Synchronization reveals correlation between oscillators on networks

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The understanding of synchronization ranging from natural to social systems has driven the interests of scientists from different disciplines. Here, we have investigated the synchronization dynamics of the Kuramoto dynamics departing from the fully synchronized regime. We have got the analytic expression of the dynamical correlation between pairs of oscillators that reveals the relation between the network dynamics and the underlying topology. Moreover, it also reveals the internal structure of networks that can be used as a new algorithm to detect community structures. Further, we have proposed a new measure about the synchronization in complex networks and scrutinize it in small-world and scale-free networks. Our results indicate that the more heterogeneous and “smaller” the network is, the more closely it would be synchronized by the collective dynamics.

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The last decade has witnessed the rapid explosion of interest in complex networks, which are found in many fields as diverse as biology, technology, and social organizations \cite{1,2,3,4}. The relation between structure and function becomes a key area in the study of complex networks. In the study of networked-dynamics, the emergent synchronization of interacting oscillators generally has occupied a privileged position because of its rich applications in variety of areas ranging from Neuroscience to Sociology \cite{3,4,5,6,7,8,9,10,11,12,13,14,15,16,17}. In neural systems, the brain dynamics is characterized by synchronization phenomena for a given topology of synapses. And in metabolic networks, hundreds of interconnected biochemical reactions are responsible for the biomass and energy fluxes which are adjusted to optimize the robustness of synchronized behavior. Other example is the synchronized or coordinated behavior of communication patterns in social organizations. The study of synchronization provides us with insights into the key issue: how the collective dynamics couple the relationships between the systematic function and the underlying topology.

One of the most successful attempts to understand synchronization phenomena is due to the Kuramoto model (KM) \cite{18,19}, which is rich enough for many different contexts, including superconducting currents in Josephson junction arrays, emerging coherence in populations of chemical oscillators, and the accuracy of central circadian pacemakers in insects and vertebrates, etc. This model describes a population of \( N \) coupled phase oscillators which evolves in time according to the following dynamics

\[
\frac{d\theta_i}{dt} = \omega_i + \sigma \sum_j A_{ij} \sin(\theta_j - \theta_i) + \xi_i(t). \tag{1}
\]

Here, \( \theta_i \) represents the phase of the \( i \)th oscillator, \( \omega_i \) the intrinsic frequency, \( \sigma \) the coupling constant, \( A_{ij} \) the effective coupling between the oscillators, and \( \xi_i(t) \) is the white noise due to the complicated environment, with expectation and variance

\[
\langle \xi_i(t) \rangle = 0,
\]

\[
\langle \xi_i(t) \xi_j(t') \rangle = 2\delta_{ij}\delta(t-t').
\]

There are many attractors that each oscillator rotates at its natural frequency when incoherent. As the coupling strength increases, some units become resonant. At sufficiently strong coupling, when coherent, there is only one attractor of the dynamics, and all oscillators rotate at the same frequency, \( \omega_i \), the average of \( \omega_i \). We call it fully synchronized.

The synchronization problem is solved mainly in the mean-field approach which, unfortunately, is not usually fulfilled in real systems. The nontrivial pattern of connectivity in complex networks brings the non mean-field properties and incorporates many new questions to the research of synchronization. Because of the elegant work of Pecora and Carroll \cite{20}, the Master Stability Function (MSF) formalism allows us to analysis the dynamical synchronizability in terms of purely topological properties, independent of details of unit dynamics. However, the MSF requires several constraints and has limited applications that it only deals with the stability of the exact synchronized states, i.e., it is used to study infinitesimal deviations from the synchronization manifold of chaotic oscillators. However, many realistic dynamics are only close to or even far from the full synchronization.

