Anisotropy of spin splitting and spin relaxation in lateral quantum dots

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Inelastic spin relaxation and spin splitting εs in lateral quantum dots are studied in the regime of strong in-plane magnetic field. Due to both g-factor energy dependence and spin-orbit coupling εs demonstrates a substantial non-linear magnetic field dependence similar to that observed by R.Hanson et al [Phys. Rev. Lett. 91, 196802 (2003)]. It also varies with the in-plane orientation of magnetic field due to crystalline anisotropy of spin-orbit coupling. Spin relaxation rate is also anisotropic, the anisotropy increasing with the field. When the magnetic length is less than the ‘thickness’ of GaAs dot, the relaxation can be order of magnitude faster for B[[100] than for B[[110].

Proposals to use electronic spin in quantum dots for quantum information processing have fuelled extensive studies of spin-orbit (SO) coupling in heterostructures as means to manipulate the electron spin and as a source of spin relaxation. Recent theories2,14–16 and experiments3,4,5 suggest that spin relaxation in quantum dots is strongly suppressed by electron confinement but may be sped-up by a magnetic field, in particular, by the field parallel to the plane of the lateral structure.

It is customary to assume that an in-plane magnetic field couples only to spin of the electron. In this approximation one can describe spin relaxation in terms of effective two-dimensional (2D) SO coupling2,14,16. This approach can be justified provided that λB ≫ λs, where λB = 2mℏ/√(eB) is magnetic length and λs is the extent of the subband wave function across the 2D plane. In the opposite limit that corresponds to a strong magnetic field, λB ≪ λs, subbands in a heterostructure transform into bulk Landau levels (magneto-subbands) thus changing parameters of the effective 2D motion. This effect has been observed in optical and FIR spectroscopy of low-density GaAs/AlGaAs heterostructures14, resonant tunnelling in double-barrier devices15, and in quantum transport characteristics of lateral dots16.

In this Letter, we propose a theory of the spin relaxation of electrons in lateral dots in a strong in-plane magnetic field. The field effect on the orbital electron motion transforms states in low-density heterostructures into magneto-subbands. We take into account this crossover as well as the Dresselhaus-type spin-orbit coupling in GaAs11. We show that at high fields both the inelastic spin-flip time T1 at low temperatures kT ≪ εs and the electron spin splitting εs depend on the magnetic field orientation with respect to crystallographic axes, which can be used to distinguish the SO coupling-induced effects from those caused by a hyperfine interaction with nuclei. We also present analytical description of εs and T1 dependences on both the magnitude and direction of the magnetic field.

The Hamiltonian of electrons in a dot made of a lateral GaAs/AlGaAs structure grown in direction lz = [001] can be written as

\[ \hat{H}_{3D} = \frac{\hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2}{2m} + V(\mathbf{r}) + \frac{g \mu_B}{2} \sigma_X + \hat{H}_{so}, \]

\[ \hat{H}_{so} = \gamma \hbar^{-3} \sum_{kij=x,y,z} \epsilon^{ijk} \hat{p}_i \hat{p}_j \sigma_j \hat{p}_i. \]  (1)

In Eq. (1) we use two systems of in-plane coordinates. Axes x and y (used in the SO coupling term \( \hat{H}_{so} \)) are determined by cristallographic directions [100] and [010], respectively. Axis Z is directed along the in-plane field B = lX B, with lX = (lX, 0, 0) and Y along lY = (−lY, lY, 0). In Eq. (1) the kinematic, \( p_\alpha = -i \partial_\alpha \) and canonical, \( \hat{p}_\alpha \) momenta are written in the coordinate system X and Y. We use the Landau gauge A = −(z − a)BlY, so that \( \hat{p}_z = p_z \), \( \hat{p}_X = p_X \) and \( \hat{p}_Y = p_Y − \frac{\hbar}{2} B (z − a) \), with a to be specified later.

In the spin part of \( \hat{H}_{3D} \), \( \mu \) is Bohr magneton, g-factor in GaAs is \( g ≈ −0.44 + \frac{4}{3} g' \), \( \epsilon^{ijk} \) is the antisymmetric tensor, and SO coupling constant \( \gamma \) according to Refs. 14,15,16 is \( \gamma = (26 \pm 6) eV \). \( \hat{H}_{so} \) in Eq. (1) is written in the form which guarantees that it is Hermitian, despite the non-commutativity of operators \( \hat{p}_x \) and \( \hat{p}_y \) with \( \hat{p}_z \).

