SYZYGIES IN EQUIVARIANT COHOMOLOGY
FOR NON-ABELIAN LIE GROUPS

MATTHIAS FRANZ

Abstract. We extend the work of Allday–Franz–Puppe on syzygies in equivariant cohomology from tori to arbitrary compact connected Lie groups $G$. In particular, we show that for a compact orientable $G$-manifold $X$ the analogue of the Chang–Skjelbred sequence is exact if and only if the equivariant cohomology of $X$ is reflexive, if and only if the equivariant Poincaré pairing for $X$ is perfect. Along the way we establish that the equivariant cohomology modules arising from the orbit filtration of $X$ are Cohen–Macaulay. We allow singular spaces and introduce a Cartan model for their equivariant cohomology. We also develop a criterion for the finiteness of the number of infinitesimal orbit types of a $G$-manifold.

1. Introduction

Let $R$ be a polynomial ring in $r$ indeterminates, and let $M$ be a finitely generated module over $R$. Then $M$ is called a $j$-th syzygy if there is an exact sequence

$$0 \rightarrow M \rightarrow F_1 \rightarrow \cdots \rightarrow F_j$$

with finitely generated free $R$-modules $F_1, \ldots, F_j$. The first syzygies are exactly the torsion-free modules, the second syzygies the reflexive ones, and the $r$-th syzygies are free. In this sense, syzygies interpolate between torsion-freeness and freeness. Allday, Puppe and the author initiated the study of syzygies in the context of torus-equivariant cohomology [2], [3]. To illustrate their results, let us focus on the second syzygies. We use Alexander–Spanier cohomology with real coefficients.

Let $T \cong (S^1)^r$ be a torus, and let $X$ be a $T$-space with finite Betti sum and satisfying some other mild assumptions. Then $R_T = H^*(BT)$ is a polynomial ring in $r$ indeterminates of degree 2, and the equivariant cohomology $H^*_T(X)$ of $X$ is an $R_T$-module. Allday–Franz–Puppe showed that the Chang–Skjelbred sequence

$$0 \rightarrow H^*_T(X) \rightarrow H^*_T(X^T) \rightarrow H^{*+1}_T(X_{1,T}, X^T)$$

is exact if and only if $H^*_T(X)$ is a reflexive $R_T$-module. Here $X^T$ denotes the fixed point set, and $X_{1,T}$ is the union of $X^T$ and all 1-dimensional orbits. The sequence (1.2) often permits an efficient computation of $H^*_T(X)$. Moreover, if $X$ is a Poincaré duality space, then the reflexivity of $H^*_T(X)$ is also equivalent to the perfection of the equivariant Poincaré pairing

$$H^*_T(X) \times H^*_T(X) \rightarrow R_T, \quad (\alpha, \beta) \mapsto \langle \alpha \cup \beta, o_T \rangle,$$

which is the composition of the cup product and evaluation on an equivariant orientation $o_T$ of $X$.

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The main purpose of the present paper is to extend the results of [2] and [3] to actions of arbitrary compact connected Lie groups. It turns out that essentially all results carry over to this more general setting. We achieve this by combining the techniques of Allday–Franz–Puppe with those of Goertsches–Rollenske [19], whose study of Cohen–Macaulay actions gave a first hint at the possibility of such an extension. Let us describe our results in more detail.

Let $G$ be a compact connected Lie group with maximal torus $T$ and corresponding Weyl group $W$. Then $r$ is the rank of $G$, and $R_G = H^*(BG)$ is again a polynomial ring in $r$ indeterminates of even degrees. Let $X$ be a $G$-space; we always assume that it admits an equivariant closed embedding into some smooth $G$-manifold or $G$-orbifold. This allows us to define a Cartan model for the equivariant cohomology $H^*_G(X)$ of $X$ and also for a suitably defined equivariant homology $H^*_G(X)$. The latter is not the homology of the Borel construction of $X$; it is related to equivariant cohomology via a universal coefficient spectral sequence (Proposition 3.11), and for manifolds and locally orientable orbifolds also via equivariant Poincaré duality (Proposition 3.13). Here “locally orientable” means that $X$ is a rational homology manifold, see Section 3.3 for the precise definition. For most results, we also assume that only finitely many infinitesimal orbit types occur in $X$. (The infinitesimal orbit type of $x \in X$ is the orbit of the Lie algebra $\mathfrak{g}_x$ of the stabilizer $G_x$ under the adjoint action.) This condition is often redundant, see below.

The $G$-orbit filtration of $X$ is defined by

\[(1.4) \quad X_{i,G} = \{ x \in X \mid \text{rank } G_x \geq r - i \} \]

for $-1 \leq i \leq r$. Then $X_{-1,G} = \emptyset$, $X_{0,G}$ is the maximal-rank stratum and $X_{r,G} = X$. All $X_{i,G}$ are $G$-stable and closed in $X$. Note that if $G = T$, then $X_{0,T} = X^T$. The maximal-rank stratum plays the role of the fixed point set in the non-abelian context.

Based on the work of Goertsches–Rollenske, we show:

**Proposition 1.1.** Let $X$ be a $G$-space. For $0 \leq i \leq r$, there is an isomorphism of $R_G$-modules

\[ H^*_G(X_{i,G}, X_{i-1,G}) = H^*(X_{i,T}, X_{i,T} \cap X_{i-1,G})^W. \]

An immediate consequence is the following:

**Corollary 1.2.** Let $X$ be a $G$-space with only finitely many infinitesimal orbit types and such that $H^*(X)$ is finite-dimensional. The $R_G$-module $H^*_G(X_{i,G}, X_{i-1,G})$ is zero or Cohen–Macaulay of dimension $r - i$ for $0 \leq i \leq r$.

In the case of a compact $G$-manifold $X$, Goertsches–Rollenske proved the first result for $i = 0$ [19, Prop. 3.3] and the second if $X_{i-1} = \emptyset$ [19, Cor. 4.3]; in particular they established the freeness of $H^*_G(X_{0,G})$ over $R_G$ [19, Cor. 3.5].

As for torus actions, we consider the Atiyah–Bredon sequence

\[(1.5) \quad 0 \rightarrow H^*_G(X) \xrightarrow{\iota^*} H^*_G(X_{0,G}) \xrightarrow{\delta_0} H^{r+1}_G(X_{1,G}, X_{0,G}) \xrightarrow{\delta_1} \ldots \]

\[\xrightarrow{\delta_{i+2}} H^{r+i}_G(X_{r-1,G}, X_{r-2,G}) \xrightarrow{\delta_{i+1}} H^{r+r}_G(X_{r,G}, X_{r-1,G}) \rightarrow 0.\]

Here $\iota^*$ is induced by the inclusion $\iota: X_0 \rightarrow X$, and $\delta_i$ for $i \geq 0$ is the connecting homomorphism in the long exact sequence for the triple $(X_{i+1,G}, X_{i,G}, X_{i-1,G})$. The
Atiyah–Bredon complex $AB^*_G(X)$ with $AB^*_G(X) = H^{*+i}_G(X_{i,G}, X_{i-1,G})$ is obtained from (1.5) by dropping the leading term.

**Theorem 1.3.** Let $X$ be a $G$-space with only finitely many infinitesimal orbit types and such that $H^*(X)$ is finite-dimensional. For any $j \geq 0$, the $j$-th cohomology of the Atiyah–Bredon complex is

$$H^j(AB^*_G(X)) = \text{Ext}^j_{R_G}(H^*_G(X), R_G).$$

This allows us to characterize syzygies in equivariant cohomology:

**Theorem 1.4.** Let $X$ be a $G$-space with only finitely many infinitesimal orbit types and such that $H^*(X)$ is finite-dimensional, and let $1 \leq j \leq r$. Then $H^*_G(X)$ is a $j$-th syzygy over $R_G$ if and only if the part

$$0 \to H^*_G(X) \to H^*_G(X_{0,G}) \to \cdots \to H^{*+j-1}_G(X_{j-1,G}, X_{j-2,G})$$

of the Atiyah–Bredon sequence is exact.

In the torus case this result is due to Allday–Franz–Puppe [2, Thm. 5.7], [3, Thm. 4.8]. For general $G$ and compact $X$, the case $j = 1$ has been established by Allday (unpublished, based on the case $G = SU(2)$ solved by Chen) and Goertsches–Rollenske [19, Thm. 3.9]. Whether it extends to higher syzygies in the non-abelian setting has been an open question, see [19, Question 4.7].

In the important special case $j = 2$ we get:

**Corollary 1.5.** Let $X$ be a compact orientable $G$-manifold or $G$-orbifold. The following are equivalent:

1. The $R_G$-module $H^*_G(X)$ is reflexive.
2. The sequence

$$0 \to H^*_G(X) \to H^*_G(X_{0,G}) \to H^{*+1}_G(X_{1,G}, X_{0,G})$$

is exact.
3. The equivariant Poincaré pairing $H^*_G(X) \times H^*_G(X) \to R_G$ is perfect.

That the freeness of $H^*_G(X)$ over $R_G$ implies the exactness of the non-abelian Chang–Skjelbred sequence is due to Brion [2, Thm. 9] for compact multiplicity-free $G$-spaces and to Goertsches–Mare [13, Thm. 2.2] for compact $G$-manifolds. The perfection of the equivariant Poincaré pairing was shown by Ginzburg [17, Cor. 3.9] and Brion [8, Prop. 1] under the assumption that $H^*_G(X)$ is free over $R_G$.

Let $X$ be a $G$-space. In [2] and [3], $T$-spaces were assumed to have finite Betti sum and also finitely many infinitesimal orbit types, and as mentioned above, we often require the same for $G$-spaces. As another application of equivariant homology, we show that the latter condition is redundant for manifolds and locally orientable orbifolds.

**Theorem 1.6.** Let $X$ be a $G$-manifold or locally orientable $G$-orbifold. If $H^*(X)$ is finite-dimensional, then only finitely many infinitesimal orbit types occur in $X$.

If $X$ is compact, it follows easily from the differentiable slice theorem that there are actually only finitely many orbit types in $X$, cf. [5, Prop. VIII.3.13]. (For the orbifold version of the differentiable slice theorem, see [24, Prop. 2.3].) By a result of Mann [25, Thm. 3.5], the same conclusion holds if $X$ is an orientable manifold (or cohomology manifold) with finitely generated integral cohomology. While Theorem [4.6] is in the same spirit as Mann’s result, the proof is very different.
The paper is organized as follows: After discussing Cohen–Macaulay modules, syzygies and the algebraic aspects of Cartan models in Section 2, we define equivariant de Rham homology and cohomology for possibly singular spaces in Section 3. In Section 4 we discuss the behaviour of syzygies under restriction to and induction from a maximal torus. To prepare for the proof of the main results, we study spaces with isotropy groups of constant rank in Section 5. The results of [2] and [3] are generalized in Section 6 where we also introduce the notion of a Cohen–Macaulay filtration of a $G$-space. In the final Section 7 we prove Theorem 1.6.

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## 2. Algebraic preliminaries

### 2.1. Notation and standing assumptions.

Throughout this paper, the letter $G$ denotes a compact connected Lie group of rank $r$ with maximal torus $T \cong (S^1)^r$ and corresponding Weyl group $W = N_G(T)/T$. The Lie algebra of $G$ is written as $\mathfrak{g}$ and its dual as $\mathfrak{g}^*$. We adopt this naming scheme for all Lie groups.

All manifolds and orbifolds are assumed to be paracompact. (See [1, Sec. 1.1] for the definition of an orbifold.) All (co)homology is taken with real coefficients. We adopt a cohomological grading for all complexes, so that differentials always have degree +1. The degree of an element $c$ of a complex is denoted by $|c|$. Homological complexes are turned into cohomological ones by grading them negatively. For example, elements in the $n$-th homology group $H_n(X)$ of a space $X$ have degree $-n$. A quasi-isomorphism is a map of complexes inducing an isomorphism in cohomology.

