BRAID GROUP ACTION ON THE MODULE CATEGORY OF QUANTUM AFFINE ALGEBRAS

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Abstract. Let $g_0$ be a simple Lie algebra of type ADE and let $U'_q(g)$ be the corresponding untwisted quantum affine algebra. We show that there exists an action of the braid group $B(g_0)$ on the quantum Grothendieck ring $K_t(g)$ of Hernandez-Leclerc’s category $C^0_g$. Focused on the case of type $A_{N-1}$, we construct a family of monoidal autofunctors $\{S_i\}_{i \in \mathbb{Z}}$ on a localization $T_N$ of the category of finite-dimensional graded modules over the quiver Hecke algebra of type $A_\infty$. Under an isomorphism between the Grothendieck ring $K(T_N)$ of $T_N$ and the quantum Grothendieck ring $K_t(A_{(N-1)}^{(1)})$, the functors $\{S_i\}_{1 \leq i \leq N-1}$ recover the action of the braid group $B(A_{N-1})$. We investigate further properties of these functors.

1. Introduction

The monoidal category $C_g$ of finite-dimensional representations of a quantum affine algebra $U'_q(g)$ has been extensively investigated because of its rich structure. Among various approaches, Nakajima ([14]), Varagnolo-Vasserot ([16]), and Hernandez ([3]) studied $t$-deformations of the Grothendieck ring of $C_g$. These $t$-deformations are interesting, because they provide a way to calculate the $q$-character of simple representations. There is a full subcategory $C^0_g$ of $C_g$, introduced by Hernandez and Leclerc in [4], which contains an essential information on $C_g$ but has a smaller set of the classes of simple modules. The Grothendieck ring of $C^0_g$ is isomorphic to the polynomial ring in countably many variables, while that of $C_g$ is the one in uncountably many variables. For the cases where $g$ is one of untwisted ADE types, a $t$-deformation $K_t(g)$ of the Grothendieck ring of $C^0_g$, called the 	extit{quantum Grothendieck ring}, was investigated from a ring theoretic point of view in [5]. It turns out that the $\mathbb{C}(t^{1/2})$-algebra $K_t(g)$
has an interesting presentation: there is a set of generators consisting of a countable infinite number of copies of Drinfeld-Jimbo generators of a half of the quantum group $U_t(g_0)$, and they satisfy the quantum Serre relations in a copy, $t$-boson relations between adjacent copies, and $t$-commutation relations between non-adjacent copies. This presentation reflects the following feature of the category $C^0_0$: for each choice of a Dynkin quiver $Q$ with an additional data, they defined a monoidal subcategory $C_Q$ of $C^0_0$ such that the quantum Grothendieck ring of $C_Q$ is isomorphic to the half $U^-_t(g_0)$ of the quantum group $U_t(g_0)$, and all the fundamental representations in $C^0_0$ can be obtained from those in $C_Q$ by taking functors $D^m (m \in \mathbb{Z})$. Here $D$ is the contravariant functor taking the right dual.

One of main results of this paper is that there exists an action of the braid group $B(g_0)$ of type $g_0$ on the quantum Grothendieck ring $K_t(g)$ (Theorem 2.3). Since we give the action explicitly, the braid relations can be checked by the presentation of $K_t(g)$. Recall that the blocks of the category $C^0_0$ is parameterized by the root lattice of $g_0$ and the tensor product is compatible with the addition on the root lattice ([10]). It turns out that the action of the generators $\sigma_i$ of $B(g_0)$ on $K_t(g)$ correspond to the reflections with respect to the simple roots $\alpha_i$ on the root lattice. Indeed, the action of $\sigma_i$ is related with Saito's reflection functor as seen in Theorem 2.4.

