Chaos-order transition in Bianchi I non-Abelian Born-Infeld cosmology

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(Dated: 09.04.05)

We investigate the Bianchi I cosmology with the homogeneous SU(2) Yang-Mills field governed by the non-Abelian Born-Infeld action. A similar system with the standard Einstein-Yang-Mills (EYM) action is known to exhibit chaotic behavior induced by the Yang-Mills field. When the action is replaced by the Born-Infeld-type non-Abelian action (NBI), the chaos-order transition is observed in the high energy region. This is interpreted as a smothing effect due to (non-perturbative in $\alpha'$) string corrections to the classical EYM action. We give a numerical evidence for the chaos-order transition and present an analytical proof of regularity of color oscillations in the limit of strong Born-Infeld non-linearity. We also perform some general analysis of the Bianchi I NBI cosmology and derive an exact solution in the case when only the U(1) component of the Yang-Mills field is excited. Our new exact solution generalizes the Rosen solution of the Bianchi I Einstein-Maxwell cosmology to the U(1) Einstein-Born-Infeld theory.

PACS numbers: 04.20.Jb

I. INTRODUCTION

One of the key questions in theoretical cosmology is whether the space-time metric near the singularity is regular or chaotic. As was shown by Belinskii, Khalatnikov and Lifshitz (BKL), the generic solution of the four-dimensional vacuum Einstein equations exhibits an oscillating behavior which was later qualified as essentially chaotic (see and references therein). Recently the issue of chaos in the early universe received a renewed attention due to discovery that the antisymmetric form fields in ten and eleven-dimensional supergravities imply chaos. Namely, it was shown that the general solution near a space-like singularity of the Einstein-dilaton-p-form field equations exhibits an oscillatory behavior of the BKL type. However the issue of chaos in superstring cosmology is not completely solved. Namely, it was shown that the antisymmetric Bianchi I, the YM chaos unambiguously persists when the usual YM action is replaced by the non-Abelian Born-Infeld action (NBI). This would probe the effect of the string non-locality on the issue of chaos.

Classical YM fields governed by the ordinary quadratic action exhibit chaotic behavior in various situations. The simplest case is that of the homogeneous YM fields depending only on time in the flat space-time: when only two YM components are excited, the problem is reduced to the well-known two-dimensional hyperbolic system $H = (p_x^2 + p_y^2 + x^2 y^2)/2$ which is chaotic. Furthermore, in the lattice simulations of the inhomogeneous YM system, one observes the energy flow from the infrared to the ultraviolet region. Therefore, it is believed that the chaotic behavior is typical for the purely classical YM equations, one of the arguments being the absence of solitons in this theory. In addition, it is known that adding the Higgs field to the YM theory leads to stabilization of chaos in the homogeneous systems. In this case the hyperbolic model is replaced by the system of coupled harmonic oscillators which is regular in the weak coupling regime.

In the context of YM fields in the presence of gravity, it is of interest to pint the following. The YM field has violent oscillating behavior near the singularity of the Einstein-Yang-Mills (EYM) black holes, but the oscillations are not chaotic. In the domain of cosmology, some homogeneous models, such as an axisymmetric Bianchi I, the YM chaos unambiguously per-
sists \[21, 22, 23\], though in principle gravity leads to a smoothing of the chaotic behavior. On the other hand, the Born-Infeld effect on the flat-space dynamics of the homogeneous axisymmetric YM field was shown to provide a chaos-order transition \[24\], so it can be expected that in the gravity coupled case this effect will be even more pronounced. Note that in the related investigation of the behavior of the YM field inside black holes \[25\], it was found that violent YM oscillations disappear once the quadratic YM action is replaced by the NBI action.

In this paper, we study in detail the axisymmetric Bianchi I cosmology with the YM field governed by the NBI action. Though the system becomes much more complicated when the ordinary YM Lagrangian is replaced by the non-Abelian Born-Infeld (NBI) Lagrangian, we show that the equations can be considerably simplified in some physically interesting limiting cases, and even admit some exact solutions. The generic solution exhibits a transition from the YM chaos to the regular oscillating regime when moving backward in time.

II. GENERAL SETTING

As was discussed recently \[15, 26, 27, 28\], the definition of the NBI action presented either in the determinant form

\[
S = \frac{1}{16\pi} \int -\det(g_{\mu\nu} - \beta^{-1}F_{\mu\nu}) \sqrt{-g} \, d^4x, \tag{1}
\]

or in the equivalent (in four space-time dimensions) “square root” form

\[
S = \frac{1}{16\pi} \int \sqrt{1 + \frac{F_{\mu\nu}F_{\mu\nu}}{2\beta^2} - \frac{(F_{\mu\nu}F_{\mu\nu})^2}{16\beta^4}} \sqrt{-g} \, d^4x. \tag{2}
\]

