DEMAZURE CRYSTALS OF GENERALIZED VERMA MODULES
AND A FLAGGED RSK CORRESPONDENCE

JAE-HOON KWON

ABSTRACT. We prove that the Robinson-Schensted-Knuth correspondence is a \( \mathfrak{g}_\infty \)-crystal isomorphism between two realizations of the crystal graph of a generalized Verma module with respect to a maximal parabolic subalgebra of \( \mathfrak{g}_\infty \). A flagged version of the RSK correspondence is derived in a natural way by computing a Demazure crystal graph of a generalized Verma module. As an application, we discuss a relation between a Demazure crystal and plane partitions with a bounded condition.

1. Introduction

The Robinson-Schensted-Knuth (simply RSK) algorithm \[15\] has been playing fundamental roles in combinatorics and representation theory with generalizations in various directions. It gives a bijection between the set \( M \) of \( \mathbb{N} \times \mathbb{N} \) matrices with non-negative integers of finite support and the set \( T \) of pairs of semistandard tableaux of the same shape, and explains in a bijective way the expansion of the Cauchy kernel into Schur functions, called the Cauchy identity:

\[
\prod_{i,j \geq 1} (1 - x_i y_j) = \sum_{\lambda} s_\lambda(X)s_\lambda(Y),
\]

where the sum is over all partitions \( \lambda \) and \( s_\lambda(X) \) (or \( s_\lambda(Y) \)) is the Schur function in \( X = \{ x_1, x_2, \ldots \} \) (or \( Y = \{ y_1, y_2, \ldots \} \)). A representation theoretic interpretation of the Cauchy identity can be given by a general principle called Howe duality \[7\], that is, \( S(\mathbb{C}^{>0} \otimes \mathbb{C}^{>0}) \), the symmetric algebra generated by \( \mathbb{C}^{>0} \otimes \mathbb{C}^{>0} \) has a multiplicity-free decomposition into irreducible \( (\mathfrak{gl}_{>0}, \mathfrak{gl}_{>0}) \)-bimodules parameterized by partitions, where \( \mathbb{C}^{>0} \) is the complex vector space with a basis \( \{ v_i \mid i \in \mathbb{N} \} \) and \( \mathfrak{gl}_{>0} = \mathfrak{gl}(\mathbb{C}^{>0}) \) is the corresponding general linear Lie algebra.

We have a more direct interpretation of the RSK map with the help of the Kashiwara’s crystal base theory of the quantum group \( U_q(\mathfrak{gl}_{>0}) \) \[10, 12, 14\]. That is, both \( M \) and \( T \) have two \( \mathfrak{gl}_{>0} \)-crystal structures commuting with each other, which are called \( (\mathfrak{gl}_{>0}, \mathfrak{gl}_{>0}) \)-bicrystals or double crystals \[20\], and the RSK map is an isomorphism of bicrystals. The decomposition as a \( (\mathfrak{gl}_{>0}, \mathfrak{gl}_{>0}) \)-bimodule follows

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immediately by considering highest weight crystal elements in $M$. We refer the readers to [4, 17, 20, 21, 22, 23] for more results on bicrystal, its application and generalization to other types of Lie algebras.

The main purpose of this paper is to give a new representation theoretic interpretation of the RSK correspondence and its applications. Let $\mathfrak{gl}_\infty$ be the general linear Lie algebra, which is spanned by the elementary matrices $E_{ij}$ ($i, j \in \mathbb{Z} \setminus \{0\}$). Let $l = \mathfrak{gl}_{<0} \oplus \mathfrak{gl}_{>0}$ be a Levi subalgebra, where $\mathfrak{gl}_{\leq 0}$ is a subalgebra spanned by $E_{ij}$ ($i, j \geq 0$), respectively, and let $u_\pm$ be the nilradical spanned by $E_{ij}$ for $i > 0, j < 0$ (resp. $i < 0, j > 0$). We may identify $S(\mathbb{C}^{>0} \otimes \mathbb{C}^{>0})$ with the enveloping algebra $U(u_\pm)$, which is a generalized Verma module induced from the maximal parabolic subalgebra $p = l \oplus u_+$ (see [3] for a quantized version of this fact and its relation with canonical basis). Motivated by this observation, we introduce $\mathfrak{gl}_\infty$-crystal structures on $M$ and $T$ extending the $(\mathfrak{gl}_{<0}, \mathfrak{gl}_{>0})$-bicrystal structures (note that $\mathfrak{gl}_{<0} \simeq \mathfrak{gl}_{>0}$), and then show that the RSK map is an isomorphism of $\mathfrak{gl}_\infty$-crystals (Theorem 3.6).

The RSK map also enables us to define a natural embedding of $\mathcal{B}(n\Lambda_0)$ into $M$, where $\Lambda_0$ is the 0th fundamental weight of $\mathfrak{gl}_\infty$ and $\mathcal{B}(n\Lambda_0)$ is the crystal graph of the irreducible $U_q(\mathfrak{gl}_\infty)$-module with highest weight $n\Lambda_0$ (Proposition 4.3). Hence we may regard $M$ as a crystal graph of the quantum group $U_q(u_-)$ since it can be realized as a limit of $\mathcal{B}(n\Lambda_0)$. In general, we give a combinatorial description of a crystal graph of a generalized Verma module with arbitrary $l$-dominant highest weight.

Next, we define Demazure crystals $M_w$ and $T_w$ for $w \in W$ following [11], where $W$ is the Weyl group for $\mathfrak{gl}_\infty$, and give explicit combinatorial descriptions of them (Theorem 5.4 and 5.7). As an interesting corollary, the resulting flagged RSK correspondence between $M_w$ and $T_w$ (Corollary 5.9) explains a nice relation between the support of a matrix in $M$ and the flag conditions of the corresponding tableaux in $T$, which was observed earlier in a purely combinatorial way (cf. [31]). In terms of characters, this can be summarized as follows; for each $w \in W$ we have

$$\sum_{S \subseteq [n]^2} \prod_{(i, j) \in S} \frac{x_i y_j}{(1 - x_i y_j)} = \sum_{\nu \in \Lambda} \hat{s}_{\nu}(X_{\alpha(w)}) \hat{s}_{\nu}(Y_{\beta(w)}),$$

where the left hand side is the sum over supports in $M$ dominated by $w$ with respect to the Bruhat order, and the right-hand side is the sum over products of flagged Schur functions with flag conditions $\alpha(w), \beta(w)$ determined by $w$ (see Section 5 for the precise definitions of these notations). We present variations by considering
symmetric matrices in \( \mathcal{M} \) as crystal graphs for affine Lie subalgebras \( \mathfrak{b}_\infty \) and \( \mathfrak{c}_\infty \) of \( \mathfrak{gl}_\infty \).

Finally, we discuss an application to plane partitions. We show that a Demazure crystal associated with \( w \) corresponds to a set of (symmetric) plane partitions whose shapes are bounded by a partition \( \lambda \) corresponding to \( w \), and obtain its trace generating function as Demazure characters.

The paper is organized as follows. In Section 2, we recall necessary background on crystal graphs. In Section 3, we define \( \mathfrak{gl}_\infty \)-crystal structures on \( \mathcal{M} \) and \( \mathcal{T} \). In Section 4, we give a combinatorial description of a crystal graph of a generalized Verma module with arbitrary \( \mathfrak{gl} \)-dominant highest weights including \( \mathcal{M} \). In Section 5, we define and compute the Demazure crystals \( \mathcal{M}_w \) an \( \mathcal{T}_w \) explicitly. In Section 6, we discuss an application of Demazure crystals to plane partitions.

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## 2. Preliminaries

### 2.1. Lie algebra \( \mathfrak{gl}_\infty \)

Let \( \mathbb{Z}^x \) denote the set of non-zero integers. Let \( \mathfrak{gl}_\infty \) denote the Lie algebra of \( \mathbb{Z}^x \times \mathbb{Z}^x \) complex matrices with finitely many non-zero entries. Let \( E_{ij} \) be the elementary matrix with 1 at the \( i \)-th row and the \( j \)-th column and zero elsewhere.

The Cartan subalgebra is given by \( \mathfrak{h} = \bigoplus_{i \in \mathbb{Z}^x} \mathbb{C}E_{ii} \). Let \( \Pi^+ = \{ h_{-i} = E_{-i-1,-i-1} - E_{-i,-i}, \ h_i = E_{ii} - E_{i+1,i+1} \ (i \in \mathbb{Z}_{>0}), \ h_0 = E_{-1,-1} - E_{11} \} \) be the set of simple co-roots, \( \Pi = \{ \alpha_{-i} = \epsilon_{-i-1} - \epsilon_{-i}, \ \alpha_i = \epsilon_i - \epsilon_{i+1} \ (i \in \mathbb{Z}_{>0}), \alpha_0 = \epsilon_1 - \epsilon_0 \} \) the set of simple roots, and \( \Delta^+ = \{ \epsilon_i - \epsilon_j \ | \ i, j \in \mathbb{Z}^x, i < j \} \) the set of positive roots, where \( \epsilon_i \in \mathfrak{h}^* \) is determined by \( \langle \epsilon_i, E_{ij} \rangle = \delta_{ij} \) and \( \langle \cdot, \cdot \rangle \) is a natural pairing on \( \mathfrak{h}^* \times \mathfrak{h} \). The Dynkin diagram associated with the Cartan matrix \( (\langle \alpha_j, h_i \rangle)_{i,j \in \mathbb{Z}} \) is

\[
\begin{array}{cccccccccccccccc}
\cdots & \circ & \cdots & \circ & \circ & \circ & \cdots & \circ & \cdots & \cdots
\end{array}
\]

Let \( Q = \bigoplus_{i \in \mathbb{Z}} \mathbb{Z}\alpha_i \) be the root lattice. Let \( P = \bigoplus_{i \in \mathbb{Z}^x} \mathbb{Z}\epsilon_i \oplus \mathbb{Z}\Lambda_0 \) be the weight lattice of \( \mathfrak{gl}_\infty \), where \( \Lambda_0 \) is given by \( \langle \Lambda_0, E_{-i,-i} \rangle = -\langle \Lambda_0, E_{i,i} \rangle = \frac{1}{2} \) for \( i \in \mathbb{Z}_{>0} \). There is a partial ordering \( \geq \) on \( P \), where \( \lambda \geq \mu \) if and only if \( \lambda - \mu \) is a non-negative integral linear combination of \( \alpha_i \)'s \( (i \in \mathbb{Z}) \). Let \( P^+ = \{ \Lambda \in P \ | \ \Lambda(h_i) \geq 0, \ i \in \mathbb{Z} \} \), the set of dominant integral weights. For \( i \in \mathbb{Z}^x \), let \( \Lambda_i \) be

\[
\Lambda_i = \begin{cases} 
\Lambda_0 - \sum_{k=i}^{-1} \epsilon_k, & \text{if } i < 0, \\
\Lambda_0 + \sum_{k=1}^{i} \epsilon_k, & \text{if } i > 0,
\end{cases}
\]

We call \( \Lambda_i \in P^+ \ (i \in \mathbb{Z}) \) the \( i \)-th fundamental weight of \( \mathfrak{gl}_\infty \).
2.2. Crystal graphs. Let us recall the notion of crystal graphs for the Lie algebra \( \mathfrak{gl}_\infty \) (cf. \cite{[10] [11] [12]}).

**Definition 2.1.** A \( \mathfrak{gl}_\infty \)-crystal is a set \( B \) together with the maps \( \varphi : B \to P \), \( \varepsilon : B \to \mathbb{Z} \cup \{0\} \) and \( \tilde{e}_i, \tilde{f}_i : B \to B \cup \{0\} \) for \( i \in \mathbb{Z} \), satisfying the following conditions;

1. for \( b \in B \), we have \( \varphi_i(b) = \langle \text{wt}(b), h_i \rangle + \varepsilon_i(b) \),
2. if \( \tilde{e}_i b \in B \) for \( b \in B \), then \( \varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) + 1, \varphi_i(\tilde{e}_i b) = \varphi_i(b) - 1, \text{wt}(\tilde{e}_i b) = \text{wt}(b) - \alpha_i \),
3. if \( \tilde{f}_i b \in B \) for \( b \in B \), then \( \varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) - 1, \varphi_i(\tilde{f}_i b) = \varphi_i(b) + 1, \text{wt}(\tilde{f}_i b) = \text{wt}(b) + \alpha_i \),
4. \( \tilde{f}_i b = b' \) if and only if \( b = \tilde{e}_i b' \) for all \( i \in \mathbb{Z}, b, b' \in B \),
5. If \( \varphi_i(b) = -\infty \), then \( \tilde{e}_i b = \tilde{f}_i b = 0 \),

where \( 0 \) is a formal symbol and \( -\infty \) is the smallest element in \( \mathbb{Z} \cup \{0\} \) such that \( -\infty + n = -\infty \) for all \( n \in \mathbb{Z} \).

A \( \mathfrak{gl}_\infty \)-crystal \( B \) becomes a colored oriented graph, where \( b \overset{i}{\rightarrow} b' \) if and only if \( b' = \tilde{f}_i b \) \( (i \in \mathbb{Z}) \), and it is called a crystal graph for \( \mathfrak{gl}_\infty \). Let \( \mathbb{C}[P] \) be the group algebra of \( P \) with basis \( \{ e^\lambda \mid \lambda \in P \} \). We define the character of \( B \) by \( \text{ch}B = \sum_{b \in B} e^{\text{wt}(b)} \).

Let \( B_1 \) and \( B_2 \) be \( \mathfrak{gl}_\infty \)-crystals. A morphism \( \psi : B_1 \to B_2 \) is a map from \( B_1 \cup \{0\} \) to \( B_2 \cup \{0\} \) such that

1. \( \psi(0) = 0 \),
2. \( \text{wt}(\psi(b)) = \text{wt}(b), \varepsilon_i(\psi(b)) = \varepsilon_i(b), \) and \( \varphi_i(\psi(b)) = \varphi_i(b) \) whenever \( \psi(b) \neq 0 \),
3. \( \psi(\tilde{e}_i b) = \tilde{e}_i \psi(b) \) for \( b \in B_1 \) such that \( \psi(b) \neq 0 \) and \( \psi(\tilde{e}_i b) \neq 0 \),
4. \( \psi(\tilde{f}_i b) = \tilde{f}_i \psi(b) \) for \( b \in B_1 \) such that \( \psi(b) \neq 0 \) and \( \psi(\tilde{f}_i b) \neq 0 \).

We call \( \psi \) an embedding and \( B_1 \) a subcrystal of \( B_2 \) when \( \psi \) is injective, and strict if \( \psi : B_1 \cup \{0\} \to B_2 \cup \{0\} \) commutes with \( \tilde{e}_i \) and \( \tilde{f}_i \) \( (i \in \mathbb{Z}) \), where we assume that \( \tilde{e}_i 0 = \tilde{f}_i 0 = 0 \).

We define the tensor product of \( B_1 \) and \( B_2 \) to be the set \( B_1 \otimes B_2 = \{ b_1 \otimes b_2 \mid b_1 \in B_1, b_2 \in B_2 \} \) with

\[
\begin{align*}
\text{wt}(b_1 \otimes b_2) & = \text{wt}(b_1) + \text{wt}(b_2), \\
\varepsilon_i(b_1 \otimes b_2) & = \max(\varepsilon_i(b_1), \varepsilon_i(b_2) - \langle \text{wt}(b_1), h_i \rangle), \\
\varphi_i(b_1 \otimes b_2) & = \max(\varphi_i(b_1) + \langle \text{wt}(b_2), h_i \rangle, \varphi_i(b_2)), \\
\tilde{e}_i(b_1 \otimes b_2) & = \begin{cases} \\
\tilde{e}_i b_1 \otimes b_2, & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\
b_1 \otimes \tilde{e}_i b_2, & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases}
\end{align*}
\]
\[ \widetilde{f}_i(b_1 \otimes b_2) = \begin{cases} \widetilde{f}_i b_1 \otimes b_2, & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes \widetilde{f}_i b_2, & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2), \end{cases} \]

for \( i \in \mathbb{Z} \), where we assume that \( 0 \otimes b_2 = b_1 \otimes 0 = 0 \).

