On Multi-dimensional Compressible Flows of Nematic Liquid Crystals with Large Initial Energy in a Bounded Domain

Fei Jiang\textsuperscript{a}, Song Jiang\textsuperscript{b}, Dehua Wang\textsuperscript{c}

\textsuperscript{a}College of Mathematics and Computer Science, Fuzhou University, Fuzhou, 350108, China.
\textsuperscript{b}Institute of Applied Physics and Computational Mathematics, Beijing, 100088, China.
\textsuperscript{c}Department of Mathematics, University of Pittsburgh, Pittsburgh, PA, 15260, USA.

Abstract

We study the global existence of weak solutions to a multi-dimensional simplified Ericksen-Leslie system for compressible flows of nematic liquid crystals with large initial energy in a bounded domain $\Omega \subset \mathbb{R}^N$, where $N = 2$ or $3$. By exploiting a maximum principle, Nirenberg’s interpolation inequality and a smallness condition imposed on the $N$-th component of initial direction field $d_0$ to overcome the difficulties induced by the supercritical nonlinearity $|\nabla d|^2 d$ in the equations of angular momentum, and then adapting a modified three-dimensional approximation scheme and the weak convergence arguments for the compressible Navier-Stokes equations, we establish the global existence of weak solutions to the initial-boundary problem with large initial energy and without any smallness condition on the initial density and velocity.

Keywords: Liquid crystals, compressible flows, weak solutions, weak convergence arguments.

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1. Introduction

We study the global existence of weak solutions to the following multi-dimensional simplified version of the Ericksen-Leslie model in a bounded domain $\Omega \subset \mathbb{R}^N$ which describes the motion of a compressible flow of nematic liquid crystals:

\begin{align}
\frac{\partial \rho}{\partial t} + \text{div}(\rho \mathbf{v}) &= 0, \\
\frac{\partial (\rho \mathbf{v})}{\partial t} + \text{div}(\rho \mathbf{v} \otimes \mathbf{v}) + \nabla P &= \mu \Delta \mathbf{v} + (\mu + \lambda)\text{div} \mathbf{v} \\
&\quad - \nu \text{div} \left( \nabla \mathbf{d} \otimes \nabla \mathbf{d} - \frac{1}{2} |\nabla \mathbf{d}|^2 \mathbb{I} \right), \\
\frac{\partial \mathbf{d}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{d} &= \theta (\Delta \mathbf{d} + |\nabla \mathbf{d}|^2 \mathbf{d}),
\end{align}

with initial conditions:

\begin{align}
\rho(x, 0) &= \rho_0(x), \quad \mathbf{d}(x, 0) = d_0(x), \quad (\rho \mathbf{v})(x, 0) = \mathbf{m}_0(x) \quad \text{in } \Omega, \\
\end{align}

and boundary conditions:

\begin{align}
\mathbf{n} \cdot \nabla \mathbf{d}(x, t) &= 0, \quad \mathbf{v}(x, t) = 0, \quad x \in \partial \Omega, \quad t > 0,
\end{align}

Email addresses: jiangfei059@163.com (Fei Jiang), jiang@iapcm.ac.cn (Song Jiang), dwang@math.pitt.edu (Dehua Wang)
where \( \mathbf{n} \) denotes the outer normal vector of \( \Omega \). The unknown function \( \rho \) is the density of the nematic liquid crystals, \( \mathbf{v} \) the velocity and \( P(\rho) \) the pressure determined through the equations of state, \( \mathbf{d} \) represents the macroscopic average of the nematic liquid crystal orientation field. The constants \( \mu, \lambda, \nu, \) and \( \theta \) denote the shear viscosity, the bulk viscosity, the competition between kinetic and potential energies, and the microscopic elastic relation time for the molecular orientation field, respectively, they satisfy the physical conditions:

\[
\mu > 0, \quad \lambda + \mu \geq 0, \quad \nu > 0, \quad \theta > 0.
\]

\( \mathbb{I} \) denotes the \( N \times N \) identity matrix. The term \( \nabla \mathbf{d} \circ \nabla \mathbf{d} \) denotes the \( N \times N \) matrix whose \((i, j)\)-th entry is given by \( \partial_{x_i} \mathbf{d} \cdot \partial_{x_j} \mathbf{d} \), for \( 1 \leq i, j \leq N \), i.e., \( \nabla \mathbf{d} \circ \nabla \mathbf{d} = (\nabla \mathbf{d})^\top \nabla \mathbf{d} \), where \( (\nabla \mathbf{d})^\top \) denotes the transpose of the \( N \times N \) matrix \( \nabla \mathbf{d} \).

In 1989, Lin [17] first derived a simplified Ericksen-Leslie system modeling liquid crystal flows when the fluid is incompressible and viscous. Subsequently, Lin and Liu [19, 20] established some analysis results on the simplified Ericksen-Leslie system, such as the existence of weak and strong solutions and the partial regularity of suitable solutions, under the assumption that the liquid crystal director field is of varying length by Leslie’s terminology, or variable degree of orientation by Ericksen’s terminology.

Since the supercritical nonlinearity \( |\nabla \mathbf{d}|^2 \mathbf{d} \) causes significant mathematical difficulties, Lin in [17] introduced a Ginzburg-Landau approximation of the simplified Ericksen-Leslie system, i.e., \( |\nabla \mathbf{d}|^2 \mathbf{d} \) in (1.3) is replaced by the Ginzburg-Landau penalty function \((1 - |\mathbf{d}|^2)/\epsilon \) or by a more general penalty function. Consequently, by establishing some estimates to deal with the direction field and its coupling/interaction with the fluid variables, a number of results on the Navier-Stokes equations can be successfully generalized to such Ginzburg-Landau approximation model. For examples, when \( \rho \) is a constant, i.e., the homogeneous incompressible case, Lin and Liu [19] proved the global existence of weak solutions in 2D and 3D. In particular, they also obtained the existence and uniqueness of global classical solutions either in 2D or in 3D for large fluid viscosity \( \mu \). In addition, the existence of weak solutions to the density-dependent incompressible flow of liquid crystals was proved in [11, 23]. Recently, Wang and Yu [30], and Liu and Qin [24] independently established the global existence of weak solutions to the three-dimensional compressible flow of liquid crystals with the Ginzburg-Landau penalty function.

In the past a few years, progress has also been made on the analysis of the model (1.1)–(1.3) by overcoming the difficulty induced by the supercritical nonlinearity \( |\nabla \mathbf{d}|^2 \mathbf{d} \). For the incompressible case, the existence of large weak solutions in 2D was established in [18] and [13] for a bounded domain and the whole space respectively, and the local existence of large strong solutions and global existence of small strong solutions in three dimensions were proved in [1, 6, 15, 21, 29]. For the 3D compressible case, the existence of strong solutions have been investigated extensively. For examples, the local existence of strong solutions and a blow-up criterion were obtained in [3, 3], while the existence and uniqueness of global strong solutions to the Cauchy problem in critical Besov spaces were proved in [7] provided that the initial data are close to an equilibrium state, and the global existence of classical solutions to the Cauchy problem was shown in [14] with smooth initial data that has small energy but possibly large oscillations with possible vacuum and constant state as far-field condition. Recently progress has also been made on the existence of weak solutions to multi-dimensional problem (1.1)–(1.3). For examples, Jiang et al [10] established the existence of global weak solutions to the two-dimensional problem in a bounded domain under a restriction imposed on the initial energy including the case of small initial energy. Moreover they also obtained the existence of global large weak solutions to the two-dimensional Cauchy problem, provided that the second component of initial data of
the direction field satisfies some geometric angle condition. At the same time, Wu and Tan \[31\] established the existence of global weak solutions to the Cauchy problem \((1.1)-(1.3)\) by using Suen and Hoff’s method \[28\], if the initial energy around equilibrium state is sufficiently small, the coefficients \(\mu\) and \(\lambda\) satisfy \(0 \leq \lambda + \mu < (3 + \sqrt{21})\mu/6\), and the initial data \((v_0, d_0)\) satisfies \(\|v_0\|_{L^p(\mathbb{R}^3)} + \|d_0\|_{L^p(\mathbb{R}^3)} < \infty\) with \(p > 6\).

To our best knowledge, however, there are no results available on weak solutions of the multi-dimensional problem \((1.1)-(1.3)\) with large initial data in a bounded domain, due to the difficulties induced by the compressibility and the supercritical nonlinearity. It seems that the only global existence of large weak solutions to \((1.1)-(1.3)\) was shown in the 1D case in \[2\]. On the other hand, there exists a global weak solution to the multi-dimensional compressible Navier–Stokes equations with large initial data (i.e., the initial energy can be arbitrarily large). A question naturally arises whether one can establish a global existence result for the problem \((1.1)-(1.5)\) without any smallness restriction imposed on the initial density and velocity. In the current paper, we give a positive answer to this question in the two-dimensional case under a restriction on the last component of initial direction field \(d_0\), while in the three-dimensional case, a somewhat weaker existence result is obtained.

Before stating our main result, we explain the notations and conventions used throughout this paper. In this paper we focus our study on the case of isentropic flows as in \[30\] and assume that

\[P(\rho) = A\rho^\gamma, \quad \text{with } A > 0, \gamma > \frac{N}{2}.\]

For the sake of simplicity, we define

\[I := I_T := (0, T), \quad Q_T = \Omega \times I,\]

\[\mathcal{F}(t) := \mathcal{F}(\rho, v, d) := \int_{\Omega} \left( \mu |\nabla v|^2 + (\lambda + \mu) |\text{div}v|^2 + \theta (|\Delta d + |\nabla d|^2|d|^2|) \right) dx,\]

and

\[\mathcal{E}(t) := \mathcal{E}(\rho, m, d) := \int_{\Omega} \left( \frac{1}{2} \frac{|m|^2}{\rho} 1_{\{\rho > 0\}} + \frac{A}{\gamma - 1} \rho^\gamma + \frac{\nu \theta |\nabla d|^2}{2} \right) dx \quad \text{with} \quad m = \rho v,\]

where \(1_{\{\rho > 0\}}\) denotes the characteristic function. We use the bold fonts to denote the product spaces, for examples,

\[L^p(\Omega) := (L^p(\Omega))^N, \quad H_0^k(\Omega) := (H_0^k(\Omega))^N = (W_0^{k,2}(\Omega))^N, \quad H^k(\Omega) := (W^{k,2}(\Omega))^N;\]

and the Sobolev space with weak topology is defined as

\[C^0(\bar{I}, L^q_{\text{weak}}(\Omega)) := \left\{ f : I \to L^q(\Omega) \right| \int_{\Omega} f \cdot g dx \in C(\bar{I}) \text{ for any } g \in L^q(\Omega) \right\}.\]

In what follows, the letter \(C_0\) will denote a generic positive constant which may depend on the dimension of space \(N\), and the letter \(C(\ldots)\) will denote a generic positive constant depending on its variables, and is nondecreasing in its variables, except for the domain \(\Omega\). It should be noted that the letter \(C(\ldots)\) may depend on the physical parameters and the dimension \(N\) in some places, however we usually omit this dependence for simplicity.

Our existence result of large weak solutions for \((1.1)-(1.5)\) reads as follows.
Theorem 1.1. Let \( N = 2 \) or \( 3 \), \( \Omega \subset \mathbb{R}^N \) be a bounded domain of class \( C^{2,\alpha} \) with \( \alpha \in (0,1) \), and the initial data \( \rho_0, m_0, d_0 \) satisfy the following conditions:

\[
\rho_0 \in L^\gamma(\Omega), \quad \rho_0 \geq 0 \text{ a.e. in } \Omega, \tag{1.8}
\]

\[
m_0 \in L^{\frac{2\gamma}{\gamma+2}}(\Omega), \quad m_0 1_{\{\rho_0 = 0\}} = 0 \text{ a.e. in } \Omega, \quad \frac{|m_0|^2}{\rho_0} 1_{\{\rho_0 > 0\}} \in L^1(\Omega), \tag{1.9}
\]

\[
d_0 \in H^2(\Omega), \quad |d_0| \leq 1 \text{ in } \Omega. \tag{1.10}
\]

Then, there exists a constant \( \varepsilon_0 := \varepsilon_0(N, \Omega) \leq 1 \) depending on \( N \) and \( \Omega \) (but independent of the physical parameters in (1.1)–(1.3) and the initial data), such that if \( \rho_{0N} := \rho_{0N}(x) \) (the \( N \)-th component of \( d_0(x) \)) satisfies

\[
1 - \rho_{0N} < \varepsilon_0, \tag{1.11}
\]

the initial-boundary value problem (1.1)–(1.3) has a global weak solution \( (\rho, v, d) \) on \( I = I_T \) for any given \( T > 0 \), with the following properties:

1. **Regularity:**

\[
0 \leq \rho \text{ a.e. in } Q_T, \quad \rho \in C^0(\bar{I}, L^\gamma_{\text{weak}}(\Omega)) \cap C^0(\bar{I}, L^p(\Omega)) \cap L^{\gamma+\eta}(Q_T),
\]

\[
v \in L^2(\bar{I}, H^1_0(\Omega)), \quad \rho v \in L^\infty(\bar{I}, L^{2\gamma}(\Omega)) \cap C^0(\bar{I}, L^{2\gamma}_{\text{weak}}(\Omega)),
\]

\[
|d| \leq 1 \text{ a.e. in } Q_T, \quad d \in L^2(\bar{I}, H^1(\Omega)) \cap C^0(\bar{I}, H^1(\Omega)), \quad \partial_t d \in L^\frac{4}{3}(\bar{I}, L^2(\Omega)),
\]

\[
(n \cdot \nabla d)|_{\partial \Omega} = 0 \text{ in the sense of trace for a.e. } t \in I,
\]

where \( p \in [1, \gamma) \), and \( \eta \in (0, (2\gamma - N)/N) \).

