Vortex Lattice Melting in 2D Superconducting Networks and Films

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Abstract

We carry out an extensive Monte Carlo study of phase transitions in 2D superconducting networks, in an applied magnetic field, for square and honeycomb geometries. We consider both systems with a dilute vortex density $1/q$, and dense systems near “full frustration” with vortex density $1/2 - 1/q$. The dilute case gives the continuum limit as $q \to \infty$, and serves as a model for a uniform superconducting film. For this dilute case, we find a transition temperature $T_c \sim 1/q$, at which the vortex lattice unpins from the network and forms a “floating solid” phase. At a higher temperature $T_m$, this floating solid melts into a vortex liquid. We analyze the transition at $T_m$ according to the Kosterlitz-Thouless theory of dislocation mediated melting in 2D. While we find a discontinuous jump in the vortex shear modulus at $T_m$ which is consistent with this theory, we find (in opposition to this theory) that the transition is weakly first order, and we find no evidence for a hexatic liquid phase. For the case near full frustration, we find that the system can be described in terms of the density of defects in an otherwise fully frustrated vortex pattern. These dilute defects result in similar behavior as that found in the dilute vortex system, with pinned, floating, and liquid defect phases.

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I. INTRODUCTION

Two dimensional (2D) periodic superconducting networks, and in particular arrays of Josephson junctions, have served as a convenient theoretical and experimental model system in terms of which one can study, in a well controlled way, the effects of thermal fluctuations and pinning, on vortex structures and phase coherence in 2D superconductors\textsuperscript{1}. Such 2D superconductors have received renewed attention recently with the observation that many of the high temperature superconductors consist of weakly coupled layers, and so for some range of parameters may display effectively two dimensional behavior\textsuperscript{2,3}. One focus of this renewed interest has been concerned with the melting of the 2D vortex lattice, induced by an applied magnetic field, in a uniform continuous superconducting film. Controversy has resulted as to whether such a vortex lattice even exists at any finite temperature, or whether a vortex liquid is the only thermodynamically stable state\textsuperscript{4}. In this work we address the thermodynamic behavior of vortex structures in 2D superconducting systems. Our focus will be on behavior in discrete periodic networks, however our results will also yield conclusions concerning the behavior of uniform films.

Despite a decade of theoretical work, many fundamental questions remain unresolved concerning the nature of the phase transitions in 2D superconducting networks. When a uniform transverse magnetic field is applied, it induces a fixed density of vortices into the network, as in the mixed state of a type II superconductor. However, unlike a uniform superconductor, for which the ground state is a periodic triangular lattice of equally spaced vortices, the discrete network structure serves as an effective periodic pinning potential, which at low temperatures confines the vortices to sit at the centers of the unit cells of the network. This can result in novel vortex structures at low temperature, determined by the competition between the repulsive vortex-vortex interaction, and the periodic pinning potential induced by the network\textsuperscript{5–7}. Finding the ground state vortex structure for an arbitrary value of vortex density, for a given periodic network, remains an unsolved problem. The phase transitions at finite temperature have remained largely unexplored except for a
few of the simplest cases. An early conjecture by Teitel and Jayaprakash (TJ) argued that the superconducting transition in such networks would be governed by commensurability effects. If one measures the dimensionless vortex density $f$ as the number of magnetic field induced vortices per unit cell of the network, they predicted that for rational $f = p/q$, the transition temperature would vary discontinuously as $T_c(p/q) \sim 1/q$. While experimental evidence for high order commensurability effects has been reported in Josephson junction arrays, simulations by Halsey have challenged this conjecture for large $q$. A similar conjecture by TJ concerning the behavior of the ground state critical current, $i_c(f)$, has since been disproven in simulations by Lobb and co-workers and by Straley who argue that as $f$ varies, $i_c(f)$ has a lower non-zero limit determined by the single body effects of a non-interacting vortex in a periodic pinning potential; this conclusion has also been arrived at analytically by Vallat and Beck. However the validity of the TJ conjecture with respect to $T_c(f)$, which is intrinsically determined by many body effects, has remained unresolved.

In this paper we attempt to study the TJ conjecture systematically, by carrying out Monte Carlo simulations of superconducting networks for two special classes of vortex density. We first consider the dilute case of vortex densities $f = 1/q$, $q$ integer, for both square and honeycomb networks. This dilute case, as $q \to \infty$, can equivalently be viewed as the continuum limit, in which the lattice spacing of the periodic network decreases to zero for a fixed areal density of vortices. Our results for this case therefore also address the problem of vortex lattice melting in a uniform continuous superconducting film. Secondly, we consider vortex densities $f = 1/2 - 1/q$, close to full frustration, on the square network.

Our results may be summarized as follows. For the dilute case with large $q$, the low temperature state is a Bravais lattice of vortices, with long range translational order, pinned commensurably to the periodic network. At a critical temperature $T_c(1/q) \sim 1/q$, there is a sharp first order phase transition to a floating triangular vortex lattice, which is depinned from the periodic potential of the network. This floating lattice displays the algebraic translational correlations characteristic of a 2D vortex lattice in a uniform continuum. The
Depinning transition $T_c(f)$ satisfies the TJ conjecture, and marks the loss of true d.c. superconductivity in the network, due to the flux flow resistance which will result from drift of the unpinned vortex lattice. At a higher $T_m$, which becomes independent of $q$ as $1/q \to 0$, this floating vortex lattice melts into an isotropic vortex liquid. We analyze this transition according to the theory of dislocation mediated melting in 2D, due to Kosterlitz and Thouless, Nelson and Halperin, and Young (KTNHY). While we find good agreement with certain predictions of this KTNHY theory, we find evidence that the second order melting transition predicted by KTNHY is pre-empted by a weak first order transition.

For the close to fully frustrated case, $f = 1/2 - 1/q$, the ground state is everywhere like that of $f = 1/2$ (a checkerboard pattern of vortices on alternating sites), except for a superimposed commensurate Bravais lattice of missing vortices, or “defects,” so as to give the desired density $f < 1/2$. The transitions in this system are then governed by the behavior of these defects. Upon heating, there is first a depinning transition $T_c(f)$ of the defect Bravais lattice into a floating triangular defect lattice; this depinning follows the TJ conjecture, $T_c(f) \sim 1/q$, and marks the loss of true d.c. superconductivity. At a higher $T_m$, the floating defect lattice melts into an isotropic defect liquid. Finally, at a higher $T_{m'}$, there is an additional sharp transition representing the disordering of the vortices forming the $f = 1/2$ like background.

The remainder of our paper is organized as follows. In Section II we present the theoretical model used to describe the superconducting network, and its relation to a uniform superconducting film. We review the KTNHY theory of 2D melting, and discuss the observables we measure and the methods we use to analyze our data. Finally we describe our Monte Carlo procedure. In Section III we present our results for the dilute case $f = 1/q$ on a honeycomb network. This corresponds to vortices on the dual triangular lattice of sites. We use finite size scaling to test in detail the predictions of KTNHY. In Section IV we present our results for the dilute case $f = 1/q$ on a square network. In Section V we present our results for the dense case of $f = 5/11$ on a square lattice, and infer the behavior for more general densities $f = 1/2 - 1/q$. In Section VI we present our discussion and conclusions.
II. MODEL AND METHODS OF ANALYSIS

A. Model for a superconducting network

A two dimensional superconducting network in a magnetic field, is described by the Hamiltonian,

$$\mathcal{H}[\theta_i] = \sum_{\langle ij \rangle} U(\theta_i - \theta_j - A_{ij})$$

(1)

where $\theta_i$ is the fluctuating phase of the superconducting wavefunction on node $i$ of a periodic network of sites. The sum is over pairs of nearest neighbor sites, representing the bonds of the network, and

$$A_{ij} = (2\pi/\Phi_0) \int_i^j A \cdot dl$$

(2)

are fixed constants, giving the integral of the magnetic vector potential across bond $\langle ij \rangle$ ($\Phi_0 = \hbar c/2e$ is the magnetic flux quantum). $U(\theta)$ is the interaction potential between neighboring nodes, and its argument is just the gauge invariant phase difference across the bond. $U(\theta)$ is periodic in $\theta$ with period $2\pi$, and has its minimum at $\theta = 0$. We will be interested here in the case of a uniform applied magnetic field $\nabla \times A = B$, transverse to the plane of the network. In this case, the sum of the $A_{ij}$ going counter clockwise around any unit cell of the network is constant, and determined by the magnetic flux through the cell,

$$\sum_{cell} A_{ij} = 2\pi AB/\Phi_0 \equiv 2\pi f,$$

(3)

where $A$ is the area of a unit cell of the network. $f$ therefore is the number of flux quanta of applied magnetic field, per unit cell.

For an array of Josephson junctions, the interaction potential in Eq.(1) is taken as $U(\theta) = -J_0 \cos(\theta)$. For a superconducting wire network, in the London approximation, a more appropriate interaction is given by the Villain function, defined by,

$$e^{-U(\theta)/T} \equiv \sum_{m=-\infty}^{\infty} e^{-J_0(\theta-2\pi m)^2/2T}$$

(4)
where we take $k_B \equiv 1$.

For the Villain interaction, one can show by duality transformation\textsuperscript{32} that the Hamiltonian of Eq.(1) can be mapped onto the following 2D classical Coulomb gas,

\begin{equation}
\mathcal{H} = \frac{1}{2} \sum_{ij} (n_i - f)V(r_i - r_j)(n_j - f),
\end{equation}

where the sum is over all sites $i, j$ of the dual lattice of the periodic network (i.e. the sites $i$ in Eq.(5) lie at the centers of the unit cells of the network). $n_i = 0, \pm 1, \pm 2, \ldots$ are integer “charges” representing vortices in the phases $\theta_i$, and the magnetic field flux density is represented by the uniform background charge $-f$. $V(r)$ is the lattice Coulomb potential in 2D, which solves the equation,

\begin{equation}
\Delta^2 V(r) = -2\pi \delta_{r,0},
\end{equation}

where $\Delta^2$ is the discrete Laplacian for the network. For large separations, $V(r) \simeq \ln |r|$.

In mapping from the network Hamiltonian given by Eqs.(1) and (4), to the Coulomb gas Hamiltonian of Eq.(5), we have followed convention\textsuperscript{33} by rescaling the temperatures so that $T_{CG} = T_{XY}/2\pi J_0$, where $T_{CG}$ refers to the temperature in the Coulomb gas model, and $T_{XY}$ refers to the temperature in the network (also referred to as a “uniformly frustrated” XY model\textsuperscript{5}). Henceforth, we will denote $T_{CG}$ as simply $T$.

Our simulations will be carried out in terms of this Coulomb gas problem, rather than in terms of the phases $\theta_i$. Although the Villain interaction may give quantitative differences when compared to the cosine interaction of a Josephson array, since the two functions have the same symmetry, we expect that they will display the same qualitative critical behavior\textsuperscript{25}

For our simulations, we work with a finite $L \times L$ grid of sites, and apply periodic boundary conditions to the Laplace Eq.(6) defining the Coulomb potential $V(r)$. In this case, $V$ can be explicitly calculated in terms of its Fourier transform\textsuperscript{34}. For a square network of lattice constant $a_0$, one finds

\begin{equation}
V(r) = \frac{\pi}{N} \sum_k \frac{e^{i k \cdot r}}{2 - \cos(k \cdot a_1) - \cos(k \cdot a_2)}.
\end{equation}
where \( N = L^2 \), \( \{a_1, a_2\} = \{a_0 \hat{x}, a_0 \hat{y}\} \) are the basis vectors, and the summation is over all wave vectors consistent with the periodic boundary conditions, i.e. the set \( \{k\} = \{(m_1/L)b_1 + (m_2/L)b_2\} \), with \( m_1, m_2 = 0, 1, 2 \ldots L - 1 \), and with \( \{b_1, b_2\} = \{(2\pi/a_0)\hat{x}, (2\pi/a_0)\hat{y}\} \) the basis vectors of the reciprocal lattice.

For a honeycomb network, the charges \( n_i \) sit on the dual triangular grid of sites, and the Coulomb potential is given by,

\[
V(\mathbf{r}) = \frac{3\pi}{2N} \sum_k \frac{e^{ik \cdot \mathbf{r}}}{3 - \cos(k \cdot \mathbf{a}_1) - \cos(k \cdot \mathbf{a}_2) - \cos(k \cdot \mathbf{a}_3)},
\]

where \( \{\mathbf{a}_1, \mathbf{a}_2\} = \{a_0 \hat{x}, a_0 (\hat{x}/2 + \sqrt{3}\hat{y}/2)\} \) are the basis vectors, \( \mathbf{a}_3 = \mathbf{a}_2 - \mathbf{a}_1 \), and the wave vectors are determined by \( \{b_1, b_2\} = \{(2\pi/a_0)(\hat{x} - (1/\sqrt{3})\hat{y}), (2\pi/a_0)(2/\sqrt{3})\hat{y}\} \).

The \( k = 0 \) terms in the summations of Eqs.(7) and (8) will cause a divergence in \( V(\mathbf{r}) \).

