Torsors under finite and flat group schemes of rank \( p \)
with Galois action

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Abstract

In this note we study the geometry of torsors under flat and finite commutative group schemes of rank \( p \) above curves in characteristic \( p \), and above relative curves over a complete discrete valuation ring of inequal characteristics. In both cases we study the Galois action of the Galois group of the base field on these torsors. We also study the degeneration of \( \mu_p \)-torsors from characteristic 0 to characteristic \( p \) and show that this degeneration is compatible with the Galois action. We then discuss the lifting of torsors under flat and commutative group schemes of rank \( p \) from positive to zero characteristics. Finally, for a proper and smooth curve \( X \) over a complete discrete valuation field of inequal characteristics we show the existence of a canonical Galois equivariant filtration on the first étale cohomology group of the geometric fibre of \( X \) with values in \( \mu_p \).

0. Introduction. In this note we study the geometry of torsors under flat and finite commutative group schemes of rank \( p \) above curves in characteristic \( p \) (cf. I), and above curves over a discrete complete valuation ring of inequal characteristics (cf. II). For a connected scheme \( Z \) over a field of characteristic \( p > 0 \) we define the group \( H^1_{\text{fppf}}(Z)_p \) of “mixed torsors” of rank \( p \), and in case \( Z \) is a semi-stable curve we define the group \( H^1_{\text{fppf}}(Z)^{\text{kum}}_p \) which classify “mixed kummerian torsors” of rank \( p \). These torsors arise naturally in the case of degeneration of \( \mu_p \)-torsors from characteristic 0 to characteristic \( p \) (cf. 1.7 and [Sa-1]). In all these cases we study the canonical action of the Galois group of the base field on these torsors.

In paragraph II we consider a discrete complete valuation ring \( R \) of inequal characteristics with fraction field \( K \) and residue field \( k \), and a formal \( R \)-scheme \( X \) of finite type which is normal flat over \( R \), and smooth of relative dimension 1. We denote by \( G_K \) the Galois group of a separable closure of \( K \). Let \( X_K \) (resp. \( X_K \)) denote the geometric generic fibre of \( X \) viewed as a rigid analytic space (resp. the geometric special fibre of \( X \)). We study the degeneration of \( \mu_p \)-torsors of \( X_K \). Our main result is the following which in
particular states that the degeneration of $\mu_p$-torsors of $X_K$ is compatible with the Galois action:

**Theorem.** (cf. 2.5.1 and 2.7) There exists a canonical specialisation group homomorphism:

$$Sp : H^1_{et}(X_K, \mu_p) \to H^1_{fppf}(X_K, \mu_p)$$

which is $G_K$-equivariant. Moreover there exists a canonical specialisation map (this is not a group homomorphism):

$$Sp : H^1_{et}(X_K, \mu_p) \to H^1_{fppf}(X_K)_p$$

which is $G_K$-equivariant.

Hier we consider the groups $H^1_{fppf}(X_K, \mu_p)$ and $H^1_{fppf}(X_K)_p$ as $G_K$-sets via the canonical quotient $G_K \to G_k$ of $G_K$, where $G_k$ is the Galois group of the residue field $k$ of $R$, and via the canonical action of $G_k$ on these groups. Also by construction the kernel of $Sp : H^1_{et}(X_K, \mu_p) \to H^1_{fppf}(X_K, \mu_p)$ corresponds to those $\mu_p$-torsors of $X_K$ which degenerate to either an $\alpha_p$ or an étale torsor of $X_K$.

In paragraph III we explain the degeneration of $\mu_p$-torsors above the boundaries of formal fibres of formal $R$-curves at closed points, and define their degeneration type. In paragraph IV we discuss the lifting of torsors under flat and finite commutative group schemes of rank $p$ from characteristic $p$ to characteristic 0, and show in particular that such torsors above proper curves can always be lifted to characteristic zero (cf. 4.5). Finally for a scheme $X$ which is proper, smooth, of relative dimension 1 over a discrete complete valuation ring of inequal characteristics we define a canonical decreasing filtration on the group $H^1_{et}(X_{\bar{\eta}}, \mu_p)$, where $X_{\bar{\eta}}$ is the geometric generic fibre of $X$ (cf. 5.1). Moreover we show that this filtration is equivariant under the action of the Galois group of the base field (cf. 5.2).

This paper is the first of a serie of papers [Sa] and [Sa-1] where we compute the vanishing cycles arising from the degeneration of $\mu_p$-torsors above curves as well as the semi-stable reduction of these torsors and the Galois action on them.

I. Torsors under finite and flat group schemes of rank $p$ in characteristic $p$.

In this section we discuss the geometry of torsors under finite and flat group schemes of rank $p$ above curves in characteristic $p$. For such torsors we introduce the notion of “conductor” and “residue” at the “critical points”, and we carefully explain the Galois action on them and the fact that conductor and residue are invariant under this action. We
also introduce for a semi-stable curve the group $H^1_{fppf}(\ )^\text{kum}$ which classifies “kummerian mixed torsors” and which appear naturally when considering the degeneration of $\mu_p$-torsors from zero to positive characteristics (cf. [Sa-1]).

1.1. Kummer theory.

Let $X$ be a scheme and let $l$ be a positif integer. The following sequence is exact on $X_{fppf}$:

$$ (1) \ 1 \to \mu_l \to \mathbb{G}_m \xrightarrow{x^l} \mathbb{G}_m \to 1 $$

Let $f : Y \to X$ be a $\mu_l$-torsor. Then there exists an open covering $U := (U_s)_s$ of $X$ and invertible fonctions $u_s \in \Gamma(U_s, \mathcal{O}_X)^*$, which are defined up to multiplication by a $p$-power, such that above $U_s$ the torsor $f$ is given by an equation $T^p_s = u_s$ (cf. [Mi], III, 4).

Assume now that $X$ is a scheme over a field $k$. Let $\kbar$ be an algebraic closure of $k$, and let $G_k$ be the Galois group of the separable closure $k_{\text{sep}}$ of $k$ in $\kbar$. The group $G_k$ acts canonically by automorphisms on $\overline{X} := X \times_k \kbar$, and this action induces in a natural way an action of $G_k$ on $\mu_l$-torsors of $\overline{X}$. There exists a canonical homomorphism:

$$ (1') \ G_k \to \text{Aut} \ H^1_{fppf}(\overline{X}, \mu_l) $$

More precisely, with the same notations as above, an element $\sigma \in G_k$ acts on $\overline{X}$ hence on coverings of $\overline{X}$. The element $\sigma$ associates to the covering $U := (U_s)_s$ the covering $U^\sigma := (U_s^\sigma)_s$. To a $\mu_l$-torsor $f : \overline{Y} \to \overline{X}$ is then associated the $\mu_l$-torsor $f^\sigma : \overline{Y}^\sigma \to X$, which is locally defined by an equation $T^p_s = u_s^\sigma$, where $u_s^\sigma$ is the image via $\sigma$ of $u_s$ which is a unit on $U_s^\sigma$.

1.2. In what follows and unless otherwise is specified, we will use the following notations: let $U$ be a smooth geometrically connected algebraic curve over a field $k$ of characteristic $p > 0$, and let $X$ be the smooth compactification of $U$. Let $G_k$ denote the Galois group of a separable closure $k_{\text{sep}}$ of $k$ in $\kbar$. The group $G_k$ acts canonically by automorphisms on $\overline{X} := X \times_k \kbar$, and denote by $\overline{U}$ the smooth compactification of $\overline{U}$. If $U \neq X$, let $S := X - U$ and let $\overline{S} := S \times_k \kbar$.