In this work we study the KM dynamics departing from the fully synchronized state. We get the analytic expression of the dynamical correlation between any two oscillators which reveals the relation between the network dynamics and the underlying topology. It could be used as a new algorithm to detect community structures. Moreover, we propose a new measure about the synchronization in complex networks and scrutinize it in
in terms of the normal modes,
\[ \frac{d\vartheta_\alpha}{dt} = -\sigma \lambda_\alpha \vartheta_\alpha + \zeta_\alpha(t), \quad (3) \]
where \( \vartheta_\alpha = \sum_j \psi_{\alpha j} \vartheta_j \), \( \zeta_\alpha = \sum_j \psi_{\alpha j} \xi_j \), and \( \psi_{\alpha j} \) denotes the \( \alpha \)th normalized eigenvector of the Laplacian, \( \lambda_\alpha \) is the corresponding eigenvalue for \( \alpha = 0, \ldots, N - 1 \). Considering \( \xi_i(t) \) is delta correlated, we can find easily that \( \zeta_\alpha(t) \) is also delta correlated: \( \langle \zeta_\alpha(t) \zeta_\beta(t') \rangle = 2\delta_{\alpha\beta} \delta(t-t') \).

Without loss of generalization, we set \( \sigma = 1, \zeta_\alpha(0) = 0 \). Using stochastic calculus methods \cite{21}, we get
\[ \vartheta_\alpha(t) = \int_0^t e^{-\lambda_\alpha(t-t')} \zeta_\alpha(t') dt', \quad (4) \]
so that
\[ \langle \vartheta_\alpha(t)^2 \rangle = \int_0^t \int_0^t e^{-\lambda_\alpha(t-t')} e^{-\lambda_\alpha(t-t'')} \langle \zeta_\alpha(t') \zeta_\alpha(t'') \rangle dt' dt'' \]
\[ = \int_0^t 2e^{-2\lambda_\alpha(t-t')} dt' \]
\[ = \frac{1}{\lambda_\alpha} (1 - e^{-2\lambda_\alpha t}) \quad (5) \]

Thus, the time scales of the dynamical relaxation of the eigenmodes are inversely proportional to the corresponding eigenvalues. And for large \( t \gg 1/2\lambda_\alpha \), we have
\[ \langle \vartheta_\alpha(t)^2 \rangle = 1/\lambda_\alpha. \quad (6) \]

Transforming it back to the basis of \( \vartheta_i \) by substituting \( \vartheta_i = \sum_\alpha \psi_{\alpha i} \vartheta_\alpha \), we find the steady-state correlation function:
\[ \langle (\vartheta_i - \overline{\vartheta})(\vartheta_j - \overline{\vartheta}) \rangle = \sum_{\alpha=1}^{N-1} \psi_{\alpha i} \psi_{\alpha j} \langle \vartheta_\alpha^2 \rangle \]
\[ = \sum_{\alpha=1}^{N-1} \frac{1}{\lambda_\alpha} \psi_{\alpha i} \psi_{\alpha j}, \quad (7) \]
where \( \overline{\vartheta} = \frac{1}{N} \sum_i \vartheta_i \) and \( (...) \) stands for the average over initial random phases. This analytic expression gives the dynamical correlation between any pair of oscillators just in terms of the properties of the underlying topology. Actually, the right term of Eq. \( 7 \) gives the pseudo-inverse of the Laplacian matrix, which acts like a Green Function. It could be used to detect the community structure in various complex networks.

One real world example with known community structure is the collaboration network of the Santa Fe Institute from ref. \cite{22}. The top row of Fig. 1 illustrates the largest component of the collaboration graph, which consists of 118 scientists denoted by nodes. The edge is drawn between a pair of scientists if they coauthored one or more papers. Obviously, the scientists of different interests group in different communities.

![Figure 1](image)
Using the same stochastic calculus method, we get to rewrite the Eq. 2 as:

$$d\theta_i/dt = -\sum_j A_{ij}(\theta_i - \theta_j) + \xi_i(t)$$

$$= -k_i(\theta_i - \overline{\theta}) + \xi_i(t).$$

(8)

Using the same stochastic calculus method, we get

$$\langle (\theta_i - \overline{\theta})^2 \rangle = \frac{1}{k_i}(1 - e^{-2k_i t}),$$

(9)

which indicates that the time scales of the dynamical relaxation of the observed phases are inversely proportional to the degrees of the corresponding oscillators. The different connectivities in the topology give rise to the corresponding order of the eigenvalues, which are illustrated in Fig. 2. The hubs with the large eigenvalues induce the fast relaxations, i.e., the evolutional pattern of synchronization starts first at the hubs and pervades almost the whole network in a hierarchical structure across the nodes with smaller degrees. And for large $t \gg 1/k_i$, we have got $\langle (\theta_i - \overline{\theta})^2 \rangle = 1/k_i$, which indicates that hubs are synchronized more closely by the collective dynamics.

