SEMIGROUP PROPERTIES OF SOLUTIONS OF SDES DRIVEN BY LÉVY PROCESSES WITH INDEPENDENT COORDINATES

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ABSTRACT. We study the stochastic differential equation \( dX_t = A(X_{t-}) dZ_t, \) \( X_0 = x, \) where \( Z_t = (Z_t^{(1)}, \ldots, Z_t^{(d)})^T \) and \( Z_t^{(1)}, \ldots, Z_t^{(d)} \) are independent one-dimensional Lévy processes with characteristic exponents \( \psi_1, \ldots, \psi_d. \) We assume that each \( \psi_i \) satisfies a weak lower scaling condition WLSC\((\alpha, \beta, 0, \nu)\) for some constants \( 0 < \alpha \leq \beta < 2, \) \( \nu \geq 0, \) \( C, \overline{C} > 0, \) and \( \alpha > (2/3)\beta. \) We also assume that the determinant of \( A(x) = (a_{ij}(x)) \) is bounded away from zero, and \( a_{ij}(x) \) are bounded and Lipschitz continuous. In both cases (i) and (ii) we prove that for any fixed \( \gamma \in (0, \alpha) \) the semigroup \( P_t \) of the process \( X_t \) satisfies \( |P_t f(x) - P_t f(y)| \leq c t^{-\gamma/\alpha} \|x - y\|^\gamma |f|_\infty \) for arbitrary bounded Borel function \( f. \) We also show the existence of a transition density of the process \( X_t. \)

1. Introduction

We study the following stochastic differential equation
\[
dX_t = A(X_{t-}) dZ_t, \quad X_0 = x \in \mathbb{R}^d.
\]
We make the following assumptions on matrices \( A(x) \) and a process \( Z_t. \)

**Assumptions (A).** \( A(x) = (a_{ij}(x)) \) is a \( d \times d \) matrix for each \( x \in \mathbb{R}^d \) \((d \in \mathbb{N}, d \geq 2).\) There are constants \( \eta_1, \eta_2, \eta_3 > 0, \) such that for any \( x, y \in \mathbb{R}^d, i, j \in \{1, \ldots, d\} \)
\[
|a_{ij}(x)| \leq \eta_1, \tag{2}
\]
\[
\det(A(x)) \geq \eta_2, \tag{3}
\]
\[
|a_{ij}(x) - a_{ij}(y)| \leq \eta_3 |x - y|. \tag{4}
\]

**Assumptions (Z).** \( Z_t = (Z_t^{(1)}, \ldots, Z_t^{(d)})^T, \) where \( Z_t^{(1)}, \ldots, Z_t^{(d)} \) are independent one-dimensional Lévy processes (not necessarily identically distributed). For each \( i \in \{1, \ldots, d\} \) the characteristic exponent \( \psi_i \) of the process \( Z_t^{(i)} \) is given by
\[
\psi_i(\xi) = \int_{\mathbb{R}} (1 - \cos(\xi x)) \nu_i(x) \, dx,
\]
where \( \nu_i(x) \) is the density of a symmetric, infinite Lévy measure \((i.e. \nu_i: \mathbb{R} \setminus \{0\} \to [0, \infty), \int_{\mathbb{R}} (x^2 \wedge 1) \nu_i(x) \, dx < \infty, \int_{\mathbb{R}} \nu_i(x) \, dx = \infty, \nu_i(-x) = \nu_i(x) \) for \( x \in \mathbb{R} \setminus \{0\}.\) There exists \( \eta_d > 0 \) such that \( \nu_i \in C^1(0, \eta_d), \nu_i'(x) < 0 \) for \( x \in (0, \eta_d) \) and \( -\nu_i'(x)/x \) is decreasing on \((0, \eta_d)\). \( \psi_i \) satisfies a weak lower scaling condition WLSC\((\alpha, \beta, 0, \nu)\) and a weak upper scaling condition WUSC\((\beta, \theta_0, \nu, \alpha)\) for some constants \( 0 < \alpha \leq \beta < 2, \theta_0 \geq 0, \overline{C}, \nu > 0 \) (the definitions of WLSC and WUSC are presented in Section 2).
It is well known that under these assumptions SDE (1) has a unique strong solution $X_t$, see e.g. [37, Theorem 34.7 and Corollary 35.3]. By [11, Corollary 3.3] $X_t$ is a Feller process.

In the paper we will consider two mutually exclusive assumptions:

**Assumptions (Q1).** Matrices $A(x)$ and the process $Z_t$ satisfy assumptions (A), (Z). All $\psi_1, \ldots, \psi_d$ are the same.

**Assumptions (Q2).** Matrices $A(x)$ and the process $Z_t$ satisfy assumptions (A), (Z). Not all $\psi_1, \ldots, \psi_d$ are the same. $\alpha > (2/3)\beta$.

Put $\nu(x) = (\nu_1(x), \ldots, \nu_d(x))$. Let $\mathbb{E}^x$ denote the expected value of the process $X_t$ starting from $x$ and $\mathcal{B}_b(\mathbb{R}^d)$ denote the set of all Borel bounded functions $f : \mathbb{R}^d \to \mathbb{R}$. For any $t \geq 0$, $x \in \mathbb{R}^d$ and $f \in \mathcal{B}_b(\mathbb{R}^d)$ we put

$$P_t f(x) = \mathbb{E}^x f(X_t).$$

The main result of this paper is the following theorem.

**Theorem 1.1.** Let $A(x)$, $Z_t$ satisfy assumptions (Q1) or (Q2), $X_t$ be the solution of (1) and $P_t$ be given by (3). Then for any $\gamma \in (0, \alpha)$, $\tau > 0$, $t \in (0, \tau]$, $x, y \in \mathbb{R}^d$ and $f \in \mathcal{B}_b(\mathbb{R}^d)$ we have

$$|P_t f(x) - P_t f(y)| \leq c t^{-\gamma/\alpha} |x - y|^\gamma ||f||_\infty,$$

where $c$ depends on $\gamma, \tau, \alpha, \beta, \theta_0, \underline{C}, \overline{C}, d, \eta_1, \eta_2, \eta_3, \eta_4, \nu$.

This gives the strong Feller property of the semigroup $P_t$. Note that the weaker result namely the strong Feller property of the resolvent $R_s f(x) = \int_0^\infty e^{-st} T_s f(x) \, dt$ ($s > 0$) follows from [43, Theorem 3.6].

We also show the existence of a transition density of the process $X_t$.

**Proposition 1.2.** Let $A(x)$, $Z_t$ satisfy assumptions (Q1) or (Q2) and $X_t$ be the solution of (1). Then the process $X_t$ has a lower semi-continuous transition density function $p(t, x, y)$, $p : (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty]$ with respect to the Lebesgue measure on $\mathbb{R}^d$.

Recently, the existence of densities for stochastic differential equations driven by Lévy processes have been studied in [17] (cf. also [14]). Our existence results and the existence results from [17] have some large intersection. However, their results do not imply ours and our results do not imply theirs. Some more comments on this are in the Remark 1.5.

One may ask about the boundedness of $p(t, x, y)$. It turns out that for some choices of matrices $A(x)$ and processes $Z_t$ (satisfying assumptions (Q1)) and for some $t > 0$ and $x \in \mathbb{R}^d$ we might have $p(t, x, \cdot) \notin L^\infty(\mathbb{R}^d)$ (see Remarks 4.23 and 4.24 in [31]). Nevertheless we have the following regularity result.

**Theorem 1.3.** Let $A(x)$, $Z_t$ satisfy assumptions (Q1) or (Q2), $X_t$ be the solution of (1) and $P_t$ be given by (3). Then for any $\gamma \in (0, \alpha/(d + \beta + 1 - \alpha))$, $\tau > 0$, $t \in (0, \tau]$, $x \in \mathbb{R}^d$ and $f \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ we have

$$|P_t f(x)| \leq c t^{-\gamma(d + \beta + 1 - \alpha)/\alpha} ||f||_{L^\infty}^{-\gamma} ||f||_\infty^\gamma,$$

where $c$ depends on $\gamma, \tau, \alpha, \beta, \theta_0, \underline{C}, \overline{C}, d, \eta_1, \eta_2, \eta_3, \eta_4, \nu$.

Note that we have been able to show only lower semi-continuity of $p(t, x, y)$. In fact, we believe that a stronger result is true.
Conjecture 1.4. Let $A(x)$, $Z_t$ satisfy assumptions (Q1) or (Q2) and $X_t$ be the solution of (1). Then the process $X_t$ has a continuous transition density function $p(t, x, y)$, $p : (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty]$ with respect to the Lebesgue measure on $\mathbb{R}^d$. If $p(t_0, x_0, y_0) = \infty$ for some $t_0 > 0$, $x_0, y_0 \in \mathbb{R}^d$ then for all $t > 0$, $x \in \mathbb{R}^d$ we have $p(t, x, y_0) = \infty$. 

The continuity should be understood here in the extended sense (as a function with values in $[0, \infty]$).

Estimates of the type $|P_t f(x) - P_t f(y)| \leq c_t |x - y|^{\gamma} \|f\|_{\infty}$ or $|\nabla_x P_t f(x)| \leq c_{p,t} \|f\|_p$ (for $p > 1$) of semigroups of solutions of SDEs

$$dX_t = A(X_{t-}) dZ_t + b(X_t) dt, \quad X_0 = x \in \mathbb{R}^d$$

(7) driven by general Lévy processes $Z_t$ with jumps have attracted a lot of attention recently. Similarly, of great interest were Hölder or gradient estimates of transition densities of the semigroups of the type $|p(t, x, y) - p(\tau, z, y)| \leq c_t \gamma |x - z|^{\gamma}$, $|\nabla_x p(t, x, y)| \leq c_t \gamma$. A lot is known about such estimates when the driving process $Z_t$ has a non-degenerate diffusion part [18]. Another well studied case is when $Z_t$ is a subordinated Brownian motion [10], or a symmetric Lévy process with its Lévy exponent satisfying a weak lower scaling condition at infinity [23]. There also results for pure-jump Lévy processes in $\mathbb{R}^d$ such that their Lévy measure satisfies $\nu(dz) \geq c_1 |z| \leq c_2 |z|^{-d-\alpha}$ for some $\alpha \in (0, 2)$ and $c, r > 0$ [30]. The typical techniques are the coupling method, the use of the Bismut-Elworthy-Li formula or the Levi (parametrix) method.

Much more demanding case is when the Lévy measure of the driving process $Z_t$ is singular. The above gradient type estimates have been studied for SDEs driven by additive cylindrical Lévy processes (i.e. when $A \equiv I$ and $b \not\equiv 0$ in (4)). The above Hölder type estimates for SDEs driven by processes $Z_t$ with singular Lévy measures were also studied in the case when matrices $A(x)$ were diagonal [30], [36]. The case when the Lévy measure of the driving process $Z_t$ is singular and matrices $A(x)$ are not diagonal is much more difficult (heuristically it corresponds to rotations of singular jumping measures). The first important step in understanding this case was done in [31] in which it was assumed that the driving process $Z_t$ is a cylindrical $\alpha$-stable process in $\mathbb{R}^d$ with $\alpha \in (0, 1)$.

The proof of the main result Theorem 1.1 is based on ideas from [31]. Similarly as in [31] we first truncate the Lévy measure of the process $Z_t$. Then, as in [31], we construct the semigroup of the solution of (1), driven by the process with truncated Lévy measure using the Levi method. Finally, we construct the semigroup of the solution of (1), driven by the not truncated process, by (roughly speaking) adding long jumps to the truncated process.

Nevertheless, there are big differences between this paper and [31]. First, in [31] the generators of processes $Z_t^{(i)}$ are operators of order smaller than 1 and in this paper they may be of order bigger than 1. This is much more difficult situation. Secondly, in [31] the processes $Z_t^{(i)}$ are stable processes and in this paper they are quite general Lévy processes. The investigation of these processes is much more complicated than stable processes (see Section 2). Thirdly, and most importantly, in [31] all components $Z_t^{(i)}$ are identically distributed and in our paper we consider the case in which $Z_t^{(i)}$ have different distributions. From technical point of view, in order to use Levi’s method, we have to apply generators of $Z_t^{(i)}$ to the density of
When $Z_t^{(i)}$ and $Z_t^{(j)}$ have different distributions this leads to major difficulties in proofs (see e.g. proofs of Lemma 2.1 and Proposition 3.10).

Let us also add that the properties of harmonic functions corresponding to the solutions of \textbf{(1)}, when the driving process $Z_t$ is just the cylindrical $\alpha$-stable process were studied in \cite{[40]}, (see also \cite{[8]} for more general results).

Now we exhibit some examples of processes for which assumptions (Q1) or (Q2) are satisfied. In all these examples we assume that matrices $A(x)$ satisfy (A).

**Example 1.** Assume that for each $i \in \{1, \ldots, d\}$ we have $Z_t^{(i)} = B_t^{(i)} S_t^{(i)}$ where $B_t^{(i)}$ is the one-dimensional Brownian motion and $S_t^{(i)}$ is a subordinator with an infinite Lévy measure $\mu$ and Laplace exponent $\varphi$. Assume also that $B_t^{(1)}, \ldots, B_t^{(d)}, S_t^{(1)}, \ldots, S_t^{(d)}$ are independent and for each $i \in \{1, \ldots, d\}$ we have $\varphi \in WUSC(\alpha/2, 0, C')$, $\varphi \in WUSC(\beta/2, \theta''_0, C')$ for some constants $\alpha, \beta \in (0, 2)$, $\theta''_0 \geq 0$, $C', C'' > 0$. Then assumptions (Q1) are satisfied.

In particular, this holds for the process $Z_t = (Z_t^{(1)}, \ldots, Z_t^{(d)})^T$ such that $Z_t^{(1)}, \ldots, Z_t^{(d)}$ are independent and for each $i \in \{1, \ldots, d\}$ $Z_t^{(i)}$ is a one-dimensional, symmetric $\alpha$-stable process, where $\alpha \in (0, 2)$.

Similarly, this holds for the process $Z_t = (Z_t^{(1)}, \ldots, Z_t^{(d)})^T$ such that $Z_t^{(1)}, \ldots, Z_t^{(d)}$ are independent and for each $i \in \{1, \ldots, d\}$ $Z_t^{(i)}$ is a one-dimensional, relativistic $\alpha$-stable process (cf. \cite{[40]}) with $\psi_i(\xi) = (m^{2/\alpha} + |\xi|^2)^{\alpha/2} - m$, where $\alpha \in (0, 2)$, $m > 0$.

**Example 2.** Assume that for each $i \in \{1, \ldots, d\}$ we have $Z_t^{(i)} = B_t^{(i)} S_t^{(i)}$ where $B_t^{(i)}$ is the one-dimensional Brownian motion and $S_t^{(i)}$ is a subordinator with an infinite Lévy measure $\mu_i$ and Laplace exponent $\varphi_i$ such that not all $\varphi_1, \ldots, \varphi_d$ are equal. Assume also that $B_t^{(1)}, \ldots, B_t^{(d)}, S_t^{(1)}, \ldots, S_t^{(d)}$ are independent and for each $i \in \{1, \ldots, d\}$ we have $\varphi_i \in WLS(\alpha/2, 0, C'_i)$, $\varphi_i \in WUSC(\beta/2, \theta''_0, C')$ for some constants $\alpha, \beta \in (0, 2)$, $\alpha > (2/3)\beta$, $\theta''_0 \geq 0$, $C'_i, C'' > 0$. Then assumptions (Q2) are satisfied.

In particular, let $Z_t = (Z_t^{(1)}, \ldots, Z_t^{(d)})^T$ be such that $Z_t^{(1)}, \ldots, Z_t^{(d)}$ are independent and for each $i \in \{1, \ldots, d\}$ $Z_t^{(i)}$ is a one-dimensional, symmetric $\alpha_i$-stable process ($\alpha_i \in (0, 2)$ and they are not all equal). Put $\alpha = \min(\alpha_1, \ldots, \alpha_d)$ and $\beta = \max(\alpha_1, \ldots, \alpha_d)$. If $\alpha > (2/3)\beta$ then assumptions (Q2) are satisfied. The SDE \textbf{(1)} driven by such $Z_t$ is of great interest see e.g. \cite{[7]}, \cite{[8]}, \cite{[17]} example (Z2) on page 2.

**Example 3.** Assume that for each $i \in \{1, \ldots, d\}$ the process $Z_t^{(i)}$ is the pure-jump symmetric Lévy process in $\mathbb{R}$ with the Lévy measure $\nu(x) \, dx$ given by the formula

$$
\nu(x) = \begin{cases} 
A_\alpha|x|^{-1-\alpha} & \text{for } x \in (-1, 1) \setminus \{0\}, \\
0 & \text{for } |x| > 1,
\end{cases}
$$

where $A_\alpha|x|^{-1-\alpha}$ is the Lévy density for the standard one-dimensional, symmetric $\alpha$-stable process, $\alpha \in (0, 2)$. Assume also that $Z_t^{(1)}, \ldots, Z_t^{(d)}$ are independent. Then assumptions (Q1) are satisfied. Clearly, $Z_t$ is not a subordinated Brownian motion.

**Remark 1.5.** In \cite{[17]} the following SDE

$$
dX_t = A(X_{t-}) \, dZ_t + b(X_t) \, dt, \quad X_0 = x \in \mathbb{R}^d.
$$
is studied, where $A(x), b(x)$ are Hölder continuous and $\{Z_t\}$ is a Lévy process in $\mathbb{R}^d$ such that $Z_t$ has a density $f_t$ and there exist $\alpha_1, \ldots, \alpha_d \in (0, 2)$ for which we have

$$\limsup_{t \to 0^+} t^{1/\alpha_k} \int_{\mathbb{R}^d} |f_t(z + e_k h) - f_t(z)\, dz| \leq c|h|, \quad h \in \mathbb{R}, \ k \in \{1, \ldots, d\}.$$

The main result in [17] states that there exists a density of $X_t$ and that the density belongs to the appropriate anisotropic Besov space. This result holds if some conditions on $\alpha_1, \ldots, \alpha_d$ are satisfied (see [17] (2.8), (2.9)).

On one hand, the existence result in [17] holds for some processes $Z_t$, some matrices $A(\cdot)$ and nonzero drifts $b(\cdot)$ which are not considered in our paper. On the other hand, there are some processes $Z_t$ for which our result holds and the result in [17] does not hold, because their conditions on $\alpha_1, \ldots, \alpha_d$ are in some cases more restrictive than our condition $\alpha > (2/3)\beta$. Take for example the process $Z_t = (Z_t^{(1)}, \ldots, Z_t^{(d)})^T$ such that $Z_t^{(1)}, \ldots, Z_t^{(d)}$ are independent and for each $i \in \{1, \ldots, d\}$ $Z_t^{(i)}$ is a one-dimensional, symmetric $\alpha_i$-stable process ($\alpha_i \in (0, 2)$). Put $\alpha = \alpha_{\min} = \min(\alpha_1, \ldots, \alpha_d)$ and $\beta = \alpha_{\max} = \max(\alpha_1, \ldots, \alpha_d)$. Assume that $\alpha = \alpha_{\min} = 1/8$ and $\beta = \alpha_{\max} = 1/6$. Then our condition $\alpha/\beta = 3/4 > 2/3$ is satisfied and the condition in [17] (2.9) $\alpha_{\min}(1/\gamma + \chi) > 1$ is not satisfied. Indeed, we have

$$\alpha_{\min} \left(\frac{1}{\gamma} + \chi\right) < \alpha_{\min} \left(\frac{1}{\alpha_{\max}} + 1\right) = \frac{7}{8} < 1.$$  

Note also that we prove that $p(t, x, y)$ is lower semi-continuous in $(t, x, y)$ and no such result is proven in [17]. Moreover, the methods in [17] do not give strong Feller property of the semigroup $P_t$.

The paper is organized as follows. In Section 2 we study properties of the transition density of a one-dimensional Lévy process with a suitably truncated Lévy measure $\nu_t$. In Section 3 we construct the transition density $u(t, x, y)$ of the solution of (1) in which the process $Z_t$ is replaced by a process with a truncated Lévy measure. We also show that it satisfies the appropriate heat equation in the approximate setting. In Section 4 we construct the transition semigroup of the solution of (1). We also prove Theorems 1.1, 1.3 and Proposition 1.2.

2. ONE-DIMENSIONAL DENSITY

First, we introduce the definition of a weak lower scaling condition and a weak upper scaling condition (cf. [4]). Let $\varphi$ be a non-negative, non-zero function on $[0, \infty)$. We say that $\varphi$ satisfies a weak lower scaling condition WLSC$(\alpha, \theta_0, C)$ if there are numbers $\alpha > 0, \theta_0 \geq 0$ and $C > 0$ such that

$$\varphi(\lambda \theta) \geq C \lambda^\alpha \varphi(\theta), \quad \text{for} \quad \lambda \geq 1, \ \theta \geq \theta_0.$$  

We say that $\varphi$ satisfies a weak upper scaling condition WUSC$(\beta, \theta_0, C)$ if there are numbers $\beta > 0, \theta_0 \geq 0$ and $C > 0$ such that

$$\varphi(\lambda \theta) \leq C \lambda^\beta \varphi(\theta), \quad \text{for} \quad \lambda \geq 1, \ \theta \geq \theta_0.$$  

In the whole paper we assume that either (Q1) or (Q2) are satisfied.

All constants appearing in this paper are positive and finite. In the whole paper we fix $\nu = (\nu_1, \ldots, \nu_d)$ and constants $\tau > 0, \alpha, \beta \in (0, 2), \theta_0 \geq 0, C, \overline{C} > 0, d \in \mathbb{N}$ $(d \geq 2), \eta_1, \eta_2, \eta_3, \eta_4 > 0$. We adopt the convention that constants denoted by $c$ (or $c_1, c_2, \ldots$) may change their value from one use to the next. In the whole paper,
unless is explicitly stated otherwise, we understand that constants denoted by $c$ (or $c_1, c_2, \ldots$) depend on $\nu, \tau, \alpha, \beta, \delta_0, C, \overline{C}, d, \eta_1, \eta_2, \eta_3$. We also understand that they may depend on the choice of the constants $\varepsilon$ and $\gamma$. We write $f(x) \approx g(x)$ for $x \in A$ if $f, g \geq 0$ on $A$ and there is a constant $c \geq 1$ such that $c^{-1} f(x) \leq g(x) \leq c f(x)$ for $x \in A$. The standard inner product for $x, y \in \mathbb{R}^d$ we denote by $xy$.

For any $t > 0$, $x \in \mathbb{R}^d$ we define the measure $\pi_t(x, \cdot)$ by

$$\pi_t(x, A) = \mathbb{P}^x(X_t \in A),$$

(8)

for any Borel set $A \subset \mathbb{R}^d$. $\mathbb{P}^x$ denotes the distribution of the process $X$ starting from $x \in \mathbb{R}^d$. For any $t > 0$, $x \in \mathbb{R}^d$ we have

$$P_tf(x) = \int_{\mathbb{R}^d} f(y)\pi_t(x, dy), \quad f \in \mathcal{B}_b(\mathbb{R}^d).$$

(9)

Let us fix $i \in \{1, \ldots, d\}$. Similarly as in [31] we will need to truncate the density of the Lévy measure $\nu_i$. The truncated density will be denoted by $\nu_t^{(\delta)}(x)$. There exists $\delta_0 \in (0, 1]$ (not depending on $i$) such that for any $\delta \in (0, \delta_0]$ the following construction of $\nu_t^{(\delta)} : \mathbb{R} \setminus \{0\} \to [0, \infty)$ is possible. For $x \in (0, \delta]$ we put $\nu_t^{(\delta)}(x) = \nu_i(x)$, for $x \in (\delta, 2\delta]$ we put $\nu_t^{(\delta)}(x) = 0$ and for $x \geq 2\delta$ we put $\nu_t^{(\delta)}(x) = 0$. Moreover, $\nu_t^{(\delta)}(x)$ is constructed so that $\mu_t^{(\delta)}(x) \in C^1(0, \infty)$, $(\mu_t^{(\delta)}(x))'(x) \leq 0$ for $x \in (0, \infty)$, $-\mu_t^{(\delta)}(x)/x$ is nonincreasing on $(0, \infty)$ and satisfies $\mu_t^{(\delta)}(-x) = \mu_t^{(\delta)}(x)$ for $x \in (0, \infty)$.

Let us choose $\delta \in (0, \delta_0]$. In this section the constants denoted by $c$ (or $c_1, c_2, \ldots$) may additionally depend on $i$ and the choice of $\delta$.

We also define

$$G_t^{(\delta)} f(x) = \frac{1}{2} \int_{\mathbb{R}} (f(x + w) + f(x - w) - 2f(x))\mu_t^{(\delta)}(w) dw.$$ 

By $g_{i,t}^{(\delta)}$ we denote the heat kernel corresponding to $G_t^{(\delta)}$ that is

$$\frac{\partial}{\partial t} g_{i,t}^{(\delta)}(x) = G_t^{(\delta)} g_{i,t}^{(\delta)}(x), \quad t > 0, \ x \in \mathbb{R},$$

$$\int_{\mathbb{R}} g_{i,t}^{(\delta)}(x) dx = 1, \quad t > 0.$$ 

It is well known that $g_{i,t}^{(\delta)}$ belongs to $C^1((0, \infty))$ as a function of $t$ and belongs to $C^2(\mathbb{R})$ as a function of $x$.

