On dually flat general \((\alpha, \beta)\)-metrics

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Abstract

Based on the previous research, in this paper we study the dually flatness of a special class of Finsler metrics called general \((\alpha, \beta)\)-metrics, which is defined by a Riemannian metric \(\alpha\) and a 1-form \(\beta\). By using a new kind of deformation technique, we construct many non-trivial explicit dually flat general \((\alpha, \beta)\)-metrics.

1 Introduction

In 2000, S.-I. Amari and H. Nagaoka introduced the notion of dual flatness for Riemannian metrics when they studied information geometry\[1\], and this notion have been extended to general Finsler metrics by Z. Shen in 2007\[9\]. A Finsler metric \(F\) on a manifold \(M\) is said to be locally dually flat if and only if in a adapted local coordinate system \((x^i)\), the function \(F=F(x,y)\) satisfies

\[
[F^2]_{x^i y^k} y^k - 2 [F^2]_{x^j} = 0.
\]

For a Riemannian metric \(\alpha = \sqrt{a_{ij}(x)y^i y^j}\), it is known that \(\alpha\) is locally dually flat if and only if in an adapted coordinate system, its fundamental tensor is the Hessian of some local smooth function \(\psi(x)\)[1], i.e.,

\[
a_{ij}(x) = \frac{\partial^2 \psi}{\partial x^i \partial x^j}(x).
\]

In fact, the dual flatness of a Riemannian metric can also be described by its spray[14]: \(\alpha\) is locally dually flat if and only if its spray coefficients can be expressed in an adapted coordinate system as

\[
G^i_\alpha = 2\theta y^i + \alpha^2 \theta^i,
\]

where \(\theta := \theta_i(x)y^i\) is a 1-form and \(\theta^i := \alpha^j \theta_j\).

The characterization for dually flat Riemannian metrics is clear. Hence, how to describe the dual flatness for Finsler metrics becomes an interesting problem. However, it is still not easy to be researched for the general case. So we begin with some special kinds of Finsler metrics.

Randers metrics, introduced by a physicist G. Randers in 1941 when he studied general relativity, is an important class of Finsler metrics. Generally, a Randers metric is given in the form \(F = \alpha + \beta\) where \(\alpha\) is a Riemannian metric and \(\beta\) is a 1-form. But it can also be expressed in another famous form as follows:

\[
F = \sqrt{(1 - \bar{b}^2)\alpha^2 + \bar{\beta}^2} - \bar{\beta} \left(\frac{1}{1 - \bar{b}^2}\right),
\]

where \(\bar{\alpha}\) is also a Riemannian metric, \(\bar{\beta}\) is a 1-form and \(\bar{b}\) is the norm of \(\bar{\beta}\) with respect to \(\bar{\alpha}\). \((\bar{\alpha}, \bar{\beta})\) is called the navigation data of the Randers metric \(F\). Based on the results in [4], the author provided a more clear description for dually flat Randers metrics: A Randers metric \(F = \alpha + \beta\) is locally dually flat if and only if \(\bar{\alpha}\) is locally dually flat and \(\bar{\beta}\) is dually related with respect to \(\bar{\alpha}\)[14]. The notion of dually related 1-forms was proposed by the author in [14]. The definition is given below:

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Definition 1.1. Let $\alpha$ be a locally dually flat Riemannian metric on a manifold $M$. Suppose that the spray coefficients $G^i_{\alpha}$ of $\alpha$ are given in an adapted coordinate system by (1.2) with some 1-form $\theta$ on $M$. Then a 1-form $\beta$ on $M$ is said to be dually related with respect to $\alpha$ if

$$b_{ij} = 2\theta b_j + c(x)a_{ij},$$

where $c(x)$ is a scalar function on $M$.

