I. INTRODUCTION

Both electron correlation and lattice dynamics are important in understanding the physical properties of many condensed matter systems. But the interplay between them has not been studied in detail because there are no reliable theoretical methods to describe the electron-phonon interaction in the strongly correlated electron systems. For weakly correlated materials we have the successful Migdal-Eliashberg theory. There is, however, no systematic theoretical extension of the theory to the strongly interacting systems. With the recent advent of the dynamical mean field theory (DMFT) various lattice problems can be mapped (or approximated) onto impurity problems. In this context it is important to understand the correlated impurity problem coupled to local phonons.

One of the simplest model which incorporate both electron correlation and phonon is the Hubbard-Holstein (HH) model. In the context of DMFT, it is mapped onto the Anderson-Holstein (AH) model. The AH model is a single-impurity Anderson model with a linear coupling to a local phonon mode as in the Holstein model. Recently, the HH model was studied with DMFT in combination with the numerical renormalization group (NRG) at half-filling at zero temperature. The NRG method has also been successfully applied to AH model and almost exact results on the electron and phonon spectral functions have been obtained. However, it is still desirable to develop an analytic scheme in spite of its approximate nature since it helps us to understand the underlying physics in more clear and intuitive way. In this paper, we will consider the regime where the electron correlation dominates over the phonon effect, which will allow us to study the phonon effects on the Kondo physics. The pure Holstein model without electron correlation is studied in Ref. using semiclassical approximation and in Ref. using NRG and DMFT.

When the local Coulomb repulsion of Anderson model is very large compared to the hybridization between the impurity and the conduction electrons, the impurity develops a magnetic moment at high temperature regime. The moment is gradually screened by the spin of the conduction electrons as the temperature is lowered below the Kondo temperature scale. The physical properties of Anderson model at temperature lower than the Kondo temperature can not be described within perturbative framework since the effective Kondo coupling constant becomes very large in that temperature regime. This low temperature state is often called the Kondo fixed point in the sense of renormalization group. We mention that the physics of Kondo fixed point can describe the correlated metallic states such as heavy fermions in the context of DMFT.

In the limit of the infinite local Coulomb repulsion the impurity develops the local moment at high temperature and the system crosses over into the (completely screened) Kondo fixed point at low temperature. Thus the physical properties of the system in the limit of infinite local Coulomb repulsion are expected to be qualitatively similar to those of other physical systems with very large but finite local Coulomb repulsion. If the local Coulomb repulsion is taken to be infinite the doubly occupied state of the impurity is not allowed in the physical Hilbert space. In general, the standard perturbative approach based on the Wick theorem on the electron operators and the associated Feynman diagram expansions does not apply to the systems with the constrained Hilbert space. This fact poses a technical difficulty for the approaches based on the electron operators. However, if we employ the slave field representation of electron operators, the constrained Hilbert space can be treated in a considerably simplified way. The slave field method also admits the Feynman diagram expansion and

Slave-boson approach to the infinite-\( U \) Anderson-Holstein impurity model

Hyun C. Lee
Department of Physics and Basic Science Research Institute, Sogang University, Seoul, 121-742, Korea

Han-Yong Choi
Department of Physics, Department of Physics, BK21 Physics Research Division, and Institute of Basic Science Research, Sung Kyun Kwan University, Suwon, 440-746, Korea.
(Dated: November 9, 2018)

The infinite-\( U \) Anderson-Holstein impurity model is studied with a focus on the interplay between the strong electron correlation and the weak electron-phonon interaction. The slave boson method has been employed in combination with the large degeneracy expansion (1/\( N \)) technique. The charge and spin susceptibilities and the phonon propagator are obtained in the approximation scheme where the saddle point configuration and the Gaussian 1/\( N \) fluctuations are taken into account. The spin susceptibility is found not to be renormalized by electron-phonon interaction, while the charge susceptibility is renormalized. From the renormalized charge susceptibility the Kondo temperature is found to increase by the electron-phonon interaction. It turns out that the bosonic 1/\( N \) Gaussian fluctuations play a very crucial role, in particular, for the phonon propagator.

PACS numbers: 71.27.+a,63.20.Kr,71.38.-k
the direct manipulations of functional integrals. This slave field method has been successfully applied to the infinite-U Anderson model\textsuperscript{13,14,15} and it is particularly adequate in the study of the low temperature properties of the Anderson model. The bose condensed state which emerges in the saddle point (mean-field) approximation embodies the Kondo resonant state, which is the crux of the Kondo fixed point. Thus the essential physics of the Kondo fixed point is captured in the leading approximation of the slave boson method. At high temperature the bose condensate disappears and it corresponds to the high temperature weak Kondo coupling regime where the perturbative approach such as the poor man's scaling\textsuperscript{16} can be applied.

Our objective in this paper is to incorporate the phonon dynamics into the infinite-U Anderson model within the scheme of the slave boson method along the line pursued by Coleman\textsuperscript{13} We will assume that the electron-phonon interaction is weak compared to the strong electron correlation, so that the physical situation at the zeroth approximation is still best described in terms of Kondo fixed point physics. We have developed a scheme in which the effect of electron-phonon interaction can be computed systematically in the large-degeneracy limit or, in other words, the $1/N$ expansion. The basic idea of the $1/N$ expansion is briefly illustrated in Appendix A.\textsuperscript{17}

The charge and spin susceptibilities of the impurity and the local phonon propagator have been obtained. Note that the dynamical correlation functions are generally very hard to compute when the electron correlation is very strong. The feasibility of the analytic computation of the dynamical correlation functions is the great advantage of the slave boson methods over other methods. It is found that the spin susceptibility of the impurity [Eq. (67)] is not modified by the electron-phonon interaction up to the approximation of Gaussian fluctuations. The charge susceptibility of the impurity up to the second order in the (weak) electron-phonon coupling is given by Eq. (71). From the charge susceptibility at zero frequency, the Kondo temperature renormalized by the phonon propagator can be extracted, and it is given by Eq. (S1). Finally the phonon propagator is found to be Eq. (S2). These are the main results of this paper. In all of these results the influence of electron-phonon interaction is small since the charge fluctuations which directly couple to the local phonon are severely suppressed by very strong electron correlations. These results manifestly demonstrate the essential interplay between the strong electron correlation and the (weak) electron-phonon interaction. For instance, the small renormalization by electron-phonon interaction in Eq. (S1) is expressed by a product of the polaron energy and the Kondo energy, where the nonperturbative physics of the electron correlation is encapsulated in the Kondo energy scale. The smallness of the influence of the electron-phonon interaction in our case is partially due to the absence of the orbital degrees of freedom. The effect of phonons can be much more substantial for the materials with multi-orbitals, where the phonons can couple to the orbital quantum number.\textsuperscript{18} Our approach can be extended to such cases, and the progresses in this direction are being made.

This paper is organized as follows. In Sec. II we introduce the AH model and the slave boson formalism. Then the rescaled effective action in the Read-Newns gauge is obtained. In Sec. III we summarize the basic results on the saddle point approximation by Coleman.\textsuperscript{13} In Sec. IV the phonon dynamics is coupled to the boson excited states in the $1/N$ expansion and the effective action of them is computed. In Sec. V the spin and the charge susceptibilities are calculated from the effective action obtained in Sec. IV and their physical properties are analyzed. In Sec. VI the phonon propagator is calculated from the effective action obtained in Sec. IV. Sec. VII is for the summary and concluding remarks. Some of details of the calculations can be found in the Appendices.

