Shape stability of a quadrature surface problem in infinite Riemannian manifolds

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Abstract

In this paper, we give a simple control on how an optimal shape can be characterized. The framework of Riemannian manifold of infinite dimension is essential. And the covariant derivative plays a key role in the computation and in the analysis of qualitative properties from the shape hessian. The control depends only on the mean curvature of the domain which is a minimum or a critical point.

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1 Introduction

The search for the notion of quadratures made a prodigious leap forward (1669-1704) thanks to Leibniz and Newton who, with the infinitesimal calculus, made the link between quadrature and derivative. A brief remind could be interesting to see this link with shape optimization. Regarding a bounded domain of $\Omega \subset \mathbb{R}^N$ with regular boundary, for instance $C^2$, $\mu$ a signed measure compactly supported in $\Omega$, it is well known there is a measure $\sigma$ called a balayage measure carried by the surface $\partial \Omega$ and having the same potential as $\mu$ outside $\overline{\Omega}$, see for instance [29], [32] for more details about this topic. And in this case, by classical approximation technique, one has the following relation:

$$\int_{\partial \Omega} h d\sigma = \langle h, \mu \rangle \quad \forall h \in H(\overline{\Omega})$$

where $H(\overline{\Omega})$ denotes the set of functions harmonic on a neighborhood of $\overline{\Omega}$. And we say that $\partial \Omega$ is a quadrature surface with respect to $\mu$ if (1) is satisfied.

This notion is closely linked with the overdetermined Cauchy elliptic problem. And one can claim

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that $\partial \Omega$ is a quadrature surface if and only if there is a solution to the following overdetermined Cauchy problem

$$
\begin{cases}
-\Delta u_\Omega &= \mu \quad \text{in } \Omega \\
u_\Omega &= 0 \quad \text{on } \partial \Omega \\
\frac{\partial u_\Omega}{\partial \vec{\nu}} &= 1 \quad \text{on } \partial \Omega
\end{cases}
$$

(2)

The above quadrature surface free boundary problem has some physical motivations and can be related to many areas such as free streamlines, jets, Hele-shaw flows, electromagnetic shaping, gravitational problems etc. It has been intensively studied at least during the last forty years, see for example [39], [19] and the references contained in these books for more details. Among these works, some authors have established an intimate link between the existence of quadrature surfaces and the solution of free boundary problems governed by overdetermined partial differential equations, see for instance [25], [37], [38], [17] and references therin.

The quadrature surface problem (2) can be tackle by a shape optimization approach when $\mu$ is regular enough, for instance by taking it in $L^2(\Omega)$, $\text{supp}(\mu) \subset \Omega$. The authors invite the readers who get interest on details to see for instance [9] and [17].

Before proceeding further, let us remind that in optimisation or in the study of minimal action, one of the essential questions is the characterization of an optimum if it exists. When one is in a differentiable environment, i.e. if the objective function is differentiable as well as its constraints if any, the first derivative and second one (hessian) play a fundamental role. In finite dimension, the characterization results are very well known even when we are in Banach spaces.

On the other hand, when we have to deal with admissible sets of regular openings of $\mathbb{R}^N, N \geq 2$ containing the optimum to be characterized, the question is to be treated in a more delicate way. Indeed, if we consider a shape optimization problem where the variable is a regular open of class $C^2$ and in which a boundary value problem of partial differential equations is posed, then the computation of the second derivative. Added to this, the equivalence of norms is to be handled if any exist. In this paper, we aim at studying these issues of characterization of critical or optimal domains in the case where the minimum of the considered shape functional exists, in infinite dimensional Riemannian structures. To do so, it is crucial to find form spaces and associated metrics.

Finding a shape space and an associated metric is a challenging task and different approaches lead to various models. One possible approach is to do as in [33], [34]. These authors proposed, a survey of various suitable inner products is given, e.g., the curvature weighted metric and the Sobolev metric. There are various types of metrics on shape spaces, e.g., inner metrics [1], [31] like the Sobolev metrics, outer metrics [6], [30], [34], [6, 31, 43], metamorphosis metrics [27], [27, 60], the Wasserstein or Monge-Kantorovic metric on the shape space of probability measures [2], [7], the Weil-Petersson metric [31], [35], current metrics [18] and metrics based on elastic deformations [23, 50, [18, 64]. However, it is a challenging task to model both, the shape space and the associated metric. There does not exist a common shape space or shape metric suitable for all applications. Different approaches lead to diverse models. The suitability of an approach depends on the requirements in a given situation.

In recent work, it has been shown that PDE constrained shape optimization problems can be embedded in the framework of optimization on shape spaces. E.g., in [12], shape optimization is considered as optimization on a Riemannian shape manifold, the manifold of smooth shapes. Moreover, an inner product, which is called Steklov- Poincaré metric, for the application of finite element (FE) methods is proposed in [49].

As pointed out in [41], shape optimization can be viewed as optimization on Riemannian shape
manifolds and the resulting optimization methods can be constructed and analyzed within this framework. This combines algorithmic ideas from [1] with the Riemannian geometrical point of view established in [4].

In [33, 34, 24, 25], a geometric structure of two-dimensional \( C^\infty \) shapes was introduced and subsequently generalized to shapes in higher dimensions in [35, 4, 5, 6, 26]. Essentially, closed curves (and closed higher-dimensional surfaces) are identified with mappings of the unit sphere to any shape under consideration. In two dimensions, this can be naturally motivated by the Riemannian mapping theorem. In this work, we focus on two-dimensional shapes as subsets of \( \mathbb{R}^2 \). And considering [9], [17], we think that it is possible to write our work in high dimensions and even if \( \Omega \) is an open set with boundary of a compact \( N \)-dimensional Riemannian noted \( \mathcal{M} \).

One of our main question is the following:

Does it possible to express the Hessian of a shape functional to get sufficient conditions so that the critical domain of the functional \( J \) should its minimum? To answer this question, we study the positiveness of the quadratic form of the functional \( J \) which is related to the quadrature surface that is never but the following free boundary problem

\[
\begin{align*}
-\Delta u_\Omega &= f \quad \text{in } \Omega \\
u_\Omega &= 0 \quad \text{on } \partial \Omega \\
-\frac{\partial u_\Omega}{\partial \nu} &= k \quad \text{on } \partial \Omega
\end{align*}
\]

\( k \) is a positive constant, and \( f \in L^2(\Omega), \text{supp} f \subset \Omega, \vec{v} \) is the exterior unit normal vector. The above quadrature surface can be formulated as the following shape optimization problem:

\[
\min_{\Omega \subset \mathbb{R}^2} J(\Omega)
\]

under the following partial differential equations contraints

\[
\begin{align*}
-\Delta u_\Omega &= f \quad \text{in } \Omega \\
u_\Omega &= 0 \quad \text{on } \partial \Omega
\end{align*}
\]

where

\[
J(\Omega) = \frac{1}{2} \int_\Omega |\nabla u_\Omega|^2 dx + \frac{k^2}{2} |\Omega| \tag{3}
\]

is a real valued shape differentiable objective function, where \( |\Omega| = \int_\Omega dx \).

In [9], [17], there are all details on existence results of quadrature surface by using shape optimization tools.

And the second aim question is the problem computation of the Hessian in the infinite Riemannian framework and how it can be related to the second shape derivative to deduce qualitative properties when the minimum of a regular enough shape functional exists or when \( \Omega \) is a critical point that is to say that the first derivative of \( J(\Omega) \) is equal to zero.

The paper is organized as follows:

In section 2, we give a brief survey, based on works in [33, 34], about the characterization of the tangent space in a framework of Riemannian manifold of infinite dimension.

The section 3 deals with the optimality condition of first order for the shape optimization and the
computation of the covariant derivative. This latter plays a key role in our final result. We shall give a direct way to compute it which appears as a simplified expression.

In section 4, we shall recall some technical but classical computations of shape second derivative and established a result (stated as a proposition) giving the expression of the quadratic form associated to the quadature surface problem.

The section 5 which contains our main contributions, is devoted to the positiveness of the shape hessian in a Riemannian point of view of infinite dimension. And, we shall propose simple control which allows to get key information on the optimal shape domains when these latters are strict local mminimum or critical point of the shape functional considered.

2 Characterization of tangent space at a point of $B_e$

The aim is to analyze the correlation of the Riemannian geometry on infinite dimensional manifolds $B_e$ with shape optimization.

The authors would like to stress on the fact that, what follow has been already done in pioneering works, see [33], [34], [35]. We only reproduce some fundamental steps related to our work.

Let $\Omega$ be a simply connected and compact subset of $\mathbb{R}^2$ with $\Omega \neq \emptyset$ and $C^\infty$ boundary $\partial \Omega$. As always in shape optimization, the boundary of the shape is all that matters. Thus we can identify the set of all shapes with the set of all those boundaries.

Let $Emb(\mathbb{S}^1, \mathbb{R}^2)$ be the set of all smooth embeddings on $\mathbb{S}^1$ in the plan $\mathbb{R}^2$, its elements are the injective mappings $c : \mathbb{S}^1 \rightarrow \mathbb{R}^2$. Let us $Diff(\mathbb{S}^1)$ be the set of all $C^\infty$ diffeomorphism on $\mathbb{S}^1$ which opere differentiably on $Emb(\mathbb{S}^1, \mathbb{R}^2)$. Let us consider $B_e$ as the quotient of $Emb(\mathbb{S}^1, \mathbb{R}^2)$ under the action of $Diff(\mathbb{S}^1)$ on $Emb(\mathbb{S}^1, \mathbb{R}^2)$. In term we whole we have

$$B_e(\mathbb{S}^1, \mathbb{R}^2) := \{ [c] / c \in Emb \} \text{ where } [c] := \{ c' \in Emb / c' \sim c \}$$

To characterize the tangent space at $B_e$ we start with the characterization of the tangent space at $Emb$ denoted $T_eEmb$ and the tangent space at the orbit of $c$ by $Diff(\mathbb{S}^1)$ at $c$ denoted $T_e(Diff(\mathbb{S}^1).c)$. Thus the tangent space to $B_e$ is then identified with an additional to $T_e(Diff(\mathbb{S}^1).c)$ in $T_eEmb$.

