PARTIAL REGULARITY FOR MINIMA OF HIGHER-ORDER QUASICONVEX INTEGRANDS WITH NATURAL ORLICZ GROWTH

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Abstract. A partial regularity theorem is presented for minimisers of $k$th-order functionals subject to a quasiconvexity and general growth condition. We will assume a natural growth condition governed by an $N$-function satisfying the $\Delta_2$ and $\nabla_2$ conditions, assuming no quantitative estimates on the second derivative of the integrand; this is new even in the $k=1$ case. These results will also be extended to the case of strong local minimisers.

1. Introduction

In this paper we will investigate the regularity of minimisers of functionals of the form

$$\mathcal{F}(w) = \int_\Omega F(\nabla^k w) \, dx,$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain and $w: \Omega \to \mathbb{R}^N$ is a vector-valued mapping. We will assume $F$ satisfies a suitable higher-order quasiconvexity condition in the sense of Morrey [52] for $k = 1$ and Meyers [49] for $k \geq 2$, and a general growth condition governed by an $N$-function $\varphi$ which we specify below. If $F$ is sufficiently regular it is known that minimisers of (1.1) are $F$-extremal, that is they satisfy the associated Euler-Lagrange system

$$(-1)^k \nabla^k : F'(\nabla^k u) := (-1)^k \sum_{|\alpha|=k} D^\alpha \left( F'(\nabla^k u) : e^\alpha \right) = 0$$

in $\Omega$, using the notation detailed in Section 1.1.

A striking feature of vector-valued problems is that minimisers need not be regular in general (see for instance [18, 48, 55, 64, 51]), however we can still hope for partial regularity results, which assert that minimisers $u$ are as regular as the data allows everywhere except on a small singular set. The first partial regularity result in the quasiconvex setting was established by Evans [28] which has been extended significantly since; see for instance [31, 32, 2, 27, 12, 3, 45, 44, 20, 8, 37, 17].

In the higher order case, the regularity theory for strongly $k$-quasiconvex integrands has been studied for instance by Guidorzi [39], Kronz [45], and Schemm [59]. Note the results of [16, 9] show that higher-order quasiconvexity reduces to ordinary quasiconvexity under quantitative Lipschitz bounds on $F'$, so while these results do not directly apply we expect the higher order case to be essentially the same as the $k=1$ setting.

We will mention that in the quasiconvex setting, some kind of minimising condition is essential to obtain regularity in general. This is illustrated in the seminal work of Müller & Šverák [54] where Lipschitz but nowhere $C^1$ solutions to the
associated Euler-Lagrange system are constructed, and this has been extended by
Kristensen & Taheri [41] and Székelyhidi [55] to weak local minimisers and
strongly polyconvex integrands respectively. As illustrated in [41] one can infer
partial regularity for strong local minimisers however, which we discuss in Section

1.1. Hypotheses and main results. We will consider higher-order integrands
satisfying a natural growth condition governed by an $N$-function, and a strict qua-
siconvexity condition. We refer the reader to Sections 2.1, 2.2 for the precise
definitions and convention we use.

Hypotheses 1.1. Let $F: \mathbb{M}_k \to \mathbb{R}$ with $n \geq 2, N, k \geq 1$ satisfy the following.

(H0) $F$ is $C^2$.

(H1) There exist $K \geq 0$ and an $N$-function $\varphi$ satisfying the $\Delta_2$ and $\nabla_2$ conditions
(as defined in Section 2.2) such that

$$|F(z)| \leq K(1 + \varphi(|z|))$$

for all $z \in \mathbb{M}_k$. By rescaling $\varphi, K$ if necessary, we will assume $\varphi(1) = 1$.

(H2) $F$ is strictly $W^{k, \tilde{\varphi}}$-quasiconvex in the sense that there is $\nu > 0$ such that

$$\int_{\mathbb{R}^n} F(z_0 + \nabla \xi) - F(z_0) \, dx \geq \nu \int_{\mathbb{R}^n} \varphi_1(|\nabla \xi|) \, dx$$

for all $z_0 \in \mathbb{M}_k$ and $\xi \in C_c^\infty(\mathbb{R}^n)$, where $\varphi_1$ is defined in [28].

For integrands of this type, we say $u \in W^{k, \tilde{\varphi}}(\Omega, \mathbb{R}^N)$ is a minimiser of (1.1) if for any $\xi \in W_0^{k, \varphi}(\Omega, \mathbb{R}^N)$ we have

$$F(u) \leq F(u + \xi).$$

Note it suffices to verify this for $\xi \in C_c^\infty(\Omega, \mathbb{R}^N)$, from which the above follows by
density (using both (H1) (H2)).

We will point out that while the partial regularity for quasiconvex integrands sat-
sifying a general growth condition have been considered by Diening, Lengeler,
Stroffolini, & Verde [20] for the $k = 1$ case, the authors assume a controlled
growth condition where quantitative estimates are assumed on the second deriva-
tives $F''$. We will relax this condition and establish the following.

Theorem 1.2 ($\varepsilon$-regularity theorem). Let $F$ satisfy Hypotheses 1.1 and let $M > 0,$
$\alpha \in (0, 1)$. Then there exists $\varepsilon > 0$ such that if $u \in W^{k, \tilde{\varphi}}(\Omega, \mathbb{R}^N)$ minimises (1.1)
with $\Omega \subset \mathbb{R}^n$ a bounded domain, and $B_R(x_0) \subset \Omega$ is such that $|\langle \nabla^k u \rangle_{B_R(x_0)}| \leq M$
and

$$\int_{B_R(x_0)} \varphi_1(|\nabla^k u - \langle \nabla^k u \rangle_{B_R(x_0)}|) \, dx \leq \varepsilon,$$

we have $u$ is of class $C^{k, \alpha}$ in $B_{R/2}(x_0)$.

Our proof largely hinges on a suitable Caccioppoli inequality and a harmonic
approximation argument, which can be traced back to the works of Giusti &
Miranda [35] and Morrey [53] in the general variational context, which were
inspired by prior works in the geometric setting. The Caccioppoli inequality will
be based on of the version in [28], which was adapted to the general growth setting
in [20]. For the harmonic approximation argument we will adapt a recent approach
of Gmeineder & Kristensen [37], which uses a duality argument to directly
estimate the excess in the approximation.

To apply this harmonic approximation argument of [37] in the general growth
setting, we will need to additionally use a version of the Lipschitz truncation lemma
of Acerbi & Fusco [1, 2]. This approach parallels the $A$-harmonic approximation
argument which has appeared in for instance [21] [20] [15] [23] [22], which traces back to arguments from geometric problems and can be found in texts of Simon [61] [62]. In [20] a direct proof of a suitable \(A\)-harmonic approximation is proven by applying a Lipschitz truncation to a suitable dual problem, which closely mirrors the strategy we adopt.

Once the above \(\varepsilon\)-regularity theorem is established, the following partial regularity theorem follows by standard means.

**Theorem 1.3** (Partial regularity of minimisers). Let \(F\) satisfy Hypotheses [14] and \(u \in W^{k,p}(\Omega, \mathbb{R}^N)\) be a minimiser of \([14]\) with \(\Omega \subset \mathbb{R}^N\) a bounded domain. Then there exists an open subset \(\Omega_0 \subset \Omega\) of full measure such that \(u\) is \(C^{k+\alpha}\) in \(\Omega_0\) for each \(\alpha \in (0,1)\). Moreover we have \(\Omega_0 = \Omega \setminus (\Sigma_1 \cup \Sigma_2)\), where

\[
\begin{align*}
\Sigma_1 &= \left\{ x \in \Omega : \limsup_{r \to 0} \int_{B_r(x)} |\nabla u| \, dx = \infty \right\}, \\
\Sigma_2 &= \left\{ x \in \Omega : \limsup_{r \to 0} \int_{B_r(x)} \varphi_{1+|\nabla w|_{B_r(x)}} \left( |\nabla u - (\nabla u)_{B_r(x)}| \right) \, dx > 0 \right\}.
\end{align*}
\]

In light of the partial regularity results in the linear growth setting [37], a natural question is whether the \(\nabla_2\)-condition is really necessary. In fact the results announced in [38] implies that partial regularity holds if we have merely the growth condition \(\varphi(t) \leq Ct^q\) with \(q < \frac{n}{n-1}\), however the general case remains open which we wish to address in future work.

## 2. Preliminaries

### 2.1. Notation.

We will briefly fix some notation which will be used throughout the text. We denote by \(\mathbb{M}_k = \text{Sym}_k(\mathbb{R}^n, \mathbb{R}^N)\) the space of symmetric \(k\)-linear maps \((\mathbb{R}^n)^k \to \mathbb{R}^n\), which is equipped with the inner product \(z : w = \sum_{i,j=1}^n z(e_i) \cdot w(e_j)\) for \(z, w \in \mathbb{M}_k\), taking tensor powers of the standard orthonormal basis \(\{e_i\}\) of \(\mathbb{R}^n\). The associated norm is denoted \(|z| = \sqrt{z : z}.\) Note it \(k = 1\) we have \(\mathbb{M}_1 = \mathbb{R}^{Nn}\) is the space of \(N \times n\) matrices. If \(u : \Omega \to \mathbb{R}^N\) is \(k\)-times continuously differentiable with \(\Omega \subset \mathbb{R}^N\) open, we denote the partial derivatives of \(u\) by \(D^k u\) using multi-index notation, and its \(k^{th}\) order gradient by \(\nabla^k u : \Omega \to \mathbb{M}_k\), given by \(\nabla^k u(e) = D^k u.\) The same notation will be used for weak derivatives.

We will equip \(\mathbb{R}^n\) with the Lebesgue measure \(\mathcal{L}^n\), and if \(A \subset \mathbb{R}^n\) is non-empty and open such that \(\mathcal{L}^n(A) < \infty\), for any \(f \in L^1(A, \mathcal{V})\) with \((\mathcal{V}, |\cdot|)\) a finite dimensional real vector space we define

\[
(f)_A := \int_A f \, dx := \frac{1}{\mathcal{L}^n(A)} \int_A f \, dx.
\]

We also denote by a \(B_R(x_0)\) the open ball in \(\mathbb{R}^n\) centred at \(x_0\) with radius \(R\).

For a differentiable map \(F : \mathbb{M}_k \to \mathbb{R}\) we define its derivative \(F' : \mathbb{M}_k \to \mathbb{M}_k\) as

\[
F'(z)w = \left. \frac{d}{dt} \right|_{t=0} F(z + tw),
\]

and if \(F\) is \(C^2\), its second derivative \(F''(z)\) will be a linear map \(\mathbb{M}_k \to \mathbb{M}_k\) satisfying

\[
F''(z)v : w = \left. \frac{d}{dt} \right|_{t=0} F'(z + tv)w.
\]

This can be viewed as a symmetric bilinear form on \(\mathbb{M}_k\).

Additionally \(C, C_1, C_2, \ldots\) will denote constants which may change from line to line, and if not specified will depend only on the parameters the resulting estimate depends on. We may also write \(C_{\alpha, \beta, \ldots}\) to emphasise the dependence on certain
parameters. We also write $A \sim B$ if there exists constants $C_1, C_2 > 0$ such that $C_1 A \leq B \leq C_2 A$.

2.2. $N$-functions. We will define the scales of growth we are interested in, and record some basic properties. This material is classical, and can be found for instance in [38, 55, 4].

