NILPOTENT ORBITS IN VARIATION OF $p$-ADIC ÉTALE COHOMOLOGY

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ABSTRACT. We formulate the analogue of the nilpotent orbits and nilpotent orbit theorem for variation of $p$-adic étale cohomology or crystalline cohomology with respect to the slope filtration. Specifically we show that any such orbit converges to semistable filtration. In continuation we discuss a generalization of $SL_2$-orbit theorem.

1. Introduction

In complex Hodge theory a variation of Hodge structure VHS, $\mathcal{V}$ over the complex manifold $S$ locally defines a period map

$$
\Phi : (\Delta^*)^n \to D/\Gamma
$$

where $D$ is the flag variety of the Hodge filtrations parametrized by the points in $S$ and $\Gamma$ is the monodromy group. $\Delta^*$ is the puncture disc in dimension one. $D$ receives a complex structure by an embedding $D \hookrightarrow \hat{D}$. By factoring with $\Gamma$ one means the period map is a $\Gamma$-equivariant map into $D$. Then a nilpotent orbit in $\hat{D}$ can be written as

$$
\psi(z_1, ..., z_n) = \exp(z_1N_1 + ... + z_nN_n) \exp(n(z_1, ..., z_n)) F_0
$$

on $U^n$ where $U$ is the upper half plane, and $n$ takes values in the nilpotent cone of $\mathfrak{g} = \text{End}(\mathcal{V}_0)$. $F_0$ is the Hodge filtration on $\mathcal{V}_0$ and is a point in $\hat{D}$. The study of the asymptotics of this map is an important subject in Hodge theory, which basically leads to the nilpotent and $SL_2$-orbit theorems.

Key words and phrases. Variation of Hodge structure, étale cohomology, Crystalline cohomology, Hodge-Tate decomposition, Slope filtration, Semistability, Nilpotent Orbits, $p$-adic Nilpotent orbit, Quasi-unipotent F-Isocrystals.
In the $p$-adic case one has the Hodge-Tate decompositions for the étale and crystalline cohomologies when the coefficient system is extended to $\mathbb{C}_p = \mathbb{Q}_p^\wedge$. The Hodge-Tate (HT)-decomposition defines the slope filtration for these vector spaces over $\mathbb{C}_p$ and one can consider the period space for these filtrations with same the Frobenius-Hodge numbers, similar to the complex case. In this case a variation of HT-structure defines a $G = Gal(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$-equivariant map

$$\Theta : S \longrightarrow \mathcal{F}(\nu)^{rig,ss}$$

called also the period map. The flag variety $\mathcal{F}(\nu)^{rig,ss}$ is assumed to be embedded by the Berkovich functor as a rigid analytic space over $\mathbb{C}_p$. $\nu = (\nu_1, ..., \nu_n) \in \mathbb{Q}^n$ are the set of slopes. It is the flag variety of semi-stable HT-filtrations embedded as an open subset in the whole flag manifold $\mathcal{F}(\nu)$. The group $G = Gal(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ plays the role of the monodromy group in this case. A well known theorem by S. Sen says that, for $L$ a $p$-adic field, a finite dimensional $\mathbb{C}_p$-representation $\rho$ of $G_L = Gal(\overline{L}/L)$ onto $V$ can locally be written as

$$\rho(\sigma) = \exp[\theta \log \chi(\sigma)]$$

where $\chi$ is a cyclotomic character. The operator $\theta \in End(V_{\mathbb{C}_p})$ is called the Sen operator associated to the representation $\rho$. We will use this terminology followed by a classical Fourier analysis on the $p$-adic disc due to Amice-Schneider [ST]. Y. Amice introduces a $p$-adic Fourier correspondence which allows to identify the $\mathbb{Q}_p$-adic disc with its character variety; i.e a space of cyclotomic-type characters. This correspondence later was generalized by P. Schneider and Teitelbaum. We consider the disc as the space of characters via the $p$-adic Fourier theory of Amice and Schneider. The identification will help us to shorten the proof of orbit theorems and also use the estimates in [ST] for the convergence in $p$-adic case. It follows that we can write a $p$-adic nilpotent orbit as

$$\exp[N \log(\kappa_z)]\Theta(\kappa_z) = \exp[N \log(\kappa_z)] \exp[n(\chi_z) \log \chi_z]F_0$$

where $\kappa_z$ and $\chi_z$ are cyclotomic type characters and $F_0$ is a reference point. The difference between the orbit in (4) and that in (2) is that the latter is parametrized by characters. In other words we reparametrize the period map of $p$-adic isocrystals in terms of cyclotomic (Lubin-Tate) characters labeled by the points of the $p$-adic disc. Then what we need is that the pointwise parametrization is analytic. This is the point we use the theory in [ST], [BSX]. The application concerns the study of semistability in the limit of a nilpotent orbit of $p$-adic Hodge-Tate structures. The theorem states that the limit filtration associated to a nilpotent element in the
Nilpotent cone of the representation $V$ is semistable. We will call it the limit slope filtration. It will play the role of the limit Hodge filtration in complex Hodge theory. This concept will open the way of doing standard Hodge geometry in the $p$-adic Hodge theory.

The motivation after nilpotent orbit theorem is the $SL_2$-orbit theorem. In complex Hodge theory it is stated as follows. Given any nilpotent orbit $\exp(zN)F$, it is possible to choose a homomorphism of complex Lie groups $\psi : SL(2, \mathbb{C}) \rightarrow G_\mathbb{C}$, and a holomorphic, horizontal, equivariant embedding $\tilde{\psi} : \mathbb{P}^1 \rightarrow \tilde{D}$, which is related to $\psi$ by

\begin{equation}
\tilde{\psi}(h \cdot i) = \psi(h) \circ F,
\end{equation}

Moreover, one can also find a holomorphic mapping $z \mapsto g(z)$ of a neighborhood of $\infty \in \mathbb{P}^1$ into the complex Lie group $G_\mathbb{C}$ such that

\begin{equation}
\exp(zN) \circ F = g(-iz)\tilde{\psi}(z)
\end{equation}

Starting from a $p$-adic nilpotent orbit $\exp(N\log\kappa_z)$ one can similarly embed $N$ into an $sl_2$-triple in $g = End(V)$ where $V$ is an isocrystal. There is 1-1 correspondence between the Nilpotent orbits in $g$ and equivalence class of $\epsilon$-Hermitian Young tableaux, $\epsilon = \pm 1$. By a Gram-Schmid type argument one can choose an orthogonal frame of the $p$-adic local system in a way that it remains orthogonal along a nilpotent orbit. This will lead us in the first step to attach an $SL_2$-orbit to a given nilpotent orbit, such that they are asymptotic to each other. We screen a sketch of the proof in [S] and [P] and a general $p$-adic Lie group argument to explain a partial generalization of $SL_2$-orbit theorem.

**Organization of the text:** Section (2) involves the explanation of Hodge-Tate structure on the étale and crystalline cohomologies. In section (3) we generalize this language to $p$-adic isocrystals and define semistability. Section (4) involves a basic discussion on the period domains of isocrystals, in order to introduce the notations and give some sense on the nature of these spaces. We mainly express the strong stratification property of these spaces following the texts [R], [R1], [R2]. In section (5) we review the Amice-Schneider-Teitelbaum $p$-adic Fourier theory. Our use of this theory is formal and it only makes the proof of the main result in the next section shorter, or more solid. It is possible to skip this language, in the proof of limit theorem however the proof would need adhoc technical facts to be proved. Thus the use of the theory in [ST] will provide a solid concentration in the proof of the nilpotent orbit theorem in $p$-adic case. Section (6) contains the definition of the $p$-adic nilpotent orbits and the analogous theorem for the Schmid nilpotent orbit theorem in the $p$-adic case. The Hilbert-Mumford criterion for semistability has
been used together with several ideas in the Schmid proof in \([S]\) together with some applications of the Schneider-Teitelbaum Fourier theory in the proof. In section (7) we sketch the ideas of \(SL_2\)-orbit theorem, and provide a partial generalization of the work of W.Schmid to the nilpotent orbits of \(p\)-adic isocrystals. Section 8 is an expository description of the monodromy transformation in \(p\)-adic Hodge theory. In Section 9 we have discussed the mixed case.

2. Hodge decomposition and classical nilpotent orbit theorem

The cohomology group \(H^m(X, \mathbb{Q})\) of a compact Ka"hler manifold \(X\) has a canonical decomposition

\[
H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q} (= \bigoplus_{p+q=k} H^q(X, \Omega^p_X); \quad H^{q,p} = \overline{H^{p,q}}
\]

for all \(0 \leq k \leq 2(n = \dim(X))\) where \(\Omega^q_X\) is the sheaf of \(q\)-differentials on \(X\), known as Hodge decomposition. This decomposition is equivalent to the existence of the Hodge filtration defined by \(F^p H^k_C := \bigoplus_{r \geq p} H^{r,s}(X)\). The combination of this filtration with the symmetry \(H^{p,q} = \overline{H^{q,p}}\) is called a Hodge structure. Complex Hodge theory associates to a smooth projective morphism \(f : X \to S\) of complex quasi-projective varieties, the variation of Hodge structure \(V_s := H^k(X_s = f^{-1}(s))\) over \(S\), for each \(0 \leq k \leq 2n\) and defines a map \(\phi_k : S \to D\). The set \(D\) is the classifying space of polarized Hodge structure, which inherits a complex structure by embedding \(D \hookrightarrow \tilde{D}\) into its compact dual. If \(S \hookrightarrow \tilde{S}\) is a partial compactification, then \(V\) degenerates along the boundary \(S \setminus S\). The data of such a variation of Hodge structure can also be encoded in a semisimple representation

\[
\rho : \pi_1(S, s) \to Aut(V = V_s)
\]

It is convenient to take \(S = \Delta^*\) and consider unipotent monodromies \(T_j = \exp(-N_j)\). Then the period map is

\[
(\Delta^*)^r \to D/\Gamma, \quad s \mapsto F_s^\bullet
\]

where \(\Gamma = Im(\rho)\). The period space \(D/\Gamma\) should be understood as a moduli of Hodge flags moduli the equivalences coming from the automorphisms of the ambient (the fibers) variety. By the Monodromy Theorem, \([S]\), the eigenvalues of the generators of the monodromy group \(\Gamma\) are roots of unity. Therefore possibly after passing to a finite cover of the punctured disc, we have the lift of the period map
\( F : \mathbb{U}^r \rightarrow D \)

where \( U \) is the upper half plane, and the covering map \( U^r \rightarrow (\Delta^\times)^r \) is defined by

\( s_j := \exp(2\pi i z_j) \)

The map

\[
\Psi(z_1, \ldots, z_r) = \exp(\sum z_j N_j) F(z_1, \ldots, z_r), \quad \Psi : \mathbb{U}^r \rightarrow \mathcal{D}
\]

is holomorphic. A classical question in Hodge theory is to study the asymptotic of the Hodge decomposition along a degenerating limit. This idea can be described by the notion of nilpotent orbit.

**Definition 2.1.** A nilpotent orbit is a holomorphic horizontal map \( \theta : \mathbb{C}^r \rightarrow \mathcal{D} \) given by

\( \theta(z) = \exp(\sum z_i N_i) F_0 \)

where \( N_j \in \mathfrak{g}_\mathbb{R} := \text{End}(V)_\mathbb{R} \) are nilpotent and \( F \in \mathcal{D} \) such that there is a constant \( \alpha \) such that \( \theta(z) \in D \) for \( \text{Im}(z) > \alpha \).

We have the following well known fact.

**Theorem 2.2.** (Nilpotent Orbit Theorem - W. Schmid, [S]) Let \( N_1, \ldots, N_r \) be monodromy logarithms and \( \psi : (\Delta^\times)^r \rightarrow \mathcal{D} \) be the untwisted period map. Then

- The limit \( F_\infty := \lim_{s \rightarrow 0} \psi(s) \) exists and \( \psi(s) \) extends holomorphically to \( \Delta^r \).
- The map \( \theta(z) = \exp(\sum z_j N_j) F_\infty \) is a nilpotent orbit.
- For any \( G \)-invariant distance on \( D \), there exists positive constants \( \beta, K \) such that for \( \text{Im}(z_j) > \alpha \), such that \( d(\theta(z), F(z)) \leq K \sum_j (\text{Im}(z_j))^{\beta} e^{-2\pi \text{Im}(z_j)} \).

