Brewster anomaly in random anisotropic media

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Anderson localization and the Brewster anomaly phenomenon, which is the delocalization of $p$-polarized waves at a special incident angle, in randomly-stratified anisotropic media are studied theoretically for two different random models. In the first case, the random parts of the transverse and longitudinal components of the dielectric tensor are assumed to be uncorrelated, while, in the second case, they are proportional to each other. The longitudinal component is the one in the stratification direction. We calculate the localization length numerically using the invariant embedding method. In addition, we derive the existence condition for the Brewster anomaly and concise analytical expressions for the localization length, which are accurate in the weak disorder limit. We find that the Brewster anomaly can occur only in the second case with positive ratio of the random parts. The incident angle at which the anomaly occurs depends sensitively on the ratio of the random parts and the average values of the tensor components. In the cases where the critical angle of total reflection exists, the angle at which the anomaly occurs can be either bigger or smaller than the critical angle. When the transverse and longitudinal components are uncorrelated, localization is dominated by the the transverse component at small incident angles and by the longitudinal component at large incident angles.

I. INTRODUCTION

Even though it has been studied extensively for over half a century, Anderson localization of quantum particles and classical waves continues to attract the interest of many researchers [1–4]. In this paper, we are interested in the localization of electromagnetic waves in randomly-stratified anisotropic media, which can be encountered frequently among both naturally-occurring media and fabricated metamaterials [5–8]. We focus especially on the unique phenomenon called Brewster anomaly (BA), which is the delocalization of $p$-polarized waves at a special incident angle $\theta$ [9–14]. Understanding the mechanism of this phenomenon is crucial in the development of polarization-insensitive reflectors and polarization-sensitive optical devices, as well as in understanding some bio-optical properties. Using a novel method based on the invariant embedding theory [15–18], we derive precise conditions for the occurrence of the BA in random anisotropic media and derive concise analytical expressions for the localization length in the weak disorder limit for two different random models. These results will be compared with more accurate numerical results obtained using the invariant embedding method and also with the previous results obtained for randomly-layered anisotropic media [6, 8].

II. MODEL

We consider a random uniaxial medium, the dielectric permittivity tensor of which is diagonalized in the coordinate system $(x, y, z)$ and is written as

$$
\epsilon = \begin{pmatrix}
\epsilon_\perp & 0 & 0 \\
0 & \epsilon_\perp & 0 \\
0 & 0 & \epsilon_\parallel
\end{pmatrix}.
$$

(1)

The medium is stratified along the $z$ axis and the transverse and longitudinal tensor components, $\epsilon_\perp$ and $\epsilon_\parallel$, are random functions of $z$ only. Plane electromagnetic waves of frequency $\omega$ and vacuum wave number $k_0 (= \omega/c)$ are assumed to propagate in the $xz$ plane. Then the wave equations for the $s$- and $p$-polarized waves are completely decoupled. For $p$ waves, the $y$ component of the magnetic field satisfies

$$
H_y'' - \frac{\epsilon_\perp}{\epsilon_\parallel} H_y' + \left( k_0^2 \epsilon_\perp - q^2 \epsilon_\parallel \right) H_y = 0,
$$

(2)

where $q$ is the $x$ component of the wave vector and a prime denotes a differentiation with respect to $z$.

We assume that an inhomogeneous anisotropic medium of thickness $L$ lies in $0 \leq z \leq L$ and the waves are incident obliquely from a uniform dielectric region ($z > L$) and transmitted to another uniform dielectric region ($z < 0$). The incident and transmitted regions are filled with an ordinary isotropic medium, where $\epsilon$ ($= \epsilon_\perp$) is a scalar quantity. When $\theta$ is the angle of incidence, $q$ is equal to $k \sin \theta$, where $k = k_0 \sqrt{\epsilon_\perp}$. From now on, we will assume that $\epsilon_\perp$ and $\epsilon_\parallel$ are always normalized by $\epsilon_1$ to simplify the notations.

