The top Yukawa coupling from 10D supergravity with $E_8 \times E_8$ matter

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Abstract

We consider the compactification of N=1, D=10 supergravity with $E_8 \times E_8$ Yang-Mills matter to N=1, D=4 model with 3 generations. With help of embedding $SU(5) \rightarrow SO(10) \rightarrow E_6 \rightarrow E_8$ we find the value of the top Yukawa coupling $\lambda_t = \frac{\sqrt{16 \pi \alpha_{GUT}}}{3}$ at the GUT scale.

1 Introduction

Although superstring theories have great success and today are the best candidates for a quantum theory unifying all interactions, they still do not predict any experimentally testable value, mostly because there is no unambiguous procedure of compactification of extra spatial dimensions.

On the other hand, many phenomenological models based on the unification group $SO(10)$ were constructed (for instance [1]). They describe quarks and leptons in representations $16_1$, $16_2$, $16_3$ and Higgses, responsible for $SU(2)$ breaking, in 10. Basic assumption of these models is that there is only one Yukawa coupling $16_3 \cdot 10 \cdot 16_3$ at the GUT scale, which gives masses of third generation. Masses of first two families and mixings are generated due to the interaction with additional superheavy ($\sim M_{GUT}$) states in $16 + 16$, 45, 54. Such models explain generation mass hierarchy and allow to express all Yukawa matrices, which well fit into experimentally observable pattern, in terms of few unknown parameters.

Models of [1] are unlikely derivable from ”more fundamental” theory like supergravity/string. Nevertheless, something similar can be constructed from D=10 supergravity coupled to $E_8 \times E_8$ matter, which is low-energy limit of heterotic string [4].

The lagrangian of N=1, D=10 supergravity with Yang-Mills matter [3] does not contain any free parameter. The gauge group is fixed to be either $SO(32)$ or $E_8 \times E_8$ due to Green-Schwarz anomaly cancellation mechanism [1]. We will consider only $E_8 \times E_8$ case, since it naturally leads to $SO(10)$ group. Furthermore, since $\text{Tr}_{E_8^{(1)} \times E_8^{(2)}} = \text{Tr}_{E_8^{(1)}} + \text{Tr}_{E_8^{(2)}}$, fields from $E_8^{(2)}$ interact with $E_8^{(1)}$ only gravitationally, so we will consider only $E_8^{(1)}$.

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After compactification to four dimensions 10D-vector produces 6 real (or 3 complex) scalars in representation 248 each. In order to break supersymmetry \( N = 4 \rightarrow N = 1 \), some of them should get \( SU(3) \)-valued VEVs \[3\]. So \( E_8 \) group is broken down to \( E_6 \): 

\[ 248 = (78, 1) + (27, 3) + (\overline{27}, 3) + (1, 8) \]

In this way we get 9 generations in 27 of \( E_6 \) with 9 mirror generations \( \overline{27} \). After \( E_6 \) is broken to \( SO(10) \), we get 12 pairs \( 16 + \overline{16} \), 18 copies of 10 and 3 ones of 45. To compute masses of all these states, one should evaluate some compactification scheme, for example Scherk-Schwarz compactification \[3\]. We shall consider this question in future publications \[4\].

Nevertheless, since in the models of \[1\] the top quark gets the mass at the tree level, in our case it is possible to find the top Yukawa coupling without going deep in the details of compactification. It indicates the fact, that gauge and Yukawa couplings are unified in higher-dimensional supersymmetric theories. It is the subject of this paper.

In section 2 we perform the reduction of the N=1, D=10 supergravity to N=1, D=4 theory and rewrite the result in conventional form in terms of Kahler- and super-potential, carefully pointing out all assumptions and conventions.

In section 3 we propose a way to choose 3 massless generations and compute the value of the top Yukawa coupling.

In Appendix we consecutively construct representations of \( SO(10) \), \( E_6 \) and \( E_8 \) groups.