The dot is formed by a potential profile \( V = V_z(z) + \frac{1}{2} m \omega_z^2 z^2 + \frac{1}{2} m \omega_x^2 (X^2 + Y^2) \), which is stronger in the heterostructure growth direction l_z than within the XY plane. We consider two particular cases: triangular well \( V_z(z) = \frac{1}{2} m \omega_z^2 z^2 \) and parabolic well \( V_z(z) = \frac{1}{2} m \omega_z^2 (z − a)^2 \). The wave functions \( \psi_n(p_x, p_Y) \) of electrons in the \( n \)-th magneto-subband and their 2D dispersion, \( \varepsilon_n(p_Y) = \frac{\hbar}{2m} \rho_0^2 \), is determined by

\[ \hat{H}_z = \frac{\hbar^2}{2m} \rho_0^2 \hat{p}_z^2 + V_z(z) + \frac{1}{2} m \omega_z^2 (z − a)^2 \]

(2)

whereas the parameter \( a \) is chosen in such a way that the lowest subband dispersion \( \varepsilon_0(p_Y) \) has minimum at \( p_Y = 0 \). This defines \( z = z − a \) and \( m \omega_z^2 = \hbar^2 / \rho_0^2 \). For a triangular well, we describe magneto-subbands using the function of a parabolic cylinder17,18 and evaluate \( a \) and \( m \alpha_Y = \eta m \) numerically. For a parabolic well, harmonic oscillator functions give \( a = 0 \) and \( \eta = 1 + \omega_z^2 / \omega_x^2 \).

Since electron confinement across the plane is much stronger than in lateral directions, \( p_x, p_Y \ll p_z \), we substitute \( \hat{p}_Y = p_Y − \frac{\hbar}{2} B z = p_Y − m \omega_z^2 z \), \( \hat{p}_z = p_z \).
\( t_x p_x - l_y p_y + l_y m \omega_c \hat{z} \) and \( \hat{p}_x = l_x p_x + l_x p_y - l_x m \omega_c \hat{z} \) into \( \hat{H}_{2D} \), expand it in powers of kinematic momenta \( p_x \) and \( p_y \) (up to quadratic terms) and derive the effective 2D Hamiltonian \( \hat{H}_{2D}(p_x, p_y, \sigma) \). In particular, when analysing SO coupling, we expand \( \hat{H}_{so} \) up to linear order in \( p_y \) and \( p_x \),

\[
\hat{H}_{so} \approx \hat{H}_{so}^0 + \hat{H}_{so}^1, \quad \text{where}
\]

\[
\frac{\hat{H}_{so}^0}{\gamma} = l_x l_y \left[ 2 \frac{\hat{p}_x \hat{p}_y}{\hbar^2 \lambda_B^2} - \frac{\hat{z}_3}{\lambda_B} \right] \sigma_x \\
+ \left( l_x^2 - l_y^2 \right) \left[ \frac{\hat{p}_x^2}{\hbar^2} - \hat{z}_3 \frac{\hat{p}_y}{\hbar^2} \sigma_y \right] \\
+ \left( l_y^2 - l_x^2 \right) \left[ \frac{\hat{p}_y^2}{\hbar^2} - \hat{z}_3 \frac{\hat{p}_x}{\hbar^2} \sigma_x \right]
\]

\[
\frac{\hat{H}_{so}^1}{\gamma} = (l_x^2 - l_y^2) \left[ \left( \frac{\hat{p}_x^2}{\hbar^2} - \hat{z}_3 \frac{\hat{p}_y}{\hbar^2} \right) \sigma_x + \frac{\hbar^2 \lambda_B^2}{p_x \sigma_Y} - \frac{\hat{p}_x^2 \sigma_Y}{h} \right] \\
- l_x l_y \left( \frac{\hat{p}_x^2}{\hbar^2} + \left( \frac{\hat{p}_y^2}{\hbar^2} - \hat{z}_3 \frac{\hat{p}_x}{\hbar^2} \sigma_x \right) \right) \\
- \left( l_y^2 - l_x^2 \right) \left[ \left( \frac{\hat{p}_y^2}{\hbar^2} - \hat{z}_3 \frac{\hat{p}_x}{\hbar^2} \sigma_x \right) \sigma_y + \frac{\hbar^2 \lambda_B^2}{p_y \sigma_X} - \frac{\hat{p}_y^2 \sigma_X}{h} \right]
\]

