The quantitative distribution of Hecke eigenvalues of Maass forms

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Abstract

Let \( f \) be a normalized Hecke–Maass cusp form of weight zero for the group \( SL_2(\mathbb{Z}) \). This article presents several quantitative results about the distribution of Hecke eigenvalues of \( f \). Applications to the \( \Omega_{\pm} \)-results for the Hecke eigenvalues of \( f \) and its symmetric square \( \text{sym}^2(f) \) are also given.

Keywords: Maass forms, Fourier coefficients, Distributions

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1 Introduction and statement of results

Let \( f \) be a normalized primitive holomorphic cusp form of even weight \( k \geq 12 \) for the group \( SL_2(\mathbb{Z}) \). The generalized Ramanujan conjecture for such \( f \) was proved by Deligne in 1974 which implies that the normalized Hecke eigenvalues \( \lambda_f(p) \), with \( p \) running through all the primes, lie in the interval \([-2, 2]\). The asymptotic distribution of the Hecke eigenvalues \( \lambda_f(p) \), as the prime \( p \) varies, is an interesting and difficult problem. Inspired by the Sato-Tate conjecture for elliptic curves, Serre in the 1960s conjectured, settled recently [2], that for any such \( f \), \( \lambda_f(p) \) for \( p \leq x \) distribute nicely: the sequence \( \{\lambda_f(p)\} \) is equidistributed in \([-2, 2]\] with respect to the Sato-Tate measure \( \frac{1}{2\pi} \sqrt{4 - t^2} \, dt \).

More precisely, given any interval \( I \subseteq [-2, 2] \),

\[
\lim_{x \to \infty} \frac{\# \{ p \leq x : \lambda_f(p) \in I \} }{\pi(x)} = \frac{1}{2\pi} \int_I \sqrt{4 - t^2} \, dt,
\]

where \( \pi(x) \) denotes the number of primes less than or equal to \( x \) for any real number \( x \geq 2 \). This is also called the Sato-Tate conjecture. It has significant implications in number theory. It is well-known that the Langlands’ functoriality conjectures imply the Sato-Tate conjecture for automorphic representations on \( GL(2) \). In absence of a proof of Langlands’ functoriality it is an interesting question, “how much” Sato-Tate one can squeeze out of the handful of known cases of functoriality. It is clear and not very deep that these cases of functoriality imply some distributional results of Hecke eigenvalues, and it leads to a challenging optimization problem to push this to the limit. This type of question was probably initiated by Serre some 20 or 30 years ago. Regarding the large values of \( \lambda_f(p) \), Serre [17, Appendix] in a letter to Shahidi has proved the following.
Theorem 1.1 (Serre) For every $\epsilon > 0$ and $c = 2 \cos(2\pi/7) \simeq 1.247$, there are infinitely many primes $p$ such that $\lambda_f(p) > c - \epsilon$ and infinitely many primes $q$ with $\lambda_f(q) < -(c - \epsilon)$. Moreover, these infinite sets have positive upper densities.

In the same letter he was also interested to the following related questions:

a. For a given $\epsilon > 0$, does there exist a non-zero lower bound, depending only on $\epsilon$, for the upper density of the set of primes $p$ with $\lambda_f(p) > c - \epsilon$?

b. Is it possible to prove an analogous result like Theorem 1.1 for any $c \in (0, 2]$ without assuming the Deligne's bound for the Fourier coefficients?

c. For a given $\epsilon > 0$, does there exist an explicit constant $N(\epsilon)$ such that there is a prime $p \leq N(\epsilon)$ with $\lambda_f(p) > c - \epsilon$?

Regarding question (b), Serre remarked that without using Deligne's result one can show that for every $\epsilon > 0$, there are infinitely many primes $p$ such that $|\lambda_f(p)| > c - \epsilon$, where $c$ is as in Theorem 1.1. It would also be worth to point out that Chiriac and Jorza in [5] studied questions (a) and (b) for the constraint $|\lambda_f(p)| > 1$ and Walji in [18] studied question (b) for $\lambda_f(p) > 0.904$.

In a similar spirit, it would also be interesting to look these questions for small values of $\lambda_f(p)$. In this direction, using Deligne's result Serre [17, Appendix] proved that for every $\epsilon > 0$, there are infinitely many primes $p$ such that $|\lambda_f(p)| < \sqrt{2/3} + \epsilon$. It makes sense to point out that the above Serre's results are obsolete for holomorphic cusp forms because now the Sato-Tate conjecture is known therefore we have a very good understanding of the distribution of Hecke eigenvalues.

In contrast with holomorphic forms, it is not known that Maass forms satisfy the Ramanujan-Petersson conjecture, although one expects it to be true because the conjecture fits correctly in the general Langlands functoriality program. It is a major unsolved problem in the general theory today and its solution would have significant consequences in number theory and related areas. However, in the literature there are some quantitative results towards this conjecture. Ramakrishnan [15] proved that for a Maass form $f$, the Ramanujan conjecture is true for primes with the lower Dirichlet density (or, analytic density) at least $9/10$. This lower Dirichlet density is later improved to $34/35$ by Kim and Shahidi [11]. Recently, Luo and Zhou [12] have refined the result of Kim and Shahidi from Dirichlet density to the natural density.

As in the case of holomorphic cusp form, the Sato-Tate conjecture is expected to hold for Maass cusp forms as well. In the same letter to Shahidi, Serre asked the analogous questions as discussed above for the Hecke eigenvalues of Maass forms. According to him, his method has an obstruction due to lack of knowledge of the Ramanujan conjecture (or, at least, bounded Hecke eigenvalues) for Maass forms.

The main purpose of this article is to study distribution of Hecke eigenvalues of a weight zero Hecke–Maass cusp form for the group $SL_2(\mathbb{Z})$ in various intervals. The main idea that we use to prove all our distribution results is to use analytic prime number theorems and known cases of functoriality of symmetric power $L$ functions of a Hecke–Maass cusp form $f$ which lead to a polynomial optimization problem. Our first result gives an affirmative answer to the Serre’s above questions when $|\lambda_f(p)| < 1$. 
Theorem 1.2  Let $f$ be a Hecke–Maass cusp form of weight zero for the group $SL_2(\mathbb{Z})$. Assume the Fourier coefficients $\lambda_f(n)$ of $f$ satisfy the Ramanujan conjecture. Then we have

$$\liminf_{x \to \infty} \frac{\# \{ p \leq x : |\lambda_f(p)| < 1 \}}{\pi(x)} \geq 0.00010077.$$  \hfill (2)

Unconditionally, we prove the following.

Theorem 1.3  There are infinitely many primes $p$ such that $|\lambda_f(p)| < 1$.

Now we give an upper bound for the smallest prime $p$ such that $|\lambda_f(p)| < 1$ which answers question (c) of Serre in this situation.

