The Space of Weak Connections in High Dimensions

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Abstract

The space of Sobolev connections, as it has been introduced for studying the variation of Yang-Mills Lagrangian in the critical dimension 4, happens not to be weakly sequentially complete in dimension larger than 4. This is a major obstruction for studying the variations of this important Lagrangian in high dimensions. The present paper generalizes the result [17] valid in 5 dimensions to arbitrary dimension and introduces a space of so called "weak connections" for which we prove the weak sequential closure under Yang-Mills energy control. We also establish a strong approximation property of any weak connection by smooth connections away from codimension 5 polyhedral sets. This last property is used in a subsequent work in preparation [18] for establishing the partial regularity property for general stationary Yang-Mills weak connections.

1 Introduction

Motivated by geometric applications of first importance, the analysis of Yang-Mills energy up to the conformal dimension 4 (the dimension at which the Lagrangian is invariant under dilations) has known a fast and spectacular development in the late 70’s early 80’s. To that purpose, the space of Sobolev Connections of a given smooth bundle have been introduced and studied (see [23], [24], [7]). This space enjoys a sequential almost weak closure property under Yang-Mills Energy control assumptions. “Almost” in the sense that, from a sequence of uniformly bounded Yang-Mills energy Sobolev connections of a given bundle, one can extract a subsequence such that it converges weakly modulo gauge transformations away from finitely many points to a limiting Sobolev connection, but on a possibly different smooth bundle.

In [17] (see also [20]) the two authors reformulated this classical “sequential almost weak closure property” into an exact “sequential weak closure property" in the following way.

Let $G$ be a compact Lie group and $(M^n, h)$ a compact riemannian manifold. Introduce the space of so called Sobolev connections defined by

$$
\mathfrak{A}_G(M^n) := \left\{ A \in L^2(\wedge^1 M^n, g) : \int_{M^n} |dA + A \wedge A|^2_h \, d\text{vol}_h < +\infty \right\}
$$

then we have proved the following result.

**Theorem 1.1** (Compactness in dimensions $\leq 4$ [20]). For $n \leq 4$ the space $\mathfrak{A}_G(M^n)$ is weakly sequentially closed below any given Yang-Mills energy level; precisely For any $A^k \in \mathfrak{A}_G(M^n)$ satisfying

$$
\limsup_{k \to +\infty} YM(A^k) = \int_{M^n} |dA^k + A^k \wedge A^k|^2_h \, d\text{vol}_h < +\infty
$$

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there exists a subsequence \( A^k \) and a Sobolev connection \( A^\infty \in \mathfrak{A}_G(M^n) \) such that

\[
d^2_{\text{conn}}(A^k, A^\infty) := \inf_{g \in W^{1,2}(M^n, G)} \int_{M^n} |A^k - (A^\infty)^g|_h^2 \, dvol_h \to 0
\]

moreover

\[
YM(A^\infty) \leq \liminf_{k \to 0} YM(A^k)
\]

Remark 1.2. Observe that the space \( \mathfrak{A}_G(M^n) \) contains for instance global \( L^2 \) one forms taking values into the Lie algebra \( \mathfrak{g} \) that correspond to smooth connections of some Yang-Mills energy of a sequence of such smooth connections is uniformly bounded, we can extract a subsequence converging weakly to a Sobolev connection and corresponding possibly to another \( G \)-bundle. This possibility of "jumping" from one bundle to another is encoded in the definition of \( \mathfrak{A}_G(M^n) \).

Because of this weak closure property the space \( \mathfrak{A}_G(M^n) \) is the ad-hoc space for studying the variations of Yang-Mills energy in dimension less or equal than 4. This is however not the case in higher dimension. We have the following proposition.

Proposition 1.3 (Sobolev connections in dimension \( > 4 \) [20]). For \( n > 4 \) the space \( \mathfrak{A}_{SU(2)}(M^n) \) is not weakly sequentially closed below any given Yang-Mills energy level: namely there exist \( A^k \in \mathfrak{A}_{SU(2)}(M^n) \) satisfying

\[
\limsup_{k \to +\infty} YM(A^k) = \int_{M^n} |dA^k + A^k \wedge A^k|^2 \, dvol_h < +\infty
\]

and a Sobolev connection \( A^\infty \in L^2 \) such that

\[
d^2_{\text{conn}}(A^k, A^\infty) := \inf_{g \in W^{1,2}(M^n, SU(2))} \int_{M^n} |A^k - (A^\infty)^g|_h^2 \, dvol_h \to 0
\]

but such that in every neighborhood \( U \) of every point of \( M^n \) there is no \( g \) such that \( (A^\infty)^g \in W^{1,2}(U) \).

In the search of the suitable formulation for variational problems involving Yang-Mills Lagrangians in higher dimensions, [22] and [21] introduced the class of so-called admissible connections, which are connections that are smooth outside a rectifiable set of codimension 4. We note here that even assuming that our connections locally coincide with a Sobolev connection outside a rectifiable set of codimension 4 still does not allow to recover the weak closure under Yang-Mills energy control denied by Proposition 1.3.

The main purpose of the present work is to propose a space of so called weak connections, which extends the space of Sobolev connections \( \mathfrak{A}_G(M^n) \) in the case \( n > 4 \) and enjoying a sequential weak closure property under the control of Yang-Mills energy. This space also includes the class of admissible connections from [22, 21]. In line with [17] where the case \( n = 5 \) is presented, we introduce the following definition.

Definition 1.4 (Weak connections). Let \( G \) be a compact Lie group and \( (M^n, h) \) a compact riemannian manifold. For \( n \leq 4 \) the space of weak connections \( \mathcal{A}_G(M^n) \) is defined to coincide with the space of Sobolev connections defined by

\[
\mathcal{A}_G(M^4) = \mathfrak{A}_G(M^4) := \left\{ A \in L^2(\wedge^1 M^n, \mathfrak{g}) : \int_{M^n} |dA + A \wedge A|^2 \, dvol_h < +\infty \right\}
\]

locally \( \exists \, g \in W^{1,2} \) s.t. \( A^g \in W^{1,2} \)

For \( n > 4 \) we define the space of weak connections \( \mathcal{A}_G(M^n) \) to be

\[
\mathcal{A}_G(M^n) := \left\{ A \in L^2(M^n, \wedge^1 TM \otimes \mathfrak{g}) : \int_{M^n} |dA + A \wedge A|^2 \, dvol_h < +\infty \right\}
\]

\[
\forall f \in C^\infty(M^n, \mathbb{R}^{n-4}), \quad \text{a.e. } y \in \text{Reg}(f), \quad \nu_{f^{-1}(y)}^* A \in \mathcal{A}_G(f^{-1}(y))
\]
where \( \text{Reg}(f) \subset \mathbb{R}^{n-4} \) denotes the regular values of \( f \), and for a submanifold \( \Sigma \subset M^n \) we denote by \( \Sigma A \) the restriction of the 1-form \( A \) to \( \Sigma \).

We next introduce a gauge-invariant pseudo-distances \( \delta \) between weak connection forms \( A, A' \in \mathcal{A}_G(M^n) \), fitting to the above definition, by setting

\[
\delta^2_{\text{conn}}(A, A') := \sup_{f \in C^\infty(M^n, \mathbb{R}^{n-4})} \inf_{g \in \text{Reg}(f)} \frac{\int_{M^n} \left| (dg + Ag - gA') \wedge f^*\omega_h \right|^2 \, d\text{vol}_h}{\left| f^*\omega_h \right|^2} \quad (1.2)
\]

where the infimum is taken over all measurable \( g \), and we use the notations \( d\omega = d\omega_{n-4} = dy_1 \wedge \cdots \wedge dy_n-4 \) for the volume form of \( \mathbb{R}^{n-4} \) and \( d\sigma_h \) for the 4-dimensional surface element of \( f^{-1}(y) \) for \( y \in \text{Reg}(f) \). The above integrals are well-defined because almost all values of \( f \) are regular by Sard’s theorem. In order to justify the good-definition of the expressions in (1.2), (1.3) under the low regularity assumption on \( g \), we note that \( dg \) can be interpreted as a distribution, and testing it against \( f^*\omega \) is a well-defined operation. Because \( G \) is bounded, the terms \( Ag, gA' \) are the product of an \( L^\infty \) and an \( L^2 \) function, and thus their wedge with \( f^*\omega \) is also well-defined. Finally, we can pass from (1.2) to (1.3) via the co-area formula.

Note that the integrand in (1.2), (1.3) is finite precisely if there exists a (measurable on \( M^n \)) gauge \( g \) which restricted to a.e. levelsets \( f^{-1}(y) \) is in \( W^{1,2}(f^{-1}(y), G) \), which is fitting to Definition 1.4.

In order to see that our definition (1.2) is a higher-dimensional extension of the Donaldson distance \( d \) given in (1.1) and used in dimension \( n \leq 4 \), we note the following alternative expression\(^\dagger\) of (1.1), paralleling (1.2) (for the proof, see Lemma A.4):

\[
d^2_{\text{conn}}(A, A') := \inf_{g \in \text{Reg}(f)} \sup_{f \in C^\infty(M^n, \mathbb{R}^{n-4})} \int_{M^n} \left| (dg + Ag - gA') \wedge f^*\omega_h \right|^2 \, d\text{vol}_h \quad (1.4)
\]

We will use the notation \( A \simeq A' \) whenever \( A, A' \in \mathcal{A}_G(M^n) \) are related by a gauge map \( g \in W^{1,2}(M^n, G) \), i.e. \( g^{-1}dg + g^{-1}Ag = A' \). Then (see [11] for example) the so-defined relation \( \simeq \) is an equivalence relation, and we may consider the quotient space \( \mathcal{A}_G(M^n) / \simeq \) formed of equivalence classes

\[
[A] := \{ A' \in \mathcal{A}_G(M^n) : A' \simeq A \}. \quad (1.5)
\]

For curvature forms \( F, F' \in L^2(\wedge^2 M^n, g) \) we also obtain the well-known equivalence relation \( \simeq \) (denoted by abuse of notation by the same symbol, because as it turns out \( A \simeq A' \) is equivalent to \( F_A \simeq F_{A'} \)), according to which \( F \simeq F' \) if there exists a measurable \( g : M^n \to G \) such that \( g^{-1}Fg = F' \). On \( g \)-valued forms we use the canonical conjugation-invariant norm, under which if \( F \simeq F' \) then pointwise a.e. there holds \( |F| = |F'| \). This means that, as is well-known, the Yang-Mills energy is constant on each equivalence class \([A] \).

Between curvature forms \( F, F' \in L^2(\wedge^2 M^n, g) \) we introduce (and by abuse of notation, we use the same notation as for connection forms), pseudo-distances defined by the following formulas, where again the infimum is taken over measurable \( g : M^n \to G \):

\[
d^2_{\text{curv}}(F, F') := \inf_{g : M^n \to G} \int_{M^n} \left| g^{-1}Fg - F' \right|^2 \, d\text{vol}_h \quad (1.6)
\]

\(^\dagger\)For two functions \( f, g : X \to \mathbb{R} \) we use the notation \( f \asymp g \) if there exists \( C > 0 \) such that for all \( x \in X \) there holds \( C^{-1}f(x) \leq g(x) \leq Cf(x) \).
and

$$\delta^2_{\text{curv}}(F, F') := \sup_{f \in C^\infty(M^n, \mathbb{R}^n)} \inf_{g : M^n \to \hat{G} M^n} \int_{M^n} |(g^{-1}Fg - F') \wedge f^*\omega|^{\frac{2}{h}} d\text{vol}_h,$$  \hspace{1cm} (1.7)$$

The evident motivation for introducing the Donaldson pseudo-distance \(d\) from (1.1) or (1.4) between connection forms (and respectively, the distance (1.6) between curvature forms), is that it induces, arguably, the simplest possible geometric distance on equivalence classes of connections (resp. curvatures).

We have the following results concerning the above distances, proved in Appendix A:

**Proposition 1.5.** The following hold:

1. For \(G = SU(2)\) there holds \(d_{\text{curv}} \asymp \delta_{\text{curv}}\).

2. For any \(G\) we have for weak connections \(A, B\) that \(d_{\text{curv}}(F_A, F_B) = 0\) if and only if \(d_{\text{conn}}(A, B) = 0\) and \(\delta_{\text{curv}}(F_A, F_B) = 0\) if and only if \(\delta_{\text{conn}}(A, B) = 0\).

The above proves that the pseudo-distance \(\delta_{\text{conn}}\) from (1.2) (respectively \(\delta_{\text{curv}}\) from (1.7)) induces a distance on equivalence classes \([A]\) as in (1.5) (resp. on equivalence classes of curvature forms) in the case \(G = SU(2)\). In the case of other simply connected Lie groups \(G\), we are tempted to conjecture that this continues to hold:

**Question 1.6.** Is it true that for general simply-connected compact Lie groups \(G\) there holds \(d_{\text{curv}} \asymp \delta_{\text{curv}}\) and \(d_{\text{conn}} \asymp \delta_{\text{conn}}\)?

**Remark 1.7.** When comparing two pseudo-distances \(\delta_1, \delta_2\) over a space \(X\) we can compare them at increasing levels of precision, and ask:

1. whether \(\delta_1 = 0 \Leftrightarrow \delta_2 = 0\), i.e. if they generate the same metric space;

2. whether the topology induced by \(\delta_1\) is the same as the topology induced by \(\delta_2\);

3. whether \(\delta_1 \asymp \delta_2\).

While it seems plausible that Question 1.6 holds and the strong statement \(d_{\text{conn}} \asymp \delta_{\text{conn}}\) is valid for general Lie groups \(G\), outside the case \(G = SU(2)\) we don’t even know a proof (or counterexample) of the assertion that, given two constant \(g\)-valued 2-forms \(F, F' \in \wedge^2 \mathbb{R}^n \otimes g\), the following conditions are equivalent:

(a) For all 4-dimensional subspaces \(H \in Gr(n, A)\) there exists \(g_H \in G\) such that the restrictions to \(H\) are conjugated by \(g_H\), i.e. \(g_H^{-1}i_H^*Fg_H = i_H^*F'\).

(b) There exists \(g \in G\) such that \(g^{-1}Fg = F'\).

The non-trivial part of the question is to prove that (a) implies (b), as the reverse implication follows by restriction.

The important property of interest for us is the fact that the pseudo-distance \(\delta_{\text{conn}}\) metrizes the weak convergence under a Yang-Mills energy control, and we have:

**Theorem 1.8** (Compactness of weak connections in dimension \(> 4\)). The space \(\mathcal{A}_G(M^n)\) is weakly sequentially closed below any Yang-Mills energy level. More precisely, let \(A^j \in \mathcal{A}_G(M^n)\) such that

$$\limsup_{j \to +\infty} \int_{M^n} |dA^j + A^j \wedge A^j|^2 d\text{Vol}_{M^n} < +\infty$$

then there exists a subsequence \(j'\) and \(A^\infty \in \mathcal{A}_G(M^n)\) such that for the equivalence classes of connections we have

$$\delta_{\text{conn}}(A_j, A) \to 0.$$
This result can be interpreted as a nonlinear version of Rellich-Kondrachov’s compactness theorem, in which the linear operation of taking the gradient of a Sobolev $W^{1,2}$-regular function, is now replaced by the nonlinear operation of taking the curvature of a weak connection.

The proof of the sequential weak closure property of $A_C(M^n)$ uses a strong approximation result of an elements in $A_C(M^n)$ by smooth connections away from polyhedral codimension 5 singularities, whose space $\mathcal{R}^\infty(M^n)$ is precisely defined in Section 5 (see (5.1)).

**Theorem 1.9.** If $A \in \tilde{A}_G(M^n)$ then there exists a sequence of connection forms $A_j \in \mathcal{R}^\infty(M^n)$ and a sequence of gauge changes $g_j \in W^{1,2}(M^n, G)$ such that if $F_j := dA_j + A_j \wedge A_j$ then as $j \to \infty$ there holds

$$\|g_j^{-1}d g_j + g_j^{-1} A_j g_j - A\|_{L^2(M^n)} \to 0, \quad \|g_j^{-1} F_j g_j - F_A\|_{L^2(M^n)} \to 0.$$  \hfill (1.8)

With the definitions (1.4) and (1.6), the existence of $g_j$ such that (1.8) holds is equivalent to saying that $d(A_j, A) \to 0$.

The main consequence of Theorem 1.9 in the study of partial regularity for stationary Yang-Mills connections, is that it paves the way to apply the partial regularity results of [14], a step which we plan to take in the paper [18] in general dimension.

In [14] it was proved that if $F$ has small Morrey norm condition on $F$ and $A$ is approximated by smooth connections with Morrey norm control rather than by elements of $\mathcal{R}^\infty(M^n)$ as in Theorem 1.9, then it is then possible to extract controlled Coulomb gauges of $A$ which allow to prove sharp $\epsilon$-regularity estimates. In [18], by the approximation procedure leading to Theorem 1.9 we prove precisely such approximating smooth connections, and thus [14] directly leads to the sharp partial regularity result for weak connections. This was done in dimension 5 in [17]. As a consequence of these results, we have in a unified variational framework in all dimensions the compactness result of Theorem 1.8 and the partial regularity theory extending [21] and [14].

A further consequence of Theorem 1.9 we also obtain the following important property of weak connections:

**Proposition 1.10** (Bianchi identity for weak curvatures). Assume that $A, F$ are the $L^2$ curvature and connection forms corresponding to a weak connection class $[A] \in A_C(\mathbb{R}^5)$. Then the equation

$$d_A F := dF + [F, A] = 0$$  \hfill (1.9)

holds in the sense of distributions.

### 1.1 Structure of the proof

In Section 2 we construct controlled gauges on the sphere by extending Uhlenbeck’s method [24] to a new setting. Using the outcome of Section 2, in Section 3 we prove a result which allows to bootstrap to higher dimensions the local regularity of connection forms. In Section 4 we introduce a local version of Definition 1.4, defining the space $\tilde{A}_G([-1, 1]^n)$; we then present a setup for the proof of our approximation result, by separating regions of concentration and non-concentration of energy. Then in Section 5, the result of Section 3 is fit to the setting prepared in Section 4, allowing to prove the strong approximation from Theorem 1.9 on the space $\tilde{A}_G([-1, 1]^n)$ (see Theorem 5.2). In Section 6, again in the local space $\tilde{A}_G([-1, 1]^n)$, we prove that the Yang-Mills energy forms a 2-weak gradient structure for the connections in the appropriate metric, which allows to prove a local version (see Theorem 1.8) of our compactness of Theorem 6.1. In Section 7, based on the outcomes of Sections 5 and 6, we complete the manifold-case of the proofs of Theorems 1.9 and 1.8. Finally, in Appendix A we prove that $\delta$ induces a distance on $A_C(M^n)/\simeq$ for $G = SU(2)$, prove equivalence of different definitions of our distance, and briefly show how Question 1.7 is connected with other related questions.
List of notations

- $G$: a compact Lie group
- $g$: the Lie algebra of $G$
- $n$: the dimension of the base space of our bundles.
- $(M^n, h)$: a compact Riemannian manifold. Sometimes the Riemannian metric $h$ is omitted in the notation.
- $A$: a $g$-valued 1-form.
- $F$: a $g$-valued 2-form.
- $g$: a map into the Lie group $G$, which can be interpreted as a “singular gauge change” of singular bundles.
- $A^g$: the new expression $A^g = g^{-1} dg + g^{-1} Ag$ of the connection form after the gauge transformation $g$.
- $\mathcal{A}_G(M^n)$: the space of Sobolev connections which locally in some $W^{1,2}$-gauge are $W^{1,2}$. Used here only for $n \leq 4$.
- $\tilde{\mathcal{A}}_G([-1,1]^n)$: the local model of the space $\mathcal{A}_G(M^n)$, described in Definition 4.2

2 Controlled gauges on the $n$-sphere

In this section we follow the overall structure of the argument from [24] to prove the following result:

**Theorem 2.1.** Let $\pi : L^n(S^n, \wedge^1 TS^n \otimes g) \to L^n(S^n, \wedge^1 TS^n \otimes g)$ be a linear operator which is bounded on $L^p$ for $p \in [n, n + \epsilon_{\pi}]$ for some $\epsilon_{\pi} > 0$, which satisfies $\pi \circ \pi = \pi$ and such that for $\omega \in L^1(S^n, \wedge^1 TS^n \otimes g)$ there holds

$$\|(id - \pi)\omega\|_{L^\infty(S^n)} \leq C_{\pi}\|\omega\|_{L^1(S^n)} \quad (2.1a)$$

and

$$d((id - \pi)\omega) = 0. \quad (2.1b)$$

There exist constants $\epsilon_0, C$ with the following properties. If $A \in L^n(S^n, \wedge^1 TS^n \otimes g)$ is a connection form over $S^n$ such that together with the corresponding curvature form $F$ satisfies

$$\|F\|_{L^{n/2}(S^n)} + \|A\|_{L^2(S^n)} \leq \epsilon_0$$

then there exists a gauge transformation $g \in W^{1,n}(S^n, G)$ such that

$$g^{-1} dg \in \text{Im}(\pi), \quad (2.2a)$$

$$d_{g^{-1}}(\pi (A^g)) = 0, \quad (2.2b)$$

$$\|A^g\|_{L^2(S^n)} \leq C(\|F\|_{L^{n/2}(S^n)} + \|A\|_{L^2(S^n)}). \quad (2.2c)$$
The proof consists in studying the case where the integrability exponents \( n/2, n \) are replaced by \( p/2, p \) for \( p > n \) first, and then obtaining the \( p = n/2 \) case as a limit. Note that for \( p > n \) the space \( W^{1,p}(S^n, G) \) embeds continuously in \( C^n(S^n, G) \), thus gauges \( g \) of small \( W^{1,p} \)-norm will be expressible as \( g = \exp(v) \) for some \( v \in W^{1,p}(S^n, g) \), due to the local invertibility of the exponential map \( \exp : G \to g \).

We then consider the space

\[
E_p := \left\{ v \in W^{1,p}(S^n, g) : \int_{S^n} v = 0, \ dv \in \text{Im}(\pi) \right\}
\]

(2.3a)

where \( x_k \) are the ambient coordinate functions relative to the canonical immersion \( S^n \to \mathbb{R}^{n+1} \). In case \( p > n \) the Banach space \( E_p \) is, by the above considerations, the local model of the Banach manifold

\[
M_p := \{ g \in W^{1,p}(S^n, G) : (2.2a) \text{ holds} \}.
\]

(2.3b)

We then consider the sets

\[
\mathcal{U}_p^\epsilon := \{ A \in L^p(S^n, \Lambda^1 T S^n \otimes g) : \|F_A\|_{L^p(S^n)} \leq \epsilon_0 \}
\]

(2.3c)

and their subsets

\[
\mathcal{V}_p^{C_p} := \left\{ A \in \mathcal{U}_p^\epsilon : \exists g \in M_n \text{ s.t. (2.2b) holds,} \right\}
\]

(2.3d)

and

\[
\|A^\delta\|_{L^n} \leq C_{\delta} \|F\|_{L^{n/2}} \text{ for } q = p, n
\]

and

\[
\|F\|_{L^{n/2}} + \|A\|_{L^2} < \epsilon
\]


2.1 Constructing modified Coulomb gauges: proof of Theorem 2.1

Like in [24] we prove theorem 2.1 by showing that if \( \epsilon_0 > 0 \) is small enough then for \( p \geq n \) we may find \( C_p \) such that

\[
\mathcal{V}_p^{C_p} = \mathcal{U}_p^\epsilon.
\]

(2.4)

We are interested in (2.4) just for \( p = n \) but we use the cases \( p > n \) in the proof: we successively prove the following statements.

1. \( \mathcal{U}_p^\epsilon \) is path-connected.

2. For \( p \geq n \) the set \( \mathcal{V}_p^{C_p} \) is closed in \( L^p(S^n, \Lambda^1 T S^n \otimes g) \).

3. For \( p > n \) there exists \( C_p, \epsilon_0 \) such that the set \( \mathcal{V}_p^{C_p} \) is open relative to \( \mathcal{U}_p^\epsilon \). In particular (2.4) is true for \( p > n \).

4. There exists \( K \) such that if \( g \in M_p, \|A^\delta\|_{L^n} \leq K \) and

\[
d_{S^n}(\pi(A^\delta)) = 0, \quad \|F\|_{L^{n/2}} + \|A\|_{L^2} < \epsilon_0
\]

then

\[
\|A^\delta\|_{L^n} \leq C_{\epsilon_0}(\|F\|_{L^{n/2}} + \|A\|_{L^2})
\]

5. The case \( p = n \) of (2.4) follows from the case \( p > n \).

Proof of step 1

Fix \( p \geq n, \epsilon, A \in \mathcal{U}_p^\epsilon \). We observe that \( 0 \in \mathcal{U}_p^\epsilon \). Moreover the connection forms \( A_t(x) := tA(tx) \) for \( t \in [0, 1] \) all belong to \( \mathcal{U}_p^\epsilon \) as well, like in [24].
Proof of step 2

Let \( A_k \in \mathcal{V}^c_{p,C_p} \) be a sequence of connection forms converging in \( W^{1,p} \) to \( A \). Consider the gauges \( g_k \) as in the definition (2.3d) of \( \mathcal{V}^c_{p,C_p} \). We may assume that the \( A_k^p \) have a weak \( W^{1,p} \)-limit \( \tilde{A} \). The bounds and equation in (2.3d) are preserved under weak limit thus we finish if we prove that there exists a gauge \( g \in M_p \) such that \( \tilde{A} = A^g \). Writing
\[
dg_k = g_k A_k^{q_k} - A_k g_k \tag{2.5}
\]
as \( G \subset \mathbb{R}^N \) is bounded it follows that \( \|dg_k\|_{L^p} \lesssim \|A_k^q\|_{L^p} + \|A_k\|_{L^p} \), thus up to extracting a subsequence we have \( g_k \overset{W^{1,p}}{\rightarrow} g \). Thus we may pass to the limit equation (2.5) and we obtain indeed \( \tilde{A} = A^g \) and also \( g \in M_p \).

