Boundedness of Intrinsic Square Functions on the Weighted Weak Hardy Spaces

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Abstract. In this paper, by using the atomic decomposition theorem for weighted weak Hardy spaces, we will show the boundedness properties of intrinsic square functions including the Lusin area integral, Littlewood–Paley $g$-function and $g^\ast\lambda$-function on these spaces.

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1. Introduction

Let $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times (0, \infty)$ and $\varphi_t(x) = t^{-n} \varphi(x/t)$. The classical square function (Lusin area integral) is a familiar object. If $u(x,t) = P_t * f(x)$ is the Poisson integral of $f$, where $P_t(x) = c_n (t^2 + |x|^2)^{(n+1)/2}$ denotes the Poisson kernel in $\mathbb{R}^{n+1}_+$. Then we define the classical square function (Lusin area integral) $S(f)$ by

$$S(f)(x) = \left( \iint_{\Gamma(x)} |\nabla u(y,t)|^2 t^{1-n} dydt \right)^{1/2},$$

where $\Gamma(x)$ denotes the usual cone of aperture one:

$$\Gamma(x) = \{(y,t) \in \mathbb{R}^{n+1}_+ : |x-y| < t\}$$

and

$$|\nabla u(y,t)|^2 = \left| \frac{\partial u}{\partial t} \right|^2 + \sum_{j=1}^n \left| \frac{\partial u}{\partial y_j} \right|^2.$$

We can similarly define a cone of aperture $\beta$ for any $\beta > 0$:

$$\Gamma_\beta(x) = \{(y,t) \in \mathbb{R}^{n+1}_+ : |x-y| < \beta t\},$$
and corresponding square function

\[ S_\beta(f)(x) = \left( \int \int_{\Gamma_\beta(x)} \left| \nabla u(y, t) \right|^2 t^{1-n} dy dt \right)^{1/2}. \]

The Littlewood–Paley $g$-function (could be viewed as a “zero-aperture” version of $S(f)$) and the $g^*_\lambda$-function (could be viewed as an “infinite aperture” version of $S(f)$) are defined respectively by

\[ g(f)(x) = \left( \int_0^\infty \left| \nabla u(x, t) \right|^2 t \, dt \right)^{1/2} \]

and

\[ g^*_\lambda(f)(x) = \left( \int \int_{\mathbb{R}^{n+1}^+} \left( \frac{t}{t + |x-y|} \right)^{\lambda n} \left| \nabla u(y, t) \right|^2 t^{1-n} dy dt \right)^{1/2}, \quad \lambda > 1. \]

The modern (real-variable) variant of $S_\beta(f)$ can be defined in the following way (here we drop the subscript $\beta$ if $\beta = 1$). Let $\psi \in C^\infty(\mathbb{R}^n)$ be real, radial, have support contained in $\{ x : |x| \leq 1 \}$, and $\int_{\mathbb{R}^n} \psi(x) \, dx = 0$. The continuous square function $S_{\psi, \beta}(f)$ is defined by

\[ S_{\psi, \beta}(f)(x) = \left( \int \int_{\Gamma_\beta(x)} \left| f * \psi_t(y) \right|^2 dy dt \right)^{1/2}. \]

In 2007, Wilson [19] introduced a new square function called intrinsic square function which is universal in a sense (see also [20]). This function is independent of any particular kernel $\psi$, and it dominates pointwise all the above defined square functions. On the other hand, it is not essentially larger than any particular $S_{\psi, \beta}(f)$. For $0 < \alpha \leq 1$, let $\mathcal{C}_\alpha$ be the family of functions $\varphi$ defined on $\mathbb{R}^n$ such that $\varphi$ has support containing in $\{ x \in \mathbb{R}^n : |x| \leq 1 \}$, $\int_{\mathbb{R}^n} \varphi(x) \, dx = 0$, and for all $x, x' \in \mathbb{R}^n$,

\[ |\varphi(x) - \varphi(x')| \leq |x - x'|^\alpha. \]

For $(y, t) \in \mathbb{R}^{n+1}_+$ and $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, we set

\[ A_\alpha(f)(y, t) = \sup_{\varphi \in \mathcal{C}_\alpha} \left| f * \psi_t(y) \right|. \]

Then we define the intrinsic square function of $f$ (of order $\alpha$) by the formula

\[ S_\alpha(f)(x) = \left( \int \int_{\Gamma(x)} \left( A_\alpha(f)(y, t) \right)^2 dy dt \right)^{1/2}. \]
We can also define varying-aperture versions of $S_\alpha(f)$ by the formula

$$S_{\alpha,\beta}(f)(x) = \left( \int_{\Gamma_\beta(x)} \left( A_\alpha(f)(y, t) \frac{2}{t^{n+1}} \right)^2 \, dy \, dt \right)^{1/2}.$$ 

The intrinsic Littlewood–Paley $g$-function and the intrinsic $g^*_\lambda$-function will be defined respectively by

$$g_\alpha(f)(x) = \left( \int_{0}^{\infty} \left( A_\alpha(f)(x, t) \frac{2}{t} \right)^2 \, dt \right)^{1/2}$$

and

$$g^*_{\lambda,\alpha}(f)(x) = \left( \int_{\mathbb{R}^{n+1}} \left( \frac{t}{t+|x-y|} \right)^{\lambda n} \left( A_\alpha(f)(y, t) \frac{2}{t^{n+1}} \right)^2 \, dy \, dt \right)^{1/2}, \quad \lambda > 1.$$ 

In [20], Wilson proved the following result.