Theorem 1.4  Let $f$ be a Hecke–Maass cusp form of weight zero for the group $SL_2(\mathbb{Z})$ with Laplacian eigenvalue $\frac{1}{4} + u^2$. Then there exists a prime

$$p \ll \exp \left( c \log^2(1 + |u|) \right)$$

such that $|\lambda_f(p)| < 1$. Here $c$ is an absolute constant and the implied constant in the $\ll$ symbol in (3) is also absolute though ineffective.

Next we obtain a more general result about the small values of $|\lambda_f(p)|$ without assuming the Ramanujan conjecture. More precisely, we prove the following.

Theorem 1.5  Let $f$ be a Hecke–Maass cusp form of weight zero for the group $SL_2(\mathbb{Z})$ and $a > 1$ be a real number. Then

$$\liminf_{x \to \infty} \frac{\# \{ p \leq x : 0.908 < |\lambda_f(p)| < 1.928 \}}{\pi(x)} \geq 0.0054\ldots$$

$$\liminf_{x \to \infty} \frac{\# \{ p \leq x : 1 < |\lambda_f(p)| < 2 \}}{\pi(x)} \geq 0.164880.$$  \hfill (4)

Remark 1.1  This result has many interesting consequences. For example, when we take $a = 4$, then it immediately shows that the set of primes satisfying the Ramanujan conjecture has lower density $> 3/4$. Now as mentioned earlier, Luo and Zhou in [12] has shown that this set of primes has density at least $34/35$. Certainly $34/35 > 3/4$, but the point is Luo and Zhou’s proof is long whereas our result which is nontrivial has been obtained by very elementary means and is comparatively very short.

Our next results are concerned with the large values of $|\lambda_f(p)|$.

Theorem 1.6  Let $f$ be a Hecke–Maass cusp form of weight zero for the group $SL_2(\mathbb{Z})$. Then

a. $\liminf_{x \to \infty} \frac{\# \{ p \leq x : 0.908 < |\lambda_f(p)| < 1.928 \}}{\pi(x)} \geq 0.0054\ldots$

and

b. $\liminf_{x \to \infty} \frac{\# \{ p \leq x : 1 < |\lambda_f(p)| < 2 \}}{\pi(x)} \geq 0.164880.$

Remark 1.2  The density ”0.164880” obtained in Part (b) of Theorem 1.6 is close to its predicted density (which is 0.3910022) by Eq. (1). Also this result refines [5, Theorem B] in two ways: first, we obtain a lower bound for the limit infimum whereas in [5, Theorem B] they have obtained an upper Dirichlet density and second, our bound is much stronger than their bound. It is also worth pointing out that Walji in [19] has proved that the set
of primes \( \{ p : \lambda_f(p) > 0.778 - \epsilon \} \) has upper Dirichlet density greater than 1/100 for any \( \epsilon > 0 \).

**Remark 1.3** If we apply the method of the proof of Theorem 1.6 to the polynomial \(-x^2(a^2 - a)(x^2 - 4)\) where \( a = (1.189)^2 \) then one can prove that the set of primes \( p \) with \( 1.189 < |\lambda_f(p)| < 2 \) has lower density at least 0.0362. The reason to include this remark here is that there is a different proof of this fact given by Murty [14, Cor. 1, p. 527] which he used to get an \( \Omega \)-result for the Fourier coefficients of \( f \). Murty’s proof implicitly assumes the Ramanujan conjecture for the Hecke eigenvalues of a Hecke–Maass cusp form and this is not known, whereas this remark gives an unconditional proof.

Next we prove a result about the phenomenon \( |\lambda_f(p)| > \sqrt{2} \) which gives Corollary 1.8 that will be used later to get an \( \Omega \)-result about \( \lambda_{\text{sym}^2f}(n) \) (see proof of Corollary 1.11).

**Theorem 1.7** Let \( f \) be a Hecke–Maass cusp form of weight zero for the group \( SL_2(\mathbb{Z}) \).  

a. Assume the Fourier coefficients \( \lambda_f(n) \) of \( f \) satisfy the Ramanujan conjecture. Then 

\[
\liminf_{x \to \infty} \frac{\# \{p \leq x : |\lambda_f(p)| > \sqrt{2} \}}{\pi(x)} \geq \frac{1}{32}.
\]

b. Unconditionally, there are infinitely many primes \( p \) such that \( |\lambda_f(p)| > \sqrt{2} \).

Combining Theorem 1.3 and part (b) of Theorem 1.7 with the identity \( \lambda_{\text{sym}^2f}(p) = \lambda_f(p^2) = \lambda_f(p)^2 - 1 \), we obtain the following immediate corollary about the sign changes for the sequence \( \{\lambda_{\text{sym}^2f}(p)\} \).

**Corollary 1.8** There are infinitely many primes \( p \) such that \( \lambda_{\text{sym}^2f}(p) > 0 \) and also there are infinitely many primes \( q \) such that \( \lambda_{\text{sym}^2f}(q) < 0 \).

**Remark 1.4** It seems rather unlikely that one can prove that the set of primes figuring in Theorem 1.3 has positive lower density by our strategy used here, namely employing suitable polynomials \( P(x) \in \mathbb{R}[x] \) and using the asymptotics of the even power moments of \( \lambda_f(p) \). We are presently working on this and we hope to report our progress in a future article.

For the other part of the paper, as a consequence of the Deligne’s result, we know \( |\tau(n)| \leq d(n)n^{11/2} \), more generally, if \( a_f(n) \) is the \( n \)th Fourier coefficient of a holomorphic normalized Hecke eigenform \( f \) of weight \( k \), then \( |a_f(n)| \leq d(n)n^{(k-1)/2} \), where \( d(n) \) is the divisor function. So, it is natural to ask, “Are the above bounds optimal as a function of \( n^\epsilon \)?” In other words, are these inequalities sharp? It is elementary to show that \( d(n) = O(n^\epsilon) \) for any \( \epsilon > 0 \) [1, p. 296]. Therefore, we have \( |\tau(n)| \leq n^{11/2+\epsilon} \). Hardy showed that \( |\tau(n)| > n^{11/2} \), holds infinitely often. The best result in this regard is due to Murty [13], who showed that there is an absolute and effective constant \( c > 0 \) such that 

\[
|a_f(n)| > n^{(k-1)/2} \exp \left( \frac{c \log n}{\log \log n} \right)
\]

holds infinitely often. This is the best possible order because the maximal order of \( d(n) \) is \( \exp \left( \frac{c' \log n}{\log \log n} \right) \), due to Wigert [20]. In the literature, the phenomenon (5) is referred to as an \( \Omega \)-result for the Fourier coefficient of a holomorphic cusp form.
Motivated from this, the final aim of this paper is to prove a $\Omega$-result for a general real sequence $\{a(n)\}$, that satisfy certain assumptions and as a consequence of that, we obtain the $\Omega$-results for the Hecke eigenvalues of a Hecke–Maass cusp form $f$ and its symmetric square $\text{sym}^2(f)$.

