Jacobi Forms of Higher Index and Paramodular Groups in $\mathcal{N} = 2$, $D = 4$ Compactifications of String Theory

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Abstract

We associate a Jacobi form over a rank $s$ lattice to $\mathcal{N} = 2$, $D = 4$ heterotic string compactifications which have $s$ Wilson lines at a generic point in the vector multiplet moduli space. Jacobi forms of index $m = 1$ and $m = 2$ have appeared earlier in the context of threshold corrections to heterotic string couplings. We emphasize that higher index Jacobi forms as well as Jacobi forms of several variables over more generic even lattices also appear and construct models in which they arise. In particular, we construct an orbifold model which can be connected to models that give index $m = 3$, 4 or 5 Jacobi forms through the Higgsing process. Constraints from being a Jacobi form are then employed to get threshold corrections using only partial information on the spectrum. We apply this procedure for index $m = 3$, 4 or 5 Jacobi form examples and also for Jacobi forms over $A_2$ and $A_3$ root lattices. Examples with a single Wilson line are examined in detail and we display the relation of Siegel forms over a paramodular group $\Gamma_m$ to these models, where $\Gamma_m$ is associated with the T-duality group of the models we study. Finally, results on the heterotic string side are used to clarify the linear mapping of vector multiplet moduli to Type IIA duals without using the one-loop cubic part of the prepotential on the Type II side, and also to give predictions for the geometry of the dual Calabi-Yau manifolds.

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1 Introduction

A variety of modular forms and Jacobi forms arise frequently in string theory. Constraints that come from having a modular or Jacobi form in a physical quantity are highly nontrivial and may even allow us to get information not accessible through direct methods. The focus in this work will be on $D = 4, \mathcal{N} = 2$ string compactifications and their relation to Jacobi forms of possibly several variables.

On the heterotic string side, the modified index of [1] is used to compute threshold corrections to gauge couplings and gravitational couplings [2]. In the case where no Wilson line is turned on, this index gives rise to a modular form [3]. The case with Wilson lines, on the other hand, involves Jacobi forms. In [4], index $m = 1$ Jacobi forms in the sense of [5] are found to be relevant to the modified index. A further example with an $m = 2$ Jacobi form is given in [6], where the computation of threshold corrections is through indirect methods which we also will study in this paper and generalize to higher indexes. [7], on the other hand, has some indications that Jacobi forms of several variables over root lattices of simple Lie algebras may be relevant for heterotic compactifications with Wilson lines. It explores the relation of such generalized Jacobi forms to Type IIA compactifications on Calabi-Yau threefolds.
Turning on generic Wilson lines is important if one wishes to come up with detailed predictions for a possible Type II dual and the geometry of its compactification space, e.g. Gromov-Witten potentials of the compactifying Calabi-Yau manifold. So, we will continue along the way of [4], [6] and extend their work by studying the relation of Jacobi forms over general even lattices to heterotic compactifications with an arbitrary number of Wilson lines. The dependence of threshold corrections on Wilson lines has close relations to toroidal compactifications and accordingly the presence of extended Jacobi forms in the context of $\mathcal{N} = 4$ compactifications has already been noted in [8].

A useful property of Jacobi forms is that for a fixed weight and a fixed lattice they live in a finite dimensional vector space. Therefore, as in the context of modular forms or ordinary Jacobi forms, one can look at the first few Fourier coefficients of a quantity known to be a Jacobi form and use the finite dimensionality property to fix the whole function. Such arguments will be used on a number of examples which are connected in hypermultiplet moduli space to an orbifold model. In this way, we will test our arguments with explicit computations using exactly calculable orbifold conformal field theory.

Section 2 is an in depth analysis of the topics noted above. It starts with a general discussion of heterotic string compactifications on $K3 \times T^2$ and associated threshold corrections. Section 2.1 discusses the overall structure of vector multiplet moduli space and explains in general terms why one would get a Jacobi form through the modified index computation. In section 2.2, we show how the general discussion given in section 2.1 works in the context of orbifold compactifications. Since our aim is to work with models connected to orbifold models on the hypermultiplet moduli space, in section 2.3 we study several symmetry breaking patterns for $\mathcal{N} = 2$ gauge theories. In section 2.4, we start with an orbifold example and use the methods of section 2.3 to get a number of interesting models through Higgsing. Also, we show how the relevant Jacobi forms can be obtained using only partial information on the spectrum. These results are checked against explicit orbifold limit computations. After that, in section 2.5 we review the overall structure of threshold corrections and give explicit expressions for the associated prepotential and gravitational coupling in terms of the Jacobi forms under study. Weyl chamber dependence is another topic studied in this section. Lastly, in section 2.6 we study models with a single Wilson line at a generic point of their vector multiplet moduli space. This is the simplest case of the formalism developed up to that point. The relevant Jacobi forms are the Jacobi forms of [5]. For an index $m$ Jacobi form associated with such models threshold corrections are given in terms of Siegel forms of the appropriate T-duality groups which turn out to be paramodular groups $\Gamma_m$. We give details for such models along the lines of [4], [6]. We first review the cases $m = 1$ and $m = 2$, and then extend this for our examples with $m = 3, 4, \text{and } 5$.

Having discussed the details on the heterotic side, we turn our attention in section 3 to Type IIA compactifications and Calabi-Yau threefolds. One needs a dictionary to map vector multiplet moduli between those two cases. A possible way to accomplish this for small $h^{1,1}$ values is to compare cubic prepotentials on both sides of the computation. A mapping without this comparison is suggested in [7]. Our results on Weyl chambers in
section 2 suggests a way to clarify some points and extend the mapping to more general settings. In this form of the conjecture the moduli mapping is unambiguously determined on the heterotic side.

Finally, relevant mathematical definitions, conventions and tools are gathered in the Appendix.

2 $\mathcal{N} = 2$ Heterotic String Compactifications

In this work, we will be interested in four dimensional string theories (on $\mathbb{R}^{3,1}$) that can be obtained as compactifications of six dimensional ($\mathbb{R}^{5,1}$), $\mathcal{N} = (1,0)$ supersymmetric heterotic string models, on a torus $T^2$. The resultant four dimensional theories have $\mathcal{N} = 2$ spacetime supersymmetry and we will only study perturbative properties of these theories.

One way to obtain such a six dimensional theory is by geometrically compactifying ten dimensional ($\mathbb{R}^{9,1}$) heterotic string on a $K3$ surface. A $K3$ surface has a holonomy group restricted to $SU(2)$ and hence halves the number of supersymmetries of the ten dimensional string theory. The massless spectrum of this six dimensional theory can be studied by starting with ten dimensional, $\mathcal{N} = (1,0)$ supergravity coupled to $E_8 \times E_8$ or $Spin(32)/\mathbb{Z}_2$, which are the low energy effective field theories of the two supersymmetric ten dimensional heterotic strings; and then dimensionally reducing these field theories on a $K3$ surface. Massless supermultiplets in the resulting six dimensional theory consist of a gravity multiplet, a tensor multiplet, $N^6_v$ vector multiplets and $N^6_h$ hypermultiplets. A tensor multiplet contains a single real scalar which is the heterotic dilaton. Hypermultiplets, on the other hand, each contain four real scalars. Vector multiplets do not contain any scalars and their bosonic part consists only of a six dimensional vector field. One should also note that we restrict to perturbative models where there are no background five-branes. Five-brane backgrounds lead to additional tensor multiplets in the massless spectrum.

In such a scheme, one should also pick a non-trivial Yang-Mills background on the $K3$ surface for consistency. This can be easily seen by looking at the gauge invariant field strength of the $B$-field, $H = dB + \omega_{3L} - \frac{1}{30} \omega_{3Y}$, where the Chern-Simons terms $\omega_{3L}$ and $\omega_{3Y}$ satisfy $d\omega_{3L} = \text{tr} R^2$ and $d\omega_{3Y} = \text{Tr} F^2$. In these equations, $R$ is the curvature two-form, $F$ is the Lie algebra valued field strength two-form, and $\text{tr}$ and $\text{Tr}$ denote traces in the fundamental and adjoint representations, respectively. The background curvature is nontrivial for the $K3$ part and satisfies a topological constraint,

$$\int_{K3} \text{tr} R^2 = 24. \quad (1)$$

Therefore, if the $H$ field is well-defined one needs to have

$$\int_{K3} dH = \int_{K3} \text{tr} R^2 - \frac{1}{30} \int_{K3} \text{Tr} F^2 = 0, \quad (2)$$

\footnote{Note that there is no $H$-flux on the $K3$ since we assume to have no background five-branes.}
in other words, one needs to have a total of 24 instantons turned on as a nontrivial gauge field background.

Finally, one can compute the massless spectrum in six dimensions using anomaly cancellation and index theorems, with respect to various ways of embedding the instantons in the $E_8 \times E_8$ or $Spin(32)/\mathbb{Z}_2$ gauge groups [9].

One can then form chains of such models where the subgroup in which the instantons are embedded gets larger step-by-step. This leads to a cascade of models where the unbroken gauge group becomes smaller at each step. This can equivalently be described by charged hypermultiplet scalars getting a vacuum expectation value (vev) and breaking the gauge group. Examples of such chains are discussed in [10, 11, 12] and reviewed in [13].

One such simple chain starts with $E_8 \times E_8$ heterotic string and embeds a $SU(2)$ bundle in the first $E_8$ with $n_1$ instantons and a $SU(2)$ bundle in the second $E_8$ with $n_2$ instantons. It is common to parametrize these instanton numbers by an integer, $k$, such that $(n_1, n_2) = (12 + k, 12 - k)$. For $k = 0, 1, 2$, this model has initially an $E_7 \times E_7$ gauge symmetry and both $E_7$ factors can be broken through the chain

$$E_7 \rightarrow E_6 \rightarrow SO(10) \rightarrow SU(5) \rightarrow SU(4) \rightarrow SU(3) \rightarrow SU(2) \rightarrow 1.$$ (3)

Throughout this chain, instantons are embedded in simple groups and the unbroken gauge group is the commutant of this simple group in $E_8$. For larger $k$, it is not possible to completely break the gauge group in the $E_8$ factor with fewer instantons since, at some point, the number of instantons becomes insufficient to support a background in a larger group. This means for $k \geq 3$, one of the gauge groups can only be broken down to a terminal gauge group.

It is simple to find the massless hypermultiplet spectrum of the $E_7 \times E_7$ theory from an index computation as

$$\left(4 + \frac{k}{2}\right)(56, 1) + \left(4 - \frac{k}{2}\right)(1, 56) + 62(1, 1),$$ (4)

where we label irreducible $E_7$ representations by their dimensions. Noting also the 266 massless vector multiplets in the adjoint representations of $E_7$'s, one gets $N_h^6 - N_v^6 = 244$ as required by the absence of gravitational anomalies for the $(1, 0)$ supergravity in six dimensions with one tensor multiplet [13].

More complicated examples can be obtained by turning on $U(1)$ backgrounds or semisimple backgrounds. For example, going through the Higgsing process described above one can obtain the following massless hypermultiplet spectrum starting with the $U(1)$ background studied in [12] (where again $(12 + k, 12 - k)$ instantons are embedded in the $E_8$ factors):

$$48(1)_q + 48(1)_{-q} + 149(1)_0,$$ (5)

with the unbroken gauge group being $U(1)$. Another example, [14], is obtained by embedding a $SU(3) \times SU(2)$ bundle in the first $E_8$ with $(10, 4)$ instanton numbers and 10
instantons in the full $E_8$ for the second $E_8$. The hypermultiplet spectrum is then easily computed as

$$99(1) + 10(6) + 10(\bar{6}) + 2(15) + 2(\bar{15})$$

with respect to the unbroken gauge group $SU(6)$. One can then break the gauge symmetry down to a $U(1)$ with the same hypermultiplet spectrum as in (5). We will describe the group theoretic details of such processes in the coming sections.

At this point, we start studying four dimensional theories that can be obtained as $T^2$ compactifications of the six dimensional theories described above, where we will denote the final unbroken gauge group of the six dimensional theory as $G$. The resulting theory has a gravity multiplet and a number of vector multiplets and hypermultiplets in its massless spectrum.

Four dimensional vector multiplets feature a complex scalar in addition to a four dimensional vector field in their bosonic part. Hypermultiplets, on the other hand, have four real scalars, similar to six dimensions. Furthermore, hypermultiplets in six dimensions give hypermultiplets in the four dimensional theory. The gravity multiplet of six dimensions gives rise to a gravity multiplet in four dimensions as well as two vector multiplets. The complex scalars of these vector multiplets, called $T$ and $U$ are moduli fields parameterizing the constant metric and constant $B$-field on $T^2$. The tensor multiplet gives another vector multiplet where its complex scalar, $S$, is the axio-dilaton field. The weak coupling limit of the heterotic string is $3S = 4\pi/g_s \rightarrow \infty$. Lastly, vector multiplets in six dimensions give rise to vector multiplets in four dimensions.

The potential associated with the vector multiplet scalars remain zero if commuting members of vector multiplet scalars gain a vev (which is equivalent to turning on Wilson lines on the torus $T^2$). Therefore, at a generic point in the moduli space of vector multiplets, scalars in the Cartan subalgebra of the gauge group gain a vev and every state that is charged gains mass. This leaves the gravity multiplet, $N_h$ neutral fields in the hypermultiplet spectrum and $N_v = 3 + \text{rank}(G)$ vector multiplets as massless fields. The number three comes from the vector multiplets carrying the scalars $S$, $T$ and $U$. In the following work, we will call the scalars of the $s \equiv \text{rank}(G)$ vector multiplets Wilson lines and denote them by $V_i$ where $i = 1, \ldots, s$, although, in general, those moduli will be some combinations of $T^2$ moduli and Wilson lines on $T^2$.

An interesting physical quantity one can compute in such theories is the modified index defined in [1]. We will compute this index in the form

$$\mathcal{I} = -\frac{i}{\eta(q)^2} \text{Tr} \left( J_0 (-1)^{J_0} q^{L_0 - c/24} q^{L_0 - c/24} \right),$$

where the trace is taken over the Ramond sector of the internal CFT and $J_0$ is the zero mode of the $U(1)$ current which lies in the $N = 2$ superconformal symmetry of the internal CFT. We will adopt the convention that $N$ denotes worldsheet supersymmetry; whereas $\mathcal{N}$ denotes spacetime supersymmetry.
Our main interest in this quantity lies in the fact that its integral over the fundamental domain of $SL(2, \mathbb{Z})$ in the form

$$\Delta_{\text{grav}} = \int \frac{d^2 \tau}{\tau_2} \left[ \frac{-i}{\eta(q)^2} \text{Tr} \left( J_0(-1) q^{L_0-c/24} \bar{q}^{\bar{L}_0-\bar{c}/24} \right) \left( E_2(q) - \frac{3}{\pi \tau_2} \right) - b_{\text{grav}} \right]$$

and

$$\Delta_{\text{gauge}} = \int \frac{d^2 \tau}{\tau_2} \left[ \frac{-i}{\eta(q)^2} \text{Tr} \left( J_0(-1) q^{L_0-c/24} \bar{q}^{\bar{L}_0-\bar{c}/24} \right) \left( Q^2 - \frac{1}{8\pi \tau_2} \right) - b_{\text{gauge}} \right]$$

(8)

(9)
gives gravitational and gauge coupling threshold corrections, respectively \[2\]. In these expressions, $E_2$ is the order two Eisenstein series, $\tau_2 = 3\tau$ and $Q$ is a generator of the gauge group for which we are computing the threshold correction. We will be more explicit about our conventions in the coming sections.

The theories we consider here have $\mathcal{N} = (1,0)$ supersymmetry in six dimensions. This means that the internal CFT for the four dimensional theory is the sum of a $N = (0,2)$ supersymmetric free theory with central charge $(2,3)$ corresponding to the torus and $N = (0,4)$ supersymmetric CFT with central charge $(20,9)$ \[15\]. Following the discussion in the section 3 of \[3\], one sees that the trace involved in $\mathcal{I}$ reduces to the Witten index for the $N = (0,4)$ supersymmetric part and hence it is invariant under its smooth deformations. Hypermultiplet moduli of the four dimensional theory comes purely from this factor and hence $\mathcal{I}$ is invariant under smooth changes of hypermultiplet moduli.

This is, of course, different for vector multiplet moduli. The scalars of vector multiplets have vertex operators of the form $\bar{\partial} X^\pm (\bar{z}) J^a(z)$ where $X^\pm$ are the free scalars of the $(c, \bar{c}) = (2,3)$ factor and $J^a$ is a gauge current. Finding the dependence of threshold corrections on vector multiplet moduli for the models we consider is one of the goals of this paper. The hypermultiplet moduli independence of the modified index allows its computation for a wide class of theories which are smoothly connected to orbifold limits where one can explicitly compute the modified index and its vector multiplet moduli dependence. This is essentially done in \[16\] where the modified index is expressed in terms of several lattice sums. Our main aim will be to emphasize the modular properties of the modified index, give its relations to Jacobi forms (of several variables) and display the power of these modular restrictions on several examples and to check these results with orbifold computations.

### 2.1 Vector Multiplet Moduli Space and Jacobi Forms from Modified Index

The classical moduli spaces of vector multiplet moduli for $\mathcal{N} = 2$, $D = 4$ supergravity theories are strictly restricted. Peccei-Quinn symmetry requires them to be a product of two factors where one is parametrized by the axio-dilaton field alone. Then, it is a theorem in supergravity \[17\] that the only such product manifold is of the form

$$\frac{SU(1,1)}{U(1)} \otimes \frac{SO(n,2)}{SO(n) \otimes SO(2)}.$$
In the models we consider, where one compactifies a six dimensional theory on a two-torus, moduli that spans the second factor are the torus moduli $T$ and Wilson lines $V^i$ so that $n = s + 2$. This picture is further refined in string theory by discrete identifications on the moduli space. The identifications on the $SO(n, 2)/SO(n) \otimes SO(2)$ part can be obtained by the action of a T-duality group. We will discuss this aspect later in our discussion. Particular examples of such duality groups include paramodular groups.

Now, it is necessary to describe the details of the final, unbroken gauge group of the six dimensional theory before compactification. (This is equivalent to describing the 4D classical gauge group enhancement one would observe when the Wilson lines on the torus are switched off.) Let the gauge group be a product of simple factors coming from simply laced Lie algebras and a number of $U(1)$’s.

$$G = \prod G_i \times [U(1)]^k.$$  \hspace{1cm} (11)

We will only deal with the case where the simple factors are generated at level 1. Then, the current algebra corresponding to the Cartan subalgebra of the simple factors together with the $U(1)$ factors can be factored out by compact scalars on a $s$-dimensional torus (Frenkel-Kac-Segal construction, [18, 19]). This is the torus generated by exponentiating this maximally commuting part of the total gauge Lie algebra [20]. In other words, we take $s$ left moving scalars $X^i$ normalized so that $\partial X^i(z) \partial X^j(w) = \delta^{ij}/(z - w)^2 + \ldots U(1)$ generators are $\vec{j}(z) = i\partial \vec{X}(z)$ and we identify points as

$$\vec{X} \sim \vec{X} + 2\pi \Lambda,$$  \hspace{1cm} (12)

where the lattice, $\Lambda$, defines the torus, $\mathbb{R}^s/(2\pi \Lambda)$, described above. The usual Euclidean metric induced by the $\partial X \partial X$ OPE gives a bilinear form on this lattice.

The lattice, $\Lambda$, is a direct sum of the simple factors’ root lattices and a number of one-dimensional lattices corresponding to $U(1)$ factors, where each of these lattice factors are orthogonal to each other.

$$\Lambda = (\oplus \Lambda_{j,R}) \oplus (\oplus_{i=1}^k \langle m_i \rangle),$$  \hspace{1cm} (13)

where $\Lambda_{j,R}$ denotes the root lattice corresponding to $G_i$, and $\langle m \rangle$ is a one dimensional lattice, $\mathbb{Z}\vec{\beta}$, with $\vec{\beta} \cdot \vec{\beta} = m$. The charges with respect to those $U(1)$’s lie on the dual lattices $\langle m_i \rangle^\ast$. Finally, we require $m_i$’s to be even integers. The fact that the lattice $\Lambda$ is integral and that the charge lattice lies in the dual lattice $\Lambda^\ast$ is what makes the theory invariant under a ‘spectral flow’.

