Fragmentation of a Filamentary Cloud Permeated by a Perpendicular Magnetic Field. II. Dependence on the Initial Density Profile

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Abstract

We examine the linear stability of a filamentary cloud permeated by a perpendicular magnetic field. The initial magnetic field is assumed to be uniform and perpendicular to the cloud axis. The model cloud is assumed to have a Plummer-like density profile and to be supported against self-gravity by turbulence. The effects of turbulence are taken into account by enhancing the effective pressure of a low-density gas. We derive the effective pressure as a function of density from the condition of hydrostatic balance. It is shown that the model cloud is more unstable against radial collapse when the radial density slope is shallower. When the magnetic field is relatively weak, radial collapse is suppressed. If the displacement vanishes in a region very far from the cloud axis, the model cloud is stabilized completely by a relatively weak magnetic field. If rearrangement of the magnetic flux tubes is permitted, the model cloud is unstable even when the magnetic field is extremely strong. The stability depends on the outer boundary condition as in the case of an isothermal cloud. The growth rate of the rearrangement mode is smaller when the radial density slope is shallower.

Key words: ISM: clouds – ISM: magnetic fields – magnetohydrodynamics (MHD)

1. Introduction

Filamentary structures are ubiquitously found in star-forming regions (see, e.g., André et al. 2014 and references therein). They are considered as an intermediate state from clouds to stars and their fragmentation is likely to be a process that forms cores. This idea is supported by observations showing that prestellar cores and newly formed stars are associated with the dense parts of filamentary clouds. Although filamentary clouds are unstable against fragmentation in general (see, e.g., Stodólkiewicz 1963; Larson 2003), a magnetic field may suppress the fragmentation if it is strong and perpendicular to the cloud axis. The effect of the magnetic field on fragmentation is a key issue for understanding core formation.

We have examined the stability of a filamentary cloud permeated by a perpendicular magnetic field against fragmentation using a simplified model (Hanawa et al. 2017, Paper I hereafter). The initial magnetic field was assumed to be uniform and the gas to be isothermal in Paper I for simplicity. These assumptions are for technical reasons; it is difficult to construct an equilibrium model for a molecular cloud permeated by a perpendicular cloud (see, e.g., Tomisaka 2014; Hanawa & Tomisaka 2015). When the magnetic field is parallel to the cloud axis or helical around it, we can build various equilibrium models by assuming symmetry around the cloud axis (see, e.g., Toci & Galli 2015b). Such cloud models have been studied extensively for many years (Stodólkiewicz 1963; Hanawa et al. 1993; Nakamura et al. 1993; Fiege & Pudritz 2000). However, the magnetic fields are perpendicular to denser clouds (see, e.g., Sugitani et al. 2011; André et al. 2014; Kusune et al. 2016; Soler et al. 2016), while less dense clouds are associated with parallel magnetic fields.

Magnetic field direction is important to fragmentation of a filamentary cloud. The magnetic force is perpendicular to the magnetic field and hence gas flow along the axis cannot be suppressed by magnetic fields parallel to the axis. The wavelength of fragmentation is shorter when the parallel magnetic field is stronger. This apparent destabilizing effect is ascribed to the fact that the magnetic field is assumed to be concentrated around the axis to support the cloud against radial collapse. Given the central density and temperature, the filament diameter is larger for a stronger magnetic field. The wavelength of the fragmentation, which is roughly twice the Jeans length, is shorter for a stronger magnetic field, when measured in units of the diameter. Moreover, the mass-to-flux ratio is infinitely large, and hence supercritical if the cloud is elongated along the magnetic field. When the magnetic field is perpendicular to the cloud axis, the magnetic force works against fragmentation (Paper I). Even relatively weak magnetic fields stabilize a filamentary cloud against fragmentation, if they are perpendicular to the cloud and their ends are fixed in the region very far from it. When the magnetic field is helical around the axis, the magnetic force works against fragmentation but induces non-axisymmetric instability (see, e.g., Hanawa et al. 1993; Fiege & Pudritz 2000).

Interestingly, perpendicular magnetic fields suppress instability less effectively if the field lines are free, i.e., allowed to move. When the magnetic fields are free in the region very far from the cloud, they are rearranged to fragment the cloud by reducing its gravitational energy. This means that the stability depends on the outer boundary condition. When the magnetic fields are parallel to the cloud axis, the boundary condition has little effect on the instability unless they are placed close to the cloud axis. This is reasonable since the instability is due to the self-gravity of the cloud and depends only on the dense central gas.

In Paper I the model cloud is assumed to be isothermal and supported by gas pressure alone against gravity in equilibrium. Accordingly, the density is assumed to decrease in proportion to $r^{-4}$ in the region very far from the cloud center, where $r$
denotes the distance from the cloud axis. However, observed clouds show much shallower radial density profiles, which is often approximated by a Plummer-like profile:

$$
\rho(r) = \rho_c \left[1 + \left(\frac{r}{R_{\text{flat}}}\right)^2\right]^{-p/2},
$$

(1)

where $\rho_c$ and $R_{\text{flat}}$ denote the central density and “radius,” respectively (Arzoumanian et al. 2011; Juvela et al. 2012; Palmeirim et al. 2013; Ohashi et al. 2018). The index $p$ denotes the slope of the density profile, $-d \ln \rho/d \ln r$, in the region far from the cloud axis. The index is estimated to be $p \approx 2$ from the model fit to the column density distribution derived from submillimeter continuum emission.

Considering the above-mentioned arguments, we examine the stability of a Plummer-like cloud against fragmentation taking account of perpendicular magnetic fields. We assume that the Plummer-like profile is supported against gravity by “effective gas pressure,” which mimics the effects of turbulence. If the effective temperature decreases with an increase in density, a Plummer-like profile with $p < 4$ is realized, as will be shown below. We examine the effects of the density profile on fragmentation of a filamentary cloud using the method developed in Paper I. When $p < 4$, the model cloud is unstable also against radial collapse, although the isothermal model is neutrally stable against it (see, e.g., the review by Larson 2003). It will be shown that the radial collapse is stabilized by relatively weak magnetic fields. The growth rate of the instability depends on $p$, but only quantitatively, except for radial collapse.

This paper is organized as follows. We describe our assumptions and methods for computation in Section 2. The results are shown in Section 3, where also the stability of an unmagnetized cloud is analyzed. We discuss the implications of our models in Section 4 and summarize our main findings in Section 5. Appendices A and B are devoted to improvements in computing long-wavelength modes and the case of no magnetic field, respectively.

2. Methods

2.1. Basic Equations

As in Paper I, we employ the ideal magnetohydrodynamic equations for our stability analysis. They are expressed as

$$
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0,
$$

(2)

$$
\rho \frac{dv}{dt} = -\nabla P + j \times B - \rho \nabla \Phi,
$$

(3)

$$
j = \frac{\nabla \times B}{4\pi},
$$

(4)

where $\rho$, $\Phi$, $\mathbf{v}$, $\mathbf{B}$, and $j$ denote the density, gravitational potential, velocity, magnetic field, and electric current density, respectively. Here the symbol $P$ denotes the pressure, which is designed to include effects of turbulence implicitly.

