LOCATING-DOMINATING SETS OF FUNCTIGRAPHS

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ABSTRACT. A locating-dominating set of a graph $G$ is a dominating set of $G$ such that every vertex of $G$ outside the dominating set is uniquely identified by its neighborhood within the dominating set. The location-domination number of $G$ is the minimum cardinality of a locating-dominating set in $G$. Let $G_1$ and $G_2$ be the disjoint copies of a graph $G$ and $f: V(G_1) \to V(G_2)$ be a function. A functigraph $F^f_G$ consists of the vertex set $V(G_1) \cup V(G_2)$ and the edge set $E(G_1) \cup E(G_2) \cup \{uv: v = f(u)\}$. In this paper, we study the variation of the location-domination number in passing from $G$ to $F^f_G$ and find its sharp lower and upper bounds. We also study the location-domination number of functigraphs of the complete graphs for all possible definitions of the function $f$. We also obtain the location-domination number of functigraph of a family of spanning subgraph of the complete graphs.

1. INTRODUCTION

Locating-dominating sets were introduced by Slater [23, 25]. The initial application of locating-dominating sets was fault-diagnosis in the maintenance of multiprocessor systems [19]. The purpose of fault detection is to test the system and locate the faulty processors. Locating-dominating sets have since been extended and applied. The decision problem for locating-dominating sets for directed graphs has been shown to be an NP-complete problem [5]. A considerable literature has been developed in this field (see [2, 6, 9, 15, 17, 22, 23, 24]). In [4], it was pointed out that each locating-dominating set is both locating and dominating set. However, a set that is both locating and dominating is not necessarily a locating-dominating set.

We use $G$ to denote a connected graph with the vertex set $V(G)$ and the edge set $E(G)$. The degree of a vertex $v$ in $G$, denoted by $deg(v)$, is the number of edges to which $v$ belongs. The open neighborhood of a vertex $u$ of $G$ is $N(u) = \{v \in V(G): uv \in E(G)\}$ and the closed neighborhood of $u$ is $N[u] = N(u) \cup \{u\}$. Two vertices $u, v$ are adjacent twins if $N[u] = N[v]$ and non-adjacent twins if $N(u) = N(v)$. If $u, v$ are adjacent or non-adjacent twins, then $u, v$ are twins. A set of vertices is called a twin-set if every two distinct vertices of the set are twins.

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Formally, we define a locating-dominating set as: A subset $L_D$ of the vertices of a graph $G$ is called a locating-dominating set of $G$ if for every two distinct vertices $u, v \in V(G) \setminus L_D$, we have $\emptyset \neq N(u) \cap L_D \neq N(v) \cap L_D \neq \emptyset$. The location-domination number, denoted by $\lambda(G)$, is the minimum cardinality of a locating-dominating set of $G$.

The functigraph has its foundations back in the idea of permutation graph [7] and mapping graph [10]. A permutation graph of a graph $G$ with $n$ vertices consists of two disjoint identical copies of $G$ along with $n$ additional edges between the two copies according to a given permutation on $n$ points. In a mapping graph, the additional $n$ edges between the two copies are defined according to a given function between the vertices of the two copies. The mapping graph was rediscovered and studied by Chen et al. [8], where it was called the functigraph. Thus, a functigraph is the generalized form of permutation graph in which the function $f$ need not necessarily a permutation. In the recent past, a number of graph variants were studied for functigraphs. Eroh et al. [12] studied that how metric dimension behaves in passing from a graph to its functigraph and investigated the metric dimension of functigraphs on complete graphs and on cycles. Eroh et al. [11] investigated the domination number of functigraph of cycles in great detail, the functions which achieve the upper and lower bounds. Qi et al. [16, 21] investigated the bounds of chromatic number of functigraph. Kang et al. [18] investigated the zero forcing number of functigraphs on complete graphs, on cycles, and on paths. Fazil et al. [13, 14] have studied fixing number and distinguishing number of functigraphs. The aim of this paper is to study the variation of location-domination number in passing from a graph to its functigraph and to find its sharp lower and upper bounds.

Formally, a functigraph is defined as: Let $G_1$ and $G_2$ be the disjoint copies of a connected graph $G$ and let $f : V(G_1) \rightarrow V(G_2)$ be a function. A functigraph $F^f_G$ of the graph $G$ consists of the vertex set $V(G_1) \cup V(G_2)$ and the edge set $E(G_1) \cup E(G_2) \cup \{uv : v = f(u)\}$. Unless otherwise specified, all the graphs $G$ considered in this paper are simple, non-trivial and connected. Throughout the paper, we will denote $V(G_1) = A_1$, $V(G_2) = A_2$, $f(V(G_1)) = I$, $|I| = k$, a locating-dominating set of $F^f_G$ with the minimum cardinality by $L^*_D$, the elements of $A_1$ and $A_2$ are denoted by $u$ and $v$, respectively and each section of this paper has different labeling for the elements of $A_1$ and $A_2$.

This paper is organized as follows. Section 2 gives the sharp lower and upper bounds for the location-domination number of functigraphs. This section also establishes the connection between the location-domination number of graphs and their corresponding functigraphs in the form of realizable result. Section 3 provides the location-domination number of functigraphs of the complete graphs for all possible definitions of the function $f$. In Section 4, we investigate the location-domination
number of the functigraph of a family of spanning subgraphs of the complete graphs for all possible definitions of constant function $f$.

2. Some basic results and bounds

By the definitions of twin vertices and twin-set, we have the following straightforward result:

**Proposition 2.1.** [20] Let $T$ be a twin-set of cardinality $m \geq 2$ in a connected graph $G$. Then, every locating-dominating set $L_D$ of $G$ contains at least $m - 1$ vertices of $T$.

**Theorem 2.2.** [23] Let $G$ be a graph of order $n \geq 2$, then $\lambda(G) = n - 1$ if and only if $G = K_n$ or $G = K_1, n - 1$, where $K_n$ and $K_1, n - 1$ are the complete graph and complete bipartite graph of order $n$.

**Lemma 2.3.** Let $G$ be a graph of order $n \geq 2$ and $F^f_G$ be its corresponding functigraph. If $\lambda$ is the location-domination number of $F^f_G$, then $2n + 1 \leq 2^\lambda + \lambda$.