In real space, this is a reflection of the infinite self energy of a point charge. Configurations with infinite total energy will carry zero weight in the partition function sum, and may therefore be excluded. To keep the energy of the Coulomb gas finite, we therefore impose the condition of overall charge neutrality

\[
\sum_i (n_i - f) = 0.
\]

If we define \( N_c \) as the total number of charges in the system, then Eq.(9) gives

\[
N_c \equiv \sum_i n_i = fN.
\]

Thus the density of magnetic flux quanta \( f \), is equal to the density of charges (vortices) \( N_c/N \). In the neutral system, the infinite self energies will exactly cancel, and in place of \( V(\mathbf{r}) \) we can use only the nonsingular part of the Coulomb potential (7) and (8) defined by \( V'(\mathbf{r}) \equiv V(\mathbf{r}) - V(\mathbf{r} = 0) \). For a given system size, we evaluate \( V'(\mathbf{r}) \) by numerically performing the summations indicated in Eqs.(7) and (8).

The ground state will therefore be a periodic vortex structure consisting of \( N_c \) sites with \( n_i = +1 \) (all other sites having \( n_i = 0 \)), spaced as equally apart as allowed by the network geometry. Understanding the behavior of this vortex structure at finite temperature will be one of the main goals of this work.
B. Relation to a uniform superconducting film

The Coulomb gas model of the preceding section can also be used to describe the melting of the vortex lattice in a uniform continuous superconducting film. For a superconducting film, the states of the system can be described by a complex wavefunction, \( \psi(\mathbf{r}) = |\psi(\mathbf{r})| e^{i\theta(\mathbf{r})} \). As shown by Pearl\(^27\) for a film of thickness \( d \), provided the sample size is smaller than the transverse magnetic penetration length \( \lambda_\perp = \lambda_\perp^2/d \), the magnetic field will be essentially uniform and constant throughout the film. In this case, the states \( \psi(\mathbf{r}) \) will be weighted in the partition function sum according to the Ginzburg-Landau free energy,

\[
\mathcal{H}_{GL}[\psi] = \int d^2r \left\{ \alpha |\psi|^2 + \frac{1}{2} \beta |\psi|^4 + \frac{1}{2m} \left| \frac{\hbar}{i} \nabla - \frac{2e}{c} \mathbf{A} \right| \psi \right|^2 \}
\]  

(11)

with \( \nabla \times \mathbf{A} = \mathbf{B} \) a fixed constant. The mean field solution that minimizes \( \mathcal{H}_{GL}[\psi] \), is similar to that found\(^28\) in three dimensions: \( (i) \) there is a triangular lattice of equally spaced vortices in the phase \( \theta(\mathbf{r}) \); \( (ii) \) the areal density of vortices is \( B/\Phi_0 \), with an average separation of \( a_v \sim \sqrt{\Phi_0/B} \); \( (iii) \) the size of the normal core of a vortex is determined by \( \xi_0 \sim 1/\sqrt{\alpha} \), where \( \alpha = 0 \) determines the \( B = 0 \) mean field transition temperature; \( (iv) \) the mean field phase transition at finite \( B \) occurs when \( \xi_0 \sim a_v \).

To include fluctuations, one should now sum the partition function over all fluctuations of \( \psi(\mathbf{r}) \) about the mean field solution. In doing so, one common approach has been to make the London approximation. Here one assumes that, outside of the normal vortex core, the amplitude of the superconducting wavefunction is kept constant, and only the phase fluctuates, i.e. \( \psi(\mathbf{r}) = |\psi| e^{i\theta(\mathbf{r})} \). The London approximation is expected to be good whenever the bare vortex core radius is very much smaller than the average separation between vortices, \( \xi_0 \ll a_v \); by \( (iv) \) above, this corresponds to temperatures well below the mean field phase transition.

Substituting \( \psi(\mathbf{r}) = |\psi| e^{i\theta(\mathbf{r})} \) into Eq.(11) results, within additive constants, in the simplified free energy

\[
\mathcal{H}[\theta] = \frac{1}{2} J_0 \int d^2r \left| \nabla \theta - \frac{2\pi}{\Phi_0} \mathbf{A} \right|^2,
\]

(12)
where \( J_0 = \Phi_0^2 / 16\pi^3 \lambda_\perp \), and the integral is implicitly cut off at the vortex cores. Eq. (12) is just a continuum version of the network Hamiltonian, Eq. (1). Following Halperin and Nelson, who considered the \( B = 0 \) case, and Huberman and Doniach, and Fisher who considered the finite \( B \) case, we note that Eq. (12) can be mapped onto a continuum Coulomb gas of logarithmically interacting charges. For finite \( B \), this can be written in the form of a one component plasma on a uniform background charge density \( B/\Phi_0 \),

\[
H = \frac{1}{2} \int d^2r d^2r' [n(r) - B/\Phi_0] V(r - r') [n(r') - B/\Phi_0]. \tag{13}
\]

Here \( n(r) \equiv (1/2\pi) \hat{z} \cdot \nabla \times \nabla \theta \) is the vorticity in the phase of the superconducting wavefunction, determined by singular integer vortices \( n_i \) at positions \( r_i \), \( n(r) = \sum_i n_i \delta (r - r_i) \). \( V(r) \) solves the 2D Laplace equation, \( \nabla^2 V = -2\pi \delta(r) \).

The Coulomb gas of Eq. (5), introduced in the preceding section as a description for a network, can now be viewed as a discrete approximation to the continuum problem of Eq. (13). For a fixed areal density of vortices, \( B/\Phi_0 \), we recover the continuum Eq. (13) from the discrete Eq. (5) as we take the network lattice constant \( a_0 \to 0 \). Since the number of vortices per unit cell in the network is \( f \sim a_0^2 B/\Phi_0 \), we see that the continuum is equivalent to the \( f \to 0 \) limit. Thus by studying the melting of dilute vortex lattices in a network, we can also learn about the melting of a vortex lattice in a uniform superconducting film. As in the previous section, the mapping between the Coulomb gas and the superconductor is obtained by measuring the Coulomb gas temperature \( T_{CG} \) in units of \( 2\pi J_0 = \Phi_0^2 / 8\pi^2 \lambda_\perp \), i.e. \( T_{\text{super}} = 2\pi J_0 T_{CG} \).

Finally, we note that the melting of the 2D vortex lattice, described by the continuum Coulomb gas Hamiltonian of Eq. (13), has been treated within the general 2D melting theory of KTNHY. Within this theory, Fisher has estimated that the melting transition occurs more than an order of magnitude below the mean field transition. This observation completes the self consistency of the argument for using the London approximation.
C. Review of the theory of 2D melting

The analysis of our results will be guided by the ideas of the theory of defect mediated melting in 2D, developed by KTNHY. Although our results are, in many aspects, in opposition to this KTNHY theory, it still represents a useful starting point in exploring the phenomenon of 2D melting.

For the 2D harmonic crystal on a smooth substrate (i.e. in the absence of any one-body potential) it is well known that fluctuations in the long-wavelength phonon modes, lead to a logarithmic divergence in the displacements of the particles, destroying translational long range order at any finite temperature. This is a consequence of the rigorous Mermin-Wagner theorem concerning long range order in 2D. The standard theory of elasticity shows however, that despite the absence of translational long range order, the low temperature phase of such a crystal is characterized by a slow power-law decay of translational correlations, very different from the fast exponential decay that one would expect in the liquid. This phenomenon has been termed “quasi-long range” order, and we shall refer to such a phase as a “2D solid”. Based on the ideas of Kosterlitz and Thouless, that the melting of such a 2D solid would be nucleated by the unbinding of topological lattice defects, Nelson and Halperin and Young formulated a theory (KTNHY) which predicted that 2D melting would occur via two separate second order KT-like transitions. In particular, they predicted that the 2D solid with algebraic translational correlations would become unstable to the unbinding of dislocation defect pairs at a temperature $T_m$, and melt into a new phase called the hexatic liquid. This hexatic phase would be characterized by short range translational order, but quasi-long range six-fold orientational order. As the temperature is increased, KTNHY predicted that this quasi-long range orientational order would eventually be destroyed by the unbinding of disclination defect pairs, and at $T_i > T_m$, the hexatic liquid would melt into a normal (isotropic) liquid with short range orientational order. We summarize this scenario by writing down the long-range limiting behavior predicted for the translational and orientational correlation functions.
For translational correlations,

\[
\langle e^{i \mathbf{G} \cdot (\mathbf{r}_i - \mathbf{r}_j)} \rangle \sim \begin{cases} 
1 & \text{in perfect crystal (} T = 0) \\
\frac{1}{r_{ij}^{-\eta_{\mathbf{G}}(T)}} & \text{in 2D solid (} 0 < T < T_m) \\
e^{-r_{ij}/\xi_6} & \text{in hexatic or normal liquid (} T > T_m) 
\end{cases}
\]  

(14)

where \( r_{ij} = |\mathbf{r}_i - \mathbf{r}_j| \) is the separation between particles \( i \) and \( j \), and \( \mathbf{G} \) is a reciprocal lattice vector of the perfect (triangular) crystal at \( T = 0 \). \( \eta_{\mathbf{G}}(T) \) is a temperature dependent exponent, which for the 2D harmonic crystal can be expressed in terms its shear modulus \( \mu \) and bulk modulus \( \lambda \), as

\[
\eta_{\mathbf{G}}(T) = \frac{k_B T |\mathbf{G}|^2 (3\mu + \lambda)}{4\pi \mu (2\mu + \lambda)}.
\]  

(15)

In the 2D vortex lattice, the bulk modulus \( \lambda \) is infinite because of the long-ranged nature of the Coulomb interaction. The expression for \( \eta_{\mathbf{G}}(T) \) thus simplifies to,

\[
\eta_{\mathbf{G}}(T) = \frac{k_B T |\mathbf{G}|^2}{4\pi \mu}.
\]  

(16)

A key prediction of the KTNHY theory is that if \( \mathbf{G}_1 \) is a shortest reciprocal lattice vector, then \( \eta_{\mathbf{G}_1}^{-1}(T) \) takes a discontinuous jump at \( T_m \) to zero from the universal value,

\[
\eta_{\mathbf{G}_1}^{-1}(T_m) = 3.
\]  

(17)

In what follows, we will directly test this prediction. We wish to stress however, that the behavior of translational correlations in the 2D solid phase, as given by Eq. (14), is a general result of continuum elastic theory, independent of all assumptions concerning the mechanism of the melting transition. It is only the universal jump in \( \eta_{\mathbf{G}_1}^{-1}(T_m) \), and the existence of the hexatic phase, which are specific predictions of KTNHY.

The six-fold orientational correlation function, according to KTNHY, behaves as

\[
\langle e^{i \theta(\mathbf{r}_i) - \theta(\mathbf{r}_j)} \rangle \sim \begin{cases} 
\alpha e^{-r_{ij}/\xi_6} + \varphi_6^\infty & \text{in 2D solid (} 0 < T < T_m) \\
\frac{1}{r_{ij}^{-\eta_{\varphi}(T)}} & \text{in hexatic liquid (} T_m < T < T_i) \\
e^{-r_{ij}/\xi_6} & \text{in normal liquid (} T > T_i) 
\end{cases}
\]  

(18)
where $\vartheta(r_i)$ is the angle of the bond from particle $i$ to its nearest neighbor, relative to some fixed reference direction. $\alpha$ is a proportionality constant of order one, and $\varphi_6^\infty$ gives the value of the long range orientational order expected in the 2D solid phase. The exponent $\eta_6^{-1}(T)$, describing the quasi-long range order of the hexatic phase, is predicted to have a universal jump to zero at $T_i$ from the value $\eta_6^{-1}(T_i^-) = 4$.

In the 2D solid, a relation between $\varphi_6^\infty$ and the vortex shear modulus $\mu$ can be derived from continuum elastic theory,

$$\varphi_6^\infty \simeq \exp \left[ -\frac{9k_B T \Lambda^2}{8\pi \mu} \right] = \exp \left[ -\frac{9\Lambda^2 \eta_G}{2|G|^2} \right],$$

(19)

where $\Lambda \sim 2\pi/a_v$ is an ultraviolet cutoff ($a_v$ is the average separation between particles). Since we will independently measure $\varphi_6^\infty$ and $\eta_G$ in our simulation, we will use this relation as a check of the consistency of our results.

For a periodic superconducting network, we have discussed how the discrete substrate of the network serves to induce a periodic pinning potential for the magnetic field induced vortices. To treat this case, we are therefore interested in how the above 2D melting scenario is altered by the presence of a periodic substrate. We shall be interested in the situation where the period of the substrate is sufficiently small compared to the spacing between particles, so that the essential features of the defect mediated melting theory remain intact. This problem has been treated by Nelson and Halperin. The main result of such a “fine-mesh” periodic perturbation is the appearance of a new phase at low temperatures, in which the 2D solid is commensurably pinned to the substrate. This phase has true long range translational order, and we shall refer to it as the “pinned solid”. At a certain depinning temperature $T_c < T_m$, there is a transition to a 2D “floating solid” phase, where the solid decouples from the substrate, and translational correlations behave identically to those of a 2D solid on a uniform substrate; this triangular floating solid may in general be incommensurate with the periodic substrate. Increasing temperature, the floating solid is expected to melt at $T_m$, via the dislocation unbinding mechanism, into a liquid phase. On a triangular substrate, this liquid will have a small (but finite) long-ranged six-fold
orientational order induced by the substrate, at all temperatures. There should, however, be a temperature $T_i$ where $\varphi_6^\infty(T)$ shows a significant drop, reminiscent of the disclination unbinding transition on the smooth substrate. This drop should become increasingly sharper as the ratio of substrate period to particle separation becomes smaller. On the square substrate, Nelson and Halperin predict that there will be a sharp Ising transition at a $T_i > T_m$, where quasi-long range six-fold orientational order in the liquid vanishes, and only the long range four-fold orientational order induced by the substrate remains. This Ising transition can be viewed as a “ghost” of the hexatic to normal liquid transition, which would occur in the absence of the periodic substrate. The four-fold orientational order, induced by the substrate, again persists at all higher temperatures.

