A Model-free Learning Algorithm for Infinite-horizon Average-reward MDPs with Near-optimal Regret

Mehdi Jafarnia-Jahromi∗
University of Southern California
mjafarni@usc.edu
Chen-Yu Wei
University of Southern California
chenyu.wei@usc.edu
Rahul Jain
University of Southern California
rahul.jain@usc.edu
Haipeng Luo
University of Southern California
haipengl@usc.edu

Abstract

Recently, model-free reinforcement learning has attracted research attention due to its simplicity, memory and computation efficiency, and the flexibility to combine with function approximation. In this paper, we propose Exploration Enhanced Q-learning (EE-QL), a model-free algorithm for infinite-horizon average-reward Markov Decision Processes (MDPs) that achieves regret bound of $\tilde{O}(\sqrt{T})$ for the general class of weakly communicating MDPs, where $T$ is the number of interactions. EE-QL assumes that an online concentrating approximation of the optimal average reward is available. This is the first model-free learning algorithm that achieves $\tilde{O}(\sqrt{T})$ regret without the ergodic assumption, and matches the lower bound in terms of $T$ except for logarithmic factors. Experiments show that the proposed algorithm performs as well as the best known model-based algorithms.

1 Introduction

Reinforcement learning (RL) studies the problem of an agent interacting with an unknown environment while trying to maximize its cumulative reward. The agent faces a fundamental exploration-exploitation trade-off: should it explore the environment to gain more information for future decisions, or should it exploit the available information to maximize the reward. Efficient exploration is a crucial property of learning algorithms evaluated with the notion of regret: the difference between the cumulative reward of the optimal policy and that of the algorithm. Regret quantifies the speed of learning, i.e., low regret algorithms can learn more efficiently.

RL algorithms can broadly be classified as model-based and model-free. Model-based algorithms maintain an estimate of the environment dynamics and plan based on the estimated model. Model-free algorithms, on the other hand, directly estimate the value function or the policy without explicitly estimating the environment model. Model-free algorithms are simpler, memory and computation efficient, and more amenable to extend to large scale problems by incorporating function approximation. Indeed, most of the recent advances in RL such as DQN [12], TRPO [17], AC3 [13], PPO [18], etc., are all in the model-free paradigm.

It was believed that model-based algorithms can better manage the trade-off between exploration and exploitation. Several model-based algorithms with low regret guarantees have been proposed in the past decade including UCRL2 [10], REGAL [5], PSRL [15], UCBVI [4], SCAL [7], EBF [22] and

∗Corresponding author.
We consider infinite-horizon average-reward MDPs described by
\[ s, a, r, p (s', a|s, a) \in \mathbb{S} \times A \to \mathbb{R} \]
and reward function \( r : \mathbb{S} \times A \to [0, 1] \) is deterministic reward function, and \( p(s'|s, a) \) is the transition kernel. Here \( \mathbb{S} \) and \( A \) are finite sets with cardinalities \( S \) and \( A \), respectively. The gain of a stationary deterministic policy \( \pi : \mathbb{S} \to A \) with the initial state \( s \) is defined as
\[ J^\pi (s) := \liminf_{T \to \infty} \frac{1}{T} \mathbb{E} \left[ \sum_{t=1}^{T} r(s_t, \pi(s_t)) \mid s_1 = s \right], \]
where \( s_{t+1} \sim p(\cdot|s_t, a_t) \) for \( t = 1, 2, 3, \ldots \). Let \( J^*(s) := \max_\pi J^\pi (s) \) be the optimal gain. The optimal gain \( J^* \) is independent of the initial state \( s \) for the standard class of weakly communicating MDPs considered in this paper. An MDP is weakly communicating if its state space \( \mathbb{S} \) can be divided into two subsets. In the first subset, all the states are transient under any stationary policy. In the second subset, every state is accessible from any other state under some stationary policy. It is known that the weakly communicating assumption is required to achieve low regret [5]. From the standard MDP theory [16], we know that for weakly communicating MDPs, there exists a function \( q^* : \mathbb{S} \times A \to \mathbb{R} \) (unique up to an additive constant) such that the following Bellman equation holds:
\[ J^* + q^*(s, a) = r(s, a) + \mathbb{E}_{s' \sim p(\cdot|s, a)}[\max_b q^*(s', b)], \]
for all \( s \in \mathbb{S} \) and \( a \in A \). The optimal gain \( J^* \) is achieved by the corresponding optimal policy \( \pi^*(s) \in \text{argmax}_\pi q^*(s, a) \) (note that such a policy may not be unique).

In this paper, we consider the reinforcement learning problem of an agent interacting with a weakly communicating MDP with unknown transition kernel \( p \) and reward function \( r \) (thus, the Bellman
Table 1: Regret comparisons of model-free and model-based RL algorithms for infinite-horizon average-reward MDPs with $S$ states, $A$ actions and $T$ steps. The lower bound is almost achieved by model-based algorithms. However, prior to this work, model-free algorithms could obtain $\tilde{O}(\sqrt{T})$ regret bound only with the strong ergodic assumption. Here $D$ is the diameter of the MDP, $sp(v^*) := \max_s v^*(s) - \min_s v^*(s) \leq D$ is the span of the optimal value function, $sp(q^*) \geq sp(v^*)$ is the span of the optimal $Q$-value function, and $V_{s,a}^* := \operatorname{Var}_{s' \sim p(\cdot|s,a)}[v^*(s')] \leq sp(v^*)^2$ denotes the variance of the optimal value function. For the exact definition of the mixing time $t_{\text{mix}}$, the hitting time $t_{\text{hit}}$, and the distribution mismatch coefficient $\rho \leq t_{\text{hit}}$, refer to [20].

| Algorithm          | Regret                     | Comment                                |
|--------------------|----------------------------|----------------------------------------|
| REGAL [5]          | $\mathcal{O}(sp(v^*)\sqrt{SAT})$ | no efficient implementation            |
| UCRL2 [10]         | $\tilde{O}(DS\sqrt{AT})$    | -                                     |
| PSRL [15]          | $\tilde{O}(sp(v^*)S\sqrt{AT})$ | -                                     |
| OSP [14]           | $\tilde{O}(\sqrt{t_{\text{mix}}SAT})$ | -                                     |
| SCAL [7]           | $\tilde{O}(sp(v^*)S\sqrt{AT})$ | -                                     |
| KL-UCRL [19]       | $\tilde{O}(\sqrt{S\sum_{s,a}V_{s,a}^*T})$ | -                                     |
| UCRIL2B [8]        | $\tilde{O}(S\sqrt{DAT})$    | -                                     |
| EBF [22]           | $\tilde{O}(\sqrt{DAT})$     | -                                     |
| Model-based        |                            |                                        |
| POLITEX [1]        | $\tilde{O}(t_{\text{max}}^3t_{\text{hit}}\sqrt{SAT}^{3/2})$ | -                                     |
| EE-POLITEX [2]     | $\tilde{O}(T^{4/5})^1$      | ergodic assumption                     |
| AAPI [9]           | $\tilde{O}(T^{2/3})$        | unchain assumption                     |
| Optimistic QL [20] | $\tilde{O}(sp(v^*)(SA)^{1/2}T^{3/2})$ | -                                     |
| MDP-OOMD [20]      | $\tilde{O}(\sqrt{t_{\text{max}}^3DAT})$ | -                                     |
| EE-QL (this work)  | $\tilde{O}(\min\{\rho^2t_{\text{hit}}, t_{\text{max}}^3\} \sqrt{SAT})$ | concentrating estimate of $J^*$       |
| lower bound [10]   | $\Omega(\sqrt{DAT})$       | -                                     |

1 The dependence of the regret on other parameters is not explicitly mentioned in the original paper.

Regret evaluates the transient performance of the learning algorithm by measuring the difference between the total gain of the optimal policy and the cumulative reward obtained by the learning algorithm up to time $T$. The goal of the agent is to maximize the total reward (or equivalently minimize the regret). If a learning algorithm achieves sub-linear regret, its average reward converges to the optimal gain. [22] proposed a model-based algorithm with regret bound of $\tilde{O}(\sqrt{DAT})$ (where $D$ is the diameter of the MDP) and matches the lower bound of [10]. The best existing regret bound of a model-free algorithm for weakly communicating MDPs is $\tilde{O}(sp(v^*)(SA)^{1/3}T^{2/3})$ by [20].

