In this paper we discuss the connection on a space of $N = 2$ TCFT’s that appears in the context of background (in)dependence. We formulate a family of target space field theories with a similar connection on it. Each theory is a gauge theory (with the gauge group being $SDiff$ in the case of 3-fold). It describes deformations of Kähler structures much like Kodaira Spencer theory describes deformations of the complex structures. It is manifestly background independent. It appears to be a target space field theory for supersymmetric quantum mechanics.
1. Introduction

Kodaira-Spencer theory [1] is a string field theory for topological $B$-model. As it was noticed in [2] in this case the string field theory reduces to a field theory. The reason for this is that topological $B$-model coupled to gravity is essentially independent of the Kähler structure. Rescaling the volume to infinity one recovers that the path integral is dominated by highly degenerate Riemann surfaces. One can think of degenerate Riemann surfaces as infinitely thin tubes attached to each other. In other words, topological $B$-model can be described as supersymmetric quantum mechanics. In the case of topological $A$-model the situation is different. It is known that nontrivial worldsheet configurations (instantons) play the crucial role in topological $A$-model. String field theory for $A$-model is defined on the loop space. In the large volume limit the instanton effects are suppressed and one can describe the semiclassical limit of string field theory as supersymmetric quantum mechanics (SQM). This SQM makes sense by itself even when the volume is not large. It also exhibits some properties of underlying string theory.

SQM in question describes deformations of Kähler structures in the same way as Kodaira-Spencer theory [1] describes deformations of complex structures. We will call this theory AKS, where A stands for topological $A$-model in Witten's terminology [4], and KS stands for Kähler structures. It is known that the perturbation theory of Chern-Simons theory can be interpreted as a perturbation theory of open strings propagating on $T^*(M)$, where $M$ is three dimensional[3]. In trying to describe the closed string sector (which is required by consistency in open string theory) E. Witten introduced the action for AKS theory [3]. In spite of the fact that AKS is very similar to Chern-Simons it is not a topological theory. Its Hamiltonian is non trivial, while the phase space is finite dimensional. On the other hand the AKS theory enjoys the properties of being independent on complex structure. It depends only on the Kähler class of the metric. We call this theory a Kähler topological theory defined on a Kähler manifold. The gauge invariant observables of Chern-Simons theory are Wilson lines. In [3] the Wilson lines were used in order to incorporate the worldsheet instantons in string theory. In the case of AKS theory we do not know any gauge invariant observable except the action. It is tempting to suggest that the would be gauge invariant observables are related to holomorphic curves in the target space, or saying differently to worldsheet instantons.

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1. This situation is very similar to the conventional theory of gravity
The plan of this paper is the following. In Section 2 we discuss the notion of background independence. This discussion is quite general and is applicable to any \( N = 2 \) topological conformal field theories (TCFT). There is a natural connection on the moduli space of TCFT. This connection allows one to identify the perturbed TCFT at certain background with unperturbed TCFT at another background. Background independence is equivalent to the statement that the connection is flat. In general, there is an obstacle known as holomorphic anomaly. In order to avoid this problem one has to consider only the holomorphic deformations of TCFT. The background independence imposes strong restrictions on the form of contact terms. In principle, these equations should fix the connection in full (quantum) theory. Semiclassically, these equations have a unique solution and supersymmetric quantum mechanics (SQM) is a theory which solves them.

Whereas the moduli space of TCFT is a complexified Kähler cone, the moduli space of SQM is a real Kähler cone. As explained in Section 2 we identify the real deformations of SQM with the holomorphic deformations of \( N = 2 \) TCFT by means of analytic continuation. Under this identification the semiclassical limit of the holomorphic connection of TCFT is mapped on the flat connection of SQM. Therefore the holomorphic anomaly does not show up in SQM. As a result the SQM is background independent. This connection has a natural geometric interpretation.

Sections 3 – 5 are devoted to AKS theory and its properties. One can construct AKS action for a given point in the moduli space of Kähler structures and the tangent vector (\( \omega \) and \( x \in H^2 \)) that serves as the background data. AKS is a gauge invariant theory with symmetries generated by the large volume limit of string BRST \( Q \). The classical equation of motion is equivalent to the condition \( Q^2 = 0 \). The solution of this equation of motion determines a perturbed Kähler structure (the precise meaning of this will be explained). The gauge group is non-abelian and in the case of 3-dimensional Kähler manifolds is isomorphic to volume preserving diffeomorphisms. In Section 3 we discuss the Batalin-Vilkovisky formalism for AKS theory. The absence of higher Massey products on the Kähler manifold makes AKS action exact at the quantum level. In Section 5 we discuss the Hamiltonian quantization of AKS theory.

AKS is a target space field theory for suitably modified (along the lines of reference [4]) \( N = 2 \) SQM. The connection discussed above in the context of SQM naturally appears in AKS. It allows to relate AKS theories at different Kähler structures. The idea of background independence can be fully applied to AKS. Under the variation of Kähler structure the AKS action minus the action evaluated on the classical trajectory scales
with volume as the second power. This scaling can be reabsorbed into the redefinition of the coupling constant \( g^2 \rightarrow g^2 \cdot \text{Vol}^2 \). This power differs from the one naively expected from the dimensional considerations.

We conjecture that the effective action \( \Gamma(x) \) for SQM is a free energy for AKS which depends on \( x \) as a parameter. In Section 6 we prove this relation at the tree level. In doing this we found a very simple mechanism that allows one to rewrite vacuum diagrams for AKS as S-matrix diagrams of effective field theory.

2. Contact terms

2.1. Background independence and the contact term algebra

Consider a family of A-twisted \( N = 2 \) superconformal \( \sigma \)-models on Calabi-Yau space. Let us start with the discussion of how the susy generators vary under the variation along the moduli. The physical operators (chiral fields) in topological A-model are given by BRST cohomology

\[
\phi = \phi_{i_1i_2...i_n\bar{j}_1...\bar{j}_m} \chi^{i_1}...\chi^{i_n}\bar{\chi}^{\bar{j}_1}...\bar{\chi}^{\bar{j}_m}.
\]

(2.1)

and are independent of the target space metric. These are 0-forms on the world sheet. Solving the descent equation [3] one obtains the operators \( \phi^{(1)} \) and \( \phi^{(2)} \), respectively 1- and 2-forms on the world sheet which are BRST-closed only modulo total derivative. Also, we will need to consider the antichiral fields

\[
\phi^\dagger = \phi_{i_1i_2...i_n\bar{j}_1...\bar{j}_m} \rho^{i_1}...\rho^{i_n}\bar{\rho}^{\bar{j}_1}...\bar{\rho}^{\bar{j}_m}.
\]

(2.2)

The operator (2.2) is a \((n,m)\)-form on the world sheet. The antichiral fields are not BRST-closed. In this paper we will restrict ourselves to studying the effects of deforming the theory by exactly marginal operators (2.1) (corresponding to the target space \((1,1)\)-forms) and by their antichiral counterparts (2.2):

\[
S \rightarrow S + \epsilon^a \int \phi^{(2)}_a + \epsilon^{a} \int [Q, [\bar{Q}, \phi^\dagger_a]].
\]

(2.3)

In (2.3) the perturbation \( \int \phi^{(2)}_a \) is exactly marginal, while \( \int [Q, [\bar{Q}, \phi^\dagger_a]] \) is BRST trivial.

For a given point \( p \) in the moduli space \( \mathcal{M} \) the perturbations are the vectors in the tangent space \( T_p \mathcal{M} \) to \( \mathcal{M} \) at \( p \). The coordinates \( \epsilon^a, \epsilon^{a} \) make \( T_p \mathcal{M} \) into a complex space.

\[2\] Talking about the moduli space here, we usually mean the complexified cone of Kähler classes \( \mathcal{K}_C \).
At the moment, we have a family of perturbed topological theories (2.3), parameterized by points \((p, \epsilon, \bar{\epsilon})\) of the tangent bundle \(T\mathcal{M}\) to the moduli space \(\mathcal{M}\). The concept of global background independence is that any perturbed theory at \((p, \epsilon, \bar{\epsilon})\) is equivalent to some unperturbed theory at \((\tilde{p}, 0, 0)\), where the coordinates of \(\tilde{p} \in \mathcal{M}\) are functions of \((\epsilon, \bar{\epsilon})\) and \(p\). In fact there is a whole family of perturbed theories \((p', \epsilon(p'), \bar{\epsilon}(p'))\) parameterized by \(p'\) such that \(\epsilon(p) = \epsilon, \bar{\epsilon}(p) = \bar{\epsilon}\) and \(\epsilon(\tilde{p}) = \bar{\epsilon}(\tilde{p}) = 0\). This implies existence of a connection \(\mathcal{D}\) on \(T\mathcal{M}\) such that

- It preserves the physical content of theory. This means that the parallel transport does not alter the correlation functions of the theory;
- Every constant section \(x = (\epsilon(p), \bar{\epsilon}(p))\) (a solution of the equation \(\mathcal{D}x = 0\)) has one and only one zero on \(\mathcal{M}\).

The first condition above guarantees that the correlators remain the same on the constant section passing through a given perturbed theory. The second condition allows one to reach the unperturbed theory unambiguously moving along the constant section. The connection should be necessarily flat. The very notion of constant section having a zero at the particular point requires this.

In general there is no flat connection with all the above properties. There is a non-zero curvature, which is given by the relation of special geometry. On the other hand the tangent bundle is holomorphic and \((0, 2)\) and \((2, 0)\) components of the curvature are equal to zero which means that holomorphic (antiholomorphic) directions are flat. Therefore, perturbing the theory only by chiral primary fields one can consistently define the path-independent parallel transport of the tangent space. This perturbation by chiral primary fields is nothing else but an analytic continuation in holomorphic direction.

In general, such connection is affine — the transformation it induces in the tangent space is not linear but rather a composition of the linear one and a translation. The linear piece provides a linear connection \(D\) on \(T\mathcal{M}\).

Leaving the discussion of the global background independence for the next sections, here we will concentrate on the local problem. By the local background independence it is usually meant that it is really possible to identify the deformations (2.3) as tangent vectors to \(\mathcal{M}\). So from now till the end of this section we assume that the parameters \(\epsilon, \bar{\epsilon}\) are infinitesimally small.

It is convenient to define the Hilbert space bundle \(\mathcal{H}\mathcal{M}\) as a bundle over the moduli space whose fiber at every point is given by Hilbert space. The space of physical states with charges \((q, \bar{q}) = (1, \bar{1})\) can be identified with the tangent space to the moduli space.
and therefore $TM$ is a subbundle in $HM$. As we will see below there are two connections $D_H$ on $HM$ and $D$ on $TM$. The connection $D$ can be obtained by restricting $D_H$ on the tangent bundle.

Let us recall the basics of the state-operator correspondence for families of topological A-models. As mentioned above, the operators are independent of the parameters $\epsilon, \bar{\epsilon}$ of deformation. Now, the state-operator correspondence implies that the states do depend on $\epsilon, \bar{\epsilon}$. Indeed, the state $|\psi\rangle$ (associated with the wave function $\psi$) is given by the path integral over hemisphere $\Sigma$ with appropriate boundary conditions. Under the variation $(2.3)$ of the action this path integral varies according to

$$\delta |\psi\rangle = \int_{\Sigma} \phi^{(2)} d^2z |\psi\rangle$$

In the case of exactly marginal deformations of conformal field theory this integral picks just a contact term contribution since the bulk term is zero

$$\phi^{(2)}(z)\psi(x) \sim \mu(\phi, \psi)\delta^2(z - x).$$

Thus we obtain the equation $\delta |\psi\rangle = |\mu(\phi, \psi)\rangle$ describing deformations of the states by $(2.3)$. As will be clear below, the contact term $\mu(\phi, \psi)$ is a chiral operator, not $Q$ closed in general even if $\psi$ is $Q$ closed. This contact term defines a connection on the Hilbert space bundle $HM$.

Before discussing the connections $D_H$ and $D$ let us first discuss the variation of susy generators. The relevant OPEs are

$$G^+(z)\phi^{(2)}(x) = \partial_x \left( \frac{1}{x - z} \phi^{(0,1)} \right) + \delta^2(x - z)\delta_\phi G^+_z,$$

$$G^+(z)\phi^{(2)}(x) = \delta^2(x - z)\delta_\phi G^+_z,$$  \hspace{1cm} (2.6a)

$$G^-(z)\phi^{(2)}(x) = \partial_x \left( \frac{1}{x - z} [\bar{Q}, \phi^{(1)}] \right) + \delta^2(x - z)\delta_\phi G^-_{zz},$$

$$G^-(z)\phi^{(2)}(x) = \delta^2(x - z)\delta_\phi G^-_{zz},$$  \hspace{1cm} (2.6b)

The contact terms in $(2.6a)$ - $(2.6d)$ (the coefficients in front of the $\delta$-functions) ensure the conservation of the perturbed currents. To interpret these OPEs we note first that the $N = 2$ susy generators $G^+_z$ and $G^-_{zz}$ explicitly depend on the target space metrics. These variations explicitly appear as coefficients in front of the $\delta$-functions.

To understand the importance of the total derivatives in $(2.6a)$ - $(2.6d)$ let us consider a BRST operator $Q$ acting on some state $|\psi\rangle$ in the perturbed theory. Its action is given by
\[ \oint d\zeta Gz^+, \] where integration runs over the boundary of the hemisphere. The perturbation does not commute with the BRST operator. It makes a difference whether we first make a perturbation and then compute the action of BRST or vice versa. The difference is given by

\[ Q|\delta\phi\psi\rangle = \epsilon \oint\partial\Sigma d\zeta Gz^+ \int_{\Sigma} d^2x \phi^{(2)}|\psi\rangle = \epsilon \oint\partial\Sigma d\bar{x}\phi^{(0,\bar{1})}|\psi\rangle + \delta\phi|Q\psi\rangle \tag{2.7} \]

where the contour integral over the boundary of \( \Sigma \) comes from the total derivative term in (2.6d) - (2.6d). One can reinterpret this contour integral as coming from the boundary variation of the action that ensures the BRST invariance of the path integral on the hemisphere \( \Sigma \). The relation (2.7) implies that in the *perturbed* theory the BRST operator depends on \( \epsilon \) as follows

\[ Q(\epsilon) = Q + \epsilon \oint \phi^{(0,\bar{1})}, \tag{2.8} \]

where \( Q \) is the BRST operator of the unperturbed theory. There are similar formulas for the other susy generators. As one can see the variation of \( G_0^- \) depends on \( \bar{\epsilon} \) and is given as follows

\[ G_0^-(\bar{\epsilon}) = G_0^- + \bar{\epsilon} \oint d\bar{\zeta} z [\bar{Q},\phi]\]

From (2.8), it is clear that the contact term \( \mu(\phi,\psi) \) is not \( Q \)-closed. In fact, there should be \( \left(Q + \epsilon \oint \phi^{(1)}\right)(|\psi\rangle + \epsilon |\mu(\phi,\psi)\rangle) = |Q\psi\rangle + \epsilon |\mu(\phi,Q\psi)\rangle \implies \]

\[ Q\mu(\phi,\cdot) - \mu(\phi,Q\cdot) = -\oint \phi^{(0,\bar{1})} \tag{2.10} \]

The variation of the chiral states (2.4) induced by the contact terms gives rise to the (infinitesimal) map of the Hilbert space \( U_{\phi} : \mathcal{H} \to \mathcal{H} \), where \( U_{\phi}|\psi\rangle = |\psi\rangle + \epsilon^a |\mu(\phi,\psi)\rangle \). This map combines with (2.8) in a way that ensures the local background independence. Namely, the representation of the susy generators changes according to

\[ U_{\phi}(Q + \epsilon \oint \phi^{(0,\bar{1})}) U_{\phi}^{-1}|\psi\rangle = \tilde{Q}|\psi\rangle \tag{2.11a} \]

\[ U_{\phi}(G_0^- + \epsilon \oint d\bar{\zeta} z [\bar{Q},\phi]) U_{\phi}^{-1}|\psi\rangle = \tilde{G}_0^- |\psi\rangle. \tag{2.11b} \]

Similar formulas are valid for the right movers. The operators \( \tilde{Q} = Q + \delta Q \) and \( \tilde{G}_0^- = G_0^- + \delta G_0^- \) are the susy generators for the *unperturbed* theory corresponding to the point \( \tilde{p} \in \mathcal{M} \) close to the point \( p \) where the original theory sits. Thus \( \delta Q \) and \( \delta G_0^- \) are the classical variations due to explicit dependence of \( N = 2 \) generators on the target space metric and should not be confused with additional terms in (2.8) and (2.9).
Operator $b_0^- = G_0^- - \overline{G_0^-}$ plays the important role in string theory. The Hilbert space $\mathcal{H}$ is defined as follows

$$\mathcal{H} = \{ \psi \in \mathcal{H} | b_0^- \psi = 0 \}$$

For $\bar{\epsilon} = 0$ the map $U_\phi = 1 + \epsilon \mu(\phi, \cdot)$ defines a flat connection on the Hilbert space bundle $\mathcal{H}\mathcal{M}$

$$|\psi\rangle \rightarrow |\psi\rangle + \epsilon |\mu(\phi, \psi)\rangle$$

Indeed, it follows from (2.11) that $U_\phi b_0^- U_\phi^{-1} = \tilde{b}_0^-$ and therefore $U_\phi$ maps $\mathcal{H}_p$ on $\mathcal{H}_{\tilde{p}}$. In the next section we will derive the semiclassical expression for $\mu(\phi, \cdot)$.

