DIVERGENT OPERATOR WITH DEGENERACY AND
RELATED SHARP INEQUALITIES

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Abstract In this paper we classify all positive extremal functions to a sharp weighted Sobolev inequality on the upper half space, which involves divergent operators with degeneracy on the boundary. We show that such a weighted Sobolev inequality can be used to derive a sharp Sobolev type inequality involving Baouendi-Grushin operator.

1. Introduction

The current work is motivated and heavily influenced by the popular work of Caffarelli and Silvestre [7], by our recent work on the extension type operators (see, for example, Dou and Zhu [10], Dou, Guo and Zhu [9], Gluck [20], Gluck and Zhu [21] and Wang and Zhu [40]). The results, among the other things, almost completely settled an open questions for years (see Theorem 1.9 below).

Throughout the paper, we denote \( \mathbb{R}_{+}^{n+1} = \{ (y, t) \in \mathbb{R}^{n+1} : t > 0 \} \) as the upper half space.

1.1. A divergent operator. In [7], Caffarelli and Silvestre study the following extension problem for \( \alpha \in (-1, 1) \):

\[
\begin{cases}
\text{div}(t^\alpha \nabla u) = 0, & \text{in } \mathbb{R}_{+}^{n+1}, \\
u(y, 0) = f(y), & \text{on } \partial \mathbb{R}_{+}^{n+1}.
\end{cases}
\] (1.1)

A nice “pointwise” view on a global defined fractional Laplacian operator is given by

\[
(-\Delta)^{\frac{1-\alpha}{2}} f(y) = -C \lim_{t \to 0^+} t^\alpha \frac{\partial u}{\partial t}(y, t)
\]

for a suitable constant \( C \).

For \( f(y) \) in a good space, solution \( u(y, t) \) to (1.1) can be represented, up to a constant multiplier, as an extension of \( f(y) \) via operator \( P_\alpha \):

\[
u(y, t) = P_\alpha(f) := \int_{\partial \mathbb{R}_{+}^{n+1}} P_\alpha(y - x, t)f(x)dx,
\]

whose positive kernel is

\[
P_\alpha(y, t) = \frac{t^{1-\alpha}}{(|y|^2 + t^2)^{(n+1)/2}}.
\]

See more discussions in the introduction part in Wang and Zhu [40] for the related study of the extension operators involving divergent operator \( \text{div}(t^\alpha \nabla u) \).

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1.2. New nonlinear equations. Our original interest is to understand the following general equation

$$- \text{div}(t^\alpha \nabla u) = f(t, u), \quad u \geq 0, \quad \text{in} \quad \mathbb{R}^{n+1}$$  \hfill (1.2)

with or without explicitly given boundary conditions. For $f(t, u) = 0$, as we mentioned above, equation (1.2) was discussed by Caffarelli and Silvestre [7] in connecting to the study of fractional Laplacian operators; the Liouville type theorems for this homogeneous equation were obtained recently by Wang and Zhu [40]. Here, we shall study equation (1.2) for $f(y, t) = t^\beta |u|^{p-1}$. After a standard scaling argument, we can see that

$$p^* = \frac{2n + 2\beta + 2}{n + \alpha - 1}$$ \hfill (1.3)

is so called critical exponent, usually associated with a Sobolev type inequality on an unbounded domain. In fact, we have such an inequality.

**Proposition 1.1.** Assume $n \geq 1$, $l > -1$, $k > 0$ and $\frac{nl}{n+1} \leq k \leq l + 1$. There is a positive constant $C_{n,k} > 0$ such that for all $u \in C^\infty_0(\mathbb{R}^{n+1})$,

$$\left( \int_{\mathbb{R}^{n+1}} t^l |u|^\frac{n+1}{n+\beta + 1} \, dy \, dt \right)^{\frac{n}{n+\beta + 1}} \leq C_{n,k} \int_{\mathbb{R}^{n+1}} t^k |\nabla u|^p \, dy \, dt. \quad (1.4)$$

Proposition 1.1 is a known result. In fact, it is true even for $k \leq 0$, see Maz’ya [33, inequality (2.1.35)]. Here we give a direct proof for $k > 0$, similar to the original one in Gagliardo [17] and Nirenberg [36]. See Section 2 for more details.

Define the weighted Sobolev space $D^{1,p}_\alpha(\mathbb{R}^{n+1})$ as the completion of the space $C^\infty_0(\mathbb{R}^{n+1})$ under the norm

$$\|u\|_{D^{1,p}_\alpha(\mathbb{R}^{n+1})} = \left( \int_{\mathbb{R}^{n+1}} t^\alpha |\nabla u|^p \, dy \, dt \right)^{\frac{1}{p}}.$$

We say $u \in D^{1,p}_\alpha(\mathbb{R}^{n+1})$ if for any compact set $K \subset \mathbb{R}^{n+1}$,

$$\int_{\mathbb{R}^{n+1} \cap K} t^\alpha |\nabla u|^p \, dy \, dt < \infty.$$

Using Hölder inequality, we can derive the following inequality from (1.4).

