COMPACT SCHUR-WEYL DUALITY AND THE TYPE B/C VW-ALGEBRA

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Abstract. We define an extension of the VW-algebra, the type B/C VW-algebra. This new algebra contains the hyperoctahedral group and it naturally acts on $\text{End}_K(X \otimes V^\otimes k)$ for Orthogonal and Symplectic groups. Thus we obtain a compact analogue of Schur-Weyl duality. We study functors $F_{\mu,k}$ from the category of admissible $O(p,q)$ or $Sp_{2n}(\mathbb{R})$ modules to representations of the type B/C VW-algebra $B^\theta_k$. Thus providing a Akawaka-Suzuki-esque link between $O(p,q)$ (or $Sp_{2n}(\mathbb{R})$) and $B^\theta_k$. Furthermore these functors take non spherical principal series modules to principal series modules for the graded Hecke algebra of type $D_k$, $C_{n-k}$ or $B_{n-k}$. With this we get a functorial correspondence between admissible simple $O(p,q)$ (or $Sp_{2n}(\mathbb{R})$) modules and graded Hecke algebra modules.

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1. Introduction

Let $G$ be an odd real orthogonal group or symplectic group, $G$ is $O(p, q)$ for $p + q = 2n + 1$ or $Sp_{2n}(\mathbb{R})$. Let $K$ denote a maximal compact subgroup of $G$. Let $\mathfrak{g}_0$ be the real Lie algebra of $G$. Define its complexification $\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{C}$. Let $X$ be an admissible $G$-module and let $V$ be the defining matrix module of the linear group $G$. The papers [1, 23, 10, 6, 14, 13] study variants of the $C$-algebra $\text{End}_G(X \otimes V \otimes k)$ of operators on $X \otimes V \otimes k$ commuting with $G$. For $G = Sp_{2n}(\mathbb{R}), O(p, q)$ there is a homomorphism of the $VW$-algebra or degenerate BMW algebra [14, 11] to $\text{End}_G(X \otimes V \otimes k)$. In this paper we focus on the larger algebra of operators which commute with $K$:

$$\text{End}_K(X \otimes V \otimes k).$$

We define an extension of the $VW$-algebra, $\mathfrak{B}_k^\theta$, by operators related to the Cartan involution $\theta$ of $G$. This new algebra $\mathfrak{B}_k^\theta$ acts on $X \otimes V \otimes k$ and commutes with the action of $K$. It is also an extension of the Cyclotomic Brauer algebra, which is unsurprising since the Author has shown [7] that the cyclotomic Brauer algebra acts on $\text{End}_K(V \otimes k)$. It is the analogue of the $VW$-algebra for operators commuting with $K$. The extension contains the Weyl group of type $B/C$, the hyperoctahedral group. This new algebra’s module category is a natural image for the functors defined by Ciubotaru and Trapa [9]:

$$F_{\mu,k}(X) = \text{Hom}_K(\mu, X \otimes V \otimes k).$$

We show that the functors $F_{\mu,k}$ take the category of admissible $O(p, q)$ or $Sp_{2n}(\mathbb{R})$-modules to $\mathfrak{B}_k^\theta$-modules. Unlike previous functors, for $G = O(p, q)$ or $Sp_{2n}(\mathbb{R})$, both categories are related to the hyperoctahedral group. Let $G = KAN$ be the Iwasawa decomposition of $G$, and $P = MAN$ be the minimal parabolic subgroup. For characters $\delta$ of $M$ and $e^\nu$ of $A$, the principal series representation $X_\delta^\nu$ (Definition 6.1) is:

$$X_\delta^\nu = \text{Ind}_{MAN}^G(\delta \otimes e^\nu \otimes 1).$$

For split real orthogonal or symplectic groups this covers all of the principal series modules. When $G = O(p, q)$ or $Sp_{2n}(\mathbb{R})$ then $M = (\mathbb{Z}_2)^n$ or $O(p - q) \times (\mathbb{Z}_2)^q$. Denote the character of $M$ which is $triv$ (or $det$) on $O(p - q)$, $-1$ on the first $k$ generators and $1$ on the remaining $n - k$ or $q - k$ by $\delta_{triv}^k$ (resp. $\delta_{det}^k$). For $Sp_{2n}(\mathbb{R})$ we drop the subscript det and triv. The graded Hecke algebra $\mathbb{H}_k(c)$ (Definition 4.1) is the graded Hecke algebra associated to the hyperoctahedral group $W(B_k)$ with a certain parameter function related to $c \in \mathbb{R}$. For $G = Sp_{2n}(\mathbb{R})$, the functors $F_{triv,k}$ and $F_{det,n-k}$ take principal series modules $X_{triv}^\nu$ to principal series modules for the graded Hecke algebra $\mathbb{H}_k(0)$ and $\mathbb{H}_{n-k}(1)$.
respectively. For \( G = O(p, q) \) the functors \( F_{\text{triv} \otimes \det, k} \) and \( F_{\det \otimes \text{triv}, q-k} \) take principal series modules \( X_{\nu_k}^{\delta_{\text{triv}}} \) to principal series modules for the graded Hecke algebra \( \mathbb{H}_k(0) \) and \( \mathbb{H}_{q-k}(1) \) respectively. A similar result holds for \( X_{\nu_k}^{\delta_{\det}} \) and functors \( F_{\text{triv} \otimes \text{triv}, k} \) and \( F_{\det \otimes \det, q-k} \). Given a particular character \( \delta \) of \( M \) we associate to it \( K \)-characters \( \mu \), and \( \mu \) (Table 6.1) with scalars \( c_{\mu} \) and \( c_{\mu} \) (Table 7.1). We prove that functors related to \( \mu \) and \( c_{\mu} \) take principal series representations to principal series representations. Thus we have defined a link between principal series of split real orthogonal or symplectic groups with principal series of certain graded Hecke algebras.

**Theorem 8.14.** For \( G = Sp_{2n}(\mathbb{R}) \) or \( O(p, q) \), \( p+q = 2n+1 \), the module \( F_{\mu, k}(X_{\delta}^{\nu}) \) is isomorphic to the \( \mathbb{H}_k(c_{\mu}) \) principal series module

\[
X(\nu_k) = \text{Ind}_{S(\theta_{k})}^{\mathbb{H}_k(c_{\mu})} \nu_k.
\]

The module \( F_{\mu, n-k}(X_{\delta}^{\nu}) \) is isomorphic to the \( \mathbb{H}_{n-k}(c_{\mu}) \) principal series module

\[
X(\bar{\nu}_{n-k}) = \text{Ind}_{S(\theta_{k})}^{\mathbb{H}_{n-k}(c_{\mu})} \bar{\nu}_{n-k}.
\]

This extends the results of Ciubotaru and Trapa [9] to non-spherical principal series modules. Importantly, if \( G \) is a split real orthogonal or symplectic group, we can describe the Hecke algebra module of the image of every principal series modules resulting from functors \( F_{\mu, k} \) and \( F_{\mu, n-k} \). Furthermore using Casselman’s subrepresentation theorem, for these split groups we have a correspondence of irreducible Harish-Chandra modules of \( G \) and graded Hecke algebra modules.

**Theorem 8.15.** Let \( G = O(n+1, n) \) or \( Sp_{2n}(\mathbb{R}) \), then \( G \) is split. Let \( X \) be an irreducible \( G \)-module. Let \( X_{\delta}^{\nu} \) be a principal series representation that contains \( X \), then the \( B^0_k \) and \( B^0_{n-k} \)-modules

\[
F_{\mu, k}(X) \text{ and } F_{\mu, n-k}(X)
\]

are submodules of the \( \mathbb{H}_k(c_{\mu}) \) and \( \mathbb{H}_{n-k}(c_{\mu}) \)-modules

\[
X(\nu_k) \text{ and } X(\bar{\nu}_{n-k}).
\]

We define two anti-involutions on \( B^0_k \) which descend to the usual anti-involutions on the graded Hecke algebra [2]. Furthermore we show that if \( X \) is a Hermitian (resp. unitary) module of \( G = Sp_{2n}(\mathbb{R}) \) then the image of \( X \) under the functor \( F_{\mu, k} \) is a Hermitian (resp. unitary) module for \( B^0_k[m_0, m_1] \). We also show that the Langlands quotient is preserved.
Theorem 9.26. Let $X^\nu_\delta$ be a principal series module for $G = O(p,q)$ or $Sp_{2n}(\mathbb{R})$. The Langlands quotient $\overline{X^\nu_\delta} = X^\nu_\delta / \text{rad}(\cdot)X^\nu_\delta$ is mapped by $F_{\mu,k}$ to the Langlands quotient of the $H_k(c_\mu)$-module, $\overline{X(\nu_k)} = X(\nu_k) / \text{rad}(\cdot)X(\nu_k)$. Similarly, $X^\nu_\delta$ is mapped by $F_{\mu,n-k}$ to the $H_{n-k}(c_\mu)$-module $\overline{X(\nu_{n-k})}$.

We then give a non-unitary test for principal series modules.

Theorem 9.29. [Non-unitary test for principal series modules] If either $X(\nu_k)$ or $X(\nu_{n-k})$ are not unitary, as $H_k(c_\mu)$ and $H_{n-k}(c_\mu)$-modules, then the Langlands quotient of the minimal principal series module $\overline{X^\nu_\delta}$, for $G = O(p,q)$ or $Sp_{2n}(\mathbb{R})$ is not unitary.

This result gives a functorial result similar to the nonunitarity criterion proved by Barbasch, Pantano, Paul and Salamanca-Riba [3, 20] We also obtain a non-unitary test for any Harish Chandra module; in the split case one could check unitarity of Hecke algebra modules however in the non-split case one would have to work with type $B/C$ Brauer algebra modules.

Theorem 9.30. [Non-unitary test for Harish-Chandra modules] Let $X$ be a Harish Chandra module. For $G = Sp_{2n}(\mathbb{R})$ or $O(p,q)$, if for any character $\mu$ and $k = 1, \ldots, n$ the $B_k^{\mu}$-module $F_{\mu,k}(X)$ is not unitary, then the Langlands quotient $\overline{X}$ of $X$ is not a unitary $G$-module.

In Section 3 we define the type $B/C$ VW-algebra $B_k^{\mu}$ and show that it acts on $X \otimes V^\otimes k$ and commutes with the action of $K$. Section 4 defines particular quotients of $B_k^{\mu}$ isomorphic to the graded Hecke algebras $H_k(c)$. In Section 5 we introduce the functors, defined in [9], $F_{\mu,k} : \mathcal{HC}(G) \to B_n^{\mu}$-mod. These functors naturally create $B_k^{\mu}$-modules. In Section 7 we show that the functors restricted to principal series modules define Hecke algebra modules. Section 8 describes the isomorphism classes of $F_{\mu,k}(X^\nu_\delta)$ and $F_{\mu,n-k}(X^\nu_\delta)$ as principal series modules of graded Hecke algebras $H_k(c_\mu)$ and $H_{n-k}(c_\mu)$. In Section 9 we prove that functors $F_{\mu,k}$ preserve unitarity and invariant Hermitian forms.
Throughout this paper we fix the following notation. Let $G$ be $O(p, q)$, $p + q = 2n + 1$ or $Sp_{2n}(\mathbb{R})$. Let $\mathfrak{g}_0$ be its Lie algebra, with complexification $\mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$. We uniformly denote a real Lie algebra by $\mathfrak{g}_0$, for a complex Lie algebra we drop the subscript. We fix a Cartan involution $\theta$ of $\mathfrak{g}_0$ and extend to $\mathfrak{g}$, let $\Theta$ be the corresponding involution of $G$. A maximal compact subgroup of $G$ is $K$, the fixed space of $\Theta$. The Lie algebra $\mathfrak{g}_0$ decomposes as $\mathfrak{k}_0 \oplus \mathfrak{p}_0$. The subspace $\mathfrak{p}_0$ is the $-1$ eigenspace of $\theta$, the subalgebra $\mathfrak{k}_0$ is the $+1$ eigenspace of $\theta$ and the Lie algebra of $K$. Similarly, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Let $\mathfrak{a}_0$ be a maximal commutative Lie subalgebra of $\mathfrak{p}_0$. Let $M$ be the centralizer of $\mathfrak{a}_0$ in $K$ under the adjoint action. We have $\text{Lie}(M) = \mathfrak{m}_0$.

**Definition 2.1.** For $G$ equal to $O(p, q)$ or $Sp_{2n}(\mathbb{R})$ write $V_0$ for the defining matrix module. That is $\rho : G \to GL(V_0)$ is the injection defining $G$ as a linear group. Write $V = V_0 \otimes_{\mathbb{R}} \mathbb{C}$ for the complexification of $V_0$.

If $G = Sp_{2n}(\mathbb{R})$ then $V = \mathbb{C}^{2n}$ and if $G = O(p, q)$ then $V = \mathbb{C}^{2n+1}$. When $G = Sp_{2n}(\mathbb{R})$, let $e_1, ..., e_{2n}$ be the standard matrix basis of $V$, then define a new basis $f_i = e_i + e_{n+i}$ for $i = 1, ..., n$ and $f'_i = e_i - e_{n+i}$ for $i = 1, ..., n$. We also label $f_i$ by $f_i^1$ and $f'_i$ by $f_i^{-1}$. When $G = O(p, q)$ then $V$ has basis $e_1, ..., e_{2n+1}$, we let $f_i = e_{p-i+1} + e_{p+i}$ and $f'_i = e_{p-i+1} - e_{p+i}$.

Following [19, Section 1.1], let $\{R, X, \hat{R}, \hat{X}, \Delta\}$ be root datum where $R$ is the set of roots, $\hat{R}$ is the set of coroots and $X$ and $\hat{X}$ are free groups that contain $R$ and $\hat{R}$ respectively. There is a perfect pairing $\langle , \rangle$ between $X$ and $\hat{X}$ which defines a pairing between $R$ and $\hat{R}$. The simple roots $\Delta$ are a subset of $R$. Let $\mathfrak{t}$ equal the complexification of $X$, and similarly $\hat{\mathfrak{t}}$ is the complexification of $\hat{X}$.

The Lie algebra $\mathfrak{g}$ decomposes as $\mathfrak{k} \oplus \mathfrak{p}$. Let $\mathfrak{a}$ be a maximal abelian Lie subalgebra of $\mathfrak{p}$. The restricted roots $\Sigma$ of $\mathfrak{g}$ are given by the eigenvalues of $\mathfrak{a}$ acting on $\mathfrak{g}$. The nilpotent Lie subalgebra $\mathfrak{n}$ is the sum of positive root spaces of the restricted roots of $\mathfrak{a}$.

**Definition 2.2.** [17, Proposition 6.46], [16] The Iwasawa decomposition of the complex vector space $\mathfrak{g}$ is

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}.$$ 

The Iwasawa decomposition of $G$ [17, Theorem 6.46] is

$$G = KAN.$$
Let $M$ be the centralizer of $a$ in $K$ and denote by $N_K(a)$ the normalizer of $a$ in $K$. Let $m_0$ be the Lie algebra of $M$ with complexification $m$. The Weyl group associated to $G$ is the group

$$W_G = N_K(a)/M.$$ 

Example 2.3. For $G = Sp_{2n}(\mathbb{R})$, a maximal abelian subalgebra $a$ of $\mathfrak{p}$ is

$$a = \left\{ \begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix} : B \text{ is diagonal}, B \in \mathfrak{gl}_n(\mathbb{C}) \right\}.$$ 

The algebra $a$ has dimension $n$, this is the real rank of $Sp_{2n}(\mathbb{R})$. Let $E_{i,j}$ be the matrix with $1$ in the $(i, j)$ position and zero elsewhere. Let $k \in \{0, ..., n\}$ The subspace $a_k$ is the span of $E_{i,n+i} + E_{n+i,i}$ for $i = 1, ..., k$. The subspace $\bar{a}_{n-k} \subset a$ is the span of the vectors $E_{k+i,n+k+i} + E_{n+k+i,k+i}$ for $i = 1, ..., n-k$. Note that

$$a = a_k \oplus \bar{a}_{n-k}.$$ 

For $G = SO(p,q)$ there is a similar decomposition of $a$ into $k$ dimensional and $q-k$ dimensional subspaces, which we label by $a_k$ and $\bar{a}_{q-k}$.

Definition 2.4. Given a finite dimensional complex Lie algebra $\mathfrak{g}$ with basis $B$ and dual basis $B^*$ with respect to the Killing form, we define the Casimir element in the enveloping algebra $U(\mathfrak{g})$ to be

$$C^g = \sum_{b \in B} bb^* \in U(\mathfrak{g}).$$

For a subalgebra $\mathfrak{h} \subset \mathfrak{g}$ we denote the Casimir element of $\mathfrak{h}$ in $\mathfrak{g}$ by $C^h = \sum_{b \in B \cap \mathfrak{h}} bb^*$ where the basis $B$ is taken such that $B \cap \mathfrak{h}$ is a basis of $\mathfrak{h}$.

3. BRAUER ALGEBRAS

For a given $\mathfrak{g}$-module $X$ and the matrix module $V$, the endomorphism ring $\text{End}_C(X \otimes V^{\otimes k})$ has been thoroughly studied. Most attention ([23, 10, 12, 6, 14]) has been on understanding the subalgebra commuting with $G$:

$$\text{End}_C(X \otimes V^{\otimes k}).$$

In the case of $\mathfrak{g} = \mathfrak{gl}_n$, $\text{End}_{\mathfrak{gl}_n}(X \otimes V^{\otimes k})$ has a map from the graded Hecke algebra associated to the symmetric group $[1]$. However, with $\mathfrak{g} = \mathfrak{so}_{2n+1}$, the relevant algebra is the VW-algebra with parameter $n$. With $\mathfrak{g} = \mathfrak{sp}_{2n}$, the corresponding algebra is the VW-algebra with parameter $-n$.

