Interactive Isogeometric Volume Visualization with Pixel-Accurate Geometry
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Abstract—A recent development, called isogeometric analysis, provides a unified approach for design, analysis and optimization of functional products in industry. Traditional volume rendering methods for inspecting the results from the numerical simulations can not be applied directly to isogeometric models. We present a novel approach for interactive visualization of isogeometric analysis results, ensuring correct, i.e., pixel-accurate geometry of the volume including its bounding surfaces. The entire OpenGL pipeline is used in a multi-stage algorithm leveraging techniques from surface rendering, order-independent transparency and numerical theory of ordinary differential equations. We showcase the efficiency of our approach on different models relevant to industry, ranging from quality inspection of the parametrization of the geometry, to stress analysis in linear elasticity, to visualization of computational fluid dynamics results.

Index Terms—Volume visualization, Isogeometric analysis, Splines, Roots of Nonlinear Equations, Ordinary Differential Equations

Fig. 1: Examples for isogeometric volume visualization in industry. Left to right: Twisted bar showing quality of parametrization; Industrial demonstrator model from TERRIFIC project showing von Mises stress of the bent model; Backstep flow from a computational fluid dynamics simulation showing turbulent viscosity.

1 INTRODUCTION

Digital 3D-models are now used through the entire lifetime of objects, from computer aided design (CAD), to numerical analysis with finite element methods (FEM), to computer-aided manufacturing (CAM). Visualization is used in all the stages, for quality inspection, studying the results from numerical analysis, and marketing. It is therefore increasingly important to offer visualization techniques that are reliable, informative and visually pleasing.

Isogeometric analysis (IGA) proposed by Hughes et al. [3] provides the integration of design and analysis by using a common representation. A main advantage of IGA is that it enables the direct feedback from numerical analysis results to the CAD model. However, for that process to work efficiently it is essential to be able to interactively inspect the results from the numerical analysis. This article extends classic volume graphics to isogeometric volumes, where both the geometry and the scalar field of the object is given in terms of splines (NURBS, B-splines, etc.), see Fig. 2.

Scientific volume visualization techniques convey information about a scalar field defined on a certain geometry. The techniques can be divided into the following approaches: simply rendering the bounding surfaces of the object; iso-surface extraction; and volume rendering. The main challenges for achieving interactive volume visualization in the setting of isogeometric volumes is that an explicit expression of the inverse function of the geometry (see Equation (2)) is not available in general, and that sampling is computationally expensive due to the need for spline evaluation, i.e., piecewise polynomial functions.

The first approach for volume visualization is to render the scalar field on the outer surfaces of the isogeometric volume only. Although no information in the interior can be retrieved, this is a popular method due to its low computational effort. Methods based on ray-casting parametric polynomial surfaces are a well-studied but challenging problem, see e.g. Kajiya [11]. B-spline surfaces can be rendered as piecewise algebraic surfaces (see Loop and Blinn [13]),
but finding corresponding scalar field values becomes difficult. An alternative to ray-casting surfaces is rasterization, in particular with the recent introduction of the tessellation shader stage in graphics processing units (GPUs). A GPU-based two-pass algorithm for pixel-accurate rendering of B-spline surfaces was presented by Yeo et al. [21]: The first pass determines a sufficient tessellation level for each patch; during the second pass the surface is actually tessellated. An alternative method is presented by Hjelmervik [9], where bounds on the second order derivatives decide sufficient tessellation levels, without querying neighboring patches, allowing a single-pass algorithm.

The second approach is to display iso-surfaces, i.e., surfaces where the scalar field has a particular value. If the scalar field is sampled discretely over a regular grid, the marching cube algorithm (see Lorensen and Klein [14]) provides an efficient implementation; see e.g., Dyken et al. [6] for an implementation on the GPU. However, for isogeometric models, where both the geometry and the scalar field are given by spline functions, the marching cube algorithm cannot be applied directly. Martin and Cohen [16] provide an algorithm for iso-surface extraction in the setting of isogeometric volumes. The method consists of iteratively diving the model into a set of Bézier volumes. When those volumes are sufficiently simple, the iso-surfaces are given as the root of a function and the Newton-Raphson method is used to approximate the surfaces. This framework has been realized as a CPU-based parallel implementation (see Martin et al. [15]). The reported timings between 1 and 7 frames per second (FPS) on a cluster, make the approach unsuitable for interactive visualization purposes.

The last approach is to model the scalar field as a participating medium, where a modifiable transfer function specifies how field values are mapped to emitted color and transparency. If the field consists of discrete samples over a regular grid, an abundance of results is available, see e.g. Levoy [12] for an early example or Engel et al. [7] for an overview. In order to make use of existing standard volume rendering methods, one possibility is to precompute a “voxelized” version of the isogeometric model: Taking the geometry into account, one can store the values of the scalar field in a texture (the algorithm proposed in this article can readily be used for that). However, there are several drawbacks of this approach: 1) A voxelized model is in general no longer geometrically pixel-accurate (see Definition 1) which becomes especially visible at the surfaces or when the camera is close to small features, see Fig. 10. Since a main reason for the introduction of IGA was that it enables geometrically exact representation of geometry, this advantage should not be lost in the visualization stage. Exact geometry is desired by designers and analysts alike. 2) As a consequence of 1), a voxelized version needs a high number of voxels in order to represent display the same geometry. The high demand on GPU memory decreases the efficiency of standard volume rendering algorithms, see Section 5.

