Well-posedness and long time behavior in nonlinear dissipative hyperbolic-like evolutions with critical exponents

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Abstract

These lectures present the analysis of stability and control of long time behavior of PDE models described by nonlinear evolutions of hyperbolic type. Specific examples of the models under consideration include: (i) nonlinear systems of dynamic elasticity: von Karman systems, Berger’s equations, Kirchhoff - Boussinesq equations, nonlinear waves (ii) nonlinear flow - structure and fluid - structure interactions, (iii) and nonlinear thermo-elasticity. A characteristic feature of the models under consideration is criticality or super-criticality of sources (with respect to Sobolev’s embeddings) along with super-criticality of damping mechanisms which, in addition, may be also geometrically constrained.

Our aim is to present several methods relying on cancelations, harmonic analysis and geometric analysis, which enable to handle criticality and also super-criticality in both sources and the damping of the underlined nonlinear PDE. It turns out that if carefully analyzed the nonlinearity can be taken ”advantage of” in order to produce implementable damping mechanism.

Another goal of these lectures is the understanding of control mechanisms which are geometrically constrained. The final task boils down to showing that appropriately damped system is ”quasi-stable” in the sense that any two trajectories approach each other exponentially fast up to a compact term which can grow in time. Showing this property- formulated as quasi-stability estimate -is the key and technically demanding issue that requires suitable tools. These include: weighted energy inequalities, compensated compactness, Carleman’s estimates and some elements of microlocal analysis.

Key words: wave and plates, critical and supercritical sources, quasi-stability, global attractors, finite-dimension.

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1 Introduction

These lectures are devoted to the analysis of stability and control of long time behavior of PDE models described by nonlinear evolutions of hyperbolic type. Specific examples of the models under consideration include: (i) nonlinear systems of dynamic elasticity: von Karman systems, Berger’s equations, Kirchhoff
- Boussinesq equations, nonlinear waves (ii) nonlinear flow - structure and fluid
structure interactions, (iii) and nonlinear thermo-elasticity. A goal to accomplish is to reduce the asymptotic behavior of the dynamics to a tractable finite dimensional and possibly smooth sets. This type of results beside having interest in its own within the realm of dynamical systems, are fundamental for control theory where finite dimensional control theory can be used in order to forge a desired outcome for the dynamics evolving in the attractor.

A characteristic feature of the models under consideration is criticality or super-criticality of sources (with respect to Sobolev’s embeddings) along with super-criticality of damping mechanisms which may be also geometrically constrained. This means the actuation takes place on a “small” sub-region only. Super-criticality of the damping is often a consequence of the “rough” behavior of nonlinear sources in the equation. Controlling supercritical potential energy may require a calibrated nonlinear damping that is also supercritical. On the other hand super-linearity of the potential energy provides beneficial effect on the long time boundedness of semigroups. From this point of view, the non-linearity does help controlling the system but, at the same time, it also does raise a long list of mathematical issues starting with a fundamental question of uniqueness and continuous dependence of solutions with respect to the given (finite energy) data. It is known that solutions to these problems can not be handled by standard nonlinear analysis-PDE techniques.

The aim of these lectures is to present several methods of nonlinear PDE which include cancelations, harmonic analysis and geometric methods which enable to handle criticality and also super-criticality in both sources and the damping. It turns out that if carefully analyzed the nonlinearity can be taken “advantage of” in order to produce implementable control algorithms.

Another aspects that will be considered is the understanding of control mechanisms which are geometrically constrained. Here one would like to use minimal sensing and minimal actuating (geometrically) in order to achieve the prescribed goal. This is indeed possible, however analytical methods used are more subtle. The final task boils down to showing that appropriately damped system is ”quasi-stable” in the sense that any two trajectory approach each other exponentially fast up to modulo a compact term which can grow in time. Showing this property- formulated as quasi-stability estimate -is the key and technically demanding issue that requires suitable tools. These include: weighted energy inequalities, compensated compactness, Carleman’s estimates and some elements of microlocal analysis.

The lecture are organized as follows.

- We start in section 2 with description of main PDE models such as wave and plate equations with both interior and boundary damping. Instead of striving for the most general formulation, we provide canonical models in the simplest possible form which however retains the main features of the problems studied.

- Section 3 deals with wellposedness of weak solutions corresponding to these models.
• Section 4 describes general and abstract tools used for proving existence of attractors and also properties of attractors such as structure, dimensionality and smoothness. Here we emphasize methods which allow to deal within non-compact environment typical when one deals with hyperbolic like dynamics and critical sources. Specifically a simple but very useful method of energy relation due to J. Ball (see [Bal04] and also [MRW98]) is presented (allowing to deal with supercritical sources), a version of "compensated compactness method" introduced first by A. Khanmamedov [Kha06] (allowing to deal with some critical sources) and a method based on "quasi-stability" estimate originated in the authors work [CL04a] (and further developed in [CL08a]) which gives in one shot several properties of the attractor such as smoothness, finite-dimensionality and also limiting properties for the family of attractors.

• Section 5 demonstrates how the abstract methods can be applied to the problems of interest. Clearly, the trust of the arguments is in derivation of appropriate inequalities. While we provide the basic insight into the arguments, the details of calculations are often referred to the literature. This way the interested reader will be able to find a complete justification of the claims made. We should also emphasize that presentation of our results is focused on main features of the dynamics and not necessarily on a full generality. However, subsections such as Generalizations or Extensions provide information on possible generalizations with a well documented literature and citations.

• Section 6 is devoted to other models such as: structurally damped plates, fluid-structures, fluid-flow interactions, thermoelastic interactions, Midlin-Timoshenko beams and plates, Quantum Zakharov system and Schrödinger-Boussinesq equations. These equations exemplify models where general abstract tools for the treatment of long behavior, presented in section 4 apply. Due to space limitations, the analysis here is brief with details deferred to the literature.

• Each section concludes with examples of further generalizations-extensions and also with a list of open problems.

Basic notations: Let \( \Omega \subset \mathbb{R}^d \), \( d = 2, 3 \), be a bounded domain with a smooth boundary \( \Gamma = \partial \Omega \). We denote by \( H^s(\Omega) \) \( L_2 \)-based Sobolev space of the order \( s \) endowed with the norm \( ||u||_{s,\Omega} \equiv ||u||_{H^s(\Omega)} \) and the scalar product \( (u,v)_{s,\Omega} \). As usual for the closure of \( C_0^\infty(\Omega) \) in \( H^s(\Omega) \) we use the notation \( H_0^s(\Omega) \). Below we use the notations:

\[
((u,v)) \equiv \int_\Omega uv\,d\Omega, \quad ||u||^2 \equiv ((u,u)), \quad \langle u,v \rangle \equiv \int_\Gamma uv\,d\Gamma,
\]

\( Q_T \equiv \Omega \times (0,T), \quad \Sigma_T \equiv \Gamma \times (0,T) \).

We also denote by \( C(0,T,Y) \) a space of strongly continuous functions on the interval \([0,T]\) with values in a Banach space \( Y \). In the case when we deal with weakly continuous functions we use the notation \( C_w(0,T,Y) \).
2 Description of the PDE models

We consider second order PDE models of evolutions with nonlinear (velocity type) feedback control represented by monotone, continuous functions, and critical-supercritical sources defined on $\Omega \subset \mathbb{R}^d$, $d = 2, 3$. In what follows, we describe models of wave and plate equations with a dissipation occurring either in the interior of the domain or on the boundary. Three examples given below constitute canonical models which admit vast array of generalizations, some of which are discussed in Section 6. We begin with the interior damping first.

2.1 Waves and plates with nonlinear interior damping and critical-supercritical sources

(A) Wave equation with nonlinear damping-source interaction. In a bounded domain $\Omega \subset \mathbb{R}^3$ we consider the following wave equation with the Dirichlet boundary conditions:
\[ w_{tt} - \Delta w + a(x)g(w_t) = f(w) \quad \text{in} \quad \Omega \times (0,T); \quad w = 0 \quad \text{on} \quad \Gamma \times (0,T), \tag{2.1} \]
where $T > 0$ may be finite or $\infty$. We suppose that the damping has the structure $g(s) = g_1 s + |s|^{m-1}s$ for some $m \geq 1$ and assume the following typical behavior for the source $f(w) \sim -|w|^{p-1}w$, where either $1 \leq p \leq 3$ or else $3 < p \leq \min \{5, \frac{6m}{m+1}\}$. The nonnegative function $a \in C^1(\Omega)$ represents the support and intensity of the damping. The associated energy function has the form:
\[ E(t) \equiv \frac{1}{2}||w_t(t)||^2 + \frac{1}{2}||\nabla w(t)||^2 - \int_{\Omega} \hat{f}(w(t))dx, \tag{2.2} \]
where $\hat{f}$ denotes the antiderivative of $f$, i.e. $\hat{f}' = f$. The energy balance relation for this model has the form
\[ E(t) + \int_s^t (ag(w_t), w_t))d\tau = E(s). \tag{2.3} \]
When $p \leq 3$ the source in the wave equation is up to critical. This is due to Sobolev’s embedding $H^1(\Omega) \subset L_6(\Omega)$ so that $f(w) \in L_2(\Omega)$ for $w$ of finite energy. In this case local (in time) well-posedness holds with no restrictions for $m$. Globality of solutions is guaranteed by suitable a priori bounds resulting from either a structure of the source $f(w)$ or from the interplay with the damping, see Theorem 3.3 below.

Long time behavior will be analyzed under the condition of weak degeneracy of the damping coefficient $a(x)$ in the so called “double critical case”, i.e. $p \leq 3$, $m \leq 5$. We use the compensated compactness criterion in Theorem 4.3 for proving existence of an attractor and quasi-stability inequality (4.9) required by Definition 4.13 in order to establish both smoothness and finite dimensionality of global attractor.
When $3 < p \leq 5$ the damping required for uniqueness of solutions needs to be superlinear. With sufficiently superlinear damping we show that the corresponding dynamical system is well posed on the finite energy phase space. However, the issue of long time behavior, in that case, is still an open problem. The known frameworks for studying supercritical sources fail due to nonlinearity of the damping. More details will be given later. The ranges of $p \in [5, 6)$ are considered in [BL08a].

We also refer to [CV02, Chapter XIV] where the case of a linear damping and a supercritical force is considered from point of view of trajectory attractors.

**B) Von Karman plate equation with nonlinear interior damping.** Let $\Omega \subset \mathbb{R}^2$ and $\alpha \in [0, 1]$. We denote $M_\alpha \equiv I - \alpha \Delta$ and consider the equation

$$M_\alpha w_{tt} + \Delta^2 w + a(x)\left[g(w_t) - \alpha \text{div}G(\nabla w_t)\right] = [F(w), w] + P(w)$$

in $\Omega \times (0, \infty)$ with the *clamped* boundary conditions:

$$w = \frac{\partial}{\partial n} w = 0 \text{ on } \Gamma \times (0, \infty),$$

where the *Airy stress function* $F(w)$ solves the elliptic problem

$$\Delta^2 F(w) = -[w, w], \text{ in } \Omega, \text{ with } F = \frac{\partial}{\partial n} F = 0 \text{ on } \Gamma,$$

and the von *Karman bracket* $[u, v]$ is given by

$$[u, v] = \partial_x^2 u \cdot \partial_x^2 v + \partial_y^2 u \cdot \partial_y^2 v - 2 \cdot \partial_x^2 u \cdot \partial_x \partial_y v,$$

The *damping* functions $g : \mathbb{R} \mapsto \mathbb{R}_+$ and $G : \mathbb{R}^2 \mapsto \mathbb{R}_+^2$ have the following form

$$g(s) = g_1 s + |s|^{m-1} s \text{ and } G(s, \sigma) = G_1 \cdot (s; \sigma) + (|s|^{m-1} s; \sigma^{m-1} \sigma),$$

where $g_1$ and $G_1$ are nonnegative constants.

The source term $P$ is assumed locally Lipschitz operator acting from $H^2(\Omega)$ into $L_2(\Omega)$ when $\alpha = 0$ and from $H^2(\Omega)$ into $H^{-1}(\Omega)$ when $\alpha > 0$.

The associated *energy function* and the *energy balance* relation have the form

$$E(t) \equiv \frac{1}{2} (|w(t)|^2 + \alpha |\nabla w(t)|^2) + \frac{1}{2} |\Delta w(t)|^2 + \frac{1}{4} |\Delta F(w)|^2$$

and

$$E(t) + \int_s^t D(w_t) d\tau = E(s) + \int_s^t ((P(w), w_t)) d\tau,$$

where the interior damping form $D(w_t)$ is given by

$$D(w_t) = ((ag(w_t), w_t)) + \alpha((aG(\nabla w_t), \nabla w_t)).$$

The case when $\alpha > 0$ is subcritical with respect to the Airy source. The well-posedness and existence of attractors in this case is more standard [CL10],.
When \( \alpha = 0 \) the Airy stress source is critical. Wellposedness will be achieved by displaying hidden regularity (see Lemma 3.9 below) for Airy’s stress function. Existence of attractors along with their smoothness will be shown by means of quasi-stability inequality (see (4.9) in Definition 4.13) which can be proved as long as \( P \) is subcritical.

(C) Kirchhoff-Boussinesq plate with interior damping. With notations as in the case of the von Karman plate we consider

\[
M_\alpha w_{tt} + \Delta^2 w + a(x) [g(w_t) - \alpha \text{div} G(\nabla w_t)] = \text{div} [||\nabla w||^2 \nabla w] + P(w) \tag{2.12}
\]

with the clamped boundary conditions (2.5). The damping functions \( g : \mathbb{R} \to \mathbb{R}_+ \), \( G : \mathbb{R}^2 \to \mathbb{R}^2_+ \) and the source \( P \) are the same as above. The associated energy function has the form

\[
E(t) \equiv \frac{1}{2} \left( ||w_t(t)||^2 + \alpha ||\nabla w_t(t)||^2 \right) + \frac{1}{2} ||\Delta w(t)||^2 + \frac{1}{4} \int_\Omega |\nabla w(x, t)|^4 \, dx \tag{2.13}
\]

The corresponding energy balance relation is given by (2.10).

The well-posedness for the case \( \alpha > 0 \) is standard. This is due to the fact that \( \text{div} |\nabla w|^2 \nabla w \in H^{-1}(\Omega) \) for finite energy solutions \( w \). The case \( \alpha = 0 \) is subtle. Its analysis requires special consideration and depends on linearity of the damping. Similarly, long time behavior dealt with by using Ball’s method presented in an abstract version in Theorem 4.4 also requires linear damping.

2.2 Nonlinear waves and plates with geometrically constrained damping and critical-supercritical sources

This class contains models in the situation when the support of the damping \( a(x) \) is strictly contained in \( \Omega \). This is to say \( \text{supp} a(x) \subset \Omega_0 \subset \Omega \). In addition, we considered singular case of interior localized damping which is the boundary damping. These models are described next.

(A) Wave equation with nonlinear boundary damping-source interaction. In a bounded domain \( \Omega \subset \mathbb{R}^3 \) we consider the following wave equation

\[
w_{tt} - \Delta w + a(x) g(w_t) = f(w) \quad \text{in} \quad \Omega \times (0, T), \tag{2.14}
\]

with the Neumann boundary conditions:

\[
\frac{\partial}{\partial n} w + g_0(w_t) = h(w) \quad \text{in} \quad \Gamma \times (0, T). \tag{2.15}
\]

Concerning the internal damping \( g \) and source \( f \) terms our hypotheses are the same as in Section 2.1(A). The internal damping coefficient \( a(x) \) satisfies relations

\[
a \in C^1(\Omega), \quad a(x) \geq 0 \text{ in } \Omega \quad \text{with} \quad \text{supp} a \subset \Omega_0 \subset \subset \Omega
\]

The boundary damping has the form \( g_0(s) = g_2 s + |s|^{q-1}s \) with \( q \geq 1 \) and \( g_2 \geq 0 \). The boundary source has the behavior \( h(w) \sim -|w|^{k-1}w \), where \( 1 \leq k \leq \max \left\{ 3, \frac{q}{q+1} \right\} \).
The associated energy function is given by

\[ E(t) \equiv \frac{1}{2}||w_t(t)||^2 + \frac{1}{2}||\nabla w(t)||^2 - \int_{\Omega} \hat{f}(w(t))dx - \int_{\Gamma} \hat{h}(w(t))dx \quad (2.16) \]

where as above \( \hat{f} \) and \( \hat{h} \) denote the antiderivative of \( f \) and \( h \). The energy balance relation in this case has the form

\[ E(t) + \int_{s}^{t} \left[ ((ag(w_t), w_t)) + < \dot{g}_0(w_t), w_t > > \right] d\tau = E(s). \quad (2.17) \]

Local and global well-posedness result will be presented for the boundary-source and damping model. One could consider larger range of boundary and interior sources \( k \in (1, 4), p \in (1, 6) \) [BL08a]. In this case, however, potential energy corresponding to the sources is not well defined. Well-posedness result requires higher integrability of initial data [BL08a, BL10]. More general structure of sources and damping can be considered. However, the required polynomial bounds are critical.

In the triple critical case \( \{p = 3, q = 3, m = 5\} \) with \( k = 2 \) the existence of a global attractor is shown by taking advantage of compensated compactness result in Theorem 4.3 (see [CL07a]). With the additional hypotheses imposed on the damping smoothness finite-dimensionality of attractor is established by proving that the system is quasi-stable (in the sense of Definition 4.13).

In the case when only boundary damping is active (or internal damping is localized) existence and smoothness of attractor require additional growth condition restrictions imposed on the damping. This is to say, we need to assume \( p \leq 3 \) and \( q \leq 1 \) [CLT08, CLT09].

(B) Von Karman plate equation with nonlinear boundary damping. In a bounded domain \( \Omega \subset \mathbb{R}^2 \) we consider the equation

\[ M_\alpha w_{tt} + \Delta^2 w + a(x) [g(w_t) - \alpha \text{div} G(\nabla w_t)] = [\mathcal{F}(w), w] + P(w) \quad (2.18) \]

with the hinged dissipative boundary conditions:

\[ w = 0, \quad \Delta w = -g_0(\frac{\partial}{\partial n}w_t), \quad \text{on} \quad \Gamma \times (0,T) \quad (2.19) \]

Here we use the same notations as in Section 2.1(B). In particular, Airy’s stress function \( \mathcal{F}(w) \) solves (2.6) and the internal damping functions \( g \) and \( G \) have the form (2.8). Concerning the boundary damping we assume that \( g_0(s) \sim g_2s + |s|^{q-1}s, \quad q \geq 1 \). The associated energy function \( E \) has the same form as in Section 2.1(B), see (2.9). The corresponding energy balance relation reads as follows

\[ E(t) + \int_{s}^{t} \left[ D(w_t) + < \dot{g}_0(\frac{\partial w_t}{\partial n}), \frac{\partial w_t}{\partial n} > > \right] d\tau = E(s) + \int_{s}^{t} ((P(w), w_t))d\tau, \quad (2.20) \]

where the internal damping term \( D(w_t) \) is given by (2.11).
Below we show the existence of a continuous semiflow corresponding to (2.18) and (2.19). The existence and smoothness of an attractor with boundary damping alone is shown under additional hypotheses restricting the growth of the boundary damping. In the case of boundary damping one could consider damping affecting other boundary conditions such as free and simply supported, see [CL10].

(C) Kirchhoff–Boussinesq plate with boundary damping. With the same notations as above in a domain $\Omega \subset \mathbb{R}^2$ we consider the equation

$$M_\alpha w_{tt} + \Delta^2 w + a(x) [g(w_t) - \alpha \text{div} G(\nabla w_t)] = \text{div} [\|\nabla w\|^2 \nabla w] + P(w) \quad (2.21)$$

with the *hinged dissipative* boundary conditions (2.19). The internal damping functions $g$ and $G$ satisfy (2.8). The boundary damping is the same as the previous case, i.e., $g_0(s) \sim g_2 s + |s|^{q-1} s$, $q \geq 1$. The source $P$ is same as above. The associated *energy function* has the form (2.13). The *energy balance* relation is given by (2.20).

When $\alpha > 0$ the wellposedness of finite energy solutions with internal damping follows from the observation that the model can be represented as a locally Lipschitz perturbation of monotone operator. This approach is also adaptable to boundary damping, like in the case of von Karman plate. In the case $\alpha = 0$ the problem is much more delicate. While existence of finite energy solutions can be established with general form of the interior-boundary damping, the uniqueness of solutions requires the linearity of the damping. In addition, continuous dependence on the data depends on time reversibility of dynamics which, in turn, does not allow for boundary damping. In view of the above most of the questions asked are open in the presence of boundary damping.

### 3 Well-posedness and generation of continuous flows

In this section we present several methods which enable to deal with Hadamard well-posedness of PDE equations with supercritical Sobolev exponents. By supercritical, we mean sources that may not belong to finite kinetic energy space for trajectories of finite energy. While existence of finite energy solutions is often handled by various variants of Galerkin method, it is the uniqueness and continuous dependence on the data that is problematic. There is no unified theory for the treatment of such problems, however there exist methods that are applicable to classes of these problems.

In this section we concentrate on three models described in Section 2.1, where each of the model displays different characteristics and associated difficulties. In order to cope with this, different methods need to be applied. We provide a show case for the following methods which are critical for proving Hadamard well-posedness in the examples cited:

- Interaction of superlinear sources via the damping.
  Illustration: wave equation.
• Cancelations-harmonic analysis and microlocal analysis methods.  
  Illustration: Von Karman equations.

• Sharp control of Sobolev’s embeddings and duality scaling.  
  Illustration: Kirchoff-Boussinesq plate.

1. In the first example the equation can be viewed as a perturbation of a monotone operator. However, the superlinearity of the source destroys locally Lipschitz character of the semilinear equation. In order to offset the difficulty, superlinear damping \( g(w_t) \) is used. The interplay between the superlinearity of potential and kinetic energy lies in the heart of the problem. The presence of this damping becomes critical in establishing well defined dynamical system evolving on a finite energy phase space.

2. In the second example, classical regularity results for von Karman nonlinearity, when \( \alpha = 0 \), fail to show that the perturbation of monotone operator is locally Lipschitz. However, in this case, more subtle methods based on harmonic analysis and compensated compactness allow to show that the Airy’s stress function has hidden regularity property. This estimate allows to prove that, contrary to the original prediction, the semilinear term is locally Lipschitz. This allows to prove, again, that the resulting dynamical system is well posed on a finite energy space.

3. In the third example, the source of the difficulties is restoring force \( \text{div} [ |\nabla w|^2 \nabla w] \) which fails to be locally Lipschitz with respect to finite energy solutions in the case \( \alpha = 0 \). By using method relying on logarithmic control of Sobolev’s embeddings and topological shift of the energy (“Sedenko’s method”) we are able to prove that the resulting system with linear damping is well posed on finite energy space. Critical role in the argument is played by time reversibility of the dynamics and linearity of the damping. This excludes superlinear damping and boundary dissipation in the case \( \alpha = 0 \).

Conclusion: These three canonical examples present three different methods on how to deal with the loss of local Lipschitz property and still be able to obtain well-posed flows defined on finite energy phase space. The methods presented are transcendental and applicable to other dynamics displaying similar properties. Some of the examples are given in Section 6.

3.1 Wave equation with a nonlinear interior damping - model in (2.1)

The statements of the results

With reference to the model (2.1) with \( a(x) \geq a_0 > 0 \) in \( \Omega \) the following assumption is assumed throughout.
Assumption 3.1  1. A scalar function \( g(s) \) is assumed to be of the form \( g(s) = g_1 s + g_2(s) \), where \( g_1 \in \mathbb{R} \) and \( g_2(s) \) is continuous and nondecreasing on \( \mathbb{R} \) with \( g_2(0) = 0 \).

2. The source \( f(s) \) is either represented by a \( C^1 \) function and such that \( |f'(s)| \leq C(1+|s|^{p-1}) \) with \( 1 \leq p \leq 3 \), or else we have that \( f \in C^2(\mathbb{R}) \) and (i) there exist \( m_{g_2}, M_{g_2} > 0 \) such that the damping function \( g_2 \) satisfies the inequality

\[
m_{g_2}|s|^{m+1} \leq g_2(s)s \leq M_{g_2}|s|^{m+1}, \quad |s| \geq 1.
\]

for some \( m > 1 \), (ii) \( |f''(s)| \leq C(1+|s|^{p-2}) \) with \( p \) satisfying the following compatibility growth condition

\[
3 \leq p < 6 \quad \text{and} \quad p \leq \frac{6m}{m+1}
\]

While in the case when the source exponent \( p \leq 3 \) we adopt classical semigroup definition of the generalized solution (see, e.g., [CL08a]), for supercritical cases \( p > 3 \) we provide the following variational definition.

Definition 3.2 (Weak solution) Let (3.1) and (3.2) be in force. By a weak solution of (2.1), defined on some interval \( (0, T) \) with initial data \((w_0; w_1)\) from \( H_1(\Omega) \times L_2(\Omega) \), we mean a function \( u \in C_w(0, T; H^1(\Omega)) \), such that

1. \( u_t \in L_{m+1}((0, T) \times \Omega) \cap C_w(0, T; L_2(\Omega)) \).

2. For all \( \phi \in C(0, T, H^1_0(\Omega)) \cap C^1(0, T; L_2(\Omega)) \cap L_{m+1}([0, T] \times \Omega) \), we have that

\[
\int_0^T \int_\Omega (-u_t \phi_t + \nabla u \nabla \phi) \, d\Omega \, dt + \int_0^T \int_\Omega g(u_t) \phi \, d\Omega \, dt = -\int_\Omega u_0 \phi \, d\Omega + \int_0^T \int_\Omega f(u) \phi \, d\Omega \, dt.
\]

3. \( \lim_{t \to 0} (u(t) - u_0, \phi)_{1, \Omega} = 0 \) and \( \lim_{t \to 0} ((u_t(t) - u_1, \phi)) = 0 \) for all \( \phi \in H^1_0(\Omega) \).

Theorem 3.3 Let Assumption 3.1 be in force. We assume that initial data \((w_0; w_1)\) possess the properties

\( w(0) = w_0 \in H^1_0(\Omega), \quad w_t(0) = w_1 \in L_2(\Omega) \)

and for \( p > 5 \), \( w_0 \in L_r(\Omega) \) with \( r = \frac{3}{2}(p - 1) \).

Then

- There exist a unique, local (in time) solution \( w(t) \) of finite energy. This is to say: there exists \( T > 0 \) such that

\[
w \in C([0, T]; H^1_0(\Omega)), \quad w_t \in C([0, T]; L_2(\Omega)),
\]

where \( w \) is a generalized solution when \( p \leq 3 \) and \( w \) is a weak solution when condition (3.1) holds with some \( m \geq 1 \) (which is the case when \( p \geq 3 \)).
In this case $3 < p \leq 5$ weak solutions satisfy the energy relation in (2.3).

If $p \leq 3$ and (3.1) holds with some $m$, then generalized solution satisfies also the variational form (3.3) with the test functions $\phi$ from the class $C(0, T; H^1_0(\Omega)) \cap C^1(0, T; L_2(\Omega)) \cap L_\infty(Q_T)$. Moreover, the said solutions are continuously dependent on the initial data.

When $p \leq m$ the obtained solutions are global, i.e., $T = \infty$. The same holds under dissipativity condition:

$$\liminf_{|s| \to \infty} \frac{-f(s)}{s} > -\lambda_1$$

where $\lambda_1$ is the first eigenvalue of $-\Delta$ with the zero Dirichlet data.

When $p > m$ the obtained solutions are global, i.e., $T = \infty$. The same holds under dissipativity condition:

$$\liminf_{|s| \to \infty} \frac{-f(s)}{s} > -\lambda_1$$

Remark 3.1 In the case when $p \in [1, 3]$ one constructs solutions as the limits of strong semigroup solutions [CL07a]. These are referred to as generalized solutions [CL07a, CL08a]. It can be shown that generalized semigroup solutions do satisfy the variational form in question and variational weak solutions are unique [CL08a]. In order to justify this, it suffices to consider only the nonlinear damping term $g(s)$ with $g_1 = 0$, since the latter provides a bounded linear perturbation—thus it does not affect the limit passage. The energy inequality provides $L_1$ bound for $g(w_n)$. This allows to apply Dunford-Pettis compactness criterion (see later) in order to transit with the weak limit in $L_1$, and thus obtaining variational form of solutions (3.3) with appropriate test function as specified in the third part of Theorem 3.3.

**Sketch of the proof**
Without loss of generality we may assume $a(x) = 1$ and $g_1 = 0$, so $g_2(s) = g(s)$ (these parameters have no impact on the proof).

**Case 1: Critical/subcritical sources and arbitrary monotone damping.**
In this case the proof is standard and follows from monotone operator theory. Setting the second order equation as a first order system

$$W_t + A(W) = F(W), \quad t > 0, \quad W(0) = (w_0; w_1) \quad (3.5)$$
on the space $H \equiv H^1_0(\Omega) \times L_2(\Omega)$ where $W = (w; w_t)$ and

$$A \equiv \begin{pmatrix} 0 & -I \\ -\Delta & g(\cdot) \end{pmatrix}, \quad F(W) \equiv \begin{pmatrix} 0 \\ f(w) \end{pmatrix},$$

allows to the use of monotone operator theory (see, e.g., [Bar76, Sho97] for the basic theory). Indeed, $A$ is maximally monotone and $F(W)$ is locally Lipschitz.
on $H$. This last statement is due to the restriction $p \leq 3$ and Sobolev’s embedding $H^1(\Omega) \subset L^6(\Omega)$. General theorem (see [CEL02] and also [CL10, Chapter 2]) allows to conclude local unique existence of semigroup solutions.

