Analysis of Cohesion in Fast-spinning Small Bodies

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Abstract

In this paper, the structural stability of a fast-spinning small body is investigated. In particular, a nonlinear yield condition in tensile stress is applied to estimate the required cohesion in a fast-spinning small body. The least upper bound of required cohesion is investigated for both ellipsoid and irregular shape models. The stress state of a fast-spinning ellipsoid is discussed analytically, and the effects of spin rates and size ratios are analyzed. For an irregularly shaped body, an element average stress method is developed to estimate the range of stress of any element in the body, where only self-gravity and centrifugal force are considered. The maximum tensile stress in the whole body is used to solve the required cohesion. Finally, the proposed methods are applied to different asteroid shape models. The result shows that the least upper bound of cohesion is mainly determined by the spin rate and length of the major axis, but an irregular shape will change the stress distribution and cause a stressed surface. The required cohesion of a fast-spinning small body varies between tens to 1000 Pa. The methods developed in this paper can rapidly provide a conservative lower bound on the cohesion in a fast-spinning body and qualitatively show the distribution of stress, which provides an effective way to study the structural stability of fast-spinning bodies of those bodies.

Unified Astronomy Thesaurus concepts: Asteroids (72); Small solar system bodies (1469); Near-Earth objects (1092); Planetary structure (1256)

1. Introduction

The structural stability of small bodies is a topic of interest in planetary science, as it reflects the inner structure of an asteroid and contains clues about its formation and dynamical evolution. Due to this, there has been a long series of studies to improve our understanding of the inner structure of small bodies. Initial studies focused on the equilibrium shape of fluid bodies with gravitational, centrifugal, and tidal forces (Chandrasekhar 1963, 1969; Lebovitz 1998; Fasso & Lewis 2001). But as these models are only warranted for the structural stability of gaseous planets, the theory for elastic bodies was developed to study cohesionless small bodies. To do this, the equilibrium of elastic triaxial ellipsoids with gravitational, rotational, and tidal forces was studied (Dobrovolskis 1982; Slyuta & Voropaev 1997). However, these elastic solutions are related to the initial stress configuration, and the consequences of structural failure are hard to determine. To overcome those difficulties, limit analyses based on elastic–plastic theories were applied, which found the maximum load a body can hold without unconstrained plastic flow. Holsapple (2001) solved the limit problem for a cohesionless ellipsoidal body with self-gravitational and rotational forces based on the Mohr–Coulomb (M-C) yield criterion. He pointed out that the structural failure of a cohesionless body is a shear failure, and the minimum friction angles for stability as a function of size ratio were investigated. Another approach that has been developed is an energy-based approach to solve the spin limit of ellipsoidal bodies (Holsapple 2004), and the volume average stresses are derived to solve the upper bound of the limit loads (Holsapple 2008). Hirabayashi & Scheeres (2013) extended these methods to partial average stress and used them to find the region sensitive to structural failure in asteroid (216) Kleopatra. Meanwhile, observation finds that cohesion exists widely in small bodies (Sánchez & Scheeres 2014). A finite element model (FEM) was proposed to study the structural stability and determine the required cohesion (Hirabayashi & Scheeres 2014). Based on that, the internal structure due to rotational instability was discussed (Hirabayashi et al. 2015). These studies provide good insight into the internal structure of asteroids with rotation periods above “the rotation barrier” (∼2.1 hr; Harris 1996), where centrifugal accelerations at the equator exceed gravitational accelerations and in compressive stress.

With the improvement of observation, more and more fast-spinning small bodies have been discovered recently (Scheirich et al. 2010; Hergenrother & Whiteley 2011; Polishook et al. 2016). They have a small size and a short rotation period, much less than the rotation barrier. The inner stress and structure stability of such small fast rotators are of interest to scientists. Due to their fast spin rates, the material cannot gravitationally bound. Therefore, tensile stress exists in the small body, and cohesion is necessary to hold the body together. The cohesion and failure modes of asteroids are discussed numerically by Hirabayashi & Scheeres (2019), where they found the minimum cohesion for structural stability by iteration with an FEM. The Drucker–Prager (D-P) yield criterion is used in the analysis. However, the D-P, as well as the M-C, yield criteria are more suitable for compressive stress. In general, the yield condition in tensile stress is nonlinear for bulk solids, such as soil and rocks (Eaves & Jones 1971). Therefore, the cohesion based on D-P or M-C yield conditions may underestimate the cohesion in fast-spinning small bodies.

In this paper, we focus on studying the structural stability and cohesion in fast-spinning small bodies. In particular, a nonlinear yield criterion in tensile stress is used to analyze the
required cohesion in a fast rotator. An element average stress method is developed to evaluate the stress state in an irregularly shaped and rapidly rotating small body. The stress distribution due to self-gravity and centrifugal force is investigated for both ellipsoid and irregularly shaped models. The least upper bound of cohesion on a fast-spinning asteroid is discussed. The critical spin rate and maximum tensile stress at different size ratios are discussed analytically in an ellipsoid. Then, the range of stress states of any element in an irregularly shaped body is analyzed and verified using an ellipsoid model. Finally, the required cohesion in several fast-spinning asteroids is computed and compared. The influence of irregular shape on stress distribution is analyzed. This study provides a simple method to estimate the required cohesion in a fast-spinning small body, contributing to studying the accurate internal structure of such fast rotators.

This paper is organized as follows. In Section 2, we describe the tensile yield criterion used in this paper, and the relation between critical cohesion and maximum stress state is discussed. The stress state of a fast-spinning ellipsoid is discussed analytically in an ellipsoid. Then, the range of stress states of any element in an irregularly shaped body is estimated. In Section 5, the cohesion in different asteroids is studied, and the influence of the shape model on stress distribution is discussed. Section 6 draws the conclusion.