Proof of step 3

Fix \( p > n \) and let \( A \in \mathcal{V}^c_{p,C_p} \). Consider the following data:
\[
g \in M_p, \\
\eta \in L^p(\mathbb{S}^n, \wedge^1 TS^n \otimes \mathfrak{g}), \\
V_p := \frac{d}{d\nu} \left( \pi \left( L^p(\mathbb{S}^n, \wedge^1 TS^n \otimes \mathfrak{g}) \right) \right).
\]
Consider the function of such \( g, \eta \), with values in \( V_p \subset W^{-1,p}(\mathbb{S}^n, \mathfrak{g}) \) defined as follows:
\[
N_A(g, \eta) := \frac{d}{d\nu} \left( \pi \left( (g^{-1}dg + g^{-1}(A + \eta)g) \right) \right) = \frac{d}{d\nu} \left( g^{-1}dg + \pi \left( g^{-1}(A + \eta)g \right) \right),
\]
where we used the fact that \( \pi = id \) on \( \text{Im}(\pi) \).

Note that \( N_A(id,0) = 0 \) and \( N_A \) is \( C^1 \). We want to apply the implicit function theorem in order to solve in \( g \) the equation \( N_A(g, \eta) = 0 \) for \( g \) in a \( W^{1,p} \)-neighborhood of \( id \in M_p \).

The implicit function theorem will imply also that the dependence of \( g \) on \( \eta \) is continuous. Note that there holds \( \exp(tv)\pm 1 = 1 \pm tv + O(t^2) \) as \( t \to 0 \). Using this and the fact that \( E_p \) is the tangent space to \( M_p \) at \( id \) we find the linearization of \( N_A \) at \( (id,0) \) in the first variable:
\[
H_A(v) := \frac{\partial}{\partial t} \left. N_A(id,0) \right|_{t=0} = \left. \frac{\partial}{\partial t} \right|_{t=0} \left[ \frac{d}{d\nu} \left( \pi \left( (\exp(tv))^{-1} d\exp(tv) + \exp(tv)^{-1}(A + \eta)\exp(tv) \right) \right) \right]
= \frac{d}{d\nu} \left( dv + \pi([A,v]) \right)
= \Delta_{\mathbb{S}^n} v + \frac{d}{d\nu} \left( \pi([A,v]) \right).
\]
To prove invertibility of \( H_A : E_p \to V_p \) we note that for all \( f \in W^{-1,p} \), we have that \( \Delta_{\mathbb{S}^n} v = f \) has a unique solution \( v \in W^{1,p} \) of zero average, thus \( \Delta_{\mathbb{S}^n} : W^{1,p} \cap \{ f = 0 \} \to W^{-1,p} \) is bijective, and moreover in this case \( \|dv\|_{L^p} \leq C_p \|f\|_{W^{-1,p}} \) by elliptic estimates.

In order to impose the extra constraints coming from \( \pi \), note that if \( f = \frac{d}{d\nu} \pi \alpha \) then we may rewrite the equation \( \Delta_{\mathbb{S}^n} v = \frac{d}{d\nu} \pi \alpha \) in the weak form
\[
\langle dv, d\varphi \rangle = \langle \pi \alpha, d\varphi \rangle, \quad \forall \varphi \in W^{1,p} \left( \mathbb{S}^n, \mathfrak{g} \right).
\]

If \( d\varphi \) represents a functional on \( L^p \) which vanishes on \( \text{Im}(\pi) \), then (2.6) gives \( \langle dv, d\varphi \rangle = 0 \). As \( \text{Im}(d) \subset L^p \) is a closed subspace, this implies that \( dv \in \text{Im}(\pi) \), thus \( v \in E_p \). Therefore the restriction of the Laplacian \( \Delta_{\mathbb{S}^n} |_{E_p} : E_p \to V_p \) is invertible with bounded inverse. To conclude that \( H_A \) is invertible with bounded inverse as well, we show that for \( A \in \mathcal{V}^c_{p,C_p} \) we have that
\[
v \mapsto \frac{d}{d\nu} \pi([A,v]) \in E_p \to V_p
\]
is a bounded linear operator of small operator norm, where the norms we consider are the ones induced by $W^{1,p}$ on $E_p$ and by $W^{-1,p}$ on $V_p$, respectively. Indeed for $v \in E_p$ and for all test functions $\varphi \in W^{1,p}$ we have

$$
\langle d_{\pi} [A, v], \varphi \rangle = \langle (\pi([A, v]), d\varphi) \leq \|d\varphi\|_{L^p} \|\pi([A, v])\|_{L^p} \leq C_{\pi,p} \|A\|_{L^p} \|v\|_{L^{p'}} \|\varphi\|_{W^{1,p}'}.
$$

which implies the desired estimate. If $\|A\|_{L^p}$ is small enough (depending only on $p, \pi$), we thus obtain that $H_A$ is invertible. In particular this condition holds for all $A \in \mathcal{V}_{p,0}^{\epsilon_0}$ for $C_p, \epsilon_0$ small enough.

**Proof of step 4**

For $g \in M_p$ and using $\pi \circ \pi = \pi$, we have $\pi(g^{-1} dg) = g^{-1} dg$, thus $d_{\pi} ([id, \pi]) = d_{\pi} (g^{-1} A g)$. Given this and (2.1a), we find

$$
\|A\|_{L^n} \leq \|\pi A^g\|_{L^n} + \|id - \pi\| (g^{-1} A g) \|_{L^n} \leq \|\pi A^g\|_{L^n} + C_{\pi} \|A\|_{L^2}. \tag{2.7}
$$

Then using (2.1b) as well as the property $d_{\pi} (\pi A^g) = 0$, by Hodge and Sobolev inequalities we have

$$
\|\pi A^g\|_{L^n} \leq \|\nabla \pi A^g\|_{L^{n/2}} \leq \|d\pi A^g\|_{L^{n/2}} + \|d_{\pi} (\pi A^g)\|_{L^{n/2}} = \|dA^g\|_{L^{n/2}}. \tag{2.8}
$$

Now just use the equation $F = dA + A \wedge A$ noting that pointwise a.e. there holds $|F_{A^g}| = |F_A|$, and then by triangle and Hölder inequalities, via (2.7) and (2.8) we conclude:

$$
\|A^g\|_{L^n} \leq \|\pi A^g\|_{L^n} + C_{\pi} \|A\|_{L^2} \leq C_n \|dA^g\|_{L^{n/2}} + C_{\pi} \|A\|_{L^2} \\
\leq C_n \|F\|_{L^{n/2}} + C_n \|A^g\|_{L^n} + C_{\pi} \|A\|_{L^2}.
$$

If $\|A^g\|_{L^n} \leq K$ small enough then the second term above is estimated by $KC_n \|A^g\|_{L^n}$ which can then be absorbed to the left side of the inequality, giving the desired estimate.

**Proof of step 5**

We may approximate $A \in \mathcal{U}_{n}^{\alpha}$ by smooth $A_k$ in $L^n$ norm such that $F_{A_k} \to F_A$ in $L^{n/2}$ as well. In particular there holds $A_k \in L^p$ for all $p > n$. We may obtain that $A_k \in \mathcal{U}_{n}^{\alpha} = \mathcal{V}_{p,0}^{\epsilon_0} \cap \mathcal{U}_{n}^{\alpha}$, $p > n$ and in particular we find $g_k \in M_p$ such that

$$
\|A_k\|_{L^n} \lesssim \|A_k\|_{L^n} \lesssim \|F_k\|_{L^{2/n}} \lesssim \epsilon_0,
$$

where the constants depend only on the exponents $p$ and $n$. By possibly diminishing $\epsilon_0$ we thus achieve $\|A_k\|_{L^n} \leq K$ for all $k$. By the closure result of Step 2 for $p = n$ we thus obtain that the same estimate holds for $A$ and for some gauge $g \in M_n$ and by Step 4 we conclude that $A \in \mathcal{V}_{p,0}^{\epsilon_0}$, as desired. \( \square \)

### 3 Replacement of nonabelian curvatures on Lipschitz domains in $n$ dimensions

In this section we prove the extension result which will help to define our approximating connections.

We consider the scale $r = 1$. Theorem 3.1 will later be used on all faces of skeleton of a cubeulation, even on the ones of higher codimension, but by a slight abuse of notation we denote still by “$n$” the dimension of our domains. The ambient dimension will be called $N$. 

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Theorem 3.1 has the role of translating the $L^n$-smallness condition (3.2) on $F$ on the $n$-dimensional boundary to a similar smallness (with a slightly worse, but still small, constant) (3.3c) in the $n+1$ dimensional interior, which will allow the theorem to be used iteratively. At each step of the iteration we use Theorem 2.1, which requests $A$ to already be $L^n$-integrable. For $n = 4$ this is a consequence of Definition 1.4, which describes the class of weak connections. In higher dimension $n > 4$ we achieve it by iterative extension on the skeleton of dimension $5 \leq k \leq n$, as a consequence of Theorem 3.1 itself. The qualitative improvement (3.3d) in $A$ is crucial for applying Theorem 2.1 at the next step.

**Theorem 3.1.** Let $D \subset \mathbb{R}^N$ be homeomorphic to the $n+1$ dimensional ball $\mathbb{B}^{n+1}$ via map $\Psi_D : \mathbb{R}^{n+1} \supset \mathbb{B}^{n+1} \to D$. Assume that $\Psi_D$ is also bi-Lipschitz with bi-Lipschitz constant $C_D$.

Let $F \in L^2(D, \wedge^2 \mathbb{R}^n \otimes \mathfrak{g})$ and $A \in L^2(D, \wedge^1 \mathbb{R}^n \otimes \mathfrak{g})$ (where the underlying measure is the Hausdorff $(n+1)$-dimensional measure on $D$) be such that in the sense of distributions $i_D^*F = di_D^*A + i_D^*A \wedge i_D^*A$ on $D$. Fix also constant forms $\overline{F} \in \wedge^2 \mathbb{R}^n \otimes \mathfrak{g}$ and $\overline{A} \in \wedge^1 \mathbb{R}^n \otimes \mathfrak{g}$.

Assume that

$$\exists \hat{\varphi} \in W^{1,2}(\partial D,G) \quad \text{such that} \quad i_{\partial D}^*\hat{\varphi} \in L^n(\partial D, \wedge^1 TD \otimes \mathfrak{g}).$$

(3.1)

Then there exists a constant $\epsilon_0 > 0$ depending only on $n,C_D,G$ such that if in the above gauge there holds

$$\int_{\partial D} |i_{\partial D}^*F|^n/2 < \epsilon_0, \quad \int_{\partial D} |i_{\partial D}^*A|^n < \epsilon_0, \quad |\overline{F}| < \epsilon_0, \quad |\overline{A}|^2 < \epsilon_0,$$

(3.2)

then there exists $\hat{A} \in L^2(D, \wedge^1 TD \otimes \mathfrak{g})$ and $\hat{\varphi} \in W^{1,2}(D,G)$ such that $\hat{A}$ is Lipschitz in the interior of $D$, $i_{\partial D}^*\hat{A} = i_{\partial D}^*\overline{A}$, $\hat{\varphi}|_{\partial D} = \varphi$ and the distribution $F_{\hat{A}} = d\hat{\varphi} + \hat{\varphi} \wedge \hat{A}$ is represented by an element of $L^2(D, \wedge^2 TD \otimes \mathfrak{g})$, and we have the approximation bounds

$$\|d\hat{A} + \hat{\varphi} \wedge \hat{A} - \overline{F}\|_{L^2(\partial D)} \lesssim \|i_{\partial D}^*\overline{F} - \overline{A}\|_{L^2(\partial D)}$$

(3.3a)

$$+ \|\overline{F}\| \|i_{\partial D}^*(\hat{A} - \overline{A})\|_{L^2(\partial D)} + \|i_{\partial D}^*\hat{\varphi}\|_{L^2(\partial D)} + \|i_{\partial D}^*\overline{F}\|_{L^{n/2}(\partial D)} + |\overline{F}|^2$$

and

$$\|\hat{A} - \overline{A}\|_{L^2(\partial D)} \lesssim \|i_{\partial D}^*(\hat{A} - \overline{A})\|_{L^2(\partial D)}.$$  

(3.3b)

Moreover we have the improved regularity

$$\|d\hat{A} + \hat{\varphi} \wedge \hat{A}\|_{L^\infty(\partial D)} \lesssim \|i_{\partial D}^*\hat{\varphi}\|_{L^n(\partial D)} + \|i_{\partial D}^*\overline{F}\|_{L^{n/2}(\partial D)} + |\overline{F}|,$$

(3.3c)

and

$$\hat{A} \in L^{n+1}(D, \wedge^1 TD \otimes \mathfrak{g}).$$

(3.3d)

**Proof.** **Preparation.** Transfer of the informations to the sphere. We do all our estimates on $S^n, \mathbb{B}^{n+1}$, and we use the bi-lipschitz homeomorphism $\Psi_D$ to link this model case with the case of general $D$. Indeed we note that for measurable $L^p$-integrable $k$-forms $\omega$ on $D$ and $\omega'$ on $\partial D$ there holds, with $C_\phi$ denoting here the Lipschitz constant of a function $\phi$,

$$C_{\Psi_D}^{-k} C_{\Psi_D}^\chi \| \omega' \|_{L^p(\partial D)} \leq \| \Psi_D^* \omega' \|_{L^p(S^n)} \leq \| \Psi_D^* \omega' \|_{L^p(\partial D)} \leq C_{\Psi_D}^k C_{\Psi_D}^\chi \| \omega' \|_{L^p(\partial D)};$$

$$C_{\Psi_D}^{-k} C_{\Psi_D}^\chi \| \omega \|_{L^p(D)} \leq \| \Psi_D^* \omega \|_{L^p(\mathbb{B}^{n+1})} \leq \| \Psi_D^* \omega \|_{L^p(D)} \leq C_{\Psi_D}^k C_{\Psi_D}^\chi \| \omega \|_{L^p(D)}.$$

(3.4)

Thus the bounds obtained on $S^n$ and $\mathbb{B}^{n+1}$ which we obtain below are comparable to those on general $D$.  

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Let $\pi_{nc,D} : L^p(\partial D, \wedge^1 T\partial D \otimes g) \to L^p(\partial D, \wedge^1 T\partial D \otimes g)$ be defined by

$$\pi_{nc,D}(A) = A - \sum_{j=1}^N \left( \int_{\partial D} (A, i_{\partial D}^* dx_j) d\mathcal{H}^n \right) i_{\partial D}^* dx_j.$$  

In other words, $\pi_{nc,D}$ removes from $A$ to the $L^2$-components parallel to $\text{Span}\{i_{\partial D}^* dx_k, k = 1, \ldots, N\}$. In particular for any $A$ as in the thesis of our theorem, we then have

$$\|\pi_{nc,D}(i_{\partial D}^* A)\|_{L^2(\partial D)} = \inf \{\|i_{\partial D}^*(A - B)\|_{L^2(\partial D)} : B \in \wedge^1 \mathbb{R}^N \otimes g\}$$

$$\leq \|i_{\partial D}^*(A - \mathcal{A})\|_{L^2(\partial D)}. \quad (3.5)$$

We are now going to define an involution $\pi_D : L^p(\mathbb{S}^n, \wedge^1 TS^a \otimes g) \to L^p(\mathbb{S}^n, \wedge^1 TS^a \otimes g)$ like in Theorem 2.1 and related to $\pi_{nc,D}$ by the following property:

$$\pi_D(\Psi_D^* i_{\partial D}^* A) = \Psi_D^* (\pi_{nc,D}(i_{\partial D}^* A)) \text{ for all } A \in L^p(\mathbb{R}^N, \wedge^1 \mathbb{R}^N \otimes g). \quad (3.6)$$

This involution can be written explicitly:

$$\pi_D(A) = A - \sum_{j=1}^N \left( \int_{\mathbb{S}^n} (A, \Psi_D^* i_{\partial D}^* dx_j) d\mathcal{H}^n \right) \Psi_D^* i_{\partial D}^* dx_j. \quad (3.7)$$

This clearly implies (2.1b). Then (2.1a) holds with $\pi_D$ in the place of $\pi$ because $\Psi_D \circ i_{\partial D}$ is Lipschitz and $dx_j$ is smooth. In fact, as $\Psi_D$ is Lipschitz, from (3.7) we also obtain for $1 \leq p < \infty$ that

$$\|(1 - \pi_D)(i_{\partial D}^* A)\|_{L^\infty(\mathbb{S}^n)} \leq C_N p \|i_{\partial D}^* A\|_{L^p(\mathbb{S}^n)} \leq C_N p \|i_{\partial D}^* A\|_{L^p(\partial D)}. \quad (3.8)$$

**Step 1.** Coulomb gauge on the boundary. Denote $\Psi_D^* A := A_D$. Let $g$ be the change of gauge $g$ given by Theorem 2.1 applied with $A$ replaced by $i_{\partial D}^* \Psi_D^* A = i_{\partial D}^* A_D$. Then $g$ satisfies

$$\pi_D(g^{-1} dg) = g^{-1} dg, \quad (3.9a)$$

$$d_{\partial D}^* \pi_D(i_{\partial D}^* A_D) = 0 \quad (3.9b)$$

and

$$\|i_{\partial D}^* A_D^g\|_{L^\infty(\mathbb{S}^n)} \leq C_n \left( \|i_{\partial D}^* F_{A_D}\|_{L^\infty(\mathbb{S}^n)} + \|i_{\partial D}^* A_D\|_{L^\infty(\mathbb{S}^n)} \right). \quad (3.9c)$$

From $i_{\partial D}^* A_D^g = g^{-1} dg + g^{-1} i_{\partial D}^* A_D g$ we obtain using the $\pi_D$-Coulomb condition on $A^g$ (where we identify 1-forms and vector fields using the metric)

$$\Delta_{\mathbb{S}^n} g = dg \cdot i_{\partial D}^* A_D^g + (g - id) d_{\partial D}^* i_{\partial D}^* A_D - d_{\partial D}^* (\pi_D(i_{\partial D}^* A_D) g)$$

$$- d_{\partial D}^* [(1 - \pi_D) A_D (g - id)] + d_{\partial D}^* [(1 - \pi_D) (i_{\partial D}^* (A_D^g - A_D))] \right).$$

The last term can be simplified using (3.9a):  

$$d_{\partial D}^* [(1 - \pi_D) (i_{\partial D}^* (A_D^g - A_D))] = d_{\partial D}^* [(1 - \pi_D) (i_{\partial D}^* (g^{-1} A_D g - A_D))] \right). \quad (3.10)$$

If $\hat{g}$ is the average of $g$ on $\mathbb{S}^n$ taken in $\mathbb{R}^{n+1}$, then by the mean value formula there exists $x \in \mathbb{S}^n$ such that $|g(x) - \hat{g}| \leq C\|g - \hat{g}\|_{L^2}$. Up to changing $g$ to $g\theta$ where $g\theta$ is a constant rotation, we may also assume $g(x) = id$. Using the embedding $W^{1,2} \to L^\infty$ valid in $n$ dimensions, (3.10), (3.8) and the Hölder inequalities we deduce (estimating the $W^{1,2}$-norm by duality with $W^{1,2}$):

$$\|dg\|_{L^1(\mathbb{S}^n)} \lesssim \|\Delta_{\mathbb{S}^n} g\|_{W^{-1,2}(\mathbb{S}^n)} \lesssim \|dg\|_{L^2(\mathbb{S}^n)} \|i_{\partial D}^* A_D^g\|_{L^\infty(\mathbb{S}^n)}$$

$$+ \|g - id\|_{L^\infty(\mathbb{S}^n)} \|i_{\partial D}^* A_D^g\|_{L^\infty(\mathbb{S}^n)} + \|\pi_D(i_{\partial D}^* A_D)\|_{L^2(\mathbb{S}^n)}$$

$$+ \|g - id\|_{L^2(\mathbb{S}^n)} \|1 - \pi_D) (i_{\partial D}^* A_D)\|_{L^\infty(\mathbb{S}^n)} \right). \quad (3.11)$$
By using Sobolev inequality \( \|g - id\|_L^p(S^n) \lesssim \|dg\|_{L^2(S^n)} \), the boundedness \( |g(x) - id| \leq \text{diam}(G) \), (3.9a) and \( \|i_{\partial D}^* F\|_{L^\infty(\partial D)} \lesssim |\bar{A}| < \epsilon_0 \), we absorb the terms not containing \( \pi_D(A_D) \) from the right hand side of (3.11) to the left hand side. Using (3.8), for \( \epsilon_0 > 0 \) small enough we reabsorb also the \((1 - \pi_D)\)-term. By (3.5), (3.6) we obtain

\[
\|dg\|_{L^2(S^n)} \lesssim C \|\pi_D(i_{\partial D}^* A_D)\|_{L^2(S^n)} \leq C \|i_{\partial D}^* (A - \bar{A})\|_{L^2(\partial D)}. \tag{3.12}
\]

**Step 2. Estimates on \( i_{\partial n}^* \Psi_D + \bar{F} \).** By (3.12), Poincaré’s inequality and the Lipschitz bounds (3.4), we obtain

\[
\|g^{-1}(i_{\partial n}^* \Psi_D^* F)g - i_{\partial n}^* \Psi_D^* F\|_{L^2(S^n)} \lesssim \|i_{\partial n}^* \Psi_D^* F\|_{L^\infty(S^n)} \|g - id\|_{L^2(S^n)} \lesssim \|i_{\partial D}^* F\|_{L^\infty(\partial D)} \|i_{\partial D}^* (A - \bar{A})\|_{L^2(\partial D)}. \tag{3.13}
\]

Since equation \( F_{Ax} = g^{-1} F g \) is invariant under pullback and the norm on \( g \)-valued forms is invariant under conjugation, using then (3.9) and the triangular inequality we obtain

\[
\|i_{\partial n}^* (F_{Ax} - \Psi_D^* F)\|_{L^2(S^n)} \lesssim \|i_{\partial D}^* F\|_{L^\infty(\partial D)} \|i_{\partial D}^* (A - \bar{A})\|_{L^2(\partial D)} \tag{3.14}
\]

Using (3.9c) and the bounded embedding \( L^4 \to L^{n/2} \) for \( n \geq 4 \) to bound \( \|A_D^\perp \ast A_D^\perp\|_{L^2} \leq \|A_D^\perp\|_{L^2}^2 \), from (3.14) and (3.4) we deduce

\[
\|i_{\partial n}^* (dA_D - \Psi_D^* F)\|_{L^2(S^n)} \lesssim \|i_{\partial D}^* F\|_{L^\infty(\partial D)} \|i_{\partial D}^* (A - \bar{A})\|_{L^2(\partial D)} \tag{3.15}
\]

\[
\|i_{\partial n}^* (d\bar{A}_D - \Psi_D^* F)\|_{L^2(S^n)} \lesssim \|i_{\partial D}^* F\|_{L^\infty(\partial D)} \|i_{\partial D}^* (A - \bar{A})\|_{L^2(\partial D)} \tag{3.16}
\]

**Step 3. Extension to the interior.** Define the following form belonging to \( C^\infty(\mathbb{R}^N, \wedge^1\mathbb{R}^N \otimes g) \):

\[
B := \sum_{i<j} \bar{F}_{ij} x_i dx_j - x_j dx_i / 2. \tag{3.17}
\]

For an 1-form \( \eta \in L^p(S^n, \wedge^1T\mathbb{R}^n \otimes g) \) we then consider the minimization problem

\[
\min \left\{ \int_{\mathbb{B}^{n+1}} |d(C - \Psi_D B)|^2 + |d^{*n+1}(C - \Psi_D^* B)|^2 \right\}
\text{ s. t. } C \in W^{\frac{2n}{n+1},1}(\mathbb{B}^{n+1}, \wedge^1\mathbb{B}^{n+1} \otimes g), \quad i_{\partial n} C = \eta \tag{3.18}
\]

A classical argument (and the fact that \( d\Psi_D B = \Psi_D dB = \Psi_D^* F \)) shows that the solution \( \tilde{\eta} \) to (3.17) is uniquely given by

\[
\begin{align*}
d^{*n+1}(\tilde{\eta} - \Psi_D B) &= 0 \quad \text{in } \mathbb{R}^{n+1}, \\
d^{*n+1}(d\tilde{\eta} - \Psi_D^* F) &= 0 \quad \text{in } \mathbb{R}^{n+1}, \\
i_{\partial n} \tilde{\eta} &= \eta \quad \text{on } \partial\mathbb{B}^{n+1},
\end{align*}
\]

and one has

\[
\|\tilde{\eta} - \Psi_D^* B\|_{L^{n+1}(\mathbb{B}^{n+1})} \leq C \|\tilde{\eta} - \Psi_D^* B\|_{W^{\frac{2n}{n+1},1}(\mathbb{B}^{n+1})} \leq C \|\eta - i_{\partial n}^* \Psi_D^* B\|_{L^p(S^n)}. \tag{3.19}
\]

Note that \( \Psi_D^* B \) is the solution to (3.17) for the choice \( \eta = i_{\partial n}^* \Psi_D^* B \). If we choose \( \eta = \pi_D(i_{\partial n}^* A_D) \) in (3.17) then we find the extension \( \pi_D(i_{\partial n}^* A_D) \). Next, for \( A \in L^3(D, \wedge^1TD \otimes g) \) we denote

\[
\bar{A} := \sum_{k=1}^{n+1} \Psi_D dx_k \frac{1}{C_n} \int_{S^n} \langle i_{\partial n}^* \Psi_D A, \Psi_D^* i_{\partial n}^* A \rangle dx_k, \tag{3.20}
\]
where $C_n$ is a normalization constant such that $i^\pm_{g_n} \Psi^*_D \alpha = i^\pm_{g_n} \Psi^*_D \alpha$ for any constant form $\alpha$. With notation (3.20) we define

$$\bar{A}_D^g := \pi(i^\pm_{g_n} A_D^g) + \bar{\nabla}.$$

**Step 4. Estimates on the extended curvatures.** Note that $d\Psi^*_D dx_b = \Psi^*_D d^2 x_b = 0$, therefore by (3.20) we have $d\bar{A}^g = 0$. Similarly we find $d(1 - \pi_D)(\alpha) = 0$ for general 1-forms $\alpha$ on $S^n$. Using this, analogously to (3.19) we obtain

$$\|d\bar{A}^g_D - \Psi^*_D \bar{F}\|^2_{L^2(S^n)} = \|d\pi_D(i^\pm_{g_n} A_D^g) - \Psi^*_D \bar{F}\|^2_{L^2(S^n)} \lesssim \|d\pi_D(i^\pm_{g_n} A_D^g) - i^\pm_{g_n} \Psi^*_D \bar{F}\|^2_{L^2(S^n)} = \|i^\pm_{g_n} (dA_D^g - \Psi^*_D \bar{F})\|^2_{L^2(S^n)}.$$  

By (3.19) and the triangle inequality we obtain

$$\|\bar{A}_D^g \wedge \bar{A}_D^g\|_{L^{1/2}((S^n)^{+1})} \lesssim \|\bar{A}_D^g\|_{L^{n+1}((S^n)^{+1})}$$

$$\lesssim \|\pi_D(i^\pm_{g_n} A_D^g) - \Psi^*_D \bar{F}\|_{L^{n+1}((S^n)^{+1})}.$$  

By (3.20) and the triangle inequality we obtain

$$\|d\bar{A}_D^g\|_{L^{n+1}((S^n)^{+1})} \lesssim \|d\pi_D(i^\pm_{g_n} A_D^g)\|_{L^{n+1}((S^n)^{+1})}$$

By (3.8) with $p = n$ combined with (3.9) and (3.4), we find

$$\|(1 - \pi_D)(i^\pm_{g_n} A_D^g)\|_{L^n(S^n)} + \|\bar{\nabla} F\|_{L^{n+1}(S^n)^{+1}} \lesssim \|i^\pm_{OD} F\|_{L^2(OD)} + \|i^\pm_{OD} A^g\|_{L^n(OD)},$$

whereas the remaining terms appear directly in (3.9). Thus from (3.22) and (3.23), respectively, we obtain the following two bounds:

$$\|d\bar{A}_D^g\|_{L^{n+1}((S^n)^{+1})} \lesssim \|i^\pm_{OD} F\|_{L^2(OD)} + \|i^\pm_{OD} A^g\|_{L^n(OD)},$$

$$\|d\bar{A}_D^g\|_{L^{n+1}((S^n)^{+1})} \lesssim \|i^\pm_{OD} F\|_{L^2(OD)} + \|i^\pm_{OD} A^g\|_{L^n(OD)} + |F|^2.$$
which will help to prove (3.3a) in a later step. Next, we perform the analogous bound where instead of (3.15) we use (3.25), and we obtain
\[
\|dA_D^g + A_D^g \wedge A_D^g\|_{L^{n+1}(B^{n+1})} \lesssim \|i_{\partial D}^g A^g\|_{L^n(\partial D)} + \|i_{\partial D}^g F\|_{L^{n/2}(\partial D)} + |F| + \|i_{\partial D}^g A^g\|_{L^n(\partial D)} + \|i_{\partial D}^g F\|_{L^{n/2}(\partial D)} + |F|^2
\lesssim \|i_{\partial D}^g A^g\|_{L^n(\partial D)} + \|i_{\partial D}^g F\|_{L^{n/2}(\partial D)} + |F| .
\] (3.27)
where due to hypothesis (3.2), we were able to absorb the second line into the first.

