TOPOLOGICAL GYROGROUPS WITH Fréchet-Urysohn Property and ωα-base

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Abstract. The concept of topological gyrogroups is a generalization of a topological group. In this work, ones prove that a topological gyrogroup $G$ is metrizable iff $G$ has an $\omega^\alpha$-base and $G$ is Fréchet-Urysohn. Moreover, in topological gyrogroups, every (countably, sequentially) compact subset being strictly (strongly) Fréchet-Urysohn and having an $\omega^\alpha$-base are all weakly three-space properties with $H$ a closed $L$-subgyrogroup.

1. Introduction

The concept of a gyrogroup was originally posed by Ungar in [30, 31]. It is obvious that a group is a gyrogroup that every groupoid automorphism is an identity mapping. Then, in 2017, Atiponrat [2] equipped the gyrogroup with a topology and gave the definition of a topological gyrogroup. At the same time, she gave some examples of topological gyrogroups, such as Möbius gyrogroups equipped with standard topology. Moreover, she posed an open problem whether the first-countability and metrizability are equivalent in topological gyrogroups. Afterwards, Cai, Lin and He in [13] gave a positive answer about this problem, since all topological gyrogroups are rectifiable spaces. In fact, this kind of spaces has been studied for many years, see [3, 11, 16, 24, 25, 26, 31, 32]. However, they all did not research the quotient spaces of topological gyrogroups. Until in [6, 7, 8, 10], Bao and Lin started to investigate the quotient spaces of strongly topological gyrogroups and achieved some good results. For example, if $H$ is an admissible $L$-subgyrogroup of a strongly topological gyrogroup $G$, then the left coset space $G/H$ is submetrizable. More important, they constructed a strongly topological gyrogroup with an infinite $L$-subgyrogroup. By the same construction, we can obtain a topological gyrogroup with an infinite $L$-subgyrogroup. Therefore, it is meaningful to research the quotient spaces of a topological gyrogroup when the left coset is an $L$-subgyrogroup. In particular, we will investigate what properties of topological groups still hold in topological gyrogroups.

This paper is aims to research topological gyrogroups with $\omega^\alpha$-base, Fréchet-Urysohn properties and weakly three-space properties. We prove that a topological gyrogroup $G$ is metrizable iff $G$ has an $\omega^\alpha$-base and $G$ is Fréchet-Urysohn. Moreover, in topological gyrogroups, every (countably, sequentially) compact subset being strictly (strongly) Fréchet-Urysohn and having an $\omega^\alpha$-base are all weakly three-space properties with $H$ a closed $L$-subgyrogroup. More precisely, if $H$ is a closed $L$-subgyrogroup of a topological gyrogroup $G$ with the first-countability of every (countably, sequentially) compact

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subset of $H$, and if every (countably, sequentially) compact subset of $G/H$ is strictly (strongly) Fréchet-Urysohn, respectively, then every (countably, sequentially) compact subset of $G$ is strictly (strongly) Fréchet-Urysohn. Finally, if a topological gyrogroup $G$ has a closed first-countable $L$-subgyrogroup $H$ with the quotient space $G/H$ having an $\omega^\omega$-base, then $G$ has an $\omega^\omega$-base.

2. Preliminaries

In this paper, we assume that all topological spaces are Hausdorff, $\mathbb{N}$ denote the set of all positive integers and $\omega$ denote the first infinite ordinal. The readers see [1, 15, 31] for more notation and terminology. Next we recall some definitions and facts.

Definition 2.1. [31] Assume that $(G, \oplus)$ is a groupoid. We call $(G, \oplus)$ a gyrogroup, if the following conditions are satisfied:

(G1) For every $a \in G$, there is a unique identity element $0 \in G$ with $0 \oplus a = a = a \oplus 0$;

(G2) for every $x \in G$, we can find a unique inverse element $\ominus x \in G$ with $\ominus x \oplus x = 0 = x \oplus (\ominus x)$;

(G3) for every $x, y \in G$, we can find $\text{gyr}[x, y] \in \text{Aut}(G, \oplus)$ with $x \oplus (y \oplus z) = (x \oplus y) \oplus \text{gyr}[x, y](z)$ for arbitrary $z \in G$, and

(G4) for every $x, y \in G$, $\text{gyr}[x \oplus y, y] = \text{gyr}[x, y]$.

Notice that a group is a gyrogroup naturally only if the $\text{gyr}[x, y]$ is the identity mapping.

Definition 2.2. [29] Assume that $(G, \oplus)$ is a gyrogroup. We call a nonempty subset $H$ of $G$ a subgyrogroup, if the restriction of $\text{gyr}[a, b]$ to $H$ is an automorphism of $H$ for every $a, b \in H$ and $H$ forms a gyrogroup under the operation inherited from $G$. Denote it by $H \leq G$.

