 \geq k_{\max} - 20\) is the same by symmetry.

Now, Let \(k_3 = k_{\max}\). In this case \(k_3 \geq k - 3\). Furthermore, in view of the above argument, we can assume that \(k_3 \geq k_{\med} + 20\), thus

\[
\eta_{k,e}(\xi)(P(\tau, \xi) + i)^{-1} \mathcal{F}_{(2+1)} [P_k(\mathcal{R}(P_{k_1} u_1 P_{k_2} u_2) P_{k_3} u_3)] \\
= \eta_{k,e}(\xi)(P(\tau, \xi) + i)^{-1} \mathcal{F}_{(2+1)} [P_k(\mathcal{R}(P_{k_1} u_1 P_{k_2} u_2) P_{k_3} u_3)]
\]

where \(\mathcal{F}(\tilde{P}_{k,e}f)(\xi, \tau) = \tilde{\eta}_{k,e}(\xi) \tilde{f}(\xi, \tau)\),

\[
\tilde{\eta}_{k,e}(\xi) = \begin{cases} 
\eta_{[k-1, k+1]}(\xi) \cdot \eta_{[k-10, k+10]}(\xi \cdot \mathbf{e}) & \text{if } k \geq 100; \\
\eta_{k}(\xi) & \text{if } k < 100.
\end{cases}
\]

and

\[
\| \tilde{P}_{k,e} P_{k_3} u_3 \|_{L^\infty_{200}} \lesssim 2^{-k_3/2} \| P_{k_3} u_3 \|_{\bar{Z}_{k_3}}.
\]

(5.12)
Thus
\[ 2^{k_1+k_2+k_3} \| \eta_{k,e}(\xi)(P(\tau, \xi) + i)^{-1} F_{(2+1)}[P_k(\mathcal{R}(P_{k_1} u_1 P_{k_2} u_2) P_{k_3} u_3)] \| z_k \leq C 2^{-\frac{k}{2}} 2^{k_1+k_2+k_3} \| P_k(\mathcal{R}(P_{k_1} u_1 P_{k_2} u_2) \tilde{P}_{k,e} P_{k_3} u_3) \| L^1_\alpha \leq C 2^{-\frac{k}{2}} 2^{k_1+k_2+k_3} \| \tilde{P}_{k,e} P_{k_3} u_3 \| L^\infty_\alpha \leq \| \mathcal{R}(P_{k_1} u_1 P_{k_2} u_2) \| L^1_\alpha \]

By \( \| P_k \mathcal{R}(f) \| L^1_\alpha \leq C \| f \| L^1_\alpha \), we can continue with
\[ C 2^{\frac{k}{2}} 2^{k_1+k_2+k_3} \| \tilde{P}_{k,e} P_{k_3} u_3 \| L^\infty_\alpha (k_{\text{med}} + 1) \| P_{k_1} u_1 P_{k_2} u_2 \| L^1_\alpha \leq C 2^{\frac{k}{2}} 2^{k_1+k_2+k_3} (k_{\text{med}} + 1) \| \tilde{P}_{k,e} P_{k_3} u_3 \| L^\infty_\alpha \| P_{k_1} u_1 \| L^2_\alpha \| P_{k_2} u_2 \| L^2_\alpha \leq C 2^{\frac{k}{2}} 2^{k_1+k_2+k_3} \sum_{k \leq k_{\text{med}} + 1} \| \mathcal{R}(P_{k_1} u_1 P_{k_2} u_2) \| L^1_\alpha \]

We finish the proof of (5.9). \( \square \)

6. Multilinear Estimates

The purpose of the this section is to estimate the nonlinear term
\[ \mathcal{N}_0(u) = \frac{2\pi}{1 + u^2} \]

with \( u \in C(\mathbb{R} : H^\infty) \). The basic tool to analysis the \( \mathcal{N}_0(u) \) term is the algebra property of the resolution spaces, say Lemma 6.3. In order to set up Lemma 6.3 we need the following two simple \( L^2 \) estimates.

Lemma 6.1. If \( k_1, k_2, k \in \mathbb{Z}_+, j_1, j_2, j \in \mathbb{Z}_+, \) and \( g_{k_1,j_1}, g_{k_2,j_2} \) are \( L^2 \) functions supported in \( D_{k_1,j_1} \) and \( D_{k_2,j_2} \) then
\[ \| 1_{D_{k,j}} \cdot (g_{k_1,j_1} * g_{k_2,j_2}) \| L^2 \leq C 2^{\min(k_1,k_2,k) - \frac{1}{2}} 2^{\min(j_1,j_2,j) - \frac{1}{2}} \| g_{k_1,j_1} \| L^2 \| g_{k_2,j_2} \| L^2. \]  

(6.1)

For any \( k, j \in \mathbb{Z}_+ \), and \( f_k \in Z_k \) we denote
\[ f_{k,j}(\xi, \tau) = f_k(\xi, \tau) \cdot \eta_{\leq j}(P(\tau, \xi)) \text{ and } f_{k,j}(\xi, \tau) = f_k(\xi, \tau) \cdot \eta_{\geq j}(P(\tau, \xi)). \]

We will use the following estimate in this section frequently.

Lemma 6.2. If \( k_1, k_2 \in \mathbb{Z}_+, k_1 \leq k_2 + C, j_1, j_2 \in \mathbb{Z}_+, \) and \( f_{k_1} \in Z_{k_1}, f_{k_2} \in Z_{k_2} \) and \( \sigma' > 1 \) then
\[ \| \tilde{f}_{k_1,j_1} * \tilde{f}_{k_2,j_2} \| L^2 \leq C (2^{j_2/2} 2^{(k_1+k_2)/2})^{-1} (2^{\sigma' k_1} \| f_{k_1} \| Z_{k_1}) \| f_{k_2} \| Z_{k_2} \]

(6.2)

where \( \mathcal{F}^{-1}(\tilde{f}_{k_1,j_1}) \in \{ \mathcal{F}^{-1}(f_{k_1,j_1}), \mathcal{F}^{-1}(f_{k_1,j_1}) \}, i = 1, 2. \)
Proof of Lemma 6.2. If $k_2 \leq 100$, By Lemma 3.3 and Lemma 3.8
\[
\| \tilde{f}_{k_1, \geq j_1} * \tilde{f}_{k_2, \geq j_2} \|_{L^2} \leq C \| \mathcal{F}^{-1}(f_{k_1, \geq j_1}) \|_{L^\infty} \| \mathcal{F}^{-1}(f_{k_2, \geq j_2}) \|_{L^2} \\
\leq C 2^{k_1} \| f_{k_1} \|_{Z_{k_1}} 2^{-j_2/2} \| f_{k_2} \|_{Z_{k_2}}.
\]
This is enough for (6.2).

