RATIONAL POINTS ON ELLIPTIC K3 SURFACES OF QUADRATIC TWIST TYPE

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Abstract. We propose a double covering method to study the density of rational points in Zariski and real topology on quadratic twist type elliptic surfaces

\[ f(t)y^2 = g(x), \]

where \( f, g \) are cubic or quartic polynomials (without repeated roots). This method, unconditional on standard conjectures, applies in particular to certain generic Mordell-Weil rank 0 cases such as the example of Cassels and Schinzel, as well as to surfaces with \( f \) of lower degree. We also address the question about the density of fibres of prescribed rank raised by Hindry and Salgado.

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1. Introduction

Throughout this article, by an elliptic surface, we mean a projective surface \( \mathcal{E} \) defined over a number field \( k \) (smooth, geometrically integral) admitting an elliptic fibration into \( \mathbb{P}^1 \) endowed with a section over \( k \) (in literature we sometimes add “Jacobian” to emphasize this extra condition, see [Huy16, Chap. 11]). This surface is K3 if moreover it has trivial canonical class and \( H^1(\mathcal{E}, \mathcal{O}_\mathcal{E}) = 0 \). The problem of (potential) density of rational points on K3 surfaces has been intensively studied by many authors. The result of Bogomolov-Tschinkel [BT00] states that rational points are always potentially Zariski dense. Turning to the density of points over the base field. There are few K3 surfaces on which rational points are shown to be dense in the current literature. As far as the author is aware, we do not know a single example of a K3 surface on which rational points exist without being Zariski dense. From a more statistical point of view, Batyrev-Manin’s conjecture [BM90] predicts the asymptotic behaviour of rational points of bounded height associated to some appropriate ample line bundle, but it seems to be still out of reach for K3 surfaces.

Let us resume some current known methods to prove the Zariski density of rational points on elliptic surfaces. One can either show that they are in fact unirational over the base field (for geometrically rational surfaces) or make use of the available elliptic fibrations to show...
that infinitely smooth fibres are of Mordell-Weil rank strictly positive. The first method works for a number of examples. See for example the work of Billard [Bil98], Desjardins [Des17], Salgado [Sal09], [Sal12a], [Sal12b]. Let us say a bit more about the latter. Let \( \pi : E \to B \) be an elliptic fibration where the base curve \( B \) is birational to the projective line \( \mathbb{P}^1 \) and for \( t \in B(k) \) let \( E_t \) be the fiber \( \pi^{-1}(t) \). Via this projection \( E \) can be viewed as families of elliptic curves \( \{E_t\}_{t\in B(k)} \). By the Mordell-Weil Theorem (see for example [Sil09]), rational points on elliptic curves \( E_{\mathbb{Q}} \) form a finitely generated abelian group. We will denote the rank of its torsion-free part by \( \text{rank}(E_{\mathbb{Q}}) \), the number that we shall call Mordell-Weil rank in what follows. In general we have the Néron-Silverman specialisation theorem ([Sil83], see also [HS17, Theorem 1.1, Proposition 3.1]) which compares the ranks of special fibers with the generic fiber \( E_{k(t)} \). More precisely, except for at most finitely many \( t \in B(k) \), we have

\[
\text{rank}(E_{k(t)}(\mathbb{Q}(t))) \leq \text{rank}(E_{k}(k)).
\]

So once the previous one is > 0, then almost all members in the family \( \{E_t\}_{t\in \mathbb{P}^1(k)} \) are of positive rank. See for example the \( K^3 \) surfaces considered by Elkies [Elk88], Ono-Trebat-Leder [OTL16] and Salgado [Sal12a], Várilly-Alvarado [VA11]. Sometimes the construction of non-torsion section requires some base change (rational or elliptic) of the initial fibration. This technique, developed by Salgado in [Sal09], [Sal12b] and later in [SvL14] and [HS17], allows us to introduce “new” section while preserving the old ones. The result of [JS10] proving the Zariski density of rational points on an elliptic surface of general type may also be interpreted in this way.

Concerning the real topology of the set of rational points over \( \mathbb{Q} \), Mazur has made the following conjecture.

**Conjecture 1.1** (Mazur [Maz92], Conjecture 4). The family \( \{E_t\} \) verifies one of the following exclusive conditions.

1. The elliptic curves \( E_t \) has Mordell-Weil rank equal to 0 for all but a finite number of elements \( t \in \mathbb{Q} \);
2. The set \( \{t \in \mathbb{Q} : \text{rank}(E_t(\mathbb{Q})) > 0\} \) is dense in \( \mathbb{R} \).

The known cases verifying (1) are split surfaces. For those whose generic Mordell-Weil rank is positive, they verify (2) by Néron-Silverman specialisation and thus the set of rational points are dense in the real locus.

We will be concerned with the isotrivial elliptic pencils defined over \( \mathbb{Q} \), arising as family of quadratic twists of an elliptic curve:

\[
(1) \quad E^{f(T)} : f(T)Y^2 = g(X),
\]

where \( f(T) \in \mathbb{Q}[T] \) and

\[
g(X) = X^3 + aX + b \in \mathbb{Q}[X]
\]

is of degree 3 with distinct complex roots. On performing a change of variables, we can write the above equation as

\[
Y^2 = X^3 + af(T)^2X + bf(T)^3.
\]

The projection in \( T \) parametrizes a family of elliptic curves with constant \( j \)-invariant.

The variation of the root number in such quadratic twist families is first studied by Rohrlich [Roh93]. He proves for the isotrivial family above the sets of fibres with positive or negative root numbers are either both dense in \( \mathbb{R} \) or are precisely the intervals where the sign of the polynomial \( f(t) \) does not change. One of the evidences which makes us to believe the Zariski density of rational points is the parity conjecture (see (7)). Morally it says that the
root number implies the parity of the Mordell-Weil rank of the elliptic curve. It can be seen as a weaker version of the (weak) Birch and Swinnerton-Dyer conjecture, asserting that the Shafarevich-Tate group is finite and that the analytic rank and the Mordell-Weil rank of any elliptic curve coincide.

Concerning general families of abelian schemes \( A \to B \) defined over a number field whose base \( B \) is a curve of genus \( \leq 1 \), in [Sal12b] and [HS17], Hindry and Salgado ask the question about the density of the number of fibres with given Mordell-Weil rank, which can be seen as a far more precise version of the Néron-Silverman specialisation. For simplicity we shall suppose that \( B \simeq \mathbb{P}^1_\mathbb{Q} \) is a rational curve. We associate a Weil height \( H \) to the line bundle \( \mathcal{O}(1) \). For example, a naive choice can be that for \( t = \frac{p}{q} \in \mathbb{Q}, H(\frac{p}{q}) = \max(|p|, |q|) \) for \( \gcd(p, q) = 1 \). For \( T > 0 \) and \( n \in \mathbb{N} \), consider the sets

\[
F_n(T) = \{ t \in \mathbb{Q} : H(t) \leq T, \text{rank}(A(\mathbb{Q})) = n \}, \quad N(T) = \{ t \in \mathbb{Q} : H(t) \leq T \}.
\]

It is easy to see that \( \#N(T) \simeq cT^2 \) for some \( c > 0 \). The question is to estimate \( \#F_n(T) \) or \( \# \cup_{n \geq n_0} F_n(T) \) for \( n_0 \in \mathbb{N} \), whereas Néron-Silverman specialisation theorem implies that

\[
\lim_{T \to \infty} \frac{\# \bigcup_{n \geq \text{rank}(A(\mathbb{Q}(\mathbb{P}^1)))} F_n(T)}{\#N(T)} = 1.
\]

So the question becomes interesting only in cases where \( n_0 \geq \text{rank}(A(\mathbb{Q}(\mathbb{P}^1))) + 1 \).

However there exist elliptic K3 surfaces with zero generic rank. In [CS82], Cassels-Schinzel constructed the following quadratic twist family defined over \( \mathbb{Q} \) whose affine model is given by the following equation

\[
E^d : y^2 = x^3 - d^2(1 + t^4)^2x.
\]

Here \( x, y, t \) are variables and \( d \in \mathbb{N} \) is the parameter. The discriminant (with respect to \( t \))

\[
\Delta_d(t) = 64d^6(1 + t^4)^6
\]

is of degree 24 and hence \( E^d \) is birational to a K3 surface by Miranda's classification [Mir89]. They prove that (CS82 Theorem 1], see also [Roh93 LEMMA §9]) there exist only finitely many solutions \((x, y) \in E^d(\mathbb{C}(t)) \) to (3) up to complex multiplication by \( \mathbb{Z}[i] \) and they are in fact \( E^d(\mathbb{Q}(t)) \). Moreover, if \( d \) is neither a square nor twice a square (i.e. \( d/2 \) a square), then \( E^d(\mathbb{Q}(t)) \) consists of only torsion points (i.e. \( y = 0 \)). So in this case

\[
\text{rank}(E^d(\mathbb{Q}(t))) = 0,
\]

and the Néron-Silverman specialisation \textit{a priori} does not give any information on fibres of positive rank. When \( d \equiv 5, 6, 7 \mod 8 \), one can show that fibres with negative root numbers are dense in \( \mathbb{R} \). Conditionally on the standard conjectures, rational points should be Zariski dense.

From now on assume that \( d \neq 2 \) and is square-free. Our approach to attack the surfaces \( E^d \) is via some sort of double coverings similar to that used by Colliot-Thélène, Skorobogatov and Swinnerton-Dyer in [CTSSD97], where their goal was to prove that rational points are not so dense on certain hyperelliptic surfaces. One of the main consequences of this method is as follows. Here efforts are made to be unconditional on standard conjectures.