2. Tensile Yield Criterion of a Bulk Solid

The yield criterion defines the limit of a range of stresses that can be applied without causing permanent deformation in a body. Here we assume the materials in the asteroid to be continuum media, and the yield condition is thought to depend on hydrostatic pressure and shear. The M-C and D-P yield criteria are widely used in the structural stability analysis of small bodies (Holsapple 2001, 2004; Hirabayashi & Scheeres 2013, 2019), mainly for compressive stress states. The M-C yield criterion is written as

\[
g(\sigma_1, \sigma_3, \varphi) = (\sigma_1 - \sigma_3) \sec \varphi + (\sigma_1 + \sigma_3) \tan \varphi \leq 2C_0, \tag{1}
\]

where \( \varphi \) is the friction angle, \( C_0 \) is the cohesive strength, \( \sigma_i (i = 1, 2, 3) \) is the principal stress component, and \( \sigma_1 < \sigma_2 < \sigma_3 \). Here we assume that the positive value means compressive stress. The yield locus is a slope attaching to the Mohr circle centered at \( \frac{\sigma_1 + \sigma_3}{2} \) with a radius of \( \frac{\sigma_1 - \sigma_3}{2} \).

However, it is found that the M-C yield criterion is inaccurate in tensile stress. In a tensile yield test on various bulk solids, they found that the yield loci satisfy a power law (Ashton et al. 1964):

\[
\left( \frac{\tau}{C_0} \right)^n = \frac{\sigma}{T} + 1. \tag{2}
\]

Equation (2) is known as the Warren Spring equation. Here \( T > 0 \) is the tensile strength limit of a material, and \( n \) is an index between 1 and 2 describing flowability: \( n = 1 \) means free-flowing, which equals the M-C yield criterion, and \( n = 2 \) corresponds to very cohesive material (Schwedes 2003). Usually, the index, tensile stress limit, and cohesion of a material are determined by a series of tests. An index close to 2 is suitable for most rocks. To find a conservative result of cohesion, we choose \( n = 2 \) in our research. Then, the yield locus becomes a parabolic:

\[
\tau^2 = \frac{\sigma C_0^2}{T} + C_0^2. \tag{3}
\]

In order to keep the structural stable, the Mohr circle should be under the yield locus. That means the cohesion should satisfy

\[
\frac{\sigma C_0^2}{T} + C_0^2 - \left( \frac{\sigma_1 - \sigma_3}{2} \right)^2 + \left( \sigma - \frac{\sigma_1 + \sigma_3}{2} \right)^2 \geq 0, \sigma \in [\sigma_1, \sigma_3] \tag{4}
\]

or

\[
f(C_0, T, \sigma) = \sigma^2 + \left( \frac{C_0^2}{T} - \sigma_1 - \sigma_3 \right) \sigma + \sigma_1 \sigma_3 + C_0^2 \geq 0. \tag{5}
\]

Denoting \( B = \frac{C_0^2}{T} - \sigma_1 - \sigma_3 \) and \( C = \sigma_1 \sigma_3 + C_0^2 \) should satisfy one of the following conditions:

1. \(-\frac{b}{2} \leq \sigma_1, f(\sigma_1) \geq 0,\)
2. \(-\frac{b}{2} \geq \sigma_3, f(\sigma_3) \geq 0,\)
3. \(-\frac{b}{2} \in [\sigma_1, \sigma_3], \Delta = B^2 - 4C \leq 0.\)

By calculation, it is easily found that condition 1 corresponds to \( C_0^2 \geq (\sigma_3 - \sigma_1)T \) and \( T \geq -\sigma_1 \). Condition 2 does not apply, as it requires \( C_0^2 \leq (\sigma_1 - \sigma_3)T < 0 \). If we assume \( \sigma_1 < 0 < \sigma_3, |\sigma_1| > \sigma_3 \), condition 3 becomes

\[
\begin{align*}
T \geq -\sigma_1 \\
C_0^2 \in [K_1, K_2] + K_3
\end{align*}
\]

where \( K_1 = 2T^2 + T(\sigma_1 + \sigma_3) \) and \( K_2 = 2T\sqrt{T^2 + (\sigma_1 + \sigma_3)T + \sigma_1 \sigma_3} \). The value of \( K_1 - K_2 \) decreases as \( T \) increases and has its maximum value at \( T = -\sigma_1 \). The limit inferior of \( \sigma_1 < 0 < \sigma_3, |\sigma_1| > \sigma_3 \) is \( \frac{\sigma_1 - \sigma_3}{2} \) when \( T \) tends to infinity. That means that at a given stress state \( \sigma_1 < 0 < \sigma_3, |\sigma_1| > \sigma_3 \), the tensile limit and cohesion to keep the structure stable should satisfy

\[
\begin{align*}
T \geq -\sigma_1 \\
C_0^2 \geq 2T^2 + T(\sigma_1 + \sigma_3) - 2T\sqrt{T^2 + (\sigma_1 + \sigma_3)T + \sigma_1 \sigma_3}.
\end{align*}
\]

There are infinite sets of \((T, C)\) that meet the constraints at a given stress state. To resolve this, two additional constraints are considered in our study. First, the tensile stress limit of a body should be as small as possible. Second, the body still follows the linear yield condition in Equation (1) in the compressive region, and the two yield loci connect at \( \sigma = 0 \) with the same slope. Then, the following constraint holds:

\[
\frac{C_0}{2T} = \tan \varphi. \tag{8}
\]
Usually, the friction angle is between 30° and 45°. That means the cohesion is between \( \frac{2 \mu k}{3} T \) and 27.