**Step 5. Correcting the restriction on the boundary.** Extend now \( g \) harmonically in \( B^{n+1} \) and denote by \( \tilde{g} \) this extension. Using (3.12) together with classical elliptic estimates, we find
\[
\|\tilde{g}^{-1}(\Psi_D^g F)\tilde{g} - \Psi_D^g F\|_{L^2(B^{n+1})} \lesssim |F| \|\tilde{g} - id\|_{L^2(B^{n+1})} \lesssim |F| \|i_{\partial D}(A - \mathcal{A})\|_{L^2(\partial D)} .
\] (3.28)
Combining (3.26) and (3.28) gives
\[
\|dA_D^g + A_D^g \wedge A_D^g - \tilde{g}^{-1}(\Psi_D^g F)\tilde{g}\|_{L^2(B^{n+1})} \lesssim \|i_{\partial D}(F - F)\|_{L^2(\partial D)}
+ |F| \|i_{\partial D}(A - \mathcal{A})\|_{L^2(\partial D)} + \|i_{\partial D} A^g\|_{L^n(\partial D)} + \|i_{\partial D} F\|_{L^{n/2}(\partial D)} + |F|^2.
\] (3.29)

Denote
\[
\hat{A}_D := (A_D^g)_{\tilde{g}^{-1}}.
\] (3.30)
With this notation one has \( F_{\hat{A}_D} = \tilde{g} F_{A_D^g} \tilde{g}^{-1} \), and after applying (3.4) to pass from \( B^{n+1} \) back to \( D \), we see that (3.29) implies the estimate (3.3a). By the same token, estimate (3.3c) follows from (3.27). Moreover we have
\[
A_D^g = A_D^g
\] (3.31)
in the notations (3.30) and \( \hat{A}^g \) harmonic, and thus is smooth in the interior of \( B^{n+1} \). Note that
\[
i_{\partial A}^g \hat{A}_D = i_{\partial A}^g (A_D^g)_{\tilde{g}^{-1}} = (i_{\partial A}^g A_D^g)_{\tilde{g}^{-1}} = i_{\partial A} A_D .
\] (3.32)
Define
\[
\hat{\mathcal{A}} := (\Psi_D^{-1})^* \hat{\mathcal{A}}.
\] (3.33)
We observe that since \( \hat{A}_D \) has \( L^{n+1} \) bounds (3.22), by the bound on \( \overline{\mathcal{A}}^g \), and by the Lipschitz bounds (3.4), it follows that \( \hat{\mathcal{A}} \) has \( L^{n+1} \) bounds as well, as requested in (3.3d).

By (3.4) we also obtain that the distributional expression \( F_{\hat{A}} = d\hat{A} + \hat{A} \wedge \hat{A} \) is well defined and \( F_{\hat{A}} \in L^2 \).

**Step 6. Verification of (3.3b).** We now use the definition (3.30) of \( \hat{A}_D \), as well as the estimates \( \|d\tilde{g}\|_{L^2(B^{n+1})} \lesssim \|d\tilde{g}\|_{L^2(\mathbb{R}^n)} \) together with (3.12), and then the bounds (3.9) on \( i_{\partial A} A^g \) from Theorem 2.1. Note also that if \( g \in G \) and \( M \in g \) then \( |g^{-1} M g - M| \lesssim |g - id| |M| \). We thus obtain the following chain of inequalities:
\[
\|\hat{A}_D - \Psi_D^g \overline{\mathcal{A}}\|_{L^2(B^{n+1})} \lesssim \|d\tilde{g}\|_{L^2(B^{n+1})} + \|\tilde{g} - id\| \|\overline{\mathcal{A}}^g - \Psi_D^g \overline{\mathcal{A}}\|_{L^2(B^{n+1})}
+ \|\tilde{g} - id\| \|\Psi_D^g \overline{\mathcal{A}}\|_{L^2(B^{n+1})} + \|\hat{A}_D - \Psi_D^g \overline{\mathcal{A}}\|_{L^2(B^{n+1})}
\lesssim \|i_{\partial D}(A - \mathcal{A})\|_{L^2(\partial D)} + \|\overline{\mathcal{A}} - \Psi_D^g \overline{\mathcal{A}}\|_{L^2(B^{n+1})}
+ \|i_{\partial D}(A - \mathcal{A})\|_{L^2(\partial D)} \|\overline{\mathcal{A}}^g - \Psi_D^g \overline{\mathcal{A}}\|_{L^2(\mathbb{R}^n)}
+ \|i_{\partial D}(A - \mathcal{A})\|_{L^2(\partial D)} \left( \|\Psi_D^g B\|_{L^{n+1}(B^{n+1})} + \|\overline{\mathcal{A}}^g B\|_{L^{n+1}(B^{n+1})} \right). 
\] (3.34)
In the last two lines from (3.34), we recognize the same expression as in the third line of (3.22), thus we can use thesame reasoning that leads from (3.22) to (3.24) to write
\[
\|A_D - \Psi^*_D\mathcal{A}\|_{L^2(\mathbb{R}^{n+1})} \lesssim \|i^*_\partial (A - \mathcal{A})\|_{L^2(\partial D)} + \|\nabla \Psi^*_D\mathcal{A}\|_{L^2(\mathbb{R}^{n+1})} + \|\nabla \Psi^*_D\mathcal{A}\|_{L^2(\partial D)} (\|i^*_\partial A\|_{L^2(\partial D)} + \|i^*_\partial F\|_{L^2(\partial D)} + |F|) \tag{3.35}
\]
The remaining estimate we need is the one below, which follows from the definition of $\mathcal{A}$. We will use also the fact that $(1 - \pi_D)(g^{-1}dg) = 0$ and (3.8) for $p = 1$ together with the Hölder inequality and (3.12), and (3.4).
\[
\|\mathcal{A} - \Psi^*_D\mathcal{A}\|_{L^2(\mathbb{R}^{n+1})} \lesssim \|(1 - \pi_D)(i^*_\partial (A - \mathcal{A}))\|_{L^2(\mathbb{R}^{n+1})} = \|(1 - \pi_D)(i^*_\partial (g^{-1}\Psi^*_DAg - \Psi^*_D\mathcal{A}))\|_{L^2(\mathbb{R}^{n+1})} \lesssim \|i^*_\partial (A - \mathcal{A})\|_{L^2(\partial D)} \|i^*_\partial A\|_{L^2(\partial D)} + \|\pi_D\|_{L^2(\partial D)} \|i^*_\partial F\|_{L^2(\partial D)} + |F| \tag{3.36}
\]
Regarding the first term in (3.36), the factor $\|i^*_\partial A\|_{L^2(\partial D)}$ is bounded by $\epsilon_0$ by hypothesis, thus we may absorb the first term into (3.34). To estimate the second term we use the fact that $\Psi^*_D$ is in fact linear on 1-forms (contrary to the case of $k$-forms for $k \geq 2$), i.e. $\Psi^*_D A - \Psi^*_D \mathcal{A} = \Psi^*_D (A - \mathcal{A})$, and thus we may use again (3.8) and obtain the strong bound
\[
\|i^*_\partial (A - \mathcal{A})\|_{L^2(\partial D)} \|i^*_\partial A\|_{L^2(\partial D)} \lesssim \|i^*_\partial (A - \mathcal{A})\|_{L^2(\partial D)} \|i^*_\partial A\|_{L^2(\partial D)} \tag{3.37}
\]
Combining this with (3.36) and inserting then into (3.35), we obtain
\[
\|A_D - \Psi^*_D\mathcal{A}\|_{L^2(\mathbb{R}^{n+1})} \lesssim \|i^*_\partial (A - \mathcal{A})\|_{L^2(\partial D)} (1 + \|i^*_\partial A\|_{L^2(\partial D)} + \|i^*_\partial F\|_{L^2(\partial D)} + |F|) \tag{3.37}
\]
and by using hypothesis (3.2) we find the bound (3.3b), as desired.  

\section{The space $\tilde{\mathcal{A}}_G([-1,1]^n)$ and the setup for tracking energy concentration}

\subsection{Local model for the space of weak connections}

We prepare now to define (in Definitions 4.1 and 4.2 below) a localized-in-space model $\tilde{\mathcal{A}}_G([-1,1]^n)$ for our space $\mathcal{A}_G(M^n)$ for the case of $M = [-1,1]^n$. The intuition is that $[-1,1]^n$ models a chart on a general manifold $M^n$, and we orient it to follow the level-sets of the functions $f$ appearing in Definition 1.4. Therefore in Definition 4.2 below we only use coordinate functions as slicing functions $f$. Our main results will be first proved in this setting in order to make the proofs clearer, and then extended to a general setting in Section 7.

Let $1 \leq i \leq n$ be an integer and denote
\[
H(k, t) := \{(x_1, \ldots, x_n) \in [-1,1]^n : x_k = t\}.
\]
We then consider the natural coordinates
\[
i_{k,t} : [-1,1]^{n-1} \to H(k, t), \quad i_{H(k,t)}(x_1, \ldots, x_{n-1}) := (x_1, \ldots, x_k, t, x_{k+1}, \ldots, x_n) - 1.
\]
More generally, for the case of $k$-dimensional coordinate subspaces we proceed as follows. Let $I = \{i_1, \ldots, i_k\}$ where we used the ordering of indices $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. Then for a $k$-tuple of real numbers $T = (t_{i_1}, \ldots, t_{i_k}), t_i \in [-1,1]$ indexed by $I$, we denote
\[
H(I, T) := \{(x_1, \ldots, x_n) \in [-1,1]^n : \forall i \in I, x_i = t_i\}.
\]

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The parameterization \( i_{H(I,T)} : [-1,1]^{n-k} \to H(I,T) \) of \( H(I,T) \) will then be given by
\[
i_{H(I,T)}(x_1, \ldots, x_k) := (y_1, \ldots, y_n) \quad \text{with} \quad y_i := \begin{cases} x_{j_i} & \text{if } i = j_i \in J \\ t_{i,i} & \text{if } i = i_j \in I \end{cases},
\]
in which we used the ordering \( 1 \leq j_1 < j_2 < \cdots < j_{n-k} \leq n \) of the indices in \( J := \{1, \ldots, n\} \setminus I \).

We now pass to define the space \( \tilde{A}_G \).

**Definition 4.1.** Let \( n = 4 \). Then we define \( \tilde{A}_G([-1,1]^4) \) as the set of \( A \in L^2([-1,1]^4, \wedge^1 \mathbb{R}^4, g) \) such that the following properties hold:
\[
\int_{[-1,1]^4} |F_A|^2 < +\infty \quad (4.1a)
\]
\[
\forall B \subset [-1,1]^4 \text{ open} , \exists g_B \in W^{1,2}(B,G) \text{ s.t. } A^{g_B} \in L^2(B, \wedge^1 \mathbb{R}^4, g). \quad (4.1b)
\]

**Definition 4.2** (Local model of weak connections, \( n \geq 5 \)). We define the space \( \tilde{A}_G([-1,1]^n) \) of \( L^2 \) weak connections on singular bundles over \( [-1,1]^n \) to be composed of all \( A \in L^2([-1,1]^n, \wedge^1 \mathbb{R}^n \otimes g) \) such that the following hold:
\[
F_A \overset{D'}{=} dA + A \wedge A \in L^2, \quad (4.2a)
\]
\[
\forall k = 1, \ldots, n, \text{ a.e. } t \in [-1,1], \quad i^{l_{H(k,t)}}_{H(I,T)} A \in \tilde{A}_G([[-1,1]^{n-1}) \quad (4.2b)
\]

We now note some important facts concerning the above definitions.

**Remark 4.3** (slicing only by 4-planes). By expanding the inductive condition \( (4.2b) \) of Definition 4.2, we may replace it by
\[
\forall I \subset \{1, \ldots, n\}, \#I = n - 4, \text{ a.e. } t \in [-1,1], \quad i^{l_{H(I,T)}}_{H(I,T)} A \in \tilde{A}_G([-1,1]^4) \quad (4.3)
\]
and we note that this condition would become equivalent to the one from Definition 4.4 for \( M^n = [-1,1]^n \) if we were to replace the class \( f \in C^\infty([-1,1]^n, \mathbb{R}^{n-4}) \) by the smaller one given just by subsets of the coordinates:
\[
C_{n,n-4} := \{ f : [-1,1]^n \to \mathbb{R}^{n-4} : \quad \exists I \subset \{1, \ldots, n\}, \#I = n - 4, \quad f(x_1, \ldots, x_n) = (x_i)_{i \in I} \}. \quad (4.4)
\]

**Remark 4.4** (About \( L^4 \) and \( W^{1,2} \)). As a consequence of the gauge extraction theorem 2.1, condition \( (4.1b) \) is equivalent to the following more classical condition (present also in [17]):
\[
\forall B \subset [-1,1]^4 \text{ open} \exists g_B \in W^{1,2}(B,G) \text{ s.t. } A^{g_B} \in W^{1,2}(B, \wedge^1 \mathbb{R}^4, g). \quad (4.5)
\]
The equivalence of condition \( (4.1b) \) and \( (4.5) \) under the condition \( (4.1a) \) in 4 dimensions can be proved as follows. First note that the proof of our gauge extraction theorem 2.1 for \( \pi = 0 \) and \( n = 4 \) remains valid in case we replace \( \mathbb{S}^4 \) by a small ball \( B \subset [-1,1]^4 \) and provides a local gauge \( g_B \) in which \( \| A^{g_B} \|_{L^4(B)} \leq \| F \|_{L^2(B)} = \| dA^{g_B} + A^{g_B} \wedge A^{g_B} \|_{L^2(B)} \), \( d^* A^0 = 0 \) thus \( \| A^{g_B} \|_{W^{1,2}} \leq C(\| F \|_{L^2} + \| A^{g_B} \|_{L^2}^2) \) if \( \| F \|_{L^2} \leq \epsilon_0 \). Then the gauge-patching reasoning similar to the one of [17]'s compactness result \( (H) \) allows to prove the fact that such good gauges can be patched over finite unions of small-energy balls covering a given compact, as desired. The existence of such covers follows by the fact that \( F \in L^2 \).

In direct analogy to \( (1.2) \), for \( A, A' \in \tilde{A}_G([-1,1]^n) \) we define the pseudo-distance \( \tilde{\delta} := \tilde{\delta}_{\text{conn}} \) by
\[
\tilde{\delta}^2(A, A') = \tilde{\delta}_{\text{conn}}^2(A, A') := \sup_{f \in C_{n,n-4}} \inf_{g \in [-1,1]^n} \int_{[-1,1]^n} |(d g + A g - g A') \wedge f^* \omega|^2 \frac{\text{dvol}}{|f^* \omega|}.
\]
which only differs from \( (1.2) \) by the fact that we have replaced the constraint \( f \in C^\infty([-1,1]^n, \mathbb{R}^{n-4}) \) by \( f \in C_{n,n-4} \). We analogously can define a distance \( \delta_{\text{curv}}(F, F') \) as in (1.7) between curvature forms \( F, F' \in L^2([-1,1]^n, \wedge^2[-1,1]^n \otimes g) \).
4.2 Choosing cubeulations

To choose well-behaved cubeulations we base ourselves mainly on Fubini’s theorem. We proceed as follows:

- Fix a small scale \( r > 0 \) which will be the size of the cubes used in our cubeulation.
- For \( i \in \{1, \ldots, n\} \) and \( t \in [0, r] \), the family of coordinate hyperplanes inside \([-1, 1]^n\) is denoted as follows:
  \[
  \mathcal{F}_{r,i,t} := \{ H(i, t') : t' \in (r \mathbb{Z} + t) \cap [-1, 1] \}.
  \]
- For \( I \subset \{1, \ldots, n\} \) with \( \# I = n - k \) and \( t_I \in [0, r]^{|I|} \) we parameterize \( k \)-dimensional cubes as follows:
  \[
  \mathcal{F}_{r,I,t} := \{ \cap_{i \in I} H(i, t_i') : \forall i \in I H(i, t_i') \in \mathcal{F}_{r,i,t} \}.
  \]
- For \( t = (t_1, \ldots, t_n) \in [0, r]^n \) and \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^d \), \( i \in \{1, \ldots, n\} \), we define the cube
  \[
  C_{r,t,\alpha} := t + r\alpha + [0, r]^n.
  \]
- Corresponding to the subdivision of \([-1, 1]^n\) in cubes from the family \( \{C_{r,t,\alpha}\}_\alpha \) we denote by \( C_{r,t} \) the polyhedral complex generated by the following under intersection:
  \[
  P_{r,t} := \{ C_{r,t,\alpha} : \alpha \in \mathbb{Z}^d, C_{r,t,\alpha} \cap [-1, 1]^n \neq \emptyset \}.
  \]
Then we denote by

- \( P_{r,t}^{(k)} \) the \( k \)-skeleton of \( P_{r,t} \) and by \( C_{r,t}^{(k)} \) the set of \( k \)-dimensional faces of cubes \( C_{r,t,\alpha} \) contributing to \( P_{r,t}^{(k)} \).

- If \( C_{\alpha} \in P_{r,t} \) and \( \omega \in L^1_{\text{loc}}([-1, 1]^n, V) \) we define by superposition
  \[
  \bar{\omega}_{C_{\alpha}} := \frac{1}{|C_{\alpha}|} \int_{C_{\alpha}} \omega(x) dx \in V.
  \]

This will be only used in the case where

- \( V = \wedge^1 \mathbb{R}^n \otimes \mathfrak{g} \) or \( V = \wedge^2 \mathbb{R}^n \otimes \mathfrak{g} \).

Note that in order to simplify the proof later, in Section (5) we will re-define the quantities \( \overline{\mathcal{A}}_{C_{\alpha}} \) and \( \overline{\mathcal{T}}_{C_{\alpha}} \) to denote a (slightly) different averaging.

- If \( \chi_{C_{\alpha}} : [-1, 1]^n \to \{0, 1\} \) is the function which equals 1 on \( C_{\alpha} \) and 0 outside it, then corresponding to the complex \( P_{r,t} \) we also define the piecewise constant \( V \)-valued function
  \[
  \bar{\omega} := \sum_{\alpha \in P_{r,t}} \chi_{C_{\alpha}}(x) \bar{\omega}_{C_{\alpha}}.
  \]

In the above notations, if no confusion arises we will often omit either one or both of the indices \( t \) and \( r \).

Next, fixing the underlying connection and curvature forms \( \omega = A \) or \( \omega = F_A \) as in (4.2), we want to find good choices of \( t \) such that the skeleta defined above do not carry too much \( L^2 \)-energy. This is done in the next proposition, which we state for general forms for clarity (cf. also [17, Prop. 2.6] for a particular case).
exists a decreasing function \( a_{\omega, \delta} : [0, 1] \rightarrow \mathbb{R}^+ \) such that \( \lim_{r \downarrow 0} a_{\omega, \delta}(r) = 0 \) with the following properties.

For all \( r > 0 \) there exist a subset \( T_{r, \delta}(\omega) \subset [0, r]^n \) with \( |T_{r, \delta}(\omega)| \geq \delta r^n \), a constant \( C \) depending only on \( \delta, n, g \) such that for all \( p \leq k \leq n-1 \) and for all \( t \in T_{r, \delta}(\omega) \) all the restrictions \( i_Q^* \omega \) appearing below are measurable and such that there holds:

\[
 r^{n-k} \sum_{C_{\alpha} \in P_{r, t} Q \in C_{\alpha}^{(k)}} \int_Q |i_Q^* \omega|^2 \leq C_1 \int_{[-1,1]^n} |\omega|^2, \tag{4.7a}
\]

\[
 r^{n-k} \sum_{C_{\alpha} \in P_{r, t} Q \in C_{\alpha}^{(k)}} \sum_{I \in \mathcal{I}_{r, t}} \int_Q |i_{r, t}^* \omega|^2 \leq \alpha_{\omega, \delta}(r). \tag{4.7b}
\]

Proof. Note that because there are \( n - k \) coordinates which are constant along any \( k \)-face, it follows that for any \( Q \in P_{r, t}^{(k)} \) there holds (and the inequality may be non-sharp only for those \( Q \) which are \( r \)-close to \( \partial[-1,1]^n \))

\[
 \# \{ C_{\alpha} \in P_{r, t} : Q \in C_{\alpha}^{(k)} \} \leq 2^{n-k}.
\]

If \( t_I \in [0, r]^I \) is the vector of \( I \)-indexed ordered coordinates of \( t \in [0, r]^n \) then we have

\[
 \sum_{C_{\alpha} \in P_{r, t} Q \in C_{\alpha}^{(k)}} \int_Q |i_Q^* \omega|^2 = 2^{n-k} \sum_{I \in \mathcal{I}_{r, t}} \int_{F_{r, t, I}} |i_{r, t, I}^* \omega|^2 := 2^{n-k} \sum_{#I=n-k} \mathcal{I}_{I, t_I}(\omega).
\]

Note that with the above notations if \( \omega_I^* \) is the form obtained from \( \omega \) by retaining only the terms \( dx_{i_1} \wedge \cdots \wedge dx_{i_p} \) with \( i_1, \ldots, i_p \in I \) then there holds

\[
 \int_{t_I \in [0, r]^I} I_{I, t_I}(\omega) dt_I = \int_{[-1,1]^n} |\omega_I|^2 := \mathcal{I}_I(\omega) \leq \int_{[-1,1]^n} |\omega|^2.
\]

By Chebychev’s inequality we obtain that since \( ||[0, r]^I|| = r^{n-k} \) there holds

\[
 |\mathcal{I}_{I, \delta}(\omega) := \left\{ t \in [0, r]^n : 2^{k-n} n! k!(n-k)! \mathcal{I}_{I, t_I}(\omega) > C_1 \mathcal{I}_I(\omega) \right\}| \leq 2^{n-1} \mathcal{I}_{I}(\omega)^{\frac{1}{k!(n-k)!}} := C_2 r^n.
\]

Then by subadditivity we obtain

\[
 \left| \bigcup_{#I=n-k} \mathcal{I}_{I, \delta}(\omega) \right| \leq \sum_{#I=n-k} |\mathcal{I}_{I, \delta}(\omega)| \leq \frac{n!}{k!(n-k)!} C_2 r^n := C_3 r^n.
\]

If we denote

\[
 T(\omega) := [0, r]^n \setminus \bigcup_{#I=n-k} \mathcal{I}_{I, \delta}(\omega)
\]

then we obtain (4.7a) for all \( t \in T(\omega) \) and

\[
 |T(\omega)| \geq (1 - C_3) r^n,
\]

which can be made arbitrarily close to \( r^n \) by choosing \( C_1 \) large enough.