We ignore ambipolar diffusion for simplicity. Ambipolar diffusion weakens the magnetic force (see, e.g., Hosseinirad et al. 2018). However, the typical timescale is a factor of 10 longer than the dynamical timescale (see, e.g., Nakano & Umebayashi 1988) and the effects are not large. Thus we do not take account of ambipolar diffusion in order to avoid further complication. Note that ambipolar diffusion does not work in our initial model since the magnetic field is uniform; it works only on a perturbation. Thus, it reduces the growth rate of instability but cannot suppress stability. See, e.g., Hosseinirad et al. (2018) for the effects of ambipolar diffusion on fragmentation. They analyzed the stability of a filamentary cloud permeated by a longitudinal magnetic field.

The equation of state is specified in the subsequent section such that the equilibrium density distribution is well approximated by the Plummer function

$$
\rho_0 = \rho_c \left(1 + \frac{r^2}{2\rho H^2}\right)^{-p/2}.
$$

(5)

where $\rho_c$ and $r$ denote the central density of the filamentary cloud and the distance from the cloud axis, respectively. The symbol $H$ denotes the length scale; this is expressed as $R_{\text{flat}} = \sqrt{2}pH$. The radial density profile is shown in Figure 1.

The self-gravity of the gas is taken into account through Poisson’s equation

$$
\Delta \Phi = 4\pi G \rho,
$$

(6)

where $G$ denotes the gravitational constant.

In the following, we use the units system where $\rho_c = 1$, $H = 1$, and $4\pi G \rho_c = 1$ in our numerical computations.

2.2. Equilibrium Model

When the density profile is expressed by Equation (5), the filamentary cloud has mass per unit length

$$
\lambda_c = 2\pi \int_0^r \rho_0(r')r'dr'
$$

(7)

$$
= \begin{cases} 
4\pi \rho_c H^2 \ln \left(1 + \frac{r^2}{2H^2}\right) & (p = 2) \\
4\pi \rho_c H^2 \left[1 - \left(1 + \frac{r^2}{2pH^2}\right)^{-p/2}\right] & (\text{otherwise})
\end{cases}
$$

(8)

Figure 1. Equilibrium density profiles for index $p = 2, 3,$ and $4$. 

Equilibrium density profiles for index $p = 2, 3,$ and $4$. 

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inside the radius \( r \). Thus the gravity is evaluated to be

\[
g_r = -\frac{d\Phi}{dr} = -\frac{2G\rho_r}{r}
\]  

(9)

\[
= \left\{ \begin{array}{ll}
\frac{8\pi G\rho_r H^2}{r} \ln \left( 1 + \frac{r^2}{4H^2} \right) & \text{(p = 2)} \\
\frac{8\pi G\rho_r H^2}{(p - 2)r} \left[ 1 - \left( 1 + \frac{r^2}{2pH^2} \right)^{1-p/2} \right] & \text{otherwise}
\end{array} \right.
\]

(10)

We assume that our model cloud is supported by pressure alone in equilibrium against gravity for simplicity. This assumption means that the magnetic field is assumed to be uniform. This may be a crude assumption and the magnetic field is likely to be concentrated in a dense cloud. However, it is very difficult to take account of a non-uniform magnetic field (see, e.g., Tomisaka 2014). The observed radial volume density profile is also derived from the projected surface density under the assumption that the cloud is symmetric around the axis. Thus, it is worth analyzing this very simplified model.

Since the pressure gradient is given by

\[
\frac{dP_0}{dr} = \rho_0 g_r,
\]

(11)

the pressure and density should satisfy the relation

\[
\frac{dP_0}{d\rho_0} = \left( \frac{dP_0}{dr} \right) \left( \frac{d\rho_0}{dr} \right)^{-1}.
\]

(12)

When \( p = 2 \), the right-hand side of Equation (12) is evaluated to be

\[
\frac{dP_0}{d\rho_0} = \frac{4\pi GH^2 \rho_c^2}{\rho_c - \rho_0} \ln \left( \frac{\rho_c}{\rho_0} \right).
\]

(13)

where Equations (5) and (10) are substituted into Equation (11). Otherwise, it is evaluated to be

\[
\frac{dP_0}{d\rho_0} = \frac{8\pi G\rho_r H^2}{p - 2} \left( \frac{2pH^2}{r^2} \right) \ln \left[ 1 + \frac{r^2}{2pH^2} \right] \left[ 1 - \left( 1 + \frac{r^2}{2pH^2} \right)^{1-p/2} \right] \left[ 1 - \left( \frac{\rho}{\rho_c} \right)^{2p} \right]^{-1} \left[ 1 - \left( \frac{\rho}{\rho_c} \right)^{1-2/p} \right].
\]

(14)

In the following, we assume that Equations (13) and (14) hold not only in equilibrium but also for a perturbation. Thus we use the term \( dP/d\rho \) instead of \( dP_0/d\rho_0 \). Figure 2 shows its value as a function of \( \log (\rho/\rho_c) \) in units of \( 4\pi G\rho_c H^2 \). The sound speed \( \sqrt{dP/d\rho} \) increases with increasing density for \( p < 4 \), while it increases for \( p > 4 \). In this paper we restrict ourselves to the case of \( p \leq 4 \), since the velocity turbulence is lower in a region of higher density in the interstellar medium. Thus it is similar to the logatrope proposed by McLaughlin & Pudritz (1996). They introduced an effective equation of state

\[
P = P_c \left[ 1 + \kappa \ln \left( \frac{\rho}{\rho_c} \right) \right],
\]

(15)

to mimic interstellar turbulence, where \( P_c, \rho_c, \) and \( \kappa \) denote model parameters. However, the dependence of the sound speed on density is weaker than that for the logatrope, since Equation (15) means

\[
\frac{dP}{d\rho} = \frac{P}{\rho_c}.
\]

(16)

We examine Equation (14) again in Section 4.

We assume that the magnetic field is uniform and runs in the \( x \)-direction in equilibrium. To specify the initial magnetic field strength in the analysis, we use the plasma beta at the cloud center:

\[
\beta = \frac{8\pi \rho_c}{B_0^2} \left( \frac{d\rho}{d\rho} \right)_{\rho = \rho_c} = \frac{32\pi G\rho_c^2 H^2}{B_0^2},
\]

(17)
as in Paper I. The plasma beta is related to the mass-to-flux ratio

\[
f = \frac{\int \rho_0(x', y, z) dx'}{\rho_0} = \frac{\sqrt{2\pi p} \Gamma \left( \frac{p-1}{2} \right)}{\Gamma \left( \frac{p}{2} \right)} \frac{B_0 \rho_c H}{\rho_0},
\]

(18)

where \( \Gamma \) denotes the gamma function. When the mass-to-flux ratio is critical, i.e., \( f = (2\pi \sqrt{G})^{-1} \), the plasma beta is \( \beta = 2/p^2, 1/3, \) and \( 4/p^3 \) for \( p = 2, 3, \) and \( 4 \), respectively.