Next, we take an integral basis to the lattice $\Lambda$ (for definiteness we will take the simple roots of the simple factors to be in this basis) and denote the members of this basis by $\vec{\beta}_1, \ldots, \vec{\beta}_k$’s induce a lattice metric as $d_{ij} = \vec{\beta}_i \cdot \vec{\beta}_j$. Further, we can define a dual basis for $\Lambda^\ast$ by $\vec{\gamma}^i = d^{ij} \vec{\beta}_j$, where $d^{ij}$ is the inverse of the lattice metric.

\footnote{In the following text, when we talk about a $U(1)$ generated by or in the direction of a vector $\vec{\beta} \in \Lambda$, we mean the current $\vec{\beta} \cdot \vec{j}(z)$.}
We can now separate the contribution of the gauge currents in the holomorphic part of the energy tensor as

\[ T(z) = \frac{1}{2} \left( \tilde{j}(z) \right)^2 + T'(z), \quad (14) \]

where \( T'(z) \) is independent of the scalars \( \tilde{X} \).

Then, we take a generic state that has charge \( \vec{q} \in \Lambda^* \) and is generated by a primary field \( \Sigma(z, \bar{z}) \) in the \((c, \bar{c}) = (20, 9)\) internal CFT plus some other components coming from the non-compact six dimensional part. Then, we separate the contribution from the charge as

\[ \Sigma(z, \bar{z}) = e^{i\vec{q}.\vec{X}}(z) : W(z, \bar{z}). \quad (15) \]

\( W(z, \bar{z}) \) denotes contributions independent of \( \tilde{X} \) (up to oscillator contributions coming from polynomials in \( \partial \tilde{X} \)) and we denote its \( L_0 \) \((\bar{L}_0)\) eigenvalues by \( h_L \) \((h_R)\). It follows that

\[ i\partial \tilde{X}(z)\Sigma(w, \bar{w}) = \frac{\vec{q}}{z - w}\Sigma(w, \bar{w}) + \ldots \quad (16) \]

or, expressed in another way,

\[ \tilde{j}_0|\Sigma\rangle = \vec{q}|\Sigma\rangle, \quad (17) \]

where \(|\Sigma\rangle\) is the state generated by \( \Sigma(w, \bar{w}) \). Also, we can see that the contribution of the charge part to \( L_0 \) eigenvalue is \( \vec{q}.\vec{q}/2 \). If one decomposes \( \vec{q} \) as \( \vec{q} = k_i\gamma^i \) with \( (k_i) \in \mathbb{Z}^s \), this contribution is equivalently \( k_id^ijk_j/2 \).

We now consider vertex operators \( V_{\lambda}(z) = e^{i\vec{\lambda}.\tilde{X}}(z) : \) for some \( \vec{\lambda} \in \Lambda \). This is a well defined operator because \( \Lambda \subset \Lambda^* \). Also, since \( V_{\lambda}(z)V_{\vec{q}}(w) \sim (z - w)^{\vec{\lambda}.\vec{q}}V_{\lambda+\vec{q}}(w) + \ldots \), this operator is local with respect to every state. That means there is a bijective pairing between states of charge \( \vec{q} \) and \( \vec{\lambda} + \vec{q} \) where \( W(z, \bar{z}) \) contributions are matched.

Now, we compactify on a two-torus with arbitrary constant metric and B-field. Then we turn on Wilson lines via deformations of the string Lagrangian with a term of the form

\[ \epsilon^{\alpha\beta}\partial_\alpha X^\mu \tilde{A}_\mu \partial_\beta \tilde{X}, \quad (18) \]

where \( X^\mu \) with \( \mu = \pm \) are coordinates on the torus.

One can now go through the same procedure as we went through in the six dimensional theory. There are now \( s + 4 \) \( U(1) \) currents and one can write currents in terms of \( s + 4 \) compact scalars. The lattice that determines the torus is constructed by rotating \( U \otimes U \otimes \Lambda \) by a member of \( SO(s + 2, 2) \) where \( U \) is the standard hyperbolic lattice and the member of \( SO(s + 2, 2) \) that rotates the standard lattice is determined by moduli.

Very schematically, \( \tilde{j}_0|m_i, n_i, k_i\rangle = \tilde{p}_L|m_i, n_i, k_i\rangle, \tilde{j}_0|m_i, n_i, k_i\rangle = \tilde{p}_R|m_i, n_i, k_i\rangle \) and the contribution of these scalars to the energy tensor is \( T_X(z) = \frac{1}{2} : \tilde{j}j(z) : \) and \( T_X(z) = \ldots \).
\[ \frac{1}{2} : \tilde{j}.\tilde{j}(z) : \] Through the parametrization of \( SO(s + 2)/SO(s + 2) \otimes SO(2) \) by a vector \( y = (T, U, \tilde{y}) \in \mathbb{C}^{17,1} \), as used in \( \mathbb{R} \), one gets

\[ \frac{p_L^2 - p_R^2}{2} = \frac{1}{2} \tilde{q} \cdot \tilde{q} - m_1 n_1 + m_2 n_2, \tag{19} \]

and

\[ \frac{p_R^2}{2} = \frac{1}{-2(y_2, y_2)} \left| \tilde{q} \cdot \tilde{y} + m_1 U + n_1 T + m_2 - n_2 \frac{(y, y)}{2} \right|^2. \tag{20} \]

The inner product \((y, y')\) appearing above is defined by \((y, y') = -TU' - UT' + \tilde{y} \cdot \tilde{y}'\). Using (integral) charges \( k_i \) defined by \( \tilde{q} = k_i \gamma^i \) and parameterizing Wilson lines by \( \tilde{y} = V^i \beta_i \) these expressions can be given in terms of the \( \Lambda \)-lattice metric \( d_{ij} \) as

\[ \frac{p_L^2 - p_R^2}{2} = \frac{1}{2} k_i d^{ij} k_j - m_1 n_1 + m_2 n_2, \tag{21} \]

and

\[ \frac{p_R^2}{2} = \frac{1}{-2(y_2, y_2)} \left| k_i V^i + m_1 U + n_1 T + m_2 - n_2 \frac{(y, y)}{2} \right|^2, \tag{22} \]

where

\[ (y, y) \equiv y^a \eta_{ab} y^b = -2TU + V^i d_{ij} V^j. \tag{23} \]

Following a similar argument to the one above we see that states with charge \((m_i, n_i, k_i) \in (U \oplus U \oplus \Lambda^*)\) are bijectively matched with respect to their \( W(z, \bar{z}) \) contributions when the charge is shifted by a member of its dual lattice, \( \Gamma \equiv U \oplus U \oplus \Lambda \) (which is the lattice that creates the torus before the moduli dependent rotation).

Using this matching, it is possible to factorize the vector multiplet moduli dependence in a number of interesting physical quantities such as the modified index or the partition function. Suppose we would like to compute a quantity which involves a trace over the internal CFT. Furthermore, suppose that only insertion in the trace that involves \((X^\pm, \bar{X})\) part of the CFT is \( q^{L_X - e_X/24} \bar{q}^{\bar{L}_X - e_X/24} \). In this case

\[ \text{Tr}_{H_{\text{int}}} \left[ \sum_{(m,n,k)\in\Gamma} q^{p_L^2/2} \bar{q}^{p_R^2/2} \text{Tr}_{H_{\text{int}}^{m,n,k}} \ldots \right] = \sum_{(m,n,k)\in\Gamma} \left( \sum_{\mu\in\Lambda^*} \left( \sum_{(m,n,k-\mu)\in\Gamma} q^{p_L^2/2} \bar{q}^{p_R^2/2} \right) \right) \text{Tr}_{H_{\text{int}}^{0,0,0}} \ldots , \tag{24} \]

\[ = \sum_{\mu\in\Lambda^*} \left( \sum_{(m,n,k-\mu)\in\Gamma} q^{p_L^2/2} \bar{q}^{p_R^2/2} \right) Z_{\Gamma,\mu}(\tau, T, U, V^i) h_\mu, \tag{25} \]

where \( H_{\text{int}}^{m,n,k} \) is the vector space of states with charge \((m_i, n_i, k_i)\),

\[ Z_{\Gamma,\mu}(\tau, T, U, V^i) \equiv \sum_{(m,n,k-\mu)\in\Gamma} q^{p_L^2/2} \bar{q}^{p_R^2/2}, \tag{26} \]

\[ 10 \]
and \( h_\mu \equiv \text{Tr}_\mathcal{H}_{\text{int}}^{0,\mu} \ldots \).

For the modified index of the theories we consider, this leads to an interesting result. \( h_\mu \) is independent of hypermultiplet moduli and \( \bar{q} \), and hence we write \( h_\mu = h_\mu(q) \). Furthermore, we note that \( \tau_2 \mathcal{I}(E_2 - 3/\pi \tau_2) \) should be modular invariant for the integral, \(|\ldots|\), to make sense. Of course, this should be correct for any value of vector multiplet moduli and in particular for \( V^i = 0 \). For \( V^i = 0 \), \( Z_{\Gamma,\mu} \) factorizes as

\[
Z_{\Gamma,\mu} \bigg|_{V^i = 0} = Z_{2,2} \vartheta_{\Lambda,\mu}(\tau, \bar{z} = 0),
\]

where \( \vartheta_{\Lambda,\mu} \) is a theta function associated to \( \Lambda \),

\[
\vartheta_{\Lambda,\mu}(\tau, z^i) = \sum_{k_i \equiv \mu_i (\text{mod } d_{ij}n_j)} q^{k_i d_{ij}k_j/2} y_1^{k_1} \cdots y_s^{k_s}, \text{ where } y_i \equiv e^{2\pi i z_i},
\]

and \( Z_{2,2} \) is a sum over torus momentum and winding numbers.

Since \( Z_{2,2} \tau_2 (E_2 - 3/\pi \tau_2) \) transforms under modular transformations with weight 2, the factor

\[
\sum_{\mu \in \Lambda^*/\Lambda} \vartheta_{\Lambda,\mu}(\tau, \bar{z} = 0) h_\mu(\tau)
\]

transforms with weight \(-2\). This finally implies that \( \phi_{-2,\Lambda}(\tau, \bar{z}) \) defined by

\[
\phi_{-2,\Lambda}(\tau, \bar{z}) = \sum_{\mu \in \Lambda^*/\Lambda} \vartheta_{\Lambda,\mu}(\tau, \bar{z}) h_\mu(\tau)
\]

is a weight \(-2\) Jacobi form with respect to the lattice \( \Lambda \).

More explicitly,

\[
\phi_{-2,\Lambda}(\tau, \bar{z}) = \sum_{n, k_i} c(n, k_i) q^n y_1^{k_1} \cdots y_s^{k_s}.
\]

Here, the variables \( y_s \) counts charges in the Cartan subalgebra of \( G \).

By the elliptic transformation property of Jacobi forms, the Fourier coefficients \( c(n, k_i) \) depend only on \( \Delta \equiv n - k_i d_{ij}k_j/2 \) and \( k_i (\text{mod } d_{ij}n_j) \), which is obvious from the theta decomposition form we have above. For the modified index, \( \Delta = N_L + h_L - 1 \), where \( N_L \) is the total left oscillator number and \( h_L \) is the contribution of \((c, \bar{c}) = (20, 9)\) internal CFT to \( L_0 \) excluding the contribution from the charge part. This means that \( c(n, k_i) \) vanishes unless \( \Delta = n - k_i d_{ij}k_j/2 \geq -1 \). In particular, the only \( q^{-1} \) term comes from the unphysical tachyon as

\[
\phi_{-2,\Lambda}(\tau, \bar{z}) = -\frac{2}{q} + O(1), \text{ as } \Im \tau \to +\infty.
\]

\footnote{Note that \( b_{\text{grav}} \) term in the integral is an IR regulator.}

\footnote{See the Appendix for the definition of Jacobi forms.}
Moreover, multiplying $\phi_{-2,\Lambda}$ by the cusp form $\Delta(q) = \eta(q)^{24}$ one gets a weight 10, holomorphic Jacobi form associated with the lattice $\Lambda$. Also, trivially extending the discussion in [3], we see that we can interpret $\phi_{-2,\Lambda}(\tau, \vec{z})$ as

$$
Z_{2,2}\phi_{-2,\Lambda}(\tau, \vec{z}) = 2 \left( \sum_{\text{Hyper BPS}} q^\Delta y_1^{k_1} \cdots y_s^{k_s} - \sum_{\text{Vector BPS}} q^\Delta y_1^{k_1} \cdots y_s^{k_s} \right),
$$

(34)

where to be exactly correct one should set the Wilson lines to zero for the $\Gamma$ momentum contribution to $L_0$ and ignore constraints coming from level matching. Also, note that the terms vector BPS and hyper BPS multiplets in this expression are used in the sense of [3], where one separates them according to their structure in the right-moving part of the internal CFT. One important point to note is that the gravity multiplet contributes to this sum as a vector BPS multiplet $q^0$ term with zero charge.

Finally, let us make a few comments on T-duality. The norm and conjugacy class preserving automorphisms of the lattice $\Gamma$ for the norm

$$
p_L^2 - p_R^2 = k_i d^j k_j - 2m_1 n_1 + 2m_2 n_2,
$$

(35)

can be compensated by a corresponding transformation of the moduli $(T, U, V^i)$ so that $p_R^2$ and hence the spectrum is also preserved. These transformations form part of the T-duality group transforming vector multiplet moduli $T, U$ and $V^i$. If there are identifications between $\mathcal{H}_{\text{int}}^{0,0,\mu}$, then it is also possible that the T-duality group can be extended by lattice automorphisms mixing some of the conjugacy classes. Notice that the form of $\Delta_{\text{grav}}$ makes it manifestly T-duality invariant.

In [21], it is shown that for $\Lambda = \langle 2m \rangle$ where $m \in \mathbb{Z}^+$, the T-duality transformations described above generate the extended paramodular group $\Gamma^+_m$. We will describe this group later in more detail. We will generically denote such discrete transformations by $O(\Gamma)$. Then, the perturbative moduli space will have the following component from $(T, U, V^i)$ moduli:

$$
O(\Gamma)/\frac{SO(n, 2)}{SO(n) \otimes SO(2)}.
$$

(36)

One should also note that a similar argument applies for the trace expression generating $24\Delta_{\text{gauge}} - \Delta_{\text{grav}}$,

$$
\mathcal{B} \equiv -\frac{i}{\eta(q)^2} \text{Tr} \left[ J_0(-1)^{L_0} q^{L_0 - c/24} q^{L_0 - c/24} \left( 24Q^2 - E_2 \right) \right],
$$

(37)

provided that $Q$ is a generator of a gauge group for which no Wilson lines are turned on. That may happen if, for example, the gauge group of the six-dimensional theory is, say, $G \times SU(2)$ but Wilson lines are switched on only for $G$. Repeating the argument above, one gets a factorization

$$
\mathcal{B} = \sum_{\mu \in \Lambda^*/\Lambda} Z_{\Gamma,\mu}(\tau, T, U, V^i) f_\mu.
$$

(38)

\footnote{Remember that when we say Wilson line we actually mean a combination of Wilson lines and torus moduli in the form $V^i$.}
Taking \( V_i \) to zero and looking at the modular properties of \( B \) one can deduce that \( f_\mu \)'s combine into a weight 0 Jacobi form for the lattice \( \Lambda \), which we will denote by \( \psi_{0,\Lambda}(\tau, \vec{z}) \).

The requirements we give for \( G \) are satisfied for geometric compactifications of the ten-dimensional heterotic string theories where the unbroken gauge group at the six-dimensional level are created by a suitable part of the original, level one, \( E_8 \times E_8 \) or \( Spin(32)/\mathbb{Z}_2 \) current algebra at ten dimensions. We will show in the next section how this factorization also works for the modified index computed at an orbifold point more explicitly.

Finally, before leaving this section, note that the 'spectral flow' property described in this section is in the spirit of [22], where, in comparison, we have the charge lattice more explicit here, and it is very similar to the 'spectral flow' described in [23, 24] for M-theory and Type-II compactifications on a Calabi-Yau threefold.

### 2.2 Modified Index at Orbifold Limits

In this section, we will be interested in \( \mathcal{N} = 2 \) orbifolds of \( \Gamma^{16} = E_8 \times E_8' \) or \( \Gamma^{16} = Spin(32)/\mathbb{Z}_2 \) heterotic string. We start by compactifying the heterotic string on \( T^4 \times T^2 \). Then, we introduce complex coordinates on \( T^4 \) and assume that it admits a \( \mathbb{Z}_N \) symmetry, \( z_1 \rightarrow \exp(2\pi ia/N)z_1 \) and \( z_2 \rightarrow \exp(-2\pi ia/N)z_2 \) where \( a = 0, 1, \cdots, N-1 \). Such tori exist for \( N = 2, 3, 4, 6 \). We will now mod out by that \( \mathbb{Z}_N \) symmetry. Since there are fixed points of those \( \mathbb{Z}_N \) transformations; although there is no curvature away from the fixed points, there will be curvature concentrated on them since there is a conical singularity with a monodromy contained in \( \mathbb{Z}_N \). These monodromies can be included in an \( SU(2) \) group and in this way we get an orbifold limit of \( K3 \) surfaces.

To preserve modularity (or equivalently to cancel spacetime anomalies) \( \mathbb{Z}_N \) should also act on the gauge bundle over the orbifold. We will be concerned with the case in which we are going to implement this action via a shift, \( \frac{a}{N} \vec{\gamma} \), on the \( \Gamma^{16} \) lattice. Here \( \vec{\gamma} \in \Gamma^{16} \) and \( a \in \mathbb{Z} \) to have an action included in \( \mathbb{Z}_N \). Note that none of the \( U(1) \) generators of the original string theory are projected out of the spectrum. However, a root, \( \vec{r} \), survives only if \( N|\vec{r} \cdot \vec{\gamma} \). Here we are using the natural Euclidean metric induced on \( \Gamma^{16} \otimes \mathbb{R} \).

Next, we vary the constant metric and B field on \( T^2 \) as well as Wilson lines on \( T^2 \) giving a vev to scalars in the Cartan subalgebra (CSA) as in the last section. Coordinates on vector multiplet moduli space can be described by a vector, \( y = (T, U, \vec{y}) \in \mathbb{C}^{17,1} \) as in the last section.

Then, as given in [16] we have explicitly

\[
\mathcal{I} = -\frac{1}{4N\eta(q)^2} \sum_{a,b=1}^{N} \zeta^{K3}_{a,b} \zeta^{\text{torus}}_{a,b},
\]  

(39)
where
\[ Z_{a,b}^{K3} = k_{ab} q^{- (a/N)^2} \eta(q)^2 \vartheta^{-2} \left( \frac{a}{N} + \frac{b}{N} \right) \tau, \]  
and
\[ Z_{a,b}^{\text{torus}} = e^{-2 \pi i \frac{ab}{N} \gamma^2} \eta(q)^{-18} \sum_{\vec{R} \in \Gamma^{16}, m,n} e^{2 \pi i \frac{\vec{R} \cdot \gamma}{\vec{R} \cdot \vec{R}}} q^{\tilde{p}_a L/2} q^{\tilde{p}_R R/2}. \]

Above \( k_{ab} \) are constants determined by modularity. They are given in [16] and [25]. \( \tilde{p}_a \) and \( \tilde{p}_R \) are defined as in equations (19) and (20) where we use \( \vec{R} \) instead of \( \vec{q} \).

When vector multiplet moduli are turned off, the gauge group of an orbifold model is as in (11). Suppose a vector \( \vec{\beta}' \in \Gamma^{16} \) creates one of those \( U(1) \) factors and it is a primitive vector of \( \Gamma^{16} \). Charges with respect to this vector then can be fractional, as there are states in the twisted sector surviving the orbifold projection such that their charges are multiples of \( \text{gcd} (\vec{\beta}', \vec{\gamma}, N) \) /\( N \). So, to make the charges lie on the dual of the lattice of the compact scalar generating this \( U(1) \), one should take \( Z_{\vec{\beta}'} \equiv Z \left( \frac{N}{\text{gcd}(\vec{\beta}', \vec{\gamma}, N)} \vec{\beta}' \right) \) as the factor appearing in \( \Lambda' \). Here, \( \Lambda' \) is the lattice of compact scalars generating the gauge current. This lattice now satisfies the hypothesis given in the previous section.

