The non-extensive version of the Kolmogorov-Sinai entropy at work

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We address the problem of applying the Kolmogorov-Sinai method of entropic analysis, expressed in a generalized non-extensive form, to the dynamics of the logistic map at the chaotic threshold, which is known to be characterized by a power law rather than exponential sensitivity to initial conditions. The computer treatment is made difficult, if not impossible, by the multifractal nature of the natural invariant distribution: Thus the statistical average is carried out on the power index $\beta$. The resulting entropy time evolution becomes a smooth and linear function of time with the non-extensive entropy. The parameter $\beta$ means. Its departure from the standard value leads to the exponentially increasing interest \textsuperscript{7} on this kind of entropic index

$$\lim_{\Delta x(0) \rightarrow 0} \frac{\Delta x(t)}{\Delta x(0)}.$$  

where $\Delta x(t)$ denotes the distance, at time $t$, between the trajectory of interest and a very close auxiliary trajectory. The initial distance between the trajectory of interest and the auxiliary trajectory, $\Delta x(0)$, is made smaller and smaller so as to let emerge the kind of sensitivity of the dynamics under examination.

The authors of Refs. \textsuperscript{3–5} related the function $\xi(t)$ to the non-extensive version of the KS entropy with heuristic arguments. With these arguments they established that the analytical form to assign to $\xi(t)$ for the non-extensive form of the KS entropy to increase linearly is

$$\xi(t) = [1 + (1 - Q)\lambda Q t^{1/(1-Q)}],$$  

where $\lambda Q$ is a sort of Lyapunov coefficient. Throughout this letter we shall be referring to $Q$, predicted with entropic arguments, as true entropic index. In the specific case $Q < 1$, of interest here, Eq. \textsuperscript{5} means that the distance between the trajectory of interest and the auxiliary trajectory increases as an algebraic power of time. However, the numerical calculations made in Refs. \textsuperscript{4–5} show that the function $\xi(t)$ exhibits wild fluctuations, although the intensity of these fluctuations fulfills the prediction of Eq. \textsuperscript{5}. In Ref. \textsuperscript{4} a theoretical prediction was made for the power index $\beta$, and consequently for the true entropic index $Q = (\beta - 1)/\beta$. This prediction reads:

$$\frac{1}{1 - 1/\beta} = \frac{1}{\alpha_{\text{min}}} - \frac{1}{\alpha_{\text{max}}},$$  

where $\alpha_{\text{min}}$ and $\alpha_{\text{max}}$ denote the crowding indices corresponding to the minimum and maximum concentration, respectively.

More recently, the same problem of evaluation of the true entropic index $Q$, resulting in the linear increase of entropy as a function of time, was dealt with by the authors of Ref. \textsuperscript{6}, by means of the numerical calculation of the distribution entropy: The authors of this paper made, in fact, the delicate assumption that if a true $Q$ exists, making the trajectory entropy increase linearly in time, then the same $Q$ makes the distribution entropy increase linear in time also. In a sense this letter aims at checking this important assumption. This is a challenging problem, as the ascertainment of the equivalence of these two

\begin{equation}
H_q = 1 - \sum_{i=1}^{W} p_i^q \frac{1}{q - 1},
\end{equation}
distinct entropy forms is the object of discussion also in the case of strong chaos.

The problem here under study is that of the calculation of

$$H_q(N) = \frac{1 - \sum_{\omega_0...\omega_{N-1}} p(\omega_0...\omega_{N-1})^q}{q - 1}, \quad q \neq 1$$

(5)

where $p(\omega_0...\omega_{N-1})$ is the probability of finding the cylinder corresponding to the sequence of symbols $\omega_0...\omega_{N-1}$. This entropy expression affords a rigorous way of defining the earlier introduced concept of true entropic index. If it exists, $Q$ is the value of the entropic index $q$ making the entropy of Eq.(5) increase linearly in time. In the case $q = Q = 1$ the $\lim_{N \to \infty} H_q(N)/N$ becomes the ordinary Kolmogorov-Sinai (KS) entropy. In an earlier work \[12\] a numerical calculation was made to establish $Q$ in the case of a text of only two symbols, with strong correlations. The case of many more symbols would be beyond the range of the current generation of computers. However, when the sequence of symbols is generated by dynamics, as in the case here under study, and the function $\xi(t,x)$ of Eq.(3) is available (for convenience, we make now explicit the dependence on the initial condition $x$), it is possible to adopt the prescription of Ref. \[12\], which writes $H_q(t)$ of Eq.(3) as

