Optical Spin Vortex in Nonlinear Anisotropic Media

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A theoretical study is given of a new type of optical vortex in nonlinear anisotropic media. This is called an “optical spin vortex”, which is realized as a special solution of the two-component nonlinear Schrödinger equation. The existence of spin vortex is inherent in the spin texture that is caused by the anisotropy of dielectric tensor, where a role of spin is played by the Stokes vector. By introducing the effective Lagrangian governing the non-linear Schrödinger equation, we derive the evolution equation for the motion of spin vortex, which exhibits a clear distinction from the conventional singular vortex inherent in the phase function that is described by the single component complex field.

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The nonlinear optics has been a major subject in physics for long time [1]. The main interest is focussed on particular solutions of the basic non-linear equation that governs the light field or eletro-magnetic field. Among others, remarkable is the optical vortex, the existence of which has been early suggested in [2]. Recently the detailed study has been carried out from both of theoretical and experimental point of view [3,4]. More recently experimental verification has been also given for the multi-vortices [4]. The basic idea of the optical vortex follows an analogy with the existence of the superfluid vortex that is described by the complex Landau-Ginzburg order parameter for bose fluid as is stated in [3]. The equation for the light field, which is known as the nonlinear Schrödinger (NLS) equation, is very similar to the Pitaevski equation. Thus it is natural to expect the occurrence of the optical counterpart of the superfluid vortex.

The purpose of this letter is to put forward a possible new type of vortex and its dynamical behavior in the nonlinear and anisotropic media. We call this new type vortex the “optical spin vortex”. Our starting equation is the two-component NLS equation. Here the basic ingredient is the state of polarization, which is realized by the two-component field. The concept of polarization is fruitful which forms a basis of modern crystal optics [2,3]. The quantity that describes polarization state is given by the Stokes parameters (or vector), which is geometrically realized as a point on the Poincaré sphere. The crucial point is that the Stokes vectors form the field of pseudo-spin, namely, we consider the special configuration of pseudo-spin. Having recognized the role of the Stokes vector as a pseudo-spin, we have the analogy with the spin vortex. This can be naturally formulated by the effective Lagrangian for the pseudo-spin field reduced from two-component NLS equation. Such a Lagrangian has not been used thus far in condensed matter physics. Notable here is that the spin vortex is non-singular in contrast to the singular vortex that appears in the conventional optical vortex, where the singularity is inherent in the phase of the complex scalar field.

By using the effective Lagrangian, we can naturally derive the evolution equation for the non-singular vortex, which reveals the topological invariant that is characteristics of the non-singular vortex. In this note, we are concerned with the basic qualitative idea and the details of numerical analysis will be given elsewhere.

Two component nonlinear Schrödinger equation.— First we shall derive the two-component nonlinear Schrödinger equation for the light wave traveling through anisotropic media. The procedure follows the one developed in a recent paper [4]. Suppose that the electro-magnetic wave of wave vector $k$ travels in the direction of $z$ with the dielectric tensor $\epsilon$. The nonlinear nature of media implies that $\epsilon$ has a field dependence in nonlinear form. The $z$-axis is prescribed to be the principal axis of the dielectric tensor, namely, the axis corresponding to one of the eigenvalues of the dielectric tensor. In this geometry, $\epsilon$ is taken to be $2 \times 2$ matrix and the Maxwell equation for the displacement field $\mathbf{D}$ is given by the second order differential equation:

$$ \frac{\partial^2 \mathbf{D}}{\partial z^2} + \nabla^2 \mathbf{D} + \omega^2 \epsilon \mathbf{D} = 0 $$

where $\nabla = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right)$ and $(x,y)$ denotes the coordinate in the plane perpendicular to $z$ axis. Now we put

