(d, σ)-VERONESE VARIETY AND SOME APPLICATIONS

N. DURANTE*, G. LONGOBARDI*, AND V. PEPE†

ABSTRACT. Let $\mathbb{K}$ be the Galois field $F_{q^2}$ of order $q^2$, $q = p^e, p$ a prime, $A = \text{Aut}(\mathbb{K})$ be the automorphism group of $\mathbb{K}$ and $\sigma = (\sigma_0, \ldots, \sigma_{d-1}) \in A^d$, $d \geq 1$. In this paper the following generalization of the Veronese map is studied:

$$\nu_{d, \sigma} : (v) \in \text{PG}(n - 1, \mathbb{K}) \rightarrow (v^{\sigma_0} \otimes v^{\sigma_1} \otimes \cdots \otimes v^{\sigma_{d-1}}) \in \text{PG}(n^d - 1, \mathbb{K}).$$

Its image will be called the $(d, \sigma)$-Veronese variety $\mathcal{V}_{d, \sigma}$. For $d = t, \sigma$ a generator of $\text{Gal}(F_{q^2}/F_q)$ and $\sigma = (1, \sigma, \sigma^2, \ldots, \sigma^{t-1})$, the $(t, \sigma)$-Veronese variety $\mathcal{V}_{d, \sigma}$ is the variety studied in [13] [14]. Such a variety is the Grassmann embedding of the Desarguesian spread of $\text{PG}(n^t - 1, F_q)$ and it has been used to construct codes $[9]$ and (partial) ovoids of quadrics, see [12] [15]. Here, we will show that $\mathcal{V}_{d, \sigma}$ is the Grassmann embedding of a normal rational scroll and any $d+1$ points of it are linearly independent. We give a characterization of $d+2$ linearly dependent points of $\mathcal{V}_{d, \sigma}$ and for some choices of parameters, $\mathcal{V}_{d, \sigma}$ is the normal rational curve; for $p = 2$, it can be the Segre’s arc of $\text{PG}(3, q^2)$; for $p = 3$ $\mathcal{V}_{d, \sigma}$ can be also a $|\mathcal{V}_{d, \sigma}|$-track of $\text{PG}(5, q^2)$. Finally, we investigate the link between such points sets and a linear code $C_{d, \sigma}$ that can be associated to the variety, obtaining examples of MDS and almost MDS codes.

1. INTRODUCTION

Let $V = V(n, \mathbb{K})$ be an $n$-dimensional vector space over a field $\mathbb{K}$, we will denote by $\text{PG}(V)$ as well as $\text{PG}(n - 1, \mathbb{K})$ the projective space induced by it.

We refer to [7] for the definition of dimension, degree, smoothness and tangent space of an algebraic variety and for the methods and techniques used to study classical varieties.

The Veronese variety $\mathcal{V}_d$ of degree $d$ and dimension $n - 1$ is a classical algebraic variety widely studied over fields of any characteristic [7] [9] and it is the image of the Veronese map

$$\nu_d : (x_0, x_1, \ldots, x_{n-1}) \in \text{PG}(n-1, \mathbb{K}) \rightarrow (\ldots, X_I, \ldots) \in \text{PG}\left(\binom{n+d-1}{d} - 1, \mathbb{K}\right)$$

where $X_I$ ranges over all the possible monomials of degree $d$ in $x_0, x_1, \ldots, x_{n-1}$. The Veronese map can be defined also by

$$\nu_d : (v) \in \text{PG}(n - 1, \mathbb{K}) \rightarrow (v \otimes v \otimes \cdots \otimes v) \in \text{PG}\left(\binom{n+d-1}{d} - 1, \mathbb{K}\right).$$

Now, let $V_i$ be $n_i$-dimensional vector spaces over the field $\mathbb{K}$, $i = 0, 1, \ldots, d - 1$. A Segre variety of type $(n_0, n_1, \ldots, n_{d-1})$ in $\text{PG}(\bigotimes_{i=0}^{d-1} V_i)$ is the set

$$(1) \Sigma_{n_0-1,n_1-1,\ldots,n_{d-1}-1} = \{ (v_0 \otimes v_1 \otimes \cdots \otimes v_{d-1}) \mid v_i \in V_i \setminus \{0\}, i = 0, 1, \ldots, d-1 \}$$
If $n_0 = \ldots = n_d - 1 = n$, we write $\Sigma_{(n-1)^d}$ instead of $\Sigma_{n-1,n-1,\ldots,n-1}$. Then it is clear that $V_d$ turns out to be a linear section of the Segre variety product of $\text{PG}(n-1, \mathbb{K})$ for itself $d$ times.

If $\zeta$ is a collineation of $\text{PG}(V^\otimes d)$ fixing $\Sigma_{(n-1)^d}$, then there exist $\zeta_i$, $i = 0, 1, \ldots, d-1$ semilinear maps of $\text{PG}(V)$, with the same companion field automorphism, and a permutation $\tau$ on $\{0, 1, \ldots, d-1\}$ such that

$$(v_0 \otimes v_1 \otimes \cdots \otimes v_{d-1})^\zeta = (v_{\tau(0)}^\zeta \otimes v_{\tau(1)}^\zeta \otimes \cdots \otimes v_{\tau(d-1)}^\zeta),$$

for a proof of this in positive characteristic see [22].

Let $L_h$ be the set of all projective subspaces of dimension $h$ of $\text{PG}(n-1, \mathbb{K})$, and consider

$$g_{n,h} : \langle v_0, v_1, v_2, \ldots, v_h \rangle \in L_h \mapsto \langle v_0 \land v_1 \land v_2 \land \cdots \land v_h \rangle \in \text{PG}(\land^{h+1} V).$$

where $\land$ is the wedge product and $\land^{h+1} V$ the $(h+1)$-th exterior power of $V$. This map is called Grassmann embedding and its image $G_{n,h}(V)$ is called Grassmanian of subspaces of dimension $h$ of $\text{PG}(V)$. It is well-known that $G_{n,h}(V)$ is an algebraic variety which is the complete intersection of certain quadrics, see [7].

