A note on volume thresholds for random polytopes

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Abstract

We study the expected volume of random polytopes generated by taking the convex hull of independent identically distributed points from a given distribution. We show that, for log-concave distributions supported on convex bodies, we need at least exponentially many (in dimension) samples for the expected volume to be significant, and that super-exponentially many samples suffice for $\kappa$-concave measures when their parameter of concavity $\kappa$ is positive.

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1 Introduction

Let $X_1, X_2, \ldots$ be independent identically distributed (i.i.d.) random vectors uniform on a set $K$ in $\mathbb{R}^n$. Let

$$K_N = \operatorname{conv}\{X_1, \ldots, X_N\}. \quad (1)$$

We are interested in bounds on the number $N$ of points needed for the volume $|K_N|$ of $K_N$ to be asymptotic in expectation to the volume $|\operatorname{conv} K|$ of the convex hull of $K$ as $n \to \infty$. In the pioneering work [12], Dyer, Füredi and McDiarmid established sharp thresholds for the vertices of the cube $K = \{-1,1\}^n$, as well as for the solid cube $K = [-1,1]^n$. More precisely, they showed that for every $\varepsilon > 0$,

$$\frac{\mathbb{E}|K_N|}{2^n} \xrightarrow{n \to \infty} \begin{cases} 0, & \text{if } N \leq (\nu - \varepsilon)^n \\ 1, & \text{if } N \geq (\nu + \varepsilon)^n \end{cases}, \quad (2)$$

where for $K = \{-1,1\}^n$, we have $\nu = 2/\sqrt{e} = 1.213...$ and for $K = [-1,1]^n$, we have $\nu = 2\pi e^{-\gamma-1/2} = 2.139...$ (see also [13]). For further generalisations establishing sharp exponential thresholds see [16] (in a situation when the $X_i$ are not uniform on a set but have i.i.d. components compactly supported in an interval).

The case of a Euclidean ball is different. Pivovarov showed in [22] (see also [7]) that when

$$K = B_2^n = \{x \in \mathbb{R}^n, \sum x_i^2 \leq 1\},$$

the threshold is superexponential, that is for every $\varepsilon > 0$,

$$\frac{\mathbb{E}|K_N|}{|K|} \xrightarrow{n \to \infty} \begin{cases} 0, & \text{if } N \leq e^{(1-\varepsilon)\frac{1}{2n} \log n} \\ 1, & \text{if } N \geq e^{(1+\varepsilon)\frac{1}{2n} \log n} \end{cases}. \quad (3)$$

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He additionally considered the situation when the $X_i$ are not uniform on a set but are Gaussian.

In recent works [7, 8], the authors study the case of the $X_i$ having rotationally invariant densities of the form const\( (1 - \sum x_i^2)^\beta \mathbf{1}_{B_2}, \beta > -1 \). This is the so-called Beta model of random polytopes attracting considerable attention in stochastic geometry. In particular, $\beta = 0$ corresponds to the uniform distribution on the unit ball and the limiting case $\beta \to -1$ corresponds to the uniform distribution on the unit sphere. As established in [7], the threshold here is as follows: for every constant $\epsilon \in (0, 1)$ and sequences $N = N(n)$, $-1 < \beta = \beta(n)$, we have

\[
\frac{E[K_N]}{|B_2^n|} \xrightarrow{n \to \infty} \begin{cases} 
0, & \text{if } N \leq e^{(1-\epsilon)(\frac{2}{\epsilon^2} + \beta) \log n} \\
1, & \text{if } N \geq e^{(1+\epsilon)(\frac{2}{\epsilon^2} + \beta) \log n} \end{cases}
\]  

(4)

which was further refined in [8]: for every positive constant $c$, the limit is $e^{-c}$ if $N$ grows like $e^{(\frac{2}{\epsilon^2} + \beta) \log n}$ as $n \to \infty$.

We would like to focus on establishing general bounds for some large natural families of distributions. Specifically, suppose that for each dimension $n$, we are given a family $\{\mu_{n,i}\}_{i \in I_n}$ of probability measures such that each $\mu_{n,i}$ is supported on a compact set $V_{n,i}$ in $\mathbb{R}^n$. We would like to find the largest number $N_0$ and the smallest number $N_1$ (in terms of $n$ and some parameters of the family) such that for every $\mu_{n,i}$ from the family, $\frac{E[K_{V_{n,i}}]}{|\text{conv}V_{n,i}|} = o(1)$ for $N \leq N_0$ and $\frac{E[K_{V_{n,i}}]}{|\text{conv}V_{n,i}|} = 1 - o(1)$ for $N \geq N_1$ as $n \to \infty$. ($K_N$ is a random polytope given by (1) with $X_1, X_2, \ldots$ being i.i.d. drawn from $\mu_{n,i}$).

