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A window on infrared QCD with small expansion parameters

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Abstract

Lattice simulations of the QCD correlation functions in the Landau gauge have established two remarkable facts. First, the coupling constant in the gauge sector—defined, e.g., in the Taylor scheme—remains finite and moderate at all scales, suggesting that some kind of perturbative description should be valid down to infrared momenta. Second, the gluon propagator reaches a finite nonzero value at vanishing momentum, corresponding to a gluon screening mass. We review recent studies which aim at describing the long-distance properties of Landau gauge QCD by means of the perturbative Curci–Ferrari model. The latter is the simplest deformation of the Faddeev–Popov Lagrangian in the Landau gauge that includes a gluon screening mass at tree-level. There are, by now, strong evidences that this approach successfully describes many aspects of the infrared QCD dynamics. In particular, several correlation functions were computed at one- and two-loop orders and compared with ab-initio lattice simulations. The typical error is of the order of ten percent for a one-loop calculation and drops to few percents at two loops. We review such calculations in the quenched approximation as well as in the presence of dynamical quarks. In the latter case, the spontaneous breaking of the chiral symmetry requires to go beyond a coupling expansion but can still be described in a controlled approximation scheme in terms of small parameters. We also review applications of the approach to nonzero temperature and chemical potential.

Keywords: quantum chromodynamics, field theory, perturbation theory

(Some figures may appear in colour only in the online journal)

1. The ultraviolet Dr Jekyll and the infrared Mr Hyde

Since the discovery of asymptotic freedom in the early 70s [Pol73, GW73], a plethora of experimental and theoretical works have firmly established quantum chromodynamics (QCD) as the fundamental theory of strong interactions. QCD seems paradoxical at first sight, however, being a description of physically observable objects (the hadrons) in terms of unobservable ones (the quarks and the gluons). The former are the relevant excitations at length scales larger than about a Fermi whereas the latter appear to be...
the relevant degrees of freedom at smaller distances [Wei96]. As is well-known, the coexistence of these two, infrared (IR) and ultraviolet (UV) faces of the theory poses a major difficulty for the detailed understanding of the properties and interactions of hadrons. On the theory side, the dichotomy takes the form of an essentially perturbative regime at short distances and/or UV momenta versus a nonperturbative IR regime. This picture actually extends to a wide class of QCD-like theories with various quark contents, including the pure Yang–Mills (YM) case, with no dynamical quarks, known as the quenched limit. It is based, on the one hand, on the phenomenon of asymptotic freedom at UV scales and, on the other hand, on the fact that standard perturbation theory predicts its own failure at IR momenta in the form of a Landau pole, a finite energy scale \( \Lambda_{\text{QCD}} \approx 300 \text{ MeV} \) at which the coupling constant diverges.

Some essential aspects of the IR regime can be tackled from first principles using numerical simulations based on lattice gauge theory. In the QCD case, for instance, these simulations are able to reproduce the hadron spectrum with great accuracy using only the coupling constant and the quark masses as input parameters [FH12]. More generally, lattice simulations give strong theoretical support in favour of two fundamental phenomena at IR scales, namely confinement—the fact that the physical excitations of the theory are massive colourless objects—and the dynamical breaking of the chiral symmetry for theories with not too many light quark flavours, including QCD [D+14]. Lattice simulations also study the rich phase structure of such theories under extreme conditions of temperature [BFH+14, B+14] and density [Sex14, Sco16, GL16, AAJS16]—relevant for quark–gluon plasma physics in high-energy nuclear collisions, astrophysics of ultra compact stars, or early-Universe cosmology—with, in particular, the possibility of a confinement–deconfinement transition and of a restoration of the chiral symmetry, depending on the details of the quark content.

Although lattice simulations are by far the most powerful tool to explore the IR sector of such theories, they remain limited in at least two ways. First, they are based on the Monte Carlo sampling technique, which requires a positive definite measure of the (discretized) functional integral. They are thus essentially limited, so far, to the calculation of static quantities, that can be accessed from the Euclidean action when the latter is real. Typical dynamical quantities, which involve Minkowskian momenta (cross sections, transport coefficients, etc.), or cases where the Euclidean action is complex (for instance for some theories—including QCD—at nonzero chemical potential6) are still largely out of reach to present-day lattice technology. The second important limitation is that Monte Carlo simulations are, to a large extent, a black box from which it is not easy to pinpoint which are the fundamental phenomena at play. For instance, confinement has not revealed all of its mysteries yet and although many facets have been unravelled [Gre], a completely satisfactory description is yet to be found.

Alternatives to lattice methods typically involve functional quantum field theory techniques, such as Dyson–Schwinger equations (DSE) [vSAH97, AB98, Avs01, Lvs02, FA02, FP07, FMP09, CFM+16], the functional renormalisation group (FRG) [EHW98, PLNvs04, FG04, FMP09, CFM+16], and the variational Hamiltonian approach (VHA) [SLR06]. Although those, in principle, do not suffer from the limitations mentioned above for lattice simulations, they come with their own caveats. First, they necessarily rely on some approximations which are often difficult to control in a systematic way in the nonperturbative regime. Second, contrarily to lattice techniques, which can directly access the physical observables of the theory, the continuum approaches6 are formulated in terms of the quark and gluon degrees of freedom and it is often quite involved to extract information for physically observable quantities such as, e.g., hadron masses.

Having brushed this broad panorama, let us now draw the main contours of this review article. The above paradigmatic picture of a weakly coupled high-energy regime versus a strongly coupled one in the IR essentially relies on the Faddeev–Popov (FP) approach to perturbation theory [FP67], whose predictions are, however, in contradiction with some ab-initio results. In particular, the aforementioned Landau pole is most probably spurious. Indeed, numerical calculations of pure YM theories in the Landau gauge show that, in the UV, the coupling7 increases with decreasing momentum scale as predicted by the FP perturbation theory, but it remains finite at all scales and even decreases in the deep IR [BIMP09], see figure 1. Even more, one observes that the actual loop-expansion parameter—that is, in \( d = 4 \) dimensions, \( N_c \alpha_s/(4\pi) \), with \( N_c = 3 \) the number of colours—never exceeds moderate values, of the order of 0.3, thus potentially opening a novel perturbative way in the IR regime. This does not imply that the IR sector is fully perturbative—spontaneous chiral symmetry breaking is one among many examples of phenomena that is not captured by a coupling expansion at any finite order—but this indicates that it is neither genuinely nonperturbative and that, at least in the Landau gauge, some aspects of the IR dynamics may admit a perturbative description. This stunning observation has, surprisingly, never reached to the broad audience it deserves and remains largely unknown even to the QCD community. It is one goal of the present article to advertise it to its full merit and to discuss in detail its far-reaching consequences.

Because the FP theory predicts a Landau pole, an IR perturbative approach necessarily involves a modified starting point. One famous possibility is the Gribov–Zwanziger (GZ) quantization scheme [Gri78, Zwa89], which aims at tackling the Gribov–Singer ambiguity [Gri78, Sin78] that plaques the FP theory in the IR. The approach reviewed in this article follows a different, more phenomenological route which exploits

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5 This problem is usually known in the literature as the ‘sign problem’.

6 In this article, for simplicity, we shall use the denomination ‘continuum’ to encompass all theoretical approaches, except for lattice simulations.

7 To be precise, the present statements concern the Taylor coupling, which characterizes the ghost–antighost–gluon vertex at vanishing ghost momentum [Tay71]; see section 3.1. For a review of the strong coupling constant, see [BDdT16].
Figure 1. Lattice results for the Landau gauge Taylor coupling $\alpha_s(q^2)$ in the SU(3) YM theory. Reproduced from [BIMPS09]. CC BY 4.0.

Figure 2. Lattice results for the gluon propagator in the SU(3) YM theory in the Landau gauge. Reproduced from [BIMPS09]. CC BY 4.0. In this reference, $D(q^2)$ stands for the gluon propagator, denoted $G(q)$ in the present article.

another important result of lattice simulations in the Landau gauge, namely, the fact that the gluon propagator reaches a finite nonzero value at vanishing momentum, corresponding to a nonzero screening mass, as shown in figure 2. This massive-like behaviour has far-reaching consequences [Rob20], many of which have yet to be unravelled.

A simple deformation of the FP Lagrangian which accounts for this feature consists in adding a tree-level gluon mass term [TW10], which actually corresponds to a particular case of the Curci–Ferrari (CF) Lagrangian [CF76a].\footnote{A possible relation between the gluon mass term and the Gribov–Singer ambiguity problem has been investigated in [ST12, RSTT21].} Note that the gluon mass is introduced in a gauge-fixed setting and thus does not, per se, break the gauge invariance of the theory\footnote{The more subtle question of the Becchi–Rouet–Stora–Tyutin symmetry of the gauge-fixed theory is discussed in section 4.2.}. Among the interesting properties of this model let us mention (i) its perturbative renormalisability in four dimensions; (ii) the fact that one recovers the standard FP theory together with the associated phenomenology in the UV regime; and (iii) the key observation [TW11] that the mass term suffices to screen the IR divergences responsible for the Landau pole of the FP theory, thereby allowing for a well-defined perturbative expansion all the way from the UV to the deep IR.

In this context, the working hypothesis is that the CF model provides a good starting point for an efficient perturbative description of various IR aspects of QCD-like theories. This is to be viewed as an effective description, where essential aspects of the IR dynamics (in the Landau gauge) are efficiently captured by a simple gluon mass term—which has to be fitted against some (e.g., lattice or experimental) data—and where the residual interactions can then be treated perturbatively. This idea has been first put forward in [TW10, TW11], where the Euclidean ghost and gluon propagators of YM theories in the Landau gauge were computed at one-loop order in the CF model and successfully compared to lattice data for different gauge groups and different spacetime dimensions. This exciting observation has triggered systematic studies of modified (IR) perturbative descriptions based on the CF model. This includes the genuine CF model, viewed as a phenomenological proxy of a—yet to be found—IR completion of the FP Lagrangian [PTW13, RSTW14, PTW14, RSTW15b, RSTW15a, PTW15, RSTW16, RST15, RSTT17, PRS+17, RSTW17, GPRT19, PRS+21, Rei20, BPRW20, Ser20, BGPR21, HK19, HK20, Kon15, KWH+20, Web12, DW20, KS21, SBHK19, SK19], as well as the screened perturbation theory approach [Sir15a, Sir15b, Sir16b, Sir16a, Sir17, CS18, CS20], where the gluon mass term is added and subtracted to the FP Lagrangian and where the subtracted mass is treated, together with the coupling, in a perturbative expansion around the CF Lagrangian. A large number of quantities of physical interest have been computed, either in the vacuum or at nonzero temperature and density, including propagators, three-point vertex functions, phase diagrams, etc. at one-loop order and, by now, also at two-loop order in many cases [GPRT19, BPRW20, BGPR21]. The results compare very well with the available lattice data, thereby confirming the above perturbative picture. The approach is also used to investigate quantities which are not directly accessible with lattice techniques. The present article aims at summarising these results and advertising them to a broad audience.

The article is organised as follows. In section 2, we introduce our notations and discuss some aspects of the gauge fixing in nonabelian gauge theories that motivate the model studied in this review. Section 3 presents an overview of the status of the YM propagators in the Landau gauge. In section 4, we briefly review the CF model and in section 5, we present the results of perturbative calculations of propagators and three-point vertex functions of YM theories in this model at one- and two-loop orders and their comparison to lattice results. Similar calculations in the case of dynamical quarks are reviewed in section 6, including the case of light quarks, where the dynamical breaking of chiral symmetry plays a key role. We discuss the results of the perturbative CF model for the propagators...
and the phase diagram of the theory at nonzero temperature and chemical potential in section 7. Section 8 reviews some results in the Minkowskian domain and section 9 briefly discusses some of the open questions. Finally, we summarise and conclude in section 10.

2. QCD and gauge fixing

2.1. The QCD Lagrangian

The aim of this section is mainly to fix our notations. QCD is a field theory which involves fermionic (Dirac bispinor) fields $\psi$ for the quarks and a gluon field $A_{\mu}$ which takes values in the Lie algebra of the SU(3) gauge group. In this review, we mainly focus on Euclidean properties and it is therefore convenient to work with the Wick-rotated Lagrangian density $L$, which can be written as the sum of a pure gauge term $L_{YM}$ and a matter term $L_{\psi}$. The first one is the YM Lagrangian

$$L_{YM} = \frac{1}{2} \text{tr} F_{\mu \nu}^2,$$  

(1)

It involves the field strength $F_{\mu \nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} - ig_{s} [A_{\mu} , A_{\nu}]$, where $g_{s}$ is the bare coupling constant. The matter term involves a sum over the quark flavours:

$$L_{\psi} = \sum_{i=1}^{N_{f}} \bar{\psi}_{i} \left( D_{\mu} \psi_{i} + M_{b,i} \right) \psi_{i},$$  

(2)

where the covariant derivative acting on a fermion is $D_{\mu} \psi = (\partial_{\mu} - ig_{s} A_{\mu}) \psi$, the Feynman notation is used ($D = \gamma_{\mu} \partial_{\mu}$ with $\gamma_{\mu}$ the Euclidean Dirac matrices whose anticommutator is $\{ \gamma_{\mu} , \gamma_{\nu} \} = 2 \delta_{\mu \nu}$), and $M_{b,i}$ are the bare masses of the different quark flavours. For completeness, we recall that the covariant derivative acting on a field $X$ which takes values in the Lie algebra reads $D_{\mu} X = \partial_{\mu} X - ig_{s} [A_{\mu} , X]$.

QCD involves, in principle, six quark flavours. For the present applications, the three heavier ones, $c$, $b$, and $t$, can be safely neglected and one considers only the three lighter ones, $u$, $d$, and $s$. It is however, interesting, for methodological purposes, to consider other gauge groups, different number of quark flavours and even change the space (time) dimension. The QCD-like theories considered here are based on the gauge groups SU($N_{c}$) and involve $N_{f}$ quark flavours with various masses. The case $N_{f} = 0$ is interesting by its simplicity and because it is expected to exhibit several key properties of QCD. This pure YM or quenched limit can be viewed as QCD with all quark masses so large that there are no contribution from fermionic fluctuations.

Under a gauge transformation, the fields transform as

$$A_{\mu} \rightarrow A'_{\mu} = U A_{\mu} U^\dagger + \frac{i}{g_{s}} U \partial_{\mu} U^\dagger,$$  

(3)

where $U(x)$ is a field which takes value in the gauge group. This causes the field strength and the covariant derivatives to transform as $F_{\mu \nu} \rightarrow U F_{\mu \nu} U^\dagger$ and $D_{\mu} \psi \rightarrow U D_{\mu} \psi$, which in turn implies that the QCD Lagrangian is gauge invariant.

In actual calculations, it is often convenient to decompose a field $X$ taking values in the Lie algebra (such as the gluon field) on a basis $\{ r^{a} \}$ of the Lie algebra: $X = X^{a} r^{a}$. We choose the normalisation of the generators $r^{a}$ such that $\text{tr} r^{a} r^{b} = \frac{1}{2} \delta^{ab}$ so that the YM Lagrangian reads (sum over indices implicit)

$$L_{YM} = \frac{1}{4} (F_{\mu \nu}^{a})^{2}.$$  

(4)

In this basis, the components of the covariant derivative, $(D_{\mu} X)^{a} = \partial_{\mu} X^{a} + g_{s} f^{abc} A_{\mu}^{b} X^{c}$, and of the field strength, $F_{\mu \nu}^{a} = \partial_{\mu} A^{a}_{\nu} - \partial_{\nu} A^{a}_{\mu} + g_{s} f^{abc} A^{b}_{\mu} A^{c}_{\nu}$, involve the structure constants $f^{abc}$ of the gauge group.

2.2. Gauge fixing

Gauge invariance is a very powerful concept that highly constrains the possible physical theories but it also comes with important drawbacks. In particular, it implies that the propagator for the gluon field, one of the building blocks of continuum quantum field theory approaches, is not well defined. Technically, gauge invariance imposes that the second derivative of the YM action is transverse, $q_{\mu} \partial_{\nu} \text{YM}(q) / \partial q_{\mu} = 0$, and thus noninvertible, which makes apparent that there is no tree-level gluon propagator associated to the YM action.

To overcome this difficulty, the strategy used in virtually all continuum approaches consists in fixing the gauge. The underlying idea consists in dividing the space of all gauge configurations in equivalence classes, called gauge orbits: two gauge configurations belong to the same equivalence class if they are related by a gauge transformation. Gauge invariance stipulates that all the field configurations within a gauge orbit bear the same physical content. In the path integral version of quantum field theory, summing over all gauge field configurations is redundant and it is enough to retain one representative per gauge orbit. The procedure which consists in restricting the path integral to one field configuration per gauge orbit is called gauge fixing and the representative is chosen according to a given gauge condition. The Landau gauge, defined by the condition

$$\partial_{\mu} A^{a}_{\mu} = 0,$$  

(5)

is a very convenient and widely used choice, in particular, for what concerns nonperturbative approaches. This whole review concerns the Landau gauge\textsuperscript{10}.

In continuum approaches, the gauge-fixing procedure is, most often, implemented through the famous FP procedure [FP67]. It boils down to adding to the Lagrangian density a gauge-fixing part expressed in terms of ghost fields $c$ and $\bar{c}$ (which are Grassmann variables) and a Lagrange multiplier $h$ (known as the Nakanishi–Lautrup field). For the Landau gauge, it reads

$$L_{FP} = \partial_{\mu} \bar{c}^{a}(D_{\mu} c)^{a} + i h \bar{c}^{a} \partial_{\mu} A^{a}_{\mu}.$$  

(6)

The YM and FP actions are invariant under the Becchi–Rouet–Stora–Tyutin (BRST) symmetry [BRS75, CDM+18].

\textsuperscript{10}We mention that the screened perturbation theory approach has also been applied to the case of linear covariant gauges [SC18, Sir19b], which have recently also been investigated with lattice techniques [CMS09, BBC+15].
The BRST symmetry is a crucial property of the FP gauge fixing procedure which is heavily used to prove renormalisability and discuss the unitarity of the theory (we will come back on these issues below). It has several interesting properties. First, the symmetry is nonlinearly realized (the variations of the fields are not linear in the fields). It is actually a supersymmetry, which transforms bosonic fields to Grassmann ones, and reciprocally. This implies that \( s \) anticommutes with the Grassmann quantities. Finally, it is nilpotent, \( s^2 = 0 \).