Let $\mathbb{K}$ be the Galois field $\mathbb{F}_{q^t}$ of order $q^t$, $A = \text{Aut}(\mathbb{K})$ be the automorphism group of $\mathbb{K}$ and $\sigma = (\sigma_0, \ldots, \sigma_{d-1}) \in A^d$, $d \geq 1$. The aim of this paper is to study the following generalization of the Veronese map

$$\nu_{d,\sigma} : \langle v \rangle \in \text{PG}(n-1, \mathbb{K}) \mapsto \langle v^\sigma_0 \otimes v^\sigma_1 \otimes \cdots \otimes v^\sigma_{d-1} \rangle \in \text{PG}(n^d - 1, \mathbb{K})$$

and some properties of its image that will be called here the $(d, \sigma)$-Veronese variety $V_{d,\sigma}$. For $d = t$, $\sigma$ a generator of $\text{Gal}(\mathbb{F}_{q^t}/\mathbb{F}_q)$ and $\sigma = (1, \sigma, \sigma^2, \ldots, \sigma^{t-1})$, the $(t, \sigma)$-Veronese variety $V_{t,\sigma}$ is the variety studied in [19] [12] [14]. Such a variety is the Grassmann embedding of the Desarguesian spread of $\text{PG}(nt - 1, \mathbb{F}_q)$ and it has been used to construct codes [6] and (partial) ovoids of quadrics, see [12] [15].

A $[\nu, \kappa]$-linear code $C$ is a subspace of the vector space $\mathbb{F}_q^\nu$ of dimension $\kappa$. The weight of a codeword is the number of its entries that are nonzero and the Hamming distance between two codewords is the number of entries in which they differ. The distance $\delta$ of a linear code is the minimum distance between distinct codewords and it is equals to the minimum weight. A linear code of length $\nu$, dimension $\kappa$, and minimum distance $\delta$ is called a $[\nu, \kappa, \delta]$-code. A matrix $H$ of order $(\nu - \kappa) \times \nu$ such that

$$xH^T = 0 \quad \text{for all } x \in C$$

is called a parity check matrix for $C$. The minimum weight, and hence the minimum distance, of $C$ is at least $w$ if and only if any $w - 1$ columns of $H$ are linearly independent [13] Theorem 10, p. 33]. Each linear $[\nu, \kappa, \delta]$-code $C$ satisfies the following inequality

$$\delta \leq \nu - \kappa + 1,$$

called Singleton bound. If $\delta = \nu - \kappa + 1$, $C$ is called maximum distance separable or MDS, while if $\delta = \nu - \kappa$ the code is called almost MDS. These can be related to some subsets of points in the projective space. More precisely, $C$ is a $[\nu, \kappa, \delta]$-linear code if and only if the columns of its parity check matrix $H$ can be seen as $\nu$ points in $\text{PG}(\nu - \kappa - 1, q)$ each $\delta - 1$ of which are in general position, [4] Theorem 1]. Then, the existence of MDS or almost MDS linear codes is equivalent to the existence of arcs or tracks in projective spaces, respectively.
Definition 1.1. A $k$-arc is a set of $k$ points in $\text{PG}(r,q)$ such that $r + 1$ of them are in general position. An $\ell$-track is a set of $\ell$ points in $\text{PG}(r,q)$ such that every $r$ of them are in general position.

Here, we study the variety $\mathcal{V}_{d,\sigma}$ and we will prove that it is the Grassmann embedding of a normal rational scroll and that any $d + 1$ points of it are in general position, i.e. any $d + 1$ points of $\mathcal{V}_{d,\sigma}$ are linearly independent. Moreover, we give a characterization of $d + 2$ linearly dependent points of this variety and investigate how such a property is interesting for a linear code $C_{d,\sigma}$ that can be associated to the variety.

2. The variety $\mathcal{V}_{d,\sigma}$

Let $V = V(n, K)$ be an $n$-dimensional vector space over the field $K$ and $\text{PG}(V) = \text{PG}(n - 1, K)$ be the induced projective space. In particular, if $K$ is the Galois field of order $q^i$, we will denote the projective space by $\text{PG}(n - 1, q^i)$.

Let $A = \text{Aut}(K)$ be the automorphism group of $K$ and $\sigma = (\sigma_0, \ldots, \sigma_{d - 1}) \in A^d$, $d \geq 1$, and define the map

$$(2) \quad \nu_{d,\sigma} : \langle v \rangle \in \text{PG}(V) \longrightarrow \langle v^{\sigma_0} \otimes v^{\sigma_1} \otimes \cdots \otimes v^{\sigma_{d - 1}} \rangle \in \text{PG}(V^{\otimes d}).$$

Up to the action of the group $\text{PGL}(V)$, we may assume that $\sigma_0 = 1$. It is clear that the map $\nu_{d,\sigma}$ is an injection of $\text{PG}(V)$ into $\text{PG}(V^{\otimes d})$ by the injectivity of the map $\nu_{2,\sigma}$.

We will call $\nu_{d,\sigma}$ the $(d,\sigma)$-Veronese embedding and, as defined before, its image $\mathcal{V}_{d,\sigma}$ the $(d,\sigma)$-Veronese variety. Then $\mathcal{V}_{d,\sigma}$ is a rational variety of dimension $n - 1$ in $\text{PG}(N - 1, K)$, $N = n^d$ and it has as many points as $\text{PG}(n - 1, q^i)$, see [9, 7].

As a consequence of [22] Theorem 3.5 and 3.8, one gets the following

Theorem 2.1. Let $\zeta$ be a collineation of $\text{PG}(V^{\otimes d})$. Then $\zeta$ fixes $\mathcal{V}_{d,\sigma}$ if and only if

$$\langle v \otimes v^{\sigma_1} \otimes \cdots \otimes v^{\sigma_{d - 1}} \rangle^\zeta = \langle v^{\zeta_0} \otimes v^{\zeta_1} \otimes \cdots \otimes v^{\zeta_{d - 1}} \rangle \quad \text{for any} \quad v \in V \setminus \{0\}$$

where $\zeta_0$, is a bijective semilinear map of $V$.

Note that applying the map

$$\langle v \otimes v^{\sigma_1} \otimes \cdots \otimes v^{\sigma_{d - 1}} \rangle^\zeta = \langle v^{\zeta_0} \otimes v^{\zeta_1} \otimes \cdots \otimes v^{\zeta_{d - 1} \otimes \sigma_{d - 1}} \rangle$$

where $\zeta_i$ is a bijective semilinear map, we get a subvariety of $\Sigma_{n - 1}^d$ projectively equivalent to $\mathcal{V}_{d,\sigma}$.

