Linear Seesaw for Dirac Neutrinos

with $A_4$ Flavour Symmetry

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Abstract

We propose a linear seesaw model to realise light Dirac neutrinos within the framework of $A_4$ discrete flavour symmetry. The additional fields and their transformations under the flavour symmetries are chosen in such a way that naturally predicts the hierarchies of different elements of the seesaw mass matrix and also keeps the unwanted terms away. For generic choices of flavon alignments, the model predicts normal hierarchical light neutrino masses with the atmospheric mixing angle in the lower octant. Apart from predicting interesting correlations between different neutrino parameters as well as between neutrino and model parameters, the model also predicts the leptonic Dirac CP phase to lie in a specific range $-\pi/2 \lesssim \delta \lesssim -\pi/5$ and $\pi/5 \lesssim \delta \lesssim \pi/2$ that includes the currently preferred maximal value. The predictions for the absolute neutrino masses in one specific version of the model can also saturate the cosmological upper bound on sum of absolute neutrino masses.

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I. INTRODUCTION

The fact that neutrinos have non-zero but tiny masses, several order of magnitudes smaller compared to the electroweak scale and large mixing [1] has been verified again and again in the last two decades. The present status of neutrino oscillation data can be found in the recent global fit analysis [2–4], which clearly indicate that we do not yet know some of the neutrino parameters namely, the mass hierarchy of neutrinos: normal \((m_3 > m_2 > m_1)\) or inverted \((m_2 > m_1 > m_3)\), leptonic CP violation as well as the octant of atmospheric mixing angle \(\theta_{23}\). While the next generation neutrino experiments will be able to settle these issues, one still can not determine the nature of neutrino: Dirac or Majorana in neutrino oscillation experiments. Though Majorana nature of neutrinos can be probed through lepton number violating signatures like neutrinoless double beta decay \((0\nu\beta\beta)\), there has not been any positive signal of it yet. For example, please refer to the latest results from KamLAND-ZEN experiment [5]. Although such null results only disfavour the quasi-degenerate regime of light Majorana neutrinos and can never rule out Majorana nature of neutrinos, this has recently motivated the particle physics community to study the scenario of Dirac neutrinos with similar interest as given to Majorana neutrinos in the last few decades. The conventional seesaw mechanism for the origin of neutrino masses [6–9] and its many descendants predict light Majorana neutrinos. On the contrary, there were fewer proposals to generate light Dirac neutrino masses initially [10, 11] but it has recently gained momentum with several new proposals to realise sub-eV scale Dirac neutrino masses [12–39]. Since the coupling of left and right handed neutrinos to the standard model (SM) Higgs field will require fine tuning of Yukawa coupling to the level of \(10^{-12}\) or even less, it is important to forbid such couplings at tree level by introducing some additional symmetries such as \(U(1)_{B-L}, Z_N, A_4\) which also make sure that the right handed singlet neutrinos do not acquire any Majorana mass terms.

There have been several discussions on other conventional seesaw mechanisms in the context of Dirac neutrinos for example, type I seesaw [32, 35], type II seesaw [33], inverse seesaw [35] and so on. Here show how light Dirac neutrinos can be realised within another seesaw scenario, known as linear seesaw mechanism. We consider the presence of \(A_4\) flavour symmetry augmented by additional discrete \(Z_N\) and global lepton number symmetries which not only dictate the neutrino mixing patterns but also keep the unwanted terms away from...
the seesaw mass matrix, in order to realise linear seesaw. Linear seesaw for Majorana
neutrino was proposed in earlier works \cite{40, 41} and further extended to radiative seesaw
models in \cite{42, 43} and hidden gauge sector models in \cite{44}. We extend it to Dirac neutrino
scenarios in a minimal way incorporating the above-mentioned flavour symmetries. Apart
from retaining the usual attractive feature of linear seesaw, like the viability of seesaw scale
at TeV naturally without much fine-tuning, the model also predicts several other aspects
of neutrinos that can be tested at upcoming experiments. Among them, the preference
for normal hierarchy, specific range of Dirac CP phase that includes the maximal value,
atmospheric mixing angle in lower octant are the ones which address the present puzzles in
neutrino physics.

Rest of the paper is organised as follows. In section II, we discuss the conventional and
Dirac linear seesaw model and its predictions for sub-eV Dirac neutrinos in details. Finally,
we conclude in section III.

II. THE LINEAR SEESAW MODEL

In the conventional linear seesaw model for Majorana neutrinos \cite{40, 41}, the standard
model fermion content is effectively extended by two different types of neutral singlet
fermions \((N, S)\) per generation and the complete neutral fermion mass matrix \((9 \times 9)\) in
the basis \((\nu_L, N, S)\) assumes the form

\[
M_\nu = \begin{pmatrix}
0 & m_D & M_L \\
m_D^T & 0 & M \\
M_L^T & M^T & 0
\end{pmatrix},
\]

which, being linear in Dirac neutrino mass matrix \(m_D\) is known as the linear seesaw \cite{40, 41}.

Here the effective light neutrino mass is roughly given by \(\sim \epsilon m_D/M\) where \(\epsilon\) is originated
from a small lepton number violating term in \(M_L\), the \((13)\) entry of the neutral fermion
mass matrix given in Eq. (1). This is a simple alternative to the usual inverse seesaw
model \cite{45–48} where we also introduce two sets of gauge singlet Majorana neutrinos at the
TeV scale to obtain light neutrino mass in sub-eV range. We now consider an extension of this simple linear seesaw model to generate sub-eV Dirac neutrino masses following a similar roadmap that was used to accommodate light Dirac neutrinos in type I seesaw [32, 35], type II seesaw [33], inverse seesaw [35] etc. Apart from introducing the right handed counterpart of the usual left handed neutrinos, the other heavy fermions introduced for seesaw purpose are of Dirac nature, having both helicities: \((N_L, N_R)\) and \((S_L, S_R)\). In such a case the complete linear seesaw mass matrix can be written in \((\nu_L, N_L, S_L)^T, (\nu_R, N_R, S_R)\) basis as

\[
m_{\nu} = \begin{pmatrix}
0 & m_{\nu N} & M_{\nu S} \\
 m'_{\nu N} & 0 & M_{NS} \\
 M'_{\nu S} & M'_{NS} & 0
\end{pmatrix}.
\]

(3)

The corresponding formula for light Dirac neutrinos can be written as

\[
m_{\nu} = m_{\nu N}(M'_{\nu S}M_{NS}^{-1}) + (M_{\nu S}M_{NS}^{-1})m'_{\nu N}.
\]

(4)

