The classical limit and the form of the hamiltonian constraint in non-perturbative quantum general relativity

Lee Smolin

Center for Gravitational Physics and Geometry
Department of Physics
The Pennsylvania State University
University Park, PA, USA 16802

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ABSTRACT
It is argued that some approaches to non-perturbative quantum general relativity lack a sensible continuum limit that reproduces general relativity. This may be true in spite of their being mathematically well defined diffeomorphism invariant quantum field theories that result from applying canonical quantization to general relativity. The basic problem is that generic physical states lack long ranged correlations, because the form of the state allows a division into spatial regions, such that no change in the physical state in one region can be measured by observables restricted to another. These disconnected regions have generically finite expectation value of physical volume, which means that the theory has no long ranged correlations or massless particles. One consequence of this is that the ADM energy is unbounded from below, at least when that is defined with respect to a natural notion of quantum asymptotic flatness and a corresponding definition of an operator that measures $E_{ADM}$ (which is given here). These problems occur in Thiemann’s new formulation of quantum gravity. Related issues arise in some other approaches such as that of Borissov, Rovelli and Smolin. A new approach to the Hamiltonian constraint, which may avoid the problem of the lack of long ranged correlations, is proposed.

* smolin@phys.psu.edu
1 Introduction

A non-perturbative quantum theory of gravity must satisfy at least two criteria to be a candidate for a description of nature: it must be well defined as a quantum theory and it must have a good classical limit that guarantees that the classical Einstein’s equations are satisfied approximately in an appropriately defined classical regime. This second criteria is necessary because a non-perturbative theory is not going to be defined in terms of classical fields, but in terms of objects defined without reference to a classical background geometry. The classical metric must emerge in an appropriate limit or regime, and it must then satisfy the appropriate equations.

The classical limit is certainly a significant issue in the family of approaches to quantum gravity that have emerged based on the joining of the Ashtekar-Sen formalism [1], or one of its variants [2, 3], with the notion of states based on Wilson loops [4, 5, 6, 7, 8, 9, 10, 11, 12]. In these formalisms, there is no background metric, instead, diffeomorphism invariant and physical states are described in terms of topological and combinatorial information encoded in embeddings of spin networks [14] in space [15], up to diffeomorphisms of the spatial three manifold Σ. Further, area and volume are discrete [10, 9] and there are straightforward graphical formulas for the action of dynamical operators such as the Hamiltonian constraint [17, 18, 23, 20, 21]. These results have turned out to be robust: they arise when the theory is defined through several different regularization procedures [24, 25, 22, 15], they are independent of coupling to matter and they are confirmed by rigorous methods [26, 27, 13, 28, 24, 25].

Because the Planck scale description is based completely on the topology and combinatorics of graphs, it is to be expected that problems with the classical limit might arise. This is because the kind of long ranged behavior characteristic of classical general relativity may be only expected to emerge from a discrete dynamics at a critical point, when the correlation length diverges. Indeed, exactly this is seen in a number of approaches to quantum gravity based on discrete structures, such as dynamical triangulations and Regge calculus [29, 30, 31, 32, 33]. One sees in these cases that there are no continuum limits that could lead to a recovery of the classical physics unless the bare Newton and cosmological constant are tuned to a critical point [34, 31, 32, 33]. This suggests that without fine tuning, non-perturbative approaches to quantum gravity in which the quantum geometry is discrete at Planck scales may lack classical limits [34].

Another piece of evidence that the existence of a good semiclassical limit
is sensitive to the Planck scale physics has emerged from the study of black holes. Several authors have studied the hypothesis that the area of any surface, and thus that of an event horizon is a discrete quantity. It turns out that whether a theory that predicts a discrete spectra for the area of the horizon has a good classical limit depends on the detailed spectrum. If the spectrum is equally spaced, as proposed in , then there is no good classical limit, because the longest wavelength radiated is proportional to the Schwarzschild radius of the black hole. This means that the continuous Hawking spectrum is not reproduced, even in the limit of large black holes. This is true, even though the level spacing in the area spectrum proposed is of the order of the Planck area. Remarkably, a small change in the formula for the spectrum, to the one which results from canonical quantization, resolves this problem.

Recently, in a remarkable series of papers, Thomas Thiemann has proposed and developed a new approach to quantum gravity along the lines of the non-perturbative canonical approaches, which makes it possible to overcome a number of problems with previous formalisms. Chief of these is that it is possible to write closed form expressions for the action of the Hamiltonian constraint and physical inner product for the theory with Lorentzian signature, eliminating the difficulty known as “the problem of the reality conditions”. Thiemann introduces completely regulated forms of the Hamiltonian constraint for both the Lorentzian and Euclidean signature case and finds the full set of solutions to the constraints for these forms, which are normalizable under a satisfactorily defined physical inner product. Other developments include a length operator with discrete spectrum and an explicit form of a transformation between the euclidean and lorentzian form of the theory. Furthermore, building on previous work with and by collaborators, this is done in the context of a completely rigorous formulation, so that all these results are presented as theorems in mathematical quantum field theory.

Whatever follows, Thiemann’s papers represent a significant achievement that contain one possible completion of the program of constructing quantum general relativity non-perturbatively. The theory presented there may indeed be considered to be a well defined diffeomorphism invariant quantum field theory gotten by applying a certain quantization procedure to general relativity. This makes it urgent to consider the second question, that of the classical limit. One goal of this paper is to investigate whether Thiemann’s formalism may have a good classical limit. Unfortunately, we find evidence that, at least as defined so far, the theory fails to have a good
classical limit.

The reasons for this are developed in the next two sections. I should note that the detailed discussion is not self contained, but is based on the definitions and results in [39, 40]. However, the basic issues are not difficult to describe on an intuitive level. The main problem is that generic states lack long distance correlations of the type expected if the theory has massless particles. This problem arises because Thiemann’s definition of the Hamiltonian constraint acts entirely within disjoint regions in a generic spin network state, and does not convey any information between them. Furthermore, these regions have finite expectation values of volume, when measured with the operators and inner product used in [39, 40]. As a result of this, generic states have arbitrary finite correlation lengths.

A consequence is that the hamiltonian of the theory, defined on an appropriate space of asymptotically flat states, is not bounded from below. As I will show in the next section, in states that have a good classical limit asymptotically, the information near infinity that is picked up in the ADM mass in no way constrains the behavior of the state in the interior, as one can generically find regions in the interior that are completely uncorrelated with the regions near infinity. This means that there are no long ranged effects of the kind that one sees at work in the proofs of the positive energy theorem that prevent initial data which are asymptotic to negative mass Schwarzschild metrics from being extended over the whole spatial manifold. As a result, one can construct exact normalizable solutions to the constraints that are eigenstates of a natural form of the ADM energy operator with any sign of the mass.

This raises several questions, which will be discussed at length in section 4. First, are the problems restricted to this one formalism, or are they more general? Second, are there ways that we might modify the quantum theory so as to avoid these difficulties and insure the existence of a classical limit? In fact, for reasons I will discuss, it seems that the existence of a good classical limit is quite sensitive to the form of the Hamiltonian constraint. Other forms which have been proposed in the literature, such as [17, 18, 19], also have a problem with correlations restricted to bounded regions. However, it is not difficult to modify the form of the quantum Hamiltonian constraint so as to eliminate this difficulty. One way to do this is proposed in section 5. An interesting and provocative fact is that it is difficult to see how the form proposed there could be deduced from a conventional point split regularization procedure. This issue is discussed in the concluding
I should also stress that there have been other suggestions as to how the problems I discuss here may be avoided. These are discussed in section 4.

Finally, before beginning, I should emphasize that the following considerations are based partly on physical arguments which are not at the same level of rigor as the original paper. It may be possible to fill in the details to obtain rigorous results along the lines sketched here, but this has not been done. I should also warn the reader that in the course of the argument a few assumptions are made, whose justification is that their negation would imply that the theory lacks a good classical limit. So the argument that follows has partly the form of a proof by contradiction, one can challenge certain of the assumptions, but only at the cost of admitting from the beginning that the theory cannot reproduce the physics of general relativity.

2 An ADM energy operator and its spectrum

The key issue involved in defining the ADM energy non-perturbatively is in giving a notion of asymptotic flatness that may apply to non-perturbative states constructed from spin networks which are also exact solutions to the constraints, when those are smeared with lapses and shifts that satisfy appropriate fall off conditions. I propose here one way in which this might be done, which has its origins in the notion of a weave[44], which is a quantum state built from spin networks that is associated with a classical metric. One of the strange things about the theory, which is shared by all of the approaches based on the connection or loop representation is that there can be states which are at the same time exact physical states and weaves that approximate a classical three geometry.

