Uniformly Weighted Star-Factors of Graphs *

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Abstract

A star-factor of a graph \(G\) is a spanning subgraph of \(G\) such that each component of which is a star. An edge-weighting of \(G\) is a function \(w : E(G) \rightarrow \mathbb{N}^+\), where \(\mathbb{N}^+\) is the set of positive integers. Let \(\Omega\) be the family of all graphs \(G\) such that every star-factor of \(G\) has the same weights under a fixed edge-weighting \(w\). In this paper, we present a simple structural characterization of the graphs in \(\Omega\) that have girth at least five.

Key words: star-factor, girth, edge-weighting

1 Introduction

Throughout this paper, all graphs considered are simple. We refer the reader to \([2]\) for standard graph theoretic terms not defined in this paper.

Let \(G = (V, E)\) be a graph with vertex set \(V(G)\) and edge set \(E(G)\). If \(G\) is not a forest, then length of the shortest cycle in \(G\) is called the girth of \(G\), denoted by \(g(G)\) and a forest is considered to have an infinite girth. If \(S \subset V(G)\), then \(G - S = G[V - S]\) is the subgraph of \(G\) obtained by deleting the vertices in \(S\) and all the edges incident with them. Similarly, if \(E' \subset E(G)\), then \(G - E' = (V(G), E(G) - E')\). We denote the degree of a vertex \(x\) in \(G\) by \(d_G(x)\), and the set of vertices adjacent to \(x\) in \(G\) by \(N_G(x)\). We also denote by \(\delta(G)\) the minimum degree of vertices in \(G\). A cycle (or path) with \(n\) vertices is denoted by \(C_n\) (or \(P_n\)). If vertices \(u\) and \(v\) are connected in \(G\), the distance between \(u\) and \(v\) in \(G\), denoted by \(d_G(u, v)\), is the length of a shortest \((u, v)\)-path in \(G\). The diameter of \(G\) is the maximum distance over all pairs of vertices in \(G\). A leaf is a vertex of degree one and a stem is a vertex which has at least one leaf as its neighbor. A star is a tree isomorphic to \(K_{1,n}\) for some \(n \geq 1\), and the vertex of degree \(n\) is called the center of the star. A star-factor of a graph \(G\) is a spanning subgraph of \(G\) such that each component of which is a star. Clearly a graph with isolated vertices has no star-factors. On the other hand, it is not hard to see that every graph without isolated vertices admits a star-factor. If one limits the size of the

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star used, the existence of such a star-factor is non-trivial. In [1], Amahashi and Kano presented a criterion for the existence of a star-factor, i.e., \{K_{1,1}, \cdots, K_{1,n}\}-factor. Yu [5] obtained an upper bound on the maximum number of edges in a graph with a unique star-factor.

An edge-weighting of a graph \( G \) is a function \( w : E(G) \rightarrow \mathbb{N}^+ \), where \( \mathbb{N}^+ \) is the set of positive integers. For a subgraph \( H \), the weight of \( H \) under \( w \) is the sum of all the weight values for edges belonging to \( H \), i.e., \( w(H) = \sum_{e \in E(H)} w(e) \). Motivated by the minimum cost spanning trees and the optimal assignment problems, Hartnell and Rall posed an interesting general question: for a given graph, does there exist an edge-weighting function \( w \) such that a certain type of spanning subgraphs always have the same weights? In particular, they investigated the following narrow version of the problem in which the spanning subgraph is a star-factor.

**Star-Weighting Problem** (Hartnell and Rall [3]): For a given graph \( G = (V, E) \), does there exist an edge-weighting \( w \) of \( G \) such that every star-factor of \( G \) has the same weights under \( w \)?

To start the investigation, one may consider a special case that \( w \) is a constant function, i.e., all edges in \( G \) are assigned with the same weights. In this case, two star-factors of \( G \) have the same weights if and only if they both have the same number of edges. For simplicity, we assume that all edges are assigned with weight one.

