First steps of a nucleation theory in disordered systems

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Abstract

We devise a field theoretical formalism for a microscopic theory of nucleation processes and phase coexistence in finite dimensional glassy systems. We study disordered $p$-spin models with large but finite range of interaction. We work in the framework of glassy effective potential theory which in mean-field is a non-convex, two minima function of the overlap. We associate metastability and phase coexistence with the existence of space inhomogeneous solutions of suitable field equations and we study the simplest of such solutions.

1 Introduction

A prominent theoretical problem in the physics of glassy systems is the comprehension of the processes that restore ergodicity in regions where approximated liquid theories or Mean-Field models predict a spurious structural arrest. It is well known that the Mode Coupling Theory [1], while describing correctly several aspects of the dynamics of moderately supercooled liquids,
predicts the divergence of the relaxation time and ergodicity breaking at a temperature $T_c$ well above the observed laboratory glass transition temperature $T_G$. In the so called “random first order phase transition scenario” [2], corresponding to “one step replica symmetry breaking” (1RSB) [3] in mean field disordered models, this ergodicity breaking appears to be related to the appearance of exponentially numerous metastable states, capable to trap the system in regions of free-energy extensively higher than the thermodynamic value. Within this scenario a Kauzmann-like entropy crisis occurs at a temperature $T_K < T_c$. Below this temperature, an ideal glassy state is thermodynamically stable.

Of course, in extended systems, strict metastability is an artifact of mean-field approximation and should disappear as soon as the finite range character of interaction is properly treated. A remarkable hypothesis states that while acquiring a finite life time, the gross structure of metastable states survives in short range glassy systems and dominates the low temperature relaxation. Unfortunately, a microscopic satisfactory description of the configuration space for liquid systems below $T_c$ is presently lacking. Based on the random first order transition scenario, Kirkpatrick, Thirumalai, Wolynes and others [2, 4, 5, 6, 7] have proposed the notion of “mosaic state” where below a well defined temperature-dependent coherence length the system appears to behave in a mean-field like glassy manner, while it behaves as a liquid on larger scales. This length should be divergent at the temperature $T_K$, where the ideal glassy phase sets in. We meet another much debated problem in the theory of disordered systems: the possibility in finite dimension of having ideal glassy phases with characteristics similar to the mean field ones [8, 9, 10].

In the past, different attempts have been made to investigate the restoration of ergodicity in the temperature region between $T_K$ and $T_c$ and the description of the low temperature phase $T < T_K$, through analogies with first order phase transitions. In the aforementioned contributions [2, 4, 5, 6, 7] a phenomenological nucleation theory was proposed, where the exponential multiplicity of metastable states would give rise to “entropic droplets” providing the main driving force for ergodicity restoration. This force would be contrasted by an interface cost proportional to a power $d - d_c$ of the droplet radius. In the simplest version of the argument, $d_c$, which can interpreted as the lower critical dimension for an ideal glassy phase, is equal to one and the resulting free-energy barrier scales as $(T - T_K)^{-2}$ in three dimensions. A
great effort was made to justify a renormalization of the value of $d_c$, in order to derive a Vogel-Fulcher scaling $(T - T_K)^{-1}$.

It was later realized by Parisi [11, 12] that a nucleation theory in glassy systems could be formalized considering an appropriate effective potential function [13, 14, 15, 16] exhibiting a characteristic double well shape below $T_c$. Postulating a finite surface tension between metastable states, in [11, 12] the mean field behavior of the critical droplet in the form anticipated in [2, 4] was derived. The first trials to compute from first principle entropic droplets were performed in [17] for a random heteropolymer model.

Biroli and Bouchaud [18] have recently rephrased some of the concepts in [2, 4, 5, 6, 7], elucidating at the physical level the notions of mosaic state and entropic droplets, and proposing a self-consistent derivation of a glassy coherence length. Again, an interface cost to put different metastable states in contact is supposed rather than derived.

In ordinary first order phase transitions with two or more phases in competition, metastable states are destabilized by nucleation processes. These are possible as soon as the interaction range is finite. A proper theory of nucleation is achieved considering a large but finite interaction range, where appropriate asymptotic expansions can be applied [19, 20, 21]. The study of interfaces and nucleation is reduced to the analysis of space-inhomogeneous solutions of the saddle point equations and fluctuations around these solutions. This method can be seen as an expansion around mean-field and becomes more and more accurate for larger and larger interaction range.

The aim of this paper is to set up a formalism allowing for a first principle ergodicity restoration and phase coexistence theory in disordered systems, starting from the theoretical analysis of microscopic models. The goal of this theory should be the computation of free-energy barriers and the critical dimension $d_c$. The crucial idea, in common with the previous phenomenological analysis, is that in disordered systems local properties can behave similarly to mean-field, while this is not necessarily so for global properties. This idea has recently received a rigorous foundation in the context of spin glass models with large but finite range Kac kind of interactions [22, 23, 24]. The analysis of these models, provides therefore a natural starting point to study finite range effects in expansions around mean field. In this paper, we use long but finite range Kac versions of the spherical $p$-spin models and analyze them by the means of the replica theory. The general framework in which we move is the setting of effective potential theory for coupled replica
systems [13, 14, 15, 16]. This allows to define field theoretical free-energies functionals of local overlaps, analogous to the familiar Landau free-energies functional of the local-order parameter in non disordered systems. In the presence of a “random first order transition” the potentials have a two minima structure similarly to systems undergoing a first order phase transition. We argue that one can devise a theory of ergodicity restoration and phase coexistence which, analogously to nucleation in first order transition, is based on the analysis of the instantonic, space inhomogeneous solutions of suitable field equations. In this paper we begin the analysis of these equations both in the metastable and in the coexistence region. This allows us to put on formal basis the extensions to disordered systems of the concepts of critical droplets, interface tension and associated free-energy barrier. In spite of the fact that the content of the theory we develop is very different from ordinary nucleation of a stable phase into an metastable one, thanks to the many formal analogies, we found it useful to use the intuitive language of first order transitions. To avoid confusion we always specify the physical meaning of the quantities involved. Due to the complexity of the resulting theory, we expect many possible solutions to exist to the instantonic equations. In this paper we always look for the simplest of such solutions. It is well possible that in the future better solutions will improve our results. In addition, while a complete theory should include the study of the matrix of small fluctuations around the instantonic saddle points, we do not perform this analysis here leaving it for future research.¹

¹After this paper appeared as a preprint (cond-mat/0412383) an interesting contribution by M. Dzero, J. Schmalian, P. G. Wolynes (cond-mat/0502011) proposed a partial analysis of the fluctuations matrix, as well as a new solution in one of the cases considered here.
2 The effective potential theory

As customary in statistical physics [25, 26] we consider coarse grained free-energy functionals as functions of the local order parameter as a starting point in the study of medium and large distance properties of our system. In mean-field disordered glassy systems the order parameter is the probability distribution of the overlap between configurations, as induced by the canonical measure and the quenched disorder [3]. In spin systems the overlap is just the normalized scalar product among spin configurations, while other notions of similarity have been proposed for particle systems [27]. Proper effective potential functions for glassy systems have been obtained considering the free-energy of identical copies of a given system for fixed values of the mutual overlaps. The constraint on the overlaps allows for suitable modifications of the Boltzmann weight so as to access otherwise hidden regions of configuration space. The method has been successfully used to study metastable states in mean field spin glass models, $p$-spin models [14, 15, 16] and ROM [28], and liquids in the HNC approximation [29, 30] with similar results. Models undergoing 1RSB and in particular $p$-spin models, are the prototype of mean-field systems displaying a Kauzmann-like entropy crisis. The configurational entropy, defined as the logarithmic number of ergodic components that contribute at each temperature to the partition function, is an increasing function of the temperature in the domain $T_K < T \leq T_c$ and vanishes at $T_K$.