In this section, we define the type $B/C$ Brauer algebra as an extension of the VW-algebra. We endow it with a natural action on
$X \otimes V^{\otimes k}$ and prove that it commutes with the action of $K$. The degenerate BMW algebra is a quotient of the VW-algebra. We choose not to use the BMW algebra [11] as we are fundamentally interested in resulting graded Hecke algebra modules, the quotients to Hecke algebras defined in Section 4 annihilate the difference between the VW-algebra and the degenerated BMW algebra.

**Definition 3.1.** [5] The rank $k$ Brauer algebra $B_k[m]$, with parameter $m \in \mathbb{C}$, is the associative $\mathbb{C}$-algebra generated by elements $t_{i,i+1}$ and $e_{i,i+1}$ for $i = 1, ..., k-1$, subject to the conditions:

- the subalgebra generated by $t_{i,i+1}$ is isomorphic to $\mathbb{C}[S_n]$, i.e., $e^2_{i,i+1} = me_{i,i+1}$,
- $t_{i,i+1}e_{i,i+1} = e_{i,i+1}t_{i,i+1} = e_{i,i+1}$,
- $t_{i,i+1}t_{i+1,i+2}e_{i,i+1}t_{i+1,i+2} = e_{i+1,i+2}$,
- $[t_{i,i+1}, e_{j,j+1}] = 0$ for $j \neq i, i+1$.

**Definition 3.2.** Let $U$ be a vector space with basis $z_1, ..., z_k$. The rank $k$ VW-algebra $B_k[m]$, with parameter $m \in \mathbb{C}$ is as a vector space equal to

$B_k[m] \cong B_k[m] \otimes S(U)$.

The multiplication satisfies the following conditions:

- $t_{i,i+1}z_i - z_{i+1}t_{i,i+1} = 1 + e_{i,i+1}$,
- $[t_{i,i+1}, z_j] = 0$, $j \neq i, i+1$,
- $e_{i,i+1}(z_i + z_{i+1}) = 0 = (z_i + z_{i+1})e_{i,i+1}$,
- $[e_{i,i+1}, z_j] = 0$, $j \neq i, i+1$,
- $[z_i, z_j] = 0$,
- $e_{12}z_i e_{12} = W_i e_{12}$ for constants $w_i \in \mathbb{C}$,

the subalgebra generated by $t_{i,i+1}$, $e_{i,i+1}$ is isomorphic to $B_k[m]$.

Let us consider $X$ and $V$ as $U(\mathfrak{g})$-modules then $X \otimes V^{\otimes k}$ has a $U(\mathfrak{g})^{\otimes k+1}$-module structure. We define operators that form the action of the Brauer algebra.

**Definition 3.3.** Given the action of $U(\mathfrak{g})^{\otimes k+1}$ on $X \otimes V^{\otimes k}$ we write $(g)_i$ for the action of $g$ on the $i+1$st tensor in $U(\mathfrak{g})^{\otimes k+1}$,

$$(g)_i = \underbrace{id \otimes \cdots \otimes id}_{i \text{ times}} \otimes g \otimes \underbrace{id \otimes \cdots \otimes id}_{k-i \text{ times}}.$$

By construction we start counting from zero. Hence $(g)_0 = g \otimes id \otimes ... \otimes id \in U(\mathfrak{g})^{\otimes k+1}$. 
Definition 3.4. Fix a basis $B$ of $\mathfrak{g}$ such that $B = (B \cap \mathfrak{k}) \cup (B \cap \mathfrak{p})$. Let $B^*$ be the dual basis with respect to the Killing form of $\mathfrak{g}$. For $0 \leq i < j \leq k$, define $\Omega_{ij}$ to be the operator

$$\Omega_{ij} = \sum_{b \in B} (b)_i \otimes (b^*)_j \in U(\mathfrak{g})^{\otimes k+1}.$$ 

Similarly we define $\Omega^k_{ij}$ and $\Omega^p_{ij}$ as

$$\Omega^k_{ij} = \sum_{b \in B \cap \mathfrak{k}} (b)_i \otimes (b^*)_j \in U(\mathfrak{g})^{\otimes k+1},$$

$$\Omega^p_{ij} = \sum_{b \in B \cap \mathfrak{p}} (b)_i \otimes (b^*)_j \in U(\mathfrak{g})^{\otimes k+1}.$$

Lemma 3.5. The operators $\Omega_{ij}, \Omega^k_{ij}$ and $\Omega^p_{ij}$ are independent of the choice of basis of $\mathfrak{g}$, $\mathfrak{k}$ and $\mathfrak{p}$ respectively.

Proof. It is sufficient to prove the statement for $\Omega_{12} \in U(\mathfrak{g})^2$. Let $C^g = \sum_{b \in B} bb^* \in U(\mathfrak{g})$ be the Casimir element and $\Delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \times U(\mathfrak{g})$ denote comultiplication. The operator $\Omega_{12}$ can be written as:

$$\Omega_{12} = \Delta(C^g) - C^g \otimes 1 - 1 \otimes C^g.$$

The Casimir element $C^g$ is independent of the choice of basis therefore $\Omega_{12}$ is also independent. Similarly replacing $\mathfrak{g}$ with the Lie subalgebra $\mathfrak{k}$, $\Omega^k_{12}$ is independent of choice of basis. Finally $\Omega^p_{12}$ is independent as it is the difference of the other two,

$$\Omega_{ij} - \Omega^k_{ij} = \Omega^p_{ij}.$$

Lemma 3.6. If $\mathfrak{g} = \mathfrak{sp}_{2n}$ or $\mathfrak{so}_{2n+1}$ then $V \otimes V$ decomposes as

$$\Lambda^2 V \oplus S^2 V \oplus 1 \oplus 1$$

for $\mathfrak{so}_{2n+1}$,

$$\Lambda^2 V \oplus S^2 V \oplus 1$$

for $\mathfrak{sp}_{2n}$.

Here 1 denotes the trivial module of $\mathfrak{g}$.

Let $\text{pr}_1$ be the projection of $V \otimes V$ onto the trivial submodule 1 in the decomposition above. Let $\text{pr}_{i,i+1}$ be the projection onto the trivial submodule of $V_i \otimes V_{i+1}$.
Lemma 3.7. [11, Theorem 2.2] Let $G$ be $O(p,q)$ or $Sp_{2n}(\mathbb{R})$. Let $X$ be a complex $G$-representation and $V$ the defining matrix module of $G$. Then there exists $m \in \mathbb{N}$ such that there is a homomorphism $\pi : \mathfrak{B}_k[m] \to \text{End}(X \otimes V^\otimes k)$:

$$\pi(z_i) = \sum_{j<i} \Omega_{ji},$$

$$\pi(t_{i,i+1}) = s_{i,i+1},$$

$$\pi(e_{i,i+1}) = id \otimes ... \otimes id \otimes m \text{pr}_{i,i+1} \otimes id \otimes .... \otimes id.$$ 

For $G = Sp_{2n}(\mathbb{R})$ the parameter is $m = -n$ and if $G = O(p,q)$ then $m = \lfloor \frac{p+q}{2} \rfloor$.

Theorem 3.8. For $G = O(p,q)$ or $Sp_{2n}(\mathbb{R})$, the VW-algebra with the action on $X \otimes V^\otimes k$ defined in Lemma 3.7 commutes with the action of $U(g)$ on $X \otimes V^\otimes k$.

Lemma 3.9. [9, Lemma 2.3.1] Let $0 < i < j \leq k$ and $G = O(p,q)$ or $Sp_{2n}(\mathbb{R})$. As operators on $X \otimes V^\otimes k$

$$\Omega_{ij} = s_{ij} + m \text{pr}_{i,i+1}, \text{ for } 1 \leq i < j \leq k$$

where $m = \lfloor \frac{p+q}{2} \rfloor$ or $-n$ respectively.

Proof. One only needs to consider the operator $\Omega_{12}$ on $V \otimes V$. By Lemma 3.6 $V \otimes V$ decomposes as

$$\Lambda^2 V \oplus S^2 V/1 \oplus 1 \text{ for } \mathfrak{g}_C = \mathfrak{so}_{2n+1}(\mathbb{R}),$$

$$\Lambda^2 V/1 \oplus S^2 V \oplus 1 \text{ for } \mathfrak{g}_C = \mathfrak{sp}_{2n}(\mathbb{C}).$$

On $V \otimes V$ $s_{12} = pr_S V - pr_{\Lambda^2 V}$. Then using the fact that $\Omega_{12} = \Delta(C) - C \otimes 1 - 1 \otimes C$ we find the operators

$$\Omega_{12} \text{ and } s_{12} + me_{12},$$

act by the same scalars on the irreducible decomposition of $V \otimes V$. □

For $G = GL_n$ the commutator $\text{End}_{GL_n}(X \otimes V^\otimes k)$ contains the same type Weyl group, the symmetric group (11). One might expect that in type $B$ and $C$ this may be the case too. However $\text{End}_{Sp_{2n}(\mathbb{R})}(X \otimes V^\otimes k)$, $\text{End}_{O(p,q)}(X \otimes V^\otimes k)$ and the VW-algebra, do not contain a copy of the hyperoctahedral group. We look to establish a theory that has this type symmetry reflected in the commutator.

We introduce the type $B/C$ VW-algebra which acts on $X \otimes V^\otimes k$ and commutes with the action of $K$ for $G = Sp_{2n}(\mathbb{R})$ or $O(p,q)$. Crucially the type $B/C$ VW-algebra contains the Weyl group of Type $B/C$, $W(B_k)$. Recall the hyperoctahedral group is generated by simple reflections $s_{e_i-e_{i+1}}$ and $s_{e_k}$. 
Definition 3.10. The type $B/C$ VW-algebra $B^\theta_k[m_0, m_1]$ is generated by the VW-algebra $B_k[m_0]$ and reflections $\theta_j$, for $j = 1, \ldots, k$, such that the subalgebra generated by $t_i, \ldots, k$, and $\theta_j$ is isomorphic to the group algebra of the $k^{th}$ hyperoctahedral group $\mathbb{C}[W(B_k)]$ and the following relations hold:

$$[e_{i,i+1}, \theta_j] = 0 \text{ for all } j,$$
$$e_{i,i+1}\theta_i\theta_{i+1} = e_{i,i+1} = \theta_i\theta_{i+1}e_{i,i+1} \text{ for } i = 1, \ldots, k - 1,$$
$$[\theta_n, x_j] = 0 \text{ for } j \neq k.$$

$$e_{i,i+1}\theta_i e_{i,i+1} = m_1 e_{i,i+1} \text{ for } i = 1, \ldots, k - 1,$$

The Lie algebra $g$ decomposes as eigenspaces of a Cartan involution $\theta$ that is $g = \mathfrak{k} \oplus \mathfrak{p}$. For $G = O(p, q)$ or $Sp_{2n}(\mathbb{R})$ there is a semisimple involutive $\xi \in g$ such that $\theta$ is equal to conjugation by $\xi$.

Remark 3.11. The subalgebra of $B^\theta_k[m_0, m_1]$ generated by $e_{i,i+1}$, $t_i$, and $\theta_i$ is equal to the cyclotomic Brauer $Br_{k,2}[m_0, m_1]$, see [15, 4, 7] for the definition of the cyclotomic Brauer algebra, its representation theory and how it acts on $\text{End}_K(X \otimes V^{\otimes k})$.

Lemma 3.12. The type $B/C$ VW-algebra $B^\theta_k[m]$ acts on $X \otimes V^{\otimes k}$. This action is defined by extending the action $\pi$ of the VW-algebra to the extra generators $\theta_i$. The generators $\theta_i$ act by $(\xi)_i \in U(\mathfrak{g})^{\otimes k+1}$. Extend $\pi$ to $B^\theta_k[m]$ by $\pi(\theta_i) = (\xi)_i \in U(\mathfrak{g})^{\otimes k+1} \subset \text{End}(X \otimes V^{\otimes k})$. That is

$$\pi : B^\theta_k[m] \rightarrow \text{End}_K(X \otimes V^{\otimes k})$$

$$\pi(\theta_i) = (\xi)_i.$$

Explicitly, $(\xi)_i = \underbrace{id \otimes \ldots \otimes id}_{i} \otimes \underbrace{\xi \otimes id \otimes \ldots \otimes id}_{k-i} \in U(\mathfrak{g})^{\otimes k+1}$. The constants $(m_0, m_1)$ equal $\left(\frac{p+q}{2}, p-q\right)$ when $G = O(p, q)$ and $(m_0, m_1) = (-n, 0)$ if $G = Sp_{2n}(\mathbb{R})$.

Proof. Since we know that the VW-algebra $B_k[m]$ acts on $X \otimes V^{\otimes k}$ and that the cyclotomic Brauer algebra $Br_{k,2}[m_0, m_1]$ acts on $\text{End}_K(V^{\otimes k})$ we only need to check the action of $\theta_j$, and $z_i$ and the relations involving them in Definition 3.10. This equates to checking $[z_i, \theta_k] = 0$ for all $i < n$.

If $i \neq j$, then $(g)_i$ and $(h)_j$ commute in $U(\mathfrak{g})^{k+1}$. Definition 3.7 states $\pi(z_i) = \sum_{j<i} \Omega_{ji}$, hence:

$$[z_i, (\xi)_k] = \sum_{j<i} [\Omega_{ji}, (\xi)_k] = 0 \text{ for } k < n.$$
Theorem 3.13. Let $G = O(p, q)$ or $Sp_{2n}(\mathbb{R})$ and $X$ a Harish-Chandra module. The type $B/C$ Brauer algebra $\mathfrak{B}_k^\theta[m]$ acts on $X \otimes V^\otimes_k$ and commutes with the action of $K$ on $X \otimes V^\otimes_k$.

Proof. The action of $\mathfrak{B}_k^\theta[m]$ commutes with $g$ and by restriction with $K$. The algebra $\mathfrak{B}_k^\theta[m] = \langle \mathfrak{B}_k^\theta[m], \theta_j : j = 1, \ldots, k \rangle$. Therefore, to verify that $\mathfrak{B}_k^\theta[m]$ commutes with the action of $K$, one only needs to check that $\pi(\theta_j) = (\xi)_j$ commutes with the action of $K$. Conjugation by $\xi$ is the Cartan involution: $\xi^{-1} K \xi = \Theta(K)$. By definition, $\Theta$ is the identity on $K$. Hence $\xi_k - k \xi = 0$ for $k \in \mathfrak{k}$. Therefore:

$$[(\xi)_i, k] = \sum_{j=0}^{k+1} (\xi)_i - (k)_j (\xi)_i (k)_j = 0.$$ 

Hence the action of $(\xi)_i$ and $K$ commute. □

4. Quotients of the type $B/C$ Brauer algebra $\mathfrak{B}_k^\theta$

In Section 5 we introduce functors, defined in [9], from the category $HC(G)$-mod to the category of $\mathfrak{B}_k^\theta[m]$ modules. However, we are aiming at graded Hecke algebra modules. In this section, we look at particular ideals and quotients of $\mathfrak{B}_k^\theta[m]$ which are isomorphic to graded Hecke algebras. This will set up Section 6 in which we focus on principal series modules and show that via the quotients defined in this section, the functors defined in Section 5 descend to take principal series modules to graded Hecke algebra modules.

Recall that $W(R)$ denotes the Weyl group associated to a root datum $(R, X, \hat{R}, \hat{X}, \Delta)$ and $\langle , \rangle : X \times \hat{X} \to \mathbb{C}$ is the pairing between dual spaces. Define the $\mathbb{C}$-spaces $t = X \otimes_\mathbb{Z} \mathbb{C}$, $t^* = \hat{X} \otimes_\mathbb{Z} \mathbb{C}$.

Definition 4.1. [19] The graded Hecke algebra $\mathbb{H}^R(c)$ associated to the root system $(R, X, \hat{R}, \hat{X}, \Delta)$ and parameter function $c$ from $\Delta$ to $\mathbb{C}$, is as a vector space

$$\mathbb{H}^R(c) \cong S(t) \otimes \mathbb{C}[W(R)],$$

such that as an algebra $S(t)$ and $\mathbb{C}[W(R)]$ are subalgebras and the following cross relations hold,

$$s_\alpha \varepsilon - s_\alpha (\varepsilon)s_\alpha = c(\alpha)(\alpha, \varepsilon), \text{ for } \varepsilon \in t \text{ and } \alpha \in \Delta.$$

If the parameter function $c : \Delta \to \mathbb{C}$ is taken to uniformly be 1, then in this case the graded Hecke algebra is entirely defined by the root system $(W, R, \Delta)$. For a Hecke algebra determined by the uniform parameter we denote it by $\mathbb{H}^{R_k}$ where $R_k$ is the root system. For
example $\mathbb{H}^{D_k}$ denotes the graded Hecke algebra associated to the root system $D_k$ with the parameter function $c : \delta \rightarrow \mathbb{C}$ such that $c(\alpha) \equiv 1$.