3 The Exploration Enhanced Q-learning Algorithm

In this section, we introduce the Exploration Enhanced Q-learning (EE-QL) algorithm (see Algorithm 1). The algorithm works for the broad class of weakly communicating MDPs. It is well-known that the weakly communicating condition is necessary to achieve sublinear regret [5].

EE-QL approximates the $Q$-value function for the infinite-horizon average-reward setting using stochastic approximation with carefully chosen learning rates. The algorithm takes greedy actions...
We have numerically verified that this choice of \( J \) satisfies Assumption 1.

Under Assumption 1, the following assumption.

This result improves the previous best known regret bound of \( O(\sqrt{T}) \) by \([20]\) and matches the lower bound of \( \Omega(\sqrt{DSAT}) \) \([10]\) in terms of \( T \) up to logarithmic factors. To the best of our knowledge, this is the first model-free algorithm that achieves \( \tilde{O}(\sqrt{T}) \) regret bound for the general class of weakly communicating MDPs in the infinite-horizon average-reward setting.

4 Analysis

In this section, we provide the proof of Theorem 1. Before we start the analysis, let’s define

\[
\alpha_t^{\tau} := \alpha_t \prod_{j=t+1}^{\tau} (1 - \alpha_j) \tag{3}
\]
for $i \geq 1$, where $\alpha_i = 1/\sqrt{i}$ is the learning rate used in Algorithm 1. $\alpha^i$ determines the effect of the $i$-th step on $\tau$-th update. This quantity has nice properties that are listed in Lemma 4 and are central to our analysis. In particular, the $\sqrt{T}$ regret bound is merely due to properties 2 and 4 in Lemma 4.

4.1 Proof of Theorem 1

Proof. We start by decomposing the regret using Lemma 7. With probability at least $1 - \delta$, the regret of any algorithm can be bounded by

$$R_T \leq \text{sp}(v^*) + \text{sp}(v^*) \sqrt{\frac{1}{2} T \ln \frac{1}{\delta}} + \sum_{i=1}^{T} \Delta(s_i, a_i),$$

(4)

where $\Delta(s, a) := v^*(s) - q^*(s, a)$. Suffices to bound $\sum_{t=1}^{T} \Delta(s_t, a_t)$. Let $n_{t+1}(s, a)$ denote the number of visits to state-action pair $(s, a)$ before time $t + 1$ (including time $t$ and excluding time $t + 1$). For notational simplicity, let $n_{t+1} := n_{t+1}(s_t, a_t)$ and $t_i(s, a)$ be the time step at which $(s, a)$ is visited for the $i$th time. We can write:

$$
\begin{align*}
\sum_{t=1}^{T} & \left[ v_t(s_t) - v^*(s_t) + \Delta(s_t, a_t) \right] = \sum_{t=1}^{T} \left[ Q_t(s_t, a_t) - v^*(s_t) + \Delta(s_t, a_t) \right] \\
& = \sum_{t=1}^{T} \left[ Q_{t+1}(s_t, a_t) - q^*(s_t, a_t) \right] \\
& = \sum_{t=1}^{T} \left[ Q_{t+1}(s_t, a_t) - q^*(s_t, a_t) \right] + \sum_{t=1}^{T} \left[ Q_t(s_t, a_t) - Q_{t+1}(s_t, a_t) \right].
\end{align*}
$$

(5)

where the first equality is by the fact that $\alpha_i = \arg\max_{a \in A} Q_t(s_t, a)$ and the second equality is by the definition of $\Delta(s_t, a_t)$. The second term on the right hand side can be bounded by $2(2B + 1)\sqrt{SAT} + cSA(1 + \ln T)$ by using line (3) of the Algorithm (see Lemma 3) where $B := \text{sp}(q^*) + cSA(1 + \ln T)$. The rest of the proof proceeds to write the first term on the right hand side in terms of $v_{t+1}(s_{t+1}) - v^*(s_{t+1})$ (to telescope with the left hand side) plus some sublinear additive terms. We can write:

$$
\begin{align*}
\sum_{t=1}^{T} & \left[ Q_{t+1}(s_t, a_t) - q^*(s_t, a_t) \right] = \sum_{t=1}^{T} \left[ Q_{t+1}(s_t, a_t) - q^*(s_t, a_t) \right] 1(n_{t+1}(s_t, a_t) \geq 1) \\
& = \sum_{s, a} \sum_{t=1}^{T} 1(s_t = s, a_t = a) \left[ Q_{t+1}(s, a) - q^*(s, a) \right] 1(n_{t+1}(s, a) \geq 1) =: R_1
\end{align*}
$$

By Lemma 6, the term $R_1$ can be written as:

$$
R_1 = \sum_{s, a} \sum_{j=1}^{T} 1(s_t = s, a_t = a) \left\{ \sum_{i=1}^{n_{t+1}(s, a)} \alpha_{n_{t+1}(s, a)}^i [J^* - J_{t_i(s, a)}] + \sum_{i=1}^{n_{t+1}(s, a)} \alpha_{n_{t+1}(s, a)}^i [q_{t_i(s, a)}(s_{t_i(s, a)} + 1) - v^*(s_{t_i(s, a)} + 1)] \\
+ \sum_{i=1}^{n_{t+1}(s, a)} \alpha_{n_{t+1}(s, a)}^i [v^*(s_{t_i(s, a)} + 1) - \mathbb{E}_{s' \sim p(\cdot \mid s, a)} v^*(s')] \right\}
$$

$$
= \sum_{s, a} \sum_{j=1}^{T} \left\{ \sum_{i=1}^{j} \alpha_{j}^i [J^* - J_{t_i(s, a)}] + \sum_{i=1}^{j} \alpha_{j}^i [q_{t_i(s, a)}(s_{t_i(s, a)} + 1) - v^*(s_{t_i(s, a)} + 1)] \\
+ \sum_{i=1}^{j} \alpha_{j}^i [v^*(s_{t_i(s, a)} + 1) - \mathbb{E}_{s' \sim p(\cdot \mid s, a)} v^*(s')] \right\}
$$
We proceed by upper bounding each term in the latter by using Lemma 4(3). Note that
\[ |J^* - J_{t_i(s,a)}| \leq \frac{c}{\sqrt{t_i(s,a)}} \leq \frac{c}{\sqrt{i}}, \]

Moreover, note that \( \max s, v^* \leq B \), where \( B := \text{sp}(q^*) + cSA(1 + \ln T) \) is the uniform bound on \( \|Q_t\|_{\infty} \) as in Lemma 2. This choice of \( v^* \) implies that \( 0 \leq v_t(s) - v^*(s) \leq 2B + \text{sp}(v^*) \) for all \( s, t \).