We conjecture that the braid group action can be lifted to the action on the monoidal category $C^0_0$. We show that it is the case when $g$ is of type $A_{N-1}^{(1)}$. A key point of view is the use of a rigid monoidal category $T_N$ which is constructed out of the category $A$ of finite-dimensional graded modules over the quiver Hecke algebra $R^{A_\infty}$ of type $A_\infty$ ([7]). It is a certain localization of $A$ and there is a monoidal functor $F_N$ from $T_N$ to $C^{0(1)}_{A_{N-1}}$ which sends simple objects to simple objects. Moreover this functor induces an isomorphism between the Grothendieck ring $K(T_N)$ and the quantum Grothendieck ring $K_t(A_{N-1}^{(1)})$. It is summarized by the diagram

$$K(T_N) \cong K_t(A_{N-1}^{(1)}) \xrightarrow{\sim} K(C^0_{A_{N-1}}).$$

Hence the category $T_N$ can be understood as a graded lift of $C^0_{A_{N-1}}$ as a rigid monoidal category.

We show that there is a family of monoidal autofunctors $\{\mathcal{S}_i\}_{1 \leq i \leq N-1}$ on the category $T_N$ which recover the action of the braid group $B(A_{N-1})$ under the isomorphism between $K(T_N)$ and $K_t(A_{N-1}^{(1)})$ (Theorem 3.1, Theorem 3.3). There is a general procedure, developed in [11], to construct monoidal functors between the categories of modules over quiver Hecke algebras, and a similar procedure can be applied for the category $T_N$. This is a main advantage in working on the category $T_N$ rather than the category $C_{A_{N-1}}^0$.

Finally we provide several consequences of the existence of such functors $\mathcal{S}_i$. For a simple object $L$ which belongs to an orbit of $L(i)$ for some $i$ under the action $B(A_{N-1})$,
one can define an automorphism $s_L$ which has similar properties with the automorphisms $s_i$ (Theorem 4.2). Moreover $s_{L(i)}$ coincides with $s_i$.

This paper is an announcement whose details will appear elsewhere.

2. Braid group action on the quantum Grothendieck rings

Let $g_0$ be a finite-dimensional simple Lie algebra of simply-laced type with a Cartan matrix $A = (a_{ij})_{i,j \in I_0}$, $g$ the untwisted affine Kac-Moody algebra associated with $g_0$, and $U'_q(g)$ the quantum affine algebra associated with $g$. We take the algebraic closure of $\mathbb{C}(q)$ inside $\bigcup_{m > 0} \mathbb{C}((q^{1/m}))$ as the base field $k$ for $U'_q(g)$. Let $\mathcal{C}_g$ be the category of finite-dimensional integrable modules over $U'_q(g)$. There is a family $\{V(\varpi_i)_{c} \mid i \in I_0, c \in k^x\}$ in $\mathcal{C}_g$, called the fundamental representations.

Following [4], we denote by $\mathcal{C}_g^0$ the smallest full subcategory of the category $\mathcal{C}_g$ which is stable under taking subquotients, extensions, tensor products and contains

$$\{V(\varpi_i)_{(q^{-p})} \mid i \in I_0, \ p \equiv d(1, i) \mod 2\},$$

where $d(i, j)$ is the distance between the vertices $i$ and $j$ in the Dynkin diagram of $g_0$. Here $1 \in I_0$ is an arbitrarily chosen element. Then the complexified Grothendieck ring $\mathbb{C} \otimes_{\mathbb{Z}} K(\mathcal{C}_g^0)$ of $\mathcal{C}_g^0$ has a $t$-deformation $\mathcal{K}_t(g)$, called the quantum Grothendieck ring of $\mathcal{C}_g^0$. To each simple module $S$ in $\mathcal{C}_g^0$, we can associate an element $[S]_t$ of $\mathcal{K}_t(g)$ and we have $\mathcal{K}_t(g) = \bigoplus_S \mathbb{C}(t^{1/2})[S]_t$. Here $S$ ranges over the set of the isomorphism classes of simple modules in $\mathcal{C}_g^0$.

Let $Q$ be a Dynkin quiver with type $g_0$, and let $\phi_Q$ be a height function, i.e., it associates an integer $\phi_Q(i)$ to each vertex $i$ of $Q$ such that $\phi_Q(i) = \phi_Q(j) + 1$ if $i \to j$. We assume further that $\phi_Q(1) \in 2\mathbb{Z}$. A pair $Q = (Q, \phi_Q)$ is called a Q-data.

