Quantum walks in two dimensions: controlling directional spreading with entangling coins and tunable disordered step operator

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Abstract
We study a 2D disordered discrete-time quantum walk (QW) based on 1D generalized elephant QW where an entangling coin operator is assumed. We show that considering a given disorder in one direction, it is possible to control the degree of spreading and entanglement in the other direction. This observation helps assert that the random QWs of this ilk serve as a controllable decoherence channel with the degree of randomness being the tunable parameter and highlight the role of dimensionality in quantum systems regarding information and transport.

Keywords: quantum walks, directional spreading, controllable decoherence

(Some figures may appear in colour only in the online journal)

1. Introduction
The establishment of the first (discrete-time) protocol for a quantum walker on a grid [1] allowed not only the theoretical description of experimental quantum phenomena like photon emission exchange in a superconducting cavity [2, 3]—among so many other physical systems
[4]—, but played a crucial role in moving from the conceptualization of quantum computation
[5, 6] into its materialization [7, 8].

Being understood as the quantum analogue to the random walk, the first quantum walk (QW) has given rise to a myriad of models. Nonetheless, classical systems differ awfully from quantum systems; namely, the latter abides by the superposition principle, the statistical notion of pure and mixture states provided by the density matrix as well as the concept of entanglement.

Albeit those subjects have been intensively explored for uni-dimensional QWs little has been made regarding higher-dimensional QW systems with disorder. Concerning standard QWs, the first systematic studies can be ascribed to [9] where it was also shown that the introduction of disorder can lead to the emergence of a classical distribution. It is worth heeding the appearance of classical-like spreading is observed in 1D QWs as well [10–16]. Moreover, by increasing the dimension of the QW it is possible to increase the noise level and still obtain more coherence [17].

In all of the aforementioned works it was employed a standard step operator; however, novel phenomena emerge when the discrete-time QW is employed with nonstandard translation operators [18–28] or bespoken coins [29, 30]. For instance, in [23] it was presented a solvable quantum model, the elephant QW (EQW), which is able to spread hyperballistically. Within this context, the introduction of a memory model [23]—which at odds with the Markovianity of the canonical QW—has set forth the possibility of introducing a random QW model yielding a broad diffusion behavior ranging from normal to hyper-ballistic [24]. Moreover, it was recently shown [31] that such a class of QWs generates maximally entangled states for almost all initial coin states and coin operators whether the walker is at first localized or not. One the other hand, there are one-dimensional QWs where dynamical disorder is introduced in the coin operator [32–35] and the same phenomenon is observed, something that was experimentally verified recently in a photonic setup [36].

That said, in this paper we aim at probing the spreading and entanglement of a 2D-quantum walker based on the generalized QW protocol [24]. The latter property has been paid little attention to [37–40], but some works focused on decoherence though [41, 42].

The remainder of the manuscript goes as follows: in section 2 we present the pertinent two-dimensional discrete time QW models, then in section 3 we present the results and in section 4 the discussion and the concluding remarks of our study.

2. Model

2.1. The standard 2D discrete time QW

Coined discrete time QWs correspond to the evolution of a quantum system, the quantum coin, in a discrete position space, \( \mathcal{H}_p \), through the association between its degree of freedom and a direction of motion. That evolution takes place by considering the action of a coin operator updating the coin state and then the action of a shift operator that changes the position state of the quantum walker accordingly. In the case of the one-dimensional lattice \( \mathcal{H}_p \equiv \text{span}(\{|x| \, x \in \mathbb{Z}\}) \) and the quantum coin is a simple two-level system \( \mathcal{H}_c \equiv \text{span}\{|\uparrow\rangle, |\downarrow\rangle\} \), with the unitary evolution given by

\[
U \equiv S(\mathbb{I}_p \otimes \mathbb{C}_2),
\]

where \( S \) is the shift operator

\[
S \equiv \sum_{x \in \mathbb{Z}} (|x + 1\rangle \langle x| \otimes |\uparrow\rangle \langle \uparrow| + |x - 1\rangle \langle x| \otimes |\downarrow\rangle \langle \downarrow|),
\]
and $C_2$ the quantum coin toss operator

$$C_2(\theta, \beta, \gamma) \equiv \begin{pmatrix} \cos \theta & \sin \theta e^{i\beta} \\ \sin \theta e^{i\gamma} & -\cos \theta e^{i(\gamma+\beta)} \end{pmatrix}. \tag{3}$$

Instances of prevalent quantum coins operators are the Hadamard with $(\theta = \pi/4, \beta = \gamma = 0)$,

$$C_2(\pi/4, 0, 0) \equiv H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \tag{4}$$

and the Kempe operators where $(\beta = \gamma = \pi/2)$,

$$C_2(\theta, \pi/2, \pi/2) \equiv C_k(\theta) = \begin{pmatrix} \cos \theta & i\sin \theta \\ i\sin \theta & \cos \theta \end{pmatrix}. \tag{5}$$

For higher dimensions, as the degree of freedom of the problem augments, it is possible to observe a wider range of ways in which the quantum system evolves. As an example, in the two-dimensional lattice case, one considers a qubit coin using it to move to one direction one step at a time, or to use different basis such as the eigenvectors of the Pauli matrices $\sigma_z$ and $\sigma_x$ to associate them to the directions of motion. Here, we consider a four-dimensional coin made by a composition of two one-dimensional coins, that is, the coin state $|\psi_c\rangle \in \mathcal{H}_c_1 \otimes \mathcal{H}_c_2 = \text{span}(\{|\uparrow, \uparrow\rangle, |\uparrow, \downarrow\rangle, |\downarrow, \uparrow\rangle, |\downarrow, \downarrow\rangle\})$ such that the motion of the quantum walker occurs on the diagonals of the lattice (see figure 1).

As we now have a composite coin, we use two types of coin operators, the separable ones, i.e.

$$C_2^\beta \equiv C_2^{(1)} \otimes C_2^{(2)}, \tag{6}$$

that acts individually in each sub-coin, without creating correlations between them, or the non-separable ones.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Representation of the 2D coined discrete time quantum walk using a composite quantum coin, made of two one-dimensional coins.}
\end{figure}
In this work we consider the entangling coin operator as the case of a non-separable coin operator, where we use the controlled NOT gate operator (CNOT) to entangle the coins at each coin toss

\[ C_{\text{entang}} \equiv \left( C_2^{(1)} \otimes C_2^{(2)} \right) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \]  

(7)

Making use of the same association between the sub-coin states with the directions of motion in the one-dimensional case for each direction in the 2D grid, the shift operator of a discrete-time QW (DTQW) over a 2D lattice is

\[ S = \sum_{x_1, x_2 \in \mathbb{Z}, \mathbb{Z}} \left( |x_1 + 1, x_2 + 1 \rangle \langle x_1, x_2| \otimes |\uparrow, \uparrow\rangle \langle \uparrow, \uparrow| + |x_1 + 1, x_2 - 1 \rangle \langle x_1, x_2| \otimes |\uparrow, \downarrow\rangle \langle \uparrow, \downarrow| \right. \\
+ \left. |x_1 - 1, x_2 + 1 \rangle \langle x_1, x_2| \otimes |\downarrow, \uparrow\rangle \langle \downarrow, \uparrow| + |x_1 - 1, x_2 - 1 \rangle \langle x_1, x_2| \otimes |\downarrow, \downarrow\rangle \langle \downarrow, \downarrow| \right). \]  

