Kramers pairs of Majorana corner states in a topological insulator bilayer

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We consider a system consisting of two tunnel-coupled two-dimensional topological insulators proximitized by a top and bottom superconductor with a phase difference of $\pi$ between them. We show that this system exhibits a time-reversal invariant second-order topological superconducting phase characterized by the presence of a Kramers pair of Majorana corner states at all four corners of a rectangular sample. We furthermore investigate the effect of a weak time-reversal symmetry breaking perturbation and show that an in-plane Zeeman field leads to an even richer phase diagram exhibiting two nonequivalent phases with two Majorana corner states per corner as well as an intermediate phase with only one Majorana corner state per corner. We derive our results analytically from continuum models describing our system. In addition, we also provide independent numerical confirmation of the resulting phases using discretized lattice representations of the models, which allows us to demonstrate the robustness of the topological phases and the Majorana corner states against parameter variations and potential disorder.

I. INTRODUCTION

Motivated by the seminal work on one-dimensional (1D) $p$-wave superconductors \cite{1}, Majorana bound states have been predicted to occur in a variety of condensed matter systems as a signature of a topologically non-trivial superconducting phase. Apart from their fundamental interest, Majorana bound states are considered to be promising building blocks for topologically protected qubits due to their non-Abelian braiding statistics. Many well-known proposals for the experimental realization of Majorana bound states rely on the competition between a strong magnetic field and proximity-induced superconducting pairing \cite{2, 3}. However, such setups suffer from the disadvantage that a strong magnetic field itself has a detrimental effect on superconductivity. To circumvent this issue, the concept of time-reversal invariant topological superconductivity has raised significant interest. In this case, Kramers pairs of Majorana bound states emerge in the absence of a magnetic field \cite{4, 5}.

In the standard proposals, Majorana bound states are realized at the zero-dimensional edges of 1D topological superconductors (TSCs). More recently, the notion of topological insulators (TIs) and TSCs has been extended to capture also their higher-order generalizations \cite{6–10}. While conventional $d$-dimensional TIs and TSCs exhibit gapless edge states at their $(d-1)$-dimensional boundaries, $n$th-order $d$-dimensional TIs or TSCs exhibit gapless edge states at their $(d-n)$-dimensional boundaries. In particular, a two-dimensional (2D) second-order topological superconductor (SOTSC) hosts Majorana bound states at the corners of a rectangular sample. By now, a large variety of platforms hosting such Majorana corner states (MCSs) has been proposed. While most of these proposals use an applied magnetic field to induce the second-order phase \cite{11–40}, the case of time-reversal invariant SOTSCs with Kramers pairs of MCSs has been studied less extensively. The few setups proposed so far rely on unconventional superconductivity as the relevant mechanism driving the transition to the second-order phase \cite{41, 42}. This motivates us to look for an alternative model realizing a time-reversal invariant SOTSC based on conventional ingredients only. The setup we propose consists of two tunnel-coupled 2D TIs proximitized by a top and bottom superconductor with a phase difference of $\pi$ between them, see Fig. 1. In the absence of interlayer tunneling and superconductivity, each TI layer hosts a pair of gapless helical edge states. Once interlayer tunneling and superconductivity are turned on, these edge states are gapped out. However, the resulting phase is not necessarily trivial. Indeed, we show that in a certain region of parameter space, the system is a SOTSC with a Kramers pair of MCSs at all four corners of a rectangular sample. These corner states are protected by particle-hole and time-reversal symmetry and cannot be removed unless one of the protecting symmetries is broken or the edge gap closes and reopens.

