RISK SHARING UNDER HETEROGENEOUS BELIEFS WITHOUT CONVEXITY

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Abstract. We consider the problem of finding Pareto-optimal allocations of risk among finitely many agents. The associated individual risk measures are law invariant, but with respect to agent-dependent and potentially heterogeneous reference probability measures. Moreover, we assume that the individual risk assessments are consistent with the respective second-order stochastic dominance relations, but remain agnostic about their convexity. A simple sufficient condition for the existence of Pareto optima is provided. The proof combines local comonotone improvement with a Dieudonné-type argument, which also establishes a link of the optimal allocation problem to the realm of “collapse to the mean” results.

1. Introduction

This paper addresses the problem of finding optimal allocations of risk among finitely many agents, i.e., optimisers for the problem

\[
\text{Minimise } \sum_{i \in I} \rho_i(X_i) \quad \text{subject to } \sum_{i \in I} X_i = X. \tag{1.1}
\]

The agents under consideration form a finite set \(I\). Each agent \(i \in I\) measures the risk of net losses \(Y\) in a space \(\mathcal{X}\) with a risk measure \(\rho_i\). Given the total loss \(X\) collected in the system, an allocation attributes a portion \(X_i \in \mathcal{X}\) to each agent, i.e., the condition \(\sum_{i \in I} X_i = X\) holds. We do not impose any restriction on the notion of an allocation, i.e., every vector whose coordinates sum up to \(X\) is hypothetically feasible. An allocation of \(X\) is optimal (or Pareto optimal) if it minimises the aggregated risk \(\sum_{i \in I} \rho_i(X_i)\) in the system. This risk-sharing problem is of fundamental importance for the theory of risk measures, for capital allocations in capital adequacy, and in the design and discussion of regulatory frameworks; cf. [30, 60, 62].

Usually, the space of net losses appearing in (1.1) is an infinite-dimensional space of random variables, which complicates finding optima. Individual risk measures are often assumed to be convex and monetary, standard axioms in the theory of risk measures since [33, 36]. Likewise, in the closely related field of utility assessment via variational preferences, concavity assumptions are common; cf. [53]. Convexity (or concavity) alone is not sufficient though. A powerful solution theory has so far

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Date: February 22, 2023.

Acknowledgements: I would like to thank two anonymous referees, Cosimo Munari, Gregor Svindland, Ruodu Wang, participants of research seminars at the University of Waterloo, the Amsterdam School of Economics, and the workshop “Risk Measures and Uncertainty in Insurance” at the University of Hannover for valuable comments and discussions related to this work.
mostly been established under the additional assumption of law invariance, meaning that the risk of a random variable $X$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ merely depends on its distribution under the reference measure $\mathbb{P}$. A rich strand of literature has studied the wide-ranging analytic consequences of law invariance in conjunction with convexity; see, e.g., [11, 20, 32, 42, 45, 48, 58] and the cited references. Studies of the risk sharing problem for convex monetary risk measures (or equivalently, concave monetary utility functions) are [1, 9, 15, 31, 43]. Risk sharing problems with special law-invariant, but not necessarily convex functionals, are considered in [27, 51].

Abstractly speaking, the solution theory in the law-invariant case can be split in two steps.

**Step 1:** Reduce the set of optimisation-relevant allocations to comonotone allocations. This reduction is mainly driven by the fact that convexity and (semi)continuity properties in conjunction with law invariance usually imply monotonicity of the involved functionals in the so-called convex order. Therefore, the comonotone improvement results of [17, 31, 44, 52] can be used.

**Step 2:** Find suitable bounds which prove that relevant comonotone allocations form a compact set. That enables approximation procedures and the selection of converging optimising sequences. Under cash-additivity of the involved functionals, Step 2 poses no problem and can be achieved by a simple exchange of cash among the involved agents; cf., e.g., [31].

Steps 1 and 2 of the previous arguments rely fundamentally on the existence of a single “objective” reference probability measure which is shared and accepted by all agents in $I$. A more recent and quickly growing strand of literature in finance, insurance, and economics, however, dispenses with this paradigm. Two economic considerations motivate these heterogeneous reference probability measures. Firstly, different agents may have access to different sources of information, resulting in information asymmetry and in different subjective beliefs. Secondly, agents may entertain heterogeneous probabilistic subjective beliefs as a result of their preferences or their use of different internal models.

Applying heterogeneous (probabilistic) beliefs in risk sharing and related problems, [6] explores the demand for insurance when the insurer exhibits ambiguity, whereas the insured is an expected-utility agent. Under heterogeneous reference probabilities for insurer and insured, [13] provides optimal reinsurance designs, and [22, 39] generalise Arrow’s “Theorem of the Deductible”. [14] studies bilateral risk sharing with exposure constraints and admits a very general relation between the two involved reference probabilities. [7, Section 6] consider optimal risk sharing under heterogeneous reference probabilities affected by exogenous triggering events, while [23] studies the existence of Pareto optima and equilibria in a finite-dimensional setting when optimality of allocations is assessed relative to individual and potentially heterogeneous sets of probabilistic priors. [28] studies risk sharing for Value-at-Risk and Expected Shortfall agents endowed with heterogeneous reference probability measures, and provides explicit formulae for the shape of optimal allocations. [50] makes similar contributions to weighted risk sharing with distortion risk measures under heterogeneous beliefs. Finally, an important point of reference is [3]. In that paper, two agents with heterogeneous reference probability models try to share risks optimally, and each of them measures individual risk with a law-invariant concave monetary utility function. However, the risk sharing problem is constrained to random variables over a finite $\sigma$-algebra – which reduces the optimisation problem to a finite-dimensional space –, one of

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1 These two steps are separated explicitly in, e.g., [48], where also numerous economic optimisation problems are analysed according to that scheme.
the two reference probabilities involved only takes rational values on the finite $\sigma$-algebra in question, and the mathematical techniques are quite different from the ones used here.

Against the outlined backdrop of the existing literature, several key features of our results stand out.

**Infinite-dimensional setting:** Whereas dealing with heterogeneous reference probability has sometimes been facilitated by restriction to a finite dimensional setting, we consider problem (1.1) for the space $\mathcal{X}$ of all bounded random variables over an atomless probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a bona fide infinite-dimensional space.

**True generalisation of most known results:** As already stated, we consider a finite set $\mathcal{I}$ of agents, each measuring risk with a functional $\rho_i : L^\infty \rightarrow \mathbb{R}$ which is law invariant with respect to a probability measure $\mathbb{Q}_i$ equivalent to $\mathbb{P}$. The latter only plays the role of a gauge and we are free to assume $\mathbb{P} = \mathbb{Q}_j$ for some $j \in \mathcal{I}$. Our main result in Section 4 provides mild sufficient conditions for the existence of optimal risk allocations without a rationality condition on involved probabilities as in [3]. Mathematically, key to our strategy is to adapt the two-step procedure of comonotone improvement outlined above to the case of heterogeneous reference probabilities. This will mostly be achieved in Appendix B. However, the simplification of Step 2 in that procedure via “rebalancing cash” is not an option anymore under heterogeneity. Instead, finding suitable bounds for optimisation-relevant allocations becomes crucial. While optimal allocations will usually not be comonotone in the heterogeneous case, our procedure nevertheless unifies various existence results under a single reference probability measure and extends them to the case of multiple reference measures.

**Nonconvex risk measures:** We do not assume convexity (or concavity) of the involved functionals. Instead, we work with three axiomatic properties of risk measures introduced and studied in the recent literature as alternatives to (quasi)convexity. While this trend is in large parts motivated by the lacking convexity of many distortion risk measures such as the Value-at-Risk, [5, Section 1.2] presents further critical remarks on subadditivity. A requirement we shall impose throughout our study is the assumption that individual risk measures be consistent, a property recently axiomatised by [54]. While many desirable characterisations are shown to be equivalent, their eponymous feature is monotonicity with respect to second-order stochastic dominance. Each law-invariant and convex monetary risk measure is consistent (up to an affine transformation), but the converse implication does not hold.

The second ingredient – which is mostly of technical relevance and makes the assumptions in the main results particularly satisfiable – is star shapedness. Star-shaped risk measures have been studied systematically in the recent working paper [18]. They are motivated, for instance, by the observation that subadditive risk measures intertwine the measurement of concentration and liquidity effects (by curves $\mathbb{R}_+ \ni t \mapsto \rho(tX)$) with diversification benefits from merging portfolios. The latter is translated by (quasi)convexity of the risk measures in question. Star-shaped risk measures are more agnostic and replace (quasi)convexity by the demand that decreasing exposition to an acceptable loss profile (having at most neutral risk) does not lead to a loss of acceptability:

$$\forall X \in \mathcal{X} \forall \lambda \in [0, 1] : \ \rho(X) \leq 0 \implies \rho(\lambda X) \leq 0.$$  

Further economic motivation is discussed in [18, Section 2].

Third, we shall require a certain compatibility of probabilistic beliefs and assume the existence of a finite measurable partition $\pi$ of $\Omega$ such that the agents agree on the associated conditional distributions. This is akin to and generalises the (much more specific) setting of [55], one of the earliest contributions on heterogeneous reference measures. While Section 3 will reveal all necessary details
and shed more light on this assumption, we anticipate here the consequence that individual risk measures fall in the most relevant class of \emph{scenario-based risk measures} recently introduced in \cite{61}. A desirable aspect of the latter perspective is that scenario-basedness is preserved under the infimal convolution operation (cf. Corollary 4.8), while the individual law invariance is lost in the general heterogeneous case.

The paper unfolds as follows. Section 1.1 collects preliminaries. Consistent and star-shaped risk measures and their admissibility for our main results are studied at length in Section 2. In Section 3, we carefully introduce our setting of heterogeneous reference probability measures and motivate the buttressing assumption from different angles. The main result, Theorem 4.1, is stated in Section 4 and presents a sufficient condition under which heterogeneous agents endowed with consistent risk measures can find optimal allocations. The remainder of Section 4 is devoted to developing a profound understanding of these assumptions and to the formulation of related results. All proofs, mathematical details, and auxiliary results can be found in Appendices A–E.

1.1. Preliminaries. We first outline terminology, notation, and conventions adopted throughout the paper.

\textbf{General terminology:} The effective domain of a function \( f : S \to [-\infty, \infty] \) defined on a nonempty set \( S \) is \( \text{dom}(f) := \{ s \in S \mid f(s) \in \mathbb{R} \} \). For an arbitrary natural number \( K \in \mathbb{N} \), we denote the set \( \{1, \ldots, K\} \) by \([K]\). Bold-faced symbols denote vectors of objects.

\textbf{Function spaces:} The absolute continuity relation between two probability measures \( Q \) and \( P \) on \((\Omega, \mathcal{F})\) is denoted by \( Q \ll P \), and equivalence of probability measures by \( Q \approx P \). Fix a probability space \((\Omega, \mathcal{F}, P)\). \( L_0^\infty \) denotes the space of equivalence classes up to \( P\)-almost-sure (\( P\)-a.s.) equality of real-valued random variables over \((\Omega, \mathcal{F}, P)\). The subspaces of equivalence classes of bounded and \( P\)-integrable random variables are denoted by \( L_\infty^\infty \) and \( L_1^1 \), respectively. All these spaces are canonically equipped with the \( P\)-a.s. order \( \leq_p \), and all appearing (in)equalities between random variables are to be understood in this sense. We denote the respective positive cones by \( L_\infty^\infty^+ \) and \( L_1^1^+ \), and the supremum norm on \( L_\infty^\infty \) by \( \| \cdot \|_\infty \). If we consider the spaces \( L_\infty^\infty \) and \( L_1^1 \) with respect to a probability measure \( Q \not\equiv P \) on \((\Omega, \mathcal{F})\), we shall write \( L_\infty^\infty_Q \) and \( L_1^1_Q \). (Conditional) expectations (given a sub-\( \sigma\)-algebra \( G \subset \mathcal{F} \)) computed with respect to a measure \( Q \) are denoted by \( \mathbb{E}_Q[\cdot] \) (\( \mathbb{E}_Q[\cdot|G] \), respectively). As usual, we shall identify the dual space of \( L_\infty^\infty \), comprised of all bounded linear functionals on that space, via the Dunford-Schwartz integral with the space \( \text{ba} \) of all finitely additive set functions with bounded total variation that are absolutely continuous with respect to \( P \), i.e., \( \mu(N) = 0 \) whenever \( P(N) = 0 \). Often, the subspace of countably additive signed measures in \( \text{ba} \) will be identified with \( L_1^1 \). At last, note that every \( \mu \) in the positive cone \( \text{ba}_+ \) has a unique \textit{Yosida-Hewitt decomposition} as the sum of a measure \( \zeta \ll P \) and a pure charge \( \tau \), i.e., \( \tau \in \text{ba}_+ \) and, for a suitable vanishing sequence of events \( (B_n) \subset \mathcal{F}, \tau(B_n^c) = 0 \) for all \( n \in \mathbb{N} \). We write \( \mu = \zeta \oplus \tau \) or \( \mu = R \oplus \tau \), where \( R := \frac{d\zeta}{d\mu} \) is the \( P\)-density of \( \zeta \). Note that we shall abuse notation slightly and denote integrals with respect to finitely additive probabilities \( \mu \) by \( \mathbb{E}_\mu[\cdot] \).

\textbf{Probability measures and distributions:} Throughout the paper, the underlying fixed probability space \((\Omega, \mathcal{F}, P)\) is \textit{atomless}, i.e., we can define a random variable on it whose cumulative distribution function under \( P \) is continuous. In particular, \( P \) has \textit{convex range}, i.e., for all \( A \in \mathcal{F} \) and all \( p \in [0, P(A)] \) we find \( B \in \mathcal{F} \) with the properties \( B \subset A \) and \( P(B) = p \). Each probability measure \( Q \ll P \) is also atomless.
II denotes the set of finite measurable partitions \( \pi \subset \mathcal{F} \) of \( \Omega \) satisfying \( P(B) > 0 \) for all \( B \in \pi \). For a probability measure \( Q \) on \( (\Omega, \mathcal{F}) \) and an event \( B \in \mathcal{F} \) with \( Q(B) > 0 \), we define the conditional probability measure \( Q^B : \mathcal{F} \to [0,1] \) by 
\[
Q^B(A) := \frac{Q(A \cap B)}{Q(B)}.
\]
If for two elements \( X, Y \in L^0 \) and a probability measure \( Q \ll P \) the distributions \( Q \circ Y^{-1} \) of \( Y \) under \( Q \) agrees with the distribution \( Q \circ X^{-1} \) of \( X \) under \( Q \), we shall write \( X \sim_Q Y \). A subset \( \mathcal{A} \subset L^0 \) is \textit{law invariant} with respect to a probability measure \( Q \ll P \) (or \( Q \)-law invariant) if
\[
X \in \mathcal{A} \text{ and } X \sim_Q Y \implies Y \in \mathcal{A}.
\]
Given nonempty sets \( \mathcal{A} \subset L^0 \) and \( S \), a function \( f : \mathcal{A} \to S \) is law invariant with respect to a probability measure \( Q \ll P \) if
\[
X, Y \in \mathcal{A} \text{ and } X \sim_Q Y \implies f(X) = f(Y).
\]
Given \( X \in L^0 \) and a probability measure \( Q \ll P \), \( q_X^Q \) denotes the quantile function of \( X \) under \( Q \) defined by
\[
q_X^Q(s) := \inf \{ x \in \mathbb{R} \mid Q(X \leq x) \geq s \}, \quad s \in (0,1).
\]
Unif(0,1) denotes the uniform distribution over the interval (0,1).

