Fractional elliptic problem involving a singularity, a critical exponent and a Radon measure

Akasmika Panda‡, Debajyoti Choudhuri‡, Ratan K. Giri‡,∗

‡Department of Mathematics, National Institute of Technology Rourkela
Rourkela - 769008, India
† Mathematics department, Technion - Israel Institute of Technology,
Amado building, Haifa 32000, Israel
Emails: akasmika44@gmail.com, dc.iit12@gmail.com

Abstract
In this paper, we prove the existence of a positive SOLA (Solutions Obtained as Limits of Approximations) to the following PDE involving fractional power of Laplacian.

\[(−\Delta)^s u = \frac{1}{u^\gamma} + \lambda u^{2^{*}_s-1} + \mu \text{ in } \Omega,\]
\[u > 0 \text{ in } \Omega,\]
\[u = 0 \text{ in } \mathbb{R}^N \setminus \Omega.\] (0.1)

Here, \(\Omega\) is a bounded domain of \(\mathbb{R}^N\), \(s \in (0,1), 2s < N, \lambda, \gamma \in (0,1), 2^*_s = \frac{2N}{N-2s}\) is the fractional critical Sobolev exponent and \(\mu\) is a bounded Radon measure in \(\Omega\).

Keywords: Fractional Sobolev spaces, SOLA, Radon measure, Marcinkiewicz space, critical exponent.

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1 Introduction

In this paper we discuss the following fractional elliptic problem with a singularity, a critical exponent and a Radon measure.

\[(−\Delta)^s u = \frac{1}{u^\gamma} + \lambda u^{2^{*}_s-1} + \mu \text{ in } \Omega,\]
\[u > 0 \text{ in } \Omega,\]
\[u = 0 \text{ in } \mathbb{R}^N \setminus \Omega.\] (P_\lambda)

∗Corresponding author: giri@campus.technion.ac.il
where $\Omega$ is a bounded domain in $\mathbb{R}^N$ with $C^2$ boundary, $s \in (0, 1)$, $N > 2s$, $0 < \gamma < 1$, $\lambda > 0$, $\mu$ is a bounded Radon measure and $(-\Delta)^s$ is the fractional Laplacian defined by

$$(-\Delta)^s u = \text{P. V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy.$$ 

Problems involving nonlocal operators have theoretical applications as well as real-life applications in various fields of science. The applications of fractional order Laplacian can be found in Lévy stable diffusion process, chemical reactions in liquids, geophysical fluid dynamics, electromagnetism etc (refer [2] for further details). Nonlocal problems containing singular or irregular data are used in dislocation problems [10], quasi-geostrophic dynamics [11], image reconstruction problems [20] etc. The problem of denoising an image is to find a clear image $u$ from a noisy $f$. In the deblurring problem, a given image $f$ is considered as a blurry version of an unknown exact image $u$, which is to be determined. For further details refer Kinermann et al. [24]. Readers may refer to the work in [36], [37], [38].

In general, the presence of a measure data in the problem weakens the class of solution space, i.e. we lose some degrees of differentiability or/and integrability of the solution space. Solutions to problems involving measure data or $L^1$ data are obtained by approximations and usually by working in Marcinkiewicz spaces. Readers may refer [4], [7], [8], [25] and the references therein for further readings on these types of problems. Bocardo et al. ([7], [8]) proved that the solution to a nonlinear elliptic equation involving a Radon measure lies in $W^{1,q}_0(\Omega)$ for every $q < \frac{N(p-1)}{N-1}$, where $1 < p < N$. Recently, in 2015, Kuusi et al. [25] considered a similar kind of problem with a fractional nonlocal operator and established the existence of a solution in $W^{s_1,q}(\Omega)$ for every $s_1 < s < 1, q < \min\{\frac{N(p-1)}{N-s}, p\}$. Purely singular problems both in the local and nonlocal cases are studied in [9], [12], [26], etc. and the references therein. In all these articles the choice of a solution space depends on the power $\gamma$ of the singular term (whether $\gamma \leq 1$ or $\gamma > 1$). Further, we refer [5], [6], [32], etc. to survey Brezis-Nirenberg type critical exponent problems (without the singular term and measure data).

The problem $(P_\lambda)$ for $\lambda = 0$ and the limiting case of $s = 1$ has been analyzed by Panda et al. in [31]. The authors have guaranteed the existence of a weak solution in $W^{1,q}_0(\Omega)$ if $0 < \gamma \leq 1$ and in $W^{1,q}_{loc}(\Omega)$ if $\gamma > 1$ for every $q < \frac{N}{N-1}$. Ghosh et al. in [16] extended this result and studied the problem $(P_\lambda)$ with $s \in (0, 1)$ and $\lambda = 0$. In the last few decades, the following problem has been studied by many researchers, both in the local and the nonlocal setup.

$$(-\Delta)_p^s u = \frac{\lambda_1 f(x)}{u^\gamma} + \lambda_2 u^r \quad \text{in } \Omega,$$

$$u > 0 \quad \text{in } \Omega,$$

$$u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega,$$

where $p \in (0, N/s), \lambda_1, \lambda_2 > 0, 1 < r \leq p_*^s, \gamma > 0$ and $f > 0$ is a bounded function. We cite [11], [13], [15], [17]-[19], [21]-[23], [28], [29], [35] and the references therein for the readers to have a glimpse of the problems of the type as in (1.2). The authors have shown the existence and multiplicity of solutions to (1.2) using different techniques like variational method, concentration compactness method, Perron method and Nehari manifold method. Amongst these works, the papers by Dhanya et al. [13], Giacomoni et al. (17,18), Haitao
Hirano et al. \([22, 23]\), etc. dealt with (1.2) for the local case, i.e. for \(s = 1\). The nonlocal case (for \(s \in (0, 1)\)) has been studied by Ghanmi & Soudi \([15]\), Giacomoni et al. \([19]\), Mukherjee & Sreenadh \([28]-[29]\), Soudi et al. \([35]\) and the references therein.

We use the relation among the fractional Sobolev space, Bessel potential space, Marcinkiewicz space to find a solution in a function space weaker than \(H^s_0(\Omega)\). Such solutions are called as SOLA (see Definition 2.9). Due to the presence of nonlinearities with a critical exponent, singularity and a measure data, difficulties arise in the study of \((P_\lambda)\). Thus, it is not easy to directly approach the problem with any commonly used tools like variational method, Nehari manifold method, etc. We study our main problem via a sequence of approximating problems. It is very challenging to prove the existence of a solution to the approximating problems and simultaneously showing the boundedness of the sequence of solution to these approximating problems in \(L^{2^*_s}(\Omega)\). To overcome these difficulties we take the help of two auxiliary problems. Precisely, we guarantee that the approximating problem admits at least one solution in a complete Hilbert manifold \(H = \{u \in H^s_0(\Omega) : \|u\|_{L^{2^*_s}(\Omega)} = 1\}\). We follow some of the arguments of \([31]\) to prove our main result stated in the following theorem.

**Theorem 1.1.** There exists \(0 < \Lambda < \infty\) such that for \(\lambda \in (0, \Lambda)\) the problem \((P_\lambda)\) admits a positive SOLA \(u \in W^{s_1, q}_0(\Omega)\) for every \(s_1 < s\) and \(q < \frac{N}{N-s}\) in the sense of Definition 2.9.

Before ending this section we describe the arrangements of the paper. In Section 2 we introduce suitable function spaces to deal with our problem and also provide some auxiliary results which will play important roles throughout the article. In Section 3 we prove the existence of a weak solution to the approximating problem for a certain range of \(\lambda\). Section 4 is devoted to the proof of Theorem 1.1. Further, in the Appendix, we show the multiplicity of solutions in the Nehari manifold.

