Generalized Conformal Symmetry in D-Brane Matrix Models

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Abstract

We study in detail the extension of the generalized conformal symmetry proposed previously for D-particles to the case of supersymmetric Yang-Mills matrix models of Dp-branes for arbitrary \( p \). It is demonstrated that such a symmetry indeed exists both in the Yang-Mills theory and in the corresponding supergravity backgrounds produced by Dp-branes. On the Yang-Mills side, we derive the field-dependent special conformal transformations for the collective coordinates of Dp-branes in the one-loop approximation, and show that they coincide with the transformations on the supergravity side. These transformations are powerful in restricting the forms of the effective actions of probe D-branes in the fixed backgrounds of source D-branes. Furthermore, our formalism enables us to extend the concept of (generalized) conformal symmetry to arbitrary configurations of D-branes, which can still be used to restrict the dynamics of D-branes. For such general configurations, however, it cannot be endowed a simple classical space-time interpretation at least in the static gauge adopted in the present formulation of D-branes.

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1 Introduction

Conformal symmetry has long been playing a prominent role both in the realm of local field theory and of string theory. Especially, in the latter, the world-sheet (super) conformal symmetry stands out as the single most important principle, at least in its perturbative formulation, which endows it the characteristic features of a unified theory of all the interactions of nature including gravity.

In the recent developments in string dualities, notably in the context of conjectured AdS/CFT correspondence \[1\] \[2\] \[3\], the conformal symmetry continues to play a pivotal role. For example, in the prototypical case of D3-brane system, the \(AdS_5 \times S^5\) space-time produced by a large number of coincident D3-branes and the \(\mathcal{N} = 4\) super Yang-Mills theory describing the low energy dynamics of them share the same symmetry group including the conformal group \(SO(4,2)\). This symmetry, together with supersymmetric non-renormalization theorems, was shown \[1\] to be powerful enough to fix the form of the effective action for a probe D3-brane in such a space-time in the near-horizon limit.

From the standpoint of the space-time uncertainty principle proposed by one of the present authors \[4\], which qualitatively captures the essence of the short-distance space-time structure in string theories, the existence of such a conformal symmetry is deeply connected to the opposite scaling laws for the “longitudinal” coordinate (including time) \(X_\parallel\) and the “transverse” coordinate \(X_\perp\). For the brane system in question, the former corresponds to the world volume space-time coordinate \(x^\alpha\) while the latter refers to the target space (spatial) coordinate \(x^m\) transverse to the brane, and the proposed principle expresses the duality between the small and the large distance scales for these two categories of coordinates. This duality is at the heart of the \(s-t\) duality, which in turn should be the basis of the AdS/CFT correspondence. It is also intimately related to the so-called UV/IR correspondence \[5\] thought to be the mechanism underlying the holographic principle \[6\].

As the viewpoint described above is not restricted to any particular D-brane, it is natural to suspect that some form of conformal invariance might exist for D\(p\)-brane systems for general value of \(p\), not just for \(p = 3\). Indeed in a previous paper \[7\], two of the present authors demonstrated that a conformal symmetry of \(SO(2,1)\) type exits for
D0-brane system both for the supergravity solution and for the super Yang-Mills matrix theory, provided that the string coupling $g_s$ (or the Yang-Mills coupling $g^2$) is also transformed like a background field of dimension 3. Although it is not a symmetry in the strict sense of the word, it is a powerful structure with which one can derive Ward identities that govern the theory\[8\]. In fact it was shown that the effective action for the probe D-particle in the near-core limit dictated by this structure coincides with the one obtained in Matrix theory calculation \[9\] in the discrete light-cone prescription. In this sense, the above symmetry structure should deserve to be called a generalized conformal symmetry (GCS).

In the same work \[7\], another important aspect of the (generalized) conformal invariance was recognized and emphasized. It is related to the difference in the forms of the special conformal transformation (hereafter referred to as SCT) on the supergravity side and on the Yang-Mills side. This difference already exists in the case of D3-brane system. For the world-volume coordinates of the Yang-Mills theory, it is of the canonical form, namely $\delta_\epsilon x^\alpha = 2\epsilon \cdot xx^\alpha - \epsilon^\alpha x^2$. Contrarily, SCT that leaves the metric of $AdS_5 \times S^5$ invariant has a non-canonical form

$$\delta_{\epsilon}^{AdS} x^\alpha = 2\epsilon \cdot xx^\alpha - \epsilon^\alpha x^2 - \epsilon^\alpha \frac{2g^2 N}{U^2},$$

(1)

with the last term depending both on the Yang-Mills coupling $g$ and on the transverse radial coordinate $U(= r/\alpha')$. Since $U$ corresponds to the expectation value of the diagonal part of the Higgs field on the Yang-Mills side, the latter dependence is field-dependent as well as non-linear. This field-dependence is of utmost importance in restricting the dynamics of D-branes, as it connects the terms in the effective action which, from the viewpoint of the Yang-Mills theory, are induced at different loop orders.

This difference in realization is formally consistent in the usual picture \[2\] of the Yang-Mills theories, where these theories are considered to live on the boundaries of the AdS space-times, since the transformation trivially reduces to the usual linear one as $U \to \infty$. In this interpretation, the information of the Yang-Mills theory is used only as the boundary condition for the theory in the bulk.

\[8\]For instance, this type of symmetry structure is successfully used in \[8\] to prove a non-renormalization theorem.
However, if one takes the super Yang-Mills theory as the dynamical theory of D-branes, then the situation becomes quite different. Since one can place the D-branes anywhere in the bulk, in order for the D-branes to correctly detect the supergravity effect, the non-linear field-dependent SCT which characterizes the gravity in the bulk must emerge within the Yang-Mills theory. It is not at all evident how such a field-dependent transformation, whose origin is the isometric diffeomorphism of supergravity, is derived from the linear conformal transformation of the Yang-Mills theory. As the (generalized) conformal symmetry is the basic underlying structure that supports the conjectured duality, understanding of this problem is clearly of prime importance. In previous studies, including the Maldacena’s original work, this issue however remained untouched.

Very recently, we have succeeded in resolving this issue in the case of D3-brane system \cite{10}. We showed that SCT law for the diagonal Higgs field in $\mathcal{N} = 4$ Yang-Mills theory is modified by what we called the “quantum metamorphosis” effect associated with the loops of off-diagonal massive fields and that, to the leading order in the velocity expansion, it takes exactly the form of the AdS transformation law (1), including the numerical coefficient. The key observation was that SCT changes the gauge orbit specified by a background gauge and the extra transformation necessary to get back to the original gauge orbit induces the desired correction in the transformation of the diagonal Higgs field.

The purpose of this article is to extend the previous discussions to the system of D$p$-branes for general $p$. Specifically, we will provide the answers to the following questions:

(i) Does there exist a generalized conformal symmetry on both the supergravity and the super Yang-Mills side for D$p$-brane system for general $p$?

(ii) If it does, how are its realizations on respective side related?

While the affirmative answer to (i) can be obtained rather straightforwardly along the lines of the previous work on the D0-brane system \cite{7}, the proper understanding of (ii) turned out to involve an intriguing subtlety compared with the D3-brane case treated in \cite{10}.