Regarding (4.7b) we first note that by mollification for any \( \epsilon > 0 \) we may obtain \( \omega_\epsilon \in C^1([-1,1]^n, \wedge^p \mathbb{R}^n \otimes g) \) such that

\[
 \|\omega_\epsilon - \omega\|_{L^2} \leq \epsilon \tag{4.8}
\]

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and then we find a set \( T(\omega_r - \omega) \) as above. For \( r > 0 \) and \( \epsilon > 0 \) we find that for all \( C_\alpha \in P_{r,t} \) and all \( Q \in C^{(k)}_\alpha \) there holds
\[
\int_Q |i_{Q}(\omega_r - \hat{\omega})| \leq r^{k+2} \| \nabla \omega_r \|_{L^\infty}. \quad (4.9)
\]
Now we note that the relation
\[
R_{r,t} := \{ (Q, C_\alpha) \in P^{(k)}_{r,t} \times P_{r,t} : Q \in C^{(k)}_\alpha \}
\]
can be partitioned into \( 2^{n-k} \) relations \( R^\beta_{r,t}, \beta \in \{1, \ldots, 2^{n-k}\} \) such that for each such \( R^\beta_{r,t} \) and for each \( Q \in P^{(k)}_{r,t} \) there exists at most one choice of \( C_\alpha \in P_{r,t} \) such that \( (Q, C_\alpha) \in R^\beta_{r,t} \). Let \( Q_\beta \) be the set of such \( Q \) such that one such \( C_\alpha \) exists. We then find that
\[
\sum_{C_\alpha \in P_{r,t} \cap C^{(k)}_\alpha} \sum_{Q \in Q_\beta} \int_Q |i_{Q}^* (\omega - \hat{\omega})|^2 = \sum_{\beta} \sum_{Q \in Q_\beta} \left( \sum_{\alpha \in \mathcal{A}(Q,C_\alpha) \in R^\beta_{r,t}} \int_Q |i_{Q}^* (\omega - \hat{\omega})|^2 \right). \quad (4.10)
\]
We find from (4.8) via Jensen’s inequality and since \( |i_{Q}^* \gamma|^2 \leq |\gamma|^2 \) if \( \gamma \) is a constant form, that for each \( \beta \)
\[
\sum_{\beta} \sum_{Q \in Q_\beta} \left( \sum_{\alpha \in \mathcal{A}(Q,C_\alpha) \in R^\beta_{r,t}} \int_Q |i_{Q}^* (\omega - \hat{\omega})|^2 \right) \leq \epsilon. \quad (4.11)
\]
Now by the inequality \( (a + b + c)^2 \leq 9(a^2 + b^2 + c^2) \) applied to the integrals over each \( Q \) appearing above and using (4.8) together with (4.7a) for \( \omega - \omega_r \), (4.9), (4.11) and (4.10), we obtain that for \( t \in T(\omega) \cap T(\omega - \omega_r) \) there holds
\[
\sum_{C_\alpha \in P_{r,t}} \sum_{Q \in Q_\beta} \int_Q |i_{Q}^* (\omega - \hat{\omega})|^2 \leq 9 \left[ (2^{n-k} + 1) C_1 r^{k-n} \epsilon + r^{k+2} \| \nabla \omega_r \|_{L^\infty} \| \omega \|_{L^\infty} \#R_{r,t} \right]
\]
\[
\leq r^{k-n} \| \nabla \omega_r \|_{L^\infty} \#C_\alpha \quad (4.12)
\]
where in the last inequality we estimate \( \#R_{r,t} \leq 2^{n-k} r^{-n} \) because \( \#P_{r,t} \leq r^{-n} \), and \( \#C^{(k)}_\alpha \leq 2^{n-k} \) for each \( C_\alpha \in P_{r,t} \).

We now fix \( C_1 \) such that \( 2C_3 < 1 - \delta \). This ensures that
\[
\| T(\omega) \cap T(\omega - \omega) \| > \delta r^n.
\]
Consider now (4.12): By fixing \( \epsilon > 0 \) small and then choosing \( r \lesssim \epsilon \| \nabla \omega_r \|^{-1}_{L^1} \), we find that the term multiplying \( r^{k-n} \) on the right can be made arbitrarily small for small \( r \), thus there exists \( o_{\omega,\delta}(r) \) such that \( o_{\omega,\delta}(r) \to 0 \) as \( r \to 0 \) and such that (4.7b) holds. We thus choose \( T_{r,\delta}(\omega) = T(\omega) \cap T(\omega - \omega_r) \), and the properties (4.7) hold, as desired.

### 4.3 Good cubes and bad cubes

Later on we will to apply Proposition 3.1 on \( r \)-dilations of \( k \)-faces of our good cubeulation of scale \( r \). Thus the hypotheses of the Proposition will have to hold for all \( k \leq n \), and the estimates (3.3a), (3.3b) should be fitting the same bounds as in the gauge extraction Theorem 2.1 for \( k + 1 \). We are lead to consider "good" those \( k \)-faces of our good cubeulation on which these iterative criteria are feasible. The fact that in Proposition 3.1 the bounds for the replacements \( \hat{A}, F \) are controlled in terms of the ones for \( i_{\partial D}^* \hat{A}, i_{\partial D}^* \hat{F} \) allows us to just require bounds once, in the starting dimension \( 4 \).

**Definition 4.6 (good \( k \)-face).** Let \( A, F \) be respectively a connection and a curvature form like in (4.2). Fix a scale-\( r \) good cubeulation \( P_{r,t} \).
We define all faces of dimension 3 or lower to be good. Let \( C \subseteq P_{r,1}^{(k)} \), \( k \geq 4 \). We say that \( C \) is a \( \delta \)-good \( k \)-face if for each \( n \)-dimensional cube \( C_\alpha := C_{r,t,\alpha} \subseteq P_{r,t} \) such that \( C \subseteq C_\alpha^{(k)} \), the following estimates hold for all \( C' \subseteq C_\alpha^{(k)} \):

\[
\begin{align*}
\frac{1}{2} \int_{C} |i_{C'}^{*} F| &\leq \delta r^{-1}, \\
\frac{1}{2} \int_{C'} |i_{C}^{*} A| &\leq \delta r^{-1}, \\
\int_{C'} |i_{C'}^{*} F|^2 &\leq \delta, \\
\int_{C'} |i_{C'}^{*} (F - F_{C_\alpha})|^2 &\leq \delta, \\
\int_{C'} |i_{C'}^{*} (A - A_{\overline{C_\alpha}})|^2 &\leq \delta r^2.
\end{align*}
\]

(4.13)

If \( C \) is not a \( \delta \)-good \( k \)-face then we call it a \( \delta \)-bad \( k \)-face.

Note that the above conditions are scale-invariant.

The following direct consequence of the above definition and of Proposition 4.5 will be used in order to allow a dominated convergence argument within the "rough approximation" step of our proof.

**Lemma 4.7.** If \( P_{r,1} \) is a good cubulation for \( A,F \) then the total number \( N_\delta \) of \( \delta \)-bad \( n \)-faces satisfies for \( r > 0 \) small enough, depending on \( A,F \),

\[
N_\delta \lesssim \frac{\|F\|^2_{L^2([-1,1]^n)}}{\delta r^{n-4}} + \frac{\|A\|^2_{L^2([-1,1]^n)}}{\delta^2 r^{n-2}} + \frac{1}{\delta r^{n-4}},
\]

(4.14)

in particular the total volume of all bad \( n \)-cubes \( r^n N_\delta \) vanishes as \( r \to 0 \).

## 5 The strong approximation theorem

In this section we prove that forms \( F_A \) corresponding to \( A \in \tilde{A}_G([-1,1]^n) \) can be strongly approximated up to gauge by smooth curvatures on bundles with controlled defects, i.e. by elements of the following space:

\[
\mathcal{R}^\infty([-1,1]^n) := \left\{ A \text{ connection form s.t. there exists a polyhedral set } \Sigma^{n-5} \subset [-1,1]^n, \right. \\
\left. \text{s.t. } A = A_\nabla \text{ for a smooth connection } \nabla \text{ on some smooth } G\text{-bundle } E \to M^n \setminus \Sigma^{n-5} \right\}.
\]

(5.1)

In our construction the set \( \Sigma^{n-5} \) is the union of a finite number of intervals of dimension \( n - 5 \) parallel to the coordinate directions.

**Remark 5.1.** Equivalent to having a smooth connection \( \nabla \) as above is to have a smooth presheaf. This means that we have a good cover \( \{U_\alpha\} \) of \( M^n \setminus \Sigma^{n-5} \) and smooth connection forms \( A_\alpha \in C^\infty(U_\alpha, \Lambda^1 U_\alpha \otimes g) \) related by smooth changes of gauges \( g_{\alpha\beta} \in C^\infty(U_\alpha \cap U_\beta, G) \) such that \( A_\alpha = (A_\beta)_{g_{\alpha\beta}}. \) See [11] for more discussion on the presheaf point of view on classical connections, and Sections 1.2.4 and Appendix A of [17] for the description of the above realization map in 4-dimensions. By a reasoning completely analogous to [17, App. A] it is possible to obtain the existence of classical bundles based only on our locally-\( L^p \)-connection forms related by \( W^{1,n} \)-gauges, as obtained in (5.18) over our good cubes.

The result which we prove is the following:

**Theorem 5.2.** If \( A \in \tilde{A}_G([-1,1]^n) \) then there exists a sequence of connection forms \( A_j \in \mathcal{R}^\infty([-1,1]^n) \) with connection forms \( F_j := F_{A_j} = dA_j + A_j \wedge A_j \) such that there exist a sequence of gauge changes \( g_j \in W^{1,2}([-1,1]^n, G) \) for which, as \( j \to \infty \), there holds

\[
\| g_j^{-1} dg_j + g_j^{-1} A_j g_j - A \|^2_{L^2([-1,1]^n)} \to 0, \quad \| g_j^{-1} F_j g_j - F_A \|^2_{L^2([-1,1]^n)} \to 0.
\]
We will construct our approximants by successive extensions starting with the restriction of a starting \( A \in \mathcal{A}_G([-1,1]^n) \) to the support of the 4-skeleton of a well-chosen cubeulation of \([-1,1]^n\). Once the above result is proved, in order to pass to the situation of a general compact Riemannian manifold \((M^n, h)\), it will suffice to approximate the connections locally in coordinate charts, and to use the fact that the coordinate transformations of change of chart, or the presence of a \(C^1\)-regular metric \( h \) do not alter the \(W^{1,k}\)-bounds which appear throughout the proof, and the final mollification away from \( \Sigma^{n-5} \) can be performed in the same way.

We note here that the theory of Sobolev presheaves as in [10], [11] can be used in order to link the setting of weak connections treated here to that of classical connections, like explain in the appendix of our paper [17] about the 5-dimensional case. In particular the same reasoning shows that having smooth connection 1-forms on local charts directly allows to create a principal bundle such that these 1-forms are the differential-geometric connection forms of a connection on the associated bundle for the adjoint representation. We do not delve onto this topic in this paper, and we refer the interested reader to the above-cited works instead.

We next set up the proof of Theorem 5.2.

5.1 Notations and framework

For the whole proof, we will use a small parameter \( \delta_0 > 0 \), whose choice will be precised during the proof, and will depend only on \( n \) and \( G \).

5.1.1 Choice of a cubelation, good and bad cubes

We choose a cubeulation \( P_{r,t} \) at scale \( r > 0 \), such that

- \( P_{r,t} \) satisfies (4.7) contemporarily for \( \omega = A \) and for \( \omega = F_A \),
- all hyperplane families \( \mathcal{F}_{r,I,t} \) with \( \#I = n - 4 \) are composed exclusively of planes such that the good gauges as assumed in (4.2) exist. In particular we have a good \( L^4 \)-gauge on all 4-faces of the relative boundary of 5-faces in \( P_{r,t}^{(5)} \),
- for any 4-plane \( H(I,T) \) which intersects some face of \( P_{r,t}^{(4)} \), condition (4.3) holds, i.e. there exists a gauge \( g(I,T) \) on \( H(I,T) \) chosen, such that \( (i_{H(I,T)}^* A)^g(I,T) \) is \( L^4 \).

In order to obtain the existence of such \( P_{r,t} \), we first apply Proposition 4.5 separately with the choices \( \omega = A \) and \( \omega = F_A \), and obtain good sets of parameters which we may denote \( T_{r,A} \) and \( T_{r,F} \), respectively. If \( \delta > 1/2 \) then we find that \( T_{r,A} \cap T_{r,F} \) has positive volume, and then any parameter \( t \) in this intersection satisfies the properties (4.7) for both \( A \) and \( F_A \). Then, using the definition 4.2 and Remark 4.3, we find that some such good \( t \) is such that each 4-plane which enters the definition of \( P_{r,t}^{(4)} \) also satisfies (4.3).

We then fix the cubeulation \( P_{r,t} \) as above. We will call an element of \( C_{r,t}^{(n)} \) a good cube provided it satisfies (for \( k = n \), the above choice of \( \delta_0 > 0 \) and for our present weak connection form \( A \) the conditions of Definition 4.6, i.e. if (4.13) holds). Any \( n \)-cube from \( C_{r,t}^{(n)} \) which is not good will be called a bad cube.

5.1.2 Bilipschitz parameterizations in intermediate dimensions

We fix bilipschitz parametrizations

\[
\Psi_k : \mathbb{B}^k \to [-1,1]^k. \tag{5.2}
\]
Then for each coordinate $k$-face $C_r \in F_r^{(k)}$ we will denote

$$C_r = \tau_{C_r} \circ \delta_r(C_1), \quad k = 5, \ldots, n.$$ \hfill (5.3)

where $\tau_{C_r}$ is a translation sending the origin to the center of $C_r$ and $\delta_r$ is a dilation by a factor of $r$. We then use a parameterization

$$\Psi_{C_1} = \Psi_k \circ R_{C_1}, \text{ where } R_{C_1} \in SO(n).$$

The estimates in our proof will not depend on the precise choices of parameterisation effectuated at this stage, and only the Lipschitz constants of the intervening maps and of their inverses will be relevant.

We may also assume that if $C_\alpha$ is a $k$-face of a cube $C_\beta$, i.e. $C_\alpha \in C_\beta^{(k)}$, then $\Psi_{C_\beta}|_{C_\alpha} = \Psi_{C_\alpha}$. Denote by $\lambda_k$ the bi-lipschitz norm of $\Psi_k$.

### 5.2 Proof of the approximation Theorem 5.2

The proof of Theorem 5.2 will proceed through the following steps:

1. We start with local gauges in which our connection is $L^4$-integrable on the 4-skeleton $P_{tr}$.

2. With a suitable choice of $\delta$, on the $\delta$-good $k$-dimensional faces, iteratively with respect to $k \geq 4$, we extend the connection forms from the boundaries of $(k+1)$-dimensional cells to the interiors, via Theorem 3.1.

3. On the $\delta$-bad $k$-dimensional faces, we extend radially, again iteratively for $k \geq 4$.

4. At the end of the extension we are able mollify our connections outside a 5-dimensional polyhedral set (which is the support of the dual skeleton to the complex of bad cubes), providing the approximation bounds as required in the statement of the theorem.

#### 5.2.1 Step 1: $L^4$-connections locally on the 4-skeleton

We start with the following result, which we need for setting up the gauges defined on the faces on our skeleton:

**Proposition 5.3 (Controlled gauges with Dirichlet boundary datum).** Assume that $A \in L^2(B^n, \wedge^1 B^n \otimes \mathfrak{g})$ and that there exist $g \in W^{1,2}(B^n, G)$ such that $A^g \in L^n(B^n, \wedge^1 B^n \otimes \mathfrak{g})$, and suppose that the curvature form $F_A$ satisfies $\|F_A\|_{L^{n/2}(B^n)} < \epsilon_0$. Then there exists a gauge change $g \in W^{1,2}(B^n, G)$ such that $A^g \in L^4(B^n, \wedge^1 B^n \otimes \mathfrak{g})$, $g|_{\partial B^n} \equiv id$ and $\|A^g\|_{L^4(B^n)} \lesssim \|F_A\|_{L^{n/2}(B^n)}$.

The above result is the same as the main result of Uhlenbeck [24], with the two differences that we work with $A$ of regularity $L^n$ rather than $W^{1,n/2}$ and that we impose on our gauges $g$ the Dirichlet boundary condition rather than the Neumann one. This can be directly implemented in the proof (as presented in [20, Thm. IV.4]) by treating the linearized operator between the so-defined spaces directly, without further essential modifications.

The application of Proposition (5.3) gives the following result. Note that the term “good cover” means that the maximal number of sets from the cover that overlap at any given point is finite.

**Corollary 5.4 (Finding $L^4$-connections on 4-faces).** Assume that $C^4$ is a 4-face of our skeleton. Then exists a finite good cover $\{U_\alpha\}_\alpha$ of $C^4$ by sets $U_\alpha$ that are bi-lipschitz equivalent to $B^4$ and gauge change maps $g_\alpha \in W^{1,2}(U_\alpha, G)$ such that for all $\alpha$ we have $i_{U_\alpha}^* A^g_\alpha \in L^4(U_\alpha, \wedge^1 U_\alpha \otimes \mathfrak{g})$, $g_\alpha|_{U_\alpha \cap \partial C^4} \equiv id$ and $\|i_{U_\alpha}^* A^g_\alpha\|_{L^4(U_\alpha)} \lesssim \|i_{U_\alpha}^* F_A\|_{L^2(U_\alpha)}$.

Moreover if $C^4$ is a good cube, then the above holds already for the trivial cover formed by only $C^4$ itself.
Proof. From the definition of a good cube, we use here only the property that \( A \in L^2(C^4, \wedge^1 C^4 \otimes \mathfrak{g}) \) and that there exists \( g \in \mathcal{W}^1,2(C^4, G) \) such that \( A^g \in L^2(C^4, \wedge^1 C^4 \otimes \mathfrak{g}) \).

From the definition of a good cube we only need that \( \|i_{C^4}F_A\|_{L^2(C^4)} \leq \delta \) for \( \delta \leq C\epsilon_0 \), where \( C = (\text{Lip} \Psi_0)^4(\text{Lip}(\Psi_0^{-1}))^4 \) is a geometric constant depending only on the bi-lipschitz constant of the map that identifies \( U_\alpha \) to a ball – and can be bounded independently on our choice of cover – and \( \epsilon_0 \) is as in Proposition 5.3.

We may find a finite clopen cover \( \{U_\alpha\}_\alpha \) of \( C^4 \), such that each \( U_\alpha \) is itself bilipschitz-equivalent to a ball \( B^4 \) via a map \( \Psi_\alpha : B^4 \to U_\alpha \) and that \( \int \Psi_\alpha^{-1}(U_\alpha) |\Psi_\alpha^*F_\alpha|^2 \leq \epsilon_0 \).

Then we can identify \( B^4 \) to the unit ball \( \mathbb{B}^4 \) by dilation, and this change of coordinates in 4-dimensions leaves the \( L^2 \)-norm of \( F \) unchanged.

Note that if \( C^4 \) is a good cube, then we can take the trivial covering \( \{C^4\} \), because the smallness condition on \( F \) is already satisfied for \( \delta \) in (4.13) chosen as in the beginning of the proof.

Then we use Proposition 5.3 and transfer the result to \( U_{j,\alpha} \) via \( \Psi_{U_{j,\alpha}} \) in order to find local gauges \( g_{j,\alpha} \in \mathcal{W}^1,2(U_{j,\alpha}, G) \) such that \( g_{j,\alpha}|_{\partial U_{j,\alpha}} \equiv \text{id} \) and \( A^{g_{j,\alpha}} \in L^2(U_{j,\alpha}, \wedge^1 U_{j,\alpha} \otimes \mathfrak{g}) \). In particular, we find that \( g_{\alpha}|_{U_{\alpha} \cap \partial C^4} \equiv \text{id} \), as desired.

Next, we proceed by induction on the skeleton, using the following Lemma:

**Lemma 5.5 (gluing gauges).** Assume that \( C^{k+1} \) is a \((k+1)\)-dimensional cube, with \( k \geq 4 \).

For each one of its \( k \)-dimensional faces \( C^k_\alpha \in (\partial C^{k+1})^{(k)} \), let \( g_\alpha \in \mathcal{W}^1,2(C^k_\alpha, G) \) and \( A_\alpha \in L^2(C^k_\alpha, \wedge^1 C^k_\alpha \otimes \mathfrak{g}) \) be such that \( (A_\alpha)^{g_\alpha} \in L^2(C^k_\alpha, \wedge^1 C^k_\alpha \otimes \mathfrak{g}) \) and such that whenever \( C^k_\alpha, C^k_\beta \in (\partial C^{k+1})^{(k)} \), then \( g_\alpha = g_\beta \) on \( C^k_\alpha \cap C^k_\beta \). If we define

\[
\begin{align*}
\Psi_{\partial C^{k+1}} := & \sum_{C^k_\alpha \in (\partial C^{k+1})^{(k)}} 1_{C^k_\alpha} g_\alpha, \\
A_{\partial C^{k+1}} := & \sum_{C^k_\alpha \in (\partial C^{k+1})^{(k)}} 1_{C^k_\alpha} A_\alpha,
\end{align*}
\]

then \( g_{\partial C^{k+1}} \circ \Psi_{k+1} \in \mathcal{W}^1,2(S^k, G) \), \( (A_{\partial C^{k+1}} \circ \Psi_{k+1}) \in L^2(\partial C^{k+1}, \wedge^1 \partial C^{k+1} \otimes \mathfrak{g}) \), and there holds

\[
\Psi_{k+1}A_{\partial C^{k+1}} \circ \Psi_{k+1} \in L^k(S^k, \wedge^2 S^k \otimes \mathfrak{g}).
\]

**Proof.** The fact that \( d(g_{\partial C^{k+1}} \circ \Psi_{k+1}) \) is \( L^2 \) as desired follows by integration by parts, using the fact that \( g_\alpha = g_\beta \) on the intersection of their domains. The property (5.5) follows by applying the chain rule to (5.4), and using the fact that the normals to the common boundary of neighboring regions \( \Psi_{k+1}(C^k_\alpha), \Psi_{k+1}(C^k_\beta) \) cancel each other.

### 5.2.2 Step 2: Extension on the good skeleton

As the extension done in this step of the proof will be by iteration on the dimension, starting from the case of 4-faces, on which the connections that we create are equal to the original one, and then we replace the initial connection iteratively on the interiors of \((k+1)\)-faces for \( 4 \leq k \leq n-1 \). The extension from 4-faces to 5-faces is slightly different than the general step, because it uses the conditions (4.13) directly, instead of using bounds obtained from previous steps. Therefore we explicitly present the following passages

- the first step of the induction, i.e. the extension from 4-faces to 5-faces,
- the passage from \( k \)-faces to \((k+1)\)-faces for general \( 5 \leq k \leq n-1 \),
- the final bound obtained for the \( n \)-faces.

**Step 2.0: Preparation.** As our skeleton is already chosen we will denote it \( P \) rather than \( P_{\alpha,t} \) for the rest of the proof. We first note that since the conditions (4.13) are dilation-invariant, we may, without loss of generality assume that \( r = 1 \) up to dilation.
Step 2.1: Base for the inductive extension. Note that by applying Corollary 5.4 on all the faces of our skeleton $P^{(3)}$, we obtain for each good 4-face $C_j$, a gauge $g_{C_j}$ such that $A^{5}_{C_j} \in \mathbb{L}^1(C_j, \Lambda^4 \otimes \mathfrak{g})$, $g_{C_j} \bigg|_{\partial C_j} \equiv id$ and $\|A^{5}_{C_j}\|_{L^1(C_j)} \lesssim \|F\|_{L^2(C_j)}$. Then by using Lemma 5.5 we find for each good 5-face $C_j$ a gauge $g_{\partial C_j} \in \mathbb{L}^{1,2}(\partial C_j, \mathfrak{g})$ such that $(i_{\partial C_j}^* A)^{g_{\partial C_j}^5} \in \mathbb{L}^1(\partial C_j, \Lambda^4 \otimes \mathfrak{g})$ and

$$\left\| \left( i_{\partial C_j}^* A \right)^{g_{\partial C_j}^5} \right\|_{L^1(\partial C_j)} \lesssim \left\| i_{\partial C_j}^* F \right\|_{L^2(\partial C_j)}.$$  

(5.6)

Step 2.2: First step of the extension, from 4-faces to 5-faces. We next apply Theorem 3.1 with the following choices (indicated by “$\mapsto$”):

- $n \mapsto 4$,
- $D \mapsto C_\delta$ and $\Psi_D \mapsto \Psi_{C_\delta}$ as defined in Section 5.1.2,
- $F \mapsto F_{C_\delta}$ and $\alpha \mapsto \alpha_{C_\delta}$, where we denote

$$F_{C_\delta} := \frac{1}{10} \sum_{C_j \in (C_\delta)'} \frac{1}{|C_j|} \int_{C_j} i_{\partial C_j}^* F,$$

$$\alpha_{C_\delta} := \frac{1}{10} \sum_{C_j \in (C_\delta)'} \frac{1}{|C_j|} \int_{C_j} i_{\partial C_j}^* \alpha,$$  

(5.7)

- $A \mapsto i_{\partial C_j}^* A$ and $F \mapsto i_{\partial C_j}^* F$,
- $g \mapsto g_{\partial C_j}$, described above.

With the above choices, we can verify that the bilipschitz constant $CD$ from Theorem 3.1 is replaced by the bilipschitz constant of $\Psi_4$ fixed depending only on $k = 5$ in Section 5.1.2, and the other hypotheses are verified as follows:

- The condition (3.1) on $g \mapsto g_{\partial C_j}$ is verified by (5.6), and the bound on $A^\delta \mapsto \left( i_{\partial C_j}^* A \right)^{g_{\partial C_j}^5}$ as required in (3.2) follows from (5.6) and from the bound on $F$ in (3.2).

- The bounds (3.2) on $F \mapsto F_{C_\delta}$ and $\alpha \mapsto \alpha_{C_\delta}$ follow by the conditions (4.13) valid in the case $k \mapsto 5$ and with $C \mapsto C_j$, due to the definitions 5.7 and by triangle inequality, provided we have $\delta < \delta^{(4)}$ for a constant $\delta^{(4)}$ which will depend only on the bilipschitz constant of $\Psi_5$, and on the value of $\epsilon_0 := \epsilon^{(4)}_0$ appearing in Theorem 3.1 in which we chose $n \mapsto 4$.

- The bounds (3.2) on $F \mapsto F_{\partial C_j}$ and on $A \mapsto A_{\partial C_j}$ follow from the analogous bounds that appear in (4.13) with the choices $k \mapsto 5$ and $C \mapsto C_j$, by triangle inequality, up to diminishing $\delta^{(4)}$ by a combinatorial factor of 10, equal to the number of 4-faces of $C_j$.

