Distribution of determinant of matrices with restricted entries over finite fields

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Abstract

For a prime power $q$, we study the distribution of determinant of matrices with restricted entries over a finite field $\mathbb{F}_q$ of $q$ elements. More precisely, let $N_d(\mathcal{A}; t)$ be the number of $d \times d$ matrices with entries in $\mathcal{A}$ having determinant $t$. We show that

$$N_d(\mathcal{A}; t) = (1 + o(1)) \frac{|\mathcal{A}|^{d^2}}{q},$$

if $|\mathcal{A}| = \omega(q^{2d-1})$, $d \geq 4$. When $q$ is a prime and $\mathcal{A}$ is a symmetric interval $[-H, H]$, we get the same result for $d \geq 3$. This improves a result of Ahmadi and Shparlinski (2007).

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1 Introduction

Throughout the paper, let $q = p^r$ where $p$ is an odd prime and $r$ is a positive integer. Let $\mathbb{F}_q$ be a finite field of $q$ elements. The prime base field $\mathbb{F}_p$ of $\mathbb{F}_q$ may then be naturally identified with $\mathbb{Z}/p$. For integer numbers $m$ and $n$, let $\mathcal{M}_{m,n}(\mathcal{A})$ denote the set of $m \times n$ matrices with components in the set $\mathcal{A}$. In [1], Ahmadi and Shparlinski studied some natural classes of matrices over a finite field $\mathbb{F}_p$ of $p$ elements ($p$ is a prime) with components in a given subinterval $[-H, H] \subseteq [-((p-1)/2), (p-1)/2]$. Let $N_d(\mathcal{A}; t)$ be the number of $d \times d$ matrices with entries in $\mathcal{A}$ having determinant $t$. Ahmadi and Shparlinski [1] proved the following result (see [1] and the references therein for the motivation and related results).
Theorem 1.1 ([1, Theorem 11]) For $1 \leq H \leq (p-1)/2$ and $t \in \mathbb{F}_p^*$, we have

$$N_d([-H, H]; t) = \frac{(2H + 1)^d}{p} + O(H^{d-2}p^{1/2}(\log p)^2).$$

Note that the proof of Theorem 11 in [1] is given only in the case $t = 1$, but it goes through without any essential changes for arbitrary $t \in \mathbb{F}_p^*$. The bound of Theorem 1.1 is nontrivial if $H \gg p^{3/4+\epsilon}$. In the case $d = 2$, they obtained a stronger result.

Theorem 1.2 ([1, Theorem 12]) For $1 \leq H \leq (p-1)/2$ and $t \in \mathbb{F}_p^*$, we have

$$N_2([-H, H]; t) = \frac{(2H + 1)^4}{p} + O(H^{2}p^{o(1)}).$$

Again, the proof of Theorem 12 in [1] is given only in the case $t = 1$, but it goes through without any changes for arbitrary $t \in \mathbb{F}_p^*$. The bound of Theorem 1.2 is nontrivial if $H \gg p^{1/2+\epsilon}$.

Covert et al. [2] studied this problem in a more general setting. More precisely, define \(\text{vol}(x^1, \ldots, x^d)\) to be the determinant of the matrix whose rows are \(x^j\)'s. The focus of [2] is to study the cardinality of the volume set

$$\text{vol}(E) = \{\text{vol}(x^1, \ldots, x^d) : x^j \in E\},$$

where \(E\) is a large subset of \(\mathbb{F}_q^d\). A subset \(E \subset \mathbb{F}_q^3\) is called a product-like set if \(|E \cap H_n| \lesssim |E|^{n/3}\) for any \(n\)-dimensional subspace \(H_n \subset \mathbb{F}_q^3\). Covert et al. [2] showed that

Theorem 1.3 ([2, Theorem 2.6]) Suppose that \(E \subset \mathbb{F}_q^3\) is product-like and \(t \in \mathbb{F}_q^*, \) then

$$|\{\text{vol}(x^1, x^2, x^3) = t : x^j \in E\}| = (1 + o(1))\frac{|E|^3}{q},$$

if \(|E| = \omega(q^{15/8})\).

