Finding Optimal Flows Efficiently

Mehdi Mhalla*, Simon Perdrix†

Abstract

Among the models of quantum computation, the One-way Quantum Computer [10] [11] is one of the most promising proposals of physical realization [12], and opens new perspectives for parallelization by taking advantage of quantum entanglement [2]. Since a one-way quantum computation is based on quantum measurement, which is a fundamentally nondeterministic evolution, a sufficient condition of global determinism has been introduced in [4] as the existence of a causal flow in a graph that underlies the computation. A $O(n^3)$-algorithm has been introduced [6] for finding such a causal flow when the numbers of output and input vertices in the graph are equal, otherwise no polynomial time algorithm[1] was known for deciding whether a graph has a causal flow or not. Our main contribution is to introduce a $O(n^2)$-algorithm for finding a causal flow, if any, whatever the numbers of input and output vertices are. This answers the open question stated by Danos and Kashefi [4] and by de Beaudrap [6]. Moreover, we prove that our algorithm produces an optimal flow (flow of minimal depth.)

Whereas the existence of a causal flow is a sufficient condition for determinism, it is not a necessary condition. A weaker version of the causal flow, called $gflow$ (generalized flow) has been introduced in [3] and has been proved to be a necessary and sufficient condition for a family of deterministic computations. Moreover the depth of the quantum computation is upper bounded by the depth of the $gflow$. However, the existence of a polynomial time algorithm that finds a $gflow$ has been stated as an open question in [3]. In this paper we answer this positively with a polynomial time algorithm that outputs an optimal $gflow$ of a given graph and thus finds an optimal correction strategy to the nondeterministic evolution due to measurements.

Keywords: Graph Algorithms, Quantum Computing

1 Introduction

A one-way quantum computation [10] consists in performing a sequence of one-qubit measurements on an initial entangled quantum state described by a graph and called graph state [8] where some vertices correspond to the input qubits of the computation, others to the output qubits and the rest of the vertices correspond to auxiliary qubits measured during the computation. Since quantum measurements are nondeterministic, a one-way quantum computation requires corrections which depend on the results of the measurements, and which should induce a minimal depth for the computation.

Because of these corrections, not all graph states can be used for deterministic computation. The measurement calculus [7] is a formal framework for one-way quantum computations, where the dependencies between measurements and corrections are precisely identified. Using this formalism, Danos and Kashefi in [4] proved that a one-way quantum computation obtained by translation from a quantum circuit is such that the underlying graph satisfies a causal flow condition (see section [2]).

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* LIG, University of Grenoble, France, mehdi.mhalla@imag.fr
† Oxford University Computing Laboratory, simon.perdrix@comlab.ox.ac.uk

1 an exponential time algorithm is proposed in [6]
In [7] a polynomial time algorithm in the size of the graph has been proposed for finding a causal flow when the numbers of outputs and inputs are equal, whereas the existence of a polynomial time algorithm in the general case, has been stated as an open question. We propose in this paper a faster and more general algorithm for finding a causal flow, whenever the numbers of inputs and outputs are different.

It turns out that the existence of a causal flow is not a necessary condition for determinism. A weaker flow condition, the gflow condition has indeed been introduced for characterizing uniform, stepwise and strong deterministic computation, where the correction strategy does not depend on the measurement basis (see [3] for a formal definition.) Here, we introduce a polynomial time algorithm for finding a gflow and thus checking whether a graph allows a uniform deterministic computation, which gives substantially more relevance to the notion of gflow introduced in [3].

We also prove that the algorithms proposed are optimal, which means that they give a minimal depth flow. This implies that the gflow algorithm gives a lower bound on the complexity of a correction strategy in a measurement-based setting for quantum computation.

2 Definitions

A graph with input and output vertices is called an open graph, and is defined as follows:

**Definition 1 (Open Graph)** An open graph is a triplet $(G, I, O)$, where $G = (V, E)$ is a undirected graph, and $I, O \subseteq V$ are respectively called input and output vertices.

During a one-way quantum computation all non output qubits (represented as non output vertices in the corresponding open graph) are measured. Since quantum measurements are nondeterministic, for each qubit measurement, a corrective strategy consists in acting on some unmeasured non input qubits, depending on the classical outcome of the measurement, in order to make the computation deterministic. Thus, a corrective strategy induces a sequential dependance between measurements. As a consequence, the depth of the quantum computation depends on the corrective strategy.

