On Probabilistic Analog Automata

Asa Ben-Hur∗ Alexander Roitershtein† Hava T. Siegelmann‡

February 1, 2008

Abstract

We consider probabilistic automata on a general state space and study their computational power. The model is based on the concept of language recognition by probabilistic automata due to Rabin [12] and models of analog computation in a noisy environment suggested by Maass and Orponen [7], and Maass and Sontag [8]. Our main result is a generalization of Rabin’s reduction theorem that implies that under very mild conditions, the computational power of the automaton is limited to regular languages.

Keywords: probabilistic automata, probabilistic computation, noisy computational systems, regular languages, definite languages.

1 Introduction

Probabilistic automata have been studied since the early 60’s [11]. Relevant to our line of interest is the work of Rabin [12] where probabilistic (finite) automata with isolated cut-point were introduced. He showed that such automata recognize regular languages, and identified a condition which restricts them to definite languages (languages for which there exists an integer \( r \) such that any two words coinciding on the last \( r \) symbols are both or neither in the language).

Paz generalized Rabin’s condition for definite languages and called it weak ergodicity. He showed that Rabin’s stability theorem holds for weakly ergodic systems as well [10] [11].

In recent years there is much interest in analog automata and their computational properties. A model of analog computation in a noisy environment was introduced by Maass and Orponen in [7]. For a specific type of noise it recognizes only regular languages (see also [2]). Analog neural networks with Gaussian-like noise were shown by Maass and Sontag [8] to be limited in their language-recognition power to definite languages. This is in sharp contrast

∗Department of Biochemistry, B400 Beckman Center, Stanford University, CA 94305-5307, USA.
†Department of Mathematics, Technion - IIT, Haifa 32000, Israel (e-mail: roiterst@tx.technion.ac.il).
‡Department of Computer Science, University of Massachusetts at Amherst, 710 N. Pleasant Street Amherst, MA 01003-9305 USA.
with the noise-free case where analog computational models are capable of simulating Turing machines, and when containing real constants, can recognize non-recursive languages [13].

In this work we propose a model which includes the discrete model of Rabin and the analog models suggested in [7, 8], and find general conditions (related to ergodic properties of stochastic kernels representing probabilistic transitions of the automaton) that restrict its computational power to regular and definite languages.

We denote the state space of the automaton by $\Omega$ and the alphabet by $\Sigma$. As usual, the set of all words of length $r$ is denoted by $\Sigma^r$ and $\Sigma^* := \cup_{r \in \mathbb{N}} \Sigma^r$. We assume that $\Omega$ is a Polish space and denote by $\mathcal{B}$ the $\sigma$-algebra of its Borel subsets.

Let $E$ be the Banach space of signed measures on $(\Omega, \mathcal{B})$ with the total variation norm

$$\|\mu\|_1 := \sup_{A \in \mathcal{B}} \mu(A) - \inf_{A \in \mathcal{B}} \mu(A),$$

and let $L$ be the space of bounded linear operators in $E$ with the norm $\|P\|_1 = \sup_{\|\mu\|_1 = 1} \|P\mu\|_1$.

**Definition 1.1.** An operator $P \in L$ is said to be a Markov operator if for any probability measure $\mu$, the image $P\mu$ is again a probability measure. A Markov system is a set of Markov operators $T = \{P_u : u \in \Sigma\}$.

With any Markov system $T$, one can associate a probabilistic computational system as follows. At each computation step the system receives an input signal $u \in \Sigma$ and updates its state. If the probability distribution on the initial state $s$ is given by the probability measure $\mu_0$, then the distribution of states after $n+1$ computational steps on inputs $w = w_0, w_1, ..., w_n$, is defined by

$$P_w \mu_0 = P_{w_n} \cdots P_{w_1} P_{w_0} \mu_0.$$ 

If the probability of moving from state $x \in \Omega$ to set $A \in \mathcal{B}$ upon receiving input $u \in \Sigma$ is given by a stochastic kernel $P_u(x, A)$, then $P_u \mu(A) = \int_{\Omega} P_u(x, A) \mu(dx)$.

Let $\mathcal{A}$ and $\mathcal{R}$ be two subsets of $\mathcal{P}$ with the property of having a $\rho$-gap

$$\text{dist}(\mathcal{A}, \mathcal{R}) = \inf_{\mu \in \mathcal{A}, \nu \in \mathcal{R}} \|\mu - \nu\|_1 = \rho > 0 \quad (1.2)$$

A Markov computational system becomes a language recognition device by agreement that an input string is accepted or rejected according to whether the distribution of states of the MCS after reading the string is in $\mathcal{A}$ or in $\mathcal{R}$.