**Theorem 1.2.** There exists an infinite set \( \mathcal{D} \) of square-free integers containing 1, 5, 7, 41 \( \cdots \) such that rational points on the elliptic surface \( E^d \) (4) are dense both in Zariski topology and in real topology. Also the set \( \{ t \in \mathbb{Q} : \text{rank}(E^d_t(\mathbb{Q})) > 0 \} \) is dense in \( \mathbb{R} \). We have moreover

\[
\# \{ t \in \mathbb{Q} : H(t) \leq T, \text{rank}(E^d_t(\mathbb{Q})) > 0 \} \gg_d \log T.
\]
We conclude that Mazur’s conjecture \[1.1\] is true for such surfaces and the estimation \[4\] gives some insight to the fibre counting problem \[2\]. In fact we actually prove such lower bound for single sets \(F_n\), especially when working with small \(d\)'s, because we can say more about the precise rank of fibres. Here we only list several examples in Section \[6\].

We now try to state this double covering method in a geometric way. Let us first introduce several notations. Let \(d \in \mathbb{N}_{\geq 1}\) be the square-free parameter for \(E^d\) as before. For \(C \in \mathbb{N}_{\geq 1}\), we define the elliptic curve \(E^d_C\) with affine equation
\[
E^d_C : Cy^2 = x^3 - d^2x.
\]
It is isomorphic over \(\mathbb{Q}\) to \(E^1_{Cd}\). We also define the hyperelliptic curve with affine equation
\[
H_C : Cs^2 = 1 + t^4.
\]
It has geometric genus 1 and admits a ramified double cover to the projective line written in affine coordinates \((s,t) \mapsto t\). These curves play a key role in classifying the fibres of \(E^d\).

**Proposition 1.3.** We have the following birational equivalence for the base change of \(E^d\):
\[
E^d_{H_C} := E^d \times t H_C \cong_{\text{bir}} E^d_C \times Q H_C.
\]
Furthermore, for \(d \in \mathcal{D}\), there exists a square-free integer \(C > 0\) such that
\[
\text{rank}(E^d_{H_C}(Q(H_C))) \times \text{rank}(E^d_C(Q(E^d_C))) > 0.
\]

This implies that after base change by the hyperelliptic curve \(6\), the surface \(E^d\) becomes a split abelian surface. One can interpret this base change as follows. The fibres which are isomorphic to \(E^d_C\) over \(\mathbb{Q}\) are parametrized by \(H_C(\mathbb{Q})\) (Proposition \[1.1\]). In particular if \(H_C(\mathbb{Q})\) is infinite, we see that \(E^d\) has infinitely many isomorphic fibres. This is also the way in which we get the lower bound \[4\] as counting rational points of bounded height on the (hyper)elliptic curve \(H_C\). If moreover \(\text{rank}(E^d_C(\mathbb{Q})) > 0\), or in the classical terminology, \(Cd\) is a congruent number, rational points are thus Zariski dense (in fact dense in real topology) on the surface \(E^d_C \times_{\mathbb{Q}} H_C\). We can summarize this argument into the following inequality
\[
\text{rank}(E^d_C(\mathbb{Q})) = \text{rank}(E^d(\mathbb{Q}(H_C))) \geq \text{rank}(E^d(\mathbb{Q}(P^1))).
\]

Then it suffices to apply Néron-Silverman specialisation theorem to the fibration \(E^d_{H_C} \to H_C\) or use Proposition \[4.5\] infra to prove Theorem \[1.2\]. From another point of view, the flexibility of choosing the square-free integer \(C\) allows us to define different elliptic fibrations over \(\mathbb{Q}\), although they are all isomorphic over \(\mathbb{R}\). Geometrically the surface \(11\) has many elliptic fibrations. But it is not \textit{a priori} clear to explicitly describe them over \(\mathbb{Q}\) besides the evident ones \((x,y,t) \mapsto t\) and \(\mapsto x\). This method gives a replacement. Its strength relies heavily on our knowledge about the rank of twisted elliptic curves. But it seems to be efficient and sufficient in treating specific examples.

Our method may also be used to proof or reprove results about families of lower degree twists. In this case where \(\text{deg } f(T) = 3\), \(E^f(T)\) is an affine model of the Kummer surface associated to the abelian surface defined by the product of the elliptic curves
\[
E : y^2 = g(x), \quad F : s^2 = f(t).
\]
Since cubic polynomials can achieve any real value, we deduce from Rohrlich’s theorem \[Roh93, \text{Theorem 2}\] (cf. Theorem \[5.2\] infra) that the surface \(E^f(T)\) always has varying root number. Grant on the parity conjecture, rational points are always Zariski dense on arbitrary such Kummer surfaces. We shall prove the following unconditional result.
Proposition 1.4. Assume deg \( f(T) = 3 \). Then rational points are Zariski dense on the surface \( \mathcal{E}^{f(T)} \) if and only if there exists \( C \in \mathbb{Z}_{\neq 0} \) square-free such that

\[
\text{rank}(E_C(\mathbb{Q})) \text{ rank}(F_C(\mathbb{Q})) > 0.
\]

We would like to mention the representative work of Kuwata and Wang \cite{KW93} Theorems 2, 3 (cf. Theorem 3.3 infra). They show, when \((j_E, j_F) \neq (0, 0)\) and any of them is different from 1728, after some rational base change, there always exists a rational section whose rational points are almost non-torsion for any of the elliptic fibrations. By iterating according to the group laws associated to different elliptic fibrations, rational points are therefore dense both in Zariski topology and in real topology. Their method, relying crucially on the third fibration \((X, Y, T) \mapsto Y\), does not \textit{a priori} work for the surfaces \( \mathcal{E}^d \). Consequently Proposition 1.4 says that in this case there always exists at least one square-free integer \( C \in \mathbb{Z}_{\neq 0} \) such that the twisted curves \( E_C, F_C \) both have positive Mordell-Weil rank (see also \cite{KW93} Theorem 4). Hence the construction of these double coverings also becomes a necessary condition to ensure the existence of Zariski-dense rational points. We also give the corresponding version for the real-density. In the light of Rohrlich’s theorem, it would be interesting to have another proof of Kuwata-Wang’s result which could also cover the remaining cases.

In the terminology of \cite{HS16}, the family \( \mathcal{E}^d \) of elliptic surfaces is a special kind of \textit{Kummer varieties}. Our result can be compared with Theorems A and B in \cite{HS16}, where, conditionally on the finiteness of certain Shafarevich-Tate groups, the Kummer varieties with affine model

\[
z^2 = g_1(x)g_2(y)
\]

satisfy Hasse principle once the quartic polynomials \( g_1, g_2 \) verify some “generic” condition. However we would like to draw attention to the unconditional Proposition 1.1 in \cite{HS16}, which is concerned with split Kummer varieties. We shall state it in dimension 2 for simplicity.

Proposition 1.5 (Harpaz-Skorobogatov \cite{HS16}, Proposition 1.1). Let \( E_1, E_2 \) be two elliptic curves defined over \( \mathbb{Q} \) such that \( E_i[2](\mathbb{Q}) = 0 \) for \( i = 1, 2 \). Let \( Y_i \) be a \( E_i[2] \)-torsor defined by a class in \( H^1(\mathbb{Q}, E_i[2]) \) whose restriction to \( H^1(\mathbb{R}, E_i[2]) \) is non-zero for \( i = 1, 2 \). Let \( X = \text{Kum}(Y_1 \times Y_2) \) be the Kummer variety attached to the split abelian surface \( Y_1 \times Y_2 \) as a torsor over \( E_1 \times E_2 \). Then the real topological closure of \( X(\mathbb{Q}) \) in \( X(\mathbb{R}) \) is a union of connected components of \( X(\mathbb{R}) \).

Note that \( E_i \) is nothing but the jacobian of \( Y_i \) and that the conclusion of this proposition is Mazur’s original conjecture \cite{Maz92}. The hypothesis on \( Y_i \) roughly says that if \( Y_i \) is defined by an affine equation \( y^2 = g(x) \), then \( g(x) = 0 \) has no real solution, and the hypothesis \( E_i[2](\mathbb{Q}) = 0 \) (and hence all its quadratic twists have no 2-torsion points defined over \( \mathbb{Q} \)) guaranties that, if \( X \) possesses a \( \mathbb{Q} \)-point in the unramified locus, some quadratic twist (by some quadratic character \( F \)) \( Y_i^F, i = 1, 2 \) have simultaneously a point of infinite order. And we have \( X \simeq \text{Kum}(Y_1^F \times Y_2^F) \). In our case however, the curve coming from the twist polynomial \( s^2 = 1 + t^4 \) is the jacobian of the curve \( y^2 = x^3 - 4x \), which possesses all its 2-torsion points over \( \mathbb{Q} \). So we cannot apply directly their result to our case here.

This text is organised as follows. In Section 2 we compute, following Cassels and Schinzel, the root number of fibres of the surfaces \( \mathcal{E}^d \) for all square-free \( d \). It contains a short resume of the topology of elliptic curves over \( \mathbb{R} \) with special reference to hyperelliptic quartics. Section 3 is devoted to geometric properties of the general twist surfaces \( \mathcal{E}^{f(T)} \) and known results such as variation of root numbers and Zariski density of rational points for \( \text{deg } f(T) \leq 3 \). We introduce the double covering method in Section 4 from several points of view: isomorphism classes of fibres, base change and \( \mathbb{Z}/2 \)-torsors. In Section 5 we develop, following \cite{Coh07}, a...
criterion for general even hyperelliptic quartic to possess infinitely many rational points by reducing to certain elliptic curves. It is also naturally related to a number of Diophantine equations. The main theorem is proven in Section 6. We also investigate the density of fibres with prescribed rank in particular $E^d$'s. The last section we apply this method to treat other quadratic twist type elliptic surfaces of lower degree.