$$ \mathbf{D}(z) = f(x,y,z) \exp[\text{i}kn_0z] $$

with $k = \frac{\omega}{c}$, and $n_0(\equiv \sqrt{\epsilon_0})$ means the refractive index for the otherwise isotropic medium. The amplitude $f(x,y,z)$ is written as $f = f_1(f_1,f_2) = f_1 \mathbf{e}_1 + f_2 \mathbf{e}_2$, which is slowly varying function of $z$ besides the $(x,y)$ dependence, and $\mathbf{e}_1$ and $\mathbf{e}_2$ denotes the basis of linear polarization. By substituting (2) into (1) and considering the short wave length
limit, i.e., \(|\frac{df}{dz}| << k\), we can derive the equation for the amplitude \(f\); namely, keeping only the first derivative \(\frac{df}{dz}\) besides the Laplacian with respect to \((x, y)\), we get

\[
i\lambda \frac{\partial f}{\partial z} + \left[\frac{\lambda^2}{n_0^2} \nabla^2 + (\hat{\epsilon} - n_0^2)\right] f = 0
\]

(3)

where \(\lambda\) is the wavelength divided by \(2\pi\). This equation is regarded as a two-state Schrödinger equation where \(\lambda\) just plays a role of the Planck constant and \(z\) is a substitute of the time variable. The components \((f_1, f_2)\) couple each other to give rise to the change of polarization which is just the effect of birefringence governed by a \(2 \times 2\) matrix “potential” \(\hat{\nu} = \hat{\epsilon} - n_0^2\). The \(\hat{\nu}\) represents the deviation from the isotropic value and if the non-absorptive media is considered, it becomes hermitian. From the hermiticity, the most general form of \(\hat{\nu}\) is written as

\[
\hat{\nu} = \begin{pmatrix}
v_0 + \alpha & \beta + i\gamma \\
\beta - i\gamma & v_0 - \alpha
\end{pmatrix}.
\]

(4)

Now for the use of later argument, it is convenient to transform the basis of linear polarization \(f\) in the circular basis, that is,

\[
e_{\pm} = \frac{1}{\sqrt{2}}(e_1 \pm ie_2)
\]

(5)

which defines the unitary transformation \((e_+, e_-) = T(e_1, e_2)\) Here \(T\) is the unitary transformation of \(2 \times 2\) matrix:

\[
T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}.
\]

(6)

Hence we can introduce the wave function and as \(\psi = Tf = \{\psi_1^*, \psi_2^*\}\). Thus we have

\[
i\lambda \frac{\partial \psi}{\partial z} = \hat{H}\psi
\]

(7)

and the transformed “Hamiltonian” becomes

\[
\hat{H} = ThT^{-1} = -\frac{\lambda^2}{n_0^2} \nabla^2 + V
\]

(8)

where the potential term is written in terms of the Pauli spin; \(V = v_0 \times 1 + \sum_1^3 v_i \sigma_i\). We here introduce the “quantum” Lagrangian leading to the Schrödinger type equation, which is given by

\[
I = \int \psi^\dagger \left(i\lambda \frac{\partial}{\partial z} - \hat{H}\right) \psi \, d^2xdz.
\]

(9)

Indeed, the Dirac variation equation \(\delta I = 0\) recovers the Schrödinger equation. For later convenience, we write \(L_C = \int \psi^\dagger i\lambda \frac{\partial}{\partial z} \psi \, d^2x\) and \(H = \int \psi^\dagger \hat{H} \psi \, d^2x\), which are called the canonical term and the Hamiltonian term respectively.

**Reduction to the pseudo-spin field.** — Now, having given the two component field function \(\psi\), we can introduce the Stokes parameter by using the Pauli spin \(\vec{S}\), namely, this is written as \(S_i = \psi^\dagger \sigma_i \psi\), \(S_0 = \psi^\dagger 1\psi\) with \(i = x, y, z\). We see that the relation \(S_0^2 = S_x^2 + S_y^2 + S_z^2\) holds. \(S_0\) gives the field strength; \(S_0^2 \equiv |D|^2\). Using the spinor representation,

\[
\psi_1 = S_0 \cos \frac{\theta}{2}, \quad \psi_2 = S_0 \sin \frac{\theta}{2} \exp[i\phi],
\]

(10)

we have the polar form for the Stokes vector \(\vec{S} = (S_x, S_y, S_z)\):

\[
S_x = S_0 \sin \theta \cos \phi, \quad S_y = S_0 \sin \theta \sin \phi, \quad S_z = S_0 \cos \theta
\]