We say $\varphi: [0, \infty) \to [0, \infty)$ is an $N$-function if $\varphi$ is increasing, continuous, convex such that $\varphi(t) = 0$ if and only if $t = 0$ and we have the limits

$$\lim_{t \to 0} \frac{\varphi(t)}{t} = 0, \quad \lim_{t \to \infty} \frac{\varphi(t)}{t} = \infty.$$  

We define the conjugate function by

$$\varphi^*(s) = \int_0^s (\varphi')^{-1}(\sigma) \, d\sigma.$$  

We say an $N$-function $\varphi$ satisfies the $\Delta_2$-condition if there is $C \geq 1$ for which $\varphi(2t) \leq C\varphi(t)$ for all $t \geq 0$, and the least such $C$ will be denoted $\Delta_2(\varphi)$. We also say $\varphi$ satisfies the $\nabla_2$-condition if $\varphi^*$ satisfies the $\Delta_2$-condition, and write $\nabla_2(\varphi) = \Delta_2(\varphi^*)$. We will use the notation $\varphi \in \Delta_2$, $\varphi \in \nabla_2$ to denote $N$-functions satisfying the $\Delta_2$ and $\nabla_2$ conditions respectively, and write $\varphi \in \Delta_2 \cap \nabla_2$ if both are satisfied.

Note that if $\varphi \in \Delta_2$, then there is $p > 1$ such that $\varphi(st) \leq s^p \varphi(t)$ for all $t > 0, s > 1$. The minimal $p$ will be denoted by $p_\varphi$, which roughly comparable to $\log(\Delta_2(\varphi))$.

Lemma 2.1. The following inequalities hold for an $N$-function $\varphi$.

(a) (Young’s inequality) For $t, s \geq 0$ we have

$$ts \leq \varphi(t) + \varphi^*(s),$$

with equality if and only if $s = \varphi'(t)$ or $t = (\varphi^*)'(s)$.

(b) For any $t \geq 0$ we have

$$t \leq \varphi(t)^{-1}(\varphi^*)^{-1}(t) \leq 2t.$$

(c) If $\varphi \in \Delta_2 \cap \nabla_2$ we have the equivalences

$$\varphi(t) \sim t \varphi'(t) \sim \varphi^*(\varphi'(t)) \sim \varphi^*(\varphi(t)/t).$$

In particular (a) implies the equivalent definition

$$\varphi^*(s) = \sup_{t>0} (ts - \varphi(t)),$$

and for $\delta > 0$ we have

$$ts \leq \delta \varphi(t) + \delta \varphi^*(s/\delta).$$

We refer the reader to [55] for a proof; note that [a] and [b] are shown in Theorem I.3 and Proposition II.1(ii) respectively, and (c) follows by convexity of $\varphi$ and [b].

If $\varphi$ is an $N$-function and $a > 0$, following [b] we also introduce the shifted $N$-function

$$\varphi_a(t) = \int_0^t \frac{\tau \varphi'(\max\{a, \tau\})}{\max\{a, \tau\}} \, d\tau.$$  

Note that $\varphi_a(t) \sim t^2$ if $t \leq a$ and $\varphi_a(t) \sim \varphi(t)$ if $t \geq a$ (constant depends on $a$) and using (2.5) we have $(\varphi_a)^* = (\varphi^*)_{\varphi(a)}$. Further if $\varphi \in \Delta_2 \cap \nabla_2$, then the same holds for each $\varphi_a$ with $1 \leq \Delta_2(\varphi_a) \leq \Delta_2(\varphi), 1 \leq \nabla_2(\varphi_a) \leq \nabla_2(\varphi)$ for all $a > 0$. Also if $\varphi \in \Delta_2$, then for each $M > 0$ we have $\varphi_{1+a} \sim \varphi_{1+M}$ for all $0 \leq a \leq M$ (constant depends on $M, \Delta_2(\varphi)$).
2.3. Orlicz-Sobolev spaces. We will also define the natural function spaces associated to the functional (1.1), and establish some basic properties which we will use later.

**Definition 2.2.** For an open set \( \Omega \subset \mathbb{R}^n \), a finite-dimensional normed space \((V, | \cdot |)\) and \( \varphi \in \Delta_2 \), we define the *Orlicz space* \( L^\varphi(\Omega, V) \) as the space of \( f \in L^1_{\text{loc}}(\Omega, V) \) for which

\[
\rho_\Omega^\varphi(|f|) := \int_\Omega \varphi(|f|) \, dx < \infty,
\]

which we equip with the *Luxemburg norm*

\[
\|f\|_{L^\varphi(\Omega, V)} = \inf \left\{ \lambda > 0 : \rho_\Omega^\varphi \left( \frac{|f|}{\lambda} \right) \leq 1 \right\}.
\]

Note that the \( \Delta_2 \)-condition ensures that \( L^\varphi(\Omega, V) \) as defined is a linear space, and that \( f \in L^1_{\text{loc}}(\Omega, V) \) lies in \( L^\varphi(\Omega, V) \) if and only if \( \|f\|_{L^\varphi(\Omega, V)} < \infty \). Also if \( \Omega \) bounded, we have \( L^\varphi(\Omega, V) = L^{\varphi_a}(\Omega, V) \) for all \( a > 0 \), with \( \varphi_a \) defined as in (2.8).

Given this, for \( N, k, \ell \geq 1 \) we can define the *Orlicz-Sobolev spaces* \( W^{k,\varphi}(\Omega, M_\ell) \) as the space of \( k \)-times weakly differentiable \( u \in L^\varphi(\Omega, M_\ell) \) such that \( \nabla u \in L^\varphi(\Omega, M_{\ell+1}) \) for all \( 1 \leq j \leq k \). We equip this space with the norm

\[
\|u\|_{W^{k,\varphi}(\Omega, M_\ell)} = \sum_{j=0}^k \|\nabla^j u\|_{L^\varphi(\Omega, M_{j+1})}.
\]

We also define \( W^{k,\varphi}_0(\Omega, M_\ell) \) to be the closure of \( C_0^\infty(\Omega, M_\ell) \) with respect to \( \|\cdot\|_{W^{k,\varphi}(\Omega, M_\ell)} \), and \( W^{k,\varphi}_{\text{loc}}(\Omega, M_\ell) \) to be the space of \( u \in W^{k,\varphi}_{\text{loc}}(\Omega, M_\ell) \) such that the restriction \( u|_\Omega \) lies in \( W^{k,\varphi}(\Omega, M_\ell) \) for each compactly contained domain \( \Omega' \subset \Omega \). These Orlicz-Sobolev spaces enjoy many of the familiar properties satisfied by the standard Sobolev spaces; see for instance [4, Section 8]. We will record some specific results we need here.

This first result we need is the following Poincaré-Sobolev inequality, which is far from sharp but will suffice for our purposes. We note that the optimal scales for the Orlicz-Sobolev embedding theorem has been identified in [15].

**Lemma 2.3.** Let \( \varphi \in \Delta_2, \ell \geq 1 \) then if \( u \in W^{1,\varphi}(\Omega, M_\ell) \) we have \( u \in L^{\varphi_a}(\Omega, M_{\ell-1}) \) for each \( 1 \leq p \leq \frac{n}{n-1} \). Moreover for any \( B_R(x_0) \subset \Omega \) we have

\[
\left( \frac{1}{B_R(x_0)} \int_{B_R(x_0)} \varphi \left( \frac{|u - (u)_{B_R(x_0)}|}{R} \right)^p \, dx \right)^{\frac{1}{p}} \leq C \int_{B_R(x_0)} \varphi(|\nabla u|) \, dx,
\]

where \( C = C(n, \Delta_2(\varphi)) > 0 \). If \( u \in W^{1,\varphi}_{\text{loc}}(B_R(x_0), M_\ell) \), the same holds without subtracting an average.

**Proof.** We will establish the result for \( R = 1 \), from which the general case follows by rescaling. The result for \( p = 1 \) follows by a standard application of the Riesz potential, as is shown in [1] (see also [19]). For general \( 1 \leq p \leq \frac{n}{n-1} \) assume that \( (u)_{B_1(x_0)} = 0 \) by translation, and noting that \( \nabla (\varphi(|u|)) = \varphi'(|u|) \nabla |u| \) we can apply Young’s inequality to bound

\[
\int_{B_1(x_0)} |\nabla (\varphi(|u|))| \, dx \leq \int_{B_1(x_0)} \varphi'(|u|) + \varphi(|\nabla u|) \, dx \leq \int_{B_1(x_0)} \varphi(2|u|) + \varphi(|\nabla u|) \, dx.
\]
From here we conclude that \( \varphi(|u|) \in W^{1,1}(B_1(x_0)) \) using the \( \Delta_p \)-condition and the \( p = 1 \) case, and so by the Gagliardo-Nirenberg inequality we have \( \varphi(|u|) \in L^p(B_1(x_0), \mathbb{R}^N) \) for each \( 1 \leq p \leq \frac{n}{n-1} \) with the associated estimate

\[
(2.14) \quad \left( \int_{B_1(x_0)} \varphi(|u|^p) \, dx \right)^{\frac{1}{p}} \leq C \int_{B_1(x_0)} |\nabla \varphi(|u|)| \, dx \leq C \int_{B_1} \varphi(|\nabla u|) \, dx,
\]

as required. The \( W^{1,\varphi}_0 \) case can be found in [4] Proposition 4.1 under more general assumptions.

\[ \square \]

**Remark 2.4.** Applying this iteratively with \( p = 1 \) we deduce for all \( 0 \leq j \leq k - 1 \) that

\[
(2.15) \quad \int_{B_1(x_0)} \varphi \left( \frac{|\nabla^j (u - a_{x_0,R})|}{R^{k-j}} \right) \, dx \leq C \int_{B_1(x_0)} \varphi \left( R^{k} |\nabla^j u| \right) \, dx,
\]

where \( a_{x_0,R} \) is the unique polynomial of degree at most \( k - 1 \) satisfying

\[
(2.16) \quad \int_{B_1(x_0)} D^\alpha (u - a_{x_0,R}) \, dx = 0
\]

for all \( |\alpha| \leq k \). Also if \( u \in W^{k,\varphi}_0(B_R(x_0), \mathbb{M}_\ell) \), we can omit the \( a_{j,u} \) term.

We will also need some results for affine functions when considering the local minimiser case in Section 4. These can be deduced by combining the results in [40] Lemmas 2.2, 2.3 and [45] Lemma 2.

**Lemma 2.5.** Let \( B_R(x_0) \subset \mathbb{R}^n \) be a ball, \( N, \ell \geq 1 \), and \( \varphi \in \Delta_2 \). Then if \( u \in W^{1,\varphi}(B_R(x_0), \mathbb{M}_\ell) \) we have the affine function \( A_{x_0,R} : \mathbb{R}^n \to \mathbb{M}_\ell \) defined to satisfy

\[
(2.17) \quad A(x_0) = \int_{B_R(x_0)} u \, dx = 0, \quad \nabla A = \frac{n+2}{R^2} \int_{B_R(x_0)} u(x) \otimes (x - x_0) \, dx
\]

satisfies

\[
(2.18) \quad \int_{B_R(x_0)} \varphi(|u - A_{x_0,R}|) \, dx \leq C \int_{B_R(x_0)} \varphi(|u - A|) \, dx
\]

for any other \( A : \mathbb{R}^n \to \mathbb{M}_\ell \) affine, and we also have the estimates

\[
(2.19) \quad \varphi \left( |\nabla A_{x_0,R} - (\nabla u)_{B_R(x_0)}| \right) \leq C \int_{B_R(x_0)} \varphi \left( |\nabla u - (\nabla u)_{B_R(x_0)}| \right) \, dx.
\]

\[
(2.20) \quad \varphi \left( |\nabla A_{x_0,R} - \nabla A_{x_0,\sigma R}| \right) \leq C \int_{B_{\sigma R}(x_0)} \varphi \left( \frac{|u - A_{x_0,R}|}{\sigma R} \right) \, dx,
\]

for all \( \sigma \in (0, 1) \).