The paper is organized as follows. In Sec. II we describe our setup, which consists of two tunnel-coupled 2D TIs in proximity to a top and a bottom superconductor of a phase difference of $\pi$, see Fig. 1. In Sec. III we obtain expressions for the gapless edge states appearing in the absence of superconductivity and interlayer tunneling. In this case, our model simply corresponds to two decoupled 2D TIs, each described by a Bernevig-Hughes-Zhang (BHZ) model \cite{43}. In Sec. IV we then perturbatively account for weak interlayer tunneling as...
well as weak proximity-induced superconductivity. As a consequence, the helical edge states found previously are gapped out. We show that there exists a regime of parameters for which the system is a time-reversal invariant SOTSC with a Kramers pair of MCSs at all four corners of a rectangular sample. In order to account for a possible complication in some experimentally relevant setups, we additionally comment on the case of unequal interlayer tunneling amplitudes for particle-like and hole-like bands in Sec. VII. Finally, in Sec. VII we discuss the case of broken time-reversal symmetry in the presence of a weak in-plane Zeeman field. We show that this enriches the phase diagram further, allowing us to access also a SOTSC phase with a single MCS per corner. We summarize our results in Sec. VII.

II. MODEL

We consider a 2D TI bilayer, where each of the two TI layers is described by a BHZ model \cite{43}. In momentum space, the Hamiltonian of a single TI layer can then be written as $H_0 = \sum_k \Psi_\mathbf{k}^\dagger H_0(\mathbf{k}) \Psi_\mathbf{k}$ in the basis $\Psi_\mathbf{k} = (\psi_{k111}, \psi_{k\bar{1}11}, \psi_{k\bar{1}\bar{1}1}, \psi_{k\bar{1}\bar{1}\bar{1}})$, where $\psi_{k\sigma s}$ ($\psi_{k\bar{1}\sigma s}$) destroys (creates) an electron with in-plane momentum $\mathbf{k} = (k_x, k_y)$, orbital degree of freedom $\sigma \in \{1, \bar{1}\}$ and spin $s \in \{1, \bar{1}\}$. The Hamiltonian density is given by

$$H_0(\mathbf{k}) = \left( \frac{\hbar^2 k_x^2}{2m_x} + \frac{\hbar^2 k_y^2}{2m_y} + \epsilon \right) \sigma_z + \mu + \lambda_x k_x \sigma_x s_z + \lambda_y k_y \sigma_y,$$

where $\sigma_i$ and $s_i$ for $i \in \{x, y, z\}$ are Pauli matrices acting in orbital and spin space, respectively. The parameters $m_x$, $m_y$, $\lambda_x$, and $\lambda_y$ are material-specific constants inherent to the BHZ model \cite{43}. For simplicity, we assume $m_x = m_y = m$ and $\lambda_x = \lambda_y = \lambda$, the presence of $\tau_z$ in the term proportional to $\lambda_x$ breaks the usual fourfold rotational symmetry of the BHZ model given by $U_{\pi/2} = e^{i\pi\eta x \tau_x (2\sigma_0 - \sigma_1)/4}$. This will turn out to be crucial to realize the second-order phase proposed in the following. However, for $\Gamma = 0$, we can define a generalized fourfold rotational symmetry $U_{\pi/2}' = e^{i\pi\eta x \tau_x (2\sigma_0 - \sigma_1)/4}$ such that $U_{\pi/2}' H(k_x, k_y) U_{\pi/2}'^{-1} = H(-k_y, k_x)$ in the isotropic case.

III. EDGE STATES IN THE FIRST-ORDER PHASE

Let us first consider the case $\Delta_{ac} = \Gamma = 0$. In this case, our model simply corresponds to two decoupled copies of the BHZ model. Furthermore, we set $\mu = 0$ to simplify our analysis. The bulk spectrum is then given by

$$E_{\pm}(\mathbf{k}) = \pm \sqrt{\left( \frac{\hbar^2 k_x^2}{2m_x} + \frac{\hbar^2 k_y^2}{2m_y} + \epsilon \right)^2 + \lambda_x^2 k_x^2 + \lambda_y^2 k_y^2}.$$  

We find that the bulk gap closes at $\mathbf{k} = 0$ for $\epsilon = 0$, separating a trivial phase for $\epsilon > 0$ from a topologically non-trivial TI phase for $\epsilon < 0$ \cite{43}. In our case, the latter is characterized by the presence of one pair of counter-propagating helical edge states per layer.

