Numerical Calculation of Hubble Hierarchy Parameters and Observational Parameters of Inflation

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Abstract

We present results obtained by a software we developed for computing observational cosmological inflation parameters: the scalar spectral index ($n_s$) and the tensor-to-scalar ratio ($r$) for a standard single field and tachyon inflation, as well as for a tachyon inflation in the second Randall-Sundrum model with an additional radion field. The calculated numerical values of observational parameters are compared with the latest results of observations obtained by the Planck Collaboration. The program is written in C/C++. The \textit{GNU Scientific Library} is used for some of the numerical computations and R language is used for data analysis and plots.

1 Introduction

The inflation theory proposes a period of extremely rapid (exponential) expansion of the universe during the very early stage of the universe. Although inflationary cosmology has successfully complemented the Standard Model, the process of inflation, in particular its origin, is still largely unknown. Over the past 35 years, numerous models of inflationary expansion of the universe have been proposed. The cosmological inflation is a process in which the size of the universe has increased exponentially at least $e^{60} \approx 10^{26}$ times. The simplest model of inflation is based on the existence of a single scalar field called inflaton. It is possible to build models with other types of fields, for example vector fields, however they have been less used \cite{1, 2}. One of the most important ways to test inflationary cosmological models is to compare the computed with the measured values of the observational parameters: the scalar spectral index ($n_s$) and the tensor-to-scalar power ratio ($r$) \cite{3, 4}. Although the analytical methods for approximate calculation of

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observation parameters is well known there is a drawback: this procedure is often not possible to use for complex models, with a complicated form of the potential, described by a non-standard Lagrangian [5, 6].

In this paper we present a quite general method for numerical calculation of the Hubble hierarchy parameters $\varepsilon_i$ and the observational parameters of inflation that can be used to validate inflationary models. Using this method, the observational parameters for an inflationary model with a tachyon scalar field, described by a non-standard Lagrangian of the Dirac-Born-Infeld (DBI) type, were calculated. The role of the tachyon field in a braneworld cosmology (Randall-Sundrum II model) was subsequently analyzed [7, 8].

2 Inflationary models

The dynamics of a classical real scalar field ($\varphi$) minimally coupled with gravity is given by

$$S = -\frac{1}{16\pi G} \int \sqrt{-g} R d^4x + \int \sqrt{-g} \mathcal{L}(X, \varphi) d^4x,$$

(1)

where $G$ is gravitational constant, $R$ is the Ricci scalar, $g$ is the determinant of the metric tensor, and $\mathcal{L}(X, \varphi)$ is the Lagrangian, with kinetic term $X \equiv \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi$. We will assume the spatially 4-dimensional flat space-time with the standard metric

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = dt^2 - a^2(t)(dr^2 + r^2 d\Omega^2).$$

(2)

In a general case the Lagrangian of a scalar field can be written as an arbitrary function of a scalar $\varphi$ field and a kinetic term $X$. Based on the form of the Lagrangian, several types can be distinguished [9].

2.1 Tachyon inflation

The software we developed can be used for a wide range of different inflationary models. In this paper we will mainly discuss the models in which inflation is driven by a tachyon field ($\theta$) originating in the string theory, described by the Lagrangian of the DBI type and the corresponding Hamiltonian [10, 11]

$$p \equiv \mathcal{L}(X, \theta) = -V(\theta) \sqrt{1 - 2X} = -V(\theta) \sqrt{1 - \dot{\theta}^2},$$

$$\rho \equiv \mathcal{H} = \frac{V(\theta)}{\sqrt{1 - \dot{\theta}^2}},$$

(3)

where $V(\theta)$ is a tachyon potential which satisfies the following properties [12, 13]

$$V(0) = \text{const}, \quad V'(\theta > 0) < 0, \quad V(|\theta| \to \infty) \to 0.$$

(4)

In this case Friedman equation takes the form

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \mathcal{H} = \frac{8\pi G}{3} \frac{V(\theta)}{\sqrt{1 - \dot{\theta}^2}},$$

(5)

where $H$ is the Hubble expansion rate and $a$ is the scale factor.
Dynamics of inflation can be described in the two equivalent ways: using the energy-momentum conservation equation or Hamilton’s equations. For a tachyonic potential $V(\theta)$ the energy-momentum conservation equation reads

$$\dot{\rho} = -3H(P + \rho) \Rightarrow \frac{\dot{\theta}}{1 - \theta^2} + 3H\dot{\theta} + \frac{1}{V} \frac{\partial V}{\partial \theta} = 0,$$

(6)

where is $V' = \partial V/\partial \theta$.