In both \( \hat{H}_{so}^0 \) and \( \hat{H}_{so}^1 \) the last term does not contribute to the effective 2D Hamiltonian: for magneto-subbands determined by \( \hat{H}_z \) in Eq. (6) \( \langle 0, p_x, p_y | \hat{z}_3 \hat{p}_2 | 0, p_x, p_y \rangle = 0 \) and \( \langle 0, p_x, p_y | \hat{z} \hat{p}_2 + p_z | 0, p_x, p_y \rangle = 0 \).

The first term in \( \hat{H}_{so}^0 \) yields an anisotropic addition to the 2D electron spin splitting linear in \( \gamma \). The second term slightly turns the spin quantization axis off the magnetic field \( \mathbf{B} = 1 \mathbf{x} \mathbf{B} \). It can be neglected as long as we restrict ourselves to the lowest order in \( \gamma \). Thus,

\[
\varepsilon = g \mu_B - l_x l_y \gamma \lambda^{-3} A_s,
\]

\[
ge \approx -0.44 + \langle 0 | p_x^2 | 0 \rangle g', \quad \lambda = (h^2/mF)^{1/3},
\]

\[
A_s = \frac{\lambda^3 \langle 0 | \hat{z}^3 | 0 \rangle}{\lambda_B^6} - 2 \frac{\lambda^3 \langle 0 | \hat{z} \hat{p}_2 | 0 \rangle}{\lambda_B^6 h^2}.
\]

The anisotropy of spin splitting is crucially sensitive to the inversion asymmetry of the confinement potential \( V_z \), thus it is a peculiarity of heterostructures. The anisotropy effect in Eq. (1) is maximal in a field oriented along crystallographic directions \([110]\) or \([1\bar{1}0]\). The field dependence of the anisotropic part of spin splitting is characterised by the parameter \( A_s \). In a weak magnetic field, \( \omega_c h < \varepsilon_1 - \varepsilon_0 \), perturbation theory analysis gives \( A_s \approx 2.46 m \lambda^2 / \omega_c = 3.42 \omega_c h / (\varepsilon_1 - \varepsilon_0) \) leading to the anisotropy of linear \( g \)-factor. The field dependence \( A_s(B) \) at high fields is shown in Fig.1(a). For GaAs/AlGaAs heterostructure with \( \lambda_c \approx 100 \AA \) and \( \gamma \approx (26 \pm 6) eV \AA^3 \), Eq. (1) predicts that the spin splitting \( \varepsilon \) is modulated by about 10\% for different orientations of the magnetic field. \( \varepsilon \) also includes an isotropic non-linear \( B \)-dependent part due to the g-factor dependence on the electron momentum, \( \langle 0 | p_x^2 | 0 \rangle \sim \text{Beh}/2c \),

\[
\Theta = -\hbar \lambda^{-1}_x (\kappa - \xi), \quad \Theta_y = -\hbar \lambda^{-1}_y (2\kappa + \xi),
\]

\[
\Gamma_x = -\hbar \lambda^{-1}_x (2\kappa - 3\zeta), \quad \Gamma_y = -\hbar \lambda^{-1}_y \kappa \xi,
\]

\[
\lambda^{-1} = \frac{g \mu_B}{h \lambda^2}, \quad \kappa = (0 | p_x^2 | 0), \quad \xi = (0 | \hat{z}^2 | 0).
\]

Fig. 1(b) shows how \( \Theta \) and \( \Gamma \) depend on the magnetic field for a triangular well \( V_z = Fz \) with \( \lambda_c = \)
\( (\hbar^2/m_F)^{1/3} \). At high fields these dependences are similar to what we found for a parabolic well \( V_z(z) = \frac{1}{2}m\omega^2z^2 \) with \( \lambda_z = \sqrt{2\hbar/m\omega} \),

\[
\Theta_X = \frac{1}{\vartheta} \Theta_Y = \frac{1}{\vartheta} - 3\vartheta; \Gamma_X = \vartheta^3 - 3\vartheta; \Gamma_Y = -\vartheta^3
\]

where \( \vartheta = \sqrt{1 + \omega_c^2/\omega^2} = \lambda_2eB/2\hbar c \) at \( \omega_c \gg \omega \). This similarity implies that at high field we can approximate \( \Theta, \Gamma, \) and \( m_Y \) in heterostructures by their values obtained for a parabolic well with the same \( \lambda_z \).