**Corollary 1.2.** Assume $n \geq 1$, $\beta > -1$, $\alpha + \beta \geq 0$, $\frac{2n+2\beta}{n+1} \leq \beta \leq \beta + 2$. There is a positive constant $C_{1,\alpha,\beta} > 0$ such that, for all $u \in D^{1,2}_{\alpha,\beta}(\mathbb{R}^{n+1})$,

$$\left( \int_{\mathbb{R}^{n+1}} t^\beta |u|^\frac{2n+2\beta+2}{n+\alpha+1} \, dy \, dt \right)^{\frac{n+\alpha+1}{2n+2\beta+2}} \leq C_{1,\alpha,\beta} \int_{\mathbb{R}^{n+1}} t^\alpha |\nabla u|^2 \, dy \, dt. \quad (1.5)$$

**Remark 1.3.** The condition of $\alpha$ and $\beta$ in Corollary 1.2 implies that $\alpha \geq 0$. Besides, if $\alpha$ is zero, $\beta$ is also zero, and it is the classical result of Sobolev inequality. Thus we only consider $\alpha > 0$.

Sobolev inequalities with monomial weights were also studied early by Cabre and Ros-Oton [5, Theorem 1.3]. In particular, for $\alpha = \beta$, inequality (1.5) and its sharp form were obtained by Cabre and Ros-Oton [5, Theorem 1.3], Bakry, Gentil and Ledoux [2] and Nguyen [37], essentially from the classical sharp Sobolev inequality in $\mathbb{R}^{n+1}$.
To study the sharp form of inequality (1.5) for general $\alpha$ and $\beta$, we define

$$S_{1,\alpha,\beta} := \inf_{u \in C^\infty_0(\mathbb{R}^{n+1}) \setminus \{0\}} \frac{\int_{\mathbb{R}^{n+1}} |t^\alpha \nabla u|^2 dy}{\int_{\mathbb{R}^{n+1}} |t^\beta |u|^{2+2\alpha+3/n+1} dy} > 0. \quad (1.6)$$

Using the concentration compactness principle, we obtain the existence of the extremal functions for $\alpha > \frac{n-1}{n+1} \beta$. The case $\alpha = \frac{n-1}{n+1} \beta$ is more complicated, see details in Section 3.

**Theorem 1.4.** Assume that $\beta > -1$, $\alpha + \beta \geq 0$ and $\frac{n-1}{n+1} \beta < \alpha < \beta + 2$, constant $S_{1,\alpha,\beta}$ is achieved by a nonnegative extremal function $u(y, t) \in D_{\alpha}^{1,2}(\mathbb{R}^{n+1})$.

Let $u(y, t) \geq 0$ be an extremal function to $S_{1,\alpha,\beta}$, then $\forall \phi(y, t) \in D_{\alpha}^{1,2}(\mathbb{R}^{n+1})$,

$$\int_{\mathbb{R}^{n+1}} t^\alpha \nabla u \cdot \nabla \phi dy dt = \int_{\mathbb{R}^{n+1}} t^\beta |u|^{2+2\alpha+3/n+1} \phi dy dt. \quad (1.7)$$

If we know that $u \in C^2(\mathbb{R}^{n+1}) \cap C^1(\mathbb{R}^{n+1})$, then $u(y, t)$ is a classical solution to the following equation

$$\begin{cases} -\text{div}(t^\alpha \nabla u) = t^\beta |u|^{2+2\alpha+3/n+1}, & (y, t) \in \mathbb{R}^n \times \mathbb{R}_+, \\ \lim_{t \to 0^+} t^\alpha \frac{\partial u}{\partial t} = 0. \end{cases} \quad (1.8)$$

**Definition 1.5.** $u \in D_{\alpha,\text{loc}}^{1,2}(\mathbb{R}^{n+1})$ is said to be a weak solution to (1.8) if equality (1.7) holds for all $\phi \in D_{\alpha}^{1,2}(\mathbb{R}^{n+1})$.

Due to the degeneracy or singularity of the operator, we cannot show that any weak solution is in $C^1(\mathbb{R}^{n+1})$. But we are able to show

**Theorem 1.6.** Let $\beta > -1$, $\alpha + \beta \geq 0$ and $\frac{n-1}{n+1} \beta < \alpha < \beta + 2$. Assume that $u \in D_{\alpha,\text{loc}}^{1,2}(\mathbb{R}^{n+1})$ is a weak solution to equation (1.8), then $u \in C^2(\mathbb{R}^{n+1}) \cap C^\gamma(\mathbb{R}^{n+1})$ for some $\gamma \in (0, 1)$.

We obtain the following Liouville theorem for positive weak solutions to equation (1.8) for $\alpha \geq 0$. In two special cases, we obtain the precise form of these solutions, thus can compute precisely the sharp constant to inequality (1.3).

**Theorem 1.7.** Let $\beta > -1$, $\alpha + \beta \geq 0$ and $\frac{n-1}{n+1} \beta < \alpha < \beta + 2$. If $n = 1$, in addition assume

$$1 - (1 - \alpha)^2 \leq \frac{\alpha(2 + \beta)}{(\alpha + \beta + 2)^2}. \quad (1.9)$$

Assume that $u \in D_{\alpha,\text{loc}}^{1,2}(\mathbb{R}^{n+1})$ is a positive weak solution to equation (1.8). Then, $u(y, t) = \frac{1}{\sqrt{|y-y^0|^2 + (t+A)^2}} \psi(\frac{(y-y^0, t+A)}{|y-y^0|^2 + (t+A)^2} - (y^0, A))$, \quad (1.10)

for some $y^0 \in \mathbb{R}^n$, $A > 0$, and $\psi(r) > 0$ satisfies an ordinary differential equation

$$\begin{cases} \psi'' + \frac{n}{r} \psi' - \frac{\alpha(\alpha-\gamma-1)}{4\gamma} r \psi = -C \psi^{\frac{n+2\alpha+3}{n+1}}, & 0 < r < \frac{1}{2A}, \\ \psi(\frac{1}{2A}) = A. \end{cases} \quad (1.11)$$

for some constant $C > 0$ independent of $A$. Further, there is only one solution to equation (1.11).
Moreover, in following two cases, the solutions can be explicitly written out.