If $k_2 \geq 100$, in view of Lemma 3.3, we can assume that: $f_{k_2}$ is supported in \{$(\xi_2, \tau_2) : |\xi_2 - v| \leq 2^{k_2-50}$\} for some $v \in I_{k_2}^{(2)}$. Let $\tilde{\nu} = \nu/|v|$, then when $k_1 + k_2 \geq j_2$, we use Lemma 3.7 and Lemma 3.3 to get
\[
\| \tilde{f}_{k_1, \geq j_1} * \tilde{f}_{k_2, \geq j_2} \|_{L^2} \leq C \| \mathcal{F}^{-1}(f_{k_1, \geq j_1}) \|_{L^\infty} \| \mathcal{F}^{-1}(f_{k_2, \geq j_2}) \|_{L^2} \\
\leq C 2^{k_1/2} (k_1 + 1)^2 \| f_{k_1, \geq j_1} \|_{Z_{k_1}} 2^{-j_2/2} \| f_{k_2, \geq j_2} \|_{Z_{k_2}} \\
\leq C 2^{-1/2} 2^{\sigma k_1} \| f_{k_1} \|_{Z_{k_1}} \| f_{k_2} \|_{Z_{k_2}}.
\]
When $k_1 + k_2 \leq j_2$, we use the definition and Lemma 3.8 to get
\[
\| \tilde{f}_{k_1, \geq j_1} * \tilde{f}_{k_2, \geq j_2} \|_{L^2} \leq C \| \mathcal{F}^{-1}(f_{k_1, \geq j_1}) \|_{L^\infty} \| \mathcal{F}^{-1}(f_{k_2, \geq j_2}) \|_{L^2} \\
\leq C 2^{k_1} \| f_{k_1, \geq j_1} \|_{Z_{k_1}} 2^{-j_2/2} \| f_{k_2, \geq j_2} \|_{Z_{k_2}} \\
\leq C 2^{-j_2/2} 2^{\sigma k_1} \| f_{k_1} \|_{Z_{k_1}} \| f_{k_2} \|_{Z_{k_2}}.
\]
Thus we finish the proof. \hfill \Box

Lemma 6.3. Assume $u, v \in F^\sigma + \overline{F}^\sigma$, then for $\sigma > 1$ we have
\[
\| u \cdot v \|_{F^\sigma + \overline{F}^\sigma} \leq C \| u \|_{F^\sigma + \overline{F}^\sigma} \| v \|_{F^\sigma + \overline{F}^\sigma} \tag{6.3}
\]

Remark 6.4. If here we define $F^\sigma$ by $X_k$ instead of $Z_k$, then the bilinear estimate (6.3) was already proved in [10].

Proof of Lemma 6.3. Let $f_k \in \{ \eta_k^{(2)}(\xi) \cdot \mathcal{F}(u), \eta_k^{(2)}(\xi) \cdot \mathcal{F}(\nu) \}$, and $g_k \in \{ \eta_k^{(2)}(\xi) \cdot \mathcal{F}(v), \eta_k^{(2)}(\xi) \cdot \mathcal{F}(\nu) \}$. It suffices to show that for any $k_1, k_2 \in \mathbb{Z}_+$, $\sigma > 1$,
\[
\sum_{k_1, k_2 \in \mathbb{Z}_+} 2^{\sigma k} \left( \sum_{k_1, k_2 \in \mathbb{Z}_+} \| \eta_k^{(2)}(\xi) \cdot (f_k \ast g_k) \|_{Z_k} \right)^2 \\
\leq C \left( \sum_{k_1 \in \mathbb{Z}_+} 2^{\sigma k_1} \| f_{k_1} \|_{Z_{k_1}}^2 \right) \left( \sum_{k_2 \in \mathbb{Z}_+} 2^{\sigma k_2} \| g_{k_2} \|_{Z_{k_2}}^2 \right). \tag{6.4}
\]
Furthermore, we need to show that, if $k_1, k_2, k \in \mathbb{Z}_+$, $k_1 \leq k_2 + 10$, $f_{k_1} \in Z_{k_1}$ and $f_{k_2} \in Z_{k_2}$, then
\[
2^{\sigma k} \| \eta_k^{(2)}(\xi) \cdot (\tilde{f}_{k_1} \ast \tilde{f}_{k_2}) \|_{Z_k} \leq C 2^{-|k_2-k|/4} (2^{\sigma k_1} \| f_{k_1} \|_{Z_{k_1}}) (2^{\sigma k_2} \| f_{k_2} \|_{Z_{k_2}}), \tag{6.5}
\]
where $\mathcal{F}^{-1}(\tilde{f}_{k_1}) \in \{ \mathcal{F}^{-1}(f_{k_1}), \overline{\mathcal{F}}^{-1}(f_{k_1}) \}$, and $1 < \sigma' < \sigma$. 
We may assume \( k \leq k_2 + 20 \). If \( k_2 \leq 99 \), the bound (6.5) follows easily from Lemma 6.1 (also see the Case 1 below). We only consider the case \( k_2 \geq 100 \). In view of Lemma 3.3, we may assume that

\[ f_{k_2} \text{ is supported in } I_{k_2}^{(2)} \times \mathbb{R} \cap \{(\xi_2, \tau_2) : |\xi_2 - v| \leq 2^{k_2 - 50}\} \text{ for some } v \in I_{k_2}^{(2)}. \]

With \( v \) as above, let \( \hat{v} = v/|v| \in S^1 \) and

\[ \hat{K} = k_1 + k_2 + 100 \]

