QUOTIENTS OF AFFINE SPACES FOR ACTIONS OF REDUCTIVE GROUPS

MIHAI HALIC

This article is devoted to the study of invariant quotients of an affine space by the action of a reductive group. There are a multitude of examples of varieties arising in this way: the most at hand are the toric varieties, obtained for representations of tori, and the quiver varieties, which include as very particular cases the flag varieties of the general linear group.

The possible linearizations of the action of a linear group on an affine space, and more generally on a normal affine variety, are given by the characters of the group. A first issue that we address is that of the existence of a fan structure on the space of characters, whose cones parameterize the possible invariant quotients which appear.

Theorem Let $X$ be a normal, affine $G$-variety. The GIT-equivalence classes in $X^*(G) \mathbb{R}$ corresponding to the $G$-action on $X$ are the relative interiors of the cones of a rational, polyhedral fan $\Delta^G(X)$, which we call the GIT-fan of $X$.

This result can be viewed as a non-abelian generalization of the Gelfand-Kapranov-Zelevinskij decomposition for linear representations of tori. The analogous problem for projective varieties has been investigated in [6, 21]. The name of GIT-fan is borrowed from [21], where the author proves a similar result for projective varieties. The strategy for proving our result is to reduce the problem to the projective case, where we can apply the result of Ressayre.

In the rest of the paper we have tried to answer various questions concerning the geometrical properties of the quotients of an affine space $V$ on which $G$ acts linearly. The first issue in this direction is that of computing the Chow ring of the invariant quotients.

Theorem Let $\chi \in X^*(G)$ be a character with the property that the corresponding semi-stable locus coincides with the properly stable one. Then

$$A_*(V//_\chi G)_\mathbb{Q} \cong \left(A^*_\mathbb{Q}\right) / \varphi\langle [E]_T; E \text{ is a } (T, \chi)\text{-unstable component of } V\rangle_\mathbb{Q},$$

where $\varphi$ denotes the projection from the $T$- to the $G$-equivariant Chow ring of a point. If moreover the ground field is that of the complex numbers and $V//_\chi G$ is projective, then its cohomology ring is isomorphic to the Chow ring.

This formula has been obtained by Ellingsrud and Strømme in [9] under the additional assumptions that the group $G$ acting on the affine space contains its homotheties, and the stability concept corresponds to a character with large weight on the subgroup of homotheties.

We also answer in positive a question raised in loc. cit., asking whether the Chow ring of the quotients is generated by the Chern classes of ‘natural’ vector bundles, induced by representations of the group $G$.

In section 6 we use the techniques developed by Seshadri in [23], to show that projective varieties of the form $V//_\chi G$ actually fit together, in families over separated and reduced schemes over $\text{Spec} \mathbb{Z}$, and moreover that the absolute case considered in the previous sections

2000 Mathematics Subject Classification. 14L24,14L30,14F43,14F17.
corresponds to base change. If the base is \( \text{Spec } \mathbb{Z} \), we obtain flat families of quotients, which will eventually allow us to extend in positive characteristic certain cohomological properties which hold in characteristic zero. In theorem 6.2 we compute the Chow ring of families of such varieties, parameterized by reduced schemes defined over algebraically closed fields, and also, in case we work over the complex numbers, the cohomology ring of the total space of the fibration.

We conclude the article by computing the Picard group and the ample cone of the geometric quotients that we obtain. Using the Hochster-Roberts theorem, we further prove the vanishing of higher cohomology groups of nef line bundles.

**Theorem** Let \( \chi \in X^*(G) \) be a character such that the codimension of the unstable locus \( \text{codim}_V \text{V}^s(G, \chi) \geq 2 \), and \( \text{V}^s(G, \chi) = \text{V}^s_0(G, \chi) \). There is a prime \( p(\chi) \) depending only on the GIT-chamber of \( \chi \), having the following property: if the characteristic of the ground field is zero or larger than \( p(\chi) \), then for any nef, invertible sheaf \( \mathcal{L} \to \mathcal{V} \), \( \mathcal{L} \to \mathcal{V} \chi G \) holds:

\[
H^i(\mathcal{V} \chi G, \mathcal{L}) = 0, \quad \forall i > 0, \quad \text{and} \quad H^i(\mathcal{V} \chi G, \mathcal{L}^{-1}) = 0, \quad \forall i \neq \kappa(\mathcal{L}),
\]

where \( \kappa(\mathcal{L}) \) denotes the Kodaira-Iitaka dimension.

In characteristic zero, we prove in theorem 7.1 that a similar vanishing result holds for the quotients of smooth, affine varieties.

**Acknowledgements** This work has been partially supported by EAGER - European Algebraic Geometry Research Training Network, contract No. HPRN-CT-2000-00099 (BBW). I am very indebted to M. Brodmann for patiently explaining me properties of the local cohomology modules, that I needed in section 7. I thank also S. Stupariu for useful discussions, which helped me to improve the presentation in section 8.

1. Stability for linear actions of reductive groups I

Let \( K \) be an algebraically closed field of arbitrary characteristic, \( G \) a reductive group over \( K \), and \( G_m := \text{Spec } K[t, t^{-1}] \) the multiplicative group of \( K \). We denote \( Z(G)^o \) the connected component of the centre of \( G \); we fix a maximal torus \( T \) of \( G \) and denote by \( W := N_G(T)/T \) the corresponding Weyl group. Further, we consider a finite dimensional \( G \)-module \( V \), such that the representation \( p : G \to \text{Gl}(V) \) has finite kernel.

Let \( \mathcal{V} := \text{Spec } \text{Sym}^* V \) be the affine space corresponding to \( V \), and consider

\[
\Sigma : G \times \mathcal{V} \to \mathcal{V}
\]

the natural \( G \)-action on it (see [20 section 1.1]). Since \( \text{Pic}(\mathcal{V}) = \{ \mathcal{O}_\mathcal{V} \} \), the possible linearizations of \( \Sigma \) are parameterized by the group of characters \( X^*(G) \) of \( G \): namely, \( \chi \in X^*(G) \) defines the linearization

\[
(1.1) \quad \Sigma_\chi : G \times (\mathcal{V} \times A^1_K) \to \mathcal{V} \times A^1_K, \quad \Sigma_\chi(g, (x, z)) := \left( \Sigma(g, x), \chi(g)z \right).
\]

In this formula, and overall in the article, we adopt the convention that for defining the linearization of a \( G \)-action in a sheaf we indicate the corresponding \( G \)-action on the geometric line bundle defined by the sheaf. In the case above, the geometric line bundle is \( \mathcal{O}_\mathcal{V} := \text{Spec } \text{Sym}^* \mathcal{O}_\mathcal{V} = \mathcal{V} \times A^1_K \). A function \( f \in K[\mathcal{V}] \) is \( \Sigma_\chi \)-invariant precisely when \( f(\Sigma(g, x)) = \chi(g)f(x) \), for all \( (g, x) \in G \times \mathcal{V} \). For \( n \geq 0 \), we denote by \( K[\mathcal{V}]_{\chi^n}^G \) the vector space of \( \Sigma_\chi^n \)-invariant functions on \( \mathcal{V} \), and define the algebra

\[
K[\mathcal{V}]_{\chi^n}^G := \bigoplus_{n \geq 0} K[\mathcal{V}]_{\chi^n}^G.
\]
The goal of this section is to describe the (semi-)stable locus $\mathcal{V}^{ss}(G, \chi)$ corresponding to $\Sigma \chi$, for $\chi \in \mathcal{X}^+(G)$. Proposition 1.2 below is stated explicitly in [18, proposition 2.5], but the proof is very sketchy. Since for our computations we need a clear description of the unstable locus, we found useful to give a thorough proof of the numerical criterion for linear actions. Another remark is that in loc. cit. the author allows the representation $\rho$ to have kernel; however, one notices that, if the $\Sigma \chi$-semi-stable locus is not empty, then the restriction to the connected component of the identity $\chi|_{\ker \rho} = 1$, by trivial reasons. Therefore the assumption that the kernel of $\rho$ is finite is not restrictive.

A one-parameter subgroup (1-PS for short) $\lambda \in \mathcal{X}_s(G)$ of $G$ decomposes $V$ into the direct sum of the weight spaces $V = \bigoplus_j V_j$, where $\lambda(t)|_{V_j} = t^j \text{Id}_{V_j}$. We observe that, except for a finite number of $j$'s, all the $V_j$'s are zero. This decomposition breaks $V$ into the direct product

$$\mathcal{V} = \text{Spec}(\text{Sym}^\bullet V) = \text{Spec}(\bigotimes_j \text{Sym}^\bullet V_j) = \times_j \mathcal{V}_j,$$

and we remark that $\lambda$ acts on $\mathcal{V}_j$ by multiplication through $t^j$. We denote $(x_j)_j$ the components of a point $x \in \mathcal{V}$ with respect to this decomposition, and we define

$$m(x, \lambda) := \begin{cases} \min \{ j \mid x_j \neq 0 \} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Lemma 1.1. (i) For $\chi \in \mathcal{X}^+(G)$, the corresponding $\chi$-unstable locus is

$$\mathcal{V}^{\text{us}}(G, \chi) = \bigcup_{\lambda \in \mathcal{X}_s(T), \langle \chi, \lambda \rangle < 0} G \cdot E(\lambda) = G \cdot \mathcal{V}^{\text{us}}(T, \chi),$$

where $E(\lambda) := \{ x \in \mathcal{V} \mid m(x, \lambda) \geq 0 \}$. It is invariant under the action of $N_{\text{Gl}(V)} G$, the normalizer of $G$ in $\text{Gl}(V)$.

(ii) For each $\lambda \in \{ \langle \chi, \cdot \rangle < 0 \}$, $E(\lambda)$ is a linear subspace of $\mathcal{V}$, which is stable under the parabolic subgroup

$$P(\lambda) := \{ g \in G \mid \lim_{t \to 0} \lambda(t)g\lambda(t)^{-1} \text{ exists in } G \}$$

of $G$. As a consequence, $G \cdot E(\lambda)$ is closed in $\mathcal{V}$.

(iii) There is a finite set $\mathcal{F}(\chi) \subset \{ \langle \chi, \cdot \rangle < 0 \}$, such that

$$\bigcup_{\lambda \in \mathcal{X}_s(T), \langle \chi, \lambda \rangle < 0} G \cdot E(\lambda) = \bigcup_{\lambda \in \mathcal{F}(\chi)} G \cdot E(\lambda).$$

It corresponds to the irreducible components of the union on the left-hand-side above.

Proof. We define the representation

$$\rho' : G \to \text{Gl}(K \oplus V), \quad \rho'(g) := \text{diag}(\chi^{-1}(g), \rho(g)),$$

and consider the induced action $\Sigma' : G \times (A^1_K \times V) \to A^1_K \times V$, $\Sigma'(g, (z, x)) = (\chi(g)z, \Sigma(g, x))$. The corresponding ring of invariants is obviously

$$K[A^1_K \times V]^G = \left\{ F = \sum_{n \geq 0} t^n f_n \mid f_n \in K[A^1_K]^G \right\}.$$
We deduce that
\[ x \in \mathcal{V}^\text{us}(G, \chi) \iff f(x) = 0, \forall f \in K[\mathcal{V}]^G_{\chi, n}, n \geq 1 \]
\[ \iff F(1, x) = F(0, x) = f_0(x), \forall F \in K[\mathbb{A}^1_K \times \mathcal{V}]^G \]
\[ \iff F(1, x) = 0, \forall F \in (\mathcal{I}[0]_{\chi, n})^G \]
\[ \iff \Sigma'(G \times (1, x)) \cap \{(0) \times \mathcal{V}\} \neq \emptyset. \]

We deduce from [16] theorem 1.4] that the last condition is equivalent to the existence of \( \lambda \in \mathcal{X}_0^*(G) \) such that \( \lim_{t \to 0} \lambda(t) \cdot 1, x \in \{0\} \times \mathcal{V} \). This happens precisely when \( \langle \chi, \lambda \rangle < 0 \) and \( m(x, \lambda) \geq 0 \). The conclusion follows now, because any such \( \lambda \) is contained in a maximal torus of \( G \), and moreover all maximal tori of \( G \) are conjugated to \( T \).

(ii) The first part is obvious and we prove the second one: we notice that \( \mathbb{E}(\lambda) = \{x \in \mathcal{V} \mid \lim_{t \to 0} \lambda(t) \cdot x \text{ exists in } \mathcal{V}\} \). For \( g \in P(\lambda) \), we may write
\[ \Sigma(\lambda(t), \Sigma(g, x)) = \Sigma(\lambda(t) g \lambda(t)^{-1}, \Sigma(\lambda(t), x)), \]
and deduce that the limit at zero exists, hence \( \Sigma(g, x) \in \mathbb{E}(\lambda) \). The last statement is proved in proposition 1.5.

(iii) The number of one parameter subgroups of a torus is countable, and a countable union of closed subvarieties of \( \mathcal{V} \), which is itself closed, is actually a finite union. \( \square \)

We are now in position to state the Hilbert-Mumford criterion corresponding to our linear problem.

**Proposition 1.2.** Fix a character \( \chi \in \mathcal{X}^*(G) \). Then:
(i) \( x \in \mathcal{V}^\text{ss}(G, \chi) \iff \{\forall \lambda \in \{\langle \chi, \cdot \rangle < 0 \} \Rightarrow m(x, \lambda) < 0\}. \)
(ii) \( x \in \mathcal{V}^\text{ss}_{(0)}(G, \chi) \iff \{\forall \lambda \in \{\langle \chi, \cdot \rangle \leq 0\} \Rightarrow m(x, \lambda) < 0\}. \)

**Proof.** The first equivalence is a straightforward consequence of the previous proposition. We are going to prove the second one.

(\( \Rightarrow \)) Suppose that there exists \( x \in \mathcal{V}^\text{ss}_{(0)}(G, \chi) \), which does not fulfill the condition on the right-hand-side. Since \( x \) is stable, it is automatically semi-stable; this fact, together with our hypothesis, implies the existence of a 1-PS \( \lambda \) such that \( \langle \chi, \lambda \rangle = 0 \) and \( m(x, \lambda) \geq 0 \). Then \( x_0 := \lim_{t \to 0} \Sigma(\lambda(t), x) \) exists in \( \mathcal{V} \), and differs from \( x \), since the stabilizer of \( x \) is finite.

We claim that \( x_0 \in \mathcal{V}^\text{ss}(G, \chi) \): indeed, there is a function \( f \in K[\mathcal{V}]^G_{\chi, n}, n \geq 1 \), such that \( f(x) \neq 0 \). Then
\[ f(x_0) = \lim_{t \to 0} f(\Sigma(\lambda(t), x)) = \chi^n(\lambda(t)) \cdot f(x) = t^n(\chi, \lambda) f(x) = f(x) \neq 0. \]

But this means that the \( G \)-orbit of \( x \) is not closed in \( \mathcal{V}^\text{ss}(G, \chi) \), which is a contradiction.

(\( \Leftarrow \)) Consider \( x \in \mathcal{V} \) which satisfies the relation on the right-hand-side: then it is automatically semi-stable. We must prove that its orbit is closed in \( \mathcal{V}^\text{ss}(G, \chi) \) and that its stabilizer is finite.

Assume that the \( G \)-orbit of \( x \) is not closed in \( \mathcal{V}^\text{ss}(G, \chi) \): then we find a 1-PS \( \lambda \) such that \( x_0 := \lim_{t \to 0} \Sigma(\lambda(t), x) \in \mathcal{V}^\text{ss}(G, \chi) \setminus \mathcal{V}^\text{ss}_{(0)}(G, \chi) \). We deduce that \( m(x, \lambda) \geq 0 \), and therefore \( \langle \chi, \lambda \rangle > 0 \). For any \( f \in K[\mathcal{V}]^G_{\chi, n}, n \geq 1 \), we have
\[ f(x_0) = \lim_{t \to 0} f(\Sigma(\lambda(t), x)) = \lim_{t \to 0} \chi^n(\lambda(t)) \cdot f(x) = \lim_{t \to 0} t^n(\chi, \lambda) f(x) = 0, \]
which contradicts the fact that \( x_0 \in \mathcal{V}^\text{ss}(G, \chi) \).
Since $Gx := \Sigma(G, x)$ is closed in $V^{ss}(G, \chi)$, it is closed in some $G$-invariant affine subset containing $x$; it follows that $Gx$ is affine, and therefore the stabilizer of $x$ is reductive, by Matsushima’s criterion (see [22]). Assume now that there is a 1-PS $\lambda$ contained in the stabilizer of $x$: then we may chose it such that $\langle \chi, \lambda \rangle \geq 0$. Obviously $m(x, \lambda) = 0$, which contradicts our hypothesis. \qed

2. Stability for linear actions of reductive groups II

In this section we will view the affine space $V$ as a $G$-invariant, Zariski open subset of the projective space $\mathbb{P}(K \oplus V) := \text{Proj}(\text{Sym}^*(K \oplus V))$, for an appropriate $G$-action on this latter one. For any character $\chi \in \mathcal{X}^*(G)$, we prove that the $G$-action on $\mathbb{P}(K \oplus V)$ can be linearized in such a way that the intersection of the corresponding semi-stable locus with $V$ coincides with the $\chi$-semi-stable locus of $V$. The goal of this approach is that of allowing us to reduce the issue of chamber structure on $\mathcal{X}^*(G)$ in the affine case to known results in the projective case.

Let us recall that for any character $\chi \in \mathcal{X}^*(G)$, the invariant quotient $V/\chi G$ corresponding to the linearization $\Sigma_\chi$ can be described as

$$V/\chi G = V^{ss}(G, \chi)/G = \text{Proj}(K[V]^{G,\chi}),$$

and this one is projective over $\text{Spec}(K[V]^G)$.

Let us denote $\pi : V = \text{Spec} K[V] \to \text{Spec} K[V]^G =: V/G$ the natural projection, and define $\hat{0} := \pi(0) \in V/G$. Further, $K[V]^{G,\chi}$ being a subalgebra of $K[V]$ inherits an additional graduation given by the degree in $K[V]$ (it becomes a bi-graded algebra); we denote by $K[V]^{G,\chi}_{nD, (p)} \subset K[V]^{G,\chi}$, the submodule homogeneous elements of degree $p$.

**Lemma 2.1.** For a character $\chi \in \mathcal{X}^*(G)$, the following statements hold:

(i) There is an integer $c > 0$ such that $K[V]^{G,\chi}_{n, \infty} = (K[V]^{G,\chi})^n$ for all $n \geq 1$.

Assume that $\chi \in \mathcal{X}^*(G)$ is such that the condition (i) holds with $c = 1$.

(ii) In the case $K[V]_\chi^G \neq K$, there are integers $D, D_\chi \geq 1$ such that the following two conditions are fulfilled:

\[
\begin{align*}
&\forall n \geq 1 \forall x \in V \setminus \pi^{-1}(\hat{0}) \exists f \in K[V]_{nD}^{G,\chi} \text{ with } f(x) \neq 0; \\
&\forall n > D_\chi \forall x \in V^{ss}(G, \chi) \setminus \pi^{-1}(\hat{0}) \exists f \in K[V]_{nD}^{G,\chi} \text{ with } f(x) \neq 0.
\end{align*}
\]

Conversely: consider $x \in V$ with the property that there are $N > 0$ and $f \in K[V]_{\chi N, (Nd)}^G$, with $d > D_\chi$, such that $f(x) = 0$. Then $x \in V^{ss}(G, \chi) \setminus \pi^{-1}(\hat{0})$.

(iii) In the case $K[V]^G = K$,

\[
\max\{\deg f \mid f \in K[V]_{\chi n}^G\} \leq n \cdot \max\{\deg f \mid f \in K[V]^G\} =: nD_\chi, \text{ for all } n \geq 1.
\]

**Proof.** The first statement is a particular case of [11, lemme 2.1.6], applied to the noetherian ring $K[V]^{G,\chi}$. The statement (iii) is obvious, since in this case $K[V]_{\chi n}^G$, which is a finite dimensional $K$-module, generates $K[V]^{G,\chi}$ as a $K$-algebra.

