LAGRANGIAN F-STABILITY OF CLOSED LAGRANGIAN SELF-SHRINKERS

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Abstract. In this paper, we study the Lagrangian F-stability of closed Lagrangian self-shrinkers immersed in complex Euclidean space. We show that any closed Lagrangian self-shrinker with first Betti number greater than one is Lagrangian F-unstable. In particular, any two-dimensional embedded closed Lagrangian self-shrinker is Lagrangian F-unstable. For a closed Lagrangian self-shrinker with first Betti number equal to one, we show that Lagrangian F-stability is equivalent to Hamiltonian F-stability. We also characterize Hamiltonian F-stability of a closed Lagrangian self-shrinker by its spectral property of the twisted Laplacian.

1. Introduction

By a self-shrinker, we mean an immersed surface in Euclidean space whose mean curvature vector is related to the normal part of the position vector by

\[ H = -c(x - x_0)_{\perp}, \quad c > 0. \]

A self-shrinker gives rise to a homothetically shrinking solution to the mean curvature flow. The geometric object becomes important since Huisken’s monotonicity formula [18] tells that any time-slice of a tangent flow (cf. Section 2) at a Type-I singularity is a self-shrinker. The tangent flow at a general singularity is a homothetically shrinking weak solution of the mean curvature flow (Brakke flow [5]), see [20, 41].

Abresch and Langer classified all immersed closed self-shrinkers in the plane. However even for the case of two dimensional self-shrinkers in \( \mathbb{R}^3 \), various examples are expected and a complete classification seems impossible, see for instance [3]. Besides using point-wise conditions, for example mean convexity [18, 19], recently Colding and Minicozzi [14] employing a new kind stability to classify self-shrinkers. That is the entropy-stability for self-shrinkers.

For an \( n \)-dimensional immersed surface \( \Sigma \hookrightarrow \mathbb{R}^N \), Colding and Minicozzi [14] introduced the entropy of \( \Sigma \) by

\[ \lambda (\Sigma) = \sup_{x_0 \in \mathbb{R}^N, t_0 > 0} \int_{\Sigma} (4\pi t_0)^{-\frac{n}{2}} e^{-\frac{|x - x_0|^2}{4t_0}} d\mu. \]

The entropy \( \lambda \) has very nice properties. For example, it is invariant under dilations and rigid motions.

A self-shrinker is called entropy-stable if it is a local minimum of the entropy functional. The entropy-stability is closely related to the singular behavior of the mean curvature flow, based on the fact that \( \lambda \) is non-increasing along the mean curvature flow in Euclidean space.
space. In fact entropy-stable self-shrinkers are considered as generic singularities of mean curvature flow in [14], i.e. those cannot be perturbed away.

In order to classify entropy-stable self-shrinkers, Colding and Minicozzi [14] introduced the notion of F-stability for self-shrinkers. The F-functional with respect to $x_0 \in \mathbb{R}^N, t_0 > 0$ of an immersed surface $\Sigma$ is defined by

\begin{equation}
F_{x_0, t_0}(\Sigma) = \int_{\Sigma} (4\pi t_0)^{-\frac{n}{2}} e^{-\frac{|x-x_0|^2}{4t_0}} d\mu.
\end{equation}

Note that the entropy $\lambda(\Sigma)$ is the supremum of the F-functional taken over all $x_0 \in \mathbb{R}^N, t_0 > 0$. A critical point of $F_{x_0, t_0}$ is self-shrinkers that becomes extinct at $(x_0, t_0)$ (cf. Section 2), and such a self-shrinker $\Sigma$ is called F-stable if for any 1-parameter family of deformations $\Sigma_s$ of $\Sigma = \Sigma_0$, there exist deformations $x_s$ of $x_0$ and $t_s$ of $t_0$ such that $(F_{x_s, t_s}(\Sigma_s))'' \geq 0$ at $s = 0$. Roughly speaking, a self-shrinker is critical point of the entropy functional, and an F-stable self-shrinker is a self-shrinker at which the second variation of entropy $\lambda$ is non-negative.

The F-stability are closely related to entropy-stability, and the classification of entropy-stable self-shrinkers relies on the classification of F-stable self-shrinkers. Colding and Minicozzi [14] showed that shrinking spheres, cylinders and planes are the only codimension one F-stable (equivalently, entropy-stable) self-shrinkers.

In this paper we carry over some of Colding and Minicozzi’s ideas to the Lagrangian mean curvature flow case. In particular we shall explore the Lagrangian F-stability of closed Lagrangian self-shrinkers. We assume our Lagrangian self-shrinkers are closed, orientable and have dimensions greater or equal to two.

An immersed surface $\Sigma^n$ in $\mathbb{C}^n$ is called Lagrangian if the standard Kähler form $\omega$ of $\mathbb{C}^n$ restricted to $\Sigma$ is zero, or equivalently the standard complex structure of $\mathbb{C}^n$ maps any tangent vector of $\Sigma$ to a normal vector. When the ambient space is released to Kähler-Einstein manifolds, Smoczyk [31] showed that along the mean curvature flow the Lagrangian condition is preserved, i.e. a Lagrangian mean curvature flow. The Lagrangian mean curvature flow was devised mainly for searching minimal Lagrangian submanifolds in a Calabi-Yau manifold, and becomes one of main tools in understanding Strominger-Yau-Zaslow’s conjecture in Mirror symmetry [36] and Thomas-Yau’s conjecture [37]; See for instance [11, 12, 27, 28, 33, 39]. For recent developments of the Lagrangian mean curvature flow in Kähler-Einstein manifolds, see for instance survey papers [26, 32, 40] and the references therein.

A Lagrangian self-shrinker by definition is a self-shrinker satisfying the Lagrangian condition. Type-I singularities along the Lagrangian mean curvature flow are modeled by self-shrinkers. The study of Lagrangian self-shrinkers has draw some attentions recently. In particular, many compact or noncompact examples are found, see for instance [1, 9, 10, 21, 23, 24].

The F-functional and the entropy of a Lagrangian surface $\Sigma$ are given by (1.3) and (1.2) respectively. However if we stick to the Lagrangian mean curvature flow, admissible deformations are Lagrangian deformations, i.e. those preserve the Lagrangian condition. A normal vector field is called a Lagrangian variation if it is an infinitesimal Lagrangian deformation. There is a correspondence between Lagrangian variations and closed 1-forms via $X \leftrightarrow \theta := -i_X \omega$. If $X$ is a normal vector field such that $-i_X \omega$ is exact, we call $X$ a Hamiltonian variation.
When restricted to Lagrangian deformations, a critical point of $F_{x_0,t_0}$ is a Lagrangian self-shrinker that becomes extinct at $(x_0,t_0)$ and a Lagrangian self-shrinker is a critical point of the entropy functional. We call a Lagrangian self-shrinker $\Sigma$ Lagrangian entropy-stable if it is a local minimum of the entropy under Lagrangian deformations, and a Lagrangian self-shrinker $\Sigma$ is called Lagrangian (resp. Hamiltonian) F-stable if for any 1-parameter family of Lagrangian (resp. Hamiltonian) deformations $\Sigma_s$ of $\Sigma = \Sigma_0$, there exist deformations $x_s$ of $x_0$ and $t_s$ of $t_0$ such that $(F_{x_s,t_s}(\Sigma_s))'' \geq 0$ at $s = 0$.

Using the correspondence between Lagrangian variations and closed 1-forms, we can rewrite the second variation of the F-functional in terms of closed 1-form. This correspondence was used by Oh [29] in studying the Lagrangian stability of minimal Lagrangian submanifold in Kähler manifold. It was first observed by Smoczyk that any closed Lagrangian self-shrinker has nontrivial Maslov class, i.e. $[-iHd\omega] \neq 0$, hence its first Betti number $b_1 \geq 1$, see [10]. In fact, the mean curvature form $-iHd\omega$ is a twisted harmonic form (cf. Section 3). We find that the Lagrangian variation associated to a twisted harmonic 1-form $\theta \notin [-iHd\omega]$ decreases the entropy. This allows us to prove the following

**Theorem 1.1.** Any closed Lagrangian self-shrinker with $b_1 \geq 2$ is Lagrangian F-unstable.

In $\mathbb{C}^2$, the embedded closed Lagrangian surface has very strict topological constraint. It has to be torus. As a corollary, in $\mathbb{C}^2$ there are no embedded closed Lagrangian F-stable Lagrangian self-shrinkers. In case that the closed Lagrangian self-shrinker has $b_1 = 1$, we are led to study the Hamiltonian F-stability.

**Theorem 1.2.** For a Lagrangian self-shrinker $\Sigma$ with $b_1 = 1$, the Lagrangian F-stability of $\Sigma$ is equivalent to the Hamiltonian F-stability of $\Sigma$.

However for $b_1 \geq 2$, there is indeed a difference between Lagrangian F-stability and Hamiltonian F-stability. For example the Clifford torus

$$T^n = \{(z^1, \ldots, z^n) : |z^1|^2 = \cdots = |z^n|^2 = 2\}$$

is Hamiltonian F-stable but not Lagrangian F-stable. It is an interesting question whether there exists a closed Lagrangian F-stable self-shrinker of dimension greater than one, i.e. a closed Hamiltonian F-stable Lagrangian self-shrinker with $b_1 = 1$. We also ask if there exists a complete noncompact Lagrangian F-stable Lagrangian self-shrinker which has polynomial volume growth, besides Lagrangian planes and Lagrangian $S^1 \times \mathbb{R}^{n-1}$? It’s also interesting to have more examples of Hamiltonian F-stable Lagrangian self-shrinkers.

