Holographic study of entanglement and complexity for mixed states

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Abstract

In this paper, we holographically quantify the entanglement and complexity for mixed states by following the prescription of purification. The bulk theory we consider in this work is a hyperscaling violating solution, characterized by two parameters, hyperscaling violating exponent \( \theta \) and dynamical exponent \( z \). This geometry is dual to a non-relativistic strongly coupled theory with hidden Fermi surfaces. For entanglement, we compute the holographic analogy of entanglement of purification (EoP), denoted as the minimal area of the entanglement wedge cross section and observe the effects of \( z \) and \( \theta \). On the other hand, in order to probe the mixed state complexity we compute the mutual complexity (\( \Delta C \)) for the BTZ black hole and the hyperscaling violating geometry by incorporating the holographic subregion complexity conjecture. Furthermore, various aspects of holographic entanglement entropy such as entanglement Smarr relation and Fisher information metric has also been studied.

1 Introduction

The gauge/gravity duality [1–3] has been employed to holographically compute quantum information theoretic quantities and has thereby helped us to understand the bulk-boundary relations. Among various observables of quantum information theory, entanglement entropy (EE) has been the most fundamental thing to study as it measures the correlation between two subsystems for a pure state. EE has a very simple definition yet sometimes it is notoriously difficult to compute. However, the holographic computation of entanglement entropy which can be denoted as the Ryu-Takayanagi (RT) prescription, is a remarkably simple technique which relates the area of a codimension-2 static minimal surface with the entanglement entropy of a subsystem [4–6]. The RT-prescription along with its modification for time-dependent scenario (HRT prescription [7]) has been playing a key role for holographic studies of information theoretic quantities as the perturbative calculations which could not be done on the field theoretic side due to its strongly coupled nature, can now be performed in the bulk side since it is of weakly coupled nature.

Another important information theoretic quantity which has gained much attention recently is the computational complexity. The complexity of a quantum state represents the minimum number of simple operations which takes the unentangled product state to a target state [8–10]. There are several proposals to compute complexity holographically. Recently, several interesting attempts has been made to define complexity in QFT [11–13]. In context of holographic computation, initially, it was suggested that the complexity of a state (measured in gates) is proportional to the volume of the Einstein-Rosen bridge (ERB) which connects two boundaries of an eternal black hole [14,15]

\[
C_V(t_L, t_R) = \frac{V_{ERB}(t_L, t_R)}{8\pi RG_{d+1}} \tag{1}
\]
where $R$ is the AdS radius and $V_{ERB}(t_L, t_R)$ is the co-dimension one extremal volume of ERB which is bounded by the two spatial slices at times $t_L$ and $t_R$ of two CFTs that live on the two boundaries of the eternal black hole. Another conjecture states that complexity can be obtained from the bulk action evaluated on the Wheeler-DeWitt patch [16–18]

$$C_A = \frac{I_{WDW}}{\pi \hbar} \text{.}$$

(2)

The above two conjectures depend on the whole state of the physical system at the boundary. In addition to these proposals, there is another conjecture which depends on the reduced state of the system. This states that the co-dimension one volume enclosed by the co-dimension two extremal RT surface is proportional to the complexity

$$C_V = \frac{V(\Gamma_{\min})}{8\pi R \mathcal{G}_{d+1}} \text{.}$$

(3)

This proposal is known as the holographic subregion complexity (HSC) conjecture in the literature [19–21]. Recently, in [22], it was shown that there exists a relation between the universal pieces of HEE and HSC. Furthermore, the universal piece of HSC is proportional to the sphere free energy $F_S$ for even dimensional dual CFTs and proportional to the Weyl a-anomaly for odd dimensional dual CFTs.

In recent times, much attention is being paid to the study of entanglement entropy and complexity for mixed states. Some recent observations in this direction can be found in [26–34]. On the other hand, recently the study of complexity for mixed states has gained appreciable amount of attention. Similar to the case of EE for mixed state, the concept of ‘purification’ is also being employed in this context [35,36]. A subsequence of EoP. Consider a density matrix $\rho_{AB}$ corresponding to mixed state in Hilbert space $\mathcal{H}$, where $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. Now the process of purification states that one can construct a pure state $|\psi\rangle$ from $\rho_{AB}$ by adding auxiliary degrees of freedom to the Hilbert space $\mathcal{H}$

$$\rho_{AB} = tr_{A'B'}|\psi\rangle \langle \psi|; \quad \psi \in \mathcal{H}_{AA'}BB' = \mathcal{H}_{AA'} \otimes \mathcal{H}_{BB'} \text{.}$$

(4)

Such states $\psi$ are called purifications of $\rho_{AB}$. It is to be noted that the process of purification is not unique and different procedures for purification for the same mixed state exists. In this set up, the definition of EoP ($E_P$) reads [23]

$$E_P(\rho_{AB}) = \min_{tr_{A'B'}|\psi\rangle \langle \psi|} S(\rho_{AA'}); \quad \rho_{AA'} = tr_{BB'}|\psi\rangle \langle \psi| \text{.}$$

(5)

In the above expression, the minimization is taken over any state $\psi$ satisfying the condition $\rho_{AB} = tr_{A'B'}|\psi\rangle \langle \psi|$, where $A'B'$ are arbitrary. In this paper we will compute the holographic analog of EoP, given by the minimal area of entanglement wedge cross section (EWCS) $E_W$ [25]. However, there is no direct proof of $E_P = E_W$ duality conjecture yet and it is mainly based on the following properties of $E_P$ which are also satisfied by $E_W$ [23,25]. These properties are as follows.

(i) $E_P(\rho_{AB}) = S(\rho_A) = S(\rho_B)$; $\rho_{AA'} = tr_{BB'}|\psi\rangle \langle \psi|$

(ii) $\frac{1}{2} I(A : B) \leq E_P(\rho_{AB}) \leq \min \{S(\rho_A), S(\rho_B)\}$

In the above properties, $I(A : B) = S(A) + S(B) - S(A \cup B)$ is the mutual information between two subsystems $A$ and $B$. Some recent observations in this direction can be found in [26–34]. On the other hand, recently the study of complexity for mixed states has gained appreciable amount of attention. Similar to the case of EE for mixed state, the concept of ‘purification’ is also being employed in this context [35,36]. The purification complexity is defined as the minimal pure state complexity among all possible purifications available for a mixed state. Preparing a mixed state on some Hilbert space $\mathcal{H}$, starting from a reference (pure) state involves the extension of the Hilbert space $\mathcal{H}$ by introducing auxiliary degrees of freedom [37,38].

In this set up, a quantity denoted as the mutual complexity $\Delta C$ has been prescribed in order to probe the concept of purification complexity. The mutual complexity $\Delta C$ satisfies the following definition

$$\Delta C = C(\rho_A) + C(\rho_B) - C(\rho_{A\cup B}) \text{.}$$

(6)
In this paper, we will incorporate the HSC conjecture in order to compute the quantities $C(\rho_A)$, $C(\rho_B)$ and $C(\rho_{A\cup B})$.