**Proof.** Let $L \subset V(F^f_G)$ be a non-empty set and $|L| = \lambda$. Let $v \in V(F^f_G) \setminus L$, then $N(v) \cap L$ is a subset of $L$. If $L$ is a locating-dominating set of $F^f_G$, then $N(v) \cap L$ for all $v \in V(F^f_G) \setminus L$ must be non-empty distinct subsets of $L$, which is possible only when the number of non-empty subsets of $L$ are greater than or equal to the number of vertices in $V(F^f_G) \setminus L$. Since the number of non-empty subsets of $L$ is $2^\lambda - 1$ and the number of vertices in $V(F^f_G) \setminus L$ is $2n - \lambda$. Therefore the result follows. \qed

![Figure 1. The functigraph of $K_{1,n-1}$ when $f$ is constant and $I = \{v_n\}$. Black vertices form a locating-dominating set with the minimum cardinality.](image-url)
Theorem 2.4. Let $G$ be a graph of order $n \geq 3$, then $3 \leq \lambda(F_G^I) \leq 2n - 2$. Both bounds are sharp.

Proof. Since $n \geq 3$, therefore by Lemma 2.3, $7 \leq 2^\lambda + \lambda$ which yields $3 \leq \lambda$. For the sharpness of the lower bound, let $G = P_3$ be the path graph of order 3 and $f$ be identity function, then $\lambda(F_G^I) = 3$. For the upper bound, we consider the most worse cases in which $\lambda(G) = n - 1$ and $f$ is a constant function. If $\lambda(G) = n - 1$, then by Theorem 2.2, $G$ is either $K_n$ or $K_{1,n-1}$. It is proved in Lemma 3.1 that $\lambda(F_G^I) = 2n - 3$, whenever $G = K_n$ and $f$ is a constant function. Therefore, we consider $G = K_{1,n-1}$ and $f$ is constant. Let $V(G_1) = A_1 = \{u_1, ..., u_{n-1}\} \cup \{u_n\}$ where each of the vertices $u_1, ..., u_{n-1}$ is adjacent to $u_n$. Similarly, label the corresponding vertices of $A_2 = \{v_1, ..., v_{n-1}\} \cup \{v_n\}$. We define a constant function $f : A_1 \rightarrow A_2$ by $f(u_i) = v_n$ for all $1 \leq i \leq n$. The corresponding functigraph $F_G^I$ is shown in the Figure 1. Our claim is $\lambda(F_G^I) = 2n - 2$. Since $\{u_1, ..., u_{n-1}\} \cup \{v_1, ..., v_{n-1}\}$ is a locating-dominating set of $F_G^I$, therefore $\lambda(F_G^I) \leq 2n - 2$. Let $L_D$ be a locating-dominating set of $F_G^I$. Since $F_G^I$ contains disjoint twin sets $\{u_1, ..., u_{n-1}\}$ and $\{v_1, ..., v_{n-1}\}$, therefore by Proposition 2.1, $L_D$ must contains at least $n - 2$ vertices from each of these twin sets and hence $\lambda(F_G^I) \geq 2n - 4$. Without loss of generality, assume $L_D$ contains $\{u_1, ..., u_{n-2}\}$ and $\{v_1, ..., v_{n-2}\}$ from each of these twin sets. We claim that $L_D$ contains at least two vertices from the set $B = \{u_{n-1}, u_n, v_{n-1}, v_n\}$. If $|L_D \cap B| = 0$, then $N(u_{n-1}) \cap L_D = N(v_{n-1}) \cap L_D = \emptyset$, a contradiction. If $|L_D \cap B| = 1$, then there are the following possible cases. If $L_D \cap B = \{u_{n-1}\}$ or $L_D \cap B = \{u_n\}$, then $N(u_{n-1}) \cap L_D = \emptyset$, a contradiction. If $L_D \cap B = \{v_{n-1}\}$, then $N(u_{n-1}) \cap L_D = N(v_{n-1}) \cap L_D$, a contradiction. If $L_D \cap B = \{v_n\}$, then $N(u_{n-1}) \cap L_D = \emptyset$, a contradiction. Thus, $|L_D \cap B| \geq 2$ and consequently $|L_D| \geq 2n - 2$. Hence, $\lambda(F_G^I) = 2n - 2$ and the result follows. \hfill $\Box$

Lemma 2.5. For any integer $t \geq 2$, there exist a connected graph $G$ such that $\lambda(F_G^I) - \lambda(G) = t$.

Proof. We construct the graph $G$ by taking the path graph $P_3$ and label its vertices as $u_1, u_2$ and $u_3$. Attach $t - 1$ pendants with $u_1$ and label them as $u_{1,i}$ where $1 \leq i \leq t - 1$. This completes the construction of the graph $G$. Take another copy of $G$ and label the corresponding vertices with $v_1, v_2, v_3$ and $v_{1,i}$ where $1 \leq i \leq t - 1$. Define a constant function $f : A_1 \rightarrow A_2$ which maps every vertex of $A_1$ to $v_1 \in A_2$. First we prove that $\lambda(G) = t$. Consider the set $\{u_1, u_3, u_{1,1}, ..., u_{1,t-2}\}$, then the reader can easily verify that this is a locating-dominating set of cardinality $t$ and hence $\lambda(G) \leq t$. Let $L_D$ be a locating-dominating set of $G$. Since $G$ contains $\{u_{1,1}, ..., u_{1,t-1}\}$ twin vertices, therefore by Proposition 2.1, $\lambda(G) \geq t - 2$. Without loss of generality, assume $L_D \cap \{u_{1,1}, ..., u_{1,t-1}\} = \{u_{1,1}, ..., u_{1,t-2}\}$. Our claim is $L_D$ contains at least two elements from $B = \{u_1, u_2, u_3, u_{1,t-1}\}$. If $|L_D \cap B| = 0$, then $N(u_{1,t-1}) \cap L_D = \emptyset$, a contradiction. If $|L_D \cap B| = 1$, then we discuss four possible
cases. If \( L_D \cap B = \{u_{1,t-1}\} \), then \( L_D \cap N(u_2) = \emptyset \), a contradiction. If \( L_D \cap B = \{u_1\} \), then \( L_D \cap N(u_3) = \emptyset \), a contradiction. If \( L_D \cap B = \{u_2\} \) or \( L_D \cap B = \{u_3\} \), then \( L_D \cap N(u_{1,t-1}) = \emptyset \), a contradiction. Thus, \(|L_D \cap B| \geq 2\) and consequently \(|L_D| \geq t\). Thus, \( \lambda(G) = t \).