For \( b_i \in B_i \) (\( i = 1, 2 \)), let \( C(b_i) \) denote the connected component of \( b_i \) in \( B_i \) as a \( \mathbb{Z} \)-colored oriented graph. We say that \( b_1 \) is equivalent to \( b_2 \) if there is an isomorphism of crystal graphs \( C(b_1) \to C(b_2) \) sending \( b_1 \) to \( b_2 \).

Let \( B \) be a \( \mathfrak{gl}_\infty \)-crystal given by

\[ \ldots \to -2 \rightarrow -1 \rightarrow -1 \rightarrow 0 \rightarrow 1 \rightarrow 2 \rightarrow 2 \rightarrow \ldots, \]

where \( \text{wt}(k) = \varepsilon_k \), and \( \varepsilon_i(k) \) (resp. \( \varphi_i(k) \)) is the number of \( i \)-colored arrows coming into \( b \) (resp. going out of \( k \)) for \( k \in B \).

Let \( \mathfrak{gl}_{<0} \) and \( \mathfrak{gl}_{>0} \) be the subalgebras of \( \mathfrak{gl}_\infty \) spanned by \{ \( E_{ij} \mid i, j \in \mathbb{Z}_{<0} \) \} and \{ \( E_{ij} \mid i, j \in \mathbb{Z}_{>0} \) \}, respectively. We can define \( \mathfrak{gl}_{<0} \)-crystals (resp. \( \mathfrak{gl}_{>0} \)-crystals) as in Definition \([2.1]\) with respect to \( \overline{e}_i \)'s and \( \overline{f}_i \)'s for \( i \in \mathbb{Z}_{<0} \) (resp. \( i \in \mathbb{Z}_{>0} \)), and view \( B_{>0} = \{ k \in B \mid k > 0 \} \) as a \( \mathfrak{gl}_{>0} \)-crystal and \( B_{<0} = \{ k \in B \mid k < 0 \} \) as a \( \mathfrak{gl}_{<0} \)-crystal. In addition, let us consider a \( \mathfrak{gl}_{<0} \)-crystal \( B_{<0}^\vee \) given as follows;

\[ -1^\vee \to -1, -2^\vee \to -2, -3^\vee \to -3, \ldots, \]

where \( \text{wt}(-k^\vee) = -\varepsilon_k \) for \( k > 0 \). Note that \( B_{<0}^\vee \) is the dual crystal of \( B_{<0} \) (cf. \([12]\)).

For \( \lambda \in P \), let \( T_\lambda = \{ t_\lambda \} \) be a \( \mathfrak{gl}_\infty \)-crystal with \( \text{wt}(t_\lambda) = \lambda, \overline{e}_i t_\lambda = \overline{f}_i t_\lambda = 0 \), and \( \varepsilon_i(t_\lambda) = \varphi_i(t_\lambda) = -\infty \) for \( i \in \mathbb{Z} \).

2.3. Semistandard tableaux. Let \( \mathcal{P} \) be the set of partitions. We identify a partition \( \lambda \) with a Young diagram or a subset \( \{ (i, j) \mid 1 \leq j \leq \lambda_i \} \) of \( \mathbb{N} \times \mathbb{N} \) (cf. \([25]\)). The number of non-zero parts of \( \lambda \) is denoted by \( \ell(\lambda) \), called the length of \( \lambda \). For \( \lambda = (\lambda_i)_{i \geq 1} \in \mathcal{P} \), \( \lambda' = (\lambda'_i)_{i \geq 1} \) denotes the conjugate of \( \lambda \). For \( \mu \in \mathcal{P} \) with \( \mu \subseteq \lambda \), \( \lambda/\mu \) denotes the skew Young diagram, and \( |\lambda/\mu| \) denotes the number of boxes in the diagram. We denote by \( \lambda^\pi \) the skew Young diagram obtained by \( 180^\circ \)-rotation of \( \lambda \), which is called of anti-normal shape. Put \( \mathcal{P}^\pi = \{ \lambda^\pi \mid \lambda \in \mathcal{P} \} \), the set of anti-normal shaped skew Young diagrams.

Let \( \mathcal{A} \) be a linearly ordered set. For a skew Young diagram \( \lambda/\mu \), a tableau \( T \) obtained by filling \( \lambda/\mu \) with entries in \( \mathcal{A} \) is called a semistandard tableau of shape \( \lambda/\mu \) if the entries in each row are weakly increasing from left to right, and the entries in each column are strictly increasing from top to bottom. We write \( \text{sh}(T) = \lambda/\mu \). We denote by \( \text{SST}_\mathcal{A}(\lambda/\mu) \) the set of all semistandard tableaux of shape \( \lambda/\mu \) with entries in \( \mathcal{A} \). Let \( \mathcal{W}_\mathcal{A} \) be the set of finite words in \( \mathcal{A} \). We associate to each \( T \in \text{SST}_\mathcal{A}(\lambda/\mu) \) a word \( w(T) \in \mathcal{W}_\mathcal{A} \) which is obtained by reading the entries of \( T \) row by row from top to bottom, and from right to left in each row.
We suppose that $A = B, B_{>0}, B_{<0}$, and $B_{\neq 0}$, with the linear ordering $<$ induced from the partial ordering on $P$. Then $W_B$ is a $gl_\infty$-crystal since we may view each non-empty finite word $w = w_1 \cdots w_r$ as $w_1 \otimes \cdots \otimes w_r \in B^{\otimes r}$. Similarly, $W_B$ (resp. $W_{B_{<0}}$ or $W_{B_{\neq 0}}$) becomes a $gl_{>0}$-crystal (resp. $gl_{<0}$-crystal). Sending $T$ to $w(T)$ gives an injective map from $SST_A(\lambda/\mu)$ to $W_A$, and the image of $SST_A(\lambda/\mu)$ together with $\{0\}$ is invariant under the operators $\tilde{e}_i, \tilde{f}_i$. Hence it is a crystal graph $[14]$.

Suppose that $\lambda/\mu \in P$ or $P^{\pi}$. Then $SST_A(\lambda/\mu)$ is connected. In particular, if $A = B_{>0}$ (resp. $B_{<0}$), then $SST_A(\lambda/\mu)$ contains a highest weight element $H_{\lambda/\mu}$, where in each $k$th column of $H_{\lambda/\mu}$, the $l$th entry from the top position is filled with $l$ (resp. $-l^\vee$).

3. The RSK correspondence and $gl_\infty$-crystals

In this section, we define two $gl_\infty$-crystal structures on the set $M$ of matrices with non-negative integral entries of finite support and the set $T$ of pairs of semistandard tableaux of the same shape. We show that the RSK correspondence, which is a bijection from $M$ to $T$, is an isomorphism of $gl_\infty$-crystals.

Since it is already known that the RSK correspondence is a morphism of $(gl_{<0}, gl_{>0})$-bicrystals $[4, 20]$, our main result in this section is to extend it as a $gl_\infty$-crystal morphism by defining the missing operators $\tilde{e}_0, \tilde{f}_0$ on $M$ and $T$, which are compatible with the RSK algorithm.

3.1. Crystal of integral matrices. Let

$$\Omega = \{ (i,j) \in W_{B_{<0}} \times W_{B_{>0}} \mid$$

(3.1)

(1) $i = i_1 \cdots i_r$ and $j = j_1 \cdots j_r$ for some $r \geq 0$,

(2) $(i_1, j_1) \leq \cdots \leq (i_r, j_r)$,

where for $(i,j)$ and $(k,l) \in B_{<0} \times B_{>0}$,

$$(i,j) < (k,l) \iff \begin{cases} j < l \quad \text{or}, \\ j = l \text{ and } i > k. \end{cases}$$

Similarly, let $\Omega'$ be the set of pairs $(k,l) \in W_{B_{<0}} \times W_{B_{>0}}$ such that (1) $k = k_1 \cdots k_r$ and $l = l_1 \cdots l_r$ for some $r \geq 0$ and (2) $(k_1, l_1) \leq' \cdots \leq' (k_r, l_r)$, where

$$(i,j) <' (k,l) \iff \begin{cases} i < k \quad \text{or}, \\ i = k \text{ and } j > l. \end{cases}$$
Then $\Omega$ is a $\mathfrak{gl}_{<0}$-crystal, where $x_i(i,j) = (x_i,i,j)$ for $(i,j) \in \Omega$, $x = \bar{e}, \bar{f}$ and $i \in \mathbb{Z}_{<0}$. Here, we assume that $x_i(i,j) = 0$ if $x_i = 0$. Similarly, $\Omega'$ is a $\mathfrak{gl}_{>0}$-crystal, where $x_j(k,l) = (k,x_j,l)$ for $(k,l) \in \Omega'$, $x = \bar{e}, \bar{f}$ and $j \in \mathbb{Z}_{>0}$.

Consider the following set of $\mathbb{N} \times \mathbb{N}$ matrices with non-negative integers of finite support:

$$
\mathcal{M} = \{ A = (a_{-i^\vee,j})_{i,j \geq 1} \mid a_{-i^\vee,j} \in \mathbb{Z}_{\geq 0}, \sum_{i,j \geq 1} a_{-i^\vee,j} < \infty \}.
$$

For $(i,j) \in \Omega$, define $A(i,j) = (a_{-i^\vee,j})$ to be the matrix in $\mathcal{M}$, where $a_{-i^\vee,j}$ is the number of $k$'s such that $(i_k,j_k) = (-i^\vee,j)$. Then the map $(i,j) \mapsto A(i,j)$ is a bijection between $\Omega$ and $\mathcal{M}$, where the pair of empty words $(\emptyset, \emptyset)$ corresponds to zero matrix, say $\mathcal{O}$. Similarly, we have a bijection $(k,l) \mapsto A(k,l)$ from $\Omega'$ to $\mathcal{M}$.

With these bijections, $\mathcal{M}$ becomes a crystal graph for both $\mathfrak{gl}_{<0}$ and $\mathfrak{gl}_{>0}$. Moreover, the operators $\bar{e}_i, \bar{f}_i$ commute with $\bar{e}_j, \bar{f}_j$ for $i \in \mathbb{Z}_{<0}$ and $j \in \mathbb{Z}_{>0}$, and hence $\mathcal{M}$ becomes a $(\mathfrak{gl}_{<0}, \mathfrak{gl}_{>0})$-bicrystal (cf. [4, 20]).

Now, for $A = (a_{-i^\vee,j}) \in \mathcal{M}$, we define

$$
\bar{e}_0 A = \begin{cases} 
A - E_{-1^\vee,1}, & \text{if } a_{-1^\vee,1} \neq 0, \\
0, & \text{otherwise},
\end{cases}
$$

$$
\bar{f}_0 A = A + E_{-1^\vee,1},
$$

where $E_{-1^\vee,1} \in \mathcal{M}$ denotes the elementary matrix with 1 at the position $(-1^\vee,1)$ and 0 elsewhere. Put $\text{wt}(A) = \sum_{i,j > 0} a_{-i^\vee,j}(-\epsilon_i + \epsilon_j)$, $\varepsilon_0(A) = a_{-1^\vee,1}$, and $\varphi_0(A) = \langle \text{wt}(A), h_0 \rangle + \varepsilon_0(A)$. Then we have the following.

**Proposition 3.1.** $\mathcal{M}$ is a $\mathfrak{gl}_\infty$-crystal, and

$$
\mathcal{M} = \{ \bar{f}_{i_1} \cdots \bar{f}_{i_r} \mathcal{O} \mid r \geq 0, i_1, \ldots, i_r \in \mathbb{Z} \} \setminus \{ \mathcal{O} \}.
$$

In particular, $\mathcal{M}$ is connected with highest weight element $\mathcal{O}$.

**Proof.** It is easy to see that $\mathcal{M}$ is a $\mathfrak{gl}_\infty$-crystal. Let $A \in \mathcal{M}$ be given. We claim that $A = \bar{f}_{i_1} \cdots \bar{f}_{i_r} \mathcal{O}$ for some $r \geq 0$ and $i_1, \ldots, i_r \in \mathbb{Z}$. We use induction on $s(A) = \sum_{i,j} a_{-i^\vee,j}$. If $s(A) = 0$, then it is clear. Let $s(A)$ be positive. First, $A$ is connected to a diagonal matrix $A^0 = (a_{2^\vee,j})$ such that $a_{-1^\vee,1}^0 \geq a_{-2^\vee,2}^0 \geq a_{-3^\vee,3}^0 \geq \ldots$ since $\mathcal{M}$ is a $(\mathfrak{gl}_{<0}, \mathfrak{gl}_{>0})$-bicrystal (cf. [3, 17]). That is, $\tilde{e}_{j_1} \cdots \tilde{e}_{j_r} A = A^0$ for some $r \geq 0$ and $j_1, \ldots, j_r \in \mathbb{Z}$ and $\tilde{e}_i A^0 = 0$ for all $i \in \mathbb{Z}^\times$. If $a_{-1^\vee,1}^0 = 0$, then $A^0 = \mathcal{O}$. If not, then $\tilde{e}_0 A^0 \neq 0$ and $s(A^0) = s(A) - 1$. Hence, the proof completes by induction hypothesis. $\square$
3.2. Crystal of bitableaux. By the RSK algorithm, each $A \in \mathcal{M}$ is in one-to-one correspondence with $(P(A), Q(A))$ in $\text{SST}_{B_{<0}}(\lambda) \times \text{SST}_{B_{>0}}(\lambda)$ for some $\lambda \in \mathcal{P}$ [15].

In what follows, we need a variation of this correspondence with anti-normal shaped tableaux. Let us describe it in detail.

Let $\nu \in \mathcal{P}$ and $T \in \text{SST}_A(\nu^\pi)$ be given. For $a \in A$, we define $T \leftarrow a$ to be the tableau of an anti-normal shape obtained from $T$ by applying the following procedure; (1) let $a'$ be the largest entry in the right-most column which is smaller than or equal to $a$, (2) replace $a'$ by $a$. If there is no such $a'$, put $a$ at the top of the column and stop the procedure, (3) repeat (1) and (2) on the next column with $a'$.

For $w = w_1 \ldots w_r \in \mathcal{W}_A$, we define $P(w)$ to be $((w_1 \leftarrow w_2) \leftarrow w_3) \cdots \leftarrow w_r$. Note that $w$ is equivalent to $P(w)$ as elements of crystals.

Let $A \in \mathcal{M}$ be given with $A = A(i, j) = A(k, l)$ for $(i, j) \in \Omega$ and $(k, l) \in \Omega'$. Let $i_{rev}$ and $l_{rev}$ be the reverse word of $i$ and $l$, respectively. We define

$$P(A) = P(i_{rev}), \quad Q(A) = P(l_{rev}).$$

Let $i = i_1 \ldots i_r$. For $1 \leq k \leq r$, let us fill the box with $c$ if it is created when $i_k$ is inserted into $(i_r \leftarrow i_{r-1}) \cdots \leftarrow i_{k+1}$ and $j_k = c$. Then we have a tableau $Q(i_{rev}) \in \text{SST}_{B_{<0}}(\nu^\pi)$ with $\nu^\pi = \text{sh}P(i_{rev}) \in \mathcal{P}^\pi$. By the symmetry of the RSK correspondence, we have $Q(i_{rev}) = P(l_{rev}) = Q(A)$. Put

$$\mathcal{T} = \bigsqcup_{\nu \in \mathcal{P}} \text{SST}_{B_{<0}}(\nu^\pi) \times \text{SST}_{B_{>0}}(\nu^\pi).$$

Hence we have a bijection

$$\kappa : \mathcal{M} \longrightarrow \mathcal{T},$$

where $\kappa(A) = (P(A), Q(A))$.