The explicit form of these edge states is readily obtained by following the standard procedure of matching decaying eigenfunctions. Let us first focus on the edges along the $x$ direction. For this, we consider a semi-infinite geometry such that the sample is finite along the $y$ direction and infinite along the $x$ direction. In this setting, $k_x$ remains a good quantum number, while we replace $k_y$ with $-i\partial_y$. For simplicity, we begin by solving for
zero-energy eigenstates at $k_x = 0$ before perturbatively including linear contributions in $k_x$. Thus, we solve
\[ \mathcal{H}(0, -i\partial_y) = \left( \epsilon - \frac{\hbar^2 \partial_y^2}{2m_y} \right) \eta_s \sigma_z - i\lambda_y \partial_y \eta_z \sigma_y \]  
(4)
for exponentially decaying eigenfunctions $\Phi(y)$ with vanishing boundary conditions $\Phi(y = 0) = 0$. As both the layer index as well as the spin-projection along the $z$ axis are good quantum numbers, we can express our solutions as eigenstates of $\tau_z$ and $s_z$. Furthermore, since \{ $\mathcal{H}(0, -i\partial_y), \sigma_z$ \} = 0 and we are looking for zero-energy eigenstates, the solutions are also eigenstates of $\sigma_z$. Therefore, we can write the solutions in terms of eigenstates $|\tau, s, a\rangle$ defined via $\tau_z s_z \sigma_z |\tau, s, a\rangle = \tau s a |\tau, s, a\rangle$, where $a \in \{1, \bar{1}\}$ is used to denote the eigenvalue of $\sigma_z$. Explicitly, we find that the solutions are given by
\[ \Phi_{\tau s}^\tau(y) = |\tau, s, 1\rangle (e^{-y/\xi_1} - e^{-y/\xi_2}) \]  
(5)
with $\xi_{1/2} = (-\lambda_y \pm \sqrt{\beta_y})/(2\epsilon)$ for $\beta_y = \lambda_y^2 + 2\hbar^2 \epsilon/m_y$ and where we have suppressed a normalization factor. Note that since $\epsilon < 0$ in the topologically non-trivial phase, we have $\text{Re}(\xi_{1/2}) > 0$, confirming that our solutions are indeed exponentially localized to the edge of the system. Furthermore, it is straightforward to see that the solutions are related by time-reversal symmetry as $\mathcal{T} \Phi_{\tau s}^\tau(y) = \tilde{s} \Phi_{\tau s}^\tau(y)$.

For the edges along the $y$ direction, a similar consideration yields
\[ \mathcal{H}(0, -i\partial_x) = \left( \epsilon - \frac{\hbar^2 \partial_x^2}{2m_x} \right) \eta_z \sigma_z - i\lambda_x \partial_x \tau_z s_z \sigma_x. \]  
(6)
In this case, the solutions for the edge states turn out to be eigenstates of $\tau_z$, $s_z$, and $\sigma_y$. Therefore, we will write them in terms of eigenstates $|\tau, s, b\rangle$ defined via $\tau_z s_z \sigma_y |\tau, s, b\rangle = \tau s b |\tau, s, b\rangle$, where $b \in \{1, \bar{1}\}$ is used to denote the eigenvalue of $\sigma_y$. We arrive at
\[ \Phi_{\tau s}^\tau(x) = |\tau, s, \tau s\rangle (e^{-x/\xi_1} - e^{-x/\xi_2}) \]  
(7)
with $\xi_{1/2} = (-\lambda_y \pm \sqrt{\beta_x})/(2\epsilon)$ for $\beta_x = \lambda_x^2 + 2\hbar^2 \epsilon/m_x$ and where we have again omitted a normalization factor. Finally, the kinetic term governing the low-energy spectrum can be found by taking into account the linear terms in $k_x$ or $k_y$, respectively. Along the $x$ direction, we find that
\[ \lambda_x k_x \Phi_{\tau s}^\tau(\tau z s z |\tau, s, s\rangle |\Phi_{\tau s}^\tau|) = \tau s \lambda_y k_y \delta_{\tau, \bar{\tau}} \delta_{s, \bar{s}}. \]  
(8)
Indeed, the structure of the edge states given in Eq. (5) makes it immediately clear that states with $\tau s = +1$ ($\tau s = -1$) propagate in the positive (negative) $x$ direction. Similarly, we find that
\[ \lambda_y k_y \Phi_{\tau s}^\tau |\Phi_{\tau s}^\tau| = \tau s \lambda_y k_y \delta_{\tau, \bar{\tau}} \delta_{s, \bar{s}} \]  
(9)
along the $y$ direction. Again, states with $\tau s = +1$ ($\tau s = -1$) propagate in the positive (negative) $y$ direction. As expected, we therefore find a pair of counterpropagating gapless edge states per layer, see also Fig. 2(a) for a numerical verification. Within each layer, counterpropagating edge states carry opposite spin projections, while counterpropagating edge states in opposite layers carry the same spin projection.