## 2 Reduction N=1, D=10 \( \rightarrow \) N=1, D=4

Bosonic lagrangian of \( N = 1, D = 10 \) supergravity with Yang-Mills matter has the form \[3\]:

\[ L^{(10)} = \frac{1}{4} R + \frac{1}{2} \phi^M \phi_M + \frac{1}{12} e^{2\phi} H^{MNP} H_{MNP} + \frac{1}{4} e^\phi \text{Tr} (F^{MN} F_{MN}) \]  \( (1) \)

We use the following conventions for the field-strength tensor \( F \) and 3-form \( H \):

\begin{align*}
F_{MN} &= 2 \partial_M A_N - 2 A_{[M} A_N] \\
H_{MNP} &= 3 \partial_M B_{NP} + 6 \text{Tr} \left( A_{[M} \partial_N A_P] - \frac{2}{3} A_{[M} A_N A_P] \right)
\end{align*}  \( (2) \)

In \( (1) \), \( (2) \) the vector field \( A_M = A_M^A T^A \), \( T^A \) are antihermitean generators (in this section the gauge group is arbitrary). In principle we could write the gauge coupling constant before the trace, but it can be removed by means of rescaling, so it’s not physical; actual value of the gauge coupling in four dimensions is determined by the VEV of the dilaton field \( \phi \).

The following index notations are used here:

| space          | dimension | flat    | world  |
|----------------|-----------|---------|--------|
| initial:       | \( D = 10 \) | \( A, B, C, \ldots \) | \( M, N, P, \ldots \) |
| our:           | \( D = 4 \)  | \( \alpha, \beta, \gamma, \ldots \) | \( \mu, \nu, \lambda, \ldots \) |
| internal:      | \( D = 6 \)  | \( a, b, c, \ldots \) | \( m, n, p, \ldots \) |

The Minkowski metric is \( \eta_{AB} = (+, -, \ldots, -) \).
Table 1: N=1, D=10 SURGA multiplet at the reduction to four dimensions. Index $I = 1...4$ is a part of 10D spinorial index; it splits on $(i, 4)$, $i = 1, 2, 3$ after supersymmetry breaking $N = 4 \rightarrow N = 1$. Numbers in brackets are physical degrees of freedom, carried by each field.

Table 1 demonstrates, how the fields of 10D supergravity form 4D-multiplets at the reduction. Not all of them can be coupled to N=1, D=4 supergravity. At first, one should vanish the part of N=4 multiplet, namely, 3 gravitinos $\psi_{\mu}^i$, 3 fermions $\chi_i$ from dilatino and 6 vectors $B_{\mu m}$. Since 2-form $B$ is not gauge invariant, $\delta B_{MN} = -2\text{Tr}(U\partial [M A_N])$ at the gauge transformations $\delta A_M = D_M U$, the condition $B_{\mu m} = 0$ does not break the gauge invariance only if vectors $A_{\mu}$ and scalars $A_m$ belong to different representations, so that $\text{Tr}(T(A_{\mu})T(A_m)) = 0$. This excludes the possibility to couple adjoint scalars with N=1, D=4 supergravity in supersymmetric way.

The fields in the very right box of the Table 1 mix left- and right-handed generations. We do not know, is it possible to couple them to N=1, D=4 SUGRA, so we put $E_{\bar{s}}^i = B_{st} = \psi_{\bar{s}}^i = 0$. Again, these conditions are supersymmetric-invariant if we consider only left- or right-handed chiral fields $A_s$, but not both of them, so that $\text{Tr}(T(A_s)T(A_{\bar{t}})) = 0$. It indicates, that reasons, responsible for $N = 4 \rightarrow N = 1$ supersymmetry breaking, also break the mirror symmetry.

We shall not consider vector multiplet $(E_{\mu}^a, \psi_m^a)$, since $SU(3)$-holonomy group, gauged by these vectors, is broken, and they become massive.

