A Survey of Algebraic Extensions of Commutative, Unital Normed Algebras

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ABSTRACT

We describe the role of algebraic extensions in the theory of commutative, unital normed algebras, with special attention to uniform algebras. We shall also compare these constructions and show how they are related to each other. [MSC: 46J05, 46J10]

Introduction

Algebraic extensions have had striking applications in the theory of uniform algebras ever since Cole used them (in [5]) to construct a counterexample to the peak-point conjecture. Apart from this, their main use has been in (a) the construction of examples of general, normed algebras with special properties and (b) the Galois theory of Banach algebras. We shall not discuss (b) here; a summary of some of this work is included in [29].

In the first section of this article we shall introduce the types of extensions and relate their applications. The section ends by giving the exact relationship between the types of extensions. Section 2 contains a table summarising what is known about the extensions’ properties.

A theme lying behind all the work to be discussed is following question:

(Q) Suppose the normed algebra \( B \) is related to a subalgebra \( A \) by some specific property or construction. (For example, \( B \) might be integral over \( A \): every element \( b \in B \) satisfies \( a_0 + \cdots + a_{n-1}b^{n-1} + b^n = 0 \) for some \( a_0, \ldots, a_{n-1} \in A \).) What properties of \( A \) (for example, completeness or semisimplicity) must be shared by \( B \)?

This is a natural question, and interesting in its own right. Many special cases of it have been studied in the literature. We shall review the related body of work in which \( B \) is constructed from \( A \) by adjoining roots of monic polynomial equations.

Throughout this article, \( A \) denotes a commutative, unital normed algebra, and \( \tilde{A} \) its completion. The fundamental construction of [1] applies to this class algebras. Algebraic extensions of more general types of topological algebras have received limited attention in the literature (see [19], [21]).

If \( E \) is a subset of a ring then \((E)\) will stand for the the ideal generated by \( E \).

1. Types of Algebraic Extensions and their Applications

1.1 Arens-Hoffman Extensions

Let \( \alpha(x) = a_0 + \cdots + a_{n-1}x^{n-1} + x^n \) be a monic polynomial over the algebra \( A \). The basic construction arising from \( A \) and \( \alpha(x) \) is the Arens-Hoffman extension, \( A_\alpha \). This was introduced in [1]. Most of the obvious questions of the type (Q)
for Arens-Hoffman extensions were dealt with in this paper and in the subsequent work of Lindberg ([18], [20], [13]). See columns two and three of Table 2.2.

All the constructions we shall meet are built out of Arens-Hoffman extensions.

DEFINITION 1.1.1. A mapping \( \theta : A \to B \) between algebras \( A \) and \( B \) is called unital if it sends the identity of \( A \) to the identity of \( B \). An extension of \( A \) is a commutative, unital normed algebra, \( B \), together with a unital, isometric monomorphism \( \theta : A \to B \).

The Arens-Hoffman extension of \( A \) with respect to \( \alpha(x) \) is the algebra \( A_\alpha := A[x]/(\alpha(x)) \) under a certain norm; the embedding is given by the map \( \nu : a \mapsto (\alpha(x)) + a \).

To simplify notation, we shall let \( \bar{x} \) denote the coset of \( x \) and often omit the indeterminate when using a polynomial as an index.

It is a purely algebraic fact that each element of \( A_\alpha \) has a unique representative of degree less than \( n \), the degree of \( \alpha(x) \). Arens and Hoffman proved that, provided the positive number \( t \) satisfies the inequality

\[
\sum_{k=0}^{n-1} \| a_k \| t^k \geq \sum_{k=0}^{n-1} \| b_k \| t^k \quad (b_0, \ldots, b_{n-1} \in A)
\]

defines an algebra norm on \( A_\alpha \).

The first proposition shows that Arens-Hoffman extensions satisfy a certain universal property which is very useful when investigating algebraic extensions. It is not specially stated anywhere in the literature; it seems to be taken as obvious.

PROPOSITION 1.1.2. Let \( A^{(1)} \) be a normed algebra and let \( \theta : A^{(1)} \to B^{(2)} \) is a unital homomorphism of normed algebras. Let \( \alpha_1(x) = a_0 + \cdots + a_{n-1}x^{n-1} + x^n \in A^{(1)}[x] \) and \( B^{(1)} = A^{(1)}_{a_1} \). Let \( y \in B^{(2)} \) be a root of the polynomial \( \alpha_2(x) := \theta(\alpha_1(x)) := \phi_0(a_0) + \cdots + \phi_0(a_{n-1})x^{n-1} + x^n \). Then there is a unique homomorphism \( \phi : B^{(1)} \to B^{(2)} \) such that

\[
A^{(1)} \xrightarrow{\nu} B^{(1)} \xrightarrow{\phi} B^{(2)}
\]

is commutative and \( \phi(\bar{x}) = y \).