Meanwhile, based on the constraints above, if \( T = -\sigma_1 \), the minimum cohesion should be \( C_0 = \sqrt{(\sigma_3 - \sigma_1)T} \), or \( C_0 / T = \sqrt{1 - \sigma_3 / \sigma_1} \). Comparing \( \sqrt{1 - \sigma_3 / \sigma_1} \) with 2 tan \( \varphi \), we can find the minimum \( T \) and the corresponding \( \sigma_0 = 210 \) Pa, and the corresponding cohesions are \( C_1 = 23.09 \) Pa and \( C_2 = 15.00 \) Pa. As comparisons, the required cohesions based on the M-C yield criterion are \( C_1 = 18.8 \) Pa and \( C_2 = 25.2 \) Pa. The nonlinear yield criterion requires larger cohesion than the traditional linear yield criterion at the same stress state and friction angle. If the stress state changes to \( \sigma_1 = -10 \) Pa and \( \sigma_3 = 5 \) Pa, the minimum tensile stress limit will increase slightly to \( 18.8 \) Pa. The difference increases with the increase of tensile stress state.

Comparing the results indicates that the traditional M-C yield criterion may underestimate cohesion when a small body is in tensile stress. Since the D-P yield criterion is similar to the M-C yield criterion, previous studies based on the M-C or D-P yield criterion may obtain a lower value of cohesion than the actual value. The difference increases with the increase of tensile stress and friction angle. As the nonlinear yield condition is in accordance with the experimental results, it might give us a better estimation of cohesion in a tensile stress body. Since the cohesion is mainly determined by the maximum tensile stress when \( \sigma_1 < -\sigma_3 < 0 \), which is common in a fast-spinning body, we focus on discussing the maximum tensile stress and use it to calculate the required cohesion of fast-spinning small bodies.

### 3. Cohesion in Fast-spinning Ellipsoids

First, the stress state and required cohesion for an ideal ellipsoid are investigated. We assume that the materials in the body follow elasticity–perfect plasticity and its plastic deformation is small. That means the material is zero-hardening and zero-softening. The principal axes of an ellipsoid are aligned with the \( x-, y-, \) and \( z-\)axes with semiaxes \( a > b > c \). The body spins around the \( z-\)axis with spin rate \( \omega \). The potential of the ellipsoid is in the form of a quadratic (Holsapple 2001):

\[
V = -V_0 + k_x x^2 + k_y y^2 + k_c z^2.
\]

If we only consider the self-gravity and centrifugal potential, the coefficients \( k_x, k_y, \) and \( k_c \) can be expressed as

\[
\begin{align*}
k_x &= \rho \pi Gabc \int_0^\infty \frac{ds}{(a^2 + s) \Delta} - \frac{\omega^2}{2} \\
k_y &= \rho \pi Gabc \int_0^\infty \frac{ds}{(b^2 + s) \Delta} - \frac{\omega^2}{2} \\
k_z &= \rho \pi Gabc \int_0^\infty \frac{ds}{(c^2 + s) \Delta}
\end{align*}
\]

with \( \Delta = \sqrt{(a^2 + s)(b^2 + s)(c^2 + s)} \) and \( V_0 = \rho \pi Gabc \int_0^\infty \frac{ds}{\Delta} \).

For the cohesionless condition, based on symmetry, the closed form can be solved by solving the equilibrium equation, stress boundary conditions, and lateral stress at the poles as

\[
\begin{align*}
\sigma_z &= \rho k_z a^2 \left[ 1 - \left( \frac{x}{a} \right)^2 - \left( \frac{y}{b} \right)^2 - \left( \frac{z}{c} \right)^2 \right] \\
\sigma_y &= \rho k_y b^2 \left[ 1 - \left( \frac{x}{a} \right)^2 - \left( \frac{y}{b} \right)^2 - \left( \frac{z}{c} \right)^2 \right] \\
\sigma_z &= \rho k_c c^2 \left[ 1 - \left( \frac{x}{a} \right)^2 - \left( \frac{y}{b} \right)^2 - \left( \frac{z}{c} \right)^2 \right]
\end{align*}
\]
and the three shear stresses are zero everywhere. The analytical solution has been used to determine the required friction angles for cohesionless bodies. If the cohesion is added, the actual stress state is related to the material properties and the actual cohesion. But Equation (12) is still a good way to evaluate the required cohesion to keep the structure stable at a fast spin rate. The actual stress should be no larger than Equation (12), as cohesion will prevent deformation. Therefore, the cohesion solved by the maximum value of Equation (12) will be a least upper bound of the required cohesion to keep the structure stable, denoted as $C_0^U$. If the body has a cohesion larger than $C_0^U$, the stress in the body will be below yield everywhere; hence, the body will remain stable.

According to Equation (12), $\sigma_x$ is independent of the spin rate $\omega$ and always in a compressive state. But with the increase of the spin rate, $\sigma_x$ and $\sigma_y$ will change to the tensile state. It is obvious that the maximum tensile stress is $\sigma_x$ at the center $o$. If we define two size ratios as $\beta = b/a$ and $\gamma = c/a$, the least upper bound of required cohesion for an ellipsoid is solved by $\sigma_x^U$ with different shape ratios. The potential parameters become

$$
\begin{align*}
    k_x &= \beta \gamma \rho G \int_0^\infty \frac{ds}{(1 + s)\Delta'} - \frac{\omega^2}{2} \\
    k_y &= \beta \gamma \rho G \int_0^\infty \frac{ds}{(\beta^2 + s)\Delta'} - \frac{\omega^2}{2}, \\
    k_z &= \beta \gamma \rho G \int_0^\infty \frac{ds}{(\gamma^2 + s)\Delta'},
\end{align*}
$$

where $\Delta' = \sqrt{(s + 1)(s + \beta^2)(s + \gamma^2)}$.

