Galois coverings of Schreier graphs of groups generated by bounded automata

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Abstract

We give a characterization of the covering Schreier graphs of groups generated by bounded automata to be Galois. We also investigate the zeta and \( L \) functions of Schreier graphs of few groups namely the Grigorchuk group, Gupta-Sidki \( p \) group, Gupta-Fabrykowski group and BSV torsion-free group.

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1 Introduction

The first appearance of the graph theory analog of Riemann zeta function was in the work of Y. Ihara on \( p \)-adic groups in \([16, 17]\). H. Bass \([2]\), T. Sunada \([22]\) have enormously contributed to the subject during 1980s and 1990s. In the last decade the significant contribution has been made by A. Terras. See \([23]\) for zeta functions of finite graphs. The discussion on zeta functions of infinite graphs by R. Grigorchuk and A. Zuk can be found in \([14]\).

Ihara zeta function, \( L \) function and Galois covering of Schreier graphs of Basilica group has been studied by the authors in \([19]\). Here we extend this work for Schreier graphs of self-similar groups generated by bounded automata.

The graph theory analogue of Riemann zeta function \( \zeta(s) \) is known as Ihara zeta function which we define as follows: Let \( Y = (V, E) \) be a connected graph and let \( t \in \mathbb{C} \), with \( |t| \) sufficiently small. Then the Ihara zeta function \( \zeta_Y(t) \) of graph \( Y \) is defined as

\[
\zeta_Y(t) = \prod_{[C] \text{ prime cycle in } Y} (1 - t^\mu(C))^{-1},
\] (1)
where a prime $|C|$ in $Y$ is an equivalence class of tailless, back-trackless primitive cycles $C$ in $Y$ and length of $C$ is $\nu(C)$. (See Chapter 2, page no. 10-21 of [23].) The Ihara-Bass’s Theorem establishes the connection between $\zeta_Y(t)$ and the adjacency matrix $A$ of the graph $Y$ which is given as

$$\zeta_Y(t)^{-1} = (1 - t^2)^{-1} \det(I - At + Qt^2), \quad (2)$$

where $r = |E| - |V| - 1 = \text{rank of fundamental group of } Y$ and $Q$ is the diagonal matrix whose $j^{th}$ diagonal entry is $Q_{jj} = d(v_j) - 1$, where $d(v_j)$ is the degree of the $j-$th vertex of $Y$. For the proof of the Ihara-Bass’s theorem see page no. 89 of [23]. Suppose the graphs $Y$ and $\tilde{Y}$ are finite and connected. We say that the graph $\tilde{Y}$ is an unramified covering of the graph $Y$ if there is a covering map $\pi : \tilde{Y} \to Y$ which is an onto map such that for every vertex $u \in Y$ and for every $v \in \pi^{-1}(u)$, the set of vertices adjacent to $v$ in $\tilde{Y}$ is mapped by $\pi$ one-to-one, onto the vertices in $Y$ which are adjacent to $u$. Note that for an unramified covering, $\pi^{-1}(u)$ has the same number of elements for all $u$ in $Y$. If this number is $d$, we refer to $\tilde{Y}$ as a $d$-sheeted covering of $Y$. Therefore the vertex set of $\tilde{Y}$ can be viewed as the set of vertices $(u,i)$, where $u$ is the vertex of the graph $Y$ and $1 \leq i \leq d$. A $d$-sheeted covering is a normal or Galois covering if there are $d$ graph automorphisms $\sigma : \tilde{Y} \to \tilde{Y}$ such that $\pi(\sigma(v)) = \pi(v), \forall v \in \tilde{Y}$. These automorphisms form the Galois group $G = \text{Gal}(\tilde{Y}|Y)$. (See [10, 23] for more details.) The Galois group $G$ acts transitively on sheets of $\tilde{Y}$. i.e. $g \circ (u,h) = (u,gh)$, for $u \in Y, g, h \in G$. It follows that $g$ moves a path in $\tilde{Y}$. Suppose that $\tilde{Y}|Y$ is normal with Galois group $G$. For a path $p$ of $Y$, Proposition 13.3 of [23] says there is a unique lift to a path $\tilde{p}$ of $\tilde{Y}$, starting on sheet $id$ (here $id$ we mean identity in $G$), having the same length as $p$. If $\tilde{p}$ has its terminal vertex on the sheet labeled $g \in G$, define the normalized Frobenius automorphism $\sigma(p) \in G$ by $\sigma(p) = g$. Fix a spanning tree $\tilde{T}$ of $Y$, suppose $e_1, e_2, \cdots, e_r$ are non-tree edges of $Y$ with directions assigned. Then Lemma 17.1 of [23] says that the $r$ normalized Frobenius automorphisms $\sigma(e_j), j = 1, \cdots, r$, generate $G$.

Let $G$ be a self-similar group generated by bounded automaton $A$. Let $G$ have level transitive actions on $d$-rooted tree $X^*$ and let $\Gamma_n$ be $n$-level Schreier graph of $G$. Recall that every element $g$ of a self-similar group $G$ can be represented by the following way.

$$g = \psi_g(g|_{x_1}, \cdots, g|_{x_d})$$

where $\psi_g \in Sym(d)$ represents the action of $g$ on the alphabet $X$ and the $g|_{x_i}$’s the restrictions of the action on the corresponding sub-trees. We call
ψ\_g as the root permutation and Ψ\_G as group generated by these root permutations. The main goal of this paper is to prove the following result. If |(Ψ\_G)| = d, then root permutations are same as the permutations corresponds to the normalized Frobenius automorphisms i.e. the covering Γ\_n+1|Γ\_n is normal with Galois group G ≃ Ψ\_G.

After a quick overview of few definitions like self-similar group, group generated by bounded automata and basic notion related to actions on rooted trees i.e Schreier graphs, we discuss the few examples of Schreier graphs, in Section 2. We define the notion of generalized replacement product Γ\_n ⊙ Γ\_r for Schreier graphs Γ\_n and Γ\_r of G in Section 3. We also prove the isomorphism f : Γ\_n ⊙ Γ\_r → Γ\_n+r in section 3 only. Section 4 deals with the general result about Galois coverings. The detail discussion on zeta and L functions of Schreier graphs of few groups namely Grigorchuk group, Gupta-Sidki p group and others is presented in Section 5.

### 2 Preliminaries

\[
\begin{array}{c}
\emptyset \\
0 \\
00 \quad 01 \\
000 \quad 001 \quad 010 \quad 011 \quad 100 \quad 101 \quad 110 \quad 111
\end{array}
\]

**Figure 1:** Binary rooted tree.

Let X be a finite set, that we call alphabet. By X\(^*\) we denote the set \{d\_n \cdots d\_2d\_1 : d\_i ∈ X\} of all finite words over the alphabet X, including the empty word 0. The length of a word v = d\_n \cdots d\_2d\_1 (i.e. the number of letters in it) is denoted |v|. The set X\(^*\) is naturally identified as a vertex set of a rooted tree in which two words are connected by an edge if and only if
they are of the form \( v \) and \( vx \), where \( v \in X^* \), \( x \in X \). The empty word \( \emptyset \) is the root of the tree \( X^* \). See Fig. 1 for the case \( X = \{0, 1\} \). The set \( X^n \subset X^* \) is called the \( n \)-th level of the tree \( X^* \). We denote by \( Aut X^* \) the group of all automorphisms of the rooted tree \( X^* \). The notation \( g(x) \), we mean that the image of \( x \) under left action of \( g \). A faithful action of a group \( G \) on the set \( X^* \) is called self-similar if for every \( g \in G \) and \( x \in X \) there exists \( g|_x \in G \) such that \( g(xw) = g(x)g|_x(w) \) for all \( w \in X^* \). Inductively one defines the restriction \( g|_{x_n \cdots x_2 x_1} = g|_{x_n}|_{x_{n-1} \cdots}|_{x_1} \) for every word \( x_n \cdots x_2 x_1 \in X^* \). Restrictions have the following properties

\[
g(vu) = g(v)g|_u(u), g|_{vu} = g|_u|_u, (g \cdot h)|_v = g|_{h(v)} \cdot h|_v
\]

for all \( g, h \in G \) and \( v, u \in X^* \). An automorphism group \( G \) (i.e. \( G \) is a subgroup of \( Aut X^* \)) of the rooted tree \( X^* \) is self-similar if for every \( g \in G \) and \( v \in X^* \) we have \( g|_v \in G \). For self-similar groups we refer to [3, 18]. In other words an automorphism group \( G \leq Aut X^* \) of the rooted tree \( X^* \) is self-similar if \( G \leq Sym(X) G \), where \( Sym(X) \) denotes the symmetric group of \( d \) symbols, \( d = |X| \) and \( \wr \) denote the wreath product of groups.