Lateral orbital states described by \( \hat{H}_{2D} \) have the spectrum \( E_{MM'} = (M + \frac{1}{2})\hbar \vartheta + (M' + \frac{1}{2})\hbar \vartheta \eta^{-1/2} \). The lowest level wave function is \( \langle 0 | \rangle = \langle \pi\lambda\lambda_Y \rangle^{-1/2} \sqrt{x^2/\lambda^2 + y^2/\lambda^2} \varphi_0(z) \), where \( \lambda = \sqrt{2\hbar/m\vartheta} \) and \( \lambda_Y = \eta^{-1/4} \lambda \). Here, \( \vartheta \) and \( \eta^{-1/2} \vartheta \) are the frequencies of electron harmonic oscillations along the \( X \) and \( Y \) axes, respectively, and the dot states \( | n \rangle = | M, M' \rangle \) are characterized by quantum numbers \( M \) and \( M' \).

The rate of the phonon-assisted spin flip \( | 0, + \rangle \rightarrow | 0, - \rangle \) in the lowest order in both the e-ph interaction and SO coupling is

\[
T^{-1}_1 = \frac{2\pi}{\hbar} \int \frac{L^2dq}{(2\pi)^3!} A^2 \delta(\varepsilon_s - \hbar qs), \quad A = \sum_{n \neq 0} \left[ \frac{\langle 0 | W | n \rangle \langle n | Y \rangle | 0 \rangle}{E_0 - E_n + \varepsilon_s} + \frac{\langle 0 | Y | n \rangle \langle n | W \rangle | 0 \rangle}{E_0 - E_n - \varepsilon_s} \right]. \tag{6}
\]

Here, \( W(r, q) = w_0 e^{iqr}/L^{3/2} \) is the phonon field with \( L^3 \) being the normalisation volume for phonons. We choose

\[
|w_0|^2 = \frac{\beta^2}{q_s + \Xi q_s^2}, \quad \text{where} \quad q_s = \varepsilon_s/\hbar s.
\]

to take into account both piezoelectric (\( \beta \)) and deformation (\( \Xi \)) phonon potentials.

Operators \( h_{\text{so}}(\hat{\mathbf{p}}, \mathbf{r}) \) can be obtained using Eq. (5),

\[
\hat{\mathbf{\sigma}} \cdot h_{\text{so}}(\hat{\mathbf{p}}, \mathbf{r}) = -\frac{\mathbf{p} \hat{\sigma} \mathbf{A}}{m} - \frac{\mathbf{p} \hat{\sigma} \mathbf{A}}{m_Y}. \tag{7}
\]

As long as the orbital part of the Hamiltonian \( \hat{H}_{2D} \) remains T-invariant, the orbital eigenstates are real, and \( \langle 0 | e^{iqr} | n \rangle = \langle n | e^{iqr} | 0 \rangle \). Moreover, \( \langle n | h_{\text{so}} | 0 \rangle = -\langle 0 | h_{\text{so}} | n \rangle \) because spin operator \( \hat{\mathbf{\sigma}} \) changes sign under the \( t \rightarrow -t \) transformation whereas the product \( \hat{\mathbf{\sigma}} \cdot h_{\text{so}} \) remains the same (as a spin-orbit part of T-invariant \( \hat{H}_{2D} \)). Consequently, two terms in the amplitude of the phonon-emission-assisted spin-flip process in Eq. (6) cancel in the limit \( \varepsilon_s \rightarrow 0 \), and the transition amplitude reads

\[
A = 2w_s \sum_{M, M' \geq 1} \frac{\langle 0 | h_{\text{so}}^* | M, M' \rangle \langle M, M' | e^{iqr} | 0 \rangle}{\hbar \vartheta M + M' \hbar \vartheta \sqrt{m/m_Y}} - \varepsilon_s^2. \tag{8}
\]

Being generic for any T-invariant Hamiltonian\(^{19}\) such a cancellation should take place in all orders in \( p_X \) and \( p_Y \), hence it is sufficient to analyse \( A \) using only the linear in momentum SO coupling in \( \hat{H}_{2D} \).

