Critical Phenomena in DIS

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Saturation in deep inelastic scattering (DIS) and deeply virtual Compton scattering (DVCS) is associated with a phase transition between the partonic gas, typical of moderate $x$ and $Q^2$, and partonic fluid appearing at increasing $Q^2$ and decreasing Bjorken $x$. In this paper we do not intend to propose another parametrization of the structure function; instead we suggest a new insight into the internal structure of the nucleon, as seen in DIS, and its connection with that revealed in high-energy nucleons and heavy-ion collisions.

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I. INTRODUCTION

The thermodynamic approach to high-energy hadronic collisions was initiated by the paper of Fermi Ref.\cite{Fermi}, subsequently successfully applied to high-energy nuclear collisions. Less familiar, although numerous, are the applications of the thermodynamical approach to deep inelastic scattering (DIS)\cite{Bialas,Koch,McLerran:1986uv}. Formally, by vector meson dominance, DIS can also fall in the category of hadron-hadron (heavy vector meson-baryon) processes. While the creation of a large number of particles, justifying the applicability of the statistical mechanics, is typical of both classes of reactions, there are subtleties that make the thermodynamics of DIS different from that of hadronic or nuclear collisions. One is connected with the reference frame: while thermalization and possible creation of ”quark-gluon plasma” is considered

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in the rest frame, the partonic picture of DIS is realized in the infinite momentum frame, \( p_z \to \infty \). The connection and transition between these frames recently was treated in Ref. [6].

The second point is the role and treatment of the temperature. Any statistical distribution in a gas of quasi-free particles (nucleons or partons) implies the existence of a "temperature" \( T \), as e.g. in the Boltzmann distribution \( e^{-E/T} \), although its physical interpretation is not unique. A limited temperature \( T \) is typical of the hadronic phase [7] but not of the partons, that can be heated indefinitely.

In Ref. [8] arguments based on parton-hadron duality lead the concept of "pseudo-thermalization", according to which, after the hadron is broken by the interaction, the inclusive spectrum reflects the "thermal" distribution holding at the parton level.

A related problem in DIS is the choice of the variable in the statistical distribution. The use of the Bjorken variable \( x \), instead of the energy (or momentum), in the above (or similar) distributions needs the introduction of a proper dimensional parameter, related to the change of the coordinate systems (see, e.g. [6]), \( \exp(-x/\bar{x}) \), \( \bar{x} = x(1+k^2/x^2m^2) \), where \( k \) is the quark transverse momentum and \( m \) is the proton mass. Note that the "temperature" \( T \) here is that of the partonic gas (or liquid/fluid) inside the nucleon and hence, unlike the hadronic systems [7], it must not be limited.

FIG. 1: Change of regime in the behavior of the SF is visible around \( x \sim 10^{-2} \). Beyond that value, Bjorken scaling holds (modulus \( \ln Q^2 \)), while at lower \( x \), the \( Q^2 \) dependence changes drastically and it cannot be considered as a small scaling violating effect anymore (see Ref. [9].)

It was suggested in Ref. [9] that saturation in DIS, predicted by QCD and observed
experimentally, corresponds to the condensation of partonic gas to a fluid. The interior of
the nucleon excited in DIS or DVCS undergoes a phase transition from a partonic gas (high-
and intermediate, \( x \sim 0.05 \)) to a partonic fluid. The division line is located roughly at
those values of \( x \) and \( Q^2 \) where Bjorken scaling is not valid any more, as shown in Fig. II.

Our idea, based on the observed behavior of the DIS structure functions, is that the
partonic matter in a nucleon (or nucleus) as seen in DIS, undergoes a change of phase from
a nearly perfect gas, typical of the Bjorken scaling region, to a liquid, where the logarithmic
scaling violation is replaced by a power (see Figure I). The presence of two regions, namely
that of Bjorken scaling and beyond it (call them for the moment "dilute" and "dense"), via
an intermediate mixed phase, are visible in this figure. They can be quantifies in various
ways that will be discussed in the next section. The relevant variable here are the fraction
of the nucleon momentum, or the Bjorken variable \( x \) and the incident photon's virtuality
\( Q^2 \).