In fact there are two equally good descriptions of the perturbed theory. In one description a BRST operator $Q(\epsilon)$ varies according to (2.8), while $b_0^-$ remains constant (we already put $\bar{\epsilon} = 0$, see (2.9)). There is another possibility to describe the perturbed theory as an unperturbed one at the new background. In this description the BRST operator $Q$ and $b_0^-$ are mapped on the operators $\tilde{Q}$ and $\tilde{b}_0^-$ at the new background. These two descriptions are equivalent to each other and related by conjugation by operator $U_\phi$.

At this point one can use (2.5) to compute the effect of the deformation (2.3) on the correlation functions. An important case is a two-point function of one chiral and one anti-chiral primary field which gives a matrix element of the Zamolodchikov metric on $\mathcal{M}$: $g_{b\overline{c}} = \langle \phi^\dagger_b | \phi_{\overline{c}} \rangle$. We remind the reader that the primary fields are the “harmonic” representatives of the BRST cohomology. This turns out to be quite important, because in the matrix element $\langle \phi^\dagger_c | \mu(\phi_a, \phi_b) \rangle$ responsible for the chiral deformation of $g_{b\overline{c}}$, only the harmonic part of the contact term contributes. Indeed, taking the “Hodge decomposition”

$$\mu(\phi_a, \phi_b) = \Gamma^c_{ab} \phi_c + Q(...) + G_0(...)$$

one sees that both $Q$- and $G_0$- exact terms decouple since $\langle \phi^\dagger_c \rangle$ is harmonic: $\partial_a g_{b\overline{c}} = \Gamma^c_{ab} g_{c\overline{c}}$. In (2.14) we decomposed the harmonic part of the contact term in the basis of chiral primaries. By definition, the coefficients $\Gamma^c_{ab}$ are the holomorphic Cristoffel symbols of the metric connection $D_a$ for the Zamolodchikov metric.

A similar argument can be applied to describe the deformation of the multipoint correlation functions, possibly on the higher genus worldsheet. Indeed, in this case one

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3 Depending on the normalization, it may be either Zamolodchikov or $tt^*$ metric. The former is more natural object for the string is theory, so we stick to it.
integrates over the moduli space of algebraic curves with punctures. The contact terms appear on the compactification divisors, when two punctures try to collide. They cannot possibly collide on the Deligne-Mamford compactified moduli space \[9\]. Instead, a node develops on the curve by splitting off a rational curve with two “colliding” punctures on it. In a language more familiar to physicists, the colliding punctures sit on a sphere connected to the rest of the worldsheet by a long tube. The length (and twist) of the tube is the right parameter to describe the closing of the punctures. In the limit when the punctures collide (\(z = x\) in the argument of the \(\delta\)-function), the tube is infinitely long. Propagating along the infinitely long tube, the state \(|\mu(\phi_a, \phi_b)\rangle\) is automatically projected onto the ground states \[6\]. In other words, again only the first term on the right hand side of (2.14) happens to be relevant.

We see that for the purposes of computing the correlation functions, it suffices to use the “projected” form of the deformation equations: \(\delta_a |\phi_b\rangle = \Gamma^c_{ab} |\phi_c\rangle\) and \(\delta_a |\phi^\dagger_b\rangle = 0\). It describes the parallel transport of the chiral ground states with respect to the Zamolodchikov connection \(D_a\). In fact one can make this projection more explicit. Let us use the description of the perturbed theory in which the BRST operator \(Q(\epsilon)\) varies, while \(b^{-0}\) remains constant. The connection on the space of physical states should be compatible with the variations of the BRST operator and \(b^{-0}\), namely

- \(Q\)-closed states should be mapped on \(Q(\epsilon)\)-closed states;
- \(Q\)-exact states should be mapped on \(Q(\epsilon)\)-exact states;
- the variation of the states should be \(b^{-0}\)-trivial

The first two conditions ensures that physical states are mapped on the physical states, while the last condition ensures the uniqueness of the connection \[10\].

Formally, for \(|\psi\rangle\) being \(Q\)-closed one can immediately write down the connection that satisfies the above conditions

\[|\psi\rangle \rightarrow |\psi\rangle - \epsilon \frac{1}{Q} b^{-0} \phi |\psi\rangle.\]  \(\text{(2.15)}\)

One can show that \(b^{-0} \phi |\psi\rangle\) is always \(Q\) exact and therefore \(Q\) is invertible\[^4\]. The first two conditions follow from the identity \(\oint \phi^{(1)} |\psi\rangle = b^{-0} \phi |\psi\rangle\) for \(\phi\) and \(\psi\) being \(Q\)-closed. The connection \(\text{(2.15)}\) can be derived using the cancel propagator arguments \[11\].

Describing the perturbed theory as an unperturbed one around the new background one obtains the connection on the tangent bundle \(\mathcal{T}\mathcal{M}\)

\[^4\] This is an analog of \(\partial \bar{\partial}\) Lemma for Kähler manifolds.
\[ |\psi\rangle \rightarrow |\psi\rangle + \epsilon |\mu(\phi, \psi)\rangle - \frac{1}{Q} b_0^- \phi |\psi\rangle . \] (2.16)

As we will see later both these connections (2.13) and (2.16) appear in the description of AKS theory.

2.2. Semiclassical calculations

In the large volume limit the Hilbert space of the theory can be identified with the space of differential forms on the target space. The left and right $U(1)$ charges can be identified with (holomorphic, anti-holomorphic) degree of the form. For this Hilbert space there is the following dictionary

\[
\begin{align*}
G^+_0 &\leftrightarrow \partial , & \bar{G}^+_0 &\leftrightarrow \bar{\partial} \\
G^-_0 &\leftrightarrow \partial^\dagger , & \bar{G}^-_0 &\leftrightarrow \bar{\partial}^\dagger
\end{align*}
\] (2.17)

between the susy generators and the differential operators on $M$. The total BRST operator $Q$ corresponds to $d$ while $b^-_0 = G^-_0 - \bar{G}^-_0$ corresponds to $d^\dagger = \partial - \bar{\partial}^\dagger$. Let us also introduce the operator $L$ of multiplication by Kähler form $\omega_{ij}$ and the operator $\Lambda$ of contraction with bivector $\omega^{ij}$. The commutation relations between $L$, $d^\dagger$ and $\Lambda$, $d$ are two of the Hodge identities

\[
[d, \Lambda] = d^\dagger \quad \text{and} \quad [d^\dagger, L] = d
\] (2.18)

(see [12] and Appendix A).

Computing the OPE in the $\sigma$-model formalism we arrive to the following formula:

\[
\mathcal{O}^{(2)}_{\phi}(z)\mathcal{O}_{\psi}(x) \sim \cdots + \delta^2(z-x)\mathcal{O}_{m(\phi, \psi)}(x) , \] (2.19)

where $m(\phi, \psi)$ is a bilinear symmetric operation on differential forms defined by

\[
m(\phi, \psi) = \Lambda(\phi \wedge \psi) - (\Lambda \phi) \wedge \psi - \phi \wedge (\Lambda \psi) . \] (2.20)

It has a degree $\deg m(\cdot, \cdot) = -2$.

For each Kähler form $\omega$, the linear operator $\Lambda$ descends on cohomology $H^*(X)$. One should just identify $H^*(X)$ with the space of harmonic forms. It is easy to see that the result (denoted by the same symbol $\Lambda$) depends only on the cohomological class $[\omega] \in H^{1,1}$ of $\omega$. Remarkably, we have a problem trying to use the same trick to descend the bilinear
operation \( m(\cdot, \cdot) \) on cohomology. Indeed, one can check that even for both \( \phi \) and \( \psi \) harmonic, the result \( m(\phi, \psi) \) is not even \( d \)-closed. The reason is that the product \( \phi \wedge \psi \) is not harmonic in general, so \( d\Lambda(\phi \wedge \psi) = d^{ct}(\phi \wedge \psi) \neq 0 \). This is the semiclassical manifestation of the relation (2.10). As it was discussed in the previous section, we introduce the operation \( s(\cdot, \cdot) \) on harmonic forms, defined as

\[
s(\phi, \psi) = m(\phi, \psi) - \frac{1}{d} d^{ct} \phi \wedge \psi,
\]

where we have to use the Hodge decomposition of the product: \( \phi \wedge \psi = h_{\phi \wedge \psi} + d c_{\phi \wedge \psi} \); where harmonic \( h_{\phi \wedge \psi} \) satisfies \( d^{ct} h_{\phi \wedge \psi} = 0 \). The result of \( s(\cdot, \cdot) \) is a \( d \)-closed form. So defined, \( s(\cdot, \cdot) \) descends to cohomology (this will be proved in Section 4). A reader can check that (2.21) is just a semiclassical version of (2.14).\(^5\)

Let us discuss the semi-classical (without instantons) \( tt^* \) and Zamolodchikov metrics on the complexified Kähler cone \( K_\mathbb{C} \subset H^{1,1} \). It is convenient to introduce a complexified Kähler class \( \omega \) so that the positive definite real (true) Kähler class is \( \Omega = \omega + \bar{\omega} \). We expect that in the absence of instantons the metrics depend only on \( \Omega \). In the tangent space to \( K \) there are “chiral” vectors \( \xi_a \) deforming \( \omega \) and their “antichiral” counterparts \( \bar{\xi}_a \) deforming \( \bar{\omega} \). Then the (classical) Zamolodchikov metric is defined by the scalar product

\[
\langle \bar{\xi}_a | \xi_a \rangle = \frac{1}{\text{Vol}_\Omega} \int (\ast \bar{\xi}_a) \wedge \xi_a = \frac{1}{\text{Vol}_\Omega} \int \Lambda s(\bar{\xi}_a, \xi_a) \Omega^n = \Lambda s(\bar{\xi}_a, \xi_a),
\]

where both \( \Lambda \) and \( s(\cdot, \cdot) \) are computed with respect to the real class \( \Omega \). The \( tt^* \) metric is given by the right hand side of (2.22) without \( \text{Vol}_\Omega^{-1} \) prefactor. It was noticed by Candelas [13] (for general discussion see also [6]) that the metric (2.22) is Kählerian:

\[
\langle \bar{\xi}_a | \xi_a \rangle = \partial_a \bar{\partial}_a \log (\text{Vol}_\Omega)^2
\]

The corresponding metric connection is

\[
D_a = \partial_a - s(\xi_a, \cdot)
\]

\[
D_{\bar{a}} = \bar{\partial}_{\bar{a}} - s(\bar{\xi}_{\bar{a}}, \cdot)
\]

Similar construction appears in topological B-models as well as in topological LG theories. The case of topological B-model will be discussed in Section 3.7. For \( \phi, \psi \in H_{(0,1)}(T_M) \) the connection is given \( (\delta \psi)' = -\psi \perp \phi' - \frac{1}{2} \partial(\phi \wedge \psi)' \), where the contraction with holomorphic 3 form is denoted by prime and operation \( \perp \) is the contraction of two holomorphic indices. In the case of topological LG theories the connection is given by \( \partial(\phi \psi/W')_+ \). Operation \( (\cdot/W')_+ \) is identified with \( 1/Q \), while \( b_0^- \) – with \( \partial \).
It satisfies \([D_a, D_b] = 0\) and \([\tilde{D}_a, \tilde{D}_b] = 0\). In general, it is not flat: \([D_a, D_b] \neq 0\).

The simple example below may be helpful. When the cohomological Kähler cone \(K\) is one-dimensional (generated by \(x \in H^{1,1}(M)\)), the complexified cone \(K_C\) is an upper half-plane of a complex parameter \(z\). The Kähler form is given by \(\Omega = 2(\text{Im } z) \, x\) and the B field by \(B = 2(\text{Re } z) \, x\). It is easy to compute the semiclassical Zamolodchikov metric on \(K_C\). Using (2.22) one finds that it is the Poincaré metric
\[
\begin{align*}
g_{zz}dzd\bar{z} &= 2n \frac{dzd\bar{z}}{(z - \bar{z})^2} = \partial \bar{\partial} \log (z - \bar{z})^{2n}, \tag{2.25}
\end{align*}
\]
where \(n = \dim_{\mathbb{C}} M\). The answer is essentially independent of any detail of geometry of the manifold \(M\). The Zamolodchikov connection is given by \(\partial + 2(z - \bar{z})^{-1}\) and \(\bar{\partial} + 2(\bar{z} - z)^{-1}\).

It has constant negative curvature.

Now let us return to the general situation. Suppose that \(\phi\) is a harmonic 2-form. The perturbation \(\epsilon^a \int d^2z \, \phi^{(2)}_a (\epsilon^a \int d^2z \, [Q_-, [Q_+, \phi^\dagger]]\) ) corresponds to deformation of the Kähler form \(\omega \to \tilde{\omega} = \omega + \epsilon^a \phi_a\) \((\tilde{\omega} \to \tilde{\omega} = \tilde{\omega} + \epsilon^a \phi^\dagger_a\)). Then it follows from the formulas derived in Appendix A that the operators \(d\) and \(d^\dagger\) satisfy
\[
\begin{align*}
U^{-1}_\phi (d - \epsilon[d^\dagger, \phi])U_\phi &= d \tag{2.26a} \\
U^{-1}_\phi (d^\dagger - \bar{\epsilon}[d, \delta \Lambda])U_\phi &= (d^\dagger - (\epsilon + \bar{\epsilon})[d, \delta \Lambda]), \tag{2.26b}
\end{align*}
\]
where \(U_\phi = 1 + \epsilon m(\phi, \cdot) + o(\epsilon^2)\). One can immediately recognize in these formulas a semi-
classical limit of (2.11a) - (2.11b). Indeed, the BRST operator \(\tilde{Q}\) for the new background \(\tilde{\Omega}\) is still \(d\). The operator \(\tilde{G}_0^-\) on the r. h. s. of (2.26) coincides with \(d^\dagger\) computed for \(\tilde{\Omega}\) as a consequence of the Hodge identities.

2.3. \(N = 2\) CFT vs. \(N = 2\) Supersymmetric Quantum Mechanics.

The two-dimensional sigma model has a little brother — the \(N = 2\) supersymmetric quantum mechanics (1-d worldline sigma-model). As a theory of topological matter, SQM is an approximation to the full-fledged 2-d sigma model. The difference between them is that SQM discards worldsheet instantons.

There are chiral and antichiral fields in the theory. It is convenient to identify the former with differential forms on the target space and the latter with polyvectors:
\[
\begin{align*}
\phi &= \phi_{i_1 \ldots i_n \bar{j}_1 \ldots \bar{j}_m} \chi^{i_1} \ldots \chi^{i_n} \chi^{j_1} \ldots \chi^{j_m}, \\
\phi^\dagger &= \phi^{i_1 \ldots i_n \bar{j}_1 \ldots \bar{j}_m} \rho_{i_1} \ldots \rho_{i_n} \bar{\rho}_{j_1} \ldots \bar{\rho}_{j_m}. \tag{2.27}
\end{align*}
\]
The SQM susy generators are given by (2.17). The \textit{chiral primaries} are harmonic forms, the \textit{antichiral primaries} can be obtained by raising the indices of the latter. Of course, up to now this was just a repetition of the previous section. The differences with the string theory begin when we look at the space of deformations. There is no antisymmetric tensor field $B_{ij}$ in SQM, so this theory is naturally defined on the real Kähler cone $K$. Our major assumption is that it possible to identify the deformations with real chiral fields. To explain this point, let us see how the theory depends on the (real) Kähler form $\omega \in K$.

