Postselected weak measurement based on thermal noise effect

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Most studies for postselected weak measurement focus on using pure Gaussian state as a pointer, which can only give an amplification limit reaching the ground state fluctuation. In this paper we use the thermal state pointer. We find that its amplification limit can reach thermal fluctuations and we also give the amplification mechanism different from the one with pure Gaussian state pointer. To illustrate the results, we propose two schemes to implement room temperature amplification of the mechanical oscillator’s displacement caused by one photon in optomechanical system. The two schemes can both enhance the original displacement by nearly seven orders of magnitude, attaining sensitivity to displacements of ~ 0.26 nm. Such amplification effect can be used to observe the impact of a single photon on a room temperature mechanical oscillator which is impossible to detect in traditional measurement.

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Introduction – Weak measurement (WM) with postselection, first proposed by Aharonov et al. 1, is an enhanced detection scheme where the system is weakly coupled to the pointer. The postselection on the system leads to an unusual effect: the average displacement of the postselection pointer is far beyond the the eigenvalue spectrum of the system observable, in contrast to von Neumann measurement. The mechanism behind this effect is the superposition (interference) between different postselection pointer states 2. Much theoretical research based of weak value is shown in 3–5. WM has been realized 6, and proven applicable to amplify tiny physical effects 7–11. More experimental protocols have been proposed 12–20. A Fock-state view for WM is given in 4, based on which a WM protocols combined with optomechanical system 22, 23 is proposed 8, 8, 4, 25, 27, and more applications of the field are reviewed in 28, 29.

In most previous studies the pointer is initialised in pure Gaussian state. It was an inherit assumption that the pointer has to be in the pure state at the inception of WM 1, 2. A pointer can be easily represented with light in pure state 8, 8, but with particles of efficient mass 30, 33, it’s difficult to initialize them in pure state due to environmental induced decoherence. The discussion of mixed state pointer in WM is given in 34, 36. However, they only focus on weak-value formalism (see 28, 29 for reviews) but not what extent the amplification value can be, i.e., the amplification limit. Thermal state is not only mixed state but also easier to prepare, especially in optomechanical systems 8, 27. One may naturally ask whether using thermal state pointer in WM can give a valid result for the amplification limit, and what kind of advantage it has than pure state pointer.

Here, we present a universal and easy method for enhancing weak amplification limit. Our paper begins with a general discussion about WM using thermal state as the initial pointer state, and show that the maximal amplification of the postselection pointer’s displacement can reach thermal fluctuation, which is much larger than the ground state fluctuation with pure state pointer 4, 8, indicating that thermal noise effect of the pointer is beneficial for weak measurement amplification. This amplification is attributed to the superposition of the number state |n⟩ and the state (c + c†)|n⟩ (unnormalized) of the postselection pointer. Such superposition is the generalization of the mechanism behind the amplification in Ref. 4, 8. Moreover, recently we find that weak measurement based of thermal state can also improve accuracy of precision measurement discussed in 11 which is very different from previous results 38–41.

We apply the general idea to the field of optomechanical system. We find that the amplification of the mirror’s displacement occurred at time near zero is very important for bad cavities with non-sideband resolved regime, and can overcome the shortcomings of difficulty observing the amplification effect due to dissipation 8. Finally we show that the unique advantage of our schemes is that the amplification at room temperature, with current experimental technologies, can be used to observe the impact of a single photon on a room temperature mechanical oscillator which is impossible to detect in traditional mea-

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measurement \[42\].

Fock-state view of WM with thermal state pointer—In the standard scenario of WM, the interaction Hamiltonian between the system and the pointer is \( \hat{H} = \chi(t) A \otimes \rho \) (setting \( \hbar = 1 \)), where \( A \) is a system observable, \( \rho \) is the position observable of the pointer and \( \chi(t) \) is a narrow pulse function with interaction strength \( \chi \). As in Ref. \[44\], if we define \( \hat{c} = \frac{1}{\sqrt{2}} \hat{q} + i \sigma \hat{p} \), where \( \sigma \) is the zero-point fluctuation, the interaction Hamiltonian can be rewritten as

\[
\hat{H} = \chi(t) \sigma A(\hat{c} + \hat{c}^\dagger). \tag{1}
\]

Suppose the initial system state is \( |\psi_i\rangle = (1/\sqrt{2})(|a_1\rangle_s + |a_2\rangle_s) \), where \( a_1 \) and \( a_2 \) are eigenvalues of \( A \). Then we consider the initial pointer state as \( \rho_{th}(z) = (1 - z) \sum_{n=0}^\infty z^n |n\rangle_m \langle n| \), \( z = e^{-\hbar \omega_m/k_B T} \), \( k_B \) is the Boltzmann constant and \( T \) is the temperature.

Because of the linearity of \( \rho_{th}(z) \), we need only look at the component number states \( |n\rangle_m \) are weakly coupled with \( |\psi_i\rangle \) using Eq. (1). Then we postselect the system into a final state \( |\psi_f\rangle = (1/\sqrt{2})(|a_1\rangle_s - e^{i\varphi} |a_2\rangle_s) \) with \( \varphi \ll 1 \), which is nonorthogonal to \( |\psi_i\rangle \), i.e., \( \langle \psi_f | \psi_i \rangle \approx i\varphi/2 \) that is imaginary \[43\], then the reduced state of the pointer after postselection for \( n \) component (see Supplemental Materials (SMs) for details \[44\]) is,

\[
|\psi_m(n)\rangle = \frac{1}{2} [D(-i\alpha_1\eta)|n\rangle_m - e^{-i\varphi} D(-i\alpha_2\eta)|n\rangle_m], \tag{2}
\]

where \( \eta = \chi \sigma \) and \( D(\alpha) = \exp[\alpha \hat{c}^\dagger - \alpha^* \hat{c}] \) is the displacement operator.

When \( \varphi \ll 1 \) and \( \eta \sqrt{2n + 1} \ll 1 \), i.e., \( \eta \ll 1 \), the approximation of \( |\psi_m(n)\rangle \) \[44\] is

\[
|\psi_m(n)\rangle_{\eta \ll 1} \approx \frac{1}{2} [i\varphi |n\rangle_m + i\eta (a_2 - a_1)(c + \hat{c}^\dagger)|n\rangle_m]. \tag{3}
\]

For Eq. (3), the average displacement of the pointer \[44\] is

\[
\langle q \rangle = \sigma^2 \eta^2 \frac{1 + z}{1 - z} (a_2 - a_1)/(\varphi^2 + \frac{1 + z}{1 - z} (a_2 - a_1)^2 \eta^2) \tag{4}
\]

and \( \langle p \rangle = 0 \).

From Eq. (3), we can see that \( \langle q \rangle \) is non-zero in position space, and get the maximal positive and negative values \( \pm (1/2)^{1/2} \sigma \) (thermal fluctuation) when \( \varphi = \pm (1/2)^{1/2} (a_2 - a_1) \eta \), respectively, which are much larger than that using pure state pointer \[1, 2, 3\], i.e., the ground state fluctuation \( \sigma \). Therefore, \( |\psi_m(n)\rangle \) components corresponding to the maximal positive and negative amplification are, respectively, \( |\psi_m(n)\rangle_{\max, n = 1} = (1/2\sqrt{2}) (1/2^{1/2} (a_2 - a_1) \eta \pm (c + \hat{c}^\dagger)|n\rangle) \). Obviously, the key to understand the amplification is the superposition of the number state \( |n\rangle \) and the state \( (c + \hat{c}^\dagger)|n\rangle \) (unnormalized) of the pointer in (3). This result reveals the more generalized law of causing amplification effect since it is regarded as a generalization of the mechanism behind the amplification in standard WM \[1, 4\], which is the superposition of the ground state \( |0\rangle \) and the one phonon state \( |1\rangle \) of the pointer (see SMs \[44\]). In a word, thermal noise effect of the pointer is beneficial for the amplification of the displacement proportional to imaginary weak value. It is surprising that in (1) the approach above can also enhance precision in quantum metrology.

Optomechanical model.—To show how the above results can be applied, we consider a Mach-Zehnder interferometer combined with optomechanical system where the optomechanical cavity (OC) A and the stationary Fabry-Perot cavity B is embedded in its one and another arm, respectively (see Fig. 1), the Hamiltonian writes

\[
\hat{H} = \hbar \omega_c (a^\dagger a + b^\dagger b) + \hbar \omega_m c^\dagger c - \hbar g a^\dagger a (c + c^\dagger), \tag{5}
\]

where \( \omega_c \) is the frequency of the optic cavity A, B of length \( L \) with corresponding annihilation operators \( a \) and \( b \), \( \omega_m \) is the angular frequency of mechanical system with corresponding annihilation operator \( \hat{c} \), and the optomechanical coupling strength \( g = \omega \sigma / L \), \( \sigma = (\hbar / 2 \omega_m)^{1/2} \) which is the zero point fluctuation and \( m \) is the mass of mechanical system.

WM Amplification using a phase shifter—As shown in Fig. 1, suppose one photon enters the interferometer, after the first beam splitter and a phase shifter \( \theta \) in the arm A of the interferometer, the initial state of the photon becomes \( |\psi_1(\theta)\rangle = (1/\sqrt{2}) (e^{i\theta}|1\rangle_A |0\rangle_B + |0\rangle_A |1\rangle_B) \) with \( \theta \ll 1 \). The mirror is initialised in thermal state \( \rho_{th}(z) \). After the interaction \[49\], the second beam splitter postselects for the photon state \( |\psi_f\rangle = (1/\sqrt{2}) (|1\rangle_A |0\rangle_B - |0\rangle_A |1\rangle_B) \), which is nonorthogonal to \( |\psi_1(\theta)\rangle \), i.e., \( \langle \psi_f | \psi_1(\theta) \rangle \approx i\theta/2 \), and when a photon is detected at the dark port, as shown in SMs \[44\], the reduced state of the mirror after postselection for \( n \) component becomes

\[
|\psi_1(n)\rangle = \frac{1}{2} [e^{i(\phi(t) + \theta)} D(\xi(t))|n\rangle - |n\rangle], \tag{6}
\]

where \( \xi(t) = k (1 - e^{-\hbar \omega_m t}) \) with \( k = g/\omega_m \) and \( \phi(t) = k^2 (\omega_m t - \sin \omega_m t) \) is Kerr phase. For Eq. (6), the average displacement \( \langle q(t) \rangle \) of the mirror (see SMs for derivation

FIG. 1: The photon enters the first beam splitter of March-Zehnder interferometer, before entering an optomechanical cavity A and a conventional cavity B. The photon weakly excites the tiny mirror. After the second beam splitter, and dark port is detected, i.e., postselection acts on the case where the mirror has been excited by a photon, and fails otherwise.
When a photon is detected at the dark port, the change of mirror's displacement caused by one photon, which means that the impact of a single photon on an mechanical oscillator is beneficial for the amplification of the mirror’s displacement caused by one photon, which allows for the impact of a single photon on an mechanical oscillator with arbitrary temperature can be observed.