The co-existence of two phases, gaseous and fluid, can be described e.g. by the van der
Waals equation, valid in a tremendous range of its variables and applicable to any system
(e.g. molecular, atomic or nuclear) with short range repulsion and long range attraction
between the constituents, see e.g. Fig. 8 from Ref. 10.

In Sec. IV we describe the properties of the partonic matter by means of the van der Waals
equation and discuss the physical meaning of its parameters. To anticipate, the equation of
state shows two phases, gaseous and fluid, with a mixed phase in-between, called spinodal.

Notice that the two phases, gaseous and fluid, here are made of the same stuff - colored
quarks and gluons. We do not consider the possibility of a (de)confinement phase transition. In this sense, the situation is similar to that "low-energy" nuclear physics or in classical substances, such as vapour condensation.

Comparison between the two pictures, the DIS structure functions and the van der Waals EOS, especially at the critical points (regions), is the subject of our discussion in Secs. IV and V.

II. DIS STRUCTURE FUNCTIONS, GEOMETRICAL SCALING AND SATURATION

Most generally, DIS structure functions (SF) are sums of a singlet (S) and non-singlet (NS) terms, $F_2(x, Q^2) = F_2^S(x, Q^2) + F_2^{NS}(x, Q^2)$, each a product of a low-$x, \sim x^\alpha$ and high-$x, (1-x)^n$ factor, more specifically (see e.g. [11] for more details):

$$F_2^S(x, Q^2) = A_0 \left( \frac{Q^2}{Q^2 + a} \right)^{1+\Delta(Q^2)} x^{-\Delta(Q^2)} (1-x)^n(Q^2)+4, \quad \Delta(Q^2) = \Delta_0 \left( 1 + \frac{bQ^2}{Q^2 + c} \right),$$ (1)

where $\Delta_0 \approx 0.1$, $b \approx 0.4$, etc. The above model is applicable in the Regge domain of small and intermediate values of $Q^2$, mimicking the apparent "hardening" of the Pomeron, manifest in the rise of $\Delta(Q^2)$ from about 0.1 to 0.4. At higher virtualities, Regge behavior should be replaced by the effects of QCD evolution, absent from the above simple model. The drastic increase of $F_2(x, Q^2)$ in $1/x$ with increasing $Q^2$, as seen in Fig. [11], indicates the onset of a new dynamical regime. Although, for virtual particles, i.e. in DIS, there is no analogue of the Froissart bound, there are arguments why a change of regime should occur.

A simple and convincing argument is a physical one, according to which, by the drastically (power-like) scaling-violating rise of $F_2(x, Q^2)$, the number of partons is increasing and their volume tends to exceed that of the nucleon (gluon saturation), thus leading to partons’ recombination (or their condensation, in terms of statistical physics, see below). To quantify this effect, one compares the number of gluons per unit of transverse area, $\rho \sim xG(x, Q^2)/\pi R^2$, and the cross section for recombination, $\sigma \sim \alpha_s/Q^2$, where $\alpha_s$ is the

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1 Due to Lorentz contraction in the longitudinal direction, soft gluons belonging to different nucleons have overlapping wave functions and thus act coherently. The effect of transverse (impact-parameter) distribution can be deduced from DVCS scattering and general parton distributions [12, 13].
QCD running coupling. Saturation occurs when \( 1 \gtrsim \rho \sigma \), or equivalently \( Q^2 \leq Q_s^2(x) \), where \( Q_s^2(x) = \alpha_s x G(x Q^2) / \pi R^2 \) is the so-called saturation momentum \[14\]. The saturation domain, shown in Fig. 3 is delineated by the equation \( Q^2 = Q_s^2(x) \); phenomenologically \[14\], \( Q_s^2(x) \sim x^{-0.3} \).