We put

$$h_i(r) = \int_{\mathbb{R}} (1 \wedge (|x|^2 r^{-2}))\nu_i(x) dx,$$

$$h_i^{(\delta)}(r) = \int_{\mathbb{R}} (1 \wedge (|x|^2 r^{-2}))\mu_i^{(\delta)}(x) dx,$$

$$K_i(r) = \int_{\{x \in \mathbb{R} : |x| \leq r\}} |x|^2 r^{-2} \nu_i(x) dx,$$

$$K_i^{(\delta)}(r) = \int_{\{x \in \mathbb{R} : |x| \leq r\}} |x|^2 r^{-2} \mu_i^{(\delta)}(x) dx.$$
Clearly, \( h_i \) and \( h_i^{(\delta)} \) are decreasing. By [4] (6), (7) we have \( h_i(r) \approx \psi_i(1/r), h_i^{(\delta)}(r) \approx \psi_i^{(\delta)}(1/r) \) for \( r \in (0, \infty) \). By scaling properties of \( \psi \) we get
\[
c_1 r^{-\alpha} \leq h_i(r) \leq c_2 r^{-\beta}, \quad r \in (0, 1].
\] (10)
Note that (10) holds also for \( r \in (0, \tilde{R}] \), where \( \tilde{R} \geq 1 \) but with constants \( c_1, c_2 \) which depends additionally on \( R \).

One can easily show that \( h_i^{(\delta)}(r) \approx \tilde{h}_i(r) \) for \( r \in (0, 1] \).

We know that \( \psi_i \in \text{WLSC}(\alpha, 0, \mathcal{C}) \). We will show that \( \psi_i^{(\delta)} \in \text{WLSC}(\alpha, 1, \tilde{C}) \) for some constant \( \tilde{C}. \) Take \( \theta \geq 1 \). We have
\[
\psi_i^{(\delta)}(\lambda \theta) \approx h_i^{(\delta)} \left( \frac{1}{\lambda \theta} \right) \approx h_i \left( \frac{1}{\lambda \theta} \right) \approx \psi_i(\lambda \theta)
\]
\[
\geq \tilde{C}_i \lambda^\alpha \psi_i(\theta) \approx \lambda^\alpha h_i \left( \frac{1}{\theta} \right) \approx \lambda^\alpha h_i(\delta) \left( \frac{1}{\theta} \right) \approx \lambda^\alpha \psi_i^{(\delta)}(\theta).
\]
Using \( \psi_i^{(\delta)} \in \text{WLSC}(\alpha, 0, \mathcal{C}_2) \) and [11 Lemma 2.3] we get \( h_i^{(\delta)}(r) \leq c_1 K_i^{(\delta)}(r) \) for \( r \in (0, c_2] \). This clearly implies that for any \( R > 0 \) we have
\[
h_i^{(\delta)}(r) \leq c_3 K_i^{(\delta)}(r), \quad \text{for} \quad r \in (0, R],
\] (11)
where \( c_3 \) depends on \( R, i, \delta, \alpha, \theta_0, \mathcal{C}_1, d, \eta_1, \eta_2, \eta_3, \eta_4 \). Note also that by definition we have \( h_i^{(\delta)}(r) \geq K_i^{(\delta)}(r) \) for any \( r \in (0, \infty) \).

Now we will show that for any \( y \geq h_i(1) \) we have \( (h_i^{(\delta)})^{-1}(y) \approx h_i^{-1}(y) \). (Note that \( h_i(1) \geq h_i^{(\delta)}(1) \).) Fix \( y \geq h_i(1) \). Assume that \( h_i(\tilde{x}) = y \) and \( h_i^{(\delta)}(x) = y \). Clearly, \( \tilde{x} \in (0, 1] \) and \( x \in (0, 1] \). We have \( h_i^{(\delta)}(\tilde{x}) \leq h_i(\tilde{x}) = y = h_i^{(\delta)}(x) \). Hence \( \tilde{x} \geq x \).

Recall that for any \( \theta \geq 1 \) and \( \lambda \geq 1 \) we have
\[
h_i^{(\delta)} \left( \frac{1}{\lambda \theta} \right) \geq c \left( \frac{1}{\lambda} \right)^{-\alpha} h_i^{(\delta)} \left( \frac{1}{\theta} \right).
\]
Using this we have
\[
y = h_i^{(\delta)}(x) = h_i^{(\delta)} \left( \frac{x}{\tilde{x}} \right) \geq c \left( \frac{x}{\tilde{x}} \right)^{-\alpha} h_i^{(\delta)}(\tilde{x})
\]
\[
\geq c_1 \left( \frac{x}{\tilde{x}} \right)^{-\alpha} h_i(\tilde{x}) = c_1 \left( \frac{x}{\tilde{x}} \right)^{-\alpha} y.
\]
Hence \( y \geq c_1 (x/\tilde{x})^{-\alpha} y \), which implies \( x \geq c_1^{1/\alpha} \tilde{x} \). So, finally we have shown that for any \( y \geq h_i(1) \) we have \( (h_i^{(\delta)})^{-1}(y) \approx h_i^{-1}(y) \). Clearly, this implies that
\[
(h_i^{(\delta)})^{-1}(y) \approx h_i^{-1}(y) \quad \text{for} \quad y \in [1 \land (1/r), \infty).
\] (12)
(Of course, \( 1 \land (1/r) \) could be changed to a different constant.)

Choose \( s \in [1/r, \infty) \). Let \( r \) be such that \( r = h_i^{-1}(s) \). Clearly, \( h_i(r) = s \). \( \psi_i(1/r) \approx h_i(x) \), so \( \psi_i(1/r) = sa \), for some \( a \in (c_1, c_2) \). We have \( \psi_i^{-1}(sa) = 1/r \). By scaling property of \( \psi_i \) and [4] Remark 4 we get
\[
h_i^{-1}(s) = r = \frac{1}{\psi_i^{-1}(sa)} = \left[ \frac{1}{c_3 s^{1/\alpha}}, \frac{1}{c_4 s^{1/\beta}} \right].
\]
Hence for any \( t \in (0, \tau) \) (by putting \( s = 1/t \)) we obtain
\[
c_5 t^{1/\alpha} \leq h_i^{-1}(1/t) \leq c_6 t^{1/\beta}.
\] (13)
For any $\varepsilon \in (0,1]$, $\tau > 0$, $t \in (0, \tau]$ and $x \in \mathbb{R}$ we define

$$
\tilde{g}_{i,t}^{(e)}(x) = \begin{cases} 
\frac{1}{h_i^{-1}(\frac{t}{\tau})} \wedge \frac{th_i(|x|)}{|x|} & \text{for } |x| < \varepsilon, \\
\frac{c_x}{t^{(d-\beta-1)/\alpha}} e^{-|x|} & \text{for } |x| \geq \varepsilon,
\end{cases}
$$

(14)

where $c_x = \left(\frac{1}{h_i^{-1}(\frac{t}{\tau})} \wedge \frac{th_i(|x|)}{|x|}\right)$ and where we understand $h_i(0)/0 = \infty$.

The constant $c_x$ is chosen so that for any $t \in (0, \tau]$ the function $x \to \tilde{g}_{i,t}^{(e)}(x)$ is nonincreasing on $[0, \infty)$.

Note that for $t \in (0, \tau]$ and $|x| \leq h_i^{-1}(1/t) \wedge \varepsilon$ we have $\tilde{g}_{i,t}^{(e)}(x) = 1/h_i^{-1}(1/t)$. We denote $g_{i,t}^{*}(x) = \left(\frac{1}{h_i^{-1}(\frac{t}{\tau})} \wedge \frac{th_i(x)}{|x|}\right)$.

We introduce the following convention. For a function $f$ and arguments $x, u \in \mathbb{R}$ we write $f(x \pm u) = f(x - u) + f(x + u)$.

**Lemma 2.1.** For any $\varepsilon \in (0,1]$, there exist $c$ such that for $\delta = \min\{\delta_0, \frac{\varepsilon a}{8(d+\beta+1)}, \frac{\varepsilon}{4d(\alpha+1)^2}\}$, and any $t \in (0, \tau]$, $x, u, w \in \mathbb{R}$, we have

$$
g_{i,t}^{(\delta)}(x) \leq c\tilde{g}_{i,t}^{(\delta)}(x),
$$

(15)

$$
|g_{i,t}^{(\delta)}(x + u) - g_{i,t}^{(\delta)}(x)| \leq \frac{c|u|}{h_i^{-1}(\frac{t}{\tau})} (g_{i,t}^{(\delta)}(x + u) + g_{i,t}^{(\delta)}(x)),
$$

(16)

$$
|g_{i,t}^{(\delta)}(x + u) + g_{i,t}^{(\delta)}(x - u) - 2g_{i,t}^{(\delta)}(x)| \leq \frac{c|u|^2}{(h_i^{-1}(\frac{t}{\tau})^2 \max_{\xi \in [x-u,x+u]} \tilde{g}_{i,t}^{(\delta)}(\xi)},
$$

(17)

$$
|g_{i,t}^{(\delta)}(x \pm u) - (g_{i,t}^{(\delta)}(x \pm w)| \leq \frac{c|u|^2 - |u|^2}{(h_i^{-1}(\frac{t}{\tau}))^2} (g_{i,t}^{*}(x \pm u) + g_{i,t}^{*}(x \pm w))
$$

(18)

**Proof.** By (11), (19) Theorem 1.1] and (12) we get

$$
g_{i,t}^{(\delta)}(x) \leq c \left(\frac{1}{h_i^{-1}(\frac{t}{\tau})} \wedge \frac{th_i(|x|)}{|x|}\right) = cg_{i,t}^{*}(x).
$$

(19)

Again, by (11), (19) Theorem 1.1], (12) and the mean value theorem we get

$$
|g_{i,t}^{(\delta)}(x + u) - g_{i,t}^{(\delta)}(x)| \leq \frac{c|u|}{h_i^{-1}(\frac{t}{\tau})} \left(\frac{1}{h_i^{-1}(\frac{t}{\tau})} \wedge \left(\frac{th_i(|x + u|)}{|x + u|} + \frac{th_i(|x|)}{|x|}\right)\right)
$$

(20)

and

$$
|g_{i,t}^{(\delta)}(x + u) + g_{i,t}^{(\delta)}(x - u) - 2g_{i,t}^{(\delta)}(x)| \leq \frac{c|u|^2}{(h_i^{-1}(\frac{t}{\tau}))^2} \left(\frac{1}{h_i^{-1}(\frac{t}{\tau})} \wedge \max_{\xi \in [x-u,x+u]} \frac{th_i(\xi)}{|\xi|}\right).
$$

(21)

Let $Z_{i,t}^{(\delta)}(t)$ be a Lévy process in $\mathbb{R}$ with a Lévy measure $\mu_{i,t}^{(\delta)}(x)\,dx$. Its transition density equals $g_{i,t}^{(\delta)}(x)$. Put $Z_{i,1,t}^{(\delta)}(t) := Z_{i,t}^{(\delta)}(t)$, $\mu_{i,1,t}^{(\delta)}(x) = \mu_{i,t}^{(\delta)}(x)$, $g_{i,1,t}^{(\delta)}(x) = g_{i,t}^{(\delta)}(x)$.

By [29] Theorem 1.5 there exists a Lévy process $Z_{i,3}^{(\delta)}(t)$ in $\mathbb{R}^3$ with the characteristic exponent $\psi_{i,3}^{(\delta)}(\xi) = \psi_{i,t}^{(\delta)}(|\xi|)$, $\xi \in \mathbb{R}^3$, and the radial, radially nonincreasing transition density $g_{i,3,t}^{(\delta)}(x) = g_{i,3,t}^{(\delta)}(|x|)$, $x \in \mathbb{R}^3$,

satisfying

$$
g_{i,3,t}^{(\delta)}(r) = \frac{-1}{2\pi r} \frac{d}{dr} g_{i,1,t}^{(\delta)}(r), \quad r > 0.
$$

(22)
The Lévy measure of \( Z_{i,3}(t) \) has a density \( \mu_{i,3}^{(\delta)}(x) = \mu_{i,3}^{(\delta)}(|x|), \, x \in \mathbb{R}^3 \setminus \{0\} \), which satisfies

\[
\mu_{i,3}^{(\delta)}(r) = \frac{-1}{2\pi r} \frac{d}{dr} \mu_{i,3}^{(\delta)}(r), \quad r > 0.
\]

In particular, by our assumptions, \( \mu_{i,3}^{(\delta)}(r) \) is nonincreasing on \((0, \infty)\) and \( \text{supp}(\mu_{i,3}^{(\delta)}) \subset B(0, 2\delta) \).

By [29, Proposition 3.1] there exists a Lévy process \( Z_{i,5}^{(\delta)}(t) \) in \( \mathbb{R}^5 \) with the characteristic exponent \( \psi_{i,5}^{(\delta)}(\xi) = \psi_{i}^{(\delta)}(|\xi|), \, \xi \in \mathbb{R}^5 \), Lévy measure \( d\mu_{i,5}^{(\delta)} \) and the radial transition density \( g_{i,5,t}^{(\delta)}(x) = g_{i,5,t}^{(\delta)}(|x|), \, x \in \mathbb{R}^5 \) satisfying

\[
g_{i,5,t}^{(\delta)}(r) = \frac{-1}{2\pi r} \frac{d}{dr} g_{i,5,t}^{(\delta)}(r), \quad r > 0.
\] (23)

We observe the following relationship between \( g_{i,3,t}^{(\delta)}, \, n = 1, 3, 5, \)

\[
\frac{d^2}{dr^2} g_{i,1,t}^{(\delta)}(r) = \frac{d}{dr} \left( -2\pi r g_{i,3,t}^{(\delta)}(r) \right)
= -2\pi g_{i,3,t}^{(\delta)}(r) - 2\pi r \frac{d}{dr} g_{i,3,t}^{(\delta)}(r)
= -2\pi g_{i,3,t}^{(\delta)}(r) + (2\pi r)^2 g_{i,3,t}^{(\delta)}(r).
\] (24)

At this moment we make a remark that \( g_{i,5,t}^{(\delta)}(r) \) does not need to be nonincreasing on \((0, \infty)\) and we can not claim directly that the measure \( d\mu_{i,5}^{(\delta)} \) is supported on \( B(0, 2\delta) \). Nevertheless this claim is true. To see this, for \( R > 2\delta \), we apply (23) to get

\[
\int_{B(0,R) \setminus B(0,2\delta)} d\mu_{i,5}^{(\delta)}(y) = \lim_{t \to 0^+} \frac{1}{t} \int_{B(0,R) \setminus B(0,2\delta)} g_{i,5,t}^{(\delta)}(y) \, dy
= c \lim_{t \to 0^+} \frac{1}{t} \int_{2\delta}^{R} r^4 g_{i,5,t}^{(\delta)}(r) \, dr
= c \lim_{t \to 0^+} \frac{1}{t} \int_{2\delta}^{R} r^3 \frac{d}{dr} g_{i,3,t}^{(\delta)}(r) \, dr
= c \lim_{t \to 0^+} \frac{1}{t} \left( R^4 g_{i,3,t}^{(\delta)}(R) - (2\delta)^3 g_{i,3,t}^{(\delta)}(2\delta) - 3 \int_{2\delta}^{R} r^2 g_{i,3,t}^{(\delta)}(r) \, dr \right)
= 0.
\]

This proves the claim about the support of the measure \( d\mu_{i,5}^{(\delta)} \).

For \( n = 1, 3, 5 \) and \( t \in (0, \tau] \) we have

\[
g_{i,n,t}^{(\delta)}(0) = c_n \int_{\mathbb{R}^n} e^{-t\psi_{i,n}^{(\delta)}(y)} \, dy = c_n' \int_{0}^{\infty} e^{-t\psi_{i}^{(\delta)}(\rho)} \rho^{n-1} \, d\rho
\leq c_n'' \int_{0}^{\infty} e^{-\rho^\alpha} \rho^{n-1} \, d\rho \leq c_n''' t^{-n/\alpha}.
\]

Denote \( d\mu_{i,n}^{(\delta)}(x) = \mu_{i,n}^{(\delta)}(x) \, dx \), for \( n = 1, 3 \).
Let $t \leq 1 \wedge \tau$. Since all the measures $d\mu_{i,n}^{(\delta)}$, $n = 1, 3, 5$ are supported on the centered balls of radius $2\delta$ we can use Lemma 4.2 from [44] to get for $n = 1, 3, 5$,

$$g_{i,n,t}^{(\delta)}(x) \leq e^{-\frac{|x|}{n\delta} \log \left( \frac{m_0}{|x|} \right)} g_{i,n,t}^{(\delta)}(0) = \left( \frac{m_0}{\delta|x|} \right)^{\frac{|x|}{n\delta}} g_{i,n,t}^{(\delta)}(0)$$

$$\leq ce\frac{|x|}{n\delta} \log \left( \frac{m_0}{|x|} \right), \quad |x| \geq \frac{m_0}{\delta} t,$$

where $m_0 = \int_{\mathbb{R}^n} |y|^2 d\mu_{i,n}^{(\delta)}(y)$. This yields

$$g_{i,n,t}^{(\delta)}(x) \leq ct^{(d+\beta-1)/\alpha} e^{-\frac{|x|}{n\delta} \log \left( \frac{m_0}{|x|} \right)},$$

provided $|x| \geq \max\{8\delta(1 + \frac{n+d+\beta-1}{\alpha}), \frac{m_0}{\delta} t\}$. We observe that there exists $c_1$ such that

$$e^{-\frac{|x|}{n\delta} \log \left( \frac{m_0}{|x|} \right)} \leq c_1 e^{-(a+d)|x|/\alpha}, \quad x \in \mathbb{R}^n,$$

so we obtain

$$g_{i,n,t}^{(\delta)}(x) \leq c_2 t^{(d+\beta-1)/\alpha} e^{-(a+d)|x|/\alpha}, \quad |x| \geq \max\{8\delta(\frac{n+d+\beta-1}{\alpha}), \frac{m_0}{\delta} t\}.$$ (25)

Let $1 \leq t \leq \tau$. Using again Lemma 4.2 from [44] we get

$$g_{i,n,t}^{(\delta)}(x) \leq e^{-\frac{|x|}{n\delta} \log \left( \frac{m_0}{|x|} \right)} g_{i,n,t}^{(\delta)}(0) \leq e^{-\frac{|x|}{n\delta}} g_{i,n,t}^{(\delta)}(0)$$

$$\leq ct^{(d+\beta-1)/\alpha} e^{-\frac{|x|}{n\delta}}, \quad |x| \geq \frac{m_0}{\delta} t.$$ (26)

Combining (25) and (26) for $t \in (0, \tau]$, $r \geq \varepsilon$ we get

$$g_{i,1,t}^{(\delta)}(r) \leq ce^{-r t^{(d+\beta-1)/\alpha}},$$

Similarly, using (25), (26), (22) and (24) for $t \in (0, \tau]$, $r \geq \varepsilon$ we get

$$\left| \frac{d}{dr} g_{i,1,t}^{(\delta)}(r) \right| \leq ce^{-r t^{(d+\beta-1)/\alpha}},$$

$$\left| \frac{d^2}{dr^2} g_{i,1,t}^{(\delta)}(r) \right| \leq ce^{-r t^{(d+\beta-1)/\alpha}}.$$

This, the mean value theorem, (19), (20), (21) and the definition of $\tilde{g}_{i,t}(x)$ give (15), (16), (17).

Finally, we proceed with the proof of (18). It is enough to prove it for $x > 0$ and $0 \leq u \leq w$. We will use the function $g_{i,3,t}^{(\delta)}$ described above. The function is symmetric and nonincreasing on the positive halfline and it enjoys the estimate

$$g_{i,3,t}^{(\delta)}(x) \leq ce^\frac{1}{(h_i^{-1}(\frac{1}{4}))^2 t} g_{i,t}^{(\delta)}(x), \quad t \leq \tau, x \in \mathbb{R}.$$

Hence, for $0 \leq x, 0 \leq w^* \leq w$ we have, by (22),

$$ |g_{i,t}^{(\delta)}(x + w) - g_{i,t}^{(\delta)}(x + w^*)| = \int_0^{w-w^*} \frac{d}{d\xi} g_{i,t}^{(\delta)}(x + w^* + \xi) d\xi $$

$$ = 2\pi \int_0^{w-w^*} (x + w^* + \xi) g_{i,t}^{(\delta)}(x + w^* + \xi) d\xi $$

$$ \leq 2\pi g_{i,t}^{(\delta)}(x + w^*) \int_0^{w-w^*} (x + w^* + \xi) d\xi $$

$$ \leq c g_{i,t}^{(\delta)}(x + w^*)(w^2 - w^*^2) $$

$$ \leq c g_{i,t}^{(\delta)}(x - w)(w^2 - w^*^2) $$

$$ \leq c \frac{1}{(h_i^{-1}(\frac{1}{\delta}))^2} g_{i,t}^*(x - w)(w^2 - w^*^2). \quad (27) $$

We first consider the case $|x - u| > |x - w|$, which can be split into two subcases. One of them is the subcase $0 \leq u < w \leq x$. Then,

$$ |g_{i,t}^{(\delta)}(x + u) + g_{i,t}^{(\delta)}(x - u) - (g_{i,t}^{(\delta)}(x + w) + g_{i,t}^{(\delta)}(x - w))| $$

$$ = \int_u^w \left| \frac{d}{d\xi} g_{i,t}^{(\delta)}(x + \xi) - \frac{d}{d\xi} g_{i,t}^{(\delta)}(x - \xi) \right| d\xi $$

$$ \leq c \sup_{x-w \leq \xi \leq x+w} \left| \frac{d^2}{d\xi^2} g_{i,t}^{(\delta)}(\xi) \right| (w^2 - u^2) $$

$$ \leq c \frac{1}{(h_i^{-1}(\frac{1}{\delta}))^2} g_{i,t}^*(x - w)(w^2 - u^2). \quad (28) $$

Next, consider the second subcase $u \leq x \leq w$ and $|x - w| \leq x - u$. Denote $w^* = 2x - w$, then $0 < u \leq w^* \leq x$ and $|x - w^*| = |x - w|$. Hence, by (27) and (28),

$$ |g_{i,t}^{(\delta)}(x + u) + g_{i,t}^{(\delta)}(x - u) - (g_{i,t}^{(\delta)}(x + w) + g_{i,t}^{(\delta)}(x - w))| $$

$$ \leq |g_{i,t}^{(\delta)}(x + w) - g_{i,t}^{(\delta)}(x + w^*)| $$

$$ + |g_{i,t}^{(\delta)}(x + w^*) + g_{i,t}^{(\delta)}(x - w^*) - (g_{i,t}^{(\delta)}(x + u) + g_{i,t}^{(\delta)}(x - u))| $$

$$ \leq c \frac{1}{(h_i^{-1}(\frac{1}{\delta}))^2} g_{i,t}^*(x - w)(w^2 - w^*^2 + w^*^2 - u^2). \quad (29) $$
Combining (28) and (29) we get (18) if $|x - w| < |x - u|$. Next, we consider the case $|x - w| \geq |x - u|$. Then $|x - u| \leq |x - w| \leq x + u \leq x + w$. By (28) we arrive at

$$
\begin{align*}
|g_{i,t}^{(\delta)}(x + u) - (g_{i,t}^{(\delta)}(x + w))| & = |g_{i,t}^{(\delta)}(x + u) + g_{i,t}^{(\delta)}(|x - u|) - (g_{i,t}^{(\delta)}(x + w) + g_{i,t}^{(\delta)}(|x - u|))| \\
& \leq |g_{i,t}^{(\delta)}(x + u) - g_{i,t}^{(\delta)}(x + w)| + |g_{i,t}^{(\delta)}(|x - u|) - g_{i,t}^{(\delta)}(|x - w|)| \\
& \leq \frac{c}{(h_i^{-1}(\frac{x}{t}))} g_{i,t}^{(\delta)}(x + u)((x + w)^2 - (x + u)^2) \\
& + \frac{c}{(h_i^{-1}(\frac{x}{t}))} g_{i,t}^{(\delta)}(|x - u|)((x - w)^2 - (x - u)^2) \\
& \leq \frac{c}{(h_i^{-1}(\frac{x}{t}))} g_{i,t}^{(\delta)}(x - u)(w^2 - u^2),
\end{align*}
$$

which completes the proof of (18). \ \Box

Lemma 2.2. Let $\varepsilon \in (0, 1]$. For any $t \in (0, \tau)$, $x, x' \in \mathbb{R}$ if $|x - x'| \leq h_i^{-1}(1/t)/4$ and $|x - x'| \leq \varepsilon/4$ then

$$
\tilde{g}_{i,t}^{(\varepsilon)}(x') \leq \tilde{g}_{i,t}^{(\varepsilon)}(x/2).
$$

Proof. Recall that $x \to \tilde{g}_{i,t}^{(\varepsilon)}(x)$ is nonincreasing and $\tilde{g}_{i,t}^{(\varepsilon)}(-x) = \tilde{g}_{i,t}^{(\varepsilon)}(x)$. Therefore we may assume that $|x| \geq |x'|$.

Assume that $|x'| \leq (h_i^{-1}(1/t)/2)\wedge(\varepsilon/2)$. Then $|x| \leq |x'| + |x - x'| \leq h_i^{-1}(1/t) \wedge \varepsilon$ so $\tilde{g}_{i,t}^{(\varepsilon)}(x') = \tilde{g}_{i,t}^{(\varepsilon)}(x) = 1/(h_i^{-1}(1/t))$.