Moreover, we call $H$ an $L$-subgyrogroup, if $H$ is a subgyrogroup and satisfies $\text{gyr}[a, h](H) = H$ for each $a \in G$ and $h \in H$. Denote it by $H \leq_L G$.

Lemma 2.3. [31] Assume that $(G, \oplus)$ is a gyrogroup. For every $x, y, z \in G$,

(1) $(\ominus x) \oplus (x \oplus y) = y$. (left cancellation law)

(2) $(x \oplus (\ominus y)) \oplus \text{gyr}[x, \ominus y](y) = x$. (right cancellation law)

(3) $(x \oplus \text{gyr}[x, y](\ominus y)) \oplus y = x$.

(4) $\text{gyr}[x, y](z) = \ominus (x \oplus y) \oplus (x \oplus (y \ominus z))$.

Definition 2.4. [2] Call $(G, \tau, \oplus)$ a topological gyrogroup if it satisfies the followings:

(1) $(G, \tau)$ is a topological space.

(2) $(G, \oplus)$ is a gyrogroup.

(3) The binary operation $\oplus : G \times G \to G$ is jointly continuous, where $G \times G$ is equipped with the product topology, and the inverse operation $\ominus(\cdot) : G \to G$, i.e. $x \to \ominus x$, is also continuous.

Remark Obviously, every topological group is a topological gyrogroup. However, every topological gyrogroup whose gyrations are not identically equal to the identity is not a topological group. Moreover, it was given an example in [2] to show that there exists a topological gyrogroup but not a topological group, such as the Einstein gyrogroup equipped with the standard topology.
Definition 2.5. \textit{\cite{11, 23, 18}} Let $X$ be a topological space and $x \in X$. We say that $x$ has a neighborhood $\omega^\omega$-base or a local $\Theta$-base if there is a base $\{U_\alpha(x) : \alpha \in \mathbb{N}^\mathbb{N}\}$ of neighborhoods at $x$ with $U_\beta(x) \subset U_\alpha(x)$ for every $\alpha \leq \beta$ in $\mathbb{N}^\mathbb{N}$, where $\mathbb{N}^\mathbb{N}$ consisted by all functions from $\mathbb{N}$ to $\mathbb{N}$ equipped with the natural partial order, i.e., $f \leq g$ iff $f(n) \leq g(n)$ for every $n \in \mathbb{N}$. We say that $X$ has an $\omega^\omega$-base or a $\Theta$-base if $X$ has a neighborhood $\omega^\omega$-base or a local $\Theta$-base at each point of $X$.

Definition 2.6. \textit{\cite{17}} If $X$ is a topological space and $A \subset X$, we call $A$ sequentially closed if there is not sequence of points of $A$ converging to a point not in $A$. Call $X$ sequential if all sequentially closed subsets of $X$ are closed.

Definition 2.7. \textit{\cite{17}} A topological space $X$ is called Fréchet-Urysohn at a point $x \in X$ if there exists a sequence $\{x_n\}_{n \in \mathbb{N}}$ in $A$ with $\{x_n\}_{n \in \mathbb{N}}$ converging to $x$ for each $A \subset X$ satisfying $x \in \bar{A} \subset X$. We call $X$ Fréchet-Urysohn if it is Fréchet-Urysohn at every point.

Definition 2.8. \textit{\cite{22, 25}} A topological space $X$ is called strictly (strongly) Fréchet-Urysohn at a point $x \in X$ if whenever $\{A_n\}_{n \in \mathbb{N}}$ is a sequence (decreasing sequence) of subsets in $X$ and $x \in \bigcap_{n \in \mathbb{N}} \overline{A_n}$, we can find $x_n \in A_n$ for all $n \in \mathbb{N}$ with the sequence $\{x_n\}_{n \in \mathbb{N}}$ converging to $x$. We call $X$ strictly (strongly) Fréchet-Urysohn if it is strictly (strongly) Fréchet-Urysohn at every point.

3. $\omega^\omega$-base and Fréchet-Urysohn property in topological gyrogroups.

In this section, ones research topological gyrogroups with $\omega^\omega$-base and Fréchet-Urysohn property. It is shown that if a topological gyrogroup $G$ is first-countable, then it has an $\omega^\omega$-base. If a topological gyrogroup $G$ has an $\omega^\omega$-base and is Fréchet-Urysohn, then it is metrizable. Therefore, we deduce that a topological gyrogroup $G$ is metrizable iff having an $\omega^\omega$-base and being Fréchet-Urysohn are both satisfied by $G$.

