Scalable Hypergraph Embedding System

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ABSTRACT

Many problems such as node classification and link prediction in network data can be solved using graph embeddings, and a number of algorithms are known for constructing such embeddings. However, it is difficult to use graphs to capture non-binary relations such as communities of nodes. These kinds of complex relations are expressed more naturally as hypergraphs. While hypergraphs are a generalization of graphs, state-of-the-art graph embedding techniques are not adequate for solving prediction and classification tasks on large hypergraphs accurately in reasonable time. In this paper, we introduce FaHyEm, a novel hierarchical framework for scalable unsupervised hypergraph embedding. FaHyEm is capable of generating high quality embeddings for real-world hypergraphs with millions of nodes and hyperedges in only a couple of minutes while existing hypergraph systems either fail for such large hypergraphs or may take days to produce the embeddings.

1 INTRODUCTION

A hypergraph is a generalization of a graph in which an edge can connect any number of nodes. Formally, a hypergraph \( H \) is a tuple \((V, E)\) where \( V \) is the set of nodes and \( E \) is a set of nonempty subsets of \( V \) called hyperedges. Nodes and hyperedges may have weights. Graphs are a special case of hypergraphs in which each hyperedge connects exactly two nodes.

Figure 1a shows a hypergraph with 6 nodes and 3 hyperedges. The hyperedges are shown as colored shapes around nodes. The degree of a hyperedge is the number of nodes it connects. In the figure, hyperedge \( h_2 \) connects nodes \( a, b \) and \( c \), and it has a degree of three. Nodes and hyperedges can have weights.

Hypergraphs arise in many application domains. For example, Giurgiu et al. [11] model protein interaction networks as hypergraphs; nodes in the hypergraph represent the proteins and hyperedges represent protein complexes formed by interactions between multiple proteins. The DisGeNET knowledge platform [26] represent a disease genomics dataset as a hypergraph in which nodes represents genes and hyperedges represent diseases associated with certain collections of genes. Algorithms for solving hypergraph problems are then used to predict new protein complexes, and to predict that a cluster of genes is associated with an as-yet undiscovered disease. Finally, Hypergraphs are also used to model data center networks [28], optimize storage sharding in distributed databases [17], and minimize the number of transactions in data centers with distributed data.

1.1 Hypergraph Embedding

Bengio et al. [1] show that one way to solve such prediction problems in graphs is to find an embedding of the graph using representation learning. Formally, embedding of network is finding a mapping \( \phi : V \rightarrow \mathbb{R}^d \) where \( d << n \) and \( n \) is the number of nodes in a network. There is a rich literature on graph embedding methods that use a variety of techniques ranging from random walks [13, 24, 29] to matrix factorization [27] and graph neural networks [14, 33]. Graph embedding techniques can be extended to hypergraphs in two ways but neither of them is satisfactory.

One approach is to represent the hypergraph as a graph by replacing each hyperedge with a clique of edges connecting the vertices of that hyperedge, and then use graph embeddings to solve the prediction problems. This approach has been explored by [9] and [34]. However, the clique expansion is lossy because the hypergraph cannot be recovered from the clique expansion in general. This information loss persists even if the dual of the hypergraph is considered [19].

Zien et al. [40] show that another approach is to work with the star expansion of the hypergraph. Given a hypergraph \( H=(V, E) \) where \( V \) is the set of nodes, \( E \) is the set of hyperedges, we create a bipartite graph \( H^* = (V^*, E^*) \) by (i) introducing a node \( u_e \) for each hyperedge \( e \in E \) in final graph \( V^* = V \cup E \), and (ii) introducing an edge between a node \( u_e \) in \( V \) and a hyperedge node \( u_e \in E \) if \( u \in e \) in the hypergraph. i.e., \( E^* = \{ (u_e, v_e) : u \in e, e \in E \} \). Figure 1(b) shows an example. Unlike the clique expansion of a hypergraph,
the star expansion is not lossy provided nodes representing hyperedges are distinguished from nodes representing hypergraph nodes. However, graph representation learning approaches do not distinguish between the two types of nodes in the bipartite graph, which lowers accuracy for prediction problems as we show in this paper.

These problems motivated us to develop FaHyEm, a parallel multi-level framework for constructing hypergraph embeddings, which allows us to use existing graph embedding algorithms for hypergraphs. FaHyEm performs hypergraph embedding in a much faster and more scalable manner than existing methods. The input hypergraph is coarsened successively until a small enough hypergraph is obtained, an embedding is found for the small hypergraph, and this embedding is interpolated back into the series of coarse graphs until an embedding is found for the input hypergraph. Crucially, this hierarchical framework differentiates between nodes and hyperedges of a hypergraph. FaHyEm considers the bipartite graph to be a heterogeneous graph and treats the two kinds of nodes in the star expansion differently, unlike previous methods, which helps to maintain the structure of the hypergraph through successive levels of coarsening. We introduce a novel coarsening strategy that not only reduces the size of a hypergraph but also utilizes the feature vectors of the nodes of a hypergraph. This coarsening strategy can be combined with an iterative refinement algorithm described in this paper, which can be used on its own to improve the quality of given hypergraph embedding.

We evaluate FaHyEm on a number of data sets for node classification and hyperedge prediction. Our experiments show that our hierarchical framework can compute the embedding of hypergraphs with millions of nodes and hyperedges in just a few minutes without loss of accuracy in downstream tasks, while all existing hypergraph embedding techniques either fail to run on such large inputs and or take days to complete. The results show that FaHyEm reduces the computation time (and therefore the energy) required to generate embeddings of hypergraphs and graphs by orders of magnitude without compromising on accuracy.

Our main technical contributions are summarized below.