Finally we record an interpolation estimate in the Orlicz scales, which is a straightforward adaptation of the \( W^{k,p} \) case (compare with the results in [4] Section 5)

**Lemma 2.6.** Let \( \varphi \in \Delta_2 \cap \Delta_2 \) and \( k \geq 0 \). Then there exists \( C = C(n, k, \Delta_2(\varphi)) > 0 \) such that for all \( x_0 \in \mathbb{R}^n \), \( R > 0 \) and \( u \in W^{k,\varphi}(B_R(x_0), \mathbb{M}_\ell) \), we have the estimate

\[
(2.21) \quad \int_{B_R} \varphi(|\nabla^j u|) \, dx \leq C \int_{B_R} \varphi(\delta^{-j}|u|) + \varphi(\delta^{k-j}|\nabla^k u|) \, dx.
\]

**Sketch of proof.** We start by observing the one-dimensional estimate

\[
(2.22) \quad \varphi(|f'(0)|) \leq \frac{C}{\delta} \int_0^\delta \varphi(\delta^{-1}|f(t)|) + \varphi(\delta|f''(t)|) \, dt,
\]
following [4] Lemma 5.4 using Jensen’s inequality with \( \varphi \). For \( x \in B_R(x_0) \) and \( \omega \in S^{n-1} \) we apply this to \( f(t) = u(x + t\omega) \), noting that the inequality holds on \([0, \infty)\) if we extend \( u \) by zero. Then integrating over \( x, \omega \) we get

\[
\int_{B_R(x_0)} \varphi(|\nabla u|) \, dx 
\]

(2.23)

\[
\leq C \int_{B_R(x_0)} \int_{S^{n-1}} \int_0^\delta \left( \delta^{-1}|u(x + t\omega)| \right) + \varphi \left( \delta \nabla^2 u(x + t\omega) \right) \, dt \, d\mathcal{H}^{n-1}(\omega) \, dx 
\]

\[
\leq C \int_{\mathbb{R}^n} \varphi \left( \delta^{-1}|u(x)| \right) + \varphi \left( \delta \nabla^2 u(x) \right) \, dx, 
\]

establishing the \( j = 1, k = 2 \) case. For the general case we proceed by induction; suppose the result holds for some \( k \geq 2 \) and \( j = k - 1 \), then we can estimate

\[
\int_{B_R(x_0)} \varphi(|\nabla^k u|) \, dx 
\]

(2.24)

\[
\leq C \int_{B_R(x_0)} \varphi(\delta^{-1}|\nabla^{k-1} u|) + \varphi(\delta|\nabla^{k+1} u|) \, dx 
\]

\[
\leq C \int_{B_R(x_0)} \varphi(\mu \delta^{-1}|u|) + \varphi(\mu|\nabla u|) + \varphi(\delta|\nabla^{k+1} u|) \, dx. 
\]

Since \( \varphi \in \nabla_2 \), choosing \( \mu > 0 \) sufficiently small so that \( C\varphi(\mu|\nabla^{k} u|) \leq \frac{1}{2} \varphi(|\nabla^{k} u|) \) the estimate follows. A similar downward induction argument extends the result to all \( 1 \leq j \leq k - 1 \).

2.4. Linear elliptic estimates. Our linearisation strategy will involve comparing our minimiser with solutions to a linearised system, for which we will need some solvability results. For this consider a bilinear form \( A \) on \( \mathbb{M}_k \) satisfying the uniform Legendre-Hadamard ellipticity condition

\[
\lambda |\xi|^2 |\eta|^{2k} \leq A(\xi \otimes \eta, \xi \otimes \eta) \leq \Lambda |\xi|^2 |\eta|^{2k} 
\]

(2.25)

for all \( \xi \in \mathbb{R}^N, \eta \in \mathbb{R}^n \), where \( 0 < \lambda \leq \Lambda < \infty \). Here we write \( \eta^k = \eta \otimes \cdots \otimes \eta \) to denote the \( k \)-fold tensor product and identify elements \( \xi \otimes \eta^k \in \mathbb{M}_k \) to send \( (x_1, \ldots, x_k) \rightarrow \xi \sum_{|\alpha|=k} x^\alpha \eta^\alpha \). These generalise the rank-one matrices from the \( k = 1 \) case.

We will consider the operator

\[
\nabla^k : A \nabla^k u = \sum_{|\alpha| = |\beta| = m} \nabla^\beta (A_{\beta, \alpha} \nabla^\alpha u), 
\]

(2.26)

where \( A_{\alpha, \beta} = \delta[e^\alpha, e^\beta] \) denotes the coefficients of \( A \).

Remark 2.7. For our harmonic approximation arguments, we will need the fact that the strict quasiconvexity condition \([\text{H2}]\) implies that \( A[v, w] = F''(z_0)[v, w] \) is Legendre-Hadamard elliptic. To see this let \( F \) satisfy Hypotheses \([\text{H4}]\) and for \( z_0 \in \mathbb{M}_k \) and \( \xi \in C_c^\infty(\mathbb{R}, \mathbb{R}^N) \) consider the functional

\[
\mathcal{J}(t) = \int_{\Omega} F(z_0 + t\nabla^k \xi) - F(z_0) - \nu \varphi_{1+|z_0|}(t\nabla^k \xi) \, dx \geq 0. 
\]

(2.27)

By \([\text{H2}]\) this is minimised when \( t = 0 \), so noting that \( \varphi''_{1+|z_0|}(0) \) exists and by differentiating under the integral sign, we have \( \mathcal{J}''(0) \) exists and is non-negative. That is we have

\[
\int_{\Omega} F''(z_0) \nabla^k \xi : \nabla^k \xi \, dx \geq \nu \int_{\Omega} \varphi'(\max\{1, |z_0|\}) \max\{1, |z_0|\} |\nabla^k \xi|^2 \, dx. 
\]

(2.28)
From this we deduce that $F''(z_0)$ satisfies
\[
F''(z_0)(\xi \otimes \eta^k) : (\xi \otimes \eta^k) \geq \frac{\nu \varphi'(1)}{1 + M} |\xi|^2 |\eta|^{2k}
\]
for all $z_0 \in M_k$ with $|z_0| \leq M$ and $\xi \in \mathbb{R}^N, \eta \in \mathbb{R}^n$. Note that since we set $\varphi(1) = 1$, the ellipticity constant only depends on $\nu, N, \Delta_2(\varphi)$.

**Proposition 2.8.** Let $\varphi \in \Delta_2 \cap \Delta_3$ be an $N$-function and $A$ be uniformly Legendre-Hadamard elliptic as above. Then if $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain, the problem
\[
\begin{cases}
(-1)^k \nabla^k : A \nabla^k u = (-1)^k \nabla^k : G & \text{ in } \Omega, \\
\partial_j^2 u = 0 & \text{ on } \partial \Omega, \quad \text{for all } 1 \leq j \leq k - 1,
\end{cases}
\]
is uniquely solvable in $W^{k,\varphi}_0(\Omega, \mathbb{R}^N)$ for all $G \in L^{\varphi}(\Omega, M_k)$, and there is $C > 0$ such that
\[
\int_\Omega \varphi\left(|\nabla^k u|\right) \, dx \leq C \int_\Omega \varphi(|G|) \, dx.
\]

We will apply this estimate on balls, on which we can note this estimate is scale invariant.

**Remark 2.9.** The boundary condition (2.30) will be interpreted as simply requiring $u \in W^{k,\varphi}_0(\Omega, \mathbb{R}^N)$. This can be made precise by defining suitable trace operators (see for instance [67, 56]), but this will be sufficient for our purposes.

This result is well-known and follows from the results in Agmon, Douglis, & Nirenberg [5], and the interpolation results of Peetre [67]. We will sketch a more elementary argument based on the interpolation results of Stampacchia [63], using a modern formulation which closely follows the argument of Krylov [46]. We note that by using pointwise estimates for the maximal function in the $L^q$ scales instead of $L^2$, more general estimates in a similar spirit have also been obtained by Dong & Kim [23].

**Sketch of proof.** We first establish the result for when $\varphi(t) = t^p$, for which we let $L = (-1)^k \nabla^k : \Lambda \nabla^k$ viewed as a linear operator $W^{k,p}_0(\Omega, \mathbb{R}^N) \to W^{-k,-p}(\Omega, \mathbb{R}^N) \simeq W^{k,p'}_0(\Omega, \mathbb{R}^N)^\ast$. When $p = 2$ we can use the Plancherel theorem to show that for any $\omega \subset \mathbb{R}^n$ open we have
\[
\lambda \int_\omega |\nabla^k u|^2 \, dx \leq \int_\omega A[\nabla^k u, \nabla^k u] \, dx
\]
for all $u \in W^{k,2}_0(\omega, \mathbb{R}^N)$, from which we can deduce unique solvability in $\omega$ using the Lax-Milgram lemma. If $p \geq 2$ we will establish an a-priori estimate, so let $u \in C_0^\infty(\Omega, \mathbb{R}^N)$. Then if $x_0 \in \overline{\Omega}$ and $0 < R < R_0$ with $R_0 > 0$ sufficiently small, we have the problem
\[
\begin{cases}
Lw = Lu & \text{ in } \Omega \cap B_R(x_0), \\
\partial_j^2 u = 0 & \text{ on } \partial \Omega \cap B_R(x_0), \quad \text{for all } 1 \leq j \leq k - 1
\end{cases}
\]
admits a unique solution $w \in W^{k,2}_0(\Omega \cap B_R(x_0), \mathbb{R}^N)$. Since the difference $v = u - w$ satisfies $Lv = 0$ in $\Omega \cap B_R(x_0)$, by standard energy methods (see for instance [65], Section 5.11) we have the uniform estimate
\[
\sup_{\Omega \cap B_{R/2}(x_0)} |\nabla^{k+1}v| \leq \frac{C}{R} \int_{\Omega \cap B_R(x_0)} |\nabla^k u|^2 \, dx.
\]
By combining these we can argue that for all \( \theta \in (0, 1) \) we have

\[
M_\Omega^{\theta} (|\nabla^k u|) (x) \leq C \left( \theta M_\Omega \left( |\nabla^k u|^2 \right) (x)^{\frac{1}{2}} + \theta^{-\frac{1}{2}} M_\Omega \left( |G|^2 \right) (x)^{\frac{1}{2}} \right)
\]

for all \( x_0 \in \Omega \), where we define the localised maximal functions

\[
M_\Omega (f) (x_0) = \sup_{R > 0} \frac{1}{|\Omega \cap B_R(x_0)|} \int_{\Omega \cap B_R(x_0)} |f| \, dx
\]

