EXISTENCE OF GROUND STATES OF HYDROGEN-LIKE ATOMS IN RELATIVISTIC QED II: THE NO-PAIR OPERATOR

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Abstract. We consider a hydrogen-like atom in a quantized electromagnetic field which is modeled by means of a no-pair operator acting in the positive spectral subspace of the free Dirac operator minimally coupled to the quantized vector potential. We prove that the infimum of the spectrum of the no-pair operator is an evenly degenerate eigenvalue. In particular, we show that the bottom of its spectrum is strictly less than its ionization threshold. These results hold true, for arbitrary values of the fine-structure constant and the ultra-violet cut-off and for all Coulomb coupling constants less than the critical one of the Brown-Ravenhall model, $2/(2/\pi + \pi/2)$. For Coulomb coupling constants larger than the critical one, we show that the quadratic form of the no-pair operator is unbounded below. Along the way we discuss the domains and operator cores of the semi-relativistic Pauli-Fierz and no-pair operators, for Coulomb coupling constants less than or equal to the critical ones.

1. Introduction

In this article we continue our study of the existence of ground states of hydrogen-like atoms and ions in (semi-)relativistic models of quantum electrodynamics (QED). The model studied here is given by the following no-pair operator,

$$H^+_\gamma := P^+_A(D_A - \gamma/|x| + H_f) P^+_A,$$

where $D_A$ is the free Dirac operator minimally coupled to the quantized vector potential, $A$, and $P^+_A$ is the spectral projection onto its positive spectral subspace,

$$P^+_A := 1_{[0,\infty)}(D_A).$$

Moreover, $H_f$ is the energy of the photon field and $\gamma \geq 0$ the Coulomb coupling constant. The quantized vector potential $A$ depends on two physical parameters, namely the fine structure constant, $\alpha$, and an ultra-violet cut-off...
parameter, \( \Lambda \). Thus, \( H_+ + \gamma \) is acting in the projected Hilbert space \( P_+ \mathcal{H} \), where \( \mathcal{H} \) denotes the Hilbert tensor product of \( L^2(\mathbb{R}^3_+, \mathbb{C}) \) and the bosonic Fock space of the photon field. The mathematical analysis of an analogue of \( H_+ + \gamma \) describing a molecular system has been initiated in \([17, 18]\) where the stability of matter of the second kind is established (under certain restrictions on the nuclear charges and \( e^2 \) and \( \Lambda \)) and an upper bound on the (positive) binding energy is given. For more information on a general class of no-pair operators with classical external electromagnetic fields and on some applications of no-pair operators in quantum chemistry and physics we refer to \([23]\) and the references therein.

Improving earlier results from \([21]\) we show in the present article that the quadratic form of \( H_+ + \gamma \) is bounded below, for arbitrary values of \( e^2 \) and \( \Lambda \), if and only if \( \gamma \leq \gamma_c^{np} \), where

\[
\gamma_c^{np} := \frac{2}{2/\pi + \pi/2}
\]

is the critical coupling constant of the electronic Brown-Ravenhall operator \([9]\). In particular, \( H_+ + \gamma \) has a self-adjoint Friedrichs extension, provided that \( \gamma \leq \gamma_c^{np} \). The main result of the present paper states that, for all \( \gamma \in (0, \gamma_c^{np}) \), the infimum of the spectrum of this Friedrichs extension is a degenerate eigenvalue. As an intermediate step we prove a binding condition for \( H_+ + \gamma \), \( \gamma \in (0, \gamma_c^{np}] \), ensuring that the ground state energy of \( H_+ + \gamma \) is strictly less than its ionization threshold. Along the way we also study the domain and essential self-adjointness of \( H_+ + \gamma \), \( \gamma \in [0, \gamma_c^{np}] \), as well as of the semi-relativistic Pauli-Fierz operator,

\[
\mathcal{H}_\gamma := \sqrt{(\sigma \cdot (-i\nabla + A))^2 + 1 - \gamma/|x| + H_f}, \quad \gamma \in [0, \gamma_c^{PF}],
\]

where \( \gamma_c^{PF} := 2/\pi < \gamma_c^{np} \) is the critical constant in Kato’s inequality. (The formal vector \( \sigma \) contains the Pauli spin matrices.) The latter discussion improves on earlier results of \([21]\) and \([24]\).

The existence of energy minimizing ground states for atoms and molecules in non-relativistic QED, where the electrons are described by Schrödinger operators, is by now a well-established fact. The first existence proofs have been given in \([3, 5]\), for small values of \( e^2 \) and \( \Lambda \). Later on the existence of ground states for a molecular Pauli-Fierz Hamiltonian has been shown in \([12]\), for all values of \( e^2 \) and \( \Lambda \), assuming a certain binding condition, which has been verified in \([18]\). In the last decade there appeared a large number of further mathematical contributions to non-relativistic QED. Here we only mention that ground state energies and projections have also been studied by means of infra-red finite algorithms and renormalization group methods \([11, 2, 3, 4, 5, 6, 11]\). These sophisticated methods give very detailed results as they rely on constructive algorithms rather than compactness arguments. They work, however, only in a regime where \( e^2 \) and/or \( \Lambda \) are sufficiently small.

In our earlier companion paper \([16]\) we have already shown that \( \mathcal{H}_\gamma \) has a degenerate ground state eigenvalue, for all \( \gamma \in (0, \gamma_c^{PF}) \). The existence of
ground states in a relativistic atomic model from QED where also the electrons and positrons are treated as quantized fields is investigated in [7]. To this end infra-red regularizations are imposed in the interaction terms of the Hamiltonian which is not necessary in the model treated here.

We like to stress one essential feature which both operators $H_\gamma$ and $H_\gamma^+$ have in common: namely their gauge invariance. In fact, the possibility to pass to a suitable gauge by means of a unitary Pauli-Fierz transformation allows to prove two infra-red estimates serving as key ingredients in a certain compactness argument introduced in [12]. We remark that when the projections $P_A^+$ in (1.1) are replaced by projections that do not contain the vector potential, that is, by $P_0^+$, then the resulting operator is not gauge invariant anymore. In this case one can still prove the existence of ground states provided that a mild infra-red regularization is imposed on $\gamma$ [15, 19]. It seems, however, unlikely that the infra-red regularization can be dropped when $P_0^+$ is used instead of $P_A^+$ [15]. It is also known that a no-pair model defined by means of $P_0^+$ becomes unstable as soon as more than one electron is considered [13].

Although in many parts the general strategy of our proofs in [16] and the present paper follows along the lines of [5] and [12] the analysis of the operators $H_\gamma$ and $H_\gamma^+$ poses a variety of new and non-trivial mathematical problems. This is mainly caused by the non-locality of $H_\gamma$ and $H_\gamma^+$ which both do not act as partial differential operators on the electronic degrees of freedom anymore as it is the case in non-relativistic QED. In this respect $H_\gamma^+$ is harder to analyze than $H_\gamma$ since also the Coulomb potential and the radiation field energy become non-local due to the presence of the spectral projections $P_A^+$.

There is one question left open in [16] and the present paper, namely whether $H_\gamma$ and $H_\gamma^+$ still possess ground state eigenvalues when $\gamma$ attains the respective critical values. Instead of going through all proofs in [16] and below and trying to adapt them to cover also the critical cases, it seems to be more convenient to approximate the ground state eigenvectors in the critical cases by those found for sub-critical values of $\gamma$. This argument requires an estimate on the spatial localization of the ground state eigenvectors which is uniform in $\gamma$. Earlier results [21] provide, however, only $\gamma$-dependent estimates. As a new derivation of a uniform bound would lengthen the present article too much we shall work out these ideas elsewhere.

The organization of this article and brief remarks on some techniques. In Section 2 we introduce the no-pair operator and state our main results more precisely. In Section 3 we derive some basic relative bounds involving $H_\gamma$ and $H_\gamma^+$ which improve on earlier results of [21]. Here we benefit from recent generalized Hardy-type inequalities [10, 27] that allow to derive these relative bounds also for critical values of $\gamma$. Moreover, we discuss the domains and the essential self-adjointness of $H_\gamma$ and $H_\gamma^+$. Section 4 is the core of this article as it
discusses the convergence of sequences of no-pair operators. The results are
new and tailor-made for the no-pair model. They allow to implement some
well-known arguments developed to prove the existence of ground states in
non-relativistic QED \cite{5, 12} in the present setting. In Section 5 we derive a
binding condition for the no-pair operator which is necessary in order to apply
the results of Section 4. In Section 6 we prove the existence of ground state
eigenvectors, \( \phi_m \), assuming that the the photons have a mass \( m > 0 \). We
employ a discretization argument and proceed along the lines of \cite{5}, where \( e^2 \)
and/or \( \Lambda \) are assumed to be small. The implementation of the discretization
procedure requires, however, our new results of Section 4. Moreover, as in \cite{10}
we add a new observation which allows to treat also large values of \( e^2 \) and \( \Lambda \).
This compactness argument makes use of some further non-trivial ingredients:
First, we need a bound on the spatial localization of \( \phi_m \), uniformly in \( m > 0 \).
Such a bound has already been derived in \cite{21}. Second, we need two infra-red
bounds \cite{5, 12} whose proofs are deferred to Section 8. Technically, our proof
of the infra-red bounds differs slightly from those in \cite{5, 12} as we first derive a
representation formula for \( a(k) \phi_m \). The infra-red bounds are then easily read
off from this formula. The main text is followed by two appendices where some
technical estimates on functions of the Dirac operator are given (Appendix A)
and some properties of \( \phi_m \) are discussed (Appendix B).

Frequently used notation. \( \mathcal{D}(T) \) denotes the domain of an operator \( T \) in some
Hilbert space and \( \mathcal{Q}(T) \) its form domain, provided that \( T = T^* > -\infty \). If \( T \)
is self-adjoint, then \( \mathbb{R} \ni \lambda \mapsto \mathbb{1}_{\lambda}(T) \) denotes its spectral family and
\( \mathbb{1}_M(T) \) the spectral projection corresponding to some measurable subset \( M \subset \mathbb{R} \).
\( C(a, b, \ldots), C'(a, b, \ldots) \) etc. denote positive constants which only depend on
the quantities \( a, b, \ldots \) displayed in their arguments. Their values might change
from one estimate to another.

2. Definition of the Model and Main Results

First, we recall some standard notation. The Hilbert space underlying the
atomic model studied in this article is a subspace of \( \mathcal{H} := \mathcal{H}_0 \), where
\begin{equation}
\mathcal{H}_m := L^2(\mathbb{R}^3, \mathbb{C}^4) \otimes \mathcal{F}_b[\mathcal{H}_m] = \int_{\mathbb{R}^3} \mathbb{C}^4 \otimes \mathcal{F}_b[\mathcal{H}_m] d^3x, \quad m \geq 0.
\end{equation}
(In some of our proofs we choose \( m > 0 \).) Here the bosonic Fock space,
\( \mathcal{F}_b[\mathcal{H}_m] = \bigoplus_{n=0}^\infty \mathcal{F}^{(n)}_b[\mathcal{H}_m] \), is modeled over the one photon Hilbert space
\[ \mathcal{H}_m := L^2(\mathcal{A}_m \times \mathbb{Z}_2, dk), \quad \int dk := \sum_{\lambda \in \mathbb{Z}_2} \int_{\mathcal{A}_m} d^3k, \quad \mathcal{A}_m := \{|k| \geq m \}. \]
The letter \( k = (k, \lambda) \) always denotes a tuple consisting of a photon wave vector, \( k \in \mathbb{R}^3 \), and a polarization label, \( \lambda \in \mathbb{Z}_2 \). The components of \( k \) are denoted as \( k = (k^{(1)}, k^{(2)}, k^{(3)}) \). We recall that \( \mathcal{F}_b^{(0)}[\mathcal{H}_m] := \mathbb{C} \) and, for \( n \in \mathbb{N} \), \( \mathcal{F}_b^{(n)}[\mathcal{H}_m] := S_n L^2((\mathcal{A}_m \times \mathbb{Z}_2)^n) \), where, for \( \psi^{(n)} \in L^2((\mathcal{A}_m \times \mathbb{Z}_2)^n) \),

\[
(S_n \psi^{(n)})(k_1, \ldots, k_n) := \frac{1}{n!} \sum_{\pi \in S_n} \psi^{(n)}(k_{\pi(1)}, \ldots, k_{\pi(n)}) ,
\]

\( S_n \) denoting the group of permutations of \( \{1, \ldots, n\} \). For \( f \in \mathcal{H}_m \) and \( n \in \mathbb{N}_0 \), we further define \( a^\dagger(f)^{(n)} : \mathcal{F}_b^{(n)}[\mathcal{H}_m] \to \mathcal{F}_b^{(n+1)}[\mathcal{H}_m] \) by \( a^\dagger(f)^{(n)} \psi^{(n)} := \sqrt{n + 1} S_{n+1}(f \otimes \psi^{(n)}) \). Then \( a^\dagger(f) := \bigoplus_{n=0}^\infty a^\dagger(f)^{(n)} \) and \( a(f) := a^\dagger(f)^* \) are the standard bosonic creation and annihilation operators satisfying the canonical commutation relations

\[
[a^\dagger(f), \ a^\dagger(g)] = 0, \quad [a(f), \ a^\dagger(g)] = (f \mid g) \mathbbm{1} , \quad f, g \in \mathcal{H}_m ,
\]

where \( a^\dagger \) is \( a^\dagger \) or \( a \). For a three-vector of functions \( f = (f^{(1)}, f^{(2)}, f^{(3)}) \in \mathcal{H}_m^3 \), we write \( a^\dagger(f) := (a^\dagger(f^{(1)}), a^\dagger(f^{(2)}), a^\dagger(f^{(3)})) \). Then the quantized vector potential associated to some measurable family \( \Gamma \in \mathcal{H}_m^3 \), \( \mathbf{x} \in \mathbb{R}^3 \), is the triple of operators \( \mathbf{A} = (A^{(1)}, A^{(2)}, A^{(3)}) \) given as

\[
\mathbf{A} \equiv \mathbf{A}[\Gamma] := \int_{\mathbb{R}^3} \mathbbm{1}_C \otimes \mathbf{A}(\mathbf{x}) \, d^3 \mathbf{x} , \quad \mathbf{A}(\mathbf{x}) := a^\dagger(\Gamma(\mathbf{x})) + a(\Gamma(\mathbf{x})) .
\]

Next, we recall that the second quantization, \( d\Gamma(\varpi) \), of some Borel function \( \varpi : \mathcal{A}_m \times \mathbb{Z}_2 \to \mathbb{R} \) is the direct sum \( d\Gamma(\varpi) := \bigoplus_{n=0}^\infty d\Gamma^{(n)}(\varpi) \), where \( d\Gamma^{(0)}(\varpi) := 0 \), and, for \( n \in \mathbb{N} \), \( d\Gamma^{(n)}(\varpi) \) is the maximal multiplication operator in \( \mathcal{F}_b^{(n)}[\mathcal{H}_m] \) associated with the symmetric function \( (k_1, \ldots, k_n) \mapsto \varpi(k_1 + \cdots + \varpi(k_n)) \).

Our main results deal with the physical choices of \( \varpi \) and \( \Gamma \) given in Example 2.2. Since many results of the technical parts of this paper are applied to modified versions of these physical choices it is, however, convenient to introduce the following more general hypothesis:

**Hypothesis 2.1.** \( \varpi : \mathcal{A}_m \to [0, \infty) \) is a measurable function such that \( 0 < \varpi(k) := \varpi(\mathbf{k}) \), for almost every \( k = (k, \lambda) \in \mathcal{A}_m \times \mathbb{Z}_2 \). For almost every \( k \in \mathcal{A}_m \times \mathbb{Z}_2 \), \( G(k) \) is a bounded, twice continuously differentiable function, \( \mathbb{R}^3 \ni \mathbf{x} \mapsto G_x(k) \in \mathbb{C}^3 \), such that the map \((\mathbf{x}, k) \mapsto G_x(k)\) is measurable. There is some \( d \in (0, \infty) \) such that, for \( \ell \in \{-1, 0, 1, \ldots, 7\} \),

\[
\int \varpi(k)^\ell \|G(k)\|_\infty^2 dk \leq d , \quad \int \varpi(k)^{-1} \| \nabla_x \wedge G(k) \|_\infty^2 dk \leq d ,
\]

where \( \|G(k)\|_\infty := \sup_{\mathbf{x}} |G_x(k)| \), etc.

**Example 2.2.** In the physical model we are interested in we have \( m = 0 \) and the radiation field energy, \( H_{\ell} \), is given by

\[
H_{\ell} := d\Gamma(\omega) , \quad \omega(k) := |k| , \quad k = (k, \lambda) \in \mathbb{R}^3 \times \mathbb{Z}_2 .
\]
A physically interesting choice for $G_x$ is given as follows: Writing
\begin{equation}
(2.6) \quad k_\perp := (k^{(2)}, -k^{(1)}, 0), \quad k = (k^{(1)}, k^{(2)}, k^{(3)}) \in \mathbb{R}^3,
\end{equation}
we introduce the following polarization vectors,
\begin{equation}
(2.7) \quad \varepsilon(k, 0) = \frac{k_\perp}{|k_\perp|}, \quad \varepsilon(k, 1) = \frac{k}{|k|} \wedge \varepsilon(k, 0),
\end{equation}
for almost every $k \in \mathbb{R}^3$, and set
\begin{equation}
(2.8) \quad G^{\varepsilon, \Lambda}_x(k) := -e^{\frac{1}{2} \mathbb{I} \{ |k| \leq \Lambda \}} e^{-ik \cdot \varepsilon(k)},
\end{equation}
for all $x \in \mathbb{R}^3$ and almost every $k = (k, \lambda) \in \mathbb{R}^3 \times \mathbb{Z}_2$. Here $\Lambda > 0$ is an ultraviolet cut-off parameter whose value can be chosen arbitrarily large. The value of $e \in \mathbb{R}$ does not affect the validity of our results either. (In nature we have $e^2 \approx 1/137$. For in the units chosen above – energies are measured in units of the rest energy of the electron and $x$ is measured in units of one Compton wave length divided by $2\pi$ – the square of the elementary charge $e > 0$ is equal to Sommerfeld’s fine-structure constant.)

Finally, we recall the definition of the Dirac operator, $D_\Lambda$, minimally coupled to $A_x$. Let $\alpha_1, \alpha_2, \alpha_3$, and $\beta = \alpha_0$ denote the hermitian $4 \times 4$ Dirac matrices obeying the Clifford algebra relations
\begin{equation}
(2.9) \quad \alpha_i \alpha_j + \alpha_j \alpha_i = 2 \delta_{ij} \mathbb{1}, \quad i, j \in \{0, 1, 2, 3\}.
\end{equation}
They act on the second tensor factor in $\mathcal{H}_m = L^2(\mathbb{R}^3_\times) \otimes \mathbb{C}^4 \otimes \mathcal{F}_b[\mathcal{H}_m]$ and, in the standard representation, they are given in terms of the Pauli matrices,
\begin{equation*}
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\end{equation*}
as
\begin{equation*}
\alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \quad j \in \{1, 2, 3\}, \quad \beta = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}.
\end{equation*}
The free Dirac operator minimally coupled to $A$ is now given as
\begin{equation}
(2.10) \quad D_A := \alpha \cdot (-i\nabla_x + A) + \beta := \sum_{j=1}^{3} \alpha_j (-i\partial_{x_j} + A^{(j)}) + \beta.
\end{equation}
Under the assumptions on $G_x$ given in Hypothesis 2.1 it is clear that $D_A$ is well-defined a priori on the dense domain
\begin{equation}
(2.11) \quad \mathcal{D}_m := C^0_\infty(\mathbb{R}^3, \mathbb{C}^4) \otimes \mathcal{C}_m. \quad \text{(Algebraic tensor product.)}
\end{equation}
Here, $\mathcal{C}_m \subset \mathcal{F}_b[\mathcal{H}_m]$ denotes the subspace of all $(\psi^{(n)})_{n=0}^{\infty} \in \mathcal{F}_b[\mathcal{H}_m]$ such that only finitely many components $\psi^{(n)}$ are non-zero and such that each $\psi^{(n)}$, $n \in \mathbb{N}$, is essentially bounded with compact support. Moreover, a straightforward
application of Nelson’s commutator theorem with test operator $-\Delta + d\Gamma(\varpi) + 1$ (see, e.g., [18]) reveals that $D_A$ is essentially self-adjoint on $\mathcal{D}_m$, for all $G_x$ fulfilling Hypothesis 2.1. We again use the symbol $D_A$ to denote its closure starting from $\mathcal{D}_m$. Then the spectrum of $D_A$ is contained in the union of two half-lines, $\sigma(D_A) \subset (-\infty, -1] \cup [1, \infty)$, and we denote the orthogonal projections onto the corresponding positive and negative spectral subspaces by

$$(2.12)\quad P^+_A := \mathbb{1}_{\mathbb{R}^+}(D_A) = \frac{1}{2} \mathbb{1} \pm \frac{1}{2} S_A, \quad S_A := D_A | D_A |^{-1}.$$ 

For later reference we recall that the sign function, $S_A$, of $D_A$ can be represented in terms of the resolvent

$$R_A(iy) := (D_A - iy)^{-1}, \quad y \in \mathbb{R},$$

as a strongly convergent principal value [14, Lemma VI.5.6],

$$(2.13)\quad S_A \varphi = \lim_{\tau \to \infty} \int_{-\tau}^{\tau} R_A(iy) \varphi \frac{dy}{\pi} = \lim_{\tau \to \infty} \int_{-\tau}^{\tau} R_A(-iy) \varphi \frac{dy}{\pi}, \quad \varphi \in \mathcal{H}_m.$$ 

The no-pair operator studied in this paper is a self-adjoint operator acting in the positive spectral subspace $P^+_A \mathcal{H}_m$. It is defined a priori on the dense domain $P^+_A \mathcal{D}_m \subset P^+_A \mathcal{H}_m$ by

$$(2.15)\quad H^+_{\gamma, \varpi, G} := P^+_A (D_A - \gamma/|x| + d\Gamma(\varpi)) P^+_A.$$ 

Thanks to [21, Proof of Lemma 3.4(ii)], which implies that $P^+_A$ maps the subspace $\mathcal{D}(D_0) \cap \mathcal{D}(d\Gamma(\varpi))$ into itself, and Hardy’s inequality, we actually know that $H^+_{\gamma, \varpi, G}$ is well-defined on $\mathcal{D}_m$. We recall the definition (1.2) and state our first result which improves on [21, Theorem 2.1], where the semi-boundedness of $H^+_{\gamma, \varpi, G}$ has been shown, for sub-critical values of $\gamma$. Its new proof is independent of [21, Theorem 2.1].