\section{II. FORMALISM}

\subsection{A. Slave boson representation}

The Hamiltonian of AH model consists of three parts.

\begin{equation}
\mathcal{H}_{AH} = \mathcal{H}_{el} + \mathcal{H}_{ph} + \mathcal{H}_{el-ph}.
\end{equation}

$\mathcal{H}_{el}$ and $\mathcal{H}_{ph}$ is the Hamiltonian for electron and phonon part, respectively, and $\mathcal{H}_{el-ph}$ is the Hamiltonian describing the interaction between the electron and the phonon.

\begin{equation}
\mathcal{H}_{el} = \sum_{k,\sigma} \epsilon_k c_k^{\dagger} c_{k\sigma} + \sum_{\sigma} \epsilon_f \sum_{\sigma} f_{\sigma}^{\dagger} f_{\sigma} + U \sum_{\sigma} f_{\uparrow}^{\dagger} f_{\uparrow} f_{\downarrow}^{\dagger} f_{\downarrow} + \sum_{k\sigma} \frac{1}{\sqrt{N_{lat}}} (V_k f_{\sigma}^{\dagger} c_{k\sigma} + V_k^{*} c_{k\sigma}^{\dagger} f_{\sigma}).
\end{equation}

\begin{equation}
\mathcal{H}_{ph} = \frac{1}{2} \left( M \Omega^2 Q^2 + \frac{P^2}{M} \right).
\end{equation}

\begin{equation}
\mathcal{H}_{el-ph} = g_0 Q \left( \sum_{\sigma} f_{\sigma}^{\dagger} f_{\sigma} - \langle \sum_{\sigma} f_{\sigma}^{\dagger} f_{\sigma} \rangle \right).
\end{equation}

$\sigma = \uparrow, \downarrow$ is the spin index, and $V_k$ is the hybridization matrix element. $f_\sigma$ is the impurity electron operator, and $c_{k\sigma}$ is the conduction electron operator. $\epsilon_f$ is the energy of the impurity level, and $U$ is the local Coulomb repulsion at the impurity site. $Q$ is local phonon coordinate. $P$ is the conjugate momentum of $Q$ satisfying $[Q, P] = i\hbar$. $M$ is ion mass and $\Omega$ is the oscillator frequency of dispersionless (Einstein) phonons. $N_{lat}$ is the number of lattice sites for the conduction electrons. $g_0$ is the (unscaled) electron-phonon coupling constant.
In the limit of infinite $U$, the doubly occupied state of impurity $f$ electrons is prohibited. In this constrained Hilbert space, the impurity electron operator $f_\sigma$ can be expressed in terms of fermion operator $s_\sigma$ and the slave boson operator $b$ satisfying:

$$
\begin{align*}
  f_\sigma^\dagger &= s_\sigma^\dagger b, & f_\sigma &= s_\sigma b^\dagger, & (5) \\
  b^\dagger b + \sum_{\sigma} s_\sigma^\dagger s_\sigma &= 1. & (6)
\end{align*}
$$

Now the remaining empty state and the singly occupied states denoted by spin quantum number $\sigma$ at the impurity site can be represented as follows.

$$
\begin{align*}
  \langle \text{empty} \rangle &= b^\dagger \langle \text{ref} \rangle, \\
  \langle \sigma \rangle &= s_\sigma^\dagger \langle \text{ref} \rangle, & (7)
\end{align*}
$$

where $\langle \text{ref} \rangle$ denotes some arbitrary reference state. The local Coulomb repulsion term $Uf_\sigma^\dagger f_\sigma f_\sigma^\dagger f_\sigma^\dagger \tau_\sigma$ which is non-zero only for the doubly occupied states can be dropped in the constrained Hilbert space. After the elimination of the doubly occupied state we can formally increase the number of spin components from 2 to arbitrary $N$.

The Hamiltonian in the constrained Hilbert space can be expressed in terms of the slave fields as follows. (the index $\sigma$ has been changed to $m$ in anticipation of $1/N$ expansion)

$$
H = \sum_{k,m=1}^N (\epsilon_k - \mu_c) c_{km}^\dagger c_{km} + \epsilon_f \sum_m s_m^\dagger s_m \\
+ \frac{1}{\sqrt{N_{\text{lat}}}} \sqrt{\frac{N}{N}} \sum_{k,m} (V_k s_m^\dagger b c_{km} + V_k^* c_{km}^\dagger b^\dagger s_m) \\
+ \frac{1}{2} \left( M\Omega^2Q^2 + \frac{P^2}{M} \right) \\
+ \frac{g_0Q}{2} \sum_m s_m^\dagger s_m - \sum_m s_m^\dagger s_m \tag{8}
$$

The electron-phonon coupling gives rise to the charge fluctuation since it couples to the total charge at the impurity site which is either 0 or 1 in the constrained Hilbert space. Note the scaling of the hybridization matrix element $V_k$ assumed to be weakly dependent on momentum near the Fermi surface of conduction electrons in the second line of Eq. (11). From now on $\text{Re}\Sigma_0$ will be neglected since it is the order of the inverse of energy cutoff. After the integration over the conduction electrons $c_{km}$, the resulting action becomes

$$
\begin{align*}
S &= \int_0^\beta d\tau \left[ \sum_m s_m^\dagger (\partial_\tau + \epsilon_f) s_m + b^\dagger \partial_\tau b \right] \\
&+ \frac{1}{N} \int_0^\beta d\tau d\tau' \sum_m \text{Re}\Sigma_0 (\tau - \tau') s_m^\dagger (\tau) b(\tau') s_m(\tau') b^\dagger(\tau') \\
&+ \int_0^\beta d\tau \frac{1}{2} (\partial_\tau Q^2 + \Omega^2Q^2) \\
&+ \int_0^\beta d\tau g_0Q \left( \sum_m s_m^\dagger s_m - \sum_m s_m^\dagger s_m \right) \\
&+ \int_0^\beta d\tau \lambda (\sum_m s_m^\dagger s_m + b^\dagger b - 1), \tag{12}
\end{align*}
$$

where $\lambda$ is a Lagrange multiplier implementing the constraint Eq. (6). In the $1/N$ expansion this constraint will be realized through the conserving property of the $1/N$ expansion. The partition function is given by the following functional integral

$$
Z = \int D[b, \lambda] \int D[s, Q] \int D[c] e^{-\int d\tau L}. \tag{10}
$$

The integration over the conduction electrons $c_{km}$ can be done exactly and it generates a self-energy $\Sigma_0(\epsilon)$ for the slave fields.

$$
\text{Re}\Sigma_0(\epsilon) = \frac{1}{N_{\text{lat}}} \sum_k \frac{|V_k|^2}{\epsilon + \mu - \epsilon_k} \\
\approx \sum_k \epsilon_k s_m^\dagger s_m - \sum_m s_m^\dagger s_m \tag{11}
$$

where $N(E_F)$ is the density of states of the conduction electrons at Fermi energy and the subscript $FS$ denotes an average over Fermi surface. The hybridization matrix element $V_k$ was assumed to be weakly dependent on momentum near the Fermi surface of conduction electrons in the second line of Eq. (11). From now on $\text{Re}\Sigma_0$ will be neglected since it is the order of the inverse of energy cutoff. After the integration over the conduction electrons $c_{km}$, the resulting action becomes

$$
\begin{align*}
S &= \int_0^\beta d\tau \left[ \sum_m s_m^\dagger (\partial_\tau + \epsilon_f) s_m + b^\dagger \partial_\tau b \right] \\
&+ \frac{1}{N} \int_0^\beta d\tau d\tau' \sum_m \text{Re}\Sigma_0 (\tau - \tau') s_m^\dagger (\tau) b(\tau') s_m(\tau') b^\dagger(\tau') \\
&+ \int_0^\beta d\tau \frac{1}{2} (\partial_\tau Q^2 + \Omega^2Q^2) \\
&+ \int_0^\beta d\tau g_0Q \left( \sum_m s_m^\dagger s_m - \sum_m s_m^\dagger s_m \right) \\
&+ \int_0^\beta d\tau \lambda (\sum_m s_m^\dagger s_m + b^\dagger b - 1), \tag{12}
\end{align*}
$$
where
\[ \Sigma_0(\tau - \tau') = T \sum_{\epsilon} e^{-i\epsilon(\tau - \tau')} \Sigma_0(i\epsilon). \] (13)

**B. Read-Newns gauge and Rescalings**

The systematic computations of 1/N fluctuations can be most clearly achieved in the Read-Newns (RN) gauge,\textsuperscript{13,14,15}
\[ b(\tau) = R(\tau) e^{i\theta(\tau)}, \quad s_m(\tau) = z_m(\tau) e^{i\theta(\tau)}. \] (14)

Note that the original f-electron operator is invariant under the following gauge transformation:

\[ s_m \rightarrow s_m e^{i\varphi}, \quad b \rightarrow be^{i\varphi}. \] (15)

\[ R(\tau) \geq 0 \] is the modulus of complex boson b. Following Coleman,\textsuperscript{13} we write the fluctuating Lagrange multiplier as
\[ \lambda = i\Omega_0 + \lambda_{sa}, \] (16)

where \( \lambda_{sa} \) is the real part of the saddle point value of the Lagrange multiplier, which is to be determined later. In RN gauge, the infrared divergences which plague the expansion in terms of the original fields b, s_m can be avoided.

