Schwarzschild-Randers solution on a Lorentz tangent bundle.

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In this work, we extend for the first time the spherically symmetric Schwarzschild and Schwarzschild-De Sitter solutions with a Randers-type perturbation which is generated by a covector \( A^\gamma \). This gives a locally anisotropic character to the metric and induces a deviation from the Riemannian models of gravity. A natural framework for this study is the Lorentz tangent bundle of a spacetime manifold. We apply the generalized field equations of this framework to the perturbed metric and derive the dynamics for the covector \( A^\gamma \). Finally, we find the timelike, spacelike and null paths on the Schwarzschild-Randers spacetime and we compare them with the geodesics of general relativity. The obtained solutions are new and they enrich the corresponding literature.

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I. INTRODUCTION

It is well known that the Schwarzschild metric constitutes a fundamental ingredient of general relativity. This metric describes the most general spherically symmetric solution to the Einstein field equations in a region of spacetime where the energy-momentum tensor vanishes.

In the framework of a general Finsler space, an extension of a locally anisotropic perturbation of a Finsler type Schwarzschild metric has been studied by different authors. In some of these works, possible observational predictions are given based on the direction-dependent structure of spacetime. We would like to point out that our study is realized in the framework of a Schwarzschild-Randers spacetime which is different from the works of other researchers on the Finsler extension of a Schwarzschild metric [1–11]. In our approach, we use sufficiently generalized Einstein field equations on a Lorentz tangent bundle of a spacetime manifold.

In the present paper we study a Schwarzschild metric in a special Finsler-like spacetime of Randers type. This study provides a locally anisotropic perturbation of the classical Schwarzschild metric of the Riemannian setting in a natural way, induced by a covector field of the base manifold. In addition, the geometrical framework that we use, namely the framework of a Lorentz tangent bundle of a Riemannian spacetime, contains additional degrees of freedom compared to classic gravity.

The generalized field equations which have been derived for this framework in [12] are applied on the perturbed metric of our Schwarzschild-Randers spacetime and are solved for the covector field.

Afterwards, we study particle paths for our generalized spacetime. We follow the approach in [12] which takes into account the effect of internal degrees of freedom on the point particle dynamics. We apply the solution of the covector derived in the previous sections and we obtain an explicit form for the path equations which is an extension of classical geodesics of general relativity.

II. PRELIMINARIES

The natural background space for a locally anisotropic gravity is the tangent bundle of a differentiable Lorentzian spacetime manifold called a Lorentz Tangent Bundle (we will refer to it as \( TM \) hereafter) [13, 14]. \( TM \) is itself an 8-dimensional differentiable manifold, so we can define coordinate charts and tensors on it in the usual way. \( TM \) is equipped with local coordinates \( \{ U^A \} = \{ x^\mu, y^a \} \) where \( x^\mu \) are the local coordinates on the base manifold \( M \) around...
Under a local coordinate transformation on the base manifold, the adapted basis vectors transform as:

\[ T \text{distribution} \]

The transformation rule for \( \{ \delta \} \) holds:

\[ \delta^{\mu} = \delta^{\alpha} \frac{\partial}{\partial x^{\mu}} - N_{\mu}^{\alpha}(x,y) \frac{\partial}{\partial y^{\alpha}}. \]  

(1)

and

\[ \frac{\partial}{\partial y^{\alpha}} \]

(2)

where \( N_{\mu}^{\alpha} \) are the components of a nonlinear connection. The curvature of the nonlinear connection is defined as

\[ \Omega^{\alpha}_{\beta\kappa} = \frac{\delta N_{\mu}^{\alpha}}{\delta x^{\beta}} - \frac{\delta N_{\mu}^{\alpha}}{\delta x^{\beta}}. \]  

(3)

The nonlinear connection induces a split of the total space \( TTM \) into a horizontal distribution \( T_{H}TM \) and a vertical distribution \( T_{V}TM \). The above-mentioned split is expressed with the Whitney sum:

\[ TTM = T_{H}TM \oplus T_{V}TM. \]  

(4)

The horizontal distribution or h-space is spanned by \( \delta^{\alpha} \), while the vertical distribution or v-space is spanned by \( \hat{\partial}_{\alpha} \).

Under a local coordinate transformation on the base manifold, the adapted basis vectors transform as:

\[ \delta^{\mu'} = \delta^{\mu} \frac{\partial}{\partial x^{\mu'}} \delta_{\mu}, \quad \hat{\partial}_{\alpha'} = \frac{\partial x^{\alpha}}{\partial x^{\alpha'}} \hat{\partial}_{\alpha}. \]  

(5)

with \( x^{\alpha} = \hat{x}^{\alpha} x^{\mu} \). The adapted dual basis of the adjoint total space \( T^*TM \) is \( \{ E_{\mu} \} = \{ dx^{\mu}, dy^{\alpha} \} \) with the definition

\[ \delta y^{\alpha} = dy^{\alpha} + N_{\mu}^{\alpha} dx^{\mu}. \]  

(6)

The transformation rule for \( \{ dx^{\mu}, dy^{\alpha} \} \) is:

\[ dx^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^{\mu}} dx^{\mu}, \quad \delta y^{\alpha'} = \frac{\partial x^{\alpha}}{\partial x^{\alpha'}} \delta y^{\alpha}. \]  

(7)

The bundle \( TM \) is equipped with a distinguished metric \((d\text{-metric}) \mathcal{G}:

\[ \mathcal{G} = g_{\mu\nu}(x,y) dx^{\mu} \otimes dx^{\nu} + v_{\alpha\beta}(x,y) \delta y^{\alpha} \otimes \delta y^{\beta} \]  

(8)

where the metric of the horizontal space (h-space) \( g_{\mu\nu} \) and the metric of the vertical space (v-space) \( v_{\alpha\beta} \) are defined to be of Lorentzian signature \((-++,+++)\). A tangent bundle of a Lorentzian spacetime manifold equipped with such a metric will be called a Lorentzian tangent bundle. In the rest of this work, the following homogeneity conditions will be assumed: \( g_{\mu\nu}(x,ky) = g_{\mu\nu}(x,y), v_{\alpha\beta}(x,ky) = v_{\alpha\beta}(x,y), k > 0. \) When these conditions are met, the following relations hold:

\[ g_{\alpha\beta} = \text{sgn}(g) \frac{1}{2} \frac{\partial^2 F_g}{\partial y^{\alpha} \partial y^{\beta}}, \quad v_{\alpha\beta} = \text{sgn}(v) \frac{1}{2} \frac{\partial^2 F_v}{\partial y^{\alpha} \partial y^{\beta}} \]  

(9)

(10)

where \( g_{\alpha\beta} = \hat{\delta}_{\alpha}^{\alpha'} \hat{\delta}_{\beta}^{\beta'} g_{\mu\nu}, \text{sgn}(g) \) is the sign of \( g_{\alpha\beta}(x,y)y^{\alpha}y^{\beta}, \text{sgn}(v) \) is the sign of \( v_{\alpha\beta}(x,y)y^{\alpha}y^{\beta} \) and

\[ F_g(x,y) = \sqrt{|g_{\alpha\beta}(x,y)y^{\alpha}y^{\beta}|}, \quad F_v(x,y) = \sqrt{|v_{\alpha\beta}(x,y)y^{\alpha}y^{\beta}|}. \]  

(11)

(12)

From the above relations, the following conditions become obvious:

\[ \hat{\delta}_{\alpha}^{\alpha'} = \hat{\delta}_{\beta}^{\beta'} = 1 \text{ for } a = \mu + 4 \text{ and equal to zero otherwise.} \]

\[ 1 \]
1. \( F_m, m = g,v, \) is continuous on \( TM \) and smooth on \( \tilde{TM} = TM \setminus \{0\} \) i.e. the tangent bundle minus the null set \( \{(x,y) \in TM | F_m(x,y) = 0\} \)

2. \( F_m \) is positively homogeneous of first degree on its second argument:

\[
F_m(x^\mu, ky^\nu) = kF_m(x^\mu, y^\nu), \quad k > 0
\]