We fix the set of simple reflections of the hyperoctahedral group $W(B_k)$ to be \( \{ s_{i,i+1}, \theta_k : i = 1, \ldots, k - 1 \} \). We also associate to the hyperoctahedral group a $k$ dimensional vector space $t$ with basis $\vartheta_1, \ldots, \vartheta_k$ and subset $\Delta = \{ \vartheta_i - \vartheta_{i+1} \text{ and } \vartheta_k : i = 1, \ldots, k - 1 \}$. Then for $c \in \mathbb{C}$ we define the parameter $c : \Delta \rightarrow \mathbb{C}$ as

$$c(\alpha) = \begin{cases} 1 & \text{if } \alpha = \vartheta_i - \vartheta_{i+1}, \\ c & \text{if } \alpha = \vartheta_k. \end{cases}$$

We denote the graded Hecke algebra associated to the Weyl group $W(B_k)$ with the parameter $c$ as $H^c_k$.

**Lemma 4.2.** The graded Hecke algebra of type $B_k$ (resp. type $C_k$) is isomorphic to $\mathbb{H}_k(1)$ (resp. $\mathbb{H}_k(\frac{1}{2})$) and the algebra $\mathbb{H}_k(0)$ is isomorphic to an extension of the Hecke algebra of type $D_k$,

$$\mathbb{H}_k(0) \cong \mathbb{H}_k \rtimes \mathbb{Z}_2.$$  

**Proof.** The isomorphism of $\mathbb{H}_k(1)$ and the graded Hecke algebra $\mathbb{H}^{B_k}$ is apparent from the definitions. The space $t_{D_k}$ is equal to the space $t$ in $\mathbb{H}_k(0)$. The Weyl group $W(D_k)$ is naturally a subgroup of $W(B_k)$. The generator $t \in \mathbb{Z}_2$ acts on $\mathbb{H}^{D_k}$ by interchanging roots $\vartheta_{k-1} - \vartheta_k$ and $\vartheta_{k-1} + \vartheta_k$ and acts by conjugation by $s_{\theta_k} \in W(B_k)$ on $W(D_k) \subset W(B_k)$. \qed

We define two ideals in the type $B/C$ VW-algebra $\mathfrak{B}_k^\theta[m]$. We then show that the quotient of $\mathfrak{B}_k^\theta[m]$ by these ideals is isomorphic to a graded Hecke algebra.

**Definition 4.3.** Let $I_e$ be the two sided ideal in $\mathfrak{B}_k^\theta[m]$ generated by the idempotents,

$$\{ e_{i,i+1} : \text{ for } i = 1, \ldots, k - 1 \}.$$  

Let $c \in \mathbb{C}$ and $r \in \mathbb{Z}$, define $I_c^r$ to be the two sided ideal,

$$I_c^r = \langle \theta_k \vartheta_k + \vartheta_k \theta_k - 2c + 2r \theta_k \rangle.$$  

The ideal $I_e$ can be generated by any idempotent since they are all in the same $S_k$ conjugation orbit. By using $c \in \mathbb{C}$ we have abused notation; however the two occurrences of $c$ will correspond to the same constant.

**Lemma 4.4.** The quotient of the algebra $\mathfrak{B}_k^\theta[m]$ by the ideal generated by $I_e$ and $I_e^r$ is isomorphic to the graded Hecke algebra

$$\mathfrak{B}_k^\theta[m_0, m_1]/(I_e, I_e^r) \cong H_k(c).$$
Proof. Consider the presentation in Definition 3.10 with generators $z_i, \theta_j, t_{i,i+1}, e_{i,i+1}$ and relations

$$\theta_j^2 = 1, s_{i,i+1}^2 = 1, (s_{i,i+1}s_{i+1,i+2})^3 = 1, (s_{k-1,k}\theta_k)^4 = 1,$$

$$t_{i,i+1}z_i - x_{i+1}t_{i,i+1} = 1 + e_{i,i+1},$$

$$[t_{i,i+1}, z_j] = 0, j \neq i, i + 1,$$

$$e_{i,i+1}(z_i + z_{i+1}) = 0 = (z_i + z_{i+1})e_{i,i+1},$$

$$[e_{i,i+1}, z_j] = 0, j \neq i, i + 1,$$

$$[z_i, z_j] = 0, j \neq i, i + 1,$$

$$[e_{i,i+1}, \theta_j] = 0 \text{ for all } j,$$

$$e_{i,i+1}\theta_i\theta_{i+1} = e_{i,i+1} = \theta_i\theta_{i+1}e_{i,i+1} \text{ for } i = 1, ..., k - 1,$$

$$[\theta_n, z_j] = 0 \text{ for } j \neq k,$$

$$e_{12}z_1^i e_{12} = w_1 e_{12}.$$

Under the quotient by $I_e$ and $I_r$ the generators $e_{i,i+1}$ and the relations $e_{i,i+1} = 0$ cancel out. Furthermore we add another relation: $z_k\theta_k + \theta_k z_k - 2c + 2r\theta_k$. Hence the presentation has generators $z_i, \theta_j, t_{i,i+1}$ with relations

$$\theta_j^2 = 1, s_{i,i+1}^2 = 1, (s_{i,i+1}s_{i+1,i+2})^3 = 1, (s_{k-1,k}\theta_k)^4 = 1,$$

$$t_{i,i+1}z_i - x_{i+1}t_{i,i+1} = 1,$$

$$[t_{i,i+1}, z_j] = 0, j \neq i, i + 1,$$

$$[z_i, z_j] = 0, j \neq i, i + 1,$$

$$[\theta_n, z_j] = 0 \text{ for } j \neq k,$$

$$z_k\theta_k + \theta_k z_k - 2c + 2r\theta_k.$$

This is a presentation of the Hecke algebra $\mathbb{H}_k(c)$; it is the modification of the presentation in Definition 4.1 by $\epsilon_i \mapsto z_i + r$. Since we have shown that the presentation of $\mathfrak{B}^+_k[p_0, m_1]/(I_e I_r^c)$ is identical to the presentation of $\mathbb{H}_k(c)$ then these algebras are isomorphic.

Remark 4.5. We could have chosen to quotient by the ideal generated by $\theta_k z_k + z_k \theta_k - c$ without the $2r\theta_k$ part. This quotient would also be isomorphic to $\mathbb{H}_k(c)$ with $\epsilon_i$ mapping to $z_i$. However, we need the modification of the affine parts by the scalar $r$ to enable our results regarding images of principal series modules descending to Hecke algebra modules. One can think of this modification by $r$ as an analogue of the $\rho$ shift.
5. Functors from $\mathcal{HC}(G)$-mod to $\mathfrak{B}_k^\theta$-mod

In this section, we introduce functors, defined in [9]. We show these functors take Harish-Chandra modules to modules of the $\mathfrak{B}_k^\theta$ algebra.

**Definition 5.1.** [9, (2.8)] Let $n$ be the real rank of $G$. If $G = Sp_{2n}(\mathbb{R})$ the real rank is $n$. If $G = O(p,q)$ then $n = q = \min(p,q)$. Let $\mu$ be an irreducible $K$-module, fix an integer $k \leq n$. The space $V$ is the matrix module of $G$. We define the functor $F_{\mu,k}$ to be:

$$F_{\mu,k} : \mathcal{HC}(G)\text{-mod} \rightarrow \mathfrak{B}_k^\theta\text{-mod}$$

$X \mapsto \text{Hom}_K(\mu, X \otimes V^\otimes k)$,

and on morphisms $f : X \rightarrow Y$ and $g \in \text{Hom}_k(\mu, X \otimes V^\otimes k)$,

$$F_{\mu,k}f(g) : \mu \rightarrow Y \otimes V^\otimes k,$$

$$F_{\mu,k}f(g)(\mu) = f \otimes \text{id}^\otimes g(\mu).$$

**Remark 5.2.** Lemma 3.12 gives an action of $\mathfrak{B}_k^\theta$ on $X \otimes V^\otimes k$. Since this action commutes with the action of $K$ then $\mathfrak{B}_k^\theta$ naturally acts on $\text{Hom}_K(\mu, X \otimes V^\otimes k)$ from the inherited action on $X \otimes V^\otimes k$.

**Lemma 5.3.** For any irreducible $K$-module $\mu$ and $k \leq n$, the functor $F_{\mu,k}$ defined in Definition 5.1 is exact.

**Proof.** Tensoring with a finite dimensional module is exact. The module $V^\otimes k$ is finite dimensional hence the functor taking $X$ to $X \otimes V^\otimes k$ is exact. Furthermore, $\mu$ is an irreducible $K$-module. Therefore the functor which takes $Y$ to $\text{Hom}_K(\mu, Y)$ is exact. The functor $F_{\mu,k}$ is the composition of these two exact functors, hence the result follows. \qed

6. Restricting functors to principal series modules

The functors (Definition 5.1) take any Harish-Chandra module to a $\mathfrak{B}_k^\theta$-module. In this section, given a principal series module we give a basis for the image of the functors $F_{\mu,k}$ and $F_{\mu,n-k}$ for particular characters $\mu, \mu$ depending on the principal series modules.

Let $G = Sp_{2n}(\mathbb{R})$ then $K \cong U(n), M \cong (\mathbb{Z}_2)^n$. The Cartan involution $\theta$ is equal to conjugation by the matrix

$$\xi = \begin{bmatrix} 0 & \text{Id}_n \\ -i \text{Id}_n & 0 \end{bmatrix}.$$

The subspace $\mathfrak{a}$ has dimension $n$ with basis $\varepsilon_i$ and corresponds to the subgroup $A$ under the exponential map. We label a character of $\mathfrak{a}$ by $\nu \in \mathfrak{a}^*$ and characters of $A$ by $e^\nu$. The matrix module $V \cong \mathbb{C}^{2n}$ has two bases: $\{e_1, \ldots, e_{2n}\}$ and $\{f_1, \ldots, f_n, f_1^{-1}, \ldots, f_n^{-1}\}$, where $f_i^n = e_i + \eta e_{n+i}$. 
Recall that the Iwasawa decomposition of $G$ is

$$G = KAN,$$

also, that $M$ is the centraliser of $a_0$ in $K$, which is isomorphic to $\mathbb{Z}_2^n$. The character $\delta^k$ is defined to be the character of $M$ which takes the first $k$ generators of $\mathbb{Z}_2^n$ to $-1$ and the last $n - k$ to $1$. We write $1$ for the trivial character of $N$.

If $G = O(p, q)$ then $K \cong O(p) \times O(q)$, $M = O(p - q) \times O(1)^q$ embedded into $O(p, q)$ as the block matrix

$$(O(p - q), x_1, x_2, ..., x_q, x_q, ..., x_1)$$

where $x_i \in O(1)$. We denote characters of $M$, $\delta^k_{\text{triv}}$ and $\delta^k_{\text{det}}$ to be

$$\delta^k_{\text{triv}} = \text{triv} \otimes (\text{sgn}^k) \otimes \text{triv}^{q-k}$$
$$\delta^k_{\text{det}} = \text{det} \otimes (\text{sgn}^k) \otimes \text{triv}^{q-k}$$

on $O(p - q) \otimes O(1)^q$.

The Cartan involution $\theta$ is equal to conjugation by the matrix

$$\xi = \begin{bmatrix} \text{Id}_p & 0 \\ 0 & -\text{Id}_q \end{bmatrix}.$$

**Definition 6.1.** [22] Let $G = KAN$ (resp. $g_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0$) be the Iwasawa decomposition of $G$ (resp. $g_0$) and let $M$ be the centraliser of $a_0$ in $K$. Given a character $e^\nu$ of $A$ and the character $\delta$ of $M$ we define the minimal principal series representation;

$$X_{e^\nu}^\delta = \text{Ind}_{MAN}^G (\delta \otimes e^\nu \otimes 1).$$

In the non-split case principal series representations may be induced from irreducible representations of $M$ which are not one dimensional. In this chapter we will only study principal series modules that are induced from a character of $M$. We write $1^e_\delta$ for the vector spanning the representation space of the character $\delta \otimes e^\nu \otimes 1$. Hence

$$X_{e^\nu}^\delta = \text{Ind}_{MAN}^G 1^e_\delta.$$

For $G = Sp_{2n}(\mathbb{R})$, we can calculate the dimension of $F_{\text{triv},k}(X_{e^\nu}^\delta)$ and $F_{\text{det},n-k}(X_{e^\nu}^\delta)$. Note that if we want to describe the trivial isotypic component we must take $F_{\text{triv},k}$ and if we wish to look at the det isotypic component then we must take the functor $F_{\text{det},n-k}$.

For $G = O(p, q)$, we can calculate the dimension of $F_{\text{triv} \otimes \text{sgn},k}$ and $F_{\text{triv} \otimes \text{triv},q-k}$. Similarly for $X_{e^\nu}^\delta_{\text{det}}$, we take the functors $F_{\text{sgn} \otimes \text{triv},k}$ and $F_{\text{sgn} \otimes \text{sgn},q-k}$.

To enable us to succinctly discuss all of the above cases we will associate a character $\mu$ and $\mu$ to each principal series modules. Note $\delta$ is a $K$-character and $\mu, \mu$ are characters of $M$. 


$G = \text{Sp}_{2n}(\mathbb{R})$  \quad $X'_{\delta}, \delta = (\text{triv})^k \otimes (\text{sgn})^{n-k}$  \quad $\mu = \text{triv}$  \quad $\mu = \text{det}$

$G = \text{O}(p,q)$  \quad $X'_{\delta}, \delta = \text{triv}_{p-q} \otimes (\text{triv})^k \otimes (\text{sgn})^{q-k}$  \quad $\mu = \text{triv} \otimes \text{det}$  \quad $\mu = \text{triv} \otimes \text{triv}$

$G = \text{O}(p,q)$  \quad $X'_{\delta}, \delta = \text{det}_{p-q} \otimes (\text{triv})^k \otimes (\text{sgn})^{q-k}$  \quad $\mu = \text{det} \otimes \text{triv}$  \quad $\mu = \text{sgn} \otimes \text{sgn}$

| Table 6.1. Characters $\mu$, $\mu$ associated to particular principal series module. |

**Lemma 6.2.** Let $G = \text{Sp}_{2n}(\mathbb{R})$ or $G = \text{O}(p,q)$. If $X'_{\delta}$ is a minimal principal series module, then $F_{\mu,k}(X'_{\delta})$ and $F_{\mu,n-k}(X'_{\delta})$ are finite dimensional. with dimensions:

$$\dim(F_{\mu,k}(X'_{\delta})) = k!2^k = |W(B_k)|,$$

Similarly,

$$\dim(F_{\mu,n-k}(X'_{\delta})) = (n-k)!2^{n-k} = |W(B_{n-k})|.$$  

This is an extension of [9, Lemma 2.5.1] to non-spherical principal series modules and we use the same arguments.

**Proof.** We explicitly calculate a basis for 

$$F_{\mu,k}(X'_{\delta}) = \text{Hom}_K(\mu, X'_{\delta} \otimes V^{\otimes k}).$$

Since $X'_{\delta}$ is an induced module from $1_{\delta}$ and $K$ is a compact group, by Frobenius reciprocity this is equal to,

$$F_{\mu,k}(X'_{\delta}) = \text{Hom}_M(\mu|_M, 1_{\delta}^{\otimes k} \otimes V_{|_M}^{\otimes k}).$$

One can tensor by $\mu^*$ to get a space fixed by $M$, hence 

$$F_{\mu,k}(X'_{\delta}) = (\mu^* \otimes 1_{\delta}^{\otimes k} \otimes V^{\otimes k})^M.$$ 

We first prove the result for $G = \text{Sp}_{2n}(\mathbb{R})$. The module $V$ has basis 

$\{f_i^{n_i} : i = 1, ... , n$ and $n_i = \pm 1\}$ and the $j^{th}$ generator of $M$ acts by $-1^\delta_{ij}$ on $f_i^{n_i}$. Therefore if we require $M$ to act trivially on $u \in X'_{\delta} \otimes V^{\otimes k}$ the generators $M_1, ..., M_k$ must act by $1$. Let us first calculate all of the elementary tensors in $X'_{\delta} \otimes V^{\otimes k}$ which are fixed by $M$. The generators $M_1, ..., M_k$ act by $-1$ on $1_{\delta}^{\otimes k}$, hence must act by $-1$ on the tensor part contributed by $V^{\otimes k}$. To satisfy this we need to have $f_i^1$ or $f_i^{-1}$ feature in the tensor of $u$, for every $i = 1, ..., k$. Since there can only be $k$ elements tensored together in $V^{\otimes k}$ then the contribution of $u$ from $V^{\otimes k}$ must be $f_1^{n_1}, ..., f_k^{n_k}$ in some order. The set of elementary tensors in $V^{\otimes k}$ which feature all the required $f_i$ is the $S_k$ orbit of $f_1 \otimes ... \otimes f_k$. Considering not necessarily elementary tensors in $u \in X'_{\delta} \otimes V^{\otimes k}$,

$$v = \sum x_0 \otimes v_1 \otimes ... \otimes v_k,$$
where $v_i \in \{f_{li}^n : l = 1, \ldots, n \text{ and } n_l = \pm 1\}$. The $j^{th}$ generator of $M$, $M_j$, acts by $-1^{dj}$ on $f_l$. Since every elementary tensor in this basis is an eigenvector of the action of $M$ then if $M$ fixes $v = \sum x_0 \otimes v_1 \otimes \ldots \otimes v_k$ then $M$ fixes each elementary tensor in $v$. Hence every $M$ fixed vector in $X_{\delta k}^\nu \otimes V^\otimes k$ is in the subspace

$$\text{span} \left\{ \sum_{w \in S_k} 1_{\delta k} \otimes f_{w(1)}^{n_1} \otimes \ldots \otimes f_{w(k)}^{n_k} : n_i = \pm 1 \right\}.$$  

The size of the basis is $|S_k| \times 2^k = k!2^k = |W(B_k)|$. The proof is almost identical for $\text{dim}(F_{\det, n-k}(X_{\delta k}^\nu))$. One needs to note that all of the generators of $M$ must act by $-1$ on the det isotypic space, since $\det |_M = \text{sgn}$.

