New Concurrent Order Maintenance Data Structure

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Abstract

The Order-Maintenance (OM) data structure maintains a total order list of items for insertions, deletions, and comparisons. As a basic data structure, OM has many applications, such as maintaining the topological order, core numbers, and truss in graphs, and maintaining ordered sets in Unified Modeling Language (UML) Specification. The prevalence of multicore machines suggests parallelizing such a basic data structure. This paper proposes a new parallel OM data structure that supports insertions, deletions, and comparisons in parallel. Specifically, parallel insertions and deletions are synchronized by using locks efficiently, which achieve up to 7x and 5.6x speedups with 64 workers. One big advantage is that the comparisons are lock-free so that they can execute highly in parallel with other insertions and deletions, which achieve up to 34.4x speedups with 64 workers. Typical real applications maintain order lists that always have a much larger portion of comparisons than insertions and deletions. For example, in core maintenance, the number of comparisons is up to 297 times larger compared with insertions and deletions in certain graphs. This is why the lock-free order comparison is a breakthrough in practice.

Keywords: order maintenance, parallel, multi-core, shared memory, compare-and-swap, lock-free, amortized constant time, core maintenance

1. Introduction

The well-known Order-Maintenance (OM) data structure [1, 2, 3] maintains a total order of unique items in an order list, denoted as \( \mathcal{O} \), by following three operations:

- \textbf{Order}(x, y): determine if \( x \) precedes \( y \) in the ordered list \( \mathcal{O} \), denoted as \( x \preceq y \), supposing both \( x \) and \( y \) are in \( \mathcal{O} \).
- \textbf{Insert}(x, y): insert a new item \( y \) after \( x \) in the ordered list \( \mathcal{O} \), supposing \( x \) is in \( \mathcal{O} \) and \( y \) is not in \( \mathcal{O} \).
- \textbf{Delete}(x): delete \( x \) from the ordered list \( \mathcal{O} \), supposing \( x \) is in \( \mathcal{O} \).

Application. As a fundamental cohesive subgraph model, the \textit{k-core} [4, 5, 6, 7, 8] is defined as the maximal subgraph such that all vertices have degrees at least \( k \). The \textit{core number} of a vertex is defined as its maximum value of \( k \). After core numbers of vertices are computed in linear time by peeling steps [4], it is time-consuming to recalculate the core numbers when new edges are inserted or old edges are removed for dynamic graphs. In this case, core maintenance algorithms [9, 10, 11, 12] are proposed to efficiently update the core numbers of vertices, which avoids traversing the whole graph. In [11, 12], the \textit{k-order} of all vertices are used for core maintenance, where \( u \) precedes \( v \) in the peeling steps of core decomposition for all vertices \( u, v \) in the graph. The idea is that after inserting an edge, the related vertices \( w \) are traversed in \( k \)-order; we can safely skip \( w \) if \( w \) has candidate in-degree plus remaining out-degree less than \( k + 1 \), and thus the number of traversed vertices can be significantly reduced.

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**Example 1.1.** Figure 1(a) shows all the core numbers of vertices computed by peeling. For example, at first, \( u_1 \) obtains a core number as 1 since its degree is 1; after peeling \( u_1, u_2 \) gets core numbers as 1 since its degree is reduced to 1; continually, \( u_3 \) to \( u_{1000} \) gets the core numbers as 1. Here, the \( k \)-order \( O_k \) is the order of vertices that obtain their core numbers. In Figure 1(b), inserting a new edge \((u_{999}, v_3)\) may cause the core number of some vertices to increase from 1 to 2. We traverse the affected vertices starting from \( u_{999} \) by \( O_1 \). In this case, we traverse only two vertices, \( u_{999} \) and \( u_{1000} \), and find that their core numbers can increase from 1 to 2. We can see a great number of vertices, \( u_1 \) to \( u_{998} \), are avoided to the traversed, which achieves high performance for maintaining the core numbers of vertices. Finally, to maintain the \( k \)-order, \( u_{999} \) and \( u_{1000} \) are removed from \( O_1 \) and appended to \( O_2 \).

Due to the prevalence of the multicore shared-memory architecture, many parallel core maintenance approaches were proposed [13, 14, 15, 16]. As one of the important uses in [16], our concurrent OM data structure is used to maintain the \( k \)-order in parallel, when multiple edges are inserted simultaneously. Such an idea about core maintenance can be applied to truss maintenance [17]. Similarly, the OM data structure can be used to maintain the topological order of vertices in directed acyclic graphs after inserting or removing edges, where for every directed edge \((u, v)\) we have \( u \) come before \( v \) in order [18, 19].

Additionally, ordered sets are widely used in Unified Modeling Language (UML) Specifications [20], e.g., a display screen (an OS’s representation) has a set of windows, but furthermore, the set is ordered, so do the ordered bag and sequence. Nowadays, shared-memory multi-core machines are widely used, which motivates the efficient parallelization of the above algorithms. As a building block, it immediately suggests itself parallelizing the OM data structure.

**Sequential.** In the sequential case, the OM data structure has been well studied. The naive idea is to use a balanced binary search tree [21]. All three operations can be performed in \( O(\log N) \) time, where there are at most \( N \) items in the ordered list \( O \). In [1][2], the authors propose an OM data structure that supports all three operations in \( O(1) \) time. The idea is that all items in \( O \) are linked as a double-linked list. Each item is assigned a label to indicate its order. We can perform the \( \text{Order} \) operation by comparing the labels of two items by \( O(1) \) time. Also, the \( \text{Delete} \) operation also costs \( O(1) \) time without changing other labels. For the \( \text{Insert}(x, y) \) operation, \( y \) can be directly inserted after \( x \) with \( O(1) \) time, if there exists label space between \( x \) and \( x \)’s successors; otherwise, a \( \text{relabel} \) procedure is triggered to rebalance the labels, which costs amortized \( O(\log N) \) time per insertion. After introducing a list of sublists structure, the amortized running time of the \( \text{relabel} \) procedure can be optimized to \( O(1) \) per insertion. Thus, the \( \text{Insert} \) operation has \( O(1) \) amortized time.

**Parallel.** In this paper, we present a new concurrent OM data structure. In terms of parallel \( \text{Insert} \) and \( \text{Delete} \) operations, we use locks for synchronization without allowing interleaving. In the average case, there is a high probability that multiple \( \text{Insert} \) or \( \text{Delete} \) operations occur in different positions of \( O \) so that these operations can execute completely in parallel. For the \( \text{Order} \) operation, we adopt a lock-free mechanism, which allows executing completely in parallel for any pair of items in \( O \). To implement the lock-free \( \text{Order} \), we devise a new algorithm for...
the Insert operation that always maintains the Order Snapshot for all items, even if many relabel procedures are triggered. Here, the Order Snapshot means the labels of items indicate their order correctly. As Insert operations always maintain the Order Snapshot, we do not need to lock a pair of items when comparing their labels in parallel. In other words, lock-free Order operations are based on Insert operations that preserve the Order Snapshot.

