A single-particle path integral for composite fermions and the renormalization of the effective mass

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(December 20, 2000)

To study composite fermions around an even denominator fraction, we adapt the phase space single-particle path integral technique for interacting electrons in zero magnetic field developed recently by D.S. Golubev and A.D. Zaikin, Phys. Rev. B 59, 9195 (1999). This path integral description gives an intuitive picture of composite fermion propagation very similar to the Caldeira-Leggett treatment of a particle interacting with an external environment. We use the new description to explain the origin of the famous cancellation between the self-energy and the vertex corrections in semi-classical transport measurements. The effective range of the cancellation is given by the size of the propagating particle, which for the Coulomb interaction scales with the temperature $T$ as $T^{-1/4} |\log T|^{-1}$ in the semi-classical limit. Using this scheme we find that the effective mass in the semi-classical limit for composite fermions in GaAs is approximately 6 times the bare mass.

I. INTRODUCTION

In this article we address the question of semi-classical propagation of an interacting system in a real space formulation. In particular we have in mind the composite fermion (CF) system which is seen in the fractional quantum Hall regime around filling factor $\frac{1}{2}$. The experimental efforts so far indicates that this system is to a high degree a classical system in the sense that it is very hard to observe any kind of interference effects.

In elementary textbooks the notion of semi-classical transport is sometimes introduced through localized wave packets, and these appear to be essential building blocks. Historically this has really not always been the case. For instance the derivation of the Boltzmann equation in the case of impurity scattering typically only uses individual quasi-particles with a given momentum i.e. delocalized plane waves. This is also the case for other weakly interacting systems like electron scattering on phonons at low temperatures. However, a Boltzmann equation for electrons scattering on hot phonons could not be derived using plane waves. Prange and Kadanoff solved that problem by shifting focus from individual quasi-particles to deformations of the Fermi surface. In real space this step corresponds to forming an electron wave packet that is localized in the direction of propagation. This localization in the direction of propagation causes the electron-phonon scattering to be very brief in time due to the huge difference between the Fermi velocity and the sound velocity. Recently Kim, Lee, and Wen showed that the Boltzmann equation for composite fermions only makes sense for smooth Fermi surface deformations. In real space the wave packet now also needs to be localized in the direction transverse to the direction of propagation, so the composite fermions is the first system where the textbook introduction actually is a proper description. For composite fermions the necessity of transverse confinement of the wave packet can be understood intuitively. The important interaction is mediated through a gauge field and the coupling is thus through the flux enclosed by an area. For a localized wave packet with a finite width propagating along a classical path the relevant area is the area swept by the wave packet. The width of the packet is thus a natural cut-off on the transverse gauge fluctuations. In this article we pursue some of the consequences of this idea.

We will study the real space propagation with the help of phase space path integrals. Originally, the Feynman path integral technique was developed for non-interacting systems. Feynman-Vernon and Caldeira-Legget expanded the idea to systems interacting with an environment. Recently Golubev and Zaikin extended the technique to cover the linear response regime for the reduced one-particle density matrix in a Coulomb interacting gas of electrons. This last step is not yet widely recognized. Nonetheless, in Sec. II we adapt it to the case of composite fermions. We summarize the main points of the path integral technique in subsection II C. In Sec. III the path integral technique is on a formal level compared with the Kubo formula description of transport and, more briefly, with the Landauer-Büttiker scattering approach to transport. In Sec. IV we discuss the typical semi-classical approximation scheme of saying that the two paths are almost identical, and the typical width between the two paths is calculated. In Sec. V the calculation of the effective mass of composite fermions is carried through in the semi-classical limit.

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We have found it necessary to introduce quite a few averages and short hand notations. As a reader's guide we list them here. \( \langle X \rangle \) is the full quantum and thermal average. A subscript 0 as in \( \langle X \rangle_0 \) means that the average is performed for the non-interacting electron gas. \( \langle X \rangle_{a,t} \) is the average over different Chern-Simon gauge field configurations, the precise definition is Eq. (13). \( \langle X \rangle = \langle X \rangle_{[0,t]} \) in Eq. (78) is the average in the time interval \([0,t]\) over different single-particle paths, where each path is given a weight that reflects its importance in a given experiment. It thus depends on what experiment is considered. \( \langle X \rangle = \langle X \rangle_{0,t} \) is the shorthand notation for the sum over single-particle paths including the weight due to the interactions. It is defined in Eq. (79). \( \langle X \rangle \) defined in Eq. (A4) is a short hand notation for the average being calculated in equilibrium. For instance \( \langle X \rangle_{a,t} \) in Eq. (11) is the equilibrium version of \( \langle X \rangle_{a,t} \).

II. THE DENSITY MATRIX

We consider a two dimensional electron gas in a static and homogeneous external magnetic field \( B_{ext} \) perpendicular to the plane. We write \( B_{ext} \) as a scalar since its direction is fixed, and likewise we consider the rotation of any vector potential in the two dimensional plane as a scalar. Furthermore, we also apply an external electrical potential \( \phi(r, t) \). The external potentials are written in four-vector form as \( A = (\phi, A) \). In this section the goal is to derive a path-integral description of the linear response of the reduced one-particle density matrix \( \rho \) to the potential \( \phi(r, t) \). For simplicity we will in this section neglect the influence of impurities, but it is straightforward to include them in the formalism as an extra external potential.

A. The formulation of the problem using electrons as the fundamental particles

We will assume the system is described by the Hamiltonian

\[
H_{el} = H_0[A] + H_{int}. \tag{1}
\]

Here

\[
H_0[A] = \int dr \left\{ \frac{1}{2m} \left[ \frac{\hbar^2}{i} \nabla + eA(r) \right]^2 - e\phi(r, t) - \mu \right\} \hat{\psi}(r) \tag{2}
\]

is the Hamiltonian for non-interacting particles in the 2D plane represented by the second quantized electron field operators \( \hat{\psi}(r) \) and \( \hat{\psi}^\dagger(r) \), while

\[
H_{int} = \frac{e^2}{2} \int dr \int d'r' \hat{\psi}^\dagger(r) \hat{\psi}^\dagger(r') \psi(r-r') \hat{\psi}(r') \hat{\psi}(r) + H_{eb} \tag{3}
\]

is the part due to electron-electron interaction. \( H_{eb} \) represents the interaction between the electrons and a fixed neutralizing positively charged background jellium. \( m \) is the mass of the electron, \( \mu \) is the chemical potential, \(-e\) is the electron charge, and \( e^2 v(r) \) is the Coulomb pair interaction between electrons. If the screening effect of the back gate is neglected \( v(r) = 1/(4\pi\epsilon|r|) \), where \( \epsilon \) is the permittivity of the medium surrounding the two dimensional electron gas.

All the physical one-particle quantities of the system, like the current and the density, can be expressed in terms of the reduced one particle density matrix

\[
\rho_c(r, r'; t) = \langle \hat{\psi}^\dagger(r', t) \hat{\psi}(r, t) \rangle \tag{4}
\]

Here \( \langle X \rangle \) denotes the full quantum mechanical and thermal average of \( X \). The subscript \( c \) stands for canonical, because upon a Fourier transformation the annihilation and creation operators correspond to states labeled by the canonical momentum. This labeling is not gauge invariant so neither is \( \rho_c(r, r') \), but it has some nice formal properties we will make use of in the following. Final physical expectation values we prefer to phrase in terms of the gauge invariant density matrix

\[
\rho(r, r') = \rho_c(r, r') e^{i \frac{\hbar}{\epsilon} \int_{r'}^{r} dr'' \cdot A(r'')} \tag{5}
\]

where the path of integration is chosen to be along the straight line from \( r' \) to \( r \). In general different choices of the integration path lead to different values of \( \rho \), but we are mainly interested in local probes like the density and the
convenient to introduce variables for the average $R = \frac{1}{2}(r + r')$, and for the difference $\Delta r = r - r'$. We use a new symbol for the density matrix expressed in these variables 

$$w(R, \Delta r) = \rho \left( R + \frac{\Delta r}{2}, R - \frac{\Delta r}{2} \right).$$  \tag{6}

The Fourier transform with respect to the difference variable is the corresponding gauge invariant Wigner distribution function 

$$w(R, p) = \int d\Delta r \, e^{-ip \cdot \Delta r} w(R, \Delta r).$$  \tag{7}

In terms of the gauge invariant Wigner distribution function the current density at $R$ is given by 

$$j(R) = -e \int \frac{dP}{(2\pi\hbar)^2} \, \rho \, w(R, p).$$  \tag{8}

Notice the momentum in the Wigner distribution $p$ in the classical local limit is the kinetic momentum.

The thermal and quantum average of $\rho_c$ and $\rho$ can be calculated with the help of a path integral over Grassmann fields $\psi_{el}$ and $\bar{\psi}_{el}$. $\psi_{el}$ corresponds to the operator $\hat{\psi}^\dagger$ and they should not be confused. Likewise, $\bar{\psi}_{el}$ corresponds to $\bar{\psi}$. Using this representation we obtain 

$$\rho(r, r'; t) = e^{i\bar{S}} \int \frac{d\sigma}{e^{i\bar{S}}} \, A(\sigma') \int D\bar{\psi}_{el} \int D\psi_{el} \psi_{el}(r', t) \psi_{el}(r, t) \exp \left( \frac{i}{\hbar} \bar{S}[\bar{\psi}_{el}, \psi_{el}; t] \right),$$  \tag{9}

where the action $S$ is given by 

$$S[\bar{\psi}_{el}, \psi_{el}; t] = \int_{C_t} d\sigma \int dr \, \bar{\psi}_{el}(r, \sigma) i\hbar \frac{\partial}{\partial \sigma} \psi_{el}(r, \sigma) - \int_{C_t} d\sigma H_{el}[\bar{\psi}_{el}, \psi_{el}].$$  \tag{10}

The contour $C_t$ runs from minus infinity up to the observation time $t$ and back again to minus infinity. The contour time $\sigma = (s, \mu)$, where $s$ is a real time and $\mu = \pm$ is a contour index that takes the value $+$ on the forward part of the contour and $-$ on the backward in time part of the contour, see Fig. 1. In the prefactor $\bar{\psi}_{el}(r', t) \psi_{el}(r, t)$ the creation Grassmann field $\bar{\psi}_{el}(r', t)$ has to come later than the annihilation Grassmann field $\psi_{el}(r, t)$ on the contour $C_t$, but whether both fields are on the forward or backward parts of the contour $C_t$ or $\psi_{el}(r, t)$ is on the forward part, while $\bar{\psi}_{el}(r', t)$ is on the backward in time part of $C_t$ is immaterial.

### B. The Chern-Simon field dependent density matrix

The first step towards obtaining an effective single-particle path integral description is to introduce fluctuating fields as extra integration variables such that the action becomes quadratic in the Grassmann fields. The electron physics is
then reduced to that of non-interacting electrons in a fluctuating field background. In the case of zero magnetic field the relevant transformation is a Hubbard-Stratonovich transformation that introduces an effective electric potential at each contour point. In the case of composite fermions the relevant transformation is a Chern-Simons transformation that introduces a Chern-Simons gauge field $a = (a_0, a)$ at each contour point, and that also performs a singular gauge transformation on the Grassmann fields attaching two magnetic flux quanta $h/e$ to each electron. The transformed Grassmann fields we denote by $\bar{\psi}$ and $\psi$, not to be confused with $\hat{\psi}^d$, $\bar{\psi}^d$, and $\psi^d$.

The action transforms into the standard action for composite fermions

$$S_{\text{CF}}[\bar{\psi}, \psi, a, A_{\text{eff}} - a; t] = S_{\text{CS}}[a; t] + e \int_{C_t} d\sigma \int dr a_0(r, \sigma) n + e \int_{C_t} d\sigma \int dr \bar{\psi}(r, \sigma) i\hbar \frac{\partial}{\partial \sigma} \psi(r, \sigma) - \int_{C_t} d\sigma H_0[A_{\text{eff}} - a].$$

(11)

Here $S_{\text{CS}}[a; t]$ is the Chern-Simons action

$$S_{\text{CS}}[a; t] = \int_{C_t} d\sigma \int dr \left( \frac{e^2}{4\pi \hbar} a_0(\sigma) \nabla \times a(\sigma) + \frac{e^2}{8\pi \hbar} a(\sigma) \times \frac{\partial a}{\partial \sigma} \right) - \frac{e^4}{2(4\pi \hbar)^2} \int_{C_t} d\sigma \int dr \int dr' \left( \nabla \times a(r, \sigma) \right) \psi(r - r') \left[ \nabla \times a(r', \sigma) \right].$$

(12)

The average mean-field magnetic field $2nh/e$ has been subtracted from both the externally magnetic field $\nabla \times A$ and the Chern-Simons gauge field. The remaining fields are denoted by $A_{\text{eff}} = (0, A_{\text{eff}}) = (0, A - (\nabla \times)^{-1} 2\hbar n)$ and $a = (a_0, a)$ respectively. For notational consistency with the subsection on linear response we have chosen to have a vanishing external potential, $\phi = 0$. If non-zero it would be added to $A_{\text{eff}}$. The last term in $S_{\text{CS}}$ originates from the interaction term $H_{\text{int}}$, Eq. (3), after applying the identity $\nabla \times a(r, \sigma) = 2\hbar \left( \bar{\psi}a_\dagger(r, \sigma) \psi_\dagger(r, \sigma) - n \right)$. For later use we notice that in the terms that couple the Chern-Simons gauge field $a$ to the Chern-Simons fields $\bar{\psi}$ and $\psi$, $a$ only enters in the combination $A_{\text{eff}} - a$.

In the first line of Eq. (11) the temporal integration contour can be extended to the Keldysh contour $C_\infty$, since without a Hamiltonian there is no real time development and consequently the integral over the $a$-field on the Keldysh contour from the observation time $t$ to infinity is one. $C_\infty$ is used below for technical reasons whenever the evolution of various quantities like effective actions and density matrices in a fixed background gauge field history $a$ is discussed. The contour $C_t$ is used when the connection to a physical density matrix as in Eq. (15) is wanted.