A related phenomenon in DIS is the so-called geometrical scaling (GS) (not to be confused with GS in hadronic processes!), implying the existence of a ”saturation radius” \( R_0(x) \), given by \[14\]

\[
R_0^2(x) = \left( \frac{x}{x_0} \right)^{\lambda} / Q_0^2,
\]

where \( Q_0^2 = 1 \text{ GeV}^2 \), \( \lambda = 0.29 \) and \( x_0 = 3 \cdot 10^{-4} \). GS was derived \[14\] from the dipole model of DIS and was shown to be compatible with the HERA data in a wide span of \( x \) and \( Q^2 \).

Substitution of (2) into the effective intercept of the low-\( x \) factor in Eq. (1) (the Pomeron contribution),

\[
F_2(x, Q^2) \sim x^{\Delta [Q_s^2(x) \sim x^{-0.3}]},
\]

produces a maximum (saturation) in this simple model for the singlet (gluonic) structure function.

Formally, saturation can be treated in the context of the BFKL equation \[15\] and its modifications \[16\]. The production of partons in a nucleon by splitting of partons, resulting in the increase of their number \( N \), is described by the asymptotic BFKL evolution equation

\[
\frac{\partial N(x, k_T^2)}{\partial \ln(1/x)} = \alpha_s K_{BFKL} \otimes N(x, K_T^2),
\]

where \( K_{BFKL} \) is the BFKL integral kernel (splitting function). It results in a power-like increase of the SF toward smaller \( x \), \( F_2(x) \sim x^{-\alpha_P + 1} \), where \( \alpha_P \) is the BFKL pomeron intercept, \( \alpha_P - 1 = (4 \alpha_s N_c \ln 2) / \pi > 0 \); \( \alpha_s \) is the (“running”) QCD coupling and \( N_c \) is the number of colours. Since the number of partons increases with energy, at certain ”saturation” values of \( x \) and \( Q^2 \), an inverse process, namely the recombination of pairs of partons comes into play, and the BFKL equation is replaced by the following one \[17\]

\[
\frac{\partial N(x, k_T^2)}{\partial \ln(1/x)} = \alpha_s K_{BFKL} \otimes N(x, K_T^2) - \alpha_s [K_{BFKL} \otimes N(x, K_T^2)]^2,
\]

based on the simple idea that the number of recombinations is roughly proportional to the the number of parton pairs, \( N^2 \). Saturation sets in when the second, quadratic term in Eq. (5) overshoots the first, linear one, thus tampering the increase of the number of produced partons and securing unitarity.
An explicit model realizing the onset of saturation, based on an interpolation between the known asymptotic – Regge and DGLAP - regimes of the structure functions, yet fitting the data, was proposed in Ref. [19]. This model will be used to match the saturation region in DIS with the predicted phase transition.

The ansatz for the small-\(x\) singlet part (labelled by the upper index \((S, 0)\) of the proton structure function, interpolating between the soft (VMD, Pomeron) and hard (DGLAP evolution) regimes, reads [19]:

\[
F_2^{(S,0)}(x, Q^2) = A \left( \frac{Q^2}{Q^2 + a} \right)^{1 + \tilde{\Delta}(Q^2)} e^{\Delta(x, Q^2)},
\]

with the "effective power"

\[
\tilde{\Delta}(Q^2) = \epsilon + \gamma_1 \ln \left( 1 + \gamma_2 \ln \left[ 1 + \frac{Q^2}{Q_0^2} \right] \right),
\]

and

\[
\Delta(x, Q^2) = \left( \tilde{\Delta}(Q^2) \ln \frac{x_0}{x} \right)^{f(Q^2)},
\]

where

\[
f(Q^2) = \frac{1}{2} \left( 1 + e^{-Q^2/Q_1^2} \right).
\]
At small and moderate values of $Q^2$ (specified by fits to the data), the exponent $\tilde{\Delta}(Q^2)$ may be interpreted as a $Q^2$-dependent "effective Pomeron intercept".