\textbf{Risk measures:} A \textit{monetary risk measure} \( \rho : L^\infty \to \mathbb{R} \) is a map that is:
(a) \textit{monotone}, i.e., for \( X, Y \in L^\infty \) with \( X \leq Y \), \( \rho(X) \leq \rho(Y) \) holds.
(b) \textit{cash-additive}, i.e.,
\[
\forall X \in L^\infty \forall m \in \mathbb{R} : \quad \rho(X + m) = \rho(X) + m.
\]
As mentioned in the introduction and reflected by the preceding definition, risk measures are applied to losses net of gains in this manuscript, not gains net of losses. In particular, nonnegative random variables correspond to pure losses. A monetary risk measure \( \rho : L^\infty \to \mathbb{R} \) is \textit{normalised} if \( \rho(0) = 0 \). Prominent normalised risk measures that will be used throughout the paper are:

(a) the \textit{entropic risk measure} with parameter \( \beta > 0 \) under \( Q \ll P \):
\[
\text{Entr}^Q_\beta(X) := \frac{1}{\beta} \log \left( \mathbb{E}_Q[e^{\beta X}] \right), \quad X \in L^\infty.
\]
(b) the \textit{Expected Shortfall} at level \( p \in [0,1] \) under \( Q \ll P \):
\[
\text{ES}_p^Q(X) = \begin{cases} 
\frac{1}{1-p} \int_p^1 q_X^Q(s) \, ds & p \in [0,1), \\
\sup_{s \in (0,1)} q_X^Q(s) & p = 1,
\end{cases} \quad X \in L^\infty.
\]
The \textit{acceptance set} \( \mathcal{A}_\rho := \{ X \in \mathcal{X} \mid \rho(X) \leq 0 \} \) of a monetary risk measure \( \rho \) collects all loss profiles that bear neutral risk. By monotonicity and cash-additivity, \( \rho \) is a norm-continuous function and the acceptance set \( \mathcal{A}_\rho \) is closed. \( \rho \) can also be recovered from \( \mathcal{A}_\rho \) via the formula
\[
\rho(X) = \inf \{ m \in \mathbb{R} \mid X - m \in \mathcal{A}_\rho \}, \quad X \in L^\infty.
\]
The \textit{asymptotic cone} of the acceptance set \( \mathcal{A}_\rho \) of a risk measure \( \rho \) is the set \( \mathcal{A}_\rho^\infty \) of all \( U \in L^\infty \) that can be represented as \( U = \lim_{k \to \infty} s_k Y_k \) for a vanishing sequence \( (s_k) \subset (0,\infty) \) and a sequence \( (Y_k) \subset \mathcal{A}_\rho \). More information on asymptotic cones in a finite-dimensional setting can be found in [8, Chapter 2].

At last, consider an arbitrary function \( f : L^\infty \to [-\infty,\infty] \). On \textbf{ba} we define the \textit{convex conjugate} of \( f \) as
\[
f^*(\mu) = \sup_{X \in L^\infty} \left\{ \int X \, d\mu - f(X) \right\}, \quad \mu \in \textbf{ba}.
\]
2. Admissible risk measures

2.1. Consistent and star-shaped risk measures. This paper proves the existence of optimal risk sharing schemes for consistent risk measures introduced in [54]. Consistency means that the risk assessment respects the second-order stochastic dominance relation between arguments. Given random variables $X, Y \in L^\infty$ and a probability measure $Q \ll P$, recall that $Y$ dominates $X$ in $Q$-second-order stochastic dominance relation if $\mathbb{E}_Q[v(X)] \leq \mathbb{E}_Q[v(Y)]$ holds for all convex and nondecreasing functions $v: \mathbb{R} \to \mathbb{R}$. A normalised monetary risk measure $\rho: L^\infty \to \mathbb{R}$ is a $Q$-consistent risk measure (cf. [54]) if, whenever $Y \in L^\infty$ dominates $X \in L^\infty$ in second-order stochastic dominance relation under $Q$, then also $\rho(X) \leq \rho(Y)$.

Each normalised, convex, and $Q$-law-invariant monetary risk measure is $Q$-consistent, but also minima of such risk measures (cf. [54, Proposition 3.2]). More precisely, by [54, Theorem 3.3], for each consistent risk measure $\rho$ there is a set $\mathcal{T}$ of $Q$-law-invariant and convex monetary risk measures $\tau$ such that, for all $X \in L^\infty$, $\rho(X) = \min_{\tau \in \mathcal{T}} \tau(X)$. A direct consequence of this representation is the following formula for the convex conjugate $\rho^*$ of $\rho$:

$$\rho^* = \sup_{\tau \in \mathcal{T}} \tau^*(\cdot). \quad (2.1)$$

For every $Q$-consistent risk measure $\rho$, $\rho \geq \mathbb{E}_Q[\cdot]$ and $\rho^*(d\mathbb{P}/d\mathbb{P}) = 0$ holds. Indeed, each $X \in L^\infty$ admits the estimate

$$\rho(X) \geq \rho(\mathbb{E}_Q[X]) = \mathbb{E}_Q[X] = \mathbb{E}_P[d\mathbb{Q}/d\mathbb{P} X].$$

A normalised monetary risk measure $\rho: L^\infty \to \mathbb{R}$ is star shaped (cf. [18]) if the acceptance set $A_\rho$ is a star-shaped set, i.e., $sY \in A_\rho$ holds for all pairs $(s, Y) \in [0, 1] \times A_\rho$. Star-shaped risk measures have recently been studied in detail in [18], and in our definition we implicitly invoke their characterisation provided in [18, Proposition 2]. Each normalised convex monetary risk measure is star shaped; like consistency, star shapedness is a weaker property than convexity. We also remark that the class of star-shaped consistent risk measures is discussed in [18, Theorem 4]. While they will mostly play a technical role in our study, further background on them is provided in Lemma A.4.

2.2. Admissibility of consistent risk measures. Given $Q \ll P$, we now isolate which $Q$-consistent risk measures $\rho$ are admissible for our main results. As preparation, the conjunction of cash-additivity and monotonicity implies that a set function $\mu \in \mathbf{ba}$ satisfies $\rho^*(\mu) < \infty$ only if $\mu$ is a finitely additive probability. We also recall that we identify countably additive elements in $\mathbf{ba}$ with $L^1$ via $D \mapsto \mu_D$, $\mu_D: \mathcal{F} \to \mathbb{R}$ being defined by $\mu_D(A) = \mathbb{E}_P[D1_A]$. (Note that the reference probability under which we integrate is always $P$, even though we may consider a $Q$-consistent risk measure, $Q \neq P$.) Hence, $D \in L^1$ lies in $\text{dom}(\rho^*)$ only if $D \in L^1_+$ and $\mathbb{E}_P[D] = 1$; i.e., these elements are probability densities.

**Definition 2.1.** A $Q$-consistent risk measure $\rho: L^\infty \to \mathbb{R}$ is admissible if there is a probability density $D \in L^1_+$ with the following two properties:

(a) $\rho^*(D) < \infty$.
(b) If $U \in A_\rho^\infty$ satisfies $\mathbb{E}_P[DU] = 0$, then $U = 0$.

$\mathcal{C}(\rho)$ denotes the set of all compatible $D$, i.e., $D$ has properties (a)–(b) above.

Definition 2.1 is best illustrated in the special case where $\rho$ is convex. Convexity entails that the asymptotic cone $A_\rho^\infty$ collects acceptable net losses $U$ of particular quality; arbitrary quantities $tU$, $t > 0$, thereof are still acceptable. If $\mathbb{E}_P[DU] = 0$ for a strictly positive probability density $D \in L^1_+$
and $U$ is not constant, then it must take negative and positive values, thus triggering net gains and net losses. If $\rho$ is sufficiently risk averse and gains can never fully compensate large losses obtaining in different states, the assertion that $U$ is acceptable in arbitrary volumes seems questionable.

It is well possible that not every density in $\text{dom}(\rho^*) \cap L^1$ is compatible with $\rho$. For instance, the entropic risk measure $\rho = \text{Entr}_1$ satisfies $\text{dom}(\rho^*) = \{D \in L^1_+ \mid \mathbb{E}_D[D] = 1\}$ and $\mathcal{A}_\rho^\infty = -L^\infty_+$. In particular, a probability density $D \in L^1_+$ lies in $\mathcal{C}(\text{Entr}_1)$ if and only if $D > 0 \mathbb{P}$-a.s.

**Lemma 2.2.** Suppose $\mathbb{Q} \ll \mathbb{P}$ and that $\rho$ is an admissible $\mathbb{Q}$-consistent risk measure. Then the following assertions hold:

1. $\mathbb{Q} \approx \mathbb{P}$.
2. Each $D \in \mathcal{C}(\rho)$ satisfies $\mathbb{P}(D > 0) = 1$.
3. For all $Z \in \text{dom}(\rho^*) \cap L^1$, $D \in \mathcal{C}(\rho)$, and $\lambda \in (0, 1]$,

$$\lambda D + (1 - \lambda)Z \in \mathcal{C}(\rho).$$  

(2.2)

The notions of admissibility and compatibility are of central importance for the second step in the quest for optimal allocations outlined in the introduction: extracting a convergent optimising sequence. They thus appear prominently in Theorem 4.1 below. The next proposition presents an exhaustive description of admissible risk measures.

**Proposition 2.3.** Suppose $\mathbb{Q} \approx \mathbb{P}$ and that $\rho$ is a $\mathbb{Q}$-consistent risk measure. Equivalent are:

1. $\rho$ is admissible.
2. $\frac{\partial \mathbb{Q}}{\partial \mathbb{P}} \in \mathcal{C}(\rho)$.
3. $\text{dom}(\rho^*)$ contains at least two elements.
4. $\text{dom}(\rho^*) \cap L^1$ contains at least two elements.
5. There is a finitely additive probability $\nu \in \text{dom}(\rho^*)$ such that, whenever $U \in \mathcal{A}_\rho^\infty$ satisfies $\mathbb{E}_U[U] = 0$, then also $U = 0$.

Moreover, statements (1)–(5) all imply

6. There is no constant $\beta > 0$ such that $\rho \leq \mathbb{E}_Q[\cdot] + \beta$.

Note that point (6) in Proposition 2.3 means that an admissible consistent risk measure is necessarily more conservative than determining the expected loss under the reference measure $\mathbb{Q}$ and adding a safety margin. While disproving (6) therefore serves as an easy check to admissibility, verification of (6) does not suffice to verify admissibility.

**Example 2.4.** Define a risk measure $\tau : L^\infty \to \mathbb{R}$ by

$$\tau(X) := \inf \left\{ \text{ES}_{1/n}^\mathbb{P}(X) + n \mid n \in \mathbb{N} \setminus \{1\} \right\}$$

and consider the $\mathbb{P}$-consistent risk measure $\rho := \min\{\text{ES}_{1/n}^\mathbb{P}, \tau\}$. Then $\rho$ is not star shaped (and therefore also not convex). Next we observe that assertion (6) in Proposition 2.3 applies. To see this, nonatomicity of $(\Omega, \mathcal{F}, \mathbb{P})$ yields a sequence $(X_k) \subset L^\infty$ satisfying $\mathbb{P}(X_k = -k^2) = 1 - \mathbb{P}(X_k = 0) = 0$, $k \in \mathbb{N}$. For $k \geq 2$,

$$\text{ES}_{1/n}^\mathbb{P}(X_k) = \begin{cases} 0 & k \leq n, \\ \frac{-k(n-k)}{n-1} & k > n. \end{cases}$$

A direct computation therefore shows $\rho(X_k) = 0$ for all $k \geq 2$, while $\mathbb{E}_\mathbb{P}[X_k] = -k$. Nevertheless, $\mathcal{C}(\rho) = \emptyset$. Indeed, for $n \geq 2$, $\text{dom}(\text{ES}_{1/n}^\mathbb{P}) = \{Z \in L^\infty_+ \mid \mathbb{E}_{\mathbb{P}}[Z] = 1, Z \leq \frac{n}{n-1}\}$. By (2.1), a
probability density $Z \in L^1_+$ lies in $\text{dom}(\rho^*)$ only if $Z \leq \frac{n}{n-1}$ for all $n \geq 2$, i.e., only if $Z = 1$. Hence, $C(\rho) = \emptyset$ follows with the equivalence between points (1) and (3) in Proposition 2.3.

An even simpler positive condition guaranteeing admissibility is available if $\rho$ is additionally star shaped. Admissibility then boils down to $\rho$ not agreeing with the expectation under $Q$. The assumptions of Theorem 4.1 are therefore particularly mild for star-shaped consistent risk measures.

**Proposition 2.5.** Let $Q \approx P$ and $\rho$ be a $Q$-consistent and star-shaped risk measure. Then $\rho$ is admissible if and only if $\rho \neq E_Q[\cdot]$.

Note that Example 2.4 also shows that Proposition 2.5 fails without the assumption of star shapedness.

We close this section with a few more technical remarks.

**Remark 2.6.**

(1) If $\rho$ is admissible, $C(\rho)$ contains uncountably many different densities and is dense in $\text{dom}(\rho^*) \cap L^1$.

This follows from (2.2) together with points (2) and (4) of Proposition 2.3.

(2) From Propositions 2.3 and 2.5 one can conclude that a $Q$-consistent risk measure $\rho$ is admissible if and only if its biconjugate $\rho^{**}: L^\infty \to \mathbb{R}$ defined by

$$
\rho^{**}(X) = \sup_{\mu \in \text{dom}(\rho^*)} \left\{ E_\mu[X] - \rho^*(\mu) \right\}
$$

satisfies $\rho^{**} \neq E_Q[\cdot]$. This is due to $\rho^{**}$ being the “convexification” of $\rho$, i.e., the largest convex $Q$-consistent risk measure dominated by $\rho$ (cf. proof of [46, Lemma F.3]).

(3) The spirit of Definition 2.1 and, more specifically, the equivalence in Proposition 2.5 establishes a link to the realm of “collapse to the mean” results. The latter subsumes incompatibility results between law invariance of a functional and the existence of “directions of linearity”. The asymptotic cone of the acceptance set of a consistent risk measure can be understood to collect such directions. In the theory of risk measures, “collapse to the mean results” go back at least to [37]. We refer to the more detailed discussion in [46].

### 3. Reference probabilities

Throughout the remainder of the paper, we shall consider $n \geq 2$ agents identified by integers $i \in [n]$. With each of them is associated a probability measure $Q_i \ll P$, and $i$ measures risks of net losses with a $Q_i$-consistent risk measure $\rho_i: L^\infty \to \mathbb{R}$. Note that for different agents $i \neq j$ both $Q_i = Q_j$ and $Q_i \neq Q_j$ are possible. The next crucial structural assumption concerns the vector $(Q_1, \ldots, Q_n)$ of reference probability measures on $(\Omega, \mathcal{F})$.

**Assumption 3.1.** For each $i \in [n]$, $Q_i$ is equivalent to $P$ and $\frac{dQ_i}{dP}$ is a simple function. That is, there is a partition $\pi \in \Pi$ such that

$$
Q_i(E) = \sum_{B \in \pi} Q_i(B)P^B(E), \quad E \in \mathcal{F}. \tag{3.1}
$$

The main results in Section 4 will make heavy use of Assumption 3.1. It injects a sufficient degree of finite dimensionality in the problem to adapt the procedure of comonotonic improvement to the present heterogeneous setting. This is discussed in detail in Remark B.4. In view of its prominence, we shall discuss Assumption 3.1 at length from five conceivable angles.

Consider the finite set $\mathcal{P} := \{P^B \mid B \in \pi\}$ of mutually singular probability measures. If two random variables $X, Y \in L^\infty$ agree in distribution under each $P^B \in \mathcal{P}$, (3.1) implies that $X \sim_{Q_i} Y$, $i \in [n]$. 
Hence, \( \rho_1, \ldots, \rho_n \) are \( \mathcal{P} \)-based risk measures, a notion recently introduced in [61]. Whenever two random variables \( X, Y \in L^\infty \) satisfy \( X \sim_{\mathcal{P}} Y \) for all \( \mathcal{P}^B \in \mathcal{P} \), then \( \rho_i(X) = \rho_i(Y), \ i \in [n] \). The present case of mutually singular measures in \( \mathcal{P} \) enjoys particular prominence in [61].

Second, it seems immediate to ask how Assumption 3.1 is reflected by the \( Q_i \)-consistent risk measures \( \rho_i \) used by the agents in the risk sharing problem. A complete answer is provided by Theorem 3.2, the main result of this section. It translates Assumption 3.1 as a relaxed notion of dilatation monotonicity common to all \( \rho_i \). We call a function \( \varphi: L^\infty \to \mathbb{R} \) \( \mathcal{P} \)-dilatation monotone if, for every \( X \in L^\infty \) and every sub-\( \sigma \)-algebra \( \mathcal{G} \subset \mathcal{F} \),

\[
\varphi(\mathbb{E}_\mathcal{P}[X|\mathcal{G}]) \leq \varphi(X).
\]

(3.2)

\( \varphi \) is called \( \mathcal{P} \)-dilatation monotone above a sub-\( \sigma \)-algebra \( \mathcal{H} \subset \mathcal{F} \) if (3.2) holds for all \( X \in L^\infty \) and all sub-\( \sigma \)-algebras \( \mathcal{G} \supseteq \mathcal{H} \). While the former notion of dilatation monotonicity is standard in the literature (see, e.g., [21, 56] and the references therein for more information), dilatation monotonicity above a threshold sub-\( \sigma \)-algebras seems to be less common.