The map \( \phi \) is continuous if and only if \( \theta \) is continuous.

Proof. This is elementary; see [7].

1.2 Incomplete Normed Algebras

A minor source of applications of Arens-Hoffman extensions fits in nicely with our thematic question (Q): these extensions are useful in constructing examples to show that taking the completion of \( A \) need not preserve certain properties of \( A \).

The method uses the fact that the actions of forming completions and Arens-Hoffman extensions commute in a natural sense. A special case of this is stated in [17]; the general case is proved in [7], Theorem 3.13, and follows easily from Proposition 1.1.2.

It is convenient to introduce some more notation and terminology here.
Let $\Omega(A)$ denote the space of continuous epimorphisms $A \to \mathbb{C}$; when $\Omega$ appears on its own it will refer to $A$. As discussed in [1], this space, with the weak *-topology relative to the topological dual of $A$, generalises the notion of the maximal ideal space of a Banach algebra. In fact, it is easy to check that $\Omega$ is homeomorphic to $\Omega(\tilde{A})$, the maximal ideal space of the completion of $A$.

The Gelfand transform of an element $a \in A$ is defined by

$$\hat{a}: \Omega \to \mathbb{C}; \, \omega \mapsto \omega(a)$$

and the map sending $a$ to $\hat{a}$ is a homomorphism, $\Gamma$, of $A$ into the algebra, $C(\Omega)$, of all continuous, complex-valued functions on the compact, Hausdorff space $\Omega$. We denote the image of $\Gamma$ by $\hat{A}$. A good reference for Gelfand theory is Chapter three of [24].

**DEFINITION 1.2.1 ([1]).** The algebra $A$ is called *topologically semisimple* if $\Gamma$ is injective.

If $A$ is a Banach algebra then this condition is equivalent to the usual notion of semisimplicity. The precise conditions under which $A_\alpha$ is topologically semisimple if $A$ is are determined in [1]. In [17] Lindberg shows that the completion of a topologically semisimple algebra need not be semisimple.

In order to illustrate Lindberg’s strategy we recall two standard properties of normed algebras.

**DEFINITION 1.2.2.** The normed algebra $A$ is called *regular* if for each closed subset $E \subseteq \Omega$ and $\omega \in \Omega - E$ there exists $a \in A$ such that $\hat{a}(E) \subseteq \{0\}$ and $\hat{a}(\omega) = 1$. The algebra is called *local* if $\hat{A}$ contains every complex function, $f$, on $\Omega$ such that every $\omega \in \Omega$ has a neighbourhood, $V$, and an element $a \in A$ such that $f|_V = \hat{a}|_V$.

It is a standard fact that regularity is stronger than localness; see Lemma 7.2.8 of [24].

**EXAMPLE 1.2.3.** Let $A$ be the algebra of all continuous, piecewise polynomial functions on the unit interval, $I$, and $\alpha(x) = x^2 - \text{id}_I \in A[x]$. Let $A$ have the supremum norm. By the Stone-Weierstrass theorem, $\tilde{A} = C(I)$ and hence $\Omega$ is identifiable with $I$. Clearly $A$ is regular. We leave it as an exercise for the reader to find examples to show that $A_\alpha$ is not local. This is not hard; it may be helpful to know that in this example the space $\Omega(A_\alpha)$ is homeomorphic to $\{(s, \lambda) \in I \times \mathbb{C}: \lambda^2 = s\}$. This follows from facts in [1].

In the present example, neither localness nor regularity is preserved by (incomplete) Arens-Hoffman extensions.

Finally we can explain the method for showing that some properties of normed algebras are not shared by their completions because, in the above, ‘non-regularity’ is not preserved by completion of $A_\alpha$ (nor is ‘non-localness’). To see this, note that $A$ is clearly regular if $A$ is and so by a theorem of Lindberg (see Table 2.2) the Arens-Hoffman extension $(\tilde{A})_\alpha$ is regular. But, by a result of [17] referred to above, this algebra is isometrically isomorphic to the completion of $A_\alpha$. Of course Lindberg’s original application was much more significant; there are simpler examples of the present result: for example the algebra of polynomials on $I$. 

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1.3 Uniform Algebras

It is curious that the application of Arens-Hoffman extensions to the construction of integrally closed extensions of normed algebras did not appear in the literature for some time after [1]. It was seventeen years later until a construction was given in [22]. Even then the author acknowledges that the construction was prompted by the work of Cole, [5], in the theory of uniform algebras.

Cole invented a method of adjoining square roots of elements to uniform algebras. He used it to extend uniform algebras to ones which contain square roots for all of its elements. Apart from feeding back into the general theory of commutative Banach algebras (mainly accomplished in [22] and [23]) his construction provided important examples in the theory of uniform algebras. We shall describe these after recalling some basic definitions.

