On the distance spectrum of minimal cages and associated distance biregular graphs

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September 14, 2021

Abstract

A $(k, g)$-cage is a $k$-regular simple graph of girth $g$ with minimum possible number of vertices. In this paper, $(k, g)$-cages which are Moore graphs are referred as minimal $(k, g)$-cages. A simple connected graph is called distance regular (DR) if all its vertices have the same intersection array. A bipartite graph is called distance biregular (DBR) if all the vertices of the same partite set admit the same intersection array. It is known that minimal $(k, g)$-cages are DR graphs and their subdivisions are DBR graphs. In this paper, for minimal $(k, g)$-cages we give a formula for distance spectral radius in terms of $k$ and $g$, and also determine polynomials of degree $\lfloor \frac{g^2}{2} \rfloor$, which is the diameter of the graph. This polynomial gives all distance eigenvalues when the variable is substituted by adjacency eigenvalues. We show that a minimal $(k, g)$-cage of diameter $d$ has $d + 1$ distinct distance eigenvalues, and this partially answers a problem posed in [5]. We prove that every DBR graph is a 2-partitioned transmission regular graph and then give a formula for its distance spectral radius. By this formula we obtain the distance spectral radius of subdivision of minimal $(k, g)$-cages. Finally we determine the full distance spectrum of subdivision of some minimal $(k, g)$-cages.

Keywords: Distance spectrum; Distance regular graph; Distance biregular graph; Minimal $(k, g)$-cage; Subdivision graph; $k$-partitioned transmission regular graph.

Subclass: 05C12, 05C50
1 Introduction and Preliminaries

In this article, by a graph we mean a finite, simple, connected and undirected graph. Let $G = (V, E)$ be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$. The adjacency matrix $A(G)$ of $G$ is an $n \times n$ matrix with $(i, j)^{th}$ entry 1 or 0 according as $v_i$ is adjacent to $v_j$ or not. The incidence matrix $R(G)$ of $G$ is an $n \times m$ matrix whose $(i, j)^{th}$ entry is 1 or 0 according as vertex $v_i$ is an end vertex of edge $e_j$ or not. The distance matrix $D(G)$ of $G$ is an $n \times n$ matrix whose $(i, j)^{th}$ entry is the distance (length of the shortest path) between the vertices $v_i$ and $v_j$. The eigenvalues of $A(G)$ (respectively $D(G)$) are called eigenvalues (respectively distance eigenvalues or D-eigenvalues) of $G$. The set of all eigenvalues (respectively distance eigenvalues) of $G$ is called the spectrum (respectively distance spectrum or D-spectrum) of $G$. If $\lambda_1, \lambda_2, \ldots, \lambda_p$ are distinct eigenvalues (respectively distance eigenvalues) of $G$ with respective multiplicities $m_1, m_2, \ldots, m_p$ then the spectrum (respectively distance spectrum) of $G$ is denoted by $\{\lambda_1^{(m_1)}, \lambda_2^{(m_2)}, \ldots, \lambda_p^{(m_p)}\}$. The largest eigenvalue of $D(G)$ is called the distance spectral radius of $G$.

The distance matrix of a graph gives several structural information of the graph. Thus the computation of the distance matrix and its characteristic polynomial is much more intense problem. Graham and Pollack [13] introduced distance matrix of a graph and established a relationship between the number of negative eigenvalues of this matrix and addressed a problem in data communication systems. The distance matrix and distance spectrum of a graph has numerous applications to chemistry [9] and other branches of science and engineering. For some recent results on the characteristic polynomials of the distance matrices and distance spectra of graphs, one may refer [2, 3, 15].

For any graph $G$ of diameter $d$, and a vertex $u \in V(G)$, $G_i(u)$ denotes the set of all vertices in $G$ of distance $i$ from $u$, $i = 0, 1, \ldots, d$. A connected graph $G$ is called distance regular (in short DR) if it is regular and for any two vertices $x, y \in G$ at distance $i$, there are precisely $c_i$ neighbors of $y$ in $G_{i-1}(x)$ and $b_i$ neighbors of $y$ in $G_{i+1}(x)$, $0 \leq i \leq d$, where $c_0$ and $b_d$ are undefined. The sequence $\{b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\}$ is called the intersection array of a distance regular graph $G$. For $i = 0, 1, \ldots, d$, the numbers $c_i, b_i,$ and $a_i$, where $a_i = l - b_i - c_i$ and $l$ is the degree of regularity of $G$, are called the intersection numbers of $G$. Biggs [6] introduced distance regular (DR) graphs. For results on DR graphs and their link with other combinatorial structures one may refer [8][19]. Every DR-graph of diameter $d$ has exactly $d + 1$ distinct adjacency eigenvalues and at most $d + 1$ distinct D-eigenvalue [5].
authors in [1] characterized some DR graphs with diameter three and four having exactly three distinct distance eigenvalues.

For an $n$-vertex graph $G$ with diameter $d$, the $i^{th}$ distance matrix $A_i$, $i = 1, 2, \ldots, d$, of $G$ is an $n \times n$ matrix whose rows and columns are indexed by vertices of $G$ and $(j, m)^{th}$ entry is 1 or 0 according as distance between $j^{th}$ and $m^{th}$ vertices is $i$ or not. Thus the distance matrix $D$ of graph $G$ can be written as

$$D = A_1 + 2A_2 + \cdots + dA_d$$

The adjacency matrix $A$ of a distance regular graph $G$ with diameter $d$ and its $i^{th}$ distance matrices satisfy the following recurrence relation [8].

$$AA_i = c_{i+1}A_{i+1} + a_iA_i + b_{i-1}A_{i-1}, \quad A_0 = I, \quad A_1 = A, \quad i = 0, 1, 2, \ldots, d \quad (2)$$

Applying equation (2) we get that the $i^{th}$ distance matrix $A_i$ of a distance regular graph $G$ with diameter $d$ can be expressed as a polynomial (of degree $i$) of its adjacency matrix $A$, $i = 1, 2, \ldots, d$. Then from (1), the distance matrix $D$ can also be written as a polynomial of $A$, say $D = p(A)$, of degree $d$. Thus for every eigenvalue $\lambda$ of $A$, $p(\lambda)$ is a distance eigenvalue of graph $G$.

A connected graph $G$ is called distance-biregular (DBR) graph if it is bi-partite and all vertices in the same partite set have the same intersection array. We denote the bi-partition of a DBR graph as $(V_1, V_2)$. The intersection arrays for vertices in $V_1$ and $V_2$ are $\{r, e_1, \ldots, e_{d_1-1}; 1, f_2, \ldots, f_{d_1}\}$ and $\{s, g_1, \ldots, g_{d_2-1}; 1, h_2, \ldots, h_{d_2}\}$ respectively, where $r$ is the degree of vertices in $V_1$, $s$ is the degree of vertices in $V_2$, $d_1 = \max \{d(x, y) : x \in V_1, y \in V(G)\}$ and $d_2 = \max \{d(x, y) : x \in V_2, y \in V(G)\}$. The diameter $d'$ of $G$ is of course $\max(d_1, d_2)$. For any $u \in V_1$ and $v \in V_2$ we take $l_i = |G_i(u)|$ and $l'_i = |G_i(v)|$, $i = 0, \ldots, d'$. We note that $l_{d'-1} \neq 0$ and $l'_{d'-1} \neq 0$ though one of $l_d$ and $l'_d$ may be zero.

Some elementary relations on the intersection arrays of a DBR graph are given below.

**Lemma 1.1.** ( [12]) For a DBR graph, the following relations hold true: $l_0 = 1$, $l_{i+1}f_{i+1} = l_ie_i$, $l'_0 = 1$, and $l'_{i+1}h_{i+1} = l'_ig_i$.

For any graph $G$ and a vertex $v$ in it, the transmission $Tr_G(v)$ of $v$ is the sum of distances from $v$ to all other vertices in $G$. A connected graph $G$ is called $p$-transmission regular if $Tr_G(v)$ is $p$ for all the vertices $v$ in $G$. It is known [3] that for any vertex $u$ in a DR graph $G$, $G_i(u)$ has a constant number of vertices, say $k_i$, $i = 0, 1, \ldots, d$. Also $k_i$ satisfies the relations
\[ k_0 = 1, \ k_1 = l, \ k_{i+1}c_{i+1} = k_ib_i \text{ for } i = 0, 1, \ldots, d - 1. \] Thus any DR graph is a \( p \)-transmission regular graph, where \( p = \sum_{i=0}^{d} i k_i \). We note that the distance spectral radius of every \( p \)-transmission regular graph is equal to \( p \).

**Definition 1.1.** [7] Suppose \( A \) is a real symmetric matrix whose rows and columns are indexed by elements in \( X = \{1, 2, \ldots, n\} \). Consider the block representation of \( A \) with respect to the partition \( \{X_1, X_2, \ldots, X_m\} \) of \( X \) as

\[
A = \begin{pmatrix}
A_{11} & A_{12} & \cdots & A_{1m} \\
A_{21} & A_{22} & \cdots & A_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
A_{m1} & A_{m2} & \cdots & A_{mm}
\end{pmatrix},
\]

where each \( A_{ij} \) denotes the sub-matrix (block) of \( A \) formed by rows indexed in \( X_i \) and the columns indexed in \( X_j \). Let \( q_{ij} \) be the average row sum of \( A_{ij} \). Then the matrix \( Q = (q_{ij}) \) is called a **quotient matrix** of \( A \). For each block \( A_{ij} \), if the row sum is constant then the partition is called **equitable**.