The theorem says; there exists an associated nilpotent orbit

\( \theta(z) = \exp(\sum z_j N_j) F_\infty \)
which is asymptotic to $F(z)$ with respect to a suitable metric on $D$. The map $\theta$ in (15) is horizontal in the sense by Griffiths. The nilpotent orbit theorem for period mapping is closely related to regularity of the Gauss-Manin connection. For a proof using regularity see [GS]. We propose to generalize this theorem on the $p$-adic domains. The correspondence between the local system of Hodge structures and flat connections on $S$ satisfying Griffiths transversality conditions is explained by the so-called Riemann-Hilbert correspondence, [KP], [P], [S].

When $X$ is defined over an $l$ or $p$-adic field $L$, ($p$ is the characteristic of the ground residue field and $l \neq p$ a prime), the groups $H^k_{dR}(X/L)$ of algebraic de Rham cohomology is defined similar to the complex case. If $X$ is proper smooth then the Hodge to de Rham spectral sequence

$$H^q(X, \Omega^p_{X/L}) \Rightarrow H^{p+q}_{dR}(X)$$

degenerates at $E_2$ page and there exists a filtration $F^\bullet$ on each $H^k_{dR}(X/L)$ namely Hodge filtration such that

$$gr^i F H^k_{dR}(X/L) = H^{k-i}(X, \Omega^i_{X/L})$$

where $\Omega^i_{X/L}$ is the sheaf of $i$-th algebraic differential forms on $X$.

The $p$-adic étale cohomologies of $X$ are defined by

$$H^k_{et}(X, \mathbb{Q}_p) = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \lim_{\leftarrow} H^k(X, \mathbb{Z}/p^n), \quad 0 \leq k \leq 2n$$

The étale cohomology with $\mathbb{C}_p$ coefficient also satisfy a decomposition property.

**Theorem 2.3. (J. Tate) [BX]** Let $X$ be a projective, non-singular variety over $L$. The $p$-adic étale cohomologies $H^k_{et}(X_{\mathbb{Q}_p}, \mathbb{Q}_p)$ satisfy the Hodge-Tate decomposition

$$H^k_{et}(X_{\mathbb{Q}_p}, \mathbb{Q}_p) \otimes \mathbb{C}_p = \bigoplus_{i=0}^k H^{k-i}(X, \Omega^i_{X/L}) \otimes \mathbb{C}_p(-i)$$

for $0 \leq k \leq 2n$.

The theorem states that the étale cohomology of smooth projective varieties decomposes over $\mathbb{C}_p$. The filtration

$$F^p H^k_{et}(X_{\mathbb{Q}_p}, \mathbb{C}_p) := \bigoplus_{r \geq p} H^{k-r}(X, \Omega^r_{X/L}) \otimes \mathbb{C}_p(-i)$$
corresponds to (8) and is called the Hodge-Tate filtration. The integers \{-i\} appearing as the twists in the decomposition are called the Hodge-Tate weights. The groups \(H^k_{et}(X_{\overline{Q}_p}, \mathbb{Q}_p)\) are \(\mathbb{Q}_p\)-vector spaces endowed with continuous action

\[
\varrho : \text{Gal}(\overline{\mathbb{Q}_p}/L) \to \text{Gl}(H^k_{et}(X_{\overline{Q}_p}, \mathbb{Q}_p))
\]

In general any representation of the \(G_L\) satisfying a decomposition like (19) is called Hodge-Tate representation with HT-weights \{-i\} appearing in the sum. The theorem says that after tensoring with \(\mathbb{C}_p\) the Galois action is diagonalizable as \(\bigoplus_i \mathbb{C}_p(-i)^{\otimes h_i,m_i-i}\) with eigenvalues given by the twists of cyclotomic character. If \(X\) is proper smooth over \(L\), there are functorial comparison morphisms

\[
\gamma_k : H^k_{dR}(X) \otimes_K \mathbb{B}_{dR} \cong H^k_{et}(X_{\overline{Q}_p}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{B}_{dR}
\]

which respects the Hodge filtrations and the Galois actions. We have

\[
F^i(\mathbb{B}_{dR} \otimes H^k_{dR}(X)) = \sum_{a+b=i} F^a \mathbb{B}_{dR} \otimes F^b H^k_{dR}(X)
\]

The field \(\mathbb{B}_{dr}\) is the quotient of the Fontaine ring

\[
\mathbb{B}_{dR}^+ = W(\text{quot}(\lim_{x \to x^p} \mathcal{O}_K/p))
\]

It is equipped with an action of a Frobenius. \(\mathbb{B}_{dR}^+\) is a complete non-discrete valued ring with perfect residue field. The filtration \(F^i\) is defined as

\[
F^i \mathbb{B}_{dR} = \mathbb{B}_{dR}^+ \xi^i, \quad \xi \text{ a uniformizer, } i \in \mathbb{Z}
\]

It is known any non-constant polynomial in \(\mathbb{Q}_p[t]\) has a root in \(\mathbb{B}_{dR}\), i.e \(\overline{\mathbb{Q}_p} \subset \mathbb{B}_{dR}\), [BX], [FO].

Assume \(L = \text{quot}(W = W(k))\) for a perfect field \(k\) of \(\text{char}(k) = p\) and we have a proper smooth variety \(X/W\) whose generic fiber is \(X\). P. Berthelot defines the crystalline functor

\[
H^k_{crys} : X \mapsto H^k_{crys}(X/W) = \lim_{\leftarrow} H^k(X/W_n), \quad W_n := W/p^nW
\]

The crystalline cohomology groups \(H^k_{crys}(X/W)\) are finite dimensional \(W\)-vector spaces with an action of a \(\sigma\)-linear map \(\Phi\) called Frobenius (\(\sigma\) is the Frobenius of \(L\)). They have the expected dimension \(\dim_{\overline{\mathbb{Q}_l}} H^k_{et}(X_{\overline{K}}, \mathbb{Q}_l), \ l \neq p\). The \(W\)-structure
of $H^k_{\text{crys}}(X/W)$ inside $H^k_{dR}(\mathcal{X})$ endowed with Frobenius action does not depend on choice of proper smooth model $\mathcal{X}$ of $X$. The crystalline cohomology is endowed with a canonical cup product $H^i \times H^j \rightarrow H^{i+j}$ and also a trace map $\text{tr} : H^{2n}(X/W) \rightarrow W$. It is related to the de Rham cohomology of $X/k$ by the long exact universal coefficient sequence

\[ \cdots \rightarrow H^i_{\text{crys}}(X/W) \xrightarrow{\partial} H^i_{\text{crys}}(X/W) \rightarrow H^i_{dR}(X/k) \rightarrow H^{i+1}_{\text{crys}}(X/W) \rightarrow \cdots \]

If $X$ has a smooth proper lifting $\mathcal{X}$ then one has $H^k_{dR}(\mathcal{X}/W) \cong H^k_{\text{crys}}(X/W)$. In general this isomorphism will hold true after extension of the scalars to a finite extension of $W$, [O2]. We have similar decomposition and functorial comparison isomorphisms

\[ H^k_{\text{crys}}(X/W) \otimes \mathbb{C}_p = \bigoplus_{r+s=k} H^{r,s}_{\text{crys}}(X)(-r) \]

\[ \gamma^\text{crys}_k : H^k_{\text{crys}}(X) \otimes L B_{\text{cris}} \cong H^k_{\text{et}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p) \otimes \mathbb{Q}_p B_{\text{cris}} \]

The twist means twisting with the cyclotomic character $\chi : \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \rightarrow \mathbb{Z}^\times_p$ to the power $-r$, (See [FO] or [BX] for the definition of $B_{\text{cris}}$). The comparison maps respect the Hodge filtration and Galois action. It follows that the $p$-adic étale and crystalline cohomology are essentially the same and determine each other. Define the descending filtration

\[ F^i := \{ x \in H | \Phi(x) = p^i.x \} \]

namely Frobenius Hodge filtration on the crystalline cohomology, [BX], [O2], [H], [Y].

**Theorem 2.4.** [O2] Assume the Hodge spectral sequence of $X/L$ degenerates at $E_1$ and the $H^*_{\text{crys}}(\mathcal{X})$ are torsion free. Then the image of the natural map

\[ F^i H^k_{\text{crys}}(X/W) \rightarrow H^k_{dR}(X/L) \]

is the Hodge filtration.

In this case the Frobenius viewed as a linear map $\Phi : \sigma^* H^k_{\text{crys}}(\mathcal{X}) \rightarrow H^k_{\text{crys}}(\mathcal{X})$ is diagonalizable, where $\sigma$ is the Frobenius of $L$. If $H^k_{\text{crys}}$ is torsion free there is a basis of this vector space with respect to which the matrix of $\Phi$ is diagonalizable. Then
the Frobenius Hodge numbers $h^{i,j}(\Phi)$ are defined as the number of diagonal terms in the matrix whose $p$-adic ordinal is $i$. This number can also be defined as

$$h^{i,j}(\Phi) = gr^i_F gr^j_F H^k_{\text{crys}}(X/W)$$

where $F$ is the opposite filtration defined by $F^i := p^i F^{n-i}$, \cite{O1}.

**Theorem 2.5. (B. Mazure) [KL]** Assume $X/W$ is smooth, then the Frobenius Hodge numbers are equal to geometric Hodge numbers.

$$h^{i,j}(\Phi) = h^{i,j}(X/k) = \dim H^j(X, \Omega^i_{X/k})$$

The similarity of the behavior of the Frobenius Hodge numbers motivates the fact that analogous period domains may be defined for étale or crystalline cohomologies with $\mathbb{C}_p$-coefficients. The strategy we follow concerns a representation theory uniformization of the $p$-adic disc to explain the variations of étale cohomology or crystalline cohomology. As a final remark let's add that the decompositions (19) and (28) can also been stated for open non-smooth separated variety over $L$ with some little modifications, (see \cite{Y} for details).

### 3. Isocrystals

The étale and crystalline cohomologies are examples of more general objects namely (overconvergent) $\Phi$-isocrystals.

**Definition 3.1. [DOP]** Assume $k$ is a field of $\text{char}(k) = p > 0$, $W(k)$ the associated ring of Witt vectors and $L$ its field of quotients. Let $\sigma \in \text{Aut}(K/\mathbb{Q}_p)$ be the Frobenius.

- A $\Phi$-crystal $M$ over $k$ is a free $W(k)$-module of finite rank with a $\sigma$-linear endomorphism $\Phi : M \to M$ such that $M/\Phi M$ has finite length.

- An $\Phi$-isocrystal $E$ over $k$ is a finite dimensional $L$-vector space with a $\sigma$-linear bijective endomorphism of $\Phi : E \to E$.

A morphism $f : E \to E'$ of $\Phi$-isocrystals is called an isogeny if there exists $g : E' \to E$ such that $fg = gf = p^n$ for some $n$. The category of $\Phi$-isocrystals is obtained from the category of $\Phi$-crystals by formally inverting isogenies. If the ground field $k$ is algebraically closed we have the following structural fact.
Theorem 3.2. (J. Dieudonné) \[R\], \[DOP\] Let \( k \) be algebraically closed. Then the category of \( \phi \)-isocrystals over \( k \) is semisimple and the simple objects are parametrised by the rational numbers. The simple object \( E_\lambda \) corresponding to \( \lambda \in \mathbb{Q} \) is

\begin{equation}
E_\lambda = (K^s, \Phi_\lambda = \begin{pmatrix} 0 & 0 & 0 & p^r \\ 1 & 0 & 0 & 0 \\ . & . & . & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \lambda = r/s, s > 0, (r, s) = 1)
\end{equation}

Furthermore, \( \text{End}(E_\lambda) = D_\lambda \) where \( D_\lambda \) is the division algebra with center \( \mathbb{Q}_p \) and invariant \( \lambda \in \mathbb{Q}/\mathbb{Z} \).