We prove now (ii): we consider finite sets of homogeneous polynomials $S_0 \subset K[V]^{G,\chi} \setminus K$ and $S_1 \subset K[V]_{\chi n}^G$ which generate respectively $K[V]^{G,\chi}$ as a $K$-algebra, and $K[V]_{\chi n}^G$ as a $K[V]^{G,\chi}$-algebra. We define $D$ to be the smallest common multiple of $\deg f$, for $f \in S_0$, and $D_\chi := \max\{\deg f \mid f \in S_1\}$.

Then, for any $x \in V \setminus \pi^{-1}(\hat{0})$, there is $f \in S_0$ such that $f(x) \neq 0$. Raising it to a suitable power, we obtain the desired polynomial.
Consider an integer \( n \geq 1 \). For \( x \in \mathbb{V}^{ss}(G, \chi) \setminus \pi^{-1}(0) \), we find \( f_1 \in S_1 \) such that \( f(x) \neq 0 \). Moreover, the previous step shows that there is a homogeneous polynomial \( f_0 \in K[V]^G \) of degree \( \deg f_0 = (n + D_\chi - \deg f_0)D \) with \( f_0(x) \neq 0 \).

We obtain that
\[
f_0f_1^D \in K[V]_{S_D}, \quad \deg(f_0f_1^D) = (n + D_\chi)D, \quad \text{and } f_0f_1^D(x) \neq 0.
\]

Conversely, assume that we have \( x \in \mathbb{V} \) with the stated property. Then, clearly, \( x \in \mathbb{V}^{ss}(G, \chi) \). We claim that the elements of \( K[V]^G \) do not vanish at \( x \). Assume the contrary, and consider a homogeneous decomposition \( f = f' + f'' \) with \( f' \in K[s; s \in S_1] \) and \( f'' \in \mathcal{I}[s; s \in S_1] \), for \( \mathcal{I} := \langle s_0; s_0 \in S_0 \rangle \subset K[V]^G \). Since \( f(x) \neq 0 \), \( f'(x) \neq 0 \) too. Thus we have found a non-zero \( f' \in K[V]^G_{\chi^N,(Nd)} \) which can be expressed as a polynomial of degree \( N \) (because of the \( \chi^N \)-homogeneity) in the elements of \( S_1 \). But, in this case
\[
Nd = \deg f' \leq N \cdot \max\{\deg f_1 \mid f_1 \in S_1\} = ND_\chi,
\]
which contradicts the choice of \( d > D_\chi \). \( \Box \)

In what follows we view \( \mathbb{P}(K \oplus V) = \text{Proj}(\text{Sym}^*(K \oplus V)) \cong ((A_K^1 \times V) \setminus \{0\})/G_m \). We consider the representation \( G \to \text{Gl}(K \oplus V), \ g \mapsto \text{diag}(1, \rho(g)) \), and observe that for the induced action
\[
\tilde{\Sigma} : G \times \mathbb{P}(K \oplus V) \to \mathbb{P}(K \oplus V), \quad \tilde{\Sigma}(g, [a, x]) := [a, \Sigma(g, x)],
\]
the inclusion \( V \subset \mathbb{P}(K \oplus V) \) is \( G \)-invariant. What we wish is to linearize the \( G \)-action in \( \mathcal{O}_{\mathbb{P}(K \oplus V)}(d) \), for some \( d > 0 \), such that the corresponding (semi-)stable points in \( V \) are precisely the \( \Sigma_\chi \)-(semi-)stable points of \( V \).

We remark that for an integer \( d \), the geometric line bundle defined by \( \mathcal{O}_{\mathbb{P}(K \oplus V)}(d) \) can be described as
\[
\mathcal{O}_{\mathbb{P}(K \oplus V)}(d) := \text{Spec}(\text{Sym}^* \mathcal{O}_{\mathbb{P}(K \oplus V)}(-d)) \cong ((A_K^1 \times V) \setminus \{0\}) \times A_K^1/G_m,
\]
where \( G_m \) acts on \( A_K^1 \times V \times A_K^1 \) by \( t \cdot (y, z) := (ty, t^d z) \) \( \forall t \in G_m \). For \( \chi \in X^*(G) \), we linearize the \( G \)-action on \( \mathbb{P}(K \oplus V) \) in \( \mathcal{O}_{\mathbb{P}(K \oplus V)}(d) \) by
\[
\Sigma_{d, \chi} : G \times \mathcal{O}_{\mathbb{P}(K \oplus V)}(d) \to \mathcal{O}_{\mathbb{P}(K \oplus V)}(d), \quad (g, [(a, x, z)]) \mapsto [(a, \Sigma(g, x), \chi(g)z)].
\]
In particular, we deduce that there is a natural isomorphism
\[
\text{Pic}^G(\mathbb{P}(K \oplus V)) \cong \text{Pic}(\mathbb{P}(K \oplus V)) \oplus X^*(G) \cong \mathbb{Z} \oplus X^*(G).
\]

Lemma 2.2. (i) The restriction of the action \( \Sigma_{d, \chi} \) to \( \mathcal{O}_{\mathbb{P}(K \oplus V)}(d)|_V \) induces the action \( \Sigma_\chi \) on \( \mathcal{O}_V \).

(ii) The restriction homomorphism \( \text{Pic}^G(\mathbb{P}(K \oplus V)) \to \text{Pic}^G(V) \) is surjective and corresponds to the projection \( \mathbb{Z} \oplus X^*(G) \to X^*(G) \).

Proof. We consider the section \( s \) in \( \mathcal{O}_{\mathbb{P}(K \oplus V)}(d) \) given by \( s[a, x] := [(a, x), a^d]; \) it defines an isomorphism \( \mathcal{O}_V \xrightarrow{\cong} \mathcal{O}_{\mathbb{P}(K \oplus V)}(d)|_V \), and we observe that \( \Sigma_{d, \chi}(g, s[1, x]) = \chi(g) \cdot s[1, \Sigma(g, x)] \) for \( x \in V \) and \( g \in G \). \( \Box \)

The problem is that this statement says nothing about the behaviour of the (semi-)stable loci under restriction: we are going to investigate this issue.
Proposition 2.3. Let $\chi \in X^*(G)$ be a character. There are positive integers $c, D_\chi$ depending on $\chi$ such that for any $d > D_\chi$ we have

$$\mathbb{P}(K \oplus V)^{ss}(\Sigma_{d,\chi}) = \mathbb{V}^{ss}(G, \chi) \cup [\mathbb{V}^{ss}(G, \chi) \setminus \pi^{-1}(0)]/G_m.$$ 

For writing this union, we view $\mathbb{P}(K \oplus V) = \mathbb{V} \cup \mathbb{P}(V)$. Moreover,

$$\mathbb{V} \cap \mathbb{P}(K \oplus V)|_{(0)}(\Sigma_{d,\chi}) = \mathbb{V}_0^{ss}(G, \chi).$$

In particular, if $K[\mathbb{V}]^G = K$, it follows that $\mathbb{P}(K \oplus V)^{ss}(\Sigma_{d,\chi}) = \mathbb{V}^{ss}(G, \chi)$.

Proof. We choose $c > 0$ such that the condition (i) in lemma 2.1 is fulfilled. Since the right-hand-side of the equality is invariant under the change $\chi \rightsquigarrow \chi^c$, we may assume that $K[\mathbb{V}]^{G,\chi}$ is generated by $K[\mathbb{V}]_\chi^G$ over $K[\mathbb{V}]^G$. We choose further $D, D_\chi > 0$ as in (ii$_a$), resp. (iii$_b$), of loc. cit.

For $x \in \mathbb{V}^{ss}(G, \chi)$, there is a homogeneous $f \in K[\mathbb{V}]_\chi^G$, with $\deg f \leq D_\chi$, such that $f(x) \neq 0$. The section

$$(2.4) \quad s \in \Gamma(\mathbb{P}(K \oplus V), \mathcal{O}_{\mathbb{P}(K \oplus V)}(d)) \text{ defined by } s[a, x] := [(a, x), a^{d-D}\chi f(x)]$$

is clearly $G$-invariant and does not vanish at $[1, x]$.

Assume that $K[\mathbb{V}]^G \neq K$, and consider $x \in \mathbb{V}^{ss}(G, \chi) \setminus \pi^{-1}(0)$; we have proved that there is $f \in K[\mathbb{V}]_{x,\chi}^{G,(dD)}$, with $f(x) \neq 0$. The section

$$(2.4) \quad s \in \Gamma(\mathbb{P}(K \oplus V), \mathcal{O}_{\mathbb{P}(K \oplus V)}(d)) \text{ defined by } s[a, x] := [(a, x), f(x)]$$

is again $G$-invariant and does not vanish at $[0, x]$.

For the converse, let us notice that an immediate consequence of the lemma 2.2 is that $\mathbb{V} \cap \mathbb{P}(K \oplus V)^{ss}(\Sigma_{d,\chi}) \subset \mathbb{V}^{ss}(G, \chi)$. Also, in the case where $K[\mathbb{V}]^G = K$, the converse to the condition (ii$_a$) in lemma 2.1 says precisely that

$$\mathbb{P}(V) \cap \mathbb{P}(K \oplus V)^{ss}(\Sigma_{d,\chi}) \subset [\mathbb{V}^{ss}(G, \chi) \setminus \pi^{-1}(0)]/G_m$$

In the case $K[\mathbb{V}]^G = K$, we must prove that the hyperplane $\mathbb{P}(V) \hookrightarrow \mathbb{P}(K \oplus V)$ consists of $\Sigma_{d,\chi}$-unstable points. Let us assume that $[0, x_0]$ is semi-stable; then there is $N > 0$ and $F \in \text{Sym}^N((K \oplus V)^\vee)$ such that

$$F(a, \Sigma(g, x)) = \chi^N(g)F(a, x) \text{ and } F(0, x_0) \neq 0.$$ 

Consider $f \in \text{Sym}^*(V^\vee)$, $f(x) := F(0, x)$. Then $F(0, x_0) \neq 0$ implies $\deg f = \deg F = N d$. On the other hand, $f(\Sigma(g, x)) = \chi^N(g)f(x)$, that is $f \in K[\mathbb{V}]_x^{G,N}$. Applying (ii$_b$) of lemma 2.1 we find that $\deg f \leq ND_\chi < Nd$, a contradiction.

For the second equality, notice that the left-hand-side is obviously included in the right-hand-side. Conversely, we observe that for $x \in \mathbb{V}_0^{ss}(G, \chi)$, the section defined by (2.4) vanishes on the hyperplane $\mathbb{P}(V) \hookrightarrow \mathbb{P}(K \oplus V)$, and therefore the orbit $G[1, x]$ is closed in $\mathbb{P}(K \oplus V)^{ss}(\Sigma_{d,\chi})$. 

Remark 2.4. We indicate a procedure which allows, possibly after taking finite covers of $G$, to linearize the $G$ action in $\mathcal{O}_{\mathbb{P}(K \oplus V)}(1)$ without altering the semi-stable locus. For $d > 0$, we consider the endomorphism $\zeta_d : Z(G)^\circ \to Z(G)^\circ$ which raises the elements to the $d$th power, and define

$$m_d : Z(G)^\circ \times [G, G] \xrightarrow{(\zeta_d, id_{[G, G]})} Z(G)^\circ \times [G, G] \xrightarrow{m_1} G,$$
where \(m_1\) is the multiplication. Then \(m_d^* : X^*(G) \to X^*(Z(G) \times [G,G])\) has finite cokernel, and for all \(\chi \in X^*(G)\) we have \(\mathcal{V}^{ss}(G,\chi) = \mathcal{V}^{ss}(Z(G) \times [G,G],m_d^* \chi)\). Moreover, the representation

\[
Z(G) \times [G,G] \to \text{Gl}(K \oplus V), \quad (z,g') \mapsto \text{diag}(\chi^{-1}(z),\chi^{-1}(z) \cdot (\rho \circ m_d)(z,g'))
\]

induces a linearization of the \(Z(G) \times [G,G]-\text{action} \Sigma \circ m_d\) in \(\text{O}(K \oplus V)(1)\).

In the next section we will compare the numerical criteria for semi-stability in the affine and in the projective case, in order to address the issue of the chamber structure on \(X^*(G)\) with respect to the GIT-equivalence relation.

We fix a norm on \(X_*(T)_{\mathbb{R}}\) invariant under the action of the Weyl group \(W\); it naturally induces a norm \(\| \cdot \|\) on \(X_*(G)\) (see [K subsection 1.1.3]). Consider \(x \in \mathcal{V}\) and \(\lambda \in X_*(G)\); an immediate computation shows that

\[
\Sigma(\lambda(t),[1,x]) = [1,(t^j x_j)_{j \to 0}] \to [0,x_{m(x,\lambda)}] \quad \text{if} \quad m(x,\lambda) < 0;
\]

\[
[1,x_0] \quad \text{if} \quad m(x,\lambda) = 0;
\]

\[
[1,0] \quad \text{if} \quad m(x,\lambda) > 0.
\]

The weight of the \(\lambda\)-action at the specialization, corresponding to \(\Sigma_d \chi\), is

\[
\mu_{d,\chi}(x,\lambda) = \begin{cases} 
\langle \chi,\lambda \rangle - d \cdot m(x,\lambda) & \text{if} \quad m(x,\lambda) < 0; \\
\langle \chi,\lambda \rangle & \text{if} \quad m(x,\lambda) \geq 0.
\end{cases}
\]

We consider the obvious extension of \(\mu_\cdot,(x,\lambda)\) to \(\mathbb{R} \oplus X^*(G)_{\mathbb{R}}\) and define the numerical functions

\[
M : X^*(G)_{\mathbb{R}} \times \mathcal{V} \to \mathbb{R}, \quad M(\chi,x) := \inf \left\{ \frac{\langle \chi,\lambda \rangle}{|\lambda|} \mid \lambda \in X_*(G), m(x,\lambda) \geq 0 \right\},
\]

\[
\bar{M} : (\mathbb{R} \oplus X^*(G)_{\mathbb{R}}) \times \mathcal{V} \to \mathbb{R}, \quad \bar{M}((d,\chi),x) := \inf_{\lambda \in X_*(G)} \frac{\mu_{d,\chi}(x,\lambda)}{|\lambda|}.
\]

If we could control the locus where \(\bar{M}\) is positive in terms of the corresponding locus of \(M\) (the converse is easy, since \(M((d,\cdot),\cdot) \leq M(\cdot,\cdot)\) for \(d > 0\), the fan structure on \(X^*(G)_{\mathbb{R}}\) was immediate. Unfortunately, at this point we are able to do this only for rational coefficients.

The equality of the positive loci of \(M\) and \(\bar{M}\) follows in [Ki] as a corollary to our discussion in the next section.

We conclude with a generality about \(G\)-actions on affine varieties. For a normal, affine, \(G\)-variety \(X\), \(\text{Pic}^G(X) \cong X^*(G)\), where the isomorphism is given by assigning to the character \(\chi\) the linearization \(\Sigma_\chi\) defined as in [KS]. The following proposition reduces the issue of computing the semi-stability subsets of \(X\) corresponding to \(\{\Sigma_\chi \}_{\chi \in X^*(G)}\) to the similar issue for an affine space.

**Proposition 2.5.** Let \(X\) be a normal, affine variety, and \(\Sigma : G \times X \to X\) a regular action on it. Then there is a \(G\)-module \(V\) and a \(G\)-equivariant embedding \(j_X : X \hookrightarrow \mathcal{V} := \text{Spec}(\text{Sym}^* V)\) with the property that for all \(\chi \in X^*(G)\) holds

\[
X^{ss}(G,\chi) = X \cap \mathcal{V}^{ss}(G,\chi).
\]
Proof. Let $G' := [G, G]$ be the commutator subgroup of $G$: it is a normal, semi-simple subgroup of $G$, and $G/G'$ is a torus. We remark that for any character $\chi \in X^*(G)$, $K[\chi]_G' \subset K[\chi]_{G'}$ and moreover

$$\bigoplus_{\chi \in X^*(G)} K[\chi]_G = \bigoplus_{\tilde{\chi} \in \hat{X}^*(G/G')} (K[\chi]_{G'})_{\tilde{\chi}} = K[\chi]_{G'}.$$

Since $K[\chi]_{G'}$ is a finitely generated $K$-algebra and also a locally finite $G$-module, there is a finite set $S_0 \subset K[\chi]_{G'}$ with the following two properties:

- $S_0$ generates $K[\chi]_{G'}$ as a $K$-algebra;
- for each $f \in S_0$ there is $\chi_f \in X^*(G)$ such that $f \in K[\chi]_{\chi_f}$. We deduce that $V_0 := \sum_{f \in S_0} Kf$ is a $G$-module, the natural ring homomorphism $\text{Sym}^\bullet V_0 \rightarrow K[\chi]_{G'}$ is an epimorphism of $G$-modules, and moreover $(\text{Sym}^\bullet V_0)_\chi \rightarrow K[\chi]_{\chi}$ is surjective for all $\chi \in X^*(G)$.

Since $K[\chi]$ is a locally finite $G$-module, we find a finite set $S \subset M$ containing $S_0$ such that

- $S$ generates $K[\chi]$ as a $K$-algebra;
- $V := \sum_{f \in S} Kf$ is a $G$-module.

The natural ring homomorphism $\text{Sym}^\bullet V \rightarrow K[\chi]$ is an epimorphism of $G$-modules, and moreover $(\text{Sym}^\bullet V)_\chi \rightarrow K[\chi]_{\chi}$ is surjective for all $\chi \in X^*(G)$. A straightforward computation shows that the $G$-module $V$ defined in this way fulfills the requirements of the proposition. □

3. The Chamber Structure on the Cone of Effective Characters

Two natural questions naturally arise in the study of the quotients $\mathbb{V}/\chi G$, as $\chi \in X^*(G)$ varies:

(i) Is there a chamber structure on $X^*(G)$ corresponding to the possible semi-stable loci $\mathbb{V}^{ss}(G, \chi)$?

(ii) For which characters $\chi$ is the corresponding semi-stable locus non-empty?

When $G$ is a torus, the answer to these questions is given by the so-called Gelfand-Kapranov-Zelevinskij (or secondary) fan.

For answering our first question, we will rely on the results obtained in [21], where the author studies the GIT-equivalence relation on the space of $G$-linearized line bundles over a projective variety. Let us recall the main result of that paper: consider a normal, projective variety $X$, acted on by a reductive, linear algebraic group $G$. We define $\text{NS}^G(X)$ to be the quotient of $\text{Pic}^G(X)$ by the $G$-algebraic equivalence relation (see [21 subsection 2.1]). It is a finitely generated module over $\mathbb{Z}$, so that $\text{NS}^G(X)_{\mathbb{R}}$ is a finite dimensional vector space over $\mathbb{R}$.

One chooses a $W$-invariant norm on $X_*(T)$, and using it defines the function $\tilde{M} : \text{NS}^G(X) \times X \rightarrow \mathbb{R}$ such that for all $l \in \text{NS}^G(X)$ holds $X^{ss}(l) = \{ \tilde{M}(l, \cdot) \geq 0 \}$. Further, one proves that $\tilde{M}$ uniquely extends to a function on $\text{NS}^G(X)_{\mathbb{R}} \times X$, which is continuous in the first argument (see [21 lemma 3.2.5]). One extends the definition of the $l$-semi-stable locus to every $l \in \text{NS}^G(X)_{\mathbb{R}}$ by the formula $X^{ss}(l) := \{ \tilde{M}(l, \cdot) \geq 0 \}$.