**Lemma 1.2.** (7) Let \( Q \) be a quotient matrix of a real symmetric matrix \( A \) corresponding to an equitable partition. Then the spectrum of \( A \) contains the spectrum of \( Q \).

**Lemma 1.3.** (4) If \( Q \) is a quotient matrix of a real symmetric matrix \( A \) corresponding to an equitable partition, then the largest eigenvalue of \( A \) is equal to the largest eigenvalue of \( Q \).

A connected graph \( G \) is called a **\( t \)-partitioned transmission regular graph** if there exists a partition \( \bigcup_{i=1}^{t} U_i \) (called a \( t \)-partition) of the vertex set of \( G \) such that for any \( i, j \) (not necessarily distinct) in \( \{1, 2, \ldots, t\} \) and for any vertex \( x \in U_i, q_{ij} = \sum_{y \in U_j} d(x, y) \) is a constant, where \( d(x, y) \) is the distance between \( x \) and \( y \) in the graph \( G \). For a \( t \)-partitioned transmission regular graph \( G \), \( \{U_i : i = 1, 2, \ldots, t\} \) is an equitable partition of \( D(G) \). Therefore the quotient matrix of \( D(G) \) with respect to this partition is \( Q^D = [q_{ij}]_{t \times t} \), and so by Lemma 1.3 the distance spectral radius of \( G \) is the largest eigenvalue of \( Q^D \).

For positive integers \( k \) and \( g \), a \((k, g)\)-cage is a \( k \)-regular simple graph of girth \( g \) on minimum possible number, say \( n(k, g) \), of vertices. An well known [6] lower bound for \( n(k, g) \) is as given below.

\[
n(k, g) \geq n_0(k, g) = \begin{cases} 
1 + k + k(k - 1) + \cdots + k(k - 1)^{d - 2} + k(k - 1)^{d - 1}, & \text{if } g \text{ is odd} \\
1 + k + k(k - 1) + \cdots + k(k - 1)^{d - 2} + (k - 1)^{d - 1}, & \text{if } g \text{ is even,}
\end{cases}
\]

where \( d = \lfloor \frac{g}{2} \rfloor \) is the diameter of the \((k, g)\)-cage. A \((k, g)\)-cage for which equality holds in the above bound is called a **Moore graph** or a **minimal** \((k, g)\)-cage.
The Lemma below gives information about the possible minimal \((k, g)\)-cages.

**Lemma 1.4.** ([1]) There exists a Moore graph (or a minimal \((k, g)\)-cage) of degree \(k\) and girth \(g\) if and only if

(i) \(k = 2\) and \(g > 3\), cycles;
(ii) \(g = 3\) and \(k > 2\), complete graphs;
(iii) \(g = 4\) and \(k > 2\), complete bipartite graphs;
(iv) \(g = 5\) and:
- \(k = 2\), the 5-cycle,
- \(k = 3\), the Petersen graph,
- \(k = 7\), the Hoffman-Singleton graph, and possibly \(k = 57\);
(v) \(g = 6, 8,\) or \(12\), and there exists a symmetric generalized \(n\)-gon of order \(k - 1\).

It is known [6] that every minimal \((k, g)\)-cage is a DR graph with intersection array \(
\begin{array}{c}
\{k, k-1, \ldots, k-1, k-1; 1, 1, \ldots, 1, 1\} \\
\end{array}
\) if \(g\) is even, and \(
\begin{array}{c}
\{k, k-1, \ldots, k-1, k-1; 1, 1, \ldots, 1, 1\} \\
\end{array}
\) if \(g\) is odd. So the intersection number \(a_i = 0\) for all minimal \((k, g)\)-cages, \(i = 0, 1, \ldots, d\).

The subdivision graph \(S(G)\) of the graph \(G\) is obtained from \(G\) by inserting a new vertex of degree 2 on each edge of \(G\).

The result below gives adjacency spectrum of minimal \((k, g)\)-cages.

**Lemma 1.5.** ([6]) Let \(G\) be a \((k, g)\)-cage with diameter \(d\) and \(n\) vertices.

(i) If \(g = 2d\) then the \(d+1\) distinct eigenvalues of \(G\) are \(\lambda = k, -k, 2\sqrt{k-1} \cos \frac{\pi j}{d}, \) \(j = 1, 2, \ldots, d - 1\), with multiplicity \(m_\lambda = \frac{nk}{g} \left[ \frac{h}{k-\lambda} \right], h = k-1, |\lambda| \neq k\).

(ii) If \(g = 2d+1\) then the \(d+1\) distinct eigenvalues of \(G\) are \(\lambda = k, 2\sqrt{k-1} \cos a_j, \) \(j = 1, 2, \ldots, d\), where \(a_1, \ldots, a_d\) are the distinct solutions in the interval \(0 < a < \pi\) of the equation \(\sqrt{k-1} \sin \left( (d-1)a + \sin \left( \frac{\pi}{d}\right) a = 0\) with multiplicity of an eigenvalue \(\lambda\) is given by \(m_\lambda = \frac{nk}{g} \left[ \frac{4h-\lambda^2}{(k-\lambda)(f+\lambda)} \right], h = k-1, f = k \pm \frac{k-2}{g}\).

In this paper, for minimal \((k, g)\)-cages we give a formula for distance spectral radius in terms of \(k\) and \(g\), and also determine polynomials of degree \(\left\lfloor \frac{g}{2} \right\rfloor\) which give all distance eigenvalues when the variable is substituted by adjacency eigenvalues. The authors in [5] proved that every DR graph with diameter \(d\) has at the most \(d+1\) distinct D-eigenvalues and then asked for characterization of DR graphs which will have exactly \(d+1\) distinct D-eigenvalues. We show that all minimal \((k, g)\)-cages of diameter \(d\) have \(d+1\) distinct distance eigenvalues. In [12] it is proved that every distance-regularized graph is either DR or DBR. The authors in [16] proved that subdivision of a minimal \((k, g)\)-cage is a DBR graph. We prove that every DBR-graph is a 2-partitioned transmission regular graph and then give a
formula for its distance spectral radius. By this formula we determine distance spectral radius of subdivision of minimal \((k, g)\)-cages. We also find D-spectrum of subdivision of minimal \((3, 5)\)-cages, minimal \((3, 6)\)-cages, and minimal \((k, g)\)-cages for \(g = 3\) and \(4\) with any values of \(k \geq 3\).

Next we state some known results which will be used in the sequel.

**Definition 1.2.** Let \(A = (a_{ij})\) be an \(m \times n\) matrix and \(B = (b_{ij})\) be a \(p \times q\) matrix then the Kronecker product of \(A\) and \(B\), denoted by \(A \otimes B\), is defined as the \(mp \times nq\) partition matrix \((a_{ij}B)\). The product of two kronecker products gives another kronecker product: \((M \otimes P)(N \otimes Q) = MN \otimes PQ\), in case where each multiplication makes sense.

Recall that for any graph \(G\), its line graph \(L(G)\) is the graph whose vertex set is \(E(G)\) and two vertices are adjacent if the corresponding edges in \(G\) share a common end vertex.

**Lemma 1.6.** Let \(G\) be an \(r\)-regular graph with adjacency matrix \(A\), incidence matrix \(R\), and line graph \(L(G)\). Then \(RR^T = A + rI\), \(R^T R = A(L(G)) + 2I\), \(JR = 2J = R^T J\) and \(JR^T = rJ = RJ\), where \(I\) is the identity matrix and \(J\) is the all-one matrix of appropriate order.

**Lemma 1.7.** Let \(G\) be an \(r\)-regular graph with \(p\) vertices, \(q\) edges, and eigenvalues \(\{\lambda_1, \lambda_2, \ldots, \lambda_p\}\). Then spectrum of \(L(G)\) is \(\{2r - 2, \lambda_2 + r - 2, \ldots, \lambda_p + r - 2, -2^{(q-p)}\}\). Also, \(Z\) is an eigenvector corresponding to the eigenvalue \(-2\) if and only if \(RZ = 0\), where \(R\) is the incidence matrix of \(G\).

## 2 Distance spectrum of minimal cages

Here we give a formula for distance spectral radius of minimal \((k, g)\)-cages.

**Theorem 2.1.** The distance spectral radius of a minimal \((k, g)\)-cage, \(k \geq 3\), is

\[
\lambda_1 = \begin{cases} 
\frac{k(1-(k-1)^d)}{(2-k)^2} - \frac{2d(k-1)^d}{(2-k)} & \text{if } g \text{ even}, \\
\frac{k(1-(k-1)^d)}{(2-k)^2} - \frac{dk(k-1)^d}{(2-k)} & \text{if } g \text{ odd},
\end{cases}
\]

where \(d = \left\lfloor \frac{g}{2} \right\rfloor\).