Instead of the energy-momentum conservation equation, the system can be also described by the Hamiltons equations

$$\dot{\theta} = \frac{\partial H}{\partial \pi_\theta},$$

$$\dot{\pi}_\theta + 3H\pi_\theta = -\frac{\partial H}{\partial \theta}$$

(7)

where $\pi_\theta$ is the conjugate momentum and the Hamiltonian $H = \dot{\theta}\pi_\theta - L$ is given by Eq. (3).

### 2.2 The second Randall-Sundrum model

The Randall-Sundrum (RS) model was originally proposed to solve the hierarchy problem [14], and later it was realized that this model, as well as any similar brane-world model, may have interesting cosmological implications. The second RS model (RSII model) [15] describes a $4 + 1$ dimensional anti de Sitter (AdS$_5$) universe containing two 3-branes with opposite tensions, separated in the fifth dimension, with observers on the positive tension brane. The fluctuation of the interbrane distance along the extra dimension implies the existence of the radion $\phi$, a massless scalar field that causes a distortion of the bulk geometry.

The total action, as seen on the observers brane, is [7, 16]

$$S = \int d^4x \sqrt{-g} \left( -\frac{R}{16\pi G} + \frac{1}{2} g^{\mu\nu} \phi_{,\mu}\phi_{,\nu} \right)$$

$$- \int d^4x \sqrt{-g} \frac{\sigma}{k^4\theta^4} (1 + k^2\theta^2\eta)^2 \sqrt{1 - \frac{g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}}{(1 + k^2\theta^2\eta)^3}},$$

(8)

where the second term is the action of the brane, and $k = 1/l$ is the inverse of AdS$_5$ curvature radius, $\sigma$ is the brane tension and $\eta$ is the rescaled radion field $\eta = \sinh^2 \left( \sqrt{4/3} \pi G \phi \right)$.

The total Lagrangian and the Hamiltonian for the brane and the radion are

$$L = \frac{1}{2} g^{\mu\nu} \phi_{,\mu}\phi_{,\nu} - \frac{\lambda \psi^2}{\theta^4} \sqrt{1 - \frac{g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}}{\psi^3}},$$

$$H = \frac{1}{2} g^{\mu\nu} \phi_{,\mu}\phi_{,\nu} + \frac{\lambda \psi^2}{\theta^4} \left( 1 - \frac{g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}}{\psi^3} \right)^{-1/2},$$

(9)
where $\psi = 1 + k^2 \theta^2 \eta$. Following the standard procedure, the Hamilton’s equations are

\[
\dot{\phi} = \frac{\partial H}{\partial \pi_\phi},
\]

\[
\dot{\theta} = \frac{\partial H}{\partial \pi_\theta},
\]

\[
\dot{\pi}_\phi + 3H \pi_\phi = -\frac{\partial H}{\partial \phi},
\]

\[
\dot{\pi}_\theta + 3H \pi_\theta = -\frac{\partial H}{\partial \theta},
\]

(10)

where $\pi_\phi = \frac{\partial L}{\partial \dot{\phi}}$ and $\pi_\theta = \frac{\partial L}{\partial \dot{\theta}}$ are the conjugate momenta. In the spatially flat Randall-Sundrum cosmology the Hubble expansion rate $H$ is related to the Hamiltonian via the modified Friedmann equation [17]

\[
H \equiv \frac{\dot{a}}{a} = \sqrt{\frac{8\pi G}{3} \mathcal{H} \left( 1 + \frac{2\pi G}{3k^2} \mathcal{H} \right)},
\]

(11)

2.3 The Observational parameters

Over the past twenty years many research missions have performed numerous observations and collected large amounts of data. The latest and the most important observational results for testing the cosmological models of inflation are provided by the Planck collaboration [3, 4].

The inflation slow roll parameters ($\epsilon_i$) [12, 18] are defined as follows

\[
\epsilon_0 \equiv \frac{H_*}{H}, \quad \epsilon_i \equiv \frac{d\ln |\epsilon_{i-1}|}{H dt}, \quad i \geq 1,
\]

(12)

where $H_*$ is the Hubble rate at an arbitrarily chosen time, and number of e-folds ($N$) is

\[
dN(t) = H(t) dt \quad \Rightarrow \quad N(t) = \int_{t_i}^{t_e} H(t) dt,
\]

(13)

where $t_i$ is the beginning and $t_e$ is the end of inflation, $\epsilon_i(t_e) = 1$.