Corrective strategies will be defined by flows on the open graphs. A flow $(g, \prec)$ consists in a partial order $\prec$ over the vertices ($i \prec j$ if $i$ is measured before $j$) and a function $g$ that associates with each vertex, the vertices used for correcting its measurement (all non output qubits are measured.) Input qubits cannot be used for correction (see [5].)

Given an open graph, two kinds of flows are considered: the causal flow and the gflow (generalized flow.) The former has been introduced by Danos and Kashefi [4] and corresponds to the computation strategy that consists in correcting each qubit measurement by acting on a single neighbor of the measured qubit. For a given open graph, a causal flow is characterized by a function $g$ which associates with each non output vertex a non input vertex used for the correction of its measurement. More formally:

**Definition 2 (causal flow)** $(g, \prec)$ is a causal flow of $(G, I, O)$, where $g : V(G) \setminus O \rightarrow V(G) \setminus I$ and $\prec$ is a strict partial order over $V(G)$, if and only if

1. $i \prec g(i)$
2. if $j \in N(g(i))$ then $j = i$ or $i \prec j$, where $N(v)$ is the neighborhood of $v$
3. $i \in N(g(i))$.

An example of causal flow is given in Figure 1. Notice that if the numbers of input and output vertices are the same, a causal flow can be reduced to a path cover and then to a standard
network flow \[^6\]. This reduction has been used to define an \(O(n^3)\)-algorithm for finding a causal flow in the case where the cardinalities of input and output qubits are the same \[^6\].

The second type of flow considered, the generalized flow, \(gflow\), has been introduced in \[^3\] and corresponds to a more general correction strategy that associates with each non output vertex a set of vertices used for the corresponding correction (instead of a single vertex.) This generalization not only leads to a reduction of the computational depth, but also provides a corrective strategy to some open graphs having no causal flow. Moreover, notice that \(gflow\) characterizes uniform, strong and stepwise deterministic computations \[^3\].

For a given open graph, a \(gflow\) \((g, \prec)\) is characterized by a function \(g\) which associates with each non output vertex, a set of non input vertices used for its correction, and a strict partial order \(\prec\):

**Definition 3 (gflow)** \((g, \prec)\) is a \(gflow\) of \((G, I, O)\), where \(g : V(G) \setminus O \to \mathcal{P}(V(G) \setminus I) \setminus \{\emptyset\}\) and \(\prec\) is a strict partial order over \(V(G)\), if and only if

1. if \(j \in g(i)\) then \(i \prec j\)
2. if \(j \in \text{Odd}(g(i))\) then \(j = i\) or \(i \prec j\)
3. \(i \in \text{Odd}(g(i))\)

Where \(\text{Odd}(K) = \{u, |N(u) \cap K| = 1 \mod 2\}\) is the odd neighborhood of \(K\), i.e. the set of vertices having an odd number of neighbors in \(K\).

A graphical interpretation of the generalised flow is given in Figure 2.

**Figure 2:** Graphical interpretation of a \(gflow\) \((g, \prec)\): for a given vertex \(u\) all the vertices larger than \(u\) are in the future of \(u\) since the corresponding qubits will be measured after the qubit \(u\), all others are in the past of \(u\). The set \(g(u)\) has to be in the future of \(u\) and such that the following parity conditions are satisfied: there is an odd number of edges between \(g(u)\) and \(u\) and there is a even number of edges between \(g(u)\) and any vertex in the past of \(u\).

A flow \((g, \prec)\) of \((G, I, O)\) induces a partition of the vertices of the open graph:

**Definition 4** For a given open graph \((G, I, O)\) and a given flow \((g, \prec)\) of \((G, I, O)\), let

\[
V_k^\prec = \begin{cases} 
\max_{\prec}(V(G)) & \text{if } k = 0 \\
\max_{\prec}(V(G) \setminus V_{k-1}^\prec) & \text{if } k > 0
\end{cases}
\]
where $\max_{\prec}(X) = \{u \in X \text{ s.t. } \forall v \in X, -(u \prec v)\}$ is the set of the maximal elements of $X$. The depth $d^\prec$ of the flow is the smallest $d$ such that $V_{d+1}^\prec = \emptyset$. $(V_k^\prec)_{k=0...d^\prec}$ is a partition of $V(G)$ into $d^\prec + 1$ layers.

