2D (4,4) HYPERMULTIPLETS

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Abstract

The structure of on-shell and off-shell 2D, (4,4) supersymmetric scalar multiplets is investigated, in components and in superspace. We reach the surprising result that there exist eight distinct on-shell versions and an even greater variety of off-shell ones. The off-shell generalised tensor and relaxed N = 4 multiplets are introduced in superspace, and their universal invariant self-interaction is constructed.

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1 Introduction. It has almost always been assumed that 2D, \( N = 4 \) supersymmetry leads to relatively unique field theory representations. Possibly for this reason, \( N = 4 \) supergravity and \( N = 4 \) superstrings have been thought to be pretty much unique too. There has even been a proposal that an \( N = 4 \) superstring is the paradigmatic generator of all string models [1]. More recently, however, we have found increasing evidence that the uniqueness may not be the case [2, 3]. We think it is useful to learn more about manifest \( N = 4 \) supersymmetry and \( N = 4 \) scalar multiplets since these play a crucial role in providing any Lagrangian and off-shell description of related supergravities and superstrings. We want to maintain in any case the \( SU(2) \) part of the maximal \( SO(4) \cong SU(2) \times SU(2)' \) internal symmetry rotating \( N = 4 \) supercharges. It is the \( SU(2) \) that is a part of the 2D, \( N = 4 \) superconformal symmetry.

2 On-shell, 2D, \( N = 4 \) hypermultiplets. Hypermultiplet theory began with 4D, \( N = 2 \) models when Fayet [4] introduced this supersymmetry representation (see ref. [5] for its description in superspace). We can directly reduce their results to 2D, \( N = 4 \) to find the following superdifferential equations. We call this the original hypermultiplet (OHM) theory,

\[
D_{\alpha i}A^{i\hat{I}} = 2\delta_i^j C^{i\hat{I}j\bar{\psi}_\alpha} , \quad D_{\alpha}^{i\hat{I}} = 2C^{ij}\bar{\psi}_\alpha^i , \quad \bar{D}_{\alpha i}\bar{\psi}_\beta^i \bar{\psi}_\alpha = 0 . \tag{1}
\]

This \((4,4)\) hypermultiplet is related to the SM-III theory in the recently introduced classification scheme for \((4,0)\) hypermultiplets [6]. The proof of this can be carried out simply. In the above equations \( A^{i\hat{I}} \) is restricted to satisfy the equation \((A^{i\hat{I}})^* = C_{ij}C_{\hat{I}j\hat{J}}A^{i\hat{J}} \). This implies that not all of \( A^{i\hat{I}} \) is independent. A solution to this algebraic constraint is given by \( A^{i\hat{I}} = (C_{ij}\mathcal{A}_j, i\hat{A}) \) for \( \hat{I} = 1, 2, \) respectively. The equations above, rewritten in terms of \( A_i \), can then be seen to be exactly equivalent to the SM-III theory of ref. [1],

2D, \( N = 4 \) SM-III

\[
D_{\alpha i}\mathcal{A}_j = C_{ij}\bar{\pi}_\alpha , \quad \bar{D}_{\alpha}^{i\hat{I}}\mathcal{A}_j = \delta_i^j \rho_\alpha , \quad \bar{D}_{\alpha i}\rho_\beta = 0 , \quad D_{\alpha i}\bar{\pi}_\beta = 0 , \quad D_{\alpha i}\pi_\beta = 0 , \quad D_{\alpha i}\rho_\beta = i2(\gamma^c)_{\alpha\beta}\partial_c\mathcal{A}_i , \quad \bar{D}_{\alpha i}\pi_\beta = i2C^{ij}(\gamma^c)_{\alpha\beta}\partial_c\mathcal{A}_j . \tag{2}
\]

However, OHM is not the only 2D, \( N = 4 \) on-shell hypermultiplet which exists. Each of the following also forms a 2D, \( N = 4 \) on-shell representation:

2D, \( N = 4 \) SM-I

\[
D_{\alpha i}\mathcal{A} = \varphi_{\alpha i} , \quad \bar{D}_{\alpha}^{i\hat{I}}\bar{\mathcal{B}} = C^{ij}\varphi_{\alpha j} , \quad \bar{D}_{\alpha}\mathcal{A} = 0 , \quad D_{\alpha i}\bar{\mathcal{B}} = 0 , \quad D_{\alpha i}\mathcal{A} = 0 . \tag{2}
\]
\[ D_{\alpha i} \varphi_{\beta j} = i2C_{ij}(\gamma^c)_{\alpha \beta} \partial_c \bar{B} \], \quad \bar{D}_{\alpha i} \varphi_{\beta j} = i2\delta_j^i(\gamma^c)_{\alpha \beta} \partial_c \alpha \] \quad \text{(3)}

2D, \( N = 4 \) SM-II

\[ D_{\alpha i} \varphi = \lambda_{\alpha i} \], \quad D_{\alpha i} \varphi_j^k = i[\delta_i^k \alpha_j - \frac{1}{2} \delta_j^k \lambda_{\alpha i}] \],
\[ D_{\alpha i} \lambda_{\beta j} = 0 \], \quad \bar{D}_{\alpha i} \lambda_{\beta j} = i\delta_j^i(\gamma^c)_{\alpha \beta} \left( \partial_a \alpha \right) + 2(\gamma^a)_{\alpha \beta} \left( \partial_a \varphi_j^i \right) \quad \text{(4)}