Figure 2: The number of OM operations for core maintenance by inserting 100,000 random edges into each graph.

Our new parallel lock-free Order operation is a breakthrough for real applications. Typically, for the OM data structure, a large portion of operations is comparing the order of two items. For example, Figure 2 shows the number of OM operations (y-axis) for the core maintenance in [12] by randomly inserting 100,000 edges over 12 tested data graphs (x-axis). We observe that the number of Order operations is magnitudes larger than the number of Insert and Delete operations, e.g., for the RMAT graph by a factor of 287. The reason is that graphs tend to have more edges than vertices. The crucial advantage of our parallel Order operation is that it can execute completely in parallel without locking items, which is essential when trying to parallelize algorithms like core maintenance.

We analyze our parallel OM operations in the work-depth model [21, 22], where the work, denoted as $W$, is the total number of operations that are used by the algorithm, and the depth, denoted as $D$, is the longest length of sequential operations [23]. The expected running time is $O(W/P + D)$ by using $P$ workers with load balancing among those. In particular, the work and depth terms are equivalent for sequential algorithms.

Table 1: The worst-case and best-case work, depth complexities of parallel OM operations, where $m$ is the number of operations executed in parallel, $P$ is the total number of workers, and $\dagger$ is the amortized complexity.

| Parallel operation | Worst-case ($O$) | Best-case ($O$) |
|--------------------|------------------|------------------|
|                    | $W$              | $D$              | $W$              | $D$              |
| Insert             | $m^\dagger$      | $m^\dagger$      | $m$              | $m$              |
| Delete             | $m^\dagger$      | $m^\dagger$      | $m$              | $m$              |
| Order              | $m$              | $m^\dagger$      | $m$              | $m$              |

Table 1 compares the worst-case and best-case work and depth complexities for the three OM operations when running the $m$ operations of the same kind in parallel. In the best case, all three operations have $O(m)$ work and $O(1)$ depth. However, Insert has worst-case $O(m)$ work and $O(m)$ depth; such a worst-case is easy to construct by
inserting \( m \) items into the same position of \( \emptyset \), and thus all insertions are reduced to running sequentially. The \texttt{Delete} operation also has worst-case \( O(m) \) work and \( O(1) \) depth; but such a worst case only happens when all deletions cause a blocking chain, which has a very low probability. Especially, since the \texttt{Order} is lock-free, it always has \( O(m) \) work and \( O(1) \) depth in the worst and best cases. This is why \texttt{Order} operations run in parallel always have a great speedup for multicore machines. The lock-free \texttt{Order} operation is an important contribution of this work.

We conduct extensive experiments on a 64-core machine over a variety of test cases to evaluate the parallelism of the new parallel OM data structure. With 64 workers our parallel \texttt{Insert} and \texttt{Delete} achieve up to 7x and 5.6x speedups; our parallel \texttt{Order} achieves up to 34.4x speedups.

The rest of this paper is organized as follows. The related work is in Section 2. The preliminaries are given in Section 3. Our parallel OM data structure is discussed in Section 4. We conduct experimental studies in Section 5 and conclude this work in Section 6.

2. Related Work

In [24], Dietz proposes the first order data structure, with \texttt{Insert} and \texttt{Delete} having \( O(\log n) \) amortized time and \texttt{Order} having \( O(1) \) time. In [25], Tsakalidis uses BB\([n]\) trees to improve the update bound to \( O(\log n) \) and then to \( O(1) \) amortized time. In [1], Dietz et al. propose the fastest order data structure, which has \texttt{Insert} in \( O(1) \) amortized time, \texttt{Delete} in \( O(1) \) time, and \texttt{Order} in \( O(1) \) time. In [2], Bender et al. propose significantly simplified algorithms that match the bounds in [1].

A special case of OM is the file maintenance problem [1, 2], which is to store \( n \) items in an array of size \( O(n) \). File maintenance has four operations, i.e., insert, delete, scan-right (scan next \( k \) items starting from \( e \)), and scan-left (analogous to scan-right).

For the parallel or concurrent OM data structure, there exists little work [26, 3] to the best of our knowledge. In [26], the order list is split into multiple parts and organized as a B-tree, which sacrifices the \( O(1) \) time for three operations; also, the relevant nodes in the B-tree are locked for synchronization. In [3], a parallel OM data structure is proposed specifically for series-parallel (SP) maintenance, which identifies whether two accesses are logically independent. Several parallelism strategies are present for the OM data structure combined with SP maintenance. We apply the strategy of splitting a full group into our new parallel OM data structure.

3. Preliminaries

In this section, for the OM data structure, we revisit the detailed steps of the sequential version [2, 1, 3]. This is the background to discuss the parallel version in the next section.

The idea is that items in the total order are assigned labels to indicate the order. Typically, each label can be stored as an \( O(\log N) \) bits integer, where \( N \) is the maximal number of items in \( O \). Assume it takes \( O(1) \) time to compare two integers. The \texttt{Order} operation requires \( O(1) \) time by comparing labels; also, the \texttt{Delete} operation requires \( O(1) \) time since after deleting one item all other labels of items are not affected.

In terms of the \texttt{Insert} operation, efficient implementations provide \( O(1) \) amortized time. First, a two-level data structure [3] is used. That is, each item is stored in the bottom-list, which contains a group of consecutive elements; each group is stored in top-list, which can contain \( \Omega(\log N) \) items. Both the top-list and the bottom-list are organized as double-linked lists, and we use \( x.pre \) and \( x.next \) to denote the predecessor and successor of \( x \), respectively. Second, each item \( x \) has a top-label \( L'(x) \), which equals to \( x \)'s group label denoted as \( L'(x) = L(x.group) \), and bottom-label \( L_b(x) \), which is \( x \)'s label. Integer \( L' \) is in the range \([0, N^2]\) and integer \( L_b \) in the range \([0, N]\).

Initially, there can be \( N' \) items in \( O \) \((N' \leq N)\), which are contained in \( N' \) groups, separately. Each group is assigned a top-label \( L \) with an \( N \) gap between neighbors, and each item is assigned a bottom-label \( L_b \) as \([N/2]\).

\textbf{Definition 3.1} (Order Snapshot). The OM data structure maintains the Order Snapshot for \( x \) precedes \( y \) in the total order, denoted as \( \forall x, y \in \emptyset : x \preceq y \iff L'(x) < L'(y) \lor (L'(x) = L'(y) \land L_b(x) < L_b(y)) \)

The OM data structure maintains the Order Snapshot defined in Definition 3.1. In other words, to determine the order of \( x \) and \( y \), we first compare their top-labels (group labels) of \( x \) and \( y \); if they are the same, we continually compare their bottom labels.
3.1. Insert

The operation Insert(\(O, x, y\)) is implemented by inserting \(y\) after \(x\)’s bottom-list, assigning \(y\) the label \(L_b(y) = \lfloor (L_b(x).next) - L_b(x) \rfloor / 2\), and setting \(y\) in the same group as \(x\) with \(y.group = x.group\) such that \(L'(y) = L'(x)\). If \(L_b(x).next) - L_b(x) > 1\), \(y\) can successfully obtain a new label, then the insertion is complete in \(O(1)\) time. Otherwise, \(x\’s group is full\), which triggers a relabel operation. The relabel operation has two steps. First, the full group is split into many new groups, each of which contains at most \(\frac{\log N}{2}\) items, and new labels \(L_b\) are uniformly assigned for items in new groups. Second, newly created groups are inserted into the top-list, if new group labels \(L^t\) can be assigned. Otherwise, we have to rebalance the group labels. That is, from the current group \(g\), we continuously traverse the successors \(g'\) until \(L(g') - L(g) > \frac{j}{j}\), where \(j\) is the number of traversed groups. Then, new group labels can be assigned to groups between \(g\) and \(g'\) with a \(\frac{L(g') - L(g)}{j}\) gap, in which newly created groups can be inserted.