- We describe FaHyEm, a hierarchical hypergraph embedding framework that is designed to handle variable-sized hyperedges.
- **Unsupervised hypergraph embedding system**: To the best of our knowledge, FaHyEm is the first unsupervised hypergraph embedding system. The embeddings obtained from this framework can be used in downstream tasks such as hyperedge prediction and node classification.
- **Scalability**: FaHyEm is the first hypergraph embedding approach that can generate embeddings of hypergraphs with millions of nodes and hyperedges. Our approach can significantly reduce run time while producing comparable and in some cases, better accuracy than state-of-the-art techniques.

The rest of this paper is organized as follows. Section 2 summarizes the related work. Section 3 introduces the FaHyEm model in detail. Section 4 presents experimental results. We conclude in Section 5.

2 RELATED WORK

There is a large body of work on graph and hypergraph embedding techniques so we discuss only the most closely related work.

2.1 Network Embedding

We divide these techniques into hypergraph embedding techniques and multi-level (hierarchical) embedding techniques.

**Hypergraph embedding.** There are relatively few efforts on hypergraph embedding that treat hyperedges as first-class entities. As mentioned above, one popular approach to hypergraph embedding [9, 31, 36, 39] is to convert the hypergraph to a graph and
then use a graph embedding technique. For example, each hyperedge can be replaced with a clique connecting the nodes of that hyperedge to produce a graph representation. Other approaches such as [34] use a graph convolution on a modified clique expansion technique where they choose what edges to keep in the graph representation. While this method keeps more structure than methods based on the clique expansion of a hypergraph, existing methods fail to scale to large networks. There have been a few recent studies such as [32, 37] on obtaining the embedding of a hypergraph in a supervised manner for tasks such as hyperedge prediction. HyperSAGNN [37], an improvement over DHNE [32], uses a self-attention based graph neural network that learns the embedding of nodes of a hypergraph in a supervised manner. However, these approaches cannot be used for hyperedge classification tasks.

### 2.2 Multi-level Embedding

Multi-level (hierarchical) approaches attempt to improve the runtime and quality of existing or a new embedding techniques. Multi-level graph embedding consists of three phases: coarsening, initial embedding, and refinement.

**Coarsening:** A coarsened graph \( G' \) is created by merging pairs of nodes in input graph \( G \). This process is applied recursively to the coarser graph, creating a sequence of graphs. In which the final graph is the coarsest graph that meets some termination criterion (e.g., its size is below some threshold).

**Initial embedding:** Any unsupervised embedding methods for networks can be used to generate an initial embedding.

**Refinement:** For graphs \( G' \) and \( G \), the embedding of \( G' \) is projected onto \( G \) and then refined, starting from the coarsest graph and finishing with the original graph.

There is a large body of research on multi-level graph embedding. For example, HARP [4] generates a hierarchy of coarsened graphs and perform embedding from the coarsest level to the original one. However, HARP only focuses on the quality of the embeddings and does not attempt to improve the run-time and scalability. MILE [21] uses heavy edge matching [3] to coarsen the graph and leverages GCN as a refinement method to improve embedding quality. However, training a GCN model is very time consuming for large graphs and leads to poor performance when multiple GCN layers are stacked together [20]. Such multi-level embedding methods only utilise structural information of a graph. However, in many datasets such as citations, nodes of a graph have node attribute information. For a high quality embedding, it is important to exploit node attributes as well as structural information of a graph. Graphzoom[6] first performs graph fusion to generate a new graph that encodes the topology of the original graph and the node attribute information and then uses a coarsening algorithm that merges nodes with high spectral similarity. Finally, they apply a local refinement algorithm. While Graphzoom outperforms previous multi-level embedding systems, it still takes hours, in some cases days, to generate embeddings for a graph with millions of nodes and edges.

Multi-level approaches for hypergraphs have considered only hypergraph partitioning [7, 18, 22]. In principle, similar ideas from multi-level graph embedding approaches can be adopted for hypergraphs. For example, for the coarsening algorithm, we can adopt similar ideas from coarsening in graphs, i.e. merging pairs of nodes that have a hyperedge in common (heavy edge matching). While this approach is able to produce coarser hypergraphs, it reduces the number of hyperedges of the coarser graphs only for those pairs of matched nodes that are connected by a hyperedge of size two. As a result, the coarsest hypergraph is still large in terms of the size of the hyperedges which will increase the running time of the overall algorithm. In general, hypergraphs are a more complicated topic and the corresponding algorithms are typically more compute and memory intensive. The coarsening and refinement algorithms proposed in multi-level graph embedding systems are not adequate for solving inference problems on hypergraphs, as we discuss in this paper.

These limitations led us to design FaHyEm, which scales to hypergraphs with millions of nodes and hyperedges while producing high-quality embeddings.

### 3 METHODOLOGY

#### 3.1 Multi-level Framework

**Algorithm 1: FaHyEm**

1. **Input:** Hypergraph \( H = (V, E) \); node feature matrix \( X \in \mathbb{R}^{V \times k} \); coarsening depth \( c \); refinement depth \( r \); base embedding function \( f \)
2. **Output:** Vector representation \( h_u, \forall u \in (V \cup E) \)
3. \( H_r^c = (V^*, E^*) \leftarrow \text{StarExpansion}(H) \)
4. for \( i = 1 \) to \( c - 1 \) do
   5. \( H^*_{i+1} = \text{Coarsening}(H^*_i) \)
   6. end for
7. \( h_c \leftarrow f(H^*_c) \) \( \{h \text{ is the embedding}\} \)
8. for \( i = c - 1 \) to \( 1 \) do
   9. \( h_i(u) \leftarrow h_{i+1}(\tilde{u}), \forall u \in \tilde{u} \{\tilde{u} \text{ is coarsened node}\} \)
   10. for \( j = 1 \) to \( r \) do
      11. \( h_i \leftarrow \text{Refinement}(H_i^c, h_i) \)
   12. end for
13. end for

Given a hypergraph \( H=(V, E) \), the algorithms described in this paper use the star expansion of the hypergraph and assign a vector representation \( h_u \) for each \( u \in (V \cup E) \). Intuitively, these embeddings attempt to preserve structural similarity in the hypergraph: if two hyperedges have many nodes in common or if two nodes are in many of the same hyperedges, the algorithm attempts to assign the two hyperedges/nodes to points that are close in the vector space. This closeness could be metrics of vector similarity or distance. Embedding should also exploit the transitivity property of similarity: if \( a \) and \( b \) are similar, and \( b \) and \( c \) are similar, we want the embedding of \( a \) and \( c \) to be close to each other as well. Finally, if nodes have features, the embeddings should also exploit functional similarity between nodes.