The function $f(Q^2)$ "switches" Regge behavior, where $f(Q^2) = 1$, to the asymptotic solution of the GLAP evolution equation, where $f(Q^2) = 1/2$.

By construction, the model has the following asymptotic limits:

a) Large $Q^2$, fixed $x$:

$$F_2^{(S,0)}(x, Q^2 \to \infty) \to A \exp \sqrt{\gamma_1 \ln \frac{Q^2}{Q_0} \ln \frac{Q}{Q_0}},$$

which is the asymptotic solution of the GLAP evolution equation.

b) Low $Q^2$, fixed $x$:

$$F_2^{(S,0)}(x, Q^2 \to 0) \to A \ e^{\Delta(x,Q^2\to0)} \left( \frac{Q^2}{a} \right)^{1+\tilde{\Delta}(Q^2\to0)},$$

with

$$\tilde{\Delta}(Q^2 \to 0) \to \epsilon + \gamma_1 \gamma_2 \left( \frac{Q^2}{Q_0^2} \right) \to \epsilon, \quad f(Q^2 \to 0) \to 1,$$

where from

$$F_2^{(S,0)}(x, Q^2 \to 0) \to A \left( \frac{x_0}{x} \right)^{\epsilon} \left( \frac{Q^2}{a} \right)^{1+\epsilon} \propto (Q^2)^{1+\epsilon} \to 0,$$

as required by gauge invariance.

c) Low $x$, fixed $Q^2$:

$$F_2^{(S,0)}(x \to 0, Q^2) = A \left( \frac{Q^2}{Q^2 + a} \right)^{1+\tilde{\Delta}(Q^2)} e^{\Delta(x\to0,Q^2)}.$$

If

$$f(Q^2) \sim 1,$$

i.e. when $Q^2 \gg Q_1^2$, we get the standard (Pomeron-dominated) Regge behavior (with a $Q^2$ dependence in the effective Pomeron intercept)

$$F_2^{(S,0)}(x \to 0, Q^2) \to A \left( \frac{Q^2}{Q^2 + a} \right)^{1+\tilde{\Delta}(Q^2)} \left( \frac{x_0}{x} \right)^{\tilde{\Delta}(Q^2)} \propto x^{-\tilde{\Delta}(Q^2)}.$$

The total cross-section for $(\gamma, p)$ scattering as a function of the center of mass energy $W$ is

$$\sigma_{\gamma,p}^{\text{tot},(0)}(W) = 4\pi^2 \alpha \left[ \frac{F_2^{(S,0)}(x, Q^2)}{Q^2} \right]_{Q^2 \to 0} = 4\pi^2 \alpha A a^{-1-\epsilon} x_0^\epsilon W^{2\epsilon}.$$
FIG. 4: Logarithmic derivative of the proton SF, $B_x(x, Q^2) = \frac{\partial \ln F_2(x, Q^2)}{\partial (\ln(1/x))}$ calculated in Ref. [19] (see the text below).

Interestingly, in this model the logarithmic derivative of the structure function,

$$B_Q(x, Q^2) = \frac{\partial F_2(x, Q^2)}{\partial (\ln Q^2)}, \quad B_x(x, Q^2) = \frac{\partial F_2(x, Q^2)}{(\partial \ln(1/x))},$$

called $B_x$ slope, the equivalent of the Pomeron intercept in the Regge domain, has a maximum, indicative of saturation, and shown in Fig. 4.

Since the saturation phenomena occur in the small-$x$, which is a gluon-dominated region, we shall be mainly interested in that component of the SF, namely in

$$F_2^S \sim xG(x, Q^2),$$

where $G$ is the gluon (singlet) distribution function, defined above.