Note that Theorem 2.6 in [2] only states that \(\mathbb{F}_q^* \subseteq \text{vol}(E)\) if \(|E| \gg q^{15/8}\) but the given proof in [2] indeed implies Theorem 1.3 above. We will use the geometry incidence machinery developed in that paper [2] and some properties of non-singular matrices to obtain the following asymptotic result for higher dimensional cases.

Theorem 1.4 For \(t \in \mathbb{F}_q^*, d \geq 2\) and \(A \subset \mathbb{F}_q^d\), we have

$$N_d(A; t) = (1 + o(1))\frac{|A|^{d^2}}{q},$$

if \(|A| = \omega(q^{d^2-d+1} + q^{d^2/2-d+2})\).
Note that the bound in Theorem 1.4 is $|A| = \omega(q^{\frac{d^2}{d+1}})$ if $d \geq 4$ and $|A| = \omega(q^{\frac{d^2}{d-1}})$ if $d = 2, 3$. When $d = 3$, Theorem 1.4 matches with the bound in Theorem 1.3, however the latter one holds for more general sets. Covert et al. \[2\] did not extend their result (Theorem 1.4 above) to higher dimensional cases as their focus is the function $|\text{vol}(E)|$. They instead showed that $|\text{vol}(E)| = \frac{1}{BY} \cdot q$ if $E = A \times A \times A \times A$ whenever $|A| > \sqrt{q}$. It seems that their proof can be extended to higher dimensional cases.

When $q = p$ is a prime and the set $A$ is an interval $[-H, H] \subset [-\frac{p-1}{2}, \frac{p-1}{2}]$, using Theorem 1.2, we obtain a stronger result for $3 \times 3$ matrices.

**Theorem 1.5** For $1 \leq H \leq \frac{p-1}{2}$ and $t \in \F_p^*$, we have

$$N_3([-H, H]; t) = (1 + o(1)) \frac{(2H + 1)^9}{p},$$

if $H = \omega(p^{\frac{d}{2}})$.

Note that the implied constants in the symbols $O, o, \Theta, \Omega, \omega,$ and $\ll$ may depend on integer parameter $d$. We recall that the notation $U = O(V)$ and $U \ll V$ are equivalent to the assertion that the inequality $|U| \leq cV$ holds for some constant $c > 0$. The notation $U = \Omega(V)$ is equivalent to the assertion that $U \geq c|V|$ holds for some constant $c > 0$. We say that $U = o(V)$ if $U = O(V)$ but $U \neq \Omega(V)$ and $U = \omega(V)$ if $U = \Omega(V)$ but $U \neq O(V)$.

## 2 Some estimates

### 2.1 Geometric Incidence Estimate

Let $f$ be a complex-valued function on $\F_q^d$, we define the $r$-norm of $f$ on $\F_q^d$ by

$$\|f\|_r = \left( \sum_{x \in \F_q^d} |f(x)|^r \right)^{1/r}.$$

The Fourier transform of $f$ on $\F_q^d$ with respect to a non-trivial principal additive character $\chi$ on $\F_q$ is given by

$$\hat{f}(m) = q^{-d} \sum_{x \in \F_q^d} f(x) \chi(-x \cdot m).$$

One of our main tools is the following geometric incidence estimate which was developed and used in \[2\] (see also \[3, 4\] for earlier versions of this estimate).

**Theorem 2.1** (\[2, Theorem 2.1\]) Let $B(\cdot, \cdot)$ be any nondegenerate bilinear form in $\F_q^d$. Let

$$\nu(t) = \sum_{B(x, y) = t} f(x)g(y),$$

then
where \( f, g \) are non-negative functions on \( \mathbb{F}^d_q \). Then
\[
\nu(t) = q^{-1}\|f\|_1\|g\|_1 + R(t),
\]
where
\[
|R(t)| \leq q^{\frac{d-1}{2}}\|f\|_2\|g\|_2,
\]
if \( t \neq 0 \). Moreover, if \((0, \ldots, 0) \notin \text{support}(f) \equiv E\), then
\[
\sum_{t \in \mathbb{F}_q} \nu^2(t) \leq q^{-1}\|f\|_2^2 \cdot |E| \cdot \|g\|_1^2 + q^{2d-1}\|f\|_2^2 \sum_{k \neq (0, \ldots, 0)} |\hat{g}(k)|^2 |E \cap l_k|,
\]
where
\[
l_k = \{tk : t \in \mathbb{F}^*_q\}.
\]