Of course, the three metric is generally not a physical operator, as it does not commute with the Hamiltonian constraint. Thus, it is unlikely that any interpretation of the physics of such states based on the correspondence to a particular three metric is going to be meaningful. One way to understand this is that even if one can associate a three metric with a physical weave state, which is an exact solution to the constraints, it cannot be meaningful

Another question that might be asked is whether the theories discussed here may have continuum limits, even if these do not give general relativity in four dimensions. This question is explored in [58] in which it is shown that a certain sector of the theories has collective coordinates that can be understood in terms of strings in three spacetime dimensions.
as one does not know which three surface that metric is to be associated with.

However, there is one case in which it may be meaningful to find a limited correspondence between a classical metric and a physical state. This is in the case that the state satisfies also some condition of asymptotical flatness. In this case, there may be gauge invariant information in the state-metric correspondence, to leading order in $1/r$, where $r$ is an appropriate radial coordinate. Some of this information is coded in the $\text{ADM}$ mass. The basic strategy we follow here is to exploit this fact, to show that positive energy is violated as easily as it is satisfied in the quantum theory defined by $\text{[39, 40]}$.

2.1 Preliminaries concerning weaves

I will need to discuss primarily weave states associated with conformally flat metrics of the form $q_{ab} = \Omega^2(x)q^0_{ab}$, where $q^0_{ab}$ is a flat metric. The weaves will be defined with respect to the volume operator, $\hat{V}[^R]$ alone, where $^R$ is any region of three space $\Sigma$. It is true classically that $V[^R]$ for every $^R$ determines $\Omega$ and hence $q_{ab}$, within this category. We will also assume the existence of orthonormal coordinates $\hat{x}$ associated to $q^0_{ab}$ that cover $\Sigma$, which will have the topology of $R^3$.

I will discuss two kinds of weaves: eigenweaves and expectation value weaves.

The first will be eigenweaves. An eigenweave will be a linear combination of spin network states $\text{[15]} |\Gamma\rangle$, all with support on the same graph. A key point to remember in the following is that associated to vertices with four or more incoming edges are generally finite dimensional state spaces $\text{[15]}$. Generally bases may be chosen for these spaces in which the volume operator is diagonal $\text{[46]}$

To define a weave we need two length scales $R$ and $L$ such that $R >> L >> l_{Planck}$. We will say that $\Omega$ is slowly varying on the scale $R$, if $\delta_{ab}\partial_aLn(\Omega)\partial_bLn(\Omega) < R^{-2}$. Then for all cubic regions $^R$ with classical volume greater than $L^3$ we require two conditions on $|\Gamma\rangle$. The first is that the state be isotropic, when averaged over regions of size $L$, with respect to the flat metric $q^0_{ab}$. This means that no operator that averages information over regions of size $L$ will be able to determine a prefered direction around any point (up to fluctuations of order $l_{Planck}/L$.) As a result, if the state is a weave state of some metric, it is one that is conformally related to $q^0_{ab}$. The second condition is that,

$$\hat{V}[^R]|\Gamma\rangle = |\Gamma\rangle (V[^R] (q_{ab}) + O(l_{Planck}/L))$$  \hspace{1cm} (1)
where \( V[\mathcal{R}](q_{ab}) \) is the classical observable that measures the volume of the region \( \mathcal{R} \) as a function of the metric \( q_{ab} \).

An expectation value weave state \(|\Psi\rangle\) associated to \( q_{ab} \) does not necessarily have support on a single graph. It must be also isotropic, while the second condition is modified to,

\[
<\Psi|\hat{V}[\mathcal{R}]|\Psi> = <\Psi|\Psi> (V[\mathcal{R}](q_{ab}) + O(l_{planck}/L)).
\]

(2)

We may ask whether the states \(|\Gamma\rangle\) and \(|\Psi\rangle\) must satisfy similar conditions with respect to other three metric observables such as areas of large surfaces, or lengths of appropriately averaged curves. It is important to note that generically weave states do not have this property. This reflects the fact that in quantum mechanics functional relationships between classical observables are not obeyed by expectation values of states. However, it is possible to construct weave states that satisfy the appropriate conditions for both area and volume\[51\]. We may call these consistent weave states. It is good that they exist, were it not the case the theory would already have a problem with a classical limit as there will be no states in which these different operators satisfy their classical relationships. However, this raises a dynamical problem, which is to understand why the ground state of the theory should be consistent in this sense. In this paper we will assume that all weave states are constructed to be consistent.

We may then construct a consistent weave state \(|\Gamma_0\rangle\) that is an eigenweave of the flat metric \( q_{ab}^0 \). We will construct it from identical vertices each with \( m \) incident edges. \( m \) might be 6, as in the case of a cubic lattice, but will be required to be at least 4, so that each node may contribute to the volume operator. By the first condition it is required to be isotropic. We will also require that, when averaged over the scale \( L \), the expectation values of \( \hat{V}[\mathcal{R}] \) will be invariant, under the Euclidean group of \( q_{ab}^0 \), up to first order in \( l_{Planck}/L \). It follows from symmetry that any state that satisfies this must be a weave for a flat metric proportional to \( q_{ab}^0 \) when measured by any other averaged observable. We will then adjust the graph and its labelings so as to have a consistent eigenweave of \( q_{ab}^0 \).

For simplicity we will assume that all the edges have the same spin \( j_0 \). The condition of average isotropy can then be met if the nodes are scattered randomly in space (as determined by \( q_{ab}^0 \)), with each one connected to the \( m \) closest, and if the spins on the edges are small so the volume contributed by each node is order the Planck volume.

Associated to the vertices with \( m \) edges of spin \( j_0 \) there is a finite dimensional Hilbert space \( \mathcal{H}_{j_0} \). \( \hat{V} \) induces in each of these a \( p \) dimensional
Hermitian matrix $\hat{v}_{m,j_0}$. We will assume that this is non-degenerate (if it is not the following argument must be altered accordingly). It will then have eigenvalues $w_1, \ldots, w_p$ ordered in terms of size. Their normalization is chosen so that eigenvalues of the volume are $l_3^3 w_i$, $i = 1, \ldots, p$. We may note that the $w_i$ are of order one for small spins [45, 46, 42]. For the following we will need to define $w' = (w_1 + w_p)/2$.

We then construct the state $|\Gamma_0\rangle$ as follows. In each cubic box of size $L$ we distribute randomly $N$ nodes and connect them up into a network with edges with spin $j$ as we described so that

$$N = \frac{1}{w'} \frac{L^3}{l_{Planck}^3}$$

We choose randomly half the nodes and give them the highest eigenvalue $w_p$, while we give the other half the lowest, $w_1$. It is straightforward to check that this satisfies the conditions of both kinds of weaves.

Now we want to construct weaves for the conformally flat metrics $q_{ab} = \Omega^2(x) q^0_{ab}$, with $\Omega$ slowly varying. This is easy to do if we restrict ourselves to cases in which

$$\frac{w_1}{w'} < \Omega^3 < \frac{w_p}{w'}$$

which will be sufficient for our purposes. We do this by keeping the same graph of $\Gamma_0$ and varying only the distribution of the two eigenstates of volume associated to $w_1$ and $w_p$. We do this in a way that preserves the isotropy of the states, to order $l_{Planck}/L$, when averaged over regions of size $L$. This implies that the resulting metric must be conformally flat.

It is then clear we can realize 1 or 2 for every box of size $L$ if we choose a distribution of eigenvalues on the nodes in each box such that

$$\frac{\bar{w}(box)}{w'} = \Omega^3(x_{box})$$

where $\bar{w}(box)$ is the average value of the nodes in the box and $x_{box}$ is its center. Let us pick one such state and call it $|\Gamma_0^\Omega\rangle$.

Now, let us consider a class of spherically symmetric metrics built the following way. Pick two radii $R_1 < R_2$ much larger than $L$, and such that one can find a mass $M$ (with $GM >> L$) such that

$$1 + \frac{GM}{2R_2} < (w_p/w')^{1/6}$$
and
\[ 1 - \frac{GM}{2R_2} > (w_1/w')^{1/6} \]  \hspace{1cm} (7)

Then consider two choices of \( \Omega \), called \( \Omega^\pm \) given by
\[ \Omega^\pm = (1 \pm \frac{GM}{2r})^2, \quad r > R_2 \]  \hspace{1cm} (8)

and
\[ \Omega^\pm = 1, \quad r < R_1 \]  \hspace{1cm} (9)

with \( \Omega^\pm \) each chosen to be some smooth, slowly varying function of \( r \) for \( R_1 < r < R_2 \).