**Proposition 2.3.** Assume that $\varpi$ and $G$ fulfill Hypothesis 2.1. Then the quadratic form of $H^+_{\gamma, \varpi, G}$ is bounded below on $P^+_A(\mathcal{D}(D_0) \cap \mathcal{D}(d\Gamma(\varpi)))$, if and only if $\gamma \leq \gamma_{pc}^\ell$.

**Proof.** This proposition is proved at the end of Section 3. \[ \square \]

In particular, $H^+_{\gamma, \varpi, G}$ has a self-adjoint Friedrichs extension provided that $\gamma \in [0, \gamma_{pc}^\ell]$. In the rest of this section we denote this extension again by the same symbol $H^+_{\gamma, \varpi, G}$. The next theorem gives a bound on the binding energy, i.e. the gap between the ground state energy and the ionization threshold of $H^+_{\gamma, \varpi, G}$ defined, respectively, as

$$(2.16)\quad E_{\gamma}(\varpi, G) := \inf \sigma(H^+_{\gamma, \varpi, G}), \quad \gamma \in (0, \gamma_{pc}^\ell], \quad \Sigma(\varpi, G) := \inf \sigma(H^+_{0, \varpi, G}).$$
**Theorem 2.4 (Binding).** Assume that $\varpi$ and $G$ fulfill Hypothesis 2.7 and that $G_x(k) = e^{-i\mu(k)\cdot x}g(k)$, for all $x \in \mathbb{R}^3$ and almost every $k \in A_m \times \mathbb{Z}_2$, where $\mu, g : A_m \times \mathbb{Z}_2 \to \mathbb{R}^3$ are measurable such that $|\mu| \leq \varpi$ almost everywhere. Let $\gamma \in (0, \gamma_c^{\text{np}})$. Then there is some $c \in (0, \infty)$, depending only on $\gamma$ and the parameter $d$, such that $\Sigma(\varpi, G) - E_\gamma(\varpi, G) \geq c$.

**Proof.** This theorem is proved in Subsection 5.2.

Next, we state the main result of this article dealing with the physical choices $m = 0$, $\varpi = \omega$, and $G_x = G_{x,\Lambda}$ as given in Example 2.2. In this case we abbreviate $H^+_\gamma := H^+_{\gamma,\omega, G_{x,\Lambda}}$ and $E_\gamma := E_\gamma(\omega, G_{x,\Lambda})$.

**Theorem 2.5 (Existence and non-uniqueness of ground states).** For $e \in \mathbb{R}$, $\Lambda > 0$, and $\gamma \in (0, \gamma_c^{\text{np}})$, $E_\gamma$ is an evenly degenerated eigenvalue of $H^+_\gamma$.

**Proof.** The fact that $E_\gamma$ is an eigenvalue is proved in Section 7. In the following we apply Kramers’ theorem to show that $E_\gamma$ is evenly degenerated. Similarly as in [24], where the same observation is made for eigenvalues of the semi-relativistic Pauli-Fierz operator, we introduce the anti-linear operator

$$\vartheta := J\alpha_2 CR = -\alpha_2 JCR, \quad J := \begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix},$$

where $C : \mathcal{H} \to \mathcal{H}$ denotes complex conjugation, $C\psi := \overline{\psi}$, $\psi \in \mathcal{H}$, and $R : \mathcal{H} \to \mathcal{H}$ is the parity transformation $(R\psi)(x) := \psi(-x)$, for almost every $x \in \mathbb{R}^3$ and every $\psi \in \mathcal{H} = L^2(\mathbb{R}_x^3, \mathbb{C}^4 \otimes \mathcal{F}_b[\mathcal{H}_0])$. Obviously, $[\vartheta, -i\partial_x] = [\vartheta, 1/|x|] = [\vartheta, H_0] = 0$, on $\mathcal{D}(D_0) \cap \mathcal{D}(H_0)$. Since $\alpha_2$ squares to one and $C\alpha_2 = -\alpha_2 C$, as all entries of $\alpha_2$ are purely imaginary, we further get $\vartheta^2 = -1$ and $[\vartheta, \alpha_2] = 0$. Moreover, the Dirac matrices $\alpha_0$, $\alpha_1$, and $\alpha_3$ have real entries and $[J\alpha_2, \alpha_j] = J\{\alpha_2, \alpha_j\} = 0$ by (2.9), whence $[\vartheta, \alpha_j] = 0$, for $j \in \{0, 1, 3\}$. Finally, $[\vartheta, e^{+ik\cdot x}] = 0$ implies $[\vartheta, A^{(j)}] = 0$ on $\mathcal{D}(H^{1/2}_t)$, for $j \in \{1, 2, 3\}$. It follows that $[\vartheta, D_A] = 0$ on $\mathcal{D}_0 = \vartheta\mathcal{D}_0$ and, since $D_A$ is essentially self-adjoint on $\mathcal{D}_0$, we obtain $\vartheta D(D_A) = D(D_A)$ and $[\vartheta, D_A] = 0$ on $\mathcal{D}(D_A)$, which implies $\vartheta R_A(iy) - R_A(-iy) \vartheta = 0$ on $\mathcal{H}$, for every $y \in \mathbb{R}$. Using the representation (2.14) we conclude that $[\vartheta, \mathcal{P}_A] = 0$ on $\mathcal{H}$. In particular, $\vartheta$ can be considered as operator acting on $\mathcal{P}_A \mathcal{H}$. Furthermore, we obtain $H^+_\gamma \vartheta = \vartheta H^+_\gamma = 0$ on $\mathcal{D}_0$. Hence, the quadratic forms of $H^+_\gamma$ and $-\vartheta H^+_\gamma \vartheta$ coincide on $\mathcal{D}_0$, which is a form core for $H^+_\gamma$. This readily implies $\vartheta \mathcal{D}(H^+_\gamma) = \mathcal{D}(H^+_\gamma)$ and $[H^+_\gamma, \vartheta] = 0$. On account of $\vartheta^2 = -1$ and the formula

$$\langle \vartheta \varphi | \vartheta \psi \rangle = \langle \psi | \varphi \rangle, \quad \langle \vartheta \varphi | \psi \rangle = -\langle \vartheta \psi | \varphi \rangle, \quad \varphi, \psi \in \mathcal{H},$$

Kramers’ degeneracy theorem now shows that every eigenvalue of $H^+_\gamma$ is evenly degenerated. (In fact, $H^+_\gamma \varphi = E_\gamma \varphi$ implies $H^+_\gamma \vartheta \varphi = E_\gamma \vartheta \varphi$, and $\varphi \perp \vartheta \varphi$ since $\langle \vartheta \varphi | \varphi \rangle = -\langle \vartheta \varphi | \varphi \rangle$ by (2.17).)

\qed
Remark 2.6. Every ground state eigenvector of $H^+_\gamma$ is exponentially localized in the $L^2$-sense with respect to the electron coordinates \cite{21}; see (4.1) below.

3. Relative bounds and essential self-adjointness

The aim of this section is to discuss the domains and essential self-adjointness of the no-pair operators defined by (2.12) and (2.15) and to provide some basic relative bounds. It is actually more convenient from a technical point of view to extend $H^+_{\gamma,\varpi,\mathbf{g}}$ to an operator acting in the full Hilbert space $\mathcal{H}_m$ by adding

$$
H_{\gamma,\varpi,\mathbf{g}}^- := P_{\mathcal{A}} (-D_{\mathcal{A}} - \gamma/|x| + d\Gamma(\varpi)) P_{\mathcal{A}},
$$
defined a priori on $P_{\mathcal{A}} \mathcal{D}_m$. A brief computation shows that

$$
H_{\gamma,\varpi,\mathbf{g}}^{np} := H^+_{\gamma,\varpi,\mathbf{g}} \oplus H^-_{\gamma,\varpi,\mathbf{g}} = \frac{1}{2} H_{\gamma,\varpi,\mathbf{g}}^{PF} + \frac{1}{2} S_{\mathcal{A}} H_{\gamma,\varpi,\mathbf{g}}^{PF} S_{\mathcal{A}} \quad \text{on } \mathcal{D}_m,
$$

where

$$
H_{\gamma,\varpi,\mathbf{g}}^{PF} := |D_{\mathcal{A}}| - \gamma/|x| + d\Gamma(\varpi) \quad \text{on } \mathcal{D}_m,
$$
is the semi-relativistic Pauli-Fierz operator. It turns out that the distinguished self-adjoint realizations of $H^+_{\gamma,\varpi,\mathbf{g}}$ and $H^-_{\gamma,\varpi,\mathbf{g}}$ found later on are unitarily equivalent. In fact, the unitary and symmetric matrix $\tau := \alpha_1 \alpha_2 \alpha_3 \beta$ leaves $\mathcal{D}_m$ invariant and anti-commutes with $D_{\mathcal{A}}$, whence $\tau P_{\mathcal{A}} = P_{\mathcal{A}} \tau$. Consequently, we have

$$
H_{\gamma,\varpi,\mathbf{g}}^- = \tau H^+_{\gamma,\varpi,\mathbf{g}} \tau \quad \text{on } \mathcal{D}_m.
$$

In the rest of this section we always assume that $\varpi$ and $\mathbf{g}$ fulfill Hypothesis \cite{21}. $C(d,a,b,\ldots), C'(d,a,b,\ldots)$, etc. denote positive constants which depend only on the parameter $d$ appearing in Hypothesis \cite{21} and the additional parameters $a,b,\ldots$ displayed in their arguments. Their values might change from one estimate to another.

To start with we collect a number of useful estimates. As a consequence of (2.9) and the $C^*$-equality we have

$$
\|\alpha \cdot v\|_{L^2(\mathbb{C}^4)} = |v|, \quad v \in \mathbb{R}^3, \quad \|\alpha \cdot z\|_{L^2(\mathbb{C}^4)} \leq \sqrt{2}|z|, \quad z \in \mathbb{C}^3,
$$

where $\alpha \cdot z := \alpha_1 z^{(1)} + \alpha_2 z^{(2)} + \alpha_3 z^{(3)}$, for $z = (z^{(1)}, z^{(2)}, z^{(3)}) \in \mathbb{C}^3$. A standard exercise using (2.4), (3.4), the Cauchy-Schwarz inequality, and the canonical commutation relations (2.2), yields

$$
\|\alpha \cdot A \psi\|^2 \leq 6d (\|d\Gamma(\varpi) + 1\|^{1/2} \psi\|^2, \quad \psi \in \mathcal{D}(d\Gamma(\varpi)^{1/2}).
$$

In particular, $\alpha \cdot A$ is a symmetric operator on $\mathcal{D}(d\Gamma(\varpi)^{1/2})$. We also employ the following consequence of \cite{21} Lemma 3.3: For every $\nu \in [0,1]$, $S_{\mathcal{A}}$ maps $\mathcal{D}(d\Gamma(\varpi)^{\nu})$ into itself and

$$
\|(d\Gamma(\varpi) + 1)^\nu S_{\mathcal{A}} (d\Gamma(\varpi) + 1)^{-\nu}\| \leq C(d), \quad \nu \in [0,1].
$$
In Lemma 3.1 we prove that \( \triangle S := S_A - S_0 \) maps \( d\Gamma(\varpi)^{1/2} \) into \( \bigcup_{\kappa < 1} \mathcal{D}(|D_0|^\kappa) \), and

\[
\|(d\Gamma(\varpi) + 1)^\mu |D_0|^\kappa \triangle S (d\Gamma(\varpi) + 1)^\nu\| \leq C(d, \kappa),
\]

for all \( \mu, \nu \in [-1, 1], \mu + \nu \leq -1/2, \) and \( \kappa \in [0, 1) \). (A similar but less general bound has been obtained in \([21]\).) Next, we recall the following strengthened version of the generalized Hardy inequality obtained in \([27]\), for \( \kappa = 1, \) and in \([10]\) in full generality (and arbitrary dimension): Let \( 0 < \varepsilon < \kappa < 3 \) and let \( h_\kappa := 2^\kappa \Gamma([3+\kappa]/4)^2 / \Gamma([3-\kappa]/4)^2 \) denote the sharp constant in the generalized Hardy inequality in three dimensions, so that \( h_1 = 2/\pi \). Then there is some \( C(\kappa, \varepsilon) \in (0, \infty) \) such that

\[
(C(\kappa, \varepsilon)/\ell^{\kappa-\varepsilon})(-\Delta)^{\varepsilon/2} \leq (-\Delta)^{\kappa/2} - h_\kappa |x|^{-\kappa + \ell - \kappa}, \quad \ell > 0.
\]

The well-known corollary, \( h_\kappa |x|^{-\kappa} \leq |D_0|^\kappa \), together with (3.7) yields

\[
\|(d\Gamma(\varpi) + 1)^\mu |D_0|^\kappa \triangle S (d\Gamma(\varpi) + 1)^\nu\| \leq C'(d, \kappa),
\]

for all \( \mu, \nu \in [-1, 1], \mu + \nu \leq -1/2, \) and \( \kappa \in [0, 1) \). Finally, we recall the following special case of \([20]\) Corollary 3.4:

\[
\|(d\Gamma(\varpi) + 1)^{1/2}[S_A, d\Gamma(\varpi)](d\Gamma(\varpi) + 1)^{-1/2}\| \leq C(d).
\]

In view of (3.11) and (3.2) we have

\[
H_{\gamma,\varpi,G}^{\text{hp}} - H_{\gamma,\varpi,0}^{\text{hp}} = X_1 - \frac{\gamma}{2} X_2 + \frac{1}{2} X_3, \quad H_{\gamma,\varpi,G}^{\text{PF}} - H_{\gamma,\varpi,0}^{\text{PF}} = X_1,
\]

where

\[
X_1 := |D_A| - |D_0| = S_A \alpha \cdot A + \triangle S D_0,
\]

\[
X_2 := S_A |x|^{-1} S_A - S_0 |x|^{-1} S_0 = S_A |x|^{-1} \triangle S + \triangle S |x|^{-1} S_0,
\]

\[
X_3 := S_A d\Gamma(\varpi) S_A - S_0 d\Gamma(\varpi) S_0 = S_A [d\Gamma(\varpi), S_A].
\]

Here we used \( d\Gamma(\varpi) = S_0 d\Gamma(\varpi) S_0 \) in the last line. We know that the operator identities (3.11)-(3.13) are valid at least on \( \mathcal{D}(D_0) \cap \mathcal{D}(d\Gamma(\varpi)) \).

**Lemma 3.1.** For all \( \varphi \in \mathcal{D}(D_0) \cap \mathcal{D}(d\Gamma(\varpi)), \varepsilon \in (0, 1], \delta > 0, \) and \( j = 1, 2, 3, \)

\[
|\langle \varphi, X_j \varphi \rangle| \leq \delta \langle \varphi, (|D_0|^\varepsilon + d\Gamma(\varpi)) \varphi \rangle + C(d, \delta, \varepsilon) \|\varphi\|^2,
\]

\[
\|X_j \varphi\| \leq \delta \|D_0|^\varepsilon \varphi\| + \delta \|d\Gamma(\varpi) \varphi\| + C'(d, \delta, \varepsilon) \|\varphi\|.
\]

**Proof.** We pick some \( \varphi \in \mathcal{D}(D_0) \cap \mathcal{D}(d\Gamma(\varpi)) \) and put \( \Theta := d\Gamma(\varpi) + 1 \). On account of (3.5) and (3.7) we have \( |\langle \varphi, X_1 \varphi \rangle| \leq C(d, \varepsilon) \|D_0|^{\varepsilon/4} \varphi\| \|\Theta^{1/2} \varphi\| \) and \( \|X_1 \varphi\| \leq C'(d, \varepsilon) \|D_0|^{\varepsilon/4} \otimes \Theta^{1/2} \varphi\| \). By the inequality between the weighted
geometric and arithmetic means we further have, for \( \nu \in [0, 1] \), \( \varepsilon \in (0, 1] \), and \( \delta > 0 \),
\[
\left\| D_0^{\varepsilon/4} \otimes \Theta^{\nu/2} \varphi \right\|^2 \leq \left\| D_0^{\varepsilon/2} \varphi \right\| \left\| \Theta^\nu \varphi \right\| \leq \left\| \varphi \right\|^2 \left\| D_0^{\varepsilon/4} \right\|^2 \left\| \Theta^\nu \varphi \right\|
\]
(3.17)
\[
\leq \left\langle \varphi \right| \left( \delta^2 \left\| D_0 \right\|^2 + \delta^2 \Theta^{2\nu} + (2\delta)^{-6} \right) \varphi \right\rangle,
\]
which yields (3.15) & (3.16), for \( j = 1 \) (and with new \( \delta \) and \( \varepsilon \)).

Since \( |D_0| \Delta S = D_A - D_0 + (|D_0| - |D_A|) S_A = \alpha \cdot A - X_1 S_A \) we obtain, using (3.5), (3.6), and (3.16) with \( j = 1 \),
\[
\left\| D_0 | \Delta S \varphi \right\| \leq \delta \left\| D_0 \right\|^2 \left\| \Delta S \varphi \right\| + \delta (1 + C(d)) \left\| d\Gamma(\varphi) \right\| + C' \left\| \varphi \right\|
\]
\[
\leq \delta \left\| D_0^{\varepsilon} \Delta S \varphi \right\| + \delta \left\| D_0^{\varepsilon} \varphi \right\| + \mathcal{O}(\delta) \left\| d\Gamma(\varphi) \right\| + C' \left\| \varphi \right\|
\]

where \( C' \equiv C'(d, \delta, \varepsilon) \). Choosing \( \varepsilon \leq 1/2 \) we further observe that, for all \( \rho > 0 \),
(3.18)
\[
\left\| D_0^{\varepsilon} \Delta S \varphi \right\| = \left\langle \Delta S \varphi \right| \left\| D_0^{2\varepsilon} \Delta S \varphi \right\|^{1/2} \leq \rho \left\| D_0^{2\varepsilon} \Delta S \varphi \right\| + C(\rho) \left\| \varphi \right\|
\]
Assuming \( \delta \leq 1 \), \( \rho \leq 1/2 \), and combining the previous two estimates we obtain
\[
2^{-1}\left\| \varphi \right\|^2 \leq \left\| D_0 \right\|^2 \left\| \Delta S \varphi \right\| \leq 2\delta \left\| D_0^{\varepsilon} \varphi \right\| + \mathcal{O}(\delta) \left\| d\Gamma(\varphi) \right\| + C' \left\| \varphi \right\|
\]
Moreover, (3.9) yields \( \left\| \Delta S \right\| \left\| \varphi \right\| \leq C(\delta, \varepsilon) \left\| D_0^{\varepsilon/4} \otimes \Theta^{1/2} \varphi \right\| \), for every \( \varepsilon > 0 \), and we readily obtain (3.10) with \( j = 2 \). To prove (3.13) with \( j = 2 \), we estimate
(3.19)
\[
\left\langle \varphi \right| S_A \left| \varphi \right\| \right\| D_0^{\varepsilon/4} S_A \varphi \right\| \left\| \Theta^{1/2} \varphi \right\| ,
\]
where \( \tilde{A} \) ist 0 or \( A \). If \( \tilde{A} = 0 \), then we use (3.17) to estimate the RHS in (3.19) from above by the RHS in (3.15). In the case \( \tilde{A} = A \) we further estimate
\[
\left\| D_0^{\varepsilon/4} S_A \varphi \right\| \leq \left\| D_0^{\varepsilon/4} \Delta S \varphi \right\| + \left\| D_0^{\varepsilon/4} \varphi \right\| \) and use (3.17) once again, as well as the following consequence of (3.7) and (3.18),
\[
C(d, \varepsilon) \left\| D_0^{\varepsilon/4} \Delta S \varphi \right\| \left\| \Theta^{1/2} \varphi \right\|
\]
\[
\leq \rho \left\| D_0^{\varepsilon/2} \Delta S \varphi \right\| \left\| \Theta^{1/2} \varphi \right\| + C(d, \varepsilon, \rho) \left\| \varphi \right\| \left\| \Theta^{1/2} \varphi \right\|
\]
\[
\leq \rho C(d, \varepsilon) \left\| \Theta^{1/2} \varphi \right\|^2 + C(d, \varepsilon, \rho) \left\| \varphi \right\| \left\| \Theta^{1/2} \varphi \right\|
\]
\[
\leq \delta \left\langle \varphi \right| \Theta \varphi \right\rangle + C'(d, \delta, \varepsilon) \left\| \varphi \right\|^2,
\]
where \( \rho > 0 \) is chosen such that \( \rho C(d, \varepsilon) = \delta/2 \).

For \( j = 3 \), (3.15) & (3.16) are simple consequences of (3.10) and (3.14). \( \Box \)

In the next theorem we write
\[
\gamma_c^{\text{FP}} := 2/(2/\pi + \pi/2), \quad \gamma_c^{\text{PF}} := 2/\pi,
\]
and we shall first use the full strength of (3.8). We shall also employ its analogue for the Brown-Ravenhall operator acting in \( L^2(\mathbb{R}^3, C^4) \).

(3.20)
\[
B_\gamma := |D_0| - (\gamma/2)|x|^{-1} - (\gamma/2) S_0 |x|^{-1} S_0, \quad \gamma \in [0, \gamma_c^{\text{FP}}].
\]
$B^\text{el}_\gamma$ is defined by means of a Friedrichs extension starting from $C^\infty_0 (\mathbb{R}^3, \mathbb{C}^4)$ and it is known that $B^\text{el}_\gamma \geq 1 - \gamma > 0$, for $\gamma \in [0, \gamma^\text{el}_c]$ [9, 28]. The analogue of (3.8) for $B^\text{el}_\gamma$ is proven in [10] in the massless case and can be written as

\[(\Delta)^{\frac{\epsilon}{2}} - (\gamma^\text{np}_c/2)|\xi|^{-1} - (\gamma^\text{np}_c/2) S_0^{(0)} |x|^{-1} S_0^{(0)} \geq (C(\epsilon)/\ell^{1-\epsilon}) (\Delta)^{\frac{\epsilon}{2}} - \ell^{-1},\]

for $\epsilon \in (0, 1)$ and $\ell > 0$, where $S_0^{(0)}$ acts in Fourier space by multiplication with $\alpha \cdot \xi / |\xi|$. Since the symbol of $S_0$ is $(\alpha \cdot \xi + \beta) / |\xi|$, we have $|(S_0^{(0)} - S_0)^\wedge (\xi)| = (\xi)^{-1} F(\xi) \hat{\psi}(\xi)$, where $\|F(\xi)\| \leq 2$. Hence, $\| |x|^{-1}(S_0^{(0)} - S_0)\| \leq 4$ by Hardy’s inequality. Of course, $(\Delta)^{1/2} \leq |D_0|$, and we conclude that, for $\epsilon \in (0, 1)$,

\[(3.21) \quad (\Delta)^{\epsilon/2} \leq (\ell^{1-\epsilon}/C(\epsilon)) B^\text{el}_{\gamma^\text{np}} + (4\gamma^\text{np}_c \ell^{1-\epsilon} + \ell^{-\epsilon})/C(\epsilon), \quad \ell > 0.