Figure 2 illustrates the high-level idea of multi-level hypergraph embedding. Pseudocode for FaHyEm is given in Algorithm 1. This framework consists of three phases: (i) **Coarsening**, which iteratively merges nodes of the hypergraph to shrink the size of the hypergraph until the hypergraph is small enough that any network embedding algorithm can quickly obtain the embedding of
the smallest hypergraph; (ii) **Initial embedding**, in which a network embedding algorithm is used on the coarsest hypergraph to generate the embedding, and (iii) **Refinement**, in which the embedding vectors of the coarser hypergraph are projected onto a finer hypergraph and a refinement algorithm is used to refine these embedding vectors. In the rest of this section, we describe these phases in more detail. FaHyEm is a parallel implementation of the multilevel approach.

### 3.2 Coarsening

Intuitively, coarsening finds nodes that are similar to each other and merges them to obtain a coarser hypergraph. To obtain a high quality embedding, we need to explore both structural similarity and functional similarity. The connectivity of nodes and hyperedges of a hypergraph determines structural similarity while node features determine functional similarity.

The first step in coarsening a hypergraph is to find nodes that are similar to each other and merge them. This is accomplished by “assigning” each node to one of its hyperedges, and then merging all nodes \{n₁, n₂, ..., nₖ\} assigned to a given hyperedge to produce a node n’ of the coarser hypergraph. We refer to n’ as the representative of node nᵢ in the coarse hypergraph, and denote it as rep(nᵢ). If all nodes of a hyperedge h_j are merged, we remove that hyperedge from the hypergraph. Otherwise, we add the hyperedge to the next level and refer to it as rep(h_j). If a node nᵢ is contained in hyperedge h_j in the finer hypergraph and h_j is present in the coarse hypergraph, then rep(nᵢ) is made a member of rep(h_j).

If nodes of a hypergraph have features, this information can be used to find similar nodes and merge them together. In a hypergraph with node features, the feature vector of a hyperedge is the mean aggregation of the features of its nodes. In this scenario, metrics of vector similarity or distance between a feature vector of a hyperedge and a node can be used for assigning nodes to hyperedges. However, if the hypergraph has no features, FaHyEm can use other measures such as weights or degrees of a hyperedge to assign nodes to hyperedges. For example, if a node belongs to a hyperedge h_j with weight w_j and degree d_j, the “importance” of that node for that hyperedge can be estimated from the value of w_j/d_j, and the node can be assigned to the hyperedge with the largest importance score. In practice, various methods can be used to find this importance score.

Algorithm 2 lists the pseudo-code of coarsening. K is a hyperparameter that determines the number of levels of coarsening. At each level of coarsening, **ComputeEdgeFeature** computes a feature vector for a hyperedge by finding the mean aggregation of the feature vectors of its nodes. **AssignHyperedge** assigns each node υ in the current hypergraph to a hyperedge c(υ), defined as c(υ) = \text{argmax}_{e \in N(υ)} \frac{f(e) \cdot f(υ)}{|f(e)| \cdot |f(υ)|}\ where N(υ) is the set of hyperedges that node υ belongs to, f(e) is the feature vector of hyperedge e and f(υ) is the feature vector of node υ. The datasets used in experiments in this paper for node classification are citation networks and the node feature vectors come from bag-of-words encoding. In these datasets, cosine similarity aligns well with class labels. However, other metrics of vector similarity or distance such as L2 norm, correlation distance, etc. can also be used.

| Algorithm 2: Coarsening |
|-------------------------|
| 1. **Input**: fineGraph, node feature matrix X ∈ ℝ^{V×k}, neighborhood function N(υ), depth K |
| 2. **Output**: coarseGraph, node feature matrix X’ ∈ ℝ^{V’×k} |
| 3. for i = 1 to K do |
| 4. for all hyperedges e ∈ fineGraph in parallel do |
| 5. ComputeEdgeFeature(e) |
| 6. end for |
| 7. for all nodes υ ∈ fineGraph in parallel do |
| 8. AssignHyperedge(υ) |
| 9. end for |
| 10. for all hyperedges e ∈ fineGraph in parallel do |
| 11. M ← FindAssignedNodes(e) |
| 12. m ← Merge(M) |
| 13. coarseGraph.addNode(m) # m is representative of each node in M |
| 14. if |M| ≠ |e| then |
| 15. coarseGraph.addHedge(e) # representative of e |
| 16. end if |
| 17. end for |
| 18. for all hyperedges e ∈ fineGraph in parallel do |
| 19. if rep(e) exists then |
| 20. for υ ∈ N(e) do |
| 21. coarseGraph.addEdge(rep(e),rep(υ)) |
| 22. end for |
| 23. end if |
| 24. end for |
| 25. end for |

Nodes that are assigned to the same hyperedge are merged together and the resulting node is added to the coarse hypergraph. In case of a tie, FaHyEm randomly chooses a hyperedge in the neighborhood of the node. The feature vector of a coarsened node is the average of all initial node feature present in the coarsened node. Figure 2 top left shows the assignment of nodes to hyperedges. The figure in the top right shows the hypergraph after one level of coarsening. Nodes with the same color are merged together and form a single node in the coarser hypergraph. Hyperedges purple and green are removed from the hypergraph since all nodes in these hypergraphs are merged together. Hyperedge blue, however, is not removed since it has nodes that are merged in other hyperedges.