For instance, the examples of the cube and the ball suggest that the family of uniform measures on convex bodies, $N_0$ is exponential and $N_1$ is super-exponential in $n$.

In fact, the latter can be quickly deduced from a classical result by Groemer from [17], combined with the thresholds for Euclidean balls established by Pivovarov in [22]. Groemer’s theorem says that for every $N > n$, we have

\[
E[|\text{conv}\{X_1, \ldots, X_N\}|] \geq E[|\text{conv}\{Y_1, \ldots, Y_N\}|],
\]

where the $X_i$ are i.i.d. uniform on a convex set $K$ and the $Y_i$ are i.i.d. uniform on a Euclidean ball with the same volume as $K$. We thus get from (3) that

\[
\frac{1}{|K|}E[|\text{conv}\{X_1, \ldots, X_N\}|] = 1 - o(1),
\]  

(5)

as long as $N \geq e^{(1+\epsilon)\frac{2}{\epsilon^2} \log n}$.

In this work, we shall establish an exponential bound on $N_0$ for the family of log-concave distributions on convex sets and extend (5) to the family of the so-called $\kappa$-concave distributions.

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## 2 Results

Recall that a Borel probability measure $\mu$ on $\mathbb{R}^n$ is $\kappa$-concave, $\kappa \in [-\infty, \frac{1}{\lambda}]$, if for every $\lambda \in [0, 1]$ and every Borel sets $A, B$ in $\mathbb{R}^n$, we have

\[
\mu(\lambda A + (1 - \lambda)B) \geq \left(\lambda \mu(A)^\kappa + (1 - \lambda)\mu(B)^\kappa\right)^{1/\kappa}.
\]
(for background on \( \kappa \)-concave measures see e.g. [9, 10] or Section 2.1.1 in [11]). We say that a random vector is \( \kappa \)-concave if its law is \( \kappa \)-concave. For example, vectors uniform on convex bodies in \( \mathbb{R}^n \) are \( 1/n \)-concave by the Brunn-Minkowski inequality. The right hand side increases with \( \kappa \), so as \( \kappa \) increases, the class of \( \kappa \)-concave measures becomes smaller. It is a natural extension of the class of log-concave random vectors, corresponding to \( \kappa = 0 \), with the right hand side in the defining inequality understood as the limit \( \kappa \to 0^+ \). Many results for convex sets have analogues for concave measures (for instance, see [4, 5, 6, 14, 18]).

Consider \( \kappa \in (0, 1/n) \). By Borell’s theorem from [9], a \( \kappa \)-concave random vector is supported on a convex body, has a density and its density is a \( 1/\kappa \)-concave function, that is of the form \( h^\beta \) for a concave function \( h \) with \( \beta = \frac{1}{1-\kappa} \). The notion of \( \kappa \)-concavity was introduced and studied by Borell in [9, 10], which are standard references on this topic.

We also recall that a random vector \( X \) in \( \mathbb{R}^n \) is isotropic if it is centred, that is \( \mathbb{E}X = 0 \) and its covariance matrix \( \text{Cov}(X) = [\mathbb{E}X_iX_j]_{i,j \leq n} \) is the identity matrix. The isotropic constant \( L_X \) of a log-concave random vector \( X \) which is isotropic and has density \( f \) on \( \mathbb{R}^n \) is defined as \( L_X = (\text{ess sup}_{\mathbb{R}^n} f)^{1/n} \) (see e.g. [11]). By Borell’s theorem, every log-concave random vector in \( \mathbb{R}^n \) is supported on an affine subspace of \( \mathbb{R}^n \) and has a density with respect to Lebesgue measure on that subspace.

Our first main result suggests a necessary condition on \( N \) (in the form of a lower bound for \( N \) exponential in the dimension \( n \)) so that \( \mathbb{E}|K_N| \) will be significant in the case of symmetric log-concave distributions supported in convex bodies. We recall that a measure \( \mu \) on \( \mathbb{R}^n \) is symmetric (sometimes also called even) if \( \mu(A) = \mu(-A) \) for every \( \mu \)-measurable set \( A \) in \( \mathbb{R}^n \).