2. **Equations (1.1) and (1.2) hold in** \( \mathcal{D}'(Q_T)^{N+1} \), **and equation (1.3) holds a.e. in** \( Q_T \).

3. **Equation (1.1) is satisfied in the sense of renormalized solutions, that is,** \( \rho, v \) **satisfy**

\[
\partial_t b(\rho) + \text{div}[b(\rho)v] + [\rho b'(\rho) - b(\rho)] \text{div} v = 0 \text{ in } \mathcal{D}'(\mathbb{R}^N \times I), \tag{1.12}
\]

**provided** \( (\rho, v) \) **is prolonged to be zero on** \( \mathbb{R}^N \setminus \Omega, \) **for any** \( b \) **satisfying**

\[
b \in C^0[0, \infty) \cap C^1(0, \infty), \quad |b'(s)| \leq c s^{-\lambda_0}, \quad s \in (0, 1], \quad \lambda_0 < 1,
\]

**and the growth conditions at infinity:**

\[
|b'(s)| \leq c s^{\lambda_1}, \quad s \geq 1, \text{ where } c > 0, \quad 0 < 1 + \lambda_1 < \frac{(N + 2)\gamma - N}{2N}.
\]

4. **Regularity estimates:**

\[
\sup_{t \in I} (\mathcal{E}(t) + \|d(t) - e_N\|_{L^2(\Omega)}) + \|\nabla v, \nabla^2 d\|_{L^2(Q_T)} + \|\nabla d\|_{L^4(Q_T)} \leq C(\mathcal{E}_0, T, \Omega),
\]

\[
\|d - e_N\|_{L^\infty(\Omega)} < C_0 \sqrt{\varepsilon_0}, \tag{1.13}
\]

where \( \mathcal{E}_0 := \mathcal{E}(0) = \mathcal{E}(\rho_0, m_0, d_0), \ e_2 = (0, 1), \ e_3 = (0, 0, 1), \) **and we have defined**

\[
\|\nabla v, \nabla^2 d\|_{L^2(Q_T)}^2 = \sum_{1 \leq i,j \leq N} \|\partial_i \partial_j v\|_{L^2(Q_T)}^2 + \sum_{1 \leq i,j,k \leq N} \|\partial_i \partial_j d_k\|_{L^2(Q_T)}^2.
\]

In particular, if \( \Omega \) is a ball \( B_R := \{x \in \mathbb{R}^N | \ |x| < R\} \) with \( R \geq 1 \), then the above constant \( \varepsilon_0 \) can be chosen to be independent of \( \Omega \) for any \( R \geq 1 \). Moreover, the constant \( C(\mathcal{E}_0, T, \Omega) \) **in (1.13) can be replaced by a constant** \( C(\mathcal{E}_0, T, \|d_0 - e_N\|_{L^2(\Omega)}) \) **independent of** \( \Omega \).
(5) In the case of $N = 2$, if, in addition, $|d_0| = 1$, then the weak solution satisfies $d \equiv 1$, and the following finite and bounded energy inequalities:

$$\frac{d\mathcal{E}(t)}{dt} + \mathcal{F}(t) \leq 0 \quad \text{in } \mathcal{D}'(I),$$

$$\mathcal{E}(t) + \int_0^t \mathcal{F}(s)\,ds \leq \mathcal{E}_0 \quad \text{for a.e. } t \in I. \quad (1.14)$$

**Remark 1.1.** The proof of Theorem 1.1 remains basically unchanged if the motion of the fluid is driven by a bounded external force, i.e., when the momentum equations (1.2) contain an additional term $\rho f(x,t)$ with $f$ being a bounded and measurable function. We remark here that we do not require any smallness condition on $f$. However, we are not clear whether the above theorem still holds with non-homogeneous boundary condition in place of Neumann boundary condition “$(n \cdot \nabla d)|_{\partial \Omega} = 0$”. In the proof of Theorem 1.1, we use the Neumann boundary condition only in order to deduce a maximum principle on $d$.

**Remark 1.2.** We mention that the regularity requirement “$d_0(x) \in H^2(\Omega)$” is not optimal, for example, if we have “$d_0(x) \in W^{1,p}(\Omega)$ with $p > N$”, then the above theorem still holds, and this can be shown by a standard approximate approach. On the other hand, we do not known whether “$d_0(x) \in H^1(\Omega)$” is the lowest regularity requirement, since it involves the problem of the Sobolev maps between two manifolds with the lower boundedness condition (1.11).

**Remark 1.3.** In view of the above regularity estimates in a ball, we can make use of a domain expansion technique to obtain a similar existence result of global weak solutions to the corresponding Cauchy problem, for which the expression of energy $\mathcal{E}(t)$ should be written in a form around some equilibrium state $(\rho_\infty, v_\infty, e_N)$ with $\rho_\infty > 0$ to make the energy integral sense (see [10, Theorem 1.2]). Of course in this case, we can also establish a similar existence result of global weak solutions to the corresponding incompressible problem.

We now describe the main idea of the proof of Theorem 1.1 For the Ginzburg-Landau approximation model to (1.1)–(1.3), based on some new estimates to deal with the direction field and its coupling/interaction with the fluid variables, Wang and Yu in [30] adopted a classical three-level approximation scheme which consists of the Faedo-Galerkin approximation, artificial viscosity, an artificial pressure and the celebrated weak continuity of the effective viscous flux to overcome the difficulty of possible large oscillations of the density, and established the existence of weak solutions. These techniques were developed in [22] and [5, 12] for the compressible Navier-Stokes equations, we refer to the monograph [26] for more details. In the proof of Theorem 1.1, we also adopt the three-level approximation scheme, so the key steps are to deduce the a priori estimates and to construct approximate solutions to the third approximate problem. Compared with the Ginzburg-Landau approximation model in [30], however, the system (1.1)–(1.3) is much more difficult to deal with, due to the supercritical nonlinearity $|\nabla d|^2$ in (1.3). Consequently, not like that in [30], one can not deduce the (sufficiently) strong estimate $\nabla^2 d \in L^2(I, L^2(\Omega))$ directly from the basic energy inequality (1.14). Recently, Ding and Wen obtained the global existence and uniqueness of strong solutions to the 2D density-dependent incompressible model with small initial energy and positive initial density away from zero in [3], where they got $\nabla^2 d \in L^2(I, L^2(\Omega))$ from the basic energy inequality under the smallness condition of the initial energy. In fact, they first deduced $\nu \lVert \nabla d \rVert_{L^2(\Omega)}^2 \leq C_0$ and $\nu \lVert \Delta d \rVert_{L^2(\Omega)}^2 \leq C_0 + \nu \lVert \nabla d \rVert_{L^4(\Omega)}^4$ by employing the basic energy inequality, and then made use of the inequality

$$\lVert \nabla d \rVert_{L^4(\Omega)}^4 \leq C(\Omega)(\lVert \nabla^2 d \rVert_{L^2(\Omega)}^2 \lVert \nabla d \rVert_{L^2(\Omega)}^2 + \lVert \nabla d \rVert_{L^2(\Omega)}^4)$$

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for some constant $C(\Omega)$ depending on $\Omega \subset \mathbb{R}^2$, which follows from the elliptic estimates and an interpolation inequality (see \cite[Lemma 2.4]{3}), to infer that

$$\|\nabla^2 d\|_{L^2(Q_T)}^2 \leq C(\mathcal{E}_0, \|\nabla^2 d_0\|_{L^2(\Omega)}),$$

(1.15)

provided that the initial energy is sufficiently small. Motivated by this study, Jiang et al \cite{10} established the global existence of weak solutions to the corresponding compressible problem. In the current paper, we shall use another version of the interpolation inequality

$$\|\nabla d\|_{L^4(\Omega)}^4 \leq C(\Omega)(\|\nabla^2 d\|_{L^2(\Omega)}^2 \|d - e_N\|_{L^\infty(\Omega)}^2 + \|d - e_N\|_{L^4(\Omega)}^4)$$

for some constant $C(\Omega)$ depending on $\Omega \subset \mathbb{R}^N$ ($N = 2$ or 3), from which the estimate (1.15) can also be deduced for $\Omega \subset \mathbb{R}^N$ if $\|d - e_N\|_{L^\infty(\Omega)}^2$ is sufficiently small. Now the question is whether the smallness of $\|d - e_N\|_{L^\infty(\Omega)}^2$ is guaranteed by smallness of the initial data $\|d_0 - e_N\|_{L^\infty(\Omega)}^2$. Fortunately, this is the case by applying the maximum principle to nonnegative lower bounds of solutions to the equations (1.3) and the condition $|d| \leq 1$. Consequently, we deduce the desired energy estimates on $d$ from the energy inequality. With these estimates in hand, we can adopt and modify the three-dimensional approximation scheme approach to show Theorem 1.1 if we can construct a solution to the following third approximate problem:

$$\partial_t \rho + \text{div}(\rho v) = \varepsilon \Delta \rho,$$

$$\partial_t d + v \cdot \nabla d = \theta(\Delta d + f_\varepsilon(|\nabla d|^2)d),$$

(1.17)

$$\int_{\Omega} (\rho v)(t) \cdot \Psi dx - \int_{\Omega} m_0 \cdot \Psi dx = \int_0^t \int_{\Omega} \left[ \mu \Delta v + (\mu + \lambda)\nabla \text{div}v - \nabla P - \delta \nabla \rho^\beta - \varepsilon (\nabla \rho \cdot \nabla v) \right] \cdot \Psi dx ds,$$

(1.18)

where the $n$-dimensional Euclidean space $X_n$ will be introduced in Section 4, $\varepsilon, \delta, \beta > 0$ are constants, and the smooth function $f_\varepsilon(x) \geq 0$ satisfying

$$f_\varepsilon(x) = x \text{ if } N = 2; \quad \begin{cases} 0 \leq f'(x) \leq 1, \\ f(x) = \varepsilon^{-1} \text{ if } x \geq \varepsilon^{-1}, \\ 0 \leq x - f_\varepsilon(x) \to 0 \text{ as } \varepsilon \to 0, \end{cases} \quad (N = 3).$$

(1.19)

It should be noted that the third approximate problem above still enjoys the desired energy estimates (see Proposition 2.1), thus it is easy to establish the unique solvability of the third approximate problem in the 2D case by following the same proof as in \cite{10}. However, the proof in \cite{10} can not be directly applied to the 3D case, and the difficulty lies in that we could not deduce a global estimate on $\|\partial_t d\|_{L^\infty(I, L^2(\Omega))}$ (see (4.26)) for the 3D approximate problem (1.14)–(1.46). To overcome this difficulty, we introduce the cut-off function (1.19) to get a global estimate of $\|\partial_t d\|_{L^\infty(I, L^2(\Omega))}$. On the other hand, we have to pay the price for this, namely, for the 3D approximate problem (1.17) based on a cut-off function with Neumann boundary condition, we cannot show $|d| = 1$ when $|d_0| = 1$. This is the reason why the solution in Theorem 1.1 does not satisfy $|d| = 1$ in three dimensions.

The rest of paper is organized as follows. In Section 2 we deduce the basic energy equalities from the third approximate problem and derive more energy estimates on $d$ under the assumption
In Section 3 we introduce the strong solvability of sub-systems in the third approximate problem, while the unique solvability of the third approximate problem is established in Section 4. Finally, we briefly sketch how to use the standard three-level approximation scheme to prove Theorem 1.1 in Section 5.