An algorithm for direct volume rendering for freeform volumes was presented by Chang et al. [2]. The method consists of subdividing B-spline volumes into a set of Bézier volumes until the geometry of the volumes is monotone. Those volumes are then depth-sorted, and scan-converted, allowing direct blending. The algorithm reaches approximately 5 FPS on a mini-supercomputer with 8192 processors. Martin and Cohen [16] outline an algorithm based on finding the roots of a function with the Newton-Raphson method. However, to the best of our knowledge, there is no implementation of this algorithm for isogeometric models.

In this paper we present a flexible framework for volumetric visualization based on volume rendering, allowing visualization of scalar fields as well as derived properties such as parametrization quality or mechanical stress. Our approach consists of several stages, leveraging the strengths of existing algorithms where possible. Several features of our approach are novel:

1) We provide an extremely robust and efficient algorithm for determining view-ray intersections with the surfaces of the volume, see Section 3.1. This is achieved by reformulating the original problem of finding zeros of a function, to be a problem related to approximation of surfaces.
2) We device novel approaches for pixel-accurate approximation of the preimage (i.e., the inverse) of the view-ray in the parameter space, suitable for efficient implementation on modern GPUs, see Section 4. In addition to an algorithm based on approximating zeros of a function, we provide an alternative based on ordinary differential equations (ODEs).
3) Degenerate cases of the parametrization of the geometry are treated in a suitable way, further increasing the robustness of our approach, see Section 4.3.

The rest of the paper is organized as follows: Section 2 provides the necessary background for volume rendering of isogeometric models, followed by a description of our approach in Section 3. Then, novel algorithms enabling geometrically pixel-accurate sampling of the volume render integral (1) are described in Section 4. Finally, we present applications and provide details of the performance of the implementation of the overall algorithm in Section 5, followed by a conclusion in Section 7.
2 Volume Rendering for Isogeometric Models

In this article we present an algorithm for volume rendering based on tracing view-rays through the volume from an imaginary observer. If such a view-ray intersects the object one obtains the color for the pixel of the screen by evaluating an integral describing the accumulated radiance along the ray. For a more detailed description of well established techniques for volume rendering see Engel et al. [7], Jensen [10] and references therein.

2.1 Continuous Model

When taking both emission and absorption into account, the accumulated radiance along a view-ray $\gamma : \mathbb{R} \rightarrow \mathbb{R}^3$ is given by the so-called volume render integral

$$I(t) = I(0)T(0,t) + \int_{\gamma|_{[0,t]}} \sigma(s)T(s,t)ds,$$

where $\int_\gamma$ denotes the line integral. The functions $\sigma$ and $T$ depend on a user adjusted transfer function $c$, specifying emission and absorption depending on the value of the scalar field $\rho$.

A major difference to classic volume rendering is that the geometry is no longer trivial. For isogeometric models, both the spline $\phi$ describing the geometry as well as the scalar field $\rho$, are defined on the same parameter domain $P$, see Fig. 2. This means that the value of the scalar field along the view-ray $\gamma$ is given by

$$\rho_\gamma = \rho \circ \phi^{-1}(\gamma)$$

As a consequence, if $\phi$ is not linear, the value of the scalar field along the straight view-ray in the geometry domain $G$, is obtained along a (not straight) curve in the parameter domain $P$.

2.2 Numerical Approximation

In general, neither the inverse of the spline $\phi$ (see Equation (2)) nor the volume render integral (1) itself have closed-form solutions. Therefore, the solutions have to be approximated and the overall error will consist of different sources due to

- numerical quadrature of integral (1) (depending on number of sample points and their location), and
- numerical approximation of the preimage of these sample points along the view-ray (method depended).

For interactive applications, a compromise between performance and accuracy must be found. The numerical quadrature of the volume render integral is described in Section 2.2.1. The approximation of the inverse is specific to isogeometric models, and the main contribution of this paper is dedicated to it (sections 3 and 4). A definition of the requirement for the accuracy of the approximation of the inverse is given in Section 2.2.2.

2.2.1 Quadrature of volume render integral

Discretizations are based on splitting the integral (1) into intervals. Efficient implementations on GPUs are so-called compositing schemes where color and opacity is accumulated iteratively (front-to-back). In applications transfer functions typically contain high frequency components, dictating a high sampling rate (Nyquist rate). Pre-integration, which is based on calculating the volume render integral for pairs of sample values in advance, is a commonly used technique to tackle this. Since sampling points are not necessarily equidistant in our approach, this method is difficult to apply due to the fact that the volume render integral (1) is non-linear in $t$. Instead we use a dense quadrature of the transfer function in the shader program, see Fig. 3.
With this approach we increase the image quality by removing wood-grain artifacts, while avoid computationally expensive evaluations of the spline $\rho$ and $\phi$. Just as with pre-integration, supersampling assumes that the scalar field is approximately linear between two sample points.

### 2.2.2 Pixel-Accurate Rendering of Geometry

In order to display the correct geometry of the isogeometric object, the approximation of the inverse of the geometry spline $\phi$ has to meet the following requirements.