As one moves along the Kähler cone, the space of harmonic forms (= SQM ground states ) changes. It is explained in section 3.5 and Appendix A that the natural parallel transport on harmonic forms is given by a flat connection $D^R_e = \partial_e - s(e, \cdot)$, where $e$ is any tangent vector to the real Kähler cone $K$, considered as a harmonic form. It is also explained in Appendix A, that $D^R$ is not a metric connection with respect to the natural (Hodge) metric on $K$. Let us give a “physical” reason for that. We identified the tangent vectors to $K$ with chiral fields. In $N = 2$ there is a nondegenerate pairing between chiral and antichiral fields:

$$\langle \phi | \psi \rangle = \frac{1}{\text{Vol}_{\omega}} \int \phi^{ij} \psi_{ij} \omega^n, \quad (2.28)$$

(here $\omega$ is a real Kähler form). Thus \textit{a priori} it defines a pairing between forms ans polyvectors but not a metric on $K$. The connection $D^R$ preserves the pairing (2.28) in the following sense. Let $\{e_a\}$ be the covariantly constant with respect to $D^R$ coordinate frame ( for each $\omega \in K$, $e_a(\omega)$ is a harmonic form ). Denote by $\{\bar{e}_a\}$ the bivectors obtained from $\{e_a\}$ by raising the indices. One can check that so defined, $\{\bar{e}_a\}$ do not depend on $\omega$:

$$\partial_a \langle \bar{e}_b|e_c \rangle - \langle \bar{e}_b|s(e_a, e_c) \rangle = \partial_a \langle \bar{e}_b|e_c \rangle + \Gamma^d_{ac} \langle \bar{e}_b|e_d \rangle = 0. \quad (2.29)$$

In fact there are two natural connections on the (real) Kähler cone. One is equal to $D^R_e = \partial_e - s(e, \cdot)$, while the second – is the \textit{metric} connection and it is given by $D_e = \partial_e - \frac{1}{2}s(e, \cdot)$. The first one appears to be flat, while the second one does not. A simple example from the previous section is rather helpful. One dimensional (real) Kähler cone is equipped with a natural metric $(ds)^2 = \frac{1}{2}\left(\frac{dy}{y}\right)^2$. The connection defined by $s(\cdot, \cdot)$ is given by $D_e = \partial_e - 2/y$, while the metric connection is $D_e = \partial_e - 1/y$.

It is important to mention that $D_e = \partial_e - s(e, \cdot)$ is a metric connection on the complexified Kähler cone. Comparing this with the previous section we conclude that it is indeed

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$^6$ Checking this is a simple yet good exercise.
possible to consistently identify the real \( \omega \) in SQM with the holomorphic \( \omega \) in \( N = 2 \) TCFT and the (real) deformations of SQM with the holomorphic deformations of \( N = 2 \) TCFT by means of analytic continuation.

3. Theory of deformations of Kähler structures

3.1. Mirror for KS theory (AKS)

The string field theory of topological B-model is related to Kodaira-Spencer theory \[1\], which describes deformations of complex structure. It is natural to ask what is the mirror of this theory \[3\]. The mirror of the string field theory clearly should be defined on the loop space \( \mathcal{L}M \). We will show below that the semiclassical approximation (SQM) to this theory is related to the theory of deformations of Kähler structures of \( M \). In the table below we summarize the relations between deformations of Kähler and complex structures that follow from the comparison of the corresponding topological theories.

| B model (KS)       | A model (AKS)       |
|--------------------|--------------------|
| BRST               | \( \bar{\partial} \) | \( d \) |
| field              | \( A \in \Omega^1(T^*_M) \) | \( K_0 \in \Omega^{(1,1)} \) |
| constraint         | \( \partial A' = 0 \) | \( d^{c\dagger}K_0 = 0 \) |

A Kodaira-Spencer equation is given by

\[
\bar{\partial} A' + \frac{1}{2} \partial (A \wedge A)' = 0 . \tag{3.1}
\]

Using the analogy with B model one can write its mirror image as

\[
dK_0 + \frac{1}{2} d^{c\dagger}(K_0 \wedge K_0) = 0 , \tag{3.2}
\]

where \( K_0 \) satisfies a constraint \( d^{c\dagger}K_0 = 0 \). Kodaira-Spencer equation (3.1) implies that the deformed BRST operator \( \bar{\partial} + A \cdot \partial \) is nilpotent. Similarly to B-model, equation (3.2) is equivalent to the condition that an operator

\[
D = d + [d^{c\dagger}, K_0]
\]

squares to zero.
Now let us suppose that the manifold $M$ is 3-dimensional. We will explain the necessary modifications in the general case later when discussing the BV formalism. Then the equation (3.2) can be derived as an equation of motion for the action

$$S[x, K] = \frac{1}{2g^2} \int K \frac{1}{d^{c\dagger}} dK + \frac{1}{6g^2} \int (x + K) \wedge (x + K) \wedge (x + K).$$

(3.3)

In (3.3), we separate the contributions of massless and massive modes. We call $x \in \text{Ker}d \cap \text{Ker}d^{c\dagger}$ massless and $K = d^{c\dagger}Z$ massive. There is an ambiguity in this definition. On the one hand, one can show that massless modes can be parameterized as $x = h_x + dd^{c\dagger}\beta$, where $h_x$ is harmonic (see Appendix A). On the other hand, the massive mode $K$ in (3.3) is not defined canonically either, since a shift of $K$ by $dd^{c\dagger}\beta$ does not affect the kinetic term. Finally, a formula $S[x, K] = S[h_x, K + dd^{c\dagger}\beta]$ shows that one may always fix $x$ to be a harmonic 2-form, once a complex structure is chosen. Below we adhere to this interpretation of the massless mode.

The massive mode $K$ is the dynamical variable in the theory, while the massless mode plays the rôle of background. The action (3.3) possesses gauge invariance discussed below. After imposing a gauge fixing condition $d^{c\dagger}K = 0$ a propagator for the massive modes can be written as

$$\Pi = d^{c\dagger} \frac{1}{\Delta} d^{c\dagger}.$$  

(3.4)

Having defined the propagator $\Pi$ we can rewrite the formula (2.21) as follows

$$s(\phi, \psi) = P_{\text{Ker}(d)} m(\phi, \psi) = m(\phi, \psi) - \Pi(\phi \wedge \psi).$$

(3.5)

In the target space field theory such as AKS the massless modes do not propagate. On the contrary, in the string theory the massless modes are physical and do propagate. The propagator of the massless modes $x$ is related to the connection (3.5) and is given as $D(\phi, \psi) = s(\phi, \psi) + (\Lambda \phi) \psi + \phi(\Lambda \psi)$. It is clear that $D(\phi, \psi)$ depends only on the product $\phi \wedge \psi$, but not on $\psi$ and $\phi$ separately.

After imposing the gauge fixing condition $d^{c\dagger}K_0 = 0$ one can solve (3.2) in perturbation series. To write the solution one needs to fix the harmonic part $x$ of $K_0$ (in the sense of the Hodge decomposition $K_0 = x + dN$). The perturbation series for (3.2) formally coincides with the perturbation series of $\phi^3$ theory coupled to the background:

$$K_0[x] = x - \frac{1}{2} \Pi(x^2) - \frac{1}{2} \Pi(x \Pi(x^2)) + \cdots.$$  

(3.6)

---

7 this decomposition requires a choice of complex structure
We see that (the gauge-fixed) solution of the equation of motion is completely determined by the massless mode $x$. In other words, the (physical) covariant phase space consists of harmonic (1, 1)-forms.

The solution $K_0[x]$ defines a deformed Kähler structure in the same way as the solution of Kodaira-Spencer equation defines a new complex structure. In fact (see section 4), for any $x$ there exist a new Kähler structure $\tilde{\omega}(x)$ and an operator $U$, such that

$$D = d + [d^\dagger, K_0[x]] = UdU^{-1}$$
$$d^\dagger = U d^\dagger U^{-1}.$$  \hspace{1cm} (3.7)

This is an analog, for deformations of Kähler structures, of Tian-Todorov\cite{14} construction.

The relations (3.7) express the global background independence of AKS the same way as (2.26) express the local background independence of string theory. There is an obvious difference between these two theories. The former (AKS) is defined on the tangent bundle $\mathcal{T}K$ to the real Kähler cone $K$ ($\omega$ gives a point on the base and $x$ a vector in the tangent space). The latter (perturbed string theory, see section 2) is defined on the tangent bundle $\mathcal{T}K_C$ to the complexified Kähler cone $K_C$. In fact, it is natural to interpret AKS as a target space field theory for $N=2$ susy quantum mechanics$^8$. The latter is a semiclassical approximation to $N=2$ CFT, in a sense specified in 3.1.

This view on AKS as a large volume string field theory is supported by a number of properties it enjoys. This theory is

- gauge invariant (for 3-fold the gauge group is the group of volume preserving diffeomorphisms);
- independent of the complex structure;
- depends only on Kähler class of the metric;
- background independent.

### 3.2. Independence of complex structure

We expect that AKS, as a target space field theory for the A type $\sigma$-model, is independent of the complex structure, for the fixed Kähler form $\omega$. The metrics $g_{ab}$, the complex structure $J^a_b$ and $\omega$ are related by

$$g_{ab} = J^a_c \omega_{cb}$$  \hspace{1cm} (3.8)

\footnote{We are grateful to C. Vafa for this suggestion.}
In particular it means that as we change the complex structure with \( \omega \) fixed, we change the metrics and consequently the operators \( d^\dagger \) and \( \Delta \). Now the point is that the particular combination of \( g_{ab} \) and \( J^a \) in \( d^\dagger \) results in dependence of \( d^\dagger \) on just \( \omega \), as can be seen from the formula
\[
d^\dagger = [d, \Lambda]
\] (3.9)
where \( \Lambda \) is a bivector such that \( \Lambda^{ab} \omega_{bc} = \delta^a_c \). Therefore both the kinetic term of (3.3) and the constraint \( K = d^\dagger Z \) are independent of \( J^a \).

The only subtlety is that the Hodge decomposition which we use to choose the harmonic representative for \( x \), depends on the complex structure. But as the complex structure changes, \( x \) changes by a \( d \)-exact form, which does not affect the kinetic term and can be reabsorbed into the massive \( K \).

3.3. Gauge invariance

Action (3.3) is invariant under the gauge transformation
\[
\delta_\alpha K = d\alpha - d^\dagger((x + K) \wedge \alpha),
\] (3.10)
where \( \alpha \) is an infinitesimal form such that \( d^\dagger \alpha = 0 \). Note that only the massive mode \( K \) gets transformed leaving the background field \( x \) unchanged.

Indeed, keeping only the linear terms the variation can be written as
\[
g^2 \delta S = \int -K \frac{1}{d^\dagger} dd^\dagger((x + K) \wedge \alpha) + \frac{1}{2} [d\alpha - d^\dagger((x + K) \wedge \alpha)] \wedge (x + K)^2
\]
\[
= \int -\frac{1}{2} d[(K + x)^2] \wedge \alpha + \frac{1}{2} [d\alpha] \wedge (x + K)^2 + \frac{1}{6} \alpha d^\dagger[(x + K)^3]
\]
\[
= \int -\frac{1}{6} d^\dagger[\alpha](x + K)^3 = 0,
\]
where we used that \( d^\dagger \) is antihermitean and a formula \( Ad^\dagger(A^2) = \frac{1}{3}(A^3) \) which holds if \( d^\dagger A = 0 \). The commutator of two gauge transformations is given as follows
\[
[\delta_\alpha, \delta_\beta] = \delta_\gamma, \quad \text{where} \quad \gamma = d^\dagger(\alpha \wedge \beta)
\] (3.12)

Define a correspondence \( \alpha \leftrightarrow \hat{\alpha} \) between the the gauge parameters and the volume preserving vector fields by the formula \( \hat{\alpha}^I = \Lambda^I \alpha_J \). Then (3.12) is equivalent to the commutation
\[
\delta K = d^\dagger w
\]

 Using the results of Appendix A one can easily show that the variation (3.10) is massive: 
\[
\delta K = d^\dagger w
\]
relation $[\hat{\alpha}, \hat{\beta}] = \hat{\gamma}$ of the corresponding vector fields. In the case of 3-fold this implies that the gauge group is isomorphic to the group $SDiff M$ of volume preserving diffeomorphisms.

Let us define field strength as $F = dK + \frac{1}{2} d^{c\dagger}(x + K)^2$. The condition of flat connection ($F = 0$) implies that $D = d + [d^{c\dagger}, (x + K)]$ is nilpotent. The field strength $F$ is not invariant under gauge transformation as it supposed to be in non-abelian gauge theories and transforms as follows $\delta F = -d^{c\dagger}(F \wedge \alpha)$.

3.4. BV quantization of AKS.

Our aim is to establish AKS as a target space field theory for $N = 2$ SQM. But the theory described so far is a one describing only 2-forms, while the states (2.27) of SQM are differential forms of all possible degrees. Thus the “superparticle field” should rather be a linear combination $\mathcal{K} = \sum_{q=0}^{n} K_q$, each component describing the sector with a particular ghost number. Also, as mentioned above, the action (3.3) works only for 3-dimensional $M$. In fact, these two problems turn out to be each other’s cure. Adding extra fields corresponding to all degrees of freedom of SQM also makes the theory well defined for any dimensional $M$.

But we don’t even have to appeal to any a fortiori connection to SQM or to the case dim $M \neq 3$. A consistent treatment of the theory with action (3.3) within the Batalin–Vilkovisky (BV) formalism [17] (for review see also [18] and in string theory [19]) requires one to relax the condition that $K$ is a 2-form and includes all possible fields with arbitrary ghost numbers. The components $K_q \in \Omega^q(M)$ with ghost numbers $q(K) \leq n - 1$ are called fields, while the components $K_q^* \in \Omega^q(M)$ with ghost numbers $q(K) > n - 1$ are called antifields. Both fields and antifields satisfy the constraint $d^{c\dagger}K_q = 0$ and can be decomposed into the sum of massive and massless modes. The massless modes $x_q$ are the harmonic forms on $M$. They are not dynamical and just create the background. The massive modes (from now on denoted by $K_q$) are dynamical. They satisfy $K_q = d^{c\dagger}Z_{q+1}$. Note that the last condition implies there is no dynamical (anti)field $K_{2n}$ of the highest rank. Thus there is the same number (n-1) of fields and antifields.

\footnote{In this section we use both terms “the ghost number” and “degree of form” with the same meaning.}
The space of fields–antifields is equipped with an odd antibracket\footnote{In fact, this antibracket is induced by a natural (even) Poisson bracket on differential forms \( \{K_p(z), K^*_q(w)\}_{PB} = \delta_{p+q,2n} \omega^n \delta(z,w) \). The latter will be used to define a measure in the path integral formulation of AKS (see Section 5).} \( \{,\} \) given by

\[
\{K_p(z), K^*_q(w)\} = \delta_{p+q,2n-1} \omega^n d^{\mathbb{C}} \delta(z,w),
\]

where \( \delta(z,w) \) is the delta function on the Kähler manifold, such that for any function \( \varphi(x) \)

\[
\int_M \varphi(y) \delta(x,y) \omega^n(x) = \varphi(x).
\]

It pairs \( \Omega^p(M) \) with \( \Omega^{2n-p-1}(M) \). This structure is promoted to a canonical antibracket on the space of functionals:

\[
\{F, L\} = \int \sum_{p+q=2n-1} d^{\mathbb{C}} \left( \frac{\delta F}{\delta K_p} \right) \frac{\delta L}{\delta K_q} - \frac{\delta F}{\delta K_q} \frac{\delta L}{\delta K_p}.
\]

(3.13)

In general BV theory, the BRST symmetry is a canonical transformation in the antibracket:

\[
\delta_{BRST} K = \{K, S\},
\]

(3.14)

where the original action ((3.3) in our case) is replaced by a full action \( S \) which depends on both fields and antifields. The full action satisfies two conditions. It reduces to the original action when all antifields are set to zero. It also satisfies a Batalin-Vilkovisky master equation

\[
\{S, S\} = \hbar \Delta S,
\]

(3.15)

where \( \Delta \) is the natural Laplacian on the space of fields–antifields to be defined below. The r.h.s. of (3.15) is a contribution coming from the path integral measure. At the classical level (\( \hbar = 0 \)), the Batalin-Vilkovisky equation is nothing else but the condition that the full action is gauge invariant. The gauged fixed action is determined by an odd functional \( \Psi(K) \) and is given by \( S_{\Psi}(K) = S(K, K^* = \delta \Psi/\delta K) \).

It is quite remarkable that the full AKS action is given by the same expression (3.3) as the original AKS action, but without any restrictions on the ghost numbers. One should simply substitute in (3.3) the linear combination \( K \) for \( y \):

\[
S[x, K] = \frac{1}{2g^2} \int K \frac{1}{d^{\mathbb{C}}} dK + \frac{1}{6g^2} \int (x + K) \wedge (x + K) \wedge (x + K),
\]

(3.16)
or written in components

$$\frac{1}{2g^2} \int \sum_{p+q=2n-2} K_p \frac{1}{d^{c\dagger}} dK_q + \frac{1}{6g^2} \int \sum_{p+q+r=2n} (x_p + K_p) \wedge (x_q + K_q) \wedge (x_r + K_r).$$

To see why this is true we first notice that if $M$ is 3-dimensional, every term in (3.16) either consists of 2-forms or contains at least one antifield (form of rank $> 2$). When all antifields are set to zero, the only contribution to the action comes from the original field $K_2 \in \Omega^2(M)$. Now let us consider the BRST symmetry (3.14) generated by (3.16) together with (3.13). One easily finds

$$\delta_{BRST} K = dK + \frac{1}{2} d^{c\dagger} ((K + x) \wedge (K + x)). \tag{3.17}$$

For 3-dimensional $M$ this formula with antifields set to zero brings us back to (3.10), where the parameter $\alpha$ is to be identified with $K_1 + x_1$.