Suppose, now, we move in the hypermultiplet moduli space away form the orbifold point by Higgsing some parts of the gauge group (giving a vev to charged hypermultiplets). Out of the compact scalars of the initial orbifold, only some linear combinations can now be used to turn on Wilson lines. The directions in which Wilson lines can be turned on should be orthogonal to the charge vectors of hypermultiplets getting a vev. This determines a \( s \)-complex dimensional subspace, \( Y^s \) of \( \mathbb{C}^{16} \). This means that the final unbroken gauge group will be generated by compact scalars living in a \( s \)-dimensional torus generated by \( \Lambda \equiv Y^s \cap \Lambda' \subset \Lambda' \).

By the construction of \( \Lambda' \) we described above, \( \Lambda \) is an even lattice, with charges lying in the dual lattice \( \Lambda^* \). Moreover, if \( \vec{\beta}_i \) is an integral basis for \( \Lambda \) (where we again use the root vectors for the simple group contributions) one has \( \vec{\beta}_i \cdot \vec{\gamma} = s_i N \) for some integers \( s_i \). This will be crucial in displaying the factorization for the modified index.

By the hypermultiplet moduli independence, we can compute the modified index for this final model using the orbifold limit expression, where now, \( y^i = V^i \vec{\beta}_i \) and we define the lattice metric \( d_{ij} \) and the dual basis \( \{ \gamma^j \} \) as before.

\[ I = -\frac{1}{4 N \eta(q)^{18}} \sum_{a,b=1}^{N} k_{ab} q^{- (a/N)^2} \vartheta^{-2} \left( \frac{a}{N} + \frac{b}{N} \right) \sum_{\vec{R} \in \Gamma^{16}, m,n} \exp \left( 2 \pi i \frac{\vec{R} \cdot \gamma}{\vec{R} \cdot \vec{R}} q^{\tilde{p}_a L/2} q^{\tilde{p}_R R/2} \right). \]

\(^6\) We will break the gauge symmetry by giving a vev to hypermultiplet pairs with charge \( \pm \lambda \in \Lambda'^* \) as will be described in the next section. It is not hard to prove inductively that with such a breaking pattern, the lattice \( \Lambda \) has rank \( s \).
Let us partition $\Gamma^{16}$ into disjoint sets according to their contribution to charges with respect to $\vec{\beta}_j$. We define
\begin{equation}
S_{k_i} \equiv \{ \vec{r} \in \Gamma^{16} | \vec{r} = k_i \text{ for } i = 1, \ldots, s \}. \tag{43}
\end{equation}
The crucial observation is that for any set of integers $n^1, \ldots, n^s$ one has
\begin{equation}
S_{k_i + d_{ij} n^j} = S_{k_i} + n^j \vec{\beta}_j, \tag{44}
\end{equation}
reminiscent of the spectral flow property described in the previous section. Further, for any $\vec{r} \in S_{k_i - a s_i}$ one has
\begin{equation}
\tilde{R}_a = \vec{r} + \frac{a}{N} \vec{\gamma} = \left( \vec{r} + \frac{a}{N} \vec{\gamma} - k_i d^{ij} \vec{\beta}_j \right) + k_i d^{ij} \vec{\beta}_j \tag{45}
\end{equation}
\begin{equation}
= \left( \vec{r} + \frac{a}{N} \vec{\gamma} - k_i \vec{\gamma}^j \right) + k_i \vec{\gamma}^j. \tag{46}
\end{equation}
Note that the second factor is in the lattice $\Lambda^*$ and the first factor is perpendicular to the space $\Lambda \otimes \mathbb{R}$. So, in comparison to equations (19) and (20), we define $p^2_L$ and $p^2_R$ via:
\begin{equation}
\frac{\tilde{p}^2_{aL} - \tilde{p}^2_{aR}}{2} = \frac{1}{2} \left( \vec{r} + \frac{a}{N} \vec{\gamma} - k_i \vec{\gamma}^j \right)^2 + \frac{1}{2} k_i d^{ij} k_j - m_1 n_1 + m_2 n_2, \tag{47}
\end{equation}
\begin{equation}
= \frac{1}{2} \left( \vec{r} + \frac{a}{N} \vec{\gamma} - k_i \vec{\gamma}^j \right)^2 + \frac{p^2_L - p^2_R}{2}, \tag{48}
\end{equation}
and
\begin{equation}
\frac{\tilde{p}^2_{aR}}{2} = \frac{|k_i V_i + m_1 U + n_1 T + m_2 + n_2 (T U - \frac{1}{2} V_2 V_2^j d_{ij})|^2}{4 (T U_2 - \frac{1}{2} V_2 V_2^j)^2} = \frac{p^2_R}{2}. \tag{49}
\end{equation}
For $f_{ab}(\tau)$ defined as
\begin{equation}
f_{ab}(\tau) = -\frac{1}{4 N \eta(q)} \sum_{a,b=1}^{N} \sum_{m_i,n_i} \tilde{q}^{p^2_L/2} q^2 \sum_{\vec{r} \in S_{k_i - a s_i}} e^{2 \pi i N \tilde{r}. \vec{\gamma}} q^{(\vec{r} + \frac{a}{N} \vec{\gamma} - k_i \vec{\gamma}^j)^2/2}, \tag{50}
\end{equation}
we get
\begin{equation}
\mathcal{I} = \sum_{a,b=1}^{N} f_{ab} \sum_{m_i,n_i} k_i \sum_{\vec{r} \in S_{k_i - a s_i}} q^{p^2_L/2} \sum_{\vec{r} \in S_{k_i - a s_i}} e^{2 \pi i N \tilde{r}. \vec{\gamma}} q^{(\vec{r} + \frac{a}{N} \vec{\gamma} - k_i \vec{\gamma}^j)^2/2}. \tag{51}
\end{equation}
Now, we separate the sum over $(k_i) \in \mathbb{Z}^s$ as a sum over $k_i \equiv \mu_i (\text{mod } d_{ij} n^j)$ and a sum over $\mu_i (\text{mod } d_{ij} n^j)$. Noting the following relation, using (44) and that $\vec{\beta}_j \vec{\gamma} = s_i N$
\begin{equation}
\sum_{\vec{r} \in S_{k_i - a s_i}} e^{2 \pi i N \tilde{r}. \vec{\gamma}} q^{(\vec{r} + \frac{a}{N} \vec{\gamma} - k_i \vec{\gamma}^j)^2/2} = \sum_{\vec{r} \in S_{\mu_i - a s_i}} e^{2 \pi i N \tilde{r}. \vec{\gamma}} q^{(\vec{r} + \frac{a}{N} \vec{\gamma} - k_i \vec{\gamma}^j)^2/2}, \tag{52}
\end{equation}
we find
\begin{equation}
\mathcal{I} = \sum_{\mu_i (\text{mod } d_{ij} n^j)} \left( \sum_{k_i \equiv \mu_i (\text{mod } d_{ij} n^j)} q^{p^2_L/2} q^2 \right) \left( \sum_{\vec{r} \in S_{\mu_i - a s_i}} e^{2 \pi i N \tilde{r}. \vec{\gamma}} q^{(\vec{r} + \frac{a}{N} \vec{\gamma} - k_i \vec{\gamma}^j)^2/2} f_{ab} \right). \tag{53}
\end{equation}
This is of the form (26) as claimed, where

\[
h_\mu(\tau) = \sum_{\vec{r} \in S_{\mu - a_\nu}} e^{2\pi i \frac{1}{N} \vec{r} \cdot \vec{\gamma}} q^{(\vec{r} + \frac{a}{N} \vec{\gamma} - \mu \vec{\gamma})^2/2} f_{ab}.
\]  

(54)

Also, note that we still have the factorization above if there is an insertion of the form \((\vec{R} \cdot \vec{Q})^2\) in the sum provided that \(\vec{Q} \cdot \vec{\beta}_i = 0\) for all \(i\).

As an example, let us write down the form of the weight \(-2\) Jacobi form induced by this procedure for an orbifold of \(E_8 \times E_8\) heterotic string.

\[
\phi_{-2,\Lambda}(\tau, \vec{z}) = -\frac{1}{4N \eta(q)} \sum_{a,b=1}^N k_{ab} q^{-\frac{1}{2}} \left( \frac{a}{N} + \frac{b}{N} | \tau \right) q^{(a/N)^2(\vec{\gamma}^2 - 2)/2} \prod_{k=1}^8 y_k^{-\frac{a_k \vec{\gamma}}{N}} \chi_{\frac{1}{2}}^{E_8} \chi_{\frac{1}{2}}^{E_8},
\]  

(55)

where \(\chi_j^{E_8}\) for \(j = 1, 2\) is defined as

\[
\chi_j^{E_8} = \sum_{i=1}^{4} \left( \prod_{n=1+8(j-1)}^{8+8(j-1)} \vartheta_i \left[ \frac{(a \tau + b) \vec{\gamma}[n]}{N} + \sum_{r=1}^8 \vec{z} \cdot \vec{\beta}_r [n] | \tau \right] \right),
\]  

(56)

to accomplish the sum over \(E_8\) lattice points.

The easiest way to find \(\psi_{0,\Lambda}(\tau, \vec{z})\) is to add \(\vec{Q}\) as another basis vector to the basis of \(\Lambda\). Then, after computing the weight \(-2\) Jacobi form with this modified lattice as described above, one can find the effect of \(Q^2\) insertion by differentiating with respect to the variable \(z^Q\) twice and then setting this variable to zero. After this, with an appropriate subtraction of \(E_2 \phi_{-2,\Lambda}, \psi_{0,\Lambda}\) can be found. In other words, if we suppose \(\hat{\phi}_{-2,\Lambda}(\tau, \vec{z}, \vec{z}_Q)\) (so that \(\phi_{-2,\Lambda}(\tau, \vec{z}) = \hat{\phi}_{-2,\Lambda}(\tau, \vec{z}, \vec{z}_Q = 0)\)) is the weight \(-2\) Jacobi form that also counts \(Q\) charges, \(\psi_0(\tau, \vec{z})\) is given by

\[
\psi_0(\tau, \vec{z}) = \frac{24}{2 \vec{Q} \cdot \vec{Q}} \left. \left( y^Q \frac{\partial}{\partial y^Q} \right)^2 \hat{\phi}_{-2,\Lambda} \right|_{\vec{z}_Q = 0} - E_2 \phi_{-2,\Lambda}.
\]  

(57)

2.3 Symmetry Breaking in \(\mathcal{N} = 2\) Gauge Theories

In this section, we will discuss how to break gauge symmetries in the \(\mathcal{N} = 2\) theories we consider here and thereby move in the hypermultiplet moduli space. Let us suppose that

\(\text{The normalization of } Q^2 \text{ can be deduced from } [16] \text{ as } Q^2 = \frac{(P \cdot \vec{Q})^2}{2(q \cdot Q)} \text{ for a state with charge } P \text{ with respect to } \Gamma^{16}.\)
we start with a $\mathcal{N} = 2$ theory which has $\mathfrak{g}$ as its gauge Lie algebra where $\mathfrak{g}$ is a direct sum of a semisimple part and a number of abelian Lie algebras. Remember that in our case gauge symmetry is generated in 6-dimensions by compact scalars on a lattice $\Lambda$.

There may be flat directions in the scalar potential of this theory that allows charged hypermultiplet scalars to gain vev. Such a vev breaks gauge symmetry to a Lie subalgebra, $\mathfrak{h}$, and reduces the rank by one. We will describe one such flat direction which we will use in following sections. Details can be found in [26] and [27].

Each hypermultiplet consists of two chiral multiplets if we use the language of $\mathcal{N} = 1$ supersymmetry. These chiral multiplets have two scalars, say $\phi$ and $\phi^*$, which are CPT conjugates of each other. In the flat direction we use for gauge symmetry breaking, one needs two hypermultiplets with scalars $(\phi_1, \phi_1^*)$ and $(\phi_2, \phi_2^*)$ such that their $\phi$ components have charges $|\vec{q}|$ and $|\vec{q} - \vec{q}|$ with respect to the Cartan subalgebra. Further, assuming that $|\vec{q}|$ can not be set into the direction of $|\vec{q} - \vec{q}|$ by a gauge transformation in the non-abelian part, one can turn on vevs of the form $\phi_1, \phi_2^* \sim v$. Out of these two hypermultiplets, one linear combination gets mass together with the vector multiplet corresponding to the $U(1)$ generator in $\vec{q}$'s direction, and the other remains massless.

Now, let $\vec{\alpha}$ be a root in the non-abelian part. A vector field $A^\mu_\vec{\alpha}$ corresponding to this root gets mass through the minimal gauge coupling if the transformation matrix $t_\alpha$ acting on $|\pm \vec{q}\rangle$ (with respect to an appropriate representation of the non-abelian part) gives a non-vanishing result. Since, the supersymmetry is unbroken, the whole vector multiplet gets mass together with this field. Furthermore, in this case, the hypermultiplet $t_\alpha|\pm \vec{q}\rangle \sim |\pm \vec{q} + \alpha\rangle$ also gets mass through the quartic scalar potential. Moreover, these are the only vector and hypermultiplets getting mass. Such $|\pm \vec{q}\rangle$ pairs can be found if one has a hypermultiplet in a $C+\bar{C}$ representation (where $C$ is complex) or if there is a hypermultiplet in a real representation $R$ such that there is no root connecting $|\vec{q}\rangle$ to $|\vec{q} - \vec{q}|$.

There is a simple way to see whether a gauge symmetry $\mathfrak{g}$ can be broken through this procedure to a Lie subalgebra, $\mathfrak{h}$, with its rank reduced by one. One starts with the adjoint of $\mathfrak{g}$ and matter representations that will be used in the breaking (either $C + \bar{C}$ or $R$). Then, they are decomposed with respect to the representations of a maximal $\mathfrak{h} + u(1)$ Lie subalgebra of $\mathfrak{g}$. Firstly, there should be singlets of $\mathfrak{h}$ in the matter decomposition which are also charged under the $u(1)$. We will denote these by $(1)_{\pm q}$. Moreover, for every factor, $(R_i)_{q_i}$, in the decomposition of $\mathfrak{g}$’s adjoint as

$$ (\text{Adj}_\mathfrak{g}) \rightarrow (1)_0 + (\text{Adj}_\mathfrak{h})_0 + \sum_i (R_i)_{q_i}, \tag{58} $$

there should be factors in the matter decomposition in either of $(R_i)_{\pm q + \bar{q}}$ but not both. Then, vector multiplets in $\sum_i (R_i)_{ \pm \bar{q}}$ representations get mass together with corresponding hypermultiplets. This gives a sufficient condition for $\mathfrak{g} \rightarrow \mathfrak{h}$ breaking. Now, let us go over some examples which will be useful in the coming sections as well.
SU(n + 1) → SU(n)

Start with \( g = \mathfrak{su}_{n+1} \), where simple roots are given in an orthonormal basis as
\[
\vec{\beta}_1 = (1, -1, 0, \ldots, 0), \quad \vec{\beta}_2 = (0, 1, -1, 0, \ldots, 0), \ldots, \vec{\beta}_n = (0, \ldots, 0, 1, -1).
\] (59)

We will display the familiar breaking pattern to \( h = \mathfrak{su}_n \) by a \((n + 1) + (n + 1)\) pair. Take the simple roots of \( \mathfrak{su}_n \) to be \( \vec{\beta}_1, \ldots, \vec{\beta}_n - 1 \). So, the sublattice of \( \mathfrak{su}_{n+1} \)'s root lattice which is orthogonal to \( \mathfrak{su}_n \)'s root lattice is generated by \( \vec{\beta} = (1, \ldots, 1, -n) \)

Then we see the following decompositions for \( g \to h + u(1) \) where \( u(1) \) is generated by \( \vec{\beta} \):
\[
(\text{Adj}^{n+1}) \to (1)_0 + (\text{Adj}^n)_0 + (n)_{n+1} + (\overline{\mathfrak{u}})_{n-1},
\] (60)
and
\[
(n + 1) + (n + 1) \to [(1)_{-n} + (n)_1] + [(1)_n + (\overline{\mathfrak{u}})_{-1}].
\] (61)

Now, giving vev to the scalars in \((1)_{-n} + (1)_n\) breaks the \( U(1) \) symmetry. Moreover, gauge symmetry is completely reduced to \( SU(n) \) as hypermultiplets \((n)_1 + (\overline{\mathfrak{u}})_{-1}\) gain mass together with vector multiplets \((n)_{n+1} + (\overline{\mathfrak{u}})_{-n-1}\).

In summary, if one has a hypermultiplet pair \((n) + (\overline{n})\), \( SU(n) \) can be broken down to \( SU(n - 1) \). After breaking, what remains in the matter spectrum can be exemplified by
\[
(n) + (\overline{\mathfrak{u}}) \to (1) \text{ if it is the symmetry breaking representation (rep.),}
\] (62)
\[
(n) \to (1) + (n - 1) \text{ otherwise,}
\] (63)
\[
(\text{Asy}^n) \to (n - 1) + (\text{Asy}^{n-1}).
\] (64)

For reference, we also gave the decomposition of the antisymmetric representation, \((\text{Asy}^n)\), as well.

Now, we give some more examples for later reference.

\[
SU(n) \to SU(n - 2) \times SU(2)
\]

This can be accomplished by a \((\text{Asy}^n) + (\overline{\text{Asy}^n})\) pair. Some examples of matter spectrum after breaking are
\[
(\text{Asy}^n) + (\overline{\text{Asy}^n}) \to (1, 1) + (\text{Asy}^{n-2}, 1) + (\overline{\text{Asy}^{n-2}}, 1) \text{ if breaking rep.,}
\] (65)
\[
(\text{Asy}^n) \to (1, 1) + (\text{Asy}^{n-2}, 1) + (n - 2, 2),
\] (66)
\[
(n) \to (n - 2, 1) + (1, 2).
\] (67)

Note that, from the breaking representation a \((\text{Asy}^{n-2}, 1) + (\overline{\text{Asy}^{n-2}}, 1)\) pair survives. So, if one starts with \( SU(2N) \) such that there is a \((\text{Asy}^{2N}) + (\overline{\text{Asy}^{2N}})\) hypermultiplet pair.

\[8\] Though not important for this example, finding the surviving sublattice of \( \Lambda \) as we break gauge symmetry will be important when \( U(1) \) factors are involved in the initial and/or final gauge group.

18
in the matter spectrum, one can use this to break the gauge symmetry down to $SU(2)^N$. Furthermore, if the initial spectrum contains two $(\text{Asy}^{2N}, 1) + (\text{Asy}^{2N}, 1)$ pairs, the final spectrum contains matter representations

$$2 \sum_{1 \leq n < k \leq N} (2^n 2^k),$$

(68)

where $(2^n)$ is a doublet with respect to the $n^{th}$ $SU(2)$.

$SU(2)^N \rightarrow U(1)$

We take the lattice, $\Lambda$, to be created by simple roots

$$\vec{\beta}_1 = (\sqrt{2}, 0, \ldots, 0), \ldots, \vec{\beta}_N = (0, \ldots, 0, \sqrt{2}).$$

(69)

We claim that it is possible to break $SU(2)^N$ down to a $U(1)$ so that the sublattice of $\Lambda$ in this $U(1)$’s direction is created by

$$\vec{\beta}_1 + \ldots + \vec{\beta}_N = (\sqrt{2}, \ldots, \sqrt{2}).$$

(70)

Note that this sublattice is $\langle 2N \rangle$, and hence if we can find such a breaking pattern in heterotic string theories we consider, the modified index will give rise to a weight $-2$, index $N$ Jacobi form.

The representation we will use for that symmetry breaking is $2 \sum_{1 \leq n < k \leq N} (2^n 2^k)$ so that with respect to the unbroken $U(1)$ we will be left with the following (this is for the symmetry breaking matter representation)

$$2 \sum_{1 \leq n < k \leq N} (2^n 2^k) \rightarrow (2N - 1)(N - 1)(1)_0 + N(N - 2)(1)_2 + N(N - 2)(1)_{-2}.$$  

(71)

We will prove this assertion by induction.

For $N = 2$, we start with the following representations in the vector and hyper multiplets:

$Vectors$: $(3^1) + (3^2)$, and $Hypers$: $2(2^1 2^2)$.  