The corresponding stresses at the center are

$$
\begin{align*}
    \sigma_x &= \rho k_x a^2 \\
    \sigma_y &= \rho k_y a^2 \beta^2, \\
    \sigma_z &= \rho k_z a^2 \gamma^2.
\end{align*}
$$

Meanwhile, the normalized spin rate and stress are defined as $\tilde{\omega} = \omega/\sqrt{\pi \rho G}$ and $\tilde{\sigma}_x = \sigma_x/\pi \rho G a^2$, where $r_c = a\sqrt{\beta \gamma}$. The critical normalized spin rate $\tilde{\omega}_c$ that induces tensile stress in an ellipsoid is discussed, as shown in Figure 2. The critical spin rate decreases with the decrease of $\beta$ and $\gamma$. The maximum critical spin rate is 1.158 for a sphere. If the spin rate is larger than 1.158, a body with any size ratios will have tensile stress.

Similarly, if we fix the spin rate, the maximum tensile stress $\sigma_x^U$ increases as the size ratios decrease. Figure 3(a) shows $\sigma_x^U$ at $\tilde{\omega} = 5.16$. The tensile stress changes from 10 as a sphere to more than 250 as an extremely elongated body. But if we choose the semimajor axis $a$ as the unit length, the normalized stress $\tilde{\sigma}_x = \sigma_x/\pi \rho^2 G a^2$ with size ratios is shown in Figure 3(b). The difference of $\tilde{\sigma}_x^U$ is less than 0.6 for different $\beta$ and $\gamma$ and hardly changes with spin rate. That means when the spin rate is larger than $\tilde{\omega} > 1.158$, the length of the semimajor axis $a$ and spin rate are two major factors that affect the value of the maximum tensile stress. The least upper bounds of cohesion of different ellipsoids are similar if they have the same spin rate and semimajor axis. But if we fix the volume or mass of an ellipsoid, size ratios still have a great impact on the maximum tensile stress by changing $a$.

Define the ratio between the semimajor axis and the mean radius as $\alpha = a/r_c = 1/\sqrt{\beta \gamma}$. Based on the nonlinear yield criterion, Figure 4 shows the change of the normalized cohesion $C_0^U = C_0^U/\pi \rho^2 G a^2$ with spin rate at different size ratios. As shown in Figure 4, the curve is nearly a parabolic with $\tilde{\omega}$ and proportional to $a$. Meanwhile, the friction angle is still an important parameter affecting $C_0^U$. For a normalized spin rate $\tilde{\omega} = 5$, the normalized cohesion is about $C_0^U = 28$ for a friction angle $\varphi = 30^\circ$ and $C_0^U = 49$ for $\varphi = 45^\circ$ at $\alpha = 1.42$. With the increase of the spin rate, the difference between two friction angles will further enlarge.

The above analysis determines the least upper bound of the required cohesion $C_0^U$. In the meantime, the volume average stress of the ellipsoid can be solved. The volume average stress is an important concept in limit analysis, in which the least upper bound theorem indicates that the work of load, which is equal to the dissipation rate, is either equal to or higher than the actual load limits that cause structural failure. It determines the condition at which structural failure must occur or the stress state is just on the yield locus. In tensile stress, we can find a required cohesion on the yield locus based on the nonlinear yield criterion above. Comparatively, it should be the lower bound of the required cohesion $C_0^U$ to avoid structural failure. Any cohesion less than $C_0^U$ must result in structural failure under the current stress state.

Denote the Cartesian coordinate system $(x_1, x_2, x_3)$ aligned with the minimum, intermediate, and maximum moment of inertia axes of the body, which are equal to the $x_1$, $y_1$, and $z_1$ axes above. The averaged stress $\bar{\sigma}$ over the whole body with any static surface traction $t$ and body force $b$ is

$$
\bar{\sigma}_t = \frac{1}{V} \int_V \sigma_t dV = \frac{1}{V} \int_V \rho x_t b dV + \frac{1}{V} \oint_S x f_i dS.
$$

Here $V$ is the volume of an ellipsoid, $S$ is the exterior surface, and $\rho$ is the bulk density, which is assumed to be constant over the whole body. The body force $b$ includes the self-gravity $b_g$ and the force induced by centrifugal acceleration $b_c$. The whole body will be meshed into small elements, and the classical two-body problem is used to calculate the interaction between any two elements. The gravitational acceleration of element $i$ can be expressed as

$$
\mathbf{b}_{ig} = -G \rho \sum_j \frac{V_i}{|r_i - r_j|^3} (r_i - r_j).
$$
4. Element Average Stress in Irregularly Shaped Bodies

4.1. Lower and Upper Bounds of Element Average Stress

The above analysis gives an estimation of the lower and upper bounds of required cohesion for a fast-spinning ellipsoid. However, if the body is irregular, the analytical solution is hard to derive, even for a cohesionless body. The lower bound of cohesion \( C_0^l \) can be solved by obtaining the volume average stress numerically. But \( C_0^l \) is related to the maximum tensile stress in the body in the absence of cohesion.

A partial volume technique has been developed to find the region that is sensitive to structural failure in compressive stress (Hirabayashi & Scheeres 2013). The slice perpendicular to the minimum moment of inertia axis is chosen as the partial volume. This technique can also be used to find the required cohesion in different slices. However, the partial volume stress still does not reflect the maximum stress state in the body. Here we extend the partial average stress to find the average stress state of any element in an irregularly shaped body.

Assume an element \( E_0 \) is chosen between \( x \in [x_1, x_2], y \in [y_1, y_2], \) and \( z \in [z_1, z_2] \). The volume of element \( E_0 \) is \( V_0 \).

According to the definition of average stress, the diagonal components of the average stress can be expressed as

\[
\bar{\sigma}_{ii} = \frac{1}{V_0} \int_{V_0} \rho \sigma_{ii} b_i dV + \frac{1}{V_0} \oint_S x_i t_i dS. \quad (19)
\]

Take \( \bar{\sigma}_i = \bar{\sigma}_x \) as an example. The cross section \( S \) of element \( E_0 \) consists of six faces. Two of them are perpendicular to the \( x \)-axis, namely, \( S_1 \) and \( S_2 \). The corresponding tractions are named \( t_1 \) and \( t_2 \). The other four faces are parallel to the \( x \)-axis and denoted \( S_t \) together, and the traction is named \( t_x \).