To obtain the desired structure of the seesaw mass matrix given in Eq. (3) and to obtain the required hierarchy among its elements, we consider \(Z_4 \times Z_3\) symmetry and a global lepton number \(U(1)_L\) symmetry in addition to \(A_4\) flavour symmetry which plays a crucial role in realising flavour structures of the corresponding mass matrices. In Table I we elaborate the transformation of SM fields as well as the additional fermions and flavons involved in the present construction of linear seesaw. Here the SM lepton doublets \((L)\) and the additional singlet neutral fermions transform as \(A_4\) triplets. On the other hand, SM charged leptons \((e_R, \mu_R, \tau_R)\) transform as \(1,1',1''\) respectively under same \(A_4\) symmetry. Likewise most \(A_4\) flavour models, two \(A_4\) triplet flavons present in the set-up \(\phi_T\) and \(\phi_S\), play an instrumental role.
role in generating the non-diagonal mass matrices for charged lepton and neutrino sectors respectively. The Yukawa Lagrangian upto leading order for charged leptons invariant under this $A_4 \times Z_4 \times Z_3 \times U(1)_L$ symmetry is

$$\mathcal{L}_{CL} = \frac{y_e}{\Lambda}(\bar{L}\phi_L)H e_R + \frac{y_\mu}{\Lambda}(\bar{L}\phi_L)_1 H \mu_R + \frac{y_\tau}{\Lambda}(\bar{L}\phi_L)_1 H \tau_R + \text{h.c.}, \quad (5)$$

where $\Lambda$ is the cut-off scale of the theory and both $y$'s are the respective dimensionless Yukawa coupling constants. Here the leading order contribution to the charged leptons via $\bar{L}H\ell_i$ (where $\ell_i$ are the right handed charged leptons) are not invariant under $A_4$ symmetry.

In presence of the triplet flavon $\phi_T$ one can easily construct $A_4$ invariant dimension five operators as shown on the right hand side of Eq. (5) which subsequently generate the relevant masses for charged leptons after flavon $\phi_T$ and the SM Higgs field acquire non-zero vacuum expectation value (vev). In appendix A, we have briefly summarised the $A_4$ product rules which dictate the flavour structure of the mass matrices. Now, for the triplet flavon $\phi_T$, considering a generic vev alignment $\langle \phi_T \rangle = (v_T, v_T, v_T)$, the charged lepton mass matrix can be written as

$$m_t = \frac{vv_T}{\Lambda} \begin{pmatrix} y_e & y_\mu & y_\tau \\ y_\mu & \omega y_\mu & \omega^2 y_\tau \\ y_e & \omega^2 y_\mu & \omega y_\tau \end{pmatrix}, \quad (6)$$

where $v$ is the vev of the SM Higgs doublet $H$ and $\omega = e^{i2\pi/3}$ is the cube root of unity. This charged lepton mass matrix now can be diagonalised by using a matrix $U_\omega$ (also known as the magic matrix), given by

$$U_\omega = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}. \quad (7)$$

Now, for neutrino sector the relevant Yukawa Lagrangian is given by

$$\mathcal{L}_\nu = Y_{\nu N}L\bar{H}N_R + \bar{\nu}_R N_L \left( y'_{\xi_2} \xi_2^\dagger + y'_{\eta_2} \eta_2^\dagger + y'_s \phi_s^\dagger + y'_a \phi_a^\dagger \right) \frac{\rho^\dagger}{\Lambda} + \bar{\nu}_R S_L N_R \left( y_{\xi_1} \xi_1 + y_{\eta_1} \eta_1 + y'_s \phi_s + y'_a \phi_a \right) \frac{\rho}{\Lambda} + \bar{S}_L N_L \left( y_{\xi_2} \xi_2 + y_{\eta_2} \eta_2 + y_{s_2} \phi_s + y_{a_2} \phi_a \right) + \text{h.c.} \quad (8)$$
Here both $\bar{L}H N_R$ and $\bar{L}H S_R$ terms, involving SM lepton doublet are generated at tree level. As both SM lepton doublets ($L$) and gauge singlet Dirac fermions ($N, S$) are $A_4$ triplets (the SM Higgs $H$ being a singlet under the same), following the $A_4$ multiplication rules given in appendix $A$, we find the associated mass matrices to be diagonal. These mass matrices can be written as

$$m_{\nu N} = Y_{\nu N} v I, \quad M_{\nu S} = Y_{\nu S} v I,$$

where $I$ is a $3 \times 3$ identity matrix. On the other hand, owing to the specific discrete $Z_4 \times Z_3$ symmetry, $S_L$-$\nu_R$ and $\nu_R$-$N_L$ couplings are generated at dimension five level, ensuring the smallness of these couplings. These contributions come via involvement of the $A_4$ singlet flavons $\xi, \eta$ and $\rho$ as well as the triplet flavon $\phi_s$. Unlike in the conventional linear seesaw mechanism for Majorana neutrinos, here we do not have any approximate global symmetry to make certain terms of the mass matrix small, from naturalness arguments. Therefore, we need to assign these additional discrete symmetries so that at least one of the mass matrices contributing to the light neutrino mass formula in Eq. (4) arises at next to leading order. In this set-up $\phi_s$ and $\xi$ share same discrete charges like $\eta$, hence all of them therefore contribute to the $S_L$-$\nu_R$ and $\nu_R$-$N_L$ couplings. This essentially leads to same non-diagonal contributions in these couplings. Now, with the vev alignment for the flavons $\phi_s, \xi, \eta$ and $\rho$ as, $\langle \phi_s \rangle = (0, v_s, 0)$, $\langle \xi \rangle = v_\xi$, $\langle \eta \rangle = v_\eta$ and $\langle \rho \rangle = v_\rho$ respectively, the most general mass matrices corresponding to these two couplings can be written as

$$M'_{\nu S} = \begin{pmatrix} x'_1 & 0 & s'_1 + a'_1 \\ 0 & x'_1 & 0 \\ s'_1 - a'_1 & 0 & x'_1 \end{pmatrix}, \quad m'_{\nu N} = \begin{pmatrix} x'_2 & 0 & s'_2 + a'_2 \\ 0 & x'_2 & 0 \\ s'_2 - a'_2 & 0 & x'_2 \end{pmatrix},$$

where $x'_1 = (y_{\xi,1} v_\xi + y_{\eta,1} v_\eta) v_\rho / \Lambda$, $s'_1 = y_{a,1} v_s v_\rho / \Lambda$, $a'_1 = y_{a,1} v_s v_\rho / \Lambda$, $x'_2 = (y_{\xi,2} v_\xi + y_{\eta,2} v_\eta) v_\rho / \Lambda$, $s'_2 = y_{a,2} v_s v_\rho / \Lambda$ and $a'_2 = y_{a,2} v_s v_\rho / \Lambda$. Note that $s'_i$ and $a'_i$ (where $i = 1, 2$) are the symmetric and anti-symmetric contributions originated from $A_4$ multiplications. Similarly, the mixing between the heavy neutrinos $S_L - N_R$ and $S_R - N_L$ are generated at dimension four level, in adhesion with the flavons $\phi_s, \xi, \eta$. Now, again with the vev alignment for the flavons $\phi_s, \xi$ and $\eta$ as, $\langle \phi_s \rangle = (0, v_s, 0)$, $\langle \xi \rangle = v_\xi$, $\langle \eta \rangle = v_\eta$, the mass matrices involved here can be written
diagonalise this general complex matrix let us first define a Hermitian matrix

