THREE CONSECUTIVE ALMOST SQUARES

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ABSTRACT. Given a positive integer \( n \), we let \( \text{sfp}(n) \) denote the squarefree part of \( n \). We determine all positive integers \( n \) for which \( \max\{\text{sfp}(n), \text{sfp}(n+1), \text{sfp}(n+2)\} \leq 150 \) by relating the problem to finding integral points on elliptic curves. We also prove that there are infinitely many \( n \) for which

\[
\max\{\text{sfp}(n), \text{sfp}(n+1), \text{sfp}(n+2)\} < n^{1/3}.
\]

1. Introduction

The positive integers 48, 49 and 50 are consecutive, and \( 48 = 3 \cdot 4^2 \), \( 49 = 7^2 \) and \( 50 = 2 \cdot 5^2 \). Does this phenomena ever occur again? That is, is there a positive integer \( n > 48 \) for which

\[
\begin{align*}
n &= 3x^2 \\
n + 1 &= y^2 \\
n + 2 &= 2z^2
\end{align*}
\]

has an integer solution? The answer is no. One perspective on this is given by Cohen in Section 12.8.2 of [3], where this problem is solved using linear forms in logarithms. Another approach is to recognize that the pair of equations \( 2z^2 - y^2 = y^2 - 3x^2 = 1 \) define an intersection of two quadrics in \( \mathbb{P}^3 \) and hence define a curve of genus 1. Siegel proved that there are only finitely many integer points on a genus 1 curve, and hence the system of equations above has only finitely many solutions. This method of solving simultaneous Pell equations is considered in [15].

For a positive integer \( n \), define \( \text{sfp}(n) \) to be the “squarefree part” of \( n \), the smallest positive integer \( a \) so that \( a|n \) and \( \frac{n}{a} \) is a perfect square. In this paper, we consider positive integers \( n \) for which \( \text{sfp}(n) \), \( \text{sfp}(n+1) \) and \( \text{sfp}(n+2) \) are all small. Our first result is a classification of values of \( n \) for which \( \text{sfp}(n) \), \( \text{sfp}(n+1) \) and \( \text{sfp}(n+2) \) are all \( \leq 150 \). We say that such an \( n \) is “non-trivial” if \( \text{sfp}(n) < n \), \( \text{sfp}(n+1) < n + 1 \) and \( \text{sfp}(n+2) < n + 2 \).

Theorem 1. There are exactly 25 non-trivial \( n \) for which \( \text{sfp}(n) \leq 150 \), \( \text{sfp}(n+1) \leq 150 \) and \( \text{sfp}(n+2) \leq 150 \), the largest of which is \( n = 9841094 \).

Remark. A table of all 25 values of \( n \) is given in Section 5.
Given positive integers $a$, $b$ and $c$, a positive integer solution to $n = ax^2$, $n+1 = by^2$, $n+2 = cz^2$ gives a solution to the system

$$C : by^2 - ax^2 = 1, \quad cz^2 - by^2 = 1.$$ 

In [2], Cipu and Mignotte show using bounds coming from linear forms in logarithms that a related system,

$$ax^2 - cz^2 = 1 \quad by^2 - dz^2 = 1$$

has at most two solutions $(x, y, z)$ in positive integers. Another approach to effectively finding solutions to this system uses the theory of elliptic curves. In particular, the Jacobian of $C$ is isomorphic to $E$:

$$y^2 = x^3 - (abc)^2 x.$$ 

We rule out many of the 778688 candidates for the triple $(a, b, c)$ by checking to see whether there are integer solutions each of the three equations $by^2 - ax^2 = 1$, $cz^2 - by^2 = 1$ and $cz^2 - ax^2 = 2$. We also test $C$ for local solvability, and use Tunnell’s theorem (see [14]) to determine if the rank of $E$ is positive. Finally, we use the surprising property that the natural map from $C$ to $E$ sends an integral solution on $C$ to an integral point on $E$. It suffices therefore to compute all the integral points on $E$, which requires computing generators of the Mordell-Weil group. In many cases this is straightforward, but a number of cases require more involved methods (12-descent, computing the analytic rank). In one very difficult case, we cannot compute the Mordell-Weil group and in this case, we use the theory of linear forms in logarithms to show the system has no solutions.