A slightly different, although related approach is that of string or parton percolation [5, 20, 21]. Gas-liquid phase transition with a percolation type model was considered recently in Ref. [22], however in that paper the “liquid” formed of coloured quarks and gluons is colourless, i.e. hadronic.
III. STATISTICAL MODELS OF SF

We assume that the interior of a nucleon (the idea can be extended to nuclei as well), as seen in (inclusive) deep inelastic scattering (DIS) or (exclusive) deeply virtual Compton scattering (DVCS), is a thermodynamic system, that, similar to the case of nuclear or heavy-ion collisions, bear collective (thermodynamic) properties governed by a relevant equation of state (EoS). The idea that DIS structure functions (SF) can be treated thermodynamically by means of statistical mechanics is not new \[2–6\], moreover it continues attract attention, although several subtle points remain to be clarified. Above all, it concerns the choice of the relevant coordinate system and of the corresponding variables.

For simplicity, we focus our attention on the small-\(x\) singlet (gluon) component of the SF, the extension to low-\(x\) and/or the non-singlet (valence quark) contributions being straightforward

\[
xG(x, Q^2) \sim \frac{X_0 x^b}{\exp \left[ \frac{(x - X_0)}{\bar{x}} \right] + 1},
\]

where \(x\) is the Bjorken (light-cone) variable, \(X_0\) is the chemical potential, that for the gluon component can be set zero, and \(\bar{x}\) is interpreted as the temperature inside the proton.

In some papers, the dimensional energy \(E\) (or momentum \(k\)) variable is used in the statistical model of the SF instead of \(x\) as in Eq. (7). This is not a simple kinematical problem, since thermodynamics implies the presence of the dimensional temperature in the statistical distribution like \(k/T\) (be it of the Fermi-Dirac, Bose-Einstein or Boltzmann type), while the appearance of \(x\) as in Eq. (7) needs some extra modification. In Ref. \[6\] this was circumvented by using a dimensionless ”temperature” \(\bar{x} = 2T/m\), where \(m\) is the proton mass, which is a consequence of the transition from the rest frame to the infinite-momentum frame (IMF). Accordingly,

\[
G(x) \sim \exp \left( -\frac{mx}{2\bar{T}} \right).
\]

The Boltzmann factor in the denominator of Eq. (7) can mimic the large-\(x\) \(1-x^n\) factor in the SF, although it should be reconciled also with the quark counting rules, appearing in the power \(n\).

The last point is connected also with the EoS expected from the statistical distribution of the type (7). Let us remind that for an ideal gas of particles

\[
P(T) = \int_0^\infty k d^3 k \exp(-k/T),
\]
which, due to radial symmetry, can be rewritten as

\[ \int_0^\infty k^3 dk \exp(-k/T), \]

and by the change of variable \( y = k/T \), one trivially arrives at the Stefan-Boltzmann (S-B) EoS, \( P \sim T^4 \). This fact can be interpreted also physically: the large-\( x \) component of the SF corresponds to a dilute perfect gas of partons. The low-\( x \) factor in Eq. (7) will affect the ideal Stefan-Boltzmann EoS only when it will be written in Eq. (7) as \((x/\bar{x})^b = (m x^2 T)^b\) instead of \( x^b \). As a consequence, the ideal Stefan-Boltzmann EoS will be modified as

\[ P(T) \sim T^{4+b}. \]  

(9)

The relative contribution of this correction is negligible at small \( x \), but it increases with \( Q^2 \) and decreasing \( x \), resulting in a gas-liquid phase transition.

In most of the paper on the subject, \( Q^2 \) dependence is neglected - either for simplicity, or "conceptually", by assuming that the statistical approach applies to the SF for some fixed, "input" value of \( Q^2 \), from which it evolves according to the DGLAP equation. We do not exclude high \( Q^2 \) evolution of the SF, however with the following caveats:

a) the structure functions show strong \( Q^2 \) dependence, already at low \( x \), below the perturbative DGLAP domain; b) at large \( Q^2 \), instead of the monotonic DGLAP evolution, due to the proliferation of partons, the inverse process of their recombination is manifest. This process is essential in our interpretation of the saturation as a gas-liquid phase transition (see the next Section). So, we prefer to keep explicit \( Q^2 \) dependence for all \( x \) and \( Q^2 \). It is mild in the "gaseous" region of point-like partons (at large \( x \)), becoming important towards the saturation region (depending on both \( x \) and \( Q^2 \)), where the point-like partons are replaced by finite-size droplets of the partonic "fluid". This transition will be treated in the next two sections by means of the classical van de Waals equation.