1.3. $\mathbb{Z}/p\mathbb{Z}$-Torsors in characteristic $p > 0$. The following sequence is exact for the étale topology:

$$ (2) \ 0 \to \mathbb{Z}/p\mathbb{Z} \to \mathbb{G}_a \xrightarrow{x^p - x} \mathbb{G}_a \to 0 $$

Let $f : V \to U$ be a non trivial $\mathbb{Z}/p\mathbb{Z}$-torsor. Let $Y$ be the smooth compactification of $V$ and let $f' : Y \to X$ be the canonical morphism, which is finite of degree $p$, generically separable, and eventually ramified above $X - U$. There exists an open covering $(U_s)_s$ of $U$ and regular fonctions $a_s \in \Gamma(U_s, \mathcal{O}_X)$, which are defined up to addition of elements of the form $b^p - b$, such that above $U_s$ the torsor $f$ is given by an equation $T^p_s - T_s = a_s$. Above
exists a canonical homomorphism:
\[ F \text{ where } U \to \text{the set of regular differential forms on } V \]

be the set of zeros of \( w \). Cartier operation
\[ C \text{ an integer prime to } m \]
the only singularities of \( V \) which are contained in \( U \). If \( m_i \) is posif then \( f' \) is étale above \( x_i \). Otherwise \( f' \) ramify above \( x_i \), and the contribution to the degree of the different of the point \( x_i \) is \((-m_i + 1)(p - 1)\). In the later case we call the integer \( \text{cond}_{x_i}(f) := -m_i \) the \textbf{conductor} of the above torsor \( f \) at the point \( x_i \) (this is the classical Hasse conductor). In the case where \( f \) is étale above \( x_i \) we define \( \text{cond}_{x_i}(f) := 0 \) to be the \textbf{conductor} of \( f \) at \( x_i \). We define the \textbf{residue} \( h_i := \text{res}_{x_i}(f) \) of \( f \) at the point \( x_i \) in any case to be 0.

The Galois group \( G_k \) acts in a natural way on \( \mathbb{Z}/p\mathbb{Z} \)-torsors of \( \mathcal{U} \). More precisely there exists a canonical homomorphism:

\[
(2') \quad G_k \to \text{Aut } H^1_{\text{ét}}(\mathcal{U}, \mathbb{Z}/p\mathbb{Z})
\]

Let \( \sigma \in G_k \). Let \( f : \mathcal{V} \to \mathcal{U} \) be an étale \( \mathbb{Z}/p\mathbb{Z} \)-torsor, and let \( f^\sigma : \mathcal{V}^\sigma \to \mathcal{U} \) be the \( \mathbb{Z}/p\mathbb{Z} \)-torsor associated to \( f \) via the above action \((2')\). If \( f \) is locally given by an equation \( T^p_s - T_s = a_s \), then \( f^\sigma \) is locally given by the equation \( T^p_s - T_s = a^\sigma_s \), where \( a^\sigma_s \) is the image of \( a_s \) via \( \sigma \) which is a regular function on \( U^\sigma_s \). Moreover for a point \( x_i \in \mathcal{S} \) we have \( \text{cond}_{x_i}(f) = \text{cond}_{x_i}^\sigma(f^\sigma) \) (resp. \( \text{res}_{x_i}(f) = \text{res}_{x_i}^\sigma(f^\sigma) \)), where \( x_i^\sigma \in \mathcal{S} \) is the image of \( x_i \) via \( \sigma \).

\[ 1.4. \quad \mu_p \text{-Torsors. } \text{Let } f : V \to U \text{ be a } \mu_p \text{-torsor. Then } f \text{ is a finite radicial morphism of degree } p. \text{ In particular } V \text{ is homeomorphic to } U. \text{ Let } (U_s)_s \text{ and } u_s \in \Gamma(U_s, \mathcal{O}_X)^* \text{ be as in } 2.1. \text{ The logarithmic differential form } \omega := du_s/u_s \text{ of } u_s \text{ is a global differential form on } X, \text{ it is the \textbf{differential form} associated to the torsor } f \text{ (it is well defined). Let } \{y_j\}_{j \in \mathbb{Z}} \subset U \text{ be the set of zeros of } w \text{ which are contained in } U, \text{ and let } m_j - 1 := \text{ord}_{y_j}(\omega), \text{ with } m_j \geq 2 \text{ an integer prime to } p. \text{ Locally for the étale topology the torsor } f \text{ is defined above the point } y_j \text{ by an equation } T^p_j = t_j^{m_j} \text{ where } t_j \text{ is a uniformising parameter at } y_j. \text{ The unique point } z_j \in V \text{ above the point } y_j \text{ is a singular point of } V, \text{ and } V \text{ is unibranche at } y. \text{ Moreover the only singularities of } V \text{ lie above the zeros of } w \text{ which are contained in } U. \text{ For a point } x \in X \text{ we call the integer } \text{cond}_x(f) := m_x := -(\text{ord}_x(\omega) + 1) \text{ the } \textbf{conductor} \text{ of the torsor } f \text{ at the point } x. \text{ We define the } \textbf{residue} \ h_x \in \mathbb{F}_p \text{ of the torsor } f \text{ at the point } x \text{ to be } 0, \text{ unless } m_x = 0, \text{ in which case } h_x := \text{res}_x(\omega). \text{ The differential form } \omega \text{ is fixed under the Cartier operation } C. \text{ In fact the following sequence is exact on } U_{\text{ét}}:
\]

\[
(3) \quad 0 \to \mathcal{O}_U^* \xrightarrow{F} \mathcal{O}_U^* \xrightarrow{d\log} \Omega_X^1 \xrightarrow{C} \Omega_X^1 \to 0
\]

where \( F \) is the Frobenius, and \( d\log(u) := du/u \). Moreover \( H^1_{\text{fppf}}(U, \mu_p) \) is identified with the set of regular differential forms on \( U \) which are fixed by \( C \) (cf. \cite{Mi}, III, 4).

The Galois group \( G_k \) acts in a natural way on \( \mu_p \)-torsors of \( \mathcal{U} \). More precisely there exists a canonical homomorphism:

\[
(3') \quad G_k \to \text{Aut } H^1_{\text{fppf}}(\mathcal{U}, \mu_p)
\]
Let $\sigma \in G_k$. Let $f : \mathcal{V} \to \mathcal{U}$ be a $\mu_p$-torsor, and let $f^\sigma : \mathcal{V}^\sigma \to \mathcal{U}$ be the $\mu_p$-torsor associated to $f$ via the above action (3'). If $f$ is locally given by an equation $T^p_s = u_s$, then $f^\sigma$ is locally given by the equation $T^p_s = u^\sigma_s$, where $u^\sigma_s$ is the image of $u_s$ via $\sigma$, which is an invertible function on $U^\sigma_s$. In particular if $\omega := du_s/u_s$ is the differential form associated to $f$, then the differential form associated to $f^\sigma$ is $\omega^\sigma := du^\sigma_s/u^\sigma_s$. The zeros of $\omega^\sigma$ contained in $\mathcal{U}$ are the images of those of $\omega$ via $\sigma$. Moreover for a point $x_i \in \mathcal{S}$ we have $\text{cond}_{x_i}(f) = \text{cond}_{x_i}(f^\sigma)$ (resp. $\text{res}_{x_i}(f) = \text{res}_{x_i}(f^\sigma)$), where $x_i^\sigma \in \mathcal{S}$ is the image of $x_i$ via $\sigma$.