## 2 Functional settings and auxiliary results

Let \(\Omega\) be a bounded domain in \(\mathbb{R}^N\), \(1 \leq p < \infty\) and \(s \in (0, 1)\). The fractional order Sobolev space (refer \([30]\)) is defined as

\[
W^{s, p}(\mathbb{R}^N) = \left\{ u \in L^p(\mathbb{R}^N) : \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \, dx \, dy < \infty \right\}
\]

and \(W^{s, p}_0(\Omega)\) is a subspace of \(W^{s, p}(\mathbb{R}^N)\) given by

\[
W^{s, p}_0(\Omega) = \left\{ u \in W^{s, p}(\mathbb{R}^N) : \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \, dy \, dx < \infty, \ u = 0 \text{ in } \mathbb{R}^N \setminus \Omega \right\}
\]

equipped with the norm

\[
\|u\|_{W^{s, p}_0(\Omega)}^p = \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \, dy \, dx.
\]

Further, the space \((W^{s, p}_0(\Omega), \| \cdot \|_{W^{s, p}_0(\Omega)})\) is a reflexive separable Banach space. The following classical theorem will be used frequently in this article.
Theorem 2.1 (Theorem 6.5, [30]). Let $0 < s < 1$ and $p \in [1, \infty)$ with $sp < N$. Then there exists constant $C = C(N, s, p) > 0$ such that for any $u \in W^{s,p}_0(\Omega)$,

$$
\|u\|_{L^r(\Omega)} \leq C\|u\|_{W^{s,p}_0(\Omega)}
$$

for any $r \in [p, p_s^*]$, where $p_s^* = \frac{Np}{N-sp}$. Moreover, the space $W^{s,p}_0(\Omega)$ is continuously embedded in $L^r(\Omega)$ for every $r \in [1, p_s^*]$ and compactly embedded in $L^r(\Omega)$ for every $r \in [1, p_s^*)$.

Denote

$$
S_{s,p} = \inf_{u \in W^{s,p}_0(\Omega) \setminus \{0\}} \frac{\|u\|_{W^{s,p}_0(\Omega)}}{\|u\|_{L^2(\Omega)}}
$$

which is the best Sobolev constant in the Sobolev embedding (Theorem 2.1). We now define some function spaces which will be further used in this article.

Remark 2.2. For $p = 2$, we denote the Sobolev space $H^s(\mathbb{R}^N) = W^{s,2}(\mathbb{R}^N)$. These spaces are Hilbert spaces. Proposition 3.6 of [30] provides the relationship between the fractional Sobolev space $H^s(\mathbb{R}^N)$ and the fractional Laplacian $(-\Delta)^s$. It states that the norms $\| \cdot \|_{H^s(\mathbb{R}^N)}$ and $\|(-\Delta)^{s/2} \cdot\|_{L^2(\mathbb{R}^N)}$ are two equivalent norms.

Definition 2.3 ([34]). For $s \in (0,1)$ and $p \in [1, \infty)$, the Bessel potential space $L^{s,p}(\mathbb{R}^N)$ is defined as

$$
L^{s,p}(\mathbb{R}^N) = \{ u \in L^p(\mathbb{R}^N) : |\nabla^s u| \in L^p(\mathbb{R}^N) \}
$$

where $\nabla^s u = \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+sp}} dy$ is the fractional gradient of order $s$.

We refer Theorem 2.2 of [34] to see the relation between the fractional Sobolev spaces and the Bessel potential spaces.

Theorem 2.4. 1. For non-negative integer $s$ and $1 < p < \infty$, $L^{s,p}(\mathbb{R}^N)$ coincides with $W^{s,p}(\mathbb{R}^N)$ and the corresponding norms of these two spaces are equivalent.

2. For $s \in (0,1)$ and $p = 2$, $L^{s,2}(\mathbb{R}^N) = W^{s,2}(\mathbb{R}^N)$.

3. For $s \in (0,1)$, $1 < p < \infty$ and $0 < \epsilon < s$, the following continuous embedding holds

$$
L^{s+\epsilon,p}(\mathbb{R}^N) \subset W^{s,p}(\mathbb{R}^N) \subset L^{s-\epsilon,p}(\mathbb{R}^N).
$$

Definition 2.5. A measurable function $u : \Omega \to \mathbb{R}$ is said to be in the Marcinkiewicz space $M^q(\Omega)$ $(0 < q < \infty)$ if

$$
m(\{ x \in \Omega : |u(x)| > t \}) \leq \frac{C}{t^q}, \text{ for } t > 0 \text{ and } 0 < C < \infty.
$$

Remark 2.6. For $\Omega$ bounded,

1. $M^{q_1}(\Omega) \subset M^{q_2}(\Omega)$ for every $q_1 \geq q_2 > 0$.

2. For $1 \leq q < \infty$ and $0 < \epsilon < q - 1$, the following continuous embedding holds

$$
L^q(\Omega) \subset M^q(\Omega) \subset L^{q-\epsilon}(\Omega). \quad (2.4)
$$
For a fixed $k > 0$, we denote the truncation functions $T_k : \mathbb{R} \to \mathbb{R}$ by

$$T_k(s) = \begin{cases} 
    s & \text{if } |s| \leq k \\
    k & \text{if } |s| > k.
\end{cases}$$

Since our problem, defined in $(P_\lambda)$, involves a measure data as a nonhomogeneous term in the right hand side, we need to introduce the notion of convergence in measure.

**Definition 2.7.** Let $\mathcal{M}(\Omega)$ be the set of all finite Radon measures on $\Omega$ and $(\mu_n)$ be a sequence of measurable functions in $\mathcal{M}(\Omega)$. Then we say $(\mu_n)$ converges to $\mu \in \mathcal{M}(\Omega)$ in the sense of measure if

$$\int_\Omega \mu_n \phi \to \int_\Omega \phi d\mu, \quad \forall \phi \in C_0(\overline{\Omega}).$$

In the following theorem we state a commonly used variational principle, introduced by Ekeland in [14].

**Theorem 2.8.** (Ekeland Variational Principle [14]) Let $V$ be a Banach space and $\Psi : V \to \mathbb{R} \cup \{+\infty\}$ is a lower semicontinuous, Gâteaux-differentiable and bounded from below function. Then for every $\epsilon > 0$, every $u \in V$ satisfying $\Psi(u) \leq \inf \Psi + \epsilon$, every $\delta > 0$, there exists $v \in V$ such that $\Psi(v) \leq \Psi(u)$, $\|u - v\| \leq \delta$ and $\|\Psi'(v)\| \leq \frac{\epsilon}{\delta}$ where $\|\cdot\|$ and $\|\cdot\|_*$ are the norm of $V$ and the dual norm of $V$, respectively.

We now introduce a suitable notion of solution to $(P_\lambda)$ that in general do not lie in the natural energy space corresponding to the operator $(-\Delta)^s$, i.e. $H^s_0(\Omega)$, but has a lower degree of differentiability and integrability. They are called SOLA (Solutions Obtained as Limits of Approximations) and the procedure of construction of SOLA is through a sequence of approximating problems.

