THE UNIQUENESS OF PSU₃(8) IN THE MONSTER

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Abstract. As a contribution to an eventual solution of the problem of the
determination of the maximal subgroups of the Monster we show that there is
a unique conjugacy class of subgroups isomorphic to PSU₃(8). The argument
depends on some computations in various subgroups, but not on computations
in the Monster itself.

1. Introduction

The maximal subgroup problem for almost simple groups became a major focus
for research in group theory in the 1980s, and remains so today. In the case of the
sporadic groups, a systematic attack on the problem began earlier, with Livingstone
and his students in the 1960s. The problem was solved in the 20th century for
25 of the 26 sporadic simple groups, and their automorphism groups, but one
case, namely the Fischer–Griess Monster group M, remains outstanding. A great
deal of work on this case has already been done. The maximal p-local subgroups
were classified in [12, 7, 8], and some theoretical work on non-local subgroups
was accomplished in [9, 10]. Following successful computer constructions of the
Monster [6, 3] other techniques became available, and more progress was made
[2, 5, 11, 15, 16, 17], including discovery of five previously unknown maximal
subgroups, isomorphic to

- PSL₂(71), PSL₂(59), PSL₂(41), PGL₂(29), PGL₂(19).

The cases left open by this published work are possible maximal subgroups with
socle isomorphic to one of the following simple groups:

- PSL₂(8), PSL₂(13), PSL₂(16), PSU₃(4), PSU₃(8).

Of these, PSL₂(8) and PSL₂(16) have been classified in unpublished work of P. E.
Holmes, although the results seem not to be publicly available. In this paper we
deal with the case PSU₃(8). Specifically, we show that, up to conjugacy, there is
a unique subgroup PSU₃(8) in the Monster. Its normalizer is the already known
maximal subgroup (A₅ × PSU₃(8):3):2. Notation follows [11, 14], where required
background information can be found.

2. Existence

Exactly one conjugacy class of subgroups of M isomorphic to PSU₃(8) is con-
tained in the known maximal subgroups. The normalizer of such a group is
(A₅ × PSU₃(8):3):2, itself a maximal subgroup of M. For details, see [9].

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3. Strategy for proving uniqueness

The group $\text{PSU}_3(8)$ can be generated from a group $\text{PSL}_2(8) \times 3$ by extending $D_{18} \times 3$ to $(9 \times 3):S_3$. Note that $9 \times 3$ contains three cyclic subgroups of order 9, which are permuted by the $S_3$. Similarly, there are three complements of order 3, which are also permuted by the $S_3$. Hence it is sufficient to extend $9 \times 3$ to $D_{18} \times 3$ normalizing one of the other two cyclic subgroups of order 9.

We note in particular that all cyclic groups of order 9 in $\text{PSU}_3(8)$ are conjugate, and hence we need only consider subgroups $\text{PSL}_2(8) \times 3$ in which the diagonal elements of order 9 are conjugate in the Monster to the elements of order 9 inside $\text{PSL}_2(8)$. We shall show that there is only one class of $\text{PSL}_2(8) \times 3$ in the Monster that satisfies this condition. Moreover, the cyclic group of order 9 extends to a unique $D_{18}$ in $\text{PSU}_3(8)$. Hence the $D_{18} \times 3$ we wish to construct is conjugate in the Monster to the one inside $\text{PSL}_2(8) \times 3$.

4. The subgroup $3 \times \text{PSL}_2(8)$

Since $\text{PSL}_2(8)$ contains elements of order 9, the elements of order 3 fuse to $M$-class $3B$. Since it contains a pure $2^3$, the involutions are in $M$-class $2B$. In [9] Norton accounts for many of the structure constants of type $(2,3,7)$ in the Monster. In particular he shows that there is no $3 \times \text{PSL}_2(8)$ in which the $\text{PSL}_2(8)$ is of type $(2B,3B,7B)$. He also shows that there are three classes of $\text{PSL}_2(8)$ of type $(2B,3B,7A)$, just two of which centralize elements of order 3. The respective normalizers are:

(1) $\text{PSL}_2(8):3 \times 3S_6$. Here the central 3 in $3A_6$ is in Monster class $3C$, as are the 3-cycles. The elements mapping to fixed-point-free 3-elements in $A_6$ are in Monster class $3C$.

(2) $\text{PSL}_2(8) \times 2 \times S_4$. Here, the elements of order 3 in $S_4$ are in Monster class $3A$.

Hence there are exactly four classes of $\text{PSL}_2(8) \times 3$ in the Monster.