Using Frobenius reciprocity one can show,

$$F_{\det, n-k}(X_{\delta k}^\nu) = \text{Hom}_M(\text{sgn}, \delta_k \otimes V^\otimes n-k),$$

which has a basis:

$$F_{\det, n-k}(X_{\delta k}^\nu) = \text{span} \left\{ \sum_{w \in S_{n-k}} 1_{\delta k} \otimes f_{w(i+1)}^{n_{i+1}} \otimes \ldots \otimes f_{w(n)}^{n_n} : n_i = \pm 1 \right\}.$$  

For $G = O(p, q)$ note that $V|_M = V_{p-q} \bigoplus_{i=1}^q \text{triv} \otimes \ldots \otimes \text{sgn} \otimes \ldots \otimes \text{triv}$ and $\mu|_M = \text{triv}_{p-q} \otimes \text{sgn}^q$. Recall the notation $f_i^{n_{i+1}} = e_{p-i+1} + n_ie_{p+i}$, the vectors $f_i^1$ and $f_i^{-1}$ are the two eigenvectors of $M$ with character $\text{triv} \otimes \text{triv} \ldots \otimes \text{sgn} \otimes \ldots \otimes \text{triv}$.

i.e. the $i^{th}$ generator of $O(1)^q$ in $M$ acts by $-1$.

We will prove that $F_{\text{triv} \otimes \text{sgn}, k}(X_{\delta \text{triv}}^\nu)$ has basis

$$\left\{ \sum_{w \in S_k} 1_{\delta k} \otimes f_{w(1)}^{n_1} \otimes \ldots \otimes f_{w(k)}^{n_k} : n_i = \pm 1 \right\}.$$  

The other four calculations are almost identical. Note that this is equivalent to giving a basis for

$$((\text{triv} \otimes \text{sgn})|_M \otimes 1_{\delta \text{triv}}^\nu \otimes V^k)^M$$

which is equal to, as a vector space,

$$(1_{\text{triv}_{p-q} \otimes \text{sgn}^q} \otimes 1_{\delta \text{triv}}^\nu \otimes (V_{p-q} \bigoplus \text{triv} \otimes \ldots \otimes \text{sgn} \otimes \ldots \text{sgn})^k)_M.$$  

The vector $1_{\text{triv}_{p-q} \otimes \text{sgn}^q} \otimes 1_{\delta \text{triv}}^\nu \otimes f_1 \otimes \ldots \otimes f_q$ is fixed by $M$ since $O(p-q)$ acts trivially on each tensor. Furthermore for $i = 1, \ldots, k$ the $i^{th}$ generator of $O(1)^q$ in $M$ acts by $-1$ on $1_{\text{triv}_{p-q} \otimes \text{sgn}^q}$, $1$ on $1_{\delta \text{triv}}^\nu$, and
−1 on $f_1 \otimes ... \otimes f_q$. For $i = k + 1, ... q$ the $i^{th}$ generator of $O(1)^q$ in $M$ acts by $−1$ on $1_{\text{triv}}^{\nu} \otimes \text{sgn}^\mu$ −1 on $1_{\text{triv}}^{\nu} \otimes 1_{\text{triv}}^{\delta}$ and 1 on $f_1 \otimes ... \otimes f_q$. Hence every generator of $M$ acts by 1. An identical argument shows that the orbit of $1_{\text{triv}}^{\nu} \otimes \text{sgn}^\mu \otimes 1_{\text{triv}}^{\nu} \otimes f_1 \otimes ... \otimes f_q$ by $W(B_q)$ is also fixed. Any elementary tensor fixed by $M$ must be of this form; if it is not, one of the generators will act by $−1$. Finally suppose that another vector $v$ is fixed by $M$, then $v$ is a sum of elementary tensors which are all eigenvalues for $O(1)^q$, hence every elementary tensor involved must be fixed. This concludes that $v$ is in the span of the vectors

$$\left\{ \sum_{w \in S_k} 1_{\text{triv}}^{\nu} \otimes f_{w(1)}^{n_1} \otimes ... f_{w(k)}^{n_k} : n_i = \pm 1 \right\}.$$ 

We state the basis for $F_{\mu,k}$ and $F_{\mu,n-k}$ Let $G = O(p,q)$

$$F_{\text{triv} \otimes \text{det},k}(X_{\delta_{\text{triv}}}^{\nu}) = \text{span} \left\{ \sum_{w \in S_k} 1_{\delta_{\text{triv}}}^{\nu} \otimes f_{w(1)}^{n_1} \otimes ... f_{w(k)}^{n_k} : n_i = \pm 1 \right\},$$

$$F_{\text{triv} \otimes \text{triv},q-k}(X_{\delta_{\text{triv}}}^{\nu}) = \text{span} \left\{ \sum_{w \in S_{q-k}} 1_{\delta_{\text{triv}}}^{\nu} \otimes f_{w(k)}^{n_k} : n_i = \pm 1 \right\},$$

$$F_{\text{det} \otimes \text{triv},k}(X_{\delta_{\text{det}}}^{\nu}) = \text{span} \left\{ \sum_{w \in S_k} 1_{\delta_{\text{det}}}^{\nu} \otimes f_{w(1)}^{n_1} \otimes ... f_{w(k)}^{n_k} : n_i = \pm 1 \right\},$$

$$F_{\text{det} \otimes \text{det},q-k}(X_{\delta_{\text{det}}}^{\nu}) = \text{span} \left\{ \sum_{w \in S_{q-k}} 1_{\delta_{\text{det}}}^{\nu} \otimes f_{w(1)}^{n_1} \otimes ... f_{w(q)}^{n_q} : n_i = \pm 1 \right\}.$$ 

\[\square\]

7. Images of principal series modules

We write the Type B/C VW-algebra as $\mathcal{B}_k^q$ and omit $m$.

We show that on minimal principal series representations the functors (Definition 5.1) which take admissible $O(p,q)$ or $Sp_{2m}$-modules to $\mathcal{B}_k^q$-modules naturally descend to graded Hecke algebra $H_k(c)$-modules, for $c$ equal to 0, 1 or $\frac{p-q}{2}$.

In Section 4 Lemma 4.4, we proved that the type $B/C$ VW-algebra has quotients isomorphic to the Hecke algebra $H_k(c)$ with parameter $c \in \mathbb{R}$. This quotient was defined by the relations $e_i, e_{i+1} = 0$ and $\theta_k x_k + x_k \theta_k = 2c - 2r \theta_k$. Hence to show that $F_{\mu,k}(X_{\nu}^{\nu})$ descends to an $H_k(c)$-module we must prove $e_i, e_{i+1} = 0$ and $\theta_k x_k + x_k \theta_k = 2c - 2r \theta_k$ as operators on $F_{\mu,k}(X_{\nu}^{\nu})$. Similarly to show $F_{\mu,n-k}(X_{\nu}^{\nu})$ is an $H_{k-n}(r_{\mu})$-module then we must show $e_i, e_{i+1} = 0$ and $\theta_{n-k} x_{n-k} + x_{n-k} \theta_{n-k} = 2c - 2r \theta_{n-k}$ on $F_{\mu,n-k}(X_{\nu}^{\nu})$. The scalars $r_{\mu}$ and $c_{\mu}$ will be defined in Table 5.1. The arguments of this section are inspired and very similar to [9] Proposition 2.4.5, Lemma 2.7.2. We extend these results to non-spherical principal series modules. We also utilise an approach from the Brauer algebra perspective not used in [9].
Lemma 7.1. c.f. [9, 2.4.5] On the $\mathfrak{B}_k^\theta$ (resp. $\mathfrak{B}_{n-k}^\theta$) module $F_{\mu,k}(X_\theta^\nu)$ (resp. $F_{\mu,n-k}(X_\theta^\nu)$) the idempotents $e_{i,i+1}$ uniformly act by zero.

Proof. Lemma 6.2 states that the basis of $F_{\mu,k}(X_\theta^\nu)$ is given by $1_{\theta}^\nu \otimes f_{w(1)} \otimes \cdots \otimes f_{w[k]}$ for $w \in S_k$. The idempotents $e_{i,i+1}$ act by the projection onto the trivial component of $V_i \otimes V_{i+1}$. The trivial component of $V \otimes V$ is one dimensional with spanning vector $\sum_{n=1}^n f_i \wedge f_i'$. The vector $1_{\theta}^\nu \otimes f_{w(1)}^\nu \otimes \cdots \otimes f_{w(k)}^\nu$ is in the subspace perpendicular to $\sum_{n=1}^n f_i \wedge f_i'$ given in Lemma 3.6. Therefore it is in the kernel of the projection $pr_{i,i+1}$.

Recall Definition 3.4, $\Omega_{i,j} = \sum_{b \in B} (b)_i \otimes (b^*)_j \in U(g)^{k+1}$, and $\Omega_{i,j}^\theta = \sum_{b \in B^{\theta,k}} (b)_i \otimes (b^*)_j$. Lemma 3.7 gives $x_k = \Omega_{0,k}^\theta + \Omega_{1,k}^\theta + \cdots + \Omega_{k-1,k}^\theta$.

As operators on $F_{\mu,k}(X_\theta^\nu)$:
\[
\theta_k x_k + x_k \theta_k = \theta_k \sum_{i<k} \Omega_{i,k}^\theta + \sum_{i<k} \Omega_{i,k}^\theta \theta_k
\]
\[
= (\xi)^k \sum_{i<k} \sum_{b \in B} (b)_i \otimes (b^*)_k + \sum_{i<k} \sum_{b \in B} (b)_i \otimes (b^*)_k (\xi)_k
\]
\[
= \sum_{i<k} \sum_{b \in B} (b)_i \otimes (\xi b^* + b^* \xi)_k.
\]

Conjugating by $\xi$ is the Cartan involution. Therefore
\[
\xi b^* + b^* \xi = \begin{cases} 0 & \text{if } b \in p, \\ 2\xi b^* & \text{if } b \in \mathfrak{k}. \end{cases}
\]

Hence,
\[
\theta_k x_k + x_k \theta_k = 2 \sum_{i<k} \sum_{b \in B^{\theta,k}} (b)_i \otimes (\xi b)_k
\]
\[
= 2 \theta_n \sum_{i<k} \Omega_{i,k}^\theta.
\]

As operators on $F_{\mu,k}(X_\theta^\nu)$
\[
\theta_k x_k + x_k \theta_k = 2\theta_k \sum_{i<k} \Omega_{i,k}^\theta.
\]

Similarly on $F_{\mu,n-k}(X_\theta^\nu)$
\[
\theta_{n-k} x_{n-k} + x_{n-k} \theta_{n-k} = 2 \theta_{n-k} \sum_{i<n-k} \Omega_{i,n-k}^\theta.
\]

Lemma 7.2. c.f. [9, 2.7.2] On the $\mathfrak{B}_k^\theta$-module $F_{\mu,k}(X_\theta^\nu)$,
\[
\theta_k x_k + x_k \theta_k = 2\xi \left( \sum_{b \in B^{\theta,k}} \mu(b) b^* - C^{\mathfrak{g}}_k \right),
\]
where $\mathfrak{g}$ is the centre of $\mathfrak{g}$. 
Proof. Recall Definition \[3.4\] \( \Omega_{ij} = \sum_{b \in B \cap k} (b_i \otimes (\xi b)_k). \) Writing \( \theta_k x_k + x_k \theta_k \) as operators on \( F_{\mu,k}(X^\nu_k) \),

\[
\theta_k x_k + x_k \theta_k = 2\theta_k \sum_{i<k} \Omega_{i,k}^k = 2 \sum_{i<k} \sum_{b \in B \cap k} (b_i \otimes (\xi b)_k).
\]

An element \( g \in g \) acts on the tensor of two modules, \( U \otimes W \), as \( g \otimes 1 + 1 \otimes g \). Extending this, we can write the action of \( b \in U(g) \) as \( \sum_{j=1}^{k+1} (b)_j \) on \( X \otimes V^{\otimes k} \). This gives

\[
\theta_k x_k + x_k \theta_k = 2\theta_k \sum_{b \in B \cap k} (b^*)_k b - \sum_{b \in B \cap k} (bb^*)_k.
\]

By definition \( F_{\mu,k}(X^\nu_k) \) is the \( \mu \) isotypic component of \( X^\nu_k \), hence

\[
\theta_k x_k + x_k \theta_k = 2\theta_k \sum_{b \in B \cap k} (b^*)_k \mu(b) - \sum_{b \in B \cap k} (bb^*)_k.
\]

The operator \( \sum_{b \in B} (bb^*)_k \) is the Casimir operator \( C^t \) on the \( k \)th tensor \( V \). We have \( \mu(b) = 0 \) unless \( b \) is in the centre of \( U(\mathfrak{t}) \) for any character \( \mu \). Let \( \mathfrak{z} \) denote the centre of \( g \). Therefore,

\[
\theta_k x_k + x_k \theta_k = 2\theta_k \left( \sum_{b \in B \cap \mathfrak{z}} \mu(b)(b^*)_k - (C^t)_k \right),
\]

\[
= 2 \left( \xi \left( \sum_{b \in B \cap \mathfrak{z}} \mu(b)b^* - C^t \right) \right)_k.
\]

□

In order to calculate the action of \( \theta_k x_k + x_k \theta_k \) we must understand the operator \( Q_{\mu} = 2\xi \left( \sum_{b \in B \cap \mathfrak{z}} \mu(b)b^* - C^t \right) \) acting on the \( k \)th tensor of \( V \).

**Lemma 7.3.** On the \( \mathfrak{B}^\theta_{n-k} \)-module \( F_{\mu,n-k}(X^\nu_k) \);

\[
\theta_{n-k} x_{n-k} + x_{n-k} \theta_{n-k} = 2 \left( \xi \left( \sum_{b \in B \cap \mathfrak{z}} \mu(b)b^* - C^t \right) \right)_{n-k}.
\]

Replacing \( \mu \) with \( \mu \), this follows the same way as Lemma 7.2.

**Lemma 7.4.** On the module \( V \) the operator \( Q_{\mu} = 2\xi \left( \sum_{b \in B \cap \mathfrak{z}} \mu(b)b^* - C^t \right) \) (resp. \( Q_{\mu} = 2\xi \left( \sum_{b \in B \cap \mathfrak{z}} \mu(b)b^* - C^t \right) \)) is equal to \( 2r_{\mu} + 2c_{\mu} \xi \) (resp. \( 2r_{\mu} + 2c_{\mu} \xi \)), where \( r_{\mu} \) and \( c_{\mu} \) are scalars given below. In fact for \( G = O(p,q) \), \( r_{\mu} \) and \( c_{\mu} \) are independent of \( \mu \).
Recall Lemma 4.2, we have isomorphisms: $H_k(1) \cong H_k$ and $H_k(\frac{1}{2}) \cong H^{Cl}$ and $H_k(0)$ is congruent to an extension of the type $D$ graded Hecke algebra $H^{Dk}$. Hence when $G$ is split, that is $G = O(n+1,n)$ or $Sp_{2n}(\mathbb{R})$ then $c_\mu = 1, \frac{1}{2}$ or 0 and we obtain correspondences between principal series modules of split real orthogonal Lie groups with graded Hecke algebras of type $C$ and split real symplectic groups with graded Hecke algebras of type $B$ and $D$.

Proof. We prove the result first for $G = Sp_{2n}(\mathbb{R})$, in this case $g = sp_{2n}$ and $\mathfrak{k} = gl_n$. The Casimir $C$ acts by the scalar $n$ on $V$. The character $\text{triv}$ is zero uniformly on $\mathfrak{k}$ hence $\text{triv}(b) = 0$ for all $b$ and there is no contribution from $\sum_{b \in B \cap \mathfrak{z}} \text{triv}(b)b^*$. For the operator $\sum_{b \in B \cap \mathfrak{z}} \text{det}(b)b^*$, we note that the centre of $\mathfrak{k} = gl_n(\mathbb{C})$ is the span of the identity matrix, also the character $\text{det}$ of $U(n)$ differentiated to $\mathfrak{k}$ is the trace character of $gl_n$. Taking the spanning vector $Id_n$ of the centre $\mathfrak{z}$ of $gl_n$ then on $V$, $\sum_{b \in B \cap \mathfrak{z}} \text{det}(b)b^*$ is equal to

$$\sum_{b \in B \cap \mathfrak{z}} \text{det}(b)b^* = \text{trace}(Id_n) Id_n^* \quad = n^{\frac{1}{2}} Id_n \quad = Id_n.$$

Since $Id_n$ is symmetric, the identity matrix in $U(\mathfrak{k})$ embedded into $\mathfrak{g}$ is

$$\begin{bmatrix} 0 & i Id_n \\ -i Id_n & 0 \end{bmatrix}.$$ 

The matrix $\xi$, defined by the Cartan involution of $Sp_{2n}(\mathbb{R})$ is equal to

$$\xi = \begin{bmatrix} 0 & i Id_n \\ -i Id_n & 0 \end{bmatrix}.$$ 

Hence

$$\sum_{b \in B \cap \mathfrak{z}} \text{det}(b)b^* = \xi,$$
as operators on $V$.