(2.36)

\[
M_\Omega^{\theta} (f) (x_0) = \sup_{R > 0} \frac{1}{|\Omega \cap B_R(x_0)|} \int_{\Omega \cap B_R(x_0)} |f - (f)_{\Omega \cap B_R(x_0)}| \, dx
\]

for \( f \in L^1_{\text{loc}}(\Omega, M_k) \). To see this, for \( R < R_0 \) note that the above estimates implies that

\[
\int_{\Omega \cap B_R(x_0)} |\nabla^k u - (\nabla^k u)_{\Omega \cap B_R(x_0)}|^2 \, dx \\
\leq 2 \int_{\Omega \cap B_R(x_0)} |\nabla^k v - (\nabla^k v)_{\Omega \cap B_R(x_0)}|^2 \, dx \\
+ 2 \int_{\Omega \cap B_R(x_0)} |\nabla^k w - (\nabla^k w)_{\Omega \cap B_R(x_0)}|^2 \, dx \\
\leq C \theta^2 \int_{\Omega \cap B_R(x_0)} |\nabla^k u|^2 \, dx + C \theta^{-n} \int_{\Omega \cap B_R(x_0)} |G|^2 \, dx.
\]

If \( R > R_0 \) similar estimates hold by a patching argument, from which the pointwise estimate (2.35) follows. Now taking \( L^{p/2} \) norms on both sides, we can use the Hardy-Littlewood maximal inequality and the Fefferman-Stein inequality in this setting (see for instance [21,23] for more general statements) to deduce the estimate

\[
\|\nabla^k u\|_{L^p(\Omega, M_k)} \leq C \left( \theta \|\nabla^k u\|_{L^p(\Omega, M_k)} + \theta^{-\frac{p}{2}} \|G\|_{L^p(\Omega, M_k)} \right),
\]

so if \( u \in C^\infty_0 (\Omega) \) we deduce \( L^p \) estimates for \( p \geq 2 \). By density these estimates extend to all \( u \in W^{k,p}_0(\Omega, \mathbb{R}^N) \), and by regularising \( G \in L^p(\Omega, M_k) \) and passing the limit using the a-priori estimate we can infer that \( L \) is an isomorphism \( W^{k,p}_0(\Omega, \mathbb{R}^N) \to W^{-k,p}(\Omega, \mathbb{R}^N) \) (injectivity follows for the \( p = 2 \) case). Hence by duality (noting the adjoint operator \( L^* \) is an isomorphism) unique solvability extends to the \( 1 < p < 2 \) range. This establishes the result for the \( L^p \) scales, from which the Orlicz setting follows by interpolation using for instance [57,50].

We will also need solvability results for when the right-hand side \( g \) lies in \( W^{k-1,p}(\Omega, M_{k-1}) \). This will follow from the above in conjunction with the following (non-optimal) representation theorem.

**Lemma 2.10.** Suppose \( \varphi \in \Delta_2 \cap \nabla_2, \ell \geq 0 \) and \( g \in L^p(B_R(x_0), M_{\ell+1}) \), where \( B_R(x_0) \subset \mathbb{R}^n \) is any ball. Then there exists \( G \in L^{\frac{p}{\ell+1}}(B_R(x_0), M_{\ell+1}) \) such that \(-\nabla \cdot G = g \) in \( B_R(x_0) \), and we have the corresponding estimate

\[
\left( \int_{B_R(x_0)} \varphi(|G|) \, dx \right)^{\frac{1}{n+1}} \leq C \int_{B_R(x_0)} \varphi(R|g|) \, dx.
\]

**Sketch of proof.** We will consider the Newtonian potential (see for instance [33, Section 7])

\[
G(x) = \frac{1}{n\omega_n} \int_{B_R(x_0)} \frac{g(y) \otimes (x - y)}{|x - y|^n} \, dy,
\]

(2.41)
which satisfies \(-\nabla \cdot G = g\) in \(B_R(x_0)\) and by singular integral estimates (extended to Orlicz scales using [57] or [50]) we have

\[
(2.42) \quad \int_{B_R(x_0)} \varphi(|\nabla G|) \, dx \leq \int_{B_R(x_0)} \varphi(|g|) \, dx.
\]

We now apply Poincaré-Sobolev inequality (Lemma 2.3), which gives an extra term arising from the average; this can be bounded using the estimate

\[
(2.43) \quad \int_{B_R(x_0)} \varphi(|G|) \, dx \leq \frac{C}{R} \int_{B_R(x_0)} \int_{B_R(x_0)} \varphi(R|g(y)|) |x - y|^{1-n} \, dx \, dy
\]

using (2.41) and Jensen’s inequality (applied to the measure \(d\mu = \frac{1}{R} |x - y|^{-n} \, dy\)) and Fubini’s theorem.

\[
\leq C \int_{B_R(x_0)} \varphi(|Rg(y)|) \, dy.
\]

2.5. A Lipschitz truncation lemma. We will establish the following higher-order version of the Lipschitz truncation lemma of Acerbi & Fusco [1, 2], which will we need for the harmonic approximation argument. The higher order case requires a more delicate extension argument, and was established in Friesecke, James, & Müller [30, Proposition A.2] for \(k = 2\) in using extension results in Ziemer [67]. The case of general \(k\) follows by a similar argument which we will record here.

**Proposition 2.11.** Let \(\varphi \in \Delta_2 \cap \nabla_2\) and \(q \in W^{k,\infty}_0(B_R(x_0), \mathbb{R}^N)\) with \(B_R(x_0) \subset \mathbb{R}^n\) a ball. Then for each \(\lambda > 0\) there exists \(q_\lambda \in W^{k,\infty}_0(B_R(x_0), \mathbb{R}^N)\) satisfying the following estimates.

\[
(2.44) \quad \|\nabla^k q\|_{L^\infty(B_R(x_0), M_k)} \leq C_1 \lambda,
\]

\[
(2.45) \quad \int_{B_R(x_0)} \varphi(|\nabla^k q\|) \, dx \leq C_2 \int_{B_R(x_0)} \varphi(|\nabla^k q|) \, dx,
\]

\[
(2.46) \quad \varphi(\lambda) \mathcal{L}^n(\{x \in B_R(x_0) : q(x) \neq q_\lambda(x)\}) \leq C_2 \int_{B_R(x_0)} \varphi(|\nabla^k q|) \, dx,
\]

where \(C_1 = C_1(n, N, k)\) and \(C_2 = C_2(n, N, k, \Delta_2(\varphi), \nabla_2(\varphi))\).

**Proof.** By rescaling we can assume \(R = 1\), and write \(B = B_1(x_0)\). Let \(L_B\) denote the set of Lebesgue points for \(q, \nabla q, \ldots, \nabla^k q\), and we will work with the precise representatives extending by zero to \(\mathbb{R}^n\) as necessary. For each \(\lambda > 0\) consider

\[
(2.47) \quad H_\lambda = \{x \in L_B : \mathcal{M}(|\nabla^j q|) \leq \lambda \text{ for all } 0 \leq j \leq k\},
\]

where \(\mathcal{M}\) is the maximal operator defined for each \(f \in L^1_{loc}(\mathbb{R}^n, M_j)\) as

\[
(2.48) \quad \mathcal{M}f(x) = \sup_{x \in \mathbb{R}^n} \int_{B_r(x)} |f(y)| \, dy.
\]

Then for all \(x \in H_\lambda\) writing

\[
(2.49) \quad P_\lambda(y) = \sum_{|\alpha| \leq k-1} \frac{D^\alpha q(x)}{\alpha!} (y - x)^\alpha
\]

for any \(|\beta| \leq k - 1\), we have by [67] Theorem 3.4.1 that

\[
(2.50) \quad \int_{B_\lambda(x)} |D^\beta q(y) - D^\beta P_\lambda(y)| \, dy \leq C \lambda^{k-|\beta|}.
\]
Hence by [67, Theorem 3.5.7] (noting this applies even if $H_{\lambda}$ is not closed) we obtain the pointwise estimate
\begin{equation}
|D^3 q(y) - D^3 P_x(y)| \leq C_{\lambda} |x - y|^{k - |\beta|}
\end{equation}
for all $x, y \in H_{\lambda}$ and $|\beta| \leq k - 1$. For the boundary values we will argue that
\begin{equation}
|D^3 q(x)| \leq C_{\lambda} \text{dist}(x, \partial B)^{k - |\beta|}
\end{equation}
for all $x \in H_{\lambda}$ and $|\beta| \leq k$. To see this let $x_0 \in \partial B$ such that $|x - x_0| = d = \text{dist}(x, \partial B)$, and note that $\mathcal{B}_d(x_0) \subset \mathcal{B}_{2d}(x)$. Since $B$ is a ball it satisfies an external measure density condition of the from $\mathcal{L}^n(\mathcal{B}_r(y) \setminus B) \geq \nu \mathcal{L}^n(\mathcal{B}_r(y))$ for all $x_0 \in \partial B$ and $0 < r < 1$, so we have $\mathcal{L}^n(\mathcal{B}_{2d}(x) \setminus B) \geq 2^{-n} \nu \mathcal{L}^n(\mathcal{B}_{d}(x))$. Hence by a suitable Poincaré inequality (extending $u$ by zero to $\mathcal{B}_{2d}(x)$ noting it vanishes on $\mathcal{B}_d(x_0)$, and using for instance [67, Corollary 4.5.2]) we have
\begin{equation}
\int_{\mathcal{B}_{2d}(x)} |D^3 q| \, dy \leq C_{\lambda} d^{k - |\beta|}.
\end{equation}
Combined with (2.50) and using pointwise bounds for $P_x$ noting $x \in H_{\lambda}$ we have
\begin{equation}
|D^3 q(x)| \leq \int_{\mathcal{B}_{2d}(x)} |D^3 P_x(x) - D^3 P_x(y)| \, dy
\end{equation}
\begin{equation}
+ \int_{\mathcal{B}_{2d}(x)} |D^3 P_x(y) - D^3 q(y)| + |D^3 q(y)| \, dy
\leq C_{\lambda} d^{k - |\beta|},
\end{equation}
Now we extend $P$ to $H_{\lambda} \cup \partial B$ by setting $P_x = 0$ for $x \in \partial B$, and also define $v$ on $H_{\lambda} \cup \partial B$ as
\begin{equation}
\tilde{q}_{\lambda}(x) = \begin{cases} q(x) & \text{if } x \in H_{\lambda} \\ 0 & \text{if } x \in \partial B. \end{cases}
\end{equation}
Then by the above estimates we have
\begin{equation}
|D^3 \tilde{q}_{\lambda}(x) - D^3 P_x(y)| \leq C_{\lambda} |x - y|^{k - |\beta|}
\end{equation}
for all $|\beta| \leq k - 1$, and so we can extend $\tilde{q}$ to $\overline{B}$ by setting
\begin{equation}
\tilde{q}_{\lambda}(y) = \sup_{x \in H_{\lambda} \cup \partial B} P_x(y) - M_{\lambda} |x - y|^k,
\end{equation}
choosing $M > 1$ sufficiently large to ensure $\tilde{q}_{\lambda}(x)$ remains unchanged on $H_{\lambda} \cup \partial B$. Increasing $M$ further if necessary, we claim this satisfies
\begin{equation}
|\tilde{q}_{\lambda}(y) - P_x(y)| \leq C_{\lambda} |x - y|^k
\end{equation}
for all $x \in H_{\lambda} \cup \partial B$ and $y \in \overline{B}$. Indeed for $z \in H_{\lambda} \cup \partial B$ we have
\begin{equation}
\begin{aligned}
(P_x(y) - M_{\lambda} |y - z|^k) - P_x(y) &
\leq |P_x(y) - P_x(z)| - M_{\lambda} |y - z|^k \\
&\leq C_{\lambda} \sum_{j=0}^{k-1} |x - z|^{k-j} (|x - y|^j + |z - y|^j) - M_{\lambda} |y - z|^k \\
&\leq C_{\lambda} |y - x|^k + (C - M_{\lambda}) |y - z|^k,
\end{aligned}
\end{equation}
where we used Young’s inequality and the fact that $|x - z| \leq |x - y| + |y - z|$. Choosing $M \geq C$ and extremising over $z$ implies the estimate (2.58), noting the lower bound is immediate form the definition.