1.5. $\alpha_p$-Torsors. The following sequence is exact for the fppf-topology:

$$1 \to \alpha_p \to \mathbb{G}_a \xrightarrow{x^p} \mathbb{G}_a \to 1$$

Let $f : V \to U$ be an $\alpha_p$-torsor. There exists an open covering $(U_s)_s$ of $U$ and regular functions $a_s \in \Gamma(U_s, \mathcal{O}_X)$, which are defined up to addition of a $p$-power, such that above $U_s$ the torsor $f$ is given by an equation $T_s^p = a_s$. The morphism $f$ is a finite radicial morphism of degree $p$, and in particular $V$ is homeomorphic to $U$. The differential form $\omega := da_s$ is a global differential form on $X$, it is the differential form associated to the torsor $f$. Let $\{y_j\}_{j \in \mathbb{Z}} \subset U$ be the set of zeros of $w$ which are contained in $U$. Then as in 1.4 it is easy to see that the only singularities of $V$ lie above the zeros of $w$ which are contained in $U$, and these singularities are unibranches. For a point $x$ of $X$ we call the integer $\text{cond}_x(f) := m_x = - (\text{ord}_x(\omega) + 1)$ the conductor of the torsor $f$ at the point $x$. We define the residue $h_x$ of the torsor $f$ at the point $x$ to be 0. The differential form $\omega$ is annihilated by the Cartier operation $C$. In fact the following sequence is exact on $U_{\text{et}}$:

$$0 \to \mathcal{O}_U \xrightarrow{\cdot \omega} \mathcal{O}_U \xrightarrow{\cdot h_x} \Omega^1_X \xrightarrow{\cdot C} \Omega^1_X \to 0$$

and $H^1_{\text{fppf}}(U, \alpha_p)$ is identified with the set of regular differential forms on $U$ which are annihilated by $C$ (cf. [Mi], III, 4).

The Galois group $G_k$ acts in a natural way on $\alpha_p$-torsors of $\mathcal{U}$. More precisely there exists a canonical homomorphism:

$$(4') \quad G_k \to \text{Aut} H^1_{\text{fppf}}(\mathcal{U}, \alpha_p)$$

Let $\sigma \in G_k$. Let $f : \mathcal{V} \to \mathcal{U}$ be an $\alpha_p$-torsor, and let $f^\sigma : \mathcal{V}^\sigma \to \mathcal{U}$ be the $\alpha_p$-torsor associated to $f$ via the above action $(4')$. If $f$ is locally given by an equation $T^p_s = a_s$, then $f^\sigma$ is locally given by the equation $T^p_s = a^\sigma_s$, where $a^\sigma_s$ is a regular function on $U^\sigma_s$. In particular if $\omega := da_s$ is the differential form associated to $f$, then the differential form associated to $f^\sigma$ is $\omega^\sigma := da^\sigma_s$. The zeros of $\omega^\sigma$ contained in $\mathcal{U}$ are the images of
those of $\omega$ via $\sigma$. Moreover for a point $x_i \in \overline{S}$ we have $\cond_{x_i}(f) = \cond_{x_i^\sigma}(f^\sigma)$ (resp. $\res_{x_i}(f) = \res_{x_i^\sigma}(f^\sigma)$), where $x_i^\sigma \in \overline{S}$ is the image of $x_i$ via $\sigma$.

1.6. The group $H^1_{\text{fppf}}(\_)_p$.

Let $Z$ be a connected scheme over a field $k$ of characteristic $p > 0$. Consider the following group:

$$(6) \quad H^1_{\text{fppf}}(Z)_p := \bigoplus G_k H^1_{\text{fppf}}(Z, G_k)$$

where the sum is taken over all isomorphism classes of finite and flat commutative $k$-group schemes of rank $p$. Let $\overline{k}$ be an algebraic closure of $k$, and let $\overline{Z} := Z \times_{\text{Spec} \ k} \text{Spec} \ \overline{k}$. Let $G_k$ be the Galois group of a separable closure of $k$ contained in $\overline{k}$. We have:

$$H^1_{\text{fppf}}(\overline{Z})_p = H^1_{\text{fppf}}(\overline{Z}, \mu_p) \oplus H^1_{\text{et}}(\overline{Z}, \mathbb{Z}/p\mathbb{Z}) \oplus H^1_{\text{fppf}}(\overline{Z}, \alpha_p)$$

The Galois group $G_k$ acts in a natural way on $H^1_{\text{fppf}}(\overline{Z})_p$ via its canonical action on $H^1_{\text{fppf}}(\overline{Z}, \mu_p)$ (resp. on $H^1_{\text{fppf}}(\overline{Z}, \alpha_p)$, and $H^1_{\text{fppf}}(\overline{Z}, \mathbb{Z}/p\mathbb{Z})$), and we have a canonical homomorphism:

$$(5') \quad G_k \to \text{Aut} \ H^1_{\text{fppf}}(\overline{Z})_p$$

In general for a scheme $Z$ with connected components $\{Z_i\}_{i \in I}$, we define $H^1_{\text{fppf}}(Z)_p := \bigoplus_{i \in I} H^1_{\text{fppf}}(Z_i)_p$.

1.7. The group $H^1_{\text{fppf}}(\_)^{\text{kum}}_p$.

Let $X$ be a connected semi-stable curve over a field $k$ of characteristic $p > 0$, with irreducible components $\{X_i\}_{i \in I}$. Let $\{z_j\}_{j \in J}$ be the set of double points of $X$, and let $\{y_t\}_{t \in T}$ be a set of marked smooth points of $X$. Let $U := X - \{\{z_j\}_{j \in J} \cup \{y_t\}_{t \in T}\}$. Then we define the group $H^1_{\text{fppf}}(U)_p^{\text{kum}}$ as the subgroup of $H^1_{\text{fppf}}(U)_p$ consisting of those elements $(f_s)_s$ with the following property, and which we call **Kummerian mixed torsors** of $U$: let $\{U_i\}_{i \in I}$ be the set of irreducible components of $U$. For each double point
$z_j \in X_i \cap X_i'$ we impose that the following holds: there exists a component $f_i : V_i \to U_i$ (resp. $f_i' : V_i' \to U_i'$) of $(f_s)_s$ which is a torsor under a finite and flat $k$-group scheme $G_{k,i}$ (resp. $G_{k,i'}$) of rank $p$, such that if $m_{i,j}$ (resp. $m_{i',j}$) is the conductor of the torsor $f_i$ (resp. $f_i'$) at the point $z_j$, and $h_{i,j}$ (resp. $h_{i',j}$) is the residue of the torsor $f_i$ (resp. $f_i'$) at the point $z_j$, then $m_{i,j} + m_{i',j} = 0$, and $h_{i,j} + h_{i',j} = 0$. It is not difficult to see that $H^1_{ippf}(U)^{kum}_p$ is indeed a subgroup of $H^1_{ippf}(U)_p$ which is $G_k$-equivariant.