Two extended parts are defined as \( E_1(x \in [x_1, x_2], y \in [y_1, y_2], z \in [z_1, z_2]) \) and \( E_2(x \in [x_2, x_3], y \in [y_1, y_2], z \in [z_1, z_2]) \). Here \( x_1 \) and \( x_2 \) are the left and right ends of the body. Figure 5 shows the illustration.

The force between \( E_0 \) and \( E_1 \) at cross section \( S_1 \) is defined as \( T_1 = \oint_{S_1} t_1 dS \). In the same manner, the force between \( E_0 \) and \( E_2 \) at cross section \( S_2 \) is defined as \( T_2 = \oint_{S_2} t_2 dS \). The body force of any part is denoted as \( B \). There is no traction on the surface.
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1. $x_1 \oint_{S} t_f dS < x_2 \oint_{S} t_f dS$, then Equation (23) becomes
   $\bar{\sigma}_x \leq \frac{1}{V_0} \int_{S} \rho b_1 dV + \frac{x_1}{V_0} B_1 - \frac{x_2}{V_0} (B_1 + B_0) - \frac{x_3 - x_1}{V_0} P_t$,
   \hspace{0.5cm} (24)

2. $x_1 \oint_{S} t_f dS > x_2 \oint_{S} t_f dS$, then Equation (23) becomes
   $\bar{\sigma}_x \geq \frac{1}{V_0} \int_{S} \rho b_1 dV + \frac{x_1}{V_0} B_1 - \frac{x_2}{V_0} (B_1 + B_0) + \frac{x_3 - x_1}{V_0} P_t$.
   \hspace{0.5cm} (25)

We assume that the stress is tensile (negative value), the average stress should be
   $\bar{\sigma}_x' = \frac{1}{V_0} \int_{S} \rho b_1 dV + \frac{x_2}{V_0} B_2 - \frac{x_1}{V_0} (B_1 + B_0) + \frac{x_3 - x_1}{V_0} P_t$ \hspace{0.5cm} (26)

and
   $\bar{\sigma}_x'' = \frac{1}{V_0} \int_{S} \rho b_1 dV + \frac{x_1}{V_0} B_1 - \frac{x_2}{V_0} (B_1 + B_0) - \frac{x_3 - x_1}{V_0} P_t$. \hspace{0.5cm} (27)

The off-diagonal components are easy to express. Take $\bar{\sigma}_{x_2} = \bar{\sigma}_{x_3}$ as an example. According to Equation (15), $\bar{\sigma}_{x_2}$ can be expressed as
   $\bar{\sigma}_{x_2} = \frac{1}{V_0} \int_{S} \rho b_1 dV + \frac{1}{V_0} \oint_{S} t_f dS$
   \hspace{0.5cm} (28)

Since $\frac{1}{V_0} \oint_{S} t_f dS$ is between $y_1 \oint_{S} t_f dS$ and $y_2 \oint_{S} t_f dS$, $\frac{1}{V_0} \oint_{S} t_f dS$ is between $y_1 \oint_{S} t_f dS$ and $y_2 \oint_{S} t_f dS$. Since $\frac{1}{V_0} \oint_{S} t_f dS$ is between $y_1 \oint_{S} t_f dS$ and $y_2 \oint_{S} t_f dS$, we can also find the lower and upper bounds of $\bar{\sigma}_{x_2}$ based on the discussion above. The latter numerical simulations show that the off-diagonal components are no longer zero for an irregularly shaped body, but they are still small values compared with the diagonal components, especially for the

Figure 5. Illustration of element average stress for irregular bodies.

of the body. Then, we have
\begin{align*}
B_1 - T_1 + P_1 &= 0 \\
B_2 - T_2 + P_2 &= 0.
\end{align*}
\hspace{0.5cm} (20)

Here $P_1$ and $P_2$ are the resultant shear force in the four lateral faces parallel to the $x$-axis:
\begin{align*}
\bar{\sigma}_x &= \frac{1}{V_0} \int_{S} \rho b_1 dV + \frac{x_1}{V_0} \oint_{S} t_f dS
+ \frac{x_2}{V_0} \oint_{S} t_f dS + \frac{1}{V_0} \oint_{S} x_t f S f_x dt dS \\
&= \frac{1}{V_0} \int_{S} \rho b_1 dV + \frac{x_1}{V_0} (B_1 + P_t)
+ \frac{x_2}{V_0} (B_2 + P_2) + \frac{1}{V_0} \oint_{S} x_t f S f_x dt dS.
\end{align*}
\hspace{0.5cm} (21)

We assume that $t_f$ does not change the direction over a short range. Then it has
\begin{align*}
\oint_{S} x_t f S f_x dt dS &\in \left[ \min \left\{ x_1 \oint_{S} t_f dS, x_2 \oint_{S} t_f dS \right\}, \right.
\max \left\{ x_1 \oint_{S} t_f dS, x_2 \oint_{S} t_f dS \right\} \right].
\end{align*}
\hspace{0.5cm} (22)

The average stress is between
\begin{align*}
\bar{\sigma}_x \in [\kappa_0 + \kappa_1, \kappa_0 + \kappa_2],
\end{align*}
\hspace{0.5cm} (23)

where
\begin{align*}
\kappa_0 &= \frac{1}{V_0} \int_{S} \rho b_1 dV + \frac{x_1}{V_0} (B_1 + P_t) + \frac{x_2}{V_0} (B_2 + P_2), \\
\kappa_1 &= \min \left\{ \frac{x_1}{V_0} \oint_{S} t_f dS, \frac{x_2}{V_0} \oint_{S} t_f dS \right\}, \\
\kappa_2 &= \max \left\{ \frac{x_1}{V_0} \oint_{S} t_f dS, \frac{x_2}{V_0} \oint_{S} t_f dS \right\}.
\end{align*}

Meanwhile, we have $B_1 + B_1 + B_0 + P_1 + P_2 + \oint_{S} t_f dS = 0$
for the whole part $E = E_0 \cup E_1 \cup E_2$. Here $B_0$ is the body force of element $E_0$.

