ON THE SIGMA FUNCTION IDENTITY

ALEXEY GAVRILOV

Abstract. We consider the known functional identity on the Weierstrass sigma function. A complete classification of odd entire functions which satisfy the same identity is obtained.

1. Introduction

Let \( \Lambda \) be a lattice in \( \mathbb{C} \) and \( \Lambda' = \Lambda - \{0\} \). Let

\[
\sigma(z) = \sigma(z, \Lambda) = z \prod_{\lambda \in \Lambda'} \left(1 - \frac{z}{\lambda}\right) e^{\frac{z}{2}(\frac{z}{\lambda})^2}
\]

be a Weierstrass sigma function. It is an odd entire quasiperiodic function with \( \Lambda \) as the set of zeros [1].

We shall deal with the identity

\[
f(x)f(y)f(z)f(w) - f\left(\frac{x+y+z+w}{2}\right) f\left(\frac{x+y-z+w}{2}\right) f\left(\frac{x-y+z+w}{2}\right) f\left(\frac{-x+y+z+w}{2}\right) -
\]

\[
- f\left(\frac{x+y+z+w}{2}\right) f\left(\frac{x+y-z-w}{2}\right) f\left(\frac{x-y+z+w}{2}\right) f\left(\frac{-x+y-z-w}{2}\right) = 0
\]

which holds for any \( x,y,z,w \in \mathbb{C} \). This identity may be easily derived from the classical Weierstrass parallelogramm formula

\[
\psi(x) - \psi(y) = \frac{-\sigma(x-y)\sigma(x+y)}{\sigma(x)^2\sigma(y)^2},
\]

or from known properties of quasiperiodic functions or from the Riemann theta identityes. It is much more difficult to find it in the literature than to prove it. The author knows only two papers where it has been written (at least in an explicit form) [2,3]. In [2, lemma 1] it appears as a new result. In [3] a reference to a book printed in 1893 was given. Unfortunately the author has never read this book.

The aim of this paper is to prove the following

**Theorem** Let \( f \neq 0 \) be an odd entire function. Let

\[
f(x)f(y)f(z)f(w) - f\left(\frac{x+y+z-w}{2}\right) f\left(\frac{x+y-z+w}{2}\right) f\left(\frac{x-y+z+w}{2}\right) f\left(\frac{-x+y+z+w}{2}\right) -
\]

\[
- f\left(\frac{x+y+z+w}{2}\right) f\left(\frac{x+y-z-w}{2}\right) f\left(\frac{x-y+z+w}{2}\right) f\left(\frac{-x+y-z-w}{2}\right) = 0
\]

for any complex numbers \( x,y,z,w \). Then there exist \( \alpha, \beta \in \mathbb{C} \) such that one of the following statements holds

1. \( f(z) = z e^{\alpha z^2 + \beta} \);
2. \( f(z) = \sin(\alpha z) e^{\alpha z^2 + \beta} \) for some \( \alpha \neq 0 \),
3. \( f(z) = \sigma(z, \Lambda) e^{\alpha z^2 + \beta} \) for some lattice \( \Lambda \).
This result may be viewed as an algebraic definition of the Weierstrass sigma function. It is clear that the statement of the theorem is true under weaker hypothesis and an interesting question is which of the conditions may be dropped.

2. Some invariant

The method of proof is the approximation of the function given by the function of a required kind. Let \( \mathcal{M} = \{ f : f(0) = 0 \} \) be the ideal in the algebra of entire functions and let \( \Omega \) be the set of odd entire functions which are not in \( \mathcal{M}^2 \). For each \( f \in \Omega \) we have
\[
f(z) \equiv a_1 z + a_3 z^3 + a_5 z^5 + a_7 z^7 \mod \mathcal{M}^9,
\]
for some complex numbers \( a_1, a_3, a_5, a_7 \), where \( a_1 \neq 0 \). Let
\[
p(f) = a_1^2 - 2a_1 a_5,
\]
\[
q(f) = 3a_1^2 a_7 - 3a_1 a_3 a_5 + a_3^2.
\]

**Lemma 1**  If \( f \in \Omega \) and \( p(f) = q(f) = 0 \) then there exist \( \alpha, \beta \in \mathbb{C} \) such that
\[
f(z) \equiv z e^{\alpha z^2 + \beta} \mod \mathcal{M}^9.
\]

The proof is a straightforward calculation.