In order to observe the amplification effects appearing at time near \( \omega_m t = 0 \), we can obtain the approximation of \( |\psi_1(n)\rangle \) as

\[
|\psi_1(n)\rangle_{\omega_m t \ll 1} \approx \frac{1}{2} [i\theta |n\rangle + ik \omega_m t (c + c^\dagger)|n\rangle]
\]

when \( k \ll 1 \) and \( \theta \ll 1 \).

In Fig. 2(b), we plot the average displacement \( \langle q(t) \rangle / \sigma \) for \( |\psi_1(n)\rangle_{\omega_m t \ll 1} \) as function of \( n \) when \( \theta = (1+z)^{-1/2} k \omega_m t \). This condition \( \theta = (1+z)^{-1/2} k \omega_m t \) is to make \( \langle q(t) \rangle \) achieve the maximal value \( (1+z)^{-1/2} \). It shows the amplification values grow with the increase of \( n \). Obviously, the superposition of \( |n\rangle \) and \( (c + c^\dagger)|n\rangle \) is the key to obtain amplification at time near \( \omega_m t = 0 \).

**WM Amplification using displaced thermal state**—Besides the above amplification scheme, as shown in Fig. 1, we can also provide an alternative where the mirror is initialised in the displaced thermal state \( |\psi_1\rangle \) using classical light pulses drive, \( \rho_{SB}(z, \alpha) = D(\alpha) \rho_{SB}(z) D^\dagger(\alpha) \). Without the phase shifter \( \theta \), the initial state of the photon after the first beam splitter is \( |\psi_1\rangle = (1/\sqrt{2}) [(|1\rangle_A |0\rangle_B + |0\rangle_A |1\rangle_B) \). Similar to the previous scheme, when a photon is detected at the dark port, the reduced state of the mirror after orthogonal postselection \( \langle \psi_f | \psi_1 \rangle = 0 \) for \( n \) component (see SMs [44]) is

\[
|\psi_2(n)\rangle = \frac{1}{2} [e^{i(\phi(t) + i\alpha(t))} D(\xi(t))|n\rangle - |n\rangle],
\]

where \( \phi(\alpha, t) = -i(\alpha \xi(t) - \alpha^* \xi(t)) \) is caused by noncommutativity of quantum mechanics [4].

Figure 3(a) shows that the average displacement \( \langle q(t) \rangle / \sigma \) of the mirror for \( |\psi_2(n)\rangle \) (see SMs for derivation [44]) versus time \( \omega_m t \). Obviously, at time near \( \omega_m t = 0 \), the maximal amplification can reach \((1+z)^{-1/2} \) (thermal fluctuation) which is \( \sqrt{19} \sigma \) when \( z = 0.9 \). The meaning of this result is the same as the one using a phase shifter and a huge impact of a single photon on a high temperature mechanical oscillator can be observed.

Similar to Eq. (5), the approximation of \( |\psi_2(n)\rangle \) is [44]

\[
|\psi_2(n)\rangle_{\omega_m t \ll 1} = \frac{1}{2} [2k|\alpha| |\xi(n)\rangle + ik \omega_m t (c + c^\dagger)|n\rangle]
\]

when \( k \ll 1 \) and \( 2k|\alpha| \xi \ll 1 \), where \( \xi = \frac{1}{2}(\omega_m t)^2 \sin \beta + \omega_m t \cos \beta \). This indicates that the superposition of \( |n\rangle \) and \( (c + c^\dagger)|n\rangle \) is the key to obtain amplification at time near \( \omega_m t = 0 \). Fig. 3(b) show that at time \( \omega_m t = 0.001 \), the average displacement \( \langle q(t) \rangle / \sigma \) of the mirror as a function of \( \alpha = |\alpha| e^{i\theta} \), i.e., different displaced thermal state \( \rho_{SB}(z, \alpha) \).

**Dissipation**—When the mirror is considered in a thermal bath characterized by a damping constant \( \gamma_m \), we have

\[
\frac{dp(t)}{dt} = -i \frac{\hbar}{\gamma_m} [H, \rho(t)] + \gamma_m \frac{c + c^\dagger}{(1 - z)} D[c] + \gamma_m \frac{c^\dagger + c}{(1 - z)} D[c^\dagger],
\]

where \( D[c] = op(t) o^\dagger - o^\dagger op(t)/2 - \rho(t) o^\dagger o/2 \). In Fig. 4(a) and Fig. 5(a), we show that at time \( t \ll 1 \), the average displacements of the mirror (see SMs for derivation [44]) from the exact solution of Eq. (11) for the first and the second proposed schemes, respectively. They show that at room temperature 300K, even if the damping coefficient \( \gamma (\gamma = \gamma_m / \omega_m) \) become very large, such as \( \gamma = 50 \), the average displacement of the mirror is the same as the
is the overall probability of a single photon successfully generating the superposition state of $|n\rangle$ and $(c + c^\dagger)|n\rangle$. Figure 4(b) and Figure 5(b) show the photon arrival rate density $D(t)$ for the first and the second proposed schemes, respectively. They show that in the bad-cavity limit $\kappa > \omega_m$, i.e., non-sideband resolved regime, as the decay rate $\kappa$ of the cavity increases, $D(t)$ becomes increasingly concentrated at time near $t = 0$.

For a repeated experimental set up with identical conditions, the "average" displacement of the pointer is given by

$$\langle q(t) \rangle = \int_0^\infty D(t) \langle q(t) \rangle dt, \quad (13)$$

where $\langle q(t) \rangle$ is the same as $\langle q(t) \rangle$ in Eq. (7). At room temperature $T = 300K$, we use a mechanical resonator with mechanical frequency $f_m = 4.5$ kHz and effective mass $m = 100$ kg [27], indicating that $z = 0.999999999$, $\sigma = 4.32$ fm (femtometer). So the maximal amplification value $(\frac{1}{2\pi})^{1/2} \sigma = 0.26$ nm. If $T = 1500K$, $(\frac{1}{2\pi})^{1/2} \sigma = 0.5$ nm [47]. For the first scheme, with $\kappa = 1.2 \times 10^4 \omega_m$, $\langle q(t) \rangle = 11577\sigma$ if $\kappa = 0.005$, $\theta = 0.005$, and for the second scheme, with $\kappa = 2 \times 10^4 \omega_m$, $\langle q(t) \rangle = 44704\sigma$ if $\kappa = 0.005$, $|\alpha| = (\frac{1+i}{\sqrt{2}})^{1/2}/2$, $\beta = 0$. Now we compare these amplification results with the maximal unamplified value $4\kappa \sigma = 86.4$ am (attometer) caused by the radiation pressure of single photon in cavity A (amplification without postselection, see SMs [44]), therefore the amplification factor is $Q = \langle q(t) \rangle/4\kappa \sigma$ which are 578850 for the first scheme and 2235200 for the second scheme.

We then give the experimental requirements for the optomechanical device at room temperature $T = 300K$. According to Eq. (13), $P$ that we need is common, though the precise value of which depend on the dark count rate of the detector and the stability of the setup. At room temperature $T = 300K$, for the first scheme, $P$ is approximately $6.94k^2 [14]$ for a device with $\kappa = 1.2 \times 10^4 \omega_m$ when $\theta = 0.005$. The window that detectors need to open for photons is approximately $1/\kappa$, requiring the dark count rate being lower than $6.94k^2\kappa$. The dark count rate of the best silicon avalanche photodiode is about $\sim 2$ Hz. So we require $k > 0$ for a 4.5 kHz device, i.e., proposed device no. 2 from [27], but with optical finesse $F$ reduced to 2800 and cavity length being 0.5 mm. For the second scheme, $P$ is approximately $5k^2 \Omega [14]$ for a device with $\kappa = 2 \times 10^4 \omega_m$ when $|\alpha| = (\frac{1+i}{\sqrt{2}})^{1/2}/2$, $\beta = 0$. Because the dark count rate 2 Hz of the detector is lower than $5k^2\kappa$, we require $k \geq 0.000026$ for the same 4.5 kHz device, but with optical finesse $F$ reduced to 3000 and cavity length being 0.3 mm. Therefore, the implementation of the schemes provided here are feasible to observe the impact of a single photon on a room-temperature mechanical oscillator in experiment.

Conclusion-In this letter, we considered using thermal state to enhance the amplification limit of the displacement of the pointer via Fock-state view of weak measurement [15,18], and the maximal amplification can reach thermal fluctuation. The mechanism behind the amplification is the superposition between the number state $|n\rangle$ and the state $(c + c^\dagger)|n\rangle$ (unnormalized) of the postselection pointer. To this end, we proposed two different schemes for experimental implementations with optomechanical system, and show that the amplification that occurs at time near $\omega_m t = 0$ is important for bad cavities with non-sideband resolved regime, which means that
our proposed schemes are feasible to observe the impact of a single photon on a room-temperature mechanical oscillator under current experimental condition. Moreover, we have provided enough theoretical toolbox [11, 48] to amplify the weaker effect in one-photon weak-coupling optomechanics, which may be employed to explore the faint gravitational effect.