To conclude, we note again that the properties of the 2D floating solid, described above, follow solely from continuum elastic theory, independent any particular theory of melting. It is the existence of the hexatic liquid phase that is a specific prediction of the KTNHY melting theory. However, as pointed out by Nelson and Halperin, it is always possible that a “premature” unbinding of disclination pairs may lead to a direct melting of the 2D solid into the normal liquid. Such a transition is then expected to be first order. In this case, the KTNHY prediction, Eq.(17), for the universal value of $\eta_{G_1}$ at melting, becomes a lower bound, $\eta_{G_1}(T_m^-) \geq 3$. Results from various numerical simulations and experiments indicate that this first order behavior might indeed be prevailing in the various 2D systems studied so far.

D. Observables and finite size scaling

We now show how the predictions of the preceding section translate into the behavior of observables which can be directly measured in our MC simulation. There are two key issues that we wish to investigate in the superconducting networks: (i) the transition from the superconducting to the normal state, and (ii) the melting of the magnetic field induced vortex lattice. For (ii), our goal is to test the KTNHY theory of 2D melting, and so we will
be interested in studying both the translational and the orientational order of the vortex lattice.

The superconducting to normal transition, marked by the loss of superconducting phase coherence, is measured by the vanishing of the *helicity modulus*, \( \Upsilon(T) \), which measures the response of the system to applying a net twist, or phase gradient, to the phases \( \theta_i \) in the Hamiltonian, Eq.(1). For the Villain interaction of Eq.(4), the helicity modulus can be shown to be identical to the inverse dielectric function of the corresponding Coulomb gas of Eq.(5), \( \Upsilon/J_0 = \epsilon^{-1} \), where \( \epsilon^{-1} \) is defined in the usual way,

\[
\epsilon^{-1}(T) = \lim_{k \to 0} \left\{ 1 - \frac{2\pi}{TNk^2} \langle n_k n_{-k} \rangle \right\},
\]

(20)

Here \( n_k = \sum_i n_i \exp(-ik \cdot r_i) \) is the Fourier-transformed charge density. The vanishing of \( \epsilon^{-1} \) signals an insulator to metal transition in the Coulomb gas. The free charges characteristic of the conducting phase correspond to freely diffusing vortices in the superconducting network, which are responsible for the loss of phase coherence.\(^{18}\) In the simulation, the \( k \to 0 \) limit is approximated by averaging \( \epsilon^{-1} \) over the smallest allowed nonzero wave vectors.

Information on the translational order in the vortex lattice can be extracted from the *structure function*

\[
S(k) = \frac{1}{N_c} \langle n_k n_{-k} \rangle = \frac{1}{N_c} \sum_{ij} e^{ik(r_i-r_j)} \langle n_i n_j \rangle,
\]

(21)

which we evaluate for all allowed wave vectors \( k = (m_1/L)b_1 + (m_2/L)b_2 \) in the first Brillouin zone (BZ) of the reciprocal lattice to the real space dual lattice of the superconducting network. A 2D intensity plot of \( S(k) \) serves as a simple tool for visualization of the different phases in the system. In analogy to the conventional X-ray scattering images, we expect \( S(k) \) to display a periodic array of sharp delta-function Bragg peaks in a state with long range translational order, and a set of smooth concentric rings in a normal liquid phase. A phase with quasi-long range translational order, characterized by algebraic translational correlations, will be distinguished by a regular array of algebraically diverging peaks of finite width.\(^ {34}\) A hexatic liquid phase should appear as a set of concentric rings with six-fold angular modulation.
Apart from providing the simple visualization described above, the scaling of the heights of the peaks in $S(k)$, as a function of system size $L$, will serve as a good quantitative indicator of translational correlations in the system. Combining the definition of $S(k)$ in Eq.(21) with Eq.(14) (note that in Eq.(21), $n_i = 1$ on a site containing a vortex and $n_i = 0$ on a site without a vortex) one easily obtains

$$\frac{S(G)}{L^2} \sim \begin{cases} 
1 & \text{in pinned solid (} T < T_c \text{)} \quad (22a) \\
L^{-\eta_G(T)} & \text{in floating solid (} T_c < T < T_m \text{)} \quad (22b) \\
(\xi_/L)^2 & \text{in hexatic or normal liquid (} T > T_m \text{)} \quad (22c)
\end{cases}$$

The finite size scaling analysis of the translational order, that we present in Section III, will be based on the above relations. In particular, a comparison of Eqs.(22) with our MC data will allow us to extract the temperature dependent exponent $\eta_G(T)$ and test the KTNHY prediction regarding the universal jump in $\eta_G^{-1}(T_m^-)$. We shall also determine the correlation length $\xi_+(T)$ in the liquid phase.

There is an independent way to extract the exponents $\eta_G$ (and thus the vortex shear modulus $\mu$) without having to use finite size scaling. One can instead, for a given system size, fit to the heights of the peaks $S(G)$, as a function of $|G|$. For the low order peaks, this dependence is roughly Gaussian, as can be seen by combining Eq.(22b) with the expression for $\eta_G$ in Eq.(16). For the higher order peaks however, we need to rederive this dependence, since the prefactor (which is not shown in Eq.(22b)) becomes important. Substituting Eq.(14) into the definition of the structure function, Eq.(21), and approximating the summations by integrations, we get

$$S(G) = c \int d^2r (r/a_v)^{-\eta_G} \simeq 2\pi c a_v^{-\eta_G} \int_{a_v}^R dr r^{1-\eta_G}, \quad (23)$$

where $R \sim L$ is a long distance cutoff, $a_v$ is the average separation between vortices, and $c$ is a proportionality constant of the order one. The integral is easily evaluated

$$S(G)/L^2 \simeq \frac{2 \pi c}{2-\eta_G} \left[ (R/a_v)^{-\eta_G} - (R/a_v)^{-2} \right], \quad (24)$$

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and $c$ is determined by the requirement that $S(0)/L^2 = 1$. One can see that for $\eta_G \ll 2$ formula (22b) remains valid, but for $\eta_G \simeq 2$ it breaks down. In practice, since $\eta_{G_1} < 1/3$ by the KTNHY bound, and any exponent $\eta_G$ at a given fixed temperature can be written as

$$\eta_G = \eta_{G_1} \cdot (|G|^2/|G_1|^2), \quad (25)$$

formula (22b) will approximately hold for the three shortest reciprocal lattice vectors $G$ only. To draw quantitative conclusions regarding the exponent $\eta_G$, one must use the more accurate relation of Eq.(24).

Information on the bond orientational order will be obtained by measuring the four-fold and six-fold orientational correlation

$$\varphi_p(T) = \frac{1}{N^2} \sum_{ij} \langle e^{ip(\vartheta_i - \vartheta_j)} \rangle, \quad p = 4, 6 \quad (26)$$

where the sum is over sites with non-vanishing charges $n_i = +1$, and $\vartheta_i$ is the bond orientation angle defined in the previous section. For a finite system, one expects to see a sharp drop in $\varphi_p(T)$ at the transition from an orientationaly ordered phase, to a disordered or possibly hexatic phase. One can deduce the scaling of the orientational correlation $\varphi_6(T)$ with system size $L$, by combining Eq.(26) with the KTNHY prediction of Eq.(18)

$$\varphi_6 \sim \begin{cases} 2\pi \alpha(\xi_6^6/L)^2 + \varphi_6^\infty \quad & \text{ in pinned or floating solid } (0 < T < T_m) \quad (27a) \\ L^{-\eta_6(T)} \quad & \text{ in hexatic liquid } (T_m < T < T_i) \quad (27b) \\ (\xi_6/L)^2 \quad & \text{ in normal liquid } (T > T_i) \quad (27c) \end{cases}$$

Relations (27a) and (27c) hold for $L \gg \xi_6$ (otherwise one must include corrections $\sim \exp(-L/\xi_6)$). These scaling relations for $\varphi_6$ will be used extensively in our analysis, to test for the existence of a hexatic phase in our model.

**E. Monte Carlo algorithm**

For the purpose of developing a fast MC algorithm, it is important to realize that the physical phenomena described in the previous section occur at temperatures which are about
one order of magnitude lower\textsuperscript{1} than the ordinary Kosterlitz-Thouless (KT) transition\textsuperscript{2} in the zero magnetic field, $f = 0$, case. This implies that the role of vortex-antivortex pair excitations is negligible in the temperature range that we study, and that in the simulation we can restrict ourselves to the excitations caused by movement of the vortices induced by the external magnetic field $B$. We have explicitly verified that the energy of an isolated vortex-antivortex pair (in the Coulomb gas language a pair of $(+, -)$ charges) is always $E_{\text{pair}} \gg k_B T$, and thus in practice such an excitation would never be accepted in the MC simulation. Consequently, our updating scheme is as follows. In each step, one charge is selected at random and moved to a different site within a radius $r_0$, which is chosen so as to maximize the acceptance rate. We find that values $r_0 \sim a_v/2$ are optimal. The energy change $\Delta E = E_{\text{new}} - E_{\text{old}}$ is then computed, and the excitation is accepted or rejected according to the standard Metropolis algorithm:

$$\text{accept if } e^{-\Delta E/T} > x,$$

where $x$ is a random number uniformly distributed on the interval $[0, 1)$. Here and henceforth, we work in units in which $k_B \equiv 1$. $N_c$ such attempts we will refer to as one MC sweep. At low temperature, we also made global moves, by attempting to shift entire rows of charges by one space. Such moves are meant to model long wavelength shear excitations, and help to accelerate equilibration near the vortex lattice melting transition.

Due to the long-ranged nature of the Coulomb potential, the most time consuming operation is the evaluation of $\Delta E$. From Eq.(23) we find that the energy change for moving a charge from the site $R_1$ to the site $R_2$ is

$$\Delta E = - \sum_j V'(R_1 - r_j)n_j + \sum_j V'(R_2 - r_j)n_j - V'(R_1 - R_2),$$

(28)

where we have used the fact that $V'(-r) = V'(r)$ and $V'(0) = 0$. In this form, each evaluation of $\Delta E$ is a computation of the order $N_c$, as $j$ sweeps through all the sites with nonvanishing charge $n_j = \pm 1$. To speed up this process, we use an algorithm developed by Grest.\textsuperscript{35} At each site of the lattice we define a potential due to all charges in the system
\[ F(r_i) = \sum_j V'(r_i - r_j)n_j. \] (29)

Now each evaluation of
\[ \Delta E = -F(R_1) + F(R_2) - V'(R_1 - R_2) \] (30)
requires computation of only \( O(1) \). Naturally, each time the excitation is accepted, it is necessary to update \( F(r_i) \) at all sites
\[ F_{\text{new}}(r_i) = F_{\text{old}}(r_i) - V'(r_i - R_1) + V'(r_i - R_2), \quad i = 1, 2, \ldots N. \] (31)

This is a computation of order \( N \). However, since the acceptance rate in the interesting temperature range is very low (typically below 1%), this method is faster than the direct approach of Eq.(28).

Data is collected by heating the system up from the ground state. At each temperature we discard 30,000 MC sweeps to equilibrate the system. Then, starting from this equilibrated configuration, we perform several (typically 4 – 6) independent runs of 100,000 sweeps each to sample physical quantities. In some cases, when evaluating quantities at the temperatures close to the critical point, substantially longer runs are carried out. Errors are estimated from the standard deviation of these independent runs. To verify the consistency of our results, we also perform cooling from a random configuration at high temperature; no substantial hysteresis is found.