**Theorem A.** Let $0 < \alpha \leq 1$, $1 < p < \infty$ and $w \in A_p$ (Muckenhoupt weight class). Then there exists a constant $C > 0$ independent of $f$ such that

$$\|S_\alpha(f)\|_{L^p_w} \leq C \|f\|_{L^p_w}.$$ 

Moreover, Lerner [12] showed sharp $L^p_w$ norm inequalities for the intrinsic square functions in terms of the $A_p$ characteristic constant of $w$ for all $1 < p < \infty$. In [10], Huang and Liu studied the boundedness of intrinsic square functions on the weighted Hardy spaces $H^1_w(\mathbb{R}^n)$. Furthermore, they obtained the intrinsic square function characterizations of $H^1_w(\mathbb{R}^n)$. Recently, in [17, 18], we have established the strong and weak type estimates of intrinsic square functions on the weighted Hardy spaces $H^p_w(\mathbb{R}^n)$ for $n/(n+\alpha) \leq p < 1$.

The main purpose of this paper is to investigate the mapping properties of intrinsic square functions on the weighted weak Hardy spaces $W^{p}_w(\mathbb{R}^n)$ (see Section 2 for the definition). We now present our main results as follows.

**Theorem 1.1.** Let $0 < \alpha \leq 1$, $n/(n+\alpha) < p \leq 1$ and $w \in A_p(1+\frac{n}{2})$. Then there exists a constant $C > 0$ independent of $f$ such that

$$\|S_\alpha(f)\|_{W^{p}_w} \leq C \|f\|_{W^{p}_w}.$$ 

**Theorem 1.2.** Let $0 < \alpha \leq 1$, $n/(n+\alpha) < p \leq 1$ and $w \in A_p(1+\frac{n}{2})$. Suppose that $\lambda > (3n+2\alpha)/n$, then there exists a constant $C > 0$ independent of $f$ such that

$$\|g^*_{\lambda,\alpha}(f)\|_{W^{p}_w} \leq C \|f\|_{W^{p}_w}.$$ 

Wilson [19] also showed that for any $0 < \alpha \leq 1$, the functions $g_\alpha(f)(x)$ and $S_\alpha(f)(x)$ are pointwise comparable. Thus, as a direct consequence of Theorem 1.1, we obtain the following
Corollary 1.3. Let $0 < \alpha \leq 1$, $n/(n + \alpha) < p \leq 1$ and $w \in A_{p(1+\frac{\alpha}{2})}$. Then there exists a constant $C > 0$ independent of $f$ such that

$$
\|g_\alpha(f)\|_{W^{L_p}_w} \leq C\|f\|_{W^{H_p}_w}.
$$

2. Notations and Preliminaries

2.1. $A_p$ Weights

The definition of $A_p$ class was first used by Muckenhoupt [14], Hunt et al. [11], and Coifman and Fefferman [1] in the study of weighted $L^p$ boundedness of Hardy–Littlewood maximal functions and singular integrals. Let $w$ be a nonnegative, locally integrable function defined on $\mathbb{R}^n$; all cubes are assumed to have their sides parallel to the coordinate axes. We say that $w \in A_p$, $1 < p < \infty$, if

$$
\left( \frac{1}{|Q|} \int_Q w(x) \, dx \right)^{p-1} \left( \frac{1}{|Q|} \int_Q w(x)^{-1/(p-1)} \, dx \right)^{-1} \leq C
$$

for every cube $Q \subseteq \mathbb{R}^n$, where $C$ is a positive constant which is independent of the choice of $Q$.

For the case $p = 1$, $w \in A_1$, if

$$
\frac{1}{|Q|} \int_Q w(x) \, dx \leq C \cdot \text{ess inf}_{x \in Q} w(x)
$$

for every cube $Q \subseteq \mathbb{R}^n$.

A weight function $w \in A_\infty$ if it satisfies the $A_p$ condition for some $1 < p < \infty$.

It is well known that if $w \in A_p$ with $1 < p < \infty$, then $w \in A_r$ for all $r > p$, and $w \in A_q$ for some $1 < q < p$. We thus write $q_w \equiv \inf\{q > 1 : w \in A_q\}$ to denote the critical index of $w$.

Given a cube $Q$ and $\lambda > 0$, $\lambda Q$ denotes the cube with the same center as $Q$ whose side length is $\lambda$ times that of $Q$. $Q = Q(x_0, r)$ denotes the cube centered at $x_0$ with side length $r$. For a weight function $w$ and a measurable set $E$, we denote the Lebesgue measure of $E$ by $|E|$ and set the weighted measure $w(E) = \int_E w(x) \, dx$.

We give the following results that will be used in the sequel.