The reduction of other fields is quite standard [8]. As usual, 4-dimensional theory contains...
the following bosons:

4D vielbein: \( e_\mu^\alpha = \Delta^{1/4} E_\mu^\alpha \)

4D dilaton: \( S = \sqrt{\Delta} e^{\phi} - 2iD \)

moduli: \( T_{st} = -e^{-\phi} g_{st} - 2 B_{st} - 2 \text{Tr}(A_s A_t) \)

where \( \Delta = \text{det}(g_{mn}) \), \( D \) is dual to \( B_{\mu\nu} \), \( e^{2\phi} \Delta H_{\alpha\beta\gamma} = \varepsilon_{\alpha\beta\gamma\delta} D_\delta \). In these terms the bosonic part of 4-dimensional lagrangian gets the form

\[
L^{(4)} = \frac{1}{4} R + \frac{1}{4} \text{Tr} \left[ (\Re S) F^{\alpha\beta} F_{\alpha\beta} - \frac{1}{2} (\Im S) \varepsilon^{\alpha\beta\gamma\delta} F_{\alpha\beta} F_{\gamma\delta} \right] + L^{(K)} - V
\]

with kinetic terms in conventional Kahler form \( L^{(K)} = G_{AB} (T_A A_B) - \frac{1}{2} \text{Tr} \left( [A_s, A_t] [A^s, A^t] \right) \)

It can be written in conventional form:

\[
V = e^{2G} \left[ \frac{3}{2} - (G^i_j)^{-1} G^i_i \right] + \frac{1}{2} (\Re f_{AB})^{-1} \left[ G_i (T_A)^{ij} \Phi^j \right] \left[ G_k (T_B)^{kl} \Phi^l \right]
\]

with kinetic function \( f_{AB} = -S G_{AB}, G_{AB} \) is Killing tensor which stands in the trace \( Tr(AB) = G_{AB} A^A B^B \), and Kahler potential:

\[
G = -\frac{1}{2} \ln (S + \overline{S}) - \frac{1}{2} \ln \det \left[ T_{st} + T_{st} + 4 \text{Tr}(A_s A_t) \right] + \frac{1}{2} \ln |W|^2
\]

\( W \) is the superpotential:

\[
W = \frac{8\sqrt{2}}{3} \varepsilon^{stu} \text{Tr}(A_s A_t A_u)
\]

We use the convention for gravitational constant \( 4\pi G_N = 1 \). In (B) indices \( A, B \) label representation, to which vectors belong.

All fermionic terms can be restored unambiguously by functions \( f_{AB} \) and \( G \).

## 3 Top Yukawa coupling

Now we consider low energy approximation to the lagrangian (4), (5) with group \( E_8 \). The structural constants of \( E_8 \) are given in appendix (A.3); the trace convention is \( Tr(AB) = \frac{1}{30} G_{AB} A^A B^B \) with Killing tensor \( G_{AB} \) from (B5) (actually, as we mentioned earlier, this convention doesn’t matter unless we know the value of the dilaton \( S \)).

Suppose that the dilaton \( S \) and metric \( g_{st} \) get some VEVs and consider the interactions of 9 copies of 27 \( A^{ia}_s, \ a = 1...27 \). We put:

\[
A^{ia}_s = 0, \quad A^{ia}_s = \frac{1}{\sqrt{2}} e^{-\phi/2} e_j^c C^{ia}_j
\]
(\(A_{s}^{i a}\) correspond to mirror generations \(2\overline{7}i\); \(e_{s}^{j}\) is 6D vielbein in complex coordinates \(g_{a i} = e_{s}^{j} \eta_{ij} e^{j}\). Minkowski metric is \(\eta_{ij} = -\frac{1}{2} \delta_{ij}\).) Discarding nonrenormalizable interactions, one can write down the lagrangian in the form

\[
L = -\frac{\text{Re} S}{4} F^{\alpha}_{\mu \nu} F^{\alpha \mu \nu} + (D_{\mu} C_{j}^{a})^* D^{\mu} C_{j}^{a} - \left( \frac{\partial W'}{\partial C_{j}^{a}} \right)^* \frac{\partial W'}{\partial C_{j}^{a}} + D\text{-terms}
\]

with the superpotential

\[
W' = \frac{1}{6 \sqrt{\text{Re} S}} \varepsilon^{ijk} \varepsilon_{lmn} d_{abc} C_{i}^{a} C_{j}^{mb} C_{k}^{nc} = \frac{1}{\sqrt{\text{Re} S}} d_{abc} \det \left[ \begin{array}{ccc} C_{1}^{1a} & C_{1}^{2b} & C_{1}^{3c} \\
C_{2}^{1a} & C_{2}^{2b} & C_{2}^{3c} \\
C_{3}^{1a} & C_{3}^{2b} & C_{3}^{3c} \end{array} \right]
\]