By Lemma 6.2 with \( j_1 = j_2 = 0 \), we obtain

\[
2^{\sigma_k} \| \eta_{\leq \hat{K} - 1}(P(\tau, \xi)) \eta_k^{(2)}(\xi) \cdot (\tilde{f}_{k_1} \ast f_{k_2}) \|_{Z_k} \\
\leq 2^{\sigma_k} 2^{\hat{K}/2} \| \tilde{f}_{k_1} \ast f_{k_2} \|_{L^2} \\
\leq 2^{\sigma_k} (2^{\sigma_k} k_1 \| f_{k_1} \|_{Z_{k_1}}) \cdot \| f_{k_2} \|_{Z_{k_2}}
\]

So for (6.5), it remains to estimate

\[
2^{\sigma_k} \| \eta_{\geq \hat{K}}(P(\tau, \xi)) \eta_k^{(2)}(\xi) \cdot (\tilde{f}_{k_1} \ast f_{k_2}) \|_{Z_k} \\
\leq C 2^{-|k_2-k|/4} (2^{\sigma'k_1} \| f_{k_1} \|_{Z_{k_1}}) (2^{\sigma_k} \| f_{k_2} \|_{Z_{k_2}})
\]

where \( 1 < \sigma' < \sigma \). By Lemma 3.1, we need to analyze several cases.

**Case 1.** \( f_{k_2} = g_{k_2, j_2} \in X_{k_2}, f_{k_1} \in Z_{k_1} \), let \( g_{k_1, j_1} = \eta_{j_1}(P(\tau, \xi)) f_{k_1} \). By the definition of \( Z_k \) and Lemma 6.1, we get

\[
2^{\sigma_k} \| \eta_{\leq \hat{K} - 1}(P(\tau, \xi)) \eta_k^{(2)}(\xi) \cdot (\tilde{g}_{k_1, j_1} \ast f_{k_2}) \|_{Z_k} \\
\leq C 2^{\sigma_k} 2^{\max(j_1 + j_2)/2} \sup_{j \leq \max(j_1, j_2) + C} \| 1_{D_{k,j}} \cdot (\tilde{g}_{k_1, j_1} \ast g_{k_2, j_2}) \|_{L^2} \\
\leq C 2^{\sigma_k} 2^{\max(j_1 + j_2)/2} 2^{\min(j_1, j_2)/2} \| g_{k_1, j_1} \|_{L^2} \| g_{k_2, j_2} \|_{L^2}.
\]

**Case 2.** \( f_{k_2} \in Y_{k_2}^{\epsilon}, f_{k_1} \in Z_{k_1} \), and \( k_2 \leq k_1 + C \), so \(|k_1 - k_2| \leq C\). By case 1, and Lemma 3.3, we can assume that \( f_{k_2} \) is supported in the set \( \{(\xi_2, \tau_2) : |P(\xi_2, \tau_2)| \leq 2^{\hat{K}-100}\} \). Thus \( j_1 \geq \hat{K} - 10 \) (unless the left hand-side of (6.6) vanish). Let \( g_{k_1, j_1} = \eta_{j_1}(P(\tau, \xi)) f_{k_1} \), we have

\[
2^{\sigma_k} \| \eta_{\leq \hat{K} - 1}(P(\tau, \xi)) \eta_k^{(2)}(\xi) \cdot (\tilde{g}_{k_1, j_1} \ast f_{k_2}) \|_{Z_k} \\
\leq C 2^{\sigma_k} 2^{j_1/2} \| \tilde{g}_{k_1, j_1} \ast f_{k_2} \|_{L^2} \\
\leq C 2^{\sigma_k} 2^{j_1/2} \| g_{k_1, j_1} \|_{L^2} \| \mathcal{F}^{-1}(f_{k_2}) \|_{L^\infty},
\]

which is suffices for (6.6) by Lemma 3.8.

**Case 3.** \( f_{k_2} \in Y_{k_2}^{\epsilon}, f_{k_1} = f_{k_1} \ast \eta_{\leq \hat{K} - 1}(P(\tau, \xi)) \in Z_{k_1}, k_1 \leq k_2 - 10 \), so \(|k - k_2| \leq 2\). It suffice to prove

\[
2^{\sigma_k} \| \eta_{\geq \hat{K}}(P(\tau, \xi)) \eta_k^{(2)}(\xi) \cdot (\tilde{f}_{k_1} \ast f_{k_2}) \|_{Z_k} \\
\leq C (2^{\sigma_k} k_1 \| f_{k_1} \|_{Z_{k_1}}) (2^{\sigma_k} \| f_{k_2} \|_{Y_{k_2}^{\epsilon}}).
\]

(6.9)
First notice that
\[ \tilde{f}_{k_1} \ast f_{k_2} \] is supported in the set \( \{ (\xi, \tau); \xi \cdot \xi_1 \in [2^{k-2}, 2^{k+2}] \} \)
and the following identity
\[
-\mathcal{F}^{-1} \left[ (P(\tau, \xi) + i) \cdot (\tilde{f}_{k_1} \ast f_{k_2}) \right] \\
= (i \partial_\xi + \Box - i)\mathcal{F}^{-1}(f_{k_2}) \cdot \mathcal{F}^{-1}(f_{k_1}) \\
+ 2\nabla \mathcal{F}^{-1}(\tilde{f}_{k_1}) \cdot \nabla \mathcal{F}^{-1}(f_{k_2}) \\
+ \mathcal{F}^{-1}(f_{k_2}) \cdot (i \partial_\xi + \Box)\mathcal{F}^{-1}(\tilde{f}_{k_1})
\] (6.10)

where \( \vec{\nabla} = (\partial_{x_1}, -\partial_{x_2}) \), we have
\[
2^{\sigma_k} \| \eta_{\geq K}(P(\tau, \xi)) \eta_k^{(2)}(\xi) \cdot (\tilde{f}_{k_1} \ast f_{k_2}) \|_{Z_k} \\
\leq C 2^{\sigma_k 2 - k/2} \| (i \partial_\xi + \Box - i)\mathcal{F}^{-1}(f_{k_2}) \cdot \mathcal{F}^{-1}(f_{k_1}) \|_{L^1_{\xi_1}} \\
+ C 2^{\sigma_k 2 - \tilde{K}/2} \| \nabla \mathcal{F}^{-1}(\tilde{f}_{k_1}) \cdot \vec{\nabla} \mathcal{F}^{-1}(f_{k_2}) \|_{L^2} \\
+ C 2^{\sigma_k 2 - \tilde{K}/2} \| \mathcal{F}^{-1}(f_{k_2}) \cdot (i \partial_\xi + \Box)\mathcal{F}^{-1}(\tilde{f}_{k_1}) \|_{L^2}. \] (6.11)