Given the existence of large solutions, it is natural to ask how large

$$\max\{\text{sfp}(n), \text{sfp}(n+1), \text{sfp}(n+2)\}$$

can be as a function of $n$. This is the subject of our next result.

**Theorem 2.** There are infinitely many positive integers $n$ for which

$$\max\{\text{sfp}(n), \text{sfp}(n+1), \text{sfp}(n+2)\} < n^{1/3}.$$ 

**Remark.** The following heuristic suggests that the exponent $1/3$ above is optimal. A partial summation argument shows that the number of positive integers $n \leq x$ with $\text{sfp}(n) \leq z$ is

$$\frac{12\sqrt{xz}}{\pi^2} + O(\sqrt{x} \log(z)).$$

Assuming that $n$, $n+1$ and $n+2$ are “random” integers it follows that $\text{sfp}(n), \text{sfp}(n+1)$ and $\text{sfp}(n+2)$ are all $\leq n^{\alpha}$ with probability about $n^{3(\alpha - 1)/2}$. Therefore, the “expected” number of $n$ for which $\max\{\text{sfp}(n), \text{sfp}(n+1), \text{sfp}(n+2)\} \leq n^{\alpha}$ is infinite if $3(\alpha - 1)/2 \geq -1$ and finite otherwise.

2. **Background**

We denote by $\mathbb{Q}_p$ the field of $p$-adic numbers, respectively. A necessary condition for a curve $X/\mathbb{Q}$ to have a rational solution is for it to have such a solution in $\mathbb{Q}_p$ for all primes $p$.

If $d$ is an integer, let $\mathbb{Z}[\sqrt{d}] = \{a + b\sqrt{d} : a, b \in \mathbb{Z}\}$. This is a (not necessarily maximal) order in the quadratic field $\mathbb{Q}[\sqrt{d}]$. Let $N : \mathbb{Z}[\sqrt{d}] \to \mathbb{Z}$ be the norm map given by $N(a + b\sqrt{d}) = a^2 - db^2$. 

We now describe some background about elliptic curves. An elliptic curve is a curve of the shape

\[ E : y^2 = x^3 + ax + b \]

where \(a, b \in \mathbb{Q}\). Let \(E(\mathbb{Q})\) be the set of pairs \((x, y)\) of rationals numbers that solve the equation, together with the “point at infinity”. This set has a binary operation on it: given \(P, Q \in E(\mathbb{Q})\), the line \(L\) through \(P\) and \(Q\) intersects \(E\) in a third point \(R = (x, y)\). The point \(P + Q\) is defined to be \((x, -y)\). This binary operation endows \(E(\mathbb{Q})\) with the structure of an abelian group.

The Mordell-Weil theorem (see [13], Theorem VIII.4.1, for example) states the following.

**Theorem 3.** For any elliptic curve \(E/\mathbb{Q}\), \(E(\mathbb{Q})\) is finitely generated. More precisely,

\[ E(\mathbb{Q}) \cong E(\mathbb{Q})_{\text{tors}} \times \mathbb{Z}^{\text{rank}(E(\mathbb{Q}))} , \]

where \(E(\mathbb{Q})_{\text{tors}}\) is the (finite) torsion subgroup.

There is in general no algorithm which is proven to compute the rank of \(E\), (see Section 3 of Rubin and Silverberg’s paper [12]) but there are a number of procedures which work well in practice for relatively simple curves \(E\). We will reduce the problem of solving \(n = ax^2, n + 1 = by^2\) and \(n + 2 = cz^2\) to finding points \((X, Y)\) on the curve

\[ E : Y^2 = X^3 - (abc)^2 X, \]

with \(X, Y \in \mathbb{Z}\). A theorem of Siegel (see Theorem IX.4.3 of [13]) states that there are only finitely many points in \(E(\mathbb{Q})\) with both coordinates integral. There are effective and practical algorithms (see [6]) to determine the set of integral points, provided the rank \(r\) can be computed and a set of generators for \(E(\mathbb{Q})\) found. Given a point \(P = (x, y) \in E(\mathbb{Q})\), the “naive height” of \(P\) is defined by writing \(x = \frac{a}{b}\) with \(a, b \in \mathbb{Z}\) with \(\gcd(a, b) = 1\) and defining \(h(P) = \log \max\{|a|, |b|\}\). The “canonical height” of \(P\) is defined to be

\[ \hat{h}(P) = \lim_{n \to \infty} \frac{h(2^n P)}{4^n} . \]