There are three important features in the implementation Insert: (1) each group, stored in the top-list, contains \(\Omega(\log N)\) items, so that the total number of insertions is \(O(N/\log N)\); (2) the amortized cost of splitting groups is \(O(1)\) per insert; (3) the amortized cost of inserting a new group into the top-list is \(O(\log N)\) per insertion. Thus, each Insert operation only costs amortized \(O(1)\) time.

![Diagram](image)

Figure 3: An example of the OM data structure with \(N = 16\). The squares are groups located in a single double-linked top-list with a head \(h^t\) and a tail \(t^t\). The circles are items with pointers to their own groups, located in a single double-linked bottom-list. The beside numbers are corresponding labels to items and groups. (a) the initial state of \(O = \{v_1, v_2, v_3, v_4\}\). (b) the intermediate state of \(O = \{v_1, v_2, v_3, v_4, v_5, v_6\}\). (c) after inserting a new item \(u\) (dashed cycles) after \(v_1\), we get \(O = \{v_1, u, v_2, v_3, v_4, v_5, v_6\}\) by inserting a new group \(g\) (dashed square).
Example 3.1 (Insert). Figure 3 shows a simple example of the OM data structure. For simplicity, we choose $N = 2^4 = 16$, so that for items the top-labels $L'$ are 8-bit integers (above groups as group labels), and the bottom labels are 4-bit integers (below items).

Figure 3(a) shows an initial state of the two-level lists and labels. The top-list has head $h'$ and tail $t'$ labeled by 0 and $16^2 - 1$, respectively, and includes four groups $g_1$ to $g_4$ labeled with gap 16. The bottom-list has head $h_b$ and tail $t_b$ without labels, and includes four items $v_1$ to $v_4$ located in four groups $g_1$ to $g_4$ with same labels 7.

Figure 3(b) shows an intermediate state after a number of Insert and Delete operations. We can see that there does not exist label space between $v_1$ and $v_2$. Both $v_1$ and $v_2$ are located in the group $g_1$. We get that $g_1$ is full when inserting a new item after $v_1$.

In Figure 3(c), a new item $u$ is inserted after $v_1$. But the group $g_1$ is full (no label space after $v_1$), which triggers a relabel process. That is, the group $g_1$ is split into two groups, $g_1$ and $g_2$; the old group $g_1$ only has $v_1$; the newly created group $g_2$ contains $v_2$ and $v_3$; also, $v_1$ to $v_3$ are assigned $L_b$ with uniform distribution within their own group. However, there is no label space between $g_1$ and $g_2$ to insert the new group $g$, which will trigger a rebalance operation.

That is, we traverse groups from $g_1$ to $g_4$, where $g_4$ is the first that satisfies $L(g_4) - L(g_1) = 15 > j^2$ ($j = 3$). Then, both $g_2$ and $g_3$ are assigned new top-labels as 20 and 25, respectively, which have a gap of 5. Now, $g$ can be inserted after $g_1$ with $L(g) = L(g_2) + (L(g_2) - L(g_1))/2 = 17$. Finally, we can insert $u$ successfully after $v_1$ in $g_1$, with $L_b(u) = L_b(v_1) + (15 - L_b(v_1))/2 = 11$.

3.2. Atomic Primitives

The compare-and-swap atomic primitive CAS($x, a, b$) atomic primitive takes a variable (location) $x$, an old value $a$ and a new value $b$. It checks the value of $x$, and if it equals $a$, it updates the variable to $b$ and returns true; otherwise, it returns false to indicate that updating failed.

3.3. Lock Implementation

OpenMP (Open Multi-Processing) [27] is an application programming interface (API) that supports multi-platform shared-memory multiprocessing programming in C, C++, and Fortran, on many platforms, instruction-set architectures, and operating systems. This paper uses OpenMP (version 4.5) as the threading library to implement the parallel algorithms. In this work, the key issue is how to implement the locks for synchronization. One solution is to use the OpenMP lock, “omp_lock” and “omp_unlock”. Each worker will suspend the working task until the specified lock is available. The OpenMP lock will be efficient if a lot of work within the locked region.

The other solution is the spin lock, which can be implemented by the atomic primitive CAS. Given a variable $x$ as a lock, the CAS will repeatedly check $x$, and set $x$ from false to true if $x$ is false. In other words, one worker will busy-wait the lock $x$ until it is released by other workers without suspension. The spin lock will be efficient if very little work within the locked region is needed. In this case, suspending has a higher cost than busy waiting for multiple workers. Typically, especially for our use cases, a large number of unsuccessful tries indicates that workers should back off longer before trying again. We exponentially increases the back off time for each try [28].

4. Parallel Order Maintenance Data Structure

In this section, we present the parallel version of the OM data structure. We start from the parallel Delete operations. Then, we discuss the parallel Insert operation and show that the Order Snapshot is preserved at any steps including the relabel process, which is the main contribution of this work. Finally, we present the parallel Order operation, which is lock-free and thus can be executed highly in parallel.

4.1. Parallel Delete

4.1.1. Algorithm

Algorithm 1 shows the detail steps of parallel Delete. For each item $x$ in $O$, we use a boolean status $x$.live to indicate if $x$ is in $O$ or has been removed. Initially $x$.live is true. We use the atomic primitive CAS to set $x$.live from true to false, and a repeated delete of $x$ will return fail (line 1). In lines 2 - 6 and 13, we remove $x$ from the bottom-list in $O$. To do this, we first lock $y = x$.pre, $x$, and $x$.next in order to avoid deadlock (lines 2 - 4). Here, after locking $y$, we have to check that $y$ still equals $x$.pre in case $x$.pre is changed by other workers (line 3). Then, we can
4.2. Parallel Insert

has a low probability to happen. $z = 0$ time is the right bound; otherwise, $N$ is the right bound supposing $L_b$ is a (log $N$)-bit integer (line 2). If there does not exist a label gap in the bottom-list between $x$ and $x_{next}$, we know that $x_{group}$ is full, and thus the $\text{Relabel}$ procedure is triggered to make label space for $y$ (line 3). Then, $y$ is inserted into the bottom-list between $x$ and $x_{next}$ (line 6), in the same group as $x$ (line 4), by assigning a new bottom-label (line 5).