For a sink $i$ of $Q$, let $s_iQ := (s_iQ, \phi_{s_iQ})$ be the Q-data consisting of the Dynkin quiver $s_iQ$ obtained from $Q$ by reversing the arrows of $Q$ adjacent to $i$ and the height function $\phi_{s_iQ}$ of $s_iQ$ given by $\phi_{s_iQ}(j) = \phi_Q(j) + 2\delta_{i,j}$.

To a Q-data $Q$, Hernandez-Leclerc ([5]) associated a full monoidal subcategory $\mathcal{C}_Q$ of $\mathcal{C}_g^0$, and a monoidal functor $\mathcal{F}_Q : R_{g_0}\text{-mod} \to \mathcal{C}_Q$ is constructed in [6, 2], and Fujita ([1, 2]) proved that $\mathcal{F}_Q$ is an equivalence of categories. Here, $R_{g_0}\text{-mod}$ is the monoidal category of finite-dimensional modules (with nilpotent actions of the generators $x_k$) over the quiver Hecke algebra $R_{g_0}$ associated with $g_0$. Note that $\mathcal{F}_Q(L(i))$ is a fundamental module for any $i \in I_0$, where $L(i) \in R_{g_0}\text{-mod}$ is the simple module associated with $i \in I_0$.

Then, for a Q-data $Q$, we have an embedding of $\mathbb{Z}[t^{\pm 1}]$-algebras

$$j_Q : K(R_{g_0}\text{-gmod}) \hookrightarrow \mathcal{K}_t(g)$$

induced by $\mathcal{F}_Q$.

Let $\mathbb{K}_t(g_0)$ be the $\mathbb{C}(t^{1/2})$-algebra generated by $\{y_{i,m} \mid i \in I_0, m \in \mathbb{Z}\}$ with the defining relations:
(a) for $m \in \mathbb{Z}$,
\[
y_{i,m}y_{j,m} = y_{j,m}y_{i,m} \quad \text{if } a_{ij} = 0,
y_{i,m}^2 - (t + t^{-1})y_{i,m}y_{j,m}y_{i,m} + y_{j,m}y_{i,m}^2 = 0 \quad \text{if } a_{ij} = -1,
\]
(b) for $m \in \mathbb{Z}$ and $i, j \in I_0$,
\[
y_{j,m+1} = t^{a_{ij}}y_{j,m+1}y_{i,m} + \delta_{ij}(1 - t^2),
\]
(c) for $p > m + 1$ and $i, j \in I_0$,
\[
y_{j,m} = t^{(-1)^{p-m+1}a_{ij}}y_{j,m}y_{i,m}.
\]

**Remark 2.1.** We change $t$ into $t^{-1}$ in the presentation in [5].

**Theorem 2.2 ([5, Theorem 7.3]).** Let $Q$ be a $Q$-data. Then there is an isomorphism $\iota_Q : \mathbb{K}_t(\mathfrak{g}_0) \cong \mathbb{K}_t(\mathfrak{g})$ such that $\iota_Q(y_{i,m})$ is equal to $[\mathcal{F}_Q(L(i))]$, where $L(i)$ is the simple module in $R_{\mathfrak{g}_0}$-mod corresponding to $i \in I_0$.

Let $B(\mathfrak{g}_0)$ be the Braid group associated with $\mathfrak{g}_0$. It is generated by $\{\sigma_i \mid i \in I_0\}$ with the defining relations
\[
\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \quad \text{if } a_{ij} = -1,
\]
\[
\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{if } a_{ij} = 0.
\]

One of our main theorems is the following.