In a parabolic dot operators \( p_X \) and \( p_Y \) couple the state \( | 0 \rangle = | 0, 0 \rangle \) only to states \( | 1 \rangle, | 1, 0 \rangle, \) and

\[
\langle 0, 0 | e^{iqr} | 1 \rangle \rightarrow \langle 1, 0 | e^{iqr} | 0 \rangle = \frac{i}{\hbar} q_s \lambda \Lambda,
\]

\[
\langle 0, 0 | e^{iqr} | 1 \rangle \rightarrow \langle 1, 0 | e^{iqr} | 0 \rangle = \frac{i}{\hbar} q_s \lambda Y \Lambda,
\]

\[
\Lambda = \langle 0 | e^{iqr} | 0 \rangle = f(q_s)e^{-\frac{i}{\hbar} (q_s \lambda \lambda_Y + q_s \lambda Y \lambda)} \tag{9}
\]

where \( f(q_s) = \int dze^{iqr \cdot z} \varphi_0(z)^2 \). As a result,

\[
A = \frac{w_s \lambda \lambda_Y}{2 \hbar} \sum_{s} \left\{ \frac{l_x l_y Y q_X}{1 - (\varepsilon_s/\hbar \vartheta)^2} \left( \frac{l_x^2 - l_y^2}{l_x^2 + l_y^2} \right) Y q_Y \right\} \tag{10}
\]

The angular distribution of the phonon emission is determined by the form-factor \( \Lambda(q) \) in Eq. (9). Depending on the magnetic field, the emitted phonon wavelength \( q_s = \varepsilon_s/\hbar s \) may fall into one of the following regimes:

A) \( q_s < \lambda^{-1} \);  B) \( \lambda^{-1} < q_s < \lambda^{-1} \);  C) \( \lambda^{-1} < q_s \).

In regime A, the phonon wavelength exceeds all dimensions of quantum dot. As a result, \( \Lambda \approx 1 \) and phonons are emitted isotropically. In regime B, most of phonons are emitted perpendicularly to the magnetic field direction, since the phonon wavelength is shorter than the lateral dot size \( \lambda \) in the direction of external field. Accordingly, \( e^{iqr \cdot z} \approx 1, e^{-iqr \lambda \lambda_Y/2} \approx 1, \) and \( 1/|\Lambda|^2 \approx e^{-iqr \lambda \lambda_Y/2} \). Finally, in the high-field regime C phonons are emitted across the heterostructure. Using the similarity between magneto-subband states\(^{19}\) and bulk Landau levels, we approximate \( f \approx \exp \left[ -(q_s \lambda_B/2)^2 \right] \) and \( |\Lambda|^2 \approx \exp \left[-(q_s \lambda/2)^2 -(q_s \lambda_Y/2)^2 \right] \).

The rate \( T^{-1}_1 \) can be evaluated\(^{20}\) using Eqs. (6), (10),

\[
T^{-1}_1 = \frac{\lambda \lambda_Y}{4\pi \hbar \vartheta s} \left( \frac{\varepsilon_s \lambda \lambda_Y}{\hbar \vartheta} \right)^2 Q \times \left\{ \frac{(l_x^2 - l_y^2)^2}{1 - (\varepsilon_s/\hbar \vartheta)^2} + \frac{(\alpha l_x l_y)^2}{1 - (\varepsilon_s/\hbar \vartheta)^2} \right\}, \tag{11}
\]

where \( w_s^2 = (\beta^2/q_s + \Xi q_s^2) \), while \( Q \) and \( \alpha \) are specific for each particular regime (A-C),

\[
Q_A = \frac{1}{T_Y^A} (\lambda_B s)^4, \quad \alpha_A = \Theta_Y / T_Y; \quad Q_B = \frac{\lambda_B s^3}{T_Y^A}, \quad \alpha_B = 2\Theta_Y / q_s \lambda_Y; \quad Q_C = (\lambda / \lambda_Y)^3 e^{-\frac{i}{\hbar} (q_s \lambda_B)^2}, \quad \alpha_C = \Theta_Y / q_s \lambda_Y \lambda.
\]

