Application of entanglement conditions to spin systems

Hongjun Zheng\textsuperscript{1}, Ho Trung Dung\textsuperscript{1,2}, and Mark Hillery\textsuperscript{1}

\textsuperscript{(1)}Department of Physics, Hunter College of CUNY
695 Park Avenue
New York, NY 10065

\textsuperscript{(2)}Institute of Physics, Academy of Sciences and Technology
1 Mac Dinh Chi Street, District 1
Ho Chi Minh City, Vietnam

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Abstract

There have been numerous studies of entanglement in spin systems. These have usually focussed on examining the entanglement between individual spins or determining whether the state of the system is completely separable. Here we present conditions that allow us to determine whether blocks of spins are entangled. We show that sometimes these conditions can detect entanglement better than conditions involving individual spins. We apply these conditions to study entanglement in spin wave states, both when there are only a few magnons present and also at finite temperature.

1 Introduction

The realization that entanglement is a resource for a number of useful tasks in quantum information has led to a tremendous interest in its properties, quantification and in methods by which it can be produced. One area, which has been fruitful, is the study of entanglement in many-body systems (for a review see [1]).

Spin systems, in particular, have received a great deal of attention, and this has led to the formulation of several conditions for determining whether the state of a spin system is entangled. For example, in an $N$ qubit system, with

$$J_l = \frac{1}{2} \sum_{k=1}^{N} \sigma_k^{(l)},$$

(1)
with \( l = 1, 2, 3 \) and \( \sigma_k^{(l)} \) being the Pauli matrices for the \( k \)-th qubit, the state is entangled if \( [2] \)

\[
\frac{(\Delta J_3)^2}{(J_1)^2 + (J_2)^2} < \frac{1}{N}.
\]

If this inequality is satisfied, the \( N \)-qubit state cannot be expressed as

\[
\rho = \sum_j p_j \rho_j^{(1)} \otimes \rho_j^{(2)} \otimes \ldots \otimes \rho_j^{(N)},
\]

(3)

where the \( 1 \geq p_j > 0 \) sum to one, and \( \rho_j^{(k)} \) is a density matrix for the \( k \)-th qubit. A state that can be expressed in this form is known as completely separable. Further criteria for entanglement in spin systems have been developed \([3, 4, 5]\). A comprehensive study of entanglement conditions employing quantities that are at most quadratic in the collective spin operators has been given in \([6, 7]\).

Recently two inequalities have been developed for the detection of entanglement \([8]\). Consider a system consisting of two subsystems, which we shall denote by \( a \) and \( b \). The Hilbert space for the total system is \( \mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_b \), where \( \mathcal{H}_a \) is the Hilbert space for subsystem \( a \) and \( \mathcal{H}_b \) is the Hilbert space for subsystem \( b \). Let \( A \) be an operator on \( \mathcal{H}_a \) and \( B \) be an operator on \( \mathcal{H}_b \). The state of the total system is entangled if either

\[
|\langle AB \rangle|^2 > \langle A^\dag AB \rangle \langle B \rangle,
\]

(4)

or

\[
|\langle AB \rangle|^2 > \langle A^\dag A \rangle \langle B^\dag B \rangle.
\]

(5)

Note that the condition (4) holds if the left-hand side is replaced by \( |\langle A^\dag B \rangle|^2 \). In the following we shall use the form (4) or \( |\langle A^\dag B \rangle|^2 > \langle A^\dag AB \rangle \langle B \rangle \) interchangeably. These are sufficient conditions for entanglement; if they are not satisfied we cannot say whether the state is entangled or not. Some of the implications of these conditions have been explored for the case that the systems are field modes \([8, 9]\), and in the case that one is a field mode and one is a collection of atoms \([10]\). Here we would like to explore their implications for spin systems. In that case, we have a collection of spins, and our subsystems are two non-overlapping subsets of the total set. We describe each subset by a collective spin, \( J_a \) for set \( a \) and \( J_b \) for set \( b \). Let \( J_{a-} \) be the angular momentum lowering operator for set \( a \) and \( J_{b-} \) be the angular momentum lowering operator for set \( b \). The corresponding raising operators are \( J_{a+} \) and \( J_{b+} \), respectively. Our entanglement conditions become

\[
|\langle J_{a-} J_{b+} \rangle|^2 > \langle J_{a+} J_{a-} J_{b+} J_{b-} \rangle,
\]

\[
|\langle J_{a-} J_{b-} \rangle|^2 > \langle J_{a+} J_{a-} \rangle \langle J_{b+} J_{b-} \rangle.
\]

(6)

These inequalities differ from the ones discussed in the previous paragraph in that they detect entanglement between two blocks of spins, and not whether the state is completely separable or not. For example, if one is studying entanglement in a spin chain, one may simply be interested in whether the state is
entangled or not, in which case Eq. (2) could be of use. However, one might instead wish to find out whether two blocks of spins are entangled, in which case the above equations, with $J_a$ and $J_b$ being collective spin operators for the respective blocks, could be useful.

As was mentioned previously, a large amount of work has been done on entanglement in spin systems, in particular spin chains with different types of interactions between adjacent spins. Typically the concurrence of two spins in the ground state of the system is found. This has been done for many variants of the Heisenberg model, both with and without an applied magnetic field. We will be interested in applying our conditions to a spin system at finite temperature, in particular one with a ferromagnetic Heisenberg interaction. Wang and Zanardi showed that in a one-dimensional ring with periodic boundary conditions, and the spins interacting via an isotropic Heisenberg interaction, there is no entanglement between any two spins at any temperature [11]. This result depends on the SU(2) symmetry of the Hamiltonian, and does not apply if there is an applied magnetic field. Asoudeh and Karimipour look at the case in which there is an applied magnetic field and only the ground state and the state with one flipped spin are populated, and found the concurrence between two spins as a function of their separation [12]. The advantage of applying the entanglement conditions in the previous paragraph is that we can study the entanglement between blocks of spins, and not just the entanglement between individual spins.