Note that the proof of Theorem 2.1 in [2] is given only in the case of dot product, but it goes through without any essential changes if the dot product is replaced by any non-degenerate bilinear form.

Theorem 2.1 has several applications in additive combinatorics (see [2, 3, 4]). We present here another application of this theorem to the problem of finding three term arithmetic progression in productsets over finite fields. Using multiplicative character sums, Shparlinski [6] showed that for any integer \( k \) with \( p > k \geq 3 \), where \( p \) is the characteristic of \( \mathbb{F}_q \), and any two sets \( A, B \subset \mathbb{F}_q \) with
\[
|A||B| \geq \frac{(k-1)^2}{q} q^{2-1/(k-1)},
\]
the productset \( AB \) contains a \( k \)-term arithmetic progression. He asked if one can relax the condition \( k < p \). We give an affirmative answer for this question in the easiest case, \( k = 3 \). It is enough to show that the following equation has solution
\[
x_0y_0 + x_2y_2 = 2x_1y_1, \ x_i \in A, y_i \in B,
\]
has a solution given that \( x_0y_0, x_2y_2 \neq x_1y_1 \). Fix some \( x_1 \in A, y_1 \in B \) such that \( x_1y_1 \neq 0 \). From (2.1), the number of quadtuples \((x_0, y_0, x_2, y_2)\) satisfying Eq. (2.3) is at least
\[
\frac{|A|^2|B|^2}{q} - \sqrt{q}|A||B|.
\]
Besides, the number of quadtuples \((x_0, y_0, x_2, y_2)\) with \( x_0y_0 = x_2y_2 = x_1y_1 \) is bounded by \( |A||B| \) (as for each \((x_0, y_0) \in A \times B\), we have at most one choice for \((y_0, x_2)\)). Therefore, the productset \( AB \) contains a 3-term arithmetic progression if \( |A||B| > q(\sqrt{q} + 1) \). Note that for \( k = 3 \), the question of [6] is indeed a question about vanishing bilinear forms, so there is no surprise that it admits a different approach using exponential sums, which however is not likely to help for \( k > 3 \).
2.2 Recursive estimates

Let \( N_d(A; t) \) be the number of \( d \times d \) matrices with entries in \( A \) having determinant \( t \). The following theorem says that \( N_d(A; t) \) can be bounded by \( N_{d-1}(A; l) \)'s.

**Theorem 2.2** For any \( t \in \mathbb{F}_q^* \) then

\[
|N_d(A; t) - \frac{|A|^{d^2}}{q}|^2 \leq q^{d-1}|A|^{2d-1}(1 + o(1)) \sum_{l \in \mathbb{F}_q^*} N_{d-1}^2(A; l).
\]

**Proof** For any \( M \in \mathcal{M}_{d-1,d}(A) \), let \( m_i \) be the \( i \)th column of \( M \) and \( M_i \) be the \((d - 1) \times (d - 1)\) minor of \( M \) by deleting \( m_i \). Define

\[
v(M) = ((-1)^i \det(M_i))_{1 \leq i \leq d} \in \mathbb{F}_q^d.
\]

For any \( x = (x_1, \ldots, x_d) \in \mathbb{F}_q^d \), let \( f(x) := A^d(x) = A(x_1) \ldots A(x_d) \) where \( A(\cdot) \) is the characteristic function of the set \( A \), and define

\[
g(x) := |\{ M \in \mathcal{M}_{d-1,d}(A) : v(M) = x \}|
\]

It follows that

\[
N_d(A; t) = \sum_{x \cdot y = t} f(x)g(y).
\]