2. A priori for the third approximate problem

This section is devoted to formal derivation of the a priori energy estimates for the third approximate equations:

\[ \partial_t \rho + \text{div}(\rho \mathbf{v}) = \varepsilon \Delta \rho, \]  
\[ \partial_t (\rho \mathbf{v}) + \text{div}(\rho \mathbf{v} \otimes \mathbf{v}) + \nabla P + \delta \nabla \rho^3 + \varepsilon (\nabla \rho \cdot \nabla \mathbf{v}) = \mu \Delta \mathbf{v} + (\mu + \lambda) \nabla \text{div} \mathbf{v} - \nu \text{div} \left( \nabla \mathbf{d} \otimes \nabla \mathbf{d} - \frac{1}{2} |\nabla \mathbf{d}|^2 \mathbb{I} \right), \]  
\[ \partial_t \mathbf{d} + \mathbf{v} \cdot \nabla \mathbf{d} = \theta (\Delta \mathbf{d} + \varepsilon (|\nabla \mathbf{d}|^2) \mathbf{d}), \]

in a bounded domain \( \Omega \subset \mathbb{R}^N \) with initial data

\[ \rho(x, 0) = \rho_0 > 0, \quad \mathbf{d}(x, 0) = \mathbf{d}_0, \quad \mathbf{v}(x, 0) = \mathbf{v}_0, \]  

and boundary conditions

\[ \nabla \rho \cdot \mathbf{n}|_{\partial \Omega} = 0, \quad \mathbf{v}|_{\partial \Omega} = 0, \quad (\mathbf{n} \cdot \nabla \mathbf{d})|_{\partial \Omega} = 0. \]

The a priori estimates will play a crucial role in the proof of existence. We consider a classical solution \((\rho, \mathbf{v}, \mathbf{d})\) of the initial-boundary problem (2.1)–(2.5) with \( \rho > 0 \).

2.1. Basic energy estimates

We first deduce some basic energy estimates without any smallness condition imposed on the initial data.

2.1.1. Maximum principle on \( |\mathbf{d}| \)

The macroscopic average of the nematic liquid crystal orientation field \( \mathbf{d} \) satisfies

\[ |\mathbf{d}| \leq 1 \text{ in } Q_T, \text{ if } |\mathbf{d}_0| \leq 1 \text{ in } \Omega \subset \mathbb{R}^N. \]  

Next, we give a proof of (2.6) for the reader’s convenience. Multiplying the \( \mathbf{d} \)-system (1.3) by \( \mathbf{d} \), we obtain

\[ \frac{1}{2} \partial_t |\mathbf{d}|^2 + \frac{1}{2} \mathbf{v} \cdot \nabla |\mathbf{d}|^2 = \theta (\Delta \mathbf{d} \cdot \mathbf{d} + \varepsilon (|\nabla \mathbf{d}|^2) |\mathbf{d}|^2). \]

From the identity \( \Delta |\mathbf{d}|^2 = 2|\nabla \mathbf{d}|^2 + 2\Delta \mathbf{d} \cdot \mathbf{d} \) it follows that

\[ \partial_t (|\mathbf{d}|^2 - 1) - \theta \Delta (|\mathbf{d}|^2 - 1) \leq - \mathbf{v} \cdot \nabla (|\mathbf{d}|^2 - 1) + 2\theta |\nabla \mathbf{d}|^2 (|\mathbf{d}|^2 - 1). \]  

Now, letting \( d = |\mathbf{d}|^2 - 1 \) and \( d_+ = \max\{d, 0\} \geq 0 \), multiplying (2.7) by \( d_+ \) and integrating over \( \Omega \), we integrate by parts and use the boundary conditions to infer that

\[ \frac{d}{dt} \int_{\Omega} d_+^2 \, dx \leq \int_{\Omega} (4\theta |\nabla \mathbf{d}|^2 + \text{div} \mathbf{v}) d_+^2 \, dx \leq 4\theta |\nabla \mathbf{d}|^2 + |\text{div} \mathbf{v}||_{L^\infty(\Omega)} \int_{\Omega} d_+^2 \, dx. \]
Assuming that \((v, d)\) satisfies the following regularity

\[
\|4\theta|\nabla d|^2 + |\text{div} v|\|_{L^1(I, L^\infty(\Omega))} < \infty,
\]

we are able to apply Gronwall’s inequality to get (2.6) immediately.

We shall see that all the couples \((v_n, d_n)\) in the third approximate solutions constructed in Section 4 satisfy the regularity required above. We remark that (2.7) becomes an equality in the two-dimensional case, and one can get by directly multiplying (2.7) with \(|d|^2 - 1\) that

\[
|d| = 1 \text{ in } Q_T, \quad \text{if } |d_0| = 1 \text{ in } \Omega \subset \mathbb{R}^2.
\]

2.1.2. Energy inequality

Integrating by parts and utilizing the boundary conditions, one easily sees that the system (2.1)–(2.2) satisfies the energy conservation:

\[
\frac{d}{dt} E_\delta(t) + \int_\Omega (\mu |\nabla v|^2 + (\lambda + \mu) |\text{div} v|^2 + \varepsilon \delta \beta \rho^{\beta - 2} |\nabla \rho|^2 + A \varepsilon \gamma \rho^{\gamma - 2} |\nabla \rho|^2) \, dx = -\nu \int_\Omega (\nabla d)^T \Delta d \cdot v \, dx,
\]

where \((\nabla d)^T \Delta d := (\partial_i d_j)_{N \times N} \Delta d\) and

\[
E_\delta(t) = \int_\Omega \left( \frac{1}{2} \frac{|m|^2}{\rho} 1_{\rho > 0} + \frac{A \rho^\gamma}{\gamma - 1} + \frac{\delta}{\beta - 1} \rho^\beta \right) \, dx \quad \text{with } m = \rho v.
\]

Multiplying (2.3) by \(-\Delta d\) and integrating by parts, we have

\[
\frac{1}{2} \frac{d}{dt} \int_\Omega |\nabla d|^2 \, dx + \theta \int_\Omega |\Delta d|^2 \, dx
\]

\[
= \int_\Omega (v \cdot \nabla d) \cdot \Delta dd \, dx + \theta \int_\Omega \left( f_\varepsilon(|\nabla d|^2) |\nabla d|^2 + 2 \sum_{1 \leq i,j \leq N} f_\varepsilon(|\nabla d|^2)(f_{\varepsilon, d_j} \nabla \partial_{x_i} d_j) \cdot v \right) \, dx
\]

\[
\leq \int_\Omega (v \cdot \nabla d) \cdot \Delta dd \, dx + \theta \int_\Omega (|\nabla d|^4 + C_0 |\nabla d|^2 |\nabla^2 d|) \, dx,
\]

which, together with (2.9), implies

\[
\frac{d}{dt} E_\delta(t) + \|\mu |\nabla v|^2 + (\lambda + \mu) |\text{div} v|^2 + \nu \theta |\Delta d|^2 + \varepsilon \delta \beta \rho^{\beta - 2} |\nabla \rho|^2 + A \varepsilon \gamma \rho^{\gamma - 2} |\nabla \rho|^2\|_{L^1(\Omega)}
\]

\[
\leq \theta \nu \int_\Omega (|\nabla d|^4 + C_0 |\nabla d|^2 |\nabla^2 d|) \, dx,
\]

where

\[
E_\delta(t) := E_\delta(\rho, m, d) := \int_\Omega \left( \frac{1}{2} \frac{|m|^2}{\rho} 1_{\rho > 0} + \frac{A}{\gamma - 1} \rho^\gamma + \frac{\delta}{\beta - 1} \rho^\beta + \nu \frac{|\nabla d|^2}{2} \right) \, dx.
\]

For the two-dimensional case, recalling \(|d| \equiv 1\) and \(f_\varepsilon(x) \equiv x\) for \(x \geq 0\), we can deduce the standard energy equality. In fact, multiplying the equations (2.3) by \(\Delta d + |\nabla d|^2 d\), integrating by parts and using the boundary conditions, one deduces that

\[
\frac{1}{2} \frac{d}{dt} \int_\Omega |\nabla d|^2 \, dx + \theta \int_\Omega |\Delta d + |\nabla d|^2 d|^2 \, dx = \int_\Omega (v \cdot \nabla d) \cdot \Delta dd \, dx,
\]
which, together with (2.9), yields

\[
\int_\Omega [\mu |\nabla v|^2 + (\lambda + \mu)|\text{div} v|^2 + \nu \theta (|\Delta d| + |\nabla d|^2)] dx + \frac{d}{dt} \mathcal{E}(t) = 0,
\]

(2.11)

whence,

\[
\int_0^t \int_\Omega [\mu |\nabla v|^2 + (\lambda + \mu)|\text{div} v|^2 + \nu \theta (|\Delta d| + |\nabla d|^2)] dx ds + \mathcal{E}(t) = \mathcal{E}(0),
\]

(2.12)

where \( \mathcal{E} := \mathcal{E}(\rho_0, m_0, d_0) \).

2.1.3. Maximum principle on lower bounds

The system (2.3) possesses the following maximum principle on nonnegative lower bounds:

\[ d_i \geq d_{\bar{d}_i} \text{ if } d_{\bar{d}_i} \geq 0 \text{ for any given constant } d_{\bar{d}_i}, \]

(2.13)

where \( 1 \leq i \leq N \), and we have denoted the \( i \)-th component of \( d \) and \( d_0 \) by \( d_i \) and \( d_{\bar{d}_i} \), respectively. This conclusion will play a crucial role in this paper, so we give its proof here for the reader’s convenience.

Letting

\[ \omega_i = d_i - d_{\bar{d}_i}, \quad \omega_i^- = \min\{\omega_i, 0\} \leq 0, \]

we can deduce from (2.3) that

\[ \partial_t \omega_i - \theta \Delta \omega_i = \theta f_\varepsilon(|\nabla d|^2)(\omega_i + d_{\bar{d}_i}) - \nabla \cdot \omega_i, \]

(2.14)

Multiplying (2.14) by \( \omega_i^- \), and using the Neumann boundary condition, (1.19) and Hölder’s inequality, we find that

\[
\frac{1}{2} \frac{d}{dt} \|\omega_i^-\|^2_{L^2(\Omega)} + \theta \|\nabla \omega_i^-\|^2_{L^2(\Omega)} = \int_\Omega \theta f_\varepsilon(|\nabla d|^2)(\omega_i + d_{\bar{d}_i}) - \nabla \cdot \omega_i \omega_i^- dx
\]

\[
= \int_\Omega \theta f_\varepsilon(|\nabla d|^2)\omega_i^- dx + \frac{1}{2} \int_\Omega \text{div} \omega_i^- dx + \int_\Omega \theta f_\varepsilon(|\nabla d|^2)d_{\bar{d}_i} \omega_i^- dx,
\]

\[
\leq \left( \|\theta |\nabla d|^2\|_{L^\infty(\Omega)} + \|\text{div} \omega_i^-\|_{L^\infty(\Omega)} \right) \|\omega_i^-\|^2_{L^2(\Omega)} + \int_\Omega \theta |\nabla d|^2 d_{\bar{d}_i} \omega_i^- dx,
\]

which, together with the fact \( d_{\bar{d}_i} \omega_i^- \leq 0 \), yields

\[
\frac{d}{dt} \|\omega_i^-\|^2_{L^2(\Omega)} \leq \left( 2 \|\theta |\nabla d|^2\|_{L^\infty(\Omega)} + \|\text{div} \omega_i^-\|_{L^\infty(\Omega)} \right) \|\omega_i^-\|^2_{L^2(\Omega)}.
\]

Hence, if we apply Gronwall’s inequality to the above inequality, we obtain

\[
\|\omega_i^- (t)\|^2_{L^2(\Omega)} \leq \|\omega_i^- (0)\|^2_{L^2(\Omega)} e^{\int_0^t \left( 2 \|\theta |\nabla d|^2\|_{L^\infty(\Omega)} + \|\text{div} \omega_i^-\|_{L^\infty(\Omega)} \right) ds} = 0,
\]

which gives (2.13).
2.2. More estimates under the small oscillation condition imposed on $d$

To obtain more estimates on $d$ under the small oscillation condition, we first introduce the well-known Nirenberg interpolation inequality (see [25, Theorem]):

Lemma 2.1. Let $u$ belong to $L^q(\mathbb{R}^N)$ and its derivatives of order $m$, $\nabla^m u$, belong to $L^r(\mathbb{R}^N)$, $1 \leq q, r \leq \infty$. Then for the derivatives $\nabla^j u, 0 \leq j < m$, the following inequality holds.

$$
\|\nabla^j u\|_{L^p(\mathbb{R}^N)} \leq C_0 \|\nabla^m u\|_{L^r(\mathbb{R}^N)}^{\alpha} \|u\|_{L^q(\mathbb{R}^N)}^{1-\alpha},
$$

(2.15)

where

$$
\frac{1}{p} = \frac{j}{n} + \alpha \left( \frac{1}{r} - \frac{m}{n} \right) + (1 - \alpha) \frac{1}{q},
$$

for all $\alpha$ in the interval

$$
\frac{j}{m} \leq \alpha \leq 1
$$

(the constant $C_0$ depends only on $n$, $m$, $j$, $q$, $r$, $\alpha$), with the following exceptional cases:

1. If $j = 0$, $rm < n$ and $q = \infty$, then we make the additional assumption that either $u$ tends to zero at infinity or $u \in L^{\tilde{q}}(\mathbb{R}^N)$ for some finite $\tilde{q} > 0$.
2. If $1 < r < \infty$, and $m - j - n/r$ is a non-negative integer, then (2.15) holds only for $\alpha$ satisfying $j/m \leq \alpha < 1$.