**Theorem 1.9** Let $\{a(n)\}$ be a real sequence such that

(i) the function $n \mapsto a(n)$ is multiplicative,
(ii) $\sum_{p \leq x} a(p)^4 \sim \alpha \pi(x)$, for some constant $\alpha \geq 2$.

Then there exists a constant $c > 0$ such that

$$a(n) = \Omega\left(\exp\left(\frac{c \log n}{\log \log n}\right)\right).$$

**Corollary 1.10** Let $f$ be a Hecke–Maass cusp form of weight zero for the group $\text{SL}_2(\mathbb{Z})$. Further assume $\lambda_f(n)$ (resp. $\lambda_{\text{sym}^2 f}(n)$) are the normalized Fourier coefficients of $f$ (resp. $\text{sym}^2(f)$). If $a(n)$ denotes either $\lambda_f(n)$ or $\lambda_{\text{sym}^2 f}(n)$, then

$$a(n) = \exp\left(\frac{c \log n}{\log \log n}\right)$$

for some $c > 0$.

We will now sharpen our $\Omega$-result and prove an $\Omega_+\,$-result for both the sequences $\lambda_f(n)$ and $\lambda_{\text{sym}^2 f}(n)$. This result explicitly shows large oscillations of both these sequences.

**Corollary 1.11** For any Hecke–Maass cusp form $f$ as in Corollary 1.10 with $a(n)$ denoting either $\lambda_f(n)$ or $\lambda_{\text{sym}^2 f}(n)$, then we have

$$a(n) = \Omega_+\left(\exp\left(\frac{d \log n}{\log \log n}\right)\right),$$

for some $d > 0$.

**Remark 1.5** To the best of our knowledge Corollaries 1.10 and 1.11 are new in our context, i.e., for a Hecke–Maass cusp form.

**Remark 1.6** Our method of proof of the $\Omega$-result here has $\text{sym}^2(f)$ as a limiting case since we don’t know the asymptotics of the power moments of $\lambda_f(p)$ beyond the eighth power. Hence we cannot prove the corresponding $\Omega$-result for $\text{sym}^2(f)$ for instance.

### 1.1 Notations

For a real valued function $f$ and a positive function $g$, the symbol “$f(y) = O(g(y))$” or “$f(y) \ll g(y)$“ means that there is a constant $c > 0$ such that $|f(y)| \leq cg(y)$ for any $y$ in the concerned domain. The dependence of this implied constant on some parameter(s) may sometimes be displayed by a suffix (or suffixes). Furthermore, the notation $f(x) \asymp g(x)$ means both the bounds $f(x) \ll g(x)$ and $g(x) \ll f(x)$ hold. The notation “$f(y) = o(g(y))$” means that $f(y)/g(y) \to 0$ as $y \to \infty$ and “$f(y) \sim g(y)$” means $f(y) - g(y) = o(g(y))$. The symbol $f = \Omega(g)$ means that $\lim \sup_{x \to \infty} \frac{f(x)}{g(x)} > 0$. We write $f = \Omega_+(g)$ if $\lim \sup_{x \to \infty} \frac{f(x)}{g(x)} > 0$ and $f = \Omega_-(g)$ if $\lim \inf_{x \to \infty} \frac{f(x)}{g(x)} < 0$. Lastly, $f = \Omega_\pm(g)$ means
that \( f = \Omega_+ (g) \) and also \( f = \Omega_- (g) \). Throughout the paper, the symbols \( p \) and \( q \) denote primes.

2 Preliminaries

This section contains a very brief account of the theory of Maass forms based primarily on [4, Sect. 1.9]. Let \( \mathbb{H} := \{ x + iy \in \mathbb{C} : y > 0 \} \) be the upper half-plane. Then the non-Euclidean Laplacian

\[
\Delta = -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)
\]

acts on functions on \( \mathbb{H} \). A Maass form \( f \) of weight zero for the group \( SL_2(\mathbb{Z}) \) is a smooth function on \( \mathbb{H} \) such that

(i) \( f(\gamma z) = f(z) \) for \( \gamma \in SL_2(\mathbb{Z}) \);

(ii) \( f \) is an eigenfunction of \( \Delta \); and

(iii) \( f(x + iy) = O(y^N) \) as \( y \to \infty \) for some \( N \).

If \( f \) is a Maass form with the Laplacian eigenvalue \( \lambda \), then one writes \( \lambda = s (1 - s) \) with \( s = 1/2 + iu \) where \( s, u \in \mathbb{C} \), \( u \) being known as the spectral parameter. In other words, we have \( \lambda = 1/4 + u^2 \). A Maass form \( f \) for \( SL_2(\mathbb{Z}) \) with spectral parameter \( u \) admits the following Fourier expansion

\[
f(x + iy) = \sqrt{y} \sum_{n \in \mathbb{Z}} a_f(n) K_{iu}(2\pi |n|y)e(nx).
\]

Here \( e(x) = e^{2\pi i x} \) and \( K_{iu} \) is the Bessel function. Note that \( f \) is called a Maass cusp form if \( a_0(f) = 0 \).

As in the case of holomorphic modular forms, there is a parallel Hecke theory for Maass forms. In particular, there is an orthonormal basis of Maass cusp forms consisting of forms that are common eigenfunctions of the Hecke operators \( T_n \), \( n \geq 1 \). The elements in the basis are known as Hecke–Maass cusp forms. If \( f \) is a Hecke–Maass cusp form with \( n \)th Hecke eigenvalue \( \lambda_f(n) \), then it is known that

\[
a_f(n) = a_f(1) \lambda_f(n) \quad (n \geq 1).
\]

We call such an \( f \) normalized if \( a_f(1) = 1 \). So the Fourier coefficients of a normalized Hecke–Maass cusp form are same as its Hecke eigenvalues. The eigenvalues \( \lambda_f(n) \) of \( f \) are multiplicative and satisfy

\[
\lambda_f(n) \lambda_f(m) = \sum_{d \mid (m,n)} \lambda_f \left( \frac{mn}{d^2} \right).
\]

In particular, for any prime \( p \), we have \( \lambda_f(p^2) = \lambda_f(p)^2 - 1 \). The general Ramanujan conjecture asserts that for a normalized Hecke–Maass cusp form \( f \) the \( p \)th Hecke eigenvalue \( \lambda_f(p) \) satisfies the bound

\[
|\lambda_f(p)| \leq 2p^{\frac{1}{2}}.
\]

Although this conjecture is wide open, we know from the works of Kim and Sarnak [9] that

\[
|\lambda_f(p)| \leq 2p^{\frac{1}{2}}.
\]