We consider two random models. In Model I, we assume that $\epsilon_\perp$ and $\epsilon_\parallel$ are independent random functions of $z$ and satisfy

$$
\epsilon_\perp = a + \delta \epsilon_\perp(z), \quad \epsilon_\parallel = b + \delta \epsilon_\parallel(z),
$$

(3)

where $a$ and $b$ are the disorder-averaged values of $\epsilon_\perp$ and $\epsilon_\parallel$ and $\delta \epsilon_\perp(z)$ and $\delta \epsilon_\parallel(z)$ are Gaussian random functions satisfying

$$
\langle \delta \epsilon_\perp(z) \delta \epsilon_\perp(z') \rangle = \bar{g}_\perp \delta(z - z'), \quad \langle \delta \epsilon_\perp(z) \rangle = 0,
$$

$$
\langle \delta \epsilon_\parallel(z) \delta \epsilon_\parallel(z') \rangle = \bar{g}_\parallel \delta(z - z'), \quad \langle \delta \epsilon_\parallel(z) \rangle = 0.
$$

(4)

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The notation \( \langle \cdots \rangle \) denotes averaging over disorder and \( \delta \) and \( \gamma \) are independent parameters characterizing the strength of disorder. In Model II, we consider the situation where the random components \( \delta \) and \( \gamma \) are independent, but proportional to each other such that \( \delta = f \delta \), where \( f \) is a real constant.

### III. INVARIANT IMBEDDING METHOD

Since the BA occurs only for \( p \) waves, we focus on that case here. We consider a \( p \) wave of unit magnitude incident on the anisotropic medium. Using the invariant imbedding method and starting from Eq. (2), we derive exact differential equations satisfied by the reflection and transmission coefficients, \( r \) and \( t \):

\[
\frac{1}{ip} \frac{dr}{dl} = 2r + \frac{1}{2} \left( \sec^2 \theta - \epsilon - \frac{\tan^2 \theta}{\epsilon} \right) (1 + r)^2, \\
\frac{1}{ip} \frac{dt}{dl} = \epsilon + \frac{1}{2} \left( \sec^2 \theta - \epsilon - \frac{\tan^2 \theta}{\epsilon} \right) (1 + r) t,
\]

where \( p = k \cos \theta \) is the negative \( \epsilon \) component of the wave vector in the incident and transmitted regions. We use Eq. (5) in calculating the localization length \( \xi \) defined by

\[
\xi = - \lim_{L \to \infty} \left( \frac{L}{\ln T} \right),
\]

where \( T \) is the transmittance given by \( T = |t|^2 \).

The invariant imbedding equations for \( r \) and \( t \), Eq. (5), are stochastic differential equations with random coefficients. In order to handle the random terms appearing in the denominators of the coefficients in Eq. (5), we need to assume that the disorder in \( \epsilon \) is sufficiently weak so that

\[
\frac{1}{\epsilon} = \frac{1}{b + \delta} \approx \frac{1}{b} - \frac{\delta}{b^2}.
\]

We point out that this is the only approximation used in the present work.

#### A. Model I

In order to obtain the localization length, we need to compute the average \( \langle \ln T(L) \rangle \) in the \( L \to \infty \) limit. The nonrandom differential equation satisfied by \( \ln T \) can be obtained using the second of Eq. (5) and Novikov’s formula \[17\] and takes the form

\[
- \frac{1}{k} \frac{d(\ln T)}{dl} = C_1 + \text{Re} \left( ic_0 - 2ic_2 \right) Z_1 + C_1 Z_2,
\]

where \( Z_n \) \((n = 1, 2)\) is equal to \( \langle \epsilon^n \rangle \) and the parameters \( C_0 \), \( C_1 \) and \( C_2 \) are defined by

\[
C_0 = a + \frac{\tan^2 \theta}{b} - \sec^2 \theta.
\]

The dimensionless disorder parameters \( g \) and \( \gamma \) are given by

\[
C_1 = g_\parallel \cos^2 \theta + g_\perp \frac{\tan^2 \theta \sin^2 \theta}{b^2}, \\
C_2 = g_\parallel \cos^2 \theta - g_\perp \frac{\tan^2 \theta \sin^2 \theta}{b^2}.
\]

The initial conditions for \( Z_n \)’s are \( Z_0 = 1 \) and \( Z_n(l = 0) = 0 \) for \( n > 0 \). In the \( l \to \infty \) limit, the left-hand sides of these equations vanish and we obtain an infinite number of coupled algebraic equations. The moments \( Z_n \) with \( n > 0 \) are coupled to one another and their magnitudes decrease rapidly as \( n \) increases. Based on this observation, we solve these equations numerically by a systematic truncation method \[18\].