Proposition 2.1 Let $c \in Emb$, then the tangent space at $c$ to $Emb$ is given by: $T_eEmb = C^\infty(\mathbb{S}^1, \mathbb{R}^2)$.

Proof. Let $h \in T_eEmb$, then $h$ is obtained by looking at a path of embeddings which passes through $c$. Let $c : I \times \mathbb{S}^1 \rightarrow \mathbb{R}^2$ be an embedding path such that $c(t, \theta) = c(\theta) + th(\theta)$ where $h : \mathbb{S}^1 \rightarrow \mathbb{R}^2$ we have : $\frac{d}{dt}_{t=0}c(t, \theta) = h(\theta)$. Since $c(t, \theta)$ is an embedding path then $c(t, \theta)$ is an immersion thus

$$T_eEmb = Im(T_0c(t, \theta)) = C^\infty(\mathbb{S}^1, \mathbb{R}^2).$$

$\blacksquare$

Proposition 2.2 The tangent space to the orbit of $c$ by $Diff(\mathbb{S}^1)$ is the subspace of $T_eEmb$ formed by vectors $m(\theta)$ of type $c_0(\theta) = c'(\theta)$ times a function.

Proof. We have $Diff(\mathbb{S}^1).c \in Emb$ because these are all the bijective reparametrizations of the same curve $c(\theta)$ therefore $T_e(Diff(\mathbb{S}^1).c) \subset T_eEmb$. Let $m \in T_e(Diff(\mathbb{S}^1).c)$ then $m$ is obtained
by looking at a family of parametrizations \( c(t, \theta) := c(\phi(t, \theta)) \) of the curve \( c(\theta) \) where
\[
\phi(t, \cdot): S^1 \rightarrow S^1
\]
is a diffeomorphism of \( S^1 \) where \( s \in S^1 \) and \( t \) is the parameter of the variation of the reparametrization \( \phi(t, s) \) of \( S^1 \). We have \( \frac{d}{dt}_{|t=0} c(t, \theta) = c'(\theta) \frac{d}{dt}_{|t=0} \phi(0, \theta) \) since \( c(t, \theta) \) is a parametrization of the curve \( c(\theta) \) so it an immersion then we have
\[
T_c(\text{Diff}(S^1).c) = \text{Im}(T_0 c(t, \theta)) = c'(\theta) \frac{d}{dt}_{|t=0} \phi(0, \theta).
\]

**Remark 2.3** The choice of the supplementary must be coherent with the action of \( \text{Diff}(S^1) \) i.e we choose a supplementary at \( T_c(\text{Diff}(S^1).c) \) in \( T_c \text{Emb} \) invariably by the action of \( \text{Diff}(S^1) \). For that it suffices to define a metric on \( \text{Emb} \) for which \( \text{Diff}(S^1) \) acts isometrically and to define the supplementary of \( T_c(\text{Diff}(S^1).c) \) as its orthogonal for this metric.

**Definition 2.4** Let us \( G^0 \) be an invariant metric by the action of \( \text{Diff}(S^1) \) on the manifold \( \text{Emb}(S^1, \mathbb{R}^2) \), defined by the application:
\[
G^0 : T_c \text{Emb} \times T_c \text{Emb} \rightarrow \mathbb{R}
\]
\[
(h, m) \mapsto \int_{S^1} \langle h(\theta), m(\theta) \rangle |c'(\theta)|d\theta
\]
where \( \langle h(\theta), m(\theta) \rangle \) is the scalar product of \( h(\theta) \) and \( m(\theta) \) in \( \mathbb{R}^2 \).

**Proposition 2.5** Let \( c \in B_e \) then \( T_c B_e \) is colinear with the vector fields following the unit normal outside the form \( \Omega \). In other words
\[
T_c B_e \simeq \{ h \ | h = \alpha \tilde{v}, \alpha \in C^\infty(S^1, \mathbb{R}) \}.
\]

**Proof.** From the results shown above the orthogonal of \( T_c(\text{Diff}(S^1).c) \) in \( T_c \text{Emb} \) is the set of \( h(\theta) \) in \( T_c \text{Emb} \) which are orthogonal for the metric \( G^0 \) to all \( m(\theta) = \frac{d}{dt}_{|t=0} \phi(0, \theta)c'(\theta) \) this means that \( h(\theta) \) must be perpendicular to \( c'(\theta) \) so \( h(\theta) = \alpha(\theta)\tilde{v}(\theta) \) where \( \alpha(\theta) \in C^\infty(S^1, \mathbb{R}) \). Therefore we have
\[
T_c B_e \simeq \{ h|h = \alpha \tilde{v}, \alpha \in C^\infty(S^1, \mathbb{R}) \}
\]
where \( \tilde{v} \) is the unit normal outside the form \( \Omega \) defined at the boundary by \( \partial \Omega = c \) such that \( \tilde{v}(\theta) \perp c'(\theta) \) for all \( \theta \in S^1 \) and \( c' \) defined the circumferential derivative. Now let us consider the following terminology:
\[
ds = |c_\theta|d\theta \quad \text{arc length.}
\]

**Definition 2.6** A Sobolev-type metric on the manifold \( B_e(S^1, \mathbb{R}^2) \) is map:
\[
G^A : T_c B_e \times T_c B_e \rightarrow \mathbb{R}
\]
\[
(h, m) \mapsto \int_{S^1} (1 + AK_e^2(\theta))\langle h(\theta), m(\theta) \rangle |c'(\theta)|d\theta
\]
where \( K_e \) is the sectional curvature of \( c \) and \( A \) a positive real.

**Remark 2.7**
1. By setting \( h = \alpha \tilde{v}, m = \beta \tilde{v} \) and by parametrizing \( c(s) \) by arc length we have:
\[
G^A(h, m) = \int_{\partial \Omega} (1 + AK_e^2(\theta))\alpha\beta ds.
\]
2. If \( A > 0 \) \( G^A \) is a Riemannian metric.
3 Optimality condition of first order and covariant derivative

The shape optimization problem we have, consists in finding the solution of the following optimization problem:

$$
\min_{\Omega} J(\Omega)
$$

$$
J(\Omega) = -\frac{1}{2} \int_\Omega |\nabla u_\Omega|^2 dx + \frac{k^2}{2} |\Omega|
$$

is a shape functional. We seek the shape derivative associated with the functional $J(\Omega)$ following the direction of the vector field $V : \mathbb{R}^2 \to \mathbb{R}^2$, $C^\infty$ class:

$$
dJ(\Omega)[V] = \int_{\partial \Omega} \left( k^2 - \left( \frac{\partial u_\Omega}{\partial \bar{\nu}} \right)^2 \right) \langle V, \nu \rangle d\sigma.
$$

If $V|_{\partial \Omega} = \alpha \bar{\nu}$ we can still write:

$$
dJ(\Omega)[V] = \int_{\partial \Omega} \left( k^2 - \left( \frac{\partial u_\Omega}{\partial \bar{\nu}} \right)^2 \right) \alpha d\sigma. \quad (4)
$$

It should be noted that there is a link between the shape derivative of $J$ and the gradient in Riemannian structures see [41] and [49]. To illustrate our claiming, let us consider the Sobolev metric $G^4$ to ease the understanding of the computations. But the authors think that it is quite possible to generalize this study in higher dimension than two and even with other metrics.

Our purpose is to calculate the gradient of $J : B_c \to \mathbb{R}$ then we have:

$$
dJ(\Omega)[V] = G^4(\nabla J(\Omega), V) \quad (5)
$$

if $V|_{\partial \Omega} = h$ we have

$$
dJ_c(h) = G^4(\nabla J(\Omega), h)
$$

$$
dJ_c(h) = \int_{\partial \Omega} (1 + AK_c^2) \nabla J \alpha.
$$

But from (5),

$$
dJ_c(h) = \int_{\partial \Omega} \left( k^2 - \left( \frac{\partial u_\Omega}{\partial \bar{\nu}} \right)^2 \right) \alpha d\sigma
$$

and thus

$$
\int_{\partial \Omega} \left( k^2 - \left( \frac{\partial u_\Omega}{\partial \bar{\nu}} \right)^2 \right) \alpha d\sigma = \int_{\partial \Omega} (1 + AK_c^2) \nabla J \alpha d\sigma
$$

so that

$$
\nabla J = \frac{1}{1 + AK_c^2} \left( k^2 - \left( \frac{\partial u_\Omega}{\partial \bar{\nu}} \right)^2 \right).
$$

The next step is to compute the explicit form of the covariant derivative $\nabla_h m \in T_c B_c$ with $h, m \in T_c B_c$.

The following result has been already established in a pioneering work, see [41]. We only bring
another way in the proof and additional details in the computations of the covariant derivative. In the last part of the paper where we think it contains our main contributions, the covariant derivative plays a key role in the study of the positiveness of the quadratic form. We shall come back to this fact.