We can give two characterizations of Hamiltonian F-stability for a closed Lagrangian self-shrinker with arbitrary $b_1(\geq 1)$ by its spectral property of the twisted Laplacian. Without loss of generality, we assume that the closed Lagrangian self-shrinker becomes extinct at $(0,1)$, i.e. $H = -\frac{1}{2} x^\perp$. The twisted Laplacian is then $\Delta_f = \Delta - \frac{1}{2} < x^\perp, \nabla >$.

**Theorem 1.3.** A closed Lagrangian self-shrinker that becomes extinct at $(0,1)$ is Hamiltonian F-stable if and only if the twisted Laplacian $\Delta_f$ has

$$\lambda_1 = \frac{1}{2}, \quad \Lambda_2 = \{< x, w > : w \in \mathbb{R}^{2n}\}; \quad \lambda_2 \geq 1.$$

**Theorem 1.4.** A closed Lagrangian self-shrinker $\Sigma$ that becomes extinct at $(0,1)$ is Hamiltonian F-stable if and only if

$$\int\int_{\Sigma} |du|^2 e^{-\frac{|z|^2}{4}} d\mu \geq \int\int_{\Sigma} u^2 e^{-\frac{|z|^2}{4}} d\mu, \quad \text{for all u s.t.} \quad \int\int_{\Sigma} u e^{-\frac{|z|^2}{4}} d\mu = \int\int_{\Sigma} u x e^{-\frac{|z|^2}{4}} d\mu = 0.$$
The classification problem of self-shrinkers with higher codimensions is much more complicated due to the complexity of the normal bundle, see for instance [2, 35]. Very recently, F-stability of self-shrinkers with higher codimensions was considered in [2, 4, 22]. The Lagrangian F-stability has also been considered by Lee and Lue [22]. In particular, they proved some of closed Lagrangian self-shrinkers in [1] are Lagrangian F-unstable.

Colding and Minicozzi’s idea of classifying self-similar solutions by employing entropy-stability and F-stability also applies to other geometric flows, for the harmonic map heat flow case see [42] and for the Yang-Mills flow case see [13]. For Ricci shrinkers and Ricci-flat manifolds, an analogous stability to the F-stability is the linear stability. A Ricci shrinker (resp. a Ricci-flat manifold) is called linearly stable if the second variation of Perelman’s \(\nu\)-entropy (resp. \(\lambda\)-entropy) [30] is non-positive at the Ricci shrinker (resp. the Ricci-flat manifold), see [7, 8]. It was proved in [15] that any compact Ricci-flat manifold admitting nontrivial parallel spinors is linearly stable. For the special class of Kähler-Ricci solitons with Hodge number \(h^{1,1} \geq 2\), Hall and Murphy [17] proved that they are linearly unstable (allowing non-Kählerian deformations), which extended the result of Cao-Hamilton-Ilmanen [7] in the Kähler-Einstein case. In the contrast, if one considers deformations of Kähler metrics in the fixed class \(c_1(M)\), Tian and Zhu [38] proved that the \(\nu\)-energy is maximized at a Kähler-Ricci soliton.

The paper is organized as follows: in the next section, we compute the first and second variation formula of the F-functional. The variation formulas of the F-functional will be applied to Lagrangian self-shrinkers and Lagrangian variations in Section 3, where we study Lagrangian F-stability of closed Lagrangian self-shrinkers and prove Theorem 1.1. In the last section we consider the Hamiltonian F-stability of closed Lagrangian self-shrinkers and prove Theorem 1.2, 1.3 and 1.4.

2. F-functional, entropy, and second variational formula

In this section we first recall Huisken’s monotonicity formula, which plays a crucial role in the formation of singularities along the mean curvature flow. At a given singularity \((x, T)\), one can extract a tangent flow at \((x, T)\) from a sequence of rescaled flows. Each time-slice of the tangent flow is a self-shrinker. We also recall the F-functional and Colding-Minicozzi’s entropy for immersed surfaces in Euclidean space. The first and second variation formula of the F-functional will be calculated, which is a slight modification of the hypersurface case [14]. The calculations were also carried out in [2, 4, 22]. A critical point of the F-functional \(F_{x_0, t_0}\) is a self-shrinkers that becomes extinct at \((x_0, t_0)\), and the second variation formula gives rise to F-stability of self-shrinkers. In the end of this section we give a characterization of the F-stability, see also [2, 4, 22]. The variation formulas of the F-functional will be applied to Lagrangian self-shrinkers in the next section.

Let \(\Sigma\) be an \(n\)-dimensional complete immersed surface in \(\mathbb{R}^N\), \(g\) the induced metric on \(\Sigma\), and \(\{e_i\}_{i=1}^n\) a local orthonormal frame of \(T\Sigma\). The second fundamental form and the mean curvature of \(\Sigma\) are respectively given by

\[
h_{ij} = (e_i e_j)^\perp, \quad H = (e_i e_i)^\perp.
\]

here the superscript \(\perp\) denotes the normal projection. The projection to the tangential space will be denoted by the superscript \(\top\). If there exist positive constants \(C_1, C_2\) and \(d\) such that \(Vol(B_r(0) \cap \Sigma) \leq C_1 r^d + C_2\), here \(r\) is the Euclidean distance, we say that \(\Sigma\) has polynomial volume growth.
Let $\Sigma_t$ be a family of immersed surfaces in $\mathbb{R}^N$, evolved by the mean curvature flow

$$\frac{\partial x}{\partial t} \perp = H.$$ 

Assume $T$ is the first singular time of the mean curvature flow. For any $x_0 \in \mathbb{R}^N, t_0 > 0$ and $t < \min\{t_0, T\}$, let

$$\rho_{x_0,t_0}(x, t) = \left[4\pi(t_0 - t)^{\frac{N}{2}} e^{-\frac{|x-x_0|^2}{4(t_0-t)}} \right]^{-\frac{N}{2}}$$

and

$$\Phi_{x_0,t_0}(t) = \int_{\Sigma_t} \rho_{x_0,t_0}(x, t) d\mu_t.$$ 

Then Huisken’s monotonicity formula [18] reads

$$\frac{d}{dt} \int_{\Sigma_t} \rho_{x_0,t_0}(x, t) d\mu_t = - \int_{\Sigma_t} |H + \frac{(x-x_0)^\top}{2(t_0-t)}|^2 \rho_{x_0,t_0}(x, t) d\mu_t.$$ 

The monotonicity formula is crucial in understanding the formation of singularities along the mean curvature flow. At a given singularity $(x, T)$ and for a given $\lambda_j > 0$, one can consider the following rescaled flow

$$(2.1) \quad \tilde{\Sigma}^\lambda_j := \lambda_j (\Sigma_{T+\lambda_j^{-2}s} - x), \quad s < 0.$$ 

Huisken [18] showed that if the mean curvature flow develops a Type-I singularity at time $T$, i.e. $(T - t) \sup_{\Sigma_t} |h_{ij}|^2$ is uniformly bounded in $t$, there exists a sequence $\lambda_j \to \infty$ such that the sequence $\tilde{\Sigma}^\lambda_j$ converges smoothly to a limiting flow $\tilde{\Sigma}_s$. In general using the monotonicity formula and Brakke’s compactness theorem [5], Ilmanen [20] and White [41] proved that at any given singularity $(x, T)$, there exists a sequence $\lambda_j \to \infty$ such that the sequence $\tilde{\Sigma}^\lambda_j$ converges weakly to a limiting flow $\tilde{\Sigma}_s$. $\tilde{\Sigma}_s$ is called a tangent flow at $(x, T)$. Moreover for each $s < 0$, $\tilde{\Sigma}_s$ is a self-shrinker.

If the mean curvature flow is initiating from a compact immersed surface, Colding and Minicozzi [14] showed that each time-slice of $\tilde{\Sigma}_s$ has polynomial volume growth. From now on we restrict ourselves to self-shrinkers which are smooth and have polynomial volume growth. More specifically than (1.1), we call an immersed surface in $\mathbb{R}^N$ a self-shrinker that becomes extinct at $(x_0, t_0)$ if it satisfies

$$(2.2) \quad H + \frac{(x-x_0)^\top}{2t_0} = 0.$$ 

For a self-shrinker that becomes extinct at $(x_0, t_0)$, $\sqrt{t_0 - t}(\Sigma-x_0)$ defines a homothetically shrinking mean curvature flow. Note that a self-shrinker that becomes extinct at $(x_0, t_0)$ is also a steady point of Huisken’s monotonicity quantity $\Phi_{x_0,t_0}$.