(13)

3. The form

\[
f_{\alpha\beta}(x, y) = \frac{1}{2} \frac{\partial^2 F_m}{\partial y^\alpha \partial y^\beta}
\]

defines a non-degenerate matrix:

\[
\det[f_{\alpha\beta}] \neq 0
\]

(14)

In this work, we consider a distinguished connection \((d-\text{connection})\) \( D \) on \( TM \). This is a linear connection with coefficients \( \{\Gamma_{\beta\gamma}^\alpha\} = \{L^\mu_{\nu\kappa}, L^\alpha_{\beta\gamma}, C^\mu_{\nu\gamma}, C^\alpha_{\beta\gamma}\} \) which preserves by parallelism the horizontal and vertical distributions:

\[
D_{\delta\kappa} \delta_\nu = L^\mu_{\nu\kappa}(x,y)\delta_\mu, \quad D_{\delta\kappa} \delta_\nu = C^\mu_{\nu\gamma}(x,y)\delta_\mu
\]

(16)

\[
D_{\delta\kappa} \delta_\beta = L^\alpha_{\beta\gamma}(x,y)\delta_\alpha, \quad D_{\delta\kappa} \delta_\beta = C^\alpha_{\beta\gamma}(x,y)\delta_\alpha
\]

(17)

From these, the definitions for partial covariant differentiation follow as usual, e.g. for \( X \in \tilde{T}TM \) we have the definitions for covariant \( h\)-derivative

\[
X^A_\nu \equiv D_\nu X^A = \delta_\nu X^A + L^A_{B\nu} X^B
\]

(18)

and covariant \( v\)-derivative

\[
X^A|_\beta \equiv D_\beta X^A = \delta_\beta X^A + C^A_{B\beta} X^B
\]

(19)

A \( d\)-connection can be uniquely defined given that the following conditions are satisfied:

- The \( d\)-connection is metric compatible
- Coefficients \( L^\mu_{\nu\kappa}, L^\alpha_{\beta\gamma}, C^\mu_{\nu\gamma}, C^\alpha_{\beta\gamma}\) depend solely on the quantities \( g_{\mu\nu}, v_{\alpha\beta} \) and \( N^\alpha_{\beta\gamma}\)
- Coefficients \( L^\mu_{\nu\kappa} \) and \( C^\alpha_{\beta\gamma}\) are symmetric on the lower indices, i.e. \( L^\mu_{\nu\kappa} = C^\alpha_{\beta\gamma} = 0 \)

We use the symbol \( D \) instead of \( D \) for a connection satisfying the above conditions, and call it a canonical and distinguished \( d\)-connection. Metric compatibility translates into the conditions:

\[
D_{\kappa} g_{\mu\nu} = 0, \quad D_{\kappa} v_{\alpha\beta} = 0, \quad D_{\gamma} g_{\mu\nu} = 0, \quad D_{\gamma} v_{\alpha\beta} = 0.
\]

(20)

The coefficients of canonical and distinguished \( d\)-connection are

\[
L^\mu_{\nu\kappa} = \frac{1}{2} g^{\mu\rho} \left( \delta_\kappa g_{\rho\nu} + \delta_\nu g_{\rho\kappa} - \delta_\rho g_{\nu\kappa} \right)
\]

(21)

\[
L^\alpha_{\beta\gamma} = \delta_\beta N^\alpha_{\gamma} + \frac{1}{2} v^{\alpha\gamma} \left( \delta_\kappa v_{\beta\gamma} - \delta_\beta v_{\gamma\kappa} - v_{\gamma\delta} \delta_\gamma N^\delta_{\kappa} - v_{\gamma\delta} \delta_\gamma N^\delta_{\kappa} \right)
\]

(22)

\[
C^\mu_{\nu\gamma} = \frac{1}{2} g^{\mu\rho} \delta_\kappa g_{\rho\nu}
\]

(23)

\[
C^\alpha_{\beta\gamma} = \frac{1}{2} v^{\alpha\delta} \left( \delta_\gamma v_{\beta\delta} + \delta_\beta v_{\delta\gamma} - \delta_\delta v_{\gamma\beta} \right)
\]

(24)

Curvature and torsion in \( TM \) can be defined as multilinear maps:

\[
\mathcal{R}(X,Y)Z = [D_X, D_Y]Z - D_{[X,Y]}Z
\]

(25)

and

\[
\mathcal{T}(X,Y) = D_X Y - D_Y X - [X,Y],
\]

(26)
where $X, Y, Z \in TTM$. We use the definitions

$$\mathcal{R} (\delta_\lambda, \delta_\kappa) \delta_\nu = R^\mu_{\nu \kappa \lambda} \delta_\mu$$

$$\mathcal{R} (\dot{\delta}_\lambda, \dot{\delta}_\kappa) \dot{\delta}_\beta = S^\gamma_{\beta \gamma \delta} \dot{\delta}_\delta$$

$$\mathcal{T} (\delta_\nu, \delta_\mu) = T^\mu_{\nu \kappa} \delta_\mu + T^\nu_{\mu \kappa} \dot{\delta}_\nu$$

$$\mathcal{T} (\dot{\delta}_\nu, \dot{\delta}_\mu) = T^\mu_{\nu \kappa} \dot{\delta}_\mu + T^\nu_{\mu \kappa} \dot{\delta}_\nu$$

The $h$-curvature tensor of the $d$–connection in the adapted basis and the corresponding $h$-Ricci tensor have, respectively, the components

$$R^\mu_{\nu \kappa \lambda} = \delta_\lambda L^\mu_{\nu \kappa} - \delta_\kappa L^\mu_{\nu \lambda} + L^\rho_{\nu \kappa} L^\mu_{\rho \lambda} - L^\rho_{\nu \lambda} L^\mu_{\rho \kappa} + C^\mu_{\nu \lambda} \Omega^\alpha_{\kappa \lambda}$$

$$R_{\mu \nu} = R^\kappa_{\mu \nu \kappa} = \delta_\kappa L^\kappa_{\mu \nu} - \delta_\nu L^\kappa_{\mu \kappa} + L^\rho_{\mu \kappa} L^\kappa_{\rho \nu} - L^\rho_{\mu \nu} L^\kappa_{\rho \kappa} + C^\kappa_{\mu \kappa} \Omega^\alpha_{\nu \kappa}$$

The $v$-curvature tensor of the $d$–connection in the adapted basis and the corresponding $v$-Ricci tensor have, respectively, the components

$$S^\alpha_{\beta \gamma \delta} = \dot{\delta}_\delta C^\alpha_{\beta \gamma} - \dot{\delta}_\gamma C^\alpha_{\beta \delta} + C^\alpha_{\beta \gamma} C^\epsilon_{\delta \epsilon} - C^\alpha_{\beta \delta} C^\epsilon_{\gamma \epsilon}$$

$$S_{\alpha \beta} = S^\gamma_{\alpha \beta \gamma} = \dot{\delta}_\gamma C^\gamma_{\alpha \beta} - \dot{\delta}_\beta C^\gamma_{\alpha \gamma} + C^\gamma_{\alpha \beta} C^\epsilon_{\gamma \epsilon} - C^\gamma_{\alpha \gamma} C^\epsilon_{\beta \epsilon}.$$
hence the minus sign under the square root. We calculate the metric tensor $g_{\alpha\beta}$ which induces an interaction between internal and external spaces. This is different from $g_{\gamma x}$ that focuses on the timelike subspace of the internal $y$.

Finally, the energy-momentum tensor $\mathcal{T}_{\mu\nu}$ which depend on just the external or internal structure respectively.

where $\delta^\alpha_\mu$ and $\delta^\alpha_\nu$ are the Kronecker symbols and

$$T^\alpha_{\beta\nu} = \partial_\beta N^\alpha_\nu - L^\alpha_{\beta\nu}$$

are torsion components, where $L^\alpha_{\beta\nu}$ is defined in (22).