Then we have two eigenweave states, which correspond to a three geometry that is either positive or negative mass Schwarzschild externally, each of which are smoothly joined to a flat metric on the interior. Let us call these weaves spin network states \( |\Gamma^\pm_0\rangle \). Note that by making \( R_1 \) and \( R_2 \) large enough we can construct these states for any \( M \), positive or negative.

We may note that all of the weave states we have been discussing are solutions of Type I (by the definition of Theorem 1.1 of [40]) to the Hamiltonian constraint. This is true for both the Euclidean and Minkowskian operators. Furthermore, for each state we have been discussing a diffeomorphism invariant state can be given which is the characteristic state of the corresponding diffeomorphism class [15, 13].

Thus, we see that there are physical, normalizable weave states associated to three metrics which are asymptotically Schwarzschild, for any values of the mass, both positive and negative. However this is not yet enough to conclude that the theory is unstable, as we cannot directly associate the three geometry to the state, as we discussed above. The question is whether we can define a gauge invariant operator which measures the ADM mass in a class of states restricted by some suitable definition of asymptotic flatness, and whether this can be done in such a way that the ADM mass extracted is in fact the mass associated with the asymptotic behavior of the metric of the weave. This is done in the next subsection. After this, the results are extended to the generic solutions of type II.

### 2.2 Quantum asymptotic flatness and the ADM operator

The main problem to define an operator for the ADM energy in non-perturbative quantum gravity is to give a definition of asymptotic flatness appropriate for the quantum theory. Here I will not attempt to be rigorous, but just to give
the main idea which is necessary to introduce an operator that measures the ADM energy of a quantum state.

Let us first recall that an asymptotically flat three metric is defined with respect to some given flat background metric, $q^0_{ab}$, which can be written in appropriate Euclidean coordinates, $\hat{x}^a$ as $\delta_{ab}$. We need also a choice of a radial coordinate $r$ defined with respect to $q^0_{ab}$, and the associated angular coordinates $\theta$ and $\phi$. Given these a metric $q_{ab}$ will be considered asymptotically flat (in isotropic coordinates) if it take the form in the coordinates $\hat{x}^a$:

$$q_{\hat{a}\hat{b}} = (1 + \frac{GM(\theta, \phi)}{2r})^4 \delta_{\hat{a}\hat{b}} + O(1/r^2)$$

(10)

where $M(\theta, \phi)$ is an angle dependent mass. We may note that for the Schwarzcchild solution in standard isotropic coordinates, $M$ is a constant and is equal to the ADM energy.

The complete condition of asymptotic flatness includes also a specification of the fall off behavior of the extrinsic curvature $k_{ai}(x)$. This is because the ADM energy can be negative, even for the positive mass Schwarzcchild solution, if it is defined on slices whose extrinsic curvatures do not vanish fast enough near infinity. In the classical theory there is then a condition like

$$|k(x)|^2 \equiv q^{ab}k_{ai}k_{bi}(x) = O(1/r^4)$$

(11)

as $r \to \infty$. Thus we require also a quantum mechanical version of this condition. How this is to be formulated and satisfied is discussed at the end of this section.

The ADM mass is defined by

$$E_{ADM} \equiv \lim_{r \to \infty} E(r)$$

(12)

where

$$E(r) \equiv \frac{1}{16\pi G} \int_{S^2(r)} d^2 S \epsilon_a q^{0ca}q^{0bd}(\partial_d q_{ab} - \partial_c q_{bd})$$

(13)

Because any asymptotically flat metric is conformally flat up to the order measured by $E_{ADM}$, we have

$$E(r) = \frac{-1}{6\pi G} \int_{S^2(r)} d^2 S \epsilon_a q^{0ca}q^{1/2}$$

(14)

\[I thank Abhay Ashtekar, Don Marolf and Thomas Thiemann for pointing out this issue.\]
up to terms of order $1/r^2$, where $q = \det(q_{ab})$. From [14] we have of course $E_{ADM} = M$ when it has no angular dependence.

The gauge invariance in the asymptotically flat case is generated by $\mathcal{H}(N)$ and $D(v)$, where the lapse $N$ and shift $v^a$ are order $1/r$ as $r \to \infty$, which guarantees the gauge invariance of $E_{ADM}$.

Now, let us make a definition of a quantum state which is asymptotically flat with respect to a flat metric $q_{ab}^0$ and a radial coordinate $r$. Recall first that we may define a weave with respect to several different functions of the three metric, the area, volume, $q_{ab}$. A state $|\Psi\rangle$ will be called area- (or volume-, or $q_{AB}$-) asymptotically flat if it is, up to terms of order $1/r^2$, an expectation value weave of the area (or volume or $q_{ab}$) corresponding three metric $q_{ab}$ which is asymptotically flat.

We will call a weave state $|\Psi\rangle$ simply *metrically* asymptotically flat if it is area, volume and $q_{ab}$ asymptotically flat for the same three metric $q_{ab}$. One way to accomplish this is with a weave that is isotropic, up to terms of order $1/r^2$, as this guarantees that all the averaged observables will have to agree that it is a weave of a metric that is, up to terms in $1/r^2$, conformally flat. One then only has to make sure that the different observables agree about the scaling of the $1/r$ piece of the metric.

We may also make such definitions for weave eigenstates, in which case the definition using eigenvalues replaces the one using expectation values. We will call a state that satisfies the same definition for eigenweaves, an eigen-asymptotically flat state.

In both cases, a complete definition of asymptotic flatness will involve a fall off condition of the expectation value of the extrinsic curvature, as I discuss below.

Given [14] we may define an operator corresponding to the ADM mass, appropriate to $q_{ab}^0$ and $r$ as follows. We define

$$E_{ADM} \equiv \lim_{r \to \infty} \hat{E}(r)$$

where $\hat{E}(r)$ is defined from [14] using the volume operator.

$$\hat{E}(r) = \frac{-1}{6\pi GL^3 r^2} \int_{S^2(r)} \partial_r \hat{V}(Box(r)^L)$$

where $\hat{V}(Box(r)^L)$ is the volume operator on any region which is a cube of volume $L^3$ centered at a radial coordinate $r$, with respect to $q_{ab}^0$ and $r$.

It follows from what we have said that $<\Psi|\hat{E}_{ADM}|\Psi>$ is gauge invariant with respect to $\mathcal{H}(N)$ and $D(v)$, with $N$ and $v$ bounded by $1/r$, where $|\Psi>$
is a volume-asymptotically flat states. Further, under the same conditions $\hat{E}_{ADM}$ commutes with $\mathcal{H}(N)$ and $\mathcal{D}(v)$ when acting on the space of eigen-volume-asymptotically flat states.

We might worry that the operator for $ADM$ energy has built into it a length scale $L$. This might be eliminated by a more sophisticated definition of the operator, however, the present one is sufficient for our purposes as we are discussing weave states associated to classical metrics. In this case it is acceptable to use a notion of course graining in the definition of the $ADM$ energy. Any definition that disagreed with this one on such states would lead itself to problems with the classical limit, as it would lead to a disagreement between the $ADM$ energy and the asymptotic form of the metric extracted by taking expectation values of appropriate metric observables.

It is now straightforward to show that the spectrum of $\hat{E}_{ADM}$ is unbounded from either above or below on the space of solutions to Thiemann’s constraints defined by Theorem 1.1 of [40]. For we may note that both of the states $|\Gamma_{0}^{\pm}\rangle$ defined above are eigen-volume-asymptotically flat. Furthermore, it follows directly that they are eigenstates of $\hat{E}_{ADM}$ with

$$\hat{E}_{ADM}|\Gamma_{0}^{\pm}\rangle = \pm M|\Gamma_{0}^{\pm}\rangle$$

(17)

2.3 Lack of positive energy on Type II states

One might wonder if this is just a problem for the solutions called Type I in Theorem 1.1 of QSD II. If this were so we might worry less, as these are solutions which are also eigenstates of the volume operator, which are thus very special. For example, while they are normalizable with respect to the inner product used in [23, 40] one might worry that that is not correct, and that there is another, physical inner product with respect to which they are unphysical. Thus, it is of interest to know how general are the existence of states with both both positive and negative expectation value of the $ADM$ mass operator. Unfortunately, there are much more general forms for such states, as we will now see.

As a first step we may note that all that is really required to get the unboundedness of the spectrum is that the state is Type I (that is constructed from eigen-vertices of the volume) up to terms of order $1/r^2$. One can contaminate the state with dressed vertices of Type II, which are not eigenstates of the volume operator in their neighborhoods, as long as the contribution to the action of the volumes of large boxes falls off as $1/r^2$.