The paper is organized as follows. In Section 2, we briefly discuss the aspects of the bulk theory which in this case is a hyperscaling violating geometry. We then consider a single strip-like subsystem and holographically compute the EE in Section 3. We also make comments on the thermodynamical aspects of the computed HEE by computing the entanglement Smarr relation satisfied by the HEE. Furthermore, we holographically compute the relative entropy in order to obtain the Fisher information metric. In Section 4, we consider two strip-like subsystems and holographically compute the EoP by using the $E_P = E_W$ conjecture. We briefly study the temperature dependent behaviour of $E_WCS$ along with the effects of $z$ and $\theta$ on the $E_W$. We then compute the HSC corresponding to a single strip-like subsystem in Section 5. In Section 6, we holographically compute the mutual complexity $\Delta C$ by incorporating the HSC conjecture for the BTZ black hole and the hyperscaling violating geometry. We then conclude in Section 7. We also have an Appendix in the paper.

2 Bulk theory: Hyperscaling violating geometry

We shall start our analysis with a bulk hyperscaling violating spacetime geometry. The solution corresponds to the following effective action of Einstein-Maxwell-scalar theory

$$ S_{\text{bulk}} = \frac{1}{16\pi G} \int d^{d+1}x \sqrt{-g} \left[ (R - 2\Lambda) - W(\phi) F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \partial_{\mu}\phi \partial^{\mu}\phi - V(\phi) \right] $$

where $F_{\mu\nu}$ is the Faraday tensor associated with the gauge field $A_\mu$, $\phi$ is the scalar field associated with the potential $V(\phi)$ and $W(\phi)$ is the coupling. Extremization of this action leads to the following black hole solution

$$ ds^2 = \frac{R^2}{r^2} \left[ -f(r) \frac{dr^2}{f(r)} + r^{2d-9} \frac{d\tau^2}{f(r)} + \sum_{i=1}^{d-1} dx_i^2 \right] . $$

The lapse function $f(r)$ has the form

$$ f(r) = 1 - \left( \frac{r}{r_H} \right)^{d-1}(1+\frac{\theta}{d-1}) $$

where $r_H$ is the event horizon of the black hole. The Hawking temperature of black hole is obtained to be

$$ T_H = \frac{(d-1)(z+d-\theta-1)}{4\pi(d-\theta-1)} \frac{1}{r_H^{z(d-1)/(d-\theta-1)}} . $$

The above mentioned metric is holographic dual to a $d$-dimensional non-relativistic strongly coupled theory with Fermi surfaces. The metric is associated with two independent exponents $z$ and $\theta$. The presence of these two exponents leads to the following scale transformations

$$ x_i \rightarrow \xi x_i \quad t \rightarrow \xi^z t \quad ds \rightarrow \xi^{\frac{z}{d-1}} ds . $$

This non-trivial scale transformation of the proper spacetime interval $ds$ is quite different from the usual AdS/CFT picture. The non-invariance of $ds$ in the bulk theory implies violations of hyperscaling in the boundary theory. Keeping this in mind, $\theta$ is identified as the hyperscaling violation exponent and $z$ is identified as the dynamical exponent. In the limit $z = 1, \theta = 0$, we recover the SAdS$_{d+1}$ solution which is dual to a relativistic CFT in $d$-dimensions and in the limit $z \neq 1, \theta = 0$, we obtain the ‘Lifshitz solutions’. The two independent exponents $z$ and $\theta$ satisfy the following inequalities

$$ \theta \leq d - 2, \quad z \geq 1 + \frac{\theta}{d-1} . $$
The ‘equalities’ of the above mentioned relations holds only for gauge theories of non-Fermi liquid states in $d = 3$. In this case, $\theta = 1$ and $z = 3/2$. For general $\theta = d - 2$, logarithmic violation of the ‘Area law’ of entanglement entropy is observed. This in turn means for $\theta = d - 2$, the bulk theory holographically describes a strongly coupled dual theory with hidden Fermi surfaces.

### 3 Holographic entanglement Smarr relation

To begin our analysis, we consider our subsystem $A$ to be a strip of volume $V_{sub} = L^{d-2}l$, where $-\frac{1}{2} < x_1 < \frac{1}{2}$ and $-\frac{1}{2} < x_{2,3,...,d-1} < \frac{1}{2}$. The amount of entanglement entropy captured by the above mentioned volume reads

$$S_{BH} = \frac{L^{d-2}l}{4G_{d+1}r_h^{d-1}}. \quad (11)$$

It is to be noted that the thermal entropy of the dual field theory is related with the temperature as $S_{th} \propto T^{d-1}\delta$. For $\theta = d - 2$, it reads $S_{th} \propto T^\frac{d}{2}$. This result is observed for compressible states with fermionic excitations.

We parametrize the co-dimension one static minimal surface as $x_1 = x_1(r)$ which leads to the following area of the extremal surface $G_{\text{min}}$

$$A(G_{\text{min}}) = 2R^{d-1}L^{d-2}r_t \left(\frac{p}{\sqrt{1-\theta}}\right)^{-p} \sum_{n=0}^\infty \frac{\Gamma(n + \frac{1}{2}) \alpha^n (1 + \frac{p}{\sqrt{1-\theta}})}{\Gamma(n + 1)} \left(\frac{u^{\frac{p}{\sqrt{1-\theta}}-p+np(1+\frac{p}{\sqrt{1-\theta}})}}{\sqrt{1-u^{2p}}}\right); \quad u = \frac{r}{r_t}, \quad \alpha = \frac{r_t}{r_h} \quad (12)$$

where $r_t$ is the turning point and $p = (d - 1)$ which we have introduced for the sake of simplicity. By substituting the area functional (given in eq. (12)) in the RT formula, we obtain the HEE

$$S_E = \frac{A(G_{\text{min}})}{4G_{d+1}}$$

$$= \frac{2L^{d-2}}{4G_{d+1} \left(\frac{p}{\sqrt{1-\theta}} - p\right)} \left(\frac{1}{\epsilon}\right)^{-p} \left(\frac{p}{\sqrt{1-\theta}}\right)^{-p} + \frac{L^{d-2}r_t}{4G_{d+1}} \left(\frac{p}{\sqrt{1-\theta}}\right)^{-p} \sum_{n=0}^\infty \frac{\Gamma(n + \frac{1}{2}) \alpha^n (1 + \frac{p}{\sqrt{1-\theta}})}{\Gamma(n + 1)} \left(\frac{\frac{p}{\sqrt{1-\theta}}+p+np(1+\frac{p}{\sqrt{1-\theta}})}{2p}\right) \frac{\Gamma \left(\frac{\frac{p}{\sqrt{1-\theta}}+p+np(1+\frac{p}{\sqrt{1-\theta}})}{2p}\right)}{\Gamma \left(\frac{\frac{p}{\sqrt{1-\theta}}+2p+2np(1+\frac{p}{\sqrt{1-\theta}})}{2p}\right)}. \quad (13)$$