**DEFINITION 1.3.1.** A uniform algebra, $A$, is a subalgebra of $C(X)$ for some compact, Hausdorff space $X$ such that $A$ is closed with respect to the supremum norm, separates the points of $X$, and contains the constant functions. We speak of ‘the uniform algebra $(A, X)$’. The uniform algebra is natural if all of its homomorphisms $\omega \in \Omega$ are given by evaluation at points of $X$, and it is called trivial if $A = C(X)$.

Introductions to uniform algebras can be found in [4], [26], and [16]. An important question in this area is which properties of $(A, X)$ force $A$ to be trivial. For example it is sufficient that $A$ be self-adjoint, by the Stone-Weierstrass theorem. In [5] an example is given of a non-trivial uniform algebra, $(B, X)$, which is natural and such that every point of $X$ is a ‘peak-point’. It had previously been conjectured that no such algebra existed.

We shall describe the use of Cole’s construction in the next section, but now we reveal some of the detail.

**PROPOSITION 1.3.2 ([5, 7]).** Let $\mathcal{U}$ be a set of monic polynomials over the uniform algebra $(A, X)$. There exists a uniform algebra $(A_{\mathcal{U}}, X_{\mathcal{U}})$ and a continuous, open surjection $\pi: X_{\mathcal{U}} \to X$ such that

(i) the adjoint map $\pi^*: C(X) \to C(X_{\mathcal{U}})$ induces an isometric, unital monomorphism $A \to A_{\mathcal{U}}$, and

(ii) for every $\alpha \in \mathcal{U}$ there exists $p_\alpha \in A_{\mathcal{U}}$ such that $\pi^*(\alpha)(p_\alpha) = 0$.

**Proof.** We let $X_{\mathcal{U}}$ be the subset of $X \times \mathbb{C}^\mathcal{U}$ consisting of the elements $(\kappa, \lambda)$ such that for all $\alpha \in \mathcal{U}$

$$f_0^{(\alpha)}(\kappa) + \cdots + f_{n(\alpha)-1}^{(\alpha)}(\kappa)\lambda_{n(\alpha)-1} + \lambda_{n(\alpha)} = 0$$

where $\alpha(x) = f_0^{(\alpha)} + \cdots + f_{n(\alpha)-1}^{(\alpha)}x^{n(\alpha)-1} + x^{n(\alpha)} \in \mathcal{U}$. The reader can easily check that $X_{\mathcal{U}}$ is a compact, Hausdorff space in the relative product topology and so the following functions are continuous:

$$\pi: X_{\mathcal{U}} \to X; (\kappa, \lambda) \mapsto \kappa$$

$$p_\alpha: X_{\mathcal{U}} \to \mathbb{C}; (\kappa, \lambda) \mapsto \lambda_\alpha \quad (\alpha \in \mathcal{U}).$$

The extension $A_{\mathcal{U}}$ is defined to be the closed subalgebra of $C(X_{\mathcal{U}})$ generated by $\pi^*(A) \cup \{p_\alpha : \alpha \in \mathcal{U}\}$ where $\pi^*$ is the adjoint map $C(X) \to C(X_{\mathcal{U}}); g \mapsto g \circ \pi$. 
It is not hard to check that $A^U$ is a uniform algebra on $X^U$ with the required properties.

We shall call $A^U$ the Cole extension of $A$ by $U$. Cole gave the construction for the case in which every element of $U$ is of the form $x^2 - f$ for some $f \in A$. It is remarked in [22] that similar methods can be used for the general case; these were independently, explicitly given in [7].

By repeating this construction, using transfinite induction, one can generate uniform algebras which are integrally closed extensions of $A$. Full details of this, including references and the required facts on ordinal numbers and direct limits of normed algebras, can be found in [7]. Again this closely follows [5]. Informally the construction is as follows.

Let $\upsilon$ be a non-zero ordinal number. Set $(A_0, X_0) = (A, X)$. For ordinal numbers $\tau$ with $0 < \tau \leq \upsilon$ we define

$$(A_\tau, X_\tau) = \begin{cases} (A^U_\sigma, X^U_\sigma) & \text{if } \tau = \sigma + 1 \text{ and} \\ \lim_{\sigma < \tau} ((A_\sigma, X_\sigma)_{\sigma < \tau}, \pi^*_{\rho, \sigma}, \pi_{\rho, \sigma})_{\rho \leq \sigma < \tau} & \text{if } \tau \text{ is a limit ordinal.} \end{cases}$$

The construction requires sets of monic polynomials, $U_\sigma \subseteq A_\sigma[x]$, to be chosen inductively. The notation $((A_\sigma, X_\sigma)_{\sigma < \tau}, \pi^*_{\rho, \sigma}, \pi_{\rho, \sigma})_{\rho \leq \sigma < \tau}$ is not standard; it means that $((X_\sigma)_{\sigma < \tau}, (\pi^*_{\rho, \sigma}, \pi_{\rho, \sigma})_{\rho \leq \sigma < \tau})$ is an inverse system of (non-empty) compact, Hausdorff spaces, while at the same time the induced direct system of uniform algebras, $((C(X_\sigma))_{\sigma < \tau}, (\pi^*_{\rho, \sigma})_{\rho \leq \sigma < \tau})$, ‘restricts’ in a natural way to the direct system $((A_\sigma)_{\sigma < \tau}, (\pi^*_{\rho, \sigma})_{\rho \leq \sigma < \tau})$.