However, what if we require that the state has a generic form also out to infinity? It might be reasonable to require this if there were a superselection
principle of some kind that ruled out type I vertices. In this case, however, one can show that the situation is no better. Now, we will no longer have eigenstates of $\hat{E}_{ADM}$, at least generically. But it is still possible to find an infinite number of states $|\Psi\rangle$, all of whose vertices are of Type II, such that $\langle \Psi | \hat{E}_{ADM} | \Psi \rangle$ is negative. This follows directly from the fact that each node of a non-extraordinary graph may be dressed completely independently of the others.

To see this, let us consider for the moment a small network with just one ordinary node of the type we have been considering with its $m$ spin $j$ lines sticking out of it. We may call it $\Gamma_\tilde{n}$ (These are not spin networks, but we can consider such non-gauge invariant states in this formalism.) We will consider as before just two of the volume eigenstates at the nodes, corresponding to the highest and lowest eigenvalue $w_p$ and $w_1$. Let us say that a node in one of these states is in the state $+$ or $-$. Let us choose a dressing of each of these, corresponding to a solution of type II involving $n$ extraordinary edges added to the one node. ($n$ may be chosen as we like, as there are generically solutions for each $n$.) We then have two states

$$|\Psi^\pm\rangle = \sum_I |\Gamma^\pm_I\rangle e^\pm_I$$  \hspace{1cm} (18)$$

where the spinnets $|\Gamma^\pm_I\rangle$ include adding $n$ extraordinary edges to the one node of $\Gamma^\pm_\tilde{n}$. There are two such states, distinguished by $\pm$ which are dressings of the two eigenstates at the node.

Let us assume we may choose them such that

$$\langle \Psi^\pm | \Psi^{\pm'} \rangle = \delta_{\pm\pm'}$$ \hspace{1cm} (19)$$

Each of these states is no longer an eigenstate of volume. However, each contributes a value to the expectation value of volume:

$$v(\pm) = \langle \Psi^\pm | \hat{V} | \Psi^\pm \rangle$$ \hspace{1cm} (20)$$

In the following we will assume $v(+) > v(-)$. (If not we will switch the designations so this is true.)

Now, let us go back to our initial network $|\Gamma_0\rangle$ for the flat metric $\delta_{ab}$. We will construct a large set of physical states $|\Psi(\epsilon_\mu)\rangle$. Here the nodes are labeled by $\mu$ and each of them is put first in the state $+$ or $-$, which will be labeled by $\epsilon_\mu = \pm$, respectively. We then dress each node according to the prescription [18]. Because the Hamiltonian constraint factors into a sum that each acts just around each non-extraordinary vertex, all of these are physical
states of type II. Now among these are many which are expectation value weaves corresponding to slowly varying conformally flat metrics. Further, to any such metric we may associate many such states.

To see this, let us note that for a large region $\mathcal{R}$

$$< \Psi(\epsilon_\mu) | \hat{V}[\mathcal{R}] | \Psi(\epsilon_\mu) > = \sum_{\mu \in \mathcal{R}} v(\epsilon_\mu)$$  \hspace{1cm} (21)$$

Given a slowly varying $\Omega$ such that

$$\frac{\Omega^3(\text{max})}{\Omega^3(\text{min})} = \frac{v(+)}{v(-)}$$  \hspace{1cm} (22)$$
we can construct an associated expectation value weave by distributing the $+$ and $-$ dressings so that for each box of volume $L^3$ in the flat background metric

$$< \Psi(\epsilon_\mu) | \frac{V(\text{box})}{L^3} | \Psi(\epsilon_\mu) > = \frac{\bar{v}}{w' |L^3_{\text{Planck}}|} = \Omega^3(x_{\text{box}})$$  \hspace{1cm} (23)$$
where $\bar{v}$ is the average of the $v(\pm)$ values over the dressed nodes in the box.

We then can construct expectation value weaves of this kind that match the two conformal factors $\Omega^\pm$ of which are asymptotically positive or negative mass Schwarchild, for any mass $M$. Let us call these states $| \Psi(\Omega^\pm) >$. We then have,

$$< \Psi(\Omega^\pm) | \hat{E}_{\text{ADM}} | \Psi(\Omega^\pm) > = \pm M$$  \hspace{1cm} (24)$$

Thus, we have shown in this section that a well defined operator that measures the $ADM$ energy has a spectrum that is unbounded from above and below, and further, even when restricting to pure Type II states, its expectation value is still unbounded above and below. This means that the classical limit must fail, in the sense that there are an infinite number of states that have a good $ADM$ energy, but whose energy is arbitrarily negative. We may note that as the $ADM$ energy is gauge invariant, we cannot be in doubt as to the interpretation of this result, as we might if it were only a result about the existence of weave states that are associated with negative mass Schwarchild.

We may note also that it would not help to try to define another operator that measures $ADM$ energy based on measurements of other operators such as areas or lengths or $q_{ab}$. This is because, if there is to be a good classical limit it must be that there are isotropic weave states that are weaves of the same conformally flat metric, which ever operator is used in the construction of the weave. If this were not the case we could not believe that the theory
adequately reproduced the classical three geometry. Furthermore, we require only that the states have a good classical limit to leading order in $1/r$, which is enough to define eigenstates of the $ADM$ energy. We have no need to require that the asymptotically flat states have a good classical limit in the interior, or to higher order in $1/r$. But then, assuming only that there are such states, we see that it is as easy to construct those that are asymptotically negative mass Schwarzschild as it is to construct those that are asymptotically positive mass Schwarzschild, when the definition given here is used. But then, as the $ADM$ energy measures a property of these states that is gauge invariant, there can be no escaping the conclusion that this means that the $ADM$ energy is unbounded in the quantum theory defined by [39, 40].

2.4 Implementing fall off conditions on the extrinsic curvature

One might still worry that the problem is that a fall off condition for the extrinsic curvature has not yet been imposed. As this is necessary for the classical positive energy theorem, it might be that implementation of this condition in the quantum theory will restrict the states that have only positive values of the $ADM$ energy [18]. To be completely sure that the theory has an unbounded Hamiltonian one must make sure that the states are restricted by a quantum analogue of the fall off condition for the extrinsic curvature. The simplest possibility is to define define an appropriately ordered and averaged operator for $k^a_b$,

$$O^a_b(R) = \int_R Tr(\tilde{E}^a k_b)$$

We may then require that

$$<\Psi|O^a_b(R)|\Psi> = O(1/r^2)$$

for regions of size $L^3$ in the background metric $q^a_{\mu\nu}$ as $r \to \infty$. (We may note that this may be superior to measuring expectation values of quadratic operators that might represent $|k|^2$, as these will have zero point fluctuations that will have to be subtracted out to define the asymptotic behavior.

It is, however difficult to imagine that it is not possible to satisfy an additional asymptotic condition such as [26], given the enormous freedom to construct states which satisfy the asymptotic conditions for the expectation values of the metric observables. For there is no reason not to construct
weave states using more than the two eigenstates of volume at each node I used above. Let us suppose we instead construct our weaves using identical nodes that have a large number, \( r \) of possible volume eigenstates. There will then be many ways to get any desired \( \Omega(r, \theta, \phi) \) that satisfies (4) by mixing the \( r \) eigenstates appropriately in each region of size \( L \). This is true for either the eigenweaves or expectation value weaves. In each region I require that \( \omega^3 = \sum_{i=1}^{r} v_i n_i \) add up to some required number, proportional to \( (1 + M(\theta, \phi)/2r)^6 \) where \( v_i \) is the contribution to the volume of node eigenstate \( i \) in Planck units and \( n_i \) is the number of the \( N \) nodes in the region that are of this type. Thus we must satisfy two conditions, \( \sum n_i = N \), and that the sum for \( \Omega \) is fixed. Given \( r \) kinds of nodes there are then an \( r-2 \) dimensional set of possibilities of achieving the required dependence on the conformal factor, for any desired \( M(\theta, \phi) \). It is difficult to see how, if \( r \) is large enough, it will not be possible to use this freedom to match any desired fall off conditions for the extrinsic curvature such as (26).

For example, we may note that by the isotropy of the weave, we must have

\[
\langle \Psi | O_6^a (R) | \Psi \rangle = C(R) \delta_a^6
\]  

(27)

Given that we have a large number of solutions to the problem of matching the fall off conditions for the volume, it should be trivial to choose among this set to get any desired fall off on the functions \( C \) as a function of \( r \). As the regions \( R \) each contain many nodes, then we will have on average \( C(R) = \sum_{i=1}^{r} c_i n_i \) where the sum is again over the nodes in \( R \) and \( c_i \) is the average contribution of a node of type \( i \) to \( C \). All that is needed to have \( C(R) = 0 \), on average, is for the \( c_i \) to span a range of both positive and negative values such that restricted to the \( r-2 \) dimensional subspace of solutions to the metric fall off conditions it is possible always to balance the positive and the negative \( c_i \)’s in each region.

