A brief note on Weyl frames and canonical transformations in geometrical scalar–tensor theories of gravity

A B Barreto\textsuperscript{1,2}*, M L Pucheu\textsuperscript{2} and C Romero\textsuperscript{2}*

\textsuperscript{1} Instituto Federal de Educação, Ciência e Tecnologia do Rio Grande do Sul, 95043-700, Caxias do Sul-RS, Brazil
\textsuperscript{2} Departamento de Física, Universidade Federal da Paraíba, C. Postal 5008, 58051-970, João Pessoa-PB, Brazil

E-mail: adriano.barreto@caxias.ifrs.edu.br, mlaurapucheu@fisica.ufpb.br and cromero@fisica.ufpb.br

Received 27 July 2017, revised 22 December 2017
Accepted for publication 4 January 2018
Published 25 January 2018

Abstract
We consider scalar–tensor theories of gravity defined in Weyl integrable space-time and show that for time-lapse extended Robertson–Walker metrics in the ADM formalism a class of Weyl transformations corresponding to change of frames induce canonical transformations between different representations of the phase space. In this context, we discuss the physical equivalence of two distinct Weyl frames at the classical level.

Keywords: Weyl transformations, scalar–tensor theories, canonical transformations

1. Introduction

As is well known, the Hamiltonian formalism has proved to be a powerful tool in the study of the dynamics of classical systems. Its application to general relativity, as well as to other theories of gravity, based on the so-called ADM formalism [1], has set up the basis for several approaches to a quantum theory of gravity [2]. Certainly, a prominent aspect of the Hamiltonian formalism is related to canonical transformations, the very special class of transformations defined in the phase space which preserves the form of Hamilton equations. Clearly, dynamical systems leading to Hamilton equations which may be related by canonical transformations are to be regarded as being physically equivalent [3].

In a somewhat different context, namely, that of scalar–tensor theories of gravity, the issue of physical equivalence appears when different frames are used to write the field equations. Usually these frames are related by a set of transformations involving the metric and the scalar field. In the case of Brans–Dicke gravity, two frames, in particular, are considered as important
for the mathematical formulation of the theory: the Einstein and Jordan frames [4].

It turns out that, with respect to scalar–tensor theories, the original approach assumes, as in general relativity, that the space-time manifold is Riemannian. On the other hand, it has been shown recently that when the Palatini variational method is applied to derive the field equations from the action, then in a wide class of scalar–tensor theories, a non-Riemannian compatibility condition between metric and affine connection appears quite naturally [6] (For a more general result, see [7]). In a certain sense, this condition seems to establish the space-time geometry from first principles, the space-time manifold being dynamically determined by the particular coupling of the scalar field in the gravitational sector. In the case of Brans–Dicke theory, the mentioned procedure leads to what has been called a Weyl integrable space-time, a particular version of the geometry conceived by Weyl in his attempt to unify gravity and electromagnetism [8].

Now, Weyl geometry is one of the simplest generalizations of Riemann geometry, in which the metric compatibility condition is weakened. This was the way Weyl devised to introduce a covariant vector field $\sigma_\mu$, which bears amazing similarity with the electromagnetic 4-potential. Weyl also introduced the tensor $F_{\mu\nu} = \partial_\mu \sigma_\nu - \partial_\nu \sigma_\mu$, which he interpreted as representing a kind of length curvature. As a consequence of the modification in the Riemannian compatibility condition, the covariant derivative of the metric tensor does not vanish, as in Riemannian geometry, and the length of vectors parallel transported along a curve may change. Weyl’s compatibility condition is given by $\nabla_\alpha g_{\mu\nu} = \sigma_\alpha g_{\mu\nu}$, and is invariant under the conformal transformation $g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} = e^f g_{\mu\nu}$ and $\sigma_\mu \rightarrow \tilde{\sigma}_\mu = \sigma_\mu + \partial_\mu f$, where $f$ is an arbitrary scalar function [9]. These findings are considered by some authors as the ‘dawning’ of modern gauge theories [10]. If $F_{\mu\nu} = 0$ (null second curvature) then there is no electromagnetic field. In this case, there exists a scalar field $\phi$, such that $\sigma_\mu = \partial_\mu \phi$, and, instead of a vector field, we are left with a scalar field $\phi$, which, in addition to the metric, is the fundamental object that characterizes this geometry. A space-time endowed with this particular version of Weyl geometry is known as Weyl Integrable Space-Time [11].