**Definition 1** (Pixel-Accurate Approximation of Geometry). Let $\hat{p}_i \in P \ (i \in \mathbb{N})$ be approximations to points in the preimage of the view-ray, i.e., $\hat{\phi}^{-1}(\gamma)$. The approximation is called pixel-accurate, if (for all view-rays) the image of those approximations $\hat{g}_i \ (i.e., \hat{g}_i = \phi(\hat{p}_i))$ lies within the so-called pixel frustum. The pixel frustum, depicted in Fig. 4, is defined as the volume containing all points belonging to a given pixel.

We will now continue to describe our approach for isogeometric volume rendering.

### 3 Approach and Implementation

We present a novel approach enabling interactive volume visualization of isogeometric models with pixel-accurate geometry. As described in Section 2, an important step for approximating the volume render integral (1) is to determine the intersections of the view-ray with the surfaces of the object. Due to singularities of the Jacobian of $\phi$ on the boundary, existing approaches often struggle with finding intersections, leading to computationally expensive algorithms. In order to device an efficient and stable algorithm, our approach for volume-rendering consists of the following stages that are executed for every frame as part of the render pipeline.

1) **View-Ray intersections with surfaces:** We determine the intersections of all view-rays with the surfaces of the object by reinterpreting the problem as an approximation of surfaces, see Section 3.1.

2) **Depth-sorting of intersections:** Once all intersections are determined, they are sorted along each view-ray according to the distance from the origin (depth), see Section 3.2.

3) **Approximate Volume Render Integral:** For each pair of entry and exit points the volume render integral (1) is approximated, using pixel-accurate approximations of the inverse of the view-rays, see Section 4.

### 3.1 View-Ray Intersections with Surfaces

Computing the intersections between a ray and a spline surface can be a computationally expensive and unstable operation. Commonly, the problem is stated as finding the zeros of a function. However, the intersection problem can be restated as the problem of finding a view-dependent approximation of the surfaces of the object. Therefore, the rasterization process of GPUs can be used as a very efficient, parallel implementation of finding all ray-surface intersections of a triangulation. Two alternative approaches to construct a view-dependent triangulation using the hardware tessellator where recently presented by Yeo et al. [21] and by Hjelmervik [9]. Both methods are applicable in our setting, guaranteeing both water tightness and that the approximation error satisfies the requirements from definition 1. Our implementation uses [9] since it provides a single-pass algorithm.

### 3.2 Depth-sorting of Intersections

Along one view-ray, there can be more than two intersections with the surfaces, due to the following two reasons:

- **Non-convex objects:** For non-convex objects, the view-ray can intersect the geometry multiple times, see Fig. 5, leading to multiple entry and exit points along the view-ray.

- **Multi-block objects:** Isogeometric models often consist of a number of so-called volume blocks. Each
block is a tensor product B-spline volume. They are often introduced to be able to model real-life features such as holes or to improve the quality of the parametrization. In our approach all boundary surfaces are rendered for all volume blocks.

For each block of the object, the first stage of our approach (see Section 3.1) will lead to an unordered list of intersections per view-ray. In addition, the number of intersections depends on view-angle, geometry and block structure. For a correct result, compositing schemes, i.e., discretizations of the volume render integral (1), need to process the semi-transparent contribution of each part of the view-ray in depth-order. In computer graphics, this problem is known as order-independent transparency and several approaches can be found in the literature, for instance based on "depth peeling" (see Everitt [8]) or using an A-buffer (see Carpenter and Li [1] and Myers and Bavoil [17]).

Since modern GPUs allow atomic operations, our approach is based on a linked list (per pixel-location), which is populated in a single pass, as detailed in Yang et al. [20]. This list is subsequently sorted before the final compositing.

4 PIXEL-ACCURATE INVERSE OF VIEW-RAYS

In the previous Section 3 we described how to obtain the intersections of the view-ray $\gamma$ with the surfaces of the volume in the parameter domain $P$, providing pairs of entry and exit points ($g_{in}$ and $g_{out}$) for the compositing scheme. In order to approximate the volume render integral (1) pixel-accurately (see definition 1), approximations for sample points between $g_{in}$ and $g_{out}$ have to be found.

This section describes two alternative methods for finding an approximation of the inverse of the function $\phi$ for all points on the view-ray between $g_{in}$ and $g_{out}$. If not otherwise mentioned, we will assume that each pair of entry and exit points

1) is given and pixel-accurate (ensured by Section 3.1),
2) belongs to only one volume-block, denoted by $P_\alpha$ (ensured by Section 3.2),
3) the line connecting the entry with the exit point does not intersect the boundary of block $P_\alpha$ (this case is described in Section 4.3), and
4) $\phi_\alpha$ is a diffeomorphism with a continuous first derivative on the block $P_\alpha$ (see Section 4.3 for the case when $J\phi$ is singular on the boundary $\partial P_\alpha$).

We would like to point out that, under assumptions 1.-4., the problem of finding the inverse is well-posed in the sense of Hadamard, since $\phi_\alpha$ is continuous, bijective and differentiable within the interior of each block. The aforementioned conditions on $\phi_\alpha$ are reflected in the requirements for the numerical analysis performed on the isogeometric object.

The two alternative methods described in the following have certain advantages and disadvantages, see Section 6 for a discussion. It is worth noting that both methods are not limited to the case where $\phi$ is a trivariate spline, but work as long as above assumptions on $\phi$ hold.