Next we prove that \( \lambda(F_G^f) = 2t \). Consider the set \( \{u_1, u_3, u_{1,1}, \ldots, u_{1,t-2}, v_1, v_3, v_{1,1}, \ldots, v_{1,t-2}\} \), then the reader can easily verify that this is a locating-dominating set of \( F_G^f \) of cardinality \( 2t \) and hence \( \lambda(F_G^f) \leq 2t \). Since \( f \) is a constant function, therefore the sets \( \{u_{1,1}, \ldots, u_{1,t-1}\} \) and \( \{v_{1,1}, \ldots, v_{1,t-1}\} \) are also disjoint twin sets of \( F_G \) each of cardinality \( t - 1 \), therefore by Proposition 2.1, \( \lambda(F_G^f) \geq 2t - 4 \).

Let \( L_D \) be a locating-dominating set of \( F_G^f \). Without loss of generality, assume \( L_D \cap \{u_{1,1}, \ldots, u_{1,t-1}\} = \{u_{1,1}, \ldots, u_{1,t-2}\} \) and \( L_D \cap \{v_{1,1}, \ldots, v_{1,t-1}\} = \{v_{1,1}, \ldots, v_{1,t-2}\} \).

As \( f \) is a constant function, therefore by using the similar arguments as in the case of graph \( G \), the locating-dominating set \( L_D \) of \( F_G^f \) must contains at least two elements from each of the sets \( \{u_{1,1}, u_{2,3}, u_{1,t-1}\} \) and \( \{v_{1,2}, v_{3}, v_{1,t-1}\} \) and consequently, \(|L_D| \geq 2t\). Thus, \( \lambda(F_G^f) = 2t \) and the result follows.

\[\square\]

3. THE LOCATION-DOMINATION NUMBER OF FUNCTIGRAPH OF THE COMPLETE GRAPHS

We find the location-domination number of functigraph of the complete graphs for all possible definitions of the function \( f \). In this section, we use the following terminology for labeling the vertices of functigraph. Let \( G \) be a complete graph of order \( n \) and \( f : A_1 \to A_2 \) be a function. Let \( v \in I \subset A_2 \), then we denote the set \( \{f^{-1}(v)\} \subset A_1 \) by \( \Psi_v \) and its cardinality by \( s = |\Psi_v| \ (1 \leq s \leq n) \). If \( s = 1 \) for some \( v \in I \), then we name the edge \( vf^{-1}(v) \in E(F_G^f) \) as a functi matching of \( F_G^f \). The discussion has two parts, the first part discuss the cases in which \( F_G^f \) does not have any functi matching and in the second part \( F_G^f \) have at least one functi matching. For the first part of discussion, let \( F_G^f \) does not have any functi matching. In this case, we label the vertices of \( I \) as: \( I = \{v_1, v_2, \ldots, v_k\} \) where the subscript index is assigned to each \( v \) according to the index of corresponding \( s_i = |\Psi_{v_i}| \ (1 \leq i \leq k) \), where \( s_i \) are assinged indices according as \( s_1 \geq s_2 \geq \ldots \geq s_k \). The set \( A_2 \setminus I \) is a twin set of \( F_G^f \) and we denote the set by \( \Phi = A_2 \setminus I \). The vertices of \( \Phi \) are labeled as \( \Phi = \{v_{k+1}, v_{k+2}, \ldots, v_n\} \). The vertices of \( A_1 \) are labeled as: \( \Psi_{v_i} = \{u_{1,i}, u_{2,i}, \ldots, u_{s_i}\} \) and for each \( i = 2, \ldots, k \), \( \Psi_{v_i} = \{u_{i,1}, u_{i,2}, \ldots, u_{i,s_i}\} \), where the indices \( l_j \) \((j = 1, 2, \ldots, s_i)\) for fix \( i \) are given by \( l_j = \sum_{m=1}^{i-1} s_m + j \). The labeling of vertices of a functigraph of the complete graph \( K_9 \) for \( k = 3 \) is illustrated in the Figure 2(a) where the functigraph does not have any functi matching. It can be seen that for each \( i \ (1 \leq i \leq k) \), \( s_i \geq 2 \), \( \Psi_{v_i} \subset A_1 \) is a twin-set of vertices. Also, \( \cup_{i=1}^{k} \Psi_{v_i} = A_1 \) and \( \sum_{i=1}^{k} s_i = n \).
Lemma 3.1. Let \( G = K_n \) be the complete graph of order \( n \geq 2 \), and \( f : A_1 \to A_2 \) be a constant function, then

\[
\lambda(F_G^f) = \begin{cases} 
2n-2, & \text{if } n = 2 \\
2n-3, & \text{if } n \geq 3
\end{cases}
\]

Proof. For \( n = 2 \), \( F_G^f \) is the complete graph \( K_3 \) with a pendant attached with any one of the vertices of \( K_3 \). Clearly, \( \lambda(F_G^f) = 2n-2 \). For \( n \geq 3 \), \( F_G^f \) with constant \( f \) has \( I = \{v_1\} \) and using the labeling as defined earlier \( \Psi_{v_1} = \{u_1, ..., u_n\} \) and \( \Phi = \{v_2, ..., v_n\} \). Let \( L^*_D \) be a locating-dominating set of \( F_G^f \) with the minimum cardinality. By Proposition 2.1, \( |L^*_D \cap \Psi_{v_1}| \geq n - 1 \) and \( |L^*_D \cap \Phi| \geq n - 2 \). Thus, \( \lambda(F_G^f) \geq 2n-3 \). Moreover, \( A_1 \setminus \{u_n\} \cup A_2 \setminus \{v_1, v_n\} \) is locating-dominating set of \( F_G^f \), and hence \( \lambda(F_G^f) \leq 2n-3 \) and the result follows.

Theorem 3.2. Let \( G \) be the complete graph of order \( n \geq 4 \) and \( F_G^f \) does not have functi matchings. If \( 1 < k < n \), then \( \lambda(F_G^f) = 2n-k-2 \)

Proof. For \( 1 < k < n \), \( I = \{v_1, v_2, ..., v_k\} \). First we prove that the set \( L = \{\cup_{i=1}^k \Psi_{v_i} \setminus \{u_{s_i}\}\} \cup \{\Phi \setminus \{v_n\}\} \) is a locating-dominating set of \( F_G^f \) with the cardinality \( |L| = \sum_{i=1}^k s_i - 1 + (n-k-1) = 2n-k-2 \). Since \( V(F_G^f) \setminus L = \{u_{s_1}, v_1, ..., v_k, v_n\} \). We prove that all the elements of \( V(F_G^f) \setminus L \) have distinct non-empty neighbors in \( L \).