1) If \( \beta = \alpha - 1, \alpha \geq \frac{1}{2} \) for \( n > 1 \) or \( \alpha \in \{ \frac{1}{2} \} \cup \left( \frac{1}{2}, \infty \right) \) for \( n = 1 \), then up to the multiple of some constant, \( u(y,t) \) must be the form of

\[
u(y,t) = \left(\frac{A}{(A + t)^2 + |y - y^0|^2}\right)^{\frac{n + \alpha - 1}{2}},
\]

where \( A > 0 \), \( y^0 \in \mathbb{R}^n \), and

\[
S_{1,\alpha,\alpha-1} = \alpha(n + \alpha - 1)\left[\frac{\pi^\frac{n}{2}}{\Gamma(n/2 + \alpha)}\right]^{\frac{n}{n + \alpha}}.
\]

2) If \( \beta = \alpha > 0 \) for \( n > 1 \) or \( \alpha \geq \sqrt{2} \) for \( n = 1 \), then up to the multiple of some constant, \( u(y,t) \) must be the form of

\[
u(y,t) = \left(\frac{A}{A^2 + t^2 + |y - y^0|^2}\right)^{\frac{n + \alpha - 1}{2}},
\]

where \( A > 0 \), \( y^0 \in \mathbb{R}^n \), and

\[
S_{1,\alpha,\alpha} = (n + \alpha - 1)(n + \alpha + 1)\left[\frac{\pi^\frac{n}{2}}{\Gamma(n/2 + \alpha)}\right]^{\frac{n}{n + \alpha}}.
\]

For regular solutions, Theorem 1.7 part 2) for \( \alpha = \beta = 0 \) follows from the classical result of Caffarelli, Gidas and Spruck \[6\]. See Zhu’s thesis \[42\] for another proof via the method of moving spheres. Here, we will use the method of moving spheres to prove Theorem 1.7. The method of moving spheres enables us to obtain the precise form of positive solutions to equation (1.8) on the boundary \( \partial \mathbb{R}^{n+1}_+ \). We then transform the equation into a new equation on a ball with constant boundary value, and successfully show that all solutions to the new equation must be radially symmetric with respect to the center of the ball, which has a unique radially symmetric solution for \( \alpha, \beta \) satisfying the conditions in Theorem 1.7. In two cases: \( \beta = \alpha - 1 \) and \( \beta = \alpha \), we can write down the precise unique solution to the ODE (1.11), which leads to the complete classification of positive solutions.

1.3. Baouendi-Grushin Operator. As an application of sharp inequality (1.5) and the classification results in Theorem 1.7 we consider the following critical semilinear equation with Baouendi-Grushin operator

\[
\Delta_x u + (\tau + 1)^2 |z|^{2\tau} \Delta_x u = -u^{\frac{n+2\tau}{n-2}}, \quad u > 0 \quad \text{in} \quad \mathbb{R}^{n+m},
\]

where \( \tau \in (0,\infty), n, m \geq 1, x \in \mathbb{R}^n, z \in \mathbb{R}^m \) and \( Q = m + n(\tau + 1) \) is the homogeneous dimension. The partial differential operator \( \mathcal{L} := \Delta_x + (\tau + 1)^2 |z|^{2\tau} \Delta_x \) is often called Baouendi-Grushin operator \( \{11\, 22\, 23\} \). For \( n = 0 \), equation (1.14) is the constant scalar curvature equation on \( \mathbb{R}^m \), which is widely studied, and well-understood through the work of Gidas, Ni and Nirenberg \[19\] and the work of Caffarelli, Gidas and Spruck \[6\] (see, Zhu’s thesis \[42\] for a simpler proof via the method of moving spheres). For \( n \geq 1 \) and \( \tau > 0 \), the operator is degenerate on \( |z| = 0 \). In particular, for \( n = 1, m = 2k \ (k \in \mathbb{N}) \) and \( \tau = 1 \), equation (1.14) is the constant Webster curvature equation on Heisenberg group \( \mathbb{H} = \mathbb{R} \times \mathbb{C}^n \) for solution \( u(x,z) \) which is radially symmetric in \( z \). Jerison and Lee \[25\, 26\] was able to classify positive solutions with decay at infinity to this equation. See also Garafalo and Vassilev \[18\] for further generalization. For \( \tau = 1 \), equation (1.14) is also related to the transonic flow problem, see, for example, Wang \[41\].
Moreover, equation (1.14) is also related to the following weighted Sobolev inequality.

Let \( \mathcal{D}^1_1(\mathbb{R}^{n+m}) \) be the Hilbert space as the completion of \( C_0^\infty(\mathbb{R}^{n+m}) \) under the norm
\[
\|u(x, z)\|_{\mathcal{D}^1_1(\mathbb{R}^{n+m})} = \left( \int_{\mathbb{R}^{n+m}} (|\nabla_x u|^2 + (\tau + 1)^2 |z|^{2\tau} |\nabla_z u|^2) \, dx \, dz \right)^{\frac{1}{2}}
\]
where \( x \in \mathbb{R}^n, z \in \mathbb{R}^m. \)

**Proposition 1.8.** For \( \tau \geq 0 \), there is an optimal positive constant \( S_\tau(n, m) \) such that, \( \forall u(x, z) \in \mathcal{D}^1_1(\mathbb{R}^{n+m}), \)
\[
\left( \int_{\mathbb{R}^{n+m}} |u|^2 \frac{2-2\tau}{2-\tau} \, dx \, dz \right) \frac{2-2\tau}{2-\tau} \leq S_\tau^{-1}(n, m) \int_{\mathbb{R}^{n+m}} (|\nabla_x u|^2 + (\tau + 1)^2 |z|^{2\tau} |\nabla_z u|^2) \, dx \, dz,
\]
where \( x \in \mathbb{R}^n, z \in \mathbb{R}^m \) and \( Q = m + n(\tau + 1) \) is the homogeneous dimension. Moreover, the equality holds for some extremal functions in \( \mathcal{D}^1_1(\mathbb{R}^{n+m}) \).