If either of \( p \) and \( q \) is not equal to zero, then we may define
\[
\mu(f) = \frac{p(f)^3}{q(f)^2} \in \mathbb{C}P^1.
\]

We shall write \( \mu(f(z)) \) instead of \( \mu(z \mapsto f(z)) \).

**Lemma 2**  Let \( f_1, f_2 \in \Omega \) and \( \mu(f_1) = \mu(f_2) \) are well-defined. Then there exist \( \alpha, \beta \in \mathbb{C} \) and \( a \in \mathbb{C}^\times \) such that
\[
f_2(z) \equiv f_1(az) e^{\alpha z^2 + \beta} \mod \mathcal{M}^9.
\]

It is easy to see that
\[
\mu(f(az)e^{\alpha z^2 + \beta}) = \mu(f(z))
\]
for any \( f \in \Omega \). Let us choose \( \alpha_i, \beta_i, i = 1, 2 \) such that
\[
\hat{f}_i(z) = f_i(z) e^{\alpha_i z^2 + \beta_i} \equiv z + A_i z^5 + B_i z^7 \mod \mathcal{M}^9.
\]

Then \( p(\hat{f}_i) = -2A_i, q(\hat{f}_i) = 3B_i \). Since \( \mu(\hat{f}_1) = \mu(\hat{f}_2) \), we have
\[
\hat{f}_2(z) = \hat{f}_1(az)
\]
for some \( a \neq 0 \).

3. Modular forms

We shall compute \( \mu(\sigma(z, \Lambda)) \). For some reasons it is more convenient to use the Jacobi function in this case instead of the Weierstrass one. It is known that for the lattice
\[
\Lambda = \rho(\mathbb{Z} + \tau \mathbb{Z}), \rho \neq 0, \Im(\tau) > 0
\]
we have
\[
\sigma(z, \Lambda) = \vartheta_1(z, \rho, \tau)e^{\alpha z^2 + \beta},
\]
where
\[ \vartheta_1(z, \tau) = 2 \sum_{n=0}^{\infty} (-1)^n e^{\pi i r(n + \frac{1}{2})^2} \sin((2n + 1)\pi z) \]
is the first Jacobi theta function and \( \alpha, \beta \) are some complex numbers [1, §2.3]. Hence \( \mu(\sigma(z, \Lambda)) = \mu(\vartheta_1(z, \tau)) \).

Let \( p(\tau) = p(\vartheta_1(z, \tau)), q(\tau) = q(\vartheta_1(z, \tau)). \)

**Lemma 3** The functions \( p(\tau) \) and \( q(\tau) \) may be written in the form
\[
\begin{align*}
p(\tau) &= \frac{\pi^2}{30} \eta^6 g_2, \\
q(\tau) &= -\frac{\pi^3}{33} \eta^8 g_3,
\end{align*}
\]
where
\[ \eta = \eta(\tau) = e^{\frac{\pi i}{12}} \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau}) \]
is the Dedekind eta function and
\[
\begin{align*}
g_2 &= g_2(\tau) = (2\pi)^4 \left( \frac{1}{12} + 20 \sum_{n=1}^{\infty} \frac{n^3}{e^{-2\pi i n \tau} - 1} \right), \\
g_3 &= g_3(\tau) = (2\pi)^6 \left( \frac{1}{216} - \frac{7}{3} \sum_{n=1}^{\infty} \frac{n^5}{e^{-2\pi i n \tau} - 1} \right)
\end{align*}
\]
are the Weierstrass modular forms.

From the known equations
\[
\vartheta_1(z, \tau + 1) = e^{\pi i} \vartheta_1(z, \tau), \\
\vartheta_1 \left( \frac{z}{\tau}, \frac{1}{\tau} \right) = -i \sqrt{z} e^{\frac{\pi i}{2}} \vartheta_1(z, \tau)
\]
we can derive
\[
\begin{align*}
p(\tau + 1) &= i p(\tau), \\
p(-\frac{1}{\tau}) &= i \tau^7 p(\tau).
\end{align*}
\]
The product \( \eta^6 g_2 \) obeys the same equations, so \( \gamma(\tau) = \frac{\tau}{\eta^6 g_2} \) is a modular function.