**Theorem 3.1** Let $\Omega \subset \mathbb{R}^2$ at least of class $C^2$, $V, W \in C^\infty(\mathbb{R}^2, \mathbb{R}^2)$ vector fields which are orthogonal to the boundaries i.e

$$V_{|\partial \Omega} = \alpha \nu$$

with $\alpha := \langle V_{|\partial \Omega}, \nu \rangle$ and

$$W_{|\partial \Omega} = \beta \nu$$

with $\beta := \langle W_{|\partial \Omega}, \nu \rangle$ such that $V_{|\partial \Omega} = h := \alpha \nu$, $W_{|\partial \Omega} = m =: \beta \nu$, belongs to the tangent space of $B_c$. Then the covariant derivative associate with the Riemannian metric $G^A$ can be expressed as follows:

$$\nabla_V W : = \nabla_h m = \partial \beta \partial \nu \alpha + \left( \frac{3AK^2_c + K_c}{1 + AK^2_c} \right) \alpha \beta$$

where $D_V W$ is the directional derivative of the vector field $W$ in the direction $V$.

**Proof.** To calculate $\nabla_V W$ we use the compatibility of the metric with the connection $\nabla$. We have $hG^A(m, l) = G^A(\nabla_h m, l) + G^A(m, \nabla_h l)$ for all $l \in T_c B_c$. Let $Z$ be a vector fields such that $l := Z_{|\partial \Omega} = \gamma \nu$, we set

$$F(c_t(\theta)) = (1 + AK^2_c(\theta)) \langle m(\theta), l(\theta) \rangle$$

thus $G^A(m, l) = \int_{S^1} F(c_t(\theta)) |c_t'(\theta)| d\theta$ then we calculate the following expression

$$h(G^A(m, l)) = \frac{d}{dt} \bigg|_{t=0} \left( \int_{S^1} F(c_t(\theta)) |c_t'(\theta)| d\theta \right) [V]$$

where $c_t(\theta)$ denotes a family of (parameterized) curves with $c_0(\theta) = c(\theta)$ and $c_t'(\theta)$ denotes the derivative with respect to $\theta$ of the curve $c_t \rightarrow c_t(\theta)$. We have

$$h(G^A(m, l)) = \int_{S^1} \left( \frac{\partial}{\partial \nu} \frac{1 + AK^2_c}{\partial \nu} \alpha \beta \gamma \alpha [c_t'(\theta)] + \frac{\partial (|c_t'(\theta)|)}{\partial \nu} \beta \gamma \alpha \right) d\theta$$

$$= \int_{S^1} \left( 2AK_c \frac{\partial K_c}{\partial \nu} \alpha \beta \gamma + (1 + AK^2_c) \frac{\partial \beta}{\partial \nu} \gamma \alpha + (1 + AK^2_c) \frac{\partial \alpha}{\partial \nu} \beta \gamma \right) d\theta$$

$$+ \int_{S^1} \left( \frac{\partial |c_t'(\theta)|}{\partial \nu} (1 + AK^2_c) \beta \gamma \alpha d\theta \right)$$

Now let us calculate $\frac{\partial K_c}{\partial \nu}$. We have:

$$\frac{\partial K_c}{\partial \nu} = \frac{\langle \nu, c_\theta \rangle}{|c_\theta|^2} K_\theta + \frac{\langle \nu, ic_\theta \rangle}{|c_\theta|} K^2 + \frac{1}{|c_\theta|} \left( \frac{1}{|c_\theta|} \frac{\langle \nu, ic_\theta \rangle}{|c_\theta|} \right) \theta.$$. 
Then we have $\langle \vec{v}, c_{\theta} \rangle = 0$ because $\vec{v} \perp c_{\theta}$ and moreover,

$$\frac{\langle \vec{v}, ic_{\theta} \rangle}{|c_{\theta}|} = \frac{\langle \vec{v}, i\theta \rangle}{|c_{\theta}|} = \langle \vec{v}, \vec{v} \rangle$$

$$\frac{\langle \vec{v}, ic_{\theta} \rangle}{|c_{\theta}|} = \frac{\langle \vec{v}, i\theta \rangle}{|c_{\theta}|} = ||\vec{v}||^2 = 1$$

Hence, we obtain that:

$$\frac{\partial K_c}{\partial \vec{v}} = K_c^2 + \frac{1}{|c_{\theta}|} \left( \frac{1}{|c_{\theta}|} \left( \frac{\langle \vec{v}, ic_{\theta} \rangle}{|c_{\theta}|} \right)_{\theta} \right).$$

Let us compute step by step the above last term in the right hand side. First, we have

$$\left( \frac{\langle \vec{v}, ic_{\theta} \rangle}{|c_{\theta}|} \right)_{\theta} = \frac{\partial}{\partial \theta} \left( \frac{\langle \vec{v}, ic_{\theta} \rangle}{|c_{\theta}|} \right)$$

$$= \frac{\partial}{\partial \theta} \frac{\langle \vec{v}, ic_{\theta} \rangle}{|c_{\theta}|} - \frac{\partial|c_{\theta}|}{\partial \theta} \frac{\langle \vec{v}, ic_{\theta} \rangle}{|c_{\theta}|^2}$$

$$= \frac{\partial}{\partial \theta} \frac{\langle \vec{v}, ic_{\theta} \rangle}{|c_{\theta}|}$$

$$= \frac{\partial}{\partial \theta} \frac{\langle \vec{v}, ic_{\theta} \rangle}{|c_{\theta}|}$$

$$\left( \frac{\langle \vec{v}, ic_{\theta} \rangle}{|c_{\theta}|} \right)_{\theta} = \frac{\langle \vec{v}', ic_{\theta} \rangle + \langle \vec{v}, i\theta \rangle}{|c_{\theta}|}$$

$$= \frac{\langle \vec{v}'(\theta), ic_{\theta} \rangle + \langle \vec{v}(\theta), ic_{\theta} \rangle}{|c_{\theta}|}$$

$$= \frac{\langle \vec{v}'(\theta), ic_{\theta} \rangle}{|c_{\theta}|} + \frac{\langle \vec{v}(\theta), ic_{\theta} \rangle}{|c_{\theta}|}$$

$$= \frac{\langle \vec{v}'(\theta), \vec{v}(\theta) \rangle}{|c_{\theta}|} + \frac{\langle \vec{v}(\theta), \vec{v}(\theta) \rangle}{|c_{\theta}|}$$

Note that $||\vec{v}||^2 = 1$ which is never but $\langle \vec{v}, \vec{v} \rangle = 1$. Therefore, by differentiation, we have:

$$\langle \vec{v}'(\theta), \vec{v}(\theta) \rangle + \langle \vec{v}(\theta), \vec{v}'(\theta) \rangle = 0$$

$$2\langle \vec{v}'(\theta), \vec{v}(\theta) \rangle = 0$$

$$\langle \vec{v}'(\theta), \vec{v}(\theta) \rangle = 0.$$
Indeed, proceeding further the computation, we have:

\[
\left( \frac{\langle \vec{\nu}, ic_\theta \rangle}{|c_\theta|} \right)_\theta = \left( \frac{\langle \vec{\nu}(\theta), ic_\theta \rangle}{|c_\theta|} \right)_\theta
\]

\[
= \left( \frac{\langle \vec{\nu}(\theta), -K_c|c_\theta|c_\theta \rangle}{|c_\theta|} \right)_\theta
\]

\[
= \left( \frac{\langle \vec{\nu}(\theta), -K_c|c_\theta| \rangle}{|c_\theta|} \right)_\theta
\]

\[
= -K_c \langle \vec{\nu}(\theta), c_\theta \rangle = 0
\]

Finally, from all the above steps, we have:

\[
\frac{1}{|c_\theta|} \left( \frac{\langle \sigma, ic_\theta \rangle}{|c_\sigma|} \right)_\theta = 0 \quad \text{and we get}
\]

\[
\frac{\partial K_c}{\partial \vec{\nu}} = K_c^2.
\]

Therefore we have

\[
h(G^A(m,l)) = \int_{\partial \Omega} \left( 2AK_c \frac{\partial K_c}{\partial \vec{v}} \right) \alpha \beta \gamma + (1 + AK_c^2) \frac{\partial^3}{\partial \theta^3} \gamma \alpha + (1 + AK_c^2) \frac{\partial^3}{\partial \theta^3} \beta \alpha d\theta
\]

\[
+ \int_{\partial \Omega} \frac{\partial |c_\theta'(\theta)|}{\partial \vec{v}} (1 + AK_c^2) \beta \gamma \alpha d\theta
\]

\[
\frac{\partial |c_\theta'(\theta)|}{\partial \vec{v}} (1 + AK_c^2) \beta \gamma \alpha d\theta
\]

(6)

Let us calculate now the following expression

\[
\frac{\partial |c_\theta'(\theta)|}{\partial \vec{v}}
\]

To do this we parametrized \( c(\theta) \) by arc length i.e \(|c'(\theta)| = 1\). Since

\[
\langle c'(\theta), c'(\theta) \rangle = 1
\]

and differentiating it, we have:

\[
\langle c''(\theta), c'(\theta) \rangle = 0.
\]

Then \( c''(\theta) = c_{\theta\theta}(\theta) \) is proportional to \( \vec{v}(s) \) so \( c''(\theta) = K_c(\theta)\vec{v}(\theta) \) (this is the definition of the curvature of the curve \( c \)).