This paper considers local versions of effective potentials for spatially extended systems. Before entering into the discussion of these versions, we briefly review the general properties of the potentials in mean field. Full details can be found in [14, 15, 16].

Two main versions are usually considered: the annealed potential and the quenched potential that allow to weigh configuration space regions in different ways and thus study different aspects of the glassy phase.

In the annealed version one considers $r$ copies of the original system and study the constrained free-energy where only the configurations such that all the mutual overlaps have a fixed value $\bar{p}$ are taken into account. In this paper we limit ourselves to considering the most commonly studied case $r = 2$, which, denoting $q(\sigma, \tau)$ the overlap between configurations $\sigma$ and $\tau$,
can be written with transparent notation as

\[ V_a(\tilde{p}) = -\frac{T}{N}E \log \left[ \frac{1}{Z^2} \sum_{\sigma,\tau} \exp \left( -\beta (H(\sigma) + H(\tau)) \right) \delta (q(\sigma,\tau) - \tilde{p}) \right]. \]  

(1)

In words, this is the average \( E \) over the quenched disorder of the logarithm of the probability distribution of the overlap among two replicas [3].

In the quenched construction one fixes an unconstrained equilibrium configuration and considers the free-energy of a second system which has a fixed overlap \( \tilde{p} \) with this reference state. This is written as

\[ V_q(\tilde{p}) = -\frac{T}{N}E \frac{1}{Z} \sum_{\tau} \exp \left( -\beta H(\tau) \right) \log \left[ \frac{1}{Z} \sum_{\sigma} \exp \left( -\beta H(\sigma) \right) \delta (q(\sigma,\tau) - \tilde{p}) \right]. \]  

(2)

In both cases the potential exhibits the same qualitative features, analogous to the ones of the mean field free-energy as a function of the order parameter in presence of a first order phase transition. At high temperature it is a convex function with a single minimum for zero value of the overlap. On decreasing the temperature it first looses convexity and then at a temperature \( T_s \) which depends on the version one is considering, it develops a secondary minimum at higher values of the overlap. Finally, at the even lower temperature \( T_K \) of the aforementioned entropy crisis, and below, both potentials exhibit two minima with degenerate free-energies (see fig. 1).

Differently from ordinary first order phase transitions however, the two minima are not associated to phases with different macroscopic characteristics but rather, the whole shape of the function is related to the existence of an exponential multiplicity of ergodic components. In particular, the difference in free-energy between the secondary and the primary minimum represents different quantities for the annealed and the quenched potential, but in both cases it is associated with the existence of a multiplicity of metastable states with different internal free-energy.

The quenched potential as a function of the temperature strictly reflects the phase structure of the model [14]. The temperature \( T_s \) where the secondary minimum first appears coincides here with the ergodicity breaking temperature \( T_c \) [31]. It is well known that below \( T_c \) the equilibrium free-energy can be decomposed in the sum of an internal free-energy, i.e. the free-energy of single ergodic components (which is equal for all the components relevant at a given temperature), and the configurational entropy,
the log-multiplicity of these metastable states [32]. While the internal free-energy calculated in the two minima is the same, the difference in absolute free-energy directly measures the configurational entropy multiplied by the temperature. At the temperature \(T_K\) where the configurational entropy vanishes, the two minima become degenerate, and remain degenerate below.

In the \(r = 2\) annealed potential, the secondary minimum appears at a temperature \(T_s = T_2 > T_c\). It was found in [16] that the secondary minimum is associated to both copies being in one of the lowest internal free-energy states: the ones of zero configurational entropy. In this case the difference between the secondary and the primary minima represents the cost needed to put a system in the lowest available internal free-energy states. This is a positive cost, since this situation implies a loss in configurational entropy. As in the quenched case, for temperatures equal to \(T_K\) and below, the two minima are degenerate. It should be noticed that close to \(T_K\), while in the quenched case one has that the free-energy difference among the minima behaves as \(T - T_K\), in the annealed case it behaves as \((T - T_K)^2\). It is less costly to constrain a system in the internal free-energy ground state then in
a single particular equilibrium state. At and below $T_K$ the minima of the two potentials give the same information. In both cases they probe the lowest free-energy components of the system. Due to this fact, we feel it important to discuss the two cases in parallel.

Both versions can be adapted to spatially extended systems to define free-energies as functionals of local order parameters. Once the free-energy functional is built, one can use it to study metastability, ergodicity restoration and phase coexistence.

An intuitive argument on the existence of a coherence glassy length has been put forward in [18], that can be directly rephrased in terms of the quenched potential. If one wishes to know over which length scale a supercooled liquid appears to be instantaneously frozen, one can consider the configurations of a system constrained to be “close” to a reference equilibrium state outside a bubble of radius $R$. For low $R$ it is free-energetically advantageous to remain close to the reference state while for large $R$ it is advantageous to get far apart from the reference. Biroli and Bouchaud identify the coherence length as the radius $R^*_q$ that separates the two regions. This radius can be computed as the critical radius for an overlap droplet in the quenched setting. Certainly, one should specify what “close” means in this context. The natural choice for the Kac model is to choose it to be at the overlap $\tilde{p}^*$ where the potential has the secondary minimum.

Obviously the same line of reasoning can be applied to the annealed case. How large can a bubble be such that two replicas that are constrained to be at overlap $\tilde{p}^*$ outside the bubble, will also be close inside the bubble? This problem is related to the existence of a phase transition at $T_K$. In case of a transition one would expect a divergent critical radius, and a divergent length (long range order) in the whole low temperature phase $T < T_K$. This divergent length is usually associated with a divergent free-energy barrier. One gets then an estimate of the lower critical dimension as the highest dimension where this barrier fails to diverge. From the above discussion we see that in the effective potential setting the problem can be formally expressed as a nucleation theory where one seeks instantonic configurations of the overlap in space. Unlike ordinary nucleation, the droplet size should not be interpreted as the size of the typical fluctuation inducing the decay of a metastable phase, but rather as the typical length over which there is a crossover from glassy-like to liquid-like behavior [2].
2.1 Technical matter

In disordered systems, the annealed and quenched potentials can be studied using the replica method. One can deal with both cases in a unified framework considering a number $n' = nr$ replicas, with $n$ going to zero in the usual way.