We have \( \|f\|_1 = \|f\|_2 = |A|^d \) and \( \|g\|_1 = |A|^{(d-1)d} \). From (2.1), we have

\[
\left| N_d(A; t) - \frac{|A|^{d^2}}{q} \right|^2 \leq q^{d-1}|A|^d\|g\|_2^2.
\]

(2.4)

Now, we estimate \( \|g\|_2^2 \). Note that \( x \cdot y = t \in \mathbb{F}_q^* \) so \( y \neq (0, \ldots, 0) \). Therefore

\[
\|g\|_2^2 = \sum_{y \neq (0, \ldots, 0)} g^2(y)
\]

\[
= \sum_{i=1}^d \sum_{y_i \in \mathbb{F}_q^*} \sum_{y_j \in \mathbb{F}_q, j > i} g^2(0, \ldots, 0, y_i, \ldots, y_d).
\]

(2.5)

We need the following lemma.

**Lemma 2.3** For any \( 1 \leq i \leq d \), then

\[
\sum_{y_i \in \mathbb{F}_q^*} \sum_{y_j \in \mathbb{F}_q, j > i} g^2(0, \ldots, 0, y_i, \ldots, y_d) \leq |A|^{d-i} \sum_{l \in \mathbb{F}_q^*} N_{d-1}^2(A; l).
\]
Proof (of the lemma) For any $M \in \mathcal{M}_{d-1,d}(A)$, let $m_i$ be the $i$th column of $M$ and $M_i$ be the $(d-1) \times (d-1)$ minor of $M$ by deleting $m_i$. For any fixed $y_i, \ldots, y_d \in \mathbb{F}_q$ and $M_i \in \mathcal{M}_{d-1,d-1}(A)$ with $\det(M_i) = (-1)^i y_i \in \mathbb{F}_q^*$. Let

$$\mathbf{y} = \frac{1}{\det(M_i)} (0, \ldots, 0, y_{i+1}, \ldots, y_d)^t \in \mathbb{F}_q^d.$$ 

We have

$$v(M) = ((-1)^i \det(M_i))_{1 \leq i \leq d} = (0, \ldots, 0, y_i, \ldots, y_d).$$

Hence, by Cramer’s rule and the non-singularity of $M_i$, we have

$$M_i \mathbf{y} = m_i.$$ 

So there is at most one possibility of $m_i$ for each fixed $y_i, \ldots, y_d$ and $M_i$. This implies that

$$g(0, \ldots, 0, y_i, \ldots, y_d) \leq N_{d-1}(A; (-1)^i y_i),$$

for any $y_i \in \mathbb{F}_q^*$. Since $\det(M_i) = (-1)^i y_i \in \mathbb{F}_q^*$, we can write (2.6) as

$$(0, \ldots, 0, y_{i+1}, \ldots, y_d)^t = \det(M_i) M_i^{-1} m_i.$$ 

By Gaussian elimination, we can remove all nonzero entries under the main diagonal in the first $i-1$ rows of $\det(M_i) M_i^{-1}$. Since $m_i \in \mathcal{A}^{d-1}$, for any fixed $M_i$, there are at most $|\mathcal{A}|^{d-i}$ possibilities for $(y_{i+1}, \ldots, y_d)$. This implies that, for any $y_i \in \mathbb{F}_q^*$ then

$$\sum_{y_j \in \mathbb{F}_q^*, j > i} g(0, \ldots, 0, y_i, \ldots, y_d) \leq |\mathcal{A}|^{d-i} N_{d-1}(A; (-1)^i y_i).$$

If $0 \leq x, y \leq A$, then $x^2 + y^2 \leq (\max\{A, x + y\}^2 + (x + y - \max\{A, x + y\})^2$. Thus, from (2.7) and (2.8), we have

$$\sum_{y_j \in \mathbb{F}_q^*, j > i} g^2(0, \ldots, 0, y_i, \ldots, y_d) \leq |\mathcal{A}|^{d-i} N_{d-1}^2(A; (-1)^i y_i).$$