Finally, we have the definition:

**Definition 1.3.** Let $\mu_0$ be an initial distribution and $\mathcal{A}$ and $\mathcal{R}$ be two bounded subsets of $\mathcal{E}$ that satisfy (1.2). Let $T = \{P_u : u \in \Sigma\}$ be a set of Markov operators on $\mathcal{E}$. We say that the Markov computational system (MCS) $\mathcal{M} = \langle \mathcal{E}, \mathcal{A}, \mathcal{R}, \Sigma, \mu_0, T \rangle$ recognizes the subset $L \subseteq \Sigma^*$ if for all $w \in \Sigma^*$:

$$w \in L \Leftrightarrow P_w \mu_0 \in \mathcal{A}$$

$$w \notin L \Leftrightarrow P_w \mu_0 \in \mathcal{R}.$$
We recall that two words \( u, v \in \Sigma^* \) are equivalent with respect to \( L \) if and only if \( uw \in L \iff vw \in L \) for all \( w \in \Sigma^* \). A language \( L \subseteq \Sigma^* \) is regular if there are finitely many equivalence classes. \( L \) is definite if for some \( r > 0 \), \( uw \in L \iff u \in L \) for all \( w \in \Sigma^* \) and \( u \in \Sigma^r \). If \( \Sigma \) is finite, then definite languages are regular.

A quasi-compact MCS can be characterized as a system such that \( \Sigma \) is finite and there is a set of compact operators \( \{ Q_w \in \mathcal{L} : w \in \Sigma^* \} \) such that \( \lim_{|w| \to \infty} \| P_w - Q_w \|_1 = 0 \). Section 2 is devoted to MCS having this property. Our main result (Theorem 1) states that quasi-compact MCS can recognize regular languages only. As a consequence of this result, we obtain the following theorem which shows that “any reasonable” probabilistic automata recognize regular languages only:

**Theorem.** Let \( \mathcal{M} \) be an MCS. Assume that \( \Sigma \) is finite, and there exist constant \( K > 0 \) and probability measure \( \mu \) such that \( P_u(x, A) \leq K \mu(A) \) for all \( u \in \Sigma \), \( x \in \Omega \), \( A \in \mathcal{B} \). Then, if a language \( L \subseteq \Sigma^* \) is recognized by \( \mathcal{M} \), it is a regular language.

A MCS is weakly ergodic if there is a set of constant operators \( \{ H_w \in \mathcal{L} : w \in \Sigma^* \} \) such that \( \lim_{|w| \to \infty} \| P_w - H_w \|_1 = 0 \). In Section 3 we carry over the theory of discrete weakly ergodic systems developed by Paz [10, 11] to our general setup. In particular, if a language \( L \) is recognized by a weakly ergodic MCS, then it is definite language.

### 2 The Reduction Lemma and Quasi-compact MCS

We prove here a general version of Rabin’s reduction theorem (Lemma 2.2) which makes the connection between a measure of non-compactness of the set \( \{ P_w \mu_0 : w \in \Sigma^* \} \) with the computational power of MCS. Then we introduce the notion of quasi-compact MCS and show that these systems satisfy the conditions stated in Lemma 2.2.

If \( S \) is a bounded subset of a Banach space \( E \), Kuratowski’s measure of non-compactness \( \alpha(S) \) of \( S \) is defined by:

\[
\alpha(S) = \inf \{ \varepsilon > 0 : S \text{ can be covered by a finite number of sets of diameter smaller than } \varepsilon \}. \tag{2.1}
\]

A bounded set \( S \) is totally bounded if \( \alpha(S) = 0 \).

**Lemma 2.2.** Let \( \mathcal{M} \) be an MCS, and assume that \( \alpha(\mathcal{O}) < \rho \), where \( \mathcal{O} = \{ P_w \mu_0 : w \in \Sigma^* \} \) is the set of all possible state distributions of \( \mathcal{M} \), and \( \rho \) is defined by \( \| P_w - Q_w \|_1 \). Then, if a language \( L \subseteq \Sigma^* \) is recognized by \( \mathcal{M} \), it is a regular language.

