EFFECTIVE EQUIDISTRIBUTION OF CLOSED
HOROCYCLES FOR GEOMETRICALLY FINITE
SURFACES

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Abstract. For a complete hyperbolic surface whose fundamental
group is finitely generated and has critical exponent bigger than
\frac{1}{2}, we obtain an effective equidistribution of closed horocycles in
its unit tangent bundle. This extends a result of Sarnak in 1981
for surfaces of finite area. We also discuss applications in Affine
sieves.

1. Introduction

Let $G = \text{PSL}_2(\mathbb{R})$ be the group of orientation preserving isometries
of the hyperbolic plane $\mathbb{H}^2 = \{ x + iy : y > 0 \}$. Let $\Gamma$ be a finitely
generated discrete torsion-free subgroup of $G$, which is not virtually
cyclic. The quotient space $X := \Gamma \backslash G$ can be identified with the unit
tangent bundle of the hyperbolic surface $\Gamma \backslash \mathbb{H}^2$. For $x \in \mathbb{R}$ and $y > 0$,
we define

\[ n_x := \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad a_y := \begin{pmatrix} \sqrt{y} & 0 \\ 0 & \sqrt{y^{-1}} \end{pmatrix}. \]

Via the multiplication from the right, the action of $a_y$ on $X$ corresponds
to the geodesic flow and the orbits of the subgroup $N := \{ n_x : x \in \mathbb{R} \}$
give rise to the stable horocyclic foliation of $X$. For a fixed closed
horocycle $[g]N$ in $X$, we consider a sequence of closed horocycles $[g]Na_y$ as $y \to 0$.

We denote by $\Lambda(\Gamma)$ the limit set of $\Gamma$, the set of accumulation points
of an orbit of $\Gamma$ in the boundary $\partial(\mathbb{H}^2) = \mathbb{R} \cup \{ \infty \}$. A point $\xi \in \Lambda(\Gamma)$
is called a parabolic limit point for $\Gamma$ if $\xi$ is the unique fixed point in
$\partial(\mathbb{H}^2)$ of an element of $\Gamma$.

A horocycle in $\mathbb{H}^2$ is simply a Euclidean circle tangent to a point in
$\partial(\mathbb{H}^2)$, called the basepoint; here a circle tangent to $\infty$ is understood
as a horizontal line. Topological behavior of a horocycle in $\Gamma \backslash \mathbb{H}^2$ is

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completely determined by its basepoint, say, \( \xi \); it is closed and non-compact (resp. compact) if and only if \( \xi \) lies outside \( \Lambda(\Gamma) \) (resp. \( \xi \) is a parabolic limit point of \( \Gamma \)) \cite{6}.

When \( \Gamma \) is a lattice in \( G \), \( \Lambda(\Gamma) \) is the entire boundary and hence a closed horocycle is necessarily based at a parabolic limit point and compact. In this case, Sarnak \cite{21} obtained a sharp effective equidistribution: a sequence of closed horocycles becomes equidistributed with respect to the \( G \)-invariant measure in \( X \) as their length tends to infinity. One can also use the mixing of the geodesic flow via thickening argument to obtain an effective equidistribution in this case, which yields less sharp result than Sarnak’s. This argument goes back to the 1970 thesis of Margulis \cite{16} and was used by Eskin and McMullen \cite{7}.

When \( \Gamma \) is not a lattice, there are always noncompact closed horocycles in \( \Gamma \setminus \mathbb{H}^2 \), and compact horocycles exist only if \( \Gamma \) contains a parabolic element, or equivalently if there is a cusp in \( \Gamma \setminus \mathbb{H}^2 \). Roblin \cite{19} showed that for a fixed compact piece \( N_0 \) of \( N \), the sequence \( [g]N_0a_y \) becomes equidistributed in \( X \), as \( y \to 0 \), with respect to an infinite locally finite measure, called the Burger-Roblin measure (corresponding to the stable horocyclic foliation). In \cite{12}, it was observed that the relevant dynamics happens only within a compact part of \( N \) even for a noncompact closed horocycle, and hence Roblin’s theorem also applies to infinite closed horocycles. But Roblin’s proof is non-effective.

**Effective equidistribution of a closed horocycle:** Let \( 0 < \delta \leq 1 \) denote the critical exponent of \( \Gamma \), which is also equal to the Hausdorff dimension of \( \Lambda(\Gamma) \). We have \( \delta = 1 \) if and only if \( \Gamma \) is a lattice.

Our main goal is to describe an effective equidistribution of a closed horocycle in \( \Gamma \setminus G \). In the rest of the introduction, we assume that \( 1/2 < \delta < 1 \) and that the horocycle \( [e]N \) is closed in \( \Gamma \setminus G \).

Set \( k_{\theta} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \), \( K = \{ k_{\theta} : 0 \leq \theta < \pi \} \) and \( A = \{ a_y : y > 0 \} \). We have the Iwasawa decomposition \( G = NAK \): any element of \( G \) can be uniquely written as \( n_xa_yk_{\theta} \) for \( n_x \in N, a_y \in A, k_{\theta} \in K \). A Casimir operator of \( G \) is given as follows:

\[
C = -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + y \frac{\partial^2}{\partial x \partial \theta}.
\]

For \( K \)-invariant functions on \( G \), \( C \) acts as the (negative of the) hyperbolic Laplacian:

\[
\Delta = -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).
\]
By the results of Patterson [18], and of Lax and Phillips [14], the Laplace operator $\Delta$ on $L^2(\Gamma \backslash \mathbb{H}^2) = L^2(\Gamma \backslash G)^K$ has only finitely many eigenvalues

$$0 < \alpha_0 = \delta(1 - \delta) < \alpha_1 \leq \cdots \leq \alpha_k < 1/4$$

lying below the continuous spectrum $[1/4, \infty)$. The existence of a point eigenvalue is the precise reason that our main theorem is stated only for $\delta > 1/2$. Writing $\alpha_1 = s_1(1 - s_1)$, a positive number

$$0 < s_1 < \delta$$

will be called a spectral gap for $\Gamma$.

The base eigenvalue $\delta(1 - \delta)$ is simple and moreover there exists a positive eigenfunction $\phi_0 \in L^2(\Gamma \backslash G)^K$ with $\Delta \phi_0 = \delta(1 - \delta)\phi_0$. Patterson gave an explicit formula:

$$\phi_0(nxa_\theta k_\theta) = \int_{\Lambda(\Gamma)} \left( \frac{(u^2 + 1)y}{(x-u)^2 + y^2} \right)^\delta d\nu_1(u)$$

where $\nu_1$ is the Patterson measure on $\Lambda(\Gamma)$ associated to $i \in \mathbb{H}^2$. We normalize $\nu_1$ so that $\|\phi_0\|_2 = 1$.

Denote by $V$ the unique $G$-subrepresentation of $L^2(\Gamma \backslash G)$ on which $\mathcal{C}$ acts by the scalar $\delta(1 - \delta)$. The $K$-fixed subspace of $V$ is spanned by $\phi_0$ and $V$ decomposes into the orthogonal sum $\bigoplus_{\ell \in \mathbb{Z}} \mathbb{C}\phi_\ell$ where $\phi_\ell \in C^\infty(\Gamma \backslash G)$ satisfies $\phi_\ell(gk_\theta) = e^{2\ell i\theta} \phi_\ell(g)$ for all $g \in \Gamma \backslash G$ and $k_\theta \in K$ and has unit norm: $\|\phi_\ell\|_2 = 1$.

We show that there exists $c_{\phi_\ell} \neq 0$ such that for all $y > 0$,

$$\int_{\Lambda(\Gamma) \backslash \mathbb{N}} \phi_\ell(nxa_\theta) dx = c_{\phi_\ell} \cdot y^{1-\delta}.$$
Theorem 1.1. Suppose that $1/2 < \delta < 1$. For $\psi \in C_c^\infty(\Gamma \setminus G)$, as $y \to 0$,

$$
\int_{(N \cap \Gamma) \setminus N} \psi(n_x a_y) \, dx = \sum_{\ell \in \mathbb{Z}} c_{\phi_\ell} \cdot \langle \psi, \phi_\ell \rangle \cdot y^{1-\delta} + O(\mathcal{S}_3(\psi)y^{1-\delta + \frac{2\pi}{5}})
$$

where $\sum_{\ell \in \mathbb{Z}} |c_{\phi_\ell} \cdot \langle \psi, \phi_\ell \rangle| \ll \mathcal{S}_2(\psi)$. Here $\mathcal{S}_m(\psi)$ denotes the Sobolev norm of $\psi$ of degree $m$.