Let \( \mathcal{W} \) be the family of all graphs \( G \) such that if \( S_1 \) and \( S_2 \) are any two star-factors of \( G \), then \( S_1 \) and \( S_2 \) have the same number of edges. Clearly, \( S_1 \) and \( S_2 \) have the same number of edges is equivalent to that they have the same number of components. Hartnell and Rall [3] classified the family \( \mathcal{W} \) when graphs in \( \mathcal{W} \) have girth at least five. In [4], the authors characterized the family \( \mathcal{W} \) when all its members have girth three and minimum degree at least two.

We denote by \( \Omega \) the family of all graphs \( G \) such that every star-factor of \( G \) has the same weights under some fixed edge-weighting \( w \). In the definition of edge-weighting, we assume that \( w(e) \neq 0 \) for every \( e \in E(G) \). We also note that if \( G \in \mathcal{W} \) then \( G \in \Omega \), but the converse is not always true. In this paper, we give a structural characterization of the graphs in \( \Omega \) that have girth at least five.

## 2 Main Result

We start with a few easy observations and lemmas.

Note that if \( H \) is a spanning subgraph of \( G \), then any star-factor of \( H \) is also a star-factor of \( G \). The following lemma will be used frequently in reducing the problem of determining membership in \( \Omega \) to its spanning subgraphs.

**Lemma 1.** Let \( F \) be a subset of \( E(G) \) such that \( G - F \) has no isolated vertices. If \( G - F \notin \Omega \), then \( G \notin \Omega \) as well.

The above lemma implies that if \( G \) is in \( \Omega \), then so is \( G - F \). The basic idea to show that a graph does not belong to \( \Omega \) is to decompose \( G \) into several components without isolated vertices and then simply find one of them not belonging to \( \Omega \).

**Observation 1.** Let \( P_6 = v_1v_2v_3v_4v_5v_6 \) be a component of \( G \) and \( G \in \Omega \), then \( w(v_3v_4) = w(v_2v_3) + w(v_4v_5) \).
Observation 2. Let $P_5 = v_1 v_2 v_3 v_4 v_5$ be a component of $G$ and $G \in \Omega$, then $w(v_2 v_3) = w(v_3 v_4)$.

Use these two observations and Lemma 1 we get the following observation.

Observation 3. Let $C$ be a cycle. If $|V(C)| = 6$ or $|V(C)| \geq 8$, then $C \notin \Omega$.

We investigate the graphs, in $\Omega$, with girth at least five but without leaves first.

Theorem 1. If $\delta(G) \geq 2$ and either $g(G) = 6$ or $g(G) \geq 8$, then $G \notin \Omega$.

Proof. Let $C$ be a cycle of order six in $G$ and $F = \{u_1 u_2 \mid u_1 \in V(C), u_2 \in V(G) - V(C)\}$. Since the girth of $G$ is six, there is no isolated vertices created in $G - F$. So $C$, as a component in $G - F$, is a cycle of order six. Hence $G \notin \Omega$ by Lemma 1 and Observation 3.

For the case of $\delta(G) \geq 2$ and $g(G) \geq 8$, the argument is very similar. □

Theorem 2. If $\delta(G) \geq 2$ and $g(G) = 7$, then $G \in \Omega$ if and only if $G$ is a 7-cycle.

Proof. Let $G$ be a 7-cycle, let $w(e) = k$ for each $e \in E(G)$ and $k \in N^+$. Then it is easy to check that $G \in \Omega$.

On the other hand, assume $G \in \Omega$ but $G$ is not a 7-cycle. Let $C = v_1 v_2 v_3 v_4 v_5 v_6 v_7$ be a cycle in $G$. Without loss of generality, assume that $v_7$ has a neighbor $u$ not on $C$. Let $F_1 = \{u_1 u_2 \mid u_1 \in V(C), u_2 \in V(G) - V(C)\}$. Since the girth of $G$ is seven, there are no isolated vertices created in $G - F_1$. We see that $C$, as a component in $G - F_1$, is a cycle of length seven. Since $G \in \Omega$, then all edges on the cycle $C$ must have the same weights. Let $F_2 = \{u_1 u_2 \mid u_1 \in \{v_1, v_2, v_3, v_4, v_5, v_6\}, u_2 \in V(G) - \{v_1, v_2, v_3, v_4, v_5, v_6\}\}$, then there are no isolated vertices created in $G - F_2$ again. But $P_6 = v_1 v_2 v_3 v_4 v_5 v_6$, as a component in $G - F_2$, is a path of order six. Since $G \in \Omega$, $w(v_3 v_4) = w(v_2 v_3) + w(v_4 v_5)$ by Observation 1. Hence all the weights of edges on the cycle $C$ must be 0, a contradiction. □

Lemma 2. Let $G$ be a graph with an induced cycle of order five such that four of the vertices are of degree two and the fifth is a stem. Then $G \notin \Omega$.