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2. Preliminaries

2.1. Root numbers. Let $\omega(E)$ be the root number of an elliptic curve $E/\mathbb{Q}$. By the work of Shimura, Wiles et al (previously the Weil-Taniyama conjecture), all elliptic curves defined over $\mathbb{Q}$ are modular and so their $L$-series $L_E(\cdot)$ admit analytic continuation to the complex plane and satisfy the functional equation

$$\zeta_E(s) = \omega(E)\zeta_E(2-s), \quad \omega(E) = \pm 1,$$

where $\zeta_E(s) = N_E^2(2\pi)^{-s}\Gamma(s)L_E(s)$, $N_E$ the conductor and $\Gamma(\cdot)$ the Gamma function. The case where $\omega(E) = -1$ implies that the $L$-series $L_E(\cdot)$ has a zero at $s = 1$ and hence the analytic rank $r_{an}(E)$ of $E$ is greater than 1. One has in fact

$$\omega(E) = (-1)^{r_{an}(E)}.$$ 

The parity conjecture asserts that

$$\omega(E) = (-1)^{\text{rank } E(\mathbb{Q})}. \tag{7}$$

In particular when $\omega(E) = -1$ one should have $\text{rank}(E(\mathbb{Q}))$ is odd and a fortiori positive.

The study of quadratic twists of the elliptic curve

$$E := E^1 : y^2 = x^3 - x$$

has a long history. For any integer $D \in \mathbb{N}_{\geq 1}$, we define the elliptic curve $E_D$ to be its quadratic twist by $D$

$$E_D : y^2 = x^3 - D^2x, \tag{8}$$

We say that $D$ is a congruent number if $\text{rank}(E_D(\mathbb{Q})) > 0$. A classical computation carried out by Birch and Stephens [BS66] shows that for the quadratic twist family of elliptic curves $E_D$ where $D > 0$ a square-free integer,

$$\omega(E_D) = \begin{cases} 
1 & D \equiv 1, 2, 3 \mod 8; \\
-1 & D \equiv 5, 6, 7 \mod 8.
\end{cases} \tag{9}$$

A folklore conjecture asserts that all such integers $D$ satisfying $\omega(E_D) = -1$ should be congruent numbers. Of course this is implied by the parity conjecture (7).

We proceed as Cassels-Schinzel [CSS2, p.347] to compute the root number for the family $E^d$. Write $t = \frac{l}{m}$, with $(l, m) \in \mathbb{Z}, \gcd(l, m) = 1$. With the change of variables

$$X = m^4x, \quad Y = m^6y,$$
the equation (3) becomes

\[ E^d : Y^2 = X^3 - (d(l^4 + m^4))^2 X. \]

For any prime \( p \mid l^4 + m^4 \), if \( p \neq 2 \), then we have \( p \equiv 1 \mod 8 \); if \( p = 2 \), then \( 2\|l^4 + m^4 \). By (9), the root numbers of the fibers are computed as follows. First we consider odd \( d \). For \( n \in \mathbb{N} \), we define \( \varepsilon(n) = \frac{n}{2} \). One has for \( d \equiv 3, 5 \mod 8 \) square-free,

\[ \omega(E^d_{(l,m)}) = \begin{cases} (-1)^{\varepsilon(d)} & \text{if } 2 \mid l^4 + m^4, \\ -(-1)^{\varepsilon(d)} & \text{if not}. \end{cases} \]

Otherwise

\[ \omega(E^d_{(l,m)}) = \begin{cases} -1 & \text{if } d \equiv 7 \mod 8; \\ 1 & \text{if } d \equiv 1 \mod 8. \end{cases} \]

Next if \( d = 2d_1 \) with \( d_1 \equiv 1, 3, 5, 7 \mod 8 \), we have similarly

\[ \omega(E^d_{(l,m)}) = \begin{cases} 1 & \text{if } 2 \mid l^4 + m^4, d_1 \equiv 1, 3 \mod 8 \text{ or } 2 \nmid l^4 + m^4, d_1 \equiv 1, 5 \mod 8; \\ -1 & \text{if } 2 \mid l^4 + m^4, d_1 \equiv 5, 7 \mod 8 \text{ or } 2 \not\mid l^4 + m^4, d_1 \equiv 3, 7 \mod 8. \end{cases} \]

We conclude from this computation that the surface \( E^d \) has constant root number \(-1\) (resp. \( 1 \)) if and only if \( d \equiv 7 \mod 8 \) or \( d = 2d_1, d_1 \equiv 7 \mod 8 \) (resp. \( d \equiv 1 \mod 8 \) or \( d = 2d_1, d_1 \equiv 1 \mod 8 \)), and in other cases we get varying root numbers so that \( \pm 1 \) appear with the same probability.

2.2. Elliptic curves over \( \mathbb{R} \). Recall that an elliptic curve defined over \( \mathbb{Q} \)

\[ E : y^2 = x^3 + ax + b \]

is connected if and only if

\[ \Delta = -16(4a^3 + 27b^2) < 0. \]

If it is not connected then the equation

\[ x^3 + ax + b = 0 \]

has 3 distinct real roots and it has precisely two connected components. We shall call the one containing the point at infinity the identity component \( E(\mathbb{R})^0 \).

**Lemma 2.1.** Let \( E_1, E_2 \) be two elliptic curves over \( \mathbb{Q} \). Consider the followings.

1. Rational points on \( E_1 \times \mathbb{Q} E_2 \) are dense in real topology,
2. Rational points on \( E_1 \times \mathbb{Q} E_2 \) are Zariski dense,
3. \( \text{rank}(E_1(\mathbb{Q})) \cdot \text{rank}(E_2(\mathbb{Q})) > 0 \).

Then we have (1) \( \Rightarrow \) (2) \( \iff \) (3). If both \( E_1, E_2 \) verify that \( E_i(\mathbb{R}) \) is connected or \( E_i \) has a 2-torsion point defined over \( \mathbb{Q} \) not on the identity component, then all of the above are equivalent.

**Proof.** The equivalence of (2) and (3) is clear. We show that under the additional hypothesis, (2) \( \Rightarrow \) (1). It is a classical fact that any infinite subgroup of a 1-dimensional Lie group has open topological closure. After Waldschmidt [Wal93] this is also true for simple abelian varieties of dimension \( g \) and rank \( > g^2 - g \). This means that once rational points are Zariski dense on \( E_i \), their topological closure contains the identity component. So if this is the case but \( E_i(\mathbb{R}) \) is not connected, then the translation by the 2-torsion point on the non-identity introduces rational points on the other component of \( E_i(\mathbb{R}) \). We conclude that \( E_i(\mathbb{Q}) = E_i(\mathbb{R}) \). \( \square \)
2.3. Hyperelliptic quartics. Recall here the well-known fact that a (smooth) hyperelliptic quartic has a smooth model as the intersection of two quadratic surfaces in $\mathbb{P}^3$. Over $\mathbb{R}$ we can choose the plane affine model as (with a singularity at infinity)

$$Y^2 = G(X) = \sum_{i=0}^{4} a_i X^i, \quad a_i \in \mathbb{Z}, a_4 \neq 0.$$  

(11) Its geometric genus is 1.

2.3.1. Birational transformation. If the curve (11) possesses a $\mathbb{Q}$-point $(x_0, y_0)$ with $y_0 \neq 0$, then it can be transformed into a Legendre cubic model and hence equipped with a group structure on the set of its rational points. The explicit birational change of variables formulas are given in [Coh07, Proposition 7.2.1] and its Legendre equation has the shape

$$y^2 + bxy + cy = x^3 + dx^2 + ex.$$  

(12) It passes through $(0,0)$. By making a further change of variables (in fact a linear transformation or a rotation) on gets a Weierstrass model.

This change of variables formula takes particularly simple form when the polynomial $G(X)$ is even. More precisely, the equation

$$Y^2 = X^4 + a_2 X^2 + a_0$$  

(13) can be transformed into the Legendre form

$$y^2 = x(x^2 + cx + d)$$  

by

$$x = 2X^2 - 2Y + a_2, \quad y = 2X(2X^2 - Y + a_2), \quad c = -2a_2, \quad d = a_2^2 - 4a_0.$$  

(15) We remark that, from (15) and valid more generally from [Coh07, Proposition 7.2.1], a point with coordinates $(0, Y_0), Y_0 \neq 0$ is sent to $(h(Y_0), 0)$, where $h(Y_0) \neq 0$ is a rational function in $Y_0$.

2.3.2. Topology of hyperelliptic quartics. In order that the curve (11) is smooth, the polynomial $G(X)$ has no repeated roots. So over $\mathbb{R}$ only three cases can happen:

1. Type I: $G(X) = 0$ has no real roots;
2. Type II: $G(X) = 0$ has two distinct real roots;
3. Type III: $G(X) = 0$ has four distinct real roots.

We now show that all these types of curves have an affine model over $\mathbb{R}$ which is almost even. And we shall give criterion in the even cases to assure that twists of (11) contain infinitely many rational points in Section 2.3.2.

Type I: The curve (11) has the $\mathbb{R}$-diffeomorphic affine model

$$y^2 = t^4 + 1.$$  

The (affine) real locus of (11) has two symmetric branches (one on the upside of the $X$-axis the other below). Each branch intersects the $Y$-axis in certain point $(0, Y_0)$. So in this case its Legendre model (12) intersects $x$-axis three times (so is its Weierstrass model). We conclude that the curve (11) is NOT connected. However the map $(X, Y) \mapsto (X, -Y)$ exchanges rational points between its two components.

Type II: the polynomial $G(X)$ factorises into the product of two degree two factors, one with two distinct roots and the other without root. In this case one gets two branches, one on the left and the other on the right with respect to the $Y$-axis. By choosing a suitable
translation, we may assume that the two real roots are opposite to each other. So the curve \([11]\) is \(\mathbb{R}\)-diffeomorphic to
\[Y^2 = (X^2 + 1)(X^2 - 1).\]
We readily check that its Legendre model is connected, and so is the curve itself.

Type III: Still by a suitable change of variables if necessary we may assume that the smallest and the biggest roots are opposite to each other say \((-X_0, 0)\) and \((X_0, 0)\) and hence on the interval \(X \geq X_0\), \([11]\) is \(\mathbb{R}\) diffeomorphic to
\[Y^2 = (X^2 - X_0^2)(X^2 - X_1^2),\]
where \(0 < X_1 < X_0\). As before one sees that this part is connected. The inner part of \([11]\) bounded by the two other roots gives rise to the second connected component.

