Homogenization of a diffusion in a high-contrast random environment and related Markov semigroups

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Abstract

The goal of the paper is to describe the large time behaviour of a Markov process associated with a symmetric diffusion in a high-contrast random environment and to characterize the limit semigroup and the limit process under the diffusive scaling.

1 Introduction

The paper focuses on the large time behaviour of a diffusion in a high contrast random statistically homogeneous environment. We also study the limit behaviour of the corresponding semigroups. Equivalently, we consider the limit behaviour of a diffusion defined in a high contrast environment with a random microstructure on a finite time interval.

Elliptic and parabolic operators with high contrast rapidly oscillating periodic coefficients have been widely studied in the homogenization theory. The first rigorous results for parabolic operators of this type were obtained in [10] and [8]. In particular, it was shown that, under proper choice of the scaling coefficient, the homogenized problem contains a non-local in time operator which reflects the so-called memory effect. Later on in [2], with the help of the two-scale convergence technique, the limit problem was written as a coupled system of parabolic PDEs in the space with higher number of variables. In the works [16], [17] high contrast problems in domains with singular or asymptotically singular periodic geometry were considered. At present, there are many works in the existing mathematical literature that describe the effective behaviour of high contrast periodic media. Under proper scaling, in parabolic problems this usually results in the memory effect while homogenization of spectral problems leads to a non-linear dependence on the spectral parameter.

In this paper we deal with second order divergence form operators in $\mathbb{R}^d$. Each such an operator is a generator of a Markov semigroup. The corresponding Markov process (generalized diffusion) has continuous trajectories. However, the presence of a non-local in temporal variable term in the effective operator means that the limit dynamics of the coordinate process is not Markov.

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The goal of this work is to equip the coordinate process with additional components in such a way that the dynamics of the enlarged process remains Markovian in the limit. We show that it is sufficient to combine the coordinate process in $\mathbb{R}^d$ with a position of the diffusion inside the rescaled inclusions for the time intervals when the diffusion is trapped by one of the inclusions.

It is interesting to observe that, although in the original processes the additional components are functions of the coordinate process, in the limit process these components are getting independent while the coordinate process becomes coupled with them.

The explicit form of the limit operator on the extended space gives us a possibility to use this operator as a generator of an approximation dynamics for the processes in high contrast random stationary dispersive porous media. The discrete version of such approximation process was constructed in the work [13], where we considered a discrete diffusion in a high-contrast random environment given by a jump random walk on the lattice $\mathbb{Z}^d$. The crucial step in this construction is to describe the ”clock process” governing transitions from the observable ”real” space to the supplementary ”astral” spaces and back. The ”clock process” is a continuous time finite Markov chain with transition rates depending on parameters of the limit operator.

In the paper we introduce proper functional spaces, construct the limit semigroup, and prove the semigroup convergence.

In addition to proving the semigroup convergence, we study the spectrum of the generator of the limit semigroup. Then the semigroup convergence in $L^2$ spaces allows us to provide some information about the limit behaviour of the spectrum of the original operators.

To our best knowledge, the questions considered in this paper have not been studied in the existing literature. In the discrete framework the results on scaling limits of symmetric random walks in a high contrast periodic environment were obtained in our previous work [13].

Our approach essentially relies on the approximation technique developed in [5] and the technique of correctors in random media. In contrast with the periodic framework, the auxiliary operators used to introduce correctors need not be of Fredholm type in the case of random inclusions. The construction of the first corrector can be found in the existing literature, see for instance [9]. However, when defining the higher order correctors we face additional difficulties.

2 Problem setup

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a standard probability space. Consider a symmetric diffusion operator in divergence form

$$A_\varepsilon^\omega f(x) = \text{div} (a_\varepsilon^\omega (x) \nabla f(x)), \quad (1)$$

where

$$a_\varepsilon^\omega(x) = \begin{cases} \mathbb{I}, & x \in \mathbb{R}^d \setminus \varepsilon G^\omega = (G_\varepsilon^\omega)^c, \\ \varepsilon^2 \mathbb{I}, & x \in \varepsilon G^\omega, \end{cases} \quad (2)$$

$I$ being the unit matrix. Here $G_\varepsilon^\omega = \varepsilon G^\omega$ and $G^\omega \subset \mathbb{R}^d$, $\omega \in \Omega$, is a classical disperse medium. That is $\mathbb{R}^d \setminus G^\omega$ is a random statistically homogeneous set such that almost surely (a.s) it is connected and unbounded and its complement $G^\omega \subset \mathbb{R}^d$ consists of a countable number of uniformly bounded simply connected domains with uniformly Lipschitz boundary. Moreover, the distance between any two such domains admits a uniform deterministic lower bound. The set $G^\omega$ corresponds to the matrix blocks (inclusions), and $\mathbb{R}^d \setminus G^\omega \subset \mathbb{R}^d$ to the fractures system, see [4].

To be more specific in this work we consider a class of disperse media that satisfy the following additional condition:

- there is a finite collection of bounded domains in $\mathbb{R}^d$ such that all these domains are uniformly $C^2$ regular and a.s. any connected component of $G^\omega$ can be obtained by a proper translation and rotation of one of these domains. The domains are denoted by $D_j$, $j = 1, \ldots, N$, and the whole collection by
\[\mathcal{D}, \mathcal{D} = \{\mathcal{D}_j\}_{j=1}^N\] with \(N \in \mathbb{Z}^+\). We assume without loss of generality that each domain \(\mathcal{D}_j, j \geq 1\), contains the origin.

An example of such a disperse medium is associated with a Bernoulli site percolation model on the lattice \(\mathbb{Z}^d\) embedded in \(\mathbb{R}^d\). Let \(\{\xi_j, j \in \mathbb{Z}^d\}, \xi_j \in \{0, 1\}\) be a sequence of i.i.d. random variables having the Bernoulli law: \(\mathbb{P}(\xi_j = 1) = p, \mathbb{P}(\xi_j = 0) = 1 - p\). We then define \(\mathcal{B}_j = j + [-\frac{1}{2}, \frac{1}{2}]^d, j \in \mathbb{Z}^d\), and consider the set \(G^{1,\omega} = \bigcup_{j: \xi_j = 1} \mathcal{B}_j\). This set is a.s. a union of countable number of bounded connected components (components) and not more than one unbounded connected component, see \([8]\). We replace each bounded connected component of \(G^{1,\omega}\) with the minimal simply connected set that contains this component. Then we choose the sets that have Lipschitz boundary and whose volume does not exceed \(M\), where \(M \geq 1\) is a given positive integer. We then smoothen these sets and obtain a finite collection \(\mathcal{D}\) of the reference sets. By the standard arguments of percolation theory, \(G^\omega\) is statistically homogeneous and ergodic.

Denote by \(G^\omega_j\) the subset of \(G^\omega\) that consists of all the components which have the same geometry as \(\mathcal{D}_j\), that is each such a component can be obtained by a proper rotation and translation of \(\mathcal{D}_j\) in \(\mathbb{R}^d\). The connected components of \(G^\omega_j\) are denoted by \(\{G^\omega_{j,i}\}_{i \in \mathbb{Z}^+}\). We also denote \(G^\omega_0 = \mathbb{R}^d \setminus G^\omega\).

Letting \(\alpha_0 = \mathbb{P}\{0 \in (G^\omega)^c\}, \alpha_0^j = \mathbb{P}\{0 \in G^\omega_j\}, \alpha_0 + \sum_{j \geq 1} \alpha_0^j = 1\), we assume without loss of generality that \(\alpha_0^j > 0\) for all \(j\). We then introduce \(\alpha_j = |\mathcal{D}_j|^{-1} \alpha_0^j = |\mathcal{D}_j|^{-1} \mathbb{P}\{0 \in G^\omega_j\}\).

For each \(\varepsilon > 0\) the operator \(A^\omega_\varepsilon\) has random statistically homogeneous coefficients in \(\mathbb{R}^d\), where the randomness is defined through the random geometry of \(G^\omega\). These operators are also called metrically transitive with respect to the unitary group of the space translations in \(\mathbb{R}^d\). In \(L^2(\mathbb{R}^d)\) we introduce a domain of \(A^\omega_\varepsilon\) by

\[
D(A^\omega_\varepsilon) = \left\{ f \in H^1(\mathbb{R}^d), f \in H^2(G^\omega_\varepsilon) \cap H^2(\mathbb{R}^d \setminus G^\omega_\varepsilon), \varepsilon^2 \nabla f(x)|_{\partial G^\omega_\varepsilon} \cdot n^+ = -\nabla f(x)|_{\partial G^\omega_\varepsilon} \cdot n^- \right\}
\]

(5)

The last relation in \([\mathbb{R}]\) represents the continuity of flux \(a_\varepsilon \nabla f\) through the boundary \(\partial G^\omega_\varepsilon\). Here \(n^-, n^+\) are respectively the internal and external normals on \(\partial G^\omega_\varepsilon\).

**Remark 2.1.** Notice that for any function \(v \in D(A^\omega_\varepsilon)\) its trace and the trace of its flux on the interface \(\partial G^\omega_\varepsilon\) is a well-defined \(L^2(\partial G^\omega_\varepsilon)\) function.

Then \((A^\omega_\varepsilon, D(A^\omega_\varepsilon))\) is almost surely a self-adjoint operator in \(L^2(\mathbb{R}^d)\), and for any \(\lambda > 0\) the operator \((\lambda - A^\omega_\varepsilon)\) is coercive. By the Hille-Yosida theorem, \(A^\omega_\varepsilon\) is a generator of a strongly continuous, positive, contraction semigroup \(T^\omega_\varepsilon(t)\) on \(L^2(\mathbb{R}^d)\) for a.e. \(\omega \in \Omega\).

# 3 The limit operator.

In this section we describe the generator of the limit Markov semigroup. Denote

\[E = \mathbb{R}^d \times \mathcal{D}^*, \quad \text{where} \quad \mathcal{D}^* = \{\ast\} \cup \mathcal{D}.
\]

We consider functions \(F\) defined on \(E\) of the following vector form

\[
F(x, \xi) = \left\{ \begin{array}{ll}
 f_0(x), & \text{if } x \in \mathbb{R}^d, \xi = \ast,
 f_j(x, \xi), & \text{if } x \in \mathbb{R}^d, \xi \in \mathcal{D}_j, j = 1, \ldots, N,
\end{array} \right.
\]

(6)
with \( f_0 \in L^2(\mathbb{R}^d), \ f_j \in L^2(\mathbb{R}^d \times D_j) \). Equipped with the norm

\[
\|F\|^2 = \alpha_0 \int_{\mathbb{R}^d} f_0^2(x) \, dx + \sum_{j=1}^N \alpha_j \int_{\mathbb{R}^d} (f_j(x, \xi) \, d\xi \, dx
\]  

where \( \alpha_0, \alpha_j \) was defined in (4), \( \alpha_0 > 0, \alpha_j > 0 \), it is a Hilbert space. We call this Hilbert space \( L^2(E, \alpha) \).