IV. KRAMERS PAIRS OF MAJORANA CORNER STATES

In the following, we take into account the effect of the superconducting and tunneling term in a perturbative
way. For this, we assume $\Delta_{sc}$ and $\Gamma$ to be small compared to the bulk gap of the first-order phase, such that their only effect will be to potentially gap out the edge states found above. In order to understand the emergence of corner states, we derive an effective Hamiltonian describing the low-energy physics for each edge.

Let us start by considering the tunneling term $\mathcal{H}_T = \Gamma \eta_z \tau_z$, while keeping $\Delta_{sc} = 0$ for the moment. For the edge states along the $x$ direction, we obtain by direct calculation

$$\langle \Phi_{\tau s}^x | \mathcal{H}_T | \Phi_{\tau' s'}^x \rangle = \Gamma \delta_{\tau \tau'} \delta_{ss'}.$$

As such, the tunneling term fully gaps out the edge states along the $x$ direction. Along the $y$ direction, however, we obtain

$$\langle \Phi_{\tau s}^y | \mathcal{H}_T | \Phi_{\tau' s'}^y \rangle = 0$$

for all $\tau, \tau', s, s'$, which may seem surprising at first. However, this is a direct consequence of the symmetries of the system. Indeed, we note that the system has an additional symmetry $\mathcal{O} = \tau_z \sigma_z$, that anticommutes with the Hamiltonian given in Eq. (5). Furthermore, we find $\mathcal{O} | \Phi_{\tau s}^y \rangle = -| \Phi_{\tau s}^y \rangle$. Together with $\{ \mathcal{H}_T, \mathcal{O} \} = 0$, we can then find $\langle \Phi_{\tau s}^y | \mathcal{H}_T | \Phi_{\tau s}^y \rangle = -\langle \Phi_{-\tau s}^y | \mathcal{H}_T | \Phi_{-\tau s}^y \rangle = 0$. The other matrix elements are trivially zero by the definition of $\mathcal{H}_T$, which confirms Eq. (11).

Let us now additionally consider the effect of superconductivity. Clearly, superconductivity will open a gap along all edges, leading to an effective edge Hamiltonian of the form

$$H_{\text{eff}}^x(k_x) = \lambda_x k_x \tau_z s_z + \Gamma \eta_z \tau_x + \Delta_{sc} \eta_y \tau_z s_y$$

for the edges along the $x$ direction and

$$H_{\text{eff}}^y(k_y) = \lambda_y k_y \tau_x s_x + \Delta_{sc} \eta_y \tau_z s_y$$