4.1 Root Finding Based Algorithm.

This first approach has been outlined by Martin and Cohen [16], but no practical details are discussed and, to the best of our knowledge, no implementation exists. We will provide a brief description of the algorithm, followed by a discussion of the approach.

Mathematically, the point $p_i$ that is the preimage of $g_i$ is in the null space of the following function

$$F_{g_i} : P \rightarrow \mathbb{R}^3 : p \mapsto \phi(p) - g_i.$$  

A standard method for finding approximations of the roots of function (3) is the Newton-Raphson method, given by

$$J_F(x_n)(x_{n+1} - x_n) = -F(x_n),$$  

where $J_F(x_n)$ is the Jacobian matrix of $F(x_n)$. The Newton-Raphson method iteratively updates the guess $x_n$ as follows:

$$x_{n+1} = x_n - J_F(x_n)^{-1}F(x_n).$$  

The iteration is stopped when the difference $|x_{n+1} - x_n|$ is below some tolerance, or after a maximum number of iterations is reached.
where $J_F$ denotes the Jacobian matrix of $F$, and $x_n$ are the approximations to the root of $F$. To solve the $3 \times 3$ linear system of equations (4) we employ the QR-algorithm, see e.g., [5]. See Fig. 6 for an example.

Note that for each iteration of (4), both $\phi$ and $J_F = J_\phi$ have to be evaluated at the same point. Since $\phi$ is a spline function the Jacobian can be evaluated cheaply by reusing calculations for $\phi$.

The Newton-Raphson method converges quadratically for "good" starting points. The method can, however, fail in certain situations. We will address each situation in the following for the problem at hand.

For a good argumentation let us note that a bijective $\phi$ induces a metric space $(P, d_P)$ on the open parameter domain $P$ of each block with

$$d_P(p_1, p_2) := \|\phi(p_1) - \phi(p_2)\|_{L^2}, \quad p_1, p_2 \in P.$$ (5)

According to our basic assumptions, Equation (3) has a unique solution and therefore a root of $F$ is a root of the norm of $F$ and vice versa. Since

$$\|F_{g_i}(p)\|_{L^2} = d_P(p, \phi^{-1}(g_i)),$$ (6)

the problem of finding the inverse is equivalent to finding the minimum distance to the preimage of $g_i$. As a consequence, $F_{g_i}$ will not have horizontal asymptotes or local extrema/stationary points, and converge for all starting points in the interior of the block. Overshooting can be an issue in case of large distance between samples or where the geometry is complicated. Since we have a bounded domain $P$ we need to clamp values to the range of the parameter domain, usually $[0, 1]^3$, and use the method of line search. We would also like to point out that since $J_F = J_\phi$ the Jacobian in Equation (4) is non-singular on $P \setminus \partial P$ due to our basic assumptions.

The Newton-Raphson method (4) needs a stopping criterion. In our case we require the method to be pixel-accurate, so we stop the iteration when the distance to the view-ray (given by $\|F_{g_i}(x_n)\|_{L^2}$) is less or equal to the minimum distance of the point $g_i$ to the frustum boundary.

4.2 ODE Based Algorithm.

We will now describe a second approach based on the fact that the view-ray $\gamma$ can be seen as an integral curve of a (first order linear) dynamical system, i.e., the solution of an ordinary differential equation (ODE). The idea is then to directly work in the parameter domain by appropriately defining an ODE for which the image of the solution coincides with the original view-ray $\gamma$. This can be achieved by defining a vector field and numerically approximating the integral curve from the point where the ray enters the domain.

In order to describe our method we start by defining an ODE on the geometry domain by

$$g'(s) = V_\parallel(g(s)), \quad g(0) = g_{In}.$$ (7)

We define the vector field to be

$$V_{\parallel}(g) = V_\parallel + cV_\perp(g),$$ (8)

consisting of two components, where the constant $c > 0$ is the relative weight between the two components. The first component is a constant velocity parallel to the view-ray given by $V_\parallel = \frac{g_{Out} - g_{In}}{\|g_{Out} - g_{In}\|}$. The second component is a velocity perpendicular to the view-ray and depends on the signed distance to the view-ray, i.e.,

$$V_\perp(g) = (p_{In} - g) - \perp p_{In} - g, \quad V_\parallel > V_\perp, \quad \text{where } \perp \text{ is the usual scalar product.}$$ (9)

The ODE (7) is a first order linear dynamical system, which can be rewritten in the standard form

$$g' = Ag + b,$$

with

$$A = c \begin{pmatrix} V_{\parallel,1}^2 - 1 & V_{\parallel,1}V_{\parallel,2} & V_{\parallel,1}V_{\parallel,3} \\ V_{\parallel,2}V_{\parallel,1} & V_{\parallel,2}^2 - 1 & V_{\parallel,2}V_{\parallel,3} \\ V_{\parallel,3}V_{\parallel,1} & V_{\parallel,3}V_{\parallel,2} & V_{\parallel,3}^2 - 1 \end{pmatrix},$$ (10)

$$b = cp_{In} + (1 - c < p_{In}, V_\parallel >) V_\parallel.$$ (11)

The dynamics are determined by the eigenstructure of $A$, see e.g., [18]. There are two negative eigenvalues $\lambda_{1,2} = -c$ and one zero eigenvalue $\lambda_3 = 0$ with corresponding eigenvalue $v_3 = V_\parallel$. We can write a vector $u \in \mathbb{R}^3$ as the sum $u = u_\perp + u_\parallel$, where $\mathbb{R}^3 = M_\perp + M_\parallel = \text{span}(v_1, v_2) \oplus \text{span}(v_3)$. Furthermore, there exists a unique vector $b_\perp \in M_\perp$ such that $Ab_\perp = b_\perp$. The solution of Equation (10) is then

$$g(t) = e^{At}(g_\parallel + b_\parallel) - b_\perp + tb_\parallel.$$ (11)
This analysis shows that the view-ray is a solution and that all solutions (irrespective of the starting point \(g(0))\) converge exponentially towards the view-ray.