Now let $G = O(p, q)$, $p + q = 2n + 1$, then $\mathfrak{g} = \mathfrak{so}_{2n+1}$ and $\mathfrak{k} = \mathfrak{so}_p \oplus \mathfrak{so}_q$.

Any character $\mu$ of $K$ differentiated and then restricted to $\mathfrak{z}$ is zero. Hence for any $\mu$,

$$\sum_{b \in \mathfrak{z}} \mu(b) b^* = 0.$$  

We are left to calculate $C^\mathfrak{k}$ on $V$. $C^\mathfrak{k}$ acts by

$$[p \text{Id}_p \ 0 \ \ 0 \ q \text{Id}_q].$$

For $G = O(p, q)$ the semisimple element defining $\theta$ is

$$\xi = \begin{bmatrix} \text{Id}_p & 0 \\ 0 & -\text{Id}_q \end{bmatrix}.$$

Hence for $G = O(p, q)$

$$Q_\mu = 2\xi (\sum_{b \in \mathfrak{z}} \mu(b) b^* - C^\mathfrak{k}) = 2\xi \left(-\frac{p+q}{2} \text{Id} - \frac{p-q}{2} \xi\right) = (q-p) \text{Id}_n - (p+q)\xi.$$

**Corollary 7.5.** For $G = O(p, q)$ or $Sp_{2n}$, consider the principal series module $X^\mu_\nu$ for particular $\mu$ and $\mu'$ given in Table 6.1. On the $\mathfrak{H}_k^\mu$-module $F_{\mu,k}(X^\mu_\nu)$, the following equality holds:

$$\theta_{n-k} x_{n-k} + x_{n-k} \theta_{n-k} = 2r_\mu - 2c_\mu \theta_{n-k}.$$  

Hence by Lemma 4.4, $F_{\mu,k}(X^\mu_\nu)$ is an $\mathfrak{H}_k(c_\mu)$-module via the quotient defined by the relations $e_{i,i+1} = 0$ and $\theta_{n-k} x_{n-k} + x_{n-k} \theta_{n-k} = 2r_\mu + 2c_\mu \theta_{n-k}$. Similarly $F_{\mu,n-k}(X^\mu_{\nu})$ is an $\mathfrak{H}_{n-k}(c_\mu)$-module.

We have shown that the image of $X^\mu_\nu$ under the function $F_{\mu,k}$ naturally descends to a module of the graded Hecke algebra $\mathfrak{H}_k(c_\mu)$.

**Theorem 7.6.** Let $X^\mu_\nu$ be a minimal principal series module of $G = Sp_{2n}(\mathbb{R})$ or $O(p, q)$. Let $\mu$ and $\mu'$ be the particular characters in Table 6.1 and $r_\mu$, $c_\mu$ be particular scalars in Table 7.1. Let $\pi$ denote the homomorphism from $\mathfrak{H}_k^\mu(|m_0, m_1|)$ to $\text{End}(F_{\mu,k}(X^\mu_\nu))$ in Lemmas 3.7 and 3.12. The graded Hecke algebra $\mathfrak{H}_k(c_\mu)$ acts on $F_{\mu,k}(X^\mu_\nu)$, by the homomorphism,

$$\psi : \mathfrak{H}_k(c_\mu) \to \text{End}(F_{\mu,k}(X^\mu_\nu)),$$

$$e_i \mapsto \pi(x_i - r_\mu),$$

$$s_{i,i+1} \mapsto \pi(s_{i,i+1}),$$

$$s_{\epsilon_i} \mapsto \pi(\theta_i).$$

Hence $F_{\mu,k}(X^\mu_\nu)$, can be considered as an $\mathfrak{H}_k(c_\mu)$-module.
Let $\pi$ denote the homomorphism from $\mathfrak{B}^\theta_{n-k}[m]$ to $\text{End}(F_{\mu, n-k}(X_\delta^\nu))$. The graded Hecke algebra $\mathbb{H}_{n-k}(c_{\mu})$ acts on $F_{\mu, n-k}(X_\delta^\nu)$, by the homomorphism,

$$\psi : \mathbb{H}_{n-k}(c_{\mu}) \rightarrow \text{End}(F_{\mu, n-k}(X_\delta^\nu)), $$

$$\epsilon_i \mapsto \pi(x_i - r_{\mu}),$$

$$s_{i,i+1} \mapsto \pi(s_{i,i+1}),$$

$$s_{\epsilon_i} \mapsto \pi(\theta_i).$$

Hence $F_{\mu, n-k}(X_\delta^\nu)$, can be considered an $\mathbb{H}_{n-k}(c_{\mu})$-module.

It should also be noted that as a $\mathfrak{B}^\theta_k$-module $F_{\mu,k}(X_\delta^\nu)$ is essentially an $\mathbb{H}_k(c_{\mu})$-module. That is, there is no element in $\mathfrak{B}^\theta_k$ that has a non-trivial action on $F_{\text{triv},k}(X_\delta^\nu)$ that does not correspond to an element in the Hecke algebra.

For $G = O(n+1, n)$ or $Sp_{2n}(\mathbb{R})$, every principal series module is induced from a character on $M$. Therefore for split real orthogonal or symplectic groups we can entirely describe the Hecke algebra modules resulting from functors $F_{\mu,k}$ and $F_{\mu, n-k}$ on principal series modules. Casselman [8] states that every irreducible representation in $\mathcal{HC}(G)$ is a subrepresentation of a principal series module. Therefore if $X$ is a subrepresentation of $X_\delta^\nu$ then $F_{\mu,k}(X)$ also descends to a Hecke algebra module.

**Theorem 7.7.** Let $G$ be a split real Lie group of type $B$ or $C$. Let $X$ be an irreducible Harish-Chandra $G$-module. Hence $X$ is a subrepresentation of a principal series module $X_\delta^\nu$, then the $\mathfrak{B}^\theta_k$ and $\mathfrak{B}^\theta_{n-k}$-modules $F_{\mu,k}(X)$ and $F_{\mu, n-k}(X)$ naturally descend to $\mathbb{H}_k$ and $\mathbb{H}_{n-k}$-modules.

**Proof.** Let $X$ be an irreducible Harish-Chandra module. Casselman’s theorem shows that $X$ is a submodule of some principal series module, let $X_\delta^\nu$ be such a principal series modules containing $X$ as a submodule. Note that this principal series module may not be unique. Then since $F_{\mu,k}(X)$ is exact and $X$ is a submodule of $X_\delta^\nu$ then $F_{\mu,k}(X)$ is a submodule of $F_{\mu,k}(X_\delta^\nu)$ which is a $\mathbb{H}_k$ module. Therefore $F_{\mu,k}(X)$ is a $\mathbb{H}_k$ module. Similarly for $\mu$ and $n - k$.  

Therefore for every Harish Chandra module of $O(n+1, n)$ and $Sp_{2n}(\mathbb{R})$ we can define two corresponding Hecke algebra modules.
8. Principal series modules map to principal series modules

In this section we take a closer look at the $\mathbb{H}(c_{\mu})$-modules obtained from $X_\delta^\nu$ under the functors $F_{\mu,k}$ and $F_{\mu,n-k}$. We fully classify these as graded Hecke algebra principal series representations related to $\nu$.

Recall that $\mathbb{H}_k(c)$, defined in 4.1, is the graded Hecke algebra associated to $W(B_k)$ with parameter function $c : \Delta \to \mathbb{C}$ such that

$$c_{\epsilon_i - \epsilon_{i+1}} = 1 \text{ and } c_{2\epsilon_i} = 2c.$$  

The algebra $\mathbb{H}_k(c)$ contains the group algebra, $\mathbb{C}[W(B_k)]$, of the hyperoctahedral group. Recall the labeling of vectors in $X \otimes V^{\otimes k}$; we label the tensor product starting at zero. A general elementary tensor in $X \otimes V^{\otimes k}$ would be written $x_0 \otimes v_1 \otimes v_2 \otimes \ldots \otimes v_k$. We begin by restricting to the action of the Weyl group $W(B_k)$ inside $\mathbb{H}(c)$ and computing the resulting $\mathbb{C}[W(B_k)]$-modules isomorphism class. Fix a $M$-character $\delta$ and recall the $K$-characters $\mu$ and $\mu'$ depending on $\delta$ from Table 6.1.

**Lemma 8.1.** As a $\mathbb{C}[W(B_k)]$-module

$$F_{\mu,k}(X_\delta^\nu) \cong \mathbb{C}[W(B_k)],$$

and as a $\mathbb{C}[W(B_{n-k})]$-module

$$F_{\mu,n-k}(X_\delta^\nu) \cong \mathbb{C}[W(B_{n-k})].$$

**Proof.** From Lemma 6.2 we have an explicit basis of $F_{\mu,k}(X_\delta^\nu)$;

$$\text{Hom}_K(\mu, X_{\delta^k} \otimes V^{\otimes k}) = \text{span}\{ \sum_{w \in S_k} 1_\delta \otimes f_{w(1)}^{n_1} \otimes \ldots \otimes f_{w(k)}^{n_k} \}.$$  

The symmetric group $\mathbb{C}[S_k] \subset \mathbb{C}[W(B_k)]$ acts by permuting the tensor product. The reflections in $\mathbb{C}[W(B_k)]$ related to $2\epsilon_i$ act by $id \otimes \ldots \otimes \theta_i \otimes \ldots \otimes id$. They take $f_i$ to $f_i'$ on the $i$th factor of the tensor product.

Take the vector $1_\delta^\nu \otimes f_1 \otimes \ldots \otimes f_k$, the $\mathbb{C}[W(B_k)]$ submodule of $F_{\text{triv},k}(X_{\delta^k})$ generated by $1_\delta \otimes f_1 \otimes \ldots \otimes f_k$ is the subspace spanned by

$$\{ 1_\delta^\nu \otimes f_{w(1)}^{n_1} \otimes \ldots \otimes f_{w(k)}^{n_k} : w \in \mathbb{C}[S_k] \},$$

The only group element of $\mathbb{C}[W(B_k)]$ that fixes $1_\delta^\nu \otimes f_1 \otimes \ldots \otimes f_k$ is the identity, hence this module has dimension equal to $k! 2^k$, the dimension of $\mathbb{C}[W(B_k)]$. The dimension is equal to the dimension of $F_{\text{triv},k}(X_{\delta^k})$, therefore we have equality. An isomorphism between the $\mathbb{C}[W(B_k)]$-module $\mathbb{C}[W(B_k)]$ and $F_{\text{triv},k}(X_{\delta^k})$ can be defined by sending the identity element $e \in \mathbb{C}[W(B_k)]$ to $1_\delta^\nu \otimes f_1 \otimes \ldots \otimes f_k$.

The decomposition of $F_{\mu,n-k}(X_\delta^\nu)$ follows in exactly the same way, sending $e \in \mathbb{C}[W(B_{n-k})]$ to $1_\delta^\nu \otimes f_{k+1} \otimes \ldots \otimes f_n$.  

We have a description of \( F_{\mu,k}(X^\nu_\delta) \) as a \( C[W(B_k)] \)-module. We would like to describe it as an \( H(c_\mu) \)-module. The algebra \( H(c_\mu) \) is generated by \( C[W(B_k)] \) and the affine operators \( \epsilon_1, \ldots, \epsilon_k \). Our calculation reduces to calculating the action of the affine operators \( \epsilon_i \). The operators \( \epsilon_i \in S(a_k) \) act on \( X^\nu_\delta \otimes V^\otimes k \) by

\[
\sum_{0<j<i\leq n} \Omega_{ji} + r_\mu.
\]

We define principal series representations for \( \mathbb{H}_k(c) \). Then we show that the image of \( X^\nu_\delta \) is isomorphic to a principal series representation defined by a particular character.

The subspace \( a_k \subset a \) defined in Example 2.3 is a dimension \( k \)-subspace of \( a \).

**Definition 8.2.** [18] Let \( \lambda \) be a character for \( S(a_k) \subset H_k(c_\mu) \), we define a principal series representation \( X(\lambda) \) for \( H_k(c_\mu) \):

\[
X(\lambda) = \text{Ind}_{S(a_k)}^{\mathbb{H}_k(c)} \lambda.
\]

We write \( 1_\lambda \) for a fixed vector in the image of the character \( \lambda : S(a) \to \mathbb{C} \). The symmetric algebra \( S(a_k) \) is generated by the affine operators \( \epsilon_1, \ldots, \epsilon_k \). The principal series representation can be described as a representation generated by, \( 1_\lambda \), a \( C[W(B_k)] \)-cyclic vector on which \( \epsilon_i \) acts by the scalar \( \lambda(\epsilon_i) \). We prove that the \( C[W(B_k)] \)-module, \( F_{\mu,k}(X^\nu_\delta) \) is as a \( \mathbb{H}_k(c_\mu) \)-module isomorphic to a principal series module for the correct character \( \lambda \).

We fix a specific basis for \( \mathfrak{sp}_{2n} \) and \( \mathfrak{so}(p,q) \). Since the operators \( \Omega_{ij} \in U(\mathfrak{g})^{k+1} \) are defined in terms of, although independent of, a basis for \( \mathfrak{g} \). This basis allows us to explicitly calculate \( \Omega_{0j} \). It should be emphasized that the following basis is a decomposition of \( \mathfrak{g} \) into reduced root spaces under the adjoint action of \( a \). Recall that \( a \subset \mathfrak{sp}_{2n}(\mathbb{R}) \) is

\[
\left\{ \begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix} : B \text{ is diagonal} \right\}.
\]

**Definition 8.3.** Recall the decomposition of the Lie algebra \( \mathfrak{g}_0 \) as

\[
\mathfrak{g}_0 = \mathfrak{n}_0^+ \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0^-,
\]

where \( \mathfrak{a}_0 \) is the maximal abelian subalgebra of \( \mathfrak{p}_0 \) and \( \mathfrak{n}_0^+ \) is the span of the positive root spaces with respect to the restricted root decomposition. Let \( B_0^+ , B_0^- , B_0 \) be fixed bases for \( \mathfrak{n}_0^+ , \mathfrak{n}_0^- \) and \( \mathfrak{a}_0 \). The restricted roots \( \Sigma \) are \( \pm \epsilon_i , \pm \epsilon_j , \pm \epsilon_j \). We will denote a vector in the positive root space \( \lambda \in \Sigma^+ \) by \( n_\lambda \) and the negative root space will be \( \hat{n}_\lambda \). For example \( n_{\epsilon_i - \epsilon_j} \) for \( i < j \) is in \( \mathfrak{n}^+ \). And \( \hat{n}_{\epsilon_i - \epsilon_j} \in \mathfrak{n}_0^- \). We will scale \( \hat{n}_\lambda \) such that

\[
\hat{n}_\lambda = n_{-\lambda} = \theta(n_\lambda).
\]
Hence $n_\lambda + \hat{n}_\lambda$ is $\theta$-invariant and hence in $\mathfrak{k}$.

**Definition 8.4.** For $1 \leq s, t \leq n$, the matrix $E_{s,t}$ is the matrix with a $1$ in the $s, t$ position and zero elsewhere. Let $i < j$. Set

\[
\begin{align*}
E_{s,t} &= E_{i,j} + E_{i,n+j} - E_{j,i} + E_{j,n+i} + E_{n+i,n+j} + E_{n+j,i} - E_{n+j,n+i}, \\
\hat{E}_{s,t} &= -E_{i,j} + E_{i,n+j} + E_{j,i} + E_{j,n+i} - E_{n+i,n+j} + E_{n+j,i} + E_{n+j,n+i}, \\
\tilde{E}_{s,t} &= E_{i,j} + E_{i,n+j} - E_{j,i} + E_{j,n+i} - E_{n+i,n+j} - E_{n+j,i} - E_{n+j,n+i}, \\
\hat{\tilde{E}}_{s,t} &= -E_{i,j} - E_{i,n+i} + E_{n+i,i} - E_{n+i,n+i}, \\
\tilde{E}_{s,t} &= -E_{i,j} - E_{i,n+i} + E_{n+i,i} + E_{n+i,n+i}, \\
\alpha_{s,t} &= E_{i,n+1} + E_{n+i,i}.
\end{align*}
\]

These vectors give a reduced root space decomposition for $\mathfrak{sp}_{2n}(\mathbb{R}) = n_0^+ \oplus a_0 \oplus n_0^-$ where $\alpha_{ei} \in a_0$, $n \in n_0^+$ and $\hat{n} \in n_0$.