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[42] Quantum optomechanical system usually refers to a high finesse cavity with a movable mirror where the light in the cavity can give a force on the mirror [22, 23]. When there is only one photon in the cavity, the displacement of the mirror caused by the photon is hard to detect in traditional measurement since it is much smaller than the spread of the mirror wave packet (quantum fluctuation). Of course, if the mirror is in thermal state (thermal fluctuation), the mirror’s displacement caused by the photon is impossible to detect in traditional measurement.
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This is a supplement for “Postselected weak measurement based on thermal noise effect”, wherein we provide the derivation of some results in main text.

Derivation of weak measurement with thermal state pointer

If the system is the initial state $|\psi_i\rangle = (1/\sqrt{2})(|a_1\rangle_s + |a_2\rangle_s)$ and the initial pointer state is the thermal state $\rho_{th}(z)$. After the interaction Eq. (1) in the main text, the time evolution of the total system is given by

$$
\rho_{sm} = e^{-i \int \hat{H} dt} |\psi_i\rangle \langle \psi_i | \rho_{th} e^{i \int \hat{H} dt}
$$

$$
= (1 - z) \sum_{n=0}^{\infty} z^n \exp[-i\eta A(\hat{c} + \hat{c}^\dagger)] |n\rangle_m \langle n|_m \langle \psi_i | \exp[i\eta \delta_z (\hat{c} + \hat{c}^\dagger)]
$$

$$
= (1 - z) \sum_{n=0}^{\infty} z^n |\psi\rangle_{sm} \langle \psi |_{sm}
$$

(14)

with

$$
|\psi\rangle_{sm} = \frac{1}{\sqrt{2}} [|a_1\rangle_s D(-i\alpha \eta) + |a_2\rangle_s D(-i\alpha_2 \eta)]|n\rangle_m,
$$

(15)

where $\eta = \chi \sigma$ assumed to be very small and $D(\alpha) = \exp[\alpha \hat{c}^\dagger - \alpha^* \hat{c}]$ is the displacement operator. This result correspond to weak measurement without postselection on the system.

The average displacement of the pointer observable $M$ ($M = \hat{q}, \hat{p}$) is

$$
\langle M \rangle = \frac{Tr(\rho_{m} M)}{Tr(\rho_m)} - Tr(\rho_{th} M),
$$

(16)

where $\rho_m$ is an updated pointer state after postselection and $\rho_{th}$ is an initial pointer state. Here we use the operator $\hat{q} = \sigma(c + c^\dagger)$ and $\hat{p} = \frac{1}{2\sigma}(-i)(c - c^\dagger)$ over all the paper.

Amplification: postselecting state $|\psi_f\rangle$ is nonorthogonal to $|\psi_i\rangle$ and $\langle \psi_f | \psi_i \rangle \approx \varepsilon$ is real number

When the postselected state of the system $|\psi_f\rangle = \cos(\frac{\pi}{4} - \varepsilon)|a_1\rangle_s - \sin(\frac{\pi}{4} - \varepsilon)|a_2\rangle_s$ with $\varepsilon \ll 1$ is performed for the total system (15), i.e., $\langle \psi_f | \psi_i \rangle \approx \varepsilon$, which is real number, the reduced state of the pointer for $n$ component becomes

$$
|\varphi_m(n)\rangle = \langle \psi_f | \psi \rangle_{sm}
$$

$$
= \frac{1}{\sqrt{2}} \langle \psi_f | [|a_1\rangle_s D(-i\alpha_1 \eta) + |a_2\rangle_s D(-i\alpha_2 \eta)]|n\rangle_m,
$$

$$
= \frac{1}{\sqrt{2}} \cos(\frac{\pi}{4} - \varepsilon) D(-i\alpha_1 \eta)|n\rangle_m - \sin(\frac{\pi}{4} - \varepsilon) D(-i\alpha_2 \eta)|n\rangle_m
$$

(17)

and therefore, over all $n$ component, then the final total state of the pointer is

$$
\rho_m = (1 - z) \sum_{n=0}^{\infty} z^n |\varphi_m(n)\rangle \langle \varphi_m(n) |
$$

(18)

Substituting Eq. (18) into Eq. (16) with $M = p$, we can obtain the analytical expression

$$
\langle p \rangle = \frac{1}{2\sigma} [(\Theta - \Theta^*) \cos^2(\frac{\pi}{4} - \varepsilon) - \exp(-\frac{(1 + z)|\Theta|^2}{2(1 - z)}) \cos(\frac{\pi}{4} - \varepsilon) \sin(\frac{\pi}{4} - \varepsilon)] / [1 - 2 \cos(\frac{\pi}{4} - \varepsilon) \sin(\frac{\pi}{4} - \varepsilon) \exp(-\frac{(1 + z)|\Theta|^2}{2(1 - z)})],
$$

(19)

where $\Theta = -i\eta(a_1 - a_2)$.
Making small quantity expansion for amplification

For the Eq. (17), when \( \varepsilon \ll 1 \) and \( \eta / \sqrt{2n+1} \ll 1 \), i.e., \( \eta \ll 1 \), we perform a small quantity expansion about \( \eta \) and \( \varepsilon \) till the second order, then leading to

\[
|\varphi_m(n)\rangle_{\eta \ll 1} \approx \frac{1}{2}[(1 + \varepsilon)(1 - i\alpha_1 \eta c^+ - i\alpha_1 \eta c)|n\rangle_m - (1 - \varepsilon)(1 - i\alpha_2 \eta c^+ - i\alpha_2 \eta c)|n\rangle_m]
\]

\[
\approx \frac{1}{2}[2\varepsilon|n\rangle_m + i\eta(a_2 - a_1)(c + c^\dagger)|n\rangle_m]
\]

and therefore, over all \( n \) component, then the final total state of the pointer is \( \rho_m \approx |\varphi_m(n)\rangle_{\eta \ll 1} \langle \varphi_m(n)|_{\eta \ll 1} \) and substituting it into Eq. (19), the average displacement of the pointer is

\[
\langle p \rangle = \frac{1}{2\sigma} \frac{4(a_2 - a_1)\varepsilon \eta}{4\varepsilon^2 + \frac{\pi}{2\sigma}(a_2 - a_1)^2 \eta^2}.
\]

which is the asymptotic solution of (19) and

\[
\langle q \rangle = 0.
\]

From Eq. (21), we can still get the maximal positive value \( (\frac{1+\varepsilon}{1-\varepsilon})^{1/2} \) when \( \varepsilon = (\frac{1+\varepsilon}{1-\varepsilon})^{1/2} (a_2 - a_1) \eta / 2 \) and the negative value \( -(\frac{1-\varepsilon}{1+\varepsilon})^{1/2} \) when \( \varepsilon = -(\frac{1-\varepsilon}{1+\varepsilon})^{1/2} (a_2 - a_1) \eta / 2 \), respectively. Because \( \frac{1+\varepsilon}{1-\varepsilon} \ll 1 \), so \( |\langle p \rangle| < \frac{1}{2\sigma} \) (zero-point fluctuation), implying that the maximal amplification of the pointer’s displacement in momentum space is less than zero-point fluctuation, in sharp contrast to Eq. (14) in the following Section II which indicate that \( |\langle p \rangle| = \pm \frac{1}{2\sigma} \) when \( \varepsilon = \pm (a_2 - a_1) \eta / 2 \). Therefore, thermal noise effect of the pointer has a negative effect for the amplification of the displacement proportional to weak value.

Although the displacement proportional to weak value has been amplified using thermal state pointer, but it is far less than the larger uncertainty (thermal fluctuation) of the pointer, indicating that mixed state pointer with larger fluctuation is infeasible for the displacement proportional to weak value. In other words, if mixed state pointer (e.g., thermal state) didn’t have any advantage over pure state pointer, it would be pointless to study amplification with mixed state pointer.

Displacement of the pointer in momentum space corresponding to real part of weak value

According to the definition of weak value [1]

\[
A_w = \frac{\langle \psi_f | A | \psi_i \rangle}{\langle \psi_f | \psi_i \rangle},
\]

where \( |\psi_i\rangle \) and \( |\psi_f\rangle \) is the preselected and postselected state, in this case of using thermal state the weak-value regime satisfies the condition \( \eta / \sqrt{2n+1} \ll \varepsilon \ll 1 \). When the postselection state of the system \( |\psi_f\rangle = \langle 1/\sqrt{2}| \cos(\frac{\pi}{2} - \varepsilon) |a_1\rangle_s - \sin(\frac{\pi}{2} - \varepsilon) |a_2\rangle_s \) is performed for the total system (16):

\[
|\Phi_f\rangle = \langle \psi_f | \exp[-i\eta A(c + c^\dagger)] |\psi_i \rangle |\rho_{th} \exp[i\eta A(c + c^\dagger)] |\psi_f \rangle
\]

\[
\approx (1 - z) \sum_{n=0}^{\infty} z^n [\langle \psi_f | \psi_i \rangle - i\eta \langle \psi_f | A | \psi_i \rangle (c + c^\dagger) |n\rangle_m < |n\rangle_m |\langle \psi_i | \psi_f \rangle + i\eta |\psi_i | A | \psi_f \rangle (c + c^\dagger)|n\rangle_m |\langle | \exp[-i\eta A_w(c + c^\dagger)] |n\rangle_m < |n\rangle_m \exp[i\eta A_w^*(c + c^\dagger)],
\]

where \( A_w = \frac{a_1 \cos(\frac{\pi}{2} - \varepsilon) - a_2 \sin(\frac{\pi}{2} - \varepsilon)}{\cos(\frac{\pi}{2} - \varepsilon) - \sin(\frac{\pi}{2} - \varepsilon)} \approx \frac{(a_1 - a_2)\varepsilon}{2\varepsilon} \) (real).