Now we can check the gauge invariance of the full action (3.14). The computation itself mostly repeats the one (3.11) done in the previous section. It gives $\delta_{BRST} S = \{S, S\} = 0$.

Note that the right hand side of the BRST transformation (3.17) coincides with equations of motion for the action (3.16). BRST triviality of the dynamical equations may not be surprising after all, if we notice that the action (3.16) can also be written in a form

$$S[x, K] = \frac{1}{2g^2} \int K \wedge \omega \wedge K + \frac{1}{6g^2} \int (x + K) \wedge (x + K) \wedge (x + K), \tag{3.18}$$

particularly useful in applications (we used the Hodge identity (2.18) and the constraint $K = d^{c\dagger} Z$). After gauge fixing the solutions of equations of motion can be expressed in terms of the massless modes $x_q$ by series similar to (3.6). Therefore the BV covariant phase space, alias the space of solutions coincides with the space of harmonic forms modulo the gauge group. In particular, it is finite-dimensional.

The BV Laplacian is defined by:

$$\Delta = \int \sum_{p+q=2n-1} \frac{\delta}{\delta K_q} d^{c\dagger} \frac{\delta}{\delta K_p}.$$

To verify that this definition is indeed covariant one has to take into account that $\delta K_p(x)/\delta K_r(y) = \delta_{p,r}(x, y) \omega^n(x)$. Now we can check the BV master equation (3.13). The gauge invariance of the full action implies that l.h.s of (3.13) is equal to zero. The r.h.s can be computed easily. It equals

$$\Delta S = \frac{1}{g^2} \int d^{c\dagger} (K_1 + x_1) \wedge \omega^n(x) = 0$$

19
due to constraint.

The above discussion implies that quantum corrections are not needed for maintaining the gauge invariance of the AKS theory. There is no 4- or higher interaction vertices in the full action (3.10). The same situation was encountered in [20] and [3] describing Chern-Simons theory. These facts have similar geometric reasons. The same reasons guarantee existence of series (3.6) for AKS (and similar series for CS theory) describing the solution of the equation of motion in terms of massless component $x$. The higher vertices are related [3] to the higher Massey products in cohomology. The nontrivial Massey products are obstructions to writing formulas like (3.6) since $d$ cannot be inverted. In the cohomological theory which appears in CS ("cohomology with coefficients in $End(E)$") the Massey products are absent. A very important fact about topology of Kähler manifolds is that there are no Massey higher products either — it is a consequence of $dd^c$-lemma (see [21] for a proof and a nice exposition on important consequences of this fact).

3.5. Kähler “topological” invariance.

The action (3.3) is invariant under the variation of the Kähler form $\omega \to \omega + d\alpha$. To prove this, it is convenient to use another form of the action (3.3)

$$S(K) = \frac{1}{2g^2} \int K \omega K + \frac{1}{6g^2} \int (K + x)^3,$$

(3.19)
equivalent to (3.3) on the constraint $K = d^{c\dagger}Z$. In (3.19), the variation of the kinetic term is due to variation both of the Kähler form and the field $K$. The field $K$ changes because the constraint $K = d^{c\dagger}Z$ explicitly depends on the metrics:

$$\delta d^{c\dagger} = [d^{c\dagger}, [\Lambda, d\alpha]] \quad (3.20a)$$

$$(K + \delta K) = (d^{c\dagger} + \delta d^{c\dagger})(Z + \delta Z) \quad (3.20b)$$

$$\delta K = -[\Lambda, d\alpha]K + d^{c\dagger}\chi \quad (3.20c)$$

where we choose $\chi = \alpha K$. In this case one can rewrite the variation as $\delta K = L_\xi K$, where $L_\xi K$ is the Lie derivative along the vector field $\xi$ dual to 1-form $\alpha$. This is a clever choice since the variation of the harmonic part $x$ is also given by Lie derivative $\delta x = L_\xi x$. Therefore, if we use $\xi = \alpha K$, we have a natural relation

$$\delta(K + x) = L_\xi(K + x) \quad (3.21)$$
The variation of the kinetic term is
\[
\delta_{\text{Kin}} S = \frac{1}{2g^2} \int K d\alpha K + 2(\mathcal{L}_\xi K)\omega K
\]
\[
= \frac{1}{2g^2} \int K d\alpha K + \mathcal{L}_\xi(K\omega K) - K(\mathcal{L}_\xi \omega)K = 0
\]
(3.22)
since \( \mathcal{L}_\xi \omega = d(i(\xi)\omega) = d\alpha \) and
\[
\int \mathcal{L}_\xi(K\omega K) = \int (d(i(\xi) + i(\xi)d)(K \wedge \omega \wedge K) = 0
\]
(3.23)
as \( K \wedge \omega \wedge K \) is a top form.

The variation of the potential term is
\[
\delta_{\text{Pot}} S = \frac{1}{2g^2} \int (\mathcal{L}_\xi(K + x))(K + x)^2 = \frac{1}{6g^2} \int \mathcal{L}_\xi(K + x)^3 = 0
\]
(3.24)
where we used (3.21) and the same argument as in (3.23).

### 3.6. Dependence on the Kähler class.

To define AKS theory one needs to fix some data — Kähler structure \( \omega \) and massless background \( x \). It turns out that this data is redundant. AKS action possess additional symmetry which acts on the background data \( \omega \to \tilde{\omega} \) and \( x \to \tilde{x}(\tilde{\omega}) \) as well as on \( K \to \tilde{K}(\tilde{\omega}) \) such that the action (3.3) is almost invariant, namely
\[
\frac{1}{\text{Vol}_\omega^2} (S[x,K; \omega] - S_0[x; \omega]) = \frac{1}{\text{Vol}_{\tilde{\omega}}^2} (S[\tilde{x},\tilde{K}; \tilde{\omega}] - S_0[\tilde{x}; \tilde{\omega}])
\]
(3.25)
where \( S_0[x,\omega] \) is the classical action evaluated on the solution \( K_0[x] \) and functions \( x(\tilde{\omega}), K(\tilde{\omega}) \) satisfy differential equations discussed below. The combination that appeared in (3.25) may be viewed as a background independent action. The second term does not depend on the dynamical variable \( K \) and therefore does not affect the equations of motion.

The appearance of the volume factor in front of the action is quite remarkable and can be viewed as volume dependence of the string coupling constant.

To prove (3.25) let us consider an infinitesimal variation \( \omega \to \omega + \delta \omega \) by harmonic form \( \delta \omega \) accompanied by the following transformation of fields
\[
x \to \tilde{x} = x + \delta \omega - s(\delta \omega, x)
\]
(3.26a)
\[
K \to \tilde{K} = K - m(\delta \omega, K + x) + s(\delta \omega, x).
\]
(3.26b)
One can check that the deformed fields satisfy the constraint $\tilde{d}^\dagger \tilde{x} = 0$ and $d\tilde{x} = 0$ as well as $\tilde{K} = d^\dagger Z$. Therefore the transformation (3.26) is consistent with the decomposition on massless and massive modes. Moreover, (3.26) preserves the gauge fixing: $d^\dagger x = d^\dagger \tilde{x} = 0$.

As the Kähler form changes by an infinitesimal harmonic form $\delta \omega$, the action (3.3) changes so that

$$g^2 S[\tilde{K}, \tilde{x}; \omega + \delta \omega] - g^2 S[K, x; \omega] =$$

$$= \int \frac{1}{2} K \delta \omega K - m(\delta \omega, K) \omega K - [-s(x, \delta \omega) + m(\delta \omega, x)] \omega K +$$

$$+ \int \frac{1}{2} \delta \omega (K + x)^2 - \frac{1}{6} m(\delta \omega, (K + x)^3) =$$

$$= - \int m(\delta \omega, \frac{1}{2} K \omega K + \frac{1}{6} (K + x)^3) + \frac{1}{2} \int \delta \omega x^2$$

$$+ \int [\delta \omega K x - (-s(x, \delta \omega) + m(\delta \omega, x)) \omega K]$$

$$= 2(\Lambda \delta \omega) g^2 S[K, x; \omega] + \frac{1}{2} \int \delta \omega x^2 \tag{3.27}$$

This derivation deserves a few comments. First, we used the fact that $m(\cdot, \cdot)$ differentiates multiplication of forms (see Appendix A). Second, $(\Lambda \delta \omega)$ is a number since $d^\dagger \delta \omega = 0$ and therefore

$$\int m(\delta \omega, \mathcal{L}) = -2(\Lambda \delta \omega) \int \mathcal{L}.$$ Third, to obtain the last identity we noticed that since $K = d^\dagger Z$ and $m(\delta \omega, x) - s(x, \delta \omega) = \Pi(\delta \omega x)$, where $\Pi$ is the massive propagator (see (3.3)) one can write

$$\int K \omega (-s(x, \delta \omega) + m(\delta \omega, x)) = \int Z[d^\dagger, \omega] \Pi(\delta \omega x) = \int Z d \Pi(\delta \omega x) = \int K \delta \omega x.$$ One can see that the variation of the action consists of two terms; the first one is just the rescaled action, while the second is $K$ independent. Therefore the difference $S[K, x; \omega] - S[Q, x; \omega]$ scales by a factor $(1 + 2(\Lambda \delta \omega))$ under the transformation (3.26). It is important that the solution $K_0[x]$ of the equation of motion (3.2) is mapped on the solution of the equation of motion for the perturbed Kähler structure $\tilde{\omega}$ which can be written as $K_0[\tilde{x}]$. Taking $Q = K_0[x]$ we conclude that $S[K, x; \omega] - S_0[x; \omega]$ scales under the variation (3.26). The scale factor $(1 + 2(\Lambda \delta \omega))$ is related to the variation of the of volume as follows

$$\delta \text{Vol}_\omega = 3 \int \omega^2 \delta \omega = -\frac{1}{2} \int m(\delta \omega, \omega^3) = (\Lambda \delta \omega) \text{Vol}_\omega.$$
This implies the infinitesimal form of the relation (3.25).

We will call a combination that appears in (3.25) the background independent action

\[ \mathcal{A}[K', x; \omega] = \frac{1}{\text{Vol}^2_\omega} (S[x, K; \omega] - S_0[x; \omega]) = \]

\[ = \frac{1}{g^2 \text{Vol}^2_\omega} \int \left( \frac{1}{2} K' \frac{1}{d c^\dagger} D K' + \frac{1}{6} K' \wedge K' \wedge K' \right), \]

where we introduce a new dynamical variable \( K' = x + K - K_0 \) shifting \( K \) by the classical solution \( K_0 \). The differential operator \( D \) is given by

\[ D = d + [d c^\dagger, K_0] . \] (3.29)

A transformation (3.26) is a symmetry of background independent action \( \mathcal{A}[K', x; \omega] \). Written as

\[ D_a x + \partial_a \omega = 0 , \quad \text{where} \quad D_a = \partial_a + s(\partial_a \omega, \cdot) \] (3.30)

(3.26a) defines the parallel transport on the space of massless modes. The properties of this system will be discussed in the next section. Similarly one can define the parallel transport \( K'(\bar{\omega}) \) of \( K' \). Consider the solution \( x(\bar{\omega}) \) of (3.30), satisfying the initial condition \( x(\omega) = x \). The action (3.28) evaluated on \( x(\bar{\omega}) \) and \( K'(\bar{\omega}) \) is independent of \( \bar{\omega} \).

Under an infinitesimal change of Kähler structure \( \omega \to \omega + \delta \omega \) the solution (3.6) of the equation of motion transforms as follows: \( K_0 \to \tilde{K}_0 = K_0 + \delta \omega - m(\delta \omega, K_0) \). It is easy to convince oneself that \( \tilde{K}_0 \) satisfies the equation of motion with respect to the new background:

\[ d \tilde{K}_0 + \frac{1}{2} \tilde{d} c^\dagger (\tilde{K}_0 \wedge \tilde{K}_0) = 0 \quad \text{and} \quad \tilde{d} c^\dagger \tilde{K}_0 = 0 . \]

At the same time the operators \( D \) and \( d c^\dagger \) transform by conjugation by the operator \( U_{\delta \omega} = 1 + m(\delta \omega, \cdot) + o(\delta \omega^2) \) as follows:

\[ D \to U_{\delta \omega}^{-1} D U_{\delta \omega} = \tilde{D} \]

\[ d c^\dagger \to U_{\delta \omega}^{-1} d c^\dagger U_{\delta \omega} = d c^\dagger + [d, \delta \Lambda] = \tilde{d} c^\dagger . \] (3.31)

As a result the perturbed \( \tilde{D} \) is given by \( \tilde{D} = d + [\tilde{d} c^\dagger, \tilde{K}_0] \). We see that the symmetry (3.26) preserves the relation between \( D \) and \( K_0 \).

\[ ^{12} \text{It is worth mentioning that it is } S[K, x; \omega] / \text{Vol}^2_\omega, \text{not } \mathcal{A}[K, x; \omega] \text{ which is related to the generating function of string amplitudes.} \]
In Section 4 we will show that for each \( x \) one can find \( \omega_0 \) such that \( \tilde{x}(\omega_0) = 0 \implies K_0 = 0 \) and therefore \( D = d \) in that background. In this case the “statement of background independence” (3.25) can be written in a form familiar from [1].

The volume dependence in (3.28) deserves a separate discussion. If one introduces the volume-dependent “running string coupling constant” \( g_\omega \) which governs the magnitude of the cubic interaction, from (3.28) it follows that

\[
\boxed{g_\omega^2 = g_0^2 V^2} \tag{3.32}
\]

The reason for growth of \( g_\omega \) with volume \( V \) is quite clear. For small enough \( V \), SQM is strongly interacting. On the other hand, the \( V \rightarrow \infty \) limit for the fixed \( g \) corresponds to free theory. Background independence means that the theory is the same for all values of \( V \), therefore we should keep increasing \( g \) as \( V \rightarrow \infty \) in order to preserve the nontriviality of cubic interaction.

What is really interesting in (3.32) is the parabolic rather then linear growth of \( g(V)^2 \). It suggests that the field \( K' \) in (3.28) scales as \( K' \rightarrow \lambda^2 K' \) as the Kähler form \( \omega \) goes to \( \lambda \omega \). To understand this better, let us notice that the scaling of \( K' \) in (3.28) should coincide with that of the operator \( \frac{D}{d^c} \) in the kinetic term. It is obvious that \( d^c \rightarrow \lambda^{-1} d^c \); the point is that the BRST operator \( D \) also changes: \( D \rightarrow \lambda D \). The overall factor \( \lambda^2 \) is in agreement with what we expect from (3.32).