IV. GAS-FLUID PHASE TRANSITION IN THE VAN DER WAALS EQUATION OF STATE

Having defined the statistical properties of the SF, we now proceed to an equation of state (EoS) describing the transition between a parton gas, via a mixed foggy phase, to the partonic liquid. To this end, we use the van der Waals equation

\[ (P + N^2 a/V^2)(V - Nb) = NT, \]  

(10)
see, e.g. [23, 24], where $a$ and $b$ are parameters depending on the properties of the system, $N$ is the number of particles and $V$ is the volume of the "container", $V(s) = \pi R^3(s)$, $R(s) \sim \ln s$ is the nucleon radius in our case. For point-like particles (perfect gas), $a = b = 0$, and Eq. (10) reduces to $pV = NT$, and, since $N/V \sim T^3$, we get in this approximation $p \sim T^4$, to be compared with $p \sim T^{4+b}$, Eq. (9), of the preceding Section.

Alternatively, Eq. (10) can be written as

$$(P + a/V^2)(V - b) = RT,$$

or, equivalently

$$P = \frac{RT}{V - b} - \frac{a}{V^2}.$$  

The parameter $b$ is responsible for the finite dimensions of the constituents, related to $1/Q$ in our case, and the term $a/V^2$ is related to the (long-range) forces between the constituents. From this cubic equation in $V$ one finds [23] the following values for the critical values $V = V_c$, $P = P_c$, and $T = T_c$ in terms of the parameters $a$ and $b$:

$$V_c = 3b, \quad P_c = a/(27b^2) \quad T_c = 8a/(27Rb).$$

The particle number $N(s)$ can be calculated as [25]

$$N(s) = \int_0^1 dx F_2(x, Q^2),$$

where $F_2(x, Q^2)$ is given by Eqs. (1) or (6). We remind the kinematics: $s = Q^2(1-x)/x + m^2$, which at small $x$ reduces to $s \approx Q^2/x$. The radius of the constituent as seen in DIS is $r_0 \sim 1/Q$, hence its two-dimensional volume is $\sim Q^{-2}$.

By introducing "reduced" volume, pressure and temperature,

$$\mathcal{P} = P/P_c, \quad \mathcal{V} = V/V_c = \rho_c/\rho, \quad \mathcal{T} = T/T_c,$$

the van der Waals equation (10) can be rewritten as

$$
\left(\mathcal{P} + 3/\mathcal{V}^3\right)(\mathcal{V} - 1/3) = 8\mathcal{T}/3.
$$

Note that Eq. (11) contains only numerical constants, and therefore it is universal. States of various substances with the same values of $\mathcal{P}$, $\mathcal{V}$ and $\mathcal{T}$ are called "corresponding states"; equation (11) is called the "van der Waals equation for corresponding states". The
universality of the liquid-gas phase transition and the corresponding principle are typical if any system with short-range repulsive and long-range attractive forces. This property is shared both by ordinary liquids and by nuclear matter. This was demonstrated, in particular, in Ref. [10], where typical van der Waals curves were shown to be similar to those derived by means of the Skyrme effective interaction, see Fig. 2.

Following Ref. [10], we present two example of EoS, one based on the Skyrme effective interaction and finite-temperature Hartree-Fock theory, and the other one is the van der Waals EoS. Jaqaman et al. [10] start with the EoS

\[ P = \rho kT - a_0 \rho^2 + a_3 (1 + \sigma) \rho^{2+\sigma}, \]

(12)

where \( \rho = N/V \) is the density and \( a_0, a_3 \) and \( \sigma \) are parameters, \( \sigma = 1 \) corresponding to the usual Skyrme interaction. Particular values of the above parameters, corresponding to various options (degenerate and non-degenerate Fermi gas) as well as to finite and infinite nuclear matter are quoted in Table 1 of Ref. [10].