Next we will define a vector field \(V_p(p)\) such that the solution of the ODE on the parameter domain \(P\)

\[
p'(s) = V_p(p(s)), \quad (s, p) \in \mathbb{R} \times [0, 1]^3, \quad p(0) = p_{In},
\]

(12)
coinsides with the preimage of the solution of Equation (7).

Using that \(\phi\) is a differentiable map on the inside of \(P\), the dual is given by

\[
\phi_* : TP \rightarrow TG, (p, v) \mapsto (\phi(p), J_\phi(p)v),
\]

(13)
where \(T\) denotes the tangent space, see e.g., Spi-vak [19]. Since \(\phi\) is a diffeomorphism on the inside of \(P\), it follows that the dual has an inverse given by,

\[
\phi_*^{-1} : (p, v) \mapsto (\phi^{-1}(p), (J_\phi(p))^{-1}v),
\]

(14)
using the inverse function theorem. By means of Equation (14) the vector field \(V_p(p)\) in Equation (12) (the "preimage of \(V_g\)") is given through the following system of linear equations

\[
J_\phi(p)V_p(p) = V_g(\phi(p)),
\]

(15)
Note that \(J_\phi\) exists and is non-singular for all \(p\) in the inside of \(P\). See Fig. 7 for an example of how the vector field becomes non-trivial in the parameter domain.

Since the vector field \(V_p\) is in general non-linear, we establish the following theorem.

**Theorem 1** (Existence and Uniqueness). Under the condition that \(\phi\) is a diffeomorphism with a continuous Jacobian, there exists a unique solution to the initial value problem (12) (with (15)) that continues up to the boundary.

**Proof**: According to Theorem 2.7 in [4] (based on the theorem of Picard-Lindelöf) it is enough to show that the right hand side of Equation (12) is continuous and locally Lipschitz-continuous. For every compact subset of \(S \subset P\) there exists a constant \(C_\phi\), such that for all \(V,W \in \mathbb{R}^3\) and \(p_1, p_2 \in S\)

\[
\|J_\phi^{-1}(p_1)V - J_\phi^{-1}(p_2)V\| \leq C_\phi \|V - W\|,\quad V,W \in \mathbb{R}^3
\]

since \(\phi\) has a continuous Jacobian. It is then easy to show that

\[
\|V_p(p_1) - V_p(p_2)\| \leq C_\phi \|A\|_{op} L_\phi \|p_1 - p_2\|,
\]

(16)
where \(\|\|_{op}\) is the operator norm and \(L_\phi\) is the Lipschitz constant of \(\phi\).

Solutions of the ODE (12) can be approximated using a wide variety of numerical methods, e.g., explicit Runge-Kutta methods (see [4]). The simplest scheme is the explicit Euler method given by

\[
p(s^{n+1}) = p(s^n) + \Delta s^n V_p(p(s^n)),
\]

(17)
where we use standard notation using superscripts indicating discrete “times” and \(\Delta s^n = (s^{n+1} - s^n)\). Note that in each step we have to solve the linear system (15).

The stiffness index of Equation (10) is \(L = \max_i(\|\lambda_i\|) = c_i\), see e.g., [4]. While larger \(c\) have the benefit of preventing the numerical approximation to deviate to far from the view-ray (see Fig. 7), the ODE will become increasingly stiff. This behavior will be inherited in the ODE on \(P\) as well. Explicit solvers, such as the one described in Equation (17), will suffer from impractically small \(\Delta s^n\). Generally, implicit schemes have larger stability regions. An A-stable method suitable for stiff equations is the implicit Euler scheme, given by

\[
p(s^{n+1}) = p(s^n) + \Delta s^n V_p(p(s^{n+1})),
\]

(18)
where a system of non-linear equations must be solved in each step. In order to achieve maximum stability, we do not solve for \(p(s^{n+1})\) directly, but rather for the difference to the previous point, see e.g., [4]. Thus, we have to solve

\[
G(z) = z - \Delta s^n V_p(z + p(s^n)) = 0,
\]

(19)
numerically, for which we apply the Newton-Raphson method in order to approximate the solution. This method depends on the Jacobian of \(G\), which can
be approximated. However, in order to increase performance we avoid further evaluations of \( G \) and instead derive an exact expression for \( J_G \). By rewriting Equation (19) as

\[
J_\phi(z + p^n)G(z) = J_\phi(z + p^n)z - \Delta s^n V_\phi(\phi(z + p^n)),
\]

where \( p^n = p(s^n) \) and applying the Jacobian operator to both sides, we derive (using the chain- and product rule) the following linear system of equations

\[
J_\phi(z + p^n)J_G(z) = H_\phi(z + p^n)(z - G(z)) + (I - \Delta s J_\phi(\phi(z + p^n))) J_\phi(z + p^n).
\]

Here, \( I \) is the identity matrix and \( H_\phi \) denotes the Hessian of \( \phi \), a tensor of order 3. Observe, that the spline \( \phi \) along with its first and second order derivatives are evaluated at the same point, allowing for an efficient calculation in a shader program. The Jacobian of the vector field \( V_\phi \) is the following constant matrix

\[
J_{V_\phi}(g) = c(C - I),
\]

where the i-th column of \( C \) is given by \( V_{\phi_i}V_j \).