We will make some comments in order to give a physical interpretation in relation to the equations (42), (43) and (44). Lorentz violations produce anisotropies in the space and the matter sector. These act as a source of local anisotropy which is produced from the metric $\nu_{\alpha\beta}$ which includes additional internal structure of spacetime. Finally, the energy-momentum tensor $\mathcal{Z}_\alpha$ reflects the dependence of matter fields on the nonlinear connection $N^\alpha_{\lambda}$, a structure which induces an interaction between internal and external spaces. This is different from $T_{\mu\nu}$ and $Y_{\alpha\beta}$ which depend on just the external or internal structure respectively.

### A. Schwarzschild-Randers spacetime

As we mentioned above, the horizontal part $g_{\mu\nu}$ of the metric (22) will be taken to be the Schwarzschild metric so that

$$g_{\mu\nu}dx^\mu dx^\nu = -\left(1 - \frac{R_s}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{R_s}{r}} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2$$

where $R_s = 2GM$ is the Schwarzschild radius (we have set the speed of light constant $c = 1$).

In the following, we assume a Lagrangian function $\mathcal{L}$ of Randers type from which we will derive $\nu_{\alpha\beta}$:

$$\mathcal{L} = \sqrt{-g_{\alpha\beta}(x) y^{\alpha} y^{\beta} + A_{\gamma}(x) y^{\gamma}}$$

(47)

where $g_{\alpha\beta} = g_{\mu\nu} \delta^\alpha_\mu \delta^\beta_\nu$ is the Schwarzschild metric and $A_{\gamma}(x)$ is a covector which will be determined by the equations. We take $A_{\gamma}(x)$ to be a weak term ($|A_{\gamma}(x)| \ll 1$) and we ignore higher than first order terms in $A_{\gamma}$ hereafter. We focus on the timelike subspace of the internal $y-$space with respect to the Schwarzschild metric $(g_{\alpha\beta}(x) y^{\alpha} y^{\beta} < 0)$ hence the minus sign under the square root. We calculate the metric tensor $\nu_{\alpha\beta}$ of (47):

$$\nu_{\alpha\beta} = -\frac{1}{2} \frac{\partial^2 \mathcal{L}^2}{\partial y^\alpha \partial y^\beta}$$

(48)

After some calculations and by omitting terms $\mathcal{O}(A^2)$ we arrive at

$$\nu_{\alpha\beta} = g_{\alpha\beta}(x) + \frac{1}{a}(A_{\beta} g_{\alpha\gamma} y^{\gamma} + A_{\gamma} g_{\alpha\beta} y^{\gamma} + A_{\alpha} g_{\beta\gamma} y^{\gamma}) + \frac{1}{a^3} A_{\gamma} g_{\alpha\beta} g_{\beta\delta} y^{\gamma} y^{\delta} y^{\epsilon},$$

(49)

where we have set $a = \sqrt{-g_{\alpha\beta} y^{\alpha} y^{\beta}}$. From (49) we see that the metric $\nu_{\alpha\beta}$ takes the form

$$\nu_{\alpha\beta}(x, y) = g_{\alpha\beta}(x) + w_{\alpha\beta}(x, y)$$

(50)

where we have set

$$w_{\alpha\beta} = \frac{1}{a}(A_{\beta} g_{\alpha\gamma} y^{\gamma} + A_{\gamma} g_{\alpha\beta} y^{\gamma} + A_{\alpha} g_{\beta\gamma} y^{\gamma}) + \frac{1}{a^3} A_{\gamma} g_{\alpha\beta} g_{\beta\delta} y^{\gamma} y^{\delta} y^{\epsilon}$$

(51)
Its inverse is $v^{\beta\gamma} = g^{\beta\gamma} - w^{\beta\gamma}$ so that $v_{\alpha\beta}v^{\beta\gamma} = g_{\alpha\beta}g^{\beta\gamma} = \delta^\gamma_\alpha$ to first order in $w_{\alpha\beta}$. The total metric over the tangent bundle is then written as

$$G = g_{\mu\nu}(x) \, dx^\mu \otimes dx^\nu + [g_{\alpha\beta}(x) + w_{\alpha\beta}(x, y)] \, \delta y^\alpha \otimes \delta y^\beta$$

(52)

We remark that, as we can see from (50), the metric $v_{\alpha\beta}(x, y)$ is a Finslerian perturbation of the Riemannian metric $g_{\alpha\beta}(x)$. We observe that if we let $w_{\alpha\beta} \to 0$ then the field equations (39)-(41) reduce to the Einstein field equations of general relativity.

Next we will calculate the terms for (39) and (40). From the definitions (37) and (38) we see that when $g_{\mu\nu}$ has no explicit dependence on $y$ then $R_{\mu\nu}$ and $R$ reduce to the classical Ricci tensor and scalar of general relativity. Additionally, since $g_{\mu\nu}(x)$ is the Schwarzschild metric, both $R_{\mu\nu}$ and $R$ are zero. Here we have assumed vacuum solutions, so the energy momentum tensors are zero and the equations (39) and (40) become

$$-\frac{1}{2} S g_{\mu\nu} + \left( \delta^{(\lambda}_\nu \delta^\mu_\alpha) - g^{\alpha\lambda} g_{\mu\nu} \right) \left( D_k T^\beta_{\lambda\beta} - T^\gamma_{\kappa\gamma} T^\beta_{\lambda\beta} \right) = 0$$

(53)

$$S_{\alpha\beta} - \frac{1}{2} S v_{\alpha\beta} + \left( v^{\gamma\delta} v_{\alpha\beta} - \delta^{(\gamma}_\alpha \delta^\delta_\beta \right) \left( D_\gamma C^\mu_{\mu\delta} - C^\nu_{\nu\gamma} C^\mu_{\mu\delta} \right) = 0$$

(54)

Field equation (41) gives us no additional information since all three terms vanish identically in our case. We can simplify equation (54) by calculating $C^\mu_{\mu\delta}$ from (23) and we find that it is zero since the metric $g_{\mu\nu}$ depends only on $x$. Then by taking the trace of the remaining terms in (54) we can show that $S_{\alpha\beta}$ and $S$ are also zero, so the field equation (53) becomes

$$\left( \delta^{(\lambda}_\nu \delta^\mu_\alpha) - g^{\alpha\lambda} g_{\mu\nu} \right) \left( D_k T^\beta_{\lambda\beta} - T^\gamma_{\kappa\gamma} T^\beta_{\lambda\beta} \right) = 0.$$ 

(55)

In order to calculate the torsion components (45) we need a specific nonlinear connection. We choose:

$$N^\alpha_{\mu} = \frac{1}{2} y^\beta g^{\alpha\gamma} \partial_\mu g_{\beta\gamma}$$

(56)

We substitute (22) and (56) in (45) and after some calculations we get

$$T^\alpha_{\nu\alpha} = -\frac{1}{2} \delta^\alpha_\nu w$$

(57)

with $w = g_{\alpha\beta}w^{\alpha\beta}$. The above relation (57) shows us that the torsion is of first order on $w_{\alpha\beta}$ so the terms $T^\gamma_{\kappa\gamma} T^\beta_{\lambda\beta}$ from (55) are omitted. Then by taking the trace of the remaining terms in (55) we have the equation that follows:

$$g^{\mu\nu} D_\mu T^\alpha_{\nu\alpha} = 0$$

(58)

Substituting the latter equation to (55) we find

$$D_{(\mu} T^\alpha_{\nu)\alpha} = 0$$

(59)

By the definition of the covariant derivative in (18), equation (59) becomes

$$\delta_{(\mu} T^\alpha_{\nu)\alpha} - L^\alpha_{\mu\nu} T^\alpha_{\kappa\alpha} = 0$$

(60)

Using equations (51) and (57) we get $T^\alpha_{\nu\alpha}$ in terms of $A_\beta$:

$$T^\alpha_{\nu\alpha} = -\frac{7}{2} \delta^\alpha_\nu \left( A_\beta \, \frac{y^\beta}{a} \right).$$

(61)