Replacing these bounds for \( M^+ \) and \( -M^- \) in Lemma 4(3) implies
\[
R_1 \leq \sum_{s,a}^{n_T(s,a)} \left\{ J^* - J_{t_i(s,a)} + \frac{5c}{2i} + \frac{c}{\sqrt{i}} \left( 1 - \frac{1}{\sqrt{i+1}} \right)^{n_T(s,a) - i+1} \right\} 
\]

To simplify the right hand side of the above inequality, observe that
\[
\sum_{s,a}^{n_T(s,a)} (J^* - J_{t_i(s,a)}) = \sum_{s,a}^{T} \sum_{t=1}^{T} \mathbb{I}(s_t = s, a_t = a)(J^* - J_t) = \sum_{t=1}^{T} (J^* - J_t). \tag{7} \]

Similarly,
\[
\sum_{s,a}^{n_T(s,a)} (v_{t_i(s,a)}(s_{t_i(s,a)+1}) - v^*(s_{t_i(s,a)+1})) = \sum_{t=1}^{T} (v_t(s_{t+1}) - v^*(s_{t+1})), \tag{8} \]
\[
\sum_{s,a}^{n_T(s,a)} (v^*(s_{t_i(s,a)+1}) - \mathbb{E}_{s' \sim p(s,a)} v^*(s')) = \sum_{t=1}^{T} (v^*(s_{t+1}) - \mathbb{E}_{s' \sim p(s,a)} v^*(s')). \tag{9} \]

Using the inequalities in Lemma 5 and Lemma 4(5), replacing the equalities (7), (8), (9) into the right hand side of (6), and adding and subtracting \( v_{t+1}(s_{t+1}) \) implies
\[
R_1 \leq \sum_{t=1}^{T} (J^* - J_t) + \frac{5cSA}{2} (1 + \ln T) + 2\sqrt{2cSA} 
\]
\[
+ \sum_{t=1}^{T} (v_{t+1}(s_{t+1}) - v^*(s_{t+1})) + \sum_{t=1}^{T} (v_t(s_{t+1}) - v_{t+1}(s_{t+1}) + 5(2B + \text{sp}(v^*))\sqrt{SAT} 
\]
\[
+ \sum_{t=1}^{T} (v^*(s_{t+1}) - \mathbb{E}_{s' \sim p(s,a)} [v^*(s')]) + 6 \text{sp}(v^*)\sqrt{SAT}. \tag{10} \]

Note that by Assumption 1, \( \sum_{t=1}^{T} |J^* - J_t| \leq \sum_{t=1}^{T} \|J^* - J_t\|_1 \leq c\sqrt{T} \). Furthermore, \( \sum_{t=1}^{T} (v^*(s_{t+1}) - \mathbb{E}_{s' \sim p(s,a)} [v^*(s')]) \) is a martingale difference sequence and can be bounded by \( \text{sp}(v^*)\sqrt{\frac{2T}{\ln 1/\delta}} \) with probability at least 1 − \( \delta \), using Azuma’s inequality. Moreover, \( \sum_{t=1}^{T} (v_t(s_{t+1}) - v_{t+1}(s_{t+1})) \leq \)
2\sqrt{SAT} + cSA(1 + \ln T) by Lemma 8. Replacing these bounds on the right hand side of the above inequality, simplifying the result and plugging back into (5) implies

\[ \sum_{t=1}^{T} \left[ v_t(s_t) - v^*(s_t) + \Delta(s_t, a_t) \right] \leq \sum_{t=1}^{T} (v_{t+1}(s_{t+1}) - v^*(s_{t+1})) + (14B + 11 \text{sp}(v^*) + 4)\sqrt{SAT} + 2c\sqrt{T} + \text{sp}(v^*) \sqrt{\frac{1}{2} T \ln \frac{1}{\delta}} + \frac{9cSA}{2} \ln T + \left( \frac{9}{2} + 2\sqrt{2} \right)cSA, \]

with probability at least \(1 - \delta\). Telescoping the left hand side with the right hand side and noting that \(v_{T+1}(s_{T+1}) - v_1(s_1) \leq 2B\) (Lemma 2) and \(v^*(s_1) - v^*(s_{T+1}) \leq \text{sp}(v^*)\), implies that

\[ \sum_{t=1}^{T} \Delta(s_t, a_t) \leq (14B + 11 \text{sp}(v^*) + 4)\sqrt{SAT} + 2c\sqrt{T} + \text{sp}(v^*) \sqrt{\frac{1}{2} T \ln \frac{1}{\delta}} + \frac{9cSA}{2} \ln T + \left( \frac{9}{2} + 2\sqrt{2} \right)cSA + 2B + \text{sp}(v^*), \]

with probability at least \(1 - \delta\). Replacing this bound into (4) implies that

\[ R_T \leq (14B + 11 \text{sp}(v^*) + 4)\sqrt{SAT} + 2c\sqrt{T} + 2\text{sp}(v^*) \sqrt{\frac{1}{2} T \ln \frac{2}{\delta}} + \frac{9cSA}{2} \ln T + \left( \frac{9}{2} + 2\sqrt{2} \right)cSA + 2B + 2\text{sp}(v^*), \]

with probability at least \(1 - \delta\) which completes the proof. \(\square\)

### 4.2 Auxiliary Lemmas

In this section, we provide some auxiliary lemmas that are used in the proof of Theorem 1. The proof for these lemmas can be found in the appendix.

**Lemma 2.** The \(Q_i\) in Algorithm 1 is bounded by \(||Q_i||_\infty \leq \text{sp}(q^*) + cSA(1 + \ln(t - 1))\).

**Lemma 3.** The second term of (5) can be bounded by

\[ \sum_{t=1}^{T} \left[ Q_t(s_t, a_t) - Q_{t+1}(s_t, a_t) \right] \leq 2(2B + 1)\sqrt{SAT} + cSA(1 + \ln T). \]

**Lemma 4.** The following properties hold:

1. \(\sum_{i=1}^{\tau} \alpha^i_{\tau} = 1\) for any \(\tau \geq 1\).

2. For any \(i \geq 1\), and any \(K \geq i\), we have \(1 - \left(1 - \frac{1}{\sqrt{i+1}}\right)^{K-i+1} \leq \sum_{\tau=i}^{K} \alpha^\tau_{\tau} \leq 1 + \frac{5}{2\sqrt{i}}\).

3. Let \(M\) be a scalar and define \(M^+ = \max(M, 0)\) and \(M^- = \max(-M, 0)\). Then, for any \(i \geq 1\), and any \(K \geq i\), we have \(M \sum_{\tau=i}^{K} \alpha^\tau_{\tau} \leq M + \frac{5M^+}{2\sqrt{i}} + M^- \left(1 - \frac{1}{\sqrt{i+1}}\right)^{K-i+1}\).

4. For any \(K \geq 0\), we have \(\sum_{i=1}^{K} \left(1 - \frac{1}{\sqrt{i+1}}\right)^{K-i+1} \leq \sqrt{K}\).

5. For any \(K \geq 1\), we have \(\sum_{i=1}^{K} \frac{1}{\sqrt{i+1}} \left(1 - \frac{1}{\sqrt{i+1}}\right)^{K-i+1} \leq 2\sqrt{2}\).

**Lemma 5 (Frequently used inequalities).** The following inequalities hold:

1. \(\sum_{t=1}^{T} \frac{1}{\sqrt{n_{t+1}(s_t, a_t)}} = \sum_{s,a} \sum_{t=1}^{T} n_{t+1}(s,a) \frac{1}{\sqrt{t}} \leq 2\sqrt{SAT}\).

2. \(\sum_{s,a} \sum_{t=1}^{T} n_{t+1}(s,a) \left(1 - \frac{1}{\sqrt{t+1}}\right)^{n_{t+1}(s,a)-i+1} \leq \sqrt{SAT}\).