Let $\Sigma$ be a complete self-shrinker which becomes extinct at $(x_0, t_0)$ and has polynomial volume growth. Colding and Minicozzi [14] introduced an operator acting on functions on $\Sigma$ by

$$(2.3) \quad \Delta u - \frac{1}{2t_0} < (x-x_0)^\top, \nabla u > = e^{-\frac{|x-x_0|^2}{4t_0}} \text{div}(e^{-\frac{|x-x_0|^2}{4t_0}} \nabla u).$$
Analogous operators also appear in other backgrounds, see for instance [16, 25]. Given a function $f$, the so-called twisted Laplacian is defined by

$$\Delta_f u = \Delta u - g(\nabla f, \nabla u).$$

The twisted Laplacian on functions is actually $-d^* f$, here $d^*$ is the adjoint operator of $d$ with respect to the measure $e^{-f} d\mu$, see for instance [16]. On the self-shrinker $\Sigma$ which becomes extinct at $(x_0, t_0)$, we choose $f = \frac{|x - x_0|^2}{4t_0}$. Then

$$\Delta_f = \Delta - \frac{1}{2t_0} < (x - x_0)^\top, \nabla \cdot >.$$

The following "weighted $W^{2,2}$ space" $W^{2,2}_w$ was also introduced in [14]

$$W^{2,2}_w = \{ u | \int_{\Sigma} (|u|^2 + |\nabla u|^2 + |\Delta f u|^2) e^{-\frac{|x - x_0|^2}{4t_0}} d\mu < \infty \}.$$

For any $u, v \in W^{2,2}_w$, it holds that

$$\int_{\Sigma} u \Delta_f v e^{-\frac{|x - x_0|^2}{4t_0}} d\mu = -\int_{\Sigma} g(\nabla u, \nabla v) e^{-\frac{|x - x_0|^2}{4t_0}} d\mu = \int_{\Sigma} v \Delta_f u e^{-\frac{|x - x_0|^2}{4t_0}} d\mu.$$  

**Lemma 2.1.** Let $\Sigma$ be an $n$-dimensional complete self-shrinker which becomes extinct at $(x_0, t_0)$ and has polynomial volume growth, $w$ a vector in $\mathbb{R}^N$. Then

1. $\int_{\Sigma} < x - x_0, w > e^{-\frac{|x - x_0|^2}{4t_0}} d\mu = 0$;
2. $\int_{\Sigma} |x - x_0|^2 e^{-\frac{|x - x_0|^2}{4t_0}} d\mu = 0$;
3. $\int_{\Sigma} |x - x_0|^4 - 4n(n+2)\ell_0^2 + 16t_0^3|H|^2 e^{-\frac{|x - x_0|^2}{4t_0}} d\mu = 0$;
4. $\int_{\Sigma} |x - x_0|^2 < x - x_0, w >^2 e^{-\frac{|x - x_0|^2}{4t_0}} d\mu = 0$;
5. $\int_{\Sigma} |x - x_0|^2 < x - x_0, w > e^{-\frac{|x - x_0|^2}{4t_0}} d\mu = 0$;
6. $\int_{\Sigma} < (x - x_0)^\top, w > e^{-\frac{|x - x_0|^2}{4t_0}} d\mu = 0$.

**Proof.** The proof is similar to the hypersurface case in [14].

1. Let $u = < x - x_0, w >$, $v = 1$.

By (2.4),

$$\Delta_f u = < H, w > - \frac{1}{2t_0} < (x - x_0)^\top, w > = -\frac{1}{2t_0} < x - x_0, w >.$$

Therefore, (1) follows from (2.5).

2. Let $u = |x - x_0|^2$, $v = 1$.
The identity then follows from (2.5) and
\[
\Delta f u = 2n + 2 < x - x_0, H > - \frac{1}{t_0} |(x - x_0)^\top|^2 \\
= 2n - \frac{1}{t_0} |x - x_0|^2.
\]

(3) Let
\[u = v = |x - x_0|^2.
\]
Then
\[
\Delta f u = 2n - \frac{1}{t_0} |x - x_0|^2,
\]
\[|\nabla u|^2 = 4 |(x - x_0)^\top|^2.
\]
Hence it follows from (2.5) that
\[
\int_{\Sigma} (2n - \frac{1}{t_0} |x - x_0|^2) |x - x_0|^2 e^{-\frac{|x - x_0|^2}{4t_0}} d\mu \\
= - \int_{\Sigma} 4 |(x - x_0)^\top|^2 e^{-\frac{|x - x_0|^2}{4t_0}} d\mu \\
= - \int_{\Sigma} 4 |x - x_0|^2 - 4t_0^2 |H|^2 e^{-\frac{|x - x_0|^2}{4t_0}} d\mu.
\]
By using (2), we get
\[
\int_{\Sigma} |x - x_0|^4 e^{-\frac{|x - x_0|^2}{4t_0}} d\mu \\
= \int_{\Sigma} ((2n + 4)t_0 |x - x_0|^2 - 16t_0^2 |H|^2) e^{-\frac{|x - x_0|^2}{4t_0}} d\mu \\
= \int_{\Sigma} (4n(n + 2)t_0^2 - 16t_0^5 |H|^2) e^{-\frac{|x - x_0|^2}{4t_0}} d\mu.
\]

(4) Let
\[u = v = < x - x_0, w >.
\]
We have
\[
\Delta f u = - \frac{1}{2t_0} < x - x_0, w >, \quad \nabla u = w^\top.
\]
Hence
\[
\int_{\Sigma} - \frac{1}{2t_0} < x - x_0, w >^2 e^{-\frac{|x - x_0|^2}{4t_0}} d\mu = - \int_{\Sigma} |w|^2 e^{-\frac{|x - x_0|^2}{4t_0}} d\mu.
\]

(5, 6) Let
\[u = < x - x_0, w >, \quad v = |x - x_0|^2.
\]
We have
\[
\Delta f u = - \frac{1}{2t_0} < x - x_0, w >, \quad \nabla u = w^\top
\]
and
\[
\Delta f v = 2n - \frac{1}{t_0} |x - x_0|^2, \quad \nabla v = 2(x - x_0)^\top.
\]
Hence by (2.5), we get

\[ \int_{\Sigma} \langle x - x_0, w \rangle > (2n - \frac{1}{t_0}) |x - x_0|^2 e^{-\frac{|x-x_0|^2}{4t_0}} d\mu \]

\[ = \int_{\Sigma} - \langle w^\top, 2(x - x_0)^\top \rangle e^{-\frac{|x-x_0|^2}{4t_0}} d\mu \]

\[ = \int_{\Sigma} - \frac{1}{2t_0} < x - x_0, w > |x - x_0|^2 e^{-\frac{|x-x_0|^2}{4t_0}} d\mu. \]

Then it follows from (1) that

\[ \int_{\Sigma} - \frac{1}{t_0} < x - x_0, w > |x - x_0|^2 e^{-\frac{|x-x_0|^2}{4t_0}} d\mu \]

\[ = \int_{\Sigma} - \frac{1}{2t_0} < x - x_0, w > |x - x_0|^2 e^{-\frac{|x-x_0|^2}{4t_0}} d\mu. \]

Hence,

\[ \int_{\Sigma} |x - x_0|^2 < x - x_0, w > e^{-\frac{|x-x_0|^2}{4t_0}} d\mu = \int_{\Sigma} < (x - x_0)^\top, w > e^{-\frac{|x-x_0|^2}{4t_0}} d\mu = 0. \]

□

For an immersed \( n \)-dimensional surface \( \Sigma \) in \( \mathbb{R}^N \), the F-functional (with respect to \( x_0 \in \mathbb{R}^N, t_0 > 0 \)) and the entropy [14] are respectively defined by

\[ F_{x_0,t_0} (\Sigma) = \int_{\Sigma} (4\pi t_0)^{-\frac{n}{2}} e^{-\frac{|x-x_0|^2}{4t_0}} d\mu \]

and

\[ \lambda (\Sigma) = \sup_{x_0,t_0} F_{x_0,t_0} (\Sigma). \]

The relation between the F-functional and Huisken’s monotonicity quantity is given by

(2.6)

\[ F_{x_0,t_0} (\Sigma_{t_1}) = \Phi_{x_0,t_0+t_1} (\Sigma_{t_1}), \]

here \( \Sigma_t \) is a 1-parameter family of immersions. The entropy functional has very nice properties, for example (i) \( \lambda \) is invariant under dilations and rigid motions; (ii) \( \lambda \) is non-increasing along the mean curvature flow. The property (ii) follows from (2.6) and Huisken’s monotonicity formula. One easily sees that the surface \( \Sigma^{s^\lambda}_{\lambda} \) in (2.1) satisfies \( \lambda (\Sigma^{s^\lambda}_{\lambda}) \leq \lambda (\Sigma_0) \). We now compute the first variation formula of the F-functional.

**Proposition 2.2.** Let \( \Sigma \) be a complete immersed surface in \( \mathbb{R}^N \) and \( \Sigma_s \) a family of deformations of \( \Sigma \) generated by normal variation \( X_s, s \in (-\epsilon, \epsilon) \). Let \( x_s, t_s \) be deformations of \( x_0 \) and \( t_0 \) respectively with velocities \( \dot{x}_s \) and \( \dot{t}_s \), then \( \frac{d}{ds} F_{x_s,t_s} (\Sigma_s) \) is given by

\[ \int_{\Sigma_s} \left[ - < X_s, H + \frac{x - x_s}{2t_s} > + \frac{1}{2t_s} < x - x_s, \dot{x}_s > + \dot{t}_s \frac{- n (|x - x_s|^2 - \frac{n}{2t_s})}{4t_s^2} \right] (4\pi t_s)^{-\frac{n}{2}} e^{-\frac{|x-x_0|^2}{4t_0}} d\mu_s. \]