Example 3.2. Let

$$A = (a_{-i,j})_{1 \leq i, j \leq 3} = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 2 & 0 \end{pmatrix} \in \mathcal{M},$$

where we assume that $a_{-i,j} = 0$ unless $1 \leq i, j \leq 3$. Then

$$i = -2^\vee -2^\vee -1^\vee -3^\vee -3^\vee -2^\vee -1^\vee, \quad l = 3^1 2^1 1^1 2^2,$$

and

$$P(A) = \begin{pmatrix} -1^\vee & -2^\vee & -2^\vee \\ -1^\vee & -2^\vee & -3^\vee & -3^\vee \end{pmatrix}, \quad Q(A) = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 \end{pmatrix}.$$

Clearly $\mathcal{T}$ is a $(\mathfrak{gl}_{<0}, \mathfrak{gl}_{>0})$-bicrystal, and for $A \in \mathcal{M}$, $P(A)$ (resp. $Q(A)$) is equivalent to $A$ as elements of $\mathfrak{gl}_{<0}$ (resp. $\mathfrak{gl}_{>0}$)-crystals. Summarizing, we have the following.
Proposition 3.3. \( \kappa \) is a \((\mathfrak{gl}_{<0}, \mathfrak{gl}_{>0})\)-bicrystal isomorphism. \( \square \)

Now, let us describe the \( \mathfrak{gl}_\infty \)-crystal structure on \( T \). Suppose that \((S, T) \in T\) is given. For each \( k \)-th column of \( \nu \) enumerated from right to left, let \( s_k \) and \( t_k \) be the smallest entries in the \( k \)-th column of \( S \) and \( T \), respectively. We assign

\[
\sigma_k = \begin{cases} 
+ & \text{if the } k \text{-th column is empty,} \\
+ & \text{if } s_k > -1^\lor \text{ and } t_k > 1, \\
- & \text{if } s_k = -1^\lor \text{ and } t_k = 1, \\
\cdot & \text{otherwise.}
\end{cases}
\]

In the sequence \( \sigma = (\ldots \sigma_2, \sigma_1) \), we replace a pair \(( \sigma_4, \sigma_3) = (-, +) \), where \( s' > s \) and \( \sigma_t = \cdot \) for \( s < t < s' \), by \((\cdot, \cdot)\), and repeat this process as far as possible until we get a sequence with no \(-\) placed to the left of \(+\). We call this sequence the 0-signature of \((S, T)\)

We call the left-most \(-\) in the 0-signature of \((S, T)\) the 0-good \(-\) sign, and define \( \overline{e}_0(S, T) \) to be the bitableaux obtained from \((S, T)\) by removing \(-1^\lor \) and 1 in the corresponding columns. If there is no 0-good \(-\) sign, then we define \( \overline{e}_0(S, T) = 0 \). We call the right-most \(+\) in the 0-signature of \((S, T)\) the 0-good \(+\) sign, and define \( \overline{f}_0(S, T) \) to be the bitableaux obtained from \((S, T)\) by adding \(-1^\lor \) and 1 on top of the corresponding columns. If there is no 0-good \(+\) sign, then we define \( \overline{f}_0(S, T) = 0 \).

Example 3.4. Let \((S, T) \in T\) be given with \( \sigma \) as follows.

\[
S = \begin{pmatrix} -1^\lor & -3^\lor \\ -1^\lor & -2^\lor & -4^\lor & -4^\lor \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 3 \\ 1 & 1 & 3 & 4 \end{pmatrix},
\]

\[
\sigma = (\ldots \sigma_5, \sigma_4, \sigma_3, \sigma_2, \sigma_1) = (\ldots +, +, -, \cdot, -, +).
\]

Then the 0-good \(-\) sign is \(\sigma_4\), and 0-good \(+\) sign is \(\sigma_5\). Hence,

\[
\overline{e}_0(S, T) = \begin{pmatrix} -1^\lor & \cdot & 3^\lor & 1 & 3 \\ -2^\lor & -4^\lor & -4^\lor & 1 & 3 & 4 \end{pmatrix},
\]

\[
\overline{f}_0(S, T) = \begin{pmatrix} 1^\lor & 3^\lor & \cdot & 1 & 3 \\ -1^\lor & -1^\lor & -2^\lor & -4^\lor & -4^\lor & 1 & 1 & 3 & 4 \end{pmatrix}.
\]

Proposition 3.5. \( T \) is a \( \mathfrak{gl}_\infty \)-crystal, and

\[
T = \{ \overline{f}_{i_1} \cdots \overline{f}_{i_r}(\emptyset, \emptyset) \mid r \geq 0, \ i_1, \ldots, i_r \in \mathbb{Z} \} \setminus \{ \emptyset \}.
\]

In particular, \( T \) is connected with highest weight element \((\emptyset, \emptyset)\).

Proof. It is straightforward to check that \( \overline{e}_0(S, T), \overline{f}_0(S, T) \in T \cup \{ \emptyset \} \). Let \((S, T)\) be given with \( \text{sh}(S) = \text{sh}(T) \) non-empty. We may assume that \( \overline{e}_i(S, T) = 0 \) for all
$i \in \mathbb{Z}^\times$. Then $S$ (resp. $T$) is a highest weight element of a $\mathfrak{gl}_{<p}$-crystal (resp. $\mathfrak{gl}_{>0}$-crystal), where in each $k$th column of $S$ (resp. $T$), the $l$th entry from the top position is filled with $-l^\vee$ (resp. $l$). Hence $\tilde{e}_0(S,T) \neq 0$. If we use induction on $|\text{sh}(S)|$, then we conclude that $\tilde{f}_{i_1} \cdots \tilde{f}_{i_r}(\emptyset,\emptyset) = (S,T)$ for some $r \geq 0$ and $i_1,\ldots,i_r \in \mathbb{Z}$. \hfill $\Box$

3.3. Isomorphism. Now we are in a position to state the main result in this section.

**Theorem 3.6.** The map $\kappa : M \rightarrow T$ is a $\mathfrak{gl}_\infty$-crystal isomorphism.

**Proof.** By Proposition [3.3], it suffices to show that $\kappa$ commutes with $\tilde{e}_0$ and $\tilde{f}_0$. More precisely, we claim that for $A \in M$ and $k \geq 1$,

$$\kappa(f_0^k A) = \kappa(k E_{-1^\vee,1} + A) = \tilde{f}_0^k \kappa(A).$$

(3.8)

We use induction on $t(A) = \sum_{k \geq 1} a_{-k^\vee,1}$. We may assume that $a_{-1^\vee,1} = 0$.

If $t(A) = 0$, then it is not difficult to see that (3.8) holds. We suppose that $t(A) > 0$. Let $-p^\vee$ be the largest one such that $a_{-p^\vee,1} \neq 0$. Let $B = A - E_{-p^\vee,1}$.

By induction hypothesis, we have for $k \geq 1$

$$\kappa(f_0^k B) = \kappa(k E_{-1^\vee,1} + B) = \tilde{f}_0^k \kappa(B).$$

For $k \geq 1$, let $\kappa(k E_{-1^\vee,1} + X) = (P^{(k)}(X), Q^{(k)}(X))$ with $X = A, B$. Then $P^{(k)}(A) = P^{(k)}(B) \leftarrow -p^\vee$ by definition of $P(\cdot)$ and $Q^{(k)}(A)$ is obtained from $Q^{(k)}(B)$ by filling the corresponding box, say $c$, in $\text{sh}(P^{(k)}(A))/\text{sh}(P^{(k)}(B))$ with $1$. For convenience, let us write $\kappa(k E_{-1^\vee,1} + A) = \kappa(k E_{-1^\vee,1} + B) \leftarrow (-p^\vee,1)$. Let

$$\sigma = (\ldots,\sigma_2,\sigma_1), \quad \sigma' = (\ldots,\sigma'_2,\sigma'_1)$$

be the sequences of signs associated with $\kappa(k E_{-1^\vee,1} + B)$ and $\kappa(k E_{-1^\vee,1} + A)$, respectively (see (3.7)), and let

$$\tilde{\sigma} = (\ldots,\tilde{\sigma}_2,\tilde{\sigma}_1), \quad \tilde{\sigma}' = (\ldots,\tilde{\sigma}'_2,\tilde{\sigma}'_1)$$

be the 0-signatures of $\kappa(k E_{-1^\vee,1} + B)$ and $\kappa(k E_{-1^\vee,1} + A)$, respectively.

Suppose that by the insertion of $-p^\vee$ into $P^{(k)}(B)$, $c$ is filled with $-q^\vee$ for some $q \leq p$, and it is located at the $t$th column enumerated from the rightmost one.

**Case 1.** $q > 1$. Let $k E_{-1^\vee,1} + B = A(i,j)$ with $(i,j) \in \Omega$. Consider the horizontal strip made by inserting the subwords of $i_{\text{rev}}$ corresponding to the first column of $k E_{-1^\vee,1} + B$.

Then we observe the following facts;

1. no $-1^\vee$ has been bumped out in the bumping path for $P^{(k)}(B) \leftarrow -p^\vee$.
2. by induction hypothesis all $-1^\vee$’s which have been added on $P(B)$ by applying $\tilde{f}_0^k$ to $\kappa(B)$ are placed to the right of $-q^\vee$ in the $t$th column, and they do not intersect with the bumping path for $P^{(k)}(B) \leftarrow -p^\vee$. 


(3) the insertion of $-p^\vee$ into $\mathbf{P}^{(k)}(B)$ does not change the sign $\sigma_k$ for $1 \leq k \leq t - 1$, and hence $\sigma_k = \sigma'_k$ for $1 \leq k \leq t - 1$.

Hence we have

$$\bar{e}_0^k[\kappa(kE_{-1}^\vee, 1 + B) \leftarrow (-p^\vee, 1)] = \kappa(B) \leftarrow (-p^\vee, 1) = \kappa(A),$$

and

$$\bar{f}_0^k\kappa(A) = \kappa(kE_{-1}^\vee, 1 + B) \leftarrow (-p^\vee, 1) = \kappa(kE_{-1}^\vee, 1 + A).$$

**Case 2.** $q = 1$. Consider the bumping path for $\mathbf{P}^{(k)}(B) \leftarrow -p^\vee$. Then there exists $1 \leq s \leq t$ such that

1. $-x^\vee$ ($x \geq 2$) has been bumped out from the $(k - 1)$th column and placed at the $k$th column for $2 \leq k \leq s$,
2. $-1^\vee$ has been bumped out from the $(k - 1)$th column and placed at the $k$th column for $s + 1 \leq k \leq t$.

As in Case 1, it follows that all $-1^\vee$'s which have been added to $\mathbf{P}(B)$ by applying $\bar{f}_0^k$ to $\kappa(B)$ are placed to the right of the $t$th column, and $\sigma_r = \sigma'_r$ for $1 \leq r \leq s$.

Since all $-1^\vee$'s in the $r$th column of $\mathbf{P}^{(k)}(B)$ for $s \leq r \leq t - 1$ have been shifted to the left by one column by the insertion of $-p^\vee$ to $\mathbf{P}^{(k)}(B)$, we have $\sigma_r = \sigma'_r$ for $s + 1 \leq r \leq t - 1$. Note that $\sigma_t = +$ and $\sigma'_t = -$.

Let $u$ be the top entry of the $s$th column in $\mathbf{Q}^{(k)}(B)$. If $u = 1$, then we have $\sigma_s = -$ and $\sigma'_s = +$. If $u > 1$, then we have $\sigma_s = +$ and $\sigma'_s = +$. Now, comparing $\sigma$ and $\sigma'$ (hence $\bar{\sigma}$ and $\bar{\sigma}$), it is not difficult to see that

$$\bar{e}_0^k\kappa(kE_{-1}^\vee, 1 + A) = \bar{e}_0^k[\kappa(kE_{-1}^\vee, 1 + B) \leftarrow (-p^\vee, 1)] = \kappa(B) \leftarrow (-p^\vee, 1) = \kappa(A).$$

This completes the proof. \(\square\)

**Example 3.7.** Consider

$$\kappa \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} -1^\vee & -3^\vee & 1 & 3 \\ -1^\vee & -2^\vee & -4^\vee & -4^\vee \end{pmatrix},$$

Applying $\bar{e}_0$ and $\bar{f}_0$ on both sides, we get

$$\kappa \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} -1^\vee & -3^\vee & 1 & 3 \\ -2^\vee & -4^\vee & -4^\vee \end{pmatrix}.$$
and

$$\kappa \begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} -1^\vee & -3^\vee \\ -1^\vee & -2^\vee & 1 \\ -2^\vee & -4^\vee & 1 \\ -4^\vee & -4^\vee & 1 \end{pmatrix},$$

respectively (see Example 3.4).

4. Crystal graphs of generalized Verma modules

Let $u_\pm$ be the subalgebra of $\mathfrak{gl}_\infty$ spanned by $E_{ij}$ for $i < 0$, $j > 0$ (resp. $i > 0$, $j < 0$). Let $p = \mathfrak{gl}_{<0} \oplus \mathfrak{gl}_{>0} \oplus u_+$ be a maximal parabolic subalgebra. Then we have $\mathfrak{gl}_\infty = u_- \oplus p$. The set of roots for the nilradical $u_-$ is given by $\Delta(u_-) = \{-\epsilon_i + \epsilon_j \mid i > 0, j < 0\}$. Let $U(u_-)$ be the universal enveloping algebra of $u_-$. By PBW theorem, $U(u_-)$ has a basis parameterized by $M$.

In this section, we prove that the $\mathfrak{gl}_\infty$-crystal $M$ is the crystal graph of a generalized Verma module $U(u_-)$ or its $q$-analogue (cf. [3]) in the sense that it is the limit of the crystal graphs of the integrable highest weight $\mathfrak{gl}_\infty$-modules with highest weight $n\Lambda_0$ as $n \to \infty$.

4.1. Crystal $\mathcal{B}(n\Lambda_0)$. Let $\mathcal{F}$ be the set of semi-infinite words

$$w = \cdots w_{-3}w_{-2}w_{-1}$$

with letters in $\mathcal{B}$ such that

1. $w_{i-1} < w_i$ for all $i < 0$,
2. there exists an integer $c \in \mathbb{Z}$ such that $w_i = i + c$ for $i \ll 0$.

For $w \in \mathcal{F}$, we define $\text{wt}(w) = \Lambda_0 + \sum_{k \in \mathcal{B}} m_k \epsilon_k \in P$, where $m_k = \left| \{ i \mid w_i = k \} \right| - \delta_{-k,|k|}$. It is well-defined since $m_k = 0$ for almost all $k \in \mathcal{B}$. For each $i \in \mathbb{Z}$, we define the operators $\tilde{e}_i, \tilde{f}_i : \mathcal{F} \to \mathcal{F} \cup \{0\}$ by the same way as we do on $W_\mathcal{B}$. Then $\tilde{e}_i$ and $\tilde{f}_i$ are well-defined, and $\mathcal{F}$ is a $\mathfrak{gl}_\infty$-crystal. For $i \leq 0$, let $H_{\Lambda_i} = \cdots i - 3 i - 2 i - 1$, and for $i > 0$, let $H_{\Lambda_i} = \cdots -2 - 11 \cdots i - 1 i$. We have the following decomposition as $\mathfrak{gl}_\infty$-crystals

$$\mathcal{F} = \bigsqcup_{i \in \mathbb{Z}} \mathcal{B}(\Lambda_i),$$

where $\mathcal{B}(\Lambda_i)$ is the connected component of $H_{\Lambda_i}$ with $\text{wt}(H_{\Lambda_i}) = \Lambda_i$. Recall that $\mathcal{F}$ is the crystal graph of the Fock space representation, which can be realized as the space of semi-infinite wedge vectors, and $\mathcal{B}(\Lambda_i)$ is the crystal graph of the irreducible highest weight $\mathfrak{gl}_\infty$-module with highest weight $\Lambda_i$ (cf. [27]).
Let \( \lambda = (\lambda_1, \cdots, \lambda_n) \) be a sequence of non-increasing \( n \) integers, called a \textit{generalized partition of length} \( n \). We call an \( n \)-tuple of semi-infinite words \( \mathbf{w} = (w^{(1)}, \cdots, w^{(n)}) \) a \textit{semi-infinite semistandard tableau of shape} \( \lambda \) if

1. \( w^{(i)} = \cdots w^{(i)}_{-3} w^{(i)}_{-2} w^{(i)}_{-1} \in \mathcal{B}(\Lambda_{\lambda_{n-i+1}}) \) for \( 1 \leq i \leq n \),
2. \( w^{(i+1)}_{k+d_i} \leq w^{(i)}_{k} \) for \( 1 \leq i < n \) and \( k < 0 \), where \( d_i = \lambda_{n-i+1} - \lambda_{n-i} \).