**Proof.** Since a minimal \((k, g)\)-cage is a DR graph which is also a \(p\)-transmission regular graph, the distance spectral radius of this graph is the transmission \(p\) of any vertex \(x\) in it. From the intersection array of the minimal \((k, g)\)-cage we get, \(p = \sum_{y \in G} d(x, y) = k + 2k(k-1) + 3k(k-1)^2 + \cdots + (d-1)k(k-1)^{d-2} + dc(k-1)^{d-1}, \)

where \(c = 1\) for \(g\) even and \(c = k\) for \(g\) odd. For \(g\) even, \(p = k + 2k(k-1) + 3k(k-1)^2 + \cdots + (d-1)k(k-1)^{d-2} + d(k-1)^{d-1} = \)
\[ k + 2k(k - 1) + 3k(k - 1)^2 + \cdots + (d - 1)k(k - 1)^{d-2} + d(k - k + 1)(k - 1)^{d-1} = k[1 + 2(k - 1) + \cdots + d(k - 1)^{d-1}] - d(k - 1)^d = kS - d(k - 1)^d, \]

where \( S = 1 + 2(k - 1) + \cdots + d(k - 1)^{d-1} \). Then we get \( S - (k - 1)S = [1 + (k - 1) + \cdots + (k - 1)^{d-1}] - d(k - 1)^d \). So \( S = \frac{1-(k-1)^d}{(2-k)} - \frac{d(k-1)^d}{(2-k)} \) and we get the result in this case. If \( g \) is odd then \( p = kS \), and hence the result. \( \square \)

The next lemma will be useful to prove some important results of this paper.

**Lemma 2.1.** For integers \( i \) and \( j \), \( i, j = 0, 1, 2, \ldots, d \), consider the recurrence relation \( a_i^j = \begin{cases} a_i^{j-1} + a_i^{j-2} & \text{if } 1 \leq j \leq \left\lfloor \frac{i}{2} \right\rfloor, \\
0, & \text{otherwise} \end{cases} \), with initial conditions

\[ a_i^0 = \begin{cases} k-1, & i=1, \ldots, d \\
k, & i=0 \end{cases}, \]

Then we get,

(i) for \( j > 0 \), \( a_{2j+b}^j = \begin{cases} k, & \text{if } b = 0 \\
k + a_{2j-1}^j + a_{2j-2}^j + \cdots + a_{2j+b-2}^j, & \text{if } b > 0 \end{cases} \)

(ii) \( a_i^j = g_i^j k - h_i^j \) for \( 1 \leq j < \left\lfloor \frac{i}{2} \right\rfloor \), \( i = 4, \ldots, d \), where \( g_i^j = 1 + g_{2j-1}^{j-1} + g_{2j-2}^{j-1} + \cdots + g_{i-2}^{j-1}, h_i^j = h_{2j-1}^{j-1} + h_{2j-2}^{j-1} + \cdots + h_{i-2}^{j-1}, g_i^0 = i-1, h_i^0 = i-2 \), and \( g_i^0 = h_i^0 = 1 \).

**Proof.** (i) First, we take \( b = 0 \), do induction on \( j \) and show that \( a_{2j}^0 = k \). For \( j = 0 \), \( a_0^0 = k \), and for \( j = 1 \), we have \( a_1^1 = a_1^0 + a_0^0 = k \), as \( a_1^1 = 0 \) from the hypothesis. We assume that the result is true up to \( j - 1 \). Now \( a_{2j}^1 = a_{2j-1}^1 + a_{2j-2}^1 + \cdots + a_{2j-b}^1 \), \( b = 0 \), since \( a_{2j-1}^0 = 0 \) from the hypothesis. Hence \( a_{2j}^1 = k \), for every \( j \). Next let \( b > 0 \). For any fixed \( j \geq 1 \) we do induction on \( b \). If \( b = 1 \), \( a_{2j+1}^1 = a_{2j}^1 + a_{2j-1}^1 = k + a_{2j-1}^1 \). We assume that the equation holds true up to \( j \), \( a_{2j+b}^j = a_{2j+b-1}^j + a_{2j+b-2}^j + \cdots + a_{2j+b-3}^j + a_{2j+b-2}^j \). This proves the first part of the lemma.

For (ii), we do induction on \( j \) for any fixed \( i \geq 2 \). By (i) of this Lemma we get, \( a_i^j = k + a_i^0 + a_i^0 + \cdots + a_{i-2}^0 = k + (i-2)(k-1) = (i-1)k-(i-2) = g_i^1 k - h_i^1 \). This proves the result for \( j = 1 \). Let the equation be true upto \( j-1 \). Now \( a_i^j = k + a_i^{j-1} + a_i^{j-1} + \cdots + a_i^{j-2} = k + (g_{2j-1}^{j-1} k - h_{2j-1}^{j-1}) + (g_{2j-2}^{j-1} k - h_{2j-2}^{j-1}) + \cdots + (g_{i-2}^{j-1} k - h_{i-2}^{j-1}) = (1 + g_{2j-1}^{j-1} + g_{2j-2}^{j-1} + \cdots + g_{i-2}^{j-1}) k - (h_{2j-1}^{j-1} + h_{2j}^{j-1} + \cdots + h_{i-2}^{j-1}) = g_i^j k - h_i^j \). Hence the result. \( \square \)
Theorem 2.2. Let $G$ be a minimal $(k, g)$-cage. The $i^{th}$ distance matrix $A_i$, $i = 0, 1, 2, \ldots, d$, of $G$ can be expressed as:

$$A_i = \frac{1}{c} [A^i - a_1^i A^{i-2} + (k - 1) a_2^i A^{i-4} - (k - 1)^2 a_3^i A^{i-6} + \cdots + (-1)^{\lfloor \frac{i}{2} \rfloor}]$$

$$(k - 1)^{\lfloor \frac{i}{2} \rfloor - 1} a_1^i A^{i-2\lfloor \frac{i}{2} \rfloor}$$

(3)

where $a_i$ are as in Lemma 2.1. $c$ is $k$ for $i = d$ and $g$ even, and is 1 otherwise.

Proof. We do induction on $i$. First, let $g$ be an odd integer. So intersection array of $G$ is $\{k, k-1, \ldots, k-1; 1, 1, \ldots, 1\}$. From recurrence relation (2), we have $AA_1 = c_2 A_2 + a_1 A_1 + b_0 A_0$, $A_0 = I$ and $A_1 = A$. Since $a_0 = 0$, $c_2 = 1$, and $b_0 = k$, we get $A^2 = A_2 + kI$. Then $A_2 = A^2 - kI = A^2 - a_1^1 I$. Thus equation (3) is true for $i = 0, 1, 2$. Let us assume that it is true up to $d - 1$. Then we consider $i = d$. Since $a_{d-1} = 0$, $c_d = 1$, and $b_{d-2} = k - 1$, we have

$$AA_{d-1} = c_d A_d + a_{d-1} A_{d-1} + b_{d-2} A_{d-2} = A_d + (k - 1) A_{d-2},$$

$$A(A^{d-1} - a_{d-1}^1 A^{d-3} + \cdots + (-1)^{\lfloor \frac{d-1}{2} \rfloor} (k - 1)^{\lfloor \frac{d-1}{2} \rfloor - 1} a_{d-2}^1 A^{d-2\lfloor \frac{d-1}{2} \rfloor}) = A_d + (k - 1)(A^{d-2} - a_{d-2}^1 A^{d-4} + \cdots + (-1)^{\lfloor \frac{d-2}{2} \rfloor} (k - 1)^{\lfloor \frac{d-2}{2} \rfloor - 1} a_{d-3}^1 A^{d-3\lfloor \frac{d-2}{2} \rfloor}).$$

Since for $d$ even, $[\frac{d-2}{2}] = [\frac{d-1}{2}] = \lfloor \frac{d}{2} \rfloor - 1$, we get

$$A_d = A^d - \{ a_{d-1}^1 + (k - 1) \} A^{d-2} + (k - 1) \{ a_{d-2}^1 + a_{d-2}^1 \} A^{d-4} - \cdots + (-1)^{\lfloor \frac{d}{2} \rfloor} (k - 1)^{\lfloor \frac{d}{2} \rfloor - 1} a_{d-3}^1 A^{d-2\lfloor \frac{d}{2} \rfloor}.$$

For $d$ odd, $[\frac{d-2}{2}] = [\frac{d-1}{2}] = \lfloor \frac{d}{2} \rfloor$. So we get

$$A_d = A^d - \{ a_{d-1}^1 + (k - 1) \} A^{d-2} + (k - 1) \{ a_{d-2}^1 + a_{d-2}^1 \} A^{d-4} - \cdots + (-1)^{\lfloor \frac{d}{2} \rfloor} (k - 1)^{\lfloor \frac{d}{2} \rfloor - 1} a_{d-3}^1 A^{d-2\lfloor \frac{d}{2} \rfloor}.$$

Hence the result holds true in this case.