Let us compute now \( \frac{d}{dt}(|c_\theta'(\theta)|) \) at \( t = 0 \) where

\[
|c_\theta'(\theta)| = |\frac{d}{dt}(c(\theta) + t\vec{v}(\theta))|
\]

\[
= |c'(\theta) + t\vec{v}'(\theta)|
\]

\[
= (|c'(\theta)|^2 + t^2|\vec{v}'(\theta)|^2 + 2t\langle c'(\theta), \vec{v}'(\theta) \rangle)^{\frac{1}{2}}
\]

(7)
From a Taylor’s expansion of the previous expression in $t$, we see that:

$$\frac{d}{dt}|_{t=0}c'_t(\theta) = \langle c'(\theta), \nu(\theta) \rangle$$

and since

$$\langle c'(\theta), \nu(\theta) \rangle = 0$$

by differentiating we have

$$\langle c''(\theta), \nu'(\theta) \rangle = -\langle c''(\theta), \nu(\theta) \rangle = K_c, $$

and hence

$$\frac{d}{dt}(|c'_t(\theta)|) = K_c.$$

We can conclude that:

$$h(G^A(k,l)) = \int_{\partial \Omega} (2AK^3_c \alpha \beta \gamma + (1 + AK^2_c) \frac{\partial \beta}{\partial \nu} \gamma \alpha) \, ds$$

$$+ \int_{\partial \Omega} (1 + AK^2_c) \frac{\partial \beta}{\partial \nu} \beta \alpha + K_c(1 + AK^2_c)\alpha \beta \gamma) \, ds. \quad (8)$$

We have

$$G^A(\nabla_h m, l) = \int_{\partial \Omega} (1 + AK^2_c)\nabla_h m \gamma$$

$$= \int_{\partial \Omega} (1 + AK^2_c)\nabla_V W \gamma$$

and

$$G^A(m, \nabla_h l) = \int_{\partial \Omega} (1 + AK^2_c)\beta \nabla_h l$$

$$= \int_{\partial \Omega} (1 + AK^2_c)\beta \nabla_V Z.$$

Therefore

$$G^A(\nabla_h m, l) + G^A(m, \nabla_h l) = \int_{\partial \Omega} (1 + AK^2_c)\nabla_V W \gamma + \int_{\partial \Omega} (1 + AK^2_c)\beta \nabla_V Z$$

$$= \int_{\partial \Omega} (1 + AK^2_c) (\nabla_V W \gamma + \beta \nabla_V Z) \, ds.$$

and

$$\int_{\partial \Omega} (1 + AK^2_c) (\nabla_V W \gamma + \beta \nabla_V Z) \, ds = \int_{\partial \Omega} (2AK^3_c \alpha \beta \gamma + (1 + AK^2_c) \frac{\partial \beta}{\partial \nu} \gamma \alpha) \, ds$$

$$+ \int_{\partial \Omega} (1 + AK^2_c) \frac{\partial \beta}{\partial \nu} \beta \alpha + K_c(1 + AK^2_c)\alpha \beta \gamma) \, ds.$$

$$= \int_{\partial \Omega} (3AK^3_c + K_c) \alpha \beta \gamma + (1 + AK^2_c) \frac{\partial \beta}{\partial \nu} \gamma \alpha$$

$$+ (1 + AK^2_c) \frac{\partial \gamma}{\partial \nu} \beta \alpha \, ds.$$
\[ \nabla_V W \gamma + \beta \nabla_V Z = \left( \frac{3AK^3_c + K_c}{1 + AK_c^2} \right) \alpha \beta \gamma + \frac{\partial \beta}{\partial \nu} \gamma \alpha + \frac{\partial \gamma}{\partial \nu} \beta \alpha \]

\[ \nabla_V W \gamma = \left( \frac{3AK^3_c + K_c}{1 + AK_c^2} \right) \alpha \beta \gamma + \frac{\partial \beta}{\partial \nu} \gamma \alpha + \frac{\partial \gamma}{\partial \nu} \beta \alpha - \beta \nabla_V Z \]

By pointing out that

\[ \nabla_V Z = \frac{\partial \gamma}{\partial \nu} \alpha, \]

we have:

\[ \nabla_V W \gamma = \left( \frac{3AK^3_c + K_c}{1 + AK_c^2} \right) \alpha \beta \gamma + \frac{\partial \beta}{\partial \nu} \gamma \alpha + \beta \frac{\partial \gamma}{\partial \nu} \alpha - \beta \frac{\partial \gamma}{\partial \nu} \alpha \]

Finally, we have:

\[ \nabla_V W = \left( \frac{3AK^3_c + K_c}{1 + AK_c^2} \right) \alpha \beta + \frac{\partial \beta}{\partial \nu} \alpha. \]

**Remark 3.2** It is quite possible to begin the proof of the above theorem by the application of shape calculus rules for volume and boundary functionals as in [12], [17], [30] on the following functional

\[ \int_{\Omega} (1 + AK^2_c) \alpha \beta d\sigma \]

The remaining computations are almost similar. We only underline that at the end it is necessary, to see that the local covariant derivative \( \nabla_X Y \) = \( \frac{d}{dt}_{t=0} (Y(x + tX(x))) \) where \( Y = X = \vec{\nu} \) and \( D\vec{\nu} \vec{\nu} = 0 \) since \( |\vec{\nu}|^2 = 1 \), \( D\vec{\nu} \) being the Jacobian matrix.

**Remark 3.3** Let us now calculate the torsion of the connection \( \nabla \). Indeed, one is wondering if the torsion of the connection \( \nabla \) coincides with the Levi-Civita connection.

We have

\[ T(V, W) = \nabla_V W - \nabla_W V - [V, W] \]

\[ T(V, W) = \langle D_V W, \vec{\nu} \rangle + \left( \frac{3AK^3_c + K_c}{1 + AK_c^2} \right) \langle V, \vec{\nu} \rangle \langle W, \vec{\nu} \rangle \]

\[ - \langle D_W V, \vec{\nu} \rangle - \left( \frac{3AK^3_c + K_c}{1 + AK_c^2} \right) \langle V, \vec{\nu} \rangle \langle W, \vec{\nu} \rangle - [V, W] \]

\[ T(V, W) = \frac{\partial \beta}{\partial \nu} \alpha - \frac{\partial \alpha}{\partial \nu} \beta - [h, m]. \]

But

\[ \frac{\partial \beta}{\partial \nu} \alpha - \frac{\partial \alpha}{\partial \nu} \beta = [h, m]. \]
Then we have:

\[ T(V, W) = [h, m] - [h, m] \]
\[ T(V, W) = 0. \]

As conclusion, we claim that \( \nabla \) is compatible with the metric \( G^A \) and its torsion is zero, so it coincides with the Levi-Civita connection.

4 Sufficient condition for the minimality of a shape functional

In this section, assuming at first that there are at least one critical point, we shall first present the sufficient condition on the existence of a local minimum for a functional \( J(\Omega) \) given as follows:

\[ J(\Omega) = \int_{\Omega} f_0(u_\Omega, \nabla u_\Omega) \] (10)

where \( f_0 \) is a function of \( \mathbb{R} \times \mathbb{R}^n \) that we suppose to be smooth and \( u_\Omega \) denotes a smooth solution of a boundary value problem.

And in the second part, in the case where \( J(\Omega) = -\frac{1}{2} \int_{\Omega} |\nabla u_\Omega|^2 dx + \frac{h^2}{2}|\Omega| \), we compute the second shape derivative.

The fundamental question is then to study the existence of the local strict minima of this functional under possible constraints that \( \Omega \) is unknown but is a critical point. That means the first derivative with respect to the domain is equal to zero at the domain \( \Omega \). We shall examine, for that, how varies this solution \( u_\Omega \) when its domain of definition \( \Omega \) moves.

Let us recall the classic method of studying a critical point. Let \((B, \| \cdot \|_1)\) be a Banach space and let \( E : (B, \| \cdot \|_1) \rightarrow \mathbb{R} \) be a function of class \( C^2 \) whose differential \( Df \) vanishes at 0. The Taylor-Young formula is then written

\[ E(u) = E(0) + D^2 E(0) . (u \cdot u) + o(\|u\|^2) \] (11)

In particular, if the Hessian form \( D^2 E(0) \) is coercive in the norm \( \| \cdot \|_1 \), then the critical point 0 is a strict local minimum of \( E \). The fundamental difficulty in the study of critical forms is caused by the appearance of a second norm \( \| \cdot \|_2 \) lower that \( \| \cdot \|_1 \) (i.e. \( \| \cdot \|_2 \leq C\| \cdot \|_1 \)). The Hessian form is not in general, coercive for the norm \( \| \cdot \|_1 \) but for the standard norm \( \| \cdot \|_2 \). If these norms are not equivalent and this is the general rule, concluding that the minimum is strict is impossible, even locally for the strong norm. It is quite possible to give several example. But let us reproduce a simple example of such a situation on the space \( H^1_0(0, 1) \) that was presented in the thesis of [12].

Let us consider the functional \( E \) defined by

\[ E(u) = \|u\|^2_{L^2(0, 1)} - \|u\|^4_{H^1_0(0, 1)}. \]

We can check that \( E \) is twice differentiable on \( H^1_0(0, 1) \) and which moreover one has in 0 :

\[
\begin{aligned}
E'(0) &= 0 \\
E''(0)(h, h) &= 2\|h\|^2_{L^2(0, 1)}.
\end{aligned}
\]
For each direction, we find that 0 is a minimum strictly local. indeed, for all nonzero \( u_0 \in H_0^1(0, 1) \) and for all \( t \in \mathbb{R} \), we have

\[
E(tu_0) = t^2 \|u_0\|_{L^2(0, 1)}^2 - t^4 \|u_0\|_{H_0^1(0, 1)}^4 > 0 \text{ if } t^2 < \frac{\|u_0\|_{L^2(0, 1)}^2}{\|u_0\|_{H_0^1(0, 1)}^4}.
\]

However, 0 is not a local minimum even for the norm \( H_0^1 \). Indeed, there is no \( r > 0 \) such that

\[
\|u\|_{H_0^1(0, 1)} < r \implies E(u) > E(0) \quad \text{i.e.} \quad \|u\|_{L^2(0, 1)}^2 > \|u\|_{H_0^1(0, 1)}^4,
\]
since we can always build a sequence in \( H_0^1(0, 1) \) such that

\[
\left\{ \begin{array}{l}
\|u_n\|_{H_0^1(0, 1)} = r/2, \\
\|u_n\|_{L^2(0, 1)} \to 0 \text{ quand } n \to +\infty.
\end{array} \right.
\]

To solve this problem, we will use the Taylor’s formula with an integral remainder, instead of (11) i.e

\[
E(u) - E(0) = \int_0^1 (1 - t) E''(tu, u) \, dt.
\]

This formula allows to express exactly the difference in energy between a critical form \( \Omega_0 \) and a neighboring form \( \Omega \) via an integral term that we can carefully estimate thanks to the study of the variations of the Hessian.