\[ M'_{NS} = \begin{pmatrix} x_1 & 0 & s_1 + a_1 \\ 0 & x_1 & 0 \\ s_1 - a_1 & 0 & x_1 \end{pmatrix}, \quad M_{NS} = \begin{pmatrix} x_2 & 0 & s_2 + a_2 \\ 0 & x_2 & 0 \\ s_2 - a_2 & 0 & x_2 \end{pmatrix}, \]

where \( x_1 = y_{kn}v_x + y_{v}v_y, s_1 = y_{s}v_s, a_1 = y_{a}v_s, x_2 = y_{k}v_x + y_{v}v_y, s_2 = y_{s}v_s \) and \( a_2 = y_{a}v_s \). Here also \( s_i \) and \( a_i \) are the symmetric and anti-symmetric contributions originated from \( A_4 \) multiplication. This unique contribution \( (a_i \) or \( a'_i) \) is a specific feature of \( A_4 \) flavour models for Dirac neutrinos and usually do not appear for Majorana neutrinos due to symmetry property of the Majorana mass matrix. It is worth mentioning that, these anti-symmetric parts, originated due to the Dirac nature of neutrinos, significantly dictate the pattern of neutrino mixing and can explain non-zero \( \theta_{13} \) in a very minimal scenario [34] compared to what is usually done with Majorana neutrinos [49]. Such anti-symmetric contribution from \( A_4 \) triplet products can also play a non-trivial role in generating nonzero \( \theta_{13} \) in Majorana neutrino scenarios (through Dirac Yukawa coupling appearing in type I seesaw) [50]. Now, substituting these mass matrices obtained in Eq. (9)-(11) in the linear seesaw formula given in Eq. (4) one can obtain the effective light neutrino mass matrix as

\[
\begin{align*}
m_{\nu} &= Y_{\nu N V} \begin{pmatrix} x'_1 & 0 & s'_1 + a'_1 \\ 0 & x'_1 & 0 \\ s'_1 - a'_1 & 0 & x'_1 \end{pmatrix} \begin{pmatrix} x_1 & 0 & s_1 + a_1 \\ 0 & x_1 & 0 \\ s_1 - a_1 & 0 & x_1 \end{pmatrix}^{-1} \\
&+ Y_{\nu S V} \begin{pmatrix} x'_2 & 0 & s'_2 + a'_2 \\ 0 & x'_2 & 0 \\ s'_2 - a'_2 & 0 & x'_2 \end{pmatrix} \begin{pmatrix} x_2 & 0 & s_2 + a_2 \\ 0 & x_2 & 0 \\ s_2 - a_2 & 0 & x_2 \end{pmatrix}^{-1} \\
&= \frac{Y_{\nu N V}}{a'_1^2 - s'_1^2 + x'_1^2} \begin{pmatrix} (a_1 - s_1)(a'_1 + s'_1) + x_1x'_1 & 0 & (a'_1 + s'_1)x_1 - (a_1 + s_1)x'_1 \\ 0 & x'_1(x'_1^2 - s'_1^2 + x'_1^2) & x_1 \\ (a_1 - s_1)x'_1 - (a'_1 - s'_1)x_1 & 0 & (a_1 + s_1)(a'_1 - s'_1) + x_1x'_1 \end{pmatrix} \\
&+ \frac{Y_{\nu S V}}{a'_2^2 - s'_2^2 + x'_2^2} \begin{pmatrix} (a_2 - s_2)(a'_2 + s'_2) + x_2x'_2 & 0 & (a'_2 + s'_2)x_2 - (a_2 + s_2)x'_2 \\ 0 & x'_2(x'_2^2 - s'_2^2 + x'_2^2) & x_2 \\ (a_2 - s_2)x'_2 - (a'_2 - s'_2)x_2 & 0 & (a_2 + s_2)(a'_2 - s'_2) + x_2x'_2 \end{pmatrix}.
\end{align*}
\]

Here all the elements appearing in the effective neutrino mass matrix are complex in general.

To diagonalise this general complex matrix let us first define a Hermitian matrix \( \mathcal{M} \), given
by
\[ \mathcal{M} = m_\nu m_\nu^\dagger \]
\[ = \begin{pmatrix}
|\lambda_1 p_1 + \lambda_2 q_1|^2 + |\lambda_1 p_2 + \lambda_2 q_2|^2 & 0 & (\lambda_1 p_1 + \lambda_2 q_1)(\lambda_1 p_4 + \lambda_2 q_4)^* + (\lambda_1 p_2 + \lambda_2 q_2)(\lambda_1 p_5 + \lambda_2 q_5)^* \\
0 & |\lambda_1 p_3 + \lambda_2 q_3|^2 & 0 \\
(\lambda_1 p_4 + \lambda_2 q_4)(\lambda_1 p_1 + \lambda_2 q_1)^* + (\lambda_1 p_5 + \lambda_2 q_5)(\lambda_1 p_2 + \lambda_2 q_2)^* & 0 & |\lambda_1 p_4 + \lambda_2 q_4|^2 + |\lambda_1 p_5 + \lambda_2 q_5|^2
\end{pmatrix} \tag{13} \]
where
\[
\lambda_1 = Y_\nu N v/(a_1^2 - s_1^2 + x_1^2), \quad \lambda_2 = Y_\nu N v/(a_2^2 - s_2^2 + x_2^2) \tag{14}
\]
\[
p_1 = (a_1 - s_1)(a_1' + s_1') + x_1x_1', \quad q_1 = (a_2 - s_2)(a_2' + s_2') + x_2x_2', \tag{15}
\]
\[
p_2 = (a_1' + s_1')x_1 - (a_1 + s_1)x_1', \quad q_2 = (a_2' + s_2')x_2 - (a_2 + s_2)x_2', \tag{16}
\]
\[
p_3 = x_1'(a_1^2 - s_1^2 + x_1^2)/x_1, \quad q_3 = x_2'(a_2^2 - s_2^2 + x_2^2)/x_2, \tag{17}
\]
\[
p_4 = (a_1 - s_1)x_1' - (a_1' - s_1')x_1, \quad q_4 = (a_2 - s_2)x_2' - (a_2' - s_2')x_2, \tag{18}
\]
\[
p_5 = (a_1 + s_1)(a_1' - s_1') + x_1x_1', \quad q_5 = (a_2 + s_2)(a_2' - s_2') + x_2x_2'. \tag{19}
\]
This structure of the Hermitian matrix \( \mathcal{M} \) suggests that it can be diagonalised by a rotation matrix \( U_{13} \) satisfying \( U_{13}^\dagger \mathcal{M} U_{13} = \text{diag}(m_{1}^2, m_{2}^2, m_{3}^2) \), where
\[
U_{13} = \begin{pmatrix}
\cos \theta & 0 & \sin \theta e^{-i\psi} \\
0 & 1 & 0 \\
-\sin \theta e^{i\psi} & 0 & \cos \theta
\end{pmatrix}, \tag{20}
\]
and \( m_{1,2,3}^2 \) are the light neutrino mass eigenvalues. Here the rotation angle \( \theta \) and phase \( \psi \) can be evaluated using the complex parameters in Eq. \( (13) \). From Eqs. \( (13)-(19) \) it is clear that there exists several parameters in \( \mathcal{M} \) (obtained from the effective light neutrino matrix) to constrain \( \theta \) and \( \psi \) satisfying correct neutrino oscillation data. Therefore due to presence of several non-trivial matrices having many complex parameters in the effective light neutrino mass matrix, it does not lead to very specific constraints on the parameters appearing in the neutrino linear seesaw mass matrix.