Every element \( g \) of a self-similar group \( G \) can be uniquely represented by the following way.

\[
g = \psi_g(g|_{x_1}, \cdots, g|_{x_d}) \quad (3)
\]

where \( \psi_g \in Sym(d) \) and \( g|_{x_i} \in G \). We call \( \psi_g \) as root permutation and \( \Psi_G \) as group generated by the root permutations.

\[
\Psi_G = \langle \psi_g : g \in S \rangle. \quad (4)
\]

An automaton is a quadruple \( \mathcal{A} = (S, X, t, o) \), where \( S \) is the set of states, \( X \) is an alphabet; \( t : S \times X \rightarrow S \) is the transition map, and \( o : S \times X \rightarrow X \) is the output map. The automaton \( \mathcal{A} \) is finite if the set of states \( S \) is finite and it is invertible if, for all \( s \in S \), the transformation \( o(s, \cdot) : X \rightarrow X \) is a permutation of \( X \). An automaton \( \mathcal{A} \) can be represented by its Moore diagram, a directed labeled graph whose vertices are identified with the states of \( S \). For every state \( s \in S \) and every letter \( x \in X \), the diagram has an arrow from \( s \) to \( t(s, x) \) labeled by \( x|o(s, x) \). This graph contains complete information about the automaton. Given an automaton, one can consider the group generated by the transformation \( o(s, \cdot) \) induced by the states of the automaton on the elements in \( X^* \). This gives rise to an automaton group. Notice that, groups generated by automata are self-similar.
A finite invertible automaton is called *bounded* if the number of left- (equivalently, right-) infinite paths in the Moore diagram of \( \mathcal{A} \) ending at (avoiding) the trivial state is finite. Equivalently it can also be defined as a finite invertible automaton is bounded if the number of directed paths of length \( n \) in the Moore diagram of \( \mathcal{A} \) avoiding the trivial state is bounded independently on \( n \) (See [20]). In the sequel we present some important examples of groups generated by bounded automata.

**Example 2.1. Grigorchuk group**

Let \( X = \{0, 1\} \). Define the automorphisms of the binary tree \( X^* \) by the wreath recursions:

\[
\begin{align*}
a &= \psi_a(1, 1), & b &= \psi_b(a, c), & c &= \psi_c(a, d), & d &= \psi_d(1, b),
\end{align*}
\]

where \( \psi_a \) is the transposition \( (0 1) \in Sym(X) \) and \( \psi_b = \psi_c = \psi_d \) is the identity permutation in \( Sym(X) \). This group was constructed by R. Grigorchuk, as an example of infinite periodic finitely generated group. Also it is the first example of a group of intermediate growth i.e. its growth function has intermediate growth between polynomial and exponential. (See [12].) As all the restrictions given in Eq. 5 belong to generating set, we can have the generating automaton for the Grigorchuk group. See Fig. 2.

![Figure 2: The automaton generating the Grigorchuk group.](image)

**Example 2.2. BSV torsion-free group**

Let \( X = \{0, 1, \cdots , d - 1\} \). Define the automorphisms of the binary tree \( X^* \)
by the wreath recursions:

\[ a = \psi_a(1, 1, \ldots, 1, a), \quad b = \psi_b(1, 1, \ldots, 1, b^{-1}), \quad b^{-1} = \psi_{b^{-1}}(1, 1, \ldots, 1, b), \]

where \( \psi_a = (0, 1, \ldots, d-1) = \psi_b, \psi_{b^{-1}} = \psi_b^{-1} \in \text{Sym}(X) \). See Fig. 3 for generating automaton. This group was introduced by S. Sidki and E. F.

\[ (d - 1)|0 \]

\[ \quad a \]

\[ 0|1, 1|2, \ldots, (d - 2)|(d - 1) \]

\[ 0|1, 1|2, \ldots, (d - 2)|(d - 1) \quad 1|0, 2|1, \ldots, (d - 1)|(d - 2) \]

\[ 0|(d - 1) \]

\[ \quad b \]

\[ \quad b^{-1} \]

\[ \text{Figure 3: The automaton generating the BSV group} \]

Silva. (See [21].) Note that \textit{BSV torsion-free} \( k \)-group has transitive action on \( d \)-rooted tree, where \( 1 < d \in \mathbb{N} \). It is not difficult to show that \textit{BSV torsion free group} generated by \textit{bounded automaton}. This group is a generalization of a just-nonsolvable torsion-free group defined on binary rooted tree by A. M. Bunner, S. Sidki and A. C. Vieira [6].

\textbf{Example 2.3.} Gupta-Sidki group.

Let \( p = 3 \), so we have \( X = \{0, 1, 2\} \). Define the automorphisms of the binary tree \( X^* \) by the wreath recursions:

\[ a = \psi_a(b, b^{-1}, a), \quad b = \psi_b(1, 1, 1) \]

where \( \psi_b = (0, 1, 2) \in \text{Sym}(X) \) and \( \psi_a \) is the identity permutation. The group is generated by \( a, b, a^{-1}, b^{-1} \) and it was introduced by N. Gupta and
S. Sidki. This group is an counter example to the general Burnside problem, which shows that a finitely generated group, all of whose elements have finite $p$-power order (for a fixed prime $p$), can be infinite. See [15].

**Example 2.4. Fabrykowski-Gupta group.**

Let $X = \{0, 1, 2\}$. Define the automorphisms of the binary tree $X^*$ by the wreath recursions:

$$a = \psi_a(1, 1, 1), \ b = \psi_b(a, b, 1)$$

where $\psi_a = (0, 1, 2) \in \text{Sym}(X)$ and $\psi_b$ is the identity permutation. The group is generated by $a, b$ and it was introduced by N. Gupta and J. Fabrykowski. See [11].

**Example 2.5. Basilica group**

Let $X = \{0, 1\}$. Define the automorphisms of the binary tree $X^*$ by the wreath recursions:

$$a = \psi_a(b, 1), \ b = \psi_b(a, 1),$$

where $\psi_a$ is the identity permutation in $\text{Sym}(X)$ and $\psi_b = (0, 1) \in \text{Sym}(X)$. For generating automaton of Basilica group see Fig. 4(a). This group was introduced by R. Grigorchuk and A. Žuk [13]. The Basilica group is related to the fractal called Basilica which is the Julia set of the polynomial $z^2 - 1$.

The next section is devoted to Schreier graphs and tile graphs of Self-similar groups generated by bounded automata.

### 2.1 Schreier graphs

Let $G$ be a group generated by a finite set $S$. If the group $G$ acts on a set $M$, then the corresponding Schreier graph $\Gamma(G, S, M)$ is the graph with the vertex set $M$, and two vertices $v$ and $u$ are adjacent if there exists $s \in S$ such that $s(v) = u$ or $s(u) = v$.

Let $G$ be a group generated by the bounded automaton $\mathcal{A}$. The levels $X^n$ of the tree $X^*$ are invariant under the action of the group $G$. Denote by $\Gamma_n$ the Schreier graph of the action of $G$ on $X^n$. Denote $\Gamma'_n$ by tile graph with vertex set $X^n$, and two vertices $v$ and $u$ are adjacent if there exists $s \in S$ such that $s(v) = u$ and $s|_v = 1$. The tile graph is therefore a sub-graph of the Schreier graph. If the tile graph is connected, then it is in fact a spanning sub-graph of Schreier graph.
A left-infinite sequence \( \cdots x_2x_1 \) over \( X \) is called *post critical* if there exists a left-infinite path \( \cdots e_2, e_1 \) in the Moore diagram of \( \mathcal{A} \) not ending at the trivial state labeled by \( \cdots x_2x_1|\cdots y_2y_1 \) for some \( y_i \in X \). If \( G \) is a group generated by a bounded automaton then the set of post critical sequences, say \( \mathcal{P} \), is finite. Suppose \( p = \cdots x_2x_1 \) is any post critical sequence over \( X \) we say that \( p_n = x_n \cdots x_2x_1 \) is post critical vertex (associated to \( p \)) of the graphs (i.e. Schreier graph \( \Gamma_n \) and tile graph \( \Gamma'_n \)). Notice, that if an edge \( \{u, s(u)\} \) for some \( s \in S \) of the graph \( \Gamma_n \) is absent in \( \Gamma'_n \), then the word \( u \) is a post critical vertex.

\[
E(\Gamma'_r) = E(\Gamma_r) \setminus \{e_{s|u} = \{u, s(u)\} : u \in X^r, \text{ there exists } s \in S \text{ with } s|u \neq 1\}.
\]

By the notation \( s|u \neq 1 \) we mean that there exists the left path \( e_n, \ldots, e_2, e_1 \) labeled by \( u \) and starting by \( s \) in the Moore diagram of \( \mathcal{A} \) (corresponding to action of \( s \) on the word \( u \)) which does not end in the trivial state.