In particular, $\mathcal{D}(B^\text{el}_\gamma) \subset \mathcal{Q}(B^\text{el}_\gamma) \subset \bigcap_{\ell < 1} \mathcal{Q}(|D_0|^{\epsilon})$.

**Theorem 3.2.** Let $\sharp \in \{\text{np, PF}\}$ and $\gamma \in [0, \gamma^\text{el}_c]$. Then $H^2_{\gamma, \infty, G}$ is infinitesimally form bounded on $\mathcal{D}_m$ with respect to $H^2_{\gamma, \infty, 0}$. More precisely, for all $\delta > 0$ and $\epsilon \in (0, 1)$, we have, in the sense of quadratic forms on $\mathcal{D}(D_0) \cap \mathcal{D}(d\Gamma(\varpi))$,

\[(3.22) \quad \pm (H^2_{\gamma, \infty, G} - H^2_{\gamma, \infty, 0}) \leq \delta |D_0|^{\epsilon} + \delta d\Gamma(\varpi) + C(d, \delta, \epsilon),
\]

\[(3.23) \quad \pm (H^2_{\gamma, \infty, G} - H^2_{\gamma, \infty, 0}) \leq \delta H^2_{\gamma, \infty, 0} + C(d, \delta),
\]

\[(3.24) \quad (\Delta)^{\epsilon/2} + \delta d\Gamma(\varpi) \leq 2\delta H^2_{\gamma, \infty, G} + C'(d, \delta, \epsilon),
\]

\[(3.25) \quad |D_\Lambda|^{\epsilon} \leq \delta H^2_{\gamma, \infty, G} + C''(d, \delta, \epsilon).
\]

Hence, by the KLMN theorem $H^2_{\gamma, \infty, G}$ has a distinguished self-adjoint extension – henceforth again denoted by the same symbol – such that $\mathcal{D}(H^2_{\gamma, \infty, G}) \subset \mathcal{Q}(H^2_{\gamma, \infty, 0})$. Furthermore, $\mathcal{Q}(H^2_{\gamma, \infty, G}) = \mathcal{Q}(H^2_{\gamma, \infty, 0})$. If $\gamma < \gamma^\text{el}_c$, then we have $\mathcal{Q}(H^2_{\gamma, \infty, G}) = \mathcal{Q}(H^2_{0, \infty, G}) = \mathcal{Q}(|D_0|^{\epsilon}) \cap \mathcal{Q}(d\Gamma(\varpi))$. In the critical case we have $\mathcal{Q}(H^2_{0, \infty, G}) \subset \mathcal{Q}(H^2_{\gamma^\text{el}, \infty, G}) \subset \bigcap_{\ell < 1} \mathcal{Q}(|D_0|^{\epsilon}) \cap \mathcal{Q}(d\Gamma(\varpi))$.

**Proof.** The form bounds (3.22)–(3.24) are consequences of (3.5), (3.13), and (3.21). (3.25) follows from (3.24) and (A.16) below.

If $\gamma < \gamma^\text{el}_c$ is sub-critical, we have $B^\text{el}_\gamma \geq (1 - \gamma/\gamma^\text{np}_c)|D_0|$ and $|D_0| - \gamma/|x| \geq (1 - \gamma/\gamma^\text{PF}_c)|D_0|$, respectively, whence $(1 - \gamma/\gamma^\text{np}_c) H^2_{0, \infty, 0} \leq H^2_{\gamma^\text{el}, \infty, 0} \leq H^2_{0, \infty, 0}$ on $\mathcal{Q}(|D_0|) \cap \mathcal{Q}(d\Gamma(\varpi))$, where $H^2_{0, \infty, 0} = |D_0| + d\Gamma(\varpi)$. In the critical case we only have $|D_0|^{\epsilon} + d\Gamma(\varpi) \leq H^2_{\gamma^\text{el}, \infty, 0} + C(\epsilon)$, for every $\epsilon \in (0, 1)$, as a lower bound. \(\square\)

**Theorem 3.3.** For $\sharp \in \{\text{np, PF}\}$ and $\gamma \in [0, \gamma^\text{el}_c]$, the following holds true:

(i) $H^2_{\gamma, \infty, G}$ and $H^2_{\gamma, \infty, 0}$ have the same domain and their operator cores coincide.
(ii) For all $\delta, \varepsilon > 0$ and $\varphi \in \mathcal{D}(H_{\gamma,\varphi,0}^1)$,

$$(3.26) \quad \|(H_{\gamma,\varphi,G}^2 - H_{\gamma,\varphi,0}^2) \varphi\| \leq \delta \|D_0\| \|\varphi\| + \delta \|d\Gamma(\varphi)\| + C(d, \delta, \varepsilon) \|\varphi\|,$$

$$(3.27) \quad \|(H_{\gamma,\varphi,G}^2 - H_{\gamma,\varphi,0}^2) \varphi\| \leq \delta \|H_{\gamma,\varphi,0}\| \varphi\| + C(d, \delta) \|\varphi\|.$$

(iii) $\mathcal{D}(H_{\gamma,\varphi,G}^2) \subset \mathcal{D}(d\Gamma(\varphi))$ and, for all $\delta > 0$ and $\varphi \in \mathcal{D}(H_{\gamma,\varphi,G}^2)$,

$$(3.28) \quad \|d\Gamma(\varphi)\| \leq \|H_{\gamma,\varphi,0}\| \varphi\| \leq (1 + \delta) \|H_{\gamma,\varphi,G}^2 \varphi\| + C(d, \delta) \|\varphi\|.$$

Proof. For $\varphi \in \mathcal{X} := \mathcal{D}(D_0) \cap \mathcal{D}(d\Gamma(\varphi))$, the bound (3.26) follows immediately from (3.11)–(3.14), and Lemma 3.1. We define $T := H_{\gamma,\varphi,G}^2 - H_{\gamma,\varphi,0}^2$ on the domain $\mathcal{D}(T) := \mathcal{X}$. We also fix some $\varepsilon \in (0, 1/2)$ in what follows. As a symmetric operator $T$ is closable. We deduce that $\mathcal{D}(\tilde{T}) \supset \mathcal{Y} := \mathcal{D}([D_0]^{\varphi}) \cap \mathcal{D}(d\Gamma(\varphi))$ and

$$(3.29) \quad \|\tilde{T} \varphi\| \leq \delta \|D_0\| \|\varphi\| + \delta \|d\Gamma(\varphi)\| + C(d, \delta, \varepsilon) \|\varphi\|, \quad \varphi \in \mathcal{Y}.$$

As a next step we estimate the RHS of (3.29) from above in the case $\sharp = \text{np}$. For $\varepsilon \in (0, 1/2)$ and an appropriate choice of $\ell > 0$ in (3.21), we obtain

$$(3.30) \quad \|D_0\| \varphi\|^2 \leq \langle \varphi, (B_{\ell}^{\text{el}} + C(\varepsilon)) \varphi \rangle \leq \langle \varphi, H_{\gamma,\varphi,0}^{\text{np}} \varphi \rangle + C(\varepsilon) \|\varphi\|^2,$$

for $\varphi \in \mathcal{Z} := \mathcal{D}(B_{\ell}^{\text{el}}) \cap \mathcal{D}(d\Gamma(\varphi)) \subset \mathcal{Y}$. In view of $B_{\ell}^{\text{el}} \otimes d\Gamma(\varphi) \geq 0$ and $d\Gamma(\varphi) = S_0 d\Gamma(\varphi) S_0$ we may further estimate

$$(3.31) \quad \|d\Gamma(\varphi)\|^2 \leq \|(B_{\ell}^{\text{el}} + d\Gamma(\varphi))/2 + S_0 d\Gamma(\varphi) S_0/2 \varphi\|^2 = \|H_{\gamma,\varphi,0}^{\text{np}} \varphi\|^2,$$

for $\varphi \in \mathcal{Z}$. Combining (3.29)–(3.31) we obtain

$$(3.32) \quad \|\tilde{T} \varphi\| \leq \delta \|H_{\gamma,\varphi,0}^{\text{np}} \varphi\| + C(d, \delta) \|\varphi\|, \quad \varphi \in \mathcal{Z},$$

where $\mathcal{Z}$ is an operator core for $H_{\gamma,\varphi,0}^{\text{np}}$. According to the Kato-Rellich theorem $\tilde{H}_{\gamma,\varphi}^{\text{np},G} := H_{\gamma,\varphi,0}^{\text{np}} + \tilde{T}$ is self-adjoint on $\mathcal{D}(H_{\gamma,\varphi,0}^{\text{np}}) \subset \mathcal{Q}(H_{\gamma,\varphi,0}^{\text{np}})$ and the operator cores of $\tilde{H}_{\gamma,\varphi}^{\text{np},G}$ and $H_{\gamma,\varphi,0}^{\text{np}}$ coincide. Furthermore, $\tilde{H}_{\gamma,\varphi}^{\text{np},G}$ and $H_{\gamma,\varphi,0}^{\text{np}}$ coincide on $\mathcal{D}_m$ and we know from Theorem 3.2 that $H_{\gamma,\varphi,0}^{\text{np},G}$ is uniquely determined by the property $\mathcal{D}(H_{\gamma,\varphi,0}^{\text{np},G}) \subset \mathcal{Q}(H_{\gamma,\varphi,0}^{\text{np}})$.

Therefore, $\tilde{H}_{\gamma,\varphi}^{\text{np},G} = H_{\gamma,\varphi,0}^{\text{np}}$ which proves (i)–(iii), for $\sharp = \text{np}$. In the case $\sharp = \text{PF}$ we use (3.8) instead of (3.21) and put $\mathcal{Z} := \mathcal{D}([D_0] - \gamma/|\varphi|) \cap \mathcal{D}(d\Gamma(\varphi))$. □

**Corollary 3.4.** (i) For $\gamma \in [0, \gamma_{\text{np}}]$, the algebraic tensor product $\mathcal{D}(B_{\ell}^{\text{el}}) \otimes \mathcal{C}_m$ is an operator core of $H_{\gamma,\varphi,0}^{\text{np},G}$. ($\mathcal{C}_m$ has been defined below (2.11).)

(ii) For $\gamma \in [0, \gamma_{\text{PF}}]$, the algebraic tensor product $\mathcal{D}([D_0] - \gamma/|\varphi|) \otimes \mathcal{C}_m$ is an operator core of $H_{\gamma,\varphi,0}^{\text{PF},G}$.

(iii) For $\gamma \in [0, 1/2)$, $H_{\gamma,\varphi,0}^{\text{np},G}$ and $H_{\gamma,\varphi,0}^{\text{PF},G}$ are essentially self-adjoint on $\mathcal{D}_m$. 

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Proof. The domains appearing in (i) (resp. (ii)) are operator cores of $H^2_{\gamma, w, 0}$ and, hence, of $H^2_{\gamma, w, G}$ by Theorem 3.3. By Hardy’s inequality, $\| x^{-1} \varphi \| \leq 2 \| S_0 | x |^{-1} S_0 \varphi \| \leq 2 \| D_0 \| \varphi \|$, whence $\| (H^2_{\gamma, w, 0} - H^{\text{PF}}_{0, w, 0}) \varphi \| \leq 2 \gamma \| H^{\text{PF}}_{0, w, 0} \varphi \|$, for all $\varphi \in D(D_0) \cap D(d\Gamma(\varphi))$. Since $H^{\text{PF}}_{0, w, 0}$ is essentially self-adjoint on $D_n$ the same holds true for $H^2_{\gamma, w, 0}$ by the Kato-Rellich theorem, provided that $\gamma < 1/2$. Hence, (iii) follows from Theorem 3.3(i), too. □

Proof of Proposition 2.3. The semi-boundedness in the case $\gamma \leq \gamma_c^\text{sep}$ follows from Theorem 3.2. Let $\gamma > \gamma_c^\text{sep}$ and pick some $\gamma \in (\gamma_c^\text{sep}, \gamma)$. Due to (9) we find normalized $\psi_n \in D(D_0)$, $n \in \mathbb{N}$, such that $\langle \psi_n | B^\text{el}_\gamma \psi_n \rangle \to -\infty$, as $n \to \infty$, where $B^\text{el}_\gamma$ now denotes the expression on the RHS of (3.1) with domain $D(D_0)$. Let $\Omega := (1, 0, 0, \ldots)$ denote the vacuum vector in $\mathcal{F}_b[X_m]$ and set $\Psi_n := \psi_n \otimes \Omega$, so that $\| \Psi_n \| = 1$ and $d\Gamma(\varphi) \Psi_n = 0$. On account of (3.15) we obtain

$$\langle \psi_n | H^\text{np}_{\gamma, w, G} \Psi_n \rangle \leq \langle \psi_n | (B^\text{el}_\gamma + d\Gamma(\varphi)) \Psi_n \rangle + \delta \langle \psi_n | (| D_0 |^{1/2} + d\Gamma(\varphi)) \Psi_n \rangle + C(d, \delta) \leq \tilde{\gamma} \langle \psi_n | B^\text{el}_\gamma \psi_n \rangle - \left( \frac{\tilde{\gamma}}{\gamma} - 1 - \delta \right) \langle \psi_n | D_0 \psi_n \rangle + C'(d, \delta).$$

Choosing $\delta := \tilde{\gamma}/\gamma - 1 > 0$ we see that $\langle \psi_n | H^\text{np}_{\gamma, w, G} \Psi_n \rangle \to -\infty$, $n \to \infty$. In view of (3.1) this implies that $\langle P^+_A \Psi_n | H^+_\gamma \Psi_n \rangle \to -\infty$ or $\langle P^-_A \Psi_n | H^-_{\gamma, w, G} \Psi_n \rangle \to -\infty$. If the latter divergence holds true, then $\langle P^+_A \tau \Psi_n | H^+_\gamma \Psi_n \rangle \to -\infty$ by (3.3) which concludes the proof. □

4. Convergence of no-pair operators

The following localization estimate [21] plays an essential role in the sequel:

**Proposition 4.1 (Exponential localization).** There exists $k \in (0, \infty)$ and, for all $w$ and $G$ fulfilling Hypothesis 2.1 and all $\gamma \in (0, \gamma_c^\text{sep})$, we find some $C \equiv C(\gamma, d) \in (0, \infty)$ such that the following holds true: Let $\lambda < \Sigma := \inf \sigma[\mathcal{H}_{0, w, G}]$ and let $a > 0$ satisfy $a \leq k(\gamma_c - \gamma)/(\gamma_c + \gamma)$ and $\varepsilon := 1 - \frac{\lambda + C}{\Sigma + a} k a^2 > 0$. Then $\text{Ran}(\mathcal{I}_\lambda(H^\text{np}_{\gamma, w, G})) \subset D(e^{a|x|})$ and

$$\| e^{a|x| \mathcal{I}_\lambda(H^\text{np}_{\gamma, w, G})} \| \leq (k/\varepsilon^2)(\Sigma + C) e^{k a(\Sigma + C)/\varepsilon}.$$

**Proof.** If $H^\text{np}_{\gamma, w, G}$ is replaced by $H^+_{\gamma, w, G}$, then the assertion follows from [21] Theorem 2.2. In view of (3.1) and (3.3) it is, however, clear that the same estimate holds also for $H^\text{np}_{\gamma, w, G}$. □

**Remark 4.2.** To apply Proposition 4.1 later on we note that, in view of (3.23), $\Sigma = \inf \sigma[\mathcal{H}_{0, w, G}] \leq C'(d) < \infty$, where $C'(d)$ depends only on $d$. 

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In the next proposition we assume that \( \varpi \) and \( G \) fulfill Hypothesis 2.1 with parameter \( d \) and that, for every \( n \in \mathbb{N} \), \( \varpi_n \) and \( G_n \) fulfill Hypothesis 2.1 with the same parameter \( d \) such that

\[
\forall a > 0 : \quad \Delta_n(a) := \int \left(1 + \frac{1}{\varpi(k)}\right) \sup_x e^{-2a|x|} |G_{n,x}(k) - G_x(k)|^2 \, dk \xrightarrow{n \to \infty} 0.
\]

Furthermore, we assume that \( |\varpi - \varpi_n| \leq \kappa_n \varpi \), for some \( \kappa_n \geq 0 \), \( \kappa_n \searrow 0 \). To simplify the notation we put

\[
H := H_{\gamma,\varpi,G}^n, \quad H_n := H_{\gamma,\varpi,G_n}^n, \quad E := \inf \sigma[H], \quad E_n := \inf \sigma[H_n],
\]

\[
\Sigma := \inf \sigma[H_{0,\varpi,G}^n], \quad \Sigma_n := \inf \sigma[H_{0,\varpi,G_n}^n],
\]

and, for some \( z \in \mathbb{C} \setminus \mathbb{R} \),

\[
\mathcal{R}(z) := (H - z)^{-1}, \quad \mathcal{R}_n(z) := (H_n - z)^{-1}.
\]

**Proposition 4.3.** For \( \gamma \in (0, \gamma_c^\text{np}) \) and under the above assumptions, the following holds:

1. Let \( \lambda < \Sigma \) and \( f \in C_0^\infty(\mathbb{R}) \). Then
   \[
   \lim_{n \to \infty} \|\{f(H_n) - f(H)\} 1_\lambda(H)\| = 0.
   \]

2. Let \( \lambda < \Sigma \) and \( \mu > \lambda \). Then we find some \( N \in \mathbb{N} \) such that, for all \( n \geq N \),
   \[
   \dim \text{Ran}(1_\lambda(H)) \leq \dim \text{Ran}(1_\mu(H_n)).
   \]

3. \( \overline{E} := \liminf E_n \leq E \).

   If, in addition, there is some \( c > 0 \) such that \( \Sigma_n - E_n \geq c \), for all \( n \in \mathbb{N} \), then the following holds true also:

4. \( \overline{E} := \liminf E_n \geq E \), thus \( \lim_{n \to \infty} E_n = E \).

5. Let \( \phi_n \in \text{Ran}(1_{E_n + 1/n}(H_n)) \), \( n \in \mathbb{N} \), be normalized and let \( \phi \in \mathcal{H}_m \) denote a weak limit of some subsequence of \( \{\phi_n\} \). If \( \phi \neq 0 \), then \( \phi \) is a ground state eigenvector of \( H \).

**Proof.** (1): Let \( z \in \mathbb{C} \setminus \mathbb{R} \), \( \varphi, \psi \in \mathcal{H} \), \( \varphi_n, z := \mathcal{R}_n(z) \varphi, \psi_n := \mathcal{R}(z) \psi \), and \( \delta S_n := S_{\Lambda} - S_{\Lambda_n} \), where \( \Lambda \equiv \Lambda[G] \) and \( \Lambda_n \equiv \Lambda[G_n] \) as defined in (2.3). Theorem 3.3(i) and the bound \((1 - \kappa_n) \varpi \leq \varpi_n \leq (1 + \kappa_n) \varpi \) imply that \( H \) and \( H_n \) have the same domain and the latter is contained in \( Q(|D_0|) \cap Q(d\Gamma(\varpi)) \), if \( \kappa_n < 1 \). For large \( n \), we thus have

\[
2 \left\langle \varphi \left| (\mathcal{R}_n(z) - \mathcal{R}(z)) \psi \right\rangle = 2 \left\langle \varphi_n, z \left| (|D_{\Lambda_n} - |D_{\Lambda_n}|) \psi \right\rangle + \left( \varphi_n, z \left| (d\Gamma(\varpi) - \varpi_n) + S_{\Lambda_n} d\Gamma(\varpi_n) \right) \psi \right\rangle \right.
\]

\[
+ \left( \varphi_n, z \left| \delta S_n (\gamma - |x|) + S_{\Lambda_n} \right) \psi \right\rangle
\]

\[
+ \left( \varphi_n, z \left| S_{\Lambda_n} (\gamma - |x|) + d\Gamma(\varpi_n) \right) \delta S_n \psi \right\rangle.
\]