The rational number \( \lambda \) is called the slope of \( E_\lambda \). It follows that there exists a unique decomposition

\begin{equation}
E = \bigoplus_{\lambda \in \mathbb{Q}} E_\lambda
\end{equation}

where \( E_\lambda \) has slope \( \lambda = r/s \). This means; there exists a \( W(k) \)-lattice \( M \subset V_\alpha \) such that \( \Phi^a M = p^r M \). The filtration \( F_\beta = \bigoplus_{\lambda \leq -\beta} E_\lambda \) is called the slope or Newton filtration.

By ordering the slopes \( \lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n \) one can define the Newton polygon of \( E \) as the graph of the function

\begin{equation}
i \mapsto \lambda_1 + \lambda_2 + ... + \lambda_i, \quad 0 \mapsto 0
\end{equation}

When \( k \) is the algebraic closure of \( \mathbb{F}_q \), \( q = p^h \) the Newton polygon is the Newton polygon of the polynomial \( \det(1 - \Phi^h t) \). The Newton polygon describes the combinatorics of the slopes of an isocrystal in a simple way. It is possible to classify \( \Phi \)-isocrystals in low dimensions with the aid of their Newton polygons.

Another fact we are going to deal with is an analogue of monodromy operator for \( p \)-adic isocrystals (see also Section 8). It will play an important role to us to state the result of this paper.

Theorem 3.3. (S. Sen) \[S\], \[FO\] Suppose \( L \) is a \( p \)-adic field. If \( \rho : G_L \to \text{Aut}_{\mathbb{C}_p}(V) \) is a \( \mathbb{C}_p \)-representation of \( G_L \) on \( V \), then there exists a unique operator \( \Theta \in \text{End}(V) \) such that for every \( \omega \in G_L \), there exists an open subset \( U_\omega \subset G_L \).

\begin{equation}
\rho(\sigma) = \exp[\Theta \log \chi(\sigma)], \quad \sigma \in U_\omega
\end{equation}
where \( \chi \) is a cyclotomic character.

The operator \( \Theta \) in this theorem is called Sen operator. We have stated this theorem in its simplest form that is over \( \mathbb{C}_p \) and is sufficient for our purpose. For more details see the reference. One can prove that two \( \mathbb{C}_p \) representations are isomorphic if and only if their corresponding Sen operators defined by Theorem 3.3 are similar. It is a famous result by Sen that the Lie algebra \( g = \text{Lie}(\rho(G_L)) \) is the smallest \( \mathbb{Q}_p \)-space \( S \subset \text{End}_{\mathbb{Q}_p} V \) such that \( \Theta \in S \otimes_{\mathbb{Q}_p} \mathbb{C}_p \). The Sen theorem philosophically explains our method to state nilpotent orbits in the \( p \)-adic Hodge theory.

The Category of isocrystals is equipped with the data of a nilpotent transformation \( N : E \rightarrow E \) which satisfies

\[
N \circ \phi = p^r \phi \circ N, \quad \text{some } r \in \mathbb{Z}_{>0}
\]

This transformation may be derived from the Theorems 3.3, but it can be independent. In this case a \( \Phi \)-isocrystal may also be called \( (\Phi, N) \)-modules.

**Example 3.4.** Take \( L = \mathbb{Q}_p(\sqrt[p]{1}) \), and \( V = L^2 \cong L e_1 \oplus L e_2 \). Consider the following data \( \phi(e_1) = e_1, \phi(e_2) = p^{k-1} e_2 \) and
\[
F^{k-1} = L(e_1 + \pi^i e_2)
\]
for \( i \in \{0, \ldots, p-2\} \) and set \( N = 0 \), and the Galois action \( g.e_1 = e_1, g.e_2 = \chi_L(g)^{-1}.e_2 \) for \( g \in \text{Gal}(L/\mathbb{Q}) \). Where \( \chi_L \) is Teichmuller lift of the cyclotomic character. Then \( V \) is a crystalline representation of \( G_L \), i.e defines \( \Phi \)-isocrystals with HT-weights \( \{0, k-1\} \). 

A basic notion related to isocrystals is that of semi-stability.

**Definition 3.5.** If \( (V, F) \) is a filtered \( L \)-vector space, then define the degree and the slope of the filtration \( F \)

\[
\text{deg}(V, F) = \sum_i i \cdot \dim gr^i F(V), \quad \mu(V, F) = \text{deg}(V, F)/\text{rank}(V, F)
\]

respectively. A filtration \( F^\bullet \) of an isocrystal \( M \) is called semistable if \( \mu(N, F^\bullet) \leq \mu(M, F^\bullet) \) for any subisocrystal \( N \) of \( M \).

The degree is an additive function on the category of filtered vector spaces and also one has \( \text{deg}(V) = \text{deg}(\bigwedge^{\text{max}} V) \). The slope satisfies \( \mu(V \otimes W) = \mu(V) + \mu(W), \mu(V^*) = -\mu(V) \). We purpose to study semistability in a variation of \( \Phi \)-isocrystals. Particularly we are interested to check out this property in the limits of special one parameter family of \( p \)-adic isocrystals.
The concept of (semi)-stability can also be defined for vector bundles which plays a central role in the theory of moduli spaces. The next theorem called the Hilbert-Mumford inequality for a semistable vector bundles in GIT explains a criteria for this. Assume that the filtration $F$ of $V$ is split semisimple, that is $V = \bigoplus V_i$ with $V_i$ of pure type. This decomposition correspond to a 1-parameter subgroup of $G = GL(V)$ namely $\lambda : \mathbb{G}_m \to Sl(V)$. By considering a suitable basis $\{s_0, ..., s_n\}$ of $V$ we can write $\lambda$ diagonally as

$$\lambda(t) = \text{diag}[t^{\rho_0}, ..., t^{\rho_n}t^{-p}], \quad \rho_0 \geq ... \geq \rho_n$$

where $\rho = \sum \rho_i/(n+1)$. Mumford introduced a subsheaf $L(\nu) = (t^{\rho_0}s_0, ..., t^{\rho_n}s_n) \subset O_{X \times \mathbb{A}^1}(1)$ generated by the sections in the parenthesis. The semistable subset $F(\nu)^{ss}$ of $F(\nu)$ can also be described in terms of geometric invariant theory.

**Theorem 3.6.** [DOP], [R2] A point $F \in F(\nu)$ is semistable if and only if

$$\mu^{L(\nu)}(F, \lambda) \leq 0$$

The line bundle $L(\nu)$ can also be defined as having the fiber $\bigotimes_{i=1}^r \text{det}(F_i/F_{i-1})^{\nu(i)}$. We will use this criterion in order to check out the semistability in the limit of a nilpotent orbit, in Section (6).

4. Filtration by slopes and Period domains over $p$-adic fields

In this section we briefly review the structure of period domains over $p$-adic fields from the references [R], [R1] and [DOP]. A short discussion in terms of the language of $p$-divisible groups and their deformation spaces has been added at the end. We begin with A direct consequence of the Theorem 2.2 that is if $k$ is algebraically closed, there is an injection

$$\{\text{Isomorphism classes of } \Phi - \text{isocrystals of rank } n\} \hookrightarrow (\mathbb{Q}^n)_+$$

$$E \mapsto \nu(E)$$

called Newton map where $(\mathbb{Q}^n)_+ := \{(\nu_1, ..., \nu_n) \in \mathbb{Q}^n | \nu_1 \geq ... \geq \nu_n\}$ where $\lambda$ occurs in $\nu(E)$ with multiplicity the dimension of the isocrystal (isotypic or isoclinic also used) component of type $\lambda$. The image of the Newton map can be described as follows. Write $\nu \in (\mathbb{Q}^n)_+$ as

$$\nu = (\nu_1^{n_1}, ..., \nu_r^{n_r}), \quad \nu_1 > ... > \nu_r$$
Then

$$\nu \in \text{Image} \iff \nu_i n_i \in \mathbb{Z}, \forall i$$

Assume for example we have a family of abelian varieties $A/S$ over a base scheme of characteristic $p > 0$. When $l \neq p$ the family of Tate modules $T_l(A_s)$, for $s \in S$ defines a local system of $\mathbb{Z}_l$-modules on $S$. If $l = p$, the Tate modules $T_l(A_s)$ are replaced by Dieudonné modules $M(A_s)$, which are $\Phi$-crystals. In this case the Dieudonné modules are not constant as $s$ varies. There exists a stratification of $S$ with locally closed strata’s where the isomorphism class of the Dieudonné module $M(A_s) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ are constant, [R], [DOP].

**Theorem 4.1.** [R], [DOP] Let $E$ be $\Phi$-isocrystals over a scheme $S$ of characteristic $p$. Then the Newton vector of $E_s$ goes down under specialization, that is when $E$ is of constant rank the function $s \mapsto \|\nu(E_s)\|$ is locally constant on $S$ and for any $\nu_0$, the set $\{s \in S \mid \nu(E_s) \leq \nu_0\}$ is locally closed in $S$.

A non-trivial consequence of this theorem is the following stratification property of period domains of $\Phi$-isocrystals.

**Corollary 4.2.** [R], [DOP] When $E$ be $\Phi$-isocrystals over a noetherian scheme $S$ of characteristic $p$, then the set of points of $S$ where the Newton vector is constant is locally closed in $S$ and defines a finite decomposition of $S$.

Suppose $V$ is an $L$-vector space of dimension $n$, and let $(\nu(1) > \ldots > \nu(r)) \in \mathbb{Q}^r$ be arbitrary. Let

$$\nu = (\nu(1)^{n_1}, \ldots, \nu(r)^{n_r})$$

Define

$$\mathcal{F}(\nu) := \{ 0 \subset V_1 \subset \ldots \subset V_r = V \mid \text{rank}(gr^i_F(V)) = n_i \}$$

Berkovich defines a natural analytification functor $\mathcal{F}(\nu) \mapsto \mathcal{F}(\nu)^{an}$ into a smooth compact $K$-manifold, which satisfies all the expected compatibility properties op. cit, [DOP]. The group $Gl(V)$ acts transitively on $\mathcal{F}(\nu)$. Let $\mathcal{F}(\nu)^{ss}$ be the locus of semistable filtrations. $\mathcal{F}(\nu)^{ss}$ is the period domain associated to $(V, \nu)$, and is Zariski open in $\mathcal{F}(\nu)$ for basic reasons. The structure theory of flag variety $\mathcal{F}(\nu)$ as a symmetric space can be studied via the root system of the Lie group $G = Gl(V)$ and its Lie algebra. The basics of this approach proceeds almost the same as the complex case. In this case $\mathcal{F}(\nu)$ has the structure of Schubert variety, that its stratification is
through the Schubert cells corresponding to closure of torus orbits, via the moment map, see [DOP] for details.

**Theorem 4.3.** [DOP] Let $E$ be an isocrystal, and $\nu \in (\mathbb{Q}^n)_+$. 

- The subset $\{ x \in \mathcal{F}(\nu)^{\text{an}} \mid F_x \in \mathcal{F}(\nu)(\overline{k(x)})^{\text{ss}} \}$, is open in $\mathcal{F}(\nu)^{\text{an}}$, and hence is the underlying space of an analytic space. (The field $k(x)$ is the residue field of the local ring at the point $x$ and bar means completion).

- The map

$$(46) \quad \mathcal{F}(\nu) \to \mathbb{R}, \quad x \mapsto \mu_{F_x}(E \otimes \overline{k(x)})$$

is upper semi-continuous and locally constant.

- There exists a rigid analytic space $\mathcal{F}(\nu)^{\text{ss,rig}}$ and a fully faithful functor $\mathcal{F}(\nu)^{\text{ss,an}} \to \mathcal{F}(\nu)^{\text{ss,rig}}$.

By rigid space we mean it is locally isomorphic to an affinoid space on which the Galois group $G$ acts. An affinoid space is a space isomorphic to a subset of $p$-adic disc defined by the zero set of some ideal of convergent power series. A rigid analytic space is the analogue of the complex analytic space over a non-archimedean field. We will always assume that $\mathcal{F}(\nu)$ is embedded into its rigid analytification via the Berkovich functor and Theorem (4.3), without mentioning it, see [DOP], [SW] for details.