Definition 3.1. (see [21 section 2]) An element $l \in \text{NS}^G(X)_{\mathbb{R}}$ is called effective if $X^{ss}(l) \neq \emptyset$. One says that $l_1, l_2 \in \text{NS}^G(X)_{\mathbb{R}}$ are GIT-equivalent if $X^{ss}(l_1) = X^{ss}(l_2)$.

The main result of [21] reads:
Theorem Let $X$ be a normal, projective $G$-variety, and denote by $\mathcal{C}^G(X)$ the set of effective classes in $\text{NS}^G(X)_{\mathbb{R}}$. Then the following hold:

(i) $\mathcal{C}^G(X)$ is closed in $\text{NS}^G(X)_{\mathbb{R}}$.

(ii) For all $l_o \in \mathcal{C}^G(X)$, $C(l_o) := \{l \in \mathcal{C}^G(X) \mid X^{ss}(l) \subseteq X^{ss}(l_o)\}$ is a closed, convex, rational polyhedral cone in $\text{NS}^G(X)_{\mathbb{R}}$.

(iii) The cones $C(l)$, $l \in \mathcal{C}^G(X)$ form a fan covering $\mathcal{C}^G(X)$.

(iv) The GIT-equivalence classes are the relative interiors of these cones.

The fan constructed in this way is called in [21] the GIT-fan, and will be denoted by $\Delta^G(X)$. Further, we let $\Delta^G(X) := \Delta^G(X) \cap \text{NS}^G(X)_{\mathbb{Q}}$ and notice that as the cones of $\Delta^G(X)$ are rational, $\Delta^G(X)$ is obtained from the fan $\Delta^G(X)_{\mathbb{Q}}$ by extending the coefficients from $\mathbb{Q}$ to $\mathbb{R}$.

What we are going to prove is that this result, which holds for projective varieties, can be adapted to our setting, where we deal with group actions on affine spaces. For $\chi \in X^*(G)$, we define $V^{ss}(\chi) := \{M(\chi, \cdot) \geq 0\}$, where $M$ is given by (2.5).

Theorem 3.2. Let $\rho : G \to \text{Gl}(V)$ be a representation of the reductive group $G$ which has finite kernel, and consider the induced $G$-action on $\mathbb{P}(K \oplus V)$ defined by (2.4). We denote by $e$ the trivial character of $G$. Then the following statements hold:

(i) $\mathbb{R}_{\geq 0}(1, e)$ is a ray in $\Delta^G(\mathbb{P}(K \oplus V))$.

(ii) The GIT-equivalence classes in $X^*(G)_{\mathbb{R}}$ corresponding to the $G$-action on $V$ are the relative interiors of the cones of the fan

$$\Delta^G(V) := \text{star}(\mathbb{R}_{\geq 0}(1, e))/\mathbb{R}(1, e).$$

This result, together with proposition 2.5, immediately yield

Theorem 3.3. Let $X$ be a normal, affine $G$-variety. The GIT-equivalence classes in $X^*(G)_{\mathbb{R}}$ corresponding to the $G$-action on $X$ are the relative interiors of the cones of a rational, polyhedral fan $\Delta^G(X)$, which we call the GIT-fan of $X$.

We will prove theorem 3.2 in two steps: first we prove the existence of the fan structure on the set of effective characters in $X^*(G)_{\mathbb{Q}}$; this part uses the results obtained in the previous section. Secondly, we prove that the induced fan in $X^*(G)_{\mathbb{R}}$, obtained by extending the coefficients from $\mathbb{Q}$ to $\mathbb{R}$, parameterizes the GIT-equivalence classes in $X^*(G)_{\mathbb{R}}$.

Proof. (theorem 3.2 with \(\mathbb{Q}\) coefficients) (i) A direct computation yields $\mathbb{P}(K \oplus V)^{ss}(\Sigma_{1,e}) = \mathbb{V}|V/\pi^{-1}(0)]/G_m$, so that $(1, e)$ is an effective class in $\text{NS}^G(\mathbb{P}(K \oplus V)) = \text{Pic}^G(\mathbb{P}(K \oplus V)) \cong \mathbb{Z} \oplus X^*(G)$. From the third part of the theorem above, we deduce that $(1, e)$ is in the relative interior of a cone in $\Delta^G(\mathbb{P}(K \oplus V))_{\mathbb{Q}}$.

We claim that $\mathbb{Q}_{\geq 0}(1, e)$ is actually a ray in the GIT-fan. Assume the contrary, that $(1, e)$ is in the relative interior of a cone $\tau_0 \in \Delta^G(\mathbb{P}(K \oplus V))_{\mathbb{Q}}$; since $\tau_0$ is rational, we find an integral point $l = (d, \chi)$ on its boundary. Then [21], proposition 8 implies that

$$\mathbb{V} \subset \mathbb{P}(K \oplus V)^{ss}(\Sigma_{1,e}) \subsetneq \mathbb{P}(K \oplus V)^{ss}(\Sigma_{d,\chi}).$$

In particular, $[1, 0] \in \mathbb{P}(K \oplus V)^{ss}(\Sigma_{d,\chi})$, so that there is a homogeneous polynomial $F$ of positive degree, such that

$$F(1, 0) \neq 0 \text{ and } F(a, \rho(g)v) = \chi^N(g) \cdot F(a, v), \text{ for all } (a, v) \in \mathbb{A}_K^1 \times \mathbb{V}.$$ 

Restricting to $a = 1$, we obtain $f \in K[\mathbb{V}]_{\chi^N}$ such that $f(0) \neq 0$. We deduce that

$$f(0) = f(\rho(g)0) = \chi^N(g) \cdot f(0),$$
and therefore \( \chi = e \). It follows that \( l = (d, e) \) is collinear with \((1, e)\), and this contradicts the choice of \( l \).

(ii) First, consider \( \chi \in X^*(G) \) such that \( V^{ss}(G, \chi) \neq \emptyset \). For \( c, d \) appropriate

\[
P(K \oplus V)^{ss}(\Sigma_{d, \chi}) = V^{ss}(G, \chi) \cup \left[ V^{ss}(G, \chi) \setminus \pi^{-1}(\hat{0}) \right] / G_m \subseteq P(K \oplus V)^{ss}(\Sigma_{1, e}),
\]

so that, on one hand, \((d, c)\) is effective and therefore contained in the relative interior of a cone \( \tau \) of \( \Delta^G(P(K \oplus V)) \); on the other hand, \cite{21} proposition 8 implies that \((1, e)\) belongs to \( \tau \). All together shows that \( \tau \) is in \( \star(Q_{\geq 0}(1, e)) \).

Using proposition \cite{23} again, we deduce that for two GIT-equivalent characters \( \chi_1, \chi_2 \in X^*(G) \), that is \( V^{ss}(G, \chi_1) = V^{ss}(G, \chi_2) \), the linearizations \( \Sigma_{d, \chi_1} \) and \( \Sigma_{d, \chi_2} \) are still GIT-equivalent, for suitable \( c \) and \( d \) sufficiently large. Therefore \((d, \chi_1)\) and \((d, \chi_2)\) are in the relative interior of the same cone of the GIT-fan of \( P(K \oplus V) \).

Conversely, consider \( \tau \in \star(Q_{\geq 0}(1, e)) \) and an integral point \((d, \chi)\) in its relative interior. Then lemma \cite{22} implies that \( \emptyset \neq V \cap P(K \oplus V)^{ss}(\Sigma_{d, \chi}) \subset V^{ss}(G, \chi) \), and therefore \( \chi \) is effective. Further, since \((1, e)\) is in \( \tau \), \((d + n, \chi)\) is in the relative interior of \( \tau \) for all \( n \geq 0 \), and therefore the \((d, \chi)\)- and \((d + n, \chi)\)-semi-stable loci coincide. On the other hand,

\[
P(K \oplus V)^{ss}(\Sigma_{d_1, \chi_1^1}) = V^{ss}(G, \chi) \cup \left[ V^{ss}(G, \chi) \setminus \pi^{-1}(\hat{0}) \right] / G_m
\]

for some appropriate \( c_1 \), and for all \( d_1 \) sufficiently large. Choosing \( d_1 \) and \( n \) in such a way that \((d + n, \chi)\) and \((d_1, \chi_1^1)\) are collinear, we deduce that \( P(K \oplus V)^{ss}(\Sigma_{d, \chi}) \) is still given by the right-hand-side of the equality above.

Consider now \( \tau \in \star(Q_{\geq 0}(1, 0)) \), and two integral points \((d_1, \chi_1), (d_2, \chi_2)\) in its relative interior. Then a similar argument shows that \( V^{ss}(G, \chi_1) = V^{ss}(G, \chi_2) \), that is \( \chi_1 \) and \( \chi_2 \) are GIT-equivalent.

For extending the coefficients from \( Q \) to \( R \), we have to adapt some of the intermediate results in \cite{21} to our context. We recall that \( T \subset G \) is a maximal torus of \( G \); for \( g \in G \), we denote \( T^g := gTg^{-1} \). Further, the \( T \)-module \( V \) decomposes into the direct sum of its weight subspaces \( V = \bigoplus_{a \in \Phi} V_a \), and we denote \( \{ \eta_a \}_{a \in \Phi} \) the corresponding weights. Then \( V = \chi_{a \in \Phi} V_a \), and \( T \) acts on \( V \) by the character \( \eta_a \). For \( x \in V \), we write \((x_a)_{a \in \Phi(x)}\) its non-zero coordinates with respect to this decomposition.

**Definition 3.4.** We define the *stability set* of a point \( x \in V \) to be

\[
\Omega(x) := \{ \chi \in X^*(G)_R \mid M(\chi, x) \geq 0 \}.
\]

**Lemma 3.5.** (i) For all \( x \in V \), the stability set \( \Omega(x) \subset X^*(G)_R \) is a closed, convex, rational polyhedral cone.

(ii) There are only finitely many stability sets.

**Proof.** (i) Let \( x \in V \): \( \Omega(x) \) is closed since \( M(\cdot, x) \) is continuous on \( X^*(G)_R \); \( \Omega(x) \) is convex since \( M(\chi_1 + \chi_2, x) \geq M(\chi_1, x) + M(\chi_2, x) \), for all \( \chi_1, \chi_2 \in X^*(G)_R \).

We prove now the rationality property. Define the set \( \mathcal{L}_x := \{ \lambda \in \mathcal{X}_x(G) \mid m(x, \lambda) \geq 0 \} \), and notice that

\[
\mathcal{L}_x = \bigcup_{g \in G} \mathcal{X}_x(T^g) \cap \mathcal{L}_x = \bigcup_{g \in G} \text{Ad}_g(\mathcal{X}_x(T) \cap \mathcal{L}_{\Sigma(g^{-1}, x)}).
\]
Moreover, for $g \in G$, $\mathcal{X}_s(T) \cap \mathcal{L}_{\Sigma(g^{-1},x)} \subset \mathcal{X}_s(T)$ is convex (if we consider $\mathbb{Q}$-coefficients), and
\[
\mathcal{X}_s(T) \cap \mathcal{L}_{\Sigma(g^{-1},x)} = \{ \lambda \in \mathcal{X}_s(T) \mid m(\Sigma(g^{-1},x), \lambda) \geq 0 \}
= \{ \lambda \in \mathcal{X}_s(T) \mid \min_{\eta \in \Phi(\Sigma(g^{-1},x))} \langle \eta_0, \lambda \rangle \geq 0 \}.
\]
Since $\Phi(\Sigma(g^{-1},x)) \subset \Phi$ for all $g \in G$, only a finite number of such sets appear as $g \in G$ varies; let $\Gamma_x \subset G$ be a set of representatives. We deduce that
\[
M(\chi, x) = \inf_{\lambda \in \mathcal{L}_x} \frac{\langle \chi, \lambda \rangle}{|\lambda|} = \inf_{g \in G} \inf \left\{ \frac{\langle \chi, \lambda \rangle}{|\lambda|} \right\} \lambda \in \text{Ad}_g(\mathcal{X}_s(T) \cap \mathcal{L}_{\Sigma(g^{-1},x)})
\]
\[
= \inf_{g \in G} \left\{ \frac{\langle \chi, \lambda \rangle}{|\lambda|} \right\} \lambda \in \mathcal{X}_s(T) \cap \mathcal{L}_{\Sigma(g^{-1},x)}
\]
\[
= \min_{g \in \Gamma_x} \left\{ \frac{\langle \chi, \lambda \rangle}{|\lambda|} \right\} \lambda \in \mathcal{X}_s(T) \cap \mathcal{L}_{\Sigma(g^{-1},x)}
\]
and therefore
\[
\chi \in \Omega(x) \iff \inf \left\{ \frac{\langle \chi, \lambda \rangle}{|\lambda|} \right\} \lambda \in \mathcal{X}_s(T) \cap \mathcal{L}_{\Sigma(g^{-1},x)} \geq 0, \ \forall g \in \Gamma_x
\]
\[
\iff \chi \in \bigcap_{g \in \Gamma_x} (\mathcal{X}_s(T) \cap \mathcal{L}_{\Sigma(g^{-1},x)})^\vee.
\]
This is a finite intersection of half-spaces defined by rational equations, so that $\Omega(x)$ is a rational, polyhedral cone.

(ii) Clearly, any stability set is the union of the GIT classes contained in it. We claim that there is a finite number of stability sets of the form $\Omega(x) \cap \mathcal{X}^*(G)_{\mathbb{Q}}$; indeed, it follows from the proof of theorem 3.2 with $\mathbb{Q}$ coefficients that
\[
\Omega(x) \cap \mathcal{X}^*(G)_{\mathbb{Q}} = \text{pr}_{\mathcal{X}^*(G)_{\mathbb{Q}}} \left[ \Omega([1, x]) \cap \text{Support of star} \left( \mathbb{Q}_{\geq 0}(1, e) \right) \right],
\]
which is clearly finite. By the previous step, $\Omega(x) = \overline{\Omega(x) \cap \mathcal{X}^*(G)_{\mathbb{Q}}}$, which concludes the lemma.

Lemma 3.6. (i) For a subset $U \subset \mathbb{V}$, we define $C(U) := \{ l \in \mathcal{X}^*(G)_{\mathbb{R}} \mid U \subseteq \mathbb{V}^{ss}(G,l) \}$. Then $C(U)$ is a closed, convex, rational cone in $\mathcal{X}^*(G)_{\mathbb{R}}$.

(ii) For any $\chi \in \mathcal{X}^*(G)_{\mathbb{Q}}$ holds
\[
\mathbb{V}^{ss}(G, \chi) = \mathbb{V}^{ss}(G, \chi_0), \ \forall \chi \in \text{rel.int.}C(\mathbb{V}^{ss}(G, \chi_0)).
\]

Proof. (i) We notice that $C(U) = \bigcap_{x \in U} \Omega(x)$; by the previous lemma, this is a finite intersection of closed, convex, rational cones, so that $C(U)$ is the same.

(ii) As $\chi \in C(\mathbb{V}^{ss}(G, \chi_0))$, it follows $\mathbb{V}^{ss}(G, \chi_0) \subset \mathbb{V}^{ss}(G, \chi)$, and therefore $C(\mathbb{V}^{ss}(G, \chi)) \subset C(\mathbb{V}^{ss}(G, \chi_0))$. We deduce that
\[
\chi \in C(\mathbb{V}^{ss}(G, \chi)) \cap \text{rel.int.}C(\mathbb{V}^{ss}(G, \chi_0)),
\]
and, using the first part of the lemma, we conclude that there is a rational point $\chi'$ in the intersection above. Then $\mathbb{V}^{ss}(G, \chi) \subset \mathbb{V}^{ss}(G, \chi')$ since $\chi' \in C(\mathbb{V}^{ss}(G, \chi))$, and also $C(\mathbb{V}^{ss}(G, \chi')) = C(\mathbb{V}^{ss}(G, \chi_0))$ by theorem 3.2 with rational coefficients. It follows that $\mathbb{V}^{ss}(G, \chi) \subset \mathbb{V}^{ss}(G, \chi_0)$. \qed
Proof. (theorem \ref{thm:main} with \(\mathbb{R}\) coefficients) We have already proved the statement for rational coefficients, and we know from \cite{21} that the GIT classes in \(\Delta^G(\mathbb{P}(K\oplus V))\) build a fan structure. Combining these facts with lemma \ref{lem:3.6}(ii), we deduce that for concluding the theorem we still need to prove the following statement: any \(\chi \in \mathcal{X}^*(G)_\mathbb{R}\) is GIT-equivalent to some rational character \(\chi_0 \in \mathcal{X}^*(G)_\mathbb{Q}\).

Let \(\chi \in \mathcal{X}^*(G)_\mathbb{R}\) such that \(\mathcal{V}^s(G, \chi) \neq \emptyset\). We know from lemma \ref{lem:3.6} that \(\mathcal{C}(\mathcal{V}^s(G, \chi)) \subset \mathcal{X}^*(G)_\mathbb{R}\) is a rational polyhedral cone. Therefore we find a sequence \(\{\chi_n\}_n \subset \mathcal{X}^*(G)_\mathbb{Q}\) such that \(\chi_n \to \chi\). Since the number of stability sets for rational classes is finite, we may assume, after possibly passing to a subsequence, that \(\mathcal{V}^s(G, \chi_n) =: U\) is independent of \(n\). Now, \(\chi_n \in \mathcal{C}(\mathcal{V}^s(G, \chi_n)) = \mathcal{C}(U)\) for all \(n\); since \(\chi_n\) converges to \(\chi\) and \(\mathcal{C}(U)\) is closed, we deduce that \(\chi \in \mathcal{C}(U)\).

There is a unique face \(\tau \subset \mathcal{C}(U)\), possibly equal to \(\mathcal{C}(U)\), such that, on one hand, \(\chi \in \text{rel.int.} \tau\); on the other hand, applying again theorem \ref{thm:main} with rational coefficients, we deduce that such a face equals \(\mathcal{C}(\mathcal{V}^s(G, \chi_0))\), for any rational point \(\chi_0 \in \text{rel.int.} \tau\). Applying lemma \ref{lem:3.6} we deduce that \(\chi\) is GIT-equivalent to \(\chi_0\).

An immediate consequence of the theorem is the following comparison result between the numerical functions defined by \cite{25}:

\begin{corollary}
For any \(\chi \in \mathcal{X}^*(G)\), there is a positive number \(d_\chi > 0\) depending on \(\chi\) such that for any \(d > d_\chi\) holds:
\[
M(\chi, x) \geq 0 \quad (\text{resp.} > 0) \iff \bar{M}(\langle d, \chi \rangle, [1, x]) \geq 0 \quad (\text{resp.} > 0), \quad \forall x \in \mathbb{V}.
\]
\end{corollary}

In this statement, the non-obvious implication is from left to the right. Our sign differs from that in \cite{6} because there the authors identify a vector space with the affine space defined by it (see \cite{6} subsection 1.1.5)).

Now we turn to our second question, namely to describe those characters of \(G\) for which the semi-stable locus is non-empty. It is immediate to see that the effective characters form a convex cone in \(\mathcal{X}^*(G)_\mathbb{R}\), and the natural question which raises is what this cone looks like. Secondly, we wish to characterise the characters \(\chi \in \mathcal{X}^*(G)_\mathbb{R}\) for which \(\mathcal{V}^s(G, \chi) \neq \mathcal{V}^s_{\mathbb{Q}}(G, \chi)\). In the terminology of \cite{6}, one says that these characters \textit{belong to a wall}; on the other hand, in \cite{21} a wall is by definition a cone of codimension one in \(\Delta^G(V)\). Here we will adopt the former definition, but we stress that there may be cones of codimension zero in \(\Delta^G(\mathbb{V})\) which are walls for this definition.

\begin{definition}
A character \(l \in \mathcal{X}^*(G)_\mathbb{R}\) is \textit{effective} if \(\mathcal{V}^s(G, l) \neq \emptyset\). A cone \(\tau \in \Delta^G(\mathbb{V})\) is a \textit{wall} if \(\mathcal{V}^s(G, l) \neq \mathcal{V}^s_{\mathbb{Q}}(G, l)\) for all \(l \in \text{rel.int.} \tau\).
\end{definition}

We start by addressing these issues in the abelian case.

\begin{lemma}
Let \(G\) be a torus and \(\rho : G \to \text{Gl}(\mathbb{V})\) a representation, whose weights are \(\{\eta_a\}_{a \in \Phi}\).