There is a one-to-one correspondence between the set \( P \) of post critical vertices of \( \Gamma_n \) and \( \Gamma'_n \) and the set \( \mathcal{P} \) post critical sequences of group \( G \). Therefore, with slight abuse in notations, we shall consider post critical sequences as post critical vertices of \( \Gamma_n \) and \( \Gamma'_n \). We use the following assumptions for the rest of this paper.

1. Let \( G \) be a self-similar group generated by the bounded automaton \( \mathcal{A} \).
2. Tile graphs \( \Gamma'_n \) are connected.

### 2.2 Inflation of graphs

We shall describe the *inflation* of graphs in Theorem 2.1. Inflation produces the \( r + 1 \)-level *tile* graph by using the \( r \)-level *tile* graph. To understand this construction one must know the associated *model* graph.

**Definition 2.1. Model graph**

The graph \( M = M_1 \) whose vertex set \( V_M = \mathcal{P}_\mathcal{A} \times X \) and the edge set \( E_M \) is given below:

\[
E_M = \left\{ (p, x), (q, y) \right\} \begin{cases} 
\text{there exists a path \( \cdots e_2e_1 \) in the Moore} \\
\text{diagram of } \mathcal{A}, \text{ which ends in the trivial state} \\
\text{and which is labelled by where } px|qy, p, q \in \mathcal{P}_\mathcal{A} \\
\text{and } x, y \in X
\end{cases}.
\]

is called the *model* graph associated to an automaton \( \mathcal{A} \).
Inflation:
The inflation was introduced by V. Nekrashevych \cite{Nekrashevych}. We recall the Theorem 2 of \cite{Nekrashevych} where inflation was described. It can be described using the first model graph $M_1 = M = (V_M, E_M)$ associated to the automaton $A$. We call the vertex $(p, x)$ of $M$ as a post-critical vertex, if the associated sequence $px$ is post-critical. Note that there is a one-to-one correspondence between the post-critical vertices of $M$ and the elements of the post-critical set $\mathcal{P}_A$. To construct the $r + 1$-level tile graph $G_{\Gamma_r}^+$, we first take $|X|$ copies of the tile graph $G_{\Gamma_r}$, having vertices $X^r \times x, \forall x \in X$. Among these copies, we first identify model vertices ($V_M = \mathcal{P}_A \times X$), then we put the model edges (i.e. elements of $E_M$) between these model vertices. This gives us the new graph with vertex set $X^{r+1}$ and any two vertices $ux$ and $vy$ are adjacent if there exists $s \in S$ such that $s(ux) = vy$ and $s|_{ux} = 1$ which is exactly the edge set of tile graph $G_{\Gamma_r}^+$. Let us state it as follows:

**Theorem 2.1.** To construct the tile graph $G_{\Gamma_r}^+$ take $|X|$ copies of the tile graph $G_{\Gamma_r}$, identify their sets of vertices with $X^r \times x$, $\forall x \in X$. Among these copies, we first identify model vertices ($V_M = \mathcal{P}_A \times X$), then we put the model edges (i.e. elements of $E_M$) between these model vertices. This gives us the new graph with vertex set $X^{r+1}$ and any two vertices $ux$ and $vy$ are adjacent if there exists $s \in S$ such that $s(ux) = vy$ and $s|_{ux} = 1$ which is exactly the edge set of tile graph $G_{\Gamma_r}^+$. Let us state it as follows:

**Proof.** See proof of the Theorem 2 of \cite{Nekrashevych}.

To construct the Schreier graph $G_{\Gamma_r}^+$ from the tile graph $G_{\Gamma_r}^+$, look at the post critical vertices say $p$ and $q$ of the tile graph $G_{\Gamma_r}^+$ and connect them by an edge if there exists $s \in S$ such that $s(p) = q$ and $s|_p \neq 1$.

![Diagram](image)

(a) Generating automaton $B$ of Basilica group  
(b) Model graph

**Figure 4:** The Basilica group, $\mathcal{P}_B = \{p_1 = 0, p_2 = (01)^{-\omega}, p_3 = (10)^{-\omega}\}$
2.3 Examples

Here we shall apply the inflation to the Schreier graphs of the Basilica group.

**Example 2.6.** Basilica group

This group is generated by three state automaton as shown in Fig. 4(a). The corresponding model graph is shown in Fig. 4(b). The Fig. 5 explains inflation.
The dashed and dark edges shown are model and Schreier edges respectively. In order to construct the Schreier graph of Basilica group $B\Gamma_n$, we take the tile graph $B\Gamma'_n$ and we add edges between post-critical vertices $u$ and $v$ if $s(u) = v$ for some $s \in S$ with $s|_u \neq 1$ as shown in Fig. 6.

In the Definition 2.1 and Theorem 2.1 one can observe the role of the alphabet $X$. After taking $|X|$ copies of $G\Gamma'_r$, Theorem 2.1 produces a next level tile graph $G\Gamma'_r$ and in order to do this, the model graph with vertex set $\mathcal{P} \times X$ has been used. That is, to construct the tile graph $G\Gamma'_r$, starting with the tile graph $G\Gamma'_r$, apply Theorem 2.1 iteratively $n$-times and produce the tile graph $G\Gamma'_r + n$. Notice that applying Theorem 2.1 iteratively is equivalent to taking $|X^n|$ copies of $G\Gamma'_r$ and using edges of the model graph $M_n$. One can connect these copies by identifying model vertices from the $|X^n|$ copies. Let us write this explicitly. For $n$-th iterated inflation graphs we follow Section 1.1 on page no. 94 of [3] and Section 3.10 on page no. 110 of [18]. More information on Schreier graphs of the Basilica group can be found in [7].

**Definition 2.2.** We denote $M_n$ as $n$-th iterated model graph associated to the bounded automaton $\mathcal{A}$ whose vertex set is $V_n = \mathcal{P}_A \times X^n$ and the edge set $E_n$ is given below:

$$E_{M_n} = \left\{ (p, x_n \cdots x_1), (q, y_n \cdots y_1) \right\} \text{ there exists a path } \cdots e_2 e_1 \text{ in the Moore diagram of } \mathcal{A}, \text{ which ends in the trivial state and which is labelled by } \begin{align*} px_n \cdots x_1 &\sim qy_n \cdots y_1, \text{ where } p, q \in \mathcal{P}_A && x_n \cdots x_1, y_n \cdots y_1 \in X^n \end{align*} \right\} \quad \text{(11)}

Now we describe the $n$th iterated inflation in the Corollary given below.

**Corollary 2.1.** To construct the tile graph $G\Gamma'_r$, take $|X^n|$ copies of the tile graph $G\Gamma'_r$, identify their sets of vertices with $X^r \times x_n \cdots x_1$ for all $x_n \cdots x_1 \in X^n$ and connect two vertices $(u, x_n \cdots x_1)$ and $(v, y_n \cdots y_1)$ by an edge if $u, v \in \mathcal{P}_A$ and $\{(u, x_n \cdots x_1); (v, y_n \cdots y_1)\} \in E_{M_n}$.

**Proof.** Proof follows immediately by applying iteratively Theorem 2.1 $n$-times.

**Example 2.7.** We shall construct the tile graph $B\Gamma'_5$ and hence Schreier graph $B\Gamma_5$ by applying Corollary 2.1. In order to do this we start with the tile graph $B\Gamma'_2$. See Figures 7 and 8. The numbered vertices and dashed edges are shown in the Fig. 7 are the vertices and the edges of third iterated model graph $M_3$. For the vertex numbering see Table 11.
Figure 7: The tile graph $B\Gamma'_5$ of Basilica group over $X^5$, $\mathcal{P}_B = \{p_1 = (0)^{-\omega}, p_2 = (01)^{-\omega}, p_3 = (10)^{-\omega}\}$.