In this article, we shall show that, when we consider the ADM formalism for scalar–tensor theories, then Weyl transformations induce canonical transformations between different representations of the phase space. We obtain a generating function corresponding to the canonical transformations, thereby showing the physical equivalence of two distinct Weyl frames at the classical level. We revisit, in this way, the discussion about the physical equivalence between Jordan and Einstein frames, now from the point of view of Weyl geometry, in which affine geodesics and the concept of proper time are invariant under frame transformations.

2. Hamiltonian formalism and Weyl frames

Let us consider the gravitational sector of Brans–Dicke action

$$S = \int d^4x \sqrt{-g} e^{-\phi} \left( R + \omega \phi_{,\mu} \phi^{,\mu} \right),$$

(1)

Brans and Dicke [4]. For a nice review of scalar–tensor theories, see Faraoni.

For a clear exposition on the problem of physical equivalence of different frames in Brans–Dicke theory see Faraoni and Gunzig [5]. See also Faraoni and Nadeau.

For a more detailed account of Weyl geometry, see Adler [9]. A more formal mathematical treatment is given by Folland. For an updated and comprehensive review on Weyl geometry, see Scholz.

In the last decades a great deal of work has gone into scalar–tensor theories of gravity in the framework WIST. See, for instance Novello et al [11].
where $R$ is the Ricci scalar calculated from the Weyl connection, and $\omega$ denotes a free dimensionless parameter.

As already mentioned, Weyl integrable geometry is an extension of Riemannian geometry. In the case of Weyl integrable space-time, the transformations mentioned above reduce to\(^7\)

\[
\begin{align*}
g_{\mu\nu} &= e^f g_{\mu\nu}, \\
\phi &= \phi + f.
\end{align*}
\]

(2)

Let us now restrict ourselves to homogeneous and isotropic cosmological models, with the Friedman–Lemaître–Robertson–Walker line element given by

\[
ds^2 = N^2(t)dt^2 - a^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right],
\]

(3)

where $N(t)$ denotes the lapse function and $a(t)$ is the cosmic scale factor. It is not difficult to see that, after neglecting surface terms, the reduced action of the Lagrangian corresponding to (3) is

\[
L = e^{-\phi} \left[ \left( \omega - \frac{3}{2} \right) a^3 \dot{\phi}^2 + 6 \left( kNa - \frac{a}{N} \dot{a}^2 + \frac{a^2}{N} \dot{a} \dot{\phi} \right) \right].
\]

(4)

On the other hand, the canonical momenta will be given by

\[
p_a = \frac{6e^{-\phi}}{N} \left( a^2 \dot{\phi} - 2a \dot{a} \right),
\]

(5)

\[
p_\phi = \frac{e^{-\phi}}{N} \left[ (2\omega - 3) a^3 \dot{\phi} + 6a^2 \dot{a} \right].
\]

(6)

Thus the total Hamiltonian of the model can be written as

\[
H = NH, \quad \mathcal{H} = \mathcal{H}_c,
\]

(7)

with the super-Hamiltonian constraint being

\[
\mathcal{H}_c = \frac{e^{\phi}}{4\omega a} \left[ \frac{3 - 2\omega}{12} p_a^2 + \frac{p_{\phi}^2}{a^2} + \frac{p_a p_{\phi}}{a} \right] - 6kae^{-\phi}.
\]

(8)

At this point, let us transform the action (1) (written in the Weyl frame $(g, \phi)$) by performing the general Weyl transformations (2). It is not difficult to verify that (1), turns into the new action

\[
\bar{S} = \int d^4x \sqrt{-\bar{g}} e^{-\bar{\phi}} \left( \bar{R} + \omega \bar{\phi}_{\mu} \bar{\phi}^{\mu} - 2\omega \bar{\phi}_{\mu} f^{\mu} + \omega f_{\mu} f^{\mu} \right),
\]

(9)

now written in the new frame $(\bar{g}, \bar{\phi})$. We now rewrite the FLRW line element as

\[
ds^2 = \bar{N}^2 \bar{d}^2 - \bar{a}^2(t) \left[ \frac{\bar{d}r^2}{1 - k\bar{r}^2} + \bar{r}^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right].
\]

(10)