A causal flow or a gflow $(g, \prec)$ of $(G, I, O)$ leads to a corrective strategy for the corresponding one-way quantum computation, which consists in measuring the non output qubits of each layer in parallel, from the layer $V_k^\prec$ to the layer $V_1^\prec$. After the measurement of a layer $V_k^\prec$, with $k > 0$, corrections are realised according to the function $g$ by acting on qubits in $\cup_{i<k} V_i^\prec$ (see [3] for details.) The depth of such a one-way quantum computation is $d^\prec$.

**Definition 5** For a given open graph $(G, I, O)$ and two given flows $(g, \prec)$ and $(g', \prec')$ of $(G, I, O)$, $(g, \prec)$ is more delayed than $(g', \prec')$ if $\forall k, |\cup_{i=0...k} V_k^\prec| \geq |\cup_{i=0...k} V_k^{\prec'}|$ and there exists a $k$ such that the inequality is strict.

A causal flow (resp. gflow) $(g, \prec)$ is maximally delayed if there exists no causal flow (resp. gflow) of the same open graph that is more delayed.

For instance, the flow $(g, \prec)$ described in Figure 1 is a maximally delayed causal flow. However, $(g, \prec)$ is not a maximally delayed gflow since $(g', \prec')$ is a more delayed gflow, where $g'(a_0) = \{b_0, b_1, b_2\}, g'(a_1) = \{b_1, b_2\}, g'(a_2) = \{b_2\}, g'(b_0) = \{c_0\}, g'(b_1) = \{c_1\}, g'(b_2) = \{b_2\}$, and $\{a_0, a_1, a_2\} \prec' \{b_0, b_1, b_2\} \prec' \{c_0, c_1, c_2\}$. One can prove that $(g', \prec')$ is a maximally delayed gflow.

The following two lemmas are proved for both kinds of flows.

**Lemma 6** If $(g, \prec)$ is a maximally delayed causal flow (gflow) of $(G, I, O)$ then $V_0^\prec = O$.

**Proof** Let $(g, \prec)$ be a maximally delayed causal flow (gflow) of $(G, I, O)$. Elements of $V_0^\prec$ have no image under $g$ because of condition 1 in both definitions thus $V_0^\prec \subseteq O$. Moreover, by contradiction, if $O \setminus V_0^\prec \neq \emptyset$, let $\prec' = \prec \setminus (O \setminus V_0^\prec) \times V(G)$, $(g, \prec')$ is a causal flow (gflow) of $(G, I, O)$: condition 1 of both definitions is satisfied by $\prec'$, because the domain of $g$ does not intersect $O$, so for any $i$ in the domain of $g$, $i \prec' j$ iff $i \prec j$; conditions 2 and 3 of both definitions are satisfied in a same way. Thus, $(g, \prec')$ is a causal flow (gflow) of $(G, I, O)$. Moreover, for any $k$, $\cup_{i=0...k} V_k^\prec \subseteq \cup_{i=0...k} V_k^{\prec'}$, and $|V_0^\prec| < |V_0^{\prec'}|$ thus $(g, \prec')$ is more delayed than $(g, \prec)$ which leads to a contradiction. \hfill $\square$

**Lemma 7** If $(g, \prec)$ is a maximally delayed causal flow (gflow) of $(G, I, O)$ then $(\bar{g}, \bar{\prec})$ is a maximally delayed causal flow (gflow) of $(G, I, O \cup V_1^\prec)$ where $\bar{g}$ is the restriction of $g$ to $V(G) \setminus (V_0^\prec \cup V_1^\prec)$ and $\bar{\prec} = \prec \setminus V_1^\prec \times V_0^\prec$.