(4.2)
We fix some $\kappa \in (3/4, 1)$ and set $\varepsilon := 1 - \kappa \in (0, 1/4)$, $\Theta := d\Gamma(\varpi) + 1$, $\Theta_n := d\Gamma(\varpi_n) + 1$, and $\Pi := 1_\lambda(H)$. In the sequel we always assume that $\psi = \Pi \psi$ and that $n$ is so large that $\varkappa_n \leq 1/2$, so that $\varpi \leq 2\varpi_n$ and, hence, $\Theta \leq 2\Theta_n$ and $\Theta_n \leq 2\Theta$. On account of Proposition 4.1 we further find some $a > 0$ such that $\|e^{a|x|/\varepsilon} \Pi\| \leq C(d, a, \varepsilon, \lambda)$. Analogously to (3.5) we then have
\begin{equation}
\| \alpha \cdot (A - A_n) \Theta^{-1/2} e^{-a|x|} \| \leq 6^{1/2} \Delta_n^{1/2}(a),
\end{equation}
and Lemma A.1 below implies the following bounds,
\begin{equation}
\| O^\kappa \Theta^\mu \delta S_n \Theta^\rho e^{-a|x|} \| + \| |x|^{-\kappa} \Theta^z \delta S_n \Theta^{-\kappa} e^{-a|x|} \| \leq C(d, \kappa) \Delta_n^{1/2}(a),
\end{equation}
for $O \in \{ |D_A|, |D_{A_n}| \}$ and $\mu, \nu \in [-1, 1]$ with $\mu + \nu \leq -1/2$ and $\mu \land \nu \leq -1/2$. Lemma A.1 further implies
\begin{equation}
\| |x|^{-\kappa} e^{-a|x|} \delta S_n \Theta^{-1} \| \leq C'(d, \kappa) \Delta_n^{1/2}(a).
\end{equation}
Using also $[\Pi, \mathcal{R}(z)] = [\Pi, S_A] = [S_A, \mathcal{R}(z)] = [S_{A_n}, \mathcal{R}_n(z)] = 0$, and $\|S_A\| = \|S_{A_n}\| = 1$, we may estimate the first term on the RHS of (4.2) as
\begin{align*}
\langle \varphi_{n,z} \mid (|D_A| - |D_{A_n}|) \psi_z \rangle & \leq \langle \alpha \cdot (A - A_n) \varphi_{n,z} \mid S_A \psi_z \rangle + \langle S_{A_n} |D_{A_n}|^\varepsilon \varphi_{n,z} \mid |D_{A_n}| \delta S_n \psi_z \rangle \\
& \leq 6^{1/2} \Delta_n^{1/2}(a) \| \mathcal{R}_n(z) \| \| \varphi \| \| e^{a|x|} \Theta^{1/2} \Pi \| \| \mathcal{R}(z) \| \| \psi \| \\
& + \| |D_{A_n}|^\varepsilon \mathcal{R}_n(z) \| \| \varphi \| \| |D_{A_n}| \delta S_n \Theta^{-1} \| \| e^{a|x|} \Theta^{1/2} \Pi \| \| \mathcal{R}(z) \| \| \psi \| .
\end{align*}
In view of $[\mathcal{R}_n(z), S_{A_n}] = 0$ the second term on the RHS of (4.2) can be estimated as
\begin{align*}
\langle \varphi_{n,z} \mid (d\Gamma(\varpi - \varpi_n) + S_{A_n} d\Gamma(\varpi - \varpi_n) S_A) \psi_z \rangle & \leq 2 \| d\Gamma(\varpi - \varpi_n) \mathcal{R}_n(z) \| \| \varphi \| \| \psi \| \leq 4 \varkappa_n \| \Theta_n \mathcal{R}_n(z) \| \| \varphi \| \| \mathcal{R}(z) \| \| \psi \| .
\end{align*}
Likewise, we obtain for the third term on the RHS of (4.2)
\begin{align*}
\langle \varphi_{n,z} \mid \delta S_n (-\gamma/|x| + d\Gamma(\varpi)) S_A \psi_z \rangle & \leq 2 \| e^{-a|x|} |x|^{-\kappa} \delta S_n \Theta^{-1} \| \| \Theta_n \mathcal{R}_n(z) \| \| \mathcal{R}(z) \| \| \psi \| \\
& + 2 \| e^{-a|x|} d\Gamma(\varpi)^{1/2} \delta S_n \Theta^{-1} \| \| \Theta_n \mathcal{R}_n(z) \| \| \mathcal{R}(z) \| \| e^{a|x|} \Theta^{1/2} \Pi \| \| \mathcal{R}(z) \| \| \psi \| ,
\end{align*}
where $\| |x|^{-\kappa} e^{a|x|} \Pi \|^2 \leq C(\varepsilon) \| D_0 \|^{2\varepsilon} \Pi \| e^{a|x|/\varepsilon} \Pi \|$, since $1/\varepsilon > 2$, and the norm of $e^{-a|x|} d\Gamma(\varpi)^{1/2} \delta S_n \Theta^{-1} = \{ \Theta^{-1} \delta S_n e^{-a|x|} d\Gamma(\varpi)^{1/2} \}$ is bounded according to (4.4). Finally, we treat the fourth term on the RHS of (4.2),
\begin{align*}
\langle \varphi_{n,z} \mid S_{A_n} (-\gamma/|x| + d\Gamma(\varpi_n)) \delta S_n \psi_z \rangle & \leq \| |x|^{-\kappa} \Theta^z \mathcal{R}_n(z) \| \| \mathcal{R}(z) \| \| \varphi \| \| |x|^{-\kappa} \Theta^{-\kappa} \delta S_n \Theta^{-\kappa} e^{-a|x|} \| \| e^{a|x|} \Theta^z \Pi \| \| \mathcal{R}(z) \| \| \psi \| \\
& + \| \Theta_n \mathcal{R}_n(z) \| \| \mathcal{R}(z) \| \| \delta S_n \Theta^{-1/2} e^{-a|x|} \| \| e^{a|x|} \Theta^{1/2} \Pi \| \| \mathcal{R}(z) \| \| \psi \| ,
\end{align*}
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where \( \| x \| \Theta \mathcal{R}_n(z) \|^2 \leq C(\varepsilon) \| D_0^{2\varepsilon} \mathcal{R}_n(z) \| \| \Theta_n \mathcal{R}_n(z) \| \) since \( 2\varepsilon < 1/2 \).

On account of (3.24), (3.25), (3.28), and \( 2\varepsilon < 1/2 \),
\[
\sup_{n \in \mathbb{N}} \| \mathcal{O} \mathcal{R}_n(z) \| \leq \frac{C(d, \varepsilon)}{1 - \text{Im } z}, \quad \text{for } \mathcal{O} \in \{ |D_A|^\varepsilon, |D_0|^{2\varepsilon}, \Theta_n \},
\]
where \( a \wedge b := \min\{a, b\} \). By virtue of (3.28) and (4.1) we further have
\[
\| e^{a|z|} \Theta^{1/2} \Pi \| \leq \| e^{a|z|} \Theta^\varepsilon \Pi \| \leq \| e^{a|z|/\varepsilon} \Pi \| \| \Theta \Pi \|^\varepsilon \leq C(d, \kappa, \lambda),
\]
and (3.24) implies \( \| D_0^{2\varepsilon} \Pi \| \leq C'(d, \kappa, \lambda) \). Combining all the above estimates with (4.1) and (4.5) we arrive at
\[
(4.6) \quad \| (\mathcal{R}_n(z) - \mathcal{R}(z)) \Pi \| = \sup_{\| \varphi \| = \| \psi \| = 1} | \langle \varphi, (\mathcal{R}_n(z) - \mathcal{R}(z)) \Pi \psi \rangle | \leq \frac{b(n)}{1 - \text{Im } z^2};
\]
where \( b(n) = O(\Delta_n^{1/2}(a)) \rightarrow 0, n \rightarrow \infty \). Now, Part (1) follows from (4.6) and the Helffer-Sjöstrand formula (see, e.g., [8]),
\[
(4.7) \quad f(T) = \frac{1}{2\pi i} \int_{\mathbb{C}} (T - z)^{-1} \partial_z \tilde{f}(z) \, dz \wedge d\bar{z},
\]
valid for every self-adjoint operator \( T \) on some Hilbert space. Here \( \tilde{f} \in C^\infty_0(\mathbb{C}) \)

is a compactly supported, almost analytic extension of \( f \) such that \( \tilde{f}|_{\mathbb{R}} = f \)

and \( |\partial_z \tilde{f}(z)| = O(\|\text{Im } z\|^N) \), for every \( N \in \mathbb{N} \).

(2): It suffices to show that
\[
\left\| \left\{ 1_\mu(H_n) - 1_\lambda(H) \right\} 1_\lambda(H) \right\| \leq 1,
\]
for all sufficiently large \( n \); see, e.g., [8, Lemma 6.8]. To this end we choose \( f \in C^\infty_0(\mathbb{R}, [0, 1]) \) such that \( f \equiv 1 \) on \( [e_0, \lambda] \), where \( e_0 := \min\{E, \inf_n E_n\} > -\infty \)
(by (3.23)). Supposing further that \( f \) is decreasing on \( [\lambda, \infty) \) with \( f(\mu) = 1/2 \) we may ensure that \( |f - 1|_{(-\infty, \mu]}| \leq 1/2 \) on \( \bigcup_{n \in \mathbb{N}} \sigma[H_n] \). Then \( 1_\lambda(H) = f(H) 1_\lambda(H) \), whence, by Part (1),
\[
\left\| \left\{ 1_\mu(H_n) - 1_\lambda(H) \right\} 1_\lambda(H) \right\| \leq \frac{1}{2} + \left\| \left\{ f(H_n) - f(H) \right\} 1_\lambda(H) \right\| \xrightarrow{n \to \infty} \frac{1}{2}.
\]
(3): Assume we had \( E < \mathcal{T} \). Then we find \( \varepsilon > 0 \) and integers \( n_1 < n_2 < \ldots \)

such that \( E + \varepsilon < E_{n_\ell} \), for all \( \ell \in \mathbb{N} \). Applying (2) with \( \lambda := E + \varepsilon/2 \) and \( \mu := E + \varepsilon \) we obtain the following contradiction, for all sufficiently large \( \ell \),
\[
0 < \text{dim Ran}(1_\lambda(H)) \leq \text{dim Ran}(1_\mu(H_n)) = 0.
\]
(4) & (5): We set \( \Pi_n := 1_{E_{n+1/n}}(H_n) \). Thanks to the additional assumption \( \Sigma_n - E_n \geq c > 0 \) and Remark [1.2] we may apply Proposition [4.1] to find some \( n \)-independent constants \( a, C \in (0, \infty) \) such that
\[
(4.8) \quad \forall n \in \mathbb{N}, \; n > 1/c : \quad \| e^{a|z|} \Pi_n \| \leq C.
\]
Let \( z \in \mathbb{C} \setminus \mathbb{R} \). We observe that in the proof of Part (1) we may interchange the roles of \( H \) and \( H_n \) and the new bound (4.8) permits to get the following analogue of (4.6),

\[
(4.9) \quad \| (\mathcal{R}(z) - \mathcal{R}_n(z)) \Pi_n \| \leq \frac{b'(n)}{1 \wedge |\Im z|^2},
\]

where \( 0 < b'(n) \to 0 \). For every \( n \in \mathbb{N} \), we pick some normalized \( \phi_n \in \operatorname{Ran}(\Pi_n) \). By the spectral calculus \((\mathcal{R}_n(z) - (E_n - z)^{-1}) \phi_n \to 0 \) strongly, as \( n \to \infty \). Furthermore, we find integers \( n_1 < n_2 < \ldots \) such that \( E_{n_\ell} \to \overline{E} \), as \( \ell \to \infty \), and such that \( \phi := \lim_{\ell \to \infty} \phi_{n_\ell} \) exists. By virtue of (4.9) we first infer that

\[
(\mathcal{R}(z) - \mathcal{R}_{n_\ell}(z)) \phi_{n_\ell} + (\mathcal{R}_{n_\ell}(z) - \frac{1}{E_{n_\ell} - z}) \phi_{n_\ell} + \frac{1}{E_{n_\ell} - z} - \frac{1}{E - z} \phi_{n_\ell} \xrightarrow{\ell \to \infty} 0,
\]

strongly. As the expression on the left equals \((\mathcal{R}(z) - (E - z)^{-1}) \phi_{n_\ell} \) we deduce that \( E \in \sigma[H] \), thus \( E \leq \overline{E} \), thus \( E = \overline{E} \) by (3). Moreover, we obtain

\[
0 = \lim_{\ell \to \infty} (\mathcal{R}(z) - \frac{1}{E - z}) \phi_{n_\ell} = (\mathcal{R}(z) - \frac{1}{E - z}) \phi.
\]

Therefore, \( \phi \in \mathcal{D}(H) \) and \( H \phi = E \phi \). \( \square \)

5. Existence of binding

In the whole Section 5 we assume that \( \varpi \) and \( G \) fulfill Hypothesis 2.1 and that \( G_x \) can be written as \( G_x = e^{-i\mu \cdot x} g \) almost everywhere on \( A_m \times \mathbb{Z}_2 \), where \( \mu, g : A_m \times \mathbb{Z}_2 \to \mathbb{R}^3 \) are measurable and \( |\mu| \leq \varpi \) almost everywhere.

5.1. Fiber decomposition. In order to prove the binding condition we replace \( H_{n,p}^{\mu,\varpi,G} \) by some suitable unitarily equivalent operator. This is the reason why we restrict our attention to coupling functions of the form \( G_x = e^{-i\mu \cdot x} g \).

Let us denote the components of \( \mu \) as \( \mu^{(j)} \), \( j = 1, 2, 3 \), and define

\[
p_{\|} := d\Gamma(\mu) := \left( d\Gamma(\mu^{(1)}), d\Gamma(\mu^{(2)}), d\Gamma(\mu^{(3)}) \right).
\]

Then a conjugation of the Dirac operator with the unitary operator \( e^{ip_{\|} \cdot x} \) – which is simply a multiplication with the phase \( e^{i(\mu^{(k_1)} + \cdots + \mu^{(k_3)}) \cdot x} \) in each Fock space sector \( F_{b}^{(n)}[M_m] \) – yields

\[
e^{ip_{\|} \cdot x} D_A e^{-ip_{\|} \cdot x} = \alpha \cdot (-i\nabla_x - p_{\|} + A(0)) + \beta.
\]

A further conjugation with the Fourier transform, \( F : L^2(\mathbb{R}^3_x) \to L^2(\mathbb{R}^3_\xi) \), with respect to the variable \( x \) turns the transformed Dirac operator into

\[
(5.1) \quad F e^{ip_{\|} \cdot x} D_A e^{-ip_{\|} \cdot x} F^* = \int_{\mathbb{R}^3} \hat{D}(\xi) \, d^3\xi,
\]

where, as usual, \( F \equiv F \otimes 1 \). Here the operators

\[
\hat{D}(\xi) := \alpha \cdot (\xi - p_{\|} + A(0)) + \beta, \quad \xi \in \mathbb{R}^3,
\]

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act in $\mathbb{C}^4 \otimes \mathcal{F}_b[\mathcal{H}_m]$. They are fiber Hamiltonians of the transformed Dirac operator in (5.1) with respect to the isomorphism
\begin{equation}
\mathcal{H}_m \cong \int_{\mathbb{R}^3} \mathbb{C}^4 \otimes \mathcal{F}_b[\mathcal{H}_m] \, d^3\xi.
\end{equation}

For every $\xi \in \mathbb{R}^3$, we introduce
\begin{equation}
\hat{S}(\xi) := \frac{1}{\hat{D}(\xi)} |\hat{D}(\xi)|^{-1}.
\end{equation}
Corresponding to (5.2) we then have the following direct integral representation of the no-pair operator without exterior potential,
\begin{equation}
\mathcal{F} e^{ipr_\times} H_{0,\infty, G} e^{-ipr_\times} \mathcal{F}^* = \int_{\mathbb{R}^3} \hat{H}(\xi) \, d^3\xi,
\end{equation}
where
\begin{equation}
\hat{H}(\xi) := |\hat{D}(\xi)| + (1/2) \, d\Gamma(\varpi) + (1/2) \hat{S}(\xi) \, d\Gamma(\varpi) \hat{S}(\xi).
\end{equation}

5.2. Proof of the binding condition. In view of (2.16), (3.1), and (3.3) we have $E_\gamma(\varpi, G) = \sigma[H_{0,\infty, G}^\mathrm{np}]$, for $\gamma \in (0, \gamma_0^\mathrm{np})$, and $\Sigma(\varpi, G) = \inf \sigma[H_{0,\infty, G}^\mathrm{np}]$. Let $\rho \in (0, 1]$ be fixed in what follows. From the general theory of direct integrals of self-adjoint operators and (5.4) it follows that there exist $\xi_\ast \in \mathbb{R}^3$ and $\varphi_\ast \in D(\hat{H}(\xi_\ast))$, $\|\varphi_\ast\| = 1$, such that
\begin{equation}
\langle \varphi_\ast | \hat{H}(\xi_\ast) \varphi_\ast \rangle < \Sigma(\varpi, G) + \rho.
\end{equation}
We abbreviate
\begin{equation}
\hat{D}_\ast(\xi) := \hat{D}(\xi + \xi_\ast), \quad \hat{S}_\ast(\xi) := \hat{S}(\xi + \xi_\ast), \quad \hat{H}_\ast(\xi) := \hat{H}(\xi + \xi_\ast),
\end{equation}
and introduce the unitary transformation
\begin{equation}
U := e^{i(p_\ast - \xi_\ast) \cdot x}.
\end{equation}
Then $H_{\gamma,\infty, G}^\mathrm{np}$ can be written as
\begin{equation}
H_{\gamma,\infty, G}^\mathrm{np} = U^* \mathcal{F}_\ast \int_{\mathbb{R}^3} \hat{H}_\ast(\xi) \, d^3\xi \, \mathcal{F}_\ast U - (\gamma/2) |x|^{-1} - (\gamma/2) S_A |x|^{-1} S_A.
\end{equation}

Proof of Theorem 2.2. Let $\rho \in (0, 1]$, $\xi_\ast$, and $\varphi_\ast$ be as above. We shall employ the following bound proven in [16]: For all real-valued $\varphi_1 \in H^{1/2}(\mathbb{R}^3)$, $\|\varphi_1\| = 1$,
\begin{equation}
\left\langle \varphi_1 \otimes \varphi_\ast \left| \int_{\mathbb{R}^3} |\hat{D}(\xi)| \, d^3\xi \, \varphi_1 \otimes \varphi_\ast \right\rangle \right.
\end{equation}
\begin{equation}
\leq \left\langle \varphi_1 | (\sqrt{1 - \Delta} - 1) \varphi_1 \right\rangle + \left\langle \varphi_\ast | |\hat{D}(\xi_\ast)| \varphi_\ast \right\rangle.
\end{equation}
Moreover, we estimate trivially
\begin{equation}
(\gamma/2) \left\langle U^* \varphi_1 \otimes \varphi_\ast \left| S_A |x|^{-1} S_A U^* \varphi_1 \otimes \varphi_\ast \right\rangle \right. \geq 0.
\end{equation}
In Lemma 5.2 we show that, for every real-valued \( \varphi_1 \in C_0^\infty(\mathbb{R}^3) \), \( \| \varphi_1 \| = 1 \),

\[
\left| \left\langle \hat{\varphi}_1 \otimes \varphi_* \big| \int_{\mathbb{R}^3} (\hat{S}_* (\xi) d\Gamma (\varpi) \hat{S}_* (\xi) - \hat{S}_* (\xi) ) d\Gamma (\varpi) \hat{S}_* (\xi) \right\rangle \right| \leq C(d) \| \nabla \varphi_1 \|^2_{L^2}.
\]

Combining (5.6)–(5.8) we obtain
\[
\Sigma(\varpi, G) + \rho - E_\gamma (\varpi, G) \\
\geq \left\langle \varphi_* \big| \hat{H} (\xi_*) \varphi_* \right\rangle - \left\langle U^* \varphi_1 \otimes \varphi_* \big| H_{\gamma, \varpi, G}^{bb} U^* \varphi_1 \otimes \varphi_* \right\rangle \\
\geq - \left\langle \varphi_1 \big| (\sqrt{1 - \Delta} - 1 - (\gamma/2) |x|^{-1}) \varphi_1 \right\rangle - C(d) \| \nabla \varphi_1 \|^2_{L^2}/2
\]

In the last step we used \( \sqrt{1 + t - 1} \leq t/2, \ t \geq 0 \). We pick some \( \theta \in C_0^\infty(\mathbb{R}, [0, 1]) \) with \( \theta \equiv 1 \) on \([-1, 1], \ \theta \equiv 0 \) on \( \mathbb{R} \setminus (-2, 2) \) and set

\[
\varphi_1 (x) := \frac{1}{2^1/2 R^{3/2}} \theta(|x|/R), \ x \in \mathbb{R}^3, \ \mathcal{L}^\nu := \int_{\mathbb{R}^3} \theta^2(|x|) \, d^3x,
\]

for some \( R \geq 1 \). Then it is straightforward to see that the first, positive term in the last line of (5.9) behaves like \( R^{-1} \) whereas the second term is some \( O(R^{-2}) \). Hence, choosing \( R \) sufficiently large, depending only on \( \gamma \) and \( d \), we obtain \( \Sigma(\varpi, G) + \rho - E_\gamma (\varpi, G) \geq c(\gamma, d) > 0 \), where \( \rho \in (0, 1] \) is arbitrarily small.

It remains to prove the bound (5.8). For this we need, however, a few preparations. In what follows we set

\[
\hat{R}_\xi (iy) := (\hat{D} (\xi + \xi_*) - iy)^{-1}, \ \xi \in \mathbb{R}^3, \ \hat{R}_0 (iy) := \hat{R}_0 (iy), \ y \in \mathbb{R},
\]

so that, analogously to (2.14),

\[
\hat{S}_* (\xi) \psi = \lim_{\tau \to \infty} \int_{-\tau}^{\tau} \hat{R}_\xi (iy) \psi \frac{dy}{\pi}, \ \psi \in C^4 \otimes \mathcal{F}_\nu.
\]

**Lemma 5.1.** There is some \( K \equiv K(d) \in (0, \infty) \), and, for all \( \xi \in \mathbb{R}^3, \nu \in (0, 1] \), and \( y \in \mathbb{R} \), we can construct \( \hat{\Gamma}_\xi (y) \in \mathcal{L}(C^4 \otimes \mathcal{F}_\nu) \), \( \| \hat{\Gamma}_\xi (y) \| \leq 2 \), such that

\[
\hat{R}_\xi (iy) (d\Gamma (\varpi) + K)^{-1/2} = (d\Gamma (\varpi) + K)^{-1/2} \hat{R}_\xi (iy) \hat{\Gamma}_\xi (y).
\]

**Proof.** We set \( \Theta := d\Gamma (\varpi) + K \). Due to [21, Lemma 3.1] we know that \( [\alpha \cdot A (0), \Theta^{-1/2}] \Theta^{1/2} \) extends to a bounded operator on \( C^4 \otimes \mathcal{F}_\nu \), henceforth denoted by \( Z \), and \( \| Z \| \leq C(d)/K^{1/2} \). We choose \( K \) so large that \( \| Z \| \leq 1/2 \). Then we readily infer (compare [21, Corollary 3.1]) that \( \Theta^{-1/2} \hat{R}_\xi (iy) \hat{\Gamma}_\xi (y) = \hat{R}_\xi (iy) \Theta^{-1/2} (1 - Z^* \hat{R}_\xi (iy)) \). Since \( \| \hat{R}_\xi (iy) \| \leq 1 \), the assertion follows with \( \hat{\Gamma}_\xi (y) := (1 - Z^* \hat{R}_\xi (iy))^{-1} \). 

\[\square\]
Lemma 5.2. The bound \((5.8)\) holds true.