(11)

which forms a pseudo spin, so to speak, and is pictorially given by the point on the Poincaré sphere. In terms of the angle variable, the canonical and Hamiltonian term in the Lagrangian are given as

\[
L_C = \int S_0^2 \lambda \frac{1 - \cos \theta}{2} \frac{\partial \phi}{\partial z} \, d^2x,
\]

\[H = H_T + \check{V}.
\]

(12)
Here the potential term in the Hamiltonian is given as

\[ \tilde{V} = \int (v_0 + \sum_{i=1}^{3} v_i S_i) \, d^2x. \]  

(13)

We note that \( v_0 \) and \( v_i \)'s are nonlinear functions of the field strength \( S_0 \) in general and this fact suggests that we have a stable special solution for the field of pseudo spin. The kinetic energy term is written as a sum of three terms; \( H_T = \frac{2}{m_0} \int \nabla \psi \psi \, d^2x \equiv H_1 + H_2 + H_3 \), where the respective terms become

\[ H_1 = \int \frac{S_0^2 \lambda^2}{m_0} (\nabla S_0)^2 \, d^2x, \]

\[ H_2 = \int \frac{S_0^2 \lambda^2}{4n_0} \{ (\nabla \theta)^2 + \sin^2(\nabla \phi)^2 \} \, d^2x, \]

\[ H_3 = \int \frac{S_0^2 \lambda^2}{4n_0} \{ (1 - \cos \theta) \nabla \phi \}^2 \, d^2x \]  

(14)

where \( H_1 \) gives the kinetic energy associating with the space modulation of the field strength, and the second term represents the intrinsic energy for pseudo-spin which exactly coincides with the continuous Heisenberg spin chain. The last term is nothing but the fluid kinetic energy inherent spin structure, where we put

\[ \mathbf{v} = (1 - \cos \theta) \nabla \phi \]  

(15)

which means the velocity field as is seen below. In this way, we have a new type of fluid, so to speak, "anisotropic optical fluid", which is naturally described by the Lagrangian written in terms of the pseudo-spin field.

**Equation of motion for optical spin vortex.**— We now examine the vortex solution by starting with the pseudo-spin Lagrangian given in the above. Taking variation leads to the coupled equations for the field variables \( (S_0, \theta, \phi) \). In what follows, we confine our argument to the case that \( S_0 \) becomes constant. Physically, this corresponds to the constant background field with a "dark" core which is controlled by the profile of the remaining variables \( (\theta, \phi) \). A static solution for one vortex is obtained with the phase function \( \phi = \tan^{-1} \frac{\theta}{\lambda} \), and the profile function \( \theta \) is given as a function of the radial variable \( r \). The explicit form of the profile function \( \theta(r) \) may be derived from the extremum condition for the Hamiltonian \( H \), with the specific boundary condition at \( r = \infty \) and \( r = 0 \). Namely, we choose such that at the origin \( r = 0 \), we can take \( \theta(0) = 0 \) in general, that is, the left-handed circular polarization at the origin [see Fig.1(a) and (b)], whereas at \( r = \infty \), there exist several possibilities; we here consider two typical cases: A) \( \theta(\infty) = \pi \) and B) \( \theta(\infty) = \pi/2 \). We here introduce the vector \( \mathbf{m}(x) = S/S_0 \). Then, we have \( m_3(0) = 1 \) for both cases, A), B), namely, the spin field directs upward. On the other hand, we have at \( r = \infty \), \( m_3(\infty) = -1 \) for case A) and \( m_3(\infty) = 0 \) for case B), that is, the spin field directs *downward* for A), which represents the right-handed circular polarization [Fig.1(a)] and *outward* for B), which represents the linear polarization [Fig.1(b)]. Here we simply assume the existence of the solution of the velocity \( \mathbf{v} \) satisfying the above boundary condition.