Two conditions are discussed:
maximum value. Therefore, we ignore the off-diagonal components and consider the diagonal components of the stress matrix in the body-fixed frame as the major stress.

Based on the above discussion, the upper and lower bounds of the average stress of any element in an irregularly shaped body $R$ can be determined. If the body is in compressive stress and cohesionless, the actual stress state of the body should be within the range. Otherwise, the calculated stress can be used to estimate the required cohesion. The maximum stress $\sigma^x_{\text{max}}$ over the whole body $\sigma^x_{\text{max}} = \max \{|\sigma^x(E_0)|, E_0 \in R\}$ will be used to calculate the least upper bound of required cohesion $C^0_U$ to keep the structure stable.

### 4.2. Verification by Ellipsoid Models

The analytical solution of an ellipsoid is adopted to verify the feasibility of the element average stress. The size of the ellipsoid is $29.0 \times 14.2 \times 14.2$ m with a constant density $\rho = 2.5 \text{ cm}^3 \text{ g}^{-1}$. Two cases are discussed. In the first case, a low spin rate is chosen: $\omega_1 = 3.737 \times 10^{-4}$ rad s$^{-1}$. Normal stresses $\sigma_x$ on the y-z cross section at $x = 0$ are compared. The analytical solution of an ellipsoid is based on Equation (12), and the element average stress is solved on small elements centered at the cross section. In this simulation and those in the next section, we choose the cuboid as the small element, which is easy to describe and calculate. Figure 6 shows the results of two methods.

The shear force in the ellipsoid is always zero, $P_1 = P_2 = 0$. It is found that the lower bound of stress $\sigma^x_{\text{max}}$ is equal to the upper bound of stress $\sigma^x_{\text{max}}$ due to symmetry. Therefore, the precise average stress for any elements in the ellipsoid can be solved. As seen in Figure 6, the body is in compressive stress at a low spin rate. The numerical results are well fit to the analytical solutions, and the relative error is less than 5% for most of the regions. Large errors only appear at the boundary of the cross section. The stresses along the y- and z-axes have similar accuracy. Since the whole body is in compressive stress, the element average stress can give a good estimation of the actual stress state.

Next, we increase the spin rate to $\omega = 3.737 \times 10^{-3}$. The stress state will transfer to tensile. The numerical results are still close to the analytical results, as shown in Figure 7. Two
methods get the same maximum value of stress. Though the result is solved in the absence of cohesion and may differ from the actual stress, the maximum value can be used to solve the required cohesion $C_0$.

The comparisons indicate that the element average stress can numerically obtain the stress state of an ellipsoid in both compressive and tensile states. For compressive stress, the element average stress can give us the real stress of the elements in a cohesionless body. For tensile stress, the element average stress can provide information on the stress state induced by gravity and centrifugal force, which can be used to estimate the least upper bound of the required cohesion of the body. If the model changes to an irregularly shaped body, the upper and lower bounds of stress, $\bar{\sigma}^u$ and $\bar{\sigma}^l$, are no longer the same due to the asymmetry. Meanwhile, the shear force $P_1$ and $P_2$ are not equal to zero. The shear force will resist the motion of the body. Therefore, it has the opposite direction of the body force, and its value is should be less than the body force. Choosing $P_1 = P_1 = 0$ will increase both the lower and upper bounds of normal stress more than the actual value. Since the upper bound of normal stress is of interest, we set $P_1 = P_1 = 0$ in the following calculation. It will give us a more conservative result of $C_0$. But the error is small, as the shear force is usually far smaller than the tensile force. Accurate results of $P_1$ and $P_2$ require an FEM and are also related to material properties (Poisson rate and Youngs modulus).

The proposed element average stress can find the boundary of stress with only the calculation of body force. It does not require one to solve the equilibrium equation and compatible equations of the whole body. A conservative estimation of the required cohesion can be found easily. Therefore, it is more efficient than the previous FEM method by ANSYS with little loss of accuracy, despite the difference in the yield criterion (Hirabayashi & Scheeres 2014). It is an effective way to make a preliminary evaluation of the cohesion in a fast-spinning body. In the following section, the element average stress is applied to several shape models, and the influence of irregular shape on the required cohesion is discussed.

5. Simulations and Discussions

The element average technique above is used to analyze the required cohesion for several asteroids. Three fast-spinning

![Image](image_url)
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Figure 8. Shape model of two asteroids: (a) 1998 KY26 and (b) 54509 YORP.

Table 1

| Parameters | 1998 KY26 | 54509 YORP | 469219 Kamo’oalewa |
|------------|-----------|------------|--------------------|
| Rotation period (hr) | 0.178 | 0.203 | 0.467 |
| Spectrum type | X | X | L |
| Size/diameter (m) | 29.3 × 27.8 × 26.6 | 149.3 × 133.5 × 95.7 | 21.59, 19.41, 0.30 |
| Density (g cm⁻³) | 5.32 | 2.5 | 2.5 |

Cohesion in Asteroids 1998 KY26 and 54509 YORP

The stress state and required cohesion for two shape models are discussed. At first, the volume average stresses of the same-sized ellipsoid models are calculated as \( \bar{\sigma}_{\text{KY}} = [-21.59, -19.41, 0.30] \) and \( \bar{\sigma}_{\text{YORP}} = [-204.65, -163.45, 1.06] \) Pa. Due to its larger size, asteroid YORP requires more cohesion than asteroid KY26 to keep stable, even at a lower density and spin rate. The corresponding lower bounds of required cohesion become \( C_{\text{EL, KY}}^{\text{i}} = 24.93 \) and \( C_{\text{EL, YORP}}^{\text{i}} = 236.31 \) Pa.