**Theorem 1.** Let \( \mu \) be a symmetric log-concave probability measure supported on a convex body \( K \) in \( \mathbb{R}^n \). Let \( X_1, X_2, \ldots \) be i.i.d. random vectors distributed according to \( \mu \). Let \( K_N = \text{conv}\{X_1, \ldots, X_N\} \). There are universal positive constants \( c_1, c_2 \) such that if \( N \leq e^{n/\beta L^2_\mu} \), then

\[
\frac{\mathbb{E}|K_N|}{|K|} \leq e^{-c_2 n/L^2_\mu},
\]

where \( L_\mu \) is the isotropic constant of \( \mu \).

Our second main result provides a sufficient condition on \( N \) so that \( \mathbb{E}|K_N| \) will be significant in the case of \( \kappa \)-concave distributions.

**Theorem 2.** Let \( \mu \) be a symmetric \( \kappa \)-concave measure on \( \mathbb{R}^n \) with \( \kappa \in (0, 1/n) \), supported on a convex body \( K \) in \( \mathbb{R}^n \). Let \( X_1, X_2, \ldots \) be i.i.d. random vectors uniformly distributed according to \( \mu \). Let \( K_N = \text{conv}\{X_1, \ldots, X_N\} \). There is a universal constant \( C \) such that for every \( \omega > C \), if \( N \geq e^{\frac{1}{2}(\log n + 2 \log \omega)} \), then

\[
\frac{\mathbb{E}|K_N|}{|K|} \geq 1 - \frac{1}{\omega}.
\]

### 3 Floating bodies

It turns out that the following quasi-concave function plays a crucial role in estimates for the expected volume of the convex hull of random points (see [2, 3, 12]): for a random vector \( X \) in \( \mathbb{R}^n \) define

\[
q_X(x) = \inf\{P(X \in H) : H \text{ half-space containing } x\}, \quad x \in \mathbb{R}^n.
\]
It is clear that \( q(\lambda x + (1 - \lambda)y) \geq \min\{q(x), q(y)\} \), because if a half-space \( H \) contains \( \lambda x + (1 - \lambda)y \), it also contains \( x \) or \( y \). Consequently, superlevel sets
\[
L_{q, \delta} = \{ x \in \mathbb{R}^n, q_X(x) \geq \delta \}
\]
of this function are convex. Another way of looking at these sets is by noting that they are intersections of half-spaces: \( L_{q, \delta} = \bigcap\{ H : H \text{ is a half-space, } \mathbb{P}(X \in H) > 1 - \delta \} \).
When \( X \) is uniform on a convex set \( K \), they are called convex floating bodies (\( K \setminus L_{q, \delta} \) is called a wet part). The function \( q_X \) in statistics is called the Tukey or half-space depth of \( X \). The two notions have been recently surveyed in [21].

A key lemma from [12] relates the volume of random convex hulls of i.i.d. samples of \( X \) to the volume of the level sets \( L_{q, \delta} \). Bounds on the latter are obtained by a combination of elementary convexity arguments and deep results from asymptotic convex geometry (notably, Paouris’ reversal of the \( L_p \)-affine isoperimetric inequality due to Lutwak, Yang and Zhang). We shall present these and all the necessary background material in Section 4. Section 5 is devoted to our proofs.

4 Auxiliary results

4.1 Log-concave and \( \kappa \)-concave measures

Theorem 4.3 from [10] provides in particular the following stability of \( \kappa \)-concavity with respect to taking marginals: if \( \kappa \in (0, \frac{1}{n}) \) and \( f \) is the density of a \( \kappa \)-concave random vector in \( \mathbb{R}^n \), then
\[
\text{the marginal } x \mapsto \int_{\mathbb{R}^{n-1}} f(x, y) \, dy \text{ is a } \frac{\kappa}{1 - \kappa} \text{-concave function.} \tag{8}
\]
We will also need the following basic estimate: if \( g : \mathbb{R} \to [0, +\infty) \) is the density of a log-concave random variable \( X \) with \( \mathbb{E}X = 0 \) and \( \mathbb{E}X^2 = 1 \), then
\[
\frac{1}{2\sqrt{3e}} \leq g(0) \leq \sqrt{2} \tag{9}
\]
(see e.g. Chapter 10.6 in [1]).

