Abstract

In symmetric groups, a two-sided cell is the set of all permutations which are mapped by the Robinson-Schensted correspondence on a pair of tableaux of the same shape. In this article, we show that the set of permutations in a two-sided cell which have a minimal number of inversions is the set of permutations which have a maximal number of inversions in conjugated Young subgroups. We also give an interpretation of these sets with particular tableaux, called reading tableaux. As corollary, we give the set of elements in a two-sided cell which have a maximal number of inversions.

1 Introduction

In this article, we consider the symmetric group $S_n$. For $w \in S_n$, the length of $w$, denoted $\ell(w)$, is the number of inversions of $w$.

The Robinson-Schensted correspondence [19] is the well-known bijection $\pi : w \mapsto (P(w), Q(w))$ between $S_n$ and pairs of standard tableaux of the same shape (a partition of $n$). For each partition $\lambda$ of $n$, we denote by $T^\lambda$ the set of all permutations which are mapped by $\pi$ on a pair of tableaux of shape $\lambda$. In the Kazhdan-Lusztig theory, which we use in this article, the sets $T^\lambda$ are called two-sided cells (see [11] [22] [1]).
Our goal is to describe, for any partition $\lambda$ of $n$, the set $T_{\lambda \min}$ of elements of minimal length in $T^\lambda$ and the set $T_{\lambda \max}$ of elements of maximal length in $T^\lambda$.

For each composition $c = (n_1, \ldots, n_k)$ of $n$ (with $n_i \geq 1$), the Young subgroup $S_c = S_{n_1} \times \cdots \times S_{n_k}$ contains a unique permutation $\sigma_c$ of maximal length. It is well-known that $\sigma_c$ is an involution, called the longest element of $S_c$. Denote $\lambda(c)$ the decreasing reordering of $c$. It is well-known that two Young subgroups $S_{c_1}$ and $S_{c_2}$ are conjugated in $S_n$ if and only if $\lambda(c_1) = \lambda(c_2)$.

Schützenberger [21] has shown that the map $T \mapsto w_T = \pi^{-1}(T, T)$ is a bijection between the standard tableaux of shape a partition of $n$ and the involutions of $S_n$ (see also [4] where another interesting description of this bijection is given).

A standard tableau $T$ is a reading tableau if it has the following property: for any $1 \leq p \leq n$, either $p$ is in the first line of $T$, or if $p$ is in the $i$th line of $T$ ($i > 1$) then $p - 1$ is in the $(i - 1)$th line of $T$.

As example, the column superstandard tableaux (which are tableaux numbered from the bottom to the top of each column, from left to right) are reading tableaux. These particular tableaux are the transposed of those defined by Garsia and Remmel in [8]. Our main result is the following:

**Theorem 1.1.** Let $\lambda$ be a partition of $n$ and $T^\lambda$ be its associated two-sided cell, then

$$T_{\lambda \min} = \{ \sigma_c \mid \lambda(c) = \lambda^t \} = \{ w_T \mid T \text{ is a reading tableau of shape } \lambda \},$$

where $\lambda^t$ denotes the conjugate partition of $\lambda$.

*Example.* Consider the partition $\lambda = (3, 2, 1, 1)$ of 7; $\lambda^t = (4, 2, 1)$. Then the reading tableaux of shape $\lambda$ are

$$T_1 = \begin{array}{cccc}
1 & 5 & 7 \\
2 & 6 & 3 \\
3 & 4 & 1 \\
\end{array} = \begin{array}{c}
4 \\
2 \\
1 \\
\end{array} \quad ; \quad T_2 = \begin{array}{cc}
1 & 3 & 7 \\
2 & 4 & 5 \\
6 & 6 & 6 \\
\end{array} = \begin{array}{c}
6 \\
6 \\
7 \\
\end{array} \quad ; \quad T_3 = \begin{array}{cc}
1 & 5 \\
2 & 4 \\
3 & 3 \\
4 & 2 \\
\end{array} = \begin{array}{c}
4 \\
3 \\
1 \\
6 \\
7 \\
\end{array} \quad ; \quad T_4 = \begin{array}{ccc}
1 & 5 & 6 \\
2 & 7 & 2 \\
3 & 3 & 4 \\
\end{array} = \begin{array}{c}
5 \\
7 \\
6 \\
\end{array}$$
The right skew tableau is taken in the plactic class of the corresponding tableau viewed in the plactic monoid \([13]\) (see also \([7]\)). The corresponding involutions are then