The arguments leading to variational characterization of weak solutions are given in the Appendix [CL07a]. It can be based on weak $L^1$ compactness criteria due to Danford and Pettis (see [DS58, Section IV.8]) which states that the set $M \subset L^1(Q)$ is relatively compact with respect to weak topology if and only if this set is bounded and uniformly absolutely continuous, i.e.,

$$\forall \varepsilon > 0 \ \exists \delta > 0 : \ \text{mes}(E) \leq \delta \ \Rightarrow \ \sup_{f \in M} \left| \int_E f dQ \right| \leq \varepsilon.$$  

We want to show that $M = \{ g(w_n) \}$ is weakly $L^1(Q)$ compact for a sequence of strong solutions which also converges almost everywhere. For this using dissipation inequality

$$\int_Q g(w_n)w_n dQ \leq C$$

which (by splitting the integration into $|w_n| \leq R$ and $|w_n| \geq R$) implies that

$$\int_E |g(w_n)|dQ \leq \frac{1}{R} \int_Q g(w_n)w_n dQ + g_R \text{mes}(E), \ \forall R > 0.$$  

**Case 2: supercritical source.** In this case superlinearity of the damping is exploited. The key role is played by energy inequality obtained first for smooth solutions corresponding to truncated problem where $f$ is approximated by locally Lipschitz function $f_K$ defined in [BL10]. Locally Lipschitz theory (Case 1 above) allows to obtain the energy estimate

$$E(t) + \int_0^t ((g(w), w_t)) ds \leq E(0) + \int_0^t ((f_K(w), w_t)) ds$$  

(3.6)

with $E(t) \equiv \frac{1}{2} \left( \|w(t)\|^2 + \|\nabla w(t)\|^2 \right)$. Exploiting the growth conditions imposed on $g$ and $f$ yields

$$E(t) + \|w_t\|^{m+1}_{L^m(Q_t)} \leq C \left[ E(0) + \|f_K(w)\|^{m+1}_{L^m(Q_t)} \right]$$

and Sobolev’s embeddings $H^1(\Omega) \subset L^6(\Omega)$ along with the condition $p(m + 1) \leq 6m$ implies

$$E(t) + \|w_t\|^{m+1}_{L^m(Q_t)} \leq C \left[ E(0) + L \left( t + \int_0^t \|w\|^{m+1}_{L^m(\Omega)} ds \right) \right].$$

The above estimate followed by a rather technical limit argument (with $K \to \infty$) [BL10] allows to establish local existence of weak solutions.

**Energy identity.** This is an important step of the argument. By using finite difference approximation for the velocity $w_t(t)$ one shows that weak solutions satisfy not only energy inequality but also energy identity. The detailed argument is given in the proof of Lemma 2.3 [BL08a].
Uniqueness of the solutions is more subtle. We need to show that any two solutions from the existence class, i.e. satisfying

\[ w \in C_w([0, T], H^1_0(\Omega) \cap C^1_w([0, T], L_2(\Omega)), \quad w_t \in L_{m+1}(Q_T) \]

is uniquely determined by the initial data. To this end let’s consider the difference of two solutions: \( z \equiv w - u \), where both \( w \) and \( u \) are finite energy weak solutions specified as above with the same initial data \((w_0; w_1)\). Denoting by

\[ E_z(t) \equiv ||\nabla z||^2 + ||z||^2 \]

owing to energy inequality satisfied for weak solutions and using bounds imposed on the source function \( f(s) \) one obtains after some calculations:

\[
E_z(t) \leq \int_0^t ((f(w) - f(u), z)) dt \equiv R_f(u, w, z, t) \quad (3.7)
\]

The following lemma is critical:

**Lemma 3.4** Let \( p < 5 \). Then \( \forall \epsilon > 0 \) and \( \forall (w_0; w_1) \in H^1_0(\Omega) \times L_2(\Omega) \) such that \( ||w_0||_{1, \Omega} + ||w_1||_{0, \Omega} \leq R \), there exist a constant \( C_\epsilon(R, T) > 0 \) such that

\[
|R_f(T)| \leq \epsilon E_z(T) + C_\epsilon(R, T) \left[ T E_z(T) + \int_0^T \left( 1 + ||u_t(t)||_{L_{m+1}(\Omega)} + ||v_t(t)||_{L_{m+1}(\Omega)} \right) E_z(t) dt \right].
\]

This lemma is specialized to the case when \( p < 5 \). For \( p \in [5, 6) \) there is another term in the inequality above which accounts for the fact that the potential energy term may not be in \( L_1 \). In order to focus presentation we omit this term and refer the reader to the source [BL08a].

**Proof.** Using integration by parts we have that

\[
R_f = ((f(w) - f(u), z)) \bigg|_0^t - \int_0^t ((f'(w)w_t - f'(u)u_t, z)) dt
\]

\[
= ((f(w) - f(u), z)) \bigg|_0^t - \int_0^t ((f'(w)z_t + (f'(w) - f'(u))u_t, z)) dt
\]

\[
= \left[ ((f(w) - f(u), z)) - \frac{1}{2} ((f'(w), z^2)) \right]_0^t
\]

\[
+ \frac{1}{2} \int_0^t ((f''(w)w_t z, z)) dt - \int_0^t (((f'(w) - f'(u))u_t, z)) dt
\]

\[
\leq C \int_\Omega z'(t)[1 + |u(t)|^{p-1} + |w(t)|^{p-1}] dx
\]

\[
+ C \int_0^t \int_\Omega z'(t)[1 + |u(t)|^{p-2} + |w(t)|^{p-2}]|u_t| + |w_t|] dx dt.
\]
It is convenient to use the following elementary estimate valid with any finite energy element $z$ such that $z(0) = 0$:

$$
||z(t)||^2 = \int_\Omega \left( \int_0^t z_t(s) \, ds \right)^2 \, dx \leq t \int_0^t ||z_t(s)||^2 \, ds \leq 2t \int_0^t E_z(s) \, ds. \tag{3.8}
$$

**Estimate for $\int_0^T \int_\Omega z^2(t)|u_t(t)| \, dx \, dt$.** We use Holder’s inequality with $p = 3$ and $q = 3/2$ and obtain:

$$
\int_0^T \int_\Omega z^2(t)|u_t(t)| \, dx \, dt \leq \int_0^T \|z(t)\|_{L^6(\Omega)}^2 \cdot \|u_t(t)\|_{L^{3/2}(\Omega)} \, dt \tag{3.9}
$$

This leads to

$$
\int_0^T \int_\Omega z^2(t)[|w_t(t)| + |u_t(t)|] \, dx \, dt \leq C(R) \int_0^T E_z(t) \, dt. \tag{3.10}
$$

**Estimate for $\int_\Omega |u(T)|^{p-1}z^2(T) \, dx$.** First, we write

$$
\int_\Omega |u(T)|^{p-1}z^2(T) \, dx \leq ||z(T)||^2 + \int_{\Omega \cap \{|u(T)| > 1\}} |u(T)|^{p-1}z^2(T) \, dx. \tag{3.11}
$$

The first term is estimated in (3.8). The argument for the second term is given below. We consider here the supercritical case $3 < p < 5$. The corresponding estimate for $p \in [5, 6)$ is given in [BL08a].

Since $|u(T)| > 1$, there exists $\epsilon_0 > 0$ such that $|u(T)|^{p-1} \leq |u(T)|^{4-\epsilon_0}$. We choose $\epsilon < \frac{3}{4}$ and apply Holder’s inequality with $q = \frac{4}{2+\epsilon}$ and $p^* = \frac{3}{1-\epsilon}$. We then use Sobolev’s embeddings, the fact that $(4-\epsilon_0)\frac{3}{2(1-\epsilon)} \leq 6$, interpolation and Young’s inequality to obtain

$$
\int_{\Omega \cap \{|u(T)| > 1\}} |u(T)|^{p-1}z^2(T) \, dx \\
\leq \left( \int_\Omega |z(T)|^\frac{6}{4+\epsilon} \, dx \right)^\frac{4+\epsilon}{2} \left( \int_\Omega |u(T)|^{\frac{3(4-\epsilon_0)}{2(4-\epsilon_0)}} \, dx \right)^\frac{2(1-\epsilon)}{3}. \tag{3.12}
$$

Combining (3.11) with (3.8) and (3.12), we obtain the final estimate in this case:

$$
\int_\Omega \int_0^T |u(T)|^{p-1}z^2(T) \, dx \, dt \leq \epsilon E_z(T) + C_e(R) T \int_0^T E_z(t) \, dt \tag{3.13}
$$

**Estimate for $\int_0^T \int_\Omega |u(t)|^{p-2}|u_t(t)|z^2(t) \, dQ$.** Since for $|u(t)| \leq 1$, we obtain the term estimated in the previous case, it is sufficient to look at

$$
\int_0^T \int_\Omega |u(t)|^{p-2}|u_t(t)|z^2(t) \, dQ \quad \text{with} \quad \tilde{\Omega} = \Omega \cap \{|u(t)| > 1\}.
$$
We start by applying Holder’s inequality with $q = 3$ and $\overline{q} = 3/2$:

$$
\int_0^T \int_{\Omega} |u(t)|^{p-2} |u_t(t)| z^2(t) \leq \int_0^T \|z(t)\|_{L_6(\Omega)}^2 \left[ \int_{\Omega} |u(t)|^{\frac{4(p-2)}{6}} |u_t(t)|^2 \, dx \right]^{\frac{3}{5}}
$$

(3.14)

We use Holder’s inequality again, with $q = \frac{4}{p-2}$ and $\overline{q} = \frac{4}{6-p}$, and notice that $\frac{6}{6-p} \leq m+1 \iff p \leq \frac{6m}{m+1}$. Hence (3.14) becomes:

$$
\int_0^T \int_{\Omega} |u(t)|^{p-2} |u_t(t)| z^2(t) \leq C \int_0^T \|z(t)\|_{L_6(\Omega)}^2 \|u_t(t)\|_{L^p(\Omega)}^2
$$

(3.15)

Since $\|u(t)\|_{1,\Omega} \leq C_{R,T}$ we obtain:

$$
\int_0^T \int_{\Omega} |u(t)|^{p-2} |u_t(t)| z^2(t) \, dQ \leq C_{R,T} \int_0^T \|z(t)\|_{L_{m+1}^1(\Omega)} \|u_t(t)\|_{L_{m+1}^1(\Omega)} \, dt.
$$

(3.16)

Similarly, following the strategy used in the previous step, we obtain the estimate

$$
\int_0^T \int_{\tilde{\Omega}_w} |u(t)|^{p-2} |u_t(t)| z^2(t) \, d\tilde{\Omega}_w \leq C_{R,T} \int_0^T \|z(t)\|_{L_{m+1}^1(\Omega)} \|u_t(t)\|_{L_{m+1}^1(\Omega)} \, dt,
$$

$$
\int_0^T \int_{\tilde{\Omega}_w} |w(t)|^{p-2} |u_t(t)| z^2(t) \, d\tilde{\Omega}_w \leq C_{R,T} \int_0^T \|z(t)\|_{L_{m+1}^1(\Omega)} \|u_t(t)\|_{L_{m+1}^1(\Omega)} \, dt
$$

with $\tilde{\Omega}_w = \Omega \cap \{|w(t)| > 1\}$. This completes the proof of Lemma 3.4.

COMPLETION OF THE PROOF: In (3.7), we use Lemma 3.4 to obtain:

$$
E_z(T) \leq \epsilon E_z(T) + C_\epsilon(R,T) \left[ TE_z(T) + \int_0^T \left( 1 + \|u(t)\|_{L_{m+1}^1(\Omega)} + \|u_t(t)\|_{L_{m+1}^1(\Omega)} \right) E_z(t) \, dt \right]
$$

(3.17)

for every $\epsilon > 0$. In (3.17), we choose $\epsilon$ and $T$ such that $\epsilon + C_\epsilon(R,T)T < 1/2$ (for uniqueness it is enough to look at a small interval for $T$, since the process can be reiterated) and apply Gronwall’s inequality to obtain that $E_z(T) = 0$ for all $T \leq T_{max}$, where $[0, T_{max}]$ is a common existent interval for both solutions $w$ and $u$.

The above argument completes the proof of uniqueness of finite energy solutions. This result along with energy equality leads to Hadamard wellposedness. Details are in [BL08a, BL10].

Finite-time blow up of solutions is established in [BL08b]. This is accomplished by showing that appropriately constructed anti-Lyapunov function [STV03, Vit99] blows up in a finite time.
Generalizations-Extensions

- In the subcritical and critical case additional regularity of weak solutions equipped with more regular initial data can be established. Quantitative statements are in [CL07a].

- The wave model discussed is equipped with zero Dirichlet data. However, the same result holds for Neumann or Robin boundary conditions.

- In the subcritical and critical case, \( p \leq 3 \), the same results hold with localized damping. This is to say when \( \text{supp} a(x) \subset \Omega_0 \subset \Omega \).

- The analysis of wellposedness for the same model in the case when solutions are confined to a potential well are given in [BRT11].

Open question: Obtain the same result with partially localized damping \( \text{supp} a(x) \subset \Omega_0 \subset \Omega \) in the supercritical case \( p > 3 \).

3.2 Von Karman equation with interior damping - model (2.4)

The statement of the results

With reference to the model (2.4) the following assumption is assumed throughout.

**Assumption 3.5**

1. Scalar function \( g(s) \) is assumed to be continuous and monotone on \( \mathbb{R} \) with \( g(0) = 0 \).

2. In the case \( \alpha > 0 \) we assume that the damping \( G \) has the form \( G(s, \sigma) = (g_1(s); g_2(\sigma)) \), where \( g_i(s), i = 1, 2 \), are continuous and monotone on \( \mathbb{R} \) with \( g_i(0) = 0 \). Moreover, they are of polynomial growth, i.e., \( |g_i(s)| \leq C(1 + |s|^{q-1}) \) for some \( q \geq 1 \).

3. The source \( P(w) \) is assumed locally Lipschitz from \( H^2(\Omega) \) into \( [H_\alpha(\Omega)]' \), where

\[
H_\alpha(\Omega) = \begin{cases} 
L^2(\Omega) & \text{in the case } \alpha = 0; \\
H_0^1(\Omega) & \text{in the case } \alpha > 0. 
\end{cases} 
\] (3.18)

Below we also use the following refinement of the hypothesis concerning the source \( P \).

**Assumption 3.6** We assume that \( P(w) = -P_0(w) + P_1(w) \), where

1. \( P_0(w) \) is a Fréchet derivative of the functional \( \Pi_0(w) \) and the property holds: there exist \( a, b \in \mathbb{R} \) and \( \epsilon > 0 \) such that

\[
\Pi_0(w) + a\|w\|_{2-\epsilon, \Omega}^2 \geq b, \quad ((P_0(w), w)) + a\|w\|_{2-\epsilon, \Omega}^2 \geq b, \quad \forall w \in H_0^2(\Omega);
\]

2. there exists \( K > 0 \) and \( \epsilon > 0 \) such that \( \|P_1(w)\| \leq K\|w\|_{2-\epsilon, \Omega} \) for all \( w \in H_0^2(\Omega) \).
Remark 3.2 A specific choice of interest in applications is $P(w) = [F_0, w] - p$, where $[,]$ denotes the von Karman bracket (see (2.7)), $F_0 \in H^{3+\delta}(\Omega) \cap H^1_0(\Omega)$, $\delta > 0$, and $p \in L_2(\Omega)$. The term $F_0$ models in-plane forces in the plate and $p$ is a transversal force. The given source complies with all the hypotheses stated. In the case $\alpha > 0$ the hypotheses concerning $P(w)$, $F_0$ and $p$ can be relaxed. However we do not pursue this generalizations and refer to [CL10].

In addition to the concept of generalized (semigroup) solutions [CL08a, CL10] we can also define weak solutions.

Definition 3.7 (Weak solution) By a weak solution of (2.4), defined on an interval $[0, T]$, with initial data $(u_0; u_1)$ we mean a function $u \in C_w(0, T; H^2_0(\Omega))$, $u_t \in C_w(0, T; H^\alpha(\Omega))$ such that

1. For all $\phi \in H^2_0(\Omega)$,

\[
((u_t(t), \phi)) + \alpha((\nabla u_t(t), \nabla \phi)) + \int_0^t ((\Delta u, \Delta \phi)) \, dt
\]

\[
+ \int_0^t \left[ ((g(u_t), \phi)) + \alpha((G(\nabla u_t(t)), \nabla \phi)) \right] \, dt
\]

\[
= ((u_1, \phi)) + \alpha((\nabla u_1, \nabla \phi)) + \int_0^t ((P(u) + [F(u), u], \phi)) \, dt.
\]

2. $\lim_{t \to 0} (u(t) - u_0, \phi)_{2, \Omega} = 0$.

Here as above $C_w(0, T; Y)$ denotes a space of weakly continuous functions with values in a Banach space $Y$.

Theorem 3.8 Under the assumption 3.5 for all initial data

\[
w(0) = w_0 \in H^2_0(\Omega), \quad w_1(0) = w_1 \in H^\alpha(\Omega)
\]

(with $H^\alpha(\Omega)$ defined by (3.18)) there exist a unique, local (in time) generalized (semigroup) solution of finite energy, i.e., there exists $T > 0$ such that

\[
w \in C([0, T]; H^2_0(\Omega)), w_t \in C([0, T]; H^\alpha(\Omega)).
\]

Moreover,

- If, in addition in the case $\alpha = 0$ the damping $g(s)$ is of some polynomial growth, generalized solution becomes weak solution. A weak solution is also continuously dependent of the initial data.

- The solutions are global provided Assumption 3.6 holds. In this case a bound for the energy of solution is independent of time horizon.
Sketch of the proof

We concentrate on a more challenging case when \( \alpha = 0 \). The case \( \alpha > 0 \) can be found in [CL10, Chapter 3].

For the proof of this result the support of the damping described by \( a(x) \) plays no role. Thus, without loss of generality we may assume that \( a(x) = 1 \).

The following two lemmas describing properties of von Karman brackets can be found in [CL10]. The first Lemma is critical for the proof of well-posedness and the second Lemma implies boundedness of solutions that is independent on time horizon.

**Lemma 3.9** Let \( \Delta^{-2} \) denotes the map defined by \( z \equiv \Delta^{-2} f \) iff

\[
\Delta^2 z = f \text{ in } \Omega \text{ and } f = \frac{\partial}{\partial n} f = 0 \text{ on } \Gamma.
\]

Then

\[
\| \Delta^{-2}[u,v] \|_{W^{2,\infty}(\Omega)} \leq C \|u\|_{2,\Omega} \|v\|_{2,\Omega},
\]

where the von Karman bracket \([u,v]\) is defined by (2.7). In particular, for the Airy stress function \( F \) we have the estimate \( \|F(u)\|_{W^{2,\infty}(\Omega)} \leq C \|w\|_{2,\Omega}^2 \).

Notice that standard result [Lio69] gives

\[
\|\Delta^{-2}[u,v]\|_{3-\epsilon,\Omega} \leq C \|u\|_{2,\Omega} \|v\|_{2,\Omega}
\]

which does not imply the result stated in the lemma.

The second lemma [CL10] has to do with a control of low frequencies by nonlinear term.

**Lemma 3.10** Let \( u \in H^2(\Omega) \cap H^1_0(\Omega) \). Then for every \( \epsilon > 0 \) there exists \( M_\epsilon > 0 \) such that

\[
\|u\|^2 \leq \epsilon \left( \|\Delta u\|^2 + \|\Delta F(u)\|^2 \right) + M_\epsilon
\]

Equipped with these two lemma, the proof of global well-posedness follows standard by now procedure:

**Step 1:** Establish maximal monotonicity of the operator

\[
A \equiv \begin{pmatrix}
0 & -I \\
\Delta^2 & g(\cdot)
\end{pmatrix}
\]

on the space \( H \equiv H^2_0(\Omega) \times L^2(\Omega) \) with

\[
D(A) = \{(u;v) \in H^2_0(\Omega) \times L^2(\Omega) : \Delta^2 u + g(v) \in L^2(\Omega)\}
\]

This follows from a standard argument in monotone operator theory [Bar76, Sho97].
Step 2: Consider nonlinear term as a perturbation of $A$:

$$F(W) = \left( \begin{array}{c} 0 \\ [F(w), w] + P(w) \end{array} \right)$$

Step 3: Show that $F(W)$ is locally Lipschitz on $H$. This can be done with the help of Lemma 3.9 which implies the following estimate

$$||[F(u), u] - [F(w), w]|| \leq C(1 + \|u\|_{2, \Omega}^2 + \|w\|_{2, \Omega}^2) \|u - w\|_{2, \Omega}$$

(3.20)

The above steps lead to well-defined semigroup solutions, defined locally in time, which are unique and satisfy local Hadamard well-posedness.

The final step is global well-posedness for which a priori bounds are handy. The needed a priori bound results from the property of von Karman bracket described in Lemma 3.10. Indeed, in order to claim global well-posedness, it suffices to notice that the relation

$$(([F(w), w] + P(w), w_t)) = -\frac{d}{dt} \left[ \frac{1}{2}|\Delta F(w)|^2 + \Pi_0(w) \right] + ((P_1(w), w_t))$$

implies the a priori bounds for the energy function $\mathcal{E}(t)$ given by (2.9) with $\alpha = 0$ via energy inequality and Gronwall’s lemma.

**Stationary solutions.** For description of the structure of the global attractor we also need consider stationary solutions to problem (2.4). We define a stationary weak solution as a function $w \in H^2_0(\Omega)$ satisfying in variational sense the following equations

$$\Delta^2 w = [F(w), w] + P(w) \text{ in } \Omega, \quad w = \frac{\partial}{\partial n} w = 0 \text{ on } \Gamma,$$

(3.21)

where the Airy stress function $F(w)$ solves the elliptic problem (2.6).

**Proposition 3.11** Let Assumption 3.6 be in force. In addition assume that $P$ is weakly continuous mapping from $H^2_0(\Omega)$ into $H^{-2}(\Omega)$. Then problem (3.21) has a solution. Moreover, the set $\mathcal{N}_*$ of all stationary solutions is compact in $H^2_0(\Omega)$.

For the proof of this proposition we refer to [CR80], see also [CL10, Cia00]. We note problem (3.21) may have several solutions [CR80]. However in a generic situation the set $\mathcal{N}_*$ is finite (see a discussion in [CL10, Chapter 1]).

**Generalizations-Extensions**

1. By assuming more regular initial data one obtains regular solutions. Precise quantitative statement of this is given in [CL10]. Regular solutions are global in time.
2. Related results hold when $\alpha > 0$. The analysis here is simpler and there is no need for sharp results in Lemma 3.9, see [CL10].

3. One can consider other boundary conditions such as simply supported or free or combination thereof, [CL10].

4. Damping can be partially supported in $\Omega$. This has no effect on the arguments.

5. A related wellposedness result holds for non-conservative models with additional energy level terms that are non-conservative (for instance $\nabla w \cdot \Psi$ for some smooth vector $\Psi$).

Open question

Full von Karman systems that consist of system of 2D elasticity coupled with von Karman equations. This model accounts for in-plane accelerations. As such, there is no decomposition using Airy’s stress function. The nonlinearities entering are super-critical (even in the rotational case when $\alpha > 0$) and hidden regularity of Airy’s stress function plays no longer any role. For such model with $\alpha > 0$ one can still prove existence and uniqueness of solutions, by appealing to “Sedenko’s method” (see [Sed91]). Even more, full Hadamard well-posedness and energy identity can also be proved in that case [Las98, KL02]. However, the problem is entirely open in the non-rotational case $\alpha = 0$. In this latter case only existence of weak solutions, obtained by Galerkin method, is known.

3.3 Kirchhoff-Boussinesq model with interior damping - model in (2.12)

As before we concentrate on the most demanding case when $\alpha = 0$. For the case $\alpha > 0$ we refer to [CL08a]. The main challenge of this model is the presence of restorative force term $\text{div} \left[ |\nabla w|^2 \nabla w \right]$ which is not in $L^2(\Omega)$ for finite energy solutions $w$. This is due to the failure of Sobolev’s embedding $H^1 \subset L^\infty$ in two dimensions. We also assume, without loss of generality for the well-posedness, that $a(x) \equiv 1$.

The statement of the results

With reference to the model (2.12) with $\alpha = 0$ the following assumption is assumed throughout.

Assumption 3.12

1. $g(s)$ is continuous and monotone on $\mathbb{R}$ with $g(0) = 0$. In addition $g(s)$ is of polynomial growth at infinity and also $a(x) \equiv 1$ (without loss of generality for the well-posedness).

2. The source $P(w)$ is assumed locally Lipschitz from $H^2(\Omega)$ into $L^2(\Omega)$.

Below we deal with weak solutions.
Definition 3.13 (Weak solution) By a weak solution of (2.12), with \( \alpha = 0 \) and \( a(x) \equiv 1 \), defined on some interval \((0, T)\), with initial data \((u_0; u_1)\) we mean a function \( u \in C_w(0, T; H^2_0(\Omega)) \) such that \( u_t \in C_w(0, T; L^2(\Omega)) \) with \( g(u_t) \in L^1(Q_T) \) and

1. For all \( \phi \in H^2_0(\Omega) \),

\[
\int_\Omega u_t(t)\phi d\Omega + \int_0^t \int_\Omega (\Delta u \Delta \phi) d\Omega dt + \int_0^t \int_\Omega g(u_t) \phi d\Omega dt = \int_\Omega u_1 \phi d\Omega + \int_0^t \int_\Omega \left[ P(u)\phi - |\nabla w|^2(\nabla w, \nabla \phi) \right] d\Omega dt \tag{3.22}
\]

2. \( \lim_{t \to 0} (u(t) - u_0, \phi)_{2,\Omega} = 0 \).

Theorem 3.14 Let \( \alpha = 0 \). Under the assumption 3.12 for all initial data \( w(0) = w_0 \in H^2_0(\Omega), \; w_t(0) = w_1 \in L^2(\Omega) \) there exist a local (in time) weak solution of finite energy. This is to say: there exists \( T > 0 \) such that

\[
w \in C_w([0, T]; H^2_0(\Omega)), \; w_t \in C_w([0, T]; L^2(\Omega)). \tag{3.23}
\]

Moreover

- Under additional assumption that \( g(s) \) is linear the said solution is unique. In addition energy identity is satisfied for all weak solutions which are also Hadamard well-posed (locally).

- The solutions are global under the following dissipativity condition: for every \( \delta > 0 \) there exists \( C_\delta > 0 \) such that

\[
\int_0^t ((P(w(s)), w_t(s))) ds \leq \delta E(t) + C_\delta \left[ E(0) + \int_0^t E(s) ds \right] \tag{3.24}
\]

for any function \( w(t) \) possessing the properties in (3.23).

Remark 3.3 A specific choice of interest in applications is Boussinesq source given by \( P(w) = \Delta [w^2] \). This source complies with all the hypotheses stated. Indeed,

\[
((\Delta w^2, w_t)) = ((\Delta w, \frac{d}{dt} w^2)) + 2(\langle |\nabla w|^2, w_t \rangle)
\]

\[
= \frac{d}{dt}((\Delta w, w^2)) - ((\Delta w_t, w^2)) + 2(\langle |\nabla w|^2, w_t \rangle),
\]

which then gives

\[
((\Delta w^2, w_t)) = -\frac{d}{dt}(\langle |\nabla w|^2, w \rangle) + \langle |\nabla w|^2, w_t \rangle). \tag{3.25}
\]
Since
\[ \|w(t)\|^2 \leq \|w(0)\|^2 + 2 \int_0^t \|w(\tau)\|\|w_t(\tau)\|d\tau \leq C \left[ E(0) + \int_0^t E(\tau)d\tau \right], \]
relation (3.25) quickly implies the conclusion desired in (3.24).

### 3.4 Sketch of the proof

As mentioned before the challenge in the proof of Theorem 3.14 is to handle the lack of local Lipschitz condition satisfied by the restorative force.

We explain the main steps. Full details are given in [CL11, CL06b]

**Step 1 (Existence):** It follows via Faedo-Galerkin method. This step is standard.

**Step 2 (Uniqueness):** For this we use Sedenko’s method which based on writing the difference of two solutions in a split form as projection and coprojection on some finite dimensional space and estimating.

We introduce the operator $A$ in $L^2(\Omega)$ by the formula $A u = \Delta^2 u$ with the domain $D(A) = H^4(\Omega) \cap H^2_0(\Omega)$. The operator $A$ is a strictly positive self-adjoint operator with the compact resolvent. Let $\{e_k\}$ be the orthonormal basis in $L^2(\Omega)$ of eigenvectors of the operator $A$ and $\{\lambda_k\}$ be the corresponding eigenvalues:

\[ A e_k = \lambda_k e_k, \quad k = 1, 2, \ldots; \quad 0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \]

Moreover, the corresponding Sobolev norms are equivalent to the graph norms of the corresponding fractional powers of $A$, i.e.

\[ c_1 \|A^s u\| \leq \|u\|_{4s} \leq c_2 \|A^s u\|, \quad u \in D(A^s), \quad (3.26) \]

for all admissible $s \in [0, 1/2]$.

The following assertion is critical for the proof.

**Lemma 3.15 ([BC98])** Let $P_N$ be the projector in $L^2(\Omega)$ onto the space spanned by $\{e_1, e_2, \ldots, e_N\}$ and $f(x) \in D(A^{1/4})$. Then there exists $N_0 > 0$ such that

\[ \max_{x \in \Omega} |(P_N f)(x)| \leq C \cdot \{\log(1 + \lambda_N)\}^{1/2} \| f \|_1 \quad (3.27) \]

for all $N \geq N_0$. The constant $C$ does not depend on $N$.

**Remark 3.4** For the first time a relation similar to (3.27) was used for uniqueness in some shell models in [Sed91]. Latter the same method was applied for coupled 2D Schrödinger and wave equations [CS05, CS11], for the inertial 2D Cahn-Hilliard equation [GSZ09] and also for some models of fluid-shell interaction [CR12]. One can show (see [CL10, Appendix A] and the discussion therein) that (3.27) is equivalent to some Brézis–Gallouet [BG80] type inequality.
Assume that \( \alpha = 0 \), \( \alpha(x) \equiv 1 \) and \( g(s) = ks \) in (2.12). Let \( w_1(t) \) and \( w_2(t) \) be two weak solutions of the original problem (2.12) with the same initial data and \( w(t) = w_1(t) - w_2(t) \). Then \( w_N(t) = P_Nw(t) \) is a solution of the linear, but nonhomogenous problem
\[
\dot{w} + kw + \Delta^2 w = (P_N M)(t), \quad x \in \Omega, t > 0, \tag{3.28}
\]
with the boundary and initial conditions
\[
\frac{\partial w}{\partial n} = 0, \quad w(x, 0) = 0, \quad \partial_t w(x, 0) = 0.
\]
Here
\[
M(t) = \text{div} \left[ |\nabla w_1(t)|^2 \nabla w_1(t) - |\nabla w_2(t)|^2 \nabla w_2(t) \right] + [P(w_1(t)) - P(w_2(t))].
\]
Multiplying the equation (3.28) by \( A^{-1/2}w_t \) and integrating we obtain that
\[
||A^{-1/4}P_Nw_t||^2 + ||A^{1/4}P_Nw(t)||^2 \leq C \int_0^t ||A^{-1/4}M(\tau)||||A^{-1/4}w_t(\tau)||d\tau
\]
for all \( t \in [0, T] \). From here after accounting for (3.26) we obtain
\[
||w(t)||^2_{-1, \Omega} + ||w(t)||^2_{1, \Omega} \leq C \int_0^t ||M(\tau)||||w(t)||_{-1, \Omega}d\tau.
\]
In particular this implies that
\[
||w(t)||_{1, \Omega} \leq C \int_0^t ||M(\tau)||_{-1, \Omega}d\tau. \tag{3.29}
\]
The inequality above is the basis for further estimates. Below we also use the estimate which follows from the definition of weak solutions:
\[
\sup_{t \in [0, T]} \{||w_1(t)||_{2, \Omega} + ||w_2(t)||_{2, \Omega}\} \leq R, \tag{3.30}
\]
where \( R > 0 \) is a constant. Using Lemma 3.15 we can estimate the quantity \( ||M(t)||_{-1} \) in the following way:
\[
||M(t)||_{-1} \leq C_1 \cdot \log(1 + \lambda_N) \cdot ||w(t)||_{1} + C_2 \cdot \lambda_N^{-s/4} \tag{3.31}
\]
for some \( 0 < s < 1 \) and with the constants \( C_1 \) and \( C_2 \) depending on \( R \) from (3.30) (for details we refer to [CL06b]). Therefore it follows from (3.29) and (3.31) that \( \psi(t) = ||w(t)||_1 \) satisfies the inequality
\[
\psi(t) \leq C_1 \cdot \log(1 + \lambda_N) \int_0^t \psi(\tau)d\tau + C_2 \cdot T \cdot \lambda_N^{-s/4}, \quad t \in [0, T],
\]
for some \( 0 < s < 1 \). Thus using Gronwall's lemma we conclude that
\[
\psi(t) \leq C_2 \cdot T \cdot \lambda_N^{-s/4} \cdot (1 + \lambda_N)^C_1 t, \quad t \in [0, T].
\]
If we let $N \to \infty$, then for $0 \leq t < t_0 \equiv s \cdot (4C_1)^{-1}$ we obtain $\psi \equiv 0$. Thus $w_1(t) \equiv w_2(t)$ for $0 \leq t < t_0$. Now we can reiterate the procedure in order to conclude that $w_1(t) \equiv w_2(t)$ for all $0 \leq t \leq T$, where $T$ is the time of existence. This completes the proof of uniqueness.