**Theorem 2.3.** The Braid group $B(\mathfrak{g}_0)$ acts on $\mathbb{K}_t(\mathfrak{g}_0)$ by the following formulas:

\[
\sigma_i(y_{j,m}) = \begin{cases} y_{j,m+1} & \text{if } a_{ij} \neq -1, \\ \frac{t^{1/2}y_{j,m}y_{i,m} - t^{-1/2}y_{i,m}y_{j,m}}{t - t^{-1}} & \text{if } a_{ij} = -1, \end{cases}
\]

\[
\sigma_i^{-1}(y_{j,m}) = \begin{cases} y_{j,m-1} & \text{if } a_{ij} \neq -1, \\ \frac{t^{1/2}y_{i,m}y_{j,m} - t^{-1/2}y_{j,m}y_{i,m}}{t - t^{-1}} & \text{if } a_{ij} = -1. \end{cases}
\]

**Theorem 2.4.** Let $i$ be a sink of a $Q$-data $Q$. Then the following diagrams commute:

\[
\begin{array}{ccc}
K(R_{\mathfrak{g}_0}\{-\text{mod}) & \xrightarrow{j_Q} & \mathbb{K}_t(\mathfrak{g}) \xrightarrow{\iota_Q} \mathbb{K}_t(\mathfrak{g}_0) \\
\downarrow \sigma_i & & \downarrow \\
\mathbb{K}_t(\mathfrak{g}) & \xrightarrow{\iota_Q} & \mathbb{K}_t(\mathfrak{g}_0)
\end{array}
\]

\[
\begin{array}{ccc}
\mathbb{K}_t(\mathfrak{g}) & \xrightarrow{j_{\sigma_i}Q} & \mathbb{K}_t(\mathfrak{g}) \\
& \searrow & \downarrow \\
& & \mathbb{K}_t(\mathfrak{g}_0) \xrightarrow{\iota_Q}
\end{array}
\]
Here, $iR\text{-gmod}$ (resp. $iR\text{-gmod}$) is the full subcategory of $R\text{-gmod}$ consisting of graded modules $M$ with $E_i^*M = 0$ (resp. $E_iM = 0$), and $T_i$ is the reflection functor due to S. Kato [12, 13] (cf. Y. Saito [15]). For $E_i$ and $E_i^*$, see for example, [8].

3. The category $T_N$ and reflection functors

Let $J$ be the index set of simple roots of the root system $A_\infty$. One can identify $J$ with $\mathbb{Z}$ and the root lattice $Q$ is the subspace of $\bigoplus_{a \in \mathbb{Z}} \mathbb{Z}_{\varepsilon_a}$ generated by $\alpha_a = \varepsilon_a - \varepsilon_{a+1}$ for $a \in \mathbb{Z}$. Let $R^{A_\infty}$ be the symmetric quiver Hecke algebra of type $A_\infty$ over $k$ with the choice of parameters

$$Q_{ij} (u, v) = \delta(i \neq j) (u - v)^{\delta(j = i + 1)} (v - u)^{\delta(j = i - 1)}$$

for $i, j \in J$. It is a family $\{R^{A_\infty}(\beta)\}_{\beta \in Q^+}$ of associative $\mathbb{Z}$-graded $k$-algebras, where $Q^+ = \sum_{i \in J} \mathbb{Z}_{\geq 0} \alpha_i$ is the positive root lattice of type $A_\infty$. Each $R^{A_\infty}(\beta)$ is generated by $\{e(\nu)\}_{\nu \in J^\beta}, \{x_k\}_{1 \leq k \leq n}$ and $\{\tau_m\}_{1 \leq m \leq n - 1}$, where $n = |\beta| = \sum_{i \in I} n_i$ with $\beta = \sum_{i \in J} n_i \alpha_i$, and $J^\beta := \{\nu \in J^n \mid \alpha_{\nu_1} + \cdots + \alpha_{\nu_n} = \beta\}$. See [7] for a set of defining relations of $R^{A_\infty}(\beta)$. Note that there is an embedding of $R^{A_\infty}(\beta) \otimes R^{A_\infty}(\gamma)$ into $R^{A_\infty}(\beta + \gamma)$. Hence the category $A = \bigoplus_{\beta \in Q^+} R^{A_\infty}(\beta)$-gmod of finite-dimensional graded $R^{A_\infty}$-modules is a monoidal category whose tensor product is given by the convolution product:

$$M \circ N := R^{A_\infty} (\beta + \gamma) \otimes_{R^{A_\infty} (\beta) \otimes R^{A_\infty} (\gamma)} (M \otimes N).$$

For $M \in R^{A_\infty}(\beta)$-gmod, we set $\text{wt}(M) := -\beta$.