The action in RN gauge is expressed as follows:
\[ S = \int_0^\beta d\tau \sum_m z_m^\dagger(\partial_\tau + \epsilon_f)z_m \]
\[ + \frac{1}{N} \int_0^\beta d\tau d\tau' \sum_m \Sigma_0(\tau - \tau') z_m^\dagger(\tau) R(\tau) z_m(\tau') R(\tau') \]
\[ + \int_0^\beta d\tau \frac{M}{2} ( (\partial_\tau Q)^2 + \Omega^2 Q^2 ) \]
\[ + \int_0^\beta d\tau g_0 Q \left( \sum_m z_m^\dagger z_m - \langle \sum_m z_m^\dagger z_m \rangle \right) \]
\[ + \int_0^\beta d\tau i(\frac{d\theta}{d\tau} + \Omega_0) ( \sum_m z_m^\dagger z_m + R^2 - qN ) \]
\[ + \int_0^\beta d\tau \lambda_{sa} (R^2 - qN), \] (17)

where
\[ \delta\epsilon_f = \epsilon_f + \lambda_{sa}, \quad q = 1/N. \] (18)

The integration of phonon \( Q \) contributes the following term to the action
\[ -\frac{g_0}{2M} \int_0^\beta d\tau \int_0^\beta d\tau' T \sum_{i\omega} \frac{e^{-i\omega(\tau - \tau')}}{\omega^2 + \Omega^2} \]
\[ \times \sum_m \delta n_m(\tau) \sum_n \delta n_n(\tau'), \] (19)

where \( \delta n_m = s_m^\dagger s_m - \langle s_m^\dagger s_m \rangle \). In order for the 1/N expansion to be applicable, each term of the action should be proportional to \( N \). The structure of Eq. (19) suggests that the electron-phonon coupling \( g_0 \) should be rescaled as follows:
\[ g_0 = g/\sqrt{N}. \] (20)

By the same reason \( R(\tau) \) should be also rescaled.
\[ R(\tau) = \sqrt{N} r(\tau). \] (21)

The rescaled action becomes
\[ S = \int_0^\beta d\tau \sum_m z_m^\dagger(\partial_\tau + \epsilon_f)z_m \]
\[ + \int_0^\beta d\tau d\tau' \sum_m \Sigma_0(\tau - \tau') z_m^\dagger(\tau) r(\tau) z_m(\tau') r(\tau') \]
\[ + \int_0^\beta d\tau \frac{M}{2} ( (\partial_\tau Q)^2 + \Omega^2 Q^2 ) \]
\[ + \frac{1}{\sqrt{N}} \int_0^\beta d\tau gQ \left( \sum_m z_m^\dagger z_m - \langle \sum_m z_m^\dagger z_m \rangle \right) \]
\[ + \int_0^\beta d\tau i(\frac{d\theta}{d\tau} + \Omega_0) ( \sum_m z_m^\dagger z_m ) \]
\[ + N \int_0^\beta d\tau \lambda_{sa} (r^2 - q) \]
\[ + N \int_0^\beta d\tau \lambda_{sa} (r^2 - q), \] (22)

Now the 1/N expansion can be directly applied to the action Eq. (22).

**III. SADDLE POINT APPROXIMATION**

The leading order approximation of the 1/N expansion amounts to the saddle point approximation or equivalently, the mean-field approximation. The mean-field theory of infinite-\( U \) Anderson model has been studied in great detail in Sec. II of Coleman’s paper.\textsuperscript{13} Here, we briefly summarize the main results of Ref.\textsuperscript{13} for the purpose of the self-contained treatment and the introduction of necessary notations.

We assume that \( r, \lambda \) take a static value \( \bar{r}, \lambda_{sa} \) in the saddle point approximation, respectively. In the saddle point approximation the following fluctuations with respect to the saddle point value
\[ i(\frac{d\theta}{d\tau} + \Omega_0), \quad \delta r = r - \bar{r}_{sa} \] (23)
are neglected. We will treat the electron-phonon interaction on an equal footing with the above fluctuations, so that the electron-phonon interaction is not included.
The saddle-point values \( r_{sa}, \lambda_{sa} \) are determined by minimizing \( \tilde{S}_{sa} \) with respect to them. The minimization with respect to \( \lambda_{sa} \) yields

\[
r_{sa}^2 - q = -T \sum_{ie} \frac{1}{ie - \epsilon_f - \lambda_{sa} + i\Delta \text{sgn}(\epsilon)}.
\]

At zero temperature the summation of Eq. (27) can be done in a closed form.

\[
r_{sa}^2 - q = -\left[ \frac{1}{2} - \frac{1}{\pi} \tan^{-1} \left( \frac{\epsilon_f + \lambda_{sa}}{\Delta} \right) \right].
\]

Likewise, the minimization with respect to \( r_{sa}^2 \) yields

\[
\lambda_{sa} = -T \sum_{ie} \frac{-i\Delta \text{sgn}(\epsilon)}{ie - \epsilon_f - \lambda_{sa} + i\Delta \text{sgn}(\epsilon)}.
\]

At zero temperature the summation of Eq. (30) can be done in a closed form.

\[
\lambda_{sa} = \frac{\Delta_0}{2\pi} \ln \left[ \frac{D^2}{\Delta^2 + (\epsilon_f + \lambda_{sa})^2} \right],
\]

where \( D \) is an energy cutoff which is the order of the bandwidth of conduction electrons. Two real-valued equations Eq. (29) and Eq. (31) can be combined into a single complex-valued equation.

\[
\frac{\Delta_0}{\pi} \ln \left[ \frac{\pi \xi}{\Delta_0} \right] + \xi = \epsilon_f^* + i\Delta_0 q,
\]

where

\[
\xi = \epsilon_f + \lambda_{sa} + i\Delta,
\]

\[
\epsilon_f^* = \epsilon_f + \Delta_0 \ln \left[ \frac{\pi D}{\Delta_0} \right].
\]

The Kondo limit is specified by the condition \( |\xi| \ll \Delta_0 \), and in this limit the first term of the right hand side of Eq. (22) dominates. In the Kondo limit, the Kondo temperature scale is given by

\[
\Delta_K \equiv |\xi| = \sqrt{\Delta^2 + \epsilon_f^2} \sim De^{\pi\epsilon_f/\Delta_0}.
\]

This result derived from the saddle point approximation coincides with the one obtaind from the scaling theory of Anderson model by Haldane except for the prefactor. A few numerical solutions of Eqs. (29,31) are presented in Table I. From the numerical solutions we find that in the case of \( q = 1/2 \), \( \Delta \gg \bar{\epsilon}_f \), while for \( q = 1/4 \) and \( 1/6 \), \( \Delta \sim \bar{\epsilon}_f \). In all cases \( \Delta_K \) and \( \Delta \) are of the same order of magnitude.