As a generalization of Randers metrics from the algebraic point of view, $(\alpha, \beta)$-metrics are also defined by a Riemannian metric and a 1-form and given in the form

$$F = \alpha \phi(\frac{\beta}{\alpha}),$$

where $\phi(s)$ is a smooth function. Because of its computability, many encouraging results about $(\alpha, \beta)$-metrics have been achieved. Recently, Q. Xia gave a local characterization of locally dually flat $(\alpha, \beta)$-metrics with dimension $n \geq 3$. Later on, the author provide a more clear characterization. The result is much similar to that of Randers metrics. If $F = \alpha \phi(\frac{\beta}{\alpha})$ is a non-trivial locally dually flat $(\alpha, \beta)$-metric with $n \geq 3$, then after some special deformations, $\alpha$ will turn to be a locally dually flat Riemannian $\bar{\alpha}$ and $\beta$ to be a 1-form $\bar{\beta}$ which is dually related with respect to $\bar{\alpha}$. In this case, $F$ can be reexpressed as the form $F = \phi(b^2, \frac{\bar{\beta}}{\bar{\alpha}})$.

One can see that the navigation expression (1.3) of Randers metrics is also given in the form

$$F = \alpha \phi(b^2, \frac{\beta}{\alpha}).$$

Actually, such kind of Finsler metrics are belong to the metrical category called general $(\alpha, \beta)$-metric, which is introduced by the author as a generalization of Randers metrics from the geometric point of view. General $(\alpha, \beta)$-metrics include not only all the $(\alpha, \beta)$-metrics and the spherically symmetric Finsler metrics naturally, but also part of Bryant’s metrics and fourth root metrics.

The main purpose of this paper is to describe and construct dually flat general $(\alpha, \beta)$-metrics. It must be declare firstly that we will assume additionally that $\alpha$ is dually flat and $\beta$ is dually related to $\alpha$. According to the discussions of Randers metrics and $(\alpha, \beta)$-metrics, one can see that the dual flatness of a Randers metric or a $(\alpha, \beta)$-metric is always arises from that of some Riemannian metric, and it is the dually related 1-forms preserve the dual flatness. Hence, we believe that the assumption here is reasonable and appropriate.

The main result is given below:

**Theorem 1.2.** Let $F = \alpha \phi(b^2, \frac{\beta}{\alpha})$ be a Finsler metric on an open domain $U \subseteq \mathbb{R}^n$. Suppose that $\alpha$ and $\beta$ satisfy

$$G^i_{\alpha} = 2\theta y^i + \alpha^2 \theta^i, \quad b_{ij} = c(x)a_{ij} + 2\theta b_j,$$

where $\theta$ is a 1-form and $c(x)$ is a scalar function such that $c(x) \neq -2\theta b^k$. Then $F$ is dually flat on $U$ if and only if the function $\phi$ satisfies the following PDE:

$$\phi_2^2 + \phi \phi_{22} + 2s\phi_1 \phi_2 + 2s\phi\phi_{12} - 4\phi\phi_1 = 0.$$  

(1.5)

It should be pointed out that if the scale function $c(x)$ satisfies $c(x) = -2\theta b^k$, then according to the proof of Theorem 1.2, $F = \alpha \phi(b^2, \frac{\beta}{\alpha})$ will be always dually flat for any function $\phi(b^2, s)$. So it will be regarded as the trivial case. In another word, such a dually related 1-form is trivial.

We have reason to conjecture that general $(\alpha, \beta)$-metrics can only be obtained by this way. More specifically, we guess that if $F = \alpha \phi(b^2, \frac{\beta}{\alpha})$ is a locally dually flat Finsler metric on a manifold with dimension $n \geq 3$, then after necessary reexpressing in a new form $F = \tilde{\alpha} \phi(\tilde{b}^2, \frac{\tilde{\beta}}{\tilde{\alpha}})$, $\tilde{\alpha}$ must be locally dually flat and $\tilde{\beta}$ dually related with respect to $\tilde{\alpha}$. Moreover, the function $\phi$ must satisfy (1.5) if $\tilde{\beta}$ is non-trivial. It is true for the $(\alpha, \beta)$-metrics. One can check it using Maple program.

The solutions of (1.5) is completely determined by Proposition. In Section 4, we will construct some dually flat Riemannian metrics and their dually related 1-forms by the data

$$\alpha = |y|, \quad \beta = \lambda \langle x, y \rangle + \langle a, y \rangle,$$  

(1.6)
where \( \lambda \) is a constant numbers and \( \alpha \) is a constant vector. The corresponding result is given by Proposition 1.2. One can get infinity many dually flat general \((\alpha, \beta)\)-metrics by this way. Two typical examples are listed below.