(72)

Decomposing the representations into the representations of $U(1)$’s generated by $\vec{\beta}_1 + \vec{\beta}_2$ and the orthogonal $\vec{\beta}_1 - \vec{\beta}_2$.

$Vectors$: $2(1)_{0,0} + (1)_{2,2} + (1)_{-2,-2} + (1)_{2,-2} + (1)_{-2,2}$,  

(73)

$Hypers$: $2(1)_{2,0} + 2(1)_{-2,0} + 2(1)_{0,2} + 2(1)_{0,-2}$.  

(74)

Giving vev to the scalars of a $(1)_{0,2} + (1)_{0,-2}$ pair, one can break down to the first $U(1)$. What remains in the matter spectrum is three $(1)_0$ representations, consistent with the claim.
Now, suppose that the claim is correct up to some $N$. We will show that it is also correct for $N + 1$. We start with

\[
\text{Vectors: } \sum_{n=1}^{N+1} (3^n) \quad \text{and} \quad \text{Hypers: } \sum_{1 \leq n < k \leq N+1} 2(2^n 2^k).
\] (75)

By the inductive hypothesis we can break this down to a $SU(2) \times U(1)$ where $U(1)$ is generated by, say $\vec{\beta} = \vec{\beta}_1 + \ldots + \vec{\beta}_N$. Then, at this stage, the spectrum is

\[
\text{Vectors: } (3)_0 + (1)_0, \quad (76)
\]

\[
\text{Hypers: } (2N-1)(N-1)(1)_0 + N(N-2)(1)_2 + N(N-2)(1)_{-2} + 2N(2)_1 + 2N(2)_{-1}.
\] (77)

Under the $U(1) \times U(1)$ generated by $\vec{\beta} + \vec{\beta}_{N+1}$ and orthogonal $\vec{\beta} - N\vec{\beta}_{N+1}$, these representations decompose as

\[
\text{Vectors: } 2(1)_{0,0} + (1)_{2,-2N} + (1)_{-2,2N},
\] (78)

\[
\text{Hypers: } (2N-1)(N-1)(1)_{0,0} + N(N-2)(1)_{2,2} + N(N-2)(1)_{-2,-2} + 2N \left[ (1)_{2,-N+1} + (1)_{0,N+1} + (1)_{0,-N-1} + (1)_{-2,N-1} \right].
\] (79)

Now, one can verify that giving vev to a $(1)_{0,N+1} + (1)_{0,-N-1}$ pair breaks the gauge symmetry down to the first $U(1)$ generated by $\vec{\beta}_1 + \ldots + \vec{\beta}_{N+1}$ with remaining matter representations

\[
\text{Hypers: } (2N+1)N(1)_0 + (N-1)(N+1)(1)_2 + (N-1)(N+1)(1)_{-2}.
\] (80)

consistent with the result we wanted to prove.

### 2.4 An Orbifold Example

Our main motivation for the last section was to find possible gauge symmetry breaking patterns, supposing we start with a heterotic string model at the orbifold limit. We can, then, make use of this orbifold limit to explicitly compute the Jacobi forms $\phi_{-2,\lambda}$ and $\psi_{0,\lambda}$ (and later to compute threshold corrections and also Gromov-Witten invariants for possible Type IIA duals). The fact that the vector space of Jacobi forms is finite dimensional will sometimes enable us to write down the whole Jacobi form by only using the massless spectrum of the six dimensional theory.

A rich example we will study is the $E_8 \times E_8$ heterotic string on a $T^4/\mathbb{Z}_6$ orbifold for which the shift vector is

\[
\vec{\gamma} = (5, 1^7; 3^2, 0^6),
\] (81)

where its components are given in an orthonormal basis. We choose the orthonormal basis so that the coordinates of a single $E_8$ lattice in this basis are either all integral or all half-integral and are also constrained to have an even integer sum.
The gauge group surviving the orbifold projection is $SU(9) \times SU(2) \times E_7$. A set of simple roots for these gauge groups can be given as in the following, where we use the same basis in which $\tilde{\gamma}$ components are given:

\[
\begin{align*}
SU(9) & : & \tilde{\alpha}_1 &= (0,1,-1,0^{13}), & \tilde{\alpha}_2 &= (0^2,1,-1,0^{12}), & \tilde{\alpha}_3 &= (0^3,1,-1,0^{11}), \\
& & \tilde{\alpha}_4 &= (0^4,1,-1,0^{10}), & \tilde{\alpha}_5 &= (0^5,1,-1,0^9), & \tilde{\alpha}_6 &= (0^6,1,-1,0^8), \\
& & \tilde{\alpha}_7 &= (1/2,-1/2^6,1/2,0^8), & \tilde{\alpha}_8 &= (1/2^6,0^8), \\
SU(2) : & & \tilde{\alpha}_9 &= (0^8,1,1,0^6), \\
E_7 : & & \tilde{\alpha}_{10} &= (0^8,1/2,-1/2^6,1/2), & \tilde{\alpha}_{11} &= (0^{14},1,-1), & \tilde{\alpha}_{12} &= (0^{13},1,-1,0^1), \\
& & \tilde{\alpha}_{13} &= (0^{12},1,-1,0^2), & \tilde{\alpha}_{14} &= (0^{11},1,-1,0^3), & \tilde{\alpha}_{15} &= (0^{10},1,-1,0^4), \\
& & \tilde{\alpha}_{16} &= (0^{14},1,1). \\
\end{align*}
\]

We further compactify this theory on a $T^2$ to get a $\mathcal{N} = 2$ theory in four dimensions. The massless spectrum for this particular example, when all Wilson lines are switched off, is given by (with respect to the $SU(9) \times SU(2) \times E_7$ gauge group)

\[
\begin{align*}
Vectors: & \ (80,1,1) + (1,3,1) + (1,1,133), \\
Hypers: & \ 2(1,1,1) + (9,2,1) + (\bar{9},2,1) + 2(36,1,1) + 2(\bar{36},1,1) \\
& + 5(9,1,1) + 5(\bar{9},1,1) + 3(1,1,56) + 10(1,2,1),
\end{align*}
\]

where we have written the matter representations in a manifestly real fashion. Among the matter multiplets, $2(9,2,1)$ comes from $\mathbb{Z}_6$ fixed points, $4(36,1,1) + 10(9,1,1)$ comes from $\mathbb{Z}_3$ fixed points and $3(1,1,56) + 10(1,2,1)$ comes from $\mathbb{Z}_2$ fixed points.

The massless spectrum has some interesting features which makes it a useful example for our purposes. Firstly, $E_7$ can be broken independently of $SU(9) \times SU(2)$ by giving a vev to hypermultiplet scalars in the $3(1,1,56)$ and through the following chain:

\[
\begin{align*}
G : E_7 & \quad Vectors: \ (133) \quad Hypers: \ 3(56), \\
G : E_6 & \quad Vectors: \ (78) \quad Hypers: \ 5(1) + 2(\bar{27}) + 2(27), \\
G : SO(10) & \quad Vectors: \ (45) \quad Hypers: \ 8(1) + 4(10) + (16) + (\bar{16}), \\
G : SU(5) & \quad Vectors: \ (24) \quad Hypers: \ 9(1) + 5(\bar{5}) + 5(\bar{\bar{5}}), \\
G : SU(4) & \quad Vectors: \ (15) \quad Hypers: \ 18(1) + 4(4) + 4(\bar{4}), \\
G : SU(3) & \quad Vectors: \ (8) \quad Hypers: \ 25(1) + 3(3) + 3(\bar{3}), \\
G : SU(2) & \quad Vectors: \ (3) \quad Hypers: \ 30(1) + 4(2), \\
G : U(1) & \quad Vectors: \ - \quad Hypers: \ 35(1).
\end{align*}
\]

Note that this is the same spectrum as one would have, if $E_7$ symmetry is broken in a smooth compactification with 10 instantons on one side of the $E_6$’s [11]. In our examples below we will usually assume that this $E_7$ is completely broken and hence no Wilson lines will be switched on for groups in this chain. Only if we work out the weight zero Jacobi form, $\psi_{0,\Lambda}$, we will assume a $SU(2)$ is left unbroken from the $E_7$ chain. So, for definiteness, we fix $\beta_Q$ to be $\tilde{\alpha}_{11}$ whenever we compute $\psi_{0,\Lambda}$ for this model. Note that, even in this case,
we will assume that the Wilson line (or more appropriately the complex modulus, $V^Q$) for this $SU(2)$ is zero.

Now, if $E_7$ is completely broken by the chain described above, one gets the following spectrum under $SU(9) \times SU(2)$:

$$\text{Vectors: } (80, 1) + (1, 3), \quad \text{(87)}$$

$$\text{Hypers: } 37(1, 1) + (9, 2) + (\bar{9}, 2) + 2(36, 1) + 2(\bar{36}, 1) + 5(9, 1) + 5(\bar{9}, 1) + 10(1, 2). \quad \text{(88)}$$

Looking at the massless spectrum, we see that $SU(2)$ can be broken by using the two hypermultiplets in the $(1, 2)$ representation. $SU(9)$ can also be broken completely by using the $(9, 1) + (\bar{9}, 1)$ hypermultiplets and through chains of $SU(N) \to SU(N-1)$. Note that antisymmetric representations $2(36, 1) + 2(\bar{36}, 1)$ provide two new $N + \bar{N}$ pairs at each step of $SU(N) \to SU(N-1)$, compensating the loss of a single pair due to symmetry breaking.

Furthermore, $(36, 1) + (\bar{36}, 1)$ can be used to break $SU(9)$ to $SU(N) \times [SU(2)]^n$ form. Also, since there are two such pairs, one obtains $2(2^i 2^j)$ representations as described in the previous section. These $2(2^i 2^j)$ representations then can be used to break $[SU(2)]^n$ to a $U(1)$ with $\Lambda = (2n)$. Interestingly, if the original $SU(2)$ of the spectrum is kept unbroken, it also provides hypermultiplets of the form $2(2^i 2^j)$ via $(9, 2) + (\bar{9}, 2)$ hypermultiplets.

Now, one can apply these three symmetry breaking patterns in varying orders to get a large class of theories with gauge symmetries of the form $SU(N) \times [SU(2)]^n \times [U(1)]^k$. We now give several such examples.

**Example 1 : $\Lambda = A_1$**

The gauge symmetry can be broken down to a $SU(2)$, where the $SU(2)$ comes either from the original $SU(2)$ or from the $SU(9)$ factor. In both cases, the massless spectrum is

$$\text{Vectors: } (3), \quad \text{(89)}$$

$$\text{Hypers: } 191(1) + 28(2). \quad \text{(90)}$$

Now, turning on Wilson lines and going to a generic point on the vector multiplet moduli space gives a theory with $(N_v, N_h - 1) = (4, 190)$, where $N_v$ is the number of massless vector multiplets and $N_h$ is the number of massless hypermultiplets in four dimensions.

The spectrum above fixes the first two Fourier coefficients of $\phi_{-2}(\tau, \bar{z})$ as

$$\phi_{-2,1} = -\frac{2}{q} + 2 \left( -y_1^{-2} + 28y_1^{-1} + 186 + 28y_1 - y_1^{-2} \right) + O(q). \quad \text{(91)}$$

The coefficient of $q^0y_1^0$ term, 186, is fixed by noting that the gravity multiplet contributes with a $-1$ to this term and the contributions from hypermultiplets and vector multiplets
are 191 and \(-4\), respectively. This fixes the full Jacobi form to be
\[
\phi_{-2,1} = -\frac{2}{\Delta(q)} \frac{14E_4(q)E_{6,1}(q, y_1) + 10E_{4,1}(q, y_1)E_6(q)}{24},
\]
(92)
where the Jacobi-Eisenstein series \(E_{6,1}\) and \(E_{4,1}\) are defined in the Appendix. An explicit computation at the orbifold point verifies this. Indeed, this example is very widely studied in the literature. It first appeared in [4], where the fact that Jacobi forms arise in the computation of threshold corrections with Wilson lines was also noted. Threshold corrections and its relation to Jacobi forms for \(\Lambda = A_1\) also appeared in [28].

The numbers 14 and 10 are interpreted as the instanton numbers in a geometric compactification so that the remaining \(SU(2)\) is in the \(E_8\) factor with 14 instantons. If the \(SU(2)\) comes from the \(SU(9)\) factor, this interpretation is also consistent with our case since this orbifold model can be matched, after Higgsing, to a geometric compactification with \((14, 10)\) instantons [25]. Of course, this argument can not be used if the unbroken \(SU(2)\) is the original \(SU(2)\) in the \(SU(9) \times SU(2)\) model, since this \(SU(2)\) is special to the orbifold limit and is broken if the orbifold is blown up to a smooth compactification. Still, interestingly, this \(SU(2)\) behaves as if it is an \(SU(2)\) on the 14 instanton side of a geometric compactification of \(E_8 \times E_8\) heterotic string, as far as threshold corrections are concerned.

Group theoretically, we can explain this symmetry in the massless spectrum between \(SU(9)\) and \(SU(2)\) representations by considering a hypothetical theory with \(SU(11)\) gauge symmetry and massless spectrum:

**Vectors:** \((120)\),

**Hypers:** \(34(1) + 5(11) + 5(\overline{11}) + 2(55) + 2(\overline{55})\).

(93)
(94)

Breaking \(SU(11)\) to \(SU(9) \times SU(2)\) using \((55) + (\overline{55})\) gives a massless spectrum precisely as in our example coming from the orbifold limit.

**Example 2 : \(\Lambda = \langle 4 \rangle\)**

Now we consider an example in which the final gauge group is a \(U(1)\) obtained by breaking \(SU(2) \times SU(2)\) as described in the previous section. The massless hypermultiplet spectrum is given by

**Hypers:** \(149(1)_0 + 48(1)_1 + 48(1)_{-1}\).

(95)

When Wilson lines are turned on, this gives a theory with \((N_v, N_h - 1) = (4, 148)\). This gives the beginning of \(\phi_{-2,2}\)’s Fourier series as

\[
\phi_{-2,2} = -\frac{2}{q} + 2 \left( 48y_1 + \frac{48}{y_1} + 144 \right) + O(q).
\]

(96)

If we form the most general weight \(-2\), index 2 nearly holomorphic Jacobi form which starts as \(-2/q\) then by matching the Fourier coefficients for \(q^{-1}\) and \(q^0\) terms we can uniquely fix
\[ \phi_{-2,2} = -\frac{2}{\Delta(q)} \frac{1}{6 \times 12^2} \left( 6\phi_{0,1}^2E_6E_4 - 7\phi_{-2,1}\phi_{0,1}E_4^3 - 5\phi_{-2,1}\phi_{0,1}E_6^2 + 6\phi_{-2,1}^2E_6E_4^2 \right). \]  

(97)

\( \phi_{-2,1} \) and \( \phi_{0,1} \) are the generators of even weight weak Jacobi forms over lattices \( \Lambda = \langle 2m \rangle \), with \( m \in \mathbb{Z}^+ \), when considered as a ring over modular forms \([5]\). Detailed definitions can be found in the Appendix.

We have checked this to order \( O(q^7) \) by comparing it to the orbifold computation where we take \( \Lambda = \langle \vec{\alpha}_1 + \vec{\alpha}_3 \rangle \). Up to order \( q \), the result is given by

\[ \phi_{-2,2} = -\frac{2}{q} + \left( 96y_1 + \frac{96}{y_1} + 288 \right) + q \left( -2y_1^4 + 96y_1^3 + 10192y_1^2 + 69280y_1 \right) + \frac{69280}{y_1} + \frac{10192}{y_1^2} + \frac{96}{y_1^3} - \frac{2}{y_1^4} + 123756 \right) + O(q^2). \]  

(98)

Threshold corrections for a \( (N_v, N_h - 1) = (4, 148) \) model (though in a smooth compactification with \( (13, 11) \) instanton embeddings) and its relation to weight \(-2\), index 2 Jacobi forms appeared in \([6]\). An indirect argument is used there to get information about the Jacobi form \( \phi_{-2,2} \) similar to the argument we have given above. However, instead of using Jacobi forms, a set of Siegel forms with correct singularity structures are matched to the expressions arising in threshold corrections. We will describe the details of threshold corrections in the next section.

**Example 3 : \( \Lambda = \langle \mathbf{6} \rangle \)**

In this example, the final gauge group is a \( U(1) \) obtained by breaking a \([SU(2)]^3 \). The massless hypermultiplet spectrum for zero Wilson lines is given by

\[ Hypers: 119(\mathbf{1})_0 + 60(\mathbf{1})_1 + 60(\mathbf{1})_{-1} + 3(\mathbf{1})_2 + 3(\mathbf{1})_{-2}. \]  

(99)

When Wilson lines are turned on, we get \( (N_v, N_h - 1) = (4, 118) \) massless multiplets. The beginning of \( \phi_{-2,3} \)’s Fourier series is given as

\[ \phi_{-2,3} = -\frac{2}{q} + 2 \left( 3y_1^2 + \frac{3}{y_1^2} + 60y_1 + \frac{60}{y_1} + 114 \right) + O(q). \]  

(100)

Again by forming the most general nearly holomorphic Jacobi form which starts with the same Fourier coefficients for \( q^{-1} \) and \( q^0 \) terms and has weight \(-2\), index 3, we can uniquely fix \( \phi_{-2,3} \) as

\[ \phi_{-2,3} = -\frac{2}{\Delta(q)} \frac{1}{4 \times 125} \left( 4\phi_{0,1}^3E_6E_4 - 5\phi_{-2,1}\phi_{0,1}E_6^2 + 12\phi_{-2,1}^2\phi_{0,1}E_6E_4^2 - 3\phi_{-2,1}^3E_6^2E_4 - 7\phi_{-2,1}\phi_{0,1}E_4^3 - \phi_{-2,1}^3E_4^4 \right). \]  

(101)
This can also be compared with the explicit orbifold computation for which we matched the coefficients to order $O(q^7)$. For the orbifold computation, we have taken $\Lambda = \langle \vec{\alpha}_1 + \vec{\alpha}_3 + \vec{\alpha}_5 \rangle$.

Up to order $q$, the result is

$$
\phi_{-2,3} = -\frac{2}{q^3} + \left( \frac{6y_1^2 + 120y_1 + 120}{y_1^3} + \frac{6}{y_1^2} + 228 \right) + q \left( \frac{6y_1^4 + 1776y_1^3 + 20292y_1^2}{y_1^4} + \frac{69072y_1 + 20292}{y_1^3} + \frac{1776}{y_1^2} + \frac{6}{y_1} + 100596 \right) + O\left(q^2\right).
$$

At this point we should note that these three examples can be obtained from an intermediate $SU(6)$ model in the breaking chain, which has the same massless spectrum as in (6). This model can be obtained as a smooth six dimensional heterotic compactification as described in [14] where a F-theory dual is also described.

**Example 4 : $\Lambda = \langle \vec{8} \rangle$**

In this example, the final gauge group is a $U(1)$ obtained from a $[SU(2)]^4$ step. The massless hypermultiplet spectrum of the six dimensional theory is given by

$$
Hypers: 101(1)_0 + 64(1)_{1} + 64(1)_{-1} + 8(1)_{2} + 8(1)_{-2}.
$$

When Wilson lines are turned on, we get $(N_v, N_h - 1) = (4, 100)$ massless multiplets. The Jacobi form $\phi_{-2,4}$'s Fourier series start as

$$
\phi_{-2,4} = -\frac{2}{q} + 2 \left( \frac{8y_1^2}{y_1^3} + \frac{8}{y_1^2} + 64y_1 + \frac{64}{y_1} + 96 \right) + O(q).
$$

Again, we form the most general nearly holomorphic Jacobi form with appropriate weight and index. Then, we uniquely fix $\phi_{-2,4}$ as

$$
\phi_{-2,4} = -\frac{2}{\Delta(q)3 \times 12^4} \left( 3\tilde{\phi}^4_{0,1}E_6E_4 - 5\tilde{\phi}_{-2,1}\tilde{\phi}^3_{0,1}E_6^2 + 18\tilde{\phi}_{-2,1}\tilde{\phi}^2_{0,1}E_6E_4 - 9\tilde{\phi}^3_{-2,1}\tilde{\phi}_{0,1}E_6^2E_4 \\
+ 2\tilde{\phi}^4_{-2,1}E_6^3 - 7\tilde{\phi}_{-2,1}\tilde{\phi}^3_{0,1}E_6^3 + 3\tilde{\phi}^3_{-2,1}\tilde{\phi}_{0,1}E_4^4 + \tilde{\phi}^4_{-2,1}E_6E_4 \right).
$$

We also perform the computation at the orbifold limit by taking $\Lambda = \langle \vec{\alpha}_1 + \vec{\alpha}_3 + \vec{\alpha}_5 + \vec{\alpha}_7 \rangle$.