It is straightforward to show that

$$\delta_\mu \left( \frac{y^\alpha}{a} \right) = -E^\gamma_\beta \frac{y^\beta}{a}$$

(62)

with

$$E^\gamma_\beta_\mu(x) \equiv \frac{1}{2} g^{\alpha\gamma} \partial_\mu g_{\alpha\beta}$$

(63)
Using (61) and (62) we get
\[
T^{\alpha}_{\nu\alpha} = -\frac{7}{2} K_{\nu}^x y^\gamma \frac{1}{a}
\]
with
\[
K_{\nu\gamma}(x) \equiv \partial_\nu A_\gamma - A_\beta E^\beta_{\nu\gamma}.
\]
from relation (64) we can calculate (60):
\[
\left( \partial_{(\mu}K_{\nu)} - E^\beta_{(\mu}K_{\nu)\beta} - L^\lambda_{\mu\nu}K_{\lambda\gamma} \right) y^\gamma = 0.
\]
Relation (66) must hold for every y. Since the expression in parentheses does not depend on y, we conclude that it must identically vanish:
\[
\partial_{(\mu}K_{\nu)\gamma} - E^\beta_{(\mu}K_{\nu)\beta} - L^\lambda_{\mu\nu}K_{\lambda\gamma} = 0
\]
Remark: By comparing (22) and (63) we get a relation of the form (66). If we use this in (65) and (67) we get the equation \(D_{(\mu}D_{\nu)}A_\gamma = 0\), which is equivalent to the system of equations (65) and (67). Contracting this with \(g^{\mu\nu}\) gives
\[
\Box A_\gamma = 0
\]
with \(\Box \equiv g^{\mu\nu}D_{\mu}D_{\nu}\).

In order to fully determine the metric (52) for the respective subspace of the internal space, we need \(g_{\mu\nu}(x)\) and \(A_{\gamma}(x)\). The first is already defined in (46), so we need to solve (67) for \(A_{\gamma}(x)\) to get a full expression for the metric in our space. If we use the definitions (63) and (65) on relation (67), we get the equation that we need to solve for \(A(x)\):
\[
\partial_\gamma \partial_\nu A_\gamma - \frac{1}{2} g^{\beta\delta} \partial_\gamma g_{\beta\gamma} \partial_\nu g_{\gamma\delta} - \frac{1}{2} g^{\beta\delta} \partial_\gamma g_{\delta\gamma} \partial_\nu A_\beta
\]
\[
+ \frac{1}{4} A_\beta \left( \frac{1}{2} g^{\beta\gamma} \delta_\gamma \partial_\nu g_{\delta\gamma} - \frac{1}{2} g^{\beta\gamma} \delta_\nu \partial_\gamma g_{\delta\gamma} - \partial_\beta g^{\beta\gamma} \partial_\nu g_{\gamma\delta} - \partial_\nu g^{\beta\gamma} \partial_\gamma g_{\delta\gamma} - 2g^{\beta\gamma} \partial_\beta \partial_\nu g_{\gamma\delta} \right)
\]
\[
- \frac{1}{2} g^{\kappa\lambda} \left( \partial_\kappa g_{\nu\kappa} + \partial_\nu g_{\kappa\mu} - \partial_\kappa g_{\mu\nu} \right) \left( \partial_\lambda A_\gamma - \frac{1}{2} A_\beta g^{\beta\gamma} \delta_\lambda g_{\gamma\delta} \right) = 0,
\]
where \(g_{\mu\nu}(x)\) is the Schwarzschild metric (46). Once we get \(A(x)\) from (69), we can calculate \(w_{\alpha\beta}(x, y)\) from (61) and then use the result to calculate the full metric (52).

A similar analysis holds for the spatial subspace of the internal space. In that case, instead of (47) and (48) we have
\[
\mathcal{L} = \sqrt{g_{\alpha\beta} y^\alpha y^\beta + A_\gamma y^\gamma}
\]
and
\[
v_{\alpha\beta} = \frac{1}{2} \frac{\partial^2 \mathcal{L}^2}{\partial y^\alpha \partial y^\beta} = g_{\alpha\beta}(x) + \frac{1}{a} \left( A_\beta g_{\alpha\gamma} y^\gamma + A_\gamma g_{\alpha\beta} y^\gamma + A_\alpha g_{\beta\gamma} y^\gamma - \frac{1}{a^2} A_\gamma g_{\gamma\delta} y^\delta y^\gamma \right),
\]
where \(a = \sqrt{g_{\mu\nu} y^\mu y^\nu}\) for the spacelike sector of \(g_{\mu\nu}\) taking into consideration the signature of the metric. Following the same steps as above, we reach the same equation for \(A_\gamma\), i.e. eq. (69). Therefore, solving this equation will give us the metric for both the timelike and spacelike (with respect to \(g_{\alpha\beta}\)) subspaces of the internal space.

B. Schwarzschild-De Sitter-Randers spacetime

We will follow the same procedure as in the previous paragraph but for a Schwarzschild-Randers spacetime with a cosmological horizon, namely a Schwarzschild-De Sitter-Randers spacetime. In this scenario, we take the horizontal part of the metric (8) to be:
\[
g_{\mu\nu}dx^\mu dx^\nu = \left( 1 - \frac{R_0}{r} - \frac{\Lambda}{3} r^2 \right) dt^2 + \frac{dr^2}{1 - \frac{R_0}{r} - \frac{\Lambda}{3} r^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2
\]
while, as before, the metric tensor $\epsilon_{\alpha\beta}$ will be derived from the Lagrangian (47) and relation (48), where $g_{\alpha\beta} = \tilde{\eta}_{\alpha}^{\gamma}\tilde{\eta}_{\beta}^{\gamma}$ is now given by (72). The latter is a static spherically symmetric vacuum solution for the classical Einstein field equations with a cosmological constant $\Lambda$:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + g_{\mu\nu}\Lambda = 0$$  \hspace{1cm} (73)

with $R_{\mu\nu}$ and $R$ the Ricci tensor and scalar of general relativity. In accordance, we introduce a cosmological constant term to the field equations (39) in vacuum:

$$\overline{R}_{\mu\nu} - \frac{1}{2}(\overline{R} + S)g_{\mu\nu} + \left(\delta^{(\lambda}_{\nu}\delta^{\mu}_{\alpha} - g^{\alpha\lambda}g_{\mu\nu}\right)\left(D_{\kappa}T^{\kappa}_{\lambda\beta} - T^{\gamma}_{\lambda\gamma}T^{\kappa}_{\lambda\beta}\right) + g_{\mu\nu}\Lambda = 0$$  \hspace{1cm} (74)

The tensors $\overline{R}_{\mu\nu}$ and $\overline{R}$ reduce to the standard Ricci tensor and scalar of general relativity for the metric (72) since the latter has no direct dependence on $y$. Additionally, from (40) in vacuum we get $S = 0$ the same way as in the previous paragraph. Therefore, using (73) in equation (74) we get relation (55) again. It is obvious that the procedure is the same as before and only the explicit form of $g_{\mu\nu}(x)$ changes. As such, we reach the same equation for $A_{\gamma}$, namely relation (69) with $g_{\mu\nu}$ given by (72).

IV. Calculation of $A_{\gamma}$

A. Solution for the Schwarzschild-Randers spacetime

In order to calculate $A_{\gamma}$ we will give values to $\mu, \nu, \gamma$ of (69) and solve the resulting equations. For $\mu = 0, \nu = 0$ and $\gamma = 4$ we get:

$$\partial_{0}^{2}A_{4} + \frac{1}{2}f\partial_{1}(-f)\left(\partial_{1}A_{4} + \frac{1}{2f}A_{4}\partial_{1}(-f)\right) = 0,$$  \hspace{1cm} (75)

where we have set $f = 1 - \frac{R_{s}}{r}$.