Proof. Suppose $G$ belongs to $\Omega$, and $w$ is an edge-weighting function such that all star-factors of $G$ have the same weights under $w$. Let $C = v_1 v_2 v_3 v_4 v_5$ be a 5-cycle in $G$ with a stem $v_5$. Let $X$ be the set of leaves adjacent to $v_5$ and $F_1 = \{u_1 v_5 \mid u_1 \in V(G) - X - v_5\}$. A component $H_1$ of $G - F_1$ is isomorphic to the graph shown in Figure 1(a). Since $G - F_1$ has no isolated vertices, so $H_1 \in \Omega$ and $w(v_3 v_4) = w(v_2 v_3) + w(v_4 v_5)$ by Observation 1. On the other hand, let $F_2 = \{u_1 v_5 \mid u_1 \in V(G) - X - \{v_1, v_4\}\}$, then a component $H_2$ of $G - F_2 - v_1 v_2$ is isomorphic to the graph shown in Figure 1(b). So $H_2 \in \Omega$ and $w(v_3 v_4) = w(v_4 v_5)$ by Observation 2. From the above two relations, we have $w(v_2 v_3) = 0$, a contradiction. □

Lemma 3. Let $G$ be a graph in $\Omega$ with an induced 5-cycle. If exactly one of the vertices on this 5-cycle has degree at least three, then all of its neighbors not belonging to this 5-cycle must be stems.

Proof. Let $v$ be a vertex on the 5-cycle of degree at least three. Assume $v$ has a neighbor $x$ not on the 5-cycle and $x$ is not a stem. By Lemma 2, $x$ is not a leaf. Let $F$ be the set of edges incident with $x$ except $vx$. Then the graph $G - F$ has no isolated vertices, and the vertex $v$ is a stem belonging to an induced 5-cycle that satisfies the hypothesis of Lemma 2. Thus $G \notin \Omega$, a contradiction. □
Theorem 3. If $\delta(G) \geq 2$ and $g(G) = 5$, then $G \in \Omega$ if and only if $G$ is a 5-cycle.

Proof. If $G$ is a 5-cycle, clearly $G \in \Omega$ under a constant weight function.

Next consider a graph $G \in \Omega$ with $\delta(G) \geq 2$ and $g(G) = 5$ but $G \not\cong C_5$. Let $C = v_1v_2v_3v_4v_5$ be a cycle in $G$. Assume, without loss of generality, that $v_5$ has a neighbor $u$ not on $C$. Let $F = \{u_1u_2 \mid u_1 \in \{v_1, v_2, v_3, v_4, v_5\}, u_2 \in V(G) - V(C) - u\}$. If we delete all edges in $F$ from $G$, then no isolated vertices created in $G - F$ since $g(G) = 5$, so $G - F \in \Omega$ and $u$ is a stem in $G - F$ by Lemma 3.

Moreover, $u$ has at most two leaves as its neighbors in $G - F$, and they are adjacent to $v_2$ or (and) $v_3$ in $G$. Without loss of generality, assume that all neighbors of $u$ except $v_5$ in $G - F$ are leaves. Let $H$ be the component containing the cycle $C$ in $G - F$, and $H'$ be the induced subgraph of $G$ with $V(H)$. Then $G - V(H')$ and $H'$ have no isolated vertices, and so $H' \in \Omega$ by Lemma 1. If there is exactly one leaf as a neighbor of $u$ in $H$, then $H'$ is shown in Figure 2(a). Otherwise, $H'$ is isomorphic to the graph shown in Figure 2(b). It is not hard to check that both graphs are not in $\Omega$, a contradiction to $G \in \Omega$. □

From the four theorems above, we obtain the following corollary.