4
The organization of the rest of the article will be as follows: In section II, we present the generalized conformal transformations both from the viewpoints of supergravity and of super Yang-Mills matrix models. We demonstrate how GCS can be used to determine the effective DBI (Dirac-Born-Infeld) actions for the probe Dp-branes. Section III deals with the problem (ii) stated above. We will first generalize the mechanism of “quantum metamorphosis” previously found for D3-brane system and establish the precise form of quantum GCS for Dp-brane super Yang-Mills theory. We then compute the form of the modified SCT for the diagonal Higgs field in one-loop approximation. This turned out to differ by a $p$-dependent factor from the one expected from supergravity. To understand this apparent discrepancy, an explicit calculation of the effective action for Dp-branes will be performed, with careful treatment of the dependence of the string coupling on the world-volume coordinates. We will find that the result contains an additional term proportional to the derivative of the coupling, which nevertheless is completely consistent with the quantum GCS. We then go on to demonstrate that appropriate redefinitions of the collective coordinates of the Dp-branes in the Yang-Mills theory remove this extra term and correct the factor in the SCT law to the desired value simultaneously. This mechanism will be shown to be understood from the supergravity side as well. Our discussion on this point will disclose some remarkable consistency between supergravity and super Yang-Mills matrix models, which to our knowledge has never been envisaged in the previous literature. As the final topic in section III, we will briefly discuss a generalization of our result to more general configurations of D-branes, taking the case of D-particles as the simplest example. In the concluding section, we discuss the remaining problems as well as possible further implications and extensions of the (generalized) conformal symmetries.

2 Generalized Conformal Symmetry (GCS) for Dp-Branes

2.1 GCS for the metric and DBI action

Consider the supergravity solution produced by $N$ coincident Dp-branes at the origin. The near-horizon (or more appropriately, ‘near-core’ for general $p$) limit of interest is
defined by \( \alpha' \rightarrow 0 \),
\begin{equation}
\alpha' \rightarrow 0 , \quad (2)
\end{equation}
\begin{equation}
g^2 = (2\pi)^{p-2} g_s \alpha'(p-3)/2 = \text{fixed} , \quad (3)
\end{equation}
\begin{equation}
U = \frac{r}{\alpha'} = \text{fixed} , \quad (4)
\end{equation}

where \( g, g_s, r \) are, respectively, the Yang-Mills coupling, the string coupling, and the transverse distance from the branes. In this limit, the metric, the dilaton and the \((p+1)\)-form RR gauge fields can be written in the following form:

\begin{equation}
ds^2 = \alpha' \left( h_p^{-1/2} dx^2 + h_p^{1/2} (dU^2 + U^2 d\Omega^2_{8-p}) \right) , \quad (5)
\end{equation}
\begin{equation}
e^\phi = g_s \left( \frac{h_p}{\alpha'^2} \right)^{(3-p)/4} , \quad (6)
\end{equation}
\begin{equation}
A_{0...p} = - \frac{1}{2g_s} \left( \frac{h_p}{\alpha'^2} \right)^{-1} , \quad (7)
\end{equation}
\begin{equation}
h_p = \frac{Q_p}{U^{7-p}} , \quad (8)
\end{equation}
\begin{equation}
Q_p = g^2 N d_p , \quad d_p = 2^{7-2p} \pi^{(9-3p)/2} \Gamma \left( \frac{7-p}{2} \right) . \quad (9)
\end{equation}

Let us introduce a convenient dimensionless variable \( \rho_p \) defined by
\begin{equation}
\rho_p \equiv \frac{Q_p}{U^{3-p}} . \quad (10)
\end{equation}

Then the metric can be written in a suggestive form as
\begin{equation}
ds^2 = \alpha' \left( \frac{U^2}{\sqrt{p_p}} dx^2 + \frac{\sqrt{p_p}}{U^2} dU^2 + \sqrt{\rho_p} d\Omega^2_{8-p} \right) . \quad (11)
\end{equation}

Except for \( p = 3, \rho_p \) is coordinate-dependent and hence the space-time is not exactly of AdS type. But if \( \rho_p \) were constant, the metric would be that of \( AdS_{p+2} \times S^{8-p} \) and this prompts us to seek a generalized conformal transformation that leaves \( \rho_p \) invariant.

Since the scale and the Lorentz invariance are trivial, we will concentrate on the special conformal transformation. Take the usual transformation law for the variable \( U \), namely,
\begin{equation}
\delta U = -2\varepsilon \cdot x U . \quad (12)
\end{equation}
Then, the requirement $\delta, \rho_p = 0$ readily leads to
\[ \delta \epsilon Q_p = -2(3 - p)\epsilon \cdot x Q_p. \]
(13)

This means that we must treat $Q_p$ (i.e. $g_s$) not as a strict constant but as a “field” on the world-volume, to the linear order in $x$, before making SCT. Once SCT is made, we may set it to a constant. As for the transformation of $x^\alpha$, we assume the AdS-like form
\[ \delta x^\alpha = 2\epsilon \cdot x x^\alpha - \epsilon^\alpha x^2 - \epsilon^\alpha \frac{k \rho_p}{U^2}, \]
(14)
with some constant $k$. It is then straightforward to check that the metric (11) is invariant under the SCT defined above, provided we take
\[ k = \frac{2}{5 - p}. \]
(15)
This indeed covers the D0-brane case previously studied in [7] as well as the D3-brane case.

Let us now demonstrate that GCS governs the DBI effective action for a radially moving probe D$p$-brane in the field of a heavy source consisting of $N$ coincident D$p$-branes placed at the origin. Rather than checking the invariance of the DBI action directly, it is instructive (just as in [1]) to start from the most general scale and Lorentz invariant effective action made out of $U$, $\partial_\alpha U$ and $\rho_p$ and see how much restriction is imposed by the invariance under the generalized SCT. Such an action must be of the form
\[ S = -\int d^{p+1}x U^{p+1} f(z, \rho_p), \]
(16)
\[ z = \frac{\partial_\alpha U \partial^\alpha U}{U^4}, \]
(17)
where $f(z, \rho_p)$ is an arbitrary function. Applying SCT for $U$, $z$ and the measure $d^{p+1}x$, the condition for invariance under SCT is worked out as
\[ 0 = \delta S = -2 \int d^{p+1}x U^{p+1} \epsilon \cdot \partial U \frac{\rho_p}{U^3} \left( f - 2(z + \rho_p^{-1}) \partial_z f \right). \]
(18)
Noting that a shift of $f(z, \rho_p)$ by an arbitrary function of $\rho_p$ does not spoil the invariance, we get a differential equation for $f$, with an arbitrary function $c(\rho_p)$:
\[ f + c(\rho_p) = 2(z + \rho_p^{-1}) \partial_z f. \]
(19)
Its general solution is

\[ f = a(\rho_p) \left( \sqrt{1 + \rho_p z} - b(\rho_p) \right), \tag{20} \]

where \( a(\rho_p) \) and \( b(\rho_p) \) are arbitrary. This is as much as GCS dictates on the form of \( S \).

The remaining two functions \( a(\rho_p) \) and \( b(\rho_p) \) can then be fixed by invoking the following non-renormalization theorems. First the BPS condition that there is no static force between the D-branes fixes \( b(\rho_p) \) to be unity. Further if the \( \mathcal{O}(z) \) term is not renormalized from the simple tree level form, then \( a(\rho_p) \) is determined to be equal to \((Nd_p/(2\pi)^2)\rho_p^{-2}\).