for the edges along the $y$ direction. From this it becomes clear that as long as $|\Delta_{sc}| > 0$, the edges along the $y$ direction are trivially gapped without superconductivity. Along the $x$ direction, on the other hand, the edge gap closes at $|\Delta_{sc}| = |\Gamma|$. Indeed, we recognize Eq. (12) to be the Hamiltonian of a time-reversal invariant 1D TSC as discussed in Ref. [6]. In the topologically nontrivial phase $|\Gamma| > |\Delta_{sc}|$, this system exhibits a Kramers pair of Majorana bound states at its edges. In our model, these edges correspond to the corners between $x$ and $y$ edges, leaving us with a Kramers pair of MCSs at all four corners of a rectangular system. In Fig. 2(b), we have verified the existence of the corner states numerically. Furthermore, we have tested the stability of the corner states against potential disorder, see Fig. 2(c). In particular, we note that the symmetry $\mathcal{O}$ used to derive the corner states can be broken as long as the edge gap remains open. Indeed, the MCSs are protected solely by particle-hole and time-reversal symmetry and do not rely on any additional spatial symmetry.

V. UNEQUAL TUNNELING AMPLITUDES FOR PARTICLE-LIKE AND HOLE-LIKE BANDS

In realistic setups we generally expect the interlayer tunneling amplitude for the particle-like and hole-like bands to be different in size. This constitutes an obstruction to the second-order topological phase presented here. In the following, we account for this by introducing a refined tunneling Hamiltonian

$$\mathcal{H}_T = \frac{\Gamma_e + \Gamma_h}{2} \eta_z \tau_x + \frac{\Gamma_e - \Gamma_h}{2} \eta_z \tau_z \sigma_z,$$

where $\Gamma_e$ ($\Gamma_h$) is used to denote the tunneling amplitude for electrons (holes). Calculating the effective Hamiltonian along the $x$ and $y$ direction using the edge state solutions given in Eqs. (5) and (7), we find

$$H_{\text{eff}}^x(k_x) = \lambda_x k_x \tau_z s_z + \frac{\Gamma_e + \Gamma_h}{2} \eta_z \tau_x + \Delta_{sc} \eta_y \tau_z s_y$$

for the edges along the $x$ direction and

$$H_{\text{eff}}^y(k_y) = \lambda_y k_y \tau_x s_x + \frac{\Gamma_e - \Gamma_h}{2} \eta_z \tau_x + \Delta_{sc} \eta_y \tau_z s_y$$

for the edges along the $y$ direction. We therefore find that the SOTSC phase persists if $|\Gamma_e - \Gamma_h| < 2|\Delta_{sc}| < |\Gamma_e + \Gamma_h|$. In our experiments we have estimated $\Gamma_h$ to be negligibly small compared to $\Gamma_e$ in the experimentally accessible parameter range, i.e., $\Gamma_h \approx 0$ [51, 52]. This excludes the double-well setup as a possible realization of the topological phase proposed here. However, other systems with similar low-energy properties may circumvent this issue. In particular, mono- and few-layer Fe(Te$_{1-x}$Se$_x$) have recently been shown to exhibit a low-energy band structure described by the BHZ Hamiltonian [51, 52]. Similarly, the 2D transition metal dichalcogenides (TMDCs) MX$_2$ with $M \in \{ W, Mo \}$ and $X \in \{ S, Se, Te \}$ have been shown to exhibit the desired low-energy effective band structure [53]. It would therefore be interesting to investigate TI bilayers built from these materials as potential experimental realizations of the SOTSC proposed in this work.

