TAU-FUNCTIONS FOR THE ABLOWITZ–LADIK HIERARCHY: THE
MATRIX-RESOLVENT METHOD

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Abstract. We extend the matrix-resolvent method for computing logarithmic derivatives of tau-functions to the Ablowitz–Ladik hierarchy. In particular, we derive a formula for the generating series of the logarithmic derivatives of an arbitrary tau-function in terms of matrix resolvents. As an application, we provide a way of computing certain integrals over the unitary group.

Contents
1. Introduction 1
2. Matrix-resolvents and the AL hierarchy 6
3. From matrix resolvents to tau-functions 12
4. Generating series for correlators related to CUE 14
References 14

1. Introduction
The Ablowitz–Ladik (AL) hierarchy has been introduced in the 1970s as a spatial discretization of the AKNS-D hierarchy [1, 2]. Nowadays, it plays an important role in many different areas of mathematical physics. Without being exhaustive, let us recall that its algebro-geometric solutions have been studied in [16, 24]. In [3, 4], the same hierarchy is studied under the name of Toeplitz lattice, and its tau functions are used to study combinatorial models and integrals over the unitary group. Tau functions for the Ablowitz-Ladik hierarchy appear also as correlation functions in quantum spin chains [20]. In [9, 10, 23], the AL hierarchy is studied to investigate the theory of equivariant Gromov-Witten invariants for the resolved conifold of $\mathbb{P}^1$ under the anti-diagonal action. In [22] the tri-hamiltonian structure of the AL hierarchy is found. Relations to melting crystal models are treated in [26].

Recently, the matrix-resolvent method for studying tau-structures (in the sense of [15]) of differential integrable systems was introduced in [6, 7, 29]. This method was extended to certain differential-difference integrable systems in [13, 14, 28]. Our aim is to further extend the matrix-resolvent method to the study of the tau-structure of the AL hierarchy.

We start by giving an algebraic construction of the hierarchy. Let us denote by $\mathcal{A}$ the polynomial ring generated by infinitely many indeterminates

$$\mathcal{A} := \mathbb{Q}[q_n, r_n, q_{n+1}, r_{n+1}, q_{n+2}, r_{n+2}, \ldots].$$
Define the shift operator $\Lambda : \mathcal{A} \to \mathcal{A}$ as the (invertible) linear operator satisfying
\[
\Lambda(q_{n+i}) = q_{n+i+1}, \quad \Lambda(r_{n+i}) = r_{n+i+1}, \quad i \in \mathbb{Z},
\]
\[
\Lambda(fg) = \Lambda(f) \Lambda(g), \quad \forall f, g \in \mathcal{A}.
\]
The original Lax operator of the AL hierarchy is given by
\[
L := \Lambda + U(z),
\]
where $z$ plays the role of the spectral parameter and
\[
U(z) := \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} + \begin{pmatrix} 0 & r_n \\ q_n & 0 \end{pmatrix}.
\]

For our purposes, it is more convenient to use another Lax operator \[24\] (see also \[16\])
\[
L := \Lambda + U(\lambda),
\]
where $\lambda$ is the new spectral parameter and
\[
U(\lambda) := \begin{pmatrix} \lambda & r_n \\ \lambda q_n & 1 \end{pmatrix}.
\]

As already observed in \[24\] (cf. also \[21\]), the two Lax operators are related through the gauge transformation
\[
U(\lambda) = G_{n+1}(z) U(z) G_n(z)^{-1}, \quad \text{with} \quad G_n(z) := z^n \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix}
\]
and $\lambda = z^2$.

We now recall the definition of matrix resolvents (cf. \[11\]) associated to $L$ (of course, analogously, one can also define the matrix resolvents associated to $L$).

**Definition 1.** A matrix $R \in \text{Mat}(2, \mathcal{A}[[\lambda]])$ is a **matrix resolvent** of $L$ if it satisfies the equation
\[
\Lambda(R(\lambda)) U(\lambda) - U(\lambda) R(\lambda) = 0.
\]

In the following, we will just speak about matrix resolvents, without further specifying that they are associated to $L$.

**Lemma 1.** There exists a unique matrix resolvent $R_-(\lambda) \in \text{Mat}(2, \mathcal{A}[[\lambda]])$ satisfying the normalization conditions
\[
R_-(\lambda) - \begin{pmatrix} 0 & r_n^{-1} \\ 0 & 1 \end{pmatrix} \in \lambda \text{Mat}(2, \mathcal{A}[[\lambda]]), \quad \text{tr} R_-(\lambda) = 1, \quad \det R_-(\lambda) = 0.
\]

Similarly, there exists a unique matrix resolvent $R_+(\lambda) \in \text{Mat}(2, \mathcal{A}[[\lambda^{-1}]])$ satisfying the normalization conditions
\[
R_+(\lambda) - \begin{pmatrix} 1 & 0 \\ q_n & 0 \end{pmatrix} \in \lambda^{-1} \text{Mat}(2, \mathcal{A}[[\lambda^{-1}]]), \quad \text{tr} R_+(\lambda) = 1, \quad \det R_+(\lambda) = 0.
\]

The proof is given in Section \[2\]. The matrix resolvents $R_\alpha(\lambda)$, $\alpha \in \{+, -\}$, uniquely defined by Lemma \[1\] will be called basic matrix resolvents. We remark that the second conditions in \(7-9\) can be equivalently written as
\[
\text{tr } (R_+(\lambda)^2) = \text{tr } (R_-(\lambda)^2) = 1.
\]
For the reader’s convenience, the first two terms of the basic matrix resolvent read as follows:

\[
R_- (\lambda) = \begin{pmatrix} 0 & r_{n-1} \\ 0 & 1 \end{pmatrix} + \lambda \left( \begin{array}{cc} q_n r_{n-1} & -q_n r_{n-1}^2 + r_{n-2}(1 - q_{n-1} r_{n-1}) \\ q_n & -q_n r_{n-1} \end{array} \right) + O(\lambda^2),
\]

\[
R_+ (\lambda) = \begin{pmatrix} 1 & 0 \\ q_{n-1} & 0 \end{pmatrix} + \frac{1}{\lambda} \left( \begin{array}{cc} q_{n-2}(1 - q_{n-1} r_{n-1}) - q_{n-1}^2 r_n & r_n \\ q_{n-2} & q_{n-1} r_n \end{array} \right) + O(\lambda^{-2}).
\]