Remark 1.2. (1) We explicitly compute $\phi_{\pm \ell}$, up to a unit, (Theorem 2.3):

$$
\phi_{\pm \ell}(n_x a_y k_0) = e^{\pm 2\ell i \theta} \sqrt{\Gamma(\delta + \ell)\Gamma(1-\delta)} \int_{\Lambda(\Gamma)} \left( \frac{(x-u) \pm iy}{(x-u)^2 + y^2} \right)^\delta d\nu_i(u).
$$

(2) We note that $\phi_{-\ell} c_{\phi_{-\ell}} = \overline{\phi_{\ell} c_{\phi_{\ell}}}$ for all $\ell \in \mathbb{Z}$. Indeed, this is an important observation which clarifies a point that the main term in Theorem 1.1 is a real number for a real-valued function $\psi$.

(3) For smooth functions on the surface $\Gamma \setminus \mathbb{H}^2$, an effective result was obtained in [12] and our proof follows the same general strategy but working with all different $K$-types of base eigenfunctions $\phi_\ell$’s as opposed to studying only the trivial $K$-type $\phi_0$.

(4) In [15], we obtain an effective equidistribution of closed horospheres in the unit tangent bundle of hyperbolic 3 manifold $\Gamma \setminus \mathbb{H}^3$ when the critical exponent of $\Gamma$ is between 1 and 2 and use this result for an effective counting of circles in Apollonian circle packings.

Define the measure $\tilde{m}_N^{BR}$ on $G$ in the Iwasawa coordinates $G = KAN$: for $\psi \in C_c(G)$,

$$
\tilde{m}_N^{BR}(\psi) = \int_{KAN} \psi(ka_y n_x) y^{\delta - 1} \, dx dy d\nu_i(k(0)).
$$

This measure is left $\Gamma$-invariant and right $N$-invariant, and the Burger-Roblin measure $m_N^{BR}$ (associated to the stable horospherical subgroup $N$) is the measure on $\Gamma \setminus G$ induced from $\tilde{m}_N^{BR}$. The Burger-Roblin measure is an infinite measure whenever $0 < \delta < 1$ [17] and coincides with a Haar measure when $\delta = 1$.

Theorem 1.1 can also be stated as follows:
Theorem 1.3. Let $1/2 < \delta \leq 1$. For any $\psi \in C^\infty_c (\Gamma \backslash G)$, as $y \to 0$,
\[
\int_{(N \cap \Gamma) \backslash N} \psi(n_x a_y) \, dx = \kappa_\Gamma \cdot m_N^{BR}(\psi) \cdot y^{1-\delta} + O(S_3(\psi) \cdot y^{(1-\delta)+\frac{2\pi r}{5}})
\]
where $\kappa_\Gamma = \frac{\sqrt{\pi} \Gamma(\delta-1/2) \Gamma(\delta)}{\Gamma(\delta)} \cdot \int_{(N \cap \Gamma) \backslash N} (x^2 + 1)^{\delta} d\nu_i(x) > 0$.

Effective orbital counting on a cone: Let $Q$ be a ternary indefinite quadratic form over $\mathbb{Q}$ and $v_0 \in \mathbb{Q}^3$ be a non-zero vector such that $Q(v_0) = 0$.

Let $G_0 := \text{SO}_Q(\mathbb{R})$ and $\Gamma < G_0(\mathbb{Z})$ be a finitely generated subgroup with $\delta > 1/2$. For a square-free integer $d$, consider the subgroup of $\Gamma$ which stabilizes $v_0$ mod $d$:
\[
\Gamma_d := \{ \gamma \in \Gamma : v_0 \gamma \equiv v_0 \mod d \}.
\]

To define a sector in the cone $\{Q = 0\}$, let $\iota : \text{PSL}_2(\mathbb{R}) \to G_0$ be an isomorphism such that $\iota(N) = \{ g \in G_0 : v_0 g = v_0 \}$. Fix a norm $\| \cdot \|$ on $\mathbb{R}^3$. For any subset $\Omega \subset K$ and $T > 0$, define the sector
\[
S_T(\Omega) := \{ v \in v_0 A\Omega : \|v\| < T \}.
\]
By a theorem of Bourgain, Gamburd and Sarnak [3], there exists a spectral gap, say $s_0$, uniform for all $\Gamma_d, d$ square free.

Theorem 1.4. Suppose that $\Omega$ has only finitely many connected components. As $T \to \infty$, we have
\[
\# \{ v \in v_0 \Gamma_d \cap S_T(\Omega) \} = \frac{\Xi_{v_0}(\Gamma, \Omega)}{[\Gamma : \Gamma_d]} \cdot T^\delta + O(T^{\delta-\frac{4s_0}{\delta}}).
\]

Identifying $\Gamma$ with its pull back in $\text{PSL}_2(\mathbb{R})$, $\Xi_{v_0}(\Gamma, \Omega)$ is given by
\[
\Xi_{v_0}(\Gamma, \Omega) := \frac{\sqrt{\pi} \Gamma(\delta-1/2)}{\delta \Gamma(\delta)} \int_{(N \cap \Gamma) \backslash N} (1 + x^2)^{\delta} d\nu_i(x) \int_{k \in \Omega^{-1}} \frac{d\nu_i(k(0))}{\|v_0 k^{-1}\|^{\delta}}.
\]
As $\nu_i$ is supported on the limit set $\Lambda(\Gamma)$, $\Xi_{v_0}(\Gamma, \Omega) > 0$ if and only if the interior of $\Omega^{-1}(0)$ intersects $\Lambda(\Gamma)$.

Remark 1.5. Theorem 1.4 is proved in [17] without an error term. When the norm is $K$-invariant and $\Omega = K$, Theorem 1.4 was proved in [12].

Using the affine linear sieves developed by Bourgain, Gamburd and Sarnak [1], this theorem has an application in studying almost prime vectors in the orbit of $\Gamma$, lying in a fixed sector. To illustrate this, consider the quadratic form $Q(x_1, x_2, x_3) := x_1^2 + x_2^2 - x_3^2$ so that the integral points in the cone $Q = 0$ are Pythagorean triples. Let $F(x_1, x_2, x_3) := x_3$ be the hypotenuse.
Theorem 1.6. Suppose that the interior of $Ω^{-1}(0)$ intersects $Λ(Γ)$. Then there exists $R > 0$ (depending on $s_0$) such that

$$\#\{ (x_1, x_2, x_3) \in v_0Γ \cap S_T(Ω) : x_3 \text{ has at most } R \text{ prime factors} \} \asymp \frac{T^δ}{\log T}$$

where $f(T) \asymp g(T)$ means that their ratio is between two positive constants uniformly for all $T \gg 1$.

The constant $R$ can be computed explicitly if $δ$ is sufficiently large, using the work of Gamburd [8]; for instance, $R = 14$ if $δ > 0.9992$.

This has been carefully worked out in [12] for Euclidean norm balls and the same analysis works for sectors, using Theorem 1.4.

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2. Base eigenfunctions of different $K$-types

Let $G = \text{PSL}_2(ℤ)$ and let $N = \{ n_x : x \in ℤ \}$, $A = \{ a_y : y > 0 \}$, $K = \{ k_θ : θ ∈ [0, π) \}$ with $n_x, a_y, k_θ$ defined as in the introduction. Throughout the paper, let $Γ < G$ be a torsion-free discrete finitely generated subgroup with critical exponent $\frac{1}{2} < δ < 1$.

Consider the Casimir operator $C$ given by

$$C = -y^2 \left( \frac{∂^2}{∂x^2} + \frac{∂^2}{∂y^2} \right) + y \frac{∂^2}{∂x∂θ}.$$  

By Lax-Phillips [14], $L^2(Γ\backslash G)$ contains the unique irreducible infinite dimensional subrepresentation $V_δ$ on which $C$ acts by the scalar $δ(1−δ)$. Moreover

$$V_δ = \bigoplus_{ℓ ∈ ℤ} Cφ_ℓ$$

where $φ_ℓ ∈ C^∞(Γ\backslash G) \cap L^2(Γ\backslash G)$ is a unit vector (unique up to a unit) such that $φ_ℓ(gk_θ) = e^{2iℓθ}φ_ℓ(g)$ for all $g ∈ Γ\backslash G$ and $k_θ ∈ K$ (cf. [1]).