**Theorem 3.2.** Suppose that, for each \( i \in [n] \), \( Q_i \approx \mathcal{P} \) and \( \rho_i \) is a \( Q_i \)-consistent risk measure. Then the following are equivalent:

1. The probability measures \( (Q_1, \ldots, Q_n) \) can be chosen to satisfy Assumption 3.1.
2. Each \( \rho_i \) is \( \mathcal{P} \)-dilatation monotone above a common finite \( \sigma \)-algebra \( \mathcal{H} \subset \mathcal{F} \).

If \( \mathcal{G} \) is coarser than \( \mathcal{F} \), the conditional expectation \( \mathbb{E}_\mathcal{P}[X|\mathcal{G}] \) displays less variability than the initial random variable \( X \). In this sense, \( \mathcal{G} \) can be seen as a gain of information while \( \mathbb{E}_\mathcal{P}[X|\mathcal{G}] \) is usually interpreted as the “best approximation” of \( X \) under the probability measure \( \mathcal{P} \) using the information provided by \( \mathcal{G} \). Dilatation monotonicity of a risk measure now rewards decreased variability by not increasing the measured risk. Item (2) in Theorem 3.2 retains this intuition provided that the information is sufficient to decide which one of a set \( \mathcal{H} \) of reference events occurs. In that case, each \( \rho_i \) rewards replacing \( X \) by its best approximation \( \mathbb{E}_\mathcal{P}[X|\mathcal{G}] \) under the universally shared probability model \( \mathcal{P} \). Otherwise, the information is deemed too coarse, and agents withdraw to their potentially heterogeneous models \( Q_i \).

Third, each event \( B \) in the finite measurable partition \( \pi \) from Assumption 3.1 can be understood as the occurrence of an exogenous shock or a test event used in a backtesting procedure. [7], for instance, speaks about “exogenous environments”. While agents disagree about the likelihood of those shocks, their respective relevance or conditional distributional implications are consensus. For instance, [55] studies optimal insurance contracts under belief heterogeneity. A random loss is modelled by a random variable \( X \geq 0 \), the decision maker expresses probabilistic beliefs with a probability measure \( \mathcal{P} \), the insurer with a probability measure \( \mathcal{Q} \). One of the case studies in that paper, cf. [55, Section 2], assumes in our terminology that

\[
\mathcal{Q}(X = 0) > \mathcal{P}(X = 0) \quad \text{and} \quad \mathcal{P}(X > 0) \circ X^{-1} = \mathcal{Q}(X > 0) \circ X^{-1}
\]

(3.3)

(3.3) means that the insurer is more optimistic about the absence of losses than the decision maker. Comparing this to (3.1), one sees that the events \( \{ X = 0 \} \) and \( \{ X > 0 \} \) could play the role of shocks whose occurrence decision maker and insurer have potentially diverging opinions about, but whose consequences for the conditional distribution of \( X \) are acknowledged by all agents. In a backtesting context, [16] impose Assumption 3.1 verbatim in their study of scenario aggregation. The latter problem is faced by a financial company validating (or rejecting) their internal probabilistic model...
on the basis of evaluating selected adverse test events sufficiently conservatively; cf. [16, Section 5].

Last, but not least, [61] motivate scenario-based risk measures in the same vein.

Fourth, by virtue of Assumption 3.1, the model displays a structure analogous to the Anscombe-Aumann model for decisions under uncertainty; cf., e.g., [40, Chapter 15]. In that model, acts are modelled as stochastic kernels; i.e., an act maps a “state” into a probability measure on a set of consequences. To our knowledge, the setting of stochastic kernels is rarely used for risk measures, exceptions being [26] and [35, Section 6]. The aforementioned probability measures (called “lotteries”) are objectively given; in particular, the likelihood of the associated events in the set of consequences do not depend on the subjective likelihood of the states. By interpreting the events in a partition \( \pi \) in Assumption 3.1 as “states”, we recover the same structure in our model. Each random variable \( X \in L^\infty \) induces a finite set of lotteries \( \{ \mathbb{P}^B \circ X^{-1} \mid B \in \pi \} \) on \( \mathbb{R} \), depending on the particular state that obtains. These lotteries are “objective” and recognised by all agents. In contrast, the likelihoods \( Q_i(B) \) of the occurrence of states \( B \in \pi \) are subjective and subject to potential disagreement.

Fifth and last, the assumption can be phrased in the language of statistics. Recall that a sub-\( \sigma \)-algebra \( \mathcal{H} \subset \mathcal{F} \) is sufficient for \( \{ Q_1, \ldots, Q_n \} \) if all bounded random variables \( f: \Omega \to \mathbb{R} \) admit a common version of the conditional expectation \( \mathbb{E}_{Q_i}[f | \mathcal{H}], i \in [n] \). Interpretationally, this conditions expresses that dimension or complexity reduction in statistical experiments is feasible without implicitly requiring information about an unknown parameter (whose estimation usually is the goal).

**Proposition 3.3.** A vector \( (Q_1, \ldots, Q_n) \) of equivalent probability measures on \( (\Omega, \mathcal{F}) \) satisfies Assumption 3.1 under \( \mathbb{P} = Q_1 \) if and only if there is a finite sub-\( \sigma \)-algebra \( \mathcal{H} \subset \mathcal{F} \) sufficient for \( \{ Q_1, \ldots, Q_n \} \).

For a more thorough discussion of the relation between statistics, decision making, and risk analysis, we refer to [19].

In view of Proposition 3.3 it should be clear that, under Assumption 3.1, \( \mathbb{P} \) only plays the role of a weak gauge, determining with its null sets the equivalence class of nonatomic probability measures within which all \( Q_i, i \in [n] \), are located. In fact, \( \mathbb{P} \) can always be chosen among the \( Q_i \)'s. For our analysis it therefore serves as a somewhat arbitrary point of reference. The previous point hinges on the equivalence of appearing reference probabilities. If all risk measures involved in the risk sharing procedure are admissible, this has already been motivated by Lemma 2.2(1). However, Theorem 4.1 does not impose this assumption, so a more intrinsic reason for the equivalence assumption among all \( Q_i \)'s is warranted. Indeed, in a risk sharing scheme where equivalence fails, an agent could otherwise take on arbitrarily bad outcomes on a null set from their own perspective that is a relevant ground for improvement for other agents involved. The existence of such splitting procedures should rightly preclude the existence of optimal risk sharing schemes. This phenomenon is illustrated in the following example:

**Example 3.4.** Suppose \( \mathbb{P}, Q \) are atomless probability measures on \( (\Omega, \mathcal{F}) \) for which we can find \( 0 < p < 1 \) and disjoint events \( N_P, N_Q \) in \( \mathcal{F} \) such that \( \mathbb{P}(N_P) = Q(N_Q) = 0 \) and \( Q(N_P), \mathbb{P}(N_Q) > p \). Consider \( \rho_1 = \mathbb{E}_{\mathbb{P}}[\cdot] \) and \( \rho_2 = \mathbb{E}_{Q}[\cdot] \). A direct computation shows that \( \rho_1(-1_{N_Q} + 1_{N_P}) < 0 \) and \( \rho_2(1_{N_Q} - 1_{N_Q}) < 0 \). Hence,

\[
\inf_{(X_1, X_2) \in \mathcal{B}_0} \{ \rho_1(X_1) + \rho_2(X_2) \} \leq \inf_{n \in \mathbb{N}} \{ \rho_1(-n1_{N_Q} + n1_{N_P}) + \rho_2(n1_{N_Q} - n1_{N_P}) \} = -\infty.
\]

No optimal allocation of \( X = 0 \) can exist.
4. The main result

All results of this section are proved in Appendix E. For $X \in L^\infty$, we denote by

$$\mathcal{A}_X := \{ X \in (L^\infty)^n \mid X_1 + \ldots + X_n = X \}$$

the set of all allocations of $X$. Our problem of interest concerns $Q_i$-consistent risk measures $\rho_i$ on $L^\infty$, where $Q_i \approx P$ is some probability measure, $i \in [n]$. For $X \in L^\infty$, we aim to solve the problem

$$\sum_{i=1}^n \rho_i(X_i) \longrightarrow \min$$

subject to $X \in \mathcal{A}_X$. \hfill (4.1)

The associated infimal convolution $\rho := \square_{i\in[n]} \rho_i$ gives precisely the optimal value of (4.1). The functional $\rho$ is known to be cash-additive and monotone, i.e., $\rho$ is a monetary risk measure if $\text{dom}(\rho)$ is nonempty. An allocation $X \in \mathcal{A}_X$, $X \in L^\infty$, is optimal if $X$ solves problem (4.1), i.e.,

$$\rho(X) = (\square_{i=1}^n \rho_i)(X) = \sum_{i=1}^n \rho_i(X_i).$$

If an optimal allocation of $X$ exists, we say that $\rho$ is exact at $X$.

**Theorem 4.1.** Suppose:

(i) Probability measures $Q_1, \ldots, Q_n$ satisfy Assumption 3.1.

(ii) For each $i \in [n]$, $\rho_i$ is a $Q_i$-consistent risk measure.

(iii) The risk measures $\rho_1, \ldots, \rho_{n-1}$ are admissible, and for all $i \in [n-1]$ there is $D_i \in \mathcal{C}(\rho_i)$ such that, for all $k \in [n]$, $\rho^*_k(D_i) < \infty$.

Then, for each $X \in L^\infty$, there is an optimal allocation $X \in \mathcal{A}_X$.

4.1. Discussion of assumption (iii). Assumptions (i) and (ii) of Theorem 4.1 have been discussed in detail in Sections 2–3. We therefore turn directly to assumption (iii) and shall discuss four aspects thereof.

First, assume for the moment that all risk measures $\rho_1, \ldots, \rho_n$ are admissible. Then (iii) is implied by the simpler condition

$$\forall i \in [n-1] \forall k \in [n] : \rho^*_k(\frac{dQ_i}{dP}) < \infty. \hfill (4.2)$$

Incidentally, in case $n = 2$, this is essentially the key Assumption 2.1 of [3]. This requirement has a clear interpretation if all $\rho_i$’s are classical convex risk measures, in which case the probability densities in $\text{dom}(\rho^*_i) \cap L^1$ are “plausible probabilistic models that are taken more or less seriously” ([35, p. 308]) by agent $i$ depending on the size of $\rho^*_i$. Each reference model $\frac{dQ_i}{dP}$ plays a fundamental role for agent $i$, which is underscored by the fact that $\rho^*_i(\frac{dQ_i}{dP}) = 0$, i.e., $\frac{dQ_i}{dP}$ enjoys maximal plausibility from the point of view of $i$. (4.2) then means that $\frac{dQ_i}{dP}$ is also deemed somewhat plausible by the other agents. We can adopt this thinking in the more general consistent case, in which formula (2.1) holds for $\rho^*_i$. However, two admissible risk measures $\rho_1$ and $\rho_2$ can satisfy all assumptions of Theorem 4.1, but fail (4.2).

**Example 4.2.** Fix an event $A \in \mathcal{F}$ with $P(A) = \frac{1}{2}$ and consider the probability measure $Q \approx P$ defined by $\frac{dQ}{dP} = \frac{1}{2} 1_A + \frac{3}{2} 1_{A^c}$. We set $\rho_1 := \mathbb{E}P_1^{\frac{1}{2}} 1_A$ and $\rho_2 := \mathbb{E}Q_1^{\frac{1}{2}}$. One then identifies

$$\text{dom}(\rho^*_1) = \{ Z \in L^\infty_+ \mid \mathbb{E}_P[Z] = 1, Z \leq 1.25 \}$$

$$\text{dom}(\rho^*_2) = \{ Z \in L^\infty_+ \mid \mathbb{E}_P[Z] = 1, Z \leq 2.41_A + 0.81_{A^c} \}. $$
It is obvious that $\rho^n_2(1) = \infty$ and that $\rho^n_1(\frac{\partial}{\partial \rho}) = \infty$. However, as $\text{dom}(\rho^n_1) \cap \text{dom}(\rho^n_2) \neq \emptyset$, Lemma 4.3 below shows that also $C(\rho_1) \cap \text{dom}(\rho^n_2) \neq \emptyset$ or $\text{dom}(\rho^n_1) \cap C(\rho_2) \neq \emptyset$.

While (4.2) is too restrictive, its perspective can inform the interpretation of assumption (iii) itself. Given $i \in [n-1]$ and the admissible risk measure $\rho_i$, its compatible densities $C(\rho_i)$ are dense in $\text{dom}(\rho^n_i) \cap L^1$, but more plausible in the sense of being less extreme than models outside $C(\rho_i)$. Assumption (iii) means that within this set, we find a density $D_i$ that enjoys a certain degree of confidence by all agents involved.

As a second remark, note that assumption (iii) is stronger than the requirement

$$\bigcap_{i=1}^n \text{dom}(\rho^n_i) \neq \emptyset. \quad (4.3)$$

On its own, (4.3) is insufficient to guarantee the existence of optimal allocations (Example 4.6). However, it is an “almost necessary” requirement: whenever $\rho_1, \ldots, \rho_n$ are convex risk measures whose infimal convolution $\rho = \square^n_{i=1} \rho_i$ takes only finite values, then (4.3) must be satisfied. In the case of binary heterogeneity, i.e., in the setting of Marshall [55] described in Section 3, we can make the following interesting observation which states that (4.3) is often both necessary and sufficient for the existence of optimal allocations.

**Lemma 4.3.** Assume that $n = 2$, that Assumption 3.1 holds, and that $\rho_i$ are $Q_i$-consistent admissible risk measures, $i = 1, 2$. Moreover, suppose that there is a partition $\pi \in \Pi$ with $|\pi| = 2$ as in (3.1). Then (4.3) holds if and only if assumption (iii) of Theorem 4.1 is satisfied.

Lemma 4.3 fails if there are more than two reference events in partition $\pi$.

**Example 4.4.** Let $\pi = \{B_1, B_2, B_3\} \in \Pi$ be a partition of $\Omega$ such that $\mathbb{P}(B_i) = \frac{1}{3}$, $i \in [3]$. Consider $Q$ defined by the $\mathbb{P}$-density $Q := \frac{1}{3}1_{A_1} + 1_{A_2} + \frac{5}{3}1_{A_3}$. We define law-invariant coherent risk measures $\rho_1$ and $\rho_2$ on $L^\infty$ by

$$\rho_1(X) = \sup\{\mathbb{E}_\mathbb{P}[DX] \mid \|D - 1\|_{\infty} \leq \frac{1}{7}\},$$

$$\rho_2(X) = \sup\{\mathbb{E}_\mathbb{P}[Z X] \mid \frac{2}{3}Q \leq Z \leq 2Q\},$$

and shall prove that $\text{dom}(\rho^n_1) \cap \text{dom}(\rho^n_2)$ contains no density that is compatible with $\rho_1$ or $\rho_2$. Indeed, one first observes that a probability density $D$ lies in $\text{dom}(\rho^n_1) \cap \text{dom}(\rho^n_2)$ only if

$$D1_{A_1} = \frac{2}{3} \quad \text{and} \quad D1_{A_3} = \frac{4}{3}. \quad (4.4)$$

Concerning $\rho_1$, set $U := 51_{A_3} - 41_{A_1 \cup A_2}$ and note that, for every $D \in \text{dom}(\rho^n_1)$,

$$\mathbb{E}_\mathbb{P}[DU] = 5\mathbb{E}_\mathbb{P}[D1_{A_3}] - 4\{1 - \mathbb{E}_\mathbb{P}[D1_{A_3}]\} \leq 5 \cdot \frac{4}{9} - 4 \cdot \frac{5}{9} = 0;$$

i.e., $U \in A_{\rho_1}^\infty$. As every $D$ satisfying (4.4) will lead to $\mathbb{E}_\mathbb{P}[DU] = 0$, we infer that no density in $\text{dom}(\rho^n_1) \cap \text{dom}(\rho^n_2)$ is compatible with $\rho_1$.