Let $A$ and $B$ be complexes. The $n$-th degree of the complex $\text{Hom}_R(A, B)$ consists of all linear maps $f : A \to B$ that raise degrees by $n \in \mathbb{Z}$. The differential is defined by

\[
 d(f) = d_B \circ f - (-1)^n f \circ d_A.
\]

This generalizes to differential graded (dg) modules over some dg algebra. The complex $A^\vee = \text{Hom}_R(A, R)$ is the dual complex of $A$. Moreover, for $k \in \mathbb{Z}$, the $n$-th degree of the shifted complex $A[k]$ is equal to $A^{n+k}$; the differential of $A[k]$ is $(-1)^k d_A$. We also write $H^{*+k}(X)$ instead of $H^*(X)[k]$.

Unless specified otherwise, all tensor products are over $\mathbb{R}$.

### 2.2. Cohen–Macaulay modules.

Let $R$ be a polynomial ring over a field in $r$ indeterminates of positive degrees. We assume that $R$ is graded by assigning a positive degree to each indeterminate. Recall that a graded $R$-module $M \neq 0$ is called Cohen–Macaulay if it is finitely generated and

\[
 \text{depth } M = \dim M.
\]

**Proposition 2.1.** Let $M$ be a finitely generated graded $R$-module, and let $d \in \mathbb{N}$.

1. $M$ is Cohen–Macaulay of dimension $d$ if and only if

\[
 \text{Ext}^i_R(M, R) \neq 0 \iff i = r - d.
\]

2. In this case, $\text{Ext}^{r-d}_R(M, R)$ is again Cohen–Macaulay of dimension $d$. 


Remark 2.3. If \( \text{orbifold to establish equivariant Poincaré duality} \), and Section 6.4, where we define cup product in the equivariant de Rham cohomology of an orientable manifold or in equivariant cohomology. The only exceptions are Section 3.3, where we use the however, most of the time we will not be concerned with the multiplicative structure

Lemma 2.4. Let \( M \) be a finitely generated \( R \)-module. The following are equivalent for any \( 0 \leq j \leq r \):

1. \( M \) is a \( j \)-th syzygy.
2. Any regular sequence \( f_1, \ldots, f_j \in R \) is \( M \)-regular.
3. \( \text{depth}_{R_p} M_p \geq \min(j, \dim R_p) \) for any prime ideal \( p \ll R \).

2.4. Cartan models. Recall that a \( G^* \)-module is a complex \( A \) of \( G \)-modules with operations \( L_\xi \) (of degree 0) and \( \iota_\xi \) (of degree -1) for \( \xi \in g \) satisfying the same relations as the Lie derivative and the contraction operator for differential forms; see [21] Def. 2.3.1] for details.

The Cartan model of a \( G^* \)-module \( A \) is denoted by

\[
C^*_G(A) = (\mathbb{R}[g^*] \otimes A)^G
\]

where \( \mathbb{R}[g^*] \) denotes the real-valued polynomials on \( g \). The differential is

\[
d(f \otimes a) = f \otimes da - \sum_{k=1}^r f x_k \otimes \iota_{\xi_k} a;
\]

here \( (\xi_k) \) is a basis for \( g \) with dual basis \( (x_k) \). The cohomology of \( C^*_G(A) \) is the \textit{equivariant cohomology} of \( A \) and denoted by \( H^*_G(A) \). The coefficient field \( \mathbb{R} \) is a \( G^* \)-module with trivial operations; we set \( R_G = H^*_G(\mathbb{R}) = C^*_G(\mathbb{R}) = \mathbb{R}[g^*]^G \). The Cartan model \((2.3)\) is a \( dg \) \( R_G \)-module, hence \( H^*_G(A) \) is an \( R_G \)-module.

Remark 2.3. If \( A \) is a \( G^* \)-algebra in the sense of [21] Def. 2.3.1], then \( C^*_G(A) \) is a \( dg \) \( R_G \)-algebra by componentwise multiplication, and \( H^*_G(A) \) is an \( R_G \)-algebra. However, most of the time we will not be concerned with the multiplicative structure in equivariant cohomology. The only exceptions are Section 5.3 where we use the cup product in the equivariant de Rham cohomology of an orientable manifold or orbifold to establish equivariant Poincaré duality, and Section 6.4 where we define the equivariant Poincaré pairing.

Lemma 2.4. Let \( A \) be a bounded below \( G^* \)-module. There is a first-quadrant spectral sequence, natural in \( A \), with

\[
E_1 = R_G \otimes H^*(A) \Rightarrow H^*_G(A).
\]

Proof. See [21] Thm. 6.5.2].

Let \( A \) be a \( G^* \)-module. The inclusion \( K \hookrightarrow G \) of a closed subgroup induces a canonical restriction map \( C^*_G(A) \rightarrow C^*_K(A) \), which is a chain map compatible with the restriction map \( R_G \rightarrow R_K \). For \( K = 1 \) we get a map \( C^*_G(A) \rightarrow A \), which induces the restriction map

\[
(2.5) \quad H^*_G(A) \rightarrow H^*(A).
\]
Let $\mathcal{W} = \mathbb{R}[g^*] \otimes \bigwedge g^*$ be the Weil algebra of $g^*$, cf. [21] Sec. 3.2, and let $A$ be a $\mathcal{W}^*$-module, see [21] Def. 3.4.1. For example, if $G$ acts locally freely on a manifold $X$, then the de Rham complex $\Omega^*(X)$ is a $\mathcal{W}^*$-module, cf. Lemma 3.4 below. There is a canonical isomorphism (with some twisted differential on the right-hand side)

\[(2.6) \quad A = \bigwedge g^* \otimes A_{\text{hor}}\]

where $A_{\text{hor}} \subset A$ denotes the subcomplex of horizontal elements, cf. [21] Thm. 3.4.1. The basic subcomplex $A_{\text{bas}} = (A_{\text{hor}})^G$ is canonically a dg module over $\mathcal{W}_{\text{bas}} = R_G$.

**Lemma 2.5.** Let $A$ be a $\mathcal{W}^*$-module. As a complex, $A_{\text{bas}}$ is a deformation retract of $C^*_G(A)$. There is a natural deformation retraction $C^*_G(A) \to A_{\text{bas}}$ which is a morphism of dg $R_G$-modules. In particular, there is a natural isomorphism of $R_G$-modules

\[H^*_G(A) = H^*(A_{\text{bas}}).\]

**Proof.** See in particular [29], or [21] Ch. 5. \hfill \Box

The dual complex $A^\vee$ of a $G^*$-module $A$ is again a $G^*$-module via the assignments

\[(2.7) \quad \langle g \varphi, a \rangle = \langle \varphi, g^{-1} a \rangle, \quad \langle L_\xi \varphi, a \rangle = -\langle \varphi, L_\xi a \rangle, \quad \langle \iota_\xi \varphi, a \rangle = -(-1)^{|\varphi|} \langle \varphi, \iota_\xi a \rangle\]

for $\varphi \in A^\vee$, $a \in A$, $g \in G$, and $\xi \in g$. Here $\langle \varphi, a \rangle$ denotes the pairing between $A^\vee$ and $A$. Moreover, if $A$ is a $\mathcal{W}^*$-module, then so is $A^\vee$, where the $\mathcal{W}$-module structure is defined by

\[(2.8) \quad \langle w \varphi, a \rangle = (-1)^{|w||\varphi|} \langle \varphi, wa \rangle\]

for $\varphi \in A^\vee$, $a \in A$ and $w \in \mathcal{W}$.

**Lemma 2.6.** For any $\mathcal{W}^*$-module $A$ there is an isomorphism of dg $R_G$-modules

\[(A^\vee)_{\text{bas}} \cong (A_{\text{bas}})^\vee [-\dim G].\]

**Proof.** Dualizing (2.6), we get an isomorphism of $\mathcal{W}^*$-modules

\[(2.9) \quad A^\vee = \bigwedge g^* \otimes (A_{\text{hor}})^\vee\]

where $\bigwedge g^* \subset \mathcal{W}^*$ acts on the right-hand side through the pairing with $\bigwedge g$, and the operators $\iota_\xi$ act on $\bigwedge g$ by exterior multiplication. Hence

\[(2.10) \quad (A^\vee)_{\text{hor}} = \bigwedge_{\dim G} g^* \otimes (A_{\text{hor}})^\vee \cong (A_{\text{hor}})^\vee [-\dim G],\]

and the claim follows by taking $G$-invariants, given that $\bigwedge_{\dim G} g^*$ is $G$-invariant. \hfill \Box

2.5. **Universal coefficient theorem.** Let $A$ be a bounded $G^*$-module. We want to relate the equivariant cohomology of $A$ to that of $A^\vee$. The following result will be crucial for us.

**Lemma 2.7** (Kostant). There is a $G$-submodule $\mathcal{H} \subset \mathbb{R}[g^*]$ such that the restricted multiplication map

\[R_G \otimes \mathcal{H} = \mathbb{R}[g^*]^G \otimes \mathcal{H} \to \mathbb{R}[g^*]\]

is bijective, hence an isomorphism of $G$-modules and $R_G$-modules.

The $R_G$-action on $R_G \otimes \mathcal{H}$ is on the first factor only. Note that we necessarily have $\mathcal{H}^G = \mathbb{R}$, the constant polynomials.
Proof. See [23] Thm. 0.2; the submodule $\mathcal{H}$ is given by the harmonic polynomials on $g$. The case of complex coefficients that Kostant considers is obtained from the real case by extension of scalars, so the result holds already with real coefficients. □

From Lemma [2.7] we get isomorphisms of $R_G$-modules
\begin{align}
(2.11) & \quad C^*_G(A) = (\mathbb{R}[g^*] \otimes A)^G = R_G \otimes (\mathcal{H} \otimes A)^G, \\
(2.12) & \quad \text{Hom}_{R_G}(C^*_G(A), R_G) = \text{Hom}_{\mathbb{R}}((\mathcal{H} \otimes A)^G, R_G).
\end{align}

Lemma 2.8. As a dg $R_G$-module, $C^*_G(A)$ is homotopy equivalent to $R_G \otimes H^*(A)$ with some twisted differential.

Proof. By (2.11) we have
\begin{equation}
\mathbb{R} \otimes_{R_G} C^*_G(A) = (\mathcal{H} \otimes A)^G
\end{equation}
with some twisted differential. We filter this complex by degree in $A$. The $E_1$ page is
\begin{equation}
E_1 = (\mathcal{H} \otimes H^*(A))^G = \mathcal{H}^G \otimes H^*(A) = H^*(A)
\end{equation}
because taking $G$-invariants and cohomology commute and $G$ acts trivially in cohomology. The spectral sequence therefore degenerates, and
\begin{equation}
H^*(\mathbb{R} \otimes_{R_G} C^*_G(A)) = H^*(A).
\end{equation}
As $C^*_G(A)$ is free over $R_G$, it is $R_G$-homotopy equivalent to $R_G \otimes H^*(A)$ with some twisted differential by [24] Cor. B.2.4, cf. also [2] Rem. 3.3.

The canonical pairing $A^v \times A \to \mathbb{R}$, $(\varphi, a) \mapsto \langle \varphi, a \rangle$, has the equivariant extension
\begin{equation}
C^*_G(A^v) \times C^*_G(A) \to C^*_G(\mathbb{R}) = R_G, \quad (f_1 \otimes \varphi, f_2 \otimes a) \mapsto \langle \varphi, a \rangle f_1 f_2,
\end{equation}
whence a morphism of dg $R_G$-modules
\begin{equation}
\Phi: C^*_G(A^v) \to \text{Hom}_{R_G}(C^*_G(A), R_G).
\end{equation}
For $G = T$ a torus, it is an isomorphism as one can see by comparing (2.11) with (2.12). In general we have the following:

Lemma 2.9. The map $\Phi$ is a quasi-isomorphism of dg $R_G$-modules.