To complete the argument the \( c_i \)’s should be computed, showing that there are nodes with the required property. This has not yet been done. However, it is clear that there is no reason to expect that the imposition of one further condition on the nodes of the weave states could save the positivity of the energy. The real problem, as I will now describe, is that we have complete freedom to choose the forms of the solutions independently in the neighborhood of each node. This means that there is no long ranged order in this quantum theory, of the kind that is imposed by the classical field equations.
3 The absence of long ranged correlations

We know in general terms, from the renormalization group as well as from analysis of many different kinds of systems, what must be the case if a discrete quantum system is to have a classical limit described by a classical field theory with massless quanta. There must be a critical point at which the correlation length diverges\(^3\). For this to be the case it must be true for generic physical states that small perturbations made in one localized region can be detected by measuring some appropriate operator arbitrarily far from the region in which the disturbance is made.

I will show here that this is not realized in Thiemann’s space of states described by Theorem 1.1 of [40]. Instead, a generic physical state describes a quantum geometry that may be decomposed into regions whose volumes, defined by taking expectation value with the physical inner product, are finite, but each of which is completely uncorrelated with the others. This may be shown by constructing an infinite number of diffeomorphism invariant, physical Hermitian operators, that by construction describe local excitations, but which all commute with each other. This means that disturbances do not propagate more than a fixed finite distance in any given state of this type. But as the general solution to the constraints found by Thiemann have this property, that theory cannot have a perturbation theory in which massless fields propagate, nor may it have a classical limit which is general relativity.

The general solutions to all forms of Thiemann’s constraints may be described in the following way[40]. We begin with the space \(W_0\) of non-extraordinary spin networks (or more precisely spin networks all of whose extraordinary edges carry spin greater than \(1/2\).) Among the nodes of such a vertex are a special class, called simple nodes, which are those that may get dressed by the action of the Hamiltonian constraint. These are nodes with at least three incident edges with spins, \(j_i, i = 1,\ldots, n\) in some arbitrary ordering, with generally an additional label at the vertex \(r\), such that the tangent vectors of the incident edges span the tangent space at the node. We will label these node \(\tilde{N}_{n,j_i,r}\), where we include also implicitly in the labeling diffeomorphism invariant information about the linear relations

\(^3\)I refer here to the application of the renormalization group to random surface theory and second order phase transitions, rather than to the problem of renormalization in conventional quantum field theories. The latter does not apply to non-perturbative formulations of quantum gravity in which diffeomorphism invariance ensures the finiteness of physical operators\[1\] but the former definitely applies.
among the tangent vectors of the edges. These include all the nodes that may contribute to the volume operator, with the definition of volume used in [39, 40]. We will also assume that the labels at the vertex \( r \) include the volume plus any additional labels required to break degeneracy.

The simple nodes of a non-extraordinary spin network may then be dressed to yield the general solution to the constraints described in Theorem 1.1 of [40]. The nodes are dressed by taking linear combinations of the original graph with those in which a finite number of extraordinary edges decorate the region around each node, each joining two vertices coming out of each simple node. The result is something like a spider web around each simple node. The source of the trouble is that each node is dressed completely independently of the others.

To see this, begin with a non-extraordinary spin network and proceed as follows. First, note that each such state is in fact a solution to the constraints of type I according to Theorem 1.1 of [40]. Then, to each representative of the diffeomorphism class of each such spinnet, \( \Gamma \), construct a partition of unity \( N_\alpha \), where each \( N_\alpha \) has support on a set \( U_\alpha \) that only includes one of its nodes, labeled by \( \alpha \). Now let us consider only the hamiltonian constraint \( \hat{H}(N_\alpha) \), which by construction acts in a neighborhood of only one of the \( \alpha \)'th node. There are an infinite number of solutions of type II to

\[
\hat{H}(N_\alpha)|\Psi> = 0
\]

which may be constructed from linear combinations of states that have support on spinnets constructed from the node \( N_{n,j_i,\alpha} \) by dressing the edges incident to it with a finite, but arbitrary number of extraordinary vertices [40]. Each of these may be seen as a linear combination of open graphs with \( n \) external lines, with spins \( j_i \), as the dressing procedure never changes the spins of the portion of the edge furthest from the original simple node. Let us give a name to such a linear combination, associated with a linear combination of open graphs dressed node, and denote it \( \hat{D}N_{n,j_i,\alpha,I} \), where \( I \) labels the solution to (28) gotten by dressing the nodes.

Note that it follows from Theorem 1.1 of [40] that there are an infinite number of such solutions to (28) for each simple node \( N_{n,j_i,\alpha} \). Further, we can construct solutions to the all the Hamiltonian constraints by dressing in this manner all of the simple nodes of a non-extraordinary spinnet \( \Gamma_0 \). If we order the simple nodes arbitrarily by \( \alpha = 1, \ldots, M \), for a graph with \( M \) simple nodes, we may call such a state \( |\Gamma_0, I_\alpha> \), where the \( I_\alpha \)'s label the dressings of the simple nodes of \( \Gamma_0 \). We will assume in what follows
that each possible $I_\alpha$ can be coded as a real number. We may note that
the edges which connect each dressed node to the rest of the graph have, in
each term in the sum making up the state, the same spin as was incident
on the corresponding simple node of $\Gamma_0$. We call these the external edges of
the dressed node.

Now we may construct an infinite number of local physical operators as
follows. Let us construct a linear operator $\hat{F}_{n,j_i}$ associated to a set of $n$
edges with spins $j_i$, $i = 1, ..., n$ as follows. Acting on a spin network state
$\Psi$ it extracts any dressed node with $n$ external lines labeled by the set of
spins $j_i$ and produces the state in which the $n$ lines of that node are tied up
in a planar vertex, so that the result is a closed spin network. We may note
that such a node will be possible, by conservation of angular momentum.
By linearity, it extracts linear combinations such as those that solved the
Hamiltonian constraint we called $\mathcal{DN}_{n,j_i,I}$. We will call the resulting state
with the ends tied up at a planar node $\mathcal{EN}_{n,j_i,I}$ for the extracted node.

The operator must test to make sure it has cut the state just outside of
a single dressed node, which means it must test that the state formed by
cutting is itself a solution to the Hamiltonian constraint.

If there is more than one such dressed node in a spin network state $\Psi$ the
operator $\hat{F}_{n,j_i}$ produces a spin network state which is a disjoint and unlinked
union of the tied up spin networks produced for each one. For example if
there are two nodes with the same external edges, $\mathcal{DN}_{n,j_i,I_1}$ and $\mathcal{DN}_{n,j_i,I_2}$
in a state $|\Psi \rangle$ then $\hat{F}_{n,j_i} |\Psi \rangle = \mathcal{DN}_{n,j_i,I_1} \cup \mathcal{DN}_{n,j_i,I_2}$.

These operators act to isolate regions of the quantum geometry con-
sisting of one dressed simple node labeled by a given set of external spins.
Furthermore, generically the extracted state has finite expectation value of
volume. For this reason the operators $\hat{F}_{n,j_i}$ may be considered local opera-
tors. It is also clear by construction that they commute with the Hamilto-
nian constraint, on the kernel, in the sense that they take solutions to the
Hamiltonian constraint to solutions of the Hamiltonian constraint.

It is clear that these operators all commute with each other. However
their action is too abrupt, each eliminates all but one kind of region from the
quantum geometry. But given that these operators exist we can make others
that do more interesting things. For example, given any two dressings $I_1$ and
$I_2$ of the simple node $\mathcal{N}_{n,j_i}$ we can construct a change operator $\mathcal{C}_{n,j_i;I_1 \rightarrow I_2}$ as
follows. Acting on a spin network state $\Psi$, this locates all instances of the
dressed nodes $\mathcal{DN}_{n,j_i,I_1}$ and replaces them by the dressed nodes $\mathcal{DN}_{n,j_i,I_2}$.
It is clear by construction that these are physical diffeomorphism invariant
operators, which also have a local meaning in space. What they do is to search for local regions of the quantum geometry characterized by certain gauge invariant and local properties and modify them locally.

It makes sense to say that the properties of a given dressed node are local because, as discussed in the previous section, such a dressed node $\mathcal{DN}_{n,j_i,I_1}$ generally has an expectation value of volume which is finite.

But it follows trivially that for any two of these change-dressing operators

$$[C_{n,j_1;I_1 \rightarrow I_2}, C'_{n',j'_1;I'_1 \rightarrow I'_2}] = 0 \quad (29)$$

whenever $n \neq n'$.

We then have acting on the space of solutions an infinite number of operators whose construction shows that they act locally in space, but which all commute with each other. Furthermore, the only way such operators can fail to commute with each other is if they act in regions with the same external edges, which means they act in regions whose volumes, defined by the expectation value, are generically finite. This shows that there are an infinite number of degrees of freedom in the theory that correspond to modes that do not propagate.