All simulations were carried out on Sparc 10 workstations. The time needed to equilibrate the system and sample the physical quantities at a given temperature \( T \) was typically several hours (depending on size) using 100% of the single processor power. For example, it took approximately 3 hours to carry out 100,000 MC sweeps at a temperature close to melting, for a medium-sized system \( N_c = 81 \) with density \( f = 1/49 \). Our longest run, to sample the energy distribution near the depinning transition, took 189 hours for \( 4 \times 10^7 \) MC sweeps on the largest system of \( N_c = 169 \) and \( f = 1/49 \).
III. SIMULATIONS ON THE TRIANGULAR GRID: DILUTE CASE

A. The results

In this section we report our results from simulations of the Coulomb gas Hamiltonian \((\mathbf{5})\) on a triangular grid of sites (corresponding to a honeycomb superconducting network), for the dilute limit \(f \ll 1\) (or equivalently \(a_0 \ll a_v\)). In this case, we expect that our discretized model will well approximate the continuum. Some of these results have been reported by us previously.\[36\] The advantage of choosing a triangular grid is that, for a given system size \(L\), one can always choose \(f\) in such a way as to accommodate a perfect, commensurate, triangular vortex lattice in the ground state. By contrast, this is never possible on a square grid. It is convenient to choose \(f = 1/m^2\), with \(m\) integer, since then each system size of the form \(L = s \cdot m\), \((s\) integer\), will accommodate a triangular ground state with \(N_c \equiv fL^2 = s^2\) vortices. We have studied systematically densities \(f = 1/m^2\), with \(m = 3\) to 12, and fixed \(N_c \simeq 100\). The results of our investigation are summarized in Figure 1: for sufficiently dilute systems \((f < 1/25)\) we find three distinct phases. At low temperatures the vortex lattice is in a “pinned solid” phase, locked to the underlying grid. Above a sharp depinning temperature \(T_c(f)\), the vortices are in a “floating solid” phase, which then melts at \(T_m\) into a normal vortex liquid. The properties of these phases will be discussed below. For denser systems with \(f > 1/25\), the two transitions at \(T_c\) and \(T_m\) merge, and there is only a single transition from a pinned solid into a liquid.

For a simple visualization of the three phases, we show in Figure 2 intensity plots of \(S(k)\) at various \(T\), for the specific case of \(f = 1/49\) and \(N_c = 63\). We also display the amplitude of \(S(k)\) along the symmetry axis \(k_y\). For \(T = 0.003\) (Figure 2a), just below \(T_c(f)\), we see a regular array of \(\delta\)-function Bragg peaks, indicating long ranged translational order induced by pinning to the triangular grid. The width of these peaks corresponds to the finite resolution of wave vectors allowed by our finite system. At \(T = 0.0065\) (Figure 2b), just below \(T_m\), we see a regular array of peaks, but they are now of finite width. We
will show that these peaks are consistent with the power law singularities characteristic of
the algebraic translational correlations expected for a 2D floating solid phase. The heights
of the peaks along the symmetry axis are well described by a Gaussian, as expected from
Eqs. (24) and (25). Thus, for \( T_c(f) < T < T_m \) we do have a floating vortex lattice, as in
the continuum limit. For \( T = 0.0075 \) (Figure 3), slightly above \( T_m \) we see a rotationaly
invariant structure, typical for a liquid with short range correlations. Thus for \( T > T_m \), the
floating vortex lattice has melted into a normal liquid. It is interesting to note that we see
no sign of angular modulation in the rings above \( T_m \). One might expect such a modulation
due to the long ranged six-fold orientational order induced in principle by the underlying
triangular grid; if the grid was too fine for this effect to be significant, modulation might
still be present if a hexatic phase existed just above \( T_m \).

In Figure 3, we plot versus \( T \) the inverse dielectric function \( \epsilon^{-1}(T) \), and the orientation
order correlation \( \varphi_6(T) \), for \( f = 1/49 \) and \( N_c = 169 \) (one of the largest systems that we
have studied). We see that \( \epsilon^{-1}(T) \) vanishes at the depinning transition \( T_c(f) \), signaling the
loss of superconducting phase coherence in the floating solid phase. This is just a reflection
of the fact that an unpinned vortex lattice, in the presence of any applied d.c. current (no
matter how small), will be free to drift transversely to the current, resulting in a finite linear
“flux flow” resistance. Our results explicitly show that the absence of phase coherence in
this \( k \to 0 \) sense, does not imply the absence of a well defined vortex lattice. Considering
the orientation order, we see that \( \varphi_6 \) sharply drops at \( T_c(f) \), but remains finite up to the
melting temperature \( T_m \), where it drops again sharply to nearly zero values. The smallness
of \( \varphi_6 \) above \( T_m \) indicates that the six-fold orientational long range order which is induced
in principle by the triangular grid, is indeed a negligibly small effect at the densities we
are concerned with. We shall discuss this point in more detail in the following section.

In the Figure 4, we show the dependence of \( T_c(f) \) and \( T_m \) on the vortex density \( f \), as
estimated from the behavior of \( \epsilon^{-1}(T) \) and \( \varphi_6(T) \), and checked against the behavior of the
structure function \( S(k) \). We see that only for sufficiently dilute systems, \( f < 1/25 \), is
there a floating solid phase; for \( f > 1/25 \) there is only a single transition from a pinned
solid to a liquid. As \( f \) decreases, \( T_c(f) \) vanishes linearly with \( f \), consistent with the TJ conjecture\(^5\) for the loss of superconducting coherence. \( T_m \) however, quickly approaches a finite constant \( T_m = 0.0070 \pm 0.0005 \). In terms of the superconductor temperature, this means a vortex lattice melting at \( T_m = 0.0070 \Phi_0^2/8\pi^2\lambda_\perp \). This is well within the bounds \( 0.0046 < T_m < 0.0086 \) estimated by Fisher\(^4\) from the KTNHY theory.

**B. The melting transition: finite size scaling analysis**

To investigate if the melting transition at \( T_m \) is indeed consistent with the KTNHY theory, we have carried out a detailed finite size scaling analysis for the density \( f = 1/49 \). This density has been chosen for two reasons. First, the estimated \( T_m \) is well separated from the depinning temperature \( T_c \), and hence the floating solid phase exists in a relatively wide interval of temperatures. Second, the density is not too small, and thus we are able to study systems with as many as \( N_c = 169 \) vortices. More dilute systems, with comparable \( N_c \) would require sizes that are currently out of reach of the computer power available to us. We have carried out extensive simulations for the system sizes \( L = 28, 35, \ldots, 91 \), and we have analyzed the size-dependencies of various physical quantities at several temperatures below and above \( T_m \). Our results are as follows.

In Figure 5 we plot \( S(G_1)/L^2 \) as a function of \( L \), on a log-log scale, for several different temperatures. Data for each temperature fall on a straight line, confirming the expected power-law behavior of Eq.(22). These straight lines fall into three distinct groups. For \( T < T_c \simeq 0.0045 \), \( S(G_1)/L^2 \sim 1 \), indicating the long range order of the pinned lattice. For \( T_c < T < T_m \simeq 0.007 \), we find algebraic decay, \( S(G_1)/L^2 \sim L^{-\eta_{G_1}(T)} \). For \( T > T_m \), we find \( S(G_1)/L^2 \sim L^{-x} \), with \( x \to 2 \) as \( T \) increases, consistent with the short range order of a liquid.

Thus, our data for the floating solid phase are consistent with the predictions of the continuum elastic theory, given by Eq.(22b), and in particular we may fit our data to this expression to obtain the translational correlation exponent \( \eta_{G_1} \). We show our results in Table
We can now make quantitative comparison with the KTNHY theory, by noting that \( \eta_G \),
first exceeds the KTNHY universal value of 1/3 (see Eq.(17) at \( T = 0.0065 \), very close to the melting temperature \( T_m \simeq 0.0070 \) as estimated from the behavior of the orientational correlation \( \varphi_6(T) \) of Figure 3. The slopes of the lines in Figure 5 also show an apparent discontinuous jump at this same \( T_m \).

As a consistency check, we have also computed \( \eta_{G_2} \), where \( G_2 = 2G_1 \). Using similar fits to \( S(G_2) \) as in Figure 3, we determine the exponent \( \eta_{G_2} \), and show the results in Table I. We see that \( \eta_{G_2} \simeq 4\eta_{G_1} \) as expected, since according to Eq.(19) \( \eta_G \sim |G|^2 \).

As an alternative way of calculating \( \eta_{G_1} \), we fit to the heights of the peaks in \( S(k) \) at all the available \( G \), for a fixed size system, as described in Eqs.(23) – (25). We found that the results are only weakly dependent on the precise value of the cutoff \( R \) of Eq.(24); we therefore take \( R = L \), which results in an excellent fit. We show one example of such a fit in Figure 6. The exponent \( \eta_{G_1}(T) \), obtained in this way, is shown in Table I for the sizes \( L = 63, 77, 91 \), with \( f = 1/49 \). We note, that despite a certain tendency to overestimation, these exponents are in reasonable agreement with those obtained from the finite size scaling.

This method of extracting the shear modulus should be useful in situations where a finite size scaling analysis is difficult to handle, such as in systems with a large unit cell in the ground state.

Let us now consider the orientational order. In Figure 7 we plot the orientational correlation \( \varphi_6(T) \) as a function of \( L \) for several temperatures. In the pinned solid, \( T < T_c \), \( \varphi_6(T) \rightarrow 1 \) as \( L \) increases, confirming the expected long-ranged orientational order of the perfect pinned triangular lattice. More interestingly, \( \varphi_6(T) \) also approaches a finite value \( \varphi_6^\infty \) in the floating lattice phase, \( T_c < T < T_m \), in agreement with continuum elastic theory. The solid lines in Figure 7 are from least squares fits to Eq.(27a). The resulting fitted values of \( \varphi_6^\infty \) are shown in Table I. Above \( T_m \) we attempt to fit to the power law of Eq.(27b) for a hexatic liquid, but we always find that using Eq.(27c) for an isotropic liquid, results in a distinctly better fit. Since the underlying triangular grid will in principle result in long range six-fold order at all temperatures, we have also fit our data above \( T_m \) to the form of
Eq. (27a), which differs from Eq. (27c) only in the constant $\phi_6^\infty$. As shown in Table 1 however, we always find $\phi_6^\infty \approx 0$. Thus the discrete grid is playing a negligible role in the orientation order. To compare our fits above $T_m$, we note that in most cases the $\chi^2$-parameter of the fit to Eq. (27a) is 5 to 10 times smaller than that of Eq. (27b). The former fit is also much more stable in the sense that fitted parameters do not change significantly when the data is restricted to different ranges of $L$. We may therefore conclude that, in agreement with our investigation of the structure function, the floating solid melts directly into a normal liquid. The hexatic phase is either absent in our system, or it occurs only in some extremely narrow interval of temperatures, which makes it difficult to detect by numerical simulation.

In order to see this another way, in Figure 8 we plot versus temperature our values of $\phi_6^\infty(T)$ and $T/\eta_{G_1}(T)$, obtained from our finite size scaling analysis. From Eq. (16) we see that $T/\eta_{G_1}(T)$ is just proportional to the vortex lattice shear modulus $\mu$. We see, that $\phi_6^\infty(T)$ starts to drop at the same temperature that $T/\eta_{G_1}(T)$ first drops below the KTNHY universal value of $3T$ (see Eq. (17)), i.e. the temperature at which the floating solid starts to melt. The temperature range over which $\phi_6^\infty$ decays to zero is identical to the range over which $T/\eta_{G_1}(T)$ decays. This suggests that the small but finite values of $\phi_6^\infty$ which we find above $T_m$ are just a finite size effect, rather than a signature of the hexatic phase. Let us also note that the exponent $\eta_{G_1}(T)$ has a physical meaning only below $T_m$. Above $T_m$ it is strictly infinite and we use it here, with some abuse of notation, simply as the exponent resulting from the fit of our data to the power law form of Eq. (22b).

Having obtained the values of $\eta_{G_1}(T)$ and $\phi_6^\infty(T)$, we can now test the relation, Eq. (19), between the orientational long range order and the vortex lattice shear modulus $\mu$, that should hold in the floating solid phase. Expressing $\mu$ in terms of the exponent $\eta_{G_1}$, Eq. (19) gives

$$\eta_{G}(T) = -K \ln[\phi_6^\infty(T)],$$

with $K = 2|G|^2/9\Lambda^2$, where $\Lambda = \lambda(2\pi/a_v)$, and $\lambda$ is a dimensionless constant of order unity. In Figure 9 we plot $\eta_{G_1}^{-1}(T)$ and $-1/K \ln[\phi_6^\infty(T)]$ versus temperature. For $K = 0.33$, which
corresponds to $\lambda = 0.94$, the two data sets lie on top of each other for all $T$ below $T_m$, providing yet another consistency check for our calculation.

By fitting our data above $T_m$ to Eqs. (22c) and (27c) we have also extracted the correlation lengths $\xi_+(T)$, associated with translational order, and $\xi_6(T)$, associated with six-fold bond orientational order. We are able to determine these from finite size scaling only up to an overall multiplicative factor. We determine this factor by assuming that at high $T$, the correlation lengths are equal to the average spacing between vortices, i.e. $\xi(T \to \infty) = a_v$. With this assumption, $\xi_+(T)$ and $\xi_6(T)$ are displayed in Figure 10. We see that both correlation lengths rapidly increase around $T \simeq 0.007$. It is also evident from Figure 10, that orientational correlations persist out to longer distances, up to higher temperatures, than translational correlations.

**C. The order of melting transition**

The absence of the hexatic phase, as deduced from our analysis of the orientational correlations, suggests the possibility that the transition is not of the KTNHY type, but is due to some other mechanism, such as domain wall proliferation. It might also be, that the unbinding of disclinations occurs simultaneously with the unbinding of dislocations. Such a possibility has been suggested in Ref. 19. In any case, it is useful to determine the order of this melting transition. To examine the possibility that the transition is first order, we have used the histogram method due to Lee and Kosterlitz. For various system sizes at $f = 1/49$, we measure the energy distribution $P(E) \sim e^{-F(E)/T}$ near the melting temperature $T_m$. In Figure 11 we plot the resulting free energy $F(E)$ versus $E$. Although our data are somewhat noisy, we see a clear double well structure with an energy barrier $\Delta F$ between two coexisting phases. The inset to Figure 11 shows the dependence of $\Delta F$ on the system size $L$. The energy barrier $\Delta F$ grows with $L$, strongly suggesting a first order transition. Our system sizes remain too small to see clearly the predicted scaling $\Delta F \sim L$.

