LECTURES ON OPEN STRINGS,
AND NONCOMMUTATIVE GAUGE THEORIES

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To Alain Connes on his 55th birthday

The background independent formulation of the gauge theories on D-branes in flat space-time is considered, some examples of the solutions of their equations of motion are presented, the solutions of Dirac equation in these backgrounds are analyzed, and the generalizations to the curved spaces, like orbifolds, conifolds, and K3 surfaces, are discussed.

1. Introduction

We suggest to look for a generalized version of the gauge fields/strings correspondence [1] which may prove easier to establish (as sometimes general problems are easier then the particular ones). In the past few years a (new) connection has been (re-)discovered between noncommutative geometry [2] and string theory [3][4][5][6]. The previous understanding [7] of intrinsic noncommutativity of open string theory was supplemented by a vast number of examples stemming from the studies of D-branes, which allowed to make the noncommutativity manifest already at the field theory (or zero slope) limit. This connection may prove useful both ways. On the one hand, the noncommutative geometry is a deeply studied subject, thanks to the work of A. Connes and his followers. On the other hand, using the intuition/results from D-brane physics one can come up with new solutions/ideas for theories on noncommutative spaces.

These lecture notes should not be considered as an introduction into the noncommutative field theories and their relation to string theories. We refer the interested reader to [8]. Instead, we shall expand on several points not covered in [8].

We shall start with a unified construction of the worldvolume theories of D-branes of unspecified dimensionality. We shall discuss mostly the flat Minkowski space closed string background. In the concluding section we present a few new results on curved backgrounds: namely the orbifolds of flat space, and their deformations, the conifolds. The more abstract approach to classification of D-brane states on rational conformal field theories will not be covered here (see cf. [9][10]).

In the mean time we shall discuss several classical solutions of the noncommutative gauge theory. They correspond to flat D-branes, D-branes at angles, D-branes with different magnetic fields turned on, D-branes of various dimensions. Generically these solutions are unstable (open string spectrum contains a tachyon). In the noncommutative gauge theory the instability is reflected by the negative modes in the expansion around the solution. In principle one should be able to study the decay of the solution towards the stable ones. However the application of this analysis to string theory is limited, as in the majority of the cases the $\alpha'$-corrections to the flow may not be negligible.

We shall also consider in some detail the case of four dimensional instanton/monopole solutions. In each case we shall construct the solutions of the Dirac equation in the background of the instanton/monopole. In the case of instantons the analysis of the solutions of the Dirac equation permits to establish the completeness of the noncommutative version of ADHM construction [11]. This is a noncommutative version of the reciprocity of [12].

In the course of these lectures we shall try to make clear that the noncommutative algebras, which should be thought of the algebras of functions on the noncommutative manifold, are not fixed, but rather arise upon a choice of classical solution in the background independent formulation of the gauge theory. We shall show that this background independent formulation is on the one hand related to the background independent open string field theory [13]. On the other hand it is related to Matrix theory.

1 On leave of absence from Institute for Theoretical and Experimental Physics, 117259 Moscow, Russia
Finally, a note for Alain. In these notes we look at but the simplest noncommutative geometries arising in open string theory on flat space-time backgrounds. Nevertheless, even this simplest case turns out to be rather rich. In the \( \alpha' \to 0 \) limit the associative algebra which governs the story is the so-called Yang-Mills algebra. Almost nothing is known about its representation theory.

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2. Background Independence

It was observed by many authors, see cf. \([14][15]\) that the Matrix theory action (or its euclidean version \([16]\)) provides a background independent formulation of the large class of gauge theories. We shall now state this more precisely, and at the same time we shall fix our notations. Consider an abstract Hilbert space \( \mathcal{H} \) and a collection of \( d \) Hermitian operators \( Y^i, i = 1, \ldots, d \), acting there. We shall sometimes use the notation \( Y \) for the vector in \( V = \mathbb{R}^d \) with values in the space \( \mathbb{H}(\mathcal{H}) \) of Hermitian operators in \( \mathcal{H} \). Let \( g_{ij} \) be a Euclidean metric on \( V \). Define a formal action:

\[
S = -\frac{1}{4g_{ik}} \sqrt{g} \, g_{ij} \, \text{Tr}_H[Y^i, Y^j][Y^k, Y^l] \tag{2.1}
\]

We said (2.1) is formal because it may happen that for a physically sensible choice of \( Y \) the value of \( S \) is infinite. We shall consider \( Y \) such that with appropriate choice of a constant \( \mu \), the action becomes finite upon subtraction of \( \mu \, \text{Tr}_H 1 \) (more precisely, we subtract \( \mu 1 \) from \( [Y, Y]^2 \) before calculating the trace).

The action (2.1) has an obvious gauge symmetry (which is consistent with the infrared regularization above):

\[
Y^i \mapsto g^i Y^i g, \quad g^i g = g g^i = 1 \tag{2.2}
\]

The equations of motion following from (2.1) are:

\[
g_{ij}[Y^k, [Y^l, Y^j]] = 0, \quad j = 1, \ldots, d \tag{2.3}
\]

The operators \( Y^i \) generate some associative algebra \( A \). Let us define the so-called *Yang-Mills algebra* \( A_{YM,d} \) to be the associative algebra generated by \( Y^i \), subject to the relations (2.3).

We shall now pause to establish the meaning of the operators \( Y^i \) in the open string theory context.

To this end let us consider open superstring propagating in the flat \( \mathbb{R}^{1,1} \). Consider the sheet of the disk topology. Let \( z = e^{-i\sigma+it} \). The fields \( X^\mu \) will have the following boundary conditions:

\[
\begin{align*}
\partial_i X^\mu &= 0, & \mu &= 0, 1, \ldots, p, & r &= 1 \\
\partial_\sigma X^\mu &= 0, & \mu &= p+1, \ldots, 25, & r &= 1
\end{align*} \tag{2.4}
\]

These boundary conditions describe the single flat \( Dp \)-brane. If we want to have several parallel \( Dp \)-branes then we should allow for the Chan-Paton factors, say \( i = 1, \ldots, N \), so that the (constant) value of \( X^\mu (r = 1, \sigma) = \varphi^\mu, \mu = p+1, \ldots, 25 \) may depend on \( i : \varphi_i^\mu \).