For \( \tau > 0 \), the above weighted Sobolev inequalities (1.15) are known for many years. For example, it can be derived from a representation formula for Baouendi-Grushin operator in Franchi, Gutierrez and Wheeden [15] and a Hardy-Littlewood-Sobolev inequality due to Folland-Stein [14], and is written precisely in R. Monti and D. Morbidelli [34, inequality (1.3)]. See, also [16] and [35]. Using inequality (1.14), we will give a self-contained and direct proof in Section 6.

On the other hand, it is a long-standing open problem to find the best constant \( S_\tau(n, m) \) for \( \tau > 0 \) in the above theorem.

The main difficulty seems to be the lack of radially symmetric property for the extremal functions. Positive answer is known only in the following cases: (1) For \( n = 1, m = 1 \) and \( \tau = 1 \), the sharp inequality was early obtained by Beckner [8] from the hyperbolic geometry point of view. (2) For \( n \geq 1, m \geq 1 \) and \( \tau = 1 \), if positive solution \( u(x, z) \) is radially symmetric about \( z \) and decays to zero at infinity, the classification was essentially obtained in the early work of Jerison and Lee [25] in their study of CR Yamabe problem (for \( n = 1, m \) is even), and by Garofalo and Vassilev [18, Theorem 1.5]. (3) The decay assumption can be removed by the work of R. Monti and D. Morbidelli [34, Theorem 2.7].

Here we will obtain the sharp constant in inequality (1.15) for all \( \tau > 0 \), and classify all positive \( C^2 \) solutions that are radially symmetric in \( z \) to equation (1.14) for \( \tau > 0 \) and \( n, m \geq 1 \) except the case of \( m = 2 \) and \( n = 1. \)

**Theorem 1.9.** 1. For \( \tau = 1 \), the equality in (1.15) holds up to the multiple of some constant for all \( u(x, z) \) given by
\[
u(x, z) = \left( \frac{A}{|x - x_0|^2 + (|z|^2 + A)^2} \right)^{\frac{2n+m-2}{n+m}},
\]
where \( x_0 \in \mathbb{R}^n, A > 0, \) and
\[
S_1(n, m) = m(2n + m - 2) \left[ \frac{\pi^{\frac{n+m}{2}} \Gamma\left(\frac{n+m}{2}\right)}{\Gamma(n + m)} \right] \left( \frac{2}{n+m} \right)^{\frac{n+m}{2}},
\]
for \( n, m \geq 1 \) except the case of \( m = 2 \) and \( n = 1. \)

Moreover, for \( n, m \geq 1 \) except the case of \( m = 2 \) and \( n = 1, \) if \( u(x, z) \in C^2(\mathbb{R}^{n+m}) \) is a positive solution to equation (1.14) and is radially symmetric in \( z, \) then \( u(x, z) \) is given by (1.16).
2. For $\tau \geq 0$, the equality in (1.15) holds for all $u(x, z)$ given by
\[
u(x, z) = \left(\frac{1}{|x - x^0|^2 + (|z|^{\tau+1} + A)^2}\right)^{2-\frac{2}{\tau+1}} \psi\left(\frac{|x - x^0| |z|^{\tau+1} + A}{|x - x^0|^2 + (|z|^{\tau+1} + A)^2} - (x^0, A)\right)
\]
where $\psi > 0$ is the unique solution to (1.14), and $n, m \geq 1$ except $m = 2, n = 1$.
Moreover, for $n, m \geq 1$ except the case of $m = 2$ and $n = 1$, if $u(x, z) \in C^2(\mathbb{R}^{n+m})$ is a positive solution to equation (1.14) and is radially symmetric in $z$, then $u(x, z)$ is given by (1.17).

Unfortunately, the case of $m = 2$ and $n = 1$ is left open (the main reason is that: in this case, condition (1.9) is not satisfied, see Section 6 for more details).

It seems to be standard to show that all extremal functions in $D^1_0(\mathbb{R}^{n+m})$ to the sharp inequality (1.15) must be $C^2(\mathbb{R}^{n+m})$ functions which satisfy equation (1.14). It is certainly the case when $\tau = 0$. But for $\tau > 0$, we have not found a reference to address this point. We shall come back to discuss the regularity of weak solutions to equation (1.14) in our future study.

For general $\tau > 0$, one can obtain similar result for the best constant if one can find one solution to the ODE (1.11). See Section 5 and 6 for more details.

1.3.1. Relation between two operators. Here, we use the case of $m = 1$ to illustrate the reason that Theorem 1.7 part 1) leads to the proof of Theorem 1.9 part 1). For $m = 1$, we are able to classify all extremal functions and compute the best constant to the sharp form of inequality (1.5) in the case of $\beta = \alpha - 1$, in particular, we can classify all extremal functions and compute the constant while $\beta = -\alpha$ (that is: $\alpha = 1/2, \beta = -1/2$). In this case, using $|z| = (2\tau)^{1/2}, x = 2y$, we obtain the sharp inequality (1.15) and an extremal function, thus can compute the best constant for $\tau = 1$. Similar argument for $m \geq 2$ also works, see more details in Section 6. To prove Theorem 1.9 part 2), we use the assumption that the positive solution is $C^1$ and then use even reflection to classify the solution in each quadrant. See Section 6 for more details.