Taking sum over all $y_i \in \mathbb{F}_q^*$, the lemma follows. □

From (2.5) and Lemma 2.3, we have

$$\|g\|_2^2 \leq (|\mathcal{A}|^{d-1} + \ldots + 1) \sum_{l \in \mathbb{F}_q^*} N_{d-1}^2(A; l) = |\mathcal{A}|^{d-1}(1 + o(1)) \sum_{l \in \mathbb{F}_q^*} N_{d-1}^2(A; l).$$

The theorem follows immediately from (2.4) and (2.9). □

Theorem 2.4 For any $d \geq 2$, then

$$\sum_{l \in \mathbb{F}_q^*} N_d^2(A; l) \leq (1 + o(1)) \frac{|\mathcal{A}|^{2d}}{q} + q^{d-1} |\mathcal{A}|^{2d} (1 + o(1)) \sum_{l \in \mathbb{F}_q^*} N_{d-1}^2(A; l).$$
**Proof** Similarly as in the proof of Theorem 2.2, for any \( x = (x_1, \ldots, x_d) \in \mathbb{F}_q^d \), let 
\( f(x) := \mathcal{A}^d(x) = \mathcal{A}(x_1) \ldots \mathcal{A}(x_d), \) and define 
\[ g(x) := |\{ M \in \mathcal{M}_{d-1,d}(\mathcal{A}) : v(M) = x\}|. \]

Let \( f_0(x) = f(x), g_0(x) = g(x) \) if \( x \neq (0, \ldots, 0) \) and \( f_0(x) = g_0(x) = 0 \) otherwise. Then 
\[ N_d(\mathcal{A}; t) = \sum_{x \cdot y = t} f_0(x) g_0(y), \]
if \( t \in \mathbb{F}_q^* \). Since \( (0, \ldots, 0) \notin \text{support}(f_0) \equiv E \subseteq \mathcal{A}^d \), from (2.2) and Plancherel’s theorem, we have

\[
\sum_{l \in \mathbb{F}_q^*} N_d^2(\mathcal{A}; l) \leq \sum_{t \in \mathbb{F}_q} \left( \sum_{x \cdot y = t} f_0(x) g_0(y) \right)^2 \\
\leq q^{-1} \|f_0\|_2^2 \cdot |E| \cdot \|g_0\|_1^2 + q^{2d-1} \|f_0\|_2^2 \sum_{k \neq (0, \ldots, 0)} |\hat{g}_0(k)|^2 |E \cap l_k| \\
\leq \frac{|\mathcal{A}|^{2d^2}}{q} + q^{d-1} |\mathcal{A}|^{d+1} q^{-d} \sum_{y \in \mathbb{F}_q^d} g_0^2(y) \\
= (1 + o(1)) \frac{|\mathcal{A}|^{2d^2}}{q} + q^{d-1} |\mathcal{A}|^{d+1} \sum_{y \neq (0, \ldots, 0)} g^2(y),
\]
(2.10)
since \( |E \cap l_k| \leq |\mathcal{A}| \) for any \( k \neq (0, \ldots, 0) \). From (2.9), we have
\[
\sum_{y \neq (0, \ldots, 0)} g^2(y) \leq (1 + o(1)) |\mathcal{A}|^{d-1} \sum_{t \in \mathbb{F}_q^*} N_d^2(\mathcal{A}; l).
\]
(2.11)

The theorem follows from (2.10) and (2.11). \( \square \)

### 3 Distribution of determinant

#### 3.1 Arbitrary sets (Proof of Theorem 1.4)

From Theorem 2.4, we have the following corollary.