Corollary 1. If $G$ is a graph with $\delta(G) \geq 2$ and $g(G) \geq 5$, then $G \in \Omega$ if and only if $G$ is a 5-cycle or 7-cycle. Moreover, all edges of $G$ must have the same weights.

Next, we attempt to determine all members in $\Omega$ which have girth at least five and with leaves. To derive our main theorem, we need the following lemmas.

Lemma 4. Let $G$ be a graph of girth five and contain a 5-cycle $C$ in which no vertex is a stem and there exist two adjacent vertices of degrees at least three. Then $G \not\in \Omega$. 
Proof. Assume \( G \in \Omega \) and let \( C = v_1v_2v_3v_4v_5 \) be the 5-cycle in \( G \) such that \( v_1 \) and \( v_2 \) both have degree at least three. Let \( F_1 \) be the set of all edges not on \( C \) but incident with one of \( v_3, v_4 \) and \( v_5 \). Since \( g(G) = 5 \), then no pair of vertices on \( C \) have a common neighbor not on \( C \), and thus \( G' = G - F_1 \) has no isolated vertices. Furthermore, neither \( v_1 \) nor \( v_2 \) is a stem in \( G' \). Let \( G'' \) be the graph obtained by deleting all edges incident with \( v_1 \) but not on \( C \) from \( G' \), then none of stems created in \( G'' \) is a neighbor of \( v_2 \). However, by Lemma 1, \( G'' \) belongs to \( \Omega \) and so by Lemma 3 all neighbors of \( v_2 \) not on \( C \), in \( G'' \), must be stems. Thus all neighbors of \( v_2 \) not on \( C \) in \( G' \) must be stems. A similar argument yields that all neighbors of \( v_1 \) on \( C \) in \( G' \) must be stems. Let \( w_1 \) and \( w_2 \) be stems adjacent to \( v_1 \) and \( v_2 \), respectively, in \( G' \). Then there exists at most one common neighbor of degree two between \( w_1 \) and \( w_2 \) since \( g(G) = 5 \). Let \( X \) be the set of leaves adjacent to \( w_1 \) or \( w_2 \) in \( G' \) and \( F_2 = \{u_1u_2 : u_1 \in \{v_1, v_2, w_1, w_2\}, u_2 \in V(G') - \{v_1, v_2, w_1, w_2, v_3, v_5\} - X\} \). Then \( G' - F_2 \) (or \( G' - F_2 \cup uw_1 \), if there exists a common neighbor \( u \) of degree two between \( w_1 \) and \( w_2 \)) has no isolated vertices and a component \( H \) in \( G' - F_2 \) (or \( G' - F_2 \cup uw_1 \)) is isomorphic to the graph shown in Figure 3(a). By Lemma 4, \( H \in \Omega \). It is easy to show that \( H \notin \Omega \) by Observation 1 and Observation 2, a contradiction.

Lemma 5. Let \( G \) be a graph of girth five and contain a 5-cycle \( C = v_1v_2v_3v_4v_5 \) in which no vertex is a stem and two nonadjacent vertices \( v_1 \) and \( v_3 \) of \( C \) have degree at least three. If \( G \in \Omega \), then all neighbors of \( v_1 \) and \( v_3 \) not belonging to \( C \) must be stems.

Proof. Assume that \( v_1 \) has a neighbor \( x \) not on \( C \) but \( x \) is not a stem. By Lemma 1, \( v_2, v_4 \) and \( v_5 \) must all have degree two in \( G \). Let \( F \) be the set of all edges other than \( xv_1 \), that are incident with \( x \), and let \( G' = G - F \). Since \( x \) is not a stem in \( G \), \( G' \in \Omega \). If \( v_3 \) is not a stem in \( G' \), let \( F' \) be the set of edges, not in \( C \), but incident with \( v_3 \), then \( G' - F' \in \Omega \) but it contains a 5-cycle satisfying the conditions of Lemma 2. Therefore \( v_3 \) is a stem in \( G' \). Moreover, we note that \( v_3 \) has exactly one leaf, say \( y \), as its neighbor in \( G' \) since \( g(G) = 5 \). Thus, we arrive at an induced subgraph \( H \) shown in Figure 3(b). As \( G - H \) has no isolated vertices, so \( H \in \Omega \) by Lemma 1. But by Theorem 3, \( H \notin \Omega \), a contradiction.