Next, we consider that $g$ is even. The result holds true for $i = 0, 1, 2, \ldots, d - 1$, because the intersection numbers agree with those in the case that $g$ is odd. Since $c_d = k$, by recurrence relation (2) we get

$$AA_{d-1} = c_d A_d + a_{d-1} A_{d-1} + b_{d-2} A_{d-2} = k A_d + (k - 1) A_{d-2},$$

$$A_d = \frac{1}{k} [A^d - a_d^1 A^{d-2} + (k - 1) a_d^2 A^{d-4} - \cdots + (-1)^{\lfloor \frac{d}{2} \rfloor} (k - 1)^{\lfloor \frac{d}{2} \rfloor - 1} a_d^1 A^{d-2\lfloor \frac{d}{2} \rfloor}].$$

$\square$
In the theorem below we find polynomials of degree $\left\lfloor \frac{d}{2} \right\rfloor$ which give all distance eigenvalues of minimal $(k, g)$-cages when the variable is substituted by adjacency eigenvalues.

**Theorem 2.3.** If $\lambda$ is an eigenvalue of a minimal $(k, g)$-cage $G$ then $p(\lambda)$ is a distance eigenvalue of $G$ with the same multiplicity as that of $\lambda$, where $p(x)$ is given below.

$$p(x) = \begin{cases} 
\sum_{i=0}^{d-1} \left( \frac{d-1}{2} \right) \sum_{j=1}^{d-i-\lfloor 1 \rfloor} (-1)^{j}(i+2j)(k-1)^{j-1}a_{i+2j}^{j} \right)x^{i} + \frac{d}{k}[A^{d} + \sum_{i=1}^{d-1} \left( \frac{d}{2} \right) \sum_{j=1}^{d-i-\lfloor 1 \rfloor} (-1)^{j}(k-1)^{j-1}a_{d}^{j}x^{d-2j}], & \text{for } \ g \text{ even} \\
2x^{2} + x - 2k, & \text{for } \ g = 3 \\
\sum_{i=1}^{d-1} \left( \frac{d}{2} \right) \sum_{j=1}^{d-i-\lfloor 1 \rfloor} (-1)^{j}(i+2j)(k-1)^{j-1}a_{i+2j}^{j} \right)x^{i} + \frac{d}{k}[A^{d} + \sum_{i=1}^{d-1} \left( \frac{d}{2} \right) \sum_{j=1}^{d-i-\lfloor 1 \rfloor} (-1)^{j}(k-1)^{j-1}a_{d}^{j}x^{d-2j}], & \text{for } \ g = 5
\end{cases}$$

where $a_{d}^{j}$ are as in Lemma 2.1.

**Proof.** Here we represent $D(G)$ as a polynomial, $p(A)$, of the adjacency matrix $A$ of $G$ and then the theorem follows from a basic result that ” If $\lambda$ is an eigenvalue of $A$ then $p(\lambda)$ is an eigenvalue of $p(A)$ with the same multiplicity as that of $\lambda.”$ First, we consider that $g$ is an even integer. From Theorem 2.2 and equation (11), the distance matrix $D$ of a minimal $(k, g)$-cage $G$ can be written as below:

$$D = A_{1} + 2A_{2} + 3A_{3} \cdots + dA_{d}$$

$$= A + 2(A^{2} - a_{1}A) + 3(A^{3} - a_{1}^{2}A) + \cdots + (d - 1)[A^{d-1} - a_{d-1}^{1}A^{d-3} + \cdots + (-1)^{\left\lfloor \frac{d}{2} \right\rfloor}(k - 1)^{\left\lfloor \frac{d}{2} \right\rfloor - 1}a_{d-1}^{\left\lfloor \frac{d}{2} \right\rfloor - 1}A^{d-2} + \cdots + (-1)^{\left\lfloor \frac{d}{2} \right\rfloor}A^{d} - a_{d}^{1}A^{d-2} + \cdots + A^{d})$$

If $d$ is even then $\left\lfloor \frac{d-1}{2} \right\rfloor = \left\lfloor \frac{d-1}{2} \right\rfloor$ for even $i$ and $\left\lfloor \frac{d-1}{2} \right\rfloor = \left\lfloor \frac{d-1}{2} \right\rfloor$ for odd $i$, $i = 1, \ldots, d - 1$. So we get,

$$D = [-2a_{1}^{2} + 4(k - 1)a_{1}^{2} + \cdots + (-1)^{\left\lfloor \frac{d}{2} \right\rfloor}(d - 2)(k - 1)^{\left\lfloor \frac{d}{2} \right\rfloor - 1}a_{d-1}^{\left\lfloor \frac{d}{2} \right\rfloor - 1} + [1 - 3a_{3}^{1} + \cdots + (-1)^{\left\lfloor \frac{d-1}{2} \right\rfloor}(d - 1)(k - 1)^{\left\lfloor \frac{d-1}{2} \right\rfloor - 1}a_{d-1}^{\left\lfloor \frac{d-1}{2} \right\rfloor - 1}A + \cdots + [(d - 3) - (d - 1)a_{d-1}^{1}A^{d-3} + (d - 2)a^{d-2} + (d - 1)a_{d}^{1}A^{d-4} + \cdots + (-1)^{\left\lfloor \frac{d}{2} \right\rfloor}(k - 1)^{\left\lfloor \frac{d}{2} \right\rfloor - 1}a_{d}^{\left\lfloor \frac{d}{2} \right\rfloor - 1}A^{d-4}]$$

$$D = \begin{pmatrix} 
A_{1} + 2A_{2} + 3A_{3} & \cdots & dA_{d} \\
2(A_{1}^{2} - a_{1}A) & \cdots & 2A_{d-1}^{2} - a_{d-1}A_{d-1} \\
3(A_{3}^{2} - a_{3}^{2}A) & \cdots & 3A_{d-2}^{2} - a_{d-2}A_{d-2} \\
\vdots & \ddots & \vdots \\
dA_{d}^{2} & \cdots & dA_{d-1}^{2} - a_{d-1}A_{d-1}
\end{pmatrix}$$
\[ d - 1 = \sum_{i=0}^{d-1} i + \sum_{j=1}^{\left\lfloor \frac{d-1}{2} \right\rfloor} (-1)^j (i + 2j)(k - 1)^{j-1} a_{i+2j}^j A^i + \frac{d}{k} A^d + \sum_{i=1}^{\left\lfloor \frac{d}{2} \right\rfloor} (-1)^i (k - 1)^{i-1} a_{d-2i}^i A^{d-2i} \]

For \( d \) odd, we get

\[ D = \left[ -2a_2^1 + 4(k - 1)a_3^2 - \cdots + (-1)^{\left\lfloor \frac{d-3}{2} \right\rfloor} (d - 1)(k - 1)^{\left\lfloor \frac{d-3}{2} \right\rfloor} a_{d-3}^1 \right] + [1 - 3 a_3 + \cdots + (d - 2)(k - 1)^{\left\lfloor \frac{d-2}{2} \right\rfloor} a_{d-2}^1] A + \cdots + [(d - 3) - (d - 1) a_{d-1}^1] A^{d-1} + \frac{d}{k} A^d - a_1^1 A^{d-2} - \cdots + (-1)^{\left\lfloor \frac{d}{2} \right\rfloor} \right] (k - 1)^{\left\lfloor \frac{d}{2} \right\rfloor - 1} a_{d-2}^1 A^{d-2} \left[ \frac{d}{2} \right] \]

\[ = \sum_{i=0}^{d-1} [i + \sum_{j=1}^{\left\lfloor \frac{d-1}{2} \right\rfloor} (-1)^j (i + 2j)(k - 1)^{j-1} a_{i+2j}^j A^i + \frac{d}{k} A^d + \sum_{i=1}^{\left\lfloor \frac{d}{2} \right\rfloor} (-1)^i \]

\[ (k - 1)^{i-1} a_{d-2i}^i A^{d-2i} \]

In the above, for both \( d \) even and odd, \( D \) is expressed as a polynomial of the adjacency matrix \( A \) of \( G \). We take this polynomial as \( p(A) \) and obtain the result.

If \( g \) is an odd integer then by Lemma 1.4, \( g = 3 \) or 5. For \( g = 3 \) the value of \( d \) is 1. So, by equation (1) we have \( D = p(A) = A \), and so is the result. Now for \( g = 5 \), the value of \( d \) is 2. Applying equation (1) and Theorem 2.2 we have, \( D = A_1 + 2A_2 = A + 2(A^2 - a_1^2 I) = A + 2(A^2 - kI) = 2A^2 + A - 2kI \), and hence the result.

**Theorem 2.4.** Every minimal \((k, g)\)-cage, \( k \geq 2 \), with diameter \( d \) has \( d + 1 \) distinct distance eigenvalues.