The prefactor in Eq. (11) transforms as

$$\bar{\psi}\dagger(r', t) \psi\dagger(r, t) e^{i\int_0^t dr'' A(r'')} \rightarrow \bar{\psi}(r', t) \psi(r, t) e^{i\int_0^t dr'' \left[ A_{\text{eff}}(r'', t) - \tilde{a}(r'', t) \right]},$$

(13)

where

$$\tilde{a}(r'', t) = \frac{1}{2} \left[ a(r'', t, +) + a(r'', t, -) \right].$$

(14)

is the average of the Chern-Simons gauge field on the two parts of $C_t$. This average value is chosen for notational consistency. It is allowed since the $a$-field is continuous on the contour $C_t$, so it does not matter whether the Chern-Simons field at the turning point $t$ is evaluated on the forward or on the backward part of the contour. The total expression for the density matrix in terms of the transformed fields is

$$\rho(r, r'; t) = \left[ \int D\bar{\psi} \int D\psi e^{i\int S_{\text{CF}}} \right]^{-1} \int D\bar{\psi} \int D\psi \bar{\psi}(r, t) \psi(r, t) e^{i\int S_{\text{CF}} + i\int_0^t dr'' \left[ A_{\text{eff}}(r'') - \tilde{a}(r'', t) \right]}.$$

(15)

The exponent is quadratic in the Grassmann CF variables $\bar{\psi}$ and $\psi$ and thus describes non-interacting particles in a gauge field history $A_{\text{eff}} - a$. A typical gauge field history is not physically realizable since the gauge field $a$ at a given time $s$ is different on the forward and backwards parts of the contour $C_t$, $a(s, +) \neq a(s, -)$. Despite this we will in analogy with the usual definition of a density matrix Eq. (4) and following Golubev and Zaikin define the Chern-Simon field dependent density matrix

$$\rho_{ac}(A_{\text{eff}} - a; r, r', t) = \left[ \int D\bar{\psi} \int D\psi \exp \left( \frac{i}{\hbar} S_{\text{CF}} \right) \right]^{-1} \int D\bar{\psi} \int D\psi \bar{\psi}(r, t) \psi(r, t) \exp \left( \frac{i}{\hbar} S_{\text{CF}} \right).$$

(16)

Notice that the right-hand side only depends on the gauge fields in the combination $A_{\text{eff}} - a$, since the contribution from $S_{\text{CS}}[a; t]$ to the action $S_{\text{CF}}$ cancels out. The subscript $a$ on $\rho_{ac}$ is to remind us that $\rho_{ac}$ depends on the field.
history, while the subscript $c$ stands for canonical in analogy with $\rho_c$, see Eq. (14). The physical density matrix is recovered from $\rho_{ac}(A_{\text{eff}} - a; \mathbf{r}, \mathbf{r}', t)$ by averaging over the different $a$-field histories

$$
\rho(\mathbf{r}, \mathbf{r}', t) = \left\langle \rho_{ac}(A_{\text{eff}} - a; \mathbf{r}, \mathbf{r}', t) \exp \left\{ \frac{i}{\hbar} \int_{s'}^{s} ds'' \cdot [A_{\text{eff}}(s'') - \bar{a}(s'', t)] \right\} \right\rangle_{a,t}.
$$

(17)

In the average

$$
\langle X \rangle_{a,t} = \frac{\int D\alpha X \exp \left(\frac{i}{\hbar} S_{\text{eff}}[\alpha; t] \right)}{\int D\alpha \exp \left(\frac{i}{\hbar} S_{\text{eff}}[\alpha; t] \right)},
$$

(18)

the weight

$$
\exp \left(\frac{i}{\hbar} S_{\text{eff}}[\alpha; t] \right) = \int D\psi \int D\bar{\psi} \exp \left(\frac{i}{\hbar} S_{\text{CF}}[\bar{\psi}, \psi, a, A_{\text{eff}} - a; t] \right).
$$

(19)

is the denominator in Eq. (18). The effective action $S_{\text{eff}}[\alpha; t]$ for the $a$ field also depends on the external gauge field $A_{\text{eff}}$, but this dependence has been suppressed in the notation, since it is not needed in the following. Eq. (17) is an important formula, since it establishes the connection between the density matrix $\rho_{ac}$ depending on a given history of the gauge field and the physically accessible density matrix $\rho$.

### C. The effective action for the $a$-field

Introducing the $a$-dependent current $j_a(A_{\text{eff}} - a; \mathbf{r}, s)$ at time $s$

$$
j_a(A_{\text{eff}} - a; \mathbf{r}, s) = -\frac{e}{2m} \left[ \frac{\hbar}{i} \frac{\partial}{\partial \mathbf{r}} + e A_{\text{eff}}(\mathbf{r}) - e \bar{a}(\mathbf{r}) - \frac{\hbar}{i} \frac{\partial}{\partial \mathbf{r}'} + e A_{\text{eff}}(\mathbf{r}') - e \bar{a}(\mathbf{r}') \right] \rho_{ac}(\mathbf{r}, \mathbf{r}'; s) \bigg|_{\mathbf{r}' = \mathbf{r}},
$$

(20)

it is shown in appendix A that the effective single-particle action $S_{\text{eff}}[\alpha; t]$ is given by

$$
S_{\text{eff}}[\alpha; t] = S_{\text{CS}}[\alpha; t] - \int_{-\infty}^{t} ds \int d\mathbf{r} \left[ e \Delta a_0(\mathbf{r}, s) (\rho_{ac}(\mathbf{r}, s) - n) + \Delta a(\mathbf{r}, s) \cdot j_a(\mathbf{r}, s) \right],
$$

(21)

where

$$
\Delta a_0(\mathbf{r}, s) = a_0(\mathbf{r}, s, +) - a_0(\mathbf{r}, s, -)
$$

(22)

is the difference of the Chern-Simons gauge field on the two parts of $C_t$. $S_{\text{CS}}[\alpha; t]$ is the usual Chern-Simons action, see Eq. (12). In the notation the explicit dependence of the density matrix $\rho_{ac}$ and the current $j_a$ on the gauge field $A_{\text{eff}} - a$ has been suppressed. In equilibrium, which is denoted by the superscript "eq", we have to second order in the $a$-field

$$
S_{\text{eff}}^{\text{eq}}[\alpha; t] \approx S_{\text{CS}}[\alpha; t] - \sum_{\alpha, \beta} \int_{-\infty}^{t} ds' \int_{-\infty}^{s'} ds'' \int d\mathbf{r}' \int d\mathbf{r}'' \Delta a_0(\mathbf{r}', s') R_{\alpha \beta}(\mathbf{r}', \mathbf{r}''; s', s'') \bar{a}_\beta(\mathbf{r}'', s'')
$$

$$
+ \frac{i}{2} \sum_{\alpha, \beta} \int_{-\infty}^{t} ds' \int_{-\infty}^{s'} ds'' \int d\mathbf{r}' \int d\mathbf{r}'' \Delta a_0(\mathbf{r}', s') F_{\alpha \beta}(\mathbf{r}', \mathbf{r}''; s', s'') \Delta a_\beta(\mathbf{r}'', s'').
$$

(23)

The summation variables $\alpha, \beta \in \{0, x, y\}$ run over the single time-like and the two spatial indices e.g. $a_0 = (a_0, a_x, a_y)$. The terms linear in $a$ vanishes, since $\rho_{ac}(A_{\text{eff}}, \mathbf{r}, s) = n$ and $j_a(A_{\text{eff}}; \mathbf{r}, s) = 0$. The retarded $3 \times 3$ response matrix $\bar{\mathbf{R}}_{\alpha \beta}$ is given by the usual Kubo formula apart from an extra minus in the part that mixes the density with the currents.

$$
\bar{\mathbf{R}}(\mathbf{r}', s'; \mathbf{r}'', s'') \equiv -\frac{i}{\hbar} \left( \langle [-\rho_{ac}(\mathbf{r}', s'), -e \rho_{ac}(\mathbf{r}'', s'')] \rangle_0 - \langle [-e \rho_{ac}(\mathbf{r}', s'), j_a(\mathbf{r}'', s'')] \rangle_0 \right) \Theta(s' - s'')
$$

$$
+ \left( \begin{array}{cc} 0 & 0 \\ \frac{e^2}{m} \rho_{ac}(\mathbf{r}', s') \delta(\mathbf{r}' - \mathbf{r}'') \delta(s' - s'') & 0 \end{array} \right)
$$

(24)
and the so-called fluctuation kernel \( \tilde{F} \) by

\[
\tilde{F}(r_1, s_1; r_2, s_2) = \frac{1}{2\hbar} \left( \{ e\rho_{ac}(r', s'), -e\rho_{ac}(r'', s'') \}_0 - \{ e\rho_{ac}(r', s'), j_\alpha(r'', s'') \}_0 \right).
\]

The dependence of the density matrix \( \rho_{ac} \) and the current \( j_\alpha \) on the gauge field \( A_{\text{eff}} - a \) has been suppressed in the notation. \( \langle X \rangle_0 \) means the average over the non interacting electron gas without the Chern-Simons gauge field \( a \), and \( \Theta \) is the step function. In equilibrium the retarded response matrix \( \tilde{R} \) is related to the fluctuation matrix \( \tilde{F} \) through the fluctuation dissipation theorem

\[
F_{\alpha\beta}(r_1, r_2; \omega) = \frac{i}{2} \coth \left( \frac{\hbar \omega}{2k_B T} \right) \left[ \Theta_{\alpha\beta}(r_1, r_2; \omega) - \Theta_{\beta\alpha}(r_2, r_1; -\omega) \right],
\]

where \( \omega \) is the Fourier frequency corresponding to the time difference \( s' - s'' \). It is used that in equilibrium \( \tilde{R} \) and \( \tilde{F} \) only depends on \( s' - s'' \). In terms of the \( 2 \times 2 \) conductivity tensor \( \bar{\sigma}(q, \omega) \) the response matrix \( \tilde{R} \) is written as

\[
\tilde{R}(q, \omega) = i \left( \frac{\mathbf{q} \cdot \mathbf{\bar{\sigma}}(q, \omega)}{-\mathbf{\bar{\sigma}}(q, \omega) \cdot \mathbf{q}} - \frac{\mathbf{q} \cdot \mathbf{\bar{\sigma}}(q, \omega)}{\omega \mathbf{\bar{\sigma}}(q, \omega)} \right).
\]

Below we are mainly going to need the retarded gauge field propagator \( D_{\alpha\beta} \) defined by

\[
D_{\alpha\beta}(r', s' \mid r'', s'') \equiv -\frac{i}{\hbar} \mathcal{E} \langle \bar{a}_\alpha(r', s') \Delta a_\beta(r'', s'') \rangle_{eq} = -\frac{i}{\hbar} \Theta (s' - s'') \mathcal{E} \langle [a_\alpha(r', s'), a_\beta(r'', s'')] \rangle_0
\]

and the correlation function \( C_{\alpha\beta} \) for fluctuations of the field

\[
C_{\alpha\beta}(r', s' \mid r'', s'') \equiv \frac{\mathcal{E}^2}{\hbar} \langle \bar{a}_\alpha(r', s') {\bar{a}}_\beta(r'', s'') \rangle_{eq} = \frac{\mathcal{E}^2}{\hbar} \frac{1}{2} \langle [a_\alpha(r', s'), a_\beta(r'', s'')] \rangle_0,
\]

while the auto-correlation function for \( \Delta a_\alpha \) vanishes:

\[
\langle \Delta a_\alpha(r', s') \Delta a_\beta(r'', s'') \rangle_{eq} = 0.
\]

The average

\[
\langle X \rangle_{eq} = \frac{\int D_{\alpha} X \exp \left( \frac{\mathcal{E} S_{\text{eq}}[a; t]}{\hbar} \right)}{\int D_{\alpha} \exp \left( \frac{\mathcal{E} S_{\text{eq}}[a; t]}{\hbar} \right)},
\]

is the average \( \langle X \rangle_{eq} \) of Eq. \( 18 \) calculated in equilibrium.

In the Coulomb gauge the most troublesome contributions arises from the low frequency limit of the so-called transverse part of the gauge field propagator \( D_{\perp\perp} \), which in Fourier space is given by

\[
D_{\perp\perp}(q, \omega) = \sum_{ijkl} \epsilon^{ik} \epsilon^{jl} \frac{q_i q_j}{q^2} D_{kl}(q, \omega).
\]

Here \( \epsilon^{xx} = \epsilon^{yy} = 0, \epsilon^{xy} = -\epsilon^{yx} = 1, \) and the indices \( i, j, k \) and \( l \) run over ‘x’ and ‘y’. In equilibrium in the limit \( \hbar \omega \ll q v_F \) the propagator \( D_{\perp\perp} \) is given by the standard approximation \cite{1}.

\[
D_{\perp\perp}(q, \omega) \approx 2\pi q \frac{\hbar}{v_F} \frac{1}{i \omega - \frac{\pi e^2 v(q) q^2}{(4\pi \hbar^2)^2 v_F}} = 2\pi q \frac{\hbar}{v_F} \frac{1}{i \omega - g_1 q^2 q + 1}.
\]

For the Coulomb interaction \( v(q) = \frac{\hbar}{2e q} \) so \( \eta = 1 \) and \( g_1 = \frac{1}{\hbar^2 e^2} 2\sqrt{\eta} \varepsilon_C \) in terms of the average Coulomb energy \( \varepsilon_C = \frac{\hbar^2}{2m} \sqrt{n} \). \( n \) is the density of the electron gas. Other values of \( 1 \leq \eta < 2 \) corresponds to a power law decay \( r^{-\eta} \).

The fluctuation correlation function \( \tilde{C} \) is given by

\[
\tilde{C}(1, 2) = \int d^3 \tilde{D}(1, 3) \tilde{F}(3, 4) \tilde{D}(2, 4),
\]
where 1,2,3 and 4 each are a short hand notation for a full set of variables \( \alpha, \mathbf{r}, s \). In equilibrium \( \bar{C} \) is determined from the fluctuation dissipation theorem \( [26] \)

\[
C_{\alpha\beta}(\mathbf{r}', \mathbf{r}''; \omega) = \frac{i}{2} \coth \left( \frac{\hbar \omega}{2k_B T} \right) \left[ D_{\alpha\beta}(\mathbf{r}', \mathbf{r}''; \omega) - D_{\beta\alpha}(\mathbf{r}'', \mathbf{r}'; -\omega) \right]
\]

(35)

In particular in the Coulomb gauge is the most singular contribution due to the transverse part of the propagator

\[
C_{\alpha\beta}(\mathbf{q}, \omega) \approx -\left( \delta_{\alpha\beta} - \frac{q_a q_{\beta}}{q^2} \right) \coth \left( \frac{\hbar \omega}{2k_B T} \right) \text{Im}D_{\perp\perp}(\mathbf{q}, \omega)
\]

(36)

D. The equation of motion for the gauge field dependent density matrix

In appendix \( A \) we have by brute force differentiation showed that \( \rho_{ac} \) fulfills the all important nonlinear equation of motion

\[
i\hbar \partial_t \rho_{ac} = (1 - \rho_{ac})H(t, +)\rho_{ac} - \rho_{ac}H(t, -)(1 - \rho_{ac}).
\]

(37)

This equation should be read as an operator identity, i.e. we have suppressed all arguments and the spatial integrals over the internal variables, 1 is a shorthand notation for the identity operator \( \delta(\mathbf{r}' - \mathbf{r}'') \), \( H(+) \) is the kernel in the quadratic action for the Grassmann CF variables on the forward part of the contour, i.e.

\[
H(\mathbf{r}', \mathbf{r}''; t, +) = \delta(\mathbf{r}' - \mathbf{r}'') \left\{ -e\phi(\mathbf{r}'', t) + e a_0(\mathbf{r}'', t, +) + \frac{1}{2m} \left[ \frac{\hbar^2}{i} \nabla'' + e A_{\text{eff}}(\mathbf{r}'') - e a(\mathbf{r}'', t, +) \right]^2 \right\}.
\]

(38)

\( a(+) = [a_0(+), a(+)] \) is the gauge field on the forward contour. Likewise is \( H(-) \) the kernel on the backward in time part of the contour. If the gauge field is identical on the forward and backward sections of the contour Eq. \( [27] \) specializes to the usual equation of motion for the density matrix \( i\hbar \partial_t \rho = [H, \rho] \). The extra complication in the general case is the term quadratic in the density matrix: \( -\rho_{ac}[H(+) - H(-)]\rho_{ac} \). However we restrict ourselves to the linear response regime.