2D, \( N = 4 \) SM-IV

\[ \bar{D}_{\alpha i} B_j = \delta_j^i \psi + i2 \psi_{\alpha j} \], \quad D_{\alpha i} B_j = 0 \], \quad \psi^\alpha = (\psi^\alpha)^* \quad \text{(5)}

\[ D_{\alpha i} \psi_{\beta j}^k = \delta_i^k (\gamma^c)_{\alpha \beta} \partial_c B_j - \frac{1}{2} \delta_{j}^{k} (\gamma^c)_{\alpha \beta} \partial_c B_i \], \quad \psi_{\alpha j}^i = (\psi_{\alpha j}^i)^* \quad \text{(6)}

Thus, we see that the classification scheme \cite{6} used for the (4,0) hypermultiplets completely carries over to the case of the full (4,4) hypermultiplets. However, with the full (4,4) supersymmetry, there appear even more such multiplets because we can apply parity twists to replace some of the scalar fields in a given hypermultiplet by pseudo-scalar fields. One such example is provided by

\[ D_{\alpha i} f = 2C_{ij} \rho_{\alpha j} \], \quad \bar{D}_{\alpha i} f = 0 \], \quad D_{\alpha i} g = 2(\gamma^3)_{\alpha \beta} \bar{\rho}_{\beta i} \], \quad \bar{D}_{\alpha i} g = 0 \],
\[ D_{\alpha i} \rho_{\beta j} = i2\delta_i^j(\gamma^3 \gamma^c)_{\alpha \beta} \left( \partial_c \bar{\gamma} \right) \], \quad D_{\alpha i} \rho_{\beta j} = iC_{ij} (\gamma^c)_{\alpha \beta} \left( \partial_c f \right) \quad \text{(6)}

This particular example represents replacing two of the scalar fields in the SM-I hypermultiplet by pseudo-scalars. A second similar example is given by

\[ D_{\alpha i} \tilde{A} = iC_{ij} \tilde{\lambda}_{\alpha j} \], \quad D_{\alpha i} \tilde{B} = - C_{ij} (\gamma^3)_{\alpha \beta} \tilde{\lambda}_{\beta j} \], \quad D_{\alpha i} \tilde{L} = i\tilde{\lambda}_{\alpha i} \], \quad D_{\alpha i} \tilde{R} = (\gamma^3)_{\alpha \beta} \tilde{\lambda}_{\beta i} \]
\[ D_{\alpha i} \tilde{\lambda}_{\alpha j} = - C_{ij} \left[ (\gamma^c)_{\alpha \beta} \left( \partial_c \tilde{A} \right) + i(\gamma^3 \gamma^c)_{\alpha \beta} \left( \partial_c \tilde{B} \right) \right] \]
\[ \bar{D}_{\alpha i} \tilde{\lambda}_{\beta j} = \delta_j^i \left[ (\gamma^c)_{\alpha \beta} \left( \partial_c \tilde{L} \right) + i(\gamma^3 \gamma^c)_{\alpha \beta} \left( \partial_c \tilde{R} \right) \right] \quad \text{(7)}

So we see that there are many distinct on-shell hypermultiplet representations. It is a challenge to attempt to classify how many such representations exist. Fortunately, there is a tool available that can be used to put an upper limit on this number. In the previous work \cite{6}, we have been able to classify all (4,0) hypermultiplets as well as (4,0) minus spinor multiplets. There are four representations of each. A full on-shell (4,4) hypermultiplet is just the sum of a (4,0) hypermultiplet plus a (4,0) minus spinor multiplet. Therefore, the maximum number of \( N = 4 \) hypermultiplets is sixteen. We are going to amplify this point later on. It should be noticed that only the SM-II
(RHM) and SM-III (OHM) exist as hypermultiplet theories in 4D. The rich profusion of hypermultiplets is therefore a solely 2D phenomenon.

2 Off-shell hypermultiplets. The problem of finding the off-shell form of each on-shell hypermultiplet formulation is an unsolved one, and we are not going to solve it in full here. The previously known off-shell formulations of \(N = 4\) hypermultiplets (with finite number of auxiliary fields) include two twisted hypermultiplet versions (TM-I and TM-II) and the ‘relaxed’ hypermultiplet (RHM) [2].