For each pair of integers $a, b$ with $a \leq b$, let $[a, b]$ be the interval $\{k \in \mathbb{Z} \mid a \leq k \leq b\}$, and call it a segment. For each segment $[a, b]$, let $L(a, b)$ be the one-dimensional simple graded $R^{A_\infty}$-module generated by a vector $u(a, b)$ such that $e(\nu)u(a, b) = \delta(\nu = (a, \ldots, b))u(a, b)$. We set $L(a) := L(a, a)$ for $a \in \mathbb{Z}$. For each $N \geq 2$, let $S_N$ be the smallest subcategory of $A$ which is stable under taking convolution, subquotients, extensions, and containing $\{L(a, b) \mid b - a + 1 > N\}$. Then the quotient category $A/S_N$ equips with a new tensor product $\star$ given by

$$X \star Y := t^{B(\text{wt}(X), \text{wt}(Y))} X \circ Y,$$

where $B(x, y) := -\sum_{k > 0} (S^k x, y)$ for $x, y \in \bigoplus_{a \in \mathbb{Z}} \mathbb{Z}_{\varepsilon_a}$ and $S$ is an automorphism on $\bigoplus_{a \in \mathbb{Z}} \mathbb{Z}_{\varepsilon_a}$ given by $S(\varepsilon_a) := \varepsilon_{a+N}$. The category $T_N$ is constructed in [7] as a localization of the
monoidal category \((\mathcal{A}/\mathcal{S}_N, \star)\). The objects of \(\mathcal{T}_N\) is the same with the ones of \(\mathcal{A}/\mathcal{S}_N\). The group of morphisms is given by

\[
\text{Hom}_{\mathcal{T}_N}(X, Y) := \lim_{\lambda, \mu} \text{Hom}_{\mathcal{A}/\mathcal{S}_N}(X \circ P^\lambda, Y \circ P^\mu),
\]

where \(P^\nu := \circ_{a \in \mathbb{Z}} L(a, a + N - 1)^{\circ a}\) for \(\nu \in (\mathbb{Z}_{\geq 0})^{\oplus J}\) and the limit runs over all the pairs \((\lambda, \mu)\) such that \(\text{wt}(X \circ P^\lambda) = \text{wt}(Y \circ P^\mu)\). It turns out that \(\mathcal{T}_N\) is an abelian rigid monoidal category with a tensor product \(\star\). We denote the right dual (resp. left dual) of \(X\) by \(\mathcal{D}(X)\) (resp. \(\mathcal{D}^{-1}(X)\)). Note that \(L(a, a + N - 1) \simeq 1\) in \(\mathcal{T}_N\) for all \(a \in \mathbb{Z}\). We have a chain of functors

\[
\mathcal{A} \xrightarrow{\Omega_N} \mathcal{A}/\mathcal{S}_N \xrightarrow{\iota_N} \mathcal{T}_N.
\]

The composition will be denoted by \(\Omega_N\). Note that the Grothendieck ring \(K(\mathcal{T}_N)\) is a \(\mathbb{Z}[t^{\pm 1}]\)-algebra on which \(t\) acts by the grading shift.