The first term in the right-hand side of (6.11) can be controlled by
\[
C 2^{\sigma_k 2 - k/2} \| (i \partial_\xi + \Box - i)\mathcal{F}^{-1}(f_{k_2}) \|_{L^1_{\xi_1}} \cdot \| \mathcal{F}^{-1}(f_{k_1}) \|_{L^\infty},
\]
which is enough for (6.9) in view of Lemma 3.8. The second and the third terms are bounded by
\[
C 2^{\sigma_k 2 - \tilde{K}/2} 2^k \| \mathcal{F}^{-1}(\tilde{f}_{k_1}) \cdot \mathcal{F}^{-1}(f_{k_2}) \|_{L^2},
\]

Which is enough for (6.9) by Lemma 6.2.

**Case 4.** \( f_{k_2} \in Y^{e}_{k_2}; f_{k_1} \in Z_{k_1}, k_1 \leq k_2 - 10 \) and \( j_1 \geq \tilde{K} \). we denote \( g_{k_1,j_1} = \eta_{j_1}(P(\tau, \xi))f_{k_1} \), for (6.6), it suffices to prove
\[
2^{\sigma_k} \| \eta_{\geq \tilde{K}}(P(\tau, \xi)) \eta_k^{(2)}(\xi) \cdot (g_{k_1,j_1} \ast f_{k_2}) \|_{Z_k} \leq C 2^{k_1 2^{j_1}/2} \| g_{k_1,j_1} \|_{L^2} (2^{\sigma_k 2} \| f_{k_2} \|_{Y^{e}_{k_2}}) \] (6.12)

By Lemma 3.1 we decompose
\[ f_{k_3} = f_{k_2,\leq j_1-10} + f_{k_2,\geq j_1+10} + X_{k_2}. \]

In view of Case 1, for (6.12), it suffices to prove that
\[
2^{\sigma_k} \| \eta_k^{(2)}(\xi) \cdot (g_{k_1,j_1} \ast f_{k_2,\leq j_1-10}) \|_{Z_k} \\
+ 2^{\sigma_k} \| \eta_{j_1}(P(\tau, \xi)) \cdot \eta_k^{(2)}(\xi) \cdot (g_{k_1,j_1} \ast f_{k_2,\geq j_1+10}) \|_{Z_k} \\
\leq C (2^{k_1 2^{j_1}/2} \| g_{k_1,j_1} \|_{L^2}) \cdot (2^{\sigma_k 2} \| f_{k_2} \|_{Y^{e}_{k_2}}). \] (6.13)

For the first term in (6.13), it suffices to prove
\[
2^{\sigma_k 2^{j_1}/2} \| f_{k_2,\leq j_1-10} \ast \tilde{g}_{k_1,j_1} \|_{L^2} \leq C (2^{k_1 2^{j_1}/2} \| g_{k_1,j_1} \|_{L^2}) \cdot (2^{\sigma_k 2} \| f_{k_2} \|_{Y^{e}_{k_2}}). \] (6.14)
By Lemma 3.4, we can assume that
\[ f_{k_2, \leq j_1-10}(\xi, \tau) = 2^{-k_2/2} \eta \chi_{k_2, \leq 90(\xi^e - t^*_e)}(\xi_1^e - t^*_e + i/2k_2) \cdot h(\xi_2, \tau), \]
where \( \|h\|_{L^2} \leq C \|f_{k_2, \leq j_1-10}\|_{L^2}. \) For (6.14), it suffices to prove
\[ \|f_{k_2, \leq j_1-10} \ast \tilde{g}_{k_1, j_1}\|_{L^2} \leq 2^{k_1} \|g_{k_1, j_1}\|_{L^2} \cdot \|h\|_{L^2}. \]  
(6.15)

We estimate the \( L^2 \) norm in the left-hand side of (6.15) by duality. The left-hand side of (6.15) is bounded by
\[
I = 2^{-k_2/2} \sup_{\|a\|_{L^2} = 1} \left| \int_{\mathbb{R}^6} \tilde{g}_{k_1, j_1}(\eta^e_1 e + \eta^e_2 e^\perp, \beta) \cdot h(\xi_2, \tau) \right. \\
\times \eta \chi_{k_2, \leq 90(\xi_1^e - t^*_e)}(\xi_1^e - t^*_e + i/2k_2) \\
\left. \times a(\xi_1^e + \eta_1^e, \xi_2^e + \eta_2^e, \tau + \beta) \, d\xi_1^e d\eta_1^e d\xi_2^e d\eta_2^e d\tau d\beta \right|
\]
\[ = 2^{-k_2/2} \sup_{\|a\|_{L^2} = 1} \left| \int_{\mathbb{R}^5} \tilde{a}(\eta_1^e + t^*_e, \xi_2^e + \eta_2^e, \tau + \beta) \, d\eta_1^e d\xi_2^e d\eta_2^e d\tau d\beta \right|. \]  
(6.16)

Here
\[
\tilde{a}(\eta_1^e, \eta_2^e, \beta) = \int_{\mathbb{R}} \eta \chi_{k_2, \leq 90(\xi_1^e)}(\xi_1^e + i/2k_2) \cdot a(\xi_1^e + \eta_1^e, \eta_2^e, \beta) \, d\xi_1^e.
\]

The boundedness of Hilbert transform gives \( \|\tilde{a}(\eta_1^e, \eta_2^e, \beta)\|_{L^2_{\eta_1^e}} \leq C \|a(\eta_1^e, \eta_2^e, \beta)\|_{L^2_{\eta_1^e}}. \)