Next, the element average stress is applied to find the stress distribution in the whole body. For asteroid 1998 KY26, the upper bound of stress \( \bar{\sigma}_{\text{KY}}^{\text{u}} \) and \( \bar{\sigma}_{\text{YORP}}^{\text{u}} \) at the cross section \( x = 0 \) is shown in Figure 9, which contains the maximum value of \( \bar{\sigma}_{\text{KY}}^{\text{u}} \). As mentioned above, the off-diagonal component is much smaller than the diagonal component, especially at the center region. That means the diagonal component of the stress matrix can roughly approximate the major term. The maximum stress will not change a lot.

Figure 10 further shows the distribution of \( \bar{\sigma}_{\text{KY}}^{\text{u}} \) at the cross section \( x = 0 \) for asteroid 54509 YORP. Two asteroids show different distributions of element average stress at the cross section. Figure 9(a) is similar to Figure 7(a), as the cross section of asteroid 1998 KY26 is approximately a circle. The maximum tensile stress is offset slightly right of the center and decreases along the radial direction. The descent rate shows...
some differences along different orientations. But stress reduces to zero at all boundaries. On the contrary, the maximum tensile stress for asteroid YORP does not locate at the center but is close to the upper boundary, which has a relatively large tensile stress. This result is related to the asymmetric mass distribution. As shown in Figure 10, the maximum length of asteroid YORP is at the plane \( z = 0.025 \) km, which is exactly the coordinate of maximum tensile stress. The coordinate of maximum tensile stress of asteroid 1998 KY26 also corresponds to the maximum length along the \( x \)-axis. The stress map at plane \( y = 0 \) can better reflect the off-center distribution (shown in Figure 11). This indicates that the upper surface of the YORP model may be more susceptible to structural failure than the lower surface and the whole body may bend downward, while the structural failure may happen at the center and break asteroid 1998 KY26 into two parts.

Based on Figure 11, the maximum of the upper bound of stress \( \sigma_{\text{KY max}}^x \) in two asteroids can be found as \( \sigma_{\text{KY max}}^x = -53.08 \) and \( \sigma_{\text{YORP max}}^x = -513.80 \) Pa. In particular, the maximum tensile stress of asteroid 1998 KY26 still exists in the \( x \)-axis, \( \sigma_{\text{KY max}}^x = -53.08 \) and \( \sigma_{\text{YORP max}}^x = -48.37 \) Pa. The results are similar in two ellipsoid models, \( \sigma_{\text{KY max}}^x = -53.98 \) and \( \sigma_{\text{YORP max}}^x = -511.63 \) Pa. Therefore, the required cohesions solved by the irregularly shaped and ellipsoid models are nearly the same, \( C_{\text{KY}}^\text{MC} = 61.30 \) and \( C_{\text{KY}}^\text{MC} = 593.30 \) Pa. That means the same-sized ellipsoid model can be used to estimate the least upper bound of the required cohesion of an irregularly shaped fast-spinning body. But the shape model is necessary if we want to have a better understanding of the stress distribution in the body.

5.2. Cohesion in Asteroid 469219 Kamo’oalewa

Next, the two shape models are used to analyze the cohesion in asteroid 469219 Kamo’oalewa. Currently, there is no detailed shape model of asteroid 469219 Kamo’oalewa. According to limited observation, its light curve has a large amplitude of \( A = 0.5 \), indicating an elongated shape (Reddy et al. 2017). Therefore, two shape models are modified to three size ratios (Li & Scheeres 2021). \( S_1: a_1 = 29.0, b_1 = 14.2, c_1 = 14.2 \) m; \( S_2: a_2 = 33.0, b_2 = 15.8, c_2 = 11.2 \) m; and \( S_3: a_3 = 44.5, b_3 = 13.6, c_3 = 9.6 \) m. The required cohesions \( C_{\text{U}}^x \) for different models are listed in Table 2, and Figure 12 shows the stress distribution at cross section \( x = 0 \) for size ratio \( S_1 \).

From Figure 12 and Table 2, the required cohesions \( C_{\text{U}}^x \) of the different shape models are nearly the same and proportional to \( a^2 \). Meanwhile, despite the value, the stress distributions in Figures 9 and 12 are similar. The mass distribution of the body is the major factor that affects the stress state. The maximum tensile stress always exists at the point where the maximum length along the \( x \)-axis passes.

In the meantime, we compare the required cohesion based on the nonlinear and M-C criteria. The ellipsoid model with the size ratio \( S_1 \) is used. The stress state at the center is \( \sigma_1 = -14.30, \sigma_2 = -3.30, \sigma_3 = 0.22 \) Pa. At the friction angle \( \varphi = 30^\circ \), the cohesion based on the nonlinear criterion is \( C_{\text{SC}}^x = 16.52 \) Pa. The cohesion based on the M-C criterion is \( C_{\text{MC}}^x = 12.45 \) Pa. At the friction angle \( \varphi = 45^\circ \), the cohesion based on the nonlinear criterion is \( C_{\text{SC}}^x = 28.30 \) Pa. The cohesion based on the M-C criterion is \( C_{\text{MC}}^x = 17.30 \) Pa. The results show that the nonlinear criterion predicts a larger cohesion than the M-C criterion, especially at a larger friction angle.
Finally, based on the YORP shape model $S_1$ in Table 2, the effect of local terrain changes on the stress state is discussed. This will reflect how stress changes if some materials on the surface are artificially moved away, such as sampling by probes. Some elements on the upper surface of asteroid 469219 Kamo‘oalewa are removed to create a divot. Two divot positions are considered separately: divot $d_n$ is near the north pole, and divot $d_b$ is close to the right edge, as illustrated in Figure 13. We neglect the force applied to the body during digging or sampling and focus on the static stress state. The upper bound of stress $\sigma_{x}^{u}$ in the chosen cross section near the surface are calculated. It is found that compared with the original state, divots on the surface do not cause stress concentrations or increases in the stress state on the whole surface. As stress in the body is generated by gravity and centrifugal force, removing some parts of the body also decreases gravity and centrifugal force accordingly. If the shear force on the surface is neglected, divots only decrease the stress in the striped region containing the divots themselves along the $x$-axis. The divot near the right edge affects more regions, as the centrifugal force is far larger than the self-gravity at the divot, in which the centrifugal force and gravity have the same magnitude near the north pole. If the average shear force is considered, stresses in more areas are affected, but it will not increase the maximum tensile stress on the surface. The simulations indicate that if the asteroid is stable before sampling, taking away some samples on the (sub)surface will not change the static structural stability. The contact force during sampling might be the major factor that impacts the