Let us consider in \( L^2(E, \alpha) \) an operator of the following form

\[
(AF)(x, \xi) = \begin{pmatrix}
\Theta \cdot \nabla \nabla f_0(x) + \frac{1}{\alpha_0} \sum_{j \geq 1} \alpha_j \int_{D_j} \frac{\partial f_j(x, \xi)}{\partial \eta_\xi} d\sigma(\xi) \\
\Delta \xi f_1(x, \xi) \\
\Delta \xi f_2(x, \xi) \\
\vdots \\
\Delta \xi f_N(x, \xi)
\end{pmatrix}
\]  

(8)

where a positive definite constant matrix \( \Theta \) will be defined later on, \( \sigma(\xi) \) is the element of the surface volume on the Lipshitz boundary \( \partial D_j^k \), \( n^{-} \) is the (inner) normal to \( \partial D_j^k \). Using the relation \( n^{-} = -n^{+} \) and the Stokes formula one can rewrite the operator (8) as follows:

\[
(AF)(x, \xi) = \begin{pmatrix}
\Theta \cdot \nabla \nabla f_0(x) - \frac{1}{\alpha_0} \sum_{j \geq 1} \alpha_j \int_{D_j} \Delta \xi f_j(x, \xi) d\xi \\
\Delta \xi f_1(x, \xi) \\
\Delta \xi f_2(x, \xi) \\
\vdots \\
\Delta \xi f_N(x, \xi)
\end{pmatrix}
\]  

(9)

We denote

\[
\Upsilon(x) = -\frac{1}{\alpha_0} \sum_{j \geq 1} \alpha_j \int_{D_j} \Delta \xi f_j(x, \xi) d\xi.
\]  

(10)

For each set \( D_j, j \geq 1 \), denote by \( D_j(\Delta) \) the domain of a self-adjoint operator in \( L^2(D_j) \) that corresponds to the Laplacian in \( D_j \) with homogeneous Dirichlet boundary conditions. Since the boundary of \( D_j \) is \( C^2 \) regular, we have \( D_j(\Delta) = H^2(D_j) \cap H_0^1(D_j) \). Notice that this operator is positive. The space \( D_j(\Delta) \) is equipped with the norm \( \|g\|_{D_j(\Delta)} = \|\Delta g\|_{L^2(D_j)} \).

Defining an operator \( A \) in \( L^2(E, \alpha) \) by formulas (8), (9), one can easily check that, with a domain

\[
\hat{D}(A) = \left\{ f_0 \in H^2(\mathbb{R}^d), \ f_j - f_0 \in L^2(\mathbb{R}^d; D_j(\Delta)), \ f_j(x, \xi)|_{\xi \in \partial D_j} = f_0(x), \right\}
\]  

(11)

the operator \( (A, \hat{D}(A)) \) is a closed symmetric operator in \( L^2(E, \alpha) \), and \( \hat{D}(A) \) is dense in \( L^2(E, \alpha) \).

We introduce the following two spaces:

\[
H^1_D(E, \alpha) = \left\{ f_0 \in H^1(\mathbb{R}^d), \ f_j - f_0 \in L^2(\mathbb{R}^d; H_0^1(D_j)) \right\}
\]  

(12)

and

\[
H^2_D(E, \alpha) = \left\{ f_0 \in H^2(\mathbb{R}^d), \ f_j - f_0 \in L^2(\mathbb{R}^d; H^2(D_j) \cap H_0^1(D_j)) \right\}.
\]  

(13)

Notice that

\[
\sum_{j=1}^N \alpha_j \int_{\mathbb{R}^d} \int_{D_j} \left| \nabla \xi f_j(x, \xi) \right|^2(\xi) \, d\xi \, dx < \infty \quad \forall F \in H^1_D(E, \alpha)
\]
\[ \sum_{j=1}^{N} \alpha_j \|f_j\|_{L^2(\mathbb{R}^d; H^1(D_j))}^2 < \infty \quad \forall F \in H^1_D(E, \alpha). \]

The space \( H^{-1}(E, \alpha) \) is defined as the dual space to \( H^1_D(E, \alpha) \) in \( L^2(E, \alpha) \).

**Lemma 3.1.** For any \( m > 0 \) the operator \( (m-A, \hat{D}(A)) \) is a coercive self-adjoint operator in \( L^2(E, \alpha) \).

**Proof.** Consider the following quadratic form in \( L^2(E, \alpha) \)

\[ \Gamma(F, F) = \alpha_0 \int_{\mathbb{R}^d} \Theta \nabla f_0(x) \cdot \nabla f_0(x) \, dx + \sum_{j=1}^{N} \int_{\mathbb{R}^d} |\nabla_x f_j|^2(x, \xi) \, d\xi \, dx + \|F\|^2_{L^2(E, \alpha)} \quad (14) \]

with a domain \( D(\Gamma) = H^1(E, \alpha) \). Notice that \( f_j(x, \cdot)|_{\partial D_j} = f_0(x) \) for any \( F \in D(\Gamma) \). According to [15] Theorem x.x there exists a unique self-adjoint operator \( \tilde{A}_m \) that has the following properties:
- its domain \( D(\tilde{A}_m) \) is dense in \( L^2(E, \alpha) \);
- \( D(A_m) \) belongs to \( D(\Gamma) \);
- \( (\tilde{A}_m F, F)_{L^2(E, \alpha)} = \Gamma(F, F) \) for any \( F \in D(\tilde{A}_m) \).

We are going to show that \( \tilde{A}_m \) coincides with \( m - A \). First we prove that \( D(\tilde{A}_m) \subset \hat{D}(A) \).

Separating the first component \( f_0 \) in \( \tilde{A}_m \) we will use the notation \( F = (f_0, V) \). Taking \( F \in D(\tilde{A}_m) \) and \( U = (0, U_1) \in D(\tilde{A}_1) \) with \( U_1 \in C_0^\infty(\mathbb{R}^d; C_0^\infty(D)) \), and using the relation \( (\tilde{A}_m F, U)_{L^2(E, \alpha)} = \Gamma(F, U) \), we obtain

\[ (\tilde{A}_m F, U)_{L^2(E, \alpha)} = \Gamma(F, U) = \sum_{j \geq 1} \alpha_j ((m - \Delta \xi) V_j, U_{1,j}), \]

where the terms \( -\Delta \xi V_j, U_{1,j} \), on the right-hand side are understood as a pairing between \( L^2(\mathbb{R}^d; H^{-1}(D_j)) \) and \( L^2(\mathbb{R}^d; H^1_0(D_j)) \). This implies that \( (m - \Delta \xi) V_j \in L^2(\mathbb{R}^d \times D_j) \) and \( (0, \{ (m - \Delta \xi) V_j \}_{j \geq 1}) \in L^2(E, \alpha) \). Therefore, \( (0, V) \in H^2(E, \alpha) \). Choosing now \( U = (u_0(x), 0) \) with \( u_0 \in C_0^\infty(\mathbb{R}^d) \) and considering the fact that \( \sum_{j \geq 1} \alpha_j \int_{D_j} \Delta \xi V_j(x, \xi) \, d\xi \in L^2(\mathbb{R}^d) \), we get \( m f_0 - \text{div}(\Theta \nabla f_0) \in L^2(\mathbb{R}^d) \). Therefore, \( f_0 \in H^2(\mathbb{R}^d) \), and \( D(\tilde{A}_m) \subset \hat{D}(A) \).

Moreover, \( \tilde{A}_m F = (m - A) F \) for any \( F \in D(\tilde{A}_m) \). Since \( \tilde{A}_m \) is self-adjoint, \( D(\tilde{A}_m) = \hat{D}(A) \). This yields the desired statement. \( \square \)

We define the following set of functions:

\[ D_A = \{ f_0(x) \in C_0^\infty(\mathbb{R}^d), \, f_j(x, \xi) - f_0(x) \in C_0^\infty(\mathbb{R}^d \times D_j(\Delta)) \}. \quad (15) \]

Notice that \( f_j(x, \xi)|_{\xi \in \partial D_j} = f_0(x) \) for any \( F = \{ f_j \}_{j \geq 0} \in D_A \).

**Corollary 1.** The set \( D_A \subset L^2(E, \alpha) \) defined in (15) is a core of \( A \), i.e. \( D_A \) is a dense subset of \( L^2(E, \alpha) \) and \( A = A|_{D_A} \), see [2] for the details.

**Proof.** Clearly, \( D_A \) is a dense subset in \( L^2(E, \alpha) \). In order to show that \( D_A \) is a core of \( A \) we should also check that for some \( m > 0 \) the set \( \{ (m - A) F, F \in D_A \} \) is dense in \( L^2(E, \alpha) \). Denote \( J^\infty = \{ (u_0, U) = (u_0(x), U_j(x, \xi)) : u_0 \in C_0^\infty(\mathbb{R}^d), \, U_j \in C_0^\infty(\mathbb{R}^d \times D_j(\Delta)) \} \). Observe that \( J^\infty \) is dense in \( L^2(E, \alpha) \). By Lemma 3.1 for an arbitrary \( U \in J^\infty \) and for any \( m > 0 \) the equation

\[ m F - AF = U \quad (16) \]

has a unique solution \( F = (f_0, V) \in \hat{D}(A) \). Then the equation for \( V \) can be rewritten as

\[ (m - \Delta \xi)(V(x, \xi) - f_0(x)) = U(x, \xi) - m f_0(x) \quad \text{in} \, \mathcal{D}, \quad (V(x, \xi) - f_0(x))|_{\xi \in \partial \mathcal{D}} = 0. \quad (17) \]
or, in the coordinate form,

\[(m - \Delta)(V_j(x, \xi) - f_0(x)) = U_j(x, \xi) - mf_0(x) \text{ in } D_j, \quad (V_j(x, \xi) - f_0(x))\big|_{\xi \in \partial D_j} = 0, \quad j \geq 1. \quad (18)\]

From this equation we derive the following relation:

\[V_j(x, \xi) = V_j^I(x, \xi) + mf_0(x)V_j^0(\xi) + f_0(x) \quad (19)\]

with \(V_j^I = (m - \Delta)^{-1}U_j \in C_0^\infty(\mathbb{R}^d; D_j(\Delta)) \) and \(V_j^0 = (m - \Delta)^{-1}1 \in D_j(\Delta)\).