The \(L\)-function of \(E/\mathbb{Q}\) is defined to be

\[ L(E, s) = \sum_{n=1}^{\infty} \frac{a_n(E)}{n^s} = \prod_p \left( 1 - a_p(E)p^{-s} + p^{1-2s} \right)^{-1} \]

where \(a_p(E) = p + 1 - \#E(\mathbb{F}_p)\) provided \(p\) does not divide the conductor \(N\) of \(E\). The Birch and Swinnerton-Dyer conjecture states that \(\text{ord}_{s=1} L(E, s) = \text{rank}(E(\mathbb{Q}))\), and moreover that

\[ \lim_{s \to 1} (s-1)^r L(E, s) = \frac{\Omega R(E/\mathbb{Q}) \Pi(E/\mathbb{Q}) \prod_p c_p}{|T|^2} , \]

where \(\Omega\) is twice the real period, \(R(E/\mathbb{Q})\) is the regulator of \(E(\mathbb{Q})\) computed using the function \(\hat{h}\) above, the \(c_p\) are the Tamagawa numbers, and \(\Pi(E/\mathbb{Q})\) is the Shafarevich-Tate group.
The best partial result in the direction of the Birch and Swinnerton-Dyer conjecture is the following.

**Theorem 4** (Gross-Zagier [7], Kolyvagin [9], et al.). Suppose that \( E/\mathbb{Q} \) is an elliptic curve and \( \text{ord}_{s=1} L(E, s) = 0 \) or 1. Then, \( \text{ord}_{s=1} L(E, s) = \text{rank}(E(\mathbb{Q})) \).

The work of Bump-Friedberg-Hoffstein [1] or Murty-Murty [11] is necessary to remove a condition imposed in the work of Gross-Zagier and Kolyvagin.

We next will state Matveev’s theorem giving lower bounds for linear forms in logarithms (see [10] or Theorem 12.1.2 of [3]). Let \( L/\mathbb{Q} \) be a number field of degree \( D \), \( \alpha_1, \ldots, \alpha_n \) be nonzero elements of \( L \) and \( b_1, \ldots, b_n \) be integers. Set \( B = \max\{|b_1|, |b_2|, \ldots, |b_n|\} \) and

\[
\Lambda^* = \alpha_1^{b_1} \alpha_2^{b_2} \cdots \alpha_n^{b_n} - 1.
\]

The absolute logarithmic height of \( \alpha \in L^\times \) is defined in terms of the minimal polynomial \( f(x) = \sum_{k=0}^d a_k x^k \in \mathbb{Z}[x] \) with \( a_d \neq 0 \). It is defined to be

\[
h(\alpha) = \frac{1}{d} \left( \log(|a_d|) + \sum_{i=1}^d \max(\log(|\alpha_i|), 0) \right),
\]

where the \( \alpha_i \) are the roots of \( f(x) \). Define the modified height to be \( h'(\alpha) = \max\{Dh(\alpha), |\log \alpha|, 0.16\} \).

**Theorem 5.** Assume the notation above, that \( L/\mathbb{Q} \) is totally real, and that \( \Lambda^* \) is nonzero. Then

\[
\log |\Lambda^*| > -1.4 \cdot 30^{n+3} n^{4.5} D^2 h'(\alpha_1) h'(\alpha_2) \cdots h'(\alpha_n) (1 + \log D)(1 + \log B).
\]

We will apply this theorem with \( n = 3 \) and with \( L/\mathbb{Q} \) a biquadratic extension.

### 3. Proof of Theorem 5

**Proof.** Motivated by the observation that 8388223 = 127 \cdot 257^2 and 8388225 = 129 \cdot 255^2, we find a parametric family of solutions where \( n = a \cdot b^2 \) and \( n + 2 = (a + 2) \cdot (b - 2)^2 \).