**Proof** First, one can prove that $(\bar{g}, \bar{\prec})$ is a causal flow (gflow) of $(G, I, O \cup V_1^\prec)$. Moreover, by contradiction, if there exists a causal flow (gflow) $(g'', \prec'')$ that is more delayed than $(\bar{g}, \bar{\prec})$ then it could be extended to $(g'', \prec'')$ where $g''(u) = g'(u)$ if $u \in V \setminus (V_0^\prec \cup V_1^\prec)$, $g''(u) = g(u)$ if $u \in V_1^\prec$ and $\prec'' = \prec' \cup \{(u, v), u \in V_1^\prec \wedge u \prec v\}$. $(g'', \prec'')$ is then a more delayed causal flow (gflow) of $(G, I, O)$ than $(g, \prec)$, which leads to a contradiction. \hfill $\square$

**Lemma 8** If $(g, \prec)$ is a maximally delayed gflow, then $V_1^\prec = \{u \in V \setminus O, \exists K \subseteq O, \text{Odd}(K) \cap (V \setminus O) = \{u\}\}$.

**Proof** First, notice that if $(g, \prec)$ is a maximally delayed gflow, then for any $u \in V_1^\prec$, $g(u) \subseteq O$ since $u \prec v$ if $v \in g(u)$ (condition 1 of definition [3]) Furthermore, by definition of $V_1^\prec$, if $u \prec v$ then $v \in O$ thus conditions 2 and 3 of definition [3] imply that $\text{Odd}(g(u)) \cap (V \setminus O) = \{u\}$. \hfill $\square$
To prove that any \( u \in V \setminus O \) such that \( \exists K \subseteq O, \text{Odd}(K) \cap V \setminus O = \{ u \} \), \( u \in V_1^\prec \), we proceed by contradiction. We prove that delaying the measurement of a vertex not in \( V_1^\prec \) satisfying the condition permits to create a more delayed gflow. Indeed, let \((g, \prec)\) be a maximally delayed flow of \((G, I, O)\) and let \( u_1 \in V \setminus V_0^\prec \) be such that \( \exists K \subseteq O, \text{Odd}(K) \cap V \setminus O = \{ u_1 \} \). Let \( g'(u) = K \) if \( u = u_1 \) and \( g'(u) = g(u) \) otherwise. Let \( \prec' \) be the strict partial order defined by \( u \prec' v \) if \( u \neq u_1 \) and \( u \prec v \) or if \( u = u_1 \) and \( v \in K \). It leads to a contradiction since \((g', \prec')\) is a more delayed gflow of \((G, I, O)\) than \((g, \prec)\).

In a similar way, one can prove that:

**Lemma 9** If \((g, \prec)\) is a maximally delayed causal flow, then \( V_1^\prec = \{ u \in V \setminus O, \exists v \in O, N(v) \cap V \setminus O = \{ u \} \} \).

Lemmas 8 and 9 show that in a maximally delayed flow, all the elements that can be corrected at the last step are in the maximal layer of \( V \setminus O \) (i.e. in \( V_1^\prec \)). Combined with the recursive structure of maximally delayed flow (lemma 7), this shows that the layers \( V_k^\prec \) of a maximally delayed flow can be iteratively constructed by finding elements that can be corrected starting from the output qubits. This gives rise to the polynomial time algorithms of the next sections.

### 3 Causal flow algorithm

The problem of finding a causal flow of a given open graph is presented in [4], and a solution has been proposed in the case where the numbers of inputs and outputs are the same. The complexity of the algorithm is in \( O(nm) \) where \( n \) is the number of vertices and \( m \) the number of edges (more precisely \( O(km) \) where \( k \) is the number of inputs (outputs) [7]). We present here a more general and faster algorithm.

**Theorem 10** For a given open graph \((G, I, O)\), finding a causal flow can be done in \( O(k.n + m) \) operations where \( n = |V(G)| \) is the number of vertices of \( G \), \( m = |V(E)| \) is the number of edges and \( k = |O| \) is the size of the output.

In order to prove Theorem 10, we introduce the algorithm which decides whether given an open graph has a causal flow, and outputs a maximally delayed causal flow if one exists. This recursive algorithm is based on the recursive structure, pointed out in the previous section, of the maximally delayed causal flows.

The algorithm recursively finds the layers \((V_k^\prec)_{k=0, \ldots, d} : \) at the \( k^{th} \) call to Flowaux, the algorithm finds the set \( V_k^\prec = \text{Out}' \) (see algorithm 1 and figure 3). To improve the complexity of the algorithm, a set \( C \) of potential correctors \((\forall u \in V_k^\prec, g(u) \in C) \) is maintained. The algorithm produces a subset \( C' \) of \( C \) of vertices that can be actually used as correctors, the set \( \text{Out}' \) of vertices that can be corrected by \( C' \) is produced as well. For the recursive call, the vertices of \( \text{Out}' \) are added to the potential correctors, whereas the vertices used as correctors (i.e. \( C' \)) are removed from the set of potential correctors since a vertex can be used to correct at most one other vertex.