In the non-Abelian case of the matrix-valued \(F_{\mu\nu}\), the trace over gauge matrices must be specified. One particular definition is due to Tseytlin \[24\]. A symmetrized trace was therein introduced, prescribing a symmetrization of all products of \(F_{\mu\nu}\) in the power expansion of the determinant \[11\] before the trace is taken. Inside the symmetrized series expansion the gauge generators effectively commute, so both the determinant \[11\] and the square root \[2\] forms are equivalent. This property does not hold for other trace prescriptions, e.g., an ordinary trace. In the latter case it is common to apply the trace to the square root form \[2\]. Note that string theory seems to require the symmetrized trace definition in the lower orders of the perturbation theory \[15, 27, 29\], while higher order corrections seem to violate this prescription \[30, 31, 32, 33\]. Here we choose the “square root/ordinary trace” Lagrangian just for its simplicity. It is worth noting that in the static case discussed recently both in the ordinary \[24\] and the symmetrized trace \[25\] versions, qualitative features of solutions turn out to be the same. Thus we choose the action of the Einstein-NBI system in the following form

\[
S = -\frac{1}{4\pi} \int \left\{ \frac{1}{4G}R + \beta^2(\mathcal{R} - 1) \right\} \sqrt{-g} \, d^4x, \tag{3}
\]

where \(R\) is the scalar curvature, \(\beta\) is the BI critical field strength and

\[
\mathcal{R} = \sqrt{1 + \frac{1}{2\beta^2}F_{\mu\nu}^aF_{a\mu\nu} - \frac{1}{16\beta^4}(F_{\mu\nu}^aF_{a\mu\nu})^2}. \tag{4}
\]

The limit \(\beta \to \infty\) corresponds to the standard EYM theory with the action

\[
S = -\frac{1}{4\pi} \int \left( \frac{1}{4G}R + \frac{1}{4}F_{\mu\nu}^aF_{a\mu\nu} \right) \sqrt{-g} \, d^4x. \tag{5}
\]

We consider an axially symmetric Bianchi I space-time described by the line element

\[
ds^2 = N^2dt^2 - b^2(dx^2 + dy^2) - c^2dz^2, \tag{6}
\]

where functions \(N, b\) and \(c\) depend on time \(t\). In the YM case this problem was studied previously by Darian and Kunzle \[21, 22\] and Barrow and Levin \[23\]. The gauge field compatible with the space-time symmetry is parameterized by two functions \(u, v\) of time

\[
A = T_1udx + T_2udy + T_3dz, \tag{7}
\]

where SU(2) generators are normalized according to \([T_1, T_2] = iT_3\). The corresponding field strength matrix-valued two-form is

\[
F = \dot{u}(T_1dt \wedge dx + T_2dt \wedge dy) + \dot{v}T_3dt \wedge dz + u^2T_3dx \wedge dy + uv(T_2dz \wedge dx + T_1dy \wedge dz). \tag{8}
\]

Integrating over the 3-space, we obtain the following one dimensional Lagrangian

\[
L = -\frac{1}{2} \left( \dot{u}(bc + 2\dot{c}b) - \beta^2Nb^2c(\mathcal{R} - 1) \right), \tag{9}
\]

where now

\[
\mathcal{R} = \sqrt{1 - \frac{\mathcal{F}}{\beta^2} - \frac{\mathcal{G}}{\beta^4}}, \tag{10}
\]

\[
\mathcal{F} = \frac{2\dot{u}^2}{N^2b^2} + \frac{\dot{v}^2}{N^2c^2} - \frac{1}{b^2} \left( \frac{2u^2v^2}{c^2} + \frac{u^4}{b^2} \right), \tag{11}
\]

\[
\mathcal{G} = \frac{u(2uv + \dot{v}u)}{Nb^2c}. \tag{12}
\]

The quantity \(\mathcal{F}\) is the YM Lagrangian, and it is convenient to present it as a difference of kinetic and potential terms

\[
\mathcal{F} = T - U, \tag{13}
\]

\[
T = \frac{2\dot{u}^2}{N^2b^2} + \frac{\dot{v}^2}{N^2c^2}, \tag{14}
\]

\[
U = \left( \frac{2u^2v^2}{b^2c^2} + \frac{u^4}{b^2} \right). \tag{15}
\]
Note that from two coupling parameters entering the action, $G$ and $\beta$, one can be eliminated by an appropriate rescaling. In what follows we set $G = 1$.

The Einstein equations can be derived by variation of the one-dimensional action over $N, b, c$. Variation over $N$ gives the Hamiltonian constraint
\[ H = \frac{\partial L}{\partial N} = 0, \] (16)
where $H$ reads in the synchronous gauge $N = 1$:
\[ H = \frac{1}{2} b (\dot{b} c + 2 \dot{c} b) + \frac{b^2 c}{\mathcal{R}} [\beta^2 (\mathcal{R} - 1) - U]. \] (17)

Fixing this gauge from now on, we obtain the remaining Einstein equations:
\[ \begin{align*}
\frac{\dot{b}}{b} + \frac{\dot{c}}{c} &= 2\beta^2 (\mathcal{R} - 1) + 2 \left( \frac{\dot{u}^2}{b^2} - \frac{u^2 v^2}{b^2 c^2} - \frac{u^4}{b^4} + \frac{G^2}{\beta^2} \right), \quad (18) \\
\frac{\dot{b}}{b} + \frac{\dot{b}^2}{2 b^2} &= \beta^2 (\mathcal{R} - 1) + \frac{1}{\mathcal{R}} \left( \frac{\dot{v}^2}{c^2} - 2 \frac{u^2 v^2}{b^2 c^2} + \frac{G^2}{\beta^2} \right). \quad (19)
\end{align*} \]