\(^7\)In the literature, these transformations are also referred to either as gauge transformations or change of frames.
In the context of the Hamiltonian formalism it seems reasonable to regard \( f \) as a function of \( a \) and \( \phi \), i.e. \( f \equiv f(a, \phi) \). It is easy to see that this assumption leads to the reduced Lagrangian

\[
\mathcal{L} = e^{-\phi} \left\{ \bar{a} \left( \omega \bar{a}^2 \bar{f}_a^2 \right) - 6 \bar{a}^2 + 2a^2 \left[ 3 - \omega \bar{a} \left( 1 - \bar{f}_a \right) \bar{f}_a \right] \right. \\
\times \left. \bar{a} \dot{\bar{a}} + \bar{a}^3 \left[ \omega \left( 1 - \bar{f}_a \right)^2 - \frac{3}{2} \right] \phi^2 + 6k \bar{a} \bar{N}^2 \right\},
\]

where we have denoted \( \bar{f}(\bar{a}, \bar{\phi}) = f(a, \phi) \) and defined \( \bar{f}_a = \frac{\partial f}{\partial a} \), \( \bar{f}_\phi = \frac{\partial f}{\partial \phi} \). The canonical momenta in the transformed frame are given by

\[
p_a = \frac{2ae^{-\phi}}{N} \left\{ \bar{a} \left( \omega \bar{a}^2 \bar{f}_a^2 \right) - 6 \bar{a}^2 \right\},
\]

\[
p_\phi = \frac{2a^2e^{-\phi}}{N} \left\{ \bar{a} \left[ 3 - \omega \bar{a} \left( 1 - \bar{f}_a \right) \bar{f}_a \right] + \bar{a}^3 \left[ \omega \left( 1 - \bar{f}_a \right)^2 - \frac{3}{2} \right] \right\}
\]

while the super-Hamiltonian constraint expressed in the new variables takes the form

\[
\bar{\mathcal{H}} = \frac{e^{\phi}}{12 \omega \bar{a} \left[ af_a + 2(f_\phi - 1) \right]} \left\{ [3 - 2\omega (f_\phi - 1)^2] p_a^2 + 2 \left( 6 - \omega \bar{a} \bar{f}_a^2 \right) p_\phi^2 \frac{p_\phi}{a^2} \right. \\
+ \left. 4 \left[ 3 + \omega \bar{a} f_a \left( f_\phi - 1 \right) \right] \frac{p_a p_\phi}{a} \right\} - 6k e^{-\phi}.
\]

Since \( \bar{\mathcal{H}} \) and \( \mathcal{H} \) come from two distinct Lagrangians related by the Weyl transformation (2), it seems interesting to investigate whether there exists a transformation in the phase space that links them. According with the Hamiltonian mechanics, such a transformation, if it exists at all, must be a canonical transformation. In the next section, we shall show that, indeed, Weyl transformations induce canonical transformations which relate the generalized coordinates in both Weyl frames.

2.1. Change of frames as canonical transformations

In what follows, we shall look for a change of frames that preserves the form of Hamiltonian equations. In other words, we look for a transformation that links the canonical coordinates and takes the total Hamiltonian \( H = N\mathcal{H} \) into \( H = \bar{N}\bar{\mathcal{H}} \). This can easily be done by computing the relevant Poisson brackets and showing that they are preserved. Let us first consider the following class of transformations

\[
\bar{a} = e^{\frac{\phi}{2}} a, \\
\bar{\phi} = \phi + f, \\
p_a = \frac{2e^{-\frac{\phi}{2}}}{2 + af_a} \left( p_a - f_a p_\phi \right), \\
p_\phi = \frac{1}{1 + f_\phi} \left( p_\phi - \frac{af_\phi}{2} p_a \right).
\]
It is not difficult to check that these transformations are canonical in two cases: (i) $f_a = 0$ and $f(\phi) \neq -\phi$, and (ii) $f_\phi \equiv \frac{df}{d\phi} = 0$ and $f(a) \neq -2\ln a^8$. These relate, in principle, two distinct gravitational theories whose actions are defined in two distinct Weylian frames connected by (2)$^9$. On the other hand, because these transformations, which originated from a change in the Weyl frames, are canonical it seems reasonable to conclude that the physical equivalence between the two theories is guaranteed, at least at the classical level. Unfortunately this equivalence between frames cannot be taken further. Indeed, at the quantum level, to say the least, it is still unclear whether classically equivalent systems (related by canonical transformations) lead to quantum equivalent systems [12].