The connection between Cole extensions and Arens-Hoffman extensions will be elucidated in Section 1.6.

**DEFINITION 1.3.3.** Let $(A, X) = (A_0, X_0)$ be a uniform algebra and $\upsilon > 0$ be an ordinal number. Then $(A_\tau, X_\tau)_{\tau \leq \upsilon}$, as above, is a system of Cole extensions of $(A, X)$.

Thus $(A_1, X_1)$ is just a Cole extension of $(A_0, X_0)$. When $U_1$ is a singleton we call $A_1$ a simple extension of $A_0$; the same adjective applies to Arens-Hoffman extensions.

An integrally closed extension, $(A_\upsilon, X_\upsilon)$, is obtained by taking $\upsilon$ to be the first uncountable ordinal. At the successor ordinals the whole set of monic polynomials is frequently used to extend the algebra, but this set is larger than necessary. The same procedure is used to obtain the integrally closed extensions in other categories (to be discussed in section 1.6).
1.4 Some Applications of Cole’s Construction

Cole’s method has been developed by others, including Karahanjan and Feinstein, to produce examples of non-trivial uniform algebras with interesting combinations of properties. We cite the following example of Karahanjan.

THEOREM 1.4.1 (from [15], Theorem 4). There is a non-trivial, antisymmetric uniform algebra, \( A \), such that (1) \( A \) is integrally closed, (2) \( A \) is regular, (3) \( \Omega \) is hereditarily unicoherent, (4) \( G(A) \) is dense in \( A \), and (5) the set of peak-points of \( A \) is equal to \( \Omega \).

In the above, \( G(A) \) is our notation for the invertible group of \( A \). We refer the reader to [15] and the literature on uniform algebras for the definitions of other terms we have not defined here.

A further example in [15] also strengthens Cole’s original counter-example. Both of these examples (of non-trivial, natural uniform algebras on compact, metrisable spaces, every point of which is a ‘peak-point’) were regular. Feinstein has varied the construction to obtain such an example which is not regular in [10].

The same author also used Cole extensions in [9] to answer a question of Wilken by constructing a non-trivial, ‘strongly regular’, uniform algebra on a compact, metrisable space.

Returning to the sample theorem quoted above, note that some of these properties (for example the topological property of ‘hereditary unicoherence’) are consequences of the combination of other properties of the final algebra. By contrast, (2) and (4) hold because they are true for the base algebra on which the example is constructed. It is therefore very useful to know exactly when specific properties of a uniform algebra are transferred to those in a system of Cole extensions of it.

The known results on this problem are summarised in the first column of Table 2.2.

Determining if an algebra’s property is shared by its algebraic extensions has lead to some interesting devices. We shall elaborate on this topic in the next section. We remark in passing that the methods used in [15] to show that the final algebra has a dense invertible group have been simplified in [8]; in particular there is no need to develop the theory of ‘dense thin systems’ in [15].

1.5 A Further Remark on Cole Extensions

The reader will notice from Table 2.2 that virtually all properties of uniform algebras are preserved by Cole extensions. The key to obtaining most of these results is the following result, originating with Cole.

PROPOSITION 1.5.1 ([5], [23]). Let \((A_\tau, X_\tau)_{\tau \leq \upsilon}\) be a system of Cole extensions of \((A, X)\). There exists a family of unital contractions \((T_{\sigma, \tau}: C(X_\tau) \to C(X_\sigma))_{\sigma \leq \tau \leq \upsilon}\), such that for all \(\sigma \leq \tau \leq \upsilon\)

(i) \(T_{\sigma, \tau}(A_\tau) \subseteq A_\sigma\), and

(ii) \(T_{\sigma, \tau} \circ \pi^{*}_{\sigma, \tau} = \text{id}_{C(X_\sigma)}\).

Proof. See [23].