In particular if \( X_s |_{s=0} = X, \dot{x}_0 = y, \dot{t}_0 = h \), then \( \frac{d}{ds} |_{s=0} F_{x_s,t_s} (\Sigma_s) \) is given by

\[ \int_{\Sigma} \left[ - < X, H + \frac{x - x_0}{2t_0} > + \frac{1}{2t_0} < x - x_0, y > + h \frac{- n (|x - x_0|^2 - \frac{n}{2t_0})}{4t_0^2} \right] (4\pi t_0)^{-\frac{n}{2}} e^{-\frac{|x-x_0|^2}{4t_0}} d\mu. \]
Proof. By definition, \( F(x_s, t_s)(Σ_s) = \int_Σ (4πt_s)^{-\frac{n}{2}} e^{-\frac{|x-x_s|^2}{4t_s}} dμ_s \). The first variation formula of the F-functional then follows from

\[
\frac{d}{ds} dμ_s = - < X_s, H > dμ_s
\]

\[
\frac{d}{ds} \left[(4πt_s)^{-\frac{n}{2}} e^{-\frac{|x-x_s|^2}{4t_s}}\right] = \left(- \frac{n}{2t_s} i_s + \frac{|x-x_s|^2}{4t_s^2} i_s - \frac{1}{2} \frac{1}{2t_s} < x-x_s, X_s-\dot{x}_s > \right)(4πt_s)^{-\frac{n}{2}} e^{-\frac{|x-x_s|^2}{4t_s}}.
\]

From the first variation formula, we see a critical point of \( F_{x_0, t_0} \) is a self-shrinker that becomes extinct at \((x_0, t_0)\). Moreover if \( Σ \) is a self-shrinker that becomes extinct at \((x_0, t_0)\), then by Lemma 2.1 one sees that \((Σ_0, x_0, t_0)\) is a critical point of the F-functional, which also means that \( Σ \) is a critical point of the entropy. We now compute the second variation of the F-functional at a self-shrinker. For a normal vector field \( X \), denote

\[
(2.7) \quad LX = ΔX + < X, h_{ij} > h_{ij} - \frac{x-x_0}{2t_0}, e_i > \nabla e_i X + \frac{1}{2t_0} X.
\]

**Theorem 2.3.** Let \( Σ \) be a self-shrinker which becomes extinct at \((x_0, t_0)\) and has polynomial volume growth. Assume \( Σ_s, x_s \) and \( t_s \) are deformations of \( Σ, x_0 \) and \( t_0 \) respectively with

\[
\frac{∂Σ_s}{∂s}|_{s=0} = X, \quad \frac{∂x_s}{∂s}|_{s=0} = y, \quad \frac{∂t_s}{∂s}|_{s=0} = h.
\]

Then

\[
\frac{d^2}{ds^2} |_{s=0} F_{x_s, t_s}(Σ_s) = \int Σ - < X, LX - \frac{1}{t_0} y + \frac{1}{t_0} hH > (4πt_0)^{-\frac{n}{2}} e^{-\frac{|x-x_0|^2}{4t_0}} dμ
\]

\[
- \int Σ \left[ \frac{1}{2t_0} |y|^2 + \frac{1}{t_0} h^2 |H|^2 \right] (4πt_0)^{-\frac{n}{2}} e^{-\frac{|x-x_0|^2}{4t_0}} dμ.
\]

Proof. For convenience, we write

\[
F'' = \frac{d^2}{ds^2}|_{s=0} F_{x_s, t_s}(Σ_s), \quad G_s(x) = (4πt_s)^{-\frac{n}{2}} e^{-\frac{|x-x_s|^2}{4t_s}}, \quad G(x) = (4πt_0)^{-\frac{n}{2}} e^{-\frac{|x-x_0|^2}{4t_0}}.
\]

It follows from the first variation formula of the F-functional that \( F'' \) is

\[
\frac{d}{ds}|_{s=0} \int Σ_s [ - < X_s, H + \frac{x-x_s}{2t_s} > + \frac{1}{2t_s} < x-x_s, \dot{x}_s > + i_s(\frac{|x-x_s|^2}{4t_s^2} - \frac{n}{2t_s})] G_s dμ_s.
\]

Under the normal deformation with \( \frac{∂x_s}{∂s} = X \), we have

\[
\frac{∂}{∂s} g_{ij} = -2 < X, h_{ij} >
\]

and

\[
(\frac{∂}{∂s} H) = ΔX + < X, h_{ij} > h_{ij}.
\]
Then it follows from Lemma 2.1 that
\[
F'' = \int_{\Sigma} -\left< \frac{\partial X_s}{\partial s} \right|_{s=0}, \frac{(x-x_0)\top}{2t_0} > + \left< X, \Delta X > + \left< X, h_{ij} > \right>^2 Gd\mu
\]
\[
+ \int_{\Sigma} \left( - \left< X, \frac{X-y}{2t_0} > + \left< X, \frac{h(x-x_0)}{2t_0^2} > + \frac{1}{2t_0} \left< X-y, y \right> Gd\mu
\]
\[
+ \int_{\Sigma} h\left( - \frac{|x-x_0|^2 h}{2t_0^3} + \frac{2}{2t_0^2} \left< x-x_0, X-y \right> + \frac{n h}{2t_0^2} \right) Gd\mu
\]
\[
+ \int_{\Sigma} \left[ \frac{1}{2t_0} < x-x_0, y > + h\left( \frac{|x-x_0|^2}{4t_0^2} - \frac{n}{2t_0^2} \right) \right]^2 Gd\mu.
\]
Note that
\[
\left< \frac{\partial X}{\partial s} \right|_{s=0}, \frac{(x-x_0)\top}{2t_0} > = - \left< \frac{x-x_0}{2t_0}, \epsilon_i > X, \nabla_{\epsilon_i} X > .
\]
We denote
\[
- \left< \frac{x-x_0}{2t_0}, \epsilon_i > X = \frac{x-x_0}{2t_0} \cdot \nabla X.
\]
By the assumption on \( \Sigma \) and Lemma 2.1,
\[
F'' = \int_{\Sigma} -\left< X, \frac{x-x_0}{2t_0} \cdot \nabla X > + \left< X, \Delta X > + \left< X, h_{ij} > \right>^2 Gd\mu
\]
\[
+ \int_{\Sigma} \left[ - \frac{1}{2t_0} |X|^2 + \frac{1}{t_0} < X, y > - \frac{1}{2t_0} |y|^2 - \frac{2h}{t_0} < X, H > - \frac{nh^2}{2t_0^2} \right] Gd\mu
\]
\[
+ \int_{\Sigma} \left[ \frac{1}{2t_0} < x-x_0, y > + h\left( \frac{|x-x_0|^2}{4t_0^2} - \frac{n}{2t_0^2} \right) \right]^2 Gd\mu.
\]
By Lemma 2.1 again,
\[
\int_{\Sigma} \left[ \frac{1}{2t_0} < x-x_0, y > + h\left( \frac{|x-x_0|^2}{4t_0^2} - \frac{n}{2t_0^2} \right) \right]^2 Gd\mu
\]
\[
= \int_{\Sigma} \left[ \frac{1}{4t_0^2} < x-x_0, y >^2 + h^2\left( \frac{|x-x_0|^2}{4t_0^2} - \frac{n}{2t_0^2} \right) \right] Gd\mu
\]
\[
= \int_{\Sigma} \left[ \frac{1}{2t_0} |y|^2 + h^2\left( \frac{n}{2t_0} - \frac{1}{t_0} |H|^2 \right) \right] Gd\mu.
\]
Hence we get
\[
F'' = \int_{\Sigma} -\left< X, \Delta X > + X, h_{ij} > h_{ij} - \frac{x-x_0}{2t_0} \cdot X > Gd\mu
\]
\[
+ \int_{\Sigma} \left< X, \frac{1}{t_0} y - \frac{2}{t_0} h H > Gd\mu - \int_{\Sigma} \left[ \frac{1}{2t_0} |y|^2 + \frac{1}{t_0} h^2 |H|^2 \right] Gd\mu.
\]
Without loss of generality, from now on we assume \( x_0 = 0, t_0 = 1 \). In particular a self-shrinker that becomes extinct at \( (0, 1) \) satisfies \( H = -\frac{1}{2} x^i \) and the Jacobi operator \( L \) becomes
\[
LX = \Delta X + \left< X, h_{ij} > h_{ij} - \frac{x}{2}, \epsilon_i > \nabla_{\epsilon_i} X + \frac{1}{2} X.
\]
Let $L^2_f$ be the Hilbert space consisting of all functions square integrable with respect to the measure $e^{-\frac{|x|^2}{4}}d\mu$. When referring to a normal variation field $X$, we assume $|X|, |\nabla X|$ and $|LX|$ are in $L^2_f$. By the second variation formula of the F-functional, we arrive at the following definition of F-stability [14].

**Definition 2.1.** A self-shrinker $\Sigma$ which becomes extinct at $(0,1)$ and has polynomial volume growth is called F-stable if for any normal variation $X$, there exist a vector $y$ and a constant $h$ such that

$$F'' = (4\pi)^{-\frac{n}{2}} \int_{\Sigma} \left[ -<X, LX> + <X, y> - 2h <X, H> - \frac{1}{2}|y|^2 - h^2 |H|^2 \right] e^{-\frac{|x|^2}{4}}d\mu \geq 0.$$

A normal vector field $X$ is called an eigenvector field of $L$ and of eigenvalue $\mu$ if there exists a constant $\mu$ such that $LX = -\mu X$. Note that $L$ is self-adjoint with respect to the measure $e^{-\frac{|x|^2}{4}}d\mu$. We now show that $H$ and $w^\perp$ are eigenvector fields of $L$, here $w$ is any vector in $\mathbb{R}^N$. We also show that a self-shrinker is F-stable if and only if $H$ and $w^\perp$ are the only eigenvector fields which have negative eigenvalues. Note that for a self-shrinker the variations $H$ and $w^\perp$ correspond to dilation and translation respectively.