The relationship between the subsystem size $l$ and turning point $r_t$ reads (with the AdS radius $R = 1$)

$$l = r_t \left(\frac{p}{\sqrt{1-\theta}}\right)^{-p} \left(\frac{1}{\epsilon}\right)^{-p} \left(\frac{p}{\sqrt{1-\theta}}\right)^{-p} \sum_{n=0}^\infty \frac{\Gamma(n + \frac{1}{2}) \alpha^n (1 + \frac{p}{\sqrt{1-\theta}})}{\Gamma(n + 1)} \left(\frac{\frac{p}{\sqrt{1-\theta}}+p+np(1+\frac{p}{\sqrt{1-\theta}})}{2p}\right) \frac{\Gamma \left(\frac{\frac{p}{\sqrt{1-\theta}}+p+np(1+\frac{p}{\sqrt{1-\theta}})}{2p}\right)}{\Gamma \left(\frac{\frac{p}{\sqrt{1-\theta}}+2p+2np(1+\frac{p}{\sqrt{1-\theta}})}{2p}\right)}. \quad (14)$$

We now proceed to probe the thermodynamical aspects of HEE. It can be observed from eq. (13) that the expression of $S_E$ contains a subsystem independent divergent piece which we intend to get rid by defining a finite quantity. We call this finite quantity as the renormalized holographic entanglement entropy ($S_{REE}$). From the point of view of the dual field theory this divergence free quantity represents the change in entanglement entropy under an excitation. In order to obtain $S_{REE}$ holographically, firstly we need to compute the HEE corresponding to the asymptotic form ($r_h \rightarrow \infty$) of the hyperscaling violating black brane solution given in eq. (8). This yields the following expression

$$S_G = \frac{2L^{d-2}}{4G_{d+1} \left(\frac{p}{\sqrt{1-\theta}} - p\right)} \left(\frac{1}{\epsilon}\right)^{-p} \left(\frac{p}{\sqrt{1-\theta}}\right)^{-p} \left(\frac{p}{\sqrt{1-\theta}}\right)^{-p} \left(\frac{\frac{p}{\sqrt{1-\theta}}+p+np(1+\frac{p}{\sqrt{1-\theta}})}{2p}\right) \frac{\Gamma \left(\frac{\frac{p}{\sqrt{1-\theta}}+p+np(1+\frac{p}{\sqrt{1-\theta}})}{2p}\right)}{\Gamma \left(\frac{\frac{p}{\sqrt{1-\theta}}+2p+2np(1+\frac{p}{\sqrt{1-\theta}})}{2p}\right)}. \quad (15)$$

\(^1\)The thermal entropy of the dual field theory is basically the Hawking entropy of the black hole given in eq. (9).
We now subtract the above expression (which represents the HEE corresponding to the vacuum of the dual field theory) from $S_E$ (given in eq. (13)) in order to get a finite quantity $S_{REE}$. This can be formally represented as

$$S_{REE} = S_E - S_G.$$  \hspace{1cm} (16)

On the other hand, the internal energy $E$ of the black hole can be obtained by using the Hawking entropy (given in eq. (11)) and the Hawking temperature (given in eq. (9)). The computed expression of $E$ can be represented as

$$E = \left(\frac{p - \theta}{z + p - \theta}\right) S_{BH}T_H.$$  \hspace{1cm} (17)

This is nothing but the classical Smarr relation of BH thermodynamics. In [43], it was shown that the quantity $S_{REE}$ and the internal energy $E$ satisfies a Smarr-like thermodynamic relation corresponding to a generalized temperature $T_g$. In this set up, this relation reads [44]

$$E = \left(\frac{p - \theta}{z + p - \theta}\right) S_{REE}T_g.$$  \hspace{1cm} (18)

It is remarkable to observe that the relation given in eq. (18) has a striking similarity with the classical Smarr relation of BH thermodynamics, given in eq. (17). In the limit $r_l \rightarrow r_h$, the leading term of the generalized temperature $T_g$ produces the exact Hawking temperature $T_H$ whereas in the limit $r_l \ll r_h$, the leading term of $T_g$ reads [44]

$$\frac{1}{T_g} = \Delta_1 l^z$$  \hspace{1cm} (19)

where the detailed expression of $\Delta_1$ reads

$$\Delta_1 = 2\pi^{3/2} \left(\frac{1}{p - z} - p + p \left(1 + \frac{z}{p - z}\right)\right) \left(\frac{p}{\sqrt{\pi}} \frac{\Gamma\left(\frac{p + p}{2p}\right)}{\Gamma\left(\frac{2p}{2p}\right)}\right)^{1+z} \left(\frac{\Gamma\left(\frac{p + p + p(1 + \frac{z}{p - z})}{2p}\right)}{\Gamma\left(\frac{2p + p + p(1 + \frac{z}{p - z})}{2p}\right)}\right).$$

From eq. (19) it can be observed that in the UV limit, $T_g$ shows the similar behaviour as entanglement temperature $T_{ent}$ (proportional to the inverse of subsystem size $l$).

### 3.1 Relative entropy and the Fisher information metric

We now proceed to compute the Fisher information metric for the hyperscaling violating geometry using the holographic proposal. The Fisher information metric measures the distance between two quantum states and is given by [45]

$$G_{F,\lambda\lambda} = \langle \delta\rho \delta\rho \rangle^{(\sigma)}_{\lambda\lambda} = \frac{1}{2} Tr \left( \delta\rho \frac{d}{d\lambda} \log(\sigma + \delta\lambda \delta\rho) \bigg|_{\delta\lambda=0} \right)$$  \hspace{1cm} (20)

where $\sigma$ is the density matrix and $\delta\rho$ is a small deviation from the density matrix. On the other hand, there exists a relation between the Fisher information metric and the relative entropy $S_{rel}$ [46]. This reads

$$G_{F,mm} = \frac{\partial^2}{\partial m^2}S_{rel}(\rho_m||\rho_0); \quad S_{rel}(\rho_m||\rho_0) = \Delta \langle H_{\rho_0} \rangle - \Delta S.$$  \hspace{1cm} (21)

In the above expression, $\Delta S$ is the change in the entanglement entropy from vacuum state, $\Delta \langle H_{\rho_0} \rangle$ is the change in the modular Hamiltonian and $m$ is the perturbation parameter. In this set up, we holographically compute the relative entropy $S_{rel}(\rho_m||\rho_0)$. We consider the background is slightly perturbed from pure
hyperscaling violating spacetime while the subsystem volume $L^{d-2}l$ is fixed. Then the inverse of the lapse function $f(r)$ (given in eq. (23)) can be expressed as

$$\frac{1}{f(r)} = 1 + mr^{p(1+\frac{1}{p-\theta})} + m^2 r^{2p(1+\frac{1}{p-\theta})}$$