In addition, for a bounded domain $\Omega$ (with smooth boundary) the above assertions hold if we add to the right side (2.15) the term

$$
C(\Omega)\|u\|_{L^{\tilde{q}}(\Omega)}
$$

for any $\tilde{q} \geq 1$. All the relevant constants thus depend also on the domain.

Next, we derive more estimates on $d$ under the assumption that the initial value of $d_N$ satisfies

$$
1 - \epsilon_0 \leq d_0N \leq 1, \quad \text{for some } \epsilon_0 \in (0, 1].
$$

(2.16)

It should be noted that the constant $C'(\Omega)$ in the following deduction will denote various positive constants depending on its variable $\Omega$, but the constants $C_0$ and $\tilde{C}_1(\Omega)$–$\tilde{C}_3(\Omega)$ are fixed.

First, one gets from the maximum principle that

$$
1 - \epsilon_0 \leq d_N(x, t) \leq 1 \quad \text{for any } t > 0 \text{ and any } x \in \Omega.
$$

(2.17)

Recalling $|d| \leq 1$, that is $\sum_{i=1}^{N} d_i^2 \leq 1$, one finds that

$$
|d_i| \leq \sqrt{1 - d_N^2} < \sqrt{2\epsilon_0} \quad \text{for } 1 \leq i \leq N - 1,
$$

which combined with (2.17) leads to

$$
\|d - e_N\|_{L^\infty(\Omega)} \leq \tilde{C}_0\sqrt{\epsilon_0}.
$$

(2.18)

Thanks to Lemma 2.1 we have

$$
\|\nabla u\|_{L^4(\Omega)} \leq C(\Omega)(\|\nabla^2 u\|_{L^2(\Omega)}^{\frac{1}{2}} \|u\|_{L^\infty(\Omega)}^{\frac{1}{2}} + \|u\|_{L^4(\Omega)}) \quad \text{for } \Omega \subset \mathbb{R}^N \text{ with } N = 2 \text{ or } 3,
$$
which yields

\[ \| \nabla d \|_{L^4(\Omega)}^4 \leq C(\Omega)(\| \nabla^2 d \|_{L^2(\Omega)}^2 \| d - e_N \|_{L^\infty(\Omega)}^2 + \| d - e_N \|_{L^4(\Omega)}^4) \quad (2.19) \]

To bound the right hand of (2.19), we shall use the following elliptic estimate: There exists a constant \( \tilde{C}_1(\Omega) \), such that

\[ \| \nabla^2 d \|_{L^2(\Omega)} \leq \tilde{C}_1(\Omega)(\| \Delta d \|_{L^2(\Omega)} + \| \nabla d \|_{L^2(\Omega)}) \quad \text{for any } \nabla d \in H^1(\Omega) \quad (2.20) \]

where \( \Omega \subset \mathbb{R}^N \), which can be deduced from [26, Lemma 4.27]. Thus, putting (2.18)–(2.20) together, we conclude that

\[ \| \nabla d \|_{L^4(\Omega)}^4 \leq \tilde{C}_2(\Omega)\tilde{C}_0\epsilon_0\| \Delta d \|_{L^2(\Omega)}^2 + C(\Omega)(\| d - e_N \|_{L^\infty(\Omega)}^2 \| \nabla d \|_{L^2(\Omega)}^2 + \| d - e_N \|_{L^4(\Omega)}^4) \leq \tilde{C}_2(\Omega)\tilde{C}_0\epsilon_0\| \Delta d \|_{L^2(\Omega)}^2 + C(\Omega)(\| \nabla d \|_{L^2(\Omega)}^2 + \| d - e_N \|_{L^2(\Omega)}^2) \quad (2.21) \]

where the constant \( \tilde{C}_2(\Omega) \geq 2^{-1} \) only depends on \( \Omega \). Utilizing (2.21), (2.20), and Cauchy’s and Hölder’s inequalities, we can deduce from (2.10) that

\[
\begin{align*}
\| [\mu|\nabla v|^2 + (\lambda + \mu)|\text{div} v|^2 + \nu\theta|\Delta d|^2 + \varepsilon \delta \rho \beta^{-2}|\nabla \rho|^2 + A \varepsilon \gamma \rho^{-2}|\nabla \rho|^2 ]\|_{L^1(\Omega)} + \frac{d}{dt} \mathcal{E}_\delta(t) \\
\leq \tilde{C}_3 \theta [\tilde{C}_2(\Omega)\tilde{C}_0\epsilon_0\| \Delta d \|_{L^2(\Omega)}^2 + C(\Omega)(\| \nabla d \|_{L^2(\Omega)}^2 + \| d - e_N \|_{L^2(\Omega)}^2) \\
+ \tilde{C}_1(\Omega)(\sqrt{\tilde{C}_2(\Omega)\tilde{C}_0\epsilon_0\| \Delta d \|_{L^2(\Omega)}^2} + \sqrt{C(\Omega)}(\| \nabla d \|_{L^2(\Omega)} + \| d - e_N \|_{L^2(\Omega)})) \\
\times (\| \Delta d \|_{L^2(\Omega)} + \| \nabla d \|_{L^2(\Omega)})] \\
\leq \tilde{C}_3 \theta \left[ \left( \tilde{C}_1(\Omega)\sqrt{\tilde{C}_2(\Omega)\tilde{C}_0 + 2\tilde{C}_2(\Omega)\tilde{C}_0} \sqrt{\epsilon_0\| \Delta d \|_{L^2(\Omega)}^2} + \frac{1}{4\tilde{C}_3}\| \Delta d \|_{L^2(\Omega)}^2 \right) \\
+ C(\Omega)(\| \nabla d \|_{L^2(\Omega)}^2 + \| d - e_N \|_{L^2(\Omega)}^2) \right),
\end{align*}
\]

where \( \tilde{C}_3 \geq 2^{-1} \) denotes a constant. Now, choosing \( \epsilon_0 := \epsilon_0(\Omega) \in (0, 1] \) such that

\[ \sqrt{\epsilon_0\tilde{C}_3} \left( \tilde{C}_1(\Omega)\sqrt{\tilde{C}_2(\Omega)\tilde{C}_0 + 2\tilde{C}_2(\Omega)\tilde{C}_0} \right) \leq \frac{1}{4}, \]

we get then

\[ \frac{d}{dt} \mathcal{E}_\delta(t) + \| [\mu|\nabla v|^2 + (\lambda + \mu)|\text{div} v|^2 + \frac{\nu\theta}{2}\| \Delta d \|_{L^1(\Omega)}^2 + \varepsilon \delta \rho \beta^{-2}|\nabla \rho|^2 + A \varepsilon \gamma \rho^{-2}|\nabla \rho|^2 ]\|_{L^1(\Omega)} \leq C(\Omega)(\mathcal{E}_\delta(t) + 1), \]

which, together with Gronwall’s inequality, implies that

\[ \mathcal{E}_\delta(t) \leq C(\mathcal{E}_{\delta,0}, T, \Omega) \quad \text{for any } t \in (0, T). \]

Consequently, we can further infer that

\[
\mathcal{E}_\delta(t) + \| [\mu|\nabla v|^2 + (\lambda + \mu)|\text{div} v|^2 + \frac{\nu\theta}{2}\| \Delta d \|_{L^1(\Omega)}^2 + \varepsilon \delta \rho \beta^{-2}|\nabla \rho|^2 + A \varepsilon \gamma \rho^{-2}|\nabla \rho|^2 ]\|_{L^1(Q_t)} \leq C(\mathcal{E}_{\delta,0}, T, \Omega),
\]
where \( Q_t := (0, t) \times \Omega \) for any \( t \in (0, T) \). Moreover, from (2.19), (2.20) and (2.25) we get
\[
\| \nabla d \|_{L^2(\Omega)}^2 + \| \nabla^2 d \|_{L^2(\Omega)} + \| \nabla v \|_{L^2(\Omega)}^2 \leq C(\mathcal{E}_{\delta, 0}, T, \Omega). \tag{2.26}
\]

Finally, using (2.26), Hölder’s and Sobolev’s inequalities, we find from the equation (2.23) that
\[
\| \partial_t d \|_{L^{4/3}(\Omega)} \leq C(\| v \cdot \nabla d \|_{L^{4/3}(\Omega)} + \| \Delta d + f_{\varepsilon}(\| d \|)^2 d \|_{L^{4/3}(\Omega)}) \leq C(\mathcal{E}_{\delta, 0}, T, \Omega). \tag{2.27}
\]

Similarly, we can also deduce that
\[
\| \partial_t d \|_{L^2(\Omega)} \leq C(\mathcal{E}_{\delta, 0}, T, \Omega), \tag{2.28}
\]
where \((H^1(\Omega))^*\) denotes the dual space of \( H^1(\Omega) \).

In addition, when \( \Omega = B_R \) with \( R \geq 1 \), we can show that all the previous estimates on \((\rho, v, d)\) are independent of \( B_R \), except for \( \partial_t d \). In fact, using (2.19) and (2.20) for \( \Omega = B_1 \), and scaling the spatial variables, we can obtain
\[
\| \nabla d \|_{L^2(B_R)}^4 \leq C_0(\| \nabla^2 d \|_{L^2(B_R)}^2 \| d - e_N \|_{L^\infty(B_R)}^2 + \| d - e_N \|_{L^4(B_R)}^4),
\]
and
\[
\| \nabla^2 d \|_{L^2(B_R)} \leq C_0(\| \Delta d \|_{L^2(B_R)} + \| \nabla d \|_{L^2(B_R)}).
\]

Hence, repeating the deduction process of (2.24), and employing the above two inequalities, one can have the following estimate:
\[
\frac{d}{dt} \mathcal{E}_{\delta}(t) + \left( \mu |\nabla v|^2 + (\lambda + \mu) |\text{div} v|^2 + \frac{\nu \theta}{2} |\Delta d|^2 + \varepsilon \delta \beta |\rho|^{\beta-2} |\nabla \rho|^2 + A \varepsilon \gamma |\rho|^{-2} |\nabla \rho|^2 \right) \leq C(\| \nabla d \|_{L^2(\Omega)}^2 + \| \nabla^2 d \|_{L^2(\Omega)}^2), \tag{2.29}
\]
where the constant \( C \) is independent of \( \Omega = B_R \) for any \( R \geq 1 \). On the other hand, using (2.18), (2.23) and Cauchy-Schwarz’s inequality, we see from (2.3) that
\[
\frac{1}{2} \frac{d}{dt} \| d - e_N \|_{L^2(B_R)}^2 + \theta \| \nabla d \|_{L^2(B_R)}^2 = \theta \int_{B_R} f_{\varepsilon}(\| d \|^2)(1 - d_N) + \frac{1}{2} |d - e_N|^2 \text{div} v \ dx
\]
\[
\leq \theta \bar{C}_0 \sqrt{\varepsilon_{\delta}} \| \nabla d \|_{L^2(B_R)}^2 + \frac{\theta \bar{C}_0 \sqrt{\varepsilon_{\delta}}}{2} \| \text{div} v \|_{L^2(B_R)} \| d - e_N \|_{L^2(B_R)}
\]
\[
\leq \frac{\theta}{2} \| \nabla d \|_{L^2(B_R)}^2 + \frac{\mu}{2} \| \text{div} v \|_{L^2(B_R)} + C(\theta, \mu^{-1}) \| d - e_N \|_{L^2(B_R)}^2.
\]

Adding the above estimate to (2.24), we get
\[
\left| \frac{\mu |\nabla v|^2 + \theta |\nabla^2 d|^2}{2} + (\lambda + \mu) |\text{div} v|^2 + \frac{\nu \theta}{2} |\Delta d|^2 + \varepsilon \delta \beta |\rho|^{\beta-2} |\nabla \rho|^2 + A \varepsilon \gamma |\rho|^{-2} |\nabla \rho|^2 \right) \leq C \left( \mathcal{E}_{\delta}(t) + \| d(t) - e_N \|_{L^2(B_R)}^2 \right),
\]
which, together with Gronwall’s inequality, yields
\[
\left| \frac{\mu |\nabla v|^2 + \theta |\nabla^2 d|^2}{2} + (\lambda + \mu) |\text{div} v|^2 + \frac{\nu \theta}{2} |\Delta d|^2 + \varepsilon \delta \beta |\rho|^{\beta-2} |\nabla \rho|^2 + A \varepsilon \gamma |\rho|^{-2} |\nabla \rho|^2 \right|_{L^1(Q_t)} + \mathcal{E}_{\delta}(t) + \| d(t) - e_N \|_{L^2(B_R)} \leq C(\mathcal{E}_{\delta, 0}, T, \| d_0 - e_N \|_{L^2(\Omega)}), \quad \text{for all } t \in I.
\]

Summing up the above estimates, we conclude that
Proposition 2.1. Let $N = 2$ or $3$, and $\Omega \subset \mathbb{R}^N$ be a bounded domain and $\epsilon_0 := \epsilon_0(N, \Omega) \in (0, 1]$ satisfy \[(2.23)\]. Then the initial-boundary value problem \[(2.1) - (2.7)\] enjoys the following a priori estimates, provided the initial data $d_0, N$ satisfies $1 - d_0 < \epsilon_0$ and $|d_0| \leq 1$.

\[
|d| \leq 1, \quad \text{in particular, } |d| = 1 \text{ if } |d_0| = 1 \text{ for } N = 2, \\
\|d - e_N\|_{L^\infty(\Omega)} \leq C_0 \sqrt{\epsilon_0}, \quad (2.30) \\
\sup_{t \in I} \|\langle d(t) - e_N, \nabla d(t) \rangle\|_{L^2(\Omega)} + \|\nabla d\|_{L^2(Q_T)} + \|\nabla^2 d\|_{L^2(Q_T)} \leq C(\mathcal{E}_{\delta, 0}, T, \Omega), \quad (2.32) \\
\sup_{t \in I} \|\nabla v(t)\|_{L^2(\Omega)} + \|\rho(t)\|_{L^\infty(\Omega)} + \|\nabla v\|_{L^2(I, L^2(\Omega))} \leq C(\mathcal{E}_{\delta, 0}, T, \Omega), \quad (2.33) \\
\|\partial_t d\|_{L^{3/2}(I, L^2(\Omega))} + \|\partial_t d\|_{L^2(I, H^1(\Omega)^*)} \leq C(\mathcal{E}_{\delta, 0}, \Omega, T). \quad (2.34)
\]

In particular, if $\Omega = B_R$ with $R \geq 1$, then $\epsilon_0$ can be chosen to be independent of the domain $\Omega = B_R$ for any $R \geq 1$, and the constant $C(\mathcal{E}_{\delta, 0}, T, \Omega)$ in \[(2.32)\] and \[(2.33)\] can be replaced by a constant $C(\mathcal{E}_{\delta, 0}, T, \|d_0 - e_N\|_{L^2(\Omega)})$ independent of $B_R$.