The canonical way of setting up a string calculation in these circumstances is to consider the dimension 1 vertex operators on the boundary, evaluate their correlation function, and integrate it over the moduli space of points on the boundary of the disk. One can also add the closed string vertex operators into the interior of the disk. These operators should have the dimension (1, 1). The open string vertex operators in general change the boundary conditions, i.e. the boundary conditions corresponding to the Chan-Paton index \( i \) to the left of the vertex operator may be followed by the \( \ell \)th boundary condition to the right of the operator. This is, of course, reflected by the contribution to the dimension of the operator of the mass squared of the stretched string: \( m_i^2 \parallel = || \varphi_i - \varphi_i^* ||^2 \).

We would like to have a setup in which there is no need to specify in advance neither the values of \( \varphi_i^* \), nor \( p \) or \( N \). All this data will be encoded in the properties of the operators \( Y^\mu, \mu = 0, \ldots, 9 \), acting in some auxiliary Hilbert space \( \mathcal{H} \).

Consider the correlation function of a closed string vertex operator \( \mathcal{O} \) inserted at the center \( z = 0 \) of the disk \( D \), and the boundary operator, generalizing the usual supersymmetric Wilson loop:

\[
Z[\mathcal{O}|Y] = \langle \mathcal{O} \exp \left( -\frac{1}{4\alpha'g} \int_D g_{ij} \left( \partial_i \varphi^* \partial_j \varphi + \varphi^* \partial_i \varphi + \varphi \partial_j \varphi^* \right) \right) \mathcal{H} \{ P \exp \oint_{\partial D} i k_i \left( Y_i \langle x^1 \rangle + \partial_i \varphi^* + \partial_i \varphi + \partial_i Y_i \right) \} \rangle \tag{2.5}
\]
Here $k_i, \vartheta_i$ are the momenta conjugate to $x^i, \Psi^i = \bar{\psi}^i + \tilde{\psi}^i; k_i = g_{ij} \partial_n x^j, \vartheta_i = \psi^i - \bar{\psi}^i$. For example, for the graviton[17]: $\mathcal{O}_h = h_{ij}(p) : \psi^i \psi^j e^{ip \cdot x} :$ (times the ghosts and superghosts):

$$\mathcal{Z}[\mathcal{O}_h] = g_{4l} \int d\kappa e^{-ip \cdot \kappa} h_{ij}(p) \int_0^1 dt \mathrm{Tr}_H e^{i \kappa \delta Y} [Y^i, Y^k] e^{i (1 - s) p \cdot Y} [Y^j, Y^l] + o(\alpha')$$

It can be shown that the condition that the boundary interaction (2.5) is consistent with the conformal invariance of the worldsheet sigma model reads as:

$$g_{ij}[Y^i, [Y^j, Y^k]] = 0(\alpha') \quad (2.6)$$

We can think of (2.6) as defining a one-parametric family of the associative algebras, generated by $Y^i$. We shall call them the algebras of functions on the D-brane worldvolume.

It should be straightforward to establish a direct relation between the background independent formulation of the noncommutative Yang-Mills algebras, where the generators $Y^i$'s, and the background independent open string field theory[13].

In the sequel we shall study the case $\alpha' \to 0$, i.e. Yang-Mills algebras.

An obvious class of Yang-Mills algebras is provided by the Heisenberg-Weyl algebras, where the generators $Y^i$ obey the stronger condition:

$$[Y^i, Y^j] \in \text{center}(\mathcal{A}_{YM,d}) \quad (2.7)$$

Clearly, for $d = 2$ all Yang-Mills algebras are at the same time Heisenberg-Weyl. For generic Heisenberg-Weyl solution let us denote

$$Z^{ij} = [Y^i, Y^j], \quad (2.8)$$

we have $[Z^{ij}, Y^k] = 0$ for any $i, j, k$. Let us diagonalize $Z^{ij}$:

$$\mathcal{H} = \bigoplus A H_A, \quad Z^{ij}|_{H_A} = i\theta^{ij}_A \cdot 1_{H_A} \quad (2.9)$$

Then each subspace $H_A$ is an irreducible representation of the Heisenberg algebra $[x^i_A, x^k_B] = i\theta^{ij}_{AB}$.

We now wish to describe the spectrum of fluctuations around the solution (2.8)(2.9). To this end we can decompose the fluctuation $y^i = \delta Y^i$ as follows:

$$y^i = \sum_{A,B} y^i_{AB}, \quad y^i_{AB} : H_B \to H_A \quad (2.10)$$

We shall also impose the following gauge condition:

$$g_{ij}[Y^i, y^j] = 0 \Leftrightarrow \sum_{A,B} g_{ij} (x^i_A y^j_{AB} - y^j_{AB} x^i_B) = 0 \quad (2.11)$$

The linearized fluctuations are governed by the quadratic approximation to the action:

$$K y^i = g_{ik}[x^i, [x^k, y^j]] + 2g_{ik}[y^j, Z^{kj}] \quad (2.12)$$

which leads to the following eigenvalue problem for the spectrum of masses:

$$\omega^2 y^i_{AB} = \left( \Delta_A y^i + y^j \Delta_B - 2g_{ik} x^i_A y^j_{AB} x^k + 2y^i_{AB} (T^j_{AB} - T^j_{ij,A}) \right)$$

$$\Delta_A = g_{ij} x^i_A x^j_A, \quad T^j_{ij,A} = \theta^{ij}_A g_{ik} \quad (2.13)$$

We now proceed with some examples.

2.1. Dolan-Nappi solutions

This solution describes two branes which could sit on top of each other, and have different magnetic fields turned on.