Altogether, we get the familiar DBI action

\[ S_{DBI} = -\int d^{p+1}x \frac{1}{(2\pi)^2 g^2} \frac{U^{7-p}}{Q_p} \left( \sqrt{1 + Q_p \frac{\partial_a U \partial^a U}{U^{7-p}}} - 1 \right). \tag{21} \]

Thus, it should now be clear that GCS for general \( p \) is just as powerful as the usual conformal symmetry for \( p = 3 \).

2.2 GCS for classical super Yang-Mills

We now turn to the \((p+1)\)-dimensional super Yang-Mills theory describing the low energy dynamics of near-coincident \( N \) D\( p \)-branes\(^1\). Such a theory can be obtained most simply by the dimensional reduction of \( \mathcal{N} = 1 \) \( U(N) \) 10-dimensional super Yang-Mills theory, the classical action of which is give by

\[ S_{10} = \int d^{10}x \text{Tr} \left\{ -\frac{1}{4g_{10}^2} F_{MN} F^{MN} + i \bar{\psi} \Gamma^M [D_M, \psi] \right\}, \tag{22} \]

\[ D_M = \partial_M - iA_M. \tag{23} \]

It is not difficult to check that the fermionic part of the action, including its reduction, is invariant under the usual conformal transformations in any dimensions provided appropriate dimensions for the fermion fields are assigned. Therefore, we will concentrate on the bosonic part. When reduced to \((p+1)\)-dimensions, it takes the form

\[ S_{bosonic} = \text{Tr} \int d^{p+1}x \left\{ -\frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2g^2} D_\mu X_\mu X^\mu + \frac{1}{4g^2} [X_m, X_n]^2 \right\}. \tag{24} \]

\(^1\)Although the discussions to follow will go through formally for any \( p \), we shall restrict ourselves to \( 0 \leq p \leq 3 \), since the quantum property of the theory above 4-dimensions is not well-understood.
where the Greek (Latin) indices run in the range $0 \sim p \ (p + 1 \sim 9)$ and $X_m$ are the Higgs scalars.

Let us describe the generalized conformal symmetry possessed by this action. Consider first the usual conformal transformations of the relevant fields, especially the dilatation $\delta^D_\epsilon$ and SCT $\delta_\epsilon$. The variations at numerically the same point $x$, which are more convenient in the following, are

$$
\delta^D_\epsilon X_m = -(\epsilon + x \cdot \partial)X_m, \\
\delta_\epsilon X_m = -2\epsilon \cdot xX_m(x) - (\delta_\epsilon x^\alpha)\partial_\alpha X_m, \\
\delta_\epsilon A^\mu = -2\epsilon \cdot xA^\mu - 2(x \cdot A\epsilon^\mu - \epsilon \cdot A x^\mu) - (\delta_\epsilon x^\alpha)\partial_\alpha A^\mu,
$$

where the SCT variation $\delta_\epsilon x^\alpha$ for the coordinate is of the “canonical” form

$$
\delta_\epsilon x^\alpha = 2\epsilon \cdot x x^\alpha - \epsilon^\alpha x^2.
$$

Because of the presence of the coupling $g^2$, the action $S_{bosonic}$ is not invariant under these transformations, except for $p = 3$. However, just as in the case of the supergravity description of the D$p$-brane system discussed in the previous subsection, we can make it invariant if we regard the coupling $g^2$ as a background field $g^2(x)$ transforming like a scalar field of mass-dimension $3 - p$, namely,

$$
\delta^D_\epsilon g^2 = -\epsilon(3 - p + x \cdot \partial)g^2, \\
\delta_\epsilon g^2 = -2(3 - p)\epsilon \cdot xg^2 - (\delta_\epsilon x^\alpha)\partial_\alpha g^2.
$$

The proof is a straightforward exercise.

Thus, we have shown that indeed the concept of GCS can be extended to the system of D$p$-branes for general $p$ both on the supergravity side and on the super Yang-Mills side. The notable difference in the form of SCT on two sides, however, exists just like in the case of ordinary conformal symmetry for D3-brane system. In the next section, we shall clarify the nature of this phenomenon and give the precise correspondence.

3 Relation between the Realizations of GCS in Super Yang-Mills and in Supergravity
3.1 Quantum form of GCS for super Yang-Mills

As was briefly reviewed in the Introduction, the apparent gap between the SCT laws in the supergravity and the Yang-Mills theory can be shown to be neatly filled by a quantum effect on the Yang-Mills side in the case of D3-brane system [10]. It is then natural to expect that the same mechanism should be at work in the case of GCS as well. It turns out, however, that there is a subtle but important difference between the two categories.

Let us first apply the logic of [10] to the Dp-brane system for general p and see what happens. Actually, instead of generalizing the argument of [10] directly, we will use a more systematic BRST approach developed by Fradkin and Palchik [12], which is suitable for dealing with the standard background gauge. Let us decompose the Higgs field $X_m$ as

$$X_m = B_m + Y_m,$$

with $B_m$ the diagonal background and $Y_m$ the quantum fluctuation. We take the gauge-fixing and the corresponding ghost actions to be (hereafter $D = p + 1$)

$$S_{gf} = -\frac{1}{2} \int \frac{d^Dx}{g^2} \text{Tr} \ G^2,$$

$$G = -\partial_\mu A_\mu + i [B_m, Y_m],$$

$$S_{gh} = i \int d^Dx \text{Tr} \left( -\bar{C} \partial^\mu D_\mu C + \bar{C} [B_m, [X_m, C]] \right).$$

The total action is invariant under the BRST transformation, with a fermionic parameter $\lambda$,

$$\delta_B X_m = -i [C, X_m] \lambda, \quad \delta_B A_\mu = -[D_\mu, C] \lambda,$$

$$\delta_B C = iC^2 \lambda, \quad \delta_B \bar{C} = \frac{i}{g^2} G \lambda.$$  

Now let us apply the generalized conformal transformations. $C$ and $\bar{C}$ will be regarded as scalar fields with dimension 0 and $D - 2$ respectively. Then one finds that, while the scale invariance is trivial, $S_{gf}$ and $S_{gh}$ are not invariant under SCT:

$$\delta_\epsilon S_{gf} = 2(D - 2) \int \frac{d^Dx}{g^2} \text{Tr} \ G A \cdot \epsilon,$$

$$\delta_\epsilon S_{gh} = -2(D - 2)i \int d^Dx \text{Tr} \ C \epsilon^\mu D_\mu C.$$

10
A remarkable fact is that one can find a compensating field-dependent BRST tran-
sformation which removes these unwanted variations. Take the fermionic parameter $\lambda$ to be

$$\lambda = -2(D - 2) \int d^D y \text{Tr} (\bar{C}(y) A(y) \cdot \epsilon) , \quad (39)$$

and denote this special BRST variation by $\Delta_\epsilon$. Then the total action is invariant but the
functional measure undergoes a non-trivial transformation. It is easy to find

$$\mathcal{D}(A + \Delta_\epsilon A) \mathcal{D}(\bar{C} + \Delta_\epsilon \bar{C}) = \mathcal{D} A \mathcal{D} \bar{C} \exp \left( i (-\delta_\epsilon S_{gf} - \delta_\epsilon S_{gh}) \right) , \quad (40)$$

where $\delta_\epsilon S_{gf}$ and $\delta_\epsilon S_{gh}$ are precisely as given in (37) and (38). (Measures for other fields
are invariant.)