Up to order $q$, the result is

$$
\phi_{-2,4} = -\frac{2}{q} + \left( \frac{16y_1^2 + 128y_1 + 128}{y_1^3} + \frac{16}{y_1^2} + 192 \right) + q \left( \frac{228y_1^4 + 4992y_1^3 + 26880y_1^2}{y_1^4} + \frac{65664y_1 + 26880}{y_1^3} + \frac{4992}{y_1^2} + \frac{228}{y_1} + 87360 \right) + O\left(q^2\right).
$$
Example 5 : $\Lambda = \langle 10 \rangle$

As noted before, we can also break $SU(9) \times SU(2)$ to a $[SU(2)]^5$ first and then use matter representations $2(2^4 2^3)$ to get a $U(1)$ with $\Lambda = \langle 10 \rangle$. The massless hypermultiplet spectrum of the six dimensional theory is given by

$$Hypers: 95(1)_0 + 60(1)_1 + 60(1)_{-1} + 15(1)_2 + 15(1)_{-2}. \tag{107}$$

In the four dimensional theory, when Wilson lines are turned on, we get $(N_v, N_h - 1) = (4, 94)$ massless multiplets. The Jacobi form $\phi_{-2.5}$’s Fourier series for this example begin by

$$\phi_{-2.5} = \frac{-2}{q} + \frac{1}{\Delta(q)} 12^6 \left( 12\tilde{\phi}_{0,1}^5 E_6 E_4 - 35\tilde{\phi}_{-2,1}\tilde{\phi}_{0,1}^4 E_3 - 25\tilde{\phi}_{-2,1}\tilde{\phi}_{0,1}^4 E_6 E_4 + 120\tilde{\phi}_{-2,1}\tilde{\phi}_{0,1}^4 E_6^2 - 30\tilde{\phi}_{-2,1}\tilde{\phi}_{0,1}^4 E_4^2 - 90\tilde{\phi}_{-2,1}\tilde{\phi}_{0,1}^4 E_6^2 E_4 + 20\tilde{\phi}_{0,1}^4 \tilde{\phi}_{0,1}^4 E_6 E_4^2 + 40\tilde{\phi}_{-2,1}\tilde{\phi}_{0,1}^4 E_6^3 + 9\tilde{\phi}_{-2,1}\tilde{\phi}_{0,1}^4 E_6^3 - 21\tilde{\phi}_{-2,1}\tilde{\phi}_{0,1}^4 E_6^2 E_4 \right). \tag{109}$$

We also perform the computation at the orbifold point by taking $\Lambda = \langle \tilde{\alpha}_1 + \tilde{\alpha}_3 + \tilde{\alpha}_5 + \tilde{\alpha}_7 + \tilde{\alpha}_9 \rangle$ and match the two results up to order $O(q^7)$. Up to order $q$, the result is

$$\phi_{-2.5} = \frac{-2}{q} + \frac{1}{\Delta(q)} 12^6 \left( 30y_1^2 + 120y_1 + \frac{120}{y_1} + \frac{30}{y_1^2} + 180 \right) + q \left( 8y_1^5 + 980y_1^4 + 8520y_1^3 + 30580y_1^2 + 62320y_1 + \frac{62320}{y_1} + \frac{30580}{y_1^2} + \frac{8520}{y_1^3} + \frac{980}{y_1^4} + \frac{8}{y_1^5} + 78072 \right) + O(q^2). \tag{110}$$

Examples 6 and 7 : $\Lambda = A_2$ and $\Lambda = A_3$

One can also break $SU(9) \times SU(2)$ to a $SU(3)$ or $SU(4)$. For the $SU(3)$ case, the massless hypermultiplets of the six dimensional theory are

$$Hypers: 162(1) + 15(3) + 15(\bar{3}), \tag{111}$$

and for the $SU(4)$ case they are

$$Hypers: 139(1) + 12(4) + 12(\bar{4}) + 4(6). \tag{112}$$

This fixes the beginning of $\phi_{-2,A_2}$ and $\phi_{-2,A_3}$ as

$$\phi_{-2,A_2} = \frac{2}{q} + \left( -\frac{2y_2^2}{y_2} - 2y_2y_1 + \frac{30y_1}{y_2} - \frac{2y_1}{y_2^2} + 30y_1 + 30y_2 + \frac{30}{y_2} - \frac{2y_2^2}{y_1} + \frac{30y_2}{y_1} \right) \frac{-2}{y_2y_1} + \frac{30}{y_1} - \frac{2y_2}{y_1^2} + 312 \right), \tag{113}$$

$$
\phi_{-2,A_3} = \frac{2}{q} + \left( -\frac{2y_2^2}{y_2} - 2y_2y_1 + \frac{30y_1}{y_2} - \frac{2y_1}{y_2^2} + 30y_1 + 30y_2 + \frac{30}{y_2} - \frac{2y_2^2}{y_1} + \frac{30y_2}{y_1} \right) \frac{-2}{y_2y_1} + \frac{30}{y_1} - \frac{2y_2}{y_1^2} + 312 \right), \tag{113}$$

26
and

\[
\phi_{-2,A_3} = -\frac{2}{q} + \left(- \frac{2y_1^2}{y_2} + \frac{8y_3y_1}{y_2} - \frac{2y_3y_1}{y_2^2} - 2y_3y_1 + \frac{24y_1}{y_2} - \frac{2y_2y_1}{y_3} - \frac{2y_1}{y_3} + \frac{8y_1}{y_3} + 24y_1 \\
- \frac{2y_3^2}{y_2} + 8y_2 + \frac{24y_3}{y_2} + 24y_3 + \frac{8}{y_2} + \frac{24y_2}{y_3} + \frac{24y_2}{y_3^2} - \frac{2y_2}{y_1} - \frac{2y_2y_3}{y_1} - \frac{2y_3}{y_2y_1} \\
+ \frac{8y_3}{y_1} - \frac{2y_2}{y_3y_1} + \frac{8y_2}{y_3y_1} - \frac{2}{y_3y_1} + \frac{24}{y_1} - \frac{2y_2}{y_1^2} + 264\right) + O(q).
\]

(114)

Note that we pick \( \Lambda \) basis to be a set of simple roots and accordingly the exponents of \( y_i \) are given by fundamental weights associated with the irreducible representations in the spectrum.

Using the generators of Weyl invariant Jacobi forms for \( A_2 \) and \( A_3 \) root lattices (given in Appendix) and these two expressions we can fix the full Jacobi form to be

\[
\phi_{-2,A_2} = -\frac{2}{\Delta(q)} \frac{1}{144} \left(6\tilde{\phi}_{0,A_2}E_4E_6 + 5\tilde{\phi}_{-2,A_2}E_6^2 + 7\tilde{\phi}_{-2,A_2}E_4^3\right),
\]

(115)

and

\[
\phi_{-2,A_3} = -\frac{2}{\Delta(q)} \frac{1}{864} \left(4\tilde{\phi}_{0,A_3}E_4E_6 + 5\tilde{\phi}_{-2,A_3}E_6^2 + 7\tilde{\phi}_{-2,A_3}E_4^3 - 8\tilde{\phi}_{-4,A_3}E_4^2E_6\right).
\]

(116)

We can also compute the Jacobi form \( \phi_{-2,\Lambda} \) for these two cases using the orbifold limit. This orbifold computation was essentially done in [29]. By taking \( \Lambda = \langle \tilde{\alpha}_1, \tilde{\alpha}_2 \rangle \) and \( \Lambda = \langle \tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3 \rangle \) in our example we match the expressions given above up to order \( q^2 \).

### 2.5 Threshold Corrections

In this section, we will discuss the perturbative gauge and gravitational coupling constants in the low energy \( \mathcal{N} = 2 \) effective field theory. Their dependence on momentum scale and vector multiplet moduli are given by (if the gauge symmetry is created at level 1) [30, 2, 31]

\[
\frac{1}{g_{\text{gauge}}^2(p^2)} = \Re \left(-iS + \frac{1}{16\pi^2}\Delta_{\text{univ}}\right) + \frac{b_{\text{gauge}}}{16\pi^2} \log \frac{M_{\text{str}}^2}{p^2} + \frac{1}{16\pi^2}\Delta_{\text{gauge}},
\]

(117)

and

\[
\frac{1}{g_{\text{grav}}^2(p^2)} = 24\Re \left(-iS + \frac{1}{16\pi^2}\Delta_{\text{univ}}\right) + \frac{b_{\text{grav}}}{16\pi^2} \log \frac{M_{\text{str}}^2}{p^2} + \frac{1}{16\pi^2}\Delta_{\text{grav}}.
\]

(118)

Here, the one loop beta function coefficients are given as

\[
b_{\text{gauge}} = 2\text{Tr}_{\text{hyper}}(Q^2) - 2\text{Tr}_{\text{vector}}(Q^2),
\]

(119)

and

\[
b_{\text{grav}} = 46 + 2(N_h - N_v),
\]

(120)
and $\Delta^{\text{univ}}$ is coming from to the Green-Schwarz term. We will mostly use the conventions of [10] in this section.

In the previous sections, we described two Jacobi forms associated with $\Delta_{\text{grav}}$ and $\Delta_{\text{gauge}}$:

$$
\phi_{-2}(\tau, \vec{z}) = \sum_{n,k_i} c(n,k_i) q^n y_1^{k_1} \cdots y_s^{k_s} = \sum_{\mu \in \Lambda^*/\Lambda} \vartheta_{\Lambda,\mu}(\tau, \vec{z}) h_\mu(\tau), \quad (121)
$$

and

$$
\psi_0(\tau, \vec{z}) = \sum_{n,k_i} d(n,k_i) q^n y_1^{k_1} \cdots y_s^{k_s} = \sum_{\mu \in \Lambda^*/\Lambda} \vartheta_{\Lambda,\mu}(\tau, \vec{z}) f_\mu(\tau), \quad (122)
$$

from which we can compute gauge and gravitational threshold corrections by performing the integrals in (8) and (9).

At this point, it will be useful to define a positivity notion on the lattice $\Lambda^*$ and more generally on $\Lambda^* \otimes \mathbb{R}$. First we divide the space $\Lambda \otimes \mathbb{R} = \Lambda^* \otimes \mathbb{R}$ into two half spaces, one positive and one negative. For our examples, we will use a lexicographic ordering as follows. We decompose a vector, $\vec{v} \in \Lambda^* \otimes \mathbb{R}$, to its components with respect to the basis $\{\vec{\gamma}_i\}$ as $\vec{v} = b_i \vec{\gamma}_i$. Then, we declare a nonzero vector, $\vec{v}$, positive if

$$
b_1 = \ldots = b_{i-1} = 0 \text{ and } b_i > 0 \quad (123)
$$

is satisfied for at least one of $i = 1, 2, \ldots, s$. Similarly, when we say $(b_1, \ldots, b_s) \in \mathbb{Z}^s$ is positive we will mean that the same condition above is satisfied.

Now, following [7], we define

$$
C_0 = \frac{1}{2} \sum_{\vec{b}} c(0, \vec{b}), \quad (124)
$$

and

$$
C_{2n} = \sum_{\vec{b} > 0} c(0, \vec{b}) (b_i d^{ij} b_j)^n \text{ for } n \in \mathbb{Z}^+, \quad (125)
$$

where $c(\ldots)$ are the Fourier coefficients of $\phi_{-2,\Lambda}$. We will also use $\tilde{\phi}_{0,\Lambda} = \phi_{-2,\Lambda} E_2$, and use a notation $\tilde{C}_{2n}$ defined in a similar way as above, but this time using the Fourier coefficients, $\tilde{c}(n, \vec{b})$, of $\tilde{\phi}_{0,\Lambda}$. In the same way, we will write $D_{2n}$ for similar sums with respect to the Fourier coefficients, $d(n, \vec{b})$, of $\psi_{0,\Lambda}$.

Lastly, we will talk about the elements $r_a = (k, l, \vec{v} = b_l \vec{\gamma}_l)$ of $U \oplus \Lambda^*$ where $k, l, b_i \in \mathbb{Z}$. We will say $r$ is positive and write $r > 0$ when

$$
k > 0, \quad (126)
$$

or $k = 0$ and $l > 0$, \quad (127)

or $k = l = 0$ and $\vec{v} > 0$ (or equivalently $\vec{b} > 0$). \quad (128)
We define the vector with upper index as \( r^a = \eta^{ab} r_b \) where \( \eta^{ab} \) is the inverse of the metric \( \eta_{ab} \) satisfying \( y^a \eta_{ab} y^b = -2TU + V^i d_j V^j \). In the following, we will mean \( c(kl, \bar{b}) \) when we write \( c(r) \) and we will use the notation \( r.y \equiv r_a y^a = kT + lU + b_i V^i \).

Integrals giving the threshold corrections,

\[
\Delta_{\text{grav}} = \int_{\mathcal{F}} \frac{d^2 \tau}{\tau_2} \left[ \left( E_2(q) - \frac{3}{\pi \tau_2} \right) \sum_{\mu \in \Lambda^* / \Lambda} Z_{\Gamma, \mu} (\tau, T, U, V^i) h_\mu - \tilde{c}(0) \right],
\]

and

\[
24\Delta_{\text{gauge}} - \Delta_{\text{grav}} = \int_{\mathcal{F}} \frac{d^2 \tau}{\tau_2} \left[ \sum_{\mu \in \Lambda^* / \Lambda} Z_{\Gamma, \mu} (\tau, T, U, V^i) f_\mu - d(0) \right],
\]

can be evaluated using the work of [32] which generalizes threshold correction integrals in [3] to a wide class of automorphic integrands. The result we need can be basically read from [7].

The gravitational threshold correction is given as

\[
\Delta_{\text{grav}} = 4 \Re \left( \sum_{r > 0} \left[ \tilde{c}(r) \mathrm{Li}_1 (e^{2 \pi i r y}) + \frac{6}{\pi (y_2, y_2)} c(r) \mathcal{P}(r y) \right] \right) + \tilde{c}(0) \left[ - \log(- (y_2, y_2)) - K \right] + \frac{6}{\pi^2 (y_2, y_2)} c(0) \zeta(3) + 4 \pi \rho_a y^a + \frac{192 \pi}{y_2} \frac{1}{6} d_{abc} y^a y^b y^c.
\]

Here, \( K = \log \left( \frac{4 \pi}{\sqrt{2 \pi}} e^{1-\gamma_E} \right) \), where \( \gamma_E \approx 0.57721 \ldots \) is the Euler - Mascheroni constant,

\[
\frac{1}{6} d_{abc} y^a y^b y^c = - \frac{C_4}{4s(s + 2)} TV^i d_j V^j + \frac{C_0}{720} U^3 - \frac{C_2}{24s} UV^i d_j V^j + \frac{1}{12} \sum_{b > 0} c(0, \bar{b}) (b_i V^i)^3,
\]

and

\[
\rho_a y^a = \left( \frac{2C_2}{s} - \frac{12C_4}{s(s + 2)} \right) T + \frac{\tilde{c}_0}{6} U - \sum_{b > 0} c(0, \bar{b}) b_i V^i.
\]

Definitions for \( \mathcal{P}(x) \) and polylogarithms, \( \mathrm{Li}_n(x) \), are given in the Appendix.

To find the gauge threshold correction, we compute the difference

\[
24\Delta_{\text{gauge}} - \Delta_{\text{grav}} = 8\pi \kappa_a y^a + d(0) \left[ - \log(- (y_2, y_2)) - K \right] + 4 \sum_{r > 0} d(r) \Re \left[ \mathrm{Li}_1 (e^{2 \pi i r y}) \right],
\]

where

\[
\kappa_a y^a = \frac{D_2}{s} T + \frac{D_0}{12} U - \frac{1}{2} \sum_{b > 0} d(0, \bar{b}) b_i V^i.
\]

\footnote{Rotational symmetry property of the Jacobi form as in equations (2.14) and (2.15) of [7] is needed for the result given above to be correct and that can be easily checked for the particular examples we work with. More general arguments can be given along the lines of [33].}
We should note that when \( s = 1 \), the sum of \( \kappa a y a^2 \) part and the polylogarithm sum term is, up to an overall constant, \( \log |\Phi_\psi(T, U, V)| \), where \( \Phi_\psi(T, U, V) \) is a Siegel form which is the exponential or Borcherds lift of the weight zero Jacobi form \( \psi_0 \) [34, 35].

We can now use these results for \( \Delta_{\text{gauge}} \) and \( \Delta_{\text{grav}} \) in one-loop expressions for \( g_{\text{gauge}} \) and \( g_{\text{grav}} \) ((117) and (118)), and then compare them with the field theoretical expressions [31, 36]

\[
\frac{1}{g_{\text{gauge}}^2(p^2)} = \Re \left( -i\tilde{S} - \frac{1}{2(s+4)\pi^2} \log \Psi_{\text{gauge}} \right) + \frac{b_{\text{gauge}}}{16\pi^2} \left( \log \frac{M_{\text{Planck}}^2}{p^2} + K \right),
\]

and

\[
\frac{1}{g_{\text{grav}}^2(p^2)} = \Re \left( F_{1\text{het}} \right) + \frac{b_{\text{grav}}}{16\pi^2} \left( \log \frac{M_{\text{Planck}}^2}{p^2} + K \right).
\]

Here, \( K \) is the Kähler potential,

\[
K = -\log \Re (-iS) - \log (- (y_2, y_2)) + \text{const.}
\]

Planck scale, \( M_{\text{Planck}} \), is related to the string scale, \( M_{\text{str}} \), by

\[
M_{\text{Planck}}^2 = M_{\text{str}}^2 \Re (-iS).
\]

Finally, \( \tilde{S} \) and \( \Delta_{\text{univ}} \) are determined through the one-loop contribution to the prepotential, \( F_0^{(1)} \), as

\[
\frac{\Delta_{\text{univ}}}{16\pi^2} = \frac{1}{-(y_2, y_2)} \Re \left( F_0^{(1)} - iy^a \frac{\partial}{\partial y^a} F_0^{(1)} \right),
\]

and

\[
-i\tilde{S} = -iS - \frac{1}{s+4} \eta^{ab} \frac{\partial}{\partial y^a} \frac{\partial}{\partial y^b} F_0^{(1)}.
\]

Setting \( \frac{1}{16\pi^2} \log \Psi_{\text{gauge}} + \frac{b_{\text{gauge}}}{16\pi^2} \log (- (y_2, y_2)) \) equal to the T-duality invariant quantity \( \frac{1}{16\pi^2} (\frac{1}{s+4} \nabla^2 - 1) \Delta_{\text{gauge}} \) as in [10], one gets a differential equation for \( F_0^{(1)} \), which can be solved by

\[
F_0^{(1)} = -i \frac{1}{4\pi} \sum_{r>0} c(r) \text{Li}_3 \left( e^{2\pi i r y} \right),
\]

which together with the tree level contribution

\[
F_0^{(0)} = \frac{i}{2} S(y, y) = -iS(TU - \frac{1}{2} V^i d_{ij} V^j),
\]

gives the prepotential, \( F_{1\text{het}}^0 \), at the perturbative level (as further contributions are non-perturbative). The Laplacian is given by

\[
\nabla^2 = -2(y_2, y_2) \left( \eta^{ab} - \frac{2}{(y_2, y_2)^2} y^a_2 y^b_2 \right) \partial_a \partial_b.
\]
However, the differential equation has homogeneous solutions as well, and in particular, the cubic terms can be changed by terms of the form $-i\tilde{\rho}_a y^a(y, y)$ where $\tilde{\rho}_a$ are arbitrary real coefficients. This corresponds to the fact that dilaton, $S$, can be shifted by linear terms in $y^a$ (as discussed in [37]) without having physical consequences. We will fix the form appearing in the equation above by requiring no $TU$ factor to appear among the cubic terms.