Suppose that a topological gyrogroup $G$ has an $\omega^\omega$-base $\{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$. Set $I_k(\alpha) = \{\beta \in \mathbb{N}^\mathbb{N} : \beta_i = \alpha_i \text{ for } i = 1, \ldots, k\}$, and $D_k(\alpha) = \bigcap_{\beta \in I_k(\alpha)} U_\beta$,

where $\alpha = (\alpha_i)_{i \in \mathbb{N}} \in \mathbb{N}^\mathbb{N}$ and $k \in \mathbb{N}$. Then $\{D_k(\alpha)\}_{k \in \mathbb{N}}$ is an increasing subset sequence of $G$ and contains the identity element 0.

Definition 3.1. \textit{\cite{14}} We call a topological space $X$ strong $\alpha_4$-space if an arbitrary subset $\{x_{p,q} : p, q \in \mathbb{N}\}$ of $X$ is such that $\lim_{q \to \infty} x_{p,q} = x \in X$ for each $p \in \mathbb{N}$, we can find strictly increasing natural number sequences $\{l_k\}_{k \in \mathbb{N}}$ and $\{m_k\}_{k \in \mathbb{N}}$ such that $\lim_{k \to \infty} x_{l_k, m_k} = x$.

Lemma 3.2. Every first-countable topological gyrogroup has an $\omega^\omega$-base.

\textit{Proof.} Let $G$ be a first-countable topological gyrogroup, and $\{V_n\}_{n \in \mathbb{N}}$ a decreasing base at the identity element 0. Put $W_\alpha = V_{\alpha_1}$ for each $\alpha \in \mathbb{N}^\mathbb{N}$. We have that $\{W_\alpha\}_{\alpha \in \mathbb{N}^\mathbb{N}}$ is an $\omega^\omega$-base in $G$. \hfill $\square$

Lemma 3.3. \textit{\cite{25}} If a topological gyrogroup $G$ is Fréchet-Urysohn, then it is a strong $\alpha_4$-space.

Lemma 3.4. \textit{\cite{20}} Suppose $\alpha = (\alpha_i)_{i \in \mathbb{N}} \in \mathbb{N}^\mathbb{N}$ and $\beta_k = (\beta^{(k)}_i)_{i \in \mathbb{N}} \in I_k(\alpha)$ for all $k \in \mathbb{N}$. Then we can find $\gamma \in \mathbb{N}^\mathbb{N}$ with $\alpha \leq \gamma$ and $\beta_k \leq \gamma$ for arbitrary $k \in \mathbb{N}$.
Theorem 3.5. If a Hausdorff topological gyrogroup \( G \) is Fréchet-Urysohn and has an \( \omega^\omega \)-base \( \{ U_\alpha : \alpha \in \mathbb{N}^\mathbb{N} \} \), then \( G \) is metrizable.

Proof. First, we establish the following:

Claim. \( D_k(\alpha) \) is a neighborhood of 0 for each \( \alpha \in \mathbb{N}^\mathbb{N} \) for some \( k \in \mathbb{N} \).

Suppose not, that is, we can find \( \alpha \in \mathbb{N}^\mathbb{N} \) such that \( D_k(\alpha) \) is not a neighborhood of 0, for any \( k \in \mathbb{N} \). It means that \( 0 \in G \setminus D_k(\alpha) \) for any \( k \in \mathbb{N} \). By the hypothesis, \( G \) is Fréchet-Urysohn, for arbitrary \( k \). Therefore, we can find a sequence \( \{ x_{n,k} \}_{n \in \mathbb{N}} \) in \( G \setminus D_k(\alpha) \) which converges to 0. Since all Fréchet-Urysohn Hausdorff topological gyrogroups are strong \( \alpha_i \)-spaces, we can choose natural numbers sequences \( (n_i)_{i \in \mathbb{N}} \) and \( (k_i)_{i \in \mathbb{N}} \) with \( \lim_{i \to \infty} x_{n_i,k_i} = 0 \), where \( (n_i)_{i \in \mathbb{N}} \) and \( (k_i)_{i \in \mathbb{N}} \) are strictly increasing.

For each \( i \in \mathbb{N} \), choose \( \beta_i \in I_{k_i}(\alpha) \) with \( x_{n_i,k_i} \not\in U_{\beta_i} \). It follows from Lemma 3.4 that, for every \( i \in \mathbb{N} \), \( \beta_i \leq \gamma \) for some \( \gamma \in \mathbb{N}^\mathbb{N} \). Therefore, for any \( i \in \mathbb{N} \), \( x_{n_i,k_i} \not\in U_\gamma \). We conclude that the sequence \( \{ x_{n_i,k_i} \}_{i \in \mathbb{N}} \) does not converge to 0 and this is a contradiction.