It will be convenient to define series \( a_\alpha (\lambda), b_\alpha (\lambda), c_\alpha (\lambda), \alpha \in \{+,-\}, \) and an infinite set of polynomials \( \{a_{\alpha,p}, b_{\alpha,p}, c_{\alpha,p} \in \mathcal{A} \mid \alpha = \pm, p \geq 0\} \) via

\[
R_-(\lambda) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} a_-(\lambda) & b_-(\lambda) \\ c_-(\lambda) & -a_-(\lambda) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \sum_{p \geq 0} \lambda^p \begin{pmatrix} a_{-,p} & b_{-,p} \\ c_{-,p} & -a_{-,p} \end{pmatrix}, \quad (11)
\]

\[
R_+(\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} a_+(\lambda) & b_+(\lambda) \\ c_+(\lambda) & -a_+(\lambda) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \sum_{p \geq 0} \frac{1}{\lambda^p} \begin{pmatrix} a_{+,p} & b_{+,p} \\ c_{+,p} & -a_{+,p} \end{pmatrix}. \quad (12)
\]

In particular, \( a_{-,0} = a_{+,0} = 0, b_{-,0} = r_{n-1}, c_{-,0} = 0, b_{+,0} = 0, c_{+,0} = q_n - 1. \)

We also define matrices \( V_{\alpha,p}(\lambda), \alpha = +,-, p \geq 0 \) by

\[
V_{-,p}(\lambda) = (\lambda^{-p} R_-(\lambda))_{\leq 0} - \begin{pmatrix} a_{-,p} & b_{-,p} \\ 0 & 0 \end{pmatrix}, \quad (13)
\]

\[
V_{+,p}(\lambda) = (\lambda^p R_+(\lambda))_{\geq 0} - \begin{pmatrix} 0 & 0 \\ c_{+,p} & -a_{+,p} \end{pmatrix}. \quad (14)
\]

where \( (\cdot)_{\leq 0} \) denotes the polynomial part in \( \lambda^{-1} \) of the argument, and \( (\cdot)_{\geq 0} \) the polynomial part in \( \lambda. \)

We now want to define an infinite set of derivations \( \{D_{\alpha,p}, \alpha = \pm, p \geq 0\} \) on the ring \( \mathcal{A}[[\lambda]] \) which are admissible, i.e. they commute with the operator \( \Lambda. \) In order to do so, it suffices to define how \( D_{\alpha,p} \) acts on \( \lambda, q_n \) and \( r_n, \) and then extend its action using the Leibnitz rule and admissibility. In the proposition below, \( D_{\alpha,p} \) acts on matrices entrywise.

**Proposition 1.** The equations

\[
D_{\alpha,p} U := \Lambda(V_{\alpha,p}) U - U V_{\alpha,p}, \quad \alpha = \pm, \ p \geq 0 \quad (15)
\]

uniquely define a set of admissible derivations whose restriction to \( \mathcal{A} \) is well defined and such that \( D_{\alpha,0} \lambda = 0 \) for \( \alpha = \pm \) and \( p \geq 0. \)

Proposition 1 will be proven in Section 2 via an explicit computation. The construction of the flows in the way above is known and can also be found in e.g. [10].

Introduce the loop operators as follows:

\[
\nabla_- (\lambda) := \sum_{p \geq 0} \frac{D_{-,p}}{\lambda^{p+1}}, \quad \nabla_+ (\lambda) := \sum_{p \geq 0} \frac{D_{+,p}}{\lambda^{p+1}}. \quad (16)
\]

By using a uniqueness argument given in [28] (see the Lemma 3 and the Proposition 2 of [28]) we will prove in Section 2 the following lemma.

**Lemma 2.** For \( \alpha = \pm, \) the following equations are verified:

\[
\nabla_- (\mu) (R_{\alpha}(\lambda)) = -\frac{\mu}{\lambda} \left[ \frac{R_- (\mu)}{\mu - \lambda}, R_{\alpha}(\lambda) \right] - \left[ R_- (\mu), R_{\alpha}(\lambda) \right], \quad (17)
\]

\[
\nabla_+ (\mu) (R_{\alpha}(\lambda)) = \frac{[R_+ (\mu), R_{\alpha}(\lambda)]}{\mu - \lambda} - \left[ R_+ (\mu), R_{\alpha}(\lambda) \right], \quad (18)
\]

where \( Q_\pm(\mu) \) are given by

\[
Q_-(\mu) := \frac{1}{\mu} \begin{pmatrix} a_-(\mu) & b_-(\mu) \\ 0 & 0 \end{pmatrix}, \quad Q_+(\mu) := \frac{1}{\mu} \begin{pmatrix} 0 & 0 \\ c_+(\mu) & -a_+(\mu) \end{pmatrix}.
\] (19)

Expanding the right-hand sides of (17) and (18) in power series, we see that these two equations can be reformulated as follows:

\[
D_{\beta,q} R_\alpha(\lambda) = [V_{\beta,q}(\lambda), R_\alpha(\lambda)] , \quad \alpha, \beta = \pm, p, q \geq 0.
\] (20)

By using Lemma 2 or, equivalently, (20), we will also prove in Section 2 the following

**Proposition 2.** The derivations \( D_{\alpha,p}, \alpha = \pm, p \geq 0 \), are mutually commuting. We call (15) the “abstract AL Hierarchy” and \( D_{\alpha,p} \), the “AL derivations”.

Let us proceed with the definition of the tau-structure for the AL hierarchy.

**Definition 2.** For any \( \alpha, \beta = \pm \) and \( p, q \geq 0 \), define the polynomial \( \Omega_{\alpha,p;\beta,q} \in \mathcal{A} \) by the generating series

\[
\sum_{p,q \geq 0} \Omega_{\alpha,p;\beta,q} \lambda^{-p-1} \mu^{-q-1} = \frac{\text{tr} (R_+(\lambda) R_+^q(\mu))}{(\lambda - \mu)^2} - \frac{1}{(\lambda - \mu)^2},
\] (21)

\[
\sum_{p,q \geq 0} \Omega_{\alpha,p;\beta,q} \lambda^{-p-1} \mu^{-q-1} = \frac{\text{tr} (R_+(\lambda) R_+^q(\mu))}{(\lambda - \mu)^2} + \frac{1}{\lambda^2} (|\lambda| > |\mu|),
\] (22)

\[
\sum_{p,q \geq 0} \Omega_{\alpha,p;\beta,q} \lambda^{-p-1} \mu^{-q-1} = \frac{\text{tr} (R_-(\lambda) R_-^q(\mu))}{(\lambda - \mu)^2} + \frac{1}{\mu^2} (|\mu| > |\lambda|),
\] (23)

\[
\sum_{p,q \geq 0} \Omega_{\alpha,p;\beta,q} \lambda^{-p-1} \mu^{-q-1} = \frac{\text{tr} (R_-^q(\lambda) R_-^q(\mu))}{(\lambda - \mu)^2} - \frac{1}{(\lambda - \mu)^2}.
\] (24)

The fact that equations (21)-(24) give a proper definition of the polynomials \( \Omega_{\alpha,p;\beta,q} \) is proven in Section 3. The defining equations (21)-(24) can be written more succinctly as

\[
\sum_{p,q \geq 0} \frac{\Omega_{\alpha,p;\beta,q}}{\lambda^{\alpha p+1} \mu^{\beta q+1}} = \alpha \beta \left( \frac{\text{tr} (R_+(\lambda) R_+^q(\mu))}{(\lambda - \mu)^2} - r_{\alpha,\beta}(\lambda, \mu) \right),
\] (25)

where \( r_{+,+}(\lambda, \mu) = r_{-,+}(\lambda, \mu) = 1/(\lambda - \mu)^2 \), \( r_{+,+}(\lambda, \mu) = 1/\lambda^2 \), \( r_{-,+}(\lambda, \mu) = 1/\mu^2 \).