The Finsler metric

\[
F = \sqrt{1 + (\mu + \sigma^2)b^2} \sqrt{(1 + \mu b^2)\alpha^2 - \mu \beta^2} + \frac{\sigma \beta}{(1 + \mu b^2) \sqrt{1 + (\mu + \sigma^2)b^2}}
\]

is a dually flat Randers metric, where \( \alpha \) and \( \beta \) is given by (1.3). \( \mu \) and \( \sigma \) are constant numbers. When \( \mu = -1 \) and \( \sigma = 1 \), the corresponding metric is the famous generalized Funk metric, whose dual flatness was first proved in [4].

The Finsler metric

\[
F = \left( \sqrt{1 + (\mu + \sigma^2)b^2} \cdot \sqrt{(1 + \mu b^2)\alpha^2 - \beta^2} + \sigma \beta \right)^{\frac{1}{2}}
\]

(1.7)

is a dually flat general \((\alpha, \beta)\)-metric, where \( \alpha \) and \( \beta \) is given by (1.3). \( \mu \) and \( \sigma \) are constant numbers. Such kind of general \((\alpha, \beta)\)-metrics are actually belong to the category of \((\alpha, \beta)\)-metrics because it can be reexpressed as

\[
F = \sqrt{\frac{(\alpha + \beta)^3}{\alpha}}.
\]

By the way, it should be pointed out that when \( \mu = -1 \), \( \lambda = \sigma = 1 \) and \( \alpha = 0 \), the above metric is given by

\[
F = \sqrt{\frac{(\sqrt{1 - |x|^2})y_2 + (x, y) + (x, y))}{(1 - |x|^2) \sqrt{(1 - |x|^2) |y|^2 + (x, y)^2}}}
\]

It is also obtained by B. Li independently in another different way [6].

Finally, when \( \alpha = 0 \), all the dually flat Finsler metrics obtained by this way are just the so-called symmetric Finsler metrics, which are given in the form \( F = \phi(|x|^2, \frac{\sqrt{b^2}}{|y|}) \) [5].

2 Preliminaries

Let \( F \) be a Finsler metric on a \( n \)-dimensional manifold \( M \). The geodesic spray coefficients of \( F \) are defined by

\[
G^i := \frac{1}{4} g^{ij} \left\{ [F^2]_{xy} y^k - [F^2]_x \right\},
\]

where \( g^{ij} \) is the inverse of the fundamental tensor \( g_{ij} := \frac{1}{4} [F^2]_{ij} y^l \). For a Riemannian metric, the spray coefficients are determined by its Christoffel symbols as \( G^i(x, y) = \frac{1}{2} \Gamma_{ijk}(x)y^j y^k \).

Suppose that \( \phi(b^2, s) \) is a positive smooth function defined on the domain \( |s| \leq b < b_o \) for some positive number (maybe infinity) \( b_o \). Then the function \( F = \alpha \phi(b^2, \frac{s}{b}) \) is a Finsler metric for any Riemannian metric

\[
\alpha = \sqrt{a_{ij}(x)y^i y^j}
\]

and any 1-form \( \beta = b_i(x) y^i \) if and only if \( \phi(b^2, s) \) satisfies

\[
\phi - s \phi_2 > 0, \quad \phi - s \phi_2 + (b^2 - s^2) \phi_{22} > 0
\]

when \( n \geq 3 \) or

\[
\phi - s \phi_2 + (b^2 - s^2) \phi_{22} > 0
\]

when \( n = 2 \) [13]. Such kind of Finsler metrics belong to the metrical category called general \((\alpha, \beta)\) metrics. For a given metric, \( b := \|\beta\|_\alpha \) is the norm of \( \beta \).