2.3. Perturbation Equation

Following Paper I, we consider a small perturbation around equilibrium in order to search for an unstable mode. The perturbation is described by the displacement, defined by

\[
\xi = e^{\xi_i} \left[ \xi_x(x, y) \cos kx \xi_y + \xi_y(x, y) \cos kx \xi_x \right.
\]

\[
+ \xi_z(x, y) \sin kx \xi_x \right],
\]

(19)
where the perturbation is assumed to be sinusoidal in the $z$-direction with wavenumber $k$ and to grow exponentially with time at a rate $\sigma$. The change in density is described as

$$
\rho(x, y, z, t) = \rho_0(x, y) + e^{\sigma t} \delta \rho(x, y, z),
$$

(20)

Substituting $v = d\xi/dt$ and Equation (19) into Equation (2) we obtain

$$
\delta \rho = -\frac{\partial}{\partial x}(\rho_0 \xi_x) - \frac{\partial}{\partial y}(\rho_0 \xi_y) - k \rho_0 \xi_z.
$$

(21)

Similarly, we obtain the perturbation in the magnetic field from the induction equation

$$
\delta B = \nabla \times (\xi \times B_0).
$$

(22)

The induction equation is further expressed as

$$
\delta B(x, y, z) = b_x(x, y) \cos kze_x + b_y(x, y) \cos kze_y + b_z(x, y) \sin kze_z,
$$

(23)

$$
b_i = -B_0 \left( \frac{\partial}{\partial y}\xi_i + k \xi_i \right),
$$

(24)

$$
b_y = B_0 \frac{\partial \xi_y}{\partial x},
$$

(25)

$$
b_z = B_0 \frac{\partial \xi_z}{\partial x}.
$$

(26)

We evaluate the change in the current density to be

$$
\delta J = \frac{1}{4\pi} \nabla \times \delta B,
$$

(27)

using Equation (4). The components of the current density are expressed as

$$
\delta J(x, y, z) = j_x(x, y) \sin kze_x + j_y(x, y) \sin kze_y + j_z(x, y) \cos kze_z,
$$

(28)

$$
j_x = \frac{1}{4\pi} \left( \frac{\partial b_x}{\partial y} + kb_y \right),
$$

(29)

$$
j_y = -\frac{1}{4\pi} \left( kb_x + \frac{\partial b_y}{\partial x} \right),
$$

(30)

$$
j_z = \frac{1}{4\pi} \left( \frac{\partial b_y}{\partial x} - \frac{\partial b_z}{\partial y} \right).
$$

(31)

Then the changes in density and current density are expressed as explicit functions of $\xi$.

The change in the gravitational potential is given as a solution of the Poisson equation

$$
\nabla^2 \delta \psi = 4\pi G \rho_0.
$$

(32)

Thus, it can be regarded as an implicit function of $\xi$.

We derive the equation of motion for the perturbation by taking account of the force balance:

$$
\left( \frac{dP}{d\rho} \right) \nabla \rho_0 + \rho_0 \nabla \psi_0 = 0,
$$

(33)

Table 1

| Variable | Evaluation Point | Symmetry x | Symmetry y |
|----------|-----------------|-------------|-------------|
| $\xi_x$  | $(i - 1/2, j)$  | A           | S           |
| $\xi_y$  | $(i, j - 1/2)$  | S           | A           |
| $\xi_z$  | $(i, j)$        | S           | S           |
| $\delta \psi$ | $(i, j)$ | S           | S           |
| $\xi_i$  | $(i, j)$        | A           | S           |
| $j_i$    | $(i - 1/2, j - 1/2)$ | A | S |
| $j_i$    | $(i, j)$        | S           | S           |
| $j_i$    | $(i, j - 1/2)$  | S           | A           |

Note. A: anti-symmetric. S: symmetric.

with no electric current density, $j_0 = 0$, in equilibrium. Then the equation of motion is expressed as

$$
\sigma^2 \rho_0 \xi = - \rho_0 \nabla \left( \frac{dP}{d\rho} \frac{\delta \rho}{\rho_0} \right) - \rho_0 \nabla \delta \psi + \delta J \times B_0,
$$

(34)

where the last term represents the magnetic force. The term $J_0 \times \delta B_0$ does not appear in Equation (34) since $J_0 = 0$ in our equilibrium model. The linear growth rate, $\sigma$, is obtained as the eigenvalue of the differential Equation (34), since the right-hand side is proportional to $\xi$.

The derived perturbation equations are the same as those derived in Paper I except for the sound speed $(\sqrt{dP/d\rho})$, which is a function of density in our analysis but constant in Paper I.

Our equilibrium model is symmetric with respect to the $x$- and $y$-axes. Thus, all eigenmodes should be either symmetric or anti-symmetric with respect to these axes. We restrict ourselves to the eigenmodes symmetric to both $x$- and $y$-axes, since the unstable mode has the same symmetry in the case of no magnetic field (Nakamura et al. 1993). The choice of this symmetry is justified since we are interested only in the unstable mode. Using this symmetry, we can reduce the region of computation to the first quadrant, $x \geq 0$ and $y \geq 0$. The variables describing the perturbation and their symmetries are summarized in Table 1.

We consider two types of boundary conditions. The first assumes that the displacement should vanish in the region very far from the filament center. We call this the fixed boundary condition. The second allows the magnetic field lines to move while remaining straight and normal to the boundary. This restriction is expressed as

$$
(B_0 \cdot \nabla) \xi = 0.
$$

(35)

Thus, we assume $\partial \xi / \partial x$ on the boundary in the $x$-direction and $\xi = 0$ in the $y$-direction. We refer to this as the free boundary condition. In both types of boundary conditions, we use the symmetries given in Table 1 to set the boundary conditions for $x = 0$ and $y = 0$.

2.4. Numerical Methods

We solve the eigenvalue problem numerically by a finite-difference approach. The differential equations are evaluated
on a rectangular grid in the xy plane. We evaluate \( \xi_\alpha, \delta \psi, \delta \psi, b_\alpha \), and \( J_z \) at the points

\[
(x_i, y_j) = (i \Delta x, j \Delta y),
\]

where \( i \) and \( j \) specify the grid points, while \( \Delta x \) and \( \Delta y \) denote the grid spacing in the \( x \)- and \( y \)-directions, respectively (see Table 1). These variables are symmetric with respect to both the \( x \)- and \( y \)-axes. Using this symmetry, we consider the range \( 0 \leq i \leq n_x \) and \( 0 \leq j \leq n_y \), where \( n_x \) and \( n_y \) specify the number of grid points in each direction. When \( i > n_x \) or \( j > n_y \), the displacement \( \xi_{\alpha,i,j} \) is assumed to vanish for the fixed boundary and to have the same values at neighboring points in the computation domain for the free boundary condition. We use the indices \( i \) and \( j \) to specify the position where the variables are evaluated, such that \( \xi_{\alpha,i,j} = \xi_{\alpha}(x_i, y_j) \).