### IV. 1/N FLUCTUATIONS

The 1/N corrections are related to the fluctuations with respect to the saddle point configuration. It is convenient to treat the fluctuations collectively, so let us define a two-component vector \( X \) as follows:

\[
X = \left( \frac{\delta r}{ir_{sa}} \right) + \frac{\delta}{d\tau} + \Omega_0.
\]

The 1/N corrections can be computed systematically by expanding the action Eq. (22) with respect to the fluctuations and by integrating out \( z_m \). The action Eq. (22) is given as

\[
S_{mf} = \int_0^\beta d\tau \sum_m \left[ \sum_{\tau'} \left( \partial_{\tau'} \tilde{z}_m + \tilde{\epsilon}_f \right) \tilde{z}_m \right] + \int_0^\beta d\tau d\tau' \sum_m \left( \sigma_0 (\tau - \tau') + r_{sa}^2 \tilde{z}_m^*(\tau) \tilde{z}_m(\tau') \right) + N \int_0^\beta d\tau \lambda_{sa} (r_{sa}^2 - q).
\]

Next we integrate over \( \tilde{z}_m \) to obtain the saddle point action \( \tilde{S}_{sa}(r_{sa}, \lambda_{sa}) \) which is

\[
e^{-\tilde{S}_{sa}(r_{sa}, \lambda_{sa})} = \int D[\tilde{z}_m] e^{-S_{mf}(z, r_{sa}, \lambda_{sa})}.
\]

The explicit form of \( \tilde{S}_{sa} \) is given by

\[
\tilde{S}_{sa} = N \int_0^\beta d\tau \lambda_{sa} (r_{sa}^2 - q) - \text{Tr} \ln[\partial_{\tau} + \tilde{\epsilon}_f + r_{sa} \Sigma_0].
\]

### TABLE I: The numerical solutions of the saddle point equations Eqs. (29,31). The input parameters are \( D = 65\Delta_0 \) and \( \epsilon_f = -3.2\Delta_0 \). \( \Delta_K \) is defined in Eq. (33).

| q    | \( \Delta/\Delta_0 \) | \( \epsilon_f/\Delta_0 \) | \( \Delta_K/\Delta_0 \) |
|------|----------------------|----------------------|----------------------|
| 1/2  | 2.80 \times 10^{-3}  | 2.46 \times 10^{-3}  | 2.80 \times 10^{-3}  |
| 1/4  | 1.95 \times 10^{-3}  | 1.90 \times 10^{-3}  | 2.79 \times 10^{-3}  |
| 1/6  | 1.39 \times 10^{-3}  | 2.41 \times 10^{-3}  | 2.78 \times 10^{-3}  |

in the saddle point approximation. Now the action appropriate for the saddle point approximation can be expressed as

\[
S_{mf} = \int_0^\beta d\tau \sum_m \left[ \sum_{\tau'} \left( \partial_{\tau'} \tilde{z}_m + \tilde{\epsilon}_f \right) \tilde{z}_m \right] + \int_0^\beta d\tau d\tau' \sum_m \left( \sigma_0 (\tau - \tau') + r_{sa}^2 \tilde{z}_m^*(\tau) \tilde{z}_m(\tau') \right) + N \int_0^\beta d\tau \lambda_{sa} (r_{sa}^2 - q).
\]
can be re-expressed as follows.

\[ S = S_X^{(0)} + S_z + S_{ph} + \delta S_z, \]  

\[ S_X^{(0)} = \frac{N}{2} \int_0^\beta d\tau X^T \Gamma^{(0)} X, \]  

\[ S_z = \sum_{m,i} \left[ -i e + \tilde{\epsilon}_f - i \Delta \text{sgn}(\epsilon) \right] \tilde{z}_m^z \tilde{z}_m^z, \]  

\[ S_{ph} = \int_0^\beta d\tau \frac{M}{2} (\partial_\tau Q)^2 + \Omega^2 Q^2, \]  

\[ \delta S_z = \sum_m \left[ \int_0^\beta d\tau \left( i \Theta + \frac{g Q}{\sqrt{N}} + \phi_m \right) \tilde{z}_m^z \tilde{z}_m^z \right. \]  

\[ \left. - \int_0^\beta d\tau \frac{g Q}{\sqrt{N}} \langle \tilde{z}_m^z \tilde{z}_m^z \rangle \right] \]  

\[ + \sum_m \int_0^\beta d\tau \int_0^\beta d\tau' \Sigma_0 (\tau - \tau') \tilde{z}_m^z (\tau) \tilde{z}_m^z (\tau') \]  

\[ \times \left( r_{sa} \delta r(\tau') + r_{sa} \delta r(\tau) + \delta r(\tau) \delta r(\tau') \right), \]  

where

\[ \Gamma^{(0)} = 2 \left( \begin{array}{cc} \lambda_{sa} & 1 \\ 1 & 0 \end{array} \right) \].

\[ \phi_m \] is the source field necessary for the computation of impurity susceptibilities. The irrelevant constants are omitted in Eq. (36).

Next step is to integrate over \( z_m \). This step involves \( S_z \) and \( \delta S_z \) of the action Eq. (36). To simplify notations, define

\[ G_0^{-1}(ie) = i e + i \Delta \text{sgn}(\epsilon) - \tilde{\epsilon}_f. \]

The integration over \( z_m \) generates

\[ - \sum_m \text{Tr} \ln \left[ - G^{(0)}(\tau - \tau') + M(\tau, \tau') \right], \]

where \( M(\tau, \tau') \) is given by

\[ M(\tau, \tau') = \left( \phi_m + \frac{g}{\sqrt{N}} Q + i \Theta \right)(\tau - \tau') \]  

\[ + \Sigma_0 (\tau - \tau') (r_{sa} \delta r(\tau) + r_{sa} \delta r(\tau') + \delta r(\tau) \delta r(\tau')). \]

In our approximation we expand the logarithm of Eq. (43) with respect to \( M(\tau, \tau') \) up to the second order. This is equivalent to reckoning in the Gaussian fluctuations but not the higher order fluctuations in \( 1/\sqrt{N} \) (see Appendix A).

The relevant part of the expansion is

\[ + \sum_m \text{Tr} \left[ G_0 M \right] + \frac{1}{2} \sum_m \left[ G_0 M G_0 M \right]. \]

The details of further calculations are presented in the appendix B. The effective action of \( X, Q, \phi_m \) is given by

\[ S_{eff}^{(a)} = \frac{N}{2} \sum_{\omega} X^T (-i \omega) \Gamma^{(1)} X(i \omega) \]

\[ + \sum_{\omega} \sum_{i \omega} \frac{g}{\sqrt{N}} (r_{sa} Q)(i \omega) \]

\[ \times \left( K^{(1)}_{\theta \theta} (i \theta r_{sa} + \phi_m r_{sa}) + K^{(1)}_{\theta r} \delta r \right)(-i \omega) \]

\[ + \sum_{\omega} \sum_{i \omega} K^{(1)}_{\theta \phi} (i \omega) (i \theta r_{sa})(i \omega) (\phi_m r_{sa})(-i \omega) \]

\[ + \frac{1}{2} \sum_{\omega} \sum_{i \omega} K^{(1)}_{\phi \phi} (i \omega) (\phi_m r_{sa})^2 \]

\[ + \sum_{\omega} \sum_{i \omega} K^{(1)}_{\theta \tau} \delta r(i \omega) r_{sa} \phi_m (-i \omega) \]

\[ + \frac{1}{2} \sum_{\omega} \sum_{i \omega} (M \omega^2 + M \Omega^2 + K^{(1)}_{\theta \theta} g^2 r_{sa}^2)(Q(i \omega) Q(-i \omega)) \]

where

\[ \Gamma^{(1)}(i \omega) = \left( \begin{array}{cc} 2 \lambda_{sa} + \sum_{i=0,2,3} K^{(1)}_{\theta \theta} 2 + K^{(1)}_{\theta \phi} \\ 2 + K^{(1)}_{\theta \phi} \end{array} \right). \]

The explicit forms of various polarization functions \( K \) can be found in Appendix B. By the saddle point condition Eq. (39),

\[ 2 \lambda_{sa} + K^{(0)}_{\theta \theta} = 0. \]

Then it follows that

\[ \Gamma^{(1)}(i \omega) = \left( \begin{array}{cc} K^{(2)}_{\theta \theta} + K^{(3)}_{\theta \theta} 2 + K^{(3)}_{\theta \phi} \\ 2 + K^{(1)}_{\theta \phi} \end{array} \right). \]

In fact, only \( \text{Re} \, K^{(2)}_{\theta \tau} \) in the (1,1) element of Eq. (48) contributes to the action due to the symmetry \( \omega \leftrightarrow -\omega \). Now the integration over phonon

\[ \exp[-S_{eff}^{(b)}] = \int D[q] \exp[-S_{eff}^{(a)}(X, Q, \phi_m)] \]

contributes the following to \( S_{eff}^{(b)} \).