The paper is arranged as follows. We will initially study some general properties of our entanglement conditions. We will first compare the application of our conditions to individual and collective spins. It will be shown that the conditions can detect entanglement between spins in angular momentum intelligent states. These are states for which the uncertainty relations for angular momentum operators are satisfied as an equality. We will then show how the above conditions can be strengthened by imposing local rotational invariance. Next, we will move on to spin systems, and use the above conditions to study entanglement in spin wave states, first for states containing a small number of spin waves, and then for states at finite temperature.

## 2 Examples of states

One of the advantages of the entanglement conditions in the Introduction is that they allow us to look at the entanglement between blocks of spins rather than between individual spins. A standard approach when studying the entanglement in spin-1/2 systems is to choose two spins and calculate their concurrence. It is, however, quite possible that there is no entanglement between individual spins, but there is between blocks of spins. In that case, the method based on concurrence will fail. This can be illustrated by an example.

Let us consider four qubits, i.e. spin-1/2 particles, with qubits 1 and 2 in block $a$, and qubits 3 and 4 in block $b$. Each qubit has an orthonormal basis $\{|0\rangle, |1\rangle\}$, and a raising operator $\sigma^+(+)$ and a lowering operator $\sigma^-(–)$, where
σ(+)|0⟩ = |1⟩, σ(+)|1⟩ = 0, and σ(−) = (σ(+))†. Let us now consider the four-qubit state

|Ψ⟩ = \frac{1}{\sqrt{2}}|00⟩_{12}|00⟩_{34} + \frac{1}{2}(|01⟩_{12}|10⟩_{34} + |10⟩_{12}|01⟩_{34}).

Tracing out qubits 2 and 4 we find the reduced density matrix for qubits 1 and 3

ρ_{13} = \frac{1}{2}|00⟩_{13}⟨00| + \frac{1}{4}(|01⟩_{13}⟨01| + |10⟩_{13}⟨10|),

which is separable. So, if we just look at qubits 1 and 3, i.e. one qubit in each block, we do not see any entanglement. However, setting

J_a = σ^{(-)}_1 + σ^{(-)}_3,
J_b = σ^{(-)}_2 + σ^{(-)}_4,

we find that

⟨J_a − J_b⟩ = \frac{1}{\sqrt{2}}, \quad ⟨J_a + J_b⟩⟨J_a + J_b⟩ = \frac{1}{4},

so that the second entanglement condition in Eq. (6) is satisfied. Therefore, by looking at entanglement between blocks, we see that the state is, in fact, entangled.

We first will proceed to examine two more complicated types of entangled states in order to see whether our entanglement conditions can show that these states are indeed entangled. In the first case, we will apply the conditions to both individual and collective spins in order to see which method yields a more sensitive test of entanglement.

### 2.1 Correlated sets of qubits

Suppose we have 2n qubits. We will divide the qubits into two blocks of n qubits each, and within each block, we will consider only those states of total spin \( j = n/2 \). In particular, we want to examine states of the form

|Ψ⟩ = \sum_{m = −j}^{j} c_{m} |j, m⟩_{a} ⊗ |j, m⟩_{b}

with \( j = n/2 \), and the state with subscript \( a \) referring to the first block and the state with subscript \( b \) referring to the second. This is clearly an entangled state, and we want to see whether the entanglement conditions we have proposed will detect the entanglement. We will do this in two different ways. First, we will apply the second entanglement condition in Eq. (6) to the collective spin of each block. Next, we will choose one qubit from each block and apply the same condition to those two qubits.

The calculations for the condition using the collective spins is straightforward, and we find

⟨Ψ|J_{a} − J_{b}|Ψ⟩ = \sum_{m = −j+1}^{j} (j + m)(j − m + 1)c_{m−1}^∗c_{m},

4
\[
\langle \Psi | J_a + J_a^- | \Psi \rangle = \langle \Psi | J_b + J_b^- | \Psi \rangle = \sum_{m=-j}^{j} |c_m|^2 (j + m)(j - m + 1). \tag{12}
\]

Therefore, the second entanglement condition in Eq. (6) becomes
\[
\left| \sum_{m=-j+1}^{j} (j + m)(j - m + 1)c_m^* c_{m-1} \right| > \sum_{m=-j}^{j} |c_m|^2 (j + m)(j - m + 1). \tag{13}
\]
One possible choice of \( c_m \) is to set \( c_m = \eta x^{j+m} \), for some \( x > 0 \) and \( \eta \) an appropriate normalization constant. This gives us
\[
\left| \sum_{m=-j+1}^{j} (j + m)(j - m + 1)x^{2(j+m)-1} \right| > \sum_{m=-j+1}^{j} (j + m)(j - m + 1)x^{2(j+m)}. \tag{14}
\]
This condition is clearly satisfied when \( x < 1 \), but not satisfied for \( x > 1 \).

Now let us see what happens if we just look at one qubit in each block. Let the qubit in the first block be qubit 1 and the one in the second block be qubit 2, and we will assume that each of these qubits is the first one in its respective block. Let us call the spin-down state of an individual qubit \( |0\rangle \) and the spin-up state \( |1\rangle \). The basis states for each block are \( n \)-fold tensor products of spin-up and spin-down states for each qubit in the block. The state \( |j, m\rangle \) of \( n \) qubits with \( j = n/2 \) is the symmetric linear combination of all basis states in which there are \( j + m \) ones and \( j - m \) zeroes. There are \( \binom{2j}{j+m} \) such states. The operator \( \sigma_1^{(+)} \sigma_1^{(-)} \), where \( \sigma_1^{(+)} \) and \( \sigma_1^{(-)} \) are the raising and lowering operators for qubit 1, is just the projection onto states in which the state of the first qubit is \( |1\rangle \). There are \( \binom{2j-1}{j+m-1} \) states, and this implies that
\[
\langle j, m | \sigma_1^{(+)} \sigma_1^{(-)} | j, m \rangle = \frac{\binom{2j-1}{j+m-1}}{\binom{2j}{j+m}} = \frac{j + m}{2j}. \tag{15}
\]
This implies that
\[
\langle \Psi | \sigma_1^{(+)} \sigma_1^{(-)} | \Psi \rangle = \sum_{m=-j}^{j} |c_m|^2 \frac{j + m}{2j}. \tag{16}
\]
The expression for \( \langle \Psi | \sigma_2^{(+)} \sigma_2^{(-)} | \Psi \rangle \) is identical.