2.2. Generating functions

Another way to show that the transformations (15) are canonical is to obtain explicitly its generating function. For this purpose, let us consider the first case examined previously, that is, when $f \equiv f(\phi)$, with $f \neq -\phi$. The Weyl canonical transformations are given by

$$\bar{a} = e^{f(\phi)/2} a,$$
$$\bar{\phi} = \phi + f(\phi),$$
$$p_a = e^{-f(\phi)/2} p_a,$$
$$p_{\bar{\phi}} = \frac{1}{1 + \frac{df}{d\phi}} \left( p_{\phi} - \frac{d^2\phi}{d\phi^2} p_a \right),$$

(16)

and it is straightforward to verify that the generating function of this transformation is

$$G_1 = f(\phi) \left( p_{\phi} + \frac{ap_a}{2} \right).$$

(17)

Let us now consider the second case, in which $f_\phi = 0$. For simplicity, let us choose as a particular example, $f \equiv f(a) = e^a$. The above transformations then reduce to

$$\bar{a} = \exp \left( \frac{e^a}{2} \right) a,$$
$$\bar{\phi} = \phi + e^a,$$
$$p_a = \frac{2 \exp \left( \frac{-e^a}{2} \right)}{2 + ae^a} \left( p_a - e^a p_{\phi} \right),$$
$$p_{\bar{\phi}} = p_{\phi},$$

(19)

and are generated by the function

$$G_2 = e^a \left( p_{\phi} + \frac{ap_a}{2} \right).$$

(20)

$^8$The first of these conditions does not include the transformation that leads to the Riemann frame. This particular and important case will be considered separately in section 2.3.

$^9$Although the lapse function $N$ plays the role of a Lagrange multiplier in the Hamiltonian formalism, its redefinition is also required for a full identification between the Hamiltonians by means of (15). Thus we have set $N = e^{-1/2} N$. 
2.3. A particular example: the Riemann frame

Let us next consider the particular Weyl transformation that leads to the frame in which the transformed geometrical scalar field $\tilde{\phi}$ vanishes, i.e. the so-called Riemann frame, in which the Riemannian compatibility condition

$$\nabla_\alpha \tilde{g}_{\mu\nu} = 0$$  \hfill (21)

is recovered (For details, see [6]). Clearly, this is carried out simply by taking $f = -\phi$ in (2). In this case, it is easy to see that the transformed action takes the form

$$\tilde{S} = \int d^4x \sqrt{-\tilde{g}} \left( \tilde{R} + \omega \tilde{g}^{\mu\nu} \phi,_{\mu} \phi,_{\nu} \right),$$  \hfill (22)

where $\tilde{g}^{\mu\nu} = e^{\phi} g^{\mu\nu}$, and the Ricci scalar $\tilde{R}$ is defined purely in terms of the metric $\tilde{g}_{\mu\nu}$. It is worth noting that when $\omega = \frac{1}{2}$ (22) is formally equivalent to the Hilbert–Einstein action with a massless minimally coupled scalar field [6].

Again, from the line element

$$ds^2 = \tilde{N}^2(t) dt^2 - \tilde{a}^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \right],$$  \hfill (23)

the reduced Lagrangian is easily seen to be given by

$$\tilde{L} = 6 \left( k\tilde{a}N - \frac{\tilde{a}}{N} \tilde{a}^2 \right) + \omega \tilde{a}^3 \tilde{\phi}^2.$$  \hfill (24)

A brief comment about (24) is in order. As $\tilde{L}$ does not depend explicitly on $\tilde{\phi}$, it follows that its conjugate canonical momentum $p_{\tilde{\phi}}^{-}$ is conserved. On the other hand, the canonical momenta are

$$p_{\tilde{a}} = -\frac{12}{N} \tilde{a} \tilde{\dot{a}},$$  \hfill (25)

$$p_{\tilde{\phi}} = \frac{2}{N} \omega \tilde{a}^3 \tilde{\phi},$$  \hfill (26)

while the super-Hamiltonian reads

$$\tilde{H} = -\frac{p_{\tilde{a}}^2}{24 \tilde{a}} + \frac{p_{\tilde{\phi}}^2}{4 \omega \tilde{a}^4} - 6k \tilde{a}.$$  \hfill (27)