After some calculations and by separation of variables $A_{4} = R_{4}(r)T_{4}(t)$ we get two equations:

$$\partial_{0}T_{4}(t) = -c_{(4)}^{2}T_{4}(t)$$  \hspace{1cm} (76)

$$\partial_{1}R_{4}(r) = \frac{1 - f}{2rf}R_{4}(r) - c_{(4)}^{2}\frac{2r}{f(1 - f)}$$  \hspace{1cm} (77)

where $c_{(4)}^{2}$ is the separation constant. For $\mu = 0, \nu = 1$ and $\gamma = 4$ we get

$$\partial_{0}\partial_{1}A_{4} + \frac{1}{f}\partial_{1}(-f)\partial_{0}A_{4} = 0.$$  \hspace{1cm} (78)

After rearranging the terms and again separating variables we have

$$\partial_{1}R_{4}(r) = \frac{\partial_{1}f}{f}R_{4}(r)$$  \hspace{1cm} (79)

So we get $R_{4}(r) = k_{4}f(r)$ with $k_{4}$ being a constant resulting from the integration. By substituting this to (77) we find that the separation constant $c_{(4)}^{2}$ must be zero. That means that in order to satisfy (76) and (77), $T_{4}(t)$ must be constant and $R_{4}(r) = \check{R}_{4}f^{1/2}(r)$ with $\check{R}_{4}$ a constant. By calculating the remaining equations for $\mu = 2, \nu = 2, \gamma = 4$ and $\mu = 3, \nu = 3, \gamma = 4$ we find that $A_{4}$ has no dependence on $\theta$ or $\phi$. Therefore, we end up with

$$A_{4} = \check{A}_{4}f^{1/2}(r)$$  \hspace{1cm} (80)

with $\check{A}_{4}$ being a constant. For $\mu = 0, \nu = 0, \gamma = 5$ we get

$$\partial_{0}^{2}A_{5} - \frac{f(1 - f)}{2r}\left(\partial_{1}A_{5} - \frac{1}{2A_{5}f\partial_{1}(f^{-1})}\right) = 0.$$  \hspace{1cm} (81)
After calculations and by separating variables like before we end up with equations

$$\partial_5^2 T_5(t) = -c_{(5)}^2 T_5(t)$$  \hfill (82)

$$\partial_1 R_5(r) = -\frac{1-f}{2rf}R_5(r) - c_{(5)}^2 \frac{2r}{f(1-f)}$$  \hfill (83)

with $c_{(5)}^2$ being the separation constant. For $\mu = 0, \nu = 1, \gamma = 5$ we get

$$\partial_6 \partial_1 A_5 - \frac{1}{2} f \partial (f^{-1}) \partial_6 A_5 - \frac{1}{2} (-f^{-1}) \partial_1 (-f) \partial_6 A_5 = 0$$  \hfill (84)

After calculations we end up with $c_{(5)} = 0$ and by substitution to (82) and (83) we find

$$R_5(r) = k_5 f^{-1/2}$$  \hfill (85)

with $k_5$ being a constant. Also, like before, if we calculate the $\mu = 2, \nu = 2, \gamma = 5$ and $\mu = 3, \nu = 3, \gamma = 5$ we get no dependence on $\theta$ and $\phi$. Therefore, we find

$$A_5 = \tilde{A}_5 f^{-1/2}(r)$$  \hfill (86)

with $\tilde{A}_5$ a constant of integration. If we put this solution in the $\mu = 1, \nu = 1$ equation, we get $\tilde{A}_5 = 0$. For $\mu = 0, \nu = 0, \gamma = 6$ we separate variables like before $A_6 = R_6(r) T_6(t)$ and we get two equations:

$$\partial_1 R_6(r) = \frac{R_6(r)}{r} - c_{(6)}^2 \frac{2r}{f(1-f)}$$  \hfill (87)

$$\partial_5^2 T_6(t) = -c_{(6)}^2 T_6(t)$$  \hfill (88)

with $c_{(6)}^2$ the separation constant. For $\mu = 0, \nu = 1, \gamma = 6$ like before we find that $c_{(6)} = 0$ and by substitution to (87) we find that $R_6(r) = \tilde{R}_6 r$ with $\tilde{R}_6$ a constant of integration. We set $\tilde{R}_6$ to zero to keep our solution finite at infinity, so we end up with $A_6 = 0$. For $A_7$ we set $\mu = 0, \nu = 0, \gamma = 7$ and we find the same equations as for $A_6$. That leads to $A_7 = 0$ as well.

To sum up, we have found the following solution for $A_\gamma$ from equation (89):

$$A_\gamma(x) = \left[ \tilde{A}_4 \left( 1 - \frac{R_S}{r} \right)^{1/2}, 0, 0, 0 \right]$$  \hfill (89)

with $\tilde{A}_4$ a constant. This is a timelike covector since to second order in $A_\gamma$ we get $g^{\alpha\beta} A_\alpha A_\beta = -\tilde{A}_4^2 < 0$. It is interesting to mention that the horizon of the Schwarzschild-Randers metric coincides with that of Schwarzschild, namely $R_s = 2GM$. Practically, the quantity $A_\gamma$ can be seen as a distortion factor which quantifies the deviation from the pure Schwarzschild solution. Obviously, the Schwarzschild-Randers solution on small spherical scales ($r \sim R_s$) tends to that of Schwarzschild. On the other hand for $r \gg R_s$ the Schwarzschild-Randers metric tends to Minkowski.

### B. Solution for the Schwarzschild-De Sitter-Randers spacetime

We will modify the solution (89) and see if it satisfies (69) for the metric (72). An obvious ansatz is to replace the term $1 - \frac{R_S}{r}$ in (89) with $1 - \frac{R_S}{r} - \frac{\Lambda}{3} r^2$. Doing this we get

$$A_\gamma(x) = \left[ \tilde{A}_4 \left( 1 - \frac{R_S}{r} - \frac{\Lambda}{3} r^2 \right)^{1/2}, 0, 0, 0 \right]$$  \hfill (90)

which is verified to be a solution of (69) for the metric (72).
V. PATHS IN THE SCHWARZSCHILD-RANDERS SPACETIME

Now that we have $A_g$ and hence the full metric, we can study particle trajectories in $TM$. A Lagrangian for point particles in the total space of the Lorentz tangent bundle has been proposed in \[12\]:

$$L(x, \dot{x}, y) = \left( a g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + b \delta^\alpha_\nu v_{\alpha\beta} \dot{x}^\mu g_{\beta\gamma} y^\gamma + c v_{\alpha\beta} y^\alpha y^\beta \right)^{1/2}$$

with $a, b, c$ constants. The associated equations of motion are

$$\left( g_{\alpha\nu} + z \delta^\alpha_\nu \delta^\beta_\nu v_{\alpha\beta} \right) \ddot{x}^\kappa + (\gamma_{\nu\kappa\lambda} + z \sigma_{\nu\kappa\lambda}) \dot{x}^\kappa \dot{x}^\lambda = 0$$

and

$$y^\alpha = \tilde{\delta}^\alpha_\mu \dot{x}^\mu$$

with

$$\sigma_{\nu\kappa\lambda} = \frac{1}{2} \left( \delta^{\beta}_\nu \delta^{\alpha}_\lambda \partial_\kappa v_{\alpha\beta} + \delta^{\beta}_\nu \delta^{\beta}_\kappa \partial_\lambda v_{\alpha\beta} - \delta^{\alpha}_\nu \delta^{\beta}_\lambda \partial_\kappa v_{\alpha\beta} \right)$$

and $z = -b^2/4ac$ is a constant. The Christoffel symbols of the first kind for the metric $g_{\kappa\nu}(x)$ are

$$\gamma_{\nu\kappa\lambda} = \frac{1}{2} (\partial_\kappa g_{\nu\lambda} + \partial_\lambda g_{\nu\kappa} - \partial_\nu g_{\kappa\lambda})$$

The term $\tilde{\delta}^\alpha_\nu \delta^\beta_\nu v_{\alpha\beta}$ is the metric of the v-space lowered down to the h-space via the generalized Kronecker symbols which are defined as $\delta^\alpha_\nu = \tilde{\delta}^\alpha_\nu = 1$ for $a = \mu + 4$ and equal to zero otherwise. We will write for convenience $\tilde{\delta}^\alpha_\nu v_{\alpha\beta}$ and similarly $\tilde{\delta}^\beta_\nu w_{\alpha\beta} = w_{\kappa\nu}$.