Lemma 2.1. [9] Let $w \in A_q$ with $q \geq 1$. Then, for any cube $Q$, there exists an absolute constant $C > 0$ such that

$$
w(2Q) \leq C \cdot w(Q).
$$

In general, for any $\lambda > 1$, we have

$$
w(\lambda Q) \leq C \cdot \lambda^{nq} w(Q),
$$

where $C$ does not depend on $Q$ nor on $\lambda$.

Lemma 2.2. [9] Let $w \in A_q$ with $q > 1$. Then, for all $r > 0$, there exists a constant $C > 0$ independent of $r$ such that

$$
\int_{|x| \geq r} \frac{w(x)}{|x|^{nq}} \, dx \leq C \cdot r^{-nq} w(Q(0,2r)).
$$
Given a weight function $w$ on $\mathbb{R}^n$, for $0 < p < \infty$, we denote by $L^p_w(\mathbb{R}^n)$ the weighted space of all functions $f$ satisfying
\[
\|f\|_{L^p_w} = \left( \int_{\mathbb{R}^n} |f(x)|^p w(x) \, dx \right)^{1/p} < \infty.
\]
When $p = \infty$, $L^\infty_w(\mathbb{R}^n)$ will be taken to mean $L^\infty(\mathbb{R}^n)$, and
\[
\|f\|_{L^\infty_w} = \|f\|_{L^\infty} = \text{ess sup}_{x \in \mathbb{R}^n} |f(x)|.
\]
We also denote by $WL^p_w(\mathbb{R}^n)$ the weighted weak $L^p$ space which is formed by all measurable functions $f$ satisfying
\[
\|f\|_{WL^p_w} = \sup_{\lambda > 0} \lambda \cdot \|w\{x \in \mathbb{R}^n : |f(x)| > \lambda\}\|^{1/p} < \infty.
\]

2.2. Weighted Weak Hardy Spaces

Let us now turn to the weighted weak Hardy spaces. The (unweighted) weak $H^p$ spaces have first appeared in the work of Fefferman, Rivièere and Sagher [7]. The atomic decomposition theory of weak $H^1$ space on $\mathbb{R}^n$ was given by Fefferman and Soria [8]. Later, Liu [13] established the weak $H^p$ spaces on homogeneous groups. For the boundedness properties of some operators on weak Hardy spaces, we refer the reader to [2–6,16]. In 2000, Quek and Yang [15] introduced the weighted weak Hardy spaces $WH^p_w(\mathbb{R}^n)$ and established their atomic decompositions. Moreover, by using the atomic decomposition theory of $WH^p_w(\mathbb{R}^n)$, Quek and Yang [15] also obtained the boundedness of Calderón-Zygmund type operators on these weighted spaces.

We write $\mathcal{S}(\mathbb{R}^n)$ to denote the Schwartz space of all rapidly decreasing smooth functions and $\mathcal{S}'(\mathbb{R}^n)$ to denote the space of all tempered distributions, i.e., the topological dual of $\mathcal{S}(\mathbb{R}^n)$. Let $w \in A_\infty$, $0 < p \leq 1$ and $N = \lfloor n(q_w/p - 1) \rfloor$. Define
\[
\mathcal{A}_{N,w} = \left\{ \varphi \in \mathcal{S}(\mathbb{R}^n) : \sup_{x \in \mathbb{R}^n} \sup_{|\alpha| \leq N+1} (1 + |x|)^{N+n+1} |D^\alpha \varphi(x)| \leq 1 \right\},
\]
where $\alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and $D^\alpha \varphi = \partial^{(\alpha)} \varphi / \partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}$. For $f \in \mathcal{S}'(\mathbb{R}^n)$, the grand maximal function of $f$ is defined by
\[
G_w f(x) = \sup_{\varphi \in \mathcal{A}_{N,w}} \sup_{|y-x| < t} |(\varphi_{t} \ast f)(y)|.
\]
Then we can define the weighted weak Hardy space by $WH^p_w(\mathbb{R}^n) = \{ f \in \mathcal{S}'(\mathbb{R}^n) : G_w f \in WL^p_w(\mathbb{R}^n) \}$. Moreover, we set $\|f\|_{WH^p_w} = \|G_w f\|_{WL^p_w}$.

Theorem 2.3. [15] Let $0 < p \leq 1$ and $w \in A_\infty$. For every $f \in WH^p_w(\mathbb{R}^n)$, there exists a sequence of bounded measurable functions $\{f_k\}_{k=-\infty}^{\infty}$ such that
(i) \( f = \sum_{k=-\infty}^{\infty} f_k \) in the sense of distributions.

(ii) Each \( f_k \) can be further decomposed into \( f_k = \sum_i b_{ki} \), where \( \{b_{ki}\} \) satisfies

(a) Each \( b_{ki} \) is supported in a cube \( Q^k_i \) with \( \sum_i w(Q^k_i) \leq c 2^{-kp} \), and \( \sum_i \chi_{Q^k_i}(x) \leq c \).

Here \( \chi_E \) denotes the characteristic function of the set \( E \) and \( c \sim \|f\|_{WH^p} \).