By using Hölder’s inequality in the variables \( (\xi_2^e, \tau, \beta) \), we get
\[
I \leq C 2^{-k_2/2} \sup_{\|a\|_{L^2} = 1} \left( \int_{\mathbb{R}^2} \tilde{g}_{k_1, j_1}(\eta_1^e e + \eta_2^e e^\perp, \beta) \|h(\xi_2^e, \tau)\|_{L^2_{\xi_2^e}} \right) \|\tilde{a}(\eta_1^e + t^*_e, \xi_2^e, \beta)\|_{L^2_{\xi_2^e, \tau}} \, d\eta_1^e d\xi_2^e. \]  
(6.17)

Here \( t^*_e \) is the same as Lemma 3.4, so we have \( |\partial_x t^*_e| \geq c2^{-k_2}. \) Then by change of variables, we have
\[
I \leq C 2^{-k_2/2} 2^{k_2/2} \int_{\mathbb{R}^2} \|\tilde{g}_{k_1, j_1}(\eta_1^e e + \eta_2^e e^\perp, \beta)\|_{L^2_{\xi_2^e}} \, d\eta_1^e d\eta_2^e \cdot \|h(\xi_2, \tau)\|_{L^2_{\xi_2, \tau}}
\]
which is sufficient for (6.15).

From (6.10), we can control the second term in the right-hand side of (6.13) by
\[
C 2^{k_2} \|(|i\partial_t + \Box_x - i)| F^{-1}(f_{k_2, j_1+10}) \cdot F^{-1}(\tilde{g}_{k_1, j_1})\|_{L^1_{\xi_1}} \]
\[ + C 2^{k_2} 2^{-j_1/2} \|F^{-1}(f_{k_2, j_1+10}) \cdot (i\partial_t + \Box_x) F^{-1}(\tilde{g}_{k_1, j_1})\|_{L^2} \]  
(6.18)
\[ + C 2^{k_2} 2^{-j_1/2} \|\nabla_x F^{-1}(f_{k_2, j_1+10}) \cdot \nabla_x F^{-1}(\tilde{g}_{k_1, j_1})\|_{L^2}. \]
We estimate the first term in the right-hand side of (6.18) by
\[ C2^{\sigma k}2^{-k/2}\| (i\partial_t + \Box_x - i)\mathcal{F}^{-1}(f_{k_2 \geq j_1 + 10})\|_{L^2} \cdot \| \mathcal{F}^{-1}(\tilde{g}_{k_1,j_1})\|_{L^\infty}, \]
which is bounded by the right-hand side of (6.13) in view of Lemma 3.8. We estimate the last two terms in the right-hand side of (6.18) by
\[ C2^{\sigma k}2^{-j_1/2} \cdot 2^{j_1} \| f_{k_2 \geq j_1 + 10} \ast \tilde{g}_{k_1,j_1}\|_{L^2} \leq C2^{\sigma k}2^{j_1/2} \| f_{k_2 \geq j_1 + 10}\|_{L^2} \| \mathcal{F}^{-1}(\tilde{g}_{k_1,j_1})\|_{L^\infty}, \]
which is bounded by the right-hand side of (6.13) in view of Lemma 3.8. □

From Lemma 6.3 and (5.6), we have

**Corollary 6.5.** If \( \sigma > 1 \) and \( u_1, \ldots, u_n \in F^\sigma \), then the product \( \tilde{u}_1 \cdot \ldots \cdot \tilde{u}_n \in F^\sigma \) and
\[ \| \tilde{u}_1 \cdot \ldots \cdot \tilde{u}_n \|_{F^\sigma} \leq (C_\sigma)^n \cdot \| u_1 \|_{F^\sigma} \cdot \ldots \cdot \| u_n \|_{F^\sigma}, \]
where \( \tilde{u}_m \in \{ u_m, \overline{u}_m \} \) for \( m = 1, \ldots, n \).

Now we analyze the term
\[ N_0(u) = \frac{2\tilde{u}}{1 + u\tilde{u}} \in C(\mathbb{R}; H^\infty) \]
with \( u \in C(\mathbb{R}; H^\infty) \).

**Lemma 6.6.** For \( \sigma > 1 \) then there is \( c(\sigma) > 0 \) with the property that
\[ \| J' (N_0(u) - N_0(v)) \|_{F^\sigma} \leq C(\sigma, \sigma', \| u \|_{F^{\sigma+\sigma'}}, \| v \|_{F^{\sigma+\sigma'}}) \| J'(u - v) \|_{F^\sigma} \]
for any \( \sigma' \in \mathbb{Z}_+ \), and \( u, v \in B_{F^\sigma}(0, c(\sigma)) \cap F^\sigma \).

**Proof of Lemma 6.6.** We write first
\[ N_0(u) - N_0(v) = \frac{u - v}{1 + u\tilde{u}(1 + v\tilde{v})} - \frac{(u - v)\overline{u\tilde{v}}}{1 + u\tilde{u}(1 + v\tilde{v})}. \]
First we expand the above to power series, then by Corollary 6.5 we can get (6.19) when \( c(\sigma) \) sufficiently small. □

7. **Proof of Theorem 1.2**

In this section we prove Theorem 1.2. Our main ingredients are Lemma 4.1, Lemma 4.2, Lemma 5.1, Lemma 6.6, and the bound
\[ \sup_{t \in \mathbb{R}} \| u \|_{H^\sigma} \leq C_\sigma \| u \|_{F^\sigma} \text{ for any } \sigma \geq 0 \text{ and } u \in F^\sigma, \]
which follows from Lemma 3.8. Assume that \( \sigma_0 > 3/2 \) and \( \phi \in H^\infty \cap B_{H^{\sigma_0}}(0, \epsilon(\sigma_0)) \), where \( \epsilon(\sigma_0) \ll 1 \) is to be fixed. We define recursively
\[ \begin{cases} u_0 = \psi(t) \cdot W(t)\phi; \\ u_{n+1} = \psi(t) \cdot W(t)\phi + \psi(t) \cdot \int_0^t W(t - s)N(u_n(s)) \, ds \text{ for } n \in \mathbb{Z}_+. \end{cases} \]
Where $\mathcal{N}$ defined in (5.4), that is

$$\mathcal{N}(u_n) = \psi(t) \cdot \left[ N_0(u_n)(\partial_{x_1} u_n)^2 - \left( \partial_{x_2} u_n \right)^2 \right] - 4bR_1(N_0(u_n)\partial_{x_1} u_n)\partial_{x_2} u_n + 4bR_2(N_0(u_n)\partial_{x_2} u_n)\partial_{x_1} u_n - 4bR_3(N_0(u_n)\partial_{x_1} u_n)\partial_{x_2} u_n + 4bR_4(N_0(u_n)\partial_{x_2} u_n)\partial_{x_1} u_n.$$ 