The equations for the YM field can be presented in the following form
\[ \begin{align*}
\frac{\mathcal{R}}{c} \frac{d}{dt} \left[ \frac{c}{\mathcal{R}} \left( \dot{u} + u \frac{\mathcal{G}}{c^3} \right) \right] &= \left( \frac{\dot{u} v + \dot{v} u}{c^3} \right) \mathcal{G} - \frac{u^3}{b^2} - \frac{u v^2}{c^2}, \quad (20) \\
\frac{c}{\mathcal{R}} \frac{d}{dt} \left[ \frac{b^2}{c} \left( \dot{v} + u v \frac{\mathcal{G}}{\beta^2} \right) \right] &= -2 u^2 v + \frac{2 c u v \mathcal{G}}{\beta^2}. \quad (21)
\end{align*} \]

The energy-momentum tensor has the following components. The energy density is given by
\[ T^0_0 = \epsilon = \frac{\beta^2}{4\pi\mathcal{R}} - \frac{2\Psi^2 + \Psi^4}{4\pi\mathcal{R}}, \] (22)
the pressure in the plane orthogonal to the symmetry axis is
\[ p_x = -T^{\alpha}_x = -T^y_y = -\frac{\Pi_1 + \Pi_2 - \beta^2}{4\pi\mathcal{R}} + \frac{\beta^2}{4\pi}, \] (23)
and the pressure along the axis of the symmetry is
\[ p_z = -T^z_z = 2\Pi_\Psi - \frac{\Psi^4 - \beta^2}{4\pi\mathcal{R}} + \frac{\beta^2}{4\pi}. \] (24)

### III. REDUCTION OF ORDER

The above system of equations look as a dynamical system of the eight-order in the presence of a constraint. However, it possesses additional scaling symmetries which can be used to reduce the system order by two (for the EYM action this possibility was noticed by Darian and Kunzle [21]). It is easy to check that under a scaling transformation
\[ b \to \lambda b, \quad c \to \lambda^{-2} c, \quad u \to \lambda u, \quad v \to \lambda^{-2} v, \]
the Lagrangian remains invariant. Moreover, under a separate rescaling in the $b, u$ sector
\[ b \to \lambda b, \quad u \to \lambda u, \] (25)
the Lagrangian scales as $\lambda^2$, and under the transformation
\[ c \to \lambda c, \quad v \to \lambda v, \] (26)
as $\lambda$. The corresponding reduction of the EYM system is achieved by an introduction of new variables invariant under the above rescalings. Following Barrow and Levin [22], whose notation we will adopt in what follows (note that in Ref. [23] another convention $8\pi G = 1$ is used), we introduce the volume and shear variables
\[ a = (b^2 c)^{1/3}, \quad \chi = \left( \frac{b}{c} \right)^{1/3}, \] (27)
together with the associated Hubble parameters
\[ H_a = \frac{\dot{a}}{a}, \quad H_\chi = \frac{\dot{\chi}}{\chi}, \] (28)
as well as the scaled Yang-Mills variables
\[ \Psi = \frac{u}{b}, \quad \Gamma = \frac{v}{c}, \] (29)
which are related to $\Psi, \Gamma$ via
\[ \Pi_\Psi = \dot{\Psi} + (H_a + H_\chi) \Psi, \] (31)
\[ \Pi_\Gamma = \dot{\Gamma} + (H_a - 2H_\chi) \Gamma. \] (32)

Note that these are not the momenta conjugate to $\Psi, \Gamma$, the corresponding canonical momenta being
\[ P_\Psi = \frac{2a^3}{\mathcal{R}} \left( \Pi_\Psi + \frac{\Psi \Gamma \mathcal{G}}{\beta^2} \right), \] (33)
\[ P_\Gamma = \frac{a^3}{\mathcal{R}} \left( \Pi_\Gamma + \frac{\Psi^2 \mathcal{G}}{\beta^2} \right). \] (34)

In terms of the new variables the Hamiltonian constraint reads
\[ \frac{3}{2} \left( H_a^2 - H_\chi^2 \right) + \beta^2 - \left[ \beta^2 + \Psi^2 (\Psi^2 + 2\Gamma^2) \right] \mathcal{R}^{-1} = 0. \] (35)

The functions $T, U$ and $\mathcal{G}$ entering $\mathcal{R}$ now take the form
\[ T = 2\Pi_\Psi + \Pi_\Gamma, \] (36)
\[ U = \Psi^4 + 2\Psi^2 \Gamma^2, \] (37)
\[ \mathcal{G} = \Psi (2\Pi_\Psi \Gamma + \Pi_\Gamma \Psi). \] (38)

From the Einstein equations one can derive two first order equations for the Hubble parameters, which are linear in derivatives. Taking the sum of the Eqs. (13), (19)
and the constraint equation $(17)$, one obtains the following simple equation for $\dot{H}_a$:

$$\dot{H}_a + 3H_a^2 + \frac{2}{3R} (T + U) = 0. \quad (39)$$

Similarly, taking twice the second Einstein equation $(19)$ and subtracting $(18)$ we get

$$\dot{H}_X + 3H_X H_a - \frac{2}{3R} (\Pi^2 \gamma - \Pi^2 \Phi + \Psi^4 - \Psi^2 \Gamma^2) = 0. \quad (40)$$

Thus the Einstein equations reduce to two first order equations in the presence of a constraint.