The $\text{Relabel}(x)$ procedure splits the full group of $x$. We lock $x$'s group $g_0$ and $g_0$'s successor $g_{next}$ (line 9). We also lock all items $y \in g_0$ except $x$, as $x$ is already locked in line 1 (line 10). To split the group $g_0$ into multiple new smaller groups, we traverse the items $y \in g_0$ in reverse order by three steps (lines 11 - 15). First, if there does not exist a label gap in the top-list between $g_0$ and $g_0_{next}$, the $\text{Relabel}$ procedure is triggered to make label space for

\begin{algorithm}
\caption{Parallel-Delete($\emptyset$, $x$)}
1: \textbf{if not} CAS($x_{live}$, true, false) \textbf{then return} fail
2: $y \leftarrow x_{pre}$; Lock($y$)
3: \textbf{if} $y \neq x_{pre}$ \textbf{then} Unlock($y$); \textbf{goto} line 1
4: Lock($x$); Lock($x_{next}$)
5: $g \leftarrow x_{group}$
6: delete $x$ from bottom-list; set $x_{pre}$, $x_{next}$, $L_b(x)$, and $x_{group}$ to $\emptyset$
7: \textbf{if} $|g| = 0 \land$ CAS($g_{live}$, true, false) \textbf{then}
8: $g' \leftarrow g_{pre}$; Lock($g'$)
9: \textbf{if} $g' \neq g_{pre}$ \textbf{then} Unlock($g'$); \textbf{goto} line 8
10: Lock($g$); Lock($g_{next}$)
11: delete $g$ from top-list; set $g_{pre}$, $g_{next}$, and $L(g)$ to $\emptyset$
12: Unlock($g_{next}$); Unlock($g$); Unlock($g'$)
13: Unlock($x_{next}$); Unlock($x$); Unlock($y$);
\end{algorithm}

We logically delete items by setting their flags (line 1). One benefit of such a method is that we can delay the physical deleting of items (lines 2 - 13). The physical deleting can be batched and performed lazily at a convenient time, reducing the overhead of synchronization. Typically, this technique is used in linked lists for delete operations [28]. In his work, we will not test the delayed physical delete operations.

Obviously, during parallel \texttt{Delete} operations, the labels of other items are not affected and the Order Snapshot is maintained.

4.1.2. Correctness

For deleting $x$, we always lock three items, $x_{pre}$, $x$, and $x_{next}$ in order. Therefore, there are no blocking cycles, and thus deletion is deadlock-free.

4.1.3. Complexities

Suppose there are $m$ items to delete in the OM data structure. The total work is $O(m)$. In the best case, $m$ items can be deleted in parallel by $P$ workers with $O(1)$ depth, so that the total running time is $O(m/P)$. In the worst-case, $m$ items have to be deleted one by one, e.g. $P$ workers are blocked as a chain, with $O(m)$ depth, so that the total running time is $O(m/P + m)$.

However, when deleting multiple items in parallel, a blocking chain is unlikely to appear, and thus the worst-case has a low probability to happen.

4.2. Parallel Insert

4.2.1. Algorithm

Algorithm [2] shows the detailed steps for inserting $y$ after $x$. Within this operation, we lock $x$ and its successor $z = x_{next}$ in that order (lines 1 and 7). For obtaining a new bottom-label for $y$, if $x$ and $z$ is in the same group, $L_b(z)$ is the right bound; otherwise, $N$ is the right bound supposing $L_b$ is a (log $N$)-bit integer (line 2). If there does not exist a label gap in the bottom-list between $x$ and $x_{next}$, we know that $x_{group}$ is full, and thus the $\text{Relabel}$ procedure is triggered to make label space for $y$ (line 3). Then, $y$ is inserted into the bottom-list between $x$ and $x_{next}$ (line 6), in the same group as $x$ (line 4), by assigning a new bottom-label (line 5).

The $\text{Relabel}(x)$ procedure splits the full group of $x$. We lock $x$'s group $g_0$ and $g_0$'s successor $g_{next}$ (line 9). We also lock all items $y \in g_0$ except $x$, as $x$ is already locked in line 1 (line 10). To split the group $g_0$ into multiple new smaller groups, we traverse the items $y \in g_0$ in reverse order by three steps (lines 11 - 15). First, if there does not exist a label gap in the top-list between $g_0$ and $g_0_{next}$, the $\text{Relabel}$ procedure is triggered to make label space for
inserting a new group with assigned labels (lines 12 and 13). Second, we split \( \frac{\log N}{2} \) items \( y \) from \( g_0 \) in reverse order to the new group \( g \), which maintains the Order Snapshot (line 14). Third, we assign new \( L_b \) to all items in the new group \( g \) by using the AssignLabel procedure (line 15), which also maintains the Order Snapshot. The for-loop (lines 11 - 15) stops if less than \( \frac{\log N}{2} \) items are left in \( g_0 \). We assign new \( L_b \) to all left items in \( g_0 \) by using the AssignLabel procedure (line 16). Finally, we unlock all locked groups and items (line 17).

In the Rebalance\((g)\) procedure, we make label space after \( g \) to insert new groups. Starting from \( g_{\text{next}} \), we traverse groups \( g' \) in order until \( w > \tilde{f} \) by locking \( g' \) if necessary (\( g \) and \( g_{\text{next}} \) are already locked in line 9), where \( j \) is the number of visited groups and \( w \) is the label gap \( L(g') - L(g) \) (lines 19 - 22). That means \( j \) items will totally share \( w > \tilde{f} \) label space. All groups whose labels should be updated are added to the set \( A \) (line 21). We assign new labels to all groups in \( A \) by using the AssignLabel procedure (line 23), which maintains the Order Snapshot. Finally, we unlock groups locked in line 21 (line 24).

Notably, in the AssignLabel\((A, L, l_0, w)\) procedure, we assign labels without affecting the Order Snapshot, where the set \( A \) includes all elements whose labels need updating. \( L \) is the label function, \( l_0 \) is the starting label, and \( w \) is the label space. Note that, \( L \) can correctly return the bounded labels, e.g., \( L_0(x, \text{next}) = N \) when \( x \) is at the tail of its group \( x_{\text{group}} \). For each \( z \in A \) in order, we first correctly assign a temporary label \( \tilde{L}(z) \) (line 27), which can replace its real label \( L(z) \) at the right time by using stack \( S \) (lines 28 - 32). Specifically, for each \( z \in A \) in order, if its temporary label \( \tilde{L}(z) \) is between \( L(z, \text{pre}) \) and \( L(z, \text{next}) \), we can safely replace its label by updating \( L(z) \) as \( \tilde{L}(z) \) (lines 29 and 30), which maintains the Order Snapshot; otherwise, \( z \) is added to the stack \( S \) for further propagation (line 32). For the propagation, when one element \( z \) replaces the labels (line 30), which means all elements in stack \( S \) can find enough label space, each \( x \in S \) can be popped out by replacing its label (line 31). This propagation still maintains the Order Snapshot.