Assume now that $|x'| > (h_i^{-1}(1/t)/2)\wedge(\varepsilon/2)$. Then we have $|x| \geq |x| - |x - x'| \geq |x| - |x'|/2 = |x|/2$. Hence $\tilde{g}_{i,t}^{(\varepsilon)}(x') \leq \tilde{g}_{i,t}^{(\varepsilon)}(x/2)$. \ \Box

Lemma 2.3. Let $\varepsilon \in (0, 1]$, $\delta = \min\{\delta_0, \frac{\varepsilon}{2d(d+\beta+1)}, \frac{\varepsilon}{2d(1+d)}\}$. For any $t \in (0, \tau)$, $x, x' \in \mathbb{R}^d$ if $\sum_{j=1}^d \frac{|x_j - x'_j|}{h_i^{-1}(1/t)} \leq \frac{1}{4}$ and $|x - x'| \leq \delta$ then

$$
\begin{align*}
\left| \prod_{i=1}^d g_i^{(\delta)}(x_i) - \prod_{i=1}^d g_i^{(\delta)}(x'_i) \right| & \leq c \left( \prod_{i=1}^d \tilde{g}_{i,t}^{(\varepsilon)}(x_i/2) \right) \left[ 1 \wedge \sum_{j=1}^d \frac{|x_j - x'_j|}{h_j^{-1}(1/t)} \right]. \quad (30)
\end{align*}
$$

Proof. By Lemma 2.1 we get

$$
\begin{align*}
\left| \prod_{i=1}^d g_i^{(\delta)}(x_i) - \prod_{i=1}^d g_i^{(\delta)}(x'_i) \right| & \leq \sum_{j=1}^d \left| g_j^{(\delta)}(x_j) - g_j^{(\delta)}(x'_j) \right| \prod_{i \neq j, 1 \leq i \leq d} g_i^{(\delta)}(|x_i| \wedge |x'_i|) \\
& \leq c \left( \prod_{i=1}^d \tilde{g}_{i,t}^{(\varepsilon)}(|x_i| \wedge |x'_i|) \right) \sum_{j=1}^d \frac{|x_j - x'_j|}{h_j^{-1}(1/t)}.
\end{align*}
$$
Clearly we have
\[
\left| \prod_{i=1}^{d} g_{i,t}^{(\delta)} (x_i) - \prod_{i=1}^{d} g_{i,t}^{(\delta)} (x'_i) \right| \leq \prod_{i=1}^{d} g_{i,t}^{(\delta)} (|x_i| \wedge |x'_i|).
\]

Now the assertion follows from Lemmas 2.4 and 2.2.

Lemma 2.4. Let \( \varepsilon \in (0,1] \), \( \delta = \min \{ \delta_{0}, \frac{\varepsilon}{8(d+\varepsilon+1)} \} \) and let \( a,b \in \mathbb{R} \). Then there exists \( c \) such that for any \( t \in (0,\tau] \), \( x \in \mathbb{R} \), \( i,j,k \in \{1,\ldots,d\} \) we have

\[
\int_{\mathbb{R}} |g_{i,t}^{(\delta)} (x+aw) + g_{i,t}^{(\delta)} (x-aw) - 2g_{i,t}^{(\delta)} (x)| \mu_j^{(\delta)} (w) \, dw \leq \frac{c(|a|^\alpha + |a|^2) (\max_{|u| \leq 2\delta} \tilde{g}_{i,t}^{(\varepsilon)} (x+au))}{t^{\beta/\alpha}}; \tag{31}
\]

\[
\int_{\mathbb{R}} |g_{i,t}^{(\delta)} (x+aw) + g_{i,t}^{(\delta)} (x-aw) - 2g_{i,t}^{(\delta)} (x)| \mu_i^{(\delta)} (w) \, dw \leq \frac{c(|a|^\alpha + |a|^2) (\max_{|u| \leq 2\delta} \tilde{g}_{i,t}^{(\varepsilon)} (x+au))}{t}; \tag{32}
\]

\[
\int_{\mathbb{R}} |g_{i,t}^{(\delta)} (x+aw) - g_{i,t}^{(\delta)} (x)| |g_{k,t}^{(\delta)} (y+bw) - g_{k,t}^{(\delta)} (y)| \mu_j^{(\delta)} (w) \, dw \leq \frac{c(|a||b|^{\alpha/2} + |a||b|) (\max_{|u| \leq 2\delta} \tilde{g}_{i,t}^{(\varepsilon)} (x+au)) (\max_{|u| \leq 2\delta} \tilde{g}_{k,t}^{(\varepsilon)} (y+bu))}{t^{\beta/\alpha}}; \tag{33}
\]

\[
\int_{\mathbb{R}} |g_{i,t}^{(\delta)} (x+aw) - g_{i,t}^{(\delta)} (x)| |g_{i,t}^{(\delta)} (y+bw) - g_{i,t}^{(\delta)} (y)| \mu_i^{(\delta)} (w) \, dw \leq \frac{c(|a||b|^{\alpha/2} + |a||b|) (\max_{|u| \leq 2\delta} \tilde{g}_{i,t}^{(\varepsilon)} (x+au)) (\max_{|u| \leq 2\delta} \tilde{g}_{i,t}^{(\varepsilon)} (y+bu))}{t}; \tag{34}
\]

Proof. We have the following scaling property. For any \( a, x \in \mathbb{R} \)

\[
h_{i} \left( \frac{x}{|a|} \right) \leq c(|a|^\alpha + |a|^2) h_{i} (x). \tag{35}
\]

For \( |a| < 1 \) this follows from the WLSC scaling property for \( \psi \) and for \( |a| \geq 1 \) it is clear that

\[
h_{i} \left( \frac{x}{|a|} \right) \leq |a|^2 h_{i} (x).
\]

Let \( |w| \leq 2\delta \). Then, by (15) and (17),

\[
|g_{i,t}^{(\delta)} (x+aw) + g_{i,t}^{(\delta)} (x-aw) - 2g_{i,t}^{(\delta)} (x)| \leq c \left( \frac{|a|^2 |w|^2}{(h_{i}^{-1} (1))^2} \wedge 1 \right) \max_{|u| \leq 2\delta} \tilde{g}_{i,t}^{(\varepsilon)} (x+au).
\]
First we show (31) and (32). We have

$$\int \frac{|g^{(\delta)}_{i,t}(x + aw) + g^{(\delta)}_{i,t}(x - aw) - 2g^{(\delta)}_{i,t}(x)|}{|a|^2 |w|^2} \mu^{(\delta)}_j(w) \, dw$$

$$\leq \int \frac{|g^{(\delta)}_{i,t}(x + aw) + g^{(\delta)}_{i,t}(x - aw) - 2g^{(\delta)}_{i,t}(x)|}{|a|^2 |w|^2} \nu_j(w) \, dw$$

$$c \max_{|u| \leq 2\delta} g^{(\epsilon)}_{i,t}(x + au) \leq \int \frac{|a|^2 |w|^2}{(h^{-1}_{i,t}(\frac{1}{t}))^2} \wedge 1 \nu_j(w) \, dw$$

$$= c \max_{|u| \leq 2\delta} g^{(\epsilon)}_{i,t}(x + au) h_j \left( \frac{h^{-1}_{i,t}(\frac{1}{t})}{|a|} \right)$$

Next, by (35),

$$h_j \left( \frac{h^{-1}_{i,t}(\frac{1}{t})}{|a|} \right) \leq c(|a|^\alpha + |a|^2) h_j \left( \frac{h^{-1}_{i,t}(\frac{1}{t})}{|a|} \right).$$

Observing that

$$h_j \left( \frac{h^{-1}_{i,t}(\frac{1}{t})}{|a|} \right) \leq c t^{-\beta/\alpha}$$

if \( i \neq j \) and

$$h_i \left( \frac{h^{-1}_{i,t}(\frac{1}{t})}{|a|} \right) = t^{-1}$$

we finish the proof of (31) and (32).

To show (33) and (34) we use (15) and (16) to obtain

$$|g^{(\delta)}_{i,t}(x + aw) - g^{(\delta)}_{i,t}(x)||g^{(\delta)}_{k,t}(y + bw) - g^{(\delta)}_{k,t}(y)|$$

$$\leq c \left( \frac{|a||b| |w|^2}{h^{-1}_{i,t}(1/t)h^{-1}_{k,t}(1/t)} \wedge 1 \right) \max_{|u| \leq 2\delta} g^{(\epsilon)}_{i,t}(y + aw) \max_{|u| \leq 2\delta} g^{(\epsilon)}_{k,t}(y + bw)$$

Therefore

$$\int \frac{|g^{(\delta)}_{i,t}(x + aw) - g^{(\delta)}_{i,t}(x)||g^{(\delta)}_{k,t}(y + bw) - g^{(\delta)}_{k,t}(y)|}{|a||b|} \mu^{(\delta)}_j(w) \, dw$$

$$\leq c h_j \left( \sqrt{\frac{h^{-1}_{i,t}(1/t)h^{-1}_{k,t}(1/t)}{|a||b|}} \right) \max_{|u| \leq 2\delta} g^{(\epsilon)}_{i,t}(x + au) \max_{|u| \leq 2\delta} g^{(\epsilon)}_{k,t}(y + bu)$$

$$\leq c \left( |a||b|^{\alpha/2} + |a||b| \right) h_j \left( \sqrt{\frac{h^{-1}_{i,t}(1/t)h^{-1}_{k,t}(1/t)}{|a||b|}} \right) \max_{|u| \leq 2\delta} g^{(\epsilon)}_{i,t}(x + au) \max_{|u| \leq 2\delta} g^{(\epsilon)}_{k,t}(y + bu)$$

which proves (33) and (34) since \( h_j \left( \sqrt{\frac{h^{-1}_{i,t}(1/t)h^{-1}_{k,t}(1/t)}{|a||b|}} \right) \) is equal to \( t^{-1} \) if \( i = j = k \) and it is smaller than \( ct^{-\beta/\alpha} \) in the general case.

\[\square\]

**Lemma 2.5.** There is a constant \( C \) such that for \( a \in \mathbb{R} \) and any \( 0 < t < \tau \),

$$\int \left( \frac{|a| + |w||w^2|}{(h^{-1}_{i,t}(1/t))^2} \wedge 1 \right) \mu^{(\delta)}_j(w) \, dw \leq C \frac{t^{\alpha/(2\beta)} + |a|^\alpha + |a|^2}{t}; \quad (36)$$
\[
\int_R \left( \frac{(|a| + |w|)|w|^2}{(h^{-1}_i(1/t))^2} \wedge 1 \right) \mu_i^{(5)}(w) \, dw \leq C \frac{\alpha/3\beta + |a|^{\alpha/2} + |a|}{t} ;
\]
(37)

\[
\int_R \left( \frac{(|a| + |w|)^2|w|^2}{h^{-1}_i(1/t)h^{-1}_j(1/t)} \wedge 1 \right) \mu_k^{(5)}(w) \, dw \leq C \left[ |a|^\beta t^{-\beta/\alpha} + t^{-\beta/(2\alpha)} \right] ;
\]
(38)

\[
\int_R \left( \frac{|a| + |w||w|^2}{(h^{-1}_i(1/t))^2} \wedge 1 \right) \mu_k^{(5)}(w) \, dw \leq C \left[ |a|^{\beta/2} t^{-\beta/\alpha} + c_2 t^{-2\beta/(3\alpha)} \right] .
\]
(39)

**Proof.** Let \( k \geq 1 \) and \( b > 0 \). Then

\[
\int_R \left( \frac{|a| + |w|^k|w|^2}{b} \wedge 1 \right) \mu_i^{(5)}(w) \, dw \leq 2^{k-1} \int_R \left( \frac{a^k|w|^2}{b} \wedge 1 \right) \nu_i(w) \, dw + 2^{k-1} \int_R \left( \frac{|w|^{k+2}}{b} \wedge 1 \right) \nu_i(w) \, dw
\]

\[
\leq 2^{k-1} \int_R \left( \frac{a^k|w|^2}{b} \wedge 1 \right) \nu_i(w) \, dw + 2^{k-1} \int_R \left( \frac{|w|^{2(k+2)}}{b^{2(k+2)}} \wedge 1 \right) \nu_i(w) \, dw
\]

\[
= 2^{k-1} \left( h_i \left( \frac{\sqrt{b}}{|a|^{k/2}} \right) + h_i \left( b^{k+2} \right) \right) .
\]
(40)

Taking \( b = (h^{-1}_i(1/t))^2 \) and \( k = 2 \) we arrive at

\[
\int_R \left( \frac{|a| + |w|^2|w|^2}{(h^{-1}_i(1/t))^2} \wedge 1 \right) \mu_i^{(5)}(w) \, dw
\]

\[
\leq c \left( h_i \left( \frac{h^{-1}_i(1/t)}{|a|} \right) + h_i \left( \sqrt{h^{-1}_i(1/t)} \right) \right) .
\]

Next, by the scaling property (10), \( h_i \left( \frac{h^{-1}_i(1/t)}{|a|} \right) \leq c \frac{a^\alpha + a^2}{t} \). Moreover, by (35) and (13), we get

\[
h_i \left( \frac{1}{(h^{-1}_i(1/t))^{1/2}h^{-1}_i(1/t)} \right) \leq c \frac{(h^{-1}_i(1/t))^{\alpha/2}}{t} \leq c t^{\alpha/(2\beta)}/t,
\]

where the last inequality follows from (13). The proof of (36) is completed.

By similar arguments, taking \( b = (h^{-1}_i(1/t))^2 \) and \( k = 1 \) in (40), we arrive at (37).

Now, we proceed with the proof of (38) and (39). First, we observe that

\[
h_i^{-1}(1/t)h_j^{-1}(1/t) \geq c t^{2/\alpha} \text{ and } \mu_k^{(5)}(w) \leq \frac{c}{|w|^{1+\alpha}}.
\]

Hence,
\[ \int_{\mathbb{R}} \left( \frac{|a| + |w|^2 |w|^2}{h_i^{-1}(1/t)h_j^{-1}(1/t)} \wedge 1 \right) \mu_k^{(\beta)}(w) \leq c \int_{\mathbb{R}} \left( \frac{|a| + |w|^2 |w|^2}{t^{2/\alpha}} \wedge 1 \right) \frac{1}{|w|^{1+\beta}} dw \]

\[ \leq c \int_{\mathbb{R}} \left( \frac{|a|^2 |w|^2}{t^{2/\alpha}} \wedge 1 \right) \frac{1}{|w|^{1+\beta}} dw \]

\[ + c \int_{\mathbb{R}} \left( \frac{|w|^4}{t^{2/\alpha}} \wedge 1 \right) \frac{1}{|w|^{1+\beta}} dw \]

\[ = c_1 |a|^{\beta} t^{-\beta/\alpha} + c_2 t^{-\beta/(2\alpha)} . \]

Similar calculations show that

\[ \int_{\mathbb{R}} \left( \frac{|a| + |w||w|^2}{h_i^{-1}(1/t)h_j^{-1}(1/t)} \wedge 1 \right) \mu_k^{(\delta)}(w) \leq c |a|^{\beta/2} t^{-\beta/\alpha} + c_2 t^{-2\beta/(3\alpha)} . \]

The proof is completed. \(\square\)

**Lemma 2.6.** Let \( \eta \geq 0 \).

\[ \int_0^1 x^\eta \left( \frac{1}{h_i^{-1}(\frac{1}{t})} \wedge \frac{th_i(|x|)}{|x|} \right) dx \leq Ct^{1\wedge(\eta/\beta)} , \eta \neq \beta \] \hspace{1cm} (41)

and

\[ \int_0^1 x^\eta \left( \frac{1}{h_i^{-1}(\frac{1}{t})} \wedge \frac{th_i(|x|)}{|x|} \right) dx \leq Ct \ln(1 + 1/t) , \eta = \beta . \] \hspace{1cm} (42)

**Proof.** Let \( t > 0 \) be such that \( h_i^{-1}(\frac{1}{t}) \leq 1 \). Note that \( h_i^{-1}(\frac{1}{s}) \leq Cs^{1/\beta} \) for \( s \leq 1 \).

\[ I = \int_0^1 x^\eta \left( \frac{1}{h_i^{-1}(\frac{1}{t})} \wedge \frac{th_i(|x|)}{|x|} \right) dx = \int_0^{h_i^{-1}(\frac{1}{t})} + \int_{h_i^{-1}(\frac{1}{t})}^1 = I_1 + I_2 . \]

We have

\[ I_1 = \frac{1}{\eta + 1} \left( h_i^{-1}(\frac{1}{t}) \right)^\eta \leq Ct^{\eta/\beta} \]

and

\[ \eta I_2 = t \int_{h_i^{-1}(\frac{1}{t})}^1 \eta x^{\eta-1} h_i(x) dx = th_i(1) - \left( h_i^{-1}(\frac{1}{t}) \right)^\eta - t \int_{h_i^{-1}(\frac{1}{t})}^1 x^{\eta} h_i'(x) dx . \]

Next, we estimate the last integral. Let \( N \) be the smallest integer such that \( h_i^{-1}(\frac{1}{(N+1)t}) \geq 1 \), then
Proof. Let $\beta$ be an arbitrary unit vector in $\mathbb{R}^n$. We denote by $B_1$ the open ball of the center $x$ and radius $r > 0$. For any matrix $M$ we denote by $||M||_\infty$ the maximum of its entries.

**Lemma 2.8.** Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz map and $x \in \mathbb{R}^n$. Suppose that for any $y \in \mathbb{R}^n$, the generalized Jacobian $\partial f(y)$ consist of the matrices which can be represented as $M(x) + R$, where matrices $M(x), R$ satisfy the following conditions: there are positive $\beta$ and $\eta$ such that $||R||_\infty \leq \eta |x - y|$ and $|vM(x)^T| \geq 2\beta$ for every $v \in \mathbb{R}^n, |v| = 1$. Then $f$ is injective on $B(x, \beta/(n\eta))$ and we have $B(f(x), \beta^2/(2n\eta)) \subset f(B(x, \beta/(n\eta)))$.

**Proof.** Let $v$ be an arbitrary unit vector in $\mathbb{R}^n$. Let $M \in \partial f(y)$ and let $z = vM(x)^T$. Since $M^T = M(x)^T + R^T$ the scalar product of $z$ and $w = vM^T = z + vR^T$ can be...
estimated as follows

\[ zw = z(z + vR^T) = |z|^2 + z(vR^T) \geq |z|^2 - n\eta|z||x - y|. \]

Next, taking \( w^* = z/|z| \) we have for \(|x - y| \leq \beta/(n\eta)\),

\[ w^*(vM^T) \geq |z| - n\eta|x - y| \geq \beta. \]

Using this fact we can apply Lemma 3 and Lemma 4 of [13] to claim that for every \( y_1, y_2 \in B(x, \beta/(n\eta)) \) we have

\[ |f(y_1) - f(y_2)| \geq \beta|y_1 - y_2|, \]

which shows that \( f \) is injective in a ball \( B(x, \beta/(n\eta)) \). Next, by similar arguments, we show that

\[ |vM^T| \geq |vM(x)|T - |vR^T| \geq 2\beta - n\eta|x - y| \geq \beta, \quad |y - x| \leq \beta/(n\eta), \]

which proves that all matrices from the set \( \partial f(y) \) are of full rank if \(|y - x| \leq \beta/(n\eta)\).

Finally, we can apply Lemma 5 of [13] to show that the \( f \) image of the ball \( B(x, \beta/(n\eta)) \) contains the ball \( B(f(x), \beta^2/(2n\eta)) \).

\[ \square \]

3. CONSTRUCTION AND PROPERTIES OF THE TRANSITION DENSITY OF THE SOLUTION OF (11) DRIVEN BY THE TRUNCATED PROCESS

The approach in this section is based on Levi's method (cf. [35] [10] [34]). This method was applied in the framework of pseudodifferential operators by Kochubei [26] to construct a fundamental solution to the related Cauchy problem as well as transition density for the corresponding Markov process. In recent years it was used in several papers to study transition densities of Lévy-type processes see e.g. [11, 23, 12, 21, 19, 6, 24, 25, 27]. Levi's method was also used to study gradient and Schrödinger perturbations of fractional Laplacians see e.g. [5] [10] [49].

We first introduce the generator of the process \( X_t \). We define \( \mathcal{K}f(x) \) by the following formula

\[ \mathcal{K}f(x) = \frac{1}{2} \sum_{i=1}^{d} \int_{\mathbb{R}} [f(x + a_i(x)w) + f(x - a_i(x)w) - 2f(x)] \mu_i(w) \, dw, \]

for any Borel function \( f : \mathbb{R}^d \to \mathbb{R} \) and any \( x \in \mathbb{R}^d \) such that all the limits on the right hand side exist. Recall that \( a_i(x) = (a_{1i}(x), \ldots, a_{di}(x)) \). It is well known that \( \mathcal{K}f(x) \) is well defined for any \( f \in C_0^2(\mathbb{R}^d) \) and any \( x \in \mathbb{R}^d \). By standard arguments if \( f \in C_0^2(\mathbb{R}^d) \) then \( f(X_t) - f(X_0) - \int_0^t \mathcal{K}f(X_s) \, ds \) is a martingale (see e.g. [32] page 120).

Let us fix \( \varepsilon \in (0, 1] \) (it will be chosen later). Recall that for given \( \varepsilon \) the constant \( \delta \) is chosen according to Lemma 2.1. For such fixed \( \varepsilon, \delta \) we abbreviate \( \mu_i(x) = \mu_i^{(\delta)}(x), \)
\[ g_i = g_i^{(\delta)}, \quad g_{i,t}(x) = g_{i,t}^{(\delta)}(x), \quad \bar{g}_{i,t}(x) = \bar{g}_{i,t}^{(\varepsilon)}(x). \]

We divide \( \mathcal{K} \) into two parts

\[ \mathcal{K}f(x) = \mathcal{L}f(x) + \mathcal{R}f(x), \quad (43) \]

where

\[ \mathcal{L}f(x) = \frac{1}{2} \sum_{i=1}^{d} \int_{\mathbb{R}} [f(x + a_i(x)w) + f(x - a_i(x)w) - 2f(x)] \mu_i(w) \, dw, \]
for any Borel function $f : \mathbb{R}^d \to \mathbb{R}$ and any $x \in \mathbb{R}^d$ such that all the limits on the right hand side exist. Our first aim will be to construct the heat kernel $u(t, x, y)$ corresponding to the operator $\mathcal{L}$. This will be done by using the Levi’s method.

For each $x \in \mathbb{R}^d$ we introduce the “freezing” operator

$$\mathcal{L}^x f(x) = \frac{1}{2} \sum_{i=1}^d \int_{\mathbb{R}} [f(x + a_i(z)w) + f(x - a_i(z)w) - 2f(x)] \mu_i(w) \, dw,$$

where $a_i(x) = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_d)$. Let $G_t(x) = g_{1,t}(x_1) \ldots g_{d,t}(x_d)$ and $\tilde{G}_t(x) = \tilde{g}_{1,t}(x_1) \ldots \tilde{g}_{d,t}(x_d)$ for $t > 0$ and $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$. We also denote $B(x) = (b_{ij}(x)) = A^{-1}(x)$. Note that the coordinates of $B(x)$ satisfy conditions (2) and (4) with possibly different constants $\eta_1^*$ and $\eta_3^*$, but taking maximums we can assume that $\eta_1^* = \eta_1$ and $\eta_3^* = \eta_3$.

For any $y \in \mathbb{R}^d$, $i = 1, \ldots, d$ we put

$$b_i(y) = (b_{i1}(y), \ldots, b_{id}(y)).$$

We also denote $\|B\|_\infty = \max\{|b_{ij}| : i, j \in \{1, \ldots, d\}\}$.

For any $t > 0$, $x, y \in \mathbb{R}^d$ we define

$$p_y(t, x) = \det(B(y)) G_t(x(B(y))^T) = \det(B(y)) g_{1,t}(b_1(y)x) \ldots g_{d,t}(b_d(y)x).$$

It may be easily checked that for each fixed $y \in \mathbb{R}^d$ the function $p_y(t, x)$ is the heat kernel of $\mathcal{L}^y$ that is

$$\frac{\partial}{\partial t} p_y(t, x) = \mathcal{L}^y p_y(t, \cdot)(x), \quad t > 0, x \in \mathbb{R}^d,$$

$$\int_{\mathbb{R}^d} p_y(t, x) \, dx = 1, \quad t > 0.$$

For any $t > 0, x, y \in \mathbb{R}^d$ we also define

$$r_y(t, x) = \tilde{G}_t(x(B(y))^T) = \tilde{g}_{1,t}(b_1(y)x) \ldots \tilde{g}_{d,t}(b_d(y)x).$$

For $x, y \in \mathbb{R}^d$, $t > 0$, let

$$q_0(t, x, y) = \mathcal{L}^x p_y(t, \cdot)(x - y) - \mathcal{L}^y p_y(t, \cdot)(x - y),$$

and for $n \in \mathbb{N}$ let

$$q_n(t, x, y) = \int_0^t \int_{\mathbb{R}^d} q_0(t - s, x, z) q_{n-1}(s, z, y) \, dz \, ds. \quad (44)$$

For $x, y \in \mathbb{R}^d$, $t > 0$ we define

$$q(t, x, y) = \sum_{n=0}^\infty q_n(t, x, y)$$

and

$$u(t, x, y) = p_y(t, x - y) + \int_0^t \int_{\mathbb{R}^d} p_z(t - s, x - z) q(s, z, y) \, dz \, ds. \quad (45)$$

In this section we will show that $q_n(t, x, y)$, $q(t, x, y)$, $u(t, x, y)$ are well defined and we will obtain estimates of these functions. First, we will get some simple properties of $p_y(t, x)$ and $r_y(t, x)$. 
Lemma 3.1. Choose $\gamma \in (0, 1]$. For any $t \in (0, \tau]$, $x, x', y \in \mathbb{R}^d$ we have

$$|p_y(t, x) - p_y(t, x')| \leq c \left( 1 \wedge \left( \sum_{j=1}^d \frac{|x_j - x'_j|}{h_j^{-1}(1/t)} \right)^\gamma \right) (r_y(t, x/2) + r_y(t, x'/2)).$$

Proof. Of course, we may assume that $\sum_{j=1}^d \frac{|x_j - x'_j|}{h_j^{-1}(1/t)} > \frac{1}{4||B||_\infty}$ or $|x - x'| \geq \delta/||B||_\infty$ then the assertion clearly holds.