Let $ν_i$ be a Patterson measure on $∂(ℍ^2)$ supported in $Λ(Γ)$ with respect to $i ∈ ℍ^2$ ([18], [22]). Up to a scaling, $ν_i$ is the weak-limit as $s → δ^+$ of the family of measures

$$ν_i(s) := \frac{1}{\sum_{γ ∈ Γ} e^{-sd(i, γi)}} \sum_{γ ∈ Γ} e^{-sd(i, γi)} δ_γ.$$  

The $K$-invariant base eigenfunction $φ_0 ∈ L^2(Γ\backslash ℍ^2) \cap C^∞(Γ\backslash ℍ^2)$ can explicitly be given as the integral of the Poisson kernel against the Patterson measure [18]:

$$φ_0(x + iy) = \int_{ℝ} \left( \frac{(u^2 + 1)y}{(x - u)^2 + y^2} \right)^δ du_i(u).$$  (2.1)
Lemma 2.2. For each \( \ell \in \mathbb{Z}_{\geq 0} \), set
\[
\psi^{(\ell)}_0 := R^\ell \phi_0, \quad \text{and} \quad \psi^{-\ell}_0 := L^\ell \phi_0.
\]

Lemma 2.2. For each \( \ell \in \mathbb{Z}_{\geq 0} \),
\[
\|\psi^{(\ell)}_0\|_2 = \sqrt{\frac{\Gamma(\delta + \ell)\Gamma(1 - \delta + \ell)}{\Gamma(\delta)\Gamma(1 - \delta)}},
\]
where \( \Gamma(x) \) denotes the Gamma function for \( x > 0 \).

Since \( \psi^{(\ell)}_0 \in V_\delta \) and \( \psi^{(\ell)}_0(gk_\theta) = e^{2i\theta} \psi^{(\ell)}_0(g) \) by [1 Sec. 2], the unit vector \( \phi_{\pm \ell} \) is now given as follows (up to a sign):
\[
\phi_{\pm \ell} = \frac{\psi^{(\ell)}_0}{\|\psi^{(\ell)}_0\|_2}.
\]

Since \( \phi_0 > 0 \) and \( R \) is the complex conjugate of \( L \), we have for each \( \ell \in \mathbb{Z} \),
\[
\phi_{-\ell} = \overline{\phi_\ell}.
\]

Theorem 2.3. For each \( \ell \in \mathbb{Z}_{\geq 0} \), we have
\[
\phi_{\pm \ell}(n_xa_yk_\theta) = e^{\pm 2i\theta} \sqrt{\frac{\Gamma(\delta + \ell)\Gamma(1 - \delta)}{\Gamma(\delta)\Gamma(1 - \delta)}} \int_\mathbb{R} \left( \frac{(u^2 + 1)y}{(x-u)^2 + y^2} \right)^\delta \left( \frac{(x-u)\mp iy}{(x-u)^2 + y^2} \right) \nu_i(u) \, du.
\]

Proof. By Lemma 2.2, it suffices to show that
\[
\psi^{(\ell)}_0(n_xa_yk_\theta) = \frac{e^{\pm 2i\theta} \Gamma(\delta + \ell)}{\Gamma(\delta)} \int_\mathbb{R} \left( \frac{(u^2 + 1)y}{(x-u)^2 + y^2} \right)^\delta \left( \frac{(x-u)\mp iy}{(x-u)^2 + y^2} \right) \nu_i(u) \, du.
\]
Fix \( u \in \mathbb{R} \). For \( x \in \mathbb{R} \) and \( y > 0 \), let
\[
f_u(x,y) := \frac{(u^2 + 1)y}{(x-u)^2 + y^2}.
\]
Then
\[
(\mathcal{R}\phi_0)(n_xa_yk_\theta) = \delta \cdot e^{2i\theta} \int_\mathbb{R} f_u(x,y) \delta \cdot \frac{(x-u) - iy}{(x-u) + iy} \, d\nu_i(u).
\]
To use an induction, we assume that
\[
(\mathcal{R}^\ell \phi_0)(n_xa_yk_\theta) = \frac{\Gamma(\delta + \ell)}{\Gamma(\delta)} \cdot e^{2i\theta} \int_\mathbb{R} \left( \frac{(u^2 + 1)y}{(x-u)^2 + y^2} \right)^\delta \left( \frac{(x-u) - iy}{(x-u) + iy} \right) \nu_i(u) \, du.
\]
We compute

\[
\left( iy \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right) \left( f^\delta_u(x, y) \cdot \frac{(x-u)-iy}{(x-u)+iy}^\ell \right)
\]

\[\]

\[= \delta \cdot f^\delta_u(x, y) \cdot \frac{(x-u)-iy}{(x-u)+iy}^{\ell+1} + \ell \cdot f^\delta_u(x, y) \cdot \frac{(x-u)-iy}{(x-u)+iy}^\ell \cdot \frac{-2iy}{(x-u)+iy}\]

and

\[\frac{1}{2i} \frac{\partial}{\partial \theta} \left( \mathcal{R}^\ell \phi_0 \right)(n_x a_y k_\theta) \]

\[\]

\[= \frac{\Gamma(\delta+\ell)}{\Gamma(\delta)} \cdot \ell \cdot e^{2i\theta} \int_{\mathbb{R}} \left( \frac{(u^2+1)y}{(x-u)^2+y^2} \right)^\delta \left( \frac{(x-u)-iy}{(x-u)+iy} \right)^\ell \, d\nu_i(u).\]

Hence

\[
\left( \mathcal{R}^{\ell+1} \phi_0 \right)(n_x a_y k_\theta) = \left( \mathcal{R} \left( \mathcal{R}^\ell \phi_0 \right) \right)(n_x a_y k_\theta)
\]

\[\]

\[= \frac{\Gamma(\delta+\ell+1)}{\Gamma(\delta)} \cdot e^{2i(\ell+1)\theta} \int_{\mathbb{R}} f^\delta_u(x, y) \left( \frac{(x-u)-iy}{(x-u)+iy} \right)^{\ell+1} \, d\nu_i(u).\]

The claim about the lowering operator follows from similar computations as above. \(\square\)

**Corollary 2.4.** For any \(\ell \in \mathbb{Z}_{\geq 0}, x \in \mathbb{R}\) and \(y > 0\),

\[|\phi_{\pm \ell}(n_x a_y)| \ll \phi_0(n_x a_y)\]

with implied constant independent of \(\ell\).

**Proof.** Since \(|\frac{(x-u)-iy}{(x-u)+iy}| = |\frac{(x-u)+iy}{(x-u)-iy}| = 1\), the claim follows from Theorem 2.3. \(\square\)

### 3. THE AVERAGE OF \(\phi_\ell\) OVER A CLOSED HOROCYCLE

We suppose that \(\Gamma \backslash \Gamma N\) is closed in \(X\) and define

\[
x_0 = \begin{cases} 
\frac{1}{2} \min\{x > 0 : n_x \in N \cap \Gamma\} & \text{if } N \cap \Gamma \neq \{e\} \\
\infty & \text{otherwise}
\end{cases}
\]

(3.1)

so that \(2x_0\) is the period of the \(N\)-orbit \(\Gamma \backslash \Gamma N\).

For a function \(\psi\) on \(\Gamma \backslash G\) and \(g \in G\), we define

\[
\psi^N(g) := \int_{n_x \in (N \cap \Gamma) \backslash N} \psi(n_x g) \, dx = \int_{-x_0}^{x_0} \psi(n_x g) \, dx.
\]

As \(\Gamma \backslash \Gamma N\) is a (stable) closed horocycle based at \(\infty\), we have either \(\infty \notin \Lambda(\Gamma) (x_0 = \infty)\), or \(\infty\) is a parabolic fixed point \((x_0 < \infty)\).
Since $\Lambda(\Gamma)$ is a closed subset of boundary $\hat{\mathbb{R}}$, $\infty \notin \Lambda(\Gamma)$ implies that $\Lambda(\Gamma)$ is a bounded subset of $\mathbb{R}$.

**Proposition 3.2.** There exists $c_0 > 0$ such that

$$\phi_0^N(a_y) = c_0 \cdot y^{1-\delta}.$$ 

**Proof.** When $N \cap \Gamma$ is trivial, and hence $\Lambda(\Gamma)$ is a bounded subset of $\mathbb{R}$, we show by a direct computation (see [10]):

$$\phi_0^N(a_y) = \int_{u \in \Lambda(\Gamma)} (u^2 + 1)^{\delta} d\nu_i(u) \cdot \int_{x \in \mathbb{R}} \left( \frac{y}{x^2 + y^2} \right)^{\delta} dx$$

$$= \omega_0 y^{1-\delta} \int_{t \in \mathbb{R}} \left( \frac{1}{1 + t^2} \right)^{\delta} dt$$

$$= \omega_0 \sqrt{\pi} \Gamma(\delta - \frac{1}{2}) \Gamma(\delta) y^{1-\delta}$$

where $\omega_0 = \int_{u \in \Lambda(\Gamma)} (u^2 + 1)^{\delta} d\nu_i(u)$.

Now suppose $N \cap \Gamma$ is non-trivial and hence $0 < x_0 < \infty$. Since $\phi_0^N(a_y)$ must satisfy the differential equation $y^2 \frac{\partial^2}{\partial y^2} \phi_0^N(a_y) = \delta(1 - \delta) \phi_0^N(a_y)$, there exist constants $c_0, d_0 \in \mathbb{R}$ such that for all $y > 0$

$$\phi_0^N(a_y) = c_0 y^{1-\delta} + d_0 y^{\delta}.$$ 

Since $\phi_0 > 0$ and the above holds for all $y > 0$, it follows that $c_0, d_0 \geq 0$. We claim that $d_0 = 0$.