We summarize our discussion into the following.

**Proposition 2.2.** Curves \([11]\) of Type II are connected and those of Type I and III are not connected. For those of Type I and II, Zariski density and real density are equivalent.

### 3. Kummer type elliptic surfaces

From now on we will restrict ourselves to the isotrivial elliptic surface over \(\mathbb{Q}\).

\[E_f(T) : f(T)Y^2 = X^3 + aX + b \subset \mathbb{A}^3_{X,Y,T}.\]

We shall fix our zero section to be the one at infinity.

#### 3.1. Kodaira dimension.

Throughout the discussion we suppose that the equation \(f(T) = 0\) has no repeated roots in \(\mathbb{C}\). Define two hyperelliptic curves (whose variables should not be confused with those before).

\[E : y^2 = x^3 + ax + b \subset \mathbb{A}^2_{x,y};\]
\[F : w^2 = f(t) \subset \mathbb{A}^2_{t,w},\]

with their respective hyperelliptic involutions

\[\varrho_1 : (x, y) \mapsto (x, -y), \quad \varrho_2 : (t, w) \mapsto (t, -w).\]

Then outside the fixed points of \(\varrho_1\) and \(\varrho_2\), the map

\[\phi : E \times \mathbb{Q} F \rightarrow \mathcal{E}^{f(T)};\]
\[(x, y) \times (t, w) \mapsto (X, Y, T) = (x, \frac{y}{w}, t)\]

establishes a (generically) double covering of the split surface \(E \times F\) to (an affine model of) \(\mathcal{E}^{f(T)}\) which is unramified outside the locus \((Y = 0)\). And in this way we recover the affine model \([11]\) of the surface \(\mathcal{E}^{f(T)}\). Each smooth fibre \(\langle \mathcal{E}^{f(T)} \rangle_{T=t}\) has constant \(j\)-invariant as \(E\).

After Miranda’s classification, we have

**Proposition 3.1 ([Mir89]).** Let \(n\) be the smallest integer such that \(2n \geq \deg f(t)\). Then

- \(n = 0:\) the surface is split, i.e. \(\mathcal{E}^{f(t)}\) is birational to \(E \times \mathbb{P}^1\);
- \(n = 1:\) the surface is (geometrically) rational;
- \(n = 2:\) the surface is (birational to) \(K3\);
- \(n \geq 3:\) the surface is of Kodaira dimension 1. We sometimes say that it is elliptic of “general type”.


3.2. Variation of root numbers. On assuming BSD, studying variation of the root number of fibres can serve as an indicator of the proportion of fibres having positive Mordell-Weil rank. A theorem of Rohrlich says that in general, root numbers can change frequently. Recall that in the preceding section we have defined
\[
T^+ = \{ t \in \mathbb{Q} : E_t \text{ smooth, } \omega(E_t) = +1 \}
\]
and respectively \( T^- \).

**Theorem 3.2** (Rohrlich, [Roh93] Theorem 2). Let \( \mathcal{E}^{f(t)} \) be the family as before. Then exclusively one of the following statement is true.

- Both of the sets \( T^+ \) and \( T^- \) are dense in \( \mathbb{R} \);
- The sets \( T^+ \) and \( T^- \) are precisely up to permutation
\[
\{ t \in \mathbb{Q} : f(t) > 0 \} \quad \text{and} \quad \{ t \in \mathbb{Q} : f(t) < 0 \}.
\]

In particular, if \( f(t) \) takes negative values, then conditionally on the parity conjecture, many fibers have positive (odd) Mordell-Weil rank. Therefore rational points are Zariski dense. Applying Rohrlich’s theorem 3.2 to the examples that we consider in this article, we may rephrase our computation in Section 2.1 as follows. Since \( f(t) = 1 + t^4 > 0 \), we have \( T^- = \mathbb{R} \) and \( T^+ = \emptyset \) (resp. \( T^+ = \mathbb{R} \) and \( T^- = \emptyset \)) precisely when \( d \equiv 7 \mod 8 \) or \( d = 2d_1, d_1 \equiv 7 \mod 8 \) (resp. \( d \equiv 1 \mod 8 \) or \( d = 2d_1, d_1 \equiv 1 \mod 8 \)). For other \( d \)'s the computation reveals that we are in the first case of the theorem.

3.3. Degree two twists. Assume \( \deg f(T) = 2 \). By Proposition 3.1, these surfaces are geometrically rational. A recent result of Desjardins [Des17] implies the Zariski density of rational points on \( \mathcal{E}^{f(T)} \) for arbitrary separable quadratic polynomial \( f(T) \), improving the result first previously obtained by Rohrlich [Roh93, Theorem 3]. Her result applies to a more general family of elliptic surfaces.

**Theorem 3.3** ([Des17], Theorem 5.4). Let \( \mathcal{E} \) be an isotrivial (geometrically) rational elliptic surface with \( j \)-invariant \( 1728 \). Then \( \mathcal{E}(\mathbb{Q}) \) is Zariski dense.

One can even show that in this case fibres of positive rank is dense in \( \mathbb{R} \) and so Mazur’s conjecture [1.1] receives a satisfactory answer. It remains to discuss the question of Hindry and Salgado.

As an application and a presentation of our method, we will treat a meta-example constructed by Lewis and Schinzel in [LS80] with affine model:
\[
S : y^2 = x^4 - (8t^2 + 5)^2.
\]
viewed as fibration in \( t \), it has zero generic rank:
\[
\text{rank}(S(\mathbb{Q}(t))) = 0.
\]
One can verify that \( \omega(S_t) = -1 \) for \( t \in \mathbb{Z} \). So the parity conjecture [7] implies the Zariski-density of rational points. We shall prove the unconditional density result in Section 7.1.

3.4. Degree three twists. Kuwata and Wang [KW93] study the case where \( f(t) \) is cubic. In this case equation (16) is a singular affine model of the Kummer surface associated to the abelian surface \( E \times F \) (see the notations (17)). By using explicit geometric construction, they prove the following. The dependence on the third fibration \( p_Y \), the fibration in \( Y \)-coordinate, will be clear from the statement. The case also receives the recent consideration of Gvirtz [Gvi].
Theorem 3.4 (Kuwata-Wang, [KW93]). Suppose \((j_E, j_F) \neq (0, 0)\) or neither of them is equal to 1728, then there exists a rational base change \(\Phi : P^1 \to P^1\) such that the surface \(E(f(T)) \times_{\Phi, P^1} P^1\) has positive generic Mordell-Weil rank with respect to this new base-change fibration.

So the fact that rational points are Zariski dense readily follows from Néron-Silverman specialisation theorem.

3.5. Degree four twists. The degree 4 cases will be the main concern in this article. We may assume that

\[ f(t) = t^4 + ct^2 + dt + e \in \mathbb{Z}[t] \]

is separable and does not have linear factors in \(\mathbb{Q}[t]\). Otherwise if \(f(t) = (t - t_0)f_0(t)\), by performing the change of variables

\[ t \mapsto \frac{1}{t - t_0}, \quad y \mapsto (t - t_0)^2 y, \]

we recover the cubic case. In some case one can prove density results by constructing non-torsion rational sections as was done by Kuwata and Wang. The following result is due to Gvirtz [Gvi, Theorem 4.1]. We keep the notations as before.

Theorem 3.5 ([Gvi], §4). Suppose \(j_E = 1728\) and that

\[ f(t) = t^4 + dt + e \]

where \(de \neq 0\). Then there exists a rational base change \(u \mapsto t(u), u \in \mathbb{Q}\) such that

\[ \text{rank}(E(f(T))(\mathbb{Q}(u))) > 0. \]

Consequently, rational points are Zariski dense in \(E(f(T))\). Assume furthermore that \(f(t)\) is non-negative for all \(t \in \mathbb{R}\), then rational points are dense in \(E(f(T))(\mathbb{R})\).

3.6. Remark on Elkie’s result. We would like to comment here that in [KW93, p. 121], it is claimed without proof that the diagonal quartic surface studied by Elkies [Elk88] as the first counterexample of Euler’s conjecture

\[ S_1 : x_0^4 + x_1^4 + x_2^4 = x_3^4, \]

is a double covering of the Cassels-Schinzel surface \((3)\) with \(d = 2\). Recall that by Miranda’s classification (Proposition 3.1), all surfaces \((3)\) are K3. It is well-known by the Torelli’s theorem for K3’s [PSS71], diagonal quartic surfaces are Kummer surfaces associated to certain split abelian surface \(E_1 \times E_2\). But it is not clear whether all these surfaces are isomorphic over \(\mathbb{Q}\).

Take an affine model of \((21)\) as

\[ x^4 + y^4 + 1 = z^4. \]

We may try the change of variable (kindly communicated to the author by Gvirtz)

\[ x \mapsto rt, \quad y \mapsto r, \quad z \mapsto s, \]

and we get another affine model of \(S_1\)

\[ S_1' : (t^4 + 1)r^4 = s^4 - 1, \]

which is a double covering of

\[ S_2 : (t^4 + 1)r^2 = s^4 - 1. \]
Note that the surface $S_2$ \[23\] is the jacobian of the twisted elliptic curve over $\mathbb{Q}(t)$ (See [Sko01, Proposition 3.3.6])

\[(24)\]

$$S_3 : (t^4 + 1)r^2 = s^3 + 4s.$$  

We henceforth get this quadratic twist family which looks like $E^d$ but it is different from the family $E^d$ in real topology. Elkies’s result [Elk88] saying that rational points are dense in $S_1(\mathbb{R})$ or equivalently in $S_1' (\mathbb{R})$ implies that rational points are Zariski dense in $S_2$. By Mazur and Merel’s uniformity theorems on torsion points for elliptic curves defined over a fixed number field, we conclude that infinitely many fibres in $S_1'$ must be of positive rank. Being $E[2]$-torsors, where $E$ is the elliptic curve $r^2 = s^3 + 4s$ (However there are points of $E[2]$ not defined over $\mathbb{Q}(t)$, so this torsor is not trivial), we get that $S_2, t \simeq S_{3, t}$ over $\mathbb{Q}$ for infinitely many $t$ and each of them have infinitely many points. This proves the Zariski density of rational points on the surface $S_3$.