We may identify each \( \mathbf{w} \) with a semistandard tableau with infinitely many rows and \( n \) columns, where each row of \( \mathbf{w} \) reads (from left to right) as follows:

\[
\begin{align*}
\mathbf{w} &= \begin{array}{cccc}
w^{(4)} & w^{(3)} & w^{(2)} & w^{(1)} \\
\vdots & \vdots & \vdots & \vdots \\
-4 & -4 & -4 & -2 \\
-3 & -2 & 1 & 2 \\
-2 & -1 & 3 & 3 \\
-1 & 1 & 2 & 2 \\
3 & 3 & & \\
5 & & & \\
\end{array},
\end{align*}
\]

\[
H_{\Lambda} = \begin{pmatrix}
-2 & -2 & -2 & -2 \\
-1 & -1 & 1 & 1 \\
2 & 2 & & \\
3 & & & \\
4 & & & \\
\end{pmatrix}.
\]

\textbf{Figure 1.} Semi-infinite semistandard tableaux of shape \( \lambda = (4,2,-1,-1) \)

Here we assume that \( w^{(i)}_{k} \) is empty if there is no corresponding entry (see Figure 1).

Let \( \Lambda_{\lambda} = \sum_{k=1}^{n} \Lambda_{\lambda_k} \in P^+ \) and let \( \mathcal{B}(\Lambda_{\lambda}) \) be the set of all semi-infinite semistandard tableaux of shape \( \lambda \). We may assume that \( \mathbf{w} = (w^{(1)} \otimes \cdots \otimes w^{(n)}) \in \mathcal{B}(\Lambda_{\lambda_n}) \otimes \cdots \otimes \mathcal{B}(\Lambda_{\lambda_1}) \). By similar arguments as in the case of usual semistandard tableaux (cf.\cite{[14]}), we can check that \( \mathcal{B}(\Lambda_{\lambda}) \) together with \( \mathbf{0} \) is stable under \( \tilde{e}_i \) and \( \tilde{f}_i \) (\( i \in \mathbb{Z} \)). Hence, we have

\textbf{Proposition 4.1.} \( \mathcal{B}(\Lambda_{\lambda}) \) is a \( \mathfrak{gl}_{\infty} \)-crystal and

\[
\mathcal{B}(\Lambda_{\lambda}) = \{ \tilde{f}_{i_1} \cdots \tilde{f}_{i_r} H_{\Lambda_{\lambda}} | r \geq 0, \; i_1, \ldots, i_r \in \mathbb{Z} \} \setminus \{ \mathbf{0} \},
\]

where \( H_{\Lambda_{\lambda}} = H_{\Lambda_{\lambda_n}} \otimes \cdots \otimes H_{\Lambda_{\lambda_1}} \).

Now, let us consider \( \mathcal{B}(n \Lambda_0) \) for \( n \in \mathbb{N} \). Given \( \mathbf{w} = (w^{(1)}, \cdots, w^{(n)}) \in \mathcal{B}(n \Lambda_0) \), let \( \mathbf{w}_{>0} \) and \( \mathbf{w}_{<0} \) be the subtableaux of \( \mathbf{w} \) consisting of positive and negative entries, respectively. Note that \( \mathbf{w}_{>0} \in \text{SST}_{\mathbf{B}_{>0}}(\mu^n) \) for some \( \mu \in \mathcal{P} \) with \( \mu_1 \leq n \), and \( \mathbf{w}_{<0} \) is a semi-infinite semistandard tableau of shape \( (-\mu'_n, \ldots, -\mu'_1) \).
Suppose that \( w_{<0} = (w_{<0}^{(1)}, \ldots, w_{<0}^{(n)}) \). For each \( 1 \leq k \leq n \), we have \( w_{<0}^{(k)} \in B(\Lambda_{-\mu_k'}) \) and \( \text{wt}(w_{<0}^{(k)}) = \Lambda_0 - \sum_{i \in I_k} e_i \) for a unique \( I_k = \{ -i_{k,1} > \ldots > -i_{k,-\mu'_k} \} \subset B_{<0} \). We let \( w_{<0}^{\vee} \) be the tableau of shape \( \mu^{\vee} \), whose \( k \)-th column (from the right) is filled with \( \{ -i_{k,1}^{\vee} < \ldots < -i_{k,-\mu'_k}^{\vee} \} \subset B_{<0}^{\vee} \). It is not difficult to see that \( w_{<0}^{\vee} \in SST_{B_{<0}^{\vee}}(\mu^{\pi}) \).

Now, we define

\begin{equation}
\Psi_n(w \otimes t_{-n\Lambda_0}) = \kappa^{-1}(w_{<0}^{\vee}, w_{>0}) \in M.
\end{equation}

**Example 4.2.** Let \( w \in B(4\Lambda_0) \) be as follows.

\[
\begin{array}{ccccccccccccccc}
\vdots & \vdots & \vdots & \vdots & \vdots \\
-5 & -5 & -5 & -5 \\
-4 & -4 & -3 & -2 \\
-3 & -3 & -2 & -1 \\
-2 & -1 & 1 & 2 \\
1 & 1 & 3 & 4
\end{array}
\]

Then

\[
w_{<0}^{\vee} = \begin{pmatrix}
-1^{\vee} & -3^{\vee} \\
-1^{\vee} & -2^{\vee} & -4^{\vee} & -4^{\vee}
\end{pmatrix}
\]

\[
w_{>0} = \begin{pmatrix}
1 & 3 \\
1 & 1 & 3 & 4
\end{pmatrix}
\]

Therefore,

\[
\kappa^{-1}(w_{<0}^{\vee}, w_{>0}) = \begin{pmatrix}
1 & 0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0
\end{pmatrix}
\]

**Proposition 4.3.** For \( n \geq 1 \), the map

\[
\Psi_n : B(n\Lambda_0) \otimes T_{-n\Lambda_0} \rightarrow M
\]

is a \( \mathfrak{gl}_{\infty} \)-crystal embedding.

**Proof.** (1) Let \( w \) be given. For \( i \in \mathbb{Z}_{>0} \) and \( x = e, f \), if \( x_i w \neq 0 \), then

\[
\Psi_n(x_i(w \otimes t_{-n\Lambda_0})) = \Psi_n((x_i w) \otimes t_{-n\Lambda_0})
\]

\[
= \kappa^{-1}(w_{<0}^{\vee}, x_i w_{>0})
\]

\[
= x_i \kappa^{-1}(w_{<0}^{\vee}, w_{>0}) \quad \text{by Proposition 3.3}
\]

\[
= x_i \Psi_n(w \otimes t_{-n\Lambda_0}).
\]
Similarly, for \( i \in \mathbb{Z}_{<0} \) and \( x = e, f \), if \( \tilde{x}_i w \neq 0 \), then
\[
\Psi_n(\tilde{x}_i(w \otimes t_{-nA_0})) = \Psi_n((\tilde{x}_i w) \otimes t_{-nA_0})
\]
\[
= \kappa^{-1}(\tilde{x}_i w_{<0}^\vee, w_{>0}) \quad \text{by Lemma 5.8} \]
\[
= \tilde{x}_i \kappa^{-1}(w_{<0}^\vee, w_{>0}) \quad \text{by Proposition 3.3}
\]
\[
= \tilde{x}_i \Psi_n(w \otimes t_{-nA_0}).
\]

Finally, comparing the definitions of \( \tilde{x}_0 (x = e, f) \) on \( B(nA_0) \) and \( T \), it is straightforward to see that
\[
\Psi_n(\tilde{x}_0(w \otimes t_{-nA_0})) = \kappa^{-1}(\tilde{x}_0(w_{<0}^\vee, w_{>0})).
\]

Since \( \kappa \) commutes with \( \tilde{e}_0 \) and \( \tilde{f}_0 \) by Theorem 3.6 we have \( \Psi_n(\tilde{x}_0(w \otimes t_{-nA_0})) = \tilde{x}_0 \Psi_n(w \otimes t_{-nA_0}). \) The other conditions for \( \Psi_n \) to be a morphism can be verified directly. \( \square \)

Remark 4.4. We have
\[
\text{Im}\Psi_n = \kappa^{-1}\left( \bigcup_{\mu \in \mathcal{P}, \mu_1 \leq n} \text{SST}_{B_{<0}}^\vee(\mu) \times \text{SST}_{B_{>0}}(\mu) \right),
\]
\[
\text{Im}\Psi_n \subseteq \text{Im}\Psi_{n+1} \quad (n \geq 1),
\]
\[
\mathcal{M} = \bigcup_{n \geq 1} \text{Im}\Psi_n.
\]

Note that there exists a strict morphism \( \Phi_n : M \otimes T_{nA_0} \rightarrow B(nA_0) \) sending \( \emptyset \otimes t_{nA_0} \) to \( H_{nA_0} \) such that \( \Phi_n(A) \neq 0 \) if and only if \( A \in \text{Im}\Psi_n \).

4.2. Crystal graphs of generalized Verma modules. Given \( \mu, \nu \in \mathcal{P} \), we put
\[
\text{M}_{\mu, \nu} = \mathcal{M} \times \text{SST}_{B_{<0}}^\vee(\mu) \times \text{SST}_{B_{>0}}(\nu),
\]
\[
\emptyset_{\mu, \nu} = (\emptyset, H_{\mu}, H_{\nu}).
\]

For \( (A, S_{<0}, S_{>0}) \in \text{M}_{\mu, \nu} \), we define
1. if \( \tilde{x}_i(A \otimes S_{<0}) = A' \otimes S'_{<0} \) for \( i \in \mathbb{Z}_{<0} \), then
   \[ \tilde{x}_i(A, S_{<0}, S_{>0}) = (A', S'_{<0}, S_{>0}) \],
2. if \( \tilde{x}_i(A \otimes S_{>0}) = A'' \otimes S''_{>0} \) for \( i \in \mathbb{Z}_{>0} \), then
   \[ \tilde{x}_i(A, S_{<0}, S_{>0}) = (A'', S_{<0}, S''_{>0}) \],
3. \( \tilde{x}_0(A, S_{<0}, S_{>0}) = (\tilde{x}_0 A, S_{<0}, S_{>0}) \),
where \( x = e, f \), and \( \tilde{x}_i(A, S_{<0}, S_{>0}) = 0 \) if any of its components is 0.

Proposition 4.5. For \( \mu, \nu \in \mathcal{P} \), \( \text{M}_{\mu, \nu} \) is a \( \mathfrak{gl}_\infty \)-crystal and
\[
\text{M}_{\mu, \nu} = \{ \tilde{f}_{i_1} \cdots \tilde{f}_{i_r} \emptyset_{\mu, \nu} | r \geq 0, \ i_1, \ldots, i_r \in \mathbb{Z} \} \setminus \{0\}. \]
First, we have \( \tilde{e}_i A = 0 \) for all \( i \in \mathbb{Z}^\times \), which implies that \( A \) is a diagonal matrix with entries \( a_{-1,1} \geq a_{-2,2} \geq \ldots \). Since \( \tilde{e}_0 A = 0 \), we have \( a_{-1,1} = 0 \) and hence \( A = \widetilde{\mathbb{O}} \). This implies that \( S_{<0} = H_{\mu^\tau} \) and \( S_{>0} = H_{\nu} \) since \( \tilde{e}_i S_{<0} = 0 \) for \( i \in \mathbb{Z}_{<0} \) and \( \tilde{e}_i S_{>0} = 0 \) for \( i \in \mathbb{Z}_{>0} \), respectively.

For \( n \geq \mu_1 + \nu_1 \), we put
\[
\Lambda_{\mu,\nu; n} = \Lambda_{\lambda},
\]
where \( \lambda = (\lambda_1, \ldots, \lambda_n) \) is a generalized partition of length \( n \) such that
\[
\lambda^+ = (\max(\lambda_1, 0), \ldots, \max(\lambda_n, 0)) = \nu',
\]
\[
\lambda^- = (\max(-\lambda_1, 0), \ldots, \max(-\lambda_1, 0)) = \mu'.
\]

**Proposition 4.6.** Let \( \mu, \nu \in \mathcal{P} \) be given. Then for \( n \geq \mu_1 + \nu_1 \), there exists an embedding of \( \mathfrak{gl}_\infty \)-crystals
\[
\Psi_{\mu,\nu; n} : B(A_{\mu,\nu; n}) \otimes T_{-n\Lambda_0} \rightarrow \mathcal{M}_{\mu,\nu},
\]
sending \( H_{\Lambda_{\mu,\nu; n}} \otimes t_{-n\Lambda_0} \) to \( \bigotimes_{\mu,\nu} \).

**Proof.** Given \( w = (w^{(1)}, \ldots, w^{(n)}) \in B(A_{\mu,\nu; n}) \), consider the subtableau of \( w \) consisting of positive entries, say \( w_{>0} \). Let \( w^+_{>0} \) be the subtableau of \( w_{>0} \) corresponding to the positions where \( H_{\Lambda_{\mu,\nu; n}} \) has positive entries, and let \( w^-_{>0} \) be the compliment of \( w^+_{>0} \) in \( w_{>0} \). Note that \( w^+_{>0} \in \text{SST}_{B_{>0}}(\nu) \) and \( w^-_{>0} \in \text{SST}_{B_{<0}}((\eta/\mu)^\pi) \) for some \( \eta \supseteq \mu \) with \( \eta_1' \leq n \).

Let \( w_{<0} \) be the subtableau of \( w \) consisting of negative entries, which is also a semi-infinite semistandard tableau of shape \( (-\eta_1', \ldots, -\eta_1') \). By the same method as in [4,1], we obtain \( w_{<0}^\vee \in \text{SST}_{B_{<0}}^\vee((\eta)^\pi) \) from \( w_{<0} \).

Given \( A = A(t, j) \in \mathcal{M} \) and \( S \in \text{SST}_{B_{<0}}((\mu)^\pi) \), we define \( P(S \leftarrow A) \) to be the tableau obtained by inserting \( i_{rev} = i_r \cdots i_1 \) to \( S \), that is,
\[
P(S \leftarrow A) = (\cdots ((S \leftarrow i_r) \leftarrow i_{r-1}) \cdots) \leftarrow i_1.
\]
Suppose that \( \text{sh} P(S \leftarrow A) = \tau^\pi \) for some \( \tau \supseteq \mu \). For \( 1 \leq k \leq r \), let us fill the box in \( (\tau/\mu)^\pi \) with \( c \) if it is created when \( i_k \) is inserted into \( ((S \leftarrow i_r) \cdots) \leftarrow i_{k+1} \) and \( j_k = c \). This defines the recording tableau \( Q(S \leftarrow A) \) of shape \( (\tau/\mu)^\pi \).

Now, we let \( A \in \mathcal{M} \) and \( S_{<0} \in \text{SST}_{B_{<0}}((\mu)^\pi) \) be the unique pair such that \( P(S_{<0} \leftarrow A) = w^\vee_{<0} \) and \( Q(S_{<0} \leftarrow A) = w^-_{>0} \), and let \( S_{>0} = w^+_{>0} \). This defines a map
\[
\Psi_{\mu,\nu; n} : B(A_{\mu,\nu; n}) \otimes T_{-n\Lambda_0} \rightarrow \mathcal{M}_{\mu,\nu}
\]
by
\[
\Psi_{\mu,\nu; n}(w \otimes t_{-n\Lambda_0}) = (A, S_{<0}, S_{>0}).
\]
Modifying the arguments in Theorem 3.6 and Proposition 4.3, we can check that \( \Psi_{\mu,\nu; n} \) is a \( \mathfrak{gl}_\infty \)-crystal embedding.
Example 4.7. Let \( w \in B(\Lambda_{(4,2,-1,-1)}) \) be as in Figure 1. Note that \( \mu = (2), \nu = (2,2,1,1) \) and \( n = 4 \). We have

\[
\begin{align*}
\omega_{<0}^\vee &= -1^\vee -1^\vee, \quad \omega_{>0}^\vee = 1^\vee 2^\vee, \quad 2^\vee 3^\vee, \quad 3^\vee 4^\vee, \\
\omega_{>0}^- &= 1^\vee, \quad 2^\vee, \quad 3^\vee, \quad 1^\vee.
\end{align*}
\]

Then we can check that \( P(S \leftarrow A) = w_{<0}^\vee \) and \( Q(S \leftarrow A) = w_{>0}^- \), where

\[
S = \begin{array}{ccc}
\square & \square & \square \\
\square & \square & \square \\
\square & -3^\vee & -4^\vee
\end{array}, \quad A = \begin{pmatrix}
1 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 1
\end{pmatrix}.
\]

Hence

\[
\Psi_{(2,),(2,2,1,1) ; 4}(w \otimes t_{-4A_0}) = \begin{pmatrix}
1 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 1
\end{pmatrix}, \quad -3^\vee - 4^\vee, \quad -3^\vee - 4^\vee
\]

Remark 4.8. (1) In case of \( \nu = 0 \), the map sending \((w_{<0}^\vee, w_{>0}^-)\) to \((A, S_{<0})\) is a skew version of the RSK correspondence introduced by Sagan and Stanley [32].