\[ - \frac{E_p}{2 N} r_{sa}^2 \frac{\Omega^2}{\omega^2 + \Omega^2 + K^{(1)}_{\theta \theta} g^2 r_{sa}^2} \]

\[ \times \left| \sum_m \left( K^{(1)}_{\theta \theta} r_{sa} (i \Theta + \phi_m)(i \omega) + K^{(1)}_{\theta \phi} \delta r(i \omega) \right) \right|^2 \]

where

\[ E_p = \frac{g^2}{M \Omega^2} \]

is the polaron energy. Since we are assuming that the electron-phonon correlation is weak, the relation

\[ E_p < \Delta_0 \]
holds. We will also assume that \(E_p > \Delta K\), which is more relevant to the real physical systems. The effect of electron-phonon interaction can be traced by following \(E_p\). Let us define the following dimensionless function

\[
D(i\omega) = \frac{\Omega^2}{\omega^2 + \Omega^2 + K^{(1)}_{\theta\theta} g^2 r^2_{sa}/M}.
\]

The explicit form of the effective action \(S_{eff}^{(b)}(X, \phi_m)\) is given by

\[
S_{eff}^{(b)} = \frac{N}{2} \sum_\omega X^T (-i\omega) \Gamma(\omega) X(i\omega)
\]

\[+ \frac{1}{2} \sum_m \sum_\omega \sum_\omega \left[K^{(1)}_{\theta\theta}(i\omega) r^2_{sa} \phi_m(i\omega) \phi_m(-i\omega)\right]
\]

\[- \frac{E_p r^4_{sa}}{2N} \sum_{m,n,\omega} \left[K^{(1)}_{\theta\theta} \right] D(i\omega) \phi_m(i\omega) \phi_n(-i\omega)
\]

\[+ \sum_\omega \sum_\omega \left(1 - E_p D(i\omega) r^2_{sa} K^{(1)}_{\theta\theta}\right)
\]

\[\times \left(r_{sa}\phi_m(i\omega) \left(K^{(1)}_{\theta\theta} r_{sa} i\Theta(-i\omega) + K^{(1)}_{r\theta} \delta r(-i\omega)\right)\right),
\]

where

\[
\Gamma(\omega) = \begin{pmatrix} K_{rr} & K_{r\theta} \\ K_{r\theta} & K_{\theta\theta} \end{pmatrix},
\]

\[K_{rr} = K^{(1)}_{rr} + E_p r^2_{sa} D(i\omega) [K^{(1)}_{\theta\theta}]^2,
\]

\[K_{r\theta} = 2 + K^{(1)}_{r\theta} - E_p r^2_{sa} D(i\omega) K^{(1)}_{\theta\theta} K^{(1)}_{rr},
\]

\[K_{\theta\theta} = K^{(1)}_{\theta\theta} - E_p D(i\omega) r^2_{sa} K^{(1)}_{\theta\theta}.
\]

Finally the integration over \(X\)

\[
\exp[-S_{eff}^{(c)}(\phi_m)] = \int D[X] \exp[-S_{eff}^{(b)}(X, \phi_m)]
\]

yields the effective action of \(\phi_m\).

\[
S_{eff}^{(c)} = \frac{1}{2} \sum_m \sum_\omega \left[K^{(1)}_{\theta\theta}(i\omega) r^2_{sa} \phi_m(i\omega) \phi_m(-i\omega)\right]
\]

\[- \frac{E_p r^4_{sa}}{2N} \sum_{m,n,\omega} \left[K^{(1)}_{\theta\theta} \right] D(i\omega) \phi_m(i\omega) \phi_n(-i\omega)
\]

\[+ \sum_\omega \sum_\omega \left(1 - E_p D(i\omega) r^2_{sa} K^{(1)}_{\theta\theta}\right)
\]

\[\times \left(\phi_m(i\omega) \phi_n(-i\omega) \left[K^{(1)}_{r\theta} \phi_m(i\omega) \phi_n(-i\omega)\right] \right),
\]

where the sum over \(m\) runs from \(-N/2\) to \(N/2\). The charge susceptibility is defined by

\[
\chi_c(\tau - \tau') = \langle \delta n_f(\tau) \delta n_f(\tau') \rangle,
\]

where

\[
\delta n_f = \sum_m s_m^+ s_m - \langle \sum_m s_m^+ s_m \rangle
\]

The spin and charge susceptibilities can be computed from

\[
\chi_{mn}(\tau, \tau') = \frac{\delta^2 \ln Z}{\delta \phi_m(\tau) \delta \phi_n(\tau')} \bigg|_{\phi=0}.
\]

Here the partition function \(Z\) is understood to be a functional of \(\phi_m\) only, namely all other fields should be integrated out. In our approximation,

\[
Z = \int D[\phi_m] \exp[-S_{eff}^{(c)}(\phi_m)],
\]

where \(S_{eff}^{(c)}(\phi_m)\) is given by Eq. (50), \(\chi_{mn}(\tau, \tau')\) in frequency space is given by

\[
\chi_{mn}(\tau, \tau') = \sum_{i\omega} \chi_{mn}(i\omega) e^{-i\omega(\tau - \tau')}.
\]

Then it follows that

\[
\chi_c(i\omega) = \sum_{m,n} \chi_{mn}(i\omega),
\]

\[
\chi_s(i\omega) = \sum_{m,n} mn \chi_{mn}(i\omega).
\]

From Eq. (62) we find

\[
\chi_{mn} = -\delta_{mn} r^2_{sa} K^{(1)}_{\theta\theta}(i\omega)
\]

\[+ \frac{r^2_{sa}}{N} \left(1 - E_p D(i\omega) r^2_{sa} K^{(1)}_{\theta\theta}\right)^2
\]

\[\times \left[K^{(1)}_{r\theta} \times K^{(1)}_{\theta\theta}\right] \Gamma^{-1} \left[K^{(1)}_{r\theta} K^{(1)}_{\theta\theta}\right]
\]

\[+ \frac{r^4_{sa}}{N} E_p D(i\omega) \left[K^{(1)}_{\theta\theta}(i\omega)\right]^2.
\]

From Eq. (60) the spin susceptibility is readily obtained

\[
\chi_s(i\omega) = \left[\sum_m m^2\right] \chi_s(i\omega),
\]

\[
\tilde{\chi}_s(i\omega) = -r^2_{sa} K^{(1)}_{\theta\theta}(i\omega).
\]

In our approximation scheme the spin susceptibility is not renormalized by the electron-phonon interaction. Clearly this is an artifact of the approximation of the saddle point and the Gaussian fluctuation. If the fluctuations beyond the Gaussian fluctuations (higher order in \(1/\sqrt{N}\))
are considered the spin susceptibility is expected to be renormalized by the electron-phonon. However, the corrections due to higher order fluctuations will be rather small. It is illuminating to compare this result with that of numerical renormalization group studies. Hewson and Meyer have computed the imaginary part of the retarded spin susceptibility for the symmetric and finite-U AH model (see the Fig. 13 of Ref. 6). Compare two cases with \( \lambda = 0.0 \) and \( \lambda = 0.02 \). Their result shows that the electron-phonon interaction does not influence the qualitative behavior of the spin susceptibility except for the decrease of the peak height at very low energy around the Kondo temperature scale. For our parameter regime the decrease of the peak height at very low energy around the Kondo temperature scale. For our parameter regime

In the Kondo regime, the relation \( \Delta_{0} \) hold, so that we have

\[
\chi_{c}(i\omega) = \frac{4r_{sa}^{2}}{ N} \left[ K_{\theta\theta}^{(1)} \right] \Gamma^{-1} \left[ K_{\theta\theta}^{(1)} \right] - K_{\theta\theta}^{(1)}.
\]

(69)

This result can be obtained from Eq. (68) by putting \( E_{p} = 0 \) everywhere and was derived by Coleman. Explicit calculation shows that

\[
\chi_{c}^{(0)}(i\omega) = 4r_{sa}^{2} \left[ K_{\theta\theta}^{(1)} \right] \frac{K_{\theta\theta}^{(1)}}{K_{\theta\theta}^{(1)} - (K_{\theta\theta}^{(1)} + 2)^{2}}.
\]

(70)

For the subsequent analyses it is helpful to examine the susceptibilities in the absence of phonons in more detail. The susceptibilities at zero frequency in the absence of phonons are given by

\[
\chi_{c}^{(0)}(i\omega) = \frac{1}{\pi \Delta_{K}},
\]

\[
\chi_{s}^{(0)}(i\omega) = \frac{\pi \Delta}{(\Delta_{0} + \pi \epsilon_{f})^{2} + (\pi \Delta)^{2}}.
\]

(71)

In the Kondo regime, the relation \( \Delta_{0} \gg \pi \sqrt{\Delta^{2} + \epsilon_{f}^{2}} \) holds, so that we have

\[
\chi_{c}^{(0)}(i\omega) \sim \frac{\pi \Delta}{\Delta_{0}}.
\]

(72)

The ratio of the charge and spin susceptibility at zero frequency is then given by

\[
\frac{\chi_{c}^{(0)}(i\omega = 0)}{\chi_{s}^{(0)}(i\omega = 0)} \sim \left( \frac{\pi \Delta_{K}}{\Delta_{0}} \right)^{2} \ll 1.
\]

(73)

In fact, over most of the frequency range of interest, the absolute magnitude of spin susceptibility is far greater than that of the charge susceptibility. This is because the charge fluctuations are severely suppressed by the very strong local Coulomb repulsion in the low temperature Kondo regime, while the spin fluctuations (spin flips) are the dominant processes there.