For a given realization of networks, we use $R$ to denote the average variance of phase variables which characterizes the collective synchronized ability of the underlying network:

$$R = \frac{1}{N} \sum_i \langle (\theta_i - \overline{\theta})^2 \rangle,$$

(10)

The smaller the value of $R$, the smaller the fluctuation of network is, i.e., the more synchronized. Substituting the Eq. 7 and considering the orthonormality of $\psi_{ai}$, we have

$$R = \frac{1}{N} \sum_{\alpha=1}^{N-1} \frac{1}{\lambda_\alpha}.$$

(11)

Therefore, independent of details of unit dynamics, we can characterize the dynamical synchronizability of networked systems just in terms of the spectra of the Laplacian matrix, i.e., the purely topological properties. It is the same as the spirit of the MSF. However, in contrast to the MSF, the new measure Eq. 11 considers the spectrum in its entirety, not only the extremal eigenvalues. What this new measure emphasizes is on the fluctuation of the process of synchronization, rather than on rigorous bounds for the threshold of desynchronization. In the following, we use this new quantity about synchronization, $R$ to measure the synchronizable ability in small-world (SW) and scale-free (SF) networks, respectively.

We first consider the Watts-Strogatz model of the SW networks $[23]$, which is constructed by a rewiring process on a regular ring graph. The probability of rewiring connection is controlled by a parameter $p$, by tuning which the obtained network possesses both short average distance and high heterogeneity that nodes no longer have the same degrees, instead they follow a Poisson distribution. We define $S$ as the standard deviation of the degree distribution and $D$ as the average network distance, averaged over all pairs of nodes. The upper layer of Fig. 2(a) shows the dependence of $S$ and $D$ on the rewiring probability $p$ in the SW network. As $p$ increases, the variance $S$ increases, which implies that the degree distribution becomes more broad and heterogeneous. And as expect, the network distance $D$ decreases as the rewiring probability $p$ (or the heterogeneity of the degree distribution)
The results shown in Fig. 3 imply that the synchronizability of SW networks of size \( N = 1000 \), \( k = 8 \). The average network distance \( D \) and the new measure \( R \) decreases with the rewiring probability \( p \) in the same trend, while the standard deviation \( S \) of the degree distribution increases. In the lower layer of Fig. 3(a), \( S \) and \( D \) increase with the scaling exponent \( \gamma \), while the values of \( S \) are decreased. Since less \( R \) means more synchronizable, all the results in SW and SF networks indicate that the network would be more synchronizable as it becomes “smaller” and more heterogeneous. All plots are averaged over 100 realizations.

FIG. 3: (a) The synchronizability of SW networks of size \( N = 1000 \), \( k = 8 \). The average network distance \( D \) and the new measure \( R \) decreases with the rewiring probability \( p \) in the same trend, while the standard deviation \( S \) of the degree distribution increases. (b) The synchronizability of SF networks of size \( N = 1000 \), \( k_0 = 4 \). The values of \( D \) and \( R \) increase with the scaling exponent \( \gamma \), while the values of \( S \) are decreased. Since less \( R \) means more synchronizable, all the results in SW and SF networks indicate that the network would be more synchronizable as it becomes “smaller” and more heterogeneous. All plots are averaged over 100 realizations.

In summary, we have investigated the synchronization dynamics of the KM departing from the fully synchronized state. We have got the analytic expression of the dynamical correlation between pairs of oscillators that reveals the relation between the network dynamics and the underlying topology. Moreover, it also reveal the internal structure of networks that can be used as a new algorithm to detect community structures. Further, we have proposed a new measure about the synchronization in complex networks and scrutinize it in SW and SF networks. Our results indicate that the more heterogeneous and “smaller” the network is, the more closely it would be synchronized by the collective dynamics. We hope our study can provide new insights into the understanding of the role of synchronization between the network structure and function. We also expect it can provide new tools to detect community structure and to analyze the ubiquitous synchronization phenomenon.

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