According to the law of the corresponding states, Eq. (12) is universal for scaled (reduced) variables, for which, with \( \sigma = 1 \), it becomes

\[ P = 3T/V - 3/V^2 + 1/V^3, \]

to be compared with the van der Waals EoS

\[ P = 8T/(3V - 1) - 3/V^2. \]

Let us now write the van der Waals EoS in the form

\[ P(T; N, V) = -\left( \frac{\partial F}{\partial V} \right)_{TN} = \frac{NT}{V - bN} - a \left( \frac{N}{V} \right)^2 = \frac{nT}{1 - bn} - an^2, \]

(13)

where \( n = N/V \) is the particle number density, \( a \) is the strength of the mean-field attraction, and \( b \) governs the short-range repulsion. We identify the particle number density with the SF \( F_2(x, Q^2) \) of Sec. II. Fig. 5 shows the pressure-density dependence calculated from Eq. (13) with \( a = 5 \text{ GeV}^{-2} \) and \( b = 0.2 \text{ GeV}^{-3} \). Notice that while Eq. (13) is sensitive to \( b \) (short-range repulsion), it is less so to \( a \) (long-range attraction). Representative isotherms are shown in this figure: the dark blue line (second from the top) is the critical one, \( T_c = 8a/(27b) \). Above this temperature (top line, in pale blue), the pressure rises uniformly with density, corresponding to a single thermodynamical state for each \( P \) and
By contrast, for subcritical temperatures \(0 < T < T_c\) (red lines) the function \(P(n)\) has a maximum followed by a minimum, see Fig. 5. Below the critical value \(P_c\), three density regimes exist. The smallest density region lies in the gaseous phase below the spinodal region, while the highest densities lie in the liquid phase, above the spinodal region. The coexistence phase can be determined by a Maxwell construction.

In the next section we attempt to match the parameters appearing in the van der Waals equation with those of the nucleon structure functions. To do so, we use also an equation of state derived from the S-matrix formulation of statistical physics which, similar to the van der Waals EoS, contains metastable states.

![Diagram](image)

**Fig. 5:** The pressure-to-density dependence calculated, in arbitrary units, from Eq. (13).

V. MAPPING THE SATURATION REGION IN DIS ONTO THE SPINODAL REGION IN THE VDW EOS; METASTABILITY, OVERHEATING AND SUPERCOOLING

The correspondence between the EoS with its variables \(P, T, \mu\) etc and the observables, depending on the reaction kinematics, is the most delicate and complicated point in the thermodynamical description of any high-energy collisions, especially of DIS. It needs caution, further studies and numerical tests. Attempts to link two different approaches to hadron
dynamics, one based on the $S$ matrix (scattering amplitude, cross sections) and the other one on their collective properties (statistical mechanics, thermodynamics, equation of state) are known from literature [27, 28].

An original EoS of the ultra-relativistic nuclear matter matter based on the $S$ matrix formulation of statistical physics [27] was derived in Ref. [28]. For vanishing chemical potential, $\mu = 0$, it reads

$$P(t) = AT^4 - BT^5 + T^6,$$

(14)
where the values of the coefficients $A$, $B$ and $C$ (all positive!) were determined by the hadron scattering data. Asymptotically, $T \gg m$, $P(T) \sim T^6$, where $m$ is the proton mass. It differs from the expected (but not confirmed) properties of the so-called quark-gluon plasma predicted by perturbative QCD. The $P(T)$ dependence of the EoS (14) is shown in Fig. 6. Similar to the van der Waals curve, Fig. 2 or Fig. 13, it has a maximum $T_{s1}$ followed by a minimum $(T_{s2})$. At the minimum, the pressure is negative and the system is metastable. Before reaching the minimum at $T_{s2}$ (from the left), the system will undergo a phase transition to $T_{c2}$ of the ”cooler” branch. The interval between these points is unphysical. Metastable super-cooling and over-heating within the bag EoS is shown in Fig. 7 from [29]. Similarities between the VdW EoS and Eq. (14) were quantified in Ref. [30], where the parameters $a$ and $b$ appearing in the van der Waals EoS were assumed to be temperature-dependent, $a = a_0/T^\alpha$, $b = T^\beta$. Note that the limiting particle density is $1/b$.