Since $\Gamma$ is finitely generated, it follows that $\Gamma$ admits a fundamental domain $F$ in $\mathbb{H}^2$ such that $(-x_0, x_0) \times [Y_0, \infty)$ injects to $F$ for some $Y_0 \gg 1$.

Then

$$\|\phi_0\|^2 \geq \int_{Y_0}^{\infty} \int_{-x_0}^{x_0} \phi_0(x + iy)^2 y^{-2} dx dy$$

$$\geq \frac{1}{2x_0} \int_{Y_0}^{\infty} \left( \int_{-x_0}^{x_0} \phi_0(x + iy) dx \right)^2 y^{-2} dy$$

$$= \frac{1}{2x_0} \int_{Y_0}^{\infty} \left( c_0 y^{1-\delta} + d_0 y^{\delta} \right)^2 y^{-2} dy$$

$$\geq \frac{d_0^2}{2x_0} \int_{Y_0}^{\infty} y^{2\delta-2} dy.$$ 

Since $\delta > \frac{1}{2}$, $\|\phi_0\| = \infty$ unless $d_0 \neq 0$. Therefore $d_0 = 0$. Since $\phi_0 > 0$, clearly $c_0 > 0$. 

\qed
Lemma 3.3. Let $\ell \in \mathbb{Z}_{\geq 0}$. For any fixed $y > 0$ and $0 \leq \theta < \pi$,

$$\int_{(N \cap \Gamma) \setminus N} \frac{\partial}{\partial x} \phi_{\pm \ell}(nx_0a_yk_{\theta}) \, dx = 0.$$ 

Proof. If $x_0 < \infty$, then

$$\int_{-x_0}^{x_0} \frac{\partial}{\partial x} \phi_{\pm \ell}(nx_0a_yk_{\theta}) \, dx = \phi_{\pm \ell}(nx_0a_yk_{\theta}) - \phi_{\pm \ell}(n_{-x_0}a_yk_{\theta}).$$

Since $\Gamma n_{x_0} = \Gamma n_{-x_0}$, the claim follows.

Suppose $x_0 = \infty$. Since $\phi_0(x + iy) = \int_{u \in \Lambda(\Gamma)} \left( \frac{(u^2 + 1)y}{(x - u)^2 + y^2} \right)^{\delta} \nu_i(u)$ and $\Lambda(\Gamma)$ is bounded, we have $\phi_0(nx_0a_y) \to 0$ as $|x| \to \infty$. On the other hand,

$$\int_{-\infty}^{\infty} \frac{\partial}{\partial x} \phi_{\pm \ell}(nx_0a_yk_{\theta}) \, dx = \lim_{t \to \infty} \int_{-t}^{t} \frac{\partial}{\partial x} \phi_{\pm \ell}(nx_0a_yk_{\theta}) \, dx = \lim_{t \to \infty} (\phi_{\pm \ell}(nta_yk_{\theta}) - \phi_{\pm \ell}(n_{-t}a_yk_{\theta})).$$

Since $|\phi_{\pm \ell}(nta_yk_{\theta})| \ll \phi_0(na_y)$ by Corollary 2.3 and $\phi_0(na_y) \to 0$ as $|t| \to \infty$, the claim follows. \hfill \Box

Theorem 3.4. For any $\ell \in \mathbb{Z}_{\geq 0}$,

$$\int_{(N \cap \Gamma) \setminus N} \phi_{\pm \ell}(nx_0a_y) \, dx = c_0 \sqrt{\frac{\Gamma(\delta)\Gamma(\ell + 1 - \delta)}{\Gamma(1 - \delta)\Gamma(\delta + \ell)}} y^{1-\delta}.$$ 

In particular, for each $y > 0$,

$$\int_{(N \cap \Gamma) \setminus N} \phi_{\pm \ell}(nx_0a_y) \, dx = O(y^{1-\delta})$$

with the implied constant independent of $\ell$.

Proof. By Theorem 2.3,

$$\left| \int_{(N \cap \Gamma) \setminus N} \phi_{\pm \ell}(nx_0a_y) \, dx \right| \leq \int_{(N \cap \Gamma) \setminus N} |\phi_{\pm \ell}(nx_0a_yk_{\theta})| \, dx \ll \int_{(N \cap \Gamma) \setminus N} \phi_0(nx_0a_y) \, dx.$$
Hence by Proposition 3.2, the integral $\psi_N^{(\pm \ell)}(a_y \kappa_\theta)$ converges absolutely. To use an induction, we assume the following

$$\psi^{(\pm \ell)}_0(n_x a_y k_\theta) = e^{\pm 2i\ell \theta} c_0 \frac{\Gamma(\ell + 1 - \delta)}{\Gamma(1 - \delta)} y^{1-\delta}$$

(3.5)

is true. Then applying Lemma 3.3,

$$\psi^{(\pm(x+1))}_0(n_x a_y k_\theta)$$

$$= e^{\pm 2i\theta} \int_{(N \cap \Gamma) \setminus N} \left( \pm iy \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \pm \frac{1}{2i} \frac{\partial}{\partial \theta} \right) \psi^{(\pm \ell)}_0(n_x a_y k_\theta) \, dx$$

$$= e^{\pm 2i\theta} \cdot \left( y \frac{\partial}{\partial y} \pm \frac{1}{2i} \frac{\partial}{\partial \theta} \right) \psi^{(\pm \ell)}_0(n_x a_y k_\theta)$$

$$= e^{\pm 2i \theta} \cdot \left( y \frac{\partial}{\partial y} \pm \frac{1}{2i} \frac{\partial}{\partial \theta} \right) \left( e^{\pm 2i \theta} \cdot \frac{\Gamma(\ell + 1 - \delta)}{\Gamma(1 - \delta)} \cdot ((1 - \delta) + \ell) y^{1-\delta} \right)$$

$$= e^{\pm 2(\ell + 1)i \theta} \cdot \left( \frac{\Gamma(\ell + 1 - \delta)}{\Gamma(1 - \delta)} \cdot ((1 - \delta) + \ell) y^{1-\delta} \right)$$

since $z \Gamma(z) = \Gamma(z + 1)$. This proves (3.5) for all positive integer $\ell$. Hence the claim follows from Lemma 2.2.

□

4. THICKENING OF $\phi_\ell^N$

The following lemma is proved in a greater generality in [10] for all $L^2$-eigenfunctions in the discrete spectrum of $L^2(\Gamma \setminus \mathbb{H}^2)$.

Lemma 4.1. Suppose that $\infty \notin \Lambda(\Gamma)$ and let $J \subset \mathbb{R}$ an open subset containing $\Lambda(\Gamma)$. For all $0 < y < 1$,

$$\int_{J^c} \phi_0(n_x a_y) \, dx \ll y^\delta$$

with the implied constant independent of $y$.

Proof. As we concern only the base eigenfunction $\phi_0$, this can be shown in a simpler way. Let $\epsilon_0 := \inf_{x \notin J \cup \Lambda(\Gamma)} |x - u| > 0$. Then by the change of variable $w = \frac{x - u}{y}$ we have

$$\int_{J^c} \phi_0(n_x a_y) \, dx \leq 2y^{1-\delta} \int_{u \in \Lambda(\Gamma)} (u^2 + 1)^\delta d\nu_i(u) \cdot \int_{w = \epsilon_0/y}^\infty \left( \frac{1}{w^2 + 1} \right)^\delta dw$$
The latter integral can be evaluated explicitly as an incomplete Beta function which has known asymptotics:
\[
\int_{\epsilon_0/y}^\infty \left( \frac{1}{w^2 + 1} \right)^\delta \, dw = c \beta_{y^2/\epsilon_0}(\delta - 1/2, 1 - \delta),
\]
where
\[
\beta_z(\alpha, \beta) \ll z^\alpha.
\]
Therefore
\[
\int_{J^c} \phi_0(n_x a_y) \, dx \ll y^{1-\delta} y^{2(\delta - 1/2)} = y^\delta.
\]

Lemma 4.2. Suppose that \( \infty \notin \Lambda(\Gamma) \) and let \( J \subset \mathbb{R} \) an open subset containing \( \Lambda(\Gamma) \). Fix \( \ell \in \mathbb{Z}_{\geq 0} \). For all \( 0 < y < 1 \),
\[
\phi^\pm_\ell(a_y) = \int_J \phi_{\pm \ell}(n_x a_y) \, dx + O(y^\delta)
\]
with the implied constant independent of \( \ell \) and \( y \).