3. \(\sum_{t=1}^{T} \frac{1}{\sqrt{n_{t+1}(s_t, a_t)}} \leq SA(1 + \ln T).\)
We compare our algorithm against Optimistic QL [20], MDP-OOMD [20], and POLITEX [1] as model-free algorithms and UCRL2 [10] and PSRL [15] as model-based benchmarks. The hyper parameters for these algorithms are tuned to obtain the best performance (see Table 2 for more details). Optimistic QL outperform other model-free algorithms and are similar to model-based algorithms. POLITEX and MDP-OOMD did not achieve sub-linear regret in RiverSwim and thus removed from the left figure. In RandomMDP, our algorithm together with Optimistic QL outperform other model-free algorithms and are similar to model-based algorithms.

Lemma 6. For a fixed \((s, a)\) ∈ \(S \times A\), let \(\tau = n_t(s, a)\), and \(t_i\) be the time step at which \((s, a)\) is taken for the \(i\)th time. Then,
\[
(\mathcal{Q}_t(s, a) - q^*(s, a))^1(\tau \geq 1) = \left\{ \sum_{i=1}^{\tau} \alpha_t^i \left[ J_t^* - J_{t_i} \right] + \sum_{i=1}^{\tau} \alpha_t^i \left[ v_t(s_{t_i+1}) - v^*(s_{t_i+1}) \right] \right. \\
\left. + \sum_{i=1}^{\tau} \alpha_t^i \left[ v^*(s_{t_i+1}) - \mathbb{E}_{s' \sim p(\cdot | s, a)} v^*(s') \right] \right\}
\]

Lemma 7. With probability at least \(1 - \delta\), the regret of any algorithm is bounded as
\[
R_T \leq \text{sp}(v^*) + \text{sp}(v^*) \sqrt{\frac{1}{2} T \ln \frac{1}{\delta}} + \sum_{t=1}^{T} \left[ v^*(s_t) - q^*(s_t, a_t) \right].
\]

Lemma 8. \(\sum_{t=1}^{T} [v_t(s_{t+1}) - v_{t+1}(s_{t+1})] \leq 2\sqrt{SAT} + cSA(1 + \ln T).\)

5 Experiments

In this section, we numerically evaluate the performance of our proposed EE-QL algorithm. Two environments are considered: RandomMDP and RiverSwim. The RandomMDP environment is an ergodic MDP with \(S = 6\) states and \(A = 2\) actions where the transition kernel and the rewards are chosen uniformly at random. The RiverSwim environment is a weakly communicating MDP with \(S = 6\) states arranged in a chain and \(A = 2\) actions (left and right) that simulates an agent swimming in a river. If the agent swims left (i.e., in the direction of the river current), it is always successful. If it decides to swim right, it may fail with some probability. The reward function can be described as follows: \(r(1, \text{left}) = 0.2\), \(r(6, \text{right}) = 1\) and \(r(s, a) = 0\) for all other states and actions. The agent starts from the leftmost state \((s_1 = 1)\). The optimal policy is to always swim right to reach the high-reward state \(s = 6\).

We compare our algorithm against Optimistic QL [20], MDP-OOMD [20], and POLITEX [1] as model-free algorithms and UCRL2 [10] and PSRL [15] as model-based benchmarks. The hyper parameters for these algorithms are tuned to obtain the best performance (see Table 2 for more details). \(J_t\) is chosen as in (2) with appropriate \(C\) (see Table 2). We numerically verified that this choice of \(J_t\) satisfies Assumption 1 with \(c = 5\). Figure 1 shows that in the RiverSwim environment, our algorithm significantly outperforms Optimistic QL, the only existing model-free algorithm with low regret for weakly communicating MDPs. The reason is that the proposed algorithm does not waste optimism for the entire \(Q^*\) function. Rather, the optimism in the face of uncertainty principle is used around a single scalar \(J^*\). Note that other model-free algorithms such as POLITEX and MDP-OOMD did

\[
\sum_{t=1}^{T} [v_t(s_{t+1}) - v_{t+1}(s_{t+1})] \leq 2\sqrt{SAT} + cSA(1 + \ln T).\]
not yield sub-linear regret in RiverSwim and thus removed from the figure. This is due to the fact that RiverSwim does not satisfy the ergodicity assumption required by these algorithms. Moreover, both in the RiverSwim and RandomMDP environments, our algorithm performs as well as the best existing model-based algorithms in practice, though with less memory.

Conclusions

We proposed EE-QL, the first model-free algorithm with $\tilde{O}(\sqrt{T})$ regret bound for weakly communicating MDPs in the infinite-horizon average-reward setting. Our algorithm has a tremendous numerical performance, significantly better than the existing model-free algorithms and similar to the best model-based algorithms, yet with less memory. The key to obtain such numerical performance is to avoid optimistic estimation of each entry of the $Q$ function. Instead, EE-QL uses optimism for a single scalar $J^*$ (the gain of the optimal policy). Our algorithm assumes that a concentrating estimate of $J^*$ is available. This assumption is verified numerically for an optimistic empirical average reward estimator. The theoretical verification of this assumption is left for future work.

References

[1] Yasin Abbasi-Yadkori, Peter Bartlett, Kush Bhatia, Nevena Lazic, Csaba Szepesvari, and Gellért Weisz. Politex: Regret bounds for policy iteration using expert prediction. In International Conference on Machine Learning, pages 3692–3702, 2019.

[2] Yasin Abbasi-Yadkori, Nevena Lazic, Csaba Szepesvari, and Gellert Weisz. Exploration-enhanced politex. arXiv preprint arXiv:1908.10479, 2019.

[3] Jinane Abounadi, D Bertsekas, and Vivek S Borkar. Learning algorithms for markov decision processes with average cost. SIAM Journal on Control and Optimization, 40(3):681–698, 2001.

[4] Mohammad Gheshlaghi Azar, Ian Osband, and Rémi Munos. Minimax regret bounds for reinforcement learning. In Proceedings of the 34th International Conference on Machine Learning-Volume 70, pages 263–272. JMLR. org, 2017.

[5] Peter L Bartlett and Ambuj Tewari. Regal: A regularization based algorithm for reinforcement learning in weakly communicating mdps. In Proceedings of the Twenty-Fifth Conference on Uncertainty in Artificial Intelligence, pages 35–42. AUAI Press, 2009.

[6] Kefan Dong, Yuanhao Wang, Xiaoyu Chen, and Liwei Wang. Q-learning with ucb exploration is sample efficient for infinite-horizon mdp. arXiv preprint arXiv:1901.09311, 2019.

[7] Ronan Fruit, Matteo Pirotta, Alessandro Lazaric, and Ronald Ortner. Efficient bias-span-constrained exploration-exploitation in reinforcement learning. In International Conference on Machine Learning, pages 1573–1581, 2018.

[8] Ronan Fruit, Matteo Pirotta, and Alessandro Lazaric. Improved analysis of ucr12b, 2019. Available at rlgammaszero.github.io/docs/ucr12b_improved.pdf.

[9] Botao Hao, Nevena Lazic, Yasin Abbasi-Yadkori, Pooria Joulani, and Csaba Szepesvari. Provably efficient adaptive approximate policy iteration. arXiv preprint arXiv:2002.03069, 2020.

[10] Thomas Jaksch, Ronald Ortner, and Peter Auer. Near-optimal regret bounds for reinforcement learning. Journal of Machine Learning Research, 11(Apr):1563–1600, 2010.