We now want to compute \( (a|j, m-1\rangle \otimes b|j, m-1\rangle) \sigma_1^{(+)} \sigma_1^{(-)}(|j, m\rangle_a \otimes |j, m\rangle_b) \).
The operator \( \sigma_1^{(-)} \sigma_1^{(-)} \) will pick out the basis states with ones in the first slot
of each block. By reasoning similar to that above, we have that
\[
\begin{align*}
(\langle a|j, m - 1\rangle \otimes \langle b|j, m - 1\rangle)\sigma_1^{(-)}\sigma_2^{(-)}(|j, m\rangle_a \otimes |j, m\rangle_b) \\
= \left(\frac{2j - 1}{j + m - 1}\right)^2 \frac{2j}{j + m} \frac{2j}{j + m} \\
= \frac{(j + m)(j - m + 1)}{(2j)^2}.
\end{align*}
\] (17)

This gives us that
\[
\langle \Psi|\sigma_1^{(-)}\sigma_2^{(-)}|\Psi\rangle = \sum_{m=-j+1}^{j} c_{m-1}^* c_m (j + m)(j - m + 1) \frac{1}{(2j)^2}.
\] (18)

Finally, the second entanglement condition in Eq. (6) with \(A = \sigma_1^{(-)}\) and \(B = \sigma_2^{(+)}\) becomes
\[
\left| \sum_{m=-j+1}^{j} (j + m)(j - m + 1)c_{m-1}^* c_m \right| \geq (2j) \sum_{m=-j}^{j} |c_m|^2 (j + m). \] (19)

Comparing Eqs. (13) and (19) we note that \(2j \geq j - m + 1\) if \(m \geq -j + 1\), which is the entire range of the sum. This implies that at least for states of the type in Eq. (11), the collective spin condition is stronger, that is, it will be satisfied by more states, than the condition for individual spins.

### 2.2 Angular momentum intelligent states

We first want to find some spin states that satisfy our entanglement conditions. One possibility is to find states in which the spins of the two subsystems are highly correlated, and states in which the uncertainty of the sum (or difference) of the two spins is small will satisfy this condition.

Let us begin by looking at the uncertainty relation for the total spin,
\[
\Delta(J_{a1} + J_{b1})\Delta(J_{a2} + J_{b2}) \geq \frac{1}{2} |(J_{a3} + J_{b3})|, \] (20)

where \(J_{a1}, J_{a2},\) and \(J_{a3}\) are the components of \(J_a\), and \(J_{b1}, J_{b2},\) and \(J_{b3}\) are the components of \(J_b\). We would like to find the states which satisfy this relation as an equality. These states were first found in [13], and here we will follow the treatment given in [14]. These satisfy the eigenvalue equation
\[
[(J_{a1} + J_{b1}) + i\lambda(J_{a2} + J_{b2})]|\Psi\rangle = \beta|\Psi\rangle, \] (21)

where \(\lambda\) is real. This equation implies that
\[
\langle \Psi|(J_{a1} + J_{b1})|\Psi\rangle = \text{Re}(\beta), \quad \langle \Psi|(J_{a2} + J_{b2})|\Psi\rangle = (1/\lambda)\text{Im}(\beta),
\] (22)
\[ \Delta(J_{a1} + J_{b1})]^2 = \frac{\lambda}{2} \langle J_{a3} + J_{b3} \rangle \quad \Delta(J_{a2} + J_{b2})]^2 = \frac{1}{2\lambda} \langle J_{a3} + J_{b3} \rangle. \] (23)

From these equations, we see that when \( \lambda \) is small, \( J_{a1} \) and \( -J_{b1} \) are highly correlated, and when it is large, \( J_{a2} \) and \( -J_{b2} \) are highly correlated. These states are spin analogs of two-mode squeezed state for light.

In order to solve Eq. (21), we first define a state
\[ |\Psi\rangle' = e^{i\theta(J_{a1} + J_{b1})} |\Psi\rangle, \] (24)

and insert the resulting expression for \( |\Psi\rangle \) into Eq. (21) to give an equation for \( |\Psi\rangle' \)
\[ \{(J_{a1} + J_{b1}) + i\lambda[(J_{a2} + J_{b2}) \cos \theta - (J_{a3} + J_{b3}) \sin \theta]\}|\Psi\rangle' = \beta|\Psi\rangle'. \] (25)

Now choose \( \lambda = -1/\cos \theta \), and \( \theta \) to be in the range \( \pi \geq \theta \geq \pi/2 \), which implies that \( \lambda > 1 \), and
\[ [(J_{a-} + J_{b-}) - i\sqrt{\lambda^2 - 1}(J_{a3} + J_{b3})]|\Psi\rangle' = \beta|\Psi\rangle'. \] (26)

We now expand \( |\Psi\rangle' \) as
\[ |\Psi\rangle = \sum_{n,m=-j}^{j} C_{nm}|n,m\rangle, \] (27)

where we have set \( |n,m\rangle = |j,n\rangle \otimes |j,m\rangle \). If we assume, for simplicity, that \( C_{n,m} = 0 \), unless \( n = m \), our equation for \( |\Psi\rangle' \) reduces to the recurrence relation
\[ C_{m+1,m+1} = \frac{\beta + 2mi\sqrt{\lambda^2 - 1}}{(j + m + 1)(j - m)} C_{m,m}, \quad m < j, \] (28)

\[ [\beta + 2ij\sqrt{\lambda^2 - 1}]C_{j,j} = 0, \quad m = j. \] (29)