Once the gauge is fixed, it becomes meaningful to compute averages of quantities which are not gauge invariant. In fact, the determination of physical observables in this context relies inevitably on the previous evaluation of such quantities: the correlation functions of the fundamental fields appearing in the Lagrangian. In the last two decades, an important activity has been devoted to characterize the basic QCD correlation functions by various methods, including lattice simulations. We stress though that it is by no means necessary to fix the gauge on the lattice in order to extract physical observables. Still, the calculation of correlation functions by means of gauge-fixed lattice simulations provides a very important insight.

The results obtained by lattice simulations are described in section 3.2, but before embarking on this discussion, let us recall why the Landau gauge is particularly useful in lattice simulations. In principle, fixing the gauge on the lattice requires to find the gauge transformation \( U \) such that the gauge constraint \( \partial_\mu A_\mu(x) = 0 \) is fulfilled. This represents a large set of nonlinear equations (there are \((N_c^2 - 1)\) such equations per lattice site) and this problem is highly nontrivial numerically. In the case of the Landau gauge, an alternative approach consists in extremising the functional [Wil80, MO87, MO90]

\[
    f[A, U] = \int d^4x \text{tr} \left[ A_\mu^c(x) A_\mu(x) \right]
\]

with respect to the gauge transformation \( U \) at fixed \( A \). It can easily been shown that such an extremum fulfils the Landau gauge condition. This extremisation problem is still quite intricate because \( f \) involves many variables but very powerful numerical methods are available if one restricts to local minima. In any case, it is way easier to address than the original root-finding problem.

### 2.3. The Gribov ambiguity

The general philosophy behind the idea of gauge fixing presented in section 2.2 suffers from a major issue. As first pointed out by Gribov [Gri78], the procedure of retaining one representative per gauge orbit is in practice more intricate than it may seem. Indeed, there exist distinct gauge-field configurations that fulfil the gauge condition (5) but that are the gauge transforms of one another. In other words, the functional (8) admits many extrema. These are called Gribov copies. Later on, Singer [Sin78] proved that this ambiguity is not restricted to the Landau gauge only but exists for a large class of gauge conditions. Moreover, Neuberger [Neu87, Neu90] studied the influence on the Gribov copies on the gauge-fixing property à la FP. He showed that, on a lattice of finite size, the FP procedure is ill defined because physical observables are given by an undetermined \( 0/0 \) ratio. The status of this gauge-fixing procedure is therefore questionable at a nonperturbative level.

In a first attempt to overcome this ambiguity, Gribov [Gri78] proposed to limit the domain of functional integration over the gluon field to what is now called the first Gribov region, characterised by minima of the functional (8). Later, Zwanziger [Zwa89] proposed a local, renormalisable field theory (the GZ theory) which implements this restriction of the domain of integration at the expense of adding a collection of auxiliary fields. It was clear from the seminal work of Gribov that the restriction to the first region successfully eliminates the infinitesimal Gribov copies, i.e., copies that are infinitesimally close to one another. Unfortunately, it was also pointed out that this may not be sufficient to fully resolve the Gribov ambiguity, that is, to completely fix the gauge. In fact, van Baal [vB92] proved that there exist Gribov copies within the first Gribov region. An unambiguous gauge fixing, called the absolute Landau gauge, would consist in restricting to gauge configurations belonging the fundamental modular region, which corresponds to absolute minima of the functional (8). This is however a very difficult numerical task and no continuum or lattice technique exists to implement this constraint in an efficient way. Nonetheless, some geometric characterisations of the fundamental modular region lead to the conclusion that constraining the path integral to such field configurations would modify correlation functions in the UV by exponentially small contributions \( \propto \exp(-a/g_b^2) \) with some constant \( a \). Taking into account the running of the coupling constant, it is therefore expected that Gribov copies have no role in the UV but may influence the IR properties of QCD.\(^{11}\)

Another strategy for tackling the Gribov ambiguity consists in summing over all Gribov copies—and by them minima, maxima or saddle points of the functional (8)—with a non-flat weight function that lifts the degeneracy between the (equivalent) Gribov copies and compensates their multiple counting in the path integral [ST12]; see also [STT14, STT15, Tis18, RSTT21]. By properly choosing the weight function, the gauge-fixed theory can be written in terms of a local, renormalisable action using standard auxiliary fields techniques.

Several other proposals have been put forward to address the Gribov issue in continuum approaches, none of which is completely satisfactory, and we refer the reader to the literature for a more detailed description [PIL90, Zwa90, FP91, Sch99, vSGW08, Maa10]. Finally, we stress that the Gribov–Singer ambiguity is easily resolved in lattice calculations, for instance by arbitrarily choosing only one of the numerous extrema.
Computing gauge-invariant quantities in continuum approaches remains a significant challenge, especially in the context of Yang–Mills correlation functions: previous semi-analytical methods and, in the following section, the results from lattice Monte-Carlo simulations. We first describe the results from the aforementioned Landau gauge, which has been the most studied in this context. We focus on the Landau gauge, which has been the most studied approach, as it heavily relies on the knowledge of correlation functions with a singular IR behaviour. The idea of ‘IR slavery’ and was based on the conjecture of a questionable at a nonperturbative level [Neu86, Neu87].

3. Yang–Mills correlation functions: previous results

3.1. Correlation functions from continuum approaches

The Landau gauge condition (5) imposes a transversality condition: any correlation function vanishes if the Lorentz index of an external gluon leg is contracted with the corresponding momentum. This drastically reduces the number of tensorial structures that may appear in a given correlation function. For instance, the gluon propagator in the Landau gauge is transverse with respect to the gluon momentum. In the Euclidean domain, it reads

$$G_{\mu\nu}(q) = \delta_{\mu\nu} \frac{1}{q^2},$$

where $G(q)$ is a scalar function of $q^2$. In general (e.g., in linear gauges), the gluon propagator also involves a longitudinal part, proportional to $P_{\mu\nu}^\perp = \delta_{\mu\nu} - q_{\mu}q_{\nu}/q^2$. The pioneering semi-analytical studies [Man79, BG80, BP88, BP89] for the IR behaviour of QCD were guided by the idea of ‘IR slavery’ and were based on the conjecture of a gluon propagator with a singular IR behaviour

$$G_{\text{IR slavery}}(q) \sim \frac{1}{q^\alpha},$$

that, in a one-gluon exchange approximation, could justify the existence of a confining linear potential between static quarks. This behaviour was indeed obtained as a solution of the DSE for the gluon propagator in a very simple approximation. However, the analysis of more elaborate truncations within the DSE, the FRG, and the VHA, including not only the gluon but, also, the ghost propagator, led to the discovery of solutions with a completely different IR behaviour [AvS01]

while the ghost propagator is more singular than the bare one in the IR regime [vSAH97, AB98, AvS01, Lvs02, FA02, PLNvS04, FG04, FP07, SLR06, HAFS08, FMP09, FP09, HlvS12, HvS13, QRH14, QR15, Hub16, CFM+16, Hub20],

$$G_{\text{IR}}(q) \sim \frac{1}{q^{2+\beta}},$$

with (in $d = 4$) $\beta = 1 + \frac{\alpha}{2}$. These two correlation functions behave as power laws in the long-distance regime, hence the name ‘scaling’ solution.12

Another class of solutions, referred to as ‘decoupling’, was identified some years later, where the gluon propagator saturates in the IR (it tends to a strictly positive constant at small momentum) and the ghost correlation function behaves just as its tree-level expression, up to nonsingular corrections

$$G_{\text{decoupling}}(q) \sim \frac{1}{q^2},$$

In the Landau gauge, the coupling constant can be defined as the ghost–antighost–gluon vertex at vanishing ghost momentum. This choice, called the Taylor scheme, is particularly interesting because it receives no loop corrections [Tay71]. As a consequence it is essentially determined from the ghost and gluon propagators:

$$\alpha_3(q) = \frac{g_0^2}{4\pi} D(q) F^2(q),$$

where we have introduced the gluon and the ghost dressing functions

$$D(q) = q^2 G(q) \quad \text{and} \quad F(q) = q^2 G_{\text{IR}}(q),$$

and where $g_0$ is the renormalised coupling defined at the same renormalisation point as $D(q)$ and $F(q)$. Both the scaling and the decoupling solutions show a regular coupling constant in the IR, which tends to a constant in the former case and decreases to zero in the latter.

The IR behaviour of correlation functions has also been studied at leading order in the GZ approach. The early implementations of the original GZ model gave a scaling solution [Gri78, Zwa89, Zwa94]. However, it was soon realized that nontivial condensates can naturally appear in this model, which, when included in a refined version of the model [DGS+08, VZ12], led to a decoupling solution. An important property of the scaling and the decoupling solutions is that they are both incompatible with the existence of a Källén–Lehmann representation with a positive spectral density for transverse gluons [AvS01, CMT05, CFM+16].

12 A nonsingular solution for the gluon propagator was observed even before in [EHW98] but the authors considered this behaviour as an artefact of their approximations.
It was argued that such positivity violations are somehow related to confinement and to the presence of negative norm states which cannot appear in the physical spectrum of a unitary theory [AvS01]. This will be discussed in more detail in section 5.3.

These positivity violations were then observed in first-principle Monte-Carlo simulations [CMT03, CMT05, CM08b, BHL+07, Maa07, BIMPS09] and there is no doubt of their existence. However, their interpretation is far from settled. In particular, it is clear that a nonpathological model with negative norm states must necessarily include some form of confinement. That is, there must exist, within the set of possible states, a subspace of physical states with positive definite norm with an $S$-matrix that involves only such states. This effective decoupling of negative (and null) norm states must occur either because there is some set of symmetries that allows to characterise a physically acceptable subspace and/or due to some selection rule of purely dynamical origin. Proving the existence of a physical space with these characteristics and the unitarity of the $S$-matrix in this space is as hard as proving confinement. What is clear, in the present state of affairs, is that, just as positivity violations cannot be invoked as an indication of confinement, neither can a model be ruled out on the sole basis of their existence. It could happen that we simply do not know the true physical subspace with positive norm and unitary $S$-matrix.

Both the scaling and decoupling solutions unveiled a remarkable surprise: far from being very strong (not to mention with Landau pole-type singularities), the featured correlations stay modest in the IR. Surprisingly, these results were initially received with some indifference by the QCD community. This may be due to various reasons. First, initial studies focussed on correlations functions which are not directly physical observables. Second, these solutions were found on the basis of approximation schemes that did not rely on a small parameter that would ensure their robustness. In fact, these were considered nonperturbative but the criterion used to retain or neglect a vertex was similar to what would be done in perturbation theory. Typically, two-point correlation functions were initially computed with either bare three- or four-point vertices or with dressed expressions based on educated guesses. Higher-order vertices were systematically neglected. More recently, richer approximations have been considered and the results have shown to be reasonably robust (see, for example, [Hub20]) but, again, the vertices of order greater than four have always been neglected, which still bears similarities in spirit to a higher-order perturbative analysis.

At this point, we mention that a plausible simple explanation of the success of approximations which, although considered as nonperturbative, closely resemble perturbation theory is the existence of a relatively moderate coupling constant. This idea actually underlies the work reviewed in this article.

### 3.2. Lattice results: propagators and three-point vertices

The development of the semi-analytical methods presented in the previous section stimulated an important activity in computing correlation functions by means of lattice Monte-Carlo techniques. Using the gauge-fixing procedure described in section 2.2, various two- and three-point correlation functions have been simulated in YM theory [MO87, BBLW00, BBL+01, CM08a, BIMPS09, BMMP10, ISI09, BLY+12, Maa13, OS12] and in QCD [BHL+04, BHL+05, SO10] (see section 6). Once the gauge has been properly fixed, the computation of gluon correlators is straightforward. It is also possible to compute correlation functions involving ghosts, assuming that the terms in the gauge-fixed action that depend on these fields have the form of the FP Lagrangian. The (Gaussian) functional integration over the ghost field can be carried out exactly, which results in a nonlocal functional measure in the gauge fields. An explicit expression given by Wick’s theorem for Grassmann variables is obtained that involves products of the inverse of the FP operator times the determinant of that same operator. The same holds for correlators involving quark fields.

The results of lattice simulations have confirmed the essential features obtained from the continuum approaches, in particular, the absence of strong correlations in the IR. Moreover, they allowed to resolve, on a first-principle basis, the controversy about the scaling versus decoupling solutions in YM or in QCD.

Simulations of the gluon and ghost two-point correlators have become very precise and clearly show a decoupling type solution in $d = 3$ and $d = 4$ dimensions both for pure YM and for QCD [BBLW00, BBL+01, CM08a, BIMPS09, BMMP10, ISI09, Maa13, OS12]. This is illustrated in figures 1–3, which show the lattice results of [BIMPS09] for the Taylor coupling (15), the gluon propagator, and the ghost dressing function in $d = 4$ for the SU(3) YM theory. Concomitantly to the saturation of the gluon propagator at small momenta, the ghost propagator behaves as that of a massless excitation in this limit, that is, the dressing function remains finite in the deep IR. Simulations have also been made for other values of the number of colours $N_c$ and of the number of quark flavours $N_f$ with similar results. In $d = 2$ dimensions, instead, the lattice results show a scaling solution [CM08a, Maa13]. Furthermore, as already mentioned, the detailed analysis of the gluon propagator reveals, with no ambiguity, that, if it admits a Källén–Lehmann representation, then the spectral function cannot be positive definite [CMT05, BHL+07] (see section 5.3 for a detailed discussion of this point).

As already mentioned, the relevant perturbative expansion parameter is not $\alpha_S(q)$ but, in $d = 4$, [Wei95]

$$\lambda(q) = \frac{N_c g^2(q)}{16\pi^2} = \frac{N_c \alpha_S(q)}{4\pi}. \quad (17)$$

Since the Taylor coupling $\alpha_S(q)$ never exceeds 1.3, see figure 1, the expansion parameter $\lambda(q)$ is bounded by 0.3. This is a remarkable observation. In the absence of any prejudice, it indicates that some sort of perturbation theory should apply in the IR, at least in the SU(3) YM theory. In the context of YM theories and QCD, this natural conclusion, however, came at odds with the common wisdom of a genuinely nonperturbative

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13 Similarly, the coupling is bounded in $d = 3$ and $d = 2$, with a moderate although slightly larger value in $d = 3$ than in $d = 4$ and significantly larger value in $d = 2$. 

IR regime and has remained largely unknown. This is the seed of a new paradigm.

Three-point correlators have also been simulated on the lattice and bring important information despite the fact that the statistical and systematic errors are much larger than for the propagators. One of the most important results is that the various coupling constants that can be extracted from the ghost–gluon vertex (in various configurations of momenta) take, again, moderate values for all momenta, including the IR limit, where they typically show a slow decrease towards zero [IMPS+07, Ste06, CMM08, BIMP09, BLY+12, Maa20, ADSS+21]. Remarkably, the smallness of the couplings is compatible with a perturbative expansion. These observations include, as a particular case, the configuration of momenta corresponding to the Taylor scheme shown in figure 1.

As for the three-gluon vertex, lattice results clearly show a ‘zero-crossing’ in $d = 3$ [CMM08, MV20] and, although not as clearly, also in $d = 4$ dimensions [CMM08, BBDS+14, BDSRQZ17, SBK+17, MV20]. That is, for the various configurations of momenta studied, there always seems to be a sufficiently small momentum at which the correlation function changes sign and becomes negative. In the regime in which all momenta tend to zero the correlation function seems to diverge towards minus infinity for $d = 3$. However, the coupling constant that can be extracted from this vertex also remains rather small for all momenta.\footnote{The couplings constant must be extracted, in general, from combinations of three- or four-point functions and propagators.}

Lattice simulations of correlators involving quarks have been performed both in the quenched limit $N_f = 0$ and in the presence of dynamical quarks $N_f > 0$ [BHL+04, BHL+05, SMMvPS12, OSSS19]. The most important result observed in these simulations is that the quark propagator shows a significant dynamical mass generation, which is a signature of the spontaneous breaking of the chiral symmetry. That is, even in cases where the running mass is very small at the microscopic level (of the order of a few MeV), its zero momentum limit—the constituent mass—is of the order of several 100 MeV.

Finally, the quark–gluon vertex has been simulated in [SK02, SBK+03, SKB+04, KLSS07]. Among the more striking results, the associated coupling constant can get up to two-to-three times larger than the one in the pure gauge sector. This has far-reaching consequences as it indicates that, unlike the YM sector, the dynamics of light quarks is strongly coupled. This is consistent with the early observation [AJ88] that spontaneous chiral symmetry breaking requires a sufficiently large quark–gluon coupling in the IR.

4. The Curci–Ferrari model

One of the striking features of the studies reported in the previous section is the saturation of the gluon propagator in the IR, which shows the dynamical generation (in the Landau gauge) of what is called a screening mass—not to be confused with a pole mass, as discussed in section 4.3 below. This phenomenon is, by now, well established. As already mentioned, the FP perturbation theory is unable to describe the generation of this screening mass and the simplest deformation of the FP Lagrangian that includes it is a particular case of a class of Lagrangians known as the CF Lagrangians. We review the latter and their main properties, including symmetries and renormalisability, in the present section.