**Theorem 4.1** Let \( f_0 : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}, \ (s, v) \to f_0(s, v) \) be a function of class \( C^3 \) and \( f \) a function in \( C^\infty(\mathbb{R}^N, \mathbb{R}), \gamma \in (0, 1) \). Let \( L_0 = \text{div}(A\nabla \cdot) \) be a strictly operator and uniformly elliptical with \( A \) in \( C^2(\mathbb{R}^N, M_N(\mathbb{R}^N)) \). Let \( E \) be the defined shape functional on the class \( \mathcal{O} \) of open class \( C^2, \gamma \) as

\[
J(\Omega) = \int_{\Omega} f_0(u_{\Omega}, \nabla u_{\Omega}),
\]

where \( M_N(\mathbb{R}^N) \) stands for the space of square matrices of order \( N \) and \( u_{\Omega} \) is the solution of the homogeneous Dirichlet problem

\[
\left\{ \begin{array}{l}
L_0u = f \quad \text{in } \Omega, \\
u = 0 \quad \text{on } \partial \Omega.
\end{array} \right.
\]

Let \( \Omega_0 \in \mathcal{O} \), then, there exists a real \( \eta_0 > 0 \) and an increasing function \( \omega : (0, \eta_0] \to (0, +\infty) \) with \( \lim_{r \searrow 0} \omega(r) = 0 \), which depend only on \( \Omega_0, \ L_0, \ f_0 \) and \( f \), such that for all \( \eta \in (0, \eta_0] \) and for all \( \theta \in C^{2,\alpha}(\mathbb{R}^N, \mathbb{R}^N) \) verifying

\[
\|\theta - Id_{\mathbb{R}^N}\|_{2,\alpha} \leq \eta,
\]

we have the following estimate valid for all \( t \in [0, 1] \),

\[
\left| \frac{d^2}{dt^2} J(\Omega_t) - \frac{d^2}{dt^2} \big|_{t=0} J(\Omega_t) \right| \leq \omega(\eta) \|V, \bar{u}\|_{H^{1/2}(\partial \Omega_0)}^2 \tag{12}
\]

where \( \Omega_t = \Phi_t(\Omega_0), t \in [0, 1] \) stands for the flow related to the vector field \( V \).

For the proof see [11, Theorem 1].

In the case where \( \Omega_0 \) is a critical point for the functional \( J \), to show that it is a strict local minimum, we have to study the positiveness of a quadratic form which we are going to denote by \( Q \). This quadratic form is obtained by computing the second derivative of \( J \) with respect to the domain. So before going on, we need some hypothesis ;

let us suppose that:
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(i) \( \Omega \) is a \( C^2 \) regular open domain.

(ii) \( V(x; t) = \alpha(x)\bar{v}(x), \ \alpha \in H^2(\partial \Omega), \ \forall \ t \in [0, \epsilon]. \)

In [13], (see also [11], [12]), the authors showed that it is not sufficient to prove that the quadratic form is positive to claim that: a critical shape is a minimum. In fact most of the time people use the Taylor Young formula to study the positiveness of the quadratic form.

For \( t \in [0, \epsilon] \),

\[ j(t) := J(\Omega_t) = J(\Omega) + tdJ(\Omega, V) + \frac{1}{2}t^2 d^2 J(\Omega, V, V) + o(t^2). \]

The quantity \( o(t^2) \) is expressed with the norm of \( C^2 \). It appears in the expression of \( d^2 J(\Omega, V, V) \) the norm of \( H^2(\partial \Omega) \). And these two norms are not equivalent. The quantity \( o(t^2) \) is not smaller than \( ||V||_{H^2(\partial \Omega)} \), see the example in [13]. Then such an argument does not insure that the critical point is a local strict minimum.

In our study, we shall see that the main result in [13] can be satisfied in a simple way thanks to the hessian obtained via the Sobolev metric \( G_A \) in which the norm of \( H^1/2(\partial \Omega) \) appears directly. And this overcomes the classical issue. In fact the study of the sign of \( \int_{\partial \Omega} H d\sigma \) becomes the alone control to get information on the optimal domain and then on the optimal shape.

**Proposition 4.1**

Let \( \Omega \) be a critical point for the functional \( J \), then

\[
Q(\alpha) = d^2 J(\Omega; V; V)
= -(N-1) \int_{\partial \Omega} H \alpha^2 d\sigma + k^2 \int_{\Omega} \left| \nabla \Lambda \right|^2 dx
= -(N-1)k^2 \int_{\partial \Omega} H \alpha^2 d\sigma + k^2 \int_{\partial \Omega} \alpha \Lambda d\sigma
\]

Where \( \Lambda \) is solution of the following boundary value problem

\[
\begin{cases}
-\Delta \Lambda = 0 & \text{in } \Omega \\
\Lambda = \alpha & \text{on } \partial \Omega.
\end{cases}
\]

\( H \) is the mean curvature of \( \partial \Omega \) and \( L \) is a pseudo differential operator which is known as the Steklov-Poincaré or capacity or Dirichlet to Neumann (see e.g. [14]) operator, defined by \( L\alpha = \frac{\partial \Lambda}{\partial \nu} \).

**Proof.** We use the definition of the derivative with respect to the domain and we apply it to \( dJ(\Omega, V) \).

Then we get

\[
2Q(\alpha) = 2d^2 J(\Omega, V, V)
= \int_{\Omega} \left( \text{div}(k^2 - |\nabla u|^2)V(x, 0) \right) dx + \int_{\Omega} \text{div}(V(x, 0)) \text{div}(k^2 - |\nabla u|^2)V(x, 0)) dx
\]

\[
2Q(\alpha) = \left[ \int_{\partial \Omega} -2\nabla u \nabla u' V(x, 0) \cdot \bar{v} + \text{div}(k^2 - |\nabla u|^2)V(x, 0)) V(x, 0) \cdot \bar{v} \right] d\sigma.
\]

Since \( \Omega \) is solution of the quadrature surface problem then \( -\frac{\partial u}{\partial \nu} = k \) on \( \partial \Omega \).

By assumption, \( \partial \Omega \) is of \( C^2 \) class and since \( u = 0 \) on \( \partial \Omega \),
we have:
\[ \nabla u = \frac{\partial u}{\partial \bar{v}} \bar{v} = -k \bar{v}. \] Hence
\[
2Q(\alpha) = \left[ \int_{\partial \Omega} 2k \nabla u \cdot \bar{v} V(x,0) \bar{v} + \text{div}((k^2 - |\nabla u|^2)V(x,0))V(x,0) \bar{v} \right] d\sigma.
\]

A classical calculus in shape optimization lead us to get
\[ u' = -\frac{\partial u}{\partial \bar{v}} V \bar{v} \text{ on } \partial \Omega. \]
Let us recall again that \(-\frac{\partial u}{\partial \bar{v}} = k \text{ on } \partial \Omega \) and \( V \bar{v} = \alpha. \) Then, we have
\[ u' = k \alpha \text{ on } \partial \Omega \text{ and } \nabla u' = \frac{\partial u}{\partial \bar{v}} \frac{\partial \alpha}{\partial \bar{v}} = k L \alpha,
\]
where \( L \) is a pseudo differential operator, defined by \( L \alpha = \frac{\partial \Lambda}{\partial \bar{v}} \) and such that
\[
\begin{cases}
-\Delta \Lambda = 0 \text{ in } \Omega \\
\Lambda = \alpha \text{ on } \partial \Omega.
\end{cases}
\tag{14}
\]
\( \Lambda \) is the extension of \( \alpha \) in \( \Omega. \)

Hence
\[
2Q(\alpha) = \int_{\partial \Omega} (2k^2 \alpha L \alpha - \text{div}((|\nabla u|^2 - k^2)\alpha \bar{v})\alpha) d\sigma
\]
Let us compute now \( \text{div}((|\nabla u|^2 - k^2)\alpha \bar{v}) \) on \( \partial \Omega. \) Since \( |\nabla u| = k \text{ on } \partial \Omega, \) we have
\[
\text{div}((|\nabla u|^2 - k^2)\alpha \bar{v}) = \alpha \nabla (|\nabla u|^2 - k^2) \bar{v} = \alpha \nabla (|\nabla u|^2) \bar{v}
\]
Since we have supposed that \( \Omega \) of class \( C^2 \), locally, \( \partial \Omega \) can be described by a curve \( \varphi \) such that \( x_N = \varphi(x'), x' \in \mathbb{R}^{N-1} \) and \( D\varphi(x') = 0. \) \( D\varphi(x') \) is the Jacobian matrix of \( \varphi. \)
Let us set \( x_0 = (x', x_N) = (x', \varphi(x')) \in \partial \Omega \) then we have \( u(x_0) = 0. \)
By differentiating with respect to \( s_j \) for all \( j \in \{1, \ldots, N-1\}, \) we have:
\[
\frac{\partial u(x_0)}{\partial s_j} + \frac{\partial \varphi(x')}{\partial s_j} \frac{\partial u(x_0)}{\partial \bar{v}} = 0
\]
Since \( \frac{\partial \varphi(x')}{\partial s_j} = 0, \) we get \( \frac{\partial u(x_0)}{\partial s_j} = 0. \)
Starting from the following equality:
\[
\frac{\partial u(x_0)}{\partial s_j} + \frac{\partial \varphi(x')}{\partial s_j} \frac{\partial u(x_0)}{\partial \bar{v}} = 0,
\tag{15}
\]
and by differentiating it from \( i \in \{1, \ldots, N-1\}, \) we have:
\[
\begin{align*}
\frac{\partial^2 u(x_0)}{\partial s_i \partial s_j} + \frac{\partial \varphi(x')}{\partial s_i} \frac{\partial^2 u(x_0)}{\partial s_j \partial \bar{v}} + \frac{\partial^2 \varphi(x')}{\partial s_i \partial s_j} \frac{\partial u(x_0)}{\partial \bar{v}} + \\
+ \frac{\partial \varphi(x')}{\partial s_j} \frac{\partial^2 u(x_0)}{\partial s_i \partial \bar{v}} + \frac{\partial \varphi(x')}{\partial s_i} \frac{\partial^2 u(x_0)}{\partial s_j \partial \bar{v}} \frac{\partial \bar{v}}{\partial \bar{v}} = 0
\end{align*}
\]
Note that \( u(x_0) = 0 \) and \( \frac{\partial u(x_0)}{\partial s_j} = 0 \) \( \forall j \in \{1, \cdots, N-1\} \), and summing over the indices \( i, j \), we have