We have now established the precise form of quantum GCS:

- Super Yang-Mills theory for $Dp$-brane system is invariant under the generalized
  conformal symmetry, with the modified SCT given by

$$\tilde{\delta}_\epsilon = \delta_\epsilon + \Delta_\epsilon . \quad (41)$$

In particular this leads to the Ward identity for the effective action $\Gamma[B, g^2]$, which is 1PI
with respect to the background field $B$:

$$\int d^{d+1} x \left( \delta_\epsilon g^2(x) \frac{\delta}{\delta g^2(x)} + (\delta_\epsilon B(x) + \Delta_\epsilon B(x)) \frac{\delta}{\delta B(x)} \right) \Gamma[B, g^2] = 0 . \quad (42)$$

We emphasize that this is an exact statement for any background.

3.2 GCS at leading order

Following the formalism developed above, let us compute the extra piece $\Delta_\epsilon B$ of the SCT
explicitly. Denote by $B_{m,i}$ the i-th diagonal component of $B$. The extra contribution is
given by

$$\Delta_\epsilon B_{m,i} = 2i(D - 2) \langle [C, X_{m,i}] \int d^D y \text{Tr} (\bar{C}(y) A(y) \cdot \epsilon) \rangle , \quad (43)$$

where $\langle \ldots \rangle$ denotes the expectation value. For ease of calculation, go to the Euclidean
formulation by making the following replacements: $d^D y = -i d^D \bar{y}$, $A(y) \cdot \epsilon = \bar{A} \cdot \bar{\epsilon}$,
\( A_0 = -iA_0, \bar{\epsilon}^0 = i\epsilon^0 \). We will be interested in the correction which is leading order in the velocity \( \partial B \). Then (13) can be approximated by

\[
\Delta \epsilon B_{m,i}(\bar{x}) = 2(D-2) \int d^D \bar{y} \left\{ \langle C_{ij}(\bar{x})\bar{C}_{ji}(\bar{y})\rangle \langle Y_{m,ji}(\bar{x})\bar{A}_{\mu,ij}(\bar{y})\rangle \bar{\epsilon}^\mu - (i \leftrightarrow j) \right\} .
\]

To the same order of accuracy, the relevant 2-point functions are given by

\[
\langle C_{ij}(\bar{x})\bar{C}_{ij}(\bar{y})\rangle = i\langle \bar{x}|\Delta_{ij}|\bar{y}\rangle ,
\]

\[
\langle Y_{m,ji}(\bar{x})\bar{A}_{\mu,ij}(\bar{y})\rangle = -2i\partial_\mu B_{m,ij}g^2 \langle \bar{x}|\Delta_{ij}^2|\bar{y}\rangle ,
\]

where the basic propagator is \( \langle \bar{x}|\Delta_{ij}|\bar{y}\rangle = \int \frac{d^D p}{(2\pi)^D} \frac{1}{(p^2 + B_{ij}^2)} \). Using the formula

\[
I_n^D(B_{ij}) = \langle \bar{x}|\Delta_{ij}^n|\bar{x}\rangle = \int \frac{d^D p}{(2\pi)^D} \frac{1}{(p^2 + B_{ij}^2)^n} = \frac{\Gamma(n-(D/2))}{(4\pi)^{D/2}\Gamma(n) B_{ij}^{2n-D}} ,
\]

and going back to the Minkowski notation, we get

\[
\Delta \epsilon B_{m,i} = \sum_j \frac{4(D-2)\Gamma(3-(D/2))g^2}{(4\pi)^{D/2} B_{ij}^{6-D}} \epsilon \cdot \partial B_{m,ij} .
\]

Let us specialize to the typical source-probe situation with \( N \) \( D \)-branes as the source at the origin and \( B_m \) the probe coordinate. Taking into account the relation between \( B \equiv \sqrt{\sum_m B_m^2} \) and the supergravity coordinate \( U \), namely, \( U = 2\pi B \), the formula (49) for this configuration becomes

\[
\Delta \epsilon U = \frac{p - 1}{2} \frac{k\rho_p}{U^2} \epsilon \cdot \partial U .
\]

Remember that we have been using the scheme in which the variation is taken at the same point with the underlying canonical transformation (28). Therefore, if we convert to the scheme where \( U \) is transformed canonically without \( \Delta \epsilon U \) piece as in the supergravity treatment, the coordinate transformation should be taken as

\[
\delta_x x^\alpha = 2\epsilon \cdot x x^\alpha - \epsilon^\alpha x^2 - \frac{p - 1}{2} \epsilon^\alpha \frac{k\rho_p}{U^2} .
\]

This agrees with the AdS-type transformation law (14) only when the factor \( (p - 1)/2 \) equals unity, i.e. for \( p = 3! \) This conforms to our previous result (10) for \( p = 3, \) but
it is quite puzzling. On one hand, the GCS as formulated in (41) must certainly be the
symmetry of the effective action for super Yang-Mills for any $p$ and hence (51) should be
the correct transformation law. On the other hand, at least for the D0-brane system, the
Yang-Mills effective action has been checked to agree with the DBI action to 2-loop order
[13] and the latter is invariant under (14), not under (51) with $p = 0$. We shall resolve
this apparent contradiction in the next two subsections.

3.3 Examination of 1-loop effective action

From the point of view of Yang-Mills theory, the key to the resolution of the puzzle lies
in the careful treatment of the coordinate dependence of the coupling $g(x)$. In computing
the effective action itself, we must carefully keep terms linear in $\partial g$, which are neglected
in the usual calculation. Under SCT defined in (30), $\partial_\alpha g$ transforms like $\delta_\epsilon \partial_\alpha g = -(3 - p)\epsilon_\alpha g + \mathcal{O}(\partial g)$ and produces a finite contribution even as we set $\partial g$ to zero after the
transformation.

Let us then investigate how the effective action is modified due to this effect. For
simplicity of presentation, we exhibit the D0-brane case in some detail. Extension to
general $p$ is entirely straightforward. In the Euclidean formulation, the total action $\tilde{S}$ for
the D0-brane system takes the form

$$
\tilde{S} = \int d\tau \text{Tr} \left\{ \frac{1}{2g_s} (D_\tau X_m)^2 - \frac{1}{4g_s} [X_m, X_n]^2 - \frac{1}{2} \partial^T D_\tau \theta - \frac{1}{2} \partial^T \gamma^m [X_m, \theta] \right\} + \tilde{S}_{gf} + \tilde{S}_{gh},
$$

(52)

$$
\tilde{S}_{gf} = \int \frac{d\tau}{2g_s} \text{Tr} \left( -\partial_\tau \tilde{A} + i [B_m, X_m]^2 \right),
$$

(53)

$$
\tilde{S}_{gh} = i \int d\tau \text{Tr} \left\{ \tilde{C} \partial_\tau D_\tau C - \tilde{C} \left[ B_m, [X_m, C] \right] \right\}.
$$

(54)

We will be interested in the dependence linear in the quantity

$$
\eta(\tau) = \frac{\partial_\tau g_s}{g_s}.
$$

(55)

Since the details of the 1-loop calculation with constant $g_s$ is well-documented (see for
example [13]) we shall only indicate the modification due to the presence of $\eta(\tau)$. When