Let \( f \) be a normalized Hecke–Maass cusp form of weight zero for \( SL_2(\mathbb{Z}) \) with \( n \)th Hecke eigenvalue \( \lambda_f(n) \). Then the associated \( L \)-function is defined by

\[
L(s,f) = \prod_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_p \left( 1 - \frac{a_f(p)}{p^s} \right)^{-1} \left( 1 - \frac{\beta_f(p)}{p^s} \right)^{-1}
\]
which converges absolutely for $\text{Re}(s) > 1$. The local parameters $\alpha_f(p)$ and $\beta_f(p)$ are related to the normalized Fourier coefficients in the following way:

$$
\alpha_f(p) + \beta_f(p) = \lambda_f(p), \quad \alpha_f(p)\beta_f(p) = 1.
$$

In addition to $L(s,f)$, we have the symmetric power $L$-functions

$$
L(s, \text{sym}^m f) = \prod_p \prod_{j=0}^m \left( 1 - \frac{\alpha_f(p)\beta_f(p)^m}{p^s} \right)^{-1},
$$

for $m \geq 1$ and the Rankin-Selberg convolution of $\text{sym}^m f$ and $\text{sym}^n f$

$$
L(s, \text{sym}^m f \times \text{sym}^n f) = \prod_p \prod_{j=0}^m \prod_{i=0}^n \left( 1 - \frac{\alpha_f(p)\beta_f(p)^m\alpha_f(p)^i\beta_f(p)^n}{p^s} \right)^{-1}.
$$

By the works of Kim [9] and Kim and Shahidi [10, 11] the symmetric $m$th power $L$-functions for $m \leq 8$ are holomorphic and non-vanishing for $\text{Re}(s) \geq 1$.

3 Intermediate results

We begin this section by recalling the following result of Brumley [3, Eq. 2.23, p. 989] which is about the asymptotics of the even power moments of $\lambda_f(p)$.

**Theorem 3.1** [3] Let $f$ be a Hecke–Maass cusp form with Fourier coefficients $\lambda_f(n)$. Then for $1 \leq j \leq 8$, we have

$$
\sum_{p \leq x} \lambda_f(p^j) = o\left( \frac{x}{\log x} \right), \quad (7)
$$

as $x \to \infty$.

As a consequence of this, we prove two consecutive results about the asymptotics of the even power moments of $\lambda_f(p)$.

**Lemma 3.2** For a Hecke–Maass cusp form $f$ of level one, we have

$$
\sum_{p \leq x} \lambda_f^2(p) \sim \frac{x}{\log x}, \quad \sum_{p \leq x} \lambda_f^4(p) \sim \frac{2x}{\log x}, \quad \sum_{p \leq x} \lambda_f^6(p) \sim 5 \frac{x}{\log x}, \quad (8)
$$

for all $x$ sufficiently large.

**Proof** From relation (6), we have

$$
\lambda_f^2(p) = \lambda_f(p^2) + 1,
$$

$$
\lambda_f^4(p) = \lambda_f(p^4) + 3\lambda_f(p^2) + 2,
$$

$$
\lambda_f^6(p) = \lambda_f(p^6) + 5\lambda_f(p^4) + 9\lambda_f(p^2) + 5,
$$

for a prime $p$. The required results follow once we use the prime number theorem and Theorem 3.1 for $j = 2, 4, 6$.  

**Lemma 3.3** For all sufficiently large \( x \), we have

\[
\sum_{p \leq x} \lambda_f^8(p) \sim \frac{14x}{\log x}, \tag{9}
\]

and consequently

\[
\sum_{p \leq x} \lambda_{\text{sym2}}^4(p) \sim \frac{3x}{\log x}.
\]

**Proof** From the proof of Lemma 3.2, we write \( \lambda_f(p^2), \lambda_f(p^4) \) and \( \lambda_f(p^6) \) as a polynomial in \( \lambda_f(p) \) and hence we have

\[
\lambda_f(p^6) \lambda_f(p^2) = \lambda_f^8(p) - 6\lambda_f^6(p) + 11\lambda_f^4(p) - 7\lambda_f^2(p) + 1.
\]

On the other hand, Eq. (6) gives

\[
\lambda_f(p^6) \lambda_f(p^2) = \lambda_f(p^8) + \lambda_f(p^6) + \lambda_f(p^4).
\]

Combining last two equations

\[
\lambda_f^8(p) = \lambda_f(p^8) + \lambda_f(p^6) + \lambda_f(p^4) + 6\lambda_f^6(p) - 11\lambda_f^4(p) + 7\lambda_f^2(p) - 1.
\]

From Theorem 3.1 and Lemma 3.2, for sufficiently large \( x \), we obtain

\[
\sum_{p \leq x} \lambda_f^8(p) \sim \frac{14x}{\log x},
\]

and this proves the first part of the lemma.

For the next claim, we know

\[
\lambda_{\text{sym2}}^4(p) = \lambda_f^4(p^2) = (\lambda_f^2(p) - 1)^4 = \lambda_f^8(p) - 4\lambda_f^6(p) + 6\lambda_f^4(p) - 4\lambda_f^2(p) + 1.
\]

Now for a large \( x \), we take sum on the both sides over all the primes \( p \leq x \) in the last identity and then use Lemma 3.2 and (9) for the asymptotics of the involved sums to obtain the required result.

Next for a Hecke–Maass cusp form \( f \), we also require a lower bound for the sum \( \sum_{1 \leq p \leq x} |\lambda_f(p)| \). Using Lemma 3.2 for the asymptotics for the even power moments of \( \lambda_f(p) \), we prove the following.

**Proposition 3.4** For a Hecke–Maass cusp form \( f \) of level one we have the bound

\[
\sum_{1 \leq p \leq x} |\lambda_f(p)| \gg f \frac{x}{\log x}, \tag{10}
\]

for all \( x \) sufficiently large.
Proof From the Cauchy–Schwarz inequality, we have

\[
\left( \sum_{x<p \leq 2x} \lambda_f^2(p) \right)^2 \leq \left( \sum_{x<p \leq 2x} |\lambda_f(p)| \right) \left( \sum_{x<p \leq 2x} |\lambda_f^2(p)| \right),
\]
i.e.,

\[
\frac{\left( \sum_{x<p \leq 2x} \lambda_f^2(p) \right)^2}{\sum_{x<p \leq 2x} |\lambda_f^2(p)|} \leq \sum_{x<p \leq 2x} |\lambda_f(p)|.
\]

From Lemma 3.2, we have

\[
\sum_{x<p \leq 2x} \lambda_f^2(p) \sim \frac{x}{\log x}.
\]

Therefore, to complete the proof of the proposition it suffices to show that \( \sum_{x<p \leq 2x} |\lambda_f^2(p)| \gg \frac{x}{\log x} \). For that, first we use the Cauchy–Schwarz inequality to write

\[
\left( \sum_{x<p \leq 2x} |\lambda_f^2(p)| \right)^2 \leq \left( \sum_{x<p \leq 2x} \lambda_f^2(p) \right) \left( \sum_{x<p \leq 2x} \lambda_f^2(p) \right)
\]

and then use Lemma 3.2 for the sums on the right hand side in the above inequality. \( \square \)

From Theorem 3.1 and Proposition 3.4, we conclude the following corollary about the sign change of the sequence \( \{\lambda_f(p)\} \), which is also known by Elliott–Kish in [6]. This result will be useful for proving the \( \Omega_{\pm} \)-result; see Corollary 1.11.