We also note that \( \chi_{c}^{(0)}(i\omega) \) is related to the propagator \( \braket{\delta r(i\omega)\delta r(-i\omega)}^{(0)} \) in the absence of phonon in the following way

\[
\braket{\delta r(i\omega)\delta r(-i\omega)}^{(0)} = \frac{1}{N} \frac{1}{4r_{sa}^{2}} \chi_{c}^{(0)}(i\omega),
\]

(74)

which can be shown by inverting the matrix \( \Gamma^{(1)}(i\omega) \). This implies that higher order corrections stemming from the bose fluctuations are suppressed in the limit of large \( N \).

Using the above results on the susceptibilities in the absence of phonon and expanding the matrix \( \Gamma^{-1} \) up to the second order of \( g \), the result Eq. (68) in the presence of phonon can be expressed in a more transparent form:

\[
\chi_{c}(i\omega) \approx \chi_{c}^{(0)}(i\omega) \left[ 1 + E_{p}D(i\omega) \chi_{c}^{(0)}(i\omega) \right],
\]

(75)

where \( D(i\omega) \) is defined by Eq. (68). Using the previous results we find

\[
\left| E_{p}D(i\omega) \chi_{c}^{(0)}(i\omega) \right| \ll 1.
\]

(76)
The structure of Eq. (75) clearly prompts us to write
\[
\tilde{\chi}_c(i\omega) \sim \frac{\chi_c(0)(i\omega)}{1 - E_p D(i\omega) \chi_c(0)(i\omega)}.
\] (77)

The imaginary part of the retarded charge susceptibility [Eq. (76)] in real frequency can be expressed as
\[
\text{Im}\tilde{\chi}_c^R(\omega) \sim \text{Im}\tilde{\chi}_c^{(0),R}(\omega) \\
\times \left( 1 + E_p \text{Re}[D^R(\omega)] \text{Re}[\tilde{\chi}_c^{R,(0)}(\omega)] \right),
\] (78)
where only the leading contribution is exhibited. As depicted in Fig. 1, the variations of \(\text{Re}[\tilde{\chi}_c^{R,(0)}(\omega)]\) is almost negligible for all frequency region below the frequency of the order of \(|\epsilon_f|\). Therefore, the major frequency dependence of the electron-phonon correction to the charge susceptibility originates from \(\text{Re}[D^R(\omega)]\):
\[
\text{Re}[D^R(\omega)] \sim \frac{\Omega^2}{-\omega^2 + \Omega^2},
\] (79)
where the singularity at \(\omega = \Omega\) is cut-off by the imaginary part of \(K_{\phi\phi}^{(1)}\), but this is irrelevant in our discussion. In view of Eq. (79) a qualitative change near the phonon frequency \(\Omega\) is expected. More specifically, one can expect that the imaginary part of the charge susceptibility is to be enhanced at frequency slightly lower than the phonon frequency while it is expected to be reduced at frequency slightly higher than the phonon frequency. This expectation is in good agreement with the results of NRG studies by Hewson and Meyer (see the left panel of Fig. 13 in Ref. 3 and compare two cases of \(\lambda = 0.0\) and \(\lambda = 0.02\)).

The charge susceptibility at zero frequency in the presence of phonon can be expressed
\[
\tilde{\chi}_c(i\omega = 0) \sim \frac{\pi \Delta}{\Delta_0} \left( 1 + \frac{\pi E_p \Delta}{\Delta_0} \right).
\] (80)
For the case of \(q = 1/2\) the relation \(\Delta_K \sim \Delta\) holds, and the result Eq. (80) can be interpreted from the viewpoint of the renormalization of Kondo temperature by electron-phonon interaction.
\[
T_K \sim T_K^{(0)} \left( 1 + \frac{\pi E_p \Delta}{\Delta_0^2} \right).
\] (81)
where \(T_K^{(0)} = \Delta_K\) is the Kondo temperature in the absence of phonons. The Eq. (81) indicates that the electron-phonon interaction increases the Kondo temperature, which is consistent with the recent results found by NRG method.\(^{31}\) Since \(E_p < \Delta_0\) and \(\Delta \ll \Delta_0\) the amount of increase is rather small. This smallness of renormalization is again due to the suppressed charge fluctuations. With Coulomb repulsion fixed to be infinite from the outset, the charge fluctuation is controlled by the \(\epsilon_f\). As the impurity level goes down deeper into the Fermi sea of conduction electron, evidently the charge fluctuation is reduced. As can be seen in Eq. (81), \(\Delta\) decreases exponentially with the decreasing \(\epsilon_f\) (in the sense of becoming more negative).

**VI. PHONON PROPAGATOR**

The phonon propagator can be obtained from the action Eq. (15) by setting the source field \(\phi_m\) to zero and then by integrating over the Bose fluctuations \(X\) and the fermions \(z_m\).
\[
D_{\phi\phi}(i\omega) = \left[ M\omega^2 + M\Omega^2 - g^2 \tilde{\chi}_c^{(0)}(i\omega) \right]^{-1}.
\] (82)
The physics of Kondo fixed point is built in \(\tilde{\chi}_c^{(0)}(i\omega)\). If the Bose fluctuations \(X\) were not taken into account, the phonon propagator would be
\[
\left[ M\omega^2 + M\Omega^2 - g^2 \chi_c^{(0)}(i\omega) \right]^{-1}.
\] (83)
This can be seen from Eq. (15) by setting the Bose fluctuation \(X\) and the source field \(\phi_m\) to zero, and then by
integrating over $z_m$. As discussed in Sec. V and can be seen Fig. 2 and Fig. 3 the charge and spin susceptibility of impurity behave essentially differently from each other. First of all, the absolute magnitude of the spin susceptibility is much larger than that of the charge susceptibility by the order of magnitudes. Second the frequency dependences are also different from each other. The imaginary part of the spin susceptibility \( \text{Im} \chi_s^{R,(0)}(\omega) \) has a sharp peak at $\omega \sim \Delta_K$, while the imaginary part of the charge susceptibility \( \text{Im} \chi_c^{R,(0)}(\omega) \) has a rather broad peak at frequency higher than $\Delta_K$. \( \text{Im} \chi_c^{R,(0)}(\omega) \) also has a peak at high frequency order of $-\epsilon_f$ as seen in Fig. 1.

If the bose fluctuations were not considered, then based on Eq. (83), one would anticipate rather substantial renormalization. To make arguments more concrete let us compare last two terms in the bracket of Eq. (83) at zero frequency. Using the result of Eq. (81) we obtain

\[ \frac{g^2 \chi^c_s(i\omega = 0)}{M \Omega^2} \simeq \frac{E_p}{\Delta_K} > 1 \tag{84} \]

for physically relevant cases.