4. THE DOUBLE COVERING METHOD

4.1. Classification of fibres and parametrization of rational points. The proof of Theorem 1.2 reduces to the key observation that the elliptic surfaces $E^d$ have many isomorphic fibres over $\mathbb{Q}$. Recall the hyperelliptic curves $E, F$ \[17\]. For any square-free integer $C \in \mathbb{Z} \neq 0$, we define the “twisted” curves whose affine equations are

\[(25)\]

$$E_C : Cy^2 = x^3 + ax + b \subset \mathbb{A}^1_{x,y,}, \quad F_C : Cw^2 = f(t) \subset \mathbb{A}^1_{t,w}.$$  

The curve $E_C$ is just the quadratic twist of $E$ by $C$. Note that for general degree 4 polynomials $f(t) \in \mathbb{Q}[t]$, the curve $F_C$ does NOT admit a structure of quadratic twist of $F$ (but for even polynomials they “almost” do, we shall discuss more in Section 5). Both curves $E_C, F_C$ are also equipped with the hyperelliptic involution \[18\] and we get also the double coverings defined in the same way as \[19\]:

\[(26)\]

$$\phi_C : E_C \times_\mathbb{Q} F_C \longrightarrow \mathcal{S},$$  

The indeterminacy locus is $(w = 0) \subset E_C \times_\mathbb{Q} F_C \subset \times \mathbb{A}^1_{t,w}$ and the unramified locus is precisely

\[(27)\]

$$U_C = (E_C \times F_C) \setminus ((w = 0) \cup (y = 0))$$  

whose image is

\[(28)\]

$$\mathcal{V} = E^{f(T)} \setminus (Y = 0).$$  

We remark that for $C, C' \in \mathbb{Z} \neq 0$ square-free such that $CC' > 0$, the surfaces $E_C \times F_C$ and $E_{C'} \times F_{C'}$ are isomorphic over $\mathbb{R}$ by the change of variables

$$y \mapsto \sqrt{|C|/|C'|} y, \quad w \mapsto \sqrt{|C|/|C'|} w.$$  

To simplify notation, we shall write $\mathcal{E}_t$ to denote the fibre of $E^{f(T)}$ at $T = t$.

**Proposition 4.1.** Two smooth fibres $\mathcal{E}_t, \mathcal{E}_{t'}$ are isomorphic over $\mathbb{Q}$ if and only if there exists $C \in \mathbb{Z} \neq 0$ square-free and $w, w' \in \mathbb{Q}$ such that $(t, w), (t', w') \in F_C(\mathbb{Q})$.

**Proof.** Quadratic twists are defined up to squares. For each $t \in \mathbb{Q}$, define $C_t \in \mathbb{Z} \neq 0$ to be the unique square-free integer in the class of $f(t)$ in $\mathbb{Q}^x/\mathbb{Q}^{x^2}$. Then $\mathcal{E}_t, \mathcal{E}_{t'}$ are isomorphic over $\mathbb{Q}$ if and only if

$$C_t \equiv C_{t'} \mod \mathbb{Q}^{x^2} \iff C_t = C_{t'}.$$
since they are assumed to be square-free. Finally this is equivalent to saying that there exist \( s, s' \in \mathbb{Q} \) such that

\[
f(t) = Cs^2, \quad f(t') = Cs'^2.
\]

Now we see that the fibres \( E_t, E_{t'} \) are all isomorphic to \( E_C \) by means of the change of variable \( Y = sy \) or \( s'y \).

Now it becomes clear that Proposition 4.3 is nothing but another way of expressing Proposition 4.1 that the base change by \( F_C \) splits the surface \( \mathcal{E}^{f(T)} \):

\[
\mathcal{E}^{f(T)} \times_{\mathbb{P}^1} F_C \simeq E_C \times_{\mathbb{Q}} F_C.
\]

The fibres over \( \mathbb{R} \) are classified according to the sign of the polynomial \( f(t) \). They can be all isomorphic over \( \mathbb{Q} \) in particular cases. This will not be used in the sequel.

**Lemma 4.2.** If \( j_E = 1728 \), that is, \( b = 0 \), then all smooth fibres of \( \mathcal{E}^{f(T)} \) are isomorphic (non-canonically) over \( \mathbb{R} \). If \( j_E \neq 1728 \), two smooth fibres \( E_t, E_{t'} \) are isomorphic over \( \mathbb{R} \) if \( f(t)f(t') > 0 \).

**Proof.** After the change of variable \( y \mapsto \sqrt{|f(t)|} \) (resp. \( \sqrt{|f(t')|} \)), \( E_t, E_{t'} \) are isomorphic to

\[
\pm y^2 = x^3 + ax + b.
\]

In particular if \( b = 0 \), the above curve is isomorphic to (via the change of variable \( x \mapsto -x \) if necessary) \( E \).

The main reason of introducing this constant \( C \) is that it gives a “parametrization” of rational points on \( \mathcal{E}^d \).

**Proposition 4.3.** Recall the quasi-projective varieties \( U_C \) \((27)\) and \( V \) \((28)\). We have that \( (U_C)_C \) square-free are 2-torsors over \( V \) together with the following decompositions

\[
V(\mathbb{Q}) = \left\{ \bigcup_{C \in \mathbb{Z}_{\neq 0} \text{ square-free}} \phi_C(U_C(\mathbb{Q})) \right\};
\]

\[
V(\mathbb{R}) = \phi_C(U_C(\mathbb{R})) \bigcup \phi_{C'}(U_{C'}(\mathbb{R})),
\]

where \( C, C' \) any square-free integers such that \( C \in \mathbb{Z}_{>0} \) and \( C' \in \mathbb{Z}_{<0} \).

**Proof.** The variety \( V \) is obtained by quotient of the fixed-point-free action \( (\varrho_1, \varrho_2) \) (operating on each factor) on the surface

\[
(E_C \setminus (y = 0)) \times (F_C \setminus (w = 0)).
\]

After Colliot-Thélène and Sansuc [CTSSD97 Théorème 2.3.1, p. 421] (see also [CTSS97 Proposition 1.1, Theorem 2.1]), for \( X \) a smooth variety, the group of isomorphism classes of \( \mathbb{Z}/2\mathbb{Z} \)-torsors over \( X \), namely \( H^1_{\text{ét}}(X, \mathbb{Z}/2\mathbb{Z}) \) is identified with subset of \( H^1_{\text{ét}}(\mathbb{Q}(X), \mathbb{Z}/2\mathbb{Z}) = \mathbb{Q}(X)^*/\mathbb{Q}(X)^{2} \) consisting of rational functions defining a double divisor. Here we consider \( f(T) \) as an element of \( \mathbb{Q}(\mathcal{E}^{f(T)}) \). Let \( c^f_\mathbb{Q} \) (resp. \( c^f_\mathbb{R} \)) be the number of roots of \( f(t) = 0 \) over \( \mathbb{Q} \) (resp. \( \mathbb{R} \)). On \( V_\mathbb{Q} \) (resp. \( V_\mathbb{R} \)), the divisor \( f(T) = 0 \) defines \( c^f_\mathbb{Q} + 1 \) (resp. \( c^f_\mathbb{R} + 1 \)) rational curves, each one being of multiplicity 2 (+1 corresponds to the point of \( E_C \) at infinity). The decompositions \((30)\) and \((31)\) are nothing but the evaluation map \((k = \mathbb{Q} \text{ or } \mathbb{R})\)

\[
V(k) \longrightarrow H^1_{\text{ét}}(k, \mathbb{Z}/2\mathbb{Z}) = k^*/k^{2},
\]

and the fact that \( \{ C \in \mathbb{Z}_{\neq 0} \text{ square-free} \} \) is a set of representatives of \( \mathbb{Q}^*/\mathbb{Q}^{2} \) and for any \( C \in \mathbb{Z}_{>0} \) and \( C' \in \mathbb{Z}_{<0} \) both square-free, \( \{ C, C' \} \) is a set of representatives of \( \mathbb{R}^*/\mathbb{R}^{2} \).

\(\square\)
Remark 4.4. We would like point out that the family of projective surfaces $(E_C \times_Q F_C)_C$ are NOT $\mathbb{Z}/2\mathbb{Z}$-torsors of any smooth proper models of $E^{f(T)}$. There are several reasons for that. First, the double cover (26) is clearly ramified. Furthermore, if they were $\mathbb{Z}/2$-torsors, then the disjoint union (30) would be finite for a proper model $W$ of $E^{f(T)}$ since the evaluation map $\mathcal{W}(\mathbb{Q}) \to H^1_{et}(\mathbb{Q}, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Q}^*/\mathbb{Q}^{*2}$ would have finite image. In our case the disjoint union is clearly not finite.

4.2. Topology of rational points on Kummer type elliptic surfaces. In the view of the decompositions of rational points and real points, we have the following criterion about density of $E^{f(T)}(\mathbb{Q})$ in Zariski topology and in real topology of $E^{f(T)}$.

**Proposition 4.5.** Assume $\deg f(T) \leq 4$. Then we have

1. In order that rational points (over $\mathbb{Q}$) are Zariski dense on $E^{f(T)}$, it suffices that there exists $C \in \mathbb{Z}_{\neq 0}$ square-free such that rational points (over $\mathbb{Q}$) are Zariski-dense in the surface $E_C \times_Q F_C$.
2. In order that rational points (over $\mathbb{Q}$) are real-dense in $E^{f(T)}$, it suffices that there exist $C \in \mathbb{Z}_{>0}$ and $C' \in \mathbb{Z}_{<0}$ square-free such that rational points (over $\mathbb{Q}$) are real-dense in the surfaces $E_C \times_Q F_C$ and $E_{C'} \times_Q F_{C'}$.