II. Degeneration of $\mu_p$-torsors from characteristic $0$ to characteristic $p > 0$.

In this section we study the degeneration of $\mu_p$-torsors from zero to positive characteristics. We fix the following notations: $R$ is a complete discrete valuation ring of unequal characteristics, with residue characteristic $p > 0$, and which contains a primitive $p$-th root of unity $\zeta$. We denote by $K$ the fraction field of $R$, $\pi$ a uniformising parameter of $R$, $k$ the residue field of $R$, and $\lambda := \zeta - 1$. We also denote by $v_K$ the valuation of $K$ which is normalised by $v_K(\pi) = 1$. Let $\overline{K}$ be an algebraic closure of $K$, and let $G_K$ be the Galois group of $\overline{K}$ of $K$. Let $\overline{R}$ be the integral closure of $R$ in $\overline{K}$ which is a valuation ring, and let $\overline{k}$ be the residue field of $\overline{R}$ which is an algebraic closure of $k$. We denote by $G_k$ the Galois group of the separable closure of $k$ contained in $\overline{k}$. It is well known that there exists a canonical exact sequence:

$$0 \to I_K \to G_K \to G_k \to 0$$

where $I_K$ is the inertia subgroup of $G_K$. Moreover the subgroup $I_K$ fits in the following exact sequence:

$$0 \to I^K_w \to I_K \to I^K_t \to 0$$

where $I^K_w$ (resp. $I^K_t$) is the wild part (resp. the tame part) of inertia. The subgroup $I^K_t$ of $I_K$ is a pro-$p$-group, and $I^K_t$ is canonically isomorphic to the prime to $p$-part $\mathbb{Z}(1)(p')$ of the Tate twist $\mathbb{Z}(1)$ (cf. [Se]).

2.1. Torsors under finite and flat $R$-group schemes of rank $p$: the group schemes $\mathcal{G}^n$ and $\mathcal{H}_n$. (See also [Oo-Se-Su] and [He]).

For any positif integer $n$ denote by $\mathcal{G}^n_R := \text{Spec } R[x, 1/(\pi^n x + 1)]$. It is a commutative affine group scheme which has a generic fibre isomorphic to the multiplicative group $\mathbb{G}_m$ and whose special fibre is the additive group $\mathbb{G}_a$ (cf. [Oo-Se-Su] for more details). For $0 < n \leq v_K(\lambda)$ the polynomial ($(\pi^n x + 1)^p - 1)/\pi^{pn}$ has coefficients in $R$ and defines a group scheme homomorphism $\varphi_n : \mathcal{G}^n_R \to \mathcal{G}^{np}_R$ given by:

$$R[x, 1/(\pi^{pn} x + 1)] \to R[x, 1/(\pi^n x + 1)]$$
The isogeny $\varphi_n$ has degree $p$, its kernel $\mathcal{H}_{n,R}$ is a finite and flat $R$-group scheme of rank $p$, its generic fibre is isomorphic to $\mu_p$, its special fibre is either the radiciel group $\alpha_p$ if $0 < n < v_K(\lambda)$, or the étale group $\mathbb{Z}/p\mathbb{Z}$ if $n = v_K(\lambda)$.

Let $\alpha^{(n)} : G^n_R \to \mathbb{G}_{m,R}$ be the group schemes homomorphism given by:

$$R[u, u^{-1}] \to R[x, 1/(\pi^n x + 1)]$$

$$u \to \pi^n x + 1$$

The following is a commutative diagram of exact sequences for the fppf-topology:

$$0 \to \mathcal{H}_{n,R} \to G^n_R \to G^{np}_R \to 0$$

The upper exact sequence above has a Kummer exact sequence (1) as generic fibre, and the exact sequence (4) if $0 < n < v_K(\lambda)$, or an exact sequence of Artin-Schreir type (2) if $n = v_K(\lambda)$, as a special fibre.

Let $U$ be an $R$-scheme, and let $f : V \to U$ be a torsor under $\mathcal{H}_{n,R}$. Then there exists an open covering $(U_i)$ of $U$ and regular functions $u_i \in \Gamma(U_i, \mathcal{O}_U)$, such that $\pi^{np} u_i + 1$ is defined up to multiplication by a $p$-power, and the torsor $f$ is defined above $U_i$ by an equation $T^p_i = (\pi^n T'_i + 1)^p = \pi^{np} u_i + 1$, where $T_i$ and $T'_i$ are indeterminates.

2.2. The Galois action.

Let $G'$ be a finite and flat commutative $R$-group scheme of rank $p$, and let $G := G' \times_R \overline{R}$ which is a finite and flat $\overline{R}$-group scheme of rank $p$. We denote by $G_{\overline{k}} := G \times_{\overline{R}} \overline{k}$ the special fibre of $G$ which is either $\mathbb{Z}/p\mathbb{Z}$, $\mu_p$, or $\alpha_p$. Let $U$ be an $R$-scheme, and let $\overline{U} := U \times_R \overline{R}$.

The Galois group $G_K$ acts in a canonical way on torsors, for the fppf-topology, above $\overline{U}$ under the group scheme $G$. More precisely there exists a canonical homomorphism:

$$(6') \quad G_K \to \text{Aut} H^1_{\text{fppf}}(\overline{U}, G)$$

The above action $(6')$ is compatible with the action of $G_k$ on torsors above $\overline{U}_k := \overline{U} \times_{\overline{R}} \overline{k}$ under the group scheme $G_{\overline{k}}$. More precisely we have the following canonical commutative diagramm:

$$\begin{array}{ccc}
G_K & \to & \text{Aut} H^1_{\text{fppf}}(\overline{U}, G) \\
\downarrow & & \downarrow \\
G_k & \to & \text{Aut} H^1_{\text{fppf}}(\overline{U}_k, G_{\overline{k}})
\end{array}$$
2.3. Degeneration of $\mu_p$-torsors.

In what follows let $X$ be a formal $R$-scheme of finite type which is normal and flat over $R$. We assume that $X$ is smooth of relative dimension 1. Let $X_K := X \times_R K$ (resp. $X_k := X \times_R k$) be the generic (resp. special) fibre of $X$. By the generic fibre of $X$ we mean the associated $K$-rigid space. Let $\eta$ be the generic point of the special fibre $X_k$, and let $\mathcal{O}_\eta$ be the local ring of $X$ at $\eta$, which is a discrete valuation ring with fractions fields $K(X)$ the function field of $X$. Let $f_K : Y_K \to X_K$ be a $\mu_p$-torsor, and let $K(X) \to L$ be the corresponding extension of function fields. One has the following:

2.4. Proposition. Assume that the ramification index above $\mathcal{O}_\eta$ in the extension $K(X) \to L$ equals 1. Then the torsor $f_K : Y_K \to X_K$ extends to a torsor $f : Y \to X$ under a finite and flat $R$-group scheme of rank $p$, with $Y$ normal. Let $\delta$ be the degree of the different above $\eta$ in the extension $K(X) \to L$. Then the following cases occur:

a) $\delta = 0$ in which case $f$ is a torsor under the group scheme $\mathcal{H}_{v_K(\lambda),R}$, and $f_k$ is then an étale torsor under $\mathbb{Z}/p\mathbb{Z}$.

b) $0 < \delta < v_K(\lambda)$ in which case $\delta = v_K(\lambda) - n(p-1)$ for a certain integer $n \geq 1$, $f$ is a torsor under $\mathcal{H}_{n,R}$ and $f_k$ is a radicial torsor under $\alpha_p$.

c) $\delta = v_K(\lambda)$, $f$ is a torsor under $\mu_p$ and $f_k$ is also a radicial torsor under $\mu_p$.