We define $\tilde{\mu}_{\kappa\nu} = g_{\kappa\nu} + z v_{\kappa\nu}$ and observe that its inverse is $\tilde{\kappa}^{\mu\nu} = \left(1 + z\right)^{-2} \left(g^{\mu\nu} + z v^{\mu\nu}\right)$ in the sense that $\tilde{\kappa}^{\mu\nu} \tilde{\mu}_{\kappa\nu} = \delta^\mu_\kappa$ to first order in $w_{\mu\nu}$. Contracting (92) with $\tilde{\mu}_{\kappa\nu}$ gives

$$\tilde{\kappa}^{\mu\nu} \tilde{\mu}_{\kappa\nu} \ddot{x}^\kappa + \left(\tilde{\kappa}^{\mu\nu} \gamma_{\nu\kappa\lambda} + z \tilde{\kappa}^{\mu\nu} \sigma_{\nu\kappa\lambda}\right) \dot{x}^\kappa \dot{x}^\lambda = 0$$

$$\Leftrightarrow \ddot{x}^\mu + \left(1 + z\right)^{-2} \left[\gamma^\mu_{\nu\kappa\lambda} + z \sigma^\mu_{\nu\kappa\lambda} + z v^{\mu\nu} \left(\gamma_{\nu\kappa\lambda} + z \sigma_{\nu\kappa\lambda}\right)\right] \dot{x}^\kappa \dot{x}^\lambda = 0$$

where $\gamma^\mu_{\nu\kappa\lambda} = g^{\mu\nu} \gamma_{\nu\kappa\lambda}$ and $\sigma^\mu_{\nu\kappa\lambda} = g^{\mu\nu} \sigma_{\nu\kappa\lambda}$. After some straightforward calculations, eq. (96) gives

$$\ddot{x}^\mu + \gamma^\mu_{\nu\kappa\lambda} \dot{x}^\kappa \dot{x}^\lambda = -\frac{z}{1 + 2 \left(\gamma^\mu_{\nu\kappa\lambda} - w^{\mu\nu} \gamma_{\nu\kappa\lambda}\right)} \dot{x}^\kappa \dot{x}^\lambda,$$

with

$$\tilde{\gamma}^\mu_{\nu\kappa\lambda} \equiv \frac{1}{2} g^{\mu\nu} \left(\partial_\kappa w_{\nu\lambda} + \partial_\lambda w_{\nu\kappa} - \partial_\nu w_{\kappa\lambda}\right)$$

The horizontal part of the tangent vector on the paths is $\dot{x}^\mu = dx^\mu/ds$ with $s$ an affine parameter along the path defined as \[12\]:

$$s = s_0 + \int_{s_0}^{\lambda_1} \sqrt{\pm \tilde{\mu}_{\mu\nu}(x, y) \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} d\lambda$$

with $s_0, \lambda_0$ and $\lambda_1$ constants and $\lambda$ is an arbitrary parameter of the path. The sign of $\tilde{\mu}_{\mu\nu}(x, y)$ is determined by the tangent vector of the path, specifically if $dx^\nu/d\lambda$ is timelike with respect to $\tilde{\mu}_{\mu\nu}(x, y) \left(\tilde{\mu}_{\mu\nu}(x, y) \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \right) < 0$ then we get “-”, likewise for a spacelike tangent vector with respect to $\tilde{\mu}_{\mu\nu}(x, y) \left(\tilde{\mu}_{\mu\nu}(x, y) \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \right) > 0$ we get “+”.

The paths $\tilde{\mu}_{\mu\nu}$ will play the role for our model that the geodesics play for general relativity. As is the case for the latter, we need a classification of path segments with respect to their character i.e. timelike, null and spacelike. We define:

---

\[2\] In general, $\delta^\alpha_\nu$ and $\tilde{\delta}^\beta_\nu$ can be used to lift an object from the horizontal to the vertical subspace of $TTM$ or lower down one from the vertical to the horizontal subspace. This allows us to perform algebraic operations between components of tensors belonging to different subspaces of $TTM$. 
• Timelike segment: $g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu < 0$ at every point

• Null segment: $g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu = 0$ at every point

• Spacelike segment: $g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu > 0$ at every point

Therefore, the character of the path is determined by the metric tensor $g_{\mu\nu}(x)$ of the horizontal subspace. We define the proper time $\tau$ as

$$\tau = \tau_0 + \int_{\lambda_0}^{\lambda_1} \sqrt{-g_{\mu\nu}(x)\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}\,d\lambda,$$  \hspace{1cm} (100)

where $\tau_0$ is constant. By comparing relations (99) and (100) we see that the parameter $s$ on the paths (97) cannot be written as an affine transformation of the proper time in general.

We remark that equation (97) reduces to the classic geodesics equation of general relativity when the perturbation $w_{\alpha\beta}$ goes to zero, as it should. To begin, we rewrite the perturbation (101) as

$$w_{\nu\lambda} = g_{\lambda\rho}A_{\nu}u^\rho + g_{\nu\rho}A_{\lambda}u^\rho + (g_{\nu\lambda} + g_{\nu\sigma}g_{\lambda\tau}u^\sigma u^\tau)A_{\rho}u^\rho$$  \hspace{1cm} (101)

with

$$u^\nu = \frac{y^\nu}{a}$$  \hspace{1cm} (102)

where $a = \sqrt{-g_{\mu\nu}y^\mu y^\nu}$ and we have lowered down $A_{\gamma}$ and $y^{\gamma}$ using the generalized Kronecker deltas. It is straightforward to show

$$\partial_\mu u^\nu = -\frac{1}{2}\partial_\mu g_{\kappa\lambda}u^\kappa u^\lambda u^\nu$$  \hspace{1cm} (103)

Using (98), (101) and (103) we calculate

$$\ddot{x}^\mu + \gamma_{\kappa\lambda}^\mu \ddot{x}^\kappa \ddot{x}^\lambda = -\frac{z}{1+z} \left\{ ag^{\mu\nu}(\partial_\nu A_\kappa - \partial_\kappa A_\nu) \dot{x}^\nu + \frac{1}{a} \left[ A^\nu \left( \partial_\nu g_{\lambda\rho} - \frac{1}{2} \partial_\nu g_{\kappa\rho} \right) + \partial_\kappa A_\lambda \right] \dot{x}^\nu \dot{x}^\lambda \right\}$$  \hspace{1cm} (104)

Now, if we take into account that $g_{\mu\nu}u^\mu u^\nu = -1$, the above relation gives

$$\ddot{x}^\mu + \gamma_{\kappa\lambda}^\mu \ddot{x}^\kappa \ddot{x}^\lambda = a^2 \left\{ g^{\mu\nu} \left( \partial_\nu A_\kappa - \partial_\kappa A_\nu \right) u^\nu \right.$$  

$$\hspace{2cm} + g^{\mu\nu} \left( g_{\nu\sigma} \partial_\kappa A_\lambda + A_\sigma \partial_\kappa g_{\nu\lambda} + \frac{3}{2} A_\nu \partial_\kappa g_{\lambda\rho} - \frac{1}{4} A_\kappa \partial_\nu g_{\lambda\rho} \right) u^\sigma u^\nu u^\lambda + A_\lambda \partial_\kappa g_{\nu\sigma}u^\sigma u^\nu u^\mu u^\kappa u^\lambda \right\}.$$  \hspace{1cm} (105)