From now on, let \(\mathfrak{g}\) be the affine Kac-Moody algebra of type \(A_{N-1}^{(1)}\). We regard \(\mathcal{T}_N\) as a \(\mathbb{Z}\)-graded lifting of \(\mathcal{C}_0^\beta\) as a rigid monoidal category. Indeed there exists a monoidal functor \(\mathcal{F}_N: \mathcal{T}_N \to \mathcal{C}_0^\beta\) which sends simples to simples. It induces an isomorphism of \(\mathbb{C}(t^{1/2})\)-algebras \([\mathcal{F}_N]: \mathbb{C}(t^{1/2}) \otimes_{\mathbb{Z}[t^{\pm 1}]} K(\mathcal{T}_N) \simeq K_t(\mathfrak{g})\) ([7, Theorem 4.33]). Under the isomorphism, the generator \(y_{i,m}\) corresponds to \([\mathbb{P}^m(L(i))]\) for \(i \in \mathbb{Z}, m \in \mathbb{Z}\).

For a pair \((M, N)\) of objects in an abelian monoidal category we denote by \(M \nabla N\) the head of \(M \otimes N\) and by \(M \Delta N\) the socle of \(M \otimes N\), respectively.

We show that there is a family of autofunctors on \(\mathcal{T}_N\) which recover the braid group action on the quantum Grothendieck ring \(K_t(\mathfrak{g})\). For this purpose, we adjoin a formal object \(t^{1/2}1\) into \(\mathcal{T}_N\) such that \(t^{1/2}1 \star t^{1/2}1 \simeq t1\). Then the grading shift by \(1/2\) of \(X\) is given by \(X \to t^{1/2}1 \star X\).

**Theorem 3.1.** For \(i \in \mathbb{Z}\), there exists a monoidal functor \(\mathcal{I}_i: \mathcal{T}_N \to \mathcal{T}_N\) satisfying

\[
\mathcal{I}_i(L(j)) \simeq \begin{cases} 
\mathcal{D}L(j) & \text{if } j \equiv i \text{ mod } N, \\
t^{1/2}(L(j \mp 1) \nabla L(j)) & \text{if } j \equiv i \pm 1 \text{ mod } N, \\
L(j) & \text{otherwise}.
\end{cases}
\]

The functor \(\mathcal{I}_i\) has an inverse \(\mathcal{I}_i^{-1}: \mathcal{T}_N \to \mathcal{T}_N\) satisfying

\[
\mathcal{I}_i^{-1}(L(j)) \simeq \begin{cases} 
\mathcal{D}^{-1}L(j) & \text{if } j \equiv i \text{ mod } N, \\
t^{1/2}(L(j) \nabla L(j \mp 1)) & \text{if } j \equiv j \pm 1 \text{ mod } N, \\
L(j) & \text{otherwise}.
\end{cases}
\]

Let us explain briefly how to construct the functors \(\mathcal{I}_i\). For each \(j \in J\), denote \(\mathcal{M}_j\) the \(R^\lambda\)-module \(t^{-1}L(j + 1, j + N - 1)\) if \(j \equiv i \text{ mod } N\), \(t^{1/2}(L(j \mp 1) \nabla L(j))\) if \(j \equiv i \pm 1 \text{ mod } N\) and \(L(j)\) otherwise. For each \(\beta \in \mathbb{Q}_+^\times\) and \(\mu = (\mu_1, \ldots, \mu_m) \in J^\beta\),
set

\[ \Delta(\mu) = M_{\mu_1} \circ \cdots \circ M_{\mu_m}, \quad \text{and} \quad \Delta(\beta) = \bigoplus_{\mu \in J^\beta} \Delta(\mu), \]

where \( M_j \) is the affinization of \( \bar{M}_j \). Then along a similar line with [11, Section 4], one can show that there exists a ring homomorphism

\[ (R^{A_{\infty}}(\beta))^{\text{opp}} \rightarrow \text{End}_{\mathcal{A}^{\text{big}}/S_N^{\text{big}}}(Q_N(\Delta(\beta))), \]

where \( \mathcal{A}^{\text{big}}/S_N^{\text{big}} \) is an infinite analogue of \( \mathcal{A}/S_N \). Let \( R'_{\beta}: R^{A_{\infty}}(\beta)-\text{gmod} \rightarrow \mathcal{A}/S_N \) be the restriction of a left adjoint of the functor \( \text{Hom}_{\mathcal{A}^{\text{big}}/S_N^{\text{big}}}(Q_N(\Delta(\beta)), -) \). Then we obtain a monoidal functor \( R: \mathcal{A} \rightarrow T_N \), the composition

\[ \mathcal{A} \xrightarrow{\bigoplus_{\beta \in \mathbb{Q}^+} R'_{\beta}} \mathcal{A}/S_N \xrightarrow{T_N} T_N. \]