**Example 8.5.** Let $\mathfrak{g} = \mathfrak{sp}_4$. We give the basis given in Definition 8.4 for $n^+$,

\[
\begin{align*}
n_{\epsilon_1 - \epsilon_2} &= \begin{bmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix}, \\
n_{\epsilon_1 + \epsilon_2} &= \begin{bmatrix} 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{bmatrix}, \\
n_{\epsilon_1} &= \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\
n_{\epsilon_2} &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 \end{bmatrix}.
\end{align*}
\]

**Definition 8.6.** Let $\mathfrak{g}_0 = \mathfrak{so}(p, q)$ we follow [17], VI, pg. 371 Example $\mathfrak{so}(p, q)$.
\[
\begin{align*}
n_{\epsilon_i - \epsilon_j} &= E_{p-j+1,p-i+1} + E_{p-j+1,p+i} - E_{p-i+1,p-j+1} + E_{p-i+1,p+j} \\
&- E_{p+i,p-j+1} - E_{p+i,p+j} - E_{p+j,p-i+1} + E_{p+j,p+i}, \\
n_{\epsilon_i + \epsilon_j} &= E_{p-j+1,p-i+1} - E_{p-j+1,p+i} - E_{p-i+1,p-j+1} + E_{p-i+1,p+j} \\
&- E_{p+i,p-j+1} + E_{p+i,p+j} + E_{p+j,p-i+1} - E_{p+j,p+i}, \\
\hat{n}_{\epsilon_i - \epsilon_j} &= E_{p-j+1,p-i+1} - E_{p-j+1,p+i} - E_{p-i+1,p-j+1} - E_{p-i+1,p+j} \\
&+ E_{p+i,p-j+1} - E_{p+i,p+j} + E_{p+j,p-i+1} + E_{p+j,p+i}, \\
n_{\epsilon_i + \epsilon_j} &= E_{p-j+1,p-i+1} + E_{p-j+1,p+i} - E_{p-i+1,p-j+1} - E_{p-i+1,p+j} \\
&+ E_{p+i,p-j+1} + E_{p+i,p+j} - E_{p+j,p-i+1} - E_{p+j,p+i}.
\end{align*}
\]

The root space for \( \epsilon_i \) is \( p - q \) dimensional. Let \( l = 1, \ldots, p - q \) then

\[
n_{\epsilon_i}^l = E_{l,p-i+1} - E_{l,p+i} - E_{p-i+1,l} - E_{p+i,l}.
\]

Finally

\[
a_{\epsilon_i} = E_{p-i+1,p+i} + E_{p+i,p-i+1}.
\]

**Example 8.7.** Let \( \mathfrak{g}_0 = \mathfrak{so}(3, 2) \). We give the basis given in Definition 8.6 for \( \mathfrak{n}_0^+ \).
Lemma 8.8. For $G = Sp_{2n}$, recall the basis $f_i = e_i + e_{n+i}$, $f'_i = e_i - e_{n+i}$ of $V = \mathbb{C}^{2n}$. For $G = O(p,q)$ we recall that $f_i = e_{p-i+1} + e_{p+i}$, $f'_i = e_{p-i+1} - e_{p+i}$. Then by left multiplication of the given matrix in Definitions 8.4 and 8.6 we can calculate the following actions on $f_i$: 

\[
\begin{align*}
n_{\epsilon_i - \epsilon_j}(f_k) &= 0 \text{ for all } k, \\
n_{\epsilon_i + \epsilon_j}(f'_k) &= \begin{cases} 2f'_j & \text{if } f'_k = f'_j, \\ 0 & \text{otherwise}. \end{cases} \\
n_{\epsilon_i}(f_k) &= 0 \text{ for all } k, \\
n_{\epsilon_i}(f'_k) &= \begin{cases} 2f_k & \text{if } f'_k = f'_i, \\ 0 & \text{otherwise}, \end{cases} \\
n_{\epsilon_i - \epsilon_j}(f_k) &= \begin{cases} 2f_i & \text{if } f_k = f_j, \\ 0 & \text{otherwise}, \end{cases}
\end{align*}
\]
\[ n_{\epsilon_i-\epsilon_j}(f'_k) = \begin{cases} 2f'_j & \text{if } f'_k = f_i, \\ 0 & \text{otherwise}, \end{cases} \]

\[ (n_{\epsilon_i-\epsilon_j} + \hat{n}_{\epsilon_i-\epsilon_j})(f_k) = \begin{cases} f_i & \text{if } f_k = f_j, \\ -f_j & \text{if } f_k = f_i, \\ 0 & \text{otherwise}. \end{cases} \]

**Proof.** This follows from left multiplication of the elements of \( \mathfrak{sp}_{2n} \) and \( \mathfrak{so}(p, q) \) on the defining module \( V \) with elements \( f_i \) and \( f'_i \) in the basis of \( V \).

To prove that the \( \mathbb{C}[W(B_k)] \)-module is in fact isomorphic to a principal series \( \mathbb{H}_k(c_\mu) \)-module we need to find a \( \mathbb{C}[W(B_k)] \) cyclic vector such that the \( \epsilon_i \) act by scalars on this cyclic vector. The cyclic vector is \( 1^\nu \otimes f_1 \otimes \ldots \otimes f_k \).

**Lemma 8.9.** On the vector \( 1^\nu \otimes f_1 \otimes \ldots \otimes f_k \) the operator \( \Omega_{0l} \) acts by

\[ \nu(\epsilon_l) - \sum_{i<l}(s_{il} + id) - \sum_{i>l} id. \]

**Proof.** Recall that \( \Omega_{0l} \) is defined to be \( \sum_{b \in B}(b)_0 \otimes (b^*)_l \) for a given basis \( B \) of \( \mathfrak{g}_0 \). We choose to use the fixed basis defined in Definition 8.3.

The subspace \( \mathfrak{a} \) is the Lie algebra of the subgroup \( A \subset G \). The basis of \( \mathfrak{a} \) defined in 8.4 and 8.6 is such that \( a_{\epsilon_i}(f_j) = \delta_{ij}f_j \). Furthermore \( a_{\epsilon_i} \) acts on the cyclic vector \( 1^\nu \) of \( X^\nu_k \) by \( \nu(x_i) \). Therefore the contribution from \( \mathfrak{a} \subset \mathfrak{g} \) is

\[ (a_{\epsilon_i})_0 \otimes (a_{\epsilon_i})_l = \delta_{il} \nu(x_i). \]

The module \( X^\nu_k \) is induced from the character \( \delta \otimes e^\nu \otimes 1 \) of \( MAN \), which is the trivial character on \( N \). The space \( \mathfrak{n}_0^+ \) is the Lie algebra of \( N \). The differential of the trivial character to \( \mathfrak{n}_0^+ \) is zero. Therefore \( (n)_0 \) acts by zero on \( 1^\nu \) for all \( n \in \mathfrak{n}_0^+ \). Hence the contribution from \( \mathfrak{n}^+ \) is:

\[ (n)_0 \otimes (n)_l^* = 0, \text{ for } n \in \mathfrak{n}^+. \]

Since \( n \in \mathfrak{n}_0^+ \) annihilates \( 1^\nu \) then \( (n)_0 \otimes (b)_l = 0 \) for any \( b \in \mathfrak{g}_{2n}, n \in \mathfrak{n}^+ \), a fact we will use later in this proof. The operator \( \hat{n}_{\epsilon_i-\epsilon_j} \) is equal to \( \frac{1}{2}(n_{\epsilon_i-\epsilon_j})_l \) which is zero on any \( f_k \) hence;

\[ (\hat{n}_{\epsilon_i-\epsilon_j})_0 \otimes (\hat{n}_{\epsilon_i-\epsilon_j})_l^* = 0. \]

Similarly \( n_{\epsilon_i} \) is zero on any \( f_k \) therefore;

\[ (\hat{n}_{\epsilon_i})_0 \otimes (\hat{n}_{\epsilon_i})_l = 0. \]

The only remaining basis elements to consider are those of the form \( n_{\epsilon_i-\epsilon_j} \) from \( \mathfrak{n}_0^- \subset \mathfrak{g}_0 \). We utilise the trick that as a \( K \)-module \( F_{\mu,k}(X^\nu_{\delta k}) \)
is just the $\mu$ isotypic component of $X^\nu_{\delta^k} \otimes V^{\otimes k}$. The contribution from $
hat_{\nu_{l_1-l_j}}$ is:

$$(\nhat_{\nu_{l_1-l_j}})_0 \otimes (\nhat^*_{\nu_{l_1-l_j}})_t.$$ We can add the operator $(n_{\nu_{l_1-l_j}})_0 \otimes (\nhat^*_{\nu_{l_1-l_j}})_t$ which since $n_{\nu_{l_1-l_j}} \in \mathfrak{n}^+$, by above, acts by zero. Therefore we are not modifying the original operator,

$$(\nhat_{\nu_{l_1-l_j}})_0 \otimes (\nhat^*_{\nu_{l_1-l_j}})_t = \frac{1}{2} (\nhat_{\nu_{l_1-l_j}} + n_{\nu_{l_1-l_j}})_0 \otimes (n_{\nu_{l_1-l_j}})_t.$$ The vector $\nhat_{\nu_{l_1-l_j}} + n_{\nu_{l_1-l_j}}$ is $\theta$-invariant, hence is in $\mathfrak{f}$. Recall that for $k \in \mathfrak{f}$ acting on the tensor $X \otimes V^{\otimes k}$ that $k = \sum_{i=0}^k (k)_i$. Since we are working with the $\mu$-isotypic space, we replace $\nhat_{\nu_{l_1-l_j}} + n_{\nu_{l_1-l_j}} \in \mathfrak{f}$ by $\mu(\nhat_{\nu_{l_1-l_j}} + n_{\nu_{l_1-l_j}})$ and subtract the difference to find,

$$(\nhat_{\nu_{l_1-l_j}})_0 \otimes (\nhat^*_{\nu_{l_1-l_j}})_t = \frac{1}{2} (\nhat_{\nu_{l_1-l_j}} - n_{\nu_{l_1-l_j}}) \otimes (n_{\nu_{l_1-l_j}})_t - \frac{1}{2} \sum_{m>0} (\nhat_{\nu_{l_1-l_j}} - n_{\nu_{l_1-l_j}})_m \otimes (n_{\nu_{l_1-l_j}})_t.$$ The character $\mu$ (or $\mu$) differentiated to $\mathfrak{a}$ is zero (or the trace character) hence $\mu(\nhat_{\nu_{l_1-l_j}} + n_{\nu_{l_1-l_j}}) = 0$. Lemma 8.8 gives the explicit action of $n_{\nu_{l_1-l_j}}$ on $f_k$, using this one can determine the action;

$$(\nhat_{\nu_{l_1-l_j}})_0 \otimes (\nhat^*_{\nu_{l_1-l_j}})_t = \begin{cases} -s_{tt} - id & \text{if } f_{tt} = f_i \text{ and } f_{ti} = f_j, \\ -id & \text{if } f_{tt} = f_i, \\ 0 & \text{otherwise}. \end{cases}$$

The only non-zero terms are contributed by $a_{\nu_{l_1}}$, and $\nhat_{\nu_{l_1-l_i}}$ and $\nhat_{l_1-l_i}$. Which act, on the cyclic vector, by $\nu(\epsilon_l), -s_{tt} - id$ and $-id$ respectively. Summing these up gives,

$$\Omega_{tt} = \nu(\epsilon_l) - \sum_{t<l} (s_{tt} + id) - \sum_{t>l} id,$$ on the $\mathbb{C}[W(B_k)]$ cyclic vector $1^n_{\delta^k} \otimes f_1 \otimes \ldots \otimes f_k$. □

The equivalent statement for $F_{\mu,n-k}(X^\nu_{\delta})$ is below. It follows from the proof of Lemma 8.9

**Lemma 8.10.** On the vector $1^n_{\delta^k} \otimes f_{k+1} \otimes \ldots \otimes f_n$ the operator $\Omega_{tt}$ acts by

$$\nu(\epsilon_{k+t}) - \sum_{t<l} (s_{k+t,k+t} + id) - \sum_{t>l} id,$$ for $l = 1, \ldots, n-k$. 
Corollary 8.11. The operator \( \varepsilon_i = \sum_{i<l} \Omega_{ul} + n \) acts by the scalar \( \nu(\varepsilon_i) \) on the vector \( 1^\nu_\delta \otimes f_1 \otimes ... \otimes f_k \).

Proof. This follows from the fact that \( \Omega_{ul} \) acts by \( \nu(\varepsilon_i) - n - \sum_{t<l} s_{ul} \) and, by Lemma 3.9, \( \sum_{i=1}^{l-1} \Omega_{ul} \) acts by \( \sum_{t<l} s_{ul} \) on \( 1^\nu_\delta \otimes f_1 \otimes ... \otimes f_k \). \( \square \)

Corollary 8.12. The operator \( \varepsilon_i = \sum_{i<l} \Omega_{ul} + n \) acts by the scalar \( \nu(\varepsilon_{k+1}) \) on the vector \( 1^\nu_\delta \otimes f_{k+1} \otimes ... \otimes f_n \).

Definition 8.13. Example 2.3 defines subspaces \( a_k \) and \( \bar{a}_{n-k} \) of \( a \) such that
\[
a = a_k \oplus \bar{a}_{n-k}.
\]
Let \( \nu \) be a character of \( a \). Define \( \nu_k \) to be the restricted character
\[
\nu|_{a_k}
\]
and \( \bar{\nu}_{n-k} \) to be \( \nu|_{\bar{a}_{n-k}} \).

For a principal series module \( X_\delta^\nu \) we have shown that as a \( W(B_k) \)-module \( F_{\mu,k}(X_\delta^\nu) \) is isomorphic to \( C[W(B_k)] \) and as a Hecke algebra module it is a principal series module induced from a character of \( S(V) \subset \mathbb{H}(c_\mu) \).

Theorem 8.14. For \( G = Sp_{2n}(\mathbb{R}) \) or \( O(p,q) \) \( p+q = 2n+1 \), the module \( F_{\mu,k}(X_\delta^\nu) \) is isomorphic to the \( \mathbb{H}(c_\mu) \) principal series module
\[
X(\nu_k) = \text{Ind}_{S(a_k)}^{\mathbb{H}(c_\mu)} \nu_k.
\]
The module \( F_{\mu,n-k}(X_\delta^\nu) \) is isomorphic to the \( \mathbb{H}_{n-k}(c_\mu) \) principal series module
\[
X(\bar{\nu}_{n-k}) = \text{Ind}_{S(\bar{a}_{n-k})}^{\mathbb{H}_{n-k}(c_\mu)} \bar{\nu}_{n-k}.
\]

For spherical principal series representations, this recovers the results of [9, Theorem 3.0.4].

Proof. One defines an isomorphism by taking the given cyclic vector \( 1^\nu_\delta \otimes f_1 \otimes ... \otimes f_k \in F_{\text{triv},k}(X_\delta^\nu) \) to the cyclic vector \( 1_{\nu_k} \) of \( X(\nu_k) \). Both vectors are \( C[W(B_k)] \) cyclic. By Corollary 8.11 the affine operators \( \varepsilon_i \) act on both vectors by \( \nu_k(\varepsilon_i) \), therefore this is a well-defined isomorphism.

Lemma 6.12 gives a basis of \( F_{\text{det},n-k}(X_\delta^\nu) \):
\[
\{ 1^\nu_\delta \otimes f_{w(1)+k}^{n_1} \otimes ... \otimes f_{w(n-k)+k}^{n_{n-k}} : w \in S_{n-k} \}.
\]
For \( F_{\mu,k}(X_\delta^\nu) \) and \( X(\bar{\nu}_{n-k}) \), both modules are \( C[W(B_{n-k})] \) cyclic, and Corollary 8.12 shows that the affine operators \( \varepsilon_i \) for \( i = 1, ..., n-k \), act on the same scalar on the cyclic vector \( 1^\nu_\delta \otimes f_{k+1} \otimes ... \otimes f_n \) and \( 1_{\bar{\nu}_{n-k}} \), respectively. \( \square \)
Casselman’s theorem [8] states that every irreducible representation in \( \mathcal{HC}(G) \) is a subrepresentation of a principal series module. If \( G \) is a split real orthogonal or symplectic group then \( M \) is abelian and every principal series module is induced from a character.

**Theorem 8.15.** Let \( G \) be \( O(n+1,n) \) or \( Sp_{2n}(\mathbb{R}) \), then \( G \) is split. Let \( X \) be an irreducible \( G \)-module. Let \( X^\theta \) be a principal series representation that contains \( X \), then the \( \mathfrak{B}_{k}^\theta \) and \( \mathfrak{B}_{n-k}^\theta \)-modules

\[
F_{\mu,k}(X) \quad \text{and} \quad F_{\nu,n-k}(X)
\]

are submodules of the \( \mathbb{H}_{k}(c_\mu) \) and \( \mathbb{H}_{n-k}(c_\mu) \)-modules

\[
X(\nu_k) \quad \text{and} \quad X(\bar{\nu}_{n-k}).
\]

9. Hermitian forms

In this section we define two star operations on \( \mathfrak{B}_{k}^\theta \). Through the quotients defined in Lemma 4.4 these star operations descend to the usual star operations on the graded Hecke algebras \( \mathbb{H}_{k}(c) \) [2]. We then show that a Harish-Chandra module \( X \in \mathcal{HC}(G) \) with invariant Hermitian form gets mapped, by \( F_{\mu,k} \), to a \( \mathfrak{B}_{k}^\theta \)-module with invariant Hermitian form. This extends the results in [9] to any Harish-Chandra module. Furthermore, if \( X \) is a unitary module, then it maps to a unitary module for \( \mathfrak{B}_{k}^\theta \). In this section we assume that \( \mu \) is a character of \( K \).

**Definition 9.1.** Let \( G \) be \( O(p,q) \) \( p+q=2n+1 \) or \( Sp_{2n}(\mathbb{R}) \), let \( \mathfrak{g}_0 \) be its Lie algebra, with complexification \( \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \). Conjugation \( \bar{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{g} \) is defined by the real form \( \mathfrak{g}_0 \). Define the star operation as the conjugate linear map \( \ast : \mathfrak{g} \to \mathfrak{g} \) such that:

\[
g^\ast = -\bar{g} \quad \text{for all} \quad g \in \mathfrak{g}.
\]

Define the operation \( \cdot : \mathfrak{g} \to \mathfrak{g} \) by:

\[
p^\cdot = \bar{p} \quad \text{for all} \quad p \in \mathfrak{p}.
\]

\[
k^\cdot = -\bar{k} \quad \text{for all} \quad k \in \mathfrak{k}.
\]

Recall Definition 4.1 of the Hecke algebra \( \mathbb{H}_{k}(c) \). We define the Drinfeld presentation of \( \mathbb{H}_{k}(c) \).