Note that starting from a torsor $f_K : Y_K \to X_K$ as in 2.3, the condition that the ramification index above $\mathcal{O}_\eta$ equals 1 is always satisfied after eventually a finite ramified extension of $R$ (cf. [Ep]).

Proof. The above result is classically known in the case where $X$ is the formal spectrum of a complete discrete valuation ring (cf. [Hy], note that this is the only case we will need in 2.4). It has been treated in the case where $X$ is factoriel in [Ma-Gr] and [He]. We explain the proof in the general case. By hypothesis there exists an open covering $(U_i)_i$ of $X$ and units $u'_i \in \Gamma(U_{i,K},\mathcal{O}_X)^*$ which are defined up to multiplication by $p$-powers, where $U_{i,K} := U_i \times_R K$, and such that the torsor $f_K$ is defined above $U_{i,K}$ by an equation $T_i^p = u'_i$. Moreover $u'_i = \pi^{p\alpha_i}u_i$ where $u_i \in \Gamma(U_i,\mathcal{O}_X)^*$ is a unit (the power of $\pi$ is a $p$-multiple since the ramification index above $\eta$ equals 1, and the fact that $u_i$ is a unit can be checked easily locally and uses the fact that $X$ is a smooth curve). If for a given $i$ the image $\bar{u}_i$ of $u_i$ in $\mathcal{O}_X(U_i)/\pi\mathcal{O}_X(U_i)$ is not a $p$-power then this is the case for each $i$ since $u_i = a_{i,j}^pu_j$ in $\mathcal{O}_X(U_i \cap U_j)$. And in this case the class of the cocycle $(a_{i,j})_{i,j}$ defines a non trivial $\mu_p$-torsor $f : Y \to X$ of $X$ which extends $f_K$ and induces a $\mu_p$-torssor of $X_k$. Otherwise $\bar{u}_i$ is a $p$-power for each $i$. Then as in [Gr-Ma], [He], and after multiplying $u_i$ by a suitable $p$-power, which does not change the $\mu_p$-torsor $f_K$, one can assume that $f_1 = 1 + \pi^{rt_i}a_i$, and the image $\bar{a}_i$ of $a_i$ in $\mathcal{O}_X(U_i)/\pi\mathcal{O}_X(U_i)$ is not a $p$-power, in which case $\delta_i := v(\lambda) - t_i(p-1)$ is the degree of the different above the generic point $\eta$ of $U_{i,k} := U_i \times_R k$. In particular $n := t_i$ is constant for all $i$. The $\mathcal{H}_{n,R}$-torsor $f : Y \to X$ above $X$, which is given above $U_i$.
by the equation $T^p_i = 1 + \pi^{nm}a_i$, extends then the $\mu_p$-torsor $f_K$.

2.5. With the same notations as in 2.3 let $X_K := X \times_R \overline{K}$, $X_{\overline{R}} := X \times_R \overline{R}$, and let $X_{\overline{K}} := X \times_R \overline{K}$. Hier we consider $X_{\overline{K}}$ as a rigid space. Then it follows easily from the proof of 2.4, that there exists canonical specialisation group homomorphisms:

$$H^1_{et}(X_{\overline{K}}, \mu_p) \rightarrow H^1_{fppf}(X_{\overline{R}}, \mu_p) \rightarrow H^1_{fppf}(X_{\overline{K}}, \mu_p)$$

whose composite gives the following:

2.5.1. Proposition. There exists a canonical specialisation group homomorphism:

$$\text{Sp} : H^1_{et}(X_{\overline{K}}, \mu_p) \rightarrow H^1_{fppf}(X_{\overline{K}}, \mu_p)$$

which is $G_K$-equivariant.

Note that the $\mu_p$-torsors of $X_{\overline{K}}$ which belongs to Ker($\text{Sp}$) correspond to thoses torsors which induce in reduction $\alpha_p$ or $\mathbb{Z}/p\mathbb{Z}$-torsors. For the proof of the second assertion of 2.5.1 cf. 2.7. Also hier we mean the action of $G_K$ on $H^1_{fppf}(X_{\overline{K}}, \mu_p)$ via its canonical quotient: $G_K \rightarrow G_k$.

2.6. Compatibility of degeneration with the Galois action.

Let $X$ be a formal $R$-scheme of finite type which is normal and flat over $R$. We suppose that $X$ is smooth of relative dimension one. Let $X_K := X \times_R K$ (resp. $X_k := X \times_R k$) be the generic (resp. special) fibre of $X$. By the generic fibre of $X$ we mean the associated $K$-rigid space. We have the Galois action of $G_K$ on $\mu_p$-torsors of $X_{\overline{K}} := X \times_K \overline{K}$ (cf. (1')):

$$G_K \rightarrow H^1_{et}(X_{\overline{K}}, \mu_p)$$

As a direct consequence of the result in 2.4 one sees that there exists a canonical specialisation set theoretical map (this is not a group homomorphism):

$$\text{Sp} : H^1_{et}(X_{\overline{K}}, \mu_p) \rightarrow H^1_{fppf}(X_{\overline{K}})$$

Where $X_{\overline{K}} := X \times_{\text{Spec} k} \text{Spec} \overline{k}$. Moreover the action of $G_k$ on $H^1_{fppf}(X_{\overline{K}})$ induces an action of $G_K$ on $H^1_{fppf}(X_{\overline{K}})$ via the canonical surjectif homomorphism: $G_K \rightarrow G_k$. We have the following:

2.7. Theorem. There exists a canonical specialisation map $\text{Sp} : H^1_{et}(X_{\overline{K}}, \mu_p) \rightarrow H^1_{fppf}(X_{\overline{K}})$ which is $G_K$-equivariant.

Proof. We explain the argument of the proof. We may assume that $\text{Pic}(X_{\overline{K}}) = 0$. Let $f_{\overline{K}} : Y_{\overline{K}} \rightarrow X_{\overline{K}}$ be a non-trivial $\mu_p$-torsor. Then $f_{\overline{K}}$ is given by an equation $T^p = u_{\overline{K}}$, where
$u_\mathcal{R}$ is an invertible function on $X_\mathcal{R}$. By 2.4 this torsor extends to a torsor $f_\mathcal{R} : Y_\mathcal{R} \to X_\mathcal{R}$ under a finite and flat $\mathcal{R}$-group scheme of rank $p$, where $X_\mathcal{R} := X \times_R \mathcal{R}$ and $\mathcal{R}$ is the integral closure of $R$ in $k$. This torsor induces on the special fibres a torsor $f_\mathcal{R} : Y_\mathcal{R} \to X_\mathcal{R}$ under a finite and flat $k$-group scheme of rank $p$, where $X_\mathcal{R} := X \times_k k$, and by definition $f_\mathcal{R}$ is the non zero component of the image of $f_\mathcal{R}$ in $H^1_{\text{fppf}}(X_k)_p$ via the map $\text{Sp}$. We treat the case where $f_\mathcal{R}$ is an étale torsor. The other cases are treated similarly. Then it follows from 2.4 that one can choose the function $u_\mathcal{R}$ of the form $1 + \lambda^p a$, where $a$ is a regular function on $X_\mathcal{R}$. Moreover the torsor $f_k$ is given by the equation $t^p - t = \sigma$, where $\sigma$ is the image of $a$ in $\Gamma(X_k, \mathcal{O}_{X_k})$. Let $\sigma \in G$. Then the torsor $f_\mathcal{R}^\sigma$, image of $f_\mathcal{R}$ under $\sigma$, is given by the equation $1 + \lambda^p a^\sigma$, where $a^\sigma$ is the transform of $a$ under $\sigma$, its special fibre is the torsor given by the equation $t^p - t = \sigma^\sigma$, where $\sigma^\sigma$ is the transform of $\sigma$ under $\sigma$ which is the image of $\sigma$ in $G_k$. But then this torsor is nothing else but the transform $f_k^\sigma$ of $f_k$ under $\sigma$.