**Definition 2.9 (SOLA for $(P_\lambda)$).** Let $\mu \in \mathcal{M}(\Omega)$ and $0 < \gamma < 1$. Then we say $u \in W^{s_1,q}_0(\Omega)$ for $s_1 < s$ and $q < \frac{N}{N-s}$ is a SOLA to $(P_\lambda)$ if

$$\int_{\mathbb{R}^N} (-\Delta)^{s/2} u \cdot (-\Delta)^{s/2} \phi = \int_{\Omega} \frac{1}{u^n} \phi + \int_{\Omega} \lambda u^{2^*_s - 1} \phi + \int_{\Omega} \phi d\mu, \quad \forall \phi \in C_0^\infty(\Omega) \quad (2.5)$$

and for every $\omega \subset \subset \Omega$, there exists a $C_\omega$ such that

$$u \geq C_\omega > 0. \quad (2.6)$$

We prove the existence of solution to the problem $(P_\lambda)$ with the help of the following sequence of approximating problem.

$$(\Delta)^s u_n = \frac{1}{(u_n + \frac{1}{n})^\gamma} + \lambda u_n^{2^*_s - 1} + \mu_n \text{ in } \Omega,$$

$$u_n > 0 \text{ in } \Omega,$$

$$u_n = 0 \text{ in } \mathbb{R}^N \setminus \Omega,$$

where $\mu_n > 0$ for each $n$ and the sequence $(\mu_n) \subset L^\infty(\Omega)$ bounded in $L^1(\Omega)$. This sequence $(\mu_n)$ converges to $\mu$ in the sense of measure as defined in Definition 2.7.
**Definition 2.10.** A function \( u_n \in H^1_0(\Omega) \) is said to be a positive weak solution of \((P_{\lambda,n})\) if for every \( \phi \in C^\infty_c(\Omega) \)
\[
\int_{\mathbb{R}^N} (-\Delta)^{s/2} u_n \cdot (-\Delta)^{s/2} \phi = \int_{\Omega} \frac{1}{(u_n + \frac{1}{n})^\gamma} \phi + \int_{\Omega} \lambda u_n^{2^*_\gamma - 1} \phi + \int_{\Omega} \mu_n \phi \tag{2.7}
\]
and for every \( \omega \subset \subset \Omega \), there exists a \( C_\omega \) such that \( u_n \geq C_\omega > 0 \).

We begin with the following sequence of problems.
\[
(-\Delta)^s w_n = \frac{1}{(w_n + \frac{1}{n})^\gamma} + \mu_n \text{ in } \Omega,
\]
\[
w_n > 0 \text{ in } \Omega,
\]
\[
w_n = 0 \text{ in } \mathbb{R}^N \setminus \Omega.
\]

We now look for a weak solution to \((P_{\lambda,n}^1)\) in the space \( \tilde{H} = \{ u \in H^s_0(\Omega) : \| u \|_{L^{2^*_\gamma}(\Omega)} < 1 \} \). Following the proof of Lemma 2.3 and Lemma 2.4 of Ghosh et al. [16], the problem \((P_{\lambda,n}^1)\) admits a positive weak solution \( w_n \) in \( \tilde{H} \cap L^\infty(\Omega) \). Furthermore, for every \( n \in \mathbb{N} \) and for every relatively compact set \( \omega \subset \Omega \), there exists a constant \( C_\omega \) independent of \( n \) such that \( w_n \geq C_\omega > 0 \).

**Remark 2.11.** The solution to the problem \((P_{\lambda,n}^1)\) is unique. To prove this, assume that the problem has two different solutions \( w_n \) and \( \bar{w}_n \). Let us consider \((w_n - \bar{w}_n)^+\) as a test function in the weak formulation of \((P_{\lambda,n}^1)\).

\[
0 \leq \int_{\mathbb{R}^N} |(-\Delta)^{s/2}(w_n - \bar{w}_n)^+|^2 \\
\leq \int_{\mathbb{R}^N} (-\Delta)^{s/2}(w_n - \bar{w}_n) \cdot (-\Delta)^{s/2}(w_n - \bar{w}_n)^+ \\
= \int_{\Omega} \left( \frac{1}{(w_n + \frac{1}{n})^\gamma} - \frac{1}{(\bar{w}_n + \frac{1}{n})^\gamma} \right) (w_n - \bar{w}_n)^+ \leq 0. \tag{2.8}
\]

This implies \((w_n - \bar{w}_n)^+ = 0 \text{ a.e in } \Omega \) and \( w_n \leq \bar{w}_n \text{ a.e in } \Omega \). In a similar manner taking \((\bar{w}_n - w_n)^+\) as a test function we can show that \( w_n \geq \bar{w}_n \text{ a.e in } \Omega \). This proves the claim.

We observe that \( u_n = w_n + v_n \) is a solution to \((P_{\lambda,n})\) if and only if \( w_n \) is a weak solution to \((P_{\lambda,n}^1)\) and \( v_n \) is a weak solution to the following problem
\[
(-\Delta)^s v_n + \frac{1}{(w_n + \frac{1}{n})^\gamma} - \frac{1}{(v_n + w_n + \frac{1}{n})^\gamma} = \lambda(w_n + v_n)^{2^*_\gamma - 1} \text{ in } \Omega,
\]
\[
v_n > 0 \text{ in } \Omega,
\]
\[
v_n = 0 \text{ in } \mathbb{R}^N \setminus \Omega.
\]

The following theorem guarantees the existence of a weak solution of \((P_{\lambda,n}^2)\) in the set \( H_n \) defined as \( H_n = \{ u \in \tilde{H} : \| u + w_n \|_{L^{2^*_\gamma}(\Omega)} = 1 \} \). We will prove this theorem in Section 3.
Theorem 2.12. There exists $\Lambda_n > 0$ such that for $\lambda \in (0, \Lambda_n)$, the problem $(P_{\lambda,n}^2)$ has a positive weak solution $v_n$ in $H_n$.

Since $v_n \in H_n$, $u_n = w_n + v_n \in H = \{ u \in H_0^s(\Omega) : \|u\|_{L^{2s}(\Omega)} = 1 \}$. We are now in a position to state the following theorem.

Theorem 2.13. There exists $0 < \Lambda_n < \infty$ such that for $\lambda \in (0, \Lambda_n)$ the problem $(P_{\lambda,n})$ admits a positive weak solutions $u_n$ in $H$ in the sense of Definition 2.10.

3 Existence of positive solution to $(P_{\lambda,n}^2)$

Define a function $g_n : \Omega \times \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ by

$$g_n(x, s) = \begin{cases} 
\frac{1}{(w_n(x) + \frac{s}{n})^\gamma} - \frac{1}{(s + w_n(x) + \frac{1}{n})^\gamma} & \text{if } s + w_n(x) + \frac{1}{n} > 0 \\
-\infty & \text{otherwise.}
\end{cases}$$

(3.9)

The properties of the function $g_n$ are same as the properties of the function $g$,

$$g(x, s) = \begin{cases} 
\frac{1}{w_n(x)} - \frac{1}{(s + w_n(x))^\gamma} & \text{if } s + w_n(x) > 0 \\
-\infty & \text{otherwise}
\end{cases}$$

(3.10)

as defined in [19]. Denote $G_n(x, s) = \int_0^s g_n(x, \tau) d\tau$ for $(x, s) \in \Omega \times \mathbb{R}$. We define the corresponding energy functional $I_{\lambda,n} : H_0^s(\Omega) \to (\mathbb{R}, \mathbb{R})$ of $(P_{\lambda,n}^2)$ by

$$I_{\lambda,n}(v_n) = \frac{1}{2} \int_{\mathbb{R}^2} \frac{|v_n(x) - v_n(y)|^2}{|x - y|^{N+2s}} dx dy + \int_{\Omega} G_n(x, v_n) dx - \frac{\lambda}{2s} \int_{\Omega} |v_n + w_n|^{2^*_s} dx \quad \text{if } G_n(., v_n) \in L^1(\Omega)$$

(3.11)

Further,

$$\langle I'_{\lambda,n}(v_n), v \rangle = \int_{\mathbb{R}^2} \frac{(v_n(x) - v_n(y))(v(x) - v(y))}{|x - y|^{N+2s}} dx dy + \int_{\Omega} g_n(x, v_n) v dx - \lambda \int_{\Omega} |v_n + w_n|^{2^*_s-1} v dx$$

for any $v \in H_0^s(\Omega)$. We now define the weak solution of $(P_{\lambda,n}^2)$ as follows.