Substituting Eq. (24) into Eq. (19), the average displacement of the pointer in momentum space is

\[
\langle p \rangle = -i \frac{1}{2\sigma}(1 - z) \sum_{n=0}^{\infty} z^n < |n\rangle_m \exp[i\eta A_w(c + c^\dagger)] (c - c^\dagger) \exp[-i\eta A_w^*(c + c^\dagger)] |n\rangle_m
\]

\[
= \chi \text{ Re } A_w
\]

which is exactly the same weak values as a pure Gaussian pointer state [1]. But the the average displacement of the pointer in position space is always

\[
\langle q \rangle = 0.
\]
Amplification: postselecting state $|\psi_f\rangle$ is nonorthogonal to $|\psi_i\rangle$ and $\langle \psi_f | \psi_i \rangle \approx i \varphi / 2$ is imaginary number

When the postselection $|\psi_f\rangle = (1/\sqrt{2})(|a_1\rangle_s - e^{i\varphi}|a_2\rangle_s)$ with $\varphi \ll 1$ is performed for the total system (15), i.e., $\langle \psi_f | \psi_i \rangle \approx i \varphi / 2$, which is an imaginary number, the reduced state of the pointer for $n$ component becomes

$$|\psi_m(n)\rangle = \langle \psi_f | \psi\rangle_{sm} = \frac{1}{\sqrt{2}}(\psi[|a_1\rangle_s D(-ia_1\eta) + |a_2\rangle_s D(-ia_2\eta)]n)_m = \frac{1}{2}[D(-ia_1\eta)|n\rangle_m - e^{-i\varphi} D(-ia_2\eta)|n\rangle_m].$$

(27)

Therefore, this is Eq. (2) in the main text.

For Eq. (27) and over all $n$ component, the final total state of the pointer is

$$\rho_m = (1 - z) \sum_{n=0}^{\infty} z^n |\psi_m(n)\rangle \langle \psi_m(n)|.$$

(28)

Substituting Eq. (28) into Eq. (16) with $M = \hat{q}$, we can obtain the analytical expression

$$\langle q \rangle = \sigma[(\Theta + \Theta^*) - \frac{1}{1 - z} \exp(-\frac{(1 + z)|\Theta|^2}{2(1 - z)}) \times [(e^{i\varphi}\Theta + e^{-i\varphi}\Theta^*) - z(e^{i\varphi}\Theta^* + e^{-i\varphi}\Theta)]/2$$

$$- \exp(-\frac{(1 + z)|\Theta|^2}{2(1 - z)})(e^{i\varphi} + e^{-i\varphi})],$$

(29)

where $\Theta = -i\eta(a_1 - a_2)$.

Making small quantity expansion for amplification

For Eq. (27), we can perform a small quantity expansion about $\eta$ and $\varphi$ till the second order when $\varphi \ll 1$ and $\eta \sqrt{2n + 1} \ll 1$, i.e., $\eta \ll 1$, then

$$|\psi_m(n)\rangle_{\eta\ll1} \approx \frac{1}{2}(1 - ia_1\eta(c + c^\dagger))|n\rangle_m - (1 - i\varphi)(1 - ia_2\eta(c + c^\dagger))|n\rangle_m).$$

$$\approx \frac{1}{2}[i\varphi |n\rangle_m + i\eta(a_2 - a_1)(c + c^\dagger)|n\rangle_m].$$

(30)

Note that we use its approximation $e^{-i\varphi} \approx 1 - i\varphi$ to get the state $i\varphi |n\rangle_m$.

Therefore, this is Eq. (3) in the main text.

Substituting it into Eq. (30), the average displacement of the pointer in position space for $n$ component is

$$\bar{q} = 2\sigma(2n + 1)(a_2 - a_1)\varphi\eta,$$

(31)

For Eq. (30), over all $n$ component, then the final total state of the pointer is

$$\rho_m \approx \frac{1}{A_T} (1 - z) \sum_{n=0}^{\infty} z^n |\psi_m(n)\rangle_{\eta\ll1} \langle \psi_m(n)|_{\eta\ll1}$$

(32)

where $A_T = \frac{1}{4}(\varphi^2 + \frac{1}{1 - z}(a_2 - a_1)^2\eta^2)$ is a normalized coefficient, and substituting it into Eq. (16), the average displacement of the pointer in position space is

$$\langle q \rangle = \frac{1}{A_T} (1 - z) \sum_{n=0}^{\infty} z^n \bar{q}$$

$$= \frac{2\frac{1 + \varphi}{1 - z}(a_2 - a_1)\varphi\eta}{\varphi^2 + \frac{1}{1 - z}(a_2 - a_1)^2\eta^2},$$

(33)
which is the asymptotic solution of \( \{23\} \). Therefore, this is Eq. (4) in the main text. Note that the result \( \{33\} \) should be twice the average results, which originate from the coherence (superposition) between the pointer states \( \{35\} \) and the classical statistical properties of thermal state itself, respectively. It is obvious that the key to understand the amplification is the superposition of the number state \(|n\rangle\) and the state \((c + c^\dagger)|n\rangle\) (unnormalized) of the pointer.

Displacement of the pointer in position space corresponding to imaginary part of weak value

According to the definition of Eq. \( \{23\} \), in this case of using thermal state, the weak-value regime satisfies the condition \( \eta \sqrt{2n + 1} \ll \phi \ll 1 \). When the postselection state of the system \(|\psi_f\rangle = (1/\sqrt{2})(|a_1\rangle_s - e^{i\phi}|a_2\rangle_s) \) is performed for the total system \( \{15\} \):

\[
|\Phi_f\rangle = \langle \psi_f | \exp[-i\eta A(\hat{c} + \hat{c}^\dagger)] | \psi_i \rangle \rho_{th} \exp[i\eta A(\hat{c} + \hat{c}^\dagger)] | \psi_f \rangle \\
\approx (1 - z) \sum_{n=0} z^n \langle \psi_f | \hat{c}^\dagger | \psi_i \rangle \rho_{th} | \psi_f \rangle + i\eta \langle \psi_i | A | \psi_f \rangle (\hat{c} + \hat{c}^\dagger) \\
\approx (1 - z) \sum_{n=0} z^n \langle \psi_f | \hat{c}^\dagger | \psi_i \rangle \rho_{th} | \psi_f \rangle + i\eta \langle \psi_i | A | \psi_f \rangle (\hat{c} + \hat{c}^\dagger),
\]

(34)

where \( A_w = \frac{a_i - e^{-i\phi} a_i}{\sqrt{2}} \approx \frac{a_i - a_0}{\sqrt{2}} \) (imaginary).

Substituting Eq. \( \{33\} \) into Eq. \( \{15\} \), the the average displacement of the pointer in position space is

\[
\langle q \rangle = (1 - z) \sum_{n=0} z^n \langle n | m \rangle \exp[i\eta A_w(\hat{c} + \hat{c}^\dagger)] \exp[-i\eta (A_w - A_w^*) (\hat{c} + \hat{c}^\dagger)] | n \rangle_m \\
+ (1 - z) \sum_{n=0} z^n \langle n | m \rangle \exp[i\eta A_w(\hat{c} + \hat{c}^\dagger)] \exp[-i\eta (A_w - A_w^*) (\hat{c} + \hat{c}^\dagger)] | n \rangle_m
\]

(35)

Changing to the \( q \) representation in rectangular coordinate this becomes

\[
\langle q \rangle = \frac{(1 - z)}{\sqrt{2\pi\sigma}} \int_{-\infty}^{\infty} dq \left( \sum_{n=0} \frac{1}{2^n n!} z^n H_n(\frac{q}{\sqrt{2}\sigma}) H_n(\frac{q}{\sqrt{2}\sigma}) \exp[-i\eta (A_w - A_w^*) q] \right) \\
\times \exp[-\frac{q^2}{2\sigma^2}]/(1 - \frac{q^2}{2\sigma^2}) \int_{-\infty}^{\infty} dq \left( \sum_{n=0} \frac{1}{2^n n!} z^n H_n(\frac{q}{\sqrt{2}\sigma}) H_n(\frac{q}{\sqrt{2}\sigma}) \exp[-\frac{q^2}{2\sigma^2}] \right),
\]

(36)

where \( H_n \) is Hermite Polynomial.

Using Mehler’s Hermite Polynomial Formula

\[
\sum_{n=0}^{\infty} H_n(x) H_n(y)(\frac{1}{2})^n = (1 - w^2)^{-1/2} \exp[\frac{2xyw - (x^2 + y^2)}{1 - w^2}]
\]

(37)

and

\[
\int_{-\infty}^{\infty} dx (x \exp[-x^2] \exp[wx]) = \frac{\partial}{\partial m} \int_{-\infty}^{\infty} dx (\exp[-x^2] \exp[wx]),
\]

(38)

then Eq. \( \{35\} \) becomes

\[
\langle q \rangle = 2\chi \text{Im} A_w \frac{1 + z}{1 - z} \sigma^2.
\]

(39)

But the the average displacement of the pointer in momentum space is always

\[
\langle p \rangle = 0.
\]

(40)

From Eq. \( \{39\} \), it can be seen that \( \langle q \rangle \) is proportional to the square of thermal fluctuation and is imaginary in position space, which is the generalization of the result of Eq. (10) in \( \{2\} \). Therefore, thermal noise effect of the pointer is beneficial for weak measurement amplification. But \( \langle q \rangle \) is not the optimal displacement, i.e., the maximal amplification value.
Fock state view of the standard weak measurement with a ground state pointer

We consider the Hamiltonian (1) in the main text. If the initial state of the system is \( |\psi_i\rangle = (1/\sqrt{2})(|a_1\rangle_s + |a_2\rangle_s) \), where \(|a_1\rangle_s \) and \(|a_2\rangle_s \) are eigenstates of \( A \). Any Gaussian can be seen as the ground state of a fictional harmonic oscillator Hamiltonian \[3\]. Suppose the initial pointer state is the ground state \(|0\rangle_m \). Then weakly couples them using the interaction Hamiltonian (1), the time evolution of the total system is given by

\[
e^{-i \hat{H} dt} |\psi_i\rangle|0\rangle_m = \exp[-i\eta A(\hat{c} + \hat{c}^\dagger)]|\psi_i\rangle|0\rangle_m = \frac{1}{\sqrt{2}}[|a_1\rangle_s D(-ia_1\eta) + |a_2\rangle_s D(-ia_2\eta)]|0\rangle_m. \tag{41}
\]