The twisted-I (TM-I) multiplet was the first off-shell description provided for a 2D, \(N = 4\) hypermultiplet. Its supersymmetry transformation laws are

\[
D_{\alpha i}F = 2C_{ij} \psi^{j}_\alpha , \quad D_{\alpha i}S = -i\bar{\psi}_{\alpha i} , \quad D_{\alpha i}P = (\gamma^3)_{\alpha}^{\beta} \bar{\psi}_{\beta i} ,
\]

\[
D_{\alpha i}\psi^{j_\beta} = \delta_i^{j_\beta} \left[ (\gamma^c)_{\alpha}^{\beta}(\partial_c S) + i(\gamma^3\gamma^c)_{\alpha}^{\beta}(\partial_c P) \right] 
+ \frac{1}{2}\left[ \delta_i^{j_\beta}(\gamma^3)_{\alpha}^{\beta} A + i\delta_{\alpha}^{\beta} A_{i_\beta} \right] ,
\]

\[
\bar{D}_{\alpha i}\psi^{j_\beta} = iC_{ij}(\gamma^c)_{\alpha}^{\beta}(\partial_c F) ,
\]

\[
D_{\alpha i}A = -2(\gamma^3\gamma^c)_{\alpha}^{\beta}\partial_c \bar{\psi}_{\beta i} ,
\]

\[
D_{\alpha i}A^{j_\beta} = 4(\delta_j^{j_\beta} - \frac{1}{2}\delta_{j_\beta}^{i_\delta}(\gamma^c)_{\alpha}^{\beta}\partial_c \bar{\psi}_{\beta i}) .
\]

All the fields are real (for \(A_{i_\beta} = (A_{j_\beta})^*\)) with the exception of \(F\) and \(\psi_{\alpha i}\). The TM-I multiplet is a parity twisted version of the SM-I multiplet where one scalar field is replaced by a pseudoscalar.

The invariant component-level action takes the form

\[
S_{TM-I} = \int d^2x \left[ \frac{1}{2}S\Box S + \frac{1}{2}P\Box P + \frac{1}{2}F\Box F + i\psi^{\alpha i}(\gamma^c)_{\alpha}^{\beta}\partial_c \bar{\psi}_{\beta i} 
- \frac{1}{2}A^2 - \frac{1}{16}A^{j_\beta}A_{i_\beta} \right] ,
\]

or in terms of unconstrained prepotentials \((V\) and \(V_{i_\beta}\)) we find

\[
S_{TM-I} = -\int d^2xd^4\zeta d^4\bar{\zeta} \left[ VA + V^{j_\beta}A_{i_\beta} \right] .
\]

The second off-shell hypermultiplet was the twisted-II (TM-II) theory discovered by Ivanov and Krivonos [8]. A description consistent with their work is given by

\[
D_{\alpha i}T = (\gamma^3)_{\alpha}^{\beta}\Psi_{\beta i} ,
\]

\[\text{\footnotesize \textsuperscript{3}}\text{A solution may require \textit{infinite} numbers of auxiliary fields in \textit{some} cases [6], but it is precisely the situation we want to avoid.}\]
\[ D_{\alpha i} \chi^j_k = i \left[ \delta^i_k \Psi_{\alpha j} - \frac{1}{2} \delta^k_j \Psi_{\alpha i} \right] , \]
\[ \chi^i_k = 0 , \quad \chi^j_k (\chi^i_k)^* = 0 , \]
\[ D_{\alpha i} \Psi_{\beta j} = \frac{i}{2} C_{\alpha \beta} C_{ij} \overline{T} , \]
\[ D_{\alpha i} \overline{T} = 0 , \quad m - (m)^* = 0 , \quad n - (n)^* = 0 , \]
\[ \bar{D}_{\alpha} i \Psi_{\beta j} = i \delta^j_k (\gamma^3 \gamma^a)_{\alpha \beta} (\partial_a T) + 2 (\gamma^a)_{\alpha \beta} (\partial_a \chi^i_k) \]
\[ + \frac{i}{2} C_{\alpha \beta} \delta^j_k m + \frac{1}{2} (\gamma^3)_{\alpha \beta} \delta^j_k n . \]
\[ D_{\alpha i} J = -i 4 C_{ij} (\gamma^a)_{\alpha \beta} (\partial_a \overline{\Psi}_{\beta j}) , \]
\[ D_{\alpha i} n = -i 2 (\gamma^3 \gamma^a)_{\alpha \beta} (\partial_a \overline{\Psi}_{\beta i}) , \]
\[ D_{\alpha i} m = -2 (\gamma^a)_{\alpha \beta} (\partial_a \overline{\Psi}_{\beta i}) . \] 

(11)

Here the complex fields are \( J \) and \( \Psi_{\alpha i} \). This multiplet is a parity-twisted version of the SM-II hypermultiplet, where again one scalar is replaced by a pseudoscalar.

An invariant component-level action is
\[ S_{\text{TM-II}} = \int d^2 x \left[ \frac{1}{2} T \Box T + \chi^i_j \Box \chi^j_i + i \Psi^a_{\alpha \gamma}(\gamma^a)_{\alpha \beta} \partial_c \overline{\Psi}_{\beta i} \right. \]
\[ \left. - \frac{1}{8} \left( m^2 + n^2 + J \overline{J} \right) \right] . \] 

(12)

The superfield form of this action is given by
\[ S_{\text{TM-II}} = -\int d^2 x d^4 \zeta d^4 \overline{\zeta} \left[ K m + L n \right] - \left[ \int d^2 x d^4 \zeta \Lambda J + \text{h.c.} \right] , \] 

(13)

in terms of the real superfield prepotentials \( K \) and \( L \) and chiral superfield prepotential \( \Lambda \).