(i) The effective cone is \(\mathcal{C}^G(\mathbb{V}) = \sum_{a \in \Phi} \mathbb{R}_{\geq 0} \eta_a\).

(ii) The walls are precisely the cones
\[
\tau_{\Phi'} := \sum_{a \in \Phi'} \mathbb{R}_{\geq 0} \eta_a, \quad \text{where} \quad \Phi' \subset \Phi \quad \text{is such that} \quad \text{codim}_{\mathcal{X}^*(G)_\mathbb{R}}(\eta_a; a \in \Phi') = 1.
\]
\end{lemma}

\begin{proof}
(i) Since the GIT-fan is rational, we may restrict ourselves to integral coefficients. Consider \(\chi = \sum_{a \in \Phi(\chi)} k_a \eta_a\), with \(\Phi(\chi) \subset \Phi\) and \(k_a > 0\). For each \(a \in \Phi(\chi)\), we consider
a linear function $f_a$ on the weight space $V_a \hookrightarrow V$. Then $f := \prod_{a \in \Phi(\chi)} f_a^\chi \in K[V]$ is $\chi$-invariant, and not identically zero. More precisely, $f$ does not vanish at general points of the form $x = (x_a)_a$, with $x_a = 0$ for $a \notin \Phi(\chi)$.

Conversely, let $\chi \in X^*(G)$ be effective, and choose $x = (x_a)_{a \in \Phi(\chi)} \in V_{\text{ss}}(G, \chi) \setminus \{0\}$. Using proposition \ref{prop:effective-ray} we deduce that

$$x \text{ is } \chi\text{-semi-stable} \iff \left[ \forall \lambda \in \{\langle \chi, \cdot \rangle < 0 \} \Rightarrow \min_{a \in \Phi(\chi)} \{\langle \eta_a, \lambda \rangle \} < 0 \right]$$

$$(3.1)$$

$$\iff \{\langle \chi, \cdot \rangle < 0 \} \cap \left( \sum_{a \in \Phi(\chi)} Z_{\geq 0}\eta_a \right)_\chi = \emptyset$$

$$\iff \left( \sum_{a \in \Phi(\chi)} Z_{\geq 0} \eta_a \right)_\chi \subset \{\langle \chi, \cdot \rangle \geq 0 \}$$

$$\iff \chi \in \sum_{a \in \Phi(\chi)} Z_{\geq 0} \eta_a.$$  

Since $\sum_{a \in \Phi(\chi)} Z_{\geq 0} \eta_a \subset \sum_{a \in \Phi} \mathbb{R}_{\geq 0} \eta_a$, the first statement follows.

(ii) A character $\chi \in X^*(G)$ is in a wall precisely if there is a point $x = (x_a)_{a \in \Phi(\chi)} \in V_{\text{ss}}(G, \chi)$ with positive dimensional stabilizer. We have proved in \ref{prop:effective-ray} that this implies $\chi \in \sum_{a \in \Phi(\chi)} Z_{\geq 0} \eta_a$. Also, there is a 1-PS $\lambda$ contained in the stabilizer of $x$, which amounts requiring that $\langle \eta_a, \lambda \rangle = 0$ for all $a \in \Phi(\chi)$. Therefore the $\{\eta_a\}_{a \in \Phi(\chi)}$’s does not span $X^*(G)_\mathbb{R}$. Conversely, one immediately sees that the point constructed in the previous part has positive dimensional stabilizer. 

We consider now the general case of a representation $\rho : G \to \text{Gl}(V)$ of a reductive group. We denote

$$X^*(G, V) := \left\{ \chi \in X^*(G) \mid \mathbb{R}_{\geq 0} \chi \text{ is a ray in } \Delta_T(G) \cap X^*(G)_\mathbb{R} \wedge \chi \text{ is } G\text{-effective} \right\}.$$  

Proposition 3.10. (i) Let $G$ be a reductive group and $\rho : G \to \text{Gl}(V)$ be a representation with finite kernel. The cone of effective characters for the action of $G$ on the affine space $V$ is convex and equals

$$C^G(V) = \sum_{\chi \in X^*(G, V)} \mathbb{R}_{\geq 0} \chi.$$  

In the case $K[V] = K$, $C^G(V)$ is strictly convex.

(ii) Let $G$ be a reductive group and $\rho : G \to \text{Gl}(V)$ be a representation with finite kernel, and let $\{\eta_a\}_{a \in \Phi}$ be the weights of its maximal torus. Assume that $X^*(G)_\mathbb{R}$ is not contained in the linear span of any $T$-wall. Then the walls in $\Delta^G(V)$ are of the form $X^*(G)_\mathbb{R} \cap \tau_{\Phi'}$, with $\Phi' \subset \Phi$ as above. In particular, in this case, all walls have codimension one in $\Delta^G(V)$.

Proof. (i) For the same reason as before, we prove the statements for integral characters. The inclusion of the right-hand-side into the left-hand-side is obvious. For the other direction, let us consider $\chi \in X^*(G)$ such that $\mathbb{R}_{\geq 0}\chi$ is an exterior ray of $C^G(V)$. As $C^G(V) \subset \text{Support}(\Delta^T(V) \cap X^*(G)_\mathbb{R})$, there is a cone $\tau \in \Delta^T(V)$ such that $\chi \in \text{rel.int.}(\tau \cap X^*(G))$. Then, for any $\chi' \in \tau \cap X^*(G)$ holds

$$V_{\text{ss}}(T, \chi) \subset V_{\text{ss}}(T, \chi') \Rightarrow V_{\text{ss}}(G, \chi) \supset V_{\text{ss}}(G, \chi'),$$

so that $\chi'$ is still $G$-effective. Since $\chi$ generates an exterior ray in $C^G(V)$, $\mathbb{R}_{\geq 0}\chi = \tau \cap X^*(G)_\mathbb{R}$; otherwise would be contained in the relative interior of a ‘larger’ cone. It follows that $\chi \in X^*(G, V)$. 

We consider the \((A_T^+)^W\)-linear endomorphism \(J_G := \sum_{w \in W} \det(w) w \) of \(A_T^+\); according to [5, lemme 4], it has the property that \(\varphi_G := J_G / \Delta_G\) is still an endomorphism of \(A_T^+\), which takes values in \((A_T^+)^W\). Since \(\varphi_G : (A_T^+)_Q \to (A_T^+)_Q^W\) is \((A_T^+)_Q^W\)-linear, and \(\varphi_G(\Delta_G) = |W|\), it is an epimorphism.

The geometrical meaning of the homomorphism \(\varphi_G\) is captured in the

**Lemma 4.1.** The composition \(A_T^* \cong A_B^* \xrightarrow{\varphi} A_G^{\dim G/B} \xrightarrow{\varphi^*} A_T^{\dim G/B} \) equals \(\varphi_G\).

**Proof.** The push-forward formula implies that the endomorphism \(h := \varphi^* \varphi_*\) is \(\varphi^*(A_G^*)\)-linear.

But \(\varphi^* : A_G^* \to (A_T^+)^W\) is injective, its cokernel is \(\mathbb{Z}\)-torsion, and therefore \(h\) is \((A_T^+)^W\)-linear. Moreover, \(h\) is vanishes in degree strictly less than \(\dim G/B = \deg \Delta_G\). Applying [5, proposition 1], we obtain that

\[|W| \cdot \Delta_G \cdot h(u) = h(\Delta_G) \cdot J_G(u) = |W| \cdot J_G(u),\]

and our claim follows. 

With these preparations, the main result of this section is
Theorem 4.2. Let $\chi \in X^*(G)$ be a character such that $\mathcal{V}_{ss}(G, \chi) = \mathcal{V}_{(0)}^s(G, \chi)$; in particular $\mathcal{V}_{/\chi} G$ is a geometric quotient. Then the Chow ring

$$A_*(\mathcal{V}_{/\chi} G)_Q \cong (A^*_T/Q)^{\mathcal{V}_G(\langle \mathcal{E}(\lambda) \rangle_T; \lambda \in \mathcal{F}(\chi))}.\]

We wish to remark that this result represents a considerable generalization of [3] theorem 4.4, where one assumes that $\rho(G)$ contains the homotheties of $\text{Gl}(V)$ (which implies that $K[\mathcal{V}]^G = K$), and moreover that $G$ acts freely on $\mathcal{V}_{(0)}^s(G, \chi)$. The authors of [3] were aware of the possibility of generalizing the results of Ellingsrud and Strømme, as they explicitly point this out on page 610 of loc. cit. The computation of $\mathcal{E}(\lambda)_T$ is immediate: if $E \hookrightarrow V$ is a sub-$T$-module, then the equivariant class of the linear space $E := \text{Spec}(\text{Sym}^*(V/E))$ is

$$\mathcal{E}|_T = \prod_a \eta^m_a \in A^*_T,$$

where $\{\eta_a\}_a$ are the weights of the $T$-module $E$, and $\{m_a\}_a$ are the corresponding multiplicities.

For the proof of this theorem we need some preparatory results.

Lemma 4.3. Let $\sigma : X \to Y$ a projective, surjective morphism between reduced and irreducible, quasi-projective varieties. Then $\sigma_* : A_*(X)_Q \to A_*(Y)_Q$ is surjective.

**Proof.** We consider an effective cycle $Z \hookrightarrow Y$ and prove that there is an effective cycle $W \hookrightarrow X$ such that $\sigma|_W : W \to X$ is finite. Since $\sigma$ is surjective, there is a component $X' \hookrightarrow \sigma^{-1}(Z)$ which maps onto $Z$; the restriction $\sigma|_{X'} : X' \to Z$ is still projective. Consider an irreducible component $W \hookrightarrow X'$ of a general hyperplane section of $X'$, of codimension equal the dimension of the general fibre of $\sigma|_W$; it will have the property that the map $\sigma|_W : W \to Z$ is finite. It follows that $\sigma_*[W]$ is a positive multiple of $[Z]$. \qed

Lemma 4.4. Let $P \subset G$ be a parabolic subgroup, $L \subset P$ its Levi component, and consider $\tilde{\eta} \in A^*_T(G/P)^W$. Further, denote by $f^T_* : A^*_T(G/P) \to A^*_T$ the proper push-forward.

(i) The fixed point set of $G/P$ under the natural $T$-action consists of finitely many points; these are

$$(G/P)^T = \{wP \mid w \in N_G(T)/N_L(T) = W/W_L\},$$

(ii) Let $i_1 : \{P\} \hookrightarrow G/P$ be the inclusion, and $\eta := i^*_1 \tilde{\eta}$. Then

$$f^T_* ([\eta]^W) = \varphi_G(\eta)_Q.$$

**Proof.** The first statement is well-known. For the second one, we will use the integration formula [S corollary 1] applied to the homogeneous variety $G/P$. For $w \in W$ we have the diagram

$$\begin{array}{ccc}
\{wP\} & \hookrightarrow & G/P \\
\downarrow & & \downarrow \text{f} \\
\text{Spec } K & \longleftarrow & \text{Spec } K
\end{array}$$

Observe that for $\tilde{\alpha} \in ([\eta]^W)_Q$, $i^*_1 \tilde{\alpha} = w(i^*_1 \tilde{\alpha})$; in particular $i^*_1 \tilde{\alpha} \in A^*_T$ is $W_L$-invariant. Choosing a set of representatives $W \subset W$ for the rest classes in $W/W_L$, with $1 \in \tilde{W}$, the integration


formula reads

$$f_s^T(\tilde{\alpha}) = \sum_{\tilde{w} \in W} (f_s^T)_* \tilde{\alpha}_{\tilde{w}} = \sum_{\tilde{w} \in W} \tilde{w} \left( (f_s^T)_* \tilde{\alpha}_{\tilde{w}} \right).$$

We observe now that the image of \((f_s^T)_* \tilde{\alpha}_{\tilde{w}} : \text{Grass}(\mathbb{P}(G/P)) \to \mathbb{P}(G/P)\) equals \((A_+^*)_{\mathbb{Q}}\), and also that \((f_s^T)_* \tilde{\alpha}_{\tilde{w}} = \eta\). It follows that \((f_s^T)_* \tilde{\alpha}_{\tilde{w}}(\mathbb{P}(G/P)) = \eta \cdot (A_+^*)_{\mathbb{Q}}\).

Secondly, we claim that \((f_s^T)_* e^T(T_P(G/P))\) equals the quotient \(\Delta_G/\Delta_L\) of the discriminants of \(G\) and \(L\) respectively. Indeed, if \(g, p\) and \(b\) are the Lie algebras of \(G, P\) and \(B\) respectively, then \((f_s^T)_* e^T(T_P(G/P))\) is the product of weights of the \(T\)-module \(g/p \cong (g/b)/(p/b)\). Remains to notice that the weights of \(T\)-module \(p/b\) are (up to sign) precisely the positive roots of \(L\).

We rewrite equality above as

$$f_s^T(\tilde{\alpha}) = \sum_{\tilde{w} \in W} \tilde{w} \left( \frac{\alpha}{\Delta_G/\Delta_L} \right), \quad \text{with} \quad \alpha := (f_s^T)_* \tilde{\alpha}_{\tilde{w}} \in \eta \cdot (A_+^*)_{\mathbb{Q}},$$

(4.2)

We are going to compute \(\varphi_G(\eta)\), and compare with this formula. For the same reason as before, \(\eta \in (A_+^*)_{\mathbb{Q}}\). For \(\eta \in (A_+^*)_{\mathbb{Q}}\), we obtain

$$\varphi_G(\eta b) = \frac{1}{\Delta_G} \sum_{\tilde{w} \in W} \sum_{w \in \tilde{w} \cdot W_L} \det(w) \cdot w(\eta b) = \frac{1}{\Delta_G} \sum_{\tilde{w} \in W} \sum_{w \in \tilde{w} \cdot W_L} \det(w) \cdot w(\eta b) = \frac{1}{\Delta_G} \sum_{\tilde{w} \in W} \sum_{w \in \tilde{w} \cdot W_L} \det(w) \cdot \frac{\alpha}{\Delta_G/\Delta_L} \cdot \varphi_L(b) = \sum_{w \in \tilde{w} \cdot W_L} \tilde{w} \left( \frac{\alpha \cdot \varphi_L(b)}{\Delta_G/\Delta_L} \right).$$

As \(\varphi_L : (A_+^*)_{\mathbb{Q}} \to (A_+^*)_{\mathbb{Q}}\) is surjective, \(\{\eta \cdot \varphi_L(b) \mid b \in (A_+^*)_{\mathbb{Q}}\} = \eta \cdot (A_+^*)_{\mathbb{Q}},\) which concludes the proof.

Proposition 4.5. Let \(\rho : G \to \text{GL}(V)\) be a representation with finite kernel, and \(E \hookrightarrow V\) a linear subspace such that its stabilizer \(P := \text{Stab}_G(E)\) is a parabolic subgroup of \(G\). Then the followings hold:

(i) \(G : E\) is closed in \(V\);

(ii) the image of the canonical homomorphism

\[ A_+^G(G : E)_{\mathbb{Q}} \to A_+^G(V)_{\mathbb{Q}} \cong (A_+^G)_{\mathbb{Q}} \]

equals \(\varphi_G([E]_T)_{\mathbb{Q}}\).

Proof. (i) Let \(e := \dim E\), and consider the action of \(G\) on the Grassmannian \(\text{Grass}(e, V)\) of \(e\)-dimensional linear subspaces of \(V\). We denote \(\mathcal{T} \to \text{Grass}(e, V)\) the tautological bundle of rank \(e\), and by \(O_E \cong G/P\) the \(G\)-orbit of \([E] \in \text{Grass}(e, V)\); since \(P \subset G\) is parabolic, \(O_E\) is closed in \(\text{Grass}(e, V)\).
In the following diagram

\[\begin{array}{ccc}
\mathcal{F} |_{O_E} & \xrightarrow{j} & O_E \times V \\
\downarrow & & \downarrow \\
O_E & \xrightarrow{f} & \text{Spec } K
\end{array}\]

all the morphisms are $G$-equivariant, and we observe that $G \cdot E = \text{pr}_V(\mathcal{F} |_{O_E})$. Since $\mathcal{F} |_{O_E}$ is closed in $O_E \times V$ and $\text{pr}_V$ is proper, $G \cdot E$ is closed in $V$.

(ii) From the commutative diagram

\[\begin{array}{ccc}
\mathcal{F} |_{O_E} & \xrightarrow{j} & O_E \times V \\
\downarrow & & \downarrow \\
G \cdot E & \xrightarrow{\sigma} & V
\end{array}\]

and the surjectivity of $\sigma_*$ proved in [23], we deduce that image of $A^*_G(G \cdot E)_Q \to A^*_G(V)_Q$ equals the image of $(\text{pr}_V)_G^* \circ f_G^*$. We consider the commutative diagram of homomorphisms between $G$-equivariant Chow groups

\[\begin{array}{cccc}
A^*_G(\mathcal{F} |_{O_E}) & \xrightarrow{f^*_G} & A^*_G(O_E \times V) & \xrightarrow{(\text{pr}_V)_G^*} & A^*_G(V) \\
\cong & & \cong & & \cong \\
A^*_{-\dim V}(O_E) & \xrightarrow{e^*(Q |_{O_E})} & A^*_{-\dim V}(O_E) & \xrightarrow{f_G^*} & A^*_G(V)
\end{array}\]

and the excess intersection theorem (see [10, example 6.3.5]) implies that the lower left homomorphism is the cap product with the equivariant Euler class of the universal quotient bundle $Q \to \text{Grass}(e, V)$. By abuse of notation, we will still write $Q \to O_E$ for the restriction of $Q$ to $O_E$. Let us consider an appropriate open subset $U$ in some representation space of $G$ (see [7, definition-proposition 1]) needed for the computation of the equivariant Chow groups. The diagram

\[\begin{array}{ccc}
U \times T O_E & \xrightarrow{f^T} & U/T \\
\varphi_{U,O_E} \downarrow & & \downarrow \varphi_U \\
U \times G O_E & \xrightarrow{f_G^*} & U/G
\end{array}\]

is cartesian, the horizontal morphisms are proper and the vertical ones are flat, and therefore the induced diagram

\[\begin{array}{ccc}
\langle e^T(Q) \rangle_Q \subset A^*_E(O_E)_Q & \xrightarrow{f^T_E} & (A^*_E)_Q \\
\varphi^* \downarrow & & \downarrow \varphi^* \\
\langle e^G(Q) \rangle_Q \subset A^*_G(O_E)_Q & \xrightarrow{f^*_G} & (A^*_G)_Q
\end{array}\]

commutes (see [10, proposition 1.7]). Now, we know from [7, proposition 6] that $\varphi^*$ is injective, and its image consists of the $W$-invariant elements, that is

\[\varphi^* [f^*_G(\langle e^G(Q) \rangle_Q)] = f^*_E(\langle e^T(Q) \rangle_W) \implies \varphi_G(\langle [E]_T \rangle_W).\]

All together shows that the image of $A^*_G(G \cdot E) \to A^*_G(V)$ is $\varphi_G(\langle [E]_T \rangle_W)$. \qedsymbol
Proof. (of the theorem) We have proved in lemma 4.1 that
\[ \mathcal{V}_{(0)}^s(G, \chi)/G = \left( \mathcal{V} \setminus \mathcal{V}_{\text{ss}}(G, \chi) \right)/G = \left( \mathcal{V} \setminus \bigcup_{\lambda \in \mathcal{F}(\chi)} G \cdot \mathcal{E}(\lambda) \right)/G, \]
where the \( \mathcal{E}(\lambda) \)'s are linear subspaces of \( \mathcal{V} \). Applying theorem 3 and proposition 5 of [7], we deduce that
\[ A_{s-\dim G}(\mathcal{V}_{(0)}^s(G, \chi))/G \cong A_{s}^G(\mathcal{V}_{(0)}^s(G, \chi)), \]
and that there is an exact sequence
\[ A_{s}^G \left( \bigcup_{\lambda \in \mathcal{F}(\chi)} G \cdot \mathcal{E}(\lambda) \right) \xrightarrow{j_{s}^G} A_{s}^G(\mathcal{V}) \longrightarrow A_{s}^G(\mathcal{V}_{(0)}^s(G, \chi)) \longrightarrow 0. \]
Further, by the very construction of \( \mathcal{F}(\chi) \), the \( \{ G \cdot \mathcal{E}(\lambda) \}_{\lambda \in \mathcal{F}(\chi)} \) are the irreducible components of the union, and therefore
\[ j_{s}^G \left[ A_{s}^G \left( \bigcup_{\lambda \in \mathcal{F}(\chi)} G \cdot \mathcal{E}(\lambda) \right) \right] = \sum_{\lambda \in \mathcal{F}(\chi)} j_{s}^G \left[ A_{s}^G(G \cdot \mathcal{E}(\lambda)) \right]. \]
For \( \lambda \in \mathcal{F}(\chi) \), \( \mathcal{E}(\lambda) \) is stabilized by the parabolic subgroup \( P(\lambda) \subset G \), so that \( \text{Stab}_G \mathcal{E}(\lambda) \) is parabolic too. It follows now from proposition 4.5 that
\[ j_{s}^G \left[ A_{s}^G(G \cdot \mathcal{E}(\lambda)) \right]_Q = \varphi_G([\mathcal{E}(\lambda)]_T)_Q. \]
This finishes the proof of theorem 4.2. \( \square \)

Remark 4.6. We have been primarily interested in the Chow ring of the quotients, but the proof shows more: for any \( \chi \in \mathcal{X}^s(G) \), the \( G \)-equivariant Chow ring of the \( \chi \)-(semi-)stable locus of \( \mathcal{V} \) is given by the formula appearing in theorem 4.2. We precise that for computing the equivariant ring of the properly stable locus, one has to divide out the (possibly) larger ideal \( \varphi_G([\mathcal{E}(\lambda)]_T) \).