One important point about the coarsening algorithm is that it is cognizant of the fact that the bipartite graph is the representation of a hypergraph, and it merges only nodes of the hypergraph to create the coarser graph so that a coarser graph represents the original hypergraph. In particular, it does not merge nodes in the star expansion that represent nodes and hyperedges in the hypergraph, since this will destroy the structure of the hypergraph and the resulting structure will not be a good approximation of the original hypergraph. The experiments in Section 4 show the importance of this aspect of FaHyEm.

**Complexity analysis.** We analyze the parallel time complexity and the parallel work of our coarsening algorithm (algorithm 2) using the well-known CREW PRAM model [16]. Let m denote the number of hyperedges and n denotes the number of nodes. Let
$\text{deg}(e_i)$ denote the degree of hyperedge $e_i$ in the hypergraph. Let $\Delta$ denote $\max_{1 \leq i \leq m}\text{deg}(e_i)$. We also assume that the size of the node feature vector, $|f|$ is constant and $|f| << m$, and $|f| << n$.

**Lemma 3.1.** In the CREW PRAM model, Algorithm 2 takes $O(m\Delta/p)$ parallel time and the total work is $O(\sum_{i=1}^{m}\text{deg}(e_i))$, where $p$ is the number of processors.

**Proof.** *Parallel Time Analysis.* The initialization step in steps 4-6 can be done in $O(m\Delta/p)$ parallel time since we need to iterate over all hyperedges and the nodes contained in them. Similarly, steps 7-9 can be done in $O(m\Delta/p)$ parallel time. For steps 10-17, since the number of hyperedges is at most $m$, the for loop runs for at most $m/p$ parallel iterations. Step 11-12 take $O(\Delta)$ sequential time and Steps 13-18 take $O(1)$ sequential time per iteration. Thus the total parallel time is $O(m\Delta/p)$. Finally, steps 18-24 takes $O(m\Delta)$ parallel time since we go over all hyperedges and we need to iterate over all nodes in their neighborhood. Thus the total parallel time of the for loop is $O(m\Delta/p)$. 

**Work Analysis.** The total work done in 4-6 is $O(\sum_{i=1}^{m}\text{deg}(e_i))$. Similarly, the total work done in 7-9 is $O(\sum_{i=1}^{m}\text{deg}(e_i))$. For a given hyperedge $e$, steps 10-17 perform $O(\text{deg}(e))$ work. Thus the total work is upper bounded by $O(\sum_{i=1}^{m}\text{deg}(e_i))$. For a given hyperedge $e$, steps 18-24 perform $O(\text{deg}(e))$ work. Thus the total work is bounded by $O(\sum_{i=1}^{m}\text{deg}(e_i))$. $\square$

### 3.3 Initial Embedding

We coarsen the hypergraph until it is small enough that any unsupervised embedding method can generate the embedding of the coarsest hypergraph in just a few seconds. We use the edgelist of the coarsest bipartite graph as the input to this embedding method. Since the coarsest graph is an approximation of the original hypergraph, we are only interested to approximate the embeddings for this graph. These results will be refined during the refinement phase. The running time for computing this initial embedding is reduced if the size of the coarsest graph is reduced.

### 3.4 Refinement

The goal of this phase is to improve embeddings by performing a variation of Laplacian smoothing [30] that we call the **refinement** algorithm. The basic idea is to update the embedding of each node $u$ using a weighted average of its own embedding and the embeddings of its immediate neighbors $N(u)$. Intuitively, smoothing eliminates high-frequency noise in the embeddings and tries to assign similar relative importance of the embeddings of the neighboring nodes of its immediate neighbors $N(u)$. Thus the total parallel time of the for loop is $O(m\Delta/p)$. 

Abstractly, this iterative scheme uses successive over-relaxation (SOR) with a parameter $\omega$ to solve the linear system $Lz = 0$ where $L$ is the Laplacian matrix of $H^*$, defined as $(D - A)$ where $D$ is the diagonal matrix with diagonal elements $d_{uu}$ equal to the degree of node $u$ for unweighted graphs (for weighted graphs, the sum of weights of outgoing edges), and $A$ is the adjacency matrix of $H^*$. To avoid oversmoothing, we do not compute the exact solution of this linear system but if we start with a good initial embedding $z_0$, a few iterations of the iterative scheme lead to significant gains in the quality of the embedding, as we show experimentally in Section 4.

Algorithm 3 shows the pseudocode for refinement. The input to this algorithm is $H^*$, the bipartite (star) representation of the hypergraph, $z_u$, the initial embedding for each node and hyperedge, and a relaxation parameter $\omega$ between 0 and 1. Embeddings of the hyperedges are updated using the embeddings of the nodes, and the embeddings of nodes are updated using the embeddings of hyperedges. Note that if $u$ represents a hyperedge, $N(u)$ is the set of nodes in that hyperedge, and if $u$ represents a node in the hypergraph, $N(u)$ represents the set of hyperedges that $u$ is contained in. Each iteration of the refinement algorithm has a linear time complexity in the size of the bipartite representation of the hypergraph.