Set $d = 2, \theta_{AB} = \theta_A e^{\epsilon A}, A, B = 1, 2, i, j = 1, 2, g_{ij} = \delta_{ij}$. The operators $Y$ corresponding to this solution are given by:

$$Y^1 + iY^2 = \begin{pmatrix} \sqrt{2\theta_1} a_1 & 0 \\ 0 & \sqrt{2\theta_2} a_2 \end{pmatrix} \quad (2.14)$$

where $a_1, a_2$ are the annihilation operators acting in two (isomorphic) copies $\mathcal{H}_1, \mathcal{H}_2$ of the Hilbert space $\mathcal{H}$. 
To facilitate the analysis let us map the operators $L_{AB} : H_A \rightarrow H_B$ to the vectors in the tensor product: $H_B \otimes H_A^* \approx H \otimes H$:

$$L_{AB} \mapsto \sum_{n_1, n_2} \langle n_2 | L_{AB} | n_1 \rangle \begin{pmatrix} 1 \\ n_2, n_1 \end{pmatrix}$$

Also, introduce the notation $\zeta = y^A_{AB} + i y^2_{AB}$; $\bar{\zeta} = y^3_{AB} - iy^4_{AB}$. First, assume that $\theta_{AB} > 0$ and introduce the creation-annihilation operators:

$$x_{AB}^i + i x_{AB}^0 = \sqrt{2 \theta_{AB} a_{1,2}}$$

and the number operators $n_{a,\bar{a}} = a_{a,\bar{a}}^* a_{a,\bar{a}}$. Then (upon the identification $\zeta \in H_1 \otimes H_2$)

$$K_{\zeta} = 2 \theta_A (n_1 + \frac{1}{2}) + 2 \theta_B (n_2 + \frac{1}{2}) - 2 \sqrt{\theta_A \theta_B} (a_{1,2}^* a_{1,2} + a_{1,2} a_{1,2}) \pm 2 \theta_{A-B} \zeta$$

This operator is conveniently diagonalized by introduction of the $SU(1,1)$ generators:

$$L_+ = a_1^+ a_2^+, L_- = a_1 a_2, L_0 = \frac{1}{2} (n_1 + n_2 + 1)$$

and the operator

$$M = \frac{1}{2} (n_1 - n_2)$$

Now, the spectrum of

$$K = 2(\theta_A - \theta_B)(M \pm 1) + 2(\theta_A + \theta_B) L_0 - 2 \sqrt{\theta_A \theta_B} (L_+ + L_-)$$

depends on whether $\theta_A$ equals $\theta_B$ or not. If $\theta_A - \theta_B \neq 0$ then upon a Bogolyubov $SU(1,1)$ transformation we can bring $K$ to the form:

$$2(\theta_A - \theta_B)(M \pm 1) + 2\theta_A - \theta_B |L_0$$

whose spectrum is (remember that the spectrum of $L_0$ is given by: $L_0 = \frac{1}{2} + |M| + k, k \in \mathbb{Z}_+$):

$$2(\theta_A - \theta_B)(|M| + 1) + |\theta_A - \theta_B|(1 + 2|M| + 2k)$$

which contains a tachyonic mode (for $\zeta$ or $\bar{\zeta}$ depending on the sign of $\theta_A - \theta_B$).

For $\theta_A = \theta_B = \theta$ the operator $K$ is manifestly positive definite:

$$K = 2\theta |n|^2$$

with the eigenvectors:

$$e^{i(n_{a,\bar{a}}^* + i a_{1,2}^* + a_{1,2} + a_{a,\bar{a}})} |0, 0\rangle$$

which in the ordinary, operator, representation correspond to the plane waves:

$$e^{i(n_1 + i n_2)}$$

2.2. Intersecting branes

As another interesting example of the solution (2.7) we shall look at the case $d = 4$:

$$\theta^0_{A,B} = \theta, \theta_{A,1}$$

$$\theta^1_{A} = \theta \theta_{A,2}$$

This solution describes two branes, each having two dimensions transverse to another. The operators $Y^i$ corresponding to this solution have the following block-diagonal form:

$$Y^1 + i Y^2 = \sqrt{2 \theta_1} \begin{pmatrix} a_1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad Y^3 + i Y^4 = \sqrt{2 \theta_2} \begin{pmatrix} 0 \\ 0 \\ a_2 \\ 0 \end{pmatrix}$$

where we denote by $a_1, a_2$ the annihilation operators acting in two copies $H_1, H_2$ of the Hilbert space $\mathcal{H}$.

In this case the spectrum of fluctuations contains in the $AB$ sector the discrete modes, corresponding to the strings localized at the intersection of the branes, and, if $\theta_1 \neq \theta_2$, starts off with the tachyonic mode:

$$\omega^2 = 2 \theta_1 n_1 + 2 \theta_2 n_2 \pm |\theta_1 - \theta_2|, n_{1,2} \in \mathbb{Z}_+$$

2.3. T-duality

The configuration from the previous example is closely related to the ‘piercing string’ solution of [18], which in our present notation is described as follows: let as before $H_1, H_2$ denote two copies of the Hilbert space $\mathcal{H}$. Now, let $H_3 = H_1 \otimes H_2$. Then the operators $Y$ will be acting in $H_3 \otimes H_2$:

$$Y^1 + i Y^2 = \sqrt{2 \theta_1} \begin{pmatrix} a_1 \otimes 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad Y^3 + i Y^4 = \sqrt{2 \theta_2} \begin{pmatrix} 1 \otimes a_2 + a_2^* \\ 0 \\ 0 \end{pmatrix}$$

The analysis of fluctuations around this solution is similar, it also exhibits a tachyonic mode for $\theta_1 \neq \theta_2$. 
3. BPS algebras

Less trivial is the case of the so-called BPS algebras (not to be confused with the algebras of BPS states), whose generators obey the following relations: let \( S \) be the spaces of chiral spinors of \( SO(d) \), let \( \gamma_i \) be the generators of the Clifford algebra,

\[
\{\gamma_i, \gamma_j\} = g_{ij} \mathbf{1},
\]

then the relations state that there exist two spinors \( \epsilon_1, \epsilon_2 \in S \) such that

\[
[Y^i, Y^j][\gamma_i, \gamma_j] \epsilon_1 + \mathbf{1} \epsilon_2 = 0
\]

If \( d = 4 \) then these relations have the following simple form:

\[
[Y^i, Y^j] \pm \frac{1}{2} \epsilon_{mabk} g^{mi} g^{nj} [Y^k, Y^l] \in \text{center}(A_{YM,4})
\]

The operators \( Y^i \) solving (3.3), considered up to a gauge transformation (2.2), define the noncommutative instanton.