(it differs from (\(8\) in normalization). Totally symmetric \(E_{6}\)-invariant tensor \(d_{abc}\) is determined in Appendix (\(30,32\)); index \(\alpha=1...78\). \(E_{6}\)-covariant derivative in (\(10\)) is:

\[
D_{\mu} C_{j}^{a} = \partial_{\mu} C_{j}^{a} - A_{\mu}^{\alpha}(T_{\alpha})^{a}_{\ b} C_{j}^{b},
\]

normalization of generators \(\text{Tr}(T_{\alpha}T_{\beta}) = -3\delta_{\alpha\beta}\), like in Appendix (A.3).

How to choose 3 generations from 9 ones in (\(11\))? As mentioned in the Introduction, if we want to build a model like (\(1\)), we should have only the interaction \(16_{3} \cdot 10 \cdot 16_{3}\) in the superpotential at the GUT scale. This term follows from \(27_{3} \cdot 27_{3} \cdot 27_{3}\), if Higgs’ 10 is the part of \(27_{3}\). If we choose

\[
C_{j}^{a} = \left( \begin{array}{ccc}
\frac{1}{\sqrt{3}} C_{3}^{a} & \frac{1}{\sqrt{2}} C_{2}^{a} & C_{1}^{a} \\
0 & \frac{1}{\sqrt{3}} C_{3}^{a} & \frac{1}{\sqrt{2}} C_{2}^{a} \\
0 & 0 & \frac{1}{\sqrt{3}} C_{3}^{a} \end{array} \right)
\]

then we get the superpotential with only third generation:

\[
W' = \frac{1}{3 \sqrt{3} \text{Re} S} d_{abc} C_{3}^{a} C_{3}^{b} C_{3}^{c}
\]

Of course the choice (\(13\)) is not unique, other variants leading to (\(14\)) are possible (although they give the same numerical factor, which determines the value of the top Yukawa coupling, discussed here). Strictly speaking, one should find the metric of 6D space, which would give masses to all states except those of (\(13\)). We shall present such solution in future publications (\(7\)). Nevertheless the superpotential does not depend on the moduli fields \(T_{a i}\), so we just accept the choice (\(13\)).

Simple reduction, considered here, does not determine the value of the dilaton \(\text{Re} S\). But it gives the relation between the top Yukawa coupling and the gauge constant, which is the evidence of the gauge–Yukawa unification. To establish the correspondence between \(\lambda_{i}\) and \(\alpha_{\text{GUT}}\), we perform two steps down \(E_{6} \rightarrow SO(10) \rightarrow SU(5)\) by means of representations, constructed in Appendix.

\(E_{6} \rightarrow SO(10)\). The multiplet \(27_{3}\) consists of \(C_{3}^{a} = (1, H^{A}, \Psi^{a})\), where \(\Psi^{a}\) are third quarks and leptons packed in 16 (\(a=1...16\)) and \(H^{A}\) are Higgses in 10 (\(A=1...10\)); we shall not consider singlets here. Concerning vector fields, we choose representation 45 from 78 one: \(A_{\mu}^{\alpha=12} = \sqrt{2} A_{\mu}^{A_{\beta}}\). To determine the value of the gauge and Yukawa couplings, we write down the following 3 elements of the theory:
• Kinetic terms:
  \[ - \frac{\text{Re } S}{4} F_{\mu \nu}^{A\beta} F^{A\beta \mu \nu} + (D_{\mu} \Psi^a)^* D^\mu \Psi^a + (D_{\mu} H^A)^* D^\mu H^A \]  
  \[ (15) \]

• Covariant derivatives:
  \[ D_{\mu} \Psi^a = \partial_{\mu} \Psi^a - \frac{1}{4} A_{\mu}^{A\beta} (\Gamma^{A\beta})_c^a \psi^c , \quad D_{\mu} H^A = \partial_{\mu} H^A - A_{\mu}^{A\beta} H^B \]  
  \[ (16) \]