[11] Chi Jin, Zeyuan Allen-Zhu, Sebastien Bubeck, and Michael I Jordan. Is Q-learning provably efficient? In Advances in Neural Information Processing Systems, pages 4863–4873, 2018.

[12] Volodymyr Mnih, Koray Kavukcuoglu, David Silver, Alex Graves, Ioannis Antonoglou, Daan Wierstra, and Martin Riedmiller. Playing atari with deep reinforcement learning. arXiv preprint arXiv:1312.5602, 2013.

[13] Volodymyr Mnih, Adria Puigdomenech Badia, Mehdi Mirza, Alex Graves, Timothy Lillicrap, Tim Harley, David Silver, and Koray Kavukcuoglu. Asynchronous methods for deep reinforcement learning. In International Conference on Machine Learning, pages 1928–1937, 2016.

[14] Ronald Ortner. Regret bounds for reinforcement learning via markov chain concentration. arXiv preprint arXiv:1808.01813, 2018.
[15] Yi Ouyang, Mukul Gagrani, Ashutosh Nayyar, and Rahul Jain. Learning unknown markov decision processes: A thompson sampling approach. In *Advances in Neural Information Processing Systems*, pages 1333–1342, 2017.

[16] Martin L Puterman. *Markov decision processes: discrete stochastic dynamic programming*. John Wiley & Sons, 2014.

[17] John Schulman, Sergey Levine, Pieter Abbeel, Michael Jordan, and Philipp Moritz. Trust region policy optimization. In *International Conference on Machine Learning*, pages 1889–1897, 2015.

[18] John Schulman, Filip Wolski, Prafulla Dhariwal, Alec Radford, and Oleg Klimov. Proximal policy optimization algorithms. *arXiv preprint arXiv:1707.06347*, 2017.

[19] Mohammad Sadegh Talebi and Odalric-Ambrym Maillard. Variance-aware regret bounds for undiscounted reinforcement learning in mdps. In *Algorithmic Learning Theory*, pages 770–805, 2018.

[20] Chen-Yu Wei, Mehdi Jafarnia-Jahromi, Haipeng Luo, Hiteshi Sharma, and Rahul Jain. Model-free reinforcement learning in infinite-horizon average-reward markov decision processes. In *International Conference on Machine Learning*, 2020.

[21] Andrea Zanette and Emma Brunskill. Tighter problem-dependent regret bounds in reinforcement learning without domain knowledge using value function bounds. In *International Conference on Machine Learning*, 2019.

[22] Zihan Zhang and Xiangyang Ji. Regret minimization for reinforcement learning by evaluating the optimal bias function. In *Advances in Neural Information Processing Systems*, 2019.
Appendix

A Proof of Lemma 2

Lemma 2 (Restated). The $Q_t$ in Algorithm 1 is bounded by

$$||Q_t||_\infty \leq \text{sp}(q^*) + c\text{SA}(1 + \ln(t - 1)).$$

Proof. We first prove for the case where $J_t = J^*$ and then extend the proof to the general case. Let $G_{sas'}$ be an operator on the space of $Q$-functions defined by

$$[G_{sas'}Q](x, u) = \begin{cases} (1 - \alpha_r)Q(s, a) + \alpha_r(r(s, a) - J^* + \max_b Q(s', b)), & \text{if } (x, u) = (s, a) \\ Q(x, u), & \text{otherwise} \end{cases}$$

where $\alpha_r \leq 1$ is arbitrary. Note that $G_{sas'}$ is a non-expansive operator because

$$G_{sas'}Q^1(s, a) - G_{sas'}Q^2(s, a) = (1 - \alpha_r)(Q^1(s, a) - Q^2(s, a)) + \alpha_r(\max_b Q^1(s', b) - \max_b Q^2(s', b))$$

$$\leq (1 - \alpha_r)(Q^1(s, a) - Q^2(s, a)) + \alpha_r(Q^1(s', b^*_1) - Q^2(s', b^*_2))$$

$$\leq (1 - \alpha_r)\|Q^1 - Q^2\|_\infty + \alpha_r\|Q^1 - Q^2\|_\infty$$

$$= \|Q^1 - Q^2\|_\infty,$$

where $b^*_1 = \arg\max_b Q^1(s', b)$. Thus, $\|G_{sas'}Q^1 - G_{sas'}Q^2\|_\infty \leq \|Q^1 - Q^2\|_\infty$. Moreover, note that $q^*$ is a fixed point of $G_{sas'}$ by the Bellman equation, i.e., $G_{sas'}q^* = q^*$. For the case that $J_t = J^*$, $Q_t$ of the algorithm can be obtained by applying a sequence of these non-expansive operators. Let $(s_{t-1}, a_{t-1}, s_t) = (s, a, s')$. We have

$$\|Q_t - q^*\|_\infty = \|G_{sas'}Q_{t-1} - G_{sas'}q^*\|_\infty \leq \|Q_{t-1} - q^*\|_\infty.$$  

A simple induction shows that $\|Q_t - q^*\|_\infty \leq \|Q_1 - q^*\|_\infty = \|q^*\|_\infty$. Therefore, $\|Q_t\|_\infty \leq \|Q_t - q^*\|_\infty + 2\|q^*\|_\infty$. Since $q^*$ is unique up to an additive constant, we can assume without loss of generality that $2\|q^*\|_\infty = \text{sp}(q^*)$. Thus, $\|Q_t\|_\infty \leq \text{sp}(q^*)$.

For the general case where $|J_t - J^*| \leq c/\sqrt{t}$, define the operator $\hat{G}_{sas'}$ on the space of $Q$-functions by

$$[\hat{G}_{sas'}Q](x, u) = \begin{cases} G_{sas'}Q(s, a) + \alpha_r(J^* - J_{t-1}), & \text{if } (x, u) = (s, a) \\ Q(x, u), & \text{otherwise} \end{cases}$$

Note that $\hat{G}_{sas'}$ is also a non-expansive operator. Let $(s_{t-1}, a_{t-1}, s_t) = (s, a, s')$ and $\alpha_r = 1/\sqrt{n_t(s, a)}$. Observe that $Q_t = \hat{G}_{sas'}Q_{t-1}$ and $\hat{G}_{sas'}q^* = q^* + \alpha_r(J^* - J_{t-1})1(x = s, u = a).$ Thus,

$$\|Q_t - q^*\|_\infty = \|\hat{G}_{sas'}Q_{t-1} - \hat{G}_{sas'}q^* + \alpha_r(J^* - J_{t-1})1(x = s, u = a)\|_\infty$$

$$\leq \|\hat{G}_{sas'}Q_{t-1} - \hat{G}_{sas'}q^*\|_\infty + \alpha_r\|J^* - J_{t-1}\|_\infty$$

$$\leq \|Q_{t-1} - q^*\|_\infty + \frac{c}{\sqrt{n_t(s, a)(t - 1)}}.$$ 

By induction we can write

$$\|Q_t - q^*\|_\infty \leq \|Q_1 - q^*\|_\infty + \sum_{t=1}^{T-1} \frac{c}{\sqrt{n_t(s', a')}},$$

where the last inequality is by Lemma 5(3) and the fact that $Q_t(s, a) = 0$ for all $s, a$. The proof completes by observing that $\|Q_t\|_\infty \leq \|Q_t - q^*\|_\infty + \|q^*\|_\infty \leq 2\|q^*\|_\infty + c\text{SA}(1 + \ln(t - 1)) = \text{sp}(q^*) + c\text{SA}(1 + \ln(t - 1)).$ \qed