The factor in curly brackets in Eq. (11) determines the relaxation rate dependence on the magnetic field orientation. If the field is so weak that \( \lambda_s < \lambda_B \), then \( \Theta_Y < 2, \lambda \approx \lambda_Y, \alpha = 2, \) and \( T^{-1}_1 \) turns out to be isotropic. The anisotropy develops when \( \lambda_B \approx \lambda_s \) [i.e., \( \eta > 1 \),...
\( \lambda > \lambda_1 \) and \( \Theta_1 / \Gamma_1 < 2 \) and increases with the field. According to Eq. (14), at high fields where \( \lambda_B \ll \lambda_z \) but \( \varepsilon_x < h\theta / \eta \), spin relaxation faster in a magnetic field oriented along [100] or [010] and slower when \( \mathbf{B} \) is parallel to [110] or [110], which was the field orientation in the experiment in Ref. 22. In the field range where \( \lambda_B \ll \lambda_z \) and \( \varepsilon_x < h\theta / \eta \), spin-flip rate for those two orientations has power-law dependence \( 22, \quad T_{1}^{-1}(\mathbf{B}|[100]) \propto B^{11/2} \) and \( T_{1}^{-1}(\mathbf{B}|[110]) \propto B^{7/2} \).

The anisotropy in \( T_1 \) is strongly enhanced in the vicinity of crossing of the level \((0,+)\) with \([0,1,\cdots)\) or \([1,0,\cdots)\), though the divergence of \( T_{1}^{-1} \) in Eq. (14) at \( \varepsilon_x = h\theta / \eta \) and \( \varepsilon_x = h\theta / \eta \) is an artifact of the lowest-order perturbation theory analysis and it is prevented by level anti-crossing due to SO coupling. For \( \mathbf{B}|[100] \) or \( \mathbf{B}|[010] \), spin relaxation is resonantly sped-up when \( \varepsilon_x = h\theta / \eta \). For \( \mathbf{B}|[110] \) and \( \mathbf{B}|[110] \) the rate \( T_{1}^{-1} \) acquires a maximum at a higher field where \( \varepsilon_x = h\theta / \eta \). For samples used in Ref. 22, \( h\theta \approx 1\text{meV} \), thus the crossing of \((0,+)\) and \([1,0,\cdots)\) was beyond the experimental field range. However, for \( \lambda_z \approx 10\text{nm} \) and \( h\theta \approx 1\text{meV} \) the crossing of levels \((0,+)\) and \([0,1,\cdots)\) should enhance spin relaxation at \( B \approx 15 / 20 \text{T} \) if \( \mathbf{B}|[100] \) or \( \mathbf{B}|[010] \). The formula in Eq. (14) is not exact when \( h\theta / \eta \approx \varepsilon_x < h\theta / \eta \), nevertheless, in that field range the anisotropic behavior of \( T_{1}^{-1} \) persists, since the spin-flip for \( \mathbf{B}|[100] \) or \( \mathbf{B}|[010] \) is enhanced due to the opening of additional relaxation channel \( (0,+) \rightarrow (0,1,\cdots) \).

To conclude, we studied the effects of the spin-orbital coupling on the spin splitting \( \varepsilon_x \) and inelastic spin relaxation rate \( T_{1}^{-1} \) in lateral quantum dots at low temperatures \( kT \ll \varepsilon_x \). We found that \( \varepsilon_x \) demonstrates a sizeable non-linearity and anisotropy in its field-dependence, Eq. (14). The anisotropy in the spin relaxation, Eq. (14) is predicted to be even stronger: if the magnetic field \( \mathbf{B} \) is high, \( \lambda_B \ll \lambda_z \), the relaxation can be order of magnitude faster for \( \mathbf{B}|[100] \) than for \( \mathbf{B}|[110] \). The latter feature of the spin relaxation due to SO coupling can be used to distinguish it from the spin relaxation involving hyperfine interaction with nuclei.