**Proof.** If \( \| P_u \mu_0 - P_v \mu_0 \|_1 < \rho \), then \( u \) and \( v \) are in the same equivalence class. Indeed, for any \( w \in \Sigma^* \),

\[
\| P_{uw} \mu_0 - P_{vw} \mu_0 \|_1 = \| P_w (P_u \mu_0 - P_v \mu_0) \|_1 \leq \| P_u \mu_0 - P_v \mu_0 \|_1 < \rho.
\]

There is at most a finite number of equivalence classes, since there is a finite covering of \( \mathcal{O} \) by sets with diameter less than \( \rho \). \( \square \)

Lemma 2.2 is a natural generalization of Rabin’s reduction theorem [12], where the state space \( \Omega \) is finite, and hence the whole space of probability measures is compact.
Example 2.3. Consider an MCS $\mathcal{M}$ such that $\Omega = \mathbb{N}$ and $\Sigma$ is a finite set. If the sums $\sum_j P_u(i,j)$ converges uniformly for each $u \in \Sigma$, then the corresponding operators $P_u \in \mathcal{L}$ are compact [3], and consequently (since $\mathcal{O} \subset \bigcup_{u \in \Sigma} P_u$) $\mathcal{M}$ recognizes regular languages.

Recall that a Markov operator $P$ is called quasi-compact if there is a compact operator $Q \in \mathcal{L}$ such that $\|P - Q\|_1 < 1$ [9].

Definition 2.4. An MCS $\mathcal{M}$ is called quasi-compact if the alphabet $\Sigma$ is finite, and there exist constants $r, \delta > 0$ such that for any $w \in \Sigma^*$ there is a compact operator $Q_w$ which satisfies $\|P_w - Q_w\|_1 \leq 1 - \delta$.

If an MCS $\mathcal{M}$ is quasi-compact, then there exists a constant $M > 0$ and a collection of compact operators $\{Q_w : w \in \Sigma^*\}$ such that $\|P_w - Q_w\|_1 \leq M(1 - \delta)^{|w|/r}$, for all $w \in \Sigma^*$.

The next theorem characterizes the computational power of quasi-compact MCSs.

Theorem 1. If $\mathcal{M}$ is a quasi-compact MCS, and a language $L \subseteq \Sigma^*$ is recognized by $\mathcal{M}$, then it is a regular language.

Proof. Fix any $\varepsilon > 0$. There exist a number $n \in \mathbb{N}$ and compact operators $Q_w$, $w \in \Sigma^n$ such that $\|P_w - Q_w\|_1 \leq \varepsilon$ for all $w \in \Sigma^n$. For any words $v \in \Sigma^*$ and $w \in \Sigma^n$, we have $\|P_{vw} - Q_{vw}(P_{vw} \mu_0)\|_1 \leq \|P_w - Q_w\|_1 \leq \varepsilon$. Since $Q_w(P_{vw} \mu_0)$ is an element of the totally bounded set $Q_w(P)$, then the last inequality implies that the set $\mathcal{O} = \{P_u \mu_0 : u \in \Sigma^*\}$ can be covered by a finite number of balls of radius arbitrarily close to $\varepsilon$. $\square$

Doeblin’s condition which follows, is a criterion for quasi-compactness (it should not be confused with its stronger version, defined in Section 3 which was used in [8]).

Definition 2.5. Let $P(x, A)$ be a stochastic kernel defined on $(\Omega, \mathcal{B})$. We say that it satisfies Condition D if there exist $\theta > 0, \eta < 1$ and a probability measure $\mu$ on $(\Omega, \mathcal{B})$ such that

$$\mu(A) \geq \theta \Rightarrow P(x, A) \geq \eta \text{ for all } x \in \Omega.$$  

Example 2.6. [7] Condition D holds if $P(x, A) \leq K \mu(A)$ for some $K > 0$ and probability measure $\mu \in \mathcal{E}$ (e.g., $P(x, A) = \int_A p(x, y) \mu(dy)$ and $|p(x, y)| < K$).

Theorem 2. Let $\mathcal{M}$ be an MCS. If $\Sigma$ is finite and for some $n \in \mathbb{N}$, all stochastic kernels $P_w(x, A)$, $w \in \Sigma^n$, satisfy Condition D, then $\mathcal{M}$ is quasi-compact.

The proof, given in Appendix A, follows the proof in [14] that Condition D implies quasi-compactness for an individual Markov operator.

The following lemma, whose proof is deferred to Appendix B, gives a complete characterization of a quasi-compact MCS in terms of its associated Markov operators.

Lemma 2.7. If an MCS $\mathcal{M}$ is quasi-compact, then $\alpha(T^*) = 0$, where $T^* = \{P_w : w \in \Sigma^*\}$.