Corollary 3.5 There are infinitely many sign changes for the sequence \( \{\lambda_f(p)\} \).

Proof It suffices to show that for any real number \( x \), sufficiently large, the interval \( I_x = (x, 2x] \) contains at least two distinct primes \( p \) and \( q \) with \( \lambda_f(p) < 0 \) and \( \lambda_f(q) > 0 \). Here we give a proof for the existence of \( p \) since the other case can be handled exactly in the same way. So, we assume the first claim is not true, i.e., \( \lambda_f(p) \geq 0 \) whenever \( p \in (x, 2x] \). Then from Proposition 3.4

\[
\sum_{x<p \leq 2x} \lambda_f(p) \gg \frac{x}{\log x}
\]

but taking \( j = 1 \) in Theorem 3.1, we know

\[
\sum_{x<p \leq 2x} \lambda_f(p) = o \left( \frac{x}{\log x} \right),
\]

which is not compatible with the above estimate. Therefore our assumption is wrong and hence it completes the proof. \( \square \)
4 Proof of Theorem 1.2

For a prime $p$, we define

$$V(p) = (g(p) + a)^2,$$

where $g(p) = -\frac{1}{14}\lambda_f^8(p) + \frac{2}{7}\lambda_f^6(p) + \frac{17}{9}\lambda_f^2(p) - 2$, an even polynomial of degree 8 in the variable $\lambda_f(p)$ and $a$ is a positive real number, chosen later. Using the prime number theorem and the asymptotics for the even power moments of $\lambda_f(p)$ given in (8) and (9), we have

$$\sum_{p \leq x} g(p) = o\left(\frac{x}{\log x}\right). \quad (11)$$

Let $S := \{ p : |\lambda_f(p)| < 1 \}$. From our assumption, $|\lambda_f(p)| \leq 2$ for all primes $p$ hence $1 \leq |\lambda_f(p)| \leq 2$ for $p \notin S$ and also for any prime $p$ one can check that $g^2(p) \leq 15.093$, so

$$\sum_{p \leq x} g^2(p) \leq 15.093 \frac{x}{\log x}. \quad (12)$$

Combine (11) and (12) to obtain

$$\sum_{p \leq x} V(p) \leq (a^2 + 15.093) \frac{x}{\log x} + o\left(\frac{x}{\log x}\right).$$

Positivity of $V(p)$ gives us

$$\sum_{p \leq x, p \notin S} V(p) \leq (a^2 + 15.093) \frac{x}{\log x} + o\left(\frac{x}{\log x}\right). \quad (13)$$

Now it is easy to check that for any $p \notin S$, $g(p) \geq 0.039$. Therefore

$$\sum_{p \leq x, p \notin S} V(p) \geq (a + 0.039)^2 \sum_{p \leq x, p \notin S} 1 = (a + 0.039)^2 \left(\frac{x}{\log x} - \sum_{p \leq x, p \in S} 1\right). \quad (14)$$

From (13) and (14), we have

$$\sum_{p \leq x, p \notin S} 1 \geq \left(1 - \frac{a^2 + 15.093}{(a + 0.039)^2}\right) \frac{x}{\log x} + o\left(\frac{x}{\log x}\right).$$

As a function of $a$, the rational function $1 - \frac{a^2 + 15.093}{(a + 0.039)^2}$ attains its maxima at $a = 387$ with maximum value 0.00010077. Therefore

$$\liminf_{x \to \infty} \frac{\#\{p \leq x : |\lambda_f(p)| < 1\}}{\frac{x}{\log x}} \geq 0.00010077$$

which completes the proof.
5 Proof of Theorem 1.3

In view of the relation $\lambda_f(p^2) = \lambda_f(p)^2 - 1$, the theorem is equivalent to the statement that there are infinitely many primes $p$ with $\lambda_f(p^2) < 0$. Therefore we will achieve the theorem by proving that for any $x$, sufficiently large, the interval $(x, 2x]$ contains a prime $p$ with $\lambda_f(p^2) < 0$.

So, let $x$ be a sufficiently large fixed real number. By way of contradiction, assume $\lambda_f(p^2) \geq 0$ for all primes $p \in (x, 2x]$. We next claim that the positivity assumption will forced us to have

$$\frac{x}{\log x} \ll \sum_{x < p \leq 2x} \lambda_f(p^2).$$

Once we prove this claim then we are done because this will contradict Theorem 3.1, namely

$$\sum_{x < p \leq 2x} \lambda_f(p^2) = o\left(\frac{x}{\log x}\right).$$

Now to prove the claim we proceed as follows. From the Cauchy–Schwarz inequality, we have

$$\left(\sum_{x < p \leq 2x} \lambda_f(p^2)^2\right)^{\frac{1}{2}} \leq \left(\sum_{x < p \leq 2x} \lambda_f(p^2)\right) \left(\sum_{x < p \leq 2x} \lambda_f^3(p^2)\right)^{\frac{1}{2}},$$

i.e.,

$$\left(\sum_{x < p \leq 2x} \lambda_f^2(p^2)^2\right) \leq \sum_{x < p \leq 2x} \lambda_f(p^2).$$

In order to obtain a lower bound of $\sum_{x < p \leq 2x} \lambda_f(p^2)$, we need to get an upper bound for $\sum_{x < p \leq 2x} \lambda_f^3(p^2)$ and a lower bound for $\sum_{x < p \leq 2x} \lambda_f^2(p^2)$. For that

$$\sum_{x < p \leq 2x} \lambda_f^2(p^2) = \sum_{x < p \leq 2x} (\lambda_f^2(p) - 1)^2 = \sum_{x < p \leq 2x} \lambda_f^4(p) + \sum_{x < p \leq 2x} 1 - 2 \sum_{x < p \leq 2x} \lambda_f^2(p).$$

From Lemma 3.2 and the prime number theorem, we have

$$\sum_{x < p \leq 2x} \lambda_f^2(p^2) \sim \frac{x}{\log x}.$$  \hspace{1cm} (16)

Similarly, we obtain

$$\sum_{x < p \leq 2x} \lambda_f^3(p^2) \sim \frac{x}{\log x}.$$  \hspace{1cm} (17)
Thus combining (15), (16) and (17) we have

\[
\frac{x}{\log x} \ll \sum_{x < p \leq 2x} \lambda_f(p^2)
\]

which proved the claim.