\[
\sum_{j=1}^{N-1} \frac{\partial^2 u(x_0)}{\partial s_j^2} + (N - 1) H \frac{\partial u(x_0)}{\partial \vec{\nu}} = 0
\]

Since \( \frac{\partial u(x_0)}{\partial s_i} = 0 \) \( \forall i \in \{1, \cdots, N-1\} \), we have also:

\[
\nabla(|\nabla u|^2(x_0)).\vec{\nu} = 2 \frac{\partial u(x_0)}{\partial \vec{\nu}} \left( -\sum_{i=1}^{N-1} \frac{\partial^2 u(x_0)}{\partial s_i^2} - f \right)
\]

In addition, we can remark that:

\[
\frac{\partial^2 u(x_0)}{\partial \vec{\nu}^2} = -\sum_{i=1}^{N-1} \frac{\partial^2 u(x_0)}{\partial s_i^2} - f \text{ on } \partial \Omega
\]

Therefore, we have

\[
\nabla(|\nabla u|^2(x_0)).\vec{\nu} = 2 \frac{\partial u(x_0)}{\partial \vec{\nu}} \left( (N - 1) H \frac{\partial u(x_0)}{\partial \vec{\nu}} - f \right)
\]

When the support of the function \( f \) is in \( \Omega \), then \( f = 0 \) on \( \partial \Omega \).

Finally we have:

\[
2Q(\alpha) = \int_{\partial \Omega} 2k^2 \alpha L \alpha - 2(N - 1)H\alpha^2 \left( \frac{\partial u(x_0)}{\partial \vec{\nu}} \right)^2 d\sigma
\]

And by the Green’s formula we get

\[
\int_{\partial \Omega} \alpha L \alpha d\sigma = \int_{\Omega} |\nabla \Lambda|^2 dx.
\]

### 5  Positiveness of the quadratic form in the infinite Riemannian point of view

**Definition 5.1** Let \( J : \Omega \to \mathbb{R} \) be an functional. One defines the hessian Riemannian shape as follows:

\[
\text{Hess}_J(\Omega)[V] := \nabla_V \text{grad} J
\]

where \( \nabla_V \) denotes the derivative following the vector field \( V \).
Theorem 5.2 The hessian Riemannian shape defined by the Riemannian metric $G^A$ verifies the following condition:

$$G^A(HessJ(\Omega)[V], W) = d(dJ(\Omega)[W])[V] - dJ(\Omega)[\nabla_V W].$$

Proof. Our purpose is to show that

$$G^A(HessJ(\Omega)[V], W) = d(dJ(\Omega)[W])[V] - dJ(\Omega)[\nabla_V W].$$

So let us use the compatibility of the metric $G^A$ with the Levi-Civita connection. We have

$$V.G^A(\text{grad}J, W) = G^A(\text{grad}J, \nabla_V W) + G^A(\nabla_V \text{grad}J, W).$$

Since $G^A(HessJ(\Omega)[V], W) = G^A(\nabla_V \text{grad}J, W)$, we have

$$G^A(HessJ(\Omega)[V], W) = V.G^A(\text{grad}J, W) - G^A(\nabla_V \text{grad}J, W).$$

Remark 5.3 In our quadrature surface case, for $W = m\vec{v}$ and $V = h\vec{v}$, we have:

$$dJ(\Omega)[V] = \int_{\partial\Omega} \left( k^2 - \left( \frac{\partial u_\Omega}{\partial v} \right)^2 \right) \alpha d\sigma$$

and then,

$$d(\ dJ(\Omega)[W]) [V] = d \left( \int_{\partial\Omega} \left( k^2 - \left( \frac{\partial u_\Omega}{\partial v} \right)^2 \right) m d\sigma \right) [V].$$

Setting

$$\psi := k^2 - \left( \frac{\partial u_\Omega}{\partial v} \right)^2,$$

then

$$\psi_t := k^2 - \left( \frac{\partial u_\Omega}{\partial v} \right)^2,$$

we have

$$d(\ dJ(\Omega)[W]) [V] = d \left( \int_{\partial\Omega_t} \psi_t m d\sigma \right) [h].$$

This is never but:

$$d(\ dJ(\Omega)[W]) [V] = \int_{\partial\Omega} \frac{\partial \psi_t}{\partial t} |_{t=0} m d\sigma + \int_{\partial\Omega} \frac{\partial (\psi_t m)}{\partial \vec{v}} h d\sigma + \int_{\partial\Omega} K_c \psi_t m h d\sigma$$

$$= \int_{\partial\Omega} \left[ \frac{\partial \psi_t}{\partial t} |_{t=0} m + \left( \frac{\partial \psi_t}{\partial \vec{v}} + K_c \psi_t \right) m h + \psi_t \frac{\partial m}{\partial \vec{v}} h \right] d\sigma.$$
Let us calculate now the following expression:

\[ \frac{\partial \psi}{\partial t} \bigg|_{t=0} = \partial \left[ k^2 - \left( \frac{\partial u_{\Omega}}{\partial t} \right)^2 \right] \bigg|_{t=0} \]

\[ \frac{\partial \psi}{\partial t} \bigg|_{t=0} = -2m \left[ \frac{\partial (\nabla u_{\Omega}, \tilde{\nu})}{\partial t} \bigg|_{t=0} + D^2 u_{\Omega} V, \tilde{\nu} + \nabla u_{\Omega}. \left( \frac{\partial \tilde{\nu}}{\partial t} \bigg|_{t=0} + D\tilde{\nu} V \right) \right] \]

\[ \frac{\partial \psi}{\partial t} \bigg|_{t=0} = -2m \left[ \nabla u_{\Omega}', \tilde{\nu} + D^2 u_{\Omega} V, \tilde{\nu} + \nabla u_{\Omega}. \left( \frac{\partial \tilde{\nu}}{\partial t} \bigg|_{t=0} + D\tilde{\nu} V \right) \right] \]

where \( D^2 u_{\Omega} \) is the hessian matrix and \( D\tilde{\nu} \) the jacobian matrix of \( \tilde{\nu} \).

Let us calculate now the following expression: \( \frac{\partial \psi}{\partial t} \bigg|_{t=0} \).

We have

\[ \frac{\partial \tilde{\nu}}{\partial t} \bigg|_{t=0} = -\nabla_\Gamma (V, \tilde{\nu}) - (D\tilde{\nu}_0, \tilde{\nu}) V, \tilde{\nu} \quad \text{on} \quad \Gamma \]

where \( \nabla_\Gamma \) is the tangential gradient, \( \Gamma = \partial \Omega \) and \( \tilde{\nu}_0 = \tilde{\nu} \) then

\[ \frac{\partial \tilde{\nu}}{\partial t} \bigg|_{t=0} = -\nabla_\Gamma (V, \tilde{\nu}) - (D\tilde{\nu}, \tilde{\nu}) V, \tilde{\nu} \quad \text{on} \quad \Gamma. \]

Since \( D\tilde{\nu}, \tilde{\nu} = 0 \), then

\[ \frac{\partial \psi}{\partial t} \bigg|_{t=0} = -\nabla_\Gamma (V, \tilde{\nu}) \quad \text{on} \quad \Gamma. \]

So

\[ \frac{\partial \psi}{\partial t} \bigg|_{t=0} = -2m \left[ \nabla u_{\Omega}', \tilde{\nu} + D^2 u_{\Omega} V, \tilde{\nu} + \nabla u_{\Omega}. (-\nabla_\Gamma (V, \tilde{\nu}) + D\tilde{\nu} V) \right] \]

And finally, we get

\[ d \left( dJ(\Omega)[W] \right)[V] = \int_{\partial \Omega} \left[ -2m \left( \nabla u_{\Omega}', \tilde{\nu} + D^2 u_{\Omega} V, \tilde{\nu} + \nabla u_{\Omega}. (-\nabla_\Gamma (V, \tilde{\nu}) + D\tilde{\nu} V) \right) + \left( \frac{\partial \psi}{\partial \tilde{\nu}} + K_c \psi \right) \right] d\sigma \]

\[ d \left( dJ(\Omega)[W] \right)[V] = \int_{\partial \Omega} \left[ -2 \langle W, \tilde{\nu} \rangle \left( \nabla u_{\Omega}', \tilde{\nu} + D^2 u_{\Omega} V, \tilde{\nu} + \nabla u_{\Omega}. (-\nabla_\Gamma (V, \tilde{\nu}) + D\tilde{\nu} V) \right) \right] d\sigma. \]

On the one hand, having the following Riemannian hessian formula

\[ G^A (HessJ(\Omega)[V], W) = d \left( dJ(\Omega)[W] \right)[V] - dJ(\Omega)[\nabla V W], \]

it is possible to bring additional details on its computation.