In the annealed case $r$ is considered to be a fixed integer number, while $n$ tends to zero. The number of real copies $r$ is usually taken to be equal to 2, and this is the case we discuss in this paper.

In the quenched case one gets the effective potential from an additional analytic continuation in $r$. What one needs to do, is to take the $r$ derivative of the replicated free-energy in the point $r = 1$ [14].

In mean-field the order parameter in the replica formalism is the $n' \times n'$ overlap matrix $Q_{\alpha,\beta}$. The free-energy as a function of $Q$ admits an expression formally identical to the one of the unconstrained problem [14]. The difference with the unconstrained problem lies in the fact that some of the elements of the matrix $Q$ are fixed to the value $\tilde{p}$ of the constrained overlap. It is useful to view $Q$ as a collection of $r^2 n \times n$ sub-matrices $Q_{u,v}^{a,b}$ with $u, v = 1, ..., r$ and $a, b = 1, ..., n$. The overlap constraint in the partition function is reflected in the fact that the elements $Q_{1,u}^{1,a}$ and $Q_{u,1}^{a,1}$ for all $u$ and $a$ are fixed to the value $\tilde{p}$. The spherical constraint implies $Q_{u,u}^{u,u} = 1$ for all $u$ and $a$. The analytic continuation for $n \to 0$ can be performed supposing Parisi Ansatz for each of the sub-matrices $Q_{u,v}^{a,b} = Q^{v,u}$ and the following “replica symmetric structure” in the upper indexes\(^2\): $Q^{1,1} = Q$, $Q^{1,u} = P$ and $Q^{u,u} = T$ for $s > 1$ and $Q^{u,v} = S$ for $u \neq v$ and $u, v > 1$. Replica symmetry breaking in the upper indexes could also be possible, but will not be discussed in this paper. Our choice is a valid Ansatz for all $r$ and allows the continuation $r \to 1$. In the annealed case $r = 2$, the matrix $S$ is obviously absent, and by symmetry one has $T = Q$.

In the following we mainly use the one step replica symmetry breaking form for the $n \times n$ replica matrices, where a given matrix $A_{ab}$ has a block structure with $n/m$ blocks of size $m$ and is parameterized by the common value $\tilde{a}$ of its diagonal elements, its value $a_1$ inside the blocks and its value $a_0$ outside the blocks [3]. It is customary to represent such a form with a

\(^2\)Since the replicas with $u = 1$ are singled out due to the constraint, a matrix respecting replica symmetry in the upper indexes is only invariant under permutation of the indexes $u = 2, ..., r$. 

9
function \(a(u) = a_0 \theta(m-u) + a_1 \theta(u-m)\) where \(u\) is a variable in the interval \([0, 1]\).

### 3 The model

In order to introduce physical space in the theory, we consider in this paper a finite-dimensional version of the spherical \(p\)-spin model [31] on a \(d\)-dimensional cubic lattice \(\Lambda\) of linear size \(L\), with spin variables \(S_i\) defined on each lattice site \(i \in \Lambda\). We use a spatially local spherical constraint. To this aim we partition \(\Lambda\) into cubes of size \(l\) and on each cube \(B_n\) \((n = 1, \ldots, (L/l)^d)\) we impose the constraint: \(\sum_{i \in B_n} S_i^2 = l^d\). The reason for this construction is that, as for the global spherical constraint in mean-field, it will allow to write the free-energy as a function of the order parameter matrix in a closed form. We then introduce the finite range \(p\)-spin Hamiltonian [23]

\[
H^{(p)}_{\Lambda}(S, J) = - \sum_{i_1, \ldots, i_p \in \Lambda} J_{i_1 \ldots i_p} S_{i_1} \cdots S_{i_p}
\]

where the couplings \(J_{i_1 \ldots i_p}\) are i.i.d. Gaussian variables with zero average and variance

\[
E(J_{i_1 \ldots i_p}^2) = \frac{1}{2} \gamma^{(p-1)d} \sum_{k \in \Lambda} \psi(\gamma|i_1 - k|) \cdots \psi(\gamma|i_p - k|)
\]

where \(\psi(|x|), x \in \mathbb{R}^d\), is a non-negative integrable function verifying the normalization \(\int d^d x \psi(|x|) = 1\). With this choice, only variables that are at distances \(r_{ij} \lesssim \gamma^{-1}\) effectively interact. The effective interaction range \(\gamma^{-1}\) will be assumed to be large throughout the paper. For technical reasons that will be discussed later, it is convenient to consider, instead of the pure \(p\) spin Hamiltonian, a compound Hamiltonian which mixes a \(p\)-spin part with \(p \geq 3\) with a (small) \(p = 2\) part, namely

\[
H_{\Lambda}(S, J) = \sqrt{a} H^{(2)}_{\Lambda}(S, J) + H^{(p)}_{\Lambda}(S, J).
\]

A useful notation in the following will be

\[
f(q) = 1/2(aq^2 + q^p).
\]
Our interest for the model derives from the fact that it has been proved that (at least for even $p$), in the thermodynamic limit, for small $\gamma$, free-energy and local order parameter are close to the values obtained in the corresponding long range model [22, 23]. In addition, it has been shown that the application of the replica method to the study of effective potential functional of Ising Kac $p$ spin for even $p$ has been shown to give rise to free-energy lower bounds for small $\gamma$ [24].

The task of the next section is to set up a formalism allowing the computation of the free-energy as a function of the overlap profile in space $\tilde{p}(x)$ on a coarse grained scale. We consider both the annealed case, where analogously to the space-less mean field case the two replicas are considered on the same foot, and the quenched case, where one of the replicas is fixed in an equilibrium reference configuration.

4 Replica analysis

Similarly to the mean-field case, one can study theoretically the annealed and the quenched potentials considering a system of $n' = r \times n$ replicas according to the procedure explained in the previous section.

The replica analysis of the Ising analogous of model (3) has been performed in [24]. As the local spherical case only requires minor modifications, the derivation will only be sketched here. One has to introduce a coarse graining length $\delta/\gamma$ such that $1 << l << \delta/\gamma << 1/\gamma$, and partition $\Lambda$ into boxes $C_x$ of that size. One then considers the “block overlap” order parameter matrix over these boxes $Q_{\alpha,\beta}(x) = (\gamma/\delta)^d \sum_{i \in C_x} S_\alpha^i S_\beta^i$ and rescales space by a factor $\gamma$. As in the previous section one can write the matrix $Q(x)$ as a collection of $r \times r \times n \times n$ matrices $Q_{a,b}^u, v(x)$ ($a, b = 1, ..., n$, and $u, v = 1, ..., r$). The free-energy for fixed overlap profile $\tilde{p}(x)$ is then obtained constraining the elements $Q_{a,a}^{1,x}(x)$ to $\tilde{p}(x)$.