By solving the linear Equation (21) we can calculate the Jacobian matrix of \( G \) and use it for the Newton-Raphson method used for the implicit Euler method. It has the same matrix as the equation we need to solve to get the vector field \( V_\phi \), see Equation (15). This means that, when using the QR-algorithm for solving both linear equations, an efficient algorithm can reuse \( Q \) and \( R \) for solving Equation (21).

4.3 Degeneracies and Points Outside Domain.

There are two prominent cases for which the methods described in sections 4.1 and 4.2 need minor adjustments. The first case is when the line between \( g_{In} \) and \( g_{Out} \) intersect the boundary of the volume block. Although this is a rare case, it can happen that the approximation of the surfaces “misses” intersections in the tessellation of the geometry (described in Section 3.1). This case is shown in Fig. 8, where the approximated surface (dashed black line in geometry) is still pixel-accurate, but the view-ray intersects the exact surface. In such a case, the Newton-Raphson method in both the implicit Euler method as well as the root finding based method will not converge, but repeatedly try to exit the parameter domain of the according block. We detect such behavior and step along the boundary of the parameter domain until the view-ray is within the domain again. For the explicit Runge-Kutta methods we simply clamp the approximated solution values to remain in \( P \).

Observe, that the resulting approximation of the view-ray (seen in Fig. 8) is still pixel-accurate, since the approximation of the surface is guaranteed to be so.

The second case is when there are degeneracies of the spline \( \phi \) along the boundary, see e.g., Fig. 9. Assume for instance that the Jacobian \( J_\phi \) is singular at the entry point \( g_{In} \). Since both the root finding method (Section 4.1) as well as the ODE based methods (Section 4.2) involve solving a system of linear equations with a singular matrix in that case (see Equations (4) and (15)), the only viable choice is to “shrink” the block in the following way. By choosing a new entry point \( \tilde{g}_{In} = g_{In} + \delta \frac{p_{Out} - p_{In}}{\| p_{Out} - p_{In} \|_2} \), with a suitable \( \delta > 0 \).

The new entry point \( p_{In} \) in the parameter domain \( P \) can be found with the root finding method with a different starting point for the iterations in the Newton-Raphson method, for instance

\[
x_0 = p_{In} + \epsilon \frac{p_{Out} - p_{In}}{\| p_{Out} - p_{In} \|_2},
\]

with an appropriately chosen \( \epsilon \). As can be seen in the middle of Fig. 1 this approach works well.

4.4 Cutting and Near Clip Planes

The root finding based method is well suited for realizing cutting planes. Before the final compositing, the depth-sorted list of intersections (from Section 3.2) is handled in the following way

- **Cutting planes**: The plane is given by a point \( g_0 \in G \) and a normal \( n \). If \( n \cdot g_{In} - g_0 \) and \( n \cdot g_{Out} - g_0 \) have opposite signs, the view-ray \( \gamma \) between \( g_{In} \) and \( g_{Out} \) intersects the cutting plane.
- **Near-Clip plane**: Since the rendering of the surfaces is water tight, an odd number of ray-surface intersections means that the near plane is inside the volume.

In both cases, trivial formulas determine the intersection point \( g_* \in G \). Given \( g_* \), we need to find the according point \( p_* \in P \) such that \( \phi(p_*) - g_* = 0 \). Thus, finding \( p_* \) is exactly solving Equation (3). Since this point can be quite far away from \( p_{In} \) or \( p_{Out} \) the root finding algorithm described in Section 4.1 is the best alternative and can readily be used.
5 Applications and Performance

In order to benchmark the performance of the proposed methods, we present the three different application scenarios shown in Fig. 1, covering a wide range of possible applications. We apply the approach described in Sections 3 and 4 in each case, and compare it with standard volume rendering algorithms, where we have precomputed a voxelized version of the model. The resulting texture has 16 bit and uses the red channel for the scalar value and the green channel to encode if the voxel is inside the object or not. Of course, many optimization strategies are established in standard volume rendering, such as adaptive sampling rates, out of core algorithms, etc. However, to allow for a fair comparison, we only use an out of the box implementation without any optimizations. In all cases we measure the performance of our algorithm on different screen resolutions on an NVIDIA Titan GPU.

For standard volume rendering the amount of GPU memory needed depends mainly on the size of the texture and is therefore independent of the screen resolution. The proposed approach on the other hand, allocates memory for the knots and the control points. In addition, a buffer is allocated for the linked list containing all view-ray intersections with the surfaces (see Section 3.2), and will therefore increase with screen resolution. We start with an application useful in the design phase of the geometry.