Although many of the results also hold in the case of a general field, from now on it will be assumed, that $K$ is the Galois field $\mathbb{F}_{q^i}$ of $q^i$ elements and $\sigma = (\sigma_0, \sigma_1, \ldots, \sigma_{d - 1}) \in G^d$ with $G = \text{Gal}(\mathbb{F}_{q^i}|K)$. Moreover, since any element $\sigma_i \in G$ is a map of the type $\sigma_i : x \mapsto x^{\alpha_i}$ with $0 \leq h_i < t$ and $0 \leq i \leq d - 1$, hereafter we will suppose that

$$\sigma = (\sigma_0, \ldots, \underbrace{\sigma_0, \sigma_1, \ldots, \sigma_1}_{d_0 \text{ times}}, \ldots, \underbrace{\sigma_m, \ldots, \sigma_m}_{d_m \text{ times}})$$

where $0 = h_0 < h_1 < \ldots < h_m < t$ and we will consider the vector $d_{\sigma} = (d_0, d_1, \ldots, d_m)$ where $d_j$ is the occurrence of $\sigma_j$ in $\sigma$, $0 \leq j \leq m$. Clearly
If $\sigma \in G^d$, the integer

$$|\sigma| = \sum_{i=0}^{d-1} q^{h_i} = \sum_{i=0}^{m} d_i q^{h_i},$$

will be called *norm* of $\sigma$.

Since we consider the ring of polynomials $\mathbb{F}_{q^t}[x_0, x_1, \ldots, x_{n-1}]$ actually as the quotient $\mathbb{F}_{q^t}[x_0, x_1, \ldots, x_{n-1}]/(x_0^{q^t} - x_0, x_1^{q^t} - x_1, \ldots, x_{n-1}^{q^t} - x_{n-1})$, from now on we assume $|\sigma| < q^t$, so that distinct polynomials will be distinct functions over $\mathbb{F}_{q^t}$. By injectivity of map in (2), it is clear that $(d, \sigma)$-Veronese variety $V_{d, \sigma}$ has as many points as $\mathrm{PG}(n-1, q^t)$.

Let $\{e_i \mid i = 0, 1, \ldots, nd - 1\}$ be the canonical basis of $V(nd, \mathbb{F}_{q^t}) = V(nd, q^t)$ and let $\Pi \cong \mathrm{PG}(n-1, q^t)$ be the subspace of $\mathrm{PG}(nd-1, q^t)$ spanned by $\{\langle e_i \rangle \mid 0 \leq i \leq n - 1\}$. Let $\phi$ be the collineation of $\mathrm{PG}(nd-1, q^t)$ such that

$$\langle e_i \rangle \mapsto \langle e_{i+n} \rangle,$$

where the subscripts are taken modulo $nd$. As done in [6, Section 4], for any $\langle v_i \rangle \in \Pi^\phi$, we can identify $v_0 \otimes v_1 \otimes v_2 \otimes \cdots \otimes v_{d-1}$ with $v_0 \wedge v_1 \wedge v_2 \wedge \cdots \wedge v_{d-1}$. Therefore, $V_{d, \sigma}$ is the Grassman embedding of the $d$-fold normal rational scroll

$$S_{n-1, \ldots, n-1}^\sigma = \{ \langle P^\phi, P^{\phi^2}, \ldots, P^{\phi^d} \rangle \mid P \in \Pi \}$$

of $\mathrm{PG}(nd-1, q^t)$, see [7, Ch.8] for a definition of normal rational scroll.

**Example 2.2.** Let $\sigma = 1$, the identity of the product group $G^d$, the $(d, \sigma)$-Veronese variety $V_{d, \sigma}$ is the classical Veronese variety of degree $d$ and $V_{d, \sigma} \subset \mathrm{PG}(N-1, q^t)$ with $N = \binom{n+d-1}{d}$. In this case, $V_{d, \sigma}$ is the Grassman embedding of $S_{n-1, \ldots, n-1} = \{ \langle P, P^\phi, P^{\phi^2}, \ldots, P^{\phi^{d-1}} \rangle \mid P \in \Pi \}$, i.e. the Segre variety $\Sigma_{n-1, d-1}$ of $\mathrm{PG}(nd-1, q^t)$, see again [7, Ch.8].

**Example 2.3.** Let $\sigma$ be a generator of $\mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_q^t)$ and $\sigma = (1, \sigma, \ldots, \sigma^{t-1})$, then we get the algebraic variety introduced in [19, 12, 14] and we will refer to it as the SLP-variety $V_{t, \sigma}$. Let $\hat{\sigma}$ be the semi-linear collineation $\phi \circ \sigma$ of $\mathrm{PG}(t-1, q^t)$ of order $t$. Then the set of points fixed by $\hat{\sigma}$, $\mathrm{Fix}(\hat{\sigma}) \subset \mathrm{PG}(nt-1, q^t)$, is a subspace isomorphic to $\mathrm{PG}(nt-1, q)$ and a subspace of $\mathrm{PG}(nt-1, q^t)$ intersects the subgeometry in a subspace of the same dimension if and only if it is set-wise fixed by $\hat{\sigma}$ (see [12, Section 3]). In this case

$$S_{n-1, \ldots, n-1}^\sigma = \{ \langle P, P^\sigma, P^{\sigma^2}, \ldots, P^{\sigma^{d-1}} \rangle \mid P \in \Pi \},$$

and hence its $(t-1)$-spaces are set-wise fixed by $\hat{\sigma}$. Also, $S_{n-1, \ldots, n-1} \cap \mathrm{Fix}(\hat{\sigma})$ is the Desarguesian $(t-1)$-spread of $\mathrm{PG}(nt-1, q) = \mathrm{Fix}(\hat{\sigma}) \subset \mathrm{PG}(nt-1, q^t)$. Therefore, $V_{t, \sigma}$ is the Grassman embedding of the Desarguesian spread of $\mathrm{PG}(nt-1, q)$. In this case, in fact, $V_{t, \sigma}$ turns out to be a variety of the subgeometry $\mathrm{PG}(n'-1, q) \subset \mathrm{PG}(n'-1, q^t)$ point-wise fixed by the semi-linear collineation of order $t$ of $\mathrm{PG}(n'-1, q^t)$ induced by $\hat{\sigma}$:

$$v_0 \otimes v_1 \otimes \cdots \otimes v_{t-1} \mapsto v_{t-1}^\sigma \otimes v_0^\sigma \otimes \cdots \otimes v_{t-2}^\sigma.$$
By (2), a point of $\PG(n-1, q^d)$ with homogeneous coordinates $(x_0, x_1, \ldots, x_{n-1})$ is mapped by $\nu_{d, \sigma}$ into a point of coordinates

$$\left(\cdots, \prod_{j=0}^{m} X_{I_j}^{\sigma_j}, \cdots\right)$$

where $X_{I_j}$ is a monomial of degree $d_j$ in the variables $x_0, x_1, \ldots, x_{n-1}$. Hence, the $(d, \sigma)$-Veronese variety $V_{d, \sigma}$ is contained in a projective space of vector space dimension

$$(4) \quad N = N_0 N_1 \cdots N_m, \quad N_j = \binom{n + d_j - 1}{d_j}, \quad j = 0, 1, \ldots, m.$$ 

Let $\sigma_0, \sigma_1, \ldots, \sigma_m$ distinct automorphisms in $\sigma$ and $d_\sigma$ the vector of their occurences and suppose that $d_i \sigma_i \neq d_j \sigma_j$ for all $i, j = 0, 1, \ldots, m$ distinct, then we get exactly $N = n^d$ distinct monomials of type $\prod_{j=0}^{m} X_{I_j}^{\sigma_j}$. This is not the case anymore if $d_i \sigma_i = d_j \sigma_j$ for some $i \neq j$. For example, if $q = 2$, $\sigma = (1, 1, 2)$, then $d_0 = 2, d_1 = 1$ and hence $d_0 = d_1 \sigma_1$. Then

$$(x_0, x_1) \otimes (x_0, x_1) \otimes (x_0^2, x_1^2) = (x_0^4, x_0^2 x_1^2, x_0 x_1^3, x_0^3 x_1, x_0 x_1^3, x_0^3 x_1^2, x_0^2 x_1, x_1^4),$$

and we get 5 distinct monomials and $V_{3, \sigma}$ is in fact contained in a projective space of vector space dimension less than $N = 6$.

Recall that an $r$-hypersurface of $\PG(n-1, q^d)$ is a variety such that its points have coordinates vanish an $r$-form of $\F_{q^d}[X_0, \ldots, X_{n-1}]$. If $r = 2$, an $r$-hypersurface is called quadric. In [20], it is shown a lower bound on the degree of an $r$-hypersurface $D$ of $\PG(n-1, q^d)$ after which $D$ could contain all points of the projective space. More precisely,

**Theorem 2.4.** [20] If an $r$-hypersurface $D$ of $\PG(n-1, q^d)$ contains all the points of the space, then $r \geq q^d + 1$.

Let $I$ be a multi-index of the form $I = I_0 I_1 \cdots I_m$, where $I_j$ is a multi-index corresponding to a monomial in $x_0, x_1, \ldots, x_{n-1}$ of degree $d_j$. Once we have labelled the coordinates of $\PG(N - 1, q^d)$ according to the multi-index $I$, we can define a natural linear map $\psi$ that sends the hyperplane of $\PG(N - 1, q^d)$ of equation $\sum_I a_I z_I = 0$ to the $\sigma$-hypersurface of equation

$$\sum_I a_I \prod_{j=0}^{m} X_{I_j}^{\sigma_j} = 0.$$

Then, by Theorem 2.4 we get the following result.

**Theorem 2.5.** Let $\sigma \in G^d$ with $d_\sigma = (d_0, d_1, \ldots, d_m)$, $|\sigma| < q^d$. The $(d, \sigma)$-Veronese variety $V_{d, \sigma}$ is not contained in any hyperplane of $\PG(N - 1, q^d)$ with $N = N_0 N_1 \cdots N_m$ and

$$N_j = \binom{n + d_j - 1}{d_j}, \quad j = 0, 1, \ldots, m.$$

In the following, we generalize some results proved in [5, Section 2] for the SLP-variety.
Theorem 2.11. Let \( \Pi_0, \Pi_1, \ldots, \Pi_{d-1} \) be proper subspaces of PG\((n-1,q^d)\) and suppose that \( P \in PG(n-1,q^d) \) is not contained in any of them. Then, \( P^{\nu,d,\sigma} \) is not contained in \( \langle \Pi_0^{\nu,d,\sigma}, \Pi_1^{\nu,d,\sigma}, \ldots, \Pi_{d-1}^{\nu,d,\sigma} \rangle \).

Proof. Recall that the dual space of \( V(n,d,q^d) \), denoted by \( V(n,d,q^d)^* \), is spanned by the simple tensors \( l_i^0 \otimes l_i^1 \otimes \cdots \otimes l_i^{d-1} \), with \( l_i^j \in V(n,q^d)^* \), and \( l_i^0 \otimes l_i^1 \otimes \cdots \otimes l_i^{d-1} \) evaluated in \( u_0 \otimes u_1 \otimes \cdots \otimes u_{d-1} \) is \( l_i^0(u_0)l_i^1(u_1) \cdots l_i^{d-1}(u_{d-1}) \in F_{q^d} \).

For every \( i \in \{0,1,\ldots,d-1\} \), take an \( l_i^* \in V(n,q^d)^* \) such that \( l_i^* \) vanishes on \( \Pi_i \) and not in \( P^{\nu,d,\sigma} \). Then the hyperplane defined by \( l_0^0 \otimes l_1^1 \otimes \cdots \otimes l_{d-1}^* \) contains the points of \( \Pi_j^{\nu,d,\sigma} \) for \( j \neq i \), \( \forall j = 0,1,\ldots,d-1 \) and it does not contain the point \( P^{\nu,d,\sigma} \).

□

Corollary 2.7. Any \( d+1 \) points of \( \mathcal{V}_{d,\sigma} \), \( d \geq 2 \), are in general position.