Now we can apply the extension result [67, Theorem 3.6.2] (extending by continuity to $\overline{H_{\lambda} \cup \partial B}$ first) to deduce the existence of $q_{\lambda} \in W^{1,\infty}_0(\mathcal{B}, \mathbb{R}^N)$ such that $\|\nabla^\beta q\|_{L^\infty(\mathcal{B} \setminus H_{\lambda})} \leq C_{\lambda}$, and such that $q = q_{\lambda}$ on $H_{\lambda}$. 

It remains to establish the estimates (2.45), (2.46). For (2.46) we apply Markov’s inequality to estimate
\[
\varphi(\lambda)\mathcal{L}^n(\{q \neq q_\lambda\}) \leq \varphi(\lambda)\mathcal{L}^n(\{H_\lambda\}) \\
\leq C \sum_{j=0}^k \int_{\mathbb{R}^n} \varphi(M(|\nabla j q|)) \, dx \\
\leq C \sum_{j=0}^k \int_B \varphi(|\nabla^j q|) \, dx \\
\leq C \int_B \varphi(|\nabla^k q|) \, dx,
\]
where we have used the boundedness of the maximal function on $L^\varphi$ (see for instance [12, Theorem 1.2.1]) and the Poincaré inequality (Lemma 2.3) in the last two lines. For (2.45) we can estimate
\[
\int_B \varphi(|\nabla q_\lambda|) \, dx \leq \int_B \varphi(|\nabla q|) \, dx + \int_{\{q \neq q_\lambda\}} \varphi \left( \|\nabla q_\lambda\|_{L^\infty(B,M_k)} \right) \, dx,
\]
from which we can conclude noting $\|\nabla q_\lambda\|_{L^\infty(B,M_k)} \leq CA$ and applying (2.46). \(\square\)

3. Proof of the regularity theorem

3.1. Caccioppoli inequality of the second kind. We will begin by establishing a Caccioppoli inequality of the second kind in this general growth setting, adapting the estimate of Evans [28]. When $k = 1$ this argument in the Orlicz setting can be found in [20], and we will show it straightforwardly extends to the case of general $k$.

For this we fix $z_0 \in M_k$, and following [2] we introduce the shifted integrand
\[
F_{z_0}(z) = F(z_0 + z) - F(z_0) - F'(z_0)z.
\]
Then for each $M \geq 1$, if $|z_0| \leq M$ by distinguishing between the cases when $|z_0| \leq M + 1$ and $|z_0| \geq M + 1$ we have
\[
\begin{align*}
|F_{z_0}(z)| \leq C \varphi_{1+|z_0|}(|z|), \\
|F'_{z_0}(z)| \leq C \varphi'_{1+|z_0|}(|z|),
\end{align*}
\]
where $C > 0$ depends on $n, N, k, \Delta_2(\varphi)$ and $G(M) := \sup_{|z| \leq 2M+1} |F''(z)|$. Here the second estimate follows from the first by rank-one convexity of $F_{z_0}$ (implied by [H2]).

Lemma 3.1 (Caccioppoli inequality). Let $u \in W^{k,\varphi}(\Omega, \mathbb{R}^N)$ be a minimiser of (1.1) where the integrand $F$ satisfies Hypotheses 1.1 and let $M > 0$. Then if $a: \mathbb{R}^n \to \mathbb{R}^N$ is a $k^{th}$ order polynomial such that $|\nabla^k a| \leq M$, there exists $C = C(n, N, k, K, \varphi, \Delta_2(\varphi), M, G(M)) > 0$ such that for any $B_R(x_0) \subset \Omega$ we have the estimate
\[
\int_{B_{R/2}(x_0)} \varphi_{1+M}(|\nabla^k u - \nabla^k a|) \, dx \leq C \int_{B_R(x_0)} \varphi_{1+M} \left( \frac{|u - a|}{R^k} \right) \, dx.
\]
Proof. We will suppress the $x_0$ dependence to simplify notation. Fix $0 < t < s < R$, and let $\eta \in C^\infty_c(B_R)$ be a radial cut-off such that $1_{B_t} \leq \eta \leq 1_{B_s}$ and $|\nabla \eta| \leq \frac{C}{(s-t)^k}$ for each $1 \leq j \leq k$. Given $a$ we set $w = u - a$, $\tilde{F} = F_{\varphi a}$ and $\tilde{\varphi} = \varphi_{1+|\nabla^j a|}$ noting...
\[ \tilde{\varphi} \sim \varphi_{1+M}, \text{ so then by the strict quasiconvexity condition } [H2] \text{ we have} \]
\[ \int_{B_t} \tilde{\varphi}(|\nabla^k(\eta u)|) \, dx \leq \int_{B_t} \tilde{F}(\nabla^k(\eta u)) \, dx \]
\[ \leq \int_{B_t} \tilde{F}(\nabla^k u) \, dx + \int_{B_t} \tilde{F}(\nabla^k(\eta u)) - \tilde{F}(\nabla^k u) \, dx. \]  

Now using the minimising property of \( u \) and noting \( \eta u = w \) on \( B_t \) we get
\[ \int_{B_t} \tilde{\varphi}(|\nabla^k(\eta u)|) \, dx \]
\[ \leq \int_{B_t} \tilde{F}(\nabla^k((1 - \eta)w)) \, dx + \int_{B_t} \tilde{F}(\nabla^k(\eta u)) - \tilde{F}(\nabla^k u) \, dx \]
\[ \leq C \int_{B_t \setminus B_s} \tilde{\varphi}(|\nabla^k((1 - \eta)w)|) + \tilde{\varphi}(|\nabla^k(\eta u)|) + \tilde{\varphi}(|\nabla^k u|) \, dx \]
\[ \leq C \int_{B_t \setminus B_s} \tilde{\varphi}(|\nabla^k u|) \, dx + C \sum_{j=0}^{k-1} \int_{B_t} \tilde{\varphi} \left( \frac{|\nabla^j w|}{(s-t)^{k-j}} \right) \, dx, \]
using the \( \Delta_2 \)-condition. By filling the hole, setting \( \theta = \frac{C}{C + 1} \in (0, 1) \) we get
\[ \int_{B_t} \tilde{\varphi}(|\nabla w|) \, dx \leq \theta \int_{B_t} \tilde{\varphi}(|\nabla w|) \, dx + C \sum_{j=0}^{k-1} \int_{B_t} \tilde{\varphi} \left( \frac{|\nabla^j w|}{(s-t)^{k-j}} \right) \, dx. \]

Now by the interpolation estimate (Lemma 2.6) we can estimate
\[ C \sum_{j=0}^{k-1} \int_{B_t} \tilde{\varphi} \left( \frac{|\nabla^j w|}{(s-t)^{k-j}} \right) \, dx \]
\[ \leq \frac{1}{2} - \frac{\theta}{2} \int_{B_t} \tilde{\varphi}(|\nabla^k w|) \, dx + C \int_{B_t} \tilde{\varphi} \left( \frac{|w|}{(s-t)^k} \right) \, dx, \]
allowing us to absorb the intermediate terms. Finally by an iteration argument (adapting [20], Lemma 3.1) we deduce that
\[ \int_{B_{R/2}} \tilde{\varphi}(|\nabla^k w|) \, dx \leq C \int_{B_R} \tilde{\varphi} \left( \frac{|w|}{R^k} \right) \, dx, \]
as required. \( \square \)

3.2. Harmonic approximation. Our second ingredient will involve approximation by solutions \( h \) to the linearised equation, adapting a recent strategy of Gmeineder & Kristensen [37], which has also been applied in [36, 29, 41]. A key feature of this argument is that we only use the Legendre-Hadamard ellipticity condition (2.29) derived in Remark 2.7 and that \( u \) is \( F \)-extremal, so it can be applied more generally to infer regularity in situations where a suitable Caccioppoli inequality holds. This was exploited by the author in [41], and we will also use this observation in Section 4 for the case of strong local minima.

We will also need some additional estimates on the integrand \( F \). Considering the shifted integrand \( F_{z_0} \) defined in (3.1) where \( z_0 \in \mathcal{M} \) with \( |z_0| \leq M \), we recall that \( F''_{z_0}(0) = F''(z_0) \) satisfies the Legendre-Hadamard ellipticity bound (2.29). Further we also need the perturbation estimate
\[ |F'_{z_0}(z) - F'_{z_0}(0)|z| \leq C \omega_M(|z|) \left( |z| + \varphi_{|z_0|}(|z|) \right). \]
Here \( \omega_M \) denotes the modulus of continuity of \( F'' \) on \(|z| \leq 2M\}, that is \( \omega_M \) is a non-negative non-increasing continuous concave function \([0, \infty) \rightarrow [0, 1]\) satisfying
\(\omega_M(0) = 0\) and

\[
(F''(z) - F''(w)) \leq 2G(M)|z - w|
\]

for all \(z, w \in \mathbb{M}_k\) such that \(|z|, |w| \leq M + 1\). The claimed estimate can be obtained by combining the above with the growth bound \((3.3)\).

**Lemma 3.2** (Harmonic approximation). Let \(u \in W^{k,\varphi}(\Omega, \mathbb{R}^N)\) be \(F\)-extremal where \(F\) satisfies Hypotheses \((1.1)\) and let \(M > 0, \delta > 0\). Then if \(a : \mathbb{R}^n \to \mathbb{R}^n\) is a \(k^{th}\) order polynomial such that \(|\nabla^k a| \leq M\), for any \(B_R(x_0) \subset \Omega\) the Dirichlet problem

\[
(3.12) \quad \left\{ \begin{array}{ll}
(-1)^k \nabla^k : F^\prime((\nabla^k a)\nabla^k h = 0 & \text{in } B_R(x_0), \\
\partial \nabla^k h = \partial \nabla^k(u - a) & \text{on } \partial B_R(x_0), \quad 1 \leq j \leq k - 1,
\end{array} \right.
\]

admits a unique solution \(h \in w + W_0^{k,\varphi}(\Omega, \mathbb{R}^N)\), and further satisfies the estimates

\[
(3.13) \quad \int_{B_R(x_0)} \varphi_{1+M}(|\nabla^k(h - a)|) \, dx \leq C \int_{B_R(x_0)} \varphi_{1+M}(|\nabla^k(u - a)|) \, dx,
\]

and

\[
(3.14) \quad \int_{B_R(x_0)} \varphi_{1+M}\left(\frac{|\nabla^{k-1}(u - a - h)|}{R}\right) \, dx \leq \delta \int_{B_R(x_0)} \varphi_{1+M}\left(\frac{|\nabla^k(u - a)|}{R}\right) \, dx + C \gamma_M \left(\int_{B_R} \varphi_{1+M}(|\nabla^k(u - a)|) \, dx\right) \int_{B_R} \varphi_{1+M}(\nabla^k(u - a)) \, dx.
\]

for all \(\delta \in (0, 1)\). Here \(\gamma_M : [0, \infty) \to [0, \infty)\) is a non-decreasing continuous function such that \(\gamma_M(0) = 0\) and we have \(C = C(n, N, k, \nu, \Delta_2(\varphi), \nabla_2(\varphi), M, G(M), \delta) > 0\).

