Entanglement, violation of Kramers-Kronig relation and curvature in spacetime

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Recent studies show that a maximally entangled Schwinger pair creates a nontraversable Einstein-Rosen (ER) bridge (wormhole) in the gravitational theory, which bridges causally not connected parts of the AdS spacetime [PRL 111, 211603 and 211602]. Same authors raise the possibility of a general correspondence between a weaker entanglement and curvature. Here, we provide some clues for such a generalized relation. First, we show that (i) entanglement and (ii) violation of Kramers-Kronig (KK) relations appear at the same critical parameters for a standard two-mode (squeezing) entanglement interaction. We also include the dampings. Second, we bring a study into attention: Presence of a spacetime-curvature in vacuum polarization, electron-positron Schwinger pairs, makes QED also violate the KK relations without violating the causality. Then, we discuss that these findings are possible to provide new clues for a generalized relation between entanglement and spacetime-curvature. Such an interpretation may also save violation of KK relations from implying the violation of the causality.

Entanglement poses a spooky action between a pair of well-separated particles. Although entanglement cannot be used for instant communication [11], choice of the measurement on one of the particles affect the state of the other instantaneously [7]. Similarly, a nontraversable wormhole, a phenomenon emerging in general relativity (gravitational theory), can connect (shortcut) two distant particles at a space-like separation. Although a nontraversable wormhole cannot be used for instant communication, as in entanglement, action on one of the particles can affect the state of the other one [4].

Recent influential works [5, 6, 11] demonstrate a concrete relation between the two phenomena: entanglement in quantum theories and Einstein-Rosen (ER) bridge (a wormhole) in the gravitational theory. In simple words, Refs. [5, 6] use equivalence (holography) principle [7, 8] between the solutions of conformal (quantum) field theory (CFT) and super-symmetric gravitational theory (as a low energy string theory), i.e. AdS$_2$ [9]. On the CFT side, they provide analytical solutions [10] to a quark-antiquark pair, in a maximally entangled state, created via Schwinger effect [11] in vacuum. The two particles are not in causal contact such that the light emitted from one particle cannot reach the other one [6]. On the AdS side, they explicitly show that this maximally entangled state, of CFT, produces a nontraversable wormhole (ER bridge) in AdS$_5$ [6, 11] via the mathematical correspondence (holography) [10] between 4-dimensional CFT and AdS$_5$. A “tunneling instanton” solution [7] accompanies the Schwinger effect [11], which corresponds to an ER bridge [11] between the horizons of the particles out of causal contact [6, 11].

Such an explicit demonstration of the correspondence between a maximally entangled Schwinger pair and emergence of a nontraversable ER bridge (wormhole) led Sonner [5] raise the following intriguing question. What if the system is not in a maximally entangled state but in a weaker inseparable state?

In this paper, we provide new clues on the entanglement-ER bridge correspondence. Our results show a possible extension of the connection between entanglement and ER bridge (very special solutions of Einstein Field equations, EFE, in curved spacetime) into a kind of correspondence between (i) a particular, small, value of entanglement and (ii) presence of curvature in spacetime.

We handle the correspondence in a completely different point of view. Instead of CFT, we use the standard second-quantized theory of quantum optics and QED in the curved spacetime [17, 18]. Similar to Refs. [5, 6, 11], QED in curved spacetime deals with electron-positron Schwinger pairs, i.e. vacuum polarization.

First, we show that the most standard, exactly solvable, two-mode squeezing $\hat{c}^\dagger = \hbar (g_1 + g_2 \hat{a} \hat{a}^\dagger + g_2^* \hat{a}^\dagger \hat{a})$, interaction [19] in a cavity manifests the violation of Kramers-Kronig (KK) relations in the input/output (transfer functions) of the cavity. More interestingly, we show that in such a damped cavity, nonclassicality [entanglement or single-mode nonclassicality (SMNc), e.g. squeezing] shows up at exactly the same critical (e.g. cavity-mirror) coupling, $g_1$, parameter. Moreover, this coincidence appears for any values of the system parameters! (see Fig. 1) We note that SMNc of a (collective) quasi-excitation is the collective entanglement of the background (particles) generating the, e.g. squeezed, excitation [20, 21], see Fig. 2 for an illustration.