**When dealing with the D0-brane system, for simplicity we shall use the often-adopted scheme; namely,
we rescale $X$ by a factor $2\pi\alpha'$ so that it carries the dimension of length and then set $l_s = \sqrt{\alpha'} = 1.$**
expanded about the background field, the quadratic parts which are modified at $O(\eta)$ are

$$\mathcal{L}_{YY} = \frac{1}{2g_s} Y_{m,ij} (-\partial^2_r + \eta(\tau) \partial_r + B^2_{ij}) Y_{m,ji}, \quad (56)$$

$$\mathcal{L}_{\tilde{A}A} = \frac{1}{2g_s} \tilde{A}_{ij} (-\partial^2_r + \eta(\tau) \partial_r + B^2_{ij}) \tilde{A}_{ji}, \quad (57)$$

$$\mathcal{L}_{Y\tilde{A}} = \frac{2i}{g_s} \left( \dot{B}_{m,ij} - \frac{1}{2} \eta(\tau) B_{m,ij} \right) Y_{m,ij} \tilde{A}_{ji}, \quad (58)$$

where $B_{m,ij}$ and $B_{ij}$ are as defined previously. Fermions and ghosts are not affected. $Y\tilde{A}$-mixing can be analyzed in exactly the same way as for the constant $g_s$ case if we make the following replacement:

$$\dot{B}_{ij} \rightarrow V_{ij}, \quad (59)$$

$$V_{ij} = \left[ \sum_m \left( \dot{B}_{m,ij} - \frac{1}{2} \eta(\tau) B_{m,ij} \right) \right]^{1/2} = \dot{B}_{ij} - \frac{1}{2} \eta \sum_m \dot{B}_{m,ij} B_{m,ij} / B_{ij} + O(\eta^2). \quad (60)$$

Then the Euclidean 1-loop effective action can be computed as

$$e^{-(\Gamma_1 + \Delta\Gamma_1)} = \prod_{i<j} \det \left( -\partial^2_r + \eta \partial_r + B^2_{ij} \right)^{-8} \times \det \left( -\partial^2_r + \eta \partial_r + B^2_{ij} + 2V_{ij} \right)^{-1} \times \det \left( -\partial^2_r + \eta \partial_r + B^2_{ij} - 2V_{ij} \right)^{-1} \times \det \left( -\partial^2_r + B^2_{ij} + \dot{B}_{ij} \right)^4 \times \det \left( -\partial^2_r + B^2_{ij} - \dot{B}_{ij} \right)^4 \times \det \left( -\partial^2_r + B^2_{ij} \right)^2, \quad (61)$$

where $\Gamma_1$ is the usual contribution and $\Delta\Gamma_1$ is the extra part linear in $\eta(\tau)$. For simplicity, we will drop the subscripts $(ij)$ and make the eikonal approximation. Namely we set

$$x \equiv B = b + \tilde{v} \tau = b + vt, \quad (62)$$

$$b \cdot \tilde{v} = 0, \quad (63)$$

$$r^2 \equiv B \cdot B = b^2 + \tilde{v}^2 \tau^2 = b^2 + \tilde{v}^2 t^2, \quad (64)$$

14
where \( \tilde{v}(v) \) is the Euclidean (Minkowski) velocity\(^{[1]} \). Then, keeping terms linear in \( \eta(\tau) \) and expanding up to \( O(\tilde{v}^4) \), \( \Delta \Gamma_1 \) is given by

\[
\Delta \Gamma_1 = 10 \text{Tr} \left\{ \Delta \eta \partial_\tau - \tilde{v}^2 \Delta \eta \partial_\tau \Delta \tau^2 + \tilde{v}^4 \Delta \eta \partial_\tau \Delta \tau^2 \Delta \tau^2 \right\} \\
+ 4 \text{Tr} \left\{ \tilde{v}^2 (\Delta^2 \eta \tau + 2 \Delta^3 \eta \partial_\tau) + \tilde{v}^4 (-2 \Delta \tau^2 \Delta^2 \eta \tau - 6 \Delta^3 \eta \partial_\tau \Delta \tau^2 + 4 \Delta^4 \eta \tau) \right\},
\]

where \( \Delta \equiv (-\partial_\tau^2 + b^2)^{-1} \) is the basic propagator. The calculation of the trace is a bit tedious but straightforward using the integration formula (48) with \( D = 1 \). The leading contribution, after converting back to Minkowski space and supplying a factor of \( N \) for the source-probe situation, turned out to be

\[
\Delta \Gamma_1 = N \int dt \eta(t) t \left( -6 v^2 T_2^1(b) - 12 v^4 T_4^1(b) \right),
\]

(65)

Assuming that the subleading corrections can be taken into account by the replacements \( b \to r \), \( v^2 t \to v \cdot x \) (see (62) \( \sim (64) \)), this corresponds to

\[
\Delta \Gamma_1 = N \int dt \frac{\dot{g}_s}{g_s} \left( -3 \frac{v \cdot x}{2 r^3} - \frac{15 v^2 v \cdot x}{8 r^7} \right).
\]

(68)

Having obtained the correction, let us see how it varies under SCT. In what follows, we will omit the infinitesimal parameter \( \epsilon_0 \) for simplicity, and denote the SCT variation by \( \delta K \). Then, the SCT’s at fixed \( t \) for the D0-brane case can be written as

\[
\delta_K x = 2tx + t^2 v, \quad \delta_K r = 2tr + t^2 \frac{v \cdot x}{r}, \quad \delta_K v = 2x + 4tv + t^2 \dot{v},
\]

(69)

\[
\delta_K \frac{\dot{g}_s}{g_s} = 6 + 2t \frac{\dot{g}_s}{g_s} + t^2 \frac{d}{dt} \left( \frac{\dot{g}_s}{g_s} \right).
\]

(70)

As was already explained before, we may set \( \dot{g}_s \) to zero after SCT is made. Thus for \( \Delta \Gamma_1 \) we only need to use the last of these formulae in the form \( \delta_K (\dot{g}_s/g_s) = 6 \). Then since \( -v \cdot x/r^3 = (d/dt)r^{-1} \), the variation of the first term in \( \Delta \Gamma_1 \) becomes a total derivative and can be dropped, and we get

\[
\delta_K \Delta \Gamma_1 = \int dt \frac{-45N v^2 v \cdot x}{4 r^7}.
\]

(71)

\(^{[1]} \)The condition in the second line can always be achieved by a constant shift of \( \tau \) and a redefinition of \( b \).
On the other hand, the usual 1-loop effective action $\Gamma_1$ is
\begin{equation}
\Gamma_1 = \frac{15N}{16} \frac{v^4}{r^7},
\end{equation}
and its SCT variation, computed using (69), takes the form
\begin{equation}
\delta_K \Gamma_1 = \int dt \frac{30N}{4} \frac{v^2 v \cdot x}{r^7}.
\end{equation}
From (71) and (73), we find
\begin{equation}
\delta_K (\Gamma_1 + \Delta \Gamma_1) = -\frac{1}{2} \delta_k \Gamma_1 = \left( \frac{p-1}{2} \right)_{p=0} \delta_k \Gamma_1.
\end{equation}
This shows that indeed, with the proper correction proportional to $\dot{g}_s$, the appearance of the extra factor $(p-1)/2$ for the SCT variation predicted by GCS is realized in D0-brane Yang-Mills theory. It is easy to confirm that the variation (74) is precisely cancelled by the variation of the lowest order action $\Gamma_0 \equiv \int dt \frac{v^2}{2g_s}$ arising from the $U$-dependent modified term in the loop-corrected SCT (51).