So, we may assume that $\sum_{j=1}^d \frac{|x_j - x'_j|}{h_j^{-1}(1/t)} \leq \frac{1}{4||B||_\infty}$ and $|x - x'| \leq \delta/||B||_\infty$. Then the assertion follows easily from Lemma 2.3.

For $x, y \in \mathbb{R}^d$ we have

$$|B(y)x|^2 = |xB(y)^T|^2 = (b_1(y)x)^2 + \ldots + (b_d(y)x)^2.$$

Lemma 3.2. For any $x, y \in \mathbb{R}^d$ and $i \in \{1, \ldots, d\}$ we have

$$\max_{1 \leq i \leq d} (b_i(y)x)^2 \geq \frac{1}{\eta_1^2 d^3} |x|^2.$$

Proof. Indeed, for any $u, x$ we have $|uA(y)^T| \leq \eta_1 d |u|$. Setting $u = xB(y)^T$ we obtain that

$$|xB(y)^T| \geq \frac{1}{\eta_1 d} |x|.$$

Since

$$|xB(y)^T|^2 = (b_1(y)x)^2 + (b_2(y)x)^2 + \ldots + (b_d(y)x)^2,$$

it follows that there is $1 \leq k \leq d$ such that $|b_k(y)x| \geq \frac{1}{\eta_1 d^2} |x|$. \hfill $\square$

Corollary 3.3. Assume that $\varepsilon \leq \frac{1}{m d^2}$. For any $t \in (0, \tau + 1]$, $x, y \in \mathbb{R}^d$, we have

$$r_y(t, x - y) \leq c_1 \left( \prod_{i=1}^d \frac{1}{h_i^{-1}(1/t)} \right) e^{-\varepsilon |x-y|}. \tag{46}$$

For any $t \in (0, \tau + 1]$, $x, y \in \mathbb{R}^d$, $|x - y| \geq \varepsilon \eta_1 d^{3/2}$, we have

$$r_y(t, x - y) \leq c_1 t e^{-\varepsilon |x-y|}. \tag{47}$$

Proof. For any $t \in (0, \tau + 1]$, $i \in \{1, \ldots, d\}$, $z \in \mathbb{R}$ by definition of $\tilde{g}_{i,t}$ we have

$$\tilde{g}_{i,t}(z) \leq \frac{c}{h_i^{-1}(1/t)} e^{-\varepsilon |z|}. \tag{48}$$

Fix $x, y \in \mathbb{R}^d$, $t \in (0, \tau + 1]$. By Lemma 3.2 there exists $i \in \{1, \ldots, d\}$ such that $|b_i(y)(x - y)| \geq \frac{1}{\eta_1 d^2} |x - y|$. Using this and (48) we get (46). For any $t \in (0, \tau + 1]$, $i \in \{1, \ldots, d\}$, $z \in \mathbb{R}$, $|z| \geq \varepsilon$ by definition of $\tilde{g}_{i,t}$ we have

$$\tilde{g}_{i,t}(z) \leq c t^{1+(d-1)/\alpha} e^{-\varepsilon |z|}. \tag{49}$$

For $t \in (0, \tau + 1]$, $i \in \{1, \ldots, d\}$ by (48) we get $1/ (h_i^{-1}(1/t)) \leq c t^{-1/\alpha}$. If $|x - y| \geq \varepsilon \eta_1 d^{3/2}$ then $\frac{1}{\eta_1 d^2} |x - y| \geq \varepsilon$ hence, by the same arguments as above, we get (47). \hfill $\square$
Using the definition of \( p_y(t,x) \) and properties of \( g_i(x) \) we obtain the following regularity properties of \( p_y(t,x) \).

**Lemma 3.4.** The function \( (t,x,y) \to p_y(t,x) \) is continuous on \((0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d\).

The function \( t \to p_y(t,x) \) is in \( C^1((0,\infty)) \) for each fixed \( x,y \in \mathbb{R}^d \). The function \( x \to p_y(t,x) \) is in \( C^2(\mathbb{R}^d) \) for each fixed \( t > 0, y \in \mathbb{R}^d \).

**Lemma 3.5.** For any \( y \in \mathbb{R}^d \) we have

\[
\left| \frac{\partial}{\partial x_i} p_y(t,x-y) \right| \leq \frac{c}{t^{(d+1)/\alpha}(1 + |x-y|)^{d+1}}, \quad i \in \{1,\ldots,d\}, \ t \in (0,\tau], \ x \in \mathbb{R}^d,
\]

\[
\left| \frac{\partial^2}{\partial x_i \partial x_j} p_y(t,x-y) \right| \leq \frac{c}{t^{(d+2)/\alpha}(1 + |x-y|)^{d+1}}, \quad i,j \in \{1,\ldots,d\}, \ t \in (0,\tau], \ x \in \mathbb{R}^d.
\]

Proof. The estimates follow from properties of \( g_i(x) \), Lemma 2.1 [13] and the same arguments as in the proof of (46). \(\square\)

**Lemma 3.6.** Let \( b^*_i(x,y), x,y \in \mathbb{R}^d; i = 1,\ldots,d \), be real functions such that there are positive \( \eta_4, \eta_5 \) and

\[
|b^*_i(x,y)| \leq \eta_4, \ x,y \in \mathbb{R}^d,
\]

\[
|b^*_i(x,y) - b^*_i(\tau,\vartheta)| \leq \eta_5(|x-\tau| + |y-\vartheta|), \ x,y,\tau,\vartheta \in \mathbb{R}^d.
\]

Let, for fixed \( x \in \mathbb{R}^d \), \( \Psi_x \) be a map \( \mathbb{R}^{d+1} \to \mathbb{R}^{d+1} \) given by

\[
\psi_x(w,y) = (w,\xi_1,\ldots,\xi_d) \in \mathbb{R}^{d+1}, \ w \in \mathbb{R}, y \in \mathbb{R}^d,
\]

where \( \xi_i = b_i(y)(x-y) + b^*_i(x,y)w \). Then there is a positive \( \varepsilon_0 = \varepsilon_0(\eta_1,\eta_3,\eta_4,\eta_5, d) \leq \frac{1}{2\eta_5} \) such that the map \( \Psi_x \) and its Jacobian determinant denoted by \( J_{\Psi_x}(w,y) \) has the property

\[
|\psi_x(w,y)| \leq 1,
\]

\[
2|\text{det}B(y)| \geq |J_{\psi_x}(w,y)| \geq (1/2)|\text{det}B(y)|,
\]

for \( |x-y| \leq \varepsilon_0, |w| \leq \varepsilon_0, (w,y) \) almost surely. Moreover the map \( \psi_x \) is injective on the set \( \{ (w,y) \in \mathbb{R}^{d+1}; |x-y| \leq \varepsilon_0, |w| \leq \varepsilon_0 \} \).

If, for fixed \( y \in \mathbb{R}^d \), \( \Phi_y \) be a map \( \mathbb{R}^{d+1} \to \mathbb{R}^{d+1} \) given by

\[
\phi_y(w,x) = \psi_x(w,y), \ w \in \mathbb{R}, x \in \mathbb{R}^d,
\]

then the Jacobian of \( \Phi_y \) denoted by \( J_{\Phi_y}(w,x) \) has the property

\[
2|\text{det}B(y)| \geq |J_{\Phi_y}(w,x)| \geq (1/2)|\text{det}B(y)|,
\]

for \( |x-y| \leq \varepsilon_0, |w| \leq \varepsilon_0, (w,x) \) almost surely. Moreover the map \( \Phi_y \) is injective on the set \( \{ (w,x) \in \mathbb{R}^{d+1}; |x-y| \leq \varepsilon_0, |w| \leq \varepsilon_0 \} \).

Proof. In the proof we assume that constants \( c \) may additionally depend on \( \eta_4, \eta_5 \). We prove the statement for the map \( \psi_x, \) only. Since \( |\psi_x(w,y)| \leq \sqrt{d}(1 + \eta_1 + \eta_4)(|w| + |x-y|) \) we have

\[
|\psi_x(w,y)| \leq 1, \text{ if } |w| + |x-y| \leq \frac{1}{\sqrt{d}(1 + \eta_1 + \eta_4)}.
\]

Next, we observe that \( (w,y) \) almost surely

\[
\frac{\partial \xi_k}{\partial y_l} = -b_{kl}(y) + (x-y) \cdot \frac{\partial b_k}{\partial y_l} + w \frac{\partial b^*_k}{\partial y_l}, \quad 1 \leq l,k \leq d.
\]
Since \( |(y - x) \cdot \frac{\partial h_k}{\partial y_l} + w \frac{\partial b_k}{\partial y_l}| \leq (\eta_3 + \eta_5)(|y - x| + |w|) \), it follows that
\[
J_{\Psi_x}(w, y) = (-)^d \det B(y) + R(x, y, w), \quad |R(x, y, w)| \leq c(|y - x| + |w|),
\]
with \( c = c(d, \eta_1, \eta_3, \eta_5) \). Since \( |\det B(y)| = \frac{1}{|\det A(y)|} \geq \frac{1}{d^{d_2}} \), we have for sufficiently small \( \varepsilon_1 = \varepsilon_1(\eta_1, \eta_3, \eta_5, d) \), \((w, y)\) almost surely
\[
J_{\Psi_x}(w, y) = (-)^d \kappa(x, y, w) \det B(y), \quad |y - x| \leq \varepsilon_1, \quad |w| \leq \varepsilon_1,
\]
where \( \frac{1}{d} \leq \kappa \leq 2 \).

Let \( J_{\Psi_x}(w, y) \) be the Jacobi matrix for the map \( \Psi_x \) which is defined \((w, y)\) almost surely. Let \( \partial \Psi_x(w, y) \) denote the generalized Jacobian of \( \Psi_x \) at the point \((w, y)\). Then from the form of \( J_{\Psi_x} \) it is clear that every matrix \( M \in \partial \Psi_x(w, y) \) can be written as
\[
M = \mathcal{B}(x) + \mathcal{R},
\]
where the coordinates \( \mathcal{B}_{kl}(x), 0 \leq k, l \leq d \) of the matrix \( \mathcal{B}(x) \) are
\[
\mathcal{B}_{kl}(x) = -b_{kl}(x), \quad k, l \geq 1,
\]
\[
\mathcal{B}_{00}(x) = 1; \quad \mathcal{B}_{l0}(x) = 0; \quad \mathcal{B}_{0l}(x) = b_l^*(x, x), \quad 1 \leq l \leq d,
\]
while all the entries of \( \mathcal{R} \) satisfy \( |\mathcal{R}_{kl}| \leq c \frac{\sqrt{w^2 + |x - y|^2}}{2} \) with \( c = c(\eta_3, \eta_5) \).

Now, for every \((u, z), u \in \mathbb{R}, z \in \mathbb{R}^d : |u|^2 + |z|^2 = 1\) we have
\[
|(u, z)\mathcal{B}(x)^T| \geq 2\beta > 0,
\]
with \( \beta = \beta(d, \eta_1, \eta_4) \). Since \( ||\mathcal{R}||_\infty \leq c \frac{\sqrt{w^2 + |x - y|^2}}{2} \) we can apply Lemma 2.8 with \( n = d + 1 \) to show on the set \( \{(w, y) ; \sqrt{w^2 + |x - y|^2} \leq \beta/(c(d + 1))\} \) the map \( \Psi_x \) is injective. This fact, combined with (51) and (52), completes the proof if we choose \( \varepsilon_0 = \varepsilon_1 \wedge \frac{1}{2\sqrt{d(1 + m + \eta_1)}} \wedge \frac{\beta}{2(d + 1)c} \).

**Remark 3.7.** Let for \( x \in \mathbb{R}^d, \Psi_x \) be the map \( \mathbb{R}^d \mapsto \mathbb{R}^d \) given by
\[
\Psi_x(y) = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d, \quad y \in \mathbb{R}^d,
\]
where \( \xi_i = b_i(y)(x - y) \). Then using the same arguments as in the above proof we can find \( \varepsilon_0 \) such that all the assertions of Lemma 3.6 are true and additionally
\[
2|\det B(y)| \geq |J_{\Psi_x}(y)| \geq (1/2)|\det B(y)|,
\]
for \( |x - y| \leq \varepsilon_0, y \) almost surely. Moreover, the map \( \Psi_x \) is injective on \( B(x, \varepsilon_0) \). We can also find \( \delta_1 = \delta_1(\eta_1, \eta_3, \eta_4, \eta_5, d) > 0 \) and \( \delta_2 = \delta_1(\eta_1, \eta_3, \eta_4, \eta_5, d) > 0 \) such that the \( \Psi_x \) image of the ball \( B(x, \delta_1) \) contains \( B(0, \delta_2) \). To this end we apply the last assertion of Lemma 2.8.

Let \( b_l^*(x, y) \) be the functions introduced in Lemma 3.6. We will use the following abbreviations
\[
z_i = B_i(x, y) = b_1(y)(x - y) = b_1(y)(x_1 - y_1) + \ldots + b_{d}(y)(x_d - y_d),
\]
\[
b_l^* = b_l^*(x, y),
\]
\[
b_l^*_{0} = b_l^*(x, x).
\]
Let for \( k, l, m \in \{1, \ldots, d\} \),
\[
A_{l,m} = A_{l,m}(x, y) = \int_{\mathbb{R}} \prod_{i \neq l} g_{l,t}(z_i + b_i^*w) |g_{l,t}(z_l \pm b_l^*w) - g_{l,t}(z_l \pm b_l^*w)| \mu_m(w) dw.
\]
For $l \neq k$ we denote
\[ B_{l,k,m} = B_{l,k}(x,y) = \int_{\mathbb{R}} \prod_{i \neq l,k} g_{i,t}(z_i + b_i^t w) |g_{l,t}(z_l + b_l^t w) - g_{l,t}(z_l + b_{l0}^t w)| |g_{k,t}(z_k + b_k^t w) - g_{k,t}(z_k - b_k^t w)| \mu_m(w) dw. \]

**Corollary 3.8.** Assume that $2\delta < \varepsilon_0$, where $\varepsilon_0$ is from Lemma 3.6. With the assumptions of Lemma 3.6 we have for $t \leq \tau$,
\[ \int_{|y-x| \leq \varepsilon_0} [A_l + B_{l,k}] dy \leq c t^{-\sigma}, \quad x \in \mathbb{R}^d, \]
and
\[ \int_{|y-x| \leq \varepsilon_0} [A_l + B_{l,k}] dx \leq c t^{-\sigma}, \quad y \in \mathbb{R}^d, \]
where $c = c(\tau, \alpha, d, \eta_1, \eta_2, \eta_3, \eta_4, \varepsilon, \delta, \nu)$.

**Proof.** In the proof we assume that constants $c$ may additionally depend on $\eta_4, \eta_5$. It is enough to prove the estimates for $l = 1$ and $k = 2$. For $x, y \in \mathbb{R}^d$ we get $|b_1^* - b_{10}^*| \leq \eta_5 |x - y|$. Hence, from (38), we have for $w \in \mathbb{R}$,
\[ |g_{1,t}(z_1 \pm b_1^t w) - g_{1,t}(z_1 \pm b_{10}^t w)| \leq c \left( \frac{|b_1^* - b_{10}^*|^2}{(h_1^{-1}(1/t))^2} \wedge 1 \right) (g_{1,t}^*(z_1 \pm b_1^t w) + g_{1,t}^*(z_1 \pm b_{10}^t w)) \leq c \left( \frac{|x - y|^2}{(h_1^{-1}(1/t))^2} \wedge 1 \right) (g_{1,t}^*(z_1 \pm b_1^t w) + g_{1,t}^*(z_1 \pm b_{10}^t w)). \]

This implies that
\[ A_{1,m} \leq c(A_{1,m}^1 + A_{1,m}^2 + A_{1,m}^3 + A_{1,m}^4), \]
where
\[ A_{1,m}^r = \int_{\mathbb{R}} \left( \prod_{i=1}^{d} g_{i,t}^*(z_i + \hat{b}_i^t w) \right) \left( \frac{|x - y|^2}{(h_1^{-1}(1/t))^2} \wedge 1 \right) \mu_m(w) dw \]
with $\hat{b}_i^t = b_i^t$, $i \geq 2$ and $\hat{b}_1^t = b_1^t$, $\hat{b}_2^t = -b_1^t$, $\hat{b}_3^t = b_{10}^t$ and $\hat{b}_4^t = -b_{10}^t$. Note that the functions $\hat{b}_i^t = \hat{b}_i^t(x,y)$ have the same properties (49, 50) as $b_i^t$. To evaluate the integral $\int_{|x-y| \leq \varepsilon_0} A_{1,m}^1 dy$ we introduce new variables in $\mathbb{R}^{d+1}$, given by $(w, \xi) = \Psi_x(w, y)$, where $\xi_i = z_i + b_i^t w, i = 1, \ldots, d$ (or $\xi_i = z_i + \hat{b}_i^t w$ if $A_{1,m}^r$ is treated for $r = 2, 3, 4$). Note that the vector $\xi = (\xi_1, \ldots, \xi_d)$ can be written as
\[ \xi = (x - y)B(y)^T + wb^*, \]
where $b^* = (b_1^*, \ldots, b_d^*)$, hence
\[ (\xi - wb^*)A(y)^T = x - y. \]
From this we infer that
\[ |w|^2|x - y| \leq c(|\xi| + |w||w|^2. \]
Let $Q_x = \{(w, y) : |y - x| \leq \varepsilon_0, |w| \leq \varepsilon_0\}$. Due to Lemma 3.6, almost surely on $Q_x$, the absolute value of the Jacobian determinant of the map $\Psi_x$ is bounded from below and above by two positive constants and $\Psi_x$ is an injective transformation.
Let $V_x = \Psi_x(Q_x)$. Observing that the support of the measure $\mu$ is contained in $[-\varepsilon_0, \varepsilon_0]$ and then applying the above change of variables, we have

$$\int_{|y-x| \leq \varepsilon_0} A_{1,m}^1 dy \leq c \int_{|y-x| \leq \varepsilon_0} \prod_{i=1}^d g_{1,i}^*(\xi_i) \left( \frac{(|\xi| + |w|)|w|^2}{(h_1^{-1}(1/t))^2} \wedge 1 \right) \mu_m(w) dw dy$$

$$\leq c \int_{|y-x| \leq \varepsilon_0} \prod_{i=1}^d g_{1,i}^*(\xi_i) \left( \frac{(|\xi| + |w|)|w|^2}{(h_1^{-1}(1/t))^2} \wedge 1 \right) \mu_m(w) dw dy$$

$$= c \int_{|y-x| \leq \varepsilon_0} \prod_{i=1}^d g_{1,i}^*(\xi_i) \left( \frac{(|\xi| + |w|)|w|^2}{(h_1^{-1}(1/t))^2} \wedge 1 \right) \mu_m(w) dw d\xi,$$

where the last equality follows from the general change of variable formula for injective Lipschitz maps (see e.g. [20, Theorem 3]). Since $|\xi| \leq 1$ for $(w, \xi) \in V_x$, we get

$$\int_{|y-x| \leq \varepsilon_0} A_{1}^1 dy \leq c \int_{|\xi| \leq 1} \prod_{i=1}^d g_{1,i}^*(\xi_i) \left( \frac{(|\xi| + |w|)|w|^2}{(h_1^{-1}(1/t))^2} \wedge 1 \right) \mu_1(w) dw d\xi.$$

Applying (37) if $m = 1$ or (39) otherwise, we have for $|\xi| \leq 1$,

$$\int_{\mathbb{R}} \left( \frac{(|\xi| + |w|)|w|^2}{(h_1^{-1}(1/t))^2} \wedge 1 \right) \mu_1(w) dw \leq c \frac{t^{\alpha/(3\beta)} + |\xi|^{\alpha/2}}{t}$$

and, for $m \geq 2$,

$$\int_{\mathbb{R}} \left( \frac{(|\xi| + |w|)|w|^2}{(h_1^{-1}(1/t))^2} \wedge 1 \right) \mu_m(w) dw \leq c \left[ |\xi|^{\beta/2} t^{-\beta/\alpha} + t^{-2\beta/(3\alpha)} \right].$$

Finally, by Lemma 2.6 we obtain for $m = 1$,

$$\int_{|y-x| \leq \varepsilon_0} A_{1,1}^1 dy \leq c \int_{|\xi| \leq 1} \prod_{i=1}^d g_{1,i}^*(\xi_i) \frac{t^{\alpha/(3\beta)} + |\xi|^{\alpha/2}}{t} d\xi \leq c t^{-(1-\alpha/(3\beta))}.$$  

Similarly we obtain

$$\int_{|y-x| \leq \varepsilon_0} A_{r,1}^1 dy \leq c t^{-(1-\alpha/(3\beta))}, \quad r = 2, 3, 4.$$  

For $m \geq 2$ in the same way we get

$$\int_{|y-x| \leq \varepsilon_0} A_{1,m} dy \leq c t^{-(2\beta/(3\alpha))}, \quad r = 1, 2, 3, 4.$$  

which completes the proof of the bound

$$\int_{|y-x| \leq \varepsilon_0} A_{1,m} dy \leq c t^{-(1-\alpha/(3\beta))}.$$  

For $x, y \in \mathbb{R}^d$ we get $|b_1^* - b_1^{|10}| \leq \eta_5 |x - y|$. Hence, from (16), we have for $w \in \mathbb{R}$,
Recall that if we fixed $\varepsilon$ where

Next, by (39), we have

Hence, by Lemma 2.6, □

This completes the proof of the first estimate.

We proceed as before and introduce new variables in $\mathbb{R}^{d+1}$, given by $(w, \xi) = \Psi_x(w, y)$, where $\xi_i = z_i + \hat{b}_i^* w$, $i = 1, \ldots, d$. Again we have that

By the same arguments as before

Next, by (39), we have

Hence, by Lemma 2.6

This completes the proof of the first estimate.

To estimate the second integral (with respect to $dx$) we proceed exactly in the same way.

For fixed $l \in \{1, \ldots, d\}$ let us consider a family of functions $b^*_i(x, y) = b_i(y) a_t(x)$, $i \in \{1, \ldots, d\}$. They satisfy the conditions (19) and (50) with $\eta_4 = d\eta_3^2$ and $\eta_5 = d\eta_1\eta_3$. Let $\varepsilon_0 = \varepsilon_0(\eta_1, \eta_3, \eta_4, \eta_5, d)$ be as found in Lemma 3.6 and Remark 3.6. Finally we choose $\varepsilon = \varepsilon(\eta_1, \eta_3, d) = \frac{\varepsilon_0}{4d^{3/2}(\eta_1)}$. From now on we keep $\varepsilon_0, \varepsilon$ fixed as above. Recall that if we fixed $\varepsilon$ we fix $\delta$ according to Lemma 2.1.
Lemma 3.9. We have

\[
\prod_{i=1}^{d} a_i + \prod_{i=1}^{d} b_i - \prod_{i=1}^{d} c_i - \prod_{i=1}^{d} d_i
\]

\[
= \sum_{j=1}^{d} \left[ \prod_{i=1}^{j-1} d_i \right] \left( c_k - d_k \right) \left( \prod_{i=k}^{j-1} a_i \right) \left( a_j - c_j \right) \left( \prod_{i=j+1}^{d} a_i \right)
\]

\[
+ \left( \prod_{i=1}^{j-1} d_i \right) \left( a_j - c_j - (d_j - b_j) \right) \left( \prod_{i=j+1}^{d} a_i \right)
\]

\[
+ \sum_{k=j+1}^{d} \left( \prod_{i=1}^{j-1} d_i \right) \left( d_j - b_j \right) \left( \prod_{i=j+1}^{k-1} b_i \right) \left( a_k - b_k \right) \left( \prod_{i=k+1}^{d} a_i \right)
\].

(53)

We understand here that for \(m > n\) we have \(\prod_{i=m}^{n} e_i = 1\) and \(\sum_{i=m}^{n} e_i = 0\).