(22)

where $m = \left(\frac{1}{\sqrt{r}}\right)^{p(1+\frac{1}{p-\theta})}$ is the holographic perturbation parameter. Since we consider a perturbation to the background geometry and also consider that the subsystem size $l$ has not changed, we can express the turning point in the following perturbed form

$$r_l = r_l(0) + m r_l(1) + m^2 r_l(2)$$

(23)

where $r_l(0)$ is the turning point for the pure hyperscaling violating geometry and $r_l(1), r_l(2)$ are the first and second order corrections to the turning point. We now write down the subsystem length $l$ up to second order in perturbation as

$$\frac{l}{r_l^{p-\theta}} = a_0 + m a_1 r_l + m^2 a_2 r_l^{2p(1+\frac{1}{p-\theta})}$$

(24)

where

$$a_0 = \frac{\sqrt{\pi}}{p} \frac{\Gamma\left(\frac{p-\theta}{2p}\right)}{\Gamma\left(\frac{p-\theta}{2p}\right)}, \quad a_1 = \frac{\sqrt{\pi}}{2p} \frac{\Gamma\left(\frac{p-\theta}{2p} + p(1+\frac{1}{p-\theta})\right)}{\Gamma\left(\frac{p-\theta}{2p} + p(1+\frac{1}{p-\theta})\right)}, \quad a_2 = \frac{3\sqrt{\pi}}{8p} \frac{\Gamma\left(\frac{p-\theta}{2p} + 2p(1+\frac{1}{p-\theta})\right)}{\Gamma\left(\frac{p-\theta}{2p} + 2p(1+\frac{1}{p-\theta})\right)}.$$

Using eq. (23) in eq. (21) and keeping in mind the consideration that $l$ has not changed, we obtain the forms of $r_l(0), r_l(1)$ and $r_l(2)$

$$r_l(0) = \left(\frac{l}{a_0}\right)^{\frac{p-\theta}{2p}}, \quad r_l(1) = -\left(\frac{2p-\theta}{2p}\right) \left(\frac{a_1}{a_0}\right) r_l(0)^{1+p(1+\frac{1}{p-\theta})}, \quad r_l(2) = \xi r_l(0)^{1+2p(1+\frac{1}{p-\theta})}$$

(25)

where

$$\xi = \left(\frac{p-\theta}{2p}\right) \left[\left(\frac{2p-\theta}{2p}\right) \left(\frac{a_1}{a_0}\right)^2 + \left(1 + \frac{2}{p-\theta}\right) \left(\frac{a_1}{a_0}\right)^2 - \left(\frac{a_2}{a_0}\right)^2\right].$$

(26)

On a similar note, the expression for area of the static minimal surface up to second order in perturbation parameter $m$ can be obtained from eq. (22). We then use eq. (23) to recast the expression for the area of the minimal surface in the form

$$A\left(\Gamma_A^{\text{min}}\right) = A\left(\Gamma_A^{\text{min}}\right)(0) + m A\left(\Gamma_A^{\text{min}}\right)(1) + m^2 A\left(\Gamma_A^{\text{min}}\right)(2).$$

(27)

It has been observed that at first order in $m$, $S_{rel}$ vanishes [46] and in second order in $m$ it reads $S_{rel} = -\Delta S$. In this set up, it yields

$$S_{rel} = -m^2 A\left(\Gamma_A^{\text{min}}\right)(2) = m^2 \frac{L^{d-2}}{4G_{d+1}} \Delta_2 \left(\frac{l}{a_0}\right)^{1+2z+(p-\theta)}$$

(28)

where

$$\Delta_2 = 2p \left(\frac{p-\theta}{p}\right) \left(\frac{a_0^2}{a_0^2}\right) + p \left(2p-\theta\right) \left(\frac{a_0^2}{a_0^2}\right) - \left(\frac{2pa_2}{2p-\theta} - p + \frac{p-\theta}{p-\theta}\right) - 2pa_0 \xi.$$ 

By substituting the above expression in eq. (21), the Fisher information is obtained to be

$$G_{F,mm} = \frac{L^{d-2}}{2G_{d+1}} \Delta_2 \left(\frac{l}{a_0}\right)^{1+2z+(p-\theta)} \propto l^{d+2z-\theta}$$

(29)

In the limit $z = 1$ and $\theta = 0$, the above equation reads $G_{F,mm} \propto l^{d+2}$ which agrees with the result obtained in [47]. The Fisher information corresponding to the Lifshitz type solutions can be found in [48].
4 Entanglement wedge cross-section and the $E_P = E_W$ duality

We now proceed to compute the holographic entanglement of purification by considering two subsystems, namely, $A$ and $B$ of length $l$ on the boundary $\partial M$. From the bulk point of view, $\partial M$ is the boundary of a canonical time-slice $M$ made in the static gravity dual. Furthermore, $A$ and $B$ are separated by a distance $D$ so that the subsystems does not have an overlap of non-zero size ($A \cap B = 0$). Following the RT prescription, we denote $\Gamma_A^{\min}$, $\Gamma_B^{\min}$ and $\Gamma_{AB}^{\min}$ as the static minimal surfaces corresponding to $A$, $B$ and $AB$ respectively. In this set up, the domain of entanglement wedge $M_{AB}$ is the region in the bulk with the following boundary

$$\partial M_{AB} = A \cup B \cup \Gamma_{AB}^{\min}.$$  \hfill (30)

It is also to be noted that if the separation $D$ is effectively large then the codimension-0 bulk region $M_{AB}$ will be disconnected. We now divide $\Gamma_{AB}^{\min}$ into two parts

$$\Gamma_{AB}^{\min} = \Gamma_A \cup \Gamma_B$$ \hfill (31)

such that the boundary $\partial M_{AB}$ of the canonical time-slice of the full spacetime $M_{AB}$ can be represented as

$$\partial M_{AB} = \Gamma_A \cup \Gamma_B$$ \hfill (32)

where $\Gamma_A = A \cup \Gamma_{AB}^A$ and $\Gamma_B = B \cup \Gamma_{AB}^B$. In this set up, it is now possible to define the holographic entanglement entropies $S(\rho_{A,\Gamma_A})$ and $S(\rho_{B,\Gamma_B})$. These quantities can be computed by finding a static minimal surface $\Sigma_{AB}^{\min}$ such that

$$\partial \Sigma_{AB}^{\min} = \partial \Gamma_A = \partial \Gamma_B.$$ \hfill (33)

There can be infinite number possible choices for the splitting given in eq.\,(31) and this in turn means there can be infinite number of choices for the surface $\Sigma_{AB}^{\min}$. The entanglement wedge cross section (EWCS) is obtained by minimizing the area of $\Sigma_{AB}^{\min}$ over all possible choices for $\Sigma_{AB}^{\min}$. This can be formally written down as

$$E_W(\rho_{AB}) = \min_{\Gamma_A \subset \partial M_{AB}} \left[ \frac{A(\Sigma_{AB}^{\min})}{4G_{d+1}} \right].$$ \hfill (34)

We now proceed to compute $E_W$ for the holographic dual considered in this paper. As we have mentioned earlier, EWCS is the surface with minimal area which splits the entanglement wedge into two domains corresponding to $A$ and $B$. This can be identified as a vertical, constant $x$ hypersurface. The time induced metric on this constant $x$ hypersurface reads

$$ds^2_{ind} = \frac{R^2}{r^2} \left[ r \frac{dr}{f(r)} + \sum_{i=1}^{d-2} dx_i^2 \right].$$ \hfill (35)