**Lemma 3.** For any \( \alpha, \beta = \pm \) and \( p, q \geq 0 \), the polynomials \( \Omega_{\alpha,p;\beta,q} \) and \( a_{\alpha,p} \in \mathcal{A} \) satisfy the following equations:

\[
\Omega_{\alpha,p;\beta,q} = \Omega_{\beta,q;\alpha,p},
\] (26)

\[
D_{\gamma,r} (\Omega_{\alpha,p;\beta,q}) = D_{\beta,q} (\Omega_{\alpha,p;\gamma,r}),
\] (27)

\[
(\lambda - 1) (\Omega_{\alpha,p;\beta,q}) = \alpha D_{\beta,q} (a_{\alpha,p}).
\] (28)

The property (26) obviously follows from (21)-(24). The proof for (27)-(28) is given in Section 3. Following [15], we call the set of polynomials

\[
\{ a_{\alpha,p}, \Omega_{\alpha,p;\beta,q}, \alpha = \pm, p, q \geq 0 \} \subseteq \mathcal{A}
\]

the **tau-structure** for the AL hierarchy. For \( k \geq 3 \), let us also define inductively

\[
\Omega_{\alpha,k;\ldots;\alpha_1,p_1} := D_{\alpha,k,p_k} \Omega_{\alpha_{k-1};\ldots;\alpha_1,p_1}.
\] (29)

Using Lemma 3 it is clear that, for any \( k \geq 2 \), \( \Omega_{\alpha,k;\ldots;\alpha_1,p_1} \) is invariant by permutations of the pairs of indices \( (\alpha_i, p_i), i = 1, \ldots, k \). The following theorem is the main result of this paper.
Corollary 1. For every $k \geq 3$, and for any fixed $\alpha_1, \ldots, \alpha_k \in \{+, -\}$, the generating series of $\Omega_{\alpha_1, p_1; \ldots; \alpha_k, p_k}$ has the following expression:

\[
\sum_{i_1, \ldots, i_k \geq 0} \frac{\Omega_{\alpha_1, i_1; \ldots; \alpha_k, i_k}}{\lambda_1^{i_1+1} \cdots \lambda_k^{i_k+1}} = -\prod_{j=1}^{k} \alpha_j \sum_{S_k/C_k} \frac{\text{tr} \left( R_{\sigma(1)}(\lambda_{\sigma(1)}) \cdots R_{\sigma(k)}(\lambda_{\sigma(k)}) \right)}{\prod_{j=1}^{k} (\lambda_{\sigma(j)} - \lambda_{\sigma(j+1)})},
\]

(30)

where $C_k$ denotes the cyclic group.

If we view $q_n$ and $r_n$ as functions of $n$ and $s = (s^{\alpha,p})_{\alpha=\pm, p \geq 0}$, and replace $D_{\alpha,p}$ in (15) by $\partial/\partial s^{\alpha,p}$, we obtain a commuting family of evolutionary differential–difference equations, the AL hierarchy. Let $(q_n = q(n,s), r_n = r(n,s))$ be an arbitrary solution to this hierarchy. Following the scheme of Dubrovin and Zhang [15], we know from Lemma 3 that there exists a function $\tau(n,s)$ satisfying

\[
\frac{\partial^2 \log \tau(n,s)}{\partial s^{\alpha,p} \partial s^{\beta,q}} = \Omega_{\alpha,p; \beta,q}(n + 1, s),
\]

(31)

\[
(\Lambda - 1) \left( \frac{\partial \log \tau(n,s)}{\partial s^{\alpha,p}} \right) = \alpha \, a_{\alpha,p}(n + 1, s),
\]

(32)

\[
\frac{\tau(n+1,s) \, \tau(n-1,s)}{\tau(n,s)^2} = 1 - q_n(s) \, r_n(s).
\]

(33)

We call $\tau(n,s)$ the tau-function of the solution $(q_n = q(n,s), r_n = r(n,s))$, although it is uniquely determined by (31)–(33) up to multiplying by a factor of the form

\[
\exp \left\{ c_0 n + \sum_{\alpha=\pm, p \geq 0} c_{\alpha,p} s^{\alpha,p} \right\}.
\]

(34)

Definition 3. Let $(q_n = q(n,s), r_n = r(n,s))$ be an arbitrary solution to the AL hierarchy, and $\tau = \tau(n,s)$ the tau-function of the solution. Let $\{ (\alpha_{\ell}, i_{\ell}), \ell = 1, \ldots, k \}$ be an arbitrary set of multi-indices, with $\alpha_{\ell} \in \{+, -\}$, $i_{\ell} \geq 0$ for every $\ell = 1, \ldots, k$. We define the $k$-point correlation functions $\langle \langle \tau_{\alpha_1, i_1} \cdots \tau_{\alpha_k, i_k} \rangle \rangle(n,s)$ of the solution by

\[
\langle \langle \tau_{\alpha_1, i_1} \cdots \tau_{\alpha_k, i_k} \rangle \rangle(n,s) := \frac{\partial^k \log \tau(n,s)}{\partial s^{\alpha_1,i_1} \cdots \partial s^{\alpha_k,i_k}}, \quad k \geq 1.
\]

(35)

From the above definitions, it is clear that

\[
\langle \langle \tau_{\alpha_1, i_1} \cdots \tau_{\alpha_k, i_k} \rangle \rangle(n,s) = \Omega_{\alpha_1, i_1; \ldots; \alpha_k, i_k}(n + 1, s), \quad k \geq 2.
\]

(36)

From formula (25), Theorem 1 and (36) we arrive at the following

Corollary 1. Let $(q_n = q(n,s), r_n = r(n,s))$ be an arbitrary solution to the AL hierarchy. For every $k \geq 2$, and for any fixed $\alpha_1, \ldots, \alpha_k \in \{+, -\}$, the generating series of $k$-point correlation functions of the solution has the following expression:

\[
\sum_{i_1, \ldots, i_k \geq 0} \frac{\langle \langle \tau_{\alpha_1, i_1} \cdots \tau_{\alpha_k, i_k} \rangle \rangle(n,s)}{\lambda_1^{i_1+1} \cdots \lambda_k^{i_k+1}} = -\prod_{j=1}^{k} \alpha_j \left( \sum_{S_k/C_k} \frac{\text{tr} \left( R_{\sigma(1)}(\lambda_{\sigma(1)}) \cdots R_{\sigma(k)}(\lambda_{\sigma(k)}) \right)}{\prod_{j=1}^{k} (\lambda_{\sigma(j)} - \lambda_{\sigma(j+1)})} \right) - \delta_{k,2} r_{\alpha_1, \alpha_2}(\lambda_1, \lambda_2),
\]

(37)
where $R_{\beta}(\lambda, n, s)$ is defined by the substitution of $(q_n = q(n, s), r_n = r(n, s))$ into $R_{\beta}(\lambda)$, and $r_{\alpha, \beta}(\lambda, \mu)$ are defined right after (25).