Second, we examine (compare with) the quantum electrodynamics (QED) solutions of vacuum-polarization in curved spacetime. Hollowood and Shore calculate the refractive index $n(\omega)$ for vacuum-polarization (Schwinger production of electron-positron pairs) of light in the pres-

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1 In a many-particle (ensemble) entangled state, measurement on a single particle can affect the state of all remaining particles. This appears in systems like single-photon superradiance or interacting Bose-Einstein condensates [2, 3].

2 A solution who can change (tunnel), e.g., between two minimum energies almost instantaneously [12, 13].
ence of background curvature in spacetime [17, 18] using an unperturbed (sigma wordline [21, 24]) formalism. They provide worldline instanton solutions describing Schwinger pair creation which can be interpreted as tunneling in this formalism [17, 18]. They show that refractive index violates the KK relations due to the interaction of electron-positron Schwinger pairs with a weak background curvature. We kindly remind that in Refs. [9, 10, 11], maximally entangled Schwinger pairs are responsible for the ER bridge (tunneling instanton).

Fortunately, Hollowood and Shore show that in the presence of background curvature in spacetime, violation of KK relations does not imply the violation of causality [17, 18]. Shortly, they argue that interaction of the generated electron-positron pair with the background “curvature” violates only the strong equivalence principle (SEP) [20, 21] in general relativity. Validity of SEP would imply the existence of global reference frames and this would indicate a superluminal propagation which could violate the causality [25]. Weak equivalence principle, however, implies only the existence of local inertial reference frames which is not sufficient to establish a link between superluminal propagation and violation of causality [25].

Both (i) a weakly-entangled system and (ii) a system, where a background spacetime curvature is present, violate the Kramers-Kronig relations. Hence, considering a possibility of a relation between entanglement and spacetime curvature could allow us to (a) circumvent the appearance of violation of KK relation to imply the causality violation in entangled devices and (b) extend the entanglement-ER bridge correspondence, bearing in Refs. [5, 6, 11], into an entanglement-curvature duality.

In advance, we state that our findings do not provide a proof for the entanglement-curvature (spacetime) relation. The reason we raise the common appearance of violation of KK relations, showing up both in the onset of (i) weak entanglement and (ii) weak spacetime curvature, as a clue is the following. Actually there are several reasons for this. First, KK violation is obviously not a common phenomenon which otherwise would not be so controversial to the audience. Second, Refs. [9, 10, 11] already provide a direct derivation between maximally entangled Schwinger pair and ER bridge. So, such a relation between weak entanglement and spacetime curvature is already something expected [5]. Third, violation of KK and onset of entanglement appear at exactly the same critical coupling $g_L > g_{crt}$, see Fig. [1]. This obviously provides a stronger clue, e.g. compared to Ref. [20] where (fourth) small fluctuations in entanglement are shown to introduce curvature (weight) in a non-dissipative system. That is, there is no critical point in the valuable work [29], but the two critical values coincide in our case. Fifth, the observation on the violation of KK relations via entanglement or SMNe (especially in a continuous regime of variables, i.e. not at some resonances), “forces” us to believe in (check the existence of) such a correspondence. Induction of curvature with inseparability can avoid the violation of causality [17, 18, 20] appearing via entanglement [5].

So this work aims to gather all clues in a single box besides providing new ones. We propose a way for circumventing acceptance of the presence of the violation of causality under the light of Refs. [5, 6, 17, 18]. We underline that the work presented here has its roots at Ref. [30].

