Direct Acceleration of SAGA using Sampled Negative Momentum

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Abstract

Variance reduction is a simple and effective technique that accelerates convex (or non-convex) stochastic optimization. Among existing variance reduction methods, SVRG and SAGA adopt unbiased gradient estimators and are the most popular variance reduction methods in recent years. Although various accelerated variants of SVRG (e.g., Katyusha and Acc-Prox-SVRG) have been proposed, the direct acceleration of SAGA still remains unknown. In this paper, we propose a direct accelerated variant of SAGA using a novel Sampled Negative Momentum (SSNM), which achieves the best known oracle complexity for strongly convex problems. Consequently, our work fills the void of direct accelerated SAGA.

1 Introduction

In this paper, we consider optimizing the following composite finite-sum problem, which arises frequently in machine learning and statistics such as supervised learning and regularized empirical risk minimization (ERM):

$$\min_{x \in \mathbb{R}^d} \left\{ F(x) \triangleq f(x) + h(x) \right\}, \quad (1)$$

where $f(x) = \frac{1}{n} \sum_{i=1}^n f_i(x)$ is an average of $n$ smooth and convex function $f_i(x)$, and $h(x)$ is a simple and convex (but possibly non-differentiable) function. Here, we also define $F_i(x) = f_i(x) + h(x)$ with $\nabla F_i(x) = \nabla f_i(x) + \partial h(x)$ and $\partial h(x)$ denotes a sub-gradient of $h(\cdot)$ at $x$, which will be used in the paper.

We focus on achieving a highly accurate solution for Problem (1), although for practical optimization tasks, such as supervised learning, low empirical risk may result in a high generalization error. In this paper, we treat Problem (1) as a pure optimization problem.

When $F(\cdot)$ in Problem (1) is strongly convex, traditional analysis shows that gradient descent (GD) yields a fast linear convergence rate but with a high per-iteration cost, and thus may not be suitable for problems with a very large $n$. As an alternative for large-scale problems, SGD [Robbins and Monro, 1951] uses only one or a mini-batch of gradients in each iteration, and thus enjoys a significantly lower per-iteration complexity than GD. However, due to the undiminished variance of the gradient estimator, vanilla SGD is shown to yield only a sub-linear convergence rate. Recently, stochastic variance reduced methods (e.g., SAG [Roux et al., 2012], SVRG [Johnson and Zhang, 2013], SAGA [Defazio et al., 2014], and their proximal variants, such as [Schmidt et al., 2017], [Xiao and Zhang, 2014] and [Konečný et al., 2016]) were proposed to solve Problem (1). All these methods are equipped with various variance reduction techniques, which help them achieve low per-iteration complexities comparable with SGD and at the same time maintain a faster linear convergence rate than GD (including accelerated GD).

1 The previous version has a mistake in proving Theorem 1, but fixing it doesn’t change Algorithm 1 and the convergence result.

2 Oracle complexity in this paper, denoted by $\mathcal{O}(\cdot)$, is the number of calls to Incremental First-order Oracle (IFO) + Proximal Oracle (PO).
Table 1: Comparison of some accelerated variants of SVRG and SAGA. Here, we regard using reductions or proximal point variants as “Indirect” acceleration.

| Method         | Indirect                  | Direct                   |
|----------------|---------------------------|--------------------------|
| SVRG (or Prox-SVRG) | APPA & Catalyst            | Katyusha & MiG           |
| SAGA           |                           | this work                |
| Point-SAGA     |                           |                          |

methods all achieve an $O((n+\kappa)\log(1/\epsilon))$ complexity\(^3\) as compared with $O(n\sqrt[3]{\kappa}\log(1/\epsilon))$ for accelerated deterministic methods (e.g., Nesterov’s accelerated gradient descent \cite{Nesterov2004}).

Inspired by the acceleration technique proposed in Nesterov’s accelerated gradient descent \cite{Nesterov2004}, accelerated variants of stochastic variance reduced methods have been proposed in recent years, such as Acc-Prox-SVRG \cite{Nitanda2014}, APCG \cite{Lin2014}, APPA \cite{Frostig2015}, Catalyst \cite{Lin2015}, SPDC \cite{Zhang2015} and Katyusha \cite{Allen-Zhu2017}. Among these algorithms, APPA and Catalyst achieve acceleration by using some carefully designed reduction techniques, which, however, result in additional log factors in their overall oracle complexities. Katyusha, as the first direct accelerated variant of SVRG, introduced the idea of negative momentum, which is a momentum provided by the snapshot of SVRG. Then, by combining it with Nesterov’s Momentum, Katyusha yields the best known oracle complexity $O((n+\sqrt[3]{\kappa}n)\log(1/\epsilon))$ for strongly convex problems. More recent work \cite{Zhou2018} shows that adding only negative momentum to SVRG is enough to achieve the best known oracle complexity for strongly-convex problems, which results in a simple and scalable algorithm called MiG.

Although a considerable amount of work has been done for accelerating SVRG, another popular stochastic variance reduced method, SAGA, does not have a direct accelerated variant until recently. Accelerating frameworks such as APPA or Catalyst can be used to accelerate SAGA, but the reduction techniques proposed in these works are always difficult to implement and may also result in additional log factors in the overall oracle complexity. A notable variant of SAGA is Point-SAGA \cite{Defazio2016}. Point-SAGA requires the proximal oracle of the entire objective and with the help of that, it can adopt a much larger learning rate than SAGA, which results in the same accelerated complexity $O((n+\sqrt[3]{\kappa}n)\log(1/\epsilon))$. Some accelerated variants of SVRG and SAGA are summarized in Table 1. However, the proximal oracle of the entire objective may not be efficiently computed in practice. Even for logistic regression, we need to run an individual loop (Newton’s method) for its proximal oracle. Therefore, a direct accelerated variant of SAGA is of real interests.

Following the idea of adding only negative momentum to SVRG \cite{Zhou2018}, we consider adding negative momentum to SAGA. However, unlike SVRG, which keeps a constant snapshot in each inner loop, the “snapshot” of SAGA is a table of points, each corresponding to the position that the component function gradient $\nabla f_i(\cdot)$ was lastly evaluated. Thus, it is non-trivial to directly accelerate SAGA. In this paper, we propose a novel Sampled Negative Momentum for SAGA. We further show that adding such a momentum has the same acceleration effect as adding negative momentum to SVRG.

Our contributions are summarized below:

- We propose a directly accelerated variant of SAGA. The acceleration technique is a combination of the negative momentum trick and a tricky double sampling scheme, which we called Sampled Negative Momentum. We further prove that this accelerated variant achieves the best known oracle complexity for strongly convex problems, which is $O((n+\sqrt{\kappa}n)\log(1/\epsilon))$.

- We discuss some subtle differences on strongly convex assumptions when applying the acceleration technique. Such differences are always neglected in previous directly accelerated methods (e.g., Katyusha

\(^3\)We denote $\kappa \triangleq \frac{L}{\mu}$ throughout the paper, which is known as the condition number of an $L$-smooth and $\mu$-strongly convex function.
and MiG). Our discussion shows that the strongly convex assumption imposed in this paper can be adapted to other strongly convex assumption using a transforming trick.

- We provide a variant of the proposed algorithm for the non-smooth setting and prove that it achieves a lower $O\left(\frac{\log(1/\epsilon)}{\sqrt{\epsilon}}\right)$ oracle complexity than the $O(\frac{1}{\epsilon})$ derived in Point-SAGA [Defazio, 2016].