Generating functions containing both infinitely many negative and positive powers of $\lambda$ appeared in [17, 18] in the computations of LUE and JUE correlators via the Riemann–Hilbert approach (cf. also [8, 12]).

The paper is organized as follows. In Section 2, we prove Lemma 1. In Section 3, we prove Lemma 3 and Theorem 1. An application related to Toeplitz determinants and integrals over the unitary is given in Section 4.

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2. Matrix-resolvents and the AL hierarchy

In this section, we prove Lemma 1, Propositions 1 and 2.

Proof of Lemma 1. As in (11)–(12), let us write

$$R_-(\lambda) = \begin{pmatrix} a_-(\lambda) & b_-(\lambda) \\ c_-(\lambda) & 1 - a_-(\lambda) \end{pmatrix},$$

$$R_+(\lambda) = \begin{pmatrix} 1 + a_+(\lambda) & b_+(\lambda) \\ c_+(\lambda) & -a_+(\lambda) \end{pmatrix}.$$ 

Substituting these expressions into (5), (6), (7), (8), (9), we find

$$(\Lambda - 1)a_-(\lambda) + q_n \Lambda b_-(\lambda) - \lambda^{-1} r_n c_-(\lambda) = 0,$$  \hspace{1cm} (38)

$$r_n (\Lambda + 1)a_- (\lambda) + (\Lambda - \lambda)b_- (\lambda) = r_n,$$ \hspace{1cm} (39)

$$q_n (\Lambda + 1)a_- (\lambda) - (\Lambda - \lambda^{-1}) c_- (\lambda) = q_n,$$ \hspace{1cm} (40)

$$(\Lambda - 1)a_- (\lambda) - r_n \Lambda c_- (\lambda) + \lambda q_n b_- (\lambda) = 0,$$ \hspace{1cm} (41)

$$a_- (\lambda) - a_-(\lambda)^2 - b_- (\lambda) c_- (\lambda) = 0,$$ \hspace{1cm} (42)

and

$$(\Lambda - 1)a_+(\lambda) + q_n \Lambda b_+ (\lambda) - \lambda^{-1} r_n c_+ (\lambda) = 0,$$ \hspace{1cm} (43)

$$r_n (\Lambda + 1)a_+ (\lambda) + (\Lambda - \lambda)b_+ (\lambda) = -r_n,$$ \hspace{1cm} (44)

$$q_n (\Lambda + 1)a_+ (\lambda) - (\Lambda - \lambda^{-1})(c_+(\lambda)) = -q_n,$$ \hspace{1cm} (45)

$$(\Lambda - 1)a_+ (\lambda) - r_n \Lambda c_+ (\lambda) + \lambda q_n b_+ (\lambda) = 0,$$ \hspace{1cm} (46)

$$a_+ (\lambda) + a_+(\lambda)^2 + b_+ (\lambda) c_+ (\lambda) = 0.$$ \hspace{1cm} (47)

Uniqueness for $R_-(\lambda)$ follows easily from (38)–(42) and (6), where we note that equations (38)–(41) and (6) already determine $R_-(\lambda)$ up to integration constants, and (42) fixes uniquely the constants; the existence follows from a careful verification of the compatibility between (38)–(42).

Similar verifications are true for $R_+(\lambda)$. The lemma is proved. \hfill $\square$
Proof of Lemma 2. We will prove identity (18) with and these equations show that (15) defines indeed how the derivations proved in a similar way. Denote by $W$ equations (5), (11), (12), we also obtain that the entries (1, 2) and (2, 1) of (15) can be equivalently written as

$$
D_{+,p}(r_n) = r_n \Lambda(a_{+,p}) + \Lambda(b_{+,p}) + \delta_{p,0} r_n, \tag{48}
$$

$$
D_{+,p}(q_n) = q_n \Lambda(a_{+,p}) - \Lambda(c_{+,p}), \tag{49}
$$

$$
D_{-,p}(r_n) = -r_n \Lambda(a_{-,p}) - \Lambda(b_{-,p}), \tag{50}
$$

$$
D_{-,p}(q_n) = -q_n \Lambda(a_{-,p}) + \Lambda(c_{-,p}) + \delta_{p,0} q_n. \tag{51}
$$

and these equations show that (15) defines indeed how the derivations $\{D_{\alpha,p}, \alpha = \pm, p \geq 0\}$ act on the variables $r_n$ and $q_n$. Since the right hand side of equations (48–51) do not depend on $\lambda$, we also proved that we can restrict these derivations to the polynomial algebra $\mathcal{A}$. \hfill \square

Let us proceed and prove Lemma 2.

Proof of Lemma 2. We will prove identity (18) with $\alpha = +$. The remaining cases can be proved in a similar way. Denote by $W(\lambda, \mu)$ the left-hand side of this identity. Clearly, $W(\lambda, \mu) \in \mathcal{A} \otimes s_{\mathbb{L}}([\lambda^{-1}, \mu^{-1}])\mu^{-1}$. It follows straightforwardly from (5) and (10) that $W(\lambda, \mu)$ satisfies the following two equations:

$$
\Lambda(W(\lambda, \mu)) U(\lambda) + \Lambda(R_+(\lambda)) \nabla_+(\mu)(U(\lambda)) - \nabla_+(\mu)(U(\lambda)) - U(\lambda) W(\lambda, \mu) = 0, \tag{52}
$$

$$
\text{tr} \left( W(\lambda, \mu) R_+(\lambda) + R_+(\lambda) W(\lambda, \mu) \right) = 0. \tag{53}
$$

One can verify that, if there exists a solution to (52–53) belonging to $\mathcal{A} \otimes s_{\mathbb{L}}([\lambda^{-1}, \mu^{-1}])\mu^{-1}$, then it is unique. Identity (18) is then proved by verifying that its right-hand side also satisfies (52–53) and belongs to $\mathcal{A} \otimes s_{\mathbb{L}}([\lambda^{-1}, \mu^{-1}])\mu^{-1}$. \hfill \square

Now let us prove Proposition 2.