The difference between the semi-stable and the properly stable locus is particularly relevant if \( G \) is semi-simple, when its character group is trivial. The semi-stable locus is then the whole \( \mathcal{V} \), and the corresponding invariant quotient is not geometric. On the other hand the quotients like \( \mathcal{V}_{(0)}^s(G)/G \) can be used to construct approximations of the algebro-geometric classifying space of \( G \).

We conclude this section with a remark about the generators of the Chow ring. In [9, section 6, last paragraph], the authors express their belief that the Chow ring of invariant quotients of affine spaces, should be generated by Chern classes of ‘naturally given vector bundles’. Actually, they make this statement for the more restrictive case that they are considering, when the representation \( G \to \text{Gl}(V) \) contains the homotheties.

Following [24, example 3, page 26] we define the representation ring \( \mathcal{R}(G) \) of \( G \) to be the ring generated by isomorphism classes of representations of \( G \) modulo the ideal generated by \( [F] - [F'] - [F''] \), where
\[ 0 \longrightarrow F' \longrightarrow F \longrightarrow F'' \longrightarrow 0 \]
is an exact sequence of \( G \)-modules; if \( \text{char } K = 0 \), reductive groups are linearly reductive, and in this case \( F = F' \oplus F'' \) as a \( G \)-module. The addition in \( \mathcal{R}(G) \) is given by direct sum, and the product by tensor product.
The restriction to the maximal torus defines a ring homomorphism
\[ \mathcal{R}(G) \rightarrow \mathcal{R}(T)^W, \]
where on the right-hand-side we consider the $W$-invariant representations of $T$. By [24, théorème 1.1, page 26], this homomorphism is actually an isomorphism. Since the representation ring of $T$ is isomorphic to the symmetric algebra of its group of characters, we deduce the isomorphism
\[ \mathcal{R}(G)_Q \isom (\text{Sym}_Q^* X^*(T)_Q)^W \cong (A_{T}^*)^W_Q \cong (A_G^*)_Q. \]
Moreover, the composition $\mathcal{R}(G)_Q \rightarrow (A_G^*)_Q$ is given by the Chern character
\[ \mathcal{R}(G) \ni (F, \rho) \mapsto \text{ch}(EG \times _F BG) \in A^*_G. \]
By abuse of notation, $EG$ stands here for an appropriate open subset of an affine space, needed to define equivariant classes, and $BG := EG/G$.

Proposition 4.7. (i) Let $\ell := \text{rank} G = \text{dim} T$. There are at most $\ell$ elements in $\mathcal{R}(G)$ whose Chern classes generate the ring $(A_{T}^*)^W_Q$.

(ii) Consider $\chi \in X^*(G)$ such that $V^\text{s}(G, \chi) = V^a(0)_G(G, \chi)$. The same statement as in (i) holds for the Chow ring of the invariant quotient $V^\parallel \chi G$.

Proof. Since $(A_{T}^*)^W_Q \rightarrow A_s(Y)_Q$ is an epimorphism, it is enough to prove the first claim. Applying [11, §5.5, théorème 4] to the $\mathbb{Q}$-vector space $X^*(T)_Q$, we deduce that there are algebraically independent, homogeneous elements $I_1, \ldots, I_\ell \in (A_{T}^*)^W_Q$, such that $(A_{T}^*)^W_Q = \mathbb{Q}[I_1, \ldots, I_\ell]$. The isomorphism [13] implies that there are elements $F_1, \ldots, F_\ell \in \mathcal{R}(G)$ such that
\[ \text{ch}_{\text{deg} I_j} (F_j) = I_j, \quad \forall \ j = 1, \ldots, \ell, \]
so that the Chern classes of $F_1, \ldots, F_\ell$ generate $(A_{T}^*)^W_Q$. \hfill \Box

A particularly comfortable situation arises when $G = \times_{j=1}^s \text{Gl}(n_j)$ is a product of linear groups; this happens, for instance, in the case of quiver representations. Then the Chow ring of the corresponding quotients will be generated by the Chern classes of the identical representations of the $s$ factors of $G$.

5. The cohomology ring of the quotients

In this section we assume that we work over the field of complex numbers, and moreover that the ring of invariants $\mathbb{C}[V]^G = \mathbb{C}$. This latter condition guarantees that the invariant quotients $V^\parallel \chi G, \chi \in X^*(G)$, are projective.

Theorem 5.1. Let $\chi \in X^*(G)$ be a character such that $V^\text{s}(G, \chi) = V^a(0)_G(G, \chi)$. Then the cohomology ring
\[ H^*(V^\parallel \chi G; \mathbb{Q}) \cong (H^*_T)_Q^W \slash \sim_{\mathcal{G}}\langle \text{F}(\lambda) | T; \lambda \in \mathcal{F}(\chi) \rangle_Q. \]

A fortiori, we deduce that the cycle map
\[ \text{cl} : A^*_s(V^\parallel \chi G)_Q \rightarrow H^*(V^\parallel \chi G; \mathbb{Q}) \]
is an isomorphism.
Proof. The idea is to use the results of section 2 and to reduce our question to a similar one for actions of reductive groups on projective varieties, for which we know the cohomology of the semi-stable locus.

We have proved in proposition 2.3 that
\[ V^{ss}(G, \chi) = P(\mathbb{C} \oplus V)^{ss}(G, \Sigma(d, \chi)) =: \Omega. \]

For the convenience of the writing, we shall denote \( \bar{V} := P(\mathbb{C} \oplus V) \).

Our first claim is that the restriction homomorphism \( H^*_G(V) \to H^*_G(\Omega) \) is surjective: indeed, the diagram
\[
\begin{array}{ccc}
H^*_G(\bar{V}) & \xrightarrow{j^*_G} & H^*_G(V) \\
\downarrow{\alpha^*} & & \downarrow{H^*_G(\Omega)} \\
H^*_G(\Omega) & & \\
\end{array}
\]
commutes, and we know from [20, page 175] that \( \alpha^* \) is surjective. Secondly, we claim that \( j^*_G \) is surjective too. This can be seen as follows: from the commutative diagram
\[
\begin{array}{ccc}
BG \cong EG \times_G [1, 0] & \xrightarrow{j_0} & EG \times_G \bar{V} \\
\downarrow{BG} & & \downarrow{\text{id}} \\
\text{id} & & id \\
\end{array}
\]
we deduce that \( (j_0)^*_G \) is surjective. As usual, \( EG \to BG \) stays for a universal \( G \)-bundle. The surjectivity of \( j^*_G \) is implied now by the commutative triangle
\[
\begin{array}{ccc}
H^*_G(\bar{V}) & \xrightarrow{j^*_G} & H^*_G(V) \\
\downarrow{(j_0)^*_G} & & \downarrow{H^*_G(\Omega)} \\
H^*_G(\bar{V}) & & \\
\end{array}
\]
corresponding to the inclusions \( \{0\} \subset V \subset \bar{V} \).

The two claims together imply that
\[
\text{Ker}(H^*_G(V) \to H^*_G(\Omega)) = j^*_G \text{Ker}(H^*_G(\bar{V}) \to H^*_G(\Omega)).
\]

Now we are going to use the result [2 corollaire 1.1] of M. Brion, which expresses the right-hand-side in terms of the \( W \)-invariant part of the \( T \)-equivariant cohomology of the \( T \)-unstable locus. For this, we consider the diagram
\[
\begin{array}{ccc}
H^*_T(V) & \xrightarrow{\varphi^*} & H^*_G(V) \\
\downarrow{j^*_T} & & \downarrow{j^*_G} \\
H^*_T(\bar{V}) & \xrightarrow{\bar{\varphi}^*} & H^*_G(\bar{V}) \\
\end{array}
\]
Applying $\varphi^*$ to both sides of (5.2), we obtain that

$$
\varphi^* \ker \left( H^*_G(\mathcal{V}) \to H^*_G(\Omega) \right) = (\varphi^* \circ \iota_G^*) \ker \left( H^*_G(\mathcal{V}) \to H^*_G(\Omega) \right)
$$

$$
= (\iota_T^* \circ \varphi^*) \ker \left( H^*_G(\mathcal{V}) \to H^*_G(\Omega) \right)
$$

$$
= \left[ \jmath_T^* \ker \left( H^*_T(\mathcal{V}) \to H^*_T(\overline{\mathcal{V}} ss(T, \Sigma_{(d, \chi^j)})) \right) \right]^W.
$$

The last equality holds because the projection onto the $W$-invariant part is a linear map (a Reynolds type homomorphism), which commutes with the pull-back. Let us point out that in order to apply Brion’s result, we must be able to linearize the $G$-action in $\mathcal{O}(1)$; according to remark 2.4, this can be achieved after replacing $G$ with a suitable finite cover (see 2.4). This modification does not affect the cohomology ring, since we work with rational coefficients.

The semi-stable locus $\overline{\mathcal{V}} ss(T, \Sigma_{(d, \chi^i)})$ corresponds to the representation

$$
\bar{\rho} : T \longrightarrow \text{Gl}(\mathbb{C} \oplus V), \quad \bar{\rho}(g) = \begin{pmatrix}
\chi(g)^{-1/d} & 0 \\
0 & \chi(g)^{-1/d} \rho(g)
\end{pmatrix},
$$

of the $d$-sheeted cover of $G$, for which the fractional power is well defined. If $\eta_1, \ldots, \eta_r$ are the characters of the $T$-action on $\mathcal{V} \cong \mathbb{A}^1_{\mathbb{C}}$ (together with their multiplicities), then $T$ acts on $\mathbb{A}^1_{\mathbb{C}} \times \mathbb{A}^r_{\mathbb{C}}$ by the characters

$$
\bar{\eta}_0 := \chi^{-1/d}, \bar{\eta}_1 := \chi^{-1/d} \eta_1, \ldots, \bar{\eta}_r := \chi^{-1/d} \eta_r.
$$

The irreducible components of the $T$-unstable locus in $\overline{\mathcal{V}}$ are of the form $\mathbb{P}(F)$, where $F \hookrightarrow \mathbb{C} \oplus \mathbb{C}^r$ runs over the set of unstable coordinate planes. For $F$ such a plane, we denote $z(F) \subset \{0, 1, \ldots, r\}$ its defining equations, that is

$$
F = \{ x_j = 0 \mid j \in z(F) \} \subset \mathbb{C} \oplus \mathbb{C}^r.
$$

Then [2] théorème 2.1 says that

$$
\ker \left( H^*_T(\overline{\mathcal{V}}) \to H^*_T \left( \overline{\mathcal{V}} ss(T, \Sigma_{(d, \chi^i)}) \right) \right) = \left\langle \prod_{j \in z(F)} (h + \bar{\eta}_j) \mid F \subset \mathbb{C} \oplus \mathbb{C}^r \text{ is unstable plane} \right\rangle,
$$

where $h := c_1 \left( \mathcal{O}(1) \right)$.

Claim \hspace{1cm} $j_T^*(h + \bar{\eta}_j) = \begin{cases} 0 \quad \text{if } 0 \in z(F); \\
\eta_j \quad \text{if } 0 \notin z(F).
\end{cases}$

Proof. (of claim) We notice first that $h + \bar{\eta}_j = c_7 \left( \mathcal{O}(1) \eta_j \right)$, where we have denoted $\mathcal{O}(1) \eta_j$ the $T$-linearized invertible sheaf $\mathcal{O}(1)$ endowed with the $T$-action through the character $\eta_j$. More precisely, the $T$-action on the geometric line bundle is

$$
T \times \mathcal{O}(1) \longrightarrow \mathcal{O}(1), \quad t \times [(a, v), z] := [t \times (a, v), \eta_j(t)z].
$$

A consequence of the commutative diagram (5.1), this time for $T$-equivariant cohomologies, is that for computing the restriction $j_T^*(h + \bar{\eta}_j) \in H^*_T(\mathcal{V})$, we must determine the character
with which $T$ operates on $\rho_0^j \Omega(1)_{\eta_j}$. An immediate computation shows that it is

$$\bar{\eta}_0^{-1} \bar{\eta}_j = \begin{cases} 1 & \text{if } j = 0; \\ \eta_j & \text{if } j \neq 0. \end{cases}$$

This finishes the proof of the claim.

We continue now the proof of the theorem. First of all, we have proved in proposition \ref{prop:intersection} that $V_{rs}(T, \chi) = V \cap V_{rs}(T, \Sigma_{(d, \lambda')})$. This implies that the intersections $V \cap \mathbb{P}(F), F$ unstable plane in $\mathbb{C} \oplus V$, coincide with the linear spaces $E(\lambda) \hookrightarrow V, \lambda \in \mathcal{F}(\chi)$. For any such $F$ we distinguish between two possibilities: either $0 \in z(F)$ or $0 \not\in z(F)$. The claim above implies that

$$J^*_T \left( \prod_{j \in z(F)} (h + \eta_j) \right) = \begin{cases} 0 & \text{if } 0 \in z(F); \\ \prod_{j \in z(F)} \eta_j & \text{if } j \neq 0. \end{cases}$$

Now, on one hand we know that $V \cap \mathbb{P}(F)$ coincides with some $E(\lambda), \lambda \in \mathcal{F}(\chi)$, and all the $E(\lambda)$'s occur in this way, and on the other hand that the defining equations of $V \cap \mathbb{P}(F)$ are $\{x_j = 0, j \in z(F)\}$. These two facts imply that

$$\prod_{j \in z(F)} \eta_j = [E(\lambda)]_T, \text{ for appropriate } \lambda \in \mathcal{F}(\chi),$$

which finishes the proof of the theorem. \hfill \Box

6. Construction of families of quotients

Let us recall that to any scheme $S$ and locally free sheaf $\mathcal{F} \to S$, one can associate the Grassmannian $\text{Grass}(d, \mathcal{F}) \to S$ of $d$-dimensional quotients of $\mathcal{F}$. With this motivation in mind, we wish to address the problem of constructing families of varieties, which are invariant quotients of affine spaces, over arbitrary bases. Our construction relies on Seshadri’s construction of quotients for actions of reductive group schemes.

We fix a complex, connected, reductive group $G_C$ and a complex representation $\rho : G_C \to \text{Gl}(V_C)$ of it with the property that $\mathbb{C}[V_C]^{G_C} = \mathbb{C}$. Then $V_C$ decomposes uniquely, as a $G_C$-module, into its isotypical components

$$V_C = \sum_{\omega \in \Omega} M_{C, \omega} \otimes_{\mathbb{C}} V_{C, \omega}, \quad \dim_{\mathbb{C}} V_{C, \omega} =: \nu_\omega \geq 1,$$

where the $M_{C, \omega}$ and the $V_{C, \omega}$'s are simple, and respectively trivial $G_C$-modules. We denote by $\rho_\omega : G_C \to \text{Gl}(M_{C, \omega})$ the corresponding representations. Further, for each $\omega$, we choose an admissible lattice $M_\omega \subset M_{C, \omega}$ (see \cite[page 225]{H}). Chevalley associates to this data, in \cite[section 4]{H}, a reductive group scheme $G \to \text{Spec} \mathbb{Z}$ which contains a maximal torus, and whose geometric fibres are connected, reductive groups having the same root system as $G_C$. Moreover, the representations $\rho_\omega$ extend to

$$\rho_\omega : G \to \text{Gl}(M_\omega) := \text{Spec} \left( \left( \text{Sym}_Z^\bullet \text{End}_Z(M_\omega) \right) [\text{det}^{-1}] \right).$$

We fix a character $\chi : G \to G_m = \text{Spec} \mathbb{Z}[t, t^{-1}]$. Then for any field $K$, we get an induced character $\chi_K : G_K \to G_{m,K} = \text{Spec} K[t, t^{-1}]$.\hfill \Box
Lemma 6.1. If $G \to \text{Gl}(\nu; \mathbb{C})$ is a representation such that $\mathbb{C}[t_1, \ldots, t_\nu]^{G_c} = \mathbb{C}$, then for any reduced ring $B$ holds $B[\mathbb{C}[t_1, \ldots, t_\nu]]^{G_B} = B$.

Proof. Assume the contrary, that there is $f \in B[\mathbb{C}[t_1, \ldots, t_\nu]]^{G_B}$, homogeneous with $\deg f > 0$. Since $B$ is reduced, there is $q \in \text{Spec} B$ such that the coefficients of $f$ are not contained in $q$. We let $K$ to be the algebraic closure of quotient field $Q(B/q)$, and we observe that the polynomial $F_K$ obtained by extending the coefficients of $f$ is non-zero, $G_K$-invariant, $\deg f_K > 0$. Using [23, theorem 1], we deduce that there is a homogeneous $F \in \mathbb{Z}[t_1, \ldots, t_\nu]^G$ with $\deg F > 0$; this contradicts our hypothesis. □

We consider further a ring $R$, which is a finite algebra over a universally Japanese ring; the examples we have in mind are $R$ (for mixed characteristic methods) and $\mathbb{Z}[[t]]$ (for the study of deformations).

Let $S \to \text{Spec} R$ be a separated and reduced scheme of finite type. For $\omega \in \Omega$, we consider a locally free sheaf $\mathcal{Y}_\omega \to S$ with $\text{rank}_{\mathcal{O}_S} \mathcal{Y}_\omega = \nu_\omega$, and define

$$\mathcal{Y} := \bigoplus_\omega M_\omega \otimes_{\mathbb{Z}} \mathcal{Y}_\omega \to S.$$ 

We denote $V := \text{Spec}(\text{Sym}^*_{\mathcal{O}_S} \mathcal{Y}) \xrightarrow{\pi} S$ the corresponding geometric vector bundle.