For example, it is easy to see from the existence of \(T: C(X^{U}) \to C(X)\) that the Cole extension \(A^{U}\) is non-trivial if \(A \neq C(X)\).
The operator $T$ was constructed in [5] for extensions by square-roots. In the case of a simple Cole extension, $(A^{(\alpha)}, X^{(\alpha)})$, there are at most two points $y_{\pm}(\kappa)$ in the fibre $\pi^{-1}(\kappa)$ for each $\kappa \in X$ and they correspond to the roots of the equation $x^2 - f(\kappa) = 0$ where $\alpha(x) = x^2 - f$. The operator is then defined by

$$T(g)(\kappa) = \frac{g(y_-(\kappa)) + g(y_+(\kappa))}{2} \quad (g \in C(X^{(\alpha)}), \kappa \in X).$$

For other sorts of monic polynomials it was not so obvious how to construct $T$. The basic techniques appeared in [22] (see the proof of Theorem 3.5) for simple extensions, and were further developed in the proof of Theorem 4 of [15], but it was not until [23] that a comprehensive construction was given. We must also mention the role of E. A. Gorin: he appears to have paved the way for [15] and [23].

1.6 Algebraic Extensions of Normed and Banach Algebras

As we have seen, algebraic extensions have had striking applications in the theory of uniform algebras. They have long been used as auxiliary constructions in the general theory of Banach algebras. Notable examples of this are in [14] and [25]; the latter explicitly uses Arens-Hoffman extensions.

However algebraic extensions for normed algebras were apparently only studied in their own right in order to generalise the work of Cole and Karahanjan. We now turn to these generalisations.

The basic extension generalising Arens-Hoffman extensions is called a standard normed extension. It is defined in the following theorem of Lindberg.

THEOREM 1.6.1 ([22]). Let $A$ be a normed algebra and $U$ a set of monic polynomials over $A$. Let $\leq$ be a well-ordering on $U$ with least element $\alpha_0$. Then there exists a normed algebra, $B_U$, with a family of subalgebras, $(B_\alpha)_{\alpha \in U}$, such that:

(i) for all $\alpha, \beta \in U$, $B_\alpha \subseteq B_\beta$ if $\alpha \leq \beta$, and,

(ii) for all $\beta \in U$, $B_\beta$ is isometrically isomorphic to an Arens-Hoffman extension of $B_{<\beta}$ by $\beta(x)$ where

$$B_{<\beta} = \begin{cases} \bigcup_{\alpha < \beta} B_\alpha & \text{if } \alpha_0 < \beta \text{ and} \\ A & \text{if } \alpha_0 = \beta. \end{cases}$$

Proof. See [22].

Lindberg shows how this leads to the construction of Banach algebras with interesting combinations of properties, one of which is integral closedness.

Let the isometric isomorphism $B_{<\beta}[x]/(\beta(x)) \to B_\beta$ in (ii) above be denoted by $\psi_\beta$ ($\beta \in U$). We shall refer to $\psi_\beta(\bar{x})$ as the standard root of $\beta(x)$ in $B_U$. This helps us when we show that standard extensions share a similar universal property to the one described in Proposition 1.1.2. Again the following lemma has not been explicitly given in the literature but is probably regarded as obvious by those working in the field. We state the result in full as we shall make use of it later.

LEMMA 1.6.2 Let $A^{(1)}$ be a normed algebra and $U$ a non-empty set of monic polynomials over $A^{(1)}$. Let $B^{(1)} = A_U^{(1)}$ be a standard extension of $A^{(1)}$ with respect to $U$ and $\theta: A^{(1)} \to B^{(2)}$ be a unital homomorphism of normed algebras.
Let $\xi_\alpha$ be the standard root of $\alpha \in \mathcal{U}$, with associated norm parameter $t_\alpha$, and suppose $(\eta_\alpha)_{\alpha \in \mathcal{U}} \subseteq B^{(2)}$ is such that $\theta(\alpha)(\eta_\alpha) = 0$ for all $\alpha \in \mathcal{U}$. Then there is a unique, unital homomorphism $\phi: B^{(1)} \to B^{(2)}$ such that the following diagram is commutative

$$
\begin{array}{ccc}
B^{(1)} & \xrightarrow{\phi} & B^{(2)} \\
\subseteq & \searrow_{\mathcal{U}} & \searrow_{\mathcal{U}} \\
A^{(1)} & \xrightarrow{\mathcal{U}} & A^{(2)}
\end{array}
$$

and for all $\alpha \in \mathcal{U}$, $\phi(\xi_\alpha) = \eta_\alpha$.

**Proof.** The result is obtained by a simple application of transfinite methods and Proposition 1.1.2.

Purely algebraic standard extensions are defined in [22] and the main content of Lemma 1.6.2 is a statement about these.

Narmania gives ([23]) an alternative construction for integrally closed extensions of a commutative, unital Banach algebra, $A$. His method is rather more conventional than the one used to define standard extensions. If $\mathcal{U}$ is a set of monic polynomials over $A$ then the Narmania extension of $A$ by $\mathcal{U}$ is equal to the Banach-algebra direct limit of $(A_S : S$ is a finite subset of $\mathcal{U})$ where each $A_S$ is isometrically isomorphic to $A$ extended finitely many times by the Arens-Hoffman construction. As this paper is not readily available in English and we shall refer to the explicit construction of Narmania’s extensions in the next result, we stop to report the precise details of this.