3.1. Noncommutative U(1) instantons

The construction of noncommutative instantons \([11]\) is a generalization of the famous ADHM procedure, which produces an anti-self-dual gauge field on \( R^4 \) (or its one-point compactification, \( S^4 \)), given a solution to a finite-dimensional version of the anti-self-duality condition. The latter is imposed on the set of complex matrices, \( B_{\alpha} \), \( \alpha = 1, 2 \), and (in the \( U(1) \) case) \( I \), where \( B_{\alpha} \) are the operators in a vector space \( V = \mathbb{C}^k \), where \( k \) is the instanton charge, while \( I \) is a vector in \( V \). The conditions, imposed on \( (B_{\alpha}, I) \) are:

\[
[B_1, B_2] = 0,
[B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger = 2 \cdot I \quad \quad (3.4)
\]

The 2 in the right-hand side of (3.4) is a convenient choice of normalization, which could be altered – what matters is whether there stands 0 or something positive. Given a solution to (3.4), one can generate another one by applying a \( U(k) \) transformation \( (B_{\alpha}, I) \rightarrow (g^{-1}B_{\alpha}g, g^{-1}I) \) with \( g \in U(k) \). Such solutions will be considered as equivalent ones.

The purpose of this note is to elucidate the meaning of the space \( V \) from the point of view of four dimensional noncommutative gauge theory. We shall see that \( V \) is nothing but the space of normalizable solutions to Dirac equation for the spinor field in the fundamental representation, in the instanton background.

We now recall the construction of the instanton gauge field. Consider the associative algebra \( R_4^k \) of operators in the Hilbert space \( H = L^2(R^4) \), which we identify with the space of states of a two-dimensional harmonic oscillator. Let \( a_{\alpha}, a_{\alpha}^\dagger \), \( \alpha = 1, 2 \) be the annihilation and creation operators, respectively, which obey the algebra:

\[
[a_{\alpha}, a_{\beta}^\dagger] = \delta_{\alpha\beta}
\]

Consider the \( R_4^k \)-module \( M = H \otimes V \), and let \( \Delta, \tilde{\Delta} \) denote the operators in \( M \):

\[
\Delta = \sum_{\alpha} (B_{\alpha} - a_{\alpha}^\dagger)(B_{\alpha} - a_{\alpha})
\]

\[
\tilde{\Delta} = \sum_{\alpha} (B_{\alpha} - a_{\alpha})(B_{\alpha} - a_{\alpha}^\dagger)
\]

\[
\Delta - \tilde{\Delta} = II^\dagger
\]

Let \( D : M \otimes C^2 \otimes H \rightarrow M \otimes C^2 \) be a morphism of \( R_4^k \) modules, given by:

\[
D = \begin{pmatrix} B_1 - a_1^\dagger & B_2 - a_2^\dagger & I \\ -B_1^\dagger + a_2 & B_1^\dagger + a_1 & 0 \end{pmatrix}
\]

It follows from (3.4) that \( D^\dagger D = \tilde{\Delta} \otimes 1_{C^2} : M \otimes C^2 \rightarrow M \otimes C^2 \). One shows \([19]\) that \( \tilde{\Delta} \) is a positive definite Hermitian operator in \( M \). Hence, the following operator is a well-defined projector in \( M \otimes C^2 \otimes H \):

\[
\Pi_1 = D \frac{1}{D^\dagger D} D^\dagger
\]
Let \( \Psi \) denote the fundamental solution to the equation \( D^1 \Psi = 0 \), i.e. a morphism of \( R_4^1 \) modules: \( \Psi : \mathcal{H} \to M \otimes \mathbb{C}^2 \otimes \mathcal{H} \). One shows that \( \Psi \) can be normalized so as to define a unitary isomorphism between \( \mathcal{H} \) and the kernel of \( D^1 \): \( \Psi^\dagger \Psi = 1_\mathcal{H} \):

\[
\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \xi \end{pmatrix},
\psi_\alpha = (B_\alpha^1 - a_\alpha) \nu
\Delta \nu = -I \xi
\xi = \Lambda^{-\frac{1}{2}} S^I
\]

(3.9)

where \( P \) is a orthogonal projection in \( \mathcal{H} \) onto a subspace, isomorphic to \( V \), which is spanned by the elements \( \eta \) which are in the image of the operator \( I^1 \exp \left( \sum_n B_n^1 a_n^1 \right) [0,0] \otimes \nu \). It is instructive to show that this image can be also characterized as the kernel of the operator \( \Lambda^{-1} \). Indeed, let \( \lambda \in \mathcal{H} \) be such that \( \lambda = I^1 \bar{\Lambda}^{-1} I \lambda \). It follows:

\[
I \lambda = I^1 \bar{\Lambda}^{-1} I \lambda \Rightarrow \bar{\Lambda}^{-1} I \lambda = \left( \bar{\Lambda} - I^1 \right) \bar{\Lambda}^{-1} I \lambda = 0
\]

(3.10)

\[
\Rightarrow \bar{\Lambda}^{-1} I \lambda = \exp \left( \sum_n B_n^1 a_n^1 \right) [0,0] \otimes \nu, \quad \nu \in V
\]

\[
\Rightarrow I \left( \lambda - I^1 \exp \left( \sum_n B_n^1 a_n^1 \right) [0,0] \otimes \nu \right) = 0
\]

In writing the formulae (3.9) we only had to use the operator \( \Lambda^{-1} \) which is an element of \( R_4^{1} \). However, for computational purposes it is useful to work with \( \Lambda \) as well. Technically one has to localize \( R_4^{1} \) over \( \Lambda \), i.e. consider the formal polynomials of the form \( \sum_n a_n \Lambda^n \) where \( a_n \) are the elements of \( R_4^{1} \).