(8)

2.2. 2D generalized elephant QW

The generalized elephant QW \cite{24} (gEQW) is a unitary randomly modified version of the one-dimensional DTQW with a non-standard shift operator where the step sizes, including others than the unit, are drawn from a probability distribution introducing a long-range coupling between the walker and the lattice nodes. In this way, the shift operator is changed to

\[ S_t = \sum_{x \in \mathbb{Z}} \left( |x + \Delta_t \rangle \langle x| \otimes |\uparrow\rangle \langle \uparrow| + |x - \Delta_t \rangle \langle x| \otimes |\downarrow\rangle \langle \downarrow| \right). \]  

(9)

where \( \Delta_t \) is the step size selected accordingly with the \( q \)-exponential distribution \cite{43}

\[ \Pr(\Delta_t) \equiv e_q(\Delta_t) = \tau_t [1 - (1 - q)\Delta_t]^{1/(1-q)}, \]  

(10)

with support given by

\[ \text{supp}(e_q(x)) = \begin{cases} \left[ 0, \frac{1}{1-q} \right], & q \leq 1 \\ \left[ 0, \infty \right), & q > 1. \end{cases} \]  

(11)

where \( q \in [0, \infty) \), \( \Delta_t \in [1, \ldots, t] \) and \( \tau_t \) is a time-dependent normalization factor.

The \( q \)-exponential is chosen for it provides us a versatile way to simulate the quantum walker evolution in such way that it is possible to change it from the standard quantum walker behavior—where no noise is present and, therefore, \( \Delta_t = 1 \)—in which the dynamical behavior is characterized by

\[ \text{Var}_x(t) \approx t^\alpha, \ t \gg 1, \]  

(12)

with the dynamical exponent \( \alpha = 2 \), when we set \( q = 1/2 \), to the elephant QW \cite{23} for the upper bound limiting case \( q \to \infty \) that yields \( \alpha = 3 \). In other words, when the step sizes are drawn from the uniform distribution, which matches the highest amount of noise possible (for more details on the behavior of the dynamical exponent as a function of \( q \) see figure 2) the quantum walker spreads hyperballistically. We interpret this unitary random evolution as an evolution in which the environment selects a random step at each time instant, e.g. because of an imperfect physical implementation of the QW, followed by a measurement from the experimenters \cite{31}.
Figure 2. Average mean dynamical exponent of the one dimensional generalized elephant quantum walk as a function of $q$ in the $q$-exponential distribution (10). The black chain line indicates the classical random walk exponent, $\bar{\alpha} = 1$, while the blue dotted line indicates the quantum walk diffusion $\bar{\alpha} = 2$.

Figure 3. Representation of one step of the two-dimensional generalized elephant quantum walk (2D gEQW).

The two-dimensional gEQW model we propose is a modified version of the 2D DTQW described previously (see figure 3). The shift operator (8) is similar to the one-dimensional version but we must consider two step size distributions for each direction. For each time step the shift operator is

$$S_t = \sum_{x_1,x_2} \left( |x_1 + \Delta_t^{(1)}, x_2 + \Delta_t^{(2)} \rangle \langle x_1, x_2 | \otimes |\uparrow, \uparrow\rangle \langle \uparrow, \uparrow| + |x_1 + \Delta_t^{(1)}, x_2 - \Delta_t^{(2)} \rangle \langle x_1, x_2 | \otimes |\uparrow, \downarrow\rangle \langle \uparrow, \downarrow| + |x_1 - \Delta_t^{(1)}, x_2 + \Delta_t^{(2)} \rangle \langle x_1, x_2 | \otimes |\downarrow, \uparrow\rangle \langle \downarrow, \uparrow| + |x_1 - \Delta_t^{(1)}, x_2 - \Delta_t^{(2)} \rangle \langle x_1, x_2 | \otimes |\downarrow, \downarrow\rangle \langle \downarrow, \downarrow| \right),$$

(13)
where \( \Delta^{(1)}_t, \Delta^{(2)}_t \) are selected independently and according to the \( q \)-exponential distribution (10). The coin operators used are the same as equations (6) and (7).

3. Results

3.1. Entangling 2D gEQW

First, let us look at the probability distribution of the position in the standard DTQW on a two-dimensional lattice. In figure 4(a), we have a discrete time QW whose initial state is localized at the origin and with both sub-coins in the equivalent superposition state. The coin operator used is that given by equation (6), with both coin operators, \( C^{(1)}_1 \) and \( C^{(2)}_2 \), as equation (5) with \( \theta = \pi/4 \). It is possible to learn that the region where it is more likely to find the quantum walker lies in the boundaries. That is an expected result bearing in mind the initial state and the coin operator we have used, and the fact that the evolution is separable for both directions. Notice that the 2D unitary operator with separable coin operators can be written as

\[
U = S_x^1 C_x^1 S_x^2 C_x^2,
\]

where

\[
S_x^1 \equiv (|x_1 + 1\rangle \langle x_1| \otimes I_{x_2} \otimes |\uparrow\rangle \langle \uparrow| \otimes I_{c_2} + |x_1 - 1\rangle \langle x_1| \otimes I_{x_2} \otimes |\downarrow\rangle \langle \downarrow| \otimes I_{c_2}),
\]

\[
S_x^2 \equiv (I_{x_1} \otimes |x_2 + 1\rangle \langle x_2| \otimes I_{c_1} \otimes |\uparrow\rangle \langle \uparrow| + I_{x_1} \otimes |x_2 - 1\rangle \langle x_2| \otimes I_{c_1} \otimes |\downarrow\rangle \langle \downarrow|),
\]

\[
C_x^1 \equiv I_{x_1} \otimes I_{x_2} \otimes C^{(1)}_2 \otimes I_{c_2},
\]

\[
C_x^2 \equiv I_{x_1} \otimes I_{x_2} \otimes C^{(2)}_2.
\]

Consequently, the QW is described by independent movements in the \( \hat{x}_1 \) and \( \hat{x}_2 \) directions with the corresponding two-dimensional probability distribution being simply the product of two one-dimensional position probability distributions obtained from walks with the same individual initial states of both directions in the 2D DTQW. (For more details, see appendix A.)

If we include the CNOT gate in the coin operator, we get a very different probability distribution as plotted in figure 4(b). First, we see that this distribution is not spatially symmetric, which indicates it is not separable as in the previous case. In order to quantify its degree of non-separability we have employed the classical trace distance measure between the joint distribution \( \Pr_{X_1, X_2}(x_1, x_2) \) and a separable distribution generated by the product of marginal distributions, \( \Pr_{X_1}(x_1)\Pr_{X_2}(x_2) \),

\[
D(\Pr_{X_1, X_2}, \Pr_{X_1, X_2}) = \frac{1}{2} \sum_{x_1, x_2} |\Pr(x_1, x_2) - \Pr(x_1, x_2)|,
\]

so that if the distribution is separable the trace distance will return zero. Figure 5 shows the trace distance between \( \Pr \) and \( \Pr \) in the separable (blue circle) and the entangling DTQW (red cross). Therein, the trace distance grows with time for the entangling case indicating the distribution is not separable, which is something that does not happen when we have used the separable coin operator. Physically, that occurs because the introduction of the CNOT gate in the coin operator creates correlations between the sub-coins at every time step that are then transferred to the position-coin system by the shift operator; that yields an inseparable position distribution for the walker in both directions.
Figure 4. Position probability distribution for the two-dimensional discrete time quantum walk with initial state $|\psi\rangle = |0,0\rangle \otimes \frac{|\uparrow\rangle + |\downarrow\rangle}{\sqrt{2}}$. In the left panel (a), the coin operator used was equation (6), with $C_1^{(1)}$ and $C_2^{(1)}$ as the Kempe coin (5) with $\theta = \pi/4$. In the right panel (b), the entangling coin operator was used, also with the same $C_1^{(1)}$ and $C_2^{(1)}$.