Proof. We put \(\delta \hat{S} := \int_{\mathbb{R}^3}(\hat{S}_s(\xi) - \hat{S}(\xi_\ast))\, d^3\xi\). Then the LHS of \((5.8)\) equals \(|2\text{Re} I_1 + I_2|\) with

\[
I_1 := \langle \delta \hat{S} \varphi_1 \otimes \varphi_* | d\Gamma(\varpi) \hat{S}(\xi_\ast) \varphi_1 \otimes \varphi_* \rangle, \quad I_2 := \|d\Gamma(\varpi)^{1/2} \delta \hat{S} \varphi_1 \otimes \varphi_*\|.
\]

Notice that the operator \(\hat{S}(\xi_\ast)\) acts only on \(\mathcal{C}^4 \otimes \mathcal{F}_b = \mathcal{F}_0^4\). By virtue of \((5.10)\) and a two-fold application of the second resolvent identity we thus obtain

\[
I_1 = - \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |\hat{\varphi}_1(\xi)|^2 \xi \cdot \langle \hat{R}(iy) \alpha \hat{R}(iy) \varphi_* | d\Gamma(\varpi) \hat{S}(\xi_\ast) \varphi_* \rangle_{\mathcal{F}_0^4} d^3\xi \, \frac{dy}{\pi}
+ \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |\hat{\varphi}_1(\xi)|^2 \langle \hat{R}(iy) \alpha \cdot \xi \hat{R}(iy) \alpha \cdot \xi \hat{R}(iy) \varphi_* | d\Gamma(\varpi) \hat{S}(\xi_\ast) \varphi_* \rangle_{\mathcal{F}_0^4} d^3\xi \, \frac{dy}{\pi}.
\]

Since \(\varphi_1\) is real-valued its Fourier transform satisfies \(|\hat{\varphi}_1(\xi)| = |\hat{\varphi}_1(\xi)|\). Substituting \(\xi \rightarrow -\xi\) we thus observe that the integral in the first line of the above formula for \(I_1\) is equal to zero. A straightforward application of Lemma 5.1 to the integral in the second line using \(\|\hat{R}_\xi(iy)\| \leq (1 + y^2)^{-1/2}\) then yields

\[
|I_1| \leq \int_{\mathbb{R}^3} (\frac{8}{\pi}) \, \frac{dy}{(1 + y^2)^{3/2}} \|d\Gamma(\varpi) + K\|^{1/2} \varphi_* \|d\Gamma(\varpi)^{1/2} \hat{S}(\xi_\ast) \varphi_*\| \int_{\mathbb{R}^3} |\xi \hat{\varphi}_1|^2.
\]

Denoting the set of all normalized \(\hat{\psi} \in \mathcal{Q}(1 \otimes d\Gamma(\varpi))\) by \(S\), we further have

\[
|I_2| \leq \sup_{\hat{\psi} \in S} |\langle d\Gamma(\varpi)^{1/2} \hat{\psi} | \delta \hat{S} \varphi_1 \otimes \varphi_* \rangle|
\leq \sup_{\hat{\psi} \in S} \int_{\mathbb{R}} |\langle d\Gamma(\varpi)^{1/2} \hat{\psi} | \hat{R}_\xi(iy) (1 \otimes \alpha) \cdot (\xi \hat{\varphi}_1) \otimes (\hat{R}(iy) \varphi_*) \rangle| \, \frac{dy}{\pi}.
\]

Applying Lemma 5.1 once more we deduce that

\[
|I_2| \leq \frac{4}{\pi} \int_{\mathbb{R}} \frac{dy}{1 + y^2} \|d\Gamma(\varpi) + K\|^{1/2} \varphi_* \| \xi \hat{\varphi}_1\|_{L^2}.
\]

Next, we observe that \(\|d\Gamma(\varpi) \varphi_* / 2 + \langle \hat{S}(\xi_\ast) \varphi_* | d\Gamma(\varpi) \hat{S}(\xi_\ast) \varphi_* \rangle / 2 \leq \Sigma(\varpi, \mathbf{G}) - 1 + \rho\), where \(\rho \ll 1\). By Remark 1.2 we have a finite upper bound on \(\Sigma(\varpi, \mathbf{G})\) depending only on \(d\). Since the LHS of \((5.8)\) is \(\leq 2|I_1| + |I_2|^2\) this concludes the proof.

6. Existence of ground states for massive photons

In this section we prove that the no-pair operator defined by means of the physical choices \(\varpi = \omega\) and \(\mathbf{G} = \mathbf{G}^{m,\Lambda}\) given in Example 2.2 has ground state eigenvectors, provided that the photons are given a mass. The photon mass, \(m > 0\), is introduced as follows:
6.1. Introduction of a photon mass. As the underlying Hilbert space we choose $H_m$. We let $\omega_m$ and $G_{e,\Lambda}^m$ denote the restrictions of $\omega$ and $G_{e,\Lambda}$ to $A_m \times \mathbb{Z}_2$ with $A_m = \{\|k\| \geq m\}$, respectively, and set

$$
H^0_{\gamma,m} := H_{\gamma,\omega_m, G_{e,\Lambda}^m}, \quad \gamma \in [0, \gamma^\text{np}_\Lambda], \ m > 0.
$$

In order to show that $H^0_{\gamma,m}$ has ground state eigenvectors we compare $H^0_{\gamma,m}$ with a modified version of it where all Fock space operators are discretized. This strategy is also used in [3, 5]. We point out, however, that the proof of Theorem 6.1 below contains a new idea which allows to deal with arbitrarily large values of $e$ and $\Lambda$.

6.2. Discretization of the photon momenta. Let $m > 0$ be fixed and let $\epsilon > 0$. We decompose $A_m$ as

$$
A_m = \bigcup_{\nu \in (\epsilon \mathbb{Z})^3} Q_\epsilon^m(\nu), \quad Q_\epsilon^m(\nu) := (\nu + [-\epsilon/2, \epsilon/2]^3) \cap A_m, \ \nu \in (\epsilon \mathbb{Z})^3.
$$

For every $k \in A_m$, we find a unique vector, $\mu^\epsilon(k) \in (\epsilon \mathbb{Z})^3$, such that $k \in Q_\epsilon^m(\mu^\epsilon(k))$, and we put

$$
\omega_\epsilon^m(k) := |\mu^\epsilon(k)|, \quad k = (k, \lambda) \in A_m \times \mathbb{Z}_2, \quad H^\epsilon_{\gamma,m} := d\Gamma(\omega_\epsilon^m),
$$

so that

$$
|\omega_m - \omega_\epsilon^m| \leq \sqrt{3} \epsilon/2 \leq (\sqrt{3} \epsilon/2m) \omega_m.
$$

We further define an $\epsilon$-average of $f \in \mathcal{H}_m$ by

$$
P^\epsilon_m f := \sum_{\nu \in (\epsilon \mathbb{Z})^3, Q_\epsilon^m(\nu) \neq \emptyset} \langle \chi_{Q_\epsilon^m(\nu)} | f \rangle \chi_{Q_\epsilon^m(\nu)},
$$

where $\chi_{Q_\epsilon^m(\nu)}$ denotes the normalized characteristic function of the set $Q_\epsilon^m(\nu)$, so that $P^\epsilon_m$ is an orthogonal projection in $\mathcal{H}_m$. Finally, we set

$$
G_{e,\Lambda}^\epsilon := e^{-i \mu^\epsilon \cdot x} P^\epsilon_m [G_{e,\Lambda}^m], \quad H^\epsilon_{\gamma,m} := H^\text{np}_{\gamma,\omega_\epsilon^m, G_{e,\Lambda}^\epsilon}.
$$

It is an easy and well-known exercise to verify that

$$
\int \left(1 + \frac{1}{\omega_m(k)}\right) \sup_x e^{-a|x|} |G_{e,\Lambda}^{\epsilon,m}(k) - G_{e,\Lambda}^{m}(k)|^2 dk \leq c_{a,m}(\epsilon),
$$

where $c_{a,m}(\epsilon) \to 0, \epsilon \searrow 0$, for all fixed $a, m > 0$. Notice that some $x$-dependent weights are required in the above estimate since we use the bound

$$
|e^{-ik \cdot x} - e^{-i \mu^\epsilon(k) \cdot x}| \leq |k - \mu^\epsilon(k)||x| \leq \sqrt{3} \epsilon |x|/2.
$$
6.3. Discrete and fluctuating subspaces. In the proof of the main result of this section, Theorem 6.1 we employ a certain tensor product representation of $\mathcal{H}_m$ we shall explain first.

We introduce the subspaces of discrete and fluctuating photon states,

$$
\mathcal{H}_m^d := P_m^e \mathcal{H}_m, \quad \mathcal{H}_m^f := (1 - P_m^e) \mathcal{H}_m,
$$

where $P_m^e$ is defined in (6.4). Corresponding to the orthogonal decomposition $\mathcal{H}_m = \mathcal{H}_m^d \oplus \mathcal{H}_m^f$ there is an isomorphism of Fock spaces, $\mathcal{F}_b[\mathcal{H}_m] = \mathcal{F}_b[\mathcal{H}_m^d] \otimes \mathcal{F}_b[\mathcal{H}_m^f]$. If $\{g_i\}$ and $\{h_j\}$ denote orthonormal bases of $\mathcal{H}_m^d$ and $\mathcal{H}_m^f$, respectively, then this isomorphism maps $a^\dagger(g_i) \ldots a^\dagger(g_u) a^\dagger(h_j) \ldots a^\dagger(h_{j_r}) \Omega$ to $a^\dagger(g_i) \ldots a^\dagger(g_u) \Omega_d \otimes a^\dagger(h_j) \ldots a^\dagger(h_{j_r}) \Omega_f$, where $\Omega_f$ is the vacuum vector in $\mathcal{F}_b[\mathcal{H}_m^f]$, for $\ell = d, f$. Let $A_m^\varepsilon := A[G_m^ε,ε]$ be defined by the formula (2.3) and let $1_f$ denote the identity in $\mathcal{F}_b[\mathcal{H}_m^f]$. Since $G_m^ε,ε \in \mathcal{H}_m^d$, for every $x$, it follows that

$$
A_m^\varepsilon = A_m^{\varepsilon,d} \otimes 1_f, \quad A_m^{\varepsilon,d} := \int_{\mathbb{R}^3} \mathbb{1}_{G^4} \otimes \left( a^\dagger(G_m^{ε,d}) + a(G_m^{ε,d}) \right) d^3x,
$$

corresponding to the isomorphism

$$(6.7) \quad \mathcal{H}_m = (L^2(\mathbb{R}^3; \mathbb{C}^4) \otimes \mathcal{F}_b[\mathcal{H}_m^d]) \otimes \mathcal{F}_b[\mathcal{H}_m^f].$$

We infer that the Dirac operator and all functions of it can be written as

$$
D_{A_m^\varepsilon} = D_{A_m^{\varepsilon,d}} \otimes 1_f, \quad |D_{A_m^\varepsilon}| = |D_{A_m^{\varepsilon,d}}| \otimes 1_f, \quad S_{A_m^\varepsilon} = S_{A_m^{\varepsilon,d}} \otimes 1_f.
$$

Since $\omega_m^\varepsilon$ commutes with $P_m^e$ and $(S_{A_m^{\varepsilon,d}})^2 = 1$ we conclude that

$$
H_{τ,m}^\varepsilon = H_{τ,m}^{\varepsilon,d} \otimes 1_f + 1 \otimes H_{1,m}^{\varepsilon,f},
$$

$$
H_{τ,m}^{\varepsilon,d} := |D_{A_m^\varepsilon}| + \frac{1}{2} \left( -\frac{γ}{|x|} + H_{τ,m}^{\varepsilon,d} \right) + \frac{1}{2} S_{A_m^\varepsilon} \left( -\frac{γ}{|x|} + H_{1,m}^{\varepsilon,d} \right) S_{A_m^\varepsilon},
$$

where $H_{1,m}^{\varepsilon,d} := d\Gamma(\omega_m^\varepsilon P_m^e)$ and $H_{1,m}^{\varepsilon,f} := d\Gamma(\omega_m^\varepsilon (1 - P_m^e))$. Tensor-multiplying minimizing sequences for $H_{τ,m}^{\varepsilon,d}$, with $Ω_f$ we finally verify that

$$(6.8) \quad \inf σ[H_{τ,m}^\varepsilon] = \inf σ[H_{τ,m}^\varepsilon].$$

6.4. Existence of ground states with photon mass.

Theorem 6.1. Let $ε \in \mathbb{R}$, $Λ > 0$, $γ \in (0, γ_{ep}^m)$, and $m > 0$. Then the spectral projection $1_{εm+m/4}(H_{τ,m}^0)$ has a finite rank.

Proof. We pick some null sequence $ε_n \searrow 0$ and apply Proposition 6.3 with $x := ω_m, G_x := G_m^{ε,m} x$ and $x_n := ω_n, G_n,x := G_m^{ε,m} x_n$, that is,

$$
H := H_{τ,m}^0, \quad H_n := H_{τ,m}^{ε_n}, \quad E := \inf σ[H], \quad E_n := \inf σ[H_n].
$$

On account of (6.2), (6.3), and (6.6) the assumptions of Proposition 4.3 are satisfied, for every fixed $m > 0$. By Theorem 2.4 we have a uniform, strictly
positive lower bound on the binding energy of $H_n$, $n \in \mathbb{N}$, so that $E_n \to E$ by Proposition 1.3(4). By virtue of Proposition 1.3(2) we further know that, for all sufficiently large $n$ with $E + 3m/8 \leq E_n + m/2$,

$$\dim \text{Ran}(1_{E+m/4}(H)) \leq \dim \text{Ran}(\Pi_n), \quad \Pi_n := 1_{E_n+m/2}(H_n).$$

It remains to show that $\Pi_n$ is a finite rank projection, for all sufficiently large $n$. To this end we employ the isomorphism (6.7) explained in the previous subsection. We denote the projection in $\mathcal{F}_n[\mathcal{K}_m]$ onto the vacuum sector by $P_{\Omega_f}$, write $P_{\Omega_f}^\perp := 1_f - P_{\Omega_f}$, and set $H_n^d := H_n^{\varepsilon \gamma, n}$. In view of $H_n^{\varepsilon \gamma, f} P_{\Omega_f} = 0$ we obtain

$$-\frac{m}{2} \Pi_n \geq \Pi_n (H_n - E_n - m) \Pi_n = \Pi_n (1 \otimes P_{\Omega_f}) (H_n - E_n - m) (1 \otimes P_{\Omega_f}) \Pi_n + \Pi_n \left\{ (H_n^d - E_n) \otimes P_{\Omega_f}^\perp + 1 \otimes (H_n^{\varepsilon \gamma, f} - m) P_{\Omega_f}^\perp \right\} \Pi_n.$$

The operator in the last line is non-negative since $H_n^d - E_n \geq 0$ by (6.8), and $H_n^{\varepsilon \gamma, f} P_{\Omega_f}^\perp \geq m P_{\Omega_f}^\perp$. In order to bound the term in the second line from below we use (3.24), that is, $H_n \geq (-\Delta)^s + (1/2) H_n^{\varepsilon \gamma, s} - C$, for some $s \in (0, 1/2)$ and $C \equiv C(e, \Lambda, s) \in (0, \infty)$. Furthermore, we observe that $T := \Pi_n (|x|^2 \otimes P_{\Omega_f}) \Pi_n$ is bounded uniformly in $n \in \mathbb{N}$ by Proposition 1.1 Remark 1.2 and our uniform lower bound on the binding energy of $H_n$. Choosing $\delta > 0$ so small that $\delta T \leq (m/4) \Pi_n$ we arrive at

$$(6.9) \quad -\frac{m}{4} \Pi_n \geq \Pi_n \left\{ (-\Delta)^s + \delta |x|^2 + \frac{1}{2} H_n^{\varepsilon \gamma, d} - E_n - m - C \right\} \otimes P_{\Omega_f} \Pi_n.$$ 

Now, both $(-\Delta)^s + \delta |x|^2$ and $H_n^{\varepsilon \gamma, d}$ have purely discrete spectrum as operators on the electron and photon Hilbert spaces and $P_{\Omega_f}$ has rank one. (Recall that $H_n^{\varepsilon \gamma, d} = d \Gamma(\omega_m^\varepsilon P_m^\varepsilon)$ and $\omega_m^\varepsilon P_m^\varepsilon$ as an operator in $\mathcal{K}_m$ has purely discrete spectrum and is strictly positive.) Let $X \geq 0$ denote the negative part of the operator $\{\cdots\}$ in (6.9). Then $X \otimes P_{\Omega_f}$ has a finite rank and $\Pi_n \leq (4/m) \Pi_n (X \otimes P_{\Omega_f}) \Pi_n$. Therefore, $\Pi_n$ is a finite rank projection, if $n$ is sufficiently large.

\[ \square \]

7. Existence of ground states
We extend $H_{\gamma,m}^0$ (defined in (6.1)) to an operator acting in $\mathcal{H} = \mathcal{H}_0$ by setting

$$(7.1) \quad G_{m,\lambda}^{\varepsilon \Lambda}(k) := 1_{\mathcal{A}_m}(k) G_{\lambda}^{\varepsilon \Lambda}(k), \quad \text{almost every } k = (k, \lambda) \in \mathbb{R}^3 \times \mathbb{Z}_2,$$

$$(7.2) \quad H_{\gamma,m} := H_{\gamma,\omega, \Lambda m}^{\varepsilon \Lambda}.$$ 

(We are abusing the notation slightly since the symbol $G_{m,\lambda}^{\varepsilon \Lambda}$ used to denote the restriction of $G_{\lambda}^{\varepsilon \Lambda}$ to $\mathcal{A}_m \times \mathbb{Z}_2$. From now on it denotes a function on $\mathbb{R}^3 \times \mathbb{Z}_2$.)
The splitting \( \mathcal{H}_0 = \mathcal{H}_m \oplus \mathcal{H}_m^\perp \) gives rise to an isomorphism
\[
\mathcal{H} = \mathcal{H}_m \otimes \mathcal{F}_b[\mathcal{H}_m^\perp],
\]
and with respect to this isomorphism we have
\[
H_{\gamma,m} = H_{0,m} \otimes 1 + 1 \otimes d\Gamma(\omega|_{(\mathbb{R}^3 \setminus A_m) \times \mathbb{Z}_2}).
\]
By Theorem 6.1 \( H_{0,m} \) has a normalized ground state eigenvector, \( \phi_m^0 \), and we readily infer that \( \phi_m = \phi_m^0 \otimes \Omega \) is a normalized ground state eigenvector of \( H_{\gamma,m} \), where \( \Omega \) denotes the vacuum vector in \( \mathcal{F}_b[\mathcal{H}_m^\perp] \). In what follows we represent \( \phi_m \) as
\[
\phi_m = (\phi_m^{(n)})_{n=0}^\infty \in \bigoplus_{n=0}^\infty L^2(\mathbb{R}^3 \times \mathbb{Z}_4) \otimes \mathcal{F}_b[\mathcal{H}_m^\perp].
\]
The aim of this section is to show that each sequence \( \{\phi_m\}_{m_j} \), \( m_j \downarrow 0 \), contains a strongly convergent subsequence. By virtue of Proposition 4.3 the limit of such a subsequence then turns out to be a ground state eigenvector of \( H_{\gamma,0} := H_{\gamma,0}^{np} \).

As in (16) we shall prove this compactness property by a suitably adapted version of an argument from (12). For this purpose we need the two infra-red bounds stated in the following proposition. Their proofs are deferred to Section 8. We recall the notation
\[
(a(k)^{(n)}(k_1, \ldots, k_n)) = (n + 1)^{1/2} \psi^{(n+1)}(k, k_1, \ldots, k_n), \quad n \in \mathbb{N}_0,
\]
almost everywhere, for \( \psi = (\psi^{(n)})_{n=0}^\infty \in \mathcal{F}_b[\mathcal{H}_0] \), and \( a(k) \Omega = 0 \).

**Proposition 7.1 (Infra-red bounds).** Let \( \epsilon \in \mathbb{R}, \Lambda > 0 \), and \( \gamma \in (0, \gamma_{\epsilon}^{np}) \).
Then there is some \( C \in (0, \infty) \), such that, for all \( m \in [0, \Lambda) \) and every normalized ground state eigenvector, \( \phi_m \), of \( H_{\gamma,m} \), we have the soft photon bound,
\[
\| a(k) \phi_m \|^2 \leq 1_{\{m \leq \|k\| \leq \Lambda\}} \frac{C}{\|k\|},
\]
for almost every \( k = (k, \lambda) \in \mathbb{R}^3 \times \mathbb{Z}_2 \), and the photon derivative bound,
\[
\| a(k, \lambda) \phi_m - a(p, \lambda) \phi_m \| \leq C \|k - p\| \left( \frac{1}{\|k\|^{1/2}} \frac{1}{\|k\|} + \frac{1}{\|p\|^{1/2}} \frac{1}{\|p\|} \right),
\]
for almost every \( k, p \in \mathbb{R}^3 \) with \( m < |k| < \Lambda, m < |p| < \Lambda \), and \( \lambda \in \mathbb{Z}_2 \). (Here we use the notation introduced in (2.6).) In particular,
\[
\sup_{m \in (0, \Lambda)} \sum_{n=1}^\infty n \|\phi_m^{(n)}\|^2 < \infty.
\]
The proof of (7.5) is actually the only place in the whole article where the special choice of the polarization vectors (2.7) is used explicitly.
Remark 7.2. Assume that $\phi_{\gamma,m}$ is some normalized ground state eigenvector of $H_{\gamma,m}$, for all $m \in [0, \Lambda)$ and every $\gamma \in (0, \gamma_{c}^{np})$. If we find $\gamma$-independent $a, C'\in (0, \infty)$ such that

\[
\forall m \in [0, \Lambda), \, \gamma \in (0, \gamma_{c}^{np}) : \, \| e^{a|x|} \phi_{\gamma,m} \| \leq C',
\]

then the constant $C$ appearing in the statement of Proposition 7.1 can be chosen independently of $\gamma \in (0, \gamma_{c}^{np})$, too. This remark shall be important in order to prove the existence of ground states at critical coupling ($\gamma = \gamma_{c}^{np}$) in a forthcoming note by two of the present authors. (Due to lack of space (7.7) cannot be derived in the present article.) A brief explanation of this remark is given at the end of Subsection 8.4.

Proof of Theorem 2.5. Let $\{m_j\}$, $m_j \searrow 0$, be some null sequence and let $\phi_{m_j}$ denote some normalized ground state eigenvector of $H_{\gamma,m_j}$, whose existence is guaranteed by the remarks at the beginning of this section. Passing to some subsequence if necessary, we may assume that $\{\phi_{m_j}\}$ converges weakly to some $\phi \in \mathcal{H}$. It suffices to show that $\phi \neq 0$.

In fact, if we set $\omega_j := \omega := \omega, \, G := G^{e,A}$, and $G_j := G_{m_j}^{e,A}$, for $j \in \mathbb{N}$, then the assumptions of Proposition 4.3 are obviously fulfilled. Since Theorem 2.4 provides a uniform, strictly positive lower bound on the binding energy of $H_{\gamma,m_j}$, $j \in \mathbb{N}$, Parts (4) and (5) of that proposition are available with

\[
H := H_{\gamma}, \quad H_j := H_{\gamma,m_j}, \quad E := \inf \sigma[H], \quad E_j := \inf \sigma[H_j].
\]

In particular, $\phi \in \mathcal{D}(H)$ and $H \phi = E \phi$. On account of (3.1) and (3.3) this proves Theorem 2.5 if $\phi \neq 0$.

In what follows we only sketch how to prove that $\{\phi_{m_j}\}$ converges actually strongly to $\phi$ along some subsequence, so that $\|\phi\| = 1$. For this proof is almost literally the same as the one of [16] Theorem 2.2], which in turn is based on the same ideas as the corresponding proof in [12]. Only different compact imbedding theorems have to be employed since we have weaker bounds on (fractional) derivatives of $\phi_{m}$ with respect to the electron coordinates than in the non-relativistic case; see (7.9) below.