If $E$ is a set, the set of all finite subsets of $E$ will be written $E^{<\omega}$. Let $S = \{\alpha_1, \ldots, \alpha_m\} \subseteq \mathcal{U}$ and let $t_\alpha (\alpha \in \mathcal{U})$ be a valid choice of Arens-Hoffman norm-parameters (see Section 1.1). It is important to insist that distinct elements $\alpha, \beta \in \mathcal{U}$ are associated with distinct indeterminates $x_\alpha, x_\beta$. Thus $S$ is an abbreviation for $\{\alpha_1(x_{\alpha_1}), \ldots, \alpha_m(x_{\alpha_m})\}$.

It is proved carefully in [23] that for $q = \sum_s q_s x_{\alpha_1}^{s_1} \cdots x_{\alpha_m}^{s_m} \in A[x_{\alpha_1}, \ldots, x_{\alpha_m}]$, the algebra of polynomials in $m$ commuting indeterminates over $A$, $(s$ is a multi-index in $\mathbb{N}_0^m$ where $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$) then $(S) + q$ has a unique representative whose degree in $x_\alpha$ is less than than $n(\alpha_j)$, the degree of $\alpha_j(x_{\alpha_j})$ $(j = 1, \ldots, m)$. For convenience we shall call such representatives minimal. Then if $q$ is the minimal representative of $(S) + q$,

$$
\| (S) + q \| := \sum_s \|q_s\| t_{\alpha_1}^{s_1} \cdots t_{\alpha_m}^{s_m}
$$

defines an algebra norm on $A_S$. The index set, $\mathcal{U}^{<\omega}$ is a directed set, directed by $\subseteq$. The connecting homomorphisms $\nu_{S,T}$ (for $S \subseteq T \in \mathcal{U}^{<\omega}$) are the natural maps; they are isometries. Thus (see [24] section 1.3) $A_D$ is the completion of the normed direct limit, $D := \bigcup_{S \in \mathcal{U}^{<\omega}} A_S / \sim$, where $\sim$ is an equivalence relation given by $(S) + q \sim (T) + r$ if and only if $q - r \in (S \cup T)$ for $S, T \in \mathcal{U}^{<\omega}$. Furthermore, the canonical map, $\nu_S$, which sends an element of $A_S$ to its equivalence class in $D$, is an isometry. Note that $A_D$ is defined to be $A$.

We can now show how the types of extensions we have been considering are related. Many of the ideas behind Proposition 1.6.3 are due to Narmania but we take the step of linking them to Cole and standard extensions.

**PROPOSITION 1.6.3.** Let $A$ be a commutative, unital Banach algebra and $\mathcal{U}$ a set of monic polynomials over $A$. Then, up to isometric isomorphism, $A_{\mathcal{U}} = \overline{B_{\mathcal{U}}}$.
If $A$ is a uniform algebra then we have

$$A^\mathcal{U} = (\overline{A_\mathcal{U}}) = (\overline{B_\mathcal{U}}),$$

where the closures are taken with respect to the supremum norm.

**Proof.** It is easily checked that if $B$ is a normed algebra then the homeomorphism $\Omega(\overline{B}) \to \Omega(B)$ induces an isometric isomorphism $\overline{B} \to (\overline{B})^\mathcal{U}$. It is therefore sufficient to prove that $A_\mathcal{U} = \overline{A_\mathcal{U}}$ and that $A^\mathcal{U} = (\overline{B_\mathcal{U}})^\mathcal{U}$. The last equality follows very quickly from the universal property of standard extensions mentioned above and the simplicity of the definition of $A^\mathcal{U}$. We shall only prove the first identification; the second can be proved by a similar approach. Although what follows is routine, we hope that it will help to clarify the details of standard and Narmania extensions.

As before let $t_\alpha (\alpha \in \mathcal{U})$ be a valid choice of Arens-Hoffman norm-parameters for the respective extensions $A_\alpha$. We shall show that there is then an isometric isomorphism between $B_\mathcal{U}$ and $D$ (when defined by these parameters); the result then follows from the uniqueness of completions.

For each $\alpha \in \mathcal{U}$ let $y_\alpha$ be the equivalence class $[(\{\alpha(x_\alpha]\}) + x_\alpha] \in D$. Since $y_\alpha$ is a root of $\nu_\beta(\alpha(x))$ in $D$ there exists, by the universal property of standard extensions, a (unique) homomorphism $\phi: B_\mathcal{U} \to D$ such that $\phi|_A = \nu_\beta$ and for all $\alpha \in \mathcal{U}$, $\phi(\xi_\alpha) = y_\alpha$. Here, $\xi_\alpha$ is the the element of $B_\mathcal{U}$ associated with $\bar{x}$ by the isometric isomorphism $\psi_\alpha : B_{<\alpha}[x]/(\alpha(x)) \to B_\alpha$ in the notation of Theorem 1.6.1. Thus $\|\xi_\alpha\| = t_\alpha$. Note that, by its definition in [22], $\psi_\alpha$ satisfies $\psi_\alpha(a) = a$ for all $a \in B_{<\alpha}$.