(b) \( \|b_{ki}\|_{L^\infty} \leq C 2^k \), where \( C > 0 \) is independent of \( i \) and \( k \);

(c) \( \int_{\mathbb{R}^n} b_{ki}^C(x^\alpha) \, dx = 0 \) for every multi-index \( \alpha \) with \( |\alpha| \leq N, N = [n(q_w/p - 1)] \).

Conversely, if \( f \in S'(\mathbb{R}^n) \) has a decomposition satisfying (i) and (ii), then \( f \in WH^p_{\mathbb{R}^n} \). Moreover, we have \( \|f\|_{WH^p_{\mathbb{R}^n}} \sim c \).

Throughout this article, \( C \) always denotes a positive constant which is independent of the main parameters involved, but it may be different from line to line.

### 3. Proof of Theorem 1.1

Before proving our main theorem in this section, let us first establish the following lemma.

**Lemma 3.1.** Let \( 0 < \alpha \leq 1 \). Then for any given function \( b \in L^\infty(\mathbb{R}^n) \) with support contained in \( Q = Q(x_0, r) \), and \( \int_{\mathbb{R}^n} b(x) \, dx = 0 \), we have

\[
S_\alpha(b)(x) \leq C \cdot \|b\|_{L^\infty} \frac{r^{n+\alpha}}{|x-x_0|^{n+\alpha}}, \quad \text{whenever} \ |x-x_0| > \sqrt{nr}.
\]

**Proof.** For any \( \varphi \in C_\alpha, \ 0 < \alpha \leq 1 \), by the vanishing moment condition of \( b \), we deduce that for any \( (y, t) \in \Gamma(x) \),

\[
|b * \varphi_t(y)| = \left| \int_Q (\varphi_t(y-z) - \varphi_t(y-x_0)) b(z) \, dz \right|
\]

\[
\leq \int_Q \frac{|z-x_0|^\alpha}{t^{n+\alpha}} |b(z)| \, dz
\]

\[
\leq C \cdot \|b\|_{L^\infty} \frac{r^{n+\alpha}}{t^{n+\alpha}}. \tag{1}
\]

For any \( z \in Q \), we have \( |z-x_0| \leq \frac{\sqrt{n}}{2}r < \frac{|x-x_0|}{2} \). Furthermore, we observe that \( \text{supp} \ \varphi \subseteq \{x \in \mathbb{R}^n : |x| \leq 1 \} \), then for any \( (y, t) \in \Gamma(x) \), by a direct computation, we can easily see that

\[
2t \geq |x-y| + |y-z| \geq |x-z| \geq |x-x_0| - |z-x_0| \geq \frac{|x-x_0|}{2}. \tag{2}
\]

Thus, for any point \( x \) with \( |x-x_0| > \sqrt{nr} \), it follows from the inequalities (1) and (2) that
\[ |S_\alpha(b)(x)| = \left( \int \int_{\mathbb{R}^n} \left( \sup_{\varphi \in C_\alpha} \left| (\varphi_t \ast b)(y) \right| \right)^2 \frac{dydt}{t^{n+1}} \right)^{1/2} \]

\[ \leq C \cdot \|b\|_{L^\infty} r^{n+\alpha} \left( \int_{\mathbb{R}^n} \int_{|x-y|<t} \frac{dydt}{t^{2(n+\alpha)+n+1}} \right)^{1/2} \]

\[ \leq C \cdot \|b\|_{L^\infty} r^{n+\alpha} \left( \int_{\mathbb{R}^n} \frac{dt}{t^{2(n+\alpha)+1}} \right)^{1/2} \]

\[ \leq C \cdot \|b\|_{L^\infty} r^{n+\alpha} \frac{r^{n+\alpha}}{|x-x_0|^{n+\alpha}}. \]

We are done. \(\square\)

We are now in a position to give the proof of Theorem 1.1.

**Proof of Theorem 1.1.** For any given \(\lambda > 0\), we may choose \(k_0 \in \mathbb{Z}\) such that \(2^{k_0} \leq \lambda < 2^{k_0+1}\). For every \(f \in WH_w^p(\mathbb{R}^n)\), then by Theorem 2.3, we can write

\[ f = \sum_{k=-\infty}^{\infty} f_k = \sum_{k=-\infty}^{k_0} f_k + \sum_{k=k_0+1}^{\infty} f_k = F_1 + F_2, \]

where \(F_1 = \sum_{k=-\infty}^{k_0} f_k = \sum_{k=-\infty}^{k_0} \sum_i b_i^k\), \(F_2 = \sum_{k=k_0+1}^{\infty} f_k = \sum_{k=k_0+1}^{\infty} \sum_i b_i^k\) and \(\{b_i^k\}\) satisfies (a)–(c) in Theorem 2.3. Then we have

\[ \lambda^p \cdot w\{ x \in \mathbb{R}^n : |S_\alpha(f)(x)| > \lambda \} \]

\[ \leq \lambda^p \cdot w\{ x \in \mathbb{R}^n : |S_\alpha(F_1)(x)| > \lambda/2 \} \]

\[ + \lambda^p \cdot w\{ x \in \mathbb{R}^n : |S_\alpha(F_2)(x)| > \lambda/2 \} \]

\[ = I_1 + I_2. \]