The direct computation shows that \( \gamma(\tau) \to \frac{\pi^2}{30} \) as \( \Im(\tau) \to \infty \). The only zeros of the denominator are \( \tau \) which belong to the orbit of \( -\frac{1+\sqrt{-3}}{2} \) under the modular group action. But this are first order zeros whereas the order of any modular function at this points is divisible by 3. Then \( \gamma \) is bounded in upper halfplane so it is a constant. The equality (5) may be proved by the same way.

As a corollary we have
\[
\mu(\sigma(z, \Lambda)) = \frac{p(\tau)^3}{q(\tau)^2} = \frac{49}{1080} \frac{g_2^3}{g_3^5} = \frac{49}{40} j(\tau) - 1728,
\]
where
\[ j(\tau) = e^{-2\pi i \tau} + 744 + 196884e^{2\pi i \tau} + \ldots \]
is the modular invariant.
4. Differential equation

**Lemma 4** Any odd entire function $f$ satisfying (3) obeys the equation
\[(f'(0))^3 f(2z) = f^4(z)(\ln f(z))'''.
\] (6)

Let us denote the left hand side of (3) by $F(x, y, z, w)$. Then
\[
\frac{1}{6} \frac{d^3}{dt^3} F(x, x + t, x + \zeta t, x + \zeta^2 t) \bigg|_{t=0} = -3f^2 f' f'' + f^3 f''' + 2f(f')^3 - (f'(0))^3 f(2z),
\]
where $f = f(z)$ and $\zeta = e^{\frac{2\pi i}{3}}$. This equals to zero, so (6) follows.

**Lemma 5** Let $f_1, f_2 \in \Omega$ be two functions satisfying (6). If $f_2 \equiv f_1 \mod \mathcal{M}^9$ then $f_2 = f_1$ identically.

We may assume $f_1'(0) = f_2'(0) = 1$. If $f_2 \neq f_1$ then for some odd $n \geq 9$
\[f_2(z) \equiv f_1(z) + b z^n \mod \mathcal{M}^{n+2},\]
where $b \neq 0$. We have
\[f_2(2z) - f_1(2z) = b 2^n z^n \mod \mathcal{M}^{n+2},
\]
\[f_2'(z)(\ln f_2(z))'' - f_1'(z)(\ln f_1(z))''' = b z^n [(n-1)(n-2)(n-3) + 8] \mod \mathcal{M}^{n+1}.
\]
Since $f_1$ and $f_2$ are both solutions of (6),
\[b z^n \psi(n) \equiv 0 \mod \mathcal{M}^{n+1},\]
where $\psi(n) = (n-1)(n-2)(n-3) + 8 - 2^n$. By assumption $b \neq 0$ so $\psi(n) = 0$ but this is impossible for $n \geq 9$.

Now we can prove the theorem. Let $f$ be a function which satisfies the conditions. By Lemma 4 it is a solution of (6). So $f \in \Omega$ because nonzero odd function may not be a solution of $f^4(z)(\ln f(z))''' = 0$. Now it is enough to prove that there exists a function of required kind approximating $f$ up to 9th order, then the conclusion of theorem holds by Lemma 5.

If $p(f) = q(f) = 0$ then this is the case by Lemma 1. If $\mu(f) = \frac{49}{49}$ then $\mu(f) = \mu(\sin(z))$. If $\mu(f) \neq \frac{49}{49}$ then $\mu(f) = \frac{49}{49} f(\tau) - \frac{1728}{49}$ for some $\tau$ from the upper halfplane so
\[\mu(f) = \mu(\sigma(z, \Lambda)), \Lambda = Z + \tau \mathbb{Z}\]
(the case $\mu = \infty$ is included). In both cases $f$ has the required approximation by Lemma 2. The proof is complete.

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E-mail address: gavrilov@lapasrv.sscc.ru

Institute of Computational Mathematics and Mathematical Geophysics; Russia, Novosibirsk