**Proposition 5.4** We have:

\[ G^A (HessJ(\Omega)[V], W) = \int_{\partial \Omega} \left[ -2 \langle W, \tilde{\nu} \rangle \left( \nabla u_{\Omega}', \tilde{\nu} + D^2 u_{\Omega} V, \tilde{\nu} + \nabla u_{\Omega}. (-\nabla_\Gamma (V, \tilde{\nu}) + D\tilde{\nu} V) \right) \right] d\sigma \]

\[ + \int_{\partial \Omega} \left[ \frac{\partial}{\partial \tilde{\nu}} \left( k^2 - \left( \frac{\partial u_{\Omega}}{\partial \tilde{\nu}} \right)^2 \right) + K_c \left( k^2 - \left( \frac{\partial u_{\Omega}}{\partial \tilde{\nu}} \right)^2 \right) \right] d\sigma \]

\[ - \frac{3AK_c^2 + K_c}{1 + AK_c^2} \left( k^2 - \left( \frac{\partial u_{\Omega}}{\partial \tilde{\nu}} \right)^2 \right) \langle V, \tilde{\nu} \rangle \langle W, \tilde{\nu} \rangle d\sigma \]  

(18)
Proof.

\[ G^A(\text{HessJ}(\Omega)[V], W) = \int_{\partial \Omega} \left[ -2 \langle W, \bar{v} \rangle (\nabla u'_{\Omega, \bar{v}} + D^2 u_{\Omega} V. \bar{v} + \nabla u_{\Omega, \bar{v}}. (-\nabla_{\Gamma}(V. \bar{v}) + D\bar{v}V)) \right. \]
\[ + \left( \frac{\partial \psi}{\partial \bar{v}} + K_c \psi \right) \langle W, \bar{v} \rangle \langle V, \bar{v} \rangle + \psi \langle D_\bar{v} W, \bar{v} \rangle \right] d\sigma \]
\[ - \int_{\partial \Omega} \psi \langle \nabla W, \bar{v} \rangle d\sigma \]
\[ = \int_{\partial \Omega} \left[ -2 \langle W, \bar{v} \rangle (\nabla u'_{\Omega, \bar{v}} + D^2 u_{\Omega} V. \bar{v} + \nabla u_{\Omega, \bar{v}}. (-\nabla_{\Gamma}(V. \bar{v}) + D\bar{v}V)) \right. \]
\[ + \left( \frac{\partial \psi}{\partial \bar{v}} + K_c \psi \right) \langle W, \bar{v} \rangle \langle V, \bar{v} \rangle + \psi \langle D_\bar{v} W, \bar{v} \rangle \right] d\sigma \]
\[ - \int_{\partial \Omega} \psi \left[ \langle D_\bar{v} W, \bar{v} \rangle + \left( \frac{3AK^2_c + K_c}{1 + AK^2_c} \right) \langle V, \bar{v} \rangle \langle W, \bar{v} \rangle \right] d\sigma \]
\[ = \int_{\partial \Omega} \left[ -2 \langle W, \bar{v} \rangle (\nabla u'_{\Omega, \bar{v}} + D^2 u_{\Omega} V. \bar{v} + \nabla u_{\Omega, \bar{v}}. (-\nabla_{\Gamma}(V. \bar{v}) + D\bar{v}V)) \right. \]
\[ + \left. \int_{\partial \Omega} \left[ \frac{\partial \psi}{\partial \bar{v}} + K_c \psi - \psi K_c \left( \frac{3AK^2_c + 1}{1 + AK^2_c} \right) \right] \langle V, \bar{v} \rangle \langle W, \bar{v} \rangle d\sigma \right] \] \[ (19) \]

Replacing \( \psi \) by its expression, we have:

\[ G^A(\text{HessJ}(\Omega)[V], W) = \int_{\partial \Omega} \left[ -2 \langle W, \bar{v} \rangle (\nabla u'_{\Omega, \bar{v}} + D^2 u_{\Omega} V. \bar{v} + \nabla u_{\Omega, \bar{v}}. (-\nabla_{\Gamma}(V. \bar{v}) + D\bar{v}V)) \right. \]
\[ + \left. \int_{\partial \Omega} \left[ \frac{\partial \psi}{\partial \bar{v}} \left( k^2 - \left( \frac{\partial u_{\Omega}}{\partial \bar{v}} \right)^2 \right) + K_c \left( k^2 - \left( \frac{\partial u_{\Omega}}{\partial \bar{v}} \right)^2 \right) \right] \langle V, \bar{v} \rangle \langle W, \bar{v} \rangle d\sigma \right] \] \[ (19) \]

On the other hand, let us compute \( G^A(\text{HessJ}(\Omega)[V], W) \) by using directly the Sobolev-type metric \( G^A \). Then we have the following proposition.

**Proposition 5.5**

\[ G^A(\text{HessJ}(\Omega)[V], W) = \int_{\partial \Omega} \left[ \frac{\partial}{\partial \bar{v}} \left( k^2 - \left( \frac{\partial u_{\Omega}}{\partial \bar{v}} \right)^2 \right) + K_c \left( k^2 - \left( \frac{\partial u_{\Omega}}{\partial \bar{v}} \right)^2 \right) \right] \langle V, \bar{v} \rangle \langle W, \bar{v} \rangle d\sigma. \] \[ (20) \]

**Proof.**

\[ G^A(\text{HessJ}(\Omega)[V], W) = \int_{\partial \Omega} \left( 1 + AK^2_c \right) \text{HessJ}(\Omega)[V] W \]
\[ = \int_{\partial \Omega} \left( 1 + AK^2_c \right) \nabla V \text{gradJ}(\Omega) W \]
\[ = \int_{\partial \Omega} \left( 1 + AK^2_c \right) \nabla h \text{gradJ}(\Omega) m \]
Since \( \text{grad} J(\Omega) = \frac{1}{1 + AK^2} \psi \), we have

\[
\nabla_h \text{grad} J(\Omega) = \frac{\partial}{\partial \vartheta} \left( \frac{1}{1 + AK^2} \psi \right) \alpha + \frac{1}{1 + AK^2} \psi \left( \frac{3AK^3 + K_c}{1 + AK^2} \right) \alpha
\]

\[
= \frac{\partial}{\partial \vartheta} \left( (1 + AK^2)^{-1} \right) \psi \alpha + \frac{\partial \psi}{\partial \vartheta} \left( \frac{1}{1 + AK^2} \right) \alpha + \frac{1}{1 + AK^2} \psi \left( \frac{3AK^3 + K_c}{1 + AK^2} \right) \alpha
\]

\[
= -2AK_c \frac{\partial K_c}{\partial \vartheta} (1 + AK^2)^{-2} \psi \alpha + \frac{\partial \psi}{\partial \vartheta} \left( \frac{1}{1 + AK^2} \right) \alpha
\]

\[
+ \frac{1}{1 + AK^2} \psi \left( \frac{3AK^3 + K_c}{1 + AK^2} \right) \alpha
\]

Note that \( \frac{\partial K_c}{\partial \vartheta} = K_c^2 \), what implies that:

\[
\nabla_h \text{grad} J(\Omega) = -2AK_c \frac{\partial K_c}{\partial \vartheta} (1 + AK^2)^{-2} \psi \alpha + \frac{\partial \psi}{\partial \vartheta} \left( \frac{1}{1 + AK^2} \right) \alpha
\]

Then, coming back to our hessian computation, we have:

\[
G^A(\text{Hess} J(\Omega)[V], W) = \int_{\partial \Omega} \left( 1 + AK^2 \right) \left[ \frac{-2AK_c^3}{(1 + AK^2)^2} \psi \alpha + \frac{\partial \psi}{\partial \vartheta} \left( \frac{1}{1 + AK^2} \right) \alpha \right] \beta d\sigma
\]

\[
+ \frac{1}{1 + AK^2} \psi \left( \frac{3AK^3 + K_c}{1 + AK^2} \right) \alpha \beta d\sigma
\]

\[
= \int_{\partial \Omega} \left[ \frac{-2AK_c^3}{1 + AK^2} \psi \alpha + \frac{\partial \psi}{\partial \vartheta} \alpha + \psi \left( \frac{3AK^3 + K_c}{1 + AK^2} \right) \alpha \right] \beta d\sigma
\]

\[
= \int_{\partial \Omega} \left[ \frac{\partial \psi}{\partial \vartheta} + \psi K_c \left( \frac{1 + AK^2}{1 + AK^2} \right) \right] \alpha \beta d\sigma
\]

Replacing \( \psi \) by its expression, we have:

\[
G^A(\text{Hess} J(\Omega)[V], W) = \int_{\partial \Omega} \left[ \frac{\partial}{\partial \vartheta} \left( k^2 - \left( \frac{\partial u_{\Omega}}{\partial \vartheta} \right)^2 \right) + K_c \left( k^2 - \left( \frac{\partial u_{\Omega}}{\partial \vartheta} \right)^2 \right) \right] \langle V, \tilde{V} \rangle \langle W, \tilde{W} \rangle d\sigma \quad (21)
\]

**Remark 5.6** Let us note first that there is a symmetry relation with respect to the hessian which is in the case of our considered Riemannian structure a self adjoint operator with respect to the metric \( G^A \).