Therefore, for each \( \alpha \in \mathbb{N}^\mathbb{N} \), set \( D_{k_\alpha}(\alpha) \) is a neighborhood of 0, where \( k_\alpha \) is a minimal natural number. It is clear that \( D_{k_\alpha}(\alpha) \subset U_\alpha \). Moreover, for \( i \in \mathbb{N} \), fix \( \alpha(i) = (i, \alpha_2, \alpha_3, ...) \in \mathbb{N}^\mathbb{N} \). Then for any \( \beta = (\beta_1, \beta_2, ...) \in I_{1}(\alpha(i)) \), \( D_1(\beta) = D_1(\alpha(i)) \). Therefore, \( \{ D_1(\alpha) : \alpha \in \mathbb{N}^\mathbb{N} \} = \{ D_1(\alpha(i)) : i \in \mathbb{N} \} \) is countable. So, \( \{ D_1(k) : k \in \mathbb{N}, \alpha \in \mathbb{N}^\mathbb{N} \} \) is countable. Furthermore, \( \{ D_{k_\alpha}(\alpha) : \alpha \in \mathbb{N}^\mathbb{N} \} \subset \{ D_k(\alpha) : k \in \mathbb{N}, \alpha \in \mathbb{N}^\mathbb{N} \} \). Therefore, the countability of the family \( \{ D_{k_\alpha}(\alpha) : \alpha \in \mathbb{N}^\mathbb{N} \} \) is obtained. In conclusion, the family \( \{ int(D_{k_\alpha}(\alpha)) : \alpha \in \mathbb{N}^\mathbb{N} \} \) is a countable open neighborhood base at 0. It follows from above that \( G \) is first-countable and hence metrizable by [12].

By Lemma 3.2 and Theorem 3.5 we deduce the following result.

Corollary 3.6. A topological gyrogroup \( G \) is metrizable iff \( G \) has an \( \omega^\omega \)-base and \( G \) is also Fréchet-Urysohn.

Let \( X \) be a topological space and \( \mathcal{N} \) a family of subsets of \( X \). We call \( \mathcal{N} \) a cs*-network at a point \( x \in X \) [19] if we can find \( N \in \mathcal{N} \) with \( x \in N \subset O_x \) and \( \{ n \in \mathbb{N} : x_n \in N \} \) is infinite, where \( (x_n)_{n \in \mathbb{N}} \) is arbitrary sequence in \( X \) converging to \( x \) and \( O_x \) is arbitrary neighborhood of \( x \).

Then we give the concept of cs*-character in topological gyrogroups.

Definition 3.7. If \( G \) is a topological gyrogroup, we call the least cardinality of \( cs^* \)-network at the identity element 0 of \( G \) cs*-character. 

Theorem 3.8. If a topological gyrogroup \( G \) has an \( \omega^\omega \)-base, then it has countable cs*-character.

Proof. Suppose that \( \{ U_\alpha : \alpha \in \mathbb{N}^\mathbb{N} \} \) is an \( \omega^\omega \)-base in \( G \). Put \( D = \{ D_k(\alpha) : \alpha \in \mathbb{N}^\mathbb{N}, k \in \mathbb{N} \} \). Then \( D \) is countable and is a cs*-network at 0. Choose a sequence \( S = (g_n)_{n \in \mathbb{N}} \) in \( G \) which converges to 0 and fix a neighborhood \( U_\alpha \) of 0. Therefore, we just need to show that \( S \cap D_k(\alpha) \) is infinite for some \( k \in \mathbb{N} \).

Claim. \( S \cap D_k(\alpha) \) is infinite for some \( k \in \mathbb{N} \).

Suppose on the contrary, \( S \cap D_k(\alpha) \) is finite for every \( k \in \mathbb{N} \). Then, for arbitrary \( k \in \mathbb{N} \), take \( n_k \in \mathbb{N} \) and \( \beta_k \in I_k(\alpha) \) such that \( n_1 < n_2 < ... \) and \( g_{n_k} \not\in U_{\beta_k} \). Then \( \{ g_{n_k} \}_{k \in \mathbb{N}} \) converges to 0. By Lemma 3.3 there exists \( \gamma \in \mathbb{N}^\mathbb{N} \) with \( \alpha \leq \gamma \) and \( \beta_k \leq \gamma \) for all \( k \in \mathbb{N} \). It is clear that \( g_{nk} \not\in U_\gamma \) for every \( k \in \mathbb{N} \). Hence, \( g_{nk} \not\rightarrow 0 \), which is a contradiction.

It was claimed in [12] that if \( G \) is a Fréchet-Urysohn topological group and \( G \) also has countable cs*-character, then \( G \) is metrizable. Now, we pose the following problem.
Question 3.9. If a topological gyrogroup $G$ is Fréchet-Urysohn and $G$ also has countable $cs^*$-character, is $G$ metrizable?

4. The weakly three-space property in topological gyrogroups with Fréchet-Urysohn property

In this section, we prove that every (countably, sequentially) compact subset being strictly (strongly) Fréchet-Urysohn is a weakly three-space property with $H$ a closed $L$-subgyrogroup of a topological gyrogroup $G$. More precisely, for a topological gyrogroup $G$ and a closed $L$-subgyrogroup $H$, if every (countably, sequentially) compact subset of $H$ is first-countable, and every (countably, sequentially) compact subset of $G/H$ is strictly (strongly) Fréchet-Urysohn, then every (countably, sequentially) compact subset of $G$ is strictly (strongly) Fréchet-Urysohn.