It is worth pointing out: we do not know whether one can show that a positive $C^2$ solution $u(x, z)$ to (1.14) for $m > 1$ is radially symmetric in variable $z$ by using the method of moving planes or not.

The paper is organized as follows: We first present a direct proof of Proposition 1.1 in Section 2; In Section 3, we prove the existence of extremal functions for inequality (1.5). We show that these extremal functions are Hölder continuous up to the boundary in Section 4. In Section 5, we prove the Liouville theorem (Theorem 1.7). In Section 6 we derive the results related to Baouendi-Grushin operator. The proofs of some technical lemmas are given in the Appendix.

2. Generalized Gagliardo-Nirenberg inequality

In this section, we shall derive the generalized Gagliardo-Nirenberg inequality (Proposition 1.1) for any $u \in C^0(\mathbb{R}^{n+m})$. We thank H. Brezis for sharing his comment on the history of the popular named Gagliardo-Nirenberg inequality. Since we are not able to verify the details first hand, we stick with the common name (the essential idea first appeared in Gagliardo’s paper [17], and shortly after it appeared in Nirenberg’s paper [36]).

We first show that the inequality holds for $l = k - 1 > -1$ (that is: $k = l + 1 > 0$, the upper bound for $k$).
Lemma 2.1. Suppose $k > 0$ and $u \in C_0^\infty(\mathbb{R}^{n+1}_+)$, then

$$
\int_{\mathbb{R}^{n+1}_+} t^{k-1} |u| dy dt \leq C(k) \int_{\mathbb{R}^{n+1}_+} t^k |\nabla u| dy dt.
$$

(2.1)

Proof. Observe that for $k > 0$,

$$
\int_0^\infty t^{k-1} u(y,t) dt = -\frac{1}{k} \int_0^\infty \frac{\partial u(y,t)}{\partial t} \cdot t^k dt.
$$

Integrating with respect to $y$ on both sides gives the desired inequality. \(\square\)

We then follow the proof for the classical Gagliardo-Nirenberg inequality to establish the inequality for $l = \frac{n+1}{n} k$ (that is: $k = \frac{n}{n+1}$, the lower bound for $k$).

Lemma 2.2. Suppose $k \geq 0$ and $u \in C_0^\infty(\mathbb{R}^{n+1}_+)$, then

$$
\left( \int_{\mathbb{R}^{n+1}_+} t^{\frac{n+1}{n}k} |u|^{\frac{n+1}{n}} dy dt \right)^{\frac{n}{n+1}} \leq C(n,k) \int_{\mathbb{R}^{n+1}_+} t^k |\nabla u| dy dt.
$$

(2.2)

Proof. For $k > 0$, integration by parts gives

$$
z^k u(y,z) = \int_z^\infty \frac{d}{dt} t^k u(y,t) dt = \int_z^\infty \left[ k t^{k-1} u(y,t) + t^k \partial_t u(y,t) \right] dt
$$

$$\leq C(k) \int_0^\infty t^k |\nabla u|(y,t) dt,
$$

where we have used Lemma 2.1. Above inequality obviously holds for $k = 0$, so is the following inequality: for $i = 1, \cdots, n$, we

$$
u(y,z) \leq \int_{-\infty}^{+\infty} |\nabla u|(y,z) dy_i.
$$

Therefore,

$$
z^\frac{1}{n} |u|^{\frac{n+1}{n}}(y,z) \leq C(k)^n \prod_{i=1}^n \left( \int_{-\infty}^{+\infty} |\nabla u|(y,z) dy_i \right)^{\frac{1}{n}} \left( \int_0^\infty t^k |\nabla u|(y,t) dt \right)^{\frac{1}{n}}.
$$

Integrating both sides with respect to the measure $z^k dy dz$ and applying the extended Hölder’s inequality with respect to such a measure yield

$$
\int_{\mathbb{R}^{n+1}_+} z^{\frac{n+1}{n}k} |u|^{\frac{n+1}{n}} dy dz
\leq C(k)^\frac{n}{n+1} \prod_{i=1}^n \left( \int_{-\infty}^{+\infty} |\nabla u|(y,z) dy_i \right)^{\frac{1}{n}} \left( \int_0^\infty t^k |\nabla u|(y,t) dt \right)^{\frac{1}{n}} dy dz
$$

$$= C(n,k) \int_{\mathbb{R}^n} \left( \int_0^\infty t^k |\nabla u|(y,t) dt \right)^{\frac{1}{n}} \prod_{i=1}^n \left( \int_{-\infty}^{+\infty} z^k |\nabla u|(y,z) dy_i \right)^{\frac{1}{n}} dz dy
$$

$$\leq C(n,k) \left( \int_0^\infty t^k |\nabla u|(y,t) dt dy \right)^{\frac{1}{n}} \left[ \int_{\mathbb{R}^n} \prod_{i=1}^n \left( \int_{-\infty}^{+\infty} z^k |\nabla u|(y,z) dy_i \right)^{\frac{1}{n}} dz \right]^{\frac{n}{n+1}} dy.$$
Proof of Proposition 1.1. Let
\[ \lambda \leq C(n,k) \left( \int_{\mathbb{R}^n_+} t^k |\nabla u(y,z)|dydz \right)^{\frac{1}{k}} \left( \int_{\mathbb{R}^n_+} \left[ \int_{0}^{\infty} \left( \int_{-\infty}^{\infty} z^k |\nabla u|dydz \right)^{\frac{1}{n+k}} dy \right] \right)^{\frac{n}{n+k}}. \]