The variables are evaluated at one of the following:

\[
(x_{i-1/2}, y_j) = \left( \left( i - \frac{1}{2} \right) \Delta x, j \Delta y \right),
\]

\[
(x_i, y_{j-1/2}) = \left( i \Delta x, \left( j - \frac{1}{2} \right) \Delta y \right),
\]

\[
(x_{i-1/2}, y_{j-1/2}) = \left( \left( i - \frac{1}{2} \right) \Delta x, \left( j - \frac{1}{2} \right) \Delta y \right)
\]

depending on the symmetry as summarized in Table 1. All these variables are evaluated in the region \( 0 \leq x \leq n_x \Delta x \) and \( 0 \leq y \leq n_y \Delta y \). In other words, we use these kinds of staggered grids to achieve second-order accuracy in space.

Using the variables defined on the grids, we rewrite the perturbation equations. Equation (21) is rewritten as

\[
\delta \psi_{i,j} = \frac{\rho_0_{i-1/2,j} \xi_{i-1/2,j} - \rho_0_{i,j} \xi_{i,j}}{\Delta x} - \frac{\rho_0_{i,j+1/2} \xi_{i+1/2,j} - \rho_0_{i,j-1/2} \xi_{i,j-1/2}}{\Delta y} - k_\rho_0_{i,j} \xi_{i,j}
\]

Equation (32), the Poisson equation, is expressed as

\[
\frac{\delta \psi_{i+1,j} + \delta \psi_{i-1,j}}{\Delta x^2} + \frac{\delta \psi_{i,j+1} + \delta \psi_{i,j-1}}{\Delta y^2} - \left( \frac{2}{\Delta x^2} + \frac{2}{\Delta y^2} + k^2 \right) \delta \psi_{i,j} = 4\pi G \delta \psi_{i,j}.
\]

The solution of Equation (41) is expressed as

\[
\delta \psi_{i,j} = \sum_j G_{i,j,j'} \delta \psi_{i,j'},
\]

where \( G_{i,j,j'} \) denotes the Green’s function and its value is obtained by solving Equation (41) numerically. The boundary condition for the Poisson equation is improved to increase the accuracy of growth at small \( k \). See Appendix A for more details.

The change in the magnetic field is evaluated as

\[
b_{\alpha,i,j} = -B_0 \left( \frac{\xi_{\alpha,i+1/2,j} - \xi_{\alpha,i-1/2,j}}{\Delta y} + k \xi_{\alpha,i,j} \right),
\]

from Equations (24) through (26). The current density is evaluated as

\[
\dot{J}_{\alpha,i,j} = -\frac{1}{4\pi} \left( k b_{\alpha,i,j} - \frac{b_{\alpha,i+1/2,j} - b_{\alpha,i-1/2,j}}{\Delta x} \right),
\]

\[
\dot{J}_{\alpha,i,j-1/2} = \frac{1}{4\pi} \left( b_{\alpha,i+1/2,j-1/2} - b_{\alpha,i-1/2,j-1/2} - \frac{b_{\alpha,i,j} - b_{\alpha,i,j-1}}{\Delta y} \right).
\]

The \( x \)-component of the current density, \( j_x \), is not evaluated, since it does not appear in the equation of motion. The fixed boundary conditions are expressed as

\[
\xi_{x,n_x+1/2,j} = 0,
\]

\[
\xi_{x,n_y+1/2,j} = 0,
\]

\[
\xi_{x,x+1/2,n_y} = 0,
\]

\[
\xi_{x,x+1/2,n_y+1} = 0,
\]

\[
\xi_{x,n_y+1/2,n_y} = 0.
\]

When the free boundary is applied, the conditions are replaced by

\[
\xi_{x,n_x+1/2,j} = \xi_{x,n_x+1/2,j},
\]

\[
\xi_{x,n_y+1/2,j} = \xi_{x,n_y+1/2,j},
\]

\[
\xi_{x,x+1/2,n_y} = \xi_{x,x+1/2,n_y},
\]

\[
\xi_{x,x+1/2,n_y+1} = \xi_{x,x+1/2,n_y+1},
\]

\[
\xi_{x,n_y+1/2,n_y} = \xi_{x,n_y+1/2,n_y}.
\]

The equation of motion (34) is expressed as

\[
\sigma^2 \rho_0_{i-1/2,j} \xi_{i-1/2,j} = -\frac{(\rho_0_{i-1/2,j})^2}{\Delta x} \left( \frac{dP}{dp} \right)_{i,j} \frac{\delta \psi_{i,j}}{\rho_{0,i,j}}
\]

\[
- \left( \frac{dP}{dp} \right)_{i,j} \frac{\delta \psi_{i,j}}{\rho_{0,i,j}} - \frac{\rho_0_{i-1/2,j}}{\Delta x} \delta \psi_{i,j} - \delta \psi_{i,j}.
\]

\[
\sigma^2 \rho_{0,i,j-1/2} \xi_{i,j-1/2} = -\frac{\rho_0_{i-1/2,j-1}}{\Delta y} \left( \frac{dP}{dp} \right)_{i,j} \frac{\delta \psi_{i,j}}{\rho_{0,i,j}}
\]

\[
- \left( \frac{dP}{dp} \right)_{i,j} \frac{\delta \psi_{i,j}}{\rho_{0,i,j}} - \frac{\rho_0_{i,j-1}}{\Delta y} \delta \psi_{i,j} - \delta \psi_{i,j-1}.
\]

\[
+ B_0 \xi_{i,j-1/2}.
\]
\[ \sigma^2 \rho_{i,j} \xi_{i,j} = -k \left( \frac{dP}{d\rho} \right)_{i,j} \delta \psi_{i,j} - k \rho_{i,j} \delta \psi_{i,j} - B_0 \delta \psi_{i,j}. \]  

Equations (60) through (62) are summarized in the form

\[ \sigma^2 B \zeta = (A + B \zeta C) \zeta, \]  

using Equations (40), and (42) through (47). Here, \( \zeta \) denotes an array of components, \( \xi_{i,j} \), \( \xi_{i,j} \), and \( \xi_{i,j} \), for all combinations of \( i \) and \( j \). The matrix elements of \( A, B, \) and \( C \) are evaluated numerically as a function of \( k \). See Appendix B of Paper I for further details. Then the growth rate is given as the solution of

\[ \det(\sigma^2 B - A - B \zeta C) = 0. \]  

We rewrite Equation (64) as

\[ \det[\sigma^2 - B^{-1/2} (A - B \zeta C) B^{-1/2}] = 0, \]  

where the matrix \( B^{-1/2} \) is easily obtained since only the diagonal elements have non-zero values in matrix \( B \). Equation (65) is an eigenvalue problem while Equation (64) is a generalized eigenvalue problem. Various library programs are available for solving the former. We use the subroutine DGEEVX of LAPACK (see Anderson et al. 1999 for details of the software) to solve Equation (65). The subroutine returns all the eigenvalues \( \sigma^2 \).