Now, \( N(u_{s_i}) \cap L = \cup_{i=1}^k \Psi_{v_i} \setminus \{u_{s_i}\} \setminus \{v_n\}, N(v_i) \cap L = \{\Psi_{v_i} \setminus \{u_{s_i}\}\} \cup \{\Phi \setminus \{v_n\}\}, \) for each \( i \) where \( 2 \leq i \leq k \). \( N(v_1) \cap L = \Psi_{v_1} \setminus \{u_{s_1}\} \cup \{\Phi \setminus \{v_n\}\}, \) \( N(v_n) \cap L = \Phi \setminus \{v_n\} \). Thus, \( L \) is a locating-dominating set of \( F_G^f \). Hence, \( \lambda(F_G^f) \leq 2n-k-2 \). Let \( L^*_D \) be a locating-dominating set of \( F_G \) with the minimum cardinality. Then by Proposition 2.1, \( L^*_D \)
must contains $s_i - 1$ vertices of the disjoint twin sets $\Psi_{v_i}$ for each $i$, $1 < i \leq k$ and $n - k - 1$ vertices of the twin set $\Phi$. Therefore, $\lambda(F^f_G) \geq 2n - 2k - 1$. Without loss of generality, assume the set $L_D \cap \Psi_{v_i} = \Psi_{v_i} \setminus \{u_{s_i}\}$, $L_D \cap \Psi_{v_i} = \Psi_{v_i} \setminus \{u_{t_i}\}$ for each $i$ ($2 \leq i \leq k$) and $L_D^* \cap \Phi = \Phi \setminus \{v_n\}$. Now consider the two element sets $\{u_{s_1}, v_i\}$ and $\{u_{t_1}, v_i\}$ ($2 \leq i \leq k$). We claim that $L_D^*$ contains atleast one element from exactly $k - 1$ sets of these two element sets. Consider $u_{s_1}, v_1 \notin L_D^*$. Next we prove that one vertex from $\{u_{s_i}, v_i\}$ ($2 \leq i \leq k$) must belongs to $L_D^*$ for all $i$ ($2 \leq i \leq k$). Suppose on contrary that both $v_i$ and $u_{s_i}$ do not belong to $L_D^*$ for some $i$ ($2 \leq i \leq k$). Then $N(u_{s_i}) \cap L_D^* = N(u_{s_i}) \cap L_D^*$, a contradiction. Similarly, by considering $u_{t_i}, v_i \notin L_D^*$ for some $i$ ($2 \leq i \leq k$) and using similar arguments we leads to a contradiction. Thus, $L_D^*$ must contains atleast one element from exactly $k - 1$ sets of these two element sets. Consequently, $|L_D^*| \geq 2n - k - 2$. Hence, $\lambda(F^f_G) = 2n - k - 2$.

For the second part of discussion, let $F^f_G$ has atleast one functi matching. In this case, we label the vertices of $I$ as: $I = \{v_1, v_2, ..., v_{k'}, v_{k'+1}, ..., v_{k}\}$ where ($1 \leq k' < k$) and the subscript index is assigned to each $v$ according to the index of corresponding $s_i$ ($1 \leq i \leq k$), where $s_i$ are assigned indices according as $s_1 \geq s_2 \geq ... \geq s_{k'} > s_{k'+1} = ... = s_k = 1$. Notations of $\Phi$ and $\Psi_{v_i}$ ($1 \leq i \leq k')$ are same as used earlier. The labeling of vertices of a functigraph of the complete graph $K_6$ for $k = 6$ and $k' = 2$ is illustrated in the Figure 2(b) where the functigraph has four functi matchings. It can be seen that for each $i$ ($k'+1 \leq i \leq k$), $s_i = 1$ and $\Psi_{v_i} = \{u_{s_i}, v_i\}$ and the edge $u_{s_i}v_i \in E(F^f_G)$ is a functi matching of $F^f_G$.

**Lemma 3.3.** Let $G = K_n$ be the complete graph of order $n \geq 2$ and $f : A_1 \rightarrow A_2$ be a bijective function, then

$$\lambda(F^f_G) = \begin{cases} n, & \text{if } n = 2, 3 \\ n-1, & \text{if } n \geq 4 \end{cases}$$

**Proof.** For $n = 2$, $F^f_G$ is a cyclic graph of order 4 and hence $\lambda(F^f_G) = 2$ by [3]. For $n = 3$, $F^f_G$ is a triangular prism of order 6 and $\lambda(F^f_G) \geq 3$ by Lemma 2.3 whereas $A_1$ is a locating-dominating set of $F^f_G$ with the cardinality 3. For $n \geq 4$, we have the labeling as defined earlier $I = A_2 = \{v_1, ..., v_n\}$ and $\Psi_{v_i} = \{u_i\}$ for all $i$ ($1 \leq i \leq n$) and hence $A_1 = \{u_1, ..., u_n\}$. Consider the set $L = \{u_1, ..., u_{n-2}, v_n\}$. We prove that $L$ forms a locating-dominating set of $F^f_G$. As $N(u_{n-1}) \cap L = \{u_1, ..., u_{n-2}\}$, $N(u_n) \cap L = \{u_1, ..., u_{n-2}, v_n\}$. Also, $N(v_i) \cap L = \{u_i, v_n\}$ for all $1 \leq i \leq n-2$ and $N(v_{n-1}) \cap L = \{v_n\}$. Thus, $L$ forms a locating-dominating set of $F^f_G$ and $\lambda(F^f_G) \leq n-1$. Next we prove that $\lambda(F^f_G) \geq n-1$. Suppose on contrary there exist a locating-dominating set $L_D$ of cardinality $n - 2$, then either $|L_D \cap A_1| \leq n - 2$ or $|L_D \cap A_2| \leq n - 2$. Assume $|L_D \cap A_1| = n - 2 - j$ and $|L_D \cap A_2| = j$ where $0 \leq j \leq n - 2$. Since $u_i, v_i$ forms functi matching of $F^f_G$ for all $i$ ($1 \leq i \leq n$). Therefore without loss of generality assume that $L_D \cap A_1 = \{u_1, u_2, ..., u_{n-2-j}\}$. If $L_D$ is a locating-dominating set of $F^f_G$, then the vertices of $A_1 \setminus L_D = \{u_{n-1-j}, u_{n-j}, ..., u_n\}$ must have distinct
neighbors in \( L_D \) which is possible only \( |L_D \cap A_2| = j + 2 \), a contradiction. Thus, \( \lambda(F_G^f) \geq n - 1 \) and the result follows. \( \square \)