4.1. The Curci–Ferrari Lagrangian

In the 70s, Curci and Ferrari proposed an alternative to the Higgs mechanism that would provide a consistent theory of massive vector bosons in the presence of nonabelian symmetries [CF76a]. Although this original motivation has been abandoned (mainly for reasons related to the issue of unitarity, see section 4.3 below), the model has received a renewed interest in the context of IR QCD [TW10, TW11]. In this section, we consider the original model, which goes beyond the case of the Landau gauge, for completeness. The Lagrangian density reads\footnote{In their original article [CF76a], Curci and Ferrari considered a more general model. We will limit ourselves here to the subset of parameters that are compatible with a massive renormalisable model.}

$$\mathcal{L} = \mathcal{L}_\mathrm{YM} + \mathcal{L}_\mathrm{CFDJ} + \mathcal{L}_m, \quad (18)$$

where $\mathcal{L}_\mathrm{YM}$ is the YM Lagrangian density (1) and where the gauge-fixing and mass contributions read, respectively,

$$\mathcal{L}_\mathrm{CFDJ} = \frac{1}{2} \partial_\mu D_\mu D_\nu c^\nu + \frac{1}{8} \langle D_\mu c \rangle^2 \partial_\mu c^\mu + \frac{\xi_s}{2} h^2 \bar{c} c,$$

$$+ \frac{\xi_s}{8} \langle f^{abc} \rangle \bar{c} \gamma^\mu \gamma^5 c^\mu c^b,$$ \quad (19)

and

$$\mathcal{L}_m = m_b \left[ \frac{1}{2} \langle A_\mu^b \rangle^2 + \xi_s \bar{c} c^b \right]. \quad (20)$$
Here, $g_b$, $\xi_b$, and $m_b$ are the bare coupling, gauge-fixing parameter, and gauge boson mass, respectively. The Lagrangian (19) is the first example in the literature of a nonlinear gauge fixing à la FP, sometimes referred to as the Curci–Ferrari–Delbourgo–Jarvis gauge [DJ82]. An important technical aspect is that the mass term (20) is introduced at tree level in a gauge-fixed version of the Lagrangian density, which, notably, guarantees its perturbative renormalisability (see section 4.2). The principal result is that the level gluon propagator $G_0(p)$ decreases as $1/p^2$ at large momentum. This is in contrast with what happens when a gauge boson mass is directly added to the YM Lagrangian, where $G_0(p) \sim \text{const}$. We present here the Euclidean version of the model with the notations that are most commonly used nowadays and which differ slightly from those originally employed in [CF76a]. In particular, we consider the model in the presence of a Nakanishi–Lautrup field $h^a$ that simplifies the writing of the (modified) BRST symmetry (see section 4.2). We do not consider matter fields for now, but their inclusion is straightforward, see section 6. The main interest of the CF Lagrangian in the form (19) is that the ghost–antighost exchange symmetry is simple and that it preserves the linear realization of some continuous symmetries [CF76b, DJ82, TW09]. This is not the case in the nonsymmetric version of the model:

$$L_{\text{GF}}^{\text{ns}} = \frac{i}{2} \bar{c} \gamma^\mu \partial_\mu c + \frac{i}{2} \bar{G} \gamma^\mu \partial_\mu G + i \epsilon^\mu_\nu \bar{a} D_\nu a - \frac{i}{2} \bar{\epsilon} g^{abc} \epsilon^b \epsilon^c - \xi \frac{Z_a}{Z_c} (f^{abc} \epsilon^b \epsilon^c)^2,$$

which is obtained from equation (19) by the field redefinition $i h^a \to i h^a + \frac{1}{2} f^{abc} \epsilon^b \epsilon^c$ and which proves more convenient for actual calculations. The case $\xi = 0$, where the ghost mass and the four-ghost interaction are absent, corresponds to the simple massive extension of the Landau gauge considered in this review.

4.2. Symmetries and renormalisability

In their original work, Curci and Ferrari observed that the Lagrangian (21) is invariant under the following generalisation of the—at the time recently discovered—BRST transformation:

$$sA^a_\mu = (D_\mu c)^a, \quad sc^a = -\frac{\xi}{2} f^{abc} c^b \epsilon^c,$$
$$s \epsilon^a = i h^a, \quad s(i h^a) = m_b \epsilon^a.$$

In the massless case, $m_b = 0$, the symmetry (22) is the standard nilpotent BRST symmetry, see equation (7). In the massive case, the Lagrangian (19) can be seen as a deformation of the standard FP Lagrangian and the transformation (22) as the corresponding deformation of the BRST symmetry. These deformations modify the behaviour of the model for momenta comparable to or smaller than the renormalised gluon mass.

In [CF76a], Curci and Ferrari proved the renormalisability of the theory by making use of the symmetries of the model, in particular, the modified BRST symmetry (22). Later, de Boer et al [dBSvNW96] computed the renormalisation factors at one-loop order, considering the five renormalisation factors

$$A^{\mu a} = \sqrt{Z_a} A_R^{\mu a}, \quad c^a = \sqrt{Z_c} c_R^a, \quad \epsilon^a = \sqrt{Z_\epsilon} \epsilon_R^a,$$
$$g_b = Z_g g, \quad m_b^2 = Z_m m^2, \quad \xi_b = \frac{Z_\xi}{Z_\epsilon} \xi$$

as independent. The symmetries of the model imply that the divergent part of the renormalisation factors are constrained by the relations

$$\sqrt{Z_a} Z_b = Z_2^2, \quad Z_a Z_{a_2} = Z_2^2.$$

This reduces the number of independent renormalisation factors to three, which can all be extracted only from the two-point correlation functions. The first relation generalizes Taylor’s nonrenormalisation theorem [Tay71], previously known in the particular case of the standard Landau gauge ($\xi_b = 0, m_b = 0$). The constraints (24), first conjectured in [BG02, Gra03], have been proven to all orders of perturbation theory [Wsc08] and were, in fact, shown to be direct consequences of gauged supersymmetries of the Lagrangian (20) [TW09]. The Landau gauge condition $\xi = 0$ is stable under renormalisation, i.e., $Z(\xi = 0) = 1$ to all orders of perturbation theory$^{16}$, so the number of independent renormalisation factors is further reduced to two.

4.3. Unitarity

Despite the interesting properties described above, the CF model was soon discarded as a pertinent description of a massive Higgs boson because of issues related with unitarity. The aim of this section is to give a critical overview of this topic and discuss the possibility of a unitary CF model for describing strong interactions. We first recall the standard proof of unitarity, in the framework of the (massless) FP gauge fixing [BRS76, BRS75, 1975, KO78b, KO79a].

One of the main goals of a field theory is to describe the scattering amplitudes between in- and out-states. In the simple cases (e.g. for the $\phi^4$ theory), the space of in-states is obtained by applying the creation operator $a_\phi(\vec{q})$ to the vacuum. It can be checked that these states have a positive norm, a necessary property for the state space to be a bona fide Hilbert space of a quantum theory. The situation is more involved when considering a (Lorentz-covariant) gauge-fixed field theory because some of the states obtained in this procedure have negative norm (this is the case of gluons with a polarization in the time direction and of the ghosts). Such a field theory makes sense at a quantum level only if one can define a subspace (called the physical subspace), which (a) contains only states with a positive norm and (b) is stable under the time evolution. Within this physical subspace, the

\footnote{For $\xi = 0$, the FP Lagrangian equation (19) is invariant under $c \to c + Cst$. The four-ghost interaction proportional to $\xi_b$ breaks this symmetry. This is at the heart of the nonrenormalisation theorem $Z(\xi = 0) = 1$.}
theory has all the necessary ingredients to represent a quantum theory.

In order to impose condition (b), a natural idea is to characterize the physical subspace by using symmetry arguments. The standard strategy consists in considering the kernel of the BRST symmetry. One however finds by inspection that there exist states with null norm, all of them belonging to the image of BRST [KO78b, KO78a, KO79b]. To account for these states, one defines the physical space as the cohomology of BRST, that is, the kernel of the BRST transformation modulo any element in the image. One must then explicitly check that the cohomology only involves states of positive norm, hence ensuring property (a). It has been shown to be the case at all orders of perturbation theory for the FP Lagrangian. This textbook construction is, however, not completely satisfactory for QCD because, as such, the physical subspace would contain coloured states, while confinement implies that only colour-neutral states should appear as acceptable states. An extra restriction, yet to be uncovered, should be used to define a physical subspace with only hadronic states.

How could this discussion be adapted to the CF model? A first difficulty is that the BRST symmetry is not nilpotent anymore (s² ≠ 0). It is, nevertheless, possible to work in the kernel of s², which is a symmetry of the CF action. In this subspace, the BRST symmetry is again nilpotent and we can adapt the procedure described above. There is, however, a more thorny issue: it has been shown that this subspace contains negative-norm states [Oji82, dBSvNW96]. On this basis, the CF model was discarded as being non-unitary. This conclusion is valid if the gauge field is associated with an observable particle (e.g., in the context of weak interactions). However, it should be mitigated if one considers the CF model as a theory for strong interactions. Indeed, it could very well be that the CF model is, in fact, confining. More precisely, there could exist a subspace, yet to be found, in which properties (a) and (b) listed above hold. We recall that, even in the standard FP case, the construction of a satisfactory physical subspace, composed of singlet states, is yet to be built and it is conceivable that a fix to this issue could also resolve the unitarity problem in the CF model. To date, the issue is still open and the CF model cannot be discarded on the basis of unitarity arguments.

The issue of unitarity discussed in this section has strong connections with the property of positivity violation discussed above. Indeed, a theory with only positive norm states would admit a Källén–Lehmann representation with a positive spectral density. Positivity violation for the transverse gluons indicates that (at least some of) these modes are unphysical and should be removed from the physical subspace.

4.4. A minimal deformation of the Faddeev–Popov Lagrangian

To conclude this section, we stress that the CF model in the Landau gauge is the simplest extension of the FP Lagrangian that:

- Preserves standard perturbation theory in the UV, in particular, remains renormalisable;
- Maintains the linearly realised symmetries of the FP Lagrangian;
- Has the same field content as the FP Lagrangian;
- Renounces to the standard, nilpotent BRST symmetry (that, anyway, seems to be broken beyond perturbation theory [Gri78, Sin78, Neu87]).

Indeed, in order to keep standard perturbation theory and linearly realised symmetries in the UV, the strictly renormalisable couplings must be identical to those of the FP Lagrangian and the only admissible modifications involve couplings with positive mass dimensions. Such deformations, which do not modify the UV, are called ‘soft’. In the Landau gauge, the only possibility is the gluon mass term. In this sense, the Landau gauge CF model is the minimal extension of the FP approach with a soft breaking of the BRST symmetry in the IR.

Here, we want to warn the reader against a common misinterpretation of the CF model in the QCD context. The gluon mass term is not meant as an explicit modification of the theory—as the resemblance of the FP and CF Lagrangians might wrongly suggest—but, rather, as an effective way to capture actual features of Landau gauge QCD that are missed by the FP perturbative approach. Although such an effective deformation of the gauge-fixed Lagrangian may induce actual modifications of the original theory, the latter must remain under control if the model is to be a good description. In the remainder of this article, we review large pieces of evidence demonstrating that the CF model indeed captures many features of the YM and QCD-like theories at a relatively low computational cost.

5. Yang–Mills correlation functions in the vacuum

The working hypothesis underlying the use of the CF model in the Landau gauge is that it provides an efficient starting point for a reliable and controllable perturbative approach to the IR dynamics of the YM fields. This hypothesis has been put to test, by now, in a large number of cases, by comparing the results of actual perturbative calculations at one- and two-loop orders to lattice data in the Landau gauge, when available. The present section reviews these results in the case of the YM correlation functions in the vacuum.

Before we proceed, a word of caution is in order, which applies in fact to the remainder of the review. Our primary aim is to review those works in the literature which actually postulate perturbation theory in the IR. The main line of investigation concerns a genuine perturbative expansion within the

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17 Note that, since the standard BRST symmetry is nilpotent, s² = 0, the image of BRST belongs to the kernel.

18 The proof consists in considering the axial gauge which is not Lorentz covariant but for which both the positivity of the state space and unitarity are explicit; see e.g. [Wei96].

19 An attempt in this direction has been undertaken in the context of the GZ model [SZ14].

20 Of course, the class of possible soft deformations is much larger if the field content of the model is enlarged as, e.g., in the GZ approach.
The CF model [TW10, TW11, Kon15, HK19, Web12, SBHK19, SK19]. The latter is considered either as an actual candidate or as a proxy for an IR completion of the FP theory. In the former case, the gluon mass term is expected to be either related to the issue of gauge fixing and of the Gribov problem, or it is to be eventually self-consistently determined. In the latter case, the gluon mass is simply an additional, phenomenological input parameter which encodes some unknown aspects of the IR physics.

Another approach [Sir16b] that revives the original screened perturbation theory [KPP97] of finite temperature field theory and extends it to the vacuum case, postulates the validity of the FP Lagrangian at all scales and uses the CF Lagrangian simply as a shifted expansion point for perturbation theory. In this case, one adds and subtracts a gluon mass term and formally treats the subtracted mass together with the coupling in a (double) perturbative expansion around the CF Lagrangian. So, although they rely on different hypothesis, the two approaches appear very similar in practice and they actually give similar results when it comes to comparing with lattice data. In order to avoid confusion in the present review, we focus on the strict perturbative CF model and we mention the results of the screened perturbation theory when appropriate.

5.1. Renormalisation and renormalisation group

As discussed in section 4.2, the CF model is renormalisable in $d \leq 4$. The divergent parts of the renormalisation factors in $d = 4$ dimensions are constrained by the two nonrenormalisation theorems (24), where $Z_4$ can be set to 1 in the case of the Landau gauge. To fully determine the renormalisation factors and, in particular, their finite parts, it is necessary to choose a renormalisation scheme. In order to unify the presentation as much as possible in this review, we choose to focus on results obtained within a single scheme. We mention, though, that other schemes have been considered as well and that the scheme dependence of the results has been—in some cases thoroughly—investigated [TW11, Web12, DW20]. As noted previously, one important feature of the CF model is that one can devise IR-safe schemes, for which the flow is regular at all scales [TW11, Web12, DW20]. We choose the one such scheme for which most of the existing CF model calculations have been performed, namely, the one originally proposed in [TW11], which we shall refer to as the IRS scheme for simplicity. It is defined by the conditions

$$G^{-1}(p = \mu) = m^2 + \mu^2, \quad F(p = \mu) = 1,$$

$$Z_\gamma \sqrt{Z_G Z_c} = 1, \quad Z_{\gamma^2} Z_G Z_c = 1,$$  \hspace{1cm} (25)

where $G(p)$ and $F(p)$ are the (renormalised) scalar part of the gluon propagator and the ghost dressing function, defined in equations (9) and (16), respectively. Note that the two last conditions in equation (25) apply to both the divergent parts and the finite parts of the renormalisation factors. It is interesting to note that these renormalisation conditions can be seen as a sort of momentum subtraction scheme (MOM), where one fixes the values of various correlation functions at a given momentum [HH80]. Indeed, the Taylor condition for $Z_4$ actually corresponds to fixing the ghost–gluon vertex at zero ghost momentum to its bare value [Tay71]. Similarly, the conditions for $Z_8$ and $Z_9$ obviously correspond to fixing the values of the ghost and the transverse gluon self-energies, respectively. As for the factor $Z_{\gamma^2}$, the last condition (25) is, in fact, equivalent to fixing the value of the longitudinal gluon self-energy at the scale $\mu$ [Web12]. There are two subtle differences with standard MOM schemes though. First, we take the renormalisation scale at an arbitrary point in order to implement an RG procedure, as opposed to the fixed renormalisation point used, say, in lattice studies. Second, having introduced a mass, the renormalisation condition for $Z_8$ is taken compatible with the tree-level expression of a massive propagator. This differs from the standard MOM condition by a trivial multiplicative factor $\mu^2/(\mu^2 + m^2)$ in the transverse gluon propagator.

In order to correctly describe the logarithmic UV tails of the correlation functions, it is important to take into account renormalisation group (RG) effects. Indeed, although the coupling stays limited, its running is significative. We thus define the standard RG beta functions and anomalous dimensions as

$$\beta_\gamma = \frac{\text{d}g}{\text{d} \ln \mu} \bigg|_{g, m_b^2}, \quad \beta_{m^2} = \frac{\text{d} m^2}{\text{d} \ln \mu} \bigg|_{g, m_b^2},$$

$$\gamma_A = \frac{\text{d} Z_A}{\text{d} \ln \mu} \bigg|_{g, m_b^2}, \quad \gamma_c = \frac{\text{d} \ln Z_c}{\text{d} \ln \mu} \bigg|_{g, m_b^2},$$  \hspace{1cm} (26)

where the $\mu$-derivatives are taken at fixed bare parameters. The renormalisation conditions (25) imply that the beta functions can be fixed in terms of the ghost and gluon anomalous dimensions as [TW11]

$$\beta_\gamma = g \left( \frac{\gamma_A}{2} + \gamma_c \right) \quad \text{and} \quad \beta_{m^2} = m^2 \left( \gamma_A + \gamma_c \right).$$  \hspace{1cm} (27)

The two-point correlation functions in the YM theory have been calculated in this model not only at one loop [TW10, TW11] but also at two loops [GPR17]. An important point to be mentioned is that due to the gluon mass the model gives a well-behaved perturbative expansion. Not only is the model renormalisable but it also features no IR divergences for non-exceptional configurations of Euclidean momenta at all orders of perturbation theory (this applies to any vertex function for any $d > 2$, see appendix B in [TW11]). This result is
not entirely trivial due to the presence of massless ghosts modes. In the case of exceptional configurations of Euclidean momenta, some IR divergences are present [TW11]. The case of Minkowskian momenta would require a separate analysis. The one-loop calculations are easily performed and already exhibit the main nontrivial features of the model [TW11, RSTW17]. In the IRS scheme (25), the one-loop anomalous dimensions read, for $d = 4$,

$$\gamma_c = -\frac{g^2 N_c}{32 \pi^4 t^4} \left[2(t+1)t - t^2 \ln t + (t+1)^2(t-2) \ln(t+1)\right],$$

(28)

$$\gamma_\lambda = -\frac{g^2 N_c}{96 \pi^4 t^3} \left[(17t^2 - 74t + 12)t - t^2 \ln tight.$$

$$+ (t-2)^2(2t-3)(t+1)^2 \ln(t+1) + (t^2 - 9t^2 + 20t - 36) \ln \frac{\sqrt{1+t} - \sqrt{1+t+4}}{\sqrt{1+t} + \sqrt{1+t+4}}\right].$$

(29)

where $t = \mu^2/m^2$. The two-loop calculation is way more involved and requires the use of symbolic programing [GPRT19]. The RG flow is obtained by integrating the beta functions (27) with initial conditions $m_0 = m(\mu_0)$ and $\gamma_0 = g(\mu_0)$ at a given scale $\mu_0$. The $d = 4$ one-loop flow is shown in figure 4 in terms of the dimensionless mass $m^2 = m^2/\mu^2 = 1/t$ and of the coupling (17). In the UV regime, $t \gg 1$, the beta functions behave as

$$\frac{\beta_t}{g} \sim \frac{11 N_c g^2}{3 \pi^2} + O(1/t) \quad \text{and} \quad \frac{\beta_{\gamma_\lambda}}{m^2} \sim \frac{-35 N_c g^2}{6 \pi^2} + O(1/t).$$

(30)

All relevant trajectories start from the Gaussian fixed point $g = t^{-1} = 0$ which is attractive in the UV. As anticipated, asymptotic freedom is recovered and the one-loop beta function for the coupling constant takes its universal form. As one flows towards the IR, three types of trajectories are observed. For a given coupling constant in the UV, if the gluon mass is large enough, the flow is driven towards a fully attractive IR fixed point, the vicinity of which is characterized by

$$\frac{\beta_t}{g} \sim \frac{1 N_c g^2}{6 \pi^2} + O(t) \quad \text{and} \quad \frac{\beta_{\gamma_\lambda}}{m^2} \sim \frac{1 N_c g^2}{3 \pi^2} + O(t).$$

(31)

In this case, the flow is IR safe (there is no Landau pole) and the propagators are regular for arbitrary momentum. As will be seen later, the trajectories that correctly describe the data from the numerical simulations are of this type. These trajectories correspond to the decoupling solutions described in section 3.1. On the other hand, if the gluon mass is taken below a certain threshold the RG flow is singular and presents a Landau pole 24. There is a limiting trajectory which separates these two behaviours. It connects the Gaussian fixed point in the UV to an IR non-Gaussian fixed point. This corresponds to a scaling solution as described in section 3.1, with, here, $\alpha = \beta = d - 2$ (at all orders of perturbation theory [RSTW17]). This is called the Gribov scaling. The structure of the flow is robust against two-loop corrections, the main difference being the location of the IR scaling fixed point [GPRT19]. We stress that the latter involves large couplings for which the present perturbative analysis is not reliable. This, for instance, is reflected in the important change of the location of this fixed point from one to two loops. In contrast, the RG trajectory that describes best the lattice results is under perturbative control. Finally, the one-loop RG flow in the IRS scheme has also been studied for general dimensions [TW11, RSTW17]. Explicit expressions for the anomalous dimensions $\gamma_\lambda$ and $\gamma_\gamma$, of similar complexity as above were obtained also for $d = 3$ and $d = 2$. For $d > 2$, the structure of the flow is the same as the one described above in $d = 4$. Instead, the case $d = 2$ is qualitatively different as there are no IR-safe trajectories, at one-loop order at least.