In order to treat the motion for a single vortex, we introduce the coordinate of the center of vortex, \( \mathbf{R}(z) = (X(z), Y(z)) \), by which the vortex solution is parameterized such that \( \theta(x - \mathbf{R}(z)) \) and \( \phi(x - \mathbf{R}(z)) \). By using the parameterization prescribed in the above, the canonical term \( L_C \), the first term in (12), is written as

\[ L_C = \int \frac{S_0^2 \lambda^2}{2} (1 - \cos \theta) \nabla \phi \cdot \dot{\mathbf{R}} \, d^2x \]

\[ = \frac{S_0^2 \lambda^2}{2} \int \mathbf{v} \cdot \dot{\mathbf{R}} \, d^2x \]  

(16)

where we have used the relation: \( \frac{\partial \phi}{\partial t} = \frac{\partial \phi}{\partial \mathbf{R}} \dot{\mathbf{R}} \). Remarkable point here is that the velocity field (12) does not bear singularity at the origin compared with the behavior of the conventional vortex for which the velocity field is simply defined by the gradient of the phase \( \mathbf{v} = \nabla \phi \). Keeping this in mind, we shall derive the equation of motion for a single vortex. In order to achieve this we consider the Euler-Lagrange equation for \( \mathbf{R} \), which gives the "balance of forces"

\[ \mathbf{F}_C \equiv \frac{d}{dz} \left( \frac{\partial L_C}{\partial \dot{\mathbf{R}}} \right) - \frac{\partial L_C}{\partial \mathbf{R}} = \frac{\partial H}{\partial \mathbf{R}}. \]  

(17)

By using eq.(13), we get
\[
\frac{S_0^2 \lambda}{2} \sigma (k \times \mathbf{R}) = -\frac{\partial H}{\partial \mathbf{R}}
\]

where the \( k \) in (18) is the unit vector perpendicular to the \( xy \)-plane. Here \( \sigma \) is defined as

\[
\sigma = \int \left( \frac{\partial v_x}{\partial x} - \frac{\partial v_y}{\partial y} \right) d^2 x.
\]

In deriving (18), we have used the relation \( \partial v \omega \). The integrand of \( \sigma \) is nothing but the vorticity which we put \( \omega \). Using eq. (15), we can write \( \omega \) in terms of the angular variable:

\[
\omega = \nabla \times \mathbf{v} = \sin \theta (\nabla \theta \times \nabla \phi)
\]

which is also rewritten in terms of the spin field \( \mathbf{m} \)

\[
\nabla \times \mathbf{v} = \mathbf{m} \cdot \left( \frac{\partial \mathbf{m}}{\partial x} \times \frac{\partial \mathbf{m}}{\partial y} \right).
\]

Equation (21) (or (20)) is an optical counterpart of a topological invariant of hydrodynamical origin \([11]\) or otherwise the Mermin-Ho relation for superfluid He3-A \([12]\). We get an integral of the differential form;

\[
\sigma = \int_S \sin \theta d\theta \wedge d\phi.
\]

Here taking into account of the boundary conditions for the profile function \( \theta(r) \) mentioned above, we have a topological interpretation for \( \sigma \). In the case A), the vortex configuration gives the continuous mapping from the compactified space \( S_2 \) to the spin configuration \( S_2 \), so the \( \sigma \) in (22) has a meaning of the mapping degree of \( S_2 \rightarrow S_2 \). Hence we get the quantization of \( \sigma \): \( \sigma = m \) (\( m \)=integer). For the case B), on the other hand, the mapping becomes \( S_2 \rightarrow S_2/2 \) (hemisphere), so intuitively we have the quantization \( \sigma = m/2 \) (\( m \)=integer). Thus it should be noted that this distinguishes the topological invariant for the conventional optical vortex \([3]\), which gives the invariant of \( \oint_C \nabla \phi ds = 2\pi \). It should be noted that the left hand side of the equation of motion (18) is considered to be an optical analog of the Magnus force in superfluids or superconductors. As is mentioned above, the “force” incorporates the topological structure inherent in the coreless vortex, which is quite different from the conventional optical vortex which is singular at the origin. The point here is that there are two distinct types vortices according to the different boundary conditions at infinity.