Figure 5. Time series for the classical trace distance between the joint distribution and the separable generated by its marginals $\Pr_{X_1, X_2} = \Pr_{X_1}(x_1)\Pr_{X_2}(x_2)$ of the standard DTQW using the separable coin operator (blue circle) and the entangling coin operator (red cross). The separable part of both coin operators walks was chosen as $C_{k}(\pi/4) \otimes C_{k}(\pi/4)$, using a localized position with equal superposition of the coin basis states as a walker initial state.

In figure 6 it is presented the spreading behavior’s analysis in the entangling walk through the position vector variance time evolution

$$\sigma_R^2 = \langle \vec{R}^2 \rangle - \langle \vec{R} \rangle^2 = \sigma_{x_1}^2 + \sigma_{x_2}^2,$$ 

(19)
where we have considered only the stationary part of its evolution. In general, it is expected that the position variance follows a polynomial form, such as $a_0 + a_1 t + a_2 t^2 + O(t^3)$ for the standard DTQW. Therefore, in order to access the dynamical exponent that characterizes the walk we have considered only the time-asymptotic regime, i.e. $t > t_a \gg 1$ the lower bound, $t_a$—corresponding to a transient scale—is defined when the second order term becomes dominant. Given that the spreading stationary regime changes with the initial state and coin operator used we have determined it for each evolution studied here.

Next, we consider the two-dimensional gEQW. First, we have centered our study on the case in which the step sizes are selected according to the uniform distribution, i.e. $q = \infty$ for both directions. Figure 7 shows us a 2D view of the position probability distribution for the evolution with the separable coin operator (6) (a) and the entangling coin operator (7) (b). The silhouette of the probability distribution turns more bell shaped when we use the entangling coin operator. Yet, we cannot assert that the 2D distribution is completely Gaussian since its marginals have returned a skewness of $\tilde{\mu}_3 \approx 0.329$ and a kurtosis of—following Fisher’s definition—$\kappa \approx -1.393$ for the first direction and of $\tilde{\mu}_3 \approx 0.330, \kappa \approx -1.383$ for the second one. They are also inseparable as depicted in figure 8.

In figure 9 we show the evolution of the variance of the position vector in the asymptotic regime. The results indicate that even when we use the entangling coin operator the dynamical exponent remains unchanged. As a means of additional comparative analysis, we present the individual average dynamical exponents for each direction in table 1.

By considering different $q$ parameters for each direction, the dynamics of the 2D-gEQW can be made anisotropic as well. In the case of a separable coin operator—i.e. no correlations between the walker’s directions of movement, we expect the results will be in accordance with those we have presented for the standard DTQW in one direction and the one-dimensional elephant QW in the other. However, what if we introduce correlations between the walker coin
Figure 7. Probability distribution of the position for the two-dimensional generalized elephant quantum walk using the $q$-exponential distribution with $q = \infty$. The initial state used was $|\psi\rangle = |0, 0\rangle \otimes |\uparrow\rangle + |\downarrow\rangle \sqrt{2} \otimes |\uparrow\rangle + |\downarrow\rangle \sqrt{2}$. In the top panel (a), the coin operator used was (6), with $C_2^{(1)}$ and $C_2^{(2)}$ as the (5) with $\theta = \pi/4$. In the bottom panel (b), the entangling coin operator was used, also with the same $C_2^{(1)}$ and $C_2^{(2)}$.

Figure 8. Time series for the classical trace distance between the joint distribution and the separable generated by its marginals $Pr_{X_1, X_2} = Pr_{X_1}(x_1)Pr_{X_2}(x_2)$ of the 2D EQW, $q_{x_1} = q_{x_2} = \infty$, using the separable coin operator (blue circle) and the entangling coin operator (red cross). The separable part of both coin operators walks was chosen as $C_k(\pi/4) \otimes C_k(\pi/4)$, using a localized position with equal superposition of the coin basis states as a walker initial state.

subs? Recalling the use of the CNOT gate together with the standard coin toss operation as a means to introduce correlations between the coins during the evolution, we proceed to the analysis of the 2D gEQW with different values of $q$ assigned to both directions.
Figure 9. Log–log graph of the position vector variance as a function of time for the 2D EQW in the asymptotic regime of figure 7 using the separable coin operator (blue circled and orange dashed lines) and the entangling coin operator (red cross and black dash-dotted lines).

Table 1. Mean dynamical exponents in the 2D-gEQW and $q = \infty$ using entangling coin toss operation. These values were obtained through 10 simulations in a $10^4 \times 10^4$ lattice with a localized initial state and equal superposition of coin basis states were used. $C_2^{(1)}$ and $C_2^{(2)}$ were chosen as equation (5) with $\theta = \pi/4$.

|       | full       | asymptotic |
|-------|------------|------------|
| $\bar{\alpha}_{x_1}$ | 2.82 ± 0.02 | 2.89 ± 0.02 |
| $\bar{\alpha}_{x_2}$ | 2.77 ± 0.02 | 2.87 ± 0.06 |

Figures 10 and 11 depict the results of a 2D-gEQW evolution with the entangling toss operation and $q_{x_1} = 0.5$ and $q_{x_2} = \infty$. Interestingly, we note that the distribution of the $\hat{x}_1$ position becomes much more localized with the wave packet spreading exponent matching the value for Brownian motion. By comparing those results with the outcome of the standard 2D entangling toss DTQW in figure 6 we hold up that the introduction of the QW with random steps sizes selected following a uniform distribution in one direction of the 2D entangling toss DTQW turns both the walker position probability distribution and wave packet spreading in the other direction to behave similarly to a classical random walk. By comparing these results with figure 7 and table 1, we also verify this feature does not appear if we consider $q = \infty$ in both directions, inasmuch as the wave packet spreading exponents are still the same as those obtained by using the separable coin toss operation. In table 2, we show the average dynamical exponent for both directions in ten simulations.

Changing the value of the parameter $q$ leads us to different results; nonetheless, the wave packet spreading suppression still occurs. Figures 12 and 13 shows the results for an evolution with $q_{x_1} = 0.5$ and $q_{x_2} = 1$. We have perceived once again that the probability distribution becomes more localized in the $\hat{x}_1$ direction. Comparing wave packet spreading suppression in both cases, figure 13 tells us that it still occurs, but to a lesser degree.
Figure 10. 2D view of the position probability distribution for the 2D gEQW. The parameters used were \( q_{x_1} = 0.5 \) for the first direction and \( q_{x_2} = \infty \) for the second. In the top panel (a) the separable coin operator was used and in the bottom panel (b) the entangling one. Both of them with \( C_{x_1}^{(1)} = C_{x_2}^{(2)} = C_1(\pi/4) \). The walker initial state was localized in the lattice origin and with the equal superposition of the coin basis states.