Table 1: Vertices of 3rd iterated model graph $M_3$.

| Numbering | Vertices       | Numbering | Vertices       |
|-----------|----------------|-----------|----------------|
| 1, 2, 3   | 01110, 00110, 10110 | 13, $p_2$, 15 | 00101, 10101, 01101 |
| 4, 5, $p_3$ | 10010, 00010, 01010 | 16, 17, 18 | 10001, 00001, 01001 |
| 7, $p_1$, 9 | 01000, 00000, 10000 | 19, 20, 21 | 01011, 00011, 10011 |
| 10, 11, 12 | 00100, 01100, 10100 | 22, 23, 24 | 10111, 00111, 01111 |

3 Generalized replacement product

The replacement product of two graphs is well known in literature. If $G_1$ and $G_2$ are two regular graphs with the regularity $d_1$ and $d_2$ respectively, then their replacement product $G_1 \boxtimes G_2$ is again a regular graph with regularity
Figure 8: The Schreier graph $B\Gamma_5$ of Basilica group over $X^5$.

$d_2 + 1$. The details about this product can be found in [11, 9]. The idea of generalized replacement product was first introduced by the first and third author for the Schreier graphs $B\Gamma_n$ of the Basilica group. (See Definition 2.1 of our paper [19]). This product was used to show the Galois covering of Schreier graphs of the Basilica group (See Proposition 2.1 of [19]). We here extend the idea of generalized replacement product for Schreier graphs $G\Gamma_n$ of self-similar group $G$ generated by a bounded automaton. Let $\Gamma = (V, E)$ be a finite connected $d$-regular graph (loops and multi-edges are allowed).

Suppose that we have a set of $d$ colors, that we identify with the set of natural numbers $[d] := \{1, 2, \ldots, d\}$. If $e = \{v, v'\}$ is an edge of the $d$-regular graph $\Gamma$ which has color say $s$ near $v$ and $s'$ near $v'$ and that any two distinct edges issuing from $v$ have a different color near $v$, then the rotation map $\text{Rot}_\Gamma : V \times [d] \to V \times [d]$ is defined by

$$\text{Rot}_\Gamma(v, s) = (v', s'), \text{ for all } v, v' \in V, \ s, s' \in [d].$$

We may have $s \neq s'$. Moreover, it follows from the definition that the composition $\text{Rot}_\Gamma \circ \text{Rot}_\Gamma$ is the identity map (See [11]). Since an edge $e =
\{v,v'\} of \(\Gamma\) is colored by some color \(s\) near \(v\) and by some color \(s'\) near \(v'\), we will say that the graph \(\Gamma\) is bi-colored. Let \(G^n\Gamma_n\) and \(G^r\Gamma_r\) be two Schreier graphs of a group \(G\) generated by a bounded automaton \(A\) with set of states \(S \cup S^{-1}\). In this case the color near \(v\) is \(s\) and near \(v'\) is \(s'\) if \(o(s,v) = v'\) and \(t(s,v) = s'\). Notice that \(s' = s^{-1}\) from the automaton group point of view. To define their generalized replacement product, we recall the edge set given in the Eq. 10 of tile graph \(G^n\Gamma_n\). \(E(G^n\Gamma_n) = E(G^r\Gamma_r) \setminus \{e_{s|u} = \{u, s(u)\} : u \in X^r\text{, there exists } s \in S \text{ with } s|u \neq 1\}\)

Note that the edge \(e_{s|u}\) mentioned above we mean the Schreier edge but not the tile edge.

**Definition 3.1.** The generalized replacement product \(G^n\Gamma_n \bigcirc G^r\Gamma_r\) is the \(|S|\)-regular graph with vertex set \(X^{n+r} = X^n \times X^r\), and whose edges are described by the following rotation map: Let \((v, u) \in X^n \times X^r\)

\[
\text{Rot}_{G^n\Gamma_n \bigcirc G^r\Gamma_r}((v, u), s) = ((v, s(u)), s^{-1}) \quad \text{if } s \in S \text{ with } s|u = 1, \quad (12)
\]

\[
\text{Rot}_{G^n\Gamma_n \bigcirc G^r\Gamma_r}((v, u), s) = ((s|v(u), s(u)), s^{-1}) \quad \text{if } s \in S \text{ with } s|_u \neq 1 \text{ and } s|_{uv} = 1, \quad (13)
\]

\[
\text{Rot}_{G^n\Gamma_n \bigcirc G^r\Gamma_r}((v, u), s) = ((s|_u(v), s(u)), s^{-1}) \quad \text{if } s \in S \text{ with } s|_u \neq 1 \text{ and } s|_{uv} \neq 1. \quad (14)
\]

One can imagine that the vertex set \(X^{n+r}\) of the graph \(G^n\Gamma_n \bigcirc G^r\Gamma_r\) is partitioned into the sheets, which are indexed by the vertices of \(G^n\Gamma_n\), where by definition of the \(v\text{-th sheet, for } v \in X^n\), consists of the vertices \(\{(v, u)|u \in X^r\}\). Within this construction the idea is to put the copy of \(G^r\Gamma_r\)(which is a tile graph) around each vertex \(v\) of \(G^n\Gamma_n\). We keep all the edges of \(G^r\Gamma_r\) as it is. The edges of the \(v\text{-th sheet } G^r\Gamma_r\) are determined by the Eq. 12 so we call them as \(v\text{-th sheet-edges}. We connect the \(|X^n|\) number of sheets by adding the edges as given in the Equations 13, 14. We call the edges given in the Eq. 13 as model lifts and the edges given in the Eq. 14 as Schreier lifts of the edges \(e_{s|u}\). The connectedness of \(G^n\Gamma_n\) and \(G^r\Gamma_r\) guarantees about the connectedness of the graph \(G^n\Gamma_n \bigcirc G^r\Gamma_r\). The Proposition 3.1 describes that the graph \(G^n\Gamma_n \bigcirc G^r\Gamma_r\) is isomorphic to a Schreier graph \(G^n\Gamma_n+1\) of \(G\). As shown in Proposition 8.1 of [13], the graph \(G^n\Gamma_n+1\) is covering of the graph \(G^n\Gamma_n\). We here extend this covering map to the \(n\text{-th level and we also prove that the covering is unramified.}

**Proposition 3.1.** If \(n, r \geq 1\), then the following holds:

1. The graphs \(G^n\Gamma_n \bigcirc G^r\Gamma_r, G^n\Gamma_n+1\) are isomorphic.
2. \( G\Gamma_{n+r} \) is an unramified, \( d^n \) sheeted graph covering of \( G\Gamma_r \).

Proof. Let \( n, r \geq 1 \).

1. Let us define the map \( f : \ G\Gamma_n \ast G\Gamma_r \to \ G\Gamma_{r+n} \) by \( f(v, u) = uv \), where \( v \in X^n \) and \( u \in X^r \). By definition of \( f \) it follows that \( f \) is bijection map. To show that \( f \) is adjacency preserving map, recall that there are three types of edges in the graph \( G\Gamma_n \ast G\Gamma_r \), i.e. first type contains sheet-edges which are described by the Eq. 12 and second contains model lifts which are described by the Eq. 13 and third contains the Schreier lifts which are described by the Eq. 14.

Sheet edges:
Note that the sheet edge connects \((v, u)\) to \((v, s(u))\), where \( s \in S \) with \( s|_u = 1 \). In other words, suppose the vertex \((v, u)\) is adjacent to the vertex \((v, s(u))\) in the graph \( G\Gamma_n \ast G\Gamma_r \). This implies

\[
\text{Rot}_{G\Gamma_n \ast G\Gamma_r}((v, u), s) = ((v, s(u)), s^{-1}),
\]

where \( s \in S \) with \( s|_u = 1 \).

As \( s|_u = 1 \), we have \( s(uv) = s(u)s|_u(v) = s(u)v \). This implies

\[
\text{Rot}_{G\Gamma_{n+r}}(uv, s) = (s(u)v, s^{-1}),
\]

So \( uv \) is adjacent to \( s(u)v \) in the graph \( G\Gamma_{n+r} \) which shows that \( f \) is adjacency preserving map.

The cases of Model and Schreier edges can be treated in the same way.

2. By the part (1), \( G\Gamma_{n+r} \simeq G\Gamma_n \ast G\Gamma_r \) contains \( |X^n| \) sheets each of which isomorphic to the tile graph \( G\Gamma'_r \). Define the map \( \pi : \ G\Gamma_{n+r} \to G\Gamma_r \) by \( \pi(uv) = u \). It is straight forward to show that \( \pi \) is a covering map.
Thus \( G\Gamma_{n+r} \) is an unramified covering of \( G\Gamma_r \).

\[\square\]

In the Definition 3.1 if we assume that \( n = 1 \) then we have the following propositions.