**Theorem 3.4.** Let \( G \) be the complete graph of order \( n \geq 3 \) and \( F_G^f \) has at least one matching. If \( 1 < k < n \), then

\[
\lambda(F_G^f) = \begin{cases} 
2n - k - 1, & \text{if } n = 3, \ k = 2, \\
2n - k - 2, & \text{if } n \geq 4, \ k \geq 2. 
\end{cases}
\]

**Proof.** (i) For \( n = 3 \) and \( k = 2 \), let \( f : A_1 \to A_2 \) be defined as \( f(u_i) = v_1 \), where \( i = 1, 2 \) and \( f(u_3) = v_2 \). By Lemma 2.3, \( \lambda(F_G^f) > 2 \). Moreover, \( \{u_1, u_3, v_2\} \) forms a locating-dominating set of \( F_G^f \). Thus, \( \lambda(F_G^f) = 3 \).

(ii) For \( n \geq 4 \) and \( 2 \leq k \leq n - 1 \). Since \( F_G^f \) has matching, therefore there exists a \( k' \) \((1 \leq k' < k)\) such that \( s_i = 1 \) for all \( i \) \((k' + 1 \leq i \leq k)\). Thus we use the labeling as described earlier for the vertices of \( A_1 \) and \( A_2 \). Let \( L_D^* \) be a locating-dominating set of \( F_G^* \) with the minimum cardinality. The proof consists of the following claims:

1. **Claim.** The set \( L = \bigcup_{i=1}^{k'} \Psi_{v_i} \setminus \{u_{s_i}\} \cup \{\Phi \setminus \{v_n\}\} \) is a locating-dominating set of the cardinality \( |L| = \sum_{i=1}^{k'} s_i - 1 + (n - \sum_{i=1}^{k'} s_i) + (n - k - 1) = 2n - k - 2 \).
   **Proof of claim:** Since \( V(F_G^f) \setminus L = \{u_{s_1}, u_{s_{k'}} + k, v_1, \ldots, v_{k'}, v_{k'+1}, \ldots, v_{k+1}, v_n\} \), we prove that all the elements of \( V(F_G^f) \setminus L \) have distinct neighbors in \( L \).

Now, \( N(u_{s_1}) \cap L = A_1 \setminus \{u_{s_1}, u_{s_{k'}} + k\}, N(u_{s_{k'}} + k) \cap L = A_1 \setminus \{u_{s_1}, u_{s_{k'}} + k\} \cup \{v_k\}, N(v_1) \cap L = \Psi_{v_1} \setminus \{u_{s_1}\} \cup \{v_k\} \cup \{\Phi \setminus \{v_n\}\}, \) for each \( i \) where \( 2 \leq i \leq k' \), \( N(v_i) \cap L = \Psi_{v_i} \cup \{v_k\} \cup \{\Phi \setminus \{v_n\}\}, \) for each \( i \) where \( k' + 1 \leq i \leq k - 1 \), \( N(v_n) \cap L = \{u_{i_{s_i} + i-k'}\} \cup \{v_k\} \cup \{\Phi \setminus \{v_n\}\}, \) and \( N(v_{n'}) \cap L = \{v_k\} \cup \{\Phi \setminus \{v_n\}\} \).

Thus, \( L \) is a locating-dominating set of \( F_G^f \) and hence, \( \lambda(F_G^f) \leq 2n - k - 2 \).

2. **Claim.** \( \lambda(F_G^f) \geq n + \sum_{i=1}^{k'} s_i - k - 2 \).
   **Proof of claim:** By Proposition 2.4, \( L_D^* \) must contains \( s_i - 1 \) vertices of the disjoint twin sets \( \Psi_{v_i} \) for each \( i, 1 < i \leq k' \) and \( n - k - 1 \) vertices of the twin set \( \Phi \). Therefore, \( \lambda(F_G^f) \geq \sum_{i=1}^{k'} (s_i - 1) + (n - k - 1) = n + \sum_{i=1}^{k'} s_i - k' - k - 1 \). Without loss of generality, assume the set \( L_D^* \cap \Psi_{v_i} = \Psi_{v_i} \setminus \{u_{s_i}\}, L_D^* \cap \Psi_{v_{i}} = \Psi_{v_{i}} \setminus \{u_{i_{s_i}}\} \) for each \( i \) \((2 \leq i \leq k')\) and \( L_D^* \cap \Phi = \Phi \setminus \{v_n\} \).

Now consider the two element sets \( \{u_{s_1}, v_i\} \) and \( \{u_{s_i}, v_i\} \) \((2 \leq i \leq k')\). We claim that \( L_D^* \) must contains at least one element from exactly \( k' - 1 \) sets of these two element sets. Consider \( \{u_{s_1}, v_i\} \not\subset L_D^* \). We prove that one vertex from \( \{u_{i_{s_i}}, v_i\} \) \((2 \leq i \leq k')\) must belong to \( L_D^* \) for all \( i \) \((2 \leq i \leq k')\). Suppose on contrary that both \( u_{i_{s_i}} \) and \( v_i \) do not belong to \( L_D^* \) for some \( i \) \((2 \leq i \leq k')\). Then \( N(u_{i_{s_i}}) \cap L_D^* = N(u_{s_1}) \cap L_D^*, \) a contradiction. Similarly, by considering \( \{u_{i_{s_i}}, v_i\} \not\subset L_D^* \) for some \( i \) \((2 \leq i \leq k')\) and using the similar arguments we lead to a contradiction. Thus \( L_D^* \) must contains at least one
element from exactly $k' - 1$ sets of these two element sets. Consequently, 
\[ \lambda(F_G^f) \geq n + \frac{k'}{2} \sum_{i=1}^{k'} s_i - k - 2. \]

(3) **Claim.** \[ |L_D^* \cap \{u_{i,k'+1}, u_{i,k'+2}, \ldots, u_k, v_{k'+1}, v_{k'+2}, \ldots, v_k\}| = n - \frac{k'}{2} \sum_{i=1}^{k'} s_i. \]

**Proof of claim:** As \( u_{i,k'+i}v_{k'+i} \in E(F_G^f) \) for each \( i \) (1 \( \leq \) \( i \) \( \leq \) \( k' \)) form the functi matchings of \( F_G^f \). We take the assumptions that we have proved in Claim 2 that \( L_D^* \cap \Psi_v = \Psi_v \setminus \{u_{s_1}\} \), \( L_D^* \cap L_\phi = L_\phi \setminus \{v_{k}\} \) for each \( i \) (2 \( \leq \) \( i \) \( \leq \) \( k' \)) and \( L_D^* \cap \Phi = \Phi \setminus \{v_{n}\} \). Now consider the two element sets \( \{u_{i,k'+i}, v_i\} \) \( (k'+1 \leq i \leq k) \) as the sets of two end vertices of the functi matchings of \( F_G^f \).