Proof. It is enough to take the \( \Pi_i \)'s of dimension 0.

□

Corollary 2.8. A set of \( d+2 \) linearly dependent points of \( \mathcal{V}_{d,\sigma} \) is the \((d,\sigma)\)-Veronese embedding of points contained in a line of PG\((n-1,q^d)\).

Proof. The statement needs to be proved for \( n > 2 \). Let \( P_0, P_1, \ldots, P_d, P_{d+1} \) be \( d+2 \) points whose embedding is linearly dependent. Let \( \Pi_i := P_i \) for \( i = 2,\ldots,d+1 \) and let \( \Pi_1 := \langle P_0, P_1 \rangle \). Suppose that \( P_i \notin \Pi_1 \), for \( i = 2,\ldots,d+1 \), then by Theorem 2.6

\[
P_i^{\nu,d,\sigma} \notin \langle \Pi_0^{\nu,d,\sigma}, \Pi_1^{\nu,d,\sigma}, \ldots, \Pi_{d-1}^{\nu,d,\sigma} \rangle
\]

but by hypothesis \( P_i^{\nu,d,\sigma} \in \langle \Pi_0^{\nu,d,\sigma}, P_1^{\nu,d,\sigma}, \ldots, P_{d-1}^{\nu,d,\sigma} \rangle \subset \langle \Pi_1^{\nu,d,\sigma}, \Pi_2^{\nu,d,\sigma}, \ldots, \Pi_{d-1}^{\nu,d,\sigma} \rangle \), a contradiction.

□

In order to prove the next Corollary, we need the following

Lemma 2.9. [11] Let \( d < |\mathbb{K}| \). Let \( S \) be a set of \( d+2 \) subspaces of PG\((2d-1,\mathbb{K})\) of dimension \( d \), pairwise disjoint, linearly dependent as points of the Grassmannian and such that any \( d+1 \) elements of \( S \) are linearly independent. Then a line intersecting 3 elements of \( S \) intersects all of them.

Since we have assumed \(|\sigma| < q^d\), Lemma 2.9 always applies to \( \mathcal{V}_{d,\sigma} \).

Corollary 2.10. A set of \( d+2 \) linearly dependent points of \( \mathcal{V}_{d,\sigma} \) is the Grassmann embedding of \((d-1)\)-subspaces of the normal rational scroll \( S_{1,1,\ldots,1} \subset PG(2d-1,q^d) \) such that a line intersecting 3 of them must intersect all of them.

Proof. By Corollary 2.8 a set \( \{P_0^{\nu,d,\sigma}, P_1^{\nu,d,\sigma}, \ldots, P_{d+1}^{\nu,d,\sigma}\} \) of \( d+2 \) linearly dependent points of \( \mathcal{V}_{d,\sigma} \) is such that \( P_0, P_1, \ldots, P_{d+1} \) are contained in the same line, hence \( \{P_0^{\nu,d,\sigma}, P_1^{\nu,d,\sigma}, \ldots, P_{d+1}^{\nu,d,\sigma}\} \) is contained in a variety \( \mathcal{V}_{d,\sigma} \) of dimension 1.

Hence, \( \{P_0^{\nu,d,\sigma}, P_1^{\nu,d,\sigma}, \ldots, P_{d+1}^{\nu,d,\sigma}\} \) is the Grassmann embedding of the \((d-1)\)-subspaces of the normal rational scroll \( S_{1,1,\ldots,1} \subset PG(2d-1,q^d) \). Then the result follows from Corollary 2.4 and Lemma 2.9.

□

Theorem 2.11. A set of \( d+2 \) linearly dependent points of \( \mathcal{V}_{d,\sigma} \) is the \( \sigma \)-Veronese embedding of points on a subline \( \cong PG(1,q^d) \), where \( F_{q^d} \) is the largest subfield of \( F_{q^d} \) fixed by \( \sigma \) in \( \sigma \).
Let \((u_i \otimes u_i^\sigma_1 \otimes \cdots \otimes u_i^\sigma_{d-1})\), \(i = 0, 1, \ldots, d + 1\) be \(d + 2\) linearly dependent points of \(V_{d,\sigma}\), and by Corollary 2.8, we can assume \(V_{d,\sigma}\) to be of dimension 1. Then, embed \(PG(1, q^2)\) as the subspace of \(PG(2d - 1, q^2)\) spanned by \(\langle \epsilon_0, \epsilon_1 \rangle\), say \(\Pi\), and hence we can write

\[
u_i \otimes u_i^\sigma_1 \otimes \cdots \otimes u_i^\sigma_{d-1} = u_i \wedge u_i^\sigma_1 \wedge \cdots \wedge u_i^\sigma_{d-1} - 1.\]

We stress out that \(\phi^j\) and \(\sigma_j\) commute and that the vectors \(u_i\)'s are pairwise not proportional. Let \(S_i := \langle u_i, u_i^\phi_1, \ldots, u_i^\phi_{d-1}\rangle\), for all \(i = 0, 1, \ldots, d + 1\), so we observe that \(S_i \cap S_j = \emptyset \forall i \neq j\). Then take a point \(P \in S_0\) such that \(\Pi \notin \langle \Pi^h, h \neq j \rangle\) for any fixed \(j \in \{0, 1, \ldots, d - 1\}\). The subspace \(\langle P, S_1 \rangle\) intersects \(S_2\) in a point, say \(R\). Let \(\ell\) be the line spanned by \(P\) and \(R\). Then \(\ell\) has non empty intersection with \(S_1\) as well. Hence, by Corollary 2.10, \(\ell\) has non empty intersection with all the \(S_i\)'s. By the choice of \(P\), the line \(\ell\) is not contained in any \(\langle \Pi^h, h \neq j \rangle\) for a fixed \(j \in \{0, 1, \ldots, d - 1\}\). If \(\ell\) intersects \(\langle \Pi^h, h \neq j \rangle\) for some \(j \in \{0, 1, \ldots, d - 1\}\), then it would be projected to a unique point of \(\Pi^\phi\) from \(\langle \Pi^h, h \neq j \rangle\). Since \(u_i \neq u_h \forall i \neq h\), then \(u_i^\phi \neq u_h^\phi \forall i \neq h\) and \(\ell\) can be projected on a unique point only if \(\ell \cap S_1\) is in \(\langle \Pi^\phi, h \neq j \rangle\) for all the \(S_i\)'s except one, a contradiction. Indeed, the point \(\ell \cap S_i = \langle \lambda_0 u_i + \lambda_1 u_i^\phi_1 + \cdots + \lambda_d u_i^\phi_{d-1}\rangle\) and the projection of \(\ell \cap S_i\) over \(\Pi^\phi\) is the point \(\langle u_i^\phi_{j_1} \rangle\), so \(h_j\) cannot be zero. Therefore, \(\ell \cap \langle \Pi^h, h \neq j \rangle = \emptyset\) for any fixed \(j \in \{0, 1, \ldots, r - 1\}\). Hence the projection of \(\ell\) on \(\Pi^\phi\) is an isomorphism of lines, say \(p_j\) and \((\ell \cap S_i)^{p_j} = \langle u_i^\phi_{j_1} \rangle\).