By Theorem 4.1, it follows that

$$(47) \quad \overline{\mathcal{F}_\nu} \subset \bigcup_{\nu' \leq \nu} \mathcal{F}_{\nu'}$$

The property (47) distinguishes the $p$-adic period domains from the complex ones. This inclusion is refereed to a strong stratification property (also called Newton stratification) of the period domains over (finite and) $p$-adic fields. It raises many structural open questions about these spaces. For instance a major open question concerns if the closure of each strata $\mathcal{F}(\nu')$ meets $\mathcal{F}(\nu)$ for $\nu' \leq \nu$.

**Example 4.4.** [R] Let $g$ be a positive integer, and $m \geq 3$ prime to $p$ an auxiliary integer. The Siegel moduli space $M_{g,m}$ of genus $g$ over $\mathbb{F}_p$ classifies the isomorphism classes of triples $(A, \lambda, \eta)$ where $A$ is an abelian scheme of relative dimension $g$ over $S$, and $\lambda : A \to \hat{A}$ is a polarization, and $\eta$ is a level $m$ structure. The existence of polarization implies that the Newton vector of the fibers lie in

$$(48) \quad (\mathbb{Q}^{2g})_+^1 = \{ (\nu_1, ..., \nu_{2g}) \in (\mathbb{Q}^{2g})_+ \mid \nu_i + \nu_{2g-i+1} = 1, \ 0 \leq \nu_i \leq 1 \}$$
Let $B_g$ be the subset satisfying the integrality condition, partially ordered. The vector $g = (1^g, 0^g)$ is the maximal element called ordinary Newton vector. In contrary $\sigma = ((1/2)^{2g})$ is the minimal one called supersingular Newton vector. One can show that

$$S_\nu = \bigcup_{\nu' \leq \nu} S_{\nu'}$$

(49)

Let $\Delta(\nu) = \{(i, j) \in \mathbb{Z}^2| 0 \leq i \leq g, \sum_{l=1}^{i} \nu_{2g-l+1} \leq j < i\}$ and $d(\nu) = \# \Delta(\nu)$. Then it is known that $S_\nu$ is equidimensional of dimension $d(\nu)$. One should notice that the inclusion in (47) is in general strict, rather (49).

There exists analogous (dual) machinery of isocrystals in terms of the language of $p$-divisible groups. A $p$-divisible group may be defined as an inverse system of finite group schemes $\{G_{p^h}\}_h$ over a base formal scheme $S$. In this case $h$ is called the height and $p$ is called the characteristic of the group.

- There is an anti-equivalence between the category of $p$-divisible groups over $\text{spec}(k)$ and the subcategory of $\Phi$-crystals over $\text{spec}(k)$ consisting $(E, \Phi)$ such that $p.E \subset F.M$.

- There exists an anti-equivalence between the category of $p$-divisible groups over $\text{spec}(k)$ up to isogeny and the full subcategory of $\Phi$-isocrystals over $\text{spec}(k)$ with slope between 0 and 1.

In this way the classifying space of the $\Phi$-isocrystals is interpreted as deformation space of the $p$-divisible group. In the following we briefly introduce this concept. Assume $\mathcal{O}_L$ is a complete discrete valuation ring with uniformizer $\pi$, we denote by $\mathcal{Nilp}(\mathcal{O}_L)$ the category of locally noetherian schemes $S$ over $\text{Spec}(\mathcal{O}_L)$ such that the ideal sheaf $\pi.\mathcal{O}_L$ is locally nilpotent. Let $\overline{S}$ be the closed subscheme defined by $\pi.\mathcal{O}_L$. The following theorem describes a typical deformation space for $p$-divisible groups.

**Theorem 4.5.** [RU] Let $X$ be a $p$-divisible group over $\text{spec}(k)$. Consider the functor $\mathcal{M} : \mathcal{Nilp}(W(k)) \to p\text{-DIV}/\overline{S}$ which associates to a $S \in \mathcal{Nilp}(W(k))$ the pair $(X, g)$ consisting of a $p$-divisible group over $\text{Spec}(k)$ and a quasi-isogeny $g : X \times_k \overline{S} \to X \times_S \overline{S}$. Then $\mathcal{M}$ is representable by a formal scheme locally formally of finite type over the formal scheme $\text{Spf}(W(k))$.

The structure sheaf of a formal scheme $\text{Spf}(A)$ where $A$ is complete with ideal of definition $m$ is $\text{lim}_{\leftarrow} A/m^n$. This representability implies that certain functors on the
category of $p$-divisible groups with endomorphisms or polarizations are also representable. We have the following analogue of the Theorems 4.1 and 4.2.

**Theorem 4.6.** [R] Assume $S$ is a regular scheme of char $= p > 0$, and $X$ is a $p$-divisible group over $S$ with constant Newton vector $\nu$. Then $X$ is isogenous to a $p$-divisible group $Y$ which admits a filtration by closed embeddings $0 = Y_0 \subset Y_1 \subset \ldots \subset Y_r = Y$ satisfying integrality conditions (43). There also exists natural numbers $r_i \geq 0$, $s > 0$ such that $\nu(i) = r_i/s_i$ and

\[ p^{-r_i}F_{r_i^{s_i}} : Y_i \to Y^{(\sigma^{s_i})}_i \]

and

\[ p^{-r_i}F_{r_i^{s_i}} : Y_i/Y_{i-1} \to (Y_i/Y_{i-1})^{(\sigma^{s_i})} \]

are isomorphisms.

The superfix in Theorem means the fixed element under the action. If $X$ is a $p$-divisible group of height $h$ over $S$ of characteristic $p$ s.t (43) holds, we can associate a lisse $p$-adic sheaf of $W(F_p)$-modules $V_X = \lim_{\leftarrow} V_{X,n}$ for the étale topology on $S$ by

\[ V_{X,n} = \{ x \in M/p^nM \mid p^{-r}F^s(x) = x \} \]

where $M$ is the Dieudonné crystal of $X$. The fibers of $V_X$ are free $W(F_p)$-modules of rank $h$. The corresponding $W(F_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$-adic sheaf depends on the isogeny class of $X$ and corresponds to a representation of the fundamental group

\[ \varrho_X : \pi_1(S) \to \text{Gl}_h(W(F_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) \]

The stratification in Theorem 4.6 decomposes the representation $\varrho = \bigoplus_i \varrho_i$ by the blocks of size $h_i = \text{height}(Y_i/Y_{i-1})$, cf. [R].

**5. Representation interpretation of the $p$-adic disc**

**($p$-adic Fourier theory)**

In the following we give a brief description on representation theory interpretation of the disc in $p$-adic theory, using a method due to Amice in 60s, [A], [ST], [BSX]. Let $B$ be the unit disc over $\mathbb{Q}_p$. Let $\mathcal{O}(B)$ (resp. $\mathcal{O}^b(B)$) be ring of holomorphic (resp. bounded holomorphic) functions on $B$. Denote by

\[ \mathcal{R}(B) = \text{functions holomorphic on the anulus } r < |z| < 1 \text{ for some } r \]
Let $\mathcal{E}^+(B)$ be the ring of bounded elements in $\mathcal{R}(B)$. It is equipped with the norm $\|\sum a_n z^n\| := \sup_n |a_n|$ and we call $\mathcal{E}(B)$ its norm completion. All of these rings carry a monoid action of $\mathbb{Z}_p \setminus 0$ by a power series

\begin{equation}
(55) \quad a_\ast(z) = [a](z) := (1 + z)^a - 1
\end{equation}

and all have the structure of a $(\Phi, \Gamma)$-module with $\Gamma = \mathbb{Z}_p^\times$, $\phi = p_*$. A $(\Phi, \Gamma)$-module with a semilinear action of $\mathbb{Z}_p \setminus 0$. Let

\begin{equation}
(56) \quad \chi : G_{\mathbb{Q}_p} \rightarrow \mathbb{Z}_p^\times, \quad H_{\mathbb{Q}_p} := \ker(\chi)
\end{equation}

be the cyclotomic character. Its action is the same as (55).

**Theorem 5.1.** (Fontaine) \cite{FO, ST} There is a one to one correspondence

\begin{equation}
(57) \quad \{\mathbb{Z}_p \text{ representations of } G_{\mathbb{Q}_p}\} \xrightarrow{\simeq} \{\text{étale } (\phi, \Gamma) - \text{modules over } \mathcal{E}(B)\}
\end{equation}

\[ T \mapsto (\mathcal{E}(B) \otimes_{\mathbb{Z}_p} T)^{H_{\mathbb{Q}_p}} \]

where the superscript means fixed elements under action of $H_{\mathbb{Q}_p}$.

Here $\Gamma = \text{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p)$. In $p$-adic Hodge theory we are involved with series of natural 1-1 correspondences between $(\Phi, \Gamma)$-modules and different $\mathbb{Z}_p$-linear representations satisfying de Rham, crystalline or semistability conditions. These correspondences become compatible with Theorem 5.1 according to \cite{BSX}. Theorem 5.1 can also be stated with the rings $\mathcal{R}(B)$ or $\mathcal{E}^+(B)$ as well. Under all these isomorphisms the étale modules correspond to each other. When $\mathbb{Q}_p$ is replaced by a finite extension $L$, there is a major difficulty appearing. The character in (56) is replaced by its analogue

\begin{equation}
(58) \quad \chi_L : G_L \rightarrow \mathcal{O}_L^\times, \quad H_L = \ker(\chi_L)
\end{equation}

called Lubin-Tate character. Taking a uniformizer $\pi \in \mathcal{O}_L$ the inverse system of residual quotients of $\mathcal{O}_L$ by powers of $\pi$ defines a formal group law or a $p$-divisible group denoted also by $T$. Then analogue of the Fontaine theorem holds with $\Gamma = \mathcal{O}_L^\times$ and $\phi = \pi_*$. In case we have an isomorphism $\mathcal{O}_L \cong \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$ as $\mathbb{Q}_p$-Lie groups which implies

\begin{equation}
(59) \quad B_L \hookrightarrow B_{\mathbb{Q}_p} \times \cdots \times B_{\mathbb{Q}_p}, \quad \mathcal{O}_L(B_L) \cong \mathcal{O}_L(B_{\mathbb{Q}_p} \times \cdots \times B_{\mathbb{Q}_p})
\end{equation}
as analytic subvarieties. We have $O_L(B_L) \hookrightarrow B_{dR}$. The geometry of the disc $B$ can be explained by the analytic characters of $O_L$ as the following.

**Theorem 5.2.** [ST] There is an isomorphism

$$B(\mathbb{C}_p) \xrightarrow{\sim} \{ \text{L-analytic characters } O_L \to \mathbb{C}_p \} =: \hat{O}_L$$

$$z \mapsto \kappa_z(x) := (1 + z)^x$$

The space $\hat{O}_L \subset D(O_L, \mathbb{C}_p)$ is the subspace of locally analytic characters on $O_L$. It is also equal to the continuous dual to the space of $\mathbb{C}_p$-valued analytic functions $C^{an}(O_L, \mathbb{C}_p)$. A simple way to stress this isomorphism for $L$ to be $\mathbb{Q}_p$ is by looking at its converse given as

$$\hat{\mathbb{Z}}_p(\mathbb{C}_p) \to B(\mathbb{C}_p), \quad \chi \mapsto \chi(1)$$

Theorem 5.2 is crucial to us, as it parametrizes the points in the $p$-adic disc with the Lubin-Tate characters (cyclotomic in case $L = \mathbb{Q}_p$). This with the formalism of the Sen Theorem 3.3 allows us to express the period morphism in terms points in the character variety $\hat{O}_L$. Another observation concerning (60) is the possibility to write down a power series expansion for $\kappa_z$. The map $\kappa(z) = \kappa_z$ is given by a formal power series of the form

$$\kappa(z) = 1 + F_{t'}(z), \quad F_{t'}(z) = \Omega(t').z + \ldots \in z.O_L[[z]]$$

depending to a choice of $t' \in T'$, where $T'$ is the (Cartier) dual of $T$. The element $\Omega(t') \in O_L$ is a fixed period depending on $t'$. different choices of $t'$ will correspond to multiplication by an element in $O_L$ on $\Omega$.