Once the running of the mass and coupling constant are determined, the RG-improved expressions for a vertex functions involving $n_\lambda$ gluon fields and $n_\gamma$ ghost fields are obtained, as usual, by solving the RG equation

$$\left(\mu \partial_\mu - \frac{n_\lambda \gamma_\lambda}{2} + \beta_t \partial_\gamma + \beta_{\gamma_\lambda} \partial_{\gamma_\lambda}\right) \Gamma^{(n_\lambda,n_\gamma)} = 0,$$

(32)

whose solution relates the vertex function at different scales:

$$\Gamma^{(n_\lambda,n_\gamma)}(\{p_i\}, \mu, g(\mu), m^2(\mu)) = z_A^{n_\lambda/2} z_{\gamma_\lambda}^{n_\gamma/2} (\mu) \Gamma^{(n_\lambda,n_\gamma)}(\{p_i\}, \mu_0, g_0, m_0^2).$$

(33)

24The general structure obtained here, with a regime of regular vs singular solutions as a function of the gluon mass parameter at fixed coupling is also observed in nonperturbative continuum approaches [PLNvS04, FG04, FMP09] although the scaling exponents at the IR scaling fixed point are different than those obtained here in the IRS scheme [RSTW17].
This relations involves the $z$ factors:

$$z_A(\mu) = \exp \int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \gamma_A(\mu') = \frac{g_0^2}{g^2(\mu)} \frac{m^4(\mu)}{m^4_0},$$  

$$z_\mu(\mu) = \exp \int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \gamma_\mu(\mu') = \frac{g^2(\mu)}{g^0} \frac{m_0^2}{m^2(\mu)},$$

where we used equation (27) in the last equalities to relate these to the running coupling constant $g(\mu)$ and mass $m(\mu)$, obtained by integrating the RG flow.

5.2. Fitting procedure

The calculation of the RG-improved correlation functions relies on integrating the RG flow. In the standard FP theory, the initialisation of the coupling constant at some RG scale $\mu_0$ is merely a scale-definition (this is the phenomenon of dimensional transmutation). For a multidimensional flow, as in the CF model, the initialisation process has far-reaching consequences because it specifies one of the infinitely-many RG trajectories (see figure 4) and a change of initial condition is in general not a simple scale redefinition.

Now, following the philosophy that the gluon mass is a phenomenological parameter that we do not try to determine from first principles, its value should be fixed by using external information. The strategy is the following. One initialise the RG flow at some scale $\mu_0$, integrates it, computes the RG-improved correlation function by using equation (33) and compares it with available lattice data. One then changes the initialisation parameters so as to minimise an error function and obtain the best agreement with lattice simulations. In general, one uses the strategy of fitting simultaneously all available data. A less stringent test would consist in fitting independently different correlation functions. In this last situation, the minimum of the error function typically lies at different points in the parameter space for different correlation functions and one would obtain better agreement with lattice simulations for each one separately.

Finally, we mention that when comparing CF results with lattice data, yet another parameter must be fixed: the overall normalisation of the correlation function under study. Consider the example of the YM propagators described in the next section. To obtain a set of curves, one needs to fix four parameters: the initial values $g_0$ and $m_0$ of the running coupling parameters as well as two multiplicative normalisation factors, for the gluon and for the ghost propagators.

5.3. Yang–Mills propagators

The ghost and gluon propagators in YM theories have been computed at one-loop order in the perturbative CF approaches in [TW10, TW11, KWH +20, DW20] and in the screened perturbation approach in [Sir15a, Sir15b, Sir16b, Sir19a]. They give a good agreement with existing lattice data for appropriate values of the gluon mass and gauge coupling parameters. Recently, two-loop corrections have been computed [GPRT19], which clearly improve the agreement with lattice data and greatly strengthen the confidence in the validity of the CF perturbative approach. We review those latest results here.

Evaluating (33) for the two-point vertices at $\mu = p$ and taking into account the renormalisation conditions (25), the gluon and ghost propagators can be expressed in terms of the running mass and coupling constant as

$$G(p) = \frac{m^4(\mu)}{m^4_0} \frac{1}{g^2(\mu) p^2 + m^2(p)}, \quad F(p) = \frac{g^2(\mu)}{g^0} \frac{m_0^2}{m^2(\mu)},$$

The one- and two-loop gluon and ghost dressing functions are compared to lattice data in figure 5 in $d = 4$ dimensions for $N_c = 3$ and $N_c = 2$. The first remarkable point is that the comparison is very good already at one-loop order, with a global error of about 7% for $N_c = 3$ and 10% for $N_c = 2$ (and a maximal error of about 15%). For $N_c = 3$, the inclusion of the two-loop contributions clearly improves the agreement (with a global error of 4%), whereas the improvement is less significant (6%) for $N_c = 2$ [BPWR20]. This can be understood from the fact that the coupling constant that controls the perturbative expansion is slightly larger for $N_c = 2$ than for $N_c = 3$. We mention that these comparisons have also been done within other schemes and generically lead to similar results [TW11, GPRT19, DW20]. Interestingly, it is also possible to devise optimized IR safe schemes which give excellent agreement with the data already at one-loop order [DW20]. Finally, one observes that the scheme dependence is reduced when going from one to two loops and that the improvement is more pronounced in the SU(3) case, possibly for the reason mentioned above.

An important point, mentioned in section 3.2, is that, as clearly seen from lattice simulations, the gluon propagator shows violations of positivity. To be precise, in a unitary, Lorentz-invariant theory with only positive norm states, one can prove the Källén–Lehmann representation, that is, the two following properties:

- The Euclidean propagator admits the integral representation
  $$G(p) = \int_0^\infty \frac{d\mu}{2\pi} \frac{\rho(\mu)}{p^2 + \mu^2}.$$  

- The spectral density $\rho(\mu)$ is positive or zero:
  $$\rho(\mu) \geq 0.$$  

It is difficult to test separately both properties by only having access to the Euclidean propagator. However, lattice simulations show that they cannot be satisfied simultaneously for the (transverse) gluon propagator. To test this, lattice simulations study the Fourier transform [CMT05, BHL +07]

$$C(t) = \int_{-\infty}^{+\infty} \frac{dp}{2\pi} e^{ipt} G(p) = \int_0^\infty \frac{d\mu}{2\pi} \frac{\rho(\mu)}{2\mu} e^{-\mu t},$$

where, in the last expression, the representation (37) was assumed. If, on top of (37), the inequality (38) is satisfied, then
Figure 5. The gluon (left) and ghost (right) dressing functions in $d = 4$ for the SU(3) (top) and SU(2) (bottom) YM theories. The lines are the one- and two-loop results in the CF model. The squares are the lattice data from [DOS17] for SU(3) and from [CMM08] for SU(2). Reproduced from [GPRT19]. CC BY 4.0.

Figure 6. The function $C(t)$ defined in equation (39) as obtained from lattice simulations [BHL*07] (triangles) compared to CF result [TW10] (dots) in the $N_c = 3$ case. The parameters are fixed by fitting the Euclidean propagators. A normalisation factor has been introduced to account for the multiplicative renormalisation of the lattice propagator.

The function $C(t)$ is positive. In figure 6 this function calculated in the lattice simulation [BHL*07] is compared to its one-loop expression in the CF model [TW10] for the $N_c = 3$ case. One observes a good agreement and, clearly, the function is not positive. Similar lattice results were obtained for the $N_c = 2$ case [CMT05]. The violation of this positivity condition on $C(t)$ can have two origins. Either the propagator cannot be expressed in the form (37), or the representation (37) is valid, but with a density $\rho(\mu)$ which takes negative values. A direct investigation of the analytic structure of the propagator is possible in the perturbative CF model and is reviewed in section 8.

Figure 7. The strong coupling constant in the Taylor scheme. The data points are those of [DOS16]. The dashed and plain lines correspond to the one-loop and two-loop CF results in the IRS scheme, respectively. In both cases, a normalisation factor is applied so that the value $\alpha_S(\mu_0)$ agrees with the lattice result at $\mu_0 = 10$ GeV. Reproduced from [GPRT19]. CC BY 4.0.

We also show, in figure 7, the Taylor coupling (15) as a function of momentum at one and two loops, compared to the lattice data for $d = 4$ and $N_c = 3$. We find a good agreement with the lattice results and an apparent convergence of the successive perturbative orders. As mentioned before, a similar analysis can be made for $N_c = 2$, where, however, the maximum value of the coupling is slightly larger. Although the perturbative analysis is still valid, this seems to slow the apparent convergence [GPRT19, BPRW20].

The propagators have also been evaluated at one loop in $d = 3$ dimensions [TW11, PTW13] and compared to existing numerical simulations for $N_c = 2$. Again, a fairly good
agreement is obtained although the results are not as good as for \( d = 4 \). Finally, simulations have also been carried out for \( d = 2 \) but, as mentioned previously, the perturbative approach considered here is not applicable in that case. Also noteworthy is the fact that the positivity violations mentioned earlier are already produced at one-loop order in \( d > 2 \) [TW10, TW11, RSTW17], thus implying that the tree-level mass term in the CF Lagrangian does not correspond to an actual massive excitation in the spectrum.

We thus see that a perturbative description is able to correctly describe nontrivial features of the YM propagators even at IR momenta. This validates—at the level of propagators for excitation in the spectrum.

...in the CF Lagrangian does not correspond to an actual massive TW11, RSTW17, thus implying that the tree-level mass term... produced since then [ADSF115].

5.4. Yang–Mills three-point functions

In order to further test the ability of the CF model to reproduce the YM correlators, and now that the parameters have been fixed from the fits to two-point functions, the next natural step is to study the predictions of the model regarding higher correlation functions. In this section, we consider the three-gluon vertex and the ghost–gluon vertex, that have both been computed in lattice simulations for the YM theory in the Landau gauge for particular configurations of momenta, for the SU(2) gauge group in \( d = 4 \) and \( d = 3 \) dimensions [CMM08, Maa20] and, more recently, also for the SU(3) gauge group in \( d = 4 \) [IMPSPOT, Ste06, SBK17, BDSRQZ17, ZBDS+19, CZB+20, ADSF+20]26.

5.4.1. The three-gluon vertex. In linear covariant gauges (including the Landau gauge), the colour structure of this vertex is proportional to the structure constants \( f^{abc} \) at all orders of perturbation theory [Smo82]. It can be decomposed in terms of six tensor components associated to the Lorentz group. We use the decomposition proposed by Ball and Chiu [BC80]:

\[
\Gamma_{\mu\nu\rho}(p, k, r) = -igf^{abc}\Gamma_{\mu\nu\rho}^{(3)}(p, k, r)
\]

with

\[
\Gamma_{\mu\nu\rho}(p, k, r) = \Lambda(p^2, k^2, r^2)\delta_{\mu\nu}(p - k)_\rho
\]

\[
\quad + B(p^2, k^2, r^2)\delta_{\mu\nu}(p + k)_\rho
\]

\[
\quad - C(p^2, k^2, r^2)(\delta_{\mu\nu}p.k - p.k_\nu)(p - k)_\rho
\]

\[
\quad + \frac{1}{3} S(p^2, k^2, r^2)(p_\mu k_\nu r_\rho + p_\nu k_\rho r_\mu)
\]

\[
\quad + F(p^2, k^2, r^2)(\delta_{\mu\nu}p.k - p.k_\nu)(p.k.r - k_\rho p.r)
\]

\[
\quad + H(p^2, k^2, r^2) \left[ - \delta_{\mu\nu}(p.k.r - k_\rho p.r) \right]
\]

\[
\quad + \frac{1}{3}(p_\mu k_\nu r_\rho - p_\nu k_\rho r_\mu) + \text{perm.}.
\]

(41)

where ‘perm.’ stands for the simultaneously cyclic permutations of the momenta \((p, k, r)\) and the indices \((\mu, \nu, \rho)\). The scalar functions \(A, C, F\) and \(S\) are symmetric under permutation of their first two arguments, whereas \(B\) is antisymmetric. The function \(H\) is completely symmetric and \(S\) is completely antisymmetric. All the tensorial components above have been computed at one-loop order in the CF model, for arbitrary momenta [PTW13]. It has been checked that the one-loop expressions reduce to the known ones in the FP model [DOT96] in the limit \(m \rightarrow 0\). The expressions are rather cumbersome and we shall not reproduce them here. Instead, we focus on the comparison with lattice data.

It is worth emphasising that what is measured in lattice simulations are correlators, which involve a contraction of the external legs of the vertex (41) with (transverse) gluon propagators. It follows that the longitudinal structures, controlled by the functions \(B\) and \(S\), are not accessible with the existing simulations. What has actually been calculated in the Monte-Carlo simulations for the SU(2) theory is the following quantity

\[
G_{\mu\nu\rho}(p, k, r) = \frac{\lambda_{\mu\nu\rho}^{\text{tree}}(p, k, r)\Gamma_{\mu\nu\rho}^{(3)}(p, k, r)}{\lambda_{\mu\nu\rho}^{\text{tree}}(p, k, r)\Lambda^{\text{tree}}(p, k, r)}
\]

(42)

where

\[
\lambda_{\mu\nu\rho}^{\text{tree}}(p, k, r) = P_{\mu\nu}(r)P_{\rho\nu}(k)P_{\rho\nu}(r) \left[ \delta_{\mu\nu}(k-r)_\alpha + \delta_{\nu\rho}(r-p)_\alpha + \delta_{\nu\rho}(p-k)_\alpha \right].
\]

(43)

Equation (42) corresponds to the transverse part of the three-gluon vertex projected along the tree-level tensorial component \([\delta_{\gamma\lambda}(k-r)\_\alpha + \text{cyclic permutations}]\), normalised by the same combination at tree level.

In figure 8, we show a comparison between the one-loop results in the CF model and the lattice data for the combination (42) for a particular configuration of momenta in both \(d = 4\) and \(d = 3\) dimensions. The overall agreement is satisfactory, although not as good as for the propagators. We stress that the lattice data show important statistical and (even larger) systematic uncertainties. This is clearly visible in figure 8 where the various lattice points are statistically incompatible among each other. For this reason, it is not obvious to assess the actual accuracy of the CF prediction. The one-loop expressions of [PTW13] have been compared to lattice data for all measured configurations of momenta, which yields a similar level of accuracy. The SU(3) case has also been treated in lattice simulations [BDSRQZ17] and compared to the one-loop predictions of the CF model [FP], with results of similar quality to those presented in figure 8. We recall that these comparison involve no fitting parameters except for an overall normalisation27.

In figure 8, one clearly observes, for \(d = 3\), what has been referred to as a zero crossing: the vertex function becomes negative for small enough values of momenta. The same behaviour is observed also for other configurations of momenta. This

26 We mention that more precise data for the three-point correlators have been produced since then [ADSF+21].

27 The normalisation is not fixed by fitting the (renormalised) propagators because the data presented here [CMM08, Maa20] concern the bare three-point correlator.
Figure 8. Three gluon correlation function $G^{AA}(p,k,r)$ for $p^2 = k^2 = r^2$ (employing the notations of [PTW13]) in $d = 4$ (left) and $d = 3$ (right). The blue curves correspond to the IRS scheme whereas the red dashed curves correspond to a vanishing momentum scheme. Lattice data from [CMM08, Maa20]. Reprinted figure with permission from [PTW13], Copyright (2013) by the American Physical Society.

feature, first observed in lattice simulations [CMM08], is a very simple prediction of the perturbative CF model at one-loop order. It simply comes from the fact that, at very low momenta, the three-gluon vertex is dominated by the ghost-loop diagram which comes with the opposite sign as compared to the tree-level term. This implies, not only a zero crossing, but a divergence towards negative infinity when all momenta vanish. The same behaviour is also predicted in $d = 4$ but for much smaller values of momenta, the divergence being only logarithmic [PTW13]. This explains why this is more difficult to see in lattice simulations [ADSF11, 21].

Finally, we note that the phenomenon of ghost dominance at low momenta, responsible for the above-mentioned zero crossing, is of a more general scope. For instance, it is also valid for other continuum approaches, such as the DSE or the FRG. It also plays a pivotal role for finite temperature physics, as discussed in section 7. Finally, it provides a simple description of the dominant low momentum behaviour of vertex functions [TW11]. Let us describe this last point here. Characterizing correlation functions with more external legs becomes very involved because more and more Feynman diagrams must be computed and because the number of independent tensorial structures increases rapidly with the number of external legs. It is, however, possible to extract some information concerning the deep IR regime (that is, when all external momenta are small compared to the gluon mass) at a low computational cost [TW11]. In this regime, the vertex is dominated by the Feynman diagrams with the smallest number of (massive) gluon propagators. Moreover, at fixed number of external legs, the Feynman diagrams with more and more loops are suppressed for IR momenta. As a result, the leading IR behaviour is given by a one-loop diagram (barring an accidental compensation of diagrams). It is then possible to predict the leading IR behaviour of a vertex. For instance, the gluon self-energy is dominated by a one-ghost-loop diagram, which behaves, in $d = 4$, as constant $+ p^2 \ln(p^2/m^2)$ in the deep IR. Similarly, the three-gluon vertex behaves as $p \ln(p^2/m^2)$, which explains the zero crossing described in above. In general, the $n$-gluon vertex with $n > 2$ behaves as $p^{4-n}$ modulo logarithms.