Example 4.1 (Parallel Insert). Figure 4(b) shows an example for parallel Insert. In Figure 4(b), we lock \( v_1 \) and \( v_2 \) in order when inserting \( u \) after \( v_1 \). However, there is no label space, and the group \( g_1 \) is full, which triggers the Relabel procedure. For the first step of relabelling, the other item \( v_3 \) in group \( g_1 \) is locked to split the group \( g_1 \).

In Figure 4(c), we lock \( g_1 \) and \( g_2 \) in order when inserting a new group \( g \) after \( g_1 \), which triggers the Rebalance procedure on the top-list. For relabelling, \( g_1 \) and \( g_4 \) are locked in order. The new temporary labels \( \tilde{L} \) of \( g_2 \) and \( g_3 \) are generated as \( 20 \) and \( 25 \). To replace real labels with temporary ones, we traverse \( g_2 \) and \( g_3 \) in order. First, we find that \( L(g_1) < \tilde{L}(g_2) < L(g_3) \) as \( 15 < 20 < 17 \) is false, so that \( g_2 \) is added to the stack such that \( S = \{g_2\} \). Second, when traversing \( g_3 \), we find that \( L(g_2) < \tilde{L}(g_3) < L(g_1) \) as \( 16 < 25 < 30 \) is true, so that \( L(g_3) \) is replaced as \( 25 \). In this case, the propagation of \( S \) begins and \( g_2 \) is popped out with \( L(g_2) \) replaced as \( 20 \). Finally, \( g_3 \) and \( g_4 \) are unlocked and the Rebalance procedure finishes.

After relabelling, the new group \( g \) can be insert after \( g_1 \) with \( L(g_1) = 17 \). Relabeling continues. The item \( v_3 \) is spitted out to \( g \) with \( L_0(v_3) = 15 \land L'(v_3) = 17 \) maintaining the Order Snapshot; similarly, \( v_1 \) is also spitted out to \( g \). Now, both \( v_1 \) and \( v_2 \) require new \( L_b \) to be assigned by the AssignLabel procedure. The new temporary label \( \tilde{L}_0 \) of \( v_2 \) and \( v_3 \) are generated as \( 5 \) and \( 10 \), respectively. For replacing, we traverse \( v_2 \) and \( v_1 \) in order by two steps. First, for \( v_2 \), we find that \( L_0(v_2) < L_0(v_3) \) is true, so that \( L_0(v_2) \) is replaced as \( 5 \). Second, for \( v_1 \), we find that \( L_0(v_2) < L_0(v_3) \) is true, so that \( L_0(v_3) \) is replaced as \( 10 \). There is no further propagation since the stack \( S \) is empty. Now, only one item \( v_1 \) is left in \( g_1 \) and \( L_0(v_1) \) is set to \( 7 \). Finally, the new item \( u \) is inserted after \( v_1 \) in \( g_1 \) with \( L_0(u) = 11 \land L'(u) = 15 \).

Example 4.2 (Assign Label). In Figure 4, we show an example how the AssignLabel procedure preserves the Order Snapshot. The label space is from \( 0 \) to \( 15 \), shown as indices. There are four items \( v_1, v_2, v_3, \) and \( v_4 \) with initial labels \( 1, 2, 3, \) and \( 4, \) respectively; also, four temporary labels, \( 3, 6, 9, \) and \( 12, \) are assigned with uniform distribution to them. We traverse items from \( v_1 \) to \( v_4 \) in order. First, \( v_1 \) and \( v_2 \) are added to the stack \( S \). Then, \( v_3 \) can safely replace its old label with its new temporary label \( 9 \), which makes space for \( v_2 \) that is at the top of \( S \). So, we pop out \( v_2 \) from \( S \) and \( v_2 \) get its new label \( 6 \), which makes space for \( v_1 \) that is at the top of \( S \). So, we pop out \( v_1 \) from \( S \) and \( v_1 \) gets its new label \( 3 \). Finally, \( v_3 \) can safely get its new label \( 12 \). In a word, updating the \( v_3 \)'s label will repeatedly make space for \( v_2 \) and \( v_1 \) in the stack. During such a process, we observe that each time an old label is updated with a new temporal label, the labels always correctly indicate the order. Therefore, the Order Snapshot is always preserved and parallel Order operations can take place.
Algorithm 2: Parallel-Insert($\emptyset, x, y$)

1. Lock($\emptyset$); $z \leftarrow x.next$; Lock($z$)
2. if $x.group = z.group$ then $b \leftarrow L_0(z)$ else $b \leftarrow N$
3. if $b - L_0(x) < 2$ then Relabel($x$); insert $y$ into bottom-list between $x$ and $x.next$
4. $L_0(y) \leftarrow L_0(x) + [(b - L_0(x))/2]$
5. $y.group \leftarrow x.group$
6. Unlock($x$); Unlock($z$)

8. procedure Relabel($x$)
9. $g_0 \leftarrow x.group$; Lock($g_0$); Lock($g_0.next$);
10. for $y \in g_0$ in reverse order until less than $\log_2 N$ items left in $g_0$
11. do insert a new group $g$ into the top-list after $g_0$ with $L(g) = (L(g_0.next) - L(g_0))/2$
12. split out $\log_2 N$ items $y$ into $g$
13. AssignLabel($g, L_0, 0, N$)
14. AssignLabel($g_0, L_0, 0, N$)
15. Unlock $g_0.next, g_0$, and all items $y \in g_0$ with $y \neq x$

18. procedure Rebalance($g$)
19. $g' \leftarrow g.next$; $j \leftarrow 1$; $w \leftarrow L(g') - L(g)$; $A \leftarrow \emptyset$
20. while $w \leq j^2$
21. do $A \leftarrow A \cup \{g'\}$; $g' \leftarrow g'.next$; Lock($g'$)
22. $j \leftarrow j + 1$; $w \leftarrow L(g') - L(g)$
23. AssignLabel($A, L', L'(g), w$)
24. Unlock all locked groups in line 21

25. procedure AssignLabel($A, L, l_0, w$)
26. $S \leftarrow$ empty stack; $k \leftarrow 1$; $j \leftarrow |A| + 1$
27. for $z \in A$ in order do $l(z) = l + k \cdot w/j$; $k \leftarrow k + 1$
28. for $z \in A$ in order do
29. if $L(z.pre) < l(z) < L(z.next)$ then
30. $L(z) \leftarrow l(z)$
31. while $S \neq \emptyset$ do $x \leftarrow S.pop(); L(x) \leftarrow l(z)$
32. else $S.push(z)$

Figure 4: An example of the AssignLabel procedure.
4.2.2. Correctness

All items and groups are locked in order. Therefore, there are no blocking cycles, and thus parallel insertions and deletions are deadlock-free.

We prove that the Order Snapshot is preserved during parallel Insert operations. In Algorithm there are two cases where the labels are updated, splitting groups (lines 11 - 15) and assigning labels (lines 15, 16, and 23) by using the AssignLabel procedure (lines 25 - 32).