By using this above mentioned induced metric, the EWCS is obtained to be

$$E_W = \frac{L^{d-2}}{4G_{d+1}} \int_{r(D)}^{r_h(D)} \frac{dr}{\sqrt{f(r)}}.$$  

$$= \frac{L^{d-2}}{4G_{d+1}} \sum_{n=0}^{\infty} \frac{1}{\sqrt{\Gamma(n+1)}} \left[ \frac{r_h(2l+D)^{np(-\frac{\theta}{D})-p+1} - r_h(D)^{np(-\frac{\theta}{D})-p+1}}{n^{p(1+\frac{\theta}{D})-p+1}} \right] \left( \frac{1}{r_h} \right)^{np(1+\frac{\theta}{D})}. $$ 

As mentioned earlier, the above expression of $E_W$ always maintains the following bound

$$E_W \geq \frac{1}{2} I(A : B); \ I(A : B) = S(A) + S(B) - S(A \cup B)$$ \hfill (36)

where $I(A : B)$ is the mutual information between two subsystems $A$ and $B$. On the other hand, in \cite{49} it was shown that there exists a critical separation between $A$ and $B$ beyond which there is no connected phase.
for any $l$. This in turn means that at the critical separation length $D_c$, $E_W$ probes the phase transition of the RT surface $\Gamma_{AB}^{\text{min}}$ between connected and disconnected phases. The disconnected phase is characterised by the fact that the mutual information $I(A : B)$ vanishes, which in this case reads

$$2S(l) - S(D) - S(2l + D) = 0 .$$

(37)

The above condition together with the bound given in eq. (36) leads to the critical separation length $D_c$.

We now write down the expression for $E_W$ (given in eq. (36)) in terms of the parameters of the boundary theory. This we do for the small temperature case ($\frac{r(D)}{r_h} \ll \frac{r(2l + D)}{r_h} \approx 1$) and for the high temperature case ($\frac{r(D)}{r_h} \ll 1$, $\frac{r(2l + D)}{r_h} \approx 1$).

4.1 $E_W$ in the low temperature limit

In the limit $\frac{r(D)}{r_h} \ll \frac{r(2l + D)}{r_h} \ll 1$, it is reasonable to consider terms up to order $m$ (where $m = \frac{1}{r_h + \frac{p}{p - \theta}}$) in the expression for $E_W$ (given in eq. (36)). On the other hand for low temperature considerations, it is possible to perturbatively solve eq. (14) which leads to the following relationship between a subsystem size $l$ and its corresponding turning point $r_t$

$$r_t(l) = l^{\frac{p - \theta}{2}} \left( \frac{\Gamma \left( \frac{2p + p - \theta}{2p} \right)}{\sqrt{\frac{\pi}{p}} \Gamma \left( \frac{p + \frac{p - \theta}{2p}}{2p} \right)} \right)^{\frac{p - \theta}{p}} \times \left[ 1 - \Delta_3 l^{(p - \theta) \left( 1 + \frac{\theta}{p - \theta} \right)} T^{1 + \frac{\theta}{p - \theta}} \right]$$

(38)

where

$$\Delta_3 = \left( \frac{p - \theta}{2p} \right) \left( \frac{4\pi}{p(1 + \frac{\theta}{p - \theta})} \right)^{1 + \frac{p - \theta}{2p}} \Gamma \left( \frac{2p + p - \theta}{2p} \right) \left( \frac{\Gamma \left( \frac{p + \frac{p - \theta}{2p}}{2p} \right)}{\Gamma \left( \frac{2p + p - \theta}{2p} \right)} \right)^{\frac{p}{2}} \Gamma \left( \frac{2p + p - \theta}{2p} \right)^{\frac{p}{2}} \Gamma \left( \frac{p + \frac{p - \theta}{2p}}{2p} \right)^{-\frac{p}{2}} \Gamma \left( \frac{2p + p - \theta}{2p} \right)^{-\frac{p}{2}} \Gamma \left( \frac{p + \frac{p - \theta}{2p}}{2p} \right)^{\frac{p}{2}}$$

Now by using eq. (38) for $r_t(2l + D)$ and $r_t(D)$, we obtain the expression for $E_W$ in the low temperature limit to be

$$E_W = E_W^{T = 0} - \frac{L^{d-2}}{4G_{d+1}} \Delta_4 \left[ (2l + D) \left( \frac{\theta}{p - \theta} \right)^{1 + \frac{\theta}{p - \theta}} \right] - \left[ D \left( \frac{\theta}{p - \theta} \right)^{1 + \frac{\theta}{p - \theta}} \right] T^{1 + \frac{\theta}{p - \theta}} + \ldots$$

(39)

where the detailed expression for $\Delta_4$ is given in the Appendix. The first term in eq. (39) is the EWCS at $T = 0$. This reads

$$E_W^{T = 0} = \frac{L^{d-2}}{4(p - 1)G_{d+1}} \left( \frac{\Gamma \left( \frac{2p + \frac{p - \theta}{2p}}{2p} \right)}{\Gamma \left( \frac{2p + \frac{p - \theta}{2p}}{2p} \right)} \right)^{(p - \theta) - \left( \frac{p - \theta}{p} \right)} \left[ \left( \frac{1}{D} \right)^{(p - \theta) - \left( \frac{p - \theta}{p} \right)} - \left( \frac{1}{2l + D} \right)^{(p - \theta) - \left( \frac{p - \theta}{p} \right)} \right].$$

(40)

It can be observed from eq. (39) that the EWCS is a monotonically decreasing function of temperature $T$ (as $1 + \frac{p - \theta}{p} > 0$). We now compute the critical separation length $D_c$, at which $E_W$ probes the phase transition of the RT surface $\Gamma_{AB}^{\text{min}}$ between the connected and disconnected phases. This to be obtained from the condition

$$2S(l) - S(D) - S(2l + D) = 0 .$$

(41)
The general expression for the HEE of a strip of length \( l \) is given in eq.\([13]\). Now similar to the above computation, in the limit \( r_l(2l + D) \ll \frac{r_l(D)}{r_h} \ll 1 \), we consider terms up to \( \mathcal{O}(m) \) in eq.\([13]\). By using this consideration, eq.\([13]\) can be expressed as

\[
\beta_1 \left( \frac{2}{l^{p-\theta-1}} - \frac{1}{D^{p-\theta-1}} - \frac{1}{(2l + D)^{p-\theta-1}} \right) + \frac{\beta_2}{r_h^{p(1+\frac{p-\theta}{2p})}} (2l^{1+z} - D^{1+z} - (2l + D)^{1+z}) = 0
\]  