When the postselection \(|\psi_f\rangle = (1/\sqrt{2})[\cos(\pi/4 - \varepsilon)|a_1\rangle_s - \sin(\pi/4 - \varepsilon)|a_2\rangle_s] \) with \( \varepsilon \ll 1 \) is performed for the total system \[41\], i.e., \( \langle \psi_f | \psi_i \rangle \approx \varepsilon \), which is real number, then the final state of the pointer is

\[
\frac{1}{2} [\cos(\pi/4 - \varepsilon)D(-ia_1\eta) - \sin(\pi/4 - \varepsilon)D(-ia_2\eta)]|0\rangle_m. \tag{42}
\]

For Eq. \[42\], when \( \varepsilon \ll 1 \) and \( \eta \ll 1 \), we can then perform a small quantity expansion about \( \eta \) and \( \varepsilon \) till the second order, and then obtain

\[
\frac{1}{2}(2\varepsilon|0\rangle_m + i(a_2 - a_1)|1\rangle_m). \tag{43}
\]

Substituting Eq. \[43\] into Eq. \[16\], in this case of the near-orthogonal postselection, i.e., \( \langle \psi_f | \psi_i \rangle \approx \varepsilon \), we can find that

\[
\langle \hat{p} \rangle = \frac{1}{2\pi} \frac{4(a_2 - a_1)\varepsilon\eta}{4\varepsilon^2 + (a_2 - a_1)^2\eta^2}. \tag{44}
\]

and

\[
\langle \hat{q} \rangle = 0. \tag{45}
\]

When \( 2\varepsilon = \pm(a_2 - a_1)\eta \), we will have the largest displacement \( \pm \frac{\eta}{2\varepsilon} \) in momentum space and when \( \varepsilon = 0 \), indicating that the postselected state of the system is orthogonal to the initial state of the system, i.e., \( \langle \psi_f | \psi_i \rangle = 0 \), the displacement of pointer in momentum space is zero. This amplification result is due to the superposition of \(|0\rangle_m \) and \(|1\rangle_m \). However, the displacement of the pointer in position space is always zero.

When the postselection \(|\psi_f\rangle = (1/\sqrt{2})(|a_1\rangle_s - e^{i\varphi}|a_2\rangle_s) \) with \( \varphi \ll 1 \) is performed for the total system \[41\], i.e., \( \langle \psi_f | \psi_i \rangle \approx i\varphi/2 \), which is an imaginary number, then the final state of the pointer is

\[
\frac{1}{2} [D(-ia_1\eta) - e^{-i\varphi}D(-ia_2\eta)]|0\rangle_m \tag{46}
\]

For Eq. \[42\], when \( \varphi \ll 1 \) and \( \eta \ll 1 \), we can perform a small quantity expansion about \( \eta \) and \( \varphi \) till the second order, and then obtain

\[
\frac{1}{2}[i\varphi|0\rangle_m + i\eta(a_2 - a_1)|1\rangle_m]. \tag{47}
\]

Substituting Eq. \[47\] into Eq. \[16\], in this case of the near-orthogonal postselection, i.e., \( \langle \psi_f | \psi_i \rangle \approx i\varphi/2 \), we can find that

\[
\langle q \rangle = \sigma \frac{2(a_2 - a_1)\varphi\eta}{\varphi^2 + (a_2 - a_1)^2\eta^2} \tag{48}
\]

and

\[
\langle p \rangle = 0. \tag{49}
\]
When \( \varphi = \pm (a_2 - a_1) \eta \) we will have the largest displacement \( \pm \sigma \) in position space and when \( \varphi = 0 \), indicating that the postselected state of the system is orthogonal to the initial state of the system, i.e., \( \langle \psi_f | \psi_i \rangle = 0 \), the displacement of pointer in position space is zero. This amplification result is due to the superposition of \( |0 \rangle_m \) and \( |1 \rangle_m \). However, the displacement of the pointer in momentum space is always zero.

Obviously, the mechanism behind the amplification with Gaussian pointer [1] is also regarded as the superposition of \( |0 \rangle \) and \( |1 \rangle \) of the pointer in Fock space. Therefore, the standard scenario of weak measurement [1] can be also shown and understood by the Fock-state view where the pointer is a ground state [3]. It gives a view of the relationship between the weak measurement and other measurement techniques.

**Amplification using a phase shifter \( \theta \).**

According to the results of Ref. [3, 4], the time evolution operator of the Hamiltonian (5) in the main text is given by

\[
U(t) = \exp[-i(r(a^\dagger a + b^\dagger b) + b^\dagger b) + i(a^\dagger a) + i(x(t)c^\dagger - \xi^*(t)c)] \exp[-ic \pm \omega_m t],
\]

where \( \phi(t) = k^2(\omega_m t - \sin \omega_m t) \), \( \xi(t) = k(1 - e^{-i \omega_m t}) \), \( r = \omega_0 / \omega_m \), \( k = g / \omega_m \) is the scaled coupling parameter.

Suppose that one photon is input into the interferometer, and after the first beam splitter and a phase shifter \( \theta \) the initial state of the photon is \( |\psi_i(\theta)\rangle = (1/\sqrt{2})(|e^{i \theta} \rangle |0 \rangle_B + |0 \rangle_A |1 \rangle_B) \). The mirror is initialised in thermal state \( \rho_{hk}(z) \). After weakly coupled interacting [50] between one photon and the mirror, the time evolution of the total system leads to a state given by

\[
\rho(z, t) = \frac{1}{2} \sum_{n=0}^{\infty} 2^n |[1]_A |0 \rangle_B e^{i(\phi(t) + \theta)} \rho_D \langle 0 | A \rangle |1 \rangle B |n \rangle_m \langle n | m \rangle |D^\dagger(\xi)|1 \rangle A |0 \rangle B
\]

\[
e^{-i(\phi(t) + \theta)} + |0 \rangle A \langle 1 | B
\]

(51)

where \( \phi(t) = k^2(\omega_m t - \sin \omega_m t) \) is Kerr phase [3, 4] of one photon.

When a photon is detected in the dark port, in the language of weak measurement the postselected state of one photon is \( |\psi_f\rangle = (1/\sqrt{2})(|1 \rangle_1 |0 \rangle_B - |0 \rangle_1 |1 \rangle_B) \), which is nonorthogonal to \( |\psi_i(\theta)\rangle \), i.e., \( |\psi_f | \psi_i(\theta) \rangle \approx i \theta/2 \). Then the reduced state of the mirror after postselection for each \( n \) component is

\[
|\psi_1(\theta)\rangle = \frac{1}{\sqrt{2}} \left[ (|\psi_f\rangle |[1]_A \rangle |0 \rangle_B e^{i(\phi(t) + \theta)} \rho_D \langle 0 \rangle_A |1 \rangle_B |n \rangle_m \langle n | m \rangle \right]
\]

\[
= \frac{1}{2} [e^{i(\phi(t) + \theta)} \rho_D \langle n | m \rangle - |n \rangle m].
\]

(52)

Therefore, this is Eq. (6) in the main text.

For Eq. (52), over all \( n \) component, then the final total state of the pointer is \( \rho_{os} = (1 - z) \sum_{n=0}^{\infty} z^n |\psi_1(\theta)) \langle \psi_1(\theta) | \). Substituting Eq. (52) into Eq. (10), we can follow a two-step procedure to obtain the average displacement of the mirror: first, calculate the numerator of equation (10), then calculate the denominator of equation (10).

For the numerator of equation (10), we obtain

\[
\langle n | D^\dagger(\xi) q D(\xi) | n \rangle = \sigma (n | D^\dagger(\xi)) (c + \xi^\dagger) D(\xi) | n \rangle = \sigma (\xi(t) + \xi^\dagger(t)),
\]

using \( D^\dagger(\alpha) c D(\alpha) = c + \alpha, \ D^\dagger(\alpha)c^\dagger D(\alpha) = c^\dagger + \alpha^* \), and

\[
\langle n | q | n \rangle = \sigma \langle n | (c + \xi^\dagger) | n \rangle = 0,
\]

using \( \langle n | q | n \rangle = \sigma \langle n | (c + \xi^\dagger) | n \rangle = 0 \),

\[
e^{i(\phi(t) + \theta)} \langle n | q D(\xi) | n \rangle = \sigma e^{i(\phi(t) + \theta)} \langle n | (c + \xi^\dagger) D(\xi) | n \rangle,
\]

\[
e^{-i(\phi(t) + \theta)} \langle n | D^\dagger(\xi) q | n \rangle = \sigma e^{-i(\phi(t) + \theta)} \langle n | D^\dagger(\xi)(c + \xi^\dagger) | n \rangle,
\]

(55)

(56)

For Eq. (55), and using

\[
\langle l | D(\alpha) | n \rangle = \sqrt{n!} / l! \alpha^{(l-n)} e^{-\alpha^2} L_n^{(l-n)} (|\alpha|^2), (l \geq n),
\]

(57)
and
\[ \langle l | D^\dagger (\alpha) | n \rangle = \sqrt{n! \over l!} (-\alpha)^{(l-n)} \exp(-{1 \over 2} - \alpha^2) L_n^{(l-n)}(|-\alpha|^2), (l \geq n), \]
(58)

where \( L_n^k(x) \) is an associated Laguerre polynomial, we can find
\[ e^{i(\phi(t)+\theta)} \langle n | qD(\xi) | n \rangle = \sigma e^{i(\phi(t)+\theta)} [ \langle n+1 | D(\xi(t)) | n \rangle + \langle n-1 | D(\xi(t)) | n \rangle ] \]
\[ = \sigma e^{i(\phi(t)+\theta)} D_{n+1, n} + \sigma e^{i(\phi(t)+\theta)} D_{n, n-1}^* \]
(59)

with
\[ D_{n+1, n} = \xi(t) \exp(-{1 \over 2} |\xi(t)|^2) L_n^1(|\xi(t)|^2), n \geq 0 \]
(60)
\[ D_{n, n-1}^* = -\xi(t) \exp(-{1 \over 2} |\xi(t)|^2) L_n^1(|\xi(t)|^2), n \geq 1. \]
(61)