From the second equation, we see that either \( \beta = -2ij\sqrt{\lambda^2 - 1} \), or \( C_{j,j} = 0 \). If \( C_{j,j} = 0 \), then it must be the case that \( \beta = -2m_0i\sqrt{\lambda^2 - 1} \) for some \( m_0 \). So,
\[ C_{m,m} = (-2i\sqrt{\lambda^2 - 1})^j (m_0 + j)! (j - m)! / (m_0 - m)!(j + m)! (2j)! C_{-j,-j}, \] (30)

for \( m \leq m_0 \) and \( C_{mm} = 0 \) for \( m > m_0 \). After grouping the \( m \)-independent constants \( (m_0 + j)! (2j)! C_{-j,-j} \) into \( C_{j,m_0}(\lambda) \), we have
\[ |\Psi(j,m_0,\lambda)\rangle = C_{j,m_0}(\lambda) e^{-i\theta(J_{a1} + J_{b1})} \sum_{m=-j}^{m_0} (-2j\sqrt{\lambda^2 - 1})^j (m_0 + j)! (j - m)! / (m_0 - m)!(j + m)! |m,m\rangle. \] (31)
We want to see if there is some range of parameters for which $|\langle J_a - J_b \rangle|^2 > \langle J_a + J_a^- \rangle \langle J_b + J_b^- \rangle$, but for these states $\langle J_a + J_a^- \rangle = \langle J_b + J_b^- \rangle$, so we just need to show that $|\langle J_a - J_b \rangle| > \langle J_a + J_a^- \rangle$. We find that

$$\langle J_a + J_a^- \rangle = |C_{j, m_0}(\lambda)|^2 \sum_{m=-j}^{m_0} [4(\lambda^2 - 1)]^{j+m} \frac{(j-m)!}{(m_0-m)!(j+m)!} \left\{ \frac{\lambda^2 + 1}{2\lambda^2} [j(j+1) - m^2] - \frac{m}{\lambda} \right\}, \quad (32)$$

and

$$\langle J_a - J_a^- \rangle = |C_{j, m_0}(\lambda)|^2 \sum_{m=-j}^{m_0} [4(\lambda^2 - 1)]^{j+m} \frac{(j-m)!}{(m_0-m)!(j+m)!} \left[ 2i\sqrt{\lambda^2 + 1} (m_0 - m) - \frac{(\lambda^2 - 1)m^2}{\lambda^2} \right]. \quad (33)$$

Consider the simple case in which $j = 1, m_0 = -1$, and $m = -1$, so that the sum has only one term. The entanglement condition becomes

$$\left| \frac{\lambda^2 - 1}{\lambda^2} \right| > \frac{(\lambda + 1)^2}{2\lambda^2}, \quad (34)$$

and the state is entangled if $\lambda > 3$.

3 Local rotational invariance

Entanglement is not affected by local unitary transformations, and so, ideally, we would like our entanglement conditions to be invariant under local unitaries as well. It is not always possible to accomplish this, but we can sometimes obtain invariance under a subgroup of the group of local unitary transformations. In Ref. [10] in which entanglement between field modes was considered, it was possible to find entanglement conditions that are invariant under local Gaussian transformations of the field modes. These new conditions were stronger than the original ones, that is they detect entanglement in a larger set of states. Thus, making the conditions invariant under a subset of local unitary transformations strengthens them.

For the entanglement conditions we are considering in this paper, the obvious group of local unitaries consists of local rotations. Under the action of the rotation $R(\alpha, \beta, \gamma) = e^{-i\alpha J_1} e^{-i\beta J_2} e^{-i\gamma J_3}$, we have that

$$R^{-1} J_+ R = \left[ \frac{1}{2} (\cos \alpha + \cos \beta) + \frac{i}{2} \sin \alpha \sin \beta \right] e^{i\gamma} J_+$$
$$+ \left[ \frac{1}{2} (\cos \beta - \cos \alpha) + \frac{i}{2} \sin \alpha \sin \beta \right] e^{-i\gamma} J_-$$
$$+ (1 - i \sin \alpha \sin \beta) e^{i\gamma} J_3. \quad (35)$$
Now suppose we start with the entanglement condition Eq. (4), and we want to find from it a condition that is invariant under local rotations of the $a$ system (finding a condition that is invariant under local rotations of both $a$ and $b$ subsystems is possible, but it results in a $9 \times 9$ matrix, which is rather unwieldy).

We note that what the local rotation on subsystem $a$ does is to send both $J_{a+}$ and $J_{a-}$ into linear combinations of $J_{a+}$, $J_{a-}$, and $J_{a3}$. This suggests that we set $A = c_1^* J_{a-} + c_2^* J_{a+} + c_3^* J_{a3}$ and $B = J_{b-}$ in Eq. (4). The entanglement condition can then be written in the form

$$\begin{pmatrix} c_1^* & c_2^* & c_3^* \end{pmatrix} M \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} > 0,$$

where $M$ is a $3 \times 3$ matrix, whose elements are linear combinations of expectation values of products of angular momentum operators. In particular,

$$
\begin{align*}
M_{11} &= |\langle J_{a-} J_{b+} \rangle|^2 - |\langle J_{a+} J_{a-} J_{b+} J_{b-} \rangle|, \\
M_{12} &= |\langle J_{a-} J_{b+} \rangle|^2 - |\langle J_{a+} J_{a-} J_{b+} J_{b-} \rangle|, \\
M_{13} &= |\langle J_{a-} J_{b+} \rangle|^2 - |\langle J_{a+} J_{a3} J_{b+} J_{b-} \rangle|, \\
M_{21} &= |\langle J_{a+} J_{b+} \rangle|^2 - |\langle J_{a-} J_{b-} J_{b+} J_{b-} \rangle|, \\
M_{22} &= |\langle J_{a+} J_{b+} \rangle|^2 - |\langle J_{a+} J_{a3} J_{b+} J_{b-} \rangle|, \\
M_{23} &= |\langle J_{a+} J_{b+} \rangle|^2 - |\langle J_{a3} J_{b+} J_{b-} \rangle|, \\
M_{31} &= |\langle J_{a+} J_{b+} \rangle|^2 - |\langle J_{a3} J_{b+} J_{b-} \rangle|, \\
M_{32} &= |\langle J_{a+} J_{b+} \rangle|^2 - |\langle J_{a3} J_{b+} J_{b-} \rangle|, \\
M_{33} &= |\langle J_{a3} J_{b+} \rangle|^2 - |\langle J_{a3} J_{b+} J_{b-} \rangle|.
\end{align*}
$$