The matrices \( A, B, \) and \( C \) have dimension \( 3(n_i, n_e + 2n_e + 2n_e + 1) \). Thus, we obtain \( 3n_i, n_e + 2n_e + 2n_e + 1 \) eigenmodes. However, we select only one unstable mode, \( \sigma^2 > 10^{-5} \), for a given \( k \) and \( B_0 \). The remaining eigenmodes denote oscillation of the filamentary cloud. In the following, we restrict ourselves to the unstable mode.

When \( B_0 = 0 \), our equilibrium model is symmetric around the \( z \)-axis and we can simplify the stability analysis using cylindrical coordinates. The numerical methods are summarized in Appendix B.

When \( k = 0 \), we need not solve Equation (62), since the \( z \)-component of the displacement, \( \xi_z \), vanishes. Accordingly, we omit the corresponding part of the matrix given in Equation (65). The dimension of the matrix to be solved reduces to \( 2n_i, n_e + n_e + n_e \). The boundary condition for the Poisson equation is given at the end of Appendix A.

3. Results

3.1. Case \( B_0 = 0 \)

Before examining the effects of magnetic field, we analyze the stability of our Plummer-like model for the case \( B_0 = 0 \). When the magnetic field vanishes, our equilibrium model is symmetric around the \( z \)-axis. Hence, the stability analysis is reduced to a 1D problem. We obtain the growth rate, \( \sigma \), as a function of the wavenumber, \( k \), for \( p = 2, 3, \) and \( 4 \) according to the method given in Appendix B. The growth rate is obtained by solving the discretized perturbation equation with the spatial resolution \( \Delta r = 0.1H \) and the boundary condition at \( r_{\text{out}} = 60H \). Thus the obtained growth rate is highly accurate. Figure 3 denotes the growth rate in units of \( \sqrt{4\pi G \rho_0} \) with the wavenumber resolution \( \Delta k = 0.01H^{-1} \).

Similar to the isothermal model, the Plummer models with \( p = 2 \) and \( 3 \) are unstable against fragmentation when the wavenumber is smaller than the critical value, \( k_{cr} \). This is smaller for lower \( p \) when measured in units of \( H^{-1} \); it is \( k_{cr} = 0.509 H^{-1}, 0.545 H^{-1}, \) and \( 0.565 H^{-1} \) for \( p = 2, 3, \) and \( 4 \), respectively. Also the wavenumber for which the growth rate takes its maximum value is also smaller for a lower index.

The maximum growth rate, \( \sigma_{\text{max}} \), is higher for lower \( p \) when measured in units of \( \sqrt{4\pi G \rho_i} \). Again the dependence of \( \sigma_{\text{max}} \) on \( p \) is weak. This is likely due to the fact that the cloud is more massive than the isothermal cloud when \( \rho_i \) and \( H \) are fixed; see Equation (8).

Figure 4 denotes the growth rate of the radial collapse (\( kH = 0 \)) mode, \( \sigma \), as a function of \( p \), for \( B_0 = 0 \). It is obtained numerically using the method shown in Appendix B with the outer boundary at \( r = 200H \). The growth rate is lower for higher \( p \) and vanishes at \( p = 4 \) (isothermal). It should be also noted that we find only one unstable mode for a given \( kH \).

Figure 4 denotes the growth rate of the radial collapse (\( kH = 0 \)) mode, \( \sigma \), as a function of \( p \), for \( B_0 = 0 \). It is obtained numerically using the method shown in Appendix B with the outer boundary at \( r = 200H \). The growth rate is lower for higher \( p \) and vanishes at \( p = 4 \) (isothermal). It should be also noted that we find only one unstable mode for a given \( kH \).

This dependence of the instability on the equation of state is well known and summarized in the review by Larson (2003). Recently, Toci & Galli (2015a) have reported a similar result.
on the radial collapse of a filamentary cloud. They assumed a polytropic equation of state, \( P = K \rho^\gamma \), where \( K \) and \( \gamma \) are a constant and the polytropic exponent, respectively. When \( \gamma < 1 \), their model cloud is also unstable against radial collapse. When \( \gamma > 1 \), the model cloud is stable against radial collapse and the density vanishes at a finite radius.

The dependence of the growth rate on \( p \) is moderate, while it is larger for a lower \( kH \). Thus we examine the eigenmode of the radial collapse for \( p = 2 \) and 3. The upper panel of Figure 5 shows the relative density perturbation, \( \delta \rho/\rho_0 \), as a function of \( r \) for the radial collapse mode, \( kH = 0 \) and \( B_0 = 0 \). The lower panel of Figure 5 shows the radial displacement, \( \xi/r \). The red curve denotes the eigenmode of \( p = 2 \) while the black curve denotes that of \( p = 3 \). The eigenmodes are normalized so that the relative density perturbation is unity, \( \delta \rho/\rho_0 = 1 \). See Equations (20) and (21) for the definition of \( \delta \rho \) and its relation to the displacement.

When the index is smaller than \( p < 4 \), the effective equation of state is “soft” in a sense that the effective sound speed decreases as the density increases. Thus the filamentary cloud is subject to radial collapse. When \( p = 2 \) and 3, the radial displacement has a maximum at \( r \approx 5H \) and \( 11H \), respectively. When \( p \) is small, an inner region around the axis collapses radially. When \( p \) is close to 4, radial collapse is realized only when the whole cloud collapses in the radial direction.

Radial collapse \((kH = 0)\) is suppressed by a relatively weak magnetic field. Figure 6 shows the growth rate as a function of the inverse of the plasma beta, i.e., the magnetic pressure normalized by the gas pressure at the cloud center. The solid curves denote the growth rates for the free boundary, while the dashed ones denote those for the fixed boundary at \( x = 32H \). The index is set to be \( p = 1.5, 2, \) and 3. A relatively weak magnetic field of \( \beta_0 = 5 \) suppresses the \( kH = 0 \) mode for \( p = 2 \). When \( p = 3 \), the radial collapse mode is completely suppressed by a very weak magnetic field of \( \beta_0 = 40 \).

The magnetic field stiffens the equation of state since the magnetic pressure increases more steeply than the gas pressure when compressed. Radial collapse is thought to be suppressed when the equation of state is “isothermal,” i.e., when the effective sound speed changes little with the increase in density.