VI. EFFECT OF ZEEMAN FIELD AND SINGLE-MCS PHASE

In this section, we additionally comment on the effects of a Zeeman field, which we again assume to be sufficiently weak compared to the bulk gap of the first-order phase. Since time-reversal symmetry is now broken, the fate of the MCSs is not a priori clear in this case. Indeed, we find that in the presence of an out-of-plane Zeeman term $\mathcal{H}_z = \Delta_{z, \perp} \eta_z s_z$, the Kramers pairs at each corner hybridize and split away from zero energy, see Fig. 3(a). Thus, the topological phase is destroyed in this case. On
displays the phase diagram of our system the other hand, however, we find that an in-plane Zeeman field does not completely destroy the topological properties of the system, but instead leads to a much richer phase diagram exhibiting two nonequivalent regions with different numbers of Kramers partners of MCSs per corner. Here, we have used $\Delta_{sc}/|\epsilon| = 0.25$, $\Gamma/|\epsilon| = 0.5$ and $\Delta_{z,\perp}/|\epsilon| \approx 0.19$. (d) For strong Zeeman fields $\Gamma + \Delta_{sc} < \Delta_{z,\parallel}$, we again find two MCSs per corner. Here, we have used $\Delta_{sc}/|\epsilon| = 0.25$, $\Gamma/|\epsilon| \approx 0.13$ and $\Delta_{z,\parallel}/|\epsilon| = 0.5$. The other numerical parameters are the same as in Fig. 2.

FIG. 3. (a) Probability density of low-energy states obtained numerically from a discretized version of Eq. (2) for a sample of $L_x = L_y = 50$ sites with $\Delta_{sc}/|\epsilon| \approx 0.31$ and $\Gamma/|\epsilon| \approx 0.63$ and the additional presence of an out-of-plane Zeeman field of strength $\Delta_{z,\perp}/|\epsilon| \approx 0.04$. We find that the two Kramers partners of MCSs at each corner hybridize and split away from zero energy, see the red dots in the inset. (b) Probability density of the lowest-energy state obtained numerically from a discretized version of Eq. (2) for a sample of $L_x = L_y = 80$ sites in the additional presence of an in-plane Zeeman field oriented along the $x$ direction. (b) For a weak Zeeman field $0 \leq \Delta_{z,\parallel} < \Gamma - \Delta_{sc}$, there are two MCSs per corner. Here, we have used $\Delta_{sc}/|\epsilon| = 0.25$, $\Gamma/|\epsilon| = 0.5$ and $\Delta_{z,\parallel}/|\epsilon| \approx 0.13$. (c) In the intermediate regime $\Gamma - \Delta_{sc} < \Delta_{z,\parallel} < \Gamma + \Delta_{sc}$, we find one MCS per corner. Here, we have used $\Delta_{sc}/|\epsilon| = \Gamma/|\epsilon| \approx 0.38$ and $\Delta_{z,\parallel}/|\epsilon| \approx 0.19$. (d) For strong Zeeman fields $\Gamma + \Delta_{sc} < \Delta_{z,\parallel}$, we again find two MCSs per corner. Here, we have used $\Delta_{sc}/|\epsilon| = 0.25$, $\Gamma/|\epsilon| \approx 0.13$ and $\Delta_{z,\parallel}/|\epsilon| = 0.5$. The other numerical parameters are the same as in Fig. 2.

We have proposed a versatile and experimentally feasible platform that realizes a 2D time-reversal invariant SOTSC phase hosting Kramers pairs of MCSs. Our setup consists of two tunnel-coupled 2D TIs proximitized by a top and bottom $s$-wave superconductor with a phase difference of $\pi$ between them. In the regime where the interlayer tunneling dominates over the proximity-induced superconductivity, we find a Kramers pair of MCSs at

\begin{equation}
H^{x}_{\text{eff}}(k_{x}) = \lambda_{s} k_{x} \tau_{3} s_{z} + \Gamma \eta_{z} \tau_{x} + \Delta_{sc} \epsilon_{y} \tau_{z} s_{y} + \Delta_{z,\parallel} \eta_{z} s_{x},
\end{equation}

while the effective Hamiltonian along the $y$ direction is still given by Eq. (16). Therefore, the edges along the $y$ direction remain trivially gapped by superconductivity, whereas the edge gap along the $x$ direction now closes at $\Delta_{z,\parallel} + \Gamma = \pm \Delta_{sc}$ and $\Delta_{z,\parallel} - \Gamma = \pm \Delta_{sc}$.