The partial order \( \prec \) of the flow found by the algorithm is defined via a labeling \( l \) which associates with each vertex the index of its layer. As a consequence, for any two vertices \( u \) and \( v \), \( u \prec v \) iff \( l(u) > l(v) \).

**Proof of Theorem 10:**
Algorithm 1: Causal flow

Figure 3: Causal flow algorithm: At the $k^{th}$ recursive call, the algorithm finds out the set $V_k^{≺_k}$ composed of the qubits that will be measured at the $d^{≺_k} - k + 1$ step of the one-way quantum computation, where $d^{≺_k}$ is the depth of the computation. At that step, all the qubits in $Out := \bigcup_{i=0..k-1} V_i^{≺_i}$ are not measured, whereas the qubits in $\bigcup_{i>k} V_i^{≺_i}$ are already measured. The correctors of the elements of $V_k^{≺_k}$ are in a set $C \subseteq Out$ of candidates composed of vertices not already assigned to the correction of some future measurement. The first stage of the algorithm to find out the set $V_k^{≺_k}$ consists in searching, among $C$, for the elements that have a unique neighbor in $V \setminus Out$. Let $C' \subseteq C$ be this set of correctors. Then, the neighborhood $Out'$ of $C'$ in $V \setminus Out$ is a set of elements that can be corrected at that step, so $Out'$ is nothing but $V_k^{≺_k}$. For the recursive call, the elements of $Out'$ are added to both $Out$ and $C$, whereas the elements of $C'$ are removed from $C$. Since at each step, a maximum number of vertices are added to $V_k^{≺_k}$, the causal flow, if it exists, produced by this algorithm is maximally delayed.
By induction on the number of non output qubits, we prove that if the given open graph has a causal flow then the algorithm outputs a maximally delayed one. Assume that the given open graph has a causal flow. First, if there is no non output qubit, then no correction is needed: the empty flow $(g,\emptyset)$ (where $g$ is a function with an empty domain) is a maximally delayed flow. Now suppose that there exist some non output vertices, according to lemma 8 the elements of $V_1^{\prec}$ satisfy the test at line 13, moreover the precondition at line 8 (that can reformulated as $g(V_1^{\prec}) \subseteq C$) implies that $V_1^{\prec}$ is composed of the elements that satisfy the test at line 13. Thus, after the loop (line 19), $Out' = V_1^{\prec}$ and $C' = g(V_1^{\prec})$. Since the existence of a causal flow is assumed, $V_1^{\prec}$ cannot be empty (all non output qubits have to be corrected), thus the algorithm is called recursively. Lemma 7 ensures the existence of a causal flow in $(G,I,O \cup V_1^{\prec})$ and since the vertices in $C'$ have no neighbor in $V \setminus (O \cup V_1^{\prec})$, they can be removed from the set of potential correctors, preserving the precondition.

The induction hypothesis ensures that the recursive calls output a maximally delayed causal flow in $(G,I,O \cup V_1^{\prec})$ and thus the causal flow $(g,\prec)$ defined is a maximally delayed causal flow of $(G,I,O)$. The termination of the algorithm is ensured by the fact that the set of output qubits strictly increases at each recursive call.

For a given open graph, if the algorithm outputs a flow $(g,\prec)$, then this flow is a valid causal flow since every output qubit has an image under $g$, moreover for any vertex $i$, $i \prec g(i)$, and finally if $j \in N(g(i))$ then $j = i$ or $i \prec j$. Thus, if the given open graph has no flow, the algorithm returns false.

To analyze the complexity of the algorithm, we consider the cost for each vertex $u$, which can be decomposed in:

- Insert $u$ in the set $C$ of potential correctors
- Create the set of vertices that $u$ might correct $N(u) \cap (V \setminus Out)$
- Check whether $u$ can be removed $|N(u) \cap (V \setminus Out)| = 1$
- Update the potential correctors sets $N(v) \cap (V \setminus Out)$ for $v \in C$ when $u$ is removed ($u$ belongs to $Out$ for the recursive call).