Proof. We filter (2.11) (with $A^v$ instead of $A$) and (2.12) by degree in $R_G$; this is compatible with the map (2.14). The differentials $d_0$ on both $E_0$ pages come from the differential in $A$, hence the $E_1$ pages are
\begin{equation}
E_1 = R_G \otimes H^*((\mathcal{H} \otimes A^v)^G) = R_G \otimes H^*(A)^v
\end{equation}
in the first case, and
\begin{equation}
E_1 = \text{Hom}_{\mathbb{R}}(H^*(A), R_G) = R_G \otimes H^*(A)^v
\end{equation}
in the second. We claim that the map $E_1(\Phi)$ between them is an isomorphism: Represent an element of (2.15) by $f \otimes \varphi$ where $\varphi$ is a $G$-invariant cycle in $A^v$ and $f \in R_G$. It maps again to $f \otimes [\varphi]$ in (2.19), proving the claim. We therefore get an isomorphism between the $E_{\infty}$ pages.

By assumption, both $C^*_G(A)$ and $C^*_G(A^v)$ are $R_G$-homotopy equivalent to dg $R_G$-modules which are bounded below. The same applies to $\text{Hom}_{R_G}(C^*_G(A), R_G)$ by Lemma [2.8]. Under the above filtration, such modules lead to first-quadrant spectral sequences with naive convergence. Because all pages of these spectral sequences
coincide with those of the original complexes, there are no convergence issues for either spectral sequence. We conclude that \( H^*(\Phi) \) is an isomorphism. \qed

**Proposition 2.10** (Universal coefficient theorem). Let \( A \) be a bounded \( G^* \)-module. There is a spectral sequence, natural in \( A \), with

\[
E_2 = \text{Hom}_{R_G}(H^*_G(A), R_G) \Rightarrow H^*_G(A^*).
\]

**Proof.** By Lemma 2.8 \( C^*_G(A) \) is \( R_G \)-homotopy equivalent to a dg \( R_G \)-module which is free over \( R_G \) on generators of bounded degrees. For such dg \( R_G \)-modules the proof of [2, Prop. 3.5] carries over and establishes a spectral sequence with the given \( E_2 \) page and converging to the cohomology of \( \text{Hom}_{R_G}(C^*_G(A), R_G) \), which is equal to \( H^*_G(A^*) \) by Lemma 2.9. \qed

### 3. Equivariant homology and cohomology

#### 3.1. Equivariant de Rham cohomology

Although we are primarily interested in manifolds, singular spaces will inevitably come up in our discussion of the orbit filtration. Because the Cartan model for equivariant cohomology has technical advantages, we discuss how to extend it to singular spaces embedded in manifolds and, more generally, in orbifolds.

**3.1.1. Closed supports.** Let \( Y \subset X \) be closed subsets of an orbifold \( Z \). We write \( H^*(X, Y) \) for the Alexander–Spanier cohomology of the pair \((X, Y)\) with closed supports. Our starting point for a de Rham model is the tautness property of Alexander–Spanier cohomology,

\[
H^*(X, Y) = \lim_{\to} H^*(U, V),
\]

where the direct limit is taken over all open neighbourhood pairs \((U, V)\) of \((X, Y)\), see [31, Cor. 6.6.3]. Since \( U \) and \( V \) are orbifolds, \( H^*(U, V) \) may be interpreted as singular cohomology or, if \( V = \emptyset \), as de Rham cohomology.

Because direct limits are exact functors, we have \( H^*(X) = H^*(\Omega^*_Z(X)) \), where the dg algebra \( \Omega^*_Z(X) \) of germs at \( X \) of differential forms on \( Z \) is defined as

\[
\Omega^*_Z(X) = \lim_{\to} \Omega^*(U);
\]

the direct limit ranges over all open neighbourhoods \( U \) of \( X \) in \( Z \); \( \Omega^*_Z(Y) \) is defined analogously. Let \( \Omega^*_Z(X, Y) \) be the kernel of the restriction map \( \Omega^*_Z(X) \to \Omega^*_Z(Y) \); it is a dg module over \( \Omega^*_Z(X) \).

**Lemma 3.1.** As a graded vector space, \( H^*(\Omega^*_Z(X, Y)) \) is naturally isomorphic to \( H^*(X, Y) \), the Alexander–Spanier cohomology of the pair \((X, Y)\). This isomorphism is compatible with the module structure over \( H^*(\Omega^*_Z(X)) = H^*(X) \).

**Proof.** The Alexander–Spanier cohomology of an open pair \((U, V)\) in \( Z \) is naturally isomorphic to the cohomology \( H^*_Z(U, V) \) of the complex \( S^*_Z(U, V) \) of smooth singular cochains. (For the manifold case see [31, Sec. 5.32], for instance. It follows for orbifolds by looking at uniformizing charts.) We therefore have

\[
H^*(X, Y) = \lim_{\to} H^*_Z(U, V) = H^*(\lim_{\to} S^*_Z(U, V)).
\]

From the de Rham theorem, we obtain a natural quasi-isomorphism

\[
\Omega_Z(X) = \lim_{\to} \Omega^*(U) \to \lim_{\to} S^*_Z(U).
\]
Naturality yields the left vertical arrow in the commutative diagram
\[
\begin{array}{cccccc}
0 & \rightarrow & \Omega^\ast_Z(X,Y) & \rightarrow & \Omega^\ast_Z(X) & \rightarrow & \Omega^\ast_Z(Y) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \lim S^\ast_\infty(U,V) & \rightarrow & \lim S^\ast_\infty(U) & \rightarrow & \lim S^\ast_\infty(V) & \rightarrow & 0.
\end{array}
\]

The top row is exact: Given a differential form \(\alpha\) on \(V \supset Y\), choose a smooth function \(f\) on \(Z\) with support in \(V\) and identically equal to 1 in a smaller neighbourhood \(V'\) of \(Y\). Then \(f\alpha\) is defined on all of \(Z\) and restricts to the same germ at \(Y\) as \(\alpha\).

Since the bottom row of (3.5) is also exact, we conclude from the five lemma that the left vertical arrow is a quasi-isomorphism. Together with (3.3) this establishes the claimed isomorphism of graded vector spaces.

That the module structures coincide follows from the fact that the isomorphisms induced by the vertical arrows in (3.5) are compatible with (sheaf-theoretic) cup products, cf. [31, Th. 5.45] and [3, Sec. II.7, Thm. III.3.1].

Now assume that \(Z\) is a \(G\)-orbifold and the pair \((X,Y)\) \(G\)-stable. We can always assume germs of differential forms at \(X\) to be defined on \(G\)-stable open neighbourhoods, which are cofinal among all open neighbourhoods of \(X\). Hence \(\Omega^\ast_Z(X)\) is a \(G\)-module, and the same applies to \(\Omega^\ast_Z(Y)\) and \(\Omega^\ast_Z(X,Y)\). So we can consider their Cartan models. (See [26, p. 245] for the Cartan model for orbifolds.)

**Lemma 3.2.** As \(R_G\)-module, \(H^\ast_G(\Omega^\ast_Z(X,Y))\) is naturally isomorphic to \(H^\ast_G(X,Y)\), the Borel \(G\)-equivariant Alexander–Spanier cohomology of the pair \((X,Y)\).

In particular, \(R_G\) is isomorphic to \(H^\ast(BG)\), the cohomology algebra of the classifying space of \(G\).

**Proof.** Recall the natural isomorphism
\[
(3.6) \quad H^\ast_G(X,Y) = H^\ast(X \times_G EG_k, Y \times_G EG_k)
\]
where \((EG_k)\) is a family of compact free \(G\)-manifolds approximating \(EG\) and \(k\) is chosen sufficiently large compared to \(n\). By Lemma 3.1 this implies
\[
(3.7) \quad H^\ast_G(X,Y) = H^\ast(\Omega^\ast_Z(X \times_G EG_k, Y \times_G EG_k))
\]
for \(k \gg n\). Also by Lemma 3.1 and naturality, this isomorphism is compatible with the module structures over \(H^\ast_G(pt) = H^\ast(BG)\).

It therefore suffices to find a family of natural morphisms of dg algebras
\[
(3.8) \quad C_G(\Omega^\ast_Z(X,Y)) \rightarrow \Omega^\ast_Z(X \times_G EG_k, Y \times_G EG_k)
\]
that induce isomorphisms in any given cohomological degree \(n \ll k\).

For the case \((X,Y) = (Z, \emptyset)\), this can be found in [21, Sec. 2.5], modulo the natural isomorphism between the Cartan model and the Weil model for equivariant cohomology, cf. [21, Thm. 4.2.1]. This implies in particular \(R_G = H^\ast(BG)\). Noting that due to the compactness of \(EG_k\) neighbourhoods of the form \(U \times_G EG_k\) are cofinal among all neighbourhoods of \(X \times_G EG_k\), we get the absolute case for arbitrary \(X\) by taking direct limits as in the proof of Lemma 3.1. The relative case again follows by the five lemma. \(\square\)
Remark 3.3. Lemma $2.3$ implies that $H^*_G(X,Y)$ is finitely generated over $R_G$ if $H^*(X,Y)$ is finite-dimensional. The converse is a consequence of the Eilenberg–Moore theorem.

Lemma 3.4. If $G$ acts locally freely on $X \setminus Y$, then $\Omega^*_Z(X,Y)$ is a $W^*$-module, and

$$
\Omega^*_Z(X,Y)_{bas} = \Omega^*_{Z/G}(X/G,Y/G).
$$

In particular, $H^*_G(X,Y) = H^*(X/G,Y/G)$.

If $G$ acts locally freely on all of $Z$, then $Z/G$ is an orbifold and $(X/G,Y/G)$ is a closed pair in $Z/G$, so that $\Omega^*_{Z/G}(X/G,Y/G)$ has been defined above. The definition of $\Omega^*_{Z/G}(X/G,Y/G)$ in the general case will be given in the proof.

Proof. Assume first that $G$ acts locally freely on $Z$, and choose a connection form for $Z$. Then the de Rham complex $\Omega^*(U)$ of any $G$-stable open subset $U \subset Z$ is naturally a $W^*$-module, hence so are $\Omega^*_Z(X)$ and $\Omega^*_Z(Y)$, and the restriction map $\Omega^*_Z(X) \to \Omega^*_Z(Y)$ is a morphism of $W^*$-module. Thus, its kernel $\Omega^*_Z(X,Y)$ is also a $W^*$-module.

We have $\Omega^*(U)_{bas} = \Omega^*(U/G)$, cf. $[21]$, Sec. 2.3.5, hence $\Omega^*_Z(X)_{bas} = \Omega^*_Z(X/G)$. The same holds again for $Y$, and by naturality we find

$$
\Omega^*_Z(X,Y)_{bas} = \Omega^*_Z(X/G,Y/G).
$$

This together with Lemma $2.3$ gives

$$
H^*_G(X,Y) = H^*(X/G,Y/G).
$$

Observe that $X$ is metrizable, hence normal. In the general case we can therefore write

$$
\Omega^*_Z(X,Y) = \lim_{\rightarrow} \Omega^*_Z(X,B)
$$

where the direct limit is taken over all $G$-stable closed neighbourhoods $B$ of $Y$ in $X$, and

$$
= \lim_{\rightarrow} \Omega^*_Z(X \setminus Y, B \setminus Y)
$$

because any differential form defined on a neighbourhood $U$ of $X \setminus Y$ and vanishing on a neighbourhood $V \subset U$ of $B \setminus Y$ can be extended to a one defined on $U \cup Y$ and vanishing on $V \cup Y$. (Note that $U \cup Y$ and $V \cup Y$ are open in $X$.) By what we have said above, (3.12) is a direct limit of $W^*$-modules, hence a $W^*$-module itself, and

$$
\Omega^*_Z(X,Y)_{bas} = \lim_{\rightarrow} \Omega^*_Z(X/G,Y/G,B' \setminus Y/G),
$$

where $B'$ runs through the closed neighbourhoods of $Y/G$ in $X/G$. The right-hand side is our definition of $\Omega^*_Z(X/G,Y/G)$.