Substituting the right-hand side of (19) for \(V\) into the first equation in (16) yields

\[mf_0 - \Theta \cdot \nabla f_0 - cmf_0 = w_0,\]

where

\[w_0 = u_0 - \sum_{j \geq 1} \alpha_j \int_{D_j} \Delta \xi (m - \Delta)^{-1}U_j(\cdot, \xi) d\xi, \quad c = - \sum_{j \geq 1} \alpha_j \int_{D_j} \Delta \xi (m - \Delta)^{-1} d\xi.\]

Under our assumptions on \(U\) we have \(w_0 \in C_0^\infty(\mathbb{R}^d)\). Also, it is straightforward to check that \(c < 1\) for any \(m > 0\). Consequently, \(f_0\) is a Schwartz class function in \(\mathbb{R}^d\). Taking a proper sequence of smooth cut-off functions \(\varphi_n\) we conclude that \((m - A)(\varphi_n f_0, V + \varphi_n f_0)\) converges in \(L^2(E, \alpha)\) to \(U\). Since \((\varphi_n f_0, V + \varphi_n f_0) \in D_A\), this yields the desired statement. \(\square\)

### 4 The semigroup convergence.

Applying the Hille-Yosida theorem, we conclude that \(A\) is a generator of a strongly continuous, positive, contraction semigroup \(T(t)\) on \(L^2(E, \alpha)\).

Define a bounded linear transformation \(\pi_\varepsilon^\omega : L^2(E, \alpha) \to L^2(\mathbb{R}^d)\) for every \(\varepsilon \in (0, 1)\) and every \(\omega \in \Omega\) as follows:

\[(\pi_\varepsilon^\omega F)(x) = \begin{cases} f_0(x), & \text{if } x \in \mathbb{R}^d \setminus G_\varepsilon^\omega; \\ \hat{f}_j(\hat{x}_j^\varepsilon, \frac{x - \hat{x}_j^\varepsilon}{\varepsilon}), & \text{if } x \in \varepsilon G_\varepsilon^\omega, \end{cases} \quad (20)\]

where \(\varepsilon^{-1}\hat{x}_j^\varepsilon\) is the vector that defines the translation which maps \(D_j\) to \(G_j^\omega\), and

\[\hat{f}_j(\hat{x}_j^\varepsilon, \xi) = \frac{1}{\varepsilon^d|D_j|} \int_{\varepsilon D_j} f_j(\hat{x}_j^\varepsilon + \eta, \xi) d\eta. \quad (21)\]

**Lemma 4.1.** Almost surely the linear operators \(\pi_\varepsilon^\omega\) are uniformly bounded in the operator norm for all \(\varepsilon \in (0, 1)\), that is

\[\|\pi_\varepsilon^\omega F\|_{L^2(\mathbb{R}^d)} \leq C\|F\|_{L^2(E, \alpha)} \quad (22)\]

for any \(F \in L^2(E, \alpha)\); the constant \(C\) is deterministic and does not depend on \(\varepsilon\). Moreover, for each \(F \in L^2(E, \alpha)\) the following relation holds a.e.

\[\|\pi_\varepsilon^\omega F\|_{L^2(\mathbb{R}^d)}^2 \to \|F\|_{L^2(E, \alpha)}^2 \quad \text{as } \varepsilon \to 0. \quad (23)\]

**Proof.** For every \(x \in \mathbb{R}^d\) and every \(\omega\) we have

\[\sum_{j \geq 0} \chi_{G_j^\omega}(x) = 1,\]

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where $G_0^\alpha = \mathbb{R}^d \setminus G_\varepsilon^\alpha$. Then we get
\[\|\pi^\alpha F\|_{L^2(\mathbb{R}^d)}^2 = \int_{\mathbb{R}^d} (\pi^\alpha F(x))^2 \, dx = \sum_{j \geq 0} \int_{\mathbb{R}^d} (\pi^\alpha F(x))^2 \chi_{G_0^\alpha}(\frac{x}{\varepsilon}) \, dx\]
\[= \int_{\mathbb{R}^d} f_0^2(x) \chi_{G_0^\alpha}(\frac{x}{\varepsilon}) \, dx + \sum_{j \geq 1} \sum_i \int_{\mathbb{R}^d} \left( \hat{f}_j(\varepsilon \hat{x}_j^x, \frac{x}{\varepsilon} - \hat{x}_j^x) \right)^2 \chi_{G_0^\alpha}(\frac{x}{\varepsilon}) \, dx. \]  
(24)

By the Jenssen inequality and the definition of $\hat{f}_j$ in (21) this implies that
\[\|\pi^\alpha F\|_{L^2(\mathbb{R}^d)}^2 \leq \int_{\mathbb{R}^d} f_0^2(x) \, dx + \sum_{j \geq 0} \frac{1}{\|D_j\|} \int_{\mathbb{R}^d} \int_{D_j} (f_j(x, \xi))^2 \, d\xi \, dx \leq \hat{C} \|F\|_{L^2(E, \alpha)}\]
with $\hat{C} = \max\{(\alpha_j^\alpha)^{-1}\}$. This yields (22).

We turn to (23) and consider the set of functions in $L^2(E, \alpha)$ which are piece-wise constant and compactly supported with respect to the first variable $x$. We denote this set by $\mathcal{E}$ and notice that it is dense in $L^2(E, \alpha)$. If $F \in \mathcal{E}$ then (23) holds by the Birkhoff ergodic theorem. Then, taking into account (22) we conclude that (23) holds for any $F \in L^2(E, \alpha)$. □

Now we are ready to formulate the main result of the work.

**Theorem 4.1** (Main theorem). For every $F \in L^2(E, \alpha)$ a.e.
\[T_{\varepsilon}^\alpha(t)\pi^\alpha F \to T(t)F, \quad i.e. \quad \|T_{\varepsilon}^\alpha(t)\pi^\alpha F - \pi^\alpha F \cdot T(t)F\|_{L^2(\mathbb{R}^d)} \to 0 \quad \text{for all } \ t \geq 0 \]  
(25)
as $\varepsilon \to 0$.

The proof of the semigroup convergence in (25) relies on the following approximation theorem [5, Theorem 6.1, Ch.1].

**Theorem** (see [5]). For $n = 1, 2, \ldots$, let $T_n(t)$ and $T(t)$ be strongly continuous contraction semigroups on Banach space $\mathcal{L}_n$ and $\mathcal{L}$, with generators $A_n$ and $A$. Let $D$ be a core for $A$. Then the following are equivalent:

a) For each $f \in \mathcal{L}$, $T_n(t)\pi_n f \to T(t)f$ for all $t \geq 0$.

b) For each $f \in D$, there exists $f_n \in \mathcal{L}_n$ for each $n \geq 1$ such that $f_n \to f$ and $A_n f_n \to Af$.

According to this theorem the semigroups convergence (25) is a consequence of the following statement:

**Theorem 4.2.** Let the generators $A$ and $A_\varepsilon$ of the strongly continuous, positive, contraction semigroups $T(t)$ and $T_\varepsilon^\alpha(t)$ be defined by (8) and (1), (2), (5), respectively, and assume that a core $D_A \subset L^2(E, \alpha)$ for the generator $A$ is defined by (11), and that a bounded linear transformation $\pi^\alpha : L^2(E, \alpha) \to L^2(\mathbb{R}^d)$ is defined by (27) for every $\varepsilon \in (0, 1)$.

Then there exists a positive definite symmetric constant matrix $\Theta$ such that a.s. for every $F \in D_A$, there exists $F_\varepsilon^\alpha \in D(A_\varepsilon^\alpha)$ such that
\[\|F_\varepsilon^\alpha - \pi^\alpha F\|_{L^2(\mathbb{R}^d)} \to 0 \quad \text{and} \quad \|A_\varepsilon^\alpha F_\varepsilon^\alpha - \pi^\alpha AF\|_{L^2(\mathbb{R}^d)} \to 0 \quad \text{as } \varepsilon \to 0. \]  
(26)
Proof. The proof relies on the correctors technique. For any $F \in D_A$, $F = (f_0, \{f_j\})$, where

$$f_0(x) \in C_0^\infty(\mathbb{R}^d), \quad f_j(x, \xi) \in C_0^\infty(\mathbb{R}^d; C^2(D_j)),$$

with

$$f_j(x, \xi)|_{\xi \in \partial D_j} = f_0(x), \quad x \in \mathbb{R}^d, \quad \forall \ j = 1, \ldots, N,$$

we construct the following family of functions $F^\omega_\epsilon$ depending on the realization $\omega$ of random environment:

$$F^\omega_\epsilon(x) = \begin{cases} f_0(x) + \epsilon(\nabla f_0(x), h^\omega_\epsilon(x)) + \epsilon^2(\nabla \nabla f_0(x), g^\omega_\epsilon(x)) + \epsilon^2 q^\omega_\epsilon(x, \frac{x}{\epsilon}), & x \in \mathbb{R}^d \setminus G^\omega_\epsilon, \\ f_1(x, \frac{x}{\epsilon}) + \epsilon \phi^\omega_1(x, \frac{x}{\epsilon}), & x \in \varepsilon G^\omega_1, \\ \vdots \\ f_N(x, \frac{x}{\epsilon}) + \epsilon \phi^\omega_N(x, \frac{x}{\epsilon}), & x \in \varepsilon G^\omega_N. \end{cases}$$

(28)

Here $h^\omega_\epsilon(\xi), g^\omega_\epsilon(x, \xi)$ are random functions of $\xi$ (so-called correctors) that also depend on $\epsilon$; $h^\omega_\epsilon(\xi)$ is the random vector function whose gradient does not depend on $\epsilon$, $g^\omega_\epsilon(\xi)$ is the random matrix function. In what follows we drop both indices $\omega$ and $\epsilon$ when refer to these functions. Correctors $\phi^\omega_j(x, \xi)$ has been introduced in order to ensure the continuity of the function $F^\omega_\epsilon$ and the fluxes on the boundary $\partial(\varepsilon G^\omega_j)$ of the corresponding inclusion.