**Corollary 3.1** For \( \mathcal{A} \subseteq \mathbb{F}_q \) and \( d \geq 2 \), we have
\[
\sum_{l \in \mathbb{F}_q^*} N_d^2(\mathcal{A}; l) = O \left( q^{-1} |\mathcal{A}|^{2d^2} + q^{d(d-1)} |\mathcal{A}|^{d(d+1)-1} \right).
\]

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Proof The proof proceeds by induction. For the base case \( d = 2 \), it follows from Theorem 2.4 that

\[
\sum_{l \in \mathbb{F}_q^*} N_2^2(A; l) \leq (1 + o(1)) \frac{|A|^8}{q} + (1 + o(1))q|A|^4 \sum_{t \in \mathbb{F}_q^*} N_1^2(A; t)
= O \left( q^{-1} |A|^8 + q |A|^5 \right).
\]

Suppose that the corollary holds for \( d - 1 \), we show that it also holds for \( d \). By induction hypothesis, we have

\[
\sum_{t \in \mathbb{F}_q^*} N_{d-1}^2(A; t) = O \left( q^{-1} |A|^{2(d-1)^2} + q \frac{(d-1)(d-2)}{2} |A|^{d(d-1)-1} \right). \tag{3.1}
\]

Theorem 2.4 implies that

\[
\sum_{l \in \mathbb{F}_q^*} N_2^2(A; l) \leq (1 + o(1)) \frac{|A|^{2d^2}}{q} + (1 + o(1))q^{d-1} |A|^{2d} \sum_{t \in \mathbb{F}_q^*} N_{d-1}^2(A; l)
= O \left( q^{-1} |A|^{2d^2} + q^{d-2} |A|^{2d^2 - 2d + 2} + q^{\frac{d(d-1)}{2}} |A|^{d(d+1)-1} \right)
= O \left( q^{-1} |A|^{2d^2} + q^{\frac{d(d-1)}{2}} |A|^{d(d+1)-1} \right),
\]

where the second line follows from (3.1) and the last line follows from

\[ q^{d-2} |A|^{2d^2 - 2d + 2} = O \left( q^{-1} |A|^{2d^2} + q^{\frac{d(d-1)}{2}} |A|^{d(d+1)-1} \right). \]

This completes the proof of the corollary.

We are now ready to give a proof of Theorem 1.4. It follows from Theorem 2.2 and Corollary 3.1 that

\[
\left| N_d(A; t) - \frac{|A|^{2d}}{q} \right| \leq q^{d-1} |A|^{2d-1} (1 + o(1)) \sum_{l \in \mathbb{F}_q^*} N_{d-1}^2(A; l)
= O \left( q^{d-2} |A|^{2d^2 - 2d + 1} + q^{\frac{d(d-1)}{2}} |A|^{d(d+1)-2} \right)
= o \left( q^{-2} |A|^{2d^2} \right),
\]

given that

\[ |A| = \omega(q^{\frac{d}{2d-1}} + q^{\frac{d^2 - d + 4}{2(d^2 - d + 2)}}). \]

This completes the proof of the theorem.
3.2 Intervals (Proof of Theorem 1.5)

It follows from Theorem 1.2 that
\[ \sum_{l \in \mathbb{F}_q^*} N^2_2([-H, H]; l) \leq (p - 1) \left( \frac{(2H + 1)^4}{p} + O(H^2 p^{o(1)}) \right)^2 \]
\[ = O\left( p^{-1} H^8 + p^{1+o(1)} H^4 \right). \quad (3.2) \]

From Theorem 2.2 and (3.2), we have
\[ \left| N_3([-H, H]; t) - \frac{(2H + 1)^9}{p} \left( \frac{2H + 1}{p} \right)^5 \right|^2 \leq p^2 (2H + 1)^5 (1 + o(1)) \sum_{l \in \mathbb{F}_q^*} N^2_2([-H; H]; l) \]
\[ = O\left( p H^{13} + p^{3+o(1)} H^9 \right). \]

This implies that
\[ N_3([-H, H]; t) = (1 + o(1)) \frac{(2H + 1)^9}{p} \]
if \( H = \omega(p^{3/5}) \), completing the proof of Theorem 1.5.

4 Remarks

Note that the quantity
\[ \sum_{l \in \mathbb{F}_q^*} N^2_2(A; l) \]
is equal to the number of matrices \( M, N \) with entries from \( A \) such that \( MN^{-1} \in \text{SL}_q(d) \). Let \( S_d(A) \) denotes this quantity, it follows from Corollary 3.1 that
\[ S_d(A) = O\left( q^{-1} |A|^{2d^2} + q^{\frac{d(d-1)}{2}} |A|^{d(d+1)-1} \right). \]

Similarly, one can get an estimate of this type for \( S_d([-H; H]) \) from Theorem 1.2 and Theorem 2.4.