If we change the state by a local rotation of system $a$, the effect on Eq. (36) is only to change the values of $c_1$, $c_2$ and $c_3$. This follows from the fact that when $A$ is conjugated by the rotation $R_a$, the form of the operator stays the same, that is, it is a linear combination of $J_{a+}$, $J_{a-}$, and $J_{a3}$, but the coefficients multiplying the operators change. If the matrix $M$ has a positive eigenvalue, then we can find values of $c_1$, $c_2$ and $c_3$ so that the above condition is satisfied, simply by choosing them to be the components of the vector corresponding to the positive eigenvalue. Therefore, our new entanglement condition becomes that $M$ has a positive eigenvalue, and this condition is invariant under local rotations on system $a$.

Let us show that this new condition is stronger than our original condition. If the state we are considering is

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|-j, -j+1\rangle + |-j+1, -j\rangle),$$

then

$$M = \begin{pmatrix} j^2 & 0 & 0 \\
0 & -2j^2 & 0 \\
0 & 0 & -j^3 \end{pmatrix},$$

as desired.
Noting that $j^2$ is positive, we see that the state is entangled. Because this condition is invariant under rotations of system $a$, it would also show that the state $R_a \otimes I_b |\Psi\rangle$ is entangled.

Now, let us see what happens if we apply our original condition to the state $R_a \otimes I_b |\Psi\rangle$. We begin by finding

$$\left|\langle R^{-1}_a J_a + R_a J_b - |\Psi\rangle\right|^2 = \frac{j^2}{4}[(\cos \alpha + \cos \beta)^2 + \sin^2 \alpha \sin^2 \beta], \quad (40)$$

and

$$\langle R^{-1}_a J_a + J_b - R_a J_b - |\Psi\rangle\rangle = \frac{j^2}{2}[(\cos \beta - \cos \alpha)^2 + \sin^2 \alpha \sin^2 \beta] + j^3(1 + \sin^2 \alpha \sin^2 \beta). \quad (41)$$

Therefore, the state is entangled according to the old condition if

$$\cos \alpha \cos \beta > j(1 + \sin^2 \alpha \sin^2 \beta) + \frac{1}{4}(\cos \alpha \cos \beta - 1)^2. \quad (42)$$

This condition can be satisfied for only a limited range of $\alpha$ and $\beta$ if $j$ is small, and it cannot be satisfied at all if $j \geq 1$, which actually allows only for $j = 1/2$. Therefore, our new condition, which is invariant under rotations of system $a$, is considerably more powerful in that it detects entanglement in a much larger set of states.

4 Spin waves

The low-lying energy states of a system of spins coupled by exchange interactions are wavelike, as shown originally by Bloch for ferromagnets. The waves are called spin waves, and they correspond to excitations of definite energy called magnons. We will study the entanglement between spins, and blocks of spins for magnon states in a ferromagnet. We will first examine entanglement in states containing a small number of magnons, and then go on to study the case of a ferromagnet at low, but finite, temperature.

4.1 Small number of magnons

The Hamiltonian describing spins on a lattice interacting via a nearest-neighbor exchange interaction and an externally applied magnetic field is

$$H = -J \sum_{j, \delta} \mathbf{S}_j \cdot \mathbf{S}_{j+\delta} - 2\mu_0 H_0 \sum_j \mathbf{S}_{jz}, \quad (43)$$

where the vectors $\delta$ connect the spin at site $j$ with its nearest neighbors on a bravais lattice, $J$ is the exchange integral, which is assumed to be positive, $\mu_0 = (g/2)\mu_B$ is the magnetic moment of the atoms, and $\mathbf{S}_j$ is the spin angular momentum operator of the atom at $j$. $H_0$ is the intensity of a static magnetic
field directed along the $z$ axis, and we will take the limit as $H_0 \to 0^+$ to make the magnetic moments line up along the positive $z$ axis when the system is in the ground state $|\Omega\rangle$. The $z$ component of the total spin, $S_z = \sum_j S_{jz}$, is a constant of the motion, and the ground state of the system simply has all of the spins pointing in the $+z$ direction.

For the case of a small number of spin waves, let us consider a line of $N$ spins with periodic boundary conditions (the spin at $N + 1$ is the same as the spin at 1). If the atoms have a spin of $1/2$, then the state containing a single magnon is a linear combination of states with one spin flipped

$$|\Psi\rangle = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} e^{i k j a} \sigma_j^{(-)} |\Omega\rangle,$$  

(44)

where $a$ is the spacing between spins, and $k = \frac{2\pi n}{(Na)}$, where $n$ is an integer in the range $-(N/2) < n \leq (N/2)$. The operator $\sigma_j^{(-)}$ is the spin lowering operator for the spin at site $j$, that is it maps the spin up state at site $j$ to the spin down state at the same site.