Proof. Note that, by Corollary 2.4,
\[
\left| \int_{x \in J^c} \phi_{\pm \ell}(n_x a_y) \, dx \right| \leq \int_{x \in J^c} |\phi_{\pm \ell}(n_x a_y)| \, dx \ll \int_{J^c} \phi_0(n_x a_y) \, dx
\]
since \( \phi_0 \) is a positive function.
Hence the claim follows from Lemma 4.1. \( \square \)

Setting \( N^- := \left\{ \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} : x \in \mathbb{R} \right\} \), the product map \( N \times A \times N^- \to G \) is a diffeomorphism at a neighborhood of \( e \).

Let \( dk \) be the invariant probability measure on \( K \) and denote by \( dg \) the Haar measure on \( G \): \( dg = \frac{1}{y^d} dx dy dk \) for \( g = n_x a_y k \). Let \( \nu \) be a smooth measure on \( AN^- \) such that \( dn_x \otimes d\nu(a_y n_x^-) = dg \). When \( \infty \notin \Lambda(\Gamma) \), fix a bounded open interval \( J \) which contains \( \Lambda(\Gamma) \) and choose a compactly supported smooth function \( 0 \leq \eta \leq 1 \) on \( N \) with \( \eta|_J = 1 \). Otherwise, let \( \eta = 1 \) on \([-x_0, x_0]\), which is a fundamental domain for \( N \cap \Gamma \backslash N \). We denote by \( U_\epsilon \) the \( \epsilon \)-neighborhood of \( e \) in \( G \). Fix \( \epsilon_0 > 0 \) so that the multiplication map
\[
supp(\eta) \times (U_{\epsilon_0} \cap AN^-) \to supp(\eta) \left( U_{\epsilon_0} \cap AN^- \right) \subset \Gamma \backslash G
\]
is a bijection onto its image. For each \( 0 < \epsilon < \epsilon_0 \), let \( r_\epsilon \) be a non-negative smooth function in \( AN^- \) whose support is contained in \( W_\epsilon := (U_\epsilon \cap A)(U_{\epsilon_0} \cap N^-) \) and \( \int_{W_\epsilon} r_\epsilon \, d\nu = 1 \).
We define the following function $\rho_{\eta, \epsilon}$ on $\Gamma \backslash G$:

$$\rho_{\eta, \epsilon}(g) = \begin{cases} \eta(n_x) \cdot r_{\epsilon}(a_y n_x^\epsilon) & \text{for } g = n_x a_y n_x^{-\epsilon} \in \text{supp}(\eta) W_\epsilon \\ 0 & \text{for } g \notin \text{supp}(\eta) W_\epsilon. \end{cases}$$

The inner product on $L^2(\Gamma \backslash G)$ is given by

$$\langle \psi_1, \psi_2 \rangle = \int_{\Gamma \backslash G} \overline{\psi_1(g)} \psi_2(g) \, dg.$$

Set $c_\ell = c_0 \frac{\Gamma(\delta) \Gamma(\ell+1-\delta)}{\Gamma(1-\delta) \Gamma(\delta+\ell)}$ so that for each $\ell \in \mathbb{Z}_{\geq 0}$,

$$\phi_{N, \ell}(a_y) = c_\ell y^{1-\delta}$$

by Theorem 3.4.

**Proposition 4.3.** For any $\ell \in \mathbb{Z}_{\geq 0}$, we have for all positive $\epsilon \ll 1$,

$$\langle a_y \phi_{\pm \ell}, \rho_{\eta, \epsilon} \rangle = c_\ell y^{1-\delta} + O_\eta(\ell y y^\delta)$$

with the implied constants independent of $\ell$ and $y$.

**Proof.** For $x \in \mathbb{R}$, set $\theta_x := -\arctan x$. Then we compute

$$n_x^- = n_\sin \theta_x a_{\cos^2 \theta_x \kappa \theta_x}.$$

For $h = a_{y_0 n_x^-} \in W_\epsilon$, $n_x \in N$ and $y > 0$, we can write

$$n_x h a_y = n_x n_\sin \theta_\kappa \cos^2 \theta_\kappa a_{y_0 y \cos^2 \theta_\kappa} \kappa \theta_\kappa.$$

Since $y_0 = 1 + O(\epsilon)$ and $x_0 = O(1)$, we have

(1) $\cos^2 \theta_\kappa = 1 + O(\epsilon)$;
(2) $y_0 \cos^2 \theta_\kappa = 1 + O(\epsilon)$
(3) $y_0 y \sin \theta_\kappa = O(\epsilon)$
(4) $e^{2\ell \kappa \theta_\kappa} = 1 + O(\ell \epsilon)$

where the implied constants are independent of $0 < y < 1$.

For $y_1 := y_0 y \cos^2 \theta_\kappa = y(1 + O(\epsilon))$ and $x_1 := -y_0 y \sin \theta_\kappa = O(y)$,

$$\phi_\ell(n_x h a_y) = e^{2\ell \kappa \theta_\kappa} \phi_\ell(n_{x+x_1} a_{y_1}) = (1 + O(\ell \epsilon)) \cdot \phi_\ell(n_{x+x_1} a_{y_1})$$

and hence

$$\int_{(N \cap \Gamma) \backslash N} \phi_\ell(n_x h a_y) \cdot \eta(n_x) \, dx = (1 + O(\ell \epsilon)) \int_{(N \cap \Gamma) \backslash N} \phi_\ell(n_x a_{y_1}) (\eta(n_x) + O(y)) \, dx$$

where the implied constants are independent of $\ell$. 
Hence using Lemma 4.2 and \( c_\ell = O(1) \), we deduce
\[
\int_{(N\cap\Gamma)\setminus N} \phi_\ell(n_xha_y) \eta(n_x) \, dx \\
= (1 + O(\ell\epsilon)) \int_{(N\cap\Gamma)\setminus N} \phi_\ell(n_xa_{y_1}) (\eta(n_x) + O(y)) \, dx \\
= \int_{(N\cap\Gamma)\setminus N} \phi_\ell(n_xa_{y_1}) \eta(n_x) \, dx + O(\ell\epsilon\phi_\ell^N(a_{y_1})) \\
= c_\ell y_1^{1-\delta} + O(\ell\epsilon y_1^{1-\delta}) + O(\ell y_1^\delta) \\
= c_\ell y_1^{1-\delta} + O(\ell\epsilon y_1^{1-\delta}) + O(\ell y_1^\delta).
\]

Since \( \int r_\epsilon d\nu(h) = 1 \), it follows that
\[
\langle a_y\phi, \rho_\eta, \epsilon \rangle = \int W_\epsilon r_\epsilon(h) \int_{\Gamma \setminus N \setminus N} \phi_\ell(n_xha_y) \eta(n_x) \, dx \, d\nu(h) \\
= c_\ell y_1^{1-\delta} + O(\ell\epsilon y_1^{1-\delta}) + O(\ell y_1^\delta).
\]

5. Equidistribution of a closed horocycle

For \( \psi \in C^\infty_c(\Gamma \setminus G) \), our goal is to compute
\[
\psi^N(a_y) := \int_{(N\cap\Gamma)\setminus N} \psi(n_xa_y) \, dx
\]
in terms of \( \phi^N_\ell(a_y) \) for \( \ell \in \mathbb{Z} \).

Let \( \{Z_1, Z_2, Z_3\} \) be a basis of the Lie algebra of \( G \). For \( \psi \in C^\infty_c(\Gamma \setminus G) \cap L^2(\Gamma \setminus G) \), and \( m \geq 1 \), we consider the following Sobolev norm \( S_m(\psi) \):
\[
S_m(\psi) = \max\{\|Z_{i_1} \cdots Z_{i_n}(\psi)\|_2 : 1 \leq i_j \leq 3, \ 0 \leq n \leq m\}.
\]

Lemma 5.1. Fix \( m \in \mathbb{N} \). For any \( \psi \in C^\infty_c(\Gamma \setminus G) \cap L^2(\Gamma \setminus G) \) and for all \( |\ell| \) large,
\[
|\langle \psi, \phi_\ell \rangle| \ll (|\ell| + 1)^{-m} S_m(\psi).
\]

In particular,
\[
\sum_{\ell \in \mathbb{Z}} |c_\ell \langle \psi, \phi_\ell \rangle| < \infty.
\]

Proof. The element \( H = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) in the Lie algebra of \( G \) corresponds to the differential operator \( \frac{\partial}{\partial \theta} \) and \( \frac{\partial}{\partial \theta} \phi_\ell = 2\ell i \phi_\ell \). Hence
\[
\left\langle \frac{\partial}{\partial \theta} \psi, \phi_\ell \right\rangle = \left\langle \psi, -\frac{\partial}{\partial \theta} \phi_\ell \right\rangle = 2\ell i \langle \psi, \phi_\ell \rangle.
\]
Similarly,
\[
\left| \left\langle \frac{\partial^m}{\partial \theta^m} \psi, \phi_\ell \right\rangle \right| = 2^m \ell^m \left| \langle \psi, \phi_\ell \rangle \right|.
\]

Hence
\[
\left| \langle \psi, \phi_\ell \rangle \right| \ll \frac{1}{(|\ell| + 1)^m} \left\| \frac{\partial^m}{\partial \theta^m} \psi \right\|_2
\]
proving the first claim. Since \( c_\ell = O(1) \), the second claim follows. \( \square \)

Fix \( 1/2 < s_1 < \delta \) so that there is no eigenvalue of the Laplacian between \( s_1(1 - s_1) \) and \( \delta(1 - \delta) \) in \( L^2(\Gamma \setminus \mathbb{H}^2) \).