It is clear that $\phi$ is surjective; we now use the transfinite induction theorem, as is customary for proving results about standard extensions, to show that $\phi$ is isometric.

Let $\mathcal{J} = \left\{ \beta \in \mathcal{U} : \phi|_{B_{\beta}} \text{ is isometric} \right\}$. It should be clear to the reader that $\alpha_0 \in \mathcal{J}$. Let $\beta \in \mathcal{U}$ and suppose that $[\alpha_0, \beta] \subseteq \mathcal{J}$. Let $b \in B_\beta$. Then, writing $n(\beta)$ for the degree of $\beta(x)$, there exist unique $b_1, \ldots, b_{n(\beta)-1}, \in B_{<\beta}$ such that $b = \sum_{j=0}^{n(\beta)-1} b_j \xi_\beta^j$. We have, by hypothesis,

$$\|b\| = \sum_{j=0}^{n(\beta)-1} \|b_j\| t_\beta^j = \sum_{j=0}^{n(\beta)-1} \|\phi(b_j)\| t_\beta^j.$$

Since the algebras $\nu_\gamma(A_\gamma)$ are directed there exists $S \in \mathcal{U}^{<\omega}$ such that $\phi(b_j) \in \nu_\gamma(A_\gamma)$ for all $j = 0, \ldots, n(\beta) - 1$. We can assume that $S = \{\alpha_1, \ldots, \alpha_m\}$ and $\alpha_1(x) = \beta(x)$. Let $q_0, \ldots, q_{n(\beta)-1} \in A[x_{\alpha_1}, \ldots, x_{\alpha_m}]$ be the minimal representatives such that $\phi(b_j) = [(S) + q_j]$ for all $j = 0, \ldots, n(\beta) - 1$. Thus $\|b_j\| = \|q_j\|$ for all $j = 0, \ldots, n(\beta) - 1$. A routine exercise in the transfinite induction theorem shows that for all $\gamma \in \mathcal{U}$, $\phi(B_{\gamma}) \subseteq \bigcup_{T \in [\mathcal{U}^{<\omega}]} \nu_T(A_T)$. It follows that the degree...
of $q_j$ in $x_{\alpha_1}$ is zero. Hence

$$\|\phi(b)\| = \left\| \left[ \sum_{j=0}^{n(\beta)-1} [(S) + q_j x_{\alpha_1}^j] \right] - \left[ \sum_{j=0}^{n(\beta)-1} q_j x_{\alpha_1}^j \right] \right\|,$$

$$= \left\| \left[ (S) + \sum_{j=0}^{n(\beta)-1} q_j x_{\alpha_1}^j \right] \right\|,$$

$$= \left\| \left[ (S) + \sum_{j=0}^{n(\beta)-1} q_j x_{\alpha_1}^j \right] \right\|,$$

$$= \sum_{j=0}^{n(\beta)-1} \|q_j\| t_{\alpha_1}^j,$$

$$= \|b\|,$$

from above. The penultimate equality above follows from noting that the representative of the coset is minimal and then expanding and collecting terms.

By the transfinite induction theorem, $J = U$ as required. □

2.1 A Survey of Properties Preserved by Algebraic Extensions

We summarise in Table 2.2 what is currently known about the behaviour of certain properties of normed algebras with respect to the types of extensions we have been considering. Some preliminary explanation of the entries is in order first.

Extra information about the polynomial(s) generating an algebraic extension can help to determine whether certain properties are preserved or not. For example if $\alpha(x)$ has degree $n$ and factorises completely over $A$ with distinct roots $\lambda_1, \ldots, \lambda_n \in A$ such that for all $\omega \in \Omega$, $\Lambda_i(\omega) \neq \Lambda_j(\omega)$ if $i \neq j$ then $\Omega(A, \alpha)$ decomposes into $n$ disjoint homeomorphs of $\Omega$ in which case very many properties of $A$, for example localness, are shared by $A, \alpha$. This property, referred to as ‘complete solvability’, is investigated in [12].

The condition on $\alpha(x)$ most frequently encountered in the literature is that it should be ‘separable’. This means that its ‘discriminant’, which is a certain polynomial in the coefficients of $\alpha(x)$, is invertible in $A$. It is interesting to compare columns two and three.

Of course one can make additional assumptions on the algebra (for example that $A$ be regular and semisimple) but the resulting table would become too large and we have restricted it to three popular categories.

References to the results follow the table. We should mention that some of the entries have trivial explanations. For example Sheinberg’s theorem, that a uniform algebra is amenable if and only if it is trivial, explains the entries for amenability in column one. Also, applying the Arens-Hoffman construction to a uniform algebra need not result in a uniform algebra so not all the entries make sense.