The following concept of the coset space of a topological gyrogroup was introduced in [6, 8].

If $H$ is an $L$-subgyrogroup of a topological gyrogroup $G$, it follows from [29, Theorem 20] that $G/H = \{ a \oplus H : a \in G \}$ forms a partition of $G$. Let $\pi : G \to G/H$ be $a \mapsto a \oplus H$ for all $a \in G$, then, it is obtained that $\pi^{-1}(\pi(a)) = a \oplus H$. Moreover, if we denote the topology of $G$ by $\tau(G)$, we define $\tau(G/H)$ in $G/H$ as the following:

$$\tau(G/H) = \{ O \subset G/H : \pi^{-1}(O) \in \tau(G) \}.$$

Furthermore, we call a property $\mathcal{P}$ three-space property in topological groups [5, 20] if topological group $G$ and a closed normal subgroup $H$ of $G$ both have $\mathcal{P}$, then $G$ enjoys $\mathcal{P}$, too.

Then we give the definitions of three-space property and weakly three-space property in topological gyrogroups.

Definition 4.1. We call a topological property $\mathcal{Q}$ three-space property in topological gyrogroups if a topological gyrogroup $G$ and a closed $L$-subgyrogroup $H$ of $G$ both have $\mathcal{Q}$, then $G$ enjoys $\mathcal{Q}$, too.

Definition 4.2. We call a topological property $\mathcal{Q}$ weakly three-space property in topological gyrogroups, if in a topological gyrogroup $G$, there is a closed $L$-subgyrogroup $H$ of $G$ having a property $\mathcal{P}$ which is stronger than $\mathcal{Q}$ and the quotient space $G/H$ has $\mathcal{Q}$, then $G$ has $\mathcal{Q}$, too.

Obviously, every three-space property in topological gyrogroups is a weakly three-space property in topological gyrogroups.

There is an open problem posed by A.V. Arhangel’ski and M. Tkachenko in [1].

Question 4.3. [21, Open problem 9.10.3] Let all compact subsets of the groups $H$ and $G/H$ be Fréchet-Urysohn. Does the same hold for compact subsets of $G$?

It is natural to pose the following problems.

Question 4.4. Let $G$ be a topological gyrogroup and $H$ a closed $L$-subgyrogroup of $G$. If every compact subset of $H$ and $G/H$ both are Fréchet-Urysohn, is any compact subset of $G$ Fréchet-Urysohn? In particular, is any compact subset being Fréchet-Urysohn a weakly three-space property in topological gyrogroups?

Question 4.5. Let $G$ be a topological gyrogroup and $H$ a closed $L$-subgyrogroup of $G$. If every compact subset of $H$ and $G/H$ both are strictly (strongly) Fréchet-Urysohn, is any compact subset of $G$ strictly (strongly) Fréchet-Urysohn? In particular, is any compact subset being strictly (strongly) Fréchet-Urysohn a weakly three-space property in topological gyrogroups?
Next, we show that all compact subsets being strictly (strongly) Fréchet-Urysohn is a weakly three-space property in topological gyrogroups. More precisely, for a topological gyrogroup $G$ and a closed $L$-subgyrogroup $H$, if every (countably, sequentially) compact subset of $H$ is first-countable, and every (countably, sequentially) compact subset of $G/H$ is strictly (strongly) Fréchet-Urysohn, then every (countably, sequentially) compact subset of $G$ is strictly (strongly) Fréchet-Urysohn, and hence give a partial answer about Question 4.5 see Theorem 4.13.

Let $f : X \to Y$ be a continuous onto mapping and the space $Y$ and the fibers of $f$ have $\mathcal{P}$, then $X$ enjoys $\mathcal{P}$ inverse fiber property \cite{[11]}. Moreover, if the domain $X$ is (countably, sequentially) compact, we call $\mathcal{P}$ an inverse fiber property for (countably, sequentially) compact sets. Moreover, if the space $X$ is regular, call $\mathcal{P}$ regular inverse fiber property.

**Lemma 4.6.** \cite{[33]} The first-countability is an inverse fiber property for (countably, sequentially) compact sets.

**Proposition 4.7.** If $\mathcal{P}$ is an inverse fiber property, then it is a three-space property in topological gyrogroups.

**Proof.** We assume that $H$ is a closed $L$-subgyrogroup of a topological gyrogroup $G$. We assume further that both gyrogroups $H$ and $G/H$ have an inverse fiber property $\mathcal{P}$. If $y \in G/H$, we can find $x \in G$ such that $\pi(x) = y$. Then $\pi^{-1}(y) = x \oplus H$ is homeomorphic with $H$, so the fiber $\pi^{-1}(y)$ has $\mathcal{P}$ for all $y \in G/H$. It follows from the inverse fiber property of $\mathcal{P}$ that $G$ enjoys $\mathcal{P}$, too.