It turns out, there is a way to have a more constrained scenario. Now along with the symmetry mentioned in Table I, for simplicity one can introduce an additional \( Z_2 \) symmetry under which both \( \eta \) and \( \rho \) are odd (with all other particles are even under this symmetry). Therefore this two flavons will always appear together and under this additional symmetry
The Lagrangian presented in Eq. (8) can be re-written in a simplified form as

\[ \mathcal{L}_\nu = Y_{\nu N} \bar{L} \tilde{H} N_R + \frac{Y_{RN}}{\Lambda} \bar{\nu}_R N_L \eta^\dagger \rho^\dagger + Y_{\nu S} \bar{L} \tilde{H} S_R + \frac{Y_{\eta S}'}{\Lambda} \bar{S}_L \nu_R \eta \rho \\
+ S_R N_L (y_s \xi + y_s \phi_S + y_a \phi_S) + S_L N_R (y'_{\xi} \xi^\dagger + y'_{\phi_S} + y'_{a} \phi_S^\dagger) + \text{h.c.} \]  

(21)

Subsequently, in the present set-up we work with this $Z_2$ symmetry to keep the analysis minimal and more predictive. Clearly, the $Y_{\nu N}$ and $Y_{\nu S}$ couplings remain unchanged and hence corresponding mass are given by Eq. (9). As the triplet flavon $\phi_s$ (and singlet $\xi$) do not share same $Z_2$ symmetry with $\eta$, the mass matrices involved in $S_L$-$\nu_R$ and $\nu_R$-$N_L$ couplings now can be written in much simpler way as

\[ M'_{\nu S} = \frac{Y'_{\nu S}}{\Lambda} v_{\eta} v_{\rho}, \quad m'_{\nu N} = \frac{Y_{RN}}{\Lambda} v_{\eta} v_{\rho}. \]  

(22)

In this simplified scenario, the mixing between the heavy neutrinos $S_L - N_R$ and $S_R - N_L$ now takes the form

\[ M'_{NS} = \begin{pmatrix} x_1 & 0 & s_1 + a_1 \\ 0 & x_1 & 0 \\ s_1 - a_1 & 0 & x_1 \end{pmatrix}, \quad M_{NS} = \begin{pmatrix} x_2 & 0 & s_2 + a_2 \\ 0 & x_2 & 0 \\ s_2 - a_2 & 0 & x_2 \end{pmatrix} \]

(23)

where $x_1 = y_{\xi_1} v_{\xi}, s_1 = y_{s_1} v_{s_1}$, $a_1 = y_{a_1} v_{a_1}, x_2 = y_{\xi_2} v_{\xi}, s_2 = y_{s_2} v_{s_2}$ and $a_2 = y_{a_2} v_{a_2}$. Clearly, presence of the same $Z_2$ symmetry forbids any contribution from the singlet flavon $\eta$ in these matrices as evident from Eq. (21). Now, in this simplified scenario, substituting these mass matrices given in Eqs. (9), (22) and (23) in the linear seesaw formula given in Eq. (4) one can obtain the effective light neutrino mass matrix as

\[ m_\nu = Y_{\nu N} v_{\nu} \frac{Y'_{\nu S}}{\Lambda} v_{\eta} v_{\rho} \begin{pmatrix} x_1 & 0 & s_1 + a_1 \\ 0 & x_1 & 0 \\ s_1 - a_1 & 0 & x_1 \end{pmatrix}^{-1} + Y_{\nu S} v_{\nu} \frac{Y_{RN}}{\Lambda} v_{\eta} v_{\rho} \begin{pmatrix} x_2 & 0 & s_2 + a_2 \\ 0 & x_2 & 0 \\ s_2 - a_2 & 0 & x_2 \end{pmatrix}^{-1} \]

\[ = \lambda_1 \begin{pmatrix} x_1 & 0 & -(a_1 + s_1) \\ 0 & a_1^2 + s_1^2 + x_1^2 & 0 \\ a_1 - s_1 & 0 & x_1 \end{pmatrix} + \lambda_2 \begin{pmatrix} x_2 & 0 & -(a_2 + s_2) \\ 0 & a_2^2 + s_2^2 + x_2^2 & 0 \\ a_2 - s_2 & 0 & x_2 \end{pmatrix}, \]  

(24)

where $\lambda_1 = \frac{Y_{\nu N} Y'_{\nu S} v_{\nu} v_{\rho}}{\Lambda (a_1^2 - s_1^2 + x_1^2)}$ and $\lambda_2 = \frac{Y_{\nu S} Y_{RN} v_{\nu} v_{\rho}}{\Lambda (a_2^2 - s_2^2 + x_2^2)}$ are dimensionless quantities. Clearly, in the present scenario the matrices involved in the heavy neutrino mixing ($M_{NS}$ and $M'_{NS}$) dictate the pattern of light neutrino mixing as all other matrices are diagonal here. Furthermore,
as mentioned earlier, the hierarchy among the different mass matrices is governed by the specific discrete symmetries in order to ensure light neutrino mass of correct order. The general structure for the light neutrino mass matrix originated from Dirac linear seesaw as given in Eq. (24), can be further analysed to satisfy correct neutrino oscillation data. This in turn puts constraints on the parameters appearing in the neutrino matrix given in Eq. (24). Besides this, using these constrains on the complex mass parameters, one can easily find the predictions involving neutrino mixing angles, Dirac CP phase and absolute masses for light neutrinos. This predictive nature of the present model makes it more interesting from the point of view of ongoing and upcoming neutrino experiments. Here we perform the analysis regarding the predictions for neutrino masses and mixing in two different frameworks. First, in a simplest scenario (Case A), we consider some equality between two terms (involving $A_4$ symmetric and anti-symmetric contributions) appearing in the linear seesaw formula. Next, in a more general scenario (Case B), we do not consider any equality among the symmetric and anti-symmetric terms in the effective light mass obtained linear seesaw formula and try to fit neutrino oscillation data. We discuss these two cases below.