Proof of Proposition 2. The proposition can be proven in two different ways. On the one hand, it can be proven using equations (17), (18) yield

$$
\nabla_+(\mu)(U(\lambda)) = \frac{\Lambda(R_+(\mu)) U(\lambda) - U(\lambda) R_+(\mu)}{\mu - \lambda} - \Lambda(Q_+(\mu)) U(\lambda) + U(\lambda) Q_+(\mu), \tag{54}
$$

$$
\nabla_-(\mu)(U(\lambda)) = \frac{\Lambda(R_-(\mu)) U(\lambda) - U(\lambda) R_-(\mu)}{\mu - \lambda} - \Lambda(Q_-(\mu)) U(\lambda) + U(\lambda) Q_-(\mu). \tag{55}
$$

The commutativity between the derivations $D_{\alpha,p}, \alpha = \pm, p \geq 0$ can then be obtained using (54–55) and (17–18). \hfill \square

We note that it follows from (54–55) and (5) that for $\alpha \in \{\pm\}$,

$$
\nabla_\alpha(\mu)(r_n) = \alpha \frac{r_n}{\mu} \Lambda(a_\alpha(\mu)) + \alpha \frac{1}{\mu} \Lambda(b_\alpha(\mu)) + \delta_{\alpha,+} \frac{r_n}{\mu}, \tag{56}
$$

$$
\nabla_\alpha(\mu)(q_n) = \alpha \frac{q_n}{\mu} \Lambda(a_\alpha(\mu)) - \alpha \frac{1}{\mu} \Lambda(c_\alpha(\mu)) + \delta_{\alpha,-} \frac{q_n}{\mu}. \tag{57}
$$

and, of course, these equations are equivalent to (18–51).
For the reader’s convenience, we list the first few examples of the flows in the AL hierarchy, obtained by replacing $D_{\alpha,p}$ by $\partial/\partial s^{\alpha,p}$:

\[
\begin{align*}
\frac{\partial r_n}{\partial s^{-,0}} &= -r_n , & \frac{\partial q_n}{\partial s^{-,0}} &= q_n , \\
\frac{\partial r_n}{\partial s^{+,0}} &= r_n , & \frac{\partial q_n}{\partial s^{+,0}} &= -q_n , \\
\frac{\partial r_n}{\partial s^{-,-1}} &= -r_{n-1} + r_n q_n r_{n-1} , & \frac{\partial q_n}{\partial s^{-,-1}} &= q_{n+1} - r_n q_n q_{n+1} , \\
\frac{\partial r_n}{\partial s^{+,1}} &= r_{n+1} - r_n q_n r_{n+1} , & \frac{\partial q_n}{\partial s^{+,1}} &= -q_{n-1} + r_n q_n q_{n-1} .
\end{align*}
\]

The combination

\[
\frac{\partial}{\partial t} := \frac{\partial}{\partial s^{+,1}} - \frac{\partial}{\partial s^{-,-1}} - \frac{\partial}{\partial s^{+,0}} + \frac{\partial}{\partial s^{-,0}}
\]

yields the complexified form of the AL equation

\[
\begin{align*}
\frac{\partial r_n}{\partial t} &= r_{n+1} - 2r_n + r_{n-1} - r_n q_n (r_{n+1} + r_{n-1}) , \\
\frac{\partial q_n}{\partial t} &= -q_{n+1} + 2q_n - q_{n-1} + r_n q_n (q_{n+1} + q_{n-1}) .
\end{align*}
\]

This is the original AL equation \[1, 2\], which is a discretization of the celebrated nonlinear Schrödinger equation.

### 3. From matrix resolvents to tau-functions

In this section, we prove Lemma 3 and Theorem 1.

Let us first show that the $\Omega_{\alpha,p;\beta,q}$ are well defined by (21), (23). Write

\[
R_{\alpha}(\mu) = R_{\alpha}(\lambda) + R'_{\alpha}(\lambda)(\mu - \lambda) + (\mu - \lambda)^2 \partial_{\lambda} \left( \frac{R_{\alpha}(\lambda) - R_{\alpha}(\mu)}{\lambda - \mu} \right) .
\]

Using this identity and equality (10), we find that the right-hand side of (21) equals

\[
\text{tr} \left( R_+(\lambda) R_+(\mu) \right) = \frac{1}{(\lambda - \mu)^2} - \frac{1}{(\lambda - \mu)^2} = -\frac{\text{tr} \left( R_+(\lambda) R'_+(\lambda) \right)}{\lambda - \mu} + \text{tr} \left( R_+(\lambda) \partial_{\lambda} \left( \frac{R_+(\lambda) - R_+(\mu)}{\lambda - \mu} \right) \right) .
\]

Differentiating equality (10) with respect to $\lambda$ we obtain that $\text{tr} \left( R_+(\lambda) R'_+(\lambda) \right)$ vanishes. From (8) it is clear that the second term of the right-hand side of (66) belongs to $\lambda^{-2}\mu^{-1}A[[\lambda^{-1}, \mu^{-1}]]$. Since the left-hand side of (66) is symmetric by exchange of $\lambda$ and $\mu$, we obtain that the right-hand side of (66) belongs to $\lambda^{-2}\mu^{-2}A[[\lambda^{-1}, \mu^{-1}]]$. Therefore, we have verified that $\Omega_{+,p;+,q}$ are well defined by (21), and that $\Omega_{+,p;+,0}$ and $\Omega_{+,0;+,q}$ vanishes. In a similar way, one can verify that $\Omega_{-,p;-,q}$ are well defined by (21), with vanishing $\Omega_{-,p;-,0}$ and $\Omega_{-,0;-,q}$. Using (8) and (6) one could easily deduce that $\Omega_{+,p;-,q}$ are well defined by (22), with vanishing $\Omega_{+,0;-,q}$, $\Omega_{+,p;-,0}$. For $\Omega_{-,p;+,q}$, by (22), the proof is the same as that for $\Omega_{+,p;-,q}$ due to the cyclicity of the trace.

Let us proceed with proving Lemma 3.
Proof of Lemma 3. We have

\[ \nabla_+(\nu) \left( \sum_{p,q \geq 0} \frac{\Omega_{\alpha,p;\beta,q}}{\lambda^{p+1} \mu^{q+1}} \right) \]

\[ = \alpha \beta \frac{\text{tr} \left( (\nu)(R_\alpha(\lambda))(R_\beta(\mu)) + R_\alpha(\lambda) \nabla_+(\nu)(R_\beta(\mu)) \right)}{(\lambda - \mu)^2} \]

\[ = \frac{\text{tr} \left( \left( \frac{[R_+(\nu), R_\alpha(\lambda)]}{\nu - \lambda} - [Q_+(\nu), R_\alpha(\lambda)] \right) R_\beta(\mu) + R_\alpha(\lambda) \left( \frac{[R_+(\nu), R_\beta(\mu)]}{\nu - \mu} - [Q_+(\nu), R_\beta(\mu)] \right) \right)}{(\lambda - \mu)^2} \]

\[ = \frac{\text{tr} \left( \left( \frac{-\frac{1}{\nu - \lambda} + \frac{1}{\nu - \mu}}{\nu - \lambda} \right) R_\alpha(\lambda) [R_+(\nu), R_\beta(\mu)] \right)}{(\lambda - \mu)^2} \]

\[ = \alpha \beta \frac{\text{tr} \left( R_\alpha(\lambda) [R_+(\nu), R_\beta(\mu)] \right)}{(\lambda - \mu)(\mu - \nu)(\nu - \lambda)}. \quad (67) \]