**Proof.** For \( k = 2 \) the minimal \((k, g)\)-cages are cycles, and the result follows by [14].

**Case 1.** In this case we consider that \( g \) is an even integer. For \( d \) even, applying Lemma 2.1 and Theorem 2.3 the distance matrix of the minimal
(\(k, g\))-cage can be written as,

\[
D = p(A) = A + 2(A^2 - a_d^2I) + \cdots + (d - 2)[A^{d-2} - a_{d-2}A^{d-4} + \cdots + (-1)^{\frac{d-2}{2}}]\cdot
\]

\[
(k - 1)\cdot [d^{\frac{d-2}{2}}a_{d-2}^2]^{-1}A^{d-2} - 2(\frac{d-2}{2})^{-1} + (1)\cdot [d^{\frac{d-2}{2}}a_{d-2}^2]^{-1}A^{d-2} - 2[(\frac{d-2}{2})^{-1}]
\]

\[
A^{d-2}(\frac{d-2}{2})^{-1}] + (d - 1)[A^{d-1} - a_{d-1}A^{d-3} + \cdots + (-1)^{\frac{d-1}{2}}(k - 1)(d^{\frac{d-1}{2}}a_{d-1}^2)
\]

\[
A^{d-2} - 2[(\frac{d-1}{2})^{-1}] + (1)\cdot [d^{\frac{d-1}{2}}a_{d-1}^2]^{-1}A^{d-2} - 2[(\frac{d-1}{2})^{-1}]
\]

\[
da_{d-1}^2A^{d-2} - 2[(\frac{d-1}{2})^{-1}]
\]

\[
(\frac{d}{2})^2 - 1)
\]

\[
\frac{d}{k}[A^d - a_d^dA^{d-2} + \cdots + (-1)^{\frac{d}{2}}k\sum_{i=0}^{\frac{d}{2}}(-1)^i\left(\left(\frac{d}{2}ight)^2 - 1\right)k^{\left(\frac{d}{2}ight)^2 - 1}]\cdot
\]

\[
\frac{d}{k}[A^d - a_d^dA^{d-2} + \cdots + (-1)^{\frac{d}{2}}k\sum_{i=0}^{\frac{d}{2}}(-1)^i\left(\left(\frac{d}{2}ight)^2 - 1\right)k^{\left(\frac{d}{2}ight)^2 - 1}]
\]

\[
= 1 \cdot \left\{ A + 2A^2 + 3A^3 + 4A^4 + \cdots + (d - 1)(A^{d-1} + h_{d-1}A^{d-2} + \cdots + h_{d-1}\frac{d-1}{2}A) - d(g^1_dA^{d-2} + (g^2_d + h_{d-1}^2)A^{d-4} + (g^3_d
\]

\[
+ 2h_{d-1}^2A^{d-6} + \cdots + (g^\frac{d}{2})_1 - 1)h_{d-1}^{\frac{d}{2}}A^2 + 1)]k + \cdots + (-1)^{\frac{d}{2}}\left(\frac{d}{2} - 1\right)h_{d-1}^{\frac{d}{2} - 1}A + d)\cdot
\]

\[
\cdot (d - 2) + g^\frac{d}{2} - 1)(d - 1)A - d)\cdot
\]

\[
\cdot [d^{\frac{d}{2}}a_{d-1}^2 - 1]
\]

since for \(d\) even \(\left|\frac{d}{2}\right| - 1 = \left|\frac{d-2}{2}\right| = \left|\frac{d-2}{2}\right|\). If possible let there be two distinct eigenvalues \(\lambda_i \neq \lambda_j\) of the minimal \((k, g)\)-cage such that \(p(\lambda_i) = p(\lambda_j)\). This implies \(k p(\lambda_i) = k p(\lambda_j)\) (since \(k \neq 0\)). Now equating the coefficients of \(k^{\frac{d}{2}}\), we get \((d - 2) + (d - 1)g_{d-1}^{\frac{d-1}{2}}\lambda_i - d = (d - 2) + (d - 1)g_{d-1}^{\frac{d-1}{2}}\lambda_j - d\), which gives \(\lambda_i = \lambda_j\), a contradiction. This proves that a minimal \((k, g)\)-cage has \(d + 1\) distinct distance eigenvalues for \(d\) and \(g\) both even.

If \(d\) is odd and \(g\) is even then by Lemma 1.3 we get that \(g\) is equal to 6. By Lemma 1.5 all distinct eigenvalues of the minimal \((k, 6)\)-cage are \(\pm k, \pm \sqrt{k - 1}\). Then the distance matrix of the minimal \((k, 6)\)-cage can be written as \(D = p(A) = \frac{2}{k}A^3 + 2A^2 + \frac{3 - 5k}{k}A - 2kI\). If possible let there exist two distinct eigenvalues \(\lambda_i \neq \lambda_j\) of minimal \((k, 6)\)-cage such
that $p(λ_i) = p(λ_j)$. Then $kp(λ_i) = kp(λ_j)$ (since $k \neq 0$). This implies $-2k^2 + (2λ_i^2 - 5λ_i)k + 3(λ_i^2 + λ_i) = -2k^2 + (2λ_j^2 - 5λ_j)k + 3(λ_j^2 + λ_j)$. Equating the coefficients of $k$ we get $2(λ_i^2 - λ_j^2) - 5(λ_i - λ_j) = 0$, and then $λ_i + λ_j = \frac{5}{2}$. Since $k$ is the largest adjacency eigenvalue, $p(k)$ is the largest distance eigenvalue $[1]$. Thus both $λ_i$ and $λ_j$ are different from $k$. Then $λ_i, λ_j \in \{-k, ±\sqrt{k} - 1\}$. Now $λ_i + λ_j = -k ± \sqrt{k} - 1 = \frac{5}{2}$, that is $4k^2 + 16k + 29 = 0$. But this equation does not give any integer solution and since $k$ is an integer, this leads to a contradiction. Hence a minimal $(k, 6)$-cage has 4 distinct distance eigenvalues.

Case 2. In this case we consider that $g$ is an odd integer. A minimal $(k, 3)$-cage is a complete graph $K_n$ and its distinct distance eigenvalues are $n - 1, -1$. Now for $g = 5$, all distinct eigenvalues of the minimal $(k, 5)$-cage are $k, -\frac{1±\sqrt{3k-3}}{2}$. Then from Theorem 2.3 the distance matrix of this graph can be written as $D = p(A) = 2A^2 + A - 2kI$. If possible let there exist two distinct eigenvalues $λ_i \neq λ_j$ of minimal $(k, 5)$-cage such that $p(λ_i) = p(λ_j)$. This implies $2(λ_i^2 - λ_j^2) + (λ_i - λ_j) = 0$, and then $λ_i + λ_j = -\frac{1}{2}$. Since $k$ is the largest adjacency eigenvalue, $p(k)$ is the largest distance eigenvalue $[1]$. Thus both $λ_i$ and $λ_j$ are different from $k$. Then $λ_i, λ_j \in \{-k, ±\sqrt{k} - 1\}$ and $λ_i + λ_j = -1$, a contradiction. Thus a minimal $(k, 5)$-cage has 3 distinct distance eigenvalues. This proves the theorem.

Remark 2.1. Theorem 2.3 supplies a class of graphs to the answer of the problem ”Characterize distance regular graphs with diameter $d$ and having exactly $d + 1$ distinct $D$-eigenvalues”, asked by Atik and Panigrahi $[5]$.

A minimal $(k, 3)$-cage is a complete graph and its distance spectrum is mentioned in Theorem 2.4. Minimal $(k, 4)$-cages and $(2, g)$-cages are complete bipartite graphs and cycles respectively, and their distance spectrum can be found in $[18]$ and $[14]$. So in the theorem below we present the distance spectrum of minimal $(k, g)$-cages, $k ≥ 3$ and $g ≥ 5$, by applying Lemma 1.5 and Theorems 2.2 and 2.3.

Theorem 2.5. 1. The distance matrix of a minimal $(k, 5)$-cage is $D = -2kI + A + 2A^2$.

(a) The distance matrix of the minimal $(3, 5)$-cage (Petersen graph) is $D = -6I + A + 2A^2$, and its distance spectrum is $\{15, -3^{(5)}, 0^{(4)}\}$.

(b) The distance matrix of the minimal $(7, 5)$-cage (Hoffman-Singleton graph) is $D = -14I + A + 2A^2$, and its distance spectrum is $\{91, -4^{(28)}, 1^{(21)}\}$.

(c) The distance matrix of the minimal $(57, 5)$-cage (if exists) is $D = -114I + A + 2A^2$, and its distance spectrum is $\{6441, -9^{(1720)}, 6^{(1520)}\}$.
2. The distance matrix of a minimal \((k, 6)\)-cage is \(D = \frac{x}{2}A^3 + 2A^2 - \frac{5k-2}{2}A - 2kI\), and its distance spectrum is \(\{5k^2 - 7k + 3, -k^2 + 3k - 3, (2(1+\sqrt{k-1}))^{(m+\sqrt{k-1})}, (2(1-\sqrt{k-1}))^{(m-\sqrt{k-1})}\}\), where \(m_{\pm}\sqrt{k-1} = \frac{nk(k-1)}{2(k^2-k+1)}\).

(Heawood graph is a minimal \((3, 6)\)-cage and its distance matrix is \(D = A^3 + 2A^2 - 4A - 6I\), and its distance spectrum is \(\{27, -3, (-2(1+\sqrt{2}))^6, (-2(1-\sqrt{2}))^6\}\)).

3. The distance matrix of a minimal \((k, 8)\)-cage is \(D = \frac{4}{3}A^4 + 3A^3 + \frac{8k}{3}A^2 - (6k - 4)A + (2k - 4)I\), and its distance spectrum is \(\{7k^3 - 16k^2 + 14k - 4, k^3 - 4k^2 + 6k - 4, (2k - 4)^{(m_0)}, (2k - 2)\sqrt{2(k - 1)}\}^{(m_0)}\), where \(m_0 = \frac{n(k-1)}{2k}, m_{\pm} = \frac{nk(k-1)}{4(k^2 - 2k + 2)}\).