Using identity
\[ \sum_{n=0}^{\infty} L_n^k(x) z^n = (1 - z)^{-k-1} \exp(-xz/(1 - z)), \]
(62)
we have the following result
\[ (1 - z) \sum_{n=0}^{\infty} z^n D_{n+1, n} = {1 \over 1 - z} \exp(-{(1 + z)|\xi(t)|^2 \over 2(1 - z)}) \xi(t) \]
(63)

Setting \( n = n' + 1 \), and using Eq. (62)
\[ (1 - z) \sum_{n=0}^{\infty} z^n D_{n, n-1}^* = (1 - z) \sum_{n'=0}^{\infty} z^{n'+1} D_{n'+1, n'}^* \]
\[ = -{z \over 1 - z} \exp(-{(1 + z)|\xi(t)|^2 \over 2(1 - z)}) \xi^*(t) \]
(64)

Then we have
\[ (1 - z)e^{i(\phi(t)+\theta)} \sum_{n=0}^{\infty} \langle n | qD(\xi) | n \rangle = \sigma [ {1 \over 1 - z} \exp(-{(1 + z)|\xi(t)|^2 \over 2(1 - z)}) + i\phi(t) + i\theta) \xi(t) \]
\[ - {z \over 1 - z} \exp(-{(1 + z)|\xi(t)|^2 \over 2(1 - z)}) + i\phi(t) + i\theta) \xi^*(t) ] \]
(65)

Next, for the denominator of equation (10), and using Eq. (57) and Eq. (58), we find
\[ e^{i(\phi(t)+\theta)} \langle n | D^\dagger (\xi) | n \rangle = e^{i(\phi(t)+\theta)} \exp(-{1 \over 2} |\xi(t)|^2) L_n^0(|\xi(t)|^2), n \geq 0 \]
(66)

and
\[ e^{-i(\phi(t)+\theta)} \langle n | D^\dagger (\xi) | n \rangle = e^{-i(\phi(t)+\theta)} \exp(-{1 \over 2} |\xi(t)|^2) L_n^0(|\xi(t)|^2), n \geq 0 \]
(67)

For Eq. (66), using identity (62), we have the following result
\[ (1 - z)e^{i(\phi(t)+\theta)} \sum_{n=0}^{\infty} \langle n | D(\xi) | n \rangle = e^{i(\phi(t)+\theta)} \exp(-{(1 + z)|\xi(t)|^2 \over 2(1 - z)}) \]
(68)
So we can obtain the average displacement of the mirror
\[ \langle q(t) \rangle = (1 - z) \sum_{n=0}^{\infty} \langle n | D^\dagger(\xi) q D(\xi) | n \rangle - e^{i(\phi(t) + \theta)} \sum_{n=0}^{\infty} \langle n | q D(\xi) | n \rangle - e^{-i(\phi(t) + \theta)} \sum_{n=0}^{\infty} \langle n | D^\dagger(\xi) q | n \rangle \]
\[ - (1 - z)e^{i(\phi(t) + \theta)} \sum_{n=0}^{\infty} \langle n | D(\xi) | n \rangle - (1 - z)e^{-i(\phi(t) + \theta)} \sum_{n=0}^{\infty} \langle n | D^\dagger(\xi) | n \rangle \]
\[ = \sigma[\xi(t) + \xi^* (t) - \frac{1}{1 - z}\{\Phi(t) + \Phi^* (t) - z(\Phi^* (t) + \Phi^* (\xi))\}]/(2 - \Phi - \Phi^*), \] (69)
where \( \Phi^* = \exp(-\frac{(1+z)|\xi(\xi)|^2}{2(1-z)} + i\phi(t) + i\Omega) \) with \( \Omega = \theta \).
Therefore, this is Eq. (7) in the main text.
Note that the denominator \( \frac{1}{4}(2 - \Phi - \Phi^*) \) is the successful postselection probability being released from optomechanical cavity after \( t \).

Small quantity expansion about time for amplification

However, in order to observe the amplification effects appearing at time near \( T = 0 \), for Eq. (69) we can then perform a small quantity expansion about time \( T \) till the second order. Suppose that \( |\omega_m t - T| \ll 1 \), i.e., \( \omega_m t \ll 1 \), \( k \ll 1 \) and \( \theta \ll 1 \), then we can obtain
\[ \psi_1(n)_{\omega_m t \ll 1} \approx \frac{1}{2}[(1 + i\theta)(1 + ik\omega_m t(c + c^\dagger))|n\rangle - |n\rangle] \approx \frac{1}{2}[i\theta|n\rangle + ik\omega_m t(c + c^\dagger)|n\rangle]. \] (70)
Therefore, this is Eq. (8) in the main text.
Substituting Eq. (70) into Eq. (16), then
\[ \langle q(t) \rangle_n = \sigma 2\theta k\omega_m t(2n + 1)/[\theta^2 + k^2(\omega_m t)^2(2n + 1)]. \] (71)
Therefore, this is the average displacement \( \langle q(t) \rangle_n \) for \( \psi_1(n)_{\omega_m t \ll 1} \) plotted in Figure 2(b) in main text.
For Eq. (70), over all \( n \) component, then the final total state of the pointer is \( \rho_{os} \approx (1 - z) \sum_{n=0}^{\infty} z^n\psi_1(n)_{\omega_m t \ll 1} \psi_1(n)_{\omega_m t \ll 1} \) and substituting it into Eq. (16), then
\[ \langle q(t) \rangle_{\omega_m t \ll 1} = \sigma 2\theta k\omega_m t\frac{1 + z}{1 - z}/[\theta^2 + k^2(\omega_m t)^2]\frac{1 + z}{1 - z}. \] (72)
Based on Eq. (72), we then obtain the maximal positive value \( (1 + z)^{1/2}/\sigma \) (thermal fluctuation) or negative value \(-(1 + z)^{1/2}/\sigma \) when \( \theta = \pm(1 + z)^{1/2}/2k\omega_m t \), respectively. Therefore, the \( |\psi(n)\rangle \) components corresponding to the maximal positive and negative amplification, respectively, are \( |\psi_1(n)\rangle_{\max,\omega_m t \ll 1} = \sqrt{2}[(1 + z)^{1/2}/2\pm (c + c^\dagger)|n\rangle] \) (unnormalized). Then the mirror state achieving the maximal positive and negative amplification, respectively, are \( \rho_{os}(z, t) = \frac{1}{2}(1 - z) \sum_{n=0}^{\infty} z^n|\psi_1(n)\rangle_{\max,\omega_m t \ll 1}\langle\psi_1(n)\rangle_{\max,\omega_m t \ll 1} \). It is obvious that the amplification with thermal state pointer is much larger than that with pure state pointer \( \rho_{os}(z, t) \) since its maximal value is the ground state fluctuation \( \sigma \). Therefore, thermal noise effect of the pointer (mirror) is beneficial for the amplification of the mirror’s displacement.

Amplification using the displaced thermal state

Suppose that one photon is input into the interferometer, and after the first beam splitter the initial state of the photon is \( |\psi_0\rangle = \sqrt{2}/2(|1\rangle_A|0\rangle_B + |0\rangle_A |1\rangle_B) \). The mirror is initialized in displaced thermal state \( \rho_{th}(z, \alpha) \). When the photon interact weakly with the optomechanical system through \( D(\xi)D(\varphi) \), the evolution state of the total system is given by
\[ \rho(z, t) = \frac{(1 - z)}{2} \sum_{n=0}^{\infty} z^n|1\rangle_A|0\rangle_B e^{i\phi(t)} D(\xi) D(\varphi) + |0\rangle_A |1\rangle_B D(\varphi)|n\rangle_m D^\dagger(\varphi)D^\dagger(\xi)|1\rangle_A |0\rangle_B e^{-i\phi(t)} + D^\dagger(\varphi)|0\rangle_A |1\rangle_B \] (73)
where \( \phi(t) = k^2(\omega_m t - \sin \omega_m t) \) is kerr phase of one photon.

When a photon is detected in the dark port, in the language of weak measurement the postselected state of the one photon is \( |\psi_f\rangle = \frac{1}{\sqrt{2}}(|1\rangle_A|0\rangle_B - |0\rangle_A|1\rangle_B) \), which is orthogonal to \( |\psi_i\rangle \), i.e., \( \langle \psi_f | \psi_i \rangle = 0 \). Then the reduced state of the mirror after postselection for each component \(|n\rangle_m\) is given by

\[
|\chi_2(n)\rangle = \frac{1}{\sqrt{2}}(|\psi_f\rangle[[|1\rangle_A|0\rangle_B e^{i\phi(t)} D(\xi) D(\varphi) + |0\rangle_A|1\rangle_B D(\varphi)]|n\rangle_m] \\
= \frac{1}{2}[e^{i\phi(t)} D(\xi) D(\varphi)|n\rangle_m - D(\varphi)|n\rangle_m],
\]

(74)

In order to make the analysis simple, we can displace the above state to the origin point in phase space, defining \( |\psi_2(n)\rangle = D^\dagger(\xi)|\chi_2(n)\rangle \) and we can obtain

\[
|\psi_2(n)\rangle = \frac{1}{2}[e^{i\phi(t)} D^\dagger(\varphi) D(\xi) D(\varphi) - D^\dagger(\varphi) D(\varphi)]|n\rangle_m] \\
= \frac{1}{2}[e^{i(\phi(t) + \phi(\alpha, t))} D(\xi)|n\rangle_m - |n\rangle_m],
\]

(75)

where \( \phi(\alpha, t) = -i[\alpha\xi(t) - \alpha^*\xi^*(t)] \) is obtained by using the property of the displacement operators \( D(\alpha)D(\beta) = \exp[\alpha\beta^* - \alpha^*\beta]D(\beta)D(\alpha) \), due to noncommutativity of quantum mechanics [3].