It is easy to see that $\alpha(\mathcal{O}) < \sup_{u \in \Sigma} \alpha(P_u \mathcal{P}) + \alpha(T)$, where $T = \{P_u : u \in \Sigma\}$. This yields a criterion for quasi-compactness in terms of the associated Markov system $T$ and also suggests generalizations to infinite alphabets, e.g. in the case if $\Sigma$ is a compact set and the map $P(u) = P_u : \Sigma \to \mathcal{L}$ is continuous.
3 Weakly Ergodic MCS

For any Markov operator $P$ define

$$
\delta(P) := \sup_{\mu, \nu \in P} \frac{1}{2} \|P\mu - P\nu\|_1 = \sup_{x, y, A \in \mathcal{B}} |P(x, A) - P(y, A)|.
$$

Then (we refer to [5, 6] for the properties of Dobrushin’s coefficient $\delta(P)$):

$$
\delta(P) = \sup_{\lambda \in \mathcal{N} \setminus \{0\}} \frac{\|P\lambda\|_1}{\|\lambda\|_1},
$$

(3.1)

where $\mathcal{N} = \{\lambda \in \mathcal{E} : \lambda(\Omega) = 0\}$.

**Definition 3.2.** A Markov system $\{P_u, u \in \Sigma\}$ is called weakly ergodic if there exist constants $r, \delta > 0$ such that $\delta(P_w) \leq 1 - \delta$ for any $w \in \Sigma^r$. An MCS $\mathcal{M}$ is called weakly ergodic if its associated Markov system $\{P_u, u \in \Sigma\}$ is weakly ergodic.

It follows from the definition and (3.1) that $\delta(P_w) \leq M(1 - \delta)^{|w|/r}$, for any $w \in \Sigma^*$ and some $M > 0$. Maass and Sontag used a strong Doeblin’s condition to prove the computational power of noisy neural networks [8]. They essentially proved (see also [11, 12]) the following result:

**Theorem 3.** Let $\mathcal{M}$ be a weakly ergodic MCS. If a language $L$ can be recognized by $\mathcal{M}$, then it is definite.

**Definition 3.3.** A Markov operator $P$ satisfies Condition $D_0$ if $P(x, \cdot) \geq c\varphi(\cdot)$ for some constant $c \in (0, 1)$ and a probability measure $\varphi \in \mathcal{P}$.

If a Markov operator $P$ satisfies Condition $D_0$ with a constant $c$, then $\delta(P) \leq 1 - c$ [4]. The following example shows that this condition is not necessary.

**Example 3.4.** Let $\Omega = \{1, 2, 3\}$ and $P(x, y) = \frac{1}{2}$ if $x \neq y$. Then $\delta(P) = \frac{1}{2}$, but $P$ does not satisfy condition $D_0$.

We next state a general version of the Rabin-Paz stability theorem [11, 12]. We first define two MCS, $\mathcal{M}$ and $\mathcal{M}'$ to be similar if they share the same measurable space $(\Omega, \mathcal{B})$, alphabet $\Sigma$, and sets $\mathcal{A}$ and $\mathcal{R}$, and differ only in their Markov operators.

**Theorem 4.** Let $\mathcal{M}$ and $\mathcal{M}'$ be two similar MCS such that the first is weakly ergodic. Then there is $\alpha > 0$, such that if $\|P_u - \tilde{P}_u\|_1 \leq \alpha$ for all $u \in \Sigma$, then the second is also weakly ergodic. Moreover, the two MCS recognize the same language.

For the sake of completeness we give a proof in Appendix [C].
Appendices

A Proof of Theorem 2

Lemma A.1. [14] Let $K(x, A)$ and $N(x, A)$ be two stochastic kernels defined by

$$K(x, A) = \int_A k(x, y)\mu(dx), \quad |k(x, y)| \leq C_K,$$

$$N(x, A) = \int_A n(x, y)\mu(dx), \quad |n(x, y)| \leq C_N,$$

where $k(x, y)$ and $n(x, y)$ are measurable and bounded functions in $\Omega \times \Omega$, and $C_K, C_N$ are constants. Then $NK \in L$ is compact.

The proof in [14] is for a special case, so we give here an alternative proof.