3.7. KS theory and dependence on the complex structure

This section probably would be more appropriate for [1]. Here we would like to discuss for KS theory a relation similar to (3.25), which is a local form of background independence. We remind the reader that the basic field \( Y \) in KS theory is a \((0, 1)\) form with coefficients in vector fields. The dynamical field in KS theory is not \( Y \) but its massive component \( A \), while the massless component \( x \) (cohomology element) plays the role of the background. For a fixed complex structure \( J \) and a cohomology element \( x \in H^{(0, 1)}(T_M) \) the KS action reads as follows

\[
S_{KS}[A, x; J] = \frac{1}{2} \int A' \frac{1}{\partial} \bar{\partial} A' + \frac{1}{6} \int (A + x)'((A + x) \wedge (A + x))' . \tag{3.33}
\]

Prime defines an isomorphism

\[
\prime : \Omega^{(0, p)}(\wedge^q T_M) \rightarrow \Omega^{(q, p)} ,
\]

24
given by the contraction with holomorphic 3-form $\Omega$. Both $x$ and $A$ satisfy the constraint $\partial A' = \partial x' = 0$. The variation of complex structure is given as

$$\bar{\partial} \rightarrow \bar{\partial} + \phi \cdot \partial ,$$

(3.34)

where $\phi \in H^{(0,1)}(T_M)$. Under this variation the holomorphic 3-form varies according to

$$\Omega \rightarrow \Omega + \phi' + (\phi \wedge \phi)' + (\phi \wedge \phi \wedge \phi)'$$

(3.35)

In fact we will only need the linear term. In this discussion we will assume that we are making an analytic continuation away from the geometric slice which allows us to relax the condition $\partial + \bar{\partial} = d$ fixed and treat $\partial$ and $\bar{\partial}$ independently. Under the variation (3.34) $\partial$ does not change. Let us postulate the following transformation law. We will see in a moment that it is indeed a symmetry of KS action for the field $Y = A + x$

$$Y' \rightarrow (Y + \delta Y)^\xi = Y' - \phi' ,$$

(3.36)

where $\xi$ defines a deformed prime operation with respect to the new holomorphic 3-form (3.35). This transformation rule implies that the variation $\delta Y \in \Omega^{(0,1)}(T_M) \oplus \Omega^{(0,2)}(\wedge^2 T_M)$. The deformed massless mode $x + \delta x$ is killed by the new operator $\partial_{\text{new}} = \bar{\partial} + \phi \cdot \partial$. Projecting on the kernel of $\partial_{\text{new}}$ we recover the transformations rules for massless and massive modes

$$(\delta x)' = -\phi' - x \perp \phi' - \frac{1}{\bar{\partial}} \partial(\phi \wedge x)'$$

$$(\delta A)' = -A \perp \phi' + \frac{1}{\bar{\partial}} \partial(\phi \wedge x)'$$

(3.37)

Operation $\perp$ defines a contraction of holomorphic vector indices and naturally replaces $m(\cdot ,\cdot )$. The above formulas define a connection on the space of massless (massive) modes which should be compared with (3.26a)-(3.26b). It is straightforward to check that (3.37) is indeed a symmetry of KS action

$$S[A + \delta A, x + \delta x; J + \delta J] = S[A, x; J] - \frac{1}{2} \int x'(x \wedge \phi)' .$$

(3.38)

This relation is just an infinitesimal form of background independence similar to (3.27). The discussion of the previous section is fully applicable to KS theory. Equation (3.37) defines a parallel transport $x(j)$ (with boundary condition $x(J) = x$) on the space of zero modes. The solution of equation $x(j) = 0$ determines a new complex structure $\tilde{J}$. The global form of background (in)dependence presented in [1] relates KS actions for $J$ and $\tilde{J}$.  

25
4. **Connection**

4.1. **Differential geometry of Kähler forms.**

The first equation of (3.26a) defines a differential equation on the space $\mathcal{K}^{1,1}(M)$ of Kähler forms on $M$. This is an infinite-dimensional vector space. In Appendix A we describe a special foliation of $\mathcal{K}^{1,1}(M)$ (the Hodge foliation). A tangent space to leaf of $\mathcal{F}$ at the point $\omega \in \mathcal{K}^{1,1}(M)$ consists of $(1,1)$-forms harmonic with respect to the Kähler structure $\omega$. Obviously, this means that the leaves are $b_{1,1}$-dimensional. Locally, one can introduce the coordinates $\{z_1, \ldots, z_{b_{1,1}}, a_1, a_2, \ldots\}$ on $\mathcal{K}^{1,1}(M)$ such that $\{z_1, \ldots, z_{b_{1,1}}\}$ are the coordinates along the leave and $\{a_1, a_2, \ldots\}$ parameterize different leaves.

Over $\mathcal{K}^{1,1}(M)$, one can consider a few vector bundles. One is the (infinite-dimensional) bundle $\mathcal{V}$ of massless modes with fibers $\mathcal{H}_\omega$ consisting of solutions of equations $dx = 0$ and $d^c x = 0$. The other is $\mathcal{H}$ — the bundle of “gauge-fixed” massless modes, satisfying an extra equation $d^i x = 0$. By definition, the solutions are the harmonic forms. Restricting ourselves to $(1,1)$-forms, we obtain a bundle $\mathcal{H}_{1,1}$ — the bundle of tangent directions to leaves of the Hodge foliation $\mathcal{F}$. All this zoo has already appeared in our discussion. The action $S_0[x, \omega]$ is defined as a function on $\mathcal{V}$ (and the AKS action $S[K, x; \omega]$ essentially is a function on $\mathcal{V} \times \Omega^3(M)$). The transformation (3.26a) can be considered as a differential equation on the section $x(\omega(z))$ of the bundle $\mathcal{T}L$

$$\partial_a x(\omega) + s(\partial_a(\omega), x(\omega)) = \partial_a(\omega) \tag{4.1}$$

where $\mathcal{T}L$ is defined as $\mathcal{H}_{1,1}$, restricted to a leaf of $\mathcal{F}$ and $\partial_a \equiv \partial/\partial z^a$ are the coordinate vectors (so that $\partial_a(\omega)$ are harmonic forms). The sections $x(\omega)$ of $\mathcal{T}L$ are vector fields on leaves, in components $x(\omega) = x^i(\omega)\partial_i$.

In principle, we could continue using the Hodge foliation $\mathcal{F}$ and the bundles $\mathcal{V}$ and $\mathcal{H}$. In particular, one can show that $\partial_a + s(\partial_a, \cdot)$ is a natural flat connection on $\mathcal{T}L$. This connection can be trivialized by choosing flat coordinates $\{t^i\}$ along the leaf. Then (4.1) defines the vector fields along the leaves, linear in $t^i$.

For our purposes it seems more natural though to make use of the symmetries to pass to more conventional finite-dimensional objects, as we do in the next section. Still, it is convenient to keep in mind the picture just described since it explains clearly the geometric meaning of $s(\delta \omega, \cdot)$.

---

13 Specifically, $\mathcal{K}^{1,1}(M)$ is foliated by $b_{1,1}$-dimensional (smooth) surfaces, called leaves. This means that every point $\omega \in \mathcal{K}^{1,1}(M)$ belongs to one and only one such surface. The leaves depend smoothly on the point of $\mathcal{K}^{1,1}(M)$.
4.2. Reduction to a finite-dimensional picture.

Although AKS is naturally defined on the infinite-dimensional space $\mathcal{V}$ of parameters $x$ and $\omega$ discussed above, one can effectively reduce $\mathcal{V}$ to a finite-dimensional object using the symmetries established in the Section 3. To fix the gauge symmetry one requires $d^\dagger x = 0$ which reduces $\mathcal{V}$ to the bundle $H$ with finite-dimensional fiber. Next, AKS is what we call in Section 3 a Kähler topological theory: essentially it depends only on the cohomological class of the Kähler form.

Let us consider the transformation (3.26a) : $x \to \tilde{x} = x + \delta\omega - s(\delta\omega, x)$. In that equation, $x$ and $\delta\omega$ are harmonic with respect to $\omega$ and $\tilde{x}$ is harmonic with respect to $\omega + \delta\omega$. Now let us interpret (3.26a) as an equation on the cohomological class of $x(\omega)$. To make sure this is a consistent interpretation one should check that the class $[\tilde{x}]$ depends only on the class $[x]$. Indeed, if $x \to x + d\alpha$ then

$$
\tilde{x} \to \tilde{x} + d\alpha - P_{\text{Ker}} d(m(d\alpha, x)) = d(\alpha + m(\alpha, x)) - P_{\text{Ker}} d^\dagger(\alpha x)
$$

$$
= \tilde{x} + d\left(\alpha + m(\alpha, x) + \frac{d^\dagger}{d}(P_{\text{Ker}} d\alpha x - h P_{\text{Ker}} d\alpha x)\right) \quad (4.2)
$$

The same argument shows that $[\tilde{x}]$ depends only on the class $[\delta\omega]$ of the variation of the Kähler form. This motivates one to consider a bundle $\mathcal{C}$ with fibers $H^*(M)$ over the Kähler cone $K \subset H^{1,1}(M)$. The bundle is defined by the connection

$$
D_a = \partial_a + s(\partial_a(\omega), \cdot), \quad (4.3)
$$

where we introduced the coordinate system $\{z^i\}$ on $K$ and $\partial_i \equiv \partial/\partial z^i$ (so that $\partial_i(\omega) \in H^{1,1}(M)$). Then the “cohomological version” of equation (3.26a) which can be written as

$$
D_a x + \partial_a(\omega) = 0 \quad (4.4)
$$

defines a section $x(\omega)$ of the tangent bundle to the Kähler cone $\mathcal{T}K \subset \mathcal{C}$. The sections $x(\omega(z))$ of $\mathcal{T}L$ are vector fields on $K$, in components $x(\omega(z)) = x^i(z) \partial_i$. The equation (4.4) written in components takes the following form:

$$
\partial_a x^i + \Gamma^i_{ab}(z)x^b = \delta^i_a, \quad (4.5)
$$

Note that the equation (4.3) can also be written as $D_a x = 0$, where $D_a x = D_a x + \partial_a(\omega)$ is the affine connection, associated with the linear connection $D_a$. ($D_a$ is called the Cartan connection.) This remark should explain what we meant in Section 2 describing the accessories of global background independence.
where $\Gamma^i_{ab}\partial_i(\omega) = s(\partial_a(\omega), \partial_b(\omega))$.

We will demonstrate that the connection $D_a$ is flat which implies that the system (4.4) is integrable. It is convenient to present (4.3) as

$$D_a = \partial_a + [\Lambda, \partial_a(\omega)] - \Lambda(\partial_a(\omega)) - \partial_a \log \text{Vol}_\omega. \tag{4.6}$$

The last term can be interpreted as a connection on the flat line bundle over the Kähler cone. It suffices to prove that $\nabla_a$ is flat.\footnote{It is worth mentioning that the operator $\nabla_a = \partial_a + [\Lambda, \partial_a(\omega)]$ is a connection preserving the intersection form while $D_a$ preserves the intersection form normalized to a unit of volume.}

$$[\nabla_a, \nabla_b] = [\partial_a \Lambda, \partial_b(\omega)] - [\partial_b \Lambda, \partial_a(\omega)] + [[\Lambda, \partial_a(\omega)], [\Lambda, \partial_b(\omega)]]. \tag{4.7}$$

Let us compute the first two terms in (4.7). Since $\partial_a \Lambda = \frac{1}{2}[[\Lambda, [\Lambda, \partial_a(\omega)]]$,

$$[\partial_a \Lambda, \partial_b(\omega)] = \frac{1}{2}[[\Lambda, \partial_b(\omega)], [\Lambda, \partial_a(\omega)]] + \frac{1}{2}[[\Lambda, [\Lambda, \partial_b(\omega)]], \partial_a(\omega)] \tag{4.8}$$

The first summand in (4.8) is antisymmetric in $a, b$ and the second one is symmetric. Substituting back to (4.7) one finally obtains

$$[\nabla_a, \nabla_b] = [[\Lambda, \partial_b(\omega)], [\Lambda, \partial_a(\omega)]] + [[\Lambda, \partial_a(\omega)], [\Lambda, \partial_b(\omega)]] = 0. \tag{4.9}$$

Since the connection $D_a$ is flat there are sections $\{e_\alpha\}$ that trivialize the bundle: each section can be expressed as linear combination of $\{e_\alpha\}$. In particular any solution of $D_a \xi = 0$ is a linear combination of $\{e_\alpha\}$ with constant coefficients. The tangent bundle to the Kähler cone can be identified with the subbundle of $(1,1)$-forms of $\mathcal{C}$. Let us consider the subset $\{e_i\}$ of $\{e_\alpha\}$ that generates $H^{(1,1)}$ in each fiber. The sections $e_i$ are the vector fields on K. As a consequence of flatness of $D_i$ all the $e_i$ commute with each other. Therefore there exists a coordinate system $\{t^i\}$ on a Kähler cone such that these vector fields are tangent to the coordinate lines: $e_i = \partial/\partial t^i$. We call $\{t^i\}$ the flat coordinates.

It is instructive to find the flat coordinate $t$ in the simplest case when dim K=1. One-dimensional Kähler cone $K$ can parameterize the linear coordinate $\omega = y \sigma$, where $\sigma$ is any fixed $(1,1)$-form. Computing the AKS connection (4.3) one finds $D_y = \partial_y + 2y^{-1}$. Solving $D_y e = 0$ one obtains $e = y^2 \partial_y$ so that $t = y^{-1}$.

Using the flat coordinates one can immediately write down the solution of (4.4). Namely,
\[
x(t) = x^i \frac{\partial}{\partial t^i} = (t^i - t^i_0) \frac{\partial}{\partial t^i},
\]

(4.10)

where \(B^i\) are the constants fixed by the initial data. Now one can see that for any initial data there is a unique point \([\tilde{\omega}] = (t^1_0, \ldots, t^n_0)\) on the Kähler cone \(K\) where the solution \(x\) vanishes.

In the previous section we used the slightly different statement that for any \(x\) and any Kähler form \(\omega\) one can find a Kähler form \(\tilde{\omega}\) for which the classical solution \(K_0\) vanishes. Now we can explain this. The classical solution \(K_0\) depends on the Kähler form \(\omega\) and is uniquely determined by cohomological class \([x]\) of \(x\). Let us find the solution (4.10) of (4.4) for the initial (cohomological) data \([\omega]\) and \([x]\). This solution can be promoted to the solution of (4.1) on the particular leaf \(L_\omega\) of the Hodge foliation specified by \(\omega \in L_\omega\). There is a one-to-one correspondence between the points on the leaf and the points on the Kähler cone. Then at the point \(\tilde{\omega} \in L_\omega\) corresponding to \([\tilde{\omega}] \in \mathcal{K}\) the harmonic representative of \([x(\tilde{\omega})]\) vanishes and so does \(K_0[x(\tilde{\omega})]\).

5. Hamiltonian approach to AKS.

5.1. Canonical variables, Hamiltonian and constraints.

In this section we return to the full AKS theory (3.16) described in Section 3. We remind the reader that the “superparticle field” is defined as a linear combination \(\mathcal{K} = \sum_{q=0}^{2n-1} K_q\), where \(K_q \in \Omega^q(M)\). The components with degrees \(0 \leq q(K) \leq n - 1\) are called fields, while the components with degrees \(n \leq q(K) \leq 2n - 1\) are called antifields. The field \(\mathcal{K}\) satisfies \(\mathcal{K} = d^{ct} Z\), or in components \(K_q = d^{ct} Z_{q+1}\). Note that there is no dynamical field of degree \(2n\). The action written in components is

\[
\frac{1}{2g^2} \int \sum_{p+q=2n-2} K_p \frac{1}{d^{ct}} dK_q + \frac{1}{6g^2} \int \sum_{p+q+r=2n} (x_p + K_p) \wedge (x_q + K_q) \wedge (x_r + K_r).
\]

It is important that the top component \(K_{2n-1}\) does not have any kinetic term. Thus it is not dynamical.

We will consider the Hamiltonian formulation of AKS theory. It is not covariant. Even worse, splitting off of 1-dimensional time spoils complex geometry. Still this is the safest way to introduce the path integral. Besides, in the Hamiltonian approach the basic

\[\text{\footnote{Thus } \mathcal{K} \text{ is massive. The massless mode is denoted by } x = \sum x_q.}\]
physics of the model appears the most clearly. Also we will be able to use the wisdom accumulated in 3-dimensional Chern-Simons theory \cite{22}, \cite{23}.

To begin, one should identify the time coordinate. We will do this in a way which is not quite general but has an advantage of preserving as much of complex geometry as possible. Assume that the manifold $M$ has a structure of a direct product $M = S \times T^2$ where $S$ is $n - 1$-dimensional complex Kähler and $T^2$ is a 1-dimensional complex torus. As a real manifold, $T^2 = T^1_\tau \times T^1_t$. We call the time the coordinate $t$ parameterizing the circle $T^1_\tau$. Then the “space” is $S \times T^1_\tau$. Also, let us choose a special Kähler structure on $M = S \times T^2$ which is $\omega = \omega_S \otimes 1 + 1 \otimes (dt \wedge d\sigma)$, where $\omega_S$ is a Kähler structure on $S$. Once we know the physics for this particular $\omega$, we can move along Kähler cone of $M$ using the methods described above.

The relation $K_{q-1} = d^c\dot{Z}_q$ now reads as $K_{q-1} = d^c_s\dot{Z}_q + i(\partial_\sigma)\dot{Z}_q - \partial_\sigma i(\partial_t)Z_q$, where $d^c_s$ is a differential operator on $S$, dot stands for differentiation with respect to time and $i(\partial_t)$, $i(\partial_\sigma)$ denote the contractions with the coordinate frame vectors. This relation contains a time derivative. The simplest way to take it into account is to write the action in terms of $Z_q$.

\begin{equation}
S[Z_p] = \frac{1}{2g^2} \int \sum_{p+q=2n} d^c Z_p \wedge dZ_q + \frac{1}{6g^2} \int \sum_{p+q+r=2n} (x_p + d^c Z_{p+1}) \wedge (x_q + d^c Z_{q+1}) \wedge (x_r + d^c Z_{r+1}).
\end{equation}

Let us decompose $Z$ as $Z = Z^0 + Z^1 \wedge d\sigma$, where $Z^1 = i(\partial_\sigma)Z$. Defined as a functional of $Z_q$ and $\partial_\mu Z_q$ the action depends on the velocities $(\dot{Z}_q^0)_{\mu_1,...,\mu_{q-1}}$ linearly. Indeed, the term $d^c Z = d^c_s Z + \dot{Z}^1 - \partial_\sigma i(\partial_t)Z$ cannot produce such velocity at all. The term $dZ$ produces the combination $\dot{Z}^0 \wedge dt$ which has to be multiplied by $(d^c_s Z^1 - \partial_\sigma i(\partial_t)Z^1) \wedge d\sigma$. Hence the momenta $\frac{\partial L}{\partial Z_q^0}$ corresponding to $Z^0$ can be written in terms of the spacial derivatives of $Z^1$ only. The Hamiltonian $H = \frac{\partial L}{\partial \dot{Z}_q} \dot{Z}_q - L$ is independent of these momenta. We conclude that $Z^0$ is conserved and serves just as a parameter\footnote{We discuss the corresponding secondary constraint below.}. This is a consequence of the obvious symmetry $Z \rightarrow Z + d^c_s W$.