(7.6) shows that the largest portion of $\phi_{m}$ belongs to Fock space sectors with low particle numbers so that the norm of $(0, \ldots, 0, \phi_{m}^{(n_0)}, \phi_{m}^{(n_0+1)}, \ldots)$ is small, for large $n_0 \in \mathbb{N}$, uniformly in small $m > 0$. Moreover, (7.4) shows that the functions $\phi_{m}^{(n)}$ – which are symmetric in the photon variables – are localized with respect to the photon momenta, again uniformly in small $m > 0$. The photon derivative bound provides uniform bounds on the weak first order derivatives w.r.t. the photon momenta in $L^p$, for every $p < 2$; see, e.g., [26] §4.8. Similar information is available also with respect to the electron coordinates: Since Theorem 2.4 gives a lower bound on the binding energy of $H_{\gamma,m}$, uniformly in
\[m > 0, \text{ it is clear from Proposition 4.1 and Remark 4.2 that there exist } C, a > 0 \text{ such that}\]
\[
\| e^{a|x|} \phi_m \| \leq C, \quad m > 0.
\]
This gives uniform localization in \(x\). Uniform \(L^2\)-bounds on fractional derivatives with respect to \(x\) of order \(s < 1/2\) follow from (3.24) which implies
\[
\phi_m^{(n)} \left\langle (-\Delta)^s \phi_m^{(n)} \right\rangle \leq \langle \phi_m^{(n)} H_j \phi_m^{(n)} \rangle + c = E_j + c \leq E + c',
\]
where the constants \(c, c' \in (0, \infty)\) do not depend on \(j, n \in \mathbb{N}\). Here we used that \(E_j \to E\) due to Proposition 4.3(4). Our aim is to exploit all this information to single out a strongly convergent subsequence from \(\{\phi_m^{(n)}\}\) by applying a suitable compact imbedding theorem. Notice that we are dealing with (fractional) derivatives of different orders in different \(L^p\)-spaces which are, moreover, defined by different means (via Fourier transformation or as weak derivatives).

The classical anisotropic function spaces \(H^{(r_1,\ldots,r_d)}(\mathbb{R}^d)\) introduced by Nikol’skii turn out to be convenient in this situation. They are defined as follows:

For \(r_1, \ldots, r_d \in [0, 1]\) and \(q_1, \ldots, q_d \geq 1\), the space \(H^{(r_1,\ldots,r_d)}(\mathbb{R}^d)\) is equal to the intersection \(\bigcap_{i=1}^d H^{r_i}_{q_i}(\mathbb{R}^d)\). For \(r_i \in [0, 1]\), a measurable function \(f : \mathbb{R}^d \to \mathbb{C}\) belongs to the class \(H^{r_i}_{q_i}(\mathbb{R}^d)\), if \(f \in L^{q_i}(\mathbb{R}^d)\) and there is some \(M \in (0, \infty)\) such that
\[
\|f(\cdot + h e_i) - f\|_{L^{q_i}(\mathbb{R}^d)} \leq M |h|^{r_i}, \quad h \in \mathbb{R},
\]
where \(e_i\) is the \(i\)-th canonical unit vector in \(\mathbb{R}^d\). If \(r_i = 1\) then (7.10) is replaced by
\[
\|f(\cdot + h e_i) - 2f + f(\cdot - h e_i)\|_{L^{q_i}(\mathbb{R}^d)} \leq M |h|, \quad h \in \mathbb{R}.
\]

\(H^{(r_1,\ldots,r_d)}(\mathbb{R}^d)\) is a Banach space with norm \(|f|_{H^{(r_1,\ldots,r_d)}} := \max_{1 \leq i \leq d} \|f\|_{L^{q_i}(\mathbb{R}^d)} + \max_{1 \leq i \leq d} M_i\), where \(M_i\) is the infimum of all constants \(M > 0\) satisfying (7.10) or (7.11), respectively.

For \(n \in \mathbb{N}\) and some fixed \(\mathbf{r} = (\zeta, \lambda_1, \ldots, \lambda_n) \in \{1, 2, 3, 4\} \times \mathbb{Z}_2^n\), we now abbreviate
\[
\phi_m^{(n)}(x, k_1, \ldots, k_n) := \phi_m^{(n)}(x, \zeta, k_1, \lambda_1, \ldots, k_n, \lambda_n),
\]
and similarly for the weak limit \(\phi\). For every \(\delta, R > 0\), we further set
\[
Q^{R}_{n, \delta} := \{ (x, k_1, \ldots, k_n) : |x| < R - \delta, \delta < |k_j| < \Lambda - \delta, \quad j = 1, \ldots, n \}.
\]
For some small \(\delta > 0\), we pick some cut-off function \(\chi \in C\infty_0(\mathbb{R}^{3(n+1)}, [0, 1])\) such that \(\chi \equiv 1\) on \(Q^{R}_{n, 2\delta}\) and \(\text{supp}(\chi) \subset Q^{R}_{n, \delta}\) and define \(\psi_m^{(n)} := \chi \phi_m^{(n)}\). Employing the ideas sketched in the first paragraphs of this proof we can now argue exactly as in [16 Proof of Theorem 2.2] to conclude that \(\{\psi_m^{(n)}\}_{j \in \mathbb{N}}\) is bounded in the Nikol’skii space \(H^{(s,s,\ldots,s,1)}(\mathbb{R}^{3(n+1)})\), for every \(p \in [1, 2]\).
We may thus apply Nikol’skii’s compactness theorem \cite[Theorem 3.2]{25} which implies that \(\{\psi^{(n)}_{m_j, \theta}\}_{j \in \mathbb{N}}\) contains a subsequence which is strongly convergent in \(L^2(Q_{n,2\delta}^R)\), provided that \(1 - 3n(p^{-1} - 2^{-1}) > 0\). For fixed \(n_0 \in \mathbb{N}\), we may choose \(p < 2\) large enough such that the latter condition is fulfilled, for all \(n = 1, \ldots, n_0\). By finitely many repeated selections of subsequences we may hence assume without loss of generality that \(\{\phi^{(n)}_{m_j, \theta}\}_{j \in \mathbb{N}}\) converges strongly in \(L^2(Q_{n,2\delta}^R)\) to \(\phi^{(n)}_{\theta}\), for \(n = 0, \ldots, n_0\) and every choice of \(\theta\). Since \(\delta > 0\) can be chosen arbitrary small and \(R > 0\) and \(n_0 \in \mathbb{N}\) arbitrary large, and since \(\{\phi_{m_j}\}\) is localized w.r.t. \(x\) and \(n\), we can further argue that \(\{\phi_{m_j}\}\) contains a strongly convergent subsequence. \(\square\)

8. Infra-red bounds

In this section we prove the soft photon and photon derivative bounds which served as two of the main ingredients for the compactness argument presented in Section 7. In non-relativistic QED a soft photon bound without infra-red regularizations has been derived first in \cite{5}. The observation which made it possible to get rid of the mild infra-red regularizations employed earlier in \cite{3} is that, after a suitable unitary gauge transformation, namely the Pauli-Fierz transformation explained in Subsection 8.1, the quantized vector potential attains a better infra-red behavior. Working in the new gauge one can thus avoid the infra-red divergent integrals that appeared in the original gauge. For this reason the gauge invariance of the no-pair operator becomes absolutely crucial to derive the results of the present section. Photon derivative bounds have been introduced in \cite{12}, where also an alternative strategy to prove the infra-red bounds has been proposed. As in our earlier companion paper \cite{16}, where we proved both infra-red bounds for the semi-relativistic Pauli-Fierz operator, our proofs rest on a suitable representation formula for \(a(k) \phi_m\).

In the whole section we always assume that \(e \in \mathbb{R}, \Lambda > 0, \gamma \in (0, \gamma_c^{(n)})\), \(m \geq 0\), and that \(\phi_m\) is a ground state eigenvector of \(H_{\gamma,m}\), where \(H_{\gamma,m}\) is defined in (7.2). Notice that we include the case \(m = 0\). We set \(A_m := A[G_m^e];\) compare (2.3) and (7.1).

We add one remark we shall use repeatedly later on: Since the unitary operator \(S_{A_m}\) commutes with \(H_{\gamma,m}\) we know that \(S_{A_m} \phi_m\) is a ground state eigenvector of \(H_{\gamma,m}\), too. In view of Proposition 4.1 we thus find some \(a \in (0, 1/2]\) and some \(F \in C^\infty(\mathbb{R}_x^3, [0, \infty))\) with \(F(x) = a|x|\), for large \(|x|\), and \(|\nabla F| \leq a\) on \(\mathbb{R}^3\), such that, uniformly in \(m \geq 0\),

\[
\|e^{2F} \phi_m\| \leq C(d, a, \gamma), \quad \|e^{2F} S_{A_m} \phi_m\| \leq C(d, a, \gamma).
\]

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We shall keep the parameter $a$ and the weight function $F$ fixed in the whole section and use them without further explanations. Moreover, we put

\begin{equation}
R_{A_m}^\pm(iy) := (D_{A_m} \pm i\alpha \cdot \nabla F - iy)^{-1}, \quad y \in \mathbb{R},
\end{equation}

which is the continuous extension of $e^{\pm F} R_{A_m}(iy) e^{\mp F}$ and satisfies

\begin{equation}
\| R_{A_m}^\pm(iy) \| \leq C (1 + y^2)^{-1/2}, \quad y \in \mathbb{R};
\end{equation}

see (A.1) and (A.2).

8.1. **Pauli-Fierz transformation.** The unitary Pauli-Fierz transformation, $U$, is given as

\begin{equation}
U := \int_{\mathbb{R}^3} 1 \otimes e^{i\mathbf{x} \cdot A_m(0)} d^3\mathbf{x}.
\end{equation}

For all $\mathbf{x} \in \mathbb{R}^3$ and almost every $k = (\mathbf{k}, \lambda) \in \mathbb{R}^3 \times \mathbb{Z}_2$, we set

\begin{equation}
\tilde{G}_{m,\mathbf{x}}^{e,\Lambda}(k) := (e^{-i\mathbf{k} \cdot \mathbf{x}} - 1) G_{m,0}^{e,\Lambda}(k),
\end{equation}

where $G_{m,0}^{e,\Lambda}(k) = 1_{\{|k| \geq m\}} G_0^{e,\Lambda}(k)$ and $G_0^{e,\Lambda}(k)$ is given by (2.8). Then the gauge transformed vector potential is

\begin{equation}
\tilde{A}_m := A_m - 1 \otimes A_m(0) = \int_{\mathbb{R}^3} 1 \otimes (a^\dagger (\tilde{G}_{m,\mathbf{x}}^{e,\Lambda}) + a(\tilde{G}_{m,\mathbf{x}}^{e,\Lambda})) d^3\mathbf{x}.
\end{equation}

In fact, using $[U, \alpha \cdot A_m] = 0$ we deduce that $U D_{A_m} U^* = D_{\tilde{A}_m}$, whence

\[ U R_{A_m}(iy) U^* = R_{\tilde{A}_m}(iy), \quad U S_{A_m} U^* = S_{\tilde{A}_m}, \quad U |D_{A_m}| U^* = |D_{\tilde{A}_m}|. \]

It is favorable to work in the new gauge since $\tilde{G}_m^{e,\Lambda}$ has a less singular infra-red behavior than $G_m^{e,\Lambda}$. In fact, we have the elementary bound

\begin{equation}
|\tilde{G}_{m,\mathbf{x}}^{e,\Lambda}(k)| \leq 1_{\{|k| \geq m\}} \min \{2, |\mathbf{k}| \} |G_0^{e,\Lambda}(k)|.
\end{equation}

In order to gain an extra power of $|\mathbf{k}|$ from the previous estimate we have to control the multiplication operator $|\mathbf{x}|$ in (8.5). In our estimates below this is possible thanks to the spatial localization of $\phi_m$. We put

\begin{equation}
\tilde{H}_{\gamma, m} := U H_{\gamma, m} U^*, \quad \tilde{H}_f := U H_f U^*, \quad \tilde{\phi}_m := U \phi_m.
\end{equation}

On $U \mathcal{D}_0$ we have

\begin{equation}
\tilde{H}_{\gamma, m} = S_{\tilde{A}_m} D_{\tilde{A}_m} + \frac{1}{2} \left( \tilde{H}_f - \gamma \frac{\gamma}{|\mathbf{x}|} \right) + \frac{1}{2} S_{\tilde{A}_m} \left( \tilde{H}_f - \gamma \frac{\gamma}{|\mathbf{x}|} \right) S_{\tilde{A}_m}.
\end{equation}

We recall that, for $f \in \mathcal{X}_0$,

\begin{equation}
a(f) U = U \left( a(f) + i \langle f | G_{m,0}^{e,\Lambda} | \mathbf{x} \rangle \right),
\end{equation}

\begin{equation}
U^* a^\dagger(f) = (a^\dagger(f) - i \langle G_{m,0}^{e,\Lambda} | f \rangle \cdot \mathbf{x}) U^*.
\end{equation}
8.2. Remarks on the commutator $[a^z(f), S^{\Lambda_{m}}_{\Lambda}]$. Let $a^2$ be $a$ or $a^\dagger$ and $f \in \mathcal{X}_0$ with $\omega^{-1/2}f \in \mathcal{X}_0$. The coupling function $G^\Lambda_{m}$ satisfies Hypothesis 2.1 with $\omega = \omega$. Thus, we again know from [21, Lemma 3.3] that $S^{\Lambda_{m}}_{\Lambda}$ maps $\mathcal{D}(H^\nu_{\tilde{\nu}})$ into itself, for $\nu \in [0,1]$. In particular, $[a^2(f), S^{\Lambda_{m}}_{\Lambda}]$ is well-defined a priori on $\mathcal{D}(H^\nu_{\tilde{\nu}})$. Using (2.2), (2.14), and (8.3) it is straightforward to show that it has an extension to an element of $\mathcal{L}(\mathcal{H})$ given as

$$
[a^2(f), S^{\Lambda_{m}}_{\Lambda}] = \pm U \int_\mathbb{R} R^{\Lambda_{m}}_{\Lambda}(iy) \alpha \cdot \langle f | \tilde{G}^{\nu,\Lambda}_{m,x} \rangle R^{\Lambda_{m}}_{\Lambda}(iy) \frac{dy}{\pi} U^*. 
$$

If $z = \dagger$, then we choose the + - sign and the superscript $z$ at the scalar product in (8.10) denotes complex conjugation. Otherwise we choose $-$ and $z$ has to be ignored. By the spectral calculus, (A.5) and (A.6) of the appendix, and Lemma A.2 we further know that, for all $y \in \mathbb{R}$, $\kappa \in [0,1)$, and $\sigma \in \{1/2, 1\}$,

$$
\| D^{\Lambda_{m}}_{\Lambda} |^{\kappa} R^{\Lambda_{m}}_{\Lambda}(iy) \| \leq C(\kappa) (1 + y^2)^{-(\kappa - 1/2)/2},
$$

$$
\| |x|^{-\kappa} R^{\Lambda_{m}}_{\Lambda}(iy) (H_{\tilde{\nu}} + 1)^{-1/2} \| \leq C(d, \kappa) (1 + y^2)^{-(\kappa - 1/2)/2},
$$

$$
\| (H_{\tilde{\nu}} + 1)^{\sigma} R^{\Lambda_{m}}_{\Lambda}(iy) (H_{\tilde{\nu}} + 1)^{-\sigma} \| \leq C(d) (1 + y^2)^{-\sigma/2}.
$$

From these bounds we readily infer that the operator in (8.10) maps $\mathcal{H}$ into $\mathcal{D}(D^{\Lambda_{m}}_{\Lambda} |^{\kappa})$ and $\mathcal{D}(\tilde{H}_{\tilde{\nu}}^\nu)$ into $\mathcal{D}(|x|^{-\kappa} \cap \mathcal{D}(\tilde{H}_{\tilde{\nu}}^\nu))$ that and that, uniformly in $m \geq 0$,

$$
\| D^{\Lambda_{m}}_{\Lambda} |^{\kappa} [a^2(f), S^{\Lambda_{m}}_{\Lambda}] \| \leq C(\kappa),
$$

$$
\| |x|^{-\kappa} [a^2(f), S^{\Lambda_{m}}_{\Lambda}] (H_{\tilde{\nu}} + 1)^{-1/2} \| \leq C(d, \kappa),
$$

$$
\| (H_{\tilde{\nu}} + 1)^{\sigma} [a^2(f), S^{\Lambda_{m}}_{\Lambda}] (H_{\tilde{\nu}} + 1)^{-\sigma} \| \leq C(d), \quad \sigma \in \{1/2, 1\}.
$$

8.3. A formula for $a(k) \phi_m$. Our aim in the following is to derive the formula $a(k) \phi_m = \Phi(k)$, for almost every $k$, where $\Phi(k)$ is defined in (8.32) below. The infra-red bounds can then be easily read off from this representation. From now on we drop the reference to $\kappa$, $\Lambda$, $\gamma$, and $m$ in the notation.

We fix some $p = (p, \mu) \in \mathbb{R}^3 \times \mathbb{Z}_2$ with $p \neq 0$ and set $\omega_p = |p|$. Moreover, we pick $k' \in U \mathcal{D}_0$ and $f \in C^\infty_0((\mathbb{R}^3 \setminus \{0\}) \times \mathbb{Z}_2)$. Then the eigenvalue equation $\tilde{H} \phi = E \phi$ implies

$$
\langle (\tilde{H} - E + \omega_p) \eta' | a(f) \phi \rangle = \langle \tilde{H} \eta' | a(f) \phi \rangle - \langle a^\dagger(f) \eta' | \tilde{H} \phi \rangle + \langle \eta' | a(\omega_p f) \phi \rangle = u_1(\eta')/2 + u_2(\eta')/2 + u_3(\eta') + u_4(\eta')/2,
$$

where the functionals $u_j$ contain contributions from various terms in (8.7). They are defined in the course of the following discussion. Using (8.8) and (8.9),

$$30$$
\[ [H_\ell, x] = 0, \text{ and } [H_\ell, a(f)] = -a(\omega_f), \] we observe that
\[
  u_1(\eta') := \langle U H_\ell U^* \eta' | a(f) U \phi \rangle - \langle U^* a^\dagger(f) \eta' | H_\ell \phi \rangle + \langle \eta' | a(\omega_p f) \tilde{\phi} \rangle
\]
(8.18)
\[
  = -\langle U^* \eta' | a((\omega - \omega_p) f) \phi \rangle + i\omega_p \langle U^* \eta' | \langle f | G_0 \rangle \cdot x \phi \rangle.
\]
Furthermore,
\[
  u_2(\eta') := \langle S_\tilde{A} H_\ell S_\tilde{A} \eta' | a(f) \tilde{\phi} \rangle - \langle a^\dagger(f) \eta' | S_\tilde{A} H_\ell S_\tilde{A} \tilde{\phi} \rangle + \langle \eta' | a(\omega_p f) \tilde{\phi} \rangle
\]
\[
  = \langle \tilde{H}_\ell S_\tilde{A} \eta' | a(f) S_\tilde{A} \tilde{\phi} \rangle - \langle a^\dagger(f) S_\tilde{A} \eta' | \tilde{H}_\ell S_\tilde{A} \tilde{\phi} \rangle + \langle \eta' | a(\omega_p f) \tilde{\phi} \rangle
\]
\[
  + \langle \tilde{H}_\ell S_\tilde{A} \eta' | [S_\tilde{A}, a(f)] \tilde{\phi} \rangle - \langle [S_\tilde{A}, a^\dagger(f)] \eta' | \tilde{H}_\ell S_\tilde{A} \tilde{\phi} \rangle.
\]
Using a computation analogous to (8.18) and writing
\[
  -S_\tilde{A} a(\omega f) S_\tilde{A} = -a(\omega f) + S_\tilde{A} [S_\tilde{A}, a(\omega f)],
\]
we arrive at
\[
  u_2(\eta') = u_1(\eta') + \langle S_\tilde{A} U^* \eta' | [S_\tilde{A}, a(\omega f)] \phi \rangle
\]
(8.19)
\[
  + \langle \tilde{H}_\ell^{1/2} S_\tilde{A} \eta' | \tilde{H}_\ell^{1/2} [S_\tilde{A}, a(f)] \tilde{\phi} \rangle - \langle [S_\tilde{A}, a^\dagger(f)] \eta' | \tilde{H}_\ell S_\tilde{A} \tilde{\phi} \rangle.
\]
To treat the remaining terms in (8.17) we pick some \( \kappa \in (1/2, 1) \) and set \( \nu := 1 - \kappa \). Since \( D_\tilde{A} = |D_\tilde{A}|^{\nu} S_\tilde{A} |D_\tilde{A}|^{\nu} \) we obtain
\[
  u_3(\eta') := \langle [a^\dagger(f), S_\tilde{A}] D_\tilde{A} \eta' | \tilde{\phi} \rangle
\]
\[
  = \langle [D_\tilde{A}]^{\nu} \eta' | S_\tilde{A} [D_\tilde{A}]^{\nu} S_\tilde{A} \tilde{\phi} \rangle - \langle \eta' | \alpha \cdot \langle f | \tilde{G}_x \rangle S_\tilde{A} \tilde{\phi} \rangle.
\]
Finally, we have
\[
  u_4(\eta') := -\gamma \langle |x|^{-\nu} S_\tilde{A} \eta' | |x|^{-\kappa} [S_\tilde{A}, a(f)] \tilde{\phi} \rangle
\]
(8.21)
\[
  -\gamma \langle |x|^{-\kappa} [a^\dagger(f), S_\tilde{A}] \eta' | |x|^{-\nu} S_\tilde{A} \tilde{\phi} \rangle.
\]
We briefly explain why \( \eta' \in U \mathcal{D}_0 \) can be replaced by any element of \( Q(\tilde{H}) \) in the above formulas: On the one hand this is due to (8.14)–(8.16) and the following consequences of (3.24), (3.25), and (3.28) (here we use \( \nu < 1/2 \)),
\[
  ||D_\tilde{A}|| (H + C(d, \nu))^{-1/2} \leq 1, \quad |||x|^{-\nu} (H + C(d, \nu))^{-1/2} \leq 1,
\]
(8.22)
\[
  ||H_\ell^\nu S_\tilde{A} (H + C(d))^\sigma|| \leq 1, \quad \sigma \in \{1/2, 1\}.
\]
Indeed, using (8.14)–(8.16) and (8.22) & (8.23) we conclude by inspection that \( u_1, \ldots, u_4 \) extend to continuous linear functionals on \( Q(\tilde{H}) \) (equipped with the form norm). On the other hand we show in Appendix 13 that \( a(f) \tilde{\phi} \in Q(\tilde{H}) \). Since \( U \mathcal{D}_0 \) is a form core for \( \tilde{H} \) this implies that the equality
\[
  \langle (\tilde{H} - E + \omega_p) \eta' | a(f) \tilde{\phi} \rangle = \overline{u}_1(\eta')/2 + \overline{u}_2(\eta')/2 + \overline{u}_3(\eta') + \overline{u}_4(\eta')/2
\]
(8.24)
holds, for all \( \eta' \in \mathcal{Q}(\tilde{H}) \). In particular, we may choose \( \eta' := U \mathcal{R}_p \eta \), for every \( \eta \in \mathcal{H} \), with
\[
\mathcal{R}_p := (H - E + \omega_p)^{-1}.
\]