As there is at most $|Out|$ potential correctors, the cost for a vertex $u$ can be decomposed in: insert in $C$ + create + $|Out|$ (remove + check).

Using a data structure for storing the sets $N(u) \cap (V \setminus Out)$ (for example an array with two pointers respectively to next and previous elements), one can remove an element in constant time and test whether the set contains exactly one element in constant time. For the creation of the structure, one needs to compute the intersection of the neighborhood with $(V \setminus Out)$. Checking whether a vertex is in $(V \setminus Out)$ can be done in constant time by maintaining an array of the vertices that are to be corrected, thus given the adjacency list of a vertex $u$ the cost of creating $N(u) \cap (V \setminus Out)$ is the degree of $u$ and the total finding cost is $O(m)$ where $m$ is the number of edges of the graph. The overall complexity is then $O(n|Out| + m)$. □

This result improves the algorithm in [4] that decides, under the precondition $|I| = |O|$, whether an open graph $(G,I,O)$ has a causal flow in $O(km)$ operations, where $k = |O|$. In [9], Pei and de Beaudrap have proved that an open graph having a causal flow has at most $(n-1)k - \binom{k}{2}$ edges. According to this result, the algorithm in [6] can be transformed (see [9]) into a $O(k^2n)$-algorithm, whereas our algorithm becomes a $O(kn)$-algorithm.
4 A polynomial algorithm for gflow

**Theorem 11** There exists a polynomial time algorithm that decides whether a given open graph has a gflow, and that outputs a gflow if it exists.

**Proof** Let \((G, I, O)\) be an open graph. The algorithm \(gFlow(\Gamma, I, O)\) (Algorithm 2), where \(\Gamma\) is the adjacency matrix of \(G\), finds a maximally delayed gflow and returns true if one exists and returns false otherwise. Given a set \(Out'\) and a subset \(X \subseteq Out'\), \(I_X\) stands for a \(|Out'|\)-dimensional vector defined by \(I_X(i) = 1\) if \(i \in X\) and \(I_X(i) = 0\) otherwise.

```
input : An open graph
output: A generalized flow
1 gFlow (\(\Gamma, I, O\)) =
2 begin
3 for all \(v \in Out\) do
4     \(l(v) := 0\);
5 end
6 gFlowaux (\(\Gamma, I, Out, Out \setminus I_n, 1\));
7 end
8 gFlowaux (\(\Gamma, I, Out, k\)) =
9 begin
10 Out' := Out \setminus I_n;
11 C := \emptyset;
12 for all \(u \in V \setminus Out\) do
13     Solve in \(\mathbb{F}_2 : \Gamma_{V \setminus Out, Out} I_X = I_{\{u\}}\);
14     If there is a solution \(X_0\) then \(C := C \cup \{u\}\) and \(g(u) := X_0\);
15     \(l(u) = k\);
16 end
17 if \(C = \emptyset\) then
18     if \(Out = V\) then
19         true
20     else
21         false
22 end
23 else
24     gFlowaux (\(\Gamma, I, Out \cup C, k+1\))
25 end
26 end
```

**Algorithm 2: Generalized flow**

At the \(k^{th}\) recursive call, the set \(C\) found by the algorithm at the end of the loop at line 16 corresponds to the layer \(V_k^\prec\) of the partition induced by the returned strict partial order. At line 13, the columns of the matrix \(\Gamma_{V \setminus Out, Out}\) correspond to the vertices that can be used for correction (vertices in \(\cup_{i<k} V_i^\prec \setminus I_n\)) and the rows to the candidates for the set \(V_k^\prec\). A solution \(X_0\) in \(\mathbb{F}_2\) to \(\Gamma_{V \setminus Out, Out} I_X = I_{\{u\}}\) corresponds to a subset of \(\cup_{i<k} V_i^\prec \setminus I_n\) that has only \(u\) as odd neighborhood in \(\cup_{i \leq k} V_i^\prec\), thus \(g(u) := X_0\) satisfies conditions 2 and 3 required by the definition of gflow (see Definition 3). Furthermore, line 10 of the algorithm ensures condition 1, thus if the algorithm returns a flow, then it satisfies the definition of gflows.