\[
\begin{align*}
w_{T_1} &= 4321657 = \sigma_{(4,2,1)} ; \quad w_{T_2} = 2165437 = \sigma_{(2,4,1)} \\
w_{T_3} &= 1543276 = \sigma_{(1,4,2)} ; \quad w_{T_4} = 4321576 = \sigma_{(4,1,2)} \\
w_{T_5} &= 2137654 = \sigma_{(2,1,4)} ; \quad w_{T_6} = 1327654 = \sigma_{(1,2,4)}
\end{align*}
\]

and \(T'_{\text{min}}^{(3,2,1,1)} = \{\sigma_{(4,2,1)}, \sigma_{(2,4,1)}, \sigma_{(1,4,2)}, \sigma_{(4,1,2)}, \sigma_{(2,1,4)}, \sigma_{(1,2,4)}\}\).

Let \(\lambda\) be a partition of \(n\) and \(T\) be a standard tableau of shape \(\lambda\). The Schützenberger evacuation of \(T\), denoted by \(\text{ev}(T)\), is a tableau of shape \(\lambda\) \([20]\) (see also \([18\ p.128-130]\)). The evacuation illustrates the conjugation and the left (and right) multiplication by the longest element \(\sigma(n)\) in \(S_n\). In particular \(Q(w\sigma(n)) = \text{ev}(Q(w)^t)\), for any \(w \in S_n\), and \(T^\lambda\sigma(n) = \sigma(n)T^\lambda = T^{\lambda^t}\). Denote \(d_c = \sigma(n)\sigma_c\). As \(\ell(\sigma(n)w) = \ell(\sigma(n)) - \ell(w)\), we obtain the following corollary:

**Corollary 1.2.** Let \(\lambda\) be a partition of \(n\) and \(T^\lambda\) be its associated two-sided cell, then

\[
\begin{align*}
T^\lambda_{\text{max}} &= \{d_c | \lambda(c) = \lambda\} \\
&= \{w | \text{ev}(Q(w)^t) = P(w\sigma(n)) \text{ is a reading tableau of shape } \lambda^t\}.
\end{align*}
\]

In the theory of Coxeter groups, the element \(d_c\) is well-known as the unique element of maximal length in the set of minimal right coset representatives of \(S_{c} [3\ Chapter\ 2]\).

As a by-product of our proof, we obtain, in Section 3 that if \(w\) is an involution, the Kazhdan-Lusztig polynomial \(P_{e,w} = 1\) if and only if \(w\) is the longest element of a Young subgroup, where \(e\) denotes the identity of \(S_n\). More precisely, we show that an involution \(w\) avoids the pattern 3412 and 4231 if and only if \(w\) is the longest element of a Young subgroup of \(S_n\).

To our knowledge, our results are the first results relating the Robinson-Schensted transformation and the length function (number of inversions) of a permutation. There is no evident link between both. Our proof is non combinatorial and uses heavily the \(a-\)function of Lusztig \([13, 14]\) (questions about the leading term of Kazhdan-Lusztig polynomials and the \(a-\)function are heavily
studied, see for instance \[23, 24\]). Trying to find a combinatorial proof (which is a challenge) leads first to the following difficulty: for any permutations \(w, x\) in a two-sided-cell \(T^\lambda\), there are permutations \(w_1, \ldots, w_k \in T^\lambda\) such that \(w_1 = w, w_k = x\) and \(w_{i+1}\) is obtained from \(w_i\) by a Knuth or a dual-Knuth relation (see \[13, 7\]). But a permutation \(w\) may be ‘locally minimal’, that is whenever \(w\) (or \(w^{-1}\)) admits a Knuth or a dual-Knuth elementary relation, this relation increases the length. The involution \(\sigma = 632541 \in S_6\) is locally minimal, but not of minimal length, in its two-sided cell.