**Definition 9.2.** Let \( R \) be a root system with pairing \( \langle \cdot, \cdot \rangle : V \times V \to \mathbb{C} \), simple roots \( \delta \), and a parameter function \( c : \Delta \to \mathbb{C} \). Denote the Weyl group of \( R \) by \( W(R) \). The Drinfeld Hecke algebra \( \mathfrak{H}_R(c) \) is a quotient of the algebra

\[
\mathbb{C}[W(R)] \rtimes T(V),
\]
by the relations
\[ w\tilde{\alpha}w^{-1} = \tilde{w}(\alpha) \text{ for all } w \in W(R), \alpha \in V, \]
\[ [\tilde{\alpha}, \tilde{\beta}] = \sum_{\gamma, \delta \in \Delta} c(\gamma)c(\delta)(\langle \tilde{\alpha}, \gamma \rangle \langle \tilde{\beta}, \delta \rangle - \langle \tilde{\beta}, \gamma \rangle \langle \tilde{\alpha}, \delta \rangle)s_\gamma s_\delta. \]

Lemma 9.3. The Drinfeld Hecke algebra and the graded Hecke algebra are defined by a root system and a parameter on simple roots. If the defining root systems and parameters are equal then these algebras are isomorphic.

Proof. One defines an isomorphism \( \phi : \mathbb{H}_R(c) \to \mathcal{H}_R(c) \) by
\[ \phi(\alpha - \frac{1}{2} \sum_{\gamma \in \Delta} c(\gamma)\langle \gamma, \alpha \rangle s_\gamma) = \tilde{\alpha}, \]
\[ \phi(w) = w, \quad \forall w \in W(R). \]

\[ \square \]

Given that the graded Hecke algebra and the Drinfeld Hecke algebra are isomorphic we omit the different notation and denote the graded Hecke algebra by \( \mathbb{H}_R(c) \). We uniformly denote a generator in the Drinfeld presentation by \( \tilde{\alpha} \) and \( \alpha \) denotes a generator in the Lusztig presentation (Definition 4.1).

Definition 9.4. Let \( * : \mathbb{H}_k(c) \to \mathbb{H}_k(c) \) be the antihomomorphism such that:
\[ \tilde{\alpha}^* = -\overline{\alpha} \text{ for all } \alpha \in \tilde{t}, \]
\[ g^* = g^{-1} \text{ for all } g \in W(B_k). \]

Let \( \cdot : \mathbb{H}_k(c) \to \mathbb{H}_k(c) \) be the antihomomorphism such that:
\[ \alpha^* = \overline{\alpha} \text{ for all } \alpha \in t \text{ (equivalently } \tilde{\alpha}^* = \overline{\alpha}), \]
\[ g^* = g^{-1} \text{ for all } g \in W(B_k). \]

Here \( v \) is the complex conjugate of \( v \).

Let \( w_0 \) be the longest element in \( W(B_k) \), it is an involution and is generated by \( k^2 \) simple reflections. It is in the centre of \( W(B_k) \). On the space of roots \( w_0 \) acts by \(-1\).

Lemma 9.5. The longest element \( w_0 \) can be written as
\[ w_0 = \theta_1\theta_2...\theta_k. \]

It is well known that the longest element \( w_0 \) relates the two star operations \( * \) and \( \cdot \) in \( \mathbb{H}_k(c) \).
Lemma 9.6. 

\[ h^* = w_0h^\bullet w_0 \text{ for all } h \in \mathbb{H}(c). \]

Lemma 9.7. The longest element \( w_0 \) is central in the finite Brauer algebra \( Br_k[m] \).

Proof. The element \( w_0 \) is central in \( W(B_k) \), therefore it is sufficient to prove that \( w_0 \) commutes with the idempotents \( e_{i,i+1} \). The reflections \( \theta_l \) commute with \( e_{i,j} \).

\[ [e_{i,j}, \theta_l] = 0 \text{ for all } i, j, l. \]

We have,

\[ w_0 e_{i,j} w_0 = \theta_1 \theta_2 \ldots \theta_k e_{i,j} \theta_k \ldots \theta_1 = e_{i,j}. \]

Hence \( w_0 \) is central in the finite Brauer algebra. \( \square \)

Since \( w_0 = \theta_1 \theta_2 \ldots \theta_k \) then as an operator on \( X \otimes V^\otimes k \)

\[ \pi(w_0) = (\xi)_1(\xi)_2 \ldots (\xi)_k = id \otimes \xi \otimes \xi \ldots \otimes \xi. \]

We calculate how \( w_0 \) and \( \Omega_{ij}, \Omega_{ij}^t, \Omega_{ij}^p \) interact.

Lemma 9.8. As operators on \( X \otimes V^\otimes k \),

\[ w_0(\Omega_{ij}^t)w_0 = \Omega_{ij}^t \text{ for all } 0 \leq i < j \leq n, \]

\[ w_0(\Omega_{ij}^p)w_0 = \begin{cases} \Omega_{ij}^p & \text{for all } 0 < i < j \leq n, \\ -\Omega_{ij}^p & \text{when } i = 0. \end{cases} \]

Proof. Recall \( \xi g \xi = \begin{cases} g & \text{if } g \in \mathfrak{t}, \\ -g & \text{if } g \in \mathfrak{p}. \end{cases} \) Therefore one finds that \( \pi(w_0) = id \otimes \xi \otimes \ldots \xi \) commutes with \( \Omega_{ij} = \sum_{b \in B \cap \mathfrak{p}}(b)_i \otimes (b^*)_j. \) For \( \Omega_{ij}^p \) we have:

\[ w_0(\Omega_{ij}^p)w_0 = (id \otimes \xi \otimes \ldots \otimes \xi) \Omega_{ij}^p (id \otimes \xi \otimes \ldots \otimes \xi), \]

\[ = (id \otimes \xi \otimes \ldots \otimes \xi) \sum_{b \in B \cap \mathfrak{p}} (b)_i \otimes (b^*)_j (id \otimes \xi \otimes \ldots \otimes \xi), \]

\[ = \begin{cases} \sum_{b \in B \cap \mathfrak{p}} (b)_i \otimes (\xi b)_j & \text{if } i = 0, \\ \sum_{b \in B \cap \mathfrak{p}} (\xi b)_i \otimes (b)_j & \text{if } i \neq 0, \end{cases} \]

\[ = \begin{cases} \sum_{b \in B \cap \mathfrak{p}} (b)_i \otimes (-b)_j & \text{if } i = 0, \\ \sum_{b \in B \cap \mathfrak{p}} (-b)_i \otimes (b)_j & \text{if } i \neq 0, \end{cases} \]

\[ = \begin{cases} -\sum_{b \in B \cap \mathfrak{p}} (b)_i \otimes (b)_j & \text{if } i = 0, \\ \sum_{b \in B \cap \mathfrak{p}} (b)_i \otimes (b)_j & \text{if } i \neq 0. \end{cases} \]

\( \square \)
Definition 9.9. Let $\cdot : B_k^\theta \to B_k^\theta$ be the conjugate linear antihomomorphism defined on the generators as follows:

$$z_i^\cdot = z_i$$

$$g^\cdot = g^{-1} \text{ for } g \in W(B_k)$$

$$e_{i,i+1}^\cdot = e_{i,i+1}.$$  

Remark 9.10. To check this antihomomorphism is well defined one must just check that the relations in Definition 3.10 are fixed.

Definition 9.11. Let $^* : B_k^\theta \to B_k^\theta$ by the antihomomorphism such that,

$$b^* = w_0 b^\cdot w_0.$$  

Remark 9.12. Since $w_0$ is central in the finite Brauer algebra then

$$g^* = g^{-1} \text{ for } g \in W(B_k) \text{ and } e_{i,j}^* = e_{i,j}.$$  

Lemma 9.13. Under the quotients in Lemma 4.14 the antihomomorphisms $^* : B_k^\theta \to B_k^\theta$ and $\cdot : B_k^\theta \to B_k^\theta$ descend to the antihomomorphisms $^* : \mathbb{H}_k(c) \to \mathbb{H}_k(c)$ and $\cdot : \mathbb{H}_k(c) \to \mathbb{H}_k(c)$ respectively.

Proof. The operation $\cdot$ fixes $e_{i,i+1}$ and

$$\theta_k z_k + z_k \theta_k = 2c - 2r \theta_k.$$  

Therefore $\cdot$ on $B_k^\theta$ descends to $\mathbb{H}_k(c)$. On the generators of $\mathbb{H}_k(c)$ it fixes the affine generators and is the inverse antihomomorphism on the group $W(B_k)$. Hence the operation $\cdot$ on $B_k^\theta$ descends to the antihomomorphism $\cdot$ on $\mathbb{H}_k(c)$. Since

$$h^* = w_0 h^\cdot w_0,$$  

in both $B_k^\theta$ and $\mathbb{H}_k(c)$ then the star operation $^*$ on $B_k^\theta$ descends to $^*$ on $\mathbb{H}_k(c)$. \hfill \Box

We give a new set of generators for $B_k^\theta$.

Definition 9.14. Define

$$\tilde{z}_i = \frac{z_i - w_0 z_i w_0}{2}, \text{ for } i = 1, \ldots, k,$$

then

$$B_k^\theta \cong \langle \tilde{z}_i, s_{j,j+1}, e_{j,j+1}, \theta_i \rangle.$$
The operators \( \tilde{z}_i \) form a Drinfeld type presentation for \( \mathfrak{B}^\theta_k \), they descend to the Drinfeld presentation of \( \mathbb{H}_k(c) \) under the quotients defined in [4.4] As operators on \( X \otimes V^\otimes k \):

\[
\pi(\tilde{z}_i) = \frac{1}{2} \pi(z_i - w_0 z_i w_0) \\
= \frac{1}{2} \left( \sum_{j<i} \Omega_{ij} - (\xi)_1(\xi)_2(\xi)_3(\xi)_k \sum_{j<i} \Omega_{ij} (\xi)_1(\xi)_2(\xi)_3(\xi)_k \right),
\]

\[
= \frac{1}{2} \left( \sum_{j<i} \Omega_{ij}^l + \Omega_{ij}^p - (\xi)_1(\xi)_2(\xi)_3(\xi)_k \sum_{j<i} \Omega_{ij}^l + \Omega_{ij}^p \Omega_{ij} (\xi)_1(\xi)_2(\xi)_3(\xi)_k \right),
\]

\[
= \frac{1}{2} \left( \sum_{j<i} \Omega_{ij}^l + \Omega_{ij}^p + \Omega_{0i} - \Omega_{0i} \sum_{0<j<i} \Omega_{ij}^l + \Omega_{ij}^p \right),
\]

\[
= \Omega_{0i}^p.
\]

**Remark 9.15.** With this presentation of \( \mathfrak{B}^\theta_k \) the operation \( * \) is defined as

\[
\tilde{z}_i^* = -\tilde{z}_i,
\]

\[
g^* = g^{-1} \text{ for all } g \in w(B_k),
\]

\[
e_{i,i+1}^* = e_{i,i+1}.
\]

**Definition 9.16.** Let \( X \) be a complex vector space, a Hermitian form \( \langle \cdot, \cdot \rangle_X \) on \( X \) is a map \( \langle \cdot, \cdot \rangle_X : X \times X \rightarrow \mathbb{C} \) such that

\[
\langle \lambda_1 x_1 + \lambda_2 x_2, x' \rangle_X = \lambda_1 \langle x_1, x' \rangle_X + \lambda_2 \langle x_2, x' \rangle_X \text{ for all } x_1, x_2, x' \in X, \lambda_1, \lambda_2 \in \mathbb{C},
\]

\[
\langle x, \lambda_1 x_1' + \lambda_2 x_2' \rangle_X = \lambda_1 \langle x, x_1' \rangle_X + \lambda_2 \langle x, x_2' \rangle_X \text{ for all } x_1', x_2', x \in X, \lambda_1, \lambda_2 \in \mathbb{C}.
\]

**Definition 9.17.** Let \( X \) be a \( \mathcal{HC}(G) \)-module. A Hermitian form \( \langle \cdot, \cdot \rangle_X \) is \( * \)-invariant if:

\[
\langle g(x_1), x_2 \rangle_X = \langle x_1, g^*(x_2) \rangle, \text{ for all } x_1, x_2 \in X \text{ and } g \in \mathfrak{g}.
\]

**Definition 9.18.** Let \( U \) be an \( \mathbb{H}_k(c) \)-module. A Hermitian form \( \langle \cdot, \cdot \rangle_U \) on \( U \) is \( * \)-invariant with respect to \( \ast \) if:

\[
\langle h(x_1), x_2 \rangle_U = \langle x_1, h^*(x_2) \rangle, \text{ for all } x_1, x_2 \in U \text{ and } h \in \mathbb{H}_k(c).
\]

Similarly for \( U \) a \( \mathfrak{B}^\theta_k \)-module, a Hermitian form \( \langle \cdot, \cdot \rangle_U \) on \( U \) is \( * \)-invariant if

\[
\langle b(x_1), x_2 \rangle_U = \langle x_1, b^*(x_2) \rangle, \text{ for all } x_1, x_2 \in U \text{ and } b \in \mathfrak{B}^\theta_k.
\]

**Definition 9.19.** A \( \mathcal{HC}(G) \)-module \( X \) is unitary if there exists a positive definite invariant Hermitian form on \( X \).

Similarly, an \( \mathbb{H}_k(c) \)-module \( U \) is unitary if \( U \) has an invariant positive definite Hermitian form and a \( \mathfrak{B}^\theta_k \)-module is unitary if it has a positive definite invariant Hermitian form.
Recall $V$ is the defining matrix module of $G$. Let $\langle \cdot, \cdot \rangle_V$ be a non-degenerate Hermitian form on $V$ such that

$$\langle kv_1, v_2 \rangle = \langle v_1, k^{-1}v_2 \rangle \text{ for all } v_1, v_2 \in V, k \in K,$$

$$\langle pv_1, v_2 \rangle = \langle v_1, pv_2 \rangle \text{ for all } v_1, v_2 \in V, p \in p.$$

This makes $V$ unitary with respect to $\cdot$.

**Definition 9.20.** c.f. [9, (4.4)] Let $X$ be in $HC(G)$ with a $\ast$-invariant Hermitian form $\langle \cdot, \cdot \rangle_X$ then we endow $X \otimes V^\otimes k$ with a Hermitian form defined on elementary tensors by

$$\langle x_0 \otimes v_1 \otimes \ldots \otimes v_k, x'_0 \otimes v'_1 \otimes \ldots \otimes v'_k \rangle_{X \otimes V^\otimes k} = \langle x_0, x'_0 \rangle_X \langle v_1, v'_1 \rangle_V \ldots \langle v_k, v'_k \rangle_V,$$

then extended to a Hermitian form. For $\mu$ a character of $K$, define a Hermitian form on $F_{\mu,k}(X) = \text{Hom}_K(\mu, X \otimes V^\otimes k)$ by:

$$\langle \phi, \psi \rangle_{F_{\mu,k}} = \langle \phi(1), \psi(1) \rangle_{X \otimes V^\otimes k}, \text{ for all } \phi, \psi \in \text{Hom}_K(\mu, X \otimes V^\otimes k).$$

**Remark 9.21.** If $X$ is a unitary space then $\langle \cdot, \cdot \rangle_{X \otimes V^\otimes k}$ endows $X \otimes V^k$ as a unitary space.

**Lemma 9.22.** Let $V$ be the complex matrix module of $G = O(p,q)$ or $Sp_{2n}(\mathbb{R})$ and $pr_{12}$ be the projection of $V \otimes V$ onto its trivial $G$ submodule. Define $\langle \cdot, \cdot \rangle_{V \otimes V}$ on $V \otimes V$ by

$$\langle v_1 \otimes v_2, v'_1 \otimes v'_2 \rangle_{V \otimes V} = \langle v_1, v'_1 \rangle_V \langle v_2, v'_2 \rangle_V,$$

and extend to a Hermitian form. Then

$$\langle pr_{12}(v_1 \otimes v_2), v'_1 \otimes v'_2 \rangle_{V \otimes V} = \langle v_1 \otimes v_2, pr_{12}(v'_1 \otimes v'_2) \rangle_{V \otimes V}.$$

**Proof.** It is sufficient to prove that the trivial submodule in $V \otimes V$ and its complement are orthogonal with the form $\langle \cdot, \cdot \rangle_{V \otimes V}$. The Peter-Weyl Theorem [21, Theorem 1.12] states that a unitary module of a compact group decomposes as an orthogonal direct sum of irreducibles. Considering $V \otimes V$ as a $\cdot$ unitary $K$-module, we have that the trivial submodule of $V \otimes V$ is orthogonal to its complement with respect to $\langle \cdot, \cdot \rangle_{V}$. □

**Lemma 9.23.** Suppose $X \in HC(G)$ with a $\ast$-invariant Hermitian form $\langle \cdot, \cdot \rangle_X$ then $F_{\mu,k}(X) \in \mathcal{B}_k^\theta$-mod has a $\ast$-invariant Hermitian form $\langle \cdot, \cdot \rangle_{F_{\mu,k}}$. Furthermore if $X$ is unitary then $F_{\mu,k}(X)$ is unitary.