Alternatively, one can introduce the Hubble factors with respect to $b$ and $c$:

$$H_b = \frac{\dot{b}}{b}, \quad H_c = \frac{\dot{c}}{c}, \quad (41)$$

**IV. YM LIMIT**

In the YM limit $\beta \to \infty$ the square-root factor in the above formulas should be replaced according to the relation

$$\lim_{\beta \to \infty} \beta^2 (R - 1) = -\frac{1}{2} \mathcal{F}. \quad (47)$$

The main qualitative difference between EYM and ENBI theories lies in the fact that the standard YM action is scale-invariant (though not the EYM one) contrary to the NBI case. This leads to a partial decoupling of the YM dynamics from that of the space-time. Given eq. $(47)$, the constraint equation simplifies to

$$\frac{1}{2} \left[ 3 \left( H_a^2 - H_X^2 \right) - (T + U) \right] = 0. \quad (48)$$

Combining this with $(50)$, one finds that one of the Einstein equations fully decouples and reduces to the vacuum form:

$$\dot{H}_a + H_a^2 + 2H_X^2 = 0. \quad (49)$$

However, the shear remains coupled to matter and obeys the equation

$$\dot{H}_X + 3H_X H_a + H_X^2 - H_X^2 = \Pi^2 \gamma + \Psi^4. \quad (50)$$

Finally, the YM field equations become

$$\Pi + (H_b + H_c) \Pi + \Psi (\Psi^2 + \Gamma^2) = 0, \quad (51)$$

$$\Pi + 2H_b \Pi + 2\Psi \Gamma = 0, \quad (52)$$

and bring the Einstein equations into the form

$$\dot{H}_b + H_b (H_b - H_c) = -\frac{2}{R} (\Pi^2 \gamma + \Psi^2 \Gamma^2), \quad (42)$$

$$\dot{H}_c + H_c (H_b - H_c) = -\frac{2}{R} (\Pi^2 \gamma + \Psi^4), \quad (43)$$

with the Hamiltonian constraint

$$\frac{1}{2} H_b (H_b + 2H_c) - \frac{1}{R} \left[ \beta^2 (1 - R) + U \right] = 0. \quad (44)$$

In addition, we have two second order equations for the YM fields which read in terms of the new variables

$$\left( \frac{d}{dt} + H_b + H_c \right) \left[ \frac{1}{R} \left( \Pi - \frac{G \Psi^2}{\beta^2} \right) \right] + \frac{1}{2} \left[ \Psi^3 + \Psi \Gamma^2 - \frac{G (\Pi \gamma \gamma + \Pi \gamma \Psi)}{\beta^2} \right] = 0, \quad (45)$$

$$\left( \frac{d}{dt} + 2H_b \right) \left[ \frac{1}{R} \left( \Pi - \frac{G \Psi^2}{\beta^2} \right) \right] + \frac{2}{R} \left[ \Psi^2 \gamma + \Psi \Gamma^2 - \frac{G \Pi \gamma \Psi}{\beta^2} \right] = 0. \quad (46)$$

where the definitions $(51), (52)$ have to be used.

The Hamiltonian form of the EYM equations can be further simplified using an exponential parametrization of the volume and shear variables

$$a = e^\alpha, \quad \chi = e^\gamma. \quad (53)$$

The canonical momenta conjugate to $\alpha, \gamma$ are

$$P_\alpha = -3e^{3\alpha} \dot{\alpha}, \quad P_\gamma = 3e^{3\alpha} \dot{\gamma}, \quad (54)$$

while the YM momenta $(55), (56)$ simplify to

$$P_\Psi = 2e^\alpha \Pi_\Psi, \quad P_T = a^2 \Pi_T \quad (55).$$

The Hamiltonian constraint $(17)$ for the EYM system in terms of the momentum variables reads

$$H = e^{-3\alpha} \left[ \frac{1}{6} \left( P_\alpha^2 - P_\gamma^2 \right) - \frac{1}{4} \left( P_\Psi^2 + 2P_T^2 \right) \right] - \frac{U}{2} = 0, \quad (56)$$

where the potential is given by the Eq. $(37)$.