First we claim that the following inequality holds:

\[ \|F_1\|_{L^2_w} \leq C \cdot \lambda^{1-p/2} \|f\|_{WH_w^p}^{p/2}. \quad (3) \]

In fact, since \(\text{supp } b_i^k \subseteq Q_i^k = Q(x_i^k, r_i^k)\) and \(\|b_i^k\|_{L^\infty} \leq C 2^k\) by Theorem 2.3, then it follows from Minkowski’s integral inequality that

\[ \|F_1\|_{L^2_w} \leq \sum_{k=-\infty}^{k_0} \sum_i \|b_i^k\|_{L^2_w} \]

\[ \leq \sum_{k=-\infty}^{k_0} \sum_i \|b_i^k\|_{L^\infty} w(Q_i^k)^{1/2}. \]
For each $k \in \mathbb{Z}$, by using the bounded overlapping property of the cubes \(\{Q^k_i\}\) and the fact that \(1 - p/2 > 0\), we thus obtain
\[
\| F_1 \|_{L^2_w} \leq C \sum_{k=-\infty}^{k_0} 2^k \left( \sum_i w(Q^k_i) \right)^{1/2}
\leq C \sum_{k=-\infty}^{k_0} 2^{k(1-p/2)} \| f \|_{WH^p_w}^{p/2}
\leq C \sum_{k=-\infty}^{k_0} 2^{(k-k_0)(1-p/2)} \cdot \lambda^{1-p/2} \| f \|_{WH^p_w}^{p/2}
\leq C \cdot \lambda^{1-p/2} \| f \|_{WH^p_w}^{p/2}.
\]

Since \(w \in A_{p(1+\frac{\alpha}{n})}\) and \(1 < p(1+\frac{\alpha}{n}) \leq 1 + \frac{\alpha}{n} \leq 2\), then \(w \in A_2\). Hence, it follows from Chebyshev’s inequality, Theorem A and the inequality (3) that
\[
I_1 \leq \lambda^p \cdot \frac{4}{\lambda^2} \| S_\alpha(F_1) \|_{L^2_w}^2
\leq C \cdot \lambda^{p-2} \| F_1 \|_{L^2_w}^2
\leq C \| f \|_{WH^p_w}^p.
\]

Now we turn our attention to the estimate of \(I_2\). We set
\[
A_{k_0} = \bigcup_{k=k_0+1}^\infty \bigcup_i \widetilde{Q}^k_i,
\]
where \(\widetilde{Q}^k_i = Q(x^k_i, \tau^{(k-k_0)/(n+\alpha)}(2\sqrt{n})r^k_i)\) and \(\tau\) is a fixed positive number such that \(1 < \tau < 2\). Thus, we can further decompose \(I_2\) as
\[
I_2 \leq \lambda^p \cdot w \left( \{ x \in A_{k_0} : |S_\alpha(F_2)(x) > \lambda/2 \} \right)
+ \lambda^p \cdot w \left( \{ x \in (A_{k_0})^c : |S_\alpha(F_2)(x) > \lambda/2 \} \right)
= I'_2 + I''_2.
\]

Since \(w \in A_{p(1+\frac{\alpha}{n})}\), then by Lemma 2.1, we can get
\[
I'_2 \leq \lambda^p \sum_{k=k_0+1}^\infty \sum_i w(\widetilde{Q}^k_i)
\leq C \cdot \lambda^p \sum_{k=k_0+1}^\infty \tau^{(k-k_0)p} \sum_i w(Q^k_i)
\leq C \| f \|_{WH^p_w}^p \sum_{k=k_0+1}^\infty \left( \frac{\tau}{2} \right)^{(k-k_0)p}
\leq C \| f \|_{WH^p_w}^p.
\]
On the other hand, an application of Chebyshev’s inequality gives us that

\[
I''_2 \leq 2^p \int_{(A_{k_0})^c} \left| S_\alpha(F_2)(x) \right|^p w(x) \, dx
\]

\[
\leq 2^p \sum_{k=k_0+1}^{\infty} \sum_i \int_{(Q_i^k)^c} \left| S_\alpha(b_i^k)(x) \right|^p w(x) \, dx.
\]

When \( x \in (Q_i^k)^c \), then a direct calculation shows that

\[
|x - x_i^k| \geq \tau^{(k-k_0)/(n+\alpha)} \sqrt{n} r_i^k > \sqrt{n} r_i^k.
\]