• The superpotential:
  \[ W' = \frac{1}{\sqrt{6 \text{Re } S}} \Psi^a (\Gamma_A)_{ab} \Psi^b H^A \]  
  \[ (17) \]

\[ \text{SO}(10) \rightarrow \text{SU}(5). \] 
Now indices \( a, b, c \ldots \) run 1...5 while \( A, B, C \ldots = 1 \ldots 24 \). As usual, we will use 24 hermitean traceless \( 5 \times 5 \) matrices \( (\lambda^A)^c_b \) normalized \( \text{Tr}(\lambda^A \lambda^B) = 2 \delta^{AB} \). \( \text{SO}(10) \)-fields produce the following \( \text{SU}(5) \) ones:

\begin{align*}
\text{SO}(10) \text{ field:} & \quad \text{IR:} & \quad \text{SU}(5) \text{ field:} & \quad \text{IR:} & \quad \text{particles:} \\
\Psi^a & \quad 16 & \psi^{ab} = \Psi^{a=ab} & \quad 10 & u_L, u_R, d_L, e_R \\
H^A & \quad 10 & H_u^a = \frac{1}{\sqrt{2}} (H_u^a + i H_u^{a+5}) & \quad 5 & \text{up-Higgs} \\
A_{\mu}^{A\beta} & \quad 45 & A^A_{\mu} = - \frac{i}{2} (\lambda^A)^c_b (A_{\mu}^{bc} - i A_{\mu}^{b,c+5} + i A_{\mu}^{b+5,c} + A_{\mu}^{b+5,c+5}) & \quad 24 & \text{SU}(5) \text{ vector}
\end{align*}

We shall not consider singlet \( \Psi^{a=1} \) and other vectors, not relevant to the \( \text{SU}(5) \)-unification model.

Elements of the theory:

• Kinetic terms:
  \[ - \frac{\text{Re } S}{4} F_{\mu \nu}^A F^{A \mu \nu} + \frac{1}{2} (D_{\mu} \psi^{ab})^* D^\mu \psi^{ab} + (D_{\mu} \varphi_a)^* D^\mu \varphi_a + (D_{\mu} H_u^a)^* D^\mu H_u^a + (D_{\mu} H_d^a)^* D^\mu H_d^a \]  
  \[ (18) \]

• Covariant derivatives:
  \[ D_{\mu} \psi^{ab} = \partial_{\mu} \psi^{ab} - \frac{i}{2} A^A_{\mu} (\lambda^A)^c_b \psi^{cb} - \frac{i}{2} A^A_{\mu} (\lambda^A)^c_b \psi^{ac} , \quad D_{\mu} \varphi_a = \partial_{\mu} \varphi_a + \frac{i}{2} A^A_{\mu} \varphi_c (\lambda^A)^c_a \]  
  \[ D_{\mu} H_u^a = \partial_{\mu} H_u^a - \frac{i}{2} A^A_{\mu} (\lambda^A)^b_a H_u^b , \quad D_{\mu} H_d^a = \partial_{\mu} H_d^a + \frac{i}{2} A^A_{\mu} H_d^b (\lambda^A)^b_a \]  
  \[ (19) \]

• The superpotential:
  \[ W' = \frac{1}{\sqrt{3 \text{Re } S}} \left( 2 H_d^a \psi^{ab} \varphi_b + \frac{1}{4} \varepsilon_{abcdf} H_u^a \psi^{bc} \psi^{df} \right) \]  
  \[ (20) \]
Figure 1: The running of the top Yukawa coupling in case of MSSM and $\alpha_{GUT} = 1/25$.

The embedding of the Standard Model group in $SU(5)$ is well known. From (18), (19) we immediately find the gauge constant $g$ and from (20) the top Yukawa coupling $\lambda_t$:

$$
g = \frac{1}{\sqrt{\text{Re} S}}, \quad \lambda_t = \frac{2}{\sqrt{3} \text{Re} S} \tag{21}
$$

Eliminating unknown $\text{Re} S$, we get the equation, which is in principle experimentally testable:

$$
\lambda_t = \sqrt{\frac{16\pi \alpha_{GUT}}{3}} \tag{22}
$$

where $\alpha_{GUT} = g^2/4\pi$. This constant determines the value of the top quark mass at low energies (or at least its upper limit, since $\tan \beta$ is unknown):

$$
m_t = \lambda_t \frac{v}{\sqrt{2}} \sin \beta, \quad v = \frac{2m_Z \sin \theta_W \cos \theta_W}{e} \approx 246 \text{ GeV} \tag{23}
$$