One of the interesting features of the hypermultiplet pair TM-I and TM-II is that they are dual to each other in such a way that they can form a supersymmetric invariant that introduces mass without the introduction of a central charge [6, 8]. It is the long-held but false belief that potentials for 2D hypermultiplets require central charges [3]. In terms of superfields, this mass term takes the form
\[ S_{\text{N=4, mass}} = M_0 \int d^2 x d^4 \zeta d^4 \overline{\zeta} \left[ V T + \frac{1}{2} V_i^j \chi^i_k \right] , \] 

(14)

or, alternatively,
\[ S_{\text{N=4, mass}} = -\tilde{M}_0 \int d^2 x d^4 \zeta d^4 \overline{\zeta} \left[ K S + L P \right] - \tilde{M}_0 \left[ \int d^2 x d^4 \zeta \Lambda F + \text{h.c.} \right] . \] 

(15)

At the component level, these are equivalent to
\[ S_{\text{N=4, mass}} = M_0 \int d^2 \sigma \left[ \frac{1}{2} m S - \frac{1}{2} n P - \frac{1}{4} \left( J \overline{F} + \overline{J} F \right) \right] . \]
\[
- \frac{1}{8} \mathcal{X}_i^j A_j^i - \frac{1}{2} \mathcal{T} A + ( \bar{\psi}^{\alpha i} \psi_{\alpha i} + \text{h.c.} ) \quad . \tag{16}
\]

The RHM at the time of its discovery appeared as a 4D, N = 2 multiplet. It exactly corresponds to the off-shell formulation of the SM-II theory! The component RHM action is given by

\[
S_{\text{RHM}, N=4} = \int \! d^2 x \left[ \varphi \Box \varphi + L_{ij} \Box L_{ij} + i \psi^{\alpha i} \partial_{\alpha \beta} \bar{\psi}^{\beta j} \right.
\]
\[
+ ( \lambda^{\alpha i} \chi_{\alpha i} + \text{h.c.} ) - 2 ( \lambda^{\alpha i j k} \chi_{\alpha i j k} + \text{h.c.} )
\]
\[
+ \frac{1}{16} N \tilde{N} + \frac{3}{8} ( K^{ij} \tilde{K}_{ij} - M^{ij} \tilde{M}_{ij} ) - \frac{5}{4} C^{ijkl} L_{ijkl}
\]
\[
- \frac{1}{36} G^{\alpha \beta} G_{\alpha \beta} - \frac{3}{8} ( A^{\alpha i j} A_{\alpha i j} - V^{\alpha i j} V_{\alpha i j} ) \right] , \tag{17}
\]

or, in terms of superfields, as

\[
S_{\text{RHM}, N=4} = \int \! d^2 x d^4 \zeta d^{\tilde{4}} \bar{\zeta} \left[ ( \lambda_{\alpha i} \rho^{\alpha i} + \bar{\chi}_{\alpha i} \bar{\rho}^{\alpha i} ) + L^{ijkl} X_{ijkl} \right] , \tag{18}
\]

where the unconstrained superfield potentials \( \rho^{\alpha i} \) and \( X_{ijkl} \) have been introduced.

In order to discuss the 2D, N = 4 OHM superspace constraints, first, let us change our notation for the superspace covariant derivatives \( D_{\alpha}^i \hat{I} \), which now carry doublet internal symmetry indices \((i, \hat{I}), i = 1, 2, \hat{I} = 1', 2'\), of the maximal automorphism group \( SO(4) \cong SU(2) \otimes SU(2)' \) of 2D, N = 4 supersymmetry, and satisfy the reality condition \((D_{\alpha}^i)^* = C_{ij} D_{\alpha}^j \bar{\rho}^{\alpha i} \) and the algebra \( \{ D_{\alpha}^i, D_{\beta}^j \} = i C^{ij} \hat{C}^{\alpha \beta} \).

Being dimensionally reduced to 2D, the N = 4 OHM (= FS hypermultiplet) complex superfields \( \mathcal{A}_i \) satisfy the constraints

\[
D^{i(\hat{I}^i \mathcal{A}^j)}_{\alpha} = 0 \quad , \tag{19}
\]

which put the theory on-shell, since they imply the equations of motion, \( \Box \mathcal{A}^i = 0 \).

One of the ways out of this problem is to introduce the \textit{generalised} off-shell 2D, N = 4 tensor multiplets \( L^{i_1 \cdots i_n} \), \( n = 2, 3, \ldots \), which are defined by the constraints

\[
D^{i(k \ L^{i_1 \cdots i_n})}_{\alpha} = 0 \quad , \tag{20}
\]

and the reality condition \((L^{i_1 \cdots i_{2p}})^* = C_{i_1 j_1} \cdots C_{i_{2p} j_{2p}} L^{j_1 \cdots j_{2p}} \) in the case of an even number of indices, \( n = 2p \). The tensors \( L^{i_1 \cdots i_n} \) are totally symmetric with respect to their \( SU(2) \) indices. In particular, when \( n = 2 \), the superfield \( L^{ij} \) just gives the standard 4D, N = 2 tensor multiplet \( \mathcal{L} \) dimensionally reduced to 2D. The other way is to use the harmonic superspace \( \mathcal{H} \).