Let \( q = p(1 + \frac{\alpha}{n}) \) for simplicity. Then for any \( n/(n+\alpha) < p \leq 1 \), \( w \in A_q \) and \( q > 1 \), we can see that \( [n(q_w/p - 1)] = 0 \). Obviously, by Theorem 2.3, we know that all the functions \( b_i^k \) satisfy the conditions in Lemma 3.1 Applying Lemma 2.2 and Lemma 3.1, we can deduce

\[
I''_2 \leq C \sum_{k=k_0+1}^{\infty} \sum_i 2^{kp} (r_i^k)^{(n+\alpha)p} \int_{|x-x_i^k| \geq \tau^{(k-k_0)/(n+\alpha)} \sqrt{n} r_i^k} \frac{w(x)}{|x-x_i^k|^{(n+\alpha)p}} \, dx
\]

\[
= C \sum_{k=k_0+1}^{\infty} \sum_i 2^{kp} (r_i^k)^{nq} \int_{|y| \geq \tau^{(k-k_0)/(n+\alpha)} \sqrt{n} r_i^k} \frac{w_i^k(y)}{|y|^{nq}} \, dy
\]

\[
\leq C \sum_{k=k_0+1}^{\infty} \sum_i 2^{kp} \left( \tau^{(k-k_0)/(n+\alpha)} \right)^{-nq} w_i^k(Q(0, \tau^{(k-k_0)/(n+\alpha)} \cdot r_i^k))
\]

\[
= C \sum_{k=k_0+1}^{\infty} \sum_i 2^{kp} \left( \tau^{(k-k_0)/(n+\alpha)} \right)^{-nq} w(Q(x_i^k, \tau^{(k-k_0)/(n+\alpha)} \cdot r_i^k)),
\]

where \( w_i^k(x) = w(x+x_i^k) \) is the translation of \( w(x) \). It is obvious that \( w_i^k \in A_q \) whenever \( w \in A_q \), and \( q_w^k = q_w \). In addition, for \( w \in A_q \) with \( q > 1 \), then we can take a sufficiently small number \( \varepsilon > 0 \) such that \( w \in A_{q-\varepsilon} \). Therefore, by using Lemma 2.1 again, we obtain

\[
I''_2 \leq C \sum_{k=k_0+1}^{\infty} \sum_i 2^{kp} \left( \tau^{(k-k_0)/(n+\alpha)} \right)^{-n\varepsilon} w(Q_i^k)
\]

\[
\leq C \|f\|_{WH^\varepsilon_q}^p \sum_{k=k_0+1}^{\infty} \left( \tau^{(k-k_0)/(n+\alpha)} \right)^{-n\varepsilon}
\]

\[
\leq C \|f\|_{WH^\varepsilon_q}^p.
\]

Summing up the above estimates for \( I_1 \) and \( I_2 \), and then taking the supremum over all \( \lambda > 0 \), we complete the proof of Theorem 1.1. \( \square \)

4. Proof of Theorem 1.2

In order to prove Theorem 1.2, we shall need the following two lemmas.
Lemma 4.1. Let $0 < \alpha \leq 1$, $n/(n + \alpha) < p \leq 1$ and $w \in A_{p(1 + \frac{\alpha}{n})}$. Then for every $\lambda > p(1 + \frac{\alpha}{n})$, we have

$$\|g_{\lambda,\alpha}^*(f)\|_{L_w^2} \leq C \|f\|_{L_w^2}$$

holds for all $f \in L_w^2(\mathbb{R}^n)$.

Proof. From the definition of $g_{\lambda,\alpha}^*$, we readily see that

$$(g_{\lambda,\alpha}^*(f)(x))^2 = \int\int_{\mathbb{R}^{n+1}_+} \left( \frac{t}{t + |x-y|} \right)^{\lambda n} \left( A_{\alpha}(f)(y,t) \right)^2 \frac{dydt}{t^{n+1}}$$

$$= \int_0^\infty \int \left( \frac{t}{t + |x-y|} \right)^{\lambda n} \left( A_{\alpha}(f)(y,t) \right)^2 \frac{dydt}{t^{n+1}}$$

$$+ \sum_{j=1}^{\infty} \int_0^{2^{j-1}t} \int \left( \frac{t}{t + |x-y|} \right)^{\lambda n} \left( A_{\alpha}(f)(y,t) \right)^2 \frac{dydt}{t^{n+1}}$$

$$\leq C \left[ S_{\alpha}(f)(x)^2 + \sum_{j=1}^{\infty} 2^{-j\lambda n} S_{\alpha,2^j}(f)(x)^2 \right]. \quad (4)$$

We are now going to estimate $\int_{\mathbb{R}^n} |S_{\alpha,2^j}(f)(x)|^2 w(x) \, dx$ for $j = 1, 2, \ldots$. Fubini’s theorem and Lemma 2.1 imply

$$\int_{\mathbb{R}^n} |S_{\alpha,2^j}(f)(x)|^2 w(x) \, dx$$

$$= \int\int_{\mathbb{R}^{n+1}_+} \left( \int_{|x-y| < 2^j t} w(x) \, dx \right) \left( A_{\alpha}(f)(y,t) \right)^2 \frac{dydt}{t^{n+1}}$$

$$\leq C \cdot 2^{j(n+\alpha)p} \int\int_{\mathbb{R}^{n+1}_+} \left( \int_{|x-y| < t} w(x) \, dx \right) \left( A_{\alpha}(f)(y,t) \right)^2 \frac{dydt}{t^{n+1}}$$