$$H_q(t) \equiv \frac{1 - \delta^q - 1}{q - 1} \int dx p(x)^q \xi(t,x)^{1-q},$$

(6)

where the symbol $t$ denotes time regarded as a continuous variable. In fact, when the condition $N \gg 1$ applies, it is legitimate to identify $N$ with $t$. The function $p(x)$ denotes the equilibrium distribution density and $\delta$ the size of the partition cells: According to Ref. \[12\] the phase space, a one-dimensional interval, has been divided into $W = 1/\delta$ cells of equal size.

There is now an important remark to make: The non-extensive form of KS entropy should read as follows,

$$h_Q = \delta^{Q-1} \lim_{t \to \infty} \frac{1}{t} \int dx p(x)^Q \xi(t,x)^{1-Q}, Q \neq 1.$$  \(6\)

(7)

This apparently means that leaving the ordinary condition $Q = 1$ has the unwanted effect of making the generalized form of KS entropy dependent on $\delta$, thereby losing what we consider to be the most attractive aspect of the KS entropy. We are inclined to believe that this apparent weakness is, on the contrary, an element of strength. We shall see that in the case under discussion in this letter, due to the multifractal character of the natural invariant distribution $p(x)$, the prescription of Eq.(6) results in a rate of entropy increase independent of the cell size. In the case of the Manneville map \[3\], $x_{n+1} = x_n + x_n^2 \mod 1$, it is shown \[3\] that in the range $3/2 < z < 2$ the stationary correlation function of the variable $x$ exists and it is not integrable, thereby suggesting a possible breakdown of the ordinary KS entropy. However, in this case the natural invariant distribution is smooth, rather than multifractal, and as a consequence of that, the request of the independence of the cell size and the adoption of the prescription of Eq.\[6\] yield the condition $Q = 1$, in agreement with the conclusions of the work of Ref. \[4\]. In the case $Q = 1$ it is straightforward to prove that Eq.(6) results in

$$H_1(t) = \int dx p(x) \ln \xi(t,x).$$

(8)

On the other hand, in the ordinary case Eq.(6) becomes

$$\xi(t,x) = \exp(\lambda(x)t),$$

(9)

thereby making Eq.(6) result in the well known Pesin relation

$$h_{KS} \equiv \lim_{t \to \infty} \frac{H_1(t)}{t} = \int dx p(x) \lambda(x).$$

(10)

In conclusion, in the case of a smooth invariant distribution, either strongly or weakly chaotic, the prescription of Eq.(6) coincides with the Pesin theorem, which allows us to replace the direct calculation of the KS entropy with the numerically easier problem of evaluating Lyapunov coefficients. The case of fractal dynamics implies the existence of the true $Q \neq 1$, which has to be properly detected looking for the value of $q$ making $H_q(t)$ linearly dependent on $t$. This letter is devoted to providing the guidelines for this search.

We shall focus our attention on the calculation of

$$\Xi_q(t,\delta) \equiv \delta^{q-1} \int dx p(x)^Q \xi(t,x)^{1-q}.$$ \(11\)

Note that if $\xi(t)$ of Eq.(6) depended on $x$, the value $\Xi_Q(t,\delta)$ resulting from the joint use of Eqs.(11) and (3) would afford a simple recipe to determine the statistical average $\langle \lambda_Q(x) \rangle$ at $Q = Q$. In principle, $\Xi_q(t,\delta)$ depends on the cell size $\delta$. However, we plan to prove that if it is properly evaluated, this quantity turns out to be independent of $\delta$. As done in the earlier work of Refs. \[3\] \[4\] \[5\] \[8\], we study the logistic map:

$$x_{n+1} = 1 - \mu |x_n|^\alpha, x \in [-1,1]$$ \(12\)

with the control parameter $\mu = 1.4011551\ldots$, namely, at the threshold of transition to chaos. In this case the invariant distribution is multifractal, and consequently, expressed as a function of $x$, looks like a set of sharp peaks, which, in turn, through repeated zooming, reveal to consist of infinitely many other, sharper, peaks. This means that the direct evaluation of Eq.(6) is hard, since it is difficult to ensure numerically that these fractal properties are reproduced at any arbitrarily small spatial scale. We have to look for a different approach.