(42)

where

\[
\beta_1 = \frac{\sqrt{\pi}}{4p} \frac{\Gamma \left( \frac{p+\theta}{2p} \right)}{\Gamma \left( \frac{p}{2p} \right)} \left( \frac{\sqrt{\pi}}{p} \frac{\Gamma \left( \frac{p+\theta}{2p} \right)}{\Gamma \left( \frac{2p+\theta}{2p} \right)} \right)^{-1+z}
\]

\[
\beta_2 = \Delta_3 \left( p - \frac{p}{p-\theta} \right) \beta_1 + \frac{\sqrt{\pi}}{8p} \left( \frac{p}{\sqrt{\pi}} \frac{\Gamma \left( \frac{p+\theta}{2p} \right)}{\Gamma \left( \frac{2p+\theta}{2p} \right)} \right)^{1+z} \left( \frac{\Gamma \left( \frac{p+\theta}{2p} + p(1+\frac{p-\theta}{2p}) \right) \Gamma \left( \frac{2p+\theta}{2p} + p(1+\frac{p-\theta}{2p}) \right)}{\Gamma \left( \frac{2p+\theta}{2p} \right)} \right).
\]

By solving the above equation we can find out the critical separation length \( D_c \) (where we substitute \( \frac{D}{T} = k = \text{constant} \)). It is worth mentioning that in the above computations of \( E_W \) and \( I(A : B) \), we have considered terms up to \( \mathcal{O}(m) \) which is the leading order term for thermal correction. Similarly, one can incorporate next-to leading order terms or more to get a more accurate result.

![Effect of hyperscaling violating exponent \( \theta \) (we set \( z = 1 \))](image1)

![Effect of dynamical exponent \( z \) (we set \( \theta = 0.2 \))](image2)

Figure 1: Effects of \( \theta \) and \( z \) on \( E_W \) and \( I(A : B) \) at low temperature (with \( d = 3, k = 0.4, L = 1 \) and \( G_{d+1} = 1 \))

### 4.2 \( E_W \) in the high temperature limit

We now consider the limit \( r_l(2l + D) \rightarrow r_h \) and \( \frac{r_l(D)}{r_h} \ll 1 \). This in turn means the static minimal surface associated with the turning point \( r_l(2l + D) \) wraps a portion of the event horizon \( r_h \). In the large \( n \) limit, the infinite sum associated with the turning point \( r_l(2l + D) \) goes as \( \frac{1}{\sqrt{\pi}} \left( \frac{1}{n} \right)^{3/2} \left( \frac{r_l(2l + D)}{r_h} \right)^{n(1+\frac{p-\theta}{2p})} \) which means it is convergent \( (\sum_{n=1}^{\infty} \frac{1}{n^{1/2}} = \zeta(1/2)) \). Further, we are considering \( \frac{r_l(D)}{r_h} \ll 1 \) and in this limit it is reasonable to keep terms only up to order \( m \) in the infinite sum associated with \( r_l(D) \). In this set up, the expression for EWCS reads

\[
E_W(T) = E_W^{T=0} + \frac{L^{d-2}}{4G_{d+1}} \Delta_1 D^{z+(\frac{p-\theta}{p})} T^{1+(\frac{p-\theta}{p})} - \frac{L^{d-2}}{4G_{d+1}} \Delta_5 T^{(\frac{p-\theta}{p})-1} \left( \frac{p-\theta}{p} \right) + \ldots
\]  

(43)
where the temperature independent term (EWCS at $T = 0$) is being given by

$$E_W^{T=0} = \frac{L^{d-2}}{4(p-1)G_{d+1}} D \left( \frac{p-\theta}{2p} \right) \left( \frac{p+\theta}{2p} \right) \Gamma \left( \frac{2p+\theta}{2p} \right) \Gamma \left( \frac{p-\theta}{2p} \right) .$$

(44)

The expressions for $\Delta_4$ and $\Delta_5$ are given in the Appendix. Now we proceed to compute the critical separation length $D_c$ in the high temperature configuration. In the limit $r_l \to r_h$, the computed result of $S_E$ (HEE of a strip like subsystem with length $l$), given in eq. (13) can be rearranged in the following form

$$S_E = \frac{L^{d-2}}{4G_{d+1}} + \frac{L^{d-2}}{4G_{d+1}} \sum_{n=0}^{\infty} P_n, \quad P_n = \left( \frac{1}{\frac{p}{p-\theta} - p + np(1 + \frac{\theta}{p-\theta})} \right) \Gamma \left( n + \frac{1}{\theta} \right) \Gamma \left( n + 1 \right) \Gamma \left( \frac{2p+\theta}{2p} + np(1 + \frac{\theta}{p-\theta}) \right) .$$

(45)

By using the above form of HEE, we can write down eq.(11) in the following form

$$\frac{\sum_{n=0}^{\infty} P_n}{r_h} - \frac{D}{r_h^{p-\theta}} - \frac{\beta_1}{D^{p-\theta-1}} - \frac{\beta_2}{r_h^{p(1+\theta/\theta-1)}} D^{1+\theta} = 0 .$$

(46)

In Fig(s.) (1) and (2), we have graphically represented the effects of $z$ and $\theta$ on the EWCS and holographic mutual information (HMI) for both low and high temperature case respectively. For the low temperature case (Fig.1) we have chose the separation length $D$ between the subsystems to be $D = 0.4 l$. From the above plots it can be observed that the EWCS always maintains the bound $E_W > \frac{1}{2} I(A : B)$. The HMI continually decays and approaches zero at a particular critical separation length $D_c$. This critical separation $D_c$ decreases with increasing $z$ and $\theta$. On the other hand $E_W$ shows a discontinuous jump at $D_c$. Upto $D_c$, $E_W$ has a finite cross-section due to the connected phase whereas beyond this critical separation length $E_W$ vanishes due to the disconnected phase of the RT surface $\Gamma^{min}_{AB}$.

5 Holographic Subregion Complexity

In this section we study the holographic subregion complexity (HSC) proposal \[19]. The subregion complexity is proportional to the volume enclosed by the co-dimension one static minimal surface with the boundary

\[2\] The first term of the expression is nothing but the thermal entropy of the boundary subsystem given in eq. (11).
coinciding with that of the subsystem. In our case, this reads

\[ V(\Gamma_{A}^{\text{min}}) = 2L^{d-2}r_t^{-\frac{d-2}{d-2}} \int_{/r_t}^{1} du \frac{u^{\frac{20-p}{p-\rho}-p}}{\sqrt{f(u)}} \int_{u}^{1} dk \frac{k^{p+\frac{\theta}{p-\sigma}}}{\sqrt{f(k)\sqrt{1-k^{2p}}}} \]

\[ = \frac{L^{d-2}l}{(p-\frac{\theta}{p-\sigma})} \epsilon \left( p-\frac{\rho}{p-\sigma} \right) + L^{d-2}r_t^{-\frac{d-2}{d-2}} \epsilon \left( p-\frac{\rho}{p-\sigma} \right) \sum_{n=0}^{\infty} V_1 \left( \frac{r_t}{r_h} \right)^{(m+n)p\left(1+\frac{\rho}{p-\sigma}\right)} \]  

(47)