The rest of the proof is organized as follows. We first analyze (7.3) with $\mathcal{N}(u_n)$ replaced by $\psi(t) N_0(u_n)(\partial_{x_1} u_n)^2$, then notice that all the results hold for the $\mathcal{N}(u_n)$ case. Finally, we use these results to conclude the proof of Theorem 1.2.

Now we define recursively

$$\begin{cases}
  u_0 = \psi(t) \cdot W(t)\phi; \\
  u_{n+1} = \psi(t) \cdot W(t)\phi + \psi(t) \cdot \int_0^t W(t-s)(\tilde{N}(u_n(s))) ds \quad \text{for } n \in \mathbb{Z}_+,
\end{cases}$$

(7.3)

where $\tilde{N}(u_n(s)) = \psi(s) N_0(u_n(s))(\partial_{x_1} u_n(s))^2$, clearly, $u_n \in C(\mathbb{R}; H^\infty)$.

We show first that

$$\|u_n\|_{F_{\sigma_0}} \leq C_{\sigma_0} \|\phi\|_{H^{\sigma_0}} \quad \text{for any } n = 0, 1, \ldots, \text{ if } \epsilon(\sigma_0) \text{ is sufficiently small.}$$

(7.4)

The bound (7.4) holds for $n = 0$, due to Lemma 4.1. Then, using Lemma 6.6 with $\sigma' = 0$, $v \equiv 0$, Lemma 5.1, and the estimates (5.6), we have

$$\|\tilde{N}(u_n)\|_{N^{\sigma_0}} \leq C_{\sigma_0} \|u_n\|^3_{F_{\sigma_0}}.$$

Using Lemma 4.2, the definition (7.3), and Lemma 4.1 it follows that

$$\|u_{n+1}\|_{F_{\sigma_0}} \leq C_{\sigma_0} \|\phi\|_{H^{\sigma_0}} + C_{\sigma_0} \|u_n\|^3_{F_{\sigma_0}},$$

which leads to (7.3) by induction over $n$.

We show now that

$$\|u_n - u_{n-1}\|_{F_{\sigma_0}} \leq 2^{-n} \cdot C_{\sigma_0} \|\phi\|_{H^{\sigma_0}} \quad \text{for any } n \in \mathbb{Z}_+ \text{ if } \epsilon(\sigma_0) \text{ is sufficiently small.}$$

(7.5)

This is clear for $n = 0$ (with $u_{-1} \equiv 0$), by Lemma 4.1. Then, using Lemma 6.6 with $\sigma' = 0$, Lemma 5.1 and the estimates (5.6) and (7.4), we have

$$\|\tilde{N}(u_{n-1}) - \tilde{N}(u_{n-2})\|_{N^{\sigma_0}} \leq C_{\sigma_0} \cdot \epsilon(\sigma_0)^2 \cdot \|u_{n-1} - u_{n-2}\|_{F_{\sigma_0}}.$$ 

Using Lemma 4.2 and the definition (7.3) it follows that

$$\|u_n - u_{n-1}\|_{F_{\sigma_0}} \leq C_{\sigma_0} \cdot \epsilon(\sigma_0)^2 \cdot \|u_{n-1} - u_{n-2}\|_{F_{\sigma_0}},$$

which leads to (7.3) by induction over $n$.

We show now that

$$\|J^{\sigma'}(u_n)\|_{F_{\sigma_0}} \leq C(\sigma_0, \sigma', \|J^{\sigma'}\|_{H^{\sigma_0}}) \quad \text{for any } n, \sigma' \in \mathbb{Z}_+. \quad \text{(7.6)}$$

We argue by induction over $\sigma'$ (the case $\sigma' = 0$ follows from (7.4)). So we may assume that

$$\|J^{\sigma'-1}(u_n)\|_{F_{\sigma_0}} \leq C(\sigma_0, \sigma', \|J^{\sigma'-1}\|_{H^{\sigma_0}}) \quad \text{for any } n \in \mathbb{Z}_+,$$ 

(7.7)
and it suffices to prove that
\[ \|\partial_{x_i}^\alpha(u_n)\|_{F^{\sigma_0}} \leq C(\sigma_0, \sigma', |J^\sigma\phi|_{H^{\sigma_0}}) \text{ for any } n \in \mathbb{Z}_+ \text{ and } i = 1, 2. \tag{7.8} \]

The bound (7.8) for \(n = 0\) follows from Lemma 4.1. We use the decomposition
\[ \mathcal{N}(u_n) = \psi(t) \cdot \mathcal{N}_0(u_n)(\partial_{x_1} u_n)^2, \]
thus
\[ \partial_{x_i}^\alpha(\mathcal{N}(u_n)) = 2\psi(t) \cdot \mathcal{N}_0(u_n) \cdot \partial_{x_1} u_n \cdot \partial_{x_i}^\alpha \partial_{x_1} u_n + E_n, \tag{7.9} \]
where
\[ E_n = \psi(t) \cdot \sum_{\sigma_i' + \sigma_2' = \sigma' \text{ and } \sigma_i' \sigma_2' < \sigma'} \partial_{x_i}^\alpha \mathcal{N}_0(u_n) \cdot \partial_{x_i}^\alpha \partial_{x_1} u_n \cdot \partial_{x_i} \partial_{x_2} u_n. \]