III. Degeneration of $\mu_p$-torsors on the boundaries of formal fibres.

In this section we explain the degeneration of $\mu_p$-torsors on the boundary $\text{Spf } R[[T]]\{T^{-1}\}$ of formal fibres of formal $R$-curves, where $R[[T]]\{T^{-1}\}$ is the ring of formal power series $\sum_{i \in \mathbb{Z}} a_i T^i$ with $\lim_{i \to -\infty} |a_i| = 0$, and where $| |$ is an absolute value of $K$ associated to its valuation. The following result will be used in [Sa] in order to prove a formula for vanishing cycles.

3.1. Proposition. Let $A := R[[T]]\{T^{-1}\}$ (cf. the definition above), and let $f : \text{Spf } B \to \text{Spf } A$ be a non trivial torsor under a finite flat $R$-group scheme $G$ of rank $p$. Let $\delta$ be the degree of the different in the above extension. We assume that the ramification index of this extension equals 1. Then the following cases occur:

- a ) $\delta = 0$, the torsor $f$ is étale and is given, after eventually a finite extention of $R$, by an equation $Z^p = \lambda^p T^m + 1$, where $m$ is a negative integer prime to $p$, for a suitable choice of the parameter $T$ of $A$.

- b ) $\delta = v_K(\lambda)$, $f$ is a torsor under $\mu_p$, and two cases can occur:
  
  - b-1 ) For a suitable choice of the parameter $T$ of $A$, and after eventually a finite extention of $R$, the torsor $f$ is given, by an equation $Z^p = T^h$ where $h \in \mathbb{F}_p^*$.

  - b-2 ) For a suitable choice of the parameter $T$ of $A$, and after eventually a finite extention of $R$, the torsor $f$ is given by an equation $Z^p = 1 + T^m$ where $m$ is a positive integer prime to $p$.

- c ) $0 < \delta < v_K(p)$ in which case $f$ is a torsor under $H_{n,R}$, and $\delta = v_K(p) - n(p - 1)$. For a suitable choice of the parameter $T$, and after eventually a finite extention of $R$, the torsor $f$ is given by an equation $Z^p = 1 + \pi^m T^m$, with $m \in \mathbb{Z}$ is prime to $p$.

The proof of the above lemma is easy. It is a direct consequence of 2.4 which in this
case is due to Hyodo (cf. [Hy] 2.16). This lemma is also stated in [He] in a slight different form in terms of automorphisms of boundaries of formal fibres.

3.2. Definition. With the same notations as in 3.1, we define the type of reduction (or the degeneration type) of the torsor $f$ to be $(G_k, m, h)$, where $G_k := G \times_R k$ is the special fibre of $G$, $m$ is the conductor associated to the torsor $f_k : \text{Spec } B/\pi B \rightarrow \text{Spec } A/\pi A$, and $h \in \mathbb{F}_p$ its residue (cf. 1.3, 1.4 and 1.5). With the same notations as in 2.4 the degeneration type is $((\mathbb{Z}/p\mathbb{Z}), m, 0)$ in case a, $(\mu_p, 0, h)$ in case b-1, $(\mu_p, -m, 0)$ in case b-2 and $(\alpha_p, -m, 0)$ in case c.

3.3. Remark. The above corollary implies in particular that given two torsors above $A := R[[T]][T^{-1}]$, under a finite flat $R$-group scheme of rank $p$, and which have the same type of reduction as defined in 3.2, and which have the same degree of the different, then after eventually “adjusting” the Galois action on the Kummer generators of these two covers, and after eventually a finite extension of $R$, one can find a Galois-equivariant isomorphism between them. Also note that the above lemma can be easily adapted to the rigid setting in order to describe torsors under group schemes of rank $p$ above the boundaries of formal fibres of curves in rigid geometry as well as their degeneration type.

IV. Lifting of rank $p$ torsors from positive to zero characteristic.

In this section we use the same notations as in 2.3. Let $X$ be a formal $R$-scheme of finite type, which is normal and flat over $R$. We assume that $X$ is smooth of relative dimension one. Let $X_k := X \times_R k$ be the special fibre of $X$. Also let $X_{\overline{R}} := X \times_R \overline{R}$, and let $X_{\overline{k}} := X \times_{\overline{k}} \overline{k}$.

4.1. Lifting of rank $p$ étale torsors.

Let $f_k : Y_k \rightarrow X_k$ be an étale $\mathbb{Z}/p\mathbb{Z}$-torsor. Then by the theorems of lifting of étale covers (cf. [Gr]) $f_k$ can be lifted, uniquely up to isomorphism, to an étale torsor of $X$. More precisely if the torsor $f_k$ is locally given by an equation $t^p - t = \bar{a}$, where $\bar{a}$ is a regular function, then the equation $((\lambda T + 1)^p - 1)/\lambda^p = a$ where $a$ is a regular function which equals $\bar{a}$ mod $\pi$ gives locally a lifting of the torsor $f_k$. In particular if $R^{nr}$ denotes the maximal unramified extension of $R$ contained in $\overline{R}$, then all étale torsors under a finite and flat $\overline{R}$-group scheme of rank $p$ are already defined over $R^{nr}$, and the Galois action of $G_K$ on those torsors factors through the Galois group of $K^{nr}$: the fractions field of $R^{nr}$, over $K$, which is canonically isomorphic to $G_k$. More precisely we have $G_k$-equivariant canonical isomorphisms:

$$H^1_{et}(X_{\overline{R}}, \mathbb{Z}/p\mathbb{Z}) \simeq H^1_{fpf}(X_{\overline{R}}, \mathcal{H}_{\lambda, \overline{R}}) \simeq H^1_{fpf}(X_{R^{nr}}, \mathcal{H}_{\lambda, R^{nr}})$$

where $X_{R^{nr}} := X \times_R R^{nr}$, $\mathcal{H}_{\lambda, R^{nr}} := \mathcal{H}_{\lambda, R} \times_R R^{nr}$, and $\mathcal{H}_{\lambda, \overline{R}} := \mathcal{H}_{\lambda, R} \times_R \overline{R}$.