(2) Note that \( M_{\mu,\nu} = \bigcup_{n \geq n_1+v_1} \text{Im} \Psi_{\mu,\nu;n} \). Hence, we may regard \( M_{\mu,\nu} \) as the limit of \( B(\Lambda_{\mu,\nu;n}) \) as \( n \to \infty \) and the crystal graph of the generalized Verma module induced from an irreducible \( t \)-module with \( t \)-dominant highest weight \(-\sum_{i<0} \mu_i \epsilon_i + \sum_{j>0} \nu_j \epsilon_j\). We also have a strict morphism \( \Phi_{\mu,\nu;n} : M_{\mu,\nu} \otimes T_{nA_0} \to B(\Lambda_{\mu,\nu;n}) \) such that \( \Phi_{\mu,\nu;n}((A, S_{<0}, S_{>0}) \otimes t_{nA_0}) = w \neq 0 \) if and only if \( \Psi_{\mu,\nu;n}(w \otimes t_{-nA_0}) = (A, S_{<0}, S_{>0}) \).

(3) Bitableaux realizations of irreducible highest weight representations of Lie (super) algebras including \( B(\Lambda_\lambda) \) and their combinatorics can be found in [18].

5. Demazure crystals and a flagged RSK correspondence

5.1. Demazure crystals. For \( i \in \mathbb{Z} \), let \( s_i \in GL(\mathfrak{h}^*) \) be the simple reflection with respect to \( \alpha_i \) defined by

\[
s_i(\lambda) = \lambda - \langle \lambda, h_i \rangle \alpha_i, \quad (\lambda \in \mathfrak{h}^*).
\]

Then \( s_i \) acts as the transposition on \( \{ \epsilon_i \mid i \in \mathbb{Z}^\times \} \) (hence on \( \mathbb{Z}^\times \)) given by

\[
s_i = \begin{cases} 
(i \ i + 1), & \text{if } i > 0, \\
(i - 1 \ i), & \text{if } i < 0, \\
(-1 \ 1), & \text{if } i = 0.
\end{cases}
\]
Let $W$ be the Weyl group of $\mathfrak{g}l_{\infty}$, which is generated by $\{s_i \mid i \in \mathbb{Z}\}$, and let $\ell(w)$ denote the length of $w \in W$. For $\Lambda \in P^+$, let $W_\Lambda$ be the stabilizer of $\Lambda$, and let $W^\Lambda = \{w \mid \ell(ws_i) > \ell(w) \text{ for } s_i \in W_\Lambda\}$. Let $\prec$ denote the Bruhat order on $W$. It induces the Bruhat order on $W^\Lambda$, which is also denoted by $\prec$.

Let $w \in W^\Lambda$ be given and $w = s_{i_1} \cdots s_{i_r}$ its reduced expression. We define the Demazure crystal of $B(\Lambda)$ associated with $w$ by

$$(5.1) \quad B_w(\Lambda) = \{\tilde{f}^{m_1}_{i_1} \cdots \tilde{f}^{m_r}_{i_r} H_\Lambda \mid m_1, \ldots, m_r \geq 0\} \setminus \{0\}.$$ 

For any $\mu, \nu \in \mathcal{P}$, we put

$$(5.3) \quad M_{\mu,\nu,w} = \{\tilde{f}^{m_1}_{i_1} \cdots \tilde{f}^{m_r}_{i_r} \mathcal{O}_{\mu,\nu} \mid m_1, \ldots, m_r \geq 0\} \setminus \{0\}.$$ 

For $w, w' \in W^\Lambda$, we have $M_{\mu,\nu,w} \subset M_{\mu,\nu,w'}$ if and only if $w \preceq w'$. Since each element in $M_{\mu,\nu,w}$ is contained in $\Psi_{\mu,\nu;n}(B_w(\Lambda_{\mu,\nu;n}) \otimes T_{-n}\Lambda_0)$ for a sufficiently large $n$, it is well-defined.

### 5.2. Grassmannian permutations.
Let $\lambda$ be a partition. The residue of a box in $\lambda$ is given by $j - i$ if it is located in the $i$th row and the $j$th column. A standard tableau of shape $\lambda$ is a tableau obtained by filling $\lambda$ with $\{1, \ldots, |\lambda|\}$ in such a way that the entries in each column (resp. row) are increasing from top to bottom (resp. left to right).

Consider $W_{\lambda_0} = \langle s_i \mid i \in \mathbb{Z}^\times \rangle$ and let $w \in W^\Lambda_0$ be given. Let $D(w) = \{(i,j) \in \mathbb{Z}^\times \times \mathbb{Z}^\times \mid i < w^{-1}(j), j < w(i)\}$ be the diagram of $w$ and let $\lambda(w) = (\lambda(w)_i)_{i \geq 1}$ be the shape of $w$, where $\lambda(w)_i = \{|j \mid (-i, j) \in D(w)\}$. Since $w(i) < w(i + 1)$ for $i \leq -2$ or $i \geq 1$, and $w(-1) > w(1)$, $\lambda(w)$ is a partition. Conversely, a partition $\lambda$ determines a unique permutation $w \in W^\Lambda_0$ such that $\lambda(w) = \lambda$. Hence, we have a bijection from $W^\Lambda_0$ to $\mathcal{P}$ sending $w$ to $\lambda(w)$, where $|\lambda(w)| = \ell(w)$. If $T$ is a standard tableau of shape $\lambda(w)$ and $a_i$ is the residue of the box corresponding to $i$ in $T$ ($1 \leq i \leq \ell(w)$), then $w = s_{a_{\ell(w)}} s_{a_{\ell(w)} - 1} \cdots s_{a_1}$ gives a reduced expression of $w$. For $w, w' \in W^\Lambda_0$, we have $w \preceq w'$ if and only if $\lambda(w) \subseteq \lambda(w')$ (cf. [24]).

**Example 5.1.** Let $w \in W^\Lambda_0$ be given by

$$w = \begin{bmatrix} \cdots & -5 & -4 & -3 & -2 & -1 & 1 & 2 & 3 & 4 & 5 & 6 & \cdots \\ \cdots & -4 & -2 & 2 & 5 & 6 & -5 & -3 & -1 & 1 & 3 & 4 & \cdots \end{bmatrix}.$$
where \( w(i) = i \) for \( i \geq 7 \) or \( i \leq -6 \). Then \( \lambda(w) = (6, 6, 4, 2, 1) \), where the residue on each box is given by

\[
\begin{array}{cccccc}
0 & 1 & 2 & 3 & 4 & 5 \\
-1 & 0 & 1 & 2 & 3 & 4 \\
-2 & -1 & 0 & 1 & & \\
-3 & -2 & & & & \\
-4 & & & & & \\
\end{array}
\]

5.3. Flagged skew Schur functions. Let \( X = \{ x_1, x_2, \ldots \} \) be the set of variables and \( X_k = \{ x_1, \ldots, x_k \} \) for \( k \geq 1 \). Let \( \phi = (\phi_1, \ldots, \phi_d) \) be a sequence of weakly increasing positive integers of length \( d \), which is called a flag of length \( d \). Given a skew Young diagram \( \lambda/\mu \) with \( \ell(\lambda), \ell(\mu) \leq d \), we define the flagged Schur function \( s_{\lambda/\mu}(X_\phi) \) by

\[
s_{\lambda/\mu}(X_\phi) = \det (h_{\lambda_i-\mu_j-i+j}(X_{\phi_i}))_{1 \leq i, j \leq d},
\]

where \( h_k(X_{\phi_i}) \) is the \( k \)th complete symmetric function in \( X_{\phi_i} \) (cf. [24]). An equivalent definition is that

\[
s_{\lambda/\mu}(X_\phi) = \sum_T x^T,
\]

where the sum ranges over all semistandard tableaux of shape \( \lambda/\mu \) such that the entries in the \( i \)th row are no more than \( \phi_i \) for \( 1 \leq i \leq d \). Here, \( x^T = \prod_i x_i^{m_i} \), where \( m_i \) is the number of occurrences of \( i \) in \( T \).

Let \( \text{SST}_{B_{>0}}(\lambda/\mu)_\phi \) (resp. \( \text{SST}_{B_{>0}}(\lambda/\mu)_\phi \)) be the set of semistandard tableaux of shape \( \lambda/\mu \) such that the entries in the \( i \)th row are no more than \( -\phi_i^\vee \) (resp. \( \phi_i \)) for \( 1 \leq i \leq d \). Let \( Y = \{ y_1, y_2, \ldots \} \) be another set of variables and \( Y_k = \{ y_1, \ldots, y_k \} \) for \( k \geq 1 \). Then we have

\[
\text{ch}\text{SST}_{B_{<0}}(\lambda/\mu)_\phi = s_{\lambda/\mu}(X_\phi), \quad \text{ch}\text{SST}_{B_{>0}}(\lambda/\mu)_\phi = s_{\lambda/\mu}(Y_\phi),
\]

where we put

\[
\begin{align*}
x_i &= e^{\text{wt}(-i^\vee)} = e^{-\epsilon_{-i}}, \\
y_j &= e^{\text{wt}(j)} = e^{\epsilon_j}
\end{align*}
\]

for \(-i^\vee \in B_{<0} \) and \( j \in B_{>0} \). When \( S \in \text{SST}_{B_{<0}}(\nu^\pi)_\phi \) (resp. \( S \in \text{SST}_{B_{>0}}(\nu^\pi)_\phi \)) is given for \( \nu \in \mathcal{P} \) with \( \ell(\nu) \leq d \), we view \( \nu^\pi = (n^d)/(n - \nu_n, \ldots, n - \nu_1) \) for some \( n \), and understand that the entries of \( S \) in each \( i \)th row from the bottom are no more than \(-\phi_d^\vee \) (resp. \( \phi_d \)) for \( 1 \leq i \leq d \).

Let \( \alpha = (\alpha_1, \ldots, \alpha_d) \) be the sequence of weakly decreasing positive integers of length \( d \). Let

\[
\tilde{s}_{\lambda/\mu}(X_\alpha) = \det (h_{\lambda_i-\mu_j-i+j}(X_{\alpha_j}))_{1 \leq i, j \leq d}.
\]
By (5.4), it is easy to check that
\[ \hat{s}_{\frac{\lambda}{\mu}}(X_\alpha) = s_{\frac{\hat{\mu}}{\hat{\lambda}}}(X_\phi), \]
where \( \hat{\lambda} = (n - \lambda_{d-i+1})_{1 \leq i \leq d}, \) \( \hat{\mu} = (n - \mu_{d-i+1})_{1 \leq i \leq d} \) for some \( n \geq \lambda_1, \mu_1, \) and \( \phi \) is the reverse sequence of \( \alpha. \) In particular, we have for \( \nu \in P \) with \( \ell(\nu) \leq d, \)
\[ s_{\nu^*}(X_\phi) = \hat{s}_\nu(X_\alpha). \]

5.4. A flagged RSK correspondence. In the sequel, we assume that \( S \) is a finite subset of \( \mathbb{N}^2 = \mathbb{N} \times \mathbb{N}. \) We define \( \theta(S) \) to be the border strip of the smallest partition \( \lambda \) such that \( S \subset \lambda. \) Recall that a border strip of a partition \( \lambda \) is a skew diagram \( \lambda/\mu, \) where \( \lambda = (\alpha_1, \ldots, \alpha_d|\beta_1, \ldots, \beta_d) \) and \( \mu = (\alpha_2, \ldots, \alpha_d|\beta_2, \ldots, \beta_d) \) following Frobenius notation. We put \( c(S) = (\alpha_1, \beta_1). \)

**Example 5.2.** Let \( S = \{ (1, 1), (1, 4), (2, 2), (3, 1), (3, 3), (4, 3) \}. \) Then
\[
S = \begin{array}{cccccc}
1 & 2 & 3 & 4 & \cdots \\
1 & \bullet & \bullet & & & \\
2 & & \bullet & & & \\
3 & & & \bullet & & \\
4 & & & & \bullet & \\
\vdots & & & & & \\
\end{array}, \quad \theta(S) = \begin{array}{cccc}
\square & \square & \ast & \ast \\
\square & \square & \ast & \ast \\
\square & \square & \ast & \ast \\
\end{array}, \quad (4, 3, 3, 3)/(2, 2, 2) = \begin{array}{cccc}
\square & \square & \ast & \ast \\
\square & \square & \ast & \ast \\
\square & \square & \ast & \ast \\
\end{array},
\]
where the skew diagram consisting of the black boxes is the border strip \( \theta(S) \) and \( \ast \) indicates the point \( c(S) = (4, 4). \)

We define inductively a finite sequence of points \( c_1 = (\alpha_1, \beta_1), \ldots, c_d = (\alpha_d, \beta_d) \) as follows;
(1) let \( S^{(1)} = S \) and put \( c_1 = c(S^{(1)}), \)
(2) for \( 1 \leq k \leq d - 1, \) let \( S^{(k+1)} = S^{(k)} \setminus \theta(S^{(k)}), \) and put \( c_{k+1} = c(S^{(k+1)}), \)
where \( d \) is the smallest one such that \( S^{(d+1)} = \emptyset. \) Note that \( c_{k+1} \) is located to the northwest of \( c_k, \) that is, \( \alpha_k > \alpha_{k+1} \) and \( \beta_k > \beta_{k+1}. \) Now, we define
\[
\lambda(S) = (\alpha_1, \ldots, \alpha_d|\beta_1, \ldots, \beta_d) \in P
\]
following Frobenius notation, where \( (\alpha_1, \ldots, \alpha_d) \) (resp. \( (\beta_1, \ldots, \beta_d) \)) corresponds to the lower (resp. upper) flag of \( \lambda(S) \) with respect to its diagonal. We define \( w(S) \) to be the permutation in \( W_{\lambda_0} \) corresponding to \( \lambda(S), \) that is,
\[
\lambda(w(S)) = \lambda(S).
\]
Example 5.3. Let $S$ be as in Example 5.2. Then

$S(1) = \begin{array}{cccc}
1 & \bullet & \bullet & \\
2 & \bullet & & \\
3 & & & \\
4 & & & \\
\end{array}$, $\theta(S(1)) = (4,3,3,3)/(2,2,2)$ =

\begin{array}{cccc}
\square & \square & \square & \\
\square & \square & & \\
\square & & & \\
\square & & & \\
\end{array}$,

$S(2) = \begin{array}{cccc}
1 & \bullet & \bullet & \\
2 & \bullet & & \\
3 & & & \\
4 & & & \\
\end{array}$, $\theta(S(2)) = (2,2,1)/(1)$ =

\begin{array}{cccc}
\square & \square & \cdot & \\
\square & \square & & \cdot \\
\square & & \cdot & \\
\square & & & \\
\end{array}$,

$S(3) = \begin{array}{cccc}
1 & \bullet & \bullet & \\
2 & \bullet & & \\
3 & & & \\
4 & & & \\
\end{array}$, $\theta(S(3)) = (1)$ =

\begin{array}{cccc}
\square & \cdot & \cdot & \\
\square & \cdot & & \\
\square & & \cdot & \\
\square & & & \\
\end{array}$,

where we have $c_1 = (4,4)$, $c_2 = (3,2)$, $c_3 = (1,1)$. Hence

$\lambda(S) = (4,3,1,4,2,1) = (4,4,3,2)$.

For $w \in W^\Lambda_0$ with a reduced expression $w = s_{i_1} \cdots s_{i_r}$, we put

$$M_w = \{ \tilde{f}_{i_1}^{m_1} \cdots \tilde{f}_{i_r}^{m_r} \oplus | m_1, \ldots, m_r \geq 0 \} \setminus \{0\},$$

that is, $M_w = M_{\emptyset, \emptyset, w}$ (see (5.3)). For $A \in M$, let $\mathrm{supp}(A) = \{ (i,j) | a_{-i',j} \neq 0 \} \subset \mathbb{N}^2$ be the support of $A$.