In Section 4, we need a special kind of metric deformations called \( \beta\)-deformations [12] [14], which are determined by a Riemannian metric \( \alpha \) and a 1-form and listed below:

\[
\tilde{\alpha} = \sqrt{\alpha^2 - \kappa(b^2)\beta^2}, \quad \tilde{\beta} = \beta;
\]

\[
\hat{\alpha} = e^{\rho(b)} \hat{\alpha}, \quad \hat{\beta} = \hat{\beta};
\]

\[
\tilde{\alpha} = \tilde{\alpha}, \quad \tilde{\beta} = \nu(b^2) \tilde{\beta}.
\]
In order to keep the positive definition of \( \tilde{\alpha} \), the deformation factor \( \kappa(b^2) \) must satisfies an additional condition:

\[
\quad 1 - \kappa b^2 > 0.
\]

(2.1)

Some basic formulas of \( \beta \)-deformations are listed below. It should be attention that the notation \( \hat{b}_{ij} \) always means the covariant derivative of the 1-form \( \hat{\beta} \) with respect to the corresponding Riemannian metric \( \hat{\alpha} \), where the symbol \( \cdot, \cdot' \) can be \( \cdot', \cdot'' \) or \( \cdot'' \) or empty in this paper. Moreover, we need the following abbreviations,

\begin{align*}
  r_{00} &:= r_{ij}y^i y^j, \; r_i := r_{ij} b^j, \; r \ := r_i b^i, \\
  s_{0} &:= s_{ij} y^i, \; s_{ij} := a_{ij} s_0, \; s_i := s_{ib} b^i, \; s_0 := s_{0b} b^b,
\end{align*}

where \( r_{ij} \) and \( s_{ij} \) are given by \( r_{ij} := \frac{1}{2}(b_{ij} + b_{ji}) \) and \( s_{ij} := \frac{1}{2}(b_{ij} - b_{ji}) \).

**Lemma 2.1.** \([12]\) Let \( \tilde{\alpha} = \sqrt{\alpha^2 - \kappa(b^2)\beta^2} \), \( \tilde{\beta} = \beta \). Then

\[
\tilde{G}^i_{\alpha} = G^i_{\alpha} - \frac{\kappa}{2(1 - \kappa b^2)} \left\{ (2(1 - \kappa b^2) \beta \delta^i_0 + r_{0b} b^i + 2 \kappa s_0 b^b) \right\}
\]

\[
+ \frac{\kappa'}{2(1 - \kappa b^2)} \left\{ (1 - \kappa b^2) \beta^2 (r^i + s^i) + \kappa \beta^2 b^i - 2(r_0 + s_0) \beta b^i \right\},
\]

\[
\tilde{b}_{ij} = b_{ij} + \frac{\kappa}{1 - \kappa b^2} \{ b^2 r_{ij} + b_i s_j + b_j s_i \} - \frac{\kappa'}{1 - \kappa b^2} \{ r b_i b_j - b^2 b_i (r_j + s_j) - b^2 b_j (r_i + s_i) \}.
\]

**Lemma 2.2.** \([12]\) Let \( \tilde{\alpha} = e^{\rho(b^2)} \tilde{\alpha}, \tilde{\beta} = \tilde{\beta} \). Then

\[
\tilde{G}^i_{\alpha} = \tilde{G}^i_{\alpha} + \rho' \left\{ 2(r_0 + s_0) y^i - (\alpha^2 - \kappa \beta^2) \left( r^i + s^i + \frac{\kappa}{1 - \kappa b^2} r b^i \right) \right\},
\]

\[
\tilde{b}_{ij} = \tilde{b}_{ij} - 2 \rho' \left\{ b_i (r_j + s_j) + b_j (r_i + s_i) - \frac{1}{1 - \kappa b^2} r (a_{ij} - \kappa b_i b_j) \right\}.
\]

**Lemma 2.3.** \([12]\) Let \( \tilde{\alpha} = \tilde{\alpha}, \tilde{\beta} = \nu(b^2) \tilde{\beta} \). Then

\[
\tilde{G}^i_{\alpha} = \tilde{G}^i_{\alpha}, \quad \tilde{b}_{ij} = \nu' b_{ij} + 2 \nu' b_i (r_j + s_j).
\]