However, regarding the result Eq. (82), for which the bose fluctuations taken into account, the ratio becomes

\[ \frac{g^2 \chi^c_s(i\omega = 0)}{M \Omega^2} \simeq \frac{E_p \Delta_K}{\Delta_0} \ll 1, \tag{85} \]

which agrees with the results by NRG. Thus, unless the electron-phonon interaction is extremely strong, the self-energy correction to the phonon propagator coming from the electronic sector is small. But if the electron-phonon interaction becomes so strong that the ratio Eq. (84) becomes order of unity, the saddle point we have started with won’t be valid any more and new saddle points will emerge. The results by NRG studies indeed demonstrate that new features emerge in the strong coupling regime of electron-phonon interaction. Unfortunately our approximation scheme does not apply in that regime. In any case we conclude that to obtain the physically correct result it is absolutely essential to include the $1/N$ bose fluctuations $X$.

The form of the phonon propagator Eq. (82) suggests that the phonon frequency softens. The renormalized phonon frequency is given by

\[ \Omega_{\text{ren}} \sim \Omega \left(1 - \frac{\pi \epsilon_f \Delta}{\Delta_0} \right). \tag{86} \]

The result Eq. (86) can be compared with that derived in the semiclassical approximation for the large but finite-$U$ and symmetric AH model:

\[ \Omega_{\text{ren}} \sim \Omega \left(1 - \frac{E_p \Delta}{U^2} \right). \tag{88} \]

The results Eqs. (86), (87) indicate that the strong Coulomb repulsion suppresses the softening of phonon frequency but the detailed form of the softening depends crucially on the details of Hamiltonian such as the symmetric or asymmetric nature of charge fluctuations of Anderson model.

**VII. SUMMARY AND CONCLUDING REMARKS**

We have studied the AH model with the infinite local Coulomb repulsion. The limit of the infinite local Coulomb repulsion eliminates the doubly occupied states in the Hilbert space of impurity. The slave boson method is very effective in treating the problems with the constrained Hilbert space, where the strong correlation is actually built in the constraint itself. In general the slave boson method is more reliable in the low temperature regime where the thermal fluctuations are small. The $1/N$ expansion scheme is employed to control approximations systematically and it is found that the electron-phonon interaction can be naturally fit into the scheme. Remarkably the nonperturbative physics of the low temperature Kondo fixed point is captured in the leading approximation. We have used the saddle point approximation supplemented with the Gaussian fluctuations. In fact, the Gaussian fluctuations play a very crucial role in determining the physical properties of impurity and phonon. The spin susceptibility is not renormalized up to this approximation. The Kondo temperature is found to increase by the electron-phonon interaction but by a rather small amount. The same is true of the charge susceptibility and phonon propagator. This is because the quantum fluctuations of phonon which couple to the charge fluctuations of the impurity electron are suppressed by the very strong local electron correlation. But the interplay between the strong electron correlation and the electron-phonon interaction is clearly displayed.

We could also have calculated the electron spectral function but all the important features such as the increase of the Kondo temperatures and the pronounced phonon effect near the phonon frequency are essentially identical with those of the charge susceptibility.

The AH model can be naturally generalized to the impurity models with the orbital degrees of freedom. As a matter of fact, this is much more realistic model than the AH model since the electron correlation effects are more prominent in the $d$ and $f$ electron systems where the orbital degrees of freedom are active. For these cases the simple Holstein phonon should be also generalized to
incorporate Jahn-Teller type phonons. Slave boson approach can be extended to the orbitally active cases, and currently the progresses in this direction are being made by the authors.

Acknowledgments

H.C.L was supported by the Korea Science and Engineering through the Center for Strongly Correlated Materials Research (CSCMR) and through Grant No. R01-1999-000-00031-0, and was also supported by the Samsung University Research Grants in 2003. H.Y.C was supported by the by the Korea Science and Engineering Foundation (KOSEF) through the grant No. R01-1999-000-00031-0 and the Ministry of Education through Brain Korea 21 SNU-SKKU Program.

APPENDIX A: ILLUSTRATION OF THE BASIC IDEA OF 1/N EXPANSION

Let us briefly illustrate the basic idea of 1/N expansion in the case of one-dimensional integral. Consider an integral of the following form defined on a certain domain $D$:

$$I = \int_D e^{-Ng(x)} , \quad N \gg 1. \quad (A1)$$

If a minimum of $g(x)$ is attained at $x_0 \in D$, then the function can be expanded

$$g(x) \sim g(x_0) + \frac{1}{2} g''(x_0)(x-x_0)^2 + \sum_{n \geq 3} \frac{g(n)(x_0)}{n!} (x-x_0)^n , \quad (A2)$$

where $g(n)$ denotes the n-th derivative and $g''(x_0) > 0$. Defining a rescaled variable $z$

$$z = \frac{1}{\sqrt{N}} \frac{1}{\sqrt{g''(x_0)}} (x-x_0) \quad (A3)$$

the integral $I$ can be approximated by

$$I \sim e^{-Ng(x_0)-n[Ng''(x_0)]/2} \int dz e^{-z^2/2-c_1z^3/ \sqrt{N} + \cdots}$$

$$\sim e^{-Ng(x_0)-n[Ng''(x_0)]/2} \left( \sqrt{2\pi} + \frac{\text{const.}}{\sqrt{N}} + \cdots \right) \quad (A4)$$

The integration over $z$ can be done systematically as a power series expansion in $1/\sqrt{N}$. In this paper we ignore the corrections of order $1/\sqrt{N}$ and higher in the bracket of Eq. (A4), but they can be computed in a controllable way if desired.

APPENDIX B: DETAILS OF THE CALCULATIONS OF 1/N CORRECTIONS

First let us specify the measure for the functional integral in the RN gauge. This is discussed in Sec. III B of Ref. [13], and here we give an equivalent but technically slightly different treatment. The measure of boson functional integral is given by

$$D[r, \theta] = \prod_\tau d(r^2(\tau) \theta(\tau)) \frac{1}{2\pi} . \quad (B1)$$

The fermion measure is invariant under the RN gauge transformation.

$$D[s, s^\dagger] = D[z, z^\dagger] . \quad (B2)$$

The angular variable $\theta$ in the sector with the winding number $m$ can be expanded

$$\theta(m)(\tau) = \frac{1}{\sqrt{\beta}} \sum_n \theta_n e^{-i\nu_n \tau} + 2\pi m \frac{\tau}{\beta} , \quad (B3)$$

where $\nu_n$ is the bosonic Matsubara frequency given by $\nu_n = 2\pi n/\beta$. This expansion automatically satisfies the winding number condition without further conditions imposed on the coefficients $\theta_n$. The zero mode term $\theta_0$ should be factored out since our action does not depend on it. For the variation which involves only the zero mode we have $d\theta_0^{(m)} = 1/\sqrt{\beta} d\theta_0$. Factoring out $\int d\theta_0^{(m)}$ we obtain

$$D[\theta] = \sum_m \sqrt{\beta} \prod_{n \neq 0} \left( \frac{d\theta_n}{2\pi} \right) \quad (B4)$$

Once the zero mode term is factored out both $\theta(\tau)$ and $d\theta(\tau)/d\tau$ are characterized by the same set of coefficients $\{\theta_n \neq 0\}$ and an integer $m$. Explicitly

$$\frac{d\theta}{d\tau} = 2\pi m/\beta + \frac{1}{\sqrt{\beta}} \sum_{n \neq 0} (-i\nu_n) \theta_n e^{-i\nu_n \tau} .$$

From the above expression we find that the constant part of $\frac{d\theta}{d\tau}$ is constrained to take a value which is an integral multiple of $2\pi/\beta$. However, as can be seen in Eq. (B2) $\frac{d\theta}{d\tau}$ always appears in combination with $\Omega_0$. Since $\Omega_0$ is a part of Lagrange multiplier implementing delta function constraint it is non-compact, in other words, it can take an arbitrary real value. Now the explicit summation over the winding number is unnecessary and we need only to consider a Bose field

$$\Theta = \frac{d\theta}{d\tau} + \Omega_0 , \quad (B5)$$

which has an ordinary mode expansion without any constraint. Effectively we can replace $D[\theta]$ with $D[\Theta]$.