A. Case A:

In this simplest scenario, we first consider $\lambda_1 = \lambda_2 = \lambda$, $a_1 = a_2 = a$, $s_1 = s_2 = s$ and $x_1 = x_2 = x$. Hence the general structure for the effective light neutrino matrix as given in Eq. (24) reduces to

$$m_\nu = 2\lambda \begin{pmatrix} x & 0 & -(a+s) \\ 0 & \frac{a^2-s^2+x^2}{x} & 0 \\ a-s & 0 & x \end{pmatrix}.$$ (25)

Here $s$ and $a$ take care of the symmetric and anti-symmetric contributions respectively originating from the two terms in the linear seesaw formula. In order to diagonalise this mass matrix, let us first define a Hermitian matrix as

$$\mathcal{M} = m_\nu m_\nu^\dagger = 4|\lambda|^2 \begin{pmatrix} |x|^2 + |s+a|^2 & 0 & x(a-s)^* - x^*(a+s) \\ 0 & \frac{a^2-s^2+x^2}{x} & 0 \\ x^*(a-s) - x(a+s)^* & 0 & |x|^2 + |a-s|^2 \end{pmatrix}.$$ (26)
This matrix, being Hermitian, can be diagonalized by a unitary matrix $U_{13}$, as given in Eq. (20) through the relation $U_{13}^\dagger M U_{13} = \text{diag}(m_1^2, m_2^2, m_3^2)$. Here we find the mass eigenvalues $(m_1^2, m_2^2, m_3^2)$ to be

$$m_1^2 = \kappa^2 \left[ 1 + \alpha^2 + \beta^2 - \sqrt{(2\alpha\beta \cos(\phi_{ax} - \phi_{sx}))^2 + 4(\alpha^2 \sin^2 \phi_{ax} + \beta^2 \cos^2 \phi_{sx})} \right],$$

(27)

$$m_2^2 = \kappa^2 \left[ 1 + \alpha^4 + \beta^4 + 2\alpha^2 \cos 2\phi_{ax} - 2\beta^2 \cos 2\phi_{sx} - 2\alpha^2 \beta^2 \cos 2(\phi_{sx} - \phi_{ax}) \right],$$

(28)

$$m_3^2 = \kappa^2 \left[ 1 + \alpha^2 + \beta^2 + \sqrt{(2\alpha\beta \cos(\phi_{ax} - \phi_{sx}))^2 + 4(\alpha^2 \sin^2 \phi_{ax} + \beta^2 \cos^2 \phi_{sx})} \right].$$

(29)

Here we have defined $\kappa^2 = 4|\lambda|^2|\chi|^2$, $\alpha = |a|/|x|$, $\beta = |s|/|x|$, $\phi_{sx} = \phi_s - \phi_x$, $\phi_{ax} = \phi_a - \phi_x$ with $s = |s|e^{i\phi_s}$, $a = |a|e^{i\phi_a}$ and $x = |x|e^{i\phi_x}$ respectively. For notational convenience, the relative phases $\phi_{sx}$ and $\phi_{ax}$ will be denoted just as $\phi_s$ and $\phi_a$ respectively from here onwards. It can be clearly seen from the expressions for mass eigenvalues that $m_3^2 > m_1^2$ implying the preference for normal hierarchical light neutrino masses. From these definitions it is clear that $\alpha$ is associated with the anti-symmetric contribution whereas $\beta$ is related to the symmetric contribution in the Dirac neutrino mass matrix. Using Eq. (7), (20), the final lepton mixing matrix in our framework is given by

$$U = U_{\omega}^\dagger U_{13}.$$  

(30)

Now, using Eq. (26) and Eq. (30), one can obtain the correlation between the rotation angle $\theta$ and phase $\psi$ as

$$\tan 2\theta = \frac{\beta \sin \phi_s \cos \psi - \alpha \cos \phi_a \sin \psi}{\alpha \beta \cos(\phi_s - \phi_a)} \quad \text{and} \quad \tan \psi = -\frac{\alpha \sin \phi_a}{\beta \cos \phi_s}. \quad \quad (31)$$

To extract the neutrino mixing angles in terms of the model parameters, we compare this with the standard parametrisation of leptonic mixing matrix known as Pontecorvo Maki Nakagawa Sakata (PMNS) mixing matrix given by

$$U_{\text{PMNS}} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix},$$

(32)

and we obtain

$$\sin \theta_{13}e^{-i\delta} = \frac{1}{\sqrt{3}}(\cos \theta + \sin \theta e^{-i\psi}).$$  

(33)
FIG. 1: Allowed regions of $\alpha$-$\phi_a$ (left panel) and $\beta$-$\phi_s$ (right panel) planes for $3\sigma$ allowed ranges of $\theta_{13}$, $\theta_{12}$, $\theta_{23}$ and the ratio ($r$) of solar to atmospheric mass squared differences [2, 4].

Now, $\sin \theta_{13}$ and $\delta$ can also be parametrised in terms of $\theta$ and $\psi$ as

$$\sin^2 \theta_{13} = \frac{1}{3}(1 + \sin 2\theta \cos \psi) \quad \text{and} \quad \tan \delta = \frac{\sin \theta \sin \psi}{\cos \theta + \sin \theta \cos \psi}.$$  \tag{34}