In the following, we comment on the different (second-order) topological phases separated by the above gap closing lines. For simplicity, we assume that $\Delta_{sc}, \Delta_{z,\perp}, \Gamma \geq 0$. Firstly, we have checked numerically that for $0 < \Delta_{z,\parallel} < \Gamma - \Delta_{sc}$, the two MCSs per corner remain intact, see Fig. 3(b). However, they are now no longer protected by time-reversal symmetry and may split away from zero energy in the presence of magnetic disorder [55, 56]. Secondly, we find that in the intermediate regime $\Gamma - \Delta_{sc} < \Delta_{z,\parallel} < \Gamma + \Delta_{sc}$, there is only one MCS per corner, see Fig. 3(c). Most interestingly, we find that for $\Gamma = \Delta_{sc}$ even an infinitesimal Zeeman field can drive the system into a SOTSC phase with one MCS per corner, as the competing tunneling and superconducting term completely cancel each other. Finally, for $\Gamma + \Delta_{sc} < \Delta_{z,\parallel}$, we again find two MCSs per corner, see Fig. 3(d). Indeed, this regime is in the same region of the phase diagram as the limit $\Gamma = 0$ and $\Delta_{sc} < \Delta_{z,\parallel}$.

In this case, we simply deal with two decoupled TI layers subjected to an in-plane Zeeman field. Indeed, a single TI layer in the presence of an in-plane Zeeman field has been shown to exhibit a SOTSC phase in Ref. [38]. To summarize, Fig. 4 displays the phase diagram of our system both in the absence [Fig. 4(a)] and presence [Fig. 4(b)] of an in-plane Zeeman field.
all four corners of a rectangular sample. Additionally, we have shown that a weak but finite in-plane Zeeman field further enriches the phase diagram. In particular, we find that there are now two nonequivalent SOTSC phases with two MCSs per corner as well as an intermediate phase with just one MCS per corner. Most interestingly, we find that for \( \Gamma = \Delta_{sc} \) even an infinitesimal Zeeman field can drive the system into a SOTSC phase with one MCS per corner, as the competing tunneling and superconducting term completely cancel each other. Finally, for \( \Gamma + \Delta_{sc} < \Delta_Z \) (shaded in dark green), we again find two MCSs per corner. Indeed, this regime can be connected to the limit \( \Gamma = 0 \) and \( \Delta_{sc} \). In this case, we simply deal with two independent copies of the system proposed in Ref. [35].

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**Appendix A: Lattice model for the BHZ Hamiltonian**

In this Appendix, we present the discretized version of the Hamiltonian given in Eq. (2). In momentum space, the discretized Hamiltonian reads

\[
\mathcal{H}(k) = [-2t_x \cos(k_x a_x) - 2t_y \cos(k_y a_y)] \eta_z \sigma_z + (\epsilon + 2t_x + 2t_y) \eta_z \sigma_z + \mu \eta_z + 2\alpha_x \sin(k_x a_x) \tau_z \sigma_x \tau_z + 2\alpha_y \sin(k_y a_y) \eta_z \sigma_y + \Delta_{sc} \eta_y \tau_z \sigma_y + \Gamma \eta_z \tau_x. \tag{A1}
\]

Here, \( a_x \) (\( a_y \)) is the lattice spacing along the \( x \) (\( y \)) direction. The spin-conserving hopping amplitude \( t_x \) (\( t_y \)) defines the effective mass along the \( x \) (\( y \)) direction via \( t_x = \hbar^2/(2m_x a_x^2) \) [\( t_y = \hbar^2/(2m_y a_y^2) \). Similarly, \( \alpha_x \) (\( \alpha_y \)) is related to \( \lambda_x \) (\( \lambda_y \)) via \( \alpha_x = \lambda_x/(2\alpha_x) \) [\( \alpha_y = \lambda_y/(2\alpha_y) \). Note that in the main part of our work, we focus on the isotropic case \( t_x = t_y = t \) and \( \alpha_x = \alpha_y = \alpha \). The strongly anisotropic case could, for example, be realized in the coupled-wire approach [61–72].
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