**Theorem 5.3.** [ST], [BSX] There exists a Fourier transform

$$\mathcal{F} : D(O_L, \mathbb{C}_p) \xrightarrow{\sim} \hat{O}_L, \quad \lambda \mapsto F_{\lambda}$$

where $F_{\lambda}$ is defined by

$$F_{\lambda} : \hat{O}_L(\mathbb{C}_p) \to \mathbb{C}_p, \quad \chi \mapsto \lambda(\chi)$$

The monoid action of $O_L \setminus 0$ now is given by multiplication:

$$a^*(\kappa_z) := \kappa_z(a,-)$$
The isomorphism \((60)\) at the first glance seems natural that the characters are parametrized by the disc around one, but the power series is of course a power series which only converges on a disc around zero, [ST], [BSX].

The isomorphisms \((60)\) and \((63)\) imply a pairing

\[
\{ , \} : \mathcal{O}(B/\mathbb{C}_p) \times C^{an}(\mathcal{O}_L) \to \mathbb{C}_p
\]

The following formulas hold for \(\{ , \}:\)

- \(\{1, f\} = f(0).\)
- \(\{F, \kappa_z\} = F(z).\)
- \(\{F, \kappa_z, f\} = \{F(z+), f\}.\)
- \(\{F, f(a.)\} = \{F \circ [a], f\}.\)
- \(\{F, f'\} = \{\Omega \log F, f\} \)
- \(\{F, x f(x)\} = \{\Omega \partial F, f\} \)
- \(\{F, P_m(\Omega)\} = (1/m!)d^m f/dz^m(0)\)

The space of analytic functions \(f : \mathcal{O}_L \to \mathbb{C}_p\) has the structure of a Hilbert space with respect to the above pairing as the following theorem states.

**Theorem 5.4.** [ST] Any analytic function \(f : \mathcal{O}_L \to \mathbb{C}_p\) has a unique representation in the form

\[
f = \sum_{n=0}^{\infty} c_n P_n(\Omega)
\]

where \(c_n = \{f, z^n\}\). Furthermore, such a series is convergent provided that there exists a real number \(r\) such that \(|c_n| r^n \to 0\) as \(n \to \infty\).

We will use this theorem in an argument on convergence for nilpotent orbits in the next section.

### 6. Nilpotent Orbits and the Limit Slope Filtration

In this section we entirely assume that the coefficient system is extended to \(\mathbb{C}_p\). When we have a variation of \(p\)-adic étale cohomology (or crystalline cohomology) over the scheme \(S\), the filtrations

\[
F^i_s = \bigoplus_{r \geq i} H^{r,s}_{et}(X)(\mathbb{C}_p), \quad \text{resp. } (H^{r,s}_{cryst}(\mathbb{C}_p))
\]
on the étale cohomologies $H^k_{et}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p) \otimes \mathbb{C}_p$ (resp. $H^k_{cris}(X/W) \otimes_W \mathbb{C}_p$) explained in Section 1, namely relations (19), (28) define the period space of Hodge-Tate structures having the same Frobenius-Hodge numbers, namely $\mathcal{F}(\nu)$ (here $\nu = (\nu_i)$ where $\nu_i = \sum_{r \geq i} h^{r,s}$, $h^{r,s} = \dim H^r_{et}(X)$) and a period map

$$\Theta : S \rightarrow \mathcal{F}(\nu), \quad s \mapsto F_s \tag{69}$$

where $\mathcal{F}(\nu)$ is assumed to be embedded by the Berkovich functor as a rigid analytic space, that is locally looks like an affinoid space. We call (69) the period map associated to the variation of étale or crystalline cohomology. We also replace $S$ by the disc $B$ later on and assume that its radius is small enough.

**Theorem 6.1.** [DOP], [HG], [HA], [DG], [SW] The period map $\Theta : B_L \rightarrow \mathcal{F}(\nu)$ is a $G_L$-equivariant rigid analytic map of rigid analytic spaces over $L$.

By this map a Hodge-Tate structure should be understood as an invariant of an algebraic manifold defined over a number field. In order to make it depend on the isomorphism class of the manifold one must identify any two HT-structure that are related by an isomorphism of the manifold. We use the analyticity of the period map to write down a generalized Mahler expansion for $\Theta$ as in Theorem 5.4, (67). We define the $p$-adic nilpotent orbit as

$$\eta(\kappa_z) = \exp[N \log(\kappa_z)]F_0, \quad F_0 \in \mathcal{F}(\nu) \tag{70}$$

where as before $N$ is a nilpotent transformation on the isocrystal satisfying $N \circ \phi = p^r \phi \circ N$, and $F_0 \in \mathcal{F}(\nu)$.

**Definition 6.2.** The map $\eta : B(\mathbb{C}_p) \rightarrow \mathcal{F}(\nu)$ given by (70) is a nilpotent orbit if

- $\eta(\kappa_z) \in \mathcal{F}(\nu)^{ss}$ for $\kappa_z \rightarrow 0$.
- $N \circ \phi = p^r \phi \circ N$, for some $r$.

Making the limit makes sense because the ground variety is an analytic space and carries a natural metric. The definition can be alternatively be stated with the variable $\kappa_z$ with its logarithm, when its radius is chosen sufficiently enough. Then the parameter should be considered to tend to infinity.

**Theorem 6.3.** Let $\Theta : B \rightarrow \mathcal{F}(\nu)$ be the period map of a variation of étale cohomology. Define $\eta(\kappa_z) = \exp[N \log(\kappa_z)]\Theta(\log \kappa_z)$. Then the followings are true
(1) The limit $F_\infty := \lim_{\kappa_z \to 0} \eta(\kappa_z)$ exists and is a semistable filtration of $V$.

(2) $\xi(\kappa_z) = \exp[N \log(\kappa_z)]F_\infty$ is a nilpotent orbit.

(3) For each non-archimedean metric $d$ on $\mathcal{F}(\nu)^{an,rig}$, there exists constants $C$, $k$, $l$ and $r > 0$ such that

\begin{equation}
    d(\xi(\kappa_z), \Theta(\log \kappa_z)) < C . p^{-k/l/q(n)}r^n, \quad \forall n
\end{equation}

is true, where $q(n) \to \infty$ as $n \to \infty$.

By non-archimedean metric we mean a metric that satisfies the following property known as triangle axiom

\begin{equation}
    d(\xi, \theta) \leq \max[d(\xi, \zeta), d(\zeta, \theta)]
\end{equation}

for any 3 points $\xi$, $\zeta$, $\theta$ in the space. Using the triangle axiom we can assume that the estimate in item (3) is enough to be considered for each component. The proof is based on several structural facts which we list as lemmas.

**Lemma 6.4.** [ST] The function

\begin{equation}
    H(x, y) = \exp(y \log x)
\end{equation}

is a rigid analytic function on $B(r)^2$ for $r$ sufficiently small.

In fact the Fourier expansion for $H(x, y)$ has a simple form that looks like the Taylor expansion in analysis. Define the norm $\| \cdot \|$ on the space of power series by

\begin{equation}
    \| \sum_i c_i(z - a)^i \|_{a,n} := \max_i \{|c_i \pi^{ni}|\}, \quad \text{Alt.} \quad \| f \|_{a,n} = \max_{z \in a + \pi^n \mathcal{O}_L} |f(z)|
\end{equation}

**Lemma 6.5.** [ST] In the Mahler expansion of the function

\[ \exp(\cdot \log z) = \sum P_m(\cdot \log z)z^m \]

for all $a \in \mathcal{O}_L$ and all $n$

\begin{equation}
    \| P_n(y, \Omega) \|_{0,n} \leq \max_{0 \leq i \leq n} \| P_i(y) \|_{0,n}
\end{equation}
Furthermore, there are constants $C_1$ and $k$ such that

\[
\| P_1(y, \Omega) \|_{0,n} < C_1 p^{-k,1/q(n)}, \quad n \geq 1
\]

where $q(n) \to \infty$ as $n \to \infty$.

We have the power series expansion

\[
\exp(y \log(z)) = \sum_{n=0}^{\infty} P_n(y) z^n
\]

where the polynomials $p_n(y)$ satisfy the following properties,

- $p_0(y) = 1, \quad p_1(y) = y$.
- $p_n(0) = 1, \quad n \geq 1$.
- $\deg(p_n) = n$ and the leading coefficient of $p_n$ is $1/n!$.
- $p_n(y + y') = \sum_{i+j=n} p_i(y)p_j(y')$
- $p_n(\delta).f(x)|_{x=0} = (1/n!)(d^n f/dx^n)|_{x=0}$, where $f(x) \in \mathbb{C}_\ast[[x]]$

The last property can be obtained by a comparison to the formal Taylor series; let $\delta = d/dx$ then the Taylor formula reads as

\[
\exp(\delta b) h(a) = \sum_n \frac{\delta^n}{n!} h(a) b^n = h(a + b)
\]

Inserting $a = \log(x), \quad b = \log(y), \quad h = f \circ \exp$ one gets

\[
\exp(\partial \log(y)) f(x) = f(x + y)
\]

Therefore the orbit limit we are going to compute with is an analogue of formal Taylor series expansion, [ST].

If $F_1$ and $F_2$ be two filtrations on the same vector space $V$, the pairing between them is defined by

\[
<F_1, F_2> := \sum_{x, y} xy \cdot \dim gr_{F_1}^x (gr_{F_2}^y (V))
\]

The GIT slope defined in 3.6 is related to this pairing as follows.
Lemma 6.6. ([DOP] page 39) Let $\lambda$ be a one parameter subgroup of $GL(V)$, that is $\lambda: \mathbb{G}_m \to GL(V)$ is a group homomorphism. Assume that $F$ is a fixed point of $\lambda$. Then

\begin{align*}
\mu^{E(\nu)}(F, \lambda) &= - < F, F_\lambda >
\end{align*}

The filtration $F_\lambda$ is called the filtration corresponding to $\lambda$. It corresponds to an eigenspace decomposition $V = \bigoplus_{x \in \mathbb{Z}} V_x$ associated to the co-character $\lambda$. Then $F^{\geq}_\lambda := \bigoplus_{y \geq x} V_x$. The projection $\pi_x: F^{\geq}_\lambda \to \text{gr} F^{\geq}_\lambda V$ has a unique $\mathbb{G}_m$-equivariant section namely $i_x: \text{gr} F^{\geq}_\lambda \to V_x$. The property in (81) is one of the structural properties of the 1-parameter subgroups in $GL(V)$ and Mumford invariant for semi-stability.

Lemma 6.7. ([DOP] page 39) Assume $F_0$ is a $\mathbb{Q}$-filtration on $V$. Let $\lambda$ be a 1-parameter subgroup of $GL(V)$. Let $F_\infty = \lim_{t \to 0} \lambda(t).F_0$. Then

\begin{align*}
<F_\infty, F_\lambda> &= <F_0, F_\lambda>
\end{align*}

The lemma asserts that the function $< ., F_\lambda >$ is constant along the orbit $\lambda$ correspondent to $F_\lambda$. This property is general for 1-parameter orbits on flag domains, see [VRS], and remark 6.8 below.

For the following assume $r$ the radius of the disc $B(r)$ is small enough such that $\log: B(r) \to B(r)$ is an isomorphism and we have

\begin{align*}
B(r) & \xrightarrow{\cong} B(r) \\
\downarrow_{z \mapsto \kappa_z} & \downarrow_{z \mapsto \Omega.z} \\
\widehat{O}_L(r \Omega) & \xrightarrow{\kappa_z \to \log \kappa_z(1)} B(r|\Omega|)
\end{align*}

The log is the logarithm of the associated formal group law. To explain this diagram first note that the map $z \mapsto \kappa_z$ maps $B(r) \to \widehat{O}_L(r|\Omega|)$ and is a rigid analytic isomorphism. Thus the diagram says under the isomorphism $B(\mathbb{C}_p) \xrightarrow{\cong} \widehat{O}_L(\mathbb{C}_p)$ the logarithm functions are compatible, see [ST] and [BSX] for details.