Ciubotaru and Trapa [9] prove this result for spherical principal series modules mapping to graded Hecke algebras. We extend this to any Harish-Chandra module which requires considering the image as a Type B/C VW-algebra module.
Proof. We show that the Hermitian form is invariant under the generators \( \tilde{z}_j, s_{i,i+1}, \theta_j \) and \( e_{i,i+1} \). For \( \tilde{z}_j, \tilde{z}_j^* = -\tilde{z}_j \) and \( \pi(\tilde{z}_j) = \Omega_{0i}^p \). The form \( \langle \cdot, \cdot \rangle_X \) is a \( * \)-invariant form on \( X \) and \( \langle \cdot, \cdot \rangle_V \) is a \( \cdot \)-invariant form on \( V \). Let \( \phi, \psi \in F_{\mu,k}(X) = \text{Hom}_K(\mu, X \otimes V \otimes k) \), then

\[
\langle \pi \tilde{z}_i(\phi), \psi \rangle_{F_{\mu,k}(X)} = \langle \pi(\tilde{z}_i)(\phi(1)), \psi(1) \rangle_{X \otimes V \otimes k}.
\]

Since \( \pi(\tilde{z}_i) = \Omega_{0i}^p = \sum_{b \in p}(b)_0 \otimes (b^*)_i \),

\[
\langle \pi \tilde{z}_i^*(\phi), \psi \rangle_{F_{\mu,k}(X)} = \langle ((\Omega_{0i}^p)^* \phi(1), \psi(1)) \rangle_{X \otimes V \otimes k},
\]

\[
= -\langle \Omega_{0i}^p \phi(1), \psi(1) \rangle_{X \otimes V \otimes k},
\]

Denote \( \phi(1) \) by \( \sum x_0 \otimes v_1 \otimes \ldots \otimes v_k \) and \( \psi(1) \) by \( \sum x'_0 \otimes v'_1 \otimes \ldots \otimes v'_k \) substituting in the definition of \( \langle \cdot, \cdot \rangle_{X \otimes V \otimes k} \) then

\[
\langle \pi \tilde{z}_i(\phi), \psi \rangle_{F_{\mu,k}(X)} = \sum_{b \in p} \sum \langle -(b)_0(x_0), x'_0 \rangle_x \langle v_1, v'_1 \rangle_v \ldots \langle (b)_i v_i, v'_i \rangle_v \ldots \langle v_k, v'_k \rangle_v.
\]

The form \( \langle \cdot, \cdot \rangle_X \) is \( * \)-invariant for \( g \) and \( \langle \cdot, \cdot \rangle_V \) is \( \cdot \)-invariant for \( g \), hence

\[
-\langle bx_0, x'_0 \rangle_x = \langle x_0, bx'_0 \rangle_x \text{ and } \langle bv_i, v'_i \rangle_V = \langle v_i, bv'_i \rangle_V \text{ for all } b \in p:
\]

\[
\langle \pi \tilde{z}_i(\phi), \psi \rangle_{F_{\mu,k}(X)} = \sum_{b \in p} \sum \langle (x_0), (b)_0 x'_0 \rangle_x \langle v_1, v'_1 \rangle_v \ldots \langle v_i, (b)_i v'_i \rangle_v \ldots \langle v_k, v'_k \rangle_v.
\]

Reversing through the definitions, we show

\[
\langle \pi \tilde{z}_i^*(\phi), \psi \rangle_{F_{\mu,k}} = \langle \phi, \pi(\tilde{z}_i) \psi \rangle_{F_{\mu,k}}.
\]

The element \( \theta_j \) acts by \( \langle \xi \rangle_j \) where \( \xi \in \mathfrak{k} \), hence \( \langle \xi v, v' \rangle_V = \langle v, \xi v' \rangle_V \). Therefore

\[
\langle x_0, x'_0 \rangle_x \langle v_1, v'_1 \rangle_v \ldots \langle (\xi) v_j, v'_j \rangle_v \ldots \langle v_k, v'_k \rangle_v = \langle x_0, x'_0 \rangle_x \langle v_1, v'_1 \rangle_v \ldots \langle v_j, (\xi) v'_j \rangle_v \ldots \langle v_k, v'_k \rangle_v.
\]

Similarly for \( s_{i,i+1} \)

\[
\langle s_{i,i+1}(x_0 \otimes v_1 \otimes \ldots v_k), x'_0 \otimes v'_1 \otimes \ldots v'_k \rangle_{X \otimes V \otimes k}
\]

\[
= \langle x_0 \otimes v_1 \otimes \ldots \otimes v_{i+1} \otimes v_i \otimes \ldots \otimes v_k, x'_0 \otimes v'_1 \otimes \ldots \otimes v'_k \rangle_{X \otimes V \otimes k}
\]

\[
= \langle x'_0 \otimes v_1 \otimes \ldots \otimes v_k, x_0 \otimes v'_1 \otimes \ldots \otimes v'_{i+1} \otimes v'_i \otimes \ldots \otimes v'_k \rangle_{X \otimes V \otimes k}
\]

\[
= \langle x_0 \otimes v_1 \otimes \ldots v_k, s_{i,i+1}(x'_0 \otimes v'_1 \otimes \ldots v'_k) \rangle_{X \otimes V \otimes k}.
\]

The projection \( e_{i,i+1} \) acts on elementary tensors \( x_0 \otimes v_1 \otimes \ldots \otimes v_k \) by

\[
e_{i,i+1}(x_0 \otimes v_1 \otimes \ldots \otimes v_k) = mx_0 \otimes v_1 \otimes \ldots \otimes v_{i-1} \otimes \text{pr}_{12}(v_i \otimes v_{i+1}) \otimes \ldots \otimes v_k.
\]
Then
\[ \langle e_{i,i+1}(x_0 \otimes v_1 \otimes \ldots \otimes v_k), x'_0 \otimes v'_1 \otimes \ldots \otimes v'_k \rangle_{X \otimes V^{\otimes k}}, \]
\[ = m(x_0 \otimes v_1 \otimes \ldots \otimes v_{i-1} \otimes \text{pr}_{12}(v_i \otimes v_{i+1}) \otimes \ldots \otimes v_k, x'_0 \otimes v'_1 \otimes \ldots \otimes v'_k)_{X \otimes V^{\otimes k}}, \]
\[ = \langle x_0, x'_0 \rangle_{X}(v_1, v'_1)_{V} \ldots \langle \text{pr}_{12}(v_i \otimes v_{i+1}), v'_i \otimes v'_{i+1} \rangle_{V} \otimes \ldots \langle v_k, v'_k \rangle_{V}. \]

Using Lemma 9.22,
\[ = \langle x_0 \otimes v_1 \otimes \ldots \otimes v_k, x'_0 \otimes v'_1 \otimes \ldots \otimes v'_{i-1} \otimes \text{pr}_{12}(v'_i \otimes v'_{i+1}) \otimes \ldots \otimes v'_k \rangle_{X \otimes V^{\otimes k}}, \]
Therefore
\[ \langle e_{i,i+1}(\phi), \psi \rangle_{F_{\mu,k}} = \langle \phi, e_{i,i+1}(\psi) \rangle_{F_{\mu,k}}. \]

The module \( F_{\mu,k}(X) \) has induced Hermitian form \( \langle \cdot, \cdot \rangle_{F_{\mu,k}} \) which is *-invariant on the generators of \( \mathfrak{B}_k^{\theta} \). Hence \( \langle \cdot, \cdot \rangle_{F_{\mu,k}} \) is a *-invariant form. If \( X \) is unitary then \( \langle \cdot, \cdot \rangle_{X \otimes V^{\otimes k}} \) is positive definite. Hence \( \langle \cdot, \cdot \rangle_{F_{\mu,k}} \) is a positive definite invariant Hermitian form and \( F_{\mu,k}(X) \) is unitary.

\[ \square \]

**Definition 9.24.** Let \( X \in \mathcal{HC}(G), \mathfrak{B}_k^{\theta}\)-mod or \( \mathbb{H}_k(c)\)-mod module with invariant Hermitian form \( \langle \cdot, \cdot \rangle_X \). We define the Langlands quotient \( \overline{X} \) to be:
\[ \overline{X} = X/\text{rad} \langle \cdot, \cdot \rangle_X, \]
where \( \text{rad} \langle \cdot, \cdot \rangle \) is the radical of the form \( \langle \cdot, \cdot \rangle \).

**Lemma 9.25.** Let \( X \) be in \( \mathcal{HC}(G)\)-mod with Hermitian invariant form \( \langle \cdot, \cdot \rangle_X \) and Langlands quotient \( \overline{X} \). The form \( \langle \cdot, \cdot \rangle_{F_{\mu,k}} \) is the endowed hermitian form of \( F_{\mu,k}(X) \) from Definition 9.20 then
\[ F_{\mu,k}(\overline{X}) = \overline{F_{\mu,k}(X)} = F_{\mu,k}(X)/\text{rad} \langle \cdot, \cdot \rangle_{F_{\mu,k}}. \]

**Proof.** One can construct an exact sequence:
\[ 0 \longrightarrow \text{rad} \langle \cdot, \cdot \rangle_X \longrightarrow X \longrightarrow \overline{X} \longrightarrow 0. \]

Exactness of the functors \( F_{\mu,k} \) is given by Lemma 5.3. Hence there is an exact sequence:
\[ 0 \longrightarrow F_{\mu,k}(\text{rad} \langle \cdot, \cdot \rangle_X) \longrightarrow F_{\mu,k}(X) \longrightarrow F_{\mu,k}(\overline{X}) \longrightarrow 0. \]

For the result it is sufficient to prove that \( F_{\mu,k}(\text{rad} \langle \cdot, \cdot \rangle_X) = \text{rad} \langle \cdot, \cdot \rangle_{F_{\mu,k}} \).

Since \( \langle \cdot, \cdot \rangle_{F_{\mu,k}} \) is an invariant form on \( F_{\mu,k}(X) \) and a non-degenerate form on \( F_{\mu,k}(\overline{X}) \) then \( F_{\mu,k} \text{rad} \langle \cdot, \cdot \rangle_X = \text{rad} \langle \cdot, \cdot \rangle_{F_{\mu,k}}. \)

\[ \square \]

**Theorem 9.26.** Let \( X^\nu_k \) be a principal series module for \( G = O(p,q) \) or \( Sp_{2n}(\mathbb{R}) \). The Langlands quotient \( \overline{X^\nu_k} = X^\nu_k/\text{rad} \langle \cdot, \cdot \rangle_{X^\nu_k} \) is mapped by \( F_{\mu,k} \), to the Langlands quotient of the \( \mathbb{H}_k(c_\mu)\)-module, \( \overline{X(v_k)} = \).
$X(\nu_k) / \text{rad}(.)_{X(\nu_k)}$. Similarly, $X^\nu_\delta$ is mapped by $F_{\mu,n-k}$, to the $\mathbb{H}_{n-k}(c_\mu)$-module $\overline{X(\nu_{n-k})}$.

**Definition 9.27.** We define subsets of $\mathfrak{a}^*$:

$$U_\delta = \{ \nu \in \mathfrak{a}^* : X^\nu_\delta \text{ is a unitary Harish-Chandra module} \}.$$  

Similarly define

$$U_k(1) = \{ \lambda \in \mathfrak{a}_k^* : X_\lambda \text{ is a unitary } \mathbb{H}_k(c_\mu) \text{ module} \}$$

and

$$U_{n-k}(1) = \{ \bar{\lambda} \in \bar{\mathfrak{a}}_{n-k}^* : X(\bar{\lambda}) \text{ is a unitary } \mathbb{H}_{n-k}(c_\mu) \text{ module} \}.$$  

Since $\mathfrak{a} = \mathfrak{a}_k \oplus \bar{\mathfrak{a}}_{n-k}$ we can associate a pair $(\lambda_k, \bar{\lambda}_{n-k}) \in \mathfrak{a}^* \times \bar{\mathfrak{a}}_{n-k}^*$ to a character of $\mathfrak{a}$ via:

$$(\lambda_k, \bar{\lambda}_{n-k}) : \mathfrak{a} \rightarrow \mathbb{C}$$  

$$(\lambda_k, \bar{\lambda}_{n-k})(\mathfrak{a}_k) = \lambda_k(\mathfrak{a}_k),$$  

$$(\lambda_k, \bar{\lambda}_{n-k})(\bar{\mathfrak{a}}_{n-k}) = \bar{\lambda}_{n-k}(\bar{\mathfrak{a}}_{n-k}).$$  

Theorem 9.26 shows that the Langlands quotients of $X^\nu_\delta$ map under $F_{\mu,k}$ and $F_{\mu,n-k}$ to Langlands quotients of principal series modules for $\mathbb{H}_k(c_\mu)$ and $\mathbb{H}_{n-k}(c_\mu)$ hence we can formulate the following non-unitary test.

**Lemma 9.28.** We have an inclusion of sets:

$$U_\delta \subseteq U_k(1) \times U_{n-k}(1).$$

This inclusion of sets states that if we take a minimal principal series module $X$ and find that, under the functor $F_{\mu,k}$, the Langlands quotient of the image is not unitary then the Langlands quotient of $X$ is not unitary.

**Theorem 9.29.** [Non-unitary test for principal series modules] If either $\overline{X(\nu_k)}$ or $\overline{X(\nu_{n-k})}$ are not unitary, as $\mathbb{H}_k(c_\mu)$ and $\mathbb{H}_{n-k}(c_\mu)$-modules, then the Langlands quotient of the minimal principal series module $\overline{X^\nu_\delta}$, for $G = O(p,q)$ or $Sp_{2n}(\mathbb{R})$ is not unitary.

The above working does not require the image to be a Hecke algebra module. Therefore, we have also proved the following theorem.

**Theorem 9.30.** [Non-unitary test for Harish-Chandra modules] Let $X$ be a Harish Chandra module. For $G = Sp_{2n}(\mathbb{R})$ or $O(p,q)$, $p+q = 2n+1$, if for any character $\mu$ and $k = 1, \ldots, n$ the $\mathfrak{g}_k^\theta$-module $\overline{F_{\mu,k}(X)}$ is not unitary, then the Langlands quotient $\overline{X}$ of $X$ is not a unitary $G$-module.

In the case when $G$ is split then $X$ is a subrepresentation of $X^\nu_\delta$ and
$F_{\mu,k}(X)$, $F_{\mu,n-k}(X)$ are Hecke algebra modules. In this case, if either $F_{\mu,k}(X)$, $F_{\mu,n-k}(X)$ is not unitary as a Hecke algebra module then $X$ is not unitary as a $G$-module.

We finish with a toy example.

**Example 9.31.** Let $G = Sp_2(\mathbb{R}) \cong SL_2(\mathbb{R})$. Then principal series modules are labelled by $\delta = \pm 1$ and $\nu \in \mathbb{C}$. The principal series modules $X_\nu^\delta$ are unitary exactly when $\nu = ib$ for $b \in \mathbb{R}$, that is $\nu$ is entirely imaginary.

In this case all principal series modules are spherical principal series modules. The root system associated to $Sp_2$ has one root $\epsilon$ and the Weyl group is $\mathbb{Z}_2$ which acts by $-1$ on $\epsilon$. Here $\mathbb{H}(c)$ will be the graded Hecke algebra associated to type $B_1$ with parameter $c$. The algebra $\mathbb{H}(c)$ is generated by $\epsilon$ and $s \in \mathbb{Z}_2$ such that $se = -s\epsilon + c$. We note that $s^* = s$ and $\epsilon^* = -\epsilon + cs$.

Our theorem gives that $F_{triv,1}(X_1^\nu) \cong \text{Ind}_{\mathbb{C}}^{\mathbb{H}(c)} 1_\nu$.

Note that $\text{Ind}_{\mathbb{C}}^{\mathbb{H}(c)} 1_\nu$ is two dimensional with basis $\{1_\nu, s1_\nu\}$ we will denote the module $\text{Ind}_{\mathbb{C}}^{\mathbb{H}(c)} 1_\nu$ by $Y_\nu$. Let $\langle , \rangle$ be a hermitian form on $Y_\nu$, for $Y_\nu$ to be unitary we require

$$\langle s(u), v \rangle = \langle u, s^*(v) \rangle = \langle u, s(v) \rangle$$

and

$$\langle \epsilon(u), v \rangle = \langle u, \epsilon^*(v) \rangle = \langle u, [-\epsilon + cs](v) \rangle.$$ 

Letting $u = 1_\nu$ and $v = 1_\nu$, then the above requirement implies

$$\nu(1_\nu, 1_\nu) = \langle \epsilon(1_\nu), 1_\nu \rangle = \langle 1_\nu, [-\epsilon + cs]1_\nu \rangle = -\bar{\nu}(1_\nu, 1_\nu) + \langle 1_\nu, s1_\nu \rangle.$$ 

For the above equation to hold $\nu = -\bar{\nu}$ and $\langle 1_\nu, s1_\nu \rangle = 0$. Hence for $Y_\nu$ to be unitary $\nu$ must be purely imaginary. Furthermore if $\nu$ is purely imaginary then we can construct a Hermitian non-degenerate form on $Y_\nu$ such that it is a unitary form. Therefore in the case of $Sp_2(\mathbb{R})$ our non-unitary test becomes an equivalence:

$X_\nu^\delta$ is unitary if and only if $F_{triv,1}(X_\nu^\delta) \cong \text{Ind}_{\mathbb{C}}^{\mathbb{H}(c)} 1_\nu$ is unitary.

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