(Levi graph is a minimal \((3, 8)\)-cage and its distance matrix is \(D = \frac{4}{3}A^4 + 3A^3 + \frac{2k}{3}A^2 - 14A + 2I\), and its distance spectrum is \(\{83, 5, 2^{(10)}, -2^{(9)}, -10^{(9)}\}\)).

4. The distance matrix of a minimal \((k, 12)\)-cage is \(D = \frac{5}{6}A^6 + 5A^5 + \frac{24-26k}{k}A^4 - (20k - 18)A^3 + \frac{27k^2 - 14k + 18}{3}A^2 + (15k^2 - 26k + 9)A - (2k^2 - 6k + 6)I\), and its distance spectrum is \(\{11k^5 - 46k^4 + 81k^3 - 72k^2 + 33k - 6, k^5 - 6k^4 + 15k^3 - 20k^2 + 15k - 6, (2k^2 + 6k - 6)^{(m_0)}, (2k - 2)(k + \sqrt{k - 1})\}^{(m_0)}\), where \(m_0 = \frac{n(k-1)}{3k}, m_{\pm} = \frac{nk(k-1)}{4(k^2 - k + 1)}, and m_{\pm} = \frac{nk(k-1)}{12(k^2 - 3k + 3)}\).

3 Distance spectrum of some distance biregular graphs

In the next theorem we show that every DBR graph is a 2-partitioned transmission regular graph.

Theorem 3.1. All distance biregular (DBR) graphs are 2-partitioned transmission regular graphs.

Proof. Let \(G\) be a DBR graph with partite sets \(V_1\) and \(V_2\). Each vertex \(u\) in \(V_1\) has \(l_i\), a constant, number of vertices at distance \(i\). From \(u\), even distance vertices are situated in \(V_1\) and odd distance vertices are in \(V_2\). Thus the number of vertices of even and odd distances is constant from each vertex \(u \in V_1\). So we get, \(q_{11} = \sum_{v \in V_1} d(u, v) = \sum_{i=0}^{\frac{k}{2}} 2il_{2i}\) and \(q_{12} = \sum_{v \in V_2} d(u, v) = \sum_{i=0}^{\frac{k}{2}} 2il_{2i-1}\).
\[ \sum_{i=0}^{d} (2i + 1)l_{2i+1} \] are constants, where \( d_1 = \max \{d(x, y) : x \in V_1, y \in V(G)\} \).

Similarly the sum of distances from each \( w \in V_2 \), \( q_{21} = \sum_{v \in V_1} d(w, v) = \sum_{i=0}^{d_2} 2il'_{2i} \) and \( q_{22} = \sum_{v \in V_2} d(w, v) = \sum_{i=0}^{d_2} (2i+1)l'_{2i+1} \) are constants, where \( l'_i \) is the number of vertices at distance \( i \) from \( w \) and \( d_2 = \max \{d(x, y) : x \in V_2, y \in V(G)\} \). Hence the result.

The larger root of the quotient matrix \( Q = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix} \) is the distance spectral radius of any DBR graphs by Lemma 1.3. The subdivision graph \( q \) and \( i = 1 \) stated, in the remaining of the paper (3.1 we determine distance spectral radius of these graphs. Unless otherwise stated, in the remaining of the paper \( V_1, V_2 \) is taken as the vertex partition of subdivision of a minimal \((k, g)\)-cage \( G \), where \( V_1 = V(G) \) and \( V_2 \) is the set of all new vertices inserted on edges of \( G \).

**Theorem 3.2.** Let \( g \) be an even integer and \( G \) be a minimal \((k, g)\)-cage with diameter \( d \). The distance spectral radius of the subdivision graph \( S(G) \) is

\[
\frac{(3k - 2)S_1' + dk(k - 1)^{d-1} + \sqrt{(k - 2)^2S_2' + 2d(k - 2)^{2d}(k - 1)^{d-2}S_1' + d^2(k - 2)^{2d}(k - 1)^{2d-2}}}{S_2'}.
\]

where \( S_1' = \frac{1}{(2-k)^2}[(kd - 2d - k + 1)(k - 1)^{d-1} + 1] \) and \( S_2' = \frac{1}{(2-k)^2}[(2kd - 4d - k)(k - 1)^{d-1} + k] \).

**Proof.** We know that [10] the intersection array of any vertex \( u \in V_1 \) is \( \{k, 1, k-1, 1, \ldots, k-1, 1; 1, 1, 1, \ldots, 1; k\} \) and the intersection array of any vertex \( v \in V_2 \) is \( \{2, k-1, 1, k-1, \ldots, 1, k-1; 1, 1, 1, \ldots, 1, 2\} \). Since \( S(G) \) is obtained by inserting a new vertex in every edge and diameter of \( G \) is \( d = \left\lfloor \frac{k}{2} \right\rfloor \), applying Theorem 3.1 we have \( l_0 = 1, l_{2i} = l_{2i-1} = k(k - 1)^{i-1}, i = 1, 2, \ldots, d-1, l_{2d-1} = k(k - 1)^{d-1}, l_{2d} = (k - 1)^{d-1}, l'_1 = 2, l'_2 = l'_{2i+1} = 2(k - 1)^i, i = 1, 2, \ldots, d-1, \) and \( l'_{2d} = (k - 1)^d \). Thus

\[
q_{11} = \sum_{i=0}^{d} 2il_{2i} = 0 + 2k + 4k(k - 1) + \cdots + (2d - 2)k(k - 1)^{d-2} + 2d(k - 1)^{d-1} \\
= 2k[1 + 2(k - 1) + 3(k - 1)^2 + \cdots + (d - 1)(k - 1)^{d-2}] + 2d(k - 1)^{d-1} \\
= 2kS_1' + 2d(k - 1)^{d-1},
\]
where $S'_1 = 1 + 2(k-1) + 3(k-1)^2 + \cdots + (d-1)(k-1)^{d-2} = \frac{1}{(2-k)^2}[(kd - 2d - k + 1)(k-1)^{d-1} + 1]$. 

\[ q_{l_2} = \sum_{i=0}^{d-1} (2i + 1)l_{2i+1} = k + 3k(k-1) + 5k(k-1)^2 + \cdots + (2d - 1)k(k-1)^{d-1} = k[1 + 3(k-1) + 5(k-1)^2 + \cdots + (2d - 1)(k-1)^{d-1}] = kS'_2, \]

where $S'_2 = 1 + 3(k-1) + 5(k-1)^2 + \cdots + (2d - 1)(k-1)^{d-1} = \frac{1}{(2-k)^2}[(kd - 4d - k)(k-1)^d + k]$. 

\[ q_{l_2} = \sum_{i=0}^{d-1} (2i + 1)l_{2i+1} = 2 + 3 \times 2(k - 1) + 5 \times 2(k - 1)^2 + \cdots + (2d - 1) \times 2 \]

\[(k-1)^{d-1} = 2S'_2. \]

\[ q_{l_2} = \sum_{i=0}^{d} 2il_{2i} = 2 \times 2(k - 1) + 4 \times 2(k - 1)^2 + 6 \times 2(k - 1)^3 + \cdots + (2d - 2) \]

\[ \times 2(k-1)^{d-1} + 2d \times (k-1)^d = 4(k-1)S'_1 + 2d(k-1)^d. \]

Thus $Q = \begin{pmatrix} q_{l_1} & q_{l_2} \\ q_{l_2} & q_{l_2} \end{pmatrix}$ is a quotient matrix of the distance matrix of $S(G)$ when $g$ is even. The characteristic polynomial of $Q$ is $x^2 - \{2(3k - 2)S'_1 + 2dk(k-1)^{d-1}\}x + 8k(k-1)S'_1^2 - 2kS'_2^2 + 4d(k+2)(k-1)^dS'_1 + 4d^2(k-1)^{2d-1} = 0$, and its larger root $(3k - 2)S'_1 + dk(k-1)^{d-1} + \sqrt{(k - 2)^2S'_1^2 + 2kS'_2^2 + 2d(k - 2)^2(k - 1)^{d-1}S'_1 + d^2(k - 2)^2(k - 1)^{2d-2}}$, where $S'_1 = \frac{1}{(2-k)^2}[(kd - 2d - k + 1)(k-1)^{d-1} + 1]$ and $S'_2 = \frac{1}{(2-k)^2}[(2kd - 4d - k)(k-1)^d + k]$ is the distance spectral radius of $S(G)$ by Lemma 1.3. □

By Lemma 1.4 if $g$ is an odd integer then minimal $(k, g)$-cages exist only for $g = 3$ and 5. So in the next theorem we determine distance spectral radius of subdivision of minimal $(k, g)$-cages for these two cases only.