Now, the observational parameters, the scalar spectral index ($n_s$) and the tensor-to-scalar power ratio ($r$), in lowest order in the slow roll parameters are

\[
r = 16\epsilon_1, \quad n_s = 1 - 2\epsilon_1 - \epsilon_2,
\]

(14)

The observational parameters in the second order in the slow roll parameters are

\[
r = 16\epsilon_1(t_i) \left[ 1 - 2\epsilon_1(t_i) + C\epsilon_2(t_i) \right],
\]

\[
n_s = 1 - 2\epsilon_1(t_i) - 2\epsilon_2(t_i) - \left[ 2\epsilon_1^2(t_i) + (2C + 3 - 2\alpha)\epsilon_1(t_i)\epsilon_2(t_i) + C\epsilon_2(t_i)\epsilon_3(t_i) \right],
\]

(15)

where $C = -0.72$, $\alpha = 1/6$ for tachyon models in the standard cosmology or $\alpha = 1/12$ in the case of the Randall-Sundrum cosmology.

According to the most restrictive results (Planck TT,TE,EE + lowW + lensing + BK14) the expected values of observational parameters for the base-ΛCDM cosmology are [4]:

\[
n_s = 0.9665 \pm 0.0038 \quad (68\% CL),
\]

\[
r_{0.002} < 0.064 \quad (95\% CL),
\]

(16)
Figure 1: Observational parameters for $V(\theta) = 1/\cosh(\theta)$ in the standard cosmology (SC) (5) (left) and the Randall-Sundrum cosmology (11) (right) for various $N$ and $\kappa$ chosen randomly, $30 \leq N \leq 150$, $0 \leq \kappa \leq 15$. The results are compared with the observational constraints from Planck collaboration [4].

Figure 2: Observational parameters for $V(\theta) = \exp(-\theta)$ for various $N$ and $\kappa$ chosen randomly, $30 \leq N \leq 150$, $0 \leq \kappa \leq 15$. Friedman equation is given for SC by (5) (left) and for RS by (11) (right).

The results we have calculated for different tachyon potentials and the RSII model are compared with the Planck results in Fig. 1 - 4.

3 Numerical results

The already mentioned software we developed for numerical calculations of the observational parameters (12)-(15) is based on the standard analytical procedure in the slow-roll regime ($\dot{\theta} \ll 1$, $\ddot{\theta} \ll 3H\dot{\theta}$). Here the slow-roll approximation is used only to determine the initial conditions, and evolution of the system is calculated numerically.

After nondimensionalization [7] of the equation of motion (6) or of the system of the Hamiltons equations (7) and (10) for tachyon and the RSII model, respectively, it is possible to obtain a system of first order differential equations with only one free parameter ($\kappa$).

The system is evolved numerically using Runge-Kutta method starting from $t = 0$ up to some large $t_{max}$ of the order of 100$\ell$. The function $N(t)$ is solved simultaneously using (13) with the initial condition $N(0) = 0$. The time evolution of the Hubble hierarchy parameters are obtained using (12). The end of inflation ($t_e$) is calculated as the solution of the equation $\varepsilon_i(t_e) = 1$. If the inflation does not end in the chosen time interval the
Figure 3: Observational parameters for $V(\theta) = \theta^{-4}$, the Friedman equation is given by [11] for various $N$ and $\kappa$ chosen randomly, $30 \leq N \leq 150$, $0 \leq \kappa \leq 15$. Each dot in the plot on the left side represents the results for chosen $(N, \kappa)$. The same results are shown in the plot on the right side, the color represents density of the results: red - higher, blue - lower density.

Figure 4: Results for observational parameters in $(n_s, r)$ plane. The color represents a density of the dots obtained for the RSII model (right). About 100,000 observational parameters were calculated for randomly chosen $N, \kappa$ and $\phi_0$ in the range $30 \leq N \leq 150$, $0 \leq \kappa \leq 15$ and $0 < \phi_0 < 1$.

order of $t_{\text{max}}$ is raised and calculation is repeated. The beginning of inflation $t_i$ is then obtained by requiring $N(t_a) - N(t_i) = N$. The simulation was repeated (about 100,000 times) for each model for randomly selected combination of $N$ and $\kappa$ in a chosen interval $(30 \leq N \leq 150, 0 \leq \kappa \leq 15)$, and results are analyzed and plotted in R language [19].