Figure 11. Log–log graph as a function of time of the \( x_1 \) position variance (blue circled) and \( x_2 \) variance (red cross) with its linear fittings (orange dashed and black dash-dotted, respectively), in the 2D-gEQW with \( q_{x_1} = 0.5 \) and \( q_{x_2} = \infty \) of figure 10.

As a means of quantifying the lessen of dynamical exponent of the quantum walker in the first direction as a function of the amount of randomness introduced in the second direction, in figure 14 we depict the average dynamical exponent for the first direction \( \bar{\alpha}_{x_1} \) as a function of \( q_{x_2} \) in the entangling gEQW. We have determined that the average dynamical exponent has a non-monotonic decaying behavior as a function of \( q_{x_2} \), increasing for some values when compared to the previous one. Nevertheless, the overall behavior is that the larger the amount of...
Table 2. Mean dynamical exponents in the 2D-gEQW entangling coin toss operation with \( q_1 = 0.5 \) and \( q_2 = \infty \). These values were obtained in 10 simulations in a \( 10^4 \times 10^4 \) lattice, using \( |\psi(0)\rangle = |0,0\rangle \otimes (|\uparrow\rangle + |\downarrow\rangle/\sqrt{2}) \otimes (|\uparrow\rangle + |\downarrow\rangle/\sqrt{2}) \). \( C_1^{(1)} \) and \( C_2^{(2)} \) were chosen as equation (5) with \( \theta = \pi/4 \).

|       | full                  | asymptotic           |
|-------|-----------------------|-----------------------|
| \( \bar{\alpha}_{x_1} \) | 1.040 ± 0.004         | 0.988 ± 0.003         |
| \( \bar{\alpha}_{x_2} \) | 2.78 ± 0.02           | 2.91 ± 0.02           |

Figure 12. 2D view of the position probability distribution for the 2D gEQW. In the first direction the standard DTQW was used, i.e. with steps with unit sizes and in the second the \( q \)-exponential distribution with \( q_2 = 1 \). The top panel (a) position distribution was obtained using the separable coin operator equation (6) with \( C_1^{(1)} = C_2^{(2)} = C_i(\pi/4) \) and the bottom panel (b) the entangling one, with the same \( C_2^{(1)} \) and \( C_2^{(2)} \) as in the (a) case.

randomness introduced on the walk in the second direction the smaller the dynamical exponent, with a lower limit given by the classical random walk dynamical exponent \( \alpha = 1 \).

In [17], it was proposed a different type of decoherent two-dimensional QW where a broken-line noise is introduced on the lattice diagonals. In this work, the position and coin space are structured in the same manner, with a 2D square lattice and a four-dimensional coin. Furthermore, [41] investigated a decoherent 2D Alternating QW—in which a one-dimensional coin is used in an alternating manner to take one step at each direction—, including a broken-line noise in the first direction and also a coin-measure type of noise. In the former work, it is also possible to control the decoherence degree that one direction goes under through the probabilities of having a missing link, which after some threshold, when they are equal for both diagonals, the classical behavior dominates. When the broken-link’s probabilities are different, a similar effect of probability lockdown can be obtained but only confining to the diagonals (see figures 9 and 10 of [17]). With respect to [41], it has more similarities with the effects that we report here due to the fact that the noise is introduced only in one direction and it affects the other through the use of only one coin. When only the coin-measure decoherence is considered, it affects more the direction in which the coin is measured. However, in both works the control cannot be as fine tuned for each direction as in our model, something that can be attributed to the use of a four-dimensional coin, and they only display diffusive and super-diffusive spreading.
Figure 13. Log–log graph of the marginal variances and its linear fittings, $\sigma^2_{x_1}$ (blue circle and orange dashed) and $\sigma^2_{x_2}$ (red cross and black dash-dotted) for the $q_{x_1} = 0.5$ and $q_{x_2} = 1$ entangling 2D-gEQW in the asymptotic regime. The initial state used was the same as in figure 12.

Figure 14. [MAIN RESULT] Averaged dynamical exponent of the first direction variance with $q_{x_1} = 0.5$ as a function of $q_{x_2}$ of the gEQW in the second direction. The data error bars indicate the standard deviation of the points obtained through ten simulations each. The blue chain line indicates the DTQW dynamical exponent and the red dotted one the classical random walk dynamical exponent. In all simulations the localized and equal superposition of the basis states was used, with $C_{\text{entang}} = C_1(\pi/4) \otimes C_1(\pi/4)$ CNOT as coin operator.

Another type of decoherent 2D QW was proposed in [33] where spatio, temporal and spatio-temporal fluctuations are added in the coin operation. Despite of also having an enhancement of the entanglement between the coin and position states, as we are going to see in section 3.2,
Figure 15. Marginal position probability distribution (without the zeros) of the first direction for the gEQW using the entangling coin operator (7) with $q_{x_1} = 0.5$. In (a) we have the standard DTQW, i.e. $q_{x_1} = q_{x_2} = 0.5$, (b) $q_{x_1} = 0.55$, (c) $q_{x_1} = 1$, (d) $q_{x_1} = \infty$, all in the same time step $t = 140$. The walker initial state used in the one localized in the origin with equal superposition of the basis states. The separable part of the coin operator used was the Kempe coin with $\theta = \pi/4$.

By looking at the marginal probability distribution for the walker in the first direction $\hat{x}_1$ we gain further insight into what is happening. In figure 15(a) we have the standard DTQW using the entangling coin operator, showing that the distribution is asymmetric despite the use of an initially symmetric coin initial state. When we increase the degree of randomness in the second direction—e.g. with $q_{x_2} = 0.55$—figure 15(b), the marginal distribution starts to be more localized around the origin, yet with long tails resembling the one-dimensional DTQW distribution obtained when using a symmetric initial coin state. By further increasing $q_{x_2}$, the distribution becomes more Gaussian-like with some remaining tail as presented in figure 15(c), for $q_{x_2} = 1$, and without it as in figure 15(d), with $q_{x_2} = \infty$. We understand it as an almost completely Gaussian distribution, with skewness of $\tilde{\mu}_3 \approx 1.22$ and kurtosis $\kappa \approx -0.06$, resembling the classical random walk distribution.

In spite of the classical-like statistical properties of the entangling 2D-gEQW with $q_{x_2} = \infty$, it is important to grasp the entangled nature of the final state of the walker in the $\hat{x}_1$ and $\hat{x}_2$ directions, coin plus position; consequently, it does not have a classical analog, as we are going to see next.
3.2. Entanglement entropy

To assess the amount of entanglement of one part of the walker system has got, we are going to use the entanglement entropy as quantifier \[ S_E(\rho) = -\text{tr}(\rho \log_2 \rho), \] (20)
where \( \rho \) is the density matrix of the system. The lower bound of the entanglement entropy equals zero in a pure—and thus separable—state and the upper bound is equal to \( \log d \), with \( d \) being the system dimension, in the case of a fully entangled system.