Now let us examine the entanglement of this state using Eq. (4). Let

$$A = S_{1+} = \sum_{j=1}^{m} \sigma_j^{(+)} ,$$

$$B = S_{2+} = \sum_{j=L+m+1}^{L+m} \sigma_j^{(+)} ,$$

(45)

where $m$ is a number such that $2m < N$. This will allow us to see if there is entanglement between two blocks of spins each of size $m$ and distanced from each other by $(L - m)$ spins. Our state is entangled if

$$|\langle S_{1+} S_{2-}\rangle|^2 > \langle S_{1-} S_{1+} S_{2-} S_{2+}\rangle.$$  

(46)

For the single magnon state above, the right-hand side is zero, so as long as the left-hand side is non-zero, we can say that the blocks of spins are entangled. In fact, we find that

$$\langle S_{1+} S_{2-}\rangle = \frac{1}{N} \sum_{j_1=1}^{m} \sum_{j_2=L+1}^{L+m} e^{i k a (j_2 - j_1)}.$$  

(47)

If the size of the blocks is small compared to the wavelength of the spin wave, the term in the sum will all have approximately the same phase, and will add coherently. This would show that the blocks of spins are entangled for this state.

If we want to look at more than one magnon, more sophisticated techniques are required. We will make use of the Holstein-Primakoff transformation, which expresses the spin operators in terms of boson creation and annihilation operators, and allows us to approximately diagonalize the Hamiltonian. The
Holstein-Primakoff transformation of the spin operator $S_j$ to boson creation and annihilation operators $a_j^\dagger$, $a_j$ is given by

$$
S_{j+} = S_{jx} + iS_{jy} = (2S - a_j^\dagger a_j)^{1/2}a_j,
$$

$$
S_{j-} = S_{jx} - iS_{jy} = a_j^\dagger(2S - a_j^\dagger a_j)^{1/2},
$$

$$
S_{jz} = S - a_j^\dagger a_j,
$$

(48)

where

$$
[a_j, a_l^\dagger] = \delta_{j,l}.
$$

(49)

If we consider only situations in which the number of flipped spins is small compared to the total number of spins, we can expand the square roots and keep only the first terms in the expansion. In addition we make a transformation from the spin operators, $a_j^\dagger$ and $a_j$, to the magnon variables, $b_k^\dagger$ and $b_k$, defined by

$$
b_k = N^{-1/2} \sum_j e^{ik_j r_j} a_j,
$$

(50)

where $r_j$ is the position of spin $j$. The magnon operators satisfy boson commutation relation:

$$
[b_k, b_{k'}^\dagger] = \delta_{k,k'}, \quad [b_k, b_{k'}] = 0.
$$

(51)

When the number of flipped spins is much less than $N$, the Hamiltonian is diagonal in the magnon operators,

$$
H = \sum_k \omega_k b_k^\dagger b_k,
$$

(52)

where

$$
\omega_k = 2J z S(1 - \gamma_k) + 2\mu_0 H_0,
$$

(53)

and

$$
\gamma_k = \frac{1}{z} \sum_\delta e^{i k_\delta}.
$$

(54)

As was mentioned before, we will work in the limit $H_0 \to 0^+$, so that in the ground state the spins are lined up along the $z$ axis. In these equations a center of symmetry is assumed so that $\gamma_k = \gamma_{-k}$, and $z$ is the number of nearest neighbors each spin has.

Now we are in a position to consider the two-magnon state. We shall again consider the one-dimensional case, that is $N$ spins in a line. We want to study the entanglement of the state

$$
|\Psi\rangle = b_{k_1}^\dagger b_{k_2}^\dagger |0\rangle = \frac{1}{N} \sum_{u,v} e^{-ik_1 x_u} e^{-ik_2 x_v} a_u^\dagger a_v^\dagger |0\rangle
$$

(55)

for $k_1 \neq k_2$. We shall examine the entanglement between two blocks consisting of $m$ spins each, one beginning at spin 1 and the other beginning at spin $L$, so
that the blocks are separated by \( L - m \) spins. Therefore, we choose

\[
A = \sqrt{2S} \sum_{j=1}^{m} a_j, \quad B = \sqrt{2S} \sum_{j=L+1}^{L+m} a_j,
\]

in Eq. (4). We find

\[
\langle A^\dagger AB^\dagger \rangle = \frac{4S^2}{N^2} \left\{ 2xy + 2xy \cos [La(k_1 - k_2)] \right\},
\]

where \( x = [\cos(k_1 ma) + 1]/[\cos k_1 a + 1] \), and \( y = [\cos(k_2 ma) + 1]/[\cos k_2 a + 1] \), and

\[
\langle A^\dagger B \rangle^2 = \frac{4S^2}{N^2} \left\{ x^2 + 2xy \cos [La(k_1 - k_2)] + y^2 \right\}.
\]

Therefore, the state is entangled if

\[
(x - y)^2 > 0,
\]

which is true as long as \( x \neq y \). One situation where \( x = y = 1 \) is when the block size is one \( m = 1 \), implying that the condition (4) does not detect entanglement between individual spins in the two-magnon state. Recall that \( k_1 a = \pi 2n_1/N \) and \( k_2 a = \pi 2n_2/N \). If the block size \( m \) is such that \( m2n_1/N = 2l_1 + 1 \) and \( m2n_2/N = 2l_2 + 1 \) where \( l_1 \) and \( l_2 \) are integers, hence \( x = y = 0 \) and no entanglement is found according to the inequality (59). The condition (59) indicates that in the ideal zero-temperature two-magnon state, entanglement is found regardless of how far the two blocks are separated. This no longer occurs in the more realistic non-zero temperature state we are going to investigate below.

### 4.2 Finite temperature

Now that we have seen that the entanglement condition, Eq. (4), is useful in detecting entanglement in states consisting of a few magnons, let us see whether it can also detect entanglement in a system of ferromagnetically interacting spins at a finite temperature, \( T \). The density matrix for the system is now given by

\[
\rho = \frac{1}{Z} e^{-\beta H},
\]

where \( \beta = 1/(k_B T) \) and \( k_B \) is Boltzmann’s constant. The partition function of the system, \( Z \) is given by

\[
Z = \text{Tr}(e^{-\beta H}) = \prod_k \sum_n e^{-\beta \omega_k n_k} = \prod_k \frac{1}{(1 - e^{-\beta \omega_k})},
\]

and \( n_k \) is the number of magnons with wave vector \( k \).