Therefore, this is Eq. (9) in main text.

For Eq. (75), over all \( n \) component, then the final total state of the pointer is \( \rho_{\alpha n} = (1 - z) \sum_{n=0}^{\infty} z^n|\psi_2(n)\rangle\langle \psi_2(n)| \) and substituting it into Eq. (16), then we show the average displacement of the mirror’s position

\[
\langle q(t) \rangle = \sigma[\xi(t) + \xi^*(t)] - \frac{1}{1-z} [\Phi\xi(t) + \Phi^*\xi^*(t)] \\
- z(\Phi^*\xi(t) + \Phi\xi^*(t))] /(2 - \Phi - \Phi^*),
\]

(76)

where \( \Phi^* = \exp(-\frac{(1+z)|\xi(t)|^2}{2(1-z)}) + i\phi(t) + i\Omega \) with \( \Omega = \phi(\alpha, t) \). In order to obtain the above result, here we use two equations,

\[
\langle l|D(\alpha)|n\rangle = \sqrt{\frac{n!}{l!}} \alpha^{l-n} \exp(-\frac{1}{2}|\alpha|^2) L_n^{(l-n)}(|\alpha|^2), (l \geq n)
\]

(77)

and

\[
\sum_{n=0}^{\infty} L_n^k(x) z^n = (1 - z)^{-k-1} \exp(-xz/(1-z)),
\]

(78)

where \( L_n^k(x) \) is an associated Laguerre polynomial [7]. Note that the denominator of Eq. (78) \( \frac{1}{4}(2 - \Phi - \Phi^*) \) is the successful postselection probability being released from optomechanical cavity after \( t \).

Therefore, Eq. (76) is the average displacement \( \langle q(t) \rangle \) of the mirror for the state \(|\psi_2(n)\rangle \) plotted in Figure 3(a) in main text.

**Small quantity expansion about time for amplification**

However, in order to observe the amplification effects appearing at time near \( T = 0 \), for Eq. (75) we can then perform a small quantity expansion about time \( T \) till the second order. Suppose that \(|\omega_m t - T| \ll 1 \), i.e., \( \omega_m t \ll 1 \), \( k \ll 1 \) and \( 2k|\alpha|\zeta \ll 1 \), then we can obtain

\[
\psi_2(n)|_{\omega_m t \ll 1} = \frac{1}{2}[(1 + i2k|\alpha|\zeta)(1 + ik\omega_m t(c + c^\dagger))|n\rangle - |n\rangle] \\
= \frac{1}{2}[i2k|\alpha|n + ik\omega_m t(c + c^\dagger)|n\rangle],
\]

(79)
where $\zeta = \frac{1}{2}(\omega_m t)^2 \sin \beta + \omega_m t \cos \beta$.

Therefore, this is Eq. (10) in main text.

For Eq. (79), over all $n$ component, then the final total state of the pointer is $(1 - z) \sum_{n=0}^{\infty} z^n \psi_2(n)_{\omega_m t \ll 1} \langle \psi_2(n) \rangle_{\omega_m t \ll 1}$ and substituting it into Eq. (10), then

$$\langle q(t) \rangle_{\omega_m t \ll 1} = \sigma 4k^2 |\alpha| |\omega_m t| \frac{1 + z}{1 - z} / [4(k|\alpha|)^2 + k^2(\omega_m t)^2(1 + z)] \text{.}$$

(80)

Based on Eq. (80), we then obtain the maximal positive value $(\frac{1 + z}{1 - z})^{1/2} \sigma$ or negative value $-(\frac{1 + z}{1 - z})^{1/2} \sigma$ when $2k|\alpha| \zeta = \pm (\frac{1 + z}{1 - z})^{1/2} k \omega_m t$, respectively. Therefore, the $|\psi_2(n)\rangle$ components corresponding to the maximal positive and negative amplification, respectively, are $|\psi_2(n)\rangle_{\max, \omega_m t \ll 1} = (1/\sqrt{2})[(\frac{1 + z}{1 - z})^{1/2}|n\rangle \pm (c + c^\dagger)|n\rangle]$ (unnormalized).

Then the mirror state achieving the maximal positive and negative amplification, respectively, are $\rho_{Z}(z, t) = \frac{1}{2}(1 - z) \sum_{n=0}^{\infty} z^n |\psi_2(n)\rangle_{\max, \omega_m t \ll 1} \langle \psi_2(n) |_{\max, \omega_m t \ll 1}$. It is obvious that the amplification with displacement thermal state pointer is much larger than that with pure state pointer $|\psi\rangle$ since its maximal value is the ground state fluctuation $\sigma$. Therefore, thermal noise effect of the pointer (mirror) is beneficial for the amplification of the mirror’s displacement.

Dissipation effect in optomechanical system

The master equation (11) in the main text is given by

$$\frac{d \rho(t)}{dt} = -\frac{i}{\hbar}[H, \rho(t)] + \frac{\gamma_m}{(1 - z)} D[\alpha] + \frac{\gamma_m z}{(1 - z)} D[\alpha^\dagger],$$

(81)

where $D[\alpha] = o \rho(t) o^\dagger - o^\dagger o \rho(t)/2 - \rho(t) o^\dagger o/2$.

For the amplification scheme using a phase shifter $\theta$, at time $t \ll 1$, if we perform a Taylor expansion about $t = 0$ till the second order, the solution of the master equation is approximately

$$\rho(t) = \rho(0) + \frac{d \rho(t)}{dt} t + \frac{d^2 \rho(t)}{2 dt^2} t^2 \text{.}$$

(82)

When the initial state of the total system is $\rho(0) = |\psi_i(\theta)\rangle \langle \psi_i(\theta)| \otimes \rho_{th}(z)$ and after the postselecting state $|\psi_f\rangle$ is performed for the system in Eq. (82) and substituting it into Eq. (10), by carefully calculation, we can obtain

$$\langle q(t) \rangle_{\omega_m t \ll 1} = \sigma |\alpha| (\frac{1 + z}{1 - z}) k \omega_m t \sin \theta + k(\omega_m t)^2 (1 - \cos \theta) - \frac{\gamma_m z}{2 (1 - z)} k(\omega_m t)^2 \sin \theta) / 2$$

$$- 2 \cos \theta + \frac{1 + z}{1 - z} k(\omega_m t)^2 \cos \theta,$$

(83)

where $\gamma = \gamma_m / \omega_m$.

This is the average displacement $\langle q(t) \rangle_{\omega_m t \ll 1}$ of the mirror after postselection plotted in Figure 4(a) in main text.

For the amplification scheme using the displaced thermal state, at time $t \ll 1$, if we perform a Taylor expansion about $t = 0$ till the third order, the solution of the master equation is approximately

$$\rho(t) = \rho(0) + \frac{d \rho(t)}{dt} t + \frac{d^2 \rho(t)}{2 dt^2} t^2 + \frac{d^3 \rho(t)}{3 dt^3} t^3 \text{.}$$

(84)

When the initial state of the total system is $\rho(0) = |\psi_i\rangle \langle \psi_i| \otimes \rho_{th}(z, \alpha)$ and after the postselecting state $|\psi_f\rangle$ is performed for the system in Eq. (84) and substituting it into Eq. (10), by carefully calculation, we can obtain

$$\langle q(t) \rangle_{\omega_m t \ll 1} = \sigma |\alpha| (\frac{1 + z}{1 - z}) k^2(\omega_m t)^2 |\alpha| \cos \theta + 4k^2(\omega_m t)^2 (|\alpha| \cos \theta)^3 - \frac{5}{3} (\frac{1 + z}{1 - z}) k^2(\omega_m t)^3 |\alpha| \cos \theta$$

$$- 3 |\alpha| (\frac{1 + z}{1 - z}) k^2(\omega_m t)^3 (|\alpha| \cos \theta)^3) / 2 (1 + z) k^2(\omega_m t)^2 + 2 k^2(\omega_m t)^2 (|\alpha| \cos \theta)^2 - \frac{1}{12} (\frac{1 + z}{1 - z}) k^2(\omega_m t)^3 |\alpha| \cos \theta,$$

(85)

This is the average displacement $\langle q(t) \rangle_{\omega_m t \ll 1}$ of the mirror after postselection plotted in Figure 5(a) in main text.
The amplification without postselection in optomechanics

The time evolution operator of the Hamiltonian (5) in the main text is given by

$$ U(t) = \exp[-ir(a^\dagger a + b^\dagger b)\omega_m t] \exp[i(a^\dagger a)^2 \phi(t)] \exp[a^\dagger a(\xi(t)c^\dagger - \xi^*(t)c)] \exp[-ic^\dagger c\omega_m t], $$  \hspace{1cm} (86)

where $\phi(t) = k^2(\omega_m t - \sin \omega_m t)$, $\xi(t) = k(1 - e^{-i\omega_m t})$, $r = \omega_0/\omega_m$, $k = g/\omega_m$ is the scaled coupling parameter.