- Some insights about the negative momentum acceleration are given. The intuition of the negative momentum trick is still unclear because unlike the hybrid momentum scheme used in Katyusha, the proposed algorithm and MiG use only negative momentum to achieve acceleration, which seems somewhat “counter-intuitive” in the algorithm design. We provide some insights by building connections between the negative momentum trick and the standard Nesterov’s momentum in [Nesterov, 2004].

2 Preliminaries

In this paper, we consider Problem (1) in standard Euclidean space with the Euclidean norm denoted by $\|\cdot\|$. We use $E$ to denote that the expectation is taken with respect to all randomness in one epoch. In order to further categorize the objective functions, we define that a convex function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be $L$-smooth if for all $x, y \in \mathbb{R}^d$, it holds that

$$f(x) \leq f(y) + \langle \nabla f(y), x - y \rangle + \frac{L}{2} \|x - y\|^2, \quad (2)$$

and $\mu$-strongly convex if for all $x, y \in \mathbb{R}^d$,

$$f(x) \geq f(y) + \langle G, x - y \rangle + \frac{\mu}{2} \|x - y\|^2, \quad (3)$$

where $G \in \partial f(y)$, the set of sub-gradient of $f(\cdot)$ at $y$ for non-differentiable $f(\cdot)$. If $f$ is differentiable, we can simply replace $G \in \partial f(y)$ with $G = \nabla f(y)$. Then we make the following assumption to identify the main objective condition (strongly convex) that is the focus of this paper:

**Assumption 1 (Strongly Convex).** In Problem (1), each $f_i(\cdot)$ is $L$-smooth and convex, $h(\cdot)$ is $\mu$-strongly convex.

3 Direct Acceleration of SAGA

Our proposed algorithm SSNM (SAGA with Sampled Negative Momentum) is formally given in Algorithm 1. As we can see, there are some unusual tricks used in Algorithm 1. Thus we elaborate some ideas behind Algorithm 1 by making the following remarks:

- **Coupled point** $y^k_{i_k}$ **correlates to the randomness of $i_k$**. Unlike the negative momentum used for Katyusha, which comes from a fixed snapshot $\tilde{x}$, the negative momentum of SAGA can only be found on a “points” table that changes over time. Thus, in SSNM, we choose to use the $i_k$th entry of the “points” table to provide the negative momentum, which makes the coupled point correlate to the randomness of sample $i_k$. In fact, all the possible coupled points $y^k_i$ form a “coupled table”. Although the table is never explicitly computed, we shall see that the concept of “coupled table” is critical in the proof of SSNM. The 3rd step in Algorithm 1 can thus be regarded as sampling a point in such a table.

- **“Biased” gradient estimator** $\bar{\nabla}_k$. The expectation of the semi-stochastic gradient estimator $\bar{\nabla}_k$ defined in Algorithm 1 is the average of the gradients computed in the “coupled table”, $\mathbb{E}_{i_k}[\bar{\nabla}_k] = \frac{1}{n} \sum_{i=1}^{n} \nabla f_i(y^k_i)$, which seems to be surprising as this expectation (except $\bar{\nabla}_1$) does not correspond to any gradient of $f(\cdot)$, but can be used to show convergence to the optimal solution of $F(\cdot)$. In some sense, $\bar{\nabla}_k$ is a “biased” gradient estimator.

\footnote{In fact, if each $f_i(\cdot)$ is $L$-smooth, the averaged function $f(\cdot)$ is itself $L$-smooth — but probably with a smaller $L$. We keep using $L$ as the smoothness constant for a consistent analysis.}
Algorithm 1 SSNM

Input: Iterations number $K$, initial point $x_1$, learning rate $\eta = \begin{cases} \sqrt{\frac{1}{3\mu nL}} & \text{if } \frac{n}{\mu} \leq \frac{3}{4}, \\ \frac{1}{2\mu n} & \text{if } \frac{n}{\mu} > \frac{3}{4}. \end{cases}$, parameter $\tau = \frac{n\eta\mu}{1+\eta\mu}$.

Initialize: “Points” table $\phi^1 = \phi^2 = \ldots = \phi^n = x_1$ and a running average for the gradients of “points” table.

1: for $k = 1, 2, \ldots, K$ do
2: 1. Sample $i^k$ uniformly in $\{1, \ldots, n\}$ and compute the gradient estimator using the running average.
3: $\nabla_k = \nabla f_{i^k}(y_k^{i^k}) - \nabla f_{i^k}(\phi_{i^k}^k) + \frac{1}{n} \sum_{i=1}^{n} \nabla f_i(\phi_i^k)$;
4: 2. Perform a proximal step.
5: $x_{k+1} = \arg \min_x \left\{ h(x) + \langle \nabla_k, x \rangle + \frac{1}{2\eta} \| x - x_k \|^2 \right\}$;
6: 3. Sample $I_k$ uniformly in $\{1, \ldots, n\}$, take $\phi_{I_k}^{k+1} = \tau x_{k+1} + (1 - \tau) \phi_{I_k}^k$. All other entries in the “points” table remain unchanged. Update the running average corresponding to the change in the “points” table.
7: end for

Output: $x_{K+1}$

- **Independent samples $I_k$ and $i_k$.** The additional sample $I_k$ is crucial for the convergence analysis of Algorithm 1 which chooses an index to store the updated point in the “points” table. The major insight of this choice is that it separates the randomness of $x_{k+1}$ and the update index in the “points” table so as to make certain inequalities valid.

- **Two learning rates for two cases.** Using different parameter settings for different objective conditions (ill-condition and well-condition) is common for accelerated methods [Shalev-Shwartz and Zhang, 2014, Allen-Zhu, 2017, Zhou et al., 2018]. If some parameters such as $L$, $\mu$ are unknown, SSNM is still a practical algorithm with tuning only $\eta$ and $\tau$, as compared with Katyusha which has potentially 4 parameters that need to be tuned. Note that we have tried to make the parameter settings in SSNM similar to Katyusha and MiG. We believe that it can help conduct some fair experimental comparisons with these methods.

- **Only one variable vector with a simple algorithm structure.** Same as MiG in [Zhou et al., 2018], SSNM only has one variable vector in the main loop. Coupled point $y_k^{i^k}$ can be computed whenever used and do not need to be explicitly stored. Moreover, SSNM has a one loop structure compared to those variants of SVRG. Such a structure is good for asynchronous implementation since algorithms with two loops in this setting always require a synchronization after each inner loop [Mania et al., 2017]. Moreover, the algorithm structure of SSNM is more elegant than Katyusha and MiG, both of which require a tricky weighted averaged scheme at the end of each inner loop.

Since algorithms such as Point-SAGA and SAGA are closely related to SSNM, in the next subsection, we compare in details these different variants of SAGA.

### 3.1 Comparison with SAGA and Point-SAGA

As summarized in Table 2, in comparison, SSNM yields the same fast $O((n + \sqrt{\kappa n}) \log(1/\epsilon))$ convergence rate as Point-SAGA without requiring additional assumptions, demonstrating the advantage of direct acceleration. Weaker assumptions on the objective function make the algorithm more implementable. However, since SSNM requires storing the “points” table, the memory complexity of SSNM is always $O(nd)$. This may

5These two algorithms can adopt an uniformly average scheme, but in this case, both algorithms require certain restarting tricks, which make them less implementable.
be a disadvantage when the objective is a linear model such as linear logistic regression and ridge regression. It is well known that for these linear models, each gradient is just a weighting of the corresponding data vector. Thus, we can simply store a scalar to represent a gradient, which allows SAGA and Point-SAGA to have an $O(n)$ memory complexity for these problems.