We consider three sets of the walker coin state, the total coin state \( \rho_c \), the \( \hat{x}_1 \) and \( \hat{x}_2 \) coin states \( \rho_c^1 \) and \( \rho_c^2 \), respectively. Figure 16 shows the entanglement entropy of the above mentioned coin systems in the entangling DTQW and show us that the two-dimensional coin has a non-zero entanglement with the two-dimensional position subsystem, going to the limit of \( S_E(\rho_c) \approx 1.87 \). The inset tells us that the \( \hat{x}_1 \) subsystem is an almost maximal entangled state \( S_E(\rho_c^1) \approx 0.97 \), whereas for \( \hat{x}_2 \) \( S_E(\rho_c^2) \approx 0.94 \). For comparison, in the one-dimensional Hadamard walk, the coin entanglement entropy reaches the asymptotic value of \( S_E(\rho_c) \approx 0.87 \) [45, 46]. That result supports our previous assertion that the entanglement between the coin subsystems generated by the CNOT gate is transferred to the sub-coin-position systems, since it gets a higher value than in the standard 1D DTQW.

Taking into account the entangling generalized elephant QW, figure 17, with \( q_{x_1} = 0.5 \) and \( q_{x_2} = 1 \), we see that the total coin system becomes completely entangled with the position degrees of freedom. Moreover, the inset shows that the coin subsystems become also maximally entangled with their respective position degrees.

When we use the uniform distribution of steps on the second direction, \( q_{x_2} = \infty \), the same features appears figure 18, i.e. a maximally entangled total coin state with its subsystems also
Figure 17. Entanglement entropy of the total coin state in the 2D-gEQW with \( q_{x_1} = 0.5 \) and \( q_{x_2} = 1 \). The inset shows the entanglement entropy for the first (blue cross) and second (red triangle) direction coin density matrix. The separable part of the entangling coin operator used was with equally balanced Kempe coin operators \( C_k(\pi/4) \) and the initial state used was the origin initially localized with equal superposition of the coin basis states.

Figure 18. Entanglement entropy of the total coin state in the 2D-gEQW with \( q_{x_1} = 0.5 \) and \( q_{x_2} = \infty \). The inset shows the entanglement entropy for the first (blue cross) and second (red triangle) direction coin density matrix. The separable part of the entangling coin operator used was with equally balanced Kempe coin operators \( C_k(\pi/4) \) and the initial state used was the origin initially localized with equal superposition of the coin basis states.
maximally entangled with their respective position degrees. Therefore, as we mentioned previously, despite the QW in the first direction having a Gaussian distribution with the dynamical exponent equal to that of the classical random walk, since it is maximally entangled with the position degree we cannot say that it is a classical state.

One might question, how do we know if the entanglement of the coin subsystems is not between them? To answer that, we have used a measure of entanglement between bipartite systems called negativity $N$ \[47, 48\]. For $2 \times 2$ systems the condition is also sufficient. It is defined as

$$N(\rho_{AB}) \equiv \frac{\|\rho_{AB}^T\|-1}{2},$$

(21)

where $T_A$ is the partial transpose over $A$ and $\|\cdot\|$ the trace norm.

The negativity follows the positive partial transpose criterion for separability which states that if the composite bipartite system parts $A$ and $B$ are entangled, then the density matrix obtained through the partial transpose of $A$ or $B$ degrees of freedom has negative eigenvalues corresponding to a violation of the semi-definite positivity property. The negativity is an entanglement monotone, going from zero for separable states to half for completely entangled states. By calculating the negativity of $\rho_c$ for all the cases mentioned above, we see that the sub-coins are not entangled after the application of the unitary operators, as it returns zero for all times.

We should mention that an extensive analysis of the asymptotic coin entanglement entropy in a 2D setting was made in \[37\]. It was found that when a separable initial state is used, the entanglement entropy of the total coin system with the position degrees is given by the sum of the entanglement of its components. However, when an entangled initial state is used this property does not hold, something that can be related to the entangling gEQW, as it can be verified in the examples that we discussed (e.g., see figure 16 where the individual entropies are near to one, $S(\rho_{c1}) \approx 0.978$, $S(\rho_{c2}) \approx 0.939$ but the total is $S(\rho_c) \approx 1.880$). In such cases, the asymptotic coin-position entanglement entropy can reach the maximum value of approximately 1.978 when the initial coin state is a maximally entangled one. This fact is attributed to the coin operator being a separable one, therefore, preserving the initial entanglement between the coins. In the entangling gEQW, we continuously introduce entanglement between the coins through the use of the CNOT gate that together with the noise introduced in the steps sizes augments the coin-position entanglement to its maximum value. Moreover, the authors also investigated the entanglement dependence on the initial conditions giving us the set of parameters that lead to a highly entangled state, something that we plan to study in the 2D gEQW in a future work.

Summarizing, these results show that the 2D gEQW has a remarkable property: it allows controlling the kind of spreading degree of dispersion of the standard QW in the first direction by means of entangling coin operator whilst enhancing entanglement with its coin subsystem and preserving the properties of the second direction as well. Taking into account the total coin system, the 2D entangling gEQW proves to be a system capable of creating maximally correlated coin states in a bidimensional setting.

### 3.3. Physical interpretation

In order to understand the reason the walker spreading in the first direction is reduced in the 2D gEQW with random step sizes in the second direction we have employed a normalized version of the $l_1$-norm coherence \[49, 50\].
Figure 19. $l_1$-norm of coherence as a function of time in the entangling 2D gEQW with different values of $q_{x_1}$. In all simulations the Kempe coin operator was used with $\theta = \pi/4$ for both directions and the initially localized walker with an equal superposition of the coin basis states. Supposing that after the initial increase the coherence decays following a power law, $C_{l_1} \propto t^{-\beta}$, a linear fitting of log-log of this graph give us the decay exponents as a function of $q_{x_2}$, $\beta(q_{x_2} = 0.5) = (0.00744 \pm 0.00005)$, $\beta(q_{x_2} = 1) = (0.219 \pm 0.002)$, $\beta(q_{x_2} = 1.5) = (0.355 \pm 0.003)$ and $\beta(q_{x_2} = \infty) = (0.433 \pm 0.005)$.

$C_{l_1}(\rho) = \frac{1}{t} \sum_{j \neq i} |\rho_{ij}|$. (22)

A normalization is made necessary since the position degree dimension is infinite, then as time passes the coherence would increase indefinitely given that more off diagonal elements would be non-zero. The $l_1$-norm coherence provides us a way to see how the state coherence of the position in the first direction evolves through time. That is likely to explain the previously mentioned properties of classical-random-walk-like behavior, since one expects that if the position density matrix is completely diagonal and one considers an equally balanced coin toss the evolution must lead to a Gaussian distribution.

Looking at the first direction coherence when using the entangling DTQW, $q_{x_1} = 0.5$, the results in figure 19 show an initial increase followed by an overshoot and a subsequent decay until it stabilizes circa $C_{l_1}(\rho_{x_1}(t)) = 0.15$. Despite the greater initial increase, as one increases the amount of randomness in the step sizes of the second direction, the decay rate of the coherence of the first direction position, eventually reaching a stable minimum one with $q_{x_2} = \infty$ and coherence around $C_{l_1}(\rho_{x_1}(t)) \approx 0.05$. As expected, that value is very small when compared to the entangling DTQW, which explains the observed behavior of having a Gaussian-like distribution and yet not a completely separable two-dimensional distribution, since the coherence is not zero. Overall, the time evolution of the coherence of the position in the first direction as a function of $q_{x_1}$, is in agreement with the observed behavior of the marginal distributions figure 15 with $0.6 < q_{x_2} < 2$ providing distributions between the standard DTQW distribution and a Gaussian one.