**Step 3 (Energy identity):** Energy inequality relies on a standard weak lower-semicontinuity argument. Since the system is *time reversible*, one obtains energy inequality for the backward problem. Combining the two inequalities: forward and backward, leads to energy *identity* (2.10) with $E$ given by (2.13) satisfied by *weak solutions*. Details are given in [CL11].

**Step 4:** Equipped with uniqueness and energy identity standard argument furnishes Hadamard well-posedness.

**Step 5:** The last step is to extend local (in time) solutions to the global ones. This is based on a priori bounds postulated by nonlinearities in (3.24).

**Generalizations-Extensions**

1. Higher regularity of solutions can be proved by assuming more regular initial data. Quantitative statements are given in [CL11].

2. Rotational models, when $\alpha > 0$, can be considered without extra difficulty. In fact, in this subcritical case one obtains full Hadamard wellposedness also in a presence of nonlinear damping subject to the same assumptions as in the von Karman case. Some details can be found in [CL08a].

3. We can also consider different boundary conditions such as hinged and free. Free boundary conditions are most challenging due to intrinsic nonlinearity on the boundary, [CL06b].

4. More general structures of restoring forces can be also considered, see [CL06a, CL06b, CL08a, CL11].

5. The support of the damping may be localized to a small (or even empty) subset of $\Omega_0 \subset \Omega$. This will not affect a finite time behavior of solutions.

**Open Problem:** Uniqueness of weak solutions with *nonlinear* damping.

**3.5 Wave equation with boundary source and damping - model in (2.14)**

With reference to the model (2.14) and (2.15), where, for simplicity, we take $g \equiv 0$ and $f \equiv 0$, the following assumption is assumed throughout.

**Assumption 3.16** 1. *Scalar function* $g_0(s)$ is assumed to be continuous and monotone on $\mathbb{R}$ with $g_0(0) = 0$. 

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2. The source \( h(s) \) is represented by a \( C^2 \) function such that \( |h''(s)| \leq C(1 + |s|^{k-2}) \), where \( 2 \leq k < 4 \), and the following growth condition is imposed on the damping \( g_0(s) \) with the constants \( m_{g_0}, M_{g_0} > 0 \):

\[
m_{g_0}|s|^{q+1} \leq g_0(s)s \leq M_{g_0}|s|^{q+1}, \quad |s| \geq 1, \quad \text{with} \quad q \geq \frac{k}{4-k} \geq 1. \quad (3.32)
\]

When the damping is sublinear i.e., \( q \in (0, 1) \), then the source is required to satisfy
\[
|h'(s)| \leq C(1 + |s|^{k-1}), \quad 1 \leq k < 4
\]
and the condition in (3.32) should be satisfied for this \( q \in (0, 1) \).

**Definition 3.17 (Weak solution)** By weak solution of problem (2.14) and (2.15) with \( g \equiv 0 \) and \( f \equiv 0 \), defined on some interval \((0, T)\) with initial data \((u_0; u_1)\), we mean a function \( u \in C_w(0, T; H^1(\Omega)) \) such that

1. \( u_t \in C_w(0, T; L^2(\Omega)) \) and \( u_t \in L^{q+1}(\Sigma_T) \), where \( \Sigma_T = [0, T] \times \Gamma \),

2. For all \( \phi \in C(0, T, H^1(\Omega)) \cap C^1(0, T; L^2(\Omega)) \cap L^{q+1}(\Sigma_T) \),

\[
\int_0^T \int_\Omega (-u_t \phi_t + \nabla u \nabla \phi) \, d\Omega dt + \int_0^T \int_{\Gamma} g_0(u_t) \phi \, d\Omega dt = - \int_\Omega u_t \phi d\Omega \bigg|_0^T + \int_0^T \int_{\Gamma} h(u) \phi \, d\Omega dt. \quad (3.33)
\]

3. \( \lim_{t \to 0} (u(t) - u_0, \phi)_{1, \Omega} = 0 \) and \( \lim_{t \to 0} ((u_t(t) - u_1, \phi)) = 0 \) for all \( \phi \in H^1(\Omega) \).

**Theorem 3.18** Let \( f \equiv 0 \), \( g \equiv 0 \) and Assumption 3.16 be in force. Let initial data \((w_0; w_1)\) be such that

\[
w(0) = w_0 \in H^1(\Omega), \quad w_t(0) = w_1 \in L^2(\Omega),
\]

and also \( w_0|_{\Gamma} \in L_{2k-2}(\Gamma) \) when \( k > 3 \). Then there exist a unique, local (in time) weak solution of finite energy. This is to say: there exists \( T > 0 \) such that

\[
w \in C([0, T]; H^1(\Omega)), \quad w_t \in C([0, T]; L^2(\Omega)).
\]

Moreover,

- When \( k \leq 3 \) the energy identity holds for weak solutions and weak solution is continuously dependent on the initial data.
- When \( k \leq q \) the obtained solutions are global, i.e., \( T = \infty \). The same holds under dissipativity condition: \( -h(s)s \geq 0 \).
- When \( 1 < k \leq 3, \, k > q \) and \( h(s) = |s|^{k-1} \) local solution blows up in a finite time for negative energy initial data.
Remark 3.5 In the case when \( h(s) = \alpha s \), no growth conditions imposed on \( g_0 \) are required. In fact, in that case the obtained solution is semigroup generalized solution.

When \( |h''(s)| \leq c \) variational form of the solution can be obtained with more relaxed hypotheses imposed on high frequencies of the damping. For instance, we can assume

\[
\lim \inf_{|s| \to \infty} \frac{g_0(s)}{s} > 0, \quad |g_0(s)| \leq c[1 + |s|^3]
\]  

(3.34)

Under the above condition one can show that the generalized solution satisfies also variational form (3.33) with the test functions \( \phi \in C(0, T; H^1(\Omega)) \cap C^1(0, T; L_2(\Omega)) \cap C(\Sigma_T) \) \[CL07a\]

Reference to the proofs

Notice that the damping \( g_0 \) is assumed active for all values of the parameter \( k > 1 \). This is unlike interior source where only supercritical values of the source require presence of the damping. In the boundary case, however, Lopatinski condition is not satisfied for the Neumann problem and this necessitates the presence of the damping which, in some sense, forces Lopatinski condition [Sak70]. This is manifested by the fact that \( H^1(Q) \) solutions of wave equation with Neumann boundary conditions do not possess \( H^1(\Sigma) \) boundary regularity (unlike Dirichlet solutions). It is the presence of the damping which, in some sense, recovers certain amount of boundary regularity [LT00, Vit02a].

The proof of Theorem 3.18 follows similar conceptual lines as in the interior case. There are however few subtle differences due to the presence of undefined traces in the equation. In the case when \( |h''(s)| \leq C \) and \( m s^2 \leq g_0(s) \leq M s^4 \) the proof is given in [CL07a]. For the remaining values of the parameters Hadamard wellposedness is proved in [BL08a, BL10]. The analysis of the boundary case is more demanding than in the interior case. The corresponding differences are clearly exposed in the structure of a counterpart to Lemma 3.4, which in the boundary case has a different form given in Lemma 4.2 in [BL08a].

The last statement in Theorem 3.18 regarding finite time blow up of energy is proved in [BL08b].

Generalizations-Extensions

1. Additional regularity of solutions corresponding to more regular initial conditions when \( q \leq 3 \) and \( |h''(s)| \leq C \) is given in [CL07a].

2. One could obtain a version of Theorem 3.18 that incorporates both dämpings: internal and boundary. This is done in [BL08a, BL10].

3. Well-posedness theory for small data taken from potential well \( (k \leq 3) \) is also available [BRT11, Vit02b].

Open questions:
1. Well-posedness of finite energy solutions theory without the boundary damping when \( k \leq 2 \) (so that \( h(u) \in L_2(\Gamma) \) for finite energy solutions).

2. Interplay between boundary and interior damping. Is it possible to show well-posedness of finite energy solutions for supercritical case \( p > 3 \) with boundary damping only (i.e. \( g = 0 \))?

3. The same question asked for finite time blow results. Does boundary source alone lead to blow up of energy in the presence of a boundary damping when \( q < k \)?

4. Boundedness of solutions when time goes to infinity and sources are generating energy. Quantitative description of the behavior of global solutions - when \( p \leq m \) and \( k \leq q \).

### 3.6 Von Karman equation with boundary damping - model (2.18)

With reference to the model (2.18) and (2.19) the following assumption is assumed throughout.

**Assumption 3.19**

1. Scalar function \( g(s) \) and \( g_0(s) \) are assumed to be continuous and monotone on \( \mathbb{R} \) with 0 at 0. The rotational damping \( G \) (the case \( \alpha > 0 \)) satisfies Assumption 3.5(2).

2. The source \( P(w) \) is assumed locally Lipschitz from \( H^2(\Omega) \) into \( H^{-1}(\Omega) \) when \( \alpha > 0 \) and from \( H^2(\Omega) \) into \( L_2(\Omega) \) when \( \alpha = 0 \) (see Remark 3.2 concerning a specific choice of interest in applications).

We concentrate on a more challenging case when \( \alpha = 0 \). In addition to the concept of generalized (semigroup) solutions we also define weak solutions.

**Definition 3.20 (Weak solution)** By a *weak solution* of (2.18) and (2.19) with \( \alpha = 0 \) and with initial data \((u_0; u_1)\), defined on some interval \((0, T)\), we mean a function \( u \in C_w(0, T; H^2(\Omega) \cap H^1_0(\Omega)) \) such that \( u_t \in L_2(\Omega) \cap H^1_0(\Omega) \) and

\[
\int_0^t \int_\Omega (\Delta u \Delta \phi) \, d\Omega d\tau + \int_0^t \int_\Omega g(u_t) \phi \, d\Omega d\tau + \int_0^t \int_\Gamma g_0(\partial_n u_t) \frac{\partial}{\partial n} \phi d\Gamma d\tau
= -\int_\Omega (u_t(t) - u_1) \phi \, d\Omega + \int_0^t \int_\Omega (P(u) + [F(u), u]) \phi \, d\Omega d\tau. \tag{3.35}
\]

1. For all \( \phi \in H^2(\Omega) \cap H^1_0(\Omega) \),

\[
\int_0^t \int_\Omega (\Delta u \Delta \phi) \, d\Omega d\tau + \int_0^t \int_\Omega g(u_t) \phi \, d\Omega d\tau + \int_0^t \int_\Gamma g_0(\partial_n u_t) \frac{\partial}{\partial n} \phi d\Gamma d\tau
= -\int_\Omega (u_t(t) - u_1) \phi \, d\Omega + \int_0^t \int_\Omega (P(u) + [F(u), u]) \phi \, d\Omega d\tau. \tag{3.35}
\]

2. \( \lim_{t \to 0} (u(t) - u_0, \phi)_{2,\Omega} = 0 \) and \( \lim_{t \to 0} ((u_t(t) - u_1, \phi) = 0. \)
Theorem 3.21 Let \( \alpha = 0 \). Under Assumption 3.19 for all initial data

\[
    w(0) = w_0 \in H^2(\Omega) \cap H^1_0(\Omega), \quad w_t(0) = w_1 \in L_2(\Omega)
\]

there exist a unique, local (in time) generalized (semigroup) solution \( w \) of finite energy, i.e.,

\[
    \exists T > 0 : \quad w \in C([0, T]; H^2(\Omega)), \quad w_t \in C([0, T]; L_2(\Omega)).
\]

Moreover

- If, in addition, the damping \( g(s) \) and \( g_0(s) \) are of some polynomial growth and \( (g_0(x) - g_0(y))(x - y) \geq c|x - y|^r \) for some \( r \geq 1 \), then a generalized solution becomes also weak solution. Weak solution is also continuously dependent on the initial data.

- The solutions are global under Assumption 3.6 concerning \( P(w) \) with the energy bound independent of the time horizon.

Reference to the proof

Well-posedness of Von Karman plates with the boundary damping occurring in the moments have been proved in [CL10]. See also [JL99, HL95].

3.7 Generalizations

1. Additional regularity of finite-time solutions for more regular initial data, [CL10, HL95].

2. Similar problems can be formulated for plate equations with the damping acting in shears and torques [Lag89]. Thus the boundary conditions under consideration are "free" with the feedback control given by \( g_0(w_t) \) acting on the highest third order boundary conditions [CL04b, CL07b].

3. Combination of interior and boundary damping can also be considered by combining the methods.

4. Model with rotational inertial forces \( \alpha > 0 \) subject to boundary damping, [CL10, HL95].

5. Regular solutions for infinite time interval can be obtained [CL10].

Open Problem: Are polynomial bounds imposed on \( g(s) \) and \( g_0(s) \) and strong coercivity condition imposed on \( g_0 \) necessary for obtaining weak solutions?
3.8 Kirchhoff-Boussinesq equation with boundary damping - model (2.21)

Rotational case $\alpha > 0$

In the case when rotational inertia are retained in the model, $\alpha > 0$, the nonlinear terms are subcritical and the well-posedness theory can be carried out along the same lines as in the case of von Karman equations. For instance, when considered (2.21) with $g = 0$, $G = 0$ and the following boundary damping

$$w = 0, \quad \Delta w = -g_0(\frac{\partial}{\partial n} w_t) \quad \text{on} \quad \Sigma,$$

where $g_0(s)$ is continuous and monotone, full Hadamard local well-posedness can be established.

Theorem 3.22 Under the assumption 3.19 with $g = 0$ and $\alpha > 0$ for all initial data

$$w(0) = w_0 \in H^2(\Omega) \cap H^1_0(\Omega), \quad w_t(0) = w_1 \in H^1_0(\Omega)$$

there exist a unique, local (in time) generalized (semigroup) solution $w$ of finite energy, i.e.,

$$\exists T > 0 : \quad w \in C([0,T]; H^2(\Omega) \cap H^1_0(\Omega)), \quad w_t \in C([0,T]; H^1_0(\Omega)).$$

Moreover

- If, in addition, the damping $g_0(s)$ are of some polynomial growth and

  $$(g_0(x) - g_0(y))(x - y) \geq c|x - y|^r \quad \text{for some} \quad r \geq 1,$$

  then a generalized solution becomes also weak solution. Weak solution is also continuously dependent on the initial data.

- The solutions are global under the dissipativity hypothesis in (3.24)

The proof of this theorem is along the same lines as in the case of von Karman equations with boundary damping.

Non-rotational case: $\alpha = 0$

As we have seen already before, this case is much more delicate even in the case of interior damping. The nonlinear source term is supercritical. While this difficulty was overcome in the case of linear interior damping, the presence of boundary damping brings new set of issues. This is both, at the level of uniqueness and continuous dependence. More specifically, in the case $g \equiv 0$:

- Existence of finite energy solutions

  $$w \in L^\infty([0,T]; H^2(\Omega) \cap H^1_0(\Omega)), \quad w_t \in L^\infty([0,T]; L_2(\Omega))$$

  defined variationally and locally in time with initial data

  $$w(0) = w_0 \in H^2(\Omega) \cap H^1_0(\Omega), \quad w_t(0) = w_1 \in L_2(\Omega).$$
can be proved by Galerkin method with a boundary damping $g_0(s)$ of some polynomial growth and such that $(g_0(x) - g_0(y))(x - y) \geq c|x - y|^r$ for some $r \geq 1$.

- Uniqueness of solutions can be established only in the case of linear damping, i.e., for $g_0(s) = bs$. This can be done by adapting Sedenko’s method [Sed91] as in [Las98, KL02] where full Von Karman system was considered. The latter displays similar difficulties when dealing with well-posedness.

- Continuous dependence on the data can be proved only in a weak topology. The difficulty lies in the fact that time reversibility of the flow is lost with the presence of boundary damping. This latter property was critical in proving energy identity for weak solutions in the case of internal linear damping.

Generalizations-Extensions

1. Existence and uniqueness of regular solutions in $H^4(\Omega) \times H^2(\Omega)$ (when $\alpha = 0$). This can be accomplished by taking advantage of higher topologies for the state, hence avoiding the problem of supercriticality. This method has been pursued in [CL11] in the case of interior damping. However, the arguments are applicable to the case of boundary damping as well.

2. Other types of boundary damping -such as occurring in “free” [CL10] boundary conditions can also be considered. In the case rotational case, well-posedness results are complete - in line with Theorem 3.22. In the non-rotational case, comments presented above apply to this case as well (with the same limitations regarding linearity of the damping). There is however an additional difficulty resulting from the fact that free boundary conditions provide intrinsically nonlinear contribution on the boundary. This, however, is of lower order and can be handled as in [CL06b, CL11]

3. Combination of internal damping and boundary damping can be treated by combining the results available for each case separately.

4. Sources that are nonconservative also can be considered, but these are outside the scope of these lectures, see [CL10].

Open Problems:

1. Hadamard well-posedness with linear boundary damping in the case $\alpha = 0$. Due to the loss of time reversibility, the arguments used for the internal damping are no longer applicable.

2. Ultimately: uniqueness and Hadamard well-posedness in the presence of nonlinear damping $g_0$ in the case $\alpha = 0$ and boundary conditions (3.36), where $g_0(s)$ is monotone and -say- of linear growth at infinity. The arguments used so far for the uniqueness of weak solutions rely critically on linearity of the damping. Thus, the case described above appears to be completely open.
4 General tools for studying attractors

In this section we describe several approaches to the study of long-time behavior of hyperbolic-like systems described above. For the general discussion of long-time behavior of systems with dissipation we refer to the monographs [BV92, Chu99, Hal88, Lad91, SY02, Tem88].

4.1 Basic notions

By definition a dynamical system is a pair of objects \((X, S_t)\) consisting of a complete metric space \(X\) and a family of continuous mappings \(\{S_t : t \in \mathbb{R}_+\}\) of \(X\) into itself with the semigroup properties:

\[ S_0 = I, \quad S_{t+\tau} = S_t \circ S_\tau. \]

We also assume that \(y(t) = S_{t}y_0\) is continuous with respect to \(t\) for any \(y_0 \in X\). Therewith \(X\) is called a phase space (or state space) and \(S_t\) is called an evolution semigroup (or evolution operator).

**Definition 4.1** Let \((X, S_t)\) be a dynamical system.

- A closed set \(B \subset X\) is said to be **absorbing** for \((X, S_t)\) iff for any bounded set \(D \subset X\) there exists \(t_0(D)\) such that \(S_tD \subset B\) for all \(t \geq t_0(D)\).
- \((X, S_t)\) is said to be (bounded, or ultimately) **dissipative** iff it possesses a bounded absorbing set \(B\). If \(X\) is a Banach space, then a value \(R > 0\) is said to be a radius of dissipativity of \((X, S_t)\) if \(B \subset \{x \in X : \|x\|_X \leq R\}\).
- \((X, S_t)\) is said to be **asymptotically smooth** iff for any bounded set \(D\) such that \(S_tD \subset D\) for \(t > 0\), there exists a compact set \(K\) in the closure \(\overline{D}\) of \(D\), such that

\[
\lim_{t \to +\infty} d_X \{S_tD \mid K\} = 0, \quad (4.1)
\]

where \(d_X \{A \mid B\} = \sup_{x \in A} \text{dist}_X(x, B)\).

A set \(D \subset X\) is said to be **forward** (or positively) **invariant** iff \(S_tD \subset D\) for all \(t \geq 0\). It is **backward** (or negatively) **invariant** iff \(S_tD \supset D\) for all \(t \geq 0\). The set \(D\) is said to be **invariant** iff it is both forward and backward invariant; that is, \(S_tD = D\) for all \(t \geq 0\).

Let \(D \subset X\). The set

\[
\gamma^t_D \equiv \bigcup_{\tau \geq t} S_{\tau}D
\]

is called the **tail** (from the moment \(t\)) of the trajectories emanating from \(D\). It is clear that \(\gamma^t_D = \gamma^0_{S_tD}\).

If \(D = \{v\}\) is a single point set, then \(\gamma^+_v \equiv \gamma^0_D\) is said to be a **positive semitrajectory** (or semiorbit) emanating from \(v\). A continuous curve \(\gamma \equiv \{u(t) : t \in \mathbb{R}\}\) in \(X\) is said to be a **full trajectory** iff \(S_tu(\tau) = u(t + \tau)\) for any \(\tau \in \mathbb{R}\) and \(t \geq 0\). Because \(S_t\) is not necessarily an invertible operator, a full trajectory
may not exist. Semitrajectories are forward invariant sets. Full trajectories are invariant sets.

To describe the asymptotic behavior we use the concept of an $\omega$-limit set. The set

$$\omega(D) \equiv \bigcap_{t>0} \overline{\bigcup_{\tau \geq t} S_\tau D}$$

is called the $\omega$-limit set of the trajectories emanating from $D$ (the bar over a set means the closure). It is equivalent to saying that $x \in \omega(D)$ if and only if there exist sequences $t_n \to +\infty$ and $x_n \in D$ such that $S_{t_n} x_n \to x$ as $n \to \infty$.

It is clear that $\omega$-limit sets (if they exist) are forward invariant.

### 4.2 Criteria for Asymptotic Smoothness

The following assertion is a generalization Ceron-Lopes criteria (see [Hal88] and the references therein).

**Theorem 4.2** Let $(X, S_t)$ be a dynamical system on a Banach space $X$. Assume that for any bounded positively invariant set $B$ in $X$ there exist $T > 0$, a continuous nondecreasing function $g : \mathbb{R}_+ \to \mathbb{R}_+$, and a pseudometric $\varrho^T_B$ on $C(0,T;X)$ such that

(i) $g(0) = 0; \ g(s) < s, \ s > 0$.

(ii) The pseudometric $\varrho^T_B$ is precompact (with respect to the norm of $X$) in the following sense. Any sequence $\{x_n\} \subset B$ has a subsequence $\{x_{n_k}\}$ such that the sequence $\{y_k\} \subset C(0,T;X)$ of elements $y_k(t) = S_{t_n} x_{n_k}$ is Cauchy with respect to $\varrho^T_B$.

(iii) The following estimate

$$\|S_T y_1 - S_T y_2\| \leq g \left( \|y_1 - y_2\| + \varrho^T_B(\{S_{t} y_1\}, \{S_{t} y_2\}) \right)$$

holds for every $y_1, y_2 \in B$, where we denote by $\{S_{t} y_1\}$ the element in the space $C(0,T;X)$ given by function $y_1(t) = S_{t} y_i$.

Then, $(X, S_t)$ is asymptotically smooth dynamical system.

Note that a precompact pseudometric is evaluated on trajectories $S_t$, rather than on initial conditions (as in the classical treatments, see, e.g., [Hal88]). This fact becomes quite useful when applying the criterion to hyperbolic-like dynamics.

**Proof.** We refer to [CL08a]. The main ingredient of the proof is the relation

$$\alpha(S_T B) \leq g(\alpha(B)), \quad (4.2)$$

where $\alpha(B)$ is the Kuratowski $\alpha$-measure of noncompactness which is defined by the formula

$$\alpha(B) = \inf \{ \delta : B \text{ has a finite cover of diameter } < \delta \} \quad (4.3)$$
for every bounded set $B$ of $X$. For properties of this metric characteristic we refer to [Hal88] or [SY02, Lemma 22.2].

The property in (4.2) implies that for every bounded forward invariant set $B$ there exists $T > 0$ such that $\alpha(S_T B) < \alpha(B)$ provided $\alpha(B) > 0$. If for some fixed $T > 0$ this property holds for every bounded set $B$ such that $S_T B$ is also bounded, then, by the definition (see [Hal88]), $S_T$ is a conditional $\alpha$-condensing mapping. It is known [Hal88] that conditional $\alpha$-condensing mappings are asymptotically smooth. Therefore Theorem 4.2 can be considered as a generalization of the results presented in [Hal88].

The above criterion is rather general, however it requires "compactness" of the sources in the equation. There are two other criteria that avoid such a requirement of a priori compactness. These are:

- *compensated compactness* criterion—an idea introduced in [Kha06] and later expanded in [CL08a];
- J. Ball’s energy method, see [Bal04] and [MRW98].

The corresponding results are presented below.

**Compensated compactness method**

**Theorem 4.3** Let $(X, S_t)$ be a dynamical system on a complete metric space $X$ endowed with a metric $d$. Assume that for any bounded positively invariant set $B$ in $X$ and for any $\epsilon > 0$ there exists $T = T(\epsilon, B)$ such that

$$d(S_T y_1, S_T y_2) \leq \epsilon + \Psi_{\epsilon, B, T}(y_1, y_2), y_1 \in B,$$

where $\Psi_{\epsilon, B, T}(y_1, y_2)$ is a functional defined on $B \times B$ such that

$$\lim_{n \to \infty} \lim_{m \to \infty} \Psi_{\epsilon, B, T}(y_n, y_m) = 0$$

for every sequence $\{y_n\}$ from $B$. Then $(X, S_t)$ is an asymptotically smooth dynamical system.

Note that in a "compact" situation, i.e., when the functional $\Psi$ is sequentially compact, the condition (4.5) is automatically satisfied. The above criterion applies in the case of critical nonlinearities.

**Proof.** The properties in (4.4) and (4.5) makes it possible to prove that $\alpha(S_t B) \to 0$ as $t \to +\infty$, where $\alpha(B)$ is the Kuratowski $\alpha$-measure defined by (4.3). The latter property implies the conclusion desired. For details we refer [CL08a] or [CL10].

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John Ball’s "energy" method

This second method applies even in the case of supercritical nonlinear terms, but there are other requirements which restrict applicability of this method. The main idea behind Ball’s method is to construct an appropriate energy type functional which can be then decomposed into exponentially decaying part and compact part. While the idea of decomposition of semigroup into uniformly stable and compact part is behind almost all criteria leading to asymptotic smoothness, the "energy" method described below postulates such a decomposition on functionals rather than operators (semigroups). This fact has far reaching consequences and allows application of the method in supercritical situations.

We follow presentation of the method given in [MRW98]. Another exposition of this method in the case of the damped wave equation can be found in [Bal04].

**Theorem 4.4** Let $S_t$ be an evolution continuous semigroup of weakly and strongly continuous operators. Assume that there exist functionals $\Phi$, $\Psi$, $L$, $K$ on phase space such that the following equality

$$[\Phi(S_t u) + \Psi(S_t u)] + \int_s^t L(S_\tau u) e^{-\omega(t-\tau)} d\tau$$

$$= [\Phi(S_s u) + \Psi(S_s u)] e^{-\omega(t-s)} + \int_s^t K(S_\tau u) e^{-\omega(t-\tau)} d\tau, \quad (4.6)$$

holds for any $u \in X$. The functionals $\Phi$, $\Psi$, $L$, $K$ are assumed to have the following properties:

- $\Phi : X \to \mathbb{R}^+$ is continuous, bounded and if $\{U_j\}_j$ is bounded in $X$, $t_j \to +\infty$, $S_{t_j} U_j \rightharpoonup U$ weakly in $X$, and $\limsup_{n \to \infty} \Phi(S_{t_n} U_j) \leq \Phi(U)$, then $S_{t_j} U_j \rightharpoonup U$ strongly in $X$.

- $\Psi : X \to \mathbb{R}$ is 'asymptotically weakly continuous' in the sense that if $\{U_j\}_j$ is bounded in $X$, $t_j \to +\infty$, $S_{t_j} U_j \rightharpoonup U$ weakly in $X$, then $\Psi(S_{t_j} U_j) \to \Psi(U)$.

- $K : X \to \mathbb{R}$ is 'asymptotically weakly continuous' in the sense that if $\{U_j\}_j$ is bounded in $X$, $t_j \to +\infty$, $S_{t_j} U_j \rightharpoonup U$ weakly in $X$, then $K(S_{s_j} U_j) \in L_1(0,t)$ and

$$\lim_{j \to \infty} \int_0^t e^{-\omega(t-s)} K(S_{s+t_j} U_j) ds = \int_0^t e^{-\omega(t-s)} K(S_s U) ds, \quad \forall t > 0.$$

- $L$ is 'asymptotically weakly lower semicontinuous' in the sense that if $\{U_j\}_j$ is bounded in $X$, $t_j \to +\infty$, $S_{t_j} U_j \rightharpoonup U$ weakly in $X$, then $L(S_{s_j} U) \in L_1(0,t)$ and

$$\liminf_{j \to \infty} \int_0^t e^{-\omega(t-s)} L(S_{s+t_j} U_j) ds \geq \int_0^t e^{-\omega(t-s)} L(S_s U) ds, \quad \forall t > 0.$$
Then $S_t$ is asymptotically smooth.

**Remark 4.1**
1. The decomposition of the functionals in (4.6) depends on validity of energy *identity* satisfied for weak solutions. This, alone, is a severe condition which may be difficult to verify (in contrast to energy *inequality*). In addition, the proof of *equality* in (4.6) hides behind the fact that the damping in the equation is very structured-typically linear.

2. The functional (typically convex) $\Phi$ plays role of the energy of linearized system - a good topological measure for the solution. In uniformly convex spaces $X$ the assumptions postulated by $\Phi$ are automatically satisfied. Indeed, this results from the fact that weak convergence and the convergence of the norms to the same element imply strong convergence.