5.4.2. The ghost–gluon vertex. The ghost–antighost–gluon vertex—usually dubbed ghost–gluon vertex for short—has a much simpler Lorentz structure than the three-gluon vertex. It can be decomposed in terms of two vectorial components as

$$\Gamma^{(3)}_{\bar{c}a\bar{b}c}(p,k,r) = -ig_0 f^{abc} \left[ k_\mu V(p^2,k^2,r^2) + r_\mu W(p^2,k^2,r^2) \right],$$

where $p$, $k$, and $r$ are the (incoming) momenta of the ghost, antighost and gluon, respectively. Only the scalar function $V(p^2,k^2,r^2)$ is measurable in Landau gauge lattice simulations for it is the only term that contributes when the vertex is contracted with a (transverse) gluon propagator.

Both vectorial components in equation (44) have been computed at one-loop order in the CF model for arbitrary momenta in $d = 3$ and $d = 4$ and for any $N_c$ [PTW13]. The various symmetries of the model [TW09] give rise to (Ward and Slavnov–Taylor) identities that, first, constrain the three-gluon and the ghost–gluon vertices separately and, second, imply nontrivial relations between those. It has been checked that the one-loop CF expressions verify these identities and that they reduce to the known one-loop expressions in the FP model [DOT96] in the limit $m \to 0$.

The one-loop CF results have been compared to the existing lattice data. In a similar way as for the three-gluon vertex, what has been measured in the Monte-Carlo simulations is the following quantity:

$$G^{\bar{c}A}(p,k,r) = k_\mu P^{\mu}_{m}^{\bar{c}A}(r) \left[ k_\nu V(p^2,k^2,r^2) + r_\nu W(p^2,k^2,r^2) \right] k_\nu P^\nu_{m}(r) k_\nu = V(p^2,k^2,r^2).$$

Again, this corresponds to the transverse part of the ghost–gluon vertex projected on the tree-level tensorial structure, normalised to the same combination at tree level. Thanks to Taylor’s nonrenormalisation theorem, the expression (45) associated with the ghost–gluon vertex is UV finite.
and is identical to its tree-level form when the ghost momentum vanishes. This is to be contrasted with the quotient (42), which has a multiplicative UV divergence and needs to be renormalised.

On top of the full one-loop expressions of [PTW13], two-loop corrections to the ghost–gluon vertex have been calculated in [BPRW20]. The calculation is quite involved for general momentum configurations and the two-loop contribution has only been computed for the case of vanishing gluon momentum, namely,

\[ v(k^2) = V(k^2, k^2, 0), \]

in \( d = 4 \) and for both \( N_c = 2 \) and \( N_c = 3 \). Even for this particular momentum configuration, the calculation requires the use of symbolic programing, similar to that used in the case of the two-loop propagators (see section 5.3). It has been explicitly verified that the two-loops expressions fulfil several nontrivial consistency checks [BPRW20]:

- The divergent parts are compatible with the Taylor nonrenormalisation theorem.
- In the limit \( m \to 0 \), the expressions reduce to those of the FP theory, obtained for the same configuration of momenta in [DOT98].
- The one- and two-loops contributions vanish in the limit of zero ghost momentum, that is, \( v(k^2 = 0) = 1 \).

We stress that this last property which is clearly observed in lattice simulations is not retrieved from the usual (massless) calculations because of IR divergences which induce a nonanalytic behaviour at small momenta. Specifically, fixing first the gluon momentum to zero and later taking the limit of vanishing ghost momentum leads to a divergence of the vertex, at odds with the lattice data. The CF mass removes this IR divergence and yields the correct nonrenormalisation behaviour.

The one- and two-loop expressions of [PTW13, BPRW20] are compared with the lattice data from [Maa20] for \( N_c = 2 \) and from [IMPS’07, Ste06] for \( N_c = 3 \) in figure 9. It should be noted that this comparison is done without any further parameter adjustment than the one used to fit the propagators (not even an overall normalisation factor). Because the fit of the latter for the SU(2) group is of lower quality than that for the SU(3) group, due, in part, to possible systematic errors in the lattice simulations, this can introduce a significant error in the parameter estimation. For this reason a second type of fit, including the ghost–gluon vertex together with the ghost and the gluon propagators, was considered in [BPRW20]. The corresponding results are shown in figure 10.

The first observation is that the one-loop results provide a good description of the data, although not as good as for the two-point functions. For both \( N_c = 2 \) and \( N_c = 3 \), the quality of the comparison with the simulations improves when going from one loop to two loops. However, the improvement is significantly better for SU(3) as was already the case for the propagators.

To conclude, the perturbative results in the CF model are quite good. In fact, in the case of three-point vertices, it is not clear that, with the level of precision achieved, it is possible to completely neglect the errors coming from the lattice simulations. Put together, these results strongly indicate that the perturbative expansion of the CF model accurately describes the YM vacuum correlation functions in the Landau gauge. This is not only true for the two-point functions but also for three-point functions. Moreover, whenever tested, the precision improves when including higher orders of perturbation theory.

6. Dynamical quarks

In the previous section, we put forward evidences which indicate that the IR regime of YM theory can be described within perturbation theory, the main nonperturbative ingredient being encapsulated in a phenomenological screening mass for the gluons. Focussing on a pure gauge theory is clearly simpler (it involves less Feynman diagrams, renormalisation factors, etc) and, at the same time, it is a first step for testing the working hypothesis under scrutiny. But, of course, these results are mainly methodological since the true QCD involves also light quarks with significant fluctuations. In this section, we discuss the attempts to include these particles to the perturbative scheme described in section 5.

Many physical phenomena occur in the presence of matter fields, which have no equivalent in the pure YM theory. Of utmost importance is the phenomenon of spontaneous chiral symmetry breaking. In a nutshell, the Dirac Lagrangian (2) for massless quarks is invariant under chiral transformations which rotate independently the left and right parts of the Dirac bispinor. This symmetry happens to be spontaneously broken by quantum fluctuations, which implies that, even if the valence quark mass (i.e., the quark mass at a scale of the order of few GeV) is small, the constituent quark mass (the one defined at an IR scale, relevant, e.g., for computing the mass of a hadron) is significantly larger.

At a technical level, dynamical quarks are taken into account by adding the Dirac Lagrangian (2) to the CF action. The theory is renormalisable and requires new renormalisation factors, for the quark field and masses, \( \bar{\psi} = \sqrt{Z_{\psi}} \psi_R, M_{\psi} = Z_M M_i \).

In what follows, we describe the perturbative calculations of the propagators and the quark–antiquark–gluon vertex of the unquenched theory in the vacuum. For not too light quarks, they compare well with existing lattice data, as reviewed in subsections 6.1 and 6.2 below. The dynamics of light quarks is strongly coupled and requires a more elaborate treatment, discussed in subsection 6.3.

6.1. Propagators

The calculation of the gluon, ghost, and quark propagators in perturbation theory is straightforward [PTW14, Sir16a, HK20, BGPR21]. At strict one-loop order, the ghost propagator is unchanged as compared to the quenched case whereas the gluon propagator receives an additional quark loop contribution. The RG running yields an additional source of normalisa
The CF model prediction for the function $v(k^2)$ in the SU(2) (left) and SU(3) case (right) in the IRS scheme, compared to the lattice data in the Taylor scheme from [Maa20] for SU(2) and from [IMPS+07, Ste06] for SU(3). The parameters $m$ and $g$ at the initial scale $\mu_0$ where previously determined from the fits of the gluon and ghost propagators. Reproduced from [BPRW20]. CC BY 4.0.

The best fit for the vertex function $v(k^2)$ in the SU(2) case at one- and two-loop orders [PTW13, BPRW20] in the IRS scheme, when compared to the lattice data in the Taylor scheme [Maa20]. Reproduced from [BPRW20]. CC BY 4.0.

dependence which affects all propagators [PTW14, PTW15]. As in the quenched case, these calculations have been recently pushed to two-loop order, first, to assess the (apparent) convergence of the perturbative expansion and, second, to elucidate the case of the quark dressing function, which receives significant two-loop contributions [BGPR21]. The details of the calculations, with and without RG improvement can be found in the mentioned references. Here, we give a brief summary of the comparison of the perturbative results for the propagators with various lattice data for different flavour contents and different quark masses [BHL⁺04, BHL⁺05, ABB⁺12, SMMPvS12, OSSS19].

The first observation is that the sensitivity of the unquenched ghost and gluon propagators to the flavour content of the theory is well described by the one-loop results. One observes though that, for a given flavour content, these propagators are rather insensitive to the precise values of the quark masses in the range of at most a few 100 MeV. The renormalised quark propagator

\[
S(p) = \frac{Z(p)}{-ip + M(p)} = Z(p) \frac{iep + M(p)}{p^2 + M^2(p)}
\]

has two independent components, the (RG invariant) mass function $M(p)$ and the dressing function $Z(p)$, which are both well measured on the lattice. One can easily find values of the parameters for which the former is well reproduced by the one-loop results, in particular, if one only fits this function, independently of the others. Of course, fitting only one function can lead to artificially good results and, in order to obtain a realistic estimate of the quality of the one-loop approximation, one should fit all possible data with a single set of parameters. This is shown in figures 11 and 12, for the lattice data of [SMMPvS12, OSSS19], corresponding to $N_f = 2$ and two values of the pion mass $M_\pi$ (or, equivalently, of the bare quark mass). One observes that the one-loop results indeed give a satisfactory description of the data, including the quark mass function. Instead, the one-loop quark dressing function compares badly to lattice data. As anticipated in [PTW14], this is because the one-loop contribution to this function is abnormally small in the CF model—it vanishes identically in the FP theory—and higher-loop effects are not negligible in comparison.

The perturbative calculations of the QCD propagators have recently been pushed up to two-loop order [BGPR21]. As in the quenched case, this allows one to really assess the reliability of the perturbative CF description. Again, we refer the reader to [BGPR21] for details and, here, we just show the quality of the results in figures 11 and 12. The two-loop corrections clearly improve the one-loop results, in particular, for what concerns the quark dressing function. One also observes that the quality of the perturbative description is better for larger quark (or pion) masses\(^{29}\). In fact, if one finds that the gluon, ghost and quark dressing functions seems always well-described by the two-loop perturbative results, this is not quite so for the quark mass function, in particular, for low pion masses. This is to be expected as the latter is directly sensitive to the dynamical breaking of the chiral symmetry in the

\(^{29}\)It may seem, at first sight, that the quality of the fits are equally good for heavy and light quarks, figures 11 and 12, respectively. However, as explained below, the detailed analysis of the errors reveals that it is not the case.
Figure 11. The various propagators at one- and two-loop orders for $N_f = 2$ degenerate flavours in $d = 4$. The upper plots present the gluon (left) and ghost (right) dressing functions whereas the lower plots show the quark mass (left) and dressing function (right). The parameters (defined at the scale $\mu_0 = 1$ GeV) are $g_0 = 4.53$, $m_0 = 430$ MeV, and $M_0 = 140$ MeV at one loop and $g_0 = 4.10$, $m_0 = 390$ MeV, and $M_0 = 160$ MeV at two loops. The points are the lattice data of [OSSS19] for $M_\pi = 426$ MeV. Reproduced with permission from [BGPR21].

chiral limit. This is clearly visible in figure 12, corresponding to a rather unfavourable case, with $M_\pi = 150$ MeV—close to the experimental value $m_\pi = 140$ MeV. Perturbative calculations give an accurate description of the data from the IR to the UV, except for the quark mass function. Indeed, the good fit of the quark mass in the IR is at the expense of a rather poor description in the UV, even at two-loop order. If one insists, instead, on correctly describing the UV for the quark mass function, the fit deteriorates in the IR, as shown in figure 13. Thus, although the two-loop corrections improve significantly the one-loop results, the perturbative results are not able to reproduce the dynamical generation of the quark mass in the IR. That is a manifestation of the fact that the phenomenon of spontaneous chiral symmetry breaking is not captured at any finite order in perturbation theory and thus requires a more involved approximation scheme. This is discussed in section 6.3 below.

6.2. The quark–gluon vertex

The quark–antiquark–gluon vertex has also been studied in lattice simulations [SBK+03, SBK+05, KLSW07, SBK+17, KOS+21, OFdPdM18] and can thus be used to further test the perturbative CF approach. This vertex has a rich Dirac structure, with twelve scalar functions of three momentum variables [SK02]. Only those which are transverse with respect to the gluon momentum are accessible to lattice simulations (in the Landau gauge) and only a restricted set (and for particular momentum configurations) have actually been measured. All have been computed at one-loop order in the CF model in [PTW15] and compared to lattice data, which we review here. One adjusts the parameters $g_0$, $m_0$, and $M_0$ of the model by fitting the correlators as in the previous subsection, following a similar procedure as the one outlined in section 5.2 for the YM case. Once this is done, the results for the various components of the three-point function are pure predictions of the model, apart from a normalisation factor, similar to the case of the three-gluon vertex. Another remark is that the results of [SBK+03, SBK+05] are quenched ($N_f = 0$) lattice data. The comparison with the perturbative expressions for the quark–antiquark–gluon vertex are still meaningful because there are no quark loop contributions at one-loop order. Flavour effects only enters through the RG running of the parameters so, in these comparisons, the quenched running was used. We focus below on the particular configurations of momenta that have been studied in lattice simulations.

For vanishing gluon momentum, the vertex has the following form

$$
\Gamma_\mu(p, -p, 0) = -i g_s \left[ \lambda_3(p) P_{\mu\nu}(p) + 4 \hat{\lambda}_3(p) P_{\mu\nu}^a(p) \right] \gamma_\nu - 2 g_s \lambda_3(p) p_\mu,
$$

(48)

30 In the previous section, the quark mass parameter is included in the fit together with $m_0$ and $g_0$. In the present and the next sections, instead, it is simply fixed to agree with the lattice value $M_\pi$ at the scale $\mu_0$.

31 Since the work of [PTW15], new lattice data have been produced with dynamical quarks [SBK+17, KOS+21, OFdPdM18].
Figure 12. The various propagators at one- and two-loop orders for $N_f = 2$ degenerate flavours in $d = 4$. The upper plots present the gluon (left) and ghost (right) dressing functions whereas the lower plots show the quark mass (left) and dressing function (right). The parameters (defined at the scale $\mu_0 = 1$ GeV) are $g_0 = 4.29$, $m_0 = 350$ MeV, and $M_0 = 60$ MeV at one loop and $g_0 = 4.35$, $m_0 = 360$ MeV, and $M_0 = 50$ MeV at two loops. The points are the lattice data of [OSSS19] for $M_\pi = 150$ MeV. Reproduced with permission from [BGPR21].

Figure 13. The quark mass function at one and two-loop orders as compared to the lattice data of [OSSS19] for $M_\pi = 150$ MeV. Here, the quark mass parameter $M_0$ is not part of the fit but is fixed to the lattice value at the UV scale $\mu_0 = 2.94$ GeV. Reproduced with permission from [BGPR21].

where, as before, $P_{\mu
u}(p) = \delta_{\mu\nu} - p_\mu p_\nu / p^2$ and $P_{\mu
u}^\parallel(p) = p_\mu p_\nu / p^2$. The function $\lambda_1(p)$ thus quantifies the renormalisation of the classical contribution $\propto \gamma_\mu$ in units of the (running) coupling $g$, defined as the ghost–antighost–gluon coupling in the Taylor scheme. The functions $\lambda_1(p)$, $\tilde{\lambda}_2(p)$, and $\lambda_3(p)$ are shown in figure 14. The agreement between the one-loop results and the lattice data is very good given the simplicity of the approximation.

The quark–gluon vertex has also been measured in lattice simulations for equal quark and antiquark momenta $p$ (so that the gluon momentum is $k = -2p$), which involves two other scalar functions $\tau_3$ and $\tau_5$:

$$\Gamma_\mu(p, p, -2p) = -ig_{\lambda_1}[\lambda_1(p)\gamma_\mu + 4\tau_3(p)p_\mu - 2i\tau_5(p)\sigma_{\mu\nu}p_\nu]$$

where $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$ and $\lambda_1(p) = \lambda_1(p) - 4p^2\tau_5(p)$. The functions $\lambda_1$, $\tau_3$, and $\tau_5$ are shown in figure 15. The functions $\lambda_1$ and $\tau_5$ are also defined for other momentum configurations and comparisons of similar quality have been obtained between one-loop results and lattice data [PTW15].

Summarising, in all the investigated cases, the agreement is good in view of the fact that there is no adjustable parameter apart from an overall normalisation in this comparison. As in the pure gauge (quenched) theory, we thus conclude that part of the IR dynamics in the presence of dynamical quarks is well described by perturbative means in the CF model. The expected nonperturbative effects in the FP theory are partly taken into account in a simple gluon mass term. An important observation though is that the agreement between lattice data and one-loop results is generally less good at very low momenta. One explanation is that the quark–gluon coupling becomes too large in the IR for a reliable perturbative treatment. For instance, defining an effective quark–gluon coupling $g_\phi$ as the coefficient of the classical structure $\propto \gamma_\mu$ of the quark–antiquark–gluon vertex at vanishing gluon momentum, i.e., $g_\phi = g_{\lambda_1}$, we see, from the lattice data presented in figure 14 that the ratio between the couplings in the quark
Figure 14. Scalar functions of the quark–gluon vertex at vanishing gluon momentum, $\lambda_1$, $\tilde{\lambda}_2$, and $\lambda_3$, as functions of the quark momentum. The lines are the one-loop results with $M_0 = 0.2$ GeV, $m_0 = 0.44$ GeV and $g_0 = 4.2$ at the scale $\mu_0 = 1$ GeV whereas the dots correspond to the lattice data of [SBK+03]. The second figure shows both $\lambda_1$ (plain line, blue dots) and $\tilde{\lambda}_2$ (dashed line, black dots). The one-loop curves are almost superimposed. Reprinted figure with permission from [PTW15], Copyright (2015) by the American Physical Society.