We still have a question to rejoin: why does the walker in the first direction goes under a decoherence process? To answer it, we recall the explanation to why the gEQW leads to a separable two-dimensional distribution in 3.1 using a separable coin operator. As we have stated,
the unitary operator with a separable coin operator is rewritten as an independent application of a walk in the first direction and in the second direction $U_i = S_{1_i}^{(l)} C_{1_i} S_{1_i}^{(r)} C_{1_i}$. A walker density matrix in the 2D model evolves through

$$\rho_{i_1,i_2}(t) = \mathcal{U}(t,0) \rho_{i_1,i_2}(0) \mathcal{U}(t,0)^\dagger,$$

where $\mathcal{U}(t,0) = \prod_{j=0}^{q} U_{i_1-j}$. By performing a partial trace over the second direction degrees we find the evolution equation for the first direction, therefore

$$\rho_{i_1}(t) = \sum_{i_2,\sigma_2} \langle x_2, \sigma_2 | \mathcal{U}(t,0) \rho_{i_1,i_2}(0) \mathcal{U}(t,0)^\dagger | x_2, \sigma_2 \rangle.$$

Given that the walker directions degrees of freedom, position and coin, are initially separable between both directions, i.e. $\rho_{i_1,i_2}(0) = \rho_{i_1}(0) \otimes \rho_{i_2}(0)$, and noting that both $S_j^{(l)}$ and $C_1$ commutes with $S_j^{(r)}$ and $C_2$ so that the evolution operator is rewritten as a product of two operators that acts only in its respective direction subspace $\mathcal{U}(t,0) = \mathcal{U}_{i_1}(t,0) \mathcal{U}_{i_2}(t,0)$

$$\rho_{i_1}(t) = \mathcal{U}_{i_1}(t,0) \rho_{i_1}(0) \mathcal{U}_{i_1}(t,0)^\dagger,$$

where we used the fact that the evolution for the second direction is unitary and therefore preserves the trace.

Now, if we use an entangling coin operator it is not possible to break the unitary operator into independent walks in both directions given that does not exists operators $A_{i_1}$ and $A_{i_2}$ such that $C_{\text{entang}} = A_{i_1} \otimes A_{i_2}$. Because of that, the directions density matrix of the walker does not stay separable during the evolution and the first direction degrees evolves through a non-unitary quantum channel

$$\rho_{i_1}(t) = \sum_{j} E_j \rho_{i_1}(0) (E_j)^\dagger,$$

where

$$E_j = \{ x_2, \sigma_2 \} = \{ x_2, \sigma_2 | \mathcal{U}(t,0) | \psi_{x_2} (0) \},$$

with $\rho_{x_2}(0) = | \psi_{x_2} \rangle \langle \psi_{x_2} |$. This quantum channel acts as a decoherence channel for the evolution of the first direction of the walker. Therefore, we use the gEQW with an entangling coin operator as a controllable decoherence channel in which by controlling the parameter $q_{x_2}$ we control the degree of decoherence that the other QW goes under (for more details please check the supplementary material section B).

4. Concluding remarks

In this work we have presented the study of a tunable 2D disordered QW, the 2D-gEQW. By making use of its greater number of degrees of freedom when compared to its 1D predecessor, we have considered another class of coin operator, the entangling coin operator, which creates correlations between the coin subsystems during the system evolution. We have shown that the 2D-gEQW is not just a simple extension of the corresponding 1D model, but it is associated with new phenomena.

Explicitly, we have found that using different levels of randomness in the step size distribution in both directions, established $q$ in the $q$-exponential distribution, it is possible to control the degree of spreading of one of the directions. Specifically, if we have the standard DTQW in the first direction, by varying the parameter $q$ of the step distribution of the second direction from 0.5 to $\infty$ its dynamical exponent changes continuously from the ballistic QW spreading, $\alpha = 2$, to the classical spreading, $\alpha = 1$, leading the position distribution to a Gaussian-like shape, not completely separable though. In respect of that, our protocol adds a new building...
block for a better understanding of the paths to classical-like behavior in QW based systems [9–12].

Looking at the time evolution of the coherence of the position in one direction, we have evidenced that when we increased the amount of randomness in the steps of the perpendicular direction the coherence decay rate increases as well, reaching a minimum value of $C_{\parallel} \approx 0.05$ for $q_z = \infty$. That explains the observed properties of the first direction position distribution and spreading behavior. Consequently, we conclude that the gEQW together with an entangling coin operator in a two-dimensional setting serve as a controllable decoherence channel, with $q$ being the tunable parameter. This is an interesting effect that highlights the role of dimensionality in quantum systems.

To understand the reason the two-dimensional distribution in the entangling gEQW is not separable, we have analyzed the coin, total and subsystems, entanglement entropy. We have found that the coin subsystems become completely entangled when using a $q_z \neq 1/2$; that indicates the generalized elephant QW in the second direction effectively transfers correlations from the sub-coins to the sub-coins and positions. To eliminate the possibility that the coins be entangled we used the negativity as a measure of entanglement in bipartite systems that returned zero in all occasions, showing that the coins are not entangled. Furthermore, the complete coin is completely correlated with the positions in the same cases, a remarkable feature in 2D settings, although we left for a future work to do an extensive analysis on the dependence of the entanglement entropy on the initial settings as in [37]. An enhancement of the coin entanglement were also reported in a two-dimensional alternating QW setting when phase disorder is introduced in the translation operator [39] and when disorder is introduced in the coin operation [33], however both of them only shows a localized and sub-diffusive spreading.

Let us mention a related 2D model [17] where the QW spreads in a lattice subjected to broken links. In that work it was shown that by controlling the probability of breaking links in one direction it is possible to engineer, in a certain range, the degree of decoherence in the perpendicular direction. However, in their protocol the QW always ends up spreading in a diffuse way whereas the present proposal is able to produce diffusion as well as superdiffusion, as exhibited in figure 14. In a 2D alternating QW setting [41], a similar effect of reduction of spreading in one direction when introducing broken-link noise in the perpendicular one is reported, nonetheless not having the possibility of a fine-tuned controlled spreading as in our model.

Recently, it was experimentally confirmed, in a photonic setup, that coin operators with dynamic disorder can increase the coin-space entanglement in QWs [36]. That said, —and to the best of our knowledge—there is no experimental realization of QWs with dynamic disorder in the step operator in the way we elaborated in this paper. Nevertheless, we mention that there are advances in optical platforms with a non-nearest-neighbor coupling that already allow the experimental implementation of simple QWs with steps of size one or two [51], which are able to pave the way to more complex systems as that we propose.

Although there is a substantial number of works analyzing entanglement features of 1D QWs [12, 16, 34, 52–63], there are far fewer studies on entanglement in 2D QWs [37–40]. Thus, this work also makes a contribution in such subject. In a previous work [31] the entanglement generation in the 1D gEQW was extensively studied, where it was demonstrated that this type of walk generates maximally entangled coin states for almost all initial parameters. Adding the above mentioned results, the present work shows that the gEQW also has the potential to generate highly entangled coin states in 2D settings, while also maintaining the ability of controlling the walker spreading rate, something that in the context of other types of random QWs is not possible. Therefore, we state that the programmable nature of the gEQW
is enriched when taking into account the results presented here, being also useful in the context of decoherence quantum channels design.