We call a topological space $X$ having a $G_\delta$-diagonal \cite{[21]} if the diagonal $\Delta = \{(x, x): x \in X\}$ of $X \times X$ is a $G_\delta$-set in $X \times X$.

**Lemma 4.8.** Let $G$ be a topological gyrogroup. The following two conditions are equivalent:

(a) all sequentially compact subspaces of $G$ have the first axioms of countability;

(b) all sequentially compact subspaces of $G$ are metrizable.

**Proof.** We only need to prove (a) $\Rightarrow$ (b). Let $X$ be a non-empty sequentially compact subset of $G$. Define a mapping $\varphi : G \times X \to G$ as $\varphi(x, y) = (\ominus x) \oplus y$ for every $x, y \in G$. Since $\varphi$ is continuous and $X \times X$ is sequentially compact, we have that $F = \varphi(X \times X)$ is sequentially compact subset of $G$ and $0 \in F$. By the first-countability of $F$, $\{0\}$ is a $G_\delta$-set in $F$. Therefore, $(\varphi|_{X \times X})^{-1}(0) = \Delta$ is the diagonal in $X \times X$, and $\Delta$ is a $G_\delta$-set in $X \times X$. Moreover, it is well-known that every sequentially compact space is a countably compact space. Then, it follows from \cite{[21]} Theorem 2.14 that if $X$ is a countably compact space and $X$ has $G_\delta$-diagonal, then $X$ is compact and metrizable. Therefore, $X$ is metrizable.

**Corollary 4.9.** The following two conditions are equivalent in topological gyrogroups:

(a) all countably compact subspaces have the first axioms of countability;

(b) all countably compact subspaces are metrizable.

**Proof.** We will show that (a) $\Rightarrow$ (b). Indeed, it is known that if $X$ is countably compact and first-countable, then it is sequentially compact. Therefore, we complete the proof by Lemma 4.8.

**Corollary 4.10.** all of the following are three-space properties in topological gyrogroups:

(a) every sequentially compact subset is closed.

(b) every sequentially compact subset is compact.
(c) every (countably, sequentially) compact subset is first-countable.
(d) every (countably, sequentially) compact subset is metrizable.

Proof. By Proposition 4.7 and [27] Lemma 2.2, (a) and (b) hold.

By Lemma 4.6 we have that every (countably, sequentially) compact subset satisfying the first axiom of countability is an inverse fiber property and by Proposition 4.7 Corollary 4.9 and [24] Theorem 3.10, (c) and (d) hold. □

Theorem 4.11. every sequentially compact subset being sequential is a three-space property in topological gyrogroups.

Proof. Assume that $G$ is a topological gyrogroup and $H$ is a closed $L$-subgyrogroup of $G$. Assume further that every sequentially compact subsets of both $H$ and $G/H$ are sequential. It follows from [27] Lemma 2.4 that every sequentially compact subset of $H$ and $G/H$ both are closed. By Corollary 4.10 every sequentially compact subset of $G$ is closed. For arbitrary sequentially compact subset $B$ of $G$, let $A$ be a sequentially closed subset of $B$. It is clear that $A$ is sequentially compact in $G$. Therefore, $A$ is closed. We obtain that $B$ is sequential by the definition. In conclusion, every sequentially compact subset of $G$ is sequential since $B$ is arbitrary. □

Lemma 4.12. If all (countably, sequentially) compact subspaces of a topological gyrogroup $G$ are Fréchet-Urysohn, then all (countably, sequentially) compact subspaces of $G$ are strongly Fréchet-Urysohn.

Proof. Let $G$ be a topological gyrogroup and all (countably, sequentially) compact subspaces of $G$ Fréchet-Urysohn. It follows from [27] Lemma 2.4 that all (countably, sequentially) compact subsets of $G$ are closed. Let $A$ be an arbitrary (countably, sequentially) compact subset of $G$. We have that $A$ is Fréchet-Urysohn and closed. Let $a$ be an accumulation point of $A$ and let $\{A_n\}_{n\in \mathbb{N}}$ be a decreasing sequence of subsets of $A$ such that $a \in \bigcap_{n \in \mathbb{N}} A_n$. Since $A$ is Fréchet-Urysohn, there exists a sequence $\{a_n\}_{n \in \mathbb{N}}$ in $A \backslash \{a\}$ converging to $a$. Set