Theorem 5.4. For $w \in W^\Lambda_0$, we have

$$M_w = \{ A \mid \lambda(\mathrm{supp}(A)) \subset \lambda(w) \} = \{ A \mid w(\mathrm{supp}(A)) \leq w \}.$$
Then we have reduced expressions \( w = s_{\alpha(w)} s_{\alpha(w)-1} \cdots s_{\alpha_1} \) and \( w' = s_{\alpha(w)-1} \cdots s_{\alpha_1} \), where \( \alpha(w) = r \). Note that

\[
(5.10) \quad M_w = \bigcup_{k \geq 0} \tilde{f}_r^k M_{w'} \setminus \{0\}.
\]

Let \( N_w = \{ A \mid \lambda(\text{supp}(A)) \subset \lambda(w) \} \). By induction hypothesis, it suffices to show that \( N_w = \bigcup_{k \geq 0} \tilde{f}_r^k N_{w'} \setminus \{0\} \).

Let \( A \in N_w \) be given. We first claim that \( A \in \tilde{f}_r^k N_{w'} \) for some \( k \geq 0 \). We will keep the previous notations \( \lambda(S) \), \( \theta(S^{(k)}) \), and \( c_k = (\alpha_k, \beta_k) \) \((1 \leq k \leq d)\) with \( S = \text{supp}(A) \). If \( \lambda(S) \) does not contain the box \( c \) corresponding to \( \lambda(w)/\lambda(w') \), then \( \lambda(S) \subset \lambda(w') \) and \( A \in N_{w'} \). So we may assume that \( c \in \lambda(S) \).

**Case 1.** Suppose that \( r = 0 \). By definition of \( \lambda(S) \), we have \( \theta(S^{(d)}) = (1,1) \).

If we choose \( k \geq 1 \) such that the entry of \( \tilde{c}_0^k A \) at \((1,1)\) is 0, then \( \text{supp}(\tilde{c}_0^k A) = \text{supp}(A) \setminus \{(1,1)\} \). Hence \( \lambda(\text{supp}(\tilde{c}_0^k A)) \subset \lambda(w') \) and \( \tilde{c}_0^k A \in N_{w'} \).

**Case 2.** Suppose that \( r \neq 0 \). We may assume that \( r > 0 \) since the argument for \( r < 0 \) is almost the same. In this case, we have \( c_s = (\alpha_s, \beta_s) \) with \( \beta_s = r+1 \) for some \( 1 \leq s \leq d \). There exists \( 1 \leq m \leq \alpha_s \) such that \( a_{-m^y, r+1} \neq 0 \) with \((m, r+1) \in S^{(s)} \) and \((i, r+1) \notin S^{(s)} \) for \( m < i \leq \alpha_s \). Since \( c \) is a removable corner of \( \lambda(w) \) and hence of \( \lambda(S) \), we have \( a_{-j^y, r} = 0 \) for \( 1 \leq i < m \) (see Figure 2). Otherwise, we have \( \beta_{s-1} = r \), which implies that \( c \) is not removable. Let \( A = A(i,j) \) and consider the subword of \( i \) consisting of \( r \) and \( r+1 \). Then the \( r+1 \)'s corresponding to the non-zero entries \( a_{-i^y, r+1} \) for \( 1 \leq i \leq m \) can be replaced by \( r \) applying \( \tilde{c}_r^k \) for some \( k \geq 1 \). Equivalently, applying \( \tilde{c}_r^k \), the entries \( a_{-i^y, r} \) (resp. \( a_{-i^y, r+1} \)) are replaced by \( a_{-i^y, r} + a_{-i^y, r+1} \) (resp. \( 0 \)) for \( 1 \leq i \leq m \). On the other hand, we can check that \( (\alpha_t, \beta_t) \) are invariant under \( \tilde{c}_r^k \) for \( 1 \leq t \leq d \) with \( t \neq s \). Hence \( \lambda(\text{supp}(\tilde{c}_r^k A)) \subset \lambda(w') \) and \( \tilde{c}_r^k A \in N_{w'} \).

Conversely, let \( A \in N_{w'} \) be given. Using similar arguments, it is not difficult to check that either \( \lambda(\text{supp}(\tilde{J}_r^k A)) = \lambda(\text{supp}(A)) \) or \( \lambda(\text{supp}(\tilde{J}_r^k A))/\lambda(\text{supp}(A)) = c \) whenever \( \tilde{J}_r^k A \neq 0 \). Hence \( \lambda(\text{supp}(\tilde{J}_r^k A)) \subset \lambda(w) \). This completes our induction. \( \square \)

**Corollary 5.5.** For \( w \in W_{A_0} \), we have

\[
\text{ch} M_w = \sum_{S \subset \text{N}^2 \atop \text{w}(S) \leq w} \prod_{(i,j) \in S} \frac{x_i y_j}{(1 - x_i y_j)}.
\]

**Remark 5.6.** Identifying \((i,j) \in \text{N}^2 \) with \( -\epsilon_i + \epsilon_j \in \Delta(u_-) \), one may write

\[
\text{ch} M_w = \sum_{S \subset \Delta(u_-) \atop \text{w}(S) \leq w} \prod_{(\alpha)} \frac{\epsilon^\alpha}{(1 - \epsilon^\alpha)}.
\]
Now, let us put
\begin{equation}
\mathcal{T}_w = \kappa(M_w) = \{ \tilde{f}_{i_1}^{m_1} \cdots \tilde{f}_{i_r}^{m_r} (\emptyset, \emptyset) | m_1, \ldots, m_r \geq 0 \} \setminus \{0\}.
\end{equation}
for \(w \in W^{\Lambda_0}\) with a reduced expression \(w = s_{i_1} \cdots s_{i_r}\). We define \(\alpha(w) = (\alpha_1, \ldots, \alpha_d)\) and \(\beta(w) = (\beta_1, \ldots, \beta_d)\) to be strict partitions of length \(d\) such that
\begin{equation}
\lambda(w) = (\alpha(w)|\beta(w)).
\end{equation}
We put \(d(w) = d\), the diagonal length of \(\lambda(w)\) and
\begin{equation}
\phi(w) = (\phi_1, \ldots, \phi_d) = (\alpha_d, \ldots, \alpha_1),
\end{equation}
\begin{equation}
\psi(w) = (\psi_1, \ldots, \psi_d) = (\beta_d, \ldots, \beta_1),
\end{equation}
which are flags of length \(d\).

**Theorem 5.7.** For \(w \in W^{\Lambda_0}\), we have
\[
\mathcal{T}_w = \bigcup_{\ell(\nu) \leq d(w)} \operatorname{SST}_{B^<_{\emptyset}}(\nu^\pi)_{|w(\nu)|} \times \operatorname{SST}_{B^>_{\emptyset}}(\nu^\pi)_{|w(\nu)|}.
\]

**Proof.** For convenience, let \(S_w\) be the right-hand side of the above identity. We will use induction on \(\ell(w) = |\lambda(w)|\). When \(\ell(w) = 1\), i.e. \(w = s_0\), it is clear. We assume that \(\ell(w) \geq 2\).

Choose \(w' \in W^{\Lambda_0}\) such that \(\ell(w') = \ell(w) - 1\), equivalently, \(\lambda(w') \subset \lambda(w)\) with \(|\lambda(w)/\lambda(w')| = 1\). Let \(r\) be the residue of \(\lambda(w)/\lambda(w')\). By (5.10) and the induction hypothesis, we have only to show that \(S_w = \bigcup_{k \geq 0} \tilde{f}_r^k S_w' \setminus \{0\}\). We assume that \(\lambda(w)\) is as in (5.12) with \(d = d(w)\).
CASE 1. Suppose that \( r = 0 \). Then we have \( d \geq 2 \) and \( \alpha_d = \beta_d = 1 \) and
\[
\lambda(w') = (\alpha_1, \ldots, \alpha_{d-1})|\beta_1, \ldots, \beta_{d-1}.
\]

Let \( (S, T) \in \mathcal{S}_w \) be given, where \( \text{sh}(S) = \text{sh}(T) = \eta^\pi \) for some \( \eta \in \mathcal{P} \) with \( \ell(\eta) \leq d(w') \). For \( k \geq 1 \), let \( (S', T') = \tilde{f}_k(S, T) \) and \( \tau^\pi = \text{sh}(S') = \text{sh}(T') \in \mathcal{P}^\pi \).

By definition of \( \tilde{f}_0 \), we have \( \ell(\tau) \leq \ell(\eta) + 1 \leq d(w) \). If \( \ell(\tau) < d(w) \), then it is clear that \( (S', T') \in \mathcal{S}_{w'} \subset \mathcal{S}_w \). Assume that \( \ell(\tau) = d(w) \), that is, \( \ell(\tau) = \ell(\eta) + 1 = d(w) \).

Then the first rows of \( S' \) and \( T' \) are filled only with \(-1 \) and \( 1 \), respectively, and the entries in the other rows of \( S' \) and \( T' \) still satisfy the flag conditions given by \( \phi(w') \) and \( \psi(w') \), respectively. Since \( \phi(w) = (1, \phi(w')) \) and \( \psi(w) = (1, \psi(w')) \), we have \( (S', T') \in \mathcal{S}_{w'} \).

Conversely, let \( (S, T) \in \mathcal{S}_w \) be given with \( \tau^\pi = \text{sh}(S) = \text{sh}(T) \in \mathcal{P}^\pi \). If \( \ell(\tau) < d(w) \), then \( (S, T) \in \mathcal{S}_{w'} \). If \( \ell(\tau) = d(w) \), then the first rows of \( S \) and \( T \) are filled only with \(-1 \) and \( 1 \), respectively, and they are all removable under successive application of \( \tilde{e}_0 \). Hence, the highest weight element \( (S', T') \) in the 0-string of \( (S, T) \) belongs to \( \mathcal{S}_{w'} \).

CASE 2. Suppose that \( r \neq 0 \). We may assume that \( r > 0 \) since the argument for \( r < 0 \) is almost the same. In this case, we have \( d(w') = d(w) \), and for some \( 1 \leq i < d \)
\[
\lambda(w') = (\alpha_1, \ldots, \alpha_i, \ldots, \alpha_d) | \beta_1, \ldots, \beta_i - 1, \ldots, \beta_d),
\]
where \( \beta_i = r + 1 \).

Let \( (S, T) \in \mathcal{S}_w \) be given. Note that the entries of \( T \) in the \( i \)th row from the bottom are no more than \( \beta_i - 1 = r \), and no \( r \) appears in the above rows. This implies that \( \tilde{f}_k(S, T) \in \mathcal{S}_w \cup \{0\} \) for \( k \geq 0 \).

Conversely, let \( (S, T) \in \mathcal{S}_w \) be given. The entries of \( T \) in the \( (i + 1) \)th row from the bottom are no more than \( \beta_{i+1} \leq \beta_i - 1 = r \). So any \( r + 1 \) in the \( i \)th row of \( T \) from the bottom, if exists, can be replaced by \( r \) applying \( \tilde{e}_k \) for some \( k \geq 0 \). This implies that \( \tilde{e}_r(S, T) \in \mathcal{S}_{w'} \).

\[ \square \]

**Corollary 5.8.** For \( w \in W_{\Lambda_0} \), we have
\[
\text{ch} J_w = \sum_{\nu \in \mathcal{P}, \ell(\nu) \leq d(w)} \delta_{\nu}(X_{\alpha(w)}) \delta_{\nu}(Y_{\beta(w)}).
\]

Combining Theorem [5.4] and Theorem [5.7] we obtain a flagged version of the RSK correspondence and the Cauchy identity.

**Corollary 5.9.** For \( w \in W_{\Lambda_0} \), the map \( \kappa \) in [3.6] gives a bijection
\[
\{ A \in \mathcal{M} \mid \lambda(\text{supp}(A)) \subset \lambda(w) \} \rightarrow \bigsqcup_{\nu \in \mathcal{P}, \ell(\nu) \leq d(w)} \text{SST}_{B_{0\nu}^\pi}(\nu^\pi)_{\phi(w)} \times \text{SST}_{B_{0\nu}^\pi}(\nu^\pi)_{\psi(w)},
\]
when restricted to $M_w$, and it commutes with $\tilde{e}_i$ ($i \in \mathbb{Z}$). In particular, we have

$$\sum_{S \subseteq \mathbb{N}^2} \prod_{(i,j) \in S} \frac{x_i y_j}{(1 - x_i y_j)} = \sum_{\nu \in \mathcal{P}} \hat{s}_\nu(X_{\alpha(w)} \hat{s}_\nu(Y_{\beta(w)})�.}
$$

**Remark 5.10.** (1) For $m, n \geq 1$, let $w_{m,n}$ be the element in $W^{\Lambda_0}$ such that $\lambda(w_{m,n}) = (n^m)$. In this case, we recover the usual RSK correspondence with $m \times n$ matrices and the Cauchy identity with variables $x_i, y_j$ ($1 \leq i \leq m, 1 \leq j \leq n$).

(2) For $S \subseteq \mathbb{N}^2$, let $r(S)$ be the diagonal length of $\lambda(S)$, i.e. the length of $\alpha$ or $\beta$ when $\lambda(S) = (\alpha|\beta)$. Then for $n \geq 1$, we have

$$\sum_{S \subseteq \mathbb{N}^2} \prod_{(i,j) \in S} \frac{x_i y_j}{(1 - x_i y_j)} = \sum_{\nu \in \mathcal{P}} s_{\nu}(X) s_{\nu}(Y).$$

When multiplied by $e^{-n\Lambda_0}$ the right-hand side of the identity is equal to the character of the irreducible highest weight representation of $\widehat{\mathfrak{gl}}_{\infty}$, a central extension of $\mathfrak{gl}_{\infty}$, with highest weight $-n\Lambda_0$, which is not integrable [9]. Hence the left-hand side gives another character formula for this highest weight module. Note that the right-hand side has a Jacobi-Trudi type formula (see [31, Ex.7.16 d] and [18] for its generalization to irreducible $\widehat{\mathfrak{gl}}_{\infty}$-modules with negative integral charges) and a Weyl-Kac type formula [19].

(3) From the correspondence between $M_w$ and $T_w$, we see that the entries in the first columns of bitableaux in $T$ is determined only by the support of the corresponding matrix in $M$. This fact was also observed by Stanley [31, Ex.7.100] in a purely combinatorial way.

### 5.5. Demazure crystal $B_w(n\Lambda_0)$.

**Proposition 5.11.** Let $w \in W^{\Lambda_0}$, let $d = d(w)$, $\phi = \phi(w)$, and $\psi = \psi(w)$. Then for $n \geq 1$, there is a bijection

$$B_w(n\Lambda_0) \longleftrightarrow \bigsqcup_{\nu \in \mathcal{P}} SST_{B_{<0}(<\nu^\pi)} \phi \times SST_{B_{>0}(<\nu^\pi)} \psi.$$

**Proof.** It follows from Remark 4.4 and Theorem 5.7. □

**Remark 5.12.** Given $A = A(i,j) \in M$ with $(i,j) \in \Omega$, let $c(A)$ be the maximal length of decreasing subwords of $i$. It is well known that $c(A)$ is equal to the number of columns in $P(A)$ or $Q(A)$ (cf. [15]). By Remark 4.4 and Theorem 5.4, the embedding $\Psi_n$ gives a bijection

$$B_w(n\Lambda_0) \longleftrightarrow \{ A \mid \lambda(\text{supp}(A)) \subseteq \lambda(w), c(A) \leq n \}.$$
For \( i \in \mathbb{Z} \) and \( \lambda \in \mathcal{P} \), let \( D_i \) be the linear operator on \( \mathbb{C}[\mathcal{P}] \) defined by
\[
D_i(e^{\lambda}) = e^{\lambda} \cdot \frac{1 - e^{-(1 + \langle \lambda, h_i \rangle)} \alpha_i}{1 - e^{-\alpha_i}}.
\]
The operators \( D_i \) satisfy the braid relations, and hence for a reduced expression of \( w = s_{i_1} \cdots s_{i_r} \in W \), the operator \( D_w = D_{i_1} \cdots D_{i_r} \) is well-defined. By [5.2], we have
\[
\text{ch} B_w(n\Lambda_0) = D_w(e^{n\Lambda_0}).
\]
Combining with Proposition 5.11, we obtain the following combinatorial identity.