3. The hypotheses required from $\Psi$, $K$ and $L$ represent some compactness property of part of the nonlinear energy describing the system. This is often the case even for supercritical nonlinearities. In fact, these terms allow to deal with non-compact sources in the equation provided that the corresponding nonlinear part of the energy is sequentially compact. Typical application involves 3D wave equation, where sources $|f(s)| \leq C(1 + |s|^p)$ with $p < 5$ can be handled (in view of compactness of the embedding $H^1(\Omega) \subset L_q(\Omega)$, $q < 6$).

### 4.3 Global attractors

The main objects arising in the analysis of long-time behavior of infinite-dimensional dissipative dynamical systems are attractors. Their study allows us to answer a number of fundamental questions on the properties of limit regimes that can arise in the systems under consideration. At present, there are several general approaches and methods that allow us to prove the existence and finite-dimensionality of global attractors for a large class of dynamical systems generated by nonlinear partial differential equations (see, e.g., [BV92, Chu99, Hal88, Lad91, Tem88] and the references listed therein).

**Definition 4.5** A bounded closed set $A \subset X$ is said to be a global attractor of the dynamical system $(X, S_t)$ iff the following properties hold.

- (i) $A$ is an invariant set; that is, $S_t A = A$ for $t \geq 0$.
- (ii) $A$ is uniformly attracting; that is, for all bounded set $D \subset X$

$\lim_{t \to +\infty} d_X \{S_t D | A\} = 0,$

where $d_X \{A | B\} = \sup_{x \in A} \text{dist}_X(x, B)$ is the Hausdorff semidistance.

It turns out that dissipativity property along with asymptotic smoothness imply an existence of global attractor. The corresponding result is standard by now and reported below (see [Hal88] and also [BV92, Lad91, Tem88]).
Theorem 4.6 Any dissipative asymptotically smooth dynamical system \((X, S_t)\) in a Banach space \(X\) possesses a unique compact global attractor \(A\). This attractor is a connected set and can be described as a set of all bounded full trajectories. Moreover \(A = \omega(B)\) for any bounded absorbing set \(B\) of \((X, S_t)\).

In the case when the dynamical system has special property—referred to as gradient system—dissipativity property is not needed (in the explicit form) in order to prove existence of a global attractor. This is a very handy property particularly when the proof of dissipativity is technically involved.

Gradient systems

The study of the structure of the global attractors is an important problem from the point of view of applications. There are no universal approaches solving this problem. It is well known that even in finite-dimensional cases an attractor can possess extremely complicated structure. However, some sets that belong to the attractor can be easily pointed out. For example, every stationary point \((S_t x = x \forall t > 0)\) belongs to the attractor of the system. One can shows that any bounded full trajectory also lies in the global attractor.

We begin with the following definition.

Definition 4.7 Let \(\mathcal{N}\) be the set of stationary points of the dynamical system \((X, S_t)\):

\[\mathcal{N} = \{v \in X : S_t v = v \text{ for all } t \geq 0\}.\]

We define the unstable manifold \(M^u(\mathcal{N})\) emanating from the set \(\mathcal{N}\) as a set of all \(y \in X\) such that there exists a full trajectory \(\gamma = \{u(t) : t \in \mathbb{R}\}\) with the properties

\[u(0) = y \text{ and } \lim_{t \to -\infty} \text{dist}_X(u(t), \mathcal{N}) = 0.\]

Now we introduce the notions of Lyapunov functions and gradient systems (see, e.g., [BV92, Chu99, Hal88, Lad91, Tem88] and the references therein).

Definition 4.8 Let \(Y \subseteq X\) be a forward invariant set of a dynamical system \((X, S_t)\).

- The continuous functional \(\Phi(y)\) defined on \(Y\) is said to be the Lyapunov function for the dynamical system \((X, S_t)\) on \(Y\) iff the function \(t \mapsto \Phi(S_t y)\) is a nonincreasing function for any \(y \in Y\).

- The Lyapunov function \(\Phi(y)\) is said to be strict on \(Y\) iff the equation \(\Phi(S_t y) = \Phi(y)\) for all \(t > 0\) and for some \(y \in Y\) implies that \(S_t y = y\) for all \(t > 0\); that is, \(y\) is a stationary point of \((X, S_t)\).

- The dynamical system \((X, S_t)\) is said to be gradient iff there exists a strict Lyapunov function for \((X, S_t)\) on the whole phase space \(X\).
A connection between gradient systems and existence of compact attractors is given below. The main result stating existence and properties of attractors for gradient systems is the following theorem (for the proof we refer to [CL08a, CL10] and the references therein; see also [Rau02, Theorem 4.6] for a similar assertion).

Theorem 4.9 Assume that \((X, S_t)\) is a gradient asymptotically smooth dynamical system. Assume its Lyapunov function \(\Phi(x)\) is bounded from above on any bounded subset of \(X\) and the set \(\Phi_R = \{x : \Phi(x) \leq R\}\) is bounded for every \(R\). If the set \(N\) of stationary points of \((X, S_t)\) is bounded, then \((X, S_t)\) possesses a compact global attractor \(A = \mathcal{M}^u(N)\). Moreover,

- The global attractor \(A\) consists of full trajectories \(\gamma = \{u(t) : t \in \mathbb{R}\}\) such that
  \[
  \lim_{t \to -\infty} \text{dist}_X(u(t), N) = 0 \quad \text{and} \quad \lim_{t \to +\infty} \text{dist}_X(u(t), N) = 0.
  \]

- For any \(x \in X\) we have
  \[
  \lim_{t \to +\infty} \text{dist}_X(S_t x, N) = 0,
  \]
  that is, any trajectory stabilizes to the set \(N\) of stationary points. In particular, this means that the global minimal attractor \(A_{\min}\) coincides with the set of the stationary points, \(A_{\min} = N\).

- If \(N = \{z_1, \ldots, z_n\}\) is a finite set, then \(A = \cup_{i=1}^n \mathcal{M}^u(z_i)\), where \(\mathcal{M}^u(z_i)\) is the unstable manifold of the stationary point \(z_i\), and also
  
  (i) The global attractor \(A\) consists of full trajectories \(\gamma = \{u(t) : t \in \mathbb{R}\}\) connecting pairs of stationary points: any \(u \in A\) belongs to some full trajectory \(\gamma\) and for any \(\gamma \subset A\) there exists a pair \(\{z, z^*\} \subset N\) such that
    
    \[
    u(t) \to z \quad \text{as} \quad t \to -\infty \quad \text{and} \quad u(t) \to z^* \quad \text{as} \quad t \to +\infty.
    \]

  (ii) For any \(v \in X\) there exists a stationary point \(z\) such that \(S_t v \to z\) as \(t \to +\infty\).

Dimension of global attractor

Finite-dimensionality is an important property of global attractors that can be established for many dynamical systems, including those arising in significant applications. There are several approaches that provide effective estimates for the dimension of attractors of dynamical systems generated by PDEs (see, e.g., [BV92, Lad91, Tem88]). Here we present an approach that does not require \(C^1\)-smoothness of the evolutionary operator (as in [BV92, Tem88]). The reason for this focus is that dynamical systems of hyperbolic-like nature do not display smoothing effects, unlike parabolic equations. Therefore, the \(C^1\) smoothness of the flows is most often beyond question, particularly in problems with a
nonlinear dissipation. Instead, we present a method which can be applied to more general locally Lipschitz flows. This method generalizes Ladyzhenskaya’s theorem (see, e.g., [Lad91]) on finite dimension of invariant sets. We also refer to [Pra02] for a closely related approach based on some kind of squeezing property. However, we wish to point out that the estimates of the dimension based on the theorem below usually tend to be conservative.

**Definition 4.10** Let $M$ be a compact set in a metric space $X$.

- The **fractal (box-counting) dimension** $\dim_f M$ of $M$ is defined by

$$
\dim_f M = \lim_{\varepsilon \to 0} \sup \frac{\ln n(M, \varepsilon)}{\ln(1/\varepsilon)},
$$

where $n(M, \varepsilon)$ is the minimal number of closed balls of the radius $\varepsilon$ which cover the set $M$.

We can also consider the Hausdorff dimension $\dim_H$ to describe complexity and embeddings properties of compact sets. We do not give a formal definition of this dimension characteristic (see, e.g., [Fal90] for some details and references) and we only note that (i) the Hausdorff dimension does not exceed (but is not equal, in general) the fractal one; (ii) fractal dimension is more convenient in calculations.

The following result which generalizes [Lad91] was proved in [CL04a], see also [CL08a].

**Theorem 4.11** Let $H$ be a separable Banach space and $M$ be a bounded closed set in $H$. Assume that there exists a mapping $V : M \to H$ such that

1. $M \subseteq VM$.
2. $V$ is Lipschitz on $M$; that is, there exists $L > 0$ such that

$$
\|Vv_1 - Vv_2\| \leq L\|v_1 - v_2\|, \quad v_1, v_2 \in M.
$$
3. There exist compact seminorms $n_1(x)$ and $n_2(x)$ on $H$ such that

$$
\|Vv_1 - Vv_2\| \leq \eta\|v_1 - v_2\| + K \cdot [n_1(v_1 - v_2) + n_2(Vv_1 - Vv_2)]
$$

for any $v_1, v_2 \in M$, where $0 < \eta < 1$ and $K > 0$ are constants (a seminorm $n(x)$ on $H$ is said to be compact iff $n(x_m) \to 0$ for any sequence $\{x_m\} \subset H$ such that $x_m \to 0$ weakly in $H$).

Then $M$ is a compact set in $H$ of a finite fractal dimension. Moreover, if $H$ is a Hilbert space and the seminorms $n_1$ and $n_2$ have the form $n_i(v) = \|P_i v\|$, $i = 1, 2$, where $P_1$ and $P_2$ are finite-dimensional orthoprojectors, then

$$
\dim_f M \leq (\dim P_1 + \dim P_2) \cdot \ln \left(1 + \frac{8(1 + L)\sqrt{2K}}{1 - \eta}\right) \cdot \left[\ln \frac{2}{1+\eta}\right]^{-1}.
$$
Remark 4.2 We note that under the hypotheses of Theorem 4.11 the mapping $V$ possesses the property (see Lemma 2.18 in [CL08a])
\[ \alpha(VB) \leq \eta \alpha(B) \quad \text{for any } B \subset M, \]
where $\alpha(B)$ is the Kuratowski $\alpha$-measure given by (4.3). This means that $V$ is $\alpha$-contraction on $M$ (in the terminology of [Hal88]). The note that the latter property is not sufficient for finite-dimensionality of the set $M$. Indeed, let
\[ X = l_2 = \left\{ x = (x_1; x_2; \ldots) : \sum_{i=1}^{\infty} x_i^2 < \infty \right\} \]
and
\[ M = \left\{ x = (x_1; x_2; \ldots) \in l_2 : |x_i| \leq i^{-2}, \ i = 1, 2, \ldots \right\} \]
We define a mapping $V$ in $X$ by the formula
\[ [Vx]_i = f_i(x_i), \quad i = 1, 2, \ldots \]
where $f_i(s) = s$ for $|s| \leq i^{-2}$, $f_i(s) = i^{-2}$ for $s \geq i^{-2}$, and $f_i(s) = -i^{-2}$ for $s \leq -i^{-2}$. One can see that $V$ is globally Lipschitz in $X$ and $VX = M = VM$.
Since $M$ is a compact set, the mapping is $\alpha$-contraction (with $\eta = 0$). On the other hand it is clear that $\dim_f M = \infty$.
We also note that this example means that the statement of Theorem 2.8.1 in [Hal88] is not true and thus it requires some additional hypotheses concerning the mapping.

In what follows we shall present a unified criterion which allows to prove both finite-dimensionality and smoothness of attractors. This criterion involves a special class of systems that we call quasi-stable.

### 4.4 Quasi-stable systems

In this section following the presentation given in [CL10, Section 7.9] we introduce a class of dissipative dynamical systems which display rather special long time behavior dynamic properties. This class will be referred to as quasi-stable systems which are defined via quasi-stability inequality given in Definition 4.13. It turns out that this is quite large class of systems naturally occurring in non-linear PDE’s of hyperbolic type of second order in time possibly interacting with parabolic equation. The interest in this class of systems stems from the fact that quasi-stability inequality almost automatically implies -in one shot- number of desirable properties such as asymptotic smoothness, finite dimensionality of attractors, regularity of attractors, exponential attraction etc. In what follows we shall provide brief introduction to the theory of quasi-stable systems.

We begin with the following assumption.
Assumption 4.12 Let \( X, Y, \) and \( Z \) be reflexive Banach spaces; \( X \) is compactly embedded in \( Y \). We endow the space \( H = X \times Y \times Z \) with the norm
\[
|y|^2 = |u_0|^2_X + |u_1|^2_Y + |\theta|^2_Z, \quad y = (u_0; u_1; \theta_0).
\]
The trivial case \( Z = \{0\} \) is allowed. We assume that \((H, S_t)\) is a dynamical system on \( H = X \times Y \times Z \) with the evolution operator of the form
\[
S_t y = (u(t); u_1(t); \theta(t)), \quad y = (u_0; u_1; \theta_0) \in H,
\]
where the functions \( u(t) \) and \( \theta(t) \) possess the properties
\[
u \in C(\mathbb{R}_+, X) \cap C^1(\mathbb{R}_+, Y), \quad \theta \in C(\mathbb{R}_+, Z).
\]
The structure of the phase space \( H \) and the evolution operator \( S_t \) in Assumption 4.12 is motivated by the study of the system generated by equation of the second order in time in \( X \times Y \) possibly interacting with some first-order evolution equation in space \( Z \). This type of interaction arises in modeling of thermoelastic plates and structural acoustic systems.

Definition 4.13 A dynamical system of the form (4.7) is said to be stable modulo compact terms (quasi-stable, for short) on a set \( B \subset H \) if there exist a compact seminorm \( \mu_X(\cdot) \) on the space \( X \) and nonnegative scalar functions \( a(t), b(t), \) and \( c(t) \) on \( \mathbb{R}_+ \) such that (i) \( a(t) \) and \( c(t) \) are locally bounded on \([0, \infty)\), (ii) \( b(t) \in L_1(\mathbb{R}_+) \) possesses the property \( \lim_{t \to \infty} b(t) = 0 \), and (iii) for every \( y_1, y_2 \in B \) and \( t > 0 \) the following relations
\[
|S_t y_1 - S_t y_2|^2_H \leq a(t) \cdot |y_1 - y_2|^2_H \quad (4.8)
\]
and
\[
|S_t y_1 - S_t y_2|^2_H \leq b(t) \cdot |y_1 - y_2|^2_H + c(t) \cdot \sup_{0 \leq s \leq t} \left[ \mu_X(u^1(s) - u^2(s)) \right]^2 \quad (4.9)
\]
hold. Here we denote \( S_t y_i = (u^i(t); u^i_1(t); \theta^i(t)), i = 1, 2 \).

Remark 4.3 The definition of quasi-stability is rather natural from the point of view of long-time behavior. It pertains to decomposition of the flow into exponentially stable and compact part. This represents some sort of analogy with the “splitting” method [BV92, Tem88], however, the decomposition refers to the difference of two trajectories, rather than a single trajectory. In addition, the quadratic dependence with respect to the semi-norm in (4.9) is essential in the definition. The relation (4.9) is called a stabilizability estimate (or quasi-stability inequality) and, in the context of long-time dynamics, was originally introduced in [CL04a] (see also [CEL04, CL04b] and the discussion in [CL08a] and [CL10]). To obtain such an estimate proves fairly technical (in critical problems) and requires rather subtle PDE tools to prove it. Illustrations of the method are given in [CL10] for some abstract models and also for a variety of von
Karman models. We also refer to [BC08, BCL07, CEL04, CL04a, CL06a, CL07a, CL08a, CLT08, CLT09, Nab09] for similar considerations for other models.

The notion of quasi-stability introduced in Definition 4.13 requires a special structure of the semiflow and a special type of a (quasi-stability) inequality. However, the idea behind this notion can be applied in many other cases (see, e.g., [CL04b, CL08a, CL11] and also [CL10]). Systems with delay/memory terms can be also included in this framework (see, e.g., [Fas07, Fas09, Pot09, Ryz05] and also [CL10]). The same idea was recently applied in [CK10] for analysis of long-time dynamics in a parabolic-type model (see below in Section 6).

**Remark 4.4** In order to write down a more explicit form of quasi-stability inequality let us consider dynamical system (say of gradient form) generated by some second order evolution equation in the space $H = X \times Y$ with an associated energy functional -say $E_y(t)$ and the damping integral denoted by $D_y(t)$, so that the energy identity reads

$$E_y(t) + \int_s^t D_y(\tau)d\tau = E_y(s), \quad s < t.$$ 

In this case a sufficient condition for quasi-stability is the following [CEL04] stabilizability inequality: there exists a parameter $T > 0$, such that the difference of two any trajectories $z(t) \equiv y_1(t) - y_2(t)$ satisfies the relation

$$E_{0,z}(T) + \int_0^T E_{0,z}(\tau)d\tau \leq C_T \int_0^T \tilde{D}_z(\tau) + C_T LOT_z, \quad (4.10)$$

where $E_{0,z}(t)$ denotes a positive part of the energy, $\tilde{D}_z$ is the damping functional for the difference $z$ and $LOT_z$ is a lower order term of the form $LOT_z = \sup_{\tau \in (0,T)} ||z(\tau)||^2_{H_1}$, where $H \subset H_1$ with a compact embedding. If one is already equipped with an existence of compact attractor, then stabilizability inequality that is required is of milder form: there exist two parameters $-\infty < s < T$ such that

$$E_{0,z}(T) + \int_s^T E_{0,z}(\tau)d\tau \leq C_{s,T} \int_s^T \tilde{D}_z(\tau)d\tau + C_{s,T} LOT_z. \quad (4.11)$$

Here $LOT_z$ denotes lower order terms measured by

$$LOT_z = \sup_{\tau \in (s,T)} ||z(\tau)||^2_{H_1},$$

and parameters $s, T$ may depend on the specific trajectories $y_1, y_2$.

The advantage of the above formulation is that these are inequalities that are obtainable by multipliers when studying stabilization and controllability (see, e.g., the monographs [Las02, LT00] and the references therein). The stabilizability inequality in (4.10) provides a direct link between controllability and long time behavior. Indeed, when considering stabilization to zero equilibria,
the lower order terms $LOT_z$ are dismissed by applying compactness-uniqueness argument. Then the inequality applied to a single trajectory $y_1 = y$

$$E_y(T) \leq C_{0,T} \int_0^T D_y(\tau) = C_{0,T}[E_y(0) - E_y(T)] \quad (4.12)$$

implies, by standard evolution method, exponential decay to zero equilibrium.

Similarly, in the case of controllability the observability inequality required [Las02] is precisely (4.12) with $D_y(t)$ denoting the quantities observed (for instance, velocity in the spatial domain or velocity on the subdomain). In either case, the crux of the matter is to establish such inequality.

In what follows our aim is to show that quasi-stable systems have far reaching consequences and enjoy many nice properties that include (i) existence of global attractors that is both finite-dimensional and smooth, (ii) exponential attractors, and so on.

**Asymptotic smoothness**

**Proposition 4.14** Let Assumption 4.12 be in force. Assume that the dynamical system $(H, S_t)$ is quasi-stable on every bounded forward invariant set $B$ in $H$. Then, $(H, S_t)$ is asymptotically smooth.

**Proof.** Let

$$\tilde{X} = \text{Closure } \{ v \in X : |v|_{\tilde{X}} \equiv \mu_X(v) + |v|_Y < \infty \}.$$  \hspace{1cm} (4.13)

One can see that $X$ is compactly embedded in the Banach space $\tilde{X}$. We have that the space

$$W^{1,2}_{\infty,2}(0, T; X, Y) = \{ f \in L_\infty(0, T; X) : f' \in L_2(0, T; Y) \}$$

is compactly embedded in $C(0, T; \tilde{X})$. This implies that the pseudometric $\rho_B'$ in $C(0, t; H)$ defined by the formula

$$\rho_B'\{S_\tau y_1, S_\tau y_2\} = c(t) \sup_{\tau \in [0, t]} \mu_X(u(t) - u(t))$$

is precompact (with respect to $H$). Here we denote by $\{S_\tau y_i\}$ the element from $C(0, t; H)$ given by function $y_i(\tau) = S_\tau y_i = (u(t), u(t), \theta(t))$. By (4.9) $\rho_B'$ satisfies the hypotheses of Theorem 4.2. This implies the result. \hfill $\blacksquare$

**Corollary 4.15** If the system $(H, S_t)$ is dissipative and satisfies the hypotheses of Proposition 4.14, then it possesses a compact global attractor.

**Proof.** By Proposition 4.14 the system $(H, S_t)$ is asymptotically smooth. Thus the result follows from Theorem 4.6. \hfill $\blacksquare$
Finite dimension of global attractors

We start with the following general assertion.

**Theorem 4.16** Let Assumption 4.12 be valid. Assume that the dynamical system \((H, S_t)\) possesses a compact global attractor \(A\) and is quasi-stable on \(A\) (see Definition 4.13). Then the attractor \(A\) has a finite fractal dimension \(\dim_A^f\) (this also implies the finiteness of the Hausdorff dimension).

**Proof.** The idea of the proof is based on the method of “short” trajectories (see, e.g., [MN96, MP02] and the references therein and also [CL08a]). We apply Theorem 4.11 in the space \(H_T = H \times W_1(0, T)\) with an appropriate \(T\). Here

\[
W_1(0, T) = \left\{ z \in L_2(0, T; X) : |z|^2_{W_1(0, T)} \equiv \int_0^T \left( |z(t)|^2_X + |z_t(t)|^2_Y \right) dt < \infty \right\}. \tag{4.14}
\]

The norm in \(H_T\) is given by

\[
||U||_{H_T}^2 = |y|^2_H + |z|^2_{W_1(0, T)}, \quad U = (y; z), \quad y = (u_0; u_1; \theta_0). \tag{4.15}
\]

Let \(y_i = (u_{0i}; u_{1i}; \theta_{0i})\), \(i = 1, 2\), be two elements from the attractor \(A\). We denote

\[
S_t y_i = (u^i(t); u^i_1(t); \theta^i(t)), \quad t \geq 0, \quad i = 1, 2,
\]

and \(Z(t) = S_t y_1 - S_t y_2 \equiv (z(t); z_t(t); \xi(t))\), where

\[
(z(t); z_t(t); \xi(t)) \equiv (u^1(t) - u^2(t); u^1_1(t) - u^2_1(t); \theta^1(t) - \theta^2(t)).
\]

Integrating (4.9) from \(T\) to \(2T\) with respect to \(t\), we obtain that

\[
\int_T^{2T} |S_t y_1 - S_t y_2|^2_H dt \leq \bar{b}_T |y_1 - y_2|^2_H + \bar{c}_T \sup_{0 \leq s \leq 2T} [\mu_X|z(s)|]^2, \tag{4.16}
\]

where

\[
\bar{b}_T = \int_T^{2T} b(t) dt \quad \text{and} \quad \bar{c}_T = \int_T^{2T} c(t) dt.
\]

It also follows from (4.9) that

\[
|S_T y_1 - S_T y_2|^2_H \leq b(T) \cdot |y_1 - y_2|^2_H + c(T) \cdot \sup_{0 \leq s \leq T} [\mu_X|z(s)|]^2
\]

and combining with (4.16) yields

\[
|S_T y_1 - S_T y_2|^2_H + \int_T^{2T} |S_t y_1 - S_t y_2|^2_H dt \leq \bar{b}_T |y_1 - y_2|^2_H + \bar{c}_T \sup_{0 \leq s \leq 2T} [\mu_X|z(s)|]^2, \tag{4.17}
\]

where

\[
b_T = b(T) + \int_T^{2T} b(t) dt \quad \text{and} \quad c_T = c(T) + \int_T^{2T} c(t) dt. \tag{4.18}
\]
Let $A$ be the global attractor. Consider in the space $H_T$ the set
\[ A_T := \{ U \equiv (u(0); u_t(0); \theta(t); u(t), t \in [0, T]) : (u(0); u_t(0); \theta(0)) \in A \}, \]
where $u(t)$ is the first component of $S_t y(0)$ with $y(0) = (u(0); u_t(0); \theta(0))$, and define operator $V : A_T \mapsto H_T$ by the formula
\[ V : (u(0); u_t(0); \theta(0); u(t)) \mapsto (S_T y(0); u(T + t)). \]
It is clear that $V$ is Lipschitz on $A_T$ and $V A_T = A_T$.

Because the space $\tilde{X}$ given by (4.13) possesses the properties $X \subset \tilde{X} \subset Y$ and $X \subset \tilde{X}$ is compact, contradiction argument yields
\[
[\mu_X(u)]^2 \leq \varepsilon |u|_X^2 + C_\varepsilon |u|_Y^2 \quad \text{for any } \varepsilon > 0. \tag{4.19}
\]
Therefore it follows from (4.8) that
\[
\sup_{0 \leq s \leq 2T} [\mu_X(z(s))]^2 \leq \frac{b_T}{c_T} |y_1 - y_2|^2_H + C(a_T, b_T, c_T) \sup_{0 \leq s \leq 2T} |z(s)|_Y^2,
\]
where $a_T = \sup_{0 \leq s \leq 2T} a(s)$ and $b_T, c_T$ given by (4.18). Consequently, from (4.17) we obtain
\[
\|V U_1 - V U_2\|_{H_T} \leq \eta_T \|U_1 - U_2\|_{H_T} + K_T \cdot (n_T(U_1 - U_2) + n_T(V U_1 - V U_2)),
\]
for any $U_1, U_2 \in A_T$, where $K_T > 0$ is a constant (depending on $a_T, b_T, c_T$, the embedding properties of $X$ into $Y$, the seminorm $\mu_X$), and
\[
\eta_T^2 = b_T = b(T) + \int_T^{2T} b(t)dt. \tag{4.20}
\]
The seminorm $n_T$ has the form $n_T(U) := \sup_{0 \leq s \leq 2T} |u(s)|_Y$. Because $W_1(0, T)$ is compactly embedded into $C(0; T; Y)$, $n_T(U)$ is a compact seminorm on $H_T$ and we can choose $T > 1$ such that $\eta_T < 1$. We also have from (4.8) that
\[
\|V U_1 - V U_2\|_{H_T} \leq L_T \|U_1 - U_2\|_{H_T} \quad \text{for } U_1, U_2 \in A_T,
\]
where $L_T = a(T) + \int_T^{2T} a(t)dt$. Therefore we can apply Theorem 4.11 which implies that $A_T$ is a compact set in $H_T$ of finite fractal dimension.

Let $\mathcal{P} : H_T \mapsto H$ be the operator defined by the formula
\[
\mathcal{P} : (u_0; u_1; \theta_0; z(t)) \mapsto (u_0; u_1; \theta_0).
\]
$A = \mathcal{P} A_T$ and $\mathcal{P}$ is Lipschitz continuous, thus $\dim_H A \leq \dim_H A_T < \infty$. Here $\dim^F$ stands for the fractal dimension of a set in the space $W$. This concludes the proof of Theorem 4.16. \[\blacksquare\]
Remark 4.5 By [CL08a] the dimension $\dim^H_A$ of the attractor admits the estimate

$$\dim^H_A \leq \left[ \frac{2}{1 + \eta_T} \right]^{-1} \cdot \ln m_0 \left( \frac{4K_T(1 + L_T^2)^{1/2}}{1 - \eta_T} \right),$$

(4.21)

Here $m_0(R)$ is the maximal number of pairs $(x_i; y_i)$ in $H_T \times H_T$ possessing the properties

$$\|x_i\|^2_{H_T} + \|y_i\|^2_{H_T} \leq R^2, \quad n_T(x_i - x_j) + n_T(y_i - y_j) > 1, \quad i \neq j.$$

It is clear that $m_0(R)$ can be estimated by the maximal number of pairs $(x_i; y_i)$ in $W_1(0, T) \times W_1(0, T)$ possessing the properties $\|x_i\|^2_{W_1(0, T)} + \|y_i\|^2_{W_1(0, T)} \leq R^2$ and $n_T(x_i - x_j) + n_T(y_i - y_j) > 1$ for all $i \neq j$. Thus the bound in (4.21) depends on the functions $a, b, c$ and seminorm $\mu_X$ in Definition 4.13 and also on the embedding properties of $X$ into $Y$.

Regularity of trajectories from the attractor

In this section we show how stabilizability estimates can be used in order to obtain additional regularity of trajectories lying on the global attractor. The theorem below provides regularity for time derivatives. The needed “space” regularity follows from the analysis of the respective PDE. It typically involves application of elliptic theory (see the corresponding results in [CL10, Chapter 9]).

Theorem 4.17 Let Assumption 4.12 be valid. Assume that the dynamical system $(H, S_t)$ possesses a compact global attractor $A$ and is quasi-stable on the attractor $A$. Moreover, we assume that (4.9) holds with the function $c(t)$ possessing the property $c_\infty = \sup_{t \in \mathbb{R}} c(t) < \infty$. Then any full trajectory $\{(u(t); u_t(t); \theta(t)) : t \in \mathbb{R}\}$ that belongs to the global attractor enjoys the following regularity properties,

$$u_t \in L_\infty(\mathbb{R}; X) \cap C(\mathbb{R}; Y), \quad u_{tt} \in L_\infty(\mathbb{R}; Y), \quad \theta_t \in L_\infty(\mathbb{R}; Z)$$

(4.22)

Moreover, there exists $R > 0$ such that

$$|u_t(t)|^2_X + |u_{tt}(t)|^2_Y + |\theta_t(t)|^2_Z \leq R^2, \quad t \in \mathbb{R},$$

(4.23)

where $R$ depends on the constant $c_\infty$, on the seminorm $\mu_X$ in Definition 4.13, and also on the embedding properties of $X$ into $Y$.

Proof. It follows from (4.9) that for any two trajectories

$$\gamma = \{U(t) \equiv (u(t); u_t(t); \theta(t)) : t \in \mathbb{R}\},$$

$$\gamma^* = \{U^*(t) \equiv (u^*(t); u^*_t(t); \theta^*(t)) : t \in \mathbb{R}\}$$

from the global attractor we have that

$$|Z(t)|^2_{H_T} \leq b(t - s)|Z(s)|^2_{H_T} + c(t - s) \sup_{s \leq \tau \leq t} [\mu_X(z(\tau))]^2$$

(4.24)
for all \( s \leq t, \ s, t \in \mathbb{R} \), where \( Z(t) = U^+(t) - U(t) \) and \( z(t) = u^+(t) - u(t) \). In the limit \( s \to -\infty \) relation (4.24) gives us that

\[
|Z(t)|_H^2 \leq c_\infty \sup_{-\infty \leq \tau \leq t} [\mu_X(z(\tau))]^2
\]

for every \( t \in \mathbb{R} \) and for every couple of trajectories \( \gamma \) and \( \gamma^* \). Using relation (4.19) we can conclude that

\[
\sup_{-\infty \leq \tau \leq t} |Z(\tau)|_H^2 \leq C \sup_{-\infty \leq \tau \leq t} |z(\tau)|_Y^2,
\]

(4.25) for every \( t \in \mathbb{R} \) and for every couple of trajectories \( \gamma \) and \( \gamma^* \) from the attractor.