Such correlation between the model parameters and neutrino mixing angles $\theta_{13}, \theta_{12}, \theta_{23},$ Dirac CP phase $\delta$ can also be found in [34, 35, 51–54]. Therefore from Eq. (31) and Eq. (34) it is clear that the neutrino mixing angles are functions of four model parameters namely, $\alpha, \beta, \phi_s$ and $\phi_a$. These are the parameters associated with symmetric and anti-symmetric part of the effective light neutrino mass matrix and corresponding relative phases. These parameters then can be constrained using the current data on neutrino mixing angles [2–4]. In addition to the bounds obtained from the mixing angles, the parameter space can be further constrained in order to satisfy correct value for mass squared differences. Here one can define a ratio for the solar to atmospheric mass squared difference as

$$r = \frac{|\Delta m^2_{12}|}{|\Delta m^2_{21}|}.$$  \tag{35}

From Eq. (27)-(29), it is evident that this ratio $r$ is a function of the model parameters $\alpha, \beta, \phi_s$ and $\phi_a$. In order to satisfy correct neutrino oscillation data, we use the $3\sigma$ allowed range of the neutrino mixing angles and mass squared differences given in global fit analysis [2, 4] to constrain these model parameters. Here in Fig. 1 we have shown the allowed regions for parameters $\alpha, \beta, \phi_s$ and $\phi_a$ satisfying $3\sigma$ ranges for neutrino mixing angles ($\theta_{13}, \theta_{12}, \theta_{23}$) and ratio of the mass squared differences $r$. In the left panel of Fig. 1 we show the allowed
FIG. 2: Predictions for lightest neutrino mass $m_1$ as a function of $\alpha$ (left panel) and Jarlskog invariant $J_{CP}$ as a function of $\phi_a$ (right panel). Here each points in both panels also satisfy $3\sigma$ allowed ranges for $\theta_{13}$, $\theta_{12}$, $\theta_{23}$ and the ratio $(r)$ of solar to atmospheric mass squared differences [2, 4].

points in $\alpha$-$\phi_a$ plane whereas in the right panel we have plotted the same in $\beta$-$\phi_s$ plane. Here we find that the parameter $\beta$, associated with the symmetric part of the neutrino mass matrix ranges between 0.7-1.2 whereas the anti-symmetric part (contained in parameter $\alpha$) remains confined within a relatively narrower region between 0.27 to 0.45. The associated phases also occupy distinct allowed regions as evident from both panels of Fig. 1. After finding the allowed regions for $\alpha$, $\beta$, $\phi_s$ and $\phi_a$, one can easily find out the common factor $\kappa$ appearing in the light neutrino mass eigenvalues using Eq. (27)-(28) and best fit value of solar mass squared difference, $\Delta m_{21}^2 = m_2^2 - m_1^2 = 7.40 \times 10^{-5} \text{ eV}^2$ [2, 4]. The estimation of $\kappa$ then enables us to find predictions for absolute neutrino masses. In the left panel of Fig. 2, we plot the predictions for lightest absolute neutrino mass $m_1$ as a function of $\alpha$ and it ranges between $(0.42 \times 10^{-2} - 0.58 \times 10^{-2})$ eV for $\alpha$ in the range 0.27 to 0.45. Similarly, one can also find the estimates for sum of all three absolute neutrino in this simplified scenario of the neutrino mass matrix and is given by $\sum m_i = (0.062 - 0.070) \text{ eV}$, lying within the cosmological bound on sum of light neutrino masses $\sum m_i \leq 0.17 \text{ eV}$ from Planck data [55].

On the other hand, in the right panel of Fig. 2, we have plotted the allowed regions for the Jarlskog CP invariant $J_{CP} = \text{Im}[U_{e1}U_{\mu2}U_{e2}^*U_{\mu1}^*]$ [56] as a function of the relative phase $\phi_a$ associated with the anti-symmetric contribution of the neutrino mass matrix and estimated to be within the range $|J_{CP}| \sim 0 - 0.024$. In Fig. 3, we show the most important among such correlations namely, the one between the Dirac CP phase $\delta$ and atmospheric mixing...
FIG. 3: Predicted correlation between Dirac CP phase $\delta$ and atmospheric mixing angle $\theta_{23}$ for Case A.

Interestingly, here we find that, the model predicts the CP phase $\delta$ to be in the range $-\pi/2 \lesssim \delta \lesssim -\pi/5$ and $\pi/5 \lesssim \delta \lesssim \pi/2$ whereas $\sin^2 \theta_{23}$ lies in the lower octant. This value of $\delta$ falls in the current preferred ballpark suggested by experiments [57] as well as global fit analysis [2, 4], predicting atmospheric mixing angle $\theta_{23}$ to be in the lower octant.

B. Case B:

In this subsection, we analyse the effective light neutrino mass matrix given in Eq. (12) in a more general canvas to illustrate the effects of contributions coming from symmetric and anti-symmetric parts appearing in the two different terms of the linear seesaw formula, without assuming any equality between two symmetric (and anti-symmetric) terms. Considering the most general structure for the light neutrino mass matrix as given in Eq. (12), we can define a Hermitian matrix as,

$$\mathcal{M} = m_\nu m_\nu^\dagger$$

$$= |\lambda|^2 \begin{pmatrix}
X_1 & 0 & X_2 \\
0 & X_3 & 0 \\
X_4 & 0 & X_5
\end{pmatrix}$$
where

\[
X_1 = 4|x|^2 + |(a_1 + a_2) + (s_1 + s_2)|^2, \\
X_2 = 2x\{(a_1 + a_2)^* - (s_1 + s_2)^*\} - 2x^*\{(a_1 + a_2) + (s_1 + s_2)\}, \\
X_3 = \frac{1}{|x|^2}[(a_1^2 + a_2^2) - (s_1^2 + s_2^2) + 2x^2]|, \\
X_4 = 2x^*\{(a_1 + a_2) - (s_1 + s_2)\} - 2x\{(a_1 + a_2)^* + (s_1 + s_2)^*\}, \\
X_5 = 4|x|^2 + |(a_1 + a_2) - (s_1 + s_2)|^2.
\]

Here for simplicity, we have considered \(x_1 = x_2 = x\) and \(\lambda_1 = \lambda_2 = \lambda\) while keeping the other terms distinct. This Hermitian matrix \(\mathcal{M}\) now can also be diagonalised by a similar rotation matrix (in the 13 plane) given in Eq. (20) with rotation angle \(\theta\) and phase factor \(\psi\). These parameters can therefore be expressed as

\[
\tan 2\theta = \frac{2[A\sin \psi - B\cos \psi]}{C_1 + C_2 + C_3 + C_4}, \quad \tan \psi = -\frac{A}{B}, \tag{38}
\]

with

\[
A = (\alpha_1 \sin \phi_{a_1} + \alpha_2 \sin \phi_{a_2}), \quad B = (\beta_1 \cos \phi_{s_1} + \beta_2 \cos \phi_{s_2}), \tag{39}
\]

\[
C_1 = \alpha_1 \beta_1 \cos(\phi_{a_1} - \phi_{s_1}), \quad C_2 = \alpha_1 \beta_2 \cos(\phi_{a_1} - \phi_{s_2}), \tag{40}
\]

\[
C_3 = \alpha_2 \beta_1 \cos(\phi_{a_2} - \phi_{s_1}), \quad C_4 = \alpha_2 \beta_2 \cos(\phi_{a_2} - \phi_{s_2}), \tag{41}
\]