4.2 Central lemma

The idea of using floating bodies to estimate volume of random polytopes goes back to [3]. The following is a key lemma from [12] (called by the authors “central”) about asymptotically matching upper and lower bounds for the volume of the random convex hull.

**Lemma 3** ([12]). Suppose \( X_1, X_2, \ldots \) are i.i.d. random vectors in \( \mathbb{R}^n \). Let \( K_N = \text{conv}\{X_1, \ldots, X_N\} \) and define \( q = q_X \), by (6). Then for every Borel subset \( A \) of \( \mathbb{R}^n \), we have
\[
\mathbb{E}|K_N| \leq |A| + N \cdot \left( \sup_{A^c} q \right) |A^c \cap \{ x \in \mathbb{R}^n, q(x) > 0 \}| \tag{10}
\]
and, if additionally \( \mu \) assigns zero mass to every hyperplane in \( \mathbb{R}^n \), then
\[
\mathbb{E}|K_N| \geq |A| \left( 1 - 2 \left( \frac{N}{n} \right) \left( 1 - \inf_{A} q \right)^{N-n} \right) \tag{11}
\]
(The proof therein concerns only the cube, but their argument repeated verbatim justifies our general situation as well – see also [16].)
4.3 Bounds related to the function \( q \)

Lemma 3 is applied to level sets \( L_{q,\delta} \) of the function \( q \) (see (7)). We gather here several remarks concerning bounds for the volume of such sets. For the upper bound, we will need the containment \( L_{q,\delta} \subset cZ_\alpha(X) \), where \( c \) is a universal constant and \( Z_\alpha \) is the centroid body (defined below). This was perhaps first observed in Theorem 2.2 in [28] (with a reverse inclusion as well). We recall an argument below.

**Remark 4.** Plainly, for the infimum in the definition (6) of \( q_X(x) \), it is enough to take half-spaces for which \( x \) is on the boundary, that is

\[
q_X(x) = \inf_{\theta \in \mathbb{R}^n} \mathbb{P}(\langle X - x, \theta \rangle \geq 0),
\]

where \( \langle u, v \rangle = \sum u_i v_i \) is the standard scalar product in \( \mathbb{R}^n \). Of course, by homogeneity, this infimum can be taken only over unit vectors. We also remark that by Chebyshev’s inequality,

\[
\mathbb{P}(\langle X - x, \theta \rangle \geq 0) \leq e^{-\langle \theta, x \rangle} E_e^{\langle \theta, X \rangle}.
\]

Consequently,

\[
q_X(x) \leq \exp \left( - \sup_{\theta \in \mathbb{R}^n} \left\{ \langle \theta, x \rangle - \log E_e^{\langle \theta, X \rangle} \right\} \right)
\]

and we have arrived at the Legendre transform \( \Lambda^*_X \) of the log-moment generating function \( \Lambda_X \) of \( X \),

\[
\Lambda_X(x) = \log E_e^{\langle x, X \rangle} \quad \text{and} \quad \Lambda^*_X(x) = \sup_{\theta \in \mathbb{R}^n} \{ \langle \theta, x \rangle - \Lambda_X(\theta) \}.
\]

Thus, for every \( \alpha > 0 \), we have

\[
\{ x \in \mathbb{R}^n, \ q_X(x) > e^{-\alpha} \} \subset \{ x \in \mathbb{R}^n, \ \Lambda^*_X(x) < \alpha \}. \tag{13}
\]

**Remark 5.** The level sets \( \{ \Lambda^*_X < \alpha \} \) have appeared in a different context of the so-called optimal concentration inequalities introduced by Latala and Wojtaszczyk in [19]. Modulo universal constants, they turn out to be equivalent to centroid bodies playing a major role in asymptotic convex geometry (see [20, 23, 24, 25, 26]). Specifically, for a random vector \( X \) in \( \mathbb{R}^n \) and \( \alpha \geq 1 \), we define its \( L_\alpha \)-centroid body \( Z_\alpha(X) \) by

\[
Z_\alpha(X) = \{ x \in \mathbb{R}^n, \sup \left\{ \langle x, \theta \rangle, \ E |\langle X, \theta \rangle|^\alpha \leq 1 \right\} \leq 1 \}
\]

(equivalently, the support function of \( Z_\alpha(X) \) is \( \theta \mapsto (E |\langle X, \theta \rangle|^\alpha)^{1/\alpha} \)). By Propositions 3.5 and 3.8 from [19], if \( X \) is a symmetric log-concave random vector \( X \) (in particular, uniform on a symmetric convex body),

\[
\{ \Lambda^*_X < \alpha \} \subset 4e Z_\alpha(X), \quad \alpha \geq 2. \tag{14}
\]

(A reverse inclusion \( Z_\alpha(X) \subset 2^{1/\alpha} e \{ \Lambda^*_X < \alpha \} \) holds for any symmetric random vector, see Proposition 3.2 therein.)