Theorem 6.2. (i) There is a natural action $G \times_{\text{Spec} \mathbb{Z}} V \to V$ of the group scheme $G \to \text{Spec} \mathbb{Z}$, which covers the trivial action on $S$.

(ii) There is a $G$-invariant open subscheme $V^{ss}(G, \chi) \subset V$ satisfying the following properties:

(ii_a) There is a categorical quotient $(Y, q)$ for the $G$-action on $V^{ss}(G, \chi)$. Moreover, there is a natural isomorphism

$$Y \cong \text{Proj} \left( \bigoplus_{n \geq 0} (\pi_* \mathcal{O}_V)^G \right) \longrightarrow S,$$

and $Y$ is projective over $S$;

(ii_b) For any algebraically closed field $K$, and any morphism $\text{Spec} K \to S$, $\text{Spec} K \times_S V^{ss}(G, \chi)$ consists of the $\chi_K$-semi-stable points of $\text{Spec} K \times_S V$;

Proof. The proof will be divided in three steps: first we show that there is a natural $G$-action on $V$, next that the quotient exists locally, and finally we glue the local quotients together.

For proving that there is a global $G$-action on $V$, is enough to prove that $\text{Gl}(M_\omega)$ acts, for each $\omega \in \Omega$. We observe that the ring homomorphism

$$\text{Sym}^*_{\mathcal{O}_S} (M_\omega \otimes_{\mathbb{Z}} \mathcal{Y}_\omega) \longrightarrow \text{Sym}^*_{\mathcal{O}_S} (\mathcal{O}_S \otimes_{\mathcal{O}_S} \text{End}_{\mathbb{Z}}(M_\omega)) \otimes_{\mathcal{O}_S} \text{Sym}^*_{\mathcal{O}_S} (M_\omega \otimes_{\mathbb{Z}} \mathcal{Y}_\omega)$$

$$m \otimes v \mapsto \mu_\omega(m) \otimes v,$$

with $\mu_\omega : \text{Sym}^*_{\mathcal{O}_S} M_\omega \to \text{Sym}^*_{\mathcal{O}_S} M_\omega \otimes \text{Sym}^*_{\mathcal{O}_S} M_\omega$ the co-multiplication, defines the $\text{Gl}(M_\omega)$ action on $\text{Spec}(\text{Sym}^*_{\mathcal{O}_S} M_\omega \otimes_{\mathbb{Z}} \mathcal{Y}_\omega)$. Since

$$V = \bigtimes_{S, \omega \in \Omega} \text{Spec}(\text{Sym}^*_{\mathcal{O}_S} M_\omega \otimes_{\mathbb{Z}} \mathcal{Y}_\omega),$$

$\text{Gl}(M_\omega)$ acts on $V$ too. Repeating the very same construction as in section 1, we deduce also the existence of a $G$-action on $\mathbb{A}_K^1 \times_S V$, where $G$ acts on $\mathbb{A}_K^1$ by $\chi$.

What prevents us from constructing the quotient globally, is that $V$ can not be embedded as a $G$-invariant closed subset of an affine space over $\text{Spec} R$; in this case, each $\mathcal{Y}_\omega \to S$ would be globally generated. What we do instead, is to make the constructions locally over $S$. 

Let us choose a finite covering \((S_i = \text{Spec } B_i)\) of \(S\) with open, affine subsets, which are moreover trivializing for all the \(y_\omega\)'s. For every index \(i\), we find then an integer \(n_i \geq 1\) such that we have \(S_i \hookrightarrow \mathbb{A}^n_{R_i}\). It follows that

\[
V_i := \left( \times_{S, \omega \in \Omega} \text{Spec} \left( \text{Sym}^*_{S} M_{\omega} \otimes_{\mathbb{Z}} \mathcal{O}_{S} \right) \right)|_{S_i}
\]

\[
\leftarrow \times_{R_i^{n_i}, \omega \in \Omega} \text{Spec} \left( \text{Sym}^*_{R_i^{n_i}} M_{\omega} \otimes_{\mathbb{Z}} \mathcal{O}_{R_i^{n_i}} \right) = \mathbb{A}^{n_i + \nu}_R,
\]

with \(\nu := \sum \omega \cdot \text{rank} M_{\omega} \cdot \nu_\omega\), and moreover this embedding is \(G\)-equivariant.

For \(R_i := \Gamma(\mathbb{A}^1_{S} \times S V_i, \mathcal{O}_{\mathbb{A}^1_{S} \times S V_i}) \cong B_i[t, t_1, \ldots, t_\nu]\), \([23, \text{ theorem 2}]\) implies that

\[
R_i^G = \bigoplus_{m \geq 0} t^{m} \cdot B_i[t_1, \ldots, t_\nu]^G \subset R_i
\]

is a finitely generated \(B_i\)-algebra. We define the \(\chi\)-unstable locus \(V^\mu_i(G, \chi) \rightarrow V_i\) to be the closed subscheme defined by the ideal

\[
\langle B_i[t_1, \ldots, t_\nu]^G_m ; m \geq 1 \rangle \subset B_i[t_1, \ldots, t_\nu]
\]

and let \(V^\mu_i(G, \chi)\) to be its complement. The glueing argument of \([20, \text{ theorem 1.10}]\) together with \([23, \text{ theorem 3 and remark 8}]\) imply that the categorical quotient \(Y_i\) of \(V^\mu_i(G, \chi)\) by \(G\) exists, and it is isomorphic to \(\text{Proj} \left( \bigoplus_{m \geq 0} B_i[t_1, \ldots, t_\nu]^G_m \right)\). It follows now from \([11, \text{ proposition 5.5.1}]\) that this quotient is projective over \(\text{Spec} B_i[t_1, \ldots, t_\nu]^G\).

Using \([23, \text{ lemma 2}]\), we remark that for any two open subsets \(S_i, S_j \subset S\), holds

\[
V^\mu(G, \chi)|_{S_i \cap S_j} = V^\mu_i(G, \chi)|_{S_i \cap S_j},
\]

so that there is a well-defined open subset \(V^\mu(G, \chi) \subset V\) of \(\chi\)-semi-stable points. Finally, using the universality property of categorical quotients, we glue together the \(V^\mu_i(G, \chi) \rightarrow Y_i\)'s into a scheme \(Y\), and this comes with a natural morphism \(V^\mu(G, \chi) \rightarrow Y\).

It remains to prove the behaviour of the semi-stable locus under base change; the question being local on \(S\), we may assume that \(S = \text{Spec } B\). The inclusion \(K \times_B V^\mu(G, \chi) \subset V^\mu_K(G_K, \chi_K)\) is trivial, so that we prove the converse. Consider a \(K\)-valued point \(x_0 \in V^\mu(G_K, \chi_K)\), and a homogeneous \(f \in B[t_1, \ldots, t_\nu]^G_K, m \geq 1\), such that \(f(x_0) \neq 0\). Then \(F_K := t^{\deg f} f \in K[t_1, \ldots, t_\nu]^G_K\), and applying \([23, \text{ theorem 1}]\) we find a homogeneous \(F \in B[t_1, \ldots, t_\nu]^G_B\) such that \(\deg F > 0\) and \(F(1, x_0) \neq 0\). By lemma \(6.1\) \(B[t_1, \ldots, t_\nu]^G_B = B\), so that \(F \in \bigoplus m \geq 1 t^{m} B[t_1, \ldots, t_\nu]^G_B\), and therefore \(x_0 \in K \times_B V^\mu(G, \chi)\).

Remark 6.3. In case \(S = \text{Spec } \mathbb{Z}\), we get a flat family of quotients parameterized by \(\text{Spec } \mathbb{Z}\), and one might hope to relate the complex geometric properties of the quotients to the arithmetical ones.

As we have already mentioned, one of the most common situation where the construction applies is that of the Grassmann bundle of \(d\)-dimensional quotients of a locally free sheaf \(\mathcal{F} \rightarrow S\). In this case, we define \(Y := \text{Hom}_\mathbb{Z}(\mathbb{A}^d, \mathcal{F}) \cong \mathbb{Z}^d \otimes \mathbb{Z} \mathcal{F}, G := \text{GL}(d), \chi := \text{det}\).

As a first application, we describe how does the cone decomposition obtained in section \(3\) varies with the characteristic of the ground field.

Corollary 6.4. Assume \(S = \text{Spec } \mathbb{Z}\), \(V = \text{Spec}(\text{Sym}^* M)\), and \(\chi, \varepsilon : G \rightarrow G_m\) are characters such that \(\varepsilon \in C(\chi_G)\). Then there is a prime \(p_0 > 0\) depending on \(\varepsilon\), such that \(\varepsilon_K \in C(\chi_K)\) for any algebraically closed field \(K\) of characteristic zero or char \(K > p_0\).
Proof. The first remark is that $V^{ss}_Q(G,\chi_Q) \subset V^{ss}_Q(G,\varepsilon_Q)$: indeed, otherwise there would exist $q \in \text{Spec}(\text{Sym}_Q^* M_Q)$ such that

$$q \in V^{ss}_Q(G,\chi_Q) \quad \text{and} \quad q \supset (\text{Sym}_Q^* M_Q)_{\varepsilon_Q} G_Q^{c}; \quad m \geq 1.$$ 

Using theorem 6.2 (ii), we obtain that $q_C \in V^{ss}_C(G,\chi_C)$ and

$$q_C \supset \sqrt{(\text{Sym}_C^* M_Q)_{\varepsilon_Q} G_Q^{c};} \quad m \geq 1,$$

which contradicts our hypothesis.

Applying theorem 6.2 (ii) again, we deduce the validity of the corollary in characteristic zero. To settle the problem in positive characteristic, let us notice that the irreducible components of $V^{us}(G,\varepsilon)_{\text{red}} \hookrightarrow V$ are divided in two groups, according to whether their generic point is mapped onto $(0) \in \text{Spec} \mathbb{Z}$ or not (i.e. the residue field has characteristic zero or strictly positive). Let $U \subset \text{Spec} \mathbb{Z}$ be the complement of the finite set of primes where we have such ‘bad reduction’. Since $V^{us}(G_Q,\varepsilon_Q) \hookrightarrow V^{us}(G_Q,\chi_Q)$, we deduce that $V^{us}(G,\varepsilon)|_U \hookrightarrow V^{us}(G,\chi)|_U$.  

We specialize now to the case where $S$ is a reduced and irreducible $K$-scheme of finite type, where $K$ is an algebraically closed field. Our construction yields in this case a locally trivial fibration $Y \to S$, whose fibres are isomorphic to $V/\chi G$, and we interested in computing its Chow ring.

Reasoning locally over $S$, we see that the $\chi|_T$-unstable locus of $V$ for the induced $T$-action is the union

$$V^{us}(T,\chi) = \bigcup_{\lambda \in \mathcal{F}(\chi)} E(\lambda),$$

where $E(\lambda) := \text{Spec}(\mathcal{V}/\mathcal{E}(\lambda)) \hookrightarrow V$, $\lambda \in \mathcal{F}(\chi)$, are sub-vector bundles associated to locally free subsheaves $\mathcal{E}(\lambda) \subset \mathcal{V}$. The $\chi$-unstable locus for the $G$-action is then

$$V^{us}(G,\chi) = \bigcup_{\lambda \in \mathcal{F}(\chi)} G \cdot E(\lambda).$$

We still denote by $\varphi_G : A_*(S)_Q \otimes_Q (A^T)_Q \to A_*(S)_Q \otimes_Q (A^T)_Q$ the homomorphism obtained from $\varphi_G$, as defined in section 4 by extending the scalars.

**Theorem 6.5.** Assume that the character $\chi \in \mathcal{X}^*(G)$ is such that $V^{ss}(G,\chi) = V^s_{\chi}(G,\chi)$. Then:

(i) the quotient $Y := V^{ss}(G,\chi)/G$ is geometric;

(ii) the Chow ring

$$A_*(Y)_Q \cong A_*(S)_Q \otimes_Q (A^T)_Q^W / \varphi_G(\mathcal{E}(\lambda)); \quad \lambda \in \mathcal{F}(\chi).$$

(iii) Assume moreover that $K = \mathbb{C}$. Then

$$H^*(Y;\mathbb{Q}) \cong H^*(S)_Q \otimes_Q (H^*_T)_Q^W / \varphi_G(\mathcal{E}(\lambda)); \quad \lambda \in \mathcal{F}(\chi).$$
Definition 7.1. Let \( \mathcal{O}(G, \chi) \rightarrow Y \) be a character, and consider the categorical quotient \( \mathcal{O}(G, \chi) \rightarrow Y = \mathcal{O}(G, \chi)/G \). For a further character \( \varepsilon \in \mathcal{X}^*(G) \), we define

\[
\mathcal{L}^\varepsilon_{Y, \chi} := (q^* \mathcal{O}(G, \chi))^G \in \mathcal{X}^*(G).
\]

In other words, for each open subset \( U \subset Y, \Gamma(U, \mathcal{L}^\varepsilon) = \Gamma(q^{-1}U, \mathcal{O}(G, \chi))^G \).
It is easy to see that $\mathcal{L}_{\chi,\varepsilon}$ is torsion-free and coherent. If $\mathcal{V}^{ss}(G,\chi) = \mathcal{V}^{s}_{(0)}(G,\chi)$, $\mathcal{L}_{\chi,\varepsilon}$ has rank one, and if moreover the $G$-action on $\mathcal{V}^{ss}(G,\chi)$ is free, then $\mathcal{L}_{\chi,\varepsilon}$ is actually invertible.

We recall from section 4 that for $\chi \in \mathcal{X}^{*}(G)$, $C(\chi) \in \Delta_{G}(\mathcal{Y})$ denotes the cone which contains $\chi$ in its relative interior.

For a character $\chi \in \mathcal{X}^{*}(G)$ such that $\mathcal{V}^{ss}(G,\chi) = \mathcal{V}^{s}_{(0)}(G,\chi) \neq \emptyset$, we define

$$\mathcal{F}'(\chi) := \{ \lambda \in \mathcal{F}(\chi) \mid \text{codim}_{\mathcal{V}} G \cdot E(\lambda) = 1 \}$$

$$= \{ \lambda \in \mathcal{F}(\chi) \mid \text{codim}_{\mathcal{E}} E(\lambda) + \text{Stab}_{G}E(\lambda) = \dim G + 1 \}.$$ 

Denoting $L_{\lambda}$ the Levi component of $\text{Stab}_{G}E(\lambda)$, we observe that for $\lambda \in \mathcal{F}'(\chi)$, $\deg[\mathcal{E}(\lambda)]_{T} - \deg \Delta_{G}/\Delta L_{\lambda} = 1$ (see section 4 for the notations). Since in this case $\mathcal{E}_{G}[\mathcal{E}(\lambda)]_{T}$ is a principal ideal, formula (4.2) implies that

$$\mathcal{E}_{G}[\mathcal{E}(\lambda)]_{T} = \langle \mathcal{E}_{G}[\mathcal{E}(\lambda)]_{T} \rangle_{Q}.$$ 

For shorthand, we will denote $\varepsilon(\lambda) := \mathcal{E}_{G}[\mathcal{E}(\lambda)]_{T} / \langle \mathcal{E}(\lambda) \rangle_{Q}$.

**Proposition 7.2.** Let $\chi \in \mathcal{X}^{*}(G)$ be a character such that $\mathcal{V}^{ss}(G,\chi) = \mathcal{V}^{s}_{(0)}(G,\chi)$. Then:

(i) $\text{Pic}(Y_{\chi})_{Q} \cong \mathcal{X}^{*}(G)_{Q} / \langle \varepsilon(\lambda) ; \lambda \in \mathcal{F}'(\chi) \rangle_{Q}$;

(ii) The ample cone of $Y_{\chi}$ is

$$\text{Pic}(Y_{\chi})_{\text{ample}} \cong \text{int.} \mathcal{C}(\chi)_{Q},$$

where $\mathcal{C}(\chi)_{Q} := \text{Image}[\mathcal{X}^{*}(G)_{Q} \cap C(\chi) \to \mathcal{X}^{*}(G)_{Q} / \langle \varepsilon(\lambda) ; \lambda \in \mathcal{F}'(\chi) \rangle_{Q}]$.

**Proof.** (i) Indeed,

$$\text{Pic}(Y_{\chi})_{Q} = \mathcal{A}(Y_{\chi})_{Q} = \left( \mathcal{A}_{G}^{1} \right)_{Q}^{W} / \left( \mathcal{A}_{G}^{1} \right)_{Q}^{W} \cap \mathcal{E}_{G}[\mathcal{E}(\lambda)]_{T} ; \lambda \in \mathcal{F}(\chi) \rangle_{Q}$$

$$= \mathcal{X}^{*}(G)_{Q} / \langle \varepsilon(\lambda) ; \lambda \in \mathcal{F}'(\chi) \rangle_{Q}.$$ 

(ii) We recall that any character $\varepsilon \in \mathcal{X}^{*}(G)_{Q} \cap \text{int.} C(\chi)$ defines the same invariant quotient as $\chi$, that is $Y$, but endows it with a possibly different polarization. Namely, for $n \geq 1$ sufficiently large, the sheaf $\mathcal{L}_{\chi}^{n} \to Y$ is invertible and ample.

For $\lambda \in \mathcal{F}'(\chi)$, let $f_{\lambda} \in \mathcal{K}[\mathcal{V}]$ be the equation of $G \cdot \mathcal{E}(\lambda) \hookrightarrow \mathcal{V}$; $G$ acts on $Q f_{\lambda}$ by a character, which is just the class $G \cdot \mathcal{E}(\lambda)_{G} \in A_{G}^{1}$, that is $\varepsilon(\lambda)$. We conclude that $f_{\lambda} \in \mathcal{K}[\mathcal{V}]_{\varepsilon(\lambda)}^{G}$. We deduce further that for $\varepsilon' = \sum_{\lambda \in \mathcal{F}(\chi)} k_{\lambda} \varepsilon(\lambda)$, with $k_{\lambda} \geq 0$,

$$\prod_{\lambda \in \mathcal{F}(\chi)} f_{\lambda}^{k_{\lambda}} : \mathcal{L}_{\varepsilon} \to \left( q_{\ast} \mathcal{O}_{\mathcal{V}}^{G}(G,\chi) \right)^{G,\varepsilon} = \left( q_{\ast} \mathcal{O}_{\mathcal{V}}^{G}(G,\chi) \right)^{G,\varepsilon'} = \mathcal{L}_{\varepsilon'}$$

is an isomorphism. The two remarks together prove the inclusion of the right hand side into the left hand side.

For the converse inclusion, let us consider an ample line bundle $\mathcal{L} \to Y$. It follows from the first part that $\mathcal{L} \cong \mathcal{L}_{\varepsilon}$ for some $\varepsilon \in \mathcal{X}^{*}(G)$. According to [4, 5, 4.5.2], there is some $n \geq 1$ such that the non-vanishing loci of the sections in $\mathcal{L}^{\otimes n}$ cover $Y$ with affine open sets. This means that for any $x \in \mathcal{V}^{ss}(G,\chi)$, there is $f \in \mathcal{K}[\mathcal{V}^{ss}(G,\chi)]_{\varepsilon}^{G}$ such that $f(x) \neq 0$. We choose finitely many such $f$’s such that the non-vanishing loci cover $\mathcal{V}^{ss}(G,\chi)$. The components of codimension one of $\mathcal{V} \setminus \mathcal{V}^{ss}(G,\chi)$ are the $G \cdot \mathcal{E}(\lambda)$’s, with $\lambda \in \mathcal{F}'(\chi)$. Therefore we find a monomial $\phi := \prod_{\lambda \in \mathcal{F}(\chi)} f_{\lambda}^{k_{\lambda}}$ with the property that the $\phi f$’s, with $f$
as above, extend to regular functions on $V$; they all belong to $K[V]^G_{\epsilon^n \cdot \prod \lambda \epsilon^{k\lambda}}$, and their non-vanishing locus contain $V^{ss}(G, \chi)$. This means that $V^{ss}(G, \chi) \subset V^{ss}(G, \epsilon^n \cdot \prod \lambda \epsilon^{k\lambda})$, and therefore $\epsilon^n \cdot \prod \lambda \epsilon^{k\lambda} \in C(\chi)$.