Demonstration of such a consistency for the Dp-brane super Yang-Mills theory for general $p$ is entirely similar: Essentially, the only difference from the D0-brane case is the use of $p$-dimensional integrals $I_2^p$ and $I_4^p$, defined in (48), in place of $I_2^1$ and $I_4^1$ in the formula corresponding to (66). In this way one obtains
\begin{equation}
\Gamma_1 + \Delta \Gamma_1 = N \int d^{p+1}x \left( \frac{C}{8} \frac{(\partial B \cdot \partial B)^2}{B^{7-p}} - \frac{C}{4} \frac{(\partial B)^2 \partial_a B \partial_a Q_p}{B^{6-p} Q_p} \right),
\end{equation}
where the second term in (73) represents the correction $\Delta \Gamma_1$. By using the SCT previously defined for general $p$, one can easily check that the relation $\delta_k (\Gamma_1 + \Delta \Gamma_1) = \frac{p-1}{2} \delta_k \Gamma_1$ is satisfied, and hence also that the total effective action including the lowest order term is invariant under the modified SCT (51).

3.4 Correspondence between super Yang-Mills and supergravity

So far we have succeeded in solving just about half of the puzzle: When carefully analyzed, the seemingly mysterious extra factor $(p-1)/2$ in the SCT on the Yang-Mills side is entirely consistent with GCS. In what follows, we shall solve the other half of the puzzle,
namely how such a modified effective action is related to the DBI action produced in supergravity, in two complementary ways. Our analysis will disclose an important aspect of the correspondence between supergravity and Yang-Mills matrix models for general $p$ in a rather explicit manner.

First we approach from the super Yang-Mills side. A crucial observation that relates the different-looking effective actions will be that the correction $\Delta \Gamma_1$ found by the 1-loop calculation can be reproduced by a simple redefinition of the probe coordinate in the usual form of the effective action. Actually, this idea naturally emerges in the effort to understand the correction term in the effective action from the supergravity side. So let us present this reasoning before we write down the precise field redefinition to be made.

For simplicity, consider the D0-brane system. Recall that, from the 11 dimensional viewpoint, the effective action for a probe D-particle in the field of a cluster of fixed source D-particles is given by

$$ S_0 = -\int d\tau p_- \frac{dx^-}{d\tau}. $$

where $dx^-/d\tau$ is determined from the massless constraint

$$ g_{\mu\nu}(x(s)) \frac{dx^\mu(s)}{ds} \frac{dx^\nu(s)}{ds} = 0. $$

Our convention for the light-cone coordinate is $x^\pm = x^{11} \pm t$, $A \cdot B = \frac{1}{2}(A^+ B^- + A^- B^+) + A_i B_i$, $2A_- = A^+$, $2A_+ = A^-$. Thus in the linearized approximation for the gravitational field, the general form of the action takes the form

$$ S_D = \int d\tau \frac{p_-}{2} \left( \left( \frac{dx^i}{d\tau} \right)^2 + h_{\mu\nu}(x)s_2^\mu s_2^\nu \right), $$

where $s_2^\mu$ is the velocity vector of the probe in the lowest order approximation, defined as $(s_2^+, s_2^-, s_2^i) = (2, -\frac{1}{2}v^2, v_i)$. For constant $g_s$, the interaction term comes from the component $h_{\mu\nu} = N_1 \kappa^2_{11} \frac{1}{2} R^\mu R^\nu$, which is the solution of the linearized Einstein equation $-\frac{1}{2} \Delta h_{\mu\nu} = \kappa^2_{11} T_{\mu\nu}$ with the energy-momentum tensor

$$ T_{\mu\nu}(x) = \frac{N_1}{2\pi R^2} \delta^9(x_\perp) s_{1\mu} s_{1\nu} $$

for the source at rest, namely for $(s_1^+, s_1^-, s_1^i) = (2s_{1-}, 2s_{1+}, s_1^i) = (2, 0, 0)$. Now for $(s_2^+, s_2^-, s_2^i)$ given above, the general form of $h_{\mu\nu}(x)s_2^\mu s_2^\nu$ is

$$ h_{\mu\nu}(x)s_2^\mu s_2^\nu = 4h_{++} - 2h_{+-} v^2 + \frac{1}{4} h_{--} v^4 + 4h_{+i} v^i - h_{-i} v^2 v^i + h_{ij} v^i v^j. $$
Recall that the relevant correction term containing $\dot{g}_s$ is of order $v^3$. Combining with the fact that the gravitational field produced by the fixed source does not depend on the velocity of the probe, we see that the only possibility for producing such a term in the effective action would be to have $h_{-i} \propto \dot{g}_s x^i / r^7$. This, however, is impossible since, in the linearized approximation, the energy-momentum tensor of the source cannot have the $-i$ component. After all, it is difficult to imagine that the time-dependence of the dilaton would produce such a component for the energy-momentum tensor for the point-like source D-particle at rest.

Blessedly, there is a way out. Suppose we make a redefinition of the time variable $t \rightarrow t + \phi(x)$ for the probe D-particle, where $\phi(x)$ is a function of the spatial coordinate of the probe. Then, this induces a shift of the metric

$$h_{-i} \rightarrow h_{-i} + \partial_i \phi,$$

which provides the missing component $h_{-i}$ capable of accounting for the modified term, with $\phi(x) \propto \dot{g}_s / r^5$. This strongly suggests that, when $g_s$ is not constant, the collective coordinate for the probe D-particle must be appropriately chosen in order for the motion of the D-particle to be described by the standard language of supergravity.

This leads us, then, to consider the following replacement of the probe coordinate

$$x \rightarrow x' = x + K \dot{g}_s \frac{v}{r^5},$$

with some constant $K$. It induces a change in velocity of the form

$$v \rightarrow v' = v + K \dot{g}_s \left( \frac{\dot{v}}{r^5} - 5 \frac{v \cdot x}{r^7} v \right).$$

Then, the tree-level action is transformed into

$$\int dt \frac{v^2}{2g_s} \rightarrow \int dt \frac{v'^2}{2g_s} = \int dt \frac{v^2}{2g_s} - \frac{5K}{2} \int dt \frac{\dot{g}_s v^2 v \cdot x}{g_s r^7},$$

to the order of interest. We see immediately that the second term reproduces the 1-loop modification if we set $K = 3N/4$. Furthermore, while the original Yang-Mills coordinate $x$ transforms under SCT like

$$\delta_K x = 2tx + t^2 v + \left( \frac{p-1}{2} \right)_{p=0} \frac{3g_s N v}{r^5},$$
the new coordinate $x'$ is easily seen to transform, to the accuracy of the present one-loop approximation, just like the one in the usual DBI action, without the factor $(p - 1)/2$:

$$\delta_K x' = 2tx' + t^2 \frac{dx'}{dt} - \frac{3g_s N}{2r^5} v' + \frac{3N}{4} \frac{v'}{r^5} = 2tx' + t^2 \frac{dx'}{dt} + \frac{3g_s N}{r^5} v'. \quad (87)$$

The same mechanism works for the general case of D$D$-brane system. The redefinition of the diagonal Higgs field is

$$X_m \rightarrow X'_m = X_m - \frac{1}{4} \frac{k \partial_{\alpha} Q_p}{U^{5-p}} \partial_{\alpha} X_m, \quad (88)$$

and again $X'_m$ can be shown to transform like the transverse coordinate in the DBI action.