Definition 3.1. A function $v_n \in H_n$ is said to be a weak solution of $(P_{\lambda,n}^2)$ if $v_n$ is a critical point of the functional $I_{\lambda,n}$.

Lemma 3.2. The functional $I_{\lambda,n}$ satisfies the Palais-Smale condition in $H_n$ for energy level

$$c < \frac{s}{N} \sum_{k,s} s N^{\frac{N-2}{2s}} - \frac{\lambda}{2s}$$

Proof. Let $(v_{n,m}) \subset H_n$ be a Palais-Smale sequence of $I_{\lambda,n}$, i.e. $I_{\lambda,n}(v_{n,m}) \to c$ and $I'_{\lambda,n}(v_{n,m}) \to 0$. Clearly, the functional $I_{\lambda,n}$ is coercive restricted to $H_n$ and hence the sequence $(v_{n,m})$ is bounded in $H_0^s(\Omega)$. Thus, there exists a $v_n \in H_0^s(\Omega)$ and a subsequence of $v_{n,m}$, which is still
denoted as $v_{n,m}$, such that $v_{n,m} \to v_n$ weakly in $H^s_0(\Omega)$.

Claim: $v_{n,m} \to v_n$ strongly in $H^s_0(\Omega)$ and $v_n \in H_n$.

Using the concentration compactness principle [Theorem 2.5 of [27]] for the case $p = 2$, there exist two positive Borel regular measures $\mu, \nu$ such that

$$
\int_{\mathbb{R}^N} \frac{|v_{n,m}(x) - v_{n,m}(y)|^2}{|x-y|^{N+2s}} dy \overset{\ast}{\rightharpoonup} \mu \geq \int_{\mathbb{R}^N} \frac{|v_n(x) - v_n(y)|^2}{|x-y|^{N+2s}} dy + \sum_{j \in I} \mu_j \delta_{x_j}, \quad \mu_j = \mu\{x_j\},
$$

(3.12)

and

$$
|v_{n,m}|^{2^*_s} \overset{\ast}{\rightharpoonup} \nu = |v_n|^{2^*_s} + \sum_{j \in I} \nu_j \delta_{x_j}, \quad \nu_j = \nu\{x_j\}
$$

(3.13)

and

$$
\mathbb{S}_{2,s} \nu_j^\frac{2}{s} \leq \mu_j, \quad \forall j \in I
$$

(3.14)

where \( \{x_j : j \in I\} \), $I$ is countable, is a set of distinct points in $\mathbb{R}^N$, \( \{\nu_j : j \in I\} \in (0, \infty)$, \( \{\mu_j : j \in I\} \in (0, \infty)$ and $\mathbb{S}_{2,s}$ is the best Sobolev constant given in (2.3). Here the symbol $\overset{\ast}{\rightharpoonup}$ denotes the tight convergence. Hence, if $I = \emptyset$ then $v_{n,m} \to v_n$ strongly in $L^{2^*_s}(\Omega)$ and $v_n \in H_n$.

Suppose $I \neq \emptyset$. Then choose $\zeta \in C_c^\infty(\mathbb{R}^N)$, $0 \leq \zeta \leq 1$, $\zeta(0) = 1$ with support in a unit ball of $\mathbb{R}^N$. Let us define for any $\epsilon > 0$, the function $\zeta_{\epsilon,j}$ as $\zeta_{\epsilon,j} = \zeta(\frac{x-x_j}{\epsilon})$. Then

\[
(I'_{\lambda,n}(v_{n,m}), \zeta_{\epsilon,j} v_{n,m}) = \int_{\mathbb{R}^{2N}} \frac{(v_{n,m}(x) - v_{n,m}(y))(\zeta_{\epsilon,j} v_{n,m}(x) - \zeta_{\epsilon,j} v_{n,m}(y))}{|x-y|^{N+2s}} dxdy
\]

\[
+ \int_{\Omega} g_n(x, v_{n,m}) \zeta_{\epsilon,j} v_{n,m} - \lambda \int_{\Omega} |v_{n,m} + w_n|^{2^*_s - 1} \zeta_{\epsilon,j} v_{n,m}
\]

\[
\geq \int_{\mathbb{R}^{2N}} \frac{(v_{n,m}(x) - v_{n,m}(y)) w_n(x) (\zeta_{\epsilon,j}(x) - \zeta_{\epsilon,j}(y))}{|x-y|^{N+2s}} dxdy
\]

\[
+ \int_{\mathbb{R}^{2N}} \frac{|v_{n,m}(x) - v_{n,m}(y)|^2 \zeta_{\epsilon,j}(x)}{|x-y|^{N+2s}} dxdy + \int_{\Omega} g_n(x, v_{n,m}) \zeta_{\epsilon,j} v_{n,m}
\]

\[
- \lambda \int_{\Omega} |v_{n,m} + w_n|^{2^*_s} \zeta_{\epsilon,j} dx
\]

\[
\geq \int_{\mathbb{R}^{2N}} \frac{(v_{n,m}(x) - v_{n,m}(y)) w_n(x) (\zeta_{\epsilon,j}(x) - \zeta_{\epsilon,j}(y))}{|x-y|^{N+2s}} dxdy
\]

\[
+ \int_{\mathbb{R}^{2N}} \frac{|v_{n,m}(x) - v_{n,m}(y)|^2 \zeta_{\epsilon,j}(x)}{|x-y|^{N+2s}} dxdy + \int_{\Omega} g_n(x, v_{n,m}) \zeta_{\epsilon,j} v_{n,m}
\]

\[
- \lambda 2^*_s \int_{\Omega} \left( |v_{n,m} + w_n|^{2^*_s - 1} w_n + v_{n,m}^{2*_s - 1} w_n \right) \zeta_{\epsilon,j} - \lambda \int_{\Omega} |v_{n,m}|^{2^*_s} \zeta_{\epsilon,j}.
\]

(3.15)

By Mosconi & Squassina [27], we have $\lim_{\epsilon \to 0} \int_{\mathbb{R}^{2N}} \frac{|v_{n,m}(y)|^2 (\zeta_{\epsilon,j}(x) - \zeta_{\epsilon,i}(y))^2}{|x-y|^{N+2s}} dxdy = 0$. Thus, on using
the Hölder’s inequality we have

\[
\lim \lim_{\epsilon \to 0 \ m \to \infty} \left| \int_{\mathbb{R}^{2N}} \frac{(v_{n,m}(x) - v_{n,m}(y))(\zeta_{\epsilon,j}(x) - \zeta_{\epsilon,j}(y))}{|x-y|^{N+2s}} \, dxdy \right|
\leq \lim \lim_{\epsilon \to 0 \ m \to \infty} \left| v_{n,m} \right|_{H^s_0(\Omega)} \left( \int_{\mathbb{R}^{2N}} \frac{|v_{n,m}(y)|^2 |\zeta_{\epsilon,j}(x) - \zeta_{\epsilon,j}(y)|^2}{|x-y|^{N+2s}} \, dxdy \right)^{1/2}
\leq \lim_{\epsilon \to 0} \left( \int_{\mathbb{R}^{2N}} \frac{|v_n(y)|^2 |\zeta_{\epsilon,j}(x) - \zeta_{\epsilon,j}(y)|^2}{|x-y|^{N+2s}} \, dxdy \right)^{1/2}
= 0.
\]