Intuitively, the updates to the embeddings of nodes and hyperedges made by the relaxation algorithm in each iteration exploit structural similarity. For example, if two hyperedges $e_1$ and $e_2$ have many nodes in common, these nodes will bring the embeddings of $e_1$ and $e_2$ closer. Performing these updates iteratively exploits transitivity of similarity. For instance, hyperedges $a$ and $c$ may not have

\begin{align*}
\text{Algorithm 3: Refinement} & \\
1: & \text{Input: Bipartite graph representation } H^* = (V^*, E^*, W) \\
2: & \text{of hypergraph } H = (V, E, W), \text{ vector representation } z_u \text{ for all } u \in (V^*) \text{, neighborhood function } N(u), \text{ parameter } \omega, \text{ parameter } k \text{ for max iteration} \\
3: & \text{Output: Refined vector representation } h_u, \forall u \in (V^*) \\
4: & \text{iter} \leftarrow 0 \\
5: & \text{while } \text{iter} < k \text{ do} \\
6: & \text{for } u \in V^* \text{ in parallel do} \\
7: & \quad z^i_u \leftarrow \sum_{v \in N(u)} (\frac{w_{uv}}{\sum_{v \in N(u)} w_{uv}}) z^{i-1}_v \\
8: & \text{end for} \\
9: & \text{iter} \leftarrow \text{iter} + 1 \\
10: & \text{end while} \\
11: & h_u \leftarrow z^k_u, \forall u \in (V^*) \\
12: & \text{end while} \\
\end{align*}

versus the embedding of the node itself, to obtain the following iterative scheme:

\begin{align*}
z^i_u &= (1 - \omega)z^{i-1}_u + \omega z^i_u 
\end{align*}
nodes in common but if each of them has many nodes in common with hyperedge b, the embeddings of a and c will become closer after a few iterations.

**Complexity analysis.** In this section we analyze the parallel time complexity and the total work of Algorithm 3 in the CREW PRAM model. Let \( |V^*| \) denote the number of nodes in a star expansion of a hypergraph i.e. total number of hyperedges and nodes. Let \( \deg(v_i) \) denote the degree of node \( v_i \) in the hypergraph. Let \( \Delta \) denote \( \max_{1 \leq i \leq |V^*|} \deg(v_i) \). We also assume that the size of the node embedding, \( |z| \) is constant and \( |z| < |V^*| \).

**Lemma 3.2.** In the CREW PRAM model, Algorithm 2 takes \( O(\frac{|V^*|+\Delta}{p}) \) parallel time and the total work is \( O(\sum_{i=1}^{|V^*|} \deg(v_i)) \), where \( p \) is the number of processors.

**Proof.** Parallel Time Analysis. Steps 6-10 can be done in \( O(\frac{|V^*|+\Delta}{p}) \) parallel time since we need to iterate over all nodes of the bipartite graph and the nodes in their neighborhood. Thus the total parallel time of the for loop is \( O(\frac{|V^*|}{p} \cdot \Delta) \).

Work Analysis. The total work done in 6-10 is \( O(\sum_{i=1}^{|V^*|} \deg(v_i)) \). Thus the total work is bounded by \( O(\sum_{i=1}^{|V^*|} \deg(v_i)) \). \( \square \)

4 EXPERIMENTS

FaHyEm provides an unsupervised method for representation learning for hypergraphs. We show these representations perform well for both node classification and hyperedge prediction. Prior works such as HyperGCN and Hyper-SAGNN have been evaluated for one or the other of these tasks but not both. Finally, we evaluate the scalability of FaHyEm on Friendster dataset, which contains 7 million nodes, and 1 million hyperedges. In addition, we conduct ablative study to understand the effectiveness of the major FaHyEm kernels.

**Experimental settings.** We implement FaHyEm in Galois 6.0. Galois is a data structure library and a runtime system that exploits parallelism in irregular graph algorithms expressed in C++ [23, 25]. All experiments are done on a machine running CentOS 7 with 4 sockets of 14-core Intel Xeon Gold 5120 CPUs at 2.2 GHz, and 187 GB of RAM in which there are 65,536 huge pages, each of which has a size of 2 MB. All the methods used in this study are parallel implementations and we use the maximum number of cores available on the machine to run the experiments. The embedding dimension is 128. \( \omega \), the hyperparameter in the refinement algorithm is set to 0.5 for all experiments. Once node embeddings are obtained, we apply logistic regression with cross-entropy loss for our downstream tasks.

4.1 Node Classification

Given a hypergraph and node labels on a small subset of nodes, the task is to predict labels on the remaining nodes. We used the standard hypergraph datasets from prior works, and these are listed in Table 1. We are given 4% of node labels and predict the remaining 96%.

**Methods Compared.** To demonstrate that FaHyEm can work with different graph embedding algorithms and produce high quality results, we explore a number of popular methods for graph embedding. We also compare our results with hypergraph convolutional networks approach for semi-supervised classification.

- **Random-walk methods:** We select DeepWalk [24] and node2vec [13] (high performance implementation[12]) for this group. These methods are properly tuned. We explored window size \{10,20\}, walk length \{20, 40, 80, 120\}, number of walks per vertex \{10,80,40\}, \( p \{1,4,0.5\} \), and \( q \{1,4,0.5\} \). The results reported in the paper are for the best hyper-parameter values, which are \(10,80,10,4,1\) respectively.

- **Multi-level based embedding methods:** We compare against unsupervised approaches MILE [21], and graphzoom [6]. MILE is a multi-level graph embedding framework. We used the default refinement technique, MD-gcn. Graphzoom is also a multi-level graph embedding framework. For the coarsening, we used simple.

- **Graph convolutional network:** We compare with GraphSAGE [14]. We use GraphSAGE in unsupervised manner with the mean aggregator model.

- **Semi-Supervised classification on hypergraphs.** We compare with HyperGCN [34]. Given a hypergraph, HyperGCN approximates the hypergraph by a graph where each hyperedge is approximated by a subgraph. A graph convolutional network (GCN) is then run on the resulting graph. We used 200 epochs and learning rate of 0.01.

For the multi-level approaches, we use node2vec, DeepWalk, and GraphSAGE as the initial embedding methods. Since MILE cannot utilise node features, we do not run GraphSAGE as an initial embedding method for MILE. We report the mean test accuracy and standard deviation over 100 different train-test splits. We optimize hyperparameters of all the baselines. For FaHyEm, we use 80 iterations of refinement.