We shall need an upper bound for the volume of centroid bodies. This was done by Paouris (see [25]). Specifically, Theorem 5.1.17 from [11] says that if \( X \) is an isotropic log-concave random vector in \( \mathbb{R}^n \), then

\[
|Z_\alpha(X)|^{1/n} \leq C \sqrt{\frac{\alpha}{n}}, \quad 2 \leq \alpha \leq n, \tag{15}
\]

where \( C \) is a universal constant.
Remark 6. Significant amount of work in [12] was devoted to showing that, for the cube, inclusion (13) is nearly tight (for correct values of $\alpha$, using exponential tilting of measures typically involved in establishing large deviation principles). We shall take a different route and put a direct lower bound on $q_X$ described in the following lemma. Our argument is based on property (8).

Lemma 7. Let $\kappa \in (0, \frac{1}{n})$. Let $X$ be a symmetric isotropic $\kappa$-concave random vector supported on a convex body $K$ in $\mathbb{R}^n$. Then for every unit vector $\theta$ in $\mathbb{R}^n$ and $a > 0$, we have

$$
\mathbb{P}(\langle X, \theta \rangle > a) \geq \frac{1}{16} \kappa \left( 1 - \frac{\alpha}{h_K(\theta)} \right)^{1/\kappa},
$$

where $h_K(\theta) = \sup_{y \in K} \langle y, \theta \rangle$ is the support function of $K$. In particular, denoting the norm given by $K$ as $\| \cdot \|_K$, we have

$$
q_X(x) \geq \frac{1}{16} \kappa \left( 1 - \| x \|_K \right)^{1/\kappa}, \quad x \in K.
$$

Proof. Consider the density $g$ of $\langle X, \theta \rangle$. Let $b = h_K(\theta)$. Note that $g$ is supported in $[-b, b]$. By (8), $g^{1/\kappa}$ is concave, thus on $[0, b]$ we can lower-bound it by a linear function whose values agree at the end points,

$$
g(t) \leq g(0) \left( 1 - \frac{t}{b} \right), \quad t \in [0, b].
$$

This gives

$$
\mathbb{P}(\langle X, \theta \rangle > a) = \int_a^b g(t) dt \geq g(0) \int_a^b \left( 1 - \frac{t}{b} \right)^{1/\kappa} dt = \kappa g(0) \left( 1 - \frac{a}{b} \right)^{1/\kappa}.
$$

Since $\langle X, \theta \rangle$ is in particular log-concave, by (9), we have $\frac{1}{2\sqrt{2\pi}} \leq g(0) \leq \sqrt{2}$. Moreover, by isotropicity,

$$
1 = \mathbb{E} \langle X, \theta \rangle^2 = \int_{-b}^b t^2 g(t) dt \leq b^2 \int_{-b}^b g(t) dt = b^2.
$$

Thus, say $g(0)b > \frac{1}{16}$ and we get (16). To see (17), first recall (12). By symmetry, $\mathbb{P}(\langle X - x, \theta \rangle \geq 0) = \mathbb{P}(\langle X, \theta \rangle \geq \| \langle x, \theta \rangle \|)$, so we use (16) with $a = \| \langle x, \theta \rangle \|$ and note that by the definition of $h_K$, $\left( \frac{\langle x, \theta \rangle}{h_K(\theta)} \right) \leq h_K(\theta), \quad \frac{\langle x, \theta \rangle}{h_K(\theta)} \leq \| x \|_K$. □