**Proof.** Suppressing the \(x_0\)-dependence, set \(w = u - a, \tilde{F} = F_{\nabla^k a}\), and \(\tilde{\varphi} = \varphi_{1+|\nabla^k a|}\) noting that \(\tilde{\varphi} \sim \varphi_{1+M}\). We know by Proposition \(2.5\) that a unique solution \(h\) exists and satisfies the modular estimate \((3.13)\) and since \(w\) is \(\tilde{F}\)-extremal, using respective weak formulations \((1.2), (3.12)\) we have

\[
(3.15) \quad \int_{B_R} \tilde{F}''(0) \nabla^k(w - h) : \nabla^k q \, dx = \int_{B_R} \left(\tilde{F}''(0) \nabla^k w - \tilde{F}''(\nabla^k w)\right) : \nabla^k q \, dx,
\]

for \(q \in W_0^{k,\infty}(B_R, \mathbb{R}^N)\). Following \([37]\) we wish to choose our test function so we obtain \(\tilde{\varphi}(R^{-1} |\nabla^{k-1}(w - h)|)\) on the left-hand side, so to achieve this we choose \(q\) to be the solution of the dual problem

\[
(3.16) \quad \left\{ \begin{array}{ll}
(-1)^k \nabla^k : F''((\nabla^k a)\nabla^k q = (-1)^{k-1} \nabla^{k-1} : g & \text{in } B_R, \\
\partial \nabla^k q = 0 & \text{on } \partial B_R, \quad 0 \leq j \leq k - 1,
\end{array} \right.
\]

where

\[
(3.17) \quad g = \tilde{\varphi}\left(\frac{|\nabla^{k-1}(w - h)|}{R}\right) \nabla^{k-1}(w - h)\left|\nabla^{k-1}(w - h)|^2\right.\
\]

Since \(t \mapsto \tilde{\varphi}(t)\) is an \(N\)-function satisfying \(\tilde{\varphi}(\tilde{\varphi}(t)/t) \sim \tilde{\varphi}(t)\) uniformly in \(t\), we have \(g\) is well-defined and satisfies the estimate

\[
(3.18) \quad \int_{B_R} \tilde{\varphi}^*(R|g|) \, dx \leq C \int_{B_R} \tilde{\varphi}\left(\frac{|\nabla^{k-1}(w - h)|}{R}\right) \, dx.
\]

Hence by Lemma \(2.10\) we can write \(g = -\nabla \cdot G\) with \(G \in L^{\frac{\infty}{\tilde{\varphi}^*}}(B_R, \mathbb{M}_k)\), so then Proposition \(2.5\) gives the existence of a unique \(q \in W_0^{k,\varphi^{\tilde{\varphi}^*}}(B_R, \mathbb{R}^N)\) and a
corresponding modular estimate, which combining with (3.18) gives

\[
(3.19) \quad \left( \int_{B_R} \tilde{\varphi}^* \left( |\nabla^k q| \right)^{\frac{m}{m-1}} \, dx \right)^{\frac{m-1}{m}} \leq C \int_{B_R} \tilde{\varphi}^* \left( \frac{|\nabla^{k-1}(w - h)|}{R} \right) \, dx.
\]

We cannot test the Euler-Lagrange system against \( q \) in general however, so we will need to take a higher order Lipschitz truncation using Proposition 2.11. Letting \( \lambda > 0 \), applying the lemma with the \( N \)-function \( (\tilde{\varphi}^*)^{\frac{m}{m-1}} \) gives \( \theta_\lambda \in W^{k,\infty}(B_R, \mathbb{R}^N) \) satisfying \( \|\nabla^k \theta_\lambda\|_{L^\infty(B_{R,M\lambda})} \leq C\lambda \) and

\[
(3.20) \quad \int_{B_R} \tilde{\varphi}^* \left( |\nabla^k \theta_\lambda| \right)^{\frac{m}{m-1}} \, dx \leq C \int_{B_R} \tilde{\varphi}^* \left( |\nabla^k q| \right)^{\frac{m}{m-1}} \, dx,
\]

\[
(3.21) \quad \tilde{\varphi}^*(\lambda \|\nabla^k q\|^{\frac{m}{m-1}}) \mathcal{L}^n \{ x \in B_R : q(x) \neq q_\lambda(x) \} \leq C \int_{B_R} \tilde{\varphi}^* \left( |\nabla^k q| \right)^{\frac{m}{m-1}} \, dx.
\]

For this choice of \( q_\lambda \) we have for each \( \delta \in (0, 1) \),

\[
\begin{align*}
\int_{B_R} \tilde{\varphi}^* \left( \frac{|\nabla^{k-1}(w - h)|}{R} \right) \, dx &= \int_{B_R} (\tilde{\varphi}^*(0) |\nabla^k w| - \tilde{\varphi}^*(|\nabla^k w|)) \, dx \\
&\quad + \int_{B_R} \tilde{\varphi}^*(0) |\nabla^k q - \nabla^k \theta_\lambda| \, dx \\
&\leq C\lambda \int_{B_R} \omega_M(|\nabla^k w|) \left( |\nabla^k w| + \tilde{\varphi}^*(|\nabla^k w|) \right) \, dx \\
&\quad + C \int_{B_R} |\nabla^k w| |\nabla^k q - \nabla^k \theta_\lambda| \, dx \\
&\leq C \int_{B_R} 2\delta \tilde{\varphi}(|\nabla^k w|) + \delta \tilde{\varphi}^*(\tilde{\varphi}(|\nabla^k w|)) \, dx \\
&\quad + C \int_{B_R} \delta \tilde{\varphi} \left( \frac{\lambda}{\delta} |\nabla^k w| \right) + \delta \tilde{\varphi}^* \left( \frac{\lambda}{\delta} \omega_M(|\nabla^k w|) \right) \, dx \\
&\quad + C \int_{B_R} \delta \tilde{\varphi}^* \left( \frac{|\nabla^k q - \nabla^k \theta_\lambda|}{\delta} \right) \, dx \\
&= C (I + II + III),
\end{align*}
\]

where we have used the perturbation estimate \( (3.10) \) and Young’s inequality \( (2.9) \). It remains to estimate each of these terms separately; for the first inequality we use the fact that \( \tilde{\varphi}^*(\tilde{\varphi}(t)) \leq C\varphi(t) \) to estimate

\[
(3.23) \quad I \leq C\delta \int_{B_R} \tilde{\varphi}(|\nabla^k w|) \, dx,
\]

and for the second term we use the fact that \( \omega_M \leq 1 \) and apply Jensen’s inequality to the concave function \( \omega_M \circ \tilde{\varphi}^{-1} \) to estimate

\[
(3.24) \quad II \leq \delta \left( \tilde{\varphi} \left( \frac{\lambda}{\delta} \right) + \tilde{\varphi}^* \left( \frac{\lambda}{\delta} \right) \right) \int_{B_R} \omega_M(|\nabla^k w|) \, dx \\
\quad \leq \delta^{1 + \max(p_u,p_u^*)} \left( \tilde{\varphi}(\lambda) + \tilde{\varphi}^*(\lambda) \right) \omega_M \circ \tilde{\varphi}^{-1} \left( \int_{B_R} \tilde{\varphi}(|\nabla^k w|) \, dx \right).
\]
For the third term we apply Hölder’s inequality along with the Lipschitz truncation estimates (3.20), (3.21) to estimate

\[
\begin{aligned}
III & \leq \delta^{1-\rho_w} \left( \frac{\mathcal{L}^n(B_R \cap \{ q \neq q_\lambda \})}{\mathcal{L}^n(B_R)} \right)^{\frac{1}{N}} \left( \int_{B_R} \tilde{\varphi}^* \left( \frac{\left| \nabla k (q - q_\lambda) \right|}{\tilde{\varphi}^* (q_\lambda)} \right)^{\frac{n-1}{n}} \right)^{\frac{n}{n-1}} \\
& \leq C_8 \left( \frac{\int_{B_R} \tilde{\varphi}^* \left( \frac{\left| \nabla k (q - q_\lambda) \right|}{\tilde{\varphi}^* (q_\lambda)} \right)^{\frac{n-1}{n}} \right)^{\frac{n}{n-1}} \left( \int_{B_R} \tilde{\varphi}^* \left( \frac{\left| \nabla k (q - q_\lambda) \right|}{\tilde{\varphi}^* (q_\lambda)} \right)^{\frac{n}{n-1}} \right)^{\frac{n}{n-1}} \\
& \leq C_8 \left( \frac{\int_{B_R} \tilde{\varphi} \left( \frac{\left| \nabla k (q - q_\lambda) \right|}{\tilde{\varphi}^* (q_\lambda)} \right) \right)^{\frac{n}{n-1}} \left( \int_{B_R} \tilde{\varphi} \left( \frac{\left| \nabla k (q - q_\lambda) \right|}{\tilde{\varphi} (q_\lambda)} \right) dx \right)^{\frac{n}{n-1}}.
\end{aligned}
\]

(3.25)

We can now choose \( \lambda > 0 \) sufficiently large so the third term can be absorbed to the left-hand side; for this we can take \( \lambda \) to satisfy

\[
(3.26) \quad C_8 \left( \frac{\int_{B_R} \tilde{\varphi} \left( \frac{\left| \nabla k (q - q_\lambda) \right|}{\tilde{\varphi}^* (q_\lambda)} \right) \right)^{\frac{n}{n-1}} = \frac{1}{2}.
\]

Now using the doubling property for \( \tilde{\varphi}, \tilde{\varphi}^* \) and writing \( \vartheta_M(t) = t + \tilde{\varphi} \circ (\tilde{\varphi}^*)^{-1}(t) \) we get

\[
\begin{aligned}
\int_{B_R} \tilde{\varphi} \left( \frac{\left| \nabla k (w - h) \right|}{R} \right) \right) dx & \leq C_8 \vartheta_M \left( \int_{B_R} \tilde{\varphi} \left( \frac{\left| \nabla k (w - h) \right|}{R} \right) \right) dx \\
& + C_8 \vartheta_M \left( \int_{B_R} \tilde{\varphi} \left( \frac{\left| \nabla k (w - h) \right|}{R} \right) \right) dx \omega_M \circ \tilde{\varphi}^{-1} \left( \int_{B_R} \tilde{\varphi} \left( \frac{\left| \nabla k (w - h) \right|}{R} \right) \right) dx.
\end{aligned}
\]

(3.27)

Finally note that since \( w - h \in W_0^{k, \varphi}(B_R, \mathbb{R}^N) \), by the Poincaré inequality and (3.13) we have

\[
\begin{aligned}
\int_{B_R} \tilde{\varphi} \left( \frac{\left| \nabla k (w - h) \right|}{R} \right) dx & \leq C \int_{B_R} \tilde{\varphi} \left( \frac{\left| \nabla k (w - h) \right|}{R} \right) dx \\
& \leq C \int_{B_R} \tilde{\varphi} \left( \frac{\left| \nabla k (w - h) \right|}{R} \right) dx,
\end{aligned}
\]

(3.28)

so setting \( \gamma_M(t) = \frac{\vartheta_M(t)}{t} \omega_M \circ \tilde{\varphi}^{-1}(t) \) the result follows. \( \square \)

3.3. **Excess decay estimate.** We now combine the above estimates to conclude, which will involve establishing estimates for the *excess energies* defined for \( M > 0 \) as

\[
E_M(x, r) = \int_{B_r(x)} \varphi_{1+M} \left( \left| \nabla k u - (\nabla k u)_{B_r(x)} \right| \right) dy.
\]

**Lemma 3.3** (Excess decay estimate). Let \( u \in W^{k, \varphi}(\Omega, \mathbb{R}^N) \) be a minimiser of (1.1) where the integrand \( F \) satisfies Hypotheses (1.1) and let \( M > 0, \delta > 0 \). Then for all \( \sigma \in (0, \frac{1}{2}) \) and \( \delta \in (0, 1) \), letting \( \gamma_M \) as in Lemma 3.2 if \( B_R(x_0) \subset \Omega \) such that \( \left| (\nabla k u)_{B_R(x_0)} \right| \leq M \) and \( E_M(x_0, R) \leq 1 \) we have

\[
E_M(x_0, \sigma R) \leq C (\sigma + C_\sigma \delta + C_{\sigma, \delta} \gamma_M(E_M(x_0, R))) E_M(x_0, R),
\]

(3.30)

where the constants depend also on \( n, N, k, \nu \Delta_2(\varphi), \nabla_2(\varphi), M, G(M) \).