B Proof of Lemma 3

Lemma 3 (Restated). The second term of (5) can be bounded by

$$\sum_{t=1}^{T} |Q_t(s_t, a_t) - Q_{t+1}(s_t, a_t)| \leq 2(2B + 1)\sqrt{SAT} + c\text{SA}(1 + \ln T)$$

11
Proof. Rearranging line (3) of Algorithm 1 implies that

\[
Q_t(s_t, a_t) - Q_{t+1}(s_t, a_t) = \frac{1}{\sqrt{n_{t+1}(s_t, a_t)}} \left[ J_t - r(s_t, a_t) + Q_t(s_t, a_t) - v_t(s_{t+1}) \right].
\]

Note that by Lemma 2, \(Q_t(s_t, a_t) - v_t(s_{t+1}) \leq 2B\) where \(B := sp(q^*) + cSA(1 + \ln T)\). Moreover, \(J_t - r(s_t, a_t) \leq J^* + c/\sqrt{T} \leq 1 + c/\sqrt{T}\) by Assumption 1. Thus,

\[
\sum_{t=1}^{T} \left[ Q_{t+1}(s_t, a_t) - Q_t(s_t, a_t) \right] \leq \sum_{t=1}^{T} \left[ \frac{2B + 1}{\sqrt{n_{t+1}(s_t, a_t)}} \right] + \sum_{t=1}^{T} \frac{c}{\sqrt{2T}}.
\]

For the first term on the right hand side, we have

\[
\sum_{t=1}^{T} \frac{2B + 1}{\sqrt{n_{t+1}(s_t, a_t)}} = \sum_{t=1}^{T} \sum_{s,a} \frac{1}{n_{t+1}(s_t, a_t)} = 2B + 1\sqrt{n_{t+1}(s, a)} = \sum_{s,a} \sum_{i=1}^{n_{T+1}(s,a)} \frac{2B + 1}{\sqrt{n_{t+1}(s, a)}} \leq 2(2B + 1)vSA,
\]

where the last step uses Lemma 5(1).

For the second term on the right hand side (11), note that \(t \geq n_{t+1}(s_t, a_t)\). Thus,

\[
\sum_{t=1}^{T} \frac{c}{\sqrt{n_{t+1}(s_t, a_t)}} \leq \sum_{t=1}^{T} \sum_{s,a} \frac{c}{n_{t+1}(s_t, a_t)} = \sum_{s,a} \sum_{i=1}^{n_{T+1}(s,a)} \frac{c}{\sqrt{n_{t+1}(s_t, a_t)}} = \sum_{s,a} \left( c + c\ln(n_{T+1}(s,a)) \right) \leq cSA(1 + \ln T).
\]

\(\square\)

C Proof of Lemma 4

Lemma 4 (Restated). The following properties hold:

1. \(\sum_{i=1}^{T} \alpha^i = 1\).
2. For any \(i \geq 1\), and any \(K \geq i\),

\[
1 - \left(1 - \frac{1}{\sqrt{i+1}}\right)^{K-i+1} \leq \sum_{r=i}^{K} \alpha^r \leq 1 + \frac{5}{2\sqrt{i}}
\]

3. Let \(M \) be a scalar and define \(M^+ = \max(M, 0)\) and \(M^- = \max(-M, 0)\). Then, for any \(i \geq 1\), and any \(K \geq i\)

\[
M \sum_{r=i}^{K} \alpha^r \leq M + \frac{5M^+}{2\sqrt{i}} + M^-\left(1 - \frac{1}{\sqrt{i+1}}\right)^{K-i+1}
\]

4. For any \(K \geq 0\),

\[
\sum_{i=1}^{K} \left(1 - \frac{1}{\sqrt{i+1}}\right)^{K-i+1} \leq \sqrt{K}.
\]

5. For any \(K \geq 1\),

\[
\sum_{i=1}^{K} \frac{1}{\sqrt{i}} \left(1 - \frac{1}{\sqrt{i+1}}\right)^{K-i+1} \leq 2\sqrt{i}.
\]
Proof. 1. We prove by induction on $\tau$. For $\tau = 1$, $\alpha_1^1 = \alpha_1 = 1$. For the induction step, note that
\[
\alpha_1^\tau = (1 - \alpha)\alpha_1^{\tau-1}.
\]
Thus,
\[
\sum_{i=1}^{\tau} \alpha_i^\tau = \alpha_1^\tau + \sum_{i=1}^{\tau-1} \alpha_i^\tau = \alpha_1 + (1 - \alpha_1) \sum_{i=1}^{\tau-1} \alpha_i^{\tau-1} = \alpha_1 + (1 - \alpha) = 1,
\]
where the third equality is by the induction hypothesis.

2. To prove the lower bound, we can write
\[
\alpha_i^\tau = \alpha_i \prod_{j=i+1}^{\tau} (1 - \alpha_j) \geq \alpha_i(1 - \frac{1}{\sqrt{i+1}})^{\tau-i}.
\]
Thus,
\[
\sum_{\tau=i}^{K} \alpha_i^\tau \geq \alpha_i \sum_{\tau=i}^{K} (1 - \frac{1}{\sqrt{\tau+1}})^{\tau-i}
\]
\[
= \alpha_i \sqrt{i+1} \left(1 - (1 - \frac{1}{\sqrt{i+1}})^{K-i+1}\right)
\]
\[
\geq 1 - (1 - \frac{1}{\sqrt{i+1}})^{K-i+1}
\]
To prove the upper bound, note that
\[
\alpha_i^\tau = \alpha_i \prod_{j=i+1}^{\tau} (1 - \alpha_j) \leq \alpha_i \exp\left(- \sum_{j=i+1}^{\tau} \alpha_j\right)
\]
\[
\leq \alpha_i \exp\left(- \int_{i+1}^{\tau+1} \frac{1}{\sqrt{x}} dx\right)
\]
\[
= \alpha_i \exp\left(-2(\sqrt{\tau+1} - \sqrt{i+1})\right),
\]
where the first inequality is by $1 + x \leq e^x$. Thus,
\[
\sum_{\tau=i}^{K} \alpha_i^\tau \leq \sum_{\tau=i}^{\infty} \alpha_i^\tau
\]
\[
\leq \alpha_i \sum_{\tau=i}^{\infty} \exp\left(-2(\sqrt{\tau+1} - \sqrt{i+1})\right)
\]
\[
= \alpha_i \left(1 + \sum_{\tau=i+1}^{\infty} \exp\left(-2(\sqrt{\tau+1} - \sqrt{i+1})\right)\right)
\]
\[
\leq \alpha_i \left(1 + \int_{i}^{\infty} \exp\left(-2(\sqrt{x+1} - \sqrt{i+1})\right) dx\right)
\]
\[
= \alpha_i \left(1 + \sqrt{i+1} + \frac{1}{2}\right)
\]
\[
\leq \alpha_i \left(1 + \sqrt{i+1} + \frac{1}{2}\right)
\]
\[
= 1 + \frac{5}{2\sqrt{i}}
\]

3. We can write $M = M^+ - M^-$. Thus, by previous part we have
\[
M \sum_{\tau=i}^{K} \alpha_i^\tau = M^+ \sum_{\tau=i}^{K} \alpha_i^\tau - M^- \sum_{\tau=i}^{K} \alpha_i^\tau
\]
\[
\leq M^+ \left(1 + \frac{5}{2\sqrt{i}}\right) - M^- + M^- \left(1 - \frac{1}{\sqrt{i+1}}\right)^{K-i+1}
\]
\[
= M^+ + \frac{5M^+}{2\sqrt{i}} + M^- \left(1 - \frac{1}{\sqrt{i+1}}\right)^{K-i+1}
\]
4. Let \( j = K - i + 1 \). We can write

\[
\sum_{i=1}^{K} \left(1 - \frac{1}{\sqrt{i + 1}}\right)^{K+i+1} = \sum_{j=1}^{K} \left(1 - \frac{1}{\sqrt{K - j + 2}}\right)^{j} \\
\leq \sum_{j=1}^{K} \left(1 - \frac{1}{\sqrt{K + 1}}\right)^{j} \\
\leq \sum_{j=1}^{\infty} \left(1 - \frac{1}{\sqrt{K + 1}}\right)^{j} \\
= \sqrt{K + 1} - 1 \leq \sqrt{K}.
\]