3. Strong solvability of sub-systems in the third approximate problem

Before proving the unique solvability of the third approximate problem, we shall introduce two preliminary results. The first result is concerned with the global solvability of the Neumann problem on the equation \[(2.1)\] for given $v$.

Proposition 3.1. Let $0 < \alpha < 1$, $\Omega$ be a bounded domain of class $C^{2,\alpha}$, and $\rho_0 := \rho_0(x)$ satisfy

\[
\rho_0 \in W^{1,\infty}(\Omega), \quad 0 < \underline{\rho} \leq \rho_0 \leq \bar{\rho} < \infty.
\]

Then, there exists a unique mapping

\[
\mathcal{S}_{\rho_0} : L^\infty(I, W^{1,\infty}_0(\Omega)) \rightarrow C^0(\bar{I}, H^1(\Omega)),
\]

such that

1. $\mathcal{S}_{\rho_0}(v)$ belongs to the function class

   \[
   \mathcal{R}_T := \{\rho \in L^2(I, W^{2,q}(\Omega)) \cap C^0(\bar{I}, W^{1,q}(\Omega)), \partial_t \rho \in L^2(I, L^q(\Omega))\}, \quad 1 < q < \infty.
   \]

2. The function $\rho = \mathcal{S}_{\rho_0}(v)$ satisfies the following initial-boundary problem:

   \[
   \begin{cases}
   \partial_t \rho + \text{div}(\rho \nabla) = \varepsilon \Delta \rho \quad \text{a.e. in } Q_T, \\
   \rho(x, 0) = \rho_0 \quad \text{a.e. in } \Omega, \\
   \nabla \rho \cdot n|_{\partial \Omega} = 0, \quad \text{in the sense of traces a.e. in } I.
   \end{cases}
   \quad (3.1)
\]

3. $\mathcal{S}_{\rho_0}(v)$ is pointwise bounded, i.e.,

   \[
   \rho e^{-\int_0^t |v|_{W^{1,\infty}(\Omega)}^2 \, dt} \leq \mathcal{S}_{\rho_0}(v)(x, t) \leq \bar{\rho} e^{\int_0^t |v(t)|_{W^{1,\infty}(\Omega)}^2 \, dt}, \quad t \in \bar{I}, \quad \text{for a.e. } x \in \Omega. \quad (3.2)
\]

4. If $\|v\|_{L^\infty(I, W^{1,\infty}(\Omega))} \leq \kappa_v$, then

   \[
   \|\mathcal{S}_{\rho_0}(v)\|_{L^\infty(I_t, H^1(\Omega))} \leq C(\Omega)\|\rho_0\|_{H^1(\Omega)} e^{\frac{C(\Omega)}{\varepsilon} (\kappa_v + \kappa_v^2) t}, \\
   \|\nabla^2 \mathcal{S}_{\rho_0}(v)\|_{L^2(Q_t)} \leq \frac{C(\Omega)}{\varepsilon} \sqrt{t} \|\rho_0\|_{H^1(\Omega)} e^{\frac{C(\Omega)}{\varepsilon} (\kappa_v + \kappa_v^2) t}, \\
   \|\partial_t \mathcal{S}_{\rho_0}(v)\|_{L^2(Q_t)} \leq C(\Omega) \sqrt{t} \|\rho_0\|_{H^1(\Omega)} e^{\frac{C(\Omega)}{\varepsilon} (\kappa_v + \kappa_v^2) t},
   \quad (3.3)
\]

for any $t \in \bar{I}$, where $I_t := (0, t)$, and $Q_t := \Omega \times I_t$. 

(5) $\mathcal{S}_{\rho_0}(v)$ depends continuously on $v$, i.e.,

$$
\|\mathcal{S}_{\rho_0}(v_1) - \mathcal{S}_{\rho_0}(v_2)\|_{L^2(\Omega)} \leq G(\kappa_v, \varepsilon, T) \|\rho_0\|_{H^1(\Omega)} \|v_1 - v_2\|_{L^2(I_t, W^{1,\infty}(\Omega))},
$$

$$
\|\partial_t[\mathcal{S}_{\rho_0}(v_1) - \mathcal{S}_{\rho_0}(v_2)](t)\|_{L^2(Q_t)} \leq G(\kappa_v, \varepsilon, T) \sqrt{t} \|\rho_0\|_{H^1(\Omega)} \|v_1 - v_2\|_{L^2(I_t, W^{1,\infty}(\Omega))}
$$

for any $t \in I$, and for any $\|v_1\|_{L^\infty(I, W^{1,\infty}(\Omega))} \leq \kappa_v$ and $\|v_2\|_{L^\infty(I, W^{1,\infty}(\Omega))} \leq \kappa_v$. The constant $G$ is nondecreasing in the first variable and may depend on $\Omega$.

**Proof.** Please refer to [26, Proposition 7.39] and [10, Proposition 3.1].

The second result is on the local solvability of the Neumann problem for the system (2.5) with given $v$.

**Proposition 3.2.** Let $0 < \alpha < 1$, $\Omega \subset \mathbb{R}^N$ be a bounded domain of class $C^{2,\alpha}$, $d_0 := d_0(x) \in H^2(\Omega)$, $K > 0$ and

$$
v \in V_K := \left\{ v \mid \|v\|_V := \left( \|v\|^2_{C^0(I, H^2(\Omega))} + \|\partial_t v\|^2_{L^2(I, H^1(\Omega))} \right)^{\frac{1}{2}} \leq K \right\}.
$$

Then there exist a finite time

$$
T^K_d := h_1(K, \|d_0\|_{H^2(\Omega)}, \Omega) \in (0, \min\{1, T\}),
$$

and a corresponding unique mapping

$$
\mathcal{D}^K_{d_0} : V_K (\text{with } T^K_d \text{ in place of } T) \to C^0(I^K_d, H^2(\Omega)),
$$

where $h_1$ is nonincreasing in its first two variables and $Q_{T^K_d} := \Omega \times I^K_d := \Omega \times (0, T^K_d)$, such that

1. $\mathcal{D}^K_{d_0}(v)$ belongs to the following function class

$$
\mathcal{R}^{K}_{T^K_d} := \{ d \mid d \in L^2(I^K_d, H^2(\Omega)), \quad \partial^2_t d \in L^2(I^K_d, (H^1(\Omega))^*), \quad \partial_t d \in C^0(I^K_d, L^2(\Omega)) \cap L^2(I^K_d, H^1(\Omega)) \}. \tag{3.4}
$$

2. $d = \mathcal{D}^K_{d_0}(v)$ satisfies the following initial-boundary problem:

$$
\begin{align*}
&\partial_t d + v \cdot \nabla d = \theta(\Delta d + f_\varepsilon(\|\nabla d\|^2)d) \quad \text{a.e. in } Q_{T^K_d}, \\
&d(x, 0) = d_0 \quad \text{a.e. in } \Omega, \\
&(n \cdot \nabla d)|_{\partial \Omega} = 0 \quad \text{in the sense of trace a.e. in } I^K_d, \tag{3.5}
\end{align*}
$$

where $f_\varepsilon \geq 0$ is defined by (2.19).

3. $\mathcal{D}^K_{d_0}(v)$ enjoys the following estimate:

$$
\|\mathcal{D}^K_{d_0}(v)\|_{V^\ast} \leq C(K, \|d_0\|_{H^2(\Omega)}, \Omega),
$$

where

$$
\|\cdot\|_{V^\ast} := \left( \|\partial_t \cdot\|^2_{L^2(I^K_d, H^1(\Omega))} + \|\partial_t \cdot\|^2_{L^\infty(I^K_d, L^2(\Omega))} + \|\cdot\|^2_{L^\infty(I^K_d, H^2(\Omega))} + \|\cdot\|^2_{L^2(I^K_d, H^3(\Omega))} \right)^{\frac{1}{2}}.
$$

Moreover (recalling (2.30) and (2.13)),

$$
|d| \leq 1, \quad \text{in particular, } |d| \equiv 1 \text{ if } |d_0| \equiv 1 \text{ for } N = 2; \\
d_i(t) \geq d_{0i} \quad \text{for any } t, \quad \text{if } d_{0i} \geq d_{0i} \geq 0 \quad \text{where } 1 \leq i \leq N.
$$
(4) $D^K_{d_0}(v)$ continuously depends on $v$ in the following sense:

$$
\|D^K_{d_0}(v_1) - D^K_{d_0}(v_2)\|_{L^2(\Omega)} + \|\nabla(D^K_{d_0}(v_1) - D^K_{d_0}(v_2))\|_{L^2(Q_t)} \leq \sqrt{t}C(K, \|d_0\|_{H^2(\Omega)}, \Omega)\|v_1 - v_2\|_{L^\infty(I_t, L^2(\Omega))}
$$

\[\text{for any } t \in I^K_d.\]

Proof. The above results on the two-dimensional non-homogeneous Dirichlet problem have been shown under the higher regularity condition $d_0 \in H^3(\Omega)$ in [10], where the appeared constants depend on $\|\nabla^3 d_0\|_{L^2(\Omega)}$. For the multi-dimensional Neumann problem considered here, we can obtain the same results by slightly modifying the proof in [10, Proposition 3.2] as follows.

(1) We utilize a semi-discrete Galerkin method from [26, Proposition 7.39] to construct a solution of the linearized Neumann problem, and a semi-discrete Galerkin method from [4] to construct a solution of the linearized non-homogeneous Dirichlet problem in [10].

(2) We make use of the elliptic estimate (2.20) for the Neumann boundary condition to replace the corresponding elliptic estimate for the non-homogeneous boundary condition in [10]. This is the reason why the constant $C$ in Proposition 3.2 is independent of $\|\nabla^3 d_0\|_{L^2(\Omega)}$.

(3) We use the following three-dimensional interpolation inequalities in Lemma 2.1

$$
\|u\|_{L^\infty(\Omega)} \leq C(\Omega)(\|\nabla^2 u\|_{L^2(\Omega)}^{3/2} \|u\|_{L^2(\Omega)}^{1/2} + \|u\|_{L^2(\Omega)}) \quad \text{for } u \in H^2(\Omega),
$$

$$
\|u\|_{L^4(\Omega)} \leq C(\Omega)(\|\nabla u\|_{L^2(\Omega)}^{3/2} \|u\|_{L^2(\Omega)}^{1/2} + \|u\|_{L^2(\Omega)}) \quad \text{for } u \in H^1(\Omega),
$$

$$
\|u\|_{L^8(\Omega)} \leq \sqrt{2}(\|\nabla u\|_{L^2(\Omega)}^{3/2} \|u\|_{L^2(\Omega)}^{1/2}) \quad \text{for } u \in H^1_0(\Omega) \quad (\text{cf. } [27, \text{Lemma 3.5}]),
$$

(3.11)

to replace the following two-dimensional interpolation inequalities

$$
\|u\|_{L^\infty(\Omega)} \leq C(\Omega)\|u\|_{H^2(\Omega)} \|u\|_{L^2(\Omega)} \quad \text{for } u \in H^2(\Omega),
$$

$$
\|u\|_{L^4(\Omega)} \leq C(\Omega)\|u\|_{H^1(\Omega)} \|u\|_{L^2(\Omega)} \quad \text{for } u \in H^1(\Omega),
$$

$$
\|u\|_{L^8(\Omega)} \leq \sqrt{2}\|\nabla u\|_{L^2(\Omega)}^{3/2} \|u\|_{L^2(\Omega)}^{1/2} \quad \text{for } u \in H^1_0(\Omega).
$$

The purpose of using (3.9)–(3.11) above in [10] is to construct the term $t^\alpha$ with $\alpha \in (0, 1)$. When we use (3.6)–(3.8) in showing solvability of the three-dimensional problem, we can still construct the term $t^\alpha$ with possible different $\alpha \in (0, 1)$, with the help of Young’s inequality in addition.