Similarly,

\[ \nabla_-(\nu) \left( \sum_{p,q \geq 0} \frac{\Omega_{\alpha,p;\beta,q}}{\lambda^{p+1} \mu^{q+1}} \right) \]

\[ = \alpha \beta \frac{\text{tr} \left( (\nu)(R_\alpha(\lambda))(R_\beta(\mu)) + R_\alpha(\lambda) \nabla_-(\nu)(R_\beta(\mu)) \right)}{(\lambda - \mu)^2} \]

\[ = \frac{\text{tr} \left( \left( -\frac{\lambda}{\nu} \frac{[R_-(\nu), R_\alpha(\lambda)]}{\nu - \lambda} - [Q_-(\nu), R_\alpha(\lambda)] \right) R_-(\mu) + R_-(\lambda) \left( -\frac{\mu}{\nu} \frac{[R_-(\nu), R_\beta(\mu)]}{\nu - \mu} - [Q_-(\nu), R_\beta(\mu)] \right) \right)}{(\lambda - \mu)^2} \]

\[ = \frac{\text{tr} \left( \left( \frac{\lambda}{\nu} - \frac{\mu}{\nu - \mu} \right) R_\alpha(\lambda) [R_-(\nu), R_\beta(\mu)] \right)}{(\lambda - \mu)^2} \]

\[ = -\alpha \beta \frac{\text{tr} \left( R_\alpha(\lambda) [R_-(\nu), R_\beta(\mu)] \right)}{(\lambda - \mu)(\mu - \nu)(\nu - \lambda)}. \quad (68) \]

From (67) and (68) we see the validity of (27).

To show (28), on the one hand, we have

\[ (\lambda - 1) \left( \sum_{p,q \geq 0} \frac{\Omega_{\alpha,p;\beta,q}}{\lambda^{p+1} \mu^{q+1}} \right) \]

\[ = \alpha \beta \frac{\text{tr} \left( (\lambda)(R_\alpha(\lambda))(\lambda)(R_\beta(\mu)) - R_\alpha(\lambda)R_\beta(\mu) \right)}{(\lambda - \mu)^2} \]

\[ = \frac{\alpha R_{\alpha,12}(\lambda)R_{\beta,21}(\mu)}{\mu(\lambda - \mu)} - \frac{\alpha R_{\alpha,21}(\lambda)R_{\beta,12}(\mu)}{\lambda(\lambda - \mu)}. \quad (69) \]

where in the last equality we used (5) and the property \( \text{tr} (R(\lambda)) = 1 \). On the other hand, for \( \beta = + \), using (18) we have

\[ \alpha \nabla_+(\mu) \left( \frac{1}{\lambda} a_\alpha(\lambda) \right) = \alpha \frac{R_{\alpha,12}(\lambda)R_{\beta,21}(\mu)}{\mu(\lambda - \mu)} - \alpha \frac{R_{\alpha,21}(\lambda)R_{\beta,12}(\mu)}{\lambda(\lambda - \mu)}; \quad (70) \]
for $\beta = -\lambda$, using (17) we have
\[
\alpha \nabla_-(\mu) \left( \frac{1}{\lambda} \alpha(\lambda) \right) = -\alpha \frac{R_{\alpha,12}(\lambda) R_{\beta,21}(\mu)}{\mu(\lambda - \mu)} + \alpha \frac{R_{\alpha,21}(\lambda) R_{\beta,12}(\mu)}{\lambda(\lambda - \mu)}. \tag{71}
\]

Identity (28) is proved. \(\square\)

We are ready to prove Theorem 1.

**Proof of Theorem 1.** Let us prove the theorem by induction. For $k = 3$, the validity of identity (30) was established in the proof of Lemma 3. Suppose (30) is true for $k = m$ ($m \geq 3$); consider now $k = m + 1$. For the case that $\alpha_{m+1} = +$, we have
\[
\sum_{i_1, \ldots, i_m, i_{m+1} \geq 0} \Omega_{\alpha_{i_1}, \ldots, \alpha_{i_m}, i_{m+1}}^\lambda \prod_{j=1}^m \alpha_j \sum_{\sigma \in S_m/C_m} \frac{1}{\lambda_{i_{m+1}}^\alpha} \sum_{l=1}^m \text{tr} \left( R_{\alpha_{\sigma(1)}}(\lambda_{\sigma(1)}) \cdots R_{\alpha_{\sigma(m)}}(\lambda_{\sigma(m)}) \right) \cdot \nabla_+(\lambda_{m+1}) \left( \prod_{j=1}^m \alpha_j \sum_{\sigma \in S_m/C_m} \prod_{j=1}^m \frac{1}{\lambda_{\sigma(j)} - \lambda_{\sigma(j+1)}} \sum_{l=1}^m \text{tr} \left( R_{\alpha_{\sigma(1)}}(\lambda_{\sigma(1)}) \cdots R_{\alpha_{\sigma(l-1)}}(\lambda_{\sigma(l-1)}) \right) \right) \\
= \prod_{j=1}^m \alpha_j \sum_{\sigma \in S_m/C_m} \prod_{j=1}^m \frac{1}{\lambda_{\sigma(j)} - \lambda_{\sigma(j+1)}} \sum_{l=1}^m \text{tr} \left( R_{\alpha_{\sigma(1)}}(\lambda_{\sigma(1)}) \cdots R_{\alpha_{\sigma(m)}}(\lambda_{\sigma(m)}) \right) \cdot \left( \frac{[R_+(\lambda_{m+1}), R_{\alpha_{\sigma(t)}}(\lambda_{\sigma(t)})]}{\lambda_{m+1} - \lambda_{\sigma(t)}} - [Q_+(\lambda_{m+1}), R_{\alpha_{\sigma(t)}}(\lambda_{\sigma(t)})] \right) \cdot R_{\alpha_{\sigma(t+1)}}(\lambda_{\sigma(t+1)}) \cdots R_{\alpha_{\sigma(m)}}(\lambda_{\sigma(m)}) \right). \tag{72}
\]