\[\text{It is worthwhile noting that this dimensional reduction is precisely equivalent to the introduction of the TM-II multiplet.} \]

\[\text{Another way is to use the harmonic superspace \( \mathcal{H} \).} \]
generalised 4D, N = 2 tensor multiplets were introduced in ref. [12]. The off-shell 2D, N = 4 generalised tensor multiplets can be used to ‘relax’ the constraints for the OHM \((n = 1)\) and the ordinary tensor multiplet \((n = 2)\), for example
\[
D^i_\alpha (A^j) = D^j_\alpha A^{ijk}, \quad D^{ij}_\alpha A^{jkl} = 0, \tag{21}
\]
or
\[
D^{i(i} L^{jk)} = D^{i(i} L^{ijkl}, \quad D^{i(i} L^{ijklm)} = 0, \tag{22}
\]
which define the relaxed multiplets of the type \((1–3)\) and \((2–4)\), respectively, according to the number of the external \(SU(2)\) indices involved. More general constructions of the type \((1–3–\ldots–(2q+1))\) or \((2–4–\ldots–(2q))\) can also be introduced [12]. In particular, the case of \((2–4)\) defines the relaxed hypermultiplet of ref. [11]. The system of tensorsuperfields with infinite relaxation \((q = \infty)\) precisely corresponds to the harmonic superfields of ref. [10], where these tensor superfields appear as the coefficients at harmonic zweibein monomials. All such constructions are just different off-shell realizations of \(N = 4\) hypermultiplet, with finite numbers of auxiliary fields.

One of the interesting tools that worked well as a way to provide a uniform classification of \((4,0)\) hypermultiplets [6] was the use of ‘spectroscopic analysis’ as a way to describe all of the \((4,0)\) hypermultiplets [3]. A simple extension of that works for the \((4,4)\) case too. The four basic hypermultiplets listed above in eqs. (1)–(5) can be thought of as

| (4, 4) HM | Spin-0 SU(2) Rep\(^\text{Parity}\) | Spin-\(\frac{1}{2}\) SU(2) Rep\(^\text{Parity}\) |
|-----------|---------------------------------|---------------------------------|
| SM – I    | \(4s^+\)                        | \(\frac{1}{2}\)                  |
| SM – II   | \(1s^+1p^+\)                    | \(\frac{1}{2}\)                  |
| SM – III  | \(\frac{1}{2}\)                 | \(4s^+\)                        |
| SM – IV   | \(\frac{1}{2}\)                 | \(1s^+1p^+\)                    |

Table I

where we use a notation with a + superscript for a scalar spin-0 field (or a spinor) and a − superscript for a pseudoscalar spin-0 field (or an axial spinor). It should be noticed that the definition of parity requires spinors of both + and − type to be in the supermultiplet. Since for the heterotic case only one handedness was present, there was no need to introduce this degree of freedom in the classification scheme.

It is now clear how we should think of the additional hypermultiplets in eqs. (6), (7), (8) and (11). These are just the cases of spin-0 combinations \(2s^+2s^−, 3s^+s^−\),

\(^6\)For a earlier and different view of these theories see ref. [13].
and $1s^{-1}p^+$, respectively. A complete enumeration of all independent multiplets consists of the spin-0 combinations

$$4s^+ , 3s^+ s^- , 2s^+ 2s^- , 1s^+ 1p^+ , 1s^+ 1p^- ,$$

as well as the spin-1/2 combinations

$$4s^+ , 1s^+ 1p^+ .$$

The spectroscopic analysis suggests that there are seven 2D independent hypermultiplets. However, there is actually a two-fold degeneracy in the $2s^+ 2s^-$ case (see equations 6 and 7). So this ultimately gives eight multiplets.

4 ($4,0$) analysis of on-shell $N = 4$ hypermultiplets. The scalars and spinors of all the ($4,0$) hypermultiplets actually form real spinor representations of $Spin(2,2)$ [6]. The same statement is true for the minus spinor multiplets (heterotic fermion multiplets) too.

The ($4,0$) hypermultiplets SM-I and SM-II can be described in terms of four real spin-0 fields denoted by $\varphi_A$ and four Majorana spinors denoted by $\Psi^{-\hat{A}}$ whose supersymmetry variations take the form

$$\delta_Q \varphi_A = i \alpha^p (L_p)_A \hat{A} \Psi^{-\hat{A}} , \quad \delta_Q \Psi^{-\hat{A}} = \alpha^p (R_p)_A \hat{A} \partial_4 \varphi_A ,$$

in terms of four real constant Grassmann parameters $\alpha^p$. The real quantities $(L_p)_A \hat{A}$ and $(R_p)_A \hat{A}$ satisfy

$$(L_p)_A ^{\hat{A}} (R_q)_\hat{B} ^{\hat{B}} + (L_q)_A ^{\hat{A}} (R_p)_\hat{B} ^{\hat{B}} = - 2 \delta_{pq} (I)_A ^{\hat{B}} ,$$

$$(R_p)_A ^{\hat{A}} (L_q)_\hat{B} ^{\hat{B}} + (R_q)_A ^{\hat{A}} (L_p)_\hat{B} ^{\hat{B}} = - 2 \delta_{pq} (I) _{\hat{A}} ^{\hat{B}} ,$$

and the L-matrices and R-matrices are thus generalised $4 \times 4$ Pauli matrices. The SM-I multiplet is associated with the set