Evidently, a comparison of the computed results with Planck data shows a reasonable agreement. A comparison of the presented results with the results published in [7, 20, 21], obtained in a different way, or using the previous version of the program, gives also a good overlap.

The best agreement with observational data provided by Planck collaboration was achieved for the potential $V(\theta) = 1/\cosh(\theta)$ and the free parameters in the interval $60 \leq N \leq 90$, $1 \leq \kappa \leq 10$ for the Randall-Sundrum cosmology (11). The statistical distributions of computed results are shown in the histograms (Fig. 5), mean $\mu(n_s) = 0.9694$, $\mu(r) = 0.062$, and median $\tilde{n}_s = 0.9745$, $\tilde{r} = 0.045$.

We would like to stress that our program provides fast numerical calculation of the observational parameters for known models.
Figure 5: Statistical distribution of the scalar spectral index $n_s$ (left) and the tensor-to-scalar ratio $r$ (right) for the $V(\theta) = 1/\cosh(\theta)$ in Randall-Sundrum cosmology \cite{6}. The free parameters are in the range $30 \leq N \leq 150$, $0 \leq \kappa \leq 15$. The solid vertical lines in the histograms mark the mean $\mu(n_s) = 0.9694$ and $\mu(r) = 0.062$ (red) and median $\tilde{n}_s = 0.9745$ and $\tilde{r} = 0.045$ (green).

4 Conclusion

The program we developed and used here has been applied to a limited set of models, mainly to the pure tachyonic and the RSII inflationary cosmological models. The theoretical input is provided by the Hamilton and the Friedmann equations for the chosen potentials and corresponding parameters. The program can readily be used for a much wider set of the models. To apply the program to a new model one must determine its corresponding equations and include these equations in the program only.

The next steps are: to extend the program to be applicable for new and different types of inflationary models, to improve the program in such a way that only the Hamiltonian or the Lagrangian of a model are their inputs. The corresponding system of differential equations would be determined by symbolic computation, after these improvements the program will be published as a free software followed by appropriate documentation.

Finally, the best fitting result is obtained for $V(\theta) = 1/\cosh(\theta)$. It opens good opportunity for further research based on this potential in the context of the RSII model and the holographic cosmology. These results will be published elsewhere.

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References

[1] A. H. Guth, Phys. Rev. D 23, 347 (1981).
[2] A. D. Linde, Phys. Lett. B 108, 389 (1982).
[3] P. A. R. Ade et al. [Planck Collaboration], astronom. astrophys. 594, a20 (2016).
[4] Y. Akrami et al. [Planck Collaboration], arXiv:1807.06211 [astro-Ph.CO] (2018).
[5] D. D. Dimitrijevic, G. S. Djordjevic, and M. Milosevic, Rom. Reports Phys. 68, 5 (2016).
[6] G. S. Djordjevic, D. D. Dimitrijevic, and M. Milosevic, Rom. J. Phys. 61, 99 (2016).
[7] N. Bilic et al., Int. J. Mod. Phys. a 32, 1750039 (2017).
[8] N. Bilic, S. Domazet, and G. S. Djordjevic, Phys. Rev. D 96, 083518 (2017).
[9] S. Li and A. R. Liddle, J. Cosmol. Astropart. Phys. 2014, 044 (2014).
[10] A. Sen, J. High Energy Phys. 2002, 065 (2002).
[11] A. Sen, Ann. Henri Poincaré 4, 31 (2003).
[12] D. Steer and F. Vernizzi, Phys. Rev. D 70, 043527 (2004).
[13] M. Fairbairn and M. H. Tytgat, Phys. Lett. B 546, 1 (2002).
[14] L. Randall and R. Sundrum, Phys. Rev. Lett. 83, 3370 (1999).
[15] L. Randall and R. Sundrum, Phys. Rev. Lett. 83, 4690 (1999).
[16] N. Bilic and G. B. Tupper, Cent. Eur. J. Phys. 12, 147 (2014).
[17] R. Maartens and K. Koyama, Living Rev. Relativ. 13, (2010).
[18] D. J. Schwarz, C. A. Terrero-Escalante, and A. A. Garcia, Phys. Lett. B 517, 243 (2001).
[19] R Development Core Team, R: a language and environment for statistical computing. 
R Foundation for Statistical Computing, (Vienna, austria, 2008), http://www.R-project.org.
[20] M. Milosevic et al., Serbian Astron. J. 192, 1 (2016).
[21] N. Bilic et al., in AIP Conf. Proc. 1722 (2016), p. 050002.