Here in the last equality we used (18). Simplifying this expression with the help of the cyclic invariance of $\text{tr}$ we obtain the validity of (30) with $k = m + 1$ (see the discussions after (4.23).
of [6]. Similarly, for the case $\alpha_{m+1} = -$, we have

$$
- \sum_{i_1, \ldots, i_m, i_{m+1} \geq 0} \frac{Q_{\alpha_1, i_1; \ldots; \alpha_m, i_m; \cdot; i_{m+1}}}{\alpha_1 i_1 + \ldots + \alpha_m i_m + 1} = \nabla_-(\lambda_{m+1}) \left( \prod_{j=1}^{m} \alpha_j \sum_{\sigma \in S_m / C_m} \frac{\text{tr} \left( R_{\alpha_{\sigma(1)}}(\lambda_{\sigma(1)}) \cdots R_{\alpha_{\sigma(m)}}(\lambda_{\sigma(m)}) \right)}{\prod_{j=1}^{m} (\lambda_{\sigma(j)} - \lambda_{\sigma(j+1)})} \sum_{l=1}^{m} \text{tr} \left( R_{\alpha_{\sigma(1)}}(\lambda_{\sigma(1)}) \cdots R_{\alpha_{\sigma(l-1)}}(\lambda_{\sigma(l-1)}) \right) \cdot \nabla_-(\lambda_{m+1})(R_{\alpha_{\sigma(l)}}(\lambda_{\sigma(l)})) \cdot R_{\alpha_{\sigma(l+1)}}(\lambda_{\sigma(l+1)}) \cdots R_{\alpha_{\sigma(m)}}(\lambda_{\sigma(m)}) \right) \\
- \sum_{\sigma \in S_m / C_m} \frac{1}{\prod_{j=1}^{m} (\lambda_{\sigma(j)} - \lambda_{\sigma(j+1)})} \sum_{l=1}^{m} \text{tr} \left( R_{\alpha_{\sigma(1)}}(\lambda_{\sigma(1)}) \cdots R_{\alpha_{\sigma(l-1)}}(\lambda_{\sigma(l-1)}) \right) \cdot \left( -\frac{\lambda_{\sigma(l)}}{\lambda_{m+1}} \frac{R_-(\lambda_{m+1}), R_{\alpha_{\sigma(l)}}(\lambda_{\sigma(l)})}{\lambda_{m+1} - \lambda_{\sigma(l)}} - |Q_-(\lambda_{m+1}), R_{\alpha_{\sigma(l)}}(\lambda_{\sigma(l)})| \right) \cdot R_{\alpha_{\sigma(l+1)}}(\lambda_{\sigma(l+1)}) \cdots R_{\alpha_{\sigma(m)}}(\lambda_{\sigma(m)}) \right),
$$

(73)

Here in the last equality we used (17). Simplifying this expression we obtain the validity of (30) with $k = m + 1$. The theorem is proved.

For the reader’s convenience, let us list the first few relations in (31)–(32):

$$
\Lambda^{-1} \frac{\partial^2 \log \tau_n}{\partial s^{+1} \partial s^{+1}} = -q_{n-2} r_n (1 - q_{n-1} r_{n-1}) , \quad \Lambda^{-1} \frac{\partial^2 \log \tau_n}{\partial s^{-1} \partial s^{-1}} = -q_n r_{n-2} (1 - q_{n-1} r_{n-1}) ,
$$

(74)

$$
\Lambda^{-1} \frac{\partial^2 \log \tau_n}{\partial s^{+1} \partial s^{-1}} = 1 - q_{n-1} r_{n-1} ,
$$

(75)

$$
\Lambda^{-1} (\Lambda - 1) \left( \frac{\partial \log \tau_n}{\partial s^{+1}} \right) = -q_{n-1} r_n , \quad \Lambda^{-1} (\Lambda - 1) \left( \frac{\partial \log \tau_n}{\partial s^{-1}} \right) = -q_n r_{n-1} ,
$$

(76)

$$
\Lambda^{-1} \frac{\partial^2 \log \tau_n}{\partial s^{+2} \partial s^{+2}} = - (1 - q_{n-1} r_{n-1}) \left[ (1 - q_{n-2} r_{n-2}) \left( q_{n-3} r_{n-3} - q_{n-2} r_{n-2} r_n - q_{n-4} r_{n-2} r_n - q_{n-3} r_{n-3} \right) \right.
+ 2 q_{n-3} (q_{n-2} r_{n-1} r_n + q_{n-1} r_n^2 - (1 - q_n r_n) r_{n+1})
- q_{n-2} (q_{n-2} r_{n-1} + q_{n-1} r_n)^2 - (1 - q_n r_n) (2 q_{n-1} r_n r_{n+1})
+ 2 q_{n-2} r_{n-2} r_{n+1} + q_{n-2} r_{n+1} \left. - r_n + 2 (1 - q_{n-1} r_{n+1}) \right) .
$$

(77)

Here $\tau_n = \tau(n, s)$. The above explicit expressions (74)–(76) agree with the unsymmetric identities given by Adler and van Moerbeke [31]. Clearly, formula (31) gives the explicit generating series of all unsymmetric identities of Adler–van Moerbeke type.
In the generic situation, we note that the above relations \((74)-(76)\) yield
\[
\frac{r_{n+1}}{r_n} = \frac{\tau_{n+2}^2}{\tau_{n+1}^2} (\Lambda - 1) \left( \frac{\partial \log \tau_n}{\partial s^{+1}} - \frac{\partial \log \tau_n}{\partial s^{-1}} \right) = \frac{\tau_{n+2}\tau_n}{\tau_{n+1}^2} (\Lambda - 1) \left( \frac{\partial \log \tau_n}{\partial s^{+1}} - \frac{\partial \log \tau_n}{\partial s^{-1}} \right),
\]
\[
\frac{q_{n+1}}{q_n} = \frac{\tau_{n+2}^2}{\tau_{n+1}^2} (\Lambda - 1) \left( \frac{\partial \log \tau_n}{\partial s^{1}} \right) = \frac{\tau_{n+2}\tau_n}{\tau_{n+1}^2} (\Lambda - 1) \left( \frac{\partial \log \tau_n}{\partial s^{+1}} - \frac{\partial \log \tau_n}{\partial s^{-1}} \right),
\]
which will be applied in the next section for concrete computations. From \((33)\) and \((16)\) we also note that \(\tau_n\) satisfies the following differential-difference equation:
\[
\frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2} = \frac{\partial^2 \log \tau_n}{\partial s^{+1}\partial s^{-1}},
\]
which is recognized as the 2-Toda equation \([27]\) and could also be applied for concrete computations.

4. Generating series for correlators related to CUE

In this section, we study an important class of solutions to the AL hierarchy related to unitary integrals.