All together, proves that

$$\text{int.} \tilde{C}(\chi)_Q \subset \text{Pic}(Y_{\chi})^{\text{ample}} \subset \tilde{C}(\chi)_Q \subset \text{Pic}(Y_{\chi})_Q.$$ 

The conclusion that $\text{Pic}(Y_{\chi})^{\text{ample}} = \text{int.} \tilde{C}(\chi)_Q$ follows, because we know from [11, corollaire 4.5.8] that the ample cone is open in the Picard variety. 

Remark 7.3. If and codim$_V V^{ss}(G, \chi) \geq 2$, there is a short exact sequence

$$0 \to \text{Pic}(Y_{\chi}) \xrightarrow{q^*} X^*_G \to T \to 0,$$

where $T$ is by definition the cokernel of $q^*$. Indeed, the pull-back by $q$ of an invertible sheaf on $Y_{\chi}$ gives rise to a $G$-linearized invertible sheaf on $V^{ss}(G, \chi)$. The codimension condition ensures that this $G$-linearized invertible sheaf uniquely extends to a $G$-linearized sheaf on $V$, which corresponds to a character of $G$.

If moreover $V^{ss}(G, \chi) = V^{ss}_{\{0\}}(G, \chi)$, the previous proposition says that $T$ is a $Z$-torsion module, because a suitably large multiple of any character defines an invertible sheaf on $Y_{\chi}$.

Now that we have described the line bundles on our quotient varieties, we wish to compute their cohomology groups. It is well-known that, over complete toric varieties, the higher cohomology groups of nef line bundles vanish. What we are going to prove is that a similar result holds in our ‘non-abelian setting’. We must say from the very beginning that our result is a more or less straightforward consequence of the Hochster-Roberts theorem.

First of all, we remark that for a pair of characters $\chi, \epsilon \in X^*_G$, with $\epsilon \in C(\chi)$, the natural inclusion $V^{ss}(G, \epsilon) \subset V^{ss}(G, \chi)$ induces a morphism on the level of quotients

$$(7.1) \quad V^{ss}(G, \chi) \xrightarrow{q_\chi} V^{ss}(G, \epsilon) \xrightarrow{q_\epsilon} Y_{\chi} \xrightarrow{\psi} Y_{\epsilon} \xrightarrow{q_\epsilon} Y_{\chi}$$

which is projective and open. In particular, $\psi$ is surjective.

Lemma 7.4. (i) For $f \in K[V]_{\chi}^G_m$, with $m \geq 1$, the preimage under $\psi$ of the affine, open subset $D(f, Y_{\epsilon}) \subset Y_{\epsilon}$ defined by $f$ is isomorphic to $\text{Proj}(\bigoplus_{n \geq 0} (f^{-1}K[V])^{G}_{\chi^n})$.

(ii) Assume char $K = 0$. Then

$$R^i \psi_* \mathcal{O}_{Y_{\chi}} = 0, \quad \forall i > 0.$$

Proof. (i) If $D(f) \subset V$ denotes the non-vanishing locus of $f$ in $V$, then clearly

$$\psi^{-1}D(f, Y_{\epsilon}) = (D(f) \cap V^{ss}(G, \chi))/G = D(f)^{ss}(G, \chi)/G.$$ 

As $D(f) = \text{Spec}(f^{-1}K[V])$, we conclude using the same trick as in lemma 1.1.

(ii) Is enough to prove that $(R^i \psi_* \mathcal{O}_{Y_{\chi}})(D(f, Y_{\epsilon})) = H^i(\psi^{-1}D(f, Y_{\epsilon}), \mathcal{O}_{Y_{\chi}}) = 0$, for any $f \in K[V]_{\chi}^G_m$, $m \geq 1$. We have just seen that $\psi^{-1}D(f, Y_{\epsilon}) = \text{Proj} R$, with

$$R = \bigoplus_{n \geq 0} R_n := \bigoplus_{n \geq 0} (f^{-1}K[V])^{G}_{\chi^n}$$

and also $\bigoplus_{n \geq 0} (f^{-1}K[V])^{G}_{\chi^n} = K[A_K^1 \times D(f)]^G$. 

Thus, for $i > 0$ the result follows from the last lemma 1.1.
Since $G$ is linearly reductive, as char $K = 0$, and $\mathbb{A}^1_K \times D(f)$ is a regular, affine variety, we conclude that we are in the situation of [14 theorem 3.4], and therefore the positively graded part of the local cohomology groups $[H^i_{R_i}(R)]_{\geq 0}$ vanish, for all $i \geq 0$, where $R_i := \oplus_{n \geq 1} R_n$. The Grothendieck-Serre correspondence [3 theorem 20.4.4] says that

$$
[H^i_{R_i}(R)]_{\geq 0} \cong \bigoplus_{j \geq 0} H^i(\text{Proj } R, \mathcal{O}_{\text{Proj } R(j)}), \quad \forall i \geq 1,
$$

and the vanishing of the degree zero part implies $H^i(\psi^{-1}D(f, Y_{\varepsilon}), \mathcal{O}_{Y_{\chi}}) = 0$.

Lemma 7.5. Let $\chi \in \mathcal{X}^*(G)$ such that $\text{codim}_Y \mathcal{V}^{\text{us}}(G, \chi) \geq 2$, and $\varepsilon \in \mathcal{X}^*(G) \cap C(\chi)$. Then the following hold:

(i) $\psi^* \mathcal{L}_{Y_{\chi}, \varepsilon} = \mathcal{L}_{Y_{\chi}, \varepsilon}$;

(ii) if $\mathcal{L}_{Y_{\chi}, \varepsilon}$ is invertible, then the natural homomorphism

$$
\psi^* \mathcal{L}_{Y_{\chi}, \varepsilon} = \psi^* \psi^* \mathcal{L}_{Y_{\chi}, \varepsilon} \rightarrow \mathcal{L}_{Y_{\chi}, \varepsilon}
$$

is an isomorphism.

We remark that for any $\varepsilon \in \mathcal{X}^*(G)$, some positive multiple of it fulfills the requirement of (ii) above.

Proof. (i) For an open subset $U' \subset Y_{\varepsilon}$ holds

$$
\Gamma(U', \psi^* \mathcal{L}_{Y_{\chi}, \varepsilon}) = \Gamma(\psi^{-1}U', \mathcal{L}_{Y_{\chi}, \varepsilon}) = \Gamma((\psi \circ q_{\chi})^{-1}U', \mathcal{O}_{Y})^G_{\varepsilon}
$$

$$
= \Gamma(q_{\varepsilon}^{-1}U' \cap \mathcal{V}^{\text{ss}}(G, \chi), \mathcal{O}_{Y})^G_{\varepsilon} \cong \Gamma(q_{\varepsilon}^{-1}U', \mathcal{O}_{Y})^G_{\varepsilon} = \Gamma(U', \mathcal{L}_{Y_{\chi}, \varepsilon}),
$$

where $(\ast)$ holds because of the assumption that $\text{codim}_Y \mathcal{V}^{\text{us}}(G, \chi) \geq 2$.

(ii) Since $\mathcal{L}_{Y_{\chi}, \varepsilon}$ is locally free, its pull-back is the same. Using remark [33], we deduce that $\psi^* \mathcal{L}_{Y_{\chi}, \varepsilon} \cong \mathcal{L}_{Y_{\chi}, \varepsilon'}$, for some character $\varepsilon'$. Consider now an affine, open subset $U' \subset Y_{\varepsilon}$ over which $\mathcal{L}_{Y_{\chi}, \varepsilon}$ is trivialized by a section $s' \in \Gamma(q_{\varepsilon}^{-1}U', \mathcal{O}_{Y})^G_{\varepsilon'}$. Then, on one hand, we have

$$
\Gamma(\psi^{-1}U', \psi^* \mathcal{L}_{Y_{\chi}, \varepsilon}) = \Gamma(\psi^{-1}U', \mathcal{L}_{Y_{\chi}, \varepsilon'}) = \Gamma(\mathcal{V}^{\text{ss}}(G, \chi) \cap q_{\varepsilon}^{-1}U', \mathcal{O}_{Y})^G_{\varepsilon'}
$$

$$
= \Gamma(q_{\varepsilon}^{-1}U', \mathcal{O}_{Y})^G_{\varepsilon'}.
$$

For the last step we use that $\text{codim}_Y \mathcal{V}^{\text{us}}(G, \chi) \geq 2$. On the other hand,

$$
\Gamma(\psi^{-1}U', \psi^* \mathcal{L}_{Y_{\chi}, \varepsilon}) = s' \cdot \Gamma(\psi^{-1}U', \mathcal{O}_{Y})^G_{\varepsilon'} = s' \cdot \Gamma(\mathcal{V}^{\text{ss}}(G, \chi) \cap q_{\varepsilon}^{-1}U', \mathcal{O}_{Y})^G_{\varepsilon'}
$$

$$
= s' \cdot \Gamma(q_{\varepsilon}^{-1}U', \mathcal{O}_{Y})^G_{\varepsilon'},
$$

since $s'$ trivializes $\mathcal{L}_{Y_{\chi}, \varepsilon}$ over $U'$. The two equalities imply $\varepsilon' = \varepsilon$.

We fix $G \rightarrow \text{Spec } \mathbb{Z}$ a reductive group scheme, which contains a maximal torus, and whose geometric fibres are connected, reductive groups, all having the same root system; we consider a representation $G \rightarrow \text{GL}(M)$, as constructed at the beginning of the previous section, and assume that $(\text{Sym}^*_K M)^G = \mathbb{Z}$. We define $V := \text{Spec} (\text{Sym}^*_K M)$, and, for a field $K$, we let $V_K := V \times_{\text{Spec } \mathbb{Z}} \text{Spec } K = \text{Spec} (\text{Sym}^*_K M \otimes_{\mathbb{Z}} K)$. Further, we fix a character $\chi : G \rightarrow \mathbb{G}_m$ such that $\text{codim}_V \mathcal{V}^{\text{us}}(G_{\mathbb{C}}, \chi_{\mathbb{C}}) \geq 2$, and we denote $Y_{\chi,K} := V_K//_{/\chi_K} \text{Proj } K[V_K]^{G_K}$. Finally, we consider a character $\varepsilon : G \rightarrow \mathbb{G}_m$ such that $\varepsilon_{\varepsilon} \in C(\chi_{\mathbb{C}})$. Corollary [34] implies that there is an open subset $U \subset \text{Spec } \mathbb{Z}$ such that $\mathcal{V}^{\text{ss}}(G, \chi)|_U \subset \mathcal{V}^{\text{ss}}(G, \varepsilon)|_U$, and consequently we obtain a commutative diagram similar to [14], defined over $U$. 

Proposition 7.6. There is a prime number $p_0 \in \mathbb{N}$ such that for any algebraically closed field $K$ of characteristic zero or $\text{char } K > p_0$ holds:

(i) $Y_{\chi,K}$ is arithmetically Cohen-Macaulay;

(ii) $H^i(Y_{\chi,K},\psi^*_K\mathcal{L}_{Y_{\chi,K}}) = 0$ for all $n \in \mathbb{Z}$, $\forall i > 0$, $i \neq \dim Y_{\varepsilon,K}$;

$H^{\dim Y_{\chi,K}}(Y_{\chi,K},\psi^*_K\mathcal{L}_{Y_{\chi,K}}) = 0$ for $\forall n \geq 0$.

We recall from [15 section 14] that a projective scheme is called arithmetically Cohen-Macaulay if it can be written as the ‘Proj’ of a graded, Cohen-Macaulay ring.

**Proof.** For the beginning, we prove the result in characteristic zero, and we will conclude using a semi-continuity argument.

So, let us fix an algebraically closed field $K$ of characteristic zero. Since in characteristic zero reductive groups are linearly reductive, the Hochster-Roberts theorem [17 theorem 0.1] implies that

$$R^{G_K} = (R^{G_K})_0 \oplus (R^{G_K})_+ := K \oplus \bigoplus_{n > 0} K[\mathbb{V}_K]_{\chi_K}^G = K[\mathbb{A}^1_K \times \mathbb{V}_K]^G_K$$

is Cohen-Macaulay, which proves the first statement.

The proof of the second statement is done in two steps. We start proving it in the case when $\varepsilon \in \text{int. } C(\chi)$, that is $C(\varepsilon_{\mathbb{C}}) = C(\chi_{\mathbb{C}})$. Corollary 6.4 implies that the equality is valid still valid for $\varepsilon_K$ and $\chi_K$, so that the quotients $Y_{\varepsilon,K}$ and $Y_{\chi,K}$ coincide. Consequently we may assume that $\varepsilon = \chi$. The Grothendieck-Serre correspondence [3 theorem 20.4.4] gives the graded isomorphisms

$$\bigoplus_{n \in \mathbb{Z}} H^i(Y_{\chi,K},\mathcal{O}_{Y_{\chi,K}}(n)) \cong H^{i+1}_{(R^{G_K})_+}(R^{G_K}), \quad \forall i > 0,$$

between the Čech and the local cohomology modules. On the other hand, since $\dim Y_{\chi,K} = \dim R^{G_K} - 1$, we deduce from [17 theorem 4.6] that

$$H^i_{(R^{G_K})_+}(R^{G_K}) = 0,$$

for $0 < i < \dim Y_{\chi,K}$, and

$$H^{\dim Y_{\chi,K}+1}_{(R^{G_K})_+}(R^{G_K})_{\geq 0} = 0,$$

and our claim follows because $\mathcal{O}_{Y_{\chi,K}}(n) \cong \mathcal{L}_{Y_{\chi,K}}^n$.

Now we consider the general case, when $\varepsilon \in C(\chi_{\mathbb{C}})$; then, corollary 6.4 implies that the same holds over $K$. We know from lemma 7.4 that $R^i\psi_{K,*}\mathcal{O}_{Y_{\chi,K}} = 0$, and we deduce

$$R^i\psi_{K,*}(\psi^*_K\mathcal{L}_{Y_{\chi,K}}^n) = \mathcal{L}_{\varepsilon_K}^n \otimes R^i\psi_{K,*}\mathcal{O}_{Y_{\chi,K}} = 0.$$ Therefore

$$H^i(Y_{\chi,K},\psi^*_K\mathcal{L}_{Y_{\chi,K}}^n) \cong H^i(Y_{\chi,K},\mathcal{L}_{\varepsilon_K}^n),$$

and our claim follows by applying the previous step.

So far we have been concerned with fields of characteristic zero, and now we wish to extend the results in large characteristic. Since the morphism $\text{pr} : Y_{\chi} \to \text{Spec } \mathbb{Z}$ is proper and flat, and $R^{G_{\mathbb{Z}}}$ is Cohen-Macaulay, [12 theorem 12.2.4 (i)] implies that there is a non-empty open subset $U(\chi) \subset \text{Spec } \mathbb{Z}$ such that the homogeneous rings of the fibres over $U(\chi)$ are still Cohen-Macaulay; by flat base change [23 lemma 2], $R^{G_K}$ is Cohen-Macaulay for all fields $K$ with $\langle \text{char } K \rangle \in U(\chi)$. This statement appears also in [15 remark 1.3].

Before starting the proof of our second claim, let us remark that the difficulty relies in showing the simultaneous vanishing of the cohomology groups of the $\mathcal{L}_{\varepsilon_K}$’s, for variable $n$. 

For fixed $n$, our claim is a direct application of the upper semi-continuity theorem [13 theorem 12.8].

The proof of our second claim is divided again in two steps: if $\varepsilon_C \in \text{int.} C(\chi_C)$, we may assume as before that $\varepsilon = \chi$. Our previous discussion implies the vanishing

$$H^i_{(R^G\mathcal{K})_+} (R^G\mathcal{K}) = 0, \text{ for } 0 < i \leq \dim Y_{\chi,K},$$

which, via the Grothendieck-Serre correspondence, translates into

$$(7.2) \quad H^i (Y_{\chi,K}, \mathcal{L}_n) = 0, \quad \forall 0 < i < \dim Y_{\chi,K}, \forall n \in \mathbb{Z}.$$  

It remains to prove the vanishing of the highest cohomology groups. Let us choose $c > 0$ large enough such that $Z[V]^G_{\chi_0} = (Z[V]^G_{\chi})^n$, and consider the corresponding projective embedding

$$\begin{align*}
\mathbb{Y}_\chi & \hookrightarrow \mathbb{P}^N_Z \\
\text{pr} & \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad

consider the diagram

\[
\begin{array}{ccc}
\psi_C^{-1}D(f_C, Y_{\varepsilon,C}) & \rightarrow & \psi_Q^{-1}D(f_Q, Y_{\varepsilon,Q}) \\
\Spec \mathbb{C} & \rightarrow & \Spec \mathbb{Q} \rightarrow U
\end{array}
\]

We have proved in lemma \[\text{[14]}\] that \(H^i(\psi_C^{-1}D(f_C, Y_{\varepsilon,C}), \mathcal{O}_{Y_{\varepsilon,C}}) = 0\), for all \(i > 0\). Applying \[\text{[13]}\] proposition 9.3] to the faithfully flat base change \(\Spec \mathbb{C} \rightarrow \Spec \mathbb{Q}\), we deduce that \(H^i(\psi_Q^{-1}D(f_Q, Y_{\varepsilon,Q}), \mathcal{O}_{Y_{\varepsilon,Q}}) = 0\), for all \(i > 0\). Now we remark that \(\mathcal{O}_{Y_{\varepsilon}}\) is flat over \(\Spec \mathbb{Q}\) (cf. remark \[\text{[8.3]}\]), so that the upper semi-continuity theorem \[\text{[13, theorem 12.8]}\] applies: it implies that there is an open subset \(U(f) \subset U \subset \Spec \mathbb{Z}\) such that \(H^i(\psi_K^{-1}D(f_K, Y_{\varepsilon,K}), \mathcal{O}_{Y_{\varepsilon,K}}) = 0\), for all \(i > 0\), as soon as \((\text{char } K) \in U(f)\). We let \(U\) to be the finite intersection of the \(U(f)\)'s obtained this way. Applying \[\text{[13]}\] proposition 9.3 and proposition 8.5], we deduce the vanishing of \(R^i\psi_K^*\mathcal{O}_{Y_{\varepsilon,K}}|_{D(f_K, Y_{\varepsilon,K})}\), for all \(i > 0\) and \(f \in S\), as soon as \((\text{char } K) \in U\).

Since \(\{D(f_K, Y_{\varepsilon,K})\}_{f \in S}\) defines an open, affine covering of \(Y_{\varepsilon,K}\), we deduce that all the higher direct images \(R^i\psi_K^*\mathcal{O}_{Y_{\varepsilon,K}}, i > 0\), vanish. \(\square\)

**Remark 7.7.** This proposition gives very satisfactory results about the vanishing of the higher cohomology groups of ample line bundles on our geometric quotients, but gives only partial answer in the case of nef line bundles. More precisely, as an immediate consequence, we deduce that for \(\varepsilon \in \partial C(\chi) \cap \mathcal{X}^*(G)\) there is a constant \(c > 0\) (depending on \(\varepsilon\)) such that

\[
H^i(Y_{\varepsilon}, \mathcal{L}_{Y_{\varepsilon},\mathcal{O}}^{\otimes n}) = 0, \quad \forall n \geq 0, \quad \text{and} \quad H^i(Y_{\varepsilon}, \mathcal{L}_{Y_{\varepsilon},\mathcal{O}}^{\otimes \varepsilon - n}) = 0, \quad \forall n \geq 0, \quad i \neq \dim Y_{\varepsilon}.
\]

Indeed, if we choose \(c > 0\) is such that \(\mathbb{Q}[Y_{\varepsilon}]^G_{Q_{\varepsilon}}\) is generated in degree one, then \(\mathcal{L}_{Y_{\varepsilon},\mathcal{O}}\) is locally free, so that \(\psi_Q^*\mathcal{L}_{Y_{\varepsilon},\mathcal{O}} = \mathcal{L}_{Y_{\varepsilon},\mathcal{O}}\) by lemma \[\text{[13]}\].