$$= C \cdot 2^{j(n+\alpha)p} \|S_{\alpha}(f)\|_{L_w^2}^2. \quad (5)$$

Since $w \in A_{p(1 + \frac{\alpha}{n})}$ and $1 < p(1 + \frac{\alpha}{n}) \leq 2$, then we have $w \in A_2$. Therefore, under the assumption that $\lambda > p(1 + \frac{\alpha}{n})$, it follows from Theorem A and the above inequalities (4) and (5) that
\[ \| g_{\lambda, \alpha}^*(f) \|_{L_w^2}^2 \leq C \left( \int_{\mathbb{R}^n} |S_\alpha(f)(x)|^2 w(x) \, dx + \sum_{j=1}^{\infty} 2^{-j \lambda n} \int_{\mathbb{R}^n} |S_{\alpha, 2j}(f)(x)|^2 w(x) \, dx \right) \]

\[ \leq C \left( \| S_\alpha(f) \|_{L_w^2}^2 + \sum_{j=1}^{\infty} 2^{-j \lambda n} \cdot 2^{j(n+\alpha)p} \| S_\alpha(f) \|_{L_w^2}^2 \right) \]

\[ \leq C \cdot \| f \|_{L_w^2}^2 \left( 1 + \sum_{j=1}^{\infty} 2^{-j \lambda n} \cdot 2^{j(n+\alpha)p} \right) \]

\[ \leq C \cdot \| f \|_{L_w^2}^2. \]

Taking square-roots on both sides of the above inequality, we are done. \[ \square \]

Following the same procedure as that of Lemma 3.1, we can also show

**Lemma 4.2.** Let \( 0 < \alpha \leq 1 \) and \( j \in \mathbb{Z}_+ \). Then for any given function \( b \in L^\infty(\mathbb{R}^n) \) with support contained in \( Q = Q(x_0, r) \), and \( \int_{\mathbb{R}^n} b(x) \, dx = 0 \), we have

\[ S_{\alpha, 2j}(b)(x) \leq C \cdot 2^{j(3n+2\alpha)/2} \| b \|_{L^\infty} \frac{r^{n+\alpha}}{|x-x_0|^{n+\alpha}}, \quad \text{whenever } |x-x_0| > \sqrt{n}r. \]

**Proof.** For any \( z \in Q(x_0, r) \), we have \( |z - x_0| < \frac{|x-x_0|}{2} \). Then for all \((y, t) \in \Gamma_{2j}(x)\) and \( |y-z| \leq t \) with \( z \in Q \), as in the proof of Lemma 3.1, we can deduce that

\[ t + 2^j t \geq |x-y| + |y-z| \geq |x-z| \geq |x-x_0| - |z-x_0| \geq \frac{|x-x_0|}{2}. \]

Thus, by using the inequalities (1) and (6), we obtain

\[ |S_{\alpha, 2j}(b)(x)| = \left( \int_{\Gamma_{2j}(x)} \left( \sup_{\varphi \in C_\alpha} \left| (\varphi_t \ast b)(y) \right| \right)^2 \frac{dydt}{t^{n+1}} \right)^{1/2} \]

\[ \leq C \cdot \| b \|_{L^\infty} \left( \int_{\frac{|x-x_0|}{2^j} + \frac{|y-z|}{2^j} < 2^j t} \int_{|y-z| < 2^j t} \frac{dydt}{t^{2(n+\alpha)+n+1}} \right)^{1/2} \]

\[ \leq C \cdot 2^{jn/2} \| b \|_{L^\infty} \left( \int_{\frac{|x-x_0|}{2^j}} \frac{dt}{t^{2(n+\alpha)+1}} \right)^{1/2} \]

\[ \leq C \cdot 2^{j(3n+2\alpha)/2} \| b \|_{L^\infty} \frac{r^{n+\alpha}}{|x-x_0|^{n+\alpha}}. \]

This finishes the proof of Lemma 4.2. \[ \square \]

We are now ready to show our main theorem of this section.
Proof of Theorem 1.2. We follow the strategy of the proof of Theorem 1.1. For any given $\sigma > 0$, we are able to choose $k_0 \in \mathbb{Z}$ such that $2^{k_0} \leq \sigma < 2^{k_0+1}$. For every $f \in \text{WH}_w^p(\mathbb{R}^n)$, we can also write

$$
\sigma^p \cdot w(\{ x \in \mathbb{R}^n : |g_{\lambda,\alpha}^*(f)(x)| > \sigma \}) \\
\leq \sigma^p \cdot w(\{ x \in \mathbb{R}^n : |g_{\lambda,\alpha}^*(F_1)(x)| > \sigma/2 \}) \\
+ \sigma^p \cdot w(\{ x \in \mathbb{R}^n : |g_{\lambda,\alpha}^*(F_2)(x)| > \sigma/2 \}) \\
= J_1 + J_2,
$$

where the notations $F_1$ and $F_2$ are the same as in the proof of Theorem 1.1. By our assumption, we know that $\lambda > (3n + 2\alpha)/n > p(1 + \alpha/n)$. Applying Chebyshev’s inequality, Lemma 4.1 and the previous inequality (3), we obtain

$$
J_1 \leq \sigma^p \cdot \frac{4}{\sigma^2} \| g_{\lambda,\alpha}^*(F_1) \|_{L_w^2}^2 \\
\leq C \cdot \sigma^{p-2} \| F_1 \|_{L_w^2}^2 \\
\leq C \| f \|_{\text{WH}_w^p}^p.
$$