Since the dependence of an AL tau-function on \(s^{+,0}, s^{-,0}\) is trivial, from now on we will denote \(s = (s^{\alpha}p)_{\alpha=\pm, p \geq 1}\) and consider only the non-trivial flows. For \(n \geq 1\), let \(Z(n,s)\) be the formal powers series of \(s\) defined via the following Toeplitz determinants:
\[
Z(n,s) = \det (\psi_{m-\ell}(s))_{\ell,m=0}^{n-1},
\]
where
\[
\psi(\zeta; s) = e^{V(\zeta, \zeta^{-1})} e^{\sum_{p \geq 1} (s^{+p} \zeta^p + s^{-p} \zeta^{-p})} = \sum_{k \in \mathbb{Z}} \psi_k(s) \zeta^k.
\]
Here \(V(\zeta, \zeta^{-1})\) is a given Laurent series with no constant terms and convergent in a neighborhood of \(S^1\). Set \(Z(0,s) \equiv 1\). It is well known (see for instance \([3, 5]\)) that these Toeplitz determinants can be written, equivalently, as matrix integrals on the unitary group \(U(n)\) with respect to the Haar measure:
\[
\det (\psi_{m-\ell}(s))_{\ell,m=0}^{n-1} = \int_{U(n)} e^{Tr(V(U,U^*) + \sum_{p \geq 1} (s^{+p} U^p + s^{-p} U u^p))} dU.
\]
It is shown in \([3]\) (see also \([19, 25]\)) that \(Z(n,s)\) coincides with a particular tau-function for the AL hierarchy for \(n \geq 0\). To use the matrix-resolvent method of computing the logarithmic derivatives of \(Z(n,s)\), one needs to solve the initial data of the solution corresponding to \(Z(n,s)\). Suppose from \((81)\) we can get \(Z(n,s^{+,1},0,\ldots)\). Then by using \((78)-(79)\) one can obtain \(q_n(s^{+,1},0,\ldots)\) and \(r_n(s^{+,1},0,\ldots)\) and so one gets \((q_n(0),r_n(0))\). With the knowledge of this initial value, formula \((37)\) then leads to a way of computing the logarithmic derivatives of \(\log Z(n,s)\) evaluated at \(s = 0\).

As an illustration of the above algorithm, let us give explicit computations for a simple example, which is given by
\[
V(\zeta) = \log \left( (1 + Q\zeta) (1 + Q\zeta^{-1}) \right),
\]
where $0 \leq Q < 1$ is a parameter. The $Q = 0$ case reduces to $V \equiv 0$ and (83) gives the CUE partition function. Computing by induction the Toeplitz determinants we find that

$$Z(n, s^+, 0, \ldots) = \sum_{\ell=0}^{n} (s^+)^{\ell} \frac{Q^\ell}{\ell!} \frac{1 - Q^{2n+2-2\ell}}{1 - Q^2}. \quad (85)$$

In particular,

$$Z(n, 0) = \sum_{k=0}^{n} Q^{2k} = \frac{1 - Q^{2n+2}}{1 - Q^2}. \quad (86)$$

We then find from (85) the initial value of the solution that corresponds to the partition function $Z(n, s)$ (as a tau-function for the AL hierarchy), as follows:

$$q(n, 0) = (-1)^n Q^n \frac{1 - Q^2}{1 - Q^{2n+2}}, \quad r(n, 0) = (-1)^n Q^n \frac{1 - Q^2}{1 - Q^{2n+2}}. \quad (87)$$

By calculating the basic matrix-resolvents we obtain from Corollary 1 the following proposition.

**Corollary 2.** For every $k \geq 2$, and for any fixed $\alpha_1, \ldots, \alpha_k \in \{+,-\}$, the following formula holds true:

$$\sum_{i_1, \ldots, i_k \geq 0} \langle \tau_{\alpha_1, i_1} \cdots \tau_{\alpha_k, i_k} \rangle (n-1) \lambda_1^{i_1+1} \cdots \lambda_k^{i_k+1} = -k \prod_{j=1}^{k} \alpha_j \left( \sum_{S_k/C_k} \frac{\text{tr} \left( R_{\alpha_{\sigma(1)}}(\lambda_{\sigma(1)}) \cdots R_{\alpha_{\sigma(k)}}(\lambda_{\sigma(k)}) \right)}{\prod_{j=1}^{k} (\lambda_{\sigma(j)} - \lambda_{\sigma(j+1)})} \right) \delta_{k,2} r_{\alpha_1,\alpha_2} (\lambda_1, \lambda_2), \quad (88)$$

where $\langle \tau_{\alpha_1, i_1} \cdots \tau_{\alpha_k, i_k} \rangle (n) = \langle \langle \tau_{\alpha_1, i_1} \cdots \tau_{\alpha_k, i_k} \rangle \rangle (n, 0)$, $r_{\alpha,\beta}(\lambda, \mu)$ are defined right after (25), and $R_{\alpha}(\lambda)$ is given by (11) - (12) with $a_{\alpha,p}, b_{\alpha,p}, c_{\alpha,p}, p \geq 0$, being explicitly given by

$$a_{-p}(n) = \frac{(-Q)^{2n-p}(1 - Q^2)(1 - Q^{2n})}{(1 - Q^{2n})(1 - Q^{2n+2})}, \quad (89)$$

$$b_{-p}(n) = \frac{(-Q)^{n-p-1}(1 - Q^2)(1 - Q^{2n+2})}{(1 - Q^{2n})(1 - Q^{2n+2})}, \quad (90)$$

$$c_{-p}(n) = \delta_{p \geq 1} \frac{(-Q)^{n-p-1}(1 - Q^2)(Q^{2p-2} - Q^{2n})}{(1 - Q^{2n})(1 - Q^{2n+2})}, \quad (91)$$

$$a_{+p}(n) = \frac{(-Q)^{2n-p}(1 - Q^2)(1 - Q^{2p})}{(1 - Q^{2n})(1 - Q^{2n+2})}, \quad (92)$$

$$b_{+p}(n) = \delta_{p \geq 1} \frac{(-Q)^{n-p-1}(1 - Q^2)(Q^{2p-2} - Q^{2n})}{(1 - Q^{2n})(1 - Q^{2n+2})}, \quad (93)$$

$$c_{+p}(n) = \frac{(-Q)^{n-p-1}(1 - Q^2)(1 - Q^{2n+2p+2})}{(1 - Q^{2n})(1 - Q^{2n+2})}. \quad (94)$$

Observe, in particular, that

$$Z(n, 0) = \frac{\mathbb{P}(X \leq n)}{1 - Q^2},$$

where $X$ is a geometric random variable:

$$\mathbb{P}(X = k) = (1 - Q^2) Q^{2k}, \quad k \geq 0. \quad (95)$$
More generally, consider
\[
V(\zeta; z, z') = \log \left( (1 + Q\zeta)^z (1 + Q\zeta^{-1})^{z'} \right),
\]
where \(z, z'\) are positive integer parameters. In this case
\[
Z(n, 0) = \frac{\mathbb{P}(L \leq n)}{(1 - Q^2)^zz'},
\]
where \(L\) is a random variable describing the directed last passage percolation in a rectangular \((z \times z')\) lattice in which each site is equipped with a geometric random variable. More precisely, take a \((z \times z')\) lattice of integer points, and associate to each point an independent and identically distributed random variable \(X_{i,j}\) of the same law as in \([95]\). Then
\[
L := \max_{\gamma:(1,1)\to(z,z')} L(\gamma); \quad L(\gamma) = \sum_{(i,j) \in \gamma} X_{i,j},
\]
where the maximum is taken on all the upright paths going from \((1,1)\) to \((z, z')\) (see for instance \([5]\)). We plan to come back to this issue in a subsequent publication.

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