Note that the family \( \{\bar{M}_j\}_{j \in J} \) of objects in \( T_N \) satisfies for any \( a \in J \) that (1) \( \bar{M}_a \ast \bar{M}_{a+1} \ast \cdots \ast \bar{M}_{a+N-1} \simeq 1 \), (2) \( \text{hd}(\bar{M}_a \ast \bar{M}_{a+1} \ast \cdots \ast \bar{M}_{a+k-1} \ast \bar{M}_{a+k}) \) is not simple for \( 1 \leq k \leq N-1 \), and (3) \( \mathcal{D}^2(\bar{M}_a) \simeq \bar{M}_{a+N} \). A similar argument as the one in [9, Section 6.1] shows that there is a monoidal functor \( S_i: T_N \rightarrow T_N \) such that \( R \simeq S_i \circ \Omega_N \).

Recall that there is an automorphism \( T: T_N \rightarrow T_N \) given by \( L(j) \mapsto L(j+1) \) for all \( j \in \mathbb{Z} \). It satisfies that \( T^N \simeq \mathcal{D}^2 \). The functors \( \{ T_i \mid i \in \mathbb{Z} \} \) satisfy the following properties.

**Proposition 3.2.** We have

(i) \( T_{i+1} \simeq T \circ T_i \circ T^{-1} \) for \( i \in \mathbb{Z} \),
(ii) \( T_i \circ T \simeq T \circ T_i \) for \( i \in \mathbb{Z} \),
(iii) \( T_i \simeq T_{i+N} \) for \( i \in \mathbb{Z} \),
(iv) \( T_1 T_2 \cdots T_{N-1} \simeq T \),
(v) \( T_i \circ T_j \simeq T_j \circ T_i \) for \( |i-j| > 2 \),
(vi) \( T_i \circ T_{i+1} \circ T_i \simeq T_{i+1} \circ T_i \circ T_{i+1} \) for \( i \in \mathbb{Z} \).

The family of functors \( \{ T_i \}_{1 \leq i \leq N-1} \) recovers the braid group action in Theorem 2.3 in the case of type \( A_{N-1} \).

**Theorem 3.3.** For each \( 1 \leq i \leq N-1 \) the \( \mathbb{Z}[t \pm 1/2] \)-algebra automorphism on \( K(T_N) \) induced by \( T_i \) is equal to \( \sigma_i \) in Theorem 2.3 under the isomorphism

\[ [T_N]: \mathbb{C}(t^{1/2}) \otimes_{\mathbb{Z}[t \pm 1]} K(T_N) \simeq K_t(\mathfrak{g}). \]
4. Reflections by root modules

Recall that for each pair of non-zero modules \((X, Y)\) of \(\mathcal{A}\), there exists a distinguished nonzero morphism \(r_{X,Y}: t^{\Lambda(X,Y)} X \circ Y \to Y \circ X\) called the \(r\)-matrix ([7]). Here, \(t\) is the grading shift functor. We have \(\Omega_N(r_{X,Y}): t^{\Lambda_N(X,Y)} X \star Y \to Y \star X\) in \(T_N\), where \(\Lambda_N(X,Y) = \Lambda(X,Y) - B(\text{wt}(X), \text{wt}(Y)) + B(\text{wt}(Y), \text{wt}(X))\).

For a pair \((X, Y)\) of objects in \(T_N\), set \(d(X, Y) := \frac{1}{2}(\Lambda_N(X, Y) + \Lambda_N(Y, X))\).

Note that \(d(X, Y) = \frac{1}{2}(\Lambda(X, Y) + \Lambda(Y, X))\) if \(\Omega(r_{X,Y}) \neq 0\).