Also, let us give a closer look to the equations of motion. One can easily see that no second time derivatives $i(\partial_t)\ddot{Z}$ of the temporal components can be found. Therefore, $i(\partial_t)Z$ is not dynamical in a usual sense. This happens because of the gauge symmetry...
To proceed, we should choose the gauge fixing. From the above discussion it follows one can consistently take the \textit{temporal gauge} \( i(\partial_t)Z = 0 \). Geometrically it means that \( Z \) is a differential form on \( S \times \mathbf{T}^1_\sigma \) depending on \( t \) as a parameter. We should also fix the massless modes \( x_q \) satisfying \( dx_q = d^c x_q = 0 \). As usual we take the harmonic representatives on \( M \). Obviously, they can be decomposed as \( x_q = x_q^{00} + x_q^{10} \wedge dt + x_q^{01} \wedge d\sigma + x_q^{11} \wedge dt \wedge d\sigma \) where \( x_q^{nm} \) are the harmonic forms on \( S \). For the sake of simplicity, let us assume that \( i(\partial_t)x_q = 0 \iff x_q^{10} = x_q^{11} = 0 \). In fact, imposing these constraints we lose a part of the information about the topology of the space-time \( M \) in \( t \) direction. In particular, the Wilson lines along \( \mathbf{T}^1_i \) are excluded.

There is a natural Poisson bracket on differential forms on \( S \times \mathbf{T}^1_\sigma \) which may formally be written as

\[
\{ K_q(z), Z_p(w) \}_{\text{P.B.}} = \delta_{p+q, 2n-1} \omega_S^{n-1}(z) \wedge d\sigma \delta(z, w) .
\]

(5.2)

This bracket pairs \( \Omega^p(S \times \mathbf{T}^1_\sigma) \) with \( \Omega^{2n-1-p}(S \times \mathbf{T}^1_\sigma) \). As usual, (5.2) means that the Poisson bracket of two functionals is given by

\[
\{ F, L \}_{\text{P.B.}} = \int \sum_{p+q=2n-1} \frac{\delta F}{\delta K_q} \frac{\delta L}{\delta Z_p} - \frac{\delta F}{\delta Z_p} \frac{\delta L}{\delta K_q} .
\]

(5.3)

Let us write down the action (3.10) in the temporal gauge. Since \( Z \) and \( x \) are differential forms on \( S \times \mathbf{T}^1_\sigma \), so is \( K = d^c_S Z + i(\partial_\sigma) \dot{Z} \). The cubic term \((K + x)^3\) equals zero as a differential form of degree \( 2n \) on \( 2n - 1 \)-dimensional manifold \( S \times \mathbf{T}^1_\sigma \). Using the above decomposition \( Z_q = Z_q^0 + Z_q^1 \wedge d\sigma \) one can write the action as:

\[
S[Z] = \frac{1}{2g^2} \int dt \int_{\mathbf{T}^1_\sigma} d\sigma \int_{p+q=2n} \dot{Z}_p^1 \wedge \dot{Z}_q^1 + 2 \dot{Z}_p^1 \wedge d^c_S Z_q^0
\]

(we used integration by parts.) The dynamical variables are \( Z_p^1 \) and their canonical momenta \( K_{2n-1-p}^0 = d^c_S Z_{2n-p}^0 + \dot{Z}_{2n-p}^1 \) — the restrictions of \( K_p \) to \( S \). The canonical momenta for \( Z_p^0 \) are zero. The constraint \( d^c_S K_{q-1}^0 = d^c_S \dot{Z}_q^1 = 0 \) and the equation of motion \( d^c_S \dot{Z}_q^0 + \dot{Z}_q^1 = 0 \) together are equivalent to \( \partial_t K = 0 \). The Hamiltonian

\[
H[K^0, Z^1] = \frac{1}{2g^2} \int_{\mathbf{T}^1_\sigma} d\sigma \int_S K^0 \wedge K^0 - 2 Z^0 \wedge d^c_S K^0
\]

(5.5)

\[18\] Clearly, this gauge does not exist for the top component \( Z_{2n} \); it cannot be made into a form on \( S \). But since the field \( K_{2n-1} = d^c Z_{2n} \) is not dynamical anyway, we can simply forget about \( Z_{2n} \).
is independent of $Z^1$.

But this is not the whole story yet. Gauge fixing $i(\partial_t)Z = 0$ produces a bunch of secondary constraints. These constraints can easily be found if we notice that together with the dynamical equation $\partial_t K = 0$ they should reproduce the AKS equations of motion $dK + d^{c\dagger}((K + x)^2) = 0$ in the temporal gauge $i(\partial_t)K = 0$. Taking the decomposition $K = K^0 + K^1 \wedge d\sigma$ where $K^1 = i(\partial_\sigma)K$ and the similar decomposition for the massless modes one finds

$$C_1 = d_S K^0 + \frac{1}{2} d_S^{c\dagger}((K^0 + x^0))^2 = 0$$
$$C_0 = \partial_\sigma K^0 + d_S K^1 + d_S^{c\dagger}((K^0 + x^0) \wedge (K^1 + x^1)) = 0$$

Together with $K^1 = d_S^{c\dagger}Z^1$, the fact that $K^0$ is the canonical conjugate for $Z^1$ and $d_S^{c\dagger}K^0 = 0$ the relations (5.6) produce all the constraints.

Both fields $K^0$ and $Z^1$ are defined over the product $S \times T^1$. There is a remaining gauge symmetry $\tilde{\mathcal{G}}_R$ — a subgroup of the AKS gauge group preserving the temporal gauge. The gauge parameter of $\tilde{\mathcal{G}}_R$ should satisfy $\dot{\alpha} = 0$ and $i(\partial_t)\alpha = 0$. It can be decomposed as $\alpha = \alpha^0 + \alpha^1 \wedge d\sigma$, both $\alpha^0$ and $\alpha^1$ being annihilated by $d_S^{c\dagger}$. Using the explicit parameterization of $\text{Ker} d_S^{c\dagger}$ introduced in Section 3, one can write $\alpha^i = d_S^{c\dagger}x^i + h^i$ where $h^i$ are harmonic. The gauge transformations written in terms of $\chi^i$, $h^i$ are:

$$\delta_0 K^0 = d_S d_S^{c\dagger} \chi^0 + d_S^{c\dagger}((d_S^{c\dagger} \chi^0 + h^0) \wedge (K^0 + x^0))$$
$$\delta_0 Z^1 = \partial_\sigma \chi^0 + (d_S^{c\dagger} \chi^0 + h^0) \wedge (d_S^{c\dagger} Z^1 + x^1)$$

---

19 This is no longer true if $i(\partial_t)x \neq 0$. In general for $x = y + z \wedge dt$ the equation of motion $\partial_t K + d_S^{c\dagger}(y \wedge z \wedge K) = 0$ shows that $H[K^0, Z^1]$ is a functional of both canonical coordinates.

20 The background fields $x^0$ and $x^1$ are independent of $\sigma$. For every $\sigma$, we can solve the first equation in (5.4) which is the equation of motion of AKS theory living on $S$. Given $x^0$, the solution is unique modulo gauge transformations. This means that as we move along $T^1_\sigma$, the field $K^0$ can only change by the gauge transformation. The second equation in (5.6) tells us this is indeed so: the infinitesimal shift along $T^1_\sigma$ is equivalent to the gauge transformation with parameter $K^1 + x^1$.

21 From now on, there will appear the whole zoo of gauge groups. Reader may find it convenient to have a glossary. By $\mathcal{G}_M$ we denote the original gauge group (3.10). Similarly, $\mathcal{G}_S$ denote the gauge group of AKS theory on the manifold $S$. Its parameters are the differential forms on $S$. We also need a group $\tilde{\mathcal{G}}_S$ which is $\mathcal{G}_S$ with $\sigma$-dependent parameters $\Omega^*(S) \otimes \Omega^0(T^1 \sigma)$. The latter has a central extension $\hat{\mathcal{G}}_S$ to be described below. The subgroup of $\mathcal{G}_M$ preserving the temporal gauge is called $\mathcal{G}_R$. Its subgroup $\mathcal{G}_R \subset \tilde{\mathcal{G}}_R$ is obtained by restricting to $\sigma$-independent parameters. One has $\mathcal{G}_M \supset \tilde{\mathcal{G}}_R \supset \mathcal{G}_S \supset \mathcal{G}_S$.

22 It is easy to see that actually $\tilde{\mathcal{G}}_R \cong \hat{\mathcal{G}}_S \triangleright \Lambda^*(R_{\tilde{\mathcal{G}}_S})$ — a semidirect product of $\hat{\mathcal{G}}_S$ with the external algebra of its adjoint representation $R_{\tilde{\mathcal{G}}_S}$.
and
\[ \begin{align*}
\delta_1 \mathcal{K}^0 &= 0 \\
\delta_1 \mathcal{Z}^1 &= d_S \chi^1 + (d_S^\dagger \chi^1 + h^1) \wedge (\mathcal{K}^0 + x^0). 
\end{align*} \tag{5.8} \]

The remaining gauge symmetry can be fixed. To fix (5.7) we impose \( d_S^\dagger \mathcal{K}^0 = 0 \) and \( \partial_\sigma \mathcal{Z}^1 = 0 \). To fix (5.8) we add \( d_S^\dagger \mathcal{Z}^1 = 0 \). Then the constraints (5.6) can be solved to find \( \mathcal{K}^0 \) and \( \mathcal{K}^1 \) in terms of the massless modes \( x^0 \) and \( x^1 \) using the series (3.6). So defined, \( \mathcal{K}^0 \) and \( \mathcal{K}^1 \) are independent of \( \sigma \). The space of solutions can be identified with the double \( \mathcal{H}_S \oplus \mathcal{H}_S d\sigma \) of the space \( \mathcal{H}_S \) of harmonic forms on \( S \).

We expect that if we did not set the temporal components \( x^{10} = x^{11} = 0 \), we would obtain the full space \( \mathcal{H}_M \sim \mathcal{H}_S \oplus \mathcal{H}_S d\sigma \oplus \mathcal{H}_S dt \oplus \mathcal{H}_S d\sigma dt \) of harmonic forms on \( M \). In fact, we have already established this in the section about the BV formalism. We see that AKS turns out to be a finite-dimensional system. Its dynamics is governed by the nonzero Hamiltonian. This is to be compared with 3-dimensional Chern-Simons theory which is also a finite-dimensional system but with a zero Hamiltonian.

5.2. Classical and quantum symplectic reduction.

The gauge transformations (5.7)–(5.8) with \( \alpha^0 = d_S^\dagger \chi^0 \) and \( \alpha^1 = d_S^\dagger \chi^1 \) generate a (normal) subgroup \( \tilde{G}^0_R \subset \tilde{G}_R \). The action of this subgroup is (almost) Poisson. Indeed, the constraints (5.6) generate the flows:
\[ \oint d\sigma \int_S \chi^i(z) \wedge \{ C_i, F[\mathcal{K}^0, \mathcal{Z}^1] \}_\text{P.B.} = \delta_i F[\mathcal{K}^0, \mathcal{Z}^1]. \tag{5.9} \]

We compute the P.B.’s between the constraints to find
\[ \begin{align*}
\left\{ \int \xi^0(x) \wedge C_0, \int \zeta^0(y) \wedge C_0 \right\}_\text{P.B.} &= \int (d_S^\dagger \xi \wedge d_S^\dagger \zeta) \wedge C_0 + \int \partial_\sigma \zeta^0 \wedge d_S d_S^\dagger \xi^0 \\
\left\{ \int \xi^0(x) \wedge C_0, \int \zeta^1(y) \wedge C_1 \right\}_\text{P.B.} &= \int (d_S^\dagger \xi \wedge d_S^\dagger \zeta) \wedge C_1 \\
\left\{ \int \xi^1(x) \wedge C_1, \int \zeta^1(y) \wedge C_1 \right\}_\text{P.B.} &= 0
\end{align*} \tag{5.10} \]

The relations (5.10) show that (5.9) furnish a representation of central extension \( \hat{G}^0_R \) of the gauge group \( \tilde{G}^0_R \). A cocycle in the right hand side of (5.10) appears due to nontrivial \( \sigma \) dependence of the gauge parameters.

We would like to leave an interesting object \( \hat{G}^0_R \) for further investigation. Our immediate aim is to obtain the physical phase space. From the above discussion it follows that
in a sense, \(\sigma\)-dependence is the pure gauge. This motivates one to consider a restriction to \(\sigma\)-independent fields and the action of (5.7)–(5.8) with \(\sigma\)-independent gauge parameters. (Essentially this returns us to AKS theory considered, however, on the manifold \(S\).)

So let us consider a phase space \(\mathcal{V} = \{\mathcal{K}^0, \mathcal{Z}^1 | \mathcal{K}^0 \in \text{Ker} d_S^\dagger, \mathcal{Z}^1 \in \Omega^*(S)/\text{Ker} d_S^\dagger\}\) with Poisson bracket given by

\[
\{\mathcal{K}^0(z), \mathcal{Z}^1(w)\}_{PB} = \omega_{S}^{n-1}(z) \delta(z, w).
\]

The transformations (5.9) with \(\sigma\)-independent parameters \(\chi^i\) act on functions on \(\mathcal{V}\) furnishing a representation of the gauge group \(G^0_R\). In particular, these transformations preserve the ideal of the set \((C_1 = 0, C_0 = 0)\).

(Indeed, doing the hamiltonian reduction, first we take the subspace of \(\mathcal{V}\) where \(C_1 = C_0 = 0\). Then we impose the secondary constraints \(\{C_1, \cdot\} = \{C_0, \cdot\} = 0\). As (5.9) shows, this is equivalent to taking \(G_R\)-invariants. As a result of the reduction, we obtain a single point, since the massless fields \(x^0\) and \(x^1\) are kept fixed all the time.)

Now we can turn to quantization. First we should choose a polarization on the phase space \(\mathcal{V}\). It is convenient to work in “\(\mathcal{K}^0\)-representation”: wave functions \(\Psi[\mathcal{K}^0]\) are functionals of \(\mathcal{K}^0\); the canonical conjugate of \(\mathcal{K}^0\) which is \(\mathcal{Z}^1 = i \delta/\delta \mathcal{K}^0\) acts on \(\Psi[\mathcal{K}^0]\) by differentiation. Wherever the ordering problem occurs, we use the “\(qp\)” prescription – put the momenta to the right.

The gauge symmetry (5.7)–(5.8) is realized by the differential operators acting on wave functions. Obviously, the generator \(\delta_1\) acts by multiplication by \(C_1 = d_S \mathcal{K}^0 + \frac{1}{2} d_S^\dagger ((\mathcal{K}^0 + x^0))^2\). The generator \(\delta_0\) is represented by the differential operator

\[
\hat{G} = d_S^\dagger d_S i \frac{\delta}{\delta \mathcal{K}^0} + d_S^\dagger \left[ (\mathcal{K}^0 + x^0) \wedge (d_S^\dagger i \frac{\delta}{\delta \mathcal{K}^0} + x^1) \right] \quad (5.11)
\]

The first constraint (5.6) means that we should consider only the wave functions with support on the space \(\mathcal{N} \equiv \{\mathcal{K}^0 | d_S^\dagger \mathcal{K}^0 = 0; C_1 = 0\}\) (\(\mathcal{N}\) consists of solutions of AKS equations of motion on \(S\)). The gauge group \(G^0_S\) acts on \(\Omega(\mathcal{N})\) by \(\hat{G}\). Then the second constraint (5.7) says that the physical wave functions satisfy

\[
\hat{G} \Psi[\mathcal{K}^0] = 0. \quad (5.12)
\]
In other words, the physical wave function is gauge invariant. Together, two relations (5.6) describe the Hamiltonian reduction of $\Omega^0(V)$ with respect to $\hat{G}_R^0$.