**Running time.** For FaHyEm and multi-level approaches (MILE, graphzoom), running time includes all three phases: coarsening, initial embedding, and refinement. For the rest of the baselines, we use the CPU time for hypergraph embedding.

For each approach to computing the initial embeddings (node2vec, DeepWalk, GraphSAGE), we have a row showing the accuracy and running times when that approach is used, and rows below those showing the accuracy and running times if that approach is used in conjunction with FaHyEm or other multilevel approaches. For example, the first row in Table 2 shows the running times and accuracy when DeepWalk is used on the star expansion of the hypergraph, while the second row (FaHyEm + DW (I=0)) shows the total running time and accuracy if FaHyEm is used without

### Table 1: Datasets used for node classification. EDGES is the size of the edges in star expansion of the hypergraph. C refers to the number of classes, and FEATURES is the size of the node feature vector.

| Data set | Nodes | Hyperedges | Edges | C | FEATURES |
|----------|-------|------------|-------|---|----------|
| CORA     | 1,434 | 1,579      | 9,572 | 7 | 1,433    |
| Citeseer | 1,458 | 1,079      | 6,906 | 6 | 3,703    |
| Pubmed   | 3,840 | 7,963      | 69,258| 3 | 500      |
| DBLP     | 41,302| 22,363     | 199,122| 6 | 1,425    |

Table 1: Datasets used for node classification. EDGES is the size of the edges in star expansion of the hypergraph. C refers to the number of classes, and FEATURES is the size of the node feature vector.
coarsening but with the output of node2vec being post-processed using our refinement algorithm. The line below that \((l=2)\) shows the results if two levels of coarsening are used in addition.

**Datasets.** We used the following standard hypergraph datasets in our study.

- **Cora (co-citation):** a computer science publication citation network dataset. All documents cited by a document are connected by a hyperedge. Each document is classified into one of seven classes based on topic. Nodes not connected to any hyperedge, as well as hyperedges containing only one node, were removed [10].
- **Citeeseer (co-citation):** scientific publications classified into six classes. All documents cited by a document are connected by a hyperedge. Nodes not connected to any hyperedge, as well as hyperedges containing only one node, were removed [10].
- **Pubmed (co-citation):** scientific publications classified into three classes. All documents cited by a document are connected by a hyperedge. Nodes not connected to any hyperedge, as well as hyperedges containing only one node, were removed [10].
- **DBLP (co-authorship):** scientific publications classified into six classes. All documents co-authored by an author are in one hyperedge [5].

These are the main takeaways from Table 2.

- FaHyEm generates the highest quality embeddings for the node classification task.
- FaHyEm outperforms HyperGCN in terms of quality for all datasets by up to 15%.
- The refinement algorithm improves the quality of embeddings for all the datasets by up to 23%. This can be seen by comparing the statistics for FaHyEm without coarsening \((l=0)\) with those for node2vec, DeepWalk, and GraphSAGE. The initial embedding for FaHyEm is obtained from node2vec, DeepWalk, and GraphSAGE so differences in the statistics arise entirely from the fact that FaHyEm performs refinement.
- FaHyEm outperforms prior multi-level graph embedding approaches (MILE and GraphZoom) for all the datasets by up to 11% for MILE and up to 9% for GraphZoom.
- Coarsening reduces the overall running time of the embedding for larger hypergraphs. Since coarsening reduces the size of the hypergraph, the initial embedding and refinement can be done faster. This can be seen by comparing the statistics for FaHyEm with 2 levels of coarsening \((l=2)\) with those for slower initial embedding approach such as GraphSAGE.

The coarsening algorithm in FaHyEm plays an important role in producing fast, and high-quality embeddings. First, FaHyEm utilizes node features during coarsening. Second, by merging only nodes that have at least one hyperedge in common, FaHyEm ensures that nodes that are structurally similar end up having similar embeddings. This also reduces the size of the hypergraph, which speeds up the initial embedding. Finally, it allows the refinement algorithm to improve the quality of embeddings successively in multiple coarse graphs. A more detailed discussion of the effect of coarsening and refinement on the running time is in 4.3.

### 4.2 Hyperedge Prediction

In hyperedge prediction, we are given a hypergraph with a certain fraction of hyperedges removed, and given a proposed hyperedge (i.e. a set of nodes) our goal is to predict if this is likely to be a hyperedge or not. Formally, given a \(k\)-tuple of nodes \((v_1, v_2, \ldots, v_k)\), our goal is to predict if this tuple is likely to be a hyperedge or not.

We compare our method with the supervised hyperedge prediction method Hyper-SAGNN [37] on four datasets listed in Table 3, and with the graph method node2vec. We did not compare our method with other hypergraph methods such as DHNE [32] since HyperSAGNN has been shown to perform better than these methods. Hyper-SAGNN is a self-attention-based approach for hyperedge prediction. We used their encoder-based approach with learning rate of 0.001 and 300 epochs.

**Datasets.** We used the following datasets in our study.

- **GPS:** a GPS network. Hyperedges are based on (user, location, activity) relations [38].
- **MovieLens:** a social network where hyperedges are based on (user, movie, tag) relations, describing peoples’ tagging activities [15].
- **drug:** a medicine network. The hyperedges are based on (user, drug, reaction) relations [8].
- **wordnet:** a semantic network from WordNet 3.0. The hyperedges are based on (head entity, relation, tail entity), expressing the relationships between words [2].
- **Friendster:** a community network. It is an on-line gaming network. Users can form a group on Friendster social network which other members can then join. These user-defined groups are considered as communities. For the social network, the induced subgraph of the nodes that either belong to at least one community or are connected to other nodes that belong to at least one community are considered too. Communities larger than 500 were removed [35].