Substituting the relations (93), (95), (101) and (105) into (97) we get

$$\ddot{x}^\mu + \gamma_{\kappa\lambda}^\mu \ddot{x}^\kappa \ddot{x}^\lambda = -\frac{z}{1+z} \left\{ ag^{\mu\nu}(\partial_\nu A_\kappa - \partial_\kappa A_\nu) \dot{x}^\nu + \frac{1}{a} \left[ A^\nu \left( \partial_\nu g_{\lambda\rho} - \frac{1}{2} \partial_\nu g_{\kappa\rho} \right) + \partial_\kappa A_\lambda \right] \dot{x}^\nu \dot{x}^\lambda \right.$$  

$$\hspace{2cm} + \frac{1}{a} \left( \frac{1}{4} g^{\mu\nu} A_\kappa \partial_\nu g_{\lambda\sigma} + g^{\mu\nu} A_\kappa \partial_\nu g_{\sigma\nu} + A^\mu \partial_\kappa g_{\lambda\rho} \right) \dot{x}^\sigma \dot{x}^\kappa \dot{x}^\lambda \right.$$  

$$\left. + \frac{1}{2a^3} A_\lambda \partial_\kappa g_{\sigma\tau} \dot{x}^\sigma \dot{x}^\tau \dot{x}^\mu \dot{x}^\kappa \dot{x}^\lambda \right\}.$$  \hspace{1cm} (106)

This is the generalized path equation for the timelike sector of the metric $g_{\mu\nu}(x)$. 
Remark: If we set \( a = 1 \) at some fixed point then (106) can be written as

\[
\ddot{x}^\mu + \gamma_{\kappa\lambda}^\mu \dot{x}^\kappa \dot{x}^\lambda - \frac{e}{m} F^\mu_{\kappa} \dot{x}^\kappa
\]

\[
= \frac{e}{m} \left\{ \left[ A^\nu \left( \partial_{\nu} g_{\kappa\lambda} - \frac{1}{2} \partial_{\kappa} g_{\nu\lambda} \right) + \partial_{\kappa} A_{\lambda} \right] \dot{x}^\mu \dot{x}^\kappa \dot{x}^\lambda 
+ \left( \frac{1}{4} g^{\mu\nu} A_\kappa \partial_{\nu} g_{\sigma\lambda} + g^{\mu\nu} A_\kappa \partial_{\nu} g_{\sigma\lambda} + A^\mu \partial_{\nu} g_{\kappa\lambda} \right) \dot{x}^\sigma \dot{x}^\kappa \dot{x}^\lambda 
+ A_{\lambda} \partial_{\nu} g_{\sigma\lambda} \dot{x}^\sigma \dot{x}^\kappa \dot{x}^\lambda \right\},
\]

(107)

with \( F_{\mu\nu} = \partial_{\nu} A_{\mu} - \partial_{\mu} A_{\nu} \) the field strength tensor of \( A_{\nu} \), and we have set \( \frac{1}{\sqrt{2}} \zeta := - \frac{m}{e} \) where \( e \) the electric charge and \( m \) the mass of the particle. If we ignore the r.h.s of the above equation then (107) will have the same form as the equation of a charged particle subject to the Lorentz force with an electromagnetic vector potential \( A_{\nu} \) in the Riemannian setting. A similar equation which is derived from a Finsler-Randers Lagrangian and contains a Lorentz path of zero length (with respect to infinitesimal distance between two neighbouring points increases following the timelike path. Therefore, the final null metric and the Lorentz force requires further investigation and goes beyond the scope of this work.

Now, it is known that we can always approach a timelike path (geodesic) with a proper time parameter broken null path with the same endpoints [24]. In this approximation it is considered that the number of null path segments with infinitesimal distance between two neighbouring points increases following the timelike path. Therefore, the final null path of zero length (with respect to \( g_{\mu\nu} \)) approaches the timelike path (106), however the parameter along them is replaced by an appropriate affine one.

Substituting to (106) the solution (89) we get the explicit form of the timelike paths components:

\[
\begin{align*}
\dot{t} + \frac{1 - f}{rf} \dot{t} & = - \frac{z}{1 + z} \dot{\lambda}_1 \left\{ \left( \frac{1}{2} a f^{-3/2} \dot{t} + \frac{1}{2} \frac{1}{rf} f^{-1/2} \dot{t}^2 + \frac{1}{2} \frac{1}{rf} f^{-5/2} \dot{t}^3 \right) \frac{1 - f}{r} - \frac{2r}{a} f^{-1/2} (i \dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2 + \frac{r}{2} \sin 2 \theta \dot{\phi} \dot{\phi}) \right. \\
& \left. + \dot{r} \left[ \frac{1}{a} \left( - f^{-3/2} + \frac{1}{2} f^{-1/2} \right) i \dot{r} \frac{1 - f}{r} \\
& \left. + \frac{1}{2a^3} \left( - \left\{ f^{1/2} i \dot{t} \dot{r} + f^{-3/2} i \dot{r}^3 \right\} \frac{1 - f}{r} + 2f^{1/2} r \{ i \dot{t} \dot{\theta}^2 + \sin^2 \theta \dot{r} \dot{\phi}^2 + \frac{r}{2} \sin 2 \theta \dot{t} \dot{\phi} \dot{\phi} \} \right) \right] \right) \\
\dot{r} + \frac{f (1 - f)}{2r} i \dot{t} - \frac{1 - f}{rf} i \dot{t}^2 - r f (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) & = - \frac{z}{1 + z} \dot{\lambda}_1 \left\{ \left( \frac{1}{2} a f^{1/2} \dot{r} - \frac{1}{4a} f^{3/2} \dot{t}^2 - \frac{5}{4a} f^{-1/2} \dot{t}^3 \right) \frac{1 - f}{r} + \frac{1}{2a} f^{3/2} r (i \dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) \right. \\
& \left. + \dot{r} \left[ \frac{1}{a} \left( - f^{-3/2} + \frac{1}{2} f^{-1/2} \right) i \dot{r} \frac{1 - f}{r} \\
& \left. + \frac{1}{2a^3} \left( - \left\{ f^{1/2} i \dot{r}^3 + f^{-3/2} i \dot{r}^3 \right\} \frac{1 - f}{r} + 2f^{1/2} r \{ i \dot{t} \dot{\theta}^2 + \sin^2 \theta \dot{r} \dot{\phi}^2 + \frac{r}{2} \sin 2 \theta \dot{t} \dot{\phi} \dot{\phi} \} \right) \right] \right) \\
\dot{\theta} + \frac{2}{r} \dot{\theta} \dot{r} - \frac{1}{2} \sin 2 \theta \dot{\phi}^2 & = - \frac{z}{1 + z} \dot{\lambda}_1 \left\{ \frac{1}{a} \left( \frac{1}{4} f^{1/2} \sin 2 \theta \dot{\phi}^2 + 2 i \dot{r} \dot{\theta} \right) \\
& \left. + \dot{r} \left[ \frac{1}{a} \left( - f^{-3/2} + \frac{1}{2} f^{-1/2} \right) i \dot{r} \frac{1 - f}{r} \\
& \left. + \frac{1}{2a^3} \left( - \left\{ f^{1/2} i \dot{r}^3 + f^{-3/2} i \dot{r}^3 \right\} \frac{1 - f}{r} + 2f^{1/2} r \{ i \dot{t} \dot{\theta}^2 + \sin^2 \theta \dot{r} \dot{\phi}^2 + \frac{r}{2} \sin 2 \theta \dot{t} \dot{\phi} \dot{\phi} \} \right) \right] \right) \\
\dot{\phi} + \frac{2}{r} \dot{\phi} \dot{r} + 2 \cot \theta \dot{\phi} & = - \frac{z}{1 + z} \dot{\lambda}_1 \left\{ \frac{1}{a} \left( i \dot{\phi} \dot{\phi} + \cot \theta \dot{t} \dot{\phi} \right) \\
& \left. + \dot{r} \left[ \frac{1}{a} \left( - f^{-3/2} + \frac{1}{2} f^{-1/2} \right) i \dot{r} \frac{1 - f}{r} \\
& \left. + \frac{1}{2a^3} \left( - \left\{ f^{1/2} i \dot{r}^3 + f^{-3/2} i \dot{r}^3 \right\} \frac{1 - f}{r} + 2f^{1/2} r \{ i \dot{t} \dot{\theta}^2 + \sin^2 \theta \dot{r} \dot{\phi}^2 + \frac{r}{2} \sin 2 \theta \dot{t} \dot{\phi} \dot{\phi} \} \right) \right] \right) \\
\end{align*}
\]
with \( f = 1 - \frac{R_S}{r} \), \( R_S \) the Schwarzschild radius.