This by itself is in conflict with the Einstein’s equations, for which there are no such non-propagating modes. Furthermore, as the expectation value of the volume of a dressed node may be as large as one likes, this is not necessarily a Planck scale phenomena. According to (29) one may alter the state of the quantum geometry on a region arbitrarily large in Planck units, in a way that does not propagate to any other regions.

It might be objected that we do not know the classical interpretation of the operators $C_{n,j_1;I_1 \rightarrow I_2}$ we have constructed. This is true, but there are classical analogues of such observables, they correspond to the relational observables described in many places [47], in which one first locates a point of spacetime by making measurements of certain fields and derivatives and then, having individuated physically a point or an event, measures other fields there. By construction, such classical observables are physical, as they are spacetime diffeomorphism invariant. To construct their explicit operator representations exactly is probably impossible, as they require the integration of Einstein’s equations.

It is clear that the operators $C_{n,j_1;I_1 \rightarrow I_2}$ are quantum mechanical analogues of these relational classical physical observables. While it will be difficult to construct the exact correspondences between them and the classical relational observables, we can discover the physical effects of some of
them directly. For example, among them are those that act on vertices which are eigenstates of the volume and replace them by other volume eigenstates. More generally the $C_{n,j;i_1 ightarrow i_2}$ will change the expectation value of the volume in the region around the dressed node in the quantum geometry. We may deduce from [29] that one can make an infinite number of transitions between physical states that each change the expectation value of the volume in a restricted region of space, in which the transitions between physically distinct regions are completely uncorrelated with each other, for all time. This certainly would not be possible in any solution to Einstein’s equations.

4 Are there ways out?

Are these problems just pathologies of the particular construction of the states in [39, 40], or do they reflect more general problems with approaches to quantum general relativity based on spin networks states? The purpose of this section is to discuss several possible answers to this question.

4.1 First remarks

We may first note several things that might at first be thought to be relevant, but which on examinination have nothing to do with the issues we are describing here.

First, the problem is not degenerate states or the fact that the Ashtekar constraint is of density weight two, (as was the case with Varadarajan’s examples of negative energy solutions to the constraints described in [49]) because Thiemann’s construction is explicitly designed to eliminate these issues.

Second, there is nothing wrong with the new identities discovered by Thiemann and exploited in his construction of the Hamiltonian constraints of the various theories. These represent important new insights into quantum general relativity that likely have many applications beyond the questions described here. Nor is there anything wrong with the use of a real connection variable of the kind Thiemann uses, which has allowed him to solve the reality condition problem. We may note also that the problems discussed apply equally to the Euclidean and Lorentzian forms of the theory.

Third, the problems are not just restricted to Thiemann’s formalism. For example, analogous problems arise with the original forms of the solutions to the constraints in [4, 7], in which the states have support only on diffeomorphism classes of spin-networks made out of closed loops, without
nodes. These have no volume, but they can be made to correspond to classical metrics by definitions of weaves based on area or surfaces or norms of fields\[14, 9\]. As such a class of such physical states could be defined that was also asymptotically flat, that included states that correspond to three metrics that are asymptotically Schwarzchild, for both positive and negative values of the mass.

Finally, the issue has nothing to do with the difference between the connection and loop representations, just as the lack of boundedness of the energy of the upside-down-harmonic oscillator has nothing to do with a choice of position or momentum representation. While the rigorous methods used by Thiemann are couched in the connection representation, what is at issue has nothing to do with subtleties of mathematical quantum field theory. It is also easily discussed in the loop representation\[4]. Indeed, one has only to notice that Thiemann’s basic identities work equally well in the loop representation so that Thiemann’s forms of the constraint operators may be constructed directly there. To see how to do this, note that in the loop representation one may also consider non-gauge invariant states corresponding to open lines\[51\]. Consider then an open or closed loop \(\gamma\) based at \(x\), with no kink or vertex there and a spherical region \(R^\delta\) around \(x\) of radius \(\delta\) in some flat background metric \(q_{ab}^0\). Then it follows right away that in the connection representation,

\[
Tr(e_a(x)\dot{\gamma}_a(0)U_\gamma) = Tr([A_a(x)\dot{\gamma}_a(0), \hat{V}]U_\gamma) = \lim_{\delta \to 0} \frac{1}{\delta} [T[\gamma], \hat{V}[R^\delta]]
\]

Thus, as long as one is interested in the end only in using Thiemann’s identities in cases in which the index on \(e_a(x)\) is tied up with the tangent vector of a curve one can use always expressions such as these to define an extension of the loop representation.

To do this one follows the philosophy of its original construction of the loop representation in quantum gravity\[6\]. We define an operator directly in the loop representation that corresponds to \(30\)

\[
\hat{T}_0[\gamma] \equiv lim_{\delta \to 0} \frac{1}{\delta} [\hat{T}[\gamma], \hat{V}[R^\delta]].
\]

This defines a new kind of loop representation operator \(T_s[\gamma]\) which has an insertion of \(e_a \dot{\gamma}_a(s)\) at the point \(s\) of the loop. By analogy we may speak of

\[\text{This is true generally, any calculation in quantum gravity so far done in one representation may easily be done in the other}\[50\].\]
the insertion of $\epsilon_\alpha \gamma^\alpha(s)$ as the “foot” of the operator, whose action on loop states is defined by [31].

Thus, the situation is exactly the same as in the case of the various definitions of volume operators [25]. One may define different regulated operators, but whatever may be done in the connection representation may be done also in the loop representation and visa versa. So the issue of the representation used may be separated completely from the issue of which regularization procedure is chosen.

This is, of course, not to say that one formalism may be more useful than the other for certain purposes. For example, at the present time, rigorous statements may be made more precisely in the connection representation, while the extension to the $q$-deformed spin network case can only be done directly in the loop representation.

Now that we have discussed where the problems do not lie, let us discuss several possible places the approach might be modified so as to avoid them.

### 4.2 Some possible loopholes

1) It might be that Thiemann’s approach is correct, but that the space of normalizable physical states described by Theorem 1.1 of [40] actually consists of more than one disjoint sector, only one of which is physically relevant. If this is the case then there may be a superselection principle which excludes those states that have the problems we have discussed

It seems, however, that this is not likely on physical grounds. First, to avoid the problem of the lack of long-ranged correlations, the allowable states would have to be dressed states of only one simple vertex. This would greatly restrict the theory and make weave states based on the volume operator impossible, as each physical state will have only one node in each term in the sum that makes it up that contributes to volume. This makes impossible also a notion of asymptotic flatness using states that are asymptotically weaves along the lines developed here. We might add also that there is no example of an interacting quantum field theory with an unbounded Hamiltonian where

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5 Along these lines we may mention that one may object that the weave states are strictly speaking not in the state space defined in [39, 40] as it includes only states constructed with finite numbers of nodes [48]. However, it is hard to imagine that there is not a straightforward extension that includes solutions with countably infinite numbers of nodes. (Again, the lack of long ranged correlations makes this seem likely.) If there is not then the theory would not be capable of incorporating an asymptotically flat regime, which would in itself be cause for worry.
the boundedness is restored by a superselection principle. Still, this is a possibility that should be explored.

2) It may be that the correct physical inner product differs from the one based on the Hilbert space used in [39, 40], in such a way that most of the solutions of both Type I and Type II are unphysical. Instead, a new physical inner product might allow only states that resolve these two issues. This would clearly be relevant, as one cannot show the boundedness of the Hamiltonian for free quantum field theory, or even the harmonic oscillator, independently of the inner product.

In this respect it is interesting to note that the one exact physical state that is known to have a good semiclassical limit [57] is the Kodama state [56] which is the exponential of the Chern-Simons invariant of the Ashtekar-Sen connection. This state does not exist in the state space defined in [39, 40] as it requires a quantum deformation of the loop representation [53, 54] based on quantum spin networks [52], such that the deformation parameter is proportional to the cosmological constant.

3) If the problem is not in the form of the inner product than it can only be resolved by changing the dynamics. In the rest of this paper I will explore the option that the problems we have discussed may be resolved by modifying the form of the Hamiltonian constraint operator.