$$L_1 = i \sigma^1 \otimes \sigma^2 ; \quad L_2 = i \sigma^2 \otimes I ; \quad L_3 = - i \sigma^3 \otimes \sigma^2 ; \quad L_4 = - I \otimes I ;$$

$$R_1 = i \sigma^1 \otimes \sigma^2 ; \quad R_2 = i \sigma^2 \otimes I ; \quad R_3 = - i \sigma^3 \otimes \sigma^2 ; \quad R_4 = + I \otimes I ,$$

and the SM-II multiplet is associated with

$$L_1 = i \sigma^2 \otimes \sigma^3 ; \quad L_2 = - i I \otimes \sigma^2 ; \quad L_3 = i \sigma^2 \otimes \sigma^1 ; \quad L_4 = + I \otimes I ;$$

$$R_1 = i \sigma^2 \otimes \sigma^3 ; \quad R_2 = - i I \otimes \sigma^2 ; \quad R_3 = i \sigma^2 \otimes \sigma^1 ; \quad R_4 = - I \otimes I .$$

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7We regard a multiplet and one of its twisted versions to be the same if one can be obtained from the other by a simple redefinition involving $\gamma^3$ acting on the spinor in the supermultiplet. If this is not the case we say the two multiplets are independent.
For the SM-III and SM-IV multiplets, the four real scalar fields are denoted by \( \varphi_\hat{A} \) and the four real spinors by \( \Psi^-_\hat{A} \) with supersymmetry variations

\[
\delta_Q \varphi_\hat{A} = i \alpha^+ \, p(\hat{R}_p)_\hat{A}^A \Psi^-_\hat{A} \quad , \quad \delta_Q \Psi^-_\hat{A} = \alpha^+ \, p(\hat{L}_p)_\hat{A} \partial_\hat{\Psi}^+ \varphi_\hat{A} .
\]

The SM-III multiplet is associated with the set in eq. (28) and SM-IV multiplet is associated with the set in eq. (29).

Very similar results follow for the spinor multiplets. In real notation, MSM-I and MSM-II take the respective forms (below \( F_\hat{A} \) denote the auxiliary fields)

\[
\delta_Q \Psi^+_\hat{A} = i \alpha^+ \, p(\hat{L}_p)_\hat{A}^A \hat{F}_\hat{A} \quad , \quad \delta_Q \hat{F}_\hat{A} = \alpha^+ \, p(\hat{R}_p)_\hat{A} \partial_\hat{\Psi}^+ \Psi^+_\hat{A} ,
\]

where MSM-I is associated with the representation in eq. (28) and MSM-II is associated with the representation in eq. (29). For MSM-III and MSM-IV we have

\[
\delta_Q \Psi^+_\hat{A} = i \alpha^+ \, p(\hat{R}_p)_\hat{A}^A \hat{F}_\hat{A} \quad , \quad \delta_Q \hat{F}_\hat{A} = \alpha^+ \, p(\hat{L}_p)_\hat{A} \partial_\hat{\Psi}^+ \Psi^+_\hat{A} ,
\]

where MSM-III is associated with the representation in eq. (28) and MSM-IV is associated with the representation in eq. (29).

Our task now is to investigate how many ways we can glue the \((4,0)\) spinor multiplets to the \((4,0)\) hypermultiplets to obtain an on-shell \((4,4)\) supersymmetry representation. Since we are only considering on-shell theories, we set the auxiliary fields to zero. Also any time the Dirac equation appears, it can be set to zero.

If we attempt to ‘glue’ the \((4,0)\) SM-I or SM-II multiplets to either MSM-I or MSM-II, the form of the supersymmetry variations can only be

\[
\delta_Q \varphi_\hat{A} = i \alpha^+ \, p(\hat{L}_p)_\hat{A}^A \hat{F}_\hat{A} \quad , \quad \delta_Q \hat{F}_\hat{A} = \alpha^+ \, p(\hat{L}_p)_\hat{A} \partial_\hat{\Psi}^+ \Psi^+_\hat{A} .
\]

In the attempt to ‘glue’ the \((4,0)\) SM-I or SM-II multiplets to either MSM-III or MSM-IV, the form of the supersymmetry variations can only be

\[
\delta_Q \varphi_\hat{A} = i \alpha^+ \, p(\hat{L}_p)_\hat{A}^A \hat{F}_\hat{A} \quad , \quad \delta_Q \hat{F}_\hat{A} = \alpha^+ \, p(\hat{L}_p)_\hat{A} \partial_\hat{\Psi}^+ \Psi^+_\hat{A} .
\]