**Lemma 2.4.** Let $\Sigma$ be a self-shrinker that becomes extinct at $(0,1)$ and $w$ a vector in $\mathbb{R}^N$. Then

$$LH = H, \quad Lw^\perp = \frac{1}{2} w^\perp.$$

**Proof.** Let $\nu_\alpha$ be a local orthonormal frame on the normal bundle of $\Sigma$ and normal at the point under consideration.

Write $H = h_{k\alpha} \nu_\alpha = -\frac{1}{2} < x, \nu_\alpha > \nu_\alpha$. Then

$$\nabla_i H = \frac{1}{2} h_{il\alpha} < x, e_l > \nu_\alpha = \frac{1}{2} < x, e_l > h_{il}$$

and

$$\Delta H = \frac{1}{2} (H + < x, h_{il} > h_{il} + < x, e_l > \nabla_l H)$$

$$= \frac{1}{2} H - < H, h_{il} > h_{il} + \frac{1}{2} < x, e_l > \nabla_l H.$$

Hence we get

$$LH = \Delta H + < H, h_{ij} > h_{ij} - < \frac{x}{2}, e_i > \nabla_i H + \frac{1}{2} H = H.$$

Write $w^\perp = < w, \nu_\alpha > \nu_\alpha$. Then

$$\nabla_i w^\perp = -h_{il\alpha} < w, e_l > \nu_\alpha = - < w, e_l > h_{il},$$

and

$$\Delta w^\perp = - < w, h_{il} > h_{il} - < w, e_l > \nabla_i H$$

$$= - < w, h_{il} > h_{il} - \frac{1}{2} < w, e_l > < x, e_k > h_{lk}$$

$$= - < w, h_{il} > h_{il} + \frac{1}{2} < x, e_k > \nabla_k w^\perp.$$

Hence we get $Lw^\perp = \frac{1}{2} w^\perp$. □
Theorem 2.5. A self-shrinker $\Sigma$ is $F$-stable if and only if $H$ and $w^\perp$ are the only eigenvector fields of $L$ which have negative eigenvalues.

Proof. Assume $\Sigma$ is Lagrangian $F$-stable and $X$ is an eigenvector field with $LX = -\mu X$ such that
\[ \int_{\Sigma} <X, H> e^{-\frac{|x|^2}{4}} d\mu = \int_{\Sigma} <X, w^\perp> e^{-\frac{|x|^2}{4}} d\mu = 0, \quad \forall w \in \mathbb{R}^N. \]
Then
\[ F'' = (4\pi)^{-\frac{n}{2}} \int_{\Sigma} [\mu |X|^2 - \frac{1}{2} |y^\perp|^2 - h^2 |H|^2] e^{-\frac{|x|^2}{4}} d\mu. \]
Therefore the $F$-stability of $\Sigma$ implies that $\mu \geq 0$.

On the other hand, any normal variation $X$ admit a unique decomposition $X = a_0 H + w^\perp + X_1$ such that $X_1$ is orthogonal to $H$ and all translations $y^\perp$. Then by Lemma 2.4,
\[ F'' = (4\pi)^{-\frac{n}{2}} \int_{\Sigma} [\mu |X|^2 - \frac{1}{2} |y^\perp|^2 - h^2 |H|^2] e^{-\frac{|x|^2}{4}} d\mu. \]
Taking $h = -a_0$ and $y = w$, we are done. \qed

The characterization of $F$-stability was proved by Lee and Lu [22]. We get the characterization independently. The necessary part of the characterization was also proved in [2]. An interesting application of the characterization was found in [22], which states that the product of two self-shrinkers which become extinct at the same $(x_0, t_0)$ is $F$-unstable.

A similar characterization for the linear stability of Ricci shrinkers was proved by Cao and Zhu [8], and a similar characterization of the $F$-stability for self-similar solutions to the harmonic map heat flow was obtained in [42].

3. Lagrangian $F$-stability of closed Lagrangian self-shrinkers

In this section we study Lagrangian $F$-stability of closed Lagrangian self-shrinkers. Our starting point is a correspondence between Lagrangian variations and closed 1-forms for Lagrangian surfaces, which was also used by Oh [29] in the studying of Lagrangian stability of minimal Lagrangian submanifolds in Kähler manifolds. Via this correspondence, we replace a Lagrangian variation by closed a 1-form in the second variation formula of the $F$-functional. Twisted harmonic 1-form will play a crucial role in the proof of Theorem 1.1. In particular, we will prove that any nontrivial twisted harmonic 1-form in a different class from the Maslov class is an obstruction to the $F$-stability.

Let $(\mathbb{C}^n, \bar{g}, J, \bar{\omega})$ be the complex Euclidean space with the standard metric, complex structure and Kähler form such that $\bar{g} = \bar{\omega}(-, J\cdot)$. An $n$-dimensional immersed surface $\Sigma$ is called a Lagrangian submanifold if $\bar{\omega}|_{\Sigma} = 0$. Let $\Sigma$ be a Lagrangian surface in $\mathbb{C}^n$,
\{e_i\}_{i=1}^n$ a local orthonormal frame of $T\Sigma$ and $\nu_i = J e_i$. Note $\{\nu_i\}_{i=1}^n$ also form a local orthonormal frame of the normal bundle. The second fundamental form is defined by

$$h_{ijk} = \langle \nabla e_i, e_j, \nu_k \rangle,$$

which is symmetric in $i, j$ and $k$. The mean curvature vector field $H$ is defined by

$$H = H_k \nu_k = \langle \nabla e_i, e_i \rangle^\perp = h_{iik} \nu_k.$$

Let $\{e^i\}_{i=1}^n$ be the dual basis of $\{e_i\}_{i=1}^n$. For any normal vector field $X = X_k \nu_k$, one can associated to it a 1-form on $\Sigma$ by

$$\theta = -i_X < - = X_k e^k.$$

A normal vector field $X$ is called a Lagrangian variation if $i_X <$ is closed. It is easy to check that Lagrangian variations are infinitesimal Lagrangian deformations. A normal variation field $X$ is called a Hamiltonian variation if $i_X <$ is exact. For example, the mean curvature vector field $H$ of a Lagrangian submanifold in $\mathbb{C}^n$ is a Lagrangian variation. The corresponding closed 1-form $-i_H <$ is called the mean curvature form, still denoted by $H$, and the cohomology class $[-i_H <]$ is called the Maslov class.

For Lagrangian surfaces, the F-functional and the entropy are the same as the definitions in the last section. The first variation formula of the F-functional in last section still holds but $\Sigma_s$ is now replaced by a Lagrangian deformation and $X_s$ is replaced by a Lagrangian variation. Note that $H$ and $(x - x_0)^\perp$ are both Lagrangian variations. Therefore a critical point of $F_{x_0, t_0}$ is a Lagrangian self-shrinker even when we are restricted to Lagrangian deformations. Similarly by the second variation formula of the F-functional, we have the following definition of Lagrangian (resp. Hamiltonian) F-stability for Lagrangian self-shrinkers.

**Definition 3.1.** Let $\Sigma$ be a complete Lagrangian self-shrinker that becomes extinct at $(0, 1)$. $\Sigma$ is called Lagrangian (resp. Hamiltonian) F-stable if for any Lagrangian (resp. Hamiltonian) variation $X$, there exist a vector $y$ and a constant $h$ such that

$$F'' = (4\pi)^{-\frac{n}{2}} \int_\Sigma - < X, LX > + < X, y > - 2h < X, H > - \frac{1}{2} |y|^2 - \frac{1}{4} h^2 |H|^2 e^{-\frac{|x|^2}{4}} d\mu \geq 0.$$

We now rewrite the second variation in terms of the closed 1-form $\theta = -i_X <$. We first decompose the closed 1-form into its harmonic part and exact part by

$$\theta = \theta_0 + du.$$

Let $d^*$ be the adjoint operator of $d$ and $\Delta_H = d^* d + d d^*$ be the Hodge Laplacian.

**Proposition 3.1.** Let $\Sigma$ be a closed Lagrangian self-shrinker that becomes extinct at $(0, 1)$ and $X$ a Lagrangian variation. Let $\theta = -i_X <$ and $\theta = \theta_0 + du$ be the decomposition. Then the second variation is given by

$$F'' = (4\pi)^{-\frac{n}{2}} \int_\Sigma [d^* du + \frac{1}{2} du(x^\top) + \frac{1}{2} \theta_0 (x^\top)]^2 - |\theta|^2 e^{-\frac{|x|^2}{4}} d\mu$$

$$+ (4\pi)^{-\frac{n}{2}} \int_\Sigma [-\theta (Jy^\top) - h \theta (Jx^\top) - \frac{1}{2} |y^\top|^2 - \frac{1}{4} h^2 |x^\top|^2] e^{-\frac{|x|^2}{4}} d\mu.$$
Proof. For $X = X_k \nu_k$, we have

$$LX = \Delta X_k \nu_k + X_p h_{ijp} h_{ijk} \nu_k - \frac{x}{2}, e_i > \nabla_i X_k \nu_k + \frac{1}{2} X_k \nu_k$$

and

$$F'' = (4 \pi)^{-\frac{n}{2}} \int_\Sigma \left[ -X_k \Delta X_k - X_k X_p h_{ijp} h_{ijk} + X_k \left< \frac{x}{2} \right., e_i > \nabla_i X_k - \frac{1}{2} |X|^2 \right] e^{-\frac{|x|^2}{4}} d\mu$$

$$+ (4 \pi)^{-\frac{n}{2}} \int_\Sigma \left< X, y > -2h < X, H > - \frac{1}{2} |y^+|^2 - h^2 |H|^2 \right] e^{-\frac{|x|^2}{4}} d\mu.$$ 