4.3 Could changing the definition of the Hamiltonian constraint help?

It is clear from the discussion of the second problem that the issue has at least partly to do with the fact that Thiemann’s Hamiltonian constraints distinguish the two kinds of vertices, extraordinary and simple. The result is that the “evolution” of any state under the Hamiltonian constraint divides into regions, each associated with a single simple vertex which is then dressed with extraordinary vertices. These different regions do not communicate with each other under the dynamics generated by the Hamiltonian constraint. Furthermore, there are an infinite number of exact solutions for dressing the vertices in which the volume of each of these regions, defined by the expectation value of the volume operator, restricted to a region containing only one simple vertex, is bounded. Thus, in all of these states there is no propagation of physical correlations beyond bounded regions. As it is clear that there is nothing like this in classical general relativity (even horizons allow one way communication) this is a feature that will have to be modified if the theory can have a good classical limit.
One may ask if this might be overcome by making a different choice of Hamiltonian constraint operator within Thiemann’s basic framework. For example, it may be that a symmetric operator may be constructed, not along the lines done in [39, 40], but by adding to the operator its hermitian conjugate with respect to the inner product. This might produce an operator whose solutions were not decomposable into regions each with finite expectation value of volume.

However, it seems likely that this will not eliminate the problem of the bounded correlations. The problem is that if the hermitian conjugate is taken in the inner product used in [39, 40] than it too is diagonal in the blocks defined by the graphs which dress the different simple vertices. This means that while the Hermitian conjugate may eliminate extraordinary edges, it can also not have any terms which connect the states of distinct simple vertices. This means that the symmetric operator is block diagonal as well, and the different sectors associated with the dressings of different simple vertices do not get mixed.

However, the problem is not only the distinction between simple and extraordinary nodes. It is not hard to find regularization procedures that lead to a form of the hamiltonian constraint that is block diagonal in the spin network basis, so that it does not propagate physical information through a whole graph, even if the distinction between different kinds of vertices is eliminated. One of these is the definition of the Hamiltonian constraint (and Hamiltonian) given in [17, 18, 19]. That form of the Hamiltonian constraint dresses any vertex, in which there are non-colinear incident edges, bivalent or higher, with two new trivalent vertices, joined by a new edge with spin 1/2. As a result, vertices that are created by the Hamiltonian constraint are themselves dressed by other vertices, in contrast to what happens in Thiemann’s definition. Furthermore, there is no requirement that only vertices with three independent tangent vectors are dressed, so that all trivalent vertices are on an equal footing.

To find solutions to this form of the constraint one may deal with the problem of the reality conditions by defining a Hermitian form of the constraint, which is

\[ \mathcal{H}(N) = \frac{1}{2} \left( \mathcal{C}(N) + \mathcal{C}^\dagger(N) \right). \]  

(32)

The details of this are discussed in [58], but the main point is easy to explain. One begins with a skeleton consisting of an arbitrary spin network. One then dresses each vertex with new trivalent nodes, each of which is dressed in turn. In this way associated to every pair of tangent vectors at a node of
the original skeleton an infinite dimensional space of states is constructed
whose basis elements form planar and fractal structures. Generic solutions
than require an infinite number of networks. However, the solutions may
still be found independently for each region which dresses a node of the
original skeleton.

One may ask whether the regions generated have finite or infinite ex-
pectation values of the volume. This depends on the form of the volume
operator used. If one uses the ordinary operator each trivalent node con-
tributes zero volume [45, 22, 59, 46, 54]. But this is remedied if the formalism
is deformed to the quantum spin network case [53], in which case all trivalent
vertices contribute to the volume [45]. One can see that in this formalism
the volume of the region affected by the initial conditions at an initial vertex
grows generically with repeated actions of the Hamiltonian constraint [55].
This means that after an infinite number of iterations of the Hamiltonian
constraint, which is necessary to produce a solution, generic states have long
ranged correlations.

At the same time, the correlations are still restricted to the regions that
are associated to each node of the original skeleton. This seems a serious
problem, whether or not the regions have finite expectation value of volume.
To avoid this one must use a definition of the Hamiltonian constraint that
does not have the feature that solutions are generated by dressing skeletons.
One way to do this is presented in the following.

5 A new proposal for the regulated Hamiltonian
constraint

Up till now most approaches to the regularization of the Hamiltonian con-
straint have followed a common methodology [6, 9], which is to construct
the operator through a regularization procedure in which the classical
expression for the constraint is expressed as a limit of point-split operators.
However, it must be emphasized that this is not necessary, as all that is
required is that the operator have a form that leads to the correct classical
limit. As the issue of the classical limit is not straightforward, as we have
seen here, we are in the position of having to find a suitable operator by
trial and error. If a given methodology fails to produce an operator with a
good classical limit, we may widen the methodology. One way to do this is
to require only that the operator, written in the connection representation,
have an action which approximates, for slowly varying connections, one of
those produced by an actual regularization procedure, up to terms small in Planck units. This is a well understood procedure in ordinary lattice gauge theory.

It is likely that in order that generic states not be decomposable into uncorrelated regions, the action of the Hamiltonian constraint must act more freely on the space of spin-networks, which means that acting on a node \( v \) of a spin network, it must alter the edges in a neighborhood of \( v \) that includes also its neighboring nodes.

It is not difficult to invent operators that do this, using Thiemann’s length operator, as I will now describe.

The action of the new form of the Hamiltonian constraint, \( \mathcal{C}_{\text{new}} \), is defined by the following four step procedure.

- **R1** \( \mathcal{C}_{\text{new}}(N) \) acts on an element \( \Gamma \) of the spin network basis at each pair of non-colinear edges \( e_1 \) and \( e_2 \) of every node \( v \). The operator acts on a node \( v \) and a pair of its edges \( e_1 \) and \( e_2 \) by modifying a subnetwork that includes these elements as well as the nodes \( v \) is connected to by \( e_1 \) and \( e_2 \), which will be called \( v_1 \) and \( v_2 \). The subnetwork to be modified includes as well the edge connecting \( v_1 \) and \( v_2 \) if it exists.

- **bf R2** Suppose that there is an edge joining \( v_1 \) and \( v_2 \), which will be

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6We may note that many of the regularization procedures so far proposed in the continuum, including Thiemann’s \([39, 40]\) and the one studied by Borissov, Rovelli and the author \([17, 18, 19]\) have steps which require additional operator dependence, besides those which are expressed in terms of the canonical variables. This implicit dependence arises in the way that the loops used in the regulated operators are chosen to conform to certain features of the geometry of the spin-networks that parameterize the spin network basis. Such operators exist naturally in the loop representation, as shown by the example of the operator \( \dot{\gamma}^a(x) \) defined by \( \langle \Gamma \rangle = \int ds \dot{\gamma}^a(s) \delta^3(x, \gamma(s)) \langle \Gamma \rangle \). But it is not known whether they can themselves be constructed through any regularization procedure from functions of the canonical variables. For this reason it is not known if they can be constructed in the connection representation formulation employed in \([21, 22]\), if one requires a construction that uses only operators that represent the classical canonical variables. Of course, this is consistent with the philosophy just enunciated for the regularization of the Hamiltonian constraint. The operator I am about to introduce takes advantage as well of the freedom to employ operators such as \( \dot{\gamma}^a(x) \) in order to make the regulated operators perform combinatorial operations directly on the spin networks.

7To my knowledge the only previous proposals for the Hamiltonian constraint that have this property are those defined on a lattice, \([5, 23, 21, 20]\).

8A slightly different version of this operator, which takes trivalent networks only to trivalent networks is used in \([28]\).

9For simplicity we may restrict attention to graphs in which there is at most one edge connecting any two nodes.
called $e_{12}$. The action of $C_{\text{new}}$ produces a sum of six terms in which the colors along $e_1$, $e_2$ and $e_{12}$, which we call $i$, $j$ and $k$ respectively, are updated by $\pm 1$. Each is multiplied by an amplitude $A_{\pm,\pm',\pm''}(i, j, k; r, s, t)$ which I give below. Here we assume that each of the nodes is written in the form in which the two edges in the problem are joined to a third edge at a trivalent vertex with an edge with definite color. In the case that the node has more than 3 incident edges, this new edge is internal to the node. But, using the recoupling identities, any higher than trivalent node can be represented in terms of trivalent nodes joined by internal edges of length zero\[15\]. The colors associated with these edges for $v, v_1$ and $v_2$, respectively, are $r, s$ and $t$. $\pm, \pm'$ and $\pm''$ refer respectively to the updating of $i, j$ and $k$. The amplitude is then,

$$A_{\pm,\pm',\pm''}(i, j, k; r, s, t) = \pm'' ij \{iii \pm 1; 112\} \{jjj \pm' 1; 112\} \{i \pm 1ir; j \pm' 1j1\} \times \{i \pm 1is; k \pm'' 1k1\} \{j \pm' 1jt; kk \pm'' 11\} \times \Theta(i, j, r)\Theta(j, k, t)\Theta(i, k, s) \frac{[r + 1][s + 1][t + 1]}{[n + 1][n + 1][n + 1]}$$

Here $\{iii \pm 1; 112\}$ are the $6-j$ symbols, and $\Theta(i, j, r)$ is the theta function defined in \[52, 54, 18\]. The formula is written in a way that is good for either the ordinary or $q$-deformed case, so $[n]$ is the quantum integer \[52\], which is equal to $n$ in the ordinary case.