In order to evaluate the renormalization running of $\lambda_t$ from the GUT scale down to weak scale, one needs to know all massless states of the theory in this range and their interactions. If we suppose, that all symmetries except $SU(3) \times SU(2) \times U(1)$ are broken somewhere near the GUT scale, and all massless states are only those of MSSM, then $\lambda_t$ runs as Figure 1 shows. In this case small $\tan \beta$ are excluded regardless of their incompatibility with $b - t$ unification.

Nevertheless, MSSM might be incomplete in all range up to the GUT scale. Indeed, initially we had 9 generations $27$ and 9 mirrors $\overline{27}$. It is possible to find a configuration, which would give masses to all extra scalars and make 6 massive Dirac fermions in representation 27 of $E_6$. Nevertheless, if 3 fermions in 27 are massless, so do 3 mirror fermions in $\overline{27}$. Whatever the compactification is, it seems difficult to make these 3 mirror generations massive, unless $SU(2)$ is broken, which happens at the weak scale. It would change the renormalization running.
4 Conclusion

Even though the main result of this paper (22) is based on several assumptions which might be incorrect, we demonstrated the predictive power of 10D supergravity coupled to $E_8 \times E_8$ matter. In order to verify them and to get other predictions, one should: 1) consider a particular anzatz for the metric of 6D-space $g_{mn}$, for instance, within the framework of Sherk-Schwarz compactification procedure and 2) explore flat directions of the potential. Then masses of superheavy states are expressed in terms of few parameters of 6D-space and VEVs of Higgses, responsible for symmetry breaking. Masses of first two families are generated due to interaction with superheavy states.

Moreover, since higher derivative corrections to the action of 10D supergravity became recently available [10], the VEV of the dilaton can be expressed in terms of the sizes of 6D space. It cannot be determined in another calculable way within the framework of second-derivative supergravity due to the dilaton runaway problem. It will be done in future publications [7].

A Appendix: groups $SO(10)$, $E_6$, $E_8$

For convenience we collected all group indices in Table 2.

A.1 $SO(10) \supset SU(5) \times U(1)$

Vector and spinorial representations of $SO(10)$ are decomposed by $SU(5)$ reps:

\[ 10 = 5 + \bar{5}, \quad 16 = 10 + \bar{5} + 1 \]

Let us denote $SO(10)$-vector index $A = (a, 5 + a) = 1...10$, $a = 1...5$ and $SO(10)$-spinorial one $a = (1, a, bc) = 1...16$ with $b < c$.

We can construct symmetric $16 \times 16$ Dirac matrices $(\Gamma^A)_{ab}$ and $(\Gamma^A)^{ab} = ((\Gamma^A)_{ab})^*$ in

| indices: | values: | representation: |
|---------|--------|-----------------|
| a, b, c,... | 1...5 | $SU(5)$ fundamental |
| A, B, C,... | 1...24 | $SU(5)$ adjoint |
| a, b, c,... | 1...16 | $SO(10)$ spinorial |
| A, B, C,... | 1...10 | $SO(10)$ vector |
| a, b, c,... | 1...27 | $E_6$ fundamental |
| A, B, C,... | 1...78 | $E_6$ adjoint |
| i, j, k,... | 1...3 | $SU(3)$ fundamental |
| Σ, Λ, Π,... | 1...8 | $SU(3)$ adjoint |
| A, B, C,... | 1...248 | $E_8$ adjoint |

Table 2: Group indices, their values and representations, which they label
euclidean 10D space in the following way:

\[
(\Gamma^a)_{bb'} = \begin{pmatrix}
0 & \delta^a_b & 0 \\
\delta^a_b & 0 & 2\delta^b_{c'd'} \\
0 & 2\delta^c_{cd'} & \varepsilon_{ac'd'}
\end{pmatrix}, \quad (\Gamma^{5+a})_{bb'} = i \begin{pmatrix}
0 & \delta^a_b & 0 \\
\delta^a_b & 0 & -2\delta^b_{c'd'} \\
0 & -2\delta^c_{cd'} & \varepsilon_{ac'd'}
\end{pmatrix}
\]  