Note that

$$(dd^* \theta)(e_k) = -\nabla_{e_k} \nabla_{e_j} \theta_j,$$

$$(d^* d \theta)(e_k) = -\nabla_{e_j} \nabla_{e_j} \theta_k + \nabla_{e_j} \nabla_{e_k} \theta_j,$$

then the Hodge Laplacian of $\theta$ is given by

$$(\Delta_H \theta)(e_k) = -\nabla_{e_k} \nabla_{e_j} \theta_j - \nabla_{e_j} \nabla_{e_j} \theta_k + \nabla_{e_j} \nabla_{e_k} \theta_j$$

$$= -\nabla_{e_j} \nabla_{e_k} \theta_k + R_{kl} \theta_l.$$ 

In $\mathbb{C}^n$, the Ricci curvature of $\Sigma$ is given by

$$R_{kl} = \left< h_{kl}, H > - \left< h_{kp}, h_{tp} > \right. \right.$$ 

Hence

$$(\Delta_H \theta)(e_k) = -\nabla_{e_j} \nabla_{e_j} \theta_k + \left< h_{kl}, H > - \left< h_{kp}, h_{tp} > \right. \right. \theta_l$$

and

$$< \theta, \Delta_H \theta > = X_k \Delta X_k - \frac{1}{2} < x, h_{kl} > X_k X_l - X_k X_l h_{kpq} h_{ltpq}.$$ 

Then

$$F'' = (4 \pi)^{-\frac{n}{2}} \int_\Sigma \left[ < \theta, \Delta_H \theta > + \frac{1}{2} < x, h_{kl} > X_k X_l + X_k \left< \frac{x}{2} \right., e_i > \nabla_i X_k \right] e^{-\frac{|x|^2}{4}} d\mu$$

$$+ (4 \pi)^{-\frac{n}{2}} \int_\Sigma \left[ -\frac{1}{2} |\theta|^2 + < X, y > -2h < X, H > - \frac{1}{2} |y^+|^2 - h^2 |H|^2 \right] e^{-\frac{|x|^2}{4}} d\mu.$$ 

We calculate the second term by

$$\int_\Sigma \left< \frac{1}{2} < x, h_{kl} > X_k X_l \right> e^{-\frac{|x|^2}{4}} d\mu$$

$$= \int_\Sigma \left[ \frac{1}{2} e_k (\left< x, e_l > X_l \right) - X_k - < x, e_l > \nabla_k X_l \right] \frac{x}{2} e_k e^{-\frac{|x|^2}{4}} d\mu$$

$$= \int_\Sigma \left[ \frac{1}{2} \theta(x^+) d^* \theta + \frac{1}{4} |\theta(x^+)|^2 - \frac{1}{2} |\theta|^2 - \frac{1}{2} X_k < x, e_i > \nabla_k X_i \right] e^{-\frac{|x|^2}{4}} d\mu.$$ 

Note that $X$ is Lagrangian, i.e. $\nabla_i X_k = \nabla_k X_i$. Therefore

$$F'' = (4 \pi)^{-\frac{n}{2}} \int_\Sigma \left[ < \theta, \Delta_H \theta > + \frac{1}{2} \theta(x^+) d^* \theta + \frac{1}{4} |\theta(x^+)|^2 - |\theta|^2 \right] e^{-\frac{|x|^2}{4}} d\mu$$

$$+ (4 \pi)^{-\frac{n}{2}} \int_\Sigma \left[ -\theta(J y^+) - h \theta(J x^+) - \frac{1}{2} |y^+|^2 - \frac{1}{4} h^2 |x^+|^2 \right] e^{-\frac{|x|^2}{4}} d\mu.$$
By the decomposition (3.1):

\[
F'' = (4\pi)^{-\frac{d}{2}} \int \Sigma [\langle \theta, \Delta_H (du) \rangle + \frac{1}{2} \theta (x^\top) d^* du + \frac{1}{4} \theta (x^\top)^2 - |\theta|^2 e^{-\frac{|x|^2}{4}} d\mu
\]

\[
+ (4\pi)^{-\frac{d}{2}} \int \Sigma [-\theta (Jy^\top) - h \theta (Jx^\top) - \frac{1}{2} |y^\top|^2 - \frac{1}{4} h^2 |x^\top|^2] e^{-\frac{|x|^2}{4}} d\mu.
\]

By integration by parts, we get

\[
\int \Sigma [\langle \theta, \Delta_H (du) \rangle + \frac{1}{2} \theta (x^\top) d^* du + \frac{1}{4} \theta (x^\top)^2 - |\theta|^2 e^{-\frac{|x|^2}{4}} d\mu
\]

\[
= \int \Sigma [\|d^* du\|^2 + \theta (x^\top) d^* du + \frac{1}{4} \theta (x^\top)^2 - |\theta|^2 e^{-\frac{|x|^2}{4}} d\mu
\]

\[
= \int \Sigma [\|d^* du\|^2 + \frac{1}{2} \theta (x^\top)^2 - |\theta|^2 e^{-\frac{|x|^2}{4}} d\mu.
\]

Hence

\[
F'' = (4\pi)^{-\frac{d}{2}} \int \Sigma [\|d^* du\|^2 + \frac{1}{2} du (x^\top) + \frac{1}{2} \theta_0 (x^\top)^2 - |\theta|^2 e^{-\frac{|x|^2}{4}} d\mu
\]

\[
+ (4\pi)^{-\frac{d}{2}} \int \Sigma [-\theta (Jy^\top) - h \theta (Jx^\top) - \frac{1}{2} |y^\top|^2 - \frac{1}{4} h^2 |x^\top|^2] e^{-\frac{|x|^2}{4}} d\mu.
\]

\[\Box\]

It is necessary to introduce the twisted Hodge Laplacian and the twisted Hodge decomposition theorem on compact Riemannian manifolds. For a given smooth function $f$ on a compact manifold $(M, g)$, let $L^2_\theta$ be the space of those differential forms which are square integrable with respect to the measure $e^{-\theta} d\mu$. Let $d^*_f$ be the adjoint operator of $d$ in the Hilbert space $L^2_\theta$. Then one has the twisted Hodge Laplacian

\[\Delta_{H,f} = d^*_f d + dd^*_f.\]

For the twisted Hodge Laplacian, Bue\ller [6] proved a Hodge decomposition theorem which states that the space of $p$-forms has an orthogonal decomposition in $L^2_\theta$ by

\[\Omega^p = \mathcal{H}^p_f \oplus imd \oplus imd^*_f,\]

here $\mathcal{H}^p_f$ is the space of twisted harmonic $p$-forms, i.e. $p$-forms in the kernel of $\Delta_{H,f}$. Hence $\mathcal{H}^p_f \cong H_{dR}(M)$ and for any closed $p$-form $\omega$ there exists a $(p-1)$-form $\alpha$ such that $\omega + d\alpha$ is a twisted harmonic $p$-form. In particular for a 1-form $\omega$, there exists a function $v$ such that $d^*_f (\omega + dv) = 0$.

We now come back to our situation. Let $\Sigma$ be a closed Lagrangian self-shrinker that becomes extinct at $(0, 1)$ and $f = \frac{|x|^2}{4}$. Note that on one forms, $d^*_f = d^* + i\gamma_f$. Hence for the $\theta = \theta_0 + du$ in (3.1),

\[d^*_f \theta = (d^* + i\gamma_f) (\theta_0 + du) = d^* du + \frac{1}{2} du (x^\top) + \frac{1}{2} \theta_0 (x^\top).\]

Therefore we can rewrite the second variation formula (3.2) as follows.
Corollary 3.1. Let $\Sigma$ be a closed Lagrangian self-shrinker that becomes extinct at $(0, 1)$ and $X$ a Lagrangian variation. Let $\theta = -i_X \omega$. Then the second variation is given by
\begin{equation}
F'' = (4\pi)^{-\frac{n}{2}} \int_{\Sigma} \left| d\theta \right|^2 - \left| \theta \right|^2 - \theta(Jy^\perp) - h\theta(Jx^\perp) - \frac{1}{2} |y^\perp|^2 - \frac{1}{4} h^2 |x^\perp|^2 e^{-\frac{|x^\perp|^2}{4}} d\mu.
\end{equation}

Lemma 3.2. For any harmonic $1$-form $\theta$, there exists a function $u_0$ such that
\begin{equation}
d^*du_0 + \frac{1}{2} du_0(x^\top) + \frac{1}{2} \theta_0(x^\top) = 0.
\end{equation}

Proof. Applying Bueßer’s twisted Hodge decomposition theorem to the harmonic $1$-form $\theta$, we see that there exists a function $u_0$ such that
\begin{equation}
0 = d^*_f(\theta_0 + du_0) = (d^* + i_{\frac{x^\top}{2}})(\theta_0 + du_0) = d^*du_0 + \frac{1}{2} du_0(x^\top) + \frac{1}{2} \theta_0(x^\top).
\end{equation}

Proposition 3.3. Let $\Sigma$ be a closed Lagrangian self-shrinker that becomes extinct at $(0, 1)$. Let $\theta_0$ be any harmonic $1$-form and $u_0$ a solution of (3.4). Then for the twisted harmonic $1$-form $\theta = \theta_0 + du_0$, we have
\begin{equation}
F'' = (4\pi)^{-\frac{n}{2}} \int_{\Sigma} \left| \theta + hH \right|^2 - \frac{1}{2} |y^\perp|^2 e^{-\frac{|x^\perp|^2}{4}} d\mu,
\end{equation}

here $H$ is the mean curvature form $-i_H \omega = -\frac{1}{2} < x, \nu_k > e^k$.