Before starting the proof of the Theorem 6.3 lets note a difference of the parameters appearing in the functions $\eta$ and $\Theta$. The variable of the first function is supposed to belong to the $p$-adic disc $B(r)$ via Theorem 5.2. The second is assumed to take a variable on the cover of the disc by the $O_L$-logarithm. It is a similar situation in Theorem 2.2. We have also assumed the radius of the disc namely $r$ is small enough.
Proof. (proof of Theorem 6.3) According to Theorem (6.1) and Lemma (6.4) the map $\eta$ is a rigid analytic map of the variable $\kappa_z$. We use the Sen theorem to write $\Theta(\log \kappa_z) = \exp[n(\log \chi z) \log \chi_z] F_0$ for a fixed point $F_0 \in \mathcal{F}(\nu)$. Then the function $n(\log \chi_z)$ is locally rigid analytic on the variable by (76). The orbit function

\[(84) \quad \eta(\kappa_z) = \exp[N \log(\kappa_z)]\Theta(\kappa_z) = \exp[N \log(\kappa_z)] \exp[n(\chi z) \log \chi_z] F_0\]

can be written as the product of two series of the form $\sum_m \tilde{P}_m(\Omega) z^m$ with coefficients to be polynomial functions in matrices. The polynomials appearing as coefficient of $z^m$ in both of these series satisfy the estimates mentioned in Lemma 6.5 when the matrix in variable may be regarded as a single variable. Therefore the criteria in Theorem 5.4 will hold for their convergence. Thus if we restrict the parameter in a sufficiently small disc $B(r)$, (84) will converge. We denote the limit by $F_\infty$ (an analogue of Hartogs theorem in complex analysis implies that $\eta$ extends analytically to 0, but we do not need this fact here). The linear group $G = Gl(V)$ acts transitively on $\mathcal{F}(\nu)$. Therefore $\mathcal{F}(\nu) = G/P$ where $P$ is a parabolic subgroup of $G$ consisting of the elements preserving a fixed flag $F$. For $\kappa_z \in B(r)(\mathbb{C}_p)$ choose elements $g, g_\Theta \in G$ lifting $\xi(\kappa_z), \Theta(\log \kappa_z)$ respectively. Then if $\|\kappa_z\| \le r$ for suitable $r$ both of the functions $g, g_\Theta$ are confined with a compact subset of $G$. Therefore if $d$ is a non-archimedean metric on the flag variety, then for a suitable constant $C$

\[(85) \quad d(\xi(\kappa_z), \Theta(\log \kappa_z)) \le C.\max(||\exp[N \log(\kappa_z)]||, ||\exp[N \log(\kappa_z)] \exp[n(\log \chi z) \log \chi_z]||)\]

Now the estimates for Fourier coefficients in the Lemma 6.5 will hold for each factor in the right hand side, which implies the inequality in item (3). Let $g \in Gl(V)$ and Set

\[(86) \quad h := \lim_{\kappa_z \to 0} \exp[N \log \kappa_z] g \exp[-N \log \kappa_z]\]

This limit exists. If $g$ is chosen such that $g.F_0 = F_\infty$ we have

\[
h F_0 = \lim_{\kappa_z \to 0} \exp[N \log \kappa_z] g \exp[-N \log \kappa_z] F_0 = \lim_{\kappa_z \to 0} \exp[N \log \kappa_z] g F_0 = \lim_{\kappa_z \to 0} \xi(\kappa_z)
\]

Let $\lambda(\log \kappa_z) = \exp[N \log \kappa_z]$ (in the situation of Lemmas 6.6, 6.7 and Theorem 3.6), which defines $F_\lambda$. Then the estimates in item (3) says that

\[
\lim_{\kappa_z \to 0} \lambda(\log \kappa_z) F_\infty = F_\infty
\]
i.e. $F_\infty$ is the fixed point of this orbit. Now assume $g \in Gl(V)$ of (79) be the specific such that $F_\infty = g.F_\lambda$. Using Lemma 6.7 we have

\begin{equation}
<F_\infty, F_\lambda> = \lim_{\kappa_z \to 0} <F_\infty, g.F_\infty> = <\hbar F_\infty, \hbar^{-1}ghF_\infty> = <F_\infty, F_\infty> \geq 0
\end{equation}

This together with Hilbert-Mumford criterion proves that $F_\infty$ is semistable. □

The filtration $F_\infty$ may be called the limit slope filtration in analogy to the complex case of limit Hodge filtration. We briefly remind this as follows. First fix $s_0 \in S$ of the ground scheme and consider the isomorphisms $u_s : V_{s_0} \to V_s$ of isocrystals by the parallel transport. We equip $V_{s_0}$ with the semistable filtration $F_\infty$ and define a new filtration on each $V_s$ by transferring $F_\infty$ via the isomorphism $u_s$. This construction equips the local system with a canonical filtration that fits in natural exact sequences in cohomology theories involved.

From the proof it can be understood that a similar limit

\begin{equation}
\eta(\kappa_{z_1}, ..., \kappa_{z_r}) := \exp[N_1 \log(\kappa_{z_1}) + ... + N_r \log(\kappa_{z_r})] \Theta(\kappa_{z_1}, ..., \kappa_{z_r})
\end{equation}

also exists and satisfies similar estimates. The different limits corresponding to the choice of elements in the nilpotent cone define boundary points in the period space $\mathcal{F}(\nu)^{ss,rig}$.

**Remark 6.8.** One can show that the function $\psi(t) = <F_t, F>$ is a Morse function and is convex along the flows of the 1-parameter family $F_t$, if may be considered over $\mathbb{R}$. Then the $\mathbb{R}$-orbit $\theta(t) = \exp(it.N)F_0$ satisfies

\begin{equation}
\theta'(t) = \nabla \psi(t)
\end{equation}

Thus $\lambda(t)$ defines the gradient flow line of $\psi(t)$. This implies that the limit

\begin{equation}
\lim_{t \to \pm \infty} (\theta(t) = \exp(it.N)F_0)
\end{equation}

always exists and should correspond to some critical points of the function $\psi$. In [VRS] the properties of the orbit function $\theta(t)$ has been studied via a moment map $\mu : \mathcal{F} \to \mathfrak{g}$, where $\mathcal{F}$ is the flag variety and $\mathfrak{g}$ is the Lie algebra of the Lie group $G$ acting on $\mathcal{F}$. Then to a point $F \in \mathcal{F}$ one can associate the invariant

\begin{equation}
w_\mu(F, u) := \lim_{t \to \infty} <\mu(\exp(itu)F), u>, \quad u \in \mathfrak{g}
\end{equation}
for a symplectic Riemanninan metric $<\cdot,\cdot>$ on $F$, called Mumford numerical invariant of $F$. It is a Theorem by Mumford that

\[(92) \quad w_\mu(gF, gug^{-1}) = w_\mu(F, u), \quad g \in G\]

This invariant plays a crucial role in geometric invariant theory on the semi-stability, [VRS].

Defining the limit slope filtration in a variation of étale or crystalline cohomology with $\mathbb{C}_p$-coefficients is a major step in doing geometric Hodge theory. For instance beginning from a proper smooth map $f : X \to S$, one can associate the $p$-adic (Hodge-Tate) local system

\[(93) \quad \mathcal{V} = R^k f_* \mathbb{Q}_p \otimes \mathbb{C}_p = \bigcup_s (H^k_{et,s} = H^k_{et}(X_s, \mathbb{Q}_p) \otimes \mathbb{C}_p)\]

We equip this local system with the limit slope filtration corresponding to some choice of nilpotent elements in the Lie algebra of $Gl(V)$, as explained above. It is a canonical semistable filtration by Theorem 6.3. Using this one can define standard Hodge theory objects similar to the complex case. For instance one may define the normal functions as

\[(94) \quad \nu : S \to J_{et} = \bigcup_s (J_{et,s} = Ext^1_{HT}(\mathbb{Z}_p(0), H^k_{et,s}))\]

where the extension is taken in the category of Hodge-Tate isocrystals. The local system $\mathcal{V}$, defines a Gauss-Manin connection

\[(95) \quad \nabla : \mathcal{V} \otimes \mathcal{O}(B)(\mathbb{C}_p) \to \mathcal{V} \otimes \Omega^1_{B/\mathbb{C}_p}\]

which satisfies the Griffiths transversality with respect to the limit slope filtration. Normally the Hodge-Tate structure degenerates along singular locus, then understanding the limit behavior of them is an important question in Hodge theory. One way to conduct with this question is by the Deligne nearby cycle functor. Consider a diagram of schemes

\[(96) \quad \begin{array}{cccccc}
X_\eta & \xrightarrow{j} & X & \xleftarrow{i} & X_s \\
\downarrow & & \downarrow f & & \downarrow \\
Spec(L) = \eta & \longrightarrow & S = Spec(\mathcal{O}_L) & \longleftarrow & s = spec(k)
\end{array}\]
where k is the residue field. The Deligne nearby functor is defined by $R^k \psi_f = i^* R^k j_*$. Applying this functor to the constant sheaf $\mathbb{C}_p$ on $X_{\eta}$ defines a sheaf concentrated on s. Therefore this sheaf may be regarded as an extension of $\mathcal{V}$ in (93) over the special fiber. There are canonical ways to equip this extension with a new filtration build up from the limit slope filtration. This fact is a major step in the theory of filtered $D$-modules, op. cit.

Remark 6.9. Let $L$ be a $p$-adic field. The Harder-Narasimhan (HN)-filtration of the isocrystal $E$ is the unique filtration such that $\text{gr}_\alpha(E)$ is semistable of slope $\alpha$ whenever nonzero. Note that $E$ is semistable iff the HN-filtration has only one jump. There is an obvious functor

$$
\text{gr}_\bullet : \text{FilIsoc}(L) \to \text{Grad}(\text{FilIsoc}(L)), \quad (E, F) \mapsto \bigoplus_\alpha (\text{gr}_\alpha^F E)[\alpha]
$$

Then $F$ is the HN-filtration iff $\text{gr}_\alpha^F E$ is semistable with slope 0. Considering this criterion one can state the nilpotent orbit theorem in terms HN-filtrations. The functor $\text{gr}_\bullet$ is an additive tensor functor which commutes with symmetric and exterior product and with duality. The Harder-Narasimhan polygon is also defined similar to the Newton polygon in (35), and analogous of Theorems 4.1 and 4.2 will hold true in this case.

7. $SL_2$-orbit Theorem for $p$-adic Hodge structures

In the complex one variable case the $SL_2$-orbit theorem roughly states that, any nilpotent orbit is asymptotic to an equivariant embedded copy of the upper half plane. More specifically, to any nilpotent orbit $e^{zN}. F$ of a polarized pure Hodge structure, there can associate an $SL_2$-orbit $e^{zN}. \hat{F}$ and a real analytic function $g : (0, \infty) \to G_\mathbb{R}$ such that

$$
e^{iyN}. F = g(y)e^{iyN}. \hat{F}$$

The $G_\mathbb{R}$-valued functions $g(y)$ and $g^{-1}(y)$ have convergent series expansions about $\infty$ of the form $(1 + \sum_{k=1}^{\infty} A_k y^{-k})$ with $A_k \in \ker(\text{ad}N)^{k+1}$. The precise statement goes as follows.