5.1 Parametrization Quality of Geometry

Before an analysis of a model can be carried out, the geometric shape has to be designed. Since the parametrization of the geometry $\phi$ is not unique, one wants to optimize the quality of the parametrization. A measure of the quality of the parametrization is given by

$$\rho = \frac{\det(J_{\phi})}{\|J_{\phi}\|_F},$$

(24)

where low values indicate that the geometry is (close to) degenerate. With this scalar field our method can be used as an inspection tool in the design phase, isolating potentially problematic areas. The parametrization quality (24) is calculated on the fly for each sample.

As an example we present a twisted bar, see left in Fig. 1. This model consists of only one volume block and the geometry is described by a quadratic B-spline. In TABLE 1 (a) we can see that both the root finding based method as well as the (3/8-rule) 4th order Runge-Kutta method have the highest frame rates. In addition they only use little GPU memory. In contrast, the precomputed voxelized model needs a texture of the size $512^3$ in order to show the same level of details. In addition to the high memory usage, this method shows a slightly lower frame rate compare with the proposed methods.

Another example is given in Fig. 9 showing the quality of the parametrization of the demonstrator part of the TERRIFIC project.

5.2 Stress Analysis in Linear Elasticity

Structural analysis is an important application area for isogeometric analysis, where external forces lead to a deformation of the object given in the form of a so-called displacement field $u: P \rightarrow \mathbb{R}^3$. The stress due to deformation is then calculated by

$$\rho = \frac{1}{2} (\sigma_{11} - \sigma_{22} + (\sigma_{22} - \sigma_{33}) + (\sigma_{33} - \sigma_{11}) + 6(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2))/2.$$  

(25)

The so-called strain tensor is $\sigma = \frac{1}{2} (J_{u\phi^{-1}} + J_{T_{u\phi^{-1}}})$, where $J_{u\phi^{-1}}(g)$ is the solution of

$$[J_{\phi}(p)]^T J_{u\phi^{-1}}(g) = [J_{u}(p)]^T.$$

As in the previous example, all those expressions are calculated on the fly for each sample of the scalar field.

In the middle of Fig. 1 we present the results for the linear elasticity simulation from the TERRIFIC project. Both geometry $\phi$ and deformation $u$ are cubic B-splines and the model consists of 15 volume blocks. All the proposed methods work well also in this case where there are some degeneracies along the boundaries, see Fig. 9. TABLE 1 (b) indicates that the root finding based method and standard rendering of the precomputed voxelized model essentially have the same frame rate. In this case Kutta’s third order solver is the fastest amongst the ODE solvers. For the precomputed voxelized approach, a 16 bit texture of the size $1024 \times 1024 \times 128$ was necessary in order to display enough details. The reason that the voxelized model performs slightly better is, that the proposed methods evaluate the von Mises stress (25) on the fly for each sample. The voxelized method has those fairly compute intensive calculations (that involve the evaluation of the splines $u$ and $\phi$) already pre-evaluated.
(a) Parametrization quality for the twisted (b) Von Mises stress for TERRIFIC model, see bar, see left in Fig. 1. middle in Fig. 1. (c) Backstep Flow from RANS simulation, see right in Fig. 1.

| Method                 | FPS | MiB | FPS | MiB | FPS | MiB |
|------------------------|-----|-----|-----|-----|-----|-----|
| Root Finding           | 67  | 63  | 36  | 87  | 89  | 109 |
| Explicit Euler (c=1)   | 4   | 1   | 109 | 1   | 189 |     |
| Midpoint Method (c=1)  | 42  | 16  | 35  | 87  | 65  | 109 |
| Kutta’s 3rd order (c=1)| 69  | 38  | 189 |     |     |     |
| Classic 4th order (c=1)| 54  | 24  | 87  |     |     |     |
| 3/8-rule 4th order (c=1)| 66 | 35  | 87  |     |     |     |
| RKF45 5th order (c=1)  | 50  | 21  | 87  |     |     |     |
| Implicit Euler (c=100) | 13  | 6   | 87  |     |     |     |
| Voxelized (512\(^2\)) | 48  | 528 | 31  | 532 | 92  | 537 |

| Method                 | FPS | MiB | FPS | MiB | FPS | MiB |
|------------------------|-----|-----|-----|-----|-----|-----|
| Root Finding           | 109 | 47  | 189 |     |     |     |
| Explicit Euler (c=1)   | 4   | 1   | 189 |     |     |     |
| Midpoint Method (c=1)  | 37  | 20  | 189 |     |     |     |
| Kutta’s 3rd order (c=1)| 69  | 38  | 189 |     |     |     |
| Classic 4th order (c=1)| 65  | 35  | 189 |     |     |     |
| 3/8-rule 4th order (c=1)| 65 | 35  | 189 |     |     |     |
| RKF45 5th order (c=1)  | 21  | 35  | 189 |     |     |     |
| Implicit Euler (c=100) | 1   | <1  | 189 |     |     |     |
| Voxelized (256\(^2\) × 128) | 92 | 537 | 60  | 542 | 93  | 47  |

**TABLE 1:** Comparison of the performance of the different algorithms for visualization of the different models on an NVIDIA Titan GPU. The root finding based method and higher order ODE based methods show interactive frame rates with low memory usage for "complicated" geometry. In the case of easy geometry (case (c)), i.e., when \( \phi \) is close to linear, the situation reverses in favour of the voxelized method.