The trouble is that this statement is unnatural, since we would like to have the cohomology vanishing for all powers, not only for the multiples of some number. This is what we are going to prove next.

So, let us assume that the character \(\varepsilon\) has the property that \(\mathcal{L}_{Y_{\varepsilon},\mathcal{O}} \rightarrow Y_{\varepsilon,C}\) is invertible, and \(\varepsilon \in \partial C(\chi_{\varepsilon}) \cap \mathcal{X}^*(G_{\varepsilon})\). Then the same holds (by faithfully flat base change) for \(\mathcal{L}_{Y_{\varepsilon},\mathcal{O}} \rightarrow Y_{\varepsilon,Q}\), which implies that \(\mathcal{L}_{Y_{\varepsilon},\mathcal{O}}|_{U'(\varepsilon)} \rightarrow Y_{\varepsilon}|_{U'(\varepsilon)}\) is still invertible over some non-empty, open subset \(U'(\varepsilon) \subset \Spec \mathbb{Z}\).

We define the representation

\[
\tilde{\rho} : \tilde{G} := G \times_{\Spec \mathbb{Z}} \mathbb{G}_m \hookrightarrow \mathfrak{g}(M \oplus \mathbb{Z}^2), \quad \tilde{\rho}(g, t) := \text{diag}(\rho(g), t^{-1}(g), t),
\]

and notice that since \((\text{Sym}_Z^* M)^G = \mathbb{Z}, (\text{Sym}_Z^* (M \oplus \mathbb{Z}^2))^G = \mathbb{Z}\) too.

After replacing \(\chi\) with a suitably large positive multiple, we may and we assume that \((\text{Sym}_Z^* M)^G, \chi\) is generated in degree one. Let us define now the character

\[
\tilde{\chi} : \tilde{G} \rightarrow \mathbb{G}_m, \quad \tilde{\chi}(g, t) := \chi(g)t,
\]

and observe that

\[
(\text{Sym}_Z^* (M \oplus \mathbb{Z}^2))^G_{\tilde{\chi}} = (\text{Sym}_Z^* M[w_1, w_2])^G_{\tilde{\chi}} = \bigoplus_{n \geq 0} (\text{Sym}_Z^* M[w_1, w_2])^G_{\tilde{\chi}^n}
\]

\[
= \bigoplus_{n \geq 0} \bigoplus_{a+b=n} \text{Sym}_Z^* M)^G_{\chi^{n}\varepsilon} \varepsilon^a w_1^b w_2^b.
\]

\[\text{(7.4)}\]
We have proved in corollary 6.4 that there is an open subset $U''(\varepsilon) \subset \text{Spec} \mathbb{Z}$ such that $\varepsilon_K \in C(\chi_K)$ for any algebraically closed field $K$ having $\langle \text{char} K \rangle \in U''(\varepsilon)$. We define $U(\varepsilon) := U'(\varepsilon) \cap U''(\varepsilon)$; it is a non-empty, open subset of $\text{Spec} \mathbb{Z}$. More precisely, $U(\varepsilon) = \text{Spec} B$, where $B$ is obtained out of $\mathbb{Z}$ by inverting finitely many primes.

Lemma 7.8. Let $\varepsilon : G \to G_m$ be a character such that $\varepsilon_C \in C(\chi_C) \cap \mathcal{X}^n(G_C)$, and moreover $\mathcal{L}_{Y,\varepsilon_C} \to Y_{\chi,C}$ is invertible. Then there is a non-empty, open subset $U(\varepsilon) \subset \text{Spec} \mathbb{Z}$ over which the following isomorphism holds

$$\mathbb{P}(\mathcal{O}_{Y,\varepsilon} \oplus \mathcal{L}_{Y,\varepsilon})|_{U(\varepsilon)} \cong \text{Proj}(\text{Sym}^B \mathcal{O}_{Y,\varepsilon} \oplus \mathcal{L}_{Y,\varepsilon}) \cong \text{Proj}(\mathbb{Z}[V \times \mathbb{A}_\mathbb{Z}^2]^G, \chi)|_{U(\varepsilon)}.$$

Proof. We prove the statement for $U(\varepsilon) = \text{Spec} B$ defined above. For shorthand, we write $\mathcal{O} := \mathcal{O}_{Y,\varepsilon}$, $\mathcal{L} := \mathcal{L}_{Y,\varepsilon}$, and $B[V] := \text{Sym}^B(M \otimes \mathbb{Z} B)$. Since we have assumed that $\mathbb{Z}[V]^G, \chi$ is generated in degree one, there is a finite subset $\{f_j\}_{j \in J} \subset \mathbb{Z}[V]^G$ such that the non-vanishing loci $\{D(f_j, Y_{\chi})\}_{j \in J}$ form an open, affine covering of $Y_{\chi}$. We notice that $D(f_j, Y_{\chi}|_{U(\varepsilon)}) = \text{Spec}((f_j^{-1}B[V])^G)$. By our previous discussion, $\mathcal{L}|_{U(\varepsilon)} \to Y_{\chi}|_{U(\varepsilon)}$ is invertible, so that we find trivializing sections $s_j \in (f_j^{-1}B[V])^G, j \in J$, such that

$$\mathcal{L}|_{D(f_j, Y_{\chi}|_{U(\varepsilon)})} = (s_j \cdot (f_j^{-1}B[V])^G)^{-1}.$$

Choosing $f$ to be one of the $f_j$'s and $s$ the corresponding $s_j$, we find that

$$\mathbb{P}(\mathcal{O}_{Y,\varepsilon} \oplus \mathcal{L}_{Y,\varepsilon})|_{D(f_j, Y_{\chi}|_{U(\varepsilon)})} = \text{Proj}\left(\bigoplus_{n \geq 0} (f_j^{-1}B[V])^G \cdot s^a \cdot 1^b \right).$$

(7.5)

On the other hand, we know that $\varepsilon_K \in C(\chi_K)$ for all algebraically closed field with $\langle \text{char} K \rangle \in U(\varepsilon)$. It follows that for all $n \geq 1$ and $0 \leq a \leq n, \chi_K^{-1}\varepsilon_K \in \text{int.} C(\chi_K)$. We deduce now from 6.4 that the $\chi$-semi-stable locus of $(V \times \mathbb{A}_\mathbb{Z}^2)|_{U(\varepsilon)}$ is

$$(V \times \mathbb{A}_\mathbb{Z}^2)|_{U(\varepsilon)}\ss (G, \chi) \cong (V|_{U(\varepsilon)})\ss (G, \chi) \times \mathbb{Z} (\mathbb{A}_\mathbb{Z}^2 \setminus \{0\}).$$

(7.6)

In particular, the $\chi$-unstable locus has codimension two; since both $\mathcal{L}_{Y,\chi}|_{U(\varepsilon)}$ and $\mathcal{L}_{Y,\varepsilon}|_{U(\varepsilon)}$ are invertible, remark 7.3 implies that

$$\mathcal{L}_{Y,\chi}|_{U(\varepsilon)} \otimes \mathcal{L}_{Y,\varepsilon}|_{U(\varepsilon)} \cong \mathcal{L}_{Y,\chi^{-1}\varepsilon}|_{U(\varepsilon)}, \quad \forall n, a \in \mathbb{Z}.$$

The categorical quotient of this scheme for the $G$-action is obtained by glueing the categorical quotients of the $G$-invariant open subschemes $D(f_j) \times \mathbb{A}_\mathbb{Z}^2 \setminus \{0\})|_{U(\varepsilon)}$. Again, taking $f$ to be one of the $f_j$'s, we observe that

$$\text{Proj}\left(\bigoplus_{n \geq 0} (f^{-1}B[V])^G \cdot w^a \cdot 1^b \right) \cong \text{Proj}\left(\bigoplus_{n \geq 0} (f^{-1}B[V])^G \cdot w^a \cdot 1^b \right).$$

(7.7)

The equalities (7.5) and (7.6) show that the varieties we are considering are locally isomorphic, and one can also check that these local isomorphisms are compatible with coordinate changes. \hfill \square

Now we are in position to prove our main cohomology vanishing result.
We claim that \( Y \) is still relatively nef. Indeed, let us consider a relatively ample, invertible sheaf \( Y \). Then for any integer \( p \)

\[
\text{Proof. Let } U(\varepsilon) \subset \text{Spec } \mathbb{Z} \text{ be as in the previous lemma, and write } L := Y_\varepsilon. \text{ Then } L|_{U(\varepsilon)} \rightarrow Y_\varepsilon|_{U(\varepsilon)} \text{ is relatively nef, in the sense that is nef on the geometric fibres of } Y_\varepsilon|_{U(\varepsilon)} \rightarrow U(\varepsilon). \text{ We claim that }
\]

\[
\mathcal{O}_{\mathbb{P}(\mathcal{O} \oplus L)}(1)|_{U(\varepsilon)} \rightarrow \mathbb{P}(\mathcal{O} \oplus L)|_{U(\varepsilon)}
\]

is still relatively nef. Indeed, let us consider a relatively ample, invertible sheaf \( \mathcal{A} \rightarrow Y_\varepsilon \). Then for any integer \( k > 0 \) holds

\[
\mathcal{A} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{O} \oplus L)}(k) = j_k^* \tilde{\mathcal{A}}_k \quad \mathcal{O}_{\mathbb{P}(\mathcal{O} \oplus L)}(1) \quad \tilde{\mathcal{A}}_k := \mathcal{O}_{\mathbb{P}(\mathcal{O} \oplus \mathcal{O} \oplus \cdots \oplus \mathcal{O})}(1)
\]

According to [14, proposition 2.2 and 3.2], \( \tilde{\mathcal{A}}_k \) is relatively ample. As \( k > 0 \) is arbitrary, we deduce that \( \mathcal{O}_{\mathbb{P}(\mathcal{O} \oplus L)}(1) \) is in the closure of the relatively ample cone, and therefore is relatively nef.

Proposition [7,6] together with lemma [18] imply that there is a constant \( c > 0 \) having the property that, after possibly shrinking \( U(\varepsilon) \) further, the following vanishing holds

\[
H^i(\mathbb{P}(\mathcal{O} \oplus L)_{K}, \mathcal{O}(1)^{nc}_{K}) = 0 \quad \forall n \geq 0,
\]

on all geometric fibres over \( U(\varepsilon) \). Denoting \( \pi : \mathbb{P}(\mathcal{O} \oplus L) \rightarrow Y_\varepsilon \) the natural projection, we have

\[
R^i(\pi_K)_* \mathcal{O}(1)^{nc}_{K} = 0, \quad \forall i > 0 \forall n \geq 0,
\]

and therefore

\[
0 = H^i(\mathbb{P}(\mathcal{O} \oplus L)_{K}, \mathcal{O}(1)^{nc}_{K}) = H^i(Y_{\pi_K}, (\pi_K)_* \mathcal{O}(1)^{nc}_{K}) = \bigoplus_{a=0}^{nc} H^i(Y_{\pi_K}, \mathcal{L}_a^{nc}).
\]

Taking \( n > 0 \) arbitrarily large, we deduce the vanishing of the positive powers of nef bundles.

For the negative powers we proceed as follows: the relative canonical sheaf of \( \pi \) is \( \omega_{rel} \cong \pi^* \mathcal{L} \otimes \mathcal{O}(-2) \), so that the relative duality implies

\[
R^i \pi_* \mathcal{O}(-1)^{nc} \cong \pi_* (\pi^* \mathcal{L} \otimes \mathcal{O}(nc - 2))^\vee = \bigoplus_{a=1}^{nc-1} \mathcal{L}^{-a}.
\]
Since \( (\pi_K)_*\mathcal{O}(-1)^{nc} = 0 \) for \( n > 0 \), the Leray spectral sequence implies that
\[
H^{i+1}(\mathbb{P}(\mathcal{O} \oplus \mathcal{L})_K, \mathcal{O}(-1)^{nc}_K) = H^i(Y_{\chi,K}, R^1(\pi_K)_*\mathcal{O}(-1)^{nc}_K)
\]
\[
= \bigoplus_{a=1}^{nc-1} H^i(Y_{\chi,K}, \mathcal{L}^{-a}).
\]

The left-hand-side vanishes for \( i \neq \dim Y_{\varepsilon,K} \). Taking again \( n > 0 \) arbitrarily large, we deduce the cohomology vanishing for the negative powers. □

Remark 7.10. It is actually possible ‘to squeeze out’ some more information, namely that the prime \( p_0 \) appearing in the previous theorem can be chosen independent of \( \varepsilon \). In other words, one is able to find a prime \( p(\chi) \), depending only on the quotient \( Y := Y_{\chi} \), for which one has vanishing of the higher cohomology groups, for all invertible, nef sheaves, in characteristic larger than \( p(\chi) \).

This can be seen as follows: consider a finite set of characters \( \{\varepsilon_j\}_{j \in J} \) such that \( \{\mathcal{L}_{\varepsilon_j}\}_{j \in J} \) generate the nef cone \( C(\chi_{\mathbb{Q}}) \) of \( Y_{\chi,\mathbb{Q}} \) over \( \mathbb{Z}_{\geq 0} \); then the same holds over a non-empty, open subset of \( \text{Spec } \mathbb{Z} \). Similarly as in lemma 7.8, one proves that
\[
\mathbb{P}(\mathcal{O} \oplus \bigoplus_{j \in J} \mathcal{L}_j) \cong [V \times_{\mathbb{Z}} A^2_{\mathbb{Z}} \times_{\mathbb{Z}} \cdots \times_{\mathbb{Z}} A^2_{\mathbb{Z}}] \big// G \times_{\mathbb{Z}} G_{\text{m}}^{[J]} \big// |J| \text{ times}
\]
for a suitable action and linearization. Applying now proposition 7.6 to large multiples of \( \mathcal{O}_{\mathbb{P}(\mathcal{O} \oplus \bigoplus_{j \in J} \mathcal{L}_j)(1)} \), just as in 7.9, one obtains the desired result.

We have focused on invariant quotients of affine spaces for actions of reductive groups. However, we observe that the Grothendieck-Serre correspondence, together with [17, theorem 0.1 and theorem 3.4] immediately imply the

Theorem 7.11. Let \( X \) be a smooth, affine variety, defined over an algebraically closed field \( K \) of characteristic zero, which is acted on by a reductive group \( G \). We assume that the \( K[X]^G = K \), and consider a character \( \chi \in X^*(G) \) such that \( X^{ss}(G,\chi) = X^{s}(G,\chi) \) and \( \text{codim } X^{us}(G,\chi) \geq 2 \). Then:
(i) \( X//_{\chi} G \) is arithmetically Cohen-Macaulay;
(ii) For an invertible and nef sheaf \( \mathcal{L} \to X \),
\[
H^i(X//_{\chi} G, \mathcal{L}) = 0 \quad \forall i > 0;
\]
\[
H^i(X//_{\chi} G, \mathcal{L}^{-1}) = 0 \quad \forall i \neq \kappa(\mathcal{L}),
\]
where \( \kappa(\mathcal{L}) \) denotes the Kodaira-Iitaka dimension of \( \mathcal{L} \).

Proof. The first statement is just the usual Hochster-Roberts theorem. For our second claim, we start by noticing that the very same argument as in remark 7.3 shows that the pull-back by the quotient map induces the monomorphism
\[
0 \longrightarrow \text{Pic}(X//_{\chi} G) \longrightarrow X^*(G).
\]
Therefore \( \mathcal{L} = \mathcal{L}_0 \) for some \( \varepsilon \in X^*(G) \) (the latter is defined similarly as in 7.1). The GIT-cone theorem 3.3 implies that \( \varepsilon \in C(\chi) \), so that a positive multiple of \( \mathcal{L} \) is globally generated.

If \( \varepsilon \in \text{int. } C(\chi) \), we may assume \( \varepsilon = \chi \), and the vanishing follows directly from the fact that the invariant ring \( K[X]^G_{\chi} \) is Cohen-Macaulay. Otherwise we consider the quotient map
ψ : X//G → X//εG; there is c > 0 and an ample line bundle L′ → X//εG such that ψ∗ L′ ∼= Lc. Using an appropriate version of lemma 4 and the previous step, we deduce that the cohomology vanishings hold for arbitrary multiples (positive and negative) of Lc.

In order to conclude, we use the same trick as before: namely, P(θ ⊕ L) is the quotient of X × K× for a suitable G × Gm-action, and we may apply the previous step to the nef line bundle θP(θ⊙L)(1).

□

References
[1] N. Bourbaki: Groupes et algèbres de Lie, Chap. V, Hermann Paris 1968
[2] M. Brion: Cohomologie équivariante des points semi-stables, J. reine angew. Math. 421 (1991), 125-140
[3] M. Brodmann, R. Sharp: Local cohomology, Cambridge University Press 1998
[4] C. Chevalley: Certains schémas de groupes semisimples, Séminaire Bourbaki Vol. 6 1960/61, Exposé no. 219, 219–234
[5] M. Demazure: Invariants symétriques entiers des groupes de Weyl et torsion, Invent. Math. 21 (1973), 287-301
[6] I. Dolgachev, Yi Hu: Variation of geometric invariant theory quotients, IHES Publ. Math. 87 (1998), 5-56
[7] D. Edidin, W. Graham: Equivariant intersection theory, Invent. Math. 131 (1998), 595–634
[8] D. Edidin, W. Graham: Localization in equivariant intersection theory and the Bott residue formula, Amer. J. Math. 120 (1998), 619-636
[9] G. Ellingsrud, S.A. Strømme: On the Chow ring of a geometric quotient, Ann. Math. 130 (1989), 159-187
[10] W. Fulton: Intersection Theory, Springer-Verlag Berlin Heidelberg 1984
[11] A. Grothendieck: Eléments de géométrie algébrique, Chap. II, IHES Publ. Math. 8 (1961)
[12] A. Grothendieck: Eléments de géométrie algébrique, Chap. IV, IHES Publ. Math. 28 (1966)
[13] R. Hartshorne: Algebraic geometry, Springer-Verlag New-York 1977
[14] R. Hartshorne: Ample vector bundles, IHES Publ. Math. 29 (1966), 63-94
[15] M. Hochster, J. Roberts: Rings of invariants of reductive groups acting on regular rings are Cohen-Macaulay, Adv. Math. 13 (1974), 115-175
[16] G. Kempf: Instability in invariant theory. Ann. Math. 108 (1978), 299-316
[17] G. Kempf: The Hochster-Roberts theorem of invariant theory, Michigan Math. J. 26 (1979), 19-32
[18] A. King: Moduli of representations of finite dimensional algebras, Quart. J. Math. Oxford 45 (1994), 515-530
[19] D. Mumford: Lectures on curves on an algebraic surface, Princeton University Press 1966
[20] D. Mumford, J. Fogarty, F. Kirwan: Geometric invariant theory, 3rd edition, Springer-Verlag Berlin New York 1994
[21] N. Ressayre: The GIT-equivalence for G-line bundles, Geometriae Dedicata 81 (2000), 295-324
[22] R. Richardson: Affine coset spaces of reductive algebraic groups, Bull. London Math. Soc. 9 (1977), 38-41
[23] C. S. Seshadri: Geometric reductivity over arbitrary base, Adv. Math. 26 (1977), 225-274
[24] SGA 6: Théorie des intersections et théorème de Riemann-Roch, P. Berthelot, A. Grothendieck, L. Illusie (ed.), Lecture Notes in Mathematics 225, Springer-Verlag Berlin Heidelberg New York 1971

Institut für Mathematik, Universität Zürich,
Winterthurerstrasse 190, CH-8057 Zürich, Switzerland