Moreover, using Lemma $2.3$ and arguments similar to the ones above, we find

$$
H^*_G(\Omega^*_Z(X,Y)) = H^*(\Omega^*_Z(X,Y)_{bas}) = \lim_{\rightarrow} H^*(X/G \setminus Y/G,B' \setminus Y/G)
$$

$$
= \lim_{\rightarrow} H^*(X/G,B') = H^*(X/G,Y/G).
$$
3.1.2. Compact supports. We write $H^*_c(X, Y)$ for the Alexander–Spanier cohomology of the pair $(X, Y)$ with compact supports. There is a natural isomorphism

$$H^*_c(X, Y) = \lim_{\rightarrow} H^*(X, B)$$

where the direct limit ranges over all neighbourhoods $B$ of $Y$ such that $X \setminus B$ has compact closure in $X$, see [30, Thm. 6.6.16]. Since $X$ is metrizable, it is enough to consider closed neighbourhoods of $B$. We therefore set

$$\Omega^*_c(X, Y) = \lim_{\rightarrow} \Omega^*_c(X, B)$$

where $B$ ranges over all closed neighbourhoods of $Y$ such that $X \setminus B$ has compact closure. Then everything said about closed supports carries over to compact supports. In particular:

**Lemma 3.5.** $H^*_c(\Omega^*_c(X, Y)) = H^*_c(X, Y)$, the Alexander–Spanier cohomology of the pair $(X, Y)$ with compact supports.

**Lemma 3.6.** $H^*_G(\Omega^*_c(X, Y)) = H^*_G(\Omega^*_c(X, Y))$, the $G$-equivariant Alexander–Spanier cohomology of the pair $(X, Y)$ with compact supports.

**Lemma 3.7.** If $G$ acts locally freely on $X \setminus Y$, then $\Omega^*_c(X, Y)$ is a $W^*$-module, and

$$\Omega^*_c(X, Y)_{bas} = \Omega^*_c(X/G, Y/G).$$

In particular, $H^*_G(\Omega^*_c(X, Y)) = H^*_c(X/G, Y/G)$.

Restriction of forms and extension by 0 give isomorphisms of $G^*$-modules

$$\Omega^*_c(X, Y) = \Omega^*_c(X \setminus Y),$$

which confirms that cohomology with compact supports is a ‘single space theory’. From it we recover the natural isomorphism (see [30, Thm. 6.6.11])

$$H^*_c(X, Y) = H^*_c(X \setminus Y)$$

and, using Lemma 3.5, its equivariant analogue,

$$H^*_G(X, Y) = H^*_c(X \setminus Y).$$

The special case $X = Z$ of (3.17) together with the exact top row of diagram (3.5) gives a natural isomorphism

$$\Omega^*_c(Y) = \Omega^*_c(Z) / \Omega^*_c(Z, Y) / \Omega^*_c(Z \setminus Y),$$

which could be used as the definition of $\Omega^*_c(Y)$; again via (3.17) it can be extended to pairs. This is the approach taken for example by Guillemin–Sternberg [21, Sec. 11.1] and also by De Concini–Procesi–Vergne [11, Sec. 1, App. A], who prove Lemmas 3.5 and 3.6 under the assumption that $X \setminus Y$ is locally contractible [11 Prop. A.4 & A.8].

3.2. Equivariant homology. We want to define equivariant homology by appropriately dualizing the Cartan model of differential forms. The most conceptual approach would be to use currents, as done by Metzler [21, p. 169] and Meinrenken [27, Sec. 4]. Because we are ultimately only interested in spaces with finite-dimensional cohomology, we will work with the algebraic instead of the topological dual of the de Rham complex.
For ease of notation, we only write out results for cohomology with closed supports and homology with compact supports in this section. All results remain valid for (co)homology with the other pair of supports.

Let \( Y \subset X \) be closed \( G \)-stable subsets of an orbifold \( Z \). We say in this case that \((X,Y)\) is a \( G \)-pair in \( Z \). Often we do not mention the ambient orbifold \( Z \) explicitly; we say that \( X \) and \( Y \) are \( G \)-spaces, and we write \( \Omega^\ast(X,Y) \) instead of \( \Omega^\ast_2(X,Y) \).

Assumption 3.8. From now on, whenever we consider the homology or equivariant homology (with compact supports) of a \( G \)-pair \((X,Y)\), we assume that \( H^\ast(X,Y) \) or, equivalently, \( H_\ast^c(X,Y) \) is finite-dimensional. In this case there is a natural isomorphism

\[
H_\ast^c(X,Y) = H^\ast_\ast(X,Y)^\vee.
\]

We make no such assumption for (equivariant) homology with closed supports, and in fact

\[
H^\ast_c(X,Y) = H^\ast_\ast(X,Y)^\vee
\]

is always true.

We define the equivariant homology \( H^G_\ast(X,Y) \) of the \( G \)-pair \((X,Y)\) as the cohomology of the Cartan model

\[
C^G_\ast(X,Y) = C^G_\ast(\Omega^\ast(X,Y)^\vee),
\]

where \( \Omega^\ast(X,Y)^\vee \) denotes the dual of the \( G^* \)-module \( \Omega^\ast(X,Y) \) as defined in Section 2.4.

Remark 3.9. Assume that \( G = T \) is a torus. As remarked after (2.17), the canonical map

\[
C^T_\ast(X,Y) \to \text{Hom}_{R_T}(C^T_\ast(X,Y), R_T)
\]

is an isomorphism of \( \text{dg } R_T \)-modules in this case. We thus recover the definition of torus-equivariant homology given in \([2]\) and \([3]\), up to the substitution of the Cartan model for the “singular Cartan model” used there. For general \( G \), the analogue of (3.24) still is a quasi-isomorphism by Lemma 2.9.

The cohomology of \( \Omega^\ast(X,Y)^\vee \) is \( H^\ast(X,Y)^\vee = H_\ast(X,Y) \) by Assumption 3.8. As a special case of Lemma 2.4, we therefore get:

Proposition 3.10. There is a first-quadrant spectral sequence, natural in \((X,Y)\), with

\[
E_1 = R_G \otimes H_\ast(X,Y) \Rightarrow H^G_\ast(X,Y).
\]

Proposition 3.11 (Universal coefficient theorem). There is a spectral sequence, natural in \((X,Y)\), with

\[
E_2 = \text{Hom}_{R_G}(H^G_\ast(X,Y), R_G) \Rightarrow H^G_\ast(X,Y).
\]

If \( H_\ast(X,Y) \) is finite-dimensional, then there is another natural spectral sequence with

\[
E_2 = \text{Hom}_{R_G}(H^G_\ast(X,Y), R_G) \Rightarrow H^\ast_\ast(X,Y).
\]
Proof. The first spectral sequence is a special case of the algebraic universal coefficient theorem (Proposition 2.10). The same result also gives a spectral sequence of the second form converging to $H^*_c(X, Y)^{\omega\omega}$. The latter is equal to $H^*_c(X, Y)$ by Lemma 2.4 because the canonical map $\Omega^*(X, Y) \to \Omega^*(X, Y)^{\omega\omega}$ is a quasi-isomorphism by assumption. □

Proposition 3.12. If $G$ acts locally freely on $X \setminus Y$, then there is an isomorphism of $R_G$-modules

$$H^*_c(X, Y) \cong H_{\ast+\dim G}(X/G, Y/G),$$

where the $R_G$-module structure of $H_\ast(X/G, Y/G)$ is dual to the one in cohomology.

Proof. By Lemma 3.1 (or Lemma 3.7, depending on supports) we have

$$(\Omega^*(X, Y)^{\omega\omega})_{bas} = \Omega^*(X/G, Y/G),$$

hence

$$(\Omega^*(X, Y)^{\omega\omega})_{bas} \cong \Omega^*(X/G, Y/G)^{[-\dim G]}$$

by Lemma 2.6. We conclude with Lemma 2.5. □

3.3. Equivariant Poincaré duality. In this section the supports of the (co)homology groups do matter, and we carefully distinguish between them.

Let $Z$ be an orbifold of dimension $n$. We say that $Z$ is locally orientable if it is locally the quotient of some $\mathbb{R}^n$ by a finite subgroup of $SO(n)$. In this case, it is a rational homology manifold, and orientable if and only if $H^*_c(Z) \cong \mathbb{R}$. Any manifold is a locally orientable orbifold.

Now assume that $Z$ is a locally orientable $G$-orbifold, and let $\pi: \tilde{Z} \to Z$ be the orientable two-fold cover of $Z$ with a fixed orientation. If $Z$ is orientable, then $\tilde{Z}$ consists of two copies of $Z$ with opposite orientations. The $G$-action on $Z$ lifts to $\tilde{Z}$, cf. [3] Lemma 2.13. The non-trivial deck transformation $\tau$ commutes with $G$ and reverses the orientation.

For any $G$-pair $(X, Y)$ in $Z$, the $(+1)$-eigenspace of the induced map $\tau^*$ on $H^*_c(\pi^{-1}(X), \pi^{-1}(Y))$ is isomorphic to $H^*_c(X, Y)$ as an $R_G$-module. We define $H^*_c(X, Y; \mathbb{R})$, the equivariant cohomology of $(X, Y)$ twisted coefficients, to be the $(-1)$-eigenspace. Equivalently, it is the equivariant cohomology of $\Omega^*_c(X, Y; \mathbb{R})$, the $(-1)$-eigenspace of $\tau^*$ on $\Omega^*_c(\pi^{-1}(X), \pi^{-1}(Y))$. There are analogous definitions for equivariant homology and for other supports.

The integration map

$$(3.27) \quad I: \Omega^*_c(\tilde{Z}) \to \mathbb{R}, \quad \alpha \mapsto \int_{\tilde{Z}} \alpha$$

(which gives 0 if $|\alpha| < n$) defines an orientation $o = [I] \in H^n_c(Z; \mathbb{R})^{\omega\omega} = H_c^n(Z; \mathbb{R})$. Moreover, it is a morphism of $G^*$-modules of degree $-n$, where $\mathbb{R}$ is considered as a trivial $G^*$-module. Hence $I_G = 1 \otimes I \in C^*_c(\Omega^*_c(\tilde{Z})^{\omega\omega})$ is a cycle and defines an equivariant homology class $o_G \in H^n_{G^*}(Z; \mathbb{R})$, which restricts to $o$ under the map $(2.3)$. In fact, it is the unique lift:

Lemma 3.13. Any orientation $o \in H^n_c(Z)$ lifts uniquely to an equivariant orientation $o_G \in H^n_{G^*}(Z)$.

Proof. The proof of [2] Prop. 3.7] carries over. □
The integration map $I$ also induces the morphism of $G^*$-modules

$$(3.28) \quad PD: \Omega^*(\tilde{Z}) \to \Omega^*_c(\tilde{Z}), \quad \alpha \mapsto (\beta \mapsto I(\alpha \wedge \beta))$$

which interchanges the $(\pm 1)$-eigenspaces. Based on it, we obtain equivariant Poincaré duality in the same way as Metzler [21, Thm. 10.10.1]:

**Proposition 3.14.** The map $PD$ induces an isomorphism $H^*_G(Z) \to H^*_{G,c}(Z; \mathbb{R})$ of $R_G$-modules of degree $-n$.

**Proof.** Between the $E_1$ pages of the spectral sequences from Lemma 2.4, the map induced by $PD$ is the $R_G$-linear extension of the non-equivariant map $H^*(PD)$, hence an isomorphism. Thus so is $PD$ itself. □

4. **Restriction and induction**

For ease of notation, we again stick to cohomology with closed supports and homology with compact supports in this section. All results remain valid for (co)homology with the other pair of supports and/or with twisted coefficients.

Recall that $T \cong (S^1)^r$ is a maximal torus of $G$ and $W$ the corresponding Weyl group.

4.1. **Restriction.** The restriction $R_G \to R_T$ is an isomorphism onto the subalgebra $(R_T)^W$ of Weyl group invariants. Moreover, there is a canonical isomorphism of $R_G$-modules

$$(4.1) \quad R_T = R_G \otimes H^*(G/T)$$

where $R_G$ acts only on the first factor of the tensor product. In particular, $R_T$ is finitely generated and free over $R_G$. As a consequence, any $R_T$-module is finitely generated over $R_T$ if and only if it is finitely generated over $R_G$.

Let $(X, Y)$ be a $G$-pair. It is canonically a $T$-pair by restriction of the action. The following proposition is a special case of general results about restrictions of $G^*$-modules, see [21, Sec. 6.8].