As shown Fig. 1 in main text, we use only single cavity A. When thermal state $\rho_{th}(z)$ is considered as a pointer in cavity A, and if one photon is weakly coupled with the mirror using (86), it can be found that the mirror will be changed from $\rho_{th}(z)$ to a displacement thermal state,

$$ \rho_{th}(z,\xi) = D(\xi(t))\rho_{th}(z)D^\dagger(\xi(t)). $$ \hspace{1cm} (87)

According to the expression of the displacement

$$ \langle \hat{q} \rangle = Tr(\rho_{th}(z,\xi)\hat{q}) - Tr(\rho_{th}(z)\hat{q}) $$

with $\hat{q} = \sigma(c + c^\dagger)$, there is the the average position displacement of the pointer without postselection,

$$ \langle \hat{q} \rangle = 2k(1 - \cos \omega_m t)\sigma. $$ \hspace{1cm} (88)

However, when displacement thermal state $\rho_{th}(z,\alpha)$ is considered as a pointer in cavity A, and if one photon is weakly coupled with the mirror (86), it can be found that the mirror will be changed from $\rho_{th}(z,\alpha)$ to a displacement thermal state,

$$ \rho_{th}(z,\varphi,\xi) = D(\xi(t))\rho_{th}(z,\varphi)D^\dagger(\xi(t)), $$ \hspace{1cm} (89)

where $\varphi(t) = \alpha e^{-i\omega_m t}$. According to the expression of the displacement

$$ \langle \hat{q} \rangle = Tr(\rho_{th}(z,\varphi,\xi)\hat{q}) - Tr(\rho_{th}(z,\varphi)\hat{q}) $$

with $\hat{q} = \sigma(c + c^\dagger)$, there is the the average position displacement of the pointer without postselection,

$$ \langle \hat{q} \rangle = 2k(1 - \cos \omega_m t)\sigma. $$ \hspace{1cm} (90)

From Eq. (86) and Eq. (92), it can be seen that the position displacement of the mirror caused by radiation pressure of one photon can not more than $4k\sigma$ for any time $t$. In the literature [10], we know that if the displacement of the mirror can be detected experimentally it should be not smaller than $\sigma$, implying that the displacement of the mirror reach strong-coupling limit, so $k = g/\omega_m$ can not be bigger than 0.25 in weak coupling condition [10]. When $k = g/\omega_m \leq 0.25$ in weak-coupling regime, the maximal displacement of the mirror $4k\sigma$ can not be more than $(1/z)^{1/2}\sigma$, i.e., thermal fluctuation of the mirror, therefore the displacement of the mirror caused by one photon can not be detected.

**Probability $P$**

The overall probability of a single photon (12) in the main text, generating the superposition state of $|n\rangle$ and $(c + c^\dagger)|n\rangle$, is given by

$$ P = \frac{1}{4} \int_0^\infty \kappa \exp(-\kappa t)(\frac{1+z}{1-\frac{z}{4}})^2 + \Omega^2)dt. $$ \hspace{1cm} (93)

where $\Omega = \theta, \phi(\alpha, t)$.

For the first scheme, $P = \frac{1+z}{1-\frac{z}{4}} \frac{k^2\omega_m^2}{4} + \beta^2$, and for the second scheme, let $|\alpha| = (\frac{1+z}{4})^{1/2}/2$ and $\beta = 0$, then

$$ P = \frac{1+z}{1-\frac{z}{4}} \frac{k^2\omega_m^2}{4} + \frac{\beta^2}{4} \frac{2\omega_m^2}{k^2\omega_m^2 + 4\omega_m^2} / 2. $$

Therefore, for the first scheme, $P$ is approximately $6.94k^2$ with $\kappa = 1.2 \times 10^4\omega_m$, $\theta = 0.005$ and $z = 0.999999999$, and for the second scheme, $P$ is approximately $5k^2$ with $\kappa = 2 \times 10^4\omega_m$, $z = 0.999999999$. 
Average displacement of the any pointer in postselected weak measurement

The interaction Hamiltonian between the system and the pointer is

\[ H_{\text{int}} = \chi(t) A \otimes q. \quad (94) \]

Suppose the initial state of the system is \( \ket{\Phi_i} = \cos \theta \ket{a_1} + \sin \theta \ket{a_2} \), and the initial state of the pointer is \( \rho_m \). The system is postselected in the state \( \ket{\Phi_f} = \cos \theta_f \ket{a_1} + e^{i\varphi} \sin \theta_f \ket{a_2} \) after the interaction \((94)\), and the pointer collapses to the state (unnormalized)

\[ \rho_{fm} = \langle \Phi_f | \exp(-i \chi A \otimes q) | \Phi_i \rangle \langle \Phi_i | \rho_m \exp(i \chi A \otimes q) | \Phi_f \rangle \]
\[ = [\cos \theta_i \cos \theta_f \exp(-i \chi a_1 q) + e^{i \phi} \sin \theta_i \sin \theta_f \exp(-i \chi a_2 q)] \rho_m [\cos \theta_i \cos \theta_f \exp(i \chi a_1 q) + e^{-i \phi} \sin \theta_i \sin \theta_f \exp(i \chi a_2 q)]. \quad (95) \]

The success probability of postselection is

\[ P_s = Tr(\rho_{fm}). \quad (96) \]

However, if \( \chi \ll 1 \) and \( \varphi \ll 1 \), and when \( \theta_i = \frac{\pi}{4} - \varepsilon \) and \( \theta_f = -\frac{\pi}{4} + \varepsilon \) with \( \varepsilon \ll 1 \), \( \rho_{fm} \quad (95) \) is approximately

\[ \rho_{fm} \approx \frac{1}{4} [4 \varepsilon + i \varphi + i \chi (a_2 - a_1) q] \rho_m [4 \varepsilon - i \varphi - i \chi (a_2 - a_1) q]. \quad (97) \]

The average displacement of the pointer observable \( M \ (M = p, q) \) is

\[ \langle M \rangle_f = \frac{Tr(M \rho_{fm})}{Tr(\rho_{fm})} - Tr(M \rho_m). \quad (98) \]

Note that

\[ Tr(M \rho_{fm}) \approx (16 \varepsilon^2 + \varphi^2) \langle M \rangle_{\rho_m} + i 4 \varepsilon \chi (a_2 - a_1) \langle [M, q] \rangle_{\rho_m} + \varphi \chi (a_2 - a_1) \langle \{ M, q \} \rangle_{\rho_m} + \chi^2 (a_2 - a_1)^2 \langle q M q \rangle_{\rho_m}, \quad (99) \]

and the normalized coefficient is

\[ A_0 = Tr(\rho_{fm}) \approx (16 \varepsilon^2 + \varphi^2) + 2 \varphi \chi (a_2 - a_1) \langle q \rangle_{\rho_m} + \chi^2 (a_2 - a_1)^2 \langle q^2 \rangle_{\rho_m}, \quad (100) \]

where \( Tr(\rho_m) \) as \( \langle \cdot \rangle_{\rho_m} \) for short throughout the paper.

By substituting \((99)\) and \((100)\) into \((98)\), we find that

\[ \langle M \rangle_f = \frac{1}{A} \left[ (16 \varepsilon^2 + \varphi^2) \langle M \rangle_{\rho_m} + i 4 \varepsilon \chi (a_2 - a_1) \langle [M, q] \rangle_{\rho_m} + \varphi \chi (a_2 - a_1) \langle \{ M, q \} \rangle_{\rho_m} + \chi^2 (a_2 - a_1)^2 \langle q M q \rangle_{\rho_m} \right], \quad (101) \]

where \([\cdot]\) and \(\{\cdot\}\) denote commutation and anticommutation rules, respectively. \((101)\) is the average displacement of the any pointer.

If the density \( \rho_m \) satisfy the symmetry condition, i.e., \( F(-x) = F(x) \), the expression \((101)\) becomes

\[ \langle M \rangle_f = \frac{1}{A} \left( i 4 \varepsilon \chi (a_2 - a_1) \langle [M, q] \rangle_{\rho_m} + \varphi \chi (a_2 - a_1) \langle \{ M, q \} \rangle_{\rho_m} \right), \quad (102) \]

where \( A = 16 \varepsilon^2 + \varphi^2 + \chi^2 (a_2 - a_1)^2 \langle q^2 \rangle_{\rho_m} \) is a normalized coefficient. It is obvious that the displacement is determined by \( i 4 \varepsilon \chi (a_2 - a_1) \langle [M, q] \rangle_{\rho_m} \) and \( \varphi \chi (a_2 - a_1) \langle \{ M, q \} \rangle_{\rho_m} \). The former and latter are both caused by interference term of this state \((97)\). In other words, the key to understand the amplification is the coherence (superposition) between the different pointers after postselection.
There are two cases for Eq. (102): (1) when \( \varphi = 0 \) and \( \varepsilon \neq 0 \), Eq. (102) becomes

\[
\langle M \rangle_f = \frac{1}{A_1} i 4 \varepsilon \chi (a_2 - a_1) \langle [M, q] \rangle_{\rho_m},
\]

where \( A_1 = 16 \varepsilon^2 + \chi^2 (a_2 - a_1)^2 \langle q^2 \rangle_{\rho_m} \) is a normalized coefficient. (103) correspond to the displacement proportional to real weak value, the result is holds up if and only if \( M = p \); (2) when \( \varphi \neq 0 \) and \( \varepsilon = 0 \), Eq. (102) becomes

\[
\langle M \rangle_f = \frac{1}{A_2} \varphi \chi (a_2 - a_1) \langle \{ M, q \} \rangle_{\rho_m},
\]

where \( A_2 = \varphi^2 + \chi^2 (a_2 - a_1)^2 \langle q^2 \rangle_{\rho_m} \) is a normalized coefficient. (104) correspond to the displacement proportional to imaginary weak value, the result is holds up if and only if \( M = q \).

Suppose that \( \rho_m = |0\rangle \langle 0| \) (ground state) or \( |\alpha\rangle \langle \alpha| \) (coherent state), the maximal amplification value of (104) is the ground state fluctuation \( \sigma \), which are exactly confirmed by Eq. (17) in Ref. [8] and Eq. (25) in Ref. [6], respectively. When \( \rho_m = S(\xi)|\alpha\rangle \langle \alpha|S^\dagger(\xi) \) with \( \xi = re^{i\theta} \), the maximal amplification value of (104) is the squeezing ground-state fluctuation \( \pm e^r \sigma \), which is exactly confirmed by Eq. (15) in Ref. [11]. However, when \( \rho_m = (1 - z) \sum_{n=0}^{\infty} z^n |n\rangle_m \langle n|_m, z = e^{-\hbar \omega_m / k_B T} \), the maximal amplification value of (103) and (104) are, respectively, \( \pm \left( \frac{1 + z}{1 - z} \right)^{1/2} \frac{1}{2 \sigma} \) (confirmed by (21) in SMs) and \( \pm \left( \frac{1 + z}{1 - z} \right)^{1/2} \frac{1}{2 \sigma} \) (thermal fluctuation), which is exactly confirmed by Eq. (4) in main text.

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