One defines the second projector:

\[
\Pi_2 = \Psi \Psi^\dagger
\]

(3.11)

It is clear from the positivity of \( \bar{\Delta} \), that:

\[
\Pi_1 + \Pi_2 = 1_{M \otimes \mathbb{C}^2 \otimes \mathcal{H}}
\]

(3.12)

This relation implies that the following identities hold:

\[
(B_1^1 - a_1^1) \frac{1}{\bar{\Delta}} (B_1^1 - a_1^1) + (B_2^1 - a_2^1) \frac{1}{\bar{\Delta}} (B_2^1 - a_2^1) = 1_k
\]

(3.13)

\[
(B_2^1 - a_2^1) \frac{1}{\bar{\Delta}} (B_2^1 - a_2^1) + (B_1^1 - a_1^1) \frac{1}{\bar{\Delta}} (B_1^1 - a_1^1) = 1_k
\]

\[
\left( I^1 \frac{1}{\bar{\Delta}} - \Lambda^{-1} I^1 \frac{1}{\bar{\Delta}} \right) (B_n - a_n^1) = 0
\]

Now, define operators in \( \mathcal{H} \)

\[
A_n = \Psi^\dagger a_n^1 \Psi, \quad A_n^1 = \Psi^\dagger a_n^1 \Psi
\]

(3.14)

One shows [11][19] that

\[
[A_1, A_2] = 0, \quad [A_1^1, A_2^1] = 0
\]

(3.15)

\[
[A_1, A_2^1] + [A_2, A_1^1] = 2
\]

3.2. Higher dimensional instantons

We now proceed with discussing BPS solutions involving more than four \( Y \)'s. In general, for the \( U(N) \) gauge theory on a \( p \) complex dimensional Kähler manifold \( X \) the natural analogues of the instanton equations are the so-called Hermitian Yang-Mills equations, which state that the curvature of the gauge field \( A \) is of the type (1,1) and that its nonabelian part is primitive, that is orthogonal to the Kähler form \( \omega \):

\[
F^{(2,0)} = 0
\]

\[
F \wedge \omega^{p-1} = \lambda \ 1 \ \omega^p
\]

(3.16)
where $\lambda$ is a constant, which can be computed from the first Chern class of the gauge bundle.

We shall now consider the noncommutative analogues of the equations (3.16). Let us introduce a complex structure on $\mathbb{R}^d$ such that the noncommutativity tensor $\theta$ is of the type $(1, 1)$. We shall for simplicity assume that it actually related to the Kähler form: $\theta = \omega^{-1}$. The equations (3.16) will now read:

\[
[Y^\alpha, Y^\beta] = 0, \quad \alpha, \beta = 1, \ldots, p \tag{3.17}
\]

(we have normalized things in such a way that $Y^\alpha = a_\alpha$, with $[a_\alpha, a^\dagger_\beta] = \delta_{\alpha\beta}$ is a solution to (3.17)) Then it is relatively easy to produce a $U(p)$ invariant solution to (3.17) with positive action:

\[
Y^\alpha = S a_\alpha \left( 1 - \frac{p}{(Np)} \right) \frac{1}{2} S^\dagger \tag{3.18}
\]

The action on this solution is finite:

\[
S_p = \text{Tr}_H \left( [Y^\alpha, Y^\beta, \dagger] - \delta^{\alpha\beta} \right) \left( [Y^\beta, Y^\alpha, \dagger] - \delta^{\alpha\beta} \right) = p(p - 1) \tag{3.19}
\]

The topological charges associated with this solution are:

\[
ch_0 = \text{Tr}_H F^{\alpha\gamma} \wedge \omega^\beta \wedge \theta^\gamma \wedge \omega^\alpha, \quad F_{ij} = \omega(\omega^{ij} [Y^k, Y^l] - i[\theta^{kl}]) \tag{3.20}
\]

\[
ch_1 = 0, \quad ch_2 = S_p = p(p - 1) \tag{4.1}
\]

4. Fermions in the $Y$ backgrounds

Given a generic $Y$ we define Dirac operator $\mathcal{D} : S_+ \otimes A \rightarrow S_- \otimes A$ by the formula:

\[
\mathcal{D}\psi = \gamma_i [Y^i, \psi] \tag{4.1}
\]

Now consider a background $Y = Y \oplus Y'$, which splits as a direct sum of two independent $Y$-backgrounds. The Dirac operator, as defined by (4.1) splits as a sum of four independent operators. The most interesting for us is the “off-diagonal” one:

\[
\mathcal{D}\chi = \gamma_i (Y^i \chi - \chi Y^i) \tag{4.2}
\]

Now suppose that $Y'$ is a frozen vacuum solution, while $Y$ is a dynamical gauge field. The off-diagonal component $\chi$ of the fermion $\psi$ which enters (4.2) is what is sometimes called the fermion in the fundamental representation (as opposed to the adjoint fermion in (4.1)).

4.1. Fermions in the instanton background

We are now interested in solution of the Dirac equation in the instanton background for the fermions in the fundamental representation:

\[
\mathcal{D}\chi = \begin{pmatrix} \nabla_1 & -\nabla_2 \end{pmatrix} \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = 0 \tag{4.3}
\]

where the covariant derivative of a field $\chi$ in the (right) fundamental representation is given by:

\[
\nabla_\alpha \chi = -A^{\dagger}_\alpha \chi + \chi a^\dagger_\alpha, \quad \nabla_\alpha \chi = A_\alpha \chi - \chi a_\alpha \tag{4.4}
\]

We claim that the fundamental solution $\chi$ to (4.3) (which should be thought of as of the morphism of the $\mathbb{R}^p$ modules:

\[
\chi : M \rightarrow \mathcal{H} \otimes \mathbb{C}^2
\]
upon identification of the space of solutions with \( V \), which is done essentially in the same way as in [12]) is given by:

\[
\chi_\alpha = \nu^i (B_\alpha - a_\alpha^i) \tilde{\Delta}^{-1} = - S A_\alpha^i \partial_\alpha \left( \nu \tilde{\Delta}^{-1} \right)
\]  