The attempt to extend SM-III and SM-IV to full on-shell theories means that \((4,4)\) supersymmetry variations must take the forms

\[
\delta_Q \varphi_\hat{A} = i \alpha^+ \, p(\hat{R}_p)_\hat{A}^A \hat{F}_\hat{A} \quad , \quad \delta_Q \hat{F}_\hat{A} = \alpha^+ \, p(\hat{R}_p)_\hat{A} \partial_\hat{\Psi}^+ \Psi^+_\hat{A} .
\]
\[ \delta_Q \Psi^-_A = \alpha^+ p (L_\mu)_A \dot{A} \partial_+ \varphi^\dot{A}, \quad \delta_Q \Psi^+_A = \beta^- p (\dot{\mathcal{P}}_\mu)_A \dot{\partial}_- \varphi^\dot{B}, \quad (35) \]

when ‘gluing’ to either MSM-I or MSM-II multiplets. Similarly, the extension of SM-III and SM-IV to full on-shell (4,4) theories means that (4,4) supersymmetry variations must take the forms

\[ \delta_Q \varphi^\dot{A} = i \alpha^+ p (R_\mu)_A \dot{A} \partial^{\dot{A}}, \quad \delta_Q \Psi^-_A = \alpha^+ p (L_\mu)_A \dot{A} \partial^\dot{A}, \quad \delta_Q \Psi^+_A = \beta^- p (\dot{\mathcal{P}}_\mu)_A \dot{\partial} \varphi^\dot{B}, \quad (36) \]

when ‘gluing’ to either MSM-III or MSM-IV multiplets. The condition for full on-shell (4,4) supersymmetry is precisely that the operator equation

\[ [\delta_Q(1), \delta_Q(2)] = i2 \delta_{pq}(\alpha^+_1 \alpha^-_2 + \beta^-_1 \beta^+_2 \partial_\pm) \quad (37) \]

is satisfied on all fields subject to the use of the Dirac equation on spinors. This will be satisfied if

\[ (J_\mu)_A^B (J_q)_B^C + (J_q)_A^B (J_\mu)_B^C = -2 \delta_{pq} (I)_B^C, \quad (a) \]

\[ (\bar{J}_\mu)_A^B (J_q)_B^C + (J_q)_A^B (\bar{J}_\mu)_B^C = -2 \delta_{pq} (I)_B^C, \quad (b) \]

\[ (K_\mu)_A^\dot{A} (\bar{K}_q)_\dot{A}^\dot{B} + (\bar{K}_q)_A^\dot{A} (K_\mu)_\dot{A}^\dot{B} = -2 \delta_{pq} (I)_A^\dot{B}, \quad (c) \]

\[ (\bar{K}_\mu)_A^\dot{A} (K_q)_\dot{A}^\dot{B} + (K_q)_A^\dot{A} (\bar{K}_\mu)_\dot{A}^\dot{B} = -2 \delta_{pq} (I)_A^\dot{B}, \quad (d) \]

\[ (\bar{P}_\mu)_A^\dot{A} (\bar{P}_q)_\dot{A}^\dot{B} + (\bar{P}_q)_A^\dot{A} (\bar{P}_\mu)_\dot{A}^\dot{B} = -2 \delta_{pq} (I)_A^\dot{B}, \quad (e) \]

\[ (\bar{P}_\mu)_A^\dot{A} (\bar{P}_q)_\dot{A}^\dot{B} + (\bar{P}_q)_A^\dot{A} (\bar{P}_\mu)_\dot{A}^\dot{B} = -2 \delta_{pq} (I)_A^\dot{B}, \quad (f) \]

\[ (Q_\mu)_A^B (Q_q)_B^C + (Q_q)_A^B (Q_\mu)_B^C = -2 \delta_{pq} (I)_B^C, \quad (g) \]

\[ (\bar{Q}_\mu)_A^B (\bar{Q}_q)_B^C + (\bar{Q}_q)_A^B (\bar{Q}_\mu)_B^C = -2 \delta_{pq} (I)_B^C, \quad (h) \]

with no other restrictions required! It is a fact that there are no set of four independent tensors (with the appropriate index structure) that satisfy equations a, b, g and h.

We thus conclude that there can be only eight on-shell 2D hypermultiplets, in agreement with the spectroscopic analysis.

5 (4,4) hypermultiplet NLSM. Remarkably, there exists the universal N = 4 supersymmetric non-linear sigma-model (NLSM) action for any kind (and number) of the generalised and/or relaxed tensor multiplets. First, let us introduce the function

\[ G(L_1^{i_1}, \ldots, L_n^{i_n}) \] as a solution to the equations

\[ \nabla^j^A G \equiv \left( D^{i_1}_A + \xi D^{i_2}_A \right) G = 0 \quad (38) \]

8Interestingly enough, these equations do have solutions for (4,3) hypermultiplets!
where a complex projective parameter $\xi$ has been introduced. It is not difficult to check that the general solution to eq. (38) can be represented in the form (cf. ref. [12])

$$G = G(\xi, Q_n(\xi)) , \quad Q_n(\xi) = \xi_1 \cdots \xi_n L_1^{i_1} \cdots L_n^{i_n} , \quad \xi_i = (1, \xi) , \quad (39)$$

where the function $G$ on the r.h.s. of this equation is now an arbitrary differentiable meromorphic function of $\xi$ and $Q_n$’s. In the case of the relaxed hypermultiplets (21) and (22), one should use

$$Q_{1R}(\xi) = Q_1(\xi) - \frac{4}{3} \frac{\partial Q_3}{\partial \xi} , \quad Q_{2R}(\xi) = Q_2(\xi) - \frac{5}{4} \frac{\partial Q_4}{\partial \xi} , \quad (40)$$