It would be interesting to find a purely combinatorial proof of the main result. It is apparently an open problem to read the length of a permutation \(w\) directly on the pair of tableaux \(\pi(w)\) (however, see \[17\], where the author gives a way to read the signature on the pair of tableaux). Fortunately, the Lusztig \(a\)-function gives us a way to avoid this problem.

2 Consequences of the main result

We denote a partition of \(n\) by \(\lambda = (\lambda_1, \ldots, \lambda_k)\), with \(\lambda_1 \geq \cdots \geq \lambda_k \geq 1\). Our reference for the general theory of the symmetric group is \[18\].

For any partition \(\lambda = (\lambda_1, \ldots, \lambda_k)\) of \(n\), we define, for \(i > 0\),

\[
m_i(\lambda) = |\{j | \lambda_j = i\}|.
\]

The number \(m_i(\lambda)\) is called the multiplicity of \(i\) in \(\lambda\) (see \[16\]). Observe that \(m_i(\lambda) = 0\) for all \(i > n\), since \(\sum \lambda_i = n\). It is well-known that the multinomial coefficient

\[
\binom{m_1(\lambda) + m_2(\lambda) + \cdots + m_n(\lambda)}{m_1(\lambda), m_2(\lambda), \ldots, m_n(\lambda)}
\]

is the number of compositions associated to \(\lambda\). Hence, we obtain the following corollary.

**Corollary 2.1.** Let \(\lambda\) be a partition of \(n\), then

\[
|T_{\min}^\lambda| = \binom{m_1(\lambda^t) + m_2(\lambda^t) + \cdots + m_n(\lambda^t)}{m_1(\lambda^t), m_2(\lambda^t), \ldots, m_n(\lambda^t)},
\]

which is the number of compositions \(c\) of \(n\) such that \(\lambda(c) = \lambda^t\).

The minimal elements in two-sided cells are linked to another important number in combinatorics

\[
n(\lambda) = \sum_{i=1}^k \binom{\lambda_i^t}{2},
\]

see \[16\] p.2-3).

**Corollary 2.2.** Let \(\lambda\) be a partition of \(n\) and write \(\lambda^t = (\lambda_1^t, \ldots, \lambda_k^t)\). Then \(\ell(w) = n(\lambda)\) for all \(w \in T_{\min}^\lambda\).
Proof. Let \( c \) be a composition of \( n \) such that \( \lambda(c) = \lambda' \). Then \( \ell(c) = \ell(\sigma_{\lambda'}) \). Let \( w_i \) be the longest element of the Young subgroup \( S_{\lambda'_1} \), then \( \ell(w_i) = \binom{\lambda'_1}{2} \). Therefore

\[
\ell(\sigma_{\lambda'}) = \sum_{i=1}^{k} \binom{\lambda'_i}{2}
\]

since \( \sigma_{\lambda'} = w_1 \ldots w_k \) (seen as a word on the letters 1, \ldots, \( n \)) and that the letters in \( w_{i+1} \) are greater than the letters in \( w_i \). The corollary follows from Theorem 1.1. \( \square \)

As in the case of minimal elements, we have the following corollaries:

Corollary 2.3. Let \( \lambda \) be a partition of \( n \), then

\[
|T^\lambda_{\text{max}}| = \binom{m_1(\lambda) + m_2(\lambda) + \cdots + m_n(\lambda)}{m_1(\lambda), m_2(\lambda), \ldots, m_n(\lambda)}
\]

which is the number of compositions \( c \) of \( n \) such that \( \lambda(c) = \lambda \).

Corollary 2.4. Let \( \lambda = (\lambda_1, \ldots, \lambda_k) \) be a partition of \( n \), then

\[
\ell(w) = \binom{n}{2} - \sum_{i=1}^{k} \binom{\lambda_i}{2}
\]

for all \( w \in T^\lambda_{\text{max}} \).