(24)

where \( b = (1, b, c, d) \), \( c < d \). Note, that one should sum over index 16 by the following

\[
\psi_b\psi^b = 1 \cdot 1 + 5_b\delta^b + \sum_{c<d} 10_{cd} 10_{cd}
\]

to avoid double count, since \( 10_{cd} \) is antisymmetric. \( \Gamma \)-matrices (24) satisfy anticommutational relations:

\[
(\Gamma^A)_{ac}(\Gamma^B)_{cb} + (\Gamma^B)_{ac}(\Gamma^A)_{cb} = 2\delta^{AB}\delta_b^a
\]

with \( \delta \)-symbol \( \delta^b_b = (1, \delta^b_b, \delta^c_c \delta^d_d) \).

**A.2 \( E_6 \supset SO(10) \times U(1) \)**

Now with help of these \( \Gamma \)-matrices we construct generators of the \( E_6 \) group using the decomposition \( E_6 \supset SO(10) \times U(1) \):

\[
27 = 1 + 10 + 16, \quad 78 = 1 + 16 + 16 + 45
\]

Let us denote 27 index as \( a = (1, A, a) \). Consider the following 27×27 antihermitean traceless matrix:

\[
T^a_b = \begin{pmatrix}
is & J^A_{a} - \frac{i}{2}s\delta^a_b & \sqrt{2}\psi_b \\
0 & -(\Gamma^A_b) & \frac{1}{4}J^{CD}(\Gamma^A_{CD})_{a,b} + \frac{i}{4}s\delta^a_b
\end{pmatrix}
\]  

(25)

Here \( s \) is real number, \( J^A_{a} \) is antisymmetric 10×10 matrix and \( \psi_a \) is 16-component complex spinor, \( \bar{\psi}^a \equiv (\psi_a)^* \). The commutator of both matrices of type (25) is also a matrix of this type:

\[
[T_1, T_2] = T_3
\]

constructed from elements:

\[
s_{(3)} = 4 \text{ Im} \left( \bar{\psi}_1(1)\psi_2(2) \right) \\
J_{(3)}^A_{B} = [J_1, J_2]_A^B - \bar{\psi}_1(1)\Gamma^A_{aB}\psi_2(2) + \bar{\psi}_2(2)\Gamma^A_{aB}\psi_1(1) \\
\psi_{(3)} = \frac{1}{4} \left( \bar{J}_1(1)\psi_2(2) - \bar{J}_2(2)\psi_1(1) + 3is_{(1)}\psi_2(2) - 3is_{(2)}\psi_1(1) \right)
\]  

(26)

where \( \bar{J} = J_{AB}\Gamma^{AB} \). This proves the algebraic structure of matrices (25). Trace of both such matrices is:

\[
\text{Tr}(T_1 T_2) = -\frac{9}{2} s_{(1)} s_{(2)} + 3\text{ Tr}(J_1 J_2) - 12 \bar{\psi}_1(1)\psi_2(2) - 12 \bar{\psi}_2(2)\psi_1(1)
\]  

(27)