instead of $Q_1$ and $Q_2$, respectively, while any dependence on $Q_3(\xi)$ or $Q_4(\xi)$ is also allowed. The function $G(\xi, Q_n(\xi))$ is chiral in the sense of eq. (38). Therefore, integrating it over the remaining superspace coordinates results in the invariant action (cf. refs. [14, 15])

$$S_{\text{NLSM}} = \int d^2 x \frac{1}{2\pi i} \oint_C \frac{d\xi}{(1 + \xi^2)^4} C_{ij} C^{\alpha\beta} \tilde{\nabla}_{\alpha}^i \tilde{\nabla}_{\beta}^j G(\xi, Q_n(\xi)) + \text{h.c.} , \quad (41)$$

where the new, linearly independent on $\nabla$’s, superspace derivatives

$$\tilde{\nabla}_{\alpha}^i = \xi D_{\alpha}^{i_1} - D_{\alpha}^{i_2} , \quad (42)$$

have been introduced. The contour $C$ in the complex $\xi$-plane should be chosen in such a way that the points $\xi_c = \pm i$, where the linear independence of $\nabla$’s and $\tilde{\nabla}$’s breaks down, will be outside the contour.

This construction of invariant NLSM action apparently breaks down one of the $SU(2)$ internal symmetries, but maintains another $SU(2)'$, which is just necessary for the full 2D, $N = 4$ superconformal symmetry to be ultimately represented by the ‘small’ linear $N = 4$ superconformal algebra, from the viewpoint of conformal field theory [10]. Still, there is a chance of having the full $SO(4)$ internal symmetry (and, hence, a larger $N = 4$ superconformal algebra in the corresponding $N = 4$ superconformal field theory), when the function $G$ and the contour $C$ are specially chosen. Indeed, $\xi$ is the inhomogeneous $CP(1)$ coordinate, whose $SU(2)$ transformation law is given by

$$\xi' = \frac{a \xi - b}{a + b \xi} , \quad \left( \begin{array}{cc} a & b \\ -b & a \end{array} \right) \in SU(2) , \quad a\bar{a} + b\bar{b} = 1 \quad . \quad (43)$$

This obviously implies $Q'_n(\xi') = (a + b\xi)^{-n} Q_n(\xi)$. Hence, the action (41) will be $SO(4)$ invariant provided that

$$G(\xi', Q'_n) = (a + b\xi)^{-2} G(\xi, Q_n) , \quad (44)$$
up to an additive total derivative.

It is presently believed \[5\] that the OHM does not allow any non-trivial off-shell formulation with a finite number of auxiliary fields. This is, however, not in conflict with our results, since (i) any theory in terms of the relaxed off-shell combination $Q_{1R}$ actually has a larger number of propagating degrees of freedom, and (ii) there is no on-shell condition in the case of the generalised tensor (or relaxed tensor) multiplets ($n \geq 2$), unlike the OHM case of $n = 1$.

Among the components of the 4D, $N = 2$ generalised tensor multiplet,

\[ L^{i_1\cdots i_n}, \quad \psi^{i_1\cdots i_{n-1}}_{\alpha}, \quad C^{i_1\cdots i_{n-2}}_{\alpha\beta}, \quad V^{i_1\cdots i_{n-2}}_{\alpha\beta}, \quad \chi^{i_1\cdots i_{n-3}}_{\alpha\beta}, \quad D^{i_1\cdots i_{n-4}}_{\alpha\beta}, \quad (45) \]

there is a 4D vector $V$, which is only conserved when $n = 2$. \[9\] The vector fields for $n > 2$ can be easily eliminated via their algebraic equations of motion in the NLSM action, whereas in the case of $n = 2$ the dimensional reduction of the 4D conserved vector results in the 2D conserved vector $V'_a$ and two auxiliary scalars. The latter also have algebraic equations of motion, whereas the former can be substituted by a propagating scalar $B$ via $V'_a = \varepsilon^{ab}_{a\beta} \partial_b B$, which results in the NLSM torsion. The situation is similar in the case of the 4D, $N = 2$ vector multiplets dimensionally reduced to 2D \[17\]. Therefore, it is the presence of the TM-I and TM-II multiplets that introduces torsion in the 2D, $N = 4$ NLSM. \[10\] Unlike the 4D, $N= 2$ NLSM \[12, 14\] which is non-renormalisable and does not always have simple geometrical interpretation, the 2D, $N=4$ NLSM of eq. (41) is either hyper-Kählerian (in the absence of torsion) or, at least, quaternionic, and it is UV finite to all orders of perturbation theory, because of $(4,4)$ supersymmetry \[17\].

The existence of many distinct 2D hypermultiplets implies the existence of many distinct $N = 4$ ‘mirror maps’ between them, as well as between the corresponding NLSM’s. They are the $N = 4$ analogues of the ‘mirror symmetry’ familiar from the $N = 2$ case.

Note added. After our paper was completed, we have been informed that the harmonic superspace description of the interacting TM-II had been recently given by E. Ivanov and A. Sutulin in Nucl. Phys. B432 (1994) 246.

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\[9\] It can be easily checked by counting the numbers of the off-shell bosonic and fermionic degrees of freedom which must coincide.

\[10\] This gives another reason to call them ‘twisted’.
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