Now we fix the trajectory \( \gamma \) and for \( 0 < |h| < 1 \) we consider the shifted trajectory \( \gamma^* \equiv \gamma_{h} = \{y(t + h) : t \in \mathbb{R}\} \). Applying (4.25) for this pair of trajectories and using the fact that all terms (4.25) are quadratic with respect to \( Z \) we obtain that

\[
\sup_{-\infty \leq \tau \leq t} |Z_{h}(\tau)|_H^2 \leq C \sup_{-\infty \leq \tau \leq t} |z_{h}(\tau)|_Y^2,
\]

(4.26)

where \( u_{h}(t) = h^{-1} \cdot [u(t + h) - u(t)] \) and \( \theta_{h}(t) = h^{-1} \cdot [\theta(t + h) - \theta(t)] \). On the attractor we obviously have that

\[
|u_{h}(t)|_Y \leq \frac{1}{h} \cdot \int_0^h |u_{t} (\tau + t)|_Y d\tau \leq C, \quad t \in \mathbb{R},
\]

with uniformity in \( h \). Therefore (4.26) implies that

\[
|u_{h}(t)|_X^2 + |u_{h}^*(t)|_Y^2 + |\theta_{h}^*(t)|_Z^2 \leq C, \quad t \in \mathbb{R}.
\]

Passing with the limit on \( h \) then yields relations (4.22) and (4.23).

**Fractal exponential attractors**

For quasi-stable systems we can also establish some result pertaining to (generalized) fractal exponential attractors.

We first recall the following definition.

**Definition 4.18** A compact set \( A_{\exp} \subset X \) is said to be inertial (or a fractal exponential attractor) of the dynamical system \((X, S_t)\) if \( A \) is a positively invariant set of finite fractal dimension (in \( X \)) and for every bounded set \( D \subset X \) there exist positive constants \( t_D, C_D \) and \( \gamma_D \) such that

\[
d_X \{S_t D | A_{\exp}\} \equiv \sup_{x \in D} \text{dist}_X(S_t x, A_{\exp}) \leq C_D \cdot e^{-\gamma_D (t - t_D)}, \quad t \geq t_D.
\]

If the dimension of \( A \) is finite in some extended space we call \( A \) a generalized fractal exponential attractor.
For more details concerning fractal exponential attractors we refer to [FEN94] and also to recent survey [MZ08]. We only note that the standard technical tool in the construction of fractal exponential attractors is the so-called squeezing property which says [MZ08], roughly speaking, that either the higher modes are dominated by the lower ones or that the semiflow is contracted exponentially. Instead the approach we use is based on the quasi-stability property which says that the semiflow is asymptotically contracted up to a homogeneous compact additive term.

Theorem 4.19 Let Assumption 4.12 be valid. Assume that the dynamical system \((H,S_t)\) is dissipative and quasi-stable on some bounded absorbing set \(B\). We also assume that there exists a space \(\tilde{H} \supseteq H\) such that \(t \mapsto S_t y\) is Hölder continuous in \(\tilde{H}\) for every \(y \in B\); that is, there exist \(0 < \gamma \leq 1\) and \(C_{B,T} > 0\) such that

\[
|S_{t_1}y - S_{t_2}y|_{\tilde{H}} \leq C_{B,T} |t_1 - t_2|^\gamma, \quad t_1, t_2 \in [0, T], \quad y \in B.
\] (4.27)

Then the dynamical system \((H,S_t)\) possesses a (generalized) fractal exponential attractor whose dimension is finite in the space \(\tilde{H}\).

In contrast with Theorem 3.5 [MZ08] Theorem 4.19 does not assume Hölder continuity in the phase space. At the expense of this we can guarantee finiteness of the dimension in some extended space only. This is why we use the notion of generalized exponential attractors.

Proof. We apply the same idea as in the proof of Theorem 4.16.

We can assume that absorbing set \(B\) is closed and forward invariant (otherwise, instead of \(B\), we consider \(B' = \bigcup_{t \geq t_0} S_t B\) for \(t_0\) large enough, which lies in \(B\)).

In the space \(H_T = H \times W_1(0,T)\) equipped with the norm (4.15), where \(W_1(0,T)\) is given by (4.14), we consider the set

\[
B_T := \{ U = (u(0); u_t(0); \theta(0); u(t), t \in [0, T]) : (u(0); u_t(0); \theta(0)) \in B \},
\]

where \(u(t)\) is the first component of \(S_t y(0)\) with \(y(0) = (u(0); u_t(0); \theta(0))\). As above we define the shift operator \(V : B_T \mapsto H_T\) by the formula

\[
V : (u(0); u_t(0); \theta(0); u(t)) \mapsto (S_T y(0); u(T + t)).
\]

It is clear that \(B_T\) is a closed bounded set in \(H_T\) which is forward invariant with respect to \(V\).

It follows from (4.8) that

\[
\|V U_1 - V U_2\|_{H_T}^2 \leq \left( a(T) + \int_T^{2T} a(t) dt \right) \|U_1 - U_2\|_{H_T}^2, \quad U_1, U_2 \in B_T.
\]

As in the proof of Theorem 4.16 we can obtain that

\[
\|V U_1 - V U_2\|_{H_T} \leq \eta_T \|U_1 - U_2\|_{H_T} + K_T \cdot \eta_T (U_1 - U_2) + n_T (V U_1 - V U_2)
\]
for any $U_1, U_2 \in B_T$ and for some $T > 0$, where $K_T > 0$ is a constant, $n_T(U) := \sup_{0 \leq s \leq T} |u(s)|$ and $\eta_T < 1$ is given by (4.20). Therefore by Theorem 7.4.2 [CL10] the mapping $V$ possesses a fractal exponential attractor; that is, there exists a compact set $A_T \subset B_T$ and a number $0 < q < 1$ such that $\dim_H^{\eta_T} A_T < \infty$, $V A_T \subset A_T$, and

$$\sup \{ \text{dist}_H(V^k U, A_T) : U \in B_T \} \leq C q^k, \quad k = 1, 2, \ldots,$$

for some constant $C > 0$. In particular, this relation implies that

$$\sup \{ \text{dist}_H(S_t y, A) : u \in B \} \leq C q^k, \quad k = 1, 2, \ldots, \quad (4.28)$$

where $A$ is the projection of $A_T$ of the first components:

$$A = \{(u(0); u_t(0); \theta(0)) \in B : (u(0); u_t(0); \theta(0); u(t), t \in [0, T]) \in A_T \}.$$

It is clear that $A$ is a compact forward invariant set with respect to $S_T$; that is, $S_T A \subset A$. Moreover $\dim_H A \leq \dim_H^{\eta_T} A_T < \infty$. One can also see that

$$A_{exp} = \cup \{ S_t A : t \in [0, T] \}$$

is a compact forward invariant set with respect to $S_T$; that is, $S_T A_{exp} \subset A_{exp}$. Moreover, it follows from (4.27) that

$$\dim_{\tilde{H}} A_{exp} \leq c \left[ 1 + \dim_H A \right] < \infty.$$

We also have from (4.28) and (4.8) that

$$\sup \{ \text{dist}_H(S_t y, A_{exp}) : u \in B \} \leq C e^{-\gamma t}, \quad t \geq 0,$$

for some $\gamma > 0$. Thus $A_{exp}$ is a (generalized) fractal exponential attractor. ■

**Remark 4.6** Hölder continuity (4.27) is needed in order to derive the finiteness of the fractal dimension $\dim_{\tilde{H}} A_{exp}$ from the finiteness of $\dim_H A$. We do not know whether the same holds true without property (4.27) imposed in some vicinity of $A_{exp}$. This is because $A_{exp}$ is an uncountable union of (finite-dimensional) sets $S_t A$. We also emphasize that fractal dimension depends on the topology. Indeed, there is an example of a set with finite fractal dimension in one space and infinite fractal dimension in another (smaller) space.

## 5 Long time behavior for canonical models described in Section 2

The goal of this section is to show how to apply abstract methods presented in the previous section in order to establish long time behavior properties of dynamical systems generated by PDE’s described in Section 2. An interesting feature is that though all the models considered are of hyperbolic type-without an inherent smoothing mechanism, the long time behavior can be “smooth” and characterized by finite dimensional structures. Our plan is to focus on the following topics.
1. Control to finite dimensional and smooth attractors of nonlinear wave equation
   • with interior fully supported dissipation,
   • with boundary or partially localized dissipation.

2. Control to finite dimensional and smooth attractors of nonlinear plates dynamics
   • von Karman plates with a nonlinear fully supported interior feedback control,
   • von Karman and Berger plates with boundary and partially localized damping,
   • Kirchhoff-Boussinesq plates with a linear interior feedback control.

For each model considered we follow the plan: (i) state the assumptions and the results, (ii) discuss possible generalizations, (iii) state open problems, (iv) provide a sketch of the proofs.

5.1 Wave dynamics

In this section we consider the existence problem for finite dimensional and smooth attractors for semilinear wave equation with critical (feedback) sources with both interior and boundary damping.

We start with interior damping and source model as described in (2.1).

Interior damping

In studying long time behavior we find convenient to collect all the assumptions required for the source and damping:

Assumption 5.1 1. The source \( f \in C^1(\mathbb{R}) \) satisfies the dissipativity condition in (3.4) and the growth condition
   \[ |f'(s)| \leq M|s|^2 \text{ for } |s| \geq 1. \]

2. The damping \( g(s) \) is an increasing function, \( g(0) = 0 \) and it satisfies the following asymptotic growth conditions:
   \[ 0 < m_g|s|^2 \leq g(s)|s| \leq M_g|s|^6 \text{ for } |s| \geq 1. \] \hspace{1cm} (5.1)

3. The damping coefficient \( a \) satisfies \( a(x) \geq a_0 > 0, x \in \Omega \).

With reference to the model (2.1) we can take
   \[ f(s) = -s^3 + cs^2 \text{ and } g(s) = g_1s + |s|^4s \text{ with } g_1 \geq 0 \text{ and } c \in \mathbb{R}. \] \hspace{1cm} (5.2)
Remark 5.1 1. The power 3 associated with the source $f$ and the power 5 associated with the damping $g$ are often referred as ”double critical" parameters for the wave equation with $n = 3$. The reason for this is that the maps $u \rightarrow f(u)$ from $H^1(\Omega)$ into $L_2(\Omega)$ and $u \rightarrow g(u)$ from $H^1(\Omega)$ into $H^{-1}(\Omega)$ are bounded but not compact.

2. More general forms of space dependent and localized damping coefficient $a(x)$ are considered in [CLT08].

Under the above conditions the wellposedness results of Theorem 3.3 assert existence of a continuous semiflow $S_t$ which defines dynamical system $(\mathcal{H}, S_t)$ with $\mathcal{H} = H_0^1(\Omega) \times L^2(\Omega)$. We are ready to undertake the study of long-time behavior. Our first main result is given below:

Theorem 5.2 (Compactness) With reference to the dynamics described by (2.1) and zero Dirichlet boundary conditions, we assume Assumption 5.1 for the source $f$ and the damping $g$. Then, there exists a global compact attractor $A \subset \mathcal{H}$ for the dynamical system $(\mathcal{H}, S_t)$. In addition $(\mathcal{H}, S_t)$ is gradient system.

To describe the structure of the attractor, we introduce the set of stationary points of $S_t$ denoted by $\mathcal{N}$:

$$\mathcal{N} = \{ V \in \mathcal{H} : S_t V = V \text{ for all } t \geq 0 \}.$$  \hspace{1cm} (5.3)

Every stationary point $W \in \mathcal{N}$ has the form $W = (w; 0)$, where $w = w(x)$ solves the problem

$$\Delta w + f(w) = 0 \text{ in } \Omega \text{ and } w = 0 \text{ in } \Gamma.$$  \hspace{1cm} (5.4)

Under the Assumption 5.1, standard elliptic theory implies that every stationary solution satisfies $w \in H^2(\Omega)$ and the set of all stationary solutions is bounded in $H^1(\Omega)$. In fact, more regularity of stationary solutions can be shown, but the above is sufficient for the analysis to follow.

The next result is a consequence of general properties of gradient systems (see, e.g., [BV92, Hal88] and also Section 4.3 above) and asserts that the attractor $A$ coincides with this unstable manifold.

Theorem 5.3 (Structure) Under the assumptions of Theorem 5.2 we have that

- $A = M^u(\mathcal{N})$;
- $\lim_{t \to +\infty} \text{dist}_H(S_tW, \mathcal{N}) = 0$ for any $W \in \mathcal{H} \equiv H^1(\Omega) \times L^2(\Omega)$;
- if (5.4) has a finite number of solutions, then for any $V \in \mathcal{H}$ there exists a stationary point $Z = (z, 0) \in \mathcal{N}$ such that $S_tV \rightarrow Z$ in $\mathcal{H}$ as $t \to +\infty$.

Our second main result read refers to the dimensionality of attractor.

Theorem 5.4 (Finite Dimensionality) Let $f \in C^2(\mathbb{R})$, $g \in C^1(\mathbb{R})$ and Assumption 5.1 be in force. In addition, we assume that for all $s \in \mathbb{R}$:

\footnote{We note that the property that (5.4) has finitely many solutions is generic.}
1. \(|f''(s)| \leq C|s|\),

2. \(0 < m \leq g'(s) \leq C[1 + sg(s)]^{2/3}\).

Then the fractal dimension of the global attractor \(A\) of the dynamical system \((H, S_t)\) is finite.

With reference to the model (2.1) we still can take \(f\) and \(g\) as in (5.2), but with \(g_1 > 0\).

The proof of Theorem 5.4 follows from quasi-stability property satisfied for the model (see Definition 4.13). The exactly same lemma allows to obtain “for free” the following regularity result for elements from the attractor.

**Theorem 5.5 (Regularity)** Under the assumptions of Theorem 5.4 the attractor is a bounded set in 

\[
V = \{(u, v) \in H^1_0(\Omega) \times H^1_0(\Omega) \mid -\Delta u + g(v) \in L^2(\Omega), \}
\]

i.e. there exist constants \(c_i > 0\) such that\(^2\)

\[
A \subset \left\{(u, v) \in H^1(\Omega) \times H^1(\Omega) \mid \|u\|_{1,\Omega} + \|v\|_{1,\Omega} \leq c_1; \|\Delta u - g(v)\| \leq c_2, \right\}
\]

A natural question that can be asked in this context is that of existence of strong (i.e., corresponding to topology of strong solutions) attractors. Though this latter property is technically related to smoothness of attractors, however the corresponding result does not follow from Theorem 5.5, unless the damping is subcritical. Additional analysis is needed for that. A first question to address in this direction is the existence of attractors for strong solutions. This is to say that when restricting solutions to regular initial data, the corresponding trajectories converge asymptotically to an attractor \(A_1\) which typically may be smaller than \(A\) the latter corresponds to weak or generalized solutions. A first step toward this goal is to show dissipativity of strong solutions (which of course does not follow at once from dissipativity of weak solutions, unless a problem is linear). Fortunately, dissipativity of strong solution is, again, direct consequence of quasi-stability. Thus, we have this property for “free”.

**Theorem 5.6 (Dissipativity of strong solutions)** Let the assumptions of Theorem 5.4 be in force. Then there exists a number \(R_0 > 0\) such that for any \(R > 0\) we can find \(t_R > 0\) such that

\[
\|w_{tt}(t)\|^2 + \|w_t(t)\|^2_{1,\Omega} \leq R_0^2 \quad \text{for all} \quad t \geq t_R
\]

for any strong solution \(w(t)\) to problem (2.1) with initial data \((w_0; w_1)\) from the set

\[
W_R = \{(w_0, w_1) \in H^1_0(\Omega) \times H^1_0(\Omega) : \|w_0\|_{1,\Omega} + \|w_1\|_{1,\Omega} \leq R, \|w_2\| \leq R\},
\]

where \(w_2 = g(w_1) - \Delta w_0\).

\(^2\)See also Theorem 5.7 below which asserts an additional regularity of attractor.
Theorem 5.6 refers to strong solutions. The existence of such, is guaranteed once initial data are taken from $W_R$. With additional calculations (using the multiplier $\Delta w_t$) one improves the statement and obtains the dissipativity for $\|w(t)\|_{2,\Omega}^2 \leq R_0^2$, $t \geq t_R$ with initial data from $H^2(\Omega)$. An interesting question is whether the same regularity is enjoyed by weak attractors. In other words, whether $H^2(\Omega) \cap H^1_0(\Omega) \times H^1_0(\Omega) \supset A$? The answer to this question is given by

**Theorem 5.7 (Regularity)** Under the same assumptions as in Theorem 5.4 the attractor is a bounded set in $H^2(\Omega) \times L^2(\Omega)$.

The proof of this theorem given in [Kha10] exploits the so called "backward smoothness on attractor".

**Decay rates to equilibrium:** Knowing that every trajectory converges to some equilibrium point one would like to know "how fast"? The answer to this question is given by decay rates of asymptotic convergence of solutions to point of equilibria. This is the topic we present next.

We introduce concave, strictly increasing, continuous function $k_0 : \mathbb{R}_+ \mapsto \mathbb{R}_+$ which captures the behavior of $g(s)$ at the origin possessing the properties

$$k_0(0) = 0 \quad \text{and} \quad s^2 + g^2(s) \leq k_0(sg(s)) \quad \text{for} \quad |s| \leq 1. \quad (5.7)$$

Such a function can always be constructed due to the monotonicity of $g$, see [LT93] or Appendix B in [CL10]. Moreover, on the strength of Assumption 5.1 there exists a constant $c > 0$ such that $k(s) = k_0(s) + cs$ is a concave, strictly increasing, continuous function $k : \mathbb{R}_+ \mapsto \mathbb{R}_+$ such that

$$k(0) = 0 \quad \text{and} \quad |s|^2 \leq k(sg(s)) \quad \text{for all} \quad s \in \mathbb{R}. \quad (5.8)$$

Given function $k$ we define

$$H_0(s) = k\left(\frac{s}{c_3}\right), \quad G_0(s) = c_1(I + H_0)^{-1}(c_2s), \quad (5.9)$$

$$Q(s) = s - (I + G_0)^{-1}(s),$$

where $c_i$ are some positive constants. It is obvious that $Q(s)$ is strictly monotone. Thus, the differential equation

$$\frac{d\sigma}{dt} + Q(\sigma) = 0, \quad t > 0, \quad \sigma(0) = \sigma_0 \in \mathbb{R}, \quad (5.10)$$

admits global, unique solution $\sigma(t)$ which, moreover, decays asymptotically to zero as $t \to \infty$. With these definitions we are ready to state our next result.

**Theorem 5.8 (Rate of convergence to equilibria)** In addition to Assumption 5.1, we assume that the set of stationary points $\mathcal{N}$ is finite, and every equilibrium $V = (v; 0)$ is hyperbolic in the sense that the problem

$$\Delta w + f'(v) \cdot w = 0 \text{ in } \Omega \quad \text{and} \quad w = 0 \text{ on } \Gamma \quad (5.11)$$

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has no non-trivial solutions. Then, for any $W_0 = (w_0; w_1) \in \mathcal{H}$, there exists a stationary point $V = (v; 0)$ such that

$$
\|S(t)W_0 - V\|_H \leq C\left(\sigma(|tT^{-1}|)\right), \quad t > 0,
$$

(5.12)

where $C$ and $T$ are positive constants depending on $W_0$, $[a]$ denotes the integer part of $a$ and $\sigma(t)$ satisfies (5.10) with $\sigma_0 = C(W_0, V)$ where $C(W_0, V)$ is a constant depending on $\|W_0\|_H$ and $\|V\|_H$ and $Q$ is defined in (5.9). In particular, if $g'(0) > 0$, then

$$
\|S(t)W_0 - V\|_H \leq Ce^{-\omega t}
$$

for some positive constants $C$ and $\omega$ depending on $W_0$.

**Remark 5.2** Since $Q(s)$ is strictly increasing and $Q(0) = 0$, the rates described by the ODE in (5.10) (see, e.g., [LT93] or [CL10, Appendix B]) decay uniformly to zero. The “speed” of decay depends on the behavior of $g'(s)$ at the origin. If $g'(s)$ decays to zero polynomially, then by solving the ODE in (5.10), one obtains algebraic decay rates for the solutions to $\sigma(t)$. If, instead $g'(0) > 0$, then $Q(\sigma) = a\sigma$ for some $a > 0$ and, consequently, the decay rates derived from (5.10) are exponential (see [LT93] for details). In the latter case using the dissipativity of strong solutions (see Theorem 5.6) one can prove by interpolation that the exponential decay rate holds for strong solutions in a stronger (than in $\mathcal{H}$) topology.

**Boundary damping**

In the case when boundary dissipation is the only active mode of damping (model in (2.14) and (2.15) with $g = 0$) one can still prove that the long time behavior is essentially the same as in the case of internal damping. The task of achieving this goal is much more technical, due to the necessity of propagating the damping from the boundary into the interior. This is done by using familiar by now “flux” multipliers. However, the resulting analysis and PDE estimates are considerably more complicated and often resort to specific unique continuation property as well as Carleman’s estimates (when dealing with critical cases). This topic has been considered in [CEL02, CEL04, CLT09]. The corresponding results is given below.

**Theorem 5.9 (Boundary Dissipation)** With reference to the equations in (2.14) and (2.15), where we take $g = 0$, $h(w) = -w$, the source $f$ satisfies dissipativity property imposed in (3.4), where $\lambda_1$ corresponds to the first eigenvalue of the Robin problem and such that $|f'(s)| \leq C(1 + |s|)$ for all $s \in \mathbb{R}$. The boundary damping $g_0$ is an increasing, differentiable function, $g_0(0) = 0$ and satisfies the asymptotic growth condition:

$$
m \leq g_0'(s) \leq M, \quad |s| > 1, \quad m, M > 0,
$$

(5.13)

Then

1. Under the above assumption the statements of Theorem 5.2 and 5.3 hold.
2. If, in addition, \( g'_0(0) > 0 \) then the statements of Theorem 5.4 and also Theorem 5.7, Theorem 5.8 (with appropriate modifications) hold true.

One fundamental difference between boundary damping and interior damping is that the restriction of linear bound at infinity for the boundary damping is essential. One dimensional examples [Van00] disprove validity of uniform stability to a single equilibrium when the damping is superlinear at infinity.

**Generalizations**

**With reference to interior damping**

1. Wave equation with homogenous Neumann or Robin boundary conditions can be considered in the same way.

2. The same wave model with additional boundary damping and source, as described in (2.14) and (2.15). The corresponding analysis is more involved. Details are given in [CL07a].

3. Possibility of degenerate damping \( g(s) \), where the density function \( a(x) \) describes possibility of degeneracy of the damping. This situation is also studied in [CL07a].

4. Strong attractors. These are attractors corresponding to strong solutions. In the case when boundary source and damping are involved and the damping may degenerate, the description of strong attractors is more involved. These are bounded sets \( V \times H^1(\Omega) \) where \( H^2(\Omega) \subset V \). The details are given in [CL07a].

5. Under additional assumptions imposed on regularity of the damping \( g(s) \) and the source \( f(s) \) one can prove that the attractors corresponding to Theorem 5.7 are infinitely smooth \( C^\infty \). This can be done by the usual boot-strap argument.

6. Attractors with localized dissipation only. This corresponds to having \( a(x) \) localized in the layer near boundary. [CLT08]. As in the case of boundary damping, requires linearly bounded damping.

**With reference to boundary damping**

1. Attractors with boundary damping which is further localized to a suitable portion of the boundary. Requires appropriate geometric conditions, see [CLT09] for details.

2. Relaxed regularity assumptions imposed on the damping function \( g_0(s) \), see [CLT09] again.
Open Problems

1. Higher regularity of attractors with minimal hypotheses on the damping and source.

2. Under which conditions weak attractors coincide with strong attractors in the case of boundary or partially localized damping?

3. Supercritical sources (e.g., $f(s) = s^5$). Ball’s method will provide existence of attractors with linear damping. However, linear damping is not sufficient for uniqueness of solutions. One needs to consider non-unique solutions.

4. Nonmonotone damping. Some particular results in this direction can be found in [CL08a].

5. Non-gradient structures. For instance models involving first order differentials as the potential sources.

Remarks

1. Theorem 5.2 - Compactness. Difficulty: double critical exponent-rules out previous methods. Use Theorem 4.3 after an appropriate rewriting of the source which then fits into the structure of the functional $\Psi$. This point is explained below.

2. Theorems 5.4 and Theorem 5.7-finite dimensionality and smoothness. Relies on proving quasi-stability inequality. The main trick is to provide appropriate decomposition of the source that takes advantage of dissipativity integral which is $L_1(\mathbb{R})$. See [CL07a, CLT09].

3. Full statement of Theorem 5.7 - [Kha10]. Smoothness of the trajectories on the attractor near $-\infty$. The existence of strong attractors (for strong solutions) is de facto consequence of quasi-stability inequality. However, in many situation proving additional spatial regularity ($H^2(\Omega)$, for instance) is problematic. It is for this task that special considerations related to backward smoothness are important.

4. Decay rates - Theorem 5.8. The difficulty relates to the fact that convergence to equilibria is a very "unstable" process. It is not possible to localize the analysis to the neighborhood of the equilibria. Decay rates are derived by using convex analysis and reducing estimates to nonlinear ODE’s. See [CL07a].

5. Boundary damping - Theorem 5.9. First results are given in [CEL02, CEL04] where finite-dimensionality was proved for subcritical sources only. Complete proof of Theorem 5.9 (in fact in a more general version) is given in [CLT09], where flux multipliers are used along with Carleman’s estimates and also backward smoothness of trajectories.
Guide through the Proofs.

We are not in a position to present complete proofs of the results. These are
given in the cited references. Here, our aim instead is to orient the reader what
are the main points one needs to go over in order to prove these results. Special
emphasis is given to more subtle technical details where ”some” tricks -most of
which recently introduced - need to be applied. It is our hope that an attentive
reader will be able to grasp the essence of the proof, the tools required so that
when guided to more specific reference he will be in a position to reconstruct a
complete proof.

Theorem 5.2 - Compactness

Step 1: The corresponding system is gradient system. This follows
from the fact that Lyapunov function
\[ V(t) \equiv \mathcal{E}(t) \]
is bounded from above and
below by the topology of the phase space. The latter results from dissipativity
condition imposed on \( f(s) \) in Assumption 5.1 (see (3.4)). Moreover \( V(t) = 0 \)
implies \( (g(w_t), w_t) = 0 \) hence \( w_t(x; t) = 0 \) in \( \Omega \) reduces the dynamics to a
stationary set defined by (5.3) and (5.4). The set \( \mathcal{N} \) of stationary points is
bounded due, again, to dissipativity condition imposed on \( f \). This part of the
argument is standard.

Step 2: Asymptotic smoothness. The main difficulty in carrying the proof
of asymptotic smoothness is double criticality of both the source and the damp-
ing. This is the reason why previous approaches (based on splitting or squeezing)
did require additional assumptions. To overcome this difficulty we use Theorem
4.3. In preparation for this theorem we are using the equation

\[ z_{tt} - \Delta z + a(x)[g(w_t) - g(u_t)] = f(w) - f(u) \quad \text{in } \Omega \times \mathbb{R}_+ \]  
(5.14)

with \( z = 0 \) on \( \Gamma \), written for the difference of two solutions \( z = w - u \), where
both \( w \) and \( u \) are solutions to the original equations confined to a bounded set
\( B \in H \), and also

- **energy identity** (multiply equation (5.14) by \( z_t \));
- **equipartition** of energy (multiply equation (5.14) by \( z \)).

In addition we recall dissipation relations for each solution \( u \) and \( w \):

\[ \int_0^t ((g(u_t), u_t)) + ((g(w_t), w_t)) \leq C_B, \quad \forall t > 0, \]  
(5.15)

and also

\[ \|u(t)\|^2 + \|u(t)\|_{L^2, \Omega}^2 + \|w_t(t)\|^2 + \|w(t)\|_{L^2, \Omega}^2 \leq C_B, \quad \forall t > 0. \]

Critical step consists of obtaining ”recovery” estimate for the energy of \( z \) in
terms of the damping and the source. The important part is that the ”re-
covery” estimate does not involve any initial data. This step is achieved by
using *equipartition* of the energy. The result of which is the following inequality [CL07a]:

\[
\frac{1}{2} T E_z(T) + \int_0^T E_z(t) \ dt \leq \int_0^T \left[ ||z_t||^2 + D(z_t) + |G(z_t, z)| \right] \ dt + \Psi(z, T), \tag{5.16}
\]

for \( T \geq 4 \), where 

\[
E_z(t) \equiv ||z_t(t)||^2 + ||\nabla z(t)||^2,
\]

\[
D(z_t) \equiv ((g(w_t) - g(u_t), z_t)), \quad G(z_t, z) \equiv ((g(w_t) - g(u_t), z)), \tag{5.17}
\]

and

\[
\Psi(z, T) \equiv - \int_0^T ((f(w) - f(u), z_t))
\]

\[
+ \int_0^T ds \int_s^T d\tau ((f(w) - f(u), z_t)) + \int_0^T |((f(w) - f(u), z))| dt
\]

Our goal is to bound the right hand side of the above relation by terms that are (i) either bounded uniformly in \( T \), or (ii) compact or (iii) small multiples of energy integral.

From the energy relation for \( z \) we have that

\[
\int_0^T D(z_t) d\tau \leq C_B + \int_0^T ((f(w) - f(u), z_t)) d\tau. \tag{5.18}
\]

One can also see from Proposition B.1.2 in [CL10] that

\[
||z_t||^2 \leq \eta + C_\eta D(z_t), \quad \forall \eta > 0.
\]

Thus the terms \( ||z_t||^2 \) and \( D(z_t) \) produce in (5.16) an admissible contribution.

The last integral in the definition of \( \Psi \) is compact. Thus the noncompact terms are the second one involving \( G \) and the first two integrals in definition of \( \Psi \). Since these two terms are similar, it suffices to analyze the main non-compact terms in (5.16) which are:

\[
\int_0^T |G(z_t, z)| dt \quad \text{and} \quad \int_0^T ds \int_s^T ((f(w) - f(u), z_t)) d\tau.
\]

These terms reflect double criticality of the damping-source exponents. We analyze these next.