If the fluctuations beyond the Gaussian fluctuation of the order $1/N$ can be ignored, namely the corrections of order $1/N^{3/2}$ and higher can be ignored, the functional measure for the $r$ integration can be approximated by

$$dr^2(\tau) = 2r(\tau)dr(\tau) \sim 2r_s a d\bar{r}(\tau) . \quad (B6)$$
The higher order corrections can be important at high temperature or energy. We also note that due to the periodicity $r(0) = r(\beta)$, $\int_0^\beta d\tau \frac{\partial r}{\partial \tau} = 0$. Thus, at low energy the measure for the boson functional integral becomes

$$D[r, \theta] \sim \prod_\tau r_{sa} d\delta r(\tau) d\theta(r).$$

This is of a simple Cartesian product form, so that we don’t have to deal with the interactions which might be generated by the nontrivial Jacobians of functional integral measure.

Now we turn to the details of the calculation of Eq. (B3). Let us rewrite Eq. (B3) as follows:

$$\delta S = -\sum_m \text{Tr} \ln(-G_0^{-1})(1 - G_0 M)$$
$$= -\sum_m \text{Tr} \ln[-G_0^{-1}] - \sum_m \text{Tr} \ln(1 - G_0 M)$$
$$\sim -\sum_m \text{Tr} \ln[-G_0^{-1}] + \sum_m \text{Tr} [G_0 M]$$
$$+ \frac{1}{2} \sum_m \text{Tr} [G_0 M G_0 M].$$

(B8)

In the absence of the source field $\phi_m$, the sum over $m$ would just give an overall factor $N$. The first term of the last line of Eq. (B8) is relevant for the mean-field approximation. The second term $\sum_m \text{Tr} \ln[G_0 M]$ contains a few parts. Among them $gQ$ part is cancelled by the very definition of electron-phonon coupling, and the part linear in $\delta r$ is cancelled by the saddle point condition. The remaining nontrivial part is given by

$$\sum_m \text{Tr} \ln [G_0 M] \to \sum_m \int d\tau \phi_r G_0$$
$$+ \frac{N}{2} \sum_i \left[ K_{rr}^{(0)} + K_{rr}^{(2)} \right] \delta r(\tau) \delta r(-\tau),$$

(B9)

where $K_{rr}^{(0)}$ and $K_{rr}^{(2)}$ are given by

$$K_{rr}^{(0)} = 2T \sum_i G_0(i\epsilon) \Sigma_0(i\epsilon),$$

$$K_{rr}^{(2)}(\tau) = 2T \sum_i G_0(i\epsilon)[\Sigma_0(i\epsilon + i\omega) - \Sigma_0(i\epsilon)].$$

(B10)

The third term $\frac{1}{2} \sum_m \text{Tr} \ln[G_0 M G_0 M]$ can be re-expressed as follows:

$$\frac{1}{2} \sum_m \text{Tr} \ln[G_0 M G_0 M] = \frac{N}{2} \sum_\omega \sum_i K_{rr}^{(3)} \delta r(\tau) \delta r(-\tau)$$
$$\frac{1}{2} \sum_m \sum_\omega K_{rr}^{(3)} \delta r(\tau) \left[ Q + r_{sa}(\phi_m + Q + \phi_m)(\tau) \right]$$
$$+ \frac{1}{2} \sum_m \sum_\omega 2K_{rr}^{(1)} \delta r(\tau) r_{sa}(\phi_m + Q + \phi_m)(\tau).$$

(B13)

APPENDIX C: THE EXPLICIT FORMS OF THE POLARIZATION FUNCTIONS $K^{(i)}(\omega)$ AT ZERO TEMPERATURE

The polarization functions $K(\omega)$ have been calculated in Appendix A of Ref. 13 in terms of the digamma function at finite temperature. We will instead calculate the polarization functions at zero temperature in a closed form. The integrations are completely elementary, and only the results and low frequency asymptotic behaviors will be displayed. First we note that only the real part of $K^{(2)}_{rr}(\omega)$ which is an even function of frequency contributes to the effective action.

$$\text{Re} K_{rr}^{(2)}(\omega) = \frac{\Delta_0}{\pi} \ln \left[ \frac{(|\omega| + \Delta)^2 + \epsilon_f^2}{\Delta^2 + \epsilon_f^2} \right].$$

(C1)

$$K_{rr}^{(3)}(\omega) = \frac{2\Delta_0 \Delta}{\pi |\omega|} \ln \left[ 1 + \frac{|\omega|(|\omega| + 2\Delta)}{\Delta^2} \right].$$

(C2)

The sum of the above two gives

$$K_{rr}^{(1)}(\omega) \equiv \text{Re} K_{rr}^{(2)} + K_{rr}^{(3)}$$
$$= \frac{\Delta_0}{\pi} \left[ \frac{|\omega| + 2\Delta}{|\omega|} \ln \left[ 1 + \frac{|\omega|(|\omega| + 2\Delta)}{\Delta^2} \right] \right].$$

(C3)

And the rest are given by

$$K_{r\theta}^{(1)}(\omega) = \frac{\Delta_0}{\pi |\omega|} \tan^{-1} \left[ \frac{\epsilon_f |\omega|}{\Delta^2 + \epsilon_f^2 + |\omega| \Delta} \right].$$

(C4)

$$K_{\theta\theta}^{(1)}(\omega) = -\frac{\Delta_0}{\pi |\omega|(|\omega| + 2\Delta)} \ln \left[ 1 + \frac{|\omega|(|\omega| + 2\Delta)}{\Delta^2} \right].$$

(C5)

The low (imaginary) frequency asymptotic behaviors are given by

$$K_{r\theta}^{(1)}(\omega) \sim \frac{4\Delta_0}{\pi} \left[ \frac{\Delta^2}{\Delta^2 K} + \frac{\Delta |\omega|^2}{\Delta^2 K} \right].$$

(C6)

$$K_{r\theta}^{(1)}(\omega) \sim \frac{2\Delta_0}{\pi} \frac{\epsilon_f}{\Delta^2 + \epsilon_f^2}. $$

(C7)
\[ K^{(1)}_{\theta\theta}(i\omega) \sim -\frac{1}{\pi r^2_s a} \frac{\Delta}{\Delta^2 + \tilde{\epsilon}_f^2} + \frac{1}{\pi r^2_s a} \frac{|\omega|\Delta^2}{(\Delta^2 + \tilde{\epsilon}_f^2)^2}. \] (C8)

* Electronic address: hyunlee@sogang.ac.kr

1 Y. Tokura and N. Nagaosa, Science 288, 462 (2000).
2 A. B. Migdal, Sov. Phys. JETP 7, 996 (1958); G. M. Eliashberg, Sov. Phys. JETP 11, 696 (1960).
3 A. Georges, G. Kotliar, W. Krauth, and M. J. Rosenberg, Rev. Mod. Phys. 68, 13 (1996).
4 G. S. Jeon, T. H. Park, J. H. Han, Hyun C. Lee, and H. Y. Choi, cond-mat/0312390.
5 A. C. Hewson, *The Kondo Problem to Heavy Fermions* (Cambridge University Press, Cambridge 1993).
6 A. C. Hewson and D. Meyer, J. Phys.:Condens. Matter 10, 196401 (2002).
7 G. S. Jeon, T. H. Park, and H. Choi, Phys. Rev. B 68, 045106 (2003).
8 T. Holstein, Ann. Phys. (Leipzig) 8, 325 (1959).
9 W. Koller, D. Meyer, Y. Ono, and A. C. Hewson, cond-mat/0312367.
10 P. Benedetti and R. Zeyher, Phys. Rev. B 58, 14320 (1998).
11 D. Meyer, A. C. Hewson, and R. Bulla, Phys. Rev. Lett. 89, 196401 (2002).
12 We consider the one-band and spin 1/2 case only, so that the underscreened and overscreened Kondo states are not relevant to our discussions.
13 P. Coleman, Phys. Rev. B 35, 5072 (1987).
14 N. Read and D. M. Newns, J. Phys. C. 16, 3273 (1983).
15 N. Read and D. M. Newns, J. Phys. C. 29, L1055 (1983).
16 F. D. M. Haldane, Phys. Rev. Lett. 40, 416 (1978).
17 We use the notation \( s_\sigma \) for the constrained electron operator, while \( f_\sigma \) is used in Coleman’s paper. \([R]\)
18 In calculation of the physical quantities for \( N = 2 \) case, the rescaling \( V_k \to \sqrt{2}V_k \) is necessary.
19 Hyun. C. Lee and H. Choi, Phys. Rev. B 69, 075109 (2004).