As explained in [24], up to terms vanishing for small $\gamma$, the replica computation of the free-energy leads to a coarse grained free-energy functional

$$F[Q] = \frac{1}{\gamma^d} S[Q] = \frac{1}{\gamma^d} \int dx \left[ K(\{Q_{a,b}\}, x) + V(Q(x)) \right]$$

(7)
with
\[
K(Q_{\alpha,\beta}, x) = -\frac{\beta}{2rn} \sum_{\alpha,\beta} [f(\hat{Q}_{\alpha,\beta}(x)) - f(Q_{\alpha,\beta}(x))]
\]
\[
V(Q) = -\frac{1}{rn} \left[ \frac{\beta}{2} \sum_{\alpha,\beta} f(Q_{\alpha,\beta}) + \frac{1}{2\beta} \text{Tr} \log Q \right]
\] (8)

and we have defined
\[
\hat{Q}_{\alpha,\beta}(x) = \int dy \psi(x - y)Q_{\alpha,\beta}(y).
\] (9)

The replicated partition function \( E(Z^{n'}) \), could in principle be evaluated as the functional integral
\[
E(Z^{n'}) = \int DQ(x) e^{-\frac{\beta \gamma}{r} S[Q]}
\] (10)

where the integration only concerns the unconstrained elements of the matrix order parameter.

We see that we are in presence of a replica field theory, with an action \( S[Q] \) that can be written as the space integral of a potential part \( V(Q) \), identical to the mean-field free-energy as a function of the overlap matrix [31], plus a kinetic part \( K(Q_{\alpha,\beta}, x) \), sensitive to space variations of the order parameter. As stated above the free-energy as a function of \( Q \) is written in a closed form.

Assuming smooth variations of \( Q \) in space, as it is customary in non-disordered cases, one can replace the kinetic term in the action by the lowest non trivial term in its gradient expansion, namely
\[
K(Q_{\alpha,\beta}, x) \approx \frac{c\beta}{4rn} \sum_{\alpha,\beta} f''(Q_{\alpha,\beta})(\nabla Q_{\alpha,\beta})^2
\] (11)

where \( c \) is the only parameter depending on the shape of the \( \psi \) function and is given by \( c = \int dx \psi(x)x^2 \). In what follows we use expression (11) for the kinetic term. With this approximation the action \( S \) becomes the space integral of a Lagrangian density, with “coordinate dependent” masses \( f''(Q_{\alpha,\beta}) \). For a pure \( p \)-spin interaction the mass would vanish at vanishing values of the overlaps. This is the reason why we introduced an additional
term with couple interactions in the Hamiltonian (5). Without this term it would be necessary to pursue the gradient expansion to higher orders to avoid singularities for small values of the overlaps. 

Thanks to the large factor $\gamma^{-d}$ in front of the action, one can estimate the integral (10) through saddle point evaluation. One needs then in principle to find solutions of the Euler-Lagrange equations for the unconstrained elements of the matrix $Q(x)$. These take the form

$$ \frac{c\beta}{2} \nabla \cdot \left[ f''(Q_{\alpha,\beta}) \nabla Q_{\alpha,\beta} \right] = \frac{c\beta}{4} f'''(Q_{\alpha,\beta})(\nabla Q_{\alpha,\beta})^2 - \frac{\beta}{2} f'(Q_{\alpha,\beta}) $$

$$ - \frac{1}{2\beta}(Q^{-1})_{\alpha,\beta}. \quad (12) $$

Solving these equation for fixed values of $Q_{aa}^1(x) = \tilde{p}(x)$ in space and with the mentioned Ansatz for the remaining elements of $Q(x)$, leads to expressions of the free-energy that can be continued analytically as required in the two cases. It was proven in [24] that the resulting expression in the annealed case, gives at least an upper bound to the exact free-energy functional up to terms that scale as $(L\gamma)^d$. We believe that as it happens in the unconstrained case the expression becomes exact in the Kac limit [22, 23].

We have now an expression that allows in principle the evaluation of the effective potential functionals. The physical free-energy is associated to the least action space homogeneous saddle point of these functionals with respect to $\tilde{p}(x)$. These are obtained just solving eq.s (12) also with respect to the elements $Q_{aa}^1(x)$. We are interested in the regions of temperature mentioned in section 2, where the minimization of the action admits two homogeneous solutions with degenerate or non-degenerate minima. In the non-degenerate case we consider spherically symmetric instantons allowing to define a free-energy barrier for nucleation. In the degenerate case, we consider instantons with planar symmetry in order to define an interface tension.

In the following we denote respectively $V_a$ and $V_q$ and $K_a$ and $K_q$ the potential and kinetic part of the action $S_a$ or $S_q$ relative to the annealed or the quenched problem.

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3We thank J.-P. Bouchaud for this observation.
5 Simple Inhomogeneous solutions

In this section we enter in the core of our analysis and study inhomogeneous solutions of the field equations (12). In both annealed and quenched cases we distinguish the region of temperatures $T_K < T < T_s$, ($T_s = T_2$ in the annealed case and $T_s = T_c$ in the quenched case), which we call region I, where the secondary minimum is higher than the primary one, and then temperatures $T < T_K$, which we call region II, for which both minima are degenerate.

Region I is where one expects “activated processes” to exist destabilizing the high free-energy minima. These kind of processes can be studied considering inhomogeneous solution with finite action difference with the metastable phase and unstable modes. One has to consider then instanton configuration such that for $r = |x| \to \infty$, the system is described by the metastable state. The simplest of such solutions are spherical droplets. Close to $T_K$ where the minima are nearly degenerate, one can use a “thin wall approximation” to derive an effective droplet model [19, 33], and estimate the droplet radius and action through the competition between a bulk free-energy gain and a surface tension loss. It has been remarked that such approximation has to break down close to the spinodal point $T = T_s$, where one finds “ramified droplets” with interface thickness comparable to the radius. In that region the leading behavior can be obtained by dimensional analysis of various degree of refinement [34, 35, 36].

In Region II the minima are degenerate and one is interested to study the “interface between the two phases”, that one can suppose to be flat. In this case one has to consider one-dimensional instantons that connect regions at $\pm \infty$ with overlaps fixed at the two minima values. The free-energy cost for the interface is related to the value of the lower critical dimension.

5.1 The annealed case

5.1.1 Region I

In region I the primary minimum is described by a solution where all the overlap parameters are equal to zero, as matrices, $P = 0$ and $Q = I$ this is the solution of the unconstrained problem, and the free-energy is just equal to twice the unconstrained free-energy. The secondary minimum which is described by a 1RSB solution with $q = p = \tilde{p} = \tilde{p}^*$, $q_0 = p_0 = 0$. The value
$m = m^*/2$ of breaking parameter in the secondary minimum starts from $m^* = 2$ at $T_2$ and decreasing monotonically for decreasing temperatures, until $m^* = 1$ at $T_K$.