For all sizes, the data have been taken at $T = 0.0065$, and then the energy distribution
is extrapolated, using the method of Ferrenberg and Swendsen \textsuperscript{15} to that temperature which gives two minima of equal depth. This criterion gives an improved estimate of the melting temperature, $T_m = 0.0066$. A total of $10^7$ MC sweeps were performed for each size to measure the energy distribution $P(E)$. We have checked the consistency of these measurements by calculating the energy of the system at various temperatures (above and below $T_m$) using the extrapolated distributions $P(E,T)$. We then compared these with energies obtained by direct simulation at those temperatures, and found good agreement for all temperatures not too far from $T_m$.

To conclude, the histogram method provides strong evidence that the melting transition is first order. This is consistent with our observation that the 2D solid melts directly into an isotropic liquid. The transition is weakly first order however, as can be seen from our result that the jump in $\eta_{G_1}(T_m^-)$ (and hence the vortex lattice shear modulus $\mu$) at melting remains very close to the KTNHY universal value.

\section*{D. The depinning transition}

Finally, we consider the order of the depinning transition at $T_c(f)$. In their work on 2D melting on a periodic substrate, Nelson and Halperin\textsuperscript{19} studied this “commensurate to floating” transition using renormalization group techniques. They concluded that the transition is most likely second order, with properties very similar to the floating solid to liquid melting transition discussed in Section IIC. To test this prediction, we use the histogram method applied at the depinning transition $T_c(f)$, just as we did in the preceding section for melting at $T_m$. Measuring the energy distribution $P(E)$ at $T_c(f)$, for $f = 1/49$ and various $L$, we show the free energy $F(E)$ versus $E$ in Figure \textsuperscript{12}. As was seen at $T_m$, we now similarly see a pronounced double well structure with barrier $\Delta F$ growing with the size of the system (see the inset). Again, this is a clear indication that the transition is first order. Due to the low acceptance rates at these low temperatures, we had to perform as many as $4 \times 10^7$ MC sweeps for each system size, in order to get reasonably accurate
energy distributions. By finding the temperature that produces minima of equal depth, we estimate that $T_c(f = 1/49) = 0.0046$.

**IV. SIMULATIONS ON THE SQUARE GRID: DILUTE CASE**

**A. The ground state**

In the present section we shall investigate the dilute limit ($f \ll 1$) of the Coulomb gas, on a square grid of sites. This corresponds to a square periodic superconducting network, which has been the predominant geometry in experimental and theoretical studies of networks.\(^1\)\(^-\)\(^7\) Qualitatively, we find similar behavior as found in Section III for the triangular grid: a depinning transition $T_c(f)$, from a commensurate pinned solid to a floating solid, followed at higher temperature by a melting transition $T_m$ to a liquid.

While the case of a square grid is more relevant to the physics of superconducting arrays, it is somewhat more difficult to study theoretically than the triangular grid of Section III. The main reason for this is the rich variety of ground state configurations that one can encounter for various system sizes and vortex densities $f$. The most extensive enumeration of such ground states, for both dilute and dense $f$, has been carried out by Straley and Barnett.\(^7\) This richness in ground state structure is due to the intrinsic competition between the repulsive vortex-vortex interaction, which prefers the formation of a perfect triangular lattice, and the geometrical constraints implied by the presence of the square grid. Since a triangular lattice is incommensurate with a square grid, for small $f$ the resulting ground states form high order commensurate approximations to a triangular lattice, that vary substantially as $f$ varies. Thus, while in the triangular grid a density $f = 1/m^2$ can always fit commensurably in a lattice of size $L = s \cdot m$ ($s$ integer), for the square grid, a density of $f = 1/q$ ($q$ integer) will require a lattice of at least $L = s \cdot q$ to contain the commensurate ground state. It thus becomes too difficult to carry out detailed finite size scaling calculations at small $f$, as the lattice sizes needed quickly become too large to simulate. We therefore must be content with
a more qualitative analysis based on simulations at a fixed size system. A second problem, related to the high order commensurability of the ground state, is the existence of excited states that are nearly degenerate in energy with the ground state. This can sometimes cause equilibration problems, or leave uncertainty as to the configuration of the true ground state. Fortunately, these difficulties occur only at low temperatures, below the depinning transition $T_c(f)$, where commensurability effects are crucial. In sufficiently dilute systems, the melting of the floating solid phase at $T_m$ is largely unaffected by such difficulties, and we find results familiar to the preceding section.

We have performed simulations for systems with a wide spectrum of densities $f = 1/q$ ($q$ integer) with $10 < q < 90$. From inspection of the inverse dielectric constant $\epsilon^{-1}(T)$ and the orientational correlation $\varphi_6(T)$, for a fixed size $L$, we estimate the depinning and melting transition temperatures, $T_c(f)$ and $T_m$, and we plot these values versus $f$ in Figure 13. We see that above $f \approx 1/30$ the depinning and melting transitions merge, and there is only a single transition from pinned solid to liquid. Due to the varying commensurability of the ground state as $f$ varies, the values $T_c(f)$ and $T_m$ no longer decrease monotonically with $f$, as was found for the triangular grid. Nevertheless, we see that $T_c(f)$ still tends linearly to zero as $f$ decreases (dashed line), in agreement with the TJ conjecture. $T_m$ appears to saturate around 0.007, in agreement with the melting temperature found for the triangular grid.

In order to find the ground state of the system for a given density $f$ and size $L$, we have devised a simple program that scans all possible periodic vortex configurations, consistent with periodic boundary conditions, and evaluates their energy. When translational and inversion symmetries are accounted for, the total number of distinct configurations is relatively small, and even for the largest of the systems that we considered it took only few minutes to execute the program. The lowest energy configuration obtained in this manner was then taken as a candidate for the ground state. In many cases we have verified that this indeed was a true ground state by performing a slow MC cooling from a random configuration at high temperature. In all cases we found that for $f = 1/q$, the ground state has a $q \times q$ peri-
odicity. In Figure 14 we display two typical examples of these ground state configurations. The almost perfect triangular lattice \((\sqrt{68} \times \sqrt{68} \times \sqrt{72})\) in Figure 14a is for \(f = 1/60\). Figure 14b shows the example of a nearly square vortex lattice \((\sqrt{50} \times \sqrt{53} \times \sqrt{89})\) with \(f = 1/51\). In what follows we shall concentrate on these two special cases as representatives of two classes of systems with slightly different physical properties.

**B. Systems with “nearly triangular” ground state: \(f = 1/60\)**

Not very surprisingly, systems with an almost triangular ground state, such as \(f = 1/60\) shown in Figure 14a, behave in a fashion similar to systems on the triangular grid studied in Section III. We display the behavior of the inverse dielectric function \(\epsilon^{-1}(T)\) for \(f = 1/60\) and \(L = 60\) in Figure 15a. In Figure 15b we show the six-fold and four-fold orientational correlations, \(\varphi_6(T)\) and \(\varphi_4(T)\). A sharp drop in \(\epsilon^{-1}(T)\) around \(T_c(f) \simeq 0.0045\) signals the loss of superconducting phase coherence. Above \(T_c(f)\), \(\epsilon^{-1}\) is zero, but \(\varphi_6(T)\) stays finite. Based on our experience from the triangular grid, we take this as a signature of a floating triangular solid with long-range orientational order. \(T_c(f)\) is thus a transition from a commensurate pinned solid, to an incommensurate floating solid. Around \(T_m \simeq 0.0075\) we see that \(\varphi_6(T)\) drops again to very small values; we take this as a signal that the floating solid has melted into a vortex liquid.

In order to confirm this scenario, we calculate the structure function \(S(k)\) at various temperatures, and display the resulting intensity plots in Figure 16. We clearly see the pinned solid (Figure 16a), the floating solid (Figure 16b), and the liquid (Figure 16c) phases. It is interesting to note that the rotational symmetry of the pinned and floating solids break the four-fold rotational symmetry of the square grid, leading to two possible degenerate orientations. In the liquid however, we see that the four-fold symmetry of the square grid is restored, with a strong four-fold angular modulation of the circular intensity peaks. This observation is also confirmed by a direct measurement of \(\varphi_4(T)\) (see Figure 14b), which is close to zero in the floating solid phase, but then rises sharply at the melting transition and
only slowly vanishes with increasing temperature. The small values of $\varphi_4(T)$ for $T_c(f) < T < T_m$ are an indication of the extent to which the commensurate, slightly distorted triangular lattice of the ground state, becomes an incommensurate perfect triangular lattice in the floating solid phase.

Since we are unable to carry out finite size scaling, we are unable to search in detail for the hexatic phase, or for the predicted Ising transition from the hexatic to the normal liquid. However, as the four-fold symmetry appears to be restored at the same temperature as the melting transition, we suggest that, as was found for the triangular grid, the hexatic phase is absent and the melting transition is first order.

Although finite size scaling is not possible, we can nevertheless still obtain the translational correlation exponent $\eta_{G_1}$ by analyzing the decay of the peaks in the structure function, using the method discussed in connection with Eqs.(23)-(25). In the present case the implementation of this method is somewhat trickier than it was for the triangular grid, since, due to the incommensurability of the floating solid, the peaks in $S(k)$ do not have well defined positions $G$ on the square reciprocal lattice. We overcome this difficulty by numerically scanning $S(k)$ for local maxima at a given distance from the center of the reciprocal lattice, and averaging over the heights of peaks of the same order. The peak heights estimated in this way are shown in Figure 17 for several temperatures $T$ in the floating lattice phase. Dashed lines are least square fits to the formula (24), and the extracted exponents $\eta_{G_1}(T)$ are summarized in Table III. The accuracy of the fit appears to be as good as in the case of the triangular grid, and we therefore have good reason to believe that our determination of $\eta_{G_1}(T)$ (and thus the vortex shear modulus $\mu$) is reasonably accurate. Once again we see from Table III that $\eta_{G_1}(T)$ first exceeds the universal KT value of $1/3$ at $T \simeq 0.007$, very close to the melting temperature $T_m \simeq 0.0075$ estimated from the behavior of $\varphi_6(T)$. 

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C. Systems with “nearly square” ground state: \( f = 1/51 \)

We now briefly describe our results from simulations on a system with density \( f = 1/51 \), which possesses a nearly square ground state (Figure 14b). Our results are for a system of size \( L = 51 \). The temperature dependence of the inverse dielectric function \( \epsilon^{-1}(T) \) is shown in Figure 18a. From the data, we estimate the depinning temperature to be \( T_c(f) \approx 0.0035 \).

We note that even though \( f = 1/51 \) here is larger than the \( f = 1/60 \) studied in the previous section, we find \( 0.0035 = T_c(1/51) < T_c(1/60) = 0.0045 \), thus illustrating the nonmonotonic behavior of \( T_c(f) \) for small \( f \). The significantly lower depinning temperature in the present case may be qualitatively understood as a result of the larger distortion of the ground state away from the perfect triangular lattice favored by the vortex-vortex interaction. This large distortion, which is favored by the pinning energy, comes at a cost in vortex-vortex interaction energy. The result is a reduced free energy difference between the pinned “distorted triangular” solid and the perfect triangular floating solid, and hence a reduced depinning temperature. A similar observation holds for the other values of \( f \) we have studied: systems with relatively lower \( T_c(f) \) compared to other nearby values of \( f \), are those with greater distortion of the ground state from a triangular lattice.

In Figure 18b we plot the temperature dependencies of the four-fold and six-fold orientational correlations \( \varphi_4(T) \) and \( \varphi_6(T) \). In contrast to the previously considered cases, the melting transition is barely visible here: one sees only a small kink in \( \varphi_4(T) \) and an inconspicuous dip in \( \varphi_6(T) \) near \( T = 0.005 \). For a clearer picture of melting, we show the structure function \( S(k) \) in Figure 19. We see again the pinned solid (Figure 19a), the floating solid (Figure 19b), and the liquid (Figure 19c). Note that the peaks in the floating solid occur at distinctly different wavevectors \( k \) than in the pinned solid; this emphasizes the fact that \( T_c(f) \) is truly a transition from a commensurate “nearly square” lattice, to an incommensurate floating triangular lattice. Inspection of these intensity plots gives a melting transition of \( T_m \approx 0.005 \), in agreement with the value hinted at in Figure 18b. We note that this value is significantly lower than the value of 0.007 found in other cases.
Thus commensurability effects can also significantly lower the melting temperature. Such commensurability effects presumably become less significant as \( f \) decreases and the ground state becomes increasingly closer to a triangular lattice. We thus expect that the melting temperature becomes \( T_m \sim 0.007 \) in the asymptotic \( f \to 0 \) limit, as is indeed suggested by Figure 13.