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1. H.-A. Engel et al., Phys. Rev. Lett. 93, 106804 (2004); G. Burkard and D. Loss, Phys. Rev. Lett. 88, 047903 (2002)
2. A.Khaetskii and Y.Nazarov, Phys. Rev. B 61, 12639 (2000); Phys. Rev. B 64, 125316 (2001)
3. I.L. Aleiner and V.I. Fal’ko, Phys. Rev. Lett. 87, 256801 (2001); J.-H. Cremers, P.W. Brouwer, and V.I. Fal’ko, Phys. Rev. B 68, 125329 (2003)
4. V.Golovach, A.Khaetskii, D.Loss, Physic. Rev. Lett. 93, 016601 (2004)
5. R.Hanson et al., Phys. Rev. Lett. 91, 196802 (2003); J.M.Elserman et al., Nature 430, 431 (2004); J.M.Elserman et al. Appl. Phys. Lett. 84, 4617 (2004); M.Krountor et al., Nature 342, 81 (2004)
6. T.Fujisawa et al., Nature 419, 278 (2002)
7. D.M. Zumbühl et al., Phys. Rev. Lett. 89, 276803 (2002)
8. In heterostructures with \( 2 \times 10^{11} \text{electrons per cm}^2 \), the reconstruction of the 2D electron spectrum has been observed in the field range \( 5 / 7 \text{T} \), whereas at the electron density \( 10^{16} \text{cm}^2 \), one electron in a 1000Å-size quantum dot [10] magneto-subbands would already form at \( B \approx 2 / 3 \text{T} \).
9. I. Kukushkin et al., JETP Lett. 51, 436 (1990); B.Meurer, D.Heitmann, K.Ploog, Phys. Rev. B 48, 11488 (1993)
10. C. Kutter et al., Phys. Rev. B 45, 8749 (1992)
11. D.M. Zumbühl et al., Phys. Rev. B 69, 121305 (2004)
12. F.Malcher, G.Lommer, U.Rössler, Phys. Rev. Lett. 60, 729 (1988); R. Eppenga and M. Shuurnans, Phys. Rev. B 37, 10923 (1988)
13. M.J.Sneling et al, Phys. Rev. B 44, 11345 (1991); R.M.Hannak et al., Solid State Commun. 93, 3132 (1995)
14. B.Jussend et al., Phys. Rev. Lett. 69, 848 (1992); B.Jussend et al., Phys. Rev. B 51, 4707 (1995); L.Wissinger et al, Phys. Rev. B 58, 15375-15377 (1998)
15. J.B. Miller et al, Phys. Rev. Lett. 90, 076807 (2003); W. Knap et al, Phys. Rev. B 53, 3912-3924 (1996)
16. In low-density structures the Rashba coupling [Y.Bychkov and E.Rashba, JETP Lett. 39, 78 (1984)] is insignificant [12,14].
17. x_{\parallel} = l_x p_x - l_y p_y, \quad \tilde{\mathcal{p}}_x = l_y \tilde{p}_y + l_x \tilde{p}_x, \quad \sigma_x = l_x \sigma_x - l_y \sigma_y
18. \( \varphi_0 \approx \exp(\lambda / 2) / \lambda \). For \( B = 0 \), \( \kappa \approx 1.24 \lambda_0^2 \).
19. The contribution from the inter-subband part of the second term in \( H_{\nu} \) is negligible, unless g-factor was engineered to be anomalously small.
20. We take into account that \( \int_{-\varepsilon_x}^{\varepsilon_x} d\lambda \langle \hat{q}_x \hat{q}_x \rangle = 0 \) and approximate \( (\Delta / 2) \langle 2\pi \rangle \Delta^2 q \delta (h\omega - \varepsilon_x) \lambda^2 \approx \begin{array}{l}
\left( 1 - (x/q_{\Delta})^2 \right) \left( 1 - (y/q_{\Delta})^2 \right) \left( 1 - (z/q_{\Delta})^2 \right) \\
\left( 1 - (x/q_{\Delta})^2 \right) \left( 1 - (y/q_{\Delta})^2 \right) \left( 1 - (z/q_{\Delta})^2 \right) \\
\left( 1 - (x/q_{\Delta})^2 \right) \left( 1 - (y/q_{\Delta})^2 \right) \left( 1 - (z/q_{\Delta})^2 \right)
\end{array}
\)
21. L.Kouwenhoven, private communication.
22. In GaAs \( m_s \approx 10 \text{meV} \), \( g_{\text{m}}/2m_e \approx 1.5 \times 10^{-2} \) \( m_e \) is the electron mass in vacuum, and \( \Lambda^2 = (q_{\Delta}/2m_e)^2 \approx 2(m_e/2m_e)^2 (h\omega_e/m_s^2) \approx 1 \). The latter conclusion differs from that in Ref. 22 predicting \( T_{1}^{-1} \) to reach maximum at \( q_{\Delta} \approx \lambda_0^2 \). In Ref. 22 the orbital effect of in-plane field has been neglected.
23. D.Bulaev and D.Loss, cond-mat/0409613