### Table 2

$c^{ud}_{0}$ of Asteroid 469219 Kamo‘oalewa with Different Shape Models ($\varphi = 30^\circ$, Unit: Pa)

| Size Ratio | 1998 KY26 | 54509 YORP | Ellipsoid |
|------------|-----------|------------|----------|
| $S_1$      | 16.52     | 16.62      | 16.52    |
| $S_2$      | 21.52     | 21.87      | 21.51    |
| $S_3$      | 39.49     | 39.66      | 39.46    |

Figure 11. The $\sigma_{x}^{u}$ at the $x$–$z$ cross section $y = 0$. (a) Asteroid 1998 KY26. (b) Asteroid 54509 YORP.

Figure 12. Upper bound of averaged stresses $\sigma_{x}^{u}$ of asteroid 469219 Kamo‘oalewa with different shape models. (a) KY model. (b) YORP model.
structural stability of a fast-spinning asteroid, which is worth further study in the future.

5.3. Discussion

The above analysis based on shape models and ellipsoids investigates the structural stability of fast-spinning asteroids. The maximum tensile stress in an irregularly shaped body is solved by the element average stress and used to find the least upper bound of required cohesion. The result indicates that the longest axis and spin rate are the major factors that determined the required cohesion. If the models have the same size and spin rate, they have a similar maximum tension, but their stress distribution varies with shape. The maximum tensile stress does not appear at the center of mass but at the position where the longest axis goes through. Meanwhile, the asymmetric shape of an asteroid will change the stress state near the surface. The surface may have a large tensile stress if the longest axis is close to it. Furthermore, the simulation of the local shape change shows that digging a hole and taking out samples on the surface will not seriously change the structural stability of the whole body. A sampling at the pole region will not significantly change the stress state. These results can provide a reference for future exploration of fast-spinning asteroids.

Note that the current value of stress solved by the element average stress method may not be the actual stress when cohesion is considered. But we can find the least upper bound of required cohesion to keep the body stable. It also qualitatively shows the distribution of stress in the irregularly shaped bodies and the most violable region. Given the nonlinear yield condition and the upper bound of the stress state, the cohesion evaluated in this paper will be more conservative than that of previous research. Even so, the results indicate that the required cohesion in a fast-spinning body is still at a low level. The cohesion varies between tens to 1000 Pa. For a 50 m asteroid such as asteroid 469219 Kamo‘oalewa, the required cohesion is $C_0 = 39.66$ Pa at a friction angle $\phi = 30^\circ$ in an elongated shape and increases to $C_0 = 68.69$ Pa at a friction angle $\phi = 45^\circ$, which is still smaller than the average state of lunar soil (about 100 Pa; Colwell et al. 2007). It will expand the range of possible material compositions of fast-spinning small bodies. But in situ exploration is required to find the accurate cohesion and material properties of such bodies. We are looking forward to seeing the potential missions to small bodies and expanding our understanding of fast-spinning small bodies. Meanwhile, the distinct spin-up of asteroid 54509 YORP has been observed due to the YORP effect (the reason for its name; Taylor et al. 2007). It would be interesting if we could observe a breakup of the body due to a faster spin rate and better estimate the cohesion in the body.

6. Conclusion

The required cohesion in fast-spinning small bodies is investigated in this paper. Due to the effect of centrifugal acceleration, asteroids are in a tensile stress state, and cohesion is necessary to keep the body stable. A nonlinear yield condition for bulk solids is used to find the required cohesion in tensile stress. It is found that the cohesion is mainly determined by the maximum tensile stress in the body and increases with the friction angle. Next, the least upper bound of required cohesions is investigated for both ellipsoid and irregular shape models, when only self-gravity and centrifugal force are considered. For an ellipsoid, the critical spin rate at different sizes is investigated. It is found that the required cohesion is affected by the spin rate and length of the major axis. Further, an element average stress method is developed to analyze the stress distribution in irregularly shaped bodies. The range of stress of any element in a body is estimated. For a fast-spinning asteroid, the maximum of the upper bound of tensile stress in the overall body is used to determine the required cohesion. Though the element average stress may not be the actual stress state when cohesion is considered, it shows the distribution of stress qualitatively and gives a conservative estimation of the required cohesion to keep a body stable. Finally, simulations on several asteroids show that the irregular shape will not affect the least upper bound of the required cohesion but will change the stress distribution. The maximum tensile stress is found at the point where the longest axis goes through, and the surface may also be stressed due to the overall asymmetric shape. The result shows that the required cohesion to keep a 50 m asteroid stable is less than 70 Pa, which is a relatively small value compared with lunar soil (averaged larger than 100 Pa). The model and technique developed in this study provide an effective way to understand the structural stability of
fast-spinning bodies and can be a useful reference for future exploration missions.

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