The difference between two types of vortex is significant from the point of view of condensed matter physics, namely, we expect that the non-singular spin vortex is converted to the usual phase vortex. This may be considered to be a kind of phase transition as a consequence of the phase transition from anisotropic and isotropic that occurs in the medium. Actually, as a result of transition the birefringence disappears and the Hamiltonian (or dielectric tensor) becomes scalar matrix, which means that the two component equation is converted to one component; \( \psi = S_0 \exp[i\phi] \), where the amplitude vanishes at the vortex center. In the process of phase transition the core structure of the optical vortex may survive and it may be possible to observe the structure change between two cores; the soft and singular [See Fig.2].

The case of multi-vortices.— We now consider the correlated motion of two or more vortices. The canonical term in Lagrangian is simply obtained by replacement as \( \mathbf{v} \cdot \mathbf{R} \rightarrow \sum_i \mathbf{v}_i \cdot \mathbf{R}_i \) where \( \mathbf{v}_i \) is the velocity field coming from the \( i \)-th vortex. The effective Hamiltonian for the assembly of vortices is given by the fluid kinetic energy, \([3]\) namely,

\[
H_{eff} = \int \sum_{ij} \omega(y') \nabla \log |y' - (x - \mathbf{R}_i)| \times \omega(y'') \nabla \log |y'' - (x - \mathbf{R}_j)| d^2 y' d^2 y'' d^2 x.
\]

We can intuitively guess that it depends on the distance between center coordinate \( \mathbf{R}_{ij} \equiv \mathbf{R}_i - \mathbf{R}_j \). The effective Lagrangian is written as

\[
L_{eff} = \frac{S_0^2 \lambda}{2} \sigma \sum_i \left( Y_i \frac{dX_i}{dz} - X_i \frac{dY_i}{dz} \right) - H_{eff}.
\]

Here a special case, we consider the equation of motion for a couple of vortices, which becomes
\[ \frac{S_0^2}{2\sigma} \left( k \times \frac{d \mathbf{R}_i}{dz} \right) = -\frac{\partial H_{\text{eff}}}{\partial \mathbf{R}_i} \]  

(25)

with \( i = 1, 2 \). By introducing the relative coordinate \( \mathbf{r} \equiv \mathbf{R}_1 - \mathbf{R}_2 \), we have a reduced form

\[ \frac{S_0^2}{2\sigma} \left( k \times \frac{d \mathbf{r}}{dz} \right) = -\frac{\partial H_{\text{eff}}}{\partial \mathbf{r}}. \]  

(26)

From this we obtain the constant of motion \( r^2 = C \), which is reminiscent of the well known conservation law of the vortex dynamics. Furthermore, we have the relation between the polar angle and the relative vector \( \mathbf{r} \), namely, \( \dot{\phi} = f(r) = \text{constant} \), which shows that the two vortices rotate around each other by constant angular velocity (c.f. [4]), [Fig. 2]. Finally, we give a remark on a possible conversion of the effective Hamiltonian in accordance with the change between two types vortices (non-singular and singular types). Namely, for the singular vortex it is given by the logarithmic form

\[ H_{\text{eff}} = -\sum_{ij} \mu_i \mu_j \log |\mathbf{R}_i - \mathbf{R}_j|. \]

On the other hand, in the limiting case that \( |y'| \ll |x - \mathbf{R}_i| \), the effective Hamiltonian (23) is shown to be reduced to the form

\[ \int \nabla \log |\mathbf{x}| \nabla \log |\mathbf{x} - \mathbf{R}_{ij}| d^2 \mathbf{x} \simeq \sigma^2 \log |\mathbf{R}_{ij}| \]  

(27)

which implies that the Hamiltonian has the same form as the singular case except the coupling constant in such a limiting case. The difference of the coupling constant may play a substantial role for detecting the possible structure change between two types of optical vortices.

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FIG. 1. Profile of the non-singular vortex; a) The case of $\theta(\infty) = \pi$ and b) The case of $\theta(\infty) = \frac{\pi}{2}$.

FIG. 2. Illustration of for evolution of vortex pairs and the possible transition between the non-singular (b) and the singular (a) vortex. The bullet • and white circle ○ in the right hand side indicate an instantaneous vortex pair at a particular value of $z$. 