**Tunneling and KK relations**—Before demonstrating our results in the next paragraphs, it is appropriate to discuss an important point. We note that violation of KK relations are observed also in “causal” interferometry devices [31–33] where multiple interferences take place. It is well-known that interference can avoid/limit electromagnetic field to occupy particular spatial domains. These domains are tunnelled at [34]. In these systems, the major problem is to define the “tunneling times” for photons [35] which are calculated to be superluminal [36–38]. Although this is discussed to appear due to the instantaneous spreading of the wavefunctions [39, 40], also relativistic equations demonstrate the superluminal tunneling times [41, 42]. It is also shown that, still open to further questions, weak Einstein causality (which states that expectations or ensemble averages may not be superluminal) is not violated in tunneling process [36]. In Ref. [43], we discuss the interference phenomenon in details. In Ref. [43], we also show that the faster-than-light “peak” velocities observed in various experiments [44] are superluminal since the group-index (mathematically can be shown to govern the peak velocity [45]) violates the KK relations. This provides a support for superluminal propagation needs to be accompanied by a violation of KK relations which Refs. [17, 18, 20] show: does not necessarily imply the violation of causality.

Actually the appearance of nonanalyticity in the upper-half of the complex-frequency-plane (CFP), in our system, is different than the ones discussed in “causal” interferometry devices [32, 46, 47]. The nonanalyticity we observe does not depend on the cavity length, unlike interferometry devices, hence possibly do not originate from interference. Our system is merely absorptive where violation of KK relations do not appear [40, 48].

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3 of the light which creates the vacuum polarization

4 Here, light interacts with the background curvature through the created electron-positron pairs.

5 SEP is not violated if the interaction depends only on the Christoffel symbols but not directly on the curvature [20, 21, 29].

6 An extended discussion can be found in Refs. [20].

7 M. Suhail Zubbairy –private communication (group meeting).

8 Actually it is not so important whether it appears via interference or not. Statement “entanglement induces (or induced with) superluminal tunneling” also lines with our discussions, which strongly accompanies the tunneling instanton in Refs. [9, 10, 17, 18] wherein tunneling time can be more superluminal (faster) compared to Refs. [20, 30].
Entanglement & violation of Kramers-Kronig relations — First, we consider an optomechanical system \([49, 50]\) in which a cavity mode \(\hat{c}\) interacts with the vibrating mirror \(\hat{a}\) placed inside the cavity. Then, we tune the cavity to favor \(\mathcal{H}_{\text{int}} = h(g_1\hat{c}^\dagger \hat{a} + g_1^* \hat{a} \hat{c})\) two-mode squeezing \([49]\) type interaction. We show that entanglement and violation of KK relations appear at the same critical coupling \(g = g_{\text{crit}}\), in Fig. 1.

Entanglement features of an optomechanical system have already been studied extensively \([51]\). Hamiltonian can be written \([30, 49]\) as
\[
\mathcal{H} = \hbar \Delta_c \hat{c}^\dagger \hat{c} + \hbar \omega_m \hat{a}^\dagger \hat{a} + \hbar g \hat{c} \hat{c}^\dagger \hat{a} \hat{c}^\dagger + i \hbar \varepsilon_L (\hat{c}^\dagger - \hat{c})
\]
in the frame rotating with the laser (pump) frequency \(\omega_L\), i.e. \(\Delta_c = \omega_c - \omega_L\). \(\omega_c\) is the frequency of the optical cavity mode. Nonclassicality, e.g. entanglement and single-mode nonclassicality (SMNC) such as squeezing, are determined by noise operators \([52, 53]\), i.e. \(\delta \hat{c} = \hat{c} - \langle \hat{c} \rangle\) and \(\delta \hat{q} = \hat{q} - \langle \hat{q} \rangle\), with \(\hat{q} = (\hat{a}^\dagger + \hat{a})/\sqrt{2}\). After the linearization, Langevin equations
\[
\delta \dot{\hat{q}} = \omega_m \delta \hat{p}
\]
\[
\delta \dot{\hat{p}} = -\gamma_c \delta \hat{p} - \omega_m \delta \hat{q} + g(\alpha_c^* \delta \hat{c} + \alpha_c \delta \hat{c}^\dagger) + g_m \hat{e}_{\text{in}}(t)
\]
\[
\delta \dot{\hat{c}} = -i(\gamma_c + \Delta) \delta \hat{c} + ig \alpha_c \delta \hat{q} + g_m \hat{a}_{\text{in}}(t)
\]
become analytically solvable, where \(\hat{a}_{\text{in}}(t)\) and \(\hat{e}_{\text{in}}(t)\) are the optical and mechanical noises, leaking in, from the two vacua \([30]\). The laser pump is used for increasing the effective coupling between the mirror and cavity mode. \(\gamma_{c,m} = \pi D(\omega_{c,m}) g_{c,m}^2\) are the damping rates and \(\Delta = \Delta_c - g |q_0|^2\) with \(q_0\) and \(\alpha_c\) are the steady-state values for \(\langle \hat{q} \rangle\) and \(\langle \hat{c} \rangle\) \([51, 55]\).