Finally, the canonical transformations in the phase space that relate the canonical variables of the two actions (1) and (22) are

$$\tilde{a} = ae^{-\phi/2},$$

$$\tilde{\phi} = \phi,$$

$$p_{\tilde{a}} = p_a e^{\phi/2},$$

$^{10}$ It is important to note that, in the action written in the Riemann frame, $\phi$ is no longer the Weyl geometrical field, and its appearance here is due simply to the particular choice of the function $f$ in the Weyl transformation, namely, $f = -\phi$. 


\[ p_\phi = p_\phi + \frac{a}{2} p_a, \]  
(28)

with the new lapse function being given by \( \tilde{N} = N e^{-\phi/2} \). The generating function in this case is simply

\[ \tilde{G} = \frac{1}{2} \tilde{\phi} \tilde{a} p_a. \]  
(29)

3. Final remarks

In this work we have investigated the problem of how Weyl transformations behave when viewed at the level of the Hamiltonian formulation of gravity in the case of scalar–tensor theories for time-lapse extended Robertson–Walker metrics. Starting from the original action of Brans–Dicke theory, in which the underlying space-time is assumed to have the geometric structure of Weyl integral space-time, we carry out a class of Weyl frame transformations which induce changes in the reduced Hamiltonian of the original action. We then obtain the result that if we consider a restrict class of Weyl transformations then the two Hamiltonians are related by a canonical transformation in the sense of Hamiltonian mechanics\(^{11}\). The physical equivalence between the actions, particularly when the Weyl transformation leads to the Riemann frame is, in a certain way, consistent with the recent interpretation of what physical equivalence\(^{12}\) means when geometrical scalar–tensor theories are viewed in different frames [6]. We would like to recall that in the context of geometrical scalar–tensor theories it is possible to define all relevant geometric and physical concepts, such as proper time, geodesics, curvature, and others, in a frame-invariant way [6]. It is interesting to note that this result may be completely carried over to the problem of equivalence between the Einstein and the Jordan frames as the transformation between these frames (found by Dicke, see [14]) is completely analogous to (2), which relates the Weyl frame to the Riemann frame when \( f = -\phi \), thereby connecting the actions (1) and (22). By no means can this equivalence be extended to the quantum level [12]. This is the case, for instance, when we are working out the canonical quantization of classical cosmological models in the framework of quantum cosmology. As far as we know, the physical equivalence between frames at the quantum level is still an open question.

Acknowledgments

The authors thank CAPES and CNPq for financial support. We thank the referees for useful and relevant comments.

Appendix. More general canonical transformations

A more general transformation than the ones obtained in section 2.1 are given by the following:

\[ \tilde{a} = e^{\xi} a, \]

\(^{11}\) In the course of the refereeing process, one of the referees found a class of canonical Weyl transformations which is more general than the ones presented in section 2.1. They are included in appendix.

\(^{12}\) For a clear and interesting discussion of the question and the meaning of physical equivalence between frames in classical scalar–tensor theories we refer the reader to [13].
\[
\tilde{\phi} = \phi + f, \\
p_a = \frac{2e^{-\frac{f}{2}}}{2 + af_\phi + 2f_\phi} \left( p_a + f_\phi p_a - f_a p_\phi \right), \\
p_\phi = \frac{1}{2 + af_\phi + 2f_\phi} \left[ 2p_\phi - a \left( f_\phi p_a - f_a p_\phi \right) \right],
\]

whenever \(2 + af_\phi + 2f_\phi \neq 0\) which, in turns, implies \(f(a, \phi) \neq f_0 e^{-\phi/2} - \phi\), where \(f_0\) is an arbitrary \(C^1\) function. Let us note that this excludes the Riemann frame.