Since, for \( x \neq x_j, \zeta_{\epsilon,j}(x) \to 0 \) as \( \epsilon \to 0, \zeta(0) = 1 \), thus on using (3.12) and (3.13) we have

\[
\lim_{\epsilon \to 0} \lim_{m \to \infty} \int_{\Omega} (|v_{n,m} + w_n|^{2s-1} w_n + |v_{n,m}|^{2s-1} w_n) \zeta_{\epsilon,j} = \lim_{\epsilon \to 0} \int_{\Omega} (|v_n + w_n|^{2s-1} w_n + |v_n|^{2s-1} w_n) \zeta_{\epsilon,j} = 0,
\]

\[
\lim_{\epsilon \to 0} \lim_{m \to \infty} \int_{\mathbb{R}^{2N}} \frac{|v_{n,m}(x) - v_{n,m}(y)|^2 \zeta_{\epsilon,j}(x)}{|x-y|^{N+2s}} \, dxdy = \lim_{\epsilon \to 0} \int_{\mathbb{R}^N} \zeta_{\epsilon,j} \, d\mu = \mu_j,
\]

\[
\lim_{\epsilon \to 0} \lim_{m \to \infty} \int_{\Omega} g_n(x, v_{n,m}) \zeta_{\epsilon,j} v_{n,m} = \lim_{\epsilon \to 0} \int_{\mathbb{R}^N} g_n(x, v_n) v_n \zeta_{\epsilon,j} = 0,
\]

\[
\lim_{\epsilon \to 0} \lim_{m \to \infty} \int_{\Omega} |v_{n,m}|^{2s} \zeta_{\epsilon,j} = \lim_{\epsilon \to 0} \int_{\Omega} \zeta_{\epsilon,j} \, dv = \nu_j.
\]

On passing the limit \( \epsilon \to 0 \) and limit \( m \to \infty \) in the inequality (3.15) we have \( 0 \geq \mu_j - \lambda \nu_j \). This further implies that \( \mu_j \leq \lambda \nu_j \). Since \( S_{2,s} \nu_j^{2s} \leq \mu_j \) from (3.14), hence we have \( \nu_j \geq \left( \frac{S_{2,s}}{\lambda} \right)^{\frac{N}{2s}} \). Therefore,

\[
c = \lim_{m \to \infty} I_{\lambda,n}(v_{n,m})
= \lim_{m \to \infty} \left( \frac{1}{2} \int_{\mathbb{R}^{2N}} \frac{|v_{n,m}(x) - v_{n,m}(y)|^2}{|x-y|^{N+2s}} \, dxdy + \int_{\Omega} G_n(x, v_{n,m}) \, dx \right) - \frac{\lambda}{2s}
\geq \frac{1}{2} \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^{2N}} \frac{|v_n(x) - v_n(y)|^2}{|x-y|^{N+2s}} \, dy + \sum_{j \in I} \mu_j \delta_{x_j} \right) \zeta_{\epsilon,i} \, dx - \frac{\lambda}{2s}
\geq \frac{s}{N} \mu_i - \frac{\lambda}{2s}
\geq \frac{s}{N} S_{2,s} \nu_j^{\frac{N}{2s}} - \frac{\lambda}{2s}
\geq \frac{s}{N} \frac{S_{2,s}^{\frac{N}{2s}}}{\lambda^{\frac{N}{2s}}} - \frac{\lambda}{2s}.
\]

which is a contradiction to our assumption \( c < \frac{s}{N} \frac{S_{2,s}^{\frac{N}{2s}}}{\lambda^{\frac{N}{2s}}} - \frac{\lambda}{2s} \). Hence, the indexing set \( I \) is empty and \( v_{n,m} \to v_n \) strongly in \( L^{2^*}(\Omega) \) and \( v_n \in H_n \).
It remains to prove that \( v_{n,m} \to v_n \) strongly in \( H^s_0(\Omega) \). We use a standard method to prove this claim. Recall

\[
\langle I'_{\lambda,n}(v_{n,m}), v_{n,m} - v_n \rangle = \int_{\mathbb{R}^{2N}} \frac{(v_{n,m}(x) - v_{n,m}(y))((v_{n,m} - v_n)(x) - (v_{n,m} - v_n)(y))}{|x - y|^{N + 2s}} \, dx \, dy
\]

\[+ \int_{\Omega} g_n(x, v_{n,m})(v_{n,m} - v_n) - \lambda \int_{\Omega} |v_{n,m} + w_n|^{2^*_s - 1}(v_{n,m} - v_n) \quad (3.16)
\]

Since \((v_{n,m})\) is a bounded Palais-Smale sequence, therefore on passing the limit \( m \to \infty \) in \((3.16)\) we have

\[
\lim_{m \to \infty} \int_{\mathbb{R}^{2N}} \frac{(v_{n,m}(x) - v_{n,m}(y))((v_{n,m} - v_n)(x) - (v_{n,m} - v_n)(y))}{|x - y|^{N + 2s}} \, dx \, dy = 0.
\]

On using a simple calculation we get

\[
\|v_{n,m} - v_n\|_{H^s_0(\Omega)}^2 = \int_{\mathbb{R}^{2N}} \frac{|(v_{n,m} - v_n)(x) - (v_{n,m} - v_n)(y)|^2}{|x - y|^{N + 2s}} \, dx \, dy
\]

\[\leq C \int_{\mathbb{R}^{2N}} \left\{ \frac{(v_{n,m}(x) - v_{n,m}(y))((v_{n,m} - v_n)(x) - (v_{n,m} - v_n)(y))}{|x - y|^{N + 2s}}
\right.

\[\left. - \frac{(v_n(x) - v_n(y))((v_{n,m} - v_n)(x) - (v_{n,m} - v_n)(y))}{|x - y|^{N + 2s}} \right\} \, dx \, dy.
\]

Thus, \( \lim_{m \to \infty} \|v_{n,m} - v_n\|_{H^s_0(\Omega)} = 0 \) and hence \( v_{n,m} \to v_n \) strongly in \( H^s_0(\Omega) \). Therefore, \( v_n \in H^s_0(\Omega) \) is a critical point of \( I_{\lambda,n} \) and hence a weak solution of \((P^2_{\lambda,n})\).

Consider the sequence \((V_\epsilon)\) which is given by

\[
V_\epsilon = \epsilon^{-\frac{N-2s}{2}} v^*(\frac{x}{\epsilon}), \quad x \in \mathbb{R}^N.
\]

Here \( v^*(x) = \tilde{v}\left(\frac{x}{s^*_{2,s}}\right) \), \( \tilde{v}(x) = \frac{\tilde{\varphi}(x)}{\|\tilde{\varphi}\|_{L^{2^*_s}(\Omega)}} \) and \( \tilde{\varphi}(x) = \beta(\alpha^2 + |x|^2)^{-\frac{N-2s}{2}} \) with two fixed constants \( \beta \in \mathbb{R}^N \setminus \{0\}, \ \alpha > 0 \). According to Servadei & Valdinoci \[33\], for each \( \epsilon > 0 \) the corresponding \( V_\epsilon \) satisfies the problem

\[
(-\Delta)^s v = |v|^{2^*_s - 2} v \quad \text{in} \ \mathbb{R}^N
\]

and

\[
\int_{\mathbb{R}^{2N}} \frac{|V_\epsilon(x) - V_\epsilon(y)|^2}{|x - y|^{N + 2s}} \, dx \, dy = \int_{\mathbb{R}^N} |V_\epsilon|^2 \, dx = S^{N/2s}_{2,s}.
\]