**Theorem 4.1.** When splitting full groups (line 14), the Order Snapshot is preserved.

**Proof.** The algorithm splits \( \frac{\log N}{k} \) items \( y \) out from \( g_0 \) into the new group \( g \) (line 14), where each \( y \in g_0 \) is traversed in reverse order within the for-loop (lines 11 - 15). For this, the invariant of the for-loop is that \( y \) has largest \( L_0 \) within \( g_0 \); the new group \( g \) has \( L(g) > L(g_0) \); also, \( y \) satisfies the Order Snapshot:

\[
(\forall x \in g_0 : x \neq y \implies L_0(y) > L_0(x))
\]

\[
\land (L(g_0) < L(g)) \land (y.pre \leq y \leq y.next)
\]

We now argue the for-loop preserve this invariant:

- \( \forall x \in g_0 : x \neq y \implies L_0(y) > L_0(x) \) is preserved as \( y \) is traversed in reserve order within \( g_0 \) and all other items \( y' \) that have \( L_0(y') \leq L_0(y) \) are already spitted out from \( g_0 \).
- \( L(g_0) < L(g) \) is preserved as \( g \) is new inserted into top-list after \( g_0 \).
- \( y.pre \leq y \) is preserved as we have \( \forall y \in g \land y.pre \in g_0 \) and \( L(g_0) < L(g) \).
- \( y \leq y.next \) is preserved as if \( y \) and \( y.next \) all in the same group \( g \), we have \( L_0(y) < L_0(y.next) \); also, if \( y \) and \( y.next \) in different groups, we have \( y \) is the first item moved to \( g \) or \( y \) is still located in \( g_0 \), which their groups indicates the correct order.

At the termination of the for-loop, the group \( g \) is split into multiple groups preserving the Order Snapshot. \( \square \)

**Theorem 4.2.** When assigning labels by using the AssignLabel procedure (lines 25 - 32), the Order Snapshot is preserved.

**Proof.** The AssignLabel procedure (lines 25 - 32) assigns labels for all items \( z \in A \). The temporal labels are first generated in advance (line 27). Then, the for-loop replaces the old label with new temporal labels (lines 28 - 32). The key issue is to argue the correctness of the inner while-loop (line 31). The invariant of this inner while-loop is that the top item in \( S \) has a temporal label that satisfies the Order Snapshot:

\[
(\forall y \in S : (y \neq S.top \implies y \leq S.top) \land y \leq z)
\]

\[
\land x = S.top \implies L(x.pre) < L(x) < L(x.next)
\]

The invariant initially holds as \( L(z) \) is correctly replaced by the temporal label \( L(x) \) in line 30 and \( z = x.next \), so that \( L(x) < L(z) \); also, we have \( (x.pre) < L(x) \) as if it is not satisfied, \( x \) should not be added into \( S \), which causes contradiction. We now argue the while-loop (line 31) preserves this invariant:

- \( \forall y \in S : (y \neq S.top \implies y \leq S.top) \land y \leq z \) is preserved as all items in \( S \) are added in order, so the top item always has the largest order; also, since all item in \( A \) are traversed in order, so \( z \) has the larger order than all item in \( S \).
- \( x = S.top \implies L(x) < L(x.next) \) is preserved as \( L(x.next) \) is already replace by the temporal label \( L(x.next) \) and \( x \) is precede \( x.next \) by using temporal labels.
- \( x = S.top \implies L(x.pre) < L(x) \) is preserved as if such invariant is not satisfied, \( x \) should not been added into \( S \) and can be safely replace the label by its temporal label, which causes contradiction.

At the termination of the inner while-loop, we get \( S = \emptyset \), so that all items that precede \( z \) have replaced new labels maintaining the Order Snapshot. At the termination of the for-loop (lines 28 -32), all items in \( A \) have been replaced with new labels. \( \square \)
4.2.3. Complexities

For the sequential version, it is proven that the amortized time is \( O(1) \). The parallel version has some refinement. That is, the AssignLabel procedure traverses the locked items two times for generating temporary labels and replacing the labels, which cost amortized time \( O(1) \). Thus, if \( m \) items are inserted in parallel, the total amortized work is \( O(m) \). In the best case, \( m \) items can be inserted in parallel by \( \mathcal{P} \) workers with amortized depth \( O(1) \), so that the amortized running time is \( O(m/\mathcal{P}) \).

The worst-case can easily happen when all insertions accrued in the same position of \( \mathcal{O} \). The relabel procedure is triggered with the constant amortized work \( W = O(1) \) for each inserted item. In the worst-case, \( m \) items have to be inserted one-by-by, e.g. \( \mathcal{P} \) workers simultaneously insert items at the head of \( \mathcal{O} \) with amortized depth \( O(m) \), and thus the amortized running time is \( O(m/\mathcal{P} + m) \).

Such worst-case can be improved by batch insertion. The idea is that we first allocate enough label space for \( m/\mathcal{P} \) items per worker, then \( \mathcal{P} \) workers can insert items in parallel. However, this simple strategy requires pre-processing of \( \mathcal{O} \) and does not change the worst-case time complexity.

4.3. Parallel Order

4.3.1. Algorithm

Algorithm 3 shows the detailed steps of Order. When comparing the order of \( x \) and \( y \), they must not have been be deleted (line 1). We first compare the top-labels of \( x \) and \( y \) (lines 2 - 5). Two variables, \( t \) and \( t' \), obtain the values of \( L(x) \) and \( L(y) \) for comparison (line 2), and the result is stored as \( r \). After that, we have to check \( L(x) \) or \( L(y) \) has been updated or not; if that is the case, we have to redo the whole procedure (line 5). Second, we compare the bottom-labels of \( x \) and \( y \), if their top-labels are equal (lines 6 - 9). Similarly, two variables, \( b \) and \( b' \), obtain the value of \( L_b(x) \) and \( L_b(t) \) for comparison (line 7), and the result is stored as \( r \). After that, we have to check whether four labels are updated or not; if anyone label is the case, we have to redo the whole procedure (lines 8 and 9). We can see our parallel Order is lock-free so that it can execute highly in parallel. During the order comparison, \( x \) or \( y \) can not be deleted (line 10). We return the result at line 11.

\[
\textbf{Algorithm 3: Parallel-Order}(\mathcal{O}, x, y) \\
1 \textbf{if } x.\text{live} = \text{false} \lor y.\text{live} = \text{false} \textbf{ then return fail} \\
2 t, t', r \leftarrow L'(x), L'(y), \emptyset \\
3 \textbf{if } t \neq t' \textbf{ then} \\
4 \quad r \leftarrow t < t' \\
5 \quad \textbf{if } t \neq L'(x) \lor t' \neq L'(y) \textbf{ then goto line 1} \\
6 \textbf{else} \\
7 \quad b, b' \leftarrow L_b(x), L_b(y); r \leftarrow b < b' \\
8 \quad \textbf{if } t \neq L'(x) \lor t' \neq L'(y) \lor b \neq L_b(x) \lor b' \neq L_b(y) \textbf{ then} \\
9 \quad \textbf{goto line 1} \\
10 \textbf{if } x.\text{live} = \text{false} \lor y.\text{live} = \text{false} \textbf{ then return fail} \\
11 \text{return } r
\]

It is true that there is an ABA problem. That is, \( L'(x) \) and \( L'(y) \) are possibly updated multiple times but remain the same values as the \( r \) and \( t' \) (line 5). In other words, \( L(x) \) and \( L(y) \) are updated but may not be identified when comparing \( t \) and \( t' \) (line 4), which may lead to a wrong result. Also, line 8 has the same problem. To solve this problem, each top-label or bottom-label, \( L' \) or \( L_b \), includes an 8-bit counter to record the version. Each time, the counter increases by one once its corresponding label is updated. With this implementation, we can safely check whether the label is updated or not merely by comparing the values (lines 5 and 8).