### 3 Proof of Theorem \([1.2]\)

Suppose \( \alpha \) and \( \beta \) satisfy \([1.4]\). It is easy to verify that

\[
  r_{ij} = c_{ij} + \theta_i b_j + \theta_j b_i,
\]

\[
  s_{ij} = \theta_i b_j - \theta_j b_i,
\]

\[
  (b^2)^{x} = 2 (r_i + s_i) = 2 \tilde{c} b_i,
\]

\[
  \alpha^{x} = y_m \frac{\partial G^i_{\alpha}}{\partial y^k} = \frac{\alpha^{-1}}{\alpha^2} (\alpha^2 \theta_i + 2 \theta y_i),
\]

\[
  \beta^{x} = b_m y^m + b_m \frac{\partial G^i_{\alpha}}{\partial y^k} = \tilde{c} y_i + 2 \beta \theta_i + 4 \theta b_i
\]

\[
  s_y^{x} = \alpha^{-2} (\alpha \beta_x - \beta \alpha_x),
\]

where \( \tilde{c} = (c + \alpha b_k b^k) \) and \( y_i = a_{ij} y^j \). Combining with the above equalities we obtain

\[
[F^2]_{x^i} = 2 \phi^2 \alpha \alpha_x^i + 2 \phi \phi_1 \alpha^2 \cdot 2 (r_i + s_i) + 2 \phi \phi_2 \alpha^2 \cdot \frac{\alpha \beta_x^i - \beta \alpha_x^i}{\alpha^2}
\]

\[
= 2 \phi \left\{ 2 (\phi - s \phi_2) (\alpha^2 \theta_i + 2 \theta y_i) + 2 \phi_1 \alpha^2 b_i + \phi_2 \alpha (\tilde{c} y_i + 2 \beta \theta_i + 4 \theta b_i) \right\}
\]

(3.1)

and

\[
[F^2]_{x^i y^j} = 2 \phi_2 \left\{ (\phi - s \phi_2) (\alpha^2 \theta_i + 2 \theta y_i) + 2 \phi_1 \alpha^2 b_i + \phi_2 \alpha (\tilde{c} y_i + 2 \beta \theta_i + 4 \theta b_i) \right\} \frac{\alpha b_i - s y_i}{\alpha^2}
\]

\[
+ 2 \phi \left\{ -2 s \phi_2 (\alpha^2 \theta_i + 2 \theta y_i) + 2 \phi_1 \alpha^2 b_i + \phi_2 \alpha (\tilde{c} y_i + 2 \beta \theta_i + 4 \theta b_i) \right\} \frac{\alpha b_i - s y_i}{\alpha^2}
\]

\[
+ 2 \phi \left\{ 4 (\phi - s \phi_2) (\theta_i y_i + b_i y_i + \theta b_i) + 4 \phi_1 \tilde{c} y_i + \phi_2 (\tilde{c}^{-1} y_i y_i + \alpha \alpha b_i + 2 s \theta b_i + 2 \alpha \theta b_i + 2 \alpha b_i b_i + 4 \alpha b_i b_i) \right\}.
\]
Hence,

\[ [F^2]_{x,y} y^k = 2\phi_2 \{ 6\phi \theta + 2\phi_1 c\beta + \phi_2 c\alpha \} (\alpha b_l - s y_l) + 2\phi \{ 2\phi_{12} c\beta + \phi_{22} c\alpha \} (\alpha b_l - s y_l) + 4\phi \{ 2(\phi - s \phi_2)(\alpha \theta_l + 2\theta y_l) + 2\phi_1 c\beta y_l + \phi_2 (c\alpha y_l + 3s \theta y_l + \alpha \theta b_l + 2\alpha \beta b_l) \}. \] (3.2)

By (5.1) and (3.2), one can see that (1.1) is equivalent to

\[ \tilde{c}(\phi_2^2 + \phi_{22} + 2s \phi_1 \phi_2 + 2s \phi \phi_{12} - 4\phi \phi_1)(\alpha b_l - s y_l) = 0, \]

hence (1.5) holds if \( \tilde{c} \neq 0 \).