Corollary 4.1 Suppose that \( p \) is a prime and \( 1 \leq H \leq (p - 1)/2 \). We have
\[ S_d([-H, H]) = O\left( p^{-1} H^{2d^2} + p^{\frac{d(d-1)}{2} + o(1)} H^{d(d+1)-2} \right). \]

Proof The proof proceeds by induction. For the base case \( d = 2 \), it follows from Theorem 1.2 that
\[ S_2([-H, H]) \leq (p - 1) \left( \frac{(2H + 1)^4}{p} + O(H^2 p^{o(1)}) \right)^2 \]
\[ = O\left( p^{-1} H^8 + p^{1+o(1)} H^4 \right). \]
Suppose that the corollary holds for $d - 1$, we show that it holds for $d$. Theorem 2.4 implies that

$$S_d([-H, H]) \leq (1 + o(1)) \frac{(2H + 1)^{2d^2}}{p} + (1 + o(1))p^{d-1}(2H + 1)^{2d}S_{d-1}([-H, H])$$

$$= O\left(p^{-1}H^{2d^2} + p^{d-2}H^{2d^2-2d+2} + p\frac{d(d-1)}{2}+o(1)H^{d(d+1)-2}\right)$$

$$= O\left(p^{-1}H^{2d^2} + p\frac{d(d-1)}{2}+o(1)H^{d(d+1)-2}\right),$$

where the second line follows from the induction hypothesis and the last line follows from

$$p^{d-2}H^{2d^2-2d+2} = O\left(p^{-1}H^{2d^2} + p\frac{d(d-1)}{2}+o(1)H^{d(d+1)-2}\right).$$

This completes the proof of the corollary.

It has been pointed out by the referee that the bound in Corollary 4.1 can be improved for small value of $H$.

**Lemma 4.2** Suppose that $q = p$ is a prime and $H = O(p^{1/2})$, then

$$S_d([-H, H]) = O\left(p\frac{d(d-1)}{2}H^{d(d+1)+o(1)}\right).$$

**Proof** The proof proceeds by induction. For the base case $d = 2$ we have

$$S_2([-H, H]) \leq |\{x_1y_2 - x_2y_1 \equiv u_1v_2 - u_2v_1 \pmod{p}\}|$$

where all variables are in $[-H, H]$. For each choice of $x_2, y_1, u_1, v_1, u_2$ and $v_2$, we get $x_1y_2 \equiv a \pmod{p}$ for some $a \in \mathbb{F}_q$.

If $a = 0$, then there are $O(H^5)$ posibilities for $x_2, y_1, u_1, v_1, u_2, v_2$ and $O(H)$ posibilities for $x_1, y_2$. If $a \in \mathbb{F}_q^*$, then the arithmetic progression $z \equiv a \pmod{p}$ contains $O(H^2/p + 1)$ elements $|z| \leq H^2$. Since $z \neq 0$ for all of them, $z = x_1y_2$ has $H^{o(1)}$ solutions. Putting everything together, we get

$$S_2([-H, H]) = O(H^6 + H^6H^2/p + 1)H^{o(1)}) = O(H^{6+o(1)})$$

if $H = O(p^{1/2})$. Suppose that the lemma holds for $d - 1$, we show that it holds for $d$. Theorem 2.4 implies that

$$S_d([-H, H]) = O(H^{2d^2}/p + p^{d-1}H^{2d}S_{d-1}([-H; H]))$$

$$= O(H^{2d^2}/p + p\frac{d(d-1)}{2}H^{d(d+1)+o(1)})$$

$$= O(p\frac{d(d-1)}{2}H^{d(d+1)+o(1)}),$$

where the second line follows from the induction hypothesis and the last line follows from

$H = O(p^{1/2})$. This completes the proof of the lemma.

\[\square\]
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