E. Formulation of linear response

We are interested in the response of the system to an external driving field \( \phi \) and in particular in the change in the density matrix: \( \delta \rho(\mathbf{r}, \mathbf{r}'; t) = \rho(\mathbf{r}, \mathbf{r}'; t) - \rho_{\text{eq}}(\mathbf{r}, \mathbf{r}'; t) \). The superscript "eq" still denotes equilibrium where \( \phi = 0 \). In general both the gauge field dependent density matrix \( \rho_{ac}(A_{\text{eff}} - a; \mathbf{r}, \mathbf{r}', t) \) and the effective action for the Chern-Simons field \( S_{\text{eff}}[a; t] \) depends on \( \phi \), giving rise to two different contributions to \( \delta \rho(\mathbf{r}, \mathbf{r}', t) \) — compare with Eqs. \( [17] \) and \( [18] \). The contribution coming from the variation of \( S_{\text{eff}}[a; t] \) can be accounted for by a change of variable from the externally applied field \( \phi \) to the total gauge field felt be the composite fermions

\[
A_{\text{tot}}(a; t) = (\phi(t), 0) + A_{\text{ind}}(a; t).
\]

(39)

Here \( A_{\text{ind}} \) is the self-consistently induced gauge field due to the change in the current and the density when \( \phi \) is applied. Technically it is defined by the demand that

\[
S_{\text{eff}}[a - A_{\text{ind}}(a); t] = S_{\text{eff}}^{\text{eq}}[a; t].
\]

(40)

From this definition it is not immediately clear that the induced field is the same on the forward and backward part of the contour and thus can be interpreted as a physical field. However a careful inspection of Eqs. \( [12] \) and \( [21] \) shows this is the case. Furthermore it is seen that the induced electromagnetic fields corresponding to \( A_{\text{ind}} \) are

\[
B_{\text{ind}}(A_{\text{eff}} - a; \mathbf{r}, t) = -\frac{4\pi \hbar}{e} \delta \rho_{ac}(A_{\text{eff}} - a; \mathbf{r}, t)
\]

(41)

\[
E_{\text{ind}}(A_{\text{eff}} - a; \mathbf{r}, t) = \frac{4\pi \hbar}{e^2} \delta v(\mathbf{r} - \mathbf{r}') \delta \rho_{ac}(A_{\text{eff}} - a; \mathbf{r}, t) + e \nabla \int d\mathbf{r}' v(\mathbf{r} - \mathbf{r}') \delta \rho_{ac}(A_{\text{eff}} - a; \mathbf{r}, t),
\]

(42)
where
\[ \delta \rho_{ac}(A_{\text{eff}} - a; r, r', t) = \rho_{ac}(A_{\text{eff}} + A_{\text{tot}} - a; r, r', t) e^{\frac{i}{\hbar} \int_r^{r'} d\rho'' \cdot A_{\text{tot}}} - \rho_{ac}^{eq}(A_{\text{eff}} - a; r, r', t) \]  
\[ \delta j_{a}(A_{\text{eff}} - a; r, t) = j_{a}(A_{\text{eff}} + A_{\text{tot}} - a; r, t) - j_{a}^{eq}(A_{\text{eff}} - a; r, t) \]  
(43)
(44)
are the changes in the density and the current respectively, induced by the total field \( A_{\text{tot}} \). The phase factor \( e^{\frac{i}{\hbar} \int_r^{r'} d\rho'' \cdot A_{\text{tot}}} \) has been introduced in order to let \( \delta \rho_{ac} \) refer to the equilibrium state. For instance \( \delta j_{a} \), results, when \( \delta \rho_{ac} \) is substituted for \( \rho_{ac}^{eq} \) in the expression for the equilibrium current Eq. (21). Eqs. (43) and (44) are what is expected on physical grounds. There is a magnetic field associated with a change in the density of composite fermions and thus the amount of attached flux. There is also a Faraday electric field associated with a current of attached flux. There is also a Faraday electric field associated with a current of attached flux. The last term in Eq. (12) is due to the electron-electron interaction.

The Jacobian associated with the variable change from \( a \) to \( a - A_{\text{ind}}(a) \) in Eqs. (17) and (18) is the identity 1, since both the density and the current, and thus \( A_{\text{ind}}(a) \), are retarded functions of the gauge field \( A_{\text{eff}} - a \). Consequently the change in the physical density matrix
\[ \delta \rho(r, r'; t) = \left( \delta \rho_{ac}(A_{\text{eff}} - a; r, r', t) \right) \exp \left\{ \frac{i}{\hbar} \int_r^{r'} d\rho'' \cdot [A_{\text{eff}}(\rho'') - \bar{a}(\rho'', t)] \right\}^{eq} \]
(45)
is the average of the change in the \( a \)-dependent density matrix over the different \( a \)-field histories with the weight \( \exp \left\{ \frac{i}{\hbar} S_{\text{eff}}(a) \right\} \) pertaining to equilibrium.

Golubev and Zaikin did not perform this change of variables, since the difference between the external and the total field is not of great bearing in their application of the formalism to weak localization. The second term in their formula (46) which they neglect is exactly what is covered by this change of variable. For composite fermions such a neglect is not permissible, since the extra term originating from the variation of \( S_{\text{eff}}(a; t) \) gives rise to for instance the Hall conductance. In our formulation the Hall effect arises from the electric field induced by the current — the first term in Eq. (12). The mechanism is completely analogous to the treatment in the case of the Boltzmann equation.

Notice that so far no linearization in the applied field has been performed and everything is exact.

The linearized equation of motion for \( \delta \rho_{ac}(A_{\text{eff}} - a; r, r', t) \) is obtained by linearizing Eq. (43) in the total external field
\[ i\hbar \partial_t \delta \rho_{ac}(r, r'; t) \simeq \int d\rho'' H^+(r, r'', t) \delta \rho_{ac}(r'', r', t) - \int d\rho'' \delta \rho_{ac}(r, r'', t) H^-(r'', r', t) + i\hbar D(r, r', t), \]
(46)
where the explicit dependence on the gauge fields \( A_{\text{eff}} - a \) and \( A_{\text{tot}} \) is suppressed in the notation. New one-particle Hamiltonians
\[ H^+ = \frac{H(+)}{2} + \left( \frac{1}{2} - \rho_{ac} \right) [H(+) - H(-)] \]  
\[ H^- = \frac{H(+)}{2} - \left( \frac{1}{2} - \rho_{ac} \right) [H(+) - H(-)] \]  
(47)
(48)
have been introduced in analogy with the equation of motion for the physical one-particle density matrix \( i\hbar \partial_t \rho = [H, \rho] \). \( H^+ \neq H^- \) because the Chern-Simons field is different on the forward and backward contours. The inhomogeneous stimulus \( D = D_1 + D_2 \) is split into two parts, each of which is gauge invariant in the total applied field \( A_{\text{tot}} \). \( D_2 \) contains all the dependence upon the difference gauge field \( \Delta a \), while \( D_1 \) is independent of \( \Delta a \). In Appendix B the explicit expressions for \( D_1 \) and \( D_2 \) are listed. In linear response \( D_1 \) and \( D_2 \) contribute additively to the physical density matrix. The response due to \( D_2 \) we will neglect, since it vanishes in the approximation where \( D \) is replaced by its average value \( \langle D \rangle^{eq}_{a,t} \). Below we do not go beyond this approximation.

In the real space representation used above \( D \) is not gauge invariant in \( A_{\text{eff}}(r) - \bar{a} \), so we find it convenient to change representation to the gauge invariant Wigner function set of variables. In analogy with Eqs. (4), (5) and (6)
\[ D(r, r', t) = \int \frac{dp}{(2\pi\hbar)^2} e^{\frac{i}{\hbar} \int_r^{r'} dp'' [p - eA_{\text{eff}}(p'') + e\bar{a}(p'', t)]} D \left[ \frac{1}{2} (r + r'), p \right]. \]
(49)
When the generalized density matrix \( \rho_{ac}(r, r') \) is peaked around \( r = r' \) on a length scale which is much smaller than the typical scale of variation of the total driving field, the gauge invariant \( D_1 \) and thus \( D \) is approximately equal to the standard Boltzmann equation driving term.
\[ D(R, p) \approx -e \left( \frac{1}{m} p \times \dot{z} \right) + E_{\text{ext}} \cdot \frac{\partial}{\partial p} w_a(A_{\text{eff}} - a; R, p). \] (50)

In the presence of the gauge field \( A_{\text{eff}} - a \) the Wigner distribution function \( w_a(A_{\text{eff}} - a; R, p, t) \) is defined in analogy with Eqs. (3) and (7)

\[ w_a(A_{\text{eff}} - a; R, p, t) = \int d\Delta r e^{-i/\hbar \int_{-\infty}^{t} dr_1 \int d\Delta r \int d\Delta r_2 U^+(r, t; r_1, t_1) D(r_1, r_2, t_1)U_l(r_2, t, r', t). \] (51)

In the semi-classical limit it corresponds to a field dependent phase space distribution function. We will not consider effects beyond the approximation where it is the Fermi distribution function.

**F. The single-particle path integral in linear response**

In this subsection we obtain in terms of a single-particle path integral the formal solution of \( \delta \rho \) in the linear response i.e. Eq. (19). The first step is to notice that the solution to Eq. (19) can be written in terms of evolution operators

\[ U^\mu(t, t') = \hat{T}^\mu \exp \left[ -i \frac{\hbar}{\mu} \int t'' H^\mu(t'') \right] , \quad \mu = \pm, \] (53)

where \( \hat{T}^+ \) and \( \hat{T}^- \) are the time and anti-time ordering operators respectively.

The corresponding change in the physical Wigner function is

\[ \delta w(R, p, t) = \int_{-\infty}^{t} dt_1 \int dR_1 \int \frac{dp_1}{(2\pi \hbar)^2} \langle J(R, p, t; R_1, p_1, t_1)D(R_1, p_1, t_1) \rangle_{\text{eq}}, \] (54)

where

\[ J(R, p, t; R_1, p_1, t_1) = \int d\Delta r \int d\Delta r_1 \int dr_1' e^{-i/\hbar \int_{-\infty}^{t} dr_1 \int d\Delta r \int d\Delta r_2 U^+(r, t; r_1, t_1) D(r_1, r_2, t_1)U_l(r_2, t, r', t). \]

is the propagator of the stimulus D in the Wigner representation. In the real space representation

\[ J(R + \Delta r, R - \Delta r, R + \Delta r_i, R - \Delta r_i, t_1) = U^+(R + \Delta r, t; R + \Delta r_i, t_1)U^-(R - \Delta r_i, t_1; R - \Delta r_i, t), \] (56)

The evolution operators \( U^+ \) and \( U^- \) can be represented as usual single-particle phase space path integrals

\[ U^\mu(r_1, r_2; t_1) = \int_{r(t_1)=r_1}^{r(t_2)=r_2} Dr \int Dp \exp \left\{ i \frac{\hbar}{\mu} \int_{t_1}^{t_2} dt \left[ (p - eA_{\text{eff}} + \bar{e} \cdot \bar{r}) - H^\mu(r(s), p(s), s)] \right. \right\}, \] (57)

where \( \mu = \pm \). The Hamiltonian kernels are given by

\[ H^+(r, p, s) = \frac{p^2}{2m} + \frac{e^2M^2(r, s)}{8m} - e\phi(r) + e\bar{a}_0(r, s) \]

\[ + e \left[ \frac{1}{2} - w_a(r, p(s^+), s) \right] \left( \Delta a_0 - \frac{p}{m} \cdot \Delta a(r, s) \right) + \frac{eM^2}{2m} \cdot \nabla w_a(a; r, p(s^+), s) \] (58)

\[ H^-(r, p, s) = \frac{p^2}{2m} + \frac{e^2M^2(r, s)}{8m} - e\phi(r) + e\bar{a}_0(r, s) \]

\[ - e \left[ \Delta a_0 - \frac{p}{m} \cdot \Delta a(r, s) \right] \left[ \frac{1}{2} - w_a(r, p(s^+), s) \right] - \frac{eM^2}{2m} \cdot \nabla w_a(a; r, p(s^+), s). \] (59)
We will calculate the average of the propagator \( J \). This procedure is gauge invariant. First notice that above we argued that the stimulus \( s \) up to \((p, t)\) to \((R_i + \Delta \sigma, p_i)\) at contour time \( \sigma = (t, +) \). Here it runs along the straight line path from \( R_i + \Delta \sigma \) to \( R_i - \Delta \sigma \) with the constant momentum \( p_i \) and at constant \( \sigma \). From here it then runs back along \([r(s, -), p(s, -)]\) to \((R_i - \Delta \sigma, p_i)\) at contour time \( \sigma = (t_i, -) \). Finally it runs along a straight line path back to \((R_i + \Delta \sigma, p_i)\) with the constant momentum \( p_i \) and at a constant time \( t_i \).

Here \( s^+ \) on both the forward and backward in time contour is a bit closer to the observation time \( t \) than \( s \). The only place where this small time shift is important is in the Wigner distribution function \( w_a(r, p(s^+), s) \), which is likely to be a rapidly varying function of the momentum \( p \) at the Fermi surface. Note also that \( p \) in the semi-classical limit is the kinetic momentum in the gauge field \( A_{eff} - \vec{a} \).