**Theorem 3.3.** Let $g$ be an odd integer and $G$ be a minimal $(k, g)$-cage. The distance spectral radius of the subdivision graph $S(G)$ is,

\[ \lambda_1(S(G)) = \begin{cases} \frac{1}{2}[2k^2 + \sqrt{2k(2k + 1)(k^2 + 1)}], & \text{if } g = 3 \\ \frac{1}{2}[(3k^3 + k - 2) + \sqrt{9k^6 + 2k^6 + 14k^6 - 40k^3 + 41k^2 - 18k + 4}], & \text{if } g = 5 \end{cases} \]

**Proof.** First, let $g$ be equal to 3. We know that $[1110]$ intersection array of any vertex in $V_1$ is $\{k, 1, k - 1; 1, 1, 2\}$ and the intersection array of any vertex in $V_2$ is $\{2, k - 1, 1, k - 2; 1, 1, 2, 2\}$. By Lemma 1.1 we have, $l_0 = 1, l_1 = l_2 = k, l_3 = \frac{k}{2}(k-1), l'_0 = 1, l'_1 = 2, l'_2 = 2(k - 1), l'_3 = (k - 1)$, and
So $Q = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}$ is a quotient matrix of the distance matrix of $S(G)$. The characteristic polynomial of $Q$ is $x^2 - 2k^2x - \frac{1}{2}k(k + 1)^2 = 0$, and its larger root $\frac{1}{2} \left[2k^2 + \sqrt{2k(2k + 1)(k^2 + 1)}\right]$ is the distance spectral radius of $S(G)$ by Lemma [1.3].

Next we take $g = 5$. Intersection array of any vertex in $V_1$ is \{1, 1, 1, 1, k - 1; 1, 1, 1, 1, 2\} and the intersection array of any vertex in $V_2$ is \{2, k - 1, 1, 1, 1, 2; 1, 1, 1, 1, 2, 2\}. By Lemma [1.1] we have, $l_0 = 1$, $l_1 = l_2 = k$, $l_3 = l_4 = k(k - 1)$, $l_5 = \frac{1}{2}k(k - 1)^2$, $l'_0 = 1$, $l'_1 = 2$, $l'_2 = l'_3 = 2(k - 1)$, $l'_4 = 2(k - 1)^2$, $l'_5 = (k - 1)^2$, and $l'_6 = \frac{1}{2}(k - 1)^2(k - 2)$. Thus $q_{11} = 2k + 4k(k - 1) = 2k(2k - 1)$, $q_{12} = k + 3k(k - 1) + \frac{5}{2}k(k - 1)^2 = \frac{1}{2}k(5k^2 - 4k + 1)$, $q_{21} = 2 + 3 \times 2(k - 1) + 5 \times (k - 1)^2 = (5k^2 - 4k + 1)$, $q_{22} = 2 \times 2(k - 1) + 4 \times 2(k - 1)^2 + 6 \times \frac{1}{2}(k - 1)^2(k - 2) = (k - 1)(3k^2 - 2k + 2)$.

So $Q = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}$ is a quotient matrix of the distance matrix of $S(G)$. The characteristic polynomial of $Q$ is $x^2 - (3k + k - 2)x - \frac{1}{2}k(k^4 + 4k^3 - 14k^2 + 20k - 7) = 0$, and its larger root $\frac{1}{2}[(3k^3 + k - 2) + \sqrt{9k^8 + 2k^5 + 14k^4 - 40k^3 + 41k^2 - 18k + 4}]$ is the distance spectral radius of $S(G)$ by Lemma [1.3].

**Example 3.1.** We know that the Heawood graph is the minimal (3,6)-cage. By Theorem 3.2, $Q = \begin{pmatrix} 54 & 81 \\ 54 & 88 \end{pmatrix}$ is a quotient matrix of the distance matrix of subdivision of Heawood graph, and its characteristic polynomial is $x^2 - 142x + 378$. So the distance spectral radius of subdivision of Heawood graph is $71 + \sqrt{4663}$. We also compute the distance characteristic polynomial of subdivision of Heawood graph, which is $(x^2 - 142x + 378)(x + 2)(x + 6)(x^4 + 20x^3 - 60x^2 - 80x + 112)^6$. So the D-spectrum of subdivision of Heawood graph is the union of \{71 + \sqrt{4663}, -2^8, -6\} and the set of roots of the polynomial $(x^4 + 20x^3 - 60x^2 - 80x + 112)^6$.

**Example 3.2.** We know that the Petersen graph is the minimal (3,5)-cage. By Theorem 3.3, $Q = \begin{pmatrix} 30 & 51 \\ 34 & 52 \end{pmatrix}$ is a quotient matrix of the distance matrix of subdivision of Petersen graph, and its characteristic polynomial is $x^2 - 82x - 174$. So the distance spectral radius of subdivision of Petersen graph is $41 + \sqrt{1855}$. We also compute the distance characteristic polynomial of subdivision of Petersen graph, which is $(x^2 - 82x - 174)(x^2 + 16x - 4)^5(x^2 - 2x - 4)^4(x + 2)^6$. So the D-spectrum of subdivision of minimal (3,5)-cage is
\[\{41 \pm \sqrt{1855}, (-8 \pm 2\sqrt{17})^{(5)}, (1 \pm \sqrt{5})^{(4)}, -2^{(5)}\}\]

In the next two theorems we find distance spectrum of subdivision of minimal \((k, 3)\)-cages (complete graphs \(K_{k,1}\)) and \((k, 4)\)-cages (complete bipartite graphs \(K_{k,k}\)). We denote the \(m \times n\) all one matrix by \(J_{m \times n}\) (or simply by \(J\) if its order is clear from the context) and an \(n\)-dimensional all one vector by \(1_n\).

**Theorem 3.4.** The distance spectrum of subdivision of a minimal \((k, 3)\)-cage is \(\{k^2 \pm 4(k+1)(2k+1), (\pm 2k)^{(k)}, 0((k+1)/2)\}\).

**Proof.** The block matrix representation of \(D(S(K_{k+1}))\) with respect to the bipartition \(V_1 \cup V_2\) of \(S(K_{k+1})\) is given by

\[
D(S(K_{k+1})) = \begin{bmatrix}
2(J_{k+1 \times k+1} - I_{k+1 \times k+1}) & 3J_{k+1 \times (k+1)} - 2R_{k+1 \times (k+1)} \\
3J_{(k+1) \times k+1} - 2R^T_{k+1 \times (k+1)} & 4J_{(k+1) \times (k+1)} - 2R^T_{(k+1) \times k+1} R_{k+1 \times (k+1)}
\end{bmatrix},
\]
where \(J\) is the all one matrix and \(R\) is the incidence matrix of \(\tilde{K}_{k+1}\). The adjacency spectrum of \(K_{k+1}\) is \(\{k, -1^{(k)}\}\). Let \(X\) be an eigenvector of \(A(K_{k+1})\) corresponding to the eigenvalue \(-1\). So \(X\) is orthogonal to the all one vector. Also by Lemma \(\ref{lemma1.6}\) \(RR^T = A(K_{k+1}) + kI\). Thus

\[
\begin{bmatrix}
2(J - I) & 3J - 2R \\
3J - 2R^T & 4J - 2R^T R
\end{bmatrix}
\begin{bmatrix}
X \\
R^T X
\end{bmatrix}
= \begin{bmatrix}
-2X - 2RR^T X \\
-2R^T X - 2R^T RR^T X
\end{bmatrix}
\]

\[
= \begin{bmatrix}
-2X - 2(k-1)X \\
-2R^T X - 2(k-1)R^T X
\end{bmatrix}
= -2k \begin{bmatrix}
X \\
R^T X
\end{bmatrix}.
\]

So \(-2k\) is an eigenvalue of \(D(S(K_{k+1}))\) with multiplicity \(k\).

Let \(Y\) be an eigenvector of \(J_{(k+1)/2 \times (k+1)}\) corresponding to the eigenvalue \(0\) with multiplicity \((k+1)/2\) – 1. Then \(Y\) is orthogonal to the all one vector. Now

\[
\begin{bmatrix}
2(J - I) & 3J - 2R \\
3J - 2R^T & 4J - 2R^T R
\end{bmatrix}
\begin{bmatrix}
RY \\
-RY
\end{bmatrix}
= \begin{bmatrix}
-2RY + 2RY \\
-2R^T RY + 2R^T RY
\end{bmatrix}
= 0 \begin{bmatrix}
RY \\
-RY
\end{bmatrix}.
\]

So \(0\) is an eigenvalue of \(D(S(K_{k+1}))\) with multiplicity \((k+1)/2\) – 1. Now the eigenvectors \(\begin{bmatrix}
X \\
R^T X
\end{bmatrix}\) and \(\begin{bmatrix}
RY \\
-RY
\end{bmatrix}\) of \(D(S(K_{k+1}))\) are orthogonal to \(\begin{bmatrix}1_{k+1} \\ 0\end{bmatrix}\) and \(\begin{bmatrix}0 \\ 1_{(k+1)/2}\end{bmatrix}\) respectively. So every other eigenvector \(Z\) of \(D(S(K_{k+1}))\) is of the
Thus $0$ is an eigenvalue of $D$, and hence the result.

**Theorem 3.5.** The distance spectrum of subdivision of a minimal $(k, 4)$-cage is \{\(2k^2 + k - 2 \pm \sqrt{(4k^4 - 2k^3 + 9k^2 - 12k + 4)}, 2k - 4, 0(\(k-1\)^2), -(k+2) + \sqrt{k^2 + 4}(2k-2), -(k+2) - \sqrt{k^2 + 4}(2k-2)\}\}.