We prove that \( L_D^* \) must contains exactly one element from each of these \( k-k' \) sets. Suppose on contrary \( \{u_{i,k'+i}, v_i\} \not\subseteq L_D^* \) for some \( i \) (\( k'+1 \leq i \leq k \)), then \( N(u_{i,k'+i}) \cap L_D^* = N(u_{i,k'+i}) \cap L_D^* \), a contradiction. Thus either \( u_{i,k'+i} \) or \( v_i \) for each \( i \) (\( k'+1 \leq i \leq k \)) (not both to maintain the minimality of \( L_D^* \)) must belong to \( L_D^* \) to make distinct neighborhood of \( u_{i,k'+i} \) in \( L_D^* \). Hence, 
\[ |L_D^* \cap \{u_{i,k'+1}, u_{i,k'+2}, \ldots, u_k, v_{k'+1}, v_{k'+2}, \ldots, v_k\}| = k - k' = n - \frac{k'}{2} \sum_{i=1}^{k'} s_i. \]

Since the sets used to prove Claim 2 and Claim 3 are disjoint subsets of \( V(F_G^f) \), therefore combining Claim 2 and Claim 3 we get \( \lambda(F_G^f) \geq 2n - k - 2 \). Combining this with Claim 1 we get the required result. \[ \square \]

**Corollary 3.5.** Let \( G \) be the complete graph of order \( n \geq 4 \) and let \( F_G^f \) contains \( p \) functi matchings, then \( \lambda(F_G^f) \geq p. \) The bound is sharp.

**Proof.** The result follows from proof of Claim 3 of Theorem 3.4. Sharpness of the bound follows from Lemma 3.3 where \( f \) is a bijective function. \[ \square \]

**Corollary 3.6.** Let \( G \) be the complete graph of order \( n \geq 4 \). Then \( \lambda(G) = \lambda(F_G^f) \) if and only if \( k = n - 1 \).

**Proof.** The result follows by Theorem 3.4 for \( n \geq 4 \). \[ \square \]

4. **Location-Domination Number of Functigraph of a family of spanning subgraphs of the Complete Graphs**

A vertex \( u \in V(G) \) is called a saturated vertex, if \( \deg(u) = |V(G)| - 1 \). Since any two saturated vertices are adjacent twins, therefore the set of all saturated vertices of a graph forms a twin set represented by \( T^s \). Let \( e' \in E(G) \) be an edge that joins two saturated vertices of \( G \), then the spanning subgraph of \( G \) which is obtained by removing the edge \( e' \) is denoted by \( G - e' \). Similarly, \( H_i = G - ie' \) \( (1 \leq i \leq \lfloor \frac{n}{2} \rfloor) \) denotes a spanning subgraph of \( G \) that is obtained by removing \( i \) edges \( e' \), where \( e' \) joins two saturated vertices of \( G \). It may also be noted that after removing the edge \( e' \), the two saturated vertices that are connected by \( e' \), are converted to non-adjacent twins and hence forms a twin set of cardinality 2. The twin set obtained after removing the \( i \)th edge \( e' \) is denoted by \( T_i^s \), \( (1 \leq i \leq \lfloor \frac{n}{2} \rfloor) \). Thus, \( H_i \) \( (1 \leq i \leq \lfloor \frac{n}{2} \rfloor) \) has \( i \) twin sets \( T_i^s \) of non-adjacent twins, each of cardinality
Further, if $n$ is even and $i = \frac{n}{2}$, then $T^s = \emptyset$. We label the vertices in $H_i$ as follows: 
$T^s_i = \{u_{i}^{1}, u_{i}^{2}\} \ (1 \leq i \leq \lfloor \frac{n}{2} \rfloor)$ and $T^s = \{u_{2i+1}, u_{2i+2}, ..., u_n\} \ (1 \leq i < \frac{n}{2})$. The following theorem gives location-domination number of functigraph of $G - i\varepsilon$.

Lemma 4.1. Let $G$ be the complete graph of order $n = 4$ and $f$ be a constant function, then $\lambda(F_{H_i}^f) = 4 \ (1 \leq i \leq 2)$.

Proof. Since $f$ is a constant function, therefore assume that $I = \{v\} \subset A_2$. If $v \in T^s_j \subset A_2$ for some $j \ (1 \leq j \leq 2)$, then $T^s_j \subset A_2$ is not twin set in $F_{H_i}^f$. Similarly if $v \in T^s \subset A_2$, then $T^s \subset A_2$ is not a twin set in $F_{H_i}^f$. Let a set $L \subset V(F_{H_i}^f)$, such that $L$ has exactly one element from each twin set of $F_{H_i}^f$. We discuss the following cases.

(1) If $i = 1$ and $v \in T^s_i \subset A_2$, then without loss of generality assume $v = v^1_1$. Also, the sets $T^s_i, T^s \subset A_1$ and $T^s \subset A_2$ are twin sets in the corresponding $F_{H_i}^f$ each with the cardinality 2. Thus by Proposition 2.1, $\lambda(F_{H_i}^f) \geq 3$. There are 8 possible choices for the set $L$. If we take $L = \{u^1_1, u_3, v_3\}$, then $N(v^1_2) \cap L = N(v_4) \cap L$ which implies that $L$ is not a locating-dominating set. Similarly, the other 7 choices for $L$ do not form locating-dominating set and hence, $\lambda(F_{H_i}^f) \geq 4$. Also, the set $\{u^1_1, u_3, v_3, v_4\}$ forms a locating-dominating set of $F_{H_i}^f$. Hence, the result follows.

(2) If $i = 1$ and $v \in T^s \subset A_2$, then without loss of generality assume $v = v^1_3$. The corresponding functigraph $F_{H_i}^f$ has twin sets $T^s, T^s_i \subset A_1$ and $T^s_i \subset A_2$ each of the cardinality 2. Then it can be seen that 8 possible choices of the set $L$ do not form locating-dominating set of $F_{H_i}^f$, therefore $\lambda(F_{H_i}^f) \geq 4$. Also, the set $\{u^1_1, u_3, v^1_1, v_4\}$ forms a locating-dominating set of $F_{H_i}^f$.