Using the path integral representation Eq. (57) for \( U^+ \) and \( U^- \) we finally arrive at the following contour path integral representation for the propagator \( J \):

\[
J(R, p; t; R_i, p_i, t_i) = \int d\Delta r \int d\Delta r_i \int dr_i \int D\sigma(+) \int Dp(+) \int D\sigma(-) \int Dp(-) \exp \left( \frac{i}{\hbar} S_0 + \frac{i}{\hbar} \Delta S \right),
\]

where

\[
S_0 = \int d\sigma \cdot \{ p(\sigma) - eA_{eff}[r(\sigma), s] \} - \int_{t_i}^{t} ds H_0[r(s, +), p(s, +), s] + \int_{t_i}^{t} ds H_0[r(s, -), p(s, -), s]
\]

is the action for non-interacting particles in an external gauge field \( A_{eff} \). The closed contour is depicted in Fig. 2. All the contributions related to the Chern-Simon gauge field are gathered in the second term in the exponent of Eq. (60):

\[
\Delta S = +e \int \{ \Delta a[r(\sigma), s] - d\sigma a_0[r(\sigma), s] \}
\]

\[
+e \int_{t_i}^{t} ds \left\{ \frac{1}{2} - w_a[r(s, +), p(s^+, +), s] \right\} \left\{ \frac{p(s^+, +)}{m} \cdot \Delta a[r(s, +), s] - \Delta a_0[r(s, +), s] \right\}
\]

\[
+e \int_{t_i}^{t} ds \left\{ \frac{1}{2} - w_a[r(s, -), p(s^-, -), s] \right\} \left\{ \frac{p(s^-, -)}{m} \cdot \Delta a[r(s, -), s] - \Delta a_0[r(s, -), s] \right\}
\]

\[
- \frac{e^2}{8m} \int_{t_i}^{t} ds \left\{ \Delta a[r(s, +), s]^2 - \Delta a[r(s, -), s]^2 \right\}
\]

Below we only consider the approximation obtained by factorizing Eq. (54)

\[
\delta w(R, p, t) = \int_{-\infty}^{t} dt_i \int dR_i \int \frac{dp_i}{(2\pi \hbar)^2} \langle J(R, p; t; R_i, p_i, t_i) \rangle^{eq}_{a,t} (D(R_i, p_i, t_i))^{eq}_{a,t}.
\]

Above we argued that the stimulus \( \langle D \rangle^{eq}_{a,t} \) in the semi-classical limit is the standard Boltzmann driving term, Eq. (50). We will calculate the average of the propagator \( J \) perturbatively to second order in the Chern-Simons gauge field \( a \). This procedure is gauge invariant. First notice that \( \langle \Delta S \rangle^{eq}_{a,t} = 0 \), since \( \langle \Delta \bar{a} \rangle^{eq}_{a,t} = 0 \) and all averages with a \( \Delta a \) at the latest time vanishes. So, to second order in the Chern-Simons field fluctuations is

\[
\langle J(R, p; t; R_i, p_i, t_i) \rangle^{eq}_{a,t} = \int d\Delta r \int d\Delta r_i \int dr_i \int D\sigma(+) \int Dp(+) \int D\sigma(-) \int Dp(-) \exp \left( \frac{i}{\hbar} S_0 + \frac{i}{\hbar} S_{int} \right),
\]

where

\[
S_{int} = \frac{i}{2\hbar} \langle \Delta S^2 \rangle^{eq}_{a,t}.
\]
To the same order it is permissible in the expression for $\Delta S$ to use the approximation $w_\alpha(a; \mathbf{R}, \mathbf{p}, t) = w_\alpha(a; R, \mathbf{p}, t)_{a=0} = w_\alpha(\mathbf{p}) = f(e_\mathbf{p})$, where $f(e)$ is the Fermi distribution function. The contribution from the gauge field to the effective action is then

$$S_{\text{int}} = \frac{i}{2\hbar} \langle \Delta S^2 \rangle_{a,t}^\text{eq}$$

$$= \frac{i}{2} \int_{t_i}^{t_f} dt' \int_{t_i}^{t_f} ds'' \sum_{\mu', \mu''} \mu' \mu'' \left( -1, \quad \bar{r}(s', \mu') \bar{C}(r(s'', \mu''), s'; r(s''\mu''), s'') \left( \frac{1}{2} - w[\mathbf{p}(s''\mu'')] \right) \right)$$

$$- \int_{t_i}^{t_f} dt' \int_{t_i}^{t_f} ds'' \sum_{\mu', \mu''} \mu' \mu'' \left( -1, \quad \bar{r}(s', \mu') \bar{D}(r(s'', \mu''), s'; r(s''\mu''), s'') \left( \frac{1}{2\pi} \mathbf{p}^{-1} s'' \mu'' \right) \right).$$

(66)

In the Coulomb gauge, which we consistently use below, the most singular part of the $2 \times 2$ matrices $\bar{C}$ and $\bar{D}$ is the transverse part. Below we often only discuss this most singular part, i.e. the transverse contribution to $S_{\text{int}}$.

$$S_{\text{int},\perp}(t_i \to t_f) = \sum_{\mu', \mu'' = \pm} \mu' \mu'' \int_{t_i}^{t_f} ds' \int_{t_i}^{t_f} ds'' L_{\text{int},\perp}(s', \mu'; s'', \mu'').$$

(67)

For later use we have incorporated the time interval $(t_i, t_f)$ in which the interaction takes place into the notation for the action. The retarded Lagrangian $L_{\text{int},\perp}(t_i, t_f) = L_{\text{int},\perp}(s', \mu'; s'', \mu'')$ for a specific pair of paths only considering the transverse contribution is given by

$$\frac{i}{\hbar} L_{\text{int},\perp} = \Theta(s' - s'') \frac{1}{\hbar} \int \frac{dq}{(2\pi\hbar)^2} \int \frac{d\omega}{2\pi} e^{-i\omega(s' - s'')} \exp \left[ \frac{i}{\hbar} \mathbf{p} \cdot (r(s', \mu') - r(s'', \mu'')) \right]$$

$$\times \frac{1}{q^2} \left[ \mathbf{q} \times \dot{r}(s', \mu') \right] \cdot \left[ \mathbf{q} \times \dot{r}(s'', \mu'') \right] \coth \left( \frac{\hbar \omega}{2k_B T} \right) \text{Im} D_{\perp}(\mathbf{q}, \omega)$$

$$- \mu' \mu'' \Theta(s' - s'') \frac{1}{\hbar} \int \frac{dq}{(2\pi\hbar)^2} \int \frac{d\omega}{2\pi} e^{-i\omega(s' - s'')} \mathbf{q} \cdot \mathbf{r}(s', \mu') - \mathbf{r}(s'', \mu'')$$

$$\times \frac{1}{mq^2} \left[ \mathbf{q} \times \dot{r}(s', \mu') \right] \cdot \left[ \mathbf{q} \times \mathbf{p}(s'', \mu'') \right] \frac{1}{2} D_{\perp}(\mathbf{q}, \omega) \left[ 1 - 2w(\mathbf{p}(s''+, \mu'')) \right].$$

(68)

In the second term $\frac{1}{2} D_{\perp}$ can be replaced by either $\text{Re} D_{\perp}$ or $\text{Im} D_{\perp}$ as long as we remember $L_{\text{int}}$ is retarded ($s' > s''$).

G. Recapitulation of the major results

Summing up, the linear response of the one particle density matrix to an external perturbation is given by Eq. (54), which schematically can be written as

$$\delta w = \int \langle J D \rangle_{a,t}^\text{eq}. \quad (69)$$

This expression we approximately factorize in order to arrive at Eq. (63), or schematically

$$\delta w = \int \langle J \rangle_{a,t}^\text{eq} \langle D \rangle_{a,t}^\text{eq}. \quad (70)$$

In the semi-classical limit the stimulus $\langle D \rangle_{a,t}^\text{eq}$ is in the classical limit given by the driving term in the Boltzmann equation Eq. (50). In Eq. (63) the propagator $\langle J \rangle_{a,t}^\text{eq}$ is represented as a double single-particle path integral with an action similar to the familiar Caldeira-Leggett influence functional. Schematically we write

$$\langle J \rangle_{a,t}^\text{eq} = \int D(\text{path} +) D(\text{path} -) \exp \left[ \frac{i}{\hbar} (S_0 + S_{\text{int}}) \right].$$

(71)

Here $S_0$ is the usual action for non-interacting particles Eq. (61). It does not couple the forward path $+$ and the backward path $-$. $S_{\text{int}}$ contains the contribution from the gauge fields and the interactions and it couples the two paths. It is given by Eq. (66). The imaginary part of $S_{\text{int}}$ is governed by the fluctuations Eq. (29), while the real part is determined by the retarded gauge field propagator Eq. (28). The distribution function also enters into the real part.
The Kubo formula

\[
\begin{array}{c|c|c}
\text{stimulus} & \text{propagation} & \text{response} \\
H_{\text{pert}} & \chi = \sum \frac{g_{j'j}}{j'-j} & X = \chi H_{\text{pert}}
\end{array}
\]

The path integral formalism

\[
\begin{array}{c|c|c}
\text{stimulus} & \text{propagation} & \text{response} \\
D[f^0] & J = \int \mathcal{D} \phi e^{iS_0/\hbar} & X = J D[f^0]
\end{array}
\]

The Boltzmann equation

\[
\begin{array}{c|c|c}
\text{stimulus} & \text{propagation} & \text{response} \\
D[f^0] & G^{-1} = \hat{p} \cdot \nabla_x + \hat{p} \cdot \nabla_p^{-1} & \delta f = G^{-1} D[f^0]
\end{array}
\]

FIG. 3. An overview of linear response theory for non-interacting particles in the Kubo formula, the path integral formalism and the Boltzmann equation. The linear response results from a stimulus combined with propagation. The figure illustrates where the distribution function \(f^0\) appears in the three different cases. Note in particular the absence of \(f^0\) in the propagation part of the path integral formalism and the Boltzmann equation.

### III. COMPARISON WITH THE KUBO FORMULA

In equilibrium a standard way to perform linear response is to use the Kubo formula. Though the path integral formalism and the Kubo formula on a formal level are similar there are some important differences in the descriptions. This is most clearly illustrated by considering a non-interacting system as is done in Fig. 3 where we also compare to the classical Boltzmann equation. The linear response is in all three cases split into a stimulus that is propagated in order to give the response. In the Kubo formula the stimulus is the applied potential, while it in the path integral formalism is the induced excitation, which in the semi-classical limit equals the usual Boltzmann stimulus Eq. (50). Despite that the stimulus is very different the Kubo formula and the path integral formulations have similar pictorial representations of the propagation: in both cases there is a forward and backward propagating electron line. However in the Kubo formula a line mathematically represents a Green’s function and the propagator in the non-interacting case is the Lindhard function, while in the non-interacting path integral formalism a line is a conventional free particle Feynman path integral. In the semi-classical limit the double Feynman path integral reduces to the classical path integral propagator for the Boltzmann equation. The difference between the Kubo formula and the path integral formalism is high lighted by the observation that the (Fermi) distribution function in the Kubo formula is contained entirely within the propagator, while it in the path integral formalism is the stimulus and not the propagation that depends on the distribution function for a non-interacting system. Notice also that in the path integral formulation no statements are made about equilibrium. The fact that for a non-interacting system the propagation is independent of the actual distribution function is well known from the Landauer-Büttiker scattering approach to transport through a non-interacting part of the system.

Within the Kubo formalism it is very natural to treat the interactions perturbatively. The effect of the interaction is traditionally divided into self-energy contributions, which alters the propagation properties of a single line or Green’s function, and vertex corrections that connects the two lines. It is possible to make the same distinction in a perturbative treatment of the interactions in the path integral formulation, and we will also use the terms self-energy and vertex corrections to distinguish between interactions that only involves one path and those that connect the forward and backward propagating paths. For an interacting system the propagator in the path integral formalism thus also depends on the actual distribution function. This is well known in the classical limit of the Boltzmann equation, since the propagator then depends on both the collision integral and the Landau interaction function, but still the stimulus has already gathered the most trivial dependence upon the distribution function.
IV. THE SEMI-CLASSICAL LIMIT AND THE SEPARATION BETWEEN THE TWO PATHS

Traditionally the semi-classical limit of a double path integral is the limit, where the important contributions are those where the forward and backward paths are almost identical. In a discussion of the semi-classical limit it is thus important to know how far apart on average the forward and backward paths are. This information is contained in the moments of the difference variable \( \Delta r (s) \). For a non-interacting system or more generally a system with instantaneous interactions the standard deviation of \( \Delta r \) can be estimated from the Heisenberg uncertainty relations between \( \Delta r \) and its conjugate variable, the average momentum \( P \). That \( P \) indeed is the conjugate variable can be read off from the relations

\[
\frac{\partial}{\partial P_t} S_0 (R_t, P_t, t; R_i, P_i, 0) = -\Delta r (t) \tag{72}
\]

\[
\frac{\partial}{\partial P_i} S_0 (R_i, P_t, t; R_i, P_i, 0) = \Delta r (0), \tag{73}
\]

where \( S_0 \) is the action for a non-interacting system, Eq. (61). We are mainly interested in Fermi liquids, so we prefer to use a polar coordinate system representation of the average momenta

\[
P (s) = P (s) \left( \cos \phi (s), \sin \phi (s) \right), \tag{74}
\]

where the angle along the Fermi surface \( \phi \) is measured relative to an arbitrarily fixed \( x \)-axis. The conjugate variable to the size of the momentum is

\[
\frac{\partial}{\partial \phi_t} S_0 (R_t, P_t, t; R_i, P_i, 0) = \frac{1}{P_t} P_t \cdot \Delta r (t). \tag{75}
\]

From the Heisenberg uncertainty relations we then expect the standard deviation \( \frac{1}{P} P \cdot \Delta r \) of the difference along the path to be given by \( \hbar \) divided by the uncertainty in the size of the momentum, i.e. \( \frac{1}{P} P \cdot \Delta r \approx \hbar v_F (k_B T)^{-1} \). This argument is carried through in more detail in appendix \( \text{[B]} \) and the coefficient to \( \hbar v_F (k_B T)^{-1} \) is also shown to be \( 1/\sqrt{3} \).

The conjugate variable to the angle \( \phi \) is

\[
\frac{\partial}{\partial \phi_t} S_0 (R_t, P_t, t; R_i, P_i, 0) = -P_t \times \Delta r (t). \tag{76}
\]

Remembering that the relevant momentum \( P_t \) approximately is the Fermi-momentum we will denote the quantity \( p_F^{-2} P_t \times \Delta r \) by the width. The Heisenberg uncertainty relation then says that the standard deviation of the width is approximately the Fermi wavelength \( \lambda_F \) divided by the angular uncertainty in the propagation direction. For a well collimated beam the width is thus big, while it is small (of the order of \( \lambda_F \)) when measuring the current response to a homogeneous field, since in this case the angular uncertainty is as big as possible (of the order 1).

The separation between the paths is not in itself directly measurable, but it tends to show up indirectly in experiments probing the semi-classical regime. It marks e.g. the cross-over between the long wave-length regime, where the vertex corrections are comparable to and partly cancels the self-energy and the short wave-length regime where the vertex corrections play a minor role. For instance in the calculation of the inverse transport time due to impurity scattering in the famous factor \( (1 - \cos \theta) \) the scattering angle \( \theta \) can be interpreted as \( \frac{1}{q} \lambda_F \), where \( q \) is the scattering vector and, as we saw above, \( \lambda_F \) is the width.