In a quantum simulation [7], one of the goals is to design quantum systems that are programmed to model chosen features of other quantum systems. In such a research field, QWs have been shown to be a versatile platform to simulate a myriad of quantum phenomena [64, 65]. In this work, we have shown a new capability with 2D QWs: without weakening the entanglement, it is possible to control the wave packet spreading in one direction by programming a parameter setup for the spreading in the perpendicular direction.

Data availability statement

The data that support the findings of this study are available upon reasonable request from the authors.

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Appendix A. Separable gEQW evolution features

To see that the 2D generalized elephant quantum walk (gEQW) using a separable coin operator leads to a separable probability distribution, we first need to take a closer look in the unitary evolution operator

\[
U_t = S_t (I_p \otimes C_3) 
\]

\[
= \sum_{x_1, x_2} \left( |x_1 + \Delta_1^{(1)} x_2 + \Delta_2^{(2)} \rangle \langle x_1, x_2 | \otimes |\uparrow, \uparrow \rangle \langle \uparrow, \uparrow | + |x_1 + \Delta_1^{(1)} x_2 - \Delta_2^{(2)} \rangle \langle x_1, x_2 | \otimes |\uparrow, \downarrow \rangle \langle \uparrow, \downarrow | \\
+ |x_1 - \Delta_1^{(1)} x_2 + \Delta_2^{(2)} \rangle \langle x_1, x_2 | \otimes |\downarrow, \uparrow \rangle \langle \downarrow, \uparrow | + |x_1 - \Delta_1^{(1)} x_2 - \Delta_2^{(2)} \rangle \langle x_1, x_2 | \otimes |\downarrow, \downarrow \rangle \langle \downarrow, \downarrow | \right) \\
\times \left( I_{c_1} \otimes I_{c_2} \otimes C_1^{(1)} \otimes C_2^{(2)} \right),
\]  

(A.1)

where \( I_{c_1}, I_{c_2} \) are the identity operators in the first and second direction sub-coin spaces, respectively. By also rewriting the coin operator as a product of two coin operations, each in its respective coin subspace, the unitary operator is rewritten as

\[
U_t = S_1^{(i)} S_2^{(i)} C_{x_1} C_{x_2},
\]  

(A.3)
where
\[ S_{x_1}^{(t)} = \sum_{x_1} \left( |x_1 + \Delta^{(1)}_1 \rangle \langle x_1 | \otimes I_{x_2} \otimes | \uparrow \rangle \langle \uparrow | \otimes I_{x_2} + |x_1 - \Delta^{(1)}_1 \rangle \langle x_1 | \otimes I_{x_2} \otimes | \downarrow \rangle \langle \downarrow | \otimes I_{x_2} \right) \] (A.4)
\[ S_{x_2}^{(t)} = \sum_{x_2} \left( |x_2 + \Delta^{(2)}_2 \rangle \langle x_2 | \otimes I_{x_1} \otimes | \uparrow \rangle \langle \uparrow | \otimes I_{x_1} + |x_2 - \Delta^{(2)}_2 \rangle \langle x_2 | \otimes I_{x_1} \otimes | \downarrow \rangle \langle \downarrow | \otimes I_{x_1} \right) \] (A.5)
\[ C_{x_1} = \left( I_{x_1} \otimes I_{x_2} \otimes \hat{C}^{(1)}_2 \otimes I_{x_2} \right) \] (A.6)
\[ C_{x_2} = \left( I_{x_1} \otimes I_{x_2} \otimes I_{x_1} \otimes \hat{C}^{(2)}_2 \right). \] (A.7)

Every operator that acts only on the first direction subspaces, position and sub-coin, commutes with the operators which acts only in the other direction, i.e.
\[ [S_{x_1}^{(t)}, S_{x_2}^{(t')}] = [S_{x_1}^{(t')}, C_{x_2}] = 0 \] (A.8)
\[ [C_{x_1}, S_{x_2}^{(t')}] = [C_{x_1}, C_{x_2}] = 0, \quad \forall t, t'. \] (A.9)

Therefore we rewrite the one-step unitary operator as
\[ U_t = S_{x_1}^{(t)} C_{x_1} S_{x_2}^{(t)} C_{x_2}, \] (A.10)
showing that at every time a 2D separable gEQW step is rewritten as two independent steps in each direction.

To show that the distribution is separable, let \( \rho_{x_1, x_2}(0) = \rho_{x_1}(0) \otimes \rho_{x_2}(0) \) be the total walker state in the initial time step. The total state at a future time \( t \) will be given by
\[ \rho_{x_1, x_2}(t) = U(t, 0) \rho_{x_1, x_2}(0) U(t, 0)^\dagger, \] (A.11)
with \( U(t, 0) = \prod_{j=0}^{t-1} U_{t-j} \). The position probability distribution is obtained through the partial trace over the coin degrees of freedom and by applying a projective measurement \( P_{x_1, x_2} = |x_1, x_2 \rangle \langle x_1, x_2| \)
\[ \Pr(x_1, x_2)(t) = \text{tr}(P_{x_1, x_2} \rho_{x_1, x_2}(t) P_{x_1, x_2}^\dagger), \] (A.12)
with \( \rho_{x_1, x_2}(t) = \text{tr}_{c_1, c_2}(\rho_{x_1, x_2}(t)) \). Given that the evolution operator applies independent steps to each direction, consequence of the separable coin operator, the coin subsystems remains separable during the entire evolution and the remaining position density operator also. Note that \( S_{x_1}^{(t)} C_{x_1} \) and \( S_{x_2}^{(t)} C_{x_2} \) acts only in \( x_1 \) and \( x_2 \), respectively. Also, remember that these operators commute with each other, even in different time steps. Consequently, the total evolution does not create correlations between the directions degrees of freedom and the total probability distribution will be
\[ \Pr(x_1, x_2)(t) = \text{tr}\left(P_{x_1, x_2} \rho_{x_1}(t) \otimes \rho_{x_2}(t) P_{x_1, x_2}^\dagger\right) = \text{tr}(x_1) \langle \rho_{x_1}(t) | x_1 \rangle \langle x_1 | x_2 \rangle \langle \rho_{x_2}(t) | x_2 \rangle \langle x_2 |) = \Pr(x_1, t) \Pr(x_2, t), \] (A.13)
proving that the separable gEQW position distribution is separable.

For completeness, let us address the task of finding the reduced dynamics of a given walker direction state. For that, first, we rewrite the unitary evolution \( U(t, 0) \) in a more appropriate form
\[ U(t, 0) = \prod_{j=0}^{t} U_{t-j} = U_{t-1} U_{t-2} \ldots U_0 = \left(S_{x_1}^{(0)} C_{x_1} S_{x_2}^{(0)} C_{x_2}\right) \left(S_{x_1}^{(-1)} C_{x_1} S_{x_2}^{(-1)} C_{x_2}\right) \ldots \left(S_{x_1}^{(0)} C_{x_1} S_{x_2}^{(0)} C_{x_2}\right) \] (A.14)
Remembering the commutation relations equations (A.8) and (A.9), we see that it is possible to permute the operators that acts only in one direction with the ones that acts only in the other so that
\[
\mathcal{U}(t, 0) = \left( \left( S_{x_1}^{(0)} C_{x_1} \right) \left( S_{x_1}^{(t-1)} C_{x_1} \right) \ldots \left( S_{x_1}^{(0)} C_{x_1} \right) \left( S_{x_2}^{(t-1)} C_{x_2} \right) \ldots \left( S_{x_2}^{(0)} C_{x_2} \right) \right)
\]
\[
= \mathcal{U}_{t_1}(t, 0) \mathcal{U}_{t_2}(t, 0),
\]
(A.15)
showing that the whole evolution can be described by an independent evolution in the second direction and then an independent evolution in the first.