In the next step we substitute a family, \( \{ f_{p,\epsilon} \}_{\epsilon > 0} \), of approximate delta functions for \( f \) and pass to the limit \( \epsilon \downarrow 0 \). So let \( h \in C^0_{\infty}(\mathbb{R}^3; [0, \infty)) \) with \( \text{supp}(h) \subset \{ |k| < 1 \} \) and \( \int_{\mathbb{R}^3} h(k) \, dk = 1 \) and set \( h_\epsilon := \epsilon^{-3} h(\cdot/\epsilon) \). Then we choose \( f := f_{p,\epsilon} \), where \( f_{p,\epsilon}(k) := h_\epsilon(k - p) \) \( \delta_{\mu,\lambda} \), for \( k = (k, \lambda) \in \mathbb{R}^3 \times \mathbb{Z}_2 \) and \( \epsilon > 0 \). Multiplying both sides of (8.24), where now \( \eta' = U \mathcal{R}_p \eta \), with some \( g \in C^0_{\infty}((\mathbb{R}^3 \setminus \{0\}) \times \mathbb{Z}_2, \mathbb{C}) \) and integrating with respect to \( p = (p, \mu) \), we obtain
\[
(8.25) \quad \int g(p) \, \langle U \, \eta \mid a(f_{p,\epsilon} \tilde{\phi}) \rangle \, dp = \sum_{i=0}^{8} C_i(\epsilon) \cdot
\]

Here \( C_0(\epsilon), \ldots, C_4(\epsilon) \) contain all contributions from \( u_1 \) and \( u_2 \), \( C_5(\epsilon) \) and \( C_6(\epsilon) \) contain those of \( u_3 \), and \( C_7(\epsilon) \) and \( C_8(\epsilon) \) account for \( u_4 \). As \( \epsilon \downarrow 0 \), the LHS of (8.25) tends to
\[
(8.26) \quad \langle U \, \eta \mid a(\overline{g}) \tilde{\phi} \rangle = \langle \eta \mid a(\overline{g}) \phi \rangle - i \langle \eta \mid \langle \overline{g} \mid G_0 \cdot x \rangle \phi \rangle,
\]

because of \( h_\epsilon * g \to g \) in \( L^2 \), Fubini’s theorem, and (8.8). The terms contained both in \( u_1 \) and \( u_2 \) give rise to (compare (8.18) and (8.19))
\[
C_0(\epsilon) := \int g(p) \, \langle \mathcal{R}_p \eta \mid a((\omega_p - \omega) f_{p,\epsilon}) \phi \rangle \, dp,
\]
\[
C_1(\epsilon) := i \int g(p) \, \omega_p \, \langle f_{p,\epsilon} \mid [G_0] \cdot (\mathcal{R}_p \eta \mid x \phi) \rangle \, dp.
\]

The remaining terms in \( u_2 \) are accounted for by (compare (8.19))
\[
C_2(\epsilon) := \frac{1}{2} \int g(p) \, \langle \mathcal{R}_p S_A \eta \mid [S_A, a(\omega f_{p,\epsilon})] \phi \rangle \, dp,
\]
\[
C_3(\epsilon) := \frac{1}{2} \int g(p) \, \langle U H_t \mathcal{R}_p S_A \eta \mid [S_A, a(f_{p,\epsilon})] \phi \rangle \, dp,
\]
\[
C_4(\epsilon) := \frac{1}{2} \int g(p) \, \langle \tilde{H}_t a(f_{p,\epsilon}), S_A \mid U \mathcal{R}_p \eta \mid S_A \phi \phi \rangle \, dp.
\]

Likewise, we have (compare (8.20))
\[
C_5(\epsilon) := \int g(p) \, \langle U S_A \mid D_A |^\nu \mathcal{R}_p \eta \mid D_A |^\nu [S_A, a(f_{p,\epsilon})] \phi \rangle \, dp,
\]
\[
C_6(\epsilon) := - \int g(p) \, \langle \mathcal{R}_p \eta \mid \alpha \cdot (f_{p,\epsilon} \mid \tilde{G}_x) S_A \phi \rangle \, dp,
\]

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and (see (8.21))

\[ C_7(\epsilon) := -\frac{\gamma}{2} \int g(p) \left< U |x|^{-\nu} R_p S_A \eta \left| |x|^{-\nu} \left[ S_{\tilde{A}}^{-1} a(f_{p,\epsilon}) \right] \phi \right> dp, \]

\[ C_8(\epsilon) := -\frac{\gamma}{2} \int g(p) \left< |x|^{-\nu} \left[ a^\dagger(f_{p,\epsilon}) \right] S_{\tilde{A}} \right| U R_p \eta \left| |x|^{-\nu} S_{\tilde{A}} \phi \right> dp. \]

To discuss the RHS of (8.25) we start with \( C_0(\epsilon) \), which converges to zero. (Almost the same term is treated in [16]. We repeat its discussion for the sake of completeness.) In fact,

\[ C_0(\epsilon) \leq \epsilon \int |g|(p) \| R_p \|^2 \| \eta \|^2 dp + \tilde{C}_0(\epsilon) \frac{4\epsilon}{\epsilon}, \]

\[ \tilde{C}_0(\epsilon) := \int |g|(p) \| a((\omega_p - \omega) f_{p,\epsilon}) \phi \|^2 dp. \]

Since \(|\omega_p - \omega| \leq \epsilon\) on \( \text{supp}(f_{p,\epsilon}) \), we further have

\[ \tilde{C}_0(\epsilon) = \int |g|(p) \left\{ \int (\omega_p - \omega_k) f_{p,\epsilon}(k) a(k) \phi dk \right\}^2 dp \]
\[ \leq \epsilon^2 \int |g|(p) \left\{ \int \frac{f_{p,\epsilon}(k')}{\omega(k')} dk' \right\} \int f_{p,\epsilon}(k) \omega(k) \| a(k) \phi \|^2 dk dp. \]

Here the integral in the curly brackets \{ \cdots \} is bounded by some \( K \in (0, \infty) \) uniformly in \( p \) as long as \( \epsilon \leq \text{dist}(0, \text{supp}(g))/2 \), whence

\[ \tilde{C}_0(\epsilon) \leq \epsilon^2 K \int (|g| * h_\epsilon)(k) \omega(k) \| a(k) \phi \|^2 dk. \]

Since \(|g| * h_\epsilon \rightarrow |g|\) in \( L^\infty \) and \( \phi \in \mathcal{D}(H_{1/2}) \) we conclude that \( C_0(\epsilon) \rightarrow 0 \).

Next, we claim that, by means of Fubini’s theorem, all the remaining expressions can be written in the form

\[ C_i(\epsilon) = \sum_{\lambda \in \mathbb{Z}_2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} g(p, \lambda) h_\epsilon(k - p) G_0(k, \lambda) \cdot s_i(k, p) d^3k d^3p, \quad i = 1, \ldots, 8, \]

where the vectors \( s_i \) are continuous on \( (\mathbb{R}^3 \setminus \{0\}) \times \mathbb{R}^3 \), so that

\[ (8.27) \quad \lim_{\epsilon \searrow 0} C_i(\epsilon) = \int g(k) G_0(k) s_i(k, k) dk, \quad i = 1, \ldots, 8. \]
In fact, using the representation \((8.11)\) it can be easily read off from the definitions of \(C_i(\epsilon)\) that

\[
\begin{align*}
  s_1(k, p) &= i|p| \langle \mathcal{R}_p \eta | x \phi \rangle, \\
  s_2(k, p) &= |k| \int \langle \mathcal{R}_p S_A \eta | T_2(y, k) e^F \phi \rangle \frac{dy}{2\pi}, \\
  s_3(k, p) &= \int \langle H_f \mathcal{R}_p S_A \eta | T_3(y, k) e^F \phi \rangle \frac{dy}{2\pi}, \\
  s_4(k, p) &= \int \langle T_4(y, k) \Theta \mathcal{R}_p \eta | e^F S_A \phi \rangle \frac{dy}{2\pi},
\end{align*}
\]

where (with the notation \((8.2)\) and \(\Theta := H_f + 1\))

\[
\begin{align*}
  T_2(y, k) &= R_A(iy) \alpha e^{-ik \cdot x} e^{-F} R_A^F(iy), \\
  T_3(y, k) &= R_A(iy) \alpha \left( e^{-ik \cdot x} - 1 \right) e^{-F} R_A^F(iy), \\
  T_4(y, k) &= \{ H_f R_A^F(iy) \Theta^{-1} \} \alpha \left( e^{ik \cdot x} - 1 \right) e^{-F} \{ \Theta R_A(iy) \Theta^{-1} \}.
\end{align*}
\]

Moreover,

\[
\begin{align*}
  s_5(k, p) &= \int \langle |D_A|^\nu \mathcal{R}_p S_A \eta | T_5(y, k) e^F \phi \rangle \frac{dy}{\pi}, \\
  s_6(k, p) &= -\langle \mathcal{R}_p \eta | \alpha \left( e^{-ik \cdot x} - 1 \right) e^{-F} (e^F S_A \phi) \rangle, \\
  s_7(k, p) &= -\gamma \frac{1}{2} \int \langle \mathcal{R}_p \eta | T_7(y, k) \Theta^{1/2} e^F \phi \rangle \frac{dy}{\pi}, \\
  s_8(k, p) &= -\gamma \frac{1}{2} \int \langle \mathcal{R}_p \eta | T_8(y, k) \Theta^{1/2} e^F \phi \rangle \frac{dy}{\pi},
\end{align*}
\]

where

\[
\begin{align*}
  T_5(y, k) &= \{ |D_A|^\nu R_A(iy) \} \alpha \left( e^{-ik \cdot x} - 1 \right) e^{-F} R_A^F(iy), \\
  T_7(y, k) &= \{ |x|^{-\nu} R_A(iy) \Theta^{-\frac{1}{2}} \} \alpha \left( e^{-ik \cdot x} - 1 \right) e^{-F} \{ \Theta^{\frac{1}{2}} R_A^F(iy) \Theta^{-\frac{1}{2}} \}, \\
  T_8(y, k) &= \{ |x|^{-\nu} R_A^F(iy) \Theta^{-\frac{1}{2}} \} \alpha \left( e^{ik \cdot x} - 1 \right) e^{-F} \{ \Theta^{\frac{1}{2}} R_A(iy) \Theta^{-\frac{1}{2}} \}.
\end{align*}
\]

On account of \((8.11)-(8.13)\) all operators in curly brackets \(\{ \cdots \}\) appearing in the definitions of \(T_4, T_5, T_7,\) and \(T_A\) are bounded and the integrals over \(y\) in the definitions of \(s_i(k, p)\) converge absolutely. In virtue of \((3.24), (3.25),\) and \(3.28\) we further have (since \(\nu < 1/2\))

\[
(8.28) \quad \| |D_A|^\nu \mathcal{R}_k \| \leq C'(d, \nu) \frac{1}{1 \wedge |k|}, \quad \| |x|^{-\nu} \mathcal{R}_k \| \leq C'(d, \nu) \frac{1}{1 \wedge |k|}, \quad \| H_f \mathcal{R}_k \| \leq C(d) \frac{1}{1 \wedge |k|}.
\]

Moreover, as a trivial consequence of \((8.1), (8.22),\) and \([S_A, H] = 0\) we have

\[
(8.29) \quad \| |x|^{-\nu} e^F \phi \| \leq C(d, a, \nu, \gamma), \quad \| |x|^{-\nu} e^F S_A \phi \| \leq C(d, a, \nu, \gamma).
\]
Thanks to \((8.28)\) and \((8.1)\) we finally know that

\[(8.30) \quad \| \Theta^{1/2} e^F \phi \|^2 \leq \| \Theta \phi \| \| e^{2F} \phi \| \leq C(d, \gamma) \left( \| H \phi \| + 1 \right) = C(d, \gamma) (|E| + 1).\]

Recall that \((8.26)\) is the limit of the LHS of \((8.25)\) and \(C_0(\epsilon) \to 0\). Taking this, \((8.27)\), and the preceding remarks into account we see that the limit of \((8.25)\) can be written as

\[(8.31) \quad \int g(k) \langle \eta | a(k) \phi \rangle dk = \int g(k) \langle \eta | \Phi(k) \rangle dk,
\]

for some function \((\mathbb{R}^3 \setminus \{0\}) \times \mathbb{Z}_2 \ni k \mapsto \Phi(k) \in H\). Indeed, writing

\[T_i(y, k) := G_0(k) \cdot T_i(y, k), \quad i = 2, 3, 4, 5, 7, 8,
\]

and using \(G_0(k) \cdot \alpha (e^{-ik \cdot x} - 1) = \alpha \cdot G_x(k)\) we find

\[\Phi(k) := i(1 + |k| \mathcal{R}_k) G_0(k) \cdot [\mathbf{x} \phi - \mathcal{R}_k \alpha \cdot \tilde{G}_x(k) e^{-F}(e^F S_A \phi)]
\]

\[+ |S_A| |\mathcal{R}_k|^2 \int_\mathbb{R} T_2(y, k) \frac{dy}{2\pi} e^F \phi + S_A \{H_1 \mathcal{R}_k\}^* \int_\mathbb{R} T_3(y, k) \frac{dy}{2\pi} e^F \phi
\]

\[+ \{\Theta \mathcal{R}_k\}^* \int_\mathbb{R} T_4(y, k) \frac{dy}{2\pi} e^F S_A \phi
\]

\[+ S_A \{|D_A| \mathcal{R}_k\}^* \int_\mathbb{R} T_5(y, k) \frac{dy}{\pi} e^F \phi
\]

\[- \frac{\gamma}{2} S_A \{|x|^{-\nu} \mathcal{R}_k\}^* \int_\mathbb{R} T_7(y, k) \frac{dy}{\pi} \Theta^{1/2} e^F \phi
\]

\[(8.32) \quad - \frac{\gamma}{2} \{\Theta^{1/2} \mathcal{R}_k\}^* \int_\mathbb{R} T_8(y, k) \frac{dy}{\pi} \{|x|^{-\nu} e^F S_A \phi\}.
\]

As \(g \in C_0^\infty((\mathbb{R}^3 \setminus \{0\}) \times \mathbb{Z}_2, C)\) is arbitrary in \((8.31)\) we have \(\langle \eta | a(k) \phi \rangle = \langle \eta | \Phi(k) \rangle\), for all \(k\) outside some set of measure zero, \(N_\eta\), which depends on \(\eta\). Choosing \(\eta\) from a countable dense subset \(H \subset H\) we conclude that \(a(k) \phi = \Phi(k)\), for all \(k \notin N\), where \(N := \bigcup_{\eta \in \mathcal{E}} N_\eta\) has zero measure. Thus, we have proved the following lemma:

**Lemma 8.1.** Let \(e \in \mathbb{R}, \Lambda > 0, m \geq 0, \gamma \in (0, \gamma^{mp})\), and \(\phi_m \in D(H_{\gamma, m})\) with \(H_{\gamma, m} \phi_m = E_{\gamma, m} \phi_m\). Then \(a(k) \phi_m = \Phi(k)\), for almost every \(k \in \mathbb{R}^3 \times \mathbb{Z}_2\), where \(\Phi(k)\) is defined in \((8.32)\).

**8.4. Derivation of the infra-red bounds.** In the following proof we again use the notation of the previous subsection. We drop the reference to \(e, \Lambda\), and \(\gamma\) in the notation, but re-introduce a subscript \(m\) when it becomes important.

**Proof of Proposition 7.1** The soft photon bound \((7.4)\) follows by combining Lemma 8.1 with the bounds \((8.28)-(8.30)\), \(|G_m, 0(k)| \leq C |k|^{-\frac{1}{2}} \mathbb{1}_{\{m \leq |k| \leq \Lambda\}}, \) and
\[
\left| \tilde{G}_{m,x}(k) \right| e^{-F(x)} \leq C' |k|^{1/2} \mathbb{1}_{\{m \leq k < \Lambda\}}, \text{ as well as}
\]
\[
\left\| k \int T_2(y, k) \frac{dy}{\pi} \right\| \leq |k| \left| G_{m,0}(k) \right| \int_R \frac{C \, dy}{1 + y^2} \leq C' |k|^{1/2} \mathbb{1}_{\{m \leq k < \Lambda\}},
\]
and, for \( i = 3, 4, 5, 7, 8 \),
\[
\left\| \int T_i(y, k) \frac{dy}{\pi} \right\| \leq \sup_x \left\{ \left| G_{m,x}(k) \right| e^{-F(x)} \right\} \int_R \frac{C(d) \, dy}{(1 + y^2)^{1-\kappa/2}} \leq C' (|k| \wedge 1)^{1/2} \mathbb{1}_{\{m \leq k < \Lambda\}}.
\]
Recall that \( T_i(y, k), i = 3, 4, 5, 7, 8 \), is defined by the same formula as \( T_i(y, k) \) except that \( \alpha (e^{-i k \cdot x} - 1) \) is replaced by \( \alpha \cdot \tilde{G}_{m,x}(k) \).

In order to prove the photon derivative bound (7.5) we again use the representation \( a(k, \phi_m) - a(p, \lambda) \phi_m = \Phi(k, \lambda) - \Phi(p, \lambda) \). Moreover, we apply the following bounds: First, by the resolvent identity,
\[
\left\| \mathcal{O} R_{k} - \mathcal{O} R_{p} \right\| \leq \frac{C |k - p|}{(1 \wedge |k|)(1 \wedge |p|)}, \quad \mathcal{O} \in \{1, H_I, \Theta^{1/2}, |D_{m}^{\nu}|, |x|^{-\nu}\}.
\]
Second, we have, for \( m < |k| < \Lambda \) and \( m < |p| < \Lambda \),
\[
\left\| \int \left( |k|T_2(y, \lambda, k) - |p|T_2(y, \lambda, p) \right) \frac{dy}{\pi} \right\| \leq \Delta'(k, p) \int_R \frac{C \, dy}{1 + y^2} = C' \Delta'(k, p),
\]
where
\[
\Delta'(k, p) := \max_{\lambda = 0, 1} \sup_x \left| k \right| G_{m,x}(\lambda, k) - \left| p \right| G_{m,x}(\lambda, p) \right| e^{-F(x)}.
\]
In view of (8.11)–(8.13) we further have, for \( i = 3, 4, 5, 7, 8 \), and \( m < |k| < \Lambda \) and \( m < |p| < \Lambda \),
\[
\left\| \int \left( T_i(y, \lambda, k) - T_i(y, \lambda, p) \right) \frac{dy}{\pi} \right\| \leq \Delta''(k, p) \int_R \frac{C(d) \, dy}{(1 + y^2)^{1-\kappa/2}} = C' \Delta''(k, p),
\]
where, again for \( m < |k|, |p| < \Lambda \),
\[
\Delta''(k, p) := \max_{\lambda = 0, 1} \sup_x |\tilde{G}_{m,x}(\lambda, k) - \tilde{G}_{m,x}(\lambda, p)| e^{-F(x)}.
\]
To obtain (7.5) it now suffices to recall the following bound from [12] (see also [16] Appendix A): For \( m < |k|, |p| < \Lambda \),
\[
\frac{1}{|k|} \left\{ \left| k \right| G_{m,0}(\lambda, k) - \left| p \right| G_{m,0}(\lambda, p) \right\} + \Delta'(k, p) + \Delta''(k, p)
\]
\[
+ \frac{|k - p|}{|k| |p|} \left\{ \left| p \right| G_{m,0}(\lambda, p) \right\} + \left| \tilde{G}_{m,x}(\lambda, p) \right| \right\}
\]
\[
\leq C |k - p| \left( \frac{1}{|k|^{1/2} |k|} + \frac{1}{|p|^{1/2} |p|} \right).
\]

Here the special form of the polarization vectors (2.7) is exploited. Notice also that some weight like $e^{-F(x)}$ is required in $\triangle'(k, p)$ and $\triangle''(k, p)$ since the RHS of $|e^{-ik\cdot x} - e^{-i p\cdot x}| \leq |k - p||x|$ in unbounded w.r.t. $x$. □

Finally, an inspection of the above proof and the preceding subsection readily shows that the assertion of Remark 7.2 holds true. In fact, all $\gamma$-dependent contributions to the constant on the RHS of the infra-red bounds stem from (8.29) and (8.30). If, however, the uniform bound (7.7) is valid, the $n$ in (8.29) and (8.30) can be replaced by $\gamma$-independent constants. Notice also that $E \equiv E_{\gamma,m}$ in (8.30) satisfies

$$E_{\gamma} \leq \Sigma(\omega, G^A_{e,\Lambda}) \leq C(e, \Lambda).$$

**Appendix A. Estimates on functions of the Dirac operator**

In this appendix we derive some technical estimates we have repeatedly referred to in the main text. To this end we always assume that $\varpi$ and $G$ fulfill Hypothesis 2.1 with constant $d$ and that $\tilde{G}$ is another coupling function such that $\varpi$ and $\tilde{G}$ fulfill Hypothesis 2.1 with the same constant $d$. Furthermore, we introduce the parameter

$$\triangle(a) := \int \left(1 + \frac{1}{\varpi(k)}\right) \sup_{x \in \mathbb{R}^3} e^{-2a|x|} |G_x(k) - \tilde{G}_x(k)|^2 \, dk, \quad a \geq 0.$$ 

We define $A$ as usual by (2.3) and $\tilde{A}$ by (2.3) with $\tilde{G}$ instead of $G$.