If we write \((4 + \sqrt{13})(649 + 180\sqrt{13})^m = x_m + y_m \sqrt{13}\) where \( x_m \) and \( y_m \) are integers, then \( x_m^2 - 13y_m^2 = 3 \) for all \( m \geq 0 \). We have that \( x_0 = 4, x_1 = 4936, y_0 = 1, y_1 = 1369 \) and

\[
x_m = 1298x_{m-1} - x_{m-2}, y_m = 1298y_{m-1} - y_{m-2}.
\]

It is easy to see that \( x_m \equiv x_{m-1} + 4 \pmod{32} \) and hence \( x_{8m+7} \equiv 0 \pmod{32} \) for all \( m \geq 0 \). Set \( a_m = x_{8m+7}/2 \) and \( n = 4a_m^3 - 3a_m - 1 \). Then we have

\[
n = (a_m - 1)(2a_m + 1)^2,
\]

\[
n + 1 = a_m(4a_m^2 - 3) = 13a_m y_{8m+7}, \text{ and}
\]

\[
n + 2 = (a_m + 1)(2a_m - 1)^2.
\]
Since $16|a_m$, $\max\{sfp(n), sfp(n+1), sfp(n+2)\} \leq \max\{a_m-1, \frac{13a_m}{16}, a_m+1\} = a_m+1 < n^{1/3}$. □

Remark. The polynomial $4x^3 - 3x$ used in the proof above is the Chebyshev polynomial $T_3(x)$. This explains why $T_3(x) - 1$ and $T_3(x) + 1$ both have a double zero.

4. Proof of Theorem [1]

Given that there are $778688 = 92^3$ possibilities for the triple $(a, b, c)$ we first test four things before searching for integral points on $E : y^2 = x^3 - (abc)^2x$. Suppose that $n = ax^2$, $n + 1 = by^2$ and $n + 2 = cz^2$ is an integral solution to the system of equations

(1) \hspace{1cm} by^2 - ax^2 = 1,
(2) \hspace{1cm} cz^2 - by^2 = 1,
(3) \hspace{1cm} cz^2 - ax^2 = 2.

4.1. Greatest common divisor conditions. If $(x, y, z)$ is an integer solution to (1), (2) and (3), then $\gcd(a, b) = 1$, $\gcd(b, c) = 1$ and $\gcd(a, c) = 1$ or 2. This reduces the number of triples to consider to 425639.

4.2. Norm equations. We next check that each of the equations $by^2 - ax^2 = 1$, $cz^2 - by^2 = 1$, and $cz^2 - ax^2 = 2$ has an integer solution. The equation $by^2 - ax^2 = 1$ has an integer solution if and only if $Y^2 - abx^2 = b$ has an integer solution. This equation has a solution if and only if $\mathbb{Z}[\sqrt{ab}]$ has an element of norm $b$. We use Magma’s routine NormEquation to test this. After these tests have been made, there are 2188 possibilities for $(a, b, c)$ that remain.

4.3. Local solvability of $C$. Let $C \subseteq \mathbb{P}^3$ be the curve defined by the two equations $by^2 - ax^2 = w^2$, $cz^2 - by^2 = w^2$. This curve must have a rational point on it in order for there to be a non-trivial solution. We verify that $C$ has points $(x : y : z : w)$ in $\mathbb{Q}_p$ for all primes $p$ dividing $2abc$. This is done via Magma’s IsLocallySolvable routine. This eliminates 244 possibilities.

4.4. Rank of the elliptic curve $E$. Let $E : y^2 = x^3 - (abc)^2x$. Define a map $M$ from non-trivial solutions to $C$, represented in the form $n = ax^2$, $n + 1 = by^2$ and $n + 2 = cz^2$ to $E$, given by

$M(x, y, z, n) = ((n + 1)(abc), (abc)^2xyz)$.

If $n > 0$, then $xyz > 0$ and so $M(x, y, z, n) \in E(\mathbb{Q})$ is an integral point, and one with a non-zero $y$-coordinate. We now use the following results about the family of elliptic curves $E$ we consider.