We used the same training and test data setups as Hyper-SAGNN (except for Friendster, which Hyper-SAGNN could not run). For this task, they randomly hide 20% of existing hyperedges and use the rest of the hypergraph for training. The negative samples are 5 times the amount of positive samples. We downloaded their code and datasets from their GitHub repository. We used the encoder-based approach to generate the features.

For FaHyEm, we use two levels of coarsening and two levels of refinement. We first obtain the embedding of the hypergraphs with node2vec as the initial embedding technique. We find the embedding of the hypergraph without seeing the hidden hyperedges. To train our classifier, we used the same positive samples as Hyper-SAGNN. For negative samples however, we used only the negative samples of a “single” epoch of Hyper-SAGNN. We note that FaHyEm is able to obtain a high-quality embedding of the hypergraph with a smaller pool of negative samples and still obtain good accuracy compared to Hyper-SAGNN. This reduces the time of training from days to minutes as we show in the paper. We then use the vector of the variances of each dimension of the embedding for hyperedge prediction. The intuition is that if nodes are spread out (high variance in the embedding), then they probably do not form a hyperedge whereas nodes that are close to each other are likely to constitute a hyperedge. Various operators such as average, min,
Table 2: Node classification. Accuracy in % and time in seconds. \( l \) is the number of coarsening levels. 0 means without coarsening.

|          | Cora | Citeseer | Pubmed | DBLP |
|----------|------|----------|--------|------|
|          | Accuracy | Time | Accuracy | Time | Accuracy | Time | Accuracy | Time |
| DeepWalk (dw) | 22.5 ±1. | 96 | 31.8 ±1. | 41 | 39.3 ±1. | 438 | 35.9 ± .4 | 1,605 |
| FaHyEm + dw (l=0) | 64.6 ±1. | 97 | 59.8 ±1. | 43 | 79.5 ±1. | 440 | 69.5 ± .5 | 1,625 |
| FaHyEm + dw (l=2) | 67.7 ±3. | 76 | 60.1 ±1. | 40 | 80.3 ±1. | 378 | 78.8 ± .4 | 757 |
| GraphZoom + dw (l=2) | 63.0 ±3. | 120 | 55.6 ±1. | 98 | 76.8 ±1. | 544 | 70.3 ± .5 | 3,230 |
| MILE + dw (l=2) | 57.5 ±2. | 50 | 52.1 ±1. | 14 | 71.9 ±1. | 253 | 71.0 ±1. | 1,100 |
| NODE2VEC (nv) | 44.6 ±3. | 24 | 51.3 ±1. | 14 | 65.3 ±2. | 66 | 64.3 ± .4 | 470 |
| FaHyEm + nv (l=0) | 67.5 ±3. | 25 | 59.1 ±1. | 16 | 79.7 ±1. | 70 | 72.4 ± .4 | 490 |
| FaHyEm + nv (l=2) | 67.7 ±3. | 20 | 60.6 ±1. | 17 | 80.7 ±1. | 69 | 78.9 ± .5 | 216 |
| GraphZoom + nv (l=2) | 60.1 ±3. | 21 | 54.4 ±1. | 15 | 74.9 ± .1 | 100 | 70.2 ± .5 | 434 |
| MILE + nv (l=2) | 59.5 ±3. | 20 | 52.2 ±1. | 14 | 68.7 ±2. | 60 | 71.8 ±1. | 402 |
| GraphSAGE (gs) | 40.9 ±2. | 511 | 45.6 ±1. | 1,167 | 60.7 ±2. | 277 | 67.7 ± .1 | 925 |
| FaHyEm + gs (l=0) | 67.2 ±3. | 515 | 60.3 ±3. | 1,170 | 80.4 ±1. | 296 | 79.9 ± .1 | 1,055 |
| FaHyEm + gs (l=2) | 68.9 ±3. | 303 | 60.2 ±1. | 843 | 80.8 ±1. | 120 | 79.4 ± .4 | 530 |
| GraphZoom + gs (l=2) | 53.9 ±1. | 360 | 52.7 ±1. | 853 | 72.4 ±1. | 137 | 75.6 ± .4 | 593 |
| HyperGCN | 57.6 ±6. | 15 | 54.1 ±10 | 12 | 64.3 ±10 | 60 | 63.3 ±10 | 480 |

and max can be used instead of variance. We used the same setting for node2vec. Hyper-SAGNN is based on a supervised method in which it learns a function to map from embeddings of nodes to hyperedges for the hyperedge prediction task.

**Experimental results.** Table 4 summarizes the hyperedge prediction results for FaHyEm, node2vec, and Hyper-SAGNN. FaHyEm achieves the best AUC and running time compared to Hyper-SAGNN. Hyper-SAGNN took almost a day for wordnet whereas FaHyEm completed the task in less than a minute. FaHyEm achieves better AUC compared to node2vec on all datasets except drug while it is always the fastest. These results show that the embedding generated by FaHyEm performs well on the second downstream task, hyperedge prediction. For the datasets used for hyperedge prediction, FaHyEm is on average 8 times faster than node2vec, and more than 1000 times faster than Hyper-SAGNN.

**FaHyEm for large hypergraphs.** We study the scalability of FaHyEm on a large hypergraph (Friendster) and compare FaHyEm’s accuracy and running time with that of MILE, and DeepWalk (Hyper-SAGNN, GraphZoom and node2vec failed to generate results for Friendster).

For the dataset Friendster, we randomly hide 20% of existing hyperedges and use the rest of the hypergraph to generate the embeddings for the nodes of the hypergraph and finally, use the variance operator to report the AUC. Since the hypergraph is large, we used five levels of coarsening and ten levels of refinement. The other baselines that was able to run Friendster was MILE with 15 levels of coarsening, and it failed for smaller numbers of coarsening levels. It took MILE 8 hours to generate embeddings for Friendster with an accuracy of 90.4 while it took FaHyEm only fifteen minutes to do the same task with better accuracy (92.3%). DeepWalk was also able to generate the embeddings for Friendster after 17 hours with accuracy 84.9%.