**V. U(1) CASE**

Consider the special case when only the $v$-component of the YM field is excited, corresponding to the U(1) subgroup of the gauge group. The Einstein equations $(42), (43)$ reduce to

$$\dot{H}_b + H_b (H_b - H_c) = 0, \quad (57)$$

$$\dot{H}_c + H_c (H_b - H_c) = -\frac{2}{R} \Pi^2. \quad (58)$$
and the Hamiltonian constraint is
\[ H_b(H_b + 2H_c) = 2\beta^2 \left( R^{-1} - 1 \right). \tag{59} \]
Integrating the BI field equation
\[ \frac{d}{dt} \left( \frac{b^2 \Pi_R}{R} \right) = 0, \tag{60} \]
one obtains
\[ \frac{b^2 \Pi_R}{R} = 2b_0, \tag{61} \]
where \( b_0 \) is an integration constant, so that
\[ R = \sqrt{1 - \frac{\Pi_R^2}{\beta^2}} = \frac{1}{\sqrt{1 + x^2}}, \quad x = \frac{2b_0}{\beta b^2}. \tag{62} \]
It is easy to see that the Einstein equation \(^{67}\) is equivalent to
\[ \frac{\dot{b}}{b} = \frac{\dot{c}}{c}, \tag{63} \]
which immediately gives a relation
\[ \dot{b} = kc, \tag{64} \]
where \( k \) is a second integration constant. Now the constraint equation becomes the following separated equation for the function \( b(t) \):
\[ \dot{H}_b + \frac{3}{2} H_b^2 = \beta^2 \left( \sqrt{1 + x^2} - 1 \right), \tag{65} \]
while the second Einstein equation \(^{65}\) is its time derivative. The right hand side of this equation is positively definite. It follows that the system has no bounces. Indeed, if \( H_b = 0 \), from the Eq. \(^{65}\) it follows that \( \dot{H}_b = 0 \), which contradicts the Eq. \(^{65}\).
We can solve the Eq. \(^{65}\) considering instead of \( b(t) \) an inverse function \( t(b) \). Then
\[ H_b = \frac{1}{bt'}, \tag{66} \]
where \( t' = dt/db \). The equation for \( t(b) \) following from \(^{65}\) reads
\[ \left( \frac{1}{t'} \right)^2 \left( \frac{t''}{t'} - \frac{1}{2b} \right) = b\beta^2 \left( 1 - \sqrt{1 + \frac{4b_0^2}{\beta^2 b^4}} \right). \tag{67} \]
This is the linear first order equation for the function
\[ z(b) = (1/t')^2, \tag{68} \]
namely,
\[ z' + \frac{z}{b} + 2b\beta^2 \left( 1 - \sqrt{1 + \frac{4b_0^2}{\beta^2 b^4}} \right) = 0. \tag{69} \]
Its solution reads
\[ z = \frac{2\beta^2}{b} \int \left( \sqrt{1 + \frac{4b_0^2}{\beta^2 b^4}} - 1 \right) b^2 db + \frac{b_1}{b}, \tag{70} \]
where \( b_1 \) is a third integration constant. An integration can be done in terms of the hypergeometric function \(^{37}\):
\[ z = \frac{2\beta^2}{3} \sqrt{b^4 + \frac{4b_0^2}{\beta^2}} - \frac{2\beta^2 b^2}{3} + \frac{8\beta b_0}{3} F\left( \frac{1}{3}, \frac{3}{4}; \frac{5}{4}; \frac{1}{1 + x^2} \right) + \frac{\tilde{b}_1}{b}, \tag{71} \]
where \( \tilde{b}_1 \neq b_1 \) is another constant. Now, according to \(^{65}\), the inverse function to the required solution is given by the integral
\[ t(b) = \int \frac{db}{\sqrt{z(b)}} + t_0, \tag{72} \]
where \( t_0 \) is the last integration constant in this process. Our solution generalizes the Rosen solution \(^{38}\) to the Einstein-Born-Infeld theory.
Near the singularity \( z \approx b_1/b \), so one has
\[ H_b = \frac{\sqrt{b_1}}{b^{3/2}}. \tag{73} \]
Integrating Eq. \(^{72}\) one obtains
\[ b = (b_1 t)^{2/3}, \tag{74} \]
and then from Eq. \(^{64}\)
\[ c = \frac{2b_1^{2/3}}{3k} t^{-1/3}. \tag{75} \]
Hence, we obtain a cigar singularity.

In the Maxwell case the situation is different. Indeed, in the limit \( \beta \to \infty \) one has
\[ z = -\frac{4b_0^2}{b^2} + \frac{b_1}{b}. \tag{76} \]
Since \( z \) should remain positive, the region of \( b \) is limited by
\[ b > b_{\text{min}} = \frac{4b_0^2}{b_1}. \tag{77} \]
Combining the Eqs. \(^{72}\), \(^{74}\) we obtain
\[ b = b_{\text{min}} + \frac{b_1 t^2}{4b_{\text{min}}}, \quad c = \frac{b_1 t}{2kb_{\text{min}}}. \tag{78} \]
This is a pancake singularity. Thus, the BI non-linearity modifies the singularity from a pancake to a cigar type.
VI. SINGULARITY STRUCTURE

Consider now the general solution near the cosmological singularity. It turns out that except for a special isotropic solution $b = c = a$, previously studied in [33, 34], generic solutions have the same metric singularities as the vacuum Bianchi I solutions. Near the pancake singularity the solution is not analytic in terms of $t$, but in terms of $t^{1/3}$. In fact, one finds the following Laurent expansion containing four free parameters $p, q, r, s$:

$$H_a = \frac{1}{3t} - \frac{2r q + sp}{9pr} t^{-2/3} + O(t^{-1/3}),$$  \hspace{1cm} (79)

$$H_x = \frac{1}{3t} + \left( \frac{ps - qr}{9pr} - \frac{p}{\sqrt{2}} \right) t^{-2/3} + O(t^{-1/3}),$$  \hspace{1cm} (80)

$$\Psi = pt^{-2/3} + q t^{-1/3} + O(1),$$  \hspace{1cm} (81)

$$\Gamma = rt^{1/3} + st^{2/3} + O(t).$$  \hspace{1cm} (82)

The $\Gamma$-component of the YM field vanishes at $t = 0$, while $\Psi$ is singular. The scale factor $a$ and the shear $\chi$ near the pancake singularity both behave as $O(t^{1/3})$.