Definition 6. A natural number $N$ is called congruent if there exists a right triangle with all three sides rational and area $N$. 

Consider the elliptic curve $E$ over $\mathbb{Q}$ given by:
$$E(\mathbb{Q}) : y^2 = x^3 - N^2 x.$$  

We have the following result.

**Theorem 7** (Proposition I.9.18 of [8]). *The number $N$ is congruent if and only if the rank of $E$ is positive.*

Proposition I.9.17 of [8] implies that the only points of finite order on $E$ are $(0, 0)$, $(\pm N, 0)$ and the point at infinity. This fact shows that $M(x, y, z, n)$ must not be in $E(\mathbb{Q})_{\text{tors}}$ and so $M(x, y, z, n)$ must be a point of infinite order. Hence the rank of $E$ is positive.

**Theorem 8** (Tunnell 1983). *If $N$ is squarefree and odd, and*

$$\# \left\{ x, y, z \in \mathbb{Z} \mid N = 2x^2 + y^2 + 32z^2 \right\} \neq \frac{1}{2} \# \left\{ x, y, z \in \mathbb{Z} \mid N = 2x^2 + y^2 + 8z^2 \right\},$$

*then $N$ is not congruent.* *If $N$ is squarefree and even, and*

$$\# \left\{ x, y, z \in \mathbb{Z} : \frac{N}{2} = 4x^2 + y^2 + 32z^2 \right\} \neq \frac{1}{2} \# \left\{ x, y, z \in \mathbb{Z} : \frac{N}{2} = 4x^2 + y^2 + 8z^2 \right\},$$

*then $N$ is not congruent.*

To use Tunnell’s theorem, we compute the generating function for the number of representations of $N$ by $2x^2 + y^2 + 8z^2$ as

$$\sum_{x,y,z \in \mathbb{Z}} q^{2x^2+y^2+8z^2} = \left( \sum_{x \in \mathbb{Z}} q^{2x^2} \right) \left( \sum_{y \in \mathbb{Z}} q^{y^2} \right) \left( \sum_{z \in \mathbb{Z}} q^{8z^2} \right),$$

as well as the representations of $N$ by $2x^2 + y^2 + 32z^2$, $4x^2 + y^2 + 8z^2$ and $4x^2 + y^2 + 32z^2$.

The use of Tunnell’s theorem rules out 530 of the 1944 remaining cases. In the case that the hypothesis of Tunnell’s theorem is false, the Birch and Swinnerton-Dyer conjecture predicts that $E(\mathbb{Q})$ does have positive rank. In this case, we proceed to the next step.

4.5. **Computing integral points.** Once the program determines that the elliptic curve $E(\mathbb{Q}) : Y^2 = X^3 - (abc)^2 X$ has positive rank, we let Magma attempt to compute the integral points on the curve. In 1377 of the 1414 cases that remain, Magma is able to determine the Mordell-Weil group and determine all of the integral points within 15 minutes. Once the integral points are determined, we check to see if they are in the image of the map $M(x, y, z, n)$ and if so, whether they correspond to a non-trivial solution.

Of the remaining 37 cases, there 32 are curves with root number $-1$ of rank $\leq 1$ for which Magma was not able to find a point of infinite order. There are 4 cases of curves with root number $-1$ of rank $\leq 3$ for which one point of infinite order is known. Finally, there is one case of a curve with root number $1$, rank $\leq 2$ where one point of infinite order is known.
For the four cases of curves with root number $-1$ and rank $\leq 3$, we numerically compute $L'(E,1)$ and show that it is nonzero. Theorem 4 proves the rank is one in these cases. Using the Saturation command, we are able to determine generators for the Mordell-Weil group. One of these cases is $a = 139$, $b = 89$ and $c = 109$. In this case, $E$ has conductor $\approx 5.82 \cdot 10^{13}$ and computing $L'(E,1)$ takes about 5 hours.