We now use the above volume to obtain the HSC. This is given by (setting AdS radius \( R = 1 \))

\[ C_V(A) \equiv \frac{V(\Gamma_{A}^{\text{min}})}{8\pi G_{d+1}} = \frac{L^{d-2}l}{8\pi (p-\frac{\theta}{p-\sigma}) G_{d+1}} \epsilon \left( p-\frac{\rho}{p-\sigma} \right) + \frac{L^{d-2}r_t^{-\frac{d-2}{d-2}} \epsilon \left( p-\frac{\rho}{p-\sigma} \right)}{8\pi G_{d+1}} \sum_{n=0}^{\infty} V_1 \left( \frac{r_t}{r_h} \right)^{(m+n)p\left(1+\frac{\rho}{p-\sigma}\right)} \]

(48)

where

\[ V_1 = \frac{\Gamma(n + \frac{1}{2})\Gamma(m + \frac{1}{2})\Gamma \left( \frac{(m+n)p(1+\frac{\rho}{p-\sigma}) + 1 + \frac{\rho}{p-\sigma}}{2p} \right)}{\sqrt{\pi}\Gamma(n + 1)\Gamma(m + 1)\Gamma \left( \frac{(m+n)p(1+\frac{\rho}{p-\sigma}) + 1 + \frac{\rho}{p-\sigma}}{2p} \right)} \left[ p \left( mp(1 + \frac{\rho}{p-\sigma}) + \frac{\rho}{p-\sigma} - p \right) \right] \]

6 Complexity for mixed states: Mutual complexity (\( \Delta C \))

The complexity for mixed states (purification complexity) is defined as the minimal (pure state) complexity among all possible purifications of the mixed state. This in turn means that one has to optimize over the circuits which take the reference state to a target state \( |\psi_{AB}\rangle \) (a purification of the desired mixed state \( \rho_A \)) and also need to optimize over the possible purifications of \( \rho_A \). This can be expressed as

\[ C(\rho_A) = \min_B C(|\psi_{AB}\rangle); \rho_A = \text{Tr}_B |\psi_{AB}\rangle \langle \psi_{AB}| \]

(49)

where \( A^c = B \). Recently a quantity denoted as the ‘mutual complexity (\( \Delta C \))’ has been defined in order to compute the above mentioned mixed state complexity. The computation of \( \Delta C \) starts with a pure state \( \rho_{AB} \) in an extended Hilbert space (including auxiliary degrees of freedom), then by tracing out the degrees of freedom of \( B \), one gets the mixed state \( \rho_A \). On the other hand, tracing out the degrees of freedom of \( A \) yields \( \rho_B \). These computed results then can then be used in the following formula to compute the mutual complexity \( \Delta C \)

\[ \Delta C = C(\rho_A) + C(\rho_B) - C(\rho_{A\cup B}) \]

(50)

The mutual complexity \( \Delta C \) is said to be subadditive if \( \Delta C > 0 \) and superadditive if \( \Delta C < 0 \).

We now choose to follow the subregion \( C = V \) conjecture to compute the quantities \( C(\rho_A) \), \( C(\rho_B) \) and \( C(\rho_{A\cup B}) \). Similarly one can follow the \( C = A \) conjecture or \( C = V2.0 \) conjecture \cite{50} to compute these quantities. We now consider that the boundary at constant t-slices is a collection of two adjacent intervals of length \( l \) (lengths corresponding to rest of the directions are fixed), namely, \( A \) and \( B \) with zero overlapping (\( A^c = B \) and \( A \cap B = 0 \)).

6.1 BTZ black hole

We first compute the HSC corresponding to a single subsystem \( A \) for the BTZ black hole. The mentioned black hole geometry is characterized by the following metric \cite{51,52}

\[ ds^2 = \frac{R^2}{z^2} \left[ -f(z)dt^2 + \frac{dz^2}{f(z)} + dx^2 \right]; \ f(z) = 1 - \frac{z^2}{z_h^2} \]

(51)
Following the prescription of $C = V$ conjecture, the HSC corresponding to a single subsystem $A$ of length $l$ in the dual field theory is obtained to be

$$C(\rho_A) = \frac{2R}{8\pi RG_{d+1}} \int_0^{z_l} \frac{x(z)}{z^2 \sqrt{f(z)}} dz = \frac{1}{8\pi} \left( \frac{l}{\epsilon} - \pi \right) \left( \text{setting } G_{d+1} = 1 \right)$$

(52)

where $\epsilon$ is the cut-off introduced to prevent the UV divergence. It is to be observed that the computed result of HSC in this case is independent of the black hole parameter (that is, event horizon $z_h$). This is a unique feature of AdS$_3$. We now consider two subsystems $A$ and $B$ of equal length $l$ in order to compute the mutual complexity $\Delta C$. This reads

$$\Delta C = C(\rho_A) + C(\rho_B) - C(\rho_{A\cup B}) = \frac{1}{8\pi} \left( \frac{l}{\epsilon} - \pi + \frac{l}{\epsilon} - \pi - \frac{2l}{\epsilon} + \pi \right) = -\frac{1}{8}.$$

(53)

It can be observed that the mutual complexity is less than zero that is the complexity is superadditive. This in turn means that the complexity of the state corresponding to the full system is greater than the sum of the complexities of the states in the two subsystems.

### 6.2 Hyperscaling violating geometry

We now proceed to compute the mixed state complexity for the hyperscaling violating geometry. By using the holographic prescription given in eq.(48), $\Delta C$ in this set up reads

$$\Delta C = \frac{1}{8\pi G_{d+1}} \left[ V(\Gamma_{A}^{\text{min}}) + V(\Gamma_{B}^{\text{min}}) - V(\Gamma_{A\cup B}^{\text{min}}) \right]$$

$$= \frac{L^{d-2}}{8\pi G_{d+1}} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} V_1 \left[ 2r_l(l) \left( \frac{r_l(l) + r_l(2l)}{r_h} \right)^{(m+n)p(1+\frac{p}{p+\theta})} - r_l(2l) \left( \frac{r_l(2l)}{r_h} \right)^{(m+n)p(1+\frac{p}{p+\theta})} \right].$$

(54)

It is to be noted that the mutual complexity $\Delta C$ in this set up is a divergence free finite quantity as the leading divergent piece $\approx \frac{L^{d-2}}{\epsilon^{\frac{p}{p+\theta}}}$ of HSC cancels out.

We now try to recast the above expression in terms of the field theoretic parameters. In order to execute this, we consider the limit $\frac{r_l(l)}{r_h} \ll \frac{r_l(2l)}{r_h} \ll 1$. In this limit, we keep terms up to order $O(m)$ in the expression of $\Delta C$. This in turn leads to the following expression

$$\Delta C = \frac{L^{d-2}}{8\pi G_{d+1}} \Delta_6 \left( \frac{r_l(l)}{r_h} \right)^{-(p-\theta)} - \frac{L^{d-2}}{8\pi G_{d+1}} \Delta_7 \left( \frac{r_l(2l)}{r_h} \right)^{1+(\frac{p}{p+\theta})},$$

(55)

where the expressions of $\Delta_6$ and $\Delta_7$ are given in the Appendix. We observe that similar to BTZ case, the above result for $\Delta C$ is less than zero. This in turn means that the mutual complexity computed using the HSC conjecture yields a superadditive result.