5.

\[
\sum_{i=1}^{K} \frac{1}{\sqrt{i + 1}} \left(1 - \frac{1}{\sqrt{i + 1}}\right)^{K+i+1} = \sum_{i=1}^{K} \frac{\sqrt{i + 1}}{\sqrt{i + 1}} \left(1 - \frac{1}{\sqrt{i + 1}}\right)^{K+i+1} \\
\leq \sqrt{2} \sum_{i=1}^{K} \frac{1}{\sqrt{i + 1}} \left(1 - \frac{1}{\sqrt{i + 1}}\right)^{K+i+1} \\
\leq \sqrt{2} \sum_{i=1}^{K} \frac{e^{-\frac{1}{\sqrt{i + 1}}}}{\sqrt{i + 1}},
\]

where the last inequality is by \( 1 - x \leq e^{-x} \). We proceed by upper bounding the latter.

\[
\sum_{i=1}^{K} \frac{1}{\sqrt{i + 1}} e^{-\frac{1}{\sqrt{i + 1}}} = \sum_{i=1}^{K} \frac{1}{\sqrt{i + 1}} e^{-\frac{1+1}{\sqrt{i + 1}} e^{-\frac{1}{\sqrt{i + 1}}}} \\
\leq e^{-\frac{1}{\sqrt{i + 1}}} \sum_{i=1}^{K} \frac{1}{\sqrt{i + 1}} e^{\frac{1+1}{\sqrt{i + 1}} e^{-\frac{1}{\sqrt{i + 1}}}} \\
\leq e^{-\frac{1}{\sqrt{i + 1}}} \int_{1}^{K+1} \frac{1}{\sqrt{x + 1}} e^{\frac{1}{\sqrt{x + 1}} e^{-\frac{1}{\sqrt{i + 1}}}} dx \\
= 2e^{-\frac{1}{\sqrt{i + 1}}} (e^{\sqrt{K+2}} - e^{\sqrt{2}}).
\]

Note that \( e^{-\frac{1+1}{\sqrt{i + 1}} e^{-\frac{1}{\sqrt{i + 1}}}} \leq e^{-\sqrt{K+2}} \). Thus,

\[
2e^{-\frac{1}{\sqrt{i + 1}}} (e^{\sqrt{K+2}} - e^{\sqrt{2}}) \leq 2e^{-\sqrt{K+2}} (e^{\sqrt{K+2}} - e^{\sqrt{2}}) \leq 2.
\]

\[\Box\]

**D Proof of Lemma 5**

**Lemma 5 (Frequently used inequalities) (Restated).** The following inequalities hold:

1. \( \sum_{t=1}^{T} \frac{1}{\sqrt{n_{t+1}(s_t, a_t)}} = \sum_{s,a} \sum_{t=1}^{n_{T+1}(s,a)} \frac{1}{\sqrt{t}} \leq 2\sqrt{SAT}. \)

2. \( \sum_{s,a} \sum_{t=1}^{n_{t+1}(s,a)} (1 - \frac{1}{\sqrt{t+1}})^{n_{T+1}(s,a) - 1} \leq \sqrt{SAT}. \)

3. \( \sum_{t=1}^{T} \frac{1}{\sqrt{n_{t+1}(s,a)}} \leq SAT(1 + \ln T). \)

**Proof.**

1. \[
\sum_{t=1}^{T} \frac{1}{\sqrt{n_{t+1}(s_t, a_t)}} = \sum_{t=1}^{T} \frac{1}{\sqrt{n_{t+1}(s_t, a_t)}} \\
= \sum_{s,a} \sum_{t=1}^{n_{t+1}(s,a)} \frac{1}{\sqrt{n_{t+1}(s,a)}} = \sum_{s,a} \sum_{t=1}^{n_{t+1}(s,a)} \frac{1}{\sqrt{t}} \\
\leq 2 \sum_{s,a} \sqrt{n_{T+1}(s,a)} \leq 2 \sqrt{SAT} \sum_{s,a} n_{T+1}(s,a) = 2\sqrt{SAT},
\]

where the last inequality is by Cauchy-Schwarz.
2. Lemma 4(4) implies that
\[
\sum_{s,a} \sum_{i=1}^{nT+1(s,a)} \left(1 - \frac{1}{\sqrt{i+1}}\right)^{nT+1(s,a)-i+1} \leq \sum_{s,a} \sqrt{nT+1(s,a)} \leq \sqrt{SAT},
\]
where the last inequality is by Cauchy-Schwarz similar to the previous part.

3. Note that \(t \geq n_{t+1}(s,a)\). Thus,
\[
\sum_{t=1}^{T} \frac{1}{\sqrt{tn_{t+1}(s,a)}} \leq \sum_{t=1}^{T} \frac{1}{n_{t+1}(s,a)}
\]
\[
= \sum_{t=1}^{T} \sum_{s,a} \mathbb{1}(s = a, a_t = a) \frac{1}{n_{t+1}(s,a)} = \sum_{s,a} \sum_{t=1}^{T} \mathbb{1}(s = a, a_t = a) \frac{1}{n_{t+1}(s,a)}
\]
\[
= \sum_{s,a} \sum_{i=1}^{nT+1(s,a)} \frac{1}{i} \leq \sum_{s,a} \left(1 + \ln \left(n_{T+1}(s,a)\right)\right) \leq SA(1 + \ln T)
\]
\[\square\]

E Proof of Lemma 6

**Lemma 6 (Restated).** For a fixed \((s,a) \in S \times A\), let \(\tau = n_t(s,a)\), and \(t_i\) be the time step at which \((s,a)\) is taken for the \(i\)-th time. Then,
\[
\left( Q_t(s,a) - q^*(s,a) \right) \mathbb{1}(\tau \geq 1) = \left\{ \sum_{i=1}^{\tau} \alpha_i^r [J^* - J_{t_i}] + \sum_{i=1}^{\tau} \alpha_i^r [v_{t_i}(s_{t_i+1}) - v^*(s_{t_i+1})] \right\}
\]
\[
+ \sum_{i=1}^{\tau} \alpha_i^r [v^*(s_{t_i+1}) - E_{s',|s,a} v^*(s')] \right\}
\]

**Proof.** If \(\tau \geq 1\), Lemma 4 implies that \(\sum_{i=1}^{\tau} \alpha_i^r = 1\). Thus, by Bellman equation
\[
q^*(s,a) = \sum_{i=1}^{\tau} \alpha_i^r [r(s,a) - J_{t_i} + v_{t_i}(s_{t_i+1})].
\]
Combining this with Lemma 9 completes the proof. \(\square\)

**Lemma 9.** For a fixed \((s,a) \in S \times A\), let \(\tau = n_t(s,a)\) and \(t_i\) be the time step at which \((s,a)\) is taken for the \(i\)-th time. Then,
\[
Q_t(s,a) = \sum_{i=1}^{\tau} \alpha_i^r [r(s,a) - J_{t_i} + v_{t_i}(s_{t_i+1})].
\]