**Proposition 4.1.** There are the following isomorphisms, natural in $(X, Y)$:

1. **As $R_G$-modules,**

$$H^*_G(X, Y) = H^*_T(X, Y)^W \quad \text{and} \quad H^*_c(X, Y) = H^*_c(X, Y)^W.$$  

2. **As $R_T$-modules,**

$$H^*_T(X, Y) = R_T \otimes_{R_G} H^*_G(X, Y) \quad \text{and} \quad H^*_T(X, Y) = R_T \otimes_{R_G} H^*_c(X, Y).$$

The cohomological parts are well-known and imply in particular that $H^*_G(X, Y)$ is finitely generated over $R_G$ if and only if $H^*_T(X, Y)$ is finitely generated over $R_T$. This also follows from the fact that both conditions are equivalent to $H^*(X, Y)$ being finite-dimensional.

**Proposition 4.2.** Let $(X, Y)$ be a $G$-pair such that $H^*_G(X, Y)$ is finitely generated over $R_G$, and let $0 \leq j \leq r$. Then $H^*_G(X, Y)$ is a $j$-th syzygy over $R_G$ if and only if $H^*_T(X, Y)$ is a $j$-th syzygy over $R_T$. 


Proof. Assume that $H^*_G(X,Y)$ is a $j$-th syzygy over $R_G$. By definition, there is an exact sequence

\[(4.2) \quad 0 \to H^*_G(X,Y) \to F_1 \to \cdots \to F_j\]

with finitely generated free $R_G$-modules $F_i$. Because $R_T$ is free over $R_G$, we obtain the exact sequence

\[(4.3) \quad 0 \to R_T \otimes_{R_G} H^*_G(X,Y) \to R_T \otimes_{R_G} F_1 \to \cdots \to R_T \otimes_{R_G} F_j\]

with finitely generated free $R_T$-modules $R_T \otimes_{R_G} F_i$. This shows that $H^*_T(X,Y) = R_T \otimes_{R_G} H^*_G(X,Y)$ is a $j$-th syzygy over $R_T$.

Now assume that $H^*_T(X,Y)$ is a $j$-th syzygy over $R_T$. This time there is an exact sequence

\[(4.4) \quad 0 \to H^*_T(X,Y) \to F_1 \to \cdots \to F_j\]

with finitely generated free $R_T$-modules $F_i$. Since the $F_i$ are also finitely generated and free over $R_G$, this shows that

\[(4.5) \quad H^*_T(X,Y) = H^*_G(X,Y) \otimes H^*(G/T)\]

is a $j$-th syzygy over $R_G$. It now follows from criterion (2) or (3) of Proposition 2.2 that the same holds for $H^*_G(X,Y)$ itself. \hfill \square

4.2. Induction. Let $(X,Y)$ be a $T$-pair, say contained in the $T$-orbifold $Z$. Then

\[(4.6) \quad \hat{X} = G \times_T X \quad \text{and} \quad \hat{Y} = G \times_T Y\]

are closed $G$-stable subsets of the $G$-orbifold $\hat{Z} = G \times_T Z$, hence $(\hat{X}, \hat{Y})$ is a $G$-pair. There is a canonical inclusion $Z \to \hat{Z}$ sending $z$ to $[1, z]$, equivariant with respect to the inclusion $T \to G$. Also note that any $R_T$-module is canonically an $R_G$-module via the restriction map $R_G \to R_T$.

Lemma 4.3. The inclusion of pairs $(X,Y) \to (\hat{X}, \hat{Y})$ induces the isomorphism of $R_G$-modules

\[H^*_G(\hat{X}, \hat{Y}) = H^*_T(X,Y)\]

Proof. See [12, Thm. 24] for a proof using equivariant de Rham theory. \hfill \square

In particular, $H^*_G(\hat{X}, \hat{Y})$ is finitely generated over $R_G$ if and only if $H^*_T(X,Y)$ is finitely generated over $R_T$.

To study the behaviour of syzygies under induction from $T$ to $G$, we need the following simple algebraic fact.

Lemma 4.4. Let $A \subset B$ be an extension of commutative rings, $q \triangleq B$ be a prime ideal and $p = q \cap A$. Then for any $B$-module $M$,

\[\text{depth}_{B_q} M_q \geq \text{depth}_{A_p} M_p.\]

Proof. Let $a_1, \ldots, a_k \in p \cap q$ be an $M_p$-regular sequence. By induction on $k$ we show that this sequence is also $M_q$-regular. The case $k = 0$ is void.

Assume

\[(4.7) \quad \frac{a_k m_k}{s_k} = \frac{a_1 m_1}{s_1} + \cdots + \frac{a_{k-1} m_{k-1}}{s_{k-1}}\]

for some $m_1, \ldots, m_k \in M$ and some $s_1, \ldots, s_k \in B \setminus q$. Then, for some $s \in B \setminus q$,

\[(4.8) \quad a_k (ss_1 \ldots s_{k-1} m_k) \in (a_1 \ldots a_{k-1})M_p.\]
Since \( a_1, \ldots, a_k \) is \( M_p \)-regular, this implies \( s s_1 \ldots s_{k-1} m_k \in (a_1 \ldots a_{k-1}) M_p \) and therefore

\[
(4.9) \quad \frac{m_k}{s_k} \in (a_1 \ldots a_{k-1}) M_q.
\]

Hence multiplication by \( a_k \) is injective on \( M_q/(a_1 \ldots a_{k-1}) M_q \), which means that the sequence is \( M_q \)-regular. \( \square \)

**Proposition 4.5.** Let \((X, Y)\) and \((\hat{X}, \hat{Y})\) be as before, and assume that \( H_*^G(X, Y) \) is finitely generated over \( R_T \). Then \( H_*^G(\hat{X}, \hat{Y}) \) is a \( j \)-th syzygy over \( R_G \) if and only if \( H_*^G(X, Y) \) is a \( j \)-th syzygy over \( R_T \).

**Proof.** Assume that \( H_*^G(X, Y) \) is a \( j \)-th syzygy over \( R_T \). Then there is an exact sequence

\[
(4.10) \quad 0 \to H_*^G(X, Y) \to F_1 \to \cdots \to F_j
\]

with finitely generated free \( R_T \)-modules \( F_i \). Since the \( F_i \) are also finitely generated and free over \( R_G \), Lemma 4.3 implies that \( H_*^G(\hat{X}, \hat{Y}) \) is a \( j \)-th syzygy over \( R_G \).

Now assume that \( H_*^G(\hat{X}, \hat{Y}) \) is a \( j \)-th syzygy over \( R_G \). Let \( q \triangleleft R_T \) be a prime ideal and set \( p = q \cap R_G \). Because \( R_T \supset R_G \) is an integral extension of commutative rings, Cohen–Seidenberg’s going-up theorem implies

\[
(4.11) \quad \dim(R_G)_q = \text{ht}_{R_G} q = \text{ht}_{R_T} p = \dim(R_T)_p;
\]

we also have

\[
(4.12) \quad \text{depth} H_*^G(X, Y)_q \geq \text{depth} H_*^G(\hat{X}, \hat{Y})_p
\]

by combining Lemmas 4.3 and 4.4. Using Proposition 2.2 and the assumption that \( H_*^G(\hat{X}, \hat{Y}) \) is a \( j \)-th syzygy, we conclude

\[
(4.13) \quad \text{depth} H_*^G(X, Y)_q \geq \min(j, \dim(R_G)_p) = \min(j, \dim(R_T)_q).
\]

Thus, \( H_*^G(X, Y) \) is a \( j \)-th syzygy over \( R_T \). \( \square \)

**Remark 4.6.** The case \( j = 1 \) of Proposition 4.5 can be shown more easily, see the proof of [20, Thm. C.70] or [19, Thm. 3.9]: If \( f \in R_T \) is non-regular for \( H_*^G(X, Y) = H_*^G(\hat{X}, \hat{Y}) \), then the product \( \prod_{w \in W} w \cdot f \) gives another such element in \( (R_T)^W = R_G \).

**Example 4.7.** Let \( X \) be a projective toric manifold of dimension \( 2r \) with moment polytope \( P = X/T \). It is well-known that \( H_*^G(X) \) is free over \( R_T \) in this case.

Choose two distinct fixed points \( x, y \in X \) and set \( Y = X \setminus \{x, y\} \). Let \( F \subset P \) be the smallest face of \( P \) containing (the images of) \( x \) and \( y \). As shown in [19, Sec. 6.1], the equivariant cohomology \( H_*^G(Y) \) with closed supports is a syzygy of order exactly \( \dim F - 1 \) over \( R_T \); the case \( X = (S^2)^r \) with two diametrically opposite vertices \( x \) and \( y \) of the \( r \)-cube \( P \) appeared already in [2, Sec. 6.1].

From Proposition 4.5 we see that the equivariant cohomology with closed supports of the induced \( G \)-manifold \( G \times_T Y \) is a syzygy of order exactly \( \dim F - 1 \) over \( R_G \). In particular, syzygies of any order can appear as the equivariant cohomology of \( G \)-manifolds.
5. The constant rank case

Let $X$ be a $G$-space such that all $G$-isotropy groups in $X$ have the same rank, say equal to $b \in \mathbb{N}$. Let
\[(5.1) \quad Y = \{ x \in X \mid \text{rank } T_x = b \}\]
be the highest-rank stratum for the $T$-action on $X$; it is $N_G(T)$-stable and, by the slice theorem, closed in $X$. For a subtorus $L \subset T$ of rank $b$, we define
\[(5.2) \quad X(L) = \{ x \in X \mid L \text{ is conjugate in } G \text{ to a maximal torus of } G_x \},
\[(5.3) \quad Y(L) = \{ x \in X \mid L \text{ is a maximal torus of } G_x \} = X^L.
\]
Note that $X$ is the union of all such $X(L)$, and $Y$ is the disjoint union of all such $Y(L)$. Each $Y(L)$ is a closed in $X$, and $Y$ is stable under $N_G(T)$.

**Lemma 5.1.**

1. The sets $X(L)$ partition $X$.
2. Each $X(L)$ is open in $X$.
3. Each $X(L)$ is a union of connected components of $X$.

**Proof.** Let $x \in X(L)$. If in addition $x \in X(L')$, then $X(L) = X(L')$ because all maximal tori of $G_x$ are conjugate. By the slice theorem, any $y$ in a sufficiently small neighbourhood of $x$ has an isotropy group that is conjugate to a subgroup of $G_x$. By our assumption on $X$, this means that $G_y$ contains a maximal torus conjugate to a maximal torus of $G_x$, whence $y \in X(L)$. The last claim follows from the first two.

Hence, in order to prove Proposition 5.5, we may assume $X = X(L)$ for some subtorus $L \subset T$ of rank $b$. Let $K = N_G(L) \supset T$. The following result is due to Goertsches–Rollenske; it appears in the proof of [19, Prop. 4.2].

**Lemma 5.2.** $G \cdot y \cap Y(L) = N_G(L) \cdot y$ for any $y \in Y(L)$.

**Proof.** $\Rightarrow$: because $Y(L)$ is stable under $N_G(L)$.

$\Leftarrow$: Assume $gy \in Y(L)$ for some $g \in G$. Then $L$ is a maximal torus of both $G_y$ and $G_{gy}$. In other words, $L$ and $gLg^{-1}$ are both maximal tori of $G_{gy}$, hence there is an $h \in G_{gy}$ such that $h^{-1}Lh = gLg^{-1}$ or $(hg)L(hg)^{-1} = L$. This shows that $hg \in N_G(L)$, and therefore $gy = hgy \in N_G(L) \cdot y$.

**Lemma 5.3.** The map $q: N_G(T) \times_{N_K(T)} Y(L) \to Y$ induced by the $G$-action is an $N_G$-equivariant homeomorphism.

**Proof.** Clearly, $q$ is $N_G$-equivariant. Since we assume $X = X(L)$, $Y$ is the disjoint union of of the $Y(L')$ where $L'$ runs through all subtori of $L$ which are conjugate in $G$ to $L$. We claim that
\[(5.4) \quad \{ L' \subset T \mid L' \text{ is conjugate in } G \text{ to } L \} = N_G(T) \cdot L = W \cdot L.
\]
$\Rightarrow$: is trivial.