The growth rate is lower for given \( p \) and \( \beta \) when the fixed boundary condition is applied. When the magnetic field is fixed in the region far from the cloud, the magnetic tension works against collapse in addition to the magnetic pressure. The difference is larger for smaller \( p \) and is negligibly small for \( p = 3 \). When \( p = 2 \), the difference is appreciable for \( \beta < 10 \). The growth rate depends somewhat on the computation domain: it depends little on the size of the computation domain \((n_x)\), when the free boundary is applied; however, it depends to some extent on \( n_x \) when the fixed boundary is applied.

3.3. Case \( B_0 = 0 \) and \( kH \neq 0 \)

First, we examine the case \( p = 2 \), since the density profile is close to the observed one.
The initial density is very low, $\rho_0/\rho_c = 3.89 \times 10^{-3}$ and $1.73 \times 10^{-3}$, at the numerical boundary at $32H$ and 48, respectively. However, the location of the boundary affects the instability through magnetic tension. When the magnetic field is fixed at a relatively short distance, the magnetic tension is strong enough to stabilize the cloud against fragmentation.

Figure 8 is similar to Figure 7 but for the free boundary. The growth rate is obtained with the same resolution, $\Delta x = \Delta y = 0.4H$, $n_x = n_y = 80$, and $\Delta k = 0.01H^{-1}$. When $\beta < 1$, the growth rate is well approximated by an empirical formula

$$
\sigma^2(k, \beta) = \sigma^2(k, 0) + \frac{d\sigma^2}{d\beta} \beta + O(\beta^2),
$$

(67)

where $\sigma(k, 0)$ and $d\sigma^2/d\beta$ are positive constants for a given $k$.

The growth rate for $\beta = 0$ shown in Figure 8 is obtained by a linear extrapolation of the growth rates at $\beta = 0.1$ and 0.5.

The growth rate depends a little on the size of the computation domain. When $p = 2, \beta = 2, kH = 0.2$, and $\Delta x = 0.4H$, the growth rate is 0.160 and 0.161 $\sqrt{4\pi G\rho_c}$ for $n_x = 80$ and 120, respectively.

The growth rate is lower for a stronger magnetic field also when the free boundary is applied. Radial collapse is suppressed by a relatively weak magnetic field ($\beta < 4$). However, the model cloud is unstable against fragmentation even when the entire cloud is subcritical. This instability is due to rearrangement of magnetic flux tubes as shown in Paper I. Although the critical wavenumber changes little, the wavenumber of the most unstable mode decreases down to $(kH)_{\text{max}} \approx 0.11$ in the limit of $\beta = 0$.

Comparison of Figures 7 and 8 tells us that the growth rate depends significantly on the boundary condition only when $\beta \lesssim 2$. When the gas pressure dominates over the magnetic pressure on the cloud axis ($\beta \gtrsim 2$), the growth rate is nearly the same for both the free and fixed boundaries except for $kH \lesssim 0.4$, i.e., when the wavenumber is close to $k_c$, the instability is weak even for $B_0 = 0$. This means that a relatively weak magnetic field does not play an important role in the region far from the axis, although the magnetic pressure dominates over the gas pressure therein. Recall that the magnetic pressure is comparable to the gas pressure at $r = 12H$ even when $\beta = 10$. The gas pressure decreases with density while the magnetic pressure remains constant in our model. The magnetic pressure dominates over the gas pressure.
near the outer boundary \((x = 32H \text{ and } y = 32H)\) of our numerical computation even for \(\beta = 100\). The magnetic field can suppress fragmentation only when the magnetic field is strong near the cloud axis and fixed in the region very far from the cloud.

The Plummer index, \(p\), can vary from cloud to cloud. We examine two cases, \(p = 1.5\) and \(p = 3\); the former is close to the observed minimum.

Figures 9 and 10 show the growth rate, \(\sigma / \sqrt{4\pi G \rho_c}\), as a function of the wavenumber, \(kH\), for the model of \(p = 1.5\). Figure 9 denotes the growth rate for the fixed boundary while Figure 10 denotes that for the free boundary. The results are qualitatively similar to those for \(p = 2\), while the growth rate is a little larger for given \(kH\) and \(\beta\). As a result, the model cloud is unstable against fragmentation for \(\beta = 1\) even when the fixed boundary is applied. The growth rate depends significantly on the boundary condition when \(\beta \lesssim 1\).

The increase in the growth rate might be due to normalization. The growth rate is normalized by the initial central density \((\rho_c)\). The wavenumber is normalized not only by the density but by the sound speed \((\sqrt{dP/d\rho})\) on the cloud axis, since the unit length can be expressed as

\[
H = \frac{1}{\sqrt{8\pi G \rho_c}} \left[ \frac{dP}{d\rho}(\rho_c) \right]^{1/2}.
\]

Equation (68) is derived from Equations (13) and (14) by taking the limit of \(r \to 0\). In other words, the growth rate is normalized by the freefall timescale at the cloud center, while the wavenumber is normalized by the Jeans length, \(\lambda_J = 2\pi H\).

When the central density and sound speed are fixed, the mass per unit inside radius \(r\) is larger for smaller \(p\). The increase in growth rate can be ascribed to the increase in the mass per unit length since this instability is due to self-gravity.

Figures 11 and 12 are similar to Figures 7 and 8, respectively, but for \(p = 3\). The result depends only quantitatively on \(p\), as expected. The growth rate is intermediate between those for \(p = 2\) and 4. When the fixed boundary is applied, the cloud is stabilized by a moderately strong magnetic field \((\beta \lesssim 1.3)\). The radial collapse \((kH \ll 0.05)\) mode is suppressed by a relatively weak \((\beta \approx 10)\) magnetic field.

The eigenfunction depends only weakly on \(p\). We do not find any qualitative change except for radial collapse in the case of a weak magnetic field.

### 4. Discussion

First we compare our equilibrium model clouds with earlier theoretical models based on the effective equation of state.

Ostriker (1964) obtained cylindrical equilibrium models by assuming a polytropic equation of state,

\[
P = K_N \rho^{1+1/N},
\]

where
where $K_N$ and $N$ denote the polytropic constant and index, respectively. His equilibrium model is symmetric around the axis and extended infinitely along it. Since he studied the case of $N \geq 1$, the effective sound speed, $(dP/d\rho)^{1/2}$, increases with density. In his model, the model cloud is truncated at a certain radius when $N$ is finite. Only when $N$ is infinite (isothermal) is the cloud extended to an infinite radius.

When $N$ is negative, the polytrope gives a similarity solution denoting radial collapse (McLaughlin & Pudritz 1997; Kawachi & Hanawa 1998). The density decreases in proportion to

$$
\rho \propto r^{-2/(1-1/N)},
$$

in the region far from the cloud axis. This radial profile is quite similar to the Plummer-like model with $p \approx 2$, when $N$ is negatively large.