V. SIMULATIONS ON THE SQUARE GRID: NEAR FULL FRUSTRATION

The square superconducting network with \( f = 1/2 \) has been the focus of extensive theoretical study in recent years. As the Hamiltonians, Eq.(1) and Eq.(3), are periodic in \( f \) with period 1, \( f = 1/2 \) represents the strongest magnetic field, and most dense vortex configuration, discernable by the network. Thus this case is usually referred to as “fully frustrated.” The ground state of this configuration is in some sense the simplest of all \( f > 0 \), consisting of a checkerboard pattern of vortices, with \( n_i = 1 \) and \( n_i = 0 \) on the two alternating sublattices of the square grid. This dense vortex lattice melts\(^{13}\) directly into a vortex liquid at \( T_m(1/2) \sim 0.13 \). Superconducting coherence vanishes\(^{39}\) at \( T_c(1/2) \sim T_m(1/2) \).

We now wish to study how the system behaves as \( f \) is varied slightly away from 1/2, in order to test the discontinuous behavior predicted by the TJ conjecture. We study in particular systems with \( f = 1/2 - 1/q \), with integer \( q \) large. While the ground states for densities of this form have been studied by Straley and co-workers\(^{7,40}\), finite temperature properties have remain unexplored.

A. \( f = 5/11 \)

We first consider the particular case of \( f = 5/11 \), which may be written as \( f = 1/2 - 1/22 \). The correct ground state for this case, which we show in Figure 20, was first found by Kolachi and Straley.\(^{40}\) It consists of a periodic superlattice of vortex vacancies superimposed on an otherwise uniform \( f = 1/2 \) like background, and is periodic with a \( 22 \times 22 \) unit cell.
Our motivation is to see whether or not this superlattice of vacancies (or “defects”) can melt independently of the $f = 1/2$ like background, and if so, whether the resulting liquid of vacancies destroys superconducting coherence. Our analysis is similar to that in the previous section.

Heating from the ground state, we show sample intensity plots of the structure function $S(k)$, at different temperatures, in Figure 21. Figure 21a shows the low temperature phase at $T = 0.010$. The bright Bragg peaks at $k = (\pm \pi/a_0, \pm \pi/a_0)$ originate from the vortex ordering in the $f = 1/2$ like background, while the periodic Bravais lattice of less intense Bragg peaks is due to the defect superlattice, which at this low temperature is pinned to the substrate. Thus we have a “pinned defect solid” phase. As the temperature is increased, we find that the defect superlattice melts at $T_m \simeq 0.015$. In Figure 21b we show the system at $T = 0.018$, just above this melting. The defects no longer give rise to Bragg peaks, but instead we see the circular rings (with strong four-fold angular modulation) characteristic of a defect liquid. However the bright Bragg peaks at $k = (\pm \pi/a_0, \pm \pi/a_0)$ remain, indicating that the $f = 1/2$ like vortex background remains ordered. Upon increasing the temperature further, the ordered $f = 1/2$ background is also eventually destroyed at $T_m' \simeq 0.040$. In Figure 21c we show the system at $T = 0.055$, above $T_m'$. The peaks at $k = (\pm \pi/a_0, \pm \pi/a_0)$ have broadened to finite width, indicating the disordering of the $f = 1/2$ like background.

To see the melting transitions more clearly, in Figure 22 we plot versus temperature the peak heights $S(q^*)$ and $S(G_1)$, with $q^* \equiv (\pi/a_0, \pi/a_0)$ giving the ordering of the $f = 1/2$ like background, and $G_1$ the shortest reciprocal lattice vector of the defect superlattice. We see that $S(G_1)$ vanishes sharply at $T_m \simeq 0.015$, where the pinned defect superlattice melts into a defect liquid. $S(q^*)$, however, remains at its $T = 0$ value of unity for all temperatures up to $T \simeq 0.020$, clearly demonstrating that the $f = 1/2$ like vortex background remains ordered throughout the defect melting transition. $S(q^*)$ starts to drop to zero around $T_m' \simeq 0.04$, where, based on the structure function intensity plots, we have estimated that the $f = 1/2$ like background melts.

To investigate superconducting coherence, in Figure 23 we show the inverse dielectric
function versus temperature. We see that $\epsilon^{-1}$ vanishes at the defect melting transition $T_m$. The diffusing defects above $T_m$ induce a diffusion of vortices, which must move to fill in the “hole” left behind by the defect as it moves. The diffusing vortices are then responsible for the destruction of superconducting phase coherence.

To summarize, we have found clear evidence that the introduction of a small concentration of defects into the fully frustrated system results in a dramatic decrease of the superconducting transition temperature from its $f = 1/2$ value. The fluctuations of the defect superlattice, on an essentially frozen $f = 1/2$ like background, result in behavior which is in many respects like that of the dilute vortex lattices studied in Section IV. In the present case, we find that the defect superlattice melts directly from a pinned solid into a liquid. In the following section we will argue, following the analogy with Section IV, that a more dilute defect superlattice would first unpin at a $T_c(f)$ into a floating defect superlattice, which would then melt at a higher temperature $T_m$ into a defect liquid.

The sharp melting transition of the $f = 1/2$ like background at a temperature $T_{m'}$ distinctly higher than the defect melting at $T_m$ is a new phenomenon, with no analogue in the dilute small $f$ systems (at small $f$ there is only a smooth crossover remnant of the vortex-antivortex unbinding transition of the $f = 0$ case). From symmetry, one would expect that the transition at $T_{m'}$ is of the Ising type. Understanding whether the melting transition in the pure $f = 1/2$ case is Ising like or not, has been the subject of much work, with the most recent simulations suggesting that it is not\textsuperscript{10} if it is not Ising, this is most likely due to the long range nature of the vortex interactions. For the $f = 5/11$ case however, the melted defect liquid will serve to screen the interactions of the vortices in the $f = 1/2$ like background, resulting in effectively short ranged interactions. An Ising transition is therefore most probable. We are unable to test this prediction, as we are unable to carry out a detailed finite size scaling analysis, for the same reasons as discussed in Section IV. However the strong screening effect of the defect liquid is evident in the substantial reduction of the background melting temperature, $T_{m'}(5/11) \simeq 0.04$, as compared to the melting transition $T_m(1/2) \simeq 0.13$ of the pure $f = 1/2$ case.
B. General case: \( f = \frac{1}{2} - \frac{1}{q} \)

In this section, we strengthen the analogy between the melting of the defect superlattice seen in the previous section, with the melting of the dilute (small \( f \)) vortex lattices studied in Section IV, in order to discuss the general case of \( f = \frac{1}{2} - \frac{1}{q} \). Our goal is to establish that for a more dilute density of defects, one will have a floating defect solid intermediate between a pinned defect solid and a defect liquid.

As seen for the \( f = \frac{5}{11} \) case, throughout a temperature range including the defect lattice melting transition \( T_m \), the \( f = \frac{1}{2} \) like background vortices remain perfectly ordered as at \( T = 0 \); domain excitations, which would reduce \( S(q^*) \) from its \( T = 0 \) value of unity, become important only above \( T_m \). In this case, one can focus solely on excitations which are due to the motion of the defects. At \( T = 0 \), these defects are seen to sit on the same sublattice of the square grid as do the \( n_i = 1 \) vortices of the \( f = \frac{1}{2} \) like background (see Figure 20); equivalently, one never has two vortices on two nearest neighbor sites. We assume that this restriction continues to hold at finite temperatures up to and including \( T_m \), i.e. the cost in energy to have two vortices on nearest neighbor sites is so high compared to \( T_m \), that such excitations may be ignored.

With this assumption, we have reduced our problem at \( f = \frac{1}{2} - \frac{1}{q} \) to that of a density \( \frac{1}{q} \) of logarithmically interacting defects, which are restricted to move on only one of the two sublattices of the original square grid. This sublattice is itself a square lattice of lattice constant \( \sqrt{2}a_0 \). As the sublattice has half the number of sites as in the original grid, our problem is thus effectively the same as a dilute density \( f' = \frac{2}{q} \) of vortices on a square grid. This mapping would be exact (within our assumptions) except for the fact that the interaction potential \( V(r) \) between the defects is still defined with respect to the original square grid, and not the sublattice to which the defects are constrained. However, as \( q \) gets large, and the average spacing between defects becomes much greater than \( a_0 \), we expect that this difference will be a negligible effect.

The assumption that the defects move only on one sublattice can be checked for the
For the $f = 5/11$ case of the previous section. Restricting the defects in real space to a sublattice whose unit cell has twice the area of that of the original square grid, means that the 1st Brillouin Zone of the effective reciprocal lattice is reduced by a factor of one half. Instead of the square shaped BZ shown in Figures 21, the effective BZ is now an inscribed diamond whose vertices bisect the edges of the squares of Figures 21. The structure function $S(k)$, as plotted over the full square shaped BZ of the original square grid, should now just be obtained by a periodic repetition of the diamond shaped BZ corresponding to the sublattice. Such periodicity is clearly seen in Figures 21a and 21b, for both the pinned defect solid, and the melted defect liquid. It is absent in Figure 21c, where $T > T_m'$, and the $f = 1/2$ like background has melted.

Having checked the validity of our assumption for $f = 5/11$, we note that that it should be even better satisfied for more dilute defect densities $f = 1/2 - 1/q$, $q > 22$. As $q$ increases, the density of defects decreases, resulting a reduced screening of the interactions between the background vortices. The background melting temperature $T_m'$ should therefore increase and approach its higher $f = 1/2$ value. At the same time, the defect superlattice unpinning temperature $T_c$ should decrease as $\sim 1/q$, while the defect superlattice melting temperature $T_m$ saturates to a lower fixed value. Thus we expect that the window of temperatures in which our assumption is valid becomes wider as $q$ increases.

We can now understand the behavior found in the preceding section. For $f = 5/11 = 1/2 - 1/22$, we have $q = 22$, and so the defects behave like an effective vortex density of $f' = 2/q = 1/11$. Comparing to our results of Figure 13 in Section IV, we see that $f'$ is large enough that we expect only a direct melting of the pinned defect solid to a defect liquid, consistent with our observation in the preceding section. In order to observe a floating defect solid, we will have to consider an $f' < 1/30$, or an $f = 1/2 - 1/q$ with $q > 60$.

To simulate a system with $f = 1/2 - 1/60$ directly, would require a grid size of at least $L = 60$, with $N_c = 1740$ vortices. This is beyond our present computational ability ($f = 5/11$, with $L = 22$ and $N_c = 220$, is about the largest system we can manage). However our conclusion, that at temperatures low compared to $T_m'$ the defects move in the presence
of an effectively frozen $f = 1/2$ like background, allows us to construct a much more efficient algorithm which will be suitable for describing behavior up to and above the defect melting transition $T_m$, provided we stay below the background melting transition $T_m'$. We do this by fixing the $f = 1/2$ like background and allowing only the vacancies to move around. This significantly reduces the number of degrees of freedom in the simulation, and we shall thus be able to treat systems with a much smaller fraction of defects $1/q$, than we could by direct MC simulation.

In order to implement the algorithm suggested above, we formally decompose the charge at site $r_i$ into two parts

$$n_i = s_i - \delta n_i,$$  \hspace{1cm} (33)

where

$$s_i \equiv \frac{1}{2}[1 + (-1)^{x_i + y_i}] = \frac{1}{2}(1 + e^{i r_i \cdot q^*})$$  \hspace{1cm} (34)

is the staggered pattern of the background vortices ($q^* \equiv (\pi/a_0, \pi/a_0)$) and $\delta n_i$ are new integer variables representing the defects in the background. Neutrality requires that $\sum_i \delta n_i = L^2/q$. Substituting Eq.(33) for $n_i$ in the Hamiltonian (5), we get

$$H = \frac{1}{2} \sum_{ij} \delta n_i V'_{ij} \delta n_j - \sum_{ij} \delta n_i V'_i (s_j - f) + \frac{1}{2} \sum_{ij} (s_i - f) V'_j (s_j - f),$$  \hspace{1cm} (35)

where $V'_{ij} \equiv V'(r_i - r_j)$. The first term in the Hamiltonian (35) gives the interaction between defects; the second term represents the interaction of the defects with a one-body potential

$$\Phi_i \equiv \sum_j V'_j (s_j - f)$$  \hspace{1cm} (36)

created by the background; the last term is just an additive constant. Substituting Eq.(34) for the $s_i$ into Eq.(33) above, gives the potential $\Phi_i$ in terms of the Fourier components of the interaction, $V_k$,

$$\Phi_i = \frac{1}{2} V_q^* e^{i r_i \cdot q^*} - \left(\frac{1}{2} - f\right) \sum_{k \neq 0} V_k,$$  \hspace{1cm} (37)
where from Eq. (7) we have $V_{q^*} = \pi/4$. Thus $\Phi_i$ oscillates with the same checkerboard pattern of the $f = 1/2$ like background. Comparing with Eq. (34), we see that the Hamiltonian of Eq. (35) can now be rewritten in the following simple form

$$\mathcal{H} = \frac{1}{2} \sum_{ij} \delta n_i V'_{ij} \delta n_j - \frac{\pi}{4} \sum_i \delta n_i s_i + E_0,$$

where $E_0$ is an additive constant.