In 28 cases, we are able to use the 12-descent routines in Magma (due to Tom Fisher [5]) to find points of large height. The most difficult case is the rank one curve where $a = 139$, $b = 107$ and $c = 101$. The curve $E$ has conductor $\approx 7.22 \cdot 10^{13}$ and finding a point on one of the 12-covers takes 90 minutes and a generator on $E$ has canonical height 1234.

4.6. Linear forms in logarithms. All but one of the 778688 original cases are handled by the methods of the previous sections. The remaining case is $a = 67$, $b = 131$ and $c = 109$. The curve $E$ has root number $-1$, rank $\leq 1$ and conductor $\approx 2.9 \cdot 10^{13}$. A long computation shows that $L'(E,1) \approx 72.604$. This would imply, assuming the Birch and Swinnerton-Dyer conjecture is true and $\text{III}(E/\mathbb{Q})$ is trivial, that a generator of the Mordell-Weil group has canonical height about 1692. Searching for points on the 12-covers up to a height of $3^{12} \cdot 10^5 \approx 10^{28}$ does not succeed in finding points.

Hence, we will approach this one case using linear forms in logarithms. We seek solutions to

$$131y^2 - 67x^2 = 1$$

and

$$109z^2 - 131y^2 = 1.$$  

Let $X = 131y$, $Y = x$ and $Z = z$. With this change of variables, we have

$$X^2 - 67 \cdot 131Y^2 = X^2 - 8777Y^2 = 131$$

and

$$X^2 - 109 \cdot 131Z^2 = X^2 - 14279Y^2 = -131.$$  

If $K_1 = \mathbb{Q}(\sqrt{8777})$, then $\mathcal{O}_{K_1}$ has class number 1 and the unique ideal above 131 in $\mathcal{O}_{K_1}$ is principal and is generated by $\gamma = 37746602 + 402907\sqrt{8777}$, which has norm 131. If $\epsilon$ is a fundamental unit, then every element in $\mathcal{O}_{K_1}$ with norm 131 has the form $\pm \gamma \epsilon^n$ for some integer $n$.

Setting $K_2 = \mathbb{Q}(\sqrt{14279})$, we have that $\mathcal{O}_{K_2}$ has class number 2 and the unique ideal above 131 is generated by $\delta = 67754549562 + 567008495\sqrt{14279}$, which has norm $-131$. If $\eta$ is a fundamental unit in $\mathcal{O}_{K_2}$ then every element in $\mathcal{O}_{K_2}$ with norm $-131$ is of the form $\pm \delta \eta^m$ for some integer $m$.

If there is a solution of (1), (2) and (3) with $a = 67$, $b = 131$ and $c = 109$, then $\gamma \epsilon^n + \overline{\gamma} \epsilon^{-n} = \delta \eta^m + \overline{\delta} \eta^{-m}$ and so $\gamma \epsilon^n - \delta \eta^m = -\overline{\gamma} \epsilon^{-n} + \overline{\delta} \eta^{-m}$. Note that $\overline{\gamma} > 0$ and $\overline{\delta} < 0$ and so the right hand side is negative and we have

$$\gamma \epsilon^n - \delta \eta^m \geq -C \max\{\epsilon^{-n}, \eta^{-m}\},$$
where $C = 1.74 \cdot 10^{-6}$. Let $\Lambda^* = (\gamma/\delta) \epsilon^m \eta - 1$ and $\Lambda = \log(\gamma/\delta) + n \log(\epsilon) - m \log(\eta)$. Lemma 12.8.2 of [3] gives a bound on both $\Lambda^*$ and $\Lambda$ in this case. In particular,

$$|\Lambda^*| \leq C \max \left( \frac{1}{\delta}, \frac{2}{\gamma} \right) \eta^{-2m}$$

and $|\Lambda| \leq 2C \log(2) \max \left( \frac{1}{\delta}, \frac{2}{\gamma} \right) \eta^{-2m}$. Applying Theorem 5 with $L = \mathbb{Q}(\sqrt{8777}, \sqrt{14279})$, we have that

$$\log |\Lambda^*| > -1.478 \cdot 10^{19}(1 + \log(\max(m, n))).$$

Simple inequalities show that $n < \frac{m \log(\eta) + \log(\delta/\gamma)}{\log(\epsilon)} < 1.477186m + 7.5$. Combining this with (4) gives

$$-1.478 \cdot 10^{19}(1 + \log(1.477186m + 7.5)) < -2m \log(\eta) - 30.7.$$  

This proves that $m < 7.2 \cdot 10^{18}$. We now apply a version of the Baker-Davenport lemma due to Dujella and Pethő (see Lemma 5 of [4]). This result states the following.