**Lemma 5.2.** For any \( \psi_1, \psi_2 \in C^\infty(\Gamma \setminus G) \) with \( S_1(\psi_i) < \infty \), and \( 0 < y < 1 \), we have
\[
\langle a_y \psi_1, \psi_2 \rangle = \sum_{\ell \in \mathbb{Z}} \langle \psi_1, \phi_\ell \rangle \langle a_y \phi_\ell, \psi_2 \rangle + O \left( y^{1-s_1} \cdot S_1(\psi_1) \cdot S_1(\psi_2) \right).
\]

**Proof.** We have \( L^2(\Gamma \setminus G) = V_\delta \oplus V_\delta^\perp \) where \( V_\delta^\perp \) does not contain any complementary series \( V_s \) with parameter \( s > \delta \). We can write
\[
\psi_1 = \sum_{\ell \in \mathbb{Z}} \langle \psi_1, \phi_\ell \rangle \phi_\ell + \psi_1^\perp
\]
with \( \psi_1^\perp \in V_\delta^\perp \) since \( \langle \psi_1 - \sum_{\ell \in \mathbb{Z}} \langle \psi_1, \phi_\ell \rangle \phi_\ell, \phi \rangle = 0 \) for any \( \phi \in V_\delta \). Hence
\[
\langle a_y \psi_1, \psi_2 \rangle = \sum_{\ell \in \mathbb{Z}} \langle \psi_1, \phi_\ell \rangle \langle a_y \phi_\ell, \psi_2 \rangle + \langle a_y \psi_1^\perp, \psi_2 \rangle.
\]

On the other hand, by the assumption on \( s_1 \), we have (cf. the proof of corollary 5.6 in [13])
\[
\langle a_y \psi_1^\perp, \psi_2 \rangle \ll y^{1-s_1} \cdot S_1(\psi_1) \cdot S_1(\psi_2).
\]
This implies the claim. \( \square \)

We refer to [12] for the next lemma:

**Lemma 5.3.** For \( \psi \in C_c^\infty(\Gamma \setminus G) \), there exists \( \widehat{\psi} \in C_c^\infty(\Gamma \setminus G) \) such that

1. for all small \( \epsilon > 0 \), and \( h \in U_\epsilon \),
\[
| \psi(g) - \psi(gh) | \leq \epsilon \cdot \widehat{\psi}(g)
\]
for all \( g \in \Gamma \setminus G \).

2. For all \( m \in \mathbb{N} \), \( S_m(\widehat{\psi}) \ll S_m(\psi) \) where the implied constant depends only on \( \text{supp}(\psi) \).

**Lemma 5.4.** Let \( \infty \notin \Lambda(\Gamma) \). For a fixed compact subset \( Q \) of \( G \), there exists a bounded subset \( J \subset \mathbb{R} \) such that \( n_x a_y \notin \Gamma Q \) for all \( x \notin J \) and any \( 0 < y < 1 \).
Proof. If not, there exist sequences \( x_j \to \infty, y_j \in \mathbb{R}, \gamma_j \in \Gamma \) and \( w_j \in Q \) such that \( n_{x_j} a_{y_j} = \gamma_j w_j \). As \( Q \) is compact, we may assume \( w_j \to w \in Q \). On the other hand, \( n_{x_j} a_{y_j}(i) = x_j + y_j i \to \infty \) as \( x_j \to \infty \). Hence \( \gamma_j(w) \to \infty \), implying that \( \infty \in \Lambda(\Gamma) \), contradiction. \( \square \)

**Theorem 5.5.** For any \( \psi \in C_c^\infty(\Gamma \setminus G) \)

\[
\psi^N(a_y) = \sum_{\ell \in \mathbb{Z}} c_\ell \langle \psi, \phi_\ell \rangle y^{1-\delta} + O(S_3(\psi) y^{1-\delta + \frac{2s \epsilon}{\delta}}).
\]

Proof. By Lemma 5.4, there exists a bounded open subset \( J \) such that \( \psi(n_{x} a_y) = 0 \) for all \( x \not\in J \) and all \( 0 < y < 1 \). When \( \infty \not\in \Lambda(\Gamma) \), we will assume that \( J \) contains \( \Lambda(\Gamma) \), by enlarging \( J \) if necessary, and otherwise \( J = (-x_0, x_0) \).

Choose a non-negative function \( \eta \in C_c^\infty(N \cap \Gamma \setminus N) \) such that \( \eta_{|J} = 1 \). Then

\[
I_\eta(\psi)(a_y) := \int_{(N \cap \Gamma) \setminus N} \psi(n_{x} a_y) \eta(n_x) \, dx = \psi^N(a_y).
\]

Let \( \epsilon_0, W_\epsilon, r_\epsilon, \) and \( \rho_{0,\epsilon} \) be as defined in section 4 with respect to this \( J \) and \( \eta \). Since \( r_\epsilon \) is the approximation of the identity in the \( A \) direction, \( S_1(\rho_{0,\epsilon}) = O_\eta(\epsilon^{-3/2}) \). For any \( 0 < y < 1 \), and any small \( \epsilon > 0 \), we have (see the proof of [13, Prop. 6.6])

\[
|I_\eta(\psi)(a_y) - \langle a_y \psi, \rho_{0,\epsilon} \rangle| \ll (\epsilon + y) \cdot I_\eta(\psi)(a_y). \tag{5.6}
\]

Fix \( 1/2 < s_1 < \delta \) as in Lemma 5.2. Let \( k \) be an integer bigger than \( \frac{s(1-\delta)}{2(\delta-s_1)} + 1 \). Setting \( \psi_0(g) := \psi(g) \), we define for \( 1 \leq j \leq k \), inductively

\[
\psi_j(g) := \psi_{j-1}(g)
\]

where \( \psi_{j-1} \) is given by Lemma 5.3. Applying Lemma 5.6 to each \( \psi_j \), we obtain for \( 0 \leq j \leq k - 1 \),

\[
I_\eta(\psi_j)(a_y) = \langle a_y \psi_j, \rho_{0,\epsilon} \rangle + O(\epsilon + y) \cdot I_\eta(\psi_{j})(a_y)
= \langle a_y \psi_j, \rho_{0,\epsilon} \rangle + O((\epsilon + y) \cdot I_\eta(\psi_{j+1})(a_y))
\]

and

\[
I_\eta(\psi_j)(a_y) = \langle a_y \psi_j, \rho_{0,\epsilon} \rangle + O_y((\epsilon + y)S_1(\psi_k)).
\]

Note

\[
|\langle \psi_j, \phi_\ell \rangle| = (|\ell| + 1)^{-3} O(S_3(\psi))
\]

by Lemmas 5.1 and 5.3.

Since \( \langle a_y \phi_\ell, \rho_{0,\epsilon} \rangle = O(\epsilon y^{1-\delta}) \) by Proposition 4.3, we deduce