For a general objective, all the three methods have the same memory complexity. In such a case, SSNM is apparently superior to the other two algorithms. Note that the exact proximal oracle for a general objective is always hard to be efficiently evaluated.

## 4 Theory

In this section, we theoretically analyze the performance of SSNM. First, we give a variance bound shown in Lemma 1. Since the stochastic gradient estimator of SSNM is computed at a coupled point that contains randomness, the variance bound for SSNM, as we can see, is unlike all the variance bounds in previous work.

**Lemma 1** (Variance Bound). Using the same notations as in Algorithm 1, we can bound the variance of stochastic gradient estimator $\tilde{\nabla}_k$ as

$$\mathbb{E}_{i_k} \left[ \left\| \tilde{\nabla}_k - \frac{1}{n} \sum_{i=1}^{n} \nabla f_i(y^k_i) \right\|^2 \right] \leq 2L \left( \frac{1}{n} \sum_{i=1}^{n} \left( f_i(\phi^k_i) - f(y^k_i) \right) - \frac{1}{n} \sum_{i=1}^{n} \langle \nabla f_i(y^k_i), \phi^k_i - y^k_i \rangle \right).$$

**Proof.**

$$\mathbb{E}_{i_k} \left[ \left\| \tilde{\nabla}_k - \frac{1}{n} \sum_{i=1}^{n} \nabla f_i(y^k_i) \right\|^2 \right] = \mathbb{E}_{i_k} \left[ \left\| \left( \nabla f_i(y^k_{i_k}) - \nabla f_i(\phi^k_{i_k}) \right) - \frac{1}{n} \sum_{i=1}^{n} \left( \nabla f_i(y^k_i) - \nabla f_i(\phi^k_i) \right) \right\|^2 \right]$$

$$\leq \mathbb{E}_{i_k} \left[ \left\| \nabla f_i(y^k_{i_k}) - \nabla f_i(\phi^k_{i_k}) \right\|^2 \right]$$

$$\leq 2L \cdot \mathbb{E}_{i_k} \left[ f_i(\phi^k_{i_k}) - f_i(y^k_{i_k}) - \langle \nabla f_i(y^k_{i_k}), \phi^k_{i_k} - y^k_{i_k} \rangle \right]$$

$$= 2L \left( \frac{1}{n} \sum_{i=1}^{n} \left( f_i(\phi^k_i) - f(y^k_i) \right) - \frac{1}{n} \sum_{i=1}^{n} \langle \nabla f_i(y^k_i), \phi^k_i - y^k_i \rangle \right),$$

where $(a)$ follows from $\mathbb{E} [\|\xi - \mathbb{E} \xi\|^2] \leq \mathbb{E} \|\xi\|^2$ and $(b)$ uses Theorem 2.1.5 in [Nesterov 2004].

Now we can formally present the main theorem of SSNM below. As stated in [Allen-Zhu 2017], the major task of the negative momentum is to cancel the additional inner product term shown in the variance bound so as to keep a close connection in each iteration. As we shall see shortly, our proposed sampled negative momentum effectively cancels the inner product term, which is where the acceleration comes from.

**Theorem 1.** Let $x^*$ be the solution of Problem 1, define the following Lyapunov function $T$, which is the same as the one in SAGA [Defazio et al., 2014]:

$$T^k \triangleq T(x_k, \phi^k) \triangleq \frac{1}{n^2 m} \left( \frac{1}{n} \sum_{i=1}^{n} F_i(\phi^k_i) - F(x^*) - \frac{1}{n} \sum_{i=1}^{n} \langle \nabla F_i(x^*), \phi^k_i - x^* \rangle \right) + \frac{1}{2nm} \|x_k - x^*\|^2.$$

| Complexity | Requirements | Memory |
|------------|--------------|--------|
| SAGA $O((n + \kappa) \log(1/\epsilon))$ | IFO of $f(\cdot)$, PO of $h(\cdot)$ | $O(nd)$ or $O(n)$ for linear models. |
| Point-SAGA $O((n + \sqrt{\kappa n}) \log(1/\epsilon))$ | PO of each $F_i(\cdot)$ | $O(nd)$ or $O(n)$ for linear models*. |
| SSNM $O((n + \sqrt{\kappa n}) \log(1/\epsilon))$ | IFO of $f(\cdot)$, PO of $h(\cdot)$ | $O(nd)$ |

* A memory issue of Point-SAGA is discussed in Appendix A.
If Assumption 1 holds, then by choosing $\tau = \frac{\eta \mu}{1 + \eta \mu}$, steps of Algorithm 1 satisfy the following contraction for the Lyapunov function in expectation (conditional on $T^k$):

$$E_{t_k, u_k} [T^{k+1}] \leq (1 + \eta \mu)^{-1} T^k.$$  

Thus, by carefully choosing $\eta$, we have the following inequalities in two cases:

(I) (For ill-conditioned problems). If $\frac{n}{\tau} \leq \frac{3}{\eta}$, with $\eta = \sqrt{\frac{1}{3n\kappa}}$ it holds that

$$E[\|x_{K+1} - x^\star\|^2] \leq \left(1 + \sqrt{\frac{1}{3nk}}\right)^{-K} \left(\frac{2}{\mu} (F(x_1) - F(x^\star)) + \|x_1 - x^\star\|^2\right).$$

The above inequality implies that in order to reduce the squared norm distance to $\epsilon$, we have an $O(\sqrt{n} \log(1/\epsilon))$ oracle complexity as $\epsilon \to 0$ in expectation.

(II) (For well-conditioned problems). If $\frac{n}{\tau} > \frac{3}{\eta}$, by choosing $\eta = \frac{1}{2\eta n}$, we have

$$E[\|x_{K+1} - x^\star\|^2] \leq \left(1 + \frac{1}{2n}\right)^{-K} \left(\frac{2}{\mu} (F(x_1) - F(x^\star)) + \|x_1 - x^\star\|^2\right).$$

This inequality implies that in this case we have an $O(n \log(1/\epsilon))$ oracle complexity as $\epsilon \to 0$ in expectation.

Thus, for strongly convex objectives, SSNM yields a fast $O((n + \sqrt{n} \mu) \log(1/\epsilon))$, which keeps up with the best known oracle complexity achieved by accelerated SVRG [Frostig et al. 2015, Allen-Zhu 2017].

### 4.1 Proof of Theorem 1

In order to prove Theorem 1, we need the following useful lemma, which can be regarded as using the 3-point equality of Bregman divergence in the Euclidean norm setting:

**Lemma 2.** If two vectors $x_{k+1}, x_k \in \mathbb{R}^d$ satisfy $x_{k+1} = \arg \min_x \{h(x) + \langle \nabla_k, x \rangle + \frac{1}{2\eta} \|x - x_k\|^2\}$ with a constant vector $\nabla_k$ and a $\mu$-strongly convex function $h(\cdot)$, then for all $u \in \mathbb{R}^d$, we have

$$\langle \nabla_k, x_{k+1} - u \rangle \leq -\frac{1}{2\eta} \|x_{k+1} - x_k\|^2 + \frac{1}{2\eta} \|x_k - u\|^2 - \frac{1 + \eta \mu}{2\eta} \|x_{k+1} - u\|^2 + h(u) - h(x_{k+1}).$$

**Proof.** This Lemma is identical to Lemma 3.5 in [Allen-Zhu 2017].

First, we analyze Algorithm 1 at the $k$th iteration, given that the randomness from previous iterations are fixed.