For completeness, we will find the spacelike paths from (97) following the same procedure as above. From (11) we get

\[
w_{\nu\lambda} = g_{\lambda\rho} A_{\nu} u^\rho + g_{\nu\rho} A_{\lambda} u^\rho + (g_{\nu\lambda} - g_{\nu\sigma} g_{\lambda\tau} u^\tau u^\sigma) A_{\rho} u^\rho
\]  

(112)

where, as before, we have lowered down \( w_{\nu\rho} \) to the horizontal space using the generalized Kronecker symbols and we have set \( u^\nu = y^\nu / a \) where \( a = \sqrt{g_{\mu\nu} y^\mu y^\nu} \). Following the same steps as for the timelike section of \( g_{\mu\nu} \) and taking into account that \( g_{\mu\nu} u^\mu u^\nu = 1 \), (97) gives

\[
\ddot{x}^\mu + \dot{\gamma}^\nu_{\mu\lambda} \dot{x}^\nu \dot{x}^\lambda = -\frac{z}{1+z} \left\{ -a g^{\mu\nu} (\partial_\nu A_\kappa - \partial_\kappa A_\nu) \dot{x}^\kappa + \frac{1}{a} \left[ A^\nu \left( \partial_\kappa g_{\nu\lambda} - \frac{1}{2} \partial_\nu g_{\kappa\lambda} \right) + \partial_\kappa A_\lambda \right] \dot{x}^\mu \dot{x}^\kappa \dot{x}^\lambda \\
+ \frac{1}{\alpha} \left( -\frac{1}{4} g^{\mu\nu} A_\kappa \partial_\nu g_{\kappa\lambda} + g^{\mu\nu} A_\kappa \partial_\nu g_{\kappa\lambda} \right) \dot{x}^\sigma \dot{x}^\tau \dot{x}^\kappa \dot{x}^\lambda \\
- \frac{1}{\alpha^3} A_{\kappa\lambda} \partial_\nu g_{\kappa\lambda} \dot{x}^\sigma \dot{x}^\tau \dot{x}^\mu \dot{x}^\nu \dot{x}^\lambda \right\}
\]  

(113)

Substituting to (113) the solution (119) we get the explicit form of the spacelike paths components:

\[
\ddot{t} + \frac{1 - f}{2r} \dot{t} = -\frac{z}{1+z} A_t \left\{ \left( -\frac{1}{2} a f^{-3/2} \dot{t} + \frac{1}{a} f^{-1/2} \dot{t} \right) \frac{1 - f}{r} \right. \\
+ \dot{f} \left( \frac{1}{a} \right) \left. \left( -f^{-3/2} + \frac{1}{2} f^{-1/2} \right) \dot{t} \right) \frac{1 - f}{r} \\
- \frac{5}{2a^3} \left\{ \left( f^{1/2} \dot{t} \dot{t} + f^{-3/2} \dot{t} \dot{t} \right) \frac{1 - f}{r} + 2f^{1/2} r \{ t i \theta^2 + \sin^2 \theta i \phi^2 + \frac{r}{2} \sin 2 \theta i \theta \phi^2 \} \right\}
\]  

(114)

\[
\ddot{\theta} + \frac{2 \dot{\theta}}{r} \dot{\theta} - \frac{1 - f}{2rf} \dot{\theta}^2 - r f (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) = -\frac{z}{1+z} A_{\theta} \left\{ \left( -\frac{1}{4} a f^{1/2} \dot{t} + \frac{1}{4a} f^{1/2} \dot{t} \dot{t} - \frac{3}{4} a f^{-1/2} \dot{t} \dot{t} \right) \frac{1 - f}{r} - \frac{1}{2a} f^{3/2} r \{ t i \theta^2 + \sin^2 \theta i \phi^2 \} \right. \\
+ \dot{f} \left( \frac{1}{a} \right) \left. \left( -f^{-3/2} + \frac{1}{2} f^{-1/2} \dot{t} \dot{t} \right) \dot{t} \right) \frac{1 - f}{r} \\
- \frac{5}{2a^3} \left\{ \left( f^{1/2} \dot{t} \dot{t} + f^{-3/2} \dot{t} \dot{t} \right) \frac{1 - f}{r} + 2f^{1/2} r \{ t i \theta^2 + \sin^2 \theta i \phi^2 + \frac{r}{2} \sin 2 \theta i \theta \phi^2 \} \right\}
\]  

(115)

\[
\ddot{\phi} + \frac{2 \dot{\phi}}{r} \dot{\phi} + 2 \cot \theta \dot{\phi} = -\frac{z}{1+z} A_{\phi} \left\{ \left( \frac{1}{a} i \dot{t} \dot{\phi} + \cot \theta i \dot{\theta} \dot{\phi} \right) \right. \\
+ \dot{f} \left( \frac{1}{a} \right) \left. \left( -f^{-3/2} + \frac{1}{2} f^{-1/2} \dot{t} \dot{t} \right) \dot{t} \right) \frac{1 - f}{r} \\
- \frac{5}{2a^3} \left\{ \left( f^{1/2} \dot{t} \dot{t} + f^{-3/2} \dot{t} \dot{t} \right) \frac{1 - f}{r} + 2f^{1/2} r \{ t i \theta^2 + \sin^2 \theta i \phi^2 + \frac{r}{2} \sin 2 \theta i \theta \phi^2 \} \right\}
\]  

(116)

\[
\ddot{\phi} + \frac{2 \dot{\phi}}{r} \dot{\phi} + 2 \cot \theta \dot{\theta} \dot{\phi} = -\frac{z}{1+z} A_{\phi} \left\{ \left( \frac{1}{a} i \dot{t} \dot{\phi} + \cot \theta i \dot{\theta} \dot{\phi} \right) \right. \\
+ \dot{f} \left( \frac{1}{a} \right) \left. \left( -f^{-3/2} + \frac{1}{2} f^{-1/2} \dot{t} \dot{t} \right) \dot{t} \right) \frac{1 - f}{r} \\
- \frac{5}{2a^3} \left\{ \left( f^{1/2} \dot{t} \dot{t} + f^{-3/2} \dot{t} \dot{t} \right) \frac{1 - f}{r} + 2f^{1/2} r \{ t i \theta^2 + \sin^2 \theta i \phi^2 + \frac{r}{2} \sin 2 \theta i \theta \phi^2 \} \right\}
\]  

(117)
VI. CONCLUSION

In this paper we derived for the first time the gravitational field as a solution of the spherically symmetric Schwarzschild-Randers and Schwarzschild-De Sitter-Randers metric. In this framework we used generalized Einstein field equations on the tangent bundle of a spacetime with zero horizontal energy-momentum tensor in which we get more degrees of freedom. In addition, we specified an appropriate timelike covector which plays a significant role in this theory differentiating our model from the traditional Schwarzschild one, giving an intrinsic anisotropic character to the ordinary Schwarzschild metric as well as for the particle paths. Moreover, we studied the correlation of the Schwarzschild-De Sitter model with the Randers one as becomes apparent from (90). We also studied the forms of paths in our spacetime and we obtained more generalized forms than the ordinary geodesic paths of the classical Schwarzschild spacetime.

It is obvious that when the covector $A_\gamma$ of our theory vanishes then we recover the ordinary form of a Schwarzschild metric and Schwarzschild-De Sitter metric respectively and their derived geodesics.

Such an approximation can be considered compatible with some current observational data and parameters with anisotropic character in cosmological models of Schwarzschild-Randers and Schwarzschild-De Sitter-Randers spacetimes. These features mean that Finsler-Randers gravity can be interesting at the astrophysical level. This study will be the goal of our next work.

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