\[
\sum_{\ell \in \mathbb{Z}} |\langle \psi_j, \phi_\ell \rangle \langle a_y \phi_\ell, \rho_{0,\epsilon} \rangle| = \sum_{\ell \in \mathbb{Z}} (|\ell| + 1)^{-2} y^{1-\delta} O(S_3(\psi)) = y^{1-\delta} O(S_3(\psi)).
\]
Hence by Lemma 5.2, we deduce that for each $1 \leq j \leq k - 1$,
\[
\langle a_y \psi_j, \rho_{\eta, \epsilon} \rangle = \sum_{\ell \in \mathbb{Z}} \langle \psi_j, \phi_\ell \rangle \langle a_y \phi_\ell, \rho_{\eta, \epsilon} \rangle + O \left( y^{1-s_1} \cdot S_1(\psi_j) \cdot S_1(\rho_{\eta, \epsilon}) \right)
\]
\[
= O(\mathcal{S}_3(\psi) \cdot y^{1-\delta}) + O(y^{1-s_1} \cdot \mathcal{S}_1(\psi_j) \cdot \mathcal{S}_1(\rho_{\eta, \epsilon}))
\]
\[
= \mathcal{S}_3(\psi) \cdot O(y^{1-\delta} + \epsilon^{-3/2} y^{1-s_1}).
\]
Hence for any $0 < y < \epsilon$, using Proposition 4.3, we deduce
\[
I_\eta(\psi)(a_y) = \langle a_y \psi, \rho_{\eta, \epsilon} \rangle + \sum_{j=1}^{k-1} O \left( \langle a_y \psi, \rho_{\eta, \epsilon} \rangle (\epsilon + y)^j \right) + O_{\psi}(\epsilon + y)^k
\]
\[
= \langle a_y \psi, \rho_{\eta, \epsilon} \rangle + O(\epsilon \cdot y^{1-\delta} + \epsilon^{-3/2} y^{1-s_1} + \epsilon^k)
\]
\[
= \sum_{\ell \in \mathbb{Z}} \langle \psi, \phi_\ell \rangle \langle a_y \phi_\ell, \rho_{\eta, \epsilon} \rangle + O(\mathcal{S}_3(\psi)(\epsilon \cdot y^{1-\delta} + \epsilon^{-3/2} y^{1-s_1} + \epsilon^k))
\]
\[
= \sum_{\ell \in \mathbb{Z}} \langle \psi, \phi_\ell \rangle c_\ell y^{1-\delta} + \mathcal{S}_3(\psi)O(y^{\delta} + \epsilon \cdot y^{1-\delta} + \epsilon^{-3/2} y^{1-s_1} + \epsilon^k).
\]
By equating $\epsilon \cdot y^{1-\delta}$ and $\epsilon^{-3/2} y^{1-s_1}$ we put $\epsilon = y^{2(\delta-s_1)/5}$ and obtain
\[
I_\eta(\psi)(a_y) = \sum_{\ell \in \mathbb{Z}} c_\ell \langle \psi, \phi_\ell \rangle y^{1-\delta} + \mathcal{S}_3(\psi)O(y^{1-\delta + 2(\delta-s_1)/5}).
\]
\]

\begin{remark}
Suppose that $\psi \in C^\infty_c(\Gamma \backslash G)$ is a real valued function. Since $c_\ell = c_{-\ell}$ and $\phi_{-\ell} = \phi_\ell$ for each $\ell \in \mathbb{Z}$, we have
\[
\sum_{\ell \in \mathbb{Z}} c_\ell \langle \psi, \phi_\ell \rangle \in \mathbb{R}
\]
as expected.
\end{remark}

6. **Comparison of main terms and Burger-Roblin measure as a distribution**

Recall the Patterson measure $\nu_i = \nu_i^\Gamma$ on the boundary and $\phi_0 = \phi_0^\Gamma$ given by
\[
\phi_0(x + iy) = \int_{\mathbb{R}} \left( \frac{k' + 1}{(x - u)^2 + y^2} \right)^{\delta} d\nu_i(u)
\]
from section 1. Note that
\[
\phi_0^\Gamma(\epsilon) = |\nu_i^\Gamma|.
\]
As before, we normalize $\nu_i$ so that $\|\phi_0\|_2 = 1$.

For $\xi \in \partial(\mathbb{H}^2)$ and $z_1, z_2 \in \mathbb{H}^2$, recall the Busemann function:
\[
\beta_\xi(z_1, z_2) = \lim_{s \to \infty} d(z_1, \xi_s) - d(z_2, \xi_s)
\]
where $\xi_s$ is a geodesic ray tending to $\xi$ as $s \to \infty$.

Using the identification of $T^1(\mathbb{H}^2)$ and $G$, we give the definition of the Bowen-Margulis-Sullivan measure $m^{BMS}$ on $\Gamma \backslash G$. For $u \in T^1(\mathbb{H}^2)$, we denote by $u^+$ and $u^-$ the forward and the backward endpoints of the geodesic determined by $u$, respectively. The correspondence

$$u \mapsto (u^+, u^-, t := \beta_{u^-}(i, \pi(u)))$$

gives a homeomorphism between the space $T^1(\mathbb{H}^2)$ with $((\xi, \xi) : \xi \in \partial(\mathbb{H}^2)) \times \mathbb{R}$ where $\pi : G \to G/K = \mathbb{H}^2$ is the canonical projection. Define the measure $\tilde{m}^{BMS}$ on $G$:

$$d\tilde{m}^{BMS}(u) = e^{\delta \beta_{u^+}(i, \pi(u))} e^{\delta \beta_{u^-}(i, \pi(u))} \nu_i(u^+) \nu_i(u^-) dt$$

This measure is left $\Gamma$-invariant and hence induces a measure $m^{BMS}$ on $\Gamma \backslash G$. As $\Gamma$ is finitely generated, we have $|m^{BMS}| < \infty$.

Roblin obtained the following in his thesis [20]:

**Theorem 6.1** (Roblin). For $\delta > 1/2$,

$$\|\phi_0\|_2^2 = |m^{BMS}| \int_{\mathbb{R}} \frac{dx}{(1 + x^2)^\delta}$$

As we have normalized $\nu_i$ so that $\|\phi_0\|_2 = 1$ and $\phi_0(i) = |\nu_i|$, we deduce $\frac{1}{|m^{BMS}|} = \int_{\mathbb{R}} \frac{dx}{(1 + x^2)^\delta}$. As $\delta > \frac{1}{2}$, we have

$$\frac{\sqrt{\pi} \Gamma(\delta - \frac{1}{2})}{\Gamma(\delta)} = \int_{-\infty}^{\infty} \frac{dx}{(1 + x^2)^\delta}.$$

To describe the equidistribution result of $N \cap \Gamma \backslash N a_y$ from [19], we recall $m^{BR}_N$ and $\mu^{PS}_N$. First, define the measure $\tilde{m}^{BR}_N$ on $G$ as follows: for $\psi \in C_c(G)$,

$$d\tilde{m}^{BR}_N(u) = e^{\delta \beta_{u^+}(i, \pi(u))} e^{\delta \beta_{u^-}(i, \pi(u))} \nu_i(u^+) \nu_i(u^-) dt$$

where $m_i$ is the $K$-invariant probability measure on $\partial(\mathbb{H}^2)$.

This measure is left $\Gamma$-invariant and right $N$-invariant, and the Burger-Roblin measure $m^{BR}_N$ (associated to the stable horocyclic subgroup $N$) is the measure on $X$ induced from $\tilde{m}^{BR}_N$.

Consider the measure $\mu^{PS}_N$ on $N$ given by

$$d\mu^{PS}_N(n_x) = e^{-\delta \beta_x(i, x+i)} \nu_i(x) = (1 + x^2)^\delta \nu_i(x).$$

This induces a measure on $(N \cap \Gamma) \backslash N$ for which we use the same notation. Since $\mu^{PS}_N$ is supported in $(N \cap \Gamma) \backslash (\Lambda(\Gamma) - \{\infty\})$, we have $\mu^{PS}_N((N \cap \Gamma) \backslash N) < \infty$.
Theorem 6.2. ([19], see also [17]) Let $\delta > 0$ and $(N \cap \Gamma) \backslash N$ be closed. For any $\psi \in C_c(\Gamma \backslash G)$,

$$\lim_{y \to 0} y^{\delta - 1} \psi^N(a_y) = \frac{\mu^N_{\text{PS}}(N \cap \Gamma \backslash N)}{|m^\text{BMS}|} m^\text{BR}_N(\psi).$$

Comparing the main terms of Theorem 6.2 and Theorem 5.5 and using Theorem 6.1, we deduce the following interesting identity of the Burger-Roblin measure considered as a distribution on $\Gamma \backslash G$:

Theorem 6.3. Let $\delta > 1/2$ and $(N \cap \Gamma) \backslash N$ be closed. For any $\psi \in C^\infty_c(\Gamma \backslash G)$,

$$\kappa_\Gamma \cdot m^\text{BR}_N(\psi) = \sum_{\ell \in \mathbb{Z}} c_\ell \langle \psi, \phi_\ell \rangle$$

where $\kappa_\Gamma = \sqrt{\frac{\pi^\Gamma(\delta - \frac{1}{2})}{\Gamma(\delta)}} \int_{(N \cap \Gamma) \backslash N} (1 + x^2)^\delta d\nu_i(x)$.

Hence Theorem 1.3 follows from Theorems 1.1 and 6.3.

7. Application to counting in sectors

7.1. Let $Q$ be a ternary indefinite quadratic form and $v_0 \in \mathbb{R}^3$ be any vector such that $Q(v_0) = 0$. Let $\Gamma_0 < \text{SO}_Q^\circ(\mathbb{R})$ be a finitely generated discrete subgroup with $\delta > 1/2$. Suppose that $v_0 \Gamma_0$ is discrete.