Finally, we remark that the conclusions in Proposition 3.2 still hold in the 3D case, provided $f_\varepsilon(x) \equiv x$. □

4. Unique solvability of the third approximate problem

In order to obtain a weak solution of the initial-boundary problem (11.1)–(11.5), we first show the existence of solutions to the third approximate problem of the original three-dimensional problem (1.1)–(1.5):

$$
\partial_t \rho + \text{div}(\rho v) = \varepsilon \Delta \rho,
$$

$$
\partial_t d + v \cdot \nabla d = \theta(\Delta d + f_\varepsilon(|\nabla d|^2)d),
$$

$$
\int_\Omega (\rho v)(t) \cdot \Psi dx - \int_\Omega m_0 \cdot \Psi dx = \int_0^t \int_\Omega \left[ \mu \Delta v + (\mu + \lambda)\nabla \text{div} v - A\nabla \rho^\gamma - \delta \nabla \rho^\beta - \varepsilon(\nabla \rho \cdot \nabla v)
\right.

- \text{div}(\rho v \otimes v) - \nu \text{div}
\left( \nabla d \otimes \nabla d - \frac{|\nabla d|^2 I}{2} \right) \right] \cdot \Psi dx ds
$$

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for all \( t \in I \) and any \( \Psi \in X_n \), with boundary conditions
\[
\nabla \rho \cdot \mathbf{n}|_{\partial \Omega} = 0, \quad \mathbf{v}|_{\partial \Omega} = 0, \quad (\mathbf{n} \cdot \nabla)|_{\partial \Omega} = 0, \tag{4.4}
\]
and modified initial data
\[
\rho(x, 0) = \rho_0 \in W^{1,\infty}(\Omega), \quad 0 < \rho_0 \leq \rho_0 \leq \rho < \infty, \tag{4.5}
\]
\[
d(x, 0) = d_0 \in H^2(\Omega), \quad \mathbf{v}(x, 0) = \mathbf{v}_0 \in X_n, \tag{4.6}
\]
where \( \mathbf{n} \) denotes the outward normal to \( \partial \Omega \), and \( \varepsilon, \delta, \beta, \rho, \tilde{\rho} > 0 \) are constants.

Here we briefly introduce the finite dimensional space \( \mathbb{X}_n \). We know from [26, Section 7.4.3] that there exist countable sets
\[
\{\lambda_i\}_{i=1}^{\infty}, \quad 0 < \lambda_1 \leq \lambda_2 \leq \ldots, \quad \text{and}
\]
\[
\{\Psi_i\}_{i=1}^{\infty} \subset W^{1,p}(\Omega) \cap W^{2,p}(\Omega), \quad 1 \leq p < \infty,
\]
such that
\[
- \mu \Delta \Psi_i - (\mu + \lambda) \nabla \text{div} \Psi_i = \lambda_i \Psi_i, \quad i = 1, 2, \ldots,
\]
and \( \{\Psi_i\}_{i=1}^{\infty} \) is an orthonormal basis in \( L^2(\Omega) \) and an orthogonal basis in \( H_0^1(\Omega) \) with respect to the scalar product \( \int_\Omega [\mu \partial_i \Psi \cdot \partial_j \Psi + (\mu + \lambda) \nabla \text{div} \Psi \nabla \text{div} \Psi] d\mathbf{x} \). We define a \( n \)-dimensional Euclidean space \( \mathbb{X}_n \) with scalar product \( < \cdot, \cdot > \) by
\[
\mathbb{X}_n = \text{span}\{\Psi_i\}_{i=1}^{n}, \quad < u, v > = \int_\Omega u \cdot v d\mathbf{x}, \quad u, v \in \mathbb{X}_n,
\]
and denote by \( \mathcal{P}_n \) the orthogonal projection of \( L^2(\Omega) \) onto \( \mathbb{X}_n \).

**4.1. Local existence**

With the help of Propositions 3.1 and 3.2 one can establish the local existence of a unique solution to the third approximate problem (4.1)–(4.6) by a contraction mapping argument. To this purpose, we rewrite the approximate momentum equations (4.3) as an operator form.

Given
\[
\rho \in C^0(\bar{I}, L^1(\Omega)), \quad \partial_t \rho \in L^1(\Omega_T), \quad \text{ess inf}_{(x,t) \in \Omega_T} \rho(x, t) \geq \rho > 0,
\]
we define, for all \( t \in \bar{I} \), that
\[
\mathcal{M}_{\rho(t)} : \mathbb{X}_n \to \mathbb{X}_n
\]
by
\[
< \mathcal{M}_{\rho(t)} v, w > := \int_\Omega \rho(t) v \cdot w d\mathbf{x}, \quad v, w \in \mathbb{X}_n.
\]

Recall that all norms on \( \mathbb{X}_n \) are equivalent, in particular,
\[
W^{k_1,p_1}(\Omega) \text{ and } (W^{k_2,p_2}_0(\Omega))^* \text{-norms are equivalent on } \mathbb{X}_n, \tag{4.7}
\]
where \( (W^{k_2,p_2}_0(\Omega))^* \) denotes the dual space of \( (W^{k_2,p_2}_0(\Omega)) \), \( k_1 \) and \( k_2 \) are integers, and \( 0 \leq k_2 < \infty, 1 \leq p_2 \leq \infty, 0 \leq k_1 \leq 1 \) and \( 1 \leq p_1 \leq \infty \) (or \( k_1 = 2, 1 \leq p_1 < \infty \)). Note that this property of equivalent norms plays an important role in the estimates of velocity \( \mathbf{v} \).

It is easy to see that
\[
\| \mathcal{M}_{\rho(t)} \|_{\mathcal{L}(\mathbb{X}_n, \mathbb{X}_n)} \leq c(n) \int_\Omega \rho(t) d\mathbf{x}, \quad t \in \bar{I}. \tag{4.8}
\]
On the other hand, we easily verify that $M_{\rho(t)}^{-1}$ exists for all $t \in \bar{I}$ and

$$
\|M_{\rho(t)}^{-1}\|_{\mathcal{L}(X_n, X_n)} \leq \frac{1}{L},
$$

(4.9)

where $\mathcal{L}(X_n, X_n)$ denotes the set of all continuous linear operators mapping $X_n$ to $X_n$. By virtue of (4.8) and (4.9), we have

$$
\|M_{\rho_{\rho_1(t)}}^{-1}M_{\rho_{\rho_1(t)}}^{-1}\|_{\mathcal{L}(X_n, X_n)} \leq \frac{c(n)}{L^2} \|\rho_1(t)\|_{L^1(\Omega)}, \quad t \in \bar{I}.
$$

(4.10)

Next, we shall look for $T_n^* \subset (0, T_d^K]$ and

$$
\mathbf{v} \in \mathbb{A} := \{\mathbf{v} \in C(T_n^*, X_n) \mid \partial_t \mathbf{v} \in L^2(T_n^*, X_n)\}, \quad I_n^* := (0, T_n^*) \subset (0, T_d^K)
$$

with $\|\mathbf{v}\|_{C(I_n^*, H^2(\Omega))} + \|\partial_t \mathbf{v}\|_{L^2(I_n^*, H^1(\Omega))} \leq K$ for some $K$, such that

$$
\int_\Omega (\rho \mathbf{v})(t) \cdot \Psi dx - \int_\Omega m_0 \cdot \Psi dx
= \int_0^t \int_\Omega \left[ \mu \Delta \mathbf{v} + (\mu + \lambda) \nabla \text{div} \mathbf{v} - A \nabla \rho^\gamma - \delta \nabla \rho^\beta - \varepsilon (\nabla \rho \cdot \nabla \mathbf{v})
- \text{div}(\rho \mathbf{v} \otimes \mathbf{v}) - \nu \text{div} \left( \nabla \mathbf{d} \otimes \nabla \mathbf{d} - \frac{\|\nabla \mathbf{d}\|^2}{2} \right) \right] \cdot \Psi dx ds
$$

(4.11)

for all $t \in [0, T_n]$ and any $\Psi \in X_n$, where $\rho(t) = [\mathcal{S}_{\rho_0}(\mathbf{v})](t)$ is the solution of the problem (3.1) constructed in Proposition 3.1 and $d(t) = D_{\rho_0}^K(\mathbf{v})(t)$ is the solution of the problem (3.5) constructed in Proposition 3.2. By the regularity of $(\rho, d)$ in Propositions 3.1 and 3.2 and the operator $M_{\rho(t)}$, the equations (4.11) can be rephrased as

$$
\mathbf{v}(t) = M_{[\mathcal{S}_{\rho_0}(\mathbf{v})](t)}^{-1} \left( \mathcal{P} m_0 + \int_0^t \mathcal{P} [\mathcal{N}(\mathcal{S}_{\rho_0}(\mathbf{v}), \mathbf{v}, D_{d_0}(\mathbf{v}))] ds \right)
$$

with $m_0 = (\rho \mathbf{v})(0)$, where $\mathcal{P} := \mathcal{P}_n$ is the orthogonal projection of $L^2(\Omega)$ to $X_n$, and

$$
\mathcal{N}(\rho, v, d) = \mu \Delta \mathbf{v} + (\mu + \lambda) \nabla \text{div} \mathbf{v} - A \nabla \rho^\gamma - \delta \nabla \rho^\beta - \varepsilon (\nabla \rho \cdot \nabla \mathbf{v})
- \text{div}(\rho \mathbf{v} \otimes \mathbf{v}) - \nu \text{div} \left( \nabla \mathbf{d} \otimes \nabla \mathbf{d} - \frac{\|\nabla \mathbf{d}\|^2}{2} \right).
$$

Moreover, one has

$$
\partial_t \mathbf{v}(t) = M_{[\mathcal{S}_{\rho_0}(\mathbf{v})](t)}^{-1} \mathcal{N}_{\rho_0}(\mathbf{v})(t) \cdot M_{[\mathcal{S}_{\rho_0}(\mathbf{v})](t)}^{-1} \left( \mathcal{P} m_0 + \int_0^t \mathcal{P} [\mathcal{N}(\mathcal{S}_{\rho_0}(\mathbf{v}), \mathbf{v}, D_{d_0}(\mathbf{v}))] ds \right)
$$

(4.12)

The authors in [10], Section 4.3 have established the local existence of the problem (4.1)–(4.6) with non-homogenous boundary condition in place of the Neumann boundary condition for the system (4.2) by using a contraction mapping argument. In view of the proof in [10] and the previous preliminary results, we can immediately find that the results in [10], Section 4.3 can be directly generalized to our problem (4.1)–(4.6) without essential changes in arguments. Thus, we have the following conclusion.
Proposition 4.1. There exist $K := K(||\mathcal{P}m_0||_{\mathcal{A}}, ||v(0)||_{\mathcal{A}}, \rho^{-1}) > 0$ and $T^*_n \leq T$, such that

$$\mathcal{T} : \mathcal{A} \to \mathcal{A}, \quad \mathcal{T}(w) := \mathcal{M}^{-1}_{(w)} \left\{ \mathcal{P}m_0 + \int_0^T [\mathcal{P}N(\mathcal{I}(w), \mathcal{P}(w))]ds \right\},$$

maps

$$B_{K, \tau_0} = \{ w \in \mathcal{A} \mid ||w||_{C(I_{\tau_0}, X_n)} + ||\partial_t w||_{L^2(I_{\tau_0}, X_n)} \leq K \} \subset C(I_{\bar{T}_n}, X_n)$$

into itself and is contractive for any $0 < \tau_0 \leq T^*_n$, where one can take $X_n = H^1_0(\Omega)$, $T^*_n$ has the form

$$0 < T^*_n = h_2(\bar{\rho}, ||\rho_0||_{H^1(\Omega)}, ||d_0||_{H^2(\Omega)}, \frac{\rho}{\rho_0} K, T, n),$$

and $h_2$ is nonincreasing in its first three variables and nondecreasing in the fourth variable.

Therefore, the map $\mathcal{T}$ possesses in $B_{K, T^*_n}$ a unique fixed point $v$ which satisfies (4.11). Thus, we have a solution $(\rho = \mathcal{I}(v), v, \mathcal{P}(v))$ which is defined in $Q_{T^*_n}$ and satisfies the initial-boundary value problem (4.1)–(4.6) for each given $n$. This means that we can find a unique maximal solution $(\rho_n, v_n, d_n)$ defined in $[0, T^*_n) \times \Omega$ for each given $n$, where $T_n \leq T$.