Proof. By (3.4), for $\theta = \theta_0 + du_0$ we have
\begin{equation}
F'' = (4\pi)^{-\frac{n}{2}} \int_{\Sigma} \left| \theta + hH \right|^2 - \theta(Jy^\perp) - h\theta(Jx^\perp) - \frac{1}{2} |y^\perp|^2 - \frac{1}{4} h^2 |x^\perp|^2 e^{-\frac{|x^\perp|^2}{4}} d\mu.
\end{equation}

We now prove that the term $\int_{\Sigma} \theta(Jy^\perp)e^{-\frac{|x^\perp|^2}{4}} d\mu$ vanishes. In fact
\begin{align*}
\int_{\Sigma} \theta(Jy^\perp)e^{-\frac{|x^\perp|^2}{4}} d\mu &= \int_{\Sigma} \theta(e_k) < Jy, e_k > e^{-\frac{|x^\perp|^2}{4}} d\mu \\
&= \int_{\Sigma} \theta(e_k) < Jy, x > e^{-\frac{|x^\perp|^2}{4}} d\mu \\
&= \int_{\Sigma} (d^*_f \theta) < Jy, x > e^{-\frac{|x^\perp|^2}{4}} d\mu \\
&= 0.
\end{align*}

Then for the twisted harmonic $1$-form $\theta = \theta_0 + du_0$, we have
\begin{align*}
F''(y, h, \theta) &= (4\pi)^{-\frac{n}{2}} \int_{\Sigma} \left| \theta + hH \right|^2 - h\theta(Jx^\perp) - \frac{1}{2} |y^\perp|^2 - \frac{1}{2} h^2 |x^\perp|^2 e^{-\frac{|x^\perp|^2}{4}} d\mu \\
&= (4\pi)^{-\frac{n}{2}} \int_{\Sigma} \left( - \sum_k (\theta_k - \frac{1}{2} h < x, \nu_k >) - \frac{1}{2} |y^\perp|^2 \right) \\
&= (4\pi)^{-\frac{n}{2}} \int_{\Sigma} \left( - \left| \theta + hH \right|^2 - \frac{1}{2} |y^\perp|^2 \right).
\end{align*}

□
Remark 3.4. The mean curvature form \( H = H_k e^k \) is twisted harmonic. In fact
\[
d^*_f H = (d^* + i_{\frac{1}{2} x^\top})(-\frac{1}{2} < x, \nu_k > e^k) = 0.
\]
By the fact that a closed self-shrinker has non-vanishing mean curvature and \( H \) is twisted harmonic, one sees that the Maslov class \([H] \neq 0\). Hence any closed Lagrangian self-shrinker must have \( b_1 \geq 1 \). This was first observed by Smoczyk, see [10].

Theorem 3.5. Any closed Lagrangian self-shrinker with \( b_1 \geq 2 \) is Lagrangian F-unstable.

Proof. If \( b_1 \geq 2 \), we can take a nontrivial class \([\theta_0]\) which is different from the class \([H]\). Let \( \theta = \theta_0 + du_0 \) be the twisted harmonic 1-form in \([\theta_0]\). Then it follows from (3.5) that \( F''(y, h, \theta) < 0 \) for all \( y, h \).

By a theorem of Whitney any closed Lagrangian embedding \( \Sigma \) in \( \mathbb{C}^n \) must have vanishing Euler characteristic \( \chi(\Sigma) \) (cf. [34]). If \( n = 2 \), the only closed Lagrangian embedding in \( \mathbb{C}^2 \) is torus.

Corollary 3.2. In \( \mathbb{C}^2 \), there are no embedded closed Lagrangian F-stable Lagrangian self-shrinkers.

4. Hamiltonian F-stability of closed Lagrangian self-shrinkers

In this section we first give the second variation formula of the F-functional under Hamiltonian deformations. We then show that for a closed Lagrangian self-shrinker \( \Sigma \) with \( b_1 = 1 \), the Lagrangian F-stability of \( \Sigma \) is equivalent to the Hamiltonian F-stability of \( \Sigma \). Finally, we characterize the Hamiltonian F-stability of a closed Lagrangian self-shrinker by its spectral property of the twisted Laplacian \( \Delta f \).

Proposition 4.1. Let \( \Sigma \) be a closed Lagrangian self-shrinker that becomes extinct at \((0, 1)\) and \( X \) a Hamiltonian variation with \(-i_X \mathbb{F} = du\). Then the second variation is given by
\[
(4.1) \quad F'' = (4\pi)^{-\frac{n}{2}} \int_\Sigma \left[ |d^*_f du|^2 - |du|^2 - du(Jy^\perp) - \frac{1}{2}|y^\perp|^2 - \frac{1}{4}h^2|x^\perp|^2\right] e^{-\frac{|x|^2}{4}} d\mu.
\]
In particular \( \Sigma \) is Hamiltonian F-stable if and only if for any Hamiltonian variation \( J \nabla u \), there exists a vector \( y \) such that
\[
\int_\Sigma \left[ |d^*_f du|^2 - |du|^2 - du(Jy^\perp) - \frac{1}{2}|y^\perp|^2\right] e^{-\frac{|x|^2}{4}} d\mu \geq 0.
\]

Proof. It follows from (3.3) and
\[
\int_\Sigma du(Jx^\perp)e^{-\frac{|x|^2}{4}} d\mu = \int_\Sigma u_k < Jx, e_k > e^{-\frac{|x|^2}{4}} d\mu
\]
\[
= \int_\Sigma -u_k < x, \nu_k > e^{-\frac{|x|^2}{4}} d\mu
\]
\[
= \int_\Sigma u[-< x, -H_pe_p > - \frac{1}{2} < x, \nu_k > < x, e_k >]e^{-\frac{|x|^2}{4}} d\mu
\]
\[
= 0.
\]
\( \blacksquare \)
We now consider the case that $b_1(\Sigma) = 1$. Note that the mean curvature form $H$ represents the nontrivial Maslov class. Hence any closed 1-form $\theta$ can be written as

\[ \theta = -2aH + du. \]

**Theorem 4.2.** For a closed Lagrangian self-shrinker $\Sigma$ with $b_1 = 1$, the Lagrangian F-stability of $\Sigma$ is equivalent to the Hamiltonian F-stability of $\Sigma$.

**Proof.** Let $X$ be any Lagrangian variation and

\[ \theta = -i_X\omega = -(a < x, \nu_k > + u_k)e^k. \]

By the formula (3.3) and the fact that the mean curvature form $H$ is twisted harmonic,

\[
F'' = (4\pi)^{-\frac{n}{2}} \int_\Sigma [||d^*f||^2 - ||\theta||^2 - \theta(Jy^\perp) - h\theta(Jx^\perp) - \frac{1}{2}||y^\perp||^2 - \frac{1}{4}h^2||x^\perp||^2] e^{-\frac{|x|^2}{4}} d\mu \\
= (4\pi)^{-\frac{n}{2}} \int_\Sigma [||d^*f||^2 - (||du||^2 + a^2||x^\perp||^2 - 2adu(Jx^\perp))] e^{-\frac{|x|^2}{4}} d\mu \\
+ (4\pi)^{-\frac{n}{2}} \int_\Sigma [-a < x, \nu_k > < Jy^\perp, e_k > + du(Jy^\perp)] e^{-\frac{|x|^2}{4}} d\mu \\
+ (4\pi)^{-\frac{n}{2}} \int_\Sigma [-h[a < x, \nu_k > < Jx^\perp, e_k > + du(Jx^\perp)] e^{-\frac{|x|^2}{4}} d\mu \\
+ (4\pi)^{-\frac{n}{2}} \int_\Sigma [-a < x, \nu_k > < Jy^\perp, e_k > + du(Jy^\perp)] e^{-\frac{|x|^2}{4}} d\mu \\
+ (4\pi)^{-\frac{n}{2}} \int_\Sigma [-\frac{1}{2}||y^\perp||^2 - \frac{1}{4}h^2||x^\perp||^2] e^{-\frac{|x|^2}{4}} d\mu. \\
\]

In the proof of Proposition 4.1, we have seen that

\[
\int_\Sigma du(Jx^\perp) e^{-\frac{|x|^2}{4}} d\mu = 0.
\]

Then

\[
F'' = (4\pi)^{-\frac{n}{2}} \int_\Sigma [||d^*f||^2 - (||du||^2 + a^2||x^\perp||^2)] e^{-\frac{|x|^2}{4}} d\mu \\
+ (4\pi)^{-\frac{n}{2}} \int_\Sigma [-a < x, \nu_k > < Jy^\perp, e_k > + du(Jy^\perp)] e^{-\frac{|x|^2}{4}} d\mu \\
+ (4\pi)^{-\frac{n}{2}} \int_\Sigma ah||x^\perp||^2 e^{-\frac{|x|^2}{4}} d\mu \\
+ (4\pi)^{-\frac{n}{2}} \int_\Sigma [-\frac{1}{2}||y^\perp||^2 - \frac{1}{4}h^2||x^\perp||^2] e^{-\frac{|x|^2}{4}} d\mu.
\]