Proposition 3.2. 1. The rotation map (12) gives \( \cup_{x \in X} (x \times G\Gamma'_r) \) i.e. the \( |X| \) disjoint copies of tile graph \( G\Gamma'_r \) indexed by \( x \in X \).

2. In addition to the rotation map (12), the rotation map (13) adds the edges between the copies of \( G\Gamma'_r \) and it produces the tile graph \( G\Gamma'_{r+1} \).
3. In addition to the rotation maps (12) and (13), the rotation map (14) adds the edges between the post critical vertices of the tile graph \( G_{\Gamma^{r+1}} \) and it produces the Schreier graph \( G_{\Gamma^{r+1}} \).

Proof. 1. For any \((x,u) \in X \times X^r\), the rotation map (12) is

\[
\text{Rot}_{G_{\Gamma^{r+1}}}((x,u),s) = ((x,s(u)), s^{-1}) \quad \text{if } s \in S \text{ with } s|_u = 1.
\]

Observe that the letter \( x \) is unchanged under the action of the generator \( s \in S \). i.e. The action of the rotation map \( \text{Rot}_{G_{\Gamma^{r+1}}} \) on \((x,u)\) is non-trivial at the second component \( u \) whereas it is acting trivially at the first component \( x \). Therefore there exists a rotation map \( \text{Rot}_{\Gamma_r} : X^r \times S \rightarrow X^r \times S \) such that

\[
\text{Rot}_{\Gamma_r}(u,s) = (s(u), s^{-1}), \quad \text{where } s|_u = 1.
\]

The edges described by the rotation map (15) are the edges of the tile graph \( G_{\Gamma_r} \). Therefore for every \( x \in X \), we have a copy of the tile graph \( G_{\Gamma_r} \).

2. Note that \( s|_u \neq 1 \) means there exists a left infinite path in the Moore diagram of automaton \( A \) which does not end in the trivial state and it is labeled by \( u|s(u) \). Note also that \( s|_{ux} = 1 \) means there exists a left infinite path in the Moore diagram of automaton \( A \) which ends in the trivial state and it is labeled by \( ux|s(u)|_u(x) \). This implies that the edges described by the rotation map (13) are the edges of the model graph \( M \). Therefore, the resultant graph has the vertex set \( X \times X^r = X^{r+1} \) and any two vertices \((x,u) \sim f = ux \) and \((x',u') \sim f = u'x' \) (which are determine using the map \( f(x,u) = ux \) as in part (1)) are adjacent if \( \exists s \in S \text{ such that } s|_u \neq 1 \) and \( s|_{ux} = 1 \) i.e. \( s(ux) = s(u)s|_u(x) = u'x' \) which is exactly the tile graph \( G_{\Gamma^{r+1}} \).

3. It is the particular case of (1) of the Proposition 3.1.

Recall that Theorem 2.1 produces the tile graph \( G_{\Gamma^{r+1}} \) by using the tile graph \( G_{\Gamma_r} \). Note that while applying the rotation maps (12) and (13) to the tile graph \( G_{\Gamma_r} \) is identical to the construction of inflation described in Theorem 2.1. For any \( n, r \geq 1 \), applying the rotation maps (12) and (13) to the tile graph \( G_{\Gamma_r} \) is identical to the construction of \( nth \) iterated inflation described in Corollary 2.1.
Example 3.1. See Figures 9 and 10 for the generalized replacement product of Schreier graphs of Grigorchuk group and BSV torsion free group respectively. Note that the continuous, dotted and dashed edges are the sheet edges, model lifts and Schreier lifts respectively.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{example3.png}
\caption{The Schreier graph $\text{Gr}\Gamma_1$, the tile graph $\text{Gr}\Gamma'_1$ of the Grigorchuk group and the graph $\text{Gr}\Gamma_1\otimes\text{Gr}\Gamma_1$.}
\end{figure}

4 Galois coverings of Schreier graphs

We now present the generalization of the Proposition 2.1.4 of [19]. In [19], the authors have studied Galois coverings of the Schreier graphs of the Basilica group and their zig-zag products.

Recall from the Eq. 4 that $\Psi_G$ is the group generated by root permutations $\psi_s$ associated to the generators $s$ of $G$, i.e. $\Psi_G = \langle \psi_s : s \in S \rangle$. We also assume that the alphabet $X$ has exactly $d$-elements. By $|G|$, we denote the order of the group $G$.

Theorem 4.1. If $|\Psi_G| = d$, then root permutations are same as the permutations corresponding to the normalized Frobenius automorphisms i.e. the covering $G\Gamma_{n+1}|G\Gamma_n$ is normal with Galois group $G = \text{Gal}(G\Gamma_{n+1}|G\Gamma_n) \simeq \Psi_G$.

Proof. By the Proposition 3.1 we have that the graph $G\Gamma_{n+1}$ is a $d$ sheeted unramified covering of the graph $G\Gamma_n$. To show that $G\Gamma_{n+1}$ is a Galois (or
normal) covering of $G_{\Gamma_n}$ with Galois group $\mathbb{G} = Gal(G_{\Gamma_{n+1}}|G_{\Gamma_n})$ we first show that $|\mathbb{G}| = d$. Denote the $d$ sheets of $G_{\Gamma_{n+1}}$ by the letters $x_i \in X = \{x_1, \ldots, x_d\}$. As $G_{\Gamma_{n+1}} \simeq G_{\Gamma_1} \otimes G_{\Gamma_n}$, the construction of the generalized replacement product says the following: if $u \in X^n, g \in S$ with $g|_u \neq 1$, then by Eq. 10, the edges $e_{g|_u} = \{u, g(u)\}$ belong to $E(G_{\Gamma_n})$ but $e_{g|_u}$ do not belong to $E(G_{\Gamma_{n+1}}')$, where $G_{\Gamma_{n+1}}'$ is the tile graph. But by the Lemma 17.1 of [23], the Galois group $\mathbb{G}$ is generated by the normalized Frobenius automorphisms $\sigma(e_{g|_u})$:

$$\mathbb{G} = Gal(G_{\Gamma_{n+1}}|G_{\Gamma_n}) = \langle \sigma(e_{g|_u}) : u \in X^n, g \in S \text{ with } g|_u \neq 1 \rangle.$$

The normalized Frobenius automorphisms $\sigma(e_{g|_u})$ are the graph automorphisms on the graph $G_{\Gamma_{n+1}}$ and they map sheets to sheets by the rule

$$\sigma(e_{g|_u})(x_i) = g|_u(x_i), \text{ for all } x_i \in X.$$

Note that the normalized Frobenius automorphisms $\sigma(e_{g|_u})$ are nothing but permutations of letters from the alphabet $X$. Therefore we can have at the most $d!$ such automorphisms. As $g \in S$ and $G$ is self-similar so $g|_u \in S$. By

![Figure 10: The Schreier graph $BSV_{\Gamma_1}$, the tile graph $BSV_{\Gamma_1}'$ of BSV torsion-free group and the graph $BSV_{\Gamma_1} \otimes BSV_{\Gamma_1}$.](image)
the Proposition 1.5.5 of [18], we have
\[
g|_u = \psi_{g|u}(g|_{ux_1}, g|_{ux_2}, \cdots, g|_{ux_d}),
\]
where \(\psi_{g|u} \in \text{Sym}(X)\) is such that \(g|_u(x) = \psi_{g|u}(x)\) and \(g|_{ux} = g|_u x\) for all \(x \in X\). Hence we have
\[
\sigma(e_{g|u})(x) = \psi_{g|u}(x), \text{ for all } x \in X.
\]
Therefore we have the following
\[
G = \langle \psi_{g|u} \mid g \in S, u \in X^n \text{ with } g|_u \neq 1 \rangle = \Psi_G
\]
But \(|\Psi_G| = d\). Thus, \(|G| = d\).
To show that \(G\Gamma_n\) is \(d\) sheeted normal covering of \(G\Gamma_n\) it remains to show that \(\pi \circ \sigma \equiv \pi\) where \(\pi : G\Gamma_{n+1} \to G\Gamma_n\) is a covering map (see Proposition 3.1) and \(\sigma \in G\) is a normalized Frobenius automorphism which maps sheets to sheets. If \(v\) is any vertex of \(G\Gamma_n\) so we denote \(vx_i\) as vertex of the \(x_i\)th sheet, where \(x_i \in X\). By Definition 13.9 of [23], we have the following
\[
(\pi \circ \sigma)(vx_i) = \pi(\sigma(vx_i)) = \pi(v\sigma(x_i)) = v
\]
and
\[
\pi(vx_i) = v, \text{ for all } x_i \in X.
\]
Therefore
\[
\pi \circ \sigma \equiv \pi.
\]
\[\square\]

**Corollary 4.1.** If the action of a bounded automaton with alphabet \(\{0, 1\}\) is transitive on the levels of the binary tree then the covering \(G\Gamma_{n+1}|G\Gamma_n\) is normal with Galois group \(G = \text{Gal}(G\Gamma_{n+1}|G\Gamma_n) \simeq \Psi_G \simeq \mathbb{Z}/2\mathbb{Z}\).