Similarly for $\rho_2$, consider $V = 71_{A_1} - 21_{A_2 \cup A_3}$. For all $D \in \text{dom}(\rho^n_2)$, $D1_{A_1} \leq \frac{2}{3}1_{A_1}$, whence

$$\mathbb{E}_\mathbb{P}[DV] = 7\mathbb{E}_\mathbb{P}[D1_{A_1}] - 2\{1 - \mathbb{E}_\mathbb{P}[D1_{A_1}]\} \leq 7 \cdot \frac{2}{9} - 2 \cdot \frac{7}{9} = 0$$

and $V \in A_{\rho_2}^\infty$ follows. As every $D \in \text{dom}(\rho^n_1) \cap \text{dom}(\rho^n_2)$ leads to $\mathbb{E}_\mathbb{P}[DV] = 0$, no such density is compatible with $\rho_2$.

A third noteworthy aspect anticipates the role that assumption (iii) plays in the proof of Theorem 4.1. Inspecting its mechanics, one notes that it suffices to demand the existence of a finitely additive
probabilities \( \nu_1, \ldots, \nu_{n-1} \in \bigcap_{i=1}^{n} \text{dom}(\rho_i) \) such that
\[
\forall k \in [n-1] : \quad U \in A_{\rho_k}^{\infty} \text{ and } \mathbb{E}_{\nu_k}[U] = 0 \implies U = 0. \tag{4.5}
\]
However, as one might suspect on the basis of Proposition 2.3(5), this does increase generality.

**Lemma 4.5.** Suppose that assumptions (i)–(ii) in Theorem 4.1 hold. Then assumption (iii) in said theorem is satisfied if and only if (4.5) holds.

Last, we re-emphasise that Theorem 4.1 only assumes the admissibility of \( \rho_1, \ldots, \rho_{n-1} \). This permits to include, for instance, expected loss assessments that price risk linearly, i.e., the \( \mathbb{Q}_n \)-consistent, but not admissible risk measure \( \rho_n = \mathbb{E}_{\mathbb{Q}_n}[\cdot] \). If \( \rho_n \) is not admissible, Proposition 2.3(3) simplifies assumption (iii) substantially to the requirement
\[
d_{\mathbb{Q}_n}d_{\mathbb{P}} \in \bigcap_{i=1}^{n-1} C(\rho_i). \tag{4.6}
\]
Moreover, condition (4.2) is precluded unless we are in a homogeneous situation. We can nevertheless also solve the optimal allocation problem if more than one agent measures risk with a non-admissible risk measure. The procedure starts by partitioning \([n]\) into two subsets \( \mathcal{I} \) – which collects all admissible agents – and \( \mathcal{J} \) – which collects all non-admissible agents. If there is to be a chance that optimal allocations exist, we must have \( \mathbb{Q}_j = \mathbb{Q}_k \) for all \( j,k \in \mathcal{J} \). Define \( \rho_0 := \square_{j \in \mathcal{J}} \rho_j \). For arbitrary \( k \in \mathcal{J} \), \( \rho \) is a \( \mathbb{Q}_k \)-consistent risk measure ([54, Theorem 4.1]). Next, we assume that for this \( k \), \( d_{\mathbb{Q}_k} d_{\mathbb{P}} \in \bigcap_{i \in \mathcal{I}} C(\rho_i) \). The latter is the correct adaptation of (4.6) and admits to apply Theorem 4.1 to allocate \( X \in L^\infty \) optimally in the new collective \( \mathcal{I} \cup \{0\} \). This optimal allocation can then by used to construct an optimal allocation for the initial collective by sharing the portion attributed to agent 0 optimally in the collective \( \mathcal{J} \). This can be done according to [54, Theorem 4.1].

### 4.2. Necessity of assumptions.

We continue our discussion of Theorem 4.1 by shedding more light on the necessity of the individual assumptions. For the sake of transparency, we shall throughout consider the case of two agents, i.e., \( n = 2 \).

**Example 4.6.**

1. **Law invariance alone or combined with star shapedness is not sufficient:** It is noteworthy that this can already be illustrated in the homogeneous case in which all reference probability measures agree. Consider constants \( \alpha, \beta > 0 \) with \( \alpha + \beta < 1 \). Let \( \rho_i : L^\infty \to \mathbb{R}, i = 1, 2 \), be defined by
\[
\rho_1(X) := \inf \{ x \in \mathbb{R} \mid \mathbb{P}(X \leq x) > 1 - \alpha \},
\rho_2(X) := \mathbb{E}_{\mathbb{P}}\mathbb{S}_{1-\beta}(X).
\]
Both functionals are \( \mathbb{P} \)-law invariant risk measures that are positively homogeneous and thus star-shaped. However, \( \rho_2 \) is a consistent risk measure, while \( \rho_1 \) is not consistent. At last, no \( X \in L^\infty \) with continuous distribution can be allocated optimally among the two agents ([49, Proposition 1]).

2. **All appearing reference probability measures have to be equivalent:** This is illustrated already in Example 3.4.

3. **Equivalence of reference probability measures alone is not enough:** To illustrate this, consider the convex monetary risk measures \( \rho_1 := \text{Entr}_{\mathbb{P}}^1 \) and \( \rho_2 := \mathbb{E}_{\mathbb{Q}}[\cdot] \), and assume \( \mathbb{Q} \approx \mathbb{P} \). If \( \rho_1 \square \rho_2 ^{2} I am indebted to Ruodu Wang for pointing out the corresponding result in [49].
were exact at each $X \in L^\infty$, one could follow the reasoning of [43, Example 6.1] and conclude that, if $\rho_1 \Box \rho_2(X) = \rho_1(X_1) + \rho_2(X_2)$ for some allocation $X \in \mathcal{A}_X$, then $\frac{dQ}{dP}$ would have to be a subgradient of $\rho_1$ at $X_1$, i.e., $\rho_1(X_1) = \mathbb{E}_P\left[\frac{dQ}{dP}X_1\right] - \rho_1^\ast\left(\frac{dQ}{dP}\right)$. By [59, Lemma 6.1], the identity
$$
\frac{dQ}{dP} = \frac{\exp(X_1)}{\mathbb{E}_P[\exp(X_1)]}
$$
must hold. As $X_1 \in L^\infty$, $\log\left(\frac{dQ}{dP}\right)$ has to be bounded from above and below, that is, there have to be constants $0 < s < S$ such that $\mathbb{P}\left(\frac{dQ}{dP} \in [s, S]\right) = 1$. Clearly, this is not satisfied for all $Q \approx P$.

(4) Assumption (iii) cannot be dropped, even if Assumption 3.1 holds: Here we use [3, Example 4.3]. Consider the probability measure $Q \approx P$ from Example 4.2 and define the convex monetary risk measures $\rho_1 := \frac{1}{2}(\mathbb{E}_P[\cdot] + \text{Entr}_P[\cdot])$ and $\rho_2 := \mathbb{E}_Q[\cdot]$. By [3, Example 4.3], $\rho_1 \Box \rho_2$ is not exact at all $X \in L^\infty$. However, as $\text{dom}(\rho_2^\ast)$ is a singleton and $\rho_2$ is not admissible, assumption (iii) in Theorem 4.1 would hold if and only if $\frac{dQ}{dP} \in \mathcal{C}(\rho_1)$. Setting $A = \{\frac{dQ}{dP} = \frac{1}{2}\}$, we consider $U := -31_A + 1_{A^c}$ and show by a direct computation that $U \in \mathcal{A}_X^\infty$. As, however, $\mathbb{E}_Q[U] = 0$, $\frac{dQ}{dP}$ cannot be compatible for $\rho_1$.

(5) Assumption (iii) without Assumption 3.1 is not enough: Let $\rho_1 := \text{Entr}_P[\cdot]$ and assume that $Q := \frac{dQ}{dP}$ is positive $\mathbb{P}$-a.s., but
$$
\forall \varepsilon > 0 : \quad \mathbb{P}(Q \leq \varepsilon) > 0.
$$
Consequently, Assumption 3.1 fails. Set
$$
\rho_2(X) := \sup \{\mathbb{E}_P[RX] : \frac{1}{2}Q \leq R \leq \frac{3}{2}Q, \mathbb{P}\text{-a.s.}, \mathbb{E}_P[R] = 1\}
$$
$$
= \sup \{\mathbb{E}_{Q_2}[ZX] : \frac{1}{2} \leq Z \leq \frac{3}{2}, \mathbb{P}\text{-a.s.}, \mathbb{E}_{Q_2}[Z] = 1\},
$$
and observe that $\rho_2$ is an admissible $Q_2$-consistent risk measure with $Q \in \mathcal{C}(\rho_1) \cap \mathcal{C}(\rho_2)$. If $\rho_1 \Box \rho_2$ were exact at some $X \in L^\infty$ with optimal allocation $(X_1, X_2) \in \mathcal{A}_X$, the argument employed in (3) would generate a subgradient $R \in \text{dom}(\rho_2^\ast)$ of $\rho_1$ at $X_1$. Again, we could find $s > 0$ such that $R \geq s \mathbb{P}$-a.s. However, we would also need that $\frac{R}{Q} \in [\frac{1}{2}, \frac{3}{2}]$ $\mathbb{P}$-a.s., meaning that $R \leq \frac{3}{2}Q$. The two constraints on $R$ are irreconcilable.

4.3. A comparison to interior-point conditions. In this subsection we address the question if there is a valid mathematical approach to establishing the existence of optimal allocations among heterogeneous agents that could serve as an alternative to Theorem 4.1. At least in the convex case, interior-point conditions come to mind that are studied in [57, Chapter 11.4.3] for convex risk functionals and that originate in the abstract study of infimal convolutions in convex analysis. For the sake of clarity, we consider $n = 2$ agents and look at the first two sufficient conditions in [57, Proposition 11.41]:
$$
\text{dom}(\rho_1^\ast) \cap \text{int}(\text{dom}(\rho_2^\ast)) \neq \emptyset \quad (4.8)
$$
For each $\mu \in \text{ba}$ and $g_\mu : \mathbb{R} \ni t \mapsto t\mu$, $0$ is an interior point of $g_\mu^{-1}(\text{dom}(\rho_1^\ast) - \text{dom}(\rho_2^\ast)).$ (4.9)

Unfortunately, conditions (4.8)–(4.9) are never satisfied by cash-additive risk measures. More promising is a condition inspired and implied by equation (11.124) in the mentioned result:

For all $(\mu_1, \mu_2) \in \text{dom}(\rho_1^\ast) \times \text{dom}(\rho_2^\ast)$, $0$ is an interior point of $g_{\mu_1 - \mu_2}^{-1}(\text{dom}(\rho_1^\ast) - \text{dom}(\rho_2^\ast)).$ (4.10)

If Assumption 3.1 holds and assumption (iii) in Theorem 4.1 is replaced by (4.10), a similar proof delivers the existence of optimal allocations. However, in addition to potential difficulties in verifying that (4.10) holds true, its value is already in a homogeneous situation extremely limited.
Proposition 4.7. Suppose that \( n = 2 \) and that \( \rho_i \) is a \( Q_i \)-consistent risk measure, \( i = 1, 2 \).

1. If (4.10) is satisfied and \( \rho_2 \) is not admissible, then also assumption (iii) holds.
2. Suppose that

\[ \text{dom}(\rho_i^*) \not\subset L^\infty, \quad \text{dom}(\rho_2^*) \cap L^1 \subset L^\infty, \quad \text{and} \quad \frac{d\rho_1}{d\beta_1}, \frac{d\rho_2}{d\beta_2} \in L^\infty. \]

Then (4.10) fails.

Proposition 4.7(2) reveals a wide array of situations in which assumption (iii) holds, but (4.10) fails; for instance, \( \rho_1 = \text{Entr}^\beta \) and \( \rho_2 = \text{ES}_p^\beta, (\beta, p) \in (0, \infty) \times (0, 1) \).

4.4. Shape of optimal allocations. One of the key findings in many (monetary) risk sharing situations under a single homogeneous reference measures is that \textit{comonotone} optimal risk allocations can be found. Let \( \mathcal{C} \) denote the set of all functions \( f: \mathbb{R} \rightarrow \mathbb{R}^n \) for which each coordinate \( f_i \) is non-decreasing and which satisfy \( \sum_{i=1}^n f_i = \text{id}_\mathbb{R} \). Consequently, each coordinate \( f_i \) of \( f \in \mathcal{C} \) is 1-Lipschitz continuous. We call the elements of \( \mathcal{C} \) \textit{comonotone} functions. For \( f \in \mathcal{C} \), we abbreviate by \( \tilde{f} \) the normalised function \( f - f(0) \). Given \( X \in L^\infty \), an allocation \( X \in \mathcal{A}_X \) is called comonotone if \( X = f(X) \) for some \( f \in \mathcal{C} \), i.e., the allocation is obtained by applying a comonotone function to the aggregated quantity \( X \).

The proofs in Appendix E demonstrate that optimal allocations will usually not be comonotone under heterogeneous reference measures, marking a substantial difference between the heterogeneous and homogeneous case. As an illustration, consider an arbitrary \( Q \approx \mathbb{P} \), positive constants \( \beta_1, \beta_2 > 0 \), and the convex risk measures \( \rho_1 = \text{ES}^\beta_{\beta_1} \) and \( \rho_2 := \text{ES}^Q_{\beta_2} \). Then \( \rho_1 \Box \rho_2 \) is exact at each \( X \in L^\infty \), and the unique optimal risk allocation \((X_1, X_2)\) is given by

\[
\begin{align*}
    X_1 &= \frac{\beta_2}{\beta_1 + \beta_2} X + \frac{1}{\beta_1 + \beta_2} \log(\frac{dQ}{d\mathbb{P}}), \\
    X_2 &= \frac{\beta_1}{\beta_1 + \beta_2} X - \frac{1}{\beta_1 + \beta_2} \log(\frac{dQ}{d\mathbb{P}}).
\end{align*}
\]

The dependence on the density \( \frac{dQ}{d\mathbb{P}} \) precludes comonotonicity unless \( \frac{dQ}{d\mathbb{P}} \equiv 1 \), which is the case if and only if \( Q = \mathbb{P} \).

In order to justify the desirability of comonotone allocations, let us decompose each risk portion \( f_i(X) \) of a comonotone allocation \( f(X) \) into two parts: \( f_i(0) \), a deterministic cost imposed on, or capital injected into the position of, agent \( i \); and the remainder \( \tilde{f}_i(X) \). In actuarial terminology, \( \tilde{f}_i \) is an indemnity function. \( \tilde{f}_i(X) \) reflects minimal rationality and fairness considerations of the agents involved in the risk sharing scheme. If \( X > 0 \), i.e., a loss is produced at the system level, then \( 0 \leq \tilde{f}_i(X) \leq X \), i.e., no agent \( i \in [n] \) makes a gain via \( \tilde{f}_i(X) \) and the portions \( \tilde{f}_i(X) \) never exceed the total net loss \( X \). Symmetrically, if \( X < 0 \) and a gain is obtained at a cumulated level, then \( X \leq \tilde{f}_i(X) \leq 0 \) and no agent incurs a loss via \( \tilde{f}_i(X) \). Moreover, in the tradition of [41], the nondecreasing nature of \( \tilde{f}_i \) is interpreted as the absence of \textit{ex post} moral hazard potentially incentivising agents to misreport their losses; cf. [41, p. 423]. By definition, the deterministic cash transfers \((f_i(0))_{i \in [n]} \in \mathbb{R}^n \) satisfy \( \sum_{i=1}^n f_i(0) = 0 \). Given that individual risk measures are cash-additive, deterministic capital transfers among agents – which could be perceived as unfair – can be eliminated without losing optimality, leading to the new optimal allocation \( \tilde{f}(X) \) of \( X \). This “rebalancing of cash” can alternatively be justified by solving a second optimisation problem and selecting an optimal comonotone function \( g \in \mathcal{C} \) whose cash transfers \( g(0) \) are closest to a uniform
distribution among agents; i.e., one additionally minimises the function

$$\Xi: \mathcal{C} \to \mathbb{R}, \quad f \mapsto \min \{ \sum_{i=1}^{n} |f_i(0) - x|^2 \mid x \in \mathbb{R} \},$$

among all $f \in \mathcal{C}$ defining an optimal allocation.