A simple object \(X\) in an abelian monoidal category is called real if \(X \otimes X\) is simple. A real simple object \(L\) in \(T_N\) is called a root module if \(d(L, D_k(L)) = \delta(k = \pm 1)\).

For example, the objects \(L(a, b)\) with \(b - a + 1 < N\) are root modules. If \(L\) is a root module, then \(\mathcal{D}(L), \mathcal{D}^{-1}(L)\) and \(\mathcal{S}_i(L)\) for \(i \in \mathbb{Z}\) are root modules.

The following is the main theorem of this section.

**Theorem 4.1.** Let \(X\) be a simple object in \(T_N\). For \(i \in \mathbb{Z}\), if 

\[ \mathfrak{b}(\mathcal{D}^k(L(i)), X) = n\delta(k = a) \]

for some \(n \geq 0\) and \(a \in \mathbb{Z}\), then

\[ \mathcal{S}_i(X) \simeq (\mathcal{D}^a L(i))^{\otimes n} \nabla X \]

up to a multiple of a power of \(t\).

The following is one of the applications of Theorem 4.1.

**Theorem 4.2.** Let \([L]\) belongs to the orbit of \(L(i)\) for some \(1 \leq i \leq N - 1\) under the braid group \(B(A_{N-1})\) action in Theorem 2.3. Then there is an automorphism \(s_L\) on \(K(T_N)\) such that

(i) \(s_{L(i)} = s_i\) for \(1 \leq i \leq N - 1\).
(ii) \(s_{(s_L(L'))} = s_L \circ s_L' \circ s_L^{-1}\) if \(L'\) also satisfies the condition in the theorem.
(iii) \(s_{\mathcal{D}^a L} = s_L\) for all \(a \in \mathbb{Z}\)
(iv) \(s_L([X]) = [([\mathcal{D}^a L]^{\otimes n} \nabla X)\) up to a power of \(t\), if \(\mathfrak{b}(\mathcal{D}^k L, X) = n\delta(k = a)\) for some \(n \geq 0\) and \(a \in \mathbb{Z}\).

5. Conjectures

Let \(U'_q(g)\) be an arbitrary quantum affine algebra. We say that a real simple module \(L\) in \(\mathcal{G}_g^0\) is a root module if \(\mathfrak{b}(\mathcal{D}^k M, M) = \delta(k = \pm 1)\) for any \(k\).
Conjecture. For any root module $L \in C_0^0$, there exists a monoidal autofunctor $\mathcal{S}_L$ of $C_0^0$ which satisfies the following conditions:

(a) $\mathcal{S}_L$ satisfies similar properties in Theorem 4.1 and Theorem 4.2.

(b) (Braid relation) For root modules $L$ and $L'$,

1. if $\delta(D^kL, L') = 0$ for any $k \in \mathbb{Z}$, then
$$\mathcal{S}_L \circ \mathcal{S}_L' \simeq \mathcal{S}_{L'} \circ \mathcal{S}_L,$$

2. if $\delta(D^kL, L') = \delta(k = 0)$ for any $k \in \mathbb{Z}$, then
$$\mathcal{S}_L \circ \mathcal{S}_L' \circ \mathcal{S}_L \simeq \mathcal{S}_{L'} \circ \mathcal{S}_L \circ \mathcal{S}_L'.$$

(c) Let $Q$ be a Q-data, and $L := F_Q(L(i))$. Then the automorphism of $\mathcal{K}_i(\mathfrak{g})$ induced by $\mathcal{S}_L$ coincides with $\sigma_i$, i.e., the following diagram commutes:

\[
\begin{array}{ccc}
\mathcal{K}_i(\mathfrak{g}_0) & \xrightarrow{\sim} & \mathcal{K}_i(\mathfrak{g}) \\
\sigma_i \downarrow & & \downarrow \mathcal{S}_L \\
\mathcal{K}_i(\mathfrak{g}_0) & \xrightarrow{\sim} & \mathcal{K}_i(\mathfrak{g}).
\end{array}
\]

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