(4.5)

It is easy to check (4.3) using the identities (3.13). Notice the following normalization of the solution (4.5):

\[
\sum_\alpha \chi_\alpha^\dagger \chi_\alpha = \sum_\alpha [a_\alpha, [a_\alpha, \tilde{\Delta}^{-1}]] \Rightarrow \text{Tr} \sum_\alpha \chi_\alpha^\dagger \chi_\alpha = 1\text{V}
\]  

(4.6)

(in the last equality we used the fact that the trace of the double commutator will not change if we set \( B_\alpha = B_\alpha^i = 0 \)) By similar calculations one shows, that:

\[
\text{Tr} \sum_\alpha \chi_\alpha^\dagger \chi_\alpha \left( \begin{array}{c} a_\beta^i \\ a_\beta^j \end{array} \right) = \left( \begin{array}{c} B_\beta^i \\ B_\beta^j \end{array} \right)
\]

(4.7)

Now, given an instanton gauge field \( Y^i = (A_\alpha, A_\alpha^i) \) we consider the associated Dirac operator \( \tilde{D} \) and the space \( V \) of its normalizable zero-modes. We shall require that \( Y^i \) obey the following asymptotics:

\[
Y^i = S x^i S^i + y^i
\]

(4.8)

where \( SS^i = 1 \), and the eigenvalues \( \lambda_\alpha \) of \( y^i \) decay faster then \( \tilde{\Delta} \) (after some reshuffling). Similarly, we are interested in the solutions \( \chi \) to the Dirac equation, which have the form:

\[
\chi_\alpha = S a_\alpha^i \frac{1}{(aa^i)^2} + \ldots
\]

(4.9)

where \( \ldots \) denote subleading (in the same sense as for the gauge field, except that with \( \tilde{\Delta} \) instead of \( \Delta \) asymptotics terms). One shows, analogously to [12] that the space of these solutions is of the dimension \( \hat{r} \), given by the instanton charge, and that the formulae (4.7) produce ADHM matrices \( B, B^i \), while the asymptotics

\[
\chi_\alpha = -i S a_\alpha^i \frac{1}{(aa^i)^2}
\]

(4.10)

gives \( I \). To define “asymptotics” of the operators properly, it is useful to employ the frozen vacuum solution \( \chi' \) and the technique of the symbols, defined with respect to this solution\(^2\), developed in [20]. By the completely analogous calculations to [12] one proves that \( B, B^i, I \) obey (3.4). This establishes the reciprocity.

4.2. Dirac field in the monopole background

We now present the solution \( \chi = \left( \begin{array}{c} \chi_+ \\ \chi_- \end{array} \right) \) to the Dirac equation

\[
2 D_\alpha \chi_+ - (D_\alpha + \Phi - z) \chi_- = 0
\]

(4.11)

\[
2 D_\alpha \chi_- - (D_\alpha + \Phi + z) \chi_+ = 0
\]

(4.12)

in the NC monopole background:

\[
\chi_\pm = \pm \Psi_\pm \frac{1}{\Delta}
\]

(4.13)

where

\[
\Delta = bb^i + c^i c
\]

\[
b^i \Psi_+ + c \Psi_- = 0
\]

\[
-c^i \Psi_+ + b \Psi_- = 0
\]

(4.14)

and the notations are from [21]. First of all, we need to invert \( \Delta \). This is easy, for (4.14) implies:

\[
0 = (bb^i + c^i c) \Psi_+ = (\Delta + 1) \Psi_+ \Rightarrow \Delta \Psi_+ = -\Psi_+
\]

\[
0 = (b^i b + c^i c) \Psi_- = (\Delta - 1) \Psi_- \Rightarrow \Delta \Psi_- = \Psi_-
\]

(4.15)

However, it is early to assume that \( \chi_\pm = \Psi_\pm \) since \( \Delta \) has a kernel:

\[
\Delta f = 0 \Rightarrow f = vK
\]

\(^2\) Given a background \( Y^i \) one can define the generalization of the Weyl symbol of the operator \( O \) by \( f_O(q) = \int dp \text{Tr}_U \left( O \exp - i p (q^i - Y^i) \right) \) and conversely, \( O_f = \int dp \text{Tr}_U (q^i \exp \left( ip (q^i - Y^i) \right) \). However, in general it is not easy to characterize the class of operators for which these formulæ make sense, and also \( O_{fO} \neq O \)
where $v$ is from $[2 l]$, $v = \sum_{n=0}^{\infty} a_n \phi^n | u \rangle | n \rangle$, and $K$ is an arbitrary $x_3$-dependent operator in the Fock space $\mathcal{H}$.

Thus, $\chi_\pm = \Psi_{\pm} - K_{\pm}^1 v^1$, and the operators $K$ must be chosen in such a way, that $\chi_{\pm} (z = 0) = 0$, which implies:

$$K_{\pm}^1 = \Psi_{\pm} (z = 0) \xi$$  \hspace{1cm} (4.16)

Recall that $\Psi_+ = cv$, $\Psi_- = -b^i v$. Hence,

$$K_{\pm}^1 = \xi^{-1} c^1 \xi, \quad K_{\pm}^2 = \xi^2$$  \hspace{1cm} (4.17)

5. Non-trivial backgrounds

So far our discussion concerned the flat closed string background. We shall now generalize the discussion to cover some non-trivial string backgrounds.

5.1. Orbifolds

The obvious starting point is to consider orbifolds. So, let $\Gamma$ be a discrete subgroup of $\text{Spin}(d) \times \mathbb{R}^d$ - the group of isometries of $\mathbb{R}^d$. For $g \in \Gamma$ let $\gamma(g) \in \text{Spin}(d)$ be the corresponding rotation matrix, and $l(g) \in \mathbb{R}^d$ the corresponding shift vector. We have the following composition rule:

$$(l(g_1), \gamma(g_1)) \times (l(g_2), \gamma(g_2)) = (l(g_1) + \gamma(g_1) l(g_2), \gamma(g_1) \gamma(g_2)) = (l(g_1 g_2), \gamma(g_1 g_2)).$$  \hspace{1cm} (5.1)

We wish to consider D-branes living in the background obtained by taking the quotient of the Minkowskian space $\mathbb{R}^d$ by $\Gamma$.