Figure 3 compares MILE, DeepWalk, and FaHyEm in terms of accuracy for different levels of coarsening for FaHyEm (for MILE, we used fifteen levels of coarsening). One reason that MILE is slower than FaHyEm is that it uses GCN as the refinement method. However, this requires training a GCN model, which is very time consuming for large graphs or hypergraphs. The main takeaway from Figure 3 is that using more levels of coarsening reduces the running time of the overall algorithm, as one would expect. However, a large number of coarsening may reduce the accuracy. While this is a fact in most multi-level approaches, Figure 3 shows that the loss of accuracy for FaHyEm is negligible and we are able to get more than 13x speed up by using 5 levels of coarsening instead of 3 levels, while losing less than 3% in accuracy.

Figure 4 shows the breakdown of the time taken by coarsening and refinement phases in FaHyEm on 1 and 14 threads, respectively. For both single thread and 14 threads, the refinement phase takes much longer than the coarsening phase. The end-to-end parallel performance of FaHyEm can be improved by limiting the number of levels for the refinement phase and by finding a faster way of computing the initial embedding.

### 4.3 Ablation Study

In this section we perform an ablation study to understand the effectiveness of the major FaHyEm kernels.

Table 5 shows the effect of different levels of refinement on node classification datasets. Performing just 5 levels of refinement improves the quality of the initial embedding by up to 12%. This table also shows that after about 100 iterations of refinement, the improvement in the quality of the embedding stops.

We also study the behaviour of FaHyEm for the largest hypergraph, Friendster. In Table 6, we see the effect of the coarsening on the size of the hypergraph as well as accuracy and running time. Coarsening improves the running time by up to 10x while the accuracy roughly stays the same.

Table 6 also shows the breakdown of time in different FaHyEm’s kernel. If the coarsest hypergraph is small, most of the time spends in refinement while for large coarsest hypergraphs, the time is mostly spend in initial embedding.
Table 3: Datasets used for hyperedge prediction.

|          | Nodes | Hyperedges | Edges |
|----------|-------|------------|-------|
| GPS      | 221   | 437        | 1,436 |
| MOVIELENS| 17,100| 46,413     | 47,957|
| DRUG     | 7,486 | 171,757    | 171,756|
| WORDNET  | 81,073| 146,433    | 145,966|
| FRIENDSTER| 7,458,099| 1,616,918 | 37,783,346|

Table 4: Area Under Curve (AUC) scores for hyperedge prediction. Time in seconds.

|          | GPS | AUC | Time | MOVIELENS | AUC | Time | DRUG | AUC | Time | WORDNET | AUC | Time | FRIENDSTER | AUC | Time |
|----------|-----|-----|------|-----------|-----|------|------|-----|------|---------|-----|------|------------|-----|------|
| FaHyEm   | 94.5| 1   | FaHyEm | 94.8 | 6.4 | FaHyEm | 96.5 | 295 | FaHyEm | 93.0 | 43.4 | FaHyEm | 93.2 | 897 |
| Hyper-SAGNN | 90.6 | 1,800 | Hyper-SAGNN | 90.8 | 11,160 | Hyper-SAGNN | 95.9 | 39,540 | Hyper-SAGNN | 87.7 | 82,800 | Hyper-SAGNN | - | - |
| Node2Vec | 94.0 | 10 | Node2Vec | 79.8 | 19 | Node2Vec | 97.4 | 895 | Node2Vec | 89.0 | 940 | Node2Vec | - | - |

Table 5: Accuracy of FaHyEm at different numbers of refinement iterations for node classification (REF-N is the number of iterations of refinement, node2vec is used as the initial embedding method.)

| REF-N | CORA | 44.5 | 58.9 | 65.6 | 66.5 | 66.9 | 67.2 | 67.4 | 67.6 | 67.7 | 67.7 |
|-------|------|------|------|------|------|------|------|------|------|------|------|
|       | Citeseer | 51.1 | 59.4 | 59.9 | 60.1 | 60.2 | 60.4 | 60.5 | 60.6 | 60.6 |
|       | Pubmed | 64.5 | 79.5 | 79.9 | 80.1 | 80.2 | 80.4 | 80.5 | 80.5 | 80.5 |
|       | DBLP | 67.1 | 74.5 | 77.3 | 77.7 | 78.0 | 78.1 | 78.3 | 78.4 | 78.5 |

Table 6: Behavior of FaHyEm at different levels of coarsening on Friendster. (COARSE-N is the number of levels of coarsening, time is in seconds.)

| COARSE-N | 3 | 4 | 5 | 6 | 7 | 8 |
|----------|---|---|---|---|---|---|
| Hyperedges | 564,262 | 460,830 | 419,588 | 404,857 | 399,194 | 396,333 |
| Nodes | 436,099 | 154,418 | 85,371 | 67,682 | 61,669 | 59,359 |
| Init Time | 31 | 34 | 36 | 43 | 41 | 39 |
| Refine Time | 11,760 | 600 | 120 | 51 | 29 |
| Accuracy | 95.8 | 95.3 | 93.2 | 92.7 | 92.3 | 92.3 |

5 CONCLUSION

We describe FaHyEm, a multi-level hypergraph embedding framework, which can process hypergraphs with millions of nodes and hyperedges in just a few minutes, producing high-quality embeddings for node classification and hyperedge prediction. Our experimental results show that it significantly outperforms multi-level graph embedding approaches such as MILE and GraphZoom in both accuracy and running time (some of these approaches fail for the large datasets in our studies). We also showed that our refinement algorithm can be used on its own to improve the quality of embeddings for hypergraphs by up to 23%. Even though FaHyEm is an unsupervised method, it outperforms supervised hyperedge prediction methods such as Hyper-SAGNN. In future work, we want to extend FaHyEm to multi-relation hypergraphs.

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