McLaughlin & Pudritz (1997) introduced a model with a slightly different equation of state, Equation (15), which is named “logatrope.” The sound speed is inversely proportional to the density as shown in Equation (16). The polytrope gives a singular equilibrium having the density profile $\rho \propto r^{-1}$. This corresponds to the polytropic model of $N = -1$.

Note that the polytropic equation of state has a steeper dependence on density compared with our model equation of state. The sound speed is proportional to a power of the density, $(dP/d\rho) \propto \rho^{1/N}$. Our model equation of state shows a weaker dependence of the sound speed on density as shown in Figure 2. Note that the effective sound speed is only 2.16 and 1.28 times at $\rho = \rho_c/100$ than at $\rho = \rho_c$ for $p = 2$ and 3, respectively. This small change produces a notable change in the radial density profile. We introduce this effective sound speed by taking account of turbulence, but a physical change in the sound speed is inversely proportional to the density as shown in Equation 6. The plasma beta can be evaluated from

$$
\beta \approx \left( \frac{c_{s,\text{eff}}}{0.3 \text{ km s}^{-1}} \right)^4 \left( \frac{w}{0.1 \text{ pc}} \right)^{-2} \left( \frac{B_0}{90 \mu \text{G}} \right)^{-2}.
$$

Equation (73) implies that a magnetic field of $\sim 100 \mu \text{G}$ plays an important role in the dynamics of a typical molecular cloud, since the numbers quoted are typical. The plasma beta can be evaluated also from $w$ and $\lambda$, if Equation (71) is valid.

It is interesting to apply the above estimate to the Musca filamentary cloud. The central 1.6 pc of the filamentary cloud shows no sign of fragmentation, although the rest of the cloud shows fragmentation consistent with stability analysis (Kainulainen et al. 2016). If the stabilization is due to a perpendicular magnetic field, the field strength should be several tens of $\mu \text{G}$. According to Kainulainen et al. (2016), the best-fit Plummer model gives $B = 2.6$ and 1.8 on the west and east sides of the filamentary cloud, respectively. They also evaluate the effective sound speed to be 22% higher than the isothermal one. The right-hand side of Equation (73) is roughly unity when $B_0 \approx 100 \mu \text{G}$.

Our stability analysis has demonstrated that the instability depends on the boundary condition, i.e., fixed or free, when the plasma beta is close to unity. This means that a filamentary cloud is not isolated and stability depends on the environment. This suggests an interesting possibility. If two filamentary clouds are threaded by the same perpendicular magnetic field lines, their fragmentation can be linked through magnetic force. Fragmentation of a cloud might be suppressed since the magnetic field lines are fixed by the other cloud. However, we cannot exclude the possibility that two clouds fragment coherently. This problem is an open question beyond the scope of this paper.

Except for radial collapse, the growth rate of the perturbation depends only quantitatively on the index, $p$. Any strong magnetic field cannot stabilize the cloud if the magnetic fields are free to move in the region far from the cloud. However, this does not mean that a filamentary cloud fragments to form cores in a short timescale. The growth rate is nearly by a factor of 10 smaller than the dynamical one, $\sqrt{4\pi G \rho}$, for $p = 2$ as shown in Figure 8. Recall that the growth rate is normalized by the freefall timescale at the cloud center:

$$
\tau_H = \frac{1}{\sqrt{4\pi G \rho}} \approx 0.18 \left( \frac{n_{H_2}}{10^4 \text{ cm}^{-3}} \right)^{-1/2} \text{Myr}.
$$

Since the growth rate is small, we need to take account of non-ideal effects, i.e., ambipolar diffusion of the magnetic field, which is ignored in our analysis for simplicity. It is well known that ambipolar diffusion increases the mass-to-flux ratio locally and causes a molecular cloud to eventually become supercritical. This competing process should be taken seriously when discussing instability due to the rearrangement of magnetic flux tubes.

We also note that the growth rate of the rearrangement instability is smaller for smaller $p$, although the maximum growth rate of $B_0 = 0$ is larger. This result is consistent with the discussion given in Paper I. Rearrangement of the magnetic fields travels along the field line as an Alfvén wave, which is proportional to $B_0/\sqrt{\rho_0}$. Thus the rearrangement takes more time when either the magnetic field weakens or the density decreases more slowly in the region far from the cloud. In our analysis, only the latter effect is taken into account, where the magnetic fields should be weaker outside the cloud than inside. Thus the rearrangement instability grows more slowly if non-uniformity of the magnetic field accounted for.
5. Summary

We have examined the stability of a filamentary cloud permeated by uniform perpendicular magnetic fields with a focus on the dependence on the initial density. Our main findings are summarized as follows.

1. The observed Plummer-like density profile can be realized if the effective sound speed increases with a decrease in density. The radial density slope is $d \ln \rho / d \ln r = -2$ if the effective sound speed is only a factor of $\sqrt{5}$ in the region where the density is a factor of 100 lower than at the cloud center. This dependence of the effective sound speed on density is much lower than that of a logatrope.

2. When the radial density slope is shallower than $d \ln \rho / d \ln r > -4$, the cloud is unstable against radial collapse. The growth rate is larger when the radial density slope is shallower. Radial collapse can be suppressed by relatively weak magnetic fields.

3. Stability of a filamentary cloud depends strongly on the boundary condition far from the filament, i.e., on the environment. If the magnetic field is fixed at a large distance from the cloud and the plasma beta is close to unity at its center, fragmentation is suppressed. If the magnetic field line can move freely, it cannot suppress the fragmentation even when the cloud is magnetically subcritical. The latter instability is induced by rearrangement of magnetic flux tubes as shown in Paper I.

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Software: LAPACK, Linear Package Algebra (Anderson et al. 1999).

Appendix A

Boundary Conditions for the Poisson Equation

We improve the boundary conditions for the Poisson equation. Equation (42) relates the change in the gravitational potential to that in the density, where the Green’s function, $G_{i,j,\vec{r},\vec{r}'}$, takes account of the boundary conditions on $x=0$ and $y=0$. In this work we removed the boundary by employing the method of the mirror image, i.e., by taking account of the change in the density in the regions at $x<0$ and/or $y<0$. Then the boundary conditions for the Poisson equation are expressed as

$$\lim_{\sqrt{x^2+y^2} \to \infty} |\phi| = 0. \quad (75)$$

The corresponding Green’s function is expressed as

$$G_{i,j,\vec{r},\vec{r}'} = 2G_{x',x} \Delta yK_0(kr), \quad (76)$$

where $K_0$ denotes the zeroth modified Bessel function of the second kind and the argument is the distance from the source multiplied by the wavenumber $k$. We solve the discretized Poisson equation:

$$\frac{G_{i+1,j,0,0} - 2G_{i,j,0,0} + G_{i-1,j,0,0}}{\Delta x^2} + \frac{G_{i,j+1,0,0} - 2G_{i,j,0,0} + G_{i,j-1,0,0}}{\Delta y^2}$$

$$- k^2G_{i,j,0,0} = \begin{cases} 1 & (i = j = 0) \\ 0 & \text{(otherwise)} \end{cases}, \quad (78)$$

with boundary condition (76), on $i = \pm(3n_x + 1)$ and $j = \pm(3n_y + 1)$ by the Gauss–Seidel iteration. Once the Green’s function $G_{i,j,0,0}$ is given, then the Green’s function $G_{i,j,\vec{r},\vec{r}'}$ is obtained by summing up the contributions from the mirror images. Thus we can save substantial computation time in solving the Poisson equation using the method of mirror image. The boundary condition (76) improves the accuracy of the Green’s function for a small wavenumber, $k\Delta x \lesssim 0.05$.