In the heterogeneous case and under Assumption 3.1, the optimal allocations whose existence is verified in Theorem 4.1 are only \textit{locally comonotone}; cf. Appendix B. More precisely, if $\pi \in \Pi$ is as in (3.1), one finds an optimal family $(f^B)_{B \in \pi} \subset \mathcal{C}$ of comonotone functions such that

$$X_i := \sum_{B \in \pi} f^B_i(X) 1_B, \quad i \in [n],$$

defines an optimal allocation. Two observations ensue: The potential loss of comonotonicity is a consequence of the heterogeneity of reference measures, not of other properties of the risk measures. Second, locally comonotone allocations split into a $\pi$-dependent cash transfer reflecting the heterogeneity structure, $\sum_{B \in \pi} f_i(0) 1_B$, and $B$-dependent indemnity schedules applied to the aggregated loss $X$, $\sum_{B \in \pi} \tilde{f}_i(X) 1_B$.

Note that the risk measures $\rho_i$, $i \in [n]$, cannot behave additively on $\text{span}\{1_B \mid B \in \pi\}$ by [46, Theorem 5.7]. Hence, rebalancing of cash and eliminating cash transfers among agents in the way described above becomes impossible. This important distinctive feature of the heterogeneous case has been emphasised multiple times in this manuscript already. As a workaround, one can take the alternative approach to rebalancing cash, select a quality criterion $\Xi$ measuring “unfairness” and trying to minimise it among all optimal allocations.

4.5. \textbf{Infimal convolution preserves $\mathcal{P}$-basedness.} Let us once again consider the situation of a homogeneous reference measure $\mathcal{P}$ in which comonotone optimal allocations can be found. As an important consequence, one can show that the infimal convolution preserves $\mathcal{P}$-law invariance, i.e., the optimal value in (4.1) depends only on the distribution of the aggregated loss $X$ under $\mathcal{P}$.\footnote{The question under which additional conditions law invariance is preserved in the homogeneous case is tackled at great generality in [51].} However, Theorem 4.1 also applies to heterogeneous reference measures, and the lack of comonotonicity of optimal allocations outlined in Section 4.4 shows that one cannot expect the infimal convolution to be law invariant with respect to some reference measure. Instead, the next corollary records that the infimal convolution operation preserves $\mathcal{P}$-basedness of the individual risk measures discussed in Section 3. This is an important addendum to [61].

\textbf{Corollary 4.8.} In the situation of Theorem 4.1 let $\pi \in \Pi$ be a finite measurable partition as in (3.1). Consider the set $\mathcal{P} := \{ \mathcal{P}^B \mid B \in \pi \}$ of conditional probability measures. Then the infimal convolution $\rho = \square_{i=1}^{n} \rho_i$ is $\mathcal{P}$-based: If for $X, Y \in L^\infty$ and all $B \in \pi$ we have $X \sim_{\mathcal{P}^B} Y$, then $\rho(X) = \rho(Y)$.

\textbf{APPENDIX A. AUXILIARY RESULTS}

This appendix collects structural properties of $\mathcal{Q}$-law-invariant risk measures where the probability measure $\mathcal{Q}$ may not agree with the gauge probability measure $\mathcal{P}$. These will be relevant for all subsequent proofs. While Lemma A.1 is standard if $\mathcal{Q} = \mathcal{P}$, we shall provide a brief sketch of the short proof in the general case $\mathcal{Q} \neq \mathcal{P}$ for the convenience of the reader.

\textbf{Lemma A.1.} Let $\mathcal{Q} \approx \mathcal{P}$ be a probability measure and $\rho$ be a $\mathcal{Q}$-consistent risk measure.
(1) Define \( \rho^s : L^1_\mathbb{Q} \to (-\infty, \infty] \) by
\[
\rho^s(D) = \sup_{X \in L^\infty} \{ E_\mathbb{Q}[DX] - \rho(X) \}.
\]

Then, \( \rho^s \) is \( \mathbb{Q} \)-law invariant and, for all \( Z \in L^1 \),
\[
\rho^s(Z) = \rho^s(\frac{d\mathbb{Q}}{d\mathbb{P}} Z).
\]

(2) For all \( Z \in L^1 \cap L^1_\mathbb{Q} \) and all sub-\( \sigma \)-algebras \( \mathcal{G} \subset \mathcal{F} \) such that \( \frac{d\mathbb{Q}}{d\mathbb{P}} \) has a \( \mathcal{G} \)-measurable version,
\[
\rho^s(E_\mathbb{Q}[Z|\mathcal{G}]) \leq \rho^s(Z).
\]

Proof. Statement (1) is standard. For (2), let \( \mathcal{G} \subset \mathcal{F} \) be a sub-\( \sigma \)-algebra such that \( \frac{d\mathbb{Q}}{d\mathbb{P}} \) has a \( \mathcal{G} \)-measurable version. Let \( Z \in L^1 \cap L^1_\mathbb{Q} \). Using the Bayes rule for conditional expectations,
\[
\frac{d\mathbb{Q}}{d\mathbb{P}} E_\mathbb{Q}[Z|\mathcal{G}] = E_\mathbb{F}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} Z|\mathcal{G}\right] = \frac{d\mathbb{Q}}{d\mathbb{P}} E_\mathbb{F}[Z|\mathcal{G}].
\]

\( \mathbb{Q} \approx \mathbb{P} \) implies that \( E_\mathbb{Q}[Z|\mathcal{G}] = E_\mathbb{F}[Z|\mathcal{G}] \). As \( \rho^s \) from (A.1) is convex, \( \mathbb{Q} \)-law invariant, proper (i.e., \( \text{dom}(\rho^s) \neq \emptyset \)), and \( \sigma(L^1_\mathbb{Q}, L^\infty_\mathbb{Q}) \)-lower semicontinuous by definition, [11, Theorem 3.6] admits to infer for all \( Z \in L^1 \cap L^1_\mathbb{Q} \) that
\[
\rho^s(Z) = \rho^s(\frac{d\mathbb{Q}}{d\mathbb{P}} Z) \geq \rho^s(E_\mathbb{Q}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} Z|\mathcal{G}\right]) = \rho^s(\frac{d\mathbb{Q}}{d\mathbb{P}} E_\mathbb{Q}[Z|\mathcal{G}]) = \rho^s(E_\mathbb{Q}[Z|\mathcal{G}]) = \rho^s(E_\mathbb{F}[Z|\mathcal{G}]).
\]

\( \square \)

Lemma A.2. Let \( \mathbb{Q} \approx \mathbb{P} \) and \( \rho \) be a \( \mathbb{Q} \)-consistent risk measure.

(1) For all \( \mu \in \text{dom}(\rho^s) \) and all \( U \in A^\infty_\rho \), \( E_\mu[U] \leq 0 \) holds.

(2) \( A^\infty_\rho \) is closed, \( \mathbb{Q} \)-law-invariant, and closed under taking \( \mathbb{Q} \)-conditional expectations.

Proof. For statement (1), let \( \mu \) and \( U \) be as described. Let \( (s_k) \subset (0, \infty) \) be a null sequence and \( (Y_k) \subset A_\rho \) such that \( \lim_{k \to \infty} s_k Y_k = U \). Recall that \( \rho^s(\mu) = \sup_{Y \in A_\rho} E_\mu[Y] \geq 0 \). Hence,
\[
E_\mu[U] = \lim_{k \to \infty} s_k E_\mu[Y_k] \leq \rho^s(\mu) \cdot \lim_{k \to \infty} s_k = 0.
\]

As for statement (2), it is straightforward to verify that \( A^\infty_\rho \) is closed. In order to see that \( A^\infty_\rho \) is closed under taking conditional expectations with respect to \( \mathbb{Q} \), fix \( U \in A^\infty_\rho \) and an arbitrary sub-\( \sigma \)-algebra \( \mathcal{G} \subset \mathcal{F} \). Let \( (s_k) \subset (0, \infty) \) and \( (Y_k) \subset A_\rho \) be sequences as in the proof of (1) with \( \lim_{k \to \infty} s_k Y_k = U \). By Jensen’s inequality and the monotonicity of \( \rho \) in second order stochastic dominance relation with respect to \( \mathbb{Q} \), \( E_\mathbb{Q}[Y_k|\mathcal{G}] \in A_\rho \) holds for all \( k \in \mathbb{N} \). Hence, \( E_\mathbb{Q}[U|\mathcal{G}] = \lim_{k \to \infty} s_k E_\mathbb{Q}[Y_k|\mathcal{G}] \in A^\infty_\rho \). For \( \mathbb{Q} \)-law invariance of \( A^\infty_\rho \), define the function \( F : L^\infty = L^\infty_\mathbb{Q} \to \mathbb{R}_+ \) defined by \( F(X) = \inf_{U \in A^\infty_\rho} \|X - U\|_\infty \) is continuous (because \( A^\infty_\rho \) is closed) and \( \mathbb{Q} \)-dilatation monotone. Indeed, for every \( X \in L^\infty \) and every sub-\( \sigma \)-algebra \( \mathcal{G} \subset \mathcal{F} \),
\[
F(E_\mathbb{Q}[X|\mathcal{G}]) \leq \inf_{U \in A^\infty_\rho} \|E_\mathbb{Q}[X|\mathcal{G}] - E_\mathbb{Q}[U|\mathcal{G}]\|_\infty \leq \inf_{U \in A^\infty_\rho} \|X - U\|_\infty = F(X).
\]

Hence, by [21, Theorem 1.1], \( F \) is \( \mathbb{Q} \)-law invariant. This translates to \( \mathbb{Q} \)-law-invariance of \( A^\infty_\rho = F^{-1}(\{0\}) \).

\( \square \)

The next lemma reveals that stability of compatible densities under certain conditional expectations.

Lemma A.3. Let \( \mathbb{Q} \approx \mathbb{P} \) and \( \rho \) be a \( \mathbb{Q} \)-consistent risk measure. Suppose \( \mathcal{G} \subset \mathcal{F} \) is a sub-\( \sigma \)-algebra such that \( \frac{d\mathbb{Q}}{d\mathbb{P}} \) has a \( \mathcal{G} \)-measurable version. Then, for all \( D \in C(\rho), \)
\[ \mathbb{E}_F[D|G] \in C(\rho). \]

**Proof.** As there is nothing to show if \( C(\rho) = \emptyset \), we may assume that \( \rho \) is admissible. Let \( G \subset F \) be a sub-\( \sigma \)-algebra as described. By Lemma A.1(2), \( \mathbb{E}_F[D|G] \in \text{dom}(\rho^*). \) Moreover, as \( Q \approx P \), we may apply the Bayes rule for conditional expectations to compute
\[
\mathbb{E}_Q[D|G] = \left( \mathbb{E}_F[\frac{dQ}{dP}] \right)^{-1} \mathbb{E}_F[\frac{dQ}{dP} D|G] = \left( \frac{dQ}{dP} \right)^{-1} \cdot \frac{dQ}{dP} \mathbb{E}_F[D|G] = \mathbb{E}_F[D|G]. \tag{A.2}
\]
Now suppose \( U \in A^\infty_\rho \) satisfies \( \mathbb{E}_F[\mathbb{E}_F[D|G]|U] = 0. \) \( \mathbb{E}_F[D|G] \in C(\rho) \) follows if we can show \( U = 0. \) Switching the conditioning and arguing as in (A.2) yields
\[
0 = \mathbb{E}_F[\mathbb{E}_F[D|G]|U] = \mathbb{E}_F[D\mathbb{E}_Q[U|G]],
\]
and \( \mathbb{E}_Q[U|G] \in A^\infty_\rho \) holds by Lemma A.2(2). As \( D \in C(\rho) \), we infer \( \mathbb{E}_Q[U|G] = \mathbb{E}_Q[U] = 0. \) As \( \frac{dQ}{dP} \in C(\rho) \) by Proposition 2.3(2), \( U = 0 \) has to hold. \( \square \)

In the next preparatory result, we consider the operation of computing the star-shaped hull of a normalised monetary risk measure \( \rho: L^\infty \rightarrow \mathbb{R} \), i.e., the functional \( \rho_*: L^\infty \rightarrow \mathbb{R} \) defined by
\[
\rho_*(X) := \inf \{ m \in \mathbb{R} \mid \exists s \in (0,1]: X - m \in sA_\rho \}. \tag{A.3}
\]

**Lemma A.4.** Suppose \( Q \approx P \) and that \( \rho \) is a \( Q \)-consistent risk measure. Then \( \rho_* \) is a star-shaped \( Q \)-consistent risk measure which satisfies
\[
\rho^*_\ast = \rho^* \quad \text{and} \quad C(\rho) = C(\rho_*). \tag{A.4}
\]

In particular, \( \rho \) is admissible if and only if \( \rho_* \) is admissible.

**Proof.** As \( \rho \geq \mathbb{E}_Q[\cdot] \), \( \rho_*(X) \in \mathbb{R} \) for all \( X \in L^\infty \) follows. Indeed, suppose \( X \in L^\infty \), \( m \in \mathbb{R} \), and \( s \in (0,1] \) are such that \( X - m \in sA_\rho \). As \( \frac{X - m}{s} \in A_\rho \), we infer \( \mathbb{E}_Q[\frac{X - m}{s}] \leq 0 \) and thus \( m \geq \mathbb{E}_Q[X] \). Now infimise over such \( m \) to obtain the same lower bound for \( \rho_*(X) \). Monotonicity, cash-additivity, and star shapedness of \( \rho_* \) are verified in a straightforward way. For \( Q \)-consistency of \( \rho_* \), suppose \( X \leq_{ssd} Y \) and that \( m \in \mathbb{R} \) is such that \( Y - m = sR \) for some \( s \in (0,1] \) and \( R \in A_\rho \). One then verifies that \( s_{-1}(X - m) \geq_{ssd} s_{-1}(Y - m) = Y \in A_\rho \), which means that \( X - m \in sA_\rho \) as well. We infer that
\[
\rho_*(X) = \inf \{ k \in \mathbb{R} \mid \exists s \in (0,1]: X - k \in sA_\rho \} \leq \inf \{ m \in \mathbb{R} \mid \exists s \in (0,1]: Y - m \in sA_\rho \} = \rho_*(Y),
\]
i.e., \( \rho_* \) is \( Q \)-consistent.

For the verification of (A.4), abbreviate first \( B := \text{cl}(\{ sY \mid s \in (0,1], Y \in A_\rho \}) \) and suppose \( \rho_*(X) \leq 0 \). This means that we must be able to find \( (s_n) \in (0,1] \) such that \( X - \frac{1}{n} \in s_nA_\rho \), \( n \in \mathbb{N} \). Hence, \( X = \lim_{n \rightarrow \infty} X - \frac{1}{n} \in B \). As one can also verify that \( B \subset A_{\rho_*} \), we obtain for all \( \mu \in \text{ba} \):
\[
\rho^*_\ast(\mu) = \sup_{R \in A_{\rho*}} \mathbb{E}_\mu[R] = \sup_{Y \in A_\rho} \sup_{s \in (0,1]} \mathbb{E}_\mu[\mathbb{E}_\mu[Y]] = \sup_{s \in (0,1]} s^{\ast}(\mu) = \rho^*(\mu). \tag{A.5}
\]

The last equality in (A.5) follows from the fact that \( \rho^*(\mu) \geq 0 \).

We finally show the second statement in (A.4), that \( C(\rho) = C(\rho_*) \). To this end, \( A_\rho \subset A_{\rho_*} \) first of all implies \( A^\infty_\rho \subset A^\infty_{\rho_*} \). Conversely, suppose that \( U \in A^\infty_{\rho_*} \), which means that for a null sequence \( (s_n) \subset (0,1) \) and \( (Y_n) \subset A_{\rho_*} \), \( U = \lim_{n \rightarrow \infty} s_nY_n \). Recalling that \( \text{cl}(\bigcup_{s \in (0,1]} sA_\rho) = A_\rho \) and choosing sequences \( (t_n) \subset (0,1) \) and \( (Y_n) \subset A_\rho \) appropriately, we can thus guarantee that \( U = \lim_{n \rightarrow \infty} s_nY_n \).