As an outcome of Theorem 3.1, we find forms $\hat{A} \in \mathbb{L}^2(C_\delta, \Lambda^4 \otimes \mathfrak{g})$, $\hat{F} = d\hat{A} + \hat{A} \wedge \hat{A} \in \mathbb{L}^2(C_\delta, \Lambda^2 C_\delta \otimes \mathfrak{g})$ and a gauge $\hat{g} \in \mathbb{L}^{1,2}(C_\delta, \mathfrak{g})$ which we rename as

$$A_{C_j} := \hat{A}, \quad A_{C_\delta} := \hat{F}, \quad g_{C_j} := \hat{g},$$

and which satisfy the boundary conditions

$$i_{\partial C_j}^* A_{C_j} = i_{\partial C_j}^* A, \quad i_{\partial C_j}^* F_{C_\delta} = i_{\partial C_j}^* F, \quad g_{C_j} \big|_{\partial C_j} = g_{\partial C_j}.$$  

(5.8)
With the above notations the bounds (3.3) then translate into the following, in which we used (5.6) to absorb the connection contributions into curvature terms:

\[ \|F_{C_j} - \hat{F}_{C_j}\|_{L^2(C_j^5)} \leq \frac{1}{10} \sum_{C_i^4 \in (C_j^5)} \|\hat{r}^2_{C_i^4} F - \hat{F}_{C_i^4}\|_{L^2(\partial C_i^4)} \]  

\[ + \frac{1}{10} |\hat{F}_{C_j}| \sum_{C_i^4 \in (C_j^5)} \|\hat{r}^2_{C_i^4} A - \hat{A}_{C_i^4}\|_{L^2(\partial C_i^4)} \]  

\[ \|A_{C_j} - \hat{A}_{C_j}\|_{L^2(C_j^5)} \leq \frac{1}{10} \sum_{C_i^4 \in (C_j^5)} \|\hat{r}^2_{C_i^4} A - \hat{A}_{C_i^4}\|_{L^2(\partial C_i^4)}, \]  

\[ \|F_{C_j}\|_{L^2(C_j^5)} \leq \|\hat{F}_{A_{C_j}}\|_{L^2(\partial C_j^5)} + |\hat{F}_{C_j}|, \]  

\[ \left( A_{C_j^5} \right)^{g_{C_j}} \in L^5(C_j^5, \Lambda^1 \mathbb{R}^5 \otimes \mathfrak{g}), \]  

where we also used the triangle inequality and the formulas (5.7).

By performing the above extension over the interiors of all \(\delta^{(4)}\)-good 5-faces \(C_j^5\), we conclude the first step of our iterative extension.

**Step 2.2': Preparation for the extension to 6-faces.** We now fix a 6-face \(C_j^6\) then, due to the condition (5.8) on the \(g_{C_j^5}\)'s, we find that \(g_{C_j^5} = g_{C_j^5}\) on \(C_j^5 \cap C_j^6\) for all \(j, j'\), thus we can apply Lemma 5.5 with \(k \mapsto 5, C_j^5 \mapsto C_j^6, C_k^{k+1} \mapsto C_k^6, g_j \mapsto g_{C_j^6}\) and \(A \mapsto \sum_j 1_{C_j^5} A_{C_j^5}\). With these choices the hypotheses of the lemma are valid due to property (5.9c). Then the lemma gives as an output a gauge \(g_{\partial C_j^6} \in W^{1,2}(\partial C_j^6, G)\) and a connection form \(A_{\partial C_j^6} \in L^2(\partial C_j^6, \Lambda^1 \partial C_j^6 \otimes \mathfrak{g})\) such that

\[ \left( A_{\partial C_j^6} \right)^{g_{\partial C_j^6}} \in L^5(\partial C_j^6, \Lambda^1 \partial C_j^6 \otimes \mathfrak{g}), \]  

which allows to start the next step in the iteration.

**Step 2.4: General step of the extension, from \(k\)-faces to \((k+1)\)-faces for \(5 \leq k \leq n - 1\).** Fix a \((k+1)\)-face \(C_j^{k+1}\). After the extension on \(k\)-faces we have for each \(k\)-face \(C_j^k \in (C_j^{k+1})^{(k)}\) a connection form \(A_{C_j^k} \in L^2(C_j^k,\Lambda^1 C_j^k \otimes \mathfrak{g})\) whose curvature form \(F = dA_{C_j^k} + A_{C_j^k} \wedge A_{C_j^k}\) satisfies \(F \in L^2(C_j^k,\Lambda^2 C_j^k \otimes \mathfrak{g})\) and constant forms \(\hat{A}_{C_j^k} \in \Lambda^1 \mathbb{R}^k \otimes \mathfrak{g}\) and \(\hat{F}_{C_j^k} \in \Lambda^2 \mathbb{R}^k \otimes \mathfrak{g}\) such that the following bounds, generalizing (4.13) to faces of dimension \(k > 4\), hold. The bounds are dilatation-invariant, but we present them in the version valid at general scale \(r\), for clarity:

\[ |\hat{F}_{C_j^k}| \leq \delta^{(k)} r^{-2}, \quad |\hat{A}_{C_j^k}| \leq \delta^{(k)} r^{-1}, \quad \int_{C_j^k} |F_{C_j^k}|^2 \leq \delta^{(k)}. \]  

At this point again we may reduce to scale \(r = 1\) by dilatation invariance. We present the justification of the above bounds (5.11) for the case \(k = 5\) with \(r = 1\).

- The bounds on \(|\hat{F}_{C_j^k}|\) and on \(|\hat{A}_{C_j^k}|\) follow under the condition \(\delta < \delta^{(5)}\) from the definition (5.7) and (4.13), by triangle inequality, due to the fact that \(C_j^{k+1} = C_j^5\) is in this a \(\delta\)-good 6-face.

- The bound on \(\|F_{C_j^k}\|^2_{L^2(C_j^5)}\) in (5.11) follows from (5.9d) provided \(\delta^{(5)} \geq C_4 \delta^{(4)},\) where \(C_4\) is a combinatorial constant, which in this case can be taken to be equal

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to $441C_{5.9c}^2$, where $C_{5.9d}$ is the implicit constant appearing in (5.9d). Then (5.9d) together with the bound on $|F_{C^2_k}|$ already discussed above, and the bounds on $A_{\partial C^2_k}$ and $F_{\partial C^2_k}$ which come from (4.13) by triangle inequality. As the number of 4-faces of $C^2_k$ is 10, we find indeed that

$$\|F_{C_k^2}\|_{L^2(C^2_k)} \leq C(5.9d) \left[ \sum_{C^2_k \in (\partial C^2_k)(4)} \|F_{C^2_k}\|_{L^2(C^2_k)} + \sqrt{\delta(4)} \right] \leq 21C(5.9d) \sqrt{\delta(4)}.$$

- The bound on $\|A_{C_k^2}\|_{L^2(C_k^2)}^2$ in (5.11) follows using (5.9b), and the already-discussed bound for $\mathcal{A}_{C_k^2}$. This term is bounded by $\delta(5) \geq (C(5.9b)+1)^2 \delta(4)$ in which the constant $\delta(4)$ comes from (4.13) and $C(5.9b)$ is the implicit constant appearing in (5.9b). The bound is as follows:

$$\|A_{C_k^2}\|_{L^2(C_k^2)} \leq \|A_{C_k^2} - \mathcal{A}_{C_k^2}\|_{L^2(C_k^2)} + \|\mathcal{A}_{C_k^2}\|_{L^2(C_k^2)} \leq C(5.9b) \max_{C^2_k \in (\partial C^2_k)(4)} \|i_{C^2_k} A - \mathcal{A}_{C^2_k}\|_{L^2(C^2_k)} + \sqrt{\delta(4)} \leq (C(5.9b)+1) \sqrt{\delta(4)}.$$

For general $5 \leq k \leq n-1$ we then assume that (5.11) holds for all $C^k_\alpha \in (C^{k+1}_j)^{(k)}$ and by using the same reasoning as in Step 2.2, we define

$$A_{\partial C^k_{j+1}} := \sum_{C^k_\alpha \in (C^{k+1}_j)^{(k)}} 1_{C^k_\alpha} A_{C^k_\alpha}, \quad (5.12)$$

whose associated curvature form is then

$$F_{\partial C^k_{j+1}} := dA_{\partial C^k_{j+1}} + A_{\partial C^k_{j+1}} \wedge A_{\partial C^k_{j+1}} = \sum_{C^k_\alpha \in (C^{k+1}_j)^{(k)}} 1_{C^k_\alpha} F_{C^k_\alpha}.$$

By using Lemma 5.5 together with the conclusion from the preceding step of our iteration (which we proved to hold for $k = 5$ and follows from (5.16c) below for the case $k \mapsto k+1 > 5$), we find that there exists $g_{\partial C^k_{j+1}} \in L^2(\partial C^k_{j+1}, G)$ such that $\left(A_{\partial C^k_{j+1}}\right)^{g_{\partial C^k_{j+1}} \in L^2(\partial C^k_{j+1}, \wedge^1 \partial C^k_{j+1} \otimes \mathfrak{g})}$

$$\left\|\left(A_{\partial C^k_{j+1}}\right)^{g_{\partial C^k_{j+1}}\in L^2(\partial C^k_{j+1})}\right\|_{L^2(\partial C^k_{j+1})} \lesssim \|F_{\partial C^k_{j+1}}\|_{L^2(\partial C^k_{j+1})}.$$  

At this point we may apply Theorem 3.1 with the following choices:

- $n \mapsto k$,
- $D \mapsto C^{k+1}_j$ and $\Psi_D \mapsto \Psi_{C^{k+1}_j}$ as defined in Section 5.1.2,
- $\mathcal{F} \mapsto \mathcal{F}_{C^{k+1}_j}$ and $\mathcal{A} \mapsto \mathcal{A}_{C^{k+1}_j}$, where we denote

$$\mathcal{F}_{C^{k+1}_j} := \frac{1}{2(k+1)} \sum_{C^k_\alpha \in (C^{k+1}_j)^{(k)}} \mathcal{F}_{C^k_\alpha},$$
$$\mathcal{A}_{C^{k+1}_j} := \frac{1}{2(k+1)} \sum_{C^k_\alpha \in (C^{k+1}_j)^{(k)}} \mathcal{A}_{C^k_\alpha}. \quad (5.14)$$
• \( A_{BD} \mapsto A_{BC_j^{k+1}} \) and \( F_{BD} \mapsto F_{BC_j^{k+1}} \).

• \( g \mapsto g_{BC_j^{k+1}} \), described above.

We now verify again that the hypotheses of Theorem 3.1 hold. We have \( C_D \mapsto \max\{\text{Lip}(\Psi_k), \text{Lip}(\Psi_k^{-1})\} \), fixed depending only on \( k \) in Section 5.1.2, and

• The condition (3.1) on \( g \mapsto g_{BC_j^{k+1}} \) was justified in (5.13).

• The bounds (3.2) on \( F \mapsto F_{BC_j^{k+1}} \) and \( A \mapsto \overline{A}_{BC_j^{k+1}} \) follow, by triangle inequality, from the definitions 5.14 and by the bounds in (5.11), provided we have \( \delta^{(k)} < \epsilon_0^{(k)} \) where \( \epsilon_0^{(k)} \) is the value of \( \epsilon_0 \) appearing in Theorem 3.1 if we chose \( n \mapsto k \).

• The bounds (3.2) on \( F \mapsto F_{BC_j^{k+1}} \) and on \( A \mapsto A_{BC_j^{k+1}} \) follow from the analogous bounds that appear in (5.11), by triangle inequality, up to diminishing \( \delta^{(k)} \) by a combinatorial factor of \( 2(k+1) \), equal to the number of \( k \)-faces of \( C_j^{k+1} \).

By applying Theorem 3.1, we find forms \( \hat{A} \in L^2(C_j^k, \Lambda^1 C_j^k \otimes g) \), \( \hat{F} = d\hat{A} + \hat{A} \wedge \hat{A} \in L^2(C_j^k, \Lambda^2 C_j^k \otimes g) \) and a gauge \( \hat{g} \in W^{1,2}(C_j^k, G) \) which we rename as

\[
A_{C_j^{k+1}} := \hat{A}, \quad A_{C_j^{k+1}} := \hat{F}, \quad g_{C_j^{k+1}} := \hat{g},
\]

and which satisfy the boundary conditions

\[
i^*_{BC_j^{k+1}} A_{C_j^{k+1}} = A_{BC_j^{k+1}}, \quad i^*_{BC_j^{k+1}} F_{C_j^{k+1}} = F_{BC_j^{k+1}}, \quad g_{C_j^{k+1}}|_{\partial C_j^{k+1}} = g_{BC_j^{k+1}}. \tag{5.15}
\]

Then the bounds (3.3) translate into:

\[
\|F_{C_j^{k+1}} - F_{BC_j^{k+1}}\|_{L^2(C_j^{k+1})} \lesssim \frac{1}{2(k+1)} \sum_{C_k^\alpha \in (C_j^{k+1})^{(k)}} \|i^*_{C_k^\alpha} F - F_{C_k^\alpha}\|_{L^2(\partial C_k^\alpha)}
\]

\[
+ \frac{1}{2(k+1)} \|F_{C_j^{k+1}}\| \sum_{C_k^\alpha \in (C_j^{k+1})^{(k)}} \|i^*_{C_k^\alpha} A - \overline{A}_{C_k^\alpha}\|_{L^2(\partial C_k^\alpha)}
\]

\[
+ \sum_{C_k^\alpha \in (C_j^{k+1})^{(k)}} \left( \|F_{C_k^\alpha}\|^2_{L^2(C_k^\alpha)} + \|F_{C_k^\alpha}\|^2 \right),
\]

\[
\|A_{C_j^{k+1}} - \overline{A}_{C_j^{k+1}}\|_{L^2(C_j^{k+1})} \lesssim \frac{1}{2(k+1)} \sum_{C_k^\alpha \in (C_j^{k+1})^{(k)}} \|i^*_{C_k^\alpha} A - \overline{A}_{C_k^\alpha}\|_{L^2(\partial C_k^\alpha)}, \tag{5.16a}
\]

\[
\|F_{C_j^{k+1}}\|_{L^\infty(C_j^{k+1})} \lesssim \sum_{C_k^\alpha \in (C_j^{k+1})^{(k)}} \left( \|F_{C_k^\alpha}\|_{L^\infty(C_k^\alpha)} + \|F_{C_k^\alpha}\| \right), \tag{5.16b}
\]

\[
\left(A_{C_j^{k+1}}\right)^{g_{C_j^{k+1}}} \in L^{k+1}(C_j^{k+1}, \Lambda^1 \mathbb{R}^{k+1} \otimes g), \tag{5.16c}
\]

where we also used the triangle inequality and the formulas (5.7).

**Step 2.5: Final bound for \( n \)-faces.** In conclusion, the bounds (5.16) together with the conditions (5.11) for \( k \geq 5 \) and (4.13) for \( k = 4 \), can be summed up to give the following:
Lemma 5.6 (estimate on a good \( n \)-face). There exists a constant \( \epsilon_0 \), depending only on the dimension, for any cubeulation \( P \) of scale \( r = 1 \), for each \( 0 < \epsilon < \epsilon_0 \) there exists \( \delta > 0 \), such that if (4.13) holds for such choice of \( \delta \) for a given \( n \)-face \( C^k_j \), then we may construct \( A_{C^k_j} \in L^2(C^k_j, \Lambda^i C^k_j \otimes g) \) and \( F_{C^k_j} := dA_{C^k_j} + A_{C^k_j} \wedge A_{C^k_j} \in L^2(C^k_j, \Lambda^i C^k_j \otimes g) \) which satisfy the bounds

\[
\|F_{C^k_j} - T_{C^k_j}\|_{L^2(C^k_j)} \lesssim \frac{1}{\epsilon_n} \sum_{C \in (C^k_j)^{(4)}} \|i_C^* F - \frac{1}{|C|} \int_C i_C^* F\|_{L^2(C)} + \frac{1}{\epsilon_n} |F_{C^k_j}| \sum_{C \in (C^k_j)^{(4)}} \|i_C^* A - \frac{1}{|C|} \int_C i_C^* A\|_{L^2(C)} + \sum_{C \in (C^k_j)^{(n)}} \|i_C^* F\|^2_{L^2(C)},
\]

(5.17a)

(5.17b)

where \( \epsilon_n := 2^{n-4} \left( \frac{n}{4} \right) \), and there exists a gauge \( g_{C^k_j} \in W^{1,2}(C^k_j, G) \) such that

\[
(A_{C^k_j} - g_{C^k_j}) \in L^n(C^k_j, \Lambda^i C^k_j \otimes g).
\]

(5.18)

5.2.3 Extension on the bad skeleton

We proceed by replacing \( A \) and \( F \) via an iterative procedure over the bad cubes. This will be performed via maps whose models, depending on the dimension \( 5 \leq k \leq n \), are denoted as follows:

\[
\pi^{(k)} : [-1,1]^k \setminus \{0\} \to \partial [-1,1]^k.
\]

(5.19)

The map \( \pi^{(k)} \) is assumed to belong to \( C^0([-1,1]^k \setminus \{0\}) \cap C^\infty([-1,1]^k \setminus \{0\}) \) and to be equal to the identity on \( \partial [-1,1]^k \). For the sake of concreteness, a possible explicit choice is \( \pi^{(k)}(x) := \frac{1}{|x_0|} \int_{|x_0|}^1 \rho_{\epsilon_0}(x-y)\pi^{(k)}(y)dy \) where \( \rho_{\epsilon_0}(x) := \frac{x_0}{|x_0|} \), \( x_0 := \max_{1 \leq j \leq k} |x_j| \) and \( \rho_{\epsilon_0}(x) = e^{-k\rho(x/\epsilon)} \) where \( \rho \) is a smooth positive radial function of integral 1 supported in \( \{ x \in \mathbb{R}^k : |x| \leq 1 \} \).

After composing with a suitable translation and rotation like in §5.1.2, we obtain smoothened radial projections

\[
\pi_{C^k_j} : C^k_j \setminus \{0\} \to \partial C^k_j,
\]

(5.20)

where \( \pi_{C^k_j} \) is the center of the \( k \)-face \( C^k_j \).

If now we consider the clopen set cover from Corollary 5.4, we can then construct from it a clopen cover of the union of all \( 4 \)-faces, still denoted by \( \{U_{\alpha}^{(4)}\}_{\alpha} \). We then extend the cover to higher-dimensional skeletons by defining iteratively

\[
\left\{ U_{\alpha}^{(k+1)} \right\}_{\alpha} := \left\{ \tilde{U}_{\alpha}^{(k+1)} := \left( \pi_{C^k_j} \right)^{-1} U_{\alpha}^{(k)} : U_{\alpha}^{(k)} \subset \partial C^k_j \right\}.
\]

With the above notation relating \( \tilde{U}_{\alpha}^{(k+1)} \) to \( U_{\alpha}^{(k)} \) and \( C^k_j \), we also iteratively define gauges \( g_{\tilde{U}_{\alpha}^{(k+1)}} \in W^{1,2}(\tilde{U}_{\alpha}^{(k+1)}, G) \) and connections \( A_{C_j^{k+1}} \in L^2(C_j^{k+1}, \Lambda^1 C_j^{k+1} \otimes g) \) by

\[
g^{\tilde{U}_{\alpha}^{(k+1)}} := g_{U_{\alpha}^{(k)}} \circ \pi_{C^k_j} \quad \text{and} \quad A_{C_j^{k+1}} := \left( \pi_{C^k_j} \right)^* A_{\partial C^k_j}.
\]

(5.21)

Where we use the same definition (5.12) as in the case of good cubes to define \( A_{\partial C^k_j} \).

We also consider the inclusion \( \tilde{i}_{\tilde{U}_{\alpha}^{(k+1)}} : \tilde{U}_{\alpha}^{(k+1)} \to C^k_j \) and define \( A_{\tilde{C}_j^{k+1}} := \tilde{i}_{\tilde{U}_{\alpha}^{(k+1)}} A_{C_j^{k+1}} \).
Then inductively from the condition described in Proposition 5.3 and Corollary 5.4, we find that there holds

\[ g_{U^{(t)}} = g_{U^{(t)}} \text{ on } U^{(t)} \cap U^{(t)} \text{ and } \left\{ \begin{array}{l}
A_{U^{(t)}} \in L^4(U^{(t)}, A, \Lambda^1 U^{(t)} \otimes g), \\
A_{U^{(t)}} = A_{U^{(t)}}.
\end{array} \right. \tag{5.22} \]

Moreover we have the bounds
\[ \|A_{C_j} \|_{L^2(C_j)} \leq c_n \sum_{C \in (C_j)^{(4)}} |i_C^* A| \|L^2(C)\|, \tag{5.24a} \]
\[ \|F_{C_j} \|_{L^2(C_j)} \leq c_n \sum_{C \in (C_j)^{(4)}} |i_C^* F| \|L^2(C)\|. \tag{5.24b} \]

Considering the above bounds together with the definition of \(A_{C_j} \) given in (5.12) and the triangle inequality, we find at stage \( n \) that there exists a constant \( c_n \) depending only on the implicit constants in (5.23), such that for each \( n \)-dimensional face \( C_j \) there holds
\[ \|A_{C_j} \|_{L^2(C_j)} \leq c_n \sum_{C \in (C_j)^{(4)}} |i_C^* A| \|L^2(C)\|, \tag{5.24a} \]
\[ \|F_{C_j} \|_{L^2(C_j)} \leq c_n \sum_{C \in (C_j)^{(4)}} |i_C^* F| \|L^2(C)\|. \tag{5.24b} \]

As a final step, after we have performed all our iterative extensions over all bad cubes, we describe more precisely what is the set over which the gauges and connection forms are not defined, which is also the set which remains not covered by the open sets \( \{U^{(n)} \}_n \). Before we proceed to the next lemma, we need some definitions.

**Definition 5.7 (dual skeleta).** Let \( r > 0 \), \( t \in \mathbb{R}^n \) and let \( S^{(k)} \) be a cube complex of dimension \( k \), generated by the cubes of the form

\[ C_j^k := C(r, J, c) := c - 1, 1] \cap \mathbb{Z}^n + c, \quad J \subset \{1, \ldots, n\}, \quad |J| = k, \quad c \in \mathbb{Z}^n + t. \tag{5.25} \]

Consider a subcomplex \( T^{(k')} \) of \( S^{(k)} \), of dimension \( k' \leq k \). Then the dual complex to \( T^{(k')} \) inside \( S^{(k)} \) is the complex \( T^{(k')} \) formed by cubes that can be written as

\[ \overline{C}^{k'} = C(r, J, c_t), \quad |J| \leq k - k', \quad c_t \in \mathbb{Z}^n + t, \]

and such that there exists a cube \( C(r, J, c_t) \in S^{(k)} \) with \( |J| = k \) and such that the cube \( C(r, J \setminus J_t, c_t) \) belongs to \( T^{(k')} \).

For our fixed \( \delta \) as in §5.2.2 we denote by

\[ P_{\delta}^{(k)} := \text{subcomplex of } P_{\delta} \text{ generated by all } \delta \text{-bad faces,} \tag{5.26} \]

and as we produce the extensions over (the support of) \( P_{\delta}^{(k)} \) of our connections via the iterative extension by the \( \pi_{C_j^k} \), \( 5 \leq k' \leq k \) as in (5.20), we define the singular set introduced in this way by

\[ \Sigma^{(k)} := P_{\delta}^{(k)} \setminus \bigcup_{\alpha} U^{(k)}_\alpha. \]

For \( A \subset \mathbb{R}^n, x \in \mathbb{R}^n \) we also denote by

\[ \text{Cone}(A, x) := \{(1 - t)a + tx : a \in A, t \in [0, 1]\}. \]

**Lemma 5.8 (structure of the introduced singular set).** The set \( \Sigma^{(k)} \cap P_{\delta}^{(k)} \) is the support of the dual complex in \( P_{\delta}^{(k)} \) to the one formed by the bad faces of dimension \( 5 \leq k' \leq k \).
Proof. The proof will proceed by induction:

- Initially, we have that all the 4-skeleton of bad cubes is covered by the sets $\{U^{(4)}_\alpha\}$. In this case there are no singular points introduced and $\Sigma^{(4)} = \emptyset$, as desired.

- After the first step of the iterative extension the part $\Sigma^{(5)}$ of the 5-skeleton which is not covered by the $\{U^{(5)}_\alpha\}$ consists of a 0-dimensional set, formed by the centers of all bad 5-dimensional faces.

- Then, assume by induction that before extending to $k + 1$-bad cubes the part $\Sigma^{(k)}$ of the $k$-dimensional skeleton which is not covered is the dual complex, inside the $k$-skeleton, of the skeleton of 5-dimensional bad faces. Then consider a $k$-dimensional face $C^k$ and denote its two $(k + 1)$-dimensional neighboring faces by $C^k_+$ and $C^k_-$ and their centers by $c_+$ and $c_-$, respectively. We note that, by the symmetry of the extension maps $\pi_{c^k_{+/-}}$ that we use, we have

$$
\Sigma^{(k+1)} \cap (\text{Cone}(C^k, c_+) \cup \text{Cone}(C^k, c_-)) = \text{Cone}(\Sigma^{(k)} \cap C^k, c_+) \cup \text{Cone}(\Sigma^{(k)} \cap C^k, c_-).
$$

(5.27)

The fact that $\Sigma^{(k+1)} \cap F^{(k+1)}_{\text{bad}}$ is the dual of the bad complex then follows, by iterating the above construction for all $(k + 1)$-dimensional faces.

\[\square\]

5.2.4 Mollification and completion of the proof

For the last stage of our construction of approximants, we have already filled all the good $n$-cubes of our cubeulation with extensions as in §5.2.2 and the bad $n$-cubes with extensions as in §5.2.3.

Together with the connection forms, we have been extending also the local gauges, outside the codimension-5 set $\Sigma^{(n)}$ from Lemma 5.8. In such gauges our connection forms have locally $L^4$-coefficients: at the base step of the iterative extension this follows from Corollary 5.4 of §5.2.1. Then in the iteration this property is preserved by (5.18) by the $k \mapsto n$ case of (5.22).

The following result, analogous to [17, Lem. 2.4], shows that if in compatible local gauges we have $L^4$-integrable connection forms, then mollifying the coefficient forms of the connection forms provides smooth approximants in our desired norms.