#### B. Model II

In Model II, \( \delta \) and \( \gamma \) are independent, but proportional to each other. This condition leads to completely different equations for \( Z_n \) and \( \langle \ln T \rangle \) for \( p \) waves. The equation for \( Z_n \) in this case is written as

\[
\frac{1}{k} \frac{dZ_n}{dl} = \ln \cos \theta \left( a - \frac{\tan^2 \theta}{b} + \sec^2 \theta \right) Z_n - \frac{i}{2} nC_0 \left( Z_{n+1} + Z_{n-1} \right) - g_\perp \left[ 3 \left( 1 + \frac{f^2}{b^2} \tan^2 \theta \right) \cos^2 \theta + 2 \frac{f^2}{b^2} \sin^2 \theta \right] n^2 Z_n + (2n + 1) nD_2 Z_{n+1} + (2n - 1) nD_2 Z_{n-1} - \frac{1}{2} n(n + 1) nD_1 Z_{n+2} - \frac{1}{2} n(n - 1) nD_1 Z_{n-2},
\]

where the parameters \( D_1 \) and \( D_2 \) are defined by

\[
D_1 = \gamma \left( 1 - \frac{f^2}{b^2} \tan^2 \theta \right)^2 \cos^2 \theta, \\
D_2 = \gamma \left( 1 - \frac{f^2}{b^2} \tan^2 \theta \right) \cos^2 \theta.
\]
The equation for the localization length takes the form
\[ -\frac{1}{k} \frac{d\ln T}{dl} = D_1 + \text{Re} \left[ (iC_0 - 2D_2) Z_1 + D_1 Z_2 \right]. \quad (13) \]

IV. EXISTENCE OF THE BREUWER ANOMALY

Next, we derive precise conditions for the occurrence of the BA. We consider our medium as consisting of a large number of very thin layers. The reflection coefficient between two neighboring layers is written as
\[ r = \frac{p/a - p'/a'}{p/a + p'/a'}, \tag{14} \]
where \( p \) (\( p' \)) is the negative \( z \) component of the wave vector in the first (second) layer with the parameters \( a \) and \( b \) (\( a' \) and \( b' \)). \( p \) satisfies \( p^2 = k^2 a - q^2 a/b \). We suppose that the wave is delocalized at an incident angle \( \theta_B \). In order for that to occur, the random variation of \( a \) and \( b \) should not cause any reflection, therefore we have the condition \( p/a = p'/a' \). We write \( a' \) and \( b' \) as \( a' = a + \delta a \) and \( b' = b + \delta b \), with \( \delta a \) and \( \delta b \) as small quantities. Substituting these into \( p/a = p'/a' \) and using a Taylor expansion, we obtain
\[ \left( 1 - \frac{\sin^2 \theta_B}{b} \right) \delta a = \left( \frac{a}{b^2} \sin^2 \theta_B \right) \delta b, \tag{15} \]
which implies that \( \delta b \) has to be proportional to \( \delta a \). Therefore, only Model II can show the BA. If we define \( \delta b = f \delta a \), the condition for the BA becomes
\[ \sin \theta_B = \left( \frac{b^2}{b + af} \right)^{1/2}. \tag{16} \]
In the case of isotropic media with \( f = 1 \) and \( b = a \), this reduces to the well-known result, \( \sin \theta_B = \sqrt{a/2} \) [9].