With the prepotential found in this way, we can also determine the Wilsonian gravitational coupling, $F_1^{\text{het}}$, as

$$F_1^{\text{het}} = 24 (\text{Re} \{ -i S \}) + i \frac{\rho_a y^a}{4\pi} \sum_{r>0} \tilde{c}(r) \text{Li}_1 \left( e^{2\pi i r.y} \right) + \text{const}. \quad (145)$$

Lastly, we note that gravitational coupling constant can be written in the form

$$\frac{1}{g_{\text{grav}}^2(p^2)} = 24 \text{Re} \left( -i S \right) + \frac{b_{\text{grav}}}{16\pi^2} \left( \log \frac{M_{\text{str}}^2}{p^2} - \log (y_1^2 + y_2^2) \right) + \text{const}$$

$$+ \frac{6}{(s+4)\pi^2} \left[ \sum_{r>0} \left( -\frac{r_a r^a}{2} c(r) + \frac{s+4}{24} \tilde{c}(r) \right) \text{Re} \left( \text{Li}_1 \left( e^{2\pi i r.y} \right) \right) \right.$$

$$+ 2\pi \left( \frac{1}{2} d^{a}_{ae} y_e^a + \frac{s+4}{48} \rho_a y_2^a \right) \left. \right] \quad (146)$$

Working out the linear term in the brackets and using the fact that $\phi^{-2,\Lambda} = -\frac{2}{q} + O(1)$ as $\tau \to i\infty$, it is easy to verify that the term in the bracket comes from the exponential lift (in particular, for $s = 1$ case it is of the form $\log |\Phi_\phi(T, U, V)|$ where $\Phi_\phi(T, U, V)$ is a Siegel form) of a weight zero Jacobi form with Fourier coefficients

$$-\frac{r_a r^a}{2} c(r) + \frac{s+4}{24} \tilde{c}(r). \quad (147)$$

This weight zero Jacobi form can be obtained by $L_{-2,\phi^{-2,\Lambda}}$, where $L_k$ is the modified heat operator

$$L_k = q \frac{\partial}{\partial q} - \frac{1}{2} \sum_{a, b} \left( y^a \frac{\partial}{\partial y^a} \right) \eta^{ab} \left( y^b \frac{\partial}{\partial y^b} \right) + \frac{s-2k}{24} E_2, \quad (148)$$

mapping a weight $k$ Jacobi form to a weight $k + 2$ Jacobi form [7, 5]. [10]

**Weyl Chambers**

In [32] a generalization of the notion of Weyl chambers is used. Weyl chambers are defined as components of the $\{y^a\}$ space where integrals such as those involved in [49]

\[10\] This way of writing the coupling constants in terms of exponential lifts $\Phi_\psi$ and $\Phi_\phi$ is in accordance with the results of section 4 in [10].
and (8) are real analytic. In particular, equations (131) and (134) have terms such as \( \sum_{r > 0} c(r) \Re[\text{Li}_n(x)] \) which are obtained by replacing the infinite series expansion of polylogarithms by \( \text{Li}_n \) functions. This is valid if the variable, \( x \), has modulus less than unity. In other words, expressions in (131) and (134) are valid if the condition
\[
r.y^2 > 0
\]
is satisfied for every \( r > 0 \) with \( c(r) \neq 0 \). We should note that expressions in (131), (134) and Weyl Chambers associated with them depend on the ordering introduced on the lattice \( U \otimes \Lambda^* \). The cubic terms in the integral (again suppose that we eliminate \( TU \) terms along with quadratic and quartic ambiguities as discussed in [3]) change when one crosses from one Weyl chamber to another. This change can also be computed by analytically continuing the polylogarithms across the boundaries of the regions introduced above.

Now, we will try to characterize a set of simpler conditions on \( y_2 \) so that it is ensured that integrals are real analytic on regions satisfying those simpler conditions. A piece of information we use is that Fourier coefficients involved in the integrals are zero unless \( r.r = -2k + b_i d_j b_j \leq 2 \). Therefore, we will be looking for \( s + 2 \) vectors \( r_\mu \) where \( \mu = -1, 0, \ldots, s \) such that \( r_\mu r_\mu \leq 2 \) and \( r_\mu > 0 \) for all \( \mu \) and moreover every vector \( r \) satisfying \( r.r \leq 2 \) and \( r > 0 \) can be decomposed as
\[
r = a_\mu r_\mu,
\]
where \( a_\mu \)'s are \( s + 2 \) nonnegative integers. The problem of existence and uniqueness of such a basis (given an ordering) is a generalization of similar problems studied in [3] and [38].

We will start by supposing that the finite set
\[
S_\Lambda = \{ \vec{b} \in \Lambda^* | \vec{b}^2 \leq 2 \text{ and } \vec{b} > 0 \}
\]
has elements \( \vec{c}_i \in S_\Lambda \ (i = 1, 2, \ldots, s) \) such that they create the elements of \( S_\Lambda \) with nonnegative coefficients and they form an integral basis for \( \Lambda^* \). Furthermore, the components of the most positive element \( \vec{\theta} \) of \( S_\Lambda \), \( \vec{\theta}[i] \), should satisfy
\[
\vec{\theta}[i] \geq \vec{b}[i]
\]
for all elements \( \vec{b} \) in \( S_\Lambda \). Such a basis is unique if it exists and it is possible to construct it as follows. First, one should put the positive elements of \( S_\Lambda \) in increasing order. Then, starting with the smallest element, one inductively adds an element in this list to the basis set if it increases the rank of the vectors chosen at this point by one. At the end, one should check whether the conditions above are satisfied.

We start the construction of \( r_\mu \) basis by noting that vectors, \( (0, 1, \vec{b}) \) and \( (0, 0, \vec{b}) \) for \( \vec{b} \in S_\Lambda \), satisfy \( r.r \leq 2 \) and \( r > 0 \). Since such vectors span a \( s + 1 \) dimensional space over reals and since no positive vector \( r \) can have negative \( k \), there is only one basis vector with nonnegative \( k \), which we will choose to be \( r_{-1} \). Moreover, vectors \( (0, 1, \vec{b}) \) and \( (0, 0, \vec{b}) \) are spanned by \( r_0, r_1, \ldots, r_s \). Similarly, since vectors \( (0, 0, \vec{b}) \) span a \( s \) dimensional space over
reals and since all of \( r_0, r_1, \ldots, r_s \) has \( k = 0 \), only one of them can have nonzero \( l \). Let us choose this basis vector to be \( r_0 \). Then, the uniqueness of \( \vec{w}_i \) basis gives
\[
    r_i = (0, 0, \vec{w}_i) \quad \text{for } i = 1, 2, \ldots, s.
\] (153)

Next, from the fact that vectors \( r = (0, 1, -\vec{b}) \) satisfy \( r . r \leq 2 \) and \( r > 0 \), and the requirement that such vectors should be spanned by \( r_0, r_1, \ldots, r_s \) with nonnegative coefficients, \( r_0 \) is fixed to be
\[
    r_0 = (0, 1, -\vec{b}).
\] (154)

Finally, since there are vectors \( r \) satisfying our conditions with negative \( l \), \( r_{-1} \) should have negative \( l \) and this is only possible if we fix
\[
    r_{-1} = (1, -1, \vec{b}).
\] (155)

At this point, the \( r_\mu \)'s chosen so far span vectors \( r \) satisfying \( r . r \leq 2 \), \( r > 0 \), and \( kl = -1 \) or 0, with positive integer coefficients. We can prove that this is also valid for vectors with \( kl > 0 \) provided that
\[
    (\vec{\xi}_i, \vec{\theta})^2 \geq (\vec{\xi}_i)^2,
\] (156)
where \( \vec{\xi}_i \) are vectors satisfying \( \vec{\xi}_i, \vec{w}_j = \delta_{ij} \). We can prove this assertion as follows. Suppose \( r = (k, l, \vec{b}) \), where \( k, l > 0 \) and \( |\vec{b}|^2 \leq 2(kl + 1) \). Since
\[
    r = kr_{-1} + (k + l)r_0 + (0, 0, \vec{b} + (k + l)\vec{\theta}),
\] (157)
we should be able to write the vector \( a_i \vec{w}_i \equiv \vec{b} + (k + l)\vec{\theta} \) in \( \vec{w}_i \) basis with nonnegative integer coefficients (\( a_i \geq 0 \)). To pick the coefficients \( a_j \), we multiply this vector with \( \vec{\xi}_j \).
\[
    a_j = \vec{\xi}_j \cdot \vec{b} + (k + l)\vec{\xi}_j \cdot \vec{\theta}.
\] (158)

By the Cauchy-Schwarz inequality
\[
    |\vec{\xi}_j \cdot \vec{b}| \leq \sqrt{\vec{\xi}_j^2 |\vec{b}|^2} \leq |\vec{\xi}_j| \sqrt{2kl + 2}.
\] (159)
Therefore,
\[
    a_j \geq (k + l)\vec{\xi}_j \cdot \vec{\theta} - |\vec{\xi}_j| \sqrt{2kl + 2},
\] (160)
which can be proved to be nonnegative for \( k, l > 0 \) given that \( (\vec{\xi}_i, \vec{\theta})^2 \geq (\vec{\xi}_i)^2 \).

Finally, having found these basis vectors, we can assert that the integral expressions in threshold corrections are valid in the chamber
\[
    r_\mu . y_2 > 0 \quad \text{for all } \mu = -1, 0, \ldots, s.
\] (161)

Note that sums of the form \( \sum_{r > 0} c(r) \text{Li}_n(.) \) can be rewritten as \( \sum_{a_\mu \in N^{s+2}} c(a_\mu r^\mu) \text{Li}_n(.) \) if we introduce the set \( N^{s+2} \equiv \mathbb{Z}_0^+ \cdot \mathbb{Z}^{s+2} - \{(0, \ldots, 0)\} \).
Example 1: $\Lambda = \langle 2m \rangle$ with $m \in \mathbb{Z}^+$

If $\Lambda = \langle 2m \rangle$, the dual lattice is $\Lambda^* = \langle 1/2m \rangle$. Let us take a generator of $\Lambda^*$, $\vec{\gamma}$, and denote any vector in $\Lambda^*$ by its component with respect to $\vec{\gamma}$. In other words, we take the generator to be the vector $\vec{\gamma} = (1)$ where it satisfies (1). Let us take a generator of $\Lambda^*$, $\vec{\gamma}$, and denote any vector in $\Lambda^*$ by its component with respect to $\vec{\gamma}$. In other words, we take the generator to be the vector $\vec{\gamma} = (1)$ where it satisfies (1). Let us take a generator of $\Lambda^*$, $\vec{\gamma}$, and denote any vector in $\Lambda^*$ by its component with respect to $\vec{\gamma}$. In other words, we take the generator to be the vector $\vec{\gamma} = (1)$ where it satisfies (1).

Now, we can check that the last requirement

$$\vec{\xi} \cdot \vec{\theta} = \lfloor \sqrt{4m} \rfloor \geq |\vec{\xi}| = \sqrt{2m}$$

is satisfied for all positive integers $m$.

In particular, for the $m = 2$ example discussed above, the cubic part of the prepotential and the linear part of the gravitational coupling is given by the following expression in the chamber $\mathfrak{T} > 3U > 23V^1 > 0$:

$$F_{0}^{\text{het,cub}} \equiv -4\pi S \frac{(y, y)}{2} + \frac{1}{6} d_{abc} y^a y^b y^c$$

$$= 4\pi S [TU - 2(V^1)^2] - 2T(V^1)^2 + \frac{U^3}{3} - 4U(V^1)^2 + 8(V^1)^3,$$

and

$$F_{1}^{\text{het,lin}} \equiv 24(4\pi S) + \rho_a y^a$$

$$= 24(4\pi S) + 24T + 44U - 96V^1.$$}

For our $m = 3$ example, we have the following quantities in the chamber $\mathfrak{T} > 3U > 33V^1 > 0$:

$$F_{0}^{\text{het,cub}} = 4\pi S [TU - 3(V^1)^2] - 3T(V^1)^2 + \frac{U^3}{3} - 6U(V^1)^2 + 14(V^1)^3,$$

and

$$F_{1}^{\text{het,lin}} = 24(4\pi S) + 24T + 44U - 132V^1.$$
Example 2 : $\Lambda = A_2$

For the $\Lambda = A_2$ model discussed previously, we can go through the same exercise and find

$$F_{0^{\text{het,cub}}} = 4\pi S \left[ TU - (V^1)^2 + V^2 V^1 - (V^2)^2 \right] - T(V^1)^2 - T(V^2)^2 + TV^1 V^2 + \frac{U^3}{3}$$

$$- 2U(V^1)^2 - 2U(V^2)^2 + 2UV^1 V^2 + \frac{10(V^1)^3}{3} + \frac{4(V^2)^3}{3} + 4V^1(V^2)^2 - 5(V^1)^2 V^2,$$

and

$$F_{1^{\text{het,lin}}} = 24(4\pi S) + 24T + 44U - 52V^1 - 4V^2,$$

in the chamber

$$\Im (T - U), \Im (U - 2V^1 + V^2), \Im (V^1 - 2V^2), \Im (V^2) > 0. \quad (172)$$

Example 3 : $\Lambda = A_3$

For our model with $\Lambda = A_3$, we have

$$F_{0^{\text{het,cub}}} = 4\pi S \left[ TU - (V^1)^2 + V^2 V^1 - (V^2)^2 - (V^3)^2 + V^2 V^3 \right] - T(V^1)^2 - T(V^2)^2$$

$$- T(V^3)^2 + TV^1 V^2 + TV^2 V^3 + \frac{U^3}{3} - 2U(V^1)^2 - 2U(V^2)^2 - 2U(V^3)^2$$

$$+ 2UV^1 V^2 + 2UV^2 V^3 + \frac{10(V^1)^3}{3} + \frac{4(V^2)^3}{3} + \frac{4(V^3)^3}{3} + 4V^1(V^2)^2 + 2V^1(V^3)^2 + 3V^2(V^3)^2 - 5(V^1)^2 V^2 - 4(V^2)^2 V^3 - 2V^1 V^2 V^3,$$

and

$$F_{1^{\text{het,lin}}} = 24(4\pi S) + 24T + 44U - 52V^1 - 4V^2 - 4V^3,$$

in the chamber

$$\Im (T - U), \Im (U - 2V^1 + V^2), \Im (V^1 - 2V^2 + V^3), \Im (V^2 - 2V^3), \Im (V^3) > 0. \quad (173)$$

2.6 Models with a Single Wilson Line and Paramodular Groups

In this section, we will gather details on the $\Lambda = \langle 2m \rangle$ case and discuss their relation to paramodular groups. Pieces of such details have already appeared in the preceding sections; however, it will be useful to put these together both because $s = 1$ case is the simplest example of $\mathcal{N} = 2$ heterotic compactifications with Wilson lines that also has partially appeared in the literature and because studying the implications of the T-duality group, which is a paramodular group in this case, on a possible Type-II dual partner would be easier for this relatively simple case.
The classical vector multiplet moduli space for this case is $SO(3, 2)/SO(3) \otimes SO(2)$ when we separate the factor coming from the axio-dilaton. This space can be parametrized by three complex moduli $T$, $U$, and $V$. To compare with the previous sections we set $y = (T, U, V \xi)$ where $\xi$ is a generator for the lattice $\Lambda = \langle 2m \rangle$, in other words $\xi \cdot \xi = 2m$. State charges, on the other hand, will be of the form $(m_1, m_2, n_1, n_2, b\gamma) \in (U \oplus U \oplus \Lambda^*)$, where $\gamma$ is a generator of the dual lattice $\Lambda^* = \langle 1/2m \rangle$, and hence charges can be parametrized by five integers $m_1$, $m_2$, $n_1$, $n_2$ and $b$. Then, automorphisms of this lattice which preserve the norm

$$\frac{p_L^2 - p_R^2}{2} = \frac{b^2}{2} \gamma \cdot \gamma - m_1 n_1 + m_2 n_2 = \frac{b^2}{4m} - m_1 n_1 + m_2 n_2,$$

as well as the conjugacy class in $\Lambda^*/\Lambda$ (or in this case $b \bmod 2m$) induce T-duality transformations together with the transformation changing the sign of all charges. We can generate these transformations from four basic operations as explained in the subsequent paragraphs [21].

The first two are generated by the automorphisms

$$m_1 \rightarrow m_1 + n_2 \quad \text{and} \quad m_2 \rightarrow n_1 + m_2,$$

and

$$m_2 \rightarrow -n_1, \quad n_1 \rightarrow m_2, \quad m_1 \rightarrow -n_2, \quad \text{and} \quad n_2 \rightarrow m_1$$

which induce

$$T \rightarrow T + 1,$$

and

$$T \rightarrow -\frac{1}{T}, \quad U \rightarrow U - m \frac{V^2}{T}, \quad V \rightarrow \frac{V}{T},$$

respectively. We see that these two transformations together generate a subgroup $SL(2, \mathbb{Z})_T$ which acts on the moduli as

$$T \rightarrow \frac{aT + b}{cT + d}, \quad U \rightarrow U - \frac{mcV^2}{cT + d}, \quad V \rightarrow \frac{V}{cT + d},$$

where $a, b, c, d$ are integers satisfying $ad - bc = 1$. Note that by picking $a = d = -1$ and $b = c = 0$ one gets the transformation $V \rightarrow -V$.

The next T-duality transformation is induced by the automorphism

$$m_2 \rightarrow m_2 - \alpha^2 mn_2 + ab, \quad n_1 \rightarrow n_1 + \lambda^2 mm_1 - 2\lambda amn_2 + \lambda b \quad b \rightarrow b + 2\lambda mm_1 - 2\alpha mn_2,$$

where $\lambda, \alpha \in \mathbb{Z}$, and is given by

$$T \rightarrow T \quad U \rightarrow U + \lambda^2 mT + 2\lambda mV \quad V \rightarrow V + \lambda T + \alpha.$$
The final transformation is induced by
\[ m_1 \leftrightarrow n_1 \] (186)
which gives
\[ T \leftrightarrow U. \] (187)

The threshold corrections, then, involve automorphic forms of the T-duality transformations induced by (183), (185) and (187). In particular, equations (134) and (146) involve Siegel forms \( \Phi_\psi(Z) \) and \( \Phi_\phi(Z) \) which are functions on the genus 2 Siegel upper half plane \( \mathcal{H}_2 \), where we parametrize the elements \( Z \) as
\[ Z = \begin{pmatrix} p & q \\ q & r \end{pmatrix} \equiv \begin{pmatrix} T & V \\ V & U/m \end{pmatrix} \in \mathcal{H}_2, \] (188)
for \( \Im p > 0, \Im r > 0 \) and \( \Im \det Z > 0 \). These two functions are Siegel forms over a discrete subgroup of \( Sp(4, \mathbb{R}) \) which can be identified with the T-duality group up to an overall sign. This discrete subgroup is a particular semi-direct product of \( \mathbb{Z}_2 \) with \( \Gamma_m \) for \( m > 1 \), where \( \Gamma_m \) is a paramodular group. We will call this semi-direct product an extended paramodular group and denote it by \( \Gamma_m^+ \). For \( m = 1 \), the action of \( \mathbb{Z}_2 \) (which comes from the \( T \leftrightarrow U \) exchange symmetry) is already included in \( \Gamma_1 \) which is isomorphic to \( Sp(4, \mathbb{Z}) \).

We can now study the particular heterotic models with \( m = 1, \ldots, 5 \) we studied in the previous sections. In equation (146) it was noted that (for \( s = 1 \))
\[ \frac{1}{g_{grav}^2(p^2)} = 24\Re \left( -iS \right) + \frac{\tilde{c}(0)}{16\pi^2} \left( \log \frac{M_{str}^2}{p^2} - \log \left( -(y_2, y_2) \right) \right) + \text{const} \]
\[ + \frac{1}{10\pi^2} \log |\Phi_m(Z)|, \] (189)
where the Siegel form, \( \Phi_m(Z) \) is the exponential lift\(^{11} \) of the weight zero, index \( m \), nearly holomorphic Jacobi form
\[ \chi_{0,m} \equiv 12L_{-2}^{\phi_{-2,m}}. \] (190)
From physical requirements it is easy to see that the Fourier coefficients of \( \chi_{0,m} \), which we will denote as \( f(n, r) \), are integral. We will also use the notation \( \Phi_m = \text{Exp-Lift}(\chi_{0,m}) \) for exponential lift.