6 proof of Theorem 1.4

As in the proof of Theorem 1.3, we address this problem for the smallest prime \( p \) such that \( \lambda_{\text{sym}f}(p) = \lambda_f(p^2) < 0 \). For that consider the function

\[
\psi(\text{sym}^2 f, x) := \sum_{n \leq x} \Lambda_{\text{sym}^2 f}(n),
\]

where \( \Lambda_{\text{sym}^2 f}(n) \) is the \( n \)th coefficient of the negative of the logarithmic derivative of the \( L \)-function \( L(s, \text{sym}^2 f) \) which are supported only on prime powers. Now since the Rankin-Selberg \( L \)-function \( L(s, \text{sym}^2 f \times \text{sym}^2 f) \) exists and has a simple pole at \( s = 1 \) (see, [7, p. 180]), then from [8, p. 110, exercise 6] we have

\[
\psi(\text{sym}^2 f, x) = \sum_{p \leq x} \lambda_{\text{sym}^2 f}(p) \log p + O(\sqrt{x}d^2 \log^2(xq(\text{sym}^2 f))),
\]

(18)

where the implied constant is absolute and \( d \) is the degree of the \( L \)-function \( L(s, \text{sym}^2 f) \) (notice that \( d = 3 \)). The \( L \)-function \( L(s, \text{sym}^2 f) \) is entire, non-vanishing at \( s = 1 \), and does not have a Siegel zero; see [7]. Using this, the prime number theorem for \( L(s, \text{sym}^2 f) \) becomes (see, [8, Eq. 5.52]),

\[
\psi(\text{sym}^2 f, x) = O\left(x \sqrt{q(\text{sym}^2 f)} e^{-\frac{c_1}{\log x}}\right),
\]

(19)

where \( c \) is an absolute constant appearing in the proof of Theorem 5.10 of [8]. After substituting these approximations in (18)

\[
\sum_{p \leq x} \lambda_{\text{sym}^2 f}(p) \log p = O(\sqrt{x}d^2 \log^2(xq(\text{sym}^2 f))) + O\left(x \sqrt{q(\text{sym}^2 f)} e^{-\frac{c_1}{\log x}}\right),
\]

i.e.,

\[
\sum_{p \leq x} \lambda_{\text{sym}^2 f}(p) \log p \ll x \sqrt{q(\text{sym}^2 f)} e^{-\frac{c_1}{\log x}}.
\]

We now apply the partial summation formula to the left hand side of the above inequality to obtain

\[
\sum_{p \leq x} \lambda_{\text{sym}^2 f}(p) \ll \frac{x}{\log x} \sqrt{q(\text{sym}^2 f)} e^{-\frac{c_1}{\log x}}.
\]

(20)

Let \( p_0 \) be the smallest prime such that \( \lambda_f(p_0^2) < 0 \). Now given any \( A > 0 \) if \( p_0 \leq A \), then we are done since \( A \) can be absorbed in the implied constant figuring in the bound occurring in the theorem. Hence we can assume that \( p_0 \) is as large as we please. Choose \( y = p_0 - \epsilon \) where \( 0 < \epsilon < 1/2 \). Then our assumption on \( p_0 \) implies that \( \lambda_f(p_0^2) \geq 0 \) for
all \( p \leq y \) and we can choose \( y \) as large as we please. Then using the same argument as for (5), we obtain
\[
\frac{y}{\log y} \ll \sum_{p \leq y} \lambda_{\text{sym}^2 f}(p).
\] (21)

Note that inequality (20) is true for any sufficiently large real number \( x \), in particular for \( y \) also. Now comparing the upper (20) and lower (21) bounds for the sum \( \sum_{p \leq y} \lambda_{\text{sym}^2 f}(p) \) and then substituting \( q(\text{sym}^2 f) = (1 + |u|)^2 \) gives the result.

7 Proof of Theorem 1.5
For a given \( a > 1 \), we consider the polynomial \( V_a(p) = \lambda_f^2(p) - a \) and let \( S_a = \{ p : |\lambda_f(p)| < \sqrt{a} \} \). Using the fact that \( \lambda_f^2(p) \geq a \) for \( p \notin S_a \), we have
\[
0 \leq \sum_{p \leq x, p \notin S_a} V_a(p) = \sum_{p \leq x} V_a(p) - \sum_{p \leq x, p \in S_a} V_a(p) = \frac{x}{\log x} - a \cdot \frac{x}{\log x} + o\left(\frac{x}{\log x}\right) - \sum_{p \leq x, p \in S_a} \lambda_f^2(p) + a \sum_{p \leq x, p \in S_a} 1.
\]

In other words, we have
\[
\sum_{p \leq x, p \in S_a} \lambda_f^2(p) + (a - 1) \cdot \frac{x}{\log x} \leq a \sum_{p \leq x, p \in S_a} 1 + o\left(\frac{x}{\log x}\right).
\]

Since \( \lambda_f^2(p) \geq 0 \) for any prime \( p \) and \( a > 1 \), hence we can ignore the sum \( \sum_{p \leq x, p \in S_a} \lambda_f^2(p) \) from the left hand side of the above inequality and then divide by \( \frac{x}{\log x} \) on both the sides to get
\[
\frac{a - 1}{a} \leq \liminf_{x \to \infty} \frac{\sum_{p \leq x, p \in S_a} 1}{\pi(x)},
\]
which completes the proof.

8 Proof of Theorem 1.6
Proof (a.) Consider the polynomial
\[
g(t) = -\frac{1}{5.7} t^6 + \frac{4}{8.5} t^4 + \frac{17}{18} t^2 - 1.
\]
Notice that \( g(t) \) is \( < 0 \) for \( t \in [0, t_0) \cup (t_1, \infty) \) and \( > 0 \) for \( t \in (t_0, t_1) \) where \( t_0 = 0.908 \) and \( t_1 = 1.928 \) and also \( g(t_0) = 0 = g(t_1) \). In the polynomial \( g(t) \), we put \( t = \lambda_f(p) \) and consider the sum \( \sum_{p \leq x} g(\lambda_f(p)) \). Using Lemmas 3.2 and 3.3 we obtain the following asymptotic,
\[
\sum_{p \leq x} g(\lambda_f(p)) \sim 0.0085 \pi(x) \quad \text{as} \quad x \to \infty.
\] (22)

Next we split the sum \( \sum_{p \leq x} g(\lambda_f(p)) \) into two parts, depending on \( g \) is positive or negative
\[
\sum_{p \leq x} g(\lambda_f(p)) = \sum_{p \in S_0(x) \cup S_1(x)} g(\lambda_f(p)) + \sum_{p \in S(x)} g(\lambda_f(p))
\]
where \( S_0(x) = \{ p \leq x : 0 \leq |\lambda_f(p)| \leq t_0 \}, \) \( S(x) = \{ p \leq x : t_0 \leq |\lambda_f(p)| \leq t_1 \} \) and \( S_1(x) = \{ p \leq x : |\lambda_f(p)| > t_1 \}. \) Since for \( p \in S_0(x) \cup S_1(x), \) \( g(\lambda_f(p)) < 0, \) so in order to get an upper bound for the sum \( \sum_{p \leq x} g(\lambda_f(p)) \) one can completely ignore the first term from the above equation. Thus

\[
\sum_{p \leq x} g(\lambda_f(p)) < \sum_{p \in S(x)} g(\lambda_f(p)) \leq 1.561 \sum_{p \in S(x)} 1.
\]