So far, the formulation above is exact. Our approximation that the background is frozen, and that defects only move on the sublattice defined by $s_i = 1$, occurs when we consider only the case where $N_c$ sites have the value $\delta n_i = 1$, and all other sites have $\delta n_i = 0$. In this approximation, the Coulomb gas near full frustration, $f = 1/2 - 1/q$, is equivalent at low temperatures to the dilute Coulomb gas of defects with integer charges $\delta n_i = 0, 1$, moving in a staggered potential of magnitude $\delta \Phi = \pi/4 = 0.7853\ldots$. As this magnitude is about two orders of magnitude greater than the relevant excitation energy scale, set by temperature $T_m$, the sites with $\delta n_i = 1$ are essentially restricted to the sublattice where $s_i = 1$; in this case they represent the vacancies in the $f = 1/2$ like background. The case where $\delta n_i = 1$ on the opposite sublattice where $s_i = 0$, represents a $(+1, -1)$ vortex-antivortex excitation, which can be ignored on energetic grounds as we had shown in earlier sections.

To check the consistency of the above procedure, we have redone our simulation of $f = 5/11$ using the new algorithm based on the Hamiltonian (38). In a fraction of the CPU time needed for the original simulation using the full Hamiltonian (5), we have recovered our original results for all quantities, at all temperatures up to about $T \simeq 0.040$, where fluctuations in the $f = 1/2$ background become significant.

Having verified the consistency of the new algorithm in this way, we now proceed to simulate systems with more dilute concentrations of defects. In Figure 24, we display the structure function $S(k)$ for the case $f = 22/45 = 1/2 - 1/90$. As expected from the discussion above, we observe a clear signature of the floating solid phase (Figure 24b) in the temperature range $0.005 < T < 0.008$. This range is identical to the range in which we found the floating solid phase for $f' = 1/45$ (see Figure 13). The low temperature phase is
a familiar “pinned defect solid” (Figure 24a); the high temperature phase is a defect liquid
with strong four-fold correlations (Figure 24b). The above scenario is confirmed by a direct
measurement of $\epsilon^{-1}(T)$ and the six-fold orientational correlation $\varphi_6(T)$ of the defects $\delta n_i$, shown in Figure 24. Both quantities behave in a way similar to those measured for dilute
vortex systems, showing a sharp drop in $\epsilon^{-1}(T)$ at the depinning transition, and a plateau
in $\varphi_6(T)$ in the floating phase.

To summarize, we conclude that for $f = 1/2 - 1/q$, with $q > 60$, there will be the
following sequence of transitions. At low temperature there is a pinned superlattice of
defects of density $1/q$, which unpins at $T_c(f) \sim 2/q$ into a floating superlattice of defects.
This floating lattice melts at $T_m \approx 0.007$ into an isotropic defect liquid. Finally, at $T_m'$,
which approaches the value of 0.13 as $q$ increases, the $f = 1/2$ like background melts via an
Ising transition, resulting in an isotropic vortex liquid of density $f$.

VI. SUMMARY AND CONCLUSIONS

We have carried out extensive Monte Carlo simulations of the Coulomb gas Hamiltonian
(5) as a model of a 2D superconducting network in an external transverse magnetic field. One
of the goals of our work was to systematically study, for the special cases of vortex density
$f = 1/q$ and $f = 1/2 - 1/q$ ($q \gg 2$) a conjecture put forward by Teitel and Jayaprakash, that
for $f = p/q$ the superconducting transition temperature scales approximately as $T_c(f) \sim 1/q$.
For the dilute case, $f = 1/q$, we have found good agreement with this conjecture, provided
one interprets the superconducting transition temperature to be the vortex lattice unpinning
temperature $T_c(f)$, where the ground state vortex lattice decouples from the superconducting
network, and is free to slide transversly to any applied d.c. current, thus producing “flux
flow” resistance.

A new result of our work is the realization that above $T_c(f)$, for sufficiently dilute systems,
a depinned “floating” vortex solid will exist. This floating vortex solid has essentially the
same properties as a vortex lattice in a uniform superconducting film, and it melts (as
\( q \to \infty \) at \( T_m \simeq 0.007 \) into an isotropic vortex liquid. While the true onset of finite linear d.c. resistivity will be \( T_c(f) \), the melting at \( T_m \) is presumably accompanied by a sharp rise in resistivity. The distinction between \( T_c \) and \( T_m \), however, may be difficult to observe experimentally, due to the existence of large energy barriers for the hopping of a vortex between neighboring cells of the superconducting network. As discussed in the introduction, the discrete nature of the network introduces an effective periodic pinning potential for vortices. For a square Josephson array, Lobb et al. have estimated the energy barrier of this pinning potential to be \( E_b \simeq 0.199/2\pi = 0.0317 \) (in our energy units). This is almost five times the vortex lattice melting temperature \( T_m \simeq 0.007! \) Thus for the square network, one is most likely to observe upon cooling only a vortex liquid, in which the vortex mobility decreases exponentially as \( e^{-E_b/T} \); the true phase transitions at \( T_m \) and \( T_c \) will be masked by the extremely slow relaxation over the energy barriers \( E_b \) at these low temperatures. Such behavior has in fact been reported in experimental studies of square Josephson arrays, where for small \( f \) near \( f = 0 \), only exponentially decreasing resistive tails are observed at low temperature; no evidence for the melting or depinning transitions at \( T_m \) and \( T_c \) has been found. For the triangular Josephson array however (a case we have not explicitly studied here), the energy barrier is estimated to be \( E_b \simeq 0.0427/2\pi = 0.0068 \) (in our energy units). This is comparable to \( T_m \), and so there might be some slight chance of experimentally observing the melting transition. Recent experimental studies of this system at small \( f \) have found surprising dynamical behavior, indicating anomalously slow diffusion of vortices. However the temperature of these experiments, \( T \simeq 0.5T_c(f = 0) \), appears to be too high for these results to be explained by any of the melting or depinning effects we have found here. For a honeycomb Josephson array (vortices on a triangular grid) we estimate the highest barrier, \( E_b \simeq 0.751/2\pi = 0.119 \).

For dense systems close to full frustration, \( f = 1/2 - 1/q \), we have argued that, at temperatures low compared to the melting temperature of the pure \( f = 1/2 \) system, the physics is dominated by defects moving on top of a quenched \( f = 1/2 \) like vortex background. We have shown how this system of defects can be mapped onto the dilute Coulomb gas of
vortices with density $f' = 2/q$. The resulting behavior is then obtained from knowledge of this dilute limit. The TJ conjecture again holds, with $T_c(f = (q-2)/2q) \sim 2/q$ marking the transition from a pinned defect superlattice to a floating defect superlattice, in which true d.c. superconductivity is lost. At a higher $T_m \simeq 0.007$, the floating defect solid melts into an isotropic defect liquid. At yet a higher $T_{m'}$, the $f = 1/2$ like background disorders. As in the $f = 0$ case, the transitions at $T_c$ and $T_m$ may be difficult to observe experimentally, due to the energy barrier for a defect to hop between nearest neighbor sites of the relevant sublattice.

For a square lattice, Dang and Győrfy have estimated this barrier to be $E_b \simeq 0.368/2\pi$ (in our energy units), even larger than that found for $f = 0$. Experimental studies of square Josephson arrays for $f$ near $f = 1/2$ have again found only exponential resistive tails as the temperature decreases.

In contrast to the transitions at $T_c$ and $T_m$, we expect that the sharp disordering transition of the $f = 1/2$ like background at $T_{m'}$ should be experimentally observable. This follows since for $f = 1/2 - 1/q$, we expect that as $q \to \infty$, $T_{m'} \to T_m(1/2) \simeq 0.13$, well above the energy scale of the barriers. This transition would presumably manifest itself as a singular increase in the linear resistivity at $T_{m'}$ (from an already finite value). The phase boundary $T_{m'}(f)$ near $f = 1/2$ is presumably a smooth function of the defect density, $1/2 - f$, however we are unable to estimate it due to our inability to simulate sufficiently large systems.

Our mapping between a dilute density of defects near $f = 1/2$, and a dilute vortex density near $f = 0$, may be extended to the more general case. Using the same arguments as in Section V, we would expect that the system with $f = 1/2 - p/q$, with $p/q$ sufficiently small, should have the same low temperature behavior as the density $f' = 2p/q$. For $p/q$ sufficiently small, we would expect that the dilute vortex lattice of density $f' = 2p/q$ behaves qualitatively like those of density $1/q$ studied here, i.e. there is first a depinning transition at $T_c(f)$ which decreases as $p/q$ decreases (whether it vanishes as $p/q$ or as $1/q$ remains to be investigated) followed by a melting transition at $T_m \simeq 0.007$. Thus for any rational fraction sufficiently close to either $f = 0$ or $f = 1/2$, we would expect behavior similar to the cases explicitly studied here.
We thus see that the TJ conjecture appears to hold, according to the following scenario. Consider a vortex density $f$ close to some simple fraction $f_0 = p_0/q_0$, $f = f_0 - 1/q$, with $q_0 \ll q$. The ground state is one which is almost everywhere like that of $f_0$, except for a pinned periodic superlattice of defects of density $1/q$. If $q$ is sufficiently large, this superlattice will unpin into a floating defect solid at $T_c(f) \sim 1/q$. Defects which are free to move lead to flux flow resistance, and destroy the superconducting phase coherence of the system. Thus an arbitrarily small concentration of defects added on top of the $f_0$ like background (i.e. for $f$ arbitrarily close to $f_0$) dramatically decreases the superconducting transition temperature, when compared to pure $f_0$ system. We have explicitly tested this scenario for the cases $f_0 = 0$ and $f_0 = 1/2$. We speculate that this behavior will be characteristic of systems near any simple fraction $f_0 = p_0/q_0$. We further speculate that his behavior may be characteristic for any sufficiently small rational fraction of defects away from a simple fraction $f_0$, i.e. $f = p_0/q_0 - p/q$ with $q_0 \ll q$.

A second goal of our work was to study in detail the melting transition of the 2D vortex lattice. This problem has been addressed previously only in the context of uniform superconducting films. Here we have addressed this issue in the context of a superconducting network. We believe, however, that our results for the dilute case we have studied in Section III are representative of the continuum limit, as treated within the London approximation. Melting within this London approximation has been treated theoretically by Huberman and Doniach, and Fisher, who applied the KTNHY theory of defect mediated melting in 2D. This theory predicts a second order melting transition at a $T_m$ well below the Ginzburg-Landau mean field transition temperature $T_{MF}$, as well as an intermediate hexatic liquid phase. We have carried out the first detailed finite size scaling analysis to check this KTNHY theory as applied to vortex lattice melting. We find a value $T_m \simeq 0.007 \pm 0.0005$ in good agreement with the value estimated by Fisher. We also find that the vortex lattice shear modulus jumps discontinuously to zero at $T_m$, with a value close to the KTNHY prediction. However we find that the melting transition is weakly first order, and we find no evidence for a hexatic phase. Our value for $T_m$, and our conclusion concerning the order of the melt-
ing transition, are in agreement with earlier simulations\footnote{13} of the continuum one component plasma model of Eq.(13).

This problem of 2D vortex lattice melting has been the focus of much renewed work recently, due to its potential connection with behavior in anisotropic high temperature superconductors. The very existence of a vortex lattice at any finite temperature has been challenged by Moore\footnote{3}, who argued that fluctuations in the phase of the order parameter $\psi(r)$, due to shear excitations of the vortex lattice, will cause the order parameter correlation function $\langle \psi^*(r)\psi(0) \rangle$ to decay exponentially at any finite temperature. From such decay, Moore argued first for the absence of a superconducting state, and then concluded as a result of this absence of superconductivity, that the vortex lattice should not exist. Support for this scenario is suggested by high temperature perturbative expansions, which also find no evidence for freezing into a vortex lattice, even when evaluated to high order\footnote{38}.

Recently, simulations have been carried out to address this question. In contrast to our work in the London approximation, these works have been carried out in the so called lowest Landau level (LLL) approximation, which is based on the Ginzburg-Landau (GL) free energy $\mathcal{H}_{GL}$ of Eq.(11). In this approximation, the complex order parameter $\psi(r)$ is expanded in terms of the lowest degenerate eigenstates of the Gaussian part of the Ginsburg-Landau free energy, and the coefficients of this expansion (or alternatively the complex positions of the vortices) are used as fluctuating variables in a Monte Carlo simulation with $\mathcal{H}_{GL}$ as the Hamiltonian. Using such simulations, and modeling a 2D system by the surface of a sphere, O’Neill and Moore\footnote{49} failed to find evidence for a vortex lattice. Other simulations in a 2D plane however\footnote{50–53} reported clear evidence for the melting of a vortex lattice at a finite temperature. Hu and MacDonald\footnote{51} and Kato and Nagaosa\footnote{52} find that this melting transition is weakly first order, in agreement with our London result. Šášik and Stroud\footnote{53} similarly find a first order transition; they further compute the vortex lattice shear modulus $\mu$ and find behavior at $T_m$ in agreement with our result. Most recently, Herbut and Tešanović\footnote{54} have developed a density functional theory of the vortex lattice melting transition, based on the LLL formalism. They again find results consistent with the above, for the order of the
transition, and the shear modulus $\mu$.