Proof. We have

\[
\prod_{i=1}^{d} a_i - \prod_{i=1}^{d} c_i = \sum_{j=1}^{d} \left( \prod_{i=1}^{j-1} c_i \right) \left( a_j - c_j \right) \left( \prod_{i=j+1}^{d} a_i \right).
\]

(54)

Similarly we get

\[
\prod_{i=1}^{d} b_i - \prod_{i=1}^{d} d_i = \sum_{j=1}^{d} \left( \prod_{i=1}^{j-1} d_i \right) \left( b_j - d_j \right) \left( \prod_{i=j+1}^{d} b_i \right)
\]

so

\[
\prod_{i=1}^{d} d_i - \prod_{i=1}^{d} b_i = \sum_{j=1}^{d} \left( \prod_{i=1}^{j-1} d_i \right) \left( d_j - b_j \right) \left( \prod_{i=j+1}^{d} b_i \right)
\]

(55)

By (54) and (55) we get

\[
\prod_{i=1}^{d} a_i + \prod_{i=1}^{d} b_i - \prod_{i=1}^{d} c_i - \prod_{i=1}^{d} d_i
\]

\[
= \left( \prod_{i=1}^{d} a_i - \prod_{i=1}^{d} c_i \right) - \left( \prod_{i=1}^{d} d_i - \prod_{i=1}^{d} b_i \right)
\]

\[
= \sum_{j=1}^{d} \left( \prod_{i=1}^{j-1} c_i \right) \left( a_j - c_j \right) \left( \prod_{i=j+1}^{d} a_i \right) - \sum_{j=1}^{d} \left( \prod_{i=1}^{j-1} d_i \right) \left( d_j - b_j \right) \left( \prod_{i=j+1}^{d} b_i \right).
\]

(56)
For any $j \in \{1, \ldots, d\}$ we have
\[
\sum_{j=1}^{d} \left( \prod_{i=1}^{j-1} c_i \right) (a_j - c_j) \left( \prod_{i=j+1}^{d} a_i \right) - \left( \prod_{i=1}^{j-1} d_i \right) (d_j - b_j) \left( \prod_{i=j+1}^{d} b_i \right)
\]
\[
= \sum_{k=1}^{j-1} \left( \prod_{i=1}^{k-1} d_i \right) (c_k - d_k) \left( \prod_{i=k+1}^{j-1} c_i \right) (a_j - c_j) \left( \prod_{i=j+1}^{d} a_i \right)
\]
\[
+ \left( \prod_{i=1}^{j-1} d_i \right) (a_j - c_j - (d_j - b_j)) \left( \prod_{i=j+1}^{d} a_i \right)
\]
\[
+ \sum_{k=j+1}^{d} \left( \prod_{i=1}^{d-1} d_i \right) (d_j - b_j) \left( \prod_{i=j+1}^{k-1} b_i \right) (a_k - b_k) \left( \prod_{i=k+1}^{d} a_i \right). \tag{57}
\]

Now, (56) and (57) give (53). \hfill \square

When assumptions (Q1) are satisfied we put $\sigma = 2/3$, when assumptions (Q2) are satisfied we put $\sigma = 2\beta/(3\alpha)$. Clearly, in both cases $\sigma \in (0, 1)$.

**Proposition 3.10.** For any $x, y \in \mathbb{R}^d$, $t \in (0, \tau]$ we have
\[
|q_0(t, x, y)| \leq \frac{1}{t^{3/\alpha + d/\alpha}}. \tag{58}
\]

For $x, y \in \mathbb{R}^d$, $t \in (0, \tau]$, $|y - x| \geq \varepsilon_0$ we have
\[
|q_0(t, x, y)| \leq c e^{-c_1|x-y|}. \tag{59}
\]

For any $t \in (0, \tau]$, $x \in \mathbb{R}^d$ we have
\[
\int_{\mathbb{R}^d} |q_0(t, x, y)| dy \leq ct^{-\sigma}. \tag{60}
\]

For any $t \in (0, \tau]$, $y \in \mathbb{R}^d$ we have
\[
\int_{\mathbb{R}^d} |q_0(t, y, x)| dx \leq ct^{-\sigma}. \tag{61}
\]

**Proof.** We have
\[
q_0(t, x, y) = \sum_{i=1}^{d} \lim_{\zeta \to 0^+} \int_{|w| > \zeta} [p_y(t, x - y + a_i(x)w) - p_y(t, x - y + a_i(y)w)] \mu_i(w) dw.
\]

For $i = 1, \ldots, d$ we put
\[
R_i = \lim_{\zeta \to 0^+} \int_{|w| > \zeta} [p_y(t, x - y + a_i(x)w) - p_y(t, x - y + a_i(y)w)] \mu_i(w) dw. \tag{62}
\]

We have $q_0(t, x, y) = R_1 + \ldots + R_d$. It is clear that it is enough to handle $R_1$ alone. Note that
\[
R_1 = \det(B(y)) \lim_{\zeta \to 0^+} \int_{|w| > \zeta} [G_t((x - y + we_1(A(x))^T)(B(y))^T)
\]
\[
- G_t((x - y + we_1(A(y))^T)(B(y))^T)] \mu_1(w) dw. \tag{63}
\]
We will use the following abbreviations
\[ z_i = B_i(x, y) = b_i(y)(x - y) = b_{i1}(y)(x_1 - y_1) + \ldots + b_{id}(y)(x_d - y_d), \]
\[ k_i = \tilde{b}_{i1}(x, y) = b_i(y)a_1(x), \]
\[ k_{i0} = \tilde{b}_{i1}(x, x). \]
Note that \( k_{10} = 1 \) and \( k_{i0} = 0, \ 2 \leq i \leq d. \)

Let
\[ \delta_t(w) = \prod_{i=1}^{d} g_{i,t}(z_i + k_i w) + \prod_{i=1}^{d} g_{i,t}(z_i - k_i w) - \prod_{i=1}^{d} g_{i,t}(z_i + k_{i0} w) - \prod_{i=1}^{d} g_{i,t}(z_i - k_{i0} w). \]
We can rewrite (63) as
\[ R_1 = \operatorname{det}(B(y)) \lim_{\zeta \to 0^+} \int_{|w| > \zeta} \delta_t(w) \mu_1(w) \, dw. \]
By Lemma 3.9 denoting
\[ a_i = g_{i,t}(z_i + k_i w), \ b_i = g_{i,t}(z_i - k_i w), \ c_i = g_{i,t}(z_i + k_{i0} w), \ d_i = g_{i,t}(z_i - k_{i0} w), \]
we have
\[
\begin{split}
\delta_t(w) & = \sum_{j=1}^{d} \left[ \sum_{k=1}^{j-1} \left( \prod_{i=1}^{k-1} d_i \right) \left( c_k - d_k \right) \left( \prod_{i=k+1}^{j-1} c_i \right) \left( a_j - c_j \right) \left( \prod_{i=j+1}^{d} a_i \right) \right] \\
& \quad + \left( \prod_{i=1}^{j-1} d_i \right) \left( a_j - c_j - (d_j - b_j) \right) \left( \prod_{i=j+1}^{d} a_i \right) \\
& \quad + \sum_{k=j+1}^{d} \left( \prod_{i=1}^{j-1} d_i \right) \left( d_j - b_j \right) \left( \prod_{i=j+1}^{k-1} b_i \right) \left( a_k - b_k \right) \left( \prod_{i=k+1}^{d} a_i \right) \right].
\end{split}
\] (64)
Hence it is enough to consider the following terms
\[ \delta_t^{k,j}(w) = \left( \prod_{i=1}^{k-1} d_i \right) \left( c_k - d_k \right) \left( \prod_{i=k+1}^{j-1} c_i \right) \left( a_j - c_j \right) \left( \prod_{i=j+1}^{d} a_i \right), \ k < j, \]
\[ \delta_t^{j,j}(w) = \left( \prod_{i=1}^{j-1} d_i \right) \left( a_j - c_j - (d_j - b_j) \right) \left( \prod_{i=j+1}^{d} a_i \right), \]
\[ \delta_t^{j,k}(w) = \left( \prod_{i=1}^{j-1} d_i \right) \left( d_j - b_j \right) \left( \prod_{i=j+1}^{k-1} b_i \right) \left( a_k - b_k \right) \left( \prod_{i=k+1}^{d} a_i \right), \ k > j. \]
We denote \( M_{i,t} = \max_{|w| \leq 2\delta} \tilde{g}_{i,t}(z_i + k^* w) \), where \( k^* = \max\{1, |k_1|, |k_2|, \ldots, |k_d|\} \).
By (33), we have
\[
\int_{\mathbb{R}} |\delta_t^{k,j}(w)| \mu_1(w) \, dw \leq \prod_{i \neq j,k} M_{i,t} \int_{\mathbb{R}} |(c_k - d_k)(a_j - c_j)| \mu_1(w) \, dw \leq c \frac{\prod_{i=1}^{d} M_{i,t}}{t^{\beta/\alpha}}
\]
and, by (31),
\[
\int_{\mathbb{R}} |\delta_t^{i,j}(w)| \mu_1(w) \, dw \leq \prod_{i \neq j} M_{i,t} \int_{\mathbb{R}} |a_j - c_j - (d_j - b_j)| \mu_1(w) \, dw \\
\leq c \prod_{i=1}^d M_{i,t} t^{-\beta/\alpha}.
\]

It follows that
\[
|R_1| \leq c \prod_{i=1}^d M_{i,t} t^{-\beta/\alpha}.
\]
Since \( M_{i,t} \leq \frac{\varepsilon}{\norm{\eta_i}^{\alpha}} \) we obtain (39) and moreover
\[
|R_1| \leq c \min_i M_{i,t} t^{-\frac{d-1+\beta}{\alpha}}.
\]
By Lemma 3.2, \( \max_i |z_i| \geq \frac{1}{\eta_i d^{3/2}} |x - y| \) and suppose that \( |z_i| \geq \frac{1}{\eta_i d^{3/2}} |x - y| \).

Then, since \( |k^x| \leq 2\eta_1^2 \), we have for \( |x - y| \geq 8d^{3/2}\eta_1^3 \delta \) and \( |w| \leq 2\delta \),
\[
|z_1 + k^x w| \geq |z_1| - |k^x w| \geq \frac{1}{\eta_1 d^{3/2}} |x - y| - 4\eta_1^2 \delta
\]
\[
= \frac{1}{\eta_1 d^{3/2}} |x - y| \left(1 - 4d^{3/2} \frac{\eta_1^3 \delta}{|x - y|} \right) \geq \frac{1}{2\eta_1 d^{3/2}} |x - y|
\]

This yields that
\[
|R_1| \leq c t^{-\beta/\alpha - (d-1)/\alpha} \widetilde{\gamma}_{1,t} \left( \frac{|x - y|}{2\eta_1 d^{3/2}} \right), \quad |x - y| \geq 8d^{3/2}\eta_1^3 \delta
\]

This proves the exponential bound
\[
|R_1| \leq ce^{-\left(\frac{|x - y|}{2\eta_1 d^{3/2}}\right)}, \quad |x - y| \geq \max\{2d^{3/2}\eta_1 \varepsilon, 8d^{3/2}\eta_1^3 \delta\}.
\]

Recall that \( \varepsilon = \frac{\varepsilon_0}{4d^{3/2}(\text{dim}(\mathbb{V}))} \) and \( \delta = \min\{\delta_0, \frac{\varepsilon_0}{8d^{3/2}(\text{dim}(\mathbb{V}))}, \frac{\varepsilon}{4d^{3/2}(\text{dim}(\mathbb{V}))} \} \). Hence \( \max\{2d^{3/2}\eta_1 \varepsilon, 8d^{3/2}\eta_1^3 \delta\} \leq \varepsilon_0 \), so finally
\[
|R_1| \leq c e^{-\left(\frac{|x - y|}{2\eta_1 d^{3/2}}\right)}, \quad |x - y| \geq \varepsilon_0,
\]
which proves (39).

The estimates (40) and (41) follow from Corollary 3.8 and (39). For example to handle the integral
\[
\int_{|y-x| \leq \varepsilon_0} \int_{\mathbb{R}} |\delta_t^{i,j}(w)| \, dw \, dy, \quad x \in \mathbb{R}^d,
\]
we take
\[
b_i^x(x, y) = -k_{i0}(x, x), i = 1, \ldots, j - 1, \quad b_i^x(x, y) = k_i(x, y), i = j, \ldots, d.
\]
Such choice of functions \( b_i^x \) enable us to apply Corollary 3.8 since they satisfy all the assumptions of Lemma 3.6. Hence
\[
\int_{|y-x| \leq \varepsilon_0} \int_{\mathbb{R}} |\delta_t^{i,j}(w)| \mu_i(w) \, dw \, dy \leq c t^{-\sigma}.
\]
The same argument (with an appropriate choice of $b^*_i$) shows that

$$
\int_{|y-x| \leq \varepsilon_0} \int_R |\delta^{i,k}_t(w)| \mu_i(w) dw dy \leq ct^{-\sigma}.
$$

This implies that

$$
\int_{|y-x| \leq \varepsilon_0} |q_0(t, x, y)| dy \leq ct^{-\sigma}.
$$

By (53) we can extend the domain of integration to the whole $\mathbb{R}^d$ keeping the upper bound as above.

Using similar arguments as in the proof of Proposition 3.10 we obtain the following result.

**Proposition 3.11.** For any $t \in (0, \tau]$, $x \in \mathbb{R}^d$ we have

$$
\int_{\mathbb{R}^d} p_y(t, x-y) dy \leq c, \quad (65)
$$

$$
\int_{\mathbb{R}^d} r_y(t, (x-y)/2) dy \leq c. \quad (66)
$$

For any $\delta_1 > 0$,

$$
\lim_{t \to 0^+} \sup_{x \in \mathbb{R}^d} \int_{B(x, \delta_1)} p_y(t, x-y) dy = 0. \quad (67)
$$

We have

$$
\lim_{t \to 0^+} \int_{\mathbb{R}^d} p_y(t, x-y) dy = 1, \quad (68)
$$

uniformly with respect to $x \in \mathbb{R}^d$.

**Proof.** For fixed $x \in \mathbb{R}^d$ we introduce new variables $u = \tilde{\Psi}_x(y)$ given by

$$
u = (x-y)B(y)^T.
$$

Note that

$$
\frac{1}{d\eta_1} |x-y| \leq |u| = |(x-y)B(y)^T| \leq d\eta_1 |x-y|. \quad (69)
$$

For $r > 0$, let $V_x(r)$ be the $\tilde{\Psi}_x$ image of the ball $B(x, r)$. By Remark 3.10 we have almost surely

$$
|J_{\tilde{\Psi}_x}(y)| \geq (1/2)|\det B(y)| \geq c, \quad |y-x| \leq \varepsilon_0,
$$

and $\tilde{\Psi}_x$ is an injective map on $B(x, \varepsilon_0)$. Hence, for $0 < \delta_1 < \varepsilon_0$, by the change of variables formula (see e.g. [20, Theorem 3]), and then by (69) we obtain

$$
\int_{\delta_1 \leq |x-y| \leq \varepsilon_0} r_y(t, (x-y)/2) dy = \int_{\delta_1 \leq |x-y| \leq \varepsilon_0} \tilde{G}_t(u/2) dy
$$

$$
\leq c \int_{\delta_1 \leq |x-y| \leq \varepsilon_0} \tilde{G}_t(u/2) |J_{\tilde{\Psi}_x}(y)| dy
$$

$$
= c \int_{V_x(\varepsilon_0) \setminus V_x(\delta_1)} \tilde{G}_t(u/2) du
$$

$$
\leq c \int_{|u| \geq \frac{\delta_1}{2\eta_1}} \tilde{G}_t(u/2) du = I(t, \delta_1).
$$
By Corollary 2.7 we have for $t \leq \tau$,
\[
\int_{\mathbb{R}^d} \tilde{G}_t(u/2) \, du \leq c.
\]
If $|x - y| \geq \varepsilon_0$ then $|x - y|/2 \geq c\varepsilon_0d^{3/2}$, hence, by (47), we obtain
\[
\int_{|x-y| \geq \varepsilon_0} r_y(t, (x - y)/2) \, dy \leq c_1 t \int_{|x-y| \geq \varepsilon_0} e^{-c|x-y|} \, dy = c_2 t.
\]
The last two inequalities prove that
\[
\sup_{x \in \mathbb{R}^d} \sup_{t \leq \tau} \int_{|x-y| \geq \delta_1} r_y(t, (x - y)/2) \, dy \leq c.
\]
Again by Corollary 2.7 we observe that $\lim_{t \to 0^+} I(t, \delta_1) = 0$. Hence, we obtain
\[
\lim_{t \to 0^+} \sup_{x \in \mathbb{R}^d} \int_{|x-y| \geq \delta_1} r_y(t, (x - y)/2) \, dy = 0.
\]
Since $p_y(t, x - y) \leq c r_y(t, (x - y)/2)$ for $t \leq \tau$, $x, y \in \mathbb{R}^d$, the proof of (65) is completed.

Note that the coordinates of the matrix $B(y)$ have partial derivatives $y$ almost surely, bounded uniformly. We can calculate the absolute value of Jacobian determinant $J_{\psi_x}(y)$, $y$ almost surely, as
\[
|J_{\psi_x}(y)| = \det B(y) + R(x, y), \quad |R(x, y)| \leq c|y - x|.
\]

Next,
\[
\int_{|x-y| \leq \delta_1} p_y(t, x - y) \, dy = \int_{|x-y| \leq \delta_1} G_t(u) \det B(y) \, dy
\]
\[
= \int_{|x-y| \leq \delta_1} G_t(u) \left| J_{\psi_x}(y) \right| \, dy - \int_{|x-y| \leq \delta_1} G_t(u) R(x, y) \, dy
\]
\[
= I_1 + I_2.
\]

Applying (70), (69) and the change of variable formula we obtain
\[
|I_2| \leq c \int_{|x-y| \leq \delta_1} |x - y|G_t(u) \, dy
\]
\[
\leq c \int_{|x-y| \leq \delta_1} |u|G_t(u) \left| J_{\psi_x}(y) \right| \, dy
\]
\[
= c \int_{V_x(\delta_1)} |u|G_t(u) \, du
\]
\[
\leq c \int_{|u| \leq d\eta \delta_1} |u|G_t(u) \, du \to 0, \text{ if } t \to 0^+.
\]

The last limit is a consequence of the fact that the probability measure $\mu_t(du) = G_t(u) \, du$ converges weakly to the Dirac measure concentrated at the origin.

Now we can pick, independently of $x$, positive $\delta_1$ and $\delta_2$ such that $B(0, \delta_2) \subset V_x(\delta_1)$ (see Remark 3.7). Applying again the change of variable formula we obtain
\[
I_1 = \int_{V_x(\delta_1)} G_t(u) \, du \geq \int_{|u| \leq \delta_2} G_t(u) \, du \to 1, \text{ if } t \to 0^+.
\]
This completes the proof that uniformly with respect to \( t \),
\[
\lim_{t \to 0^+} \int_{|x-y| \leq \delta} p_y(t, x - y) dy = 1,
\]
which combined with (67) proves (68). \( \square \)

In the sequel we will use the following standard estimate. For any \( \gamma \in (0, 1] \), \( \theta_0 > 0 \) there exists \( c = c(\gamma, \theta_0) \) such that for any \( \theta \geq \theta_0, t > 0 \) we have
\[
\int_0^t (t-s)^{\gamma-1} s^{\theta-1} ds \leq \frac{c}{\theta^\gamma} t^{(\gamma-1)+((\theta-1)+1)}.
\]  

(71)

Lemma 3.12. For any \( t > 0 \), \( x \in \mathbb{R}^d \) and \( n \in \mathbb{N} \) the kernel \( q_n(t, x, y) \) is well defined. For any \( t \in (0, \tau] \), \( x \in \mathbb{R}^d \) and \( n \in \mathbb{N} \) we have
\[
\int_{\mathbb{R}^d} |q_n(t, x, y)| dy \leq \frac{c_1^{n+1} t^{(n+1)(1-\sigma)-1}}{(n!)^{1-\sigma}},
\]  

(72)

\[
\int_{\mathbb{R}^d} |q_n(t, y, x)| dy \leq \frac{c_1^{n+1} t^{(n+1)(1-\sigma)-1}}{(n!)^{1-\sigma}}.
\]  

(73)

For any \( t \in (0, \tau] \), \( x, y \in \mathbb{R}^d \) and \( n \in \mathbb{N} \) we have
\[
|q_n(t, x, y)| \leq c_1 \frac{c_2^{n(n-1)} t^{n(1-\sigma)-1}}{(n!)(1-\sigma) t^{1+(d+\beta+1)/\alpha}}.
\]  

(74)

For any \( t \in (0, \tau] \), \( x, y \in \mathbb{R}^d \) and \( n \in \mathbb{N} \), \( |x-y| \geq n+1 \) we have
\[
|q_n(t, x, y)| \leq c_1 \frac{c_2^{n(n-1)} t^{n(1-\sigma)-1}}{(n!)(1-\sigma) e^{-\lambda |x-y|}}.
\]  

(75)

where \( \lambda = \varepsilon/\varepsilon_0 \).

Proof. By Proposition 3.10 there is a constant \( c^* \) such that for any \( x, y \in \mathbb{R}^d \), \( t \in (0, \tau] \) we have
\[
|q_0(t, x, y)| \leq c^* \frac{1}{t^{(d+\beta+1)/\alpha}},
\]  

(76)

|q_0(t, x, y)| \leq c^* e^{-\lambda |x-y|}, |x - y| \geq 1. \hspace{1cm} (77)

\[
\int_{\mathbb{R}^d} |q_0(t, x, u)| du \leq c^* t^{-\sigma},
\]  

(78)

\[
\int_{\mathbb{R}^d} |q_0(t, u, x)| du \leq c^* t^{-\sigma}.
\]  

(79)

It follows from (71) there is \( p \geq 1 \) such that for \( n \in \mathbb{N} \),
\[
\int_0^t (t-s)^{-\sigma} s^{(n+1)(1-\sigma)-1} ds \leq \frac{p}{(n+1)^{1-\sigma}} t^{(n+2)(1-\sigma)-1},
\]  

\[
\int_0^{t/2} (t-s)^{-\sigma} s^{n(1-\sigma)-1} ds \leq \frac{p}{(n+1)^{1-\sigma}} t^{(n+1)(1-\sigma)-1},
\]  

\[
\int_0^t (t-s)^{-1/2} s^{n/2} ds \leq \frac{p}{(n+1)^{1/2}} t^{(n+1)/2}.
\]  

We define \( c_1 = p c^* \geq c^* \) and \( c_2 = 2^{(d+\beta+1)/\alpha} c_1 ((1-\sigma)^{-1} + p) > c_1 \).

We will prove (72), (73), (74) simultaneously by induction. They are true for \( n = 0 \) by (76), (78), (79) and the choice of \( c_1 \). Assume that (72), (73), (74) are true for
$n \in \mathbb{N}$, we will show them for $n + 1$. By the definition of $q_n(t, x, y)$ and the induction hypothesis we obtain

$$
|q_{n+1}(t, x, y)| \leq c_1 \frac{2^{(d+\beta+1)/\alpha}}{t^{(d+\beta+1)/\alpha}} \int_0^{t/2} \int_{\mathbb{R}^d} |q_n(s, z, y)| \, dz \, ds \\
+ c_1 \frac{c_2^{(d+\beta+1)/\alpha}}{(n!)^{1-\sigma} t^{-1+(d+\beta+1)/\alpha}} \int_0^t \int_{\mathbb{R}^d} |q_0(t-s, x, z)| \, dz \, s^{n(1-\sigma)-1} \, ds \\
\leq c_1 \frac{c_1^{n+1} (d+\beta+1)/\alpha}{(n!)^{1-\sigma} t^{(d+\beta+1)/\alpha}} \int_0^{t/2} \int_{\mathbb{R}^d} (t-s)^{(1-\sigma)-1} s^{n(1-\sigma)-1} \, ds \\
+ c_1 \frac{c_2^{n+1} (d+\beta+1)/\alpha c_1}{(n!)^{1-\sigma} t^{-1+(d+\beta+1)/\alpha}} \int_0^t \int_{\mathbb{R}^d} (t-s)^{(1-\sigma)-1} s^{n(1-\sigma)-1} \, ds \\
\leq c_1 \frac{c_1^{n+1} (d+\beta+1)/\alpha}{((n+1)!)^{1-\sigma}} \left( c_1 \frac{2^{(d+\beta+1)/\alpha}}{1-\sigma} + c_2^{(d+\beta+1)/\alpha} \right) \\
= c_1 \frac{c_2^{n+1} (n+1)/2}{((n+1)!)^{1/2} t^{d/\alpha+1}}.
$$

Hence we get (74) for $n + 1$. In particular this gives that the kernel $q_{n+1}(t, x, y)$ is well defined.

By the definition of $q_n(t, x, y)$, (78) and the induction hypothesis we obtain

$$
\int_{\mathbb{R}^d} |q_{n+1}(t, x, y)| \, dy \leq \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |q_0(t-s, x, z)||q_n(s, z, y)| \, dz \, dy \, ds \\
\leq c^* \frac{c_1^{n+1}}{(n!)^{1-\sigma}} \int_0^t (t-s)^{-\sigma} s^{(n+1)(1-\sigma)-1} \, ds \\
\leq c^* \frac{c_1^{n+1}}{(n!)^{1-\sigma}} \frac{p}{(n+1)^{1-\sigma}} t^{(n+2)(1-\sigma)-1} \\
= \frac{c_1^{n+2}}{(n+1)!} t^{(n+2)(1-\sigma)-1},
$$

which proves (72) for $n + 1$. Similarly we get (73).

Now we will show (75). For $n = 0$ this follows from (77). Assume that (75) is true for $n \in \mathbb{N}$, we will show it for $n + 1$. 