## 4 Dually flat Riemannian metrics and dually related 1-forms

Let \( \alpha \) be a Riemannian metric with constant sectional curvature \( \mu \), and \( \beta \) be a closed 1-form which is also conformal with respect to \( \alpha \), then there is a local coordinate system in which \( \alpha \) and \( \beta \) can be expressed as

\[
\alpha = \frac{\sqrt{(1 + \mu |x|^2)|y|^2 - \mu(x,y)^2}}{1 + \mu |x|^2}, \quad (4.1)
\]

\[
\beta = \frac{\lambda \langle x, y \rangle + (1 + \mu |x|^2)(a,y) - \mu(a,x)\langle x, y \rangle}{(1 + \mu |x|^2)^{\frac{3}{2}}}, \quad (4.2)
\]

where \( \lambda \) is a constant number and \( a \) is a constant vector. Moreover,

\[
b_{ij} = \frac{\lambda - \mu(a,x)}{\sqrt{1 + \mu |x|^2}} a_{ij}.
\]

These special Riemannian metrics and 1-forms play an important role in projective Finsler geometry\[12, 15\].

In the rest of this section, we will use the above data to construct some dually flat Riemannian metrics and their dually related 1-forms by \( \beta \)-deformations. Firstly, it seems impossible to obtain dually flat Riemannian metrics by data \[4.1\] and \[4.2\] without any additional condition. The reason is given below.

**Lemma 4.1.** Given \( \alpha \) and \( \beta \) as \[4.1\] and \[4.2\]. Then \( \alpha \) can’t turn to be a dually flat Riemannian metric by \( \beta \)-deformations unless \( \mu = 0 \) or \( a = 0 \).

**Proof.** It is known that \( G^i_{\alpha} = Py^i \), where

\[
P = \frac{-\mu(x,y)}{1 + \mu |x|^2}.
\]

Set \( \tau(x) = \frac{\lambda - \mu(a,x)}{\sqrt{1 + \mu |x|^2}} \), then \( b_{ij} = \tau(x) a_{ij} \).

Carry out the first step of \( \beta \)-deformations. By Lemma 2.1 we have

\[
\dot{G}^i_{\alpha} = Py^i - \frac{\tau}{2(1 - \kappa b^2)}(\kappa a^2 + \kappa' b^2) b^i.
\]

Combining with Lemma 2.2 one can see that \( \dot{\alpha} \) can’t turn to be dually flat by the second step of \( \beta \)-deformations unless \( \dot{G}^i_{\alpha} \) is in the form \( \dot{G}^i_{\alpha} = Py^i + Q \dot{a}^2 b^i \). That is to say, \( \kappa \) must satisfy the following equation

\[
\kappa' = -\kappa^2. \quad (4.4)
\]

So we assume the deformation factor \( \kappa \) satisfies \[4.4\] from now on.

Carry out the second step of \( \beta \)-deformations. By Lemma 2.2 we have

\[
\dot{G}^i_{\alpha} = (P + 2\tau \rho' \beta) y^i - \frac{\tau}{2(1 - \kappa b^2)}(\kappa + 2\rho'e^{-2\rho} \dot{a}^2 b^i).
\]
Hence, $\tilde{\alpha}$ is dually flat if and only if
\[ P + 2\lambda\rho' \beta = -\frac{\tau}{1 - \kappa b^2}(\kappa + 2\rho')e^{-2\rho'b^i\tilde{y}_i}, \]
where $\tilde{y}_i := \tilde{a}_{ij}y^j$. The above equality is equivalent to
\[ P = -\tau(\kappa + 4\rho')\beta. \tag{4.5} \]
Combining with (4.2) and (4.3), it is not hard to find that the equality (4.5) will not hold unless $\mu = 0$ (in this case $P = 0$) or $a = 0$ (in this case $\beta = \frac{\lambda x, y}{(1 + \mu|x|^2)^{2/3}}$ is parallel to $P$).

When $a = 0$, it have been proven by $\beta$-deformations in [44] that the Riemannian metrics
\[ \alpha = \sqrt{(1 + \mu|x|^2)/2 - \mu(x, y)x^2(1 + \mu|x|^2)^{2/3}} \tag{4.6} \]
are dually flat on the ball $B^n(r_\mu)$ and the 1-forms
\[ \beta = \frac{\lambda(x, y)}{(1 + \mu|x|^2)^{2/3}} \tag{4.7} \]
are dually related to $\alpha$ for any constant number $\mu$ and $\lambda$, where the the radius $r_\mu$ is given by $r_\mu = \frac{1}{\sqrt{1 - \mu}}$ if $\mu < 0$ and $r_\mu = +\infty$ if $\mu \geq 0$. The above result was obtained by taking the deformation factor $\kappa = 0$.