Theorem 7.1. It is possible to choose

- a homomorphism of complex Lie groups $\psi : SL(2, \mathbb{C}) \to G_\mathbb{C}$,
- a holomorphic, horizontal, equivariant embedding $\tilde{\psi} : \mathbb{P}^1 \to \tilde{D}$, which is related to $\psi$ by
(99) \[ \tilde{\psi}(g \circ i) = \psi(g) \circ F_0 \]

- a holomorphic mapping \( z \mapsto g(z) \) of a neighborhood \( W \) of \( \infty \in \mathbb{P}^1 \) into the complex Lie group \( G_\mathbb{C} \) such that
  (a) \( \exp(zN) \circ a = g(-iz)\tilde{\psi}(z), \ z \in W - \{\infty\} \); 
  (b) \( \psi(SL(2,\mathbb{R}) \subset G_\mathbb{R}, \text{ and } \tilde{\psi}(U) \subset D \); 
  (c) \( \psi_* : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{g} = \text{Lie}(G) \) is of type \((0,0)\) for Hodge maps, that is

(100) \[ \psi_*(X_+) \in \mathfrak{g}^{-1,1}, \quad \psi_*(Z) \in \mathfrak{g}^{0,0}, \quad \psi_*(X_-) \in \mathfrak{g}^{1,-1} \]

where \( X_+, Z, X_- \) are

(101) \[ Z = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad X_+ = \begin{pmatrix} -i & 1 \\ 1 & i \end{pmatrix}, \quad X_- = \begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix} \]

the generators of \( \mathfrak{sl}_2 \);

- \( g(y) \in G_\mathbb{R} \) for \( iy \in W \cap i\mathbb{R} \);
- \( \text{Ad } g(\infty)^{-1}(N) \) is the image under \( \psi_* \) of \( \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \);
- for \( iy \in W \cap i\mathbb{R}, \ y > 0, \) let \( h(y) \) be defined by

(102) \[ \psi_*(Y) \in \text{Hom}(H_\mathbb{C}, \mathbb{C}) \]

then

(103) \[ (104) \quad g(z) = g(\infty)(1 + g_1z^{-1} + \ldots), \quad g(z)^{-1} = g(\infty)^{-1}(1 + f_1z^{-1} + \ldots) \]

be the power series expansion of \( g(z) \) and \( g(z)^{-1} \) around \( \infty \). Then the maps \( f_n \) and \( g_n \) map the \( l \)-eigenspace of \( \psi_*(Y) \) into linear span of the the eigenspaces corresponding eigenvalues equal or less \( l + n - 1 \).
We sketch some details from [P] and [S]. The subgroup $G_\mathbb{R}$ acts transitively on the classifying space $D$ and the $G_\mathbb{R}$-valued function $h(y)$ satisfies a differential equation of the form

\begin{equation}
  h^{-1} \frac{dh}{dy} = -L \text{Ad}(h^{-1}(y))N
\end{equation}

relative to a suitable $L \in \mathfrak{h} = \text{Lie}(G_\mathbb{R})$. After some change of variables (105) is written in the Lax form

\begin{equation}
  -2 \frac{d}{dy} X^+(y) = [Z(y), X^+(y)], \quad 2 \frac{d}{dy} X^-(y) = [Z(y), X^-(y)]
\end{equation}

\begin{equation}
  \frac{d}{dy} Z(y) = [X^+(y), X^-(y)]
\end{equation}

A solution for this system is given as $X^-(y) = \Phi(y)x^-$, $Z(y) = \Phi(y)\mathfrak{z}$, $X^+(y) = \Phi(y)x^+$ where

\begin{equation}
  \Phi : (a, \infty) \to Hom(\mathfrak{sl}_2(\mathbb{C}), \mathfrak{g}_\mathbb{C}), \quad \Phi(y) = \sum_n \Phi_n y^{-1-n/2}
\end{equation}

and

\begin{equation}
  x^- = \frac{1}{2} \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}, \quad \mathfrak{z} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad x^+ = \frac{1}{2} \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}
\end{equation}

Now lets check out that how much can be stated from the above in the $p$-adic case. The existence of the map $\psi$ in Theorem 7.1 is based on a general theorem of Jacobson-Morosov; that is: given any nilpotent element $x \in \mathfrak{g}$ of a semisimple Lie algebra $\mathfrak{g}$ (over any field of char = 0) can be embedded in a three-dimensional subalgebra.

Lets start with a nilpotent orbit $\exp(N \log \kappa_z) \circ F$, where $F \in \mathcal{F}(\nu)$ (may be not semistable). As we explained before, we have a homomorphism of lie groups

\begin{equation}
  \psi_p : SL(2, \mathbb{C}_p) \to G_{\mathbb{C}_p}
\end{equation}

which determines an analytic equivariant embedding $\tilde{\psi} : \mathbb{P}^1 \to \mathcal{F}(\nu)$ with $\tilde{\psi}(g \circ \omega) = \psi(g) \circ F_0$, where $\omega$ is any fixed point of $\mathbb{P}^1(\mathbb{C}_p)$. Then
ψ_p : sl(2, C_p) → g_{C_p}, \quad ψ_p, (sl(2, C_p)) \subset b \oplus g_{-1}

where g_j := \{x \in g \mid ad(Z)x = j.x\}. Then define the function g by the following

\[ \exp(N \log \kappa_z) F = g(\log \kappa_z) \tilde{\psi}(\log \kappa_z) \]

This is well defined. We can state the following analogue of Theorem 7.1.

**Theorem 7.2.** It is possible to choose

- a homomorphism of complex Lie groups ψ_p : SL(2, C_p) → G_{C_p},
- an analytic, horizontal, equivariant embedding \tilde{\psi}_p : P^1(C_p) → F(ν), which is related to ψ_p by

\[ \tilde{\psi}_p(g \circ i) = ψ_p(g) \circ F_0 \]

- \psi_p, : sl_2(C_p) → g = Lie(G) satisfies

\[ ψ_p, (X_+) \in g_{-1}, \quad ψ_p, (Z) \in g_0, \quad ψ_p, (X_− = N) \in g_1 \]

where X_+, Z, X_− = N are sl_2-triples.
- an analytic mapping \kappa_z → g(\log \kappa_z) of a neighborhood W of 0 into the p-adic Lie group G_{C_p} such that
  (a) \exp(N \log \kappa_z) F = g(\kappa_z) \tilde{\psi}(\kappa_z);
  (b) Ad g(0)^{-1}(N) is the image under \psi_p, of \left( \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right)
  (c) Let h(\kappa_z) be defined by

\[ h(\kappa_z) = g(\kappa_z) \exp(-\frac{1}{2} \log \kappa_z \psi_*(Z)) \]

then \[ h(\kappa_z)^{-1} \frac{d}{d\kappa_z} h(\kappa_z) \in g. \] The G-valued functions g(\kappa_z) and g^{-1}(\kappa_z) have convergent series expansions about 0 of the form \( 1 + \sum_{k=1}^{\infty} A_k \kappa_z^{-k/2} \) with \[ A_k \in \ker(adN)^{k+1}. \]

**Proof.** (sketchy adoption of [S]) The existence of the of the map ψ_p, ψ_p,* and \tilde{\psi}_p is based on general sl_2-theory as explained in the introduction and above. We also defined g by (111). Property can always be fixed by conjugating the representation ψ_p by an element in G. We proceed to prove the properties of the functions h and g. The point \exp(N \log \kappa_z) F defines the Hodge Tate decomposition.
of the étale cohomology with $\mathbb{C}_p$-coefficients. we have $H^i_{\ell \! \! \! \et}(\mathbb{C}_p) = \bigoplus_{i=0}^{k} H^{i,k-i}(-i)(y)$

As we mentioned in the introduction the nilpotent orbit $\exp(N \log \kappa_z) \circ F$ provides an $\epsilon$-Hermitian class $(H^k_{\ell \! \! \! \et}(\mathbb{C}_p), B)$ for the representation $H^k_{\ell \! \! \! \et}(\mathbb{C}_p)$. We will choose a basis $(w_1, ..., w_r)$ such that they successively generate the Frobenius-Hodge piec es in the slope filtration. By the same method as in [S], page 298, It is possible to modify the given basis such that $(w_1(\kappa_z), ..., w_r(\kappa_z))$ represent a basis with similar property and remain orthogonal with respect to the Hodge filtration. Then $w_j(\kappa_z)$ are rational functions of $\kappa_z$. The function $h(\kappa_z)$ is a lifting of the 1-parameter family $\exp(N \log \kappa_z)$. Its logarithmic derivative $A = -2h^{-1}h'$ takes values in $\mathfrak{g}_{\mathbb{C}_p}$. Setting

$$A(\kappa_z) = -2h^{-1}h', \quad F(\kappa_z) = Ad \ h(\kappa_z)^{-1}N, \quad E(\kappa_z) = -\theta F(\kappa_z)$$

where $\theta$ is a Cartan involution of $\mathfrak{g}$. One is needed to solve the similar Lax system in analytic coordinates

$$2E'(\kappa_z) = -[A(\kappa_z), E(\kappa_z)], \quad 2F'(\kappa_z) = [A(\kappa_z), F(\kappa_z)], \quad A'(\kappa_z) = -[E(\kappa_z), F(\kappa_z)]$$

In order to prove the last assertion, according to the Iwasawa decomposition $G = UAK$ the $G = GL(H^k_{\ell \! \! \! \et})$-valued function $h(\kappa_z)$ can be written as

$$h(\kappa_z) = u(\kappa_z)a(\kappa_z)k(\kappa_z)$$

In suitable orthonormal basis $u$ becomes lower triangular with diagonal to be only 1, and $a$ a diagonal matrice, however these entries depend rationally to the parameter variable. Then a similar argument as [S] page 299 proves that the entries of $u$ and $a$ have the series expansions

$$u_{ij}(\kappa_z) = a_m(\log(\kappa_z)^{-m} + ..., \quad a_{ij}(\kappa_z) = b_n(\log(\kappa_z)^{-(n+1)/2} + ...$$

being convergent near 0. The situation for the function $k(\kappa_z)$ is separated. However a repeated argument similar that for $a$ and $u$ shows that $k$ can be extended analytically to 0, see [S] pages 300-305 for details. Using (114) proves the result for the power series expansion of $g$. Property (b) can always be fixed by conjugating the representation $\psi_p$ by an element in $G$. We are done. $\square$
8. Monodromy in $p$-adic Hodge theory

Suppose $V_l$ is an $l$-adic representation of the absolute Galois group $G_K$ over $p$-adic field $K$ namely

$$
\rho : G_K \to V_l
$$

We have the short exact sequence

$$
1 \to I \to G_K \to Gal(\bar{k}/k) \to 1
$$

where $k$ is the residue field and $I$ is the inertia group. As always our example is the étale cohomology groups $H^n(X_{\bar{K}}, \mathbb{Q}_l)$. If $l \neq p$ the semistability of $V_l$ means that the inertia subgroup $I(p)$ at the prime $p$ of $\mathbb{K}$ acts unipotently on $V_l$. Moreover this action is given by the exponential of a morphism

$$
N : V_l(1) \to V_l
$$

of $l$-adic representations of $G_K$. It follows that

$$
\rho(g) = \exp(N.t_\ell(g)), \quad g \in I(p)
$$

This is always the case for the $l$-adic étale cohomologies. The endomorphism $N$ satisfies

$$
NF = q.FN
$$

where $q$ is the order of the residue field of $K$. The endomorphism $N$ defines the local monodromy filtration: That is the unique increasing filtration $\ldots M_i \subset M_{i+1} \subset \ldots$ such that

- $N.M_i V(1) \subset M_{i-2} V$
- $gr_k^M V(k) \xrightarrow{\cong} gr_k^{M+1} V$

By the work of Deligne the eigenvalues $\alpha$ of the Frobenius $F$ are algebraic integers, Moreover there exists integers $w(\alpha)$ such that all complex conjugates of $\alpha$ have absolute value $q^{w(\alpha)/2}$. Let $W_j$ be the sum of all the generalized eigenspaces of $F$ with eigenvalue $\alpha$ such that $w(\alpha) = j$. The filtration $P_\bullet := \{ \oplus_{j \leq i} W_j \}$ is called the weight filtration. One can show that the trace of the Frobenius $F_q$ is independent of $l$ in this case, see [D].

When $l = p$ in the log-cristalline setting there is an analogue of this map namely

$$
N : V_p(1) \to V_p
$$
of \(p\)-adic representations or the local Galois group \(G_K\) that is called the \(p\)-adic monodromy morphism. There is no hope to have such a \(p\)-adic monodromy morphism for every semistable \(p\)-adic Galois representation. This analogy is explained by the notion of Fontaine or \((\Phi, N)\)-modules. A \((\Phi, N)\)-module is a \(K\)-vector space \(D\) equipped with

- a \(\sigma\)-linear map (called Frobenius) \(\Phi: D \to D\)
- a \(K\)-endomorphism (called monodromy) \(N: D \to D\)
- \(N\Phi = p\Phi N\)
- \(D\) has an exhaustive separated decreasing filtration \((\text{Fil}_i D_K)_{i \in \mathbb{Z}}\).