It is also found in \([35]\) that the divisors of \( \Phi_m \) on \( \mathcal{H}_2/\Gamma_m^+ \) are Humbert surfaces
\[ H_D(b) = \pi_m \left( \{ Z \in \mathcal{H}_2 | aT + bV + U = 0 \} \right), \] (191)
where \( \pi_m \) projects from \( \mathcal{H}_2 \) to \( \mathcal{H}_2/\Gamma_m^+ \), the discriminant \( D > 0 \) is defined as \( D = b^2 - 4ma \), \( b \) can be restricted to particular representatives of \( \pm b (\text{mod } 2m) \) and divisor multiplicities are given by
\[ m_{D,b} = \sum_{n>0} f(n^2a, nb). \] (192)

---

\(^{11}\)See theorem 2.1 of \([35]\).
Physically, these are T-duality inequivalent surfaces on the vector multiplet moduli space on which there are BPS states that become massless hence creating a singularity in $\log \Phi_m$.

$m = 1$ Example

From equation (92) we can compute $\chi_{0,1}$ as

$$\chi_{0,1} = \frac{19}{q} + \left( 19y^2 - 28y + 1050 - \frac{28}{y} + \frac{19}{y^2} \right) + \left( 1050y^2 + 617088y + 2504520 + \frac{617088}{y} + \frac{1050}{y^2} \right)q + \ldots. \quad (193)$$

The Siegel form it lifts to has its divisor as $19H_4 - 9H_1$. In the notation of [35] and [41], $\Delta_{30}(Z)$ and $\Delta_5(Z)$ are Siegel forms of $Sp(4, \mathbb{Z})$ with divisors $H_4$ and $H_1$, respectively. This gives

$$\log \Phi_1 = 19 \log \Delta_{30} - 9 \log \Delta_5 + \text{const.} \quad (194)$$

Moreover $\Delta_{30} = \text{Exp-Lift}(\rho_{0,1})$ where

$$\rho_{0,1} = \frac{1}{q} + \left( y^2 - y + 60 - \frac{1}{y} + \frac{1}{y^2} \right) + \ldots, \quad (195)$$

and $\Delta_5 = \text{Exp-Lift}(\kappa^1_{0,1})$ where

$$\kappa^1_{0,1} = \tilde{\phi}_{0,1} = \left( y + 10 + \frac{1}{y} \right) + \frac{2(y - 1)^2 (5y^2 - 22y + 5)}{y^2} + \ldots. \quad (196)$$

This finally implies that

$$12\mathcal{L}_{-2}\phi_{-2,1} = 19\rho_{0,1} - 9\kappa^1_{0,1}. \quad (197)$$

This relation was first noticed in the context of threshold corrections in [4].

$m = 2$ Example

Equation (97) gives $\chi_{0,2}$ as

$$\chi_{0,2} = \frac{19}{q} + \left( 96y + 840 + \frac{96}{y} \right) + \left( 19y^4 + 96y^3 + 86632y^2 + 894880y + 1777542 \right) + \left( \frac{894880}{y} + \frac{86632}{y^2} + \frac{96}{y^3} + \frac{19}{y^4} \right)q + \ldots. \quad (198)$$

The Siegel form it lifts to has its divisor as $19H_8 + 96H_1$ on $\mathcal{H}_2 / \Gamma^+_2$. [35] introduces $\Psi_{12}^{(2)}(Z)$ and $\Delta_2(Z)$ which are Siegel forms of $\Gamma^+_2$ with divisors $H_8$ and $H_1$, respectively. This gives

$$\log \Phi_2 = 19 \log \Psi_{12}^{(2)} + 96 \log \Delta_2 + \text{const.} \quad (199)$$
Moreover, $\Psi_{12}^{(2)} = \text{Exp-Lift}(\rho_{0,2})$ where

$$\rho_{0,2} = \frac{1}{q} + 24 + \ldots,$$

and $\Delta_2 = \text{Exp-Lift}(\kappa_{0,2}^1)$ where

$$\kappa_{0,2}^1 = \left( y + 4 + \frac{1}{y} \right) + \ldots.$$ (201)

This finally implies that

$$12 \mathcal{L}_{-2} \phi_{-2,2} = 19 \rho_{0,2} + 96 \kappa_{0,2}^1.$$ (202)

In fact, [6] uses exactly this argument to find out the complete form of the threshold corrections in $m = 2$ case. This argument is closely related to the arguments we used in the previous sections to match Jacobi forms coming from the index computation.

**m = 3 Example**

Equation (101) gives $\chi_{0,3}$ as

$$\chi_{0,3} = \frac{19}{q} + \left( -9y^2 + 180y + 690 + \frac{180}{y} - \frac{9}{y^2} \right) + \left( -9y^4 + 9768y^3 + 212706y^2 + 925272y + 1445322 + \frac{925272}{y} + \frac{212706}{y^2} + \frac{9768}{y^3} - \frac{9}{y^4} \right)q + \ldots.$$ (203)

The Siegel form it lifts to has its divisor as $19H_{12} + 171H_1 - 9H_4$ on $H_2/\Gamma_3^+$. [35] introduces $\Psi_{12}^{(3)}(Z), \Delta_{1}(Z), \text{and } F_{2}^{(3)}(Z)$ which are Siegel forms of $\Gamma_3^+$ with divisors $H_{12}, H_1$ and $H_4$, respectively. This gives

$$\log \Phi_3 = 19 \log \Psi_{12}^{(3)} + 171 \log \Delta_1 - 9 \log F_2^{(3)} + \text{const.}$$ (204)

Furthermore, $\Psi_{12}^{(3)} = \text{Exp-Lift}(\rho_{0,3})$ where

$$\rho_{0,3} = \frac{1}{q} + 24 + \ldots,$$ (205)

$\Delta_1 = \text{Exp-Lift}(\kappa_{0,3}^1)$ where

$$\kappa_{0,3}^1 = \left( y + 2 + \frac{1}{y} \right) + \ldots,$$ (206)

and $F_2^{(3)} = \text{Exp-Lift}(\kappa_{0,3}^2)$ where

$$\kappa_{0,3}^2 = \left( y^2 - y + 12 - \frac{1}{y} + \frac{1}{y^2} \right) + \ldots.$$ (207)

This finally implies that

$$12 \mathcal{L}_{-2} \phi_{-2,3} = 19 \rho_{0,3} + 171 \kappa_{0,3}^1 - 9 \kappa_{0,3}^2,$$ (208)

which can be checked with the explicit expressions of Jacobi forms we have computed.
m = 4 Example

Equation (105) gives $\chi_{0,4}$ as

$$\chi_{0,4} = \frac{19}{q} + \left( -8y^2 + 224y + 600 \frac{224}{y} - \frac{8}{y^2} \right) + \left( 570y^4 + 38688y^3 + 308160y^2 + 895200y ight. $$

$$+ 1255560 \frac{895200}{y} + \frac{308160}{y^2} + \frac{38688}{y^3} + \frac{570}{y^4} \Bigg) q + \ldots$$

(209)

The Siegel form it lifts to has its divisor as $19\mathcal{H}_{16}(0) + 216\mathcal{H}_1 - 8\mathcal{H}_4$ on $\mathcal{H}_2/\Gamma_4^+$. It introduces $\Psi_{12}^{(4)}(Z)$, $\Delta_{1/2}(Z)$, and $F_2^{(4)}(Z)$ which are Siegel forms of $\Gamma_4^+$ with divisors $\mathcal{H}_{16}(0)$, $\mathcal{H}_1$ and $\mathcal{H}_4$, respectively. This gives

$$\log \Phi_4 = 19\log \Psi_{12}^{(4)} + 216\log \Delta_{1/2} - 8\log F_2^{(4)} + \text{const.}$$

(210)

Further, $\Psi_{12}^{(4)} = \text{Exp-Lift}(\rho_{0,4})$ where

$$\rho_{0,4} = \frac{1}{q} + 24 + \ldots,$$

(211)

$\Delta_{1/2} = \text{Exp-Lift}(\kappa_{0,4}^1)$ where

$$\kappa_{0,4}^1 = \left( y + 1 + \frac{1}{y} \right) + \ldots,$$

(212)

and $F_2^{(4)} = \text{Exp-Lift}(\kappa_{0,4}^2)$ where

$$\kappa_{0,4}^2 = \left( y^2 - y + 9 - \frac{1}{y} + \frac{1}{y^2} \right) + \ldots$$

(213)

This implies that

$$12\mathcal{L}_{-2} \phi_{-2,4} = 19\rho_{0,4} + 216\kappa_{0,4}^1 - 8\kappa_{0,4}^2,$$

(214)

which can again be checked with the explicit expressions of Jacobi forms we have computed.

m = 5 Example

Equation (109) gives $\chi_{0,5}$ as

$$\chi_{0,5} = \frac{19}{q} + \left( 3y^2 + 228y + 570 \frac{228}{y} + \frac{3}{y^2} \right) + \left( -4y^5 + 4802y^4 + 77532y^3 + 368218y^2 ight. $$

$$+ 859048y + 1121604 \frac{859048}{y} + \frac{368218}{y^2} + \frac{77532}{y^3} + \frac{4802}{y^4} - \frac{4}{y^5} \Bigg) q + \ldots$$

(215)
For \( m = 5 \), a nearly holomorphic Jacobi form with at most \( 1/q \) pole can lift to a Siegel form which has the Humbert surface \( H_5 \) with \( b = 5 \) and \( a = 1 \) among its divisors. The multiplicity in the divisor associated with this surface is given as

\[
m_{5,5} = g(1, 5) + g(4, 10) + g(9, 15)
\]

(216)

if \( g(n, r) \) are Fourier coefficients of the lifting Jacobi form, which are integers.\(^{12}\) The divisor of the Siegel form it lifts to is \( 19H_{20} + 231H_1 + 3H_4 + 15H_5 \) on \( \mathcal{H}_2/\Gamma_5^+ \). In contrast to the previous cases, there are no index 5 Jacobi forms with a divisor purely on \( H_{20} \), \( H_1 \) or \( H_4 \). However, one can still find Jacobi forms \( \rho_{0,5}, \kappa_{0,5}^1 \) and \( \kappa_{0,5}^2 \) of the form

\[
\rho_{0,5} = \frac{1}{q} + 24 + \ldots,
\]

(217)

\[
\kappa_{0,5}^1 = \left(5y + 2 + \frac{5}{y}\right) + \ldots,
\]

(218)

and

\[
\kappa_{0,5}^2 = \left(5y^2 - 5y + 36 - \frac{5}{y} + \frac{5}{y^2}\right) + \ldots.
\]

(219)

The divisors of the Siegel forms they exponentially lift to are \( H_{20} + 3H_5, 5H_1 - H_5 \) and \( 5H_4 + 7H_5 \). Then, one concludes that

\[
5 \log \Phi_5 = 95 \log \text{Exp-Lift}(\rho_{0,5}) + 231 \log \text{Exp-Lift}(\kappa_{0,5}^1) + 3 \log \text{Exp-Lift}(\kappa_{0,5}^2) + \text{const}.
\]

(220)

This then implies

\[
12L_{-2}\phi_{-2,5} = 19\rho_{0,5} + \frac{231}{5} \kappa_{0,5}^1 + \frac{3}{5} \kappa_{0,5}^2.
\]

(221)

which we compare and check with the explicit expressions of Jacobi forms we have computed.

### 3 Type IIA - Heterotic String Duality

In the previous chapter, our discussion was exclusively on \( \mathcal{N} = 2, D = 4 \) heterotic string models. Now, we can shift our focus to the dual story for a Type IIA theory compactified on an appropriate Calabi-Yau threefold. First examples of this duality are discussed in [10] and [42]. After the first examples, various chains of heterotic models with duals on Calabi-Yau threefolds have been constructed [11, 43, 12, 44]. Our discussion in this section will be conjectural as compared to the previous sections.

\(^{12}\)Terms of the form \( g(n^2, 5n) \) with \( n > 3 \) are zero and hence do not contribute to \( m_{5,5} \). This can be basically proved by the elliptical transformation property [5]. For a nearly holomorphic Jacobi form with no \( q \) pole more severe than \( 1/q \), \( g(n, r) \) becomes vanishing as soon as \( r^2 - 4nm > m^2 + 4m \).
The numbers of vector multiplet and hypermultiplet moduli in a Type II A compactification on a Calabi-Yau is determined by the topology of the manifold as 

\[ N_v = h^{1,1}, \quad N_h = h^{2,1} + 1. \]

In particular, if the heterotic models described in the previous chapter have duals on Calabi-Yau threefolds we can identify

\[ h^{1,1} = s + 3 \quad \text{and} \quad c(0) = 2(N_h - N_v - 1) = 2(h^{2,1} - h^{1,1}) = -\chi, \]

where \( \chi \) is the Euler number associated with the compactification manifold.

Vector moduli are given by expanding the complexified Kähler form in terms of integral Kähler class generators \( J_1, \ldots, J_{s+3} \) as

\[ B + iJ = \sum_i t_i J_i, \]

where \( B \) is the antisymmetric field, \( J \) is the Kähler form, and vector multiplet moduli satisfy \( \exists (t_i) \geq 0 \) in the Kähler cone.

The prepotential and gravitational coupling are given in terms of the topological properties of the Calabi-Yau (CY) manifold and hence one can study the duality by comparing the results on heterotic side with this topological data as was done in [45, 4, 29, 28, 46] for models with Wilson lines. To be more concrete, let us give some definitions. We define triple intersection numbers as

\[ \kappa_{ijk} = \int J_i \wedge J_j \wedge J_k, \]

(224)

We will also need \( \kappa_i \) which are defined as

\[ \kappa_i = \int J_i \wedge C_2, \]

(225)

where \( C_2 \) is the second Chern class of the associated CY threefold.

The prepotential and gravitational couplings for the low energy effective field theory then can be given in terms of genus-0 and genus-1 Gromov-Witten potentials as in [47, 28, 7]

\[ F_{II}^0 = -\frac{i}{6} \sum_{i,j,k} \kappa_{ijk} t_i t_j t_k - \frac{1}{16\pi^3} \chi \zeta(3) + \frac{1}{8\pi^3} \sum_{d \in N^{s+3}} N^r(d) \text{Li}_3 (e^{2\pi i d.t}), \]

(226)

and

\[ F_{II}^1 = -i\pi \sum_i \kappa_i t_i + \sum_{d \in N^{s+3}} N^{r,e}(d) \text{Li}_1 (e^{2\pi i d.t}). \]

(227)

Here, \( N^r(d) \) are the worldsheet instanton numbers for rational curves of degree \( d \). \( N^{r,e}(d) \) is defined as

\[ N^{r,e}(d) = N^r(d) + 12 \sum_{d' \in N^{s+3}} N^e(d'), \]

(228)
where $d'|d$ if $d = nd'$ for a positive integer $n$, and $N^e(d)$ is the worldsheet instanton numbers for elliptic curves of degree $d$.

From the factorization of the classical moduli space into two parts, one can show that the Calabi-Yau manifold on the dual side is a K3 fibration [47, 48, 49]. Moreover, the area of the base of this fibration is controlled by the $SL(2, \mathbb{R})/U(1)$ factor in the moduli space or in other words by the axio-dilaton, $S$, up to a piece linear in the other vector multiplet moduli. Therefore, the perturbative limit $\Im S \to \infty$ corresponds to the case where the area of the fibration base is becoming large. Therefore, among the instanton corrections only those having degree 0 on the base survive.

In most of the duality papers cited above, the vector multiplet moduli mapping is accomplished by comparing the cubic parts in the prepotential. The work of [7], however, conjectures a mapping provided that the lattice $\Lambda$ is either a root lattice of a simple Lie algebra or a scaled version of a root lattice by a positive integer. In particular, Kähler class for the base is mapped as

$$t_2 = 4\pi S - U - \frac{n}{2}(T - U).$$

This is obtained by assuming that the CY threefold is also an elliptic fibration over the Hirzebruch surface $\mathbb{F}_n$. On the heterotic side, $n$ can be read from the distribution of instantons $(12 + n, 12 - n)$ if it is a smooth compactification. In the following, we will restrict to the models that can be obtained starting from the $SU(6)$ model in [6]. Since this can also be obtained in a smooth compactification with $n = 2$, we will take $n = 2$ in the examples we discuss below.

The form of $F_0^{II}$ and $F_1^{II}$ suggests that we identify them with the perturbative heterotic results as

$$F_0^{II}|_{\Im t_2 \to \infty} = 4\pi F_0^{het} \quad \text{and} \quad F_1^{II}|_{\Im t_2 \to \infty} = 4\pi^2 F_1^{het},$$

and that we identify vector multiplet moduli as

$$t_1 = r_0.y = U - \vec{\theta}.(V^j \vec{\beta}_j),$$
$$t_3 = r_{-1}.y = T - U,$$
$$t_i + 3 = r_i.y = \vec{\alpha}_i.(V^j \vec{\beta}_j) \quad \text{for} \quad i = 1, 2, \ldots, s.$$ 

In the following, we will use these mappings for some of the heterotic models we studied.

**Example 1 :** $\Lambda = \langle 4 \rangle$

$$t_1 \to U - 2V^1, t_2 \to S - T, t_3 \to T - U, t_4 \to V^1,$$

$$\sum_i \kappa_i t_i = 92t_1 + 24t_2 + 48t_3 + 88t_4,$$

13 This is up to an ambiguity left in equation 5.27.
\[
\frac{1}{6} \sum_{i,j,k} \kappa_{ijk} t_i t_j t_k = \frac{4t_1^3}{3} + t_2 t_1^2 + 2t_3 t_1^2 + 8t_4 t_1^2 + t_3^2 t_1 + 8t_4^2 t_1 + t_2 t_3 t_1 + 4t_2 t_4 t_1 \\
+ 8t_3 t_4 t_1 + \frac{8t_4^3}{3} + 2t_2 t_4^2 + 4t_3 t_4^2 + 2t_3^2 t_4 + 2t_2 t_3 t_4. \quad (236)
\]

One can now read the instanton numbers \( N^\nu(d) \) and \( N^c(d) \) using the weight -2 Jacobi form \( \phi_{-2,2} \). This example was previously studied in [7].