We have already met most of the properties listed in the table. We end this section by discussing the ones which have not yet been specially mentioned.

1. Denseness of the invertible group. Although this property is self-explanatory it might not be obvious why it is listed. However, the condition $G(A) = A$ appears in the literature in various contexts; see for example [8].
2. The Banach algebra, $A$, is called \textit{sup-norm closed} if $\hat{A}$ is uniformly closed in $C(\Omega)$ (and therefore a uniform algebra). It is called \textit{symmetric} if $\hat{A}$ is self-adjoint.

3. For the definitions of ‘amenability’ and ‘weak amenability’ we refer the reader to section 2.8 of [6].

All the properties in the table are preserved by forming the standard unitisation of a normed algebra. Most of these results are standard facts or easy exercises; some are true by definition. However this question does not fit into our scheme because the embedding is not unital in this case.

2.2 Table

Cole extensions have only been defined for uniform algebras; the algebra is therefore assumed to be a uniform algebra throughout column one. Columns two and three, as mentioned above, refer to Arens-Hoffman extensions of a normed algebra, $A$, by a monic polynomial $\alpha(x)$; in column three it is given that $\alpha(x)$ is separable.

| Type of Extension: | Cole $A_{\alpha}$ | A.-H. $A_{\alpha}$ | standard Narmania $\alpha$ sep. |
|--------------------|-------------------|-------------------|-----------------------------|
| Property:          |                   |                   |                             |
| 1. complete        | •                 | •                 | •                           |
| 2. topologically semisimple | • | o | • | o | o |
| 3. non-local       | •                 | •                 | •                           |
| 4. local           | ?                 | o                 | ?                           |
| 5. regular         | •                 | o                 | ?                           |

\textit{for normed algebras}

\begin{tabular}{lcccc}
1. complete & • & • & • & o \\
2. topologically semisimple & • & o & • & o \\
3. non-local & • & • & • & • \\
4. local & ? & o & ? & ? \\
5. regular & • & o & ? & o \\
\end{tabular}

\textit{for Banach algebras}

\begin{tabular}{lcccc}
6. local & ? & ? & ? & ? \\
7. regular & • & • & • & ? \\
8. dense invertible group & • & • & • & • \\
9. sup-norm closed & • & o & • & o \\
10. symmetric & • & o & • & o \\
11. amenable & • & o & ? & o \\
12. weakly amenable & ? & o & ? & o \\
\end{tabular}

\textit{for uniform algebras}

\begin{tabular}{lcccc}
13. non-trivial & • & - & - & - \\
14. trivial & • & - & - & - \\
15. natural & • & - & - & - \\
\end{tabular}

\textbf{Key}

- property is always preserved
- property is sometimes, but not always preserved
- not yet determined
- it doesn’t always make sense to consider this property here
References for the Entries

If we do not mention an entry here, it can be taken that the result is an immediate consequence of the definition or was proved in the same paper in which the relevant extension was introduced (that is in [5], [1], [22], or [23]).

The results of row three are not hard to obtain, using appropriate versions of Proposition 1.5.1.

Localness and regularity were discussed in Section 1.2. The main result about this is due to Lindberg in [18]; the same section of his paper also deals with the results on the symmetry of Arens-Hoffman extensions. That regularity passes to direct limits of such extensions has been widely noted by many authors, for example in [15].

Results of row eight follow from [8]; the case of Cole extensions was partially covered in [15], but the reasoning is not clear.

The property of being sup-norm closed was investigated in [13]; this work was generalised in [28].

Finally, examples of amenable Banach algebras which do not have even weakly amenable Arens-Hoffman extensions have been known for a long time. For example, the algebra \( C \oplus C \) under the multiplication \((a, b)(c, d) = (ac, bc + ad)\) is realisable as an Arens-Hoffman extension of \( C \). Examples with both \( A \) and \( A_\alpha \) semisimple have been found by the author. However the entries marked ‘?’ in rows eleven and twelve represent intriguing open problems.

3. Conclusion

The table in section 2.2 still has gaps, and there are many more rows which could be added. For example it would be interesting to be able to estimate various types of ‘stable ranks’ (see [2]) of the extensions in terms of the stable ranks of the original algebras. (The condition \( G(A) = A \) is equivalent to the ‘topological stable rank’ of \( A \) not exceeding 1.) Remember too that there are many more questions which can be asked, of the form: ‘if \( \Omega \) has the topological property \( P \), does \( \Omega(A_\alpha) \) have property \( P \)?’

By way of a conclusion we repeat that algebraic extensions have proved immensely useful in the construction of examples of uniform algebras. There is therefore great scope for and potential usefulness in augmenting Table 2.2. It might also be valuable to reexamine the techniques used to obtain the entries to produce more general results (of the kind in [28] for example) in the context of question (Q).

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