\[	ext{By } (\ell \cap S_i)^{p_j} = (\ell \cap S_i)^{p_0} = (\ell \cap S_i)^{p_0} \text{ we get that } (\ell \cap S_i)^{p_0} \text{ is fixed by the semi-linear collineation } \sigma_j \phi^j p_j.\]

If a semi-linear collineation of \(\Pi \cong PG(1, q^2)\) fixes at least 3 points, then it fixes a subline \(\cong PG(1, q')\), where \(F_{q'}\) is the subfield of \(F_{q^2}\) fixed by \(\sigma_j\). This is true for all \(\sigma_j\) in \(\sigma\).

Finally, since the algebraic variety \(\Sigma_{n-1}d\) has dimension \((n - 1)\) and degree \(\frac{d(n - 1)!}{(n - 1)!} = \frac{d(n - 1)!}{(n - 1)!}\), a general subspace of \(PG(N - 1, q^e)\) of codimension \(d(n - 1)\) contains at most \(\frac{d(n - 1)!}{(n - 1)!}\) points of \(V_{d,\sigma}\).

Moreover, the Segre variety is smooth and hence the tangent space \(T_P(\Sigma_{n-1}d)\) to \(\Sigma_{n-1}d\) at a point \(P = \langle u_0 \otimes v_1 \otimes \cdots \otimes v_{d-1} \rangle\) has dimension \(d(n - 1)\) and it spanned by the \(d\) subspaces

\[
\langle \langle u_0 \otimes v_1 \otimes \cdots \otimes v_{d-1} \rangle \otimes u_i \otimes v_{i+1} \otimes \cdots \otimes v_{d-1} \rangle \mid \langle u_i \rangle \in PG(n - 1, q^e) \rangle \cong PG(n - 1, q^e).
\]

These subspaces pairwise intersect only in \(P\) and they are the maximal subspaces contained in \(\Sigma_{n-1}d\) through the point \(P\), and \(\Sigma_{n-1}d\) does not share with \(V_{d,\sigma}\) the property proved in Corollary 2.7. We have, in fact, \(T_P(\Sigma_{n-1}d) \cap V_{d,\sigma} = P\) for each \(P \in V_{d,\sigma}\).

3. THE CODE \(C_{d,\sigma}\)

As we have seen in Example 2.8 the SLP-variety turns out to be a variety of a subgeometry of order \(q\), even though the array \(\sigma\) is defined on a finite field of order \(q^e\), hence among all the possible choice of \(\sigma\) and \(n\), for \(q\) 'big enough' \(V_{1,\sigma}\) is the variety with the most 'dense' set of points of a projective space with the property that any \(d + 1\) points are independent. In this case, since \(d = t\) and, as proved in [5], \(t + 2\) linearly dependent points are contained in a normal rational curve of
degree \(t\) of \(\text{PG}(t, q)\), \(q > t\).

For the classical Veronese variety of degree \(d\), hence for \(\sigma = 1\), Theorem 2.11 implies that \(d + 2\) linearly dependent points are contained in the Veronese embedding of degree \(d\) of a line, hence in a normal rational curve of degree \(d\) of \(\text{PG}(d, q')\).

Finally, for a general \((d, \sigma)\)-Veronese variety, if \(d + 2 > q' + 1\), with \(q'\) defined as in Corollary 2.11 every \(d + 2\) points of \(\mathcal{V}_{d, \sigma}\) are linearly independent, hence, for ‘small’ \(q'\), it provides a dense set of points with that property. More precisely, we get \(d' = \frac{d'^2 - 1}{q' - 1}\)

points in \(\text{PG}(N - 1, q')\) such that any \(d + 2\) of them are in general position. Sets of points with properties of this sort are studied for their connections with linear codes.

If \(H\) is the matrix whose columns are the coordinates vectors of the points of the variety \(\mathcal{V}_{d, \sigma}\), we get a code \(\mathcal{C}_{d, \sigma}\) and we may study the minimum distance of it and characterize the codewords of minimum weight (for an overview on this topic, see, e.g., [2]).

**Definition 3.1.** Let \(\mathcal{V}_{d, \sigma}\) be a \((d, \sigma)\)-Veronese variety and denote by \(\mathcal{C}_{d, \sigma}\) the code whose parity check matrix \(H\) of order \(N \times (\frac{q^m - 1}{q - 1})\) has columns that are the coordinate vectors of the points of the variety \(\mathcal{V}_{d, \sigma}\).

Clearly, the order of the columns of \(H\) is arbitrary, so that Definition 3.1 makes sense only up to code equivalence, as a permutation of the columns that is not usually an automorphism of the code, see [5, Remark 3.3].

**Definition 3.2.** The support of a codeword \(w \in \mathcal{C}_{d, \sigma}\) is the set of the points of the variety \(\mathcal{V}_{d, \sigma}\) corresponding to the non-zero positions of \(w\).

As showed in [5, Theorem 3.5], the following result holds:

**Theorem 3.3.** Let \(\sigma \in \mathbb{G}^d\) with \(d_\sigma = (d_0, d_1, \ldots, d_m)\), \(|\sigma| < q'\) and \(\mathbb{F}_{q'}\) be the largest subfield fixed by \(\sigma\)’s. If \(d < q'\) then the code \(\mathcal{C}_{d, \sigma}\) has length \(r = \frac{q'^r - 1}{q' - 1}\) and parameters \([r, r - N, d + 2]\).