We start with the convexity of $f_i(\cdot)$ at $(y_{ik}^k, x^\star)$. By definition, we have

$$f_i(y_{ik}^k) - f_i(x^\star) \leq \langle \nabla f_i (y_{ik}^k), y_{ik}^k - x^\star\rangle$$

$$= (\star) \frac{1 - \tau}{\tau} \langle \nabla f_i (y_{ik}^k), \phi_{ik}^k - y_{ik}^k \rangle + \langle \nabla f_i (y_{ik}^k) - \nabla_k, x_k - x^\star \rangle + \langle \nabla_k, x_{k+1} - x^\star \rangle,$$

where $(\star)$ uses the definition of the $i_k$th entry of “coupled table” that $y_{ik}^k = \tau x_k + (1 - \tau) \phi_{ik}^k$.

As we will see, the first term in the right hand side is used to cancel the inner product term in the variance bound.

By taking expectation with respect to sample $i_k$ and using the unbiasedness that $E_{i_k} [\nabla f_i (y_{ik}^k) - \nabla_k] = 0$, we obtain

$$\frac{1}{n} \sum_{i=1}^n f_i(y_{ik}^k) - f(x^\star) \leq \frac{1 - \tau}{\tau n} \sum_{i=1}^n \langle \nabla f_i (y_{ik}^k), \phi_{ik}^k - y_{ik}^k \rangle + E_{i_k} [\langle \nabla_k, x_k - x_{k+1} \rangle] + E_{i_k} [\langle \nabla_k, x_{k+1} - x^\star \rangle].$$

(4)
In order to bound $\mathbb{E}_{i_k}[\langle \nabla_k, x_k - x_{k+1} \rangle]$, we use the $L$-smoothness of $f_{i_k}(\cdot)$ at $(\phi_{i_k}^{k+1}, y_{i_k}^k)$, which is

$$f_{i_k}(\phi_{i_k}^{k+1}) - f_{i_k}(y_{i_k}^k) \leq \langle \nabla f_{i_k}(y_{i_k}^k), \phi_{i_k}^{k+1} - y_{i_k}^k \rangle + \frac{L}{2} \| \phi_{i_k}^{k+1} - y_{i_k}^k \|^2.$$ 

Taking expectation with respect to sample $I_k$ and using our choice of $\phi_{i_k}^{k+1} = \tau x_{k+1} + (1 - \tau) \phi_{i_k}^k$ as well as the definition of “coupled table”, we conclude that

$$\mathbb{E}_{i_k}[f_{i_k}(\phi_{i_k}^{k+1})] = \frac{1}{n} \sum_{i=1}^n f_i(y_i^k) - \frac{1}{\tau} \mathbb{E}_{i_k}[f_{i_k}(\phi_{i_k}^{k+1})] + \frac{L}{2} \mathbb{E}_{i_k}[\|x_{k+1} - x_k\|^2].$$

$$\langle \nabla_k, x_k - x_{k+1} \rangle \leq \frac{1}{\tau n} \sum_{i=1}^n f_i(y_i^k) - \frac{1}{\tau} \mathbb{E}_{i_k}[f_{i_k}(\phi_{i_k}^{k+1})] + \frac{L}{2} \mathbb{E}_{i_k}[\|x_{k+1} - x_k\|^2].$$

Taking expectation with respect to sample $i_k$, we obtain

$$\mathbb{E}_{i_k}[\langle \nabla_k, x_k - x_{k+1} \rangle] \leq \frac{1}{\tau n} \sum_{i=1}^n f_i(y_i^k) - \frac{1}{\tau} \mathbb{E}_{i_k}[f_{i_k}(\phi_{i_k}^{k+1})] + \frac{L}{2} \mathbb{E}_{i_k}[\|x_{k+1} - x_k\|^2].$$

Here we see the effect of the independent sample $I_k$. It decouples the randomness of $x_{k+1}$ and the update position so as to make the above inequalities valid.

By upper bounding (1) using (3) and Lemma 2 (with $h(\cdot)$ $\mu$-strongly convex and $u = x^*$), we obtain

$$\frac{1}{n} \sum_{i=1}^n f_i(y_i^k) - f(x^*) \leq \frac{1 - \tau}{\tau n} \sum_{i=1}^n \langle \nabla f_i(y_i^k), \phi_i^k - y_i^k \rangle + \frac{1}{\tau n} \sum_{i=1}^n f_i(y_i^k) - \frac{1}{\tau} \mathbb{E}_{i_k}[f_{i_k}(\phi_{i_k}^{k+1})]
\quad + \frac{L}{2} \mathbb{E}_{i_k}[\|x_{k+1} - x_k\|^2] + \frac{1}{2\eta} \mathbb{E}_{i_k}[\|x_k - x^*\|^2] - \frac{1 + \eta \mu}{2\eta} \mathbb{E}_{i_k}[\|x_{k+1} - x^*\|^2]
\quad + h(x^*) - \mathbb{E}_{i_k}[h(x_{k+1})].$$

Here we add a constraint that $L \tau \leq \frac{1}{\eta} - \frac{\beta}{L^2 \tau}$, which is identical to the one used in [Zhou et al., 2018].

Using Young’s inequality $a \leq \frac{1}{2} a^2 + \frac{\beta}{2} b^2$ to upper bound $\mathbb{E}_{i_k}[\langle \nabla f_i(y_i^k) - \nabla_k, x_{k+1} - x_k \rangle]$ with $\beta = \frac{L \tau}{\eta} > 0$, we can simplify the above inequality as

$$\frac{1}{n} \sum_{i=1}^n f_i(y_i^k) - f(x^*) \leq \frac{1 - \tau}{\tau n} \sum_{i=1}^n \langle \nabla f_i(y_i^k), \phi_i^k - y_i^k \rangle + \frac{1}{\tau n} \sum_{i=1}^n f_i(y_i^k) - \frac{1}{\tau} \mathbb{E}_{i_k}[f_{i_k}(\phi_{i_k}^{k+1})]
\quad + \frac{1 - \tau}{2L^2} \mathbb{E}_{i_k}[\|\frac{1}{n} \sum_{i=1}^n \nabla f_i(y_i^k) - \nabla_k\|^2] + \frac{1}{2\eta} \mathbb{E}_{i_k}[\|x_k - x^*\|^2] - \frac{1 + \eta \mu}{2\eta} \mathbb{E}_{i_k}[\|x_{k+1} - x^*\|^2]
\quad + h(x^*) - \mathbb{E}_{i_k}[h(x_{k+1})].$$

By applying Lemma 3 to upper bound the variance term, we see that the additional variance term in the
variance bound is canceled by the sampled momentum, which comes to

\[
\frac{1}{n} \sum_{i=1}^{n} f_i(y_i^k) - f(x^*) \leq \frac{1}{n} \tau \sum_{i=1}^{n} f_i(y_i^k) - \frac{1}{\tau} \mathbb{E}_{i_k, I_k} [f_{i_k}^k(\phi_{I_k}^{k+1})] + \frac{1}{\tau n} \sum_{i=1}^{n} (f_i(\phi^k) - f(y_i^k)) + \frac{1}{2\eta} \|x_k - x^*\|^2 - \frac{1 + \eta \mu}{2\eta} \mathbb{E}_{i_k} [\|x_{k+1} - x^*\|^2]
\]