Without loss of generality we can assume \( 0 \in \Omega \). Consider the function \( \zeta \in C_0^\infty(\mathbb{R}^N) \) such that \( 0 \leq \zeta \leq 1 \) and for a fixed \( \delta > 0 \) with \( B_{4\delta} \subset \Omega, \) \( \zeta \equiv 0 \) in \( \mathbb{R}^N \setminus B_{2\delta}, \) \( \zeta \equiv 1 \) in \( B_\delta \). Let us define a function \( \Psi_\epsilon(x) = \zeta(x)V_\epsilon(x) \), which is zero in \( \mathbb{R}^N \setminus \Omega \). By Giacomoni et al. \[19\], there
Lemma 3.3. There exists $\Lambda_n > 0$ such that for sufficiently small $\epsilon > 0$ and for $\lambda \in (0, \Lambda_n)$,

$$
\sup \{ I_{\lambda,n}(t\Psi_\epsilon) : t \geq 0 \} < \frac{s}{N} \frac{\frac{N}{2}}{\lambda^{\frac{N-2s}{2}}} - \frac{\lambda}{2^s}.
$$

Proof. Clearly for $\lambda < \left( \frac{s}{N} \frac{N}{2} \frac{S_{2,s}}{S_{2,s}} \right)^{2s/N}$, we have $\left( \frac{s}{N} \frac{N}{2} \frac{S_{2,s}}{S_{2,s}} \right) > 0$. Consider $\epsilon > 0$ to be sufficiently small. Then for any $t \geq 0$,

$$
I_{\lambda,n}(t\Psi_\epsilon) = \frac{t^2}{2} \int_{\mathbb{R}^{2N}} \frac{|\Psi_\epsilon(x) - \Psi_\epsilon(y)|^2}{|x-y|^{N+2s}} \, dx \, dy + \int_{\Omega} G_n(x, t\Psi_\epsilon) \, dx - \frac{\lambda}{2^s} \int_{\Omega} |t\Psi_\epsilon + w_n|^2 \, dx
$$

$$
= \frac{t^2}{2} \int_{\mathbb{R}^{2N}} \frac{|\Psi_\epsilon(x) - \Psi_\epsilon(y)|^2}{|x-y|^{N+2s}} \, dx \, dy + \int_{\Omega} \frac{|t\Psi_\epsilon|}{(w_n + 1/n)^\gamma} \, dx
$$

$$
- \frac{1}{1 - \gamma} \int_{\Omega} (t\Psi_\epsilon + w_n + 1/n)^{1-\gamma} - (w_n + 1/n)^{1-\gamma} - \frac{\lambda}{2^s} \int_{\Omega} |t\Psi_\epsilon + w_n|^2 \, dx
$$

$$
\leq \frac{t^2}{2} \left( \frac{N}{2} \frac{S_{2,s}}{S_{2,s}} + a_1 \epsilon^{N-2s} \right) + tn^\gamma \int_{\Omega} |\Psi_\epsilon| + \frac{\lambda}{2^s} - \frac{\lambda}{2^s}
$$

$$
- \frac{1}{1 - \gamma} \int_{\Omega} (t\Psi_\epsilon + w_n + 1/n)^{1-\gamma} - (w_n + 1/n)^{1-\gamma} - \frac{\lambda t^2}{2^s} \int_{\Omega} |\Psi_\epsilon|^2 \, dx
$$

$$
\leq \frac{t^2}{2} \left( \frac{N}{2} \frac{S_{2,s}}{S_{2,s}} + a_1 \epsilon^{N-2s} \right) + tn^\gamma a_3^{1/q} \epsilon^{(N-2s)/2} + \frac{\lambda}{2^s} - \frac{\lambda}{2^s}
$$

$$
- \frac{1}{1 - \gamma} \int_{\Omega} (t\Psi_\epsilon + w_n + 1/n)^{1-\gamma} - (w_n + 1/n)^{1-\gamma} - \frac{\lambda t^2}{2^s} \left( \frac{N}{2} \frac{S_{2,s}}{S_{2,s}} - a_2 \epsilon^N \right). \quad (3.17)
$$

Assume $\lambda \leq 1$ and denote a function $h : \mathbb{R}^+ \to \mathbb{R}$ as follows.

$$
h(t) = \frac{\lambda}{2^s} - \frac{1}{1 - \gamma} \int_{\Omega} (t\Psi_\epsilon + w_n + 1/n)^{1-\gamma} - (w_n + 1/n)^{1-\gamma}
$$

$$
\leq \frac{1}{2^s} - \frac{1}{1 - \gamma} \int_{\Omega} (t\Psi_\epsilon + w_n + 1/n)^{1-\gamma} - (w_n + 1/n)^{1-\gamma}. \quad (3.18)
$$

Clearly as $t \to \infty$, $h(t) \to -\infty$. Hence, there exists $T_n > 0$ such that for every $t \geq T_n$, $h(t) \leq 0$. Thus, for $t \geq T_n$ we get

$$
I_{\lambda,n}(t\Psi_\epsilon) \leq \frac{t^2}{2} \left( \frac{N}{2} \frac{S_{2,s}}{S_{2,s}} + a_1 \epsilon^{N-2s} \right) + tn^\gamma a_3^{1/q} \epsilon^{(N-2s)/2} - \frac{\lambda t^2}{2^s} \left( \frac{N}{2} \frac{S_{2,s}}{S_{2,s}} - a_2 \epsilon^N \right) - \frac{\lambda}{2^s}
$$

$$
= \bar{h}_\epsilon(t) - \frac{\lambda}{2^s}.
$$
A simple use of basic calculus yields that the maximum value of \( \bar{h}_\epsilon \) is attained at

\[
t_\lambda = \left( \frac{1}{\lambda} \right) \frac{N-2s}{4s} + o(\epsilon(N-2s)/2).
\]

Thus, we have

\[
I_{\lambda,n}(t\Psi_\epsilon) \leq \frac{s}{N} \frac{S_{2,s}^N}{\lambda^{N-2s}} + o(\epsilon(N-2s)/2) - \frac{\lambda}{2s} < \frac{s}{N} \frac{S_{2,s}^N}{\lambda^{N-2s}} - \frac{\lambda}{2s}.
\]

(3.19)

For any \( t < T_n \),

\[
I_{\lambda,n}(t\Psi_\epsilon) \leq \frac{t^2}{2} \int_{\mathbb{R}^N} \frac{|\Psi_\epsilon(x) - \Psi_\epsilon(y)|^2}{|x - y|^{N+2s}} \, dxdy + \int_{\Omega} (w_n + 1/n)^\gamma \left( |t\Psi_\epsilon| \right) \leq \frac{t^2}{2} \left( S_{2,s}^{N/2s} + a_1 \epsilon^{N-2s} \right) + T_n n^{a_3} \epsilon^{(N-2s)/2}
\]

\[
< \frac{T_n^2}{2} \left( S_{2,s}^{N/2s} + a_1 \epsilon^{N-2s} \right) + T_n n^{a_3} \epsilon^{(N-2s)/2}.
\]

Choose \( \lambda_n^* > 0 \) depending on \( T_n, N, 2s, S_{2,s} \) such that for \( \lambda \in (0, \lambda_n^*) \) we obtain

\[
I_{\lambda,n}(t\Psi_\epsilon) < \frac{s}{N} \frac{S_{2,s}^N}{\lambda^{N-2s}} - \frac{\lambda}{2s}.
\]

Denote \( \Lambda_n = \min\{1, (\frac{s}{N} 2s S_{2,s})^{2s/N}, \lambda_n^*\} \). Then for \( 0 < \lambda < \Lambda_n \) we have

\[
\sup\{I_{\lambda,n}(t\Psi_\epsilon) : t \geq 0\} < \frac{s}{N} \frac{S_{2,s}^N}{\lambda^{N-2s}} - \frac{\lambda}{2s}.
\]

Hence the result. \qed

**Proof of Theorem 2.12.** We at first need to produce a Palais-Smale sequence named \((v_{n,m})\) in \( H_n \) of the functional \( I_{\lambda,n} \) using the Ekeland variational principle (see Theorem 2.8). Then by Lemma 3.2 and Lemma 3.8 there exists a subsequence of \((v_{n,m})\) (still denoted as \((v_{n,m})\)) that converges strongly to \( v_n \in H_n \). This guarantees the existence of a critical point \( v_n \) of \( I_{\lambda,n} \) in \( H_n \) for \( \lambda \in (0, \Lambda_n) \).