Proof. Let $\{n_m(x, y) : m \in \mathbb{N}\}$ be a set of simple and measurable functions such that

$$\int_\Omega \int_\Omega |n_m(x, y) - n(x, y)|\mu(dx)\mu(dy) \leq \frac{1}{m},$$

and define stochastic kernels $N_m(x, A) = \int_A n_m(x, y)\mu(dy)$. Since the corresponding operators $N_m \in L$ have finite dimensional ranges they are compact. On the other hand

$$\|NK - N_mK\|_1 = \sup_{\|\phi\|_1 = 1} \|NK\phi - N_mK\phi\|_1 \leq C_K/m,$$

thus, $NK = \lim_{m \to \infty} N_mK$ is a compact operator. \qed

Since operators $P_u, u \in \Sigma$ satisfy Condition D, they can be represented as $P_u = Q_u + R_u$, where $Q_u$ is defined by a stochastic kernels having bounded and measurable on $\Omega \times \Omega$ densities $q_u(x, y)$ with respect to $\mu$, and $\|R_u\|_1 \leq 1 - \eta$ [14]. Consider the expansion of $P_w = \prod_{k=0}^m (Q_{w_k} + R_{w_k}), w \in \Sigma^{m+1}$ in $2^{m+1}$ terms:

$$P_w = \prod_{k=0}^m Q_{w_k} + \sum_{j=0}^m \left( \prod_{k=1}^{j-1} Q_{w_k} R_{w_j} \prod_{k=j+1}^m Q_{w_k} \right) + \ldots + \prod_{k=0}^m R_{w_k}.$$  

By Lemma A.1 the terms contains $Q_{w_i}$ at least twice as factor are all compact operators in $L$. Since there are at most $m+2$ terms where $Q_{w_i}$ appear at most once, then we obtain that for any $w \in \Sigma^{m+1}$ there is a compact operator $Q_w$ such that $\|P_w - Q_w\|_1 \leq (m+2) \cdot (1-\eta)^m$.

B Proof of Lemma 2.7

We need the following proposition suggested to us by Leonid Gurvits.

Proposition B.1. Let $Q_1, Q_2 \in L$ be two compact operators, and let $H = \{P_j\} \subseteq L$ be a bounded set of operators. Then, the set $Q = \{Q_2PQ_1 : P \in H\}$ is totally bounded.
Proof. Let \( \mathcal{K} = \{ \mu \in \mathcal{E} : \|\mu\|_1 \leq 1 \} \) and \( X_i \subseteq \mathcal{E} : i = 1, 2 \) be two compact sets such that \( Q_i \mathcal{K} \subseteq X_i \). Define a bounded family \( \mathcal{F} = \{ f_j \} \) of continuous linear functions from \( X_1 \) to \( X_2 \) by setting \( f_j = Q_2 P_j \). Since \( H \) is bounded, then \( \mathcal{F} \subseteq C(X_1, X_2) \) is bounded and equicontinuous, that is by Ascoli’s theorem it is conditionally compact. Fix any \( \varepsilon > 0 \) and consider a finite covering of \( \mathcal{F} \) by balls with radii \( \varepsilon \). If \( f_i \) and \( f_j \) are included in the same ball, then
\[
\| Q_2 P_i Q_1 - Q_2 P_j Q_1 \|_1 \leq \sup_{x \in X_1} \| f_i(x) - f_j(x) \|_1 \leq 2\varepsilon.
\]
Therefore \( \alpha(Q) \leq 2\varepsilon \). This completes the proof since \( \varepsilon \) is arbitrary. \( \square \)

From Proposition B.1 it follows that the set \( \{ Q_u P Q_v : u, v \in \Sigma^n, P \in \mathcal{L}, \|P\|_1 = 1 \} \) is totally bounded.

Fix any \( \varepsilon > 0 \). There exist a number \( \eta \in \mathbb{N} \) and compact operators \( Q_w, w \in \Sigma^n \) such that \( \| P_w - Q_w \|_1 \leq \varepsilon \) for all \( w \in \Sigma^n \). Since any word \( w \in \Sigma^{2n+1} \) can be represented in the form \( w = u\hat{u}w \), where \( u, v \in \Sigma^n \), and
\[
\| P_w - Q_v P_\hat{w} Q_u \|_1 = \| P_v P_\hat{w} P_u - Q_v P_\hat{w} Q_u \|_1 \leq \| P_v P_\hat{w} P_u - P_v P_\hat{w} Q_u \|_1 + \| P_v P_\hat{w} Q_u - Q_v P_\hat{w} Q_u \|_1 \leq \| P_u - Q_u \|_1 + \| P_v - Q_v \|_1 \leq 2\varepsilon,
\]
we can conclude that \( \alpha(T^{2n+1}) \leq 2\varepsilon \), where \( T^{2n+1} = \{ P_w : w \in \Sigma^{2n+1} \} \). It follows that \( \alpha(T^*) = \alpha(T^{2n+1}) \leq 2\varepsilon \), completing the proof since \( \varepsilon > 0 \) is arbitrary.