**Proof.** It follows immediately from the Theorem 4.1 \(\square\)

### 4.1 Examples

This section is devoted to examples of Galois/normal and non-normal coverings of Schreier graphs.
Example 4.1. Galois coverings of Schreier graphs of groups generated by bounded automaton

For all the following groups, the \( n+1 \) level Schreier graph is \( d \)-sheeted normal covering of the \( n \) level Schreier graph.

1. \( d = 2 \), for the Grigorchuk group. See Fig. 2

2. Any \( d > 1 \), for the BSV torsion-free \( d \)-group. See Fig. 3

3. \( d = p \), for Gupta-Sidki \( p \)-group. See Eq. 7

4. \( d = 3 \), for the Fabrykowski-Gupta group. See Eq. 8

5. \( d = 2 \), for the basilica group. See Eq. 9 or Fig. 4(a)

Example 4.2. Non-normal covering of Schreier graphs of Tower of Hanoi group

Let \( X = \{0, 1, 2\} \). Define the automorphisms of the binary tree \( X^* \) by the wreath recursions:

\[
a = \psi_a(a, 1, 1), \quad b = \psi_b(1, b, 1), \quad c = \psi_c(1, 1, c) \quad (16)
\]

where \( \psi_a = (1 2), \psi_b = (0 2) \) and \( \psi_c = (0 1) \in Sym(X) \). See also [8] for further details about Schreier graphs of this group.

Notice that, the Tower of Hanoi group \( \mathbb{T} \) is generated by bounded automaton but the group \( \Psi_T \), generated by associated root permutations \( \psi_a = (1 2), \psi_b = (0 2) \) and \( \psi_c = (0 1) \) has order 6. The Tower of Hanoi group has transitive action on 3-rooted tree. Thus, Tower of Hanoi group does not satisfy the condition \( (|\mathbb{T}| = d) \) given in the Theorem 4.1. Therefore the \( n+1 \) level Schreier graph is 3-sheeted non-normal covering of the \( n \) level Schreier graph.

5 \( L \) functions and zeta functions of graphs

Let \( \tilde{Y} \) be the normal covering of \( Y \). Then one can calculate the Ihara zeta function of the covering graph \( \tilde{Y} \) in terms of the Artin \( L \)-functions. To calculate Artin \( L \)-functions we need the irreducible representations of the Galois group \( \mathbb{G} = Gal(\tilde{Y}|Y) \). Associated to every irreducible representation \( \rho \) of \( \mathbb{G} \), there is a Artin \( L \)-function. The Ihara zeta function \( \zeta_{\tilde{Y}}(t) \) is the product of these Artin \( L \)-functions.
**Definition 5.1.** The Artin $L$ function associated to the representation $\rho$ of $G = \text{Gal}(\tilde{Y}|Y)$ can be defined by a product over prime cycles in $Y$ as

$$L(u, \rho, \tilde{Y}|Y) = \prod_{[C] \text{ prime in } Y} \det(I - \rho(\sigma(C)))^{-1},$$

where $\tilde{C}$ is the lift of $C$ in the graph $\tilde{Y}$ that starts on sheet $id$ and ends on the sheet $l$ and $\sigma(\tilde{C})$ denotes the Frobenius automorphism as defined in the Definition 16.3 in [23] by

$$\sigma(\tilde{C}) = l^{-1},$$

where $id$ is the identity in $G$ and $l$ is the group element of $G$, $\rho(\sigma(\tilde{C}))$ is the $d_\rho \times d_\rho$ matrix associated to the irreducible representation $\rho$ and $d_\rho = \text{degree of } \rho$.

**Definition 5.2.** The $m \times m$ matrix $A(g)$ for $g \in G$ is the matrix whose $i, j$ entry is

$$A(g)_{i,j} = \text{the number of edges in } \tilde{Y} \text{ between } (i, id) \text{ to } (j, g),$$

where $id$ denotes the identity in $G$ and $m$ is the number of vertices of the graph $Y$.

We now recall the Proposition 2.1 of [10].

**Proposition 5.1.** Suppose $\tilde{Y}|Y$ is normal covering with Galois group $G = \text{Gal}(\tilde{Y}|Y)$. The adjacency matrix of $\tilde{Y}$ can be block diagonalized where the blocks are of the form

$$A_\rho = \sum_{g \in G} A(g) \otimes \rho(g),$$

each taken $d_\rho (= \text{dimension of the irreducible representation } \rho)$ times, where $\rho \in \hat{G}$ is an irreducible representation of $G$.

**Proof.** See [10] or see proof of Theorem 18.14 in [23].

By setting $Q_\rho = Q \otimes I_{d_\rho}$, with $d_\rho = \text{degree of } \rho$, we have the following analogue of formula given in the Eq. 2

$$L(t, \rho, \tilde{Y}|Y)^{-1} = (1 - t^2)^{r-1}d_\rho \det(tA_\rho + t^2Q_\rho).$$

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We use this description in some examples of automaton groups (i.e. groups generated by automata) described before.

### 5.1 Examples

We now investigate the Ihara zeta and $L$ functions of Schreier graphs of automaton groups. If $G$ is an automaton group then by $G\Gamma_n$, we shall denote the associated $n$ level Schreier graph.

**Example 5.1. BSV-group.**

By Theorem 4.1, the covering $\tilde{Y} = BSV\Gamma_2 \simeq BSV\Gamma_1 \otimes BSV\Gamma_1$ over the graph $Y = BSV\Gamma_1$ given in the Fig. 10 is 4-sheeted normal covering. We obtain spanning sub-graph of the $Y = \Gamma_1$ by deleting edges $e_a = \{4, a(4) = 1\}, e_b = \{4, b(4) = 1\}$. This gives the tile graph $BSV\Gamma'_1$ of the $BSV\Gamma_1$. So we draw the covering graph $\tilde{Y}$ by placing 4 sheets $BSV\Gamma'_1$ of $Y$ and labeling each sheet given in Table 2. Connections between 4 sheets in the cover graph $\tilde{Y}$ are given in the Table 3. In this case the irreducible characters of cyclic Galois group $\mathbb{G} = \langle g = (1, 2, 3, 4) \mid g^4 = id \rangle \simeq \mathbb{Z}/4\mathbb{Z}$ are $\chi_i(j) = \exp\left(\frac{2\pi ij}{4}\right)$ for $1 \leq i, j \leq 4$.

We now write all matrices $A(g), g \in \mathbb{G}$ given in Eq. 17.

$$A(id) = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{pmatrix}, A(g) = A(g^3) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, A(g^2) = O_{4 \times 4}.$$ 

Using Eq. 18 we write down all matrices $A_{\chi_i}$, where $\chi_i$ is an irreducible character of $\mathbb{G}$. The matrices $A_{\chi_i}$'s are called as Artinized adjacency matrices. (Note that, the Galois group $\mathbb{G}$ is cyclic, so instead of irreducible representation $\rho_i$, we can take irreducible character $\chi_i$.)

$$A_{\chi_i} = \begin{pmatrix} 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 \end{pmatrix}.$$
$$A_{\chi_2} = A_{\chi_4} = A(id),$$

$$A_{\chi_3} = \begin{pmatrix}
0 & 2 & 0 & -2 \\
2 & 0 & 2 & 0 \\
0 & 2 & 0 & 2 \\
-2 & 0 & 2 & 0
\end{pmatrix}. $$

Note that $A_{\chi_1} = A_Y$.