The same conclusions can be deduced from the expressions for the localization length, Eqs. (8) and (13). In one dimension, waves are localized in the presence of even an infinitesimally weak randomness, except for in some special cases. The fact that a \( p \) wave is delocalized at \( \theta = \theta_B \) implies that disorder does not play any role in the wave propagation process and the reflection coefficient \( r \) is the same as the value in the absence of disorder, \( r_0 \), given by
\[ r_0 = \frac{\sqrt{a} \cos \theta - [1 - (\sin^2 \theta)/b]^1/2}{\sqrt{a} \cos \theta + [1 - (\sin^2 \theta)/b]^{1/2}}. \tag{17} \]

By substituting \( Z_1 = r_0 \) and \( Z_2 = r_0^2 \) into Eq. (13), we find that \( \xi \) diverges when
\[ (r_0 - 1)^2 = \left( \frac{f}{b^2} \right) (r_0 + 1)^2 \tan^2 \theta, \tag{18} \]
from which we conclude that only Model II with \( f > 0 \) can display the BA.

V. ANALYTICAL EXPRESSIONS FOR THE LOCALIZATION LENGTH IN THE WEAK DISORDER REGIME

Starting from Eqs. (8), (10), (11) and (13), it is possible to derive analytical expressions for the localization length in the weak disorder limit. We write \( r \) as \( r = r_0 + \delta r \). From numerical calculations, we have verified that \( \langle \delta r \rangle \) and \( \langle (\delta r)^2 \rangle \) are of the first order in disorder, while \( \langle (\delta r)^3 \rangle \) is of the second order, except at incident angles close to the critical angle. From this consideration, we substitute
\[ Z_1 = r_0 + \langle \delta r \rangle, \quad Z_2 = r_0^2 + 2r_0\langle \delta r \rangle + \langle (\delta r)^2 \rangle, \]
\[ Z_3 \approx r_0^3 + 3r_0^2\langle \delta r \rangle + 3r_0\langle (\delta r)^2 \rangle \tag{19} \]
into Eq. (10) in the \( l \to \infty \) limit when \( n = 1 \) and 2 and obtain two coupled equations for \( \langle \delta r \rangle \) and \( \langle (\delta r)^2 \rangle \). We solve them analytically and substitute the results into Eq. (8) to the leading order in the disorder parameters. The expression for the localization length for Model I is
\[ (k\xi)^{-1} = 2\sqrt{w} \Theta(w) + g_{||} \frac{b - \sin^2 \theta}{ab} \]
\[ + g_{||} \frac{a \sin^4 \theta}{b^3 (b - \sin^2 \theta)}, \tag{20} \]
where \( w = a \left[ (\sin^2 \theta)/b - 1 \right] \) and \( \Theta \) is the step function, \( \Theta(x) = 1 \) for \( x > 0 \) and \( 0 \) for \( x < 0 \). Similarly, we obtain the localization length for Model II as
\[ (k\xi)^{-1} = 2\sqrt{w} \Theta(w) \]
\[ + g_{\perp} \frac{[b (b - \sin^2 \theta) - f a \sin^2 \theta]^2}{ab^3 (b - \sin^2 \theta)}. \tag{21} \]

We have found numerically that both of these equations are quite accurate when the disorder parameters are sufficiently small, except near the region where \( w = 0 \). In the isotropic case with \( f = 1 \) and \( b = a \), the second term of Eq. (21) reduces to \( g_{\perp} (a - 2\sin^2 \theta)^2/[a^2 (a - \sin^2 \theta)] \) derived in Ref. 9.

VI. NUMERICAL RESULTS

In Fig. 1, we show the normalized localization length, \( k\xi \), as a function of the incident angle for Model I, when \( a = 2 \) and \( b = \pm 1.5 \). We note that the case with \( a > 0 \) and \( b < 0 \) corresponds to a type I hyperbolic medium [19]. We consider three cases, where only \( \epsilon_{\perp} \) is random, only \( \epsilon_{||} \) is random and both \( \epsilon_{\perp} \) and \( \epsilon_{||} \) are random. In the first case, \( \xi \) increases (decreases) monotonically as \( \theta \) increases when \( b = 1.5 \) (\( b = -1.5 \)), while, in the second case, it diverges as \( \theta = 0 \) and decreases monotonically as \( \theta \) increases for both \( b = \pm 1.5 \). The third case is a combination of the first two cases.