And the second fact is that it is important to underline that the formulas obtained from the formula in Theorem 5.2 and (21) computed by a direct method with the metric \( G^A \) in two different ways have to give the same expression even if \( \Omega \) is not a critical point. And then from
these computations, one deduces that
\[
\int_{\partial \Omega} \left[ -2 \langle W, \bar{v} \rangle \left( \nabla u_{\Omega}, \bar{v} + D^2 u_{\Omega} V, \bar{v} + \nabla u_{\Omega}, (-\nabla_{\Gamma}(V, \bar{v}) + D \bar{v} V) \right) \right] d\sigma
\]
\[
= \int_{\partial \Omega} \frac{3AK_c^3 + K_c}{1 + AK_c^2} \left( k^2 - \left( \frac{\partial u_{\Omega}}{\partial \bar{v}} \right)^2 \right) \langle V, \bar{v} \rangle \langle W, \bar{v} \rangle d\sigma
\]

Remark 5.7 In this remark, we compute \( G^A(V, \text{HessJ}(\Omega)[W]) \) to show the symmetry relation with respect to the hessian with the computation of the direct method with the metric \( G^A \).

\[
G^A(V, \text{HessJ}(\Omega)[W]) = \int_{\partial \Omega} (1 + AK_c^2) \text{HessJ}(\Omega)[W] V
\]
\[
= \int_{\partial \Omega} (1 + AK_c^2) \nabla_W \text{gradJ}(\Omega)V
\]
\[
= \int_{\partial \Omega} (1 + AK_c^2) \nabla_{m \text{gradJ}(\Omega)}h
\]
where \( V = h = \alpha \bar{v} \) and \( W = m = \beta \bar{v} \). Since \( \text{gradJ}(\Omega) = \frac{1}{1 + AK_c^2} \psi \), we have:

\[
\nabla_{m \text{gradJ}(\Omega)} = \frac{\partial}{\partial \bar{v}} (\text{gradJ}(\Omega)) \beta + \frac{3AK_c^3 + K_c}{1 + AK_c^2} \text{gradJ}(\Omega) \beta
\]
\[
= \frac{\partial}{\partial \bar{v}} \left( \frac{1}{1 + AK_c^2} \psi \right) \beta + \frac{1}{1 + AK_c^2} \psi \left( \frac{3AK_c^3 + K_c}{1 + AK_c^2} \right) \beta
\]

As previously, by the same computations, we get:

\[
\nabla_{m \text{gradJ}(\Omega)} = \frac{-2AK_c^3}{(1 + AK_c^2)^2} \psi \beta + \frac{\partial \psi}{\partial \bar{v}} \left( \frac{1}{1 + AK_c^2} \right) \beta + \frac{1}{1 + AK_c^2} \psi \left( \frac{3AK_c^3 + K_c}{1 + AK_c^2} \right) \beta.
\]

And finally, we have:

\[
G^A(\text{HessJ}(\Omega)[W], V) = \int_{\Omega} (1 + AK_c^2) \left[ \frac{-2AK_c^3}{(1 + AK_c^2)^2} \psi \beta + \frac{\partial \psi}{\partial \bar{v}} \left( \frac{1}{1 + AK_c^2} \right) \beta + \frac{1}{1 + AK_c^2} \psi \left( \frac{3AK_c^3 + K_c}{1 + AK_c^2} \right) \beta \right] d\sigma
\]
\[
+ \frac{1}{1 + AK_c^2} \psi \left( \frac{3AK_c^3 + K_c}{1 + AK_c^2} \right) \beta \alpha d\sigma
\]
\[
= \int_{\partial \Omega} \left[ \frac{\partial}{\partial \bar{v}} \left( k^2 - \left( \frac{\partial u_{\Omega}}{\partial \bar{v}} \right)^2 \right) + K_c \left( k^2 - \left( \frac{\partial u_{\Omega}}{\partial \bar{v}} \right)^2 \right) \right] \langle V, \bar{v} \rangle \langle W, \bar{v} \rangle d\sigma.
\]

Let us have a look at on the two formulas of the second derivation when \( V = W = \alpha \bar{v} \).

On the one hand, by Proposition 4.11 we get:

\[
Q(\alpha) = d^2 J(\Omega; V; V)
\]
\[
= -(N - 1) \int_{\partial \Omega} H \alpha^2 d\sigma + k^2 \int_{\Omega} |\nabla \Lambda|^2 dx
\]
\[
= -(N - 1)k^2 \int_{\partial \Omega} H \alpha^2 d\sigma + k^2 \int_{\partial \Omega} \alpha L d\sigma.
\]
On the other hand by Theorem 5.2, we have:

$$G^A (\text{Hess} J(\Omega)[V], W) = d(\text{d} J(\Omega)[W]) [V] - d J(\Omega)[\nabla_V W]$$

Then for $V = W$ we derive:

$$d(\text{d} J(\Omega)[V]) [V] = 2 J(\Omega)[V] = \text{d}^2 J(\Omega)[V] = G^A (\text{Hess} J(\Omega)[V], V) + d J(\Omega)[\nabla_V V]$$

- If the quadrature surface problem has a solution $\Omega$, then $d(\text{d} J(\Omega)[V]) [V] = G^A (\text{Hess} J(\Omega)[V], V)$.
- In previous works, the second author studied the stability and positiveness of the quadratic form, see [44] for more details. He established a similar proposition as Proposition 4.1 and gave necessary and sufficient qualitative properties in the theoretical point of view.

The one obtained involves the study of a generalized spectral Steklov problem that is reminded in the following corollary.

**Corollary 5.8** Let us consider the following generalized spectral Steklov problem:

$$\Delta \phi_n = 0 \quad \text{in} \quad \Omega \backslash K$$
$$\phi_n = 0 \quad \text{on} \quad \partial K$$
$$(L + (N - 1) H I) \phi_n = \left( \frac{1}{\mu_n} - \|H^-\|_\infty \right) \phi_n \quad \text{on} \quad \partial \Omega,$$

where $I$ is the identity map, $H$ is the mean curvature of $\Omega$, $K$ is a compact regular enough (let us say $C^2$) subset of $\Omega$, $H^- = \max\{-H, 0\}$ and $\mu_n$ is a decreasing sequence of eigenvalues depending also on $H$ which goes to 0. And one must have the sign of the first eigenvalue $\lambda_0 := \frac{1}{\mu_0} - \|H^-\|_\infty = \inf\{(N - 1) \int H v^2 d\sigma + \int_{\Omega \backslash K} |\nabla \Lambda|^2 dx, v \in H^{1/2}(\partial \Omega), \int_{\partial \Omega} v^2 d\sigma = 1\},$ where

$$\Delta \Lambda = 0 \quad \text{in} \quad \Omega \backslash K$$
$$\Lambda = 0 \quad \text{on} \quad \partial K$$
$$\frac{\partial \Lambda}{\partial \nu} = v \quad \text{on} \quad \partial \Omega.$$

And the minimum is reached for $\phi_0$ satisfying

$$\Delta \phi_0 = 0 \quad \text{in} \quad \Omega \backslash K$$
$$\phi_0 = 0 \quad \text{on} \quad \partial K$$
$$(L + (N - 1) H I) \phi_0 = \lambda_0 \phi_0 \quad \text{on} \quad \partial \Omega.$$

From our work we can deduce the following conclusions as a corollary.

**Corollary 5.9**

- What is obtained with the Riemannian hessian formula is easier to derive simple control for the characterization of the optimal shape in a number of ways.

- In the case of minimum, $G^A (\text{Hess} J(\Omega)[V], V) \geq 0$. And this inequality is equivalent to

$$\int_{\partial \Omega} \left[ \frac{\partial}{\partial \nu} \left( k^2 - \left( \frac{\partial u_0}{\partial \nu} \right)^2 \right) \right] \alpha^2 \, d\sigma \geq 0 \quad \forall \alpha \in C^\infty(\mathbb{R}^2, \mathbb{R}) \cap H^{1/2}(\partial \Omega).$$


This is reduced to
\[ \int_{\partial \Omega} \frac{\partial}{\partial \nu} \left( k^2 - \left( \frac{\partial u}{\partial \nu} \right)^2 \right) d\sigma \geq 0. \]

One can deduce also another control, since
\[ \int_{\partial \Omega} \left[ \frac{\partial}{\partial \nu} \left( k^2 - \left( \frac{\partial u}{\partial \nu} \right)^2 \right)^2 \right] \alpha^2 d\sigma = -2k^2(N-1) \int_{\partial \Omega} H \alpha^2 d\sigma = -2k^2(N-1) \int_{\partial \Omega} K_c \alpha^2 d\sigma. \]

And knowing that \( \alpha \in C^\infty(\mathbb{R}^2, \mathbb{R}) \cap H^{1/2}(\partial \Omega) \), the control becomes \( \int_{\partial \Omega} K_c d\sigma \leq 0 \). Before proceeding further, let us underline that in two dimension \( H = K_c \).

And from this, we have key information to set up algorithm in order to get a good approximation of the optimal shape.

- Now, when \( \Omega \) is only a critical point, to get a strict local minimum, we need the following sufficient condition:
\[ \int_{\partial \Omega} \left[ \frac{\partial}{\partial \nu} \left( k^2 - \left( \frac{\partial u}{\partial \nu} \right)^2 \right)^2 \right] \alpha^2 d\sigma = -2k^2(N-1) \int_{\partial \Omega} K_c \alpha^2 d\sigma \geq C_0 \| \alpha \|^2, C_0 > 0. \]

One can say also that there is \( x_0 \in \partial \Omega, -2k^2(N-1)K_c(x_0) \int_{\partial \Omega} \alpha^2 d\sigma \geq C_0 \| \alpha \|^2 \). And if \( K_c(x_0) < 0 \), then \( \Omega \) is a strict local minimum.

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