**Proof.** We let \( a_1 : \mathbb{R}^n \rightarrow \mathbb{R}^N \) be a \( k \)th order polynomial satisfying

\[
\int_{B_R(x_0)} D^k(u - a_1) dx = 0
\]

(3.31)
for all \(|\alpha| \leq k\) and apply Lemma 3.2 on \(B_R(x_0)\) with \(a_1\), obtaining the unique solution \(h\) to

\[
(3.32) \quad \begin{cases}
(-1)^k \nabla^k : F''(\nabla^k a_1) \nabla^k h = 0 & \text{in } B_R \\
\partial^j\nabla^k h = \partial^j\nabla^k (u - a_1) & \text{on } \partial B_R, \quad 0 \leq j < k - 1.
\end{cases}
\]

Combining the Poincaré inequality with the approximation estimate (3.14) we have

\[
\int_{B_R(x_0)} \varphi_{1+M} \left( \frac{|u - a_1 - h|}{R^\gamma} \right) \, dx \\
\leq \int_{B_R(x_0)} \varphi_{1+M} \left( \frac{\nabla^{k-1}(u - a_1 - h)}{R} \right) \, dx \\
\leq (\delta + C_\delta \gamma M(E_M(x_0, R))) E_M(x_0, R).
\]

Now let \(a_2(x) = \sum |\alpha| \leq k \frac{D^\alpha h(x_0)}{\alpha!} (x - x_0)^\alpha\), and put \(a = a_1 + a_2\). This satisfies \(\varphi_{1+M}(\nabla^k a_2) \leq C(M + 1)\), \(|\nabla^k a| \leq C(M)\) and hence \(\varphi_{1+\nabla^k a} \sim \varphi_{1+M}\). Therefore applying the Caccioppoli inequality (Lemma 3.1) with \(a\) on \(B_{2\sigma R}(x_0)\) we obtain

\[
E_M(x_0, \sigma R) \leq C \int_{B_{2\sigma R}(x_0)} \varphi_{1+M} \left( \frac{|u - a|}{(2\sigma R)^\gamma} \right) \, dx \\
\leq C_\sigma \int_{B_{R}(x_0)} \varphi_{1+M} \left( \frac{|u - a_1 - h|}{R^\gamma} \right) \, dx \\
+ C \int_{B_{2\sigma R}(x_0)} \varphi_{1+M} \left( \frac{|h - a_2|}{(2\sigma R)^\gamma} \right) \, dx.
\]

Now as \(h\) is \(F''(\nabla^k a)\)-harmonic, by interior regularity estimates (see for instance [96, Section 5.1]) we have

\[
\sup_{B_{2\sigma R}} \frac{|h(x) - a_2(x)|}{(2\sigma R)^\gamma} \leq \sigma \sup_{B_{R/2}(x_0)} |\nabla^2 h| \leq \varphi_{1+M}^{-1}(\sigma E_M(x, R)),
\]

so together with (3.33) we obtain

\[
(3.36) \quad E_M(x_0, \sigma R) \leq C_\sigma (\delta + C_\delta \gamma M(E_M(x_0, R))) E_M(x_0, R) + C\sigma E_M(x_0, R),
\]

which is the desired estimate.

We can now conclude in the usual way.

**Proof of Theorem 1.2.** Suppose \(B_R(x_0) \subset \Omega\) such that \(|\nabla^k u|_{B_R(x_0)}| \leq M\), and let \(x \in B_{R/2}(x_0)\). Then since \(|\nabla^k u|_{B_{R/2}(x)}| \leq 2^n M\) and \(E_M(x, R/2) \leq 2^n \varepsilon \leq 1\) shrinking \(\varepsilon\) if necessary, we can apply Lemma 3.3 above with \(M\) replaced by \(2^n M\) obtain

\[
(3.37) \quad E_M(x, \sigma R/2) \leq C(\sigma + C_\delta \delta + C_\delta \gamma M(\varepsilon)) E_M(x, R/2),
\]

for \(\sigma \in (0, \frac{1}{4})\) and \(\delta > 0\). We now choose \(\sigma > 0\) sufficiently small to ensure \(C\sigma < \frac{1}{4} \sigma^{2a}\) and \(\delta > 0\) small enough so \(C\sigma \delta < \frac{1}{4} \sigma^{2a}\), and finally \(\varepsilon > 0\) so that \(CC\delta \gamma M(\varepsilon) < \frac{1}{4} \sigma^{2a}\), which gives

\[
(3.38) \quad E_M(x, \sigma R/2) \leq \sigma^{2a} E_M(x, R/2).
\]

Now we iterate the above and argue by induction that we can ensure that

\[
(3.39) \quad |(\nabla^k u)|_{B_{\sigma R/2}(x)}| \leq 2^{n+1} M,
\]

\[
(3.40) \quad E_M(x, \sigma^j R/2) \leq \sigma^{2^j a} E_M(x, R/2)
\]

for all \(|\alpha| \leq k\) and apply Lemma 3.2 on \(B_R(x_0)\) with \(a_1\), obtaining the unique solution \(h\) to

\[
(3.32) \quad \begin{cases}
(-1)^k \nabla^k : F''(\nabla^k a_1) \nabla^k h = 0 & \text{in } B_R \\
\partial^j\nabla^k h = \partial^j\nabla^k (u - a_1) & \text{on } \partial B_R, \quad 0 \leq j < k - 1.
\end{cases}
\]
for all \( j \geq 1 \), provided \( \varepsilon > 0 \) is sufficiently small. If this holds for all \( 1 \leq i \leq j \), then note first that for such \( i \) we use Jensen’s inequality to bound

\[
|((\nabla^k u)_{B_\varepsilon(x,1/2)} - (\nabla^k u)_{B_{\sigma^{-1}R/2}(x)}| 
\leq \frac{\sigma^{-n}}{B_{\sigma^{-1}R/2}(x)} |\nabla^k u - (\nabla^k u)_{B_{\sigma^{-1}R/2}(x)}| \, dx
\]

(3.41)

Hence by shrinking \( \varepsilon > 0 \) further if necessary to ensure that \( \varphi_1^{-1}(\delta \varepsilon) \leq C \sqrt{\delta} \) for all \( \delta \in (0,1) \), we can estimate

\[
|((\nabla^k u)_{B_\varepsilon(x,1/2)} - (\nabla^k u)_{B_{\sigma^{-1}R/2}(x)} - (\nabla^k u)_{B_{\sigma^{-1}R/2}(x)}| 
\leq \frac{\sigma^{-n}}{B_{\sigma^{-1}R/2}(x)} (E_M(x, \sigma^{i-1}R/2)) 
\leq \frac{\sigma^{-n}}{B_{\sigma^{-1}R/2}(x)} (\sigma^{2(i-1)\alpha \varepsilon}).
\]

Since this holds for all \( x \in B_{R/2}(x) \) and \( 0 < r < R/2 \), by the Campanato-Meyers characterisation of Hölder continuity (see for instance [14, Theorem 2.9]) it follows that \( \nabla^k u \) is \( C^{0,\alpha} \) in \( B_{R/2}(x_0) \).

\[ \Box \]

4. Extension to strong local minimisers

We will conclude our discussion with a straightforward extension of the above result to the setting of \( W^{k,\psi} \)-local minimisers. To be more precise, we consider the following.

**Definition 4.1.** Let \( F \) satisfy Hypotheses [11] and \( \psi \) be an \( N \)-function. Then we say \( u \in W^{k,\psi}(\Omega,\mathbb{R}^N) \) is a strong \( W^{k,\psi} \)-local minimiser if there exists \( \delta > 0 \) such that if \( \xi \in W^{k,\psi}_0(\Omega,\mathbb{R}^N) \) with \( \int_{\Omega} \psi(|\nabla^k \xi|) \, dx < \delta \), then we have

\[
F(u) \leq F(u + \xi).
\]

(4.1)

We also say \( u \) is a \( W^{k,\infty} \)-local minimiser if [11] is satisfied whenever \( \xi \in W^{k,\infty}_0(\Omega,\mathbb{R}^N) \) such that \( \|\nabla^k \xi\|_{L^\infty(\Omega,\mathbb{R}^N)} < \delta \).

The regularity of strong \( W^{1,p} \)-minimisers was first considered by Kristensen \& Taheri [44], who established a partial regularity theorem in the case of superquadratic \((p \geq 2)\) growth. These results have been extended for instance in [13, 60, 10, 11].

**Theorem 4.2** (\( \varepsilon \)-regularity theorem of local minimisers). Let \( F \) satisfy Hypotheses [11] and let \( u \in W^{k,\psi}(\Omega,\mathbb{R}^N) \) be a strong \( W^{k,\psi} \)-local minimiser in a bounded
Lemma 4.4

suffice to complete the argument. However one can still establish a weaker form which will
inequality; to apply the minimising condition we need to work on balls of sufficiently

\[ \lambda > 0 \] and \( \Omega' \subseteq \Omega \). For \( W^{k,\infty} \)-local minimisers we additionally require that
\( u \in W^{k,\infty}_{\text{loc}}(\Omega, \mathbb{R}^N) \) and

\[ \limsup_{r \to 0} \left( \text{ess sup}_{y \in B_r(x)} |\nabla^k u - (\nabla^k u)_{B_r(x)}| \right) < \delta \]

uniformly in \( x \in \Omega \). Then for each \( M > 0 \) and \( \alpha \in (0,1) \), there exists \( \varepsilon > 0 \) and
\( R_0 > 0 \) such that for any \( B_R(x_0) \subseteq \Omega \) with \( R < R_0 \) for which \( |(\nabla^k u)_{B_R(x_0)}| \leq M \)
and

\[ E_M(x_0, R) = \int_{B_R(x_0)} \varphi_{1+M}(|\nabla^k u - (\nabla^k u)_{B_R(x_0)}|) \, dx \leq \varepsilon, \]

we have \( u \) is of class \( C^{k,\alpha} \) in \( B_{R/2}(x_0) \).

Remark 4.3. If \( \psi \) satisfies the \( \Delta_2 \) condition then (4.2) is equivalent to requiring that
\( u \in W^{k,\infty}_{\text{loc}}(\Omega, \mathbb{R}^N) \), however extra care is needed in the absence of the
\( \Delta_2 \)-condition due the failure of suitable density results in \( L^\psi \). To the best of our
knowledge is not known if this additional condition (4.2) is necessary, even in the
polynomial case when \( \psi = t^q \).