Using Lemma 5.1,
\[ \|E_n\|_{N^{\sigma_0}} \leq C_{\sigma_0} \sum_{\sigma_i' + \sigma_2' = \sigma' \text{ and } \sigma_i' \sigma_2' < \sigma'} \|J^{-1}\partial_{x_i}^\alpha \mathcal{N}_0(u_n)\|_{\tilde{F}^{\sigma_0}} \cdot \|J^{-1}\partial_{x_i}^\alpha \partial_{x_1} u_n\|_{\tilde{F}^{\sigma_0}} \cdot \|J^{-1}\partial_{x_i} \partial_{x_2} u_n\|_{\tilde{F}^{\sigma_0}}. \]

Using now Lemma 6.6 with \(v = 0\), the bound (5.6), and the induction hypothesis (7.7), we have
\[ \|E_n\|_{N^{\sigma_0}} \leq C(\sigma_0, \sigma', |J^\sigma\phi|_{H^{\sigma_0}}). \tag{7.10} \]

In addition, using again Lemma 5.1 Lemma 6.6 with \(v = 0\), (5.6) and (7.4),
\[ \|\psi(t) \cdot \mathcal{N}_0(u_n) \cdot \partial_{x_1} u_n \cdot \partial_{x_i}^\alpha \partial_{x_1} u_n\|_{N^{\sigma_0}} \leq C_{\sigma_0} \cdot \epsilon(\sigma_0)^2 \cdot \|\partial_{x_i}^\alpha u_n\|_{\tilde{F}^{\sigma_0}}. \tag{7.11} \]

We use now the definition (7.3), together with Lemma 4.1 Lemma 4.2 and the bounds (7.10) and (7.11) to conclude that
\[ \|\partial_{x_i}^\alpha u_{n+1}\|_{\tilde{F}^{\sigma_0}} \leq C(\sigma_0, \sigma', |J^\sigma\phi|_{H^{\sigma_0}}) + C_{\sigma_0} \cdot \epsilon(\sigma_0)^2 \cdot \|\partial_{x_i}^\alpha u_n\|_{\tilde{F}^{\sigma_0}}. \]
The bound (7.8) follows by induction over \(n\) provided that \(\epsilon(\sigma_0)\) is sufficiently small.

Finally, we show that
\[ \|J^\sigma(u_n - u_{n-1})\|_{F^{\sigma_0}} \leq 2^{-n} \cdot C(\sigma_0, \sigma', |J^\sigma\phi|_{H^{\sigma_0}}) \text{ for any } n, \sigma' \in \mathbb{Z}_+. \tag{7.12} \]

As before, we argue by induction over \(\sigma'\) (the case \(\sigma' = 0\) follows from (7.5)). So we may assume that
\[ \|J^{\sigma' - 1}(u_n - u_{n-1})\|_{F^{\sigma_0}} \leq 2^{-n} \cdot C(\sigma_0, \sigma', |J^{\sigma' - 1}\phi|_{H^{\sigma_0}}) \text{ for any } n \in \mathbb{Z}_+. \tag{7.13} \]

and it suffices to prove that
\[ \|\partial_{x_i}^\alpha(u_n - u_{n-1})\|_{F^{\sigma_0}} \leq 2^{-n} \cdot C(\sigma_0, \sigma', |J^{\sigma'}\phi|_{H^{\sigma_0}}) \text{ for any } n \in \mathbb{Z}_+ \text{ and } i = 1, 2. \tag{7.14} \]
The bound (7.14) for \(n = 0\) follows from Lemma 4.1. For \(n \geq 1\) we use the decomposition
\[ \mathcal{N}(u_{n-1}) - \mathcal{N}(u_{n-2}) = \psi(t) \cdot (\mathcal{N}_0(u_{n-1}) - \mathcal{N}_0(u_{n-2})) \cdot (\partial_{x_1} u_{n-1})^2 + \psi(t) \cdot \mathcal{N}_0(u_{n-2}) \cdot \partial_{x_1}(u_{n-1} - u_{n-2}) \cdot \partial_{x_1}(u_{n-1} + u_{n-2}). \tag{7.15} \]
The same argument as before, which consists of expanding the $\sigma'$ derivative, and combining Lemma 5.1, Lemma 6.6, (7.6), and (7.13), shows that
\[
\left\| \partial_{x_1}^\prime \left( \psi(t) \cdot (\mathcal{N}_0(u_{n-1}) - \mathcal{N}_0(u_{n-2})) \cdot (\partial_{x_1} u_{n-1})^2 \right) \right\|_{H^{s_0}} 
\leq 2^{-n} \cdot C(\sigma_0, \sigma', ||J^\prime \phi||_{H^{s_0}}),
\] (7.16)

To estimate the $\sigma'$ derivative of the term in the second line of (7.13), we expand again the $\sigma'$ derivatives. Using again the combination of Lemma 5.1, Lemma 6.6, (7.6), and (7.13), the $N^{s_0}$ norm of most of the terms that appear is again dominated by $2^{-n} \cdot C(\sigma_0, \sigma', ||J^\prime \phi||_{H^{s_0}})$. The only remaining terms are
\[
\psi(t) \cdot \mathcal{N}_0(u_{n-2}) \cdot \partial_{x_1}^\prime \partial_{x_1}(u_{n-1} - u_{n-2}) \cdot \partial_{x_1}(u_{n-1} + u_{n-2}),
\]
and we can estimate
\[
\left\| \psi(t) \cdot \mathcal{N}_0(u_{n-2}) \cdot \partial_{x_1}^\prime \partial_{x_1}(u_{n-1} - u_{n-2}) \cdot \partial_{x_1}(u_{n-1} + u_{n-2}) \right\|_{N^{s_0}} 
\leq C_{s_0} \cdot \epsilon(\sigma_0)^2 \cdot ||\partial_{x_1}^\prime(u_{n-1} - u_{n-2})||_{F^{s_0}}.
\]

As before, it follows that
\[
||\partial_{x_1}^\prime(u_{n-1})||_{F^{s_0}} \leq 2^{-n} \cdot C(\sigma_0, \sigma', ||J^\prime \phi||_{H^{s_0}}) 
+ C_{s_0} \cdot \epsilon(\sigma_0)^2 \cdot ||\partial_{x_1}^\prime(u_{n-1} - u_{n-2})||_{F^{s_0}}.
\]

The bound (7.14) follows by induction provided that $\epsilon(\sigma_0)$ is sufficiently small.