Dividing the above inequality by \(n\) and using the definition that \(\phi_{I_k}^{k+1} = \tau x_{k+1} + (1 - \tau)\phi_{I_k}^k\), we have

\[
h(\phi_{I_k}^{k+1}) \leq \tau h(x_{k+1}) + (1 - \tau)h(\phi_{I_k}^k).
\]

Using the convexity of \(h(\cdot)\) and that \(\phi_{I_k}^{k+1} = \tau x_{k+1} + (1 - \tau)\phi_{I_k}^k\), we have

\[
h(\phi_{I_k}^{k+1}) \leq \tau h(x_{k+1}) + (1 - \tau)h(\phi_{I_k}^k).
\]

After taking expectation with respect to sample \(I_k\) and sample \(i_k\), we obtain

\[-\mathbb{E}_{i_k} [h(x_{k+1})] \leq \frac{1 - \tau}{\tau n} \sum_{i=1}^{n} h(\phi_i^k) - \frac{1}{\tau} \mathbb{E}_{i_k, I_k} [h(\phi_{I_k}^{k+1})].\]

Combining the above inequality with (6) and using the definition that \(F_i(\cdot) = f_i(\cdot) + h(\cdot)\), we can write (6) as

\[
\frac{1}{\tau} \mathbb{E}_{i_k, I_k} [F_{I_k}^k(\phi_{I_k}^{k+1}) - F_{I_k}(x^*)] \leq \frac{1 - \tau}{\tau n} \left( \frac{1}{n} \sum_{i=1}^{n} F_i(\phi_i^k) - F(x^*) \right) + \frac{1}{2\eta} \|x_k - x^*\|^2 - \frac{1 + \eta \mu}{2\eta} \mathbb{E}_{i_k} [\|x_{k+1} - x^*\|^2].
\]

Dividing the above inequality by \(n\) and adding both sides by \(\frac{1}{\tau n} \mathbb{E}_{I_k} \left[ \sum_{i \neq I_k} (F_i(\phi_i^k) - F_i(x^*)) \right]\), we obtain

\[
\frac{1}{\tau} \mathbb{E}_{i_k, I_k} \left[ \frac{1}{n} \sum_{i=1}^{n} F_i(\phi_i^{k+1}) - F(x^*) \right] \leq \frac{1 - \tau}{\tau n} \left( \frac{1}{n} \sum_{i=1}^{n} F_i(\phi_i^k) - F(x^*) \right) + \frac{1}{\tau n} \mathbb{E}_{i_k} \left[ \sum_{i \neq I_k} (F(i(\phi_i^k)) - F_i(x^*)) \right] + \frac{1}{2\eta n} \|x_k - x^*\|^2 - \frac{1 + \eta \mu}{2\eta n} \mathbb{E}_{i_k} [\|x_{k+1} - x^*\|^2]
\]

\[
\leq \frac{1 - \tau}{\tau} \left( \frac{1}{n} \sum_{i=1}^{n} F_i(\phi_i^k) - F(x^*) \right) + \frac{1}{2\eta} \|x_k - x^*\|^2 - \frac{1 + \eta \mu}{2\eta n} \mathbb{E}_{i_k} [\|x_{k+1} - x^*\|^2].
\]

Since \(\frac{1}{n} \sum_{i=1}^{n} F_i(\phi_i^k) - F(x^*)\) may not be positive, we need to involve the following term in our Lyapunov function:

\[
-\frac{1}{n} \sum_{i=1}^{n} \langle \nabla F_i(x^*), \phi_i^{k+1} - x^* \rangle = -\frac{1}{n} \langle \nabla F_{I_k}(x^*), \phi_{I_k}^{k+1} - x^* \rangle - \frac{1}{n} \sum_{i \neq I_k} \langle \nabla F_i(x^*), \phi_i^k - x^* \rangle
\]

\[
= -\frac{\tau}{n} \langle \nabla F_{I_k}(x^*), x_{k+1} - x^* \rangle + \frac{\tau}{n} \langle \nabla F_{I_k}(x^*), \phi_{I_k}^k - x^* \rangle - \frac{1}{n} \sum_{i=1}^{n} \langle \nabla F_i(x^*), \phi_i^k - x^* \rangle.
\]
After taking expectation with respect to sample \( I_k \) and \( i_k \), we obtain

\[
E_{i_k,i_k} \left[ - \frac{1}{n} \sum_{i=1}^{n} \langle \nabla F_i(x^*), \phi_i^{k+1} - x^* \rangle \right] = - \left( 1 - \frac{\tau}{n} \right) \left( \frac{1}{n} \sum_{i=1}^{n} \langle \nabla F_i(x^*), \phi_i^k - x^* \rangle \right).
\] (8)

In order to give a clean proof, we denote \( D_k \triangleq \frac{1}{n} \sum_{i=1}^{n} F_i(\phi_i^k) - F(x^*) - \frac{1}{n} \sum_{i=1}^{n} \langle \nabla F_i(x^*), \phi_i^k - x^* \rangle \) and \( P_k \triangleq \|x_k - x^*\|^2 \), then by combining (7), (8), we can write the contraction as

\[
\frac{1}{\tau} E_{i_k,i_k}[D_{k+1}] + \frac{1 + \eta \mu}{2 \eta \mu} E_{i_k}[P_{k+1}] \leq \frac{1 - \frac{\tau}{n} n \eta \mu}{\tau} D_k + \frac{1}{2 \eta \mu} P_k.
\] (9)

**Case I:** Consider the first case with \( \frac{n}{\kappa} \leq \frac{3}{4} \), choosing \( \eta = \sqrt{\frac{1}{3 \mu L}} \) and \( \tau = \frac{n \mu}{1 + \eta \mu} = \sqrt{\frac{\kappa}{2 \eta \mu}} < \frac{1}{2} \), first evaluate the parameter constraint:

\[
L \tau \leq \frac{1}{\eta} - \frac{L \tau}{1 - \tau} = \frac{2 - \frac{3}{\kappa}}{\frac{1}{\eta} - \frac{1 - \tau}{1 - \frac{3}{\kappa}}} \leq \sqrt{\frac{3 \kappa}{\kappa}}.
\]

which means that the constraint is satisfied by our parameter choices.

Moreover, with this choice of \( \tau \), we have

\[
\frac{1}{\tau(1 + \eta \mu)} = \frac{1 - \frac{\tau}{n} n \eta \mu}{\tau} = \frac{1}{\eta \mu}. \]

Thus, the contraction (9) can be written as

\[
\frac{1}{\eta \mu} E_{i_k,i_k}[D_{k+1}] + \frac{1}{2 \eta \mu} E_{i_k}[P_{k+1}] \leq (1 + \eta \mu)^{-1} \cdot \left( \frac{1}{\eta \mu} D_k + \frac{1}{2 \eta \mu} P_k \right).
\]

After telescoping the above contraction from \( k = 1 \ldots K \) and taking expectation with respect to all randomness, we have

\[
\frac{1}{\eta \mu} E[D_{K+1}] + \frac{1}{2 \eta \mu} E[P_{K+1}] \leq (1 + \eta \mu)^{-K} \cdot \left( \frac{1}{\eta \mu} D_1 + \frac{1}{2 \eta \mu} P_1 \right).
\]