Here 1.561 is the maximum of \( g(\lambda_f(p)) \) for \( p \in S(x). \) We now use asymptotic (22) for \( \sum_{p \leq x} g(\lambda_f(p)) \) in the last inequality

\[
\frac{0.0085}{1.561} \leq \liminf_{x \to \infty} \frac{\#\{p \leq x : 0.908 \leq |\lambda_f(p)| \leq 1.928\}}{\pi(x)},
\]

which completes the proof. \( \square \)

**Proof (b.)** Consider the polynomial

\[
g_a(t) = t^2(-t^2 + 1)(t^2 - 4).
\]

Note that \( g_a(t) \) is an even polynomial of degree 6 whose positive roots are 0, 1 and 2. Also, \( g_a(t) > 0 \) for \( t \in [0, 1) \cup (2, \infty) \) and \( g_a(t) < 0 \) for \( t \in (1, 2). \)

Now consider \( g_a \) as a polynomial in \( \lambda_f(p). \) Then Lemma 3.2 yields

\[
\sum_{p \leq x} g_a(\lambda_f(p)) \sim \pi(x) \text{ as } x \to \infty.
\]  \hspace{1cm} (23)

On the other hand, if we split the sum \( \sum_{p \leq x} g_a(\lambda_f(p)) \) into two parts, depending on the sign of \( g_a, \) then

\[
\sum_{p \leq x} g_a(\lambda_f(p)) < M \#\{p \leq x : 1 < |\lambda_f(p)| < 2\},
\]

where \( M := \max\{g_a(\lambda_f(p)) : |\lambda_f(p)| \in (1, 2)\} = 6.065. \) This together with asymptotic (23) gives

\[
\frac{1}{6.065} \leq \liminf_{x \to \infty} \frac{\#\{p \leq x : 1 < |\lambda_f(p)| < 2\}}{\pi(x)},
\]  \hspace{1cm} (24)

and hence we complete the proof. \( \square \)

**9 Proof of Theorem 1.7**

**Proof (a.)** Let \( A = \{ p : |\lambda_f(p)| > \sqrt{2} \}. \) Then we use Lemma 3.2 to say

\[
0 \leq - \sum_{p \leq x, p \notin A} \lambda_f^2(p) + 2 \sum_{p \leq x, p \notin A} \lambda_f^4(p) = -5 \frac{x}{\log x} + \sum_{p \leq x, p \notin A} \lambda_f^6(p) + 4 \frac{x}{\log x} - 2 \sum_{p \leq x, p \notin A} \lambda_f^4(p) + o \left( \frac{x}{\log x} \right).
\]
In other words,
\[
\frac{x}{\log x} \leq \sum_{p \leq x, p \in A} \lambda_f^2(p)(\lambda_f^2(p) - 2) + o \left( \frac{x}{\log x} \right).
\]

By our assumption, for any prime \( p \), \(|\lambda_f(p)| \leq 2\) therefore we have
\[
\frac{x}{\log x} \leq 32 \sum_{p \leq x, p \in A} 1 + o \left( \frac{x}{\log x} \right).
\]

Now first divide by \( \pi(x) \) on both the sides of the above inequality and then taking \( \liminf_{x \to \infty} \), we obtain
\[
\frac{1}{32} \leq \liminf_{x \to \infty} \frac{\sum_{p \leq x, p \in A} 1}{\pi(x)},
\]
which completes the proof. \( \square \)

Proof (b.) For this we consider the following even polynomial
\[
V(p) = -\lambda_f(p)^6 + 2\lambda_f(p)^4 + 1.
\]

As in the proof of Theorem 1.3, we prove the theorem by showing that for every \( x \), sufficiently large, the interval \( (x, 2x] \) contains at least one prime \( p \) such that \(|\lambda_f(p)| > \sqrt{2}\). By way of contradiction, assume \(|\lambda_f(p)| \leq \sqrt{2}\) whenever \( p \in (x, 2x] \). This shows that for \( p \in (x, 2x] \), \( V(p) \geq 1 \) and hence we have
\[
\sum_{x < p \leq 2x} V(p) \geq \frac{x}{\log x}.
\]

But from Lemma 3.2, we know
\[
\sum_{p \leq x} V(p) = o \left( \frac{x}{\log x} \right) \text{ as } x \to \infty
\]
which contradicts (25). Therefore our assumption, that \(|\lambda_f(p)| \leq \sqrt{2}\) whenever \( p \in (x, 2x] \) is false and hence this completes the proof. \( \square \)

10 Proof of Theorem 1.9

We achieve this result by showing that for each \( x \) sufficiently large, the interval \( (x, 2x] \) gives a square-free integer \( N \) such that
\[
|a(N)| \geq A \exp \left( \frac{c \log N}{\log \log N} \right), \text{ for some constant } A > 0.
\]

For a sufficiently large \( x \), consider the interval \( I_x := (x, 2x] \). Suppose for all primes \( p \in I_x \)
\[
a(p) = O \left( \exp \left( \frac{c_0 \log p}{\log \log p} \right) \right), \text{ for some constant } c_0 > 0.
\]
otherwise our claim is true. Now, fix \( \delta \) such that \( 1 < \delta < 2^{\frac{1}{4}} \), i.e., \( 1 < \delta^4 < 2 \). Therefore

\[
\sum_{p \in I_x, \left| a(p) \right| < \delta} a(p)^4 \leq \frac{x}{\log x}.
\]  

(26)

Combining this with assumption (ii) in the statement gives

\[
\sum_{p \in I_x, \left| a(p) \right| \geq \delta} a(p)^4 = \sum_{p \in I_x} a(p)^4 - \sum_{p \in I_x, \left| a(p) \right| < \delta} a(p)^4 \\
\geq (\alpha - \delta^4) \frac{x}{\log x} \\
\geq \frac{x}{\log x}.
\]  

(27)

Since \( \alpha \) is greater than or equals to 2 therefore the constant that appears in the right hand side of the above inequality is non-zero. Let \( T := \# \{ p \in I_x : \left| a(p) \right| \geq \delta \} \). Then from (27), we have

\[
T \geq c_1 \frac{x}{\log x} \exp \left( -4c_0 \frac{\log x}{\log \log x} \right).
\]  

(28)

where \( c_1 \) is a positive constant. Now, put \( N = \prod_{p \in I_x, \left| a(p) \right| \geq \delta} p \). Because the coefficients \( a(n) \) are multiplicative, we have

\[
\left| a(N) \right| = \prod_{p \mid N} \left| a(p) \right| \geq \delta^T = \exp(T \log \delta).
\]