Now suppose that the graph admits a gflow \((g, \prec)\), then it also admits a maximally delayed gflow \((g', \prec')\). The algorithm finds the set \(V'_1\) (in the loop at line 12), and by induction (similarly to the induction in the proof of Theorem 10) it also finds a maximally delayed gflow in \((G, I, O \cup V'_1^\prec)\) with the recursive call. Thus the algorithm finds a maximally delayed gflow. \(\square\)
In order to analyse the complexity, notice that lines 10 to 16 consists in solving a system $Ax = b_i$ for $n - \ell$ different $b_i$s where $n = |V|$, $\ell = |\text{Out}|$ and $A$ is a $(n - \ell) \times \ell$ matrix. In order to solve these $n - \ell$ systems, the $(n - \ell) \times n$-matrix $M = [A|b_1 \ldots b_{n-\ell}]$ is transformed into an upper triangular form within $O(p^3)$ operations using gaussian eliminations for instance, then for each $b_i$ a back substitution within $O(n^2)$ operations is used to find $x_i$, if it exists, such that $Ax_i = b_i$ (see [1]). The back substitutions costs $O(n^3)$ operations at each call of the function. Since there are at most $n$ recursive calls, the overall complexity is $O(n^4)$.

5 Depth Optimality

We consider in this section the depth of the flows found by the algorithms, which corresponds to the number of steps required by the correction strategy, and we show that both algorithms find minimal depth flows, hence optimal correction strategies.

**Theorem 12** The previous algorithms find an optimal depth flow

**Proof** First, notice that if $(g, \prec)$ is more delayed then $(g', \prec')$ then $|\cup_{i=0}^{d-\prec'} V_k^{\prec'}| \geq |\cup_{i=0}^{d-\prec} V_k^{\prec}|$. Thus $\forall k > d-\prec, V_k^{\prec} = \emptyset$, so $d-\prec \leq d-\prec'$.

Given an open graph $(G, I, O)$, and an optimal depth flow $(g, \prec)$, we can define a maximally delayed optimal depth flow $(g', \prec')$: If $(g, \prec)$ is maximally delayed then $(g', \prec') = (g, \prec)$ otherwise $(g', \prec')$ is a maximally delayed flow that is more delayed than $(g, \prec)$. According to the previous remark $(g', \prec')$ is of optimal depth as well and the optimal depth $d(G, I, O) = d-\prec = d-\prec'$. By lemma [3] $V_1^{\prec'} = \{u, \exists K \subseteq O, Odd(K) \cap V \setminus O = \{u\}\} = V_1^{\prec''}$ where $(g'', \prec'')$ is the flow found by the algorithm. Thus $d(G, I, O) = 1 + d(G, I, O \cup V^{\prec''})$. By induction on the number of output vertices the algorithm gives an optimal flow for $(G, I, O \cup V^{\prec''})$ so the depth given by the algorithm is the depth of $(g', \prec')$ which is optimal.

The optimality of the previous algorithms have several implications in one-way quantum computation. First, the depth (optimal or not) of a flow is an upper bound on the depth of the corresponding deterministic one-way quantum computation. Moreover, if the one-way quantum computation is uniformly, stepwise and strongly deterministic (which mainly means that if the measurements are applied with an error in the angle which characterises the measurement, then the computation is still deterministic), then the correction strategy must be described by a gflow [3]. As a consequence the algorithm 2 produces the optimal correction strategy, and the depth of the gflow produced by the algorithm is a lower bound on the depth of a uniformly, stepwise and strongly deterministic one-way quantum computation.

6 Conclusion

Starting from quantum computational problems (determinism in one-way quantum computation), interesting graph problems have arisen like the graph flow dependence of the depth of correcting strategies for measurement-based quantum computation.

We have defined in this paper two algorithms for finding optimal causal flow and gflow. The key points are: the simplification of the structure of the flows considering only the maximally delayed flows which have a nice recursive structure; a backward analysis (start from the outputs) which allows to take advantage of this structure and avoids backtracking.

From a complexity point of view, an important question is: given a graph state and a fixed set of measurements (we relax the uniformity condition) what would be the depth of an optimal
correction strategy. One direction to answer this question would be to define a weaker flow that is still polynomially computable.

One can also consider the characterization and the depth of computation in more generalized measurement-based models where other planes of measurements are allowed.

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