First, we collect some necessary prerequisites: We recall that, for $y \in \mathbb{R}$, $a \in [0, 1/2]$, and $F \in C^\infty(\mathbb{R}^3; \mathbb{R})$ with fixed sign and satisfying $|\nabla F| \leq a$, we have $iy \in \mathcal{A}(\Delta_A + i\alpha \cdot \nabla F)$,

$$R^F_A(iy) := e^F R^A_A(iy) e^{-F} = (\Delta_A + i\alpha \cdot \nabla F - iy)^{-1} \text{ on } \mathcal{D}(e^{-F}),$$

and

$$\|R^F_A(iy)\| \leq C (1 + y^2)^{-1/2}.$$ 

The bound (A.2) is essentially well-known. For instance, its proof given in [22] for classical vector potentials works for quantized ones as well. Next, we set

$$\Upsilon_F^\nu(y) := \sum_{\ell=0}^{\infty} \{ -Z_{\nu,\delta}^* R^F_A(iy) \}^\ell, \quad \text{and, say,} \quad \|\Upsilon_F^\nu(y)\| \leq 2.$$

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It is straightforward (compare [21, Corollary 3.1] and [20, Lemma 3.3] for negative \(\nu\)) to verify that

\[
R_\Lambda^F(iy) \tilde{H}^{-\nu}_t = \tilde{H}^{-\nu}_t R_\Lambda^F(iy) \Upsilon_\nu^F(y).
\]

In particular, \(R_\Lambda^F(iy)\) maps \(\mathcal{D}(d\Gamma(\omega)^\nu)\) into itself. If \(\Lambda\) is replaced by \(\tilde{\Lambda}\) in (A.3)-(A.6), then we denote the corresponding operator as \(\tilde{\Upsilon}_\nu^F(y)\).

**Lemma A.1.** Let \(\mu, \nu \in [-1, 1]\) with \(\mu \land \nu \leq -1/2\) and \(\mu + \nu \leq -1/2\), and \(\kappa \in [0, 1]\). Assume that \(a \in [0, 1/2]\) and \(F \in C^\infty(\mathbb{R}_x^3, [0, \infty))\) satisfies \(|\nabla F| \leq a\) and \(F(x) \geq a|x|\), for all \(x \in \mathbb{R}^3\). Then

\[
\begin{align*}
&|D_A|^{\kappa} \tilde{H}_t^\mu (S_A - S_{\tilde{A}}) \tilde{H}_t^\nu e^{-F} | \leq C(d, \kappa) \Delta^{1/2}(a), \\
&|D_A|^{\kappa} \tilde{H}_t^\mu e^{-F} (S_A - S_{\tilde{A}}) \tilde{H}_t^\nu | \leq C(d, \kappa) \Delta^{1/2}(a), \\
&|x|^{-\kappa} e^{-F} (S_A - S_{\tilde{A}}) \tilde{H}_t^{-1} | \leq C(d, \kappa) \Delta^{1/2}(a), \\
&|x|^{-\kappa} \tilde{H}_t^{\nu}^{-1} (S_A - S_{\tilde{A}}) \tilde{H}^{-\kappa}_t e^{-F} | \leq C(d, \kappa) \Delta^{1/2}(a), \quad \kappa \in (1/2, 1).
\end{align*}
\]

**Proof.** It is easy to verify that the following resolvent formula,

\[
\tilde{H}_t^\mu (R_\Lambda(iy) - R_\Lambda(iy)) \tilde{H}_t^\nu e^{-F} = R_\Lambda(iy) \Upsilon_\nu^F(y) T_{\mu,\nu}^F R_\Lambda^F(iy) \tilde{\Upsilon}_\nu^F(y),
\]

where \(T_{\mu,\nu} \in \mathcal{L}(\mathcal{H}_m)\) is the closure of \(\tilde{H}_t^\mu \alpha \cdot (\tilde{\Lambda} - \Lambda) e^{-F} \tilde{H}_t^\nu\) and satisfies \(\|T_{\mu,\nu}\| \leq C(d) \Delta^{1/2}(a)\) by a standard estimate and an interpolation argument or (A.4). Likewise, we have

\[
e^{-F} \tilde{H}_t^\mu (R_\Lambda(iy) - R_\Lambda(iy)) \tilde{H}_t^\nu = R_\Lambda^F(iy) \Upsilon_\mu^F(y) T_{\mu,\nu}^F R_\Lambda(iy) \tilde{\Upsilon}_\nu^F(y).
\]

Applying (2.14) and (A.11) we obtain, for all \(\varphi \in \mathcal{D}(\tilde{H}_t^{\nu} | D_A|^{\kappa})\) and \(\psi \in \mathcal{H}_m\),

\[
\begin{align*}
&\langle \tilde{H}_t^\mu | |D_A|^{\kappa} \varphi | (S_A - S_{\tilde{A}}) \tilde{H}_t^\nu e^{-F} \psi \rangle | \\
&\leq \int_{\mathbb{R}} \langle \varphi | |D_A|^{\kappa} R_\Lambda(iy) \Upsilon_\nu^F(y) T_{\mu,\nu}^F R_\Lambda^F(iy) \tilde{\Upsilon}_\nu^F(y) \psi \rangle dy \\
&\leq C'(d) \Delta^{1/2}(a) \int_{\mathbb{R}} \frac{dy}{(1 + y^2)^{1-\kappa/2}} \|\varphi\| \|\psi\|.
\end{align*}
\]

Here we also used (A.2), \(\| |D_A|^{\kappa} R_\Lambda(iy) \| \leq C(\kappa) (1 + y^2)^{(\kappa-1)/2}\), and (A.5). This shows that \(S_A - S_{\tilde{A}}\) maps \(\text{Ran}(e^{-F} \otimes \tilde{H}_t^\nu)\) into \(\mathcal{D}(|D_A|^\kappa \tilde{H}_t^\mu\})\). Since \(|D_A|^{\kappa}\) is continuously invertible we have \(|\tilde{H}_t^\mu | |D_A|^{\kappa} \tilde{H}_t^\nu\) and we obtain (A.7). (A.8) is proved in the same way using (A.12) and the bound \(\| |D_A|^{\kappa} R_\Lambda^F(iy) \| \leq C'(\kappa) (1 + y^2)^{(\kappa-1)/2}\), which is an easy consequence of (A.2) and the second resolvent identity. In order to derive (A.9) we employ the identity

\[
e^{-F} (R_\Lambda(iy) - R_\Lambda(iy)) \tilde{H}_t^{-1} = R_\Lambda^F(iy) \tilde{H}_t^{-1/2} T_{1/2,-1}^F R_\Lambda(iy) \tilde{\Upsilon}_0^F(y),
\]

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which together with the generalized Hardy inequality and (A.14) below yields
\[
\left| \langle |x|^{-\kappa} \varphi \mid e^{-F} (S_\mathcal{A} - S_{\overline{\mathcal{A}}}) \hat{H}^{-\frac{1}{2}} \psi \rangle \right|
\leq C(\kappa) \int_{\mathbb{R}} \| \varphi \| \| D_0^\kappa R_\mathcal{A}^F (iy) \hat{H}^{-\frac{1}{2}} \| \| T_{1,2,-1}^F \| \| R_{\overline{\mathcal{A}}} (iy) \| \| \tilde{\Upsilon}^0 (y) \| \| \psi \| \frac{dy}{\pi}
\leq C(d, \kappa) \triangle^{1/2} (a) \int_{\mathbb{R}} \frac{dy}{(1 + y^2)^{1-\kappa/2}} \cdot \| \varphi \| \| \psi \| ,
\]
for all \( \varphi \in \mathcal{D}(\| x \|^{-\kappa}) \) and \( \psi \in \mathcal{H}_m \). In a similar fashion we prove (A.10) by means of (A.14) and the identity
\[
\hat{H}^{-\kappa-1} (R_{\mathcal{A}} (iy) - R_{\overline{\mathcal{A}}} (iy)) \hat{H}^{-\kappa} e^{-F} = \hat{H}^{-\kappa-1} R_{\mathcal{A}} (iy) \hat{H}^{-\delta} T_{\delta,-\kappa}^F R_{\mathcal{A}}^F (iy) \tilde{\Upsilon}^F (y) ,
\]
where \( \delta := \kappa - 1/2 \), so that, with \( \nu := 1 - \kappa = 1/2 - \delta \),
\[
\begin{align*}
\hat{H}^{-\kappa-1} R_{\mathcal{A}} (iy) \hat{H}^{-\delta} &= R_{\mathcal{A}} (iy) \hat{H}^{-\frac{1}{2}} + [\hat{H}^{-\nu}, R_{\mathcal{A}} (iy)] \hat{H}^{-\delta} \\
&= R_{\mathcal{A}} (iy) \hat{H}^{-\frac{1}{2}} + [\hat{H}^{-\nu}, R_{\mathcal{A}} (iy)] R_{\mathcal{A}} (iy) \hat{H}^{-\delta} \\
&= R_{\mathcal{A}} (iy) \hat{H}^{-\frac{1}{2}} \left( 1 + \hat{H}^{-\frac{1}{2}} [\alpha \cdot A, \hat{H}^{-\nu}] \hat{H}^{-\delta} (\hat{H}^{\delta} R_{\mathcal{A}} (iy) \hat{H}^{-\delta}) \right) \\
&= R_{\mathcal{A}} (iy) \hat{H}^{-1/2} \left( 1 - \mathcal{Z}_{\nu,1/2} R_{\mathcal{A}} (iy) \tilde{\Upsilon}^F (y) \right) .
\end{align*}
\]
Here the operator \( X := \{ \cdots \} \) is bounded due to (A.13) and (A.5). Therefore, the computation (A.13), which is justified at least on the domain \( \mathcal{D}_m \), shows that \( B := \hat{H}^{-\kappa-1} R_{\mathcal{A}} (iy) \hat{H}^{-\delta} \) maps \( \mathcal{H}_m \) into the domain of \( D_0^{\kappa} \) and \( \| D_0^{\kappa} B \| = \| D_0^{\kappa} R_{\mathcal{A}} (iy) \hat{H}^{-1/2} X \| \leq C(d, \kappa) (1 + y^2)^{(\kappa-1)/2} \), by (A.14).

Lemma A.2. Let \( a \in [0, 1/2] \) and \( F \in C^\infty(\mathbb{R}^3, \mathbb{R}) \) have a fixed sign with \( |\nabla F| \leq a \). Then \( R_\mathcal{A}^F (iy) \) maps \( \mathcal{D}(d\Gamma(\varpi)^{1/2}) \) into \( \mathcal{D}(D_0^{\kappa}) \), \( \kappa \in [0, 1] \), and
\[
\| D_0^{\kappa} R_\mathcal{A}^F (iy) (d\Gamma(\varpi) + 1)^{-1/2} \| \leq C(d, \kappa) (1 + y^2)^{(\kappa-1)/2} , \quad y \in \mathbb{R} .
\]

Proof. Using \( \| D_0^{\kappa} R_\mathcal{A}^F (iy) \| \leq f_\kappa (y) := C(\kappa) (1 + y^2)^{(\kappa-1)/2} \), \( \| \alpha \cdot \nabla F \| \leq a \leq 1/2 \), and the notation (A.3), we obtain, for all \( \varphi \in \mathcal{D}_m \),
\[
\| D_0^{\kappa} \varphi \| \leq f_\kappa (y) (\| (D_\mathcal{A} + i\alpha \cdot \nabla F + iy) \varphi \| + C(d) \| \hat{H}^{-1/2} \varphi \| ) .
\]
Now, let \( \eta \in \mathcal{H}_m \) and put \( \psi := R_\mathcal{A}^F (iy) \tilde{\Upsilon}^{1/2} \eta \), so that \( \psi \in \mathcal{D}(D_\mathcal{A}) \). Since \( D_\mathcal{A} \) is essentially self-adjoint on \( \mathcal{D}_m \) we find \( \psi_n \in \mathcal{D}_m \), \( n \in \mathbb{N} \), such that \( \psi_n \rightarrow \psi \) in the graph norm of \( D_\mathcal{A} \). By (A.4) \( [D_\mathcal{A}, \hat{H}^{-1/2}] \), defined on \( \mathcal{D}_m \), extends to a bounded operator on \( \mathcal{H}_m \). We deduce that \( \varphi_n := \hat{H}^{-1/2} \psi_n \rightarrow \hat{H}^{-1/2} \psi = R_\mathcal{A}^F (iy) \hat{H}^{-1/2} \eta \) in the graph norm of \( D_\mathcal{A} \), too. Thus, we may plug \( \varphi_n \in \mathcal{D}_m \) into (A.15), pass to the limit \( n \rightarrow \infty \), and apply (A.5) to arrive at \( \| D_0^{\kappa} R_\mathcal{A}^F (iy) \hat{H}^{-1/2} \eta \| \leq C'(d) f_\kappa (y) \| \eta \| . \)
Lemma A.3. Let $\kappa \in (0, 1)$ and $\delta > 0$. Then
\[
|D_{A}|^{2\kappa} \leq 2^{\kappa} |D_{0}|^{2\kappa} + C(d, \kappa) (d \Gamma(\varpi) + 1)^{\kappa}
\]
(A.16)
\[
\leq 2^{\kappa} |D_{0}|^{2\kappa} + \delta d \Gamma(\varpi) + C(d, \kappa, \delta).
\]

Proof. We put $\Theta := d \Gamma(\varpi) + 1$. For every $\varphi \in \mathcal{D}_{m}$, we have $\|D_{A} \varphi\| = \|(D_{0} + \alpha \cdot A) \varphi\| \leq \|D_{0} \varphi\| + C(d) \|\Theta^{1/2} \varphi\|$, thus $|D_{A}|^{2} \leq 2|D_{0}|^{2} + 2C(d) \Theta$ on $\mathcal{D}(D_{0}) \cap \mathcal{D}(d \Gamma(\varpi)^{1/2})$. For $\kappa \in (0, 1)$, the map $t \rightarrow t^{\kappa}$ is operator monotone. Together with $(a + b)^{\kappa} \leq a^{\kappa} + b^{\kappa}$, $a, b \geq 0$, this implies (A.16). \hfill \Box

Appendix B. Some properties of ground state eigenvectors

In this appendix we always assume that $e \in \mathbb{R}$, $\Lambda > 0$, $m \geq 0$, $\gamma \in (0, \gamma_{0}^{op})$, and that $\phi_{m}$ is a ground state eigenvector of the operator $H_{\gamma, m}$ defined in (7.2), so that $H_{\gamma, m} \phi_{m} = E_{\gamma, m} \phi_{m}$. Our aim is to show that, for every $f \in \mathcal{H}_{0}$ with $\omega^{-1/2} f \in \mathcal{H}_{0}$, the vector $a(f) \phi_{m} = a(f) U \phi_{m}$ belongs to the form domain of the unitarily transformed operator $\tilde{H}_{\gamma, m} = U H_{\gamma, m} U^{*}$ defined in Subsection 8.1. This result has been used in order to derive the infra-red bounds.

Lemma B.1. $H_{1/2}^{1/2} \phi_{m} \in Q(H_{\gamma, m})$.

Proof. We set $\tilde{H}_{t} := H_{t} + K$ and
\[
f_{\varepsilon}(t) := (t + K)/(1 + \varepsilon t + \varepsilon K), \quad t \geq 0, \quad F_{\varepsilon} := f_{\varepsilon}^{1/2}(\tilde{H}_{t}),
\]
for all $\varepsilon > 0$ and some $K \geq 1$. Moreover, we put $Y := H_{\gamma, m} - E_{\gamma, m} + 1$. In [20] Proof of Theorem 6.1] one of the present authors proved that, for every sufficiently large value of $K$ and for all $\varphi_{1}, \varphi_{2} \in \mathcal{D}_{0}$ and $\varepsilon > 0$,\n\[
\langle Y \varphi_{1} | F_{\varepsilon} \varphi_{2} \rangle - \langle F_{\varepsilon} \varphi_{1} | Y \varphi_{2} \rangle \leq C \left( \langle \varphi_{1} | Y \varphi_{1} \rangle + \langle \varphi_{2} | Y \varphi_{2} \rangle \right).
\]
Moreover, it follows from [20] Proof of Theorem 6.1] that $F_{\varepsilon}$ maps the form domain of $H_{\gamma, m}$ continuously (with respect to the form norm) into itself. In particular, the inequality (B.1) extends to all $\varphi_{1}, \varphi_{2} \in Q(H_{\gamma, m})$. Using (B.1) with $\varphi_{1} = (2C)^{-1/2} F_{\varepsilon} \phi_{m} \in Q(H_{\gamma, m})$ and $\varphi_{2} = (2C)^{1/2} \phi_{m}$ we then obtain
\[
\langle F_{\varepsilon} \phi_{m} | Y F_{\varepsilon} \phi_{m} \rangle \leq \left| \langle F_{\varepsilon}^{2} \phi_{m} | Y \phi_{m} \rangle \right| + \frac{1}{2} \langle F_{\varepsilon} \phi_{m} | Y F_{\varepsilon} \phi_{m} \rangle + 2C^{2} \langle \phi_{m} | Y \phi_{m} \rangle,
\]
for all $\varepsilon > 0$. Since $Y \phi_{m} = \phi_{m}$ and $\|F_{\varepsilon} \phi_{m}\| \not\rightarrow \|\tilde{H}_{t}^{1/2} \phi_{m}\|$, as $\varepsilon \searrow 0$, because of $\phi_{m} \in \mathcal{D}(H_{1/2})$, we obtain, for $\varepsilon > 0$,
\[
\langle F_{\varepsilon} \phi_{m} | Y F_{\varepsilon} \phi_{m} \rangle \leq 2 \|\tilde{H}_{1/2}^{1/2} \phi_{m}\|^{2} + 4C^{2} \|\phi_{m}\|^{2} =: B.
\]
In particular, the densely defined functional $u(\eta) := \langle \tilde{H}_{1/2}^{1/2} \phi_{m} | Y^{1/2} \eta \rangle$, $\eta \in Q(H_{\gamma, m})$, is bounded, $|u(\eta)| = \lim_{\varepsilon \searrow 0} |\langle Y^{1/2} F_{\varepsilon} \phi_{m} | \eta \rangle| \leq B^{1/2} \|\eta\|$, whence $\tilde{H}_{1/2}^{1/2} \phi_{m} \in \mathcal{D}(Y^{1/2}) = Q(H_{\gamma, m})$ since $Y^{1/2}$ is self-adjoint. \hfill \Box
Lemma B.2. \( a(f) \phi_m \in \mathcal{Q}(H_{\gamma,m}) \), for all \( f \in \mathcal{K}_0 \) with \( \omega^{-1/2} f \in \mathcal{K}_0 \).

**Proof.** By Theorem 3.2 \( \mathcal{Q}(H_{\gamma,m}) = \mathcal{D}(|D_0|^{1/2}) \cap \mathcal{D}(H_t^{1/2}) \). It well-known and not difficult to show that \( a(f) \) maps \( \mathcal{D}(H_t) \) into \( \mathcal{D}(H_t^{1/2}) \). Thus, \( a(f) \phi_m \in \mathcal{D}(H_t^{1/2}) \) as we know from Theorem 3.3 (ii) that \( \phi_m \in \mathcal{D}(H_t) \). Moreover, by Lemma B.1 \( H_t^{1/2} \phi_m \in \mathcal{Q}(H_{\gamma,m}) \subset \mathcal{D}(|D_0|^{1/2}) \). By a simple standard estimate we can check directly that \( a(f) \psi \in \mathcal{D}(|D_0|^{1/2}) \), for every \( \psi \in \mathcal{D}(H_t^{1/2}) \) with \( H_t^{1/2} \psi \in \mathcal{D}(|D_0|^{1/2}) \). In particular, \( a(f) \phi_m \in \mathcal{D}(|D_0|^{1/2}) \). \( \Box \)

Lemma B.3. \( a(f) \tilde{\phi}_m \in \mathcal{Q}(\tilde{H}_{\gamma,m}) \), for all \( f \in \mathcal{K}_0 \) with \( \omega^{-1/2} f \in \mathcal{K}_0 \).

**Proof.** On account of (8.8) and \( U a(f) \phi_m = \mathcal{Q}(\tilde{H}_{\gamma,m}) \) (by Lemma B.2) it remains to show that \( U x_j \phi_m \in \mathcal{Q}(\tilde{H}_{\gamma,m}) \), or equivalently, \( x_j \phi_m \in \mathcal{Q}(H_{\gamma,m}) = \mathcal{D}(|D_0|^{1/2}) \cap \mathcal{D}(H_t^{1/2}) \), for every component \( x_j, j \in \{1, 2, 3\} \), of \( x \). To this end we recall the following bounds from \([21, \text{Lemma 5.4}]\): Let \( F \in \mathcal{F} \) and \( \text{Lemma B.3.} \)

We readily infer from (B.3) that, for sufficiently small \( a > 0 \),

(\( \Box \))

In view of (3.1) and the sub-criticality of \( \gamma \in (0, \gamma_c) \) we have the following straightforward consequence of (3.2),

(3.3)

for all \( \varphi \in \mathcal{D}_0 \). Moreover, as in \([21, \text{Lemma 5.5}]\) we can show that \( e^{\tilde{F}} \) maps the form domain of \( H_{\gamma,m} \) continuously into itself. Since \( \mathcal{D}_0 \) is a form core for \( H_{\gamma,m} \) we conclude that \( \varphi \) can be replaced by \( e^{\tilde{F}} \phi_m \) in (3.3). Using \( H_{\gamma,m} \phi_m = E_{\gamma,m} \phi_m \) we readily infer from (3.3) that, for sufficiently small \( a > 0 \),

(3.4)

where \( Y := H_{\gamma,m} - E_{\gamma,m} + 1 \). Next, we assume that \( F \in C^\infty(\mathbb{R}_x^3, [0, \infty)) \) equals \( a|x| \), for large \( |x| \), and pick a suitable monotonically increasing sequence, \( \{F_n\}_{n \in \mathbb{N}} \), of bounded, smooth functions \( F_n \) such that \( |\nabla F_n| \leq a \). Since \( \phi_m \in \mathcal{D}(e^{\tilde{F}}) \), for sufficiently small \( a > 0 \), it makes sense to introduce the densely defined functional \( u : \mathcal{D}(Y^{1/2}) \to \mathbb{C} \),

\( u(\eta) := (e^F \phi_m | Y^{1/2} \eta) \), \( \eta \in \mathcal{D}(Y^{1/2}) \).
By virtue of (B.4) and \( e^{F_n} : Q(H_{\gamma,m}) \rightarrow Q(H_{\gamma,m}) \) we conclude that \( u \) is bounded. In fact,

\[
|u(\eta)| = \lim_{n \rightarrow \infty} |\langle Y^{1/2} e^{F_n} \phi_m | \eta \rangle| \leq C^{1/2} \lim_{n \rightarrow \infty} \| e^{F_n} \phi_m \| \| \eta \| = C^{1/2} \| e^F \phi_m \| \| \eta \|
\]

for all \( \eta \in Q(H_{\gamma,m}) \). As \( Y^{1/2} \) is self-adjoint this implies \( e^F \phi_m \in Q(H_{\gamma,m}) \subset D(|D_0|^{1/2}) \cap D(H_{1/2}^{1/2}) \). Since multiplication with \( x_j e^{-F} \) leaves \( D(|D_0|^{1/2}) \) invariant we arrive at \( x_j \phi_m \in D(|D_0|^{1/2}) \cap D(H_{1/2}^{1/2}) \). \( \Box \)

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