These behaviors can be readily understood from the form of the function \( \eta \) given by
\[ \eta = 1 - a - \delta\epsilon_{\perp} + \left( \frac{a}{b} + \frac{1}{b} \delta\epsilon_{\perp} - \frac{a}{b^2} \delta\epsilon_{||} \right) \sin^2 \theta, \tag{22} \]
the BA if $a > 1$ and $b > 1$, which is also in consistence with the original BA. The BA can occur for both $a > 0$ and $a < 0$, with the numerical results obtained using the invariant imbedding results are compared with those obtained from Eq. (21). The discrepancy between the two is visible only in the case of Fig. 2(b) [2(c)], where the random variations of $c_\parallel$ and $c_\perp$ are directly proportional to each other with a positive ratio, in consistence with our results.

Next, we consider the situation where $0 < b < 1$. There exists a critical angle of total reflection, $\theta_c$, given by $\sin \theta_c = \sqrt{b}$. Then the BA can occur for both $a > 1$ and $a < 1$. The BA is also observed in the case of Fig. 2(a) of Ref. 6, where the longitu-
dinal component of the refractive index is uniform and $\theta$ will diverge, as $\theta$ approaches $90^\circ$. This case corresponds to that shown in Fig. 1(a) of Ref. 6, where the longitudinal component of the refractive index is uniform and matched to that of the surrounding medium. In the case of Fig. 1(b), the strength of the $\delta c_\perp$ term increases from zero monotonically as $\theta$ increases from zero, regardless of the sign of $b$, which is again in consistence with the behavior of $\xi$. When only $c_\parallel$ is random, all normally incident waves are delocalized. We find that localization is dominated by the randomness of $c_\perp$ at small incident angles and by that of $c_\parallel$ at large incident angles. The nonmonotonic behavior of $\xi$ shown in Fig. 1(c) when $a = 2$ and $b = 1.5$ is qualitatively similar to that shown in Fig. 2(b) of Ref. 6. We point out that the system called mixed stack in Ref. 6 corresponds to Model I and will not show the BA, while that called binary stack can show it.

In Fig. 2, we plot $k\xi$ versus $\theta$ for Model II, when $a = 2$, $b = 1.5$ and $c_\perp = 0.1$, for various values of $f$. The angle $\theta_B$ defined by Eq. (11) exists when $0 < b^2/(b + af) < 1$ and $f > 0$. In the case of Fig. 2(a) [2(b)], we get the BA if $f > 0.375$ ($f > 1.875$), in a perfect agreement with the numerical results. The entire curves as well as
and \(a < -1\) cases. In Fig. 3, we plot \(k\xi\) versus \(\theta\) for Model II, when \(a = \pm 2\), \(b = 0.5\) and \(g_\perp = 0.01\), for various values of \(f\). In the case of Fig. 3(a), the BA is possible for any value of \(f > 0\). In the case of Fig. 3(b), it is possible only if \(0 < f < 0.125\). We note that \(\theta_B < \theta_c = 45^\circ\) if \(a > 0\), while \(\theta_B > \theta_c\) if \(a < 0\). Interestingly, in the corresponding non-disordered case with \(g_\perp = 0\), the ordinary Brewster angle, \(\theta_b\), which is given by \(\sin \theta_b = [b(1 - a)/(1 - ab)]^{1/2}\), exists only when \(a\) is negative. When \(a = -2\) and \(b = 0.5\), \(\theta_b\) is equal to 60\(^\circ\) (> \(\theta_c\)) and has no direct relationship to \(\theta_B\). When \(a = 2\) and \(b = 0.5\), no Brewster effect occurs in the clean case, still the BA can occur at an angle smaller than \(\theta_c\) in the random case.

VII. CONCLUSION

In conclusion, we have studied Anderson localization and the BA of electromagnetic waves in random anisotropic media theoretically. We have derived the existence condition for the BA and analytical expressions for the localization length and elucidated several interesting physical aspects. Our results can provide valuable insights in understanding the unique properties of some biological reflectors and designing novel photonic devices based on anisotropic media [20].

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