**Proof.** We take the vertex partition of $S(K_{k,k})$ as $V_1 \cup V_2 \cup V_3$, where $(V_1, V_2)$ is bipartition of $K_{k,k}$ and $V_3$ is the set of all new vertices inserted on edges of $K_{k,k}$. The distance matrix of $S(K_{k,k})$ can be written as, $D(S(K_{k,k})) = \begin{bmatrix} 4(J_{k \times k} - I_{k \times k}) & 2J_{k \times k} & (3J_{k \times k} - 2I_{k \times k}) \otimes 1_k^T \\ 2J_{k \times k} & 4(J_{k \times k} - I_{k \times k}) & 1_k^T \otimes (3J_{k \times k} - 2I_{k \times k}) \\ (3J_{k \times k} - 2I_{k \times k}) \otimes 1_k & 1_k \otimes (3J_{k \times k} - 2I_{k \times k}) & 4(J_{k \times k} - I_{k \times k}) - 2A(L(K_{k,k}))(k-2 \times k) \end{bmatrix}$.

Adjacency matrix $A(K_{k,k})$ and incidence matrix $R$ of $K_{k,k}$ are \( \begin{bmatrix} 0 & J_{k \times k} \\ J_{k \times k} & 0 \end{bmatrix} \) and \( \begin{bmatrix} I_{k \times k} \otimes 1_k^T \\ 1_k^T \otimes I_{k \times k} \end{bmatrix} \) respectively. Thus $3J_{k \times k} - 2R_{k \times k} = \left[ (3J_{k \times k} - 2I_{k \times k}) \otimes 1_k^T \\ 1_k^T \otimes (3J_{k \times k} - 2I_{k \times k}) \right]$. Let $X$ be an eigenvector of $A(L(K_{k,k}))$ corresponding to the eigenvalue $-2$ with multiplicity $(k-1)^2$. Applying Lemma 1.7 we have $RX = 0$. So $X$ is orthogonal to the all one vector. Now

\[
\begin{bmatrix}
4(J - I) & 2J & (3J - 2I) \otimes 1_k^T \\
2J & 4(J - I) & 1_k^T \otimes (3J - 2I) \\
(3J - 2I) \otimes 1_k & 1_k \otimes (3J - 2I) & 4(J - I) - 2A(L(K_{k,k}))
\end{bmatrix}
\begin{bmatrix}
X \\
Y \\
Z
\end{bmatrix}
\]

\[
= \begin{bmatrix}
((3J - 2I) \otimes 1_k^T)X \\
(1_k^T \otimes (3J - 2I))X \\
-4X + 4Z
\end{bmatrix}
= \begin{bmatrix}
(3J - 2R)X \\
0 \\
0
\end{bmatrix}.
\]

Thus $0$ is an eigenvalue of $D(S(K_{k,k}))$ with multiplicity $(k-1)^2$.

Let $Z$ be an eigenvector of $A(K_{k,k})$ corresponding to the eigenvalue $0$ with multiplicity $2k - 2$. Also let $X'$ and $Y$ be vectors orthogonal to the all one vector $1_k$. If $\begin{bmatrix} X' \\ Y \\ Z \end{bmatrix}$ happens to be an eigenvector of $D(S(K_{k,k}))$ corresponding to the eigenvalue $(k+2) + \sqrt{k^2 + 4}(2k-2)$.
to an eigenvalue $\lambda$, then it must satisfy,

$$
\begin{bmatrix}
4(J - I) & 2J & (3J - 2I) \otimes 1_k^T \\
2J & 4(J - I) & 1_k^T \otimes (3J - 2I) \\
(3J - 2I) \otimes 1_k & 1_k \otimes (3J - 2I) & 4J - 2R^T R
\end{bmatrix}
\begin{bmatrix}
X' \\
Y \\
Z
\end{bmatrix}
= \lambda
\begin{bmatrix}
X' \\
Y \\
Z
\end{bmatrix}.
$$

This implies,

$$
-4X' + ((3J - 2I) \otimes 1_k^T)Z = \lambda X',
$$

$$
-4Y + (1_k^T \otimes (3J - 2I))Z = \lambda Y
$$

$$(3J - 2I) \otimes 1_k \{X' + \{1_k \otimes (3J - 2I)\}Y - 2R^T RZ = \lambda Z.
$$

Let $W = \begin{bmatrix} X' \\ Y \end{bmatrix}$. Combining the relations we get,

$$-4W + (3J - 2R)Z = \lambda Z,$$

$$(3J^T - 2R^T)W - 2R^T RZ = \lambda Z.
$$

From the first equation we get, $-4W - 2RZ = \lambda Z, \quad RZ = -\frac{1}{2}(\lambda + 4)W$.

Applying Lemma 1.6 in the second equation we get

$$-2RR^T W - 2RR^T RZ = \lambda RZ, \quad -2(A + kI)W - 2(A + kI)RZ = \lambda RZ,$$

$$-2kW + k(\lambda + 4)W = -\frac{1}{2}\lambda(\lambda + 4)W.$$

Thus $\lambda^2 + (2k + 4)\lambda + 4k = 0$. So $-(k + 2) \pm \sqrt{k^2 + 4}$ are two eigenvalues of

$D(S(K_{k,k}))$ with multiplicity $2k - 2$. Now $\begin{bmatrix} 0 \\ 0 \\ 1_k \end{bmatrix}$ is orthogonal to $\begin{bmatrix} 0 \\ 0 \\ 1_{k^2} \end{bmatrix}$ and

$$\begin{bmatrix} X' \\ Y \\ Z \end{bmatrix}$$

is orthogonal to both $\begin{bmatrix} 1_k \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1_k \\ 0 \end{bmatrix}$. So every other eigenvector $U$ of $D(S(K_{k,k}))$ is of the form $\begin{bmatrix} a1_k \\ b1_{k^2} \\ c1_{k^2} \end{bmatrix}, \quad a, b, c \neq 0$. Now $D(S(K_{k^2,k}))U = \mu U$ implies,

$$4(k - 1)a + 2kb + k(3k - 2)c = \mu a$$

$$2ka + 4(k - 1)b + k(3k - 2)c = \mu b$$

$$(3k - 2)a + (3k - 2)b + 4k(k - 1)c = \mu c.$$

Since $a, b, c \neq 0$, solving the above equations we get $\mu^3 - (4k^2 + 4k - 8)\mu^2 + (14k^3 - 28k^2 - 8k + 16)\mu - (12k^4 - 56k^3 + 80k^2 - 32k) = 0$. So $(\mu + 2k + 4)(\mu^2 - (4k^2 + 2k - 4)\mu + (6k^3 - 16k^2 + 8k) = 0$, and hence the result.  \[\square\]
Remark 3.1. Theorem 3.2 also gives that the distance spectral radius of subdivision of minimal \((k, 4)\)-cage is \(2k^2 + k - 2 + \sqrt{(4k^4 - 2k^3 + 9k^2 - 12k + 4)}\).

4 Concluding Remarks

It is known that a distance regular graph of diameter \(d\) has exactly \(d + 1\) distinct eigenvalues. However this is not the case for distance eigenvalues. The authors in [5] proved that every distance regular graph of diameter \(d\) has at the most \(d + 1\) distinct distance eigenvalues and asked for characterization of distance regular graphs which have exactly \(d + 1\) distinct distance eigenvalues. In this paper we proved that all minimal cages have exactly \(d + 1\) distinct distance eigenvalues. We also found distance spectral radius of DBR graphs and determined the full distance spectrum for some DBR graphs associated with minimal \((k, g)\)-cages. For the remaining, the following matrix representation of distance matrix of subdivision of a minimal \((k, g)\)-cage \(G\) may be useful. Here we consider \((V_1, V_2)\) as the vertex partition of subdivision of a minimal \((k, g)\)-cage \(G\), where \(V_1 = V(G)\) and \(V_2\) is the set of all new vertices inserted on edges of \(G\).

So for \(g\) even, \(D(S(G)) = \begin{bmatrix} 2D(G) & \frac{1}{2}D(G)R(G) \\ \frac{1}{2}R(G)^TD(G)^T & 2D(L(G)) \end{bmatrix}\), and for \(g\) odd, \(D(S(G)) = \begin{bmatrix} 2D(G) & \frac{1}{2}D(G)R(G) + E \\ \frac{1}{2}R(G)^TD(G)^T + ET & 2D(L(G)) \end{bmatrix}\), where \(R(G), D(G)\) and \(D(L(G))\) are incidence matrix, distance matrix and distance matrix of the line graph of \(G\) respectively. \(E\) is a matrix whose rows are indexed by vertices of \(G\) and columns are indexed by vertices on \(V_2\) and \((i, j)^{th}\) entry of \(E\) is 1 if \(d(v_i, u_j) = \max \{d(v, u) : v \in V_1, u \in V_2\}\) and 0 otherwise.

5 Acknowledgement

The first author is grateful to Council of Scientific and Industrial Research (CSIR), India [Grant number: 09/081(1283)/2016 – EMR – I], for funding the research.
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