where we have defined the parameters as \(\alpha_j = |a_j|/|x|\), \(\beta_j = |s_j|/|x|\), \(\phi_{a,j} = \phi_{a_j} - \phi_x\), \(\phi_{s,j} = \phi_{s_j} - \phi_x\), \(s_j = |s|e^{i\phi_{s,j}}\), \(a_j = |a|e^{i\phi_{a,j}}\) and \(x = |x|e^{i\phi_x}\) with \(j = 1, 2\). For notational compactness we have written the relative phases \(\phi_{a,j}, \phi_{s,j}\) in equation (39–41) as \(\phi_{a_j}, \phi_{s_j}\) with \(j = 1, 2\). Hence, diagonalising the Hermitian matrix via \(U_{13}^\dagger \mathcal{M} U_{13} = \text{diag}(m_1^2, m_2^2, m_3^2)\), we obtain the light neutrino masses as

\[
m_1^2 = \kappa^2 \left[4 + \alpha_1^2 + \alpha_2^2 + \beta_1^2 + \beta_2^2 + C_5 - \sqrt{4(C_1 + C_2 + C_3 + C_4)^2 + 4^2(A^2 + B^2)}\right], \tag{42}
\]

\[
m_2^2 = \kappa^2 \left[4 + \alpha_1^4 + \alpha_2^4 + \beta_1^4 + \beta_2^4 + C_6\right], \tag{43}
\]

\[
m_3^2 = \kappa^2 \left[4 + \alpha_1^2 + \alpha_2^2 + \beta_1^2 + \beta_2^2 + C_5 + \sqrt{4(C_1 + C_2 + C_3 + C_4)^2 + 4^2(A^2 + B^2)}\right], \tag{44}
\]

where

\[
C_5 = 2\{\alpha_1 \alpha_2 \cos(\phi_{a_1} - \phi_{a_2}) + \beta_1 \beta_2 \cos(\phi_{s_1} - \phi_{s_2})\},
\]

\[
C_6 = 4(\alpha_1^2 \cos 2\phi_{a_1} + \alpha_2^2 \cos 2\phi_{a_2}) - 4(\beta_1^2 \cos 2\phi_{s_1} + \beta_2^2 \cos 2\phi_{s_2}) \]

\[
+ 2\alpha_1^2 \alpha_2^2 \cos 2(\phi_{a_1} - \phi_{a_2}) - 2\alpha_1^2 \beta_1^2 \cos 2(\phi_{a_1} - \phi_{s_1}) - 2\alpha_2^2 \beta_2^2 \cos 2(\phi_{a_1} - \phi_{s_2}) \]

\[
- 2\alpha_1^2 \beta_2^2 \cos 2(\phi_{a_2} - \phi_{s_1}) - 2\alpha_2^2 \beta_1^2 \cos 2(\phi_{a_2} - \phi_{s_2}) + 2\beta_1^2 \beta_2^2 \cos 2(\phi_{s_1} - \phi_{s_2}).
\]
Considering the contributions from both charged lepton and neutrino sectors, the complete lepton mixing matrix in this general case also is given by

\[ U = U_{\omega}^\dagger U_{13}. \]  

(45)

FIG. 4: Allowed regions of \( \alpha_1-\alpha_2 \) (left panel) and \( \beta_1-\beta_2 \) (right panel) planes for \( 3\sigma \) allowed ranges of \( \theta_{13}, \theta_{12}, \theta_{23} \) and the ratio \( (r) \) of solar to atmospheric mass squared differences [2, 4]. These points additionally also satisfy the upper limit for sum of the three absolute neutrino masses \( \sum m_i \leq 0.17 \text{ eV} \) [57].

Comparing this mixing matrix with \( U_{\text{PMNS}} \) as given in Eq. (32), one can obtain the correlations between the the mixing angles \( (\theta_{13}, \theta_{12} \text{ and } \theta_{23}) \) and Dirac CP phase \( \delta \) as
previously given in Eq. (34). Further using Eq. (38) we find the correspondence between neutrino mixing angles and the relevant model parameters. Here the parameters $\alpha_j$, $\beta_j$, $\phi_{a_j}$ and $\phi_{s_j}$ with $j = 1, 2$ essentially dictate the neutrino mixing patterns. Obviously, the number of parameters controlling neutrino mixing in this general case is more than what it was in the simple scenario described earlier. Using Eq. (42)-(44) we define a ratio

\[ r = \frac{\Delta m^2_{21}}{|\Delta m^2_{31}|} = \frac{\Delta m^2_{21}}{|\Delta m^2_{32}|} \]

in terms of very same parameters $\alpha_j$, $\beta_j$, $\phi_{a_j}$ and $\phi_{s_j}$. Here also to satisfy correct neutrino oscillation data, we use the $3\sigma$ range of the neutrino mixing angle and mass squared differences [2, 4] to constrain these parameters and we find the correlations among them. In addition to the bounds from neutrino oscillation experiments, these parameters can also get constrained in order to satisfy the cosmological upper limit on sum of the three absolute neutrino mass, given by $\sum m_i \leq 0.17$ eV [57]. Therefore, using all these constraints, in Fig. 4 we have all the allowed points in $\alpha_1$-$\alpha_2$ (left panel) plane and $\beta_1$-$\beta_2$ (right panel) plane respectively. In the left panel, we find that the contributions involving the anti-symmetric parts ($\alpha_1$ and $\alpha_2$) are mostly confined within 0-2. On the other hand, as it is evident from the right panel of Fig. 4, the contributions involving the symmetric parts ($\beta_1$ and $\beta_2$) take relatively larger values satisfying correct neutrino oscillation data. Then in Fig. 5, we show the allowed parameter space in the $\phi_{a_1}$-$\phi_{a_2}$ plane (left panel) and $\phi_{b_1}$-$\phi_{b_2}$ plane (right panel) respectively which clearly show distinct correlations between relative phases associated with symmetric and anti-symmetric contributions. After finding
the allowed regions for $\alpha_{1,2}$, $\beta_{1,2}$, $\phi_{a_{1,2}}$ and $\phi_{s_{1,2}}$, one can again find out the common factor $\kappa$ appearing in the light neutrino mass eigenvalues involved in this general case using Eq. (42)-(43) and best fit value of solar mass squared difference, $\Delta m^2_{21} = m^2_2 - m^2_1 = 7.40 \times 10^{-5}$ eV$^2$ [2, 4] as mentioned earlier. The estimate for $\kappa$ enables us to find predictions for absolute neutrino masses using Eq. (42)-(43). In the left panel of Fig. 6, we show the predictions for lightest absolute neutrino mass $m_1$ as a function of $\alpha_1$ (parameter involved in one of the anti-symmetric contribution). It can be seen from this plot that $m_1$ can be as large as 0.05 eV for $\alpha_1$ within the limit of 2. Such values of the lightest neutrino mass correspond to sum of all three absolute neutrino masses $\sum m_i = (0.06 - 0.17)$ eV saturating the cosmological upper limit. In the right panel of Fig. 6, we have again plotted the allowed regions for the Jarlskog CP invariant $J_{CP}$ as a function of the relative phase $\phi_{a_1}$ associated with one of the anti-symmetric contribution of the neutrino mass matrix and estimated to be within the range $|J_{CP}| \sim 0 - 0.024$, analogous to the previous result. In Fig. 7, we have now plotted the correlation between the Dirac CP phase $\delta$ and atmospheric mixing angle $\theta_{23}$. Similar to the previous case, here also we find that, the model predicts the Dirac CP phase $\delta$ to be in the range $-\pi/2 \lesssim \delta \lesssim -\pi/5$ and $\pi/5 \lesssim \delta \lesssim \pi/2$ whereas $\theta_{23}$ lies in the lower octant.