4.2. Global existence

In order to show the maximal time $T_n = T$ for any $n$, it suffices to derive uniform bounds for $\rho_n$, $v_n$, $d_n$ and $\mathcal{P}(\rho_n v_n)$. However, we need to impose an additional smallness condition as in (2.16) to get the uniform boundedness of $||d_n||_{L^\infty(I_n, H^2(\Omega))}$. For simplicity, we denote

$$(\rho, v, d, \mathcal{P}m) := (\rho_n, v_n, d_n, \mathcal{P}(\rho_n v_n)).$$

We mention that in the estimates that follow, the letters $G_1(\ldots)$, $G_2(\ldots)$ and $G(\ldots)$ will denote various positive constants depending on its variables.

First, one has the energy estimates as in Proposition 2.4. In fact, by virtue of the regularity of $(\rho, v, d)$, we can deduce that $(\rho, v, d)$ satisfies (2.10) for $N \geq 2$, and (2.11) and (2.12) for $N = 2$. Then, letting $\epsilon_0$ satisfy (2.23) and the initial data $d_0$ satisfy

$$1 - \epsilon_0(\Omega) \leq d_0 \leq 1$$

arguing in the same manner as in the derivation of (2.30)–(2.34), we get

$$|d| \leq 1, \quad \text{in particular, } |d| = 1 \text{ if } |d_0| = 1 \text{ for } N = 2,$$

$$||d - e_N||_{L^\infty(\Omega)} \leq C_0 \sqrt{\epsilon_0},$$

$$||\sqrt{\epsilon_0} v||_{L^\infty(I_n, L^2(\Omega))} \leq G(\epsilon\delta_0, T, \Omega),$$

$$||\nabla d||_{L^2(Q_{T_n})} + ||\nabla d||_{L^4(Q_{T_n})} + ||\nabla d||_{L^\infty(I_n, L^2(\Omega))} \leq G(\epsilon\delta_0, T, \Omega),$$

$$||\rho||_{L^\infty(I_n, L^7(\Omega))} + ||\partial_t d||_{L^{4/3}(I_n, L^2(\Omega))} + ||\partial_t d||_{L^2(I_n, H^1(\Omega)^s)} \leq G(\epsilon\delta_0, T, \Omega).$$

In particular, if $\Omega = B_R$ with $R \geq 1$, the constant $\epsilon_0$ can be chosen to be independent of $\Omega$, and the above constant $G(\epsilon\delta_0, T, \Omega)$ can be replaced by a constant $G(\epsilon\delta_0, T, ||d_0 - e_N||_{L^2(\Omega)})$ independent of $\Omega$.

With the help of (4.15) and (4.16), we can derive more uniform bounds on $(\rho, v)$. Using (3.2) and (4.16), thanks to the norm equivalence on $X_n$ (see (1.7)), we find that

$$G_1(\rho, \epsilon\delta_0, n, T) \leq \rho \leq G_2(\bar{\rho}, \epsilon\delta_0, n, T, \Omega),$$

(4.18)
from which, (4.15) and (4.7), it follows that
\[
\|v\|_{C^0(I_0, x_0)} \leq G(\rho, \mathcal{E}_{\delta, 0}, n, T, \Omega) \tag{4.19}
\]
and
\[
\|\mathcal{P}(\rho v)\|_{C^0(I_0, x_0)} \leq c(n)\|\rho v\|_{C^0(I_0, L^2(\Omega))} \leq G(\rho, \mathcal{E}_{\delta, 0}, n, T, \Omega). \tag{4.20}
\]
Applying (4.19) to (3.3), one gets
\[
P(1) \text{ The two-dimensional case: noting that}
\]
\[
\|\rho\|_{C^0(I_0, H^1(\Omega))} \leq G(\rho, \mathcal{E}_{\delta, 0}, n, T, \Omega). \tag{4.21}
\]
Utilizing (4.9), (4.10), (4.17), (4.18)–(4.20) and (4.7), we obtain from (4.12) that
\[
\|\partial_t v\|_{L^2(I_0, x_0)} \leq G(\rho, \mathcal{E}_{\delta, 0}, n, T, \Omega), \|d\|_{H^2(\Omega)}, \|x\|_{\mathcal{E}_{\delta, 0}}, n, T, \Omega). \tag{4.22}
\]
Therefore, we have shown the uniform boundedness of \(\|\rho\|_{L^\infty(Q_T), H^1(\Omega)}\), \(\|\rho\|_{C^0(I_0, x_0)}\), \(\|v\|_{C^0(I_0, x_0)}\), \(\mathcal{P}(\rho v)\|_{C^0(I_0, x_0)}\) and \(\|\partial_t v\|_{L^2(I_0, x_0)}\). It remains to show the uniform boundedness of \(\|d\|_{L^\infty(I_0, H^2(\Omega))}\).

Differentiating (4.2) with respect to \(t\), multiplying the resulting equations by \(\partial_t d\), recalling \(|d| \leq 1\), we integrate by parts to infer that
\[
\frac{d}{dt} \int_\Omega |\partial_t d|^2 dx = 2 \int_\Omega \partial_t d \cdot (\theta \Delta \partial_t d + \theta f_\varepsilon |\nabla d|^2 \partial_t d + \theta f'_\varepsilon |\nabla d|^2 (\nabla d : \nabla \partial_t d) d - \partial_t v \cdot \nabla d - v \cdot \nabla \partial_t d \cdot \partial_t d) dx
\]
\[
= -2 \theta |\nabla \partial_t d|^2_{L^2(\Omega)} + 2 \int_\Omega [(\partial_t \nabla v) \cdot \partial_t d + \partial_t v \cdot \nabla \partial_t d - v \cdot \nabla \partial_t d \cdot \partial_t d]|d| dx
\]
\[
+ 2 \theta \int_\Omega \partial_t d \cdot |f_\varepsilon |\nabla d|^2 |\partial_t d + f'_\varepsilon |\nabla d|^2 (\nabla d : \nabla \partial_t d) d|d| dx
\]
\[
\leq -\theta |\nabla \partial_t d|^2_{L^2(\Omega)} + C(\theta) (\|\partial_t v\|_{H^1(\Omega)}^2 + \|v\|_{L^\infty(Q_T)}^2 \|\partial_t d\|^2_{L^2(\Omega)} + \|\partial_t d\|^2_{L^2(\Omega)})
\]
\[
+ C(\theta) (\|\partial_t d\|^2_{L^2(\Omega)} + \|\partial_t d\|^2_{L^2(\Omega)} (1 + \|\nabla d\|^2_{L^2(\Omega)})) \tag{4.23}
\]
where the last term on the right-hand side of (4.23) can be bounded as follows.

(1) The two-dimensional case: noting that \(f_\varepsilon(x) = x\) for the 2D case, we make use of Lemma 2.1 and Hölder’s and Young’s inequalities to see that
\[
C(\theta) (\|\partial_t d\|^2_{L^2(\Omega)}) \leq C(\theta) (\|\partial_t d\|^2_{L^2(\Omega)} + \|\partial_t d\|^2_{L^2(\Omega)}) \leq C(\theta) \|\partial_t d\|^2_{L^2(\Omega)} \leq C(\theta) (\|\partial_t d\|^2_{L^2(\Omega)} + \|\partial_t d\|^2_{L^2(\Omega)})(1 + \|\nabla d\|^2_{L^2(\Omega)}) \tag{4.24}
\]

(2) The three-dimensional case: recalling the definition of \(f_\varepsilon(x)\) in (1.19), we use (3.7), and Hölder’s and Young’s inequalities to get
\[
C(\theta) (\|\partial_t d\|^2_{L^2(\Omega)}) \leq C(\theta) (\|\partial_t d\|^2_{L^2(\Omega)} + \|\partial_t d\|^2_{L^2(\Omega)}) \leq C(\theta) (\|\partial_t d\|^2_{L^2(\Omega)} + \|\partial_t d\|^2_{L^2(\Omega)})(1 + \|\nabla d\|^2_{L^2(\Omega)}) \tag{4.25}
\]
for \(\varepsilon \in (0, 1)\).
Inserting (4.24) and (4.25) into (4.23), we conclude that
\[
\frac{d}{dt} \| \partial_t d \|^2_{L^2(\Omega)} + \frac{\theta}{2} \| \nabla \partial_t d \|^2_{L^2(\Omega)} \leq C(\theta, \epsilon^{8 - 4N}, \Omega) \| \partial_t d \|^2_{L^2(\Omega)} \left( \| \nabla d \|^4_{L^4(\Omega)} + \| v \|^2_{L^\infty(Q_T)} + 1 \right) + C(\theta) \| \partial_t v \|^2_{H^1(\Omega)},
\]
which, by applying Gronwall’s inequality, gives
\[
\| \partial_t d \|^2_{L^2(\Omega)} \leq \left( \| \partial_t d(0) \|^2_{L^2(\Omega)} + C(\theta) \| \partial_t v \|^2_{L^2(I_n, H^1(\Omega))} \right) e^{C(\theta, \epsilon^{8 - 4N}, \Omega)(\| \nabla d \|^4_{L^4(Q_T)} + \| v \|^2_{L^\infty(Q_T)} + 1)}.
\]
Noting that
\[
\| \partial_t d(0) \|^2_{L^2(\Omega)} = \| \theta(\Delta d_0 + | \nabla d_0 |^2 d_0) - v_0 \cdot \nabla d_0 \|^2_{L^2(\Omega)},
\]
we use (4.19), (4.22) and (4.17) to arrive at
\[
\| \partial_t d \|^2_{L^\infty(I_n, L^2(\Omega))} + \| \nabla \partial_t d \|^2_{L^2(I_n, L^2(\Omega))} \leq G(\rho, \bar{\rho}, \| \rho_0 \|_{H^1(\Omega)}, \| d_0 \|_{H^2(\Omega)}, \bar{E}_{\delta, 0}, \alpha, \epsilon^{8 - 4N}, T, \Omega).
\]
Recalling \(| d | \leq 1\), from (4.2) we get
\[
\theta^2 \int_\Omega | \Delta d |^2 dx \leq 3 \theta^2 \int_\Omega | \nabla d |^4 dx + 3 \int_\Omega | \partial_t d |^2 dx + 3 \int_\Omega | v |^2 | \nabla d |^2 dx. \quad (4.27)
\]
Similar to the derivation of (2.21), the first term on the right-hand side of (4.27) can be bounded as follows.
\[
3 \theta^2 \int_\Omega | \nabla d |^4 dx \leq 3 \theta^2 \tilde{C}_2(\Omega) \tilde{C}_0 \epsilon_0 \| \Delta d \|^2_{L^2(\Omega)} + C(\Omega)(\| \nabla d \|^2_{L^2(\Omega)} + \| d - e_N \|^2_{L^2(\Omega)}), \quad (4.28)
\]
where the constants \(\tilde{C}_2(\Omega)\) and \(\tilde{C}_0\) are the same as in (2.23). Noting that \(\tilde{C}_2(\Omega) \tilde{C}_0 \epsilon_0 \leq (8 \tilde{C}_3)^{-1} \leq 4^{-1}\) by (2.23), using (4.11), (4.17) and (4.26), we find from (4.27) and (4.28) that
\[
\frac{\theta^2}{4} \int_\Omega | \Delta d |^2 dx \leq 3 \theta^2 C(\Omega)(\| \nabla d \|^2_{L^2(\Omega)} + 1) \quad + 3 \| \partial_t d \|^2_{L^2(\Omega)} + 3 \| v \|^2_{L^\infty(\Omega)} \| \nabla d \|^2_{L^2(\Omega)} \quad \leq G(\rho, \bar{\rho}, \| \rho_0 \|_{H^1(\Omega)}, \| d_0 \|_{H^2(\Omega)}, \bar{E}_{\delta, 0}, \alpha, \epsilon^{8 - 4N}, T, \Omega). \quad (4.29)
\]
Hence, by virtue of (2.20), (4.17), (4.26), (4.29) and the fact \(| d | \leq 1\),
\[
\| d \|^2_{L^\infty(I_n, H^2(\Omega))} \leq G(\rho, \bar{\rho}, \| \rho_0 \|_{H^1(\Omega)}, \| d_0 \|_{H^2(\Omega)}, \bar{E}_{\delta, 0}, \alpha, \epsilon^{8 - 4N}, T, \Omega). \quad (4.30)
\]
The inequalities (4.18), (4.19), (4.21), (4.22) and (4.30) furnish the desired estimates which, in combination with Proposition 4.1 give a possibility to repeat the above fixed point argument to conclude that \(T_n = T\), and moreover, the global solution \((\rho_n, v_n, d_n)\) is unique. To end this section, we summarize our previous results on the global existence and uniqueness of a solution \((\rho_n, v_n, d_n)\) to the third approximate problem (4.11)-(4.16) as follows.