Observe that for any $F \in D_A$ as well as for any $F \in C(E)$ and any $x \in \varepsilon G^\omega_j$, we have:

$$(\pi^\omega_\epsilon F)(x) = \hat{f}_j(\hat{x}^\omega_{j,i}, \frac{x - \hat{x}^\omega_{j,i}}{\epsilon}) = f_j(x, \frac{x - \hat{x}^\omega_{j,i}}{\epsilon}) + O(\epsilon),$$

(29)

where the $L^\infty$ norm of $O(\epsilon)$ does not exceed $C\epsilon$. Our goal is to choose the correctors in such a way that the function $F^\omega_\epsilon$ defined in (28) belongs to $D(A^\omega_\epsilon)$, and both relations in (26) are fulfilled. Denote by $B_0$ the ball in $\mathbb{R}^d$ centered at 0 that contains the supports in $x$ of all the functions $f_j$, $j = 0, 1, \ldots, N$.

In order to introduce the correctors in (28) we substitute for $F^\omega_\epsilon$ in the expression $A^\omega_\epsilon F^\omega_\epsilon - \pi^\omega_\epsilon AF$ the right-hand side of (28). Using repeatedly the formula

$$\frac{\partial}{\partial x} f(x, \frac{x}{\epsilon}) = \left( \frac{\partial}{\partial x} f(x, \xi) + \frac{1}{\epsilon} \frac{\partial}{\partial \xi} f(x, \xi) \right)|_{\xi = \frac{x}{\epsilon}},$$

(30)

for $x \in \mathbb{R}^d \setminus G^\omega_\epsilon$ after straightforward computation we obtain

$$\left( A^\omega_\epsilon F^\omega_\epsilon \right)(x) = \Delta x \left( f_0(x) + \epsilon \nabla f_0(x) \cdot h(\frac{x}{\epsilon}) + \epsilon^2 \nabla \nabla f_0(x) \cdot g(\frac{x}{\epsilon}) + \epsilon^2 q(x, \frac{x}{\epsilon}) \right)$$

$$= \left( \Delta f_0(x) + 2 \nabla \nabla f_0(x) \nabla \xi h(\xi) + \frac{1}{\epsilon} \nabla f_0(x) \Delta \xi h(\xi) + \nabla \nabla f_0(x) \Delta \xi g(\xi) \\ + \epsilon^2 \Delta x q(x, \xi) + \Xi^\omega_\epsilon(x, \xi) \right)|_{\xi = \frac{x}{\epsilon}},$$

(31)

with

$$\Xi^\omega_\epsilon(x, \xi) = \Delta \nabla f_0(x) \cdot \nabla h(\xi) + 2 \nabla \nabla f_0(x) \cdot \nabla \xi g(\xi) + \Delta \nabla f_0(x) \cdot \epsilon^2 g(\xi)$$

In a similar way for $x \in \varepsilon G^\omega_j$ we have

$$\left( A^\omega_\epsilon F^\omega_\epsilon \right)(x) = \epsilon^2 \Delta x \left( f_j(x, \frac{x}{\epsilon}) + \epsilon \phi^\omega_1(x, \frac{x}{\epsilon}) \right) = \left( \Delta x f_j(x, \xi) + \Psi^\omega_j(x, \xi) \right)|_{\xi = \frac{x}{\epsilon}}.$$

(32)

with

$$\Psi^\omega_j(x, \xi) = \epsilon^2 \Delta x f_j(x, \xi) + 2 \epsilon \nabla x \cdot \nabla \xi f_j(x, \xi)$$

$$+ \epsilon^3 \Delta x \phi^\omega_j(x, \xi) + 2 \epsilon^2 \nabla x \cdot \nabla \xi \phi^\omega_j(x, \xi) + \epsilon \Delta x \phi^\omega_j(x, \xi).$$
In order to make \( F_{\varepsilon}^\omega \) belong to \( D(A_{\varepsilon}^\omega) \) we should design it in such a way that the following conditions are fulfilled on \( \partial D_{\varepsilon}^\omega \):

1) continuity condition on \( \partial(\varepsilon D_{\varepsilon}^\omega) \)

\[
\left( f_0(x) + \varepsilon \nabla f_0(x) \cdot h\left(\frac{x}{\varepsilon}\right) + \varepsilon^2 \nabla \nabla f_0(x) \cdot g\left(\frac{x}{\varepsilon}\right) + \varepsilon^2 q(x, \frac{x}{\varepsilon}) \right) \bigg|_{x \in \partial(\varepsilon D_{\varepsilon}^\omega)} = \left( f_j\left(x, \frac{x}{\varepsilon}\right) + \varepsilon \phi_j^\varepsilon\left(x, \frac{x}{\varepsilon}\right) \right) \bigg|_{x \in \partial(\varepsilon D_{\varepsilon}^\omega)}; \tag{33}
\]

2) continuity of fluxes condition

\[
\nabla x \left( f_0(x) + \varepsilon \nabla f_0(x) \cdot h\left(\frac{x}{\varepsilon}\right) + \varepsilon^2 \nabla \nabla f_0(x) \cdot g\left(\frac{x}{\varepsilon}\right) + \varepsilon^2 q(x, \frac{x}{\varepsilon}) \right) \bigg|_{x \in \partial(\varepsilon D_{\varepsilon}^\omega)} \cdot n^- = -\varepsilon^2 \nabla x \left( f_j\left(x, \frac{x}{\varepsilon}\right) + \varepsilon \phi_j^\varepsilon\left(x, \frac{x}{\varepsilon}\right) \right) \bigg|_{x \in \partial(\varepsilon D_{\varepsilon}^\omega)} \cdot n^+, \tag{34}
\]

The main purpose of the functions \( \phi_j^\varepsilon(x, \frac{x}{\varepsilon}) \) is to compensate the discrepancy between the inner and outer expansions for the function \( F_{\varepsilon}^\omega \) at the boundary \( \partial D_{\varepsilon}^\omega \), see Proposition 4.2 below. It follows from \( 27 \) that continuity condition \( 33 \) leads to the relation

\[
\phi_j^\varepsilon\left(x, \frac{x}{\varepsilon}\right) \bigg|_{x \in \partial(\varepsilon D_{\varepsilon}^\omega)} = \left( \nabla f_0(x) \cdot h\left(\frac{x}{\varepsilon}\right) + \varepsilon \nabla \nabla f_0(x) \cdot g\left(\frac{x}{\varepsilon}\right) + \varepsilon q(x, \frac{x}{\varepsilon}) \right) \bigg|_{x \in \partial(\varepsilon D_{\varepsilon}^\omega)} \bigg|_{x \in \partial(\varepsilon D_{\varepsilon}^\omega)}. \tag{35}
\]

Notice that equality \( 35 \) defines the functions \( \phi_j^\varepsilon(x, \frac{x}{\varepsilon}) \) only for \( x \in \partial(\varepsilon D_{\varepsilon}^\omega) \).

With the help of \( 30 \) the relation \( 33 \) can be rewritten as

\[
\left( \nabla f_0(x) + \varepsilon \nabla \nabla f_0(x) h(\xi) + \varepsilon^2 \nabla \nabla f_0(x) g(\xi) \right) + \varepsilon^2 \nabla \phi_j^\varepsilon(x, \xi) \bigg|_{\xi = \frac{x}{\varepsilon} \in \partial \phi_j^\varepsilon} \cdot n^- = -\varepsilon^2 \nabla x f_j(x, \xi) + \varepsilon^2 \nabla \phi_j^\varepsilon(x, \xi) \bigg|_{\xi = \frac{x}{\varepsilon} \in \partial \phi_j^\varepsilon} \cdot n^+. \tag{36}
\]

We first consider the ansatz in \( 28 \) in the set \( \mathbb{R}^d \setminus G_{\varepsilon}^\omega \). Collecting power-like terms in \( 31 \) and \( 36 \) and considering the terms of order \( \varepsilon^{-1} \) in \( 31 \) and of order \( \varepsilon^0 \) in \( 36 \), we conclude that \( h(\cdot) \) should satisfy the equation

\[
\nabla f_0(x) \triangle h(\xi) = 0, \quad \xi \in \phi_j^\varepsilon, \quad \left( \nabla f_0(x) + \nabla \phi_j^\varepsilon(x, \xi) \right) \cdot n_\varepsilon^- = 0, \quad \xi \in \partial \phi_j^\varepsilon;
\]

here \( x \) is a parameter. Since \( f_0 \) does not depend on \( \xi \), this problem can be rewritten as follows:

\[
\triangle h(\xi) = 0, \quad \xi \in \phi_j^\varepsilon, \quad \nabla \phi_j^\varepsilon(\xi) \cdot n_\varepsilon^- = -n_\varepsilon^-, \quad \xi \in \partial \phi_j^\varepsilon. \tag{37}
\]

This suggests the choice of \( h(\cdot) \), it should coincide with the standard corrector used for homogenization of the Neumann problem in a random perforated domain, see \( 9 \). We recall that the gradient of \( h(\xi) \) is a statistically homogeneous matrix function that does not depend on \( \varepsilon \), and \( h \) satisfies equation \( 37 \). Moreover, \( h \) shows a sublinear growth in \( L^2 \). Namely, assuming that \( \int_{B_0} h(\frac{x}{\varepsilon}) dx = 0 \), we have

\[
\| h(\frac{x}{\varepsilon}) \|_{L^2(B_0)} \longrightarrow 0, \quad \text{a.s. as } \varepsilon \to 0. \tag{38}
\]

We also have

\[
\| \nabla \phi_j^\varepsilon(\frac{x}{\varepsilon}) \|_{L^2(B_0)} \leq C \]

a.s. with a constant \( C \) that does not depend on \( \varepsilon \), see \( 9 \).
Lemma 4.2.  

Problem

\[ \text{The matrix } \Theta \text{ in (39) is then defined by} \]

\[
\Theta = \mathbb{E}[(\mathbb{I} + \nabla \xi h(\xi)) \chi_{\mathcal{G}^0_\varepsilon}(\xi)], \quad \text{i.e. } \Theta^{ij} = \mathbb{E}[(\delta_{ij} + \nabla^i h^j(\xi)) \chi_{\mathcal{G}^0_\varepsilon}(\xi)],
\]

(39)

where \( \chi_{\mathcal{G}^0_\varepsilon}(\cdot) \) is the characteristic function of \( \mathcal{G}^0_\varepsilon = \mathbb{R}^d \setminus \mathcal{G}^\varepsilon_0 \). It is proved in [9] that \( \Theta \) is positive definite.