$\Leftarrow$: Assume $L' = gLg^{-1}$ or, equivalently, $L' = \text{Ad}_g \cdot L$ for some $g \in G$. This means that $L'$ lies in the $G$-orbit of $L$ for the induced action on the Grassmannian $Gr_b(g)$. Since $L$ and $L'$ are fixed by $T$, Lemma 5.2 (with $Gr_b(g)$ instead of $X$ and $L = T$) implies that $L' \in N_G(T) \cdot L$.

Coming back to the original claim, we observe that $N_K(T) = N_G(T) \cap K$ is the isotropy group of $L$ for the conjugation action of $N_G(T)$ on the subtori of $T$. 
Hence $Y$ consists, like $N_G(T) \times_{N_K(T)} Y(L)$, of copies of $Y(L)$ indexed by the cosets $N_G(T)/N_K(T)$. Thus, $q$ is a homeomorphism. \hfill \Box

**Lemma 5.4.** The map $f_1: G \times_K Y(L) \to X$ induced by the $G$-action is a quasi-isomorphism (for cohomology with closed or compact supports and possibly twisted coefficients).

For compact $X$ this is again contained in the proof of [19, Prop. 4.2].

**Proof.** We are going to use the Vietoris–Begle theorem, see [30, Thm. 6.9.15] for a precise statement for Alexander–Spanier cohomology with closed supports. We extend it to relative cohomology by the five lemma and then to cohomology with compact supports by (6.15). The case of twisted coefficients follows by looking at the eigenspaces of the deck transformation $\tau$.

Because $G$ is compact, the action map $G \times X \to X$ is closed, hence so is $f_1$. Surjectivity is clear by our assumption $X = X(L)$. To apply the Vietoris–Begle theorem, it is therefore enough to show that the fibres of $f_1$ are acyclic. By $G$-equivariance it suffices to study the fibre over some $x \in Y(L)$.

Assume $f_1([g, y]) = x$, so that $gy = x = hy$ for some $h \in K$ by Lemma 5.2. Then $[g, y] = [gh^{-1}, hy] = [gh^{-1}, x]$ and $gh^{-1} \in G_x$. Hence

\begin{equation}
(5.5) \quad f_1^{-1}(x) \cong G_x/(K \cap G_x) = G_x/N_{G_x}(L),
\end{equation}

which is acyclic by [19, Lemma 3.2] given that $L$ is a maximal torus of $G_x$. \hfill \Box

**Proposition 5.5.** The inclusion $Y \hookrightarrow X$ induces an isomorphism of $R_G$-modules

\[ H^*_G(X) = H^*_T(Y)^W \]

(for equivariant cohomology with closed or compact supports and possibly twisted coefficients).

**Proof.** Consider the commutative diagram

\begin{equation}
\begin{array}{ccc}
G \times_{N_K(T)} Y(L) & \xrightarrow{f_2} & G \times_K Y(L) \\
\downarrow f_3 & & \downarrow f_1 \\
G \times_{N_G(T)} Y & \xrightarrow{f} & X,
\end{array}
\end{equation}

where $f_3$ is induced from the $N_G(T)$-equivariant map $q$ defined in Lemma 5.3

\begin{equation}
(5.7) \quad f_3: G \times_{N_K(T)} Y(L) = G \times_{N_G(T)} \left( N_G(T) \times_{N_K(T)} Y(L) \right) \xrightarrow{(id,q)} G \times_{N_G(T)} Y.
\end{equation}

Like $q$, the map $f_3$ is a homeomorphism.

Again by [19, Lemma 3.2], the fibre $K/N_K(T)$ of the bundle map $f_2$ is acyclic, so that $f_2$ is a quasi-isomorphism. We finally know from Lemma 5.4 that $f_1$ is a quasi-isomorphism, too. Hence $f$ is a $G$-equivariant quasi-isomorphism. It follows that

\begin{equation}
(5.8) \quad H^*_G(X) = H^*_G(G \times_{N_G(T)} Y) = H^*_{N_G(T)}(Y) = H^*_T(Y)^W. \hfill \Box
\end{equation}

6. **Cohen–Macaulay filtrations**

The results in this section hold for (co)homology with either pair of supports and/or with twisted coefficients. For simplicity we only state them for cohomology with closed supports/homology with compact supports and constant coefficients.
6.1. **Definition.** Let $X$ be a $G$-space, and let $\mathcal{F}$ be a filtration

$$\emptyset = X_{-1} \subset X_0 \subset \cdots \subset X_r = X$$

of $X$ by $G$-stable closed subsets. We call $\mathcal{F}$ Cohen–Macaulay if $H^G_*(X_i, X_{i-1})$ is zero or Cohen–Macaulay of dimension $r - i$ for $0 \leq i \leq r$. Because of the following observation, we could substitute equivariant cohomology for equivariant homology in the definition.

**Lemma 6.1.** Let $(X, Y)$ be a $G$-pair and $d \in \mathbb{N}$. Then $H^G_*(X, Y)$ is Cohen–Macaulay of dimension $d$ if and only if $H^G_*(X, Y)$ is so.

**Proof.** It follows from the universal coefficient theorem (Proposition 3.11) that $H^G_*(X, Y)$ is zero if and only if $H^G_*(X, Y)$ is so. Moreover, if $H^G_*(X, Y)$ is Cohen–Macaulay of dimension $d$, then the spectral sequence converging to $H^G_*(X, Y)$ collapses at the $E_2$ page by Proposition 2.1 and

$$H^G_*(X, Y) = \text{Ext}^{-r-d}_{R_G}(H^G_*(X, Y), R_G)[r - d]$$

is again Cohen–Macaulay of dimension $d$. The other direction is analogous. \qed

We will see in Section 6.3 that Cohen–Macaulay filtrations exist. For the moment, we record several properties.

**Proposition 6.2.** Let $(X_i)$ be a Cohen–Macaulay filtration of $X$. For any $0 \leq i \leq r$ there is a short exact sequence

$$0 \to H^G_*(X_i) \to H^G_*(X) \to H^G_*(X, X_i) \to 0.$$

**Proof.** As in [3, Prop. 4.3]. \qed

**Proposition 6.3.** Let $(X_i)$ be a Cohen–Macaulay filtration of $X$. The associated spectral sequence converging to $H^G_*(X)$ degenerates at $E^1 = H^G_*(X, X_{i-1})$.

**Proof.** As in [2, Cor. 4.4]. \qed

**Proposition 6.4.** Let $\mathcal{F}$ be a $G$-stable filtration of $X$. Then $\mathcal{F}$ is a Cohen–Macaulay filtration for $X$, considered as a $G$-space, if and only if it is also Cohen–Macaulay for $X$, considered as a $T$-space.

**Proof.** Write $\mathcal{F} = (X_i)$. Proposition 4.1 gives, for $0 \leq i \leq r$,.

$$H^T_*(X_i, X_{i-1}) = R_T \otimes_{R_G} H^G_*(X_i, X_{i-1}),$$

which as an $R_G$-module consists of finitely many copies of $H^G_*(X_i, X_{i-1})$. Hence $H^G_*(X_i, X_{i-1})$ is zero or Cohen–Macaulay of dimension $r - i$ over $R_G$ if and only if $H^T_*(X_i, X_{i-1})$ is so. But $H^T_*(X_i, X_{i-1})$ is Cohen–Macaulay of dimension $r - i$ over $R_G$ if and only if it is so over $R_T$, cf. [19, Lemma 2.6]. \qed

6.2. **The Atiyah–Bredon sequence.** Let $\mathcal{F} = (X_i)$ be a Cohen–Macaulay filtration of $X$. It gives rise to an *Atiyah–Bredon sequence*

$$0 \to H^G_*(X) \xrightarrow{\iota^*} H^G_*(X_0) \xrightarrow{\delta_0} H^G_*(X_1, X_0) \xrightarrow{\delta_1} \cdots \xrightarrow{\delta_{r-2}} H^G_*(X_{r-1}, X_{r-2}) \xrightarrow{\delta_{r-1}} H^G_*(X_r, X_{r-1}) \to 0.$$
Here \( \iota^* \) is induced by the inclusion \( \iota \colon X_0 \hookrightarrow X \), and \( \delta_i \) for \( i \geq 0 \) is the connecting homomorphism in the long exact sequences for the triples \( (X_{i+1}, X_i, X_{i-1}) \). We consider this sequence as a dg \( R_G \)-module \( \overline{AB}^*_G(\mathcal{F}) \) with

\[
\overline{AB}^i_G(\mathcal{F}) = \begin{cases} 
H^i_G(X) & \text{if } i = -1, \\
H^{i+1}_G(X_i, X_{i-1}) & \text{if } 0 \leq i \leq r 
\end{cases}
\]

and differentials \( \delta_i \) as above plus \( \delta_{-1} = \iota^* \). We call \( \overline{AB}^*_G(\mathcal{F}) \) the augmented Atiyah–Bredon complex of \( \mathcal{F} \). The (non-augmented) Atiyah–Bredon complex \( AB^*_G(\mathcal{F}) \) is obtained by dropping the leading term \( \overline{AB}^{-1}_G(\mathcal{F}) = H^*_G(X) \). It is the \( E_1 \) page of the spectral sequence arising from the filtration \( \mathcal{F} \) and converging to \( H^*_G(X) \).

**Theorem 6.5.** The cohomology of the Atiyah–Bredon complex is

\[
H^j(\overline{AB}^*_G(\mathcal{F})) = \text{Ext}^j_{R_G}(H^*_G(X), R_G)
\]

for \( j \geq 0 \). Under this isomorphism, the map \( H^*_G(X) \to H^0(\overline{AB}^*_G(\mathcal{F})) \) corresponds to the canonical map \( H^*_G(X) \to \text{Hom}_{R_G}(H^*_G(X), R_G) \). In particular, the cohomology of \( AB^*_G(\mathcal{F}) \) and \( \overline{AB}^*_G(\mathcal{F}) \) is independent of the Cohen–Macaulay filtration \( \mathcal{F} \).

**Proof.** This was proven in [2] Thm. 5.1 for the orbit filtration of a \( T \)-space; see [3] Sec. 5 for an alternative argument. Both proofs generalize to the present context. The independence of the Cohen–Macaulay filtration \( \mathcal{F} \) was already pointed out in [2] Rem. 4.9.

---

6.3. The orbit filtration. Let \( X \) be a \( G \)-space such that \( H^*(X) \) is finite-dimensional and such that only finitely many infinitesimal orbit types occur in \( X \). We will see in Section 7 that the second assumption is redundant if \( X \) is a manifold or locally orientable orbifold.

**Remark 6.6.** We note that \( X \) has finitely many infinitesimal \( G \)-orbit types if and only if it has finitely many infinitesimal \( T \)-orbit types. That each infinitesimal \( G \)-orbit type restricts to only finitely many infinitesimal \( T \)-orbit types can be seen as follows, cf. [5] Prop. VIII.3.14: Let \( K \subset G \) be an isotropy group occurring in \( X \). Then \( G/K \) is a compact differentiable \( T \)-manifold, hence has only finitely many \( T \)-orbit types. Moreover, the infinitesimal \( T \)-orbit types in \( G/K \) depend only on the infinitesimal \( G \)-orbit type determined by \( K \).

The converse is due to Mostow [28] Lemma 5.

The \( G \)-orbit filtration \( (X_{i,G}) \) of \( X \) is defined by

\[
X_{i,G} = \{ x \in X \mid \text{rank } G_x \geq r - i \}
\]

for \( -1 \leq i \leq r \). Each \( X_{i,G} \) is \( G \)-stable and, by the slice theorem, closed in \( X \). If \( G = T \) is a torus, then the strata \( X_{i,T} \setminus X_{i-1,T} \) are disjoint unions of fixed point sets of subtori, hence smooth if \( X \) is so. This may fail for non-commutative \( G \), see [19] Rem. 3.1 for an example.

We need a relative version of Proposition 5.5.