**Table 2: Notations for sheets**

| Vertex set of $\bar{Y}$ | Vertex set of $\Gamma_2$ | Group element |
|-------------------------|-------------------------|---------------|
| $\{1, 2, 3, 4\} \sim \{(1, 1), (1, 2), (1, 3), (1, 4)\}$ | $\{11, 21, 31, 41\}$ | id |
| $\{1', 2', 3', 4'\} \sim \{(2, 1), (2, 2), (2, 3), (2, 4)\}$ | $\{21, 22, 23, 24\}$ | $g = (1234)$ |
| $\{1'', 2'', 3'', 4''\} \sim \{(3, 1), (3, 2), (3, 3), (3, 4)\}$ | $\{31, 32, 33, 34\}$ | $g^2 = (13)(24)$ |
| $\{1''', 2''', 3''', 4'''\} \sim \{(4, 1), (4, 2), (4, 3), (4, 4)\}$ | $\{41, 42, 43, 44\}$ | $g^3 = (1432)$ |

**Table 3: Connections between sheet $e$ in $\bar{Y}$ and other sheets**

| Vertex | Adjacent vertices in $\bar{Y}$ |
|--------|-------------------------------|
| 1      | 2, 2', 4', 4''               |
| 2      | 1, 1, 3, 3                   |
| 3      | 2, 2, 4, 4                   |
| 4      | 3, 3, 1', 1'''               |

We now determine the reciprocals of $L$ functions for $\bar{Y}|Y$.

1) For $A_{\chi_1}$

$$\zeta_\bar{Y}(t)^{-1} = L(t, A_{\chi_1}, \bar{Y}|Y)^{-1} = (1 - t^2)^4(t - 1)(t + 1)(3t - 1)(3t + 1)(3t^2 + 1)^2.$$  

2) As $A_{\chi_2} = A_{\chi_4}$

$$L(t, A_{\chi_2}, \bar{Y}|Y)^{-1} = L(t, A_{\chi_4}, \bar{Y}|Y)^{-1} = (1 - t^2)^4\left(9t^4 - 6t^3 + 2t^2 - 2t + 1\right)\left(9t^4 + 6t^3 + 2t^2 + 2t + 1\right)$$

3) For $A_{\chi_3}$

$$L(t, A_{\chi_3}, \bar{Y}|Y)^{-1} = (1 - t^2)^4 (9t^4 - 2t^2 + 1)^2$$
By Eq. 20 we have

\[ \zeta_\Gamma(t)^{-1} = \prod_{\chi_i \in \{\chi_1, \ldots, \chi_4\}} L(t, A_{\chi_i}, \tilde{Y}|Y)^{-1} \]

\[ = (1 - t^2)^{16}(t - 1)(t + 1)(3t - 1)(3t + 1) \left(3t^2 + 1\right)^2 \left(9t^4 - 2t^2 + 1\right)^2 \left(9t^4 - 6t^3 + 2t^2 - 2t + 1\right)^2 \left(9t^4 + 6t^3 + 2t^2 + 2t + 1\right)^2. \]

Note that Eq. 20 gives an iterative method of calculating \( \zeta_\Gamma_{n+1} \) in terms of \( \zeta_\Gamma_n \), once the matrices \( A_\rho \)'s are determined.

Similar calculations can be done to investigate the Artin \( L \) and Ihara zeta functions of Schreier graphs of Fabrykowski-Gupta (FG), Basilica (B), Grigorchuk (Gr) and Gupta-Sidki (GS) groups.

**Example 5.2.** The graph \( \tilde{Y} = F^2 \tilde{\Gamma}_2 \simeq F^2 \Gamma_1 \otimes F^2 \Gamma_1 \) is 3-sheeted normal covering of the graph \( Y = F^2 \Gamma_1 \).

Reciprocals of \( L \) functions for \( \tilde{Y}|Y \) are as follows

1) For \( \chi_1 \)

\[ \zeta_\tilde{Y}(t)^{-1} = L(t, A_{\chi_1}, \tilde{Y}|Y)^{-1} = (1 - t^2)(t^2 - t + 1)^2 \]

2) As \( \chi_2 = \chi_3 \)

\[ L(t, A_{\chi_2}, \tilde{Y}|Y)^{-1} = L(t, A_{\chi_3}, \tilde{Y}|Y)^{-1} \]

\[ = (1 - t^2)^2 \left(3t^2 - t + 1\right)^2 \left(9t^4 - 6t^3 + t^2 - 2t + 1\right)^2 \]

By Eq. 20 we have

\[ \zeta_\tilde{Y}(t)^{-1} = (1 - t^2)^9(t - 1)(3t - 1) \left(3t^2 - t + 1\right)^4 \left(9t^4 - 6t^3 + t^2 - 2t + 1\right)^2. \]

**Example 5.3.** The graph \( \tilde{Y} = B^3 \tilde{\Gamma}_3 \simeq B^3 \tilde{\Gamma}_1 \otimes B^3 \tilde{\Gamma}_2 \) is 2-sheeted normal covering of the graph \( Y = B^3 \tilde{\Gamma}_2 \).

Reciprocals of \( L \) functions for \( \tilde{Y}|Y \) are as follows

1) For \( \chi_1 \)

\[ \zeta_\tilde{Y}(t)^{-1} = L(t, A_{\chi_1}, \tilde{Y}|Y)^{-1} = (1 - t^2)(t^2 - t + 1)^2 \left(9t^4 - 2t^2 + 1\right). \]
2) For $A_{\chi_2}$

$$L(t, A_{\chi_2}, \tilde{Y}|Y)^{-1} = (1-t^2)^4 \left( 3t^2 - 2t + 1 \right) \left( 27t^6 - 18t^5 + 3t^4 - 4t^3 + t^2 - 2t + 1 \right).$$

By Eq. 20 we have

$$\zeta_{\Gamma}(t)^{-1} = (1-t^2)(t-1)(3t^2 + 1)(3t^2 - 2t + 1)(9t^4 - 2t^2 + 1)
\left( 27t^6 - 18t^5 + 3t^4 - 4t^3 + t^2 - 2t + 1 \right).$$

**Example 5.4.** The graph $\tilde{Y} = Gr_2 \simeq Gr_1 \hat{\otimes} Gr_1$ is 2-sheeted normal covering of the graph $Y = Gr_1$.

Reciprocals of $L$ functions for $\tilde{Y}|Y$ are as follows

1) For $A_{\chi_1}$

$$\zeta_{\tilde{Y}}(t)^{-1} = L(t, A_{\chi_1}, \tilde{Y}|Y)^{-1} = (1-t^2)(t-1)(3t^2 - 2t + 1)(3t^2 - 2t + 1).$$

2) For $A_{\chi_2}$

$$L(t, A_{\chi_2}, \tilde{Y}|Y)^{-1} = (1-t^2)^3 \left( 9t^4 - 6t^3 + 2t^2 - 2t + 1 \right).$$

By Eq. 20 we have

$$\zeta_{\tilde{Y}}(t)^{-1} = (1-t^2)(t-1)(3t^2 - 2t + 1)(3t^2 - 2t + 1)(9t^4 - 6t^3 + 2t^2 - 2t + 1).$$

**Example 5.5.** The covering $\tilde{Y} = GS_2 \simeq GS_1 \hat{\otimes} GS_1$ is 3-sheeted normal covering the graph $Y = GS_1$. Therefore we have the following reciprocals of $L$ functions for the covering $\tilde{Y}|Y$:

1) For $A_{\chi_1}$

$$\zeta_{\tilde{Y}}(t)^{-1} = L(t, A_{\chi_1}, \tilde{Y}|Y)^{-1} = (1-t^2)(t-1)(3t^2 - t + 1)^2$$

2) As $A_{\chi_2} = A_{\chi_3}$

$$L(t, A_{\chi_2}, \tilde{Y}|Y)^{-1} = L(t, A_{\chi_3}, \tilde{Y}|Y)^{-1}
= (1-t^2)^3 \left( 3t^2 + 2t + 1 \right) \left( 9t^4 - 6t^3 + 4t^2 - 2t + 1 \right)$$

By Eq. 20 we have

$$\zeta_{\tilde{Y}}(t)^{-1} = (1-t^2)^3(t-1)(3t^2 - t + 1)^2 \left( 3t^2 + 2t + 1 \right)^2
\times \left( 9t^4 - 6t^3 + 4t^2 - 2t + 1 \right)^2.$$
6 Non-bounded automaton groups

In this section we shall see the Schreier graphs of the group generated by a non-bounded automaton.

Example 6.1. Group generated by a non-bounded automaton.
Let \( X = \{0, 1\} \). Define the automorphisms of the binary tree \( X^* \) by the wreath recursions:
\[
\begin{align*}
    a &= \psi_a(a, 1), \\
    b &= \psi_b(b, a),
\end{align*}
\]
where \( \psi_a \) is the transposition \((0 1) \in \text{Sym}(X)\) and \( \psi_b \) is the identity permutation in \( \text{Sym}(X) \). This group generated by \( a, b \). This group is the simplest example of a group generated by a polynomial but non bounded automaton. See Fig. 11. As the generated automaton is linear, we say that this group is generated by linear automaton and we denote it by \( L^G \). More details about this group can be found in \([4]\). For the Schreier graphs \( L\Gamma_1, L\Gamma_2 \) and \( L\Gamma_3 \) of the group \( L^G \) see Figures 12 and 13.