We first look for entanglement between two individual spins having radius vectors \( \mathbf{r}_1 \) and \( \mathbf{r}_2 \) by employing the inequality in Eq. (4) with

\[
A = S_{j_1+} = \sqrt{2S} a_1, \quad B = S_{j_2+} = \sqrt{2S} a_2.
\]
We then have that
\[
\langle AB^\dagger \rangle = \langle S_{j1} + S_{j2} \rangle = \frac{2S}{N} \sum_{k_1} \sum_{k_2} e^{-ik_1 \cdot r_1 + ik_2 \cdot r_2} \text{Tr}(b_{k_1} b_{k_2}^\dagger \rho).
\] (63)

Using the relationship \[\sum_{n=0}^{\infty} (n+1)x^n = \frac{1}{(1-x)^2},\] one obtains
\[
\text{Tr}(b_{k_1} b_{k_2}^\dagger \rho) = \frac{\delta_{k_1,k_2}}{1 - e^{-\beta \omega_{k_1}}},
\] (64)

so that
\[
\langle AB^\dagger \rangle = \frac{2S}{N} \sum_{k_1} e^{-ik_1 \cdot (r_1 - r_2)} \frac{1}{1 - e^{-\beta \omega_{k_1}}}.
\] (65)

This gives us the left-hand side of our inequality, and we now need to find the right-hand side. Using
\[
\langle n_{k_1}, n_{k_2}, n_{k_3}, n_{k_4}, \ldots | b_{k_1}^\dagger b_{k_2}^\dagger b_{k_3}^\dagger b_{k_4} | n_{k_1}, n_{k_2}, n_{k_3}, n_{k_4}, \ldots \rangle = \delta_{k_1,k_2} \delta_{k_3,k_4} n_{k_1} n_{k_2} n_{k_3} + \delta_{k_1,k_3} \delta_{k_2,k_4} n_{k_1} n_{k_2} + 1,
\] (66)

we obtain for the right hand side of Eq. (4)
\[
\langle A^\dagger A B^\dagger B \rangle = \langle S_{j1} - S_{j1} + S_{j2} - S_{j2} \rangle = \left( \frac{2S}{N} \right)^2 \sum_{k_1,k_2} \left[ \frac{e^{-\beta (\omega_{k_1} + \omega_{k_2})}}{(1 - e^{-\beta \omega_{k_1}})(1 - e^{-\beta \omega_{k_2}})} + \frac{e^{i(k_1 - k_2) \cdot (r_1 - r_2)} e^{-\beta \omega_{k_1}}}{(1 - e^{-\beta \omega_{k_1}})(1 - e^{-\beta \omega_{k_2}})} \right].
\] (67)

Equation (67) shows that \(\langle A^\dagger A B^\dagger B \rangle\) can be separated into two parts: the first one represents the self correlation of the particles and is distance independent, while the second represents interparticle correlations and depends on the distance between the particles.

Let us now specialize to a cubic lattice with lattice constant \(a\) and \(z = 6\). If \(|ka| \ll 1\), then
\[
1 - \gamma_k = 1 - \frac{1}{3} (\cos k_x a + \cos k_y a + \cos k_z a) \simeq \frac{1}{6} k^2 a^2,
\] (68)

and the magnon energy can be expressed as \(\omega_k = Dk^2\), where \(D = 2JSa^2\). To tackle the sums over \(k\) we note that
\[
-\frac{\pi}{a} < k_j \leq \frac{\pi}{a}
\] (69)

and approximate the cube by a sphere, so that \(k_j \leq \sqrt{\frac{3\pi}{a}}\). We replace the sums by integrals in a spherical coordinate system, and our entanglement inequality, Eq. (4) becomes, upon using Eqs. (65) and (67) and carrying out the angular integrations,
\[
Q = I_1^2 - (I_2^2 + I_1 I_3) > 0,
\] (70)
where $|\langle AB^\dagger \rangle|^2 = I_1^2$, $\langle A^\dagger AB^\dagger B \rangle = I_2^2 + I_1 I_3$, 

$$I_1 = \int_0^{y_0} dy f\left(\frac{y}{y_0} \frac{\Delta r}{a} \pi \sqrt{3}\right) \frac{y^2}{1 - e^{-y^2}}, \quad (71)$$

$$I_2 = \int_0^{y_0} dy \frac{y^2 e^{-y^2}}{1 - e^{-y^2}}, \quad (72)$$

$$I_3 = \int_0^{y_0} dy f\left(\frac{y}{y_0} \frac{\Delta r}{a} \pi \sqrt{3}\right) \frac{y^2 e^{-y^2}}{1 - e^{-y^2}}, \quad (73)$$

and $f(x)$ is the familiar function

$$f(x) = \frac{\sin x}{x}. \quad (74)$$

Here $\Delta r = |r_1 - r_2|$ is the interatomic distance, $y_0 = \sqrt{\beta D} \sqrt{3\pi} / a$ and the dimensionless integration variable $y$ is related to the wave vector component $k$ by $y = \sqrt{\beta D} k$. Due to the presence of the exponentially decaying factor $e^{-y^2}$ in the numerators of the integrands in $I_2$ and $I_3$, small values of $y$, $y \lesssim 1$, contribute most to these integrals. In the case of $I_3$, the fact that $y^2 e^{-y^2} / (1 - e^{-y^2})$ is a decreasing function, causes that integral to be positive.

As $T$ increases, the upper limit of the integrals, $y_0$, which is proportional to $1 / \sqrt{T}$, tends to zero, and, as a result, $e^{-y^2} \to 1$ and $I_3 \to I_1$. Hence the inequality (70) becomes $I_1^2 - (I_2^2 + I_3^2) > 0$, which cannot be fulfilled. This means that our condition does not show the existence of entanglement in the high temperature limit, which is consistent with what we expect on physical grounds, i.e. that there is no entanglement at high temperature. As the temperature decreases, the upper integral limit $y_0$ increases. The integrand in $I_1$ is an oscillating function of $y$ with a varying sign and an increasing magnitude, and the sign and value of $I_1$ are determined mostly by the contribution near $y_0$. For short distances and low temperatures, the absolute value of $I_2$ is typically much larger than those of $I_1$ and $I_3$, which makes it the leading factor in deciding the sign of $Q$.