5. Fusion of elements of order 9

Consider first the case where the central elements of $3 \times \text{PSL}_2(8)$ are in class $3C$ in the Monster. We restrict the character of degree 196883 to $S_3 \times \text{Th}$. Using the character values on $3C$ and $2A$, we obtain a decomposition as

$$2 \otimes 65628 + 1^+ \otimes 34999 + 1^- \otimes 30628$$

where the first factor denotes the representation of $S_3$. The values on classes $2A$ and $7A$ of $\text{Th}$ are easily computed:

$$\begin{array}{ccc}
1A & 2A & 7A \\
34999 & 183 & 13 \\
30628 & -92 & 3 \\
65628 & 92 & 17
\end{array}$$

from which it is easy to see that the decomposition into irreducibles of $\text{Th}$ is given by

$$\begin{align*}
34999 &= 30875 + 4123 + 1 \\
30628 &= 30628 \\
65628 &= 61256 + 4123 + 248 + 1
\end{align*}$$
It then follows that the values of the character of degree 196883 on elements of \( \text{Th-class } 9A, 9B, \text{ and } 9C \) are respectively \(-1, -1, 26\), while the values on the corresponding diagonal elements in \( 3 \times \text{Th} \) are \(26, 26, \text{ and } -1\) respectively. In other words, the diagonal elements are always in a different conjugacy class from the elements in \( \text{Th} \). Hence this case is eliminated. (In fact, in this case the PSL\(_2\)(8) contains elements of Th class 9C, that is, Monster class 9A.)

The remaining three classes of PSL\(_2\)(8) \( \times \) 3, namely the ones with a central 3A-element, are contained in the double cover of the Baby Monster. The work in [13] then shows that in these cases the elements of order 9 in PSL\(_2\)(8) are in Baby Monster class 9B, so Monster class 9A. Moreover, in two of the three cases, the diagonal elements of order 9 are in Baby Monster class 9A, so Monster class 9B. But in PSU\(_3\)(8) these two classes of elements of order 9 are fused. Hence these cases cannot extend to PSU\(_3\)(8) in the Monster.

The remaining case therefore is a 3A-type, with normalizer PSL\(_2\)(8):3 \( \times \) 3\( S_6 \). We know there exists such a subgroup PSU\(_3\)(8) in the Monster, so all elements of order 9 fuse to 9A.

6. The Centralizer of a 9A Element

From [12], the centralizer of a 9A-element in the Monster has shape [3\( 7 \).PSU\(_4\)(2)]. Looking more closely, we see that the structure is the central product of the cyclic group of order 9 with a group of shape 3\( 6 \).\( \Omega \_5\)(3), in which the action of \( \Omega \_5\)(3) on 3\( 6 \) is uniserial, with a trivial submodule and a natural module as quotient. Moreover, since this group contains 9 \( \times \) 3\( \times \)\( S_6 \), the extension is non-split, in the sense that \( C(9)/9 \cong 3^5.\Omega \_5\)(3).

These facts can be checked computationally, using the construction of the subgroup 3\( 1+12 \).\( 2.\text{Suz}\).2 described in [6]. But in fact the proof below does not depend on any of the subtleties, so the sceptical reader can ignore them.

7. The Centralizer of 9 \( \times \) 3

Centralizing the additional element of order 3 reduces the group from 9\( \circ \)3\( 6 \).\( \Omega \_5\)(3) to 9 \( \circ \) 3\( 6 \).A\(_6\). The structure of the latter group is very subtle, and in particular it contains several conjugacy classes of 3\( \cdot \)A\(_6\), and it is not obvious which one centralizes PSL\(_2\)(8).

In any case, the group of elements which either centralize 9 \( \times \) 3 or extend it to \( D_{18} \times 3 \) is of shape \((9 \circ 3^6) \cdot (A_6 \times 2) = (9 \times 3).3^4,(2 \times A_6)\). We must adjoin an involution in the conjugacy class which maps to the central involution in the quotient \( A_6 \times 2 \). But this conjugacy class contains only 3\( 6 \) = 729 elements, while the group of symmetries is 9 \( \times \) 3\( A_6 \), of order 9720. Hence every group generated in the prescribed fashion has non-trivial centralizer in the Monster. Indeed, this counting argument implies that such a centralizer has order at least 9720/729 = 13\( \frac{1}{3} \).

8. Proof of the Theorem

The centralizer of an element of order 19 is 19 \( \times \) A\(_5\), containing elements of classes 2A, 3C and 5A. The only subgroup of A\(_5\) with order at least 14 is A\(_5\) itself. Hence every PSU\(_3\)(8) in the Monster has centralizer conjugate to this A\(_5\).

As a corollary we obtain new proofs of the uniqueness of PSU\(_3\)(8) as a subgroup of the Baby Monster, the Thompson group, and the Harada–Norton group.
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