Example 4.3 (Order). In Figure 4, we show an example to determine the order of \( v_2 \) and \( v_3 \) by comparing their labels. Initially, both \( v_2 \) and \( v_3 \) have old labels, 2 and 3. After the Relabel procedure is triggered, both \( v_2 \) and \( v_3 \) have new labels, 6 and 9, in which the Order Snapshot is preserved. However, it is possible that Relabel procedures are triggered in parallel. We first get \( L(v_2) = 3 \) (old label) and second get \( L(v_2) = 6 \) (new label), but it is incorrect for \( L(v_2) > L(v_3) \). After we get \( L(v_2) = 6 \), the value of \( L(v_2) \) has to be already updated to 9 since the Order Snapshot is
maintained. In this case, we redo the whole process until $L(v_2)$ and $L(v_1)$ are not updated during comparison. Thus, we can get the correct result of $L(v_2) < L(v_1)$ even the relabel procedure is executed in parallel.

4.3.2. Correctness

We have proven that parallel Insert preserves the Order Snapshot even though relabel procedures are triggered, by which labels correctly indicate the order. In this case, it is safe to determine the order for $x$ and $y$ in parallel. We first argue the top-labels (lines 2 - 5). The problem is that we first get $t \leftarrow L(x)$ and second get $t' \leftarrow L(y)$ successively (line 2), by which $t$ and $t'$ may be inconsistent, due to a Relabel procedure may be triggered. To argue the consistency of labels, there are two cases: 1) both $t$ and $t'$ obtain old labels or new labels, which can correctly indicate the order; 2) the $t$ first obtains an old label and $t'$ second obtains a new label, which may not correctly indicate the order as $x$ may already update with a new label, and vice versa; if that is the case, we redo the whole process.

On the termination of parallel Order, the invariant is that $t$ and $t'$ are consistent and thus correctly indicate the order. The bottom-labels are analogous (lines 6 - 9).

4.3.3. Complexities

For the sequential version, the running time is $O(1)$. For the parallel version, we have to consider the frequency of redo. It has a significantly low probability that the redo will be triggered. This is because the labels are changed by the Relabel procedure, which is triggered when inserting $\Omega(\log N)$ items. Even if the labels of $x$ and $y$ are updated when comparing their order, it still has a tiny probability that such label updating happens during the comparison of labels (lines 4 and 7).

Thus, supposing $m$ items are comparing orders in parallel, the total work is $O(m)$, and the depth is $O(1)$ with a high probability. So that the running time is $O(m/P)$ with high probability.

5. Experiments

We report on experimental studies for our three parallel order maintenance operations, Order, Insert, and Delete. We generate four different test cases to evaluate their parallelized performance. All the source code is available on GitHub.

5.1. Experiment Setup

The experiments are performed on a server with an AMD CPU (64 cores, 128 hyperthreads, 256 MB of last-level shared cache) and 256 GB of main memory. The server runs the Ubuntu Linux (22.04) operating system. All tested algorithms are implemented in C++ and compiled with g++ version 11.2.0 with the -O3 option. OpenMP version 4.5 is used as the threading library. We choose the number of workers exponentially increasing as 1, 2, 4, 8, 16, 32 and 64 to evaluate the parallelism. With different numbers of workers, we perform every experiment at least 100 times and calculate the mean with 95% confidence intervals.

In our experiment, for easy implementation, we choose $N = 2^{32}$, a 32-bit integer, as the capacity of $\mathbb{O}$. In this case, the bottom-lables $L_0$ are 32-bit integers, and the top-lables $L'$ are 64-bit integers. One advantage is that reading and writing such 32-bit or 64-bit integers are atomic operations in modern machines. There are initially 10 million items in the order list $\mathbb{O}$. To test our parallel OM data structure, we do four experiments:

- **Insert**: we insert 10 million items into $\mathbb{O}$.
- **Order**: for each inserted item, we compare its order with its successive item, so that it has 10 million Order operations.
- **Delete**: we delete all inserted items, a total of 10 million times.

\[\text{https://github.com/Itisben/Parallel-OM.git}\]
\[\text{https://www.openmp.org/}\]
– **Mixed**: again, we insert 10 million items, mixed with 100 million Order operations. For each inserted item, we compare its order with its ten successive items, a total of 100 million times order comparison. This experiment is to test how often “redo” occurs in the Order operations when there are parallel Insert operations. The reason for this experiment is that many Order operations are mixed with few Insert and Delete operations in real applications, as shown in Figure 2.

For each experiment, we have four test cases by choosing different numbers of positions for inserting:

– **No Relabel** case: we have 10 million positions, the total number of initial items in $O$, so that each position averagely has 1 inserted item. Thus, it almost has no Relabel procedures triggered when inserting.

– **Few Relabel** case: we randomly choose 1 million positions from 10 million items in $O$, so that each position averagely has 10 inserted items. Thus, it is possible that a few Relabel procedures are triggered when inserting.

– **Many Relabel** case: we randomly choose 1,000 positions from 10 million items in $O$, so that each position averagely has 10,000 inserted items. Thus, it is possible that many Relabel procedures are triggered when inserting.

– **Max Relabel** case: we only choose a single position (at the middle of $O$) to insert 10,000,000 items. In this way, we obtain a maximum number of triggered relabel procedures.

All items are inserted on-the-fly without preprocessing. In other words, 10 million items are randomly assigned to multiple workers, e.g. 32 workers, even if in the Max case all insertions are reduced to sequential execution.

### 5.2. Evaluating Relabelling

In this test, we evaluate the Relabel procedures that is triggered by Insert operations over four test cases, No, Few, Many and Max. For this, different numbers of workers will have the same trend, so we choose 32 workers for this evaluation.

| Case | Relabel# | $L_b$# | $L_t$# | AvgLabel# | OrderRedo# |
|------|----------|--------|--------|-----------|-------------|
| No   | 0        | 10,000,000 | 0      | 1         | 0           |
| Few  | 2,483    | 10,069,551 | 4,967  | 1         | 0           |
| Many | 356,624  | 19,985,472 | 5,754,501 | 2.6       | 0           |
| Max  | 357,142  | 19,999,976 | 99,024,410 | 11.8      | 0           |

Table 2: The detailed numbers of the relabel procedure.