In the \( W^{k,\infty} \) case however, the construction in [14, Section 7] implies that it is
insufficient to assume that \( u \) is a \( W^{k,\infty} \)-local minimiser that lies in \( W^{k,\infty}_{\text{loc}}(\Omega, \mathbb{R}^N) \).

The key difference from the minimising setting is the lack of a full Caccioppoli
inequality; to apply the minimising condition we need to work on balls of sufficiently
small radii, which prevents us from applying the iteration argument used in the
proof of Lemma 3.1. However one can still establish a weaker form which will suffice
to complete the argument.

Lemma 4.4 (Caccioppoli-type inequality). Assume the setup of Theorem 4.1 and
let \( \kappa \in (0,1) \). Then there is \( R_0 > 0 \) and \( \varepsilon > 0 \) such that if \( B_r(x) \subseteq B_{R_0}(x_0) \subseteq \Omega \)
for which \( |(\nabla^k u)_{B_r(x)}| \leq M \) and \( E_M(x,r) \leq \varepsilon \), then define \( a_{x,r} \) such that \( \nabla^{k-1} a_{x,r} \)
satisfies Lemma 2.3 with \( \nabla^{k-1} u \) and that

\[ \int_{B_r} D^\alpha (u - a_{x,r}) \, dy = 0 \]

for all \( |\alpha| \leq k - 2 \). Then we have

\[ \int_{B_{r/2}(x)} \varphi_{1+M}(|\nabla^k u - \nabla^k a_{x,r}|) \, dy \]

\[ \leq \kappa \int_{B_r(x)} \varphi_{1+M}(|\nabla^k u - \nabla^k a_{x,r}|) \, dy + C \int_{B_r(x)} \varphi_{1+M} \left( \frac{|a_{x,r}|}{r^k} \right) \, dy. \]

Proof. Fix \( \zeta \in (0,1) \) to be specified later, and let \( 0 < t < s < r \) such that
\( r \leq \zeta(s-t) \). Then let \( \eta \in C_\infty^\infty(B_R(x_0)) \) be a cutoff such that \( 1_{B_r(x)} \leq \eta \leq 1_{B_s(x)} \)
and \( |\nabla^j \eta| \leq \frac{C}{(s-t)^j} \) for all \( 0 \leq j \leq k \). We wish to use the minimising condition with
\( \xi = \eta(u - a_{x,r}) \), so we need to verify this is possible provided \( R \) is sufficiently small.

In the case of an \( N \)-function \( \psi \), we will use the Poincaré inequality proved in
[7, Lemma 1] which is valid for general \( N \)-functions. We claim this allows us to
estimate
\[
\int_{\Omega} \psi(|\nabla \xi|) \, dy \leq \sum_{j=0}^{k} \int_{B_r(x)} \psi \left( C \frac{|\nabla^j (u - a_{x,r})|}{(s-t)^{k-j}} \right) \, dy
\]
(4.7)
\[
\leq C \sum_{j=0}^{k} \int_{B_r(x)} \psi \left( \frac{C \zeta^{k-j}}{(s-t)^{k-j}} |\nabla^k (u - a_{x,r})| \right) \, dy
\]
\[
\leq C \int_{B_{\gamma}(x)} \psi \left( C \zeta^{-k} (|\nabla^k u| + M) \right) \, dy,
\]
which can be made less than \( \delta \) by choosing \( R \) sufficiently small. Indeed we have used the additivity of \( \psi(s + t) \leq \psi(2s) + \psi(2t) \) in the first line, followed by iteratively applying the Poincaré inequality using (4.5) which applies for all \( |\alpha| \leq k - 1 \) by choice of \( \nabla^{k-1} a_{x,r} \). For the \( W^{k,\infty} \) case we can estimate
\[
\left\| \nabla^k \xi \right\|_{L^\infty(\Omega,M_k)} \leq \left\| \nabla^k u - \nabla^k u_{B(x)} \right\|_{L^\infty(B_{(x)},M_k)} + \left| \nabla^k a_{x,r} - \nabla^k u_{B(x)} \right|
\]
\[
\leq C \sum_{j=0}^{k-1} \left\| \nabla^j (u - a_{x,r}) \right\|_{L^\infty(B_{(x)},M_j)}
\]
(4.8)
\[
+ C \sum_{j=0}^{k-1} \zeta^{j-k} \left\| r^{k-j} (\nabla^j (u - a_{x,r})) \right\|_{L^\infty(B_{(x)},M_j)},
\]
for \( r < R_0 \) sufficiently small, where we have used the additional condition (4.3).
Now an analogous argument as in [44] allows us to prove the pointwise estimate
\[
\left\| r^{k-j} (\nabla^j (u - a_{x,r})) \right\|_{L^\infty(B_{(x)},M_j)} \leq (2N)^{k-j} \left\| \nabla^k (u - a_{x,r}) \right\|_{L^\infty(B_{(x)},M_k)},
\]
(4.9)
and we can also apply (2.19) from Lemma 2.5 to estimate
\[
\left| \nabla^k a_{x,r} - \nabla^k u_{B(x)} \right| \leq \varphi^{-1}_{1+M}(CEM(x_0,R)).
\]
(4.10)
Hence an application of Arzelà-Ascoli gives the interpolation estimate
\[
\left\| r^{k-j} (\nabla^j (u - a_{x,r})) \right\|_{L^\infty(B_{(x)},M_j)} \leq \gamma \left\| \nabla^k (u - a_{x,r}) \right\|_{L^\infty(B_{(x)},M_k)} + C \gamma \varphi^{-1}_{1+M}(\varepsilon),
\]
(4.11)
for all \( \gamma > 0 \), which can be chosen to sufficiently small by shrinking \( \gamma, \varepsilon > 0 \) as necessary. Combining the above gives \( \left\| \nabla^k \xi \right\|_{L^\infty(\Omega,M_k)} < \delta \).
Now we can argue as in Lemma 3.1 from the minimizing case; writing \( \tilde{\varphi} = \varphi^{-1}_{1+M} \), noting that \( w_{x,r} = u - a_{x,r} \) is \( F_{\nabla^k a_{x,r}} \)-extremal we can estimate
\[
\int_{B_{s_l}(x)} \tilde{\varphi} (|\nabla^k w_{x,r}|) \, dy \leq \theta \int_{B_{s_l}(x)} \tilde{\varphi} (|\nabla^k w_{x,r}|) + C \int_{B_{s_l}} \tilde{\varphi} \left( \frac{|w_{x,r}|}{(s-t)^{k}} \right) \, dy,
\]
(4.12)
analogously to (3.16) - (3.18), where \( \theta > 0 \) is given and independent of \( s, t \). Then there is some \( \ell \geq 1 \) such that \( \theta^\ell \leq \kappa \), so we will iteratively apply this estimate \( \ell \)-times.
For this take \( s_j = (1 - \frac{j}{\ell}) r, t_j = (1 - \frac{j-1}{\ell}) r \) for \( 1 \leq j \leq \ell \), and note that \( t_0 = r, s_\ell = \frac{r}{\ell} \), and \( (t_j - s_j) = \frac{r}{\ell} \). Therefore taking \( \zeta = \frac{r}{\ell} \), we can apply (4.12) with \( s_j, t_j \) and chain the inequalities to get
\[
\int_{B_{s_j}(x)} \tilde{\varphi} (|\nabla^k w_{x,r}|) \, dy \leq \kappa \int_{B_{s_j}(x)} \tilde{\varphi} (|\nabla^k w_{x,r}|) + C \int_{B_{s_j}} \tilde{\varphi} \left( \frac{|w_{x,r}|}{R^\alpha} \right) \, dy,
\]
(4.13)
Proof of Theorem 4.2. From here the proof of Theorem 4.2 proceeds analogously to the minimising case. Note that both strong $W^{k,\psi}$ and $W^{k,\infty}$-local minimisers are $F$-extremal, so the harmonic approximation result (Lemma 3.2) can be applied to $u$. We will sketch the remaining argument, pointing out the key modifications.

**Proof of Theorem 4.2** For $B_r(x) \subset B_R(x_0)$ we will consider the slightly modified excess energy

$$E_M(x, r) = \int_{B_r(x)} \varphi_{1+M}(|\nabla^k(u-a_{x,r})|) \, dy,$$

with $a_{x,r}$ as in Lemma 4.4, which satisfies $E_M(x,r) \leq C E_M(x, r)$ by convexity of $\varphi$ and (2.19). Then similarly as in Lemma 3.3 we claim that if $|\nabla^k a_{x,2\sigma r}|, |\nabla^k a_{x,r}| \leq M$ and $E_M(x,r) \leq 1$, then for each $\kappa, \delta \in (0, 1)$, $\sigma \in (0, \frac{1}{2})$ we have the excess decay estimate

$$E_M(x, \sigma r) \leq C \left( \sigma + C_\alpha (k+\delta) + C_\sigma \delta \gamma_M(E_M(x,r)) \right) E_M(x,r),$$

where $\gamma_M$ is as in Lemma 3.2. Taking $a_1 = a_{x,r}$ we apply Lemma 3.2 to obtain the unique solution $h \in u - a_1 + W^{k,\psi}_0(B_r(x), \mathbb{R}^N)$ to the problem

$$(-1)^k \nabla^k : F''(\nabla^k a_1) \nabla^k h = 0$$

in $B_r(x)$. Then we define $a_2 = \sum_{|\alpha| \leq k} \frac{\partial^{\alpha} h(x)}{\alpha!} (x-x_0)^\alpha$ as before, then we can apply Lemma 4.4 to estimate

$$E(x, \sigma r) \leq \kappa \int_{B_{2\sigma r}} \varphi_{1+M}(|\nabla^k(u-a_{x,2\sigma r})|) \, dx$$

$$+ C \int_{B_{2\sigma r}} \varphi_{1+M} \left( \frac{|u-a_{x,2\sigma r}|}{(2\sigma r)^k} \right) \, dx$$

$$\leq C \kappa \int_{B_{2\sigma r}} \varphi_{1+M} \left( \frac{|\nabla^{k-1}(u-a_{x,r})|}{2\sigma r} \right) \, dx$$

$$+ C \int_{B_{2\sigma r}} \varphi_{1+M} \left( \frac{|\nabla^{k-1}(u-a_{x,2\sigma r})|}{2\sigma r} \right) \, dx$$

$$\leq C_\sigma \kappa E_M(x,r) + C \int_{B_{2\sigma r}} \varphi_{1+M} \left( \frac{\nabla^{k-1}|u-a_1-a_2|}{2\sigma r} \right) \, dx,$$

where we have used (2.20) followed by the Poincaré inequality on $B_r(x)$ for the first term, and the quasi-minima property (2.15) of $\nabla^{k-1} a_{x,2\sigma r}$. From here the rest follows by arguing exactly as in Lemma 3.3.

Given the above excess decay estimate, the theorem follows by an analogous iteration argument. A slight modification is needed to ensure that $|\nabla^k a_{x,2\sigma r}|, |\nabla^k a_{x,r}| \leq C(M)$, which can be done using (2.19) and a similar argument as in 3.41. We also choose $\kappa, \delta$ so that $CC_\sigma (k+\delta) \leq \frac{1}{2} \sigma^{2\alpha}$ and the rest follows as in the proof of Theorem 4.2, we will leave this for the reader to verify.

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