**References**

[1] Arnowitt R, Deser S and Misner C W 1959 *Phys. Rev.* **116** 1322

[2] Gambini R and Pullin J 2011 *A First Course in Loop Quantum Gravity* (Oxford: Oxford University Press)

Henneaux M and Teitelboim C 1994 *Quantization of Gauge Systems* (Princeton, NJ: Princeton University Press)

Woodhouse N M J 1997 *Geometric Quantization* (Oxford Mathematical Monographs) (New York: Clarendon)

Rovelli C 2004 *Quantum Gravity* (Cambridge Monographs on Mathematical Physics) (Cambridge: Cambridge University Press)

Baez J and Munian J P 1994 *Gauge Fields, Knots and Gravity* (Series on Knots and Everything vol 4) (Singapore: World Scientific)

Bojowald M 2010 *Canonical Gravity and Applications: Cosmology, Black Holes, and Quantum Gravity* (Cambridge: Cambridge University Press)

[3] Goldstein H 1965 *Classical Mechanics* (Reading, MA: Addison-Wesley)

[4] Brans C H and Dicke R H 1961 *Phys. Rev.* **124** 925

Faraoni V 2004 *Cosmology in Scalar–Tensor Gravity* (New York: Kluwer)

Fujii Y and Maeda K 2003 *The Scalar–Tensor Theory of Gravitation* (Cambridge: Cambridge University Press)

Capozziello S and Faraoni V 2011 *Beyond Einstein Gravity* (Berlin: Springer)

[5] Faraoni V and Gunzig E 1999 *Int. J. Theor. Phys.* **38** 217

Faraoni V and Nadeau S 2007 *Phys. Rev. D* **75** 023501

SK N and Sanyal A K 2017 *Int. J. Mod. Phys. D* **26** 1750162

Capozziello S, Martin-Moruno P and Rubano C 2010 *Phys. Lett. B* **689** 117

Banerjee N and Majumder B 2016 *Phys. Lett. B* **754** 129

Pandey S 2017 *Eur. Phys. J. Plus* **132** 107

[6] Almeida T S, Pucheu M L, Romero C and Formiga J B 2014 *Phys. Rev. D* **89** 064047

[7] Burton H and Mann R B 1998 *Phys. Rev. D* **57** 4754

[8] Weyl H 1918 *Sitzungsber Deustch. Akad. Wiss.* **465** 24–37

Weyl H 1952 *Space, Time, Matter* (Dover Books on Advanced Mathematics) (New York: Dover)

[9] Adler R, Bazin M and Schiffer M 1975 *Introduction to General Relativity* (New York: McGraw-Hill) ch 15

Dahia F, Gomez G A T and Romero C 2008 *J. Math. Phys.* **49** 102501

Scholz E 2017 *Einstein Stud.* **13** 171

Folland G B 1970 *J. Diff. Geom.* **4** 145

Scholz E 2017 arXiv:1703.03187
[10] O’Raifeartaigh L 1997 *The Dawning of Gauge Theory* (Princeton, NJ: Princeton University Press)
Afriat A 2009 Stud. Hist. Phil. Sci. B **40** 20

[11] Novello M, Oliveira L A R, Salim J M and Elbas E 1993 *Int. J. Mod. Phys.* D **1** 641–77
Salim J M and Sautú S L 1996 *Class. Quantum Grav.* **13** 353
Oliveira H P, Salim J M and Sautú S L 1997 *Class. Quantum Grav.* **14** 2833
Melnikov V 1995 Classical solutions in multidimensional cosmology *Proc. of the 8th Brazilian School of Cosmology and Gravitation II* ed M Novello (Editions Frontières) pp 542–60
Bronnikov K A, Konstantinov M Yu and Melnikov V N 1995 *Grav. Cosmol.* **1** 60

Miritzis J 2004 *Class. Quantum Grav.* **21** 3043
Miritzis J 2005 *J. Phys.: Conf. Ser.* **8** 131
Aguilar J E M and Romero C 2009 *Found. Phys.* **39** 1205
Aguilar J E M and Romero C 2009 *Int. J. Mod. Phys.* A **24** 1505
Miritzis J 2013 *Int. J. Mod. Phys.* D **22** 1350019
Poulis F P and Salim J M 2014 *Int. J. Mod. Phys.* D **23** 1450091
Vazirian R, Tanhayi M R and Motahar Z A 2015 *Adv. High Energy Phys.* **7** 902396
Lobo I P, Barreto A B and Romero C 2015 *Eur. Phys. J.* C **75** 448
Pucheu M L, Alves-Junior F A P, Barreto A B and Romero C 2016 *Phys. Rev.* D **94** 064010

[12] Anderson A 1993 *Phys. Lett.* B **305** 67
Jan Lacki 2004 Stud. Hist. Phil. Mod. Phys. **35** 317

[13] Flanagan E 2004 *Class. Quantum Grav.* **21** 3817

[14] Dicke R H 1962 *Phys. Rev.* **125** 2163