For the **damping term** \( G \) one uses split \( \Omega \) into \( \Omega_1 = \{ |u_t(x, t)| > 1 \} \) and the complement \( \Omega_2 = \Omega \setminus \Omega_1 \). With the use of critical growth condition for \( g \) and the fact that \( ||z|| \leq C_B \) we obtain

\[
\int_{\Omega_1} |g(u_t)| ||z|| dx \leq \left[ \int_{\Omega_1} |g(u_t)|^{6/5} dx \right]^{5/6} \||z||_{L^6(\Omega)}
\]

\[
\leq \left[ \int_{\Omega_1} g(u_t) u_t dx \right]^{5/6} \||z||_{L^1(\Omega)} \leq C_B \int_{\Omega} g(u_t) u_t dx.
\]
On $\Omega_2$ the damping is bounded -contributing to lower order terms. Thus we obtain for $G(z_t, z)\int_0^T G(z_t, z)dt \leq C_B \int_0^T [(g(u_t), u_t) + (g(w_t), w_t)] dt + C_B T \sup_{t \in [0, T]} ||z(t)||$
\leq C_B + C_B T \sup_{t \in [0, T]} ||z(t)||,$
which gives the inequality in terms of the universal constant (independent on time $T$) and a lower order term $||z(t)||$.

For the source, the key point is the following decomposition that allows to prove sequential convergence of the functional $\Psi$ in Theorem 4.3:
\[
\int_0^T ((f(w) - f(u), z_t)) = \int_\Omega [F(w(T)) + F(u(T)) - F(w(0)) - F(u(0))]
- \int_0^T [((f(u), w_t)) + ((f(w), u_t))] dt,
\]
where $F(s)$ denotes the antiderivative of $f$. In line with Theorem 4.3 we consider these terms on sequences $u = w_n, w = w_m$ converging weakly in $H^1(\Omega)$ to a given element. We want to conclude that the corresponding functional converges sequentially to zero. Convergence on the first four terms is strong due to compactness while convergence on the last term is sequential with $w_n \to w$ weakly in $H^1(\Omega)$ and $w_m t \to w_t$ weakly in $L_2(\Omega)$. Applying (5.16) gives
\[
E_z(T) \leq \frac{\epsilon}{2} + \frac{C_B}{T} + C_\epsilon [\Psi(z, T) + LOT(z, T)] \leq \epsilon + C_\epsilon [\Psi(z, T) + LOT(z, T)],
\]
for $T = T_\epsilon$ large enough, where $LOT(z, T)$ comprise of all compact terms. Now we use the fact that
\[
\hat{\Psi}(w_n, w_m, T) \equiv \int_0^T ((f(w_n) - f(w_m), w_n t - w_m t))
+ \int_0^T ds \int_s^T d\tau ((f(w_n) - f(w_m), w_n t - w_m t))
\]
is weakly sequentially compact - see the details in [CL07a, CL08a].

**Step 3: Completion.** We apply Theorem 4.9 to deduce the results stated in Theorem 5.2 and Theorem 5.3.

**Theorem 5.3 - Structure** - follows from asymptotic smoothness and gradient structure via Theorem 4.9.

**Theorems 5.4 and 5.7 - Finite Dimensionality and Regularity.** The key tool is the quasi-stability inequality formulated in Definition 4.13. This
inequality, as in the case of asymptotic smoothness, should hold for the difference of two solutions \( z = w - u \) taken from a bounded set—say an absorbing ball \( B \). As before, we apply two multipliers \( z \) and \( z_t \).

For the damping term \( G(z_t, z) \) given in (5.17) we have the following estimate (see Proposition 5.3 in [CL07a] for details):

**Lemma 5.10 (Damping)** For every \( \epsilon > 0 \) there exists \( C_\epsilon \) such that

\[
\int_0^T |G(z_t, z)| dt \leq \epsilon \sup_{[0,T]} \|z\|_{i,\Omega}^2 \left( T + \int_0^T \left[ \left( (g(w_t), w_t) \right) + \left( (g(u_t), u_t) \right) \right] d\tau \right) + C_\epsilon \int_0^T \left( (g(u_t) - g(w_t), z_t) \right) dt.
\]

The following decomposition of the source is also essential (see Proposition 5.4 in [CL07a]).

**Lemma 5.11 (Source)** The following estimate holds true:

\[
\int_0^T \left( (f(u) - f(w), z_t) \right) \leq C_{B,T} \|z\|_{i,\Omega}^2 + \epsilon \int_0^T \|z\|_{i,\Omega}^2 + C_{\epsilon,B} \int_0^T \|z(t)\|_{i,\Omega}^2 K(t) dt,
\]

where \( K(t) = \|w_t(t)\|^2 + \|u_t(t)\|^2 \) which under the condition \( g'(s) \geq m > 0, \forall s \), by (5.15) satisfies the following dissipativity property

\[
\int_0^\infty K(s) ds \equiv \int_0^\infty \left[ \|w_t(s)\|^2 + \|u_t(s)\|^2 \right] ds < C_B. \tag{5.19}
\]

By using dissipation properties in (5.15) and in (5.19) and combining with the results of Lemma 5.10 and Lemma 5.11 one obtains (see [CL07a] for details) the inequality

\[
E_z(t) \leq C_1 \left[ e^{-\omega t} E_z(0) + \|z\|_{L^\infty(0,t;H^1_\Omega)}^2 \right] \exp \left\{ C_2 \int_0^t \tilde{K}(s) ds \right\}
\]

\[
\leq C \left[ e^{-\omega t} E_z(0) + \|z\|_{L^\infty(0,t;H^1_\Omega)}^2 \right]
\]

where

\[
\tilde{K}(t) = K(t) + \left( (g(u_t(t), u_t(t)) \right) + \left( (g(w_t(t), w_t(t)) \right).
\]

This leads to the desired quasi-stability inequality.

**Theorem 5.7 - Strong attractors** - proof explores smoothness of backward trajectories on the attractor, see [Kha10].

**Theorem 5.8 - Decay rates.** The details given in [CL07a, Section 8].

**Theorem 5.9 - Boundary Dissipation.** The proof is more technical here. Let us point out the main differences.

**Compactness:**
**Gradient structure:** In the boundary case the question of gradient structure has one additional aspect with respect to the interior damping. It involves a familiar in control theory question of **unique continuation** through the boundary. This property is known to hold due to various extensions of Holmgren-type theorems. This aspect of the problem is central in inverse problems when one is to reconstruct the source from the boundary measurements. Under suitable geometric conditions unique continuation results hold for the wave equation with a potential and $H^1$ solutions [Isa06, Rui92, EIN02].

**Reconstruction of the energy:** Since the damping is localized near the boundary, reconstruction of the energy involves propagation of the damping from a boundary into the interior. While in the case of interior damping equipartition of energy was sufficient for recovery of potential energy, this is not enough in the present case. Additional multiplier is needed which is referred as "flow multiplier" given by $h \nabla w$, where $h$ is a suitable vector field. This multiplier -of critical order for the wave equation- is successful in reconstructing the energy of a single solution -provided time $T$ is sufficiently large and the support of the damping on the boundary sufficiently large. However, when dealing with the differences of two trajectories (functions $z$), the method is no longer successful. There is a competition of two critical terms. A major detour is needed when one estimates the difference of two solutions. In the boundary case we have two critical multipliers $h \nabla z$ and $z_t$ both of energy level:

$$((f(u) - f(w), h \nabla z)) \text{ and } ((f(u) - f(w), z_t)) .$$

While this is not a problem with subcritical sources, it becomes a major issue in the case of critical sources. The remedy for the first term to this issue is by invoking Carleman’s estimates with large parameter which allows for balancing the terms. This step is very technical-details can be found in [CLT09].

**Dissipativity integrals:** Note that in this case (5.18) requires additional integration over the boundary in the right hand side. However, flux multiplier - again - with Carleman’s estimates allows to get the estimate on this term. At the end of the day, with the use of restrictive growth conditions assumed on $g$ (linear bound at infinity) one obtain reconstruction of the form as in (5.16 without $||z_t||^2$ term. The estimate of $G(z, z_t)$ is reasonably straight due to linear bound hypotheses assumed.

**Functional $\Psi$:** The estimate for the compactness part of $\Psi$ is the same. It depends on the source only. This allows for the applicability of the same Theorem 4.3 to conclude compactness of attractors with boundary dissipation.

**Quasi-stability Consequences: Finite dimensionality and Smoothness of Attractors.** Here the presence of the boundary damping presents another major challenge. This is at the level of the inequality in Lemma 5.11. The result stated there depends critically on finiteness of $K(t)$. This is no longer valid in the pure boundary case, where dissipativity integral involves boundary integrals only. In that case one resorts to the analysis of "backward" smoothness on the attractor. This procedure consists of the following four steps.
STEP 1: Prove quasi-stability inequality for solutions on the attractor near stationary points. This leads to consideration of negative time \( t \to -\infty \). Such estimate is possible due to the fact that the system is gradient system and the velocities \( w(t) \) and \( u(t) \) in (5.19) are "small" near equilibria.

STEP 2: Quasi-stability inequality provides additional smoothness of the trajectories at some negative time \( T_f < 0 \). This smoothness is propagated forward on the strength of regularity theorem. The consequence of this is that the elements in the attractor \( A \) are contained in \( H^2(\Omega) \times H^1(\Omega) \).

STEP 3: The ultimate smoothness of attractors is achieved by claiming that the attractor is bounded in the topology of \( H^2(\Omega) \times H^1(\Omega) \). The above conclusion is obtained by exploiting compactness of attractors and finite net coverage.

STEP 4: Having obtained smoothness of attractor, one proceed to prove finite-dimensionality. This becomes a straightforward consequence of quasi-stability property on the attractor resulting from the aforementioned smoothness.

It should be noticed that the method described above is fundamentally based on gradient property of the dynamics. The full details are given in [CLT09].

Remark 5.3 In the case of boundary damping (linearly bounded) there is another method that leads to compactness of attractors also in the presence of critical sources. This method relies on a suitable split of the dynamics [CEL02]. In such case there is no need for Carleman’s estimates. However, properties such as regularity and finite dimensionality of attractors intrinsically depend on quasi-stability estimate. The latter requires consideration of the differences of two solutions. It is this step that forces Carleman’s estimates to operate. In the subcritical case this is not necessary and more direct estimates provide full spectrum of results also in the boundary case, see [CEL04].

5.2 Von Karman plate dynamics

Internal damping

We consider system (2.4) with clamped boundary conditions (2.5) and discuss long time behavior under the following hypothesis.

Assumption 5.12 We assume that Assumption 3.5 holds and, in addition, \( g_i \in C^1 \) and \( g'_i(s) \geq m \) for \( |s| \geq s_0 \) in the case \( \alpha > 0 \) and in the case \( \alpha = 0 \): \( g \in C^1 \) and \( g'(s) \geq m \) for \( |s| \geq s_0 \).

We also want to be more specific about the nature of the source \( P(w) \). For sake of simplicity, we do not consider non-conservative force cases which lead to systems that are no longer of gradient structure. Instead we limit ourselves to conservatively forced models. We suppose that Assumption 3.6 holds with \( P_1(w) \equiv 0 \), i.e., \( P(w) \) is a Fréchet derivative of the functional \(-\Pi_0(w)\) and the following property holds: there exist \( a \in \mathbb{R} \) and \( \epsilon > 0 \) such that

\[
\Pi_0(w) + a\|w\|_{2-\epsilon,\Omega}^2 \geq 0, \quad ((P_0(w), w)) + a\|w\|^2_{2-\epsilon,\Omega} \geq 0, \quad \forall w \in H^2_0(\Omega). \quad (5.20)
\]
Remark 5.4 In the specific example given in Remark 3.2 the functional $\Pi_0$ takes the form

$$\Pi_0(u) = -\frac{1}{2}([F_0, u], u) - (p, u), \quad F_0 \in H^{3+\delta}(\Omega) \cap H^1_0(\Omega), \quad p \in L^2(\Omega).$$

Thus, the assumption (5.20) is satisfied (in the case $\alpha > 0$ we can take less regular $F_0$).

Under these conditions concerning $P$ the energy function can be written as

$$E(u, u_t) = \frac{1}{2} \left[ \|u_t\|^2 + \alpha \|\nabla u_t\|^2 \right] + \frac{1}{4} \|\Delta F(u)\|^2 + \Pi_0(u).$$

Remark 5.5 One can easily obtain the same result for a more general class of nonlinear sources $P(w)$ which can be axiomatized by (5.20) with an additional requirement that the map is compact.

It is easy to verify (internal damping) that the energy $E(u, u_t)$ is a strict Lyapunov function for $(H_\alpha, S_t)$. Hence $(H_\alpha, S_t)$ is a gradient system (see Definition 4.8 in Section 4.3), and the application of Theorem 4.9 yields the following result.

Theorem 5.13 (Compactness) Let Assumption 5.12 be in force. Assume that $P(w) = [F_0, w] + p$, where $F_0 \in H^{3+\delta}(\Omega) \cap H^1_0(\Omega)$ and $p \in L^2(\Omega)$. Then the dynamical system $(H_\alpha, S_t)$ generated by equations (2.4) with the clamped boundary conditions (2.5) possesses a global compact attractor $A$ in the space $H_\alpha \equiv H^1_0(\Omega) \times H^0_0(\Omega)$, where $H_\alpha$ is given by (3.18).

In the case $\alpha > 0$ the proof of Theorem 5.13 follows from general abstract results for second order equations with subcritical source (see [CL08a]). In the case $\alpha = 0$ the force is critical and the proof relies on Theorem 4.3 in the same manner as in the case of wave equations. We refer to [CL10] for details.

Remark 5.5 One can easily obtain the same result for a more general class of nonlinear sources $P(w)$ which can be axiomatized by (5.20) with an additional requirement that the map is compact.

It is easy to verify (internal damping) that the energy $E(u, u_t)$ is a strict Lyapunov function for $(H_\alpha, S_t)$. Hence $(H_\alpha, S_t)$ is a gradient system (see Definition 4.8 in Section 4.3), and the application of Theorem 4.9 yields the following result.

Theorem 5.14 (Structure) Let Assumption 5.12 be in force. Assume that $P$ in (2.4) satisfies (5.20). Let $(H_\alpha, S_t)$ be the dynamical system generated by equations (2.4) with clamped boundary conditions Assume that $(H_\alpha, S_t)$ possesses a compact global attractor $A$. Let $N_*$ be a set of stationary solutions to (2.4) (see Proposition 3.11). Then

- $A = \mathcal{M}^u(N)$, where $N = \{ (w; 0) : w \in N_* \}$ is the set of stationary point of $(H_\alpha, S_t)$ and $\mathcal{M}^u(N)$ is the the unstable manifold $\mathcal{M}^u(N)$ emanating
from $N$ which is defined as a set of all $U \in H_\alpha$ such that there exists a full trajectory $\gamma = \{U(t) = (u(t); u_t(t)) : t \in \mathbb{R}\}$ with the properties

$$U(0) = U \text{ and } \lim_{t \to -\infty} \text{dist}_{H_\alpha}(U(t), N) = 0.$$ 

- The global attractor $A$ consists of full trajectories $\gamma = \{(u(t); u_t(t)) : t \in \mathbb{R}\}$ such that

$$\lim_{t \to \pm \infty} \left\{ \|u_t(t)\|_{H_\alpha}^2 + \inf_{w \in N_*} \|u(t) - w\|_2^2 \right\} = 0.$$

- Any generalized solution $u(t)$ to problem (2.4) stabilizes to the set of stationary points; that is,

$$\lim_{t \to +\infty} \left\{ \|u_t(t)\|_{H_\alpha}^2 + \inf_{w \in N_*} \|u(t) - w\|_2^2 \right\} = 0. \quad (5.21)$$

This theorem along with generic-type results on the finiteness of the number of solutions to the stationary problem allow us to obtain the following result.

**Corollary 5.15** Under the hypotheses of Theorem 5.14 there exists an open dense set $\mathcal{R}_0$ in $L_2(\Omega)$ such that for every load function $p \in \mathcal{R}_0$ the set $N$ of stationary points for $(H_\alpha, S)$ is finite. In this case $A = \bigcup_{i=1}^N M^{\ast}(z_i)$, where $z_i = (w_i; 0)$ and $w_i$ is a solution to the stationary problem. Moreover,

- The global attractor $A$ consists of full trajectories $\gamma = \{(u(t); u_t(t)) : t \in \mathbb{R}\}$ connecting pairs of stationary points; that is, any $W \in A$ belongs to some full trajectory $\gamma$ and for any $\gamma \subset A$ there exists a pair $\{w_-, w_+\} \subset N^*$ such that

$$\lim_{t \to -\infty} \left\{ \|u_t(t)\|_{H_\alpha}^2 + \|u(t) - w_-\|_2^2 \right\} = 0$$

and

$$\lim_{t \to +\infty} \left\{ \|u_t(t)\|_{H_\alpha}^2 + \|u(t) - w_+\|_2^2 \right\} = 0.$$

- For any $(u_0; u_1) \in H_\alpha$ there exists a stationary solution $w \in N^*$ such that

$$\lim_{t \to +\infty} \left\{ \|u_t(t)\|_{H_\alpha}^2 + \|u(t) - w\|_2^2 \right\} = 0, \quad (5.22)$$

where $u(t)$ is a generalized solution to problem (2.4) with initial data $(u_0; u_1)$ and with the clamped boundary conditions.

**Remark 5.6** We note that the generic class $\mathcal{R}_0$ of loads in the statement of Corollary 5.15 can be rather thin. The standard example (see, e.g., [Chu99, Section 2.5]) is the set $\mathcal{R}_0$ in the interval $[0, 1]$ of the form

$$\mathcal{R}_0 = \left\{ x \in (0, 1) : \exists r_k \in \mathbb{Q}, \ |x - r_k| \leq \varepsilon 2^{-k-1} \right\},$$
where $\mathbb{Q}$ is the sequence of all rational numbers. This set is open and dense in $[0, 1]$, but its Lebesgue measure less than $\varepsilon$. However using analyticity of the von Karman force terms and the Lojasiewicz-Simon inequality we can obtain a result (see [Chu11b]) on convergence for all loads $p$ under some additional non-degneracy at the origin type conditions concerning damping terms. By assuming this additional non-degeneracy condition (always satisfied when $g'(0) > 0$ or in the hyperbolic case when $g(s)s \geq ms^2$, $|s| \leq 1$) the result of Corollary 5.15 holds for all loads $p \in L_2(\Omega)$. For general discussion of Lojasiewicz-Simon method we refer to [HJ99, HJ09] and to the references therein.

It follows from Assumption 5.12 (see [CL10], for instance) that there exists a strictly increasing continuous concave function $H_0(s)$ such that

$$s_1^2 + s_2^2 \leq H_0(s_1g_1(s_1) + s_2g_2(s_2)) \text{ for all } s_1, s_2 \in \mathbb{R},$$

in the case $\alpha > 0$ and $s^2 \leq H_0(sg(s))$ for $s \in \mathbb{R}$ in the case $\alpha = 0$.

The above facts are used in formulation of the result on the rate of convergence to an equilibrium.

**Theorem 5.16 (Rates of convergence to equilibria)** Assume that $P$ satisfies (5.20) in (2.4) and the hypotheses above that guarantee the existence of a compact global attractor $A$ for the dynamical system $(H_\alpha, S_t)$ generated by problem (2.4) with the clamped boundary conditions hold.

If the set of stationary solutions consists of finitely many isolated equilibrium points that are hyperbolic, then for any initial condition $y \in H_\alpha$ there exists an equilibrium point $e = (w; 0)$, $w \in H_0^2(\Omega)$, such that

$$\|S_t y - e\|_{H_\alpha}^2 \leq C \cdot \sigma ([tT^{-1}]), \quad t > 0,$$

where $C$ and $T$ are positive constants depending on $y$ and $e$, $[a]$ denotes the integer part of $a$ and $\sigma(t)$ satisfies the following ODE,

$$\frac{d\sigma}{dt} + Q(\sigma) = 0, \quad t > 0, \quad \sigma(0) = C(y, e). \quad (5.24)$$

Here $C(y, e)$ is a constant depending on $y$ and $e$,

$$Q(s) = s - (I + G_0)^{-1} (s) \quad \text{with} \quad G_0(s) = c_1 (I + H_0)^{-1} (c_2 s), \quad (5.25)$$

where positive numbers $c_1$ and $c_2$ depend on $y$ and $e$. If in Assumption 5.12 $s_0 = 0$, then the rate of convergence in (5.23) is exponential.

**Remark 5.7** It is also possible to state a result on the convergence rate without assuming hyperbolicity of equilibrium points (see [Chu11b]). However as in the case described in Remark 5.6 this requires additional non-degeneracy type properties of the damping terms. Moreover, the lack of hyperbolicity is compromised by slowing down the decay rates.

\(^3\text{In the sense that the linearization of the stationary problem with the corresponding boundary conditions around every stationary solution has a trivial solution only.}\)
The following result describes finite dimensionality of attractors.

**Theorem 5.17 (Dimension)** Assume the hypotheses which guarantee the existence of a compact global attractor $A$ for the system $(H, S)$ generated by problem (2.4) with the boundary conditions (2.5) hold (see Theorem 5.13) and $\alpha \geq 0$. In addition, we assume that

(i) In the case $\alpha = 0$ there exist $m, M_0 > 0$ such that

\[ 0 < m \leq g'(s) \leq M_0 [1 + sg(s)], \quad s \in \mathbb{R}, \quad (5.26) \]

(ii) When $\alpha > 0$ we assume that (i) $g(s)$ is either of polynomial growth at infinity or else satisfies (5.26), and (ii) there exists $0 \leq \gamma < 1$ such that the functions $g_i$ satisfy the inequality

\[ 0 < m \leq g_i'(s) \leq M [1 + sg_i(s)]^\gamma, \quad s \in \mathbb{R}, \quad i = 1, 2, \quad (5.27) \]

Then the fractal dimension of the attractor $A$ is finite.

**Remark 5.8** One can see that in the case $\alpha > 0$ the condition imposed on $g_i$ in (5.27) are valid if in addition to Assumption 5.12 we assume that there exists $m, M_1, M_2 > 0$ such that

\[ 0 \leq g_i'(s) \leq M_1 [1 + |s|^{p-1}], \quad sg_i(s) \geq m(|s|^{(p-1)/\gamma} - M_2), \quad s \in \mathbb{R}, \quad (5.28) \]

for some $p \geq 1$ (see Remark 9.2.7 [CL10]).

The following assertion on regularity of elements in the attractor is a straightforward consequence of Theorem 4.17.

**Theorem 5.18 (Regularity)** Let the hypotheses of Theorem 5.17 be valid. Then any full trajectory $\gamma = \{(u(t); u_t(t)) : t \in \mathbb{R}\}$ from the attractor $A$ of the dynamical system $(H, S)$ generated by problem (2.18) with the boundary conditions (2.5) possesses the properties

\[ u(t) \in C_r(\mathbb{R}; W), \quad (u_t(t), u_{tt}(t)) \in C_r(\mathbb{R}; H), \quad (5.28) \]

where $C_r$ means right-continuous functions and

- in the case $\alpha > 0$: $H = H^2_\gamma(\Omega) \times H^1_\gamma(\Omega)$ and $W = H^3(\Omega) \cap H^2_\gamma(\Omega)$.
- in the case $\alpha = 0$: $H = H^2_\gamma(\Omega) \times L^2(\Omega)$ and $W = H^4(\Omega) \cap H^2_\gamma(\Omega)$.

Moreover,

- in the case $\alpha > 0$: $A \subset H^3(\Omega) \times H^2(\Omega)$ and

\[ \sup_{t \in \mathbb{R}} \left( \|u(t)\|^2 + \|u_t(t)\|^2 + \|u_{tt}(t)\|^2 \right) \leq C_A < \infty, \]

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in the case $\alpha = 0$: $A \subset H^4(\Omega) \times H^2(\Omega)$ and
\[
\sup_{t \in \mathbb{R}} (\|u(t)\|_4^2 + \|u_t(t)\|_2^2 + \|u_{tt}(t)\|_1^2) \leq C_A < \infty.
\]

The validity of quasi-stability property for the system allows us to deduce, without an additional effort, another important property of dynamical system. This is existence of strong attractors that is, attractors in a strong topology determined by the generators of the (linearized) dynamical system. The corresponding result is formulated below.

**Theorem 5.19 (Strong attractors)** We assume that the hypotheses of Theorem 5.18 are in force.

**Case** $\alpha > 0$: Let the rotational damping be linear; that is, $g_i(s) = g_i \cdot s$ for $i = 1, 2$. Then the global attractor $A$ is also strong: for any bounded set $B$ from $H_{st} = (H^3(\Omega) \cap H^2_0(\Omega)) \times H^2_0(\Omega)$ we have that
\[
\lim_{t \to \infty} \sup_{y \in B} \text{dist}_{H_{st}}(S ty, A) = 0.
\]

**Case** $\alpha = 0$: Let $H_{st} = (H^4(\Omega) \cap H^2_0(\Omega)) \times H^2_0(\Omega)$. The global attractor $A$ of the system $(H_0, S_t)$ generated by the generalized solutions to problem (2.4) with clamped b.c. (2.5) is also strong, i.e. $A$ is a global attractor for the system $(H_{st}, S_t)$.

Moreover, in both cases $\alpha = 0$ and $\alpha > 0$ the global attractor $A$ has a finite dimension as a compact set in $H_{st}$.

**Proof.** We apply the idea of Theorem 8.8.4[CL10]. The hypotheses of Theorem 5.18 guarantees that the system $(H_0, S_t)$ is quasi-stable.

An important property of the dynamical system is an existence of exponential attractors, which attract trajectories at the exponential rate. The quasi-stability property allows us to deduce an existence of exponential attractors. The corresponding result is formulated below.

**Theorem 5.20 (Exponential attractors)** Let $(H_\alpha, S_t)$ be the dynamical system generated by problem (2.4) and (2.5).

**Case** $\alpha > 0$: Under the condition (5.27) the dynamical system $(H_\alpha, S_t)$ has a (generalized) fractal exponential attractor $A$ (see Definition 4.18) whose dimension is finite in the space $H^4(\Omega) \times W'$, where $W'$ is a completion of $H^1_0(\Omega)$ with respect to the norm $\| \cdot \|_{W'} = \| (1 - \alpha \Delta) \cdot \|_2$.

**Case** $\alpha = 0$: Assume the validity of the hypotheses above which guarantee the existence of a global finite-dimensional attractor. Assume in addition that
\[
|g(s)| \leq C (1 + sg(s))^\gamma, \quad s \in \mathbb{R}, \tag{5.29}
\]
for some $0 \leq \gamma < 1$. Then the system $(H_0, S_t)$ has a (generalized) fractal exponential attractor $A$ whose dimension is finite in the space $L_2(\Omega) \times H^{-2}(\Omega)$.
Main ingredients for the proofs. The rotational case $\alpha > 0$ corresponds to the situation where von Karman nonlinearity is compact, hence subcritical. This alleviates number of technical issues when proving validity of quasi-stability inequality which, in turn, leads to the existence of smooth and finite-dimensional attractors. Indeed, build in compactness turns critical integrals into lower order terms.

We thus focus here on the more subtle non-rotational case, $\alpha = 0$. Here, the main challenge is to establish quasi-stability inequality. Once this is proved, finite-dimensionality of attractors - Theorem 5.17, smoothness of attractor - Theorem 5.18 and existence of strong and exponential attractors - Theorem 5.19 and also Corollary 5.20 follow from the abstract tools presented in Section 4.4.

Thus, the core of the difficulty relates to the quasi-stability estimate obtained for a difference of two solutions $z = u - w$ defined on $Q = \Omega \times (0, T), \Sigma = \Gamma \times (0, T)$. The solutions under consideration are confined to a bounded invariant set (attractor, for instance), so we can assume

$$\|u(t)\|^2 + \|u(t)\|^2 \leq R^2, \quad \|w(t)\|^2 + \|w(t)\|^2 \leq R^2.$$  

To present the main idea we assume that $P(u) \equiv 0$ in (2.4). In this case the equation for $z$ can be written as:

$$z_{tt} + \Delta z = f \quad \text{in} \quad \Omega, \quad z = 0, \quad \frac{\partial}{\partial n} z = 0 \quad \text{on} \quad \Gamma, \quad (5.30)$$

where $f = -a(g(u_t) - g(w_t)) + [v(u), u] - [v(w), w]$.

We have the following energy equality (satisfied for strong solutions)

$$E_z(T) + D^T_t(z) = E_z(t) + \int_t^T (R(z), z_t)dxdt, \quad (5.31)$$

where

$$E_z(t) = \frac{1}{2} \int_\Omega [\|z_t\|^2 + \|\Delta z\|^2]dx, \quad R(z) = [v(u), u] - [v(w), w],$$

and

$$D^T_t(z) = \int_t^T \int_\Omega a[g(u_t) - g(w_t)]z_t dxdt.$$  

Our goal is to prove the following recovery inequality:

$$E_z(T) + \int_0^T E_z(t)dt \leq C_R D^T_0(z) + LOT(z), \quad (5.32)$$

where

$$LOT(z) \leq C_R \sup_{t \in [0, T]} \|z(t)\|_{2, \Omega}^2.$$  

Inequality (5.32), via reiteration on the intervals $[mT, (m+1)T]$, leads to quasi-stability inequality.
To prove (5.32) one applies standard multipliers $z_t$ and $z$. After some calculations (as for the wave equation) one obtains
\[
E_z(T) + \int_0^T E_z(t) \, dt \leq C_R D_0^T (z) + C_R L O T(z) + C_T \int_0^T (\mathcal{R}(z), z_t) \, dt. \quad (5.33)
\]
The difficulty, however, is in handling the critical term
\[
\int_0^T (\mathcal{R}(z), z_t) \Omega dt.
\]
In order to obtain quasi-stability inequality, this critical term should be represented in terms of the damping $D_0(T)(z)$, small multiples of the integrals of the energy, i.e., $\epsilon \int_0^T E_z(t) \, dt$ and lower order terms (subcritical quantities) in a quadratic form. The key to this argument is "compensated compactness structure" of the von Karman bracket which leads to the following estimate:
\[
| \int_0^T (\mathcal{R}(z), z_t) \Omega dt | \leq C_R \max_{t \in [0, T]} \| z(t) \|_2^2 + C_R \int_0^T (\| u_t \|_2 + \| w_t \|_2) \| z \|_2^2 \Omega dt \quad (5.34)
\]
The critical role is played by the presence in (5.34) of the velocities $\| u_t \|, \| w_t \|$ which represent the damping and obey the estimate
\[
\int_0^{+\infty} (\| u_t \|^2 + \| w_t \|^2) dt \leq C_R \quad (5.35)
\]
Indeed, once (5.34) is proved and inserted into (5.33), then (5.35) along with Cronwall’s inequality leads to the desired quasi-stability estimate. Thus, the key is to be able to prove (5.34). To this end the following "compensated compactness" decomposition of the von Karman bracket is used:
\[
(\mathcal{R}(z), z_t) = \frac{1}{4} \frac{d}{dt} Q(z) + \frac{1}{2} P(t)
\]
where
\[
Q(z) = (v(u) + v(w), [z, z]) - |\Delta v(u + w, z)|^2,
\]
\[
P(z) = -(u_t, [u, v(z, z)]) - (w_t, [w, v(z, z)])
\]
\[
-(u_t + w_t, [z, v(u + w, z)]),
\]
where $v(u, w) = \Delta^{-2}([u, w])$. By using sharp Airy’s regularity
\[
|v(u, v)|_{W^{2, \infty}} \leq C \| u \|_{2, \Omega} \| v \|_{2, \Omega}
\]
given by Lemma 3.9, one obtains the estimates
\[
|Q(z)| \leq C \| z \|^2_{2-\varepsilon, \Omega} \quad \text{and} \quad |P(z)| \leq C_R (\| u_t \| + \| w_t \|) \| z \|^2_{2, \Omega}.
\]
The above yields
\[
\int_0^T (R(z), z_t) dt \leq C(R)[Q(T) - Q(0)] + C(R) \int_0^T P(z) dt \\
\leq C_R \int_0^T (\|u_t\| + \|w_t\|)\|z\|_{2,\Omega}^2 + C_R \max_{t \in [0,T]} \|z(t)\|_{2-\epsilon}^2
\]
as desired for (5.34), hence for quasi-stability inequality.