In this work, in difference to Refs. \([49, 50]\), we are particularly interested in single-mode nonclassicality (SMNC), e.g. squeezing, of the \(\hat{c}\)-mode. The reason becomes apparent in the following section. We quantify the SMNC of the cavity mode — stays Gaussian due to becomes analytically solvable, where  \(\hat{a}_{\text{in}}(t)\)  and \(\hat{e}_{\text{in}}(t)\)  are the optical and mechanical noises, leaking in, from the two vacua \([30]\). The laser pump is used for increasing the effective coupling between the mirror and cavity mode. \(\gamma_{c,m} = \pi D(\omega_{c,m}) g_{c,m}^2\) are the damping rates and \(\Delta = \Delta_c - g |q_0|^2\) with \(q_0\) and \(\alpha_c\) are the steady-state values for \(\langle \hat{q} \rangle\) and \(\langle \hat{c} \rangle\) \([51, 55]\).

Next, we leave the second-quantized picture and work with the expectation values of the operators \([57, 59]\), e.g.,
\[
c = c_0 + c_+ \alpha_p e^{-i \Delta \gamma_{\text{in}} t} + c_- \alpha_p^* e^{i \Delta \gamma_{\text{in}} t},
\]
similarly for \(p\) and \(q\). We assume an extremely small probe field \(\alpha_p\), \(\alpha_p^2\) is the number of probe photons, of frequency \(\Delta_p\), in the rotating frame. Since the probe field \(\alpha_p\) is extremely small, system can be safely described with the linear response, i.e. \(\alpha_p^2 \sim 0\). In both treatments second order terms, e.g. \((\delta \hat{q})^2\) and \(\alpha_p^2\), are neglected. The nonanalyticities of the transfer function can be obtained from the roots of \(c_+ = 0\) \([30]\), i.e.
\[
c_+ \propto \left[\gamma_c - i(\Delta + \Delta_p)\right](\Delta_p^2 - \omega_m^2 + i \gamma_m \Delta_p) - i \omega_m |G|^2 = 0.
\]
We also checked if the zeros from the denominator of \( c_+ \) cancels the ones from the nominator and we observed that they are completely different. Denominator does not have a zero in the upper half of the complex frequency place (CFP). In our cavity system, there appears some nonanalyticities also due to interference [36, 46, 47], e.g. 
\[ -(2\gamma_c\hat{c}_+ - 2)^2 + (2\gamma_c\hat{c}_+)^2 e^{2ik_pL} + \ldots = 0 \]
with \( k_p = \omega_p/\gamma \), which we do not consider here. \( c_+ = 0 \) has 3 complex roots \( \Delta_p^{(1,2,3)} \), one of them relies in the upper half of the CFP for \( g \geq g_{\text{crit}} \), see Fig. 1.