**Corollary 5.13.** Let \( n, d \geq 1 \), and let \( \alpha, \beta \) be two strict partitions of length \( d \). Then
\[
D_w(e^{n\Lambda_0})e^{-n\Lambda_0} = \sum_{\nu \in \hat{\mathcal{P}} \subset (n^d)} \hat{s}_\nu(X_\alpha)\hat{s}_\nu(Y_\beta),
\]
where \( w \) is the unique element in \( W^{\Lambda_0} \) such that \( \lambda(w) = (\alpha|\beta) \).

### 5.6. Crystals of symmetric matrices.
From now on, let \( \epsilon \) denote either 1 or 2. We put
\[
\hat{\mathcal{M}}^\epsilon = \{ A \in \mathcal{M} | a_{-i^\nu,j} = a_{-j^\nu,i} \text{ for } i, j \geq 1, \epsilon \text{ divides } a_{-i^\nu,i} \text{ for } i \geq 1 \}.
\]
For \( i \in \mathbb{Z}_{\geq 0} \), let
\[
\tilde{E}_0 = e_0, \quad \tilde{F}_0 = f_0,
\]
\[
\tilde{E}_i = e_i e_{-i}, \quad \tilde{F}_i = f_i f_{-i} \quad (i \in \mathbb{Z}_{>0}).
\]
By similar arguments as in Proposition 3.1, we can check the following.

**Proposition 5.14.**
(1) \( \hat{\mathcal{M}}^\epsilon \cup \{ 0 \} \) is invariant under \( \tilde{E}_i \) and \( \tilde{F}_i \) for \( i \in \mathbb{Z}_{\geq 0} \).
(2) \( \hat{\mathcal{M}}^\epsilon = \{ \tilde{F}_{i_1} \cdots \tilde{F}_{i_r} | \forall r \geq 0, i_1, \ldots, i_r \in \mathbb{Z}_{\geq 0} \} \setminus \{ 0 \} \).

Put
\[
\hat{P} = \{ \lambda \in \mathcal{P} | \frac{1}{\epsilon} \langle \lambda, h_0 \rangle \in \mathbb{Z}, \langle \lambda, h_i \rangle = \langle \lambda, h_{-i} \rangle \text{ (} i \in \mathbb{Z}_{>0} \text{) } \},
\]
\[
\hat{\Pi} = \{ \hat{\alpha}_0 = \epsilon \alpha_0, \quad \hat{\alpha}_i = \alpha_i + \alpha_{-i} \text{ (} i \in \mathbb{Z}_{>0} \text{) } \} \subset \hat{P}.
\]
Then \( \hat{\Pi} \) is a set of simple roots for the root system associated to the affine Lie algebra \( \mathfrak{b}_\infty \text{ (resp. } \mathfrak{c}_\infty \text{) when } \epsilon = 1 \text{ (resp. } \epsilon = 2 \) (cf. [S]), where \( \hat{P} \) is a weight lattice. The associated Dynkin diagrams are
\[
\mathfrak{b}_\infty : \quad \circ \leftrightarrow \bullet \quad \circ \cdots \quad \circ \quad \circ \cdots \quad \circ
\]
For $i \in \mathbb{Z}_{\geq 0}$, let $\hat{h}_i \in \hat{P}^*$ be determined by $\langle \lambda, \hat{h}_i \rangle = \langle \lambda, h_i \rangle = \langle \lambda, h_{-i} \rangle$ if $i > 0$, and $\langle \lambda, \hat{h}_0 \rangle = \frac{1}{\epsilon}(\lambda, h_0)$ for $\lambda \in \hat{P}$. Then $\hat{P}^\vee = \{ \hat{h}_i | i \in \mathbb{Z}_{\geq 0} \}$ is a set of simple coroots. As in Definition 2.1, one may define an $x_{\infty}$-crystal $(x = b, c)$ with respect to $\tilde{E}_i, \tilde{F}_i, \tilde{e}_i, \tilde{f}_i (i \in \mathbb{Z}_{\geq 0})$ and $\hat{w}t$. By Proposition 5.14, $\hat{\mathcal{M}}^\epsilon$ is an $x_{\infty}$-crystal with highest weight element $\emptyset$. Here, for $A \in \hat{\mathcal{M}}^\epsilon$, $\hat{w}t(A) = \hat{w}t, \tilde{e}_i(A) = \epsilon_i(A)$ and $\hat{f}_i(A) = \varphi_i(A)$.

Since each $A$ in $\hat{\mathcal{M}}^\epsilon$ is symmetric, we have $\kappa(A) = (-S^\vee, S)$, where $S \in SST_{B>0}(\nu^\pi)$ for some $\nu \in P$ with $\epsilon|\nu$, that is, $\epsilon_1 \nu_i$ for $i \geq 1$ (cf. [15, 17]), and $-S^\vee$ is the semistandard tableau obtained by replacing each entry $i$ in $S$ with $-i^\vee$. Hence the map $\hat{\kappa} : A \mapsto S$ gives a bijection

$$\hat{\kappa} : \hat{\mathcal{M}}^\epsilon \longrightarrow \bigsqcup_{\nu \in P, \epsilon|\nu} SST_{B>0}(\nu^\pi).$$

**Proposition 5.15.** Put $\hat{\Lambda}_0 = \epsilon \Lambda_0$. For $n \geq 1$, let

$$\hat{\mathcal{B}}^\epsilon(n\hat{\Lambda}_0) = \{ \hat{F}_{i_1} \cdots \hat{F}_{i_r} H_{n\hat{\Lambda}_0} \mid r \geq 0, i_1, \ldots, i_r \in \mathbb{Z}_{\geq 0} \} \setminus \{ 0 \}.$$ 

Then $\Psi_{en}(\hat{\mathcal{B}}^\epsilon(n\hat{\Lambda}_0) \otimes T_{-n\hat{\Lambda}_0}) = \hat{\mathcal{M}}^\epsilon \cap \text{Im} \Psi_{en}$.

**Proof.** Let $w \in \mathcal{B}(n\hat{\Lambda}_0)$ be given with $\kappa \left( \Psi_{en}(w \otimes t_{-n\hat{\Lambda}_0}) \right) = (w^{\vee}_{<0}, w^{\vee}_{>0})$. Then we can check that

$$w \in \hat{\mathcal{B}}^\epsilon(n\hat{\Lambda}_0) \iff w^{\vee}_{<0} = -w^{\vee}_{>0} \iff \kappa^{-1}(w^{\vee}_{<0}, w^{\vee}_{>0}) \in \hat{\mathcal{M}}^\epsilon,$$

where $-w^{\vee}_{>0}$ is the tableau obtained from $w^{\vee}_{>0}$ by replacing each entry $i$ with $-i^\vee$. By Remark 4.3, we obtain the required identity. \[ \square \]

**Remark 5.16.** By Proposition 5.14, 5.16 and Proposition 5.15, we have a one-to-one correspondence

$$\hat{\mathcal{B}}^\epsilon(n\hat{\Lambda}_0) \leftrightarrow \bigsqcup_{\nu \in P, \epsilon|\nu, \epsilon_1 \leq en} SST_{B>0}(\nu^\pi).$$

By [13, Theorem 5.1], $\hat{\mathcal{B}}^\epsilon(n\hat{\Lambda}_0)$ is isomorphic to the crystal graph of the integrable highest weight $x_{\infty}$-module with highest weight $n\hat{\Lambda}_0$, and Proposition 5.15 implies that $\hat{\mathcal{M}}^\epsilon$ is the limit of $\hat{\mathcal{B}}^\epsilon(n\hat{\Lambda}_0)$.
Let $\sigma$ be the linear automorphism on $P$ defined by $\sigma(\Lambda_0) = \Lambda_0$ and $\sigma(\epsilon_i) = -\epsilon_{-i}$ for $i \in \mathbb{Z}$. Then $\sigma(\alpha_i) = \alpha_{-i}$ for $i \in \mathbb{Z}$. Let

$$\hat{W} = \{ w \in W \mid w\sigma = \sigma w \}.$$ 

Put $\hat{s}_0 = s_0$ and $\hat{s}_i = s_is_{-i}$ for $i \in \mathbb{Z}_{>0}$. Then $\hat{W}$ is the Coxeter group generated by $\hat{s}_i$ ($i \in \mathbb{Z}_{>0}$), which is isomorphic to the Weyl group of $x_\infty$ (see Section 5.2). Let $\hat{W}_{\Lambda_0}$ be the set of minimal length left coset representatives of $\hat{W}_{\Lambda_0}$. We have

$$\hat{W}_{\Lambda_0} = \hat{W} \cap W_{\Lambda_0},$$

and $\lambda(w) = \lambda(w)'$ or $\phi(w) = \psi(w)$ for $w \in \hat{W}_{\Lambda_0}$.

For $w \in \hat{W}_{\Lambda_0}$ with a reduced expression $w = \hat{s}_{i_1} \cdots \hat{s}_{i_r}$ ($i_1, \ldots, i_r \in \mathbb{Z}_{>0}$), let

$$\hat{M}^e_w = \{ \hat{F}^m_{i_1} \cdots \hat{F}^m_{i_r} \mathbb{O} \mid m_1, \ldots, m_r \geq 0 \} \setminus \{ 0 \},$$

$$\hat{B}_w(n\Lambda_0) = \{ \hat{F}^m_{i_1} \cdots \hat{F}^m_{i_r} \hat{H}_{n\Lambda_0} \mid m_1, \ldots, m_r \geq 0 \} \setminus \{ 0 \}.$$ 

Since $\hat{B}_w(n\Lambda_0)$ is the crystal graph of an integrable highest weight $x_\infty$-module (see Remark 5.16), the Demazure crystal $\hat{B}_w(n\Lambda_0)$ is well-defined and so is $\hat{M}^e_w$ by Proposition 5.15.

**Proposition 5.17.** For $w \in \hat{W}_{\Lambda_0}$, we have

$$\hat{M}^e_w = \hat{M}^e \cap M_w, \quad \hat{\kappa}(\hat{M}^e_w) = \bigcup_{\nu \in \bar{\nu} \cap \nu} SST_{\mathcal{B}_{>0}}(\nu^\pi)_{\phi(w)}.$$ 

**Proof.** Let us prove the first identity. Then the second one follows from Proposition 5.7 and (5.16). First, it is clear by definition that $\hat{M}^e_w \subset \hat{M}^e \cap M_w$. Suppose that $A \in \hat{M}^e \cap M_w$ is given. Let $w' = \hat{s}_{i_2} \cdots \hat{s}_{i_r}$, where $w = \hat{s}_{i_1} \cdots \hat{s}_{i_r}$ is a reduced expression. If $i_1 > 0$, then we have

$$\hat{e}_{i_1}A \neq 0 \iff \hat{e}_{-i_1}A \neq 0 \iff \hat{E}_{i_1}A \neq 0,$$

since $A$ is symmetric. If $i_1 = 0$, then we have $\hat{e}_0A \neq 0$ if and only if $\hat{E}_0A \neq 0$ by definition. From (5.10), it follows that $\hat{E}_{i_1}A \in M_{w'}$ for some $k \geq 0$. If we use induction on $\ell(w)$, then we have $\hat{E}_{i_1}A \in \hat{M}^e \cap M_{w'} = \hat{M}^e_{w'}$, and hence $A \in \hat{M}^e_{w'}$. $\square$

Put $Z = \{ z_i = x_iz_i \mid i \geq 1 \}$. For $S \subset \mathbb{N}^2$, let $S' = \{ (i, j) \mid (j, i) \in S \}$. Then Proposition 5.17 yields the following identity.

**Corollary 5.18.** For $w \in \hat{W}_{\Lambda_0}$, we have

$$\sum_{S = S' \subset \mathbb{N}^2, (i, j) \in S} \prod_{w(S) \leq w} \frac{z_i^e}{1 - z_i^e} \prod_{i < j} \frac{z_i z_j}{1 - z_i z_j} = \sum_{\nu \in \bar{\nu} \cap \nu} \hat{s}_\nu(Z_{\alpha(w)}).$$
Corollary 5.19. For \( w \in \hat{W}^\Lambda_0 \) and \( n \geq 1 \), there is in one-to-one correspondence
\[
\hat{B}_w^n(\Lambda_0) \leftrightarrow \bigcup_{\nu \in \mathcal{P}, \lambda | \nu, \nu \subset \pi} \text{SST}_{\text{B} \geq 0}(\nu^\pi)_{\phi(w)}.
\]

Proof. It follows from Proposition 5.11 and (5.17).

For \( i \in \mathbb{Z}_{\geq 0} \) and \( \lambda \in \hat{P} \), let \( \hat{D}_i \) be the linear operator on \( \mathbb{C}[\hat{P}] \subset \mathbb{C}[P] \) defined by
\[
\hat{D}_i(e^\lambda) = e^\lambda \frac{1 - e^{-(1 + \langle \lambda, \alpha_i \rangle)\tilde{\alpha}_i}}{1 - e^{-\tilde{\alpha}_i}}.
\]
For a reduced expression of \( w = \tilde{s}_{i_1} \cdots \tilde{s}_{i_r} \in \hat{W} \), we put \( \hat{D}_w = \hat{D}_{i_1} \cdots \hat{D}_{i_r} \). Combining the Demazure character formula \( \text{ch} \hat{B}_w^n(\Lambda_0) = \hat{D}_w(e^n\Lambda_0) \) with Corollary 5.19 we obtain the following identity.

Corollary 5.20. Let \( n, d \geq 1 \), and let \( \alpha \) be a strict partition of length \( d \). Then
\[
\hat{D}_w(e^n\Lambda_0)e^{-n\Lambda_0} = \sum_{\nu \in \mathcal{P}, \lambda | \nu, \nu \subset (en)^d} \tilde{s}_\nu(Z_\alpha),
\]
where \( w \) is the unique element in \( \hat{W}^\Lambda_0 \) such that \( \lambda(w) = (\alpha | \alpha) \).

6. Plane partitions

A plane partition is a collection of non-negative integers \( \pi = (\pi_{ij})_{i,j \geq 1} \) such that \( \pi_{ij} \neq 0 \) for only finitely many \( i, j \), and \( \pi_{ij} \geq \pi_{i+1,j} \) and \( \pi_{ij} \geq \pi_{i,j+1} \) for \( i, j \geq 1 \). A shape of \( \pi \) denoted by \( \text{sh}(\pi) \) is a Young diagram determined by the support of \( \pi \), i.e. \( \{ (i, j) | \pi_{ij} \neq 0 \} \). We may identify \( \pi \) with a tableau of shape \( \text{sh}(\pi) \) with entries in \( \mathbb{N} \) weakly decreasing in each row and column from left to right and top to bottom, respectively. Let \( \mathcal{P} \) denote the set of all plane partitions.

Let us recall the correspondence between \( M \) and \( \mathcal{P} \) [2]. Let \( A \in \mathcal{M} \) be given with \( \kappa(A) = (\text{P}(A), \text{Q}(A)) \). For each \( k \), let \( \lambda^{(k)} = (\alpha(k) | \beta(k)) \) be a partition where \( \alpha(k) \) and \( \beta(k) \) are strict partitions given by reading the entries of the \( k \)-th columns of \( \text{P}(A) \) and \( \text{Q}(A) \) from bottom to top (ignoring \( - \) and \( \lor \) in \( \text{P}(A) \)), respectively. Note that \( \lambda^{(1)} \supset \lambda^{(2)} \supset \cdots \). We define \( \pi(A) = (\pi(A)_{ij})_{i,j \geq 1} \) by \( \pi(A)_{ij} = | \{ k | (i, j) \in \lambda^{(k)} \} | \). It is easy to check that \( \pi(A) \) is a plane partition, and the mapping \( A \mapsto \pi(A) \) yields a bijection from \( M \) to \( \mathcal{P} \). One may identify a plane partition \( \pi \) with a set of unit cubes, where \( \pi_{ij} \) cubes are stacked vertically at each position \( (i, j) \in \text{sh}(\pi) \). Then \( \lambda^{(k)} \) is the \( k \)-th layer of \( \pi(A) \) from the bottom.

For \( \pi \in \mathcal{P} \), let \( |\pi| = \sum_{i,j} \pi_{ij} \) and for each \( r \in \mathbb{Z} \), let \( \text{tr}_r(\pi) = \sum_{i \geq 1} \pi_{ii+r} \), which is called the \( r \)-trace of \( \pi \) [2] [30]. Let \( q \) and \( v_1, v_2, \ldots \) be formal variables. For a subset
X of \( \mathcal{P} \), the norm (resp. trace) generating function of \( X \) is defined to be
\[
\sum_{\pi \in X} q^{|\pi|} \text{ and } \sum_{\pi \in X} \prod_{r \in \mathbb{Z}} v_r^{tr_r(\pi)},
\]
respectively. Note that a norm generating function can be obtained from a trace generating function by specializing \( v_r = q \) for \( r \in \mathbb{Z} \).