In Table 2 columns 2 - 4 show the details in the Insert experiment, where Relabel# is the times of triggered Relabel procedures, $L_b$# is the number of updated bottom-labels for items, $L_t$# is the number of updated top-labels for items, and AvgLabel# is the average number of updated labels for each inserted items when inserting 10 million items. We can see that, for four cases, the amortized numbers of updated labels increase slowly, where the average numbers of inserted items for each position increase by 1, 10, 10 million, and 10 billion. This is because our parallel Insert operations have $O(1)$ amortized work.

– The No case not triggers Relabel, updating only one $L_b$ per insert.

– The Few case triggers 2.5 thousand Relabel, updating 1.007 $L_b$, 0.005 $L_t$, and totally about one labels per inserted items.

– The Many case triggers 0.36 million Relabel, updating 2 $L_b$, 0.6 $L_t$, and totally about 2.6 labels per inserted items.

– The Max case triggers 0.36 million Relabel, which is the same as Many the case. But, it updates 2 $L_b$, 9.9 $L_t$, totally about 11 labels per inserted items.
In Table 2, the last column shows the numbers of redo for Order operations in the Mixed experiment, which are all zero. Since the Mixed has mixed Order and Insert operations, we may redo the Order operation if the corresponding labels are being updated. However, Relabel happens with a low probability; also, it is a low probability that related labels are changed when comparing the order of two items. This is why the numbers of redo are zero, leading to high parallel performance.

5.3. Evaluating the Running Time

In this test, we exponentially increase the number of workers from 1 to 64 and evaluate the real running time. We perform Insert, Order, Delete, and Mixed over four test cases, No, Few, Many and Max.

![Figure 5: Evaluating the running times.](image)

The plots in Figure 5 depict the performance. The x-axis is the number of workers and the y-axis is the execution time (millisecond). Note that, we compare the performance by using two kinds of lock: the OpenMP lock (denoted by solid lines) and the spin lock (denoted by dash lines). A first look reveals that the running times normally decrease with increasing numbers of workers, except for the Max case over Insert and Mixed experiments. Specifically, we make several observations:

- Three experiments, Insert, Delete and Mixed, that use the spin lock are much faster than using the OpenMP lock. This is because the lock regions always have few operations, and busy waiting (spin lock) is much faster than the suspension waiting (OpenMP lock). Unlike the above three experiments, the Order experiment does not show any differences since Order operations are lock-free without using locks for synchronization.

- For the Max case of Insert and Mixed, abnormally, the running time is increasing with an increasing number of workers. The reason is that the Insert operations are reduced to sequential in the Max Case since all items are inserted into the same position. Thus it has the highest contention on shared positions when multiple workers accessing at the same time, especially for 64 workers.

- For the Many case of Insert and Mixed, the running times are decreasing until using 4 workers. From 8 workers, however, the running times begin to increase. This is because the Insert operations have only 1,000 positions in the Many case, and thus it may have high contention on shared positions when using more than 4 workers.
– Over the Order and Delete experiments, we can see the Many and Max cases are always faster than the Few and cases. This is because the Few and No cases have 1,000 and 1 operating positions, respectively; all of these positions can fit into the CPU cache with high probability, and accessing the cache is much faster than accessing the memory.

5.4. Evaluating the Speedups

The plots in Figure 6 depict the speedups. The x-axis is the number of workers, and the y-axis is the speedups, which are the ratio of running times (by using spin locks) between the sequential version and using multiple workers. The dotted lines show the perfect speedups as a baseline. The numbers beside the lines indicate the maximal speedups.

A first look reveals that all experiments achieve speedups when using multiple cores, except for the Max case over Insert and Mixed experiments. Specifically, we make several observations:

– For all experiments, we observe that the speedups are around 1/4 to 1 when using 1 worker in all cases. This is because, for all operations of OM, the sequential version has the same work as the parallel version. Especially, for Delete, such speedups are low as 1/2 - 1/4, as locking items for deleting costs much running time.

– For Insert and Mixed, we achieve around 7x speedups using 32 workers in No and Few cases, and around 2x speedups using 4 workers in Many cases. This is because all CPU cores have to access the shared memory by the bus, which connects memory and cores, and the atomic CAS operations will lock the bus. Each Insert operation may have many atomic CAS operations for spin lock and many atomic read and write operations for updating labels and lists. In this case, the bus traffic is high, which is the performance bottleneck for Insert operations.

– For Order, all four cases achieve almost perfect speedups from using 1 to 32 workers, as Order operations are lock-free.

– For Delete, it achieves around 4x speedups using 64 workers in four cases. This is because, for parallel Delete operations, the worst case, which is all operations are blocking as a chain, is almost impossible to happen.
5.5. Evaluating the Stability

In this test, we compare 100 testing times for the Insert, Order, and Delete operations by using 32 workers. Each time, we randomly choose positions and randomly insert items for the NO, Few and Many cases, so that the test is different. However, it is always the same for the Max case, since there is only one position to insert all items.

The plots in Figure 7 depict the running time by performing the experiments 100 times. The x-axis is the index of repeating times and the y-axis is the running times (millisecond). We observe that the performance of Insert, Order, Delete and Mixed are all well bounded for all four cases. We observe that the Max case has wider variation then other cases over Insert and Mixed. This is because the parallel Insert operations always have contention on shared data in memory. Such contention causes the running times to vary within a certain range.

5.6. Evaluating the Scalability

In this test, we increase the scale of the initial order list from 10 million to 100 million and evaluate running times with fixed 32 workers. We test three cases, No, Few, and Many, by fixing the average number of items per insert position. For example, given an initial order list with 20 million items, the No case has 20 million insert positions, the Few case has 2 million positions, and the Many case has 2,000 insert positions. Since the Max case is reduced to sequential and can be optimized by using a single worker, we omit it in this test.

The plots in Figure 8 depict the performance. The x-axis is the initial size of the order list, and the y-axis is the time ratio of the current running time to the “10 million” running time. The dotted lines show the perfect time ratio as a baseline. Obviously, the beginning time ratio is one. We observe that the time ratios are roughly close to linearly increasing with the scales of the order list. This is because all parallel Insert, Delete, and Order have best-case time complexity $O(\frac{n}{p})$ and averagely their running times are close to the best case.

Specifically, for Order, we can see the time ratio is up to 20x with a scale of 100 million in No case. This is because No case has 100 million positions for random Order operations, which is not cache friendly; also, by increasing the scale of data, the cache hit rate decreases, so the performance is affected.
6. Conclusions and Future Work

We present a new parallel order maintenance (OM) data structure. The parallel Insert and Delete are synchronized with locks efficiently. Notably, the parallel Order is lock-free, and can execute highly in parallel. Experiments demonstrate significant speedups (for 64 workers) over the sequential version on a variety of test cases.

In future work, we will attempt to reduce the synchronization overhead, especially for parallel Insert. We will investigate insertions and deletions in batches by preprocessing the inserted or deleted items, respectively, which can reduce the contention for multiple workers. We also intend to apply our parallel OM data structure to many applications.

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