### 7 Conclusion

In this paper, we compute the entanglement entropy and complexity for mixed states by using the gauge/gravity correspondence. We start our analysis by considering a hyperscaling violating solution as the bulk theory. This geometry is associated with two parameters, namely, hyperscaling violating exponent $\theta$ and dynamical exponent $\theta$. It is dual to a non-relativistic, strongly coupled theory with hidden Fermi surfaces. We then consider a single strip-like subsystem in order to compute the HEE of this gravitational solution. We observe that the computed result of HEE along with the internal energy $E$, satisfies a Smarr-like thermodynamics relation associated with a generalized temperature $T_0$. This thermodynamic relation naturally emerges by
demanding that the generalized temperature $T_g$ reproduces the Hawking temperature $T_H$ as the leading term in the IR ($r_t \to r_h$) limit. In UV limit ($\frac{r_t}{r_h} \ll 1$), it is found that $T_g \propto \frac{1}{T}$, that is, $T_g$ is inversely proportional to subsystem size $l$. This behaviour is compatible with the definition of entanglement temperature given in the literature. We then holographically compute the relative entropy $S_{rel}$, by incorporating the perturbative approach. Using this the Fisher information metric is computed. We find that in this case the power of $l$ carries both the exponents $\epsilon$ and $\theta$. We then consider two strip-like subsystems $A$ and $B$ separated by a length $D$, in order to compute the EWCS ($E_W$) which is the holographic analogy of EoP. We compute $E_W$ for both low and high temperature conditions. In both cases, there is a temperature independent term (denoted as $E_W^{(0)}$) which is independent of the hyperscaling violating exponent $\epsilon$ but depends on the dynamical exponent $\theta$. On the other hand for a large enough value of $D$ (critical separation length $D_c$), the RT surface $\Gamma_{minAB}^{min}$ becomes disconnected and $E_W$ should vanish. This in turn means that $E_W$ probes the phase transition between the connected and disconnected phases of the RT surface $\Gamma_{minAB}^{min}$. We evaluate these critical separation point $D_c$ by using the property that at $D_c$ the mutual information between $A$ and $B$ becomes zero as they become disconnected. This behaviour for $I(A : B)$ and $E_W$ is shown in Fig(s). \ref{fig:1} \& \ref{fig:2} for both low and high temperature cases. We observe that $E_W$ always satisfies the property $E_W > \frac{1}{2} I(A : B)$. We then compute the HSC by considering again a single strip-like subsystem. The complexity for mixed state is computed by following the concept of mutual complexity $\Delta C$. We have used the HSC conjecture to compute the $\Delta C$ for both BTZ black hole and hyperscaling violating geometry. We observe that the computed result of mutual complexity is superadditive that is $\Delta C < 0$. This in turn means that the complexity of the state corresponding to the full system is greater than the sum of the complexities of the states in the two subsystems. We observe that for BTZ black hole $\Delta C$ is independent of temperature however for hyperscaling violating solution it contains a temperature independent term as well as a temperature dependent term. It is to be kept in mind that this nature of $\Delta C$ is observed for the HSC conjecture and similarly one can use ‘Complexity=Action’ conjecture or ‘CV2.0’ conjecture to compute $\Delta C$ in this context.

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8 Appendix

In this appendix, we give the expressions of quantities that appear in the main text. These are as follows.

\[
\Delta_4 = \Delta_3 \left( \frac{p}{\sqrt{\pi}} \frac{\Gamma \left( \frac{2p+\frac{p+\theta}{p-\theta}}{2p} \right)}{\Gamma \left( \frac{p+\frac{p+\theta}{p-\theta}}{2p} \right)} \right)^{(p-\theta)\left(1+p-\theta\right)} - \left( \frac{4\pi}{p(1+\frac{p-\theta}{p-\theta})} \right)^{1+p-\theta} \left( \frac{2}{p(1+\frac{p-\theta}{p-\theta})-p+1} \right) \left( \frac{p}{\sqrt{\pi}} \frac{\Gamma \left( \frac{2p+\frac{p+\theta}{p-\theta}}{2p} \right)}{\Gamma \left( \frac{p+\frac{p+\theta}{p-\theta}}{2p} \right)} \right)^{(p-\theta)\left(1+p-\theta\right)}
\]

\[
\Delta_5 = \left( \frac{4\pi}{p-1} \right)^{(p-1)(\frac{p+\theta}{p-\theta})} \left( \frac{p}{p(1+\frac{p-\theta}{p-\theta})} \right)^{(p-1)(\frac{p+\theta}{p-\theta})} \left( \frac{1+p-\theta}{p(1+\frac{p-\theta}{p-\theta})} \right) \left( \frac{2p+\frac{p+\theta}{p-\theta}}{p(1+\frac{p-\theta}{p-\theta})} \right)^{(p-1)(\frac{p+\theta}{p-\theta})}
\]

\[
\Delta_6 = 2 \left[ 1 - 2 \left( \frac{p+\theta}{p-\theta} \right)^{-(p-\theta)-1} \right] \lambda_1 \lambda_2 \left( \frac{p+\theta}{p-\theta} \right)^{-(p-\theta)}
\]

\[
\Delta_7 = \left[ 2^{2+p+\theta} - 2 \right] \left[ (p - \frac{p+\theta}{p-\theta}) \Delta_3 \lambda_1 \lambda_2 \left( \frac{p+\theta}{p-\theta} \right)^{-(p-\theta)} + \lambda_2 \left( \frac{p+\theta}{p-\theta} \right)^{z} \left( \frac{4\pi}{p(1+\frac{p-\theta}{p-\theta})} \right)^{(p-1)(\frac{p+\theta}{p-\theta})} \lambda_3 \right]
\]
where
\[
\lambda_1 = -\frac{\Gamma\left(\frac{1+\frac{2\theta}{p}}{2p}\right)}{\Gamma\left(\frac{1+p+\frac{2\theta}{2p}}{2p}\right)} \left[ \frac{\sqrt{\pi}}{p(p-\frac{\theta}{p-\theta})} \right] \\
\lambda_2 = \frac{p}{\sqrt{\pi}} \frac{\Gamma\left(\frac{2p+\frac{p}{2p}}{2p}\right)}{\Gamma\left(\frac{p+\frac{p}{2p}}{2p}\right)} \\
\lambda_3 = \frac{\sqrt{\pi}}{2p} \frac{\Gamma\left(\frac{p(1+\frac{1}{\sqrt{\pi}}+1+\frac{2\theta}{p^2})}{2p}\right)}{\Gamma\left(\frac{p(1+\frac{1}{\sqrt{\pi}}+1+\frac{2\theta}{p^2})}{2p}\right)} \left[ \frac{1}{\left(\frac{p^2}{p-\theta} + \frac{\theta}{p-\theta} \right)} - \frac{1}{(p-\frac{\theta}{p-\theta})} \right]
\]

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