For values of $\tilde{p}$ different from the values 0 and $\tilde{p}^*$ that it takes in the two minima respectively, the potential is described by the maximum over $q$, $p$ and $m$ of the 1RSB form

$$V_a(\tilde{p}, p, q, m) = -\frac{\beta}{2}[f(\tilde{p}) - (1 - m)(f(q) + f(p))]$$

$$-\frac{1}{2\beta}(1 - \frac{1}{m})[\log(1 - \tilde{p} - q + p) + \log(1 + \tilde{p} - q - p)]$$

$$-\frac{1}{4\beta m}[\log(1 - \tilde{p} - (1 - m)(q - p))$$

$$+ \log(1 + \tilde{p} - (1 - m)(q + p))]$$ (13)

We would like to construct a spherically symmetric solution which has finite difference in action with respect to the uniform metastable solution, and that takes the values of the parameters in this solution as boundary values for the radial distance $r$ going to infinity.

We confine ourselves to 1RBS solutions with $m$ constant in space, and given by the value $m^*/2$ it takes in the secondary minimum. This choice is motivated by the fact that the breaking parameter $m$ appears as a parameter in the distribution of the free-energies of the different ergodic components, and should not depend on space. In addition, 1RSB matrices with values of $m$ depending on space would not form a closed algebra and could not verify the field equations. Within this 1RSB Ansatz we will only consider solutions such that $q_0 = p_0 = 0$. This is consistent with the fact that in the mean-field model different states have vanishing overlap. We need therefore to find a solution of the space dependent equations departing from the secondary minimum with vanishingly small velocities for the parameters $\tilde{p}, p$ and $q$. Unfortunately, for $T$ smaller but close to $T_2$ one can see that for $m$ fixed to the value $m^*/2$, the point $q = p = \tilde{p} = \tilde{p}^*$ is a maximum and not a local minimum of the function $V_a(\tilde{p}, p, q, m^*)$ so that this Ansatz simply gives a vanishing surface tension. While this fact could signal some inconsistency of the present approach, it is not clear to us at present how to avoid the hypothesis that $m = m^*/2$. Below a second temperature $T'_2$ however, the point $q = p = \tilde{p} = \tilde{p}^*$ becomes a minimum of $V_a(\tilde{p}, p, q, m^*)$ and we can use that Ansatz to estimate the surface tension.
The simplest inhomogeneous solution we can look for, being described by the metastable saddle point at infinity and having a finite action, is described by \( \tilde{\rho} = \rho = q \) in all points \( x \) in space.

With this choice, the potential part of the action becomes

\[
W_a(\tilde{\rho}) = V_a(\tilde{\rho}, \tilde{\rho}, \tilde{\rho}, m^*) = \frac{\beta}{2}(1 - m^*) f(\tilde{\rho}) + \frac{1}{2 \beta m^*} [(1 - m^*) \log(1 - \tilde{\rho}) - \log(1 - \tilde{\rho} + \tilde{\rho}m^*)] \tag{14}
\]

If we make the thin wall approximation\([19, 33]\) close to \( T_K \), the problem becomes a one dimensional mechanical problem with a kinetic energy term which reduces simply to

\[
K_a = \frac{\beta c}{4} (m^* - 1) f''(\tilde{\rho}) \left( \frac{d\tilde{\rho}}{dr} \right)^2 \tag{15}
\]

where we remember that \( m^* \) takes values in the interval \([1, 2]\) and \((m^* - 1) > 0\) in the region we are considering.

Considering \( \tilde{\rho}_1 \) the value of \( \tilde{\rho} < \tilde{\rho}^* \) such that \( W_a(\tilde{\rho}_1) = W_a(\tilde{\rho}^*) = W^* \), the 1D integral of the action density along the instanton direction defines the surface tension \( \sigma \), which through elementary mechanics is seen to be equal to:

\[
\sigma = \int_{\tilde{\rho}_1}^{\tilde{\rho}^*} d\tilde{\rho} \sqrt{2c \beta (m^* - 1) f''(\tilde{\rho})[W_a(\tilde{\rho}) - W_a^*]} \tag{16}
\]

It is easy to see that close to \( T_K \) the value of \( \sigma \) vanishes linearly in \( T - T_K \). This is due to the fact that quite generically in the parameters that define the Hamiltonian, one has \((m^* - 1) \sim W_a(\tilde{\rho}) - W_a^* \sim T - T_K \) uniformly \( \tilde{\rho} \). This estimate of the surface tension can then be exploited in a standard way to complete the calculation and compute the radius and free-energy of the critical droplet.

In dimension \( d \) the action of a droplet with radius \( R \) is given by

\[
\Delta S_a(R) = -\Delta V_a v_d R^d + \sigma s_d R^{d-1} \tag{17}
\]

where we have denoted as \( s_d \) and \( v_d = s_d / d \) respectively the unitary spherical surface and volume. \( \Delta V_a \) is the free-energy difference between the secondary and the primary minimum, which, as already noted, close to \( T_K \) scales like
\[ \Delta V \approx (T - T_K)^2. \] Maximization with respect to \( R \), gives the radius \( R^* \) of the critical droplet where the free-energy (17) is maximum: \( R^* = \frac{(d-1)\sigma}{\Delta V}. \) Given the scaling of \( \Delta V \) and \( \sigma \), one sees that close to \( T_K \) it behaves as \( R^* \approx (T - T_K)^{-1} \) leading for the free-energy of the critical droplet

\[ \Delta F^* \approx \frac{\text{const}}{(T - T_K)^{d-2}}. \]  

This formula has the attractive feature of having the structure of the Vogel-Fulcher law in three dimensions, and, despite a finite surface tension in region I, would indicate a lower critical dimension for 1RSB transitions equal to 2. Notice that this divergence of the free-energy stems from the fact that both the difference of free-energy between the two minima and the surface tension vanish with different exponents at the transition. Unfortunately, it is not clear to us at present how and if \( \Delta F^* \) could be related to a free-energy barrier governing the relaxation time of the liquid phase. One would expect \( \Delta F^* \) to be the free-energy barrier to equilibration for a system prepared in one of the internal free-energy ground states. The equilibrium relaxation time should rather be associated with the quenched potential barrier, and being shorter than the one necessary to relax an internal free-energy ground state. We will see that in that more physical case our solution surprisingly gives a higher free-energy barrier.