\[ = C(n,k) \left( \int_{\mathbb{R}^n_+} z^k |\nabla u(y,z)|dydz \right)^{\frac{1}{k}} \cdot \left[ \int_{\mathbb{R}^{n-1}} \left( \int_{0}^{\infty} \left( \int_{-\infty}^{\infty} z^k |\nabla u|dydz \right)^{\frac{1}{n+k}} dy \right] \right] \frac{n-1}{n+k}. \]

\[ \leq C(n,k) \left( \int_{\mathbb{R}^n_+} z^k |\nabla u(y,z)|dydz \right)^{\frac{1}{k}} \]

\[ \times \left[ \int_{\mathbb{R}^{n-1}} \left( \int_{-\infty}^{\infty} \prod_{i=2}^{n} \left( \int_{-\infty}^{\infty} z^k |\nabla u|dydz \right)^{\frac{1}{n+k}} dy \right] \right] \frac{n-1}{n+k}. \]

\[ \leq \ldots \]

\[ \leq C(n,k) \left( \int_{\mathbb{R}^n_+} z^k |\nabla u(y,z)|dydz \right)^{\frac{1}{k}} \]

\[ \cdots \int_{\mathbb{R}^n_{j-1}} \left( \int_{-\infty}^{\infty} \prod_{i=j+1}^{n} \left( \int_{-\infty}^{\infty} z^k |\nabla u|dydz \right)^{\frac{1}{n+k}} dy \right] \right] \frac{n-j}{n+k} \frac{n-j-1}{n+k}. \]

\[ \leq C(n,k) \left( \int_{\mathbb{R}^n_+} z^k |\nabla u(y,z)|dydz \right)^{\frac{1}{k}}. \]

The proof is completed. \( \square \)

Proof of Proposition 1.1. Let \( \theta = \frac{(n+1)k-n}{n+k} \) and \( p = \frac{n+l+1}{n+k} \). Since \( k-1 \leq l \leq \frac{n+1}{n}k \), we know \( \theta \in [0,1] \) and \( \theta + \frac{n+1}{n}(1-\theta) = p \). For \( k > 0 \), using inequalities (2.1) and (2.2), we conclude that

\[ \int_{\mathbb{R}^n_+} t^l |u|^p dydt \leq \left( \int_{\mathbb{R}^n_+} t^{k-1} |u| dydt \right)^{\theta} \left( \int_{\mathbb{R}^n_+} t^{\frac{n+1}{n}} |u|^{\frac{n+1}{n}} dydt \right)^{1-\theta}. \]

\[ \leq C(n,k) \left( \int_{\mathbb{R}^n_+} t^k |\nabla u| dydt \right)^{p}. \]

Remark 2.3. For \( k = l = 0 \), the proof of Lemma 2.2 is the same as that of the classical Gagliardo-Nirenberg inequality, see, for example, Evans book [12]. However, for \( l > -1 \) and \( k = 0 \), our proof does not work, though we do know inequality (1.4) is still true for \( k = 0 \) from Maz’ya [33] inequality (2.1.35). See the proof of Corollary 1.2 below.

Remark 2.4. If we write \( p = \frac{n+l+1}{n+k} \), we show that condition \( l \leq \frac{n+1}{n}k \) (that is: \( p \leq \frac{n+1}{n} \)) is necessary.

Suppose that Theorem 1.1 is true for some \( k \) and \( l \). Then for any \( \lambda, t_0 > 0 \) satisfying \((1 - \lambda^{-1})t_0 \geq 0 \), we consider the rescaled functions \( u_{\lambda,t_0}(y,t) = \lambda(y^{-1}y_0 + \lambda^{-1}(t-t_0)) \). Plugging \( u_{\lambda,t_0} \) to (1.1), we have

\[ \left( \int_{\mathbb{R}^n_+} t^l |u_{\lambda,t_0}|^p dydt \right)^{\frac{1}{p}} = \lambda^{l+(n+1)/p} \left( \int_{\mathbb{R}^n_+} [z - t_0 + \lambda^{-1}t_0]^{l} \int_{\mathbb{R}^n_+} u(y,z)^p dydz \right)^{\frac{1}{p}} \]

\[ \int_{\mathbb{R}^n_+} t^k |\nabla u_{\lambda,t_0}| dydt = \lambda^{k} \int_{\mathbb{R}^n_+} [z - t_0 + \lambda^{-1}t_0]^k |\nabla u(y,z)| dydz. \]
If we let $t_0 \to \infty$, then we must have $l/p \leq k$, which is equivalent to $l \leq \frac{n+1}{n}k$, and indicates that $p \leq (n+1)/n$.

**Proof of Corollary 1.2** Without loss of generality, we assume $u \geq 0$. Applying Theorem 1.1 to $u^{2(n+1)}$, by Hölder inequality, we have

$$
\left( \int_{\mathbb{R}^{n+1}+1} t^l |u|^{\frac{2(n+1)}{n+2k-l-1}} dydt \right)^{\frac{n+1}{n+2k-l-1}} 
\leq C \int_{\mathbb{R}^{n+1}+1} t^k |u|^{\frac{n+1}{n+2k-l-1}} \nabla u dydt 
$$

$$
\leq C \left( \int_{\mathbb{R}^{n+1}+1} t^l |u|^{\frac{2(n+1)}{n+2k-l-1}} dydt \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^{n+1}+1} t^{2k-l} |\nabla u|^2 dydt \right)^{\frac{1}{2}} 
$$

then

$$
\left( \int_{\mathbb{R}^{n+1}+1} t^l |u|^{\frac{2(n+1)}{n+2k-l-1}} dydt \right)^{\frac{n+2k-l-1}{n+2k-l-1}} \leq C \int_{\mathbb{R}^{n+1}+1} t^{2k-l} |\nabla u|^2 dydt. \quad (2.3)
$$

Taking $\alpha = 2k - l$ and $\beta = l$, we obtain the desired result for $\alpha + \beta > 0$.