In this two different limits of Dirac linear seesaw discussed above, we have observed that the allowed range of the light neutrino mass is different in the two cases. In Case A, due to much constrained scenario of the neutrino mass matrix the correlation between $m_1$ and $\sum m_i$ is mainly concentrated within a narrow region. This is shown in the left panel of Fig. 8. However, for the Case B, due to presence of more number of parameters originated
Planck Bound
\[ a_1 = a_2, \quad s_1 = s_2 \]

from both the contributions of linear seesaw formula being distinct, the constrains are much more relaxed. In this case neutrino masses can vary from a very small values to larger ones saturating the cosmological upper bound \( \sum m_i \leq 0.17 \text{ eV} \). Here the whole region is basically allowed, as shown in the right panel of Fig. 8. Finally, it is important to mention that, within these two different limits, inverted hierarchy of neutrino mass is not allowed, another important prediction of our model.

III. CONCLUSION

We have proposed a linear seesaw model for Dirac neutrinos within the framework of \( A_4 \) flavour symmetry, augmented by additional discrete and global lepton number symmetry in order to make sure that the correct hierarchy between different terms appearing in the complete neutral fermion mass matrix is naturally obtained without making any ad hoc assumptions. The interesting feature of the conventional linear seesaw framework where a small lepton number breaking term in seesaw formula, linear in Dirac neutrino mass, can give rise to correct neutrino mass with heavy neutrinos lying in TeV scale, is retained in the Dirac version of it by appropriately generating hierarchical terms at different orders (dimension four and dimension five). Since lepton number is present as a global unbroken symmetry in the model, all the mass matrices involved are of Dirac type and hence the \( A_4 \) triple products contain the anti-symmetric components which play a crucial role in generating the correct neutrino phenomenology. Since we use the \( S \) diagonal basis of \( A_4 \) for Dirac neutrino
case, the charged lepton mass matrix is also non trivial in our scenarios and hence can contribute to the leptonic mixing matrix. For generic choices of $A_4$ flavon alignments, we find that the model remains very predictive in terms of neutrino mass hierarchy, leptonic CP phase, octant of atmospheric mixing angle as well as absolute neutrino masses. While the neutrino mass hierarchy is predicted to be the normal one, the Dirac CP phase $\delta$ is found to lie in the range $-\pi/2 \lesssim \delta \lesssim -\pi/5$ and $\pi/5 \lesssim \delta \lesssim \pi/2$ whereas the atmospheric mixing angle $\theta_{23}$ lies in the lower octant. The predictions for lightest neutrino mass, in one of the scenarios, can saturate the cosmological upper bound on the sum of absolute neutrino masses $\sum m_i \leq 0.17$ eV. Apart from these, being a model predicting Dirac neutrinos, it can also predict the absence of lepton number violation and hence can not be tested in ongoing and future neutrinoless double beta decay experiments. These aspects keep the detection prospects of the model very promising at experiments ranging from neutrino oscillations, cosmology to rare decay ones.

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**Appendix A: $A_4$ Multiplication Rules**

$A_4$, the symmetry group of a tetrahedron, is a discrete non-abelian group of even permutations of four objects. It has four irreducible representations: three one-dimensional and one three dimensional which are denoted by $1, 1', 1''$ and $3$ respectively, being consistent with the sum of square of the dimensions $\sum_i n_i^2 = 12$. We denote a generic permutation $(1, 2, 3, 4) \rightarrow (n_1, n_2, n_3, n_4)$ simply by $(n_1n_2n_3n_4)$. The group $A_4$ can be generated by two basic permutations $S$ and $T$ given by $S = (4321), T = (2314)$. This satisfies

$$S^2 = T^3 = (ST)^3 = 1$$

which is called a presentation of the group. Their product rules of the irreducible representations are given as

$$1 \otimes 1 = 1$$
\[1' \otimes 1' = 1'' \]
\[1' \otimes 1'' = 1 \]
\[1'' \otimes 1'' = 1' \]

\[3 \otimes 3 = 1 \otimes 1' \otimes 1'' \otimes 3_a \otimes 3_s \]

where \(a\) and \(s\) in the subscript corresponds to anti-symmetric and symmetric parts respectively. Denoting two triplets as \((a_1, b_1, c_1)\) and \((a_2, b_2, c_2)\) respectively, their direct product can be decomposed into the direct sum mentioned above. In the \(S\) diagonal basis, the products are given as

\[1 \sim a_1 a_2 + b_1 b_2 + c_1 c_2 \]
\[1' \sim a_1 a_2 + \omega^2 b_1 b_2 + \omega c_1 c_2 \]
\[1'' \sim a_1 a_2 + \omega b_1 b_2 + \omega^2 c_1 c_2 \]
\[3_s \sim (b_1 c_2 + c_1 b_2, c_1 a_2 + a_1 c_2, a_1 b_2 + b_1 a_2) \]
\[3_a \sim (b_1 c_2 - c_1 b_2, c_1 a_2 - a_1 c_2, a_1 b_2 - b_1 a_2) \]

In the \(T\) diagonal basis on the other hand, they can be written as

\[1 \sim a_1 a_2 + b_1 c_2 + c_1 b_2 \]
\[1' \sim c_1 c_2 + a_1 b_2 + b_1 a_2 \]
\[1'' \sim b_1 b_2 + c_1 a_2 + a_1 c_2 \]
\[3_s \sim \frac{1}{3}(2a_1 a_2 - b_1 c_2 - c_1 b_2, 2c_1 c_2 - a_1 b_2 - b_1 a_2, 2b_1 b_2 - a_1 c_2 - c_1 a_2) \]
\[3_a \sim \frac{1}{2}(b_1 c_2 - c_1 b_2, a_1 b_2 - b_1 a_2, c_1 a_2 - a_1 c_2) \]

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