**Lemma 9.** Let $N$ be a positive integer and $\kappa, \mu \in \mathbb{R}$. Let $\frac{p}{q}$ be a convergent of the continued fraction of $\kappa$ such that $q > 6N$. Put $\tau := \|\mu q\| - N\|\kappa q\|$, where $\| \cdot \|$ denotes the distance from the nearest integer. If $\tau > 0$, the inequality

$$0 < m\kappa - n + \mu < AB^{-m}$$

where $A > 0$, $B > 1$ are real numbers, has no solutions in the range

$$\log \left( \frac{4q}{\tau} \right) \leq m < N.$$  

Using the bounds derived above on $\Lambda$ and setting $\kappa = \frac{\log(\eta)}{\log(\epsilon)}$ and $\mu = \frac{\log(\delta/\gamma)}{\log(\epsilon)}$ we obtain that

$$0 < m\kappa - n + \mu < AB^{-m}$$

where

$$A = \frac{2C \log(2) \max \left( \frac{1}{\delta}, \frac{2}{\gamma} \right) \eta^{-2m}}{\log(\epsilon)}, B = \eta^2.$$ 

We have that $\kappa \approx 1.4771859$ and when we apply the Lemma with $p = 27067703202482171484$ and $q = 183238298295734633107$ we find that there are no solutions with $0 < m < 7.2 \cdot 10^{18}$ and hence there no solutions at all. This completes the proof of Theorem 1.

5. **Table of $n$**

The following is a table of all 25 positive integers $n$ with $\text{sfp}(k) < \max\{k, 150\}$ for $k = n, n + 1$ and $n + 2$. 

| $n$ | $\text{SFP}(n)$ |
|-----|----------------|
| 1 | 1 |
| 2 | 2 |
| 3 | 3 |
| 4 | 4 |
| 5 | 5 |
| 6 | 6 |
| 7 | 7 |
| 8 | 8 |
| 9 | 9 |
| 10 | 10 |
| 11 | 11 |
| 12 | 12 |
| 13 | 13 |
| 14 | 14 |
| 15 | 15 |
| 16 | 16 |
| 17 | 17 |
| 18 | 18 |
| 19 | 19 |
| 20 | 20 |
| 21 | 21 |
| 22 | 22 |
| 23 | 23 |
| 24 | 24 |

$p = 27067703202482171484$ and $q = 183238298295734633107$.
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\begin{align*}
\begin{array}{cccc}
 n & \text{sfp}(n) & \text{sfp}(n+1) & \text{sfp}(n+2) \\
 48 & 3 & 1 & 2 \\
 98 & 2 & 11 & 1 \\
 124 & 31 & 5 & 14 \\
 242 & 2 & 3 & 61 \\
 243 & 3 & 61 & 5 \\
 342 & 38 & 7 & 86 \\
 350 & 14 & 39 & 22 \\
 423 & 47 & 106 & 17 \\
 475 & 19 & 119 & 53 \\
 548 & 137 & 61 & 22 \\
 845 & 5 & 94 & 7 \\
 846 & 94 & 2 & 53 \\
 1024 & 1 & 41 & 114 \\
 1375 & 55 & 86 & 17 \\
 1519 & 31 & 95 & 1 \\
 1680 & 105 & 1 & 2 \\
 3724 & 19 & 149 & 46 \\
 9800 & 2 & 1 & 58 \\
 31211 & 59 & 3 & 13 \\
 32798 & 62 & 39 & 82 \\
 118579 & 19 & 5 & 141 \\
 629693 & 53 & 46 & 55 \\
 1294298 & 122 & 19 & 7 \\
 8388223 & 127 & 26 & 129 \\
 9841094 & 134 & 55 & 34 \\
\end{array}
\end{align*}

References

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