Similarly to CS theory, one can write the solution to (5.12) in terms of the functional integral. Consider a functional

$$\Psi[K_0] = e^{\frac{i}{g} \int (K_0 + x^0) \wedge x^1} \epsilon_N[K_0] \int DZ e^{\frac{i}{g} \int dS^c_3 \wedge dS Z + dS^c_3 \wedge (K_0 + x^0) \wedge dS^c_3 Z}$$

(5.13)

where $\epsilon_N[K_0]$ is a characteristic function of the set $\mathcal{N}$ and

$$\int DZ e^{\frac{i}{g} \int dS^c_3 \wedge dS Z + dS^c_3 \wedge (K_0 + x^0) \wedge dS^c_3 Z} = \left[ \int d\Omega \text{Ber}' (dS^c_3 + dS^c_3 ((K_0 + x^0) \wedge dS^c_3)) \right]^{1/2}.$$ 

(5.14)

Here $\text{Ber}'$ denotes the Berezinian computed for nonzero modes and $d\Omega$ is a supermeasure on the space of zero modes of the operator $L = dSdS^c_3 + dS^c_3 ((K_0 + x^0) \wedge dS^c_3)$. Obviously, (5.13) has a support on $\mathcal{N}$. It is easy to see that $\hat{G} \epsilon_N[K_0] = 0$ (this should already be clear from (5.9)). Using “the equations of motion” for the functional in the exponent of the integral representation for the Berezinian one can see that (5.12) is indeed satisfied.

The scalar product of two wave functions is defined as

$$\langle \Psi_1 | \Psi_2 \rangle = \int D\mathcal{K}^0 \delta [C_1[\mathcal{K}^0]] \delta [dS^c_3 \mathcal{K}^0] \overline{\Psi}_1[\mathcal{K}^0] \Psi_2[\mathcal{K}^0]$$

where the $\delta$–functionals can also be written as

$$\delta [C_1[\mathcal{K}^0]] \delta [dS^c_3 \mathcal{K}^0] = \int D\lambda D\mu e^{\frac{i}{g} \int (\mathcal{K}^0 + x^0)^2 \wedge dS^c_3 \mu + \mathcal{K}^0 \wedge (dS^c_3 \lambda - dS \mu)}.$$

The gauge invariant measure on the configuration space which appears in the scalar product can also be interpreted as the invariant measure on the gauge group $\mathcal{G}_S$. Indeed, the set $\mathcal{N}$ consists of the solutions $\kappa^0$ of the AKS equations of motion. For a given $x^0$, they

23 If we do not restrict to $\sigma$-independent fields, we obtain $\hat{G} \Psi[\mathcal{K}^0] = (\partial_\sigma \mathcal{K}^0) \Psi[\mathcal{K}^0]$. This means that $\Psi[\mathcal{K}^0]$ furnishes a projective 1-dimensional representation of the gauge group: the gauge transformation changes only a phase of the wavefunction. Note that the nontrivial phase variation is a consequence of presence of the parameter $\sigma$. This type of phenomena is usually referred to as a Berry phase. Equivalently, one can say that $\Psi[\mathcal{K}^0]$ is invariant with respect to action of the central extension of the gauge group.

24 The functional integral should be computed over $\Omega^*(S)/\text{Ker} dS^c_3$.
constitute the orbit of that gauge group. Introducing the coordinates $\kappa^0$ along $N$ and the transversal coordinates $\eta$ we can write

$$
\delta [C_1[K^0]] \delta [d_S^\dagger \eta] D\kappa^0 D\eta = \epsilon_N[K^0] \left[ \text{Ber}' \left( \frac{\delta C_1}{\delta \eta} \right) \text{Ber}' (d_S^\dagger) \right]^{-1} D\kappa^0 
= \epsilon_N[K^0] \left[ \text{Ber}' (d_S + \kappa^0 \wedge d_S^\dagger) \right]^{-1} \left[ \text{Ber}' (d_S^\dagger) \right]^{-1} D\kappa^0.
$$

(5.15)

On the other hand, using the gauge parameter $\alpha^0$ of (5.7) as a coordinate along $N$, we find

$$
D\kappa^0 = \left[ \text{Ber}' (d_S + \kappa^0 \wedge d_S^\dagger) \right] D\alpha^0,
$$

so finally the measure is just

$$
\left[ \text{Ber}' (d_S^\dagger) \right]^{-1} D\alpha^0.
$$

Note that this is indeed a natural measure on the gauge group. Since the gauge parameter should satisfy the constraint $d_S^\dagger \alpha^0 = 0$, we can write the measure as $\delta [d_S^\dagger \alpha] D\alpha$ and extend integration over the whole $\Omega^*(S)$. Integrating out the transversal coordinates, one recovers the factor $\left[ \text{Ber}' (d_S^\dagger) \right]^{-1}$.

5.3. Path integral for AKS

The measure for the Hamiltonian path integral in temporal gauge is determined by the Poisson bracket (5.2). It can be written as

$$
D\mu = \left[ DK_{2n-1} \right] \prod_{p=0}^{2n-2} \left[ DK_p \right] \left[ DZ_{2n-1-p}^1 \right] \delta [C_0] \delta [C_1] \delta [d_S^\dagger K^0] .
$$

All the fields here are $\sigma$–dependent. As in the relatively simpler case just considered, our aim is to reduce $D\mu$ to the measure on the gauge group $\tilde{G}_R$ generated by $\delta_0$ and $\delta_1$. By $\kappa^0$ and $z^1$ we denote the solutions of the constraints. These solutions are parameterized by $\alpha^0$ and $\alpha^1$ of (5.7)–(5.8). The transversal coordinates are $\eta$ and $\xi$. Following the same steps as above, one finds:

$$
D\mu = \left[ \frac{\delta (C_1, C_0)}{\delta (\eta, \xi)} \right]^{-1} \text{Ber} \left[ \frac{\delta (\kappa^0, z^1)}{\delta (\alpha^0, \alpha^1)} \right] \text{Ber} D\alpha^0 D\alpha^1
= \left[ \text{Ber}' (d_S^\dagger) \right]^{-2} D\alpha^0 D\alpha^1
$$

(5.16)

which is a natural measure on the gauge group.

Since the action is gauge invariant, we can compute the path integral of any gauge invariant observable $O$ to obtain the “localization formula”:

$$
\langle O \rangle = \int D\mu O[K^0, Z^1] e^{-S[K^0, Z^1]} = \text{Vol}(G) O[K^0, z^1] e^{-S[K^0, z^1]}
$$

(5.17)
Let us briefly discuss the general situation when $i(\partial_t)x \neq 0$. For $x = y + w \wedge dt$ the equation of motion $\partial_t K = -d^\dagger_S (z \wedge (y + K))$ shows that the time evolution is a $G_S$ gauge transformation. Thus the gauge classes of $K^0$ and $Z^1$ are $t$-independent. The action then can be written as

$$S[K^0, Z^1] = \frac{T}{2g^2} \int K^0 \wedge \dot{Z}^1 \, d\sigma + (K^0 + d^\dagger Z^1 \, d\sigma + y)^2 \wedge w + (K^0 + d^\dagger Z^1 \, d\sigma + y) \wedge y \wedge w ,$$

where $T$ is the length of $T^1_t$ and integration runs over $S \times T^1_o$. We see that $t$-dependence of gauge invariant observables is trivial. Thus one can expect that is true independently of the assumption $i(\partial_t)x = 0$.

The localization formula (5.17) is specific for the factor structure of the target space $M = S \times T^2$. For such target spaces the interaction can be removed by choosing an appropriate gauge. This situation is very similar to Chern-Simons.

6. Relation to $N = 2$ topological strings

6.1. AKS theory and complexified Kähler cone

There is a crucial difference between AKS theory and $N = 2$ topological strings. As described, AKS theory is defined on the real Kähler cone. Similarly, it is natural to think of $x$ and $K$ as real. On the other hand, the $N = 2$ TCFT is naturally defined on the complexified Kähler cone, where to a real positive Kähler form one adds an imaginary antisymmetric tensor $B_{ij}$ (the $\theta$-term).

As we just have seen, AKS is background independent. Essentially, this is a consequence of flatness of the bundle of massless states. The flat connection $s(\partial_a(\omega), \partial_b(\omega))$ naturally appears in the transformation (3.26) which leaves the action invariant. Equally important is that $s(\partial_a(\omega), \partial_b(\omega))$ preserves the constraint $d^\dagger x = 0$. A characteristic feature of this connection is that the covariantly constant vector fields $x(\omega)$ scale as $x(\lambda \omega) \rightarrow \lambda^2 x(\omega)$ with respect to it. In turn, this leads to the parabolic law (3.32) for the volume-dependent string coupling constant $g(V)$.

---

26 In general, the covariantly constant sections of $H^{p,q}$-bundle scale as $\lambda^{p+q}$ as $\omega \rightarrow \lambda \omega$.  
27 To be rigorous, one has to refer here to the similar statement about the massive field $K'$ (see (3.28)).
6.2. Classical action

In general there is no localization formula. The (Lagrangian) path integral for AKS is given by

\[ e^{-\Gamma[x;\omega]} = e^{-S_0[x;\omega]} \int \mathcal{D}K e^{-\mathcal{A}[K,x;\omega]} , \]

where we extracted the classical action \( S_0[x;\omega] \) and \( \mathcal{A}[K,x;\omega] \) is given by (3.28). \( \Gamma[x;\omega] \) is nothing else but an effective action for SQM (or semiclassical limit for TCFT). Below we will prove this statement at the tree level. Also, we will explain how the global background independence of AKS is translated into the global background independence of SQM.

It is instructive to compare the perturbation theory for TCFT coupled to gravity and AKS. The latter is formulated in terms of Yukawa couplings \( C_{abc} \) and propagators \( \Delta_{nm}, \Delta_n \) and \( \Delta \) (for the discussion on perturbation theory see \[1\] \[28\]). It was suggested in \[1\] that introducing a dilaton field \( y \) the tadpole \( \Delta_n \) and “blob” \( \Delta \) can be interpreted as ordinary propagators \( \Delta_n = \Delta_{ny} \) and \( \Delta = \Delta_{yy} \).

Below we will see how the analog of these operations appear in the perturbation theory for AKS.

On one hand the AKS action evaluated on the classical trajectory is written in terms of massive propagator \( \Pi(\cdot) \). On the other hand, the SQM amplitudes should be expressed entirely in terms of string propagator \( D(\cdot) \), defined as

\[ D(x) = \Lambda(x) - \Pi(x) \]

It is clear that \( D \) is well defined on cohomology, namely for \( x \) being \( d \)-closed \( D(x) \) is also \( d \)-closed, while for \( x \) being \( d \)-exact \( D(x) \) is also \( d \)-exact.

Let us compute the classical action in perturbation series \( S = S^{(3)} + S^{(4)} + S^{(5)} + ... \) and rewrite it in terms of \( D(\cdot) \). The solution of the classical equation of motion (3.2) is given by (3.6). Plugging this expression into the action we obtain

\[ S_0[x;\omega] = \frac{1}{g_5^2} \left( \frac{1}{6} \int x^3 - \frac{1}{8} \int x^2 \Pi(x^2) + \frac{1}{8} \int x^2 \Pi(x\Pi(x^2)) + ... \right) . \]

The classical action is given as series in terms of massive propagator \( \Pi \). Taking into account the relation (6.2) one can rewrite the second term as

\[ S^{(4)} = \frac{1}{8} \int x^2 \Pi(x^2) = \frac{1}{4!} \left( 3 \int x^2 D(x \wedge x) - 4(\Lambda x) \int x^3 \right) \]

\[ \text{In \[1\] the propagators are denoted as } S_{nm}, S_n \text{ and } S. \]
It is important that $\Lambda x$ is a number and therefore it can be taken outside the integral. This expression should be compared with perturbation series of $N = 2$ TCFT \[1\] which is given by

$$S^{(4)}_{N=2} = \frac{x^a x^b x^c x^d}{4!} \left( \sum_3 C_{ab}^{m} \Delta_{nm} C_{cd}^{m} - \sum_4 \partial_a K C_{bcd} \right) \quad (6.5)$$

Comparing these two equations we can indeed identify term by term. $C_{abc}$ are Yukawa couplings, while $\Delta_{nm}$ are matrix elements of propagator $\Delta(\cdot)$. The relation $\partial_a K = \Lambda x_a$ follows the fact that Kähler potential is given by $-\text{Log}(\text{Vol}_\omega)$.

There are some new features which appear at the next order in the perturbation series. Namely, there is a diagram whose contribution can be interpreted as coming from a tadpole $D_n$. Indeed the fifth order contribution to the action is

$$S^{(5)} = \frac{1}{5!} \left( 15 \int (D(x \wedge x))^2 x - 30(\Lambda x) \int x^2 D(x \wedge x) \right) + \frac{1}{5!} \left( 20(\Lambda x)^2 \int x^3 + 10 \left[ \frac{1}{2} \Lambda^2(x^2) - \Lambda D(x^2) \right] \int x^3 \right) \quad (6.6)$$

In deriving (6.6) it was important that $\frac{1}{2} \Lambda^2(x^2) - \Lambda D(x^2)$ is a number and therefore it can be taken outside the integral. Again, this expression should be compared with perturbation series of $N = 2$ TCFT

$$S^{(5)}_{N=2} = \frac{x^a x^b x^c x^d x^e}{5!} \left( \sum_4 C_{ab}^{m} \Delta_{nm} C_{cd}^{m} \Delta_{kp} C_{de}^{m} - 2 \sum_5 \partial_a K \sum_3 C_{ab}^{m} \Delta_{nm} C_{cd}^{m} + \sum_5 \partial_a K \partial_b K C_{cd} + \sum_5 (\Delta_n - \partial_m K \Delta_n^m) C_{ab}^{m} C_{cd} \right) \quad (6.7)$$

Again, one can identify term by term

$$\frac{1}{2} \Lambda^2(x^2) - \Lambda D(x^2) = (\Delta_n - \partial_m K \Delta_n^m) C_{ab}^{m} x^a x^b. \quad (6.8)$$

The existence of tadpole $\Delta_n$ and dilaton-dilaton propagator $\Delta$ are related to the possibility of constructing the numbers out of $\Lambda(\cdot), D(\cdot)$ and $x$. In fact there are only three irreducible possibilities (for a 3-fold) $\Lambda(x), -\frac{1}{2} \Lambda^2(x^2) + \Lambda D(x^2)$ and $\frac{3}{2} \Lambda^3(x^3) - \Lambda^2 D(x^3)$. The dilaton-dilaton propagator $\Delta$ is related to the last combination (the explicit expression is quite complicated and we won’t present it here). These calculations suggest that at every order in perturbation theory the classical action is expressible entirely in terms of massless propagators and Yukawa couplings.

39
The perturbation theory for AKS is identical to perturbation theory of $N = 2$ TCFT at least at the tree level. It is tempting to suggest that this similarity persist at the loop level and one may just borrow the perturbation series for $N = 2$ TCFT in order to construct loop corrections to AKS theory. For example the one loop correction to the one point correlation function should be given as follows $\langle x \rangle_1 = x^a C_{abc} \Delta^{bc}$. We do not know how to prove this suggestion.

There seems to be a contradiction. The interpretation of AKS as SQM seems to be at odds with the appearance of the dilaton field in the perturbation theory. There is nothing like a dilaton field in SQM. We can suggest the following resolution of this puzzle. Let us introduce an $x$-dependent factor

$$F(x) = 1 + \Lambda(x) + \frac{1}{2} \left[ -\frac{1}{2} \Lambda^2(x^2) + \Lambda D(x^2) \right] + \ldots$$ \hspace{1cm} (6.9)

Using this factor one can rearrange the perturbation series for the classical action as

$$S_0[x; \omega] = F(x)^2 \left\{ \frac{1}{3!} \left( \frac{1}{F(x)} \right)^3 \int x^3 + \frac{3}{4!} \left( \frac{1}{F(x)} \right)^4 \int x^2 D(x^2) + \frac{15}{5!} \left( \frac{1}{F(x)} \right)^5 \int x(D(x^2))^2 + \ldots \right\}$$ \hspace{1cm} (6.10)

Now it is clear that one can make field-dependent renormalization of the external legs and the coupling constant

$$x \rightarrow \phi = \frac{x}{F(x)} \quad \text{and} \quad g^2 \rightarrow \frac{g^2}{F(x)^2}$$ \hspace{1cm} (6.11)

and recover the conventional perturbation series for SQM.

6.3. Yukawa couplings

We are going to prove that classical action $S_0[x; \omega]$ is in fact a generating functions for string amplitudes. It is convenient to write a relation not for the classical action but for perturbed Yukawa coupling.

$$\partial_x^i \partial_x^j \partial_x^k \frac{1}{g_\omega^2} S_0[x; \omega] = C_{ijk} [x; \omega] = \sum_{N \geq 0} \frac{1}{N!} x^{p_1} \ldots x^{p_N} A_{ijkp_1 \ldots p_N} [0; \omega]$$ \hspace{1cm} (6.12)

\[29\] The similar statement is valid in topological string theory. The function $F(x)$ is given as follows $F(x) = 1 + x^i \partial_i + \frac{1}{2} x^a x^b C_{ab}^m (\Delta_n - \partial_m K \Delta^n) + \ldots$. 