At the next step we collect the terms of order \( \varepsilon^0 \) on the right-hand side of (31) and equate them to \( \Theta \cdot \nabla \nabla f_0(x) + \Upsilon(x) \) in order to make the difference \( (A^\varepsilon \nabla f^\varepsilon_0 - \pi^\varepsilon \mathcal{A} F) = (A^\varepsilon f^\varepsilon_0(x) - (\Theta \cdot \nabla \nabla f_0(x) + \Upsilon(x)) \) small in \( L^2(\mathcal{G}^\varepsilon_0) \) norm. This yields

\[
\left( \triangle f_0(x) + 2\nabla \nabla f_0(x) \cdot \nabla \xi h(\xi) + \nabla \nabla f_0(x) \cdot \triangle \xi g(\xi) + \varepsilon^2 \triangle_x q(x, \xi) \right) |_{x=\varepsilon} = \Theta \cdot \nabla \nabla f_0(x) + \Upsilon(x), \quad (40)
\]

where \( x \in (\mathcal{G}^\varepsilon_0 \cap \partial B_0) \); the function \( \Upsilon(x) \) is defined in (10). We also collect the terms of order \( \varepsilon^1 \) in (39):

\[
\varepsilon \left( \nabla \nabla f_0(x) h(\xi) + \nabla \xi (\nabla \nabla f_0(x) \cdot g(\xi)) + \nabla \xi g(x, \xi) \right) |_{x=\varepsilon} = -\varepsilon \nabla \xi f_j(x, \xi) |_{x=\varepsilon} \in \partial \mathcal{G}^\varepsilon_0 \cdot n^+. \quad (41)
\]

Selecting all the terms in (40)-(41) that contain the second order derivatives of \( f_0 \), we arrive at the following problem for the random matrix valued function \( g(\varepsilon \xi) = \{g_{ij}(\varepsilon \xi)\} \):

\[
\left( \triangle f_0(x) + 2\nabla \nabla f_0(x) \cdot \nabla \xi h(\xi) + \nabla \nabla f_0(x) \cdot \triangle \xi g(\xi) \right) |_{x=\varepsilon} = \Theta \cdot \nabla \nabla f_0(x), \quad x \in \varepsilon \mathcal{G}^\varepsilon_0 \cap \partial B_0,
\]

\[
\nabla \xi g(\xi) \cdot n^- |_{x=\varepsilon} = -h(\xi) \otimes n^- |_{x=\varepsilon}, \quad x \in \varepsilon \partial \mathcal{G}^\varepsilon_0 \cap \partial B_0. \quad (42)
\]

In addition to these two equations we impose the homogeneous Dirichlet boundary condition on the boundary of \( B_0 \)

\[
g \left( \frac{x}{\varepsilon} \right) = 0 \quad \text{on } \partial B_0.
\]

Finally, \( g \left( \frac{x}{\varepsilon} \right) \) is introduced as a solution to the following problem:

\[
\varepsilon^2 \triangle x g \left( \frac{x}{\varepsilon} \right) = \mathbb{E}[(\mathbb{I} + \nabla \xi h(\xi)) \chi_{\mathcal{G}^\varepsilon_0}(\xi)] - \mathbb{I} - 2\varepsilon \nabla x h \left( \frac{x}{\varepsilon} \right), \quad x \in V^\varepsilon := \varepsilon \mathcal{G}^\varepsilon_0 \cap \partial B_0,
\]

\[
\varepsilon \nabla x g \left( \frac{x}{\varepsilon} \right) \cdot n^- = -h \left( \frac{x}{\varepsilon} \right) \otimes n^-, \quad x \in \varepsilon \partial \mathcal{G}^\varepsilon_0 \cap \partial B_0,
\]

\[
g \left( \frac{x}{\varepsilon} \right) = 0, \quad x \in \partial B_0; \quad (43)
\]

here \( \mathbb{I} \) stands for the unit \( d \times d \) matrix.

**Lemma 4.2.** Problem (43) has a unique solution. Moreover, a.s.

\[
\lim_{\varepsilon \to 0} \| \varepsilon^2 g \left( \frac{x}{\varepsilon} \right) \|_{H^2(\varepsilon \mathcal{G}^\varepsilon_0 \cap \partial B_0)} = 0. \quad (44)
\]

The proof of this lemma is provided in Appendix 1, Section 6.

Next, collecting the remaining terms in (10) and (11), we arrive at the following problem for the function \( q(x, \varepsilon \xi) \):

\[
\varepsilon^2 \triangle x q \left( \frac{x}{\varepsilon} \right) = \Upsilon(x), \quad x \in \varepsilon \mathcal{G}^\varepsilon_0 \cap \partial B_0,
\]

\[
\nabla \xi q(x, \xi) \cdot n^- \big|_{x=\varepsilon} = -\nabla \xi f_j(x, \xi) \cdot n^+ \big|_{x=\varepsilon}, \quad x \in \varepsilon \partial \mathcal{G}^\varepsilon_0 \cap \partial B_0. \quad (45)
\]
where the function $\Upsilon(x) \in C_0^\infty(\mathbb{R}^d)$ is defined in (10). We then equip system (12) with the homogeneous Dirichlet boundary condition at $\partial B_0$:

$$q(x, \frac{x}{\varepsilon}) = 0 \quad \text{for} \ x \in \partial B_0. \quad (46)$$

Denote $\Phi_\varepsilon(x) = \varepsilon^2 q(x, \frac{x}{\varepsilon})$. Let $\phi_\varepsilon(\cdot) \in C_0^\infty(B_0)$ be a function such that

$$\phi_\varepsilon \geq 0 \quad \text{and} \quad \phi_\varepsilon = 1 \quad \text{for all} \ x \in \{ x \in \mathbb{R}^d : \ \text{there exist} \ j \ \text{and} \ \xi \ \text{such that} \ f_j(x, \xi) \neq 0 \}.$$

**Proposition 4.1.** The following limit relations hold a.s.:

$$\lim_{\varepsilon \to 0} \|\Phi_\varepsilon\|_{H^1(\varepsilon G_\varepsilon \cap B_0)} = 0, \quad (48)$$

$$\lim_{\varepsilon \to 0} \|\phi_\varepsilon \Phi_\varepsilon\|_{H^1(\varepsilon G_\varepsilon \cap B_0)} = 0. \quad (49)$$

Moreover,

$$\lim_{\varepsilon \to 0} \|\Delta_x (\phi_\varepsilon \Phi_\varepsilon) - \Upsilon\|_{L^2(\varepsilon G_\varepsilon \omega)} = 0. \quad (50)$$

The proof of this statement is given in Appendix 2.

We now turn to the correctors $\varepsilon \phi_j^\varepsilon(x, \frac{x}{\varepsilon})$, $j = 1, \ldots, N$. Our goal is to define them in such a way that

$$\hat{f}_j(x_j^{\varepsilon, i}, \frac{x_j^{\varepsilon, i}}{\varepsilon}) + \varepsilon \phi_j^\varepsilon(x, \frac{x}{\varepsilon}) = f_0(x) + \varepsilon(\nabla f_0(x), h(\frac{x}{\varepsilon})) + \varepsilon^2(\nabla \nabla f_0(x), g(\frac{x}{\varepsilon})) + \varepsilon^2 q(x, \frac{x}{\varepsilon}) \quad \text{on} \ \varepsilon \partial G_j^\varepsilon, \quad (51)$$

$$\varepsilon^2 \nabla \left[ \hat{f}_j(x_j^{\varepsilon, i}, \frac{x_j^{\varepsilon, i}}{\varepsilon}) + \varepsilon \phi_j^\varepsilon(x, \frac{x}{\varepsilon}) \right] \cdot n = \nabla \left[ f_0(x) + \varepsilon(\nabla f_0(x), h(\frac{x}{\varepsilon})) + \varepsilon^2(\nabla \nabla f_0(x), g(\frac{x}{\varepsilon})) + \varepsilon^2 q(x, \frac{x}{\varepsilon}) \right] \cdot n \quad \text{on} \ \varepsilon \partial G_j^\varepsilon, \quad (52)$$

$$\| \varepsilon \phi_j^\varepsilon(x, \frac{x}{\varepsilon}) \|_{L^2(\varepsilon G_j^\varepsilon)} + \varepsilon^2 \| \Delta_x (\varepsilon \phi_j^\varepsilon(x, \frac{x}{\varepsilon})) \|_{L^2(\varepsilon G_j^\varepsilon)} \to 0, \quad \text{as} \ \varepsilon \to 0. \quad (53)$$

**Proposition 4.2.** There exists a family of functions $\phi_j^\varepsilon$ with $j = 1, \ldots, N$ and $\varepsilon \in (0, 1)$ such that the relations in (51) - (53) are fulfilled.

For the proof, see Appendix 2.

We turn back to the **Proof of Theorem 4.2**. The statement of this Theorem is now a straightforward consequence of (37), (38), Lemma 4.2 and Propositions 4.1 - 4.2. Indeed, due to (51) and (52), we have $F_\varepsilon^\omega \in D(A_\varepsilon^\omega)$. Then the convergence

$$\| F_\varepsilon^\omega - \pi_\varepsilon^\omega F \|_{L^2(\mathbb{R}^d)} \to 0 \quad \text{as} \ \varepsilon \to 0$$

follows from (29), (33), (14), (19) and (53). Finally, by (12), (50) and (53) we obtain

$$\| A_\varepsilon^\omega F_\varepsilon^\omega - \pi_\varepsilon^\omega AF \|_{L^2(\mathbb{R}^d)} \to 0 \quad \text{as} \ \varepsilon \to 0. \quad \square$$

This completes the proof of Theorem 4.2.
5 Spectrum of the limit operator

We proceed with the description of the spectrum of the limit operator $A$ given by $[1]$, and then using the strong convergence of Markov semigroup $T^ω(t)$ in $L^2(E)$ we describe the limit behaviour of the spectra of operators $A^ω$, as $ε → 0$ almost surely.