Taking $\mu = 0$ in (4.1) and (4.2), we get the standard Euclidean metric and its closed conformal 1-form,
\[ \alpha = |y|, \quad \beta = \lambda(x, y) + (a, y). \tag{4.8} \]
In this case, $G^{ij}_\alpha = 0$ and $b_{ij} = \lambda a_{ij}$. It is obviously that $\alpha$ is dually flat and $\beta$ is dually related to $\alpha$.

Assume the deformation factor $\kappa$ satisfies (4.4), by Lemma 2.1 again we get
\[ \tilde{G}^{ij}_{\tilde{\alpha}} = -\frac{\lambda\kappa}{2(1 - \kappa b^2)} a^2 b^i, \quad \tilde{b}_{ij} = \frac{\lambda}{1 - \kappa b^2} a_{ij} + \lambda \kappa b_i b_j. \]
By (4.5), $\tilde{\alpha}$ can turn to be dually flat by the second deformation if and only if
\[ \kappa + 4\rho' = 0. \]
So we assume the deformation factor $\rho$ satisfies the above equation. In this case, by Lemma 2.2 again we get
\[ \tilde{G}^{ij}_{\tilde{\alpha}} = 2\tilde{\theta} y^i + \tilde{\alpha}^2 \tilde{\theta}^i, \quad \tilde{b}_{ij} = \frac{\lambda}{1 - \kappa b^2}(1 - \frac{1}{2}\kappa b^2)e^{-2\rho} a_{ij} + 2\kappa b_i b_j, \]
where $\tilde{\theta} = -\frac{1}{4}\lambda\kappa\beta$ and $\tilde{\theta}^i := \tilde{a}^i j \tilde{\theta}_j$.

Carry out the third step of $\beta$-deformations, then
\[ \tilde{G}^{ij}_{\tilde{\alpha}} = 2\tilde{\theta} y^i + \tilde{\alpha}^2 \tilde{\theta}^i, \quad \tilde{b}_{ij} = \frac{\lambda}{1 - \kappa b^2}(1 - \frac{1}{2}\kappa b^2)e^{-2\rho} a_{ij} + 2\lambda(\kappa + \nu')b_i b_j, \]
where $\tilde{\theta} := \tilde{\theta}$ and $\tilde{\theta}^i := \tilde{a}^i j \tilde{\theta}_j$. Hence, $\tilde{\beta}$ is dually related to $\tilde{\alpha}$ if and only if
\[ \lambda(\kappa + \nu')b_i = \tilde{\theta}_i, \]
which means that the deformation factor $\nu$ must satisfies
\[ \frac{\nu'}{\nu} = -\frac{5}{4} \kappa. \]

Obviously, $\kappa = 0$ is a solution of (4.1). As a result, $\rho$ and $\nu$ are both constants. When $\kappa \neq 0$, the solutions of (4.4) are given by
\[ \kappa = \frac{1}{b^2 + \text{constant}}. \]
In consideration of the additional condition of \( \kappa \) \( (2.1) \), \( \kappa \) can only be chosen as \( \kappa = \frac{1}{c + b^2} \) or \( \kappa = \frac{1}{c - b^2} \) where \( c \) is a positive number. Hence, the Riemannian metric \( (c \pm b^2)^{-\frac{3}{4}} \sqrt{\alpha^2 + (c \pm b^2)^{-1} \beta^2} \) is dually flat and the 1-form \( (c \pm b^2)^{-\frac{3}{4}} \beta \) is dually related. It can be verified easily that such a 1-form is a trivial dually related 1-form if and only if \( \lambda = 0 \).

After scaling, we have proven the following result.