For \( n \geq 1 \) and \( \lambda \in \mathcal{P} \), we put
\[
\mathcal{P}(\lambda) = \{ \pi \in \mathcal{P} | \text{sh}(\pi) \subset \lambda \},
\]
(6.1)
\[
\mathcal{P}_{\leq n} = \{ \pi \in \mathcal{P} | \pi_{11} \leq n \},
\]
\[
\mathcal{P}(\lambda)_{\leq n} = \mathcal{P}_{\leq n} \cap \mathcal{P}(\lambda).
\]

Note that \( \mathcal{P}_{\leq 1} \) is the set of ordinary partitions \( \mathcal{P} \), and hence a \( \mathfrak{gl}_\infty \)-crystal (cf.\cite{27}), where for \( \lambda \in \mathcal{P} \) and \( r \in \mathbb{Z} \), \( \tilde{f}_r \lambda \) is defined by a partition \( \mu \) such that \( \mu/\lambda \) is a single box with residue \( r \), or 0 if such \( \mu \) does not exist. If we assign the weight of the empty partition to 0, then it is easy to see that \( \mathcal{P}_{\leq 1} \) is isomorphic to \( B(\Lambda_0) \otimes T^{-\Lambda_0} \). In general, we define a \( \mathfrak{gl}_\infty \)-crystal structure on \( \mathcal{P} \) by identifying \( \pi = (\lambda^{(1)}, \lambda^{(2)}, ..., \lambda^{(n)}) \) (\( \pi \in \mathcal{P} \)) with an element of \( B(\Lambda_0)^{\otimes n} \otimes T^{-n\Lambda_0} \) for a sufficiently large \( n \), where \( \lambda^{(k)} \) is the \( k \)th layer of \( \pi \).

**Proposition 6.1.** As \( \mathfrak{gl}_\infty \)-crystals, we have

(1) \( \mathcal{P} \simeq M \),
(2) \( \mathcal{P}_{\leq n} \simeq B(n\Lambda_0) \otimes T^{-n\Lambda_0} \) for \( n \geq 1 \).

**Corollary 6.2.**

(1) \( \text{ch} M \) is the trace generating function of \( \mathcal{P} \).
(2) For \( n \in \mathbb{N} \), \( e^{-n\Lambda_0} \text{ch} B(n\Lambda_0) \) is the trace generating function of \( \mathcal{P}_{\leq n} \).

**Proof.** For \( \pi \in \mathcal{P} \), suppose that \( \text{wt}(\pi) = -\sum_{r \in \mathbb{Z}} k_r \alpha_r \). Then \( k_r \) is equal to the \( r \)-trace of \( \pi \). Identifying \( e^{-\alpha_r} \) with \( v_r \) in \( \text{ch} M \) and \( \text{ch} B(n\Lambda_0) \), we obtain the trace generating functions for \( \mathcal{P} \) and \( \mathcal{P}_{\leq n} \), respectively. \( \square \)

**Remark 6.3.** By the celebrated Weyl-Kac character formula \cite{8}, the trace generating function of \( \mathcal{P}_{\leq n} \) is
\[
\sum_{w \in W} e^{w(\Lambda_0 + \rho) - n\Lambda_0 - \rho} \prod_{\alpha \in \Delta^+} (1 - e^{-\alpha}),
\]
where \( \rho \in \mathfrak{h}^* \) is given by \( \langle \rho, h_i \rangle = 1 \) for all \( i \in \mathbb{Z} \). The norm generating function for \( \mathcal{P} \) and \( \mathcal{P}_{\leq n} \) are the corresponding principal \( q \)-characters, which are obtained by putting \( e^{-\alpha_i} = q \) for \( i \in \mathbb{Z} \). Then we recover
\[
\text{ch}_q M = \frac{1}{\prod_{i \geq 1} (1 - q^i)^i} \text{ and } \text{ch}_q B(n\Lambda_0) = \frac{1}{\prod_{i \geq 1} (1 - q^i)^{\min(i,n)}}
\]
(see \cite{8} for evaluating \( q \)-characters), which are originally due to MacMahon \cite{26}. 


Now, the results in the previous section give the following representation theoretic interpretations on plane partitions with bounded conditions.

**Proposition 6.4.** Let $\lambda \in \mathcal{P}$ and $n \geq 1$ be given. Let $w \in W^{\Lambda_0}$ be such that $\lambda(w) = \lambda$. As $\mathfrak{gl}_\infty$-crystals, we have

1. $\mathcal{P}(\lambda) \simeq M_w$,
2. $\mathcal{P}(\lambda)_{\leq n} \simeq B_w(n\Lambda_0) \otimes T_{-n\Lambda_0}$.

**Proof.** Let $A \in M$ be given. By Theorem 5.7, we see that $A \in M_w$ if and only if $\lambda^{(1)} \subset \lambda(w)$. This implies that $A \in M_w$ if and only if $\text{sh}(\pi(A)) \subset \lambda(w)$. The second isomorphism follows from Proposition 5.11. □

By Corollary 6.2 and Proposition 6.4, we obtain the generating functions for plane partitions bounded by a given shape as follows.

**Corollary 6.5.** Let $\lambda \in \mathcal{P}$ and $n \geq 1$ be given. Let $w \in W^{\Lambda_0}$ be such that $\lambda(w) = \lambda$.

1. The trace generating function of $\mathcal{P}(\lambda)$ is
   \[
   \sum_{S \subseteq \Delta(u_w)} \prod_{\lambda(S) \subseteq \lambda} \frac{e^\alpha}{1 - e^\alpha}.
   \]
2. The trace generating function of $\mathcal{P}(\lambda)_{\leq n}$ is $e^{-n\Lambda_0}D_w(e^{n\Lambda_0})$.

**Remark 6.6.** (1) There are determinantal formulas for the norm and trace generating functions of various classes of plane partitions including $\mathcal{P}(\lambda)$ and $\mathcal{P}(\lambda)_{\leq n}$ (see [16] for a most general form and the references therein for the previous works by other people). Also there are evaluations of those determinants into nice product forms for some special classes of plane partitions. But there is no such formula for $\mathcal{P}(\lambda)$ and $\mathcal{P}(\lambda)_{\leq n}$ as far as we know.

(2) A representation theoretic approach to plane partitions was first introduced by Proctor [28], where the norm generating functions for $\mathcal{P}((u^v))_{\leq n}$ was proved to be the $q$-dimension of the irreducible $\mathfrak{sl}_{u+v}$-module with highest weight $n\omega$ ($\omega$ is the $u$th fundamental weight).

A plane partition $\pi = (\pi_{ij})$ is called symmetric if $\pi_{ij} = \pi_{ji}$ for all $i, j \geq 1$. Similarly for $\epsilon = 1, 2, n \geq 1$ and $\lambda \in \mathcal{P}$ with $\lambda = \lambda'$, we put

\[
\tilde{\mathcal{P}}^\epsilon = \{ \pi \in \mathcal{P} \mid \pi \text{ is symmetric and } \epsilon \text{ divides } \pi_{ii} \text{ for all } i \geq 1 \},
\]

$$\tilde{\mathcal{P}}^\epsilon = \mathcal{P}^\epsilon \cap \tilde{\mathcal{P}}^\epsilon,$$

$$\tilde{\mathcal{P}}^\epsilon \leq n = \mathcal{P} \leq n \cap \tilde{\mathcal{P}}^\epsilon,$$

$$\tilde{\mathcal{P}}^\epsilon \leq n = \mathcal{P} \leq n \cap \tilde{\mathcal{P}}^\epsilon.$$
It is clear that \( \widehat{M}^\epsilon \) is in one-to-one correspondence with \( \widehat{P}^\epsilon \). We define for a subset \( X \) of \( \widehat{P}^\epsilon \) the norm (resp. trace) generating function of \( X \) by

\[
\sum_{\pi \in X} q^{|\pi|} \text{ and } \sum_{\pi \in X} \prod_{r \geq 0} v^{tr_r(\pi)},
\]

respectively, where \( tr'_r(\pi) = tr_r(\pi) \) for \( r \geq 1 \) and \( tr'_0(\pi) = \epsilon^{-1} tr_0(\pi) \).

As in Section 5.6, we assume that \( x = b \) if \( \epsilon = 1 \) and \( x = c \) if \( \epsilon = 2 \). We define an \( x_\infty \)-crystal structure on \( \widehat{P}^\epsilon \subset P \) with \( \widehat{E}_i, \widehat{F}_i \) for \( i \geq 0 \) [5.15]. Similar to Proposition 6.1 and Proposition 6.4, we can prove the following.

**Proposition 6.7.** Let \( n \geq 1 \) and \( \lambda \in \mathcal{P} \) with \( \lambda = \lambda' \) be given, and let \( w \in \widehat{W}^\lambda_0 \) be such that \( \lambda(w) = \lambda \). As \( x_\infty \)-crystals, we have

1. \( \widehat{P}^\epsilon \simeq \widehat{M}_w \).
2. \( \widehat{P}^\epsilon(\lambda) \simeq \widehat{M}_w^\lambda' \).
3. \( \widehat{P}^\epsilon_{\leq n} \simeq \widehat{B}^\epsilon(n\lambda_0) \otimes T_{-n\lambda_0} \).
4. \( \widehat{P}(\lambda)_{\leq n} \simeq \widehat{B}_w^\epsilon(n\lambda_0) \otimes T_{-n\lambda_0} \).

**Remark 6.8.** (1) For \( \pi \in \widehat{P}^\epsilon \), we have \( wt(\pi) = - \sum_{r \geq 0} k_r \alpha_r \), where \( k_r = tr'_r(\pi) \). Hence, identifying \( e^{-\alpha_r} \) with \( v_r \), we obtain the trace generating functions for symmetric plane partitions in Proposition [6.7] as the characters of the corresponding \( x_\infty \)-crystals. The norm generating functions can be obtained by specializing \( e^{-\alpha_r} = q^2 \) for \( r \geq 1 \) and \( e^{-\alpha_0} = q^\epsilon \).

(2) The norm generating function of \( \widehat{P}(m^m)_{\leq n} \) was conjectured by MacMahon [26], and it was proved by Andrews [1] and Macdonald [25]. It was observed by Proctor [28, 29] that the norm generating function of \( \widehat{P}(m^m)_{\leq n} \) is the \( q \)-dimension of irreducible representation of the complex simple Lie algebra \( \mathfrak{so}(2m+1) \subset \mathfrak{b}_\infty \) or \( \mathfrak{sp}(2m) \subset \mathfrak{c}_\infty \) with highest weight corresponding to \( n\lambda_0 \) (following our notation).

(3) Recently Tingley [33] gave a nice representation theoretic interpretation of cylindric plane partitions in terms of crystal graphs for affine Lie algebra \( \widehat{\mathfrak{sl}}_n \) and its generating function. It would be interesting to find an application of affine Demazure crystals to cylindric plane partitions.

**References**

[1] G. E. Andrews, *Plane partitions. I. The MacMahon conjecture*, Studies in foundations and combinatorics, pp. 131–150, Adv. in Math. Suppl. Stud., 1, Academic Press, New York-London, 1978.

[2] E. A. Bender, D. E. Knuth, *Enumeration of plane partitions*, J. Combinatorial Theory Ser. A 13 (1972), 40–54.

[3] J. Brundan, *Dual canonical bases and Kazhdan-Lusztig polynomials*, J. Algebra 306 (2006), 17–46.
[4] V. I. Danilov, G. A. Koshevoy, Bi-crystals and crystal \((GL(V), GL(W))\) duality, RIMS preprint, (2004) no. 1458.
[5] J. Fuchs, A. N. Schellekens, C. Schweigert, From Dynkin diagram symmetries to fixed point structures, Comm. Math. Phys. 180 (1996), 39–97.
[6] E. R. Gansner, The Hillman-Grassl correspondence and the enumeration of reverse plane partitions, J. Combin. Theory Ser. A 30 (1981), 71–89.
[7] R. Howe, Remarks on classical invariant theory, Trans. AMS 313 (1989), 539–570.
[8] V. G. Kac, Infinite-dimensional Lie algebras, Third edition, Cambridge University Press, Cambridge, 1990.
[9] V. G. Kac, A. Radul, Representation theory of the vertex algebra \(W_{1+\infty}\), Transform. Groups 1 (1996), 41–70.
[10] M. Kashiwara, Crystalizing the \(q\)-analogue of universal enveloping algebras, Comm. Math. Phys. 133 (1990), 249–260.
[11] M. Kashiwara, Crystal bases and Littelmann’s refined Demazure character formula, Duke Math. J. 71 (1993), 839–858.
[12] M. Kashiwara, On crystal bases, Representations of groups, CMS Conf. Proc., 16, Amer. Math. Soc., Providence, RI, (1995), 155–197.
[13] M. Kashiwara, Similarity of crystal bases, Contemp. Math. 194 (1996), 177–186.
[14] M. Kashiwara, T. Nakashima, Crystal graphs for representations of the \(q\)-analogue of classical Lie algebras, J. Algebra 165 (1994), 295–345.
[15] D. E. Knuth, Permutations, matrices, and generalized Young tableaux, Pacific J. Math. 34 (1970), 709–727.
[16] C. Krattenthaler, Generating functions for plane partitions of a given shape, Manuscripta Math. 69 (1990), 173–201.
[17] J.-H. Kwon, Crystal graphs for Lie superalgebras and Cauchy decomposition, J. Algebraic Combin. 25 (2007), 57–100.
[18] J.-H. Kwon, Rational semistandard tableaux and character formula for the Lie superalgebra \(g\hat{\mathfrak{su}}(1|\infty)\), Adv. Math. 217 (2008), 713-739.
[19] J.-H. Kwon, A combinatorial proof of a Weyl-type formula for hook Schur polynomials, J. Algebraic Combin. 28 (2008), 439–459.
[20] A. Lascoux, Double crystal graphs, Studies in Memory of Issai Schur, Progress in Math. 210, Birkhäuser (2003), 95–114.
[21] C. Lecouvey, Crystal bases and combinatorics of infinite rank quantum groups, preprint, arXiv:math/0604636.
[22] C. Lenart, A unified approach to combinatorial formulas for Schubert polynomials, J. Algebraic Combin. 20 (2004), 263–299.
[23] M. A. A. van Leeuwen, Double crystals of binary and integral matrices, Electron. J. Combin. 13 (2006).
[24] I. G. Macdonald, Notes on Schubert polynomials, Laboratoire de combinatoire et d’informatique mathématique (LACIM), Université du Québec à Montréal, Montreal, 1991.
[25] I. G. Macdonald, Symmetric functions and Hall polynomials, Oxford University Press, 2nd ed., 1995.
[26] P. A. MacMahon, Combinatory analysis, vols. 1 and 2, Cambridge University Press, Cambridge, 1915, 1916; reprinted in one volume by Chelsea, New York, 1960.
[27] K. Misra, T. Miwa, Crystal base for the basic representation of $U_q(\widehat{\mathfrak{sl}(n)})$, Comm. Math. Phys. 134 (1990), 79–88.

[28] R. A. Proctor, Bruhat lattices, plane partition generating functions, and minuscule representations, European J. Combin. 5 (1984), 331–350.

[29] R. A. Proctor, New symmetric plane partition identities from invariant theory work of De Concini and Procesi, European J. Combin. 11 (1990), 289–300.

[30] R. P. Stanley, The conjugate trace and trace of a plane partition, J. Combinatorial Theory Ser. A 14 (1973), 53–65.

[31] R. P. Stanley, Enumerative Combinatorics, vol. 2, Cambridge University Press, 1998.

[32] B. E. Sagan, R. Stanley, Robinson-Schensted algorithms for skew tableaux, J. Combin. Theory Ser. A 55 (1990), 161–193.

[33] P. Tingley, Three combinatorial models for affine $\mathfrak{sl}_n$ crystals, with applications to cylindric plane partitions, Int. Math. Res. Not. 143 (2007).

Department of Mathematics, University of Seoul, Seoul 130-743, Korea

E-mail address: jhkwon@uos.ac.kr