**Proof.** Since \(|\mathcal{V}_{d, \sigma}| = |\text{PG}(n - 1, q')|\) the code \(\mathcal{C}_{d, \sigma}\) has length \(\frac{q'^r - 1}{q' - 1}\). Moreover, since \(\mathcal{V}_{d, \sigma}\) is not contained in any hyperplane of \(\text{PG}(N - 1, q')\), the vector space dimension of the \(N \times r\) matrix \(H\) is maximal and so the dimension of the code is \(r - N\). By Corollary 2.11 guarantees that any \(d + 1\) columns of \(H\) are linearly independent; thus, by [13, Theorem 10, p. 33], the minimum distance of \(\mathcal{C}_{d, \sigma}\) is at least \(d + 2\). The image under \(\nu_{d, \sigma}\) of the canonical subline \(\text{PG}(1, q')\) of \(\text{PG}(n - 1, q')\) determines a submatrix \(H'\) of \(H\) with many repeated rows; indeed, the points represented in \(H\) constitute a normal rational curve \(\text{PG}(d, q')\) and it follows that any \(d + 2\) such points are necessarily dependent. Hence, the minimum distance is exactly \(d + 2\). \(\square\)

Now, as in [5, Theorem 3.7], by the characterizations of sets of \(d + 2\) points of \(\mathcal{V}_{d, \sigma}\) which are linearly dependent yields a characterization of the minimum weight codewords of the associated code. More precisely,

**Theorem 3.4.** A codeword \(w \in \mathcal{C}_{d, \sigma}\) has minimum weight if and only if its support consists of \(d + 2\) points contained in the image of a subline \(\text{PG}(1, q')\), \(d < q'\), where \(\mathbb{F}_{q'}\) is the largest subfield of \(\mathbb{F}_{q'}\) fixed by \(\sigma\) for all \(\sigma\) in \(\sigma\).
Suppose \( d \geq q' \) where \( \mathbb{F}_{q'} \) is the largest subfield of \( \mathbb{F}_{q^t} \) fixed by \( \sigma_i \), for all \( \sigma_i \) in \( \sigma \). By Theorem 2.11, the code \( C_{d, \sigma} \) is a linear code with minimum distance \( d + 3 \leq \delta \leq N + 1 \). If the Singleton bound is reached, then it is an MDS code. Let \( N \) be as in [1] with \( \sum_{i=0}^{m} d_i = d \). If \( n = 2 \), then

\[
N = \prod_{i=0}^{m} (d_i + 1)
\]

and the minimum is reached for \( m = 1, d_0 = d - 1, d_1 = 1 \), so \( N = 2d \).

If \( \sigma \) is such that \( \text{Fix}(\sigma) \cap \mathbb{F}_{q'} = \mathbb{F}_p \), where \( p \) is the characteristic of the field, since we should have \( d \geq p \), the smallest possible \( d = p \) and in this case

\[
\sigma = (1, 1, \ldots, 1, \sigma)
\]

getting that \( V_{d, \sigma} \) is a set of \( q' + 1 \) points in \( \text{PG}(2p - 1, q') \) such that any \( p + 2 \) of them are in general position. So the code \( C_{d, \sigma} \) is a \([q' + 1, q' - 2p + 1]-\)linear code with minimum distance at least \( p + 3 \) and the Singleton bound \( 2p + 1 \). Now, if \( \sigma : x \mapsto x^p \), then \( V_{p, \sigma} \) is the normal rational curve of \( \text{PG}(2p - 1, q') \); hence \( C_{p, \sigma} \) is an MDS code.

Furthermore for \( p \in \{2, 3\} \), the following cases can also occur

- for \( p = 2 \), \( \sigma : x \mapsto x^{2h} \), \( 1 < h < et \), \( V_{2, \sigma} \) is either the Segre arc or the normal rational curve (for \( h = et - 1 \)), hence \( C_{2, \sigma} \) is an MDS code.
- for \( p = 3 \), \( \sigma : x \mapsto x^{3h} \), \( 1 < h < et \), \( V_{3, \sigma} \) is a \((3^{et+1})\)-track of \( \text{PG}(5, 3^{et}) \); hence \( C_{3, \sigma} \) is a so called almost MDS code. [4], see next Theorem 3.6.

Clearly, as \( p \) gets larger, the minimum distance gets smaller than the Singleton bound. Before showing the announced result, we recall the following theorem due to Thas [21] and of which Kaneta and Maruta gives an elementary proof.

**Theorem 3.5.** [10] Theorem 1] In \( \text{PG}(r, q), r \geq 2 \) and \( q \) odd, every \( k \)-arc with

\[
q - \sqrt{q}/4 + r - 1/4 \leq k \leq q + 1
\]

is contained in one and only one normal rational curve of the space. In particular, if \( q > (4r - 5)^2 \), then every \((q + 1)\)-arc is a normal rational curve.

**Theorem 3.6.** Let \( q = 3^r \) and \( \sigma : x \in \mathbb{F}_{q^t} \mapsto x^{3h} \in \mathbb{F}_{q^t} \), \( 1 < h < et \), \( \gcd(h, et) = 1 \) with \( et > 4 \). Then \( V_{3, \sigma} \) with \( \sigma = (1, 1, \sigma) \) is a \((3^{et+1})\)-track of \( \text{PG}(5, 3^{et}) \) and \( C_{3, \sigma} \) is an almost MDS.

**Proof.** By the previous considerations, since the \([q' + 1, q' - 5]-\)code \( C_{d, \sigma} \) has distance at least 6, the result follows showing the existence of 6 columns of \( H \) linearly dependent or equivalently that there exists 6 points linearly dependent of the set

\[
V_{3, \sigma} = \{(1, z, z^2, z^3, z^{3h+1}, z^{3h+2}) : z \in \mathbb{F}_{q^t}\} \cup \{(0, 0, 0, 0, 0, 0, 1)\}.
\]

Suppose that any 6 points of \( V_{3, \sigma} \) with \( \sigma = (1, 1, \sigma) \) are linearly independent, hence \( V_{3, \sigma} \) is an arc of \( \text{PG}(5, q^t) \). By Theorem 3.3 \( V_{3, \sigma} \) must be projectively equivalent to rational normal curve

\[
\{(1, y, y^2, y^3, y^4, y^5) : y \in \mathbb{F}_{q^t}\} \cup \{(0, 0, 0, 0, 0, 1)\}.
\]
Since the normal rational curve has a 3-transitive automorphisms group, we can always assume that there is a collineation of $\text{PG}(5, q^t)$ fixing $(0,0,0,0,0,1)$ and $(1,0,0,0,0,0)$. Moreover, w.l.o.g. we can assume that this collineation has the identity as companion automorphism.