Now let us enumerate 78 independent matrices (25) by index \( \mathfrak{A} \), \( T_{\mathfrak{A}} = (s, \psi_a, \bar{\psi}^a, J^{AB}) \). \( E_6 \)-metric \( g_{\mathfrak{A}B} \equiv \text{Tr}(T_{\mathfrak{A}}T_{B}) \) can be found by formula (27).
The $E_6$-structural constants
\[ [T_3, T_{3b}] = f^{ε}_{αβ} T_ε \]  
(28)
can be found by (23) with help of $f^{ε}_{αβ} = g^{εδ} \text{Tr}(T_δ [T_3, T_{3b}])$, where $g^{αβ}$ is inverse to $g_{αβ}$. With help of these expressions one can find the trace in adjoint representation:
\[ f^{ε}_{αβ} f^{δ}_{εγ} = 4 g^{γβ} \]  
(29)
There exists totally symmetric invariant tensor $d_{abc}$ in $E_6$. The invariance condition is:
\[ d_{d(εT_ε)(C)} = 0 \]  
(30)
In the representation (25) all nonzero components of this tensor are:
\[ d_{1AB} = δ_{AB}, \quad d_{ABC} = \frac{1}{\sqrt{2}} (Γ_A)_{bc} \]  
(31)
It satisfies the normalization condition:
\[ d_{acδ} d^{δbc} = 10 δ_{a}^{b} \]  
(32)
The following Fiertz identity can be proved:
\[ g^{αβ} (T_3)^ε_a (T_3)^δ_b = \frac{1}{18} δ_{a}^{ε} δ_{b}^{δ} + \frac{1}{6} δ_{a}^{ε} δ_{b}^{δ} - \frac{1}{6} d^{ει} d_{ιβ} \]  
(33)
Note, that Fiertz identities of all groups (except $E_8$) are given in [9].

A.3 $E_8 \supset E_6 \times SU(3)$
Fundamental representation of $E_8$ coincides with adjoint one, so we only need to find the structural constants with help of decomposition $E_8 \supset E_6 \times SU(3)$:
\[ 248 = (78, 1) + (27, 3) + (\overline{27}, \overline{3}) + (1, 8) \]
Let us choose $E_6$-generators normalized as $\text{Tr}(T_3 T_{3b}) = -3 δ_{αβ}$. The $SU(3)$ generators are 8 traceless hermitean $3 \times 3$ Gell-Mann matrices $(λ_7)^i_j$, $Σ = 1...8, i, j = 1, 2, 3$:
\[ [λ_Σ, λ_Λ] = 2i c^{Π}_{ΠΣ} λ_Π, \quad \text{Tr}(λ_Σ λ_Λ) = 2 δ_{ΣΛ}, \quad c^{Π}_{ΣΠ} c^{Π}_{ΑΞ} = -3 δ_{ΣΛ} \]
Generators of $E_8$ are $T_α = (X_α, Y_α, Q_α, Q^α)$, where $X_α, Y_α$ are $E_6$ and $SU(3)$ generators, $Q_α$ and $Q^α \equiv (Q_α)^{+}$. The $E_8$ commutational relations, closed with respect to Jacobi identities, are:
\[ [X_α, X_β] = f^{ε}_{αβ} X_ε \quad [Y_α, Y_β] = c^{Π}_{ΠΣ} Y_Π \quad [X_α, Y_β] = 0 \]
\[ [X_α, Q_ια] = (T_3)^b_α Q_ιb \quad [X_α, Q^ια] = - (T_3)^a_ι Q_ιb \]
\[ [Y_α, Q_ια] = - \frac{i}{2} (λ_Σ)^j_i Q_ιa \quad [Y_α, Q^ια] = \frac{i}{2} (λ_Σ)^j_i Q^ιa \]
\[
\begin{align*}
[Q_{ia}, Q_{jb}] &= \frac{1}{\sqrt{2}} \varepsilon_{ijk} d_{abc} Q^{kc} \quad [Q^{ia}, Q^{jb}] = -\frac{1}{\sqrt{2}} \varepsilon_{ijk} d^{abc} Q_{k\ell} \\
[Q_{ia}, Q_{jb}] &= \delta^i_j (T_A)^a_b X_{ab} - \frac{i}{2} \delta^a_b (\lambda_\Sigma)^i_j Y_{\Sigma}
\end{align*}
\]

From here the \(E_8\) structural constants \([T_A, T_B] = C^{CAB} T_C\) can be found. The Killing tensor \(G_{\lambda\Sigma} = C^{CAD} C^{DBC}\) has the nonzero components:

\[
G_{\lambda\Sigma} = -30 \delta_{\lambda\Sigma}, \quad G_{\Sigma\Lambda} = -30 \delta_{\Sigma\Lambda}, \quad G^{jb} e^i_a = G^{ji} e^b_a = 30 \delta^i_j \delta^b_a
\]

Representation 248 is real; so its element \(A = A^A T_A\) is antihermitean \(A^+ = -A\) if \(A_{ia} = -(A^a)^*\).

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