Next, let us show that the same conclusion can actually be reached from the supergravity side without the knowledge of the 1-loop calculation in the super Yang-Mills theory. Recall that the SCT for the world-volume coordinate $x^\alpha$ in the supergravity solution takes the non-canonical form $\delta_\epsilon x^\alpha = 2\epsilon \cdot x x^\alpha - \epsilon^\alpha x^2 - \epsilon^\alpha (kQ_p/U^{5-p})$. The fact that the corresponding coordinate in the Yang-Mills theory, in contrast, transforms canonically suggests that we should identify the latter as a new coordinate $y^\alpha(x)$ on supergravity side which transforms canonically. To find such $y^\alpha(x)$, let us set $y^\alpha = x^\alpha + \zeta^\alpha$ and write down the condition on $\zeta^\alpha$. It is given by

$$\delta_\epsilon \zeta^\alpha = 2\epsilon \cdot x \zeta^\alpha + 2(\epsilon \cdot \zeta x^\alpha - x \cdot \zeta \epsilon^\alpha) + \epsilon^\alpha \frac{kQ_p}{U^{5-p}} - \epsilon^\alpha \zeta^2. \quad (89)$$

Apart from the last term non-linear in $\zeta$, this transformation law is recognized to be identical to that of $\partial^\alpha \chi$, where $\chi$ is a scalar field of dimension $= -2$ given by

$$\chi = \frac{1}{4} \frac{kQ_p}{U^{5-p}}. \quad (90)$$

Therefore we get

$$\zeta^\alpha = \frac{1}{4} \partial^\alpha \left( \frac{kQ_p}{U^{5-p}} \right) = k \frac{1}{4} \left( \frac{\partial^\alpha Q_p}{U^{5-p}} + (p - 5) \frac{Q_p \partial^\alpha U}{U^{6-p}} \right)$$

$$\simeq \frac{1}{4} \frac{k \partial^\alpha Q_p}{U^{5-p}}, \quad (91)$$

where we dropped the second term since $\partial^\alpha U$ is supposed to be small. Also since we agree to neglect $(\partial^\alpha Q_p)^2$, the omission of $\zeta^2$ part is a posteriori justified. Thus, to the leading order we find

$$y^\alpha = x^\alpha + \frac{1}{4} \frac{k \partial^\alpha Q_p}{U^{5-p}}. \quad (92)$$
If we use \( y^\alpha \) as our coordinate, we must then regard the transverse coordinate \( X'_m(x) \) of the probe Dp-brane as a function of \( y \). Then

\[
X'_m(x) = X'_m(y - \zeta) = X'_m(y) - \zeta^\alpha \partial_\alpha X'_m(y) + \cdots \\
\simeq X_m(y) - \frac{1}{4} \frac{k \partial_\alpha Q_p}{U^{5-p}} \partial_\alpha X_m(y),
\]  

(93)

where in the last line we renamed the field by dropping the prime, to emphasize that it is considered to be a field different from \( X'_m \). The relation (93) is identical in form to the redefinition found previously by using the information obtained from 1-loop calculation, except for a slight change in the argument. This difference, however, is of higher order in the present approximation. To see this, let us compute how \( X_m(y) \) transforms under SCT at a fixed point. Using \( \delta_x X'_m(x) = -2 \varepsilon \cdot x X'_m(x) - (\delta_c x^\alpha + \Delta_c x^\alpha) \partial_\alpha X'_m(x) \), we get

\[
\delta_x X_m(y) = \delta_x \left( X'_m(x) + \frac{1}{4} \frac{k \partial_\alpha Q_p}{U^{5-p}} \partial_\alpha X'_m(x) \right) \\
= -2 \varepsilon \cdot x X'_m(x) - (\delta_c x^\alpha + \Delta_c x^\alpha) \partial_\alpha X'_m(x) \\
+ \frac{1}{4} \frac{k \partial_\alpha \delta_c Q_p}{U^{5-p}} \partial_\alpha X'_m(x) + \mathcal{O}(\partial Q_p).
\]

(94)

Remembering that after the transformation we may set \( \partial Q = 0 \), hence \( y = x \), we find after a little calculation,

\[
\delta_x X_m(x) = -2 \varepsilon \cdot x X_m(x) - \delta_c x^\alpha \partial_\alpha \tilde{X}_m(x) + \frac{p - 1}{2} \varepsilon^\alpha \frac{1}{4} \frac{k \partial_\alpha Q_p}{U^{5-p}} \partial_\alpha X_m(x).
\]

(95)

This shows that \( X_m \) exhibits precisely the transformation law of the Higgs field in super Yang-Mills description.

Thus, we have reached the same result in two complementary ways: The realizations of GCS, which strongly controls the D-brane dynamics on both the supergravity and super Yang-Mills side take apparently different forms for general \( p \), but they are related by a subtle redefinition of the Higgs field or by a corresponding transformation of the world-volume coordinate. We emphasize that this is not a technical detail. It is of utmost importance for the consistency of the powerful concept of GCS.

To summarize, we have shown that, when the non-constancy of the coupling is taken into account, the effective action in general contains terms which depend on the world-volume derivatives of the coupling-constant fields. However, there is a field redefinition
which eliminates the terms of first order in the derivatives of the coupling-constant fields and make the SCT laws to be those of the (pseudo) AdS space-times. Such a redefinition is in fact compulsory, as our first argument told us, in order that the dynamics of the probe D-brane be described by supergravity. It is quite remarkable that while this effect appears on the Yang-Mills side as a quantum loop effect, the same field redefinition is dictated in the classical structure of supergravity. This provides us with a justification of the construction of the effective actions made in section II, assuming implicitly that there are no terms which are of first order with respect to the world-volume derivatives of the coupling-constant field. It also provides yet another fine example supporting the correspondence between the classical supergravity and loop-corrected super Yang-Mills theory for general \( p \).

3.5 Generalized conformal transformation for general configurations of D-branes

It should by now be fairly clear that our method of deriving the generalized conformal transformation in super Yang-Mills theory can be extended to backgrounds of arbitrary configurations of D-branes. As a matter of fact, study of such an extension turns out to shed a further light on the meaning of the generalized conformal symmetry. So let us briefly discuss this generalization, taking the system of D-particles as the simplest example.

Consider a system of \( n \) clusters, each of which consists of a large number of coincident D-particles, and denote by \( x_a \) \((a = 1, 2, \ldots, n)\) and \( N_a \) the coordinate and the number of D-particles, respectively, of the \( a \)-th cluster. The quantum-modified SCT before the field redefinition is

\[
\delta_K x_a = 2t x_a + t^2 \frac{d x_a}{dt} - \sum_b \frac{3 g_s N_b}{2 r_{ab}^5} v_{ab}, \tag{96}
\]

and the effective action up to the first order in \( \dot{g}_s \) is

\[
\Gamma = \int dt \left[ \sum_a \frac{N_a}{2 g_s} v_a^2 + \frac{1}{2} \sum_{a,b} \frac{15 N_a N_b}{16} v_{ab}^4 - \frac{1}{2} \sum_{a,b} \frac{15 N_a N_b \dot{g}_s v_{ab}^2 v_{ab} \cdot x_{ab}}{g_s r_{ab}^5} \right]. \tag{97}
\]

The field redefinition \( x_a \to x_a' \) of the D-particle collective coordinates which eliminates the dependence on \( \dot{g}_s \) takes the form

\[
x_a = x_a' - \sum_b \frac{3 N_b \dot{g}_s v_{ab}}{4 r_{ab}^5}. \tag{98}
\]
SCT for the new coordinate $x'_a$ then becomes

$$\delta_K x'_a = 2tx'_a + t^2 \frac{dx'_a}{dt} + \sum_b \frac{3g_s N_b}{r^3_{ab}} v'_{ab}. \quad (99)$$

It is a simple matter to check that the usual form of the effective action with constant string coupling is invariant under the modified transformation law (99) to the accuracy of the present approximation.