**Proposition 6.7.** Let \( 0 \leq i \leq r \). The inclusion of pairs \( (X_{i,T}, X_{i,T} \cap X_{i-1,G}) \to (X_{i,G}, X_{i-1,G}) \) induces an isomorphism of \( R_G \)-modules

\[
H^*_G(X_{i,G}, X_{i-1,G}) \cong H^*_T(X_{i,T}, X_{i,T} \cap X_{i-1,G})^W.
\]
Proof. For cohomology with compact supports, the first claim follows immediately from Proposition 6.8 and the natural isomorphism \( H^*_G(X_i,G) \rightarrow H^*_G(X_i,G,U) \) as stated above, this implies that the RH Cohen–Macaulay of dimension \( H^*_G(X_i,G) \) is finite-dimensional and therefore also the relative cohomology \( H^*_G(X_i,G,X_{i-1,G}) \) are finite-dimensional and hence so is \( H^*_G(X_{i-1,G}) \). Back to the more general case we are considering, we have by the tautness of Alexander–Spanier cohomology

\[
\begin{equation}
H^*_G(X_i,G,X_{i-1,G}) = \lim_{\rightarrow} H^*_G(X_i,G,U)
\end{equation}
\]

where the direct limit is taken over all \( G \)-stable open neighbourhoods of \( X_{i-1,G} \) in \( X_i,G \). By excision and Proposition 6.7 for the pair \( (X_i,G,U) \),

\[
\begin{equation}
= \lim_{\rightarrow} H^*_G(X_i,G \setminus X_{i-1,G},U \setminus X_{i-1,G})
\end{equation}
\]

this time the direct limit being over all \( N_G(T) \)-stable open neighbourhoods \( V \) of \( X_i,T \cap X_{i-1,G} \) in \( X_i,T \). Interchanging the direct limit and taking \( W \)-invariants and reversing the previous arguments, we finally arrive at

\[
= H^*_T(X_i,T \cap X_{i-1,G})^W.
\]

Lemma 6.8. The vector spaces \( H^*(X_i,T,X_{i-1,G}) \) and \( H^*(X_i,G,X_{i-1,G}) \) are finite-dimensional for \( 0 \leq i \leq r \).

Proof. By induction we can assume that \( H^*(X_j,X_{j-1}) \) is finite-dimensional for \( j < i \) and hence so is \( H^*(X_{i-1}) \). Thus, \( X_{i-1} \) satisfies again our assumptions for \( X \). By [4 Prop. 4.1.14], both \( H^*(X_i,T) \) and

\[
\begin{equation}
H^*(X_i,T \cap X_{i-1,G}) = H^*((X_{i-1})i,T)
\end{equation}
\]

are finite-dimensional and therefore also the relative cohomology \( H^*(X_i,T,X_{i-1,G}) \). Hence \( H^*_T(X_i,T,X_{i-1,G}) \) is finitely generated over \( R_T \) and \( R_G \), and so is the submodule of \( W \)-invariants over \( R_G \). The claim now follows from Proposition 6.7 and Remark 6.6.

Proposition 6.9. The orbit filtration of \( X \) is Cohen–Macaulay.

Proof. The torus case was established in [3 Prop. 4.1]. (The local contractability assumptions made there were needed to ensure that singular and Alexander–Spanier cohomology coincide. This is not necessary in our setting, cf. [3 Rem. 2.17].) In fact, the proof in [3] shows the following: Let \( Y \subset X_i,T \) be a \( T \)-stable closed subset containing \( X_{i-1,T} \) and such that \( H^*(X_i,T,Y) \) is finite-dimensional. Then \( H^*_T(X_i,T,Y) \) and \( H^*_T(X_i,T,Y') \) are zero or Cohen–Macaulay of dimension \( r - i \).

By Remark 6.6 only finitely many infinitesimal \( T \)-orbit types occur in \( X_i,T \), and \( H^*(X_i,T,X_{i-1,G}) \) is finite-dimensional by Lemma 6.8. By what we have said above, this implies that the \( R_T \)-module \( H^*_T(X_i,T,X_{i-1,G}) \) is zero or Cohen–Macaulay of dimension \( r - i \). By [19] Lemma 2.7 the same holds for the \( R_G \)-submodule of \( W \)-invariants. Proposition 6.7 now shows that also \( H^*_G(X_i,X_{i-1}) \) is Cohen–Macaulay of dimension \( r - i \). Hence the filtration is Cohen–Macaulay by Lemma 6.1.
6.4. **Partial exactness.** Let $X$ be a $G$-space such that $H^*(X)$ is finite-dimensional and such that only finitely many infinitesimal orbit types occur in $X$. Moreover, let $\mathcal{F} = (X_i)$ be a Cohen–Macaulay filtration of $X$, for example the orbit filtration.

**Theorem 6.10.** Let $0 \leq j \leq r$. Then $H^*_G(X)$ is a $j$-th syzygy over $R_G$ if and only if $H^i(\overline{AB^*_G}(\mathcal{F})) = 0$ for all $i \leq j - 2$, i.e., if and only if the part

$$0 \to H^*_G(X) \to H^*_G(X_0) \to \cdots \to H^*_G(X_{j-2})$$

of the Atiyah–Bredon sequence for $\mathcal{F}$ is exact.

**Proof.** The torus case was established in [2, Thm. 5.7] (or [3, Thm. 4.8]). It is stated there for the orbit filtration of a $T$-space, but we know from Theorem 6.5 that $H^*(\overline{AB^*_T}(\mathcal{F}))$ is independent of the Cohen–Macaulay filtration $\mathcal{F}$.

To reduce the general case to the torus case, we recall from Proposition 6.4 that $\mathcal{F}$ is a also a Cohen–Macaulay filtration of $X$, considered as a $T$-space. In addition, we observe that

$$H^i(\overline{AB^*_T}(\mathcal{F})) = R_T \otimes_{R_G} H^i(\overline{AB^*_G}(\mathcal{F}))$$

for $0 \leq i \leq r$, which follows from the isomorphism

$$H^*_T(X_i, X_{i-1}) = R_T \otimes_{R_G} H^*_G(X_i, X_{i-1})$$

given by Proposition 4.4 and the fact that $R_T$ is free over $R_G$. By Proposition 4.2 $H^*_G(X)$ is a $j$-th syzygy over $R_G$ if and only if $H^*_T(X)$ is a $j$-th syzygy over $R_T$. As remarked above, this latter condition is equivalent to the vanishing of $H^i(\overline{AB^*_T}(\mathcal{F}))$ for $0 \leq i \leq r$, which in turn is equivalent to the vanishing of $H^i(\overline{AB^*_G}(\mathcal{F}))$ for all $i \leq j - 2$ by (6.11) and again the freeness of $R_T$ over $R_G$. □

**Corollary 6.11.** Assume that $X$ is a compact oriented $G$-orbifold. The $R_G$-module $H^*_G(X)$ is torsion-free (reflexive) if and only if equivariant Poincaré pairing

$$H^*_G(X) \times H^*_G(X) \to R_G, \quad (\alpha, \beta) \mapsto \int_X \alpha \beta$$

is non-degenerate (perfect).

This characterization of the non-degeneracy of the equivariant Poincaré pairing has been given by Guillemin–Ginzburg–Karshon [20, Thm. C.7] under the assumption that the maximal-rank stratum $X_{0,G}$ is smooth and orientable.

**Proof.** This follows from Theorem 6.10 by observing that the map

$$H^*_G(X) \to \text{Hom}_{R_G}(H^*_G(X), R_G)$$

induced by the equivariant Poincaré pairing is the composition of the canonical map $H^*_G(X) \to \text{Hom}_{R_G}(H^*_G(X), R_G) = H^0(\overline{AB^*_G}(\mathcal{F}))$ with the $R_G$-transpose of the Poincaré duality isomorphism $H^*_G(X) \to H^*_G(X)$ from Proposition 6.14.

It can alternatively be deduced Proposition 6.1 together with the observation that the $G$-equivariant Poincaré pairing is non-degenerate or perfect if and only if the $T$-equivariant one is so. We leave the details to the reader. □

**Proposition 6.12.** Assume that $X$ is a compact orientable $G$-orbifold. If $H^*_G(X)$ is a syzygy of order $\geq r/2$, then it is free over $R_G$.

**Proof.** This follows from Theorems 6.8 and 6.10 in the same way as for the torus case [2, Prop. 5.12]. Using Proposition 4.2 one could also deduce it directly from the torus case. □
The bound “$r/2$” in Proposition 6.12 is optimal for any $G$:

**Example 6.13.** Let $a$, $b \geq 1$ and $0 \leq m \leq (r - 1)/2$. Let $Y$ be the big polygon space defined by

\begin{align}
    \|u_j\|^2 + \|z_j\|^2 &= 1 \quad (1 \leq j \leq r), \\
    u_1 + \cdots + u_{m+1} &= 0
\end{align}

where $u_1, \ldots, u_r \in \mathbb{C}^a$ and $z_1, \ldots, z_r \in \mathbb{C}^b$. The torus $T \cong (S^1)^r$ acts on $Y$ by rotating the variables $z_j$. In [10, Sec. 5] it is shown that $Y$ is a compact orientable $T$-manifold and that $H^*_T(Y)$ is a syzygy of order exactly $m$ over $R_T$.

The induced $G$-manifold $X = G \times T Y$ is again compact orientable. By Proposition 4.5, $H^*_G(X)$ is a syzygy of order exactly $m$ over $R_G$. In particular, we see that any syzygy order less than $r/2$ can occur among the equivariant cohomology modules of compact orientable $G$-manifolds.

7. Infinitesimal Orbit Types

7.1. Preliminaries on torus actions. Recall that $T \cong (S^1)^r$ is a torus, and let $X$ a $T$-space. Several properties of $T$-equivariant cohomology with compact supports and homology with closed supports were established in [3] for $T$-spaces $X$ with finite-dimensional cohomology and/or finitely many infinitesimal orbit types[1]. We now drop these restrictions and indicate how to adapt the proofs.

Let $X$ be a $T$-space.

**Proposition 7.1.** Let $K \subset T$ be a subtorus and set $L = T/K$. If $K$ acts trivially on $X$, then there are isomorphisms of $R_T$-modules

\[ H^{T,c}_L(X) = R_T \otimes_{R_K} H^*_L(X), \quad H^{T,c}_c(X) = R_T \otimes_{R_K} H^*_c(X). \]

**Proof.** Choose a splitting $T \cong K \times L$, which induces an isomorphism $R_T \cong R_K \otimes R_L$. Since $K$ acts trivially on $X$, we have

\[ \Omega^{T,c}_L(X) = R_K \otimes \Omega^*_L(X) = R_T \otimes_{R_K} \Omega^*_L(X), \]

which gives the first isomorphism. The homological case is analogous. \[\square\]

Let $X_\alpha$, $\alpha \in A$, be the pieces of the partition of $X$ into infinitesimal $T$-orbit types. For $\alpha \in A$ let $T_\alpha \subset T$ be the common identity component of the $T_x$ with $x \in X_\alpha$. Moreover, let $t_\alpha$ be the Lie algebra of $T_\alpha$, and let $r_\alpha = r - \dim T_\alpha$ be the common dimension of the $T$-orbits in $X_\alpha$.

Regarding the orbit filtration $(X_\alpha) = (X_{\alpha,T})$ of $X$, we have

\[ X_p = \bigcup_{r_\alpha \leq p} X_\alpha \]

for $-1 \leq p \leq r$. Hence $X_p \setminus X_{p-1}$ is the disjoint union of the $X_\alpha$ with $r_\alpha = p$; it is a closed suborbifold of $X \setminus X_{p-1}$.

---

[1] The stronger assumption of finitely many orbit types made in [3, Sec. 2.1] only serves to allow the use of singular cohomology for the quotient $X/T$ instead of Alexander–Spanier cohomology.
Lemma 7.2. We have
\[ H^*_T,c(X_p \setminus X_{p-1}) = \bigoplus_{\alpha \in A \atop r_\alpha = p} H^*_T,c(X_\alpha), \]
\[ H^*_{T,c}(X_p \setminus X_{p-1}) = \prod_{\alpha \in A \atop r_\alpha = p} H^*_T,c(X_\alpha). \]

Proof. The first isomorphism is a consequence of the fact that \( \Omega^*_c(X_p \setminus X_{p-1}) \) is the direct sum of the \( \Omega^*_c(X_\alpha) \). Its dual \( \Omega^*_{c}(X_p \setminus X_{p-1})^\vee \) therefore is the direct product of the \( \Omega^*_c(X_\alpha)^\vee \), which implies the second identity. \( \square \)

For a multiplicative subset \( S \subset R_T \), set
\[ (7.3) \quad A(S) = \{ \alpha \in A \mid S \cap \ker(R_T \to \mathbb{R}[t_\alpha^*]) = \emptyset \}, \]
and let \( X^S \) be the union of the \( X_\alpha \) with \( \alpha \in A(S) \). Again by the slice theorem, \( X^S \) is closed in \( X \).