Lemma 6. Let \( G \) be a graph with an induced cycle of order six such that five of the vertices are of degree two and the sixth is a stem. Then \( G \notin \Omega \).

Proof. Suppose \( G \in \Omega \) under an edge-weighting function \( w \). Let \( C = v_1v_2v_3v_4v_5v_6 \) be a 6-cycle in \( G \) such that all vertices on \( C \) are of degree two except a stem \( v_1 \).
Let $X$ be the set of leaves adjacent to $v_1$ and $F_1 = \{u_1v_1 \mid u_1 \in V(G) - X - \{v_2, v_6\}\}$. A component $H_1$ of $G - F_1 - v_5v_6$ is isomorphic to the graph shown in Figure 4(a). Since $G - F_1 - v_5v_6$ has no isolated vertices, so $H_1 \in \Omega$ and $w(v_2v_3) = w(v_1v_2) + w(v_3v_4)$ by Observation 1. On the other hand, let $F_2 = \{u_1v_1 \mid u_1 \in V(G) - X - v_2\}$. A component $H_2$ of $G - F_2 - v_4v_5$ is isomorphic to the graph shown in Figure 4(b). So $H_2 \in \Omega$ and $w(v_1v_2) = w(v_2v_3)$ by Observation 2. From the above two relations, we have $w(v_3v_4) = 0$, a contradiction.

\[ \text{Figure 4} \]

We now characterize the graphs in $\Omega$ of girth at least five. The results above reduce the problem to considering those graphs that have at least one leaf. The following lemma is true regardless of girth.

**Lemma 7.** If $G$ is a graph in which every vertex is either a leaf or a stem, then $G$ belongs to $\Omega$.

**Proof.** Let $L$ be the set of leaves of $G$ and $W$ be the set of stems. Any star-factor of $G$ must contain all edges joining a vertex in $L$ and a vertex in $W$ but cannot contain any edge incident with two vertices of $W$. Hence $G$ has a unique star-factor and $G \in \Omega$.

Now we prove our main result.

**Theorem 4.** Let $G$ be a connected graph of girth at least five. Then $G \in \Omega$ if and only if $G$ is

1) a $C_5$, or
2) a $C_7$, or
3) $G$ has leaves and each vertex in $G$ is either a leaf or a stem, or
4) $G$ has leaves but contains at least one vertex which is neither a leaf nor a stem, then each component of the graph obtained by removing the leaves and stems from $G$ is one of the following:
   a) a 5-cycle with at most two vertices of degree three or more in $G$. Furthermore, if there are two such vertices, then they are non-adjacent on the 5-cycle;
   b) a star $K_{1,m}$ $(m \geq 1)$. Moreover, the center of $K_{1,m}$ has degree $m$ in $G$ for $m \geq 2$;
   c) an isolated vertex.

**Proof.** If $g(G) \geq 5$ and $\delta(G) \geq 2$, then the theorem follows from Corollary 1.

If $\delta(G) = 1$ and each vertex in $G$ is either a leaf or a stem, then $G \in \Omega$ by Lemma 7.

Next, suppose $\delta(G) = 1$ and $G \in \Omega$, but $G$ contains at least one vertex which is neither a leaf nor a stem. Let $L$ be the set of leaves and $S = N(L)$ the set of stems. If $G - (L \cup S)$ has a component which is a 5-cycle $C$, then, by Lemmas 4 and 5 there are at most two vertices of $C$ with degree greater than two in $G$ (and if there are two such vertices, they are non-adjacent on $C$) and all of their neighbors not on $C$ must be stems.

Hence we now consider those components of $G - (L \cup S)$ that have no 5-cycles. Let $H$ be such a component, then $g(H) \geq 6$. We shall show that the diameter of $H$ is at most two.
Suppose the diameter of $H$ is at least three, then there exists a path $P = abcd$ in $H$ such that $a$ is adjacent to a stem $s$ of $G$.

![Diagram](attachment:figure5.png)

Figure 5

**Claim 1.** There exist no common neighbors of degree two between $a$ and $d$, or $s$ and $c$, or $s$ and $d$ in $G$.

Since $g(H) \geq 6$, there is no common neighborhood of degree two between $a$ and $d$.