In Fig. 1 we give a representative example of the distance dependence of $Q$. Positive values of $Q$ indicate entanglement. It can be seen from Eq. (71) that $I_1$ is an oscillating function of the interatomic distance $\Delta r$, with a damping envelop. This shows up in the behavior of $Q$: If we allow for continuous values of $\Delta r / a$ we will see the damped oscillations more clearly. For short interatomic distances, entanglement is clearly observed. $Q$ turns negative for the first time at $\Delta r / a = 13$. However, it can again become positive, meaning the reappearance of detectable entanglement at much larger distances before becoming permanently negative. In Fig. 1 the temperature is fixed. For lower temperatures, the shortest distance at which $Q$ is found to be negative and the overall range over which $Q$ is found to be positive increases.

The temperature dependence of $Q$ is illustrated in Fig. 2 for different values of the inter-particle distance. It can be seen that as the temperature increases, $Q$ monotonically decreases, and the shorter the interparticle distance, the later $Q$ crosses into the negative range. In other words, as we would expect on
Figure 1: The quantity $Q$ as a function of the interatomic distance, scaled by the lattice constant. A positive $Q$ indicates entanglement. Some large (positive) values of $Q$ are beyond the scope of the figure. The temperature is fixed at $\sqrt{2JS/(k_B T)} = 7$.

Figure 2: The quantity $Q$ as a function of the temperature for different inter-particle distances $\Delta r/a = (a) 1$ (solid line), (b) 3 (dashed line), (c) 10 (dotted line), (d) 20 (dot-dashed line).
physical grounds, lower temperatures and shorter inter-particle distances are more favorable for entanglement generation, and this parameter region is where our condition shows the presence of entanglement. If we assume some typical parameters for ferromagnets $D \sim 0.5 \times 10^{-28}$ erg cm$^2$ and $a \sim 4\AA$, using $k_B = 1.38 \times 10^{-16}$ erg K$^{-1}$, the temperatures at which $Q$ turns negative, which is where entanglement is no longer detected, are 420K, 150K, 20K, and 8K for $\Delta r/a = 1$ (solid line), 3 (dashed line), 10 (dotted line), and 20 (dot-dashed line), respectively. Thus when the atoms are closer located, entanglement can be detected at higher temperatures. We are, of course, assuming that these temperatures are still considerably below the critical temperature, so that the spin-wave description remains valid.

Let us now proceed to use the condition (4) to investigate entanglement between blocks of $m$ spins each, one beginning at spin 1 and the other beginning at spin $L$. With $A$ and $B$ chosen as in Eqs. (50), (51), calculations similar to the derivation of Eqs. (70)-(74) show that the entanglement condition now takes on the form

$$Q = \left( \sum_{i=1}^{m} \sum_{j=L+1}^{L+m} I_{1ij} \right)^2 - \left\{ mI_2 + 2 \sum_{i=1}^{m} \sum_{i'=i+1}^{m} I_{3ii'} \right\}^2 + \left( \sum_{i=1}^{m} \sum_{j=L+1}^{L+m} I_{1ij} \right) \left\{ \sum_{i=1}^{m} \sum_{j=L+1}^{L+m} I_{3ij} \right\} > 0,$$

(75)

where $I_{1ij}$ and $I_{3ij}$ are given by the respective Eqs. (71) and (73) with $\Delta r = |r_1 - r_2|$ being replaced by $\Delta r_{ij} = |r_i - r_j|$. Again $(A^1AB^1B)$ (the term in the curly brackets) consists of two parts, the first representing the correlations between spins within a block and the second representing the correlations between blocks. Equation (75) is general in that the atoms can be arranged in an arbitrary manner in space. The only assumption used is that the two blocks do not overlap. As in the case of individual spins, in the high temperature limit $y_0 \to 0$, $I_{3ij} \to I_{1ij}$, indicating explicitly that the inequality (75) cannot be satisfied. At low temperatures, whether $Q$ is positive or negative depends on the details of the terms in the sums over $I_{1ij}$.

In Fig. 3 we plotted $Q$ as a function of the block size $m$, $m$ being the number of spins contained in each block. It is assumed that each block consists of neighboring spins located along a straight line, one beginning at spin 1 and the other beginning at spin $L$ so that the blocks are separated by $L - m$ spins. For the parameters used in Fig. 3, the case of individual spins $m = 1$ exhibits no entanglement (cf. Fig. 1). As the size of the blocks $m$ increases (Fig. 3 inset), $Q$ acquires positive values indicating a presence of inter-block entanglement. The change is not monotonic, however. As $m$ increases further, the entanglement detected by our condition can disappear and reappear, being particularly strong for $m = 11, 12, 13$. The sign of $Q$ obviously depends on whether the $I_{1ij}$ add constructively or destructively. An examination of inter-block entanglement may thus offer much richer physics than simply a study of the entanglement between individual spins.
Figure 3: The quantity $Q$ as a function of the block size $m$ – the number of spins in each block for a fixed dimensionless temperature $\sqrt{2JS/(k_BT)} = 7$ and for $L = 13$. The inset zooms in the part of the plot for small $m$.

5 Conclusions

We have presented two entanglement conditions for spin systems that allow us to study the entanglement between blocks of spins. Most tests for entanglement in spin systems test for either complete separability or for entanglement between individual spins, and the results in this paper complement those. We have shown that in some cases the conditions involving blocks of spins can detect entanglement when tests of individual spins cannot. It was shown that our entanglement conditions can detect entanglement in intelligent spin states and in states of a spin chain containing a small number of spin waves. This latter result was then extended to show that entanglement in spin waves at finite temperature can also be detected.

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