**Proposition 4.2.** Let the constant \(\epsilon_0 > 0\) (depending on \(\Omega\)) satisfy (2.23),
\[
\delta > 0, \beta > 0, \epsilon > 0, \text{ and } 0 < \rho < \infty.
\]
Assume that \( \Omega \subset \mathbb{R}^N \) is a bounded \( C^{2,\alpha} \)-domain (\( \alpha \in (0,1) \)), and the initial data \((\rho_0, \mathbf{m}_0, \mathbf{d}_0)\) satisfies

\[
1-d_{0N} < \epsilon_0, \quad |\mathbf{d}_0| = 1, \quad \mathbf{d}_0 \in H^2(\Omega),
\]

\[
0 < \rho \leq \rho_0 \leq \bar{\rho}, \quad \rho_0 \in W^{1,\infty}(\Omega), \quad \mathbf{v}_0 \in \mathbf{X}_n.
\]

Then the third approximate problem (4.1)–(4.6) possesses a unique triple \((\rho_n, \mathbf{v}_n, \mathbf{d}_n)\) with the following properties:

1. Regularity: \( \rho_n \) satisfies the same regularity as in Proposition [3.1], \( \mathbf{v}_n \in C^0([I, \mathbf{X}_n]), \partial_t \mathbf{v}_n \in L^2(I, \mathbf{X}_n), \mathbf{d}_n \) satisfies the same regularity as in Proposition [3.2] with \( T \) in place of \( T \).

2. \((\rho_n, \mathbf{v}_n, \mathbf{d}_n)\) solves (4.1) and (4.2) a.e. in \( Q_T \), and satisfies (4.3) and \((\rho_n, \mathbf{v}_n, \mathbf{d}_n)|_{t=0} = (\rho_0, \mathbf{v}_0, \mathbf{d}_0)\).

3. Finite and bounded energy inequalities hold in the 2D case:

\[
\frac{d}{dt}\mathcal{E}_\delta^n(t) + \mathcal{F}^n(t) + \int_\Omega \varepsilon \delta \beta \rho_n^{\beta - 2} |\nabla \rho_n|^2(t)dx \leq 0 \quad \text{in } D'(I),
\]

and

\[
\mathcal{E}_\delta^n(t) + \int_0^t \left( \mathcal{F}^n(s) + \int_\Omega \varepsilon \delta \beta \rho_n^{\beta - 2} |\nabla \rho_n|^2(s)dx \right) ds \leq \mathcal{E}_\delta(\rho_0, \mathbf{m}_0, \mathbf{d}_0) := \mathcal{E}_{\delta,0}
\]

a.e. in \( I \), where \( \mathcal{F}^n(t) := \mathcal{F}(\rho_n, \mathbf{v}_n, \mathbf{d}_n) \) and \( \mathcal{E}_\delta^n(t) := \mathcal{E}_\delta(\rho_n, \mathbf{m}_n, \mathbf{d}_n) \) with \( \mathbf{m}_n = \rho_n \mathbf{v}_n \).

4. Additional uniform estimates:

\[
|\mathbf{d}| \leq 1 \text{ in } \tilde{Q}_T, \text{ in particular, } |\mathbf{d}| = 1 \text{ if } |\mathbf{d}_0| = 1 \text{ for } N = 2,
\]

\[
\|\mathbf{d}_n - \mathbf{e}_N\|_{L^\infty(\Omega)} < C_0 \sqrt{\epsilon_0},
\]

\[
\sup_{t \in I} \|\mathbf{d}(t) - \mathbf{e}_N, \nabla \mathbf{d}(t)\|_{L^2(\Omega)} + \|\nabla^2 \mathbf{d}\|_{L^2(Q_T)} + \|\nabla \mathbf{d}\|_{L^4(Q_T)} \leq G(\mathcal{E}_{\delta,0}, \Omega),
\]

\[
\|\partial_t \mathbf{d}\|_{L^4(I, L^2(\Omega))} + \|\partial_x \mathbf{d}\|_{L^2(I, H^1(\Omega)^*)} \leq G(\mathcal{E}_{\delta,0}, \Omega),
\]

\[
\sup_{t \in I} \|\nabla \mathbf{v}(t)\|_{L^2(\Omega)} + \|\rho(t)\|_{L^1(\Omega)} + \|\nabla \mathbf{v}\|_{L^2(I, L^2(\Omega))} \leq G(\mathcal{E}_{\delta,0}, \Omega),
\]

\[
\sqrt{\epsilon} \|\nabla \rho_n\|_{L^2(Q_T)} \leq G(\mathcal{E}_{\delta,0}, \delta, \Omega),
\]

\[
\|\rho_n\|_{L^4_\rho(Q_T)} \leq G(\mathcal{E}_{\delta,0}, \epsilon, \delta, \Omega),
\]

(see [26, Section 7.7.5.2] for the proof of (4.40) and (4.41)), where \( G \) is a positive constant which is independent of \( n \) and nondecreasing in its arguments, and may depend on \( T \). Moreover, if \( \epsilon \) is not explicitly written in the argument of \( G \), then \( G \) is independent of \( \epsilon \) as well.

5. In particular, if \( \Omega = B_R \) with \( R \geq 1 \), then \( \epsilon_0 \) can be chosen to be independent of the domain \( \Omega = B_R \) for any \( R \geq 1 \), and the constant \( G \) in (4.37) and (4.39) can be replaced by a constant \( C(\mathcal{E}_{\delta,0}, T, \|\mathbf{d}_0 - \mathbf{e}_N\|_{L^2(\Omega)}) \) independent of \( B_R \).

5. Proof of Theorem [1.1]

Once we have established Proposition 4.2, we can obtain Theorem [1.1] by employing the standard three-level approximation scheme and the method of weak convergence in a manner similar to that in [3, 22] for the compressible Navier-Stokes equations. These arguments have
also been successfully used to establish the existence of weak solutions to other models from fluid dynamics, see the 2D problem of (1.1)–(1.3) in [10], and the 3D Ginzburg-Landau approximation model to (1.1)–(1.3) in [30] for example.

First we can construct a solution sequence \((\rho_n, v_n, d_n)\) by Proposition 4.2 using the related uniform estimates in Proposition 4.2 and standard compactness arguments, we can obtain the weak limit \((\rho_\varepsilon, v_\varepsilon, d_\varepsilon)\) of the solution sequence \((\rho_n, v_n, d_n)\) as \(n \to \infty\), taking subsequences if necessary, which is a weak solution of the following second approximate problem:

\[
\begin{align*}
\partial_t \rho_\varepsilon + \text{div}(\rho_\varepsilon v_\varepsilon) - \varepsilon \Delta \rho_\varepsilon &= 0 \quad \text{in } \mathcal{D}'(Q_T), \\
\partial_t (\rho_\varepsilon v_\varepsilon) + \partial_j (\rho_\varepsilon v_\varepsilon v_j^\varepsilon) - \mu \Delta v_\varepsilon - (\mu + \lambda) \text{div} v_\varepsilon + \nabla A \rho_\varepsilon^2 + \delta \nabla \rho_\varepsilon^\beta + \nu \text{div} \left( \nabla d_\varepsilon \otimes \nabla d_\varepsilon - \frac{[\nabla d_\varepsilon]^2}{2} \right) &= \varepsilon \nabla \rho_\varepsilon \cdot \nabla v_\varepsilon \quad \text{in } (\mathcal{D}'(Q_T))^N, \\
\partial_t d_\varepsilon + v_\varepsilon \cdot \nabla d_\varepsilon &= \theta(\Delta d_\varepsilon + f_\varepsilon(|\nabla d_\varepsilon|^2)d_\varepsilon), \quad \text{a.e. in } Q_T.
\end{align*}
\]

with boundary conditions

\[
\nabla \rho_\varepsilon \cdot n_{|\partial \Omega} = 0, \quad v_\varepsilon |_{\partial \Omega} = 0, \quad (n \cdot \nabla d_\varepsilon)|_{\partial \Omega} = 0,
\]

and modified initial data (4.31)–(4.32), where \(\delta > 0, \beta \geq \max\{\gamma, 8\}\), and \(\varepsilon > 0\). Moreover, the solution \((\rho_\varepsilon, v_\varepsilon, d_\varepsilon)\) enjoys the finite and bounded energy inequalities (4.33)–(4.34), and uniform estimates (4.35)–(4.41).

We proceed to utilize the related uniform estimates and standard compactness arguments to obtain the weak limit \((\rho_\delta, v_\delta, d_\delta)\) of the weak solution sequence \((\rho_\varepsilon, v_\varepsilon, d_\varepsilon)\) to the second approximate problem as \(\varepsilon \to 0\), taking subsequences if necessary, which is a weak solution of the following first approximate problem:

\[
\begin{align*}
\partial_t \rho_\delta + \text{div}(\rho_\delta v_\delta) &= 0 \quad \text{in } \mathcal{D}'(Q_T), \\
\partial_t (\rho_\delta v_\delta) + \partial_j (\rho_\delta v_\delta v_j^\delta) - \mu \Delta v_\delta - (\mu + \lambda) \text{div} v_\delta + \nabla A \rho_\delta^2 + \delta \nabla \rho_\delta^\beta + \nu \text{div} \left( \nabla d_\delta \otimes \nabla d_\delta - \frac{[\nabla d_\delta]^2}{2} \right) &= 0 \quad \text{in } (\mathcal{D}'(Q_T))^2, \\
\partial_t d_\delta + v_\delta \cdot \nabla d_\delta &= \theta(\Delta d_\delta + |\nabla d_\delta|^2)d_\delta, \quad \text{a.e. in } Q_T.
\end{align*}
\]

with boundary conditions (5.1) and modified initial data (4.31)–(4.32). Moreover, the solution \((\rho_\delta, v_\delta, d_\delta)\) enjoys the estimates (4.33)–(4.39) and inequalities (4.33)–(4.34) with \(\varepsilon = 0\). Here we remark that it is easy to verify the convergence of \(f_\varepsilon(|\nabla d_\varepsilon|^2)\) to \(|\nabla d_\delta|^2\) as \(\varepsilon \to 0\) in three dimensions.

Using the uniform bounds given in (4.36)–(4.38) with \(d_\varepsilon\) in place of \(d_n\), applying the Arzelà-Ascoli theorem and Aubin-Lions lemma, and taking subsequences if necessary, we deduce that

\[
d_\varepsilon \to d_\delta \text{ strongly in } C^0(I, L^2(\Omega)) \cap L^p(I, \mathcal{H}^1(\Omega)) \cap L^r(I, W^{1,q}(\Omega)) \tag{5.2}
\]

for any \(p \in [1, \infty), q \in [1, 6)\) and \(r \in [1, 2)\), which, recalling the definition of \(f_\varepsilon\), implies that (taking subsequences if necessary)

\[
f_\varepsilon(|\nabla d_\varepsilon|^2)d_\varepsilon \to |\nabla d_\delta|^2d_\delta \quad \text{as } \varepsilon \to 0 \quad \text{for a.e. } \mathbf{x} \in \Omega. \tag{5.3}
\]

Thus, using Vitali’s convergence theorem, and recalling the uniform in \(\varepsilon\) boundedness of \(|\nabla d_\varepsilon|_{L^2(Q_T)}\), we infer that

\[
f_\varepsilon(|\nabla d_\varepsilon|^2)d_\varepsilon \to |\nabla d_\delta|^2d_\delta \quad \text{strongly in } L^r(Q_T) \quad \text{for any } r \in [1, 2), \tag{5.4}
\]

and weakly in \(L^2(Q_T)\).
In addition, we have the regularity $d_\delta \in L^2(I, H^2(\Omega))$ and $\partial_t d_\delta \in L^2(I, (H^1(\Omega))^*)$. In view of [26, Proposition 7.31], we get consequently

$$d_\delta \in C^0(I, H^1(\Omega)). \quad (5.5)$$

Finally, we can also obtain a weak solution $(\rho, v, d)$ of the original problem (1.1)–(1.4) with boundary conditions “$v|_{\partial \Omega} = 0$ and $(n \cdot \nabla d)|_{\partial \Omega} = 0$”, and modified initial data (4.31)–(4.32), which is the weak limit as $\delta \to 0$ of the weak solution sequence $(\rho_\delta, v_\delta, d_\delta)$ of the second approximate problem. It should be noted that the modified initial energy in (4.34)–(4.39) can be further chosen to be independent of $\delta$, in other words, the term $E_\delta, 0$ in (4.34)–(4.39) can be replace by a positive constant $\bar{E}_0 := \sup_{0 \leq \delta \leq 1} \{E_\delta, 0\}$. Hence, the weak solution $(\rho, v, d)$ enjoys the same estimates as in Theorem 1.1. Applying an approximation argument to the initial data, the modified initial data (4.31) and (4.32) can be relaxed to (1.8)–(1.11). Consequently, we can obtain the desired Theorem 1.1. We refer to [10, 30] or [5, 26] for the omitted details of the proof of the limit process and the renormalized solutions (1.12).

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