This is sufficient for \( U \in A^\infty_\rho \). By (A.4), \( \text{dom}(\rho^*_\ast) = \text{dom}(\rho^*) \). Together with \( A^\infty_\rho = A^\infty_{\rho_*}, C(\rho) = C(\rho_*), \) follows.

The following is an adaptation of [2, Lemma 1] to our more general setting.
Lemma A.5. Suppose \( \rho \) is a \( \mathbb{Q} \)-law-invariant risk measure, \( \mathbb{Q} \approx \mathbb{P} \), and that two finitely additive probabilities \( \mu = \zeta \oplus \tau \) and \( \nu = \zeta' \oplus \tau' \) satisfy \( \zeta(A) \leq \zeta'(A), A \in \mathcal{F} \), and \( \rho^*\mu < \infty \). Then also \( \rho^*(\nu) < \infty \).

Proof. Without loss of generality, we can assume \( \mathbb{Q} = \mathbb{P} \). As \( \mu(\Omega) = \nu(\Omega) = 1 \), \( \tau(\Omega) - \tau'(\Omega) = \zeta'(\Omega) - \zeta(\Omega) \) follows. Let \( R := \frac{d\mu}{d\mathbb{P}} \) and \( R' := \frac{d\nu}{d\mathbb{P}} \). Then the first assumption translates as \( R' \geq R \).

Using \([24, Proposition 3.9]\) once more, we further estimate
\[
\mathbb{E}_\nu[X] \leq (\tau'(\Omega) + \mathbb{E}_\mathbb{P}[R' - R]) \cdot \text{ES}_1^\mathbb{P}(X) + \int_0^1 q_X(s) q_R(s) \, ds = (\tau(\Omega) + \mathbb{E}_\mathbb{P}[R]) \cdot \text{ES}_1^\mathbb{P}(X) + \int_0^1 q_X(s) q_R(s) \, ds.
\]

Using \([24, Proposition 3.9]\) once more,
\[
\mathbb{E}_\nu[X] \leq \sup_{Y \sim \mathbb{P}} \mathbb{E}_\mu[Y] \leq \sup_{Y \sim \mathbb{P}} \left\{ \rho(Y) + \rho^*_i(\mu) \right\} = \rho(X) + \rho^*_i(\mu).
\]
This suffices to prove that \( \rho^*(\nu) < \infty \).

\[\square\]

Appendix B. Local comonotone improvement

Recall the definition of the set \( \mathcal{C} \) of comonotone functions \( f : \mathbb{R} \to \mathbb{R}^n \) from Section 4.4. Given \( X \in L^\infty \) and a finite measurable partition \( \pi \in \Pi \), a vector \( Y \in \mathcal{A}_X \) is a locally comonotone allocation of \( X \) over \( \pi \) if there are comonotone functions \( (f^B)_{B \in \pi} \subset \mathcal{C} \) such that, for all \( i \in [n] \),
\[
Y_i = \sum_{B \in \pi} f^B_i(X) \mathbf{1}_B.
\]

Given \( \mathbb{Q} \ll \mathbb{P} \), in the sequel we denote by \( \preceq_\mathbb{Q} \) the \( \mathbb{Q} \)-convex order on \( L^\infty \): \( X \preceq_\mathbb{Q} Y \) holds if and only if, for all convex functions \( v : \mathbb{R} \to \mathbb{R}, \mathbb{E}_\mathbb{Q}[v(X)] \leq \mathbb{E}_\mathbb{Q}[v(Y)] \). In particular, \( X \preceq_\mathbb{Q} Y \) implies that \( Y \) also dominates \( X \) in second-order stochastic dominance relation under \( \mathbb{Q} \).

Lemma B.1. Suppose a vector \((\mathbb{Q}_1, \ldots, \mathbb{Q}_n)\) of probability measures satisfies Assumption 3.1, that \( \rho_i \) is a \( \mathbb{Q}_i \)-consistent risk measure, \( i \in [n] \), and that a partition \( \pi \in \Pi \) is as in (3.1). Let \( X \in L^\infty \) and \( X \in \mathcal{A}_X \) be arbitrary. Then there exists a locally comonotone allocation \( Y \in \mathcal{A}_X \) over \( \pi \) such that, for all \( i \in [n] \),
\[
Y_i \preceq_\mathbb{Q}, X_i \quad \text{and} \quad \rho_i(Y_i) \leq \rho_i(X_i).
\]

Proof. For each \( B \in \pi \) consider the nonatomic probability space \((B, \mathcal{F}_B, \mathbb{P}|_B)\), where \( \mathcal{F}_B := \{B \cap A \mid A \in \mathcal{F} \} \). As \( \sum_{i=1}^n X_i|_B = X|_B \), there is a comonotone function \( f^B \in \mathcal{C} \) such that \( f^B_i(X|_B) \preceq_{\mathbb{P}|_B} X_i|_B \) holds for all \( i \in [n] \); cf. [17, Theorem 3.1]. In particular, setting
\[
Y_i := \sum_{B \in \pi} f^B_i(X) \mathbf{1}_B, \quad i \in [n],
\]
defines an allocation \( Y \in \mathcal{A}_X \) which we claim to be as described in (1). In order to verify this, let \( v : \mathbb{R} \to \mathbb{R} \) be an arbitrary convex function satisfying \( v(0) = 0 \) without loss of generality, and use (3.1)
to compute
\[ \mathbb{E}_{Q_i}[v(Y_i)] = \mathbb{E}_{Q_i} \left[ \sum_{B \in \pi} v(f^B_i(X))1_B \right] = \sum_{B \in \pi} Q_i(B) \mathbb{E}_{P^B} \left[ v(f^B_i(X|B)) \right] \leq \sum_{B \in \pi} Q_i(B) \mathbb{E}_{P^B}[v(X_i|B)] = \mathbb{E}_{Q_i}[v(X_i)]. \]

The same allocation \( Y \) also satisfies \( \rho_i(Y_i) \leq \rho_i(X_i) \) for all \( i \in [n] \) because each consistent risk measure \( \rho_i \) is monotone with respect to \( \preceq_{Q_i} \).

The previous lemma is the first step in the adaptation of the comonotone improvement procedure to our setting of heterogeneous reference probability measures. Under its assumptions, it implies in particular that for all vectors \( X \in \prod_{i=1}^n A_{\rho_i} \) consisting of acceptable net losses, we find a locally comonotone allocation \( Y \) of \( \sum_{i=1}^n X_i \) over \( \pi \) such that \( \rho_i(Y_i) \leq \rho_i(X_i) \leq 0 \), i.e., \( Y \in \prod_{i=1}^n A_{\rho_i} \) as well.

The next proposition provides the bounds necessary for the second step of comonotone improvement.

**Proposition B.2.** In the situation of Theorem 4.1 let \( \pi \in \Pi \) be a measurable partition of \( \Omega \) as in (3.1). Let \( (Y_k) \subset \sum_{i=1}^n A_{\rho_i} \) be a bounded sequence and suppose that, for all \( k \in \mathbb{N} \), \( Y^k \in \prod_{i=1}^n A_{\rho_i} \) is a locally comonotone allocation of \( Y_k \) over \( \pi \). Then the sequence \( (Y^k) \) is bounded as well.

**Proof.** For each \( k \in \mathbb{N} \), there are \( (f^{B,k})_{B \in \pi} \in \mathfrak{C} \) such that, for all \( i \in [n] \),
\[
Y^k_i := \sum_{B \in \pi} f^{B,k}_i(Y_k)1_B. \tag{B.1}
\]
Let us first discuss boundedness of the sequence \( (Y^k_1) \) in detail. Towards a contradiction, suppose \( (Y^k_1) \) is unbounded in norm. Again, we abbreviate by \( f^{B,k} := f^{B,k} - f^{B,k}(0) \). As
\[
Y^k_1 = \sum_{B \in \pi} \overline{f}^{B,k}_1(Y_k)1_B + \sum_{B \in \pi} f^{B,k}_1(0)1_B \tag{B.2}
\]
and the first summand is uniformly bounded in \( k \) by Lipschitz continuity of \( f^{B,k}_1 \) and boundedness of the sequence \( (Y_k) \), we obtain that the sequence \( (\beta_k) \) defined by \( \beta_k := \max_{B \in \pi} |f^{B,k}_1(0)| \) is divergent and \( \lim_{k \to \infty} \frac{\|Y^k_1\|}{\beta_k} = 1 \). As the subspace of all \( \sigma(\pi) \)-measurable random variables is of finite dimension, we can assume up to passing to a subsequence that the norm limit \( V := \lim_{k \to \infty} \sum_{B \in \pi} \beta_k^{-1} f^{B,k}_1(0)1_B \) exists. As \( \lim_{k \to \infty} \| \sum_{B \in \pi} \beta_k^{-1} f^{B,k}_1(0)1_B \| = 1 \), \( V \neq 0 \).

Notice that due to (B.2) and boundedness of the sequence \( (\sum_{B \in \pi} \overline{f}^{B,k}_1(Y_k)1_B) \) in \( k \), we also have \( V = \lim_{k \to \infty} \frac{Y^k_1}{\beta_k} \), meaning that \( V \) lies in the asymptotic cone \( A_{\rho_i}^\infty \). Let \( D_1 \in \mathfrak{C}(\rho_1) \) with the property \( \rho^*_i(D_1) < \infty \), \( i \in [n] \). By Lemma A.2(1),
\[
\mathbb{E}_{P}[D_1V] \leq 0. \tag{B.3}
\]
Now note that, as \( k \to \infty \),
\[
\sum_{i=2}^n \frac{Y_i-Y^k_i}{\beta_k} \to -V.
\]
Moreover, for i = 2, ..., n, \( Y^k_i \in \mathcal{A}_{\rho_i} \) implies that \( \mathbb{E}_p[D_1Y^k_i] \leq \rho^*(D_1) \). Combining these observations leads to the estimate

\[
\mathbb{E}_p[D_1V] = -\mathbb{E}_p[D_1(-V)] = -\lim_{k \to \infty} \sum_{i=2}^{n} \mathbb{E}_p \left[ D_1 \frac{Y^k_i}{\beta_k} \right] \geq -\lim_{k \to \infty} \frac{1}{\beta_k} \sum_{i=2}^{n} \rho_i^*(D_1) = 0.
\]

In conjunction with (B.3), \( \mathbb{E}_p[Z_1V] = 0 \) follows. Using that \( V \in \mathcal{A}_{\rho_1}^\infty \) and that \( Z_1 \in \mathcal{C}(\rho_1), V = 0 \) is a consequence in direct contradiction to \( V \neq 0 \) established above. The assumption that \( (Y^k_1) \) is unbounded has to be absurd.

Arguing analogously for the coordinate sequences \( (Y^k_i), i = 2, \ldots, n-1 \), one infers their norm boundedness. At last,

\[
\sup_{k \in \mathbb{N}} \|Y^k_n\|_\infty \leq \sup_{k \in \mathbb{N}} \|Y_k\|_\infty + \sum_{i=1}^{n-1} \sup_{k \in \mathbb{N}} \|Y^k_i\|_\infty < \infty.
\]

\[ \Box \]

The most important consequence of Proposition B.2 in the sequel is the following corollary establishing closedness of the Minkowski sum of individual acceptance sets.

**Corollary B.3.** In the situation of Theorem 4.1, the Minkowski sum \( \sum_{i=1}^{n} \mathcal{A}_{\rho_i} \) is closed.

**Proof.** Abbreviate \( \mathcal{A}_+ := \sum_{i=1}^{n} \mathcal{A}_{\rho_i} \) and let \( (Y_k) \subset \mathcal{A}_+ \) be a sequence that converges in norm to \( Y \in L^\infty \) as \( k \to \infty \). We need to prove that \( Y \in \mathcal{A}_+ \). To this effect, for all \( k \in \mathbb{N} \) let \( Y^k \) be a locally comonotone allocation of \( Y_k \) as in (B.1) such that, for all \( i \in [n] \), \( Y^k_i \in \mathcal{A}_{\rho_i} \). Such an allocation exists because of Lemma B.1. By Proposition B.2, \( \sup_{k \in \mathbb{N}} \max_{i \in [n]} \|Y^k_i\|_\infty < \infty \). By (B.2)

\[
\kappa := \sup \left\{ |f^B_{i,k}(0)| : k \in \mathbb{N}, B \in \pi, i \in [n] \right\} \leq \sup \left\{ \|Y_k\|_\infty + \max_{i \in [n]} \|Y^k_i\|_\infty : k \in \mathbb{N} \right\} < \infty.
\]

The set \( \mathcal{C}_\kappa := \{ f \in \mathcal{C} : \max_{i \in [n]} |f_i(0)| \leq \kappa \} \) is sequentially compact in the topology of pointwise convergence; cf. [31, Lemma B.1]. Hence, there are \( (g^B)_{B \in \pi} \subset \mathcal{C}_\kappa \) such that, up to switching to a subsequence \( |\pi| \) times, \( \lim_{k \to \infty} f^B_{i,k}(x) = g^B(x) \) for all \( B \in \pi \) and all \( x \in \mathbb{R} \). For each \( i \in [n] \), we further observe that

\[
\left\| \sum_{B \in \pi} (f^B_{i,k}(Y_k) - g^B_i(Y))1_B \right\|_\infty \leq \sum_{B \in \pi} \left\| (f^B_{i,k}(Y_k) - g^B_i(Y))1_B \right\|_\infty \\
\leq \sum_{B \in \pi} \left\| \left( f^B_{i,k} - g^B_i \right)(Y_k) \right\|_\infty + \|Y_k - Y\|_\infty + |f^B_{i,k}(0) - g^B_i(0)|.
\]

As \( \tilde{f}^B_{i,k} \) converges to \( g^B \) uniformly on the compact interval \( \left[ -\sup_{k \in \mathbb{N}} \|Y_k\|_\infty, \sup_{k \in \mathbb{N}} \|Y_k\|_\infty \right] \), we infer that

\[
\lim_{k \to \infty} \left\| \sum_{B \in \pi} (f^B_{i,k}(Y_k) - g^B_i(Y))1_B \right\|_\infty = 0.
\]

As \( \mathcal{A}_{\rho_i} \) is closed, \( \sum_{B \in \pi} g^B_i(Y)1_B \in \mathcal{A}_{\rho_i} \) must hold, \( i \in [n] \). It remains to observe that the latter defines a vector in \( \mathcal{A}_Y \) and that therefore \( Y \in \mathcal{A}_+ \).

\[ \Box \]

**Remark B.4.** Proposition B.2 and Corollary B.3 are the key to verifying the existence of optimal allocations claimed in Theorem 4.1. The role that Proposition B.2 – and thereby admissibility of individual risk measures – plays for Corollary B.3 and Theorem 4.1 is best understood as adaptation of the spirit of Dieudonné’s [25] famous theorem on closedness of the Minkowski sum of convex
The proof of Proposition B.2 justifies the reliance on Assumption 3.1 of our approach based on comonotone improvement. A priori, the comonotone functions \((f^B)_{B \in \pi}\) which describe the allocation on events \(B \in \pi\) have no apparent relation to each other. This problem is circumvented by the use that the proof of Proposition B.2 makes of the norm compactness of the set of \(\sigma(\pi)\)-measurable elements in the unit ball of \(L^\infty\). We do not see how to generalise this to infinite dimensions, i.e., to heterogeneous reference measures without simple densities. One should also recall from Example 4.6(5) that such a generalisation without additional constraints on the reference measures is impossible.

APPENDIX C. PROOFS FROM SECTION 2

Proof of Lemma 2.2.

(1) Assume \(\mathbb{Q} \not\subset \mathbb{P}\) and fix \(Y \in \mathcal{A}_\rho\) as well as a \(\mathbb{Q}\)-null set \(N \in \mathcal{F}\) which satisfies \(\mathbb{P}(N) > 0\). For all \(r \in \mathbb{R}, Y + r 1_N \sim Q Y\), whence \(Y + \text{span}\{1_N\} \subset \mathcal{A}_\rho\) follows from the \(\mathbb{Q}\)-law invariance of \(\rho\). If \(D \in \text{dom}(\rho^*) \cap L^1\), then \(\sup_{\rho \in \mathbb{R}} \mathbb{E}_\rho[D(Y + r1_N)] \leq \rho^*(D)\) must hold for all \(r \in \mathbb{R}\). This entails \(\mathbb{E}_\rho[D1_N] = 0\). However, we also observe that \(1_N = \lim_{k \to \infty} \frac{1}{k}(Y + k1_N)\) lies in the asymptotic cone \(\mathcal{A}_\rho^\infty\). Thus, no \(D \in \text{dom}(\rho^*) \cap L^1\) can satisfy requirements (a)–(b) in Definition 2.1.