Note that \( D_1 = F(x_1) - F(x^*) \) and \( E[D_{K+1}] \geq 0 \) based on convexity. After substituting the parameter choices, we have

\[
E[\|x_{K+1} - x^*\|^2] \leq \left( 1 + \sqrt{\frac{1}{3 \mu \kappa}} \right)^{-K} \cdot \left( \frac{2}{\mu} (F(x_1) - F(x^*)) + \|x_1 - x^*\|^2 \right).
\]

**Case II:** Consider another case with \( \frac{n}{\kappa} > \frac{3}{4} \), choosing \( \eta = \frac{1}{\mu L}, \tau = \frac{n \mu}{1 + \eta \mu} = \frac{3}{2} \) and \( \frac{3}{4} < \frac{1}{2} \). Again, we first evaluate the constraint:

\[
L \tau \leq \frac{1}{\eta} - \frac{L \tau}{1 - \tau} \Rightarrow \tau \cdot \frac{2 - \tau}{1 - \frac{3}{4}} < \frac{3}{\kappa} \leq \frac{2 \eta \mu}{2 \eta \mu}.
\]

Then by rewriting the contraction (9), telescoping from \( k = 1 \ldots K \) and taking expectation with respect to all randomness, we obtain

\[
2 E[D_{K+1}] + \frac{1}{2 \eta \mu} E[P_{K+1}] \leq (1 + \eta \mu)^{-K} \cdot \left( 2 D_1 + \frac{1}{2 \eta \mu} P_1 \right).
\]

By substituting the parameter choices, we have

\[
E[\|x_{K+1} - x^*\|^2] \leq \left( 1 + \frac{1}{2 n} \right)^{-K} \cdot \left( \frac{2}{\mu} (F(x_1) - F(x^*)) + \|x_1 - x^*\|^2 \right).
\]
4.2 Some subtle differences on strongly convex assumption

Recall that the strongly convex assumption for SAGA is imposed on each \( f_i(\cdot) \) (or the average \( f(\cdot) \) as an extension) \cite{Defazio2014}. In comparison, SSNM requires the strong convexity of \( h(\cdot) \) (in Assumption 1), which seems to be critical in the proof. Below we show that the strong convexity assumption of each \( f_i(\cdot) \) can be efficiently transformed into Assumption 1.

**Transforming the strong convexity assumption from holding for all \( f_i(\cdot) \) to Assumption 1**

Suppose we have an objective in the form (1) with each \( f_i(\cdot) \) \(-\)smooth and \( \mu\)\(-\)strongly convex, \( h(\cdot) \) convex and proper (the main assumption of SAGA). By defining \( f'_i(\cdot) = f_i(\cdot) - \frac{\mu}{2} \| \cdot \|^2 \) for each \( f_i(\cdot) \) and \( h'(\cdot) = h(\cdot) + \frac{\mu}{2} \| \cdot \|^2 \), the optimal solution of minimizing \( f'(\cdot) = \frac{1}{n} \sum_{i=1}^n f'_i(\cdot) + h'(\cdot) \) is equivalent to that of (1) and it can be verified that each \( f'_i(\cdot) \) is \((L - \mu)\)-smooth and convex, \( h'(\cdot) \) is \( \mu \)\(-\)strongly convex. Moreover, the proximal operator \( \text{prox}_{h'_i}(v) = \arg \min_x \{ h'(x) + \frac{1}{2\mu} \| x - v \|^2 \} \), \( \forall v \in \mathbb{R}^d \) can be efficiently computed as

\[
\text{prox}_{h'_i}(v) = \text{prox}_{h(1 + \mu\eta)} \left( \frac{v}{1 + \mu\eta} \right).
\]

Conversely, Assumption 1 may not be reducible to the strong convexity assumption of each \( f_i(\cdot) \) using the above trick, since the modified regularizer \( h(\cdot) - \frac{\mu}{2} \| \cdot \|^2 \) may not be as “proper” as \( h(\cdot) \).

Directly accelerated variants of SVRG (e.g., Katyusha and MiG) also require a strongly convex regularizer to achieve acceleration. This requirement can be weakened by adopting a restarting scheme for MiG (Algorithm 3 with Option II in \cite{Zhou2018}) which only requires \( F(\cdot) \) to be strongly convex and thus keeps the same assumption as in Prox-SVRG \cite{Xiao2014}. Unfortunately, we found that the similar trick does not work for SSNM. The best we can achieve is to slightly weaken the strong convexity assumption to be imposed on each \( F_i(\cdot) \), but it requires an additional upper bound \( F(x) - F(x^*) \leq \frac{2\mu}{\mu - L} \| x - x^* \|^2 \) for all \( x \in \mathbb{R}^d \), where \( L \) is potentially much larger than \( L \) (\( L = L_F = L \) when \( h(\cdot) \equiv 0 \)). Moreover, the algorithm structure will be much more complicated than Algorithm 1. Thus, we decide not to include the variant here.

The main limitation of the Lyapunov function used to prove the convergence of SSNM (and many SAGA-like algorithms) is that it does not contain an additive error term for \( F(\cdot) \), unlike the convergence of SVRG (and its variants). This subtle difference somehow explains why the SVRG-like variance reduction technique is more favorable in theory than that of SAGA.

4.3 Non-smooth extension

Problem (1) with non-smooth but \( L_1 \)-Lipschitz continuous \( f_i(\cdot) \), strongly convex \( h(\cdot) \) is also prevalent in machine learning, e.g., L2-SVM. To solve this type of problems, the most direct solution is using sub-gradient methods (e.g., Pegasos \cite{Shalev2011} with an \( O(\frac{1}{\epsilon}) \) rate). As an accelerated variant of SAGA, Point-SAGA also obtains an \( O(\frac{1}{\epsilon}) \) rate for a similar type of objectives \cite{Defazio2014}. In comparison, Point-SAGA requires the exact proximal oracle of each \( f_i(\cdot) \) but does not show improvement on the bound. In this subsection, we consider extending SSNM into this setting by utilizing the proximal information of each \( f_i(\cdot) \), which results in a convergence rate faster than \( O(\frac{1}{\epsilon}) \).

Following \cite{Orabona2012}, we apply Moreau-Yosida regularization for each \( f_i(\cdot) \), which results in a smooth approximation \( f_i^\beta(\cdot) \) (with \( \beta > 0 \)) defined as

\[
\forall v \in \mathbb{R}^d, f_i^\beta(v) = \inf_{x \in \mathbb{R}^d} \left\{ f_i(x) + \frac{1}{2\beta} \| x - v \|^2 \right\}.
\]

Then, it is clear that \( \text{prox}_{f_i}(v) \) returns the point that attains the infimum in \( f_i^\beta(v) \). As proven in Proposition 12.29 \cite{Bauschke2011}, \( f_i^\beta(\cdot) \) is \( \frac{1}{\beta} \)-smooth and its gradient can be computed as \( \nabla f_i^\beta(x) = \frac{1}{\beta}(x - \text{prox}_{f_i}(x)), \forall x \in \mathbb{R}^d \). Moreover, we have the following properties to further bound the error in this smooth approximation:

\footnote{Similar restarting trick can be used for Katyusha to weaken the strongly convex assumption.}
Lemma 3 (Lemma 2.2, Orabona et al., 2012). Let \( f_i(\cdot) \) be an \( L_1 \)-Lipschitz continuous and convex function, then for any \( x \in \mathbb{R}^d \), \( \beta > 0 \)

\[
f_i^\beta(x) \leq f_i(x) \leq f_i^\beta(x) + \frac{\beta L_i^2}{2}.
\]

Thus, by defining a “smoothed” objective \( F^\beta(\cdot) = \frac{1}{n} \sum_{i=1}^n f_i^\beta(\cdot) + h(\cdot) \), we can use SSNM to minimize \( F^\beta(\cdot) \), which leads to the following corollary:

**Corollary 1.** Using Algorithm 7 to minimize \( F^\beta(\cdot) \) defined above, and by choosing \( \beta = \frac{\mu}{4 L_i^2} \), where \( \epsilon > 0 \) (small enough) is the required accuracy, in order to achieve \( \|x_{K+1} - x^\star\|^2 \leq \epsilon \) at the output point \( x_{K+1} \), where \( x^\star \) is the solution of minimizing the original \( F(\cdot) \), we need an \( O \left( \left( n + \frac{\sqrt{n L_i}}{\sqrt{\mu \epsilon}} \right) \log(1/\epsilon) \right) \) oracle complexity in expectation.