When $k = 0$, Equation (76) is replaced with

$$G_{i,j,\vec{r},\vec{r}'} = -2G_{x',x} \Delta y \ln r, \quad (79)$$

where the asymptotic form of the modified Bessel function near $z = 0$ is applied. Although Equation (79) contains an offset proportional to $\ln(k/2)$, the $x$- and $y$-components of gravity are not affected by the offset.

Appendix B

Case $B_0 \neq 0$

When $B_0 = 0$, our equilibrium model is symmetric and the perturbation equation is reduced to an ordinary differential equation if we use the cylindrical coordinate $(r, \varphi, z)$. Following Appendix C of Paper I, we express the density, displacement, and potential in the form

$$\rho = \rho_0 + \delta \rho(r) \cos kz, \quad (80)$$

$$\xi = \xi_0(r) \cos kze, + \xi_z(r) \sin kze, \quad (81)$$

$$\psi = \psi_0 + \delta \psi(r) \cos kz, \quad (82)$$

The perturbation equations are written as

$$\delta \theta = -\frac{1}{r} \frac{\partial}{\partial r}(r \rho_0 \xi_r) - k \rho_0 \xi_z, \quad (83)$$

$$\sigma^2 \xi_r = -\frac{\partial}{\partial r} \left[ (\frac{\partial P}{\partial \rho}) \frac{\delta \rho}{\rho_0} \right] - \frac{\partial}{\partial r} \delta \psi, \quad (84)$$

$$\sigma^2 \xi_z = k \left( \frac{\partial P}{\partial \rho} \frac{\delta \rho}{\rho_0} + k \delta \psi \right), \quad (85)$$

$$4\pi G \delta \theta = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \delta \psi}{\partial r} \right) - k^2 \delta \psi. \quad (86)$$

From Equations (84) and (85), we obtain

$$\xi_z = -\int k \xi_r dr. \quad (87)$$
We solve Equations (83), (84), (86), and (87) in the discretized form.

We express the perturbation using the radial displacement 
\( \xi_r = (\xi_{r,1/2}, \xi_{r,3/2}, ..., \xi_{r,n-1/2}) \), where \( \xi_{r,j-1/2} \) denotes the radial displacement at \( r = (j - 1/2)\Delta r \). We will show that Equation (84) can be expressed in the discretized form

\[
\sigma^2 \xi_{r,j-1/2} = \sum_i F_{ji} \xi_{r,j-1/2},
\]

where the matrix elements \( F_{ji} \) are obtained by the following procedure. When evaluating \( F_{ji} \), we set

\[
\xi_{r,j-1/2} = \begin{cases} 
1 & \text{if } j = i \\
0 & \text{otherwise}
\end{cases}.
\]

Using Equation (87) we obtain the longitudinal displacement

\[
\xi_{ij} = \begin{cases} 
-k \Delta r & (j < i) \\
0 & (j \geq i)
\end{cases}.
\]

The change in the density is evaluated to be

\[
\delta \varrho_j = \begin{cases} 
\frac{4\rho_{0,1/2} \xi_{r,1/2}}{\Delta r} - k \rho_{0,0} \xi_{c,0} & (j = 0) \\
- \frac{1}{r_j \Delta r} (r_{j+1/2} \rho_{0,j+1/2} \xi_{r,j+1/2} - r_{j-1/2} \rho_{0,j-1/2} \xi_{r,j-1/2}) - k \rho_{0,j} \xi_{c,j} & (j \neq 0)
\end{cases},
\]

by discretizing Equation (83), where \( \delta \varrho_j \) and \( \rho_{0,j+1/2} \) denote the values at \( r = r_j \) and \( r_{j+1/2} \), respectively. The change in the gravitational potential, \( \delta \psi \), is obtained by solving the discretized Poisson equation:

\[
4\pi G \delta \varrho_j = \begin{cases} 
\frac{2 \delta \psi_1 - \delta \psi_0 - k^2 \delta \psi_0}{\Delta r^2} & (j = 0) \\
- \frac{1}{r_{j+1/2} \Delta r} (2 r_j \delta \psi_j + r_{j+1/2} \delta \psi_{j+1}) - k^2 \delta \psi_j & (j = 1, 2, ..., n)
\end{cases},
\]

with the boundary condition

\[
\delta \psi_{n+1} = \frac{K_0[k(n + 1)\Delta r]}{K_0(k\Delta r)} \delta \psi_n,
\]

where \( K_0 \) denotes the modified Bessel function (see Appendix A). By discretizing Equation (84) we obtain

\[
\sigma^2 \xi_{r,j-1/2} = - \frac{1}{\Delta r} \left[ \left( \frac{d\rho}{d\rho} \right)_j \delta \varrho_j - \left( \frac{d\rho}{d\rho} \right)_{j-1} \right] \\
\times \left[ \frac{\delta \varrho_{j-1}}{\rho_{0,j-1}} + \delta \psi_j - \delta \psi_{j-1} \right] + 4\pi G \rho_{0,j} \delta \psi_j
\]

When \( k = 0 \), we replace Equation (94) with

\[
\sigma^2 \xi_{r,j-1/2} = - \frac{1}{\Delta r} \left[ \left( \frac{d\rho}{d\rho} \right)_j \delta \varrho_j - \left( \frac{d\rho}{d\rho} \right)_{j-1} \right] \\
\times \left[ \frac{\delta \varrho_{j-1}}{\rho_{0,j-1}} + \delta \psi_j - \delta \psi_{j-1} \right] + \frac{4\pi G \rho_{0,j} \delta \psi_j}{\rho_{0,j-1}}
\]

for \( k = 0 \).

The growth rate, \( \sigma \), is obtained as the eigenvalue of the matrix \( F_{ji} \). The spatial resolution and outer boundary are set to be \( \Delta r = 0.1H \) and \( r_n \geq 60H \). The outer boundary should be set very far (\( r_n \geq 100H \)) for \( p < 1 \). Otherwise the growth rate is underestimated.

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