Remind that each component $f_j(x,ξ)$ of $F ∈ D_A$ can be written as the sum

$$f_j(x,ξ) = f_0(x) + r_j(x,ξ) \quad \text{with } r_j(x,ξ)|_{ξ ∈ ∂D_j} = 0 \quad ∀ j = 1, \ldots, N.$$  

Then $[4]$ takes the form

$$(-AF)(x,ξ) = \begin{pmatrix} -Θ ⋅ ∇∇f_0(x) + \frac{1}{α_0} \sum_{j=1}^N α_j \int_{D_j} \triangleξr_j(x,ξ)dξ \\ −\triangleξr_1(x,ξ) \\ ⋮ \\ −\triangleξr_N(x,ξ) \end{pmatrix} \quad (54)$$

For each $j$ the operator $−\triangleξ$ on $D_j$ with homogeneous Dirichlet boundary condition has a discrete positive spectrum $\{λ^j_m\}_{m ∈ N}$. $β^j_m > 0$, $b^j_m → ∞$. We denote by $x^j_m(ξ)$, $m = 1, 2, \ldots$, the corresponding normalized eigenfunctions and by $M$ the set of all indices $(j, m)$. We introduce the set $M^* ⊂ M$ of indices $(j, m)$ such that $∫_{D_j} x^j_m(ξ)dξ = ⟨x^j_m⟩ ≠ 0$. Let $B$ be a (countable) set of all $β^j_m$:

$$B = \bigcup_{(j,m) ∈ M} β^j_m,$$

and

$$b_1 = \min_{(j,m) ∈ M} β^j_m = \min_{(j,m) ∈ M^*} β^j_m, \quad b_1 > 0.$$  

Lemma 5.1. The continuous spectrum $σ_{cont}(−A)$ of the operator $−A$ is a countable set of non-overlapping segments

$$σ_{cont}(−A) = \bigcup_{(j,m) ∈ M^*} [\hat{λ}^j_m, β^j_m],$$

where $\hat{λ}_1 = 0$, and $\hat{λ}^j_m < β^j_m$ is the nearest to $β^j_m$ solution of equation

$$\frac{1}{α_0} \sum_j α_j \sum_m (u^j_m)^2 β^j_m + 1 = 0 \quad \text{with } u^j_m = ⟨x^j_m⟩.$$  

The point spectrum of the operator $−A$ is the union of eigenvalues $β^j_m$ with $(j, m) ∈ M \setminus M^*$:

$$σ_p(−A) = \bigcup_{(j,m) ∈ M \setminus M^*} β^j_m.$$  

Each eigenvalue $β^j_m ∈ σ_p(−A)$ has infinite multiplicity, so that $σ_p(−A)$ belongs to the essential spectrum of $−A$.

Proof. Each line in the equation $−AF = λF$ except of the first one reads

$$−A(f_0(x) + r_j(x,ξ)) = −\triangleξr_j(x,ξ) = λ(f_0(x) + r_j(x,ξ)), \quad ξ ∈ D_j. \quad (55)$$
The function $f_0(x)$ does not depend on $\xi$, its Fourier series w.r.t. $\{\zeta_m^j(\xi)\}$ for every $j$ takes the form
\[
f_0(x) \cdot 1 = f_0(x) \sum_m u^j_m \zeta_m^j(\xi), \quad \text{with} \quad u^j_m = \int \zeta_m^j(\xi) \, d\xi.
\] (56)

Denoting by $\gamma_m^j = \gamma_m(x)$ the Fourier coefficients of $r_j$, from (55) - (56) we get
\[-\Delta \zeta r_j(x, \xi) = \sum_m \beta_m^j \gamma_m \zeta_m^j(\xi) = \lambda f_0(x) \sum_m u^j_m \zeta_m^j(\xi) + \lambda \sum_m \gamma_m \zeta_m^j(\xi).
\]

Consequently, for any $\lambda \notin \mathbb{B}$ we have $\gamma_m^j = \lambda f_0(x) \frac{u^j_m}{\beta_m^j - \lambda}$, and thus the function
\[r_j(x, \xi) = \sum_m \gamma_m \zeta_m^j(\xi) = \lambda f_0(x) \sum_m \frac{u^j_m}{\beta_m^j - \lambda} \zeta_m^j(\xi),
\] (57)
is a solution of equation $-A(f_0 + r_j) = -\Delta \zeta r_j = \lambda (f_0 + r_j)$ for any $j$ and any $\lambda \notin \mathbb{B}$.

Inserting (57) in the first line of the equation $-AF = \lambda F$ with $-A$ given by (54) yields
\[-\Theta : \nabla \nabla f_0(x) - \lambda f_0(x) \frac{1}{\alpha_0} \sum_j \alpha_j \sum_m \frac{u^j_m}{\beta_m^j - \lambda} \int_{\mathcal{D}_j} \zeta_m^j(\xi) d\xi = \lambda f_0(x).
\]

Consequently
\[-\Theta : \nabla \nabla f_0(x) = \lambda f_0(x) \left( \frac{1}{\alpha_0} \sum_j \alpha_j \sum_m \frac{(u^j_m)^2}{\beta_m^j - \lambda} + 1 \right).
\] (58)

Since the spectrum of the operator $-\Theta : \nabla \nabla$ fills up the positive half-line, we obtain that all $\lambda > 0$ such that
\[\frac{1}{\alpha_0} \sum_j \alpha_j \sum_m \frac{(u^j_m)^2}{\beta_m^j - \lambda} + 1 \geq 0
\]
belong to the spectrum of the operator $-A$. One can easily check that the segment $[0, \beta_1], \beta_1 = \min_{(j,m) \in \mathbb{M}^*} \beta_m^j > 0$ belongs to the continuous spectrum of $-A$. This implies the desired statement on $\sigma_{\text{cont}}(-A)$.

It is straightforward to check that the functions $F^{(j,m)} = (0, \ldots, \varphi(x) \zeta_m^j(\xi), 0, \ldots, 0)$ with $\varphi(x) \in L^2(\mathbb{R}^d)$ for all $(j, m) \in \mathbb{M} \setminus \mathbb{M}^*$, i.e. such that $(\zeta_m^j) = 0$, are the eigenfunctions of $-A$ with corresponding eigenvalue $\beta_m^j$. This completes the proof.

Notice that the operators $A_\varepsilon^\omega$ for every $\varepsilon$ have statistically homogeneous coefficients, i.e. they are metrically transitive with respect to the unitary group of the space translations in $\mathbb{R}^d$. Then from the general results, see e.g. [12], it follows that the spectra of the operators $A_\varepsilon^\omega$ are non-random for a.e. $\omega$.

**Proposition 5.1.** For any $\lambda \in \sigma(-A)$ a.s. there exists a sequence $\lambda_\varepsilon$, $\lambda_\varepsilon \in \sigma(A_\varepsilon^\omega)$, that converges to $\lambda$ as $\varepsilon \to 0$.

**Proof.** Since $\lambda \in \sigma(-A)$, there exist functions $F_n \in \mathcal{D}_A$, $\|F_n\|_{L^2(E, \alpha)} = 1$ such that $\|(A + \lambda)F_n\|_{L^2(E, \alpha)} \to 0$ as $n \to \infty$. Using Theorem [4.2] we additionally have that for any $F_n \in \mathcal{D}_A$ there exists $F_{n, \varepsilon}^\omega \in \mathcal{D}(A_\varepsilon^\omega)$ for a.e. $\omega$ such that
\[\|F_{n, \varepsilon}^\omega - \pi_\varepsilon^\omega F_n\|_{L^2(\mathbb{R}^d)} \to 0 \quad \text{and} \quad \|A_\varepsilon^\omega F_{n, \varepsilon}^\omega - \pi_\varepsilon^\omega AF_n\|_{L^2(\mathbb{R}^d)} \to 0 \quad \text{as} \quad \varepsilon \to 0.
\]
Thus using Lemma 4.1 we obtain that for any (small) \( \delta > 0 \) there exists \( \varepsilon_0 = \varepsilon_0(\lambda, \delta) > 0 \) such that for all \( \varepsilon < \varepsilon_0 \) there exists \( F_{n,\varepsilon}^\omega \in L^2(\mathbb{R}^d) \) with \( \| F_{n,\varepsilon}^\omega \| = 1 \), and

\[
\| A_\varepsilon^\omega F_{n,\varepsilon}^\omega + \lambda F_{n,\varepsilon}^\omega \|_{L^2(\mathbb{R}^d)} < \delta. \tag{59}
\]

This implies that there is a point of the spectrum of \( -A_\varepsilon^\omega \) in the \( \delta \)-neighbourhood of \( \lambda \).

\[\square\]

6 Appendix 1. The second corrector \( g \). Proof of Lemma 4.2.

Recall the matrix valued function \( g(\xi) \) was defined as a solution to the following problem:

\[
\varepsilon^2 \Delta_x g(\frac{x}{\varepsilon}) = \mathbb{E} \left[ (1 + \nabla \xi h(\xi)) \chi^\omega(x) \right] - I - 2\varepsilon \nabla_x h(\frac{x}{\varepsilon}), \quad x \in V^\varepsilon := \varepsilon G_0^\omega \cap B_0,
\]

\[
\varepsilon \nabla_x g(\frac{x}{\varepsilon}) \cdot n^- = -h(\frac{x}{\varepsilon}) \otimes n^-, \quad x \in \varepsilon \partial G_0^\omega \cap B_0,
\]

\[
g^\omega(\frac{x}{\varepsilon}) = 0, \quad x \in \partial B_0; \tag{60}
\]

in this section we will use notation \( \chi^\omega(\cdot) = \chi_{G_0^\omega}(\cdot) \) for the characteristic function of the random set \( G_0^\omega \). We also recall that the matrix \( \Theta \) defined by \( \Theta \) is positive definite, see [9]. In the coordinate form the above problem reads

\[
\varepsilon^2 \nabla_x g^{ij}(\frac{x}{\varepsilon}) = \mathbb{E} \left[ (\delta_{ij} + \nabla \xi^i h^j(\xi)) \chi^\omega(x) \right] - \delta_{ij} - 2\varepsilon \nabla_x h^j(\frac{x}{\varepsilon}), \quad x \in V^\varepsilon,
\]

\[
\varepsilon \nabla_x g^{ij}(\frac{x}{\varepsilon}) \cdot n^- = -h(\frac{x}{\varepsilon}) \otimes n^j, \quad x \in \varepsilon \partial V^\varepsilon \cap B_0,
\]

\[
g^{ij}(\frac{x}{\varepsilon}) = 0, \quad x \in \partial B_0. \tag{61}
\]

Each component of \( g(\frac{x}{\varepsilon}) = \{ g^{ij}(\frac{x}{\varepsilon}) \} \) can be considered separately and in what follows we omit a super index \( ij \).