The prescription for modifying the action (2.1) to reflect the orbifoded nature of the ambient space-time is the following. We demand that the Hilbert space $\mathcal{H}$ forms a representation of $\Gamma$. Let $\Omega : \Gamma \rightarrow \text{End}(\mathcal{H})$ be the corresponding homomorphism. If $\mathcal{R}_i$, $i = 0, \ldots, r$, are irreducible unitary representations of $\Gamma$, $\mathcal{R}_0 = \mathbb{C}$ being the trivial representation, then

$$\mathcal{H} = \bigoplus_i \mathcal{H}_i \otimes \mathcal{R}_i$$

is the decomposition of $\mathcal{H}$. We demand that the operators $Y$ are equivariant with respect to $\Gamma$ in the sense that:

$$\Omega^{-1}(g) Y^i \Omega(g) = \gamma^i_j (g) Y^j + l^i(g)$$  \hspace{1cm} (5.2)

In practice it is convenient to introduce the quiver diagram. We shall skip some standard issues like the reduced gauge invariance, and as a consequence more freedom in the action, coming from the twisted sector fields couplings. The latter are enumerated by the tensors

$$\theta^{ij} (g) = \gamma^i_j (h) \gamma^j_k (h) \theta^{kj} (h^{-1} gh)$$  \hspace{1cm} (5.3)

which will alter the ground state solutions as follows:

$$[Y^i, Y^j] = i \frac{1}{\# \Gamma} \sum_{g \in \Gamma} \theta^{ij} (g) \Omega(g)$$  \hspace{1cm} (5.4)

(for infinite $\Gamma$ the normalization factor should be discussed separately)

---

3 Proof: use the identities: $\xi v = \delta_{1,1}^{\omega, x} \frac{1}{2} \partial_1 (\xi v) = \left( \partial_3 + x_3 - \frac{1}{2} \xi^2 \right) (\xi v), \chi_{\pm} = 2 x_3 \zeta - n \zeta_{n-1}$. Then: $\chi_+ = \left( \partial_3 + x_3 - \frac{1}{2} \xi^2 \right) v = \xi^{-1} (\partial_3 - z) (\xi v) = 1 \frac{1}{2} \left( \partial_3 + \Phi - z \right) \xi v, \chi_+ = \xi v - \xi^{-1} c^1 \xi v = \xi^{-1} \partial_1 (\xi v) = (\partial_3 + A_3) v$ From this (4.11) follows, with the help of the Bogomolny equations obeyed by $A_3, \Phi$ in [21]. It remains to check (4.12). Indeed, if we denote $\eta = \xi^2, \lambda_\pm (z) = \frac{1}{2 b^i \phi};$ then: $\left( \partial_3 - z \right) \lambda_\pm = 2 \eta_\pm (\lambda_{\pm - 1} - \lambda_\pm), \eta_\pm (\partial_3 + z) \frac{1}{2} (\partial_3 - z) \lambda_\pm = 4 \eta_\pm \left[ (\lambda_{\pm - 2} - \lambda_{\pm - 1}) + (z - \eta_\pm) (\lambda_{\pm - 1} - \lambda_\pm) \right]$ and at the same time: $\frac{1}{2 b^i \phi} \left( \eta^{-1} (\partial_3) \right) = \frac{1}{\# \Gamma + 1} (n + 1) (\lambda_{n+1} - \lambda_n) - n (\lambda_n - \lambda_{n-1})$, which implies (4.12)
5.2. Example: $\Gamma = \mathbb{Z}_2$

Let $\Gamma = \mathbb{Z}_2$ act on $\mathbb{R}^d$ as $y \mapsto -y$. We have:

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$$  \hspace{1cm} (5.5)

The solution to (5.2) reads:

$$Y^i = \begin{pmatrix} 0 & Y^i_+ \\ Y^i_- & 0 \end{pmatrix}$$  \hspace{1cm} (5.6)

where $Y^i_\pm : \mathcal{H}_\mp \to \mathcal{H}_\pm$. To simplify our problem let us assume very special form of the twisted sector field:

$$\theta^i(-1) = \zeta\theta^i(+1)$$

Then, by going to the complex notations we can rewrite (5.4) as follows:

$$[A_\alpha, A^0_\beta] = \frac{i}{2}\delta_{\alpha\beta}(1 + \zeta F). \quad [A_\alpha, A_\beta] = 0, \alpha, \beta = 1, \ldots, d/2$$  \hspace{1cm} (5.7)

where $F$ is the parity operator: $F|_{\mathcal{H}_\pm} = \pm 1$.

We shall now study the representation theory of the algebra (5.4). Clearly, it is sufficient to study the case $d = 2$. Let us rewrite (5.6) in terms of $A_i^\pm$:

$$A_\alpha = \begin{pmatrix} 0 & b_\alpha \\ a_\alpha & 0 \end{pmatrix},$$  \hspace{1cm} (5.8)

then (5.7) becomes the condition (we now drop the index $\alpha$):

$$a_\alpha a^\dagger_\alpha - b^\dagger_\alpha b = \frac{i}{2} (1 - \zeta), \quad bb^\dagger - a^\dagger a = \frac{i}{2} (1 + \zeta)$$  \hspace{1cm} (5.9)

We can assume $\zeta \geq 0$ (otherwise we exchange $a$ and $b$). Notice that if $\zeta = 0$ we can take as a solution

$$b = \frac{1}{\sqrt{2}} (1 - P) a P, a = \frac{1}{\sqrt{2}} P a (1 - P), \quad P = \frac{1}{2} (1 + F), \quad F = (-1)^{a^\dagger a}$$

and $a, a^\dagger$ are the standard creation-annihilation operators.