In view of Lemma 5.1, we notice that $\mathcal{N}(u)$ and $\mathcal{N}(u)$ share the same nonlinear estimate, so the argument above for system (7.3) can be used to system (7.2). Thus, (7.4), (7.5), (7.6), and (7.12) also hold for $u_n$ defined by system (7.2).

We can now use (7.12) and (7.11) to construct
\[
u = \lim_{n \to \infty} u_n \in C(\mathbb{R} : H^\infty).
\]

In view of (7.3),
\[
u = \psi(t) \cdot W(t) \phi + \psi(t) \cdot \int_0^t W(t - s)(\mathcal{N}(u(s))) ds \text{ on } \mathbb{R}^d \times \mathbb{R},
\]
so $\tilde{S}^{\infty}(\phi)$, the restriction of $\nu$ to $\mathbb{R}^d \times [-1, 1]$, is a solution of the initial-value problem (1.4).

For Theorem 1.2 (b) and (c), it suffices to show that if $\sigma' \in \mathbb{Z}_+$ and $\phi, \phi' \in B_{H^{s_0}}(0, \epsilon(\sigma_0)) \cap H^\infty$ then
\[
\sup_{t \in [-1, 1]} ||\tilde{S}^{\infty}(\phi) - \tilde{S}^{\infty}(\phi')||_{H^{s_0} + \sigma'} \leq C(\sigma_0, \sigma', ||\phi||_{H^{s_0} + \sigma'}) \cdot ||\phi - \phi'||_{H^{s_0} + \sigma'}. \] (7.17)

Part (b) corresponds to the case $\sigma' = 0$. To prove (7.17), we define the sequences $u_n$ and $u'_n$, $n \in \mathbb{Z}_+$, as in (7.3). Using Lemma 4.1,
\[
||u_0 - u'_0||_{F^{s_0}} \leq C_{s_0} ||\phi - \phi'||_{H^{s_0}}.
\]
Then we decompose \( N(u_n) - N(u_n') \) in the same way as in (7.15). As before, we combine Lemma 4.1, Lemma 4.2, Lemma 5.1, Lemma 6.6, and the uniform bound (7.4) to conclude that

\[
||u_{n+1} - u_{n+1}'||_{F^{\sigma_0}} \leq C_{\sigma_0} ||\phi - \phi'||_{H^{\sigma_0}} + C_{\sigma_0} \cdot \epsilon(\sigma_0)^2 \cdot ||u_n - u_n'||_{F^{\sigma_0}}.
\]

By induction over \( n \) it follows that

\[
||u_n - u_n'||_{F^{\sigma_0}} \leq C_{\sigma_0} ||\phi - \phi'||_{H^{\sigma_0}} \text{ for any } n \in \mathbb{Z}_+.
\]

In view of (7.1) this proves (7.17) for \( \sigma' = 0 \).

Assume now that \( \sigma' \geq 1 \). In view of (7.1), for (7.17) it suffices to prove that

\[
||J^{\sigma'}(u_n - u_n')||_{F^{\sigma_0}} \leq C(\sigma_0, \sigma', ||J^{\sigma'}(\phi)||_{H^{\sigma_0}}) \cdot ||J^{\sigma'}(\phi - \phi')||_{H^{\sigma_0}},
\]

(7.18)

for any \( n \in \mathbb{Z}_+ \). We argue, as before, by induction over \( \sigma' \): we decompose \( N(u_n) - N(u_n') \) as in (7.15), and combine Lemma 4.1, Lemma 4.2, Lemma 5.1, Lemma 6.6, and the uniform bound (7.6). The proof of (7.18) is similar to the proof of (7.12). This completes the proof of Theorem 1.2.

Acknowledgment. This work was finished under the patient advising of Prof. Carlos E. Kenig while the author was visiting the Department of Mathematics at the University of Chicago under the auspices of China Scholarship Council. The author is deeply indebted to Prof. Kenig for the many encouragements and precious advices. The author is also very grateful to Prof. Baoxiang Wang for encouragements and supports.

References

[1] I. Bejenaru, Global results for Schrödinger maps in dimensions \( n \geq 3 \), Comm. Partial Differential Equations 33 (2008), no. 3, 451–477.
[2] I. Bejenaru, A. D. Ionescu, and C. E. Kenig, Global existence and uniqueness of Schrödinger maps in dimensions \( d \geq 4 \), Adv. Math. 215 (2007), 263-291.
[3] I. Bejenaru, A. D. Ionescu, and C. E. Kenig, On the stability of certain spin models in 2+1 dimensions, arXiv:0906.1312.
[4] I. Bejenaru, A. D. Ionescu, C. E. Kenig and D.Tataru, Global Schrödinger maps in dimensions \( d \geq 2 \): small data in the critical Sobolev spaces, arXiv:0807.0265.
[5] Y. Ishimori, Multi-vortex solutions of a two dimensional nonlinear wave equation, Prog. Theor. Phys. 72 (1984), 33-37.
[6] A. D. Ionescu and C. E. Kenig, Global well-posedness of the Benjamin-Ono equation in low-regularity spaces, J. Amer. Math. Soc. 20 (2007), no. 3,753–798.
[7] A. D. Ionescu and C. E. Kenig, Low-regularity Schrödinger maps, Diff. Int. Eq. 19 (2006), 1271-1300.
[8] A. D. Ionescu and C. E. Kenig, Low-regularity Schrödinger maps, II: global well-posedness in dimensions \( d \geq 3 \), Comm. Math. Phys. 271 (2007), 523-559.
[9] C. E. Kenig and A. Nahmod, The Cauchy problem for the hyperbolic-elliptic Ishimori system and Schrödinger maps, Nonlinearity 18 (2005), 1987–2009.
[10] E. Onodera, Bilinear Estimates associated to the Schrödinger equation with a nonelliptic principal part, J. Ana. and App. 27(2008), 1-10.
[11] A. Soyeur, The Cauchy problem for the Ishimori equations, J. Funct. Anal. 105 (1992), 233-255.

LMAM, School of Mathematical Sciences, Peking University, Beijing 100871, China

E-mail address: wangyuzhao2008@gmail.com