Let us replace the average over $x$ with the average over the crowding power index $\alpha$. In the long-time limit we obtain \[11\]
\[ \Xi_q(t) \equiv \beta^{q-1} \int d\alpha \delta^{\alpha} - f(\alpha) e^{\beta(t,\alpha)(1-q)}. \]  

(13)

This equation rests on assuming dependence on the initial condition only through the power law index \( \beta(t,\alpha) \) itself which, in fact, according to Anania and Politi Ref. [17], reads

\[ \beta(t,\alpha) = \frac{1}{\alpha(t)} - \frac{1}{\alpha}. \]

(14)

We shall show with theoretical and numerical arguments that this relation yields Eq. (13) for the exact value of \( \Xi \).

The symbol \( \alpha \) denotes the crowding index corresponding to a given initial condition \( x \), namely the position of the trajectory at \( t = 0 \), and the symbol \( \alpha(t) \) denotes the crowding index corresponding to the position of the same trajectory at a later time \( t > 0 \). According to Anania and Politi [17]

\[ \alpha(t) = \frac{\ln(1/t)}{\ln|x(t + 2^k) - x(t)|}, \]

(15)

where \( k \) indicates the \( k \)-th generation of the Feigenbaum attractor.

Before proceeding, let us make an assumption which has the effect of accomplishing, within the non-extensive perspective, the Kolmogorov program of an entropy independence of the size of the partition cells. First of all, let us rewrite Eq. (13) in the following equivalent form:

\[ \Xi_q(t) = \int d\alpha \ e^{W(f(\alpha) - q\alpha - q - 1)} e^{V(q-1)\beta(\alpha,V)}, \]

(16)

where:

\[ W \equiv -\ln \delta \]

(17)

and

\[ V \equiv -\ln(1/t). \]

(18)

Let assume now:

\[ W << V. \]

(19)

Under the plausible condition that the functions \( f(\alpha) - q\alpha - q - 1 \) and \( \beta(\alpha,V) \) are not divergent, this assumption has the nice effect of producing

\[ \Xi_q(t) = \int d\alpha \ e^{V(q-1)\beta(\alpha,V)}. \]

(20)

At this stage we make another crucial step. This is suggested by the work of Hata, Horita and Mori [18]. The idea is that of using \( \beta(\alpha,V) \) as independent variable so as to write Eq. (20) either as:

\[ \Xi_q(V) = \int d\beta P(\beta,V) e^{V(q-1)\beta(\alpha,V)}. \]

(21)

or under the equivalent form:

\[ \Xi_q(t) = \int d\beta P(\beta,t) t^{\beta(q-1)}. \]

(22)

We follow Ref. [18] again and we adopt the asymptotic property [19]:

\[ P(\beta,t) = t^{-\psi(\beta)} P(\beta,0). \]

(23)

The numerical calculation of the function \( \psi(\beta) \) is done with a criterion different from that adopted in Ref. [18]. The authors of Ref. [18] fix a window of a given size \( t \) and move it along the sequence for the purpose of evaluating the frequency of presence within this window of a given algebraic index \( \beta \). This means that they make an average over many different initial conditions. We, on the contrary, fix a given initial condition, and we increase the size of the window, the left border of which coincides with the initial time condition \( t = 0 \), whereas the right border, at the distance \( t \) from the former, runs over the whole range of observation times. This different criterion is dictated by the specific purpose of evaluating the quantity of Eq. (22) in a way compatible with using only one single trajectory, while apparently the authors of Ref. [18] do not feel the need of fitting this constraint.