**Proof.** Denote the optimal solution of minimizing \( F^\beta(\cdot) \) as \( x^\star_\beta \). With the strong convexity of \( F(\cdot) \), we can bound the difference between \( x^\star_\beta \) and \( x^\star \) as

\[
\|x^\star_\beta - x^\star\|^2 \leq \frac{2}{\mu} (F(x^\star_\beta) - F(x^\star)).
\]

Based on Lemma 3, we have the following inequalities:

\[
F(x^\star_\beta) \leq F^\beta(x^\star_\beta) + \frac{\mu \epsilon}{8} \leq F^\beta(x^\star) + \frac{\mu \epsilon}{8} \leq F(x^\star) + \frac{\mu \epsilon}{8},
\]

where (\( \ast \)) holds due to the optimality of \( x^\star_\beta \).

Thus, we conclude that \( \|x^\star_\beta - x^\star\|^2 \leq \frac{\epsilon}{8} \), which is based on the choice of \( \beta \).

Following Theorem \( \Box \) in order to reduce the squared norm distance \( \|x_{K+1} - x^\star_\beta\|^2 \) at the output point \( x_{K+1} \) to \( \frac{\epsilon}{8} \), we need an \( O \left( \left( n + \frac{\sqrt{n L_i}}{\sqrt{\beta \mu \epsilon}} \right) \log(1/\epsilon) \right) \) oracle complexity. Note that the above results imply that \( x_{K+1} \) satisfies

\[
\|x_{K+1} - x^\star\|^2 \leq 2 \|x_{K+1} - x^\star_\beta\|^2 + 2 \|x^\star_\beta - x^\star\|^2 \leq \epsilon.
\]

The above results imply an \( O \left( \frac{\log(1/\epsilon)}{\sqrt{\epsilon}} \right) \) bound to solve the non-smooth objectives, which is superior to the \( O(\frac{1}{\epsilon}) \) obtained by Point-SAGA. In order to avoid the log factor in the bound, we can use the AdaptSmooth in Allen-Zhu and Hazan, 2016. However, as mentioned in Section 4, in order to satisfy the HOOD property in Allen-Zhu and Hazan, 2016, we need an additional upper bound \( F(x) - F(x^\star) \leq \frac{4 \mu}{L} \|x - x^\star\|^2 \) for all \( x \in \mathbb{R}^d \), which rules out certain choices of \( h(\cdot) \), such as the indicator function of a closed convex set. Moreover, a \( \log(\frac{L_F}{\mu}) \) factor will appear in the oracle complexity bound after using the AdaptSmooth. Thus, we omit further discussions about eliminating the log factor here.

5 Some insights about the negative momentum trick

In Allen-Zhu, 2017, the negative momentum (or Katyusha momentum) is described as a “magnet” that reduces the error of the semi-stochastic gradient estimator for variance reduced algorithms. Thus, the author combined this idea with Nesterov’s momentum (or “positive” momentum) to achieve acceleration. However, as shown in Zhou et al., 2018, as well as this work, it seems that merely using the negative momentum trick is enough to obtain the same accelerated convergence rate, which makes this acceleration somewhat “counter-intuitive”. In theory, it is clear that with the help of negative momentum, we can adopt a much tighter variance bound. However, this theoretical effect does not explain the source of acceleration. In this section, we try to build a connection between the negative momentum and the standard Nesterov’s momentum in Nesterov, 2004.
For simplicity, we mainly focus on the objective with $h(\cdot) \equiv 0$ in this section. First, consider the deterministic case with $n = 1$, Algorithm degenerates into an algorithm with the following key steps (with $z \in \mathbb{R}^d$ denoting the one item “points” table $\phi$):

$$y_k = \tau x_k + (1 - \tau) z_k;$$
$$x_{k+1} = x_k - \eta \nabla f(y_k);$$
$$z_{k+1} = \tau x_{k+1} + (1 - \tau) z_k.$$

Note that we can completely eliminate the sequence $\{x_k\}$, which results in a simple scheme below.

$$z_{k+1} = y_k - \eta \tau \nabla f(y_k);$$
$$y_{k+1} = z_{k+1} + (1 - \tau)(z_{k+1} - z_k).$$

By carefully choosing parameters $\eta$ and $\tau$, we recover the original Nesterov’s accelerated gradient method with constant scheme $\text{Nesterov, 2004}$. This observation motivates us to formulate the key steps in SSNM (Algorithm 1) and MiG into the following schemes (outer loops are omitted for simplicity):

| SSNM          | MiG                      |
|---------------|--------------------------|
| $\nabla^{(1)}_k = \nabla f_{i_k}(y^k_{i_k}) - \nabla f_{i_k}(\phi^k_{i_k}) + \frac{1}{n} \sum_{i=1}^n \nabla f_i(\phi^k_i);$ | for $k = 1 \ldots m$ :
| $\phi^{k+1}_{i_k} = y^k_{i_k} - \eta \tau \nabla^{(1)}_k;$ | $\nabla^{(2)}_k = \nabla f_{i_k}(y^k_{i_k}) - \nabla f_{i_k}(\tilde{x}_s) + \nabla f(\tilde{x}_s);$ |
| $y^{k+1}_{i_k = \phi^{k+1}_{i_k} + (1 - \tau)(\phi^{k+1}_{i_k} - \phi^k_{i_k});}$ | $y^s_{k+1} = y^s_k - \eta \nabla^{(2)}_k;$ |
| $\tilde{x}_{s+1} = \frac{1}{m} \sum_{k=1}^m y^s_k;$ | $y^s_{m+1} + (1 - \tau)(\tilde{x}_{s+1} - \tilde{x}_s);$ |
| $y^{s+1} = y^s_{m+1} + (1 - \tau)(\tilde{x}_{s+1} - \tilde{x}_s);$ |

The underlined parts of both algorithms can be regarded as the source of acceleration, since setting $\tau = 1$ makes both algorithms degenerate into SAGA or Prox-SVRG. A more careful analysis shows that: For MiG, the momentum $\tilde{x}_{s+1}$ is provided every $m$ stochastic steps, where $m = \Theta(n)$ as suggested by the analysis in Zhou et al. [2018]; for SSNM, although a little bit messy in randomness, we can observe that in expectation, every $n$ steps, the momentum is provided by the newly computed iterate. In comparison, the momentum in Acc-Prox-SVRG [Nitanda, 2014] is added in every stochastic step. However, as analyzed in Nitanda [2014], in pure stochastic setting (mini-batch size is 1), no acceleration can be guaranteed for Acc-Prox-SVRG in theory. The intuition here is that we may not trust the momentum provided in every stochastic step; instead, we trust the momentum provided by the average information of $n$ stochastic steps.