**Proof of Propositions 4.5.** This follows directly from the decompositions (30) and (31) in Proposition 4.3 and the fact that $\phi_C$ is étale on $U_C$ over $\mathbb{Q}$ or $\mathbb{R}$. □

Remark 4.6. If $\deg f(T) \geq 5$, the hyperelliptic curve $F$ has geometric genus $\geq 2$ so by Falting’s theorem rational points can never be dense in the surface $E_C \times F_C$. Otherwise, if $\deg f(T) \leq 4$, then $F$ is either rational or elliptic. The weaker condition that the existence of one square-free $C \in \mathbb{Z}_{\neq 0}$ such that rational points are Zariski dense on $E_C \times_Q F_C$ readily implies that rational point are dense in at least one connected component of $E^{f(T)}(\mathbb{R})$. With some extra conditions (on $E$ and $F$), one can prove rational points are actually dense on whole of $E^{f(T)}(\mathbb{R})$. For example, if $\deg f(T) = 4$ and it always takes positive values, then the real locus of $E_{-1} \times_Q F_{-1}$ is empty. In this case even though $F$ is not connected, taking into account of the symmetry (cf. Section 2.3.2), the conclusion of Proposition 4.5 still holds. This will be case for the family $E^d$ (see Section 6).

5. Rational points on hyperelliptic quartics and Diophantine equations

Hyperelliptic quartics have geometric genus 1. Whenever it has a rational point, it can be transformed into an elliptic curve over the base field by selecting this point (at infinity) as the zero element (see Section 2.3). However this point is torsion. We have shown in section 2.3.2 that, from real topological point of view, even hyperelliptic quartic are natural affine models. Our aim here is to describe an algorithmic criterion for “quadratic” twists of such hyperelliptic quartics to have infinitely many rational points.

5.1. General criterion. Fix throughout this section $b, c, d, e \in \mathbb{Z}$ such that $bde \neq 0$. We will be interested in more general hyperelliptic quartics

$$H : ew^2 = bt^4 + ct^2 + d. \tag{32}$$

It can be written as the Diophantine equation via the substitution $t = \frac{X}{Y}, w = \frac{Z}{Y^2}$,

$$cZ^2 = bX^4 + cX^2Y^2 + dY^4. \tag{33}$$
We will say that two solutions \((X, Y, Z), (X', Y', Z') \in \mathbb{Q}_>^3\) of (33) are equivalent if \(\exists \lambda \in \mathbb{Q}_>^X\) such that
\[
X = \lambda X', \quad Y = \lambda Y', \quad Z = \lambda^2 Z'.
\]
Clearly in each equivalence class there exists a unique triple \((X, Y, Z) \in \mathbb{N}_>^3\) with \(\gcd(X, Y) = 1\) and we shall call it coprime. It is related to the elliptic curve
\[
E : y^2 = x^3 + cex^2 + bde^2x
\]
whenever \((X, Y, Z)\) is a solution verifying \(Y \neq 0\). This can be extended naturally by sending a solution \((X, 0, Z)\) with \(XZ \neq 0\) to the point at infinity. The elliptic curve \(E\) has the evident 2-torsion point \((0, 0)\). We can define the associate 2-descent map \(\alpha : E(\mathbb{Q}) \to \mathbb{Q}^\times / \mathbb{Q}^{\times 2}\). For \(P = (x, y) \in E(\mathbb{Q})\) (where we denote by \(O\) the point at infinity), let
\[
\alpha(P) = \begin{cases} 1 \mod \mathbb{Q}^{\times 2} & \text{if } P = O; \\ bd \mod \mathbb{Q}^{\times 2} & \text{if } P = (0, 0); \\ x \mod \mathbb{Q}^{\times 2} & \text{otherwise.} \end{cases}
\]

The following proposition gives a criterion for which (33) has infinitely many coprime solutions (see also [Col07] §6.5.3).

**Proposition 5.1.** The followings are equivalent.

1. The hyperelliptic quartic \(H\) (32) has infinitely many rational points;
2. The Diophantine equation \(G\) (33) has infinitely many coprime solutions;
3. The elliptic curve \(E\) has positive Mordell-Weil rank and the class of \(eb\) in \(\mathbb{Q}^\times / \mathbb{Q}^{\times 2}\) is contained in \(\alpha(E(\mathbb{Q}) \setminus E(\mathbb{Q})_{\text{tor}})\).

**Proof.** The map \(\alpha\) induces an injective group homomorphism \(E(\mathbb{Q})/2E(\mathbb{Q}) \to \mathbb{Q}^\times / \mathbb{Q}^{\times 2}\). It is clear that (1) \(\iff\) (2) and any one implies (3) in the view of (35) and the fact that the number torsion points are uniformly bounded. To see the converse, assume \(\text{rank}(E(\mathbb{Q})) > 0\) and \(P \in E(\mathbb{Q})\) such that \(\alpha(P) = eb \mod \mathbb{Q}^{\times 2}\). Then we have
\[
P + 2E(\mathbb{Q}) \subseteq \alpha^{-1}(\alpha(P)).
\]

So pick any \(Q = (x, y) \in P + 2E(\mathbb{Q})\). Define \((X, Y, Z) \in \mathbb{N}_>^3\) with \(\gcd(X, Y) = 1\) by
\[
\frac{X}{Y} = \sqrt{\frac{x}{eb}}, \quad Z = \sqrt{\frac{bX^4 + cX^2Y^2 + dY^4}{e^2b}}.
\]

We readily check that this defines a coprime solution of (33). \(\square\)

The following corollary is useful in treating general degree 4 even twist families \(E^{f(T)}\).

**Corollary 5.2.** Let \(C \in \mathbb{Z}_{\neq 0}\) be square-free and let \(S_C\) be the abelian surface defined by the product of elliptic curves
\[
E_C : Cy^2 = x^3 + ax + b, \quad F_C : Cw^2 = t^4 + dt^2 + e.
\]

Then the followings are equivalent.

1. \(S_C(\mathbb{Q})\) is Zariski dense;
2. \(\text{rank}(E_C(\mathbb{Q})) \cdot \text{rank}(F_C(\mathbb{Q})) > 0\);
3. \(\text{rank}(E_C(\mathbb{Q})) > 0\) and \(F_C\) verifies any of the equivalent statements in Proposition 5.1 (with \(H = F_C\)).
5.2. The Diophantine equation $X^4 + Y^4 = CZ^2$. As a natural generalization of the Fermat’s Last Theorem, similar methods can be adapted to attack Diophantine equations of the form

$$AX^r + BY^s + CZ^t = 0, \quad A, B, C \in \mathbb{Z}; \quad r, s, t \in \mathbb{N}_{\geq 1}$$

and show that most of them have no integer solution, especially when $r, s, t$ are sufficiently large:

$$\frac{1}{r} + \frac{1}{s} + \frac{1}{t} > 1.$$ 

However for the intermediate types

$$\frac{1}{r} + \frac{1}{s} + \frac{1}{t} = 1$$

including the type that we are interested in here $(r, s, t) = (4, 4, 2)$, their solutions can be always related to rational points on certain elliptic curves. The quartic equation

$$Q_C : X^4 + Y^4 = CZ^2, \quad C \in \mathbb{N}_{\geq 1} \text{ square-free}$$

is essentially another way of writing the hyperelliptic curve $H_C$ by putting

$$t = \frac{X}{Y}, \quad s = \frac{Z}{Y^2}.$$ 

We define the associated elliptic curve to be

$$E_C : y^2 = x^3 + C^2 x,$$

together with the 2-descent map $\alpha$ defined as before.

Now we restate Proposition 5.1 with respect to this particular case (cf. also [Coh07, Proposition 6.5.5]). Note that in statement (1) we only require the existence of one non-zero solution and in (3) we add a local condition on $C$.

**Proposition 5.3.** Suppose that $C > 2$. The followings are equivalent.

1. There exists a solution $(X, Y, Z) \in \mathbb{Q}^3$ of (36) such that $XYZ \neq 0$;
2. There exist infinitely many coprime solutions of (36);
3. All odd prime factors of $C$ are $\equiv 1 \mod 8$, $2C$ is a congruent number and the image of $\alpha$ contains the class of $C$ in $\mathbb{Q}^2/\mathbb{Q}^2$.

**Proof.** We first observe that

$$\text{rank}(E_C(\mathbb{Q})) = \text{rank}(E_{2C}(\mathbb{Q}))$$

thanks to the following 2-isogeny

$$\psi : E_C \longrightarrow E_{2C}$$

$$(x, y) \longmapsto \left(\frac{y^2}{x^2}, \frac{y(C^2 - x^2)}{x^2}\right)$$

with kernel $\text{ker}(\psi) = \{O, (0,0)\}$. Furthermore, since $C > 2$ and is supposed to be square-free, $E(\mathbb{Q})_{\text{tor}} \simeq \mathbb{Z}/2\mathbb{Z}$ ([Sil09, p. 323]) and it is generated by the point $(0, 0)$. Fix a coprime solution $(X, Y, Z) \in \mathbb{N}_{\geq 1}^3$ of (36). Observe that for any prime $p \neq 2$ dividing $X^4 + Y^4$, we have $p \equiv 1 \mod 8$. Also if $2 \mid X^4 + Y^4$, then necessarily $X$ and $Y$ are both odd and an easy computation shows that $4 \mid X^4 + Y^4$. Since global solubility trivially implies local solubility, in order that (36) admits non-zero solutions, $C$ must be a product of prime factors $\equiv 1$ or $2$. 

mod 8. Through the change of variable (35) we get a non-torsion point of \( E_C(\mathbb{Q}) \). So \( 2C \) is necessarily a congruent number. It follows also that
\[
\alpha \left( C \frac{X^2}{Y^2} \right) = C \mod \mathbb{Q}^{\times 2}.
\]

The converse is proven in Proposition 5.1.

\[\Box\]

**Remark 5.4.** Local solubility and \( \text{rank}(E_{2C}(\mathbb{Q})) > 0 \) in general are NOT sufficient to guarantee the solubility of (36). As H. Cohen pointed out, for \( C = 9266 = 2 \times 41 \times 113 \), \( 2C \) is a congruent number but \( C \mod \mathbb{Q} \times 2 \) is not in the image of \( \alpha \). But nevertheless all these exceptional \( C \) seem to be very few. We extract several values of \( C \) for which (36) is solvable from [Coh07, p. 395]:
\[
(38) \quad 17, \quad 82 = 2 \times 41, \quad 113, \quad 257, \quad 4001.
\]

6. Demonstrations of the main theorem and particular examples

6.1. The hyperelliptic curves \( \mathcal{H}_C \). Recall in the beginning we have defined the hyperelliptic curve \( \mathcal{H}_C \) [6]. The discussion in Section 5.2 leads to

**Proposition 6.1.** For \( C \) in the list (38), \( \mathcal{H}_C(\mathbb{Q}) \) is dense in \( \mathcal{H}_C(\mathbb{R}) \).