![FIG. 1. The distribution density \( P(\beta,t) \) as a function of \( \beta \). The distribution on the left has been evaluated using as initial condition \( \alpha_{min} \); the distribution on the right uses as initial condition \( \alpha_{max} \). The time \( t \) has been set \( 2^{15} < t < 2^{18} \). The initial condition chosen is \( x = 1 \). The reason for this choice is widely discussed in Refs. [3,4]. This is so because \( \alpha(x = 1) = \alpha_{min} \), therefore ensuring the condition of maximum expansion. In Fig. 1 we show that the choice of \( \alpha_{min} \) rather than \( \alpha_{max} \) shifts the distribution \( P(\beta,n) \) from the right to the left, namely from a condition close to that of Ref. [18], to a condition favorable for the emergence of \( Q \). We note that the \( \beta \)-distribution does not drop to zero beyond the value \( \beta \approx 1.3 \), which, according to Eq. (14), is the maximum possible value of the power index \( \beta \). This is a consequence of the fact that the theoretical prescription of Eq. (13) refers to the time asymptotic limit \( t \rightarrow \infty \), whereas the numerical calculation is carried out with an upper bound on time: The maximum value of time explored is in fact \( t_{max} = 2^{18} \). It is expected that with the increase of the time upper bound the distribution tends to drop to zero for values of \( \beta \) larger than the maximum possible value.

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The result of the corresponding numerical calculation is shown in Fig. 2. To make more evident that the central curve, with \( q \approx 0.25 \), is that corresponding to the true \( Q \), we adopt the same procedure as that used by the authors of Ref. \[8\]. We have fitted the curves \( H_q(t) \) of Fig. 3 in the interval \([ t_1, t_2 ]\) with the polynomial \( H(t) = a + bt + ct^2 \). We define \( R = |c|/b \) as a measure of the deviation from the straight line. We expect that \( Q = R \) results in \( R = 0 \). We choose \( t_1 = 100 \) and \( t_2 = 1000 \) for all \( q \)'s. In the insert of Fig. 2 we show that \( R \) becomes virtually equal to zero for \( q = 0.25 \), which is very close to the value \( q = 0.24 \) found by the authors of Ref. \[8\].

The shift from the left to the right distribution shown in Fig. 1 is of fundamental importance to find the correct \( Q \). Further evidence of this fact is obtained by using analytical arguments to evaluate the integral of Eq. (22).

If we neglect the logarithm term, the entropy growth be-
comes equivalent to observing only one trajectory moving from
the initial condition \( x = 1 \). This central result is made possible by the use of the average over \( \beta \) suggested by the important work of Hata et al. \[18\] as well as by the result of Ref. \[12\]. The method of Ref. \[12\] proves to be an efficient way of expressing the dependence of the trajectory entropy on the sensitivity to the initial conditions. We are convinced that Eq. (4) can be regarded as the proper non-extensive generalization of the Pesin theorem. Consequently, under the assumption that the dependence on initial conditions is realized only through \( \beta(t, \alpha) \), the slope of the curve of Fig. 2, corresponding to \( q = Q = 0.25 \), is the genuine extensive KS entropy of this archetypical condition of weak chaos.

We thank C. Tsallis for reading the draft of this paper and illuminating suggestions.

\[\text{FIG. 2. The function } H_q(t)^* \text{ of Eq. (4) as a function of time for three different values of the entropic index } q. \text{ The values of } q, \text{ from the bottom to the top curve, are } q = 0.35, 0.25, 0.15. \text{ The insert shows } R \text{ vs } q. \text{ The values of } R \text{ have been multiplied by } 10^3.\]

The importance of this paper goes much beyond checking the prediction of Ref. \[8\]. The result obtained is equivalent to observing only one trajectory moving from the initial condition \( x = 1 \). This central result is made possible by the use of the average over \( \beta \) suggested by the important work of Hata et al. \[18\] as well as by the result of Ref. \[12\]. The method of Ref. \[12\] proves to be an efficient way of expressing the dependence of the trajectory entropy on the sensitivity to the initial conditions. We are convinced that Eq. (4) can be regarded as the proper non-extensive generalization of the Pesin theorem. Consequently, under the assumption that the dependence on initial conditions is realized only through \( \beta(t, \alpha) \), the slope of the curve of Fig. 2, corresponding to \( q = Q = 0.25 \), is the genuine extensive KS entropy of this archetypical condition of weak chaos.

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