Lemma 5.9 (mollification and local $L^4$-gauges). Assume that $\Omega \subset \mathbb{R}^n$ is a compact set with open interior. Let $A \in L^2(\Omega, \Lambda^1 \Omega \otimes \mathfrak{g})$ and $F := dA + A \wedge A \in L^2(\Omega, \Lambda^2 \Omega \otimes \mathfrak{g})$. If $\{U_\alpha\}$ is a cover of an open set $U \subset \mathbb{R}^n$ such that for each $U_\alpha$ there exists $g_\alpha \in W^{1,2}(U_\alpha, G)$ such that $(A|_{U_\alpha})_{g_\alpha} \in L^2(U_\alpha, \Lambda^1 U_\alpha \otimes \mathfrak{g})$ and $g_\alpha = g_\beta$ over $U_\alpha \cap U_\beta$ for all $\alpha, \beta$, then there exists a sequence $A_n \in L^2(U, \Lambda^1 U \otimes \mathfrak{g})$ such that for all $\alpha$ there holds

$$\forall U_\alpha \quad (A_n|_{U_\alpha})_{g_\alpha} \in C^\infty(U_\alpha, \Lambda^1 U_\alpha \otimes \mathfrak{g}) \quad \text{and} \quad \|A_n - A\|_{L^2(U)} \to 0.
$$

(5.28a)

and furthermore if $F_n := dA_n + A_n \wedge A_n$ then we have

$$\|F_n - F\|_{L^2(U)} \to 0.
$$

(5.28b)

Proof. We fix a sequence $\eta_n \downarrow 0$ for the rest of the proof. For any connection form $\tilde{A} \in L^2(U_\alpha, \Lambda^1 U_\alpha \otimes \mathfrak{g})$, we define its smoothing by

$$
\tilde{A}_{\eta_n}(x) := \int \rho_{\min\{\eta_n, \text{dist}(x, \partial U_\alpha)\}}(x - y)\tilde{A}(y)dy.
$$

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where \((\rho_\alpha)_{\alpha > 0}\) is a family of mollifiers with \(\rho_1\) supported on the ball of radius 1.

Then we apply the above for \(A_n\) by fixing a partition of unity \(\{\theta_\alpha\}_\alpha\) subordinated to the cover \(\{U_\alpha\}_\alpha\). We define

\[
A_n := \sum_\alpha \theta_\alpha \left( \left( (A|_{U_\alpha})^{g_{\alpha}} \right)^{g_{\alpha}^{-1}} \right).
\]

We can check directly by the properties of mollification that the smoothness condition in (5.28a) holds. The convergence required in (5.28a) follows then by triangle inequality from the formula \(A^\eta = g^{-1} \eta g + g^{-1} A g\) using the fact that \(g_{\alpha}^{-1}\) is in \(W^{1,2} \cap L^\infty\) and

\[
\left\| \left( (A|_{U_\alpha})^{g_{\alpha}} \right)^{g_{\alpha}^{-1}} - (A|_{U_\alpha})^{g_{\alpha}} \right\|_{L^2(U_\alpha)} \to 0.
\]

We next prove (5.28b). In this case we may write

\[
F_n = \sum_\alpha \theta_\alpha g_\alpha F((A|_{U_\alpha})^{g_\alpha}) g_{\alpha}^{-1} + \sum_\alpha d\theta_\alpha \left( \left( (A|_{U_\alpha})^{g_{\alpha}} \right)^{g_{\alpha}^{-1}} \right)
\]

\[
+ \sum_\alpha (\theta_\alpha^2 - \theta_\alpha) g_\alpha \left( (A|_{U_\alpha})^{g_{\alpha}} \right)^{g_{\alpha}^{-1}} \wedge (A|_{U_\alpha})^{g_{\alpha}} g_{\alpha}^{-1}
\]

\[
+ \sum_{\alpha \neq \beta} \theta_\alpha \theta_\beta \left( \left( (A|_{U_\alpha})^{g_{\eta}} \right)^{g_{\alpha}^{-1}} \wedge \left( (A|_{U_\beta})^{g_{\alpha}} \right)^{g_{\alpha}^{-1}} \right),
\]

\[
F = \sum_\alpha \theta_\alpha g_\alpha F(A|_{U_\alpha}) g_{\alpha}^{-1} + \sum_\alpha d\theta_\alpha \left( (A|_{U_\alpha})^{g_{\alpha}} \right)^{g_{\alpha}^{-1}}
\]

\[
+ \sum_\alpha (\theta_\alpha^2 - \theta_\alpha) g_\alpha (A|_{U_\alpha})^{g_{\alpha}} \wedge (A|_{U_\alpha})^{g_{\alpha}} g_{\alpha}^{-1}
\]

\[
+ \sum_{\alpha \neq \beta} \theta_\alpha \theta_\beta \left( (A|_{U_\alpha})^{g_{\alpha}} \wedge (A|_{U_\beta})^{g_{\alpha}} \right)^{g_{\alpha}^{-1}},
\]

and we can bound \(\|F_n - F\|_{L^2(U)}\) by controlling the following quantities. From the above first terms, using the fact that \(\|F_{A^1} - F_{A^2}\|_{L^2} = \|F_{A^1} + F_{A^2}\|_{L^2}\), summed over \(\alpha\), and with the lighter notation \(\tilde{A} := (A|_{U_\alpha})^{g_{\alpha}}\) and \(\eta := \eta_\alpha\), we have the terms

\[
\left\| F_{\tilde{A}_\eta} - F_{\tilde{A}} \right\|_{L^2(U_\alpha)} \leq \left\| F_{\tilde{A}_\eta} - (F_{\tilde{A}})_{\eta} \right\|_{L^2(U_\alpha)} + \left\| (F_{\tilde{A}})_{\eta} - F_{\tilde{A}} \right\|_{L^2(U_\alpha)}
\]

\[
= \left\| \tilde{A}_\eta \wedge \tilde{A} - (\tilde{A} \wedge \tilde{A})_{\eta} \right\|_{L^2(U_\alpha)} + \left\| (F_{\tilde{A}})_{\eta} - F_{\tilde{A}} \right\|_{L^2(U_\alpha)}
\]

\[
\leq \left\| \tilde{A} \right\|_{L^4(U_\alpha)} \left\| \tilde{A}_\eta \wedge \tilde{A} - \tilde{A} \right\|_{L^4(U_\alpha)} + \left\| (F_{\tilde{A}})_{\eta} - F_{\tilde{A}} \right\|_{L^2(U_\alpha)}
\]

(5.29a)

and the two terms converge to zero as \(\eta \to 0\).

Next, we have, again summed over \(\alpha\) and with the notations above,

\[
\left\| d\theta_\alpha \left( \left( (\tilde{A}_\eta)^{g_{\alpha}^{-1}} - (\tilde{A})^{g_{\alpha}^{-1}} \right) \right) \right\|_{L^2(U_\alpha)} \leq \left\| d\theta_\alpha \right\|_{L^\infty} \left\| g_{\alpha} \left( \tilde{A}_\eta - \tilde{A} \right) g_{\alpha}^{-1} \right\|_{L^2(U_\alpha)}
\]

\[
\leq \left\| d\theta_\alpha \right\|_{L^\infty} \left\| \tilde{A}_\eta - \tilde{A} \right\|_{L^2(U_\alpha)},
\]

(5.29b)

and

\[
\left\| (\theta_\alpha^2 - \theta_\alpha) g_\alpha \left( \tilde{A}_\eta \wedge \tilde{A}_\eta - \tilde{A} \wedge \tilde{A} \right) g_{\alpha}^{-1} \right\|_{L^2(U_\alpha)} \leq \left\| \theta_\alpha^2 - \theta_\alpha \right\|_{L^\infty(U_\alpha)} \left\| \tilde{A} \right\|_{L^4(U_\alpha)} \left\| \tilde{A}_\eta - \tilde{A} \right\|_{L^4(U_\alpha)}
\]

(5.29c)
and finally with the further notation \( \tilde{A} := (A|_{U_a})^{\theta_3} \) and with a required sum also over \( \beta \), we have the terms
\[
\left\| \theta_\alpha \theta_\beta \left( \tilde{A}_\eta^{-1} \wedge \tilde{A}_\eta^{\theta_1} - \tilde{A}_\eta^{\theta_1} \wedge \tilde{A}_\eta^{-1} \right) \right\|_{L^2(U_a)} \\
\leq \| \theta_\alpha \theta_\beta \|_{L^\infty(U_a)} \left( \| \tilde{A}_\eta - \tilde{A} \|_{L^4(U_a)} \| \tilde{A} \|_{L^4(U_a)} + \| \tilde{A} \|_{L^4(U_a)} \| \tilde{A}_\eta - \tilde{A} \|_{L^4(U_a)} \right).
\]

(5.29d)

Then summing all the terms (5.29) over \( \alpha \) and \( \beta \), and using the convergence proved in the previous step as well, we see that \( \| F_n - F \|_{L^2(U)} \to 0 \) too, as desired.

We now are ready to conclude the proof of Theorem 5.2. By working first on the grid \( P_{r,t} \) rescaled to scale \( r = 1 \), we consider the final result of performing the extensions as in Lemma 5.6 on all the good cubes and the ones leading to (5.24) on all the bad cubes. Then we rescale back to scale \( r = 1 \). We denote by \( \tilde{A}_r, \tilde{F}_r \) the connection and curvature forms obtained in this way.

Next, we consider the piecewise constant forms defined inductively as the averages (5.14) scaled back to scale \( r \), and thus, with \( c_n = 2^{n-4} \frac{a}{4(n-4)} \),
\[
\mathcal{T}_r(x) := \sum_{C \in P_r(x)} 1_{C}(x) \frac{1}{c_n r^d} \sum_{C \in (C^*_r)^{(n)}} \int_C i_C^* F,
\]
\[
\mathcal{A}_r(x) := \sum_{C \in P_r(x)} 1_{C}(x) \frac{1}{c_n r^d} \sum_{C \in (C^*_r)^{(n)}} \int_C i_C^* A.
\]

(5.30)

On the set of good cubes we scale to scale \( r \) and then sum up the conclusions (5.17) of Lemma 5.6. We find, denoting, like in (5.26), \( P_{r,t}^{(n)} \) good, \( P_{r,t}^{(n)} \) bad to be the subcomplexes generated by the good cubes and the bad cubes, respectively,
\[
\left\| \tilde{F}_r - F \right\|_{L^2(P_{r,t}^{(n)})} \leq \left\| \tilde{F}_r - \mathcal{T}_r \right\|_{L^2(P_{r,t}^{(n)})} + \left\| \mathcal{T}_r - F \right\|_{L^2(P_{r,t}^{(n)})} \\
\leq r^{n-4} \sum_{C \in P_{r,t}^{(n)}} \left\| i_C^* F - \frac{1}{|C|} \int_C i_C^* F \right\|_{L^2(C)}^2 + \sum_{C \in P_{r,t}^{(n)}} \left( \sum_{C \in (C^*_r)^{(n)}} \left\| \mathcal{T}_{C^*_r} \right\|_{L^2(C)}^2 \left\| i_C^* A - \frac{1}{|C|} \int_C i_C^* A \right\|_{L^2(C)}^2 \right) \\
+ r^{n-4} \sum_{C \in P_{r,t}^{(n)}} \left\| i_C^* F \right\|_{L^2(C)}^2 + \left\| \mathcal{T}_r - F \right\|_{L^2([-1,1]^n)}^2.
\]

(5.31a)

\[
\left\| \tilde{A}_r - A \right\|_{L^2(P_{r,t}^{(n)})} \leq \left\| \tilde{A}_r - \mathcal{A}_r \right\|_{L^2(P_{r,t}^{(n)})} + \left\| \mathcal{A}_r - A \right\|_{L^2(P_{r,t}^{(n)})} \\
\leq r^{n-4} \sum_{C \in P_{r,t}^{(n)}} \left\| i_C^* A - \frac{1}{|C|} \int_C i_C^* A \right\|_{L^2(C)}^2 + \left\| \mathcal{A}_r - A \right\|_{L^2([-1,1]^n)}^2.
\]

(5.31b)

Using the bounds (4.7b) valid for our choice of \( P_{r,t} \) for \( A \) and \( F \) contemporarily, we then find \( \left\| \tilde{F}_r - \mathcal{T}_r \right\|_{L^2([-1,1]^n)} \to 0 \) and \( \left\| \tilde{A}_r - \mathcal{A}_r \right\|_{L^2([-1,1]^n)} \to 0 \) as \( r \to 0 \), and the first terms

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on the right in (5.31a) and (5.31b) tend to zero as well. Concerning the forecast line in (5.31a), the assumed bound on \( \int_C \left| i_C^* \left( A - \overline{A}_{C_j^p} \right) \right|^2 \) in (4.13), together with the inversion of the order of summation and the Cauchy-Schwartz inequality, implies that

\[
\sum_{C \in \mathcal{P}^{(4)}_{\text{good}}} \sum_{C_j^p \in \mathcal{C}^{(4)}_j} |F_{C_j^p}|^2 \left\| i_C^* A - \frac{1}{|C|} \int_C i_C^* A \right\|_{L^2(C)}^2 
\lesssim \delta(n) \sum_{C_j^p \in \mathcal{P}_{\text{good}}^{\ell}} \left\| \int_{P_{\text{good}}} F \right\|^2 \leq \delta(n) \int_{P_{\text{good}}} |F|^2 \leq \delta(n) \|F\|^2_{L^2([-1,1]^n)},
\]

and by a similar estimate using again (4.13), we find for the remaining term in (5.31a) that

\[
\sum_{C \in \mathcal{P}^{(4)}_{\text{good}}} \left\| i_C^* F \right\|^4_{L^2(C)} \leq \delta(n) \int_{P_{\text{good}}} |F|^2 \leq \delta(n) \|F\|^2_{L^2([-1,1]^n)}.
\]

Thus we found

\[
\left\| \tilde{F}_r - F \right\|^2_{L^2(P_{\text{good}})} \lesssim \delta(n) \|F\|^2_{L^2([-1,1]^n)} + o_{r \to 0}(1),
\]

\[
\left\| \tilde{A}_r - A \right\|^2_{L^2(P_{\text{good}})} = o_{r \to 0}(1).
\]

For the bad cubes we use the bounds (5.24) and we may apply (4.7a) to the forms \( 1_{P_{\text{bad}}} A \) and \( 1_{P_{\text{bad}}} F \) to obtain

\[
\int_{P_{\text{bad}}} \left\| \tilde{A}_r \right\|^2 \lesssim \sum_{C_j^p \in \mathcal{P}_{\text{bad}}^{\ell}} \left\| A_{C_j^p} \right\|^2_{L^2(C_j^p)} \lesssim \sum_{C \in \mathcal{C}^{(4)}_j} \int_C \left| i_C^* A \right|^2 \lesssim \int_{P_{\text{bad}}} |A|^2,
\]

\[
\int_{P_{\text{bad}}} \left\| \tilde{F}_r \right\|^2 \lesssim \sum_{C_j^p \in \mathcal{P}_{\text{bad}}^{\ell}} \left\| F_{C_j^p} \right\|^2_{L^2(C_j^p)} \lesssim \sum_{C \in \mathcal{C}^{(4)}_j} \int_C \left| i_C^* F \right|^2 \lesssim \int_{P_{\text{bad}}} |F|^2,
\]

Now note that the bound implied (4.14) on the total measure of \( P_{\text{bad}} \):

\[
|P_{\text{bad}}| = r^n N_{\delta(n)} \lesssim \delta(n)^{4} \|F\|^2_{L^2([-1,1]^n)} + r^4 \|A\|^2_{L^2([-1,1]^n)} + r^4 \|F\|^2_{L^2([-1,1]^n)},
\]

thus if \( \delta(n)^{-1} r \to 0 \) then we find that the right hand sides of the equations (5.33) also tend to zero, by dominated convergence. In particular, by summing up (5.32) and (5.33), we find that given two sequences of positive numbers \( \delta^{(n)}_t \to 0 \) and\( r_t \to 0 \), there holds

\[
\left\| \tilde{F}_{r_t} - F \right\|_{L^2([-1,1]^n)} \to 0, \quad \left\| \tilde{A}_{r_t} - A \right\|_{L^2([-1,1]^n)} \to 0,
\]

provided \( \delta^{(n)}_t \to 0 \) and \( r_t / \delta_t \to 0 \).

We then apply the mollification as in Lemma 5.9 to such choices of \( \tilde{A}_{r_t}, \tilde{F}_{r_t} \), and the desired smooth approximants are constructed, completing the proof of Theorem 5.2.

6 Strong compactness for weak connections

Our aim is to prove the following theorem:
Theorem 6.1 (sequential weak closure of $\tilde{\mathcal{A}}_G$). Let $A_j \in \tilde{\mathcal{A}}_G([-1,1]^n)$ be a sequence of connections such that the corresponding curvature forms $F_j$ are equibounded in $L^2$ and converge weakly in $L^2$ to a 2-form $F$. Then $F$ corresponds to $A \in \tilde{\mathcal{A}}_G([-1,1]^n)$, and furthermore there holds
\begin{equation}
\delta(A_j, A) \to 0,
\end{equation}
where for $A, B \in \tilde{\mathcal{A}}_G([-1,1]^n)$, the pseudo-distance $\delta(A, B)$ is defined in (4.6).

We recall that here again like for the pseudo-distance $\delta$ defined in (1.2), for $G = SU(2)$ the pseudo-distance $\delta$ on $\tilde{\mathcal{A}}_G([-1,1]^n)$ induces a distance on gauge-equivalence classes of connections from this space, as a consequence of Corollary A.7, due to the fact that $\delta$ is equivalent to $\delta_1$ as defined in (A.7c).

The above result is mainly due to Uhlenbeck in dimensions $n \leq 4$ and it is one of the main results in [17] for $n = 5$. We aim here at proving it by induction on $n$, and thus we first describe how the proof of the $n = 4$ in [17] allows our definition of $\tilde{\mathcal{A}}_G([-1,1]^4)$ based on the $L^4$ norm rather than the $W^{1,2}$ norm, and then we use the theorem’s statement for dimension $n - 1$ in order to prove it in dimension $n$.

As it will be more befitting to the overall proof, we frame the result in terms of an abstract compactness theorem which is the tool allowing the induction on the dimension. Versions of the same tool were successfully used for proving results in the theory of metric currents and in the one of scans, see [2], [8], [5] and the references therein.

6.1 An abstract compactness result

We employ as an abstract tool Proposition 6.3 below, which is the multi-dimensional substitute of the abstract compactness result used in [17, Prop. 3.1]. The Hölder continuity which was used in [17] now does not hold for more general slicings, and thus we need a different approach. The natural candidate is a metric space valued Sobolev embedding theorem, inspired by [13, Thm. 1.13]. The difference between our case and [13] is that the metric space in which our sliced connections take values is not locally compact, unlike what assumed in [13], thus the coercivity of the Yang-Mills energy has to be used, like in [17] and [8, Thm. 9.1].

We find it useful to introduce, following the spirit in which in [2] the notion of metric-$BV$-functions was used in the proof of compactness by slicing, the following notion of metric upper gradient structure (which extends the definition [1, 53] to the case of metric-space-valued maps).

Recall that for $p > 1$ the $p$-modulus $\text{Mod}_p(\Gamma)$ of a family $\Gamma$ of absolutely continuous curves $\gamma : [a,b] \to \mathbb{R}^n$ is defined by
\[ \text{Mod}_p(\Gamma) := \inf \left\{ \int_{\mathbb{R}^n} f^p : f : \mathbb{R}^d \to [0,\infty] \text{ Borel, } \int_{\gamma} f \geq 1 \text{ for all } \gamma \in \Gamma \right\}. \]

We recall that $\text{Mod}_p$ is an outer measure on absolutely continuous curves, and we say that a property holds for $p$-a.e. curve if the set of curves for which it fails is $\text{Mod}_p$-negligible. We will use the following definition:

Definition 6.2 (metric upper gradient structures). Let $(\mathcal{Y}, \text{dist})$ be a metric space and $p > 1$. Consider a measurable map $f : \mathbb{R}^n \to \mathcal{Y}$ and a map $N : \mathcal{Y} \to \mathbb{R}$. Further, if $v \in \text{AC}([0,1], X)$ is an absolutely continuous curve, then let $|\gamma'|(t) = \lim_{s \to t} \frac{d_X(\gamma(s), \gamma(t))}{|s-t|}$ be its metric derivative. We say that $N$ gives a $p$-upper gradient structure for $f$ if for $p$-almost every curve $\gamma$ we have that $N \circ \gamma$ is Borel and
\begin{equation}
\text{dist}(f \circ \gamma(s), f \circ \gamma(t)) \leq \int_{s}^{t} N \circ \gamma(r) |\gamma'(r)| \, dr, \quad \forall \ 0 < s \leq t < 1.
\end{equation}

Next, we state the following abstract compactness result:
Proposition 6.3. Consider a metric space \( Y \) and let \( K := [-1,1]^n \). Suppose that the function \( N : Y \to \mathbb{R}^+ \) satisfies the following condition:

\[
\forall C > 0 \text{ the sublevels } \{ N \leq C \} \text{ are compact in } Y .
\]  

(6.3)

Suppose that \( f_j : K \to Y \) are measurable maps such that \( N \) gives a \( p \)-weak upper gradient structure for the \( f_j \) and that

\[
\sup_j \int_K (N \circ f_j)^p < C .
\]  

(6.4)

Then \( f_j \) have a subsequence which converges pointwise almost everywhere to a function \( f : K \to Y \) for which \( N \) gives a \( p \)-weak upper gradient structure for \( f \), and such that

\[
\int_K (N \circ f)^p < C .
\]  

(6.5)

Moreover, there holds, up to passing to the above subsequence,

\[
\int_K \text{dist}^p((f_j(x), f(x)) \, dx \to 0 \text{ as } j \to \infty .
\]  

(6.6)

Remark 6.4. Note that in Proposition 6.3, we don't assume the metric space \( Y \) to be complete or separable.

Proof. By (6.3), for each \( \epsilon > 0 \) we may find a countable \( \epsilon \)-net of \( Y' := Y \cap \{ N < \infty \} \), i.e. \( N \subset Y' \) is a countable set such that \( \min_{q,q' \in N} \text{dist}(q,q') \leq \epsilon \).

Next, consider the functions

\[
d_{j,a}(x) := \text{dist}(f_j(x), a) , \quad \text{for } a \in Y .
\]

We then note that, by triangle inequality,

\[
|d_{j,a}(x) - d_{j,a}(y)| \leq \text{dist}(f_j(x), f_j(y)) ,
\]  

(6.7)

therefore the function \( N \) also gives a common \( p \)-weak upper gradient structure for the functions \( d_{j,a} : K \to \mathbb{R}^+ \). We obtain that by (6.4) and due to the fact that sublevels of \( N \) are compact, there exists a point \( x_0 \in K \) such that up to subsequence \( f_j(x_0) \) forms a dist-Cauchy sequence, converging to \( y \in Y \). Then due to (6.4) and (6.7), we see that all the functions \( d_{j,a} \) are bounded in \( W^{1,p}(K, \mathbb{R}) \), by the same proof as in [9, Thm. 7.6], and thus by Rellich embedding and a diagonal extraction, we find a subsequence (denoted still by \( j \), by abuse of notation) and maps \( d_a \in W^{1,p}(K, \mathbb{R}) \) for \( a \in \cup_{j \geq 1} N_{1/j} \) such that

\[
d_{j,a}(x) \to d_a(x) , \quad \text{for } a \in \cup_{j \geq 1} N_{1/j} .
\]  

(6.8)

Next, we claim that for all \( x \) such that (6.8) holds, there exists a unique point \( f(x) \in Y \) such that

\[
\forall a \in \cup_{j \geq 1} N_{1/j} , \quad \text{there holds } d_a(x) = \text{dist}(f(x), a) .
\]  

(6.9)

To prove (6.9), we note that by definition of \( N_{1/j} \), for all \( x \) as in (6.8) and all \( j \), there exists \( a_j \in N_{1/j} \) such that \( \text{dist}(f_j(x), a_j) = d_{j,a_j}(x) < 1/j \). Due to (6.8), we also find that \( d_{j,a_j}(x) \) is Cauchy. By triangle inequality, \( a_j(x) \) forms a Cauchy sequence, thus it has a limit in \( Y \), and it converges to a point \( f(x) \in Y \), where \( Y \) is the completion of \( Y \). Now by (6.4) and Fatou’s lemma, we find that for a.e. \( x \in K \) the sequence \( f_j(x) \) has a subsequence \( f'(x) \) depending on \( x \) so that \( \sup_{f'(x)} \text{dist}(f_j(x)) < \infty \). Then the hypothesis (6.3) implies that \( f'(x) \) has a subsequence which converges to a point in \( Y \). But as we saw, all limit points in \( Y \) coincide with \( f(x) \), in particular \( f(x) \in Y \), proving (6.9).

The function \( x \mapsto f(x) \) is clearly measurable, and by construction \( f_j(x) \to f(x) \) for a.e. \( x \in K \). Property (6.3) implies the lower semicontinuity of \( N \) and thus we find that \( N \) is also a weak upper gradient for \( f \) and that (6.4) then gives (6.5).

In order to obtain the property (6.6) we then use the pointwise convergence and conclude by dominated convergence via \( \text{dist}(f_j(x), f(x)) \leq d_{j,a}(x) + d_a(x) \), using the fact that \( d_{j,a} \) and \( d_a \) are bounded in \( L^p(K) \), which implies the \( L^p \)-convergence from (6.6).
6.2 Scheme of proof of the closure theorem

For applying the Proposition 6.3, we use the following specializations:

- The well known [6] geometric distance on 1-forms for
  \[ A, A' \in L^2([-1,1]^k, \wedge^1[-1,1]^k \otimes g) \quad \text{with} \quad 4 \leq k \leq n, \]
  then we define the pseudo-distance
  \[ \text{Dist}_k(A, A') := \min \{ \| A - g^{-1}dg - g^{-1}A'g \|_{L^2([-1,1]^k)} : g \in W^{1,2}([-1,1]^k, G) \} \tag{6.10} \]
  and we define the equivalence relation \( \sim_k \) on \( L^2([-1,1]^k, \wedge^1[-1,1]^k \otimes g) \) according to which \( A \sim_k A' \) if \( \text{Dist}_k(A, A') = 0 \). The facts that \( \text{Dist}_k \) satisfies reflexivity and triangle inequality (and that as a consequence \( \sim_k \) is an equivalence relation) follow from the fact that \( W^{1,2}([-1,1]^k, G) \) is a group (for which see [10, Appendix]).

- On the quotient \( L^2([-1,1]^k, \wedge^1[-1,1]^k \otimes g) / \sim_k \) the pseudo-distance \( \text{Dist}_k \) induces a distance between \( \sim_k \)-equivalence classes which we denote by \( \text{dist}_k \). We denote the so-obtained metric spaces by
  \[ \mathcal{Y}_k := (\mathcal{A}_G([-1,1]^k]) / \sim_k, \text{dist}_k \) . \tag{6.11} \]

- Let \( [A] \) denote the \( \sim_k \)-equivalence class of a given \( A \in \mathcal{A}_G([-1,1]^k] \), namely the set of of all \( A' \in \mathcal{A}_G([-1,1]^k] \) such that \( A' = g^{-1}dg + g^{-1}Ag \) for \( g \in W^{1,2}([-1,1]^k, G) \).