**Proof.** This follows from Propositions 5.1, 5.3 and 2.2.

This proposition implies that the hyperelliptic projection \( \mathcal{H}_C \to \mathbb{P}^1 \) has dense image for any \( C \) listed in (38).

6.2. Density estimate. Recall the local condition (3) of Proposition 5.3, the associated elliptic curve \( E_C \) always have positive root number, so does the hyperelliptic quartic \( \mathcal{H}_C \). We can indeed verify that for all \( C \) listed in (38), \( \text{rank}(\mathcal{H}_C(\mathbb{Q})) = 2 \). It follows that
\[
\# \{(s, t) \in \mathcal{H}_C(\mathbb{Q}) : H(t) \leq T\} \asymp \log T.
\]

In this way we obtain lower bound for the density of fibres of prescribed rank (see (2)) since
\[
\# \{t \in \mathbb{Q} : H(t) \leq T, \text{rank}(E^d_t(\mathbb{Q})) = \text{rank}(E^d_C(\mathbb{Q}))\} = \frac{1}{2} \# \{(s, t) \in \mathcal{H}_C(\mathbb{Q}) : H(t) \leq T\}.
\]

6.3. The family \( \mathcal{E}^d \). By Proposition 4.3, we have
\[
(\mathcal{E}^d \setminus \{(Y = 0)\})(\mathbb{Q}) = \bigcup_{C \in \mathbb{Z}_{\neq 0} \text{ square-free}} \phi_C(E^d_C \times \mathcal{H}_C)(\mathbb{Q}).
\]

In order to prove that rational points are dense in real topology, we use Proposition 4.5. Note that for any \( C' < 0 \), the real locus of \( E^d_{C'} \times \mathcal{H}_{C'} \) is empty. We will restrict ourselves to positive square-frees. We dispose already a list of values \( C \) [38] at hand for the curve \( \mathcal{H}_C \) to have non-trivial hence infinitely many solutions. It remains to consider the rank of \( E^d_C \), that is, to determine whether \( Cd \) is a congruent number or not. We are free to cite the table of Elkies [Elk]. He shows that all square-free \( D \equiv 7 \mod 8 \) up to \( 10^6 \) the associated elliptic curve \( E_D \) has the same analytic and Mordell-Weil rank and he lists all such \( D \) for which \( E_D \) is of rank 3. By using this table one can say a bit more by looking at explicit examples of elliptic surfaces \( \mathcal{E}^d \). Some of them can have special fibres of “jumping” Mordell-Weil rank, and all such fibres will in fact be dense in \( \mathbb{R} \) because the hyperelliptic curve \( \mathcal{H}_C \) is of type I (See Section 2.3.2). All elliptic surfaces in this section will be proved to have real-dense rational points and all sets below parametrizing fibres with specific rank will satisfy (4) by the argument in §6.2.
6.3.1. $d = 5$. The first theorem is concerned with an elliptic surface with varying root number.

**Theorem 6.2.** For the elliptic surface $E_5^5$, the sets

$$\{t \in \mathbb{Q} : \text{rank}(E_5^5(t)(\mathbb{Q})) = 1\}, \quad \{t \in \mathbb{Q} : \text{rank}(E_5^5(t)(\mathbb{Q})) = 0\}$$

are both dense in $\mathbb{R}$.

Theorem 6.2 provides an example in the direction of a question addressed by Fried ([CS82, p. 346]).

**Proof.** The number $85 = 5 \times 17$ is a congruent number whereas $410 = 5 \times 82$ is not. It suffices to select from the list (38) $C = 17$ and $82$ and considering the surfaces $E_{85 \times 17}$ and $E_{410 \times 82}$. The density in $\mathbb{R}$ follows from Proposition 6.1.

The fact that on this elliptic surface smooth fibres having positive Mordell-Weil rank are dense in $\mathbb{R}$ also follows from Mazur’s observation ([Maz92, Proposition]). Note that we cannot apply Rohrlich’s Theorem 3.2 to deduce that fibres of zero rank are dense for this constant $j = 1728$ family.

6.3.2. $d = 7$. The surface

$$E^7 : y^2 = x^3 - (7(1 + t^4))^2x.$$ (39)

is original one considered by Cassels and Schinzel [CSS2]. It has constant root number $-1$. The parity conjecture implies that all fibres have positive rank. We show that fibres with rank 1 or 3 are both dense in $\mathbb{R}$.

**Theorem 6.3.** For the elliptic surface $E^7$, the sets

$$\{t \in \mathbb{Q} : \text{rank}(E^7_t(\mathbb{Q})) = 1\}, \quad \{t \in \mathbb{Q} : \text{rank}(E^7_t(\mathbb{Q})) = 3\}$$

are both dense in $\mathbb{R}$.

**Proof.** We have

$$7 \times 17 \equiv 7 \times 4001 \equiv 7 \mod 8.$$ 

By Elkies’ table [Elk],

$$\text{rank}(E_{7 \times 17}(\mathbb{Q})) = 1, \quad \text{rank}(E_{7 \times 4001}(\mathbb{Q})) = 3,$$

it suffices to consider (36) with $C = 17$ and $4001$. \hfill $\square$

6.3.3. **Surfaces with positive constant root number.** While the theorem of Gross does not cover the case $d \equiv 1 \mod 8$, where the surface $E^d$ has constant root number $+1$, we supply an two examples showing that this case is non-empty.

**Theorem 6.4.** The set

$$\sharp\{t \in \mathbb{Q} : \text{rank}(E_t^d(\mathbb{Q})) > 0\}$$

is dense in $\mathbb{R}$ for $d = 1, 41$.

**Proof.** For $d = 1$, we can choose $C = 257$ in the list (38), which is also a congruent number. For $d = 41$, since $41 \times 113$ is a congruent number (see Remark 5.4 in fact $E_{41 \times 113}$ has rank 2), it suffices to consider (36) with $C = 113$. \hfill $\square$

**Remark 6.5.** As we see from the proof, a necessary condition to prove Zariski density of rational points is the existence of a $C$ such that all its prime factors are $\equiv 1 \mod 8$ and both $C$ and $2C$ are congruent numbers. They form a very sparse subset of square-free integers. It is not known whether there exists infinitely many of them.
6.3.4. An extreme meta-example. For the quadratic twist family $\{E_D\}$, the highest rank known is 7 with smallest

$$D = 5 \times 7 \times 17 \times 97 \times 173 \times 79873.$$  

(see for instance [WDE+15]). Conditionally on GRH, the growth of the Mordell-Weil rank can be controlled by (due to Brumer)

$$\text{rank}(E_D(\mathbb{Q})) = O \left( \frac{\log D}{\log \log D} \right).$$

But in the same paper a heuristic due to Granville predicts that $\text{rank}(E_D(\mathbb{Q})) < 8$ for all $D \in \mathbb{N}$. In principle we can construct elliptic surfaces with infinitely many fibres with highest known rank. Here is an example.

**Theorem 6.6.** For the elliptic surface $\mathcal{E}^d$ with $d = 5 \times 7 \times 97 \times 173 \times 79873$, we have that the set

$$\{t \in \mathbb{Q} : \text{rank}(\mathcal{E}^d_t(\mathbb{Q})) = 7\}$$

is dense in $\mathbb{R}$.

**Proof.** It suffices to choose $C = 17$. \qed

6.4. **End of the proof of Theorem 1.2.** Up to now we have exhibited several $d$’s such that $\mathcal{E}^d$ verifies Theorem 1.2. It remains to show that the set $\mathcal{D}$ of such $d$ is infinite, we may apply a classical result of Stephens [Ste75] which states that any prime number $\equiv 5, 7 \pmod{8}$ is a congruent number. Fix such prime $p$ and consider $d = 17p$. We have $E_{17}^d \simeq E_p^1$ and so

$$\text{rank}(E_{17}^d(\mathbb{Q})) \text{ rank}(\mathcal{H}_{17}(\mathbb{Q})) > 0.$$  

Consequently, by Propositions 4.5, 5.3 and Corollary 5.2

$$\{17p : p \text{ prime number } \equiv 5, 7 \pmod{8}\} \subset \mathcal{D}.$$  

This subset clearly has infinite cardinality.

The parity conjecture (7) suggests that the set $\mathcal{D}$ should be much bigger. The recent breakthrough of Y. Tian pushes it much further by affirming the existence of infinitely many congruent numbers with arbitrarily many given number of prime factors.