For the composite fermions, where the force is transmitted via a magnetic field and thus a flux through an area, we expect the width to mark the regime where the vertex corrections sets in. We can thus use the width as an effective cut-off in the conventional calculations of the effective mass. The above considerations regarding the width are only true for a system with a time local action. For composite fermions this is not the case, since the important transverse fluctuations have a characteristic frequency around \( g_2 q^2 \), see Eq. (33). We thus have to take the lowest order correction due to the terms in the action that are not time-local into account. We do this perturbatively.

To this end let us define the average width squared \( \left( \frac{\hbar}{q} \right)^2 \) at time \( s \) as

\[
\left( \frac{\hbar}{q} \right)^2 \equiv \langle p_F^{-2} |P (s) \times \Delta r (s)|^2 \rangle_{[0,1]}, \tag{77}
\]

where the average \( \langle X \rangle_{[0,1]} \) is defined as
\[ \langle X \rangle_{[0,t]} = \frac{\int \frac{dP}{(2\pi \hbar)^2} \int dR_t \int \frac{dP}{(2\pi \hbar)^2} \int dR_i \mathcal{R}(P_t, R_t) D(P_t, R_t) \{X \exp \left[ \frac{i}{\hbar} S_0(R_t, P_t, t; R_i, P_i, 0) \right]\}_{[0,t]}}{\int \frac{dP}{(2\pi \hbar)^2} \int dR_t \int \frac{dP}{(2\pi \hbar)^2} \int dR_i \mathcal{R}(P_t, R_t) D(P_t, R_t) \{\exp \left[ \frac{i}{\hbar} S_0(R_t, P_t, t; R_i, P_i, 0) \right]\}_{[0,t]}}. \]  

Here \( \langle X \rangle_{[0,t]} \) is a short hand notation for the path integral over paths starting at time 0 and ending at time \( t \) with the Wigner function variables as boundary conditions. \( \langle X \rangle_{[0,t]} \) is defined such as to include the action \( S_{\text{int}}(t, 0) \) arising from the interactions within the interval \([0, t] \)

\[ \langle X \rangle_{[0,t]} = \int_{[0,t]} D\mathbf{R} \int_{[0,t]} D\mathbf{P} \int_{[0,t]} D\Delta p \int_{[0,t]} D\Delta r e^{iS_{\text{int}}(t, 0)}/X. \]  

When no confusion is possible we will the index \([0, t] \). In the semi-classical limit the stimulus \( D(P_t, R_t) \) is given by Eq. (78). For a current measurement the response \( \mathcal{R}(P_t, R_t) = -e\nu \Omega^{-1} = -e\frac{\hbar}{\pi} \partial R \Omega^{-1} \). \( \Omega \) is the volume of the sample. The time \( t \) is the time-interval during which the paths evolve. Below we will show that in the long time limit the cut-off is independent of \( t \) and \( s \). We only consider this limit below and for our purpose \( t \) can be taken to infinity or rather be put equal to a scattering time. Notice that according to Eq. (74) the last factor in the denominator is

\[ \left\langle \exp \left[ \frac{i}{\hbar} S_0(t, R_t, P_t, t; R_i, P_i, 0) \right] \right\rangle_{[0,t]} = \langle J(R_t, P_t; t; R_i, P_i, 0) \rangle_{a,t}. \]  

It could easily be argued that it is better to use the absolute value of both the response \( \mathcal{R}(P_t, R_t) \) and the stimulus \( D(P_t, R_t) \) in the definition of \( \langle X \rangle \), Eq. (78), since the response might be the sum of a positive contribution from paths originating in one part of the stimulus phase space and a negative contribution from other parts of the phase space. The width squared of the positively contributing paths taken as a group is then canceled by the width of the negatively contributing paths taken as a group, the overall result might even be a negative width squared. This does not seem to be terribly important in the examples discussed below so for simplicity we stick with the above definition.

We next use the conjugate variable relationship Eq. (70) to perform the rewriting

\[ p = \frac{1}{\hbar} \begin{pmatrix} P(s) \times \Delta r (s) \end{pmatrix}^2 e^{i\frac{\hbar}{\pi} S_0[R_t, P_t, t; R_i, P_i, 0]} = \frac{\partial}{\partial \phi(s)} e^{i\frac{\hbar}{\pi} S_0[R(s), P(s), s; R_i, P_i, 0]} \frac{\partial}{\partial \phi(s)} e^{i\frac{\hbar}{\pi} S_0[R, P, t; R(s), P(s), s]} \]  

inside the definition of the width squared Eq. (77). The action due to the interactions \( S_{\text{int}} \) is split into three parts

\[ S_{\text{int}}(t, 0) = \sum_{\mu', \mu'' = \pm} \mu' \mu'' \int_0^t ds' \int_0^t ds'' L_{\text{int}} \]  

\[ = S_{\text{int}}(s, 0) + S_{\text{int}}(t, s) + \sum_{\mu', \mu'' = \pm} \mu' \mu'' \int_s^t ds' \int_0^t ds'' L_{\text{int}}, \]  

where the Lagrangian \( L_{\text{int}} \) only considering the transverse contribution is given by Eq. (68). In total

\[ \left( \frac{\hbar}{\pi \hbar} \right)^2 = \lambda^2 \mathcal{N} \int \frac{dP_t}{(2\pi \hbar)^2} \int dR_t \int \frac{dP_i}{(2\pi \hbar)^2} \int dR_i \mathcal{R}(P_t, R_t) D(P_t, R_t) \int dP(s) \int dR(s) \]  

\[ \times \left\langle \left\langle e^{i\frac{\hbar}{\pi} \sum_{\mu', \mu'' = \pm} \mu' \mu'' \int_s^t ds' \int_0^t ds'' L_{\text{int}}} \frac{\partial}{\partial \phi(s)} e^{i\frac{\hbar}{\pi} S_0[R(s), P(s), s; R_i, P_i, 0]} \frac{\partial}{\partial \phi(s)} e^{i\frac{\hbar}{\pi} S_0[R, P, t; R(s), P(s), s]} \right\rangle_{[0,s]} \right. \left\rangle_{[s,t]} \right\rangle, \]  

where the normalization is contained inside the new constant \( \mathcal{N} \),

\[ \mathcal{N}^{-1} = \int \frac{dP_t}{(2\pi \hbar)^2} \int dR_t \int \frac{dP_i}{(2\pi \hbar)^2} \int dR_i \mathcal{R}(P_t, R_t) D(P_t, R_t) \{\exp \left[ \frac{i}{\hbar} S_0(R_t, P_t, t; R_i, P_i, 0) \right]\}_{[0,t]}. \]  

The contribution due to the interaction is treated perturbatively; in particular we perform the expansion in the part that contains the scattering events that begin before the observation time \( s \) and finish later than \( s \):

\[ \sum_{\mu', \mu'' = \pm} \mu' \mu'' \int_s^t ds' \int_0^t ds'' L_{\text{int}}. \]
A. Zeroth order or the time-local limit revisited

To lowest order or in the so-called time-local approximation, the last term in Eq. (83) vanishes and the width squared is

\[
\left(\frac{\hbar}{q_\perp}\right)^2 = \lambda_\nu^2 N \int \frac{dP_i}{(2\pi\hbar)^2} \int dR_i \int \frac{dP_i}{(2\pi\hbar)^2} \int dR_i \mathcal{R} (P_i, R_i) D (P_i, R_i) \int \frac{dP (s)}{(2\pi\hbar)^2} \int dR (s)
\]

\[
\times \frac{\partial}{\partial \phi (s)} \langle J [R (s), P (s), s; R_i, P_i, 0] \rangle_{a,t} \frac{\partial}{\partial \phi (s)} \langle J [R_f, P_f; R (s), P (s), s] \rangle_{a,t}.
\]

Let us study two distinct examples. The first is the case of homogeneous driving, where \(D (P_i, R_i) = D (P_i)\). We furthermore assume the system is rotation-invariant such that

\[
\int dR_i \langle J [R (s), P (s), s; R_i, P_i, 0] \rangle_{a,t}
\]

only depends on the size of the momenta \(P_i, P (s)\) and the relative momentum angle \(\phi (s) - \phi_i\). In this case it is nice to expand the angular dependence of the stimulus on the eigenfunctions \(\exp (i\nu \phi_i)\) for the angular momentum operator in the \(z\)-direction

\[
D (P_i) = \sum_\nu \exp (i\nu \phi_i) D_\nu (P_i).
\]

The width squared from the \(\nu\)'th mode is then \(\nu^2 \lambda_\nu^2\). In a path integral calculation of the impurity scattering integral in the Boltzmann equation the well known factor \(1 - \cos (\nu \theta)\) factor originates from an average of \(2 \sin^2 \left(\frac{\theta}{2} \frac{\partial}{\partial \phi (s)}\right)\), so as expected it is the width \(\nu \lambda_\nu\) which sets the boundary between the small angle scattering regime, where the vertex corrections given by \(\cos (\nu \theta)\) are important and a large scattering regime, where the vertex correction average out. Notice that for an elastic scattering system this averaging has to be performed via the boundary conditions, while in general an interacting system is self-averaging.

As another example we consider the gedanken experiment typically considered in connection with the interpretation of a single-particle Greens function. A particle is fed into the system at the Fermi surface and in a direction parallel to the \(x\)-axis. In order to make it a bit more realistic we give the direction a finite spread \(\Delta_i\) i.e. we take \(\mathcal{R} \propto \exp \left(\frac{1}{2 \Delta_i^2} \frac{\phi_i^2}{2}\right)\). A time \(t\) later we measure the probability the particle is still moving along the \(x\)-axis with a directional spread of \(\Delta_f\) i.e. we take \(D \propto \exp \left(\frac{1}{2 \Delta_f^2} \phi_f^2\right)\). The width squared is in this case \(\lambda_\nu^2 \left(\frac{1}{\Delta_i^2} + \frac{1}{\Delta_f^2}\right)\). In particular the width diverges as the angular spread to the power \(-1\).

B. The first order non time-local correction

Let us move on to discuss the first order correction to the width squared due to the non time-local part of the interaction \(\sum_{\mu', \mu'' = \pm} \mu' \mu'' \int \int ds' \int ds'' \frac{1}{\hbar} L_{\text{int}}\)

\[
\Delta \left(\frac{\hbar}{q_\perp}\right)^2 = \lambda_\nu^2 N \int \frac{dP_i}{(2\pi\hbar)^2} \int dR_i \int \frac{dP_i}{(2\pi\hbar)^2} \int dR_i \mathcal{R} (P_i, R_i) D (P_i, R_i) \int \frac{dP (s)}{(2\pi\hbar)^2} \int dR (s)
\]

\[
\times \sum_{\mu', \mu'' = \pm} \mu' \mu'' \int \int ds' \int ds'' \left\langle L_{\text{int}} (s', \mu'; s'', \mu'') \frac{\partial}{\partial \phi (s)} e^{\frac{iS_0 [R (s), \mathcal{P} (s), \mathcal{R} (s), P (s), R (s), s, t]}} \right\rangle_{[0, t]} \left\langle L_{\text{int}} (s', \mu'; s'', \mu'') \frac{\partial}{\partial \phi (s)} e^{\frac{iS_0 [R (s), \mathcal{P} (s), \mathcal{R} (s), P (s), R (s), s, t]}} \right\rangle_{[s, t]}.
\]

We have here neglected the contribution from the expansion of the normalization \(N\), since it turns out to cancel a term in the expansion of Eq. (83) we anyway neglect in our discussion below. Let us here specialize to the contribution from the long wave-length fluctuations, where \(q \cdot \Delta r \ll 1\), i.e.

\[
\frac{i}{\hbar} L_{\text{int}} (s', \mu'; s'', \mu'') = \frac{1}{\hbar} \int \frac{dq}{(2\pi\hbar)^2} \int d\omega \coth \left(\frac{\hbar \omega}{2k_B T}\right) \text{Im} D_{\perp \perp} (q, \omega) \left[\frac{1}{q} q \times \hat{R} (s') \right] \cdot \left[\frac{1}{q} q \times \hat{R} (s'')\right] \times e^{-\omega (s'' - s')} e^{i q [R (s') - R (s'')]} + \frac{1}{q} q [\mu' \Delta r (s') - \mu'' \Delta r (s'')].
\]
To lowest order in the interaction the extra contribution from $L_{\text{int}}$ to the width squared is in the long wave-length limit

$$
\Delta \left( \frac{h}{q_{s\perp}^2} \right)^2 = N \int \frac{dP}{(2\pi h)^2} \int dR \int \frac{dP}{(2\pi h)^2} \int dR \left( P, R, P, R \right) \frac{1}{p_F^2} \int \frac{dP}{(2\pi h)^2} \int dR \left( s \right)
$$

\[
\times \frac{1}{h} \int ds' \int ds'' \int dq \int d\omega \left( \frac{h\omega}{2k_BT} \right) \text{Im} \mathcal{D}_{s\perp} (q, \omega) e^{-i\omega(s'-s'')}
\]

\[
\times \left\langle \left\langle \left\langle \left\langle \right\rangle \right\rangle \right\rangle \right\rangle_{[0,s]} \left\langle \left\langle \left\langle \right\rangle \right\rangle \right\rangle_{[s,t]}
\]

Equation (91)

To lowest order we will neglect the effect of $S_{\text{int}} (s, 0)$ and $S_{\text{int}} (t, s)$ inside the path integrals $\left\langle \left\langle \left\langle \right\rangle \right\rangle \right\rangle_{[0,s]}$ and $\left\langle \left\langle \left\langle \right\rangle \right\rangle \right\rangle_{[s,t]}$ respectively. The product $\Delta r (s') \Delta r (s'')$ will replace by its average value. The most important contribution comes from the longitudinal part of $\Delta r (s') \Delta r (s'')$. Furthermore it turns out that the relevant time-scale in the above integral $\hbar (v_F q)^{-1}$ for the important fluctuations is smaller than the correlation time of $\Delta r (s') \Delta r (s'')$, which is of the order $\hbar (k_BT)^{-1} \approx h (k_BT)^{-1}$, so we replace $\Delta r (s') \Delta r (s'')$ by its equal time longitudinal average value $\hbar^2 q_{s\perp}^2 / (k_BT)^2$, see appendix C. Within these approximations the correction to the width squared is

$$
\Delta \left( \frac{h}{q_{s\perp}^2} \right)^2 = \frac{h^2 v_F^2}{6 (k_BT)^2 p_F} \frac{1}{p_F} \int \frac{dq}{2\pi h} \int d\omega \left( \frac{h\omega}{2k_BT} \right) \left[ \text{Im} \mathcal{D}_{s\perp} (q, \omega) \right] \left[ \left\langle \left\langle \left\langle \right\rangle \right\rangle \right\rangle_{[0,s]} \left\langle \left\langle \left\langle \right\rangle \right\rangle \right\rangle_{[s,t]} \right].
$$

Equation (92)

For the composite fermions the spectral function $-\text{Im} \mathcal{D}_{s\perp} (q, \omega)$ is thus of the order 1 and the width is

$$
\Delta \left( \frac{h}{q_{s\perp}^2} \right)^2 \approx \frac{h^2 v_F^2}{6 (k_BT)^2 p_F} \left( \frac{v_F p_F}{2\hbar g_1 p_F^2} \right)^{(2\eta-1)/(1+\eta)} \left( \frac{v_F p_F}{2\hbar g_1 p_F^2} \right)^{(3/(1+\eta) - 2^{3/(1+\eta)} \frac{1}{9} \csc \frac{3\pi}{2(1+\eta)} \left[ (\frac{1}{2} - \frac{3\pi}{2(1+\eta)}) \right]^{1/3}}
$$

Equation (93)

Here $\eta = 1$ corresponds to our prime example, the Coulomb interaction

$$
\frac{h^2}{q_{s\perp}^2} \approx \frac{2}{2} v_F p_F \left( \frac{v_F p_F}{2\hbar g_1 p_F^2} \right)^{3/2} \frac{4}{9} \left( \frac{4}{\pi} \right)^{3/4}
$$

Equation (94)

We expect the Fermi energy $\frac{1}{2} v_F p_F$ to be of the same order of magnitude as the energy scale set by the interaction i.e. the average Coulomb energy $\varepsilon C = \frac{e^2}{\sqrt{n}}$. The factor $\left( \frac{v_F p_F}{2\hbar g_1 p_F^2} \right)^{3/2} \frac{4}{9} \left( \frac{4}{\pi} \right)^{3/4}$ is thus of the order 1 and the width is approximately given by the geometric mean of the Fermi wave-length and the de Broglie wave-length.