- Like in [17] we will study the functional
  \[ \mathcal{N}_4 : \mathcal{Y}_4 \to \mathbb{R}^+, \quad \mathcal{N}_4([A]) = \int_{[-1,1]^4} |F_A|^2 . \tag{6.12} \]
  Note that because the curvature satisfies \( F_{g^{-1}dg + g^{-1}Ag} = g^{-1}F_Ag \) and since the norm on \( g \) is \( \text{ad}G \)-invariant, we have that \( \mathcal{N}_4([A]) \) does not depend on the representative \( A \) employed to compute \( F_A \).

- The \( f_j : [-1,1]^{n-4} \to \mathcal{Y}_4 \) will be 4-dimensional sliced connection forms corresponding to a sequence of connection forms \( A_j \in \mathcal{A}_G([-1,1]^n] \), defined as follows, with the notation of \( \S 4 \). We fix a multi-index \( I = \{i_1, \ldots, i_{n-4}\} \) and for \( T \in [-1,1]^I \) we define
  \[ \tilde{f}_j(T) := i_{H(I,T)}^* A_j, \quad f_j(T) := [\tilde{f}_j(T)] . \tag{6.13} \]
  Then the \( \tilde{f}_j \) take a.e. values in \( \mathcal{A}_G([-1,1]^{n-4}) \) by the definition (4.2) of \( \mathcal{A}_G([-1,1]^{n}] \). If \( A \sim_n A' \) then we find that for a.e. \( T \in [-1,1]^I \) the \( H(I,T) \)-trace of the differential of the gauge \( g \in W^{1,2}([-1,1]^{n}], G \) relating \( A, A' \) is defined and \( L^2 \)-integrable, and \( g|_{H(I,T)} \) relates \( i_{H(I,T)}^* A \) to \( i_{H(I,T)}^* A' \), thus \( f_j \) is well-defined up to negligible sets.

The assertion that the weak limit \( A \) of the \( A_j \) has \( H(I,T) \)-slices in \( \mathcal{A}_G([-1,1]^{n}] \) for all \( i \) and almost every \( T \) is equivalent to the thesis of the theorem 6.1.

We note that the pseudo-distance \( \tilde{\delta}(A, B) \) between local weak connections in \( \mathcal{A}_G([-1,1]^{n}] \) defined in (4.6), can now be rewritten in terms of the distance (6.10) as follows:

\[ \tilde{\delta}(A, B) = \max_{I \subset \{1, \ldots, n\}} \int_{[-1,1]^I} \text{Dist}_4 \left( i_{H(I,T)}^* A, i_{H(I,T)}^* B \right)^2 dT. \tag{6.14} \]
6.3 The compactness in dimension 4

For \( n \geq 5 \) the the compactness in \( \mathcal{Y}_n \) of sublevels of \( \mathcal{N}_n \) is precisely the compactness result which we desire to prove. Since we know that it holds for \( n = 4 \) we may proceed as for the closure theorem for rectifiable chains, and prove it by induction on \( n \), assuming that it’s true for \( n - 1 \).

**Proposition 6.5.** Let \( \mathcal{Y}_4 \) and \( \mathcal{N}_4 \) be as above. Then \( \mathcal{N}_4 \) has sublevels which are compact with respect to the distance \( \text{dist}_4 \) defined in (6.11).

*Modification of the proof of [17] Prop. 3.3.* The difference between the definition of \( \tilde{\mathcal{A}}_G([-1, 1]^4) \) defined in (4.1) and the version used in [17, Prop. 3.3] is that there a local gauges \( g \) such that \( A^0 \in L^4 \) are assumed to exist, rather than ones such that \( A^0 \in W^{1,2} \).

The way in which such hypothesis is used in [17] Prop. 3.3 is however just via ss theorem in regions where the \( L^2 \) norm of \( F \) is small. Theorem 2.1 for \( n = 4, \pi = 0 \) however works under the hypothesis that such \( A^0 \in L^4 \) locally and then we obtain

\[
d^*A^0 = 0 \quad \text{and} \quad dA^0 = F_{A^0} - A^0 \wedge A^0 \in L^2,
\]

which implies that \( A \in W^{1,2} \) by Hodge inequality. This reduces us to the situation of [17] Prop. 3.3, and the rest of the proof follows like in that proposition.

6.4 The Yang-Mills energy gives a weak gradient structure

We provide a new version of [17] Lem. 3.4 for the case of parallel slices instead of spherical slices. The main ingredient is a new version of [17] Coroll. 1.13 which we now state:

**Lemma 6.6** (controlled solutions to the gauge fixing ODE). Assume that to \( A \in \tilde{\mathcal{A}}_G([-1, 1]^n) \) and fix \( g_0 \in W^{1,2}([-1, 1]^{n-1} \times \{0\}, G) \). Then there exists a solution \( g \in W^{1,2}([-1, 1]^n, G) \) to the following ODE, where \( A = \sum A_i dx_i : \)

\[
\begin{align*}
\partial_n g &= -A_n g, &\text{on } [-1, 1]^n, \\
g(x', 0) &= g_0(x') &\text{for } x' \in [-1, 1]^{n-1}.
\end{align*}
\]

In particular the form \( A^0 := g^{-1}dg + g^{-1}Ag \) is \( L^2 \)-integrable and has zero component in the direction \( \partial/\partial x_n \). Moreover we have

\[
\|g\|_{W^{1,2}([-1, 1]^n)} \lesssim \|A\|_{L^2([-1, 1]^n)} + \|F, \epsilon_n\|_{L^2([-1, 1]^n)}.
\]

**Proof.** By Theorem 5.2 applied to the cube \([-1, 1]^n\) we have a sequence of connections \( A_j \in \mathbb{R}^\infty([-1, 1]^n) \) such that

\[
A_j \to A \quad \text{in } L^2, \quad F_{A_j} \to F_A \quad \text{in } L^2.
\]

We then solve, with notation \( A_j = \sum_i (A_j)_i dx_i \),

\[
\begin{align*}
\partial_n g_j(x', t) &= -(A_j)_n(x', t)g_j(x', t) &\text{for } t \in [-1, 1], \quad x' \in [-1, 1]^{n-1}, \\
g_j(x', 0) &= g_0(x') &\text{for } x' \in [-1, 1]^{n-1},
\end{align*}
\]

where the solution \( g_j \) is well defined on all segments \( x' = \text{const} \) except for the ones which contain one of the singular set \( \Sigma_j \) of \( A_j \). The union of all such segments is negligible, therefore \( g_j \) is defined almost everywhere. We have the following, with the further notation...
$F^{g_j}$ := $F_j := \sum_{a<b} (F^{g_j}_{ab})_a dx_a \wedge dx_b$ and for indices $i \in \{1, \ldots, n-1\}$:

$$\left( A_j^{g_j} \right)_{i|_{x_n=0}} = \left( A_j^{g_0} \right)_{i|_{x_n=0}}, \quad (6.17)$$

$$\left( A_j^{g_j} \right)_n = \left( \frac{\partial}{\partial x_n} \right) \left( A_j^{g_j} \right)_i = 0, \quad (6.16)$$

$$\left( F^{g_j}_{ni} \right) = \partial_n (A_j^{g_j})_i - \partial_i (A_j^{g_j})_n + \left( (A_j^{g_j})_n, (A_j^{g_j})_i \right), \quad (6.18)$$

$$\partial_n g_j = g_j (A_j^{g_j})_i - (A_j)_i g_j, \quad (6.19)$$

Integrating (6.17), (6.19) we find that $(A_j^{g_j})_i, i > 1$ are $L^2$-integrable with bounds depending on $\|F_{\cdot, e_n}\|_{L^2}$ only, thus we find

$$\|g_j\|_{W^{1,2}([-1,1]^n)} \lesssim \|A_j\|_{L^2([-1,1]^n)} + \|F_{\cdot, e_n}\|_{L^2([-1,1]^n)} \lesssim C. \quad (6.21)$$

Up to extracting a subsequence we may assume

$$g_j \rightharpoondown g \quad \text{weakly in } W^{1,2}$$

and thus $g_j \rightharpoonup g$ a.e. and strongly in all $L^p, p < \infty$ by interpolation between $L^{2^*}$ and $L^\infty$. From this, the rest of the reasoning proceeds precisely like for [17, Cor. 1.13], as this convergence allows to conclude the proof by approximation.

The possibility to solve an ODE such as (6.15) allows to proceed to the proof of the second hypothesis that $\mathcal{N}$ gives a 2-weak upper gradient structure for the slices $f_j$, as required for the application of Proposition 6.3. This is done by the following two Corollaries. The first result is obtained by just applying Lemma 6.6 along a curve:

**Corollary 6.7.** Let $n \geq 5$ and let $I \subset \{1, \ldots, n\}$ of cardinality 4 and let $J := \{1, \ldots, n\}\setminus I$. Assume that $A \in \mathcal{A}_G([-1,1]^n)$ and an injective rectifiable curve $\gamma : [0,1] \rightarrow [-1,1]^J$ such that for almost all $t \in [0,1]$ and for $t = 0,1$ the slices $i_{\mathcal{U}(J,\gamma(t))}^* A \in L^2(H_{\mathcal{J},\gamma(t)}, \Lambda^1 \mathbb{R}^J \otimes \mathfrak{g})$ are well-defined and satisfy the curvature bound

$$\int_0^1 \left\| i_{\mathcal{U}(J,\gamma(t))}^* F \right\|_{L^2(H_{\mathcal{J},\gamma(t)}, \Lambda^1 \mathbb{R}^J \otimes \mathfrak{g})}^2 |\dot{\gamma}(t)| \, dt < \infty.$$  

Fix $g_0 \in W^{1,2}([\gamma(0)] \times [-1,1]^4, G)$. Then there exists a solution $g_\gamma \in W^{1,2}([0,1] \times [-1,1]^J, G)$ to the ODE

$$\begin{cases}
\partial_t g_\gamma(\gamma(t), x) = -A(\gamma(t))[\gamma(t), x] g_\gamma(\gamma(t), x) & \text{for } t \in [0,1], \quad x \in [-1,1]^J, \\
g_\gamma|_{H(I, \gamma(0))} = g_0.
\end{cases} \quad (6.22)$$

Moreover we have that the component $(A^{g_\gamma})_{\gamma(t)} = 0$ for $t \in [0,1]$ and

$$\|g_\gamma\|_{W^{1,2}(\gamma([0,1]) \times [-1,1]^J)} \lesssim \|A\|_{L^2(\gamma([0,1]) \times [-1,1]^J)} + \left( \int_0^1 \left\| i_{\mathcal{U}(J,\gamma(t))}^* F \right\|_{L^2(H_{\mathcal{J},\gamma(t)}, \Lambda^1 \mathbb{R}^J \otimes \mathfrak{g})}^2 |\dot{\gamma}(t)| \, dt \right)^{\frac{1}{2}}. \quad (6.23)$$

The next Corollary can be viewed as an adaptation to the current setting (translated now in the language of weak upper gradient structures, for clarity) of the study done in the abelian case in [16]:

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Corollary 6.8 (The curvature gives a 2-weak upper gradient structure for the slices). Let \( n \geq 5 \) and let \( I \subset \{1, \ldots, n\} \) of cardinality \( n - 4 \). Assume that \( A \in \tilde{A}_G([-1,1]^n) \). Then for

\[
 f : [-1,1]^I \to Y_4 \quad \text{defined by} \quad f_I(T) = \left[ \hat{u}_{H(1,T)}^I A \right]
\]

the function \( \mathcal{N}_4 : Y_4 \to \mathbb{R}^+ \) gives a 2-weak upper gradient structure for \( f \).

Proof. We first find \( T_0 \) such that the slice \( \hat{u}_{H(1,T_0)}^I A \) is well-defined and we may start with \( g_0 \equiv id \in W^{1,2}(H(I,T_0),G) \), then apply Corollary 6.7 to extend \( g_0 \) to \( g \in W^{1,2}(H(I,T_0),G) \) such that \( A' := g^{-1}dg + g^{-1}Ag \) satisfies (6.22). Then we find that \( F' := F_A \) satisfies for \( i \in I, j \in J := \{1, \ldots, n\} \setminus I \), similarly to (6.19),

\[
(F_A)'_{i,j} = \partial_t (A'_i) \tag{6.24}
\]

Then by the definition of the distance \( \text{dist}_4 \) of \( Y_4 \), and by the expression of the curvature as the gradient of an \( L^2 \)-function on the space \([-1,1]^I \) given in (6.24), for a rectifiable curve \( \gamma : [a,b] \to [-1,1]^I \) such that all terms below are finite there holds, using definition (6.10),

\[
\text{Dist}_4 \left( \hat{u}_{H(I,\gamma(a))}^I A', \hat{u}_{H(I,\gamma(b))}^I A' \right) \leq \left\| \hat{u}_{H(I,\gamma(a))}^I A' - \hat{u}_{H(I,\gamma(b))}^I A' \right\|_{L^2([-1,1]^I)} \leq \int_a^b \left\| \nabla_T \hat{u}_{H(I,\gamma(t))} A' \right\|_{L^2([-1,1]^I)} |\dot{\gamma}(t)| \, dt \leq \int_a^b \left\| \hat{u}_{H(I,\gamma(t))} F' \right\|_{L^2([-1,1]^I)} |\dot{\gamma}(t)| \, dt, \tag{6.25}
\]

where \( \nabla_T \) represents the gradient taken in the \( T \)-variables, belonging to \([-1,1]^I \). The bound (6.25) coincides with the inequality that is required in (6.2), for \( f, \mathcal{N} \) as in the statement of the lemma.

Now we can follow the reasoning from [9, §7] valid for gradients of \( W^{1,2} \)-functions, in order to obtain that the same bound (6.25) also holds for 2-a.e. curve \( \gamma : [a,b] \to [-1,1]^I \), concluding the proof. \( \square \)

As a direct consequence of Corollary 6.8 applied to the \( f_j \) defined as in the beginning of the section (see (6.13)), and of Proposition 6.5, we have that the hypotheses of Proposition 6.3 hold for \( \mathcal{N} \Rightarrow \mathcal{N}_4 \) and \( p \mapsto 2 \).

6.5 Proof of the Closure Theorem 6.1

We first note the following lemma, analogous to [17, Lem. 3.5]:

Lemma 6.9 (cf. [17, Lem. 3.5]). Let \( n \geq 5 \), \( I \subset \{1, \ldots, n\} \) of cardinality \( n - 4 \). Let \( A_j \in \tilde{A}_G([-1,1]^n) \), and consider the gauges \( g_j(I) \) as given in Corollary 6.7. Assume that \( \sup_j \|F_{A_j}\|_{L^2([-1,1]^n)} \leq C \) and that

\[
(A_j)^{g_j(I)} \rightharpoonup A(I) \quad \text{weakly in } L^2([-1,1]^n, \Lambda^1[-1,1]^n \otimes g). \tag{6.26}
\]

Then there exists a subsequence \( j' \) such that

\[
\text{for a.e. } T \in [-1,1]^I \text{ there holds } \hat{u}_{H(I,T)} (A_{j'})^{g_{j'}(I)} \rightharpoonup \hat{u}_{H(I,T)} A(I) \text{ weakly in } L^2. \tag{6.27}
\]

The proof follows roughly the same method as the one of [17, Lem. 3.5], but with several changes, including the use of weak upper gradient structures, and therefore we present it in full.
Proof. We denote, for $T \in [-1,1]^I$, by $A_j(T) := i_{H(I,T)}^* (A_j)^{g_j(I)}$. We again consider a test form, now of the form $\beta := \omega_T \wedge \phi := \left( i_{H(I,T)}^* \omega \right) \wedge \phi$, with $\omega \in L^2([-1,1]^I, \wedge^3 \mathbb{R}^4 \otimes \mathfrak{g})$ and $\phi \in C^\infty([-1,1]^I, \wedge^{n-4} \mathbb{R}^I \otimes \mathfrak{g})$, and we define

$$f_j^\omega (T) := \int_{[-1,1]^I} A_j(T) \wedge \omega_T,$$

and we find from Corollary 6.7 that the maps $f_j^\omega : [-1,1]^I \to \mathbb{R}$ have a 2-weak upper gradient structure given by $A \mapsto \|F_A\|_{L^2([-1,1]^I)} \|\omega\|_{L^2([-1,1]^I)}$, and due to the assumed bound on $\|F_A\|_{L^2([-1,1]^I)}$, we may apply the abstract result of Proposition 6.3 to obtain the thesis.

\end{proof}

The above lemma allows to complete the proof of Theorem 6.1 proceeding precisely like for [17, Thm. 1.11].

End of proof of Theorem 6.1: We work under the hypothesis of the theorem, and we consider the global weak limit connection of the $A_j$’s, and denote it by $A \in L^2([-1,1]^n, \wedge^1 [-1,1]^n \otimes \mathfrak{g})$.

Fix first $I \subset \{1,\ldots,n\}$ of cardinality $n-4$ and first apply Proposition 6.3, to the slice functions $f_j$ as defined in §6.2: we find that pointwise a.e. $T \in [-1,1]^I$, up to subsequence the sliced connection equivalence classes as defined in §6.2 $i_{H(I,T)}^* A_j$ converge in $\mathcal{Y}_4$ and that there holds, due to (6.6), that for some forms $A(I,T) \in \overline{\mathcal{A}_G([-1,1]^I)}$ for $T \in [-1,1]^I$, there holds

$$\int_{[-1,1]^I} \text{Dist}_4^2 \left( i_{H(I,T)}^* A_j, A(I,T) \right) dT \to 0 \quad \text{as} \quad j \to \infty. \quad (6.29)$$

Now we apply Corollary 6.7, and Lemma 6.9, and find that in the $g_j(I)$-gauges up to yet another subsequence, the sliced connection forms converge as in (6.27).

We consider a sequence $A_j \in \tilde{\mathcal{A}}_G([-1,1]^n)$ as in Theorem 6.1. For $I \subset \{1,\ldots,n\}$ of cardinality $n-4$, we may find a change of gauge $g_j(I)$ as described in Corollary 6.7. Then we have in particular, due to (6.22), (6.23), that

$$\|A_j^{g_j(I)}\|_{L^2([-1,1]^n)} \leq C \|F_j\|_{L^2([-1,1]^n)}^\prime,$$

$$\|g_j(I)\|_{W^{1,2}([-1,1]^n)} \leq C \left( \|A_j\|_{L^2([-1,1]^n)} + \|F_j\|_{L^2([-1,1]^n)} \right). \quad (6.31)$$

We thus have that up to extracting a subsequence there holds

$$A_j^{g_j(I)} \rightharpoonup A(I) \text{ in } L^2([-1,1]^n, \wedge^1 [-1,1]^n \otimes \mathfrak{g}), \quad g_j(I) \rightharpoonup g(I) \text{ in } W^{1,2}([-1,1]^n, G). \quad (6.32)$$

We claim that if we denote $A(I), A(J)$ and $g(I), g(J)$ the above limit connection forms and gauges for two sets of coordinates $I \neq J \subset \{1,\ldots,n\}$ of cardinality $n-4$, we have, for

$$g(IJ) := g(I)^{-1} g(J),$$

that then

$$(A(I))^g(IJ) = A(J). \quad (6.34)$$

To see this, we introduce the notation $g_j(IJ) := (g_j(I))^{-1} g_j(J)$ and we find a $W^{1,2}$-bound for $g_j(IJ)$ similar to (6.31), as follows. In order to bound $\partial_{\alpha} g_j(IJ)$ we separately consider the cases (a) $\alpha \in I \cup J$ – in which case we assume up to exchanging the roles of $I, J$ that $\alpha \in I$ and (b) $\alpha \notin I \cup J$. In the case (a) we use

$$\left( A_j^{g_j(IJ)} \right)_\alpha = g_j(IJ)^{-1} \left( A_j^{g_j(I)} \right)_\alpha g_j(IJ) + g_j(IJ)^{-1} \partial_{\alpha} g_j(IJ),$$

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and using the bounds (6.30), (6.31) and (6.33), we find that $|\partial_\alpha g_j(IJ)|$ is controlled by $L^2$-integrable quantities. For the case (b), take a third index $\tilde{I}$ containing $\alpha$ and use the cocycle condition $g_j(IJ) = g_j(II)g_j(IJ)$, valid due to (6.33):

$$\partial_\alpha g_j(IJ) = \partial_\alpha \left( g_j(II)g_j(IJ) \right) = \partial_\alpha g_j(II)g_j(IJ) + g_j(II)\partial_\alpha g_j(IJ).$$

By triangle inequality, we thus reduce to case (a). Thus

$$\|g_j(IJ)\|_{W^{1,\infty}([-1,1]^n)} \lesssim \|F_j\|_{L^2([-1,1]^n)}.$$  \hspace{1cm} (6.35)

Since we are assuming that the right-hand side of (6.35) is bounded, we find that $g_j(IJ)$ is bounded in $W^{1,2}$, and therefore we can extract a subsequence that converges weakly in $W^{1,2}$ to a limit $g(IJ)$. The relation (6.33) also passes to the limit, and we find that (6.34) holds.

Because, by (6.33) and (6.34), the connection forms $A(I)$ obtained as weak limits for different indices $I$ as above are connected by the gauges $g(IJ)$, we find that these connection forms come from a global connection form, which is gauge-equivalent to the weak limit $A$.

Combining the outcome of the last two paragraphs, we find that for all $I$ for almost all $T \in [-1,1]^I$ the classes of slices $[i_H^*(\tilde{A})]$ of the weak limit belong to $\mathcal{Y}_4$, and thus for such $T$ we find $i_H^*(\tilde{A}) \in \mathcal{A}_G([-1,1]^I)$, as desired. Moreover by construction $i_H^*(\tilde{A})$ is gauge equivalent to $A(I,T)$ obtained in (6.29) and therefore we have also $\delta(A_j,A) \to 0$ as $j \to 0$, as desired.

7 The case of general base manifolds

In this section we extend the strong closure and compactness results of Theorems 5.2 and 6.1 to the results stated in the introduction in Theorems 1.9 and 1.8, respectively, where the base space is a general Riemannian manifold $(M^n,h)$ rather than the Euclidean cube $[-1,1]^n$ and where the slices we take of our connection forms are by regular levelsets of general functions $f \in C^\infty(M^n,\mathbb{R}^{n-4})$, like in Definition 1.4.

7.1 Locality and $C^1$-invariance of the space of weak connections

We start by noting that our definition of space of weak connections is localizable, and that it is robust under perturbation by regular diffeomorphisms, and even by bi-Lipschitz homeomorphisms.

In fact more generally the structures we study are also invariant under perturbation by bilipschitz transformations, but for this paper we concentrate on regular manifolds $M^n$, for which such more general statement is not needed. The question about what is the lowest regularity assumption on $M_n$ which allows to prove the closure mentioned in Theorem 6.1 is left for future work.

**Lemma 7.1** (Localization of $\mathcal{A}_G(M^n)$). Let $U_\alpha, \alpha \in I$ be an atlas of a compact $n$-dimensional Riemannian manifold $(M^n,h)$. Then the following hold:

1. If a differential form $A \in L^2(M^n,\Lambda^1M^n \otimes g)$ is such that its restriction $A|_{U_\alpha}$ to each $U_\alpha$ is a weak connection, $A|_{U_\alpha} \in \mathcal{A}_G(U_\alpha)$, then $A \in \mathcal{A}_G(M^n)$.

2. If $I$ is finite, then there exists a constant $C > 0$ depending only on $(M^n,h)$ such that if $A, A' \in L^2(M^n,\Lambda^1M^n \otimes g)$ are such that, with the notation of Definition 1.4, for each $\alpha \in I$ we have $\delta(A|_{U_\alpha}, A'|_{U_\alpha}) < \infty$ in $\mathcal{A}_G(U_\alpha)$, then $\delta(A,A') < \infty$ and there holds

$$C^{-1} \sum_{\alpha \in I} \delta(A|_{U_\alpha}, A'|_{U_\alpha}) \leq \delta(A,A') \leq C \sum_{\alpha \in I} \delta(A|_{U_\alpha}, A'|_{U_\alpha}).$$  \hspace{1cm} (7.1)
As the proof reasoning is rather standard we only indicate the overall reasoning, omitting the details.

**Sketch of proof:** For the point (i) note that, indeed, if $M^n$ is compact, then the bounds on the distributional curvature forms $dA|_{U_\alpha} + A|_{U_\alpha} \land A|_{U_\alpha}$ imply the corresponding bound on $dA + A \land A$, whereas the slice condition from Definition 1.4 holding on each $U_\alpha$ implies that it also holds globally on $M^n$.

For point (ii), we may proceed by classical compactness methods, and note that one may pass from $f_\alpha \in C^\infty(U_\alpha, \mathbb{R}^{n-4})$, $\alpha \in I$ to $f \in C^\infty(M^n, \mathbb{R}^{n-4})$ by restriction or by using partitions of unity, conserving information about the local structure of the levelsets.

**Lemma 7.2 (Invariance under $C^1$-diffeomorphisms).** If $\Psi : \Omega \to \Omega'$ is a $C^1$-diffeomorphism with $\Omega, \Omega' \subset \mathbb{R}^n$, then we claim that $\Psi$ establishes a correspondence between $\mathcal{A}_G(\Omega)$ and $\mathcal{A}_G(\Omega')$ in the sense that

1. there holds
   \[
   \mathcal{A}_G(\Omega) = \left\{ A \in L^2(\Omega, \Lambda^1 \mathbb{R}^n \otimes g) : \exists A' \in \mathcal{A}_G(\Omega'), A = \Psi^* A' \right\},
   \] (7.2)

2. there exists $C > 0$ depending only on $(M^n, h)$ and on the bi-Lipschitz constant of $\Psi$, such that
   \[
   C^{-1} \delta(\Psi^* A, \Psi^* A') \leq \delta(A, A') \leq C \delta(\Psi^* A, \Psi^* A').
   \] (7.3)

Note that for an $L^2$-form $A'$ and $\Psi$ Lipschitz, the form $\Psi^* A'$ is well-defined in $L^2(\Omega, \Lambda^1 \mathbb{R}^n \otimes g)$.