(2) Let \(D \in \mathbb{C}(\rho)\). As \(\mathcal{A}_\rho^\infty\) contains the cone \(-L^\infty_+, -1_{\{D=0\}} \in \mathcal{A}_\rho^\infty\). As \(\mathbb{E}_\rho[D(-1_{\{D=0\}})] = 0\), \(\mathbb{P}(D = 0) = 0\) follows.

(3) Let \(D, Z,\) and \(\lambda\) be as described and abbreviate \(R := \lambda D + (1 - \lambda)Z\). For property (a) in Definition 2.1, convexity of \(\text{dom}(\rho^*)\) implies \(R \in \text{dom}(\rho^*)\). As for property (b), suppose \(U \in \mathcal{A}_\rho^\infty\) satisfies \(\mathbb{E}_\rho[RU] = 0\). By Lemma A.2(1), both \(\mathbb{E}_\rho[DU] \leq 0\) and \(\mathbb{E}_\rho[ZU] \leq 0\). This forces \(\mathbb{E}_\rho[DU] = \mathbb{E}_\rho[ZU] = 0\). As \(D \in \mathbb{C}(\rho), U = 0\) follows.

For reasons that will become clear momentarily, we shall first give the proof of Proposition 2.5.

Proof of Proposition 2.5. As \(\rho\) is admissible, \(\rho = \mathbb{E}_\mathbb{Q}[\cdot]\) is impossible. Assuming the latter and using that \(\mathbb{Q}\) is atomless, we could select \(A \in \mathcal{F}\) with \(\mathbb{Q}(A) = \frac{1}{2}\), define \(U := 1_A - 1_{A^c} \in \mathcal{A}_\rho^\infty \setminus \{0\}\) satisfying \(\mathbb{E}_\mathbb{Q}[U] = 0\), and note that \(\text{dom}(\rho^*) = \{\mathbb{d}_\mathbb{Q}\rho\}\), contradicting that \(\mathbb{C}(\rho) \neq \emptyset\). Conversely, star-shapeness of \(\rho\) implies that the asymptotic cone \(\mathcal{A}_\rho^\infty\) is given by \(\mathcal{A}_\rho^\infty = \{U \in L^\infty : \forall t \geq 0 : tU \in \mathcal{A}_\rho\}\).

By [46, Theorem 5.7], \(\rho \neq \mathbb{E}_\mathbb{Q}[\cdot]\) if and only if, for all \(U \in \mathcal{A}_\rho^\infty, \mathbb{E}_\mathbb{Q}[U] = 0\) only holds if \(U = 0\). The latter is equivalent to \(\mathbb{d}_\mathbb{Q}\rho \in \mathbb{C}(\rho)\) and implies admissibility of \(\rho\).

Proof of Proposition 2.3. The implications (4) \(\implies\) (3) and (2) \(\implies\) (1) \(\implies\) (5) are trivial.

(1) \(\implies\) (4): \(\rho\) is admissible if and only if the star-shaped hull \(\rho_*\) introduced in (A.3) is admissible (Lemma A.4). By [46, Theorem 5.7] and (A.4), the latter is the case if and only if dom(\(\rho^*\) \(\cap L^1 = \text{dom}(\rho^*_*) \cap L^1\)) contains at least two elements.

(3) \(\implies\) (2): If (3) holds, then the identity \(\rho^*_* = \rho^*\) proved in (A.5) renders \(\rho_* = \mathbb{E}_\mathbb{Q}[\cdot]\) impossible. By the argument from the proof of Proposition 2.5 and Lemma A.4, \(\mathbb{d}_\mathbb{Q}\rho \in \mathbb{C}(\rho_* = \mathbb{C}(\rho)\).

(5) \(\implies\) (1): If the probability \(\nu\) is countably additive, the assertion is trivial. Else, let \(\nu = R \oplus \tau\) be the Yosida-Hewitt decomposition of \(\nu\). The random variable \(D := \tau(\Omega) + R\) is a probability density and satisfies \(\rho^*(D) < \infty\) by Lemma A.5. Now, if \(U \in \mathcal{A}_\rho^\infty\), then each \(U' \sim_\rho U\) also lies in \(\mathcal{A}_\rho^\infty\) and satisfies \(\mathbb{E}_\nu[U'] \leq 0\) (Lemma A.2). Together with \(\mathbb{E}_\rho[DU] \leq \sup_{U' \sim_\rho U} \mathbb{E}_\nu[U']\), we infer that \(\mathbb{E}_\nu[U] = 0\) has to hold. Consequently, \(U = 0\) and \(D\) has to be compatible with \(\rho\).
APPENDIX D. PROOFS FROM SECTION 3

Lemma D.1. A function $\varphi: L^\infty \to \mathbb{R}$ is $\mathbb{P}$-dilatation monotone above a finite $\sigma$-algebra $\mathcal{H}$ if and only if $\varphi$ is $\mathbb{P}$-dilatation monotone above $\sigma(\pi)$ for some $\pi \in \Pi$.

Proof. We only have to prove necessity. Let $\mathcal{H}$ be the finite sub-$\sigma$-algebra in question. The set $\Pi(\mathcal{H}) := \{ \pi \in \Pi \mid \pi \subset \mathcal{H} \}$ is nonempty as $\{ \Omega \} \in \Pi(\mathcal{H})$. Select $\pi^* \in \Pi(\mathcal{H})$ with maximal cardinality and prove that every $A \in \mathcal{H}$ has to satisfy $1_A = 1_E$ for some $E \in \sigma(\pi^*)$. Hence, $\sigma(\pi^*) \subset \mathcal{H} \subset \sigma(\pi^* \cup N)$, where $N$ denotes the collection of all $\mathbb{P}$-null sets. As dilatation monotonicity of $\varphi$ above $\sigma(\pi^*)$ is equivalent to dilatation monotonicity above $\sigma(\pi^* \cup N)$, the proof is concluded. \qed

Proof of Theorem 3.2. (1) implies (2): If probability measures $(Q_1, \ldots, Q_n)$ satisfy Assumption 3.1, we may fix a vector $Q$ of versions $Q_i$ of $\frac{dQ}{dP}$ which are simple functions. The set
$$\Sigma := \{ q \in \mathbb{R}^n \mid P(Q = q) > 0 \}$$
is finite. The event $\bigcap_{q \in \Sigma} \{ Q = q \}$ is a null set which we can assume to be empty by suitably manipulating the choice of densities. Hence, setting $\pi := \{ \{ Q = q \} \mid q \in \Sigma \} \in \Pi$, each density has a $\sigma(\pi)$-measurable version. Set $\mathcal{H} := \sigma(\pi)$, let $\mathcal{G} \supseteq \mathcal{H}$ be another sub-$\sigma$-algebra of $\mathcal{F}$, and note that for all $i \in [n]$ and all $X \in L^\infty$ (by the Bayes rule for conditional expectations)
$$\frac{dQ_i}{dP}\mathbb{E}[Q_i|G] = \mathbb{E}[\frac{dQ_i}{dP}X|G] = \frac{dQ_i}{dP}\mathbb{E}[X|G].$$
In view of $\frac{dQ_i}{dP} > 0$ by equivalence between the two probability measures, $\mathbb{E}_{Q_i}[X|G] = \mathbb{E}_P[X|G]$. As each $Q_i$-consistent risk measure is additionally $Q_i$-dilatation monotone, we have
$$\rho_i(\mathbb{E}_P[X|G]) = \rho_i(\mathbb{E}_{Q_i}[X|G]) \leq \rho_i(X).$$
Each $\rho_i$ is $\mathbb{P}$-dilatation monotone above $\mathcal{H}$.

(2) implies (1): Suppose that each $\rho_i$ is dilatation monotone above a common finite sub-$\sigma$-algebra $\mathcal{H} \subset \mathcal{F}$. By Lemma D.1, we can assume without loss of generality that $\mathcal{H} = \sigma(\pi)$ for some $\pi \in \Pi$. The further proof proceeds in several steps.

Step 1: Fix $i \in [n]$, $E \in \pi$, $Y \in L^\infty$, and consider the continuous function $\varphi: L^\infty_{P,E} \to \mathbb{R}$ defined by
$$\varphi(X) := \rho_i(Y1_{E^c} + X1_E).$$
Then $\varphi$ is $\mathbb{P}^E$-law invariant. By [21, Theorem 1.1], it suffices to show $\mathbb{P}^E$-dilatation monotonicity. Let $\mathcal{G} \subset \mathcal{F}$ be an arbitrary sub-$\sigma$-algebra and consider the sub-$\sigma$-algebra
$$\mathcal{G}_E := \{(A \cap E) \cup B \mid A \in \mathcal{G}, B \in \mathcal{F}, B \subset E^c \} \supseteq \mathcal{H}.$$Note that, for all $A \in \mathcal{G}$ and all $B \in \mathcal{F}$ with $B \subset E^c$,
$$\mathbb{E}_P[(X1_E + Y1_{E^c})1_{(A \cap E) \cup B}] = \mathbb{E}_P[X1_{A \cap E}] + \mathbb{E}_P[Y1_B] = \mathbb{P}(E)\mathbb{E}_{P,E}\mathbb{E}_P[X|G]1_A + \mathbb{E}_P[Y1_B]$$
$$= \mathbb{E}_P[X|G]1_{A \cap E} + \mathbb{E}_P[Y1_B].$$
As $\mathbb{E}_{P,E}[X|G]1_E + Y1_{E^c}$ can also be verified to be $\mathcal{G}_E$-measurable, we obtain
$$\mathbb{E}_P[X1_E + Y1_{E^c}|\mathcal{G}_E] = \mathbb{E}_{P,E}[X|G]1_E + Y1_{E^c}.\\ \text{ }^{4} L^\infty_{P,E} \text{ is isometrically isomorphic to } \{X1_E \mid X \in L^\infty \}.$
Hence, by $P$-dilatation monotonicity of $\rho_i$ above $\mathcal{H}$,
\[ \varphi(\mathbb{E}_P[X|G]) = \rho_i(\mathbb{E}_P[X1_E + Y1_{E^c}|G_E]) \leq \rho_i(X1_E + Y1_{E^c}) = \varphi(X). \]
Steps 2 and 3 are devoted to proving that we can, if necessary, alter the $Q_i$’s such that $\rho_i$ is still consistent and the new reference measures lie in the convex hull of $\{P^E \mid E \in \pi\}$.

**Step 2:** If $Q_i^E = P^E$ for all $E \in \pi$, then $Q_i$ is a convex combination of the $P^E$’s, or equivalently, (3.1) holds.

**Step 3:** Assume otherwise that there is $i \in [n]$ and $E \in \pi$ such that $P^E \neq Q_i^E$. We claim that
\[ \rho_i = \text{ES}_i^P, \quad \text{(D.1)} \]
i.e., $\rho_i$ assigns to $X \in \mathcal{L}_\infty$ its essential supremum under $P$ – and thus under every probability measure $Q \approx P$. Once (D.1) is verified, $\rho_i$ is therefore $Q$-consistent no matter which $Q \approx P$ one chooses. In particular, we can replace $Q_i$ by $P$ and retain the properties of the vector $(Q_1, \ldots, Q_n)$.
As both $\rho_i$ and ES$_i^P$ are $\|\cdot\|_\infty$-continuous, it suffices to verify identity (D.1) for arbitrary simple random variables. Also, as (D.1) clearly holds on the subspace of constant random variables, we shall prove $\rho_i(X) = \text{ES}_i^P(X)$ if $X$ is any nonconstant simple random variable. Using the Dubins-Spanier Theorem [4, Theorem 13.34] we can switch to $X' \sim Q_i X$ if necessary and assume that $X$ attains all its finitely many possible values on $E$ with positive probability. Abbreviate $u := -\text{ES}_i^P(-X)$, $v := \text{ES}_i^P(X)$, and observe that $Y, Z \in \mathcal{L}_\infty$ defined by
\[ Y := X1_{E^c} + v1_{\{X-u\}\cap E} + u1_{\{X<v\}\cap E} \quad \text{and} \quad Z := X1_{E^c} + v1_{E} \]
satisfy $Y \leq X \leq Z$ and thus $\rho_i(Y) \leq \rho_i(X) \leq \rho_i(Z)$. We now claim that
\[ \rho_i(Y) = \rho_i(X) = \rho_i(Z), \quad \text{(D.2)} \]
which holds true once we can verify $\rho_i(Z) \leq \rho_i(Y)$. To this end, set
\[ p^* := \sup\{P^E(A) \mid A \in \mathcal{F}, A \subset E, \rho_i(X1_{E^c} + v1_{A} + u1_{E\setminus A}) \leq \rho_i(Y)\}. \]
We have $p^* \geq P^E(X = v) > 0$. Assume towards a contradiction that $p^* < 1$. By, e.g., [44, Lemma A.2],
\[ \min_{P^E(A)=p^*} Q_i^E(A) = \min_{P^E(A)=p^*} \mathbb{E}_P \left[ \frac{dQ_i^E}{dP} 1_A \right] < p^* \]
\[ < \max_{P^E(A)=p^*} \mathbb{E}_P \left[ \frac{dQ_i^E}{dP} 1_A \right] = \max_{P^E(A)=p^*} Q_i^E(A), \]
whence we deduce the existence of $A, B \in \mathcal{F}, A, B \subset E$, satisfying $P^E(A) = P^E(B) = p^*$ and
\[ Q_i^E(A) < p^* < Q_i^E(B). \]
**Step 3:** Assume otherwise that there is $i \in [n]$ and $E \in \pi$ such that $P^E \neq Q_i^E$. We claim that
\[ Q_i^E(B) = Q_i^E(C). \]
Combining (D.3) and (D.4), $Q_i^E(C \setminus A) > 0$ and thus also $P^E(C \setminus A) > 0$. We deduce $P^E(C) > p^*$. By monotonicity of $\rho_i$, $\rho_i(X1_{E^c} + v1_{C} + u1_{E\setminus C}) \geq \rho_i(X1_{E^c} + v1_{A} + u1_{E\setminus A})$. In view of the definition of $p^*$,
\[ \rho_i(X1_{E^c} + v1_{C} + u1_{E\setminus C}) > \rho_i(X1_{E^c} + v1_{A} + u1_{E\setminus A}) = \rho_i(X1_{E^c} + v1_{B} + u1_{E\setminus B}). \]

However, we also have $X1_{E^c} + v1_C + u1_{E\setminus C} \sim_{Q_i} X1_{E^c} + v1_B + u1_{E\setminus B}$, whence the $Q_i$-law invariance of $\rho_i$ yields
\[
\rho_i(X1_{E^c} + v1_C + u1_{E\setminus C}) = \rho_i(X1_{E^c} + v1_B + u1_{E\setminus B}).
\]
(D.6) (D.5) and (D.6) pose a contradiction, and the assumption $p^* < 1$ must be absurd. We thus find a sequence $(A_n) \subset \mathcal{F}$ of events $A_n \subset E$ with $\mathbb{P}^E(A_n) \uparrow 1$ and
\[
\rho_i(X1_{E^c} + v1_{A_n} + u1_{E\setminus A_n}) \leq \rho_i(Y).
\]
Without loss, we can assume that the sequence $(A_n)$ is nondecreasing. Using the Fatou property of $\rho_i$ ([54, Theorem 3.5]), $\rho_i(Y) \geq \rho_i(Z)$. We have proved (D.2).

Now fix an arbitrary event $B \in \mathcal{F}$ such that $B \subset E^c$ and $0 < Q_i(B) < Q_i(E)$. Use the Dubins-Spanier Theorem once more to find an event $C \subset E$ satisfying $Q_i(B) = Q_i(C)$ and a simple random variable $Z' \sim_{Q_i} Z$ with the following properties: (1) $Z' = X$ on $(B \cup E)^c$; (2) $Z' = v$ on $B$; and (3) $Q_i^C \circ (Z')^{-1} = Q_i^B \circ Z^{-1}$. However, $Q_i(B) < Q_i(E)$ implies that $v$ is still the largest value $Z'$ takes on $E$. Going through the arguments above once more,
\[
\rho_i(X) = \rho_i(Z') = \rho_i(X1_{(E\cup B)^c} + v1_{E\cup B}).
\]
Continuing in this manner finally yields $\rho_i(X) = \rho_i(v) = ES^\rho_i(X)$.