In perturbative calculations of the self-energy the cut-off is often formulated in terms of a frequency cut-off. The frequency cut-off corresponding to Eq. (94) is

$$
\omega_{\text{cut}} = g_1 q_{s\perp}^2 \approx \hbar^{-1} \sqrt{\varepsilon C k_BT} \left( \frac{2\varepsilon C}{v_F p_F} \right)^{5/4}
$$

Equation (95)

We want to emphasize that this transverse cut-off is considerably larger than the longitudinal cut-off, which in appendix C is found to be $\sqrt{3\hbar^{-1} k_BT}$. In the next section we show that in case of Coulomb interaction the effective mass depends logarithmically on $\omega_{\text{cut}}$, so the temperature dependence of $\omega_{\text{cut}}$ and thus the inverse width squared is $\sqrt{\omega_{\text{cut}}} \log T^2$, which tends much slower to zero than the more conventional linear temperature scaling.

V. THE EFFECTIVE MASS

In the last section we calculated the width between the forward and backward path $\frac{h}{q_{s\perp}^2}$, Eq. (94), and we hinted at how this width might play the role of an effective cut-off in the calculation of the effective mass. In this section we elaborate a bit more on this connection.
Experimentally the effective mass of a particle is determined by simultaneously measuring the momentum and the velocity. In a Fermi liquid the momentum is typically the Fermi momentum. The velocity measurement typically involves measuring how far the particle has propagated in a known time-span \( t \). This is a little bit at odds with the semi-classical limit, where the external fields are considered to be smooth, so it is hard to define, yet measure, distances shorter than a transport mean free path length. This is necessary in a mass measurement, since the time \( t \) must be comparable to or smaller than the transport time to make the path reasonably well defined. Having said this one might hope for an intermediate regime, where it is possible to at least put some limits on the effective mass, while still maintaining the semi-classical line of thought.

One attempt at this is done by Willett, West, and Pfeiffer, who uses a surface acoustic wave (SAW) to generate an external electric field with a wave-length comparable to the mean free path. They then apply an external magnetic field and looks for resonances between the cyclotron radius and the wave-length of the SAW field. In order to observe these resonances the cyclotron frequency cannot be orders of magnitude larger than frequency of the SAW wave. This leads to an upper estimate of the elapsed time and thus the effective mass.

We will denote this sort of experiment as a semi-classical mass determination, since the forward and backward paths together perform the cyclotron motion. This is in contrast to experiments probing the single-particle energy spectrum, like Shubnikov-de Haas measurements. In such experiments according to the Bohr-Sommerfeld quantization scheme it is the phase of a single path that causes the resonance.

Below we show that in the semi-classical experiments the width between the two paths or in other words the enclosed area acts as an effective cut-off on the contribution from the magnetic field fluctuations to the effective mass. In the Bohr-Sommerfeld resonance scheme the cut-off is supplied by the area enclosed by the path itself. This area is much bigger than the semi-classical area and the cut-off is correspondingly smaller.

In order to extract the semi-classical mass from the path integral description it is in principle necessary to analyze the concrete experiment in detail. Luckily it turns out that in the semi-classical limit it is possible to come up with an approximately local relation between the velocity and the momentum. This implies the effective mass is almost independent of the concrete mass measuring experiment in the semi-classical limit.

For a non-interacting system the derivative of the action with respect to the momentum yields the Hamilton equation for the velocity \( \frac{dp}{dt} = \dot{r} - \frac{\partial H}{\partial p} = 0 \). For the two-path action it is the derivative with respect to the difference momentum \( \Delta p(s) \)

\[
\frac{\partial S}{\partial \Delta p(s)} = \mathbf{R}(s) - \frac{1}{m} \mathbf{P}(s) - \int_s^t ds' \int \frac{dq}{(2\pi \hbar)^2} \int \frac{d\omega}{2\pi} D_{\perp\parallel}(q, \omega) \exp \left\{ i \frac{q}{\hbar} \mathbf{R}(s') - \mathbf{R}(s) \right\} \exp [-i \omega (s' - s)] \\
\times \sum_{\mu, \mu' = \pm} \mu \mu' e^{i \mu \frac{q}{2} \Delta x(s') - \Delta x(s)} \frac{1}{q} \times \left[ \mathbf{R}(s') + \frac{1}{2} \mu' \Delta \mathbf{r}(s') \right] \\
\times \left\{ \frac{1}{2} - w \left[ \mathbf{P}(s^+) + \mu \frac{1}{2} \Delta \mathbf{p}(s^+) \right] \right\} \frac{1}{m q} \times \left[ \mathbf{P}(s) + \mu \frac{1}{2} \Delta \mathbf{p}(s) \right].
\]

that yields the relation between the velocity and the momentum, at least in a stationary phase approximation where \( \frac{\partial S}{\partial \Delta p(s)} = 0 \). On average the stationary phase condition is exact, i.e.

\[
\left\langle \frac{\partial S}{\partial \Delta p(s)} \right\rangle = 0.
\]

Notice that after performing the integral over \( \mathbf{R}(s) \) and \( \Delta x(s) \) inside the average as defined in Eq. (78) we have \( \Delta \mathbf{p}(s^+) = \Delta \mathbf{p}(s) - \mathbf{q} \) and \( \mathbf{P}(s^+) = \mathbf{P}(s) - \frac{1}{2} \mu \mathbf{q} \). The terms with \( \mu = \mu' \) in Eq. (96) i.e. those terms where the interaction connect either the forward or backward path with itself gives rise to that part of the renormalization of the effective mass that is attributed to the self-energy in the usual perturbative treatments. This renormalization mainly depends on the local paths of a single path and it is thus independent of which pair of paths is relevant in the given experiment. On the other hand the terms with \( \mu \neq \mu' \) in Eq. (96), which are associated with the vertex corrections, depend on both the forward and the backward paths. The renormalization due to these terms thus depends on the exact experiment. For instance in an interference experiment the two paths are almost independent and the vertex contribution is expected to vanish due to destructive interference in the phases associated with each path individually. In the semi-classical limit, where the forward and backward paths typically are close to each other it is fortunately possible to derive a universal expression for the renormalization of the mass. Below it will be shown the interaction modes with a wave-length longer than the separation between the paths does not renormalize the mass, while the renormalization due to the interaction modes with a shorter wave-length is given by the usual self-energy contribution. A possible interpretation is that the particle cannot distinguish between the field from a long wave-length interaction and an external potential.
For simplicity we neglect the extra terms arising from the velocity dependence of the force, i.e. we treat the interaction as scalar. With this small neglect the average of the velocity, denoted \( \mathbf{v}_F = \{ \mathbf{R} \} \), is

\[
\mathbf{v}_F = \frac{1}{m} \mathbf{p}_F - \int_s^t \int_s^{t'} ds' \int \frac{d\mathbf{q}}{(2\pi \hbar)^2} \int \frac{dw}{2\pi} \mathbf{v}_F m^{-1} \mathbf{p}_F D_{\perp\perp} (\mathbf{q}, \omega) e^{-i\omega(s'-s)} \times \left\{ e^{\frac{i}{\hbar} \mathbf{q} [\mathbf{R}(s') - \mathbf{R}(s)]} 2i \sin \left[ \frac{1}{2\hbar} \mathbf{q} \cdot \Delta \mathbf{r} (s') \right] \sum_{\mu = \pm} \mu e^{\mp i\mathbf{q} \cdot \Delta \mathbf{r} (s)} \frac{\partial}{\partial \mathbf{P}} \left[ \mathbf{P} (s) + \frac{1}{2} \Delta \mathbf{p} (s) - \mu \mathbf{q} \right] \right\},
\]

where in the first term it is used that the average momentum \( \langle \mathbf{P} \rangle \) is the Fermi momentum \( \mathbf{p}_F = \mathbf{p}_F \mathbf{v}_F \) (see appendix C). At least for a non-interacting system \( \Delta \mathbf{p} (s) \) is of the order of the typical wave-vector of the external force field. Our interest is the semi-classical limit so we will assume \( \Delta \mathbf{p} (s) \cdot \mathbf{p}_F < k_B T \) and neglect its contribution, i.e. we put \( \Delta \mathbf{p} (s) = 0 \). The average velocity \( \mathbf{v}_F \) is then

\[
\mathbf{v}_F = \frac{1}{m} \mathbf{p}_F - \int_s^t \int_s^{t'} ds' \int \frac{d\mathbf{q}}{(2\pi \hbar)^2} \int \frac{dw}{2\pi} \mathbf{v}_F m^{-1} \mathbf{p}_F D_{\perp\perp} (\mathbf{q}, \omega) e^{\frac{i}{\hbar} \mathbf{q} [\mathbf{R}(s') - \mathbf{R}(s)]} e^{-i\omega(s'-s)} \times \left\{ 2 \sin \left[ \frac{1}{2\hbar} \mathbf{q} \cdot \Delta \mathbf{r} (s') \right] \left\{ \sin \left[ \frac{1}{2\hbar} \mathbf{q} \cdot \Delta \mathbf{r} (s) \right] \sum_{\mu = \pm} \frac{\partial}{\partial \mathbf{P}} \left[ \mathbf{P} (s) - \mu \mathbf{q} \right] + i \cos \left[ \frac{1}{2\hbar} \mathbf{q} \cdot \Delta \mathbf{r} (s) \right] \sum_{\mu = \pm} \mu \frac{\partial}{\partial \mathbf{P}} \left[ \mathbf{P} (s) - \mu \mathbf{q} \right] \right\}. \tag{99}
\]

We are mainly interested in the long wave-length limit, so we will neglect terms of order \( q^2 \) compared to terms of order \( q \). We also assume that both \( |\mathbf{P} (s)| \) and the corresponding velocity \( \hat{\mathbf{R}} (s) \) are distributed evenly around the Fermi surface. In this approximation the second term inside the parenthesis can be neglected. We assume the relative variation in the velocity around the Fermi velocity \( \mathbf{v}_F \) is small, i.e. the temperature is much smaller than Fermi energy. It is thus possible to approximate the path locally by a straight line i.e. \( \mathbf{R} (s') - \mathbf{R} (s) \approx \mathbf{v}_F (s' - s) \). In the long wave-length limit the transverse part of \( \mathbf{q} \cdot \Delta \mathbf{r} \) is bigger than the longitudinal part. The averaging procedure is approximated by independently carrying out the longitudinal and the transverse average. In the longitudinal average we use the time-local approximation outlined in appendix C so the longitudinal average of \( \sum_{\mu = \pm} - \frac{\partial}{\partial \mathbf{P}} \left[ \mathbf{P} (s) - \mu \mathbf{q} \right] \) is approximately

\[
\left\langle \sum_{\mu = \pm} - \frac{\partial}{\partial \mathbf{P}} \left[ \mathbf{P} (s) - \mu \mathbf{q} \right] \right\rangle \approx \mathbf{v}_F \int_{-\infty}^{\infty} ds \frac{\partial}{\partial \varepsilon} f_F (\varepsilon) \sum_{\mu = \pm} \frac{\partial}{\partial \varepsilon} f_F (\varepsilon - \mu \mathbf{v}_F \cdot \mathbf{q}),
\]

where we have used \( \mathbf{v}_F = \frac{\partial \mathbf{p}_F}{\partial \mathbf{p}_F} \). In the transverse average we approximate

\[
2 \sin \left[ \frac{1}{2\hbar} \mathbf{q} \cdot \Delta \mathbf{r} (s') \right] \sin \left[ \frac{1}{2\hbar} \mathbf{q} \cdot \Delta \mathbf{r} (s) \right] \approx \left\{ 1 - \cos \left[ \frac{1}{\hbar} \mathbf{q} \cdot \Delta \mathbf{r} (s) \right] \right\} \approx \left[ 1 - \exp \left( -\frac{1}{2} q_{\perp}^{-2} q^2 \right) \right], \tag{101}
\]

with the result that the effective mass \( m^* \), defined as \( \mathbf{p}_F = m^* \mathbf{v}_F \), is given by

\[
\frac{1}{m^*} \approx \frac{1}{m} \left\{ 1 + 2 \mathbf{v}_F^2 \int_{-\infty}^{\infty} ds \int \frac{d\mathbf{q}}{(2\pi \hbar)^2} \int \frac{d\omega}{2\pi} \text{Re} D_{\perp\perp} (\mathbf{q}, \omega) \left[ 1 - e^{-i\mathbf{q} \cdot \Delta \mathbf{r} (s')} \right] \right\} \times \left\{ e^{\frac{i}{\hbar} \mathbf{q} \cdot \mathbf{v}_F (s') \cdot \mathbf{v}_F \cdot (s')} \int_{-\infty}^{\infty} d\varepsilon \frac{\partial}{\partial \varepsilon} f_F (\varepsilon) \frac{\partial}{\partial \varepsilon} f_F (\varepsilon - \mathbf{v}_F \cdot \mathbf{q}) \right\}. \tag{102}
\]

Apart from the extra factor of \( 1 - \exp \left( -\frac{1}{2} q_{\perp}^{-2} q^2 \right) \) this relation is identical to the usual relation derived from the self-energy. Thus the contribution from the short wave-length modes of the interaction to the renormalization of the mass is identical to the standard result, while the long wave-length modes do not contribute. The cross-over is at the width \( \hbar q_{\perp}^{-1} \) and this is why we above have alluded to the width as an effective cut-off.