**Boundary damping**

We consider now the model in (2.18) with the boundary conditions in (2.19). This type of problems is often referred to as control problem with reduced number of controls.

We impose the following hypotheses.

**Assumption 5.21**

• Assumption 3.5(1,2) is in force, \(a \in L_\infty(\Omega)\), \(a(x) \geq 0\) almost everywhere and \(P(w) = p \in L_2(\Omega)\).

• The boundary damping \(g_0 \in C^1(\mathbb{R})\) is increasing, with the property \(g_0(0) = 0\) and there exist positive constants \(m, M,\) and \(s_0\) such that
\[
0 < m \leq g_0'(s) \leq M \quad \text{for } |s| \geq s_0. \tag{5.36}
\]

**Case \(\alpha > 0\):**

We use the state space \(H \equiv [H^2(\Omega) \cap H^1_0(\Omega)] \times H^1_0(\Omega)\).

Using energy equality one can easily prove the following assertion.

**Theorem 5.22** Let Assumption 5.21 be in force and \(\alpha > 0\). Then

• There exists \(R_* > 0\) such that the set
\[
\mathcal{W}_R = \{y = (u_0; u_1) \in H : E(u_0, u_1) \leq R\} \tag{5.37}
\]
is a nonempty invariant set with respect to semiflow \(S_t\) generated by equations (2.18) with hinged b.c. (2.19) for every \(R \geq R_*\). Moreover, the set \(\mathcal{W}_R\) is bounded for every \(R \geq R_*\) and any bounded set is contained in \(\mathcal{W}_R\) for some \(R\).

• If in addition \(a(x) > 0\) almost everywhere in \(\Omega\), the system \((H, S_t)\) is gradient.

• The set \(\mathcal{N}\) of all stationary points of the semiflow \(S_t\) is bounded in \(H\). Thus there exists \(R_{**} > 0\) such that \(\mathcal{N} \subset \mathcal{W}_R\) for all \(R \geq R_{**}\).

We recall that
\[
\mathcal{N} = \{V \in H : S_t V = V \text{ for all } t \geq 0\}.
\]
and every stationary point $W$ has the form $W = (u; 0)$, where $u = u(x) \in H^2(\Omega)$ is a weak (variational) solution to the problem

$$\Delta^2 u = [v(u), u] + p \text{ in } \Omega; \quad u = 0, \quad \Delta u = 0 \text{ on } \Gamma, \quad (5.38)$$

with function $v(u)$ satisfying (2.6) with $w = u$, i.e., $v(u) = \Delta^{-2}([u, u])$.

**Theorem 5.23 (Compact attractors)** Let $\alpha > 0$ and Assumption 5.21 be in force. Then

- The restriction $(W_R, S_t)$ of the dynamical system $(H, S_t)$ on $W_R$ given by (5.37) has a compact global attractor $A_R \subset W_R$ for every $R \geq R^*$, where $R^*$ is the same as in Theorem 5.22.

- If, in addition, $a(x) > 0$ almost everywhere in $\Omega$, then there exists a compact global attractor $A$ for the system $(H, S_t)$. Moreover, we have $A = M^u(N)$, where $M^u(N)$ is the unstable manifold (see the definition in Section 4.3) emanating from the set $N$ of equilibria for the semiflow $S_t$.

- The two attractors $A_R$ and $A$ have finite fractal dimension provided relation (5.36) holds for all $s \in \mathbb{R}$. Moreover, in this latter case the attractors are bounded sets in the space $H^3(\Omega) \times H^2(\Omega)$ and for any trajectory $(u(t), u_t(t))$ we have the relation

$$\|u_{tt}(t)\|_{1,\Omega} + \|u_t(t)\|_{2,\Omega} + \|u(t)\|_{3,\Omega} \leq C, \quad t \in \mathbb{R}.$$

An important ingredient of the argument is

**Proposition 5.24 (Observability estimate)** Assume that $\alpha > 0$ and Assumption 5.21 is in force. Let

$$U(t) = (u(t); u_t(t)) = S_t y_1 \quad \text{and} \quad W(t) = (w(t); w_t(t)) = S_t y_2$$

be two solutions corresponding to initial conditions $y_1$ and $y_2$ from the set $W_R$ given by (5.37). Then there exist $T_0 > 0$ and constants $C_1(T)$ and $C_2(R, T)$ such that

$$T E_z(T) + \int_0^T E_z(t) dt \leq C_1(T) \int_{\Sigma_1} \left| \frac{\partial}{\partial n} z_t \right|^2 d\Sigma + C_2(R, T) \cdot \text{LOT}(z) \quad (5.39)$$

for any $T \geq T_0$, where $z \equiv u - w$,

$$E_z(t) = \frac{1}{2} \int_{\Omega} \left[ |z_t|^2 + \alpha |\nabla z_t|^2 + |\Delta z|^2 \right] dx,$$

and the lower-order terms have the form

$$\text{LOT}(z) = \sup_{0 \leq \tau \leq T} \|z(\tau)\|_{0,\Omega}^2 + \int_0^T \|z_t(\tau)\|_{0,\Omega}^2 d\tau. \quad (5.40)$$
We can obtain results on convergence rates of individual trajectories to equilibrium under the condition that the set $\mathcal{N}$ of stationary points is discrete.

As in the previous section we introduce a concave, strictly increasing, continuous function $H_0 : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ which captures the behavior of $g_0(s)$ at the origin possessing the properties

$$H_0(0) = 0 \quad \text{and} \quad s^2 \leq H_0(s_0(s)) \quad \text{for} \quad |s| \leq 1. \quad (5.41)$$

We define a function $Q(s)$ by relations (5.25) and consider the differential equation

$$\frac{d\sigma}{dt} + Q(\sigma) = 0, \quad t > 0, \quad \sigma(0) = \sigma_0 \in \mathbb{R}^+, \quad (5.42)$$

which admits the global unique solution $\sigma(t)$ decaying asymptotically to zero as $t \to \infty$.

With these preparations we are ready to state our result.

**Theorem 5.25 (Rate of stabilization)** Let the hypotheses of Theorem 5.23 be valid with $a(x) > 0$ a.e. Assume that there exist $\gamma > 0$ and $s_0 > 0$ such that the interior damping $g(s)$ satisfies the relation $sg(s) \geq \gamma s^2$ for $|s| \leq s_0$. In addition assume that problem (5.38) has a finite number of solutions. Then for any $V \in H$ there exists a stationary point $E = (e; 0)$ such that $S^t V \rightarrow E$ as $t \to +\infty$. Moreover, if the equilibrium $E$ is hyperbolic in the sense that the linearization of (5.38) around each of its solutions has the trivial solution only, then there exist $C, T > 0$ depending on $V, E$ such that the following rates of stabilization

$$\|S^t V - E\|_H \leq C\sigma([tT^{-1}]), \quad t > 0,$$

hold, where $[a]$ denotes the integer part of $a$ and $\sigma(t)$ satisfies (5.42) with $\sigma_0$ depending on $V, E \in H$ (the constants $c_i$ in the definition of $Q$ also depend on $V$ and $E$). In particular, if $g'_0(0) > 0$, then

$$\|S^t V - E\|_H \leq Ce^{-\omega t}$$

for some positive constants $C$ and $\omega$ depending on $V, E \in H$.

**Case $\alpha = 0$ and $H \equiv [H^2(\Omega) \cap H^1_0(\Omega)] \times L_2(\Omega)$:**

We consider a model with nonlinear boundary dissipation acting via hinged boundary conditions which does not account for regularizing effects of rotational inertia. Thus, the corresponding solutions are less regular than in the case of rotational models.

The following analog of Theorem 5.22 is valid.

**Theorem 5.26** Let Assumption 5.21 hold and $\alpha = 0$. Then

- There exists $R_* > 0$ such that the set

$$\mathcal{W}_R = \{y = (u_0; u_1) \in H : E(u_0, u_1) \leq R\} \quad (5.43)$$

is a nonempty bounded set in $H$ for all $R \geq R_*$. Moreover, any bounded set $B \subset H$ is contained in $\mathcal{W}_R$ for some $R$ and the set $\mathcal{W}_R$ is invariant with respect to the semiflow $S_t$. 

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If in addition \( a(x) > 0 \) then the system \((H, S_t)\) is gradient.

Our main goal is to prove the global attractiveness property for the dynamical system \((H, S_t)\). This property requires additional hypotheses imposed on the data of the problem.

**Assumption 5.27** The interior damping function \( g(s) \) is globally Lipschitz, i.e., \( |g(s_1) - g(s_2)| \leq M|s_1 - s_2| \) for all \( s_1, s_2 \in \mathbb{R} \).

This global Lipschitz requirement is due to the fact that only one boundary condition is used as a source of dissipation.

Our main result is the following.

**Theorem 5.28 (Compact attractors)** We suppose that Assumptions 5.21 and 5.27 are in force. Then

- For any \( R \geq R_* \) there exists a global compact attractor \( \mathcal{A}_R \) for the restriction \((W_R, S_t)\) of the dynamical system \((H, S_t)\) on \( W_R \), where \( W_R \) is given by (5.43).

- If we assume additionally that \( a(x) > 0 \) a.e. in \( \Omega \) and \( g(s)s > 0 \) for all \( s \neq 0 \), then there is \( R_0 > 0 \) such that \( \mathcal{A}_R \) does not depend on \( R \) for all \( R \geq R_0 \). In this case \( \mathcal{A} \equiv \mathcal{A}_{R_0} \) is a global attractor for \((H, S_t)\) and \( \mathcal{A} \) coincides with the unstable manifold \( M^u(N) \) emanating from the set \( N \) of stationary points for \( S_t \). Moreover, \( \lim_{t \to +\infty} \text{dist}_H(S_tW, N) = 0 \) for any \( W \in H \).

- The global attractors \( \mathcal{A}_R \) and \( \mathcal{A} \) are bounded sets in \( H^3(\Omega) \times H^2(\Omega) \) and have a finite fractal dimension provided the relation in (5.36) holds for all \( s \in \mathbb{R} \).

From Theorem 5.28 we obtain the following corollary.

**Corollary 5.29** Let the hypotheses of Theorem 5.28 be in force. Assume that \( a(x) > 0 \) a.e. in \( \Omega \) and \( g(s)s > 0 \) for all \( s \neq 0 \). Then the global attractor \( \mathcal{A} \) consists of full trajectories \( \gamma = \{W(t) : t \in \mathbb{R}\} \) such that

\[
\lim_{t \to -\infty} \text{dist}_H(W(t), N) = 0 \quad \text{and} \quad \lim_{t \to +\infty} \text{dist}_H(W(t), N) = 0.
\]

In particular, if we assume that equation (5.38) has a finite number of solutions, then the global attractor \( \mathcal{A} \) consists of full trajectories \( \gamma = \{W(t) : t \in \mathbb{R}\} \) connecting pairs of stationary points: any \( W \in \mathcal{A} \) belongs to some full trajectory \( \gamma \) and for any \( \gamma \subset \mathcal{A} \) there exists a pair \( \{Z, Z^*\} \subset N \) such that \( W(t) \to Z \) as \( t \to -\infty \) and \( W(t) \to Z^* \) as \( t \to +\infty \). In the latter case for any \( V \in H \) there exists a stationary point \( Z \) such that \( S_tV \to Z \) as \( t \to +\infty \).

In analogy with the previous cases one can also provide statements for the rate of stabilizations to equilibria. The statement (and the proofs) do not depend on the specific boundary conditions. However, in this case we do not need to
assume additional coercivity estimates for the interior damping. The point is that our basic Assumptions 5.21 and 5.27 are sufficient to conclude that generalized solutions are weak (variational).

**Main ingredients for the proofs:** As in the case of interior damping, the rotational case \( \alpha > 0 \) is less involved. We shall thus focus on the non-rotational model \( \alpha = 0 \) where additional subtleties are present. Similar to the interior damping case, Theorem 4.3 provides the main tool for establishing asymptotic smoothness. For sake of avoiding repetitions, we shall mainly emphasize the parts of the proof that are more specific to the boundary damping. The proof is divided into several steps which are presented below. As before, we use notation \( Q \equiv \Omega \times (0, T), \Sigma \equiv \Gamma \times (0, T) \) and assume that \( P(u) \equiv 0 \) in (2.18).

We can assume that \( u \) and \( w \) are strong (smooth) solutions. Then the difference \( z \equiv u - w \) solves the following problem

\[
\begin{aligned}
zh_{tt} + \Delta^2 z &= f \quad \text{in } \Omega, \\
z_{tt} &= 0, \quad \Delta z = \psi \quad \text{on } \Gamma,
\end{aligned}
\]  

where \( f = -a(g(u_t) - g(w_t)) + [v(u), u] - [v(w), w] \) and the boundary conditions are given by

\[
\psi = -\left[ g_0 \left( \frac{\partial}{\partial n} u_t \right) - g_0 \left( \frac{\partial}{\partial n} w_t \right) \right].
\]

We have the following energy equality (satisfied for strong solutions)

\[
E_z(T) + D^T_z(z) = E_z(t) + \int_t^T ((R(z), z_t)) dt,
\]

where

\[
E_z(t) = \frac{1}{2} \int_{\Omega} \left( |z_t|^2 + |\Delta z|^2 \right) dx,
\]

and

\[
D^T_z(z) = \int_t^T \int_{\Omega} a[g(u_t) - g(w_t)] z_t dx d\tau + D^T_\Gamma(z),
\]

with

\[
D^\Gamma_\tau(z) = \int_t^T \int_{\Gamma} \left[ g_0 \left( \frac{\partial}{\partial n} u_t \right) - g_0 \left( \frac{\partial}{\partial n} w_t \right) \right] \frac{\partial}{\partial n} z_t d\Gamma d\tau.
\]

We also use the following assertion.

**Lemma 5.30** Let \( T > 0 \) be given. Let \( \phi \in C^2(\mathbb{R}) \) be a given function with support in \( [\delta, T - \delta] \), where \( \delta \leq T/4 \), such that \( 0 \leq \phi \leq 1 \) and \( \phi \equiv 1 \) on \( [2\delta, T - 2\delta] \). Then any strong solution \( z \) to problem (5.44) satisfies the following inequality

\[
\begin{aligned}
\int_0^T E_z(t) \phi(t) dt &\leq C_1 \int_0^T E_z(t) |\phi(t)| dt + C_2 \int_\Sigma \left[ |\psi|^2 + \left( \frac{\partial}{\partial n} z \right)^2 \right] d\Sigma \\
&+ \int_0^T \int_\Omega f h \nabla z \phi dx dt + \int_0^T \int_\Omega R(z) z_t dx dt + C_3 \cdot B_T(z),
\end{aligned}
\]

(5.46)
where the constants $C_i$ do not depend on $T$ and the boundary terms\footnote{They are not defined on the energy space. These are higher-order boundary traces of solutions.} $B_T(z)$ is given by

$$B_T(z) = \int_\Sigma \left[ \left( \frac{\partial}{\partial n} \right)^2 z \right]^2 + \left( \frac{\partial}{\partial n} \frac{\partial}{\partial \tau} z \right)^2 \phi d\Sigma + \int_0^T \left\| \frac{\partial}{\partial n} \Delta z \right\|^2_{L^2(\Gamma)} \phi dt. \quad (5.47)$$

On the next step we eliminate the second- and third-order boundary traces on the boundary in the expression (5.47) for $B_T(z)$. These second-order “supercritical” boundary traces are due to the fact that the dissipation is allowed to affect the system via only one boundary condition. Traces of the second and third order (see (5.47)) are above the energy level, so these cannot enter the estimates. It is the microlocal analysis argument, again, that allows for the elimination of these terms. Handling of the boundary terms requires delicate trace estimates used already in the case $\alpha > 0$. However the treatment of critical source (giving rise to the term $R(z)$) is more troublesome now. The dissipativity integral is supported on the boundary and not in the interior. Here are few details. For the trace result we use a more general trace estimate proved in [JL99] (see Proposition 1 and Lemma 4) valid for the linear Kirchhoff problem (note that the estimates in [JL99] are independent of the parameter representing rotational forces). Thus, these estimates are applicable to both Kirchhoff and Euler–Bernoulli (5.44) models. As a consequence we have the following

**Lemma 5.31** Let $z$ be a solution to linear problem (5.44) with given $f$ and $\psi$ and $B_T(z)$ be given by (5.47). Then there exist constants $C_T > 0$ such that for any $0 < \eta < \frac{1}{2}$ the following estimate holds

$$B_T(z) \leq C_{1,T} \int_\Sigma \left( |\psi|^2 + \left| \frac{\partial}{\partial n} z_t \right|^2 \right) \phi d\Sigma + C_{2,T} \left[ \|z\|^2_{L^2([0,T];H^{1-\eta}(\Omega))} + \|z_t\|^2_{L^2([0,T];H^{-\eta}(\Omega))} + \int_0^T \|f\|_{H^{-\eta}(\Omega)} dt \right].$$

The above calculations along with the Compactness Theorem 4.3 imply existence of compact attractor -first statement in Theorem 5.28. As for smoothness and finite dimensionality, one needs to prove quasi-stability estimate -which amounts to the proof of the estimate in Proposition 5.24 -but with $\alpha = 0$. This means that von Karman bracket is no longer subcritical and the term $(R(z), z_t)$ needs to be estimated by appropriate damping on the boundary. To achieve this we proceed as in the internal case- by decomposing the bracket into $Q(z)$ and $P(z)$. However, in the boundary case the additional difficulty is due to the fact that dissipativity estimate (5.35) no longer holds. What one has, instead, is similar quantity on the boundary. In order to take advantage of this relaxed dissipation, one proceeds as in the case of wave equation by considering...
(i) smoothing effect of backward trajectories on the attractor, (ii) propagating it forward and (iii) using the compactness of the attractor. This is technical part of the argument with full details given in [CL07b, Section 4.2].

Generalizations

1. The case of nonconservative forces can also be considered. This is to say that the force $P(u)$ is not potential. In that case the system has no longer gradient structure and one needs first to establish existence of an absorbing set. This can be done under some additional restrictions on the damping, see [CL10], where the case “potential operator + ($\psi, \nabla$)w” is considered.

2. More general sources $P(w)$ can be considered, including nonlinear sources. However, in general, one needs to make appropriate compactness hypotheses, see the Remark 5.5.

3. Dissipation in other (e.g., free) boundary conditions, see [CL10, CL04b, CL07b].

Open questions

1. General theory of non-conservative forces which destroy gradient structure.

2. Localized dissipation, i.e., dissipation localized in a suitable layer near the boundary.

3. Boundary damping problems without the light damping. This has to do with the existence of strict Lyapunov function and related unique continuation across the boundary. While various unique continuation results from the boundary are available also for plates [Alb00], the non-local nature of the von Karman bracket prevents Carleman’s estimates [Alb00] from applicability.

5.3 Kirchhoff-Boussinesq plate

Interior damping. Case $\alpha > 0$:

In this case our main results presented in the following theorem (for the proofs we refer to [CL08a, Chapter 7]).

**Theorem 5.32** Let $\alpha > 0$ and $P(w) = \Delta[w^2] - \varrho|w|^{l-1}w$ for some $\varrho \geq 0$ and $l \geq 1$. Assume that the damping functions $g$ and $G$ have the structure described in (2.8). Then problem (2.12) with $\alpha > 0$ and the clamped boundary conditions (2.5) generate a continuous semiflow $S_t$ in the space $H \equiv H_0^2(\Omega) \times H_0^1(\Omega)$.  

\[\text{see Remark 5.9(2) below.}\]
Assume in addition that either
\[
\begin{aligned}
m &\leq 3, \quad \rho > 0, \quad l \leq m, \quad \inf_{x \in \Omega} \{ a(x) \} > 0 \\
\end{aligned}
\]
or else
\[
\begin{aligned}
m &< 3, \quad \rho = 0, \quad \inf_{x \in \Omega} \{ a(x) \} > 0 \text{ is sufficiently large}.
\end{aligned}
\]
Then the semiflow \( S_t \) possesses a global compact attractor \( A \). Moreover, if the constant \( G_1 \) in (2.8) is positive, then
- the fractal dimension of the attractor \( A \) is finite;
- the system possesses a fractal exponential attractor (see Definition 4.18) whose dimension is finite in the space \( H_0^1(\Omega) \times W \), where \( W \) is a completion of \( H_0^1(\Omega) \) with respect to the norm \( \| \cdot \|_W = \| (1 - \alpha \Delta) \cdot \|_2 \).

**Remark 5.9** (1) As we see the additional potential term \( \rho |u|^{l-1} u \) in the force \( P(w) \) allows us to dispense with a necessity of assuming large values for the damping parameter.

(2) The requirements concerning the damping terms in Theorem 5.32 are not optimal. We choose them for the sake of some transparency. The same results under much more general damping functions can be found in [CL08a, Chapter 7].

**Interior damping. Case \( \alpha = 0 \) with linear damping function:**
We assume that \( g(s) = s \) and consider the source term of the form
\[
P(w) = \sigma \Delta [w^2] - \rho |w|^{l-1} w \quad \text{with} \quad \sigma, \rho \geq 0.
\]
As a phase space we take \( H \equiv H_0^2(\Omega) \times L_2(\Omega) \).

**Theorem 5.33 (Compact attractor)** Let
\[
\sigma^2 < \frac{1}{4} k \min \{1, k\} \quad \text{with} \quad k \equiv \inf_{\Omega} a(x) > 0.
\]
Then the dynamical system \((H, S_t)\) generated by equations (2.12) with \( \alpha = 0 \) possesses a compact global attractor \( \mathfrak{A} \).

If \( \sigma = 0 \), then the system \((H, S_t)\) is gradient and thus \( \mathfrak{A} = \mathcal{M}^u(\mathcal{N}) \) is unstable manifold in \( H \) emerging from the set \( \mathcal{N} \) of equilibria.

We note that the system under consideration is not a gradient system when \( \sigma \neq 0 \). As a consequence, the proof of existence of global attractor can not be just reduced to the proof of asymptotic smoothness. One needs to establish first existence of an absorbing ball. For this we use Lyapunov type function of the form
\[
V_\varepsilon(t) = \mathcal{E}(t) + \varepsilon \int_{\Omega} w(t)w(t)dx,
\]
(5.50)
with $\varepsilon = \frac{1}{4}\min\{1,k\}$, see [CL06b, CL11] for details in the case when $a(x) \equiv k$ is a constant. Here the energy functional has the form

$$E(t) \equiv \frac{1}{2}||w_t(t)||^2 + \frac{1}{2}||\Delta w(t)||^2 + \frac{1}{4} \int_\Omega |\nabla w(x,t)|^4\,dx + \sigma \int_\Omega w(x,t)|\nabla w(x,t)|^2\,dx + \frac{\theta}{l+1} \int_\Omega |w(x,t)|^{l+1}\,dx.$$

In order to complete the proof of the existence of global attractor, hence of Theorem 5.33, we use Ball’s method (see Theorem 4.4).

Let us take $V_\varepsilon(t)$ with $\varepsilon = k/2$ and denote $\Psi(t) = V_{k/2}(t)$. Using the fact that energy identity holds for all weak solutions one can easily see that the following equality is satisfied for this Lyapunov’s function $\Psi(t)$:

$$\partial_t \Psi(t) + k \Psi(t) + \int_\Omega (a(x) - k)|w_t|^2\,dx = K(w(t)),$$

where

$$K(w(t)) = \int_\Omega \left( \sigma|\nabla w|^2 - \frac{k}{2}((a(x) - k)w)w_t \right)\,dx - k\sigma \int_\Omega w|\nabla w|^2\,dx - \frac{k}{4} \int_\Omega |\nabla w|^4\,dx - \frac{k\sigma(l-1)}{2l+2} \int_\Omega |w(x,t)|^{l+1}\,dx.$$

It is clear that the term $K(w(t))$ is subcritical with respect to strong energy topology, therefore using the representation

$$\Psi(T) + \int_0^T e^{-k(T-t)}L(t)\,dt = e^{-kT}\Psi(0) + \int_0^T e^{-k(T-t)}K(w(t))dt,$$

with

$$L(t) = \int_\Omega (a(x) - k)|w_t(t)|^2\,dx,$$

we can apply Theorem 4.4 to prove the existence of global attractor.

To establish the statement in the case $\sigma = 0$ we note that in this case $E(w, w')$ is a strict Lyapunov function and thus the system is gradient.

Now we consider long-time dynamics of strong solutions.

Let $H_{st} = [H^4(\Omega) \cap H^2_0(\Omega)] \times H^2_0(\Omega)$. The argument given in the well-posedness section shows that the restriction $S^t_{st}$ of the semiflow $S_t$ on $H_{st}$ is a continuous (nonlinear) semigroup of continuous mappings in $H_{st}$.

**Theorem 5.34 (Compact attractor for strong solutions)** Let $\sigma = 0$ and $a(x) \equiv k > 0$ be a constant (for simplicity). Then semiflow $S^t_{st}$ has a compact global attractor $\mathfrak{A}^t_{st}$ in $H_{st}$. This attractor $\mathfrak{A}^t_{st}$ possesses the properties:

- $\mathfrak{A}^t_{st} \subseteq \mathfrak{A}$,
• $\mathcal{A}^{st} = \mathcal{M}^{\nu}_{st}(N)$ is unstable manifold in $H_{st}$ emerging from the set $N$ of equilibria;

• $\mathcal{A}^{st}$ has a finite fractal dimension as a compact set in $H_{st}$.

Moreover, if we assume in addition $l \equiv 2m - 1$ is an odd integer, then any trajectory $(w(t): w_t(t))$ from the attractor $\mathcal{A}^{st}$ possesses the property

$$\sup_{t \in \mathbb{R}} \|w^{(n)}(t)\|_{m, \Omega} \leq C_{n,m} < \infty, \quad n, m = 0, 1, 2, \ldots,$$

where $w^{(n)}(t) = \partial_t^nw(t)$. In particular, $\mathcal{A}^{st}$ is a bounded set in $C^m(\overline{\Omega}) \times C^m(\overline{\Omega})$ for each $m = 0, 1, 2, \ldots$, and thus $\mathcal{A}^{st} \subset C^\infty(\overline{\Omega}) \times C^\infty(\overline{\Omega})$.

The proof of Theorem 5.34 consists of several steps. The situation corresponding to gradient flows (we deal with the case $\sigma = 0$).

**Step 1 (dissipativity in $H_{st}$) by the “barrier” method:** Due to dissipativity in $H$ and energy relation it is sufficient to consider dissipativity of strong solutions $w(t)$ possessing properties

$$\|w_t(t)\|^2 + \|w(t)\|^2 \leq R, \quad t \geq 0, \quad \int_0^\infty \|w_t(\tau)\|^2 d\tau \leq C_R < \infty. \tag{5.52}$$

Let $w(t)$ be strong solution satisfying (5.52) and $v(t) = w_t(t)$. Consider the functional

$$G(v, v') \equiv \frac{1}{2} \left( \|v'\|^2 + \|\Delta w\|^2 + \|v\|^2 \right. + \left. \int_\Omega \left( |\nabla v|^2 |\nabla v(x)|^2 + 2 \left| (\nabla w, \nabla v)_{\mathbb{R}^2} \right|^2 \right) dx \right), \tag{5.53}$$

and denote

$$H(t) \equiv H(v, v_t) = a_0 + G(v, v_t) + \varepsilon(v, v_t), \tag{5.54}$$

where $a_0$ and $\varepsilon$ are positive parameters. Then using Brézis–Gallouet–Sedenko type inequality (see Lemma 3.15 and Remark 3.4) and choosing $\varepsilon$ small enough, we obtain that

$$\frac{dH(t)}{dt} + \gamma H(t) \leq C_R^1 \|w_t(t)\|^2 H(t) \ln (1 + H(t)) + C_R^2$$

with positive $\gamma$ (see [CL11] for details). The above inequality leads to the dissipativity property by barrier’s method.

**Step 2 (attractor in $H_{st}$):** To prove the existence of the attractor we need to establish asymptotic compactness of the system in $\mathcal{L}$. For this we use again Ball’s method. Let $\varepsilon = k/2$ and $a_0 = 0$ in (5.54). Then one can see that

$$\frac{dH(t)}{dt} + kH(t) = K_{st}(w(t), w_t(t)),$$
where

\[ K_{st}(w, v) = 3 \int_{\Omega} (\nabla w, \nabla v) \nabla v^2 dx \]

\[ - \int_{\Omega} \left[ f'(w)v - v\nu \right] dx - \frac{k}{2} \int_{\Omega} \left[ f'(w)v - v\nu \right] v dx, \]

where \( f(s) = \rho |s|^{l-1}s \). The term \( K_{st}(w, v) \) is obviously subcritical with respect to the topology in \( H_{st} \), therefore we can apply the same Ball’s argument to prove the existence of global attractor \( A^{st} \). The inclusion \( A^{st} \subseteq A \) is evident.

One can also see that \( A^{st} = M_{st}^{\nu}(N) \).

**Step 3 (Finite dimension of the attractor in \( H_{st} \)):** To prove finite dimensionality of the strong attractor in \( H_{st} \) we use smoothness properties of trajectories and again the methods developed in [CL08a]. For this we need to establish the quasi-stability estimate in \( H_{st} \).