Proof. As \( \ell(\sigma_{(n)}) = \binom{n}{2} \) and \( \ell(\sigma_{(n)}w) = \ell(\sigma_{(n)}) - \ell(w) \), for any \( w \in S_n \), the corollary follows from same argument than in the proof of Corollary 2.2. \( \square \)

3 Proof of Theorem 1.1

The following proposition implies that

\[
\{ c \mid \lambda(c) = \lambda' \} = \{ w_T \mid T \text{ is a reading tableau of shape } \lambda \} \subset T^\lambda.
\]

Proposition 3.1. Let \( \lambda \) be a partition of \( n \); then the following conditions are equivalent:

i) \( T \) is a reading tableau of shape \( \lambda \);

ii) \( w_T = \sigma_c \), where \( c \) is a composition of \( n \) such that \( \lambda(c) = \lambda' \).

Proof. Recall that the longest element of a Young subgroup is an involution, since it is unique.

Assume (i). As \( T \) is a reading tableau, if \( n \) is in the row \( T_i \), one has \( 1 \leq p \leq n - 1 \) such that \( p + 1 \) is in the first row of \( T \), \( p + i = n \) and \( p + j \) is at the end
of the row $T_j$, for all $1 \leq j \leq i$. One applies the $i$ first steps of the inverse of Robinson-Schensted correspondence, hence

$$w_T = w_{T'} n \ldots p + 1,$$

where $T'$ is the standard Young tableau obtained by deleting $p + 1, \ldots, n$ in $T$. Thus $w_{T'}$ is a permutation on the set $\{1, \ldots, p\}$. Observe that $T'$ is also a reading tableau. The shape of $T'$ is denoted by $\lambda'$. By induction on $n$, $w_{T'}$ is the longest element of the Young subgroup $S_{c'}$, where $\lambda(c') = \lambda'$. Then $w_T$ is the longest element of the Young subgroup $S_c \times S_I$. Let $c = (c', i)$; it is now easy to see that $\lambda(c) = \lambda'_i$.

Conversely, let $c = (n_1, \ldots, n_k)$ and use induction and similar arguments with direct Robinson-Schensted correspondence on the permutation

$$w_T = n_1 \ldots 1 w',$$

where $n_1 \ldots 1$ is the longest element of the Young subgroup $S_{n_1}$ and $w'$ is the longest element of the Young subgroup $S_{n_2} \times \cdots \times S_{n_k}$.

Now, it remains to prove that $\{\sigma_c | \lambda(c) = \lambda'_i\} = T_{\min}^\lambda$, to end the proof of Theorem 1.1.

**The Lusztig $\alpha$-function:** We consider the symmetric group $S_n$ as a Coxeter system $(W, S)$ of type $A_{n-1}$ with $W = S_n$ and generating set $S$ consisting of the $n-1$ simple transpositions $\tau_i = (i, i+1)$, where $i = 1, \ldots, n-1$. Then $\ell(w)$ is also the length of $w$ as a reduced word in the elements of $S$. A classical bijection between subsets of $S$ and compositions of $n$ is obtained as follows: Let $I \subset S$ and $S \setminus I = \{\tau_{i_1}, \ldots, \tau_{i_k}\}$ with $1 \leq i_1 < i_2 < \cdots < i_k \leq n - 1$. Set $n_1 = i_1$, $n_2 = i_2 - i_1 + 1$, ..., $n_k = n - i_k$, then $n_i$ are non-negative integers. By this way, we have obtained a unique composition $c_I = (n_1, \ldots, n_k)$ of $n$ associated to $I$. Moreover,

$$W_I = S_{n_1} \times \cdots \times S_{n_k}.$$ 

Therefore, as is well-known the Young subgroups of $S_n$ are precisely the parabolic subgroups of $S_n$ (see [3, Proposition 2.3.8]).

Our basic references for the work of Kazhdan and Lusztig are [11], [15] (see also [5]). We denote by $\leq$ the Bruhat order on $S_n$.