The transformation law obtained above for the generic configuration of D-branes exhibits some notable new features, which were not seen in the special background often considered, namely that of a single heavy source consisting of coincident D-branes.

First, the transformation law for the generic background no longer admits a simple space-time interpretation. This is evident from the fact that (99) involves, in general, more than one relative velocities between D-particles. It means that, at least in our present formulation of the multi-cluster system, an overall shift of time cannot simultaneously be responsible for the redefinitions of coordinates for different clusters which move with relative velocities. This strongly suggests that a proper space-time interpretation of the (generalized) conformal transformation for general backgrounds, if it is possible at all, would require a completely covariant formulation of D-brane dynamics in which a world volume is introduced for each D-brane in a reparametrization invariant way.

Secondly, apart from the problem of space-time interpretation, it should be pointed out that the system with multi-centered heavy D$p$-brane sources does not in general behave in a simple manner under the (generalized) conformal transformation for any $p$. This is due to the fact that (even without the modified term) SCT for the relative velocity $v_{ab}$ contains an inhomogeneous term. For instance, the relevant transformation law for the D-particle case is $\delta_K v_{ab} = 2x_{ab} + 4tv_{ab} + t^2 \frac{dv_{ab}}{dt} + \cdots$, which involves $2x_{ab}$, not proportional to $v_{ab}$. Thus, even if one starts with the sources which are relatively at rest ($v_{ab} = 0$), they inevitably acquire nonzero relative velocities after the SCT, except for the case of coincident sources. This implies that in order to discuss the conformal symmetry appropriately in such a general case, we would have to treat the the motions of all the D-branes on equal footing.

A related question of importance is how and to what extent the generalized conformal symmetry can constrain the many-body dynamics of D-branes. One possibility, suggested
by our discussions in section II, is that GCS may play an important role in extending the non-renormalization theorems, so far checked to 2-loop order for special configurations, to higher loops and to general many-body systems. Since these theorems are believed to be the basis for the correspondence between supergravity and matrix models for D-branes, at least in the weak coupling region, this question is of great interest. For example, it would be important to examine the general 3-body actions given in [14] from this viewpoint. This problem will be discussed elsewhere.

4 Discussions

In this paper, we have investigated the symmetry structure of general Dp-brane systems in both their YM and the supergravity descriptions, with the aim to establish further the conjectured correspondence between these two seemingly different theories. As we have already summarized the outcome of this research in the Introduction and at appropriate places in the text, we will, in the remainder of this article, offer some further observations which would be of importance in future investigations.

1. One of the key ideas of our study is to regard the coupling constant of the theory, both in the Yang-Mills and in supergravity/string descriptions, as a “field” which transforms non-trivially under the conformal-type transformations. On the Yang-Mills side, it allowed us to extend the notion of conformal symmetry that is normally thought to exist only in 4-dimensions to other dimensions. This in turn, through quantum corrections, produced in the effective action new terms which explicitly depend on the derivative of the coupling constant, and they are crucial in reconciling the apparently disparate transformation laws between Yang-Mills and supergravity descriptions. While we have demonstrated the agreement including the exact coefficients, the full understanding and the interpretation of the new terms and the notion of dynamical YM coupling constant remains to be given. On the supergravity side, (up to reparametrizations) the extra degrees of freedom and the induced terms in the effective Lagrangian are associated with the space-time dependent dilaton field, which is reflected in the variable radius of the AdS-like space-time. Thus a more precise description and comparison of this phenomenon would require proper
inclusion of the dynamics of the dilaton degrees of freedom in supergravity.

2. The quantum effect, which was so important in connecting the Yang-Mills and the supergravity descriptions, has been computed only in the weak coupling regime at one loop. Nevertheless, our result robustly supports the conjectured relation between supergravity and the Yang-Mills matrix models. Since the Maldacena’s original conjecture is supposed to hold at large $g^2 N$, this means that there must exist significant non-renormalization theorems at work in the weak-coupling region, which protect the coefficients of the generalized conformal transformation obtained in one-loop calculation as we go to the strong-coupling regime. Understanding of these theorems and their relation to GCS is an important future problem.

3. In relation to the effects of higher loops, we wish to point out the strong similarity of our GCS Ward identities, which explicitly involves the derivative with respect to the coupling constant, and the standard renormalization group (RG) equations. RG equations are capable of organizing and summing the infinite series of logarithms and as a result the coupling constant is turned into an effective running coupling. In particular in asymptotically free theories, the essential nature of such a running coupling is determined at short distance by the one-loop effect. The fact that our one-loop calculation gives the correct coefficients for the modified terms in the generalized conformal transformation in the bulk is analogous to this phenomenon and suggests that a similar mechanism is operating in the present case. This analogy may be of importance in further elucidating the structure of the GCS formulated in this work.

4. Among the possible applications of GCS, calculations of various correlation functions would be an important challenge. In general, conformal transformations imply constraints on correlation functions and in fact they completely specify the form of lower point functions. We expect that the generalized transformations established in the present work are similarly useful. We have already seen the effectiveness of the symmetry in deriving the form of the DBI action for $p$-brane probes in a fixed source. We emphasize that since our transformations contain in a nonlinear manner
the extra radial dimension, they probe the bulk of the supergravity background. The generalized conformal transformations are then expected to be very useful for addressing questions such as the bulk to boundary or bulk to bulk correlators from the symmetry point of view. Since at present time almost all the comparisons in the literature between the bulk and the boundary using the AdS/CFT correspondence involve only correlations and sources at the boundary, the extension to the full bulk region is of major interest. It would also be useful for “proving” the correspondence between the correlation functions in the bulk and on the boundary from within the logic intrinsic to YM theory.

5. As we have already suggested in subsection IIIE, to obtain a proper space-time picture of GCS for general backgrounds, some reparametrization invariant formulation on the Yang-Mills side is likely to be required. In this regard, we wish to mention a suggestive structure already seen in the Yang-Mills action. Take for example the case of D-particles. It is not difficult to see that by scaling the (time-dependent) coupling in the kinetic and the potential terms appropriately one can obtain a form that exhibits (time) reparametrization symmetry. In this picture the usual Yang-Mills theory with a constant coupling may be regarded as a gauge-fixed version. This suggests that an extension of the theory with symmetry structure encompassing reparametrization of D-brane world volume as well as the GCS advocated in this work may indeed be possible.

We hope to address these and some other related problems in future publications.

The work of A.J. is supported in part by the Department of Energy under contract DE-FG02-91ER40688-Task A. The work of Y. K. and T.Y. is supported in part by Grant-in-Aid for Scientific Research (No. 09640337) and Grant-in-Aid for International Scientific Research (Joint Research, No. 10044061) from the Ministry of Education, Science and Culture. The final stage of the present work was completed during the visit of one (T. Y.) of the authors to Brown University. T. Y. thanks Phys. Dept. of Brown University for hospitality during his stay.
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