There is also the case in which there is in $\Gamma$ no edge joining $v_1$ and $v_2$. In this case the operator adds one and gives it a color 1. The topology of the edge is chosen so the loop it forms with $e_1$ and $e_2$ links or intersects no other edge of the network. One then applies the above formula with $\pm'' = +$, and $k = 0$, producing in this case four terms.

What this combinatorial formula corresponds to is adding a loop as usual to represent the $F_{ab}$ in the plane of the tangent vectors of the two edges. The combinatorics are as in \[17, 18, 19\] except that the new loop is taken to go around the triangle $e_1, e_2, e_{12}$.

**R3** To complete the definition of the operator we must divide by the area of the triangle $e_1, e_2, e_{12}$. We may note that, as determined by the area operator, this will often vanish, but it may instead be defined using Thiemann’s length operator \[11\] as follows. If we call $\hat{L}_1, \hat{L}_2$ and $\hat{L}_3$ the length operators of the edges of a triangle $\Delta$ of a spin-network,
we may define an operator that measures its area as,

\[ \hat{A}^2 = \frac{1}{4} \left( \frac{\hat{L}_1^2 \hat{L}_2^2}{2} + \frac{\hat{L}_2^2 \hat{L}_3^2}{2} + \frac{\hat{L}_3^2 \hat{L}_4^2}{2} - \frac{\hat{L}_1^4}{4} - \frac{\hat{L}_2^4}{4} - \frac{\hat{L}_3^4}{4} \right) \]  

where we have used the standard formula from Euclidean geometry for the area \[10\]. (If the operators fail to commute we take symmetric ordering.)

We may note that this operator will generally yield a different answer than a direct measurement of the area, using the standard area operator \[16\]. This is an inevitable consequence that we are working with a quantum field theory, in which functional relationships between classical observables may not be preserved.

We may then define this step as follows: If there is a term with no triangle corresponding to the three original edges we do nothing. If there is we multiply the state gotten by the first two steps by the operator \( \hat{A}^{-1} \). 

- **R4** Finally, we need to do something that corresponds to integrating the constraint against a lapse \( N(x) \). As we are defining the operator combinatorically, we must make an ansatz that is equivalent to doing this. Using the criteria that the action must agree on slowly varying-non-diffeomorphism invariant states in the connection representation, we see that the result of the usual definitions is to multiply by an independent \( N(v) \) at each node of a graph. To define this combinatorically we must make use of the recognition problem for subgraphs of graphs. We will multiply the action so far defined of the operator on each node \( v \) by numbers \( N(v) \), which are assumed to be assigned independently to all nodes of all networks, subject to the following restriction. When it is the case that a network \( \Gamma \) may be identified as a subnetwork of \( \Gamma' \), such that a given vertex \( v \) of \( \Gamma \) is identified uniquely with a vertex \( v' \) of \( \Gamma' \) then \( N(v) \) in the action on \( \Gamma \) must be taken equal to \( N(v') \) of \( \Gamma' \).

\[10\] We may note that postulating that a formula Euclidean geometry holds in the microscopic level is of course justified only by the fact that it is the simplest unique choice.

\[11\] It is possible that there is a zero eigenvalue of the area. To avoid this we need to define the inverse so that those states do not contribute. We do so by defining \( \hat{A}^{-1} \equiv \hat{A}^{-2} \hat{A} \), where \( \hat{A}^{-2} \) is defined on the subspace orthogonal to the kernel of \( \hat{A} \), so that terms that might contribute zero area are projected out.
One might worry that there are many examples in which a given graph \( \Gamma \) may be identified in more than one way with a subgraph of \( \Gamma' \), or that the subgraphs may have symmetries that prevent the unique identification of the node. However, the combinatorics of the graph recognition problem tells us that the proportion of such cases goes to zero rapidly as the graphs become large. As we are interested in the classical limit, and hence large complex graphs, this is sufficient.

The results of these four steps, applied to every node of a basis element \( \Gamma \) then gives a definition of the operator \( C(N) \) on the space of diffeomorphism invariant spin network states.

It is easy to see that this definition eliminates the problem of bounded correlations, so that a perturbation in a solution in one region of a network will generally propagate over the whole. We may note also that there are cases in which the adjacent nodes and edges are eliminated by the action of the constraint. For example adjacent edges with color 1 may be eliminated. Also, kinks in lines with color 1 will be eliminated by the second rule. Thus, the repeated action of the operator to any vertex will eventually produce terms that eliminate either or both that vertex and its adjacent edges. As a result, the adjoint of the operator, \( C^\dagger \) will add nodes and edges.

It is not known if the Hermitian form of this operator, \( C \) has solutions, but if it does it is then likely that they do not leave regions of a state uncorrelated.

Finally, we may note that there are still other approaches to the Hamiltonian constraint in which there may emerge long ranged correlations. These include the approaches of \([11, 12, 23]\). In fact, any approach that allows the Kodama state\([56]\) (the exponential of the Chern-Simons invariant) as a solution does generate long ranged correlations, as that state is known to be both an exact physical state, with cosmological constant, \textit{and} a semiclassical state associated to DeSitter spacetime\([57]\).

6 Concluding remarks

In this paper we have described two problems that different formulations of non-perturbative quantum gravity may suffer from. We were able to illustrate them with Theimann’s formalism\([39, 40]\) as it allows a complete description of the space of solutions. However we saw also that at least one of the problems, that of the lack of correlations which propagate over whole graphs, is likely shared by other approaches. Because of this we described a
new approach to the form of the Hamiltonian constraint, that eliminates this
difficulty, but at the cost of not following what has become the canonical
procedure to construct diffeomorphism invariant operators from point split
regularization procedures.

This is certainly progress. But we may wonder if it will be enough. What
if it is the case that a definition of the Hamiltonian constraint that generates
long-ranged correlations is not enough to restore either a good classical limit
or the positivity of the $\text{ADM}$ energy?

Indeed, while the existence of long-ranged correlations is a necessary
condition for there to be a description of the dynamics in terms of classical
geometry, there are reasons to think it may not be sufficient. As mentioned
in the introduction, experience with the renormalization group, random sur-
face theory and dynamical triangulations suggest that to define a good con-
tinuum limit that reproduces classical general relativity it is also necessary
to tune the bare parameters of the theory $[34]$.

We may note that in dynamical triangulations and Regge calculus the
continuum limit only exists (if it exists at all) at a fixed point in the bare
cosmological constant-Newton’s constant plane $[30, 31, 32, 33]$, where the
bare cosmological constant is nonvanishing. This suggests that at the very
least, approaches to quantum gravity that succeed without including a bare
cosmological constant may not have a good continuum limit, at least one that
may be related to any path integral description. Of course, this follows from
general renormalization group considerations as well, as one can generally
never have a good continuum limit in a theory without tuning the parameter
of lowest dimension.

So the choice of a good hamiltonian constraint will most likely need
to be complemented by fine tuning of the bare Newton and cosmological
constants. But what if even such fine tuning is not enough? It may also
be necessary to introduce supersymmetry and other degrees of freedom,
in order to guarantee a good continuum limit. Indeed, it would not be
surprising were this to turn out to be the case. This might mean that
in the end non-perturbative quantum gravity discovers conditions for the
existence of a continuum limit which are related to those conditions known
to be necessary for the existence of a sensible perturbative quantum theory
of gravity. (The point is that as far as we know all such theories are string
theories.) Supersymmetry will furthermore help with positive energy as the
$\text{ADM}$ operator becomes the square of the supersymmetry generators. It
would be indeed interesting were fermionic behavior, which is necessary to
stabilize ordinary matter, also necessary to stabilize the quantum geometry
of space.

Indeed, there are only two alternatives to this scenario. One is that non-perturbative quantum general relativity has by itself a continuum limit whose perturbative description resembles a supersymmetric string theory, which is the only way we know to have a good perturbative description of the interactions of gravitons. The other is that a new perturbative description, so far undiscovered, would have to emerge from the continuum limit of non-perturbative general relativity.

At the very least, the issues discussed here show that the problems of the existence of the continuum limit and its correspondence to the classical theory are key problems for non-perturbative quantum gravity. Further, these problems are closely connected with the issues already discovered by other discrete approaches such as dynamical triangulations and Regge calculus\[30, 31, 32, 33\]. The moral of all of these stories is that a quantum theory of gravity according to which Planck scale physics is discrete cannot correspond to our world only if there is a natural reason for the system to arrange itself into a critical state in which correlation lengths diverge and massless particles may emerge.

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