Let $\mathcal{A} = \mathbb{Z}[q^{1/2}, q^{-1/2}]$ where $q^{1/2}$ is an indeterminate. Let $\mathcal{H}$ be the Hecke algebra over $\mathcal{A}$ corresponding to $S_n$. Let $(T_w)_{w \in S_n}$ be the standard basis of $\mathcal{H}$ and $(\bar{T}_w)_{w \in S_n}$ the basis defined as follows:

$$\bar{T}_w = q^{-\ell(w)/2} T_w.$$ 

In [11] Theorem 1.1, Kazhdan and Lusztig have shown that there is a basis $(b_w)_{w \in S_n}$ of $\mathcal{H}$, called the Kazhdan-Lusztig basis, such that

$$b_w = \sum_{y \leq w} (-1)^{\ell(w) - \ell(y)} q^{(\ell(w) - \ell(y))/2} P_{y,w}(q^{-1}) \bar{T}_y,$$
where $P_{y,w} \in \mathcal{A}$ are the Kazhdan-Lusztig polynomials. Moreover, they have defined three equivalence relations on $S_n$, with equivalence classes that are called left cells, right cells and two-sided cells. In our case, the following result of Vogan and Jantzen result on $S_n$ gives the link with the Robinson-Schensted correspondence (see also [1]): the set $T^\lambda$ is a two-sided cell for all partitions $\lambda$ of $n$; and any two-sided cell of $S_n$ arises by this way.

Following Lusztig [14,15], let $h_{x,y,w}$ be the structure constants of the Kazhdan-Lusztig base $(b_w)_{w \in W}$, that is

$$b_x b_y = \sum_{w \in W} h_{x,y,w} b_w.$$ 

Denote $\delta(w)$ the degree of the Kazhdan-Lusztig polynomial $P_{e,w}$ as a polynomial in $q$. Write $u = q^{1/2}$. Let $a(w)$ be the smallest integer such that for any $x, y \in S_n$, $u^{a(w)} h_{x,y,w} \in \mathcal{A}^+$, where $\mathcal{A}^+ = \mathbb{Z}[u]$ (this is well defined for any Weyl group). In [14,15], Lusztig has shown the following properties about the $a$–function:

a) $a(w) \leq \ell(w) - 2\delta(w)$ ([15 Section 1.3]);

b) The $a$–function is constant on two-sided cells ([14 Theorem 5.4]).

c) For any $I \subset S$, $a(\sigma_{c_I}) = \ell(\sigma_{c_I})$ ([15 Corollary 1.9 (d) and Theorem 1.10]).

In other words, for any composition $c$ of $n$, $a(\sigma_c) = \ell(\sigma_c)$.

d) Let $D = \{w \in W \mid a(w) = \ell(w) - 2\delta(w)\}$, then each element in $D$ is an involution, called a Duflo involution ([15 Proposition 1.4]). In symmetric groups, all involutions are Duflo involutions. Indeed, each left cell contains a unique Duflo involution ([15 Proposition 1.4]); left cells are precisely coplactic classes (the sets of permutations having the same right tableau under $\pi$, see for instance [1]), and each coplactic class contains a unique involution.

Let $\lambda$ be a partition of $n$ and $T^\lambda$ be its associated two-sided cells. Properties (b) and (c) imply that $a_{\lambda} := a(\sigma_{\lambda'}) = a(w)$, for all $w \in T^\lambda$. Therefore, by (a),

$$\ell(\sigma_c) = a_{\lambda} = a(w) \leq \ell(w),$$

for any $w \in T^\lambda$. Thus

$$\{\sigma_c \mid \lambda(c) = \lambda'\} \subset T^\lambda_{\min}.$$ 

Now, let $w \in T^\lambda_{\min}$, then $a(w) = a_{\lambda} = \ell(\sigma_{\lambda'}) = \ell(w)$, since $\sigma_{\lambda'} \in T^\lambda_{\min}$. Property (d) implies that $w$ is a Duflo involution and $\delta(w) = 0$.