The case $\alpha + \beta = 0$ is more subtle, since we proved inequality (1.4) is true only for $k > 0$. See Remark 2.3. But we can prove this case as follows.

Write $q = \frac{2n-2\alpha+2}{n+\alpha-1}$. First, we observe that for $\alpha < 1$,

$$
\int_0^\infty t^{-\alpha} u^q dt = -\frac{q}{1-\alpha} \int_0^\infty t^{1-\alpha} u^{q-1} u dt.
$$

Thus

$$
\int_{\mathbb{R}^{n+1}+1} t^{-\alpha} u^q dydt 
\leq C \int_{\mathbb{R}^{n+1}+1} t^{1-\frac{2\alpha}{n+1}} |u|^{q-1} t^\frac{2\alpha}{n+1} |\nabla u| dydt 
$$

$$
\leq C \left( \int_{\mathbb{R}^{n+1}+1} t^{2-3\alpha} |u|^{2q-2} dydt \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^{n+1}+1} t^\alpha |\nabla u|^2 dydt \right)^{\frac{1}{2}} 
$$

$$
\leq \tilde{C} \left( \int_{\mathbb{R}^{n+1}+1} t^\alpha |\nabla u|^2 dydt \right)^{\frac{q-1}{q}} \left( \int_{\mathbb{R}^{n+1}+1} t^\alpha |\nabla u|^2 dydt \right)^{\frac{1}{2}}.
$$

We used inequality (2.3) in the last step, but in this step, we need $\frac{2\alpha}{n+1} \leq \alpha < 1$. For $n = 1$, Corollary 1.2 for $\alpha + \beta = 0$ has been done. But for $n \geq 2$, we still need to consider the following case.
For \( \alpha \in (0, \frac{n-1}{2n-1}) \) and \( n \geq 2 \), let \( s = \frac{2n}{n-1} \), then we have
\[
\left( \int_{\mathbb{R}^{n+1}_+} t^{-\alpha} |u|^{\frac{2}{\alpha}} \, dy \, dt \right)^\frac{\alpha}{2} = \left( \int_{\mathbb{R}^{n+1}_+} t^{-\alpha} |u|^{\frac{2n}{n-1}(\frac{n-1}{n+1})} \, dy \, dt \right)^{\frac{n(n+1)}{n-1(n+1)}}
\leq C \int_{\mathbb{R}^{n+1}_+} t^{\frac{n-1}{\alpha}} |u|^{\frac{2n}{n+1}} |\nabla u| \, dy \, dt
\leq C \left( \int_{\mathbb{R}^{n+1}_+} t^{\alpha} |\nabla u|^2 \, dy \, dt \right)^\frac{\alpha}{2} \left( \int_{\mathbb{R}^{n+1}_+} t^{\frac{n-1}{\alpha}} |u|^{2(\frac{n+1}{n-1})} \, dy \, dt \right)^\frac{1}{2}
\leq C \left( \int_{\mathbb{R}^{n+1}_+} t^{\alpha} |\nabla u|^2 \, dy \, dt \right)^\frac{\alpha}{2} \left( \int_{\mathbb{R}^{n+1}_+} t^{\alpha} |\nabla u|^2 \, dy \, dt \right)^{\frac{n+1}{2n-1}} = C \left( \int_{\mathbb{R}^{n+1}_+} t^{\alpha} |\nabla u|^2 \, dy \, dt \right)^\frac{\alpha}{2}.
\]

We used inequality (1.4) in the second step and (2.3) in the fourth step. Notice that \( 0 < \alpha < \frac{n-1}{2n-1} \) is needed in (1.4). Then all the cases of Corollary 1.2 have been done.

3. Existence of extremal functions

In this section, we prove the existence of extremal functions to the sharp form of (1.3) by the concentration-compactness principle. Throughout this section, we always assume that \( \beta > -1 \), \( \alpha + \beta \geq 0 \) and \( \frac{n-1}{n+1} \beta < \alpha < \beta + 2 \), and write \( p^* = \frac{2(n+\beta+1)}{n-\alpha-1} \).

We remark that the case \( \alpha = \frac{n-1}{n+1} \beta \) is more complicated. In fact, for \( \alpha = \frac{n-1}{n+1} \beta \), \( p^* = \frac{2(n+1)}{n-1} \). In this case, the existence (for \( n \geq 3 \)), as well as the non-existence results (for \( n = 2 \)) were obtained by Tertikas and Tintarev [39], and Benguria, Frank and Loss [4], respectively.