In figure 1 we show the result of formula (16) as a function of temperature in the whole temperate range \( T_K < T < T_2 \). It should be remembered (see below) that the flat interface approximation breaks down at \( T_2' \). Analogously, the space dependence of the instantonic solution \( \tilde{p}(x) \) can easily be obtained, and its typical shape is displayed in fig. 3. The typical spatial extension of the instanton is given by

\[ \xi = \left( \frac{1}{\beta c(m^* - 1)f''(\tilde{p}^*)} \right)^{-1/2} \]  

As in usual cases this diverges as \( \xi \sim |T - T_2'|^{-1/2} \) close to the “spinodal temperature” \( T_2' \) and the thin walls approximation breaks down in \( D > 1 \). This is at variance with what would be the typical spinodal behavior \( \xi \sim |T_2' - T|^{-1/4} \). In fact the reason for this anomaly is the mentioned fact that the point \( T_2' \) is the point where the true minimum of the potential passes from being a maximum to a minimum when the parameter \( m \) has been fixed.
Figure 2: 1D instanton action in region I in the annealed case. We considered $p = 4$ and $a = 0.1$. In this case the critical temperatures are: $T_2 = 0.727$, $T'_{2} = 0.673$ and $T_K = 0.542$. The surface tension vanishes both at $T'_{2}$ and $T_K$. The surface tension vanishes linearly at $T_K$.

to $m^*$. As a result, around $\tilde{p}_t = \tilde{p}^*|_{T = T'_{2}}$, the potential can be approximated as $W(\tilde{p}) \approx A + a(\tilde{p} - \tilde{p}_t)^3 - b(T - T'_{2})^2(\tilde{p} - \tilde{p}_t)$, with $A$, $a$ and $b$ smoothly varying functions of $T$. This gives rise to a free-energy barrier scaling as $|T - T'_{2}|^{3-d/2}$, which, while giving an upper critical dimension equal to 6, differs from the usual spinodal scaling with an exponent $3/2 - d/4$ [34, 35].

As in non-disordered systems we expect that higher order in the expansion of the free-energy functional should be taken into account above dimension 6 [36].
Figure 3: Shape of the instanton for the same parameters as in fig. 2 and $T = T_K = 0.542$. At this temperature the annealed and quenched instanton coincide. In the inset the annealed instanton for $T = 0.667$. 
5.1.2 Region II

Let us now consider briefly the region II. Below $T_K$ both the primary minimum and the secondary minimum are described by 1RSB solutions. The primary minimum has $\tilde{p} = 0$, $q(u) = q_{EA}(u - m^*)$ and $p(u) = 0$. The secondary minimum has $\tilde{p} = q_{EA}$ and $q(u) = p(u) = q_{EA}(u - m^*/2)$. $q_{EA}$ and $m^*$ are respectively the value of the overlap and the breaking parameter that appear in the 1RSB solution of the unconstrained problem. In order to interpolate in space between these two solutions it is natural to consider a two step RSB form of the kind

$$q(u) = \begin{cases} 
0 & 0 \leq u < m^*/2 \\
q_0 & m^*/2 \leq u < m^* \\
q_1 & u \geq m^*
\end{cases}$$

$$p(u) = \begin{cases} 
0 & 0 \leq u < m^*/2 \\
p_0 & m^*/2 \leq u < m^* \\
p_1 & u \geq m^*
\end{cases}$$ (20)

where $q_0, q_1, p_0, p_1$ and $\tilde{p}$ depend on space, while $m^*$ is fixed. Correspondingly, the potential takes the form:

$$V_a(q_0, q_1, p_0, p_1, \tilde{p}) = -\frac{\beta}{2} \left[-(1 - m^*) f(q_1) - \frac{m^*}{2} f(q_0) + f(\tilde{p}) - (1 - m^*) f(p_1) - \frac{m^*}{2} f(p_0) \right]$$

$$-\frac{1}{2\beta m^*} \left\{ \log[(1 - (1 - m^*) q_1 - \frac{m^*}{2} q_0)^2 - (\tilde{p} - (1 - m^*) p_1 - \frac{m^*}{2} p_0)^2] \right\}$$

$$-\frac{1}{2} \log[(1 - (1 - m^*) q_1 - m^* q_0)^2 - (\tilde{p} - (1 - m^*) p_1 - m^* p_0)^2] +$$

$$\frac{1}{2} (m^* - 1) \log[(1 - q_1)^2 - (\tilde{p} - p_1)^2] \right\}$$ (21)

while the kinetic part of the action is:

$$K_a(q_0, q_1, p_0, p_1, \tilde{p}) = \frac{c \beta}{2} \left[-(1 - m^*) f''(q_1)(\nabla q_1)^2 - \frac{m^*}{2} f''(q_0)(\nabla q_0)^2 + f''(\tilde{p})(\nabla \tilde{p})^2 - (1 - m^*) f''(p_1)(\nabla p_1)^2 - \frac{m^*}{2} f''(p_0)(\nabla p_0)^2 \right]$$ (22)
Simple differentiation reveals the existence of a solution such that in all points in space \( q_0 = p_0 = p_1 = \tilde{p} \), while \( q_1 \) is independent of position and equal to the value \( q_{EA} \) of the Edwards-Anderson parameter of the model in the low temperature phase. Assuming a flat interface with space variations in the direction \( x \), the equation verified by \( \tilde{p} \) is:

\[
 c\beta f''(\tilde{p}) \frac{\partial^2 \tilde{p}}{\partial x^2} = c\beta f'''(\tilde{p}) \left( \frac{\partial \tilde{p}}{\partial x} \right)^2 - \beta f'(\tilde{p}) - \frac{1}{2\beta m} \left\{ \frac{1}{1 - (1 - m)q_{EA} - m\tilde{p}} - \frac{1}{1 - (1 - m)q_{EA}} \right\} .
\]

This equation has a solution that corresponds to a well defined instanton in space, continuing smoothly the form found in the region I in the thin wall approximation. Interestingly enough, one readily verifies that the various terms compensate each other in such a way that the solution has an identically vanishing interface cost. In the replica formalism this is related to the fact that the number of matrix elements equal to \( \tilde{p}(x) \) is proportional to \( n \) and is reminiscent of a similar result of [37] for the Edwards-Anderson model, where it was found a replica matrix with vanishing excess action. The important difference though is that here the matrix with vanishing excess action is a true solution of the saddle point equations. At present we do not have a clear interpretation of this result. Probably the computation of the determinant of the small fluctuation matrix around the solution would give more informations about its nature. It is also possible that the vanishing of the surface tension is a signal that another solution to the Euler-Lagrange equations exists, that has an action scaling as \( L^{d-d_c} \) with \( d_c > 1 \). Unfortunately for the time being we did not find such a solution. We will see in the next paragraph as the quenched formalism gives rise to the same instanton, but this time with a finite action.

### 5.2 The quenched case

Though with a different physical meaning, the analysis of this case parallels closely the one for the annealed case. If we interpret the instanton action as a free-energy barrier, in this case we directly relate to the equilibrium
relaxation time of the system. The computation of the critical droplet in the quenched case, will allow a fundamental computation of the glass coherence length as proposed by Biroli and Bouchaud.