\[\text{C} \quad \text{Proof of Theorem 4}\]

This result is implied by the following lemma:

**Lemma C.1.** Let \( \mathcal{M} \) and \( \tilde{\mathcal{M}} \) be two similar MCS, such that the first is weakly ergodic and the second is arbitrary. Then, for any \( \beta > 0 \) there exists \( \varepsilon > 0 \) such that \( \| P_u - \tilde{P}_u \|_1 \leq \varepsilon \) for all \( u \in \Sigma \) implies \( \| P_w - \tilde{P}_w \|_1 \leq \beta \) for all words \( w \in \Sigma^* \).

**Proof.** It is easy verify by using the representation (3.1) that:

(i) For any Markov operators \( P, Q, \) and \( R \), we have \( \| PQ - PR \|_1 \leq \delta(P) \| Q - R \|_1 \).

(ii) For any Markov operators \( P, \tilde{P} \) we have \( \delta(\tilde{P}) \leq \delta(P) + \| P - \tilde{P} \|_1 \).

Let \( r \in \mathbb{N} \) be such that \( \delta(P_w) \leq \beta / r \) for any \( w \in \Sigma^r \), and let \( \varepsilon = \beta / r \). If \( \| P_u - \tilde{P}_u \|_1 \leq \varepsilon \) for any \( u \in \Sigma \), then \( \| P_w - \tilde{P}_w \|_1 \leq n \varepsilon \) for any \( w \in \Sigma^n \). It follows that \( \| P_w - \tilde{P}_w \|_1 \leq \beta \) for any \( w \in \Sigma^{\leq r} \). Moreover, for any \( v \in \Sigma^r \) and \( w \in \Sigma^* \), we have
\[
\| P_{vw} - \tilde{P}_{vw} \|_1 \leq \| P_{vw} - P_v \|_1 + \| P_v - \tilde{P}_v \|_1 + \| \tilde{P}_v - \tilde{P}_{vw} \|_1 \leq 2\delta(P_v) + \| P_v - \tilde{P}_v \|_1 + 2\delta(\tilde{P}_v) \leq 4\delta(P_v) + 3\| P_v - \tilde{P}_v \|_1 \leq \beta,
\]
completing the proof. \( \square \)
Acknowledgments

We are grateful to Leonid Gurvits for valuable discussions.

References

[1] J. Banaś and K. Goebel, Measures of Noncompactness in Banach Spaces , Marcel Dekker, New York, 1980.

[2] M. Casey, The Dynamics of discrete-time computation, with application to recurrent neural networks and finite state machine extraction , Neural Computation, 8 1996, 1135–1178.

[3] L. W. Cohen and N. Dunford, Transformations on sequence spaces, Duke. Math. J. 3 (1937), 689–701.

[4] J. L. Doob, Stochastic Processes, John Wiley and Sons, 1953.

[5] M. Iosifescu, On two recent papers on ergodicity in non-homogeneous Markov chains, Ann. Math. Statist., 43 1972, 1732–1736.

[6] M. Iosifescu and R. Theodoresku, Random Process and Learning, Springer-Verlag, Berlin-Heidelberg, 1969.

[7] W. Maass and P. Orponen, On the effect of analog noise in discrete time computation, Neural Computation 10 (1998), no. 5, 1071–1095.

[8] W. Maass and E. Sontag, Analog neural nets with Gaussian or other common noise distribution cannot recognize arbitrary regular languages, Neural Computation 11 (1999), 771–782.

[9] J. Neveu, Mathematical Foundations of the Calculus of Probability, Holden Day, San Francisco, 1964.

[10] A. Paz, Ergodic theorems for infinite probabilistic tables, Ann. Math. Statist. 41 (1970), 539–550.

[11] A. Paz, Introduction to Probabilistic Automata, Academic Press, London, 1971.

[12] M. Rabin, Probabilistic automata, Information and Control 3 (1963), 230–245.

[13] H. T. Siegelmann, Neural Networks and Analog Computation: Beyond the Turing Limit, Birkhauser, Boston, 1999.

[14] K. Yosida and S. Kakutani, Operator-theoretical treatment of Markoff’s process and mean ergodic theorem, Annals of Mathematics 42(1) (1941), 188–228.