Let $\| \cdot \|$ be any norm in $\mathbb{R}^3$. We can find a representation $\iota : G = \text{PSL}_2(\mathbb{R}) \to \text{SO}_Q(\mathbb{R})$ so that the stabilizer of $v_0$ in $\text{PSL}_2(\mathbb{R})$ via $\iota$ is the upper triangular subgroup $N$. Let $\Omega \subset K$ be a Borel subset and consider the sector

$$S_T(\Omega) := \{ v \in v_0 \cdot A \Omega : \| v \| < T \}$$

given by $\Omega$.

Theorem 7.1. Suppose that $\Omega$ has only finitely many connected components. Then

$$\# \{ v \in v_0 \Gamma_0 \cap S_T(\Omega) \} = \frac{\kappa_{\iota^{-1}(\Gamma_0)}^{-1}}{\delta} \left( \int_{k^{-1} \in \Omega} \frac{d\nu_i(k(0))}{\| v_0 k^{-1} \|^\delta} \right) T^\delta + O(T^\delta \cdot \frac{4s_5}{55}).$$

Theorem 7.1 was obtained in [17] but without an error term. Deducing this theorem from Theorem 1.1 is a verbatim repetition of the arguments in the section 8 of [15]. For the sake of completeness, we give a brief sketch here.

Let $\Gamma := \iota^{-1}(\Gamma_0)$, and let $U_\epsilon$ be an $\epsilon$-neighborhood of $e$ in $G$.

By the assumption on $\Omega$, the boundary of $\Omega^{-1}(0)$ consists of finitely many points and hence $\nu_i(\partial(\Omega^{-1}(0))) = 0$, as $\nu_i$ is atom-free. Moreover
for all sufficiently small \( \epsilon > 0 \), there exists an \( \epsilon \)-neighborhood \( K_\epsilon \) of \( e \) in \( K \) such that for \( \Omega_{\epsilon^+} = \Omega K_\epsilon \) and \( \Omega_{\epsilon^-} = \cap_{k \in K_\epsilon} \Omega k \),

\[
\nu_e(\Omega_{\epsilon^+}^{-1}(0) - \Omega_{\epsilon^-}^{-1}(0)) \leq \epsilon. \tag{7.2}
\]

By the strong wave front lemma \cite[Theorem 4.1]{9}, there exists \( 0 < \ell_0 < 1 \) such that for \( T \gg 1 \),

\[
S_T(\Omega) U_{\ell_0} \subset S_{(1+\epsilon)T} (\Omega_{\epsilon^+}) \quad \text{and} \quad S_{(1-\epsilon)T} (\Omega_{\epsilon^-}) \subset \cap_{u \in U_{\ell_0}} S_T(\Omega)u.
\]

Let \( \psi \in C_\infty^c(G) \) be a non-negative function supported in \( U_{\ell_0} \) with integral one, and set \( \Psi(\epsilon) = \sum_{\gamma \in \Gamma} \psi(\gamma g) \).

Define the counting function \( F^\Omega_T \) on \( \Gamma \setminus G \) by

\[
F^\Omega_T(g) = \sum_{\gamma \in \mathbb{N} \cap \Gamma \setminus \Gamma} \chi_{S_T(\Omega)}(v_0 \gamma g).
\]

Then

\[
\langle F^\Omega_{(1-\epsilon)T}, \Psi \rangle \leq F^\Omega_T(\epsilon) \leq \langle F^\Omega_{(1+\epsilon)T}, \Psi \rangle.
\]

Let \( \Psi^k(\epsilon) := \Psi \epsilon(\epsilon k) \).

The following is a special case of \cite[Prop. 6.2]{17} (see also \cite[Sec. 7]{13}):

**Proposition 7.3.** Suppose that \( \nu_e(\partial(\Omega^{-1}(0))) = 0 \). For all small \( \epsilon > 0 \),

\[
\int_{k \in \Omega} \frac{m_N^B(\Psi^k)}{\|v_0 k\|^{\delta}} dk = \int_{k \in \Omega^{-1}} \frac{d\nu_e(k(0))}{\|v_0 k^{-1}\|^{\delta}}(1 + O(\epsilon)).
\]

**Proposition 7.4.** For all \( T \gg 1 \)

\[
\langle F^\Omega_T, \Psi \epsilon \rangle_{L^2(\Gamma \setminus G)} = \frac{\kappa_T}{\delta} T^{\delta} \int_{k \in \Omega^{-1}} \frac{d\nu_e(k(0))}{\|v_0 k^{-1}\|^{\delta}} + O(\epsilon T^\delta + \epsilon^{-9/2} T^{\delta - 2s_\Gamma/5}).
\]

**Proof.** We only sketch a proof here and refer to \cite{15} for details.

\[
\langle F^\Omega_T, \Psi \epsilon \rangle_{L^2(\Gamma \setminus G)} = \int_{k \in \Omega} \int_{y > \|v_0 k\|^{\delta}} \left( \int_{N \cap \Gamma \setminus N} \Psi \epsilon (n_x a_y k) dx \right) y^{-2} dy dk.
\]

By Theorem \ref{thm:main} and Corollary \ref{cor:main}

\[
\int_{N \cap \Gamma \setminus N} \Psi \epsilon (n_x a_y k) dx = \kappa_T \cdot m_N^B(\Psi^k) \cdot y^{1-\delta} + O(\epsilon^{-9/2} y^{(1-\delta) + 2s_\Gamma/5})
\]

as \( S_\delta(\Psi \epsilon) = \epsilon^{-9/2} \). We then deduce, using Proposition \ref{prop:main}

\[
\langle F^\Omega_T, \Psi \epsilon \rangle_{L^2(\Gamma \setminus G)} = \frac{\kappa_T}{\delta} T^{\delta} \int_{k \in \Omega^{-1}} \frac{d\nu_e(k(0))}{\|v_0 k^{-1}\|^{\delta}}(1 + O(\epsilon)) + O(\epsilon^{-9/2} T^{\delta - 2s_\Gamma/5}).
\]
By Proposition [7.4]
\[ \langle F_{\Omega, \pm}^{\Gamma}, \Psi \rangle = \frac{k_{\Gamma} \cdot T^\delta}{\delta} \int_{k \in \Omega, \pm^{-1}} \frac{d\nu_i(k(0))}{\|v_0 k^{-1}\|^\delta} + O(\epsilon T^\delta + \epsilon^{-9/2} T^\delta - 2s_1/5). \]

Therefore by solving \( \epsilon^{-9/2} T^{-2s_1/5} = \epsilon \) for \( \epsilon \), we finish the proof of Theorem 7.1.

7.2. Let \( Q \) be a ternary indefinite quadratic form over \( \mathbb{Q} \) and \( v_0 \in \mathbb{Q}^3 \) be any vector such that \( Q(v_0) = 0 \). Let \( \Gamma < \text{SO}_Q(\mathbb{Z}) \) be a finitely generated subgroup with \( \delta > 1/2 \).

For a square-free integer \( d \), consider the subgroup of \( \Gamma \) which stabilizes \( v_0 \mod d \):
\[ \Gamma_d := \{ \gamma \in \Gamma : v_0 \gamma \equiv v_0 \mod d \}. \]

Clearly it satisfies \( \text{Stab}_{\Gamma} v_0 = \text{Stab}_{\Gamma_d} v_0 \).

By Bourgain, Gamburd and Sarnak [3], \( L^2(\Gamma_d \setminus \mathbb{H}^2) \) has a uniform spectral gap, that is, there exists \( s_0 > 0 \) such that if the second smallest eigenvalue of the Laplacian in \( L^2(\Gamma_d \setminus \mathbb{H}^2) \) is \( s_1(d)(1 - s_1(d)) \), then \( s_1(d) + s_0 < \delta \) for all square-free \( d \) (note that \( \delta(1 - \delta) \) is the bottom of the spectrum for all \( \Gamma_d \)).

Set
\[ \Xi_{v_0}(\Gamma, \Omega) := \frac{\kappa_{\text{inv}}^{-1}(\Gamma)}{\delta} \int_{k \in \Omega, \pm^{-1}} \frac{d\nu_i^{\Gamma}(k(0))}{\|v_0 k^{-1}\|^\delta}. \]

Since the Patterson measure \( \nu_i^{\Gamma} \) is normalized so that \( \phi_0^{\Gamma}(e) = |\nu_i^{\Gamma}| \) and \( \|\phi_0^{\Gamma}\|_2 = 1 \), we note that
\[ \nu_i^{\Gamma_d} = \frac{1}{\sqrt{[\Gamma : \Gamma_d]}} \nu_i^{\Gamma}. \]

Therefore \( \kappa_{\text{inv}}(\Gamma_d) = \frac{1}{\sqrt{[\Gamma : \Gamma_d]}} \kappa_{\text{inv}}(\Gamma) \) and hence
\[ \Xi_{v_0}(\Gamma_d, \Omega) = \frac{\Xi_{v_0}(\Gamma, \Omega)}{[\Gamma : \Gamma_d]}. \]

Hence Theorem 7.1 implies Theorem 1.4.

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