The theorem of lifting of étale covers above can also be interpreted as follows:
4.1.1. Lemma. There exists a canonical $G_K$-equivariant injective group homomorphism:

$$H^1_{et}(X_K, \mathbb{Z}/p\mathbb{Z}) \rightarrow H^1_{et}(X_K^p, \mu_p)$$

Moreover the Galois group $G_K$ acts on the subgroup $H^1_{et}(X_K, \mathbb{Z}/p\mathbb{Z})$ via its canonical quotient $G_{K_{nr}/K}$: the Galois group of $K^{nr}$ over $K$, which is canonically isomorphic to $G_k$.

In other words the inertia subgroup $I_K$ of $G_K$ acts trivially on the subgroup $H^1_{et}(X_K, \mathbb{Z}/p\mathbb{Z})$.

4.2. Lifting of rank $p$ radiciel torsors.

4.3. Proposition. With the same notations as in IV above, let $f_k : Y_k \rightarrow X_k$ be a torsor above $X_k$ under a finite and flat $k$-group scheme $G_k$ of rank $p$ which is radicial. Assume that $X := \text{Spf } A$ is affine and that $H^1_{et}(X_k, \mathbb{G}_m) = H^1_{fppf}(X_k, \mathbb{G}_m) = \text{Pic}(X_k) = 0$. Then the torsor $f_k$ can be lifted, non uniquely, after eventually a finite ramified extension of $R$, to a torsor $f : Y \rightarrow X$ under a finite and flat $R$-group scheme of rank $p$.

More precisely we have the following two cases:

case 1: $G_k = \mu_p$. In this case the torsor $f_k$ is given by an equation $t^p = \bar{u}$, where $\bar{u} \in A_k := A/\pi A$ is a unit which is uniquely determined up to multiplication by a $p$-power. Let $u \in A$ be an invertible function such that $u$ equals $\bar{u}$ mod $\pi$. Then the equation $T^p = u$ defines a $\mu_p$-torsor $f : Y \rightarrow X$ above $X$ which lifts the $\mu_p$-torsor $f_k$. The above lifting is however not unique. More precisely for $n \leq v_K(\lambda)$ a positif integer, the equation $T^{np} = u(1 + \pi^m u')$, where $u' \in A$ defines another lifting $f' : Y' \rightarrow X$ of $f_k$. Moreover $f$ and $f'$ are not isomorphic $\mu_p$-torsors since $(1 + \pi^m u')$ is not a $p$-power in $A$. All possible liftings of the torsor $f_k$ are in fact defined up to elements in $1 + \pi^m A$, for $n \leq pv_K(\lambda)$ a positif integer. Moreover in this case the composite of the canonical homomorphisms:

$$H^1_{fppf}(X_K^p, \mu_p) \rightarrow H^1_{fppf}(X_K^p, \mu_p) \rightarrow H^1_{fppf}(X_K, \mu_p)$$

is surjective, and we have a canonical commutative diagramm:

$$\begin{array}{ccc}
G_K & \longrightarrow & \text{Aut } H^1_{fppf}(X_K^p, \mu_p) \\
\downarrow & & \downarrow \\
G_K & \longrightarrow & \text{Aut } H^1_{fppf}(X_K^p, \mu_p) \\
\downarrow & & \downarrow \\
G_k & \longrightarrow & \text{Aut } H^1_{fppf}(X_K, \mu_p)
\end{array}$$

case 2: $G_k = \alpha_p$. In this case the torsor $f_k$ is given by an equation $t^p = \bar{a}$, where $\bar{a} \in A_k := A/\pi A$ is a function which is uniquely determined up to addition of a $p$-power. Let $n < v_K(\lambda)$ be a positif integer. Such an integer exists if and only if $v_K(\lambda) > 1$. Note that for any positif integer $n$, one can perform a totally wildly ramified extention of $R$
such that the condition $n < v_K(\lambda)$ is satisfied. Assume that such an $n$ as above exists. Let $a \in A$ be a function which equals $\bar{a} \mod \pi$. Then the equation $T^p = 1 + \pi^{np}a$ defines an $\mathcal{H}_{n,R}$-torsor $f : Y \to X$ above $X$ which lifts the $\mu_p$-torsor $f_k$. The above lifting is however not unique. More precisely for $m$ a positif integer such that $m \leq v_K(\lambda)$, and $pn < pm$, the equation $T^{mp} = (1 + \pi^{pm}a)(1 + \pi^{pm}a')$, where $a' \in A$ defines another lifting $f' : Y' \to X$ of $f_k$. Moreover $f$ and $f'$ are not isomorphic $\mathcal{H}_{n,R}$-torsors since $(1 + \pi^{pm}a')$ is not a $p$-power in $A$.

Suppose there exists a positif integer $n < v_K(\lambda)$, and let $\mathcal{H}_{n,R} := \mathcal{H}_{n,R} \times_R \overline{R}$ which is a finite and flat commutatif $\overline{R}$-group scheme of rank $p$, whose special fibre is the radicial $\overline{R}$-group scheme $\alpha_p$. Then the canonical homomorphism: $H^1_{fppf}(X_{\overline{R}}, \mathcal{H}_{n,R}) \to H^1_{fppf}(X_{\overline{R}}, \alpha_p)$ is surjective, and we have a commutative diagram:

$$
\begin{array}{ccc}
G_K & \longrightarrow & \text{Aut } H^1_{fppf}(X_{\overline{R}}, \mathcal{H}_{n,R}) \\
\downarrow & & \downarrow \\
G_k & \longrightarrow & \text{Aut } H^1_{fppf}(X_{\overline{R}}, \alpha_p)
\end{array}
$$

4.4. Let $G_k$ be a radicial finite and flat $k$-group scheme of rank $p$. Let $f_k : Y_k \to X_k$ be a torsor above $X_k$ under $G_k$. We say that this torsor is admissible if, eventually after a finite extension of $R$, there exists a finite and flat commutatif $R$-group scheme $G$, whose special fibre is isomorphic to $G_k$, and a torsor $f : Y \to X$ above $X$ under $G$ which lifts $f_k$. By the considerations above we have seen that if $X$ is affine and $\text{Pic}(X) = 0$, then every torsor $f_k : Y_k \to X_k$ above $X_k$ under $G_k$ is admissible. This is for example the case if $X$ is a formal affine open of the formal $R$-projective line. Another important example of admissible torsors is the case where $X$ is a proper and smooth formal $R$-curve. More precisely we have the following:

4.5. Proposition. Let $X$ be a proper and smooth formal $R$-curve. Let $G_k$ be a commutative finite and flat $k$-group scheme of rank $p$ which is radicial. Then every torsor $f_k : Y_k \to X_k$ above $X_k$ under the group scheme $G_k$ is admissible.

Proof. Let $J := \text{Pic}^0(X)$ be the jacobian of $X$ which is an abelian formal scheme. Let $G$ be a finite and flat commutatif $R$-group scheme, and let $G'$ be its Cartier dual. Then by [Ra] there exists a canonical isomorphism $H^1_{fppf}(X, G) \simeq \text{Hom}(G', J)$. Let $G'_k$ be the Cartier dual of $G_k$ which by the geometric abelian class field theory is a subgroup of $J_k := J \times_R k$ the special fibre of $J$. To lift then the torsor $f_k$ is equivalent by the above isomorphism to lift the subgroup $G'_k$ of $J_k$ to a subgroup $G'$ of $J$ which is finite and flat over $R$ of rank $p$, and this is indeed always possible.