Observe that \( H_n \subset H_0^s(\Omega) \) is a complete Hilbert manifold. Since the functional \( I_{\lambda,n} \) is \( C^1 \) and bounded from below on \( H_n \), we denote \( k_n = \inf_{v_n \in H_n} I_{\lambda,n}(v_n) \). Hence, there exists a sequence \((u_{n,m}) \subset H_n \) such that \( I_{\lambda,n}(u_{n,m}) \to k_n \) as \( m \to \infty \) and for every \( \epsilon > 0 \) there exists \( m_0 \in \mathbb{N} \) such that \( I_{\lambda,n}(u_{n,m}) < k_n + \epsilon \) for every \( m \geq m_0 \). The functional \( I_{\lambda,n} \) satisfies the hypotheses of Ekeland variational principle stated in Theorem 2.8. By choosing \( \delta = \sqrt{\epsilon} \) in Theorem 2.8 we guarantee that there exists a sequence \((v_{n,m}) \subset H_n \) such that \((I_{\lambda,n}(v_{n,m}))\) is uniformly bounded and \( I'_{\lambda,n}(v_{n,m}) \to 0 \). This implies \((v_{n,m})\) is a Palais-Smale sequence and we conclude our proof. \qed
4 Existence of solution to \((P_\lambda)\)

In this section we discuss about the boundedness of the sequence of solution \((u_n)\) to \((P_{\lambda,n})\) in a suitable fractional Sobolev space and also prove the existence of solution to \((P_\lambda)\).

**Lemma 4.1.** Let \(0 < \gamma < 1\) and \(u_n \in H\) be a solution to \((P_{\lambda,n})\) as given in Theorem 2.13. Then the sequence \((u_n)\) is bounded in \(W_{0}^{s_1,q}(\Omega)\) for every \(0 < s_1 < s\) and \(1 \leq q < \frac{N}{N-s} - 1\).

**Proof.** We follow Panda et al. [31] to prove this lemma. Let \(u_n \in H\) be a solution of \((P_{\lambda,n})\). Then for any \(k \geq 1\), consider \(\phi = T_k(u_n)\) as a test function in the weak formulation (2.7) of \((P_{\lambda,n})\) and we get

\[
\int_{\mathbb{R}^N} |(-\Delta)^{s/2} T_k(u_n)|^2 \leq \int_{\mathbb{R}^N} (-\Delta)^{s/2} u_n \cdot (-\Delta)^{s/2} T_k(u_n)
= \int_{\Omega} \frac{1}{(u_n + \frac{1}{n})^\gamma} T_k(u_n) + \int_{\Omega} \lambda u_n^{2^*_\gamma} - 1 T_k(u_n) + \int_{\Omega} \mu_n T_k(u_n).
\] (4.20)

Clearly, \(\frac{T_k(u_n)}{(u_n + \frac{1}{n})^\gamma} \leq \frac{u_n}{(u_n + \frac{1}{n})^\gamma} \leq u_n^{1-\gamma}\). Since \(\|u_n\|_{L^{2^*_\gamma}(\Omega)} = 1\) and the sequence \((\mu_n)\) is \(L^1\) bounded we have

\[
\int_{\mathbb{R}^N} |(-\Delta)^{s/2} T_k(u_n)|^2 \leq \int_{\Omega} u_n^{1-\gamma} + k\|\mu_n\|_{L^1(\Omega)} + \lambda k\|u_n\|_{L^{2^*_\gamma - 1}(\Omega)}^{2^*_\gamma - 1} \\
\leq C_1\|u_n\|_{L^{2^*_\gamma}(\Omega)}^{1-\gamma} + C_2 k + \lambda C_3\|u_n\|_{L^{2^*_\gamma}(\Omega)}^{2^*_\gamma - 1} \\
\leq C k.
\] (4.21)

Thus, \((T_k(u_n))\) is bounded in \(H_{0}^{s}(\Omega)\). Consider the sets \(I = \{x \in \Omega : |(-\Delta)^{s/2} u_n| \geq t\}\), \(I_1 = \{x \in \Omega : |(-\Delta)^{s/2} u_n| \geq t, u_n \leq k\}\) and \(I_2 = \{x \in \Omega : u_n > k\}\). Then, \(I \subset I_1 \cup I_2\), which implies \(m(I) \leq m(I_1) + m(I_2)\), where \(m\) is the Lebesgue measure. Using the Sobolev inequality stated in Theorem 2.11 we establish

\[
\left(\int_{\mathbb{R}^N} |T_k(u_n)|^{2^*_\gamma} \right)^{\frac{2}{2^*_\gamma}} \leq C' \int_{\mathbb{R}^N} |(-\Delta)^{s/2} T_k(u_n)|^2 \leq C k.
\] (4.22)

Then on \(I_2\), the equation (4.22) becomes

\[
k^2 m(I_2)^{\frac{2}{2^*_\gamma}} \leq C k
\]

\[
m(I_2) \leq \frac{C}{k^{N - 2s}}, \forall k \geq 1.
\] (4.23)

This proves \((u_n)\) is bounded in \(M_{-2s}^N(\Omega)\). Similarly on \(I_1\), the equation (4.21) becomes

\[
l^2 m(I_1) \leq C k
\]

\[
m(I_1) \leq \frac{C k}{l^2}, \forall k > 1.
\] (4.24)
On combining (4.23) and (4.24) we have

\[ m(I) \leq \frac{Ck}{t^2} + \frac{C}{k^{N-2s}}, \forall k > 1. \]

Choose \( k = t^{\frac{N-2s}{N-s}} \) and we obtain

\[ m(I) \leq \frac{C}{t^{N-s}}, \forall t \geq 1. \]

Thus, the sequence \((-\Delta)^s u_n\) is bounded in \( M^{\frac{N}{N-s}}(\Omega) \) and hence using the continuous embedding (2.4), \((u_n)\) and \((-\Delta)^s u_n\) are bounded in \( L^q(\Omega) \) for every \( q < \frac{N}{N-s} \). From Theorem 2.3 we conclude that \((u_n)\) is bounded in \( W^{s_1,q}_0(\Omega) \), for every \( s_1 < s \) and \( q < \frac{N}{N-s} \). \( \square \)

We now pass the limit \( n \to \infty \) in the weak formulation (2.7) and prove the existence of a SOLA.

**Proof of Theorem 1.1** Let \( \mu \in M(\Omega), 0 < \gamma < 1 \) and \( u_n \in H \) be a solution of \((P_{\lambda,n})\) for \( \lambda \in (0, \Lambda_n) \). Define \( \Lambda = \inf \Lambda_n \). Then, for \( \lambda \in (0, \Lambda) \), \((P_{\lambda,n})\) has at least one solution \( u_n \in H \).