(3) If $i = 2$ and $v \in T^s \subset A_2$, then $T^s = \emptyset$. Without loss of generality assume $v = v^1_1$. The corresponding functigraph $F_{H_2}^f$ has twin sets $T^s, T^s_i \subset A_1$ and $T^s \subset A_2$ each of the cardinality 2. By Proposition 2.1, $\lambda(F_{H_2}^f) \geq 3$. There are 8 possible choices for the set $L$. If $L = \{u^1_1, u_2, v^1_2\}$, then $N(v^1_2) \cap L = \emptyset$. If $L = \{u^1_1, u_2, v^1_2\}$, then $N(v^1_2) \cap L = \emptyset$. Thus, $L$ is not locating-dominating set. Similarly, the remaining 6 choices for the set $L$ do not form locating-dominating set. Hence, $\lambda(F_{H_2}^f) \geq 4$. Also the set $\{u^1_1, u^1_2, v^1_2, v^1_3\}$ forms a locating-dominating set of $F_{H_2}^f$. The case when $i = 2$ and $v \in T^s \subset A_2$ can also be proved by the similar arguments.
Figure 3. The labeling of the vertices of $F^f_{H_i}$ when $n = 7$, $i = 2$, $f$ is a constant function, $I = \{v\}$ and $v$ is a non-saturated vertex of $A_2$. Black vertices form a locating-dominating set of $F^f_{H_i}$ with the minimum cardinality.

**Theorem 4.2.** Let $G$ be the complete graph of order $n \geq 5$ and $f$ be a constant function such that $I = \{v\} \subset A_2$, then

$$\lambda(F^f_{H_i}) = \begin{cases} 
2n - 2i - 3, & \text{if } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1 \text{ and } v \text{ is a saturated vertex of } A_2 \\
2n - 2i - 2, & \text{if } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1 \text{ and } v \text{ is a non-saturated vertex of } A_2 \\
n - 1 & \text{if } n \text{ is even and } i = \frac{n}{2} \\
2\left\lfloor \frac{n}{2} \right\rfloor & \text{if } n \text{ is odd and } i = \left\lfloor \frac{n}{2} \right\rfloor.
\end{cases}$$

**Proof.** Since $f$ is a constant function, therefore the collection $\{T_1^f, T_2^f, \ldots, T_i^f, T^s\}$ of twin subset of $A_2$ are also twin sets in the corresponding $F^f_{H_i}$. If $v \in T_j^f \subset A_2$ for some $j$ ($1 \leq j \leq i$), then $T_j^f \subset A_2$ is not twin set in $F^f_{H_i}$. Similarly if $v \in T^s \subset A_2$, then $T^s \subset A_2$ is not twin set in $F^f_{H_i}$. We discuss the following cases

1. When $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$ and $v \in T_j^f \subset A_2$ for some $j$ ($1 \leq j \leq i$). Then without loss of generality assume that $v = v_j^1$. The corresponding $F^f_{H_i}$ have twin sets $T_1^f, T_2^f, \ldots, T_i^f, T^s \subset A_1$ and $T_1^f, \ldots, T_{j-1}^f, T_j^f, T_{j+1}^f, \ldots, T_i^f, T^s \subset A_2$. By Proposition 2.1 $\lambda(F^f_{H_i}) \geq 2i - 1 + 2(n - 2i - 1) = 2n - 2i - 3$. Let a set $L \subset V(F^f_{H_i})$, such that $L$ has all the elements except one element of each of these twin subsets. There are $2^{2i-1}(n - 2i - 1)^2$ choices for choosing the elements of the set $L$. Each choice for the set $L$ does not contain an element of the set $T^s \subset A_2$. Without loss of generality assume that $v_n \not\in L$, 

then $N(v_f^2) \cap L = N(v_n) \cap L$. Thus, $L$ is not a locating-dominating set of $F^t_{H_i}$ for all choices of the set $L$. Hence, $\lambda(F^t_{H_i}) \geq 2n - 2i - 2$. Also the set $\{u_1, u_2, \ldots, u_{n-1}, v_1, v_1, v_2, \ldots, v_{j-1}, v_j, v_{j+1}, \ldots, v_n\}$ is a locating-dominating set with cardinality $2n - 2i - 2$. Thus $\lambda(F^t_{H_i}) = 2n - 2i - 2$.

(2) When $1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1$ and $v \in T^s \subset A_2$. Then without loss of generality assume that $v = v_n$. The corresponding $F^t_{H_i}$ have twin sets $T_1^t, T_2^t, \ldots, T_i^t, T^s \setminus \{v_n\} \subset A_1$ and $T_1^t, \ldots, T_i^t, T^s \setminus \{v_n\} \subset A_2$. By Proposition 2.1 $\lambda(F^t_{H_i}) \geq 2i + (n - 2i - 1) + (n - 2i - 2) = 2n - 2i - 3$. Also the set $L = \{u_1, u_2, \ldots, u_{n-1}, v_1, v_2, \ldots, v_{j-1}, v_j, v_{j+1}, \ldots, v_n\}$ is a locating-dominating set of $F^t_{H_i}$ with the cardinality $2n - 2i - 3$. Thus $\lambda(F^t_{H_i}) = 2n - 2i - 3$.

(3) When $i = \frac{n}{2}$ and $n$ is even, then $T^s = \emptyset$ and $v \in T_j^t \subset A_2$ for some $j$ ($1 \leq j \leq i$). Then without loss of generality assume that $v = v_j$. The corresponding $F^t_{H_i}$ have twin sets $T_1^t, T_2^t, \ldots, T_i^t \subset A_1$ and $T_1^t, \ldots, T_j^t, T_{j+1}^t, \ldots, T_i^t \subset A_2$. By Proposition 2.1 $\lambda(F^t_{H_i}) \geq i + (i - 1) = n - 1$. Also, the set $L = \{u_1, u_2, \ldots, u_{n-1}, v_1, v_2, \ldots, v_{j-1}, v_j, v_{j+1}, \ldots, v_1\}$ is a locating-dominating set of $F^t_{H_i}$ with cardinality $n - 1$. Thus $\lambda(F^t_{H_i}) = n - 1$.