By Proposition 3.1, $\sigma_c \in T^\lambda$ implies $\lambda(c) = \lambda'$ Therefore, Theorem 1.1 is a direct consequence of the following result, which gives a surprising criterion about the degree $\delta(w)$ of the Kazhdan-Lusztig polynomial $P_{e,w}$, for $w \in S_n$ an involution.

**Proposition 3.2.** Let $w \in S_n$, then the following conditions are equivalent:

i) $w$ is an involution and $\delta(w) = 0$;

ii) $w = \sigma_c$, for some composition $c$ of $n$. 

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KL Polynomials and smoothness of Schubert Varieties: We say that a permutation $w \in S_n$, seen as a word $w = x_1 \ldots x_n$, avoids the pattern 4231 (resp. avoids the pattern 3412) if there is no $1 \leq i < j < k < l \leq n$ such that $x_l < x_j < x_k < x_i$ (resp. $x_k < x_l < x_i < x_j$). In other words, there is no subword of $w$ with the same relative order as the word 4231 (resp. 3412).

Here, we link these definitions with Kazhdan-Lusztig polynomials by the way of the following well-known criterion: Let $w \in S_n$, then

$$(\circ) \quad P_{e,w} = 1 \iff w \text{ avoids the patterns 4231 and 3412}.$$  

Indeed, on one hand, Lakshmibai and Sandhya have shown that a Schubert variety $X(w)$, $w \in S_n$, is smooth if and only if $w$ avoids the patterns 4231 and 3412 ([12] or see [2, Theorem 8.1.1]).

On the other hand, Deodhar [6] has shown a useful characterization of the smoothness by the way of Kazhdan-Lusztig polynomials: Let $w \in S_n$ then $P_{e,w} = 1$ if and only if $X(w)$ is smooth.

**Proof of Proposition 3.2** By the above discussion, Proposition 3.2 is a direct consequence of the following lemma.

**Lemma 3.3.** Let $w \in S_n$ an involution, then the following statements are equivalent

i) $w$ avoids the patterns 4231 and 3412;

ii) there is a composition $c$ of $n$ such that $w = \sigma_c$;

Proof. (ii) $\Rightarrow$ (i): write $c = (c_1, \ldots, c_k)$. If $k = 1$ then $\sigma_c = \sigma_{(n)}$ avoids the patterns 4231 and 3412. If $k > 1$ then $\sigma_c$ is the image of $(\sigma_{(c_1)}, \ldots, \sigma_{(c_k)})$ under the canonical isomorphism between $S_{c_1} \times \cdots \times S_{c_k}$ and $S_{c}$. Conclude by induction on $n$.

(i) $\Rightarrow$ (ii): one sees $w = x_1 \ldots x_n$ as a word on the letters $1, \ldots, n$.

One proceeds by induction on $n$. Therefore, one may suppose that (i) $\Rightarrow$ (ii) for all proper Young subgroups of $S_n$. If $n \leq 4$, it is readily seen. Suppose $n > 4$.

If $x_1 = 1$, then $w \in S_1 \times S_{n-1}$, and the lemma follows by induction.

If $n > x_1 = p > 1$, then $x_p = 1$ and $1 \leq x_i \leq p$, for all $1 \leq i \leq p$. Otherwise, there is $1 < i < p$ such that $x_i > p$. In other words, there is $1 < i < p < x_i$ such that $x_p = 1 < x_i = i < x_1 = p < x_i$, that is, $w$ has the pattern 3412 which is a contradiction.

Hence $w \in S_p \times S_{n-p}$ and the lemma follows by induction.

If $x_1 = n$, then $x_n = 1$ one just has to show that $w = w_0$. Otherwise, there is $1 < i < n-1$ such that $x_i < x_{i+1}$ (since if $i = 1$, $x_1 = n < x_2$ and if $i + 1 = n$, $x_{n-1} < x_n = 1$ which are contradictions). Thus there is $1 < i < i + 1 < n$ such that $x_n < x_i < x_{i+1} < x_1$, that is, $w$ has the pattern 4231 which is a contradiction.  

$\blacksquare$
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