Using our induction hypothesis, \(12\) and \(13\) we get for \(|x - y| \geq n + 2\)

\[
|q_{n+1}(t, x, y)| = \left| \int_0^t \int_{|z-t| \geq \frac{|x-y|}{n+2}} q_0(t - s, x, z) q_n(s, z, y) \, dz \, ds \right|
\]

\[
+ \int_0^t \int_{|z-t| \leq \frac{|x-y|}{n+2}} q_0(t - s, x, z) q_n(s, z, y) \, dz \, ds
\]

\[
\leq c_1 e^{-\frac{\lambda |x-y|}{n+2}} \int_0^t \int_{\mathbb{R}^d} |q_n(s, z, y)| \, dz \, ds
\]

\[
+ c_1 \frac{c_2^n}{(n!)^{1-\sigma}} e^{-\frac{\lambda |x-y|}{n+2}} \int_0^t \int_{\mathbb{R}^d} |q_0(t - s, x, z)| \, dz \, s^{n(1-\sigma)} \, ds
\]

\[
\leq \frac{c_1}{1-\sigma} c_1 c_2^n \frac{t(n+1)(1-\sigma)}{(n+1)!^{1-\sigma}} + c_1 c_2^n e^{(n+1)(1-\sigma)\frac{t}{(n+1)!^{1-\sigma}} - \frac{\lambda |x-y|}{n+2}}
\]

which proves \(15\) for \(n + 1\) since by the choice of constants \(\frac{c_1}{1-\sigma} c_1^{n+1} + c_2^n c_2^2 \leq c_1 c_2^{n+1}\).

By standard estimates one easily gets

\[
\sum_{n=0}^{\infty} \frac{C^n}{(n!)^{(1-\sigma)}} \leq \frac{C^k}{(k!)^{(1-\sigma)}} \sum_{n=0}^{\infty} \frac{C^{n-k}}{((n-k)!)^{(1-\sigma)}} \leq C_1 e^{-k}, \quad k \in \mathbb{N}, \quad (80)
\]

where \(C_1\) depends on \(C\).

**Proposition 3.13.** For any \(t \in (0, \infty)\), \(x, y \in \mathbb{R}^d\) the kernel \(q(t, x, y)\) is well defined. For any \(t \in (0, \tau]\), \(x, y \in \mathbb{R}^d\) we have

\[
|q(t, x, y)| \leq \frac{c}{t^{(d+\beta+1)/\alpha}} e^{-c_3 \sqrt{|x-y|}} \leq \frac{c}{t^{(d+\beta+1)/\alpha}} (1 + |x-y|)^{\beta+1}.
\]

There exists \(a > 0\) (a depends on \(\tau, \alpha, \beta, \theta, \gamma, d, \eta_1, \eta_2, \eta_3, \eta_4\)) such that for any \(t \in (0, \tau]\), \(x, y \in \mathbb{R}^d\), \(|x - y| \geq a\) we have

\[
|q(t, x, y)| \leq c e^{-c_3 \sqrt{|x-y|}}.
\]

For any \(t \in (0, \tau]\) and \(x \in \mathbb{R}^d\) we have

\[
\int_{\mathbb{R}^d} |q(t, x, y)| \, dy \leq ct^{-\alpha}, \quad (81)
\]

\[
\int_{\mathbb{R}^d} |q(t, y, x)| \, dy \leq ct^{-\alpha}. \quad (82)
\]

**Proof.** By \(14\) we clearly get \(\sum_{n=0}^{\infty} |q_n(t, x, y)| \leq ct^{-(d+\beta+1)/\alpha}\). This gives that \(q(t, x, y)\) is well defined and we have \(|q(t, x, y)| \leq ct^{-(d+\beta+1)/\alpha}\).
For \( |x - y| \geq 1 \) by (121), (125) and (80) we get
\[
|q(t, x, y)| = \left| \sum_{n=0}^{\lfloor|x-y|-1\rfloor} q_n(t, x, y) + \sum_{n=\lfloor|x-y|\rfloor}^{\infty} q_n(t, x, y) \right|
\]
\[
\leq c_1 \sum_{n=0}^{\lfloor|x-y|-1\rfloor} e^{\lambda n(1-\sigma)} \left( \frac{c}{n!} \right)^{1-\sigma} e^{-\lambda \sqrt{|x-y|}} + c_1 \sum_{n=\lfloor|x-y|\rfloor}^{\infty} \left( \frac{c}{n!} \right)^{1-\sigma} (n+1/d+\beta+1/\alpha)
\]
\[
\leq c \frac{c}{t^{1+(d+\beta+1)/\alpha}} e^{-c_3 \sqrt{|x-y|}},
\]
where \( \lfloor z \rfloor \) denotes the integer part of \( z \). Take the smallest \( n_0 \in \mathbb{N} \) such that \( n_0(1-\sigma) - 1 \geq (d+\beta+1)/\alpha \) and \( a = n_0^2 \). For \( \sqrt{|x-y|} \geq \sqrt{a} = n_0 \) we get
\[
|q(t, x, y)| \leq c_1 \sum_{n=0}^{\lfloor|x-y|-1\rfloor} e^{\lambda n(1-\sigma)} \left( \frac{c}{n!} \right)^{1-\sigma} e^{-\lambda \sqrt{|x-y|}} + c_1 \sum_{n=\lfloor|x-y|\rfloor}^{\infty} \left( \frac{c}{n!} \right)^{1-\sigma} (n+1/d+\beta+1/\alpha)
\]
\[
\leq c e^{-c_3 \sqrt{|x-y|}}.
\]
(81) and (82) follows easily from (72) and (73). □

By (15), Corollary 3.3, Proposition 3.11 and Proposition 3.13 we immediately obtain the following result.

**Corollary 3.14.** For any \( t \in (0, \infty) \), \( x, y \in \mathbb{R}^d \) the kernel \( u(t, x, y) \) is well defined. For any \( t \in (0, \tau) \), \( x, y \in \mathbb{R}^d \) we have
\[
|u(t, x, y)| \leq \frac{c}{t^{1+(d+\beta+1)/\alpha}} e^{-c_1 \sqrt{|x-y|}} \leq \frac{c}{t^{1+(d+\beta+1)/\alpha} (1 + |x - y|)^{d+1}}.
\]
There exists \( a > \varepsilon > 0 \) (\( a \) depends on \( \tau, \alpha, \beta, \theta_0, C, \sigma, d, \eta_1, \eta_2, \eta_3, \eta_4 \)) such that for any \( t \in (0, \tau) \), \( x, y \in \mathbb{R}^d \), \( |x - y| \geq a \) we have
\[
|u(t, x, y)| \leq c e^{-c_2 \sqrt{|x-y|}}.
\]

For any \( t \in (0, \tau) \) and \( x \in \mathbb{R}^d \) we have
\[
\int_{\mathbb{R}^d} |u(t, x, y)| \, dy \leq c, \quad (83)
\]
\[
\int_{\mathbb{R}^d} |u(t, y, x)| \, dy \leq c. \quad (84)
\]

**Proof.** By Corollary 3.3, Proposition 3.11 (15), we only need to prove the corresponding bounds for
\[
I(t, x, y) = \int_0^t \int_{\mathbb{R}^d} p_z(t-s, x-z) |q(s, z, y)| \, dz \, ds.
\]
For \( 0 < s < t/2 \) we have
\[
p_z(t-s, x-z) \leq \frac{c}{t^d/\alpha},
\]
and, by Proposition 3.13 for \( t/2 < s < t \),
\[
|q(s, z, y)| \leq \frac{c}{t^{(d+\beta+1)/\alpha}}.
\]
Hence,
\[
I(t, x, y) = \int_0^{t/2} \int_{\mathbb{R}^d} p_z(t - s, x - z)|q(s, z, y)| \, dz \, ds
\]
\[
+ \int_{t/2}^{t} \int_{\mathbb{R}^d} p_z(t - s, x - z)|q(s, z, y)| \, dz \, ds
\]
\[
\leq \frac{c}{t^{d/\alpha}} \int_0^{t/2} \int_{\mathbb{R}^d} |q(s, z, y)| \, dz \, ds + \frac{c}{t^{(d+\beta+1)/\alpha}} \int_{t/2}^{t} \int_{\mathbb{R}^d} p_z(t - s, x - z) \, dz \, ds
\]
\[
\leq \frac{c}{t^{1+(d+\beta+1)/\alpha}},
\]
where (82) and Proposition 3.11 were applied to estimate the integrals with respect to the space variable.

Let \( a \) the constant found in Proposition 3.13. Assume that \( |x - y| \geq 2 + 2a \). By Corollary 3.3 for \( 0 < s < t \) we have
\[
p_z(t - s, x - z) \leq ce^{-c_1|x-y|}, \quad |x - z| > |x - y|/2 > 1.
\]
Proposition 3.13 implies that for \( 0 < s < t \),
\[
|q(s, z, y)| \leq ce^{-c_1\sqrt{|x-y|}}, \quad |y - z| > |x - y|/2 > a.
\]
Hence,
\[
I(t, x, y) \leq \int_0^{t} \int_{|x-z|>|x-y|/2} \ldots \, dz \, ds + \int_0^{t} \int_{|y-z|>|x-y|/2} \ldots \, dz \, ds
\]
\[
\leq ce^{-c_1|x-y|} \int_0^{t} \int_{\mathbb{R}^d} |q(s, z, y)| \, dz \, ds
\]
\[
+ ce^{-c_1\sqrt{|x-y|}} \int_0^{t} \int_{\mathbb{R}^d} p_z(t - s, x - z) \, dz \, ds
\]
\[
\leq ce^{-c_1} + cte^{-c_1\sqrt{|x-y|}}
\]
\[
\leq ce^{-c_1\sqrt{|x-y|}}.
\]
Combining (85) and (86) we obtain the desired pointwise estimates of \( u(t, x, y) \).

Next, (83) and (84) immediately follow from (81), (82) and Proposition 3.11.

For any \( \zeta > 0 \) and \( x, y \in \mathbb{R}^d \) we put
\[
\mathcal{L}_\zeta f(x) = \sum_{i=1}^{d} \int_{|w|>\zeta} [f(x + a_i(x)w) - f(x)] \mu(w) \, dw,
\]
\[
\mathcal{L}_\zeta^y f(x) = \sum_{i=1}^{d} \int_{|w|>\zeta} [f(x + a_i(y)w) - f(x)] \mu(w) \, dw.
\]
Lemma 3.15. For any \( \xi \in (0, 1] \), \( \zeta > 0 \), \( x, y, v \in \mathbb{R}^d \) and \( t \in (\xi, \tau + \xi] \) we have
\[
\sum_{i=1}^{d} \int_{\mathbb{R}} |p_y(t, x - y + a_i(v)w) - p_y(t, x - y)| \mu(w) \, dw \leq c(\xi) e^{-c|x-y|}, \tag{87}
\]
and
\[
\sum_{i=1}^{d} \int_{|w| \leq \zeta} |p_y(t, x - y + a_i(v)w) - p_y(t, x - y)| \mu(w) \, dw \leq c(\xi) \zeta^{-\alpha}. \tag{88}
\]
where \( c(\xi) \) is a constant depending on \( \xi, \tau, \alpha, d, \eta_1, \eta_2, \eta_3, \varepsilon, \delta \).

Proof. We estimate the term for \( i = 1 \). By Lemma 3.11 for \( \gamma = 1 \) we get for \( w \in \mathbb{R} \)
\[
|p_y(t, x - y + a_1(v)w) - p_y(t, x - y)| \leq c |t|^{-1/\alpha} |w| \left(r_y \left(t, \frac{x - y}{2}\right) + r_y \left(t, \frac{x - y + a_1(v)w}{2}\right)\right).
\]
Recall that if \( |w| \geq 2\delta \) then \( \mu(w) = 0 \). So we may assume that \( |w| \leq 2\delta \). By Corollary 3.3 we get
\[
r_y \left(t, \frac{x - y}{2}\right) + r_y \left(t, \frac{x - y + a_1(v)w}{2}\right) \leq c_1 t^{d/\alpha} e^{-c|x-y|}.
\]
Now (87) and (88) follow by the fact that \( \mu(w) \leq c_1 |\mathbb{B}_{2\delta}| |w|^{-1-\alpha} \). \( \square \)

Lemma 3.16. Let \( \tau_2 > \tau_1 > 0 \) and assume that a function \( f_t(x) \) is bounded and uniformly continuous on \( [\tau_1, \tau_2] \times \mathbb{R}^d \). Then
\[
\sup_{t \in [\tau_1, \tau_2], x \in \mathbb{R}^d} \left| \int_{\mathbb{R}^d} p_y(\varepsilon_1, x - y) f_t(y) \, dy - f_t(x) \right| \to 0 \quad \text{as} \quad \varepsilon_1 \to 0^+.
\]

Proof. The lemma follows easily by Proposition 3.11. \( \square \)

For any \( t > 0 \), \( x, y \in \mathbb{R}^d \) we define
\[
\varphi_y(t, x) = \int_{0}^{t} \int_{\mathbb{R}^d} p_z(t - s, x - z) q(s, z, y) \, dz \, ds.
\]
Clearly we have
\[
u(t, x, y) = p_y(t, x - y) + \varphi_y(t, x).
\]
For any \( t > 0 \), \( x, y \in \mathbb{R}^d \), \( f \in \mathcal{B}_b(\mathbb{R}^d) \) we define
\[
\Phi_t f(x) = \int_{\mathbb{R}^d} \varphi_y(t, x) f(y) \, dy,
\]
\[
U_t f(x) = \int_{\mathbb{R}^d} u(t, x, y) f(y) \, dy.
\]
\[
Q_t f(x) = \int_{\mathbb{R}^d} q(t, x, y) f(y) \, dy.
\]
Now following ideas from [25] we will define the so-called approximate solutions. For any \( t \geq 0 \), \( \xi \in [0, 1] \), \( t + \xi > 0 \), \( x, y \in \mathbb{R}^d \) we define
\[
\varphi_y^{(\xi)}(t, x) = \int_{0}^{t} \int_{\mathbb{R}^d} p_z(t - s + \xi, x - z) q(s, z, y) \, dz \, ds
\]
and
\[
u^{(\xi)}(t, x, y) = p_y(t + \xi, x - y) + \varphi_y^{(\xi)}(t, x).
\]
For any $t \geq 0$, $\xi \in [0, 1]$, $t + \xi > 0$, $x, y \in \mathbb{R}^d$, $f \in \mathcal{B}_b(\mathbb{R}^d)$ we define

$$\Phi_t^{(\xi)} f(x) = \int_{\mathbb{R}^d} \varphi_y^{(\xi)}(t, x) f(y) \, dy,$$

$$U_t^{(\xi)} f(x) = \int_{\mathbb{R}^d} u^{(\xi)}(t, x) f(y) \, dy,$$

$$\Phi_0 f(x) = 0, \quad U_0^{(0)} f(x) = U_0 f(x) = f(x).$$

By the same arguments as Corollary 3.14 we obtain the following result.

**Corollary 3.17.** For any $t \in [0, \infty)$, $\xi \in [0, 1]$, $t + \xi > 0$, $x, y \in \mathbb{R}^d$ the kernel $u^{(\xi)}(t, x, y)$ is well defined. For any $t \in (0, \tau]$, $\xi \in [0, 1]$, $x, y \in \mathbb{R}^d$ we have

$$|u^{(\xi)}(t, x, y)| \leq \frac{c}{(t + \xi)^{d/\alpha}(1 + |x - y|)^{d+1}}.$$

For any $t \in (0, \tau]$, $\xi \in [0, 1]$ and $x \in \mathbb{R}^d$ we have

$$\int_{\mathbb{R}^d} |u^{(\xi)}(t, x, y)| \, dy \leq c,$$

$$\int_{\mathbb{R}^d} |u^{(\xi)}(t, x, y)| \, dy \leq c.$$

**Lemma 3.18.** Let $f \in C_0(\mathbb{R}^d)$ and $\tau \geq \tau_2 > \tau_1 > 0$. Then $Q_t f(x)$ as a function of $(t, x)$ is uniformly continuous on $[\tau_1, \tau_2] \times \mathbb{R}^d$. We have $\lim_{|x| \to \infty} Q_t f(x) = 0$ uniformly in $t \in [\tau_1, \tau_2]$. For each $t > 0$ we have $Q_t f \in C_0(\mathbb{R}^d)$.

**Proof.** For any $\zeta > 0$, $y \in \mathbb{R}^d$ by Lemma 3.14 we obtain that

$$(t, x) \to \mathcal{L}_\zeta y p_y(t, \cdot)(x - y) - \mathcal{L}_\zeta x p_x(t, \cdot)(x - y)$$

is continuous on $(0, \infty) \times \mathbb{R}^d$. Using this and (88) we obtain that

$$(t, x) \to q_0(t, x, y)$$

is continuous on $(0, \infty) \times \mathbb{R}^d$. (89)

By Proposition 3.10 we have

$$|q_0(t, x, y)| \leq \frac{c}{t^{1+d/\alpha}} e^{-c_1 |x - y|}.$$  (90)

For any $n \in \mathbb{N}$, $t > 0$, $x \in \mathbb{R}^d$ denote

$$Q_{n, t} f(x) = \int_{\mathbb{R}^d} q_n(t, x, y) f(y) \, dy.$$  

By (89), (90) and the dominated convergence theorem we obtain that $(t, x) \to Q_{0, t} f(x)$ is continuous on $(0, \infty) \times \mathbb{R}^d$. By Lemma 3.12 for any $t \in (0, \tau]$, $x \in \mathbb{R}^d$, $n \in \mathbb{N}$ we have

$$|Q_{n, t} f(x)| \leq \frac{c_1^{n+1} (n+1)^{2-1}}{(n!)^{1/2}} ||f||_\infty.$$  (91)

Note that for any $t > 0$, $x \in \mathbb{R}^d$, $n \in \mathbb{N}$, $n \geq 1$ we have

$$Q_{n, t} f(x) = \int_0^t \int_{\mathbb{R}^d} q_0(t - s, x, z) Q_{n-1, s} f(z) \, dz \, ds.$$  

For any $\varepsilon_1 \in (0, \tau_1/2)$ using (89), (90) and (91) we obtain that

$$(t, x) \to \int_0^{t-\varepsilon_1} \int_{\mathbb{R}^d} q_0(t - s, x, z) Q_{n-1, s} f(z) \, dz \, ds$$
is continuous on \([\tau_1, \tau_2] \times \mathbb{R}^d\). Note also that for any \(\varepsilon_1 \in (0, \tau_1/2)\), \(t \in [\tau_1, \tau_2]\), \(x \in \mathbb{R}^d\), \(n \in \mathbb{N}\), \(n \geq 1\) we have

\[
\left| \int_{t-\varepsilon_1}^{t} \int_{\mathbb{R}^d} g_0(t-s, x, z) Q_{n-1, s} f(z) \, dz \, ds \right| \leq c\|f\|_{\infty} \int_{t-\varepsilon_1}^{t} (t-s)^{-1/2} s^{-1/2} \, ds
\]

\[
\leq c t^{-1/2} \varepsilon_1^{1/2} \|f\|_{\infty}.
\]

This implies that \((t, x) \rightarrow Q_{n, t} f(x)\) is continuous on \([\tau_1, \tau_2] \times \mathbb{R}^d\). Using this and (91) we obtain that \((t, x) \rightarrow Q_t f(x) = \sum_{n=0}^{\infty} Q_{n, t} f(x)\) is continuous on \([\tau_1, \tau_2] \times \mathbb{R}^d\).

By Proposition 3.13 we obtain that \(\lim_{|x| \to \infty} Q_t f(x) = 0\) uniformly in \(t \in [\tau_1, \tau_2]\).

This implies the assertion of the lemma.

**Proposition 3.19.** Choose \(\gamma \in (0, \alpha)\). For any \(t \in (0, \tau]\), \(x, x' \in \mathbb{R}^d\), \(f \in \mathcal{B}_b(\mathbb{R}^d)\) we have

\[
|U_t f(x) - U_t f(x')| \leq c t^{-\gamma/\alpha} |x - x'|^\gamma \|f\|_{\infty}.
\]

**Proof.** We have

\[
U_t f(x) - U_t f(x') = \int_{\mathbb{R}^d} (p_y(t, x - y) - p_y(t, x' - y)) f(y) \, dy
\]

\[
+ \int_{0}^{t} \int_{\mathbb{R}^d} (p_z(t-s, x - z) - p_z(t-s, x' - z)) Q_s f(z) \, dz \, ds
\]

\[
= I + II.
\]

By Lemma 3.11 and Proposition 3.11 we get

\[
|I| \leq c\|f\|_{\infty} |x - x'|^\gamma t^{-\gamma/\alpha} \int_{\mathbb{R}^d} (r_y(t, x-y)/2 + r_y(t, x'-y)/2) \, dy \leq c\|f\|_{\infty} |x - x'|^\gamma t^{-\gamma/\alpha}.
\]

By Lemma 3.11 and Propositions 3.11, 3.13 we obtain

\[
|II| \leq c\|f\|_{\infty} |x - x'|^\gamma \int_{0}^{t} \int_{\mathbb{R}^d} (t-s)^{-\gamma/\alpha} (r_z(t-s, x-z)/2 + r_z(t-s, x'-z)/2) s^{-1/2} \, dz \, ds
\]

\[
\leq c\|f\|_{\infty} |x - x'|^\gamma \int_{0}^{t} (t-s)^{-\gamma/\alpha} s^{-1/2} \, ds
\]

\[
\leq c\|f\|_{\infty} |x - x'|^\gamma t^{1/2-\gamma/\alpha}.
\]

□

Note that by Lemma 3.15 for any \(x \in (0, 1]\), \(t \in [\xi, \tau + \xi]\), \(x, z \in \mathbb{R}^d\) we have

\[
\left| \frac{\partial p_z(t, x-z)}{t} \right| = |\mathcal{L} p_z(t, \cdot)(x-z)| \leq c(\xi)e^{-c|x-z|},
\]

where \(c(\xi)\) is a constant depending on \(\xi, \tau, \alpha, d, \eta_1, \eta_2, \eta_3, \varepsilon, \delta\).

The next lemma is similar to [25, Lemma 4.1].

**Lemma 3.20.** (i) For every \(f \in C_0(\mathbb{R}^d)\), \(x \in (0, 1]\) the function \(U_t^{(\xi)} f(x)\) belongs to \(C^1((0, \infty))\) as a function of \(t\) and to \(C^2_0(\mathbb{R}^d)\) as a function of \(x\). Moreover we have

\[
\left| \frac{\partial}{\partial t} (U_t^{(\xi)} f)(x) \right| \leq c(\xi)\|f\|_{\infty},
\]

for each \(f \in C_0(\mathbb{R}^d)\), \(t \in (0, \tau]\), \(x \in \mathbb{R}^d\), \(\xi \in (0, 1]\), where \(c(\xi)\) depends on \(\xi, \tau, \alpha, d, \eta_1, \eta_2, \eta_3\).
(ii) For every \( f \in C_0(\mathbb{R}^d) \) we have
\[
\lim_{t, \xi \to 0^+} \|U_t^{(\xi)} f - f\|_\infty = 0.
\]

(iii) For every \( f \in C_0(\mathbb{R}^d) \) we have
\[
U_t^{(\xi)} f(x) \to 0, \quad \text{as} \quad |x| \to \infty,
\]
uniformly in \( t \in [0, \tau], \xi \in [0, 1] \).

(iv) For every \( f \in C_0(\mathbb{R}^d) \) we have
\[
\|U_t^{(\xi)} f - U_t f\|_\infty \to 0, \quad \text{as} \quad \xi \to 0^+,
\]
uniformly in \( t \in [0, \tau] \).

Proof. (i) Let \( f \in C_0(\mathbb{R}^d), t \in (0, \tau], \xi \in (0, 1] \) and \( x \in \mathbb{R}^d \). We have
\[
\lim_{h \to 0^+} \frac{\Phi_{t+h}^{(\xi)} f(x) - \Phi_t^{(\xi)} f(x)}{h} = \lim_{h \to 0^+} \frac{1}{h} \int_t^{t+h} \int_{\mathbb{R}^d} p_z(t + h - s + \xi, x - z) Q_s f(z) \, dz \, ds
\]
\[
+ \lim_{h \to 0^+} \frac{1}{h} \int_0^t \int_{\mathbb{R}^d} \frac{p_z(t + h - s + \xi, x - z) - p_z(t - s + \xi, x - z)}{h} Q_s f(z) \, dz \, ds
\]
\[
= \text{I} + \text{II}.
\]

By Lemmas 3.3, 3.18 Corollary 3.3 and Proposition 3.13 we get
\[
\text{I} = \int_{\mathbb{R}^d} p_z(\xi, x - z) Q_t f(z) \, dz.
\]

By Lemma 3.4 the dominated convergence theorem, (92) and Proposition 3.13 we get
\[
\text{II} = \int_0^t \int_{\mathbb{R}^d} \frac{\partial p_z(t - s + \xi, x - z)}{\partial t} Q_s f(z) \, dz \, ds.
\]

By similar arguments we get the analogous result for \( \lim_{h \to 0^+} \left( \Phi_{t+h}^{(\xi)} f(x) - \Phi_t^{(\xi)} f(x) \right) /h \).