The most general expression for the potential that we use is a 1RSB form with $T = Q$ and $S = P$. One can check that is is always a solution to the saddle point equations and that both minima of the potential are correctly described by this form in all temperature regions. With this Ansatz we have:

$$V_q(\tilde{p}, p, q, m) = -\frac{\beta}{2} [f(\tilde{p}) - (1 - m)(f(q) + f(p))]$$

$$-\frac{1}{2\beta} \left\{ \frac{1}{m} \log(1 - \tilde{p} - (1 - m)(q - p)) + (1 - \frac{1}{m}) \log(1 - \tilde{p} - q + p) \right\}$$

$$\frac{1}{m} \frac{(\tilde{p} - (1 - m)p)}{1 - (1 - m)q}$$

$$+(1 - \frac{1}{m}) \frac{\tilde{p} - p}{(1 - q)}$$

(24)

$$K_q(\tilde{p}, p, q, m) = \frac{e\beta}{2} [f''(\tilde{p})(\nabla\tilde{p})^2 - (1 - m)(f''(q)(\nabla q)^2 + f''(p)(\nabla p)^2)]$$

(25)

We start from the analysis of the minima of the potential as a function of the overlap. The quenched potential develops the secondary minimum at $T_c$, the temperature of the dynamical transition. In the temperature range $T_K < T < T_c$, that defines region I, the minima have the following structure: the primary minimum is described by $Q = T = I$, $P = S = 0$, the secondary minimum has the same $Q$ and $T$, while $P$ and $S$ have the diagonal form $P = S = \tilde{p}^* I$.

At low temperature, in the region II $T < T_K$ on the other hand both the primary and the secondary minima are described by 1RSB solutions. In the primary minimum, one has that $Q = T$ and both matrices are parameterized by the function $q(u)$ which takes the 1RSB form $q(u) = q_{EA} \theta(u - m^*)$, while $P = S = 0$. The secondary minimum, has the same $Q$ and $T$, while $P = S$ are parameterized by a diagonal element $\tilde{p} = q_{EA}$ and the function $p(u) = q_{EA} \theta(u - m^*)$. For both minima the values of $m^*$ and $q_{EA}$ are the same and coincide with their equilibrium values in the unconstrained free-energy. In both regions we now seek solutions that connect the two minima.
5.2.1 Region I

Since in this case both the primary and the secondary minima are described a replica symmetric saddle points, we choose a RS solution to describe the instanton. It is easy to see that the matrix $Q$ verifies equations that are uncoupled from the remaining matrices and coincide with the ones for the unconstrained system. This is coherent with the fact that the reference state used in the quenched potential is unaffected by the coupling. One should then choose for $Q$ the space independent solution which is appropriate for the temperature range at hand. In this region one has $Q = I$. A simple inspection shows that the equations admit solutions with $T = Q$ and $S = P$ in all points of space, and this will be our choice, with with $T = Q = I$ and $P = S = \tilde{p}I$. The typical shape of the surface tension in the thin wall approximation as a function of the temperature is given in figure 4. We see that in this case the surface tension stays finite at $T_K$. The computation of the radius and free-energy of the critical droplet, follows the same route as in the annealed case. Taking into account that the difference between the two minima is the configurational entropy, which vanishes linearly as $T - T_K$, at $T_K$, one finds that the radius of the critical droplet is given by $R^* \sim \sigma/(\Sigma) \sim (T - T_K)^{-1}$, while its free-energy is $\Delta F^* \approx (T - T_K)^{-(d-1)}$ as in conventional first order transitions. It is interesting to compare in detail the behavior of the various quantities close to $T_K$ with the corresponding one in the annealed case. To the leading order in $T - T_K$ one finds $\Delta V_a = (m^* - 1)\Delta V_q$, $W_a = (m^* - 1)W_q$. Consequently, $\sigma_a = (m^* - 1)\sigma_q$ and $\Delta F_a = (m^* - 1)\Delta F_q$, one verifies also that remarkably, despite a different surface tension and bulk free-energy gain $\Delta V$, the characteristics of the droplets, instanton shape and radius are the same in the two approaches. We notice that given the difference in $\Delta V_a$ and $\Delta V_q$ the behavior we find for $\sigma_a$ and $\sigma_q$ is the only one compatible with the fact of having the same droplet describing the two cases.

Close to $T_c$, one finds in the quenched case the conventional mean-field spinodal scaling, where the instanton has a width $\xi \sim |T - T_c|^{-1/4}$ while the free-energy barrier scales like $\Delta F^* \sim |T - T_c|^{3/2-d/4}$. One indeed can verify that all this holds because in this case one has the usual form of the potential close to the high minimum: $W(\tilde{p}) \approx A + a(\tilde{p} - \tilde{p}_t)^3 - b(T - T_c)(\tilde{p} - \tilde{p}_t)$. 

5.2.2 Region II

In the low temperature phase, region II, the solution for the matrix $Q$ is of the 1RSB type with parameters $m^*$ and $q = q_{EA}$ independent of position. Again we choose the solution which has $T = Q$ independent of position and $S = P$. The 1RSB solution for $P$ will be parameterized by the value of the breaking parameter, which naturally will be taken equal to $m^*$ and the values of the space dependent diagonal overlap $\tilde{p}$ and off-diagonal overlap $p$. Once again, if we choose $p = \tilde{p}$ the problem becomes one dimensional and can be trivially integrated. The field equation for $\tilde{p}$ and thus the resulting instanton solution coincide with eq. (23) found in the annealed case in all region II. Despite that, thanks to the different kind of analytic continuation, the value of the surface tension is different from zero in all region II, and connects continuously with the value found in region I at $T_K$. The typical behavior of the quenched surface tension is seen in figure 4 and indicates a critical dimension $d_c = 1$. In order to investigate the possibility that this result is an artifact of our simple solution, we have looked for a different 1RSB saddle point, $t(u) = s(u) = \tilde{s}\theta(u - m)$ and $p(u) = \tilde{p}\theta(u - m)$. Unfortunately, this solution, while giving rise to different values of $\sigma$ for $T < T_K$, does not affect the scaling of the surface tension.

6 Summary and Conclusions

In this paper we set up a general formalism to study nucleation and phase coexistence in terms of inhomogeneous solution of field theoretical equations for disordered models with long but finite interaction range. We presented here the simplest instantonic solutions to the replica saddle point equations for a $p$-spin model with a local spherical constraint. We used two different kinds of set-ups: the annealed potential and the quenched potential, for which we have critical droplet to nucleation in region I and flat coexistence interfaces in region II.