When the long-wavelength expression for the propagator \( D_{\perp\perp} \), Eq. (33) in the case of Coulomb interaction, is inserted into Eq. (102) the integral is ultraviolet divergent. We use \( p_F \) as an approximative upper cut-off and get the effective mass

\[
m^* \approx m + m \frac{4}{\pi^{3/2}} \frac{p_F^2}{2m \varepsilon_C} \log \frac{p_F^2}{2q_{\perp}^2} \tag{103}
\]
For composite fermions the renormalization of the mass is quite strong so the effective mass

\[ m^* \approx \frac{2}{\pi^{3/2}} \frac{p_F^2}{\varepsilon_C} \log \frac{p_F^2}{2q_{C\perp}} \]  

(104)

is approximately independent of the bare mass \( m \).

The prefactor

\[ m \frac{4}{\pi^{3/2}} \frac{p_F^2}{2m \varepsilon_C} \approx 1.9 \, m, \]  

(105)

where \( \varepsilon_C = \frac{\varepsilon^2}{\pi^2} \sqrt{\pi} \) is the average Coulomb energy and we use typical values \([13]\) have used \( n = 1.6 \times 10^{15} \text{m}^{-2}, \varepsilon = 13 \varepsilon_0 \), and \( m \approx 0.067 \, m_0 \), where \( \varepsilon_0 \) and \( m_0 \) are the vacuum values of the permittivity and the electron mass respectively.

The argument of the logarithm is given by Eq. (93) so self-consistently the effective mass is

\[ \frac{m^*}{m} \approx 1 + \frac{4}{\pi^{3/2}} \frac{p_F^2}{2m \varepsilon_C} \log \left[ \frac{1}{2} + \frac{9/4}{18} \sqrt{\varepsilon_C \frac{\pi^{3/2}}{2m \varepsilon_C}} \left( \frac{4}{\pi^{3/2}} \frac{p_F^2}{2m \varepsilon_C} \right)^2 \left( \frac{m}{m^*} \right)^2 \right] \approx 6.3, \]  

(106)

where the estimate is valid at a temperature of \( \sim 0.13 \, \text{K} \). The cut-off at this temperature happens at distances \( hq_{C\perp}^{-1} \approx 6 \lambda_F \approx 40 \, \text{nm} \).

As expected the semi-classical effective mass Eq. (106) is smaller than the mass deduced from the activation energy and Shubnikov-de Haas measurements \([13]\), actually about half as big. In the experiment by Willett et al. \([14]\) the estimate Eq. (106) leads to a cyclotron frequency that at the secondary resonance position is about twice as big as the used SAW frequency 10.7 GHz. A naive argument \([13]\) leads to the expectation that the secondary resonance can only be observed if the cyclotron frequency is much bigger than the SAW frequency. It is thus a matter of taste whether an ‘apparent inconsistency’ between the geometric resonance experiments and the size of the effective mass arises. We here want to stress the word ‘apparent’ since Mirlin and Wölfle \([16]\) has shown that the secondary resonance can be observed even when the cyclotron frequency equals the SAW frequency.

VI. CONCLUSION

In this paper we have developed a single-particle path integral description of transport for composite fermions, thereby extending the Caldeira-Leggett formalism for a single-particle interacting with an environment. Here the environment is the Chern-Simons gauge field. Our approach is modeled over the work by Golubev and Zaikin on the electron gas without a magnetic field. As in the Caldeira-Leggett formalism there is no one, but two paths in the path integral, one propagating forward in time and one backward. A new feature compared to the Caldeira-Leggett formalism is that the distribution function enters into the real part of the single-particle action. The imaginary part of the action is determined by the fluctuation spectrum, just as in the Caldeira-Leggett formalism.

In a perturbative description of the CF’s the long wave-length magnetic field fluctuations give rise to an infrared divergence of the self-energy and the vertex corrections, separately. In a transport measurement these divergences are known to cancel. In our formalism the infrared divergence does not appear because the vertex and self-energy contributions are never split. Not only is the infrared divergence absent, but the renormalization in the infrared limit is negligible. This result is obtained in a natural way by working in a real space formulation, since the magnetic interaction acts via fluxes, i.e. areas. In the path integral formalism the size of the particle is defined as the separation between the forward and backward propagating paths. Only the magnetic field fluctuations with a wave-length shorter than this particle size contributes noticeably to the renormalization of the mass. In GaAs at \( T \sim 0.13 \, \text{K} \) we find the mass renormalization due to the magnetic field fluctuations to be roughly 6 times the bare mass. Furthermore we have shown that in the semi-classical limit the separation between the two paths, i.e. the size of the particle, scale as \( T^{-1/4} |\log T|^{-1} \). This implies the effective cut-off in frequency scale as roughly \( T^{1/2} |\log T|^2 \). As usual the mass scales approximately with the logarithm of the frequency cut-off.

The linear response regime around equilibrium is the major focus of our work, but the formalism is still valid out of equilibrium. In the non-interacting limit, as shown in Fig. 3, the single-particle path integral technique possesses the feature that the effect of the non-equilibrium distribution function is gathered completely within the stimulus, while the propagation is identical to the propagation in equilibrium. This feature is shared with other out of equilibrium techniques i.e. the Landauer-Büttiker description of transport through a non-interacting part of the system and the Boltzmann equation. In fact the classical limit of the single-particle path integral formalism leads very naturally to the Boltzmann equation. The single-particle path integral technique is thus a promising technique for the description of interacting systems both in and out of equilibrium, especially it may prove useful for studying systems out of equilibrium not easily treated by other methods.
VII. ACKNOWLEDGEMENT

K.A.E. and H.B. both acknowledge support from the Danish Natural Science Research Council through Ole Rømer Grant no. 9600548.

APPENDIX A: EQUATIONS OF MOTION

In this appendix we are going to derive differential equations for the composite fermion contribution to the effective action $S_{\text{eff}}[a; t]$ and the density matrix $\rho_{\text{ac}}(A_{\text{eff}} - a)$ in the presence of the Chern-Simons field $A_{\text{eff}} - a$ by differentiating the composite fermion part of Eqs. (19) and (16).

In order to differentiate quantities in a fixed gauge field background at time $s$ it is necessary to let the Chern-Simons field and the Grassmann fields $\bar{\psi}$ and $\psi$ be defined on $C_t$, where $t > s$. The effective action for both the Chern-Simons field and the Grassmann variables defined in Eq. (11) is then

$$S_{\text{eff}}[\bar{\psi}, \psi, a; s] = S_{\text{CS}}[a; t] - e \int_{C_t} d\sigma' \int d\mathbf{r} a_0(\mathbf{r}, \sigma') n + S_{\text{ComFer}}[\bar{\psi}, \psi, a; s; t],$$  \hspace{1cm} (A1)

where

$$S_{\text{ComFer}}[\bar{\psi}, \psi, a; s; t] = \int_{C_t} d\sigma' \int d\mathbf{r} \bar{\psi}(\mathbf{r}, \sigma') i\hbar \frac{\partial}{\partial \sigma'} \psi(\mathbf{r}, \sigma') - \int_{C_s} d\sigma' H_0[A_{\text{eff}} - a].$$  \hspace{1cm} (A2)

The second term in Eq. (A1) can be gauged away for $t > s' > s$, since there is no Hamiltonian. The terms that depend on the composite fermion Grassmann variables are gathered in $S_{\text{ComFer}}[\bar{\psi}, \psi, a; s; t]$ and it is its contribution we are going to consider in this appendix.

Notice the effective action $S_{\text{CF}}$ in the definition of $\rho_{\text{ac}}$, Eq. (14), can be replaced by $S_{\text{ComFer}}$, since the difference cancels between the numerator and denominator in Eq. (14).

1. The effective action

The composite fermion contribution to the effective action $S_{\text{eff}}[a; s]$ is

$$S_{\text{eff}}[a; s] - S_{\text{CS}}[a; t] + e \int_{-\infty}^{s} ds' \int d\mathbf{r} \Delta a_0(\mathbf{r}, s') n = \frac{\hbar}{i} \log \| \mathbb{X} \|,$$  \hspace{1cm} (A3)

where $\Delta a_0(s') = a_0(s', +) - a_0(s', bi)$ is the difference field and the short-hand notation

$$\| X \| = \int D\bar{\psi} \int D\psi \exp \left( \frac{i}{\hbar} S_{\text{ComFer}}[\bar{\psi}, \psi, a; s; t] \right) X$$  \hspace{1cm} (A4)

for the Grassmann integral has been introduced. The derivative of Eq. (A3) is

$$\partial_s \{ \text{Eq. (A3)} \} = -\| 1 \|^{-1} \| H_0[A_{\text{eff}}(s) - a(s, +)] - H_0[A_{\text{eff}}(s) - a(s, -)] \|.$$  \hspace{1cm} (A5)

Per definition of the path integral is the right-hand side equal to the expectation value of $-\{ H_0[A_{\text{eff}}(s) - a(s, +)] - H_0[A_{\text{eff}}(s) - a(s, -)] \}$ in the presence of the contour gauge field $A_{\text{eff}} - a$. It can be expressed in terms of the density matrix as

$$\partial_s \text{Eq. (A3)} = -\int d\mathbf{r} \left( e \Delta a_0(\mathbf{r}, s) \rho_{\text{ac}}(A_{\text{eff}} - a; \mathbf{r}, \mathbf{r}; s) + \Delta a(\mathbf{r}, s) \cdot j_a(A_{\text{eff}} - a; \mathbf{r}, s) \right),$$  \hspace{1cm} (A6)

where the current in the presence of the gauge field $A_{\text{eff}} - a$: $j_a(A_{\text{eff}} - a; \mathbf{r}, s)$ is defined in Eq. (21). Eq. (21) then follows upon integration using the conventional assumption of adiabatic turning on of the gauge field.
2. The density matrix

The density matrix in the presence of the Chern-Simons gauge field $\rho_{ac}$ is defined in Eq. (10). At a time $s < t$ the action $S_{\text{CF}}[\bar{\psi}, \psi, a; s]$ can be replaced by $S_{\text{ComFer}}[\bar{\psi}, \psi, a; s; t]$. In terms of the above short-hand notation

$$\rho_{ac}(r, r', s) = \langle 1 \rangle^{-1} \langle \bar{\psi}(r', s) \psi(r, s) \rangle. \quad (A7)$$

When $s$ is varied both the numerator and the denominator varies:

$$i\hbar \partial_s \rho_{ac}(r, r', s) = \langle 1 \rangle^{-1} i\hbar \partial_s \langle \bar{\psi}(r', s) \psi(r, s) \rangle - \langle 1 \rangle^{-1} \langle \bar{\psi}(r', s) \psi(r, s) \rangle i\hbar \partial_s \langle 1 \rangle. \quad (A8)$$

In the above subsection the derivative of the denominator $i\hbar \partial_s \langle 1 \rangle$ was calculated. The contribution from the first term (the numerator) is

$$\langle 1 \rangle^{-1} i\hbar \partial_s \langle \bar{\psi}(r', s) \psi(r, s) \rangle = \langle 1 \rangle^{-2} \{ H_0[A_{\text{eff}}(s) - a(s, +)]\bar{\psi}(r', s)\psi(r, s) - \bar{\psi}(r', s)\psi(r, s)H_0[A_{\text{eff}}(s) - a(s, -)] \}. \quad (A9)$$

We here used

$$\langle \bar{\psi}(r', s + \Delta s)\psi(r, s + \Delta s) \rangle = \langle \bar{\psi}(r', s)\psi(r, s) \rangle, \quad (A11)$$

since without a Hamiltonian the system does not evolve. The action $S_{\text{ComFer}}$ is quadratic in the Grassmann variables $\bar{\psi}$ and $\psi$, so Eq. (A10) can be evaluated with the help of Wick’s theorem. The Hartree disconnected diagram like pairings exactly cancels the contribution from the variation of the denominator in Eq. (A8), $\langle 1 \rangle^{-2} \langle \bar{\psi}(r', s)\psi(r, s) \rangle i\hbar \partial_s \langle 1 \rangle$, so paying due attention to the contour ordering

$$i\hbar \partial_s \rho_{ac}(r, r', t) = \int dr'' \int dr''' \left( \delta(r - r'') - \rho_{ac}[r, r''; t] \right) H(r'', r''' + r', t) \rho_{ac}[r'''', r'; t]$$

$$\quad - \int dr'' \int dr''' \rho_{ac}[r, r''; t] H(r'', r'''', t, -) \left( \delta(r''' - r'') - \rho_{ac}[r'''', r'; t] \right). \quad (A12)$$

This is Eq. (17) written out.