Denote \( \Psi_\varepsilon(\frac{x}{\varepsilon}) = \varepsilon^2 g(\frac{x}{\varepsilon}) \). Our first goal is to prove that the set of functions \( \Psi_\varepsilon(\frac{x}{\varepsilon}) \) is bounded in \( H^1(V^\varepsilon) \). Integrating by parts and using the second equality in \( (61) \) we get

\[
\int_{V^\varepsilon} \varepsilon^2 \Delta_x g(\frac{x}{\varepsilon}) \varepsilon^2 g(\frac{x}{\varepsilon}) dx = -\varepsilon^4 \int_{V^\varepsilon} |\nabla_x g(\frac{x}{\varepsilon})|^2 dx + \varepsilon^2 \int_{\partial V^\varepsilon} \frac{\partial g(\frac{x}{\varepsilon})}{\partial n} g(\frac{x}{\varepsilon}) d\sigma(x)
\]

\[
= -\varepsilon^4 \int_{V^\varepsilon} |\nabla_x g(\frac{x}{\varepsilon})|^2 dx - \varepsilon^3 \int_{\partial V^\varepsilon} h(\frac{x}{\varepsilon}) n^- g(\frac{x}{\varepsilon}) d\sigma(x). \tag{62}
\]

On the other hand, using the first equality in \( (61) \) and integrating by parts we transform the left hand side of \( (62) \) as follows:

\[
\int_{V^\varepsilon} \varepsilon^2 \Delta_x g(\frac{x}{\varepsilon}) \varepsilon^2 g(\frac{x}{\varepsilon}) dx
\]

\[
= \varepsilon^2 \int_{V^\varepsilon} \mathbb{E} \left[ (1 + \nabla \xi h) \chi^\omega \right] g(\frac{x}{\varepsilon}) dx - \varepsilon^2 \int_{V^\varepsilon} (1 + \nabla \xi h^2(\xi)) g(\frac{x}{\varepsilon}) dx - \varepsilon^3 \int_{\partial V^\varepsilon} \nabla_x h(\frac{x}{\varepsilon}) g(\frac{x}{\varepsilon}) dx
\]

\[
= \varepsilon^2 \int_{V^\varepsilon} \left( \mathbb{E} \left[ (1 + \nabla \xi h) \chi^\omega \right] - (1 + \nabla \xi h(\frac{x}{\varepsilon})) \right) g(\frac{x}{\varepsilon}) dx
\]

\[
- \varepsilon^3 \int_{\partial V^\varepsilon} \nabla_x h(\frac{x}{\varepsilon}) g(\frac{x}{\varepsilon}) d\sigma(x) + \varepsilon^3 \int_{V^\varepsilon} h(\frac{x}{\varepsilon}) \nabla_x g(\frac{x}{\varepsilon}) dx. \tag{63}
\]
Thus, (62) - (63) imply
\[ \varepsilon^4 \int_{V^\varepsilon} |\nabla_x g(x)\varepsilon| dx = -\varepsilon^2 \int_{V^\varepsilon} \left( E \left[ (1 + \nabla_x h) \chi^\omega \right] - (1 + \nabla_x h(x)) \right) g(x) dx - \varepsilon^3 \int_{V^\varepsilon} h(\varepsilon x) \nabla_x g(x) dx. \] (64)

We get from (64) that
\[ \|\nabla_x \Psi\varepsilon\|^2_{L^2(\varepsilon)} \leq A\|\Psi\varepsilon\|_{L^2(\varepsilon)} + \|\varepsilon h\|_{L^2(\varepsilon)} \|\nabla_x \Psi\varepsilon\|_{L^2(\varepsilon)}. \] (65)

We have used here the fact that \( \Lambda^\omega_\varepsilon = E \left[ (1 + \nabla_x h^\omega) \chi^\omega \right] - (1 + \nabla_x h^\omega(x)) \chi^\omega(x) \) is a stationary random field with finite second moment \( E(\Lambda^\omega_\varepsilon)^2 < +\infty \). Moreover, by the Birkhoff’ theorem
\[ \lim_{\varepsilon \to 0} E(\Lambda^\omega_\varepsilon) = 0, \] (66)
and thus \( \Lambda^\omega_\varepsilon \) a.s. weakly converges to zero in \( L^2(B_0) \) as \( \varepsilon \to 0 \).

Next we apply the results on extensions in random perforated domains, see [1], [9]. According to these results there exists a linear extension operator \( L : H^1(\varepsilon) \to H^1(B_0) \) such that for any \( f \in H^1(\varepsilon) \)
\[ Lf|_{H^1(\varepsilon)} = f, \quad \|Lf\|_{L^2(B_0)} \leq C\|f\|_{L^2(\varepsilon)}, \quad \|\nabla Lf\|_{L^2(B_0)} \leq \tilde{C}\|\nabla f\|_{L^2(\varepsilon)}, \]
where the constants \( C \) and \( \tilde{C} \) do not depend on \( \varepsilon \). Keeping for the extended function \( L\Psi\varepsilon \) the same notation \( \Psi\varepsilon \) and considering the Dirichlet boundary condition on \( \partial B_0 \) in (61), by the Friedrichs inequality we obtain
\[ \|\nabla_x \Psi\varepsilon\|^2_{L^2(\varepsilon)} \leq \|\Psi\varepsilon\|_{L^2(B_0)}^2 \leq c_1\|\nabla_x \Psi\varepsilon\|^2_{L^2(B_0)} \leq C\|\nabla_x \Psi\varepsilon\|^2_{L^2(\varepsilon)}. \] (67)

Combining this with (65) yields
\[ \|\nabla_x \Psi\varepsilon\|_{L^2(\varepsilon)} \leq A_1, \quad \|\Psi\varepsilon\|_{L^2(\varepsilon)} \leq A_2 \] (68)
with the constants \( A_1 \) and \( A_2 \) that do not depend on \( \varepsilon \). Thus a.s. the family of functions \( \{\Psi\varepsilon\} \) is bounded in \( H^1(\varepsilon) \) and in \( H^1(B_0) \). Due to the compactness of embedding of \( H^1(B_0) \) in \( L^2(B_0) \) we can pass to the limit in the product \( \Lambda^\omega_\varepsilon \Psi\varepsilon \) as \( \varepsilon \to 0 \). Thus the integral
\[ \varepsilon^2 \int_{V^\varepsilon} \left( E \left[ (1 + \nabla_x h) \chi^\omega \right] - (1 + \nabla_x h(x)) \right) g(x) dx = \int_{B_0} \left( E \left[ (1 + \nabla_x h) \chi^\omega \right] - (1 + \nabla_x h(x)) \right) \chi^\omega(x) \Psi\varepsilon(x) dx \]
tends to zero as \( \varepsilon \to 0 \) a.s. Taking into account relations (38) we derive from (64) that
\[ \|\nabla_x \Psi\varepsilon\|_{L^2(B_0)} \to 0 \] (69)
and, by the Friedrichs inequality,
\[ \|\Psi\varepsilon\|_{L^2(B_0)} \to 0. \] (70)

7 Appendix 2. Proofs of Propositions 4.1 and 4.2

We begin this section by proving Proposition 4.1. Denote \( \Phi_\varepsilon(x) := \varepsilon^2 q(x, \frac{x}{\varepsilon}) \). Then \( \Phi_\varepsilon(x) \) is a solution of the following problem:
\[ \nabla_x \Phi_\varepsilon(x) \cdot n^- = -\varepsilon^2 \nabla_x f_j(x, \frac{x}{\varepsilon}) \cdot n^+, \quad x \in \partial(\varepsilon G_j) \cap B_0, \] (71)
\[ \Phi_\varepsilon(x) = 0, \quad x \in \partial B_0. \]
In what follows we will use the following notations:

\[ \nabla f(x, \frac{x}{\epsilon}) = \nabla_x f(x, \frac{x}{\epsilon}), \quad \nabla h(\xi) = \nabla_\xi h(\xi), \quad \nabla \Phi_\varepsilon(x) = \nabla_x \Phi_\varepsilon(x). \]

In order to show that the functions \( \Phi_\varepsilon(x) \) are bounded in \( H^1(V^\varepsilon) \) we follow the line of the proof in the previous section. Multiplying the equation in (71) by \( \Phi_\varepsilon(x) \) and integrating the resulting relation over \( V^\varepsilon \) after integration by paths we obtain

\[
\int_{V^\varepsilon} \Upsilon(x) \Phi_\varepsilon(x) dx = \int_{V^\varepsilon} \Delta \Phi_\varepsilon(x) \Phi_\varepsilon(x) dx = \int_{\partial V^\varepsilon} \Phi_\varepsilon(x) \nabla \Phi_\varepsilon(x) \cdot n^- ds(x) - \int_{V^\varepsilon} |\nabla \Phi_\varepsilon(x)|^2 dx - \sum_j \int_{\partial G_j^\varepsilon} \Phi_\varepsilon(x) \varepsilon^2 \nabla f_j(x, \frac{x}{\varepsilon}) \cdot n^+ ds(x). \tag{72}
\]

By the Friedrichs inequality

\[
\left| \int_{V^\varepsilon} \Delta \Phi_\varepsilon(x) \Phi_\varepsilon(x) dx \right| \leq C_1 \| \Upsilon(x) \|_{L^2(B_0)} \| \nabla \Phi_\varepsilon(x) \|_{L^2(V^\varepsilon)}. \tag{73}
\]

with a constant \( C_1 \) that does not depend on \( \varepsilon \).