**Proof.** If $S^4$ is a generic embedded submanifold, in $\Omega'$, then the slice $i_{S^4}^* A'$ of $A'$ are transferred to slices of $A$ by the $C^1$-submanifold $\Psi(S^4)$, defined by $\Psi^* i_{S^4}^* A'$. If $\Psi$ is a $C^1$-diffeomorphism, then these slices are along $C^1$ submanifolds, as the ball boundaries appearing in Definition 1.4. We consider the case of $S^4$ from now on, the other case being treated similarly.

We may use $\Psi^*$ and composition with $\Psi$ applied to $\tilde{A}$ and $g$, respectively, to transfer the equations $A^g = g^{-1}dg + g^{-1}\tilde{A}g$ to $\Omega$ in the case of $\tilde{A}$ equal to $i_{S^4}^* A'$. The fact that $\|D\Psi\|_{L^\infty}$, $\|D\Psi\|_{L^\infty} < C$ shows that bounds on $g \in W^{1,2}(U_\alpha, G)$ defined locally on elements of a good cover $\{U_\alpha\}$ of such slices $S^4$ can, by chain rule, be transferred to $g \in W^{1,2}(\Psi^{-1}(U_\alpha), G)$, which form a good cover of $\Psi^{-1}(S^4)$. Thus the version of Definition 1.4 as indicated in the discussion following that definition, holds for $\mathcal{A}_G(\Omega)$ as defined by the right-hand side in (7.2), as claimed in point (i).

For proving point (ii), we compose $f$ from Definition 1.4 with $\Psi$ or $\Psi^{-1}$, and use the fact that taking the infimum in (1.2) over $f \in C^\infty$ or over $f \in C^1$ does not change its value. □

### 7.2 Proof of the compactness theorem for $\mathcal{A}_G(M^n)$

In this section, we indicate how to extend the proof of Theorem 6.1 from $\tilde{\mathcal{A}}_G([-1, 1]^n)$ to prove Theorem 1.8.

**Proof of Theorem 1.8, given Theorem 6.1:** We consider separately every family of slicing submanifolds $S^4$ as described in the statement of Definition 1.4 Theorem 1.8. We will find that the weak limit of the $A_j$ coincides on such family with a connection which has, on almost all slices that form a neighborhood $U_{S^4}$ of a given slice $S^4$, local gauges in which it becomes $L^4$-integrable.

**Step 1.** Weak closure in $\tilde{\mathcal{A}}_G([-1, 1]^n)$ with a tame background metric. We first note that the proof of Theorem 6.1 holds as well when the base manifold $[-1, 1]^n$ is endowed with a $C^1$-regular Riemannian metric $h$ such that $\|h - id\|_{C^1([-1, 1]^n)}$ is small enough. Indeed, the
only changes to be applied are in the computation of integrals, in which the volume form \( \text{Vol}_h \) replaces the volume element, and in the computation of norms, where \(|| \cdot |||\) has to be replaced by \(|| \cdot |||_h\). This still allows to find good cubeulations such as in Proposition 4.5. In the proof of the approximation theorem 5.2, the hypothesis that \( h \) is close to the identity allows to still obtain the needed bounds (3.4) for domains making up the given cubeulation. The rest of the proofs are easily adaptable to the present case.

**Step 2. Deformation and localization.** We note, that up to perturbing the \( f \) appearing in Definition 1.4, we may assume that for a.e. \( y \in \mathbb{R}^{n-4} \) with corresponding levelset \( S^4 = f^{-1}(y) \) corresponding to a regular value \( y \in \text{Reg}(f) \), we have for \( r > 0 \) small enough, that a neighborhood \( U_{S^4} = f^{-1}(B_r(y)) \) is foliated by levelsets corresponding to regular values of \( f \) as well. Then \( U_{S^4} \) is \( C^1 \)-diffeomorphic to \( S^4 \times B_r(0) \) and is thus the union of finitely many charts \( U_\alpha \) which are \( C^1 \)-diffeomorphic to \([-1,1]^4 \) with a Riemannian metric close to the Euclidean one. In these charts the slices by \( f^{-1}(y') \cap U_\alpha \) with \( y' \in B_r(y) \) which we need to consider are sent to the sets \([-1,1]^4 \times \{T\} \), for \( T \in [-1,1]^{n-4} \). By using Lemmas 7.2 and 7.1, we then reduce to the case considered in Step 1, and this concludes the proof. \( \square \)

A Distances and equivalence relations on connection and curvature forms

In this section we use the notation from (1.2), (1.4), (1.6) and (1.7), but for simplicity of notations we drop the subscripts “conn” and “curv”.

A.1 Geometric distances on 2-forms

Below we use the notation \( F = \sum_{i<j} F_{ij} dx_i \wedge dx_j \) for a \( g \)-valued 2-form, where \( F_{ij} \in g \). We then define the following pointwise distances between such forms:

\[
d_{pw}(F, F')^2 := \min_{g \in G} |g^{-1} F g - F'|^2 = \sum_{i<j} |g^{-1} F_{ij}^{(1)} g - F_{ij}^{(2)}|^2, \quad (A.1a)
\]

\[
\delta_{pw}(F, F')^2 := \frac{2}{(n-2)(n-3)} \sum_{J \subseteq \{1, \ldots, n\}} \min_{g(J) \in G} \sum_{i<j} |g(J)^{-1} F_{ij} g(J) - F'_{ij}|^2. \quad (A.1b)
\]

We see easily that \( \delta_{pw} \leq d_{pw} \), keeping in mind that each pair \( ij \) belongs to the 4-ple \( J \) for \( \frac{(n-2)(n-3)}{2} \) distinct 4-ples \( J \subset \{1, \ldots, n\} \). The above pointwise definitions directly extend by integration to distances \( d, \delta_1 \) on \( L^2 \)-forms \( F, F' \in L^2(M^n, \wedge^2 TM \otimes g) \). In the case of \( d_{pw} \), we find again the definition (1.6)

\[
d(F, F')^2 = \int_{M^n} d_{pw}(F(x), F'(x))^2 d\text{vol}_h(x) = \inf_{g: M^n \to G} \int_{M^n} |g^{-1} F g - F'|^2 d\text{vol}_h, \quad (A.2a)
\]

and from \( \delta_{pw} \) we define

\[
\delta(F, F')^2 := \int_{M^n} \delta_{pw}(F(x), F'(x))^2 d\text{vol}_h. \quad (A.2b)
\]
In the case $M^n = [-1,1]^n$ we may re-express the above directly via (4.4) and find a distance which is equivalent to $\delta$ defined like (4.6) and to $\delta$ as defined in (1.7):

$$\delta_1(F,F')^2 = \sum_{f \in C_{n, n-4}} \inf_{g: [-1,1]^n \rightarrow G} \int_{[-1,1]^n} |(g^{-1} F g - F') \wedge f^* \omega|^2 \frac{dvol}{|f^* \omega|}$$

$$\simeq \delta(F,F')^2 \quad \text{(A.3a)}$$

$$\simeq \sup_{f \in \text{Lip}([-1,1]^n, \mathbb{R}^{n-4})} \inf_{g: [-1,1]^n \rightarrow G} \int_{[-1,1]^n} |(g^{-1} F g - F') \wedge f^* \omega|^2 \frac{dvol}{|f^* \omega|} \quad \text{(A.3b)}$$

$$= \delta(F,F')^2. \quad \text{(A.3c)}$$

In the above, the equivalence (A.3a) follows by comparison between the supremum and the sum, with implicit constant depending only on $n$, and the equivalence (A.3b),(A.3c) follows by localizing the pointwise distance equivalence

$$\delta_{pw}(F(x),F'(x))^2 \simeq \sup_{H \in \text{Gr}(n,n-4)} \inf_{g \in H} |g^{-1} \omega_H^{-1} F g_H - \omega_H^{-1} F'H|^2. \quad \text{(A.4)}$$

From the equivalences (A.3) we can find the equivalence between the distances defined in terms of all the intermediate classes $C$ of slicing functions $f$ such that $C_{n,n-4} \subseteq C \subseteq \text{Lip}([-1,1]^n, \mathbb{R}^{n-4})$.

While as a direct consequence of the definition $d(F,F') = 0$ if and only if $F,F'$ are gauge-equivalent by a measurable gauge transformation, on the other hand, we couldn’t prove that for general $G$ the same is true under the a priori weaker equivalent conditions that $\delta_1(F,F') = 0 \Leftrightarrow \delta(F,F') = 0 \Leftrightarrow (\delta(F,F') = 0$. In the next subsection however, we prove this in the case of $G = SU(2)$.

### A.1.1 The case of SU(2)

We recall a series of very well-known identifications concerning the groups $SU(2)$, $Sp(1)$ and $SO(3)$, that unfold as follows. Recall the bijective maps

$$Sp(1) \ni w + ix + jy + kz = \alpha + j\beta \simeq \left( \begin{array}{c} \alpha \\ \beta \\ \bar{\alpha} \end{array} \right) \in SU(2)$$

and

$$\mathbb{R}^3 \ni (a_1, a_2, a_3) \simeq ia_1 + ja_2 + ka_3 \in \text{Im} \mathbb{H} \simeq \left( \begin{array}{c} ia_1 \\ -a_2 + ia_3 \\ a_2 + ia_3 \\ -ia_1 \end{array} \right) \in \mathfrak{su}(2).$$

Then one directly verifies that the actions

$$SU(2) \times \mathfrak{su}(2) \ni (g,A) \mapsto g^{-1} Ag \in \mathfrak{su}(2), \quad Sp(1) \times \text{Im} \mathbb{H} \ni (q,v) \mapsto q^{-1} v q \in \text{Im} \mathbb{H}$$

are in fact the same action, if viewed under the above identifications. Moreover if $q = w + ix + jy + kz \in Sp(1)$, $\text{Im} \mathbb{H} \ni v \simeq \bar{a} \in \mathbb{R}^3$ as above, then

$$q^{-1} v q = (-q)^{-1} v q = R_q \bar{a},$$

where the map $Sp(1) \ni q \mapsto R_q \in SO(3)$ is a $2:1$ covering of $SO(3)$ by $Sp(1)$, and $R_q = R_{w+ix+jy+kz}$ is the rotation by $2\theta$ around $(x,y,z) \in \mathbb{R}^3$, where $\cos \theta = w$.

We then find that if a set of vectors in $\mathbb{R}^3$ are identified under $SO(3)$-rotation, then the corresponding matrices in $\mathfrak{su}(2)$ are identified under $SU(2)$-conjugation action, where the identification is uniquely determined modulo a $\mathbb{Z}/2\mathbb{Z}$-action.
Proposition A.1. If $G = SU(2)$ then the pseudo-distances $\delta_{pw}$ and $d_{pw}$ are equivalent.

Proof. We note that since the space $\wedge^2\mathbb{R}^n \otimes g$ is finite-dimensional, thus it suffices to show that $\delta_{pw}(F,G) = 0$ if and only if $d_{pw}(F,G) = 0$. One implication follows directly from the previous observation of the stronger fact $\delta_{pw} \leq d_{pw}$. The implication $\delta_{pw}(F,G) = 0 \Rightarrow d_{pw}(F,G) = 0$ is based on an argument already present in [12, Lem. 3.7], which we slightly extend. If $A,B \in \mathfrak{su}(2)$ are represented by vectors $\vec{a}, \vec{b} \in \mathbb{R}^3$ under the identification $\mathfrak{su}(2) \simeq \mathbb{R}^3$, then $\text{tr}(AB) = 2\vec{a} \cdot \vec{b}$. We denote $F_{ij}, G_{ij}$ by $\mathbb{R}^3$ vectors identified to the 2-form coefficients $F_{ij}, G_{ij}$. If $\delta_{pw}(F,G) = 0$ then for any 4-ple $J \subset \{1, \ldots, n\}$ there exist $g_1 \in SU(2)$ such that $g^{-1}F_{ij}g = G_{ij}$ for all $i < j \in J$, and by thus there exists $R_I \in SO(3)$ such that $R_I F_{ij} = G_{ij}$ for all $i < j \in J$. In particular, $F_{ij} = F_{kl} = g_{ij}$ for all $i < j \in J$ and for any two pairs $i < j, k < l \in \{1, \ldots, n\}$. Due to the general fact that the set of pairwise scalar products form a complete $SO(\ell)$-invariant for $\ell$-ples of vectors in $\mathbb{R}^k$ (as can be easily see by Gram-Schmidt orthonormalization, see also [26, Ch.14]), in the case at hand we find that there exists $R \in SO(3)$ for which $RF_{ij} = G_{ij}$ for all $i < j \in \{1, \ldots, n\}$. By the above identifications, such $R$ determines a $g \in SU(2)$ and up to a factor $\varepsilon \in \{\pm 1\}$, such that $g^{-1}F_{ij}g^2 = G_{ij}$ for all $i < j \in \{1, \ldots, n\}$. This shows that $d_{pw}(F,G) = 0$, completing the proof of the proposition. 

From the above proposition and the definition (A.2a) (and its analogue for the distance $\delta$ on connection forms) we directly have the following:

Corollary A.2. If $G = SU(2)$ then with the above notations $d \approx \delta$, with implicit constant depending only on $n$.

Proving the same result for $G$ other than $SU(2)$ would answer Question 1.6 from the introduction, and we leave this as an open question:

Question A.3. Under which conditions on $G,n$ are the pseudo-distances $d_{pw}$ and $\delta_{pw}$ equivalent over $g$-valued 2-forms in $\mathbb{R}^n$?

This type of question seems to be related to the theory of invariants on Lie groups. Indeed, a seemingly closely related question is what are minimal conditions that allow to infer that for two $m$-ples of matrices $(M_1, \ldots, M_m)$ and $(\bar{M}_1, \ldots, \bar{M}_m)$ with $M_j, \bar{M}_j \in \mathfrak{su}(n)$ for $j = 1, \ldots, m$, there exists $g \in SU(n)$ such that $(\bar{M}_1, \bar{M}_2, \ldots, \bar{M}_m) = (g^{-1}M_1g, g^{-1}M_2g, \ldots, g^{-1}M_mg)$.

This type of question appears in the theory of polynomial invariants, see e.g. [19] or [25], in which we have to replace the role of $O(n)$ by $SU(n)$. However the tools which connect such study to Question A.3 seem to not be sufficiently developed yet.

A.2 Distances on connection 1-forms and on curvature 2-forms

We start by proving that our alternative definitions of Donaldson-type distances between connection forms are actually equivalent:

Lemma A.4. Let $(M^n, h)$ be a compact Riemannian manifold. For $A,A' \in L^2(\Lambda^1M^n, g)$ and for $k \geq 1$, the following holds, with an implicit constant depending only on $(M^n, h)$:

$$\inf_{g \in W^{1,2}(M^n,G)} \int_{M^n} |g^{-1}dg + g^{-1}Ag - A'|^2_h \text{dvol}_h \leq \inf_{g : M^n \rightarrow G} \sup_{f \in C^\infty(M^n, \mathbb{R}^k)} \int_{M^n} |(dg - gA') \wedge f^*\omega^k|^2 \text{dvol}_h.$$  \(A.5\)
Proof. We first show, for the case $g \in W^{1,2}(M^n, G)$, the equivalence

$$\int_{M^n} |g^{-1} dg + g^{-1} Ag - A'|^2_h \, dvol_h$$

$$\simeq \sup_{f \in C^\infty(M^n, \mathbb{R}^k)} \int_{M^n} |(g^{-1} dg + g^{-1} Ag - A') \wedge f^* \omega|^2_h \, dvol_h. \quad (A.6)$$

After establishing (A.6) for $M^n = [-1, 1]^n$, we can pass to the case of generic compact manifolds $M^n$ by the covering argument of Section 7. For $M^n = [-1, 1]^n$ (A.6) follows by estimating the supremum above and below by a finite sum, as done for two-forms in (A.3). Together with the co-area formula, this completes the proof of (A.6) for $g \in W^{1,2}(M^n, G)$.

Again, reducing without loss of generality to the case $M^n = [-1, 1]^n$, we next note that if the distributionally defined form $dg + Ag - gA'$ is represented by an $L^2$-form, then due to the fact that $A, A' \in L^2(f^{-1}(y), \wedge^1 \mathbb{R}^n \otimes g)$ we have $Ag, gA' \in L^2(f^{-1}(y), \wedge^1 \mathbb{R}^n \otimes g)$ as well, and thus by triangle inequality $dg \in L^2$ and thus $g \in W^{1,2}(f^{-1}(y), G)$. Also for $g \in G$ and $a \in g$, our norm satisfies $|a| = |ga| = |ag|$, and in particular

$$|(g^{-1} dg + g^{-1} Ag - A') \wedge f^* \omega| = |(dg + Ag - gA') \wedge f^* \omega|.$$

By testing the second line of (A.5) against coordinate functions $f \in C_{n,n-4}$, we then find that $g \in W^{1,2}(H)$ contemporarily for all coordinate hyperplanes $H$, and thus $g \in W^{1,2}([-1, 1]^n, G)$ like in the first line of (A.5), and we are then justified to use interchangeably (A.6) for $g \in W^{1,2}$ only, and this completes the proof.

In order to define the analogues of $d, \delta$ of from (A.2a) and (A.2b) for connection forms, due to the non-pointwise dependence on $g$ of the gauge-transformed connection forms $g^{-1} dg + g^{-1} Ag$, we can only use the integral formulations directly, and we find again the Donaldson distance and, respectively, a distance equivalent to $\tilde{\delta}$ on $[-1, 1]^n$ to $\delta$ on general manifolds $M^n$:

$$d(A, A')^2 := \min \left\{ \left\| g^{-1} dg + g^{-1} Ag - A' \right\|^2_{L^2([-1, 1]^n)} : g : [-1, 1]^n \to G \text{ measurable} \right\}. \quad (A.7a)$$

and denoting

$$(*) := \min_{g : [-1, 1]^n \to G} \left\| g^{-1} dg + g^{-1} i^*_{H(J,T)} Ag - i^*_{H(J,T)} B \right\|^2_{L^2([-1, 1]^4)} \quad (A.7b)$$

we have

$$\delta^2(A, B) := \frac{2}{(n-2)(n-3)} \sum_{\# J = n-4} \int_{[-1,1]^2} (*_{J \in \mathcal{J}}} \, dT \quad (A.7c)$$

We can directly see by comparing definitions, that $\delta$ from (4.6) is equivalent to $\tilde{\delta}_2$, thus making a link to the study from the previous sections, and to the distance $\delta$ described in the introduction in (1.2).

The following useful approximation result will be proved in a forthcoming work [15]:

**Lemma A.5.** If $A, B, B_k \in \tilde{\mathcal{A}}_G([-1, 1]^n, \mathbb{R}^n \otimes g)$ are weak connection forms such that $F_A = F_B$, then for almost all 2-dimensional surfaces $S^2 \subset [-1, 1]^n$ there exist smooth forms $A_k, B_k \in \Omega^1(S^2, g)$ such that $A_k \to A$ and $B_k \to B$ in $L^2$ and furthermore $F_{A_k} = F_{B_k}$.

Using the above approximation result we can prove the following:
Proposition A.6. Let $A, B \in \tilde{A}_{G}([-1, 1]^{n}, \mathbb{R}^{n} \otimes g)$ be two weak connection 1-forms with curvature forms $F_{A}, F_{B}$, respectively. There exists a measurable function $h : [-1, 1]^{n} \to G$ such that $h^{-1}F_{A}h = F_{B}$ if and only if there exists a measurable function $g : [-1, 1]^{n} \to G$ such that $g^{-1}dg + g^{-1}Ag = B$.

Proof. The existence of $g$ implies the existence of $h$ as above, because

$$F_{g^{-1}dg + g^{-1}Ag} = g^{-1}F_{Ag} ,$$

and we can then take $h := g$.

We now concentrate on the opposite implication: assuming that there exists $h$ such that $h^{-1}F_{A}h = F_{B}$, we prove that there exists $g$ such that $g^{-1}dg + g^{-1}Ag = B$. We may assume that $h \equiv \text{id}$ without loss of generality, up to replacing $A, g$ by $h^{-1}dh + h^{-1}Ah, \text{gh}^{-1}$, respectively.

We first note that for any Lipschitz injective curve $\gamma$, if the $g$-valued 1-forms $A, B$ are integrable along $\gamma$ then we can always explicitly solve the equation

$$g^{-1}\partial_{t}g + g^{-1}Ag = B_{\gamma} , \quad g(\gamma(0)) = \text{id} . \tag{A.8}$$

Indeed, the solution is explicitly expressed as

$$g(\gamma(t)) = P(\gamma|_{[0,t]}, B)^{-1}P(\gamma|_{[0,t]}, A) , \tag{A.9}$$

where the time-ordered path integrals $P(\gamma, A)$ appearing in (A.9) are defined as follows. For a curve $\gamma$, in order to define $P(\gamma, A)$ we associate to each Riemann sum $R_{N} := \sum_{j=1}^{N} \int_{\gamma_{j}} A \in g$ corresponding to a partition of $\gamma$ into a concatenation of injective curves $\gamma_{j}$, the parameter-ordered product

$$\exp(R_{N}) := \prod_{j=1}^{N} \exp_{G} \int_{\gamma_{i}} A , \tag{A.10}$$

where now $\exp_{G}$ equals the usual exponential map of $G$, which is well-defined for $\int_{\gamma_{j}} A \in g$ small enough. Then taking the limit of the expressions (A.10) along any sequence of refining Riemann sums $R_{N} \to \int_{\gamma} A$, we obtain the definition $P(\gamma, A) := \lim_{R_{N} \to \int_{\gamma} A} \exp(R_{N})$. The fact that $g$ as defined in (A.9) solves (A.8) follows directly by differentiation. The fact that the solution to (A.8) is unique follows from the classical theory of ODEs.

If we consider two different injective paths $\gamma^{(1)}, \gamma^{(2)} : [0, 1] \to [-1, 1]^{n}$ along which $A$ and $B$ are integrable and such that $\gamma^{(1)}(0) = \gamma^{(2)}(0) = 0$ and $\gamma^{(1)}(1) = \gamma^{(2)}(1) = p \in [-1, 1]^{n}$ that meet only in 0 and $p$, then the condition for the solutions to the corresponding equations (A.8) to coincide at the common point $p$ is

$$P\left(\gamma^{(1)}, B\right)^{-1}P\left(\gamma^{(1)}, A\right) = P\left(\gamma^{(2)}, B\right)^{-1}P\left(\gamma^{(2)}, A\right) , \tag{A.11}$$

which is equivalent to

$$P\left(\gamma^{(2)}, B\right)P\left(\gamma^{(1)}, B\right)^{-1} = P\left(\gamma^{(2)}, A\right)P\left(\gamma^{(1)}, A\right)^{-1} . \tag{A.12}$$

By coming back to the expressions as limits of (A.10), we see that (A.12) is directly re-expressed in terms of the solutions along the loop $\gamma := \gamma^{(1)} * (\gamma^{(2)})^{-1}$, where by $*$ we denote the concatenation of paths, and $\gamma^{-1}$ represents the path $\gamma$ parameterized backwards, i.e. $\gamma^{-1}(t) = \gamma(1 - t)$. In this notation, equation (A.12) becomes the following:

$$P(\gamma, B) = P(\gamma, A) , \tag{A.13}$$
which in geometric terms is nothing else but the condition that the holonomies of $B$ and $A$ coincide along the loop $\gamma$ starting from 0. By considering a surface $S^2 \subset [-1,1]^n$ such that $\partial S^2$ is parameterized by $\gamma$, and along which $F_A$ and $F_B$ are integrable, we claim that

$$F_A = F_B \Rightarrow P(\gamma, A) = P(\gamma, B).$$  \hfill (A.14)

To prove the above we may first use Lemma A.5 and for the purposes of (A.14) we may assume that $A, B$ are smooth, and up to reparameterization we assume $S^2 = [0,1]^2$. In this case we subdivide $S^2 = [0,1]^2$ into small squares of size $\epsilon$ and consider the discrete homotopy between the loop $\gamma$ based at 0 and with image $\partial [0,1]^2$, and the trivial loop. We note that the homotopy can be subdivided into steps each of which consists in applying the inverse of the holonomy along the polygonal loop along a square of size $\epsilon$. We denote this loop by $\gamma_p$ and let the corresponding square be $\{p, p + (\epsilon, 0), p + (\epsilon, \epsilon), p + (0, \epsilon)\}$. Then there holds

$$P(\gamma_p, A) = 1_G + \epsilon^2 F_A(p)[e_1 \wedge e_2] + o(\epsilon^2).$$  \hfill (A.15)

As $F_A = F_B$, we find that the error between the compositions of all the above elementary homotopies for $A, B$ differs by a quantity bounded by

$$o_{\epsilon \to 0}(1) \int_{[0,1]^2} |F_A| dH_2,$$

Therefore as $\epsilon \to 0$ this error tends to zero, thus (A.14) holds.

As (A.14) allows to prove $P(\gamma, A) = P(\gamma, B)$ for almost all $\gamma$ we conclude the proof that (A.11) also holds, and thus for any two paths $\gamma^{(1)}, \gamma^{(2)}$ along which $A$ and $B$ are integrable we have, with the notation (A.9) for the solution of (A.8),

$$\gamma^{(1)}(t) = \gamma^{(2)}(t) \Rightarrow g(\gamma^{(1)}(t)) = g(\gamma^{(2)}(t)).$$  \hfill (A.16)

This means that the solutions of (A.8) uniquely define a global $g$ over $[-1,1]^n$ on a full measure set. The fact that such $g$ satisfies (A.8) along all paths implies in particular (by taking $\gamma = \gamma_{p,j}$ such that $\gamma(t) = p, \dot{\gamma}(t) = e_j$ for arbitrary $p \in [-1,1]^n, j \in \{1, \ldots, n\}$) there holds

$$\forall j \in \{1, \ldots, n\}, \quad g^{-1} \partial_j g + g^{-1} A_j g = B_j.$$  \hfill (A.17)

Thus $g^{-1} dg + g^{-1} A g = B$, and the proof is complete.

From the above we directly have the following:

**Corollary A.7.** Let $A, B \in A_G([-1,1]^n)$, and consider the pseudo-distances $\delta, d$ be defined over curvature forms as in (A.2) and over connection forms as in (A.7). Then there holds $d(F_A, F_B) = 0$ if and only if $d(A, B) = 0$, and $\delta(F_A, F_B) = 0$ if and only if $\delta(A, B) = 0$.

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