**Theorem 6.7** (Ye Tian, [Tia14], Theorem 3). Let $D \equiv 5, 6, 7 \pmod{8}$ be a square-free positive integer. Define

$$\mathcal{D} = \begin{cases} \frac{D}{2} & \text{if } 2 \mid D; \\ D & \text{otherwise.} \end{cases}$$

Suppose that all but one of the prime factors of $\mathcal{D}$ are $\equiv 1 \pmod{8}$ and that the 4-rank of the imaginary quadratic field $\mathbb{Q}(\sqrt{-D})$ is zero. Then we have

$$\text{rank}(E_D(\mathbb{Q})) = 1.$$  

One can use this theorem to “enlarge” the set $\mathcal{D}$ so that it contains integers with arbitrarily many given number of prime factors. The technical condition about the 4-rank should not cause problems in counting thanks to the works of Gerth [Ger83] and Fouvry-Klüners [FK07]. They show that quadratic fields with specified 4 rank have positive density. We will not go into details.
7. Applications to quadratic twist type elliptic surfaces of lower degree

7.1. Application to degree two twist families. We keep the notations in §3. By suitable change of variables, any degree 2 quadratic twist families can be written in the following form

\[(dT^2 + e)Y^2 = X^3 + aX + b.\]

As shown by Rohrlich [Roh93] and Ulas [Ula07], using explicit construction of rational sections, and by Kollár-Mella [KM17] and Desjardins [Des17], these surfaces admit conic bundle structure and are unirational over \(\mathbb{Q}\), we get Zariski density result. In this section we are going to use the double covering method to revisit this situation. With the notations in §4, the split family we get consists of products of a twisted elliptic curve \(E_C\) with a conic \(F_C\) with one parameter. It is a classical fact dated back to Legendre that local-global principle is true for conics. So it is relatively easier to determine whether a conic has a point (hence rational over \(\mathbb{Q}\)). Instead of proving statements in full generality, we shall only exhibit one example constructed by Lewis and Schinzel [LS80] in order to illustrate the method. We will draw attention in forthcoming works to more general setting as well as pursuing the question of Hindry and Salgado mentioned in the beginning concerning the proportion of the sets \(\mathcal{S}\) amongst fibres of bounded height. Indeed, we actually proved a stronger polynomial density estimate than (4) since \(F_C\) is a rational curve.

We now consider the (hyper)elliptic surface with affine model

\[(40)\quad y^2 = x^4 - (8t^2 + 5)^2.\]

According to the birational transformation in Section 2.3, this surface is birational equivalent to the one with affine model

\[y^2 = x^3 + 4(8t^2 + 5)^2x,\]

or equivalently, to the isotrivial families with \(j = 1728\),

\[(41)\quad S : (8t^2 + 5)y^2 = x^3 + 4x.\]

**Proposition 7.1.** Rational points on the surface \((40)\) is dense in real topology. In particular fibres with positive rank are dense in \(\mathbb{R}\). We have moreover that there exist \(\delta > 0\) such that

\[(42)\quad \#\{t \in \mathbb{Q} : H(t) \leq T, \text{rank}(S_t(\mathbb{Q})) = 1\} \gg T^\delta.\]

**Proof.** For any \(C \in \mathbb{N}_{\geq 1}\) square-free, we define two curves

\[F_C : 8t^2 + 5 = Cs^2, \quad E_C : Cy^2 = x^3 + 4x.\]

We then get a double covering map

\[\phi_C : F_C \times \mathbb{Q} E_C \to S,\]

which is clearly surjective in real topology. The surface \(S(\mathbb{R})\) is then connected. Note that \(F_C\) is a conic. To prove that rational points are (Zariski or real) dense it suffices to find one \(C\) such that \(F_C(\mathbb{Q}) \neq \emptyset\) and that the elliptic curve \(E_C\) has positive Mordell-Weil rank (Proposition 4.5). We can choose for example \(C = 13\). Then the point \((s, t) = (13, 1) \in F_{13}(\mathbb{Q})\) and the curve \(E_{13}\), being isogenous to the curve \(13y^2 = x^3 - x\), has positive rank because 13 is a congruent number. The real-density statement follows from Mazur’s argument [Maz92, Proposition] or Rohrlich’s theorem (Theorem 3.3). It remains to count fibres with Mordell-Weil rank one. Following the argument in Section 6.2, we see that this reduces to counting...
rational points on the rational conic $F_{13}$.

$$\# \{ t \in \mathbb{Q} : H(t) \leq T, \text{rank}(S_t(\mathbb{Q})) = 1 \} \geq \frac{1}{2} \{ t \in \mathbb{Q} : H(t) \leq T, \exists s \in \mathbb{Q} : (s,t) \in F_{13}(\mathbb{Q}) \} \gg T$$

In particular we can take $\delta = 1$ because conics in $\mathbb{P}^2$ have $O(1)$ degree 2 and hence the height $H$ restricted to $F_{13}$ is equivalent to an $O(2)$ height on $\mathbb{P}^1$.

**Remark 7.2.** We may adapt Tian’s Theorem 6.7 and improve the estimate (42) for the single sets $F_0(T), F_1(T)$ by some careful analytic computation. We hope address this problem in future work.

### 7.2. Application to degree three twists.

We shall apply the double covering method developed in Section 4 to study the surfaces $E^f(T)$ with $\deg f(T) = 3$, aiming at proving Proposition 1.4. We keep the notations in Sections 3 and 4. After lifting into double coverings, the key input is an observation due to Rohrlich.

**Lemma 7.3** ([Roh93], §8 LEMMA). Let $E$ be an elliptic curve with Weierstrass equation

$$y^2 = x^3 + ax + b.$$ 

We fix the equation for $E$ the quadratic twist by $C \in \mathbb{Q}^\times$ to be

$$E_C : Cy^2 = x^3 + ax + b.$$ 

Then

$$\# \{ x \in \mathbb{Q} : \exists C \in \mathbb{Q}^\times, y \in \mathbb{Q}, (x,y) \in E_d(\mathbb{Q})_{\text{tor}} \} < \infty.$$ 

**Proposition 7.4.** Assume that $\deg f(T) = 3$. The followings are equivalent.

1. Rational points are Zariski dense on $E^f(T)$;
2. Rational points are dense on at least one (hence half of) connected component of $V(\mathbb{R})$;
3. There exists $C \in \mathbb{Z}_{\neq 0}$ square-free such that rational points are Zariski dense on $E_C \times F_C$.
4. The set of $w \in \mathbb{Q}$ such that $\text{rank}(E_w(\mathbb{Q})) \text{rank}(F_w(\mathbb{Q})) > 0$ is dense in one of the intervals $R_{>0}$ and $R_{<0}$.

If any one these hypothesis is satisfied, then we have

$$\# \{ t \in \mathbb{Q} : H(t) \leq T, \text{rank}(E_t^f(T)(\mathbb{Q})) > 0 \} \gg (\log T)^{\frac{1}{2}}.$$

**Proof.** We shall prove the implications (4) $\iff$ (3) $\iff$ (2) $\iff$ (1) $\implies$ (3). It is clear that (4) $\implies$ (3) since it suffices to take $C_w \in \mathbb{Z}_{\neq 0}$ square-free $\equiv w \mod \mathbb{Q}^\times$. To see (3) $\implies$ (4), without loss of generality we may assume $C > 0$. The implication follows from the observation that the set

$$\{ w \in \mathbb{Q} : w \equiv C \mod \mathbb{Q}^\times \}$$

is dense in $R_{>0}$. (2) $\implies$ (1) is easy.

We now show (1) $\implies$ (3). By Proposition 4.5 we have the decomposition of rational points on $V$:

$$V(\mathbb{Q}) = \bigsqcup_{C \in \mathbb{Z}_{\neq 0} \text{ square-free}} \phi_C(U_C(\mathbb{Q})).$$

If rational points are not dense on any of the $U_C$, then for any $C$ at least one of the groups $E_C(\mathbb{Q}), F_C(\mathbb{Q})$ can only contain torsion points. According to Lemma 7.3 and the expression of the morphism $\phi_C$ ([19] [26]), there exists a Zariski closed subset $Z$ of $E^f(T)$ consisting of points
(X, Y, T) with finitely possible X-coordinates or T-coordinates such that for any \( C \in \mathbb{Z} \neq 0 \) square-free,
\[
\phi_C(U_C(\mathbb{Q})) \subset \mathbb{Z} \cup (X = 0) \cup (Y = 0) \cup (T = 0).
\]
Hence this contradicts the fact the union of all such \( \phi_C(U_C(\mathbb{Q})) \) is dense.

Finally, the cardinality of fibre counting set is bounded from below by that of \( F_C(\mathbb{Q}) \) of bounded height, which goes like \( \asymp (\log T)^{\frac{3}{2}} \).

From the point of view of real topology, we can formulate the above proposition similarly as follows.

**Proposition 7.5.** Same hypothesis as in Proposition 7.4. We have that followings are equivalent

1. Rational points are dense in \( E^{f(T)}(\mathbb{R}) \);
2. There exists \( C \in \mathbb{Z}_{>0} \) and \( C' \in \mathbb{Z}_{<0} \) square-free such that rational points are Zariski dense on both \( E_C \times \mathbb{Q} F_C \) and \( E_{C'} \times \mathbb{Q} F_{C'} \).
3. The set of \( w \in \mathbb{Q} \) such that \( \text{rank}(E_w(\mathbb{Q})) \text{rank}(F_w(\mathbb{Q})) > 0 \) is dense in \( \mathbb{R} \).

**Proof.** \( E \) and \( F \) are either connected or equipped with the 2-torsion point \((0, 0)\) on the non-identity component. The same argument as in the proof of Proposition 7.4 works. It remains to use the following simple lemma about the topology of split Kummer surfaces. \( \square \)

**Lemma 7.6.** Suppose \( \deg f(t) = 3 \) and the leading coefficient is positive. Then the affine surface \( V \) has

\[
\begin{cases}
8 \text{ connected components, if } \Delta_E, \Delta_F > 0; \\
4 \text{ connected components, if } \Delta_E \Delta_F < 0; \\
2 \text{ connected components, if } \Delta_E, \Delta_F < 0;
\end{cases}
\]

**Proof.** This is done by explicit sign-change arguments. \( \square \)

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