$$B = (A \oplus a) \cap A_n, \text{ and } B_n = (A \oplus a_n) \cap A_n$$

Obviously, $B$ is closed, $0 \in (A \oplus a) \cap A_n = \overline{A_n} \cap B$, $b_n \in B \backslash \{0\}$ for all $n \in \mathbb{N}$ and $\{b_n\}_{n \in \mathbb{N}}$ converges to 0. Let $\{V_n\}_{n \in \mathbb{N}}$ be a symmetric open neighborhood sequence of 0 in $G$ with $b_n \notin V_n \oplus 0$ for each $n \in \mathbb{N}$. Let $C_n = (B_n \cap V_n) \ominus b_n$ for every $n \in \mathbb{N}$. Since $0 \in \overline{B_n} \cap V_n$, we have $b_n \in C_n$. Moreover, it follows from $V_n \cap C_n \subset V_n \cap (V_n \ominus b_n) = \emptyset$ that 0 $\notin \overline{C_n}$.

Now set

$$D = \bigcup_{n \in \mathbb{N}} \{C_n : n \in \mathbb{N}\}, \text{ and } S = \{0\} \cup \{b_n : n \in \mathbb{N}\}.$$ 

Then $D \subset \bigcup_{n \in \mathbb{N}} (B_n \ominus b_n) \subset B \ominus S$.

Claim. The subspace $B \oplus S$ of $G$ is Fréchet-Urysohn and closed.

It is obvious that $S$ is compact and sequentially compact.

Case 1. Suppose that $A$ is compact. By the compactness of $A$, $B$ is also compact. Therefore, the Cartesian product $B \times S$ is compact. Moreover, since the binary operation in $G \times G$ is jointly continuous, it is obtained that $B \oplus S$ is compact as the continuous image of $B \times S$. Thus, $B \oplus S$ is Fréchet-Urysohn and closed.

Case 2. We assume that $A$ is countably compact or sequentially compact. If $A$ is countably compact, by the Fréchet-Urysohn property of $A$, we obtain that $A$ is sequentially compact. Since $L_{A\oplus a}$ is homeomorphic, it is obtained that $B$ is also sequentially compact. Then the Cartesian product $B \times S$ is sequentially compact. Furthermore,
since the binary operation in $G \times G$ is jointly continuous, it is achieved that $B \oplus S$ is sequentially compact as the continuous image of $B \times S$. Thus, $B \oplus S$ is Fréchet-Urysohn and closed.

Since $b_n \in \overline{C_n}$ for each $n \in \mathbb{N}$ and $b_n$ converges to 0, we have that $0 \in \overline{D} \subset B \oplus S$, and we can find a sequence $\{d_k\}_{k \in \mathbb{N}}$ converging to 0 in $D$. For all $n \in \mathbb{N}$, since $0 \notin \overline{C_n}$, $C_n$ contains only finitely many terms of the sequence $\{d_k\}_{k \in \mathbb{N}}$. There exists a subsequence $\{C_{n_k}\}_{k \in \mathbb{N}}$ of the sequence $\{C_n\}_{n \in \mathbb{N}}$ with $d_k \in C_{n_k}$ for each $k \in \mathbb{N}$. Since $C_{n_k} \subseteq B_{n_k} \oplus b_{n_k} = ((\oplus a) \oplus A_{n_k}) \oplus b_{n_k}$, there exists $x_{n_k} \in A_{n_k}$ such that $d_k = ((\oplus a) \oplus x_{n_k}) \oplus b_{n_k}$, for all $k \in \mathbb{N}$. Then $\oplus x_{n_k} = d_k \oplus \text{gyr}((\oplus a) \oplus x_{n_k}, b_{n_k})(\oplus b_{n_k})$, so $x_{n_k} = a \oplus (d_k \oplus \text{gyr}((\oplus a) \oplus x_{n_k}, b_{n_k})(\oplus b_{n_k}))$. Since $\text{gyr}(x, y)(z) = (\oplus (x \oplus y) \oplus (y \oplus z))$ for all $x, y, z \in G$, it follows that $\text{gyr}(x, y)(0) = (\oplus (x \oplus y) \oplus (y \oplus z)) = 0$.

Thus, $x_{n_k} = a \oplus (d_k \oplus \text{gyr}((\oplus a) \oplus x_{n_k}, b_{n_k})(\oplus b_{n_k})) \rightarrow a$ whenever $k \rightarrow \infty$. When $n_{k-1} < n \leq n_k$, fix $y_n = x_{n_k}$. Then $y_n \in A_n$ for each $n \in \mathbb{N}$ and $y_n \rightarrow a$. In conclusion, $A$ is strongly Fréchet-Urysohn.

\begin{theorem}
Let $H$ be a closed $L$-subgyrogroup of a topological gyrogroup $G$ with every (countably, sequentially) compact subset of $H$ being first-countable. If $G/H$ has one of the following conditions, then $G$ has the same property:

(a) every (countably, sequentially) compact subset is strongly Fréchet-Urysohn.

(b) every (countably, sequentially) compact subset is strictly Fréchet-Urysohn.