(4) When $i = \lfloor \frac{n}{2} \rfloor$, $n$ is odd and $v \in T_j^t \subset A_2$ for some $j$ ($1 \leq j \leq i$). In this case, $T^s = \{v_n\}$ is not a twin set. Without loss of generality assume that $v = v_j$. The corresponding $F^t_{H_i}$ have twin sets $T_1^t, T_2^t, \ldots, T_i^t \subset A_1$ and $T_1^t, \ldots, T_{j-1}^t, T_{j+1}^t, \ldots, T_i^t \subset A_2$. By Proposition 2.1 $\lambda(F^t_{H_i}) \geq i + (i - 1) = 2i - 1$. Let a set $L \subset V(F^t_{H_i})$, such that $L$ has all the elements except one element of each of these twin subsets. Each choice for the set $L$ does not contain $v_n$. Then, $N(v_j^2) \cap L = N(v_n) \cap L$ for all choices of the set $L$. Thus, $L$ is not a locating-dominating set of $F^t_{H_i}$ for all choices of the set $L$ and hence, $\lambda(F^t_{H_i}) \geq 2i$. Also, the set $\{u_1, u_2, \ldots, u_{n-1}, v_1, v_2, \ldots, v_{j-1}, v_j, v_{j+1}, \ldots, v_1\}$ is a locating-dominating set with cardinality $2i$. Thus $\lambda(F^t_{H_i}) = 2\lfloor \frac{n}{2} \rfloor$.

(5) When $i = \lceil \frac{n}{2} \rceil$, $n$ is odd and $v \in T^s \subset A_2$. In this case, $T^s = \{v_n\}$ is not a twin set and $v = v_n$. The corresponding $F^t_{H_i}$ have twin sets $T_1^t, T_2^t, \ldots, T_i^t \subset A_1$ and $T_1^t, \ldots, T_i^t \subset A_2$. By Proposition 2.1 $\lambda(F^t_{H_i}) \geq 2i$. Also, the set $L = \{u_1, u_2, \ldots, u_{n-1}, v_1, v_2, \ldots, v_1\}$ is a locating-dominating set with cardinality $2i$. Thus $\lambda(F^t_{H_i}) = 2\lceil \frac{n}{2} \rceil$.

□

References

[1] T. Y. Berger-Wolf, W. E. Hart and J. Saia, Discrete sensor placement problems in distribution networks, J. Math. Comp. Modeling, 42(13)(2005), 1385-1396.

[2] N. Bertrand, I. Charon, O. Hudry and A. Lobstein, Identifying and locating-dominating codes on chains and cycles, European J. Combin., 25(2004), 969-987.
[3] J. Cáceres, C. Hernando, M. Mora, I. M. Pelayo and M. L. Puertas, On locating and dominating sets in graphs, *Workshop de Matemática Discreta Algarve/Andaluca–VI Encuentro Andaluz de Matemática Discreta*, (2009), 19-22.

[4] J. Cáceres, C. Hernando, M. Mora, I. M. Pelayo and M. L. Puertas, Locating dominating codes, *Appl. Math. Comput.*, 220(2013), 38-45.

[5] I. Charon, O. Hudry and A. Lobstein, Identifying and locating-dominating codes: NP-completeness results for directed graphs, *IEEE Trans. Inform. Theory*, 48(2002), 2192-2200.

[6] I. Charon, O. Hudry and A. Lobstein, Minimizing the size of an identifying or locating-dominating code in a graph is NP-hard, *Theor. Comput. Sci.*, 290(2003), 2109-2120.

[7] G. Chartrand and F. Harary, Planar permutation graphs, *Ann. Inst. H. Poincare*, 3(1967), 433-438.

[8] A. Chen, D. Ferrero, R. Gera and E. Yi, Functigraphs: An extension of permutation graphs, *Math. Bohem.*, 136(1)(2011), 27-37.

[9] C. J. Colbourn, P. J. Slater and L. K. Stewart, Locating-dominating sets in series parallel networks, *Congr. Numer.*, 56(1987), 135-162.

[10] W. Dorfler, On mapping graphs and permutation graphs, *Math. Slovaca*, 28(3)(1978), 277-288.

[11] L. Eroh, R. Gera, C. X. Kang, C. E. Larson and E. Yi, Domination in functigraphs, *arXiv preprint arXiv:1106.1147*.

[12] L. Eroh, C. X. Kang and E. Yi, On metric dimension of functigraphs, *Discrete Math., Alg. and Appl.*, 5(04) (2013), 1250060.

[13] M. Fazil, I. Javaid and M. Murtaza, On fixing number of functigraphs, *arXiv preprint arXiv:1611.03346*.

[14] M. Fazil, M. Mutaza, U. Ali and I. Javaid, On distinguishing number of functigraphs, *arXiv preprint arXiv:1612.00971*.

[15] A. Finbow and B. L. Hartnell, On locating-dominating sets and well-covered graphs, *Congr. Numer.*, 56(1987), 135-162.

[16] W. Gu and G. Qi, Attainability of the chromatic number of functigraphs, *In Pervasive Sys., Alg. and Net. (ISPAN), 2012 12th International Symposium*, (2012), 143-148.

[17] I. Honkala, T. Laihonen and S. Ranto, On locating-dominating codes in binary hamming spaces, *Disc. Math. Theor. Comput. Sci.*, 6(2004), 265-282.

[18] C. X. Kang and E. Yi, On zero forcing number of functigraphs, *arXiv preprint arXiv:1204.2238*.

[19] M. G. Karpovsky, K. Chakrabarty and L. B. Levitin, On a new class of codes for identifying vertices in graphs, *IEEE Transactions on Information Theory* 44(1998), 599-611.

[20] M. Murtaza, I. Javaid and M. Fazil, Locating-dominating sets and identifying codes of a graph associated to a finite vector space, *arXiv preprint arXiv:1701.08537*.

[21] G. Qi, S. Wang and W. Gu, On the chromatic number of functigraphs, *J. of Interconn. Net.*, 13(2012), 1250011.

[22] D. F. Rall and P. J. Slater, On location-dominination numbers for certain classes of graphs, *Congr. Numer.*, 45(1984), 97-106.

[23] P. J. Slater, Dominating and reference sets in a graph, *J. Math. Phys. Sci.*, 22(1988), 445-455.

[24] P. J. Slater, Fault-tolerant locating-dominating sets, *Disc. Math.*, 249(2002), 179-189.

[25] P. J. Slater, Domination and location in acyclic graphs, *Networks*, 17(1987), 55-64.