Figure 15. The scalar components $\lambda'_1$ and $\tau_5$ of the quark–antiquark–gluon vertex for equal momentum $p$ of the quark and antiquark; see equation (49). The parameters are $M_0 = 0.2$ GeV, $m_0 = 0.44$ GeV and $g_0 = 4.2$ at the scale $\mu_0 = 1$ GeV. Reprinted figure with permission from [PTW15], Copyright (2015) by the American Physical Society.

and in the pure gauge sectors $g_q/g_g = \lambda_1$ can be as large as 2 in the deep IR. This implies that the perturbative expansion parameter in the quark sector $\lambda_q = N_c g_q^2/(16\pi^2)$ can reach up to four times that of the pure gauge sector. With the typical value $g_q \sim 4$ for $N_c = 3$, that is, $\lambda_q \sim 0.3$, the parameter $\lambda_q$ can reach up to 1.2 in the deep IR, a value for which a perturbative expansion is clearly invalid\footnote{This argument combines estimates from the heavy-quark limit for $g_q$ and from the quenched limit for the quark–gluon vertex $g_g$ and has to be updated when (light) quarks fluctuations are taken into account; see below.}.

6.3. The rainbow-improved perturbative expansion and the spontaneous breaking of chiral symmetry

The above observation that a straightforward coupling expansion is inadequate to treat the dynamics of light quarks goes along with the fact that the phenomenon of dynamical chiral symmetry breaking is not captured at any finite loop order in the CF—let alone in the FP—model. This, together with the fact that the coupling that governs the pure gauge sector remains small-to-moderate, motivates a perturbative expansion in the pure gauge coupling $g_g$ alone, keeping all orders
in the quark coupling $g_q$. One can further exploits another effectively small parameter in $\text{SU}(N_c)$ gauge theories, namely $1/N_c$. Although $N_c = 3$ in QCD, it is well-known that a $1/N_c$ expansion successfully captures essential aspects of the dynamics [th74, wit79, dl16]. We will assume that this also happens in the CF model. This hypothesis gives rise to results that agree very well with the Monte-Carlo simulations as will happen in what follows.

In [prs+17, prs+21], a controlled approximation scheme for IR QCD has been proposed, based on a double expansion of the CF model in the two parameters $g_q$ and $1/N_c$. For instance, at leading order, the propagators in the gauge sector (ghost and gluon) are given by their tree-level expression whereas the quark self-energy includes the infinite set of so-called rainbow diagrams with a massive one-gluon exchange. This results in coupled integral equations for the quark renormalisation and mass functions which read, in terms of bare quantities,

$$Z^{-1}(p) = 1 - g_{q,b}^2 C_F \int_{\Lambda} \frac{d^4q}{(2\pi)^4} \frac{Z(q)}{q^2 + M^2(q)} \times 2\ell(\rho^2 + q_\perp^2)(p^2 + q_\perp^2 + \ell^2) p^2 \ell^2 (p^2 + m_b^2),$$

(50)

$$Z^{-1}(p)M(p) = M_b + 3g_{q,b}^2 C_F \int_{\Lambda} \frac{d^4q}{(2\pi)^4} \frac{M(q)Z(q)}{q^2 + M^2(q)} \frac{1}{\ell^2 + m_b^2},$$

(51)

where $\ell = p + q$ and $\int_{\Lambda}$ denotes an appropriately regulated momentum integral. Here, $M_b$, $m_b$, and $g_{q,b}$ are the (bare) quark mass, gluon mass, and quark gluon coupling, respectively, and $C_F = (N_c^2 - 1)/(2N_c)$ is the Casimir of the fundamental representation of the $\text{SU}(N_c)$ gauge group.

The rainbow resummation is known to capture the essential features of chiral symmetry breaking and the associated dynamical quark mass generation and has been widely used in the context of nonperturbative approaches to IR QCD (the bibliography on this topic is extremely large; for a review, see [mr03]). The benefit of the double expansion proposed in [prs+17, prs+21]—dubbed the rainbow-improved (RI) loop expansion—is that it avoids ad hoc modelling for the quark–gluon vertex and for the gluon propagator entering the rainbow diagrams. At leading order, these are simply given by their tree-level expressions in the CF model. Another asset is that the actual expansion in small parameters underlying the approximation scheme allows for a systematic implementation of RG improvement, which is crucial for a proper description of chiral symmetry breaking.

The RI loop expansion at first nontrivial order has been implemented in [prs+17] using a toy-model RG running and in [prs+21] using a consistent implementation of the RG running within the RI expansion. We refer the reader to these articles for the technical details and we review the main results here. Figure 16 illustrates some aspects of the chiral symmetry breaking phenomenon through the quark mass function $M(p)$ (see equation (47)) using a toy model for the running quark–gluon coupling parametrized, in particular, by a finite IR value $g_{IR}$. A nonzero value $M(p = 0)$ is dynamically generated in the chiral limit—obtained by decreasing the value of $M(\Lambda_1)$ at the UV scale $\Lambda_1$—above a critical value of $g_{IR}$. Also shown in figure 16 is the fact that the logarithmic UV tail of the theory with massive quarks turns into a power law in the chiral limit, $M(p) \propto \langle \bar{\psi}\psi \rangle/p^\Lambda$ (up to calculable logarithmic corrections), controlled by the dynamically generated quark–antiquark condensate.

The results of the RI expansion for the quark and gluon propagators at next-to-leading order, using a self-consistent RG running, have been computed in [prs+21] and compared to lattice data close to the chiral limit. At this order, the quark propagator is unchanged as compared to its leading-order expression, i.e., it is given by the resummation of rainbow diagrams with tree-level one-gluon exchange, whereas the gluon propagator receives the usual perturbative one-loop corrections in the gauge sector and an effective quark-loop with rainbow-resummed quark propagators.

The comparison with lattice data is made by adjusting the various parameters, namely, the quark and gluon masses and the coupling$^{33}$ at the scale $\Lambda_1 = 10$ GeV. The gluon mass and

$^{33}$ It is worth emphasising that, despite the different treatment of the (IR) couplings in the quark and in the pure glue sectors, there is, in $\text{free}$, only one coupling parameter to specify in actual implementations of the RI loop expansion because both couplings are related by Slavnov–Taylor identities. In the UV, this relation is under perturbative control and one can fix the value of $g_q$ in terms of $g_1$. In this section, $g_0$ refers to the pure gauge coupling at the scale $\mu_0 = g_1(\mu_0)$. 

Figure 16. Left: constituent quark mass $M(p = 0)$ as a function of the coupling parameter $g_{IR}$ (see text) for two values of the UV mass $M(\Lambda_1)$. The variation of $g_{IR}$ is done by keeping $\Lambda_{\text{QCD}}$ fixed. Right: mass function $M(p)$ in log–log scale for decreasing (from top to bottom) values of $M(\Lambda_1)$ at the UV scale $\Lambda_1 = 10$ GeV. We observe the onset of the power law behaviour at large momentum, characteristic of the spontaneous breaking of the chiral symmetry, as the chiral limit is approached. Reprinted figure with permission from [prs+17], Copyright (2017) by the American Physical Society.
Figure 17. Left: the quark mass function $M(p)$ compared to the lattice data from [OSSS19]. The quark mass at the scale $\Lambda_1 = 10$ GeV is fixed at its lattice value $M(\Lambda_1) = 3$ MeV. The best fit parameters (see text) are $m_0 = 0.12$ GeV, and $g_0 = 2.42$ at $\mu_0 = 1$ GeV. Right: the RG-improved one-loop expression of [PTW14] for $M(p)$ is also able to describe well lattice data. This, however, necessitates artificially large values of the gluon mass and of the coupling, here, $m_0 = 1.6$ GeV and $g_0 = 13$ at $\mu_0 = 1$ GeV. In particular, the one-loop approximation makes no sense. Reproduced from [PRS+21]. CC BY 4.0.

Figure 18. The gluon dressing function $p^2 G(p)$ (left) and the quark mass function $M(p)$ (right) compared with lattice data from [SMMPS12]. Here the fit is adjusted on the gluon dressing function alone and the best fit parameters (see text) are $m_0 = 0.39$ GeV and $g_0 = 4.67$ at $\mu_0 = 1$ GeV. The quark mass at the scale $\Lambda_1 = 10$ GeV is fixed at its lattice value $M(\Lambda_1) = 3$ MeV. Reproduced from [PRS+21]. CC BY 4.0.

the coupling are included as fit parameters and, in order to ease the comparison with the previous sections, we give their values $m_0$ and $g_0$ at the scale $\mu_0 = 1$ GeV, obtained through the proper RG running. The quark mass is not included in the fits but is, instead, fixed to the lattice value at the scale $\Lambda_1 = 10$ GeV, namely, $M(\Lambda_1) = 3$ MeV in all the figures presented here. This corresponds to the (renormalised) current quark mass and plays the role of the control parameter for the chiral limit. The corresponding value $M_0$ of the running quark mass at $\mu_0 = 1$ GeV can be read off the plots of $M(p)$ for each case under consideration. Figure 17 shows that one can obtain an excellent agreement for the quark mass function alone. It was noticed that a similar agreement can be obtained from the one-loop calculation presented in the subsection 6.1 above, however, at the price of an uncomfortably large value of the gluon mass parameter. Good descriptions of the data with the RI expressions typically favour comparatively small values of the gluon mass parameter. Similarly, fitting the gluon propagator alone leads to very good agreement as shown in figure 18. This, however, favours relatively large values of the gluon mass parameter, which thus deteriorates the quality of the agreement for the quark mass function, as shown in the same figure.

As mentioned before in the one-loop analysis, fitting one function alone typically leads to artificially good fits, not representative of the actual quality of the approximation. It is thus desirable to fit the quark mass function and the gluon propagator together, which, as just mentioned, are in tension with respect to the value of the gluon mass parameter. The best fit is presented in figure 19, which still shows a reasonably good agreement, at the 15% level [PRS+21]. Finally, the quark dressing function, shown in figure 20 is badly described as was already the case at one-loop order, see figures 11 and 12. Although the present approximation includes part of the two-loop contributions that are necessary for a proper description of this function, there remain significant two-loop corrections from the quark–gluon vertex corrections, that are not included. The difference between the complete two-loop result shown in section 6.1 and the one shown in figure 20 gives a measure of the size of such vertex corrections.

We end this section with a comment on the order of magnitude of the expansion parameters used here. For the best fit
RI loop expansion. At leading order, it is given by the infinite series of ladder diagrams with one-(massive)-gluon-exchange rungs and rainbow-resummed quark propagators [PRS+17]. This is not a surprise because this ladder resummation for the quark–antiquark-meson vertex is clearly able to produce good fits of the latter. The function $z_\psi(p, \mu_0) = Z(p)/Z(\mu_0)$ normalised to $z_\psi = 1$ at $p = 2$ GeV against the lattice data of [OSSS19], for the set of parameters $M(\Lambda_1) = 3$ MeV, $m_0 = 0.21$ GeV, and $g_0 = 2.45$ at $\Lambda_1 = 10$ GeV and $\mu_0 = 1$ GeV. Reproduced from [PRS+21]. CC BY 4.0.

Figure 19. Combined fit of the gluon dressing function $p^2 G(p)$ (left) and the quark mass function $M(p)$ (right) against the lattice results of [OSSS19] and [SMMPvS12], respectively. The quark mass at the scale $\Lambda_1 = 10$ GeV is fixed at its lattice value $M(\Lambda_1) = 3$ MeV. The best fit parameters (see text) are $m_0 = 0.21$ GeV, and $g_0 = 2.45$ at $\mu_0 = 1$ GeV. Reproduced from [PRS+21]. CC BY 4.0.

values, the typical gluon coupling is at most $\lambda_g = g^2 N_c/(16\pi^2) \approx 0.12$ for $N_c = 3$, whereas the running quark–gluon coupling reaches $\lambda_q = g^2 N_c/(16\pi^2) \approx 0.68$ [PRS+21]. Although these values are slightly smaller than those obtained for heavy quarks in the previous section, we see that a perturbative treatment of the quark–gluon coupling remains, a posteriori, questionable. Finally, it is interesting to note that the two expansion parameters that control the RI-loop expansion are roughly of the same order in the case of the pion in the RI loop expansion [Ser20]. In the chiral limit, the relevant integral equation for the vertex can be greatly simplified and one ends up with a set of coupled one-dimensional integral equations which are easy to solve numerically. The pion decay constant $f_\pi$ can then be systematically computed as a function of the parameters of the Lagrangian, namely, the gauge coupling and the gluon mass. Preliminary results show that there are regions of parameter space which give very good values of $f_\pi$. This is to be expected because, thanks to the chiral Ward identities, the value of $f_\pi$ is, to a large extent, determined by the quark mass function [PS79] and, as described above, the CF model equipped with the RI expansion is clearly able to produce good fits of the latter.

Interestingly, we mention that the CF model can serve as a precise definition of a gluon mass (in the Landau gauge) which can be assigned a physically measurable value, e.g., the experimental value of $f_\pi$. The constraints on a possible gluon mass in the particle data book [Z+20] refer to an outdated and, in fact, theoretically not precise definition of the gluon mass [Ynd95]. The work reported here brings the possibility of a precise, well-defined—necessarily gauge and scale dependent—gluon mass, similar to what is done for the quark masses [Z+20]. For discussions in this direction; see [Rob20].

6.4. Hadronic observables

The RI expansion can also be used for the calculation of physical observables in QCD. The simplest ones are the properties—masses and decay constants—of meson bound states, out of which the pion plays a particular role, being the Goldstone mode associated to the spontaneous breaking of the chiral symmetry. An important ingredient for this calculation using continuum approaches is the quark–antiquark-meson vertex. The latter can be consistently calculated in the

Figure 20. The function $z_\psi(p, \mu_0) = Z(p)/Z(\mu_0)$ normalised to $z_\psi = 1$ at $p = 2$ GeV against the lattice data of [OSSS19], for the set of parameters $M(\Lambda_1) = 3$ MeV, $m_0 = 0.21$ GeV, and $g_0 = 2.45$ at $\Lambda_1 = 10$ GeV and $\mu_0 = 1$ GeV. Reproduced from [PRS+21]. CC BY 4.0.
7. Nonzero temperature and density: the confinement–deconfinement transition

Lattice simulations have established that YM theories undergo a confinement–deconfinement phase transition at nonzero temperature [Sve86, KKPZ02, LTW05, Gre12, SDPvS13]. The latter is controlled by the spontaneous breaking of a symmetry specific to the nonzero temperature problem, the centre symmetry [Sve86, Gre, Pis02]. One possible order parameter for the latter is the Polyakov loop \( \ell \) [Pol78], defined as the average of a traced temporal Wilson loop:

\[
\ell = \frac{1}{N_c} \text{tr} \left\{ P \exp \left\{ i g \int_0^\beta d\tau A_0(\tau, x) \right\} \right\}. \tag{52}
\]

Here, the inverse temperature \( \beta \equiv 1/T \) (the Boltzmann constant is set to \( k_B = 1 \)) sets the extent of the compact Euclidean time interval over which the fields are defined and the path ordering operator \( P \) orders the (matrix-valued) fields from left to right according to the decreasing value of their time argument. The gauge field is periodic in Euclidean time with period \( \beta \)—and so are the ghost and antighost fields in a gauge-fixed setting—whereas quark fields are antiperiodic. The Polyakov loop is directly related to the free energy \( F_\ell \) of the system in the presence of a static colour charge as [Pol78, Sve86]

\[
\ell \propto e^{-\beta F_\ell}. \tag{53}
\]

In particular, a phase with \( \beta = 0 \) implies an infinite free-energy cost for the colour charge, characterising a confined phase. Clearly, the latter involves large field configurations with \( A_0 \sim 1/g \), which are not captured at any finite order in perturbation theory around the trivial configuration \( A_0 = 0 \). One way to cope with this issue is to expand around a nontrivial background field configuration \( \langle A_0 \rangle \neq 0 \), to be determined dynamically. Doing so in the Landau gauge is, however, problematic because the latter explicitly breaks the centre symmetry, which clearly plays a key role here. Convenient ways to encode both a nontrivial background and the essential aspects of the centre symmetry have been put forward in [BGP10] and, more recently, in [VERST21], which uses background-field techniques [Abb81, Abb82] and, in particular, the background-field generalisation of the Landau gauge, the Landau–DeWitt (LDW) gauge [Wei96]. Similar to the massive extension of the former, discussed so far in this article, the massive extension of the latter or, in other words, the LDW version of the CF model has been worked out in [RSTW15b] and used as a starting point for a perturbative analysis of the nonzero temperature physics, in the presence of a nontrivial Polyakov loop\(^{34}\).

7.1. The Landau–DeWitt gauge

The background field approach [Wei96] introduces an \textit{a priori} arbitrary background field configuration \( A_\mu \), through a modified gauge-fixing condition. In terms of \( a_\mu = A_\mu - \bar{A}_\mu \), the LDW gauge condition reads

\[
D_\mu a_\mu = 0, \tag{54}
\]

where \( D_\mu \equiv \partial_\mu + g f^{abc} A_\mu^a \phi^b \). One can construct the corresponding FP Lagrangian using standard techniques. One important property of the resulting gauge-fixed theory is a formal gauge invariance with respect to gauge transformations of the background field. The simplest background-field generalisation of the CF action which respects this essential property is [RSTW15b]

\[
S_A = \int_x \left\{ \frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a + \frac{m^2}{2} a_\mu^a a^\mu_\mu + D_\mu c^a D^\mu c^a + i \bar{c} \sigma^a \partial_\mu c^a \right\}, \tag{55}
\]

which is clearly invariant under the transformation \( A_\mu^U \rightarrow A_\mu^U = U A_\mu U^{-1} + \frac{i}{g} U \partial_\mu U^{-1}, \varphi \rightarrow U \varphi U^{-1} \), with \( \varphi \equiv (a_\mu, c, \bar{c}, h) \) and where \( U \) is an element of the gauge group. This linear symmetry is inherited by the effective action,

\[
\Gamma_A[\varphi] = \Gamma_{\tilde{A}}[U \varphi U^{-1}], \tag{56}
\]

provided the transformation \( U \) preserves the periodic boundary conditions of the fields. Such transformations, which form a group denoted \( G \), do not need to be periodic themselves, however, but only periodic up to any element of the centre of the gauge group, \( e^{i2\pi k/kN} 1 \) \((k = 0, \ldots, N_c - 1) \) in the case of SU\((N_c)\). Their relevance is that they act by multiplying the Polyakov loop \( \ell \) by the corresponding centre phase. In particular, this relates the confining phase, where \( \ell \) vanishes, to the phase where the symmetry group \( G/G_0 \) is explicitly realised, where the subgroup \( G_0 \) of periodic transformations \((k = 0)\) needs to be quotiented away because it has no impact on (and is therefore not probed by) the Polyakov loop. The quotient group \( G/G_0 \) is isomorphic to the centre of the gauge group and is known as the centre symmetry group [Pis02].