40
Let us show that $A_{ijkp_1...p_N}[0; \omega]$ are $(N+3)$ point correlation functions, given as

$$A_{ijkp_1...p_N}[0; \omega] = (-1)^N \partial_{p_1}...\partial_{p_N} C_{ijk} \quad (6.13)$$

where the derivatives $\partial_{p_i}$ are with respect to the flat coordinates $(\tau^1, \ldots, \tau^n)$ of the point $\omega$ on the Kähler cone $K$. This is equivalent to

$$C_{ijk}[x; \omega] = C_{ijk}(x^1-\tau^1, \ldots, x^n-\tau^n) \quad (6.14)$$

Following the discussion in [3] one can represent the perturbed Yukawa couplings as

$$C_{ijk}[x; \omega] = \partial_x \partial_x \partial_x \gamma \chi \frac{1}{g_\omega^2} S_0 [x; \omega] = \frac{1}{g_\omega^2} \int \partial_x K_0 \partial_x K_0 \partial_x K_0 . \quad (6.15)$$

Let us consider the solution $x(t)$ of the equation $(4.14)$ such that $x(\tau) = x$ at the point $\omega = (\tau^1, \ldots, \tau^n)$. As it was discussed in Section 4, for the flat coordinates $x(t) = x^i(t)e_i = (t^i - t^i_0)e_i$, where $e_i$ are the coordinate vectors (cf. $(4.10)$). Obviously, the parameters $t^i_0$ are related to $x^i$, $\tau^i$ by

$$x^i = \tau^i - t^i_0 . \quad (6.16)$$

Consider the Yukawa coupling evaluated on the solution $x(t)$. Under the small variation of Kähler structure $K_0$ transforms according to $(3.247)$ and therefore

$$\partial_{x^i} K_0 \rightarrow \partial_{x^i} \tilde{K}_0 = \partial_{x^i} K_0 - m(\delta \omega, \partial_{x^i} K_0) \quad (6.17)$$

It is easy to see that $C_{ijk}[x(t); \omega(t)]$ does not depend on $t$. Indeed,

$$C_{ijk}[x; \omega] = \frac{1}{g_\omega^2} \int \left( \partial_{x^i} \tilde{K}_0 \partial_{x^j} \tilde{K}_0 \partial_{x^k} \tilde{K}_0 + \sum \text{perm} m(\delta \omega, \partial_{x^i} \tilde{K}_0) \partial_{x^j} \tilde{K}_0 \partial_{x^k} \tilde{K}_0 \right) =$$

$$= \frac{1 + 2\Lambda(\delta \omega)}{g_\omega^2} \int \partial_{x^i} \tilde{K}_0 \partial_{x^j} \tilde{K}_0 \partial_{x^k} \tilde{K}_0 = C_{ijk}[\tilde{x}; \tilde{\omega}] . \quad (6.18)$$

(It is important that $C_{ijk}[x; \omega]$ is written in flat coordinates. Otherwise the additional Jacobian factors would appear in $(6.18)$.)

Now we can apply the argument used already in Section 3 (and explained in detail in Section 4). There is a point $\tilde{\omega} = (t^1_0, \ldots, t^n_0)$ on the Kähler cone $K$ such that $x^i(\tilde{\omega}) = 0$. At this point, obviously, $\partial_{x^i} K_0[x(t_0)] = \partial_{x^i} x(t_0) = e_i$. Therefore

$$C_{ijk}[0; \tilde{\omega}] = \frac{1}{g_\omega^2} \int e_i \wedge e_j \wedge e_k \quad (6.19)$$

can be interpreted in terms of intersections in $H^{1,1}(M)$. Moreover, $(6.18)$ together with $(6.16)$ show that the perturbed Yukawa coupling $C_{ijk}[x; \omega]$ computed for the Kähler structure $\omega = (\tau^1, \ldots, \tau^n)$ coincides with Yukawa coupling $C_{ijk}[0; \tilde{\omega}]$ for the Kähler structure $\tilde{\omega} = (\tau^1 - x^1, \ldots, \tau^n - x^n)$. Therefore as a function, $C_{ijk}[x; \omega]$ depends only on $\tau^i - x^i$. This implies $(6.13)$. 

41
7. Concluding remarks

One of the main motives of this paper is the connection on the Hilbert space bundle. It first appears it in the context of TCFT. The notion of background independence is formulated in terms of this connection (to be precise, in terms of affine connection). Background independence of TCFT is equivalent to flatness of this connection. The condition of supersymmetry imposes strong constraints on the form of the connection which have a simple solution in the semiclassical regime. This solution is constructed in terms of geometric operation $\Lambda$. We suggest that operation $\Lambda$ can be defined for any $N = 2$ field theory (not only in the semiclassical limit) and the connection is given in terms of $\Lambda$. The semiclassical limit of TCFT is SQM (details of identification discussed in the main body of the paper). The states in SQM are identified with harmonic forms and therefore the connection in question is the connection on the leaves of Hodge foliation. This connection turns out to be flat.

The same connection appears in AKS. It allows one to relate theories for different Kähler structures. It enters into the formulation of background independence. AKS is a gauge theory with the gauge group $SDiff$. The gauge symmetry is free of anomalies which can be checked by direct computations. Unfortunately, we were not able to construct the gauge invariant observables which may be a good direction to pursue.

It is natural to compare AKS with Chern-Simons theory. Chern-Simons is a topological theory and its hamiltonian is equal to zero. AKS is Kähler topological theory and its hamiltonian differs from zero. This hamiltonian determines a unique dynamics on the phase space. In case when the target space has factor structure $M = T^2 \times S$, AKS reduces to a free theory with constraints. In this case AKS can be quantized and one can derive a simple localization formula.

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Appendix A. Differential Geometry of $m(A, B)$

A.1. Hodge identities and $sl(2)$

Let us introduce the operator $L$ of multiplication by Kähler form $\omega_{ij}$, the operator $\Lambda$ of contraction with bivector $\omega^{ij}$ and the operator $J$ which acts on $(p, q)$ forms as $\dim M - (p + q)$. The operators $\Lambda$, $L$ and $J$ form the $sl(2)$ algebra:

$$
[\Lambda, L] = J
$$
$$
[J, L] = 2L
$$
$$
[J, \Lambda] = -2\Lambda
$$

(A.1)

The commutation relations between $L$, $\Lambda$ and $\partial$, $\bar{\partial}$, $\partial^\dagger$, $\bar{\partial}^\dagger$ are known as the Hodge identities

$$
[\partial, \Lambda] = -\bar{\partial}^\dagger \quad \text{and} \quad [\partial^\dagger, L] = \bar{\partial}
$$

$$
[\bar{\partial}, \Lambda] = \partial^\dagger \quad \text{and} \quad [\bar{\partial}^\dagger, L] = -\partial
$$

(A.2)

(see [12]). These relations imply the Hodge identities for the Laplacians:

$$
2\Delta_\partial = 2\Delta_{\bar{\partial}} = \Delta_d.
$$

(A.3)

Also, (A.2) means that $sl(2)$ defined by (A.1) preserves $\ker \Delta$, i.e. sends harmonic forms to harmonic ones. Thus the space of all harmonic forms has a decomposition (Lefshetz decomposition) as a direct sum of irreducible representations of this algebra. The highest weight vectors annihilated by $\Lambda$ are called primitive forms. The following formula is useful in applications. If $p$ is a primitive form of rank $r$, then ( $\omega$ is the Kähler form )

$$
\Lambda(\omega^k p) = k(n - k - r + 1)\omega^{k-1} p.
$$

(A.4)

A.2. Properties of $m(A, B)$

A bilinear symmetric operation on differential forms $m(A, B)$ is defined as follows

$$
m(A, B) = \Lambda(A \wedge B) - (\Lambda A) \wedge B - A \wedge \Lambda B
$$

(A.5)

This operation has several remarkable properties, summarized below.

Since $\Lambda$ makes contraction of two indices, there is a formula

$$
\Lambda(ABC) = \Lambda(AB)C + \Lambda(CA)B + \Lambda(BC)A - \Lambda(A)BC - A\Lambda(B)C - ABA\Lambda(C),
$$

43
which is equivalent to
\[ m(AB, C) = Am(B, C) + m(A, C)B. \] (A.7)

Thus the operation \( m(\cdot, \cdot) \) differentiates the multiplication of forms.

It is convenient for us to introduce four differential operators
\[ d = \partial + \bar{\partial} \quad \text{and} \quad d^c = \partial - \bar{\partial}, \]
\[ d^\dagger = \partial^\dagger + \bar{\partial}^\dagger \quad \text{and} \quad d^{c\dagger} = \partial^\dagger - \bar{\partial}^\dagger \] (A.8)
instead of \( \partial, \bar{\partial}, \partial^\dagger \) and \( \bar{\partial}^\dagger \). Acting on \( m(A, B) \) by \( d \) one gets
\[ dm(A, B) = d^{c\dagger}(A \wedge B) - (d^{c\dagger}A) \wedge B - A \wedge d^{c\dagger}B + m(dA, B) + m(A, dB) \] (A.9)

Suppose now that \( K \) is a harmonic form; then (A.9) means that
\[ [d^{c\dagger}, K]B = dm(K, B) - m(K, dB) \]
\[ [d^\dagger, K]B = d^c m(K, B) - m(K, d^c B) \] (A.10)

If we deform the Kähler form \( \omega \) by adding \( K \) to it, \( \Lambda \) changes. To the first order,
\[ \delta \Lambda = \frac{1}{2} [\Lambda, [\Lambda, K]]. \] (A.11)

Then one has a relation, “dual” to (A.10):
\[ [d, \delta \Lambda]B = d^{c\dagger} m(K, B) - m(K, d^{c\dagger}B) \]
\[ [d^c, \delta \Lambda]B = d^\dagger m(K, B) - m(K, d^\dagger B) \] (A.12)

We will prove here only the first relation:
\[ [d, \delta \Lambda]B = \frac{1}{2} [d, [\Lambda, [\Lambda, K]]]B = \frac{1}{2} d(\Lambda^2(KB) - 2\Lambda(K\Lambda B) + K\Lambda^2 B) - \frac{1}{2} (\Lambda^2(KdB) - 2\Lambda(K\Lambda dB) + K\Lambda^2 dB) \]
\[ = \frac{1}{2} (2\Lambda d^{c\dagger}(KB) - 2d^{c\dagger}(K\Lambda B) - 2\Lambda(Kd^{c\dagger}B) + 2K\Lambda d^{c\dagger}B) \] (A.13)

On the other hand,
\[ d^{c\dagger}(\Lambda(KB) - (\Lambda K)B - K(\Lambda B)) = \]
\[ \Lambda d^{c\dagger}(KB) - d^{c\dagger}(K(\Lambda B)) - [d^{c\dagger}, \Lambda K]B + (\Lambda K)d^{c\dagger}B \]
\[ = \Lambda d^{c\dagger}(KB) - d^{c\dagger}(K(\Lambda B)) + (\Lambda K)d^{c\dagger}B \] (A.14)

(We used that \( K \) is the harmonic 2-form, so \( d^{c\dagger}, \Lambda K] = 0. \))
Appendix B. A space of massless modes $\mathcal{H}$

By definition, the massless modes $x$ satisfy two equations:

$$dx = 0 \quad \text{and} \quad d^c \beta = 0$$

The space of solutions of (B.1) we call $\mathcal{H}$. It is defined for every Kähler form $\omega$ and depends on it. Since the differential operators $d$ and $d^c$ commute with each other the space $\mathcal{H}$ has to be infinite-dimensional: it contains a subspace $\mathcal{T}$ which consists of forms $dd^c t$.

A quotient $\mathcal{H}/\mathcal{T}$ is finite dimensional. For a given complex structure $J$ it can be identified with harmonic forms canonically.

Indeed, given a complex structure, one has a Hodge decomposition of a $d$-closed form $x$:

$$x = h_x + d\beta,$$  \hspace{1cm} (B.2)

where $h_x$ is harmonic and $d^\dagger \beta = 0$. The second equation in (B.1) tells us that $dd^c \beta = 0$, so $d^c \beta = h_{d^c \beta} + d\gamma$, where $h_{d^c \beta}$ is harmonic. Let us prove that this harmonic piece is always zero. Indeed, for any harmonic form $\omega$ we have

$$\int \omega \wedge h_{d^c \beta} = \int \omega \wedge d^c \beta = 0.$$  

Thus $d^c \beta = d\gamma$. To prove that $d\gamma = 0$ we act by $d^\dagger$ on both sides of this formula to obtain $d^\dagger d\gamma = 0 \implies d\gamma = 0$. Indeed,

$$0 = \langle \Gamma, d^\dagger d\gamma \rangle = \langle d\gamma, d\gamma \rangle \geq 0 \implies d\gamma = 0.$$  

So always $d^c \beta = 0$ and $d^\dagger \beta = 0$, hence $d \ast \beta = d^c \ast \beta = 0$. Now we can write $\ast \beta = h_{\ast \beta} + d\chi$ where $\ast \beta$ is harmonic. The harmonic component can be disregarded, in fact, since $\beta$ itself is defined modulo harmonic forms. So finally $\ast \beta = d\chi$; then it follows from the $dd^c$-lemma that $\ast \beta = dd^c \gamma$ and $\beta = d^c d^\dagger \ast \gamma$. The decomposition (B.2) gives $x = h_x + dd^c t_x$ and the lemma is proved.
Appendix C. Hodge foliation on the space of Kähler forms.

Let us consider the infinite-dimensional linear space $K_{1,1}(M)$ of Kähler forms on $M$. There is a natural distribution on $K_{1,1}(M)$ with a fiber at point $\omega \in K_{1,1}(M)$ defined as a space $H_\omega$ of harmonic forms with respect to $\omega$. Here we want to prove that this distribution is integrable — i.e. produces a foliation of $K_{1,1}(M)$ by finite-dimensional leaves. Each leaf is, loosely speaking, a lift to $K_{1,1}(M)$ of the cohomological Kähler cone $K$.

In particular, this fact implies that one can introduce the coordinates on $K_{1,1}(M)$ such that $n = \dim H_{1,1}(M)$ coordinate lines always have the harmonic tangent vectors $\partial_a$ (we use the canonical identification $K_{1,1}(M) \cong T K_{1,1}(M)$).

To prove the statement we check the Frobenius integrability condition for a pair of vector fields $\xi$ and $\eta$ such that $\xi(\omega)$ and $\eta(\omega)$ are harmonic forms for all $\omega \in K_{1,1}(M)$. Let us introduce (global) coordinates on $K_{1,1}(M)$ using the Hodge decomposition with respect to some particular $\omega_0$. It suffices to show that the commutator $[\xi, \eta](\omega_0)$ is harmonic with respect to $\omega_0$. For the infinitesimal variation $\omega = \omega_0 + \delta \omega$ one can write:

$$\xi(\omega) = \xi(\omega_0) - s(\delta \omega, \xi(\omega_0)) + h,$$

where $h$ is harmonic w.r.t. $\omega_0$ and the operation $s(\cdot, \cdot)$ is defined as $s(A,B) = P_{\text{Ker}(d)} m(A,B) = m(A,B) - \Pi(AB)$ (cf. (4.6)). The relation (C.1) follows from the properties of $m(A,B)$ discussed above. Indeed, $\xi(\omega)$ is $d$-closed ($\iff (A.10)$) and annihilated by $d^1$ ($\iff (A.12)$) modulo $o(\delta \omega^2)$ terms. Therefore, the commutator $[\xi, \eta]$ equals

$$[\xi, \eta](\omega_0) = s(\xi(\omega_0), \eta(\omega_0)) - s(\eta(\omega_0), \xi(\omega_0)) + h_{\eta,\xi} = h_{\eta,\xi}$$

i.e. it is harmonic.

In the tangent bundle $T L$ to the leaf $L$ one can introduce a connection

$$D_\xi = \partial_\xi - s(\xi, \cdot).$$

Arguments similar to ones in the Section 4.2 show that $D_\xi$ is flat. The connection $D_\xi$ is not a metric one. The connection compatible with the Riemann metrics

$$\langle \eta | \xi \rangle = \frac{1}{\text{Vol}_\omega} \int (\ast \eta \wedge \xi)$$

on (real) forms is given by $\partial_\xi - \frac{1}{2} s(\xi, \cdot)$. Unfortunately, this is not a connection on $T L$ since the parallel transport by it does not send harmonic forms to harmonic ones. Then, though the metrics connection descends to cohomology as a connection in the tangent bundle to the cohomological Kähler cone, it is not flat.
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