Based on the above observation, we may understand the negative momentum in SSNM and MiG as the Nesterov’s momentum based on average information, in addition to attaining tighter variance bounds.

## 6 Experiments

In this section, we conducted experiments to examine the practical performance of SSNM as well as to justify our theoretical results. All the algorithms were implemented in C++ and executed through a MATLAB interface for fair comparison. We ran experiments on an HP Z440 machine with a single Intel Xeon E5-1630v4 with 3.70GHz cores, 16GB RAM, Ubuntu 16.04 LTS with GCC 4.9.0, MATLAB R2017b.

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7 We adopt the uniform averaged scheme of MiG (Algorithm 3 with Option II in Zhou et al. [2018]) for simplicity.

8 In fact, setting $\tau = 1$ does not make SSNM and MiG exactly the same as SAGA and Prox-SVRG. For SSNM, the update index for the “points” table is different; for MiG, the initial point $y_1^{s+1}$ for the new epoch is different.

9 Pure stochastic setting is important since it is proven that in order to achieve the optimal convergence rate per data access, we should always choose a mini-batch size of 1 for a family of variance reduction methods Liu and Hsieh [2018].
Figure 1: Evaluations of SAGA, SSNM, Katyusha and MiG on the a9a dataset with $\lambda = 10^{-6}$ and $10^{-7}$ (the first two figures) and the covtype dataset with $\lambda = 10^{-8}$ and $10^{-9}$ (the last two figures).

We are optimizing the following binary problem with $a_i \in \mathbb{R}^d$, $b_i \in \{-1, +1\}$, $i = 1 \ldots m$:

$$\text{Logistic Regression: } \frac{1}{n} \sum_{i=1}^{n} \log (1 + \exp (-b_i a_i^T x)) + \frac{\lambda}{2} \|x\|^2,$$

where $\lambda$ is the regularization parameter and all the datasets used were normalized before the experiments.

The experiments were designed as some ill-conditioned problems (with very small $\lambda$), since ill-condition is where all the accelerated first-order methods take effect. We tested the following algorithms with their corresponding parameter settings:

- **SAGA.** We set the learning rate as $\eta = \frac{1}{2(\mu n + L)}$, which is analyzed theoretically in [Defazio et al., 2014].
- **SSNM.** We used the same settings as suggested in Algorithm 1, which are $\eta = \sqrt{\frac{1}{\mu n L}}$ and $\tau = \frac{n \mu \eta}{1 + \eta \mu \mu}$. (In the notations of the original work).
- **Katyusha.** As suggested by the author, we fixed $\tau_2 = \frac{1}{2}$, set $\eta = \frac{1}{\sqrt{3}L}$ and chose $\tau_1 = \sqrt{\frac{2}{3}} L$ [Allen-Zhu, 2017].
- **MiG.** We set $\eta = \frac{1}{\sqrt{3}L}$ and chose $\theta = \sqrt{\frac{2}{3}}$ as analyzed in [Zhou et al., 2018].

We report the results in Figure 1. From the results, we can make the following observations to justify the accelerated convergence rate:

- **Similar convergence results comparing with other accelerated algorithms.** In fact, we are surprised by the excellent performance of SSNM on the covtype dataset. For this dataset, SSNM is even significantly faster than Katyusha and MiG in terms of the number of epochs (though in theory, Katyusha and MiG yield the same convergence rate as SSNM). The fast convergence of SSNM in practice may imply that the algorithm could potentially benefit many applications.

- **Around 3 times slow-down when $\kappa$ is 10 times larger.** It can be observed that using the same dataset, when we divide $\lambda$ by 10 (the same as multiply $\kappa$ by 10), approximately $\sqrt{10}$ times slow-down ($\sqrt{10}$ times more oracle calls required to achieve the same accuracy) is recorded for all the accelerated methods. In comparison, SAGA shows significant slow-down when $\kappa$ is increased in both experiments. This observation justifies the $\sqrt{\kappa}$ dependency for accelerated methods.

However, as also reported in Figure 1, the convergence of SSNM, though very fast, is somewhat unstable compared with the other three methods. This can be explained by the double sampling trick used in SSNM, which greatly increases the uncertainty inside each iteration.

An empirical comparison with Point-SAGA for ridge regression is also given in Appendix A for reference.

### 6.1 Effectiveness of sample $I_k$

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A natural question is that: can we use sample $i_k$ (the sample of stochastic gradient) instead of an independent sample $I_k$ in the 7th step of Algorithm II. We empirically evaluated the effect of sample $I_k$ as shown in Figure 2. As we can see, using sample $i_k$ makes the algorithm even more unstable and slower in convergence comparing with using an independent sample $I_k$. This effect can probably be explained by some kind of variance cumulation when using the sample $i_k$.  

7 Conclusions
In this paper, we proposed SSNM, an accelerated variant of SAGA, which uses the Sampled Negative Momentum trick. Our theoretical results show that SSNM achieves the best known bound for strongly convex problems and our experiments justified such improvements for the ill-conditioned problems. Although memory consumption of SSNM is higher than SAGA and some other variants, considering its good performance and general objective assumption, SSNM is still potentially beneficial in practice.

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A An empirical comparison with Point-SAGA

Here we report an experiment comparing the performance of SAGA, Point-SAGA and SSNM with respect to iteration counter. The detailed experimental setting is given in Section 6 in the main paper. Since Point-SAGA requires the exact proximal oracle of each $F_i(\cdot)$ in theory, we focus on training ridge regression in this section:

\[
\text{Ridge Regression: } \frac{1}{n} \sum_{i=1}^{n} \frac{1}{2} (a_i^T x + b_i)^2 + \frac{\lambda}{2} \|x\|^2.
\]

Note that the proximal oracle of each $F_i(\cdot) = \frac{1}{2} (a_i^T x + b_i)^2 + \frac{\lambda}{2} \|x\|^2$ can be efficiently computed as mentioned in [Defazio, 2016]

A memory issue of Point-SAGA: In fact, when we involve an $\ell_2$-regularizer in each $F_i(\cdot)$\footnote{An $\ell_2$-regularizer is always the source of strong convexity for real world problems.} we cannot use the trick of representing a gradient by a vector since the update equation of the new table entry $g_k^{k+1}$ (in original notations) contains a term that correlates to the weight $x_k$, which leads to an $O(nd)$ memory complexity. A possible solution is to separate the proximal computations for the component functions and the regularizer, but it does not fit in the analysis of Point-SAGA.

Figure 3: Comparison of SAGA, Point-SAGA and SSNM for solving ridge regression on covtype with $\lambda = 10^{-8}$. 

\[
\text{Objective minus best}
\]

\[
\text{Number of iterations}
\]
We used the same parameter settings for SAGA and SSNM as in Section 6 in the main paper. For Point-SAGA, we chose the learning rate $\gamma$ suggested by the original work [Defazio, 2016],

$$\gamma = \frac{\sqrt{(n-1)^2 + \frac{4nL}{n}}}{2Ln} - \frac{1}{2L}.$$

The result is shown in Figure 3. As we can see, the convergence rates of Point-SAGA and SSNM are quite similar and consistently faster than SAGA. Although Point-SAGA is shown to be slightly faster than SSNM in this experiment, considering the general objective assumption and the memory issue of Point-SAGA mentioned above, SSNM is a more favorable accelerated variant of SAGA than Point-SAGA in practice. Interestingly, both accelerated variants are more unstable than SAGA in this experiment.