Near the cigar singularity the solution has a Laurent expansion in terms of $t$:

$$H_a = \frac{1}{3t} + \frac{4r^2 \bar{p}^2 - s^2}{3R_1} + O(t),$$  \hspace{1cm} (83)

$$H_x = -\frac{1}{3t} + \frac{2r^2 \bar{p}^2 + s^2}{3R_1} + O(t),$$  \hspace{1cm} (84)

$$\Psi = \bar{p} + \left( \bar{q} - \frac{2r^2 \bar{p}}{R_1} \right) t + O(t^2),$$  \hspace{1cm} (85)

$$\Gamma = \bar{r} t^{-1} + \bar{s} + \frac{\bar{r} \bar{s}^2}{R_1} + O(t),$$  \hspace{1cm} (86)

where the quantity $R_1$ is the leading term in an expansion of the NBI square root:

$$R = R_1 t^{-1} + O(1), \hspace{1cm} R_1 = \sqrt{\frac{2r^2 \bar{p}^2 - s^2}{\beta^2} - \frac{\bar{p}^2 (\bar{p} \bar{s} + 2 \bar{r} \bar{q})^2}{\beta^4}}.$$  \hspace{1cm} (87)

The scale factor and the shear have the following expansions:

$$a = a_1 \left( t^{1/3} + \frac{4r^2 \bar{p}^2 - s^2}{3R_1} t^{4/3} + O(t^{7/3}) \right),$$  \hspace{1cm} (88)

$$\chi = \chi_1 \left( t^{-1/3} + \frac{2r^2 \bar{p}^2 + s^2}{3R_1} t^{2/3} + O(t^{5/3}) \right).$$  \hspace{1cm} (89)

The quantities $p, q, r, s$ ($\bar{p}, \bar{q}, \bar{r}, \bar{s}$) are independent free parameters which, together with an arbitrariness associated with a time shift, provide five constants needed to specify the generic solution for both singularity types.

VII. SOLUTION IN THE LIMIT $\beta = 0$

In order to better understand the effect of the BI non-linearities on the gauge field dynamics let us first study the strong field limit $F \gg \beta$, or, formally, $\beta \to 0$. The leading term in the square root $\beta$ containing the pseudoscalar invariant $G$ is negative definite. Therefore, imposing the square root $R$ to be real-valued in the limit $\beta \to 0$ may be ensured only if $G$ tends to zero, in which case

$$\Psi \Pi_\Gamma + 2 \Pi_\Psi = 0.$$  \hspace{1cm} (90)

One can show that this condition is compatible indeed with the equations of motion as $\beta \to 0$.

Given the condition (90), the square root term will read

$$R = \frac{\sqrt{\Psi^2 + 2 \Gamma^2 \Psi^2 - \Pi_\Gamma^2 - 2 \Pi_\Psi^2}}{\beta}.$$  \hspace{1cm} (91)

The right hand sides of the Einstein equations (39), (40) tend to zero, so the gravitational degrees of freedom decouple

$$\dot{H}_a = -3 H_a^2, \hspace{1cm} \dot{H}_x = -3 H_x H_a,$$  \hspace{1cm} (92)

and the gravitational constraint assumes the vacuum form as well

$$H_a^2 - H_x^2 = 0.$$  \hspace{1cm} (93)

Decoupling of gravity means that in the limit $\beta \to 0$ the metric is given by the vacuum Kasner solution either of a cigar type

$$H_a = H_x = \frac{1}{3t},$$  \hspace{1cm} (94)

or a pancake type

$$H_a = -H_x = \frac{1}{3t},$$  \hspace{1cm} (95)

where we set the singularity at $t = 0$.

Substituting the explicit expressions for the Hubble and shear parameters one finds another constraint

$$\frac{\Psi^2 \Gamma}{H_a} = C = \text{const},$$  \hspace{1cm} (96)

and thus in the remaining equations one can express all the gauge field variables either in terms of $\Gamma, \Pi_\Gamma$, or in terms of $\Psi, \Pi_\Psi$. One simple consequence of this constraint is that in the non-trivial case $C \neq 0$ the variables $\Psi$ and $\Gamma$ can not have zeroes except for the singularity, and thus should preserve their signs. From the NBI field equations one then finds

$$\Pi_\Psi = \dot{\Psi}, \hspace{1cm} \text{cigar},$$  \hspace{1cm} (97)

$$\Pi_\Psi = \dot{\Psi} - \frac{2 \Psi}{3t}, \hspace{1cm} \text{pancake}.$$  \hspace{1cm} (98)

In both cases the dynamical equation for $\Psi$ will be of the form

$$\ddot{\Psi} = f(\Psi, \dot{\Psi}, C, t).$$  \hspace{1cm} (99)
with some function \( f \). It describes oscillations with a decreasing amplitude. The second YM variable \( \Gamma \) is related to \( \Psi \) algebraically via constraints (96), and therefore oscillates with the same frequency exactly in an antiphase. Oscillations are fully regular, so no YM chaos can persist in the regime of the strong BI non-linearity.