Hence there must be $f_i(y) \in \mathbb{F}_{q^t}[y]$ of degree at most 5 and linearly independent such that

$$(f_0(y), f_1(y), f_2(y), f_3(y), f_4(y), f_5(y)) = (1, z, z^2, z^{3^h}, z^{3^h+1}, z^{3^h+2})$$

with $f_i(y)$ vanishing in 0 for $i \in \{1, 2, 3, 4, 5\}$ and $f_0(0) = 1$ up to a nonzero scalar. So, $f_0(y) = 1$ for all $y \in \mathbb{F}_{q^t}$ and since $\deg f_0(y) \leq 5 < q^t$, then $f_0(y) = 1$. Note that $\deg f_i(y) \neq 0$ for $i = 1, 2, 3, 4, 5$ and

$$f_2(y) = f_1(y)^2 \mod y^{q^t} - y,$$

but $2 \deg f_1(y) \leq 10 < q^t$, and hence $f_2(y) = f_1(y)^2$ and $\deg f_1(y) \leq 2$. Similarly,

$$f_4(y) = f_1(y)^{3^h} \mod y^{q^t} - y,$$

but $3^h \deg f_1(y) \leq 3^h \cdot 2 < q^t$, so $f_4(y) = f_1(y)^{3^h}$ and $3^h \deg f_1(y) \leq 5$, obtaining $3^h \leq 5$, a contradiction. \hfill \square

Actually, the result above holds for $q^t = 27, 81$ as well, this is verified by the software MAGMA, obtaining an infinite family of almost MDS codes or, equivalently, an infinite family of $(3^{et} + 1)$-tracks of PG$(5, 3^{et})$ with $et > 2$.

4. Acknowledgement

The authors thank Italian National Group for Algebraic and Geometric Structures and their Applications (GNSAGA - INdAM) for having supported this research.

Data Deposition Information: No datasets have been used

References

[1] S. Ball, M. Lavrau, Arcs in finite projective space, Ems Surveys in Mathematical Sciences 6(1–2) (2019), 133–172.
[2] E.R. Berlekamp, Algebraic Coding Theory, Mc Graw-Hill (1968).
[3] L.R.A. Casse, A solution to B. Segre’s problem $I_{r,q}$. Atti Accad. Naz. Lincei Rend. 46 (1969), 13–20.
[4] M.A. De Boer, Almost MDS codes Des. Codes Cryptogr. 9 (1996) 143-155.
[5] L. Giuzzi, V. Pepe, Families of twisted tensor product codes, Des. Codes Cryptogr. 67 (2013), 375–384.
[6] L. Giuzzi, V. Pepe, On some subvarieties of the Grassmann variety, Linear Multilinear Algebra 63 (2015), 2121–2134.
[7] J. Harris, Algebraic Geometry Book, A First Course, Graduate Texts in Mathematics 133, Springer-Verlag (1992).
[8] J.W.P. Hirschfeld, Projective Geometries over finite fields (1990).
[9] J.W.P. Hirschfeld, J.A. Thas, General Galois Geometries, Springer Monographs in Mathematics (2016).
[10] H. Kaneta, T. Maruta, An elementary proof and an extension of Thas’ theorem on k-arcs, Math. Proc. Cambridge Philos. Soc. 105(3) (1989), 459–462.
[11] G. Lunardon, Planar fibrations and algebraic subvarieties of the Grassmann variety, Geom. Dedicata 16 (1984), 291–313.
[12] G. Lunardon, Normal Spreads, Geom. Dedicata 75 (1999), 245–261.
[13] F.J. MacWilliams, N.J.A Sloane, The Theory of Error Correcting Codes, North-Holland, Amsterdam (1977).
(d, σ)-VERONESE VARIETY AND SOME APPLICATIONS

[14] V. Pepe, On the algebraic variety \( V_{r,t} \), *Finite Fields and Their Applications* **17**(4) (2011), 343–349.

[15] V. Pepe, Desarguesian and Unitary complete partial ovoids, *J. Algebr. Comb.* **37** (2013), 503–522.

[16] B. Segre, Curve razionali normali e \( k \)-archi negli spazi finiti, *Ann. Mat. Pura Appl.* **39** (1955), 357–379.

[17] B. Segre, Ovals in a finite projective plane. *Canadian Journal of Mathematics* **7**(10) (1955), 414–416.

[18] B. Segre, Ovali e curve nei piani di Galois di caratteristica due, *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat.* **8** (1962).

[19] B. Segre, Teoria di Galois, fibrazioni proiettive e geometrie non desarguesiane, *Ann. Mat. Pura Appl.* **64** (1964), 1–76.

[20] G. Tallini, Sulle ipersuperfici irriducibili d’ordine minimo che contengono tutti i punti di uno spazio di Galois \( S_{r,q} \), *Rend. Mat. Appl.* **20**(5) (1961), 431–479.

[21] J. A. Thas, Normal rational curves and \( k \)-arcs in Galois spaces, *Rend. Mat. Appl.* **1** (1968), 331–334.

[22] R. Westwick, Transformations on tensor spaces, *Pac. J. Math.* **23** (1967), 613–620.

*Dipartimento di Matematica e Applicazioni “Renato Caccioppoli”, Università degli Studi di Napoli “Federico II”, Via Vicinale Cintia, 26, 80126 Napoli, Italy
Email address: {giovanni.longobardi,ndurante}@unina.it

†Dipartimento di Scienze di Base ed Applicate per l’Ingegneria, ‘Sapienza’ Università di Roma, Via Antonio Scarpa, 14, 00161 Roma, Italy
Email address: valepepe@sbai.uniroma1.it