**APPENDIX B: LINEARIZATION OF THE DENSITY MATRIX DIFFERENTIAL EQUATION**

The stimulus in the linearized equation of motion for $\delta \rho_{ac}(A_{\text{eff}} - a; r, r', t)$, Eq. (16), $D = D_1 + D_2$ is split into two parts. The explicit form for $D_1$, which is independent of $\Delta a$, is

$$i\hbar D_1(r, r', t) = \left[ -\frac{\partial}{\partial t} \int_{r'}^{r} dr'' \cdot A_{\text{tot}}(r'') + A_{\text{tot}}(r) \right] \cdot \frac{1}{m} \left[ \frac{\hbar}{i} \frac{\partial}{\partial r} + eA_{\text{eff}}(r) - e\bar{a}(r) \right] \rho_{ac}(r, r')$$

$$+ \left[ -\frac{\partial}{\partial r'} \int_{r'}^{r} dr'' \cdot A_{\text{tot}}(r'') - A_{\text{tot}}(r) \right] \cdot \frac{1}{m} \left[ -\frac{\hbar}{i} \frac{\partial}{\partial r'} + eA_{\text{eff}}(r') - e\bar{a}(r') \right] \rho_{ac}(r, r')$$

$$+ \rho_{ac}(r, r') \frac{\hbar}{2im} \frac{\partial}{\partial r} \left[ -\frac{\partial}{\partial r} \int_{r'}^{r} dr'' \cdot A_{\text{tot}}(r'') - A_{\text{tot}}(r) \right]$$

$$- \rho_{ac}(r, r') \frac{\hbar}{2im} \frac{\partial}{\partial r'} \left[ -\frac{\partial}{\partial r'} \int_{r'}^{r} dr'' \cdot A_{\text{tot}}(r'') - A_{\text{tot}}(r') \right]$$

$$- \int_{r'}^{r} dr'' \left( \partial_t A_{\text{tot}} + \frac{\partial \phi_{\text{tot}}}{\partial r'} \right) \rho_{ac}(r, r'). \quad (B1)$$

The explicit form of $D_2$, which contains all the dependence upon $\Delta a$, is
\[ i\hbar D_2(r, r', t) = \int dr''(1 - \rho_{ac}(r, r'')) \cdot \left[ -\frac{\partial}{\partial r''} \int_{r'}^{r''} \frac{e^{\Delta a(r''')}}{2m} \cdot \left[ -\Delta a(r''') \cdot A_{tot}(r''') + A_{tot}(r''') \right] \right] \rho_{ac}(r'', r') \]
\[ + \int dr'' \rho_{ac}(r, r'') \cdot \left[ \frac{\partial}{\partial r''} \int_{r'}^{r''} \frac{e^{\Delta a(r''')}}{2m} \cdot \left[ -\Delta a(r''') \cdot A_{tot}(r''') + A_{tot}(r''') \right] \right] (1 - \rho_{ac}(r'', r')) \]
\[ - \frac{i}{2\hbar} \int dr'' \rho_{ac}(r, r'') \left[ \int_{r'}^{r''} dr''' \cdot A_{tot}(r''') - \int_{r'}^{r''} dr''' \cdot A_{tot}(r''') - \int_{r'}^{r''} dr''' \cdot A_{tot}(r''') \right] \]
\[ \times \left[ \frac{1}{2m} \Delta a - \frac{i}{\hbar} \frac{\partial}{\partial r''} + A_{eff}(r'') - \Delta a_0 \right] \rho_{ac}(r'', r') \]
\[ - \frac{i}{2\hbar} \int dr'' \rho_{ac}(r, r'') \left[ \int_{r'}^{r''} dr''' \cdot A_{tot}(r''') - \int_{r'}^{r''} dr''' \cdot A_{tot}(r''') - \int_{r'}^{r''} dr''' \cdot A_{tot}(r''') \right] \]
\[ \times \left[ \frac{1}{2m} \Delta a - \frac{i}{\hbar} \frac{\partial}{\partial r''} + A_{eff}(r'') - \Delta a_0 \right] \rho_{ac}(r', r''). \]  

**APPENDIX C: THE LONGITUDINAL SEPARATION**

In this appendix we are interested in the distribution of the longitudinal separation \( \frac{1}{2} \mathbf{P} \cdot \Delta \mathbf{r} \) and its conjugate variable the size of the momentum \( P \). For simplicity we will only consider the case of a scalar interaction.

The first step is to rewrite the free particle action such that the longitudinal and transverse directions appear explicitly.

To this end we use polar coordinates for the momentum

\[ \mathbf{p} = \left( \frac{p \cos \frac{\mathbb{p}_n}{P}}{p \sin \frac{\mathbb{p}_n}{P}} \right), \]  

\( \frac{\mathbb{p}_n}{P} \) is the angle with respect to an arbitrarily defined \( x \)-axis. \( \mathbb{p}_n \), loosely speaking the arc-length along the Fermi surface, has been chosen as our variable instead of the angle \( \frac{\mathbb{p}_n}{P} \) in order to avoid a complicated Jacobian. In the scalar approximation the interaction is independent of \( \mathbb{p}_n \), which therefore can be integrated out.

The free particle action on the forward part of the contour is

\[ \int_0^t ds \left( \dot{\mathbf{r}} \cdot \mathbf{p} \cdot \frac{\mathbf{p}^2}{2m} \right). \]  

The stationary phase approximation for \( \mathbb{p}_n \) reads

\[ \mathbb{p}_n = p \arctan \frac{\dot{y}}{\dot{x}}. \]  

There are two solutions to this equation corresponding to the momentum and the velocity being parallel or anti-parallel. Here we only keep the parallel solution. The resulting stationary phase action on the forward contour is

\[ \int_0^t ds \left( p v - \frac{\mathbf{p}^2}{2m} \right), \]

where \( v = \sqrt{\dot{x}^2 + \dot{y}^2} \) is the velocity. A similar result holds on the backward path, but with the opposite overall sign. We are interested in discussing the separation between the two paths, so we change representation from the position and momentum along each path individually to the average \( \mathbf{R}(s) = \frac{1}{2} [\mathbf{r}(s, +) + \mathbf{r}(s, -)] \) and the difference \( \Delta \mathbf{r}(s) = \mathbf{r}(s, +) - \mathbf{r}(s, -) \). In a translation invariant system one has the choice between using \( \mathbf{R} \) or \( \hat{\mathbf{R}} \) as the variables in the path integral. Here we choose \( \hat{\mathbf{R}} \) and we parameterize it using the velocity \( V(s) = |\hat{\mathbf{R}}(s)| \) and the polar angle \( \frac{V_0(s)}{V(s)} \)

\[ \hat{\mathbf{R}}(s) = \left( \frac{V(s) \cos \frac{V_0(s)}{V(s)}}{V(s) \sin \frac{V_0(s)}{V(s)}} \right) = V(s) n(s). \]  

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The unit vector \( \mathbf{n}(s) \) points along the direction of the average propagation. \( \mathbf{n}(s) \) can also be thought of as a unit vector on the Fermi surface, since above in Eq. (C3) we only used the stationary phase condition where the momentum is aligned with the velocity.

Neglecting third order terms in the difference variables the free particle action is

\[
\sum_{\mu} \int_{0}^{t} ds \left[ p(s, \mu) v(s, \mu) - \frac{\mathbf{p}(s, \mu)^{2}}{2m} \right] \approx \int_{0}^{t} ds \left[ P(s) \mathbf{n}(s) \cdot \Delta \mathbf{r}(s) + \Delta p(s) V(s) \right] - \int_{0}^{t} ds \frac{P(s) \Delta p(s)}{m} + \mathbf{k} \cdot \left[ \left( \mathbf{R}(t) - \mathbf{R}(0) \right) - \int_{0}^{t} ds V(s) \mathbf{n}(s) \right],
\]

where \( \mathbf{k} \) is a Lagrange multiplier vector that has been introduced in order to deal with the boundary condition in the transition from \( \mathbf{R} \) to \( \mathbf{R} \). \( \frac{1}{2} \mathbf{k} |V_{\tau r}| \) is a small number, when the feeding-in and the pick-up are not local in space. As an illustration assume that the initial and final position is distributed according to isotropic Gaussians centered around \( \mathbf{R}^{0} \) and \( \mathbf{R}^{0} \) respectively. The sum of the variances is denoted by \( \left( \frac{\Delta \mathbf{r}}{\Delta \mathbf{r}} \right)^{2} \). The term \( \mathbf{k} \cdot (\mathbf{R}_{f} - \mathbf{R}_{i}) \) in the action is then replaced by

\[
\mathbf{k} \cdot (\mathbf{R}_{f}^{0} - \mathbf{R}_{i}^{0}) + \frac{i \hbar k^{2}}{2 \Delta k^{2}}
\]

after the integration with respect to \( \mathbf{R}_{i} \) and \( \mathbf{R}_{f} \) has been performed. The typical size of \( \mathbf{k} \) is now given by the smallest of the numbers \( \frac{\hbar}{|\mathbf{R}_{i} - \mathbf{R}_{f}|} \) and \( \Delta \lambda \). The last number is in the semi-classical limit small. In the case of a periodic driving \( \mathbf{k} \) is the wave-vector of the external field. Anyway in the semi-classical limit we will neglect the contribution from \( \mathbf{k} \).

1. The interaction part: general formulation

We first Taylor expand the interaction part of the action \( S_{\text{int}} \), the scalar part of Eq. (E), to second order in the difference variable \( \Delta \mathbf{r} \). Furthermore, we use a time local approximation. That is we approximate the quadratic terms \( \Delta \mathbf{r} \Delta \mathbf{r}' \) as \( \Delta \mathbf{r}^{2} \). Using this

\[
S_{\text{int}} \approx \frac{1}{2} \int_{0}^{t} ds \Delta \mathbf{r}(s) \cdot \mathbf{F}[\mathbf{R}, s] \cdot \Delta \mathbf{r}(s) - \int_{0}^{t} ds \Delta \mathbf{r}(s) \cdot \mathbf{h}[\mathbf{R}, P, s],
\]

where

\[
\mathbf{h}(\mathbf{R}, P, s) = h(V(s), P, s) \mathbf{n}(s)
\]

\[
\mathbf{F}[\mathbf{R}, s]_{\alpha \beta} = (\delta_{\alpha \beta} - \mathbf{n}_{\alpha} \mathbf{n}_{\beta}) F_{T}[V(s)] + \mathbf{n}_{\alpha} \mathbf{n}_{\beta} F_{L}[V(s)]
\]

with

\[
h(V(s), P, s) = \int_{0}^{t} ds'' \frac{\partial D_{00}}{\partial |\mathbf{R}|} (V(s)(s-s''), s-s'') \sum_{\mu''} \left[ \frac{1}{2} - w \left( P(s''+)(-1)^{\mu''} \frac{\Delta p(s'')}{2} \right) \right]
\]

\[
F_{L}[V(s)] = \int_{0}^{t} ds' \frac{\partial^{2} C_{00}}{\partial |\mathbf{R}|^{2}} (V(s)|s'-s|, s'-s)
\]

\[
F_{T}[V(s)] = \int_{0}^{t} ds' \frac{1}{V(s)|s'-s|} \frac{\partial C_{00}}{\partial |\mathbf{R}|} (V(s)|s'-s|, s'-s).
\]

As a further simplification we have above also approximated the average path by a straight line inside the kernels. In fact we will below treat both \( F_{T} \) and \( F_{L} \) as constants.

In this approximation the action separate into a longitudinal and a transverse part. We define

\[
\Delta \mathbf{r}(s) = \Delta r_{||}(s) \mathbf{n}(s) + \Delta r_{\perp}(s) \mathbf{z} \times \mathbf{n}(s).
\]

In the Wigner variables basis of the propagator we use a polar coordinate representation of the initial and final momenta.

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\[
P_t = P_t \begin{pmatrix} \cos \phi_t \\ \sin \phi_t \end{pmatrix} \quad (C14)
\]
\[
P_t = P_t \begin{pmatrix} \cos \phi_t \\ \sin \phi_t \end{pmatrix} \quad (C15)
\]

In the full propagator we now split out all the factors related to the transverse direction into the so-called transverse propagator
\[
J_T = \int D\phi \int D\Delta r_\perp 2\pi \hbar \delta \left[ \frac{\phi_l - V_\phi(t)}{V(t)} \right] 2\pi \hbar \delta \left[ \frac{\phi_l - V_\phi(0)}{V(0)} \right] e^{-\frac{\hbar}{2} \int_0^t ds \left[ \frac{1}{2} F_T \Delta r_\perp(s)^2 + i \Delta r_\perp(s) P(s) \frac{\phi_l - V_\phi(s)}{V(s)} \right]} \quad (C16)
\]

In terms of \( J_T \) the full propagator is after integrating out \( \Delta r(0) \) and \( \Delta r(t) \)
\[
J = \int D\phi \int D\Delta r_\perp \int D\phi \int D\Delta r_\parallel 2\pi \hbar \delta \left[ P_l - P(t) \right] 2\pi \hbar \delta \left[ P_l - P(0) \right] J_T 
\times e^{\frac{\hbar}{2} \int_0^t ds \left[ \Delta p(s) V(s) - \frac{\hbar}{2} P(s) \Delta p(s) \right] - \frac{\hbar}{2} \int_0^t ds \left[ \frac{1}{2} \Delta r_\perp(s)^2 F_L + i \Delta r_\perp(s) \left( \frac{1}{2} P(s) + \frac{1}{2} [V(s), P, s] \right) \right]} . \quad (C17)
\]

To lowest order the transverse propagator is independent of the dynamics of the longitudinal variables, so we will treat \( J_T \) as a constant and absorb it in the overall normalization of the path integral. The study of the remaining longitudinal propagator is the topic of the next subsection.

2. The interaction part: the longitudinal contribution

In the propagator Eq. (C17), we will neglect the dependence of \( h \) upon \( \Delta p \). \( V(s) \) thus equals \( m^{-1} P(s) \). The propagation described by Eq. (C17) is then equivalent to the quantum propagation problem
\[
-\hbar \partial_s = \frac{1}{2} \left( \frac{\hbar}{i} \frac{\partial}{\partial P} \right)^2 F_L(P) - \hbar \frac{\partial}{\partial P} h(P) \quad (C18)
\]

We neglect the momentum dependence of \( F_L \). In equilibrium is
\[
h(P) \approx \alpha \frac{1}{2} \tanh \left[ \frac{v_F}{2 k_B T} (P - p_F) \right] , \quad (C19)
\]

where the constant \( \alpha \) in the high temperature limit is related to the fluctuations through
\[
\frac{\hbar F_L}{\alpha} = \frac{k_B T}{v_F} . \quad (C20)
\]

The last term in Eq. (C18) is gauged away by the transformation \( \phi(P) \left( -\frac{\partial f}{\partial P} \right)^{-1/2} \) and the propagation problem Eq. (C18) is equivalent to
\[
-2 \frac{\hbar F_L}{\alpha^2} \partial_s \phi = -\frac{\hbar^2 F_L^2}{\alpha^2} \frac{\partial^2 \phi}{\partial P^2} + \left( -\frac{\hbar F_L}{\alpha} \frac{\partial}{\partial P} \right) \tanh \left[ \frac{1}{2} \frac{v_F}{k_B T} (P - p_F) \right] + \tanh^2 \left[ \frac{1}{2} \frac{v_F}{k_B T} (P - p_F) \right] \phi . \quad (C21)
\]

The ground state of the quantum problem Eq. (C21) is the derivative of the Fermi distribution function and the ground state energy is zero. The typical decay time of the excited states is given by \( \frac{\hbar F_L}{\alpha} \). This can be identified as the correlation time scale of \( \Delta r_\parallel \) or \( P \) with themselves. It is also the time scale it takes before the boundary conditions cannot be felt anymore. For the Coulomb interaction this time-scale is approximately \( \frac{\hbar}{k_B T} \approx (v_F p_F)^{-1} \). The ground-state wave-function of the original quantum propagation problem Eq. (C18) is proportional to \( -\frac{\partial f}{\partial P} \). This implies the distribution of the longitudinal momentum or energy is given by the derivative of the Fermi distribution function, while the average of the width squared is
\[
\left\{ \int_{-\infty}^{\infty} dp \left( -\frac{\partial f}{\partial p} \right)^{-1} \right\}^{-1} \int_{-\infty}^{\infty} dp \left( -\frac{\partial f}{\partial p} \right)^{-1} \left( -\frac{\hbar^2 v_F^2}{2} \right)^{-1} \left( -\frac{\hbar^2 v_F^2}{2} \right)^{-1} = \frac{1}{3} \frac{\hbar^2 v_F^2}{(k_B T)^2} . \quad (C22)
\]
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