Eq. (14) was derived from an on-shell hadronic scattering amplitude ($Q^2 = m^2$). Its off-shell modification, based on DIS or DVCS, might tell us much more about the connection between structure functions (partonic distributions) and the corresponding EoS.

VI. DISCUSSION AND CONCLUSIONS

Below we list our (temporary) conclusions and mention some open questions left behind this paper:

• The aim of our paper is not just another parametrization of DIS structure functions; instead we propose a new insight into the properties of the interior of the nucleon.

• For simplicity, we have concentrated on the singlet component of the SF (gluons and sea quarks), dominating the low-$x$ region, where saturation takes place. The inclusion the non-singlet and higher-$x$ components (valence quarks), according to the prescriptions given in Sec. II, are straightforward. Relevant fits to the data can be found in Refs. [2–4, 6] and will be treated elsewhere.

• We omitted any discussion of the large-$x$ $(1 - x)^n$ factor in the SF. This is because the statistical approach is not valid in the $x \to 1$ limit, on the one hand, and for simplicity, on the other hand.

• We did not consider the possible (a)symmetry between ”heating” and ”cooling” of the excited nucleon. During deep inelastic scattering, the nucleon (nucleus) gets excited
(heated), whereafter it releases its energy (heat) by producing secondaries. The energy distribution of these secondaries may reveal the temperature of the system, however this "ultimate" temperature [7], created after the confinement transition, will be different from that of the partonic system inside the proton, which may increase indefinitely.

- Any phase transition may produce fluctuations in the observed spectra of produced particles. These fluctuations may originate either from the gas-fluid-gas phase transition under discussion, or from the (de)confinement transition, beyond the scope of the present paper.

- The gas-fluid transition does not necessarily follow the van der Waals EoS. Possible alternatives are e.g. percolation [20, 21] or clustering of partons similar to the case of the molecule. A possible, molecule-like mechanism for hadron aggregation, was proposed for crossover compatible with color confinement in Ref. [22], and a toy model to realize this mechanism was constructed. A kind of grape-shaped quark gluon plasma (gQGP) was obtained. Pair distribution functions for gQGP were calculated exhibits the character of a liquid. Quark recombination or coalescence, extending the concept of single parton fragmentation, which has been used in elementary collisions since the '70s, was studied recently in Ref. [31] in the context of heavy ion collisions. It should be remembered however,
that the above models include (de)confinement transition, absent from our approach.

- Experimentally, the onset of the gas-fluid phase transition may be verified by the observed spectrum of the produced particles. The $p_{\perp}$ distribution of produced from the dilute gaseous state, i.e. below the saturation region, can be computed perturbatively, while beyond the saturation border line they result from the collision of a very large number of constituent, as in the color glass condensate \[32\]. The observation of any correlation between the transverse distribution of the particles produced in DIS (or DVCS) below- or beyond the saturation border line (see Sec. II) will bring evidence for or against the picture presented in this paper.

- Whatever the details of the transition, the important point to realize is the existence of a dense partonic substance, different from the perfect partonic gas, associated with Bjorken scaling in DIS or the so-called quark-gluon plasma, predicted by perturbative QCD and expected in high-energy hadronic and/or nuclear collisions. Instead, experimental data on DIS and on high-energy heavy ion collisions show that the partonic matter may appear as a fluid, called by L. McLerran et al. "color glass condensate" \[32\]. The nature of the strongly interacting matter under extreme conditions should be universal, be it produced in hadron-hadron, heavy nuclei or in deep inelastic lepton-hadron scattering.

- A general remark, concerning collective properties of the nuclear matter: matter is made of quarks, while gluons are binding forces between them. In that sense, strictly speaking, statistics should be applied to quarks rather than gluons.

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