By (92) we get
\[
\frac{\partial}{\partial t} \int_{\mathbb{R}^d} p_z(t + \xi, x - z) f(z) \, dz = \int_{\mathbb{R}^d} \frac{\partial}{\partial t} p_z(t + \xi, x - z) f(z) \, dz.
\]

Hence we have
\[
\frac{\partial}{\partial t} (U_t^{(\xi)} f)(x) = \int_{\mathbb{R}^d} \frac{\partial}{\partial t} p_z(t + \xi, x - z) f(z) \, dz
\]
\[
+ \int_{\mathbb{R}^d} p_z(\xi, x - z) Q_t f(z) \, dz
\]
\[
+ \int_0^t \int_{\mathbb{R}^d} \frac{\partial p_z(t - s + \xi, x - z)}{\partial t} Q_s f(z) \, dz \, ds \quad (94)
\]

Using this, (92), Propositions 3.11 and 3.13 we obtain (93). We also obtain that for every \( f \in C_0(\mathbb{R}^d), \xi \in (0, 1] \) the function \( U_t^{(\xi)} f(x) \) belongs to \( C^1((0, \infty)) \) as a function of \( t \).

The fact that \( U_t^{(\xi)} f \in C_0^2(\mathbb{R}^d) \) for \( \xi \in (0, 1] \) follows by Lemmas 3.4, 3.5, Proposition 3.13 and Lemma 3.18.
(ii) Fix \( f \in C_0(\mathbb{R}^d) \). For any \( \xi \in [0, 1] \), \( t \geq 0 \), \( t + \xi > 0 \), \( x \in \mathbb{R}^d \) we have

\[
U_t^{(\xi)}(x) = \int_{\mathbb{R}^d} p_y(t + \xi, x - y) f(y) \, dy + \Phi_t^{(\xi)}(x).
\]

For any \( \xi \in [0, 1] \), \( t \in [0, \tau] \), \( \xi + t > 0 \), \( x \in \mathbb{R}^d \) by Proposition 3.13 and Proposition 3.11 we get

\[
\left| \Phi_t^{(\xi)}(x) \right| \leq c \| f \|_{\infty} \int_0^t s^{-1/2} \, ds \leq c \| f \|_{\infty} t^{1/2}.
\]  

(95)

By (67), Proposition 3.11 and the fact that \( f \) is uniformly continuous on \( \mathbb{R}^d \) we obtain

\[
\lim_{t, \xi \to 0^+} \int_{\mathbb{R}^d} p_y(t + \xi, x - y) f(y) \, dy - f(x) = 0
\]

uniformly with respect to \( x \in \mathbb{R}^d \). This and (95) gives (ii).

(iii) This follows easily from (ii) and Corollary 3.17.

(iv) Fix \( f \in C_0(\mathbb{R}^d) \). By Lemma 3.4, Corollary 3.3 and the dominated convergence theorem we obtain that

\[
(t, x) \to \int_{\mathbb{R}^d} p_y(t, x - y) f(y) \, dy
\]

is continuous on \((0, \tau + 1] \times \mathbb{R}^d\). It follows that

\[
(\xi, t, x) \to \int_{\mathbb{R}^d} p_y(t + \xi, x - y) f(y) \, dy
\]

is continuous on \([0, 1] \times (0, \tau] \times \mathbb{R}^d\). Using Lemma 3.4 Corollary 3.3 Proposition 3.13 and the dominated convergence theorem we obtain that for any \( s \in (0, \tau) \)

\[
(\xi, t, x) \to \int_{\mathbb{R}^d} p_z(t + \xi - s, x - z) Q_s f(z) \, dz
\]

is continuous on \([0, 1] \times (s, \tau] \times \mathbb{R}^d\). Using this, Corollary 3.3 Proposition 3.13 and the dominated convergence theorem we obtain that

\[
(\xi, t, x) \to \Phi_t^{(\xi)}(x) = \int_0^\tau 1_{(0, t]}(s) \int_{\mathbb{R}^d} p_z(t + \xi - s, x - z) Q_s f(z) \, dz \, ds
\]

is continuous on \([0, 1] \times (0, \tau] \times \mathbb{R}^d\). Hence \((\xi, t, x) \to U_t^{(\xi)} f(x)\) is continuous on \([0, 1] \times (0, \tau] \times \mathbb{R}^d\). Using (95) we obtain that

\[
(\xi, t, x) \to U_t^{(\xi)} f(x) \quad \text{is continuous on } [0, 1] \times [0, \tau] \times \mathbb{R}^d.
\]  

(96)

This and (iii) implies (iv).

□

By the same arguments as in the proof of Lemma 3.20 (iv) we obtain the following result.

**Lemma 3.21.** For any \( f \in \mathcal{B}_b(\mathbb{R}^d) \) the function \((t, x) \to U_t f(x)\) is continuous on \((0, \infty) \times \mathbb{R}^d\). For any \( \xi \in (0, 1] \), \( f \in \mathcal{B}_b(\mathbb{R}^d) \) the function \((t, x) \to U_t^{(\xi)} f(x)\) is continuous on \([0, \infty) \times \mathbb{R}^d\).

Heuristically, now our aim is to show that if \( \xi \) is small then \( \frac{\partial}{\partial t}(U_t^{(\xi)} f)(x) - \mathcal{L}(U_t^{(\xi)} f)(x) \) is small. For any \( t > 0 \), \( \xi \in (0, 1] \), \( x \in \mathbb{R}^d \) we put

\[
\Lambda_t^{(\xi)} f(x) = \frac{\partial}{\partial t}(U_t^{(\xi)} f)(x) - \mathcal{L}(U_t^{(\xi)} f)(x).
\]
Lemma 3.22. \( \mathcal{L}(U^\xi_1 f)(x) \) is well defined for every \( f \in C_0(\mathbb{R}^d) \), \( t \in (0, \tau] \), \( \xi \in (0, 1] \) and \( x \in \mathbb{R}^d \) and we have

\[
\Lambda_t^\xi f(x) = \int_{\mathbb{R}^d} p_z(\xi, x-z)Q_t f(z) \, dz - Q_{t+\xi} f(x) + \int_t^{t+\xi} \int_{\mathbb{R}^d} q_0(t-s+\xi, x, z)Q_s f(z) \, dz \, ds.
\]  

Moreover we have

\[
\left| \mathcal{L}_\xi(U^\xi_1 f)(x) \right| \leq c(\xi)\|f\|_{\infty}, \quad \xi > 0,
\]

\[
\left| \mathcal{L}(U^\xi_1 f)(x) \right| \leq c(\xi)\|f\|_{\infty}.
\]

for each \( f \in C_0(\mathbb{R}^d) \), \( x \in \mathbb{R}^d \), \( t \in (0, \tau] \), \( \xi \in (0, 1] \), where \( c(\xi) \) is a constant depending on \( \xi, \tau, \alpha, d, \eta_1, \eta_2, \eta_3, \varepsilon, \delta \).

Proof. Let \( f \in C_0(\mathbb{R}^d) \), \( t \in (0, \tau] \), \( \xi \in (0, 1] \), \( x \in \mathbb{R}^d \) and \( \xi > 0 \). We have

\[
\mathcal{L}_\xi(U^\xi_1 f)(x) = \int_{\mathbb{R}^d} \mathcal{L}_\xi^\xi p_{t}(t + \xi, \cdot)(x-z) f(z) \, dz
\]

\[
+ \int_0^t \int_{\mathbb{R}^d} \mathcal{L}_\xi^\xi p_{t-s+\xi}(t-s+\xi, \cdot)(x-z) Q_s f(z) \, dz \, ds.
\]  

Using this, Lemma 3.15 and Proposition 3.13 we obtain (98). By (100), the dominated convergence theorem, Lemma 3.15 and Proposition 3.13 one gets

\[
\mathcal{L}(U^\xi_1 f)(x) = \lim_{\xi \to 0^+} \mathcal{L}_\xi(U^\xi_1 f)(x)
\]

\[
= \int_{\mathbb{R}^d} \mathcal{L}_\xi^\xi p_{t}(t + \xi, \cdot)(x-z) f(z) \, dz
\]

\[
+ \int_0^t \int_{\mathbb{R}^d} \mathcal{L}_\xi^\xi p_{t-s+\xi}(t-s+\xi, \cdot)(x-z) Q_s f(z) \, dz \, ds.
\]  

Using this and again Lemma 3.15 and Proposition 3.13 we obtain (99). Note that for \( s \in [0, t] \), \( z \in \mathbb{R}^d \) we have

\[
\frac{\partial p_t(t-s+\xi, x-z)}{\partial t} - \mathcal{L}_\xi^\xi p_{t}(t-s+\xi, \cdot)(x-z) = -q_0(t-s+\xi, x, z).
\]

Using this, (94) and (101) we get

\[
\Lambda_t^\xi f(x) = \int_{\mathbb{R}^d} p_z(\xi, x-z)Q_t f(z) \, dz
\]

\[
- \int_{\mathbb{R}^d} q_0(t + \xi, x, z) f(z) \, dz
\]

\[
- \int_0^t \int_{\mathbb{R}^d} q_0(t-s+\xi, x, z)Q_s f(z) \, dz \, ds.
\]  

For \( \xi \in (0, 1] \), \( t \in (0, \tau] \), \( x \in \mathbb{R}^d \) by the definition of \( q(t, x, y) \) we obtain

\[
\int_{\mathbb{R}^d} q_0(t + \xi, x, z) f(z) \, dz = Q_{t+\xi} f(x) - \int_0^{t+\xi} \int_{\mathbb{R}^d} q_0(t-s+\xi, x, z)Q_s f(z) \, dz \, ds.
\]  

Using this and (102) we obtain (97). \( \square \)

The next lemma is similar to [25, Lemma 4.2].
Lemma 3.23. (i) For any $f \in C_0(\mathbb{R}^d)$ we have
\[
\Lambda_t^{(\xi)} f(x) \to 0, \quad \text{as} \quad \xi \to 0^+,
\]
uniformly in $(t, x) \in [\tau_1, \tau_2] \times \mathbb{R}^d$ for every $\tau \geq \tau_2 > \tau_1 > 0$.

(ii) For any $f \in C_0(\mathbb{R}^d)$ we have
\[
\int_0^t \Lambda_s^{(\xi)} f(x) \, ds \to 0, \quad \text{as} \quad \xi \to 0^+,
\]
uniformly in $(t, x) \in (0, \tau) \times \mathbb{R}^d$.

Proof. Let $f \in C_0(\mathbb{R}^d)$ and $0 < \tau_1 < \tau_2 \leq \tau$. For any $t > 0$, $x \in \mathbb{R}^d$, $\xi \in (0, 1]$ we put
\[
\Lambda_t^{(\xi,1)} f(x) = \int_{\mathbb{R}^d} p_s(\xi, x - z) Q_t f(z) \, dz - Q_{t+\xi} f(x).
\]
\[
\Lambda_t^{(\xi,2)} f(x) = \int_{t}^{t+\xi} \int_{\mathbb{R}^d} q_0(t - s + \xi, x, z) Q_s f(z) \, dz \, ds.
\]
By Lemma 3.18 we get
\[
\sup_{t \in [\tau_1, \tau_2], x \in \mathbb{R}^d} |Q_{t+\xi} f(x) - Q_t f(x)| \to 0 \quad \text{as} \quad \xi \to 0^+.
\]
By Lemmas 3.16 and 3.18 we obtain
\[
\sup_{t \in [\tau_1, \tau_2], x \in \mathbb{R}^d} \left| \int_{\mathbb{R}^d} p_s(\xi, x - z) Q_t f(z) \, dz - Q_t f(x) \right| \to 0 \quad \text{as} \quad \xi \to 0^+.
\]
This gives (i) for $\Lambda_t^{(\xi,1)} f(x)$ instead of $\Lambda_t^{(\xi)} f(x)$.

By Proposition 3.13 for any $t \in (0, \tau]$, $x \in \mathbb{R}^d$, $\xi \in (0, 1]$ we get
\[
\left| \Lambda_t^{(\xi,1)} f(x) \right| \leq c \| f \|_\infty t^{-1/2}. \tag{104}
\]
This allows to use the dominated convergence theorem in the integral (103) with $\Lambda_t^{(\xi)} f(x)$ replaced by $\Lambda_t^{(\xi,1)} f(x)$. So (ii) for $\Lambda_t^{(\xi,1)} f(x)$ follows from (i) for $\Lambda_t^{(\xi,1)} f(x)$.

For any $t \in (0, \tau]$, $x \in \mathbb{R}^d$, $\xi \in (0, 1]$ by Propositions 3.10 and 3.13 we get
\[
\left| \Lambda_t^{(\xi,2)} f(x) \right| \leq c \| f \|_\infty \int_{t}^{t+\xi} ((t - s + \xi)s)^{-1/2} \, ds. \tag{105}
\]
This implies (i) and (ii) for $\Lambda_t^{(\xi,2)} f(x)$. \qed

4. Construction and Properties of the Semigroup of $X_t$

In this section we will construct the semigroup corresponding to the solution of (1). This will be done by, heuristically speaking, adding the impact of long jumps to the semigroup $U_t$, constructed in the last section, corresponding to the solution of (1) in which the process $Z_t$ is replaced by the process with truncated Lévy measure. From technical point of view we will construct the semigroup $T_t$ from the semigroup $U_t$, then we will show that $T_t f$ for $f \in C_0(\mathbb{R}^d)$ satisfies the appropriate heat equation and using this we will show that indeed $T_t = P_t$, where $P_t f(x) = \mathbb{E}^x f(X_t)$ is defined in (2). Finally, we will prove Theorems 1.1, 1.3 and Proposition 1.2. The construction of the semigroup $T_t$ is rather standard. It is similar to the construction made in [91]. Some proofs are similar to the analogous proofs in [31]. Such proofs will be omitted.
Let us introduce the following notation
\[ \lambda = \sum_{i=1}^{d} \int_{\mathbb{R}} (\nu_i(x) - \mu_i(x)) \, dx < \infty. \]

Note that by (111), for any \( x \in \mathbb{R}^d \) and \( f \in \mathcal{B}_b(\mathbb{R}^d) \), we have
\[ Rf(x) = \sum_{i=1}^{d} \int_{\mathbb{R}} (f(x + a_i(x)w) - f(x)) (\nu_i(w) - \mu_i(w)) \, dw. \]

We denote, for any \( x \in \mathbb{R}^d \) and \( f \in \mathcal{B}_b(\mathbb{R}^d) \),
\[ Nf(x) = \sum_{i=1}^{d} \int_{\mathbb{R}} f(x + a_i(x)w) (\nu_i(w) - \mu_i(w)) \, dw. \]

It is clear that
\[ ||Nf||_{\infty} \leq \lambda ||f||_{\infty}. \tag{106} \]

For any \( t \geq 0, x \in \mathbb{R}^d \) and \( n \in \mathbb{N}, n \geq 1, f \in \mathcal{B}_b(\mathbb{R}^d) \) we define
\[ \Psi_{0,t}f(x) = U_{t}f(x), \tag{107} \]
\[ \Psi_{n,t}f(x) = \int_{0}^{t} U_{t-s}(N(\Psi_{n-1,s}f))(x) \, ds, \quad n \geq 1. \tag{108} \]

For any \( t \geq 0, \xi \in [0, 1], x \in \mathbb{R}^d \) and \( n \in \mathbb{N}, f \in \mathcal{B}_b(\mathbb{R}^d) \) we define
\[ \Psi_{0,t}^{(\xi)}f(x) = U_{t}^{(\xi)}f(x), \tag{109} \]
\[ \Psi_{n,t}^{(\xi)}f(x) = \int_{0}^{t} U_{t-s}^{(\xi)}(N(\Psi_{n-1,s}^{(\xi)}f))(x) \, ds, \quad n \geq 1. \tag{110} \]

We remark that
\[ \Psi_{n,t} = \Psi_{n,t}^{(0)}. \]

**Lemma 4.1.** \( \Psi_{n,t}f(x) \) and \( \Psi_{n,t}^{(\xi)}f(x) \) are well defined for any \( t > 0, f \in \mathcal{B}_b(\mathbb{R}^d) \), \( x \in \mathbb{R}^d, n \in \mathbb{N} \) and \( \xi \in [0, 1] \). For any \( f \in \mathcal{B}_b(\mathbb{R}^d) \), \( x \in \mathbb{R}^d, n \in \mathbb{N} \) we have
\[ |\Psi_{n,t}f(x)| \leq \frac{c_{n+1}f^{n}}{n!} ||f||_{\infty}, \quad t \in (0, \tau], \tag{111} \]
\[ |\Psi_{n,t}^{(\xi)}f(x)| \leq \frac{c_{n+1}f^{n}}{n!} ||f||_{\infty}, \quad \xi \in (0, 1], t \in [0, \tau]. \tag{112} \]

**Proof.** We will only show the result for \( \Psi_{n,t}f(x) \) using the induction. The proof for \( \Psi_{n,t}^{(\xi)}f(x) \) is almost the same.

Let \( c \) be the constant from (83) and put \( c_{1} = (\lambda \vee 1)c. \) For \( n = 0 \) (111) follows from (83). Assume that (111) holds for \( n \geq 0, \) we will show it for \( n + 1. \) Indeed, applying (83) and (106), we get
\[ |\Psi_{n+1,t}f(x)| \leq \int_{0}^{t} \int_{\mathbb{R}^d} |u(t-s, x, z)| dz \frac{c_{n+1}s^n}{n!} \, ds \leq \frac{c_{n+2}t^{n+1}}{(n+1)!}. \]

\( \square \)
For any $x \in \mathbb{R}^d$ we define

$$T_t f(x) = e^{-\lambda t} \sum_{n=0}^{\infty} \Psi_{n,t} f(x), \quad t > 0,$$
$$T_0 f(x) = f(x),$$
$$T_t^{(\xi)} f(x) = e^{-\lambda t} \sum_{n=0}^{\infty} \Psi_{n,t}^{(\xi)} f(x), \quad t \geq 0, \; \xi \in [0,1].$$

Our ultimate aim will be to show that for any $t > 0$ we have $T_t = P_t$, where $P_t$ is given by [5].

By Lemma 4.1 we obtain

**Corollary 4.2.** $T_t f(x)$ and $T_t^{(\xi)} f(x)$ are well defined for any $t \geq 0$, $f \in \mathcal{B}_b(\mathbb{R}^d)$, $x \in \mathbb{R}^d$, $\xi \in [0,1]$ and for $t \in [0,\tau]$ we have $\max\{|T_t f(x)|,|T_t^{(\xi)} f(x)|\} \leq c \|f\|_{\infty}$.

Next, we obtain the following regularity results concerning operators $T_t$.

**Theorem 4.3.** For any $\gamma \in (0, \alpha/(d + \beta + 1 - \alpha))$, $t \in (0,\tau)$, $x \in \mathbb{R}^d$ and $f \in L^1(\mathbb{R}^d) \cap L^{\infty}(\mathbb{R}^d)$ we have

$$|T_t f(x)| \leq ct^{-\gamma(d + \beta + 1 - \alpha)/\alpha} \|f\|_{\infty}^{1-\gamma} \|f\|_{1}^{\gamma}.$$

**Proof.** For any $t \in (0,\tau)$, $x \in \mathbb{R}^d$ by Corollary 3.14 we get $|U_t f(x)| \leq c \|f\|_{\infty}$, $|U_t f(x)| \leq ct^{-\gamma(d + \beta + 1 - \alpha)/\alpha} \|f\|_{1}$, $\gamma \in (0, \alpha/(d + \beta + 1 - \alpha))$. It follows that for any $t \in (0,\tau)$, $x \in \mathbb{R}^d$ we have $|U_t f(x)| \leq ct^{-\gamma(d + \beta + 1 - \alpha)/\alpha} \|f\|_{\infty}^{1-\gamma} \|f\|_{1}^{\gamma}$. Hence $|\Psi_{t,x} f(x)| \leq c \|f\|_{\infty}^{1-\gamma} \|f\|_{1}^{\gamma}$. Using the same arguments as in Lemma 4.1 for any $t \in (0,\tau)$, $x \in \mathbb{R}^d$, $n \in \mathbb{N}$, $n \geq 1$ one gets $|\Psi_{n,t} f(x)| \leq c^n t^{n-1} \|f\|_{\infty}^{1-\gamma} \|f\|_{1}^{\gamma}/(n-1)!$, which implies the assertion of the theorem.

**Theorem 4.4.** Choose $\gamma \in (0, \alpha)$. For any $t \in (0,\tau)$, $x, x' \in \mathbb{R}^d$, $f \in \mathcal{B}_b(\mathbb{R}^d)$ we have

$$|T_t f(x) - T_t f(x')| \leq ct^{-\gamma/\alpha} \|x - x'|\|f\|_{\infty}.$$

The proof of this result is similar to the proof of Theorem 4.4 in [31] and it is omitted.

Clearly, we have, by applying [83] and [106], the following lemma.

**Lemma 4.5.** There exists $a \geq 1$ such that for any $f \in \mathcal{B}_b(\mathbb{R}^d)$, $x \in \mathbb{R}^d$, $\xi \in [0,1]$, $t \in (0,\tau)$ we have

$$\left| \int_0^t U_{t-s}^{(\xi)}(\mathcal{N}(f))(x) \, ds \right| \leq a \|f\|_{\infty},$$

where $a$ depends on $\tau, \alpha, d, \eta_1, \eta_2, \eta_3$.

**Lemma 4.6.** Assume that $f \in \mathcal{B}_b(\mathbb{R}^d)$. Put $\gamma_1 = \alpha(1-\sigma)/(2d+2\beta+2)$. For any $t \in (0,\tau)$, $x \in \mathbb{R}^d$, we have

$$|Q_t f(x)| \leq \frac{c \|f\|_{\infty}}{t^{(1+\sigma)/2}(\text{dist}(x, \text{supp}(f)) + 1)^{\gamma_1}}.$$

**Proof.** Let $t \in (0,\tau)$ be arbitrary. By Proposition 3.13 we get for $x \in \mathbb{R}^d$

$$|Q_t f(x)| \leq ct^{-\sigma} \|f\|_{\infty},$$
$$|Q_t f(x)| \leq \frac{c \|f\|_{\infty}}{t^{(d+\beta+1)/\alpha}(\text{dist}(x, \text{supp}(f)) + 1)}.$$
It follows that
\[ |Q tf(x)|^{1-\gamma_i} \leq ct^{(-\sigma)(1-\gamma_i)}\|f\|^{1-\gamma_i} \leq ct^{-\sigma}\|f\|^{1-\gamma_i}, \]
\[ |Q tf(x)|^{\gamma_i} \leq \frac{c\|f\|_{\infty}^{\gamma_i}}{t^{(1-\sigma)/2}(\text{dist}(x, \text{supp}(f)) + 1)^{\gamma_i}}. \]

This implies the assertion of the lemma. \qed

**Lemma 4.7.** Assume that \( f \in B_b(R^d) \). For any \( \varepsilon_1 > 0 \) there exists \( r \geq 1 \) (depending on \( \varepsilon_1, \tau, \alpha, d, \eta_1, \eta_2, \eta_3 \)) such that, for any \( x \in [0, 1] \), \( t \in [0, \tau] \), \( x \in R^d \), if \( \text{dist}(x, \text{supp}(f)) \geq r \) then \( \left| U_t^{(\xi)} f(x) \right| \leq \varepsilon_1 \|f\|_{\infty} \).

The proof of this result is similar to the proof of Lemma 4.7 in [31] and it is omitted.

The proof of the next lemma is standard and it is also omitted.

**Lemma 4.8.** Assume that \( f \in B_b(R^d) \). For any \( \varepsilon_1 > 0 \) there exists \( r \geq 1 \) (depending on \( \varepsilon_1, \tau, \alpha, d, \eta_1, \eta_2, \eta_3 \)) such that, for any \( x \in \mathbb{R}^d \), if \( \text{dist}(x, \text{supp}(f)) \geq r \), then \( \|N f(x)\| \leq \varepsilon_1 \|f\|_{\infty} \).

**Lemma 4.9.** Assume that \( f \in B_b(R^d) \). For any \( \varepsilon_1 > 0 \) there exists \( r \geq 1 \) (depending on \( \varepsilon_1, \tau, \alpha, d, \eta_1, \eta_2, \eta_3 \)) such that, for any \( x \in [0, 1] \), \( t \in [0, \tau] \), \( x \in \mathbb{R}^d \), if \( \text{dist}(x, \text{supp}(f)) \geq r \) then \( \left| \int_0^t U_{t-s}^{(\xi)}(N(f))(x) \, ds \right| \leq \varepsilon_1 \|f\|_{\infty} \).

The proof of this result is similar to the proof of Lemma 4.9 in [31] and it is omitted.

**Lemma 4.10.** Assume that \( f \in B_b(R^d) \). For any \( \varepsilon_1 > 0 \) there exists \( r \geq 1 \) (depending on \( \varepsilon_1, \tau, \alpha, d, \eta_1, \eta_2, \eta_3 \)), such that, for any \( x \in [0, 1] \), \( t \in [0, \tau] \), \( x \in \mathbb{R}^d \), if \( \text{dist}(x, \text{supp}(f)) \geq r \), then \( |T_t^{(\xi)} f(x)| \leq \sum_{n=0}^{\infty} |\Psi_{n, t}^{(\xi)} f(x)| \leq \varepsilon_1 \|f\|_{\infty} \).

The proof of this result is similar to the proof of Lemma 4.10 in [31] and it is omitted.

By Lemma 4.10 and Theorem 4.3 one easily obtains the following result.

**Corollary 4.11.** Assume that \( f \in B_b(R^d) \), for any \( n \in \mathbb{N} \), \( n \geq 1 \) we have \( f_n \in B_b(R^d) \), \( \sup_{n \in \mathbb{N}, n \geq 1} \|f_n\|_{\infty} < \infty \) and \( \lim_{n \to \infty} f_n(x) = f(x) \) for almost all \( x \in \mathbb{R}^d \) with respect to the Lebesgue measure. Then, for any \( t > 0 \), \( x \in \mathbb{R}^d \), we have \( \lim_{n \to \infty} T_t f_n(x) = T_t f(x) \).