According to Lemma 4.1, \((u_n)\) is bounded in \( W^{s_1,q}_0(\Omega) \), for every \( 0 < s_1 < s \) and \( q < \frac{N}{N-s} \). Hence, there exists \( u \in W^{s_1,q}_0(\Omega) \) such that \( u_n \to u \) weakly in \( W^{s_1,q}_0(\Omega) \). Thus, \( u_n \to u \) a.e. in \( \mathbb{R}^N \) and \( u \equiv 0 \) in \( \mathbb{R}^N \setminus \Omega \).

Denote

\[ \Phi(x,y) = \phi(x) - \phi(y), \bar{U}_n(x,y) = u_n(x) - u_n(y), \]

\[ \bar{U}(x,y) = u(x) - u(y) \]

and \( d\nu = \frac{dxdy}{|x-y|^{N+2s}} \).

For every \( \phi \in \mathcal{C}_c^\infty(\Omega) \), from the weak formulation of \((P_{\lambda,n})\) we have

\[ \int_{\mathbb{R}^{2N}} \bar{U}_n(x,y)\Phi(x,y)d\nu = \int_{\Omega} \frac{1}{(u_n + \frac{1}{n})^\gamma} \phi + \int_{\Omega} \lambda u_n^{2s-1} \phi + \int_{\Omega} \mu_n \phi. \]

We can rewrite the above equation as

\[ \int_{\mathbb{R}^{2N}} \bar{U}(x,y)\Phi(x,y)d\nu + \int_{\mathbb{R}^{2N}} (\bar{U}_n(x,y) - \bar{U}(x,y))\Phi(x,y)d\nu = \int_{\Omega} \frac{1}{(u_n + \frac{1}{n})^\gamma} \phi + \int_{\Omega} \lambda u_n^{2s-1} \phi + \int_{\Omega} \mu_n \phi. \]

(4.25)

Clearly,

\[ \lim_{n \to \infty} \int_{\Omega} \mu_n \phi = \int_{\Omega} \phi d\mu, \]

\[ \lim_{n \to \infty} \lambda \int_{\Omega} u_n^{2s-1} \phi = \lambda \int_{\Omega} u^{2s-1} \phi. \]

On using the Dominated Convergence Theorem we get

\[ \lim_{n \to \infty} \int_{\Omega} \frac{1}{(u_n + \frac{1}{n})^\gamma} \phi = \int_{\Omega} \frac{1}{u^\gamma} \phi. \]
Now the integral can be expressed as
\[
\int_{\mathbb{R}^{2N}} (\bar{U}_n(x, y) - \bar{U}(x, y)) \Phi(x, y) d\nu = \int_{\Omega \times \Omega} (\bar{U}_n(x, y) - \bar{U}(x, y)) \Phi(x, y) d\nu \\
+ \int_{\Omega \times (\mathbb{R}^N \setminus \Omega)} (\bar{U}_n(x, y) - \bar{U}(x, y)) \Phi(x, y) d\nu \\
+ \int_{(\mathbb{R}^N \setminus \Omega) \times \Omega} (\bar{U}_n(x, y) - \bar{U}(x, y)) \Phi(x, y) d\nu \\
= J_{1,n} + J_{2,n} + J_{3,n}.
\]
Observe $\bar{U}_n \to \bar{U}$ a.e. in $\mathbb{R}^N$. Since $\Omega$ is bounded, using Lemma 3.1 and Vitali's lemma we have $\bar{U}_n \to \bar{U}$ strongly in $L^1(\Omega \times \Omega, d\nu)$. Hence, $J_{1,n} \to 0$ as $n \to \infty$.

Let $(x, y) \in \Omega \times (\mathbb{R}^N \setminus \Omega)$, then
\[
\sup_{(x, y) \in \Omega \times (\mathbb{R}^N \setminus \Omega)} \frac{1}{|x - y|^{N+2s}} \leq C < \infty.
\]
Hence, $J_{2,n} \to 0$ and similarly $J_{3,n} \to 0$ as $n \to \infty$. Thus on passing limit $n \to \infty$ in (4.25), we obtain
\[
\int_{\mathbb{R}^{2N}} \bar{U}(x, y) \Phi(x, y) d\nu = \int_{\Omega} \frac{1}{u^\gamma} \phi + \int_{\Omega} \lambda u^{2s-1} \phi + \int_{\Omega} \phi d\mu.
\]
(4.26)
Thus, $u$ is a weak solution to $(P_\lambda)$.

**Appendix**

We now discuss about the multiplicity of solution to the problem $(P_\lambda)$ for $0 < \gamma < 1$. Let $I_\lambda$ be the corresponding energy functional given by
\[
I_\lambda(u) = \frac{1}{2} \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dxdy - \frac{1}{1-\gamma} \int_{\Omega} u^{1-\gamma} - \frac{\lambda}{2s} \int_{\Omega} u^{2s} - \int_{\Omega} u d\mu.
\]
For $0 < \gamma < 1$, $I_\lambda$ is a $C^1$ functional and $I_\lambda(0) = 0$. From Theorem 1.1, $u \in W_0^{s_1,q}(\Omega)$ is a weak solution of $(P_\lambda)$ for every $s_1 < s$, $q < \frac{N}{N-s}$. Then $u$ is also a Nehari solution of $(P_\lambda)$, i.e. $u \in N_\lambda = \{u \in W_0^{s_1,q}(\Omega) : \langle I_\lambda(u), u \rangle = 0\}$. Here
\[
\langle I_\lambda(u), u \rangle = \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dxdy - \int_{\Omega} u^{1-\gamma} - \frac{\lambda}{2s} \int_{\Omega} u^{2s} - \int_{\Omega} u d\mu.
\]
Consider the fibre map $\psi : (0, \infty) \to \mathbb{R}$ defined by
\[
\psi(t) = \frac{t^2}{2} \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dxdy - \frac{t^{1-\gamma}}{1-\gamma} \int_{\Omega} u^{1-\gamma} - \frac{\lambda t^{2s}}{2s} \int_{\Omega} u^{2s} - t \int_{\Omega} u d\mu.
\]
Then
\[
\psi'(t) = At - Bt^{-\gamma} - \lambda C t^{2s-1} - D
\]
and

$$\psi''(t) = A + \gamma B t^{-\gamma - 1} - (2^* - 1) \lambda C t^{2^* - 2}$$

where $A = \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2}} \, dx \, dy$, $B = \int_{\Omega} u^{1-\gamma}$, $C = \int_{\Omega} u^{2^*}$ and $D = \int_{\Omega} u \, d\mu$. Since $u \in N_\lambda$, $\psi'(1) = A - B - \lambda C - D = 0$. Clearly $\psi'(t) \to -\infty$ as $t \to 0$ and $t \to \infty$.

Case 1: If $\Lambda > \lambda > \frac{A + \gamma B (2^* - 1)}{(2^*_2 - 1)C}$, then $\psi''(1) < 0$. Hence, there exists at least one Nehari solution to $(P_\lambda)$.

Case 2: If $\lambda < \min\{\frac{A + \gamma B}{(2^*_2 - 1)C}, \Lambda\}$, then $\psi''(1) > 0$ and we guarantee the existence of at least three nontrivial Nehari solution to $(P_\lambda)$.

Case 3: If $\lambda = \frac{A + \gamma B}{(2^*_2 - 1)C}$, then $\psi''(1) = 0$. Thus, we obtain a saddle point and hence there exists at least one Nehari solution to $(P_\lambda)$.

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