Thus, with the exception of Ref. 49, results from the London and LLL approximations seem to be in agreement. This is as one might expect from the principle of universality in phase transitions. Although the London approximation at the “microscopic” level ignores fluctuations in the amplitude of $\psi(r)$ (such as are included in the LLL formalism), upon coarse graining the London model, phase fluctuations at the microscopic length scale will generate amplitude fluctuations on the coarse grained length scale. On this coarse grained scale, the system will be described by some effective Ginzburg-Landau free energy, complete with amplitude fluctuations, although higher order terms in $\psi$ beyond those given in Eq. (11) may be present. In contrast to the London approximation, the Ginzburg-Landau free energy of Eq. (11) includes amplitude fluctuations at the “microscopic” scale, and it is thus often viewed as a more fundamental description. However, the GL form of Eq. (11) represents only the lowest terms in an expansion of the free energy in powers of $|\psi|$, and hence is valid only near the mean field transition $T_{MF}$ where $|\psi|$ is small. Since vortex lattice melting occurs at $T_m \ll T_{MF}$, higher order terms in $\psi$ may well be important for a quantitative description. These higher order terms, however, are presumably irrelevant in determining the critical behavior, thus leading to agreement between the London and LLL simulations.

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to transition temperatures $T_c$ and $T_m$ for the Villain interaction between nodes. For the cosine interaction however, the corresponding transitions will occur at lower temperatures than in the Villain case (due to the additional reduction in junction coupling resulting from the vortex-spinwave interaction which is present in the cosine model). Thus these transitions will be even more obscured by high energy barriers than our comparison might indicate. Using the same method we have also computed the barriers for the Villain interaction and we have found that for all types of networks these are even much higher (typically by one order of magnitude) than those cited for the cosine model. For example, we have computed $E_b^{\text{Villain}} = 0.517$ for a square network, at low temperature. This large difference comes predominantly from the bond that is crossed by the vortex when climbing to the neighboring site, and it can be easily understood by comparing Villain and cosine potentials. We note also that, although these barriers are important for the dynamics of real networks, they do not effect the equilibration of our simulation; this is because in our Coulomb gas MC, we move vortices in discrete steps to neighboring cells, without the need to climb over the energy barrier. However had we done typical Metropolis MC in terms of the phases $\theta_i$ and the Hamiltonian of Eq.(1), these barriers would cause equilibration problems; this is because when a vortex moves from one cell to its neighbor, the $\theta_i$ evolve along a continuous path in phase space, that therefore must take the system over the energy barrier separating the two cells.

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| $T$      | $\eta_{G_1}(T)$   | $\eta_{G_2}(T)$   | $\varphi_6^\infty$ |
|----------|-------------------|-------------------|-------------------|
| 0.00475  | 0.188 ±0.008      | 0.704 ±0.055      | 0.571 ±0.007      |
| 0.00500  | 0.207 ±0.007      | 0.806 ±0.032      | 0.529 ±0.005      |
| 0.00525  | 0.211 ±0.007      | 0.852 ±0.028      | 0.504 ±0.004      |
| 0.00550  | 0.248 ±0.005      | 0.998 ±0.019      | 0.476 ±0.003      |
| 0.00575  | 0.255 ±0.008      | 0.999 ±0.029      | 0.458 ±0.003      |
| 0.00600  | 0.296 ±0.006      | 1.065 ±0.028      | 0.426 ±0.007      |
| 0.00625  | 0.319 ±0.010      | 1.191 ±0.016      | 0.403 ±0.004      |
| 0.00650  | 0.4 ±0.16         | 1.4 ±0.22         | 0.33 ±0.030       |
| 0.00675  | 1.4 ±0.31         | 2.0 ±0.31         | 0.20 ±0.041       |
| 0.00750  | 3.4 ±0.37         | 3.4 ±0.44         | 0.03 ±0.046       |
| 0.01100  | 2.8 ±0.23         | 2.9 ±0.30         | -0.01 ±0.032      |
| 0.01500  | 2.2 ±0.12         | 2.1 ±0.22         | 0.00 ±0.020       |

**TABLE I.** Temperature dependence of the exponents $\eta_{G_1}(T)$ and $\eta_{G_2}(T)$ for $f = 1/49$ on the triangular grid, as obtained from finite size scaling. Also displayed are the limiting values $\varphi_6^\infty$ of the orientational correlation $\varphi_6(T)$ for $L \to \infty$. 
TABLE II. Comparison of the exponents $\eta_{G_1}(T)$ obtained using two different methods. Columns 2-4 show results from fitting of $S(G)$ to Eq.-(24), for system sizes $L = 63, 77, 91$. Column 5, labelled FSS, restates the results from the finite size scaling analysis. All exponents are for density $f = 1/49$. 

| $T$ | $\eta_{G_1}(T)$ |
|----|----------------|
| | $L = 63$  | $L = 77$  | $L = 91$  | FSS  |
| 0.00475 | 0.164±0.026 | 0.161±0.025 | 0.195±0.018 | 0.188 ±0.008 |
| 0.00500 | 0.216±0.012 | 0.221±0.009 | 0.219±0.007 | 0.207 ±0.007 |
| 0.00525 | 0.249±0.009 | 0.235±0.009 | 0.247±0.006 | 0.211 ±0.007 |
| 0.00550 | 0.282±0.007 | 0.272±0.006 | 0.275±0.007 | 0.248 ±0.005 |
| 0.00575 | 0.298±0.008 | 0.292±0.009 | 0.290±0.006 | 0.255 ±0.008 |
| 0.00600 | 0.326±0.008 | 0.326±0.007 | 0.329±0.007 | 0.296 ±0.006 |
| 0.00625 | 0.351±0.014 | 0.350±0.017 | 0.352±0.016 | 0.319 ±0.010 |
| 0.00650 | 0.677±0.342 | 0.473±0.223 | 0.568±0.131 | 0.4 ±0.16  |
| 0.00675 | 1.755±0.789 | 1.678±0.453 | 2.458±0.911 | 1.4 ±0.31  |
TABLE III. Temperature dependence of the exponents $\eta_{G_1}(T)$ of the floating vortex lattice on the square grid as obtained by fitting the height of peaks in the structure function, for $f = 1/60$ and $L = 60$. 

| $T$  | $\eta_{G_1}(T)$    |
|------|-------------------|
| 0.0040 | 0.0024±0.001      |
| 0.0045 | 0.111±0.016       |
| 0.0050 | 0.198±0.012       |
| 0.0055 | 0.224±0.009       |
| 0.0060 | 0.270±0.011       |
| 0.0065 | 0.33±0.04        |
| 0.0070 | 0.49±0.10        |
FIGURES

FIG. 1. Phase diagram of the sufficiently dilute system, as found by our Monte Carlo calculation.

FIG. 2. Structure function $S(k)$ in the first Brillouin zone, (upper portion of the figure), and the profile of the peak heights along the vertical symmetry axis $k_y$ (lower portion); for $f = 1/49$ and $N_c = 63$, and three different temperatures $T$: (a) $T = 0.003$, just below $T_c$, in the “pinned solid” phase, (b) $T = 0.0065$, just below $T_m$, in the “floating solid” phase, (c) $T = 0.0075$, just above $T_m$, in the liquid. Intensities in the density plots are plotted nonlinearly to enhance features.

FIG. 3. Inverse dielectric function $\epsilon^{-1}(T)$ and orientational order correlation $\varphi_6(T)$ versus $T$ for $f = 1/49$ and $N_c = 169$. Solid and dashed lines are guides to the eye only.

FIG. 4. The dependence of the depinning and melting temperatures, $T_c$ and $T_m$, on vortex density $f$. Errors are estimated from the width in the apparent drop in $\epsilon^{-1}(T)$ and $\varphi_6(T)$. Solid and dashed lines are guides to the eye only.

FIG. 5. Finite size scaling of $S(G_1)/L^2$ (note the log-log scale) for $f = 1/49$. Solid and dashed lines are fits to Eq. (22b).

FIG. 6. Heights of the peaks $S(G)$ versus $|G|$ for $f = 1/49$ and $L = 63$. Dashed lines represent the best fit to Eq. (24), and are used to extract the exponent $\eta_{G_1}(T)$.

FIG. 7. Finite size scaling of $\varphi_6(T)$ for $f = 1/49$. Solid and dashed lines are fits to Eq. (27a).

FIG. 8. $T/\eta_{G_1}(T)$ (proportional to the shear modulus $\mu$) and orientational correlation $\varphi_6^\infty$ versus $T$, as extracted from finite size scaling. The intersection of $T/\eta_{G_1}(T)$ with the dashed line $3T$ determines the KTNHY upper bound on the melting transition $T/\eta_{G_1}(T) > 3T$. 

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FIG. 9. Comparison of $\eta_{G_1}^{-1}(T)$ and $-1/K \ln[\varphi_6^\infty(T)]$ as a test of Eq. (19). Below $T_m$ the two quantities coincide, as expected for a 2D harmonic lattice. Dashed line determines the KTNHY upper bound $\eta_{G_1}^{-1}(T) > 3$.

FIG. 10. Translational and orientational correlation lengths $\xi_+(T)$ and $\xi_6(T)$ versus $T$ for $f = 1/49$, as extracted from finite size scaling. Both correlation lengths sharply increase at the melting temperature $T_m \approx 0.0070$.

FIG. 11. Free energy distribution $F(E)$ versus $E$, at melting $T_m$, for $f = 1/49$ and several system sizes $L$. The growth in the energy barrier $\Delta F$ with increasing $L$ (see inset) indicates a first order transition. Curves for different $L$ are offset from each other by a constant, for the sake of clarity.

FIG. 12. Free energy distribution $F(E)$ versus $E$, at the depinning transition $T_c$, for $f = 1/49$ and several system sizes $L$. The growth in energy barrier $\Delta F$ with increasing $L$ (see inset) indicates a first order transition. Curves for different $L$ are offset from each other by a constant, for the sake of clarity. The abrupt ending of the distributions at the low energy side of the graph is because the lower minimum represents the ground state energy.

FIG. 13. Dependence of $T_c$ and $T_m$ on vortex density $f$ for the dilute system on a square grid. Dashed and solid lines are guides to the eye only.

FIG. 14. Two types of ground state configurations for a dilute system on the square grid: (a) nearly triangular vortex lattice $f = 1/60$; (b) nearly square vortex lattice $f = 1/51$. Solid squares denote positions of vortices.

FIG. 15. (a) Inverse dielectric constant $\epsilon^{-1}(T)$ and (b) orientational correlations $\varphi_6(T)$ and $\varphi_4(T)$ versus $T$, for the system with nearly triangular ground state with $f = 1/60$ and $L = 60$. 
FIG. 16. Melting of a nearly triangular vortex lattice on the square grid. Intensity plots of $S(k)$ for $f = 1/60$, $L = 60$ and several temperatures: (a) $T = 0.003$ in the pinned solid; (b) $T = 0.006$ in the floating solid; (c) $T = 0.009$ in the liquid.

FIG. 17. Heights of the peaks $S(G)$ versus $|G|$ for $f = 1/60$ and $L = 60$. Dashed lines represent the best fit to Eq. (24), and are used to extract the exponent $\eta_{G_1}(T)$.

FIG. 18. (a) Inverse dielectric constant $\epsilon^{-1}(T)$ and (b) orientational correlations $\varphi_6(T)$ and $\varphi_4(T)$ versus $T$, for the system with nearly square ground state with $f = 1/51$ and $L = 51$.

FIG. 19. Melting of a nearly square vortex lattice on the square grid. Intensity plots of $S(k)$ for $f = 1/51$, $L = 51$ and several temperatures: (a) $T = 0.003$ in the pinned solid; (b) $T = 0.0045$ in the floating solid; (c) $T = 0.006$ in the liquid.

FIG. 20. Ground state for $f = 5/11$ on a $22 \times 22$ unit cell. Solid squares represent vortex positions. Crosses (+) indicate vacancies (defects) in the otherwise perfect checkerboard pattern of $f = 1/2$.

FIG. 21. Melting of $f = 5/11$ for $L = 22$. $S(k)$ is shown for: (a) $T = 0.010$ in the pinned defect solid; (b) $T = 0.018$ in the defect liquid; (c) $T = 0.055$ in the completely disordered high temperature phase.

FIG. 22. Peak heights $S(q^*)$ with $q^* \equiv (\pi/a_0, \pi/a_0)$, and $S(G_1)$ where $G_1 = (2\pi/L)(1, 5)$ is the shortest reciprocal lattice vector of the defect superlattice, plotted versus $T$ for $f = 5/11$ and $L = 22$.

FIG. 23. Inverse dielectric constant $\epsilon^{-1}(T)$ versus $T$ for $f = 5/11$ and $L = 22$. 

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FIG. 24. Melting of the defect superlattice for \( f = 22/45 \) and \( L = 90 \). Intensity plot of \( S(\mathbf{k}) \) for: (a) \( T = 0.0040 \) in the pinned defect solid; (b) \( T = 0.0065 \) in the defect floating solid; (c) \( T = 0.0085 \) in the defect liquid. Intensities at \( (\pm \pi/a_0, \pm \pi/a_0) \) and \( (0,0) \), that arise from the fixed staggered background, have been subtracted for the sake of clarity.

FIG. 25. Inverse dielectric function \( \epsilon^{-1}(T) \) and orientational correlation \( \varphi_6(T) \) versus \( T \), for \( f = 22/45 \) and \( L = 90 \).