Let $X$ be the set of leaves adjacent to vertices $s$ in $G$. Suppose there is a common neighbor $u$ of degree two between $s$ and $c$. Let $F_1 = \{u_1u_2 \mid u_1 \in \{a,b,c,u,s\}, u_2 \in \{V(G) - \{a,b,c,u,s\} - X\}\}$. Then the graph $G_1 = G - F_1$ has no isolated vertices and has a component satisfying the hypothesis of Lemma 6, a contradiction to $G \not\in \Omega$, a contradiction.

Suppose $v$ is a common neighbor of degree two between $s$ and $d$. Let $F_2 = \{u_1u_2 \mid u_1 \in \{a,b,c,d,s,v\}, u_2 \in \{V(G) - \{a,b,c,d,s,v\} - X\}\}$. Then $G_2 = G - F_2$ has no isolated vertices but has a component satisfying the hypothesis of Lemma 6 a contradiction to $G \in \Omega$ by Lemma 1.

Claim 1 yields that there are no common neighbors of degree two between any two vertices of $\{s,a,b,c,d\}$. Let $F_3 = \{u_1u_2 \mid u_1 \in \{a,b,c,s\}, u_2 \in \{V(G) - \{a,b,c,s\} - X\}\}$. Since $g(G) \geq 5$, then $G_3 = G - F_3$ has no isolated vertices and has a component $H'$ isomorphic to the graph shown in Figure 5(a). By Observation 2 we have

\[ w(ab) = w(as), \] (1)

On the other hand, let $F_4 = \{u_1u_2 \mid u_1 \in \{a,b,c,d,s\}, u_2 \in \{V(G) - \{a,b,c,d,s\} - X\}\}$, then $G_4 = G - F_4$ has no isolated vertices and has a component $H''$ isomorphic to the graph shown in Figure 5(b). Then $H'' \in \Omega$ and

\[ w(ab) = w(as) + w(bc). \] (2)

Equations (1) and (2) imply that $w(bc) = 0$, a contradiction.

Hence the diameter of $H$ is at most two, i.e., $H$ is either an isolated vertex or isomorphic to a star, say $K_{1,m}$. For $m \geq 2$, let the vertices of $H$ be $c,b_1,b_2,\ldots,b_m$ where $c$ has degree $m$ in $H$. For each $1 \leq i \leq m$, let $s_i$ be a stem of $G$ adjacent to $b_i$.

**Claim 2.** If $m \geq 2$, then $c$ does not have a neighbor, in $G$, which is a stem.

Otherwise, let $s$ be one of such neighbors and let $L_i$ and $L_s$ be the sets of leaves adjacent to $s_i$ and $s$ in $G$, respectively. If there exists a vertex $u$ adjacent only to vertices in $\{s_1,s_2,\ldots,s_m,b_1,b_2,\ldots,b_m\}$ and to at least one vertex of $\{s_1,s_2,\ldots,s_m\}$, then we can delete all edges which are adjacent to $u$ except one $s_ku$ (for some $1 \leq k \leq m$). Thus we obtain a spanning subgraph of $G$ without isolated
vertices, and \( u \) is a leaf adjacent to \( s_k \). Hence we may assume that there are no vertices only adjacent to vertices \( \{s_1, s_2, \ldots, s_m, b_1, b_2, \ldots, b_m\} \). For the same reason, we may assume no vertices only adjacent to vertices \( s \) and \( s_1, 1 \leq i \leq m \). Let \( F_5 = \{u_1u_2 \mid u_1 \in \{c, b_1, b_2, \ldots, b_m, s_1, s_2, \ldots, s_m, s\}, u_2 \in \{V(G) - \{c, b_1, b_2, \ldots, b_m, s_1, s_2, \ldots, s_m, s\} - L_1 - L_2 - \cdots - L_m - L_s\}\). Then \( G_5 = G - F_5 \) has no isolated vertices, since \( g(G) \geq 5 \), but has a component \( H'' \) isomorphic to the graph shown in Figure 6. Let the total weights of all edges incident with the leaves in \( L_1 \cup L_2 \cup \cdots \cup L_m \cup L_s \) be \( w' \). Since \( H'' \in \Omega \), then by Observation 1 we have

\[
w(cb_i) = w(b_is_i) + w(cs), \quad 1 \leq i \leq m. \tag{3}\]

Since \( G \in \Omega \), we also have

\[
w' + w(cb_1) + w(cb_2) + \cdots + w(cb_m) = w' + w(cs) + w(b_1s_1) + w(b_2s_2) + \cdots + w(b_ms_m). \tag{4}\]

Equations (3) and (4) imply \( m = 1 \), a contradiction to \( m \geq 2 \).