A convenient way to study the spontaneous breaking of \( G/G_0 \), and in turn the deconfinement transition, is from the functional

\[
\tilde{\Gamma}[\bar{A}] = \Gamma_{\tilde{A}}[\varphi = 0] \tag{57}
\]

which is also invariant under the transformations \( U \in G \). A given state of the system is represented by a minimum of \( \tilde{\Gamma}[\bar{A}] \) denoted \( \bar{A}_{\text{min}} \), or by any other minimum obtained from it using a transformation \( U_0 \in G_0 \). In other words, a physical state corresponds to a \( G_0 \)-orbit of minima [RSTW16, Rei20].

The center-symmetric states are those \( G_0 \)-orbits that are invariant under the action of \( G/G_0 \) and the deconfinement transition occurs when the \( G_0 \)-orbit of minima of \( \tilde{\Gamma}[\bar{A}] \) moves away from its center-symmetric configuration.

The above considerations are greatly simplified if one restricts to constant temporal background fields along the diagonal or commuting part of the algebra \( A_\mu(x) = \delta_{\mu 0} A_0^I t^I \), with \( [t^I, t^J] = 0 \). In this case the functional \( \tilde{\Gamma}[\bar{A}] \) can be traded for the potential

\[
V(r) = \frac{\tilde{\Gamma}[\bar{A}]}{\beta \Omega} - V_{\text{vac}}, \tag{58}
\]

where \( \Omega \) is the spatial volume, \( V_{\text{vac}} \) the vacuum (zero temperature) contribution, and \( r \in \mathbb{R}^{N_c-1} \) is the vector of components.

---

\(^{34}\) We mention that a different approach to nonzero temperature physics has been pursued in the context of the screened perturbation theory, working directly in the Landau gauge with vanishing background [CS18].
72. The background-field effective potential in perturbation theory

The potential (58) has been computed at one-loop order in the (background-field-extended) CF model for a large variety of gauge groups [RSTW15b, RSTW16]. Two-loop corrections have also been computed for the SU(2) and SU(3) groups [RSTW15a, RSTW16]. We briefly describe the salient aspects of these calculations and we review the essential results here.

Calculations in the LDW gauge with a constant temporal background as specified above are greatly simplified if one switches from the usual Cartesian colour bases \( \{ r^i \} \) to the Cartan–Weyl bases \( \{ r^\kappa \} \) [RSTW16]. The labels \( \kappa \) are vectors that gather the adjoint colour charges \( \kappa^I \) of each colour mode such that \( [r^i, r^\kappa] = \kappa^I r^I \), where the generators \( r^I \) span the commuting part of the algebra, also known as the Cartan subalgebra. In the SU(2) case, for instance, a Cartan–Weyl basis is \( \{ \bar{t}, r^+, r^- \} \), where \( \bar{t} = \sigma_3/2 \) and \( r^\pm = (\sigma_1 \pm i\sigma_2)/(2\sqrt{2}) \) are the well-known raising and lowering operators, with \( \sigma_I \) the Pauli matrices. A convenient property of the Cartan–Weyl bases is that they allow for a one-to-one correspondence between the Feynman rules (and, consequently, of many calculation steps) with and without the background field. In particular, the only effect of the background is to shift the Euclidean four-momentum \( p \) of the propagator lines associated to a colour mode \( \kappa \) as \( p_\mu \rightarrow p^\kappa_\mu = p_\mu + T(r \cdot \kappa)\delta_{\mu 0} \). This generalised momentum is conserved at vertices owing to momentum and colour conservation.

The calculation of the background-field potential \( V(r) \) at one-loop order is straightforward and yields [RSTW15b]

\[
V_{\text{one-loop}}(r) = \frac{T}{2\pi} \sum_{\kappa} \int_0^\infty dq q^2 \left[ 3 \ln \left( 1 - e^{-\beta q + i\kappa_\mu r^\mu} \right) - \ln \left( 1 - e^{-\beta q - i\kappa_\mu r^\mu} \right) \right].
\]

with \( \varepsilon_q = \sqrt{q^2 + m^2} \). By setting \( r \) to zero, one recognises the free-energy density of a gas of free massive gluons and massless ghosts. The factor of 3 relates to the fact that there are three massive transverse gluonic modes. The longitudinal gluonic mode is massless and cancels with one of the two ghost degrees of freedom. The background lifts the degeneracy between the various colour modes and shifts the dispersion relations by an imaginary amount \( i(r \cdot \kappa)T \) that can be interpreted as an imaginary chemical potential for the colour charge [FS17].

The essential features of the one-loop potential can be unveiled by looking at the two asymptotic regimes \( T \gg m \) and \( T \ll m \). In the high temperature limit, all modes can be considered approximately massless and, for each colour state, the ghost contribution in (60) cancels against one gluon mode, leaving only the two ‘physical’ polarisations of the massless gluons. This results in the known Weiss deconfining potential [Wei81, PY80], which displays maxima at the confining points of the Weyl chambers. At low temperatures, instead, the massive gluon modes in (60) are exponentially suppressed and the potential is dominated by the (massless) ghost
located at model, for temperatures \( m \) propagators at zero temperature, corresponding to \( m \approx 510 \text{ MeV} \) and \( m \approx 710 \text{ MeV} \), respectively\(^{37} \). This translates to the transition temperature scales, mentioned in section 5.4. As in the case of correlation functions, this phenomenon is not restricted to the CF approach but applies to a wide class of continuum approaches using background field techniques [BGP10, FLP15, QR17]. For pure YM theories, one finds an actual phase transition between the high and low temperature phases.

Figure 22 shows the one-loop background-field potentials \( \mathcal{V}(r) = V(r)/T^4 \) for the SU(2) and the SU(3) theories at one-loop order in the CF model, for temperatures \( T = T_c \) (black), \( T < T_c \) (blue), and \( T > T_c \) (red). The abscissae represent the charge conjugation invariant directions in the corresponding Weyl chambers, along which the confinement–deconfinement transition takes place. The confining point is located at \( r = \pi \) for SU(2) and at \( r_3 = 4\pi/3 \) for SU(3). Reproduced from [RSTW15b]. CC BY 4.0.

The evaluation of the potential is done by using the same zero-temperature approaches using background field techniques [BGP10, FLP15, QR17]. For pure YM theories, one finds an actual phase transition between the high and low temperature phases.

Figure 22 shows the one-loop background-field potentials for the SU(2) and the SU(3) theories in \( d = 4 \) in the relevant directions—along which the transition takes place—in the respective Weyl chambers. One finds a continuous transition in the SU(2) case and a first order transition for SU(3), in agreement with lattice simulations [Sve86] and other approaches [BGP10, QR17]. We find that the transitions for SU(2) and SU(3) occur, respectively, at \( T_c/m \approx 0.336 \) and \( T_c/m \approx 0.364 \). Using the values fitted against the lattice propagators at zero temperature, corresponding to \( m = 710 \text{ MeV} \) and \( m = 510 \text{ MeV} \) for SU(2) and SU(3) respectively\(^{37} \), this translates to the transition temperatures reported in table 1, which are in remarkably good agreement with the lattice values [LP13] given the simplicity of the one-loop approximation. The inclusion of two-loop corrections, again, with parameters adjusted to reproduce the vacuum propagators at the relevant order, yields the values summarised in table 1. The improvement is clear.

The corresponding Polyakov loops, at the same order of approximation, are shown in figure 23. As was also pointed out in [DGH+12] in the context of matrix model calculations, the rise of the Polyakov loop from its value right above the transition temperature to its maximum value is too fast as compared to lattice results. This compromises any direct comparison of this quantity to Monte-Carlo simulations. We shall come back to this below.

7.3. The issue of massless unphysical modes in the confined phase

We have seen that the ghost dominance at low temperatures plays a pivotal role in the confinement–deconfinement mechanism described above. Although this mechanism is quite general and goes in fact beyond the particular case of the CF model considered in this review, it poses certain challenges that we now describe.

First, the fact that ghost degrees of freedom dominate in the low temperature phase may lead to inconsistent thermodynamics\(^{38} \). For instance, in the absence of the background, the ghost contribution results in a negative thermal pressure or a negative entropy in the low temperature limit [CS18, SR12]. Fortunately, the presence of the nontrivial background cures this pathological behaviour [RSTW16, QR17]. The reason is that, in the \( T \rightarrow 0 \) regime, the confining gluonic background operates a transmutation of the ghost thermal distribution functions, such that the net contribution to the pressure or entropy density remains positive. As an illustration, consider the SU(2) pressure which, at low temperature, can be written

\[
    p_h \sim \frac{1}{6\pi^2} \int_0^\infty dq \; q^3 \left[ -n_{q-\imath \kappa T} - n_q - n_{q+\imath \kappa T} \right], \quad (61)
\]

where \( n_q \equiv 1/(\exp[qT] - 1) \) is the Bose–Einstein distribution function. The bracket contains the three colour mode contributions, corresponding to \( \kappa \in \{-, 0, +\} \). The neutral mode

\(^{37}\) The evaluation of the potential is done by using the same zero-temperature renormalisation scheme as that used for the evaluation of the zero temperature propagators. More precisely, at one-loop order, the background dependent part of the potential is finite and the mass can be considered as the bare one, fixed by fitting tree-level expressions for the propagators to lattice data. At two-loop order, the background dependent part of the potential diverges and its renormalisation requires rescaling the mass parameter using one-loop renormalisation factors that should be taken equal to those used in the one-loop propagators.

\(^{38}\) More generally, this is related to the question of the proper identification of the physical space of the model, discussed in section 4.3, over which thermal averages are to be taken. In the FP theory, the nilpotent BRST symmetry guarantees that states with negative (or null) norm do not contribute to thermal averages and thus to thermodynamic observables (except through loop effects). For instance, at one-loop order, the ghost contribution cancels that of the two ‘unphysical’ gluon modes.
$\kappa = 0$ is blind to the background and contributes negatively to the thermal pressure, as expected for a ghost degree of freedom. Were it not for the imaginary shift $\pm i\pi T$ of their energies, the two other modes would give a similar negative contribution\(^{39}\). However, because of the presence of the confining background, these modes contribute instead with $-n_q \pm i\pi T = 1/(e^{\beta q} + 1)$, that is, a positive Fermi–Dirac distribution, leading eventually to a positive thermal pressure and a positive entropy at low temperature [RSTW15a]. These considerations extend to SU(3) [RSTW16].

The transmutation mechanism described here is also visible (but with the opposite effect) as one approaches the transition from below. In this limit, the gluon degrees of freedom are not exponentially suppressed, while the background remains confining, turning some of their positive distribution functions into negative ones, which tend to bring the thermal pressure or the entropy density down to a slightly negative value at one-loop order. This feature is clearly visible in the thermodynamical observables as illustrated in figure 24 for the entropy density. It has been shown, however, that the two-loop result corrects this unphysical feature [RSTW15a, RSTW16].

The low-temperature phase is plagued by yet another major problem. Namely, the dominant ghost degrees of freedom, be they surrounded or not by a confining gluonic background, remain massless. This leads to power law behaviour of the thermodynamical observables as $T \to 0$, at odds with the observations on the lattice. Again, it is worth emphasising that these issues are not restricted to the perturbative CF model and actually encompass all current continuum approaches to YM/QCD [QR17, SR12, CDJ+15]. This issue points to the inability (to date) of continuum approaches to provide a fully consistent picture of confinement.

### 7.4. Center-symmetric background field approach

The results described above rely on the use of the background functional (57) (or the corresponding potential (58)),

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\(^{39}\) The $T \to 0$ expression in the absence of the background is given by equation (61) with $q \pm i\pi T$ replaced by $q$. We recover here that the Landau gauge CF model predicts a negative pressure $p = -\pi^2 T^4/30$ at low temperatures [RSTW15a, CS18].

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**Table 1.** Estimates for the confinement–deconfinement transition temperatures at one- and two-loop orders within the CF model with either the self-consistent (CF$^{(s)}$) [RSTW15b, RSTW15a, RSTW16] or the center-symmetric (CF$^{(c)}$) [VERST21] background field approaches—see text—and their comparison to the corresponding lattice results [LP13].

| Theory | Lattice | One-loop CF$^{(s)}$ | Two-loop CF$^{(s)}$ | One-loop CF$^{(c)}$ |
|--------|---------|---------------------|---------------------|---------------------|
| SU(2)  | 295     | 237                 | 284                 | 265                 |
| SU(3)  | 270     | 185                 | 254                 | 267                 |
which involve self-consistent background fields defined as \( \hat{A} = \langle \hat{A} \rangle \). It is to be emphasised that this quantity is not, strictly speaking, an effective action in the sense that it is not a standard Legendre transform and that it is a functional of the background field (which is a gauge-fixing device). Because of this, such a self-consistent background field approach suffers from various technical difficulties—not to be described here, see e.g. [RSTW16, Rei20, VERST21]—when approximations are involved and for its possible implementation in lattice simulations.

An alternative approach has been recently proposed, that avoids these difficulties [Rei20, VERST21]. It is based on using a fixed background field \( \hat{A} \) that is invariant under the action of \( G/G_0 \). The corresponding action \( \Gamma_{\hat{A}}[\hat{A}] \) is a genuine gauge-fixed effective action which properly encodes the centre symmetry and from which one directly obtains the gluon correlation functions. Also, it can be implemented in lattice simulations with techniques currently used for the Landau gauge. It has been evaluated at one-loop order in the CF model for constant, center-symmetric background field configurations in the temporal direction and in the Cartan subalgebra of the gauge group [VERST21]. This yields the correct phase structure for the SU(2) and SU(3) YM theories with transition temperatures reported in table 1, in remarkable agreement with lattice results, much better than the corresponding one-loop results in the previous self-consistent background field approach.

Another interesting improvement concerns the temperature dependence of the Polyakov loop. In particular, it shows a moderate rise in the deconfined phase, as compared to the self-consistent background field approach, which compares well with lattice results [GHK08, LFK+13], see figure 25. Here, it is important to stress that the lattice data correspond to the renormalised Polyakov loop \( \ell(T) \), related to the bare one computed here as \( \ell(T) = Z_{\ell}(T) \), where the renormalisation factor \( Z_{\ell} \) is a function of the coupling. At first sight, it might seem sufficient to simply normalise the perturbative results to the lattice data at a given temperature. Doing so, a rather good fit is obtained in the range \([0, 2T_c]\). The agreement deteriorates for larger temperatures, where important effects not taken into account in this simple one-loop calculation, such as the resummation of hard thermal loops [HBA+14] or the RG running [HLP15, KPR21], become important.

### 7.5. Dynamical quarks

The success of the CF model in describing the finite temperature confinement–deconfinement transition in the pure YM case naturally leads one to investigate how well it can capture the phase structure in the presence of quarks. As in the vacuum case, two regimes must be distinguished, depending on the values of the quark masses. For large quark masses, the quark–gluon coupling does not differ substantially from the one in the pure gauge sector and one can rely on perturbation theory. Admittedly, this regime does not correspond to the physical QCD case. However, it possesses a rich phase structure which has been studied using various approaches in the literature. It should also be mentioned that, in this range of masses, the phase transition is still akin to a description in terms of the Polyakov loop or the self-consistent background.

At vanishing quark chemical potential, there exists a critical phase boundary in the space of quark masses, separating a region of first-order phase transitions for large quark masses (including the YM case) from a crossover region for lower quark masses (including the physical QCD point), see the left panel of figure 26. For \( N_f \) degenerate flavours, the boundary is characterised by the ratios \( R_{N_f} = M_c(N_f)/T_c(N_f) \), with \( M_c(N_f) \) and \( T_c(N_f) \) the critical quark mass and critical temperature respectively. Those have been computed at one- and two-loop orders in the CF model within the self-consistent background field approach described previously. The one-loop results, shown in the left panel of table 2, are in remarkable agreement with lattice estimates. It is important to notice that, at one-loop order, these dimensionless ratios do not involve the gluon mass parameter or the gauge coupling and are, therefore, a parameter-free prediction of the model.

We mention that for any model of the YM sector, such as the CF model considered here, the ratios \( R_{N_f} \) for different \( N_f \) at one loop are related by the universal relation [KPS12, MRS18b]

\[
N_f R_{N_f}^2 K_2(R_{N_f}) = N_f' R_{N_f'}^2 K_2(R_{N_f'}),
\]

with \( K_2 \) the modified Bessel function of the second kind. Again, two-loop corrections have been computed [MRS18a]. Although small, their general tendency is to approach the lattice results. As explained in this reference, the ratios \( R_{N_f} \) cannot be directly compared with the lattice data beyond one loop because of the different meanings of the (regularisation and renormalisation dependent) quark masses. One way to (approximately) cope with this issue is to consider the ratios \( Y_{N_f} = (R_{N_f} - R_1)/(R_2 - R_1) \), which compare pretty well with the lattice result, known for \( N_f = 3 \) [MRS18a, MRS18b].

The phase structure of the theory has also been investigated in the presence of a quark chemical potential [RST15, MRS18a, MRS18b]. For SU(3), this requires considering backgrounds with two nonvanishing components \( r_3 \) and \( r_8 \), a nonzero \( r_8 \) component being dictated by the breaking of charge conjugation invariance due the presence of a chemical
potential. For imaginary values of the chemical potential, one obtains the Roberge–Weiss (RW) transition \([RW86]\), characterised by a first order jump of the phase of the Polyakov loop at \(\mu/T = i\pi/3\) (and large enough temperatures). The RW transition is not disconnected from the phase boundary at \(\mu = 0\). In fact, as the quark masses are decreased below their critical values, the critical boundary enters the imaginary \(\mu\) region. As it reaches the particular value \(\mu/T = i\pi/3\) where the RW transition takes place, one finds a tricritical point characterised by the scaling law \([dFP10]\)

\[
R_{N_f}(\mu = 0) = R_{N_f}^{\text{tric}} + K_{N_f} \left( \frac{\pi}{3} \right)^2 + \left( \frac{\mu}{T} \right)^2 \right)^{2/5},
\]

(63)

for \(\mu/T \rightarrow i\pi/3\). The (parameter-free) one-loop estimates of \(R_{N_f}^{\text{tric}} \equiv R_{N_f}(\mu/T = i\pi/3)\) and \(K_{N_f}\) in the CF model \([RST15]\) are, again, in fairly good agreement with the lattice results \([FLLP12]\); see table 2 and the right panel of figure 26. Also, as before, two-loop corrections tend to improve these values \([MRS18a]\). One also finds that the scaling law (63) extrapolates deep into the real chemical potential region \(\mu^2 > 0\), as also seen in nonperturbative continuum approaches. Let us mention that a proper treatment of real chemical potentials requires the background \(r_5\) to be chosen purely imaginary. This surprising result relates to the QCD sign problem: for a real chemical potential, the QCD action is not real and there is \textit{a priori} no reason to find real self-consistent backgrounds—i.e., that solve the equation \(\hat{A} = (A)\hat{A}\). On the other hand, it has been shown \([RST15]\) that backgrounds of the form \((r_3, r_5) \in \mathbb{R} \times i\mathbb{R}\) are compatible with the self-consistency assumption.