**Proposition 4.2.** Let \( \alpha \) and \( \beta \) are given by \( (4.8) \). Then the Riemannian metric \( \bar{\alpha} = \sqrt{(1 + \mu b^2)\alpha^2 - \mu \beta^2} \) is dually flat and the 1-form \( \bar{\beta} = \frac{\sigma \beta}{(1 + \mu b^2)^{\frac{5}{4}}} \) is dually related to \( \bar{\alpha} \), where \( \mu \) and \( \sigma \) are constant numbers.

It is obvious that the above result cover \( (4.6) \) and \( (4.7) \).

5 Solutions of Equation \( (1.5) \)

**Proposition 5.1.** The solutions of the basic equation \( (1.5) \) are given by

\[
\phi(b^2, s) = \sqrt{g(b^2) + 2g'(b^2)s^2 + sf(b^2 - s^2) + \int_0^s (s^2 + \sigma^2)f'(b^2 - \sigma^2)\,d\sigma},
\]

where \( f(t) \) and \( g(t) > 0 \) are any \( C^\infty \) functions.

**Proof.** Let \( \psi := \phi^2 \), then \( \psi \) satisfies

\[
\psi_{22} + 2s\psi_{12} - 4\psi_1 = 0. \quad (5.1)
\]

Differentiating the above equation with respect to \( s \), one obtain

\[
\psi_{222} + 2s\psi_{122} - 2\psi_{12} = 0.
\]

Let \( \varphi := \psi_2 \), then \( \varphi \) satisfies

\[
\varphi_{22} + 2s\varphi_{12} - 2\varphi_1 = 0. \quad (5.2)
\]

Considering the variable substitution

\[
u = b^2 - s^2, \quad v = s,
\]
then \( b^2 = u + v^2, \quad s = v \). So by \( (5.2) \) we have

\[
\frac{\partial}{\partial u}(\varphi - s\varphi_2) = 0,
\]

which means \( \varphi - s\varphi_2 = f(b^2 - s^2) \) is just a function of \( b^2 - s^2 \). Hence, \( \psi_2 \) is given by

\[
\psi_2 = h(b^2)s + f(b^2 - s^2) + 2s\int_0^s f'(b^2 - \sigma^2)\,d\sigma
\]

for some function \( h(b^2) \). As a result, \( \psi \) is given by

\[
\psi = g(b^2) + \frac{1}{2}h(b^2)s^2 + sf(b^2 - s^2) + \int_0^s (s^2 + \sigma^2)f'(b^2 - \sigma^2)\,d\sigma \quad (5.3)
\]

for some function \( g(b^2) \). Putting \( (5.3) \) into \( (5.1) \) yields \( h = 4g' \).

Finally, by taking \( s = 0 \) we know that the function \( g \) must be positive.
Some typical solutions are listed below.

- \( f(t) = 0 \),
  \[ \phi(b^2, s) = \sqrt{g(b^2)} + 2g'(b^2)s^2. \]

- \( f(t) = \frac{2\varepsilon}{(1-\kappa t)^2} \) and \( g(t) = \frac{1}{1-\kappa t} \),
  \[ \phi(b^2, s) = \frac{\sqrt{1-\kappa b^2 + 2\varepsilon s^2}}{1-\kappa b^2}. \]

In particular, \( \phi(b^2, s) = \sqrt{1-\kappa b^2 + 2\varepsilon s^2} \) when \( \kappa = 1 \) and \( \varepsilon = 1 \), in this case, the corresponding general \((\alpha, \beta)\)-metrics are Randers metrics. \( \phi(b^2, s) = \sqrt{1 + s} \) when \( \kappa = 0 \) and \( \varepsilon = \frac{1}{2} \), in this case, the corresponding general \((\alpha, \beta)\)-metrics are those \((\alpha, \beta)\)-metrics given in the form \( F = \sqrt{\alpha(\alpha + \beta)} \).

- \( f(t) = 3(1 + t) \), \( g(t) = (1 + t)^{\frac{3}{2}} \),
  \[ \phi(b^2, s) = (\sqrt{1 + b^2} + s)^{\frac{3}{2}}. \]

In this case, the corresponding dually flat Finsler metrics determined by Theorem 1.2 and Proposition 1.2 are given by 1.7.

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