### 5.3 Computational Fluid Dynamics

Another important application of isogeometric analysis is computational fluid dynamics (CFD). On the right of Fig. 1 we present a visualization of an approximation of the solution of the Reynolds-averaged Navier-Stokes (RANS) equations for a backstep flow. The scalar field \( \rho \) represents turbulent viscosity and comes directly from the simulation. The model uses quadratic B-splines to represent both the geometry and the scalar field and consists of 140 blocks. As can be seen from **TABLE 1** (c) the precomputed voxelized approach has the highest frame rates and the lowest GPU memory usage. Amongst the proposed methods the explicit Euler method has the highest frame rates, followed by the second order midpoint method and the root finding based method. Since the geometry is (close to) linear, the first order ODE solver is the most efficient compared with higher order ODE solvers. Furthermore, the fact that it is enough to represent the model with only \( 256 \times 256 \times 128 \) grid points, indicates that the scalar field is fairly simple. Part of the reason why standard rendering of the voxelized model is faster lies in the fact that the proposed methods have to evaluate the splines \( \rho \) and \( \phi \) at each sample point.

We would like to comment that this example is only a preliminary simulation of the backstep flow. Optimally, the geometry would be represented with much less blocks, leading to a higher framerate and lower memory usage for the proposed methods.

### 6 Discussion of Results

The results presented in Section 5 show high visual quality with pixel-accurate geometry at interactive frame rates. Although a standard volume rendering algorithm for the precomputed voxelized version of the model seems a tempting alternative, there are several drawbacks of this approach. The most severe one is that the method does in general not guarantee a pixel-accurate geometry. As depicted in Fig. 10 there is in general no bound on the number of voxels needed to display correct geometry. However, in most cases, if the scalar field does not change heavily, there might not be any visual artifacts. The examples in the previous section (see **TABLE 1** (a) and (b)) suggest that a high number of voxels is needed for displaying the same visual result as with the proposed methods.

We would also like to point out that it is not straightforward to display a visually pleasing outer surface of the object with voxelized methods, unless the voxel grid is aligned with the object boundaries.

The comparison between the two proposed methods for approximating the inverse of the view-ray suggests the following. The number of sample points is a good indicator for the efficiency of a volume rendering method. As mentioned in Section 2.2 the number of sample points depends on the numerical quadrature of integral (1) and the numerical approximation of the preimage of these sample points. Therefore, a lower bound for the necessary number of samples is given by the numerical quadrature of the volume render integral (1). Overall, the needed number of sample points for a pixel-accurate approximation of the preimage depends on the method.

- The accuracy of the root finding based method is essentially independent of the distance between the sample points. However, the number of iterations in the underlying Newton-Raphson method increases in general for large distances, since the method only converges quadratically in a neighbourhood of the solution.
- For the ODE based methods the accuracy is a function of the distance between the sample points and depends on the order and the error constant of the method. Therefore, in order to be pixel-accurate, the ODE based methods might require a smaller sampling distance than required by the quadrature of the integral. For the TERRIFIC model (see Section 5.2) for instance the explicit Euler method uses 100 times more sample points compared with the 3rd order method in...
Fig. 10: The proposed method is geometrically pixel-accurate, the voxelized method is not. The effect becomes particularly pronounced when the camera is close to small features. Top row: Close up of twisted bar, see left in Fig. 1. Bottom row: Schematic view of methods. The precomputed voxelized model will in general not be pixel-accurate, depending on how densely the model is sampled.

For optimal performance, the sampling distances should solely be determined by the quadrature of the volume render integral (1). The potential improvements are therefore:

- **Adaptive sampling distance**: For optimal efficiency, an adaptive sampling distance for the quadrature is required.
  - For the root finding based method, an adaptive sampling distance has to take the geometry into account, making the algorithmic design more complicated.
  - The ODE based methods are defined and solved directly in the parameter space \( P \), where also the scalar field is defined. Therefore, it essentially makes it unnecessary to take the geometry into account when deciding for the sampling distance.

- **Adaptive order**: In TABLE 1 the ODE method with the highest frame rate depends on how "complicated" the geometry is. For the twisted bar, a 4th order method is optimal, for the demonstrator part from the TERRIFIC project a 3rd order method is optimal, whereas for the backstep flow a first order method is optimal. An adaptive control of the numerical schemes for the ODE based methods might therefore further increase the efficiency, leading to a higher frame rate: Use computationally expensive high order methods only when required by large sampling distances and/or in places of "complicated" geometry.

Finally, observe that the ODE based methods stand on solid theoretical ground, see Theorem 1, proving existence and uniqueness.

### 7 Conclusion

The presented approach allows interactive inspection of volumetric models used in isogeometric analysis. In the spirit of isogeometry, the algorithms operate directly on the spline models and therefore demand very little GPU memory. For models with "easy" geometry and simple scalar field, a standard ray-casting of a precomputed voxelized method might be faster. However, isogeometric analysis is typically used when geometry becomes non-trivial. Furthermore, in contrast to voxelized methods, the proposed algorithms ensure pixel-accurate geometry of both surfaces and volume irrespective of the zoom level, making it an asset during design, analysis and marketing phase. We applied our approach in three use cases relevant to industry showing good performance at interactive frame rates. In the future we plan to increase the efficiency of the presented methods through adaptive sampling and exploring adaptive order for the ODE based methods.

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