To find the quantum walker first direction state evolution we need to realize a partial trace over the second directions degrees of freedom \( \mathcal{H}_{x_2} \) and \( \mathcal{H}_{x_2} \)
\[
\rho_{t_i}(t) = \text{tr}_{x_1} \left( \mathcal{U}(t, 0) \rho_{t_i, x_1}(0) \mathcal{U}(t, 0)^\dagger \right),
\]
(A.16)
using equation (A.15)
\[
\rho_{t_i}(t) = \text{tr}_{x_1} \left( \mathcal{U}_{t_1}(t, 0) \mathcal{U}_{t_2}(t, 0) \rho_{t_i, x_1}(0) \mathcal{U}_{t_2}(t, 0)^\dagger \mathcal{U}_{t_1}(t, 0)^\dagger \right)
\]
\[
= \sum_{x_2, \sigma_2} \langle x_2, \sigma_2 | \left( \mathcal{U}_{t_1}(t, 0) \mathcal{U}_{t_2}(t, 0) \rho_{t_i, x_1}(0) \mathcal{U}_{t_2}(t, 0)^\dagger \mathcal{U}_{t_1}(t, 0)^\dagger \right) | x_2, \sigma_2 \rangle,
\]
(A.17)
where \( \sigma_2 = \{ \uparrow, \downarrow \} \). Let again that the initial state be separable between the directions. Knowing that the operator \( \langle x_2, \sigma_2 | \) only is going to act on the identities in \( \mathcal{H}_{x_2} \) and \( \mathcal{H}_{x_2} \) of \( \mathcal{U}_{t_1}(t, 0) \), we see its effect over the evolution operator noting that, for instance
\[
\sum_{x_2} \langle x_2 | x_2 \rangle = \sum_{x_2, \sigma_2} \langle x_2 | \sigma_2 \rangle \langle \sigma_2 | x_2 \rangle = \sum_{x_2} \langle x_2 |,
\]
(A.18)
such that the action of the partial trace over \( \mathcal{H}_{x_2} \) passes through \( \mathcal{U}_{t_2}(t, 0) \) and the same goes for the partial trace over \( \mathcal{H}_{x_2} \). Consequently
\[
\rho_{t_i}(t) = \sum_{x_2, \sigma_2} \langle x_2, \sigma_2 | \left( \mathcal{U}_{t_1}(t, 0) \mathcal{U}_{t_2}(t, 0) \rho_{t_i, x_1}(0) \mathcal{U}_{t_2}(t, 0)^\dagger \mathcal{U}_{t_1}(t, 0)^\dagger \right) \rangle
\]
\[
= \mathcal{U}_{t_1}(t, 0) \rho_{t_i, x_1}(0) \mathcal{U}_{t_2}(t, 0)^\dagger \sum_{x_2, \sigma_2} \langle x_2, \sigma_2 | \mathcal{U}_{t_2}(t, 0) \rho_{t_i, x_1}(0) \mathcal{U}_{t_2}(t, 0)^\dagger | x_2, \sigma_2 \rangle
\]
\[
= \mathcal{U}_{t_1}(t, 0) \rho_{t_i, x_1}(0) \mathcal{U}_{t_2}(t, 0)^\dagger \text{tr}(\rho_{t_2}(t)) = \mathcal{U}_{t_1}(t, 0) \rho_{t_i, x_1}(0) \mathcal{U}_{t_1}(t, 0)^\dagger,
\]
(A.19)
where in the third line we used the fact that the trace of the second direction density matrix is preserved since \( \mathcal{U}_{t_2}(t, 0) \) is unitary. This show us that in the separable gEQW the evolution of the first direction is obtained through the elimination of the second direction degrees and that it is simply given by \( \mathcal{U}_{t_1}(t, 0) \).

**Appendix B. Entangling coin operator gEQW evolution features**

When using a non-separable coin operator, such as equation (7), the unitary operator cannot be broken into the application of unitary and independent quantum walk steps as in equation (A.10), since the coin operator now acts jointly on both coin subspaces. This means that even if we start with a separable walker state in both directions, the unitary evolution does not necessarily will maintain this property, such that \( \rho_{t_i, x_1}(t) \neq \rho_{t_i}(t) \otimes \rho_{t_2}(t) \). Therefore equation (A.13) will not be obeyed
\[
\text{Pr}(x_1, x_2)(t) \neq \text{Pr}(x_1, t) \cdot \text{Pr}(x_2, t).
\]
(B.1)
The entangling coin operator (7) creates correlations between the coins through the use of a controlled NOT gate, that uses the first degree of freedom as a control and the second as a target. Its action on the basis states is CNOT{\{\uparrow, \uparrow\}, \{\uparrow, \downarrow\}, \{\downarrow, \uparrow\}, \{\downarrow, \downarrow\}} = \{\{\uparrow, \uparrow\}, \{\uparrow, \downarrow\}, \{\downarrow, \uparrow\}, \{\downarrow, \downarrow\}\}. On the other hand, if we have the following superposition state (\{| \uparrow \rangle + | \downarrow \rangle \rangle\}, through the action of the CNOT gate the resulting state is an entangled one \{| \uparrow \rangle + | \downarrow \rangle \rangle\}. After the application of \( c_{\text{entang}} \), the correlations are then transferred to the position-coin subsystems through the shift operation while also correlating the position degrees, as we saw in figures 5 and 8.

Because of this inseparability of the unitary operator, the evolution for the quantum walker in a given direction will not be unitary anymore. Consider again the equation (A.17) that applying the partial trace over the second direction degrees of freedom provides the walker state in the first direction. Supposing again that \( \rho_{x_1, x_2}(0) = \rho_{x_1}(0) \otimes \rho_{x_2}(0) \) and that \( \rho_{x_2}(0) = | \psi_{x_2}(0) \rangle \langle \psi_{x_2}(0) | \), then, given that the unitary operator is not separable anymore, we find a set of Kraus operators that defines the walker first direction evolution

\[
\rho_{x_1}(t) = \sum_{x_2, \sigma_2} \langle x_2, \sigma_2 | \hat{U}(t, 0)(\rho_{x_1}(0) \otimes | \psi_{x_2}(0) \rangle \langle \psi_{x_2}(0) |) \hat{U}(t, 0)^\dagger | x_2, \sigma_2 \rangle
\]

\[
= \sum_j E_j(t, 0) \rho_{x_1}(0) E_j^\dagger(t, 0),
\]

with \( j = \{x_2, \sigma_2\} \) and each Kraus operator given by

\[
E_{j=x_2, \sigma_2}(t, 0) = \langle x_2, \sigma_2 | \hat{U}(t, 0) | \psi_{x_2}(0) \rangle.
\]

For the evolution to be unitary only one Kraus operator should appear, something that only is possible if the total unitary operator is separated into two operations each in its respective subspace, as we saw in the case of a separable coin operator appendix A.

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