From Claim 2, we conclude that the center \( c \) of the star \( H = K_{1,m} \) \((m \geq 2)\) has degree \( m \) in \( G \).

![Figure 6](image-url)

Therefore every component of \( G - (L \cup S) \) is one of 4a), 4b) or 4c).

Conversely, assume \( G \) has the specified structure. In the following, we present an edge-weighting function such that every star-factor of \( G \) has the same weights.

**Case 1.** No component of \( G - (L \cup S) \) is \( K_{1,1} \).

In this case, all edges of \( G \) are assigned the same weight. We only need to show that all star-factors of \( G \) have the same number of edges. Let \( T \) be any star-factor of \( G \). Then \( T \) contains exactly one edge incident to one leaf of \( G \). Let \( H \) be a component of \( G - (L \cup S) \). If \( H \) is a 5-cycle, then \( T \) either contains three edges of \( H \) or two edges of \( H \) and exactly one edge of \( T \) joining \( H \) to a stem of \( G \). If \( H = K_{1,m} \) \((m \geq 2)\), then \( T \) contains precisely \( m \) edges incident with a vertex of \( H \). In particular, for each leaf \( x \) of \( H \) either the edge joining \( x \) to the center of \( H \) or exactly one edge joining \( x \) to a stem of \( G \) must be in \( T \). Also note that at least one edge of \( H \) must be in \( T \). Finally, if \( H \) is an isolated vertex \( u \), then \( T \) contains exactly one edge joining \( u \) to a stem of \( G \).

**Case 2.** \( K_{1,1} \) appears as a components of \( G - (L \cup S) \).

For each \( K_{1,1} = uv \) of \( G - (L \cup S) \), assign edge-weights for the edges adjacent to \( N_G(u) \cup N_G(v) \) as follows:
where \( a, b > 0 \). All other edges are assigned the same weight.

Let \( T \) be any star-factor of \( G \). Then \( T \) contains exactly one edge incident to one leaf of \( G \). For each component which is a 5-cycle or an isolated vertex, it can be dealt with as in Case 1. For \( H = uv \), then \( T \) contains an edge \( uv \) or an edge joining a stem \( s_1 \) to \( u \) and an edge joining another stem \( s_2 \) to \( v \). Since \( w(uv) = w(us_1) + w(vs_2) \), we conclude that every star-factor of \( G \) has the same weights.

This completes the proof.

From the proof of theorem above, we obtain the following corollary.

**Corollary 2.** If \( G \) is a tree, then \( G \in \Omega \) if and only if each component of the graph obtained by removing the leaves and stems from \( G \) is empty or a star \( K_{1,m} \) \((m \geq 1)\) with center having degree \( m \) in \( G \) for \( m \geq 2 \) or an isolated vertex.

**Remark 1.** The graph \( G \) shown in Figure 7 is an example which is in \( \Omega \) but requiring non-constant edge-weight function. To see this, assume that all the edges have the same weights, then we can find two star-factors with 10 edges and 7 edges, respectively. So \( G \notin \mathcal{W} \). But if we give a non-constant edge-weight function \( w \) as follows:

\[
w(e) = \begin{cases} 
2k & e \in \{a, b, c\} \\
k & e \in \{E(G) - \{a, b, c\}\}
\end{cases}
\]

where \( k > 0 \). It is not hard to verify that all the star-factors of \( G \) have the same weights under \( w \).

![Figure 7](image.png)

**Remark 2.** The main theorem has classified all graphs in \( \Omega \) with girth at least five. The families remaining to be determined are graphs of girth three or four. It seems that the structures of both families are much more complicated, but it would be an interesting problem to investigate.

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