To estimate the other term $J_2$, as before, we also set

$$
A_{k_0} = \bigcup_{k=k_0+1}^{\infty} \bigcup_{i} Q_k^i,
$$

where $Q_k^i = Q(x_i^k, r_i^k)$, $\tau^{(k-k_0)(n+\alpha)}(2\sqrt{n}) r_i^k$, $\tau$ is also a fixed real number such that $1 < \tau < 2$ and $\text{supp} b_k^i \subseteq Q_k^i = Q(x_i^k, r_i^k)$. Again, we shall further decompose $J_2$ as

$$
J_2 \leq \sigma^p \cdot \frac{4}{\sigma^2} \| g_{\lambda,\alpha}^*(F_2) \|_{L_w^2}^2 \\
+ \sigma^p \cdot w(\{ x \in (A_{k_0})^c : |g_{\lambda,\alpha}^*(F_2)(x)| > \sigma/2 \}) \\
= J_2' + J_2''.
$$

Using the same arguments as that of Theorem 1.1, we can see that

$$
J_2' \leq \sigma^p \sum_{k=k_0+1}^{\infty} \sum_i w(\widetilde{Q}_i^k) \\
\leq C \cdot \sigma^p \sum_{k=k_0+1}^{\infty} \tau^{(k-k_0)p} \sum_i w(Q_i^k) \\
\leq C \| f \|_{\text{WH}_w^p}^p.
$$

Noting that $0 < p \leq 1$. Then by Chebyshev’s inequality and the inequality (4), we have

$$
J_2'' \leq 2^p \int_{(A_{k_0})^c} |g_{\lambda,\alpha}^*(F_2)(x)|^p w(x) \, dx \\
\leq 2^p \sum_{k=k_0+1}^{\infty} \sum_i \int_{(\widetilde{Q}_i^k)^c} |S_{\alpha}(b_i^k)(x)|^p w(x) \, dx
$$
\[ +2^p \sum_{j=1}^{\infty} 2^{-j\lambda np/2} \sum_{k=k_0+1}^{\infty} \sum_{i} \int_{(Q_j^k)^c} \left| S_{\alpha,2j}(b^k_i)(x) \right|^p w(x) \, dx \]

\[ = K_0 + \sum_{j=1}^{\infty} 2^{-j\lambda np/2} K_j. \]

Note that \([n(q_w/p - 1)] = 0\) by the hypothesis. Clearly, in view of Theorem 2.3, we can easily see that all the functions \(b^k_i\) satisfy the conditions in Lemma 3.1 or Lemma 4.2. In the proof of Theorem 1.1, we have already showed that

\[ K_0 \leq C \left\| f \right\|_{WH^p_w}^p. \]

Below we shall give the estimates of \(K_j\) for every \(j = 1, 2, \ldots\). Observe that

\[ \text{for any } x \in (\tilde{Q}^k_i)^c, \text{ we have} \]

\[ |x - x^k_i| \geq \tau^{(k-k_0)/(n+\alpha)} \sqrt{nr^k_i} > \sqrt{nr^k_i}. \]

Since \(\|b^k_i\|_{L^\infty} \leq C2^k\), then, using Lemma 4.2 and following the same lines as in Theorem 1.1, we can also deduce

\[ K_j \leq C \cdot 2^{j(3n+2\alpha)p/2} \sum_{k=k_0+1}^{\infty} \sum_{i} \left\| b^k_i \right\|_{L^\infty} \left( r^k_i \right)^{(n+\alpha)p} \]

\[ \times \int_{|x-x^k_i| \geq \tau^{(k-k_0)/(n+\alpha)} \sqrt{nr^k_i}} \frac{w(x)}{|x - x^k_i|^{(n+\alpha)p}} \, dx \]

\[ \leq C \cdot 2^{j(3n+2\alpha)p/2} \sum_{k=k_0+1}^{\infty} \sum_{i} 2^{kp} \left( r^k_i \right)^{(n+\alpha)p} \]

\[ \times \int_{|x-x^k_i| \geq \tau^{(k-k_0)/(n+\alpha)} \sqrt{nr^k_i}} \frac{w(x)}{|x - x^k_i|^{(n+\alpha)p}} \, dx \]

\[ \leq C \cdot 2^{j(3n+2\alpha)p/2} \left\| f \right\|_{WH^p_w}^p. \]

Hence, we finally obtain

\[ J''_2 \leq C \left\| f \right\|_{WH^p_w}^p \left( 1 + \sum_{j=1}^{\infty} 2^{-j\lambda np/2} \cdot 2^{j(3n+2\alpha)p/2} \right) \]

\[ \leq C \left\| f \right\|_{WH^p_w}^p, \]

where the last series is convergent since \(\lambda > (3n + 2\alpha)/n\). Therefore, combining the above estimates for \(J_1\) and \(J_2\), and then taking the supremum over all \(\sigma > 0\), we conclude the proof of Theorem 1.2.

\[ \square \]

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