To estimate the second integral on the right-hand side of (72) we extend the functions \( \Phi_\varepsilon(x) \) on \( B_0 \), denote the extended functions by \( \bar{\Phi}_\varepsilon(x) \) and apply the Stokes formula. This yields

\[
\int_{\partial G_j^\varepsilon} \varepsilon^2 f_j(x, \frac{x}{\varepsilon}) \cdot n^+ \Phi_\varepsilon(x) ds(x) = \int_{G_j^\varepsilon} \varepsilon^2 \Delta f_j(x, \frac{x}{\varepsilon}) \bar{\Phi}_\varepsilon(x) dx + \int_{\partial G_j^\varepsilon} \varepsilon^2 \nabla f_j(x, \frac{x}{\varepsilon}) \cdot \nabla \bar{\Phi}_\varepsilon(x) dx. \tag{74}
\]

From this relation by the Friedrichs inequality we derive the following upper bound:

\[
\sum_j \left| \int_{\partial G_j^\varepsilon} \varepsilon^2 \nabla f_j(x, \frac{x}{\varepsilon}) \cdot n^+ \Phi_\varepsilon(x) ds(x) \right| \leq C \| \nabla \Phi_\varepsilon \|_{L^2(V^\varepsilon)}. \tag{75}
\]

Finally, (72), (73) and (74) imply the desired upper bound:

\[
\| \nabla \Phi_\varepsilon \|_{L^2(V^\varepsilon)} \leq \tilde{C}, \tag{76}
\]

i.e. the functions \( \Phi_\varepsilon(x) \) are bounded in \( H^1(V^\varepsilon) \). Consequently, the extensions \( \bar{\Phi}_\varepsilon(x) \) are also bounded in \( H^1(B_0) \) and form a compact set in \( L^2(B_0) \). Thus, there exists \( \Phi_0 \in H^1(B_0) \) such that, for a subsequence,

\[
\| \bar{\Phi}_\varepsilon - \Phi_0 \|_{L^2(B_0)} \to 0.
\]

Our goal is to prove that \( \Phi_0 \equiv 0 \), or equivalently

\[
\| \Phi_\varepsilon \|_{L^2(B_0)} \to 0 \quad \text{as} \quad \varepsilon \to 0. \tag{77}
\]

For an arbitrary \( \hat{\psi} \in C^\infty(G_0^\varepsilon) \) with a compact support in \( B_0 \) we have

\[
\int_{V^\varepsilon} \Delta \Phi_\varepsilon(x) \hat{\psi}(x) dx = -\int_{V^\varepsilon} \nabla \Phi_\varepsilon(x) \cdot \nabla \hat{\psi}(x) dx + \sum_j \int_{\partial G_j^\varepsilon} \varepsilon^2 \nabla f_j(x, \frac{x}{\varepsilon}) \cdot n^+ \hat{\psi}(x) ds(x). \tag{78}
\]
On the other hand,
\[ \int_{V_{\varepsilon}} \Delta_{x} \Phi_{\varepsilon}(x) \hat{\psi}(x) dx = \int_{V_{\varepsilon}} \Upsilon(x) \hat{\psi}(x) dx = \int_{B_0} \Upsilon(x) \chi_{\{ \varepsilon G_0^{\varepsilon} \}}(x) \hat{\psi}(x) dx. \]  
(79)

Therefore,
\[ \int_{B_0} \nabla \Phi_{\varepsilon}(x) \cdot \nabla \hat{\psi}(x) \chi_{\{ \varepsilon G_0^{\varepsilon} \}}(x) dx \]
\[ = \sum_{j} \int_{\varepsilon \bar{G}_j^{-\varepsilon}} \varepsilon^2 \nabla f_j(x, \frac{x}{\varepsilon}) \cdot n^+ \hat{\psi}(x) d\sigma(x) - \int_{V_{\varepsilon}} \Upsilon(x) \hat{\psi}(x) dx. \]  
(80)

For an arbitrary \( \psi \in C_0^\infty(B_0) \), substituting in the last relation the function \( \psi(x) + \varepsilon h(\frac{x}{\varepsilon}) \nabla \psi(x) \) for \( \hat{\psi} \) we obtain
\[ \int_{B_0} \nabla \Phi_{\varepsilon}(x) \cdot \left( \nabla \psi(x) + \nabla h(\frac{x}{\varepsilon}) \nabla \psi(x) \right) \chi_{\{ \varepsilon G_0^{\varepsilon} \}}(x) dx + o(1) \]
\[ = \sum_{j} \int_{\varepsilon \bar{G}_j^{-\varepsilon}} \varepsilon^2 \nabla f_j(x, \frac{x}{\varepsilon}) \cdot n^+ \psi(x) d\sigma(x) - \int_{V_{\varepsilon}} \Upsilon(x) \psi(x) dx \]
\[ = - \sum_{j} \int_{\varepsilon \bar{G}_j^{-\varepsilon}} \varepsilon^2 \nabla f_j(x, \frac{x}{\varepsilon}) \psi(x) dx - \int_{V_{\varepsilon}} \Upsilon(x) \psi(x) dx + o(1), \]  
(81)

where \( o(1) \) a.s. tends to zero as \( \varepsilon \to 0 \); here we have used the inequality \( \| h(\frac{x}{\varepsilon}) \|_{H^1(\varepsilon \bar{G}_0^{\varepsilon})} \leq C \) and the fact that \( \| h(\frac{x}{\varepsilon}) \|_{L^2(\varepsilon \bar{G}_0^{\varepsilon})} \) vanishes as \( \varepsilon \to 0 \). Using representation \( (\text{III}) \) of the function \( \Upsilon(x) \), the Stokes formula and the Birkhoff ergodic theorem we conclude that the right-hand side in \( (\text{III}) \) tends to \( 0 \) as \( \varepsilon \to 0 \) for any \( \psi \in C_0^\infty(B_0) \) and thus
\[ \lim_{\varepsilon \to 0} \int_{B_0} \nabla \Phi_{\varepsilon}(x) \cdot \left( \nabla \psi(x) + \nabla h(\frac{x}{\varepsilon}) \nabla \psi(x) \right) \chi_{\{ \varepsilon G_0^{\varepsilon} \}}(x) dx = 0 \]  
(82)

The subsequence of \( \nabla \Phi_{\varepsilon} \) converges weakly in \( (L^2(B_0))^d \) to \( \nabla \Phi_0 \), as \( \varepsilon \to 0 \). By the definition of matrix \( \Theta \) the sequence \( \nabla \psi + \nabla h(\frac{x}{\varepsilon}) \nabla \psi \) \( \chi_{\{ \varepsilon G_0^{\varepsilon} \}} \) converges weakly in \( (L^2(B_0))^d \) to \( \Theta \nabla \psi \). Since the function \( h(\cdot) \) satisfies equation \( (\text{III}) \), we have
\[ \text{div} \left[ \left( \nabla \psi(x) + \nabla h(\frac{x}{\varepsilon}) \nabla \psi(x) \right) \chi_{\{ \varepsilon G_0^{\varepsilon} \}}(x) \right] = \left( \Delta \psi(x) + \nabla h(\frac{x}{\varepsilon}) \nabla \nabla \psi(x) \right) \chi_{\{ \varepsilon G_0^{\varepsilon} \}}(x). \]

The right-hand side here is bounded in \( L^2(B_0) \) and thus compact in \( H^{-1}(B_0) \). By the compensated compactness theorem, see \( (\text{III}) \), we obtain
\[ 0 = \lim_{\varepsilon \to 0} \int_{B_0} \nabla \Phi_{\varepsilon}(x) \cdot \left( \nabla \psi(x) + \nabla h(\frac{x}{\varepsilon}) \nabla \psi \right) \chi_{\{ \varepsilon G_0^{\varepsilon} \}}(x) dx = \int_{B_0} \nabla \Psi_0 \cdot \Theta \nabla \psi dx. \]

Since \( \Phi_0 = 0 \) on \( \partial B_0 \), this implies that \( \Phi_0 = 0 \), and \( (\text{II}) \) follows. This convergence to \( \Phi_0 = 0 \) holds for the whole family \( \Phi_{\varepsilon}^{\varepsilon} \).

The proof of other statements of Proposition \( (\text{I}) \) is now straightforward.

We turn to the proof of Proposition \( (\text{II}) \). Denote
\[ \hat{\Xi}^{\varepsilon}(x) = f_0(x) + \varepsilon(\nabla f_0(x), h(\frac{x}{\varepsilon})) + \varepsilon^2(\nabla \nabla f_0(x), g(\frac{x}{\varepsilon})) + \varepsilon^2 q(x, \frac{x}{\varepsilon}). \]
For each \((j, i)\) with \(j \in \{1, \ldots, N\}\) and \(i \in \mathbb{Z}\) we define an open set \(\varepsilon \mathcal{Q}_{\omega, ji}^\varepsilon = \{ x \in \mathbb{R}^d : \text{dist}(x, \varepsilon \partial \mathcal{G}_{ji}^\omega) < \varepsilon \mathcal{N} \}\) and introduce in this set coordinates \(y\) such that \(y' = (y_2, \ldots, y_d)\) are smooth coordinates on \(\varepsilon \partial \mathcal{G}_{ji}^\omega\), and \(y_1\) is directed along the exterior normal, \(y_1 = \text{dist}(x, \varepsilon \partial \mathcal{G}_{ji}^\omega)\) if \(x \in \varepsilon \mathcal{G}_{ji}^\omega\) and \(y_1 = \text{dist}(x, \varepsilon \partial \mathcal{G}_{ji}^\omega)\) if \(x \in \mathbb{R}^d \setminus \varepsilon \mathcal{G}_{ji}^\omega\). Under our assumptions on the geometry of \(\mathcal{G}_{ji}^\omega\) there exists \(\varepsilon > 0\) such that

- \(\varepsilon \mathcal{Q}_{\omega, ji}^\varepsilon\) do not intersect with \(\varepsilon \mathcal{Q}_{\omega, km}^\varepsilon\), if \((j, i) \neq (k, m)\).
- Coordinates \(y = y(x)\) are well defined in \(\varepsilon \mathcal{Q}_{\omega, ji}^\varepsilon\), that is \(y = y(x)\) is an invertible diffeomorphism.

Letting \(\Xi(x) = \tilde{\Xi}(x(y))\) and \(\bar{f}_j(y) = \tilde{f}_j(\tilde{x}_{ji}^\varepsilon, \frac{x(y) - \tilde{x}_{ji}^\varepsilon}{\varepsilon})\), in \(\varepsilon \mathcal{Q}_{\omega, ji}^\varepsilon \cap \varepsilon \mathcal{G}_{ji}^\omega\) we then define

\[
\varepsilon \partial^j_1(y) = \left(\Xi(0, y') - \bar{f}_j(0, y') + y_1 \left[ \frac{\partial}{\partial y_1} \Xi(0, y') - \frac{\partial}{\partial y_1} \bar{f}_j(0, y') \right] - \Xi \right) \theta \left( - \frac{y_1}{\varepsilon} \right) + \Xi,
\]

where \(\theta(s)\) is a \(C^\infty\) cut-off function such that \(0 \leq \theta \leq 1\), \(\theta = 1\) for \(s < \frac{\varepsilon}{3}\) and \(\theta = 0\) for \(s > \frac{2\varepsilon}{3}\); \(\Xi\) is the mean value of \(\Xi(x)\) over \(\varepsilon \mathcal{Q}_{\omega, ji}^\varepsilon \cap (\mathbb{R}^d \setminus \varepsilon \mathcal{G}_{ji}^\omega)\).

By construction (51) and (52) are fulfilled. Relation (53) follows from the properties of the correctors and elliptic estimates, see [7, Chapter X].

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