**Example 2 : \( \Lambda = \langle 6 \rangle \)**

\[
t_1 \to U - 3V^1, t_2 \to S - T, t_3 \to T - U, t_4 \to V^1, \\
\sum_i \kappa_i t_i = 92t_1 + 24t_2 + 48t_3 + 144t_4, \quad (237)
\]

\[
\frac{1}{6} \sum_{i,j,k} \kappa_{ijk} t_i t_j t_k = \frac{4t_1^3}{3} + t_2 t_1^2 + 2t_3 t_1^2 + 12t_4 t_1^2 + t_3^2 t_1 + t_2 t_3 t_1 + 6t_2 t_4 t_1 \\
+ 12t_3 t_4 t_1 + 14t_4^3 + 6t_2 t_4^2 + 12t_3 t_4^2 + 3t_3^2 t_4 + 3t_2 t_3 t_4. \quad (239)
\]

**Example 3 : \( \Lambda = A_2 \)**

\[
t_1 \to U - 2V^1 + V^2, t_2 \to S - T, t_3 \to T - U, t_4 \to V^1 - 2V^2, t_5 \to V^2, \\
\sum_i \kappa_i t_i = 92t_1 + 24t_2 + 48t_3 + 132t_4 + 168t_5, \quad (240)
\]

\[
\frac{1}{6} \sum_{i,j,k} \kappa_{ijk} t_i t_j t_k = \frac{4t_1^3}{3} + t_2 t_1^2 + 2t_3 t_1^2 + 8t_4 t_1^2 + 12t_5 t_1^2 + t_3^2 t_1 + 12t_4^2 t_1 + 24t_5^2 t_1 \\
+ t_2 t_3 t_1 + 4t_2 t_4 t_1 + 8t_3 t_4 t_1 + 6t_2 t_5 t_1 + 12t_3 t_5 t_1 + 36t_4 t_5 t_1 + 6t_4^3 + 16t_5^3 \\
+ 3t_2 t_4^2 + 6t_3 t_4^2 + 6t_2 t_5^2 + 12t_3 t_5^2 + 36t_4 t_5^2 + 2t_3^2 t_4 + 2t_2 t_3 t_4 + 3t_3^2 t_5 \\
+ 27t_2 t_5 + 3t_2 t_3 t_5 + 9t_2 t_4 t_5 + 18t_3 t_4 t_5. \quad (242)
\]

Intersection numbers for \( \mathbb{P}^4(1,1,2,6,8)[18] \), which has Hodge numbers \((5,161)\), is given in [45] matching the result above. Heterotic computation at an orbifold point and moduli mapping is also given in [29], where the map was found by comparison to topological information given in [45].
Example 4 : $\Lambda = A_3$

\[t_1 \to U - 2V^1 + V^2, t_2 \to S - T, t_3 \to T - U, t_4 \to V_1 - 2V^2 + V^3, t_5 \to V^2 - 2V^3, t_6 \to V^3,\]
\[\sum_i \kappa_i t_i = 92t_1 + 24t_2 + 48t_3 + 132t_4 + 168t_5 + 200t_6, \quad (243)\]
\[\frac{1}{6} \sum_{i,j,k} \kappa_{ijk} t_i t_j t_k = \frac{4t_1^3}{3} + t_2t_1^2 + 2t_3t_1^2 + 8t_4t_1^2 + 12t_5t_1^2 + 16t_6t_1^2 + t_3^3 + 12t_4^2 t_1 + 24t_5^2 t_1
\]
\[+ 40t_6^2 t_1 + t_2t_3t_1 + 4t_2t_4t_1 + 8t_3t_4t_1 + 6t_2t_5t_1 + 12t_3t_5t_1 + 36t_4t_5t_1
\]
\[\begin{align*}
+ 8t_2t_6t_1 + 16t_3t_6t_1 + 48t_4t_6t_1 + 64t_5t_6t_1 & + 6t_3^3 + 16t_4^3 + 100t_5^3 + 3t_2^4 \\
+ 6t_4^2 t_2 + 12t_3t_5^2 + 36t_4t_5^2 + 10t_2t_5^2 & + 20t_3t_6^2 + 60t_4t_6^2 + 80t_5t_6^2 \\
+ 2t_3^4 & + 2t_2t_3t_4 + 3t_5^3 t_5 & + 27t_4^2 t_5 & + 3t_2t_5^3 & + 9t_2t_4t_5 & + 18t_3t_4t_5 & + 4t_4^3 t_6 \\
+ 36t_4^2 t_6 & + 4t_2t_3t_6 & + 12t_2t_4t_6 & + 24t_3t_4t_6 & + 16t_2t_5t_6 & + 32t_3t_5t_6 & + 96t_4t_5t_6. & (245)
\end{align*}\]

4 Discussion

In this work, we investigated the relation of the threshold corrections for $\mathcal{N} = 2$, $D = 4$ heterotic string compactifications with Wilson lines to Jacobi forms, where the Jacobi forms are over an even lattice and are possibly Jacobi forms of many variables. We showed that there are two kinds of Jacobi forms relevant in this context, a weight $-2$ Jacobi form coming from $\Delta_{\text{grav}}, \phi_{-2,\Lambda}(\tau, \bar{z})$, and a weight 0 Jacobi form coming from $24\Delta_{\text{gauge}} - \Delta_{\text{grav}}, \psi_{0,\Lambda}(\tau, \bar{z})$. The condition of being a Jacobi form is highly constraining since the vector space of Jacobi forms over a lattice is finite dimensional. If one can determine some of the coefficients in Fourier expansions of $\phi_{-2,\Lambda}(\tau, \bar{z})$ or $\psi_{0,\Lambda}(\tau, \bar{z})$, even without any information about the full BPS spectrum it may be possible to find out completely what these functions are using the finite dimensionality. We explored this idea in a number of examples which are connected to an orbifold model in hypermultiplet moduli space and tested the results using explicit computations of these Jacobi forms at the orbifold limit. An interesting future problem would be to generalize these methods to more general settings in which gauge symmetries of the low energy theory can arise. One then can test how constraining the condition of having Jacobi forms would be on the low energy effective theory. One should note that theories we consider are toroidal compactifications of $\mathcal{N} = 1$, $D = 6$ theories and finding constraints on such six dimensional theories using their toroidal compactifications is similar in spirit to the work [50].

We also computed threshold corrections and gave expressions for prepotential and gravitational coupling in terms of the Fourier coefficients of an appropriate Jacobi form. A detailed analysis of the Weyl chambers suggests extensions and clarifications on [7]'s conjectures.
on mapping heterotic vector multiplet moduli to the vector multiplet moduli of a possible
Type IIA dual. We studied this aspect using the examples we had on the heterotic side
and worked out the resulting moduli mappings and cubic prepotentials. This side certainly
deserves more attention to better understand the action of heterotic side’s T-duality group
in terms of more geometrical ideas on Calabi-Yau manifolds and to test Gromov-Witten
potentials obtained this way with explicit geometrical realizations. A possible first step for
this, which we worked out in detail on heterotic side, may be the rank one case where the
T-duality groups are extended paramodular groups, \( \Gamma^+_m \), and associated Jacobi forms are
Jacobi forms in the sense of [5].

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Appendix: Definitions and Conventions

In the main text and in the following we frequently use the notation \( q, y, y_i \) where these
mean
\[
q \equiv e^{2\pi i \tau}, \quad y \equiv e^{2\pi i z}, \quad \text{and} \quad y_i \equiv e^{2\pi iz^i}.
\]  
(246)

Basic Functions

Our conventions for classical theta functions are as follows:
\[
\vartheta_1(\tau, z) = i \sum_{k \in \mathbb{Z}} (-1)^k q^{(k+1/2)^2/2} y^{k+1/2},
\]  
(247)
\[
\vartheta_2(\tau, z) = \sum_{k \in \mathbb{Z}} q^{(k+1/2)^2/2} y^{k+1/2},
\]  
(248)
\[
\vartheta_3(\tau, z) = \sum_{k \in \mathbb{Z}} q^{k^2/2} y^{k},
\]  
(249)
\[
\vartheta_4(\tau, z) = \sum_{k \in \mathbb{Z}} (-1)^k q^{k^2/2} y^{k}.
\]  
(250)

The Dedekind eta function is defined as
\[
\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n).
\]  
(251)
Polylogarithm Function

Polylogarithm is defined by the infinite series
\[
\text{Li}_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s},
\]
when \( |z| < 1 \). It can also be defined for \( |z| \geq 1 \) by analytic continuation. We also use the function, \( \mathcal{P}(z) \), as introduced in [3]
\[
\mathcal{P}(z) = \Im(z) \text{Li}_2(e^{2\pi i z}) + \frac{1}{2\pi} \text{Li}_3(e^{2\pi i z}).
\]

Modular Forms

A modular form of weight \( k \in \mathbb{Z} \) is a holomorphic function
\[
\phi_k : \mathbb{H} \rightarrow \mathbb{C}
\]
which satisfies the following two conditions where \( \mathbb{H} \) is the complex upper plane:

- For any \( \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL_2(\mathbb{Z}) \)
  \[
  \phi_k \left( \frac{a\tau + b}{c\tau + d} \right) = (c\tau + d)^k \phi_k(\tau).
  \]
- \( \phi_k \) has a Fourier expansion of the form
  \[
  \phi_k(\tau) = \sum_{n \in \mathbb{Z}} c(n)q^n,
  \]
  where \( c(n) \) is zero unless \( n \geq 0 \). If, moreover, \( c(0) = 0 \) the modular form is called a cusp form.

The unique weight 12 cusp form (up to an overall multiplicative constant) is
\[
\Delta(\tau) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = q - 24q^2 + 252q^3 - 1472q^4 + 4850q^5 + \ldots
\]

The ring of modular forms is freely generated by Eisenstein series \( E_4 \) and \( E_6 \) which are given by
\[
E_4(\tau) = 1 + 240 \sum_{n=1}^{\infty} \frac{n^3q^n}{1 - q^n} = 1 + 240q + 2160q^2 + 6720q^3 + 17520q^4 + \ldots,
\]
and

$$E_6(\tau) = 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1 - q^n} = 1 - 504q - 16632q^2 - 122976q^3 - 532728q^4 + \ldots \quad (259)$$

The Eisenstein series $E_2$ defined by

$$E_2(\tau) = 1 - 24 \sum_{n=1}^{\infty} \frac{nq^n}{1 - q^n} = 1 - 24q - 72q^2 - 96q^3 - 168q^4 + \ldots \quad (260)$$

is not a modular form itself, however the non-holomorphic combination $E_2(\tau) - 3/(\pi \Im \tau)$ transforms under $SL(2, \mathbb{Z})$ as if it is a weight 2 modular form.

**Jacobi Forms**

The theory of Jacobi forms is worked out in detail in [5]. In this work, we are using a generalization of [5], following [51, 52, 53, 54, 55, 56, 33].

Let $L$ be a lattice endowed with a positive definite, symmetric and non-degenerate bilinear form $(,): L \times L \to \mathbb{Z}$ and let $L$ be an even lattice with respect to this bilinear form. By linearly extending this bilinear form, we also define the dual lattice $L^*$, which consists of all elements of $L \otimes \mathbb{Q}$ having integral product with all elements of $L$.

A holomorphic (respectively weak or nearly holomorphic) Jacobi form of weight $k \in \mathbb{Z}$ associated with the lattice $L$ is a holomorphic function

$$\phi_k : \mathbb{H} \times (L \otimes \mathbb{C}) \to \mathbb{C} \quad (261)$$

which satisfies the following conditions:

- For any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$

$$\phi_k \left( \frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d} \right) = (c\tau + d)^k \exp \left( \frac{2\pi i c(z, z)}{2(c\tau + d)} \right) \phi_k (\tau, z). \quad (262)$$

- For any $\lambda, \mu \in L$

$$\phi_k (\tau, z + \lambda \tau + \mu) = \exp \left[ -2\pi i \left( \frac{(\lambda, \lambda)}{2} + (\lambda, z) \right) \right] \phi_k (\tau, z). \quad (263)$$

- $\phi_k$ has a Fourier expansion of the form

$$\phi_k (\tau, z) = \sum_{n \in \mathbb{Z}, \alpha \in L^*} c(n, \alpha) q^n \exp (2\pi i (\alpha, z)), \quad (264)$$

where $c(n, \alpha)$ is zero unless $2n - (\alpha, \alpha) \geq 0$ (respectively unless $n \geq 0$ or unless $n \geq -N$ for a positive integer $N$).
From the second property, one can show that Fourier coefficients depend only on the discriminant $\Delta = n - (\alpha, \alpha)/2$ and on $\alpha \pmod{L^*/L}$.

Another important property of Jacobi forms is that the space of weight $k$ holomorphic Jacobi forms for any $k$ and over an even lattice, $\Lambda$, is finite dimensional. This allows one to write the most general weight $k$ Jacobi form over a lattice, $\Lambda$, and determine the whole function using only a few of its Fourier coefficients.

In the main text, we usually choose a particular basis for $\Lambda$ and $\Lambda^*$ and then write $\phi_k$ in terms of this basis. More explicitly, any $\alpha \in L^*$ can be expanded as $\alpha = k_i \gamma_i$ and $\vec{z}$ can be expanded as $\vec{z} = z^i \vec{\beta}_i$, using the basis vectors $\{\vec{\beta}_i\}$ of $\Lambda$ and its dual basis $\{\vec{\gamma}_i\}$. Then, one can write

$$\phi_k(\tau, \vec{z}) = \sum_{n,k} c(n, k_i) q^n y_1^{k_1} \cdots y_s^{k_s}, \quad (265)$$

where the sum is over integers $n$ and $k_i$. We will indicate that a function is a weight $k$ Jacobi form over the lattice $\Lambda$ using subscripts $(k, \Lambda)$.

For $L = \langle 2m \rangle$, one can check that the definition above reduces to the Jacobi from definition of [5] where $m$ is called the index of the Jacobi form. To denote a weight $k$, index $m$ Jacobi form we use subscripts $(k, m)$. Two important examples are Eisenstein series $E_{4,1}(\tau, z)$ and $E_{6,1}(\tau, z)$ which have Fourier expansions [5]:

$$E_{4,1}(\tau, z) = 1 + (y^2 + 56y + 126 + 56y^{-1} + y^{-2})q + \ldots, \quad (266)$$

and

$$E_{6,1}(\tau, z) = 1 + (y^2 - 88y - 330 - 88y^{-1} + y^{-2})q + \ldots. \quad (267)$$

Eisenstein series are constructed by starting with the constant, 1, and then summing over all terms that can be obtained by acting on 1 with the members of the Jacobi group fixing the cusp at infinity. Therefore, in this sense, Eisenstein series comprise the simplest examples of Jacobi forms.

Let us denote the space of weight $k$, index $m$ weak Jacobi forms by $J_{k,m}^{\text{weak}}$. Then, an important structure theorem in [5] tells that the ring of even weight weak Jacobi forms is freely generated by two weak Jacobi forms $\tilde{\phi}_{-2,1}$ and $\tilde{\phi}_{0,1}$ over the ring of modular forms, where

$$\tilde{\phi}_{-2,1} = -\left(\frac{\vartheta_1(\tau, z)^2}{\eta(\tau)^6}\right) = \frac{(y - 1)^2}{y} - \frac{2(y - 1)^4 q}{y^2} + \frac{(y - 1)^4 (y^2 - 8y + 1) q^2}{y^3} + \frac{8(y - 1)^4 (y^2 - 3y + 1) q^3}{y^3} - \frac{(y - 1)^4 (2y^4 - 31y^3 + 72y^2 - 31y + 2) q^4}{y^4} + \ldots \quad (268)$$

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and
\[
\tilde{\phi}_{0,1} = 4 \left( \frac{\varphi_2(\tau, z)^2}{\varphi_2(\tau, 0)^2} + \frac{\varphi_3(\tau, z)^2}{\varphi_3(\tau, 0)^2} + \frac{\varphi_4(\tau, z)^2}{\varphi_4(\tau, 0)^2} \right)
= \left( y + 10 + \frac{1}{y} \right) + \frac{2(y - 1)^2 (5y^2 - 22y + 5)}{y^2} \cdot \frac{(y - 1)^2 (y^4 + 110y^3 - 294y^2 + 110y + 1)q^2}{y^3} + \ldots
\]

(269)

Similarly, an explicit set of generators can be given for the case \( \Lambda = m\Lambda_R \) where \( m \) is a positive integer and \( \Lambda_R \) is the root lattice of a simple Lie algebra (except for \( E_8 \)). More explicitly, [57] and [58] give generators for Weyl invariant Jacobi forms over \( \Lambda_R(m) \) over the ring of modular forms. Weyl invariance requires the Jacobi form to be invariant under the action of the Weyl group on \( \mathbb{Z} \). Since in our physical examples states form irreducible representations of the gauge Lie algebra, Weyl invariance condition is naturally satisfied. In the main text we give examples using generators for \( A_2 \) and \( A_3 \). To define those generators, it will be useful to define

\[
\alpha(\tau, z) = \frac{i}{\eta(\tau)^3} \partial_1(\tau, z)
\]

(270)

and

\[
\beta(\tau, z) = -\frac{1}{2\pi} \frac{\partial}{\partial z} \left( \frac{\varphi_1(\tau, z)}{\eta(\tau)^3} \right).
\]

(271)

For \( A_2 \), when we pick the lattice basis as

\[
\tilde{\beta}_1 = (1, -1, 0) \text{ and } \tilde{\beta}_2 = (0, 1, -1),
\]

the generators of Weyl invariant Jacobi forms read

\[
\tilde{\phi}_{-3, A_2} = \alpha(\tau, z^1) \alpha(\tau, z^2 - z^1) \alpha(\tau, -z^2)
= \left( \frac{y_1}{y_2} y_1 + y_2 - \frac{1}{y_2} - \frac{y_2}{y_1} + \frac{1}{y_1} \right) + \left( -\frac{y_1^2}{y_2^2} + y_2^2 + \frac{8y_1}{y_2} - \frac{8y_1 - y_2^2 + 8y_2}{y_1} \right) + \frac{8}{y_2} + \frac{1}{y_2} - \frac{8y_2}{y_1} + \frac{8}{y_1} + \frac{y_2^2}{y_1^2} - \frac{1}{y_1^2} \right) \cdot q + \ldots
\]

(273)

\[
\tilde{\phi}_{-2, A_2} = \beta(\tau, z^1) \alpha(\tau, z^2 - z^1) \alpha(\tau, -z^2) + \alpha(\tau, z^1) \beta(\tau, z^2 - z^1) \alpha(\tau, -z^2)
+ \alpha(\tau, z^1) \alpha(\tau, z^2 - z^1) \beta(\tau, -z^2)
= \left( -\frac{y_1}{2y_2} - \frac{y_1}{2} - \frac{1}{2y_2} - \frac{y_2}{2y_1} - \frac{1}{2y_1} \right)
+ \left( \frac{3y_2^2}{2y_2} + \frac{y_2^2}{2} + 3y_2y_1 - \frac{7y_1}{y_2} + \frac{3y_1}{y_2^2} - \frac{7y_1 - y_2^2 - 7y_2 - \frac{1}{y_2}}{y_2} \right)
+ 27 \left( \frac{3y_2^2}{y_1} - \frac{7y_2}{y_1} + \frac{3}{y_2y_1} - \frac{7}{y_1} - \frac{y_2^2}{2y_1^2} + \frac{3y_2}{y_1^2} - \frac{1}{2y_1^2} \right) \cdot q + \ldots,
\]

(274)

\( \Lambda_R(m) \) is the lattice \( \Lambda_R \) rescaled by \( \sqrt{m} \).
and
\begin{align}
\tilde{\phi}_{0,A_2} &= 24L_{-2}\tilde{\phi}_{-2,A_2} \\
&= \left(\frac{y_1}{y_2} + y_1 + y_2 + \frac{1}{y_2} + 18 + \frac{y_2}{y_1} + \frac{1}{y_1}\right) + \left(\frac{18y_1^2}{y_2} + \frac{y_1^2}{y_2} + y_1^2 + 18y_2y_1\right) \\
&\quad - \frac{82y_1}{y_2} + \frac{18y_1}{y_2^2} - 82y_1 + y_2^2 - 82y_2 - \frac{82}{y_2} + 1 + 378 + \frac{18y_2^2}{y_1} - \frac{82y_2}{y_1} \\
&\quad + \frac{18}{y_2y_1} - \frac{82}{y_1} + \frac{y_2^2}{y_1^2} + \frac{18y_2}{y_1^2} + \frac{1}{y_1^2}\right)q + \ldots.
\end{align}
(275)

Note that the differential operator $L_k$ is defined in equation (148).

For $A_3$, when we pick the lattice basis as
\begin{align}
\vec{\beta}_1 &= (1, -1, 0, 0), \quad \vec{\beta}_2 = (0, 1, -1, 0) \quad \text{and} \quad \vec{\beta}_3 = (0, 0, 1, -1),
\end{align}
(276)
the generators of Weyl invariant Jacobi forms read
\begin{align}
\tilde{\phi}_{-4,A_3} &= \alpha(\tau, z^1)\alpha(\tau, z^2 - z^1)\alpha(\tau, z^3 - z^2)\alpha(\tau, -z^3),
\end{align}
(277)
\begin{align}
\tilde{\phi}_{-3,A_3} &= \beta(\tau, z^1)\alpha(\tau, z^2 - z^1)\alpha(\tau, z^3 - z^2)\alpha(\tau, -z^3) \\
&\quad \quad + \alpha(\tau, z^1)\beta(\tau, z^2 - z^1)\alpha(\tau, z^3 - z^2)\alpha(\tau, -z^3) \\
&\quad \quad + \alpha(\tau, z^1)\alpha(\tau, z^2 - z^1)\beta(\tau, z^3 - z^2)\alpha(\tau, -z^3) \\
&\quad \quad + \alpha(\tau, z^1)\alpha(\tau, z^2 - z^1)\alpha(\tau, z^3 - z^2)\beta(\tau, -z^3),
\end{align}
(278)
\begin{align}
\tilde{\phi}_{-2,A_3} &= 24L_{-4}\tilde{\phi}_{-4,A_3},
\end{align}
(279)
and
\begin{align}
\tilde{\phi}_{0,A_3} &= 24L_{-2}\tilde{\phi}_{-2,A_3}.
\end{align}
(280)

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