In Fig. 1 we clearly observe that for an entangler Hamiltonian \( \hat{H}_{\text{ent}} = \hbar g_1(\hat{c}^\dagger \hat{a}^\dagger + \hat{a} \hat{c}) \), violation of KK and SMNC onset at the same critical point. Moreover, this is independent of \( \gamma_{c,m} \). In Fig. 1, system becomes unstable where the plot ends. We recall that for SMNC of the cavity mode to emerge, \( \hat{H}_{\text{ent}} \) interaction is not sufficient. \( \hat{H}_{\text{ent}} \) only generates the entanglement, but \( \hat{H}_{\text{BS}} \) distributes the entanglement into SMNC.

Hence, for the simplest entanglement generator (namely the two-mode squeezing) Hamiltonian, SMNC and violation of KK relations appear together [7]. We note that, the final Hamiltonian

\[ \hat{H} = \hbar \Delta \hat{c}^\dagger \hat{c} + \hbar \omega_m \hat{a}^\dagger \hat{a} + \hbar g_1(\hat{c}^\dagger \hat{a}^\dagger + \hat{a} \hat{c}) + \hbar g_2(\hat{c}^\dagger \hat{a}^\dagger + \hat{a} \hat{c}) \]

does not include any gain. We introduce only the losses into Langevin equations. There is no physical phenomenon (restriction) which avoids the achievement of \( g_1 \geq g_{\text{crit}} \) in principle. Our work bases on the solutions of the simplest (standard) exactly solvable nonclassicality (entanglement and SMNC) generator Hamiltonian [3].

We put the optomechanics Hamiltonian as an example physical system where one can approximately obtain the interaction [8]. Other systems could also result similar interaction. For instance, in Ref. [62] it is discussed that a radiation pressure like Hamiltonian is responsible for the Stokes and anti-Stokes shifts in surface enhanced Raman scattering (SERS).

Besides its fundamental implications, such a phenomenon can also be used to “guess” the onset regime for measurements below the standard quantum limit (squeezing) by measuring the transfer functions [9, 10].

Why SMNC? Now we are in the right position to state the physical reasoning for considering the degree of SMNC of the cavity field, instead of \( \hat{c} - \hat{a} \) entanglement. As explicitly demonstrated in Ref. [2], the quasiparticle excitations of an ensemble become nonclassical (SMNC) when the particles, in the ensemble, are entangled “collectively”. For a better visualization, in Fig. 2 we plot a squeezed “phonon” wavepacket. The findings of Ref. [2] state that if the phonon is squeezed [63], vibrational (motional) degree of freedom of the vibrating atoms are collectively entangled within the extent of the phonon wavepacket. A similar visualization can be made for a squeezed photon, e.g. the cavity mode.

![FIG. 2. If, for instance, a phonon is squeezed the motional degree of freedom of the vibrating atoms is collectively entangled within the extent of the phonon wavepacket. A similar visualization can be made for a squeezed photon, e.g. the cavity mode.](image)

9 Even though probe field and frequency \( \Delta_p \) appear explicitly in the calculation of the transfer functions, violation of KK relations is related with the complete frequency response of the transfer functions.

10 The credit for this idea belongs to Peter Zoller at IQOQI of Innsbruck.

11 This can be checked via coupled QED-EFE equations.
the violation of causality in a curved spacetime background [17, 18, 25]; makes us consider the possibility of a generalization of the (into weak) entanglement-curvature relation. Actually, this is better than accepting the violation of causality with entanglement in Fig. 1 or in other systems.

We believe that the conjectures we raise and the direction we point out in this work will stimulate new and fundamental works on QED in curved spacetime.

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Flow Chart for Connection/Relations

- **single-mode nonclassicality (SMNc)**
  - Entanglement of the background (ensemble generating the excitation)
  - Violates KK relations
  - presence of $g_{\mu\nu}$ = small curvature in QED
  - vacuum polarization ($e^+ - e^-$ pairs)
  - not a common phenomenon

- **a maximally entangled Schwinger pair (QFT)**
  - ER-bridge (wormhole)
  - general relativity + string
  - tunneling instanton solution

- **SMNc**
  - Violates KK relations
  - superluminal propagation
  - does not imply violation of causality (only local inertial frame)

- **avoids** violation of causality