Denote by \(M_K(\Phi, N)\) the category of these modules. Let \(V\) be a continuous finite dimensional representation of \(G_K\). Define

(126) \[ D_{st}: \text{Rep}_{st}(G_k) \to M_K(\Phi, N), \quad D_{st}(V) := (B_{st} \otimes_{\mathbb{Q}_p} V)^{G_K} \]

where the superfix means the fixed points, and \(B_{st}\) is the Fontaine period ring. The image of this functor are called admissible filtered \((\Phi, N)\)-module. We always have \(\dim_{\mathbb{Q}_p} V \geq \dim_K D_{st}(V)\) and \(V\) is called semistable if

(127) \[ \dim_{\mathbb{Q}_p} V = \dim_K D_{st}(V) \]

If this equality holds for \(K\) replaced by a finite extension then it is called potentially semistable. We say an object \(D \in M_K(\Phi, N)\) has transverse monodromy if

(128) \[ N.Fil^i D_K \subset Fil^{i-1} D_K \]

If \(D\) is admissible then the monodromy \(N: D \to D\) induces a \(K\)-linear map

(129) \[ N_{st}: D(1) \to D \]

that depends on the trivialization of the Tate twist. The morphism \(N_{st}\) is called the monodromy morphism of \(D\). One can show that \(N_{st}\) is a morphism of filtered \((\Phi, N)\)-modules if and only if \(D\) has transverse monodromy. Lets \(M^{\text{st,tm}}_K(\Phi, N)\) be the category of such modules. It is a tannakian subcategory of \(M_K^\circ(\Phi, N)\). The corresponding representations would be denoted by \(\text{Rep}_{st,tm}(G_K)\). The map \(N_{st}\) corresponds to

(130) \[ N_p: V(1) \to V \]

via this correspondence.

The category \(\text{Vect}_{\mathbb{Q}_p}(N)\) of \(\mathbb{Q}_p\)-vector spaces \(V\) and a nilpotent transformation \(N: V \to V\) is a tannakian category with fiber functor \(\eta: (V, N) \to V\). The associated group scheme \(Aut^\otimes(\eta)\) is the additive group \(\mathbb{G}_{a, \mathbb{Q}_p}\). This gives a tensor equivalence.
(131) \[ \eta : Vect_{\mathbb{Q}_p}(N) \cong \text{Rep}(\mathbb{G}_{a,\mathbb{Q}_p}) \]

Using the discussion of the former paragraph one obtains a functor

(132) \[ w : \text{Rep}_{st,tm}(G_K) \to Vect_{\mathbb{Q}_p}(N), \quad V \mapsto (V, N_p) \]

which depends to a fixed trivialization of the Tate twist. This gives a morphism

(133) \[ e^N : \mathbb{G}_{a,\mathbb{Q}_p} \to \text{Aut}^{\otimes}(w_p) \]

where \( \text{Aut}^{\otimes}(w_p) \) is the tannakian fundamental group of the fiber functor \( w_p = \eta \circ w \).

The map \( e^N \) is called the exponential of the \( p \)-adic monodromy up to the Tate twist.

It can be presented as

(134) \[ e^N : \mathbb{G}_{a,\mathbb{Q}_p}(\mathbb{Q}_p[T]) \to \text{Gl}(V_p)(\mathbb{Q}_p[T]), \quad T \mapsto \sum_{i>0} \frac{N^i T^i}{i!} \]

where \( T = \text{id}_{\mathbb{G}_a} \), see [Pâ].

9. Nilpotent orbits in the mixed case

The definitions of period domain and period map can also be similarly stated for variation of mixed Hodge structure (MHS). However the asymptotic behavior of the period maps and the nilpotent orbits are essentially different from the pure case. Also the period map can have essential (non-removable) singularities which is never the case in pure HS. We try to discuss this notion for the mixed Hodge-Tate structure which we define in the proceeding paragraphs.

let \((V, F^\bullet, W_\bullet, Q)\) be a polarized MHS with \( F^\bullet \) and \( W_\bullet \) be the Hodge and weight filtrations respectively. The classifying space \( M \), for this MHS consists of all filtrations \( F^\bullet \) such that \((F^\bullet, W_\bullet)\) is a MHS on \( V \) which is polarized by \( Q \). The isomorphism class of a variation of polarized MHS \( \mathcal{V} \to S \) is determined by its period map

(135) \[ \phi : S \to M/\Gamma, \quad \Gamma = \text{Image}(\rho) \]

and its monodromy representation \( \rho : \pi_1(S, s_0) \to \text{Gl}(V) \) on a fixed reference fiber \( V = \mathcal{V}_{s_0} \).

Unlike the pure case, the existence of the limiting mixed Hodge structure is not guaranteed for MHS. This means that a nilpotent orbit may not be in general asymptotic to the period map. Thus one requires to make this assumption at the beginning. More specifically a variation of MHS on the punctured disc \( \Delta^* \) with unipotent monodromy is admissible if
the limiting MHS $F_\infty$ exists.
- the relative weight filtration $W^{rel}(N, W_\bullet)$, i.e. the filtration induced by the nilpotent transformation $N$ on the $Gr^W_k$ exists.

where $N$ is the logarithm of the monodromy. In this case $(F_\infty, W^{rel}(N, W_\bullet))$ is called the limiting MHS. One can analogously define admissible nilpotent orbits, $[P]$.

**Theorem 9.1.** (Nilpotent Orbit Theorem, G. Pearlstein $[P]$) If $V \to \Delta^*$ is an admissible variation of MHS with unipotent monodromy, then,
- $\eta(z) = e^{zN}F_\infty$ is an admissible nilpotent orbit.
- There are constants $\alpha$, $\beta$ and $K$ such that

\begin{equation}
\label{eq:136}
d_M(F(z), \eta(z)) < K \text{Im}(z)^\beta e^{-2\pi \text{Im}(z)}
\end{equation}

and $\eta(z) \in M$ whenever $\text{Im}(z) > \alpha$.

Now lets go back to the $p$-adic geometry. We make the following definition.

**Definition 9.2.** (Mixed Hodge-Tate (MHT) structure) A $\mathbb{Q}_p$-vector space is said to have a mixed Hodge-Tate structure if it is endowed an increasing filtration $P_\bullet$ defined over $\mathbb{Q}_p$ (indexed over integers called weight filtration) and a decreasing filtration $F^\bullet$ defined over $\mathbb{C}_p$ (indexed over rational numbers called slope filtration) such that the graded pieces $Gr^P_j V$ together with the induced filtration by $F^\bullet$ are pure Hodge-Tate structure in the sense explained in Section 2.

If $N : V \to V$ is a nilpotent transformation defined over $\mathbb{Q}_p$ then we call the MHT structure $(V, F^\bullet, P_\bullet)$ to be $N$-admissible if
- the limiting slope filtration $F_\infty$ exists.
- the relative weight filtration $P^{rel}(N, P_\bullet)$ exists

In this case $(F_\infty, P^{rel}(N, P_\bullet))$ is called the limit mixed Hodge-Tate structure. We state the following analogue for the Theorem 9.1.

**Theorem 9.3.** (Nilpotent orbit theorem for mixed Hodge-Tate structure) Assume $V \to B(r)^*$ is a variation of mixed Hodge-Tate structure over the punctured $p$-adic disc and $N : V \to V$ is a nilpotent transformation on $V_s = V$. Then
- $\eta(\kappa_z) = \exp(N \log(\kappa_z))F_\infty$ is an $N$-admissible nilpotent orbit.
- For each non-archimedean metric $d$ on $\mathcal{F}^{an,\text{rig}}$, we have the following distance estimate

\begin{equation}
\label{eq:137}
d(\Theta(\log \kappa_z), \eta(\kappa_z)) < C_r^p < q(n)^{-1/n}, \quad \forall n, \quad (r \ll 1)
\end{equation}

where $q(n) \to \infty$ as $n \to \infty$.

**Proof.** The first assertion is a consequence of admissibility, and the second follows from (85) and the uniform bound in (76). \qed
REFERENCES

[A] Y. Amice, Interpolation $p$-adique, Bull. soc. math. France 92, 117-180 (1964)
[B] C. Breuil, $p$-adic Hodge theory, deformations and local Langlands, Barcelona 18-28 July 2001
[BP] L. Berger, P. Colmez, Familles de representations de de Rham et monodromie $p$-adique, Asterisque, 2008, 319, pp.303-337.
[BO] P. Berthelot, A. Ogus, Notes on crystalline cohomology, Annales of Math. studies, Princeton University press, Princeton, NJ, 1978
[BX] L. Berger, X. Caruso, $p$-adic comparison theorems’ Statements, lectures given at the spring school "Classical and $p$-adic Hodge theory" in Rennes, May 2014.
[BSX] L. Berger, P. Schneider, B. Xie, Rigid character groups, Lubin-Tate theory, and $(\Phi, \Gamma)$-modules, preprint, 2015
[C] B. Conrad, Survey of Kisin paper crystalline representations and $F$-crystals, preprint
[D] P. Deligne, Poids dans la cohomologie des varietes algebrique, Actes du congres International des mathematiciens, Vancouver, 1974
[DG] O. Demchenko, A. Gurevich, $p$-adic period map for the moduli space of deformations of a formal group, Journal of Algebra, 288 (2005) 445-462
[DOP] J. Dat, S. Orlik, M. Rapoport, Period domains over finite and $p$-adic fields. Cambridge tracts in mathematics 183
[FO] J. Fontaine, Y. Ouyang, Theory of $p$-adic Galois representations, Preprint
[GS] P. Griffiths, W. Schmid, Recent developments in Hodge theory, a discussion of techniques and results. Proceedings of the international colloquium on discrete subgroups of lie groups, Bombay, 1973
[HA] U. Hartl, On period space for $p$-divisible groups, ArticleinComptes Rendus Mathematique 346(21-22) 2007
[H] O.Hyodo, On variation of Hodge-Tate structures, Math. Ann. 284, 7-22 (1989)
[HG] M. Hopkins, B. Gross, The rigid analytic period mapping, Lubin-Tate space, and stable Homotopy theory, Bulletin of AMS, Vol 30, No 1, (1994) 76-86
[I] L Illusie, Crystals and Barsotti-Tate groups, Grothendieck at Pisa, Talk at the colloquium de Giorgi, Pisa, April 2013
[K] K. Kedlaya, Semistable reduction for overconvergent $F$-crystals, I: Unipotence and logarithmic extensions, Compositio Math. 143(5), 1164-1212
[KL] M. Kisin, G. Lehrer, Eigenvalues of Frobenius and Hodge numbers, Pure and applied Math. Quarterly, Vol 2, No 2, 199-219, 2006
[KP] M. Kerr, G. Pearlstein, An exponential history of functions with logarithmic growth, in "Topology of Stratified Spaces", MSRI Pub. 58, Cambridge University Press, New York, 2011
[O1] A. Ogus, Frobenius and Hodge spectral sequences, Advances in mathematics, doi:10.1006/aima.2001, available online at http://www.idealibrary.com
[O2] A. Ogus, periods of integrals in characteristic $p$, Proceedings of international congress of mathematics, August 16-24,1983, Warszawa
[Pa] F. Pangam, Galois representations, Mumford-Tate groups and good reduction of abelian varieties, Mathematische Annalen May 2004, Volume 329, Issue 1, pp 119160
[P] G. Pearlstein, $SL_2$-orbits and degenerations of mixed Hodge structure, Differential Geometry, 74 (2006) 1-67
[R] M. Rapoport, On the Newton Stratification, Seminaire Bourbaki, 54eme, 2001-2002, no. 903
[R1] M. Rapoport, Non-Archimedean period domains, Proceedings of the international congress of mathematics, Zurich, Switzerland 1994
[R2] M. Rapoport, period domains over finite and $p$-adic fields, Proceedings of Symposia in pure mathematics, Vol 62.1, 1997
[S] W. Schmid, Variation of Hodge structure: The singularities of the period map, Inventiones math. 22, 211-319 (1973)

[SE] S. Sen, Continuous cohomology and p-adic Galois representations, Inventiones math. 62, 89-116 (1980)

[ST] P. Schneider, J. Teitelbaum, p-adic Fourier theory, documenta mathematica (6), 2001, 447-481

[SW] P. Scholze, J. Weinstein, Moduli of p-divisible groups, Cambridge Journal of mathematics I, 2013, 145-137

[VRS] V. Georgoulas, J. W. Robbin, D. A. Salomon, The moment-weight inequality and the Hilbert-Mumford criterion, Preprint, ETH-Zrich, November 2013, revised 5 April 2016.

[Y] G. Yamashita, P-adic étale and crystalline cohomology for open varieties, preprint

[Z] T. Zink, lectures on p-divisible groups, preprint

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