By Lemma 2.1, we have

\[
\int_\Sigma < x, \nu_k > < Jy^\perp, e_k > e^{-\frac{|x|^2}{4}} d\mu = \int_\Sigma < Jx^\perp, Jy^\perp > e^{-\frac{|x|^2}{4}} d\mu \\
= \int_\Sigma < x^\perp, y > e^{-\frac{|x|^2}{4}} d\mu \\
= 0.
\]
Hence

\[ F'' = (4\pi)^{-\frac{n}{2}} \int_{\Sigma} \left[ |d_f^\perp du|^2 - |du|^2 - du(Jy^\perp) - \frac{1}{2} |y^\perp|^2 e^{-\frac{|x|^2}{4}} d\mu \right] + (4\pi)^{-\frac{n}{2}} \int_{\Sigma} - \frac{1}{2} h^2 |x^\perp|^2 e^{-\frac{|x|^2}{4}} d\mu. \]

Comparing this second variation formula with (4.1), we get the equivalence. □

Let

\[ \lambda_1 = \inf \frac{\int_{\Sigma} |d\varphi|^2 e^{-\frac{|x|^2}{4}} d\mu}{\int_{\Sigma} |\varphi|^2 e^{-\frac{|x|^2}{4}} d\mu}, \]

where the infimum is taken over all non-zero \( \varphi \) with \( \int_{\Sigma} \varphi e^{-\frac{|x|^2}{4}} d\mu = 0 \). Then the first eigenfunction \( \varphi_1 \) satisfies

\[ d^*d\varphi_1 + \frac{1}{2} d\varphi_1(x^\top) := -\Delta_f \varphi_1 = \lambda_1 \varphi_1. \]

Let \( \lambda_1 < \lambda_2 < \lambda_3 < \cdots \) be the nonzero eigenvalues of \( \Delta_f \).

**Theorem 4.3.** A closed Lagrangian self-shrinker \( \Sigma \) that becomes extinct at \((0,1)\) is Hamiltonian F-stable if and only if the twisted Laplacian \( \Delta_f \) has

\[ \lambda_1 = \frac{1}{2}, \quad \Lambda_2 = \{ < x, w >, w \in \mathbb{R}^{2n} \}; \quad \lambda_2 \geq 1. \]

**Proof.** Note that for any vector \( w \in \mathbb{R}^{2n} \),

\[ \Delta_f < x, w > = -\frac{1}{2} < x, w >, \]

hence

\[ 0 < \lambda_1 \leq \frac{1}{2}. \]

The formula (4.1) can be rewritten as

\[ F'' = (4\pi)^{-\frac{n}{2}} \int_{\Sigma} \left[ |d_f^\perp du|^2 - \frac{1}{2} |du|^2 - \frac{1}{2} |\nabla u + Jy^\perp|^2 - \frac{1}{4} h^2 |x^\perp|^2 e^{-\frac{|x|^2}{4}} d\mu. \]

If \( \lambda_1 < \frac{1}{2} \), by taking \( u = \varphi_1 \) we get

\[ F'' = (4\pi)^{-\frac{n}{2}} \int_{\Sigma} \lambda_1 (\lambda_1 - \frac{1}{2}) \varphi_1 e^{-\frac{|x|^2}{4}} d\mu \]

\[ + (4\pi)^{-\frac{n}{2}} \int_{\Sigma} [\frac{1}{2} |\nabla \varphi_1 + Jy^\perp|^2 - \frac{1}{4} h^2 |x^\perp|^2] e^{-\frac{|x|^2}{4}} d\mu < 0. \]

Hence \( \lambda_1 = \frac{1}{2} \) is necessary for the Hamiltonian F-stability.

For any first eigenfunction \( \varphi_1 \) with \( -\Delta_f \varphi_1 = \frac{1}{2} \varphi_1 \), we get

\[ F'' = (4\pi)^{-\frac{n}{2}} \int_{\Sigma} [\frac{1}{2} |\nabla \varphi_1 + Jy^\perp|^2 - \frac{1}{4} h^2 |x^\perp|^2] e^{-\frac{|x|^2}{4}} d\mu. \]

Note that

\[ Jy^\perp = (Jy)^\top = \nabla < x, Jy >, \]
hence the following is necessary for the Hamiltonian F-stability

\[ \varphi_1 + \langle x, Jy \rangle = 0. \]

Therefore Hamiltonian F-stability implies that the first eigenfunction space is

\[ \Lambda_{\uparrow} = \{ \langle x, w \rangle, w \in \mathbb{R}^{2n} \}. \]

Here \( \langle x, w \rangle = 0 \) may happen for some \( w \neq 0 \). Note also that the corresponding Hamiltonian variations generated by the first eigenfunction space are the translations \( w^\perp \).

Assuming that \( \lambda_1 = \frac{1}{2} \) and the first eigenfunction space is \( \{ \langle x, w \rangle, w \in \mathbb{R}^{2n} \} \), we now show that the Hamiltonian F-stability is equivalent to \( \lambda_2 \geq 1 \). Given a function \( u \), it admits a decomposition

\[ u = a + \langle x, w \rangle + u_2, \]

such that

\[ \int_{\Sigma} u_2 e^{-\frac{|x|^2}{4}} d\mu = \int_{\Sigma} u_2 < x, z > e^{-\frac{|x|^2}{4}} d\mu = 0, \ \forall z \in \mathbb{R}^{2n}. \]

Then by (4.1) and the above orthogonal condition,

\[ F'' = (4\pi)^{-\frac{n}{2}} \int_{\Sigma} \left[ \frac{1}{4} < x, w >^2 + |d_f^* d u_2|^2 - \frac{1}{2} < x, w >^2 - |d u_2|^2 \right] e^{-\frac{|x|^2}{4}} d\mu \]

\[ + (4\pi)^{-\frac{n}{2}} \int_{\Sigma} \left[ - \langle J y^\perp, w \rangle - d u_2 (J y^\perp) - \frac{1}{2} |y^\perp|^2 - \frac{1}{4} h^2 |x^\perp|^2 \right] e^{-\frac{|x|^2}{4}} d\mu. \]

By Lemma 2.1

\[ \int_{\Sigma} < x, w >^2 e^{-\frac{|x|^2}{4}} d\mu = \int_{\Sigma} 2 |w^\top|^2 e^{-\frac{|x|^2}{4}} d\mu, \]

and by the above mentioned orthogonal condition,

\[ \int_{\Sigma} d u_2 (J y^\perp) e^{-\frac{|x|^2}{4}} d\mu = \int_{\Sigma} < d u_2, d < J y, x >> e^{-\frac{|x|^2}{4}} d\mu \]

\[ = \int_{\Sigma} u_2 d_f^* d < J y, x > e^{-\frac{|x|^2}{4}} d\mu \]

\[ = \int_{\Sigma} u_2 \frac{1}{2} < J y, x > e^{-\frac{|x|^2}{4}} d\mu \]

\[ = 0. \]

Hence

\[ F'' = (4\pi)^{-\frac{n}{2}} \int_{\Sigma} \left[ |d_f^* d u_2|^2 - |d u_2|^2 - \frac{1}{2} |w^\top|^2 + J y^\perp|^2 - \frac{1}{4} h^2 |x^\perp|^2 \right] e^{-\frac{|x|^2}{4}} d\mu. \]

Taking \( y = J w \) and \( h = 0 \), we get

\[ F'' = (4\pi)^{-\frac{n}{2}} \int_{\Sigma} \left[ |d_f^* d u_2|^2 - |d u_2|^2 \right] e^{-\frac{|x|^2}{4}} d\mu, \]

which is nonnegative for all such \( u_2 \) if and only if \( \lambda_2 \geq 1 \). \square

From the proof of Theorem 4.3 we see the following
Theorem 4.4. A closed Lagrangian self-shrinker \( \Sigma \) that becomes extinct at \((0,1)\) is Hamiltonian F-stable if and only if

\[
\int_{\Sigma} |u|^2 e^{-\frac{|x|^2}{4}} d\mu \geq \int_{\Sigma} u^2 e^{-\frac{|x|^2}{4}} d\mu, \quad \text{for all } u \text{ s.t. } \int_{\Sigma} u e^{-\frac{|x|^2}{4}} d\mu = \int_{\Sigma} u^2 e^{-\frac{|x|^2}{4}} d\mu = 0.
\]

We have seen that the Lagrangian F-stability of a closed self-shrinker with \( b_1 = 1 \) is equivalent to the Hamiltonian F-stability. However for \( b_1 \geq 2 \), there is a difference between two stabilities. A simple example is the (Lagrangian) Clifford torus

\[
T^n = \{(z^1, \ldots, z^n) : |z^1|^2 = \cdots = |z^n|^2 = 2\},
\]

whose mean curvature is \( H = -\frac{1}{2} x = -\frac{1}{2} x^\perp \). Hence the Clifford torus is a Lagrangian self-shrinker that becomes extinct at \((0,1)\). By Theorem 3.5 the Clifford torus is Lagrangian F-unstable. On the other hand it is well-known that the first non-zero eigenvalue of \( T^n \) is \( \lambda_1(\Delta) = \frac{1}{2} \), and the first eigenfunction space is spanned by \( \{x^k, y^k\}_{k=1}^m \); the second eigenvalue is \( \lambda_2(\Delta) = 1 \). Hence the Clifford torus is Hamiltonian F-stable.

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