The general solution near the pancake singularity can be expanded with respect to the variable \( \tau \equiv t^{1/3} \):

\[
\Gamma = p_1^2 \tau + \sqrt{6C} p_1 \tau^2 + \frac{3C}{2} \tau^2 + q_1 \tau^4 + O(\tau^5), \quad (100)
\]

\[
\Psi = \frac{\sqrt{C}}{\sqrt{3p_1}} \tau^{-2} + \frac{C}{2p_1^2} \tau^{-1} + \frac{\sqrt{3C^{3/2}}}{2p_1^3} + O(\tau), \quad (101)
\]

where \( p_1 \) and \( q_1 \) are free parameters.

Near the cigar singularity the solution can be expanded in terms of \( t \):

\[
\Gamma = p_1 t^{-1} + q_1 + \frac{3q_1^2 - p_1 C}{6p_1} t + O(t^2), \quad (102)
\]

\[
\Psi = \frac{\sqrt{C}}{\sqrt{3p_1}} - \frac{\sqrt{3C} q_1}{2 \sqrt{3p_1^{3/2}}} t + O(t^2). \quad (103)
\]

**VIII. CHAOS-ORDER TRANSITION**

Now we address the problem numerically. Various methods were suggested to study a chaotic behavior in the context of gravity, where the absence of the canonical time variable prevents a straightforward use of such convenient tools as the Lyapunov exponents (however, see [42]). In the case of the conformally invariant YM Lagrangian, one can use the approach of Ref. [23] to separate the dynamics of the YM field from the gravitational expansion and then apply the invariant technique of chaotic scattering. For systems exhibiting chaotic behavior the set of all periodic orbits has fractal structure invariant under coordinate reparameterizations.

However these methods become problematic in our case, where the conformal invariance is absent from the matter action. Apart from the special asymptotic regimes, it is not possible to separate the YM dynamics from the metric evolution. Though it can be done for the high YM intensity, it turns out that the time interval in the actual evolution where this regime holds is rather small for \( \beta \) of the order of unity or greater. The YM variables perform only a small number of oscillations during the epoch of high field intensity, the field then being fast diluted. However, if we set the parameter \( \beta \) sufficiently small, the time spent by the system in the highly nonlinear region will be large enough, and in this case the chaos-order transition is unambiguously manifest.

A numerical analysis of the system for small values of \( \beta \) reveals the following. While the gauge field strength is considerably greater than the critical field \( \beta \), both conditions (60) and (76) approximately hold, and the dynamics of the gauge field qualitatively coincides with that discussed in the previous section for \( \beta = 0 \). Both variables \( \Psi \) and \( \Gamma \) perform nonlinear oscillations with decreasing amplitude in an antiphase with respect to each other and without crossing zero. In this region the dynamics is fully regular, while the evolution of the metric is mainly governed by purely vacuum terms and is close to the vacuum Kasner solution.

To test the validity of an approximate description of the system in terms of the limiting \( \beta = 0 \) solution we check the constraint equations (90, 96) in the case of small but finite \( \beta \). The first constraint is the condition of smallness of the pseudoscalar YM invariant \( q^2 \ll \beta^2 F \). The second constraint was directly checked numerically. Fig. 4 illustrates the situation for the cigar solution with \( \beta = 10^{-4} \). From this figure one can see that the right hand side of Eq. (96) evolves on time scales much larger than the period of oscillations of the gauge field variables. Under the overall volume expansion, the YM energy density falls down and the role of the BI nonlinearity decreases. At the same time the matter terms in the Einstein equation become more significant force, in particular, to decrease the shear anisotropy \( H_\chi \) much faster than the Hubble parameter \( H_a \).

These features are illustrated in Figs. 2, 4. Fig. 2 shows the early regular evolution for \( \beta = 2 \cdot 10^{-3} \) and the cigar-type singularity. Both variables \( \Psi \) and \( \Gamma \) oscillate in the positive region. The behavior of the shear anisotropy \( H_\chi \) is smooth. Fig. 4 demonstrates the same solution at late time. One can see that the dynamics of the gauge field becomes essentially chaotic. The function \( H_\chi \) coupled to matter performs chaotic oscillations with decreasing (as compared to the Hubble parameter \( H_a \)) amplitude. The first zeroes of the gauge functions \( \Psi \) and \( \Gamma \) serve as an approximate boundary separating highly nonlinear evolution from the region of the chaotic regime. The actual type of singularity (pancake or cigar), plays a relatively small role in both the regular and strongly chaotic phases except for the small vicinity of the singularity. This is illustrated in Fig. 4 which shows the solution with pancake singularity obtained from the solution shown in Fig. 2 by changing the sign of \( H_\chi \) with initial conditions which were set at \( t = 10 \).