**Proposition 5.35 (quasi-stability inequality in \( H_{st} \))** Let the hypotheses of Theorem 5.34 be in force. Let \( w(t) \) and \( w^*(t) \) be strong solutions to the problem in question satisfying the estimates

\[ \|w(t)\|_4 + \|w_t(t)\|_2 + \|w_{tt}(t)\|_4 + \|w^*(t)\|_4 + \|w^*_{t}(t)\|_2 + \|w^*_{tt}(t)\|_2 \leq R \]

for all \( t \in \mathbb{R} \) and for some constant \( R > 0 \). Let \( \tilde{w}(t) = w(t) - w^*(t) \). Then

\[ \|\tilde{w}(t)\|_4^2 + \|\tilde{w}_t(t)\|_2^2 + \|\tilde{w}_{tt}(t)\|_2^2 \leq C_1 e^{-\gamma t} \left( \|\tilde{w}(0)\|_4^2 + \|\tilde{w}_t(0)\|_2^2 \right) \]

\[ + C_2 \int_0^t e^{-\gamma (t-\tau)} \left( \|\tilde{w}(\tau)\|_4^2 + \|\tilde{w}_t(\tau)\|_2^2 \right) d\tau \]

for all \( t > 0 \), where \( \gamma > 0 \) and \( C_1, C_2 > 0 \) may depend on \( R \).

The proof (see [CL11]) follows the strategy applied to von Karman evolution equations and presented in [CL10, Sect.9.5.3].

By Theorem 4.16 (see also Theorem 4.3 in [CL08a]) the relation in Proposition 5.35 is sufficient to prove that the fractal dimension of \( A^{st} \) as a compact set in \( H_{st} \) is finite.

**Boundary damping**

In the case of \( \alpha > 0 \) we can use the same methods (see Section 10.3 in [CL10] for the details) as for von Karman model (2.18) with boundary damping (2.19). The case of the boundary damping with \( \alpha = 0 \) is still open.

### 6 Other models covered by the methods presented

In this section we shortly describe several models whose long-time dynamics can be studied within the framework presented above. These include: (i) nonlocal
Kirchhoff wave model with strong and structural damping; (ii) finite dimensional and smooth attractors for thermoelastic plates; (iii) control to finite dimensional attractors in thermal-structure, flow-structure and fluid-plate interactions. We also consider dissipative wave models which arise in plasma physics.

6.1 Kirchhoff wave models

In a bounded smooth domain \(\Omega \subset \mathbb{R}^d\) we consider the following Kirchhoff wave model with a strong nonlinear damping:

\[
\begin{aligned}
& \ddot{u} - \sigma(\|\nabla u\|^2) \Delta u - \phi(\|\nabla u\|^2) \Delta \dot{u} + f(u) = h(x), \quad x \in \Omega, \ t > 0, \\
& u|_{\partial \Omega} = 0, \quad u(0) = u_0, \quad \dot{u}(0) = u_1.
\end{aligned}
\]  

(6.1)

Here \(\Delta\) is the Laplace operator, \(\sigma\) and \(\phi\) are scalar functions specified later, \(f(u)\) is a given source term, and \(h\) is a given function in \(L^2(\Omega)\).

This kind of wave models goes back to G. Kirchhoff (\(d = 1, \phi(s) = \phi_0 + \phi_1 s, \sigma(s) \equiv 0, f(u) \equiv 0\)) and has been studied by many authors under different sets of hypotheses (see, e.g., [Lio78] and also [Chu12] and the references therein). The model in (6.1) is characterized by the presence of three nonlinearities: the source, the damping and the stiffness.

We assume that the source nonlinearity \(f(u)\) is a \(C^1\) function possessing the following properties:

\[
\mu_f := \liminf_{|s| \to \infty} \left\{ s^{-1} f(s) \right\} > -\infty,
\]  

(6.2)

and also (a) if \(d = 1\), then \(f\) is arbitrary; (b) if \(d = 2\) then

\[
|f'(u)| \leq C (1 + |u|^{p-1}) \quad \text{for some } p \geq 1;
\]  

(c) if \(d \geq 3\) then either

\[
|f'(u)| \leq C (1 + |u|^{p-1}) \quad \text{with some } 1 \leq p \leq p_* \equiv \frac{d + 2}{d - 2},
\]  

(6.3)

or else

\[
c_0 |u|^{p_1} - c_1 \leq f'(u) \leq c_2 (1 + |u|^{p_1}) \quad \text{with some } p_1 < p < p_{**},
\]  

(6.4)

where \(p_{**} \equiv \frac{d + 4}{(d - 1)\tau}, c_i > 0\) are constants and \(s_+ = (s + |s|)/2\).

We note that the conditions above covers subcritical, critical and supercritical cases.

Under rather mild hypotheses concerning \(C^1\) functions \(\phi\) and \(\sigma\) we can prove that the problem is well-posed. This holds even without the requirement that \(\phi\) is non-negative (see the details in [Chu12]). However in order to study long-time dynamics of the problem (6.1) we need to assume that the functions \(\sigma\) and \(\phi\) from \(C^1(\mathbb{R}_+)\) are positive and either

\[
\phi(s)s \to +\infty \text{ as } s \to +\infty \quad \text{and} \quad \mu_f = \liminf_{|s| \to \infty} \left\{ s^{-1} f(s) \right\} > 0.
\]
The attractor \( A \subset H \) have that for any full trajectory \( \gamma \)

To prove this result we first establish quasi-stability properties of \( S \) and for any bounded set \( B \), non-supercritical force of the form of the topological scales (see [Chu12] for details).

Then the semi group \((6.1)\) possesses a global partially strong attractor for all \( t \geq 0 \) if \( (\text{i}) \) \( A \) is closed with respect to the partially strong topology, \( (\text{ii}) \) \( A \) is strictly invariant \( (S_t A = A \text{ for all } t > 0) \), and \( (\text{iii}) \) \( A \) uniformly attracts in the partially strong topology all other bounded sets: for any (partially strong) vicinity \( \mathcal{O} \) of \( A \) and for any bounded set \( B \) in \( H \) there exists \( t_* = t_*(\mathcal{O}, B) \) such that \( S_t B \subset \mathcal{O} \) for all \( t \geq t_* \).

The main result in [Chu12] reads as follows:

**Theorem 6.1**  Assume in addition that \( f'(s) \geq -c \) for all \( s \in \mathbb{R} \) in the non-supercritical case (when \( (6.4) \) does not hold). Then the semigroup \( S_t \) given by \((6.1)\) possesses a global partially strong attractor \( A \) in the space \( H \). Moreover, \( A \subset H_1 = [H^2(\Omega) \cap L^1(\Omega)] \times H^1_0(\Omega) \) and

\[
\sup_{t \in \mathbb{R}} \left( \| \Delta u(t) \|^2 + \| \nabla u(t) \|^2 + \| u_{tt}(t) \|^2,_{-1, \Omega} + \int_t^{t+1} \| u_{tt}(\tau) \|^2 \, d\tau \right) \leq C_A
\]

for any full trajectory \( \gamma = \{ (u(t); u_t(t)) : t \in \mathbb{R} \} \) from the attractor \( A \). We also have that \( A = M_{++}(N) \), where \( N = \{ (u; 0) \in H : \phi(\| A^{1/2} u \|^2) \lambda + F(u) = h \} \).

The attractor \( A \) has a finite fractal dimension as a compact set in the space \([H^{1+r}(\Omega) \cap H^1_0(\Omega)] \times H^r(\Omega) \) for every \( r < 1 \).

To prove this result we first establish quasi-stability properties of \( S_t \) on different topological scales (see [Chu12] for details).

In similar way we can also study a model with structural damping with a non-supercritical force of the form

\[
\begin{cases}
  u_{tt} + \phi(\| A^{1/2} u \|^2) A u + \sigma(\| A^{1/2} u \|^2) A^2 u + F(u) = 0, & t > 0, \\
  u(0) = u_0, & u_t(0) = u_1,
\end{cases}
\]

or else

\[
\hat{\mu}_\phi = \lim \inf_{s \to +\infty} \phi(s) > 0 \quad \text{and} \quad \hat{\mu}_\phi \lambda + \mu_f > 0,
\]

where \( \lambda_1 \) is the first eigenvalue of the minus Laplace operator in \( \Omega \) with Dirichlet boundary conditions.

As a phase space we consider \( H = [H^1_0(\Omega) \cap L^{p+1}(\Omega)] \times L^2(\Omega) \) endowed with partially strong\(^6\) topology: a sequence \( \{ (u^n_0; u^n_1) \} \subset H \) is said to be partially strongly convergent to \( (u_0; u_1) \in H \) if \( u^n_0 \to u_0 \) strongly in \( H^1_0(\Omega) \), \( u^n_0 \to u_0 \) weakly in \( L^{p+1}(\Omega) \) and \( u^n_1 \to u_1 \) strongly in \( L^2(\Omega) \) as \( n \to \infty \) (in the case when \( d \leq 2 \) we take \( 1 < p < \infty \) arbitrary).

Under the conditions stated above (see [Chu12]) problem \((6.1)\) generates an evolution semigroup \( S_t \) in the space \( H \). The action of the semigroup is given by the formula \( S_t y = (u(t); u_t(t)) \), where \( y = (u_0; u_1) \in H \) and \( u(t) \) is a weak solution to \((6.1)\).

To describe the dynamical properties of \( S_t \) it is convenient to introduce the following notion.

A bounded set \( A \subset H \) is said to be a **global partially strong attractor** for \( S_t \) if \( (\text{i}) \) \( A \) is closed with respect to the partially strong topology, \( (\text{ii}) \) \( A \) is strictly invariant \( (S_t A = A \text{ for all } t > 0) \), and \( (\text{iii}) \) \( A \) uniformly attracts in the partially strong topology all other bounded sets: for any (partially strong) vicinity \( \mathcal{O} \) of \( A \) and for any bounded set \( B \) in \( H \) there exists \( t_* = t_*(\mathcal{O}, B) \) such that \( S_{t_*} B \subset \mathcal{O} \) for all \( t \geq t_* \).

\[ \Box \]

\[6\] It is obvious that the partially strong convergence becomes strong below supercritical level \( (H^1_0(\Omega) \subset L^{p+1}(\Omega)) \).
with \( \theta \in [1/2, 1) \), where \( A \) is a linear positive self-adjoint operator with domain \( D(A) \) and with a compact resolvent in a separable infinite dimensional Hilbert space \( H \). In this case we assume that the damping \( \sigma \) and the stiffness \( \phi \) factors are positive \( C^1 \) functions and

\[
\Phi(s) = \int_0^s \phi(\xi)d\xi \to +\infty \text{ as } s \to +\infty.
\]

The nonlinear operator \( F \) is locally Lipschitz in the following sense

\[
\|A^{-\theta}[F(u_1) - F(u_2)]\| \leq L(\varrho)\|A^{1/2}(u_1 - u_2)\|, \quad \forall \|A^{1/2}u_1\| \leq \varrho,
\]

and potential, i.e., \( F(u) = \Pi'(u) \), where \( \Pi(u) \) is a \( C^1 \)-functional on \( D(A^{1/2}) \), and \( \Pi \) stands for the Fréchet derivative. We assume that \( \Pi(u) \) is locally bounded on \( D(A^{1/2}) \) and there exist \( \eta < 1/2 \) and \( C \geq 0 \) such that

\[
\eta\Phi(\|A^{1/2}u_1\|^2) + \Pi(u) + C \geq 0, \quad u \in H_1 = D(A^{1/2}).
\]

This hypotheses cover critical case of the source \( F \), but not supercritical. For details concerning the model in (6.5) we refer to [Chu10].

### 6.2 Plate models with structural damping

We consider a class of plate models with the strong nonlinear damping the abstract form of which is the following Cauchy problem in a separable Hilbert space \( H \):

\[
u_{tt} + D(u, u_t) + A u + F(u) = 0, \quad t > 0; \quad u|_{t=0} = u_0, \quad u_t|_{t=0} = u_1. \tag{6.6}
\]

In the case of plate models with hinged boundary conditions, \( A = (-\Delta_D)^2 \), where \( \Delta_D \) is the Laplace operator in a bounded smooth domain \( \Omega \) in \( \mathbb{R}^2 \) with the Dirichlet boundary conditions. We have then that \( H = L_2(\Omega) \) and

\[
D(A) = \{ u \in H^4(\Omega) : u = \Delta u = 0 \text{ on } \partial \Omega \}.
\]

The damping operator \( D(u, u_t) \) may have the form

\[
D(u, u_t) = \Delta \sigma(u)\Delta u_t - \text{div} \left[ \sigma_1(u, \nabla u) \nabla u_t \right] + g(u, u_t), \tag{6.7}
\]

where \( \sigma_0(s_i), \sigma_1(s_1, s_2, s_3) \) and \( g(s_1, s_2) \) are locally Lipschitz functions of \( s_i \in \mathbb{R}, i = 1, 2, 3, \) such that \( \sigma_0(s_1) > 0, \sigma_1(s_1, s_2, s_3) \geq 0 \) and \( g(s_1, s_2)s_2 \geq 0 \). Also the functions \( \sigma_1 \) and \( g \) satisfy some growth conditions. We note that every term in (6.7) represents a different type of damping mechanisms. The first one is the so-called viscoelastic Kelvin–Voight damping, the second one represents the structural damping and the term \( g(u, u_t) \) is the dynamical friction (or viscous damping). We refer to [LT00, Chapter 3] and to the references therein for a discussion of stability properties caused by each type of the damping terms in the case of linear systems. We should stress that the conditions concerning \( \sigma_i \) above allow to have a pure viscoelastic damping (i.e., we can take \( \sigma_1 \equiv 0 \) and
\( g \equiv 0 \). However a similar results remains valid for the case when \( \sigma_0 \equiv 0 \) and \( \sigma_1 \) is independent of \( \nabla u \). In this case the presence (or absence) of the friction \( g(u, u_t) \) has also no importance for long-time dynamics.

The nonlinear feedback (elastic) force \( F(u) \) may have one of the following forms (which represent different plate models):

(a) **Kirchhoff model:** \( F(u) \) is the Nemytskii operator

\[
\begin{align*}
  u & \mapsto -\kappa_1 \cdot \text{div} \left\{ |\nabla u|^q \nabla u - \mu_1 |\nabla u|^2 \nabla u \right\} + \kappa_2 \left\{ |u|^l u - \mu_2 |u|^m u \right\} - p(x),
\end{align*}
\]

where \( \kappa_i \geq 0, q > r \geq 0, l > m \geq 0, \mu_i \in \mathbb{R} \) are parameters, \( p \in L_2(\Omega) \).

(b) **Von Karman model:** \( F(u) = -[u, \mathcal{F}(u)] + p(x) \), where \( F_0 \in H^4(\Omega) \) and \( p \in L_2(\Omega) \) are given functions, the von Karman bracket \([u, v]\) is given by (2.7) and the Airy stress function \( \mathcal{F}(u) \) solves (2.6).

(c) **Berger Model:** In this case the feedback force has the form

\[
F(u) = - \left[ \kappa \int_\Omega |\nabla u|^2 dx - \Gamma \right] \Delta u - p(x), \tag{6.8}
\]

where \( \kappa > 0 \) and \( \Gamma \in \mathbb{R} \) are parameters, \( p \in L_2(\Omega) \); for some details and references see, e.g., [Chu99, Chapter 4] and [CL08a, Chapter 7].

One can show that the system generated by (6.6) is quasi-stable. Thus methods presented in these notes apply. They allow to show the existence of global attractors of finite fractal dimension which in addition posses some smoothness properties. See [CK10, CK12] for more details.

### 6.3 Mindlin-Timoshenko plates and beams

Let \( \Omega \subset \mathbb{R}^2 \) be a bounded domain with a sufficiently smooth boundary \( \Gamma \). Lets \( v(x, t) = (v_1(x, t), v_2(x, t)) \) be a vector function and \( w(x, t) \) be a scalar function on \( \Omega \times \mathbb{R}_+ \). The system of Mindlin-Timoshenko equations describes dynamics of a plate taking into account transverse shear effects (see, e.g., [LL88, Chap.1] and the references therein). This system has the form

\[
\alpha_v v_t + \kappa \cdot g(v_t) - Av + \kappa \cdot (v + \nabla w) = -f_0(v) + \nabla_x [f_1(w)],
\]

\[
w_{tt} + \kappa \cdot g_0(w_t) - \kappa \cdot \text{div}(v + \nabla w) = -f_2(w). \tag{6.10}
\]

We supplement this problem with the Dirichlet boundary conditions

\[
v_1(x, t) = v_2(x, t) = 0, \quad w(x, t) = 0 \quad \text{on} \quad \Gamma \times \mathbb{R}_+. \tag{6.11}
\]

Here the functions \( v_1(x, t) \) and \( v_2(x, t) \) are the angles of deflection of a filament (they are measures of transverse shear effects) and \( w(x, t) \) is the bending component (transverse displacement). The vector \( f_0(v) = (f_{01}(v_1, v_2); f_{02}(v_1, v_2)) \) and scalar \( f_1 \) and \( f_2 \) functions represents (nonlinear) feedback forces, whereas \( g(v_1, v_2) = (g_1(v_1), g_2(v_2)) \) and \( g_0 \) are monotone damping functions describing
resistance forces (with the intensity $k > 0$). The parameter $\alpha > 0$ describes rotational inertia of filaments. The factor $\kappa > 0$ is the so-called shear modulus (from mechanical point of view the limiting situation $\kappa \to +0$ corresponds to plane strain and the case $\kappa \to +\infty$ corresponds to absence of transverse shear).

The operator $A$ has the form

$$A = \begin{bmatrix} \partial^2_{x_1} + \frac{1+\nu}{2} \partial^2_{x_2} + \frac{1+\nu}{2} \partial^2_{x_1 x_2} \\ \frac{1+\nu}{2} \partial^2_{x_1 x_2} + \frac{1+\nu}{2} \partial^2_{x_1} + \partial^2_{x_2} \end{bmatrix},$$

where $0 < \nu < 1$ is the Poisson ratio. In 1D case ($\dim \Omega = 1$) the corresponding problem models dynamics of beams under the Mindlin-Timoshenko hypotheses. For details concerning the Mindlin-Timoshenko hypotheses and governing equations see, e.g. [Lag89] and [LL88]. We also note that in the limit $\kappa \to +\infty$ the system in (6.9) and (6.10) becomes Kirchhoff-Boussinesq type equation, see [Lag89, CL06a].

We refer to [CL08a] and [CL06a] for an analysis (based on quasi-stability technology) of long time behaviour of the Mindlin-Timoshenko plate under different sets of assumptions concerning nonlinear feedback forces, damping functions and parameters. By the same method long-time dynamics in thermoelastic Mindlin-Timoshenko model was studied in [Fas07, Fas09].

6.4 Thermal-structure interactions

This problem has the form

$$\begin{cases} u_{tt} - \alpha A \theta + A^2 u = B(u), & u|_{t=0} = u_0, \ u_t|_{t=0} = u_1, \\ \theta_t + \eta A \theta + \alpha A u_t = 0, & \theta|_{t=0} = \theta_0, \end{cases} \tag{6.12}$$

where $A$ is a linear positive self-adjoint operator with a compact inverse in a separable infinite dimensional Hilbert space $H$. The nonlinear term $B(u)$ can model von Karman (as in (2.4)) and Berger (see (6.8)) nonlinearities. In this case the variable $\theta$ represents the temperature of the plate.

For application the methods presented we refer to [CL08b] and [CL10, Chapter 11] in the von Karman case, see also [BC08] for the Berger nonlinearity.

6.5 Structural acoustic interactions

The mathematical model under consideration consists of a semilinear wave equation defined on a bounded domain $\Omega$, which is strongly coupled with the Berger or von Karman plate equation acting only on a part of the boundary of the domain $\Omega$. This kind of models, known as structural acoustic interactions, arise in the context of modeling gas pressure in an acoustic chamber which is surrounded by a combination of hard (rigid) and flexible walls.

More precisely, let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a sufficiently smooth boundary $\partial \Omega$. We assume that $\partial \Omega = \partial \Omega \cup \overline{S}$, where $\Omega \cap S = \emptyset$,

$$\Omega \subset \{x = (x_1; x_2; 0) : x' \equiv (x_1; x_2) \in \mathbb{R}^2\}$$
with the smooth contour $\Gamma = \partial \Omega$ and $S$ is a surface which lies in the subspace $\mathbb{R}^2 = \{ x_3 \leq 0 \}$. The exterior normal on $\partial \Omega$ is denoted by $n$. The set $\Omega$ is referred to as the elastic wall, whose dynamics is described some plate equation. The acoustic medium in the chamber $\mathcal{O}$ is described by a semilinear wave equation. Thus, we consider the following (coupled) PDE system

$$\begin{cases}
  z_{tt} + g(z_t) - \Delta z + f(z) = 0 & \text{in } \mathcal{O} \times (0,T), \\
  \frac{\partial z}{\partial n} = 0 & \text{on } S \times (0,T), \\
  \frac{\partial z}{\partial n} = \alpha v_t & \text{on } \Omega \times (0,T),
\end{cases} \quad (6.13)$$

and

$$\begin{cases}
  v_{tt} + b(v_t) + \Delta^2 v + B(v) + \beta z_t|_{\Omega} = 0 & \text{in } \Omega \times (0,T), \\
  v = \Delta v = 0 & \text{on } \partial \Omega \times (0,T),
\end{cases} \quad (6.14)$$

endowed with initial data

$$z(0,\cdot) = z^0, \quad z_t(0,\cdot) = z^1 \text{ in } \Omega, \quad v(0,\cdot) = v^0, \quad v_t(0,\cdot) = v^1 \text{ in } \Omega.$$ 

Here above, $g(s)$ and $b(s)$ are non-decreasing functions describing the dissipation effects in the model, while the term $f(z)$ represents a nonlinear force acting on the wave component and $B(v)$ is (nonlinear) von Karman or Berger force; $\alpha$ and $\beta$ are positive constants; The part $S$ of the boundary describes a rigid (hard) wall, while $\Omega$ is a flexible wall where the coupling with the plate equation takes place. The boundary term $\beta z_t|_{\Omega}$ describes back pressure exercised by the acoustic medium on the wall.

Well-posedness issues of the abstract second order system, which is a particular case of the one studied in [Las02, Sect. 2.6] (see also [CL10, Chapter 6]).

Long-time dynamics from point of view of quasi-stable systems were investigated in [CL10, Chapter 12] in the von Karman case and in [BCL07] for the Berger nonlinearity. We also note that flow structure model (6.13) and (6.14) in combination with thermoelastic system (6.12) was studied in [BC08] and [CL10, Chapter 12].

### 6.6 Fluid-structure interactions

Our mathematical model is formulated as follows (for details, see [CR11]).

Let $\mathcal{O} \subset \mathbb{R}^3$, $\Omega \subset \mathbb{R}^2$ and the surface $S$ be the same as in the case of the model in (6.13) and (6.14). We consider the following linear Navier-Stokes equations in $\mathcal{O}$ for the fluid velocity field $v = v(x,t) = (v^1(x,t); v^2(x,t); v^3(x,t))$ and for the pressure $p(x,t)$:

$$v_t - \nu \Delta v + \nabla p = G_f \quad \text{and} \quad \text{div } v = 0 \text{ in } \mathcal{O} \times (0, +\infty), \quad (6.15)$$

where $\nu > 0$ is the dynamical viscosity and $G_f$ is a volume force. We supplement (6.15) with the (non-slip) boundary conditions imposed on the velocity field $v = v(x,t)$:

$$v = 0 \text{ on } S; \quad v \equiv (v^1; v^2; v^3) = (0; 0; u_t) \text{ on } \Omega. \quad (6.16)$$
Here \( u = u(x, t) \) is the transversal displacement of the plate occupying \( \Omega \) and satisfying the following equation:

\[
\ddot{u} + \Delta^2 u + F(u) = p|_\Omega \quad \text{in} \quad \Omega \times (0, \infty). \tag{6.17}
\]

The nonlinear feedback (elastic) force \( F(u) \) as above may have one of the forms (Kirchhoff, Karman or Berger) which are present for plates models with structural damping (see (6.6)). We also impose clamped boundary conditions on the plate

\[
u|_{\partial \Omega} = \frac{\partial u}{\partial n}|_{\partial \Omega} = 0 \tag{6.18}
\]

and supply (6.15)–(6.18) with initial data of the form

\[
v(0) = v_0, \quad u(0) = u_0, \quad u_t(0) = u_1, \tag{6.19}
\]

We note that (6.15) and (6.16) imply the following compatibility condition

\[
\int_\Omega u(x', t)dx' = \text{const} \quad \text{for all} \quad t \geq 0, \tag{6.20}
\]

which can be interpreted as preservation of the volume of the fluid (we can choose this constant to be zero).

It was shown in [CR11] that equations (6.15)–(6.20) generates an evolution operator \( S_t \) in the space

\[
\mathcal{H} = \left\{(v_0; u_0; u_1) \in X \times \check{H}^2_0(\Omega) \times \check{L}^2(\Omega) : (v_0, n) \equiv v_0^3 = u_1 \text{ on } \Omega \right\},
\]

where

\[
X = \left\{v = (v^1; v^2; v^3) \in [L^2(\Omega)]^3 : \text{div} v = 0; \ (v, n) = 0 \text{ on } S\right\},
\]

and \( \check{L}^2(\Omega) \) (resp. \( \check{H}^2_0(\Omega) \)) denotes the subspace in \( L^2(\Omega) \) (resp. in \( H^2_0(\Omega) \)) consisting of functions with zero averages.

This evolution operator \( S_t \) is quasi-stable and thus possesses a compact global attractor, see [CR11]. We emphasize that we do not assume any kind of mechanical damping in the plate component. Thus this results means that dissipation of the energy in the fluid due to viscosity is sufficient to stabilize the system.

In a similar way (see [Chu11a]) the model which deals only with longitudinal deformations of the plate neglecting transversal deformations can be also considered (in contrast with the model (6.15)–(6.19) which takes into account the transversal deformations only). This means that instead of (6.16) the following boundary conditions are imposed on the velocity fluid field:

\[
v = 0 \quad \text{on} \ S; \quad v \equiv (v^1; v^2; v^3) = (u_1^1; u_1^2; 0) \quad \text{on} \ \Omega,
\]

where \( u = (u^1(x, t); u^2(x, t)) \) is the in-plane displacement vector of the plate which solves the wave equation of the form

\[
\ddot{u}_{tt} - \nabla \left[ \text{div} u + \nu(v_{23}^1; v_{23}^2)|_{x_3=0} + f(u) \right] = 0 \quad \text{in} \quad \Omega; \quad u^i = 0 \quad \text{on} \ \Gamma.
\]
This kind of models arises in the study of blood flows in large arteries (see the references in [Gro08]).

One can also analyze the corresponding model based on the full Karman shell model with rotational inertia (see [CR12]). In this case to obtain well-posedness we need to apply Sedenko’s method. The study of long-time dynamics is based on J.Ball’s method (see Theorem 4.4).

6.7 Quantum Zakharov system

In a bounded domain $\Omega \subset \mathbb{R}^d$, $d \leq 3$, we consider the following system

\[
\begin{cases}
    n_{tt} - \Delta (n + |E|^2) + h^2 \Delta^2 n + \alpha n_t = f(x), & x \in \Omega, \ t > 0, \\
    iE_t + \Delta E - h^2 \Delta^2 E + i\gamma E - nE = g(x), & x \in \Omega, \ t > 0.
\end{cases}
\]

(6.21)

Here $E(x, t)$ is a complex function and $n(x, t)$ is a real one, $h > 0$, $\alpha \geq 0$ and $\gamma \geq 0$ are parameters and $f(x), g(x)$ are given (real and complex) functions.

This system in dimension $d = 1$ was derived in [GHG05], by use of a quantum fluid approach, to model the nonlinear interaction between quantum Langmuir waves and quantum ion-acoustic waves in an electron-ion dense quantum plasma. Later a vector 3D version of equations (6.21) was suggested in [HS09]. In dimension $d = 2, 3$ the system in (6.21) is also known (see, e.g., [SSS09] and the references therein) as a simplified "scalar model" which is in a good agreement with the vector model derived in [HS09] (see a discussion in [SSS09]).

The Dirichlet initial boundary value problem for (6.21) is well-posed and generates a dynamical system in an appropriate phase space (see [Chu11c]). Using quasi-stability methods one can prove the existence of a finite-dimensional global attractor, for details we refer to [Chu11c].

We also note that in the case $h = 0$ we arrive to the classical Zakharov system (see [Zak72]). Global attractors in this case were studied in [Fla91, GM98] in the 1D case and in [CS05] in the 2D case. In the latter case to obtain uniqueness the Sedenko method was applied and Ball’s method was used to prove the existence of a global attractor.

6.8 Schrödinger–Boussinesq equations

The methods similar to described above can be also applied in study of qualitative behavior of the system consisting of Boussinesq and Schrödinger equations coupled in a smooth (2D) bounded domain $\Omega \subset \mathbb{R}^2$. The resulting system takes the form:

\[
\begin{align*}
    w_{tt} + \gamma_1 w_t + \Delta^2 w - \Delta (f(w) + |E|^2) &= g_1(x), \\
    iE_t + \Delta E - wE + i\gamma_2 E &= g_2(x), & x \in \Omega, \ t > 0,
\end{align*}
\]

(6.22a, b)

where $E(x, t)$ and $w(x, t)$ are unknown functions, $E(x, t)$ is complex and $w(x, t)$ is real. Here above $\gamma_1$ and $\gamma_2$ are nonnegative parameters and $g_1(x)$ and $g_2(x)$
are given (real and complex) $L_2$-functions. We equip equations (6.22) with the boundary conditions
\[ w_{|\partial \Omega} = \Delta w_{|\partial \Omega} = 0, \quad E_{|\partial \Omega} = 0, \tag{6.23} \]
and with the initial data
\[ w_t(x,0) = w_1(x), \quad w(x,0) = w_0(x), \quad E(x,0) = E_0(x). \tag{6.24} \]
Long-time dynamics in this system was studied in [CS11] by the methods described above under the following hypotheses concerning the (nonlinear) function $f$: $f \in C^1(\mathbb{R})$, $f(0) = 0$ and
\[ \exists c_1, c_2 \geq 0 : \quad F(r) = \int_0^r f(\xi)d\xi \geq -c_1 r^2 - c_2, \quad \forall |r| \geq r_0, \tag{6.25} \]
\[ \exists M \geq 0, p \geq 1 : \quad |f'(s)| \leq M(1 + |s|^{p-1}), \quad s \in \mathbb{R}. \tag{6.26} \]
We also note that the ideas which were used in the study of the Kirchhoff-Boussinesq system (see (2.12) with $\alpha = 0$ can be also applied to the following Schrödinger-Boussinesq-Kirchhoff model:
\[ w_{tt} - \Delta w_t + \Delta^2 w - \text{div} \left\{ |\nabla u|^3 \nabla u \right\} - \Delta \left( w^2 + |E|^2 \right) = g_1, \]
\[ iE_t + \Delta E - wE + i\gamma_2 E = g_2, \quad x \in \Omega, \quad t > 0, \]
equipped with the boundary conditions (6.23) and initial data (6.24). For details we refer to [CS12].

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