4.6. Remark. I do not know whether for any formal affine $R$-curve $X$, and a finite flat $k$-group scheme $G_k$ of rank $p$, any $G_k$-torsor of $X_k$ is admissible?.

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V. A canonical filtration on $H_{\text{et}}^1(\cdot, \mu_p)$.

In this section we use the same notations as in III. Let $X$ be an $R$-scheme of finite type, which is smooth, proper and flat over $R$ of relative dimension one. Let $X_k := X \times_R k$ be the special fibre of $X$. Also let $X_K := X \times_R K$, and let $X_{\overline{k}} := X \times_k \overline{k}$. We will establish in this section the existence of a canonical $G_K$-equivariant filtration $\text{Fil} := (\text{Fil}_i)_i$ on the group $H_{\text{et}}^1(X_K, \mu_p)$.

Recall first (cf. 2.5) that we have a canonical specialisation group homomorphism which is $G_K$-equivariant:

$$\text{Sp}: H_{\text{et}}^1(X_K, \mu_p) \to H_{\text{fppf}}^1(X_K, \mu_p)$$

and the kernel of $\text{Sp}$ consist of those $\mu_p$-torsors of $X_K$ which induces in reduction an $\alpha_p$ or an étale torsor of $X_{\overline{k}}$.

The group $H_{\text{et}}^1(X_K, \mu_p)$ is finite since $X$ is proper. In particular all $\mu_p$-torsors of $X_K$ are defined over a finite extension of $K$. More precisely there exists a finite extension $K'$ of $K$ with the following properties:

(a) All $\mu_p$-torsors of $X_K$ are defined over $K'$.

(b) Let $f_{K'} : Y_{K'} \to X_{K'} := X \times_R K'$ be a $\mu_p$-torsor, and let $Y'$ be the integral closure of $X'$ in $Y_{K'}$, where $X' := X \times_R R'$, and $R'$ is the integral closure of $R$ in $K'$. Then the special fibre of $Y'$ is reduced.

Moreover there exists such a finite extension $K'$ of $K$ which is minimal for the above properties. Let $\pi'$ be a uniformiser of $K'$, and let $v_{K'}$ be the valuation of $K'$ which is normalised by $v_{K'}(\pi') = 1$. It follows from 2.3.4 that if $f_{K'} : Y_{K'} \to X_{K'} := X \times_R K'$ is a non trivial $\mu_p$-torsor, then it extends to a torsor $f : Y' \to X' := X \times_R R'$ under a finite commutative and flat $R'$-group scheme. Let $\delta_f := \delta_{f_{K'}}$ be the degree of the different, in the above torsor $f$, above the generic point of the special fibre of $X'$. We have two cases: either $f$ is a non trivial $\mu_p$-torsor, or a torsor under the group scheme $\mathcal{H}_{n,R'}$ for $0 < n \leq v_{K'}(\lambda)$, the case $n = v_{K'}(\lambda)$ corresponds to the case of an étale torsor. In the first case we have $\delta_f = v_{K'}(p)$, and in the second case we have $\delta_f = v_{K'}(p) - n(p - 1)$.

5.1. Definition. With the same notations as above we define the following decreasing filtration $\text{Fil} := (\text{Fil}_n)_{0 \leq n \leq v_{K'}(p)}$ of the group $H_{\text{et}}^1(X_K, \mu_p)$: $\text{Fil}_n H_{\text{et}}^1(X_K, \mu_p)$ is the subgroup of $H_{\text{et}}^1(X_K, \mu_p)$ consisting of those, isomorphism classes, of $\mu_p$-torsor $f_{K'} : Y_{K'} \to X_{K'} := X \times_R K'$ with $\delta_{f_{K'}} \leq v_{K'}(p) - n(p - 1)$. In particular we have:

$\text{Fil}_0 H_{\text{et}}^1(X_K, \mu_p) = H_{\text{et}}^1(X_K, \mu_p)$, and $\text{Fil}_{v_{K'}(\lambda)} H_{\text{et}}^1(X_K, \mu_p) \simeq H_{\text{et}}^1(X_K, \mathbb{Z}/p\mathbb{Z})$.

The following proposition is easily verified:

5.2. Proposition. The filtration $\text{Fil} := (\text{Fil}_n)_{0 \leq n \leq v_{K'}(p)}$ is a decreasing filtration by subgroups of $H_{\text{et}}^1(X_K, \mu_p)$. Moreover this filtration is $G_K$-equivariant.
5.3. With the same notations as above, let $g$ be the genus of $X\overline{K} := X \times_R \overline{K}$, and let $r$ be the $p$-rank of $X_k := X \times_R k$ which equals $\dim_{\mathbb{Z}/p\mathbb{Z}} H^1_{\text{et}}(X_k, \mathbb{Z}/p\mathbb{Z})$, and which is an integer smaller or equal to $g$. Then it is well known that: $H^1_{\text{et}}(X\overline{K}, \mu_p) \simeq (\mathbb{Z}/p\mathbb{Z})^{2g}$, $H^1_{\text{fppf}}(X_k, \alpha_p) \simeq (\mathbb{Z}/p\mathbb{Z})^{2(g-r)}$, and $H^1_{\text{fppf}}(X_k, \mu_p) \simeq (\mathbb{Z}/p\mathbb{Z})^r$. Moreover in this case the canonical $G_K$-equivariant homomorphism:

$$\text{Sp} : H^1_{\text{et}}(X\overline{K}, \mu_p) \to H^1_{\text{fppf}}(X_k, \mu_p)$$

is surjectif. Moreover we have: Fil$_0 H^1_{\text{et}}(X\overline{K}, \mu_p) = H^1_{\text{et}}(X\overline{K}, \mu_p)$, and Fil$_1 H^1_{\text{et}}(X\overline{K}, \mu_p) = \text{Ker}(\text{Sp})$. In particular Fil$_1 H^1_{\text{et}}(X\overline{K}, \mu_p) \simeq (\mathbb{Z}/p\mathbb{Z})^{2g-r}$. Also: Fil$_{v_K}(\lambda) H^1_{\text{et}}(X\overline{K}, \mu_p) \simeq H^1_{\text{et}}(X\overline{K}, \mathbb{Z}/p\mathbb{Z}) \simeq (\mathbb{Z}/p\mathbb{Z})^r$.

In the case where $X\overline{K}$ is ordinary, namely if $r = g$, then Fil$_0 H^1_{\text{et}}(X\overline{K}, \mu_p) = H^1_{\text{et}}(X\overline{K}, \mu_p)$, Fil$_1 H^1_{\text{et}}(X\overline{K}, \mu_p) = \text{Ker}(\text{Sp}) \simeq H^1_{\text{et}}(X\overline{K}, \mathbb{Z}/p\mathbb{Z})$, and we have a canonical $G_K$-equivariant exact sequence:

$$0 \to H^1_{\text{et}}(X\overline{K}, \mathbb{Z}/p\mathbb{Z}) \to H^1_{\text{et}}(X\overline{K}, \mu_p) \to H^1_{\text{et}}(X_k, \mu_p) \to 0.$$
[Sa-1] M. Saâdi, *Galois covers of degree p: semi-stable reduction and Galois action*, math.AG/0106249.

[Se] J. P. Serre, Local Fields, 3 ´ edit., Herman, Paris, (1980).

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