The Hubble parameter \( H_a \) does not exhibit chaotic behavior. It can be presented as \( H_a = h(t)t^{-1} \), with some slowly varying smooth function \( h(t) \). Numerical curves \( h(t) \) are shown in Fig. 5 for various \( \beta \). This function interpolates between the value 1/3 at \( t = 0 \) (vacuum Kasner solution) and the value 1/2 at \( t = \infty \) corresponding to the isotropic “hot universe” cosmology. However, for small values of \( \beta \), when the system stays in a highly nonlinear regime for a considerable time interval, there is a region where \( h(t) \) is greater than 1/2. This feature can be explained using the results of the FRW-BI model and was firstly presented in ref. [35] (Subsequent analysis can be found in ref. [34, 40, 41]). Eq. (90) implies that once the contribution of the anisotropy term \( H_\chi \) decreases (i.e. the solution undergoes isotropisation), the Hamiltonian constraint tends to the Friedmann equation. In the FRW case one can derive an equation of state...
for the NBI matter which interpolates between that for conformal matter $\epsilon = 1/3p$ and the “string fluid” equation $\epsilon = -1/3p$ in the highly nonlinear regime. The latter corresponds to the value $\hbar = 1$, which is, however, never achieved in the anisotropic case which we investigate here.

**IX. DISCUSSION**

The main goal of this paper was to test the non-perturbative effects of superstring theory on the issue of chaos in cosmology. At least three different patterns of chaotic behavior in cosmology were identified. The first is the billiard-type behavior which is manifest in the Bianchi I pure gravity and its supergravity (including multidimensional cases) generalizations. The second is the bouncing behavior of the FRW-scalar field cosmology. The third type is the matter-dominated chaos of the Bianchi I EYM cosmology, and it is this type of chaos which was investigated here. Recently an interesting analysis was performed of the YM field behavior in more general type A Bianchi space-times showing that basic features of the YM chaos persist there as well. From these three patterns the last one is the most appropriate for testing the superstring non-locality effects accumulated in the BI non-Abelian action. Our results clearly demonstrate disappearance of chaos in the high energy regime.

From a mathematical viewpoint, it is worth noting that the Einstein-NBI system of equations admits a reduction of order due to the presence of scaling symmetries similarly to the EYM case. Moreover, in the strong BI regime the axisymmetric Bianchi I NBI system can be reduced further due to existence of two additional asymptotic integrals of motion. This limit is characterized by the dynamical vanishing of the pseudoscalar quadratic invariant of the YM field. This simplifies dynamics considerably and leads to a decoupling of the gravitational degrees of freedom. Color oscillations are still governed by the BI non-linearity and are reducible to the one-variable second order system predicting perfectly regular behavior.

Numerical experiments shows that the system behavior for sufficiently small $\beta$ consists of a regular phase in the high energy region near the singularity and the chaotic phase at later time. The regular phase is qualitatively similar to that described by the $\beta = 0$ approximate description. The chaos-order transition is observed when one is moving backward in time towards the singularity. The singularity itself is either of a cigar or a pancake type, as in the vacuum Bianchi I case, though the YM field does not tend to the vacuum configuration. Thus the non-perturbative in $\alpha'$ string corrections to the YM action suppress the YM chaos which takes place at lower energies where dynamics of the YM field is governed by the ordinary quadratic action.

In the case of only an Abelian component excited, an exact analytic solution of the Einstein-BI system was found which generalizes the Rosen solution to the Einstein-Maxwell equations. It also exhibits a different behavior in the singularity as compared with the Einstein-Maxwell case.

**Acknowledgments**

D.V.G. thanks the University of Beira Interior for hospitality and the grant NATO CR(RU)05/C/03 PO - GRICES - 03675 for support in the summer 2004, when this research was initiated. This work was also supported in part by the RFBR grant 02-04-16949, POCI(FEDER) P - FIS - 57547/2004 and CERN P - FIS - 49529/2003. PVM is supported by the grant FCT (FEDER) SFRH - BSAB 396/2003. He also thanks QMUL for hospitality and R. Tavakol for conversations.

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FIG. 1: The phase portrait $\Psi^{2/3}t^{2/3}$ vs. $\Gamma^{t^{1/3}}$ for $\beta = 10^{-4}$.

FIG. 2: The solution for the $\beta = 2 \cdot 10^{-3}$, regular phase, a cigar singularity. The solid line — $\Gamma^{t^{1/3}}$, the dashed line — $\Psi^{t^{1/3}}$, the dotted line — $H_x/H_a$. 
FIG. 3: Further development of the solution from Fig. 2 — chaotic oscillations.

FIG. 4: The solution for $\beta = 2 \cdot 10^{-3}$, a regular phase, a pancake singularity. The solid line — $\Gamma t^{1/3}$, the dashed line — $\Psi t^{1/3}$, the dotted line — $-H_x/H_a$. 
FIG. 5: The behaviour of $h(t) = tH_a$ for various values of $\beta$. 

$\beta = 10 \quad \beta = 0.05 \quad \beta = 0.02 \quad \beta = 0.01 \quad \beta = 0.005 \quad \beta = 10$