Example 6.2. Galois coverings of Schreier graphs of \( L^G \).
From the Eq. 21 we have
\[
\begin{align*}
    a &= \psi_a(a, 1), \\
    b &= \psi_b(b, a),
\end{align*}
\]
where \( \psi_a \) is the transposition \((0 1) \in \text{Sym}(X)\) and \( \psi_b \) is the identity permutation in \( \text{Sym}(X) \).

We shall construct the spanning tree \( T_n \) of the graph \( L\Gamma_n \): \( V(T_n) = X^n \). Let \( u_0 = \{0\}^n \) be word of length \( n \) and containing all 0’s. Note that if \( g \in L^G \) then \( g^{2^k} \) we mean \( g \circ \cdots \circ g \) (\( 2^k \) times). \( e = \{u, v\} \) is an edge in the graph \( L\Gamma_n \) if there exists a generator \( s \in S \) of \( L^G \) such that \( s(u) = v \). We call \( e \) as \( s \)-edge at \( u \). Note that the action of \( a \) on the set \( X^n \) has order \( 2^n \). Hence this action produces the spanning tree \( T_n \) of the graph \( L\Gamma_n \).
\[
\begin{align*}
    E(T_n) &= \{u_0, a(u_0)\}; \{a(u_0), a^2(u_0)\}; \cdots ; \{a^{2^n-2}(u_0), a^{2^n-1}(u_0)\} \\
    V(T_n) &= \{u_0, a(u_0), a^2(u_0), \cdots , a^{2^n-1}(u_0)\}
\end{align*}
\]
Proposition 6.1. Let $L\Gamma_m$ and $L\Gamma_n$ be any two Schreier graphs of the group $L^G$. Then following holds:

1. If $m > n$, then the graph $L\Gamma_m$ is unramified covering of the graph $L\Gamma_n$.

2. If $m = n + 1$, then the graph $L\Gamma_m$ is Galois covering of the graph $L\Gamma_n$ with $\mathbb{G} = Gal(L\Gamma_{n+1}|L\Gamma_n) = \mathbb{Z}/2\mathbb{Z}$.

Proof. 1. Let $m = n + k$, where $k \geq 1$.

We define a map $\pi : L\Gamma_m \rightarrow L\Gamma_n$ such that $\pi$ forgets the last $k$
components, i.e.
\[ \pi(x_1 \cdots x_m) = \pi(x_1 \cdots x_n x_{n+1} \cdots x_{n+k}) = x_1 \cdots x_n \]
By Proposition 3.1, \( \pi \) is a covering map.

2. Let \( u_0 = \{0\}^n \) be word of length \( n \) and containing all 0’s and \( T_n \) be the spanning tree of the graph \( L\Gamma_n \). The following sets
\[
T_n 0 = \{u_0 0, a(u_0) 0, a^2(u_0) 0, \cdots, a^{2^n - 1}(u_0) 0\}
\]
and
\[
T_n 1 = \{u_0 1, a(u_0) 1, a^2(u_0) 1, \cdots, a^{2^n - 1}(u_0) 1\}
\]
form the partition to vertex set of the graph \( L\Gamma_{n+1} \). Hence, we can say that \( T_n 0 \) and \( T_n 1 \) are the sheets and \( L\Gamma_{n+1} \) is two sheeted unramified covering of the graph \( L\Gamma_n \). To prove that it is a Galois covering, we now define the Galois group element \( \gamma \). Let \( y \in \{0, 1\} \)
\[
\gamma(x_1 \cdots x_n y) = a^{2^n + 1}(x_1 \cdots x_n y)
\]
As the action of \( a \) on the set \( X^n \) has order \( 2^n \), we have
\[
\gamma(x_1 \cdots x_n y) = x_1 \cdots x_n a(y)
\]
But by the definition of \( a \) (See Eq. 21), \( a(y) \neq y \). This implies that \( \gamma \) has order 2. It follows from the definition of \( \pi \) and \( \gamma \) that \( \pi \circ \gamma = \pi \).

Example 6.3. Ihara zeta and Artin L functions of Schreier graphs of the group generated by linear automaton.

By Theorem 4.1, the covering \( \tilde{Y} = L\Gamma_3 \simeq L\Gamma_1 \circ L\Gamma_2 \) over \( Y = L\Gamma_2 \) given in the Figures 12 and 13 is a normal covering. We draw the covering graph \( \tilde{Y} \) by placing 2 sheets of the spanning tree \( T \) of \( Y \), we label the vertices using the Table 4 and the labeling of each sheet \( T \) is given in the Table 5. Connections between 2 sheets in the cover graph \( \tilde{Y} = \Gamma_3 \) are given in the Table 6. In this case the Galois group \( G \) is cyclic group of order 2. So we have trivial character \( \chi_1(g) = 1 \) and the non-trivial \( \chi_2(g) = -1 \). We now write the matrices \( A(id) \) and \( A(g) \) given in the Eq. 17.

\[
A(id) = \begin{pmatrix}
2 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 \\
0 & 1 & 1 & 0
\end{pmatrix}, \quad A(g) = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 2 & 0 \\
1 & 1 & 0 & 0
\end{pmatrix}.
\]
Using Eq. 18 we write the Artinized adjacency matrices $A_{\chi_i}$, $i = 1, 2$ as:

\[
A_{\chi_1} = \begin{pmatrix}
2 & 1 & 0 & 1 \\
1 & 0 & 1 & 2 \\
0 & 1 & 2 & 1 \\
1 & 2 & 1 & 0
\end{pmatrix},
A_{\chi_2} = \begin{pmatrix}
2 & 1 & 0 & -1 \\
1 & 0 & 1 & 0 \\
0 & 1 & -2 & 1 \\
-1 & 0 & 1 & 0
\end{pmatrix}.
\]

**Table 4:** Vertex labeling for $\tilde{L}_2$

| Label | 1 | 2 | 3 | 4 |
|-------|---|---|---|---|
| Vertex| 00| 10| 01| 11|

**Table 5:** Notations for sheets

| Vertex set of $Y$ | Vertex set of $\tilde{L}_3$ | Group element |
|-------------------|-----------------------------|---------------|
| \{1, 2, 3, 4\}    | \{000, 100, 010, 110\}     | \text{id}     |
| \{1', 2', 3', 4'\}| \{001, 101, 011, 111\}     | \text{g} \neq \text{id} |

**Table 6:** Connections between sheet $e$ and $g$ in $\tilde{Y}$

| Vertex | Adjacent vertices in $Y$ |
|--------|--------------------------|
| 1      | 1, 2, 4'                 |
| 2      | 1, 2, 4, 4'              |
| 3      | 2, 4, 3'                 |
| 4      | 2, 3, 3', 4'             |

Reciprocals of $L$ functions for $\tilde{Y}|Y$ are as follows

1) For $A_{\chi_1}$

\[
\zeta_Y(t)^{-1} = L(t, A_{\chi_1}, \tilde{Y}|Y)^{-1} = (1-t^2)^4 (t-1)(3t^2-1) (3t^2+1) (3t^2-2t+1) (3t^2+2t+1).
\]

2) For $A_{\chi_2}$

\[
L(t, A_{\chi_2}, \tilde{Y}|Y)^{-1} = (1-t^2)^4 (9t^4-6t^3+4t^2-2t+1) (9t^4+6t^3+4t^2+2t+1).
\]
By Eq. 20 we have
\[
\zeta_\tilde{Y}(t)^{-1} = L(t, A_{x_1}, \tilde{Y}|Y)^{-1} L(t, A_{x_2}, \tilde{Y}|Y)^{-1}
\]
\[
= (1 - t^2)^8 (t - 1)(3t - 1) (3t^2 + 1) (3t^2 - 2t + 1) (3t^2 + 2t + 1) (9t^4 - 6t^3 + 4t^2 - 2t + 1)
\times (9t^4 + 6t^3 + 4t^2 + 2t + 1) .
\]

This study raises the following interesting problem

Problem : If $G$ is a self-similar group generated by polynomial automaton $A$ and $G$ has a transitive action the alphabet $X$ with $|X| = d$. If $\Psi_G$ is a group generated by root permutations $\psi_g$, where $g \in S$ and if $|\Psi_G| = d$, then is the covering $G\Gamma_{n+1}|G\Gamma_n$ normal with Galois group $\mathbb{G} \simeq \Psi_G$?

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