**Step 4:** Combine Steps 2 and 3 and modify vector $(Q_1, \ldots, Q_n)$ if necessary to guarantee that (3.1) holds.

\[\square\]

**Proof of Proposition 3.3.** If $\pi \in \Pi$ is such that every density $\frac{dQ_i}{d\pi}$ has a $\sigma(\pi)$-measurable version, then
\[
\mathbb{E}_{Q_i}[X|\sigma(\pi)] = \sum_{B \in \pi} \mathbb{E}_{Q_i}[X1_B] \frac{Q_i(B)}{Q_i(B)} 1_B = \sum_{B \in \pi} \mathbb{E}_{Q_i}[X] 1_B.
\]
The latter is independent of $i$. Conversely, we can argue as in Lemma D.1 to find $\pi \in \Pi$ such that $\sigma(\pi)$ is sufficient for $\{Q_1, \ldots, Q_n\}$. In that case, for $A \in \mathcal{F}$,
\[
Q_i(A) = \mathbb{E}_{Q_i}[\mathbb{E}_{Q_i}[1_A|\sigma(\pi)]] = \mathbb{E}_{Q_i}[\mathbb{E}_{Q_i}[1_A|\sigma(\pi)]] = \sum_{B \in \pi} Q_i(B)Q_i^B(A).
\]
As (3.1) holds, also Assumption 3.1 must hold.

\[\square\]

**Appendix E. Proofs from Section 4**

**Proof of Theorem 4.1.** First of all, we prove that $\rho$ does not attain the value $-\infty$. To this end, let $Z_1 \in \mathcal{C}(\rho_1)$ be as described in assumption (iii), let $X \in L^\infty$, and fix $Y \in \mathcal{A}_X$. By definition of the convex conjugate,
\[
\sum_{i=1}^n \rho_i(Y_i) \geq \sum_{i=1}^n \mathbb{E}_F[Z_iY_i] - \rho_i^*(Z_1) = \mathbb{E}_F[Z_1X] - \sum_{i=1}^n \rho_i^*(Z_1).
\]
Taking the infimum over $Y \in \mathcal{A}_X$ on the left-hand side proves
\[
\rho(X) \geq \mathbb{E}_F[Z_1X] - \sum_{i=1}^n \rho_i^*(Z_1) > -\infty.
\]
Also, the Minkowski sum $\mathcal{A}_+ := \sum_{i=1}^n \mathcal{A}_{\rho_i}$ is closed by Corollary B.3. The remainder of the proof is standard, but shall be included for the sake of completeness. In its next step, we verify $\mathcal{A}_+ = \mathcal{A}_\rho$.

The inclusion of the left-hand set in the right-hand set is immediate. Conversely, assume without
loss of generality $\rho(X) = 0$ and consider a sequence $(Y^k) \subset \mathbb{A}_X$ such that $\lim_{k \to \infty} \sum_{i=1}^n \rho_i(Y^k_i) = 0$. Let $U^k_i := Y^k_i - \rho_i(Y^k_i) \in \mathcal{A}_{\rho_i}, i \in [n]$. As $\mathcal{A}_+$ is closed, $X = X - \lim_{k \to \infty} \sum_{i=1}^n \rho_i(Y^k_i) = \lim_{k \to \infty} \sum_{i=1}^n U^k_i \in \mathcal{A}_+$. Now, let $X \in L^\infty$. As $\rho(X - \rho(X)) = 0$, $X - \rho(X) \in \mathcal{A}_+$ and we can find $Y \in \prod_{i=1}^n \mathcal{A}_{\rho_i}$ such that $X - \rho(X) = \sum_{i=1}^n Y_i$. Defining an allocation $X \in \mathbb{A}_X$ by $X_i := Y_i + \frac{1}{n} \rho(X), i \in [n]$, we obtain
\[
\rho(X) \leq \sum_{i=1}^n \rho_i(X_i) = \sum_{i=1}^n \rho_i(Y_i + \frac{1}{n} \rho(X)) = \sum_{i=1}^n \rho_i(Y_i) + \frac{1}{n} \rho(X) \leq \rho(X).
\]
Hence, $X$ is an optimal allocation of $X$.

**Proof of Lemma 4.3.** If all $Q_1 = Q_2$, then $\frac{dQ_1}{dP} \in C(\rho_1) \cap C(\rho_2)$. Hence, let us assume in the following that the two references probability measures are not equal. If $\pi = \{B, B^c\}$ is a partition as in the assertion of the lemma, we can assume without loss that $\mathbb{P}(B) = \frac{1}{2}$. Abbreviate $\frac{dQ_1}{dP} = q_11_B \in (2 - q_1)1_{B^c}$. By Lemma A.5, dom($\rho_1^0$) $\cap$ dom($\rho_2^0$) must contain a probability density $Z$. Noting that $\mathbb{E}_P[Z|\sigma(\pi)] = \mathbb{E}_{Q_1}[Z|\sigma(\pi)] \in$ dom($\rho_1^0$), $i = 1, 2$, we can assume that $Z = z1_B \in (2 - z)1_{B^c}$.

**Case 1:** $z \notin \text{int}(\{q_1, q_2\})$. We can then assume that either (i) $q_1 < q_2 \leq z$, or (ii) $z \leq q_2 < q_1$.

Choose $\lambda := (q_2 - z)/(q_1 - z)$ and note that $\frac{dQ_1}{dP} = \lambda \frac{dQ_1}{dP} + (1 - \lambda)Z \in C(\rho_1) \cap \text{dom}(\rho_1^0)$.

**Case 2:** $z \in \text{int}(\{q_1, q_2\})$. By switching the roles of $B$ and $B^c$, we can guarantee that $q_1 < q_2$ and $z < 1$. Recall the definition of $\rho_1^0$ in (A.1), consider $\tilde{Z} := \frac{dP}{dQ_1}Z$, and note that $\tilde{Z} \in \text{dom}(\rho_1^0)$. As $Q_1$ is atomless, we can find events $E, F \in \mathbb{F}$ such that $E \subset B$ and $F \subset B^c$, and $T$ of shape
\[
T = \frac{z}{q_1}1_{E \cup F} + \frac{2 - z}{q_1}1_{(E \cup F)^c}
\]
satisfying $T \sim_{Q_1} \tilde{Z}$. As dom($\rho_1^0$) is $Q_1$-law invariant, $T \in \text{dom}(\rho_1^0)$. Consequently, we also have $D := \frac{dQ_1}{dP}\mathbb{E}_{Q_1}[T|\sigma(\pi)] \in \text{dom}(\rho_1^0)$. By manipulating $E, F$, we can even assume $D = v1_B \in (2 - v)1_{B^c}$ with $z < v < q_2$. In particular, for $\lambda = (z - v)/(q_1 - v), Z = \lambda \frac{dQ_1}{dP} + (1 - \lambda)D$, i.e., $Z \in C(\rho_1) \cap \text{dom}(\rho_2^0)$. In either case, assumption (iii) of Theorem 4.1 is satisfied.

**Proof of Lemma 4.5.** Fix $k \in [n - 1]$ and $\nu_k \in \bigcap_{i=1}^n \text{dom}(\rho_1^0)$ as described by (4.5). Without loss, we can assume that $\nu_k$ is not countably additive. Let $\nu_k = R_k \oplus \tau_k$ be Yosida-Hewitt decomposition and set $s := 1 - \mathbb{E}_P[R_k] > 0$. If $s = 1$, then Lemma A.5 implies that the $\rho_k$-compatible density $\frac{dQ_k}{dP}$ lies in each dom($\rho_1^0$). Else, applying Lemma A.5 to each $i \in [n], D := s \frac{dQ_k}{dP} + R_k$ and $\frac{1}{1-s} R_k$ lie in $\bigcap_{i=1}^n \text{dom}(\rho_1^0)$. Thus, if $U \in \mathcal{A}_{\rho_k}^\infty$ satisfies $\mathbb{E}_P[DU] = 0$, then also $\mathbb{E}_P[R_k U] \leq 0$ by Lemma A.2. Hence, $\mathbb{E}_{Q_k}[U] = 0$ must hold. As $\frac{dQ_k}{dP} \in C(\rho_k), U = 0$ follows, and we have shown that $D$ is a $\rho_k$-compatible density in $\bigcap_{i=1}^n \text{dom}(\rho_1^0)$.

**Lemma E.1.** Suppose $\rho$ is a $\mathbb{P}$-consistent risk measure satisfying dom$(\rho^*) \setminus L^\infty \neq \emptyset$. Then dom$(\rho^*)$ contains an unbounded probability density.

**Proof.** The assertion is clear if dom$(\rho^*) \subset L^1$. Else, select a finitely additive probability $\mu \in$ dom$(\rho^*)$ that is not countably additive and let $\mu = R \oplus \tau$ be its Yosida-Hewitt decomposition. Fix an unbounded random variable $L$ with $\mathbb{E}[L] = 1 - \mathbb{E}[R] > 0$. By [34, Lemma A.32], there is a random variable $U$ with $\mathbb{P} \circ U^{-1} = \text{Unif}(0,1)$ such that $R = q_R(U)$. The density $D := q_L(U) + R$ is then unbounded and lies in dom$(\rho^*)$ by Lemma A.5.

**Proof of Proposition 4.7.** For assumption (1), suppose that (4.10) is satisfied and note that dom($\rho_2^0$) = $\{\frac{dQ_1}{dP}\}$ by assumption. Let $D \in C(\rho_1)$. By (4.10), we find $s \in (0,1)$ and $\mu \in$ dom($\rho_1^0$) such that
Theorem 4.1, Let 

That is, assumption (iii) from Theorem 4.1 holds true.

For assertion (2), Lemma E.1 yields that \( \text{dom}(\rho_2^*) \subset \{ \frac{dQ_2}{dP} Z \mid Z \in L^\infty \} \subset L^\infty \). Similarly, we can use Lemma E.1 to find an unbounded random variable \( R \in L^1_{Q_1} \) such that \( \rho_1^*(R) < \infty \). As \( \frac{dQ_2}{dP} \in L^\infty \), we get that \( D := \frac{dQ_2}{dP} R \in L^1 \) is unbounded as well and satisfies \( \rho_1^*(D) = \rho_2^*(R) < \infty \). Using (4.10) with \( (\mu_1, \mu_2) = (D, \frac{dQ_2}{dP}) \), we find a bounded density \( Z \in \text{dom}(\rho_2^*) \) such that \( V := Z + s(\frac{dQ_2}{dP} - D) \in \text{dom}(\rho_1^*) \).

\[ \text{However, } \rho_1 \text{ is unbounded.} \]

Proof of Corollary 4.8. Let \( X,Y \in L^\infty \) and suppose that \( X \sim_{\mathbb{P}} Y \) holds for all \( B \in \pi \). Moreover, let \( (f^B)_{B \in \pi} \subset \mathcal{C} \) and denote the resulting locally comonotonic partitions of \( X \) and \( Y \) by \( X \) and \( Y \), respectively. We compute for any \( i \in [n] \) and \( x \in \mathbb{R} \):

\[ Q_i(X_i \leq x) = \sum_{B \in \pi} Q_i(B) \mathbb{P}^B \left( f_i^B(X) \leq x \right) = \sum_{B \in \pi} Q_i(B) \mathbb{P}^B \left( f_i^B(Y) \leq x \right) = Q_i(Y_i \leq x). \]

That is, \( X_i \sim_{Q_i} Y_i \) holds for all \( i \in [n] \). Given the \( Q_i \)-law-invariance of each \( \rho_i \) and the proof of Theorem 4.1,

\[ \rho(X) = \min_{(f^B)_{B \in \pi} \subset \mathcal{C}} \sum_{i=1}^{n} \rho_i \left( \sum_{\pi \in B} f_i^B(X) \mathbb{1}_B \right) = \min_{(f^B)_{B \in \pi} \subset \mathcal{C}} \sum_{i=1}^{n} \rho_i \left( \sum_{\pi \in B} f_i^B(Y) \mathbb{1}_B \right) = \rho(Y). \]

\[ \square \]

References

[1] Acciaio, B. (2007), Optimal risk sharing with non-monotone monetary functionals. Finance & Stochastics 11(2):267–289.
[2] Acciaio, B. (2009), Short note on inf-convolution preserving the Fatou property. Annals of Finance 5:261–287.
[3] Acciaio, B., and G. Svindland (2009), Optimal risk sharing with different reference probabilities. Insurance: Mathematics and Economics 44(3):426–433.
[4] Aliprantis, C. C., and K. C. Border (2006), Infinite Dimensional Analysis: A Hitchhiker’s Guide, 3rd edition, Springer.
[5] Amarante, M. (2016), A representation of risk measures. Decisions in Economics and Finance 39:95–103.
[6] Amarante, M., M. Ghossoub, and E. Phelps (2015), Ambiguity on the insurer’s side: The demand for insurance. Journal of Mathematical Economics 58:61–78.
[7] Asimit, V. A., T. J. Boonen, Y. Chi, and W. F. Chong (2021), Risk sharing with multiple indemnity environments. European Journal of Operational Research 295(2):587–603.
[8] Auslander, A., and M. Teboulle (2003), Asymptotic Cones and Functions in Optimization and Variational Inequalities. Springer.
[9] Barrieu, P., and N. El Karoui (2005), Inf-convolution of risk measures and optimal risk transfer. Finance and Stochastics 9(2):269–298.
[10] Baes, M., P. Koch-Medina, and C. Munari (2019), Existence, uniqueness, and stability of optimal portfolios of eligible assets. Mathematical Finance 30(1):128–166.
[11] Bellini, F., P. Koch-Medina, C. Munari, and G. Svindland (2021), Law-invariant functionals on general spaces of random variables. SIAM Journal on Financial Mathematics 12(1):318–341.
[12] Bellini, F., P. Koch-Medina, C. Munari, and G. Svindland (2021), Law-invariant functionals that collapse to the mean. Insurance: Mathematics and Economics 98:83–91.
[13] Boonen, T. J. (2016), Optimal reinsuror with heterogeneous reference probabilities. Risks 4(3):26.
[14] Boonen, T. J., and M. Ghossoub (2020), Bilateral risk sharing with heterogeneous beliefs and exposure constraints. ASTIN Bulletin 50(1):293–323.
[15] Burgert, C., and L. Rüschendorf (2008), Allocation of risks and equilibrium in markets with finitely many traders. Insurance: Mathematics and Economics 42:177–188.
[16] Cambou, M., and D. Filipović (2017), Model uncertainty and scenario aggregation. Mathematical Finance 27(2):534–567.
[17] Carlier, G., R.-A. Dana, and A. Galichon (2012), Pareto efficiency for the concave order and multivariate comonotonicity. Journal of Economic Theory 147(1):207–229.
[54] Mao, T., and R. Wang (2020), Risk aversion in regulatory capital principles. SIAM Journal on Financial Mathematics 11(1):169–200.
[55] Marshall, J. M. (1992), Optimum insurance with deviant beliefs. In: Dionne, G. (ed.), Contributions to Insurance Economics, pp. 255–274. Kluwer Academic Publishers.
[56] Rahsepar, M., and F. Xanthos (2020), On the extension property of dilatation monotone risk measures. Statistics & Risk Modeling 37(3–4):107–119.
[57] Rüschendorf, L. (2013), Mathematical Risk Analysis. Springer.
[58] Svindland, G. (2010), Continuity properties of law-invariant (quasi-)convex risk functions on $L^\infty$. Mathematics and Financial Economics 3(1):39-43.
[59] Svindland, G. (2010), Subgradients of law-invariant convex risk measures on $L^1$. Statistics & Decisions 27(2):169-199.
[60] Tsanakas, A. (2009), To split or not to split: Capital allocation with convex risk measures. Insurance: Mathematics and Economics 44(2):268–277.
[61] Wang, R., and J. F. Ziegel (2021), Scenario-based risk evaluation. Finance & Stochastics 25:725–756.
[62] Weber, S. (2018), Solvency II, or how to sweep the downside risk under the carpet. Insurance: Mathematics and Economics 82:191–200.