Now

\[
T \log x \leq \log N = \sum_{x < p \leq 2x, \left| a(p) \right| \geq \delta} \log p \leq T \log 2x,
\]

i.e., \( \log N \approx T \log x \). Thus

\[
\left| a(N) \right| \geq \exp \left( (\log \delta) \frac{\log N}{\log x} \right).
\]

We now want to analyse \( \log x \) as a function of \( N \). From the prime number theorem, we have \( \log N \ll x \) whereas from (28)

\[
\log x \geq T \log x \geq c_0 x \exp \left( -4c_0 \frac{\log x}{\log \log x} \right).
\]

Putting all together, we obtain

\[
\log x \gg \log \log N \gg \log c_0 + \log x + O \left( \frac{\log x}{\log \log x} \right).
\]

In other words, \( \log x \approx \log \log N \), so that

\[
\left| a(N) \right| \gg \exp \left( (\log \delta) \frac{\log N}{\log \log N} \right).
\]

This completes the theorem. \( \square \)
11 Proof of Corollary 1.10 and Corollary 1.11
Since both the sequences \( \lambda_f(n) \) and \( \lambda_{\text{sym}^2f}(n) \) are multiplicative and from Lemmas 3.2 and 3.3 it is also clear that they satisfy the second assumption of Theorem 1.9 with \( \alpha = 2 \) and 3, respectively. Hence from Theorem 1.9 we complete the proof of Corollary 1.10.

The proof of Corollary 1.11 goes as follows: From Corollary 1.10, we know
\[
a(m) > c_1 \exp \left( \frac{c \log m}{\log \log m} \right)
\]
or
\[
a(m) < -c_2 \exp \left( \frac{c \log m}{\log \log m} \right)
\]
holds infinitely often for square-free integers \( m \) with some positive constants \( c_1 \) and \( c_2 \).

Without loss of generality, assume (29) is true. So it is sufficient to prove that there is a sequence of positive integers \( \{n_m\}_{m \in \mathbb{N}} \) and a positive constant \( M \) such that
\[
-a(n_m) > M \exp \left( \frac{c \log n_m}{\log \log n_m} \right)
\]
for all \( m \in \mathbb{N} \). Let \( q \) be a prime with \( a(q) < 0 \), such \( q \) exists by Corollary 3.5 or Corollary 1.8 depending on \( f \) or \( \text{sym}^2f \), respectively. Then for any \( m \) satisfying inequality 29, we define \( n_m = qm \) if \( (q, m) = 1 \) and \( n_m = \frac{m}{q} \) otherwise. In the first scenario, since \( (q, m) = 1 \) we have
\[
-a(n_m) = (-a(q))a(m) \gg_q \exp \left( \frac{c \log n_m}{\log \log n_m} \right).
\]
otherwise, we know \( (n_m, q) = 1 \) as \( m \) is square-free, hence
\[
-a(n_m) = \left( \frac{-1}{a(q)} \right) a(m) \gg_q \exp \left( \frac{c \log n_m}{\log \log n_m} \right).
\]
This completes the proof. \( \square \)

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Data availability
Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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References
1. Apostol, T.M.: Introduction to Analytic Number Theory. Springer, Berlin (1998)
2. Barnet-Lamb, T., Geraghty, D., Harris, M., Taylor, R.: A family of Calabi-Yau varieties and potential automorphy II. Publ. Res. Inst. Math. Sci. 47(1), 29–98 (2011)
3. Brumley, F.: Maass cusp forms with quadratic integer coefficients. Int. Math. Res. Not. 18, 983–997 (2003)
4. Bump, D.: Automorphic Forms and Representations. Cambridge Studies in Advanced Mathematics, vol. 55. Cambridge University Press, Cambridge (1998)
5. Chiriac, L., Jorza, A.: Comparing Hecke coefficients of automorphic representations. Trans. Am. Math. Soc. 372, 8871–8896 (2019)
6. Elliott, P.D.T.A., Kish, J.: Harmonic analysis on the positive rationals II: multiplicative functions and Maass forms. J. Math. Sci. Univ. Tokyo 23(3), 615–658 (2016)
7. Hoffstein, J., Lockhart, P.: Coefficients of Maass forms and the Siegel zero. Ann. Math. (2) 140(1), 161–181 (1994)
8. Iwaniec, H., Kowalski, E.: Analytic Number Theory, vol. 53. American Mathematical Society Colloquium Publications. American Mathematical Society, Providence (2004)
9. Kim, H.: Functoriality for the exterior square of $GL_2$ and symmetric fourth of $GL_2$. With Appendix 1 by D. Ramakrishnan, and Appendix 2 by H. Kim and P. Sarnak. J. Am. Math. Soc. 16, 139–183 (2003)
10. Kim, H.H., Shahidi, F.: Functorial products for $GL_2 \times GL_3$ and functorial symmetric cube for $GL_2$, C. R. Acad. Sci. Paris Sér. I Math. 331, 599–604 (2000)
11. Kim, H.H., Shahidi, F.: Cuspidality of symmetric powers with applications. Duke Math. J. 112(1), 177–197 (2002)
12. Luo, W., Zhou, F.: On the Hecke eigenvalues of Maass forms. Am. J. Math. 141(2), 485–501 (2019)
13. Ram Murty, M.: Oscillations of Fourier coefficients of modular form. Math. Ann. 262, 431–446 (1983)
14. Ram Murty, M.: On the estimation of eigenvalues of Hecke operators. Rocky Mountain J. Math. 15(2), 521–533 (1985)
15. Ramakrishnan, D.: On the coefficients of cusp forms. Math. Res. Lett. 4(2–3), 295–307 (1997)
16. Serre, J.-P.: Abelian $l$-adic Representations and Elliptic Curve. Research Notes in Mathematics, vol. 7. A K Peters, Ltd., Wellesley (1998). With the collaboration of Willem Kuyk and John Labute, Revised reprint of the 1968 original
17. Shahidi, F.: Symmetric Power $L$-Functions for $GL(2)$. Elliptic Curves and Related Topics. CRM Proceedings Lecture Notes, vol. 4, pp. 159–162. American Mathematical Society, Providence (1994)
18. Walji, N.: On the distribution of Hecke eigenvalues for cuspidal automorphic representations for $GL(2)$. Int. Math. Res. Not. 10, 3155–3172 (2018)
19. Walji, N.: On the occurrence of large positive Hecke eigenvalues for $GL(2)$, Automorphic forms and related topics. Contemporary Mathematics, vol. 732, pp. 259–266. American Mathematical Society, Providence (2019)
20. Wigert, S.: Sur l’ordre de grandeur du nombre de diviseurs d’un entier. Ark. Mat. 3, 1–9 (1907)

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