For $0 \leq \zeta < 1$ we have the following (essentially unique) representation:

$$b^\dagger e^+_n = \sqrt{n} e^+_{n-1}, \quad a^\dagger e^+_n = \sqrt{n + \frac{1}{2} (1 - \zeta)} e^+_n$$

$$b e^-_n = \sqrt{n + \frac{1}{2} (1 - \zeta)} e^-_{n+1}, \quad a e^-_n = \sqrt{n + \frac{1}{2} (1 + \zeta)} e^-_n$$  \hspace{1cm} (5.10)

For $\zeta \geq 1$

$$b^\dagger e^+_n = \sqrt{n + \frac{1}{2} (\zeta - 1)} e^+_n, \quad a^\dagger e^+_n = \sqrt{n} e^+_{n-1},$$

$$b e^-_n = \sqrt{n + \frac{1}{2} (\zeta + 1)} e^-_{n+1}, \quad a e^-_n = \sqrt{n} e^-_{n+1},$$  \hspace{1cm} (5.11)

and the vector $e^+_0$ should be dropped from the representation.

5.3. Example: instantons on $K3$

The closed string background corresponding to the $K3$ surface can be realized as a marginal deformation of the orbifold $T^4/\mathbb{Z}_2$ where $\mathbb{Z}_2$ acts by reflection on all four flat coordinates on the four-torus. In turn, the four-torus is a quotient of the Euclidean space $\mathbb{R}^4$ by a lattice $\Gamma_0 \cong \mathbb{Z}_4$. Thus we may hope to realize the noncommutative gauge theory on $K3$ by deforming the orbifold of $\mathbb{R}^4$ with respect to $\Gamma = \Gamma_0 \times \mathbb{Z}_2$. Thus we arrive at the following algebra (here $B$ is an anti-symmetric form on $\Gamma_0$ corresponding to the $B$-field):

$$U_i U_{i'} = U_{i'} U_i e^{B(i,i')}, \quad I, I' \in \Gamma_0$$

$$U_{-i} Y^i U_i = Y^i + l^i$$

$$\Omega^{-1} Y^i \Omega = -Y^i$$

$$\Omega^2 = 1, \quad \Omega^{-1} U_i \Omega = U_{-i}$$  \hspace{1cm} (5.12)

and the vacuum equations

$$[Y^i, Y^j] = \mu^{ij} + \sum_{l \in \Gamma_0} \mu^{ij}_l U^i_l \Omega$$  \hspace{1cm} (5.13)

For consistency the parameters $\mu^{ij}_l$ must obey: $\sum_l \mu^{ij}_l U^i_l U^j_l = \sum_{l'} \mu^{ij}_{l'} U^i_{l'} U^j_{l'} \Omega_y^{(ij)},$ i.e. $\mu^{ij}_l = \mu^{ij}_{-l'}$ for any $l' \in \Gamma_0$. Thus instead of the infinite number of deformation parameters we end up with the finite number. The twisted deformations $\mu^{ij}_l$ are in one-to-one correspondence with the elements of the coset $\Gamma_0/2\Gamma_0$, which also label the fixed points of the $\mathbb{Z}_2$ action on the four-torus.

For the instanton solutions the resulting equations look a bit like the self-duality equations with codimension four impurity:

$$[Y^i, Y^j] + \frac{1}{2} \sum_{l \in \Gamma_0} \mu^{ij}_l \sqrt{g} \sqrt{g}^{ij}[Y^k, Y^l] = \zeta^{ij} + \sum_{e \in \Gamma_0 / 2 \Gamma_0} \zeta^{ij}_e \sum_{l \in \Gamma_0} U^i_{e+l} \Omega$$  \hspace{1cm} (5.14)

where $\zeta^{ij}_e$ are the self-dual projections of $\mu^{ij}_l$.

It would be nice to obtain any explicit solution of (5.14) with $\zeta_e \neq 0$. If $\zeta_e = 0$ then the solutions to (5.14) are easy to construct. They correspond to $\mathbb{Z}_2$ equivariant instantons on the noncommutative torus [22].
5.4. Conifold

To get from the orbifold to the conifold background we shall mimic the strategy in [23]. The only difference compared to the [23] setup is that there one dealt with the $U(N)$ gauge theory, while here we operate with the $U(H)$ gauge fields. But the rest of the story is the same. One starts with the theory on the orbifold background, turns on the twisted sector fields, preserving the 8 supercharges ($\mathcal{N} = 2$ susy in 4d) and then turns on the superpotential giving masses to the chiral multiplets in the $\mathcal{N} = 2$ vector multiplets. For example, in the $\mathbb{Z}_{k+1}$ orbifold of $\mathbb{C}^2$ case the vacuum equations will now change to:

$$
U^{-1} B_1 U = \omega B_1 \\
U^{-1} B_2 U = \omega^{-1} B_2 \\
U^{-1} \Phi U = \Phi
$$

(5.15)

where $m_l, \zeta_l$ are the parameters of the generalized $A_k$ conifold.

6. Conclusions and outlook

The gauge fields/strings duality is a fascinating long-standing problem[1]. We have considered a slightly generalized version of this duality, which includes noncommutative gauge fields in various dimensions, and, as a limit, the ordinary gauge theories. We have defined Yang-Mills algebras and considered several interesting examples of their representations. These arise as $\alpha' \rightarrow 0$ limits of the algebras of functions on the D-branes in flat space-time. We have also presented an analysis of the D-branes in curved backgrounds, namely in those, obtained by orbifolding from the flat space-time.

The topics left outside of this short note include the generalizations to the non-trivial $H$-fields (in which case any universal enveloping algebra of a simple Lie algebra may arise as an example of Yang-Mills algebra, at least if the dimension of the latter permits), time-dependent backgrounds (with applications to cosmology), explicit examples of instantons on the curved spaces, and, most interestingly, the consequences of the decoupling of the null-vectors on the closed string side for the open string gauge invariant quantities [1]. We plan to present some of these considerations elsewhere.

Note added. As the lecture notes were being prepared we received a manuscript [24] which contained independently obtained higher-dimensional instanton solutions (3.18), together with the generalizations involving multiple instantons sitting on top of each other.
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