As in usual nucleation theory, in region I we have found -in the thin wall approximation- spherically symmetric solutions that leave the unstable minimum of the potential function at infinity with vanishing velocity. Differently from cases with scalar order parameter where at most one solution with a given spatial symmetry can exist, here multiple solutions are in principle
Figure 4: 1D instanton's action in regions I and II in the quenched case. We considered $p = 4$ and $a = 0.1$. In this case the critical temperatures are: $T_c = 0.575$ and $T_K = 0.542$. In this case the instanton has non vanishing action in both regions.
possible. Indeed in the replica formalism one has to choose an Ansatz to parameterize the overlap matrix, and different parameterizations could lead to different solutions. In this paper, we have always performed the simplest choice. In the case of the annealed potential we find a solution such that the surface tension has non-zero value in a temperature range $T_K < T < T'_2$ while is zero below the static transition temperature $T_K$. Its linear behavior close to $T_K$ implies a critical droplet with free-energy scaling as $(T - T_K)^{-(d-2)}$. In the quenched case we find different results: the surface tension is different from zero for all temperatures below the dynamic transition temperature $T_c$. It has a finite value at $T_K$ where there is a discontinuity in its derivative. Correspondingly, the free-energy of the critical droplet scales as $(T - T_K)^{-(d-1)}$. In region II one finds that the same instantonic solution describes the quenched and annealed interface. This is coherent with the fact that (differently from region I) in the two different set-ups, the minima of the potential functions are associated to the same family of states. Unfortunately, due to different analytic continuations, one finds different values of the interface tension. In the annealed case the interface tension is vanishing in all region II, in the quenched case it has a finite value. This is a paradoxical situation indicating possibly that more complicate replica solutions should be used to describe one or both situations. Further research and a deeper analysis of the field equations are needed to clarify the issue.

There are some obvious issues that we have not addressed in this paper. The first one concerns the physical interpretation of space-dependent overlap matrices. In Mean Field, overlap matrices are interpreted as describing global correlation functions [3]. It is of course tempting to interpret local matrices as generators of local correlation. We did not attempt in this paper an analysis along that lines. A detailed study should be devised to substantiate this hypothesis.

The second issue concerns the relation between the free-energy barriers derived within the present approach and the relaxation time to equilibrium. In Ising like transitions, one can model the transition kinetic by time-dependent Landau-Ginzburg equations with noise and in this way relate the nucleation barrier to the rate of relaxation of metastable phases [19, 20]. This kind of approach has been rigorously shown to represent the spinodal decomposition dynamics in the scaling regime for Kac kind interactions [38]. In the disordered case it is at present not clear if one could write the time evolution of the space-dependent overlap as a gradient equation in
the free-energy functionals defined in this paper. Consequently, one can not say which, if any, between the annealed or the quenched droplet free-energy can be associated to the main relaxation time of the system. A proper theory of relaxation rates should start from a dynamical approach similar to the the mean-field one, but in which the dynamical order parameters are allowed to depend on space. Again the use of Kac models should allow to obtain the theory as an expansion around Mean Field.

Summarizing, this paper indicates a route to study ergodicity restoration and the possibility of ideal glassy phases in finite dimension. Further studies are necessary to overcome the many obstacles found in the pathway.

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7 Appendix

In this appendix we derive some of the formulae used in our computations.

The form for the $rn \times rn$ replica matrix $Q$ discussed in section 2 reads

$$Q = \begin{pmatrix}
Q & P & P & P & P & \ldots \\
P & T & S & S & S & \ldots \\
P & S & T & S & S & \ldots \\
P & S & S & T & S & \ldots \\
P & S & S & S & T & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots 
\end{pmatrix} \quad \text{(26)}$$

where each symbol represents an $n \times n$ matrix.

Using this form one readily finds for the potential part of the free-energy:

$$V(Q) = -\frac{\beta}{2} \sum_{a,b}^{1,n} [f(Q_{a,b}) + 2(r-1)f(P_{a,b}) + (r-1)f(T_{a,b}) + (r-1)(r-2)f(S_{a,b})]$$
\[-\frac{1}{2\beta} \left[(r-2)\text{Tr} \log(T-S) + \text{Tr} \log[Q(T + (r-2)S) - (r-1)P^2]\right] \quad (27)\]

and for the kinetic one

\[K(Q) = c\beta \frac{1}{2} \sum_{a,b}^n \left[ f''(Q_{a,b})(\nabla Q_{ab})^2 + 2(r-1)f''(P_{a,b})(\nabla P_{ab})^2 + (r-1)f''(T_{a,b})(\nabla T_{ab})^2 + (r-1)(r-2)f''(S_{a,b})(\nabla S_{ab})^2 \right] \quad (28)\]

From this form the Euler-Lagrange equations for the various matrices can easily be written.

The most general Ansatz we use is such that for all points in space the structure of the various matrices is of the 1RSB type described in section II. The overlap parameters are space dependent, while the breaking point \( m \) is equal for all points in space. The detailed parameterization is such that each matrix is parameterized by a single diagonal element, an single off-diagonal element, and a common breaking parameter. The matrices \( Q \) and \( T \) have diagonal element equal to 1 thanks to the local spherical constraint and non diagonal elements \( q \) and \( t \) respectively. The matrices \( P \) and \( S \) have respectively diagonal elements \( \tilde{p} \) and \( \tilde{s} \) and non diagonal elements \( p \) and \( s \).

Inserting, with self-evident notation the 1RSB Ansatz and introducing the notation \( \langle q \rangle = (1 - m)q \) one has

\[V = -\frac{\beta}{2} \left[ f(1) - \langle f(q) \rangle + 2(r-1)(f(\tilde{p}) - \langle f(p) \rangle) + (r-1)(f(\tilde{t}) - \langle f(t) \rangle) \right. \]
\[+ \left. (r-1)(r-2)(f(\tilde{q}) - \langle f(s) \rangle) \right] \]
\[= -\frac{1}{2\beta} \left[ (r-2) \left( \frac{1}{m} \log(1 - \tilde{s} - \langle t - s \rangle) + (1 - \frac{1}{m}) \log(1 - \tilde{s} - (t - s)) \right) \right. \]
\[\left. + \frac{1}{m} \log[(1 - \langle q \rangle)(1 - \langle t \rangle) + (r-2)(\tilde{s} - \langle s \rangle)) - (r-1)(\tilde{p} - \langle p \rangle)^2 \right] \]
\[+ (1 - \frac{1}{m}) \log[(1 - \tilde{q})(1 - t + (r-2)(\tilde{s} - s)) - (r-1)(\tilde{p} - p)^2] \right\} \quad (29)\]

\[K = \frac{c\beta}{2} \left\{ -\langle f''(q)(\nabla q)^2 \rangle + 2(r-1)[f''(\tilde{p})(\nabla \tilde{p})^2 - \langle f''(p)(\nabla p)^2 \rangle] \right. \]
\[+ (r-1)[f''(\tilde{t})(\nabla \tilde{t})^2 - \langle f''(t)(\nabla t)^2 \rangle] + (r-1)(r-2)[f''(\tilde{s})(\nabla \tilde{s})^2 - \langle f''(s)(\nabla s)^2 \rangle] \right\} \quad (30)\]

One can specify to the annealed two replica case just setting \( r = 2 \) and to the quenched case taking the derivative with respect to \( r \) and setting \( r = 1 \).
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