\end{theorem}

\begin{proof}
It follows from Theorem 3 in [2] that every $T_0$ topological gyrogroup is regular. Suppose that $C$ is a (countably, sequentially) compact subset of $G$. It follows from Proposition 4.7, [27, Lemma 2.2] and [27, Lemma 2.4] that $C$ is closed in $G$. Set $\varphi = \pi|_C : C \rightarrow \pi(C)$, we obtain that $\pi(C)$ is (countably, sequentially) compact. Moreover, $\varphi$ is a closed mapping by [27, Lemma 2.4], and $\varphi^{-1}(\varphi(c)) = \pi^{-1}(\pi(c)) \cap C = (c \oplus H) \cap C$ is first-countable for every $c \in C$. We complete the proof by Lemma 4.12, [1, Proposition 4.7.18] and [27, Lemma 2.11].

\end{proof}

5. THE WEAKLY THREE-SPACE PROPERTY IN TOPOLOGICAL GYROGROUPS WITH \(\omega^\omega\)-BASE

Finally, we study whether having an $\omega^\omega$-base is a (weakly) three-space property in topological gyrogroups. In particular, we discuss the following question.

\begin{question}
If a closed $L$-subgyrogroup $H$ of a topological gyrogroup $G$ and the quotient space $G/H$ both have an $\omega^\omega$-base, does $G$ have an $\omega^\omega$-base?

\end{question}

We will show that having an $\omega^\omega$-base is a weakly three-space property in topological gyrogroups. More precisely, if a closed $L$-subgyrogroup $H$ of a topological gyrogroup $G$ is first-countable and $G/H$ has an $\omega^\omega$-base, then $G$ has an $\omega^\omega$-base.

\begin{theorem}
If a closed $L$-subgyrogroup $H$ of a topological gyrogroup $G$ is first-countable and the quotient space $G/H$ has an $\omega^\omega$-base, then $G$ has an $\omega^\omega$-base.

\end{theorem}

\begin{proof}
Suppose that $\{W_n\}_{n \in \mathbb{N}}$ is a family of open symmetric neighborhoods of 0 and is decreasing with $W_{n+1} \oplus W_{n+1} \subseteq W_n$ for each $n \in \mathbb{N}$. Since $H$ is first-countable, suppose further that $\{W_n \cap H\}_{n \in \mathbb{N}}$ is an open neighborhood base of 0 in $H$. Assume that $V = \{V_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ is an $\omega^\omega$-base in $G/H$. Without loss of generality, let $V_\alpha$ be symmetric for every $V_\alpha \in V$. Set $\{U_\alpha = \pi^{-1}(V_\alpha) : \alpha \in \mathbb{N}^\mathbb{N}\}$.
Put $R_\beta = W_n \cap U_\alpha$, for each $n \in \mathbb{N}$ and $\alpha = (\alpha_i) \in \mathbb{N}^{\mathbb{N}}$, where $\beta = (n, \alpha_1, \alpha_2, \ldots)$. Then $\mathcal{R} = \{ R_\beta : \beta \in \mathbb{N} \times \mathbb{N}^{\mathbb{N}} \}$ is a family of open symmetric neighborhoods of $0$ with $R_\beta \subset R_\gamma$ if $\beta \geq \gamma$.

**Claim.** $\mathcal{R}$ is an $\omega^\omega$-base for $G$.

Since $G$ is homogenous, it suffices to show that $\mathcal{R}$ is an $\omega^\omega$-base at $0$ for $G$. Let $U$ be an arbitrary open neighborhood of $0$. We can find an open symmetric neighborhood $V$ of $0$ with $V \oplus V \subset U$. By the construction of $W_n$, there exists $n \in \mathbb{N}$ with $W_n \cap H \subset V$. Then, as $\mathcal{V}$ is a symmetric $\omega^\omega$-base in $G/H$, we can choose $V_\alpha \in \mathcal{V}$ such that $V_\alpha = \pi(U_\alpha) \subset \pi(V \cap W_{n+1})$. Put

$$R_\beta = W_{n+1} \cap \pi^{-1}(V_\alpha) = W_{n+1} \cap U_\alpha.$$ 

For an arbitrary $g \in R_\beta$, since $R_\beta = W_{n+1} \cap U_\alpha$, $g \in W_{n+1}$ and $g \in U_\alpha \subset (V \cap W_{n+1}) \oplus H$. There exist $a \in V \cap W_{n+1}$ and $b \in H$ satisfying $g = a \oplus b$. Therefore, $b = (\oplus a) \oplus g \in (W_{n+1}) \oplus W_{n+1} \subset W_n$. So $g \in (V \cap W_{n+1}) \oplus (H \cap W_n) \subset V \oplus V \subset U$. In conclusion, $R_\beta \subset U$ and we complete the proof. \hfill $\square$

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