5 Proofs

5.1 Proof of Theorem 1

Since the quantity $\frac{\mathbb{E}|K_N|}{|K|}$ does not change under invertible linear transformations applied to $\mu$, without loss of generality we can assume that $\mu$ is isotropic. Let $q = q_X$ be defined by (6). Fix $\alpha > 0$ and apply (10) to the set $A = \{ x, \ q(x) > e^{-\alpha} \}$. We get

$$
\frac{\mathbb{E}|K_N|}{|K|} \leq \frac{|A|}{|K|} + Ne^{-\alpha}
$$

(we have used $\{ x, \ q(x) > 0 \} \subset K$ to estimate the last factor in (10) by 1). Combining (13), (14) and (15),

$$
|A| \leq |4eZ_{\alpha}(X)| \leq \left( 4eC \sqrt{\frac{\alpha}{n}} \right)^n.
$$
Moreover, by the definition of the isotropic constant of $\mu$,

$$1 = \int_K d\mu \leq L_n^\alpha |K|.$$  

Thus,

$$\frac{|A|}{|K|} \leq \left( 4eCL\mu \sqrt{\frac{\alpha}{n}} \right)^n.$$  

We set $\alpha$ such that $4eCL\mu \sqrt{\frac{\alpha}{n}} = e^{-1}$ and adjust the constants to finish the proof. \hfill \square

5.2 Proof of Theorem 2

As in the proof of Theorem 1, we can assume that $\mu$ is isotropic. Let $q = q\mathcal{X}_n$ be defined by (6). Consider $0 < \beta < 1$ (to be fixed shortly). By (11) which we apply to the set $A = \{x \in K, \ q(x) > \beta^{1/\kappa}\}$, we have

$$E|K|N|K| \geq |A| |K| \left( 1 - 2\frac{n}{N} \left( 1 - \beta^{1/\kappa} \right)^{N-n} \right).$$  

(The extra assumption needed in (11) is satisfied: by Borell’s theorem from [9], $\mu$ has a density on its support which by our assumption is $n$-dimensional, hence $\mu(H) = 0$ for every hyperplane $H$ in $\mathbb{R}^n$.) By the lower bound on $q$ from (17),

$$A \supset \{x \in \mathbb{R}^n, \ ||x||_K \leq 1 - (16\kappa^{-1})^{\kappa}\beta\},$$

hence, as long as $(16\kappa^{-1})^{\kappa}\beta < 1$,

$$\frac{|A|}{|K|} \geq 1 - (16\kappa^{-1})^{\kappa}\beta \geq 1 - n(16\kappa^{-1})^{\kappa}\beta \geq 1 - 32n\beta.$$  

We choose $\beta$ such that $32n\beta = 1/e^\omega$ and crudely deal with the second term,

$$\left( \frac{n}{N} \right) \left( 1 - \beta^{1/\kappa} \right)^{N-n} \leq N^n e^{-\beta^{1/\kappa}(N-n)},$$

which is nonincreasing in $N$ as long as $N \geq n\beta^{-1/\kappa}$. This holds for $\omega$ large enough if, say $N \geq n^{1/\kappa}\omega^{2/\kappa}$. Then we easily conclude that the dominant term above is $e^{-\beta^{1/\kappa}N}$ which yields, say

$$E|K|N|K| \geq \left( 1 - \frac{1}{2\omega} \right) \left( 1 - 2e^{-\omega^{2/\kappa}} \right) \geq 1 - \frac{1}{\omega},$$

provided that $n$ and $\omega$ are large enough. \hfill \square

6 Final remarks

**Remark 8.** Groemer’s result used in (5) for uniform distributions has been substantially generalised by Paouris and Pivovarov in [27] to arbitrary distributions with bounded densities. We remark that in contrast to (5), using the extremality result of the ball from [27] does not seem to help obtain bounds from Theorem 2 for two reasons. For one, it concerns bounded densities and rescaling will cost an exponential factor. Moreover, for the example of $\beta$-polytopes from [7], we have that they are generated by $\kappa$-concave measures with $\kappa = \frac{1}{\beta+n}$ and the sharp threshold for the volume is of the order $n^{(\beta+n/2)}$ (see (3)). The ball would give that $N_1 = n^{(1+\epsilon)n/2}$ points is enough.
Remark 9. The example of beta polytopes from (3) shows that the bound on $N$ in Theorem 2 has to be at least of the order $n^{\beta_2+n/2} = n^{1+n/2} \geq n^{1/2}$. Our bound $n^{1/2}$ is perhaps suboptimal. It is not inconceivable that as in the uniform case, the extremal example is supported on a Euclidean ball.

Remark 10. It is reasonable to ask about sharp thresholds like the ones in (2), (3) and (4) for other sequences of convex bodies, say simplices, cross-polytopes, or in general $\ell_p$-balls. This is a subject of ongoing work. We refer to [15] for recent results establishing exponential nonsharp thresholds for a simplex (i.e. with a gap between the constants for lower and upper bounds).

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