**Lemma 4.12.** (i) For every \( f \in C_0(\mathbb{R}^d) \) we have
\[ \lim_{t, \xi \to 0^+} \|T_t^{(\xi)} f - f\|_{\infty} = 0. \]

(ii) For every \( f \in C_0(\mathbb{R}^d) \) we have
\[ T_t^{(\xi)} f(x) \to 0, \text{ as } |x| \to \infty, \]
uniformly in \( t \in [0, \tau] \), \( \xi \in [0, 1] \).

(iii) For every \( f \in C_0(\mathbb{R}^d) \) we have
\[ \|T_t^{(\xi)} f - T_t f\|_{\infty} \to 0, \text{ as } \xi \to 0^+, \]
uniformly in \( t \in [0, \tau] \).
The proof of this result is similar to the proof of Lemma 4.12 in [31] and it is omitted.

By the same arguments as in the proof of Lemma 4.20 (iv) we obtain the following result.

**Lemma 4.13.** For any \( f \in \mathcal{B}_b(\mathbb{R}^d), n \in \mathbb{N} \), the function \((t, x) \rightarrow \Psi_{n,t}f(x)\) is continuous on \((0, \infty) \times \mathbb{R}^d\). For any \( \xi \in (0, 1], f \in \mathcal{B}_b(\mathbb{R}^d), n \in \mathbb{N} \), the function \((t, x) \rightarrow \Psi^{(\xi)}_{n,t}f(x)\) is continuous on \([0, \infty) \times \mathbb{R}^d\).

**Lemma 4.14.** \( \frac{\partial}{\partial t} \left( \Psi^{(\xi)}_{n,t}f \right)(x), \mathcal{L} \left( \Psi^{(\xi)}_{n,t}f \right)(x) \) are well defined for any \( t > 0, \xi \in (0, 1], x \in \mathbb{R}^d, n \in \mathbb{N}, n \geq 1 \) and \( f \in C_0(\mathbb{R}^d) \) and we have

\[
\frac{\partial}{\partial t} \left( \Psi^{(\xi)}_{n,t}f \right)(x) - \mathcal{L} \left( \Psi^{(\xi)}_{n,t}f \right)(x) = \int_{\mathbb{R}^d} p_z(\xi, x - z) \mathcal{N} \left( \Psi^{(\xi)}_{n-1,t}f \right)(z) \, dz + \int_0^t \Lambda^{(\xi)}_{t-s} \left( \mathcal{N} \left( \Psi^{(\xi)}_{n-1,s}f \right) \right)(x) \, ds.
\]

Moreover, \( \frac{\partial}{\partial \xi} \Psi^{(\xi)}_{n,t}f(x) \) is continuous as a function of \( t \) for \( t > 0 \).

The proof of this result is similar to the proof of Lemma 4.14 in [31] and it is omitted.

**Lemma 4.15.** For any \( t > 0, \xi \in (0, 1], x \in \mathbb{R}^d, i, j \in \{1, \ldots, d\}, k \in \mathbb{N} \) and \( f \in C_0(\mathbb{R}^d) \) we have \( \Psi^{(\xi)}_{k,t}f(x) \in C^2(\mathbb{R}^d) \) (as a function of \( x \)) and

\[
\frac{\partial}{\partial t} \left( \sum_{n=0}^{\infty} \Psi^{(\xi)}_{n,t}f \right)(x) = \sum_{n=0}^{\infty} \frac{\partial}{\partial t} \left( \Psi^{(\xi)}_{n,t}f \right)(x), \quad (113)
\]

\[
\frac{\partial}{\partial x_i} \left( \sum_{n=0}^{\infty} \Psi^{(\xi)}_{n,t}f \right)(x) = \sum_{n=0}^{\infty} \frac{\partial}{\partial x_i} \left( \Psi^{(\xi)}_{n,t}f \right)(x), \quad (114)
\]

\[
\frac{\partial^2}{\partial x_i \partial x_j} \left( \sum_{n=0}^{\infty} \Psi^{(\xi)}_{n,t}f \right)(x) = \sum_{n=0}^{\infty} \frac{\partial^2}{\partial x_i \partial x_j} \left( \Psi^{(\xi)}_{n,t}f \right)(x), \quad (115)
\]

\[
\mathcal{L} \left( \sum_{n=0}^{\infty} \Psi^{(\xi)}_{n,t}f \right)(x) = \sum_{n=0}^{\infty} \mathcal{L} \left( \Psi^{(\xi)}_{n,t}f \right)(x), \quad (116)
\]

\[
\mathcal{N} \left( \sum_{n=0}^{\infty} \Psi^{(\xi)}_{n,t}f \right)(x) = \sum_{n=0}^{\infty} \mathcal{N} \left( \Psi^{(\xi)}_{n,t}f \right)(x). \quad (117)
\]

The proof of this result is similar to the proof of Lemma 4.15 in [31] and it is omitted.

**Corollary 4.16.** For every \( f \in C_0(\mathbb{R}^d), \xi \in (0, 1] \) the function \( T^{(\xi)}_t f(x) \) belongs to \( C^1((0, \infty)) \) as a function of \( t \) and to \( C^2_0(\mathbb{R}^d) \) as a function of \( x \).

The proof of this result is similar to the proof of corollary 4.16 in [31] and it is omitted.

Heuristically, now our aim is to show that if \( \xi \) is small then \( \frac{\partial}{\partial \xi} (T^{(\xi)}_t f)(x) - \mathcal{K}(T^{(\xi)}_t f)(x) \) is small. For any \( t > 0, \xi \in (0, 1], x \in \mathbb{R}^d \) and \( f \in C_0(\mathbb{R}^d) \) let us denote

\[
\Upsilon^{(\xi)}_t f(x) = \frac{\partial}{\partial \xi} \left( T^{(\xi)}_t f \right)(x) - \mathcal{K} \left( T^{(\xi)}_t f \right)(x),
\]
\[
\Upsilon_t^{(\xi,1)} f(x) = e^{-t} \sum_{n=1}^{\infty} \left[ \int_{\mathbb{R}^d} p_z(\xi, x-z) N \left( \Psi_{n-1,t}^{(\xi)}(z) \right) dz - N \left( \Psi_{n-1,t}^{(\xi)}(x) \right) \right],
\]

\[
\Upsilon_t^{(\xi,2)} f(x) = e^{-t} \sum_{n=1}^{\infty} \int_0^t \Lambda_{t-s}^{(\xi)} \left( N \left( \Psi_{n-1,s}^{(\xi)} \right) \right) (x) ds.
\]

By Lemma 4.1, (104), (105) and the boundedness of \( N : L^\infty(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d) \) the above series are convergent.

**Lemma 4.17.** For any \( t > 0, \xi \in (0, 1], x \in \mathbb{R}^d \) and \( f \in C_0(\mathbb{R}^d) \) we have

\[
\Upsilon_t^{(\xi)} f(x) = e^{-t} \sum_{n=1}^{\infty} \int_0^t \Lambda_{t-s}^{(\xi)} \left( N \left( \Psi_{n-1,s}^{(\xi)} \right) \right) (x) ds.
\]

The proof of this result is similar to the proof of Lemma 4.17 in [31] and it is omitted.

**Lemma 4.18.** (i) For any \( f \in C_0(\mathbb{R}^d) \) we have

\[
\Upsilon_t^{(\xi)} f(x) \to 0, \text{ as } \xi \to 0^+,
\]

uniformly in \((t, x) \in [\tau_1, \tau) \times \mathbb{R}^d\) for every \( \tau_1 \in (0, \tau) \).

(ii) For any \( f \in C_0(\mathbb{R}^d) \) we have

\[
\int_0^t \Upsilon_s^{(\xi)} f(x) ds \to 0, \text{ as } \xi \to 0^+,
\]

uniformly in \((t, x) \in (0, \tau) \times \mathbb{R}^d\).

**Proof.** The lemma follows from Lemma 3.23, Proposition 3.11, Lemma 4.10, Lemma 4.11, (104), (105) and the boundedness of \( N : L^\infty(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d) \).

The next result (positive maximum principle) is based on the ideas from [25, Section 4.2]. Its proof is very similar to the proof of [25, Lemma 4.3] and it is omitted.

**Lemma 4.19.** Let us consider the function \( v : [0, \infty) \times \mathbb{R}^d \to \mathbb{R} \) and the family of functions \( v^{(\xi)} : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}, \xi \in (0, 1] \). Assume that for each \( \xi \in (0, 1] \)
\[
\sup_{t \in (0, \tau], x \in \mathbb{R}^d} |v^{(\xi)}(x, t)| < \infty, v^{(\xi)} \text{ is } C^1 \text{ in the first variable and } C^2 \text{ in the second variable.}
\]
We also assume that (for any \( \tau > 0 \))

(i) \( v^{(\xi)}(t, x) \to v(t, x) \text{ as } \xi \to 0^+ \),

uniformly in \( t \in [0, \tau], x \in \mathbb{R}^d \);

(ii) \( v^{(\xi)}(t, x) \to 0 \text{ as } |x| \to \infty \),

uniformly in \( t \in [0, \tau], \xi \in (0, 1] \);

(iii) for any \( 0 < \tau_1 < \tau_2 \leq \tau \)
\[
\frac{\partial}{\partial t} v^{(\xi)}(t, x) - \mathcal{K} v^{(\xi)}(t, x) \to 0 \text{ as } \xi \to 0^+,
\]

uniformly in \( t \in [\tau_1, \tau_2], x \in \mathbb{R}^d \);

(iv) \( v^{(\xi)}(t, x) \to v(0, x) \text{ as } (\xi \to 0^+ \text{ and } t \to 0^+) \),

uniformly in \( x \in \mathbb{R}^d \);

(v) for any \( x \in \mathbb{R}^d \) \( v(0, x) \geq 0 \).

Then for any \( t \geq 0, x \in \mathbb{R}^d \) we have \( v(t, x) \geq 0 \).
Proposition 4.20. $T_t : \mathcal{B}_b(\mathbb{R}^d) \to \mathcal{B}_b(\mathbb{R}^d)$ is a linear, bounded operator for any $t \in (0, \tau]$. For each $t \in (0, \tau]$, $f \in \mathcal{B}_b(\mathbb{R}^d)$ and $R \geq 1$ there exists a sequence $f_k \in C_0(\mathbb{R}^d)$, $k \in \mathbb{N}$ such that $\lim_{k \to \infty} f_k(x) = f(x)$ for almost all $x \in B(0, R)$; for any $k \in \mathbb{N}$ we have $\|f_k\|_\infty \leq \|f\|_\infty$ and for any $x \in B(0, R)$ we have $\lim_{k \to \infty} T_t f_k(x) = T_t f(x)$.

Proof. Fix $t \in (0, \tau]$. The fact that $T_t : \mathcal{B}_b(\mathbb{R}^d) \to \mathcal{B}_b(\mathbb{R}^d)$ is a linear, bounded operator follows by the definition of $T_t$ and Lemma 4.11.

Fix $f \in \mathcal{B}_b(\mathbb{R}^d)$, $R \geq 1$ and $k \in \mathbb{N}$, $k \geq 1$. By Lemma 4.10 there exists $R_k \geq R$ such that for any $x \in B(0, R)$ we have

$$|T_t(f 1_{B^c(0, R_k)})(x)| \leq \frac{1}{k}. \quad (119)$$

Put $g_{1,k}(x) = 1_{B(0, R_k)}(x) f(x)$, $g_{2,k}(x) = 1_{B^c(0, R_k)}(x) f(x)$. By standard methods there exists $f_k \in C_0(\mathbb{R}^d)$ such that

$$\|f_k - g_{1,k}\|_1 \leq \frac{1}{k},$$

and supp($f_k$) $\subset B(0, R_k + 1)$, $\|f_k\|_\infty \leq \|f\|_\infty$. By Theorem 4.3, for any $x \in \mathbb{R}^d$, we have

$$|T_t(f_k - g_{1,k})(x)| \leq \frac{c\|f\|_1^{1-\alpha/(2d+2\beta+2-2\alpha)}}{k^{\alpha/(2d+2\beta+2-2\alpha)}}.$$ 

This and (119) imply that for any $x \in B(0, R)$ we have $\lim_{k \to \infty} T_t f_k(x) = T_t f(x)$.

We also have $\|f_k 1_{B(0, R)} - f 1_{B(0, R)}\|_1 \leq 1/k$. Hence, there exists a subsequence $k_m$ such that $\lim_{m \to \infty} f_{k_m}(x) = f(x)$ for almost all $x \in B(0, R)$.

Proposition 4.21. For any $t \in (0, \infty)$, $x \in \mathbb{R}^d$ and $f \in C_0^0(\mathbb{R}^d)$ we have

$$T_t f(x) = f(x) + \int_0^t T_s(\mathcal{K} f)(x) \, ds. \quad (120)$$

Proof. For any $t \geq 0$, $x \in \mathbb{R}^d$, $\xi \in (0, 1]$ put

$$v(t,x) = T_t f(x) - f(x) - \int_0^t T_s(\mathcal{K} f)(x) \, ds,$$

$$v^{(\xi)}(t,x) = T_t^{(\xi)} f(x) - f(x) - \int_0^t T_s^{(\xi)}(\mathcal{K} f)(x) \, ds.$$

Note that $\mathcal{K} f \in C_0(\mathbb{R}^d)$. By Lemmas 4.12, 4.18 and Corollary 4.16 we obtain that $v(t,x)$, $v^{(\xi)}(t,x)$ satisfy the assumptions of Lemma 4.19. Note that $v(0, x) = 0$ for all $x \in \mathbb{R}^d$. The assertion of the proposition follows from Lemma 4.19.

The following result shows that $\{T_t\}$ is a Feller semigroup.

Theorem 4.22. We have

(i) $T_t : C_0(\mathbb{R}^d) \to C_0(\mathbb{R}^d)$ for any $t \in (0, \infty)$,

(ii) $T_t f(x) \geq 0$ for any $t > 0$, $x \in \mathbb{R}^d$ and $f \in C_0(\mathbb{R}^d)$ such that $f(x) \geq 0$ for all $x \in \mathbb{R}^d$,

(iii) $T_{1R} f(x) = 1$ for any $t > 0$, $x \in \mathbb{R}^d$,

(iv) $T_{t+s} f(x) = T_t (T_s f)(x)$ for any $s, t > 0$, $x \in \mathbb{R}^d$, $f \in C_0(\mathbb{R}^d)$,

(v) $\lim_{t \to 0^+} \|T_t f - f\|_\infty = 0$ for any $f \in C_0(\mathbb{R}^d)$.

(vi) there exists a nonnegative function $p(t, x, y)$ in $(t, x, y) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$; for each fixed $t > 0$, $x \in \mathbb{R}^d$ the function $y \to p(t, x, y)$ is Lebesgue measurable, $\int_{\mathbb{R}^d} p(t, x, y) \, dy = 1$ and $T_t f(x) = \int_{\mathbb{R}^d} p(t, x, y) f(y) \, dy$ for $f \in C_0(\mathbb{R}^d)$. 

Proof. (i) This follows from Theorem 4.14 and Lemma 4.12 (ii).

(ii) Let \( f \in C_0(\mathbb{R}^d) \) be such that \( f(x) \geq 0 \) for all \( x \in \mathbb{R}^d \). For \( t \geq 0, x \in \mathbb{R}^d, \xi \in (0,1] \) put \( v(t,x) = T_t f(x), v(t^\xi(x), t, x) = T_t^{(\xi)} f(x) \). By Lemmas 4.12, 4.18 and Corollary 4.16 we obtain that \( v(t,x), v^{(\xi)}(t,x) \) satisfy the assumptions of Lemma 4.19. The assertion of Theorem 4.22 (ii) follows from Lemma 4.19.

(iii) The proof is very similar to the proof of [25, Lemma 4.5 b]. Let \( f \in C_0^2(\mathbb{R}^2) \) be such that \( f \equiv 1 \) on \( B(0,1) \subset \mathbb{R}^d \) and let \( f_n(x) = f(x/n), x \in \mathbb{R}^d, n \in \mathbb{N}, n \geq 1 \). For any \( x \in \mathbb{R}^d \) we have \( \lim_{n \to \infty} f_n(x) = 1, \lim_{n \to \infty} K f_n(x) = 0 \) and \( \sup_{n \in \mathbb{N}, n \geq 1} (\| f_n \| \vee \| K f_n \|) < \infty \). By Corollary 4.11 for any \( s,t > 0 \) and \( x \in \mathbb{R}^d \), we get

\[
\lim_{n \to \infty} T_t f_n(x) = T_t \mathbb{1}_{\mathbb{R}^d}(x), \quad \lim_{n \to \infty} T_s (K f_n)(x) = 0. \tag{121}
\]

Using (120) for \( f_n \) and (121) we obtain (iii).

(iv) Let \( f \in C_0(\mathbb{R}^d) \). For \( s,t \geq 0, x \in \mathbb{R}^d, \xi \in (0,1] \) put \( v(t,x) = T_{t+s} f(x) - T_t (T_s f)(x), v^{(\xi)}(t,x) = T_{t+s}^{(\xi)} f(x) - T_t^{(\xi)} (T_s f)(x) \). By Lemmas 4.12, 4.18 and Corollary 4.16 we obtain that \( v(t,x), v^{(\xi)}(t,x) \) satisfy the assumptions of Lemma 4.19. Note that \( v(0,x) = 0 \) for all \( x \in \mathbb{R}^d \). The assertion of Theorem 4.22 (iv) follows from Lemma 4.19.

(v) Choose \( \varepsilon_1 > 0 \). Since \( f \in C_0(\mathbb{R}^d) \) there exists \( \delta_1 > 0 \) such that

\[
\forall x,y \in \mathbb{R}^d \quad |x - y| < \delta_1 \Rightarrow |f(x) - f(y)| < \varepsilon_1.
\]

Fix arbitrary \( x \in \mathbb{R}^d, t \in (0,\tau] \). Put \( f_1(y) = 1_{B(x,\delta_1)}(y)(f(y) - f(x)), f_2(y) = 1_{B(x,\delta_1)}(y)(f(y) - f(x)), y \in \mathbb{R}^d \). By (iii) we have

\[
T_t f(x) - f(x) = T_t f_1(x) + T_t f_2(x).
\]

We also have

\[
|T_t f_1(x)| < c \varepsilon_1, \quad |T_t f_2(x)| \leq 2 \| f \| x T_t 1_{B(x,\delta_1)}(x)
\]

and

\[
T_t 1_{B(x,\delta_1)}(x) = e^{-\lambda t} \int_{B(x,\delta_1)} p_y(t,x-y) \, dy + e^{-\lambda t} \Phi_t 1_{B(x,\delta_1)}(x)
\]

\[
+ e^{-\lambda t} \sum_{n=1}^{\infty} \Psi_n t 1_{B(x,\delta_1)}(x).
\]

By Proposition 4.11 there exists \( \tau_1 \in (0,\tau] \) such that

\[
\forall t \in (0,\tau] \quad \int_{B(x,\delta_1)} p_y(t,x-y) \, dy < \varepsilon_1.
\]

By Proposition 4.13 we obtain that

\[
\forall t \in (0,\tau] \quad |\Phi_t 1_{B(x,\delta_1)}(x)| \leq c \tau_1^{1-\sigma}.
\]

By Lemma 4.1 we obtain that

\[
\forall t \in (0,\tau] \quad |e^{-\lambda t} \sum_{n=1}^{\infty} \Psi_n t 1_{B(x,\delta_1)}(x)| \leq ct.
\]

This implies (v).

(vi) This follows from (i), (ii), (iii) and Theorem 4.3.

We are now in a position to provide the proof of Theorems 1.1 and 1.3.

\[ \square \]
proof of Theorem 1.1. From Theorem 4.22 we conclude that there is a Feller process $X_t$ with the semigroup $T_t$ on $C_0(\mathbb{R}^d)$. Let $\mathbb{P}^x$, $\tilde{\mathbb{E}}^x$ be the distribution and expectation of the process $X_t$ starting from $x \in \mathbb{R}^d$.

By Theorem 4.22 (vi), Proposition 4.20 and Lemma 4.10 we get

$$\tilde{\mathbb{E}}^x f(X_t) = T_t f(x) = \int_{\mathbb{R}^d} p(t, x, y) f(y) \, dy \quad f \in \mathcal{B}_b(\mathbb{R}^d), \ t > 0, \ x \in \mathbb{R}^d. \quad (122)$$

By Proposition 4.21, for any function $f \in C_c^2(\mathbb{R}^d)$, the process

$$M^x_t = f(\tilde{X}_t) - f(\tilde{X}_0) - \int_0^t \mathcal{K} f(\tilde{X}_s) \, ds$$

is a $(\tilde{\mathbb{E}}^x, \mathcal{F}_t)$ martingale, where $\mathcal{F}_t$ is a natural filtration. That is $\tilde{\mathbb{E}}^x$ solves the martingale problem for $(\mathcal{K}, C_c^2(\mathbb{R}^d))$. On the other hand, by standard arguments, the unique solution $X$ to the stochastic equation (1) has the law which is the solution to the martingale problem for $(\mathcal{K}, C_c^2(\mathbb{R}^d))$ (see e.g. [32, page 120]).

By Lipschitz property of $a_{ij}(x)$ and by the Yanada-Watanabe theorem (see [37, Theorems 37.5 and 37.6]) the equation (1) has the weak uniqueness property. By this and [32, Corollary 2.5] weak uniqueness holds for the martingale problem for $(\mathcal{K}, C_c^2(\mathbb{R}^d))$.

Hence $\tilde{X}$ and $X$ have the same law so for any $t > 0$, $x \in \mathbb{R}^d$ and any Borel bounded set $A \subset \mathbb{R}^d$ we have

$$\pi_t(x, A) = \int_A p(t, x, y) \, dy,$$

where $\pi_t(x, A)$ is defined by (8). Using this, (11) and (122) we obtain

$$P_t f(x) = T_t f(x), \quad t > 0, \ x \in \mathbb{R}^d, \ f \in \mathcal{B}_b(\mathbb{R}^d). \quad (123)$$

Now the assertion of Theorem 1.1 follows from Theorem 4.3 and (123).

proof of Theorem 1.2. The result follows from Theorem 4.3 and (123).

proof of Proposition 1.2. From Theorem 4.22 (vi) and (123) we infer that transition densities $p(t, x, y)$ for $X_t$ exists. By arguments similar to the proof of Theorem 4.22 one can show that for any $t > 0$, $x \in \mathbb{R}^d$ and almost all $y \in \mathbb{R}^d$ we have $u(t, x, y) \geq 0$. By Lemma 3.4, $u(t, x, y) \to p_0(t, x)$ is continuous on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$. By (52), (63) and (58) we obtain that $(t, x, y) \to q_0(t, x, y)$ is continuous on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$. It follows that $(t, x, y) \to q(t, x, y)$ and $(t, x, y) \to u(t, x, y)$ are continuous on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$.

For $n \in \mathbb{N}$, $n \geq 1$, $t > 0$, $x, y \in \mathbb{R}^d$ let us define by induction

$$u_n(t, x, y) = \sum_{i=1}^d \int_0^t \int_{\mathbb{R}} u_0(t, s, x, z) \int_{\mathbb{R}} u_{n-1}(s, z + a_i(z)w, y) \nu_i(w) \, dw \, dz \, ds.$$

By (108) we have

$$\Psi_{n,t} f(x) = \int_{\mathbb{R}^d} u_n(t, x, y) f(y) \, dy.$$

It follows that $p(t, x, y) = \sum_{n=0}^{\infty} u_n(t, x, y)$ for any $t > 0$, $x, y \in \mathbb{R}^d$. For any $k \in \mathbb{N}$, $i \in \{1, \ldots, d\}$, $w \in \mathbb{R} \setminus \{0\}$ put $\nu_i^{(k)}(w) = \nu_i(w) \wedge k$. For any $t > 0$, $x, y \in \mathbb{R}^d$, $k \in \mathbb{N}$
put \( u_0^{(k)}(t)(t, x, y) = u_0(t, x, y) \wedge k \). For \( n \in \mathbb{N} \), \( n \geq 1 \), \( k \in \mathbb{N} \), \( t > 0 \), \( x, y \in \mathbb{R}^d \) let us define by induction

\[
u_n^{(k)}(t, x, y) = \sum_{i=1}^d \int_0^t \int_{\mathbb{R}} u_0^{(k)}(t - s, x, z) \int_{\mathbb{R}} u_{n-1}^{(k)}(s, z + a_i(z)w, y) \nu_i^{(k)}(w) \, dw \, dz \, ds.
\]

It follows that \((t, x, y) \to u_n^{(k)}(t, x, y)\) are continuous on \((0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d\). Clearly, for any \( t > 0 \), \( x \in \mathbb{R}^d \) and almost all \( y \in \mathbb{R}^d \) we have \( u_n^{(k)}(t, x, y) \geq 0 \). We also have \( \lim_{k \to \infty} u_n^{(k)}(t, x, y) = u_n(t, x, y) \). Hence \( p(t, x, y) = \lim_{k \to \infty} \sum_{n=0}^{\infty} u_n^{(k)}(t, x, y) \). Therefore \((t, x, y) \to p(t, x, y)\) is lower semi-continuous on \((0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d\).  

\[\Box\]

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