In the case of light quarks, the situation is not different from that in the vacuum: the quark–gluon coupling is a few times larger than the pure gauge coupling, thus preventing the use of perturbation theory. One can exploit the strategy developed in the vacuum based on the double expansion in powers of the inverse number of colours and the pure gauge coupling. As already explained above, this involves the resummation of rainbow diagrams in the quark propagator. Solving the corresponding equations can be done with present day technology \([Fis19]\) but has not been attempted yet in the context of the CF model. Instead, as a first step in this direction, a drastically simplified version of the rainbow resummation at nonzero temperature and quark chemical potential has been implemented in \([MRS20]\), which gives encouraging results.

7.6. Propagators

The results presented in the previous subsections concern gauge-invariant quantities. As in the vacuum, lattice simulations also provide results for gauge-dependent correlation functions at nonzero temperature and density. In particular, many results have been produced for the YM and QCD propagators in the Landau gauge \([HKR95, HKR98, CKP01a, CKP01b, CMM07, FMM10, CM10, CM11, ABI+12, MPvSS12, SOBC14, SO16, KS21, SBHK19, SK19]\).
Figure 27. The magnetic gluon (left) and the ghost (right) propagators at vanishing (Matsubara) frequency as a function of spatial momentum in the SU(2) YM theory at $T = T_c$, in the Landau gauge. The curves are the one-loop results (without RG improvement) in the CF model and the blue points are the data from [MPvSS12]. Reprinted figure with permission from [RSTW14], Copyright (2014) by the American Physical Society.

The Landau gauge ghost and gluon propagators have been computed at one-loop order in the CF model—i.e., with no background—at nonzero temperature and compared to lattice data for the SU(2) YM theory [RSTW14]. The magnetic40 and ghost propagators are rather well reproduced, see figure 27. In contrast, the temperature dependence of the electric propagator differs substantially from the existing lattice data around the transition temperature [FMM10, MPvSS12]. We stress, however, that, first, this discrepancy is not specific to the perturbative CF approach, which produces in fact results very similar to those of nonperturbative continuum approaches [FP11] and, second, that the lattice results in the electric sector suffer from large uncertainties [MC14]. It has been suggested [FP11] that the discrepancy between continuum and lattice results may originate from the fact that the order parameter associated to the deconfinement transition is not properly accounted for in the perturbative Landau gauge calculations. Also, perturbative calculations in the CF model suggest that the limit of vanishing background field, corresponding to the Landau gauge, is unstable against small deviations [RSTW15b], which may explain the large numerical uncertainties mentioned here.

This has opened the way to the evaluation of correlation functions in background Landau gauges using the CF model [RSTT17, VERST21] with the idea that the proper inclusion of the order parameter may stabilize the comparison to lattice results. Although data for two-point functions in such gauges are not available yet, some interesting results have already been obtained within the CF model. In the self-consistent background gauge, the zero-momentum limit of the SU(2) electric propagator features a relatively sharp peak at the transition [RSTT17] which turns into a divergence in the case of the center-symmetric background gauge [VERST21]. In fact it has been argued that this divergence should also be there in the self-consistent approach [Rei20] were it not for the use of (inevitable) approximations that jeopardise certain properties of the gauge-fixing. In this sense, the results obtained with the center-symmetric background gauge, because they do not rely on these properties, should be more robust.

Similar one-loop calculations (without background) have been performed and compared to lattice calculations in the Landau gauge for two-colour QCD at nonzero quark chemical potential41 [SK19, KS21]. In this case, the physics is that of a possible Bardeen–Cooper–Schieffer (BCS) phase with a nontrivial pairing between quarks. The mentioned references consider the effect of a phenomenological BCS gap $\Delta$ on the electric and magnetic components of the gluon propagator through the quark loop contribution. As for the nonzero temperature case above, one allows for a variation of the CF parameters (coupling and gluon mass) with the chemical potential, to account for possible in medium effects. In a nutshell—we refer the reader to these references for details—one obtains good agreement between the one-loop expressions and the lattice data, as illustrated in figure 28 for the electric propagator. A similar quality is achieved for the magnetic sector as well.

8. Results in the Minkowskian domain

So far, we have reviewed the large piece of evidences accumulated over the past decade which strongly support the idea of a valid perturbative description of (some aspects of) the IR dynamics of YM and QCD-like theories. The many successful comparisons to lattice data in the Euclidean domain encourage one to use the perturbative approach in situations where lattice techniques are not available. One example, described above, is the phase diagram of QCD at nonzero quark chemical potential. Another important example concerns the study of correlation functions in the Minkowskian domain.

40 In the Landau gauge, the gluon propagator is transverse with respect to the gluon four-momentum. In the vacuum, Lorentz invariance guarantees that there is only one possible scalar function. At nonzero temperature, however, the Lorentz symmetry group is explicitly broken into its rotation subgroup and there exist two independent 3D-longitudinal (electric) and 3D-transverse (magnetic) components.

41 For $N_c = 2$, lattice simulations are possible at nonzero chemical potential because there is no sign problem: the integration measure under the Euclidean path integral is positive definite.
Figure 28. The electric gluon propagator in two-colour QCD in the Landau gauge at zero temperature and nonzero quark chemical potential $\mu_q$. The various curves correspond to the one-loop result in the CF model (with no RG improvement) for different values of the coupling $[SK19]$. The chemical potential increases in the plots from left to right. The vacuum ($\mu_q = 0$) result is also shown (dotted line) for reference.

The quark gap $\Delta$ is held fixed and the gluon mass parameter is adjusted by hand to obtain a good overall description of the lattice data (squares) of $[BHMS19]$. Reproduced from $[SK19]$. CC BY 4.0.

Figure 29. Real (left) and imaginary (right) parts of the vacuum gluon propagator at one-loop order in the (pure gauge) CF model as a function of the square momentum. The calculation is done within the IRS renormalisation scheme and does not include RG running. Here, the CF gluon mass is denoted $M$ and $\lambda = g^2 N_c / (16 \pi^2)$, with $N_c = 3$. The values of the parameters are chosen to provide a good fit of the lattice data for the real part in the Euclidean domain $s = k^2 < 0$. The imaginary part of the propagator, the so-called spectral function is negative, indicating that the (massive) gluon is not an asymptotic state in the theory, in line with the expectation from confinement. Reproduced from $[KWH+20]$. CC BY 4.0.

The analytic structures of the gluon, ghost, and quark propagators in the complex momentum plane have been studied at one-loop order in the Landau gauge CF model $[HK19, KWH+20, HK20, HK21]$ as well as in the screened perturbation theory approach $[Sir16a, Sir17, SC21]$ in the vacuum and at nonzero temperature and density, with and without RG improvement. These studies are based on analytically continuing the Euclidean propagators$^{42}$ to the whole complex plane of square momentum $s = -p^2$. The most important results are that both the gluon and the quark propagator possess pairs of complex conjugate poles and that their spectral functions are not positive definite. The spectral functions mentioned here are defined as the imaginary parts of the corresponding propagators along the real $s$ axis. They vanish identically in the Euclidean domain $s \leq 0$ and are nonzero for Minkowskian momenta $s > 0$. It is worth emphasising that the spectral functions defined in this way are not exactly those entering the (assumed) spectral representations discussed in section 5.3, due to the presence of poles away from the real $s$ axis; see, e.g., $[HK20]$. So, although related, the nonpositive spectral functions reported here are not to be put in one-to-one correspondence with the positivity violations mentioned in section 5.3, observed in lattice calculations (figure 29).

This being said, both the nonpositive spectral functions and the presence of pairs of complex-conjugate poles in the $s$-plane are in line with the fact that neither the massive gluon field nor the quark field correspond to actual asymptotic states. That is sometimes viewed as a sign of confinement although, as explained in section 3.1, this interpretation is subject to caution. First, the nonzero imaginary part of the poles results in a finite lifetime of possible gluonic or quark excitations. Second, the nonpositive spectral functions show the absence of

$^{42}$ Note that this is not quite the same thing, in general, as working with the Minkowskian version of the CF model. In particular, the presence of pairs of complex poles (see below) jeopardises the usual Wick rotation. It is not known to us whether a clear link exists between the (analytically continued) Euclidean CF model and its Minkowskian version beyond perturbation theory.
a proper Källen–Lehmann representation of asymptotic states with positive norm.

9. Open questions

At this point, we hope to have convinced the reader that the CF model provides a very efficient framework for a valid description of many aspects of IR QCD based on controllable expansion schemes. Confronted to the many successful results reviewed here, a natural question that comes to mind is: how can such a simple model be so efficient? Can this be accidental? If one views the gluon mass parameter as a mere phenomenological IR deformation of the FP theory, the model works beyond expectations, almost unbelievably well. This calls one to wonder whether the CF model could have a deeper connection to (Landau gauge) IR QCD. This line of thought requires one to seriously address the various open problems of the CF model. Let us briefly mention—and speculate about—some of the most pressing issues.

One essential question is the status of the CF mass parameter. An interesting possibility is that it could be related to the issue of properly fixing the gauge in nonabelian theories [ST12, KKS15]. A recent proposal along these lines in the Landau gauge, exploiting the early ideas of [ST12] to handle the Gribov problem, has been shown to accommodate for a gluon mass term [RSTT21], although the resulting gauge-fixed action slightly differs from that of the CF model. Such gauge fixings do not exactly correspond to those realised in lattice simulations, but the successful comparisons described in this review suggest that the mass parameter can be adjusted to mimic the essential features of the latter [ST12, Maa12]. Another attractive possibility would be that the gluon mass is a dynamical consequence of the gauge-fixing procedure, whose value is fixed from the sole knowledge of the gauge coupling at a given scale, similar to what happens in the GZ approach [VZ12]. A concrete realisation of this scenario has been proposed in a general class of nonlinear gauges in [Tis18] which, unfortunately, does not survive the Landau gauge limit. Finally, it could also be that the gluon screening mass is an actual feature of the FP theory in the Landau gauge at a nonperturbative level. This has been investigated in the context of the screened perturbation theory, where the mass term is introduced as a variational parameter that must eventually be fixed by an extremisation procedure. A first interesting attempt in this direction has been worked out in [CS18] which, however, lacks the systematics of a genuine loop expansion and sometimes leads to unphysical results (like a spurious first order phase transition at nonzero temperature for the SU(2) YM theory). Finally, we mention that the hypothesis that the gluon mass is dynamically generated in the FP theory underlies all the studies based on nonperturbative continuum approaches.

A more technical question is that of IR divergences in the Minkowskian domain. As mentioned in section 5, IR divergences in the Euclidian domain were studied in [TW11] and it was shown that, thanks to the presence of the gluon mass, there are no IR divergences for non-exceptional Euclidean momenta (for $d > 2$). The case of exceptional Euclidean momenta is more delicate [TW11]. Also, the analysis of IR divergencies in the Minkowskian domain has not yet been carried out. This remains an open question because, as in QED, and despite the fact that the gluons are massive in the present model, the analysis of on-shell IR divergences is far from trivial due to the presence of massless modes in the CF model.

Other major issues are the construction of a proper physical space and the question of unitarity. As explained in section 4.3, and contrary to a widespread idea, this is still an open question within the CF model. In fact, it is important to stress that it is not even understood in the standard FP approach beyond perturbation theory. All continuum approaches have to face this issue, which directly affects the calculation of physical observables. A simple but generic example is the thermodynamic pressure at low temperatures. In all cases, the difficult task is the identification of the actual (confined) physical space.

An intricately related question, the mother of all, is that of confinement. The lack of perturbative unitarity in the CF model has led to its disproval as an alternative to the Higgs’ mechanism for a consistent theory of massive gauge bosons. However, it could very well be that the model is confining, which would completely change the game. Despite its importance, the precise definition of confinement has not yet been clearly established in the case of QCD (see e.g. the discussion in [Gre]). In the case of pure YM theory, the commonly accepted criterion for characterizing confinement corresponds to the Wilson loop area law. Although the Wilson loop is a perfectly well-defined quantity in the Euclidean domain, the area law corresponds to a linear behaviour for the static potential of a very distant quark–antiquark pair. In this limit, this potential is dominated by the behaviour of correlation functions with momenta out of the Euclidean domain near the singularities of various correlation functions. Accordingly, this behaviour is plagued by IR divergences that, at the moment, have not been put under control in the CF model. Controlling these IR divergences and reproducing the area low behaviour would clearly be of utmost importance.

10. Conclusions

The CF model offers a promising avenue for investigating IR properties of continuum nonabelian gauge theories in the Landau gauge beyond the textbook FP gauge-fixing prescription, which is limited to the UV. The systematic analysis of the two- and three-point vacuum correlation functions of the model and their comparison with results of ab initio lattice simulations in YM and QCD-like theories strongly support the idea that a tree-level gluon mass parameter in one way or another [RSTW17] and are thus effectively based on the CF Lagrangian rather than on the FP one. The basic assumption of these approaches is that there exists a particular value of the CF mass which actually corresponds to the FP theory.
the inclusion of a gluon mass operator beyond the FP Landau gauge action suffices to capture most of the qualitative and many of the quantitative features of the Landau gauge correlations functions in the vacuum, from which one can extract physical observables.

The remarkable point is that many of these results (those for YM theory but also those for QCD with heavy quarks) rest on a purely perturbative approach within the CF model. For one thing, the model admits renormalisation group trajectories defined over the whole range of scales, in blatant contrast to the FP approach, which features an IR Landau pole, and in good agreement with lattice data. For another, the trajectories that allow one to best reproduce the results of simulations correspond to a running gauge coupling that never gets excessively large, thus justifying a posteriori the use of a perturbative approach. This, in turn, not only allows for a simple computational setup at leading order, in comparison to more demanding nonperturbative methods, but also offers the possibility to systematically investigate higher-order corrections and, thereby, the validity of the approach. Various two-loop corrections have been evaluated for YM and (heavy-quark) QCD two- and three-point correlation functions and have been found to be globally small while improving the quality of the comparison to the lattice data.

The perturbative approach is not a panacea, however, not even within the CF model, and certain questions require one to go beyond the simple coupling expansion. This is the case of the light quark sector of QCD controlled by the spontaneous breaking of chiral symmetry and characterized by an enhanced quark–gluon coupling in the IR, as compared to the typical couplings in the pure gauge sector. The perturbative nature of the pure gauge sector of the CF model allows nonetheless for the construction of a systematic expansion scheme that rests on two small parameters, the pure gauge coupling on the one side, and the inverse number of colours on the other side. Already the leading orders of this expansion scheme suffice to capture the general features of the spontaneous breaking of chiral symmetry while providing a consistent picture of the various two-point correlation functions. It would be interesting in the future to extend this analysis to the three-point correlators as well. The ultimate goal is of course to investigate low energy observables, within nonabelian gauge theories in general, and within QCD in particular. A preliminary determination of the pion decay constant $f_\pi$ within this approach is very encouraging and gives good confidence that other observables, such as the spectrum of low-lying hadrons are within reach.

Various relevant observables at nonzero temperature and chemical potential have also been evaluated, including transition temperatures and nontrivial order parameters involving nonperturbatively large field configurations. As in the vacuum case, the perturbative CF model seems to capture most qualitative features of the phase structure such as the confinement–deconfinement transition in YM theories, the critical line in the heavy-quark region of the Columbia plot, as well as the RW transition for imaginary chemical potential.

In most cases, the description is even quantitative, with, for instance, transition temperatures that gives results comparing well with simulations at one loop and improving, significantly, at two-loop order. Similar conclusions emerge from the evaluation of quark mass-to-temperature ratios along the upper critical line in the Columbia plot. In the case of QCD with light quarks, perturbation theory is again not enough but one can extend the approach mentioned above for the vacuum case at nonzero temperature and chemical potential. Preliminary results show that the CF model has the potential to access features of the phase structure in this case too, in particular the possible existence of a critical end point in the QCD phase diagram. This needs to be confirmed by more refined studies.

Of course, there are still many open questions concerning the CF model and its use as (part of) a nonperturbative completion of the gauge fixing beyond the FP construction. Its many successes should serve as an incentive to better understand the nonperturbative gauge-fixing procedure, with the potential reward of granting a relatively simple access to some of the low-energy observables of QCD. Important issues concern, for instance, unitarity and the proper definition of the physical space of the model, the study of the potential between static colour sources and the possibility that the model is confining, the behaviour of thermodynamic observables at low temperature and the role of massless modes. Also, it would be of interest to investigate formal properties of the perturbative series at high order, for instance, how the gluon mass term modifies the infrared renormalon issue of the perturbative series in the FP approach.

Finally, it is interesting to study to what extent the results discussed in this review remain valid in other gauges than the Landau gauge, in particular, the key observation that the coupling in the pure gauge sector is small. Developments along these lines are underway and recent years have seen the first lattice calculations of the gluon propagator in linear covariant gauges [CMS09, BBC+15, CDM+18]. These gauges, which are continuously connected to the Landau gauge have been also implemented using analytical methods such as the optimized perturbation theory [SC18, Sir19b] and the GZ approach [CFG+15]. The class of nonlinear covariant gauges known as the Curci–Ferrari–Delbourgo–Jarvis gauges, also continuously connected to the Landau gauge, together with some of their massive deformations, including the generalized CF model, have also been studied within perturbation theory [STT14, STT15, Gra02], although they have not yet been implemented with lattice techniques.

To conclude, lattice simulations have firmly established (see figure 1) that at least part of the IR regime of QCD is governed by a coupling of perturbative size in the Landau gauge. Although it has remained largely unknown to a broad audience, this is an observation of paramount importance with far-reaching consequences. We believe that the work reviewed in this article clearly establishes that many facets of the IR QCD dynamics admit a perturbative description in the Landau gauge and that the CF model is an efficient framework for the latter, be it at a fundamental or at a more phenomenological level. We hope that the present article will convince the reader of the usefulness of a change of paradigm concerning the IR dynamics of nonabelian theories and will motivate QCD practitioners.
to include the CF model as one serious option in their toolbox [HLP20, SBHK19].

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Data availability statement

The data that support the findings of this study are available from the ANII-FCE-126412 project.

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