Proposition 7.3 (Localization theorem). Let \( S \subset R_T \) be a multiplicative subset. The inclusion \( X^S \to X \) induces isomorphisms of \( S^{-1}R_T \)-modules
\[ S^{-1}H^*_T,c(X) \to S^{-1}H^*_T,c(X^S), \]
\[ S^{-1}H^*_{T,c}(X^S) \to S^{-1}H^*_{T,c}(X). \]

Note that we put no restriction on the number of infinitesimal orbit types in \( X \).

Proof. For the cohomological claim we assume first \( X = X_\alpha \) and \( A(S) = \emptyset \), say \( f \in S \cap \ker(R_T \to \mathbb{R}[t_\alpha^*]) \). Since \( L = T/T_\alpha \) acts locally freely on \( X_\alpha \), the quotient \( X_\alpha/L \) is again an orbifold and \( H^*_{L,c}(X_\alpha) = H^*_c(X_\alpha/L) \), see Lemma 3.7. In particular, \( H^*_{L,c}(X_\alpha) \) is bounded. Because the kernel of the restriction map \( R_T \to \mathbb{R}[t_\alpha^*] \) is generated by \( t_\alpha^* \subset R_T \), this together with Proposition 7.1 shows that a power of \( f \) annihilates \( H^*_{T,c}(X_\alpha) \), so that \( S^{-1}H^*_{T,c}(X_\alpha) = 0 \).

For general \( X \), the orbit filtration gives rise to the decreasing filtration \( \Omega^*_T,c(X \setminus X_{p-1}) \) of \( \Omega^*_T,c(X) \), hence to a spectral sequence converging to \( H^*_{T,c}(X) \) with
\[ (7.4) \quad E^0_T(X) = C^0_T(\Omega^*_T,c(X_p, X_{p-1})) = \Omega^*_T,c(X_p, X_{p-1}), \]
\[ (7.5) \quad E^1_T(X) = H^*_T,c(X_p, X_{p-1}) = H^*_T,c(X_p \setminus X_{p-1}), \]
where we have used the isomorphism 3.10. Note that \( H^*_T,c(X_p \setminus X_{p-1}) \) is given by Lemma 7.2. Everything said about \( X \) applies equally to \( X^S \) with \( A(S) \) taking the role of \( A \). Since \( X^S \) is closed in \( X \), the inclusion \( X^S \to X \) induces a map of spectral sequences \( E_*(X) \to E_*(X^S) \). Localizing \( E_1(X) \) at \( S \) eliminates all terms \( H^*_T,c(X_\alpha) \) with \( \alpha \notin A(S) \). The localized map \( S^{-1}E_1(X) \to S^{-1}E_1(X^S) \) therefore is an isomorphism, hence so is the map \( S^{-1}H^*_T,c(X) \to S^{-1}H^*_T,c(X^S) \).

The homological claim follows from this and the universal coefficient theorem (Proposition 3.11) as in [3], Prop. 2.5. \( \square \)

Lemma 7.4. For \( 0 \leq p \leq r \), there is a multiplicative subset \( S \subset R_T \) such that \( A(S) = \{ \alpha \in A \mid r_\alpha \leq p \} \). For such an \( S \), the localization map \( H^*_T,c(X_q, X_{q-1}) \to S^{-1}H^*_T,c(X_q, X_{q-1}) \) is injective for \( q \leq p \), and \( S^{-1}H^*_T,c(X_q, X_{q-1}) = 0 \) for \( q > p \).
dimensional, then only finitely many infinitesimal orbit types occur in Theorem 7.8. Let \( \dim A = r_\alpha \). Assume \( r_\alpha > p \). Since \( \mathbb{R} \) is uncountable,

\[
\ell_\alpha^r \cap \bigcup_{r \leq p} \ell_\beta^r
\]

cannot be empty. Pick a \( t_\alpha \) from this set and let \( S \subset R_T \) be the multiplicative subset generated by all such \( t_\alpha \). Then, by construction, \( A(S) \) is of the claimed form.

From Lemma 7.2 and Proposition 7.1 we see that \( H^*_{T,c}(X_q, X_{q-1}) \) is \( S \)-torsion-free for \( q \leq p \), which means that the localization map is injective. Moreover, Proposition 7.3 gives \( S^{-1}H^*_{T,c}(X_q, X_{q-1}) = 0 \) for \( q > p \) as \((X_q \setminus X_{q-1})^S = \emptyset \).

The following result generalizes [1, Cor. 4.4] to our setting. Note that unlike [2] and [3], the proof below does not use the theory of Cohen–Macaulay modules.

**Proposition 7.5.** The spectral sequence induced by the orbit filtration and converging to \( H^*_{T,c}(X) \) degenerates at the \( E_1 \) page, which is given by Lemma 7.3.

**Proof.** We prove the claim by contradiction. Let \( d_k, k \geq 1 \), be the first non-vanishing differential, say with non-zero value \( a = d_k(b) \in H^*_{T,c}(X_p, X_{p-1}) \). Choose an \( S \subset R_T \) as in Lemma 7.3 and localize the \( E_k \) page of the spectral sequence at \( S \). In the localized spectral sequence the differential vanishes as \( S^{-1}H^*_{T,c}(X_q, X_{q-1}) = 0 \) for all \( q > p \). But \( H^*_{T,c}(X_p, X_{p-1}) \) injects into its localization \( S^{-1}H^*_{T,c}(X_p, X_{p-1}) \), which implies \( a = 0 \).

**Remark 7.6.** We could alternatively state this result by saying that certain short sequences are exact, cf. Proposition 6.2.

Everything done so far in this section goes through for twisted coefficients with respect to a fixed orientation cover \( \tilde{Z} \to Z \). In addition, one has the following:

**Lemma 7.7.** Let \( Z \) be a locally orientable \( T \)-orbifold. Then each component \( Y \) of \( Z^T \) is again locally orientable. Moreover, the restriction of the orientation cover for \( Z \) to \( Y \) is the orientation cover for \( Y \).

**Proof.** The first part is well-known, cf. [1, Thm. V.3.2]. With our tools, it can be seen as follows: Choose a uniformizing chart \( \tilde{U} \to U \) of \( Z \) at \( y \in Y \) such that \( \tilde{U} \) is an open ball with linearized torus action. Because \( Z \) is locally orientable, \( H^*_c(U) \) is non-zero and concentrated in top degree. Thus \( H^*_{T,c}(U) = R_T \otimes H^*_c(U) \). By the localization theorem, this implies that \( H^*_{T,c}(U^T) = R_T \otimes H^*_c(U^T) \) is non-zero. Hence \( U^T \) is the quotient of \( \tilde{U}^T \) by an orientation-preserving action since otherwise we would have \( H^*_c(U^T) = 0 \).

For the second claim it suffices to observe that \( Y \) is orientable if and only if \( Z \) is so. This is a consequence of the fact that the normal (orbi)bundle of \( Y \) in \( Z \) is always orientable. See [13, Cor. 2] for the case of manifolds, from which the orbifold case follows.

7.2. The number of infinitesimal orbit types.

**Theorem 7.8.** Let \( X \) be a locally orientable \( G \)-orbifold. If \( H^*(X) \) is finite-dimensional, then only finitely many infinitesimal orbit types occur in \( X \).
Proof. By Mostow’s argument we may assume that $G = T$ is a torus, see Remark 6.6

Assume that there are infinitely many infinitesimal orbit types. By Proposition 7.7 and the previous section, the spectral sequence induced by the orbit filtration and converging to $H^*_r(X; \mathbb{R})$ degenerates at the $E_1$ page. Each of the infinitely many terms $H^*_r(X; \mathbb{R})$ appearing in the twisted version of Lemma 7.2 is non-zero as it contains an equivariant orientation by Lemma 7.4. Thus, $E_\infty(X) = E_1(X)$ is not finitely generated. This implies that $H^*_r(X; \mathbb{R})$ cannot be finitely generated either because $E_\infty(X)$ arises from a filtration of it and $R_T$ is Noetherian.

On the other hand, $H^r(X)$ is finite-dimensional by assumption, hence $H^*_r(X)$ is finitely generated over $R_T$. By equivariant Poincaré duality (Proposition 3.14), the same holds for $H^*_r(X; \mathbb{R})$. Contradiction.

We conclude with several examples. The first one, due to Montgomery [32, §3], illustrates the difference between our result and Mann’s [25, Thm. 3.5]. Recall that Mann showed that any orientable cohomology manifold with finitely generated integral cohomology has only finitely many orbit types.

Example 7.9. Consider a circle acting on $\mathbb{R}^3$ via rotations about some axis. For each $m \in \mathbb{N}$, choose an open solid torus equivariantly diffeomorphic to $A_m \times S^1$, where $A_m \subset \mathbb{R}^3$ is an open disc with centre $b_m$. (These tori have to be disjoint.) Remove the central circle $b_m \times S^1$ of each torus and glue the remaining manifold to copies of the $A_m \times S^1$ via the diffeomorphisms $f_m$ of $(A_m \setminus b_m) \times S^1 \cong (0,1) \times S^1 \times S^1$ given by $f_m(r,\alpha,\beta) = (r,\alpha \beta^{-m},\beta^m/\alpha)$. Each subgroup $\mathbb{Z}_m \subset S^1$ occurs as an isotropy group in the orientable manifold $X$ thus obtained. One can check that $H_1(X;\mathbb{Z})$ is isomorphic to the direct sum of all $\mathbb{Z}_m$ and that $H_2(X;\mathbb{Z})$ vanishes. In particular, $H_*(X;\mathbb{Z})$ is not finitely generated. On the other hand, $H_*(X;\mathbb{R})$ is finite-dimensional, and Theorem 7.8 holds (trivially).

Example 7.10. We now imitate this idea for the standard action of $T = S^1 \times S^1$ on $\mathbb{C}^2$. In the free part of the action, choose disjoint open subsets of the form $A_m \times T$, where the slice $A_m \subset \mathbb{C}^2$ is again an open disc with centre $b_m$. Remove $b_m \times T$ from the manifold and glue the rest to a copy of $A_m \times T$ via the diffeomorphism $f_m$ of $(A_m \setminus b_m) \times T \cong (0,1) \times (S^1)^3$ given by $f_m(r,\alpha,\beta,\gamma) = (r,\alpha \beta^{-m},\beta^m \gamma,\alpha)$. Each such operation increases the Betti sum of the manifold by 4, and it produces the isotropy group $\{(g,g^{-m}) \mid g \in S^1\} \subset T$. If this is done for all $m \in \mathbb{Z}$, we obtain a manifold with infinitely many distinct infinitesimal isotropy groups and infinite Betti sum.

We finally recall an example of Kister and Mann [22, Sec. 1] showing that Theorem 7.8 cannot be naively extended to actions on more general spaces.

Example 7.11. Take countably many closed discs, connected by a line through their centres. This space is properly homotopy-equivalent to $\mathbb{R}$, hence contractible. On the other hand, the discs can be rotated independently via characters $T \to S^1$ to produce infinitely many distinct infinitesimal orbit types. Note that an argument as in the proof of Theorem 7.8 fails here because the non-fixed points do not contribute to $H^*_r(X)$. In fact, for a closed disc $D$ with centre $x$ one has $H^*_r(D,\{x\}) = 0$, hence also $H^*_r(D,\{x\}) = 0$ by Proposition 3.10.

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Department of Mathematics, University of Western Ontario, London, Ont. N6A 5B7, Canada

E-mail address: mfranz@uwo.ca