**Proof.** Note that \(Q_t(s,a)\) remains unchanged during \([t_{j-1} + 1, t_j]\). Thus, suffices to prove
\[
Q_{t_j+1}(s,a) = \sum_{i=1}^{j} \alpha_i^j [r(s,a) - J_{t_j} + v_{t_j}(s_{t_j+1})],
\]
for \(j \geq 0\) with the convention that \(t_0 = 0\). We proceed by induction on \(j\). For \(j = 0\), \(Q_I(s,a) = 0\) by the initialization of the algorithm. For the induction step, we write
\[
Q_{t_j+1}(s,a) = (1 - \alpha_j)Q_{t_j}(s,a) + \alpha_j [r(s,a) - J_{t_j} + v_{t_j}(s_{t_j+1})]
\]
\[
= (1 - \alpha_j)Q_{t_j-1}(s,a) + \alpha_j [r(s,a) - J_{t_j} + v_{t_j}(s_{t_j+1})]
\]
\[
= (1 - \alpha_j) \left( \sum_{i=1}^{j-1} \alpha_i^{j-1} [r(s,a) - J_{t_i} + v_{t_i}(s_{t_i+1})] + \alpha_j [r(s,a) - J_{t_j} + v_{t_j}(s_{t_j+1})] \right)
\]
\[
= \sum_{i=1}^{j} \alpha_i^j [r(s,a) - J_{t_i} + v_{t_i}(s_{t_i+1})],
\]
where the first equality is by line 3 of the algorithm, the second equality is by the fact that \(Q_t(s,a)\) remains unchanged during \([t_{j-1} + 1, t_j]\), the third equality is by the induction hypothesis, and the last equality follows from \(\alpha_j^j = (1 - \alpha_j)\alpha_j^{j-1}\) and \(\alpha_j^j = \alpha_j^j\). \(\square\)
F  Proof of Lemma 7

Lemma 7 (Restated). The regret of any algorithm is bounded as

\[ R_T \leq \text{sp}(v^*) + \text{sp}(v^*) \sqrt{\frac{1}{2} T \ln \frac{1}{\delta}} + \sum_{t=1}^{T} \left[ v^*(s_t) - q^*(s_t, a_t) \right], \]

with probability at least \( 1 - \delta \).

Proof. Write the regret \( R_T \) as

\[ R_T = \sum_{t=1}^{T} \left( J_t - r(s_t, a_t) \right) = \sum_{t=1}^{T} \left( E_{s', \sim p}(s_t, a_t) \left[ v^*(s') - q^*(s_t, a_t) \right] \right) \]

\[ = \sum_{t=1}^{T} \left( E_{s', \sim p}(s_t, a_t) \left[ v^*(s') - v^*(s_{t+1}) \right] \right) + \sum_{t=1}^{T} \left( v^*(s_{t+1}) - v^*(s_t) \right) \]

\[ = R_1 + R_2 \]

where the second equality is by Bellman equation. Note that \( R_2 = v^*(s_{T+1}) - v^*(s_1) \leq \text{sp}(v^*) \). The summands in \( R_1 \) constitute a martingale difference sequence. Thus, by Azuma-Hoeffding inequality \( R_1 \leq \text{sp}(v^*) \sqrt{\frac{1}{2} T \ln \frac{1}{\delta}} \) with probability at least \( 1 - \delta \) which completes the proof.

G  Proof of Lemma 8

Lemma 8 (Restated).

\[ \sum_{t=1}^{T} \left[ v_t(s_{t+1}) - v_{t+1}(s_{t+1}) \right] \leq 2\sqrt{SA^T} + cSA(1 + \ln T). \]

Proof. Note that \( v_t \) and \( v_{t+1} \) only differ at state \( s_t \). So, only terms with \( s_{t+1} = s_t \) contribute to the summation. Moreover, if \( s_{t+1} = s_t \), then

\[ Q_{t+1}(s_t, a_t) = (1 - \alpha_{n_{t+1}(s_t, a_t)}) Q_t(s_t, a_t) + \alpha_{n_{t+1}(s_t, a_t)} [r(s_t, a_t) - J_t + v_t(s_{t+1})] \]

\[ = (1 - \alpha_{n_{t+1}(s_t, a_t)}) v_t(s_t) + \alpha_{n_{t+1}(s_t, a_t)} [r(s_t, a_t) - J_t + v_t(s_{t+1})] \]

\[ = v_t(s_t) + \alpha_{n_{t+1}(s_t, a_t)} [r(s_t, a_t) - J_t]. \]

Thus, \( v_t(s_t) - Q_{t+1}(s_t, a_t) = \alpha_{n_{t+1}(s_t, a_t)} J_t \) and

\[ \sum_{t=1}^{T} \left[ v_t(s_{t+1}) - v_{t+1}(s_{t+1}) \right] = \sum_{t=1}^{T} \sum_{s_{t+1}=s_t} [v_t(s_{t+1}) - v_{t+1}(s_{t+1})] \]

\[ = \sum_{t=1}^{T} \sum_{s_{t+1}=s_t} [v_t(s_t) - v_{t+1}(s_t)] \]

\[ \leq \sum_{t=1}^{T} \sum_{s_{t+1}=s_t} [v_t(s_t) - Q_{t+1}(s_t, a_t)] \]

\[ \leq \sum_{t=1}^{T} \alpha_{n_{t+1}(s_t, a_t)} J_t \]

\[ \leq \sum_{t=1}^{T} \frac{1}{\sqrt{n_{t+1}(s_t, a_t)}} + \sum_{t=1}^{T} \frac{c}{\sqrt{n_{t+1}(s_t, a_t)}}, \]

where the last inequality is by the fact that \( J_t \leq J^* + c/\sqrt{t} \leq 1 + c/\sqrt{t} \). Using Lemma 5 (1) and (3) completes the proof. \( \square \)
Table 2: The hyper parameters used in the algorithms. These hyper parameters are optimized to obtain the best performance of each algorithm. We simulate 10 Monte Carlo independent runs over the horizon of $T = 5 \times 10^6$ steps. For the UCRL2 algorithm, $C$ is a coefficient that scales the confidence interval. $\tau$ and $\tau'$ for the POLITEX algorithm, are the lengths of the two stages defined in Figure 3 of [1].

| Algorithm | Parameters |
|-----------|------------|
| RandomMDP | $J_t = 1/t \sum_{t=1}^{T} r(s_t, a_t) + 1.2/\sqrt{t}$ |
| EE-QL     | $H = 2, c = 0.1, b_{\tau} = c \sqrt{H/\tau}$ |
| MDP-OOMD  | $N = 2, B = 4, \eta = 0.01$ |
| POLITEX   | $\tau = 1000, \tau' = 1000, \eta = 0.2$ |
| UCRL2     | $C = 0.1$ |
| PSRL      | Dirichlet prior with parameters $[0.1, \cdots, 0.1]$ |

| RiverSwim | Parameters |
|-----------|------------|
| EE-QL     | $J_t = 1/t \sum_{t=1}^{T} r(s_t, a_t) + 2/\sqrt{t}$ |
| Optimistic Q-learning | $H = 1000, c = 1, b_{\tau} = c \sqrt{H/\tau}$ |
| UCRL2     | $C = 0.1$ |
| PSRL      | Dirichlet prior with parameters $[0.1, \cdots, 0.1]$ |

H Experiments

In this section, more details about the experiments are provided. The confidence intervals of the optimistic algorithms (such as UCRL2 and Optimistic Q-learning) are tightened by scaling the original confidence interval with a constant that is tuned as a hyper parameter. The parameters of other algorithms are also tuned for the best performance in each environment. The details can be found in Table 2.