Recall that the weighted Sobolev space \( D^{1,2}_\alpha(\mathbb{R}^{n+1}_+) \) is defined as the completion of the space \( C_0^\infty(\mathbb{R}^{n+1}_+) \) endowed with the norm
\[
\|u\|_{D^{1,2}_\alpha(\mathbb{R}^{n+1}_+)} = \left( \int_{\mathbb{R}^{n+1}_+} t^\alpha |\nabla u|^2 \, dy \, dt \right)^{\frac{1}{2}}.
\]

And we define
\[
L^p_\beta(\mathbb{R}^{n+1}_+) = \{ u : \mathbb{R}^{n+1}_+ \to \mathbb{R} \mid \|u\|_{L^p_\beta(\mathbb{R}^{n+1}_+)} = \int_{\mathbb{R}^{n+1}_+} t^\beta |u|^p \, dy \, dt < \infty \}.
\]

Define \( B_R(x) = \{ z \in \mathbb{R}^{n+1} \mid |z - x| < R \} \) and \( B_R^+(x) = B_R(x) \cap \mathbb{R}^{n+1}_+ \). We denote by \( \mathcal{M}(\mathbb{R}^{n+1}_+) \) the space of positive, bounded measures in \( \mathbb{R}^{n+1}_+ \). The sharp constant inequality (1.3) can also be classified by
\[
S_{1,\alpha,\beta} = \inf_{u \in C_0^\infty(\mathbb{R}^{n+1}_+)} \left\{ \int_{\mathbb{R}^{n+1}_+} t^\alpha |\nabla u|^2 \, dy \, dt : u \in L^p_\beta(\mathbb{R}^{n+1}_+), \int_{\mathbb{R}^{n+1}_+} t^\beta |u|^p \, dy \, dt = 1 \right\}.
\]

The aim of this section is to show that \( S_{1,\alpha,\beta} \) is attained by some functions. For \( \lambda > 0 \) and \( (z, 0) \in \partial \mathbb{R}^{n+1}_+ = \mathbb{R}^n \), define
\[
u(\lambda, t) = \lambda^{-\frac{n-1}{2n-1}} u\left( \frac{y - z}{\lambda}, \frac{t}{\lambda} \right).
\]
It is easy to verify that
\[ \|u^{\lambda,z}\|_{L^p_\beta}\left(\mathbb{R}^{n+1}_+\right) = \|u\|_{L^p_\beta}\left(\mathbb{R}^{n+1}_+\right), \text{ and } \|u^{\lambda,z}\|_{D^{1,2}_{\alpha}\left(\mathbb{R}^{n+1}_+\right)} = \|u\|_{D^{1,2}_{\alpha}\left(\mathbb{R}^{n+1}_+\right)}. \]

**Proposition 3.1.** Assume \( n \geq 1, \ \beta > -1, \ \alpha + \beta \geq 0 \) and \( \frac{n+1}{n+1} \beta < \alpha < \beta + 2 \). Let \( \{u_m\} \) be a minimizing sequence of functions for \( S_{1,\alpha,\beta} \), then after passing to a subsequence, there exists \( \lambda_j \) and \( z_j \in \mathbb{R}^n \) such that \( u_m^{\lambda_j,z_j} \to u \) in \( L^p_\beta\left(\mathbb{R}^{n+1}_+\right) \). In particular, there exists at least one minimizer for \( S_{1,\alpha,\beta} \).

Apparently, Theorem 1.4 follows from this proposition immediately. To prove Proposition 3.1, we first establish the concentration-compactness principle similar to that in P.L. Lions [30, 31].

**Lemma 3.2.** Assume \( n \geq 1, \ \beta > -1, \ \alpha + \beta \geq 0 \) and \( \frac{n+1}{n+1} \beta < \alpha < \beta + 2 \). Let \( \{u_m\} \) be a bounded sequence in \( D^{1,2}_{\alpha}\left(\mathbb{R}^{n+1}_+\right) \), and \( \mu, \nu \) be two Radon measures and a function \( u \in D^{1,2}_{\alpha}\left(\mathbb{R}^{n+1}_+\right) \), such that

1. \( u_m \to u \) weakly in \( D^{1,2}_{\alpha}\left(\mathbb{R}^{n+1}_+\right) \);
2. \( u_m \to u \) a.e. in \( \mathbb{R}^{n+1}_+ \);
3. \( \nu_m = t^\beta|u_m - u|^p dydt \to \nu \) weakly in \( \mathcal{M}(\mathbb{R}^{n+1}_+) \);
4. \( \mu_m = t^\beta|\nabla(u_m - u)|^2 dydt \to \mu \) weakly in \( \mathcal{M}(\mathbb{R}^{n+1}_+) \).

Define
\[
\mu_\infty = \lim_{R \to \infty} \lim_{m \to \infty} \int_{\mathbb{R}^{n+1}_+ \setminus B_R(0)} t^\beta|\nabla u_m|^2 dydt,
\]
\[
\nu_\infty = \lim_{R \to \infty} \lim_{m \to \infty} \int_{\mathbb{R}^{n+1}_+ \setminus B_R(0)} t^\beta|u_m|^p dydt.
\]

Then,

(\text{i}) \[ \|\mu\| \geq S_{1,\alpha,\beta}\|\nu\|^{\frac{\beta}{p}}, \]

(\text{ii}) \[ \mu_\infty \geq S_{1,\alpha,\beta}\nu_\infty^{\frac{\beta}{p}}, \]

(\text{iii}) \[ \lim_{m \to \infty} \int_{\mathbb{R}^{n+1}_+} t^\beta|\nabla u_m|^2 dydt = \int_{\mathbb{R}^{n+1}_+} t^\beta|\nabla u|^2 dydt + \|\mu\| + \mu_\infty, \]

(\text{iv}) \[ \lim_{m \to \infty} \int_{\mathbb{R}^{n+1}_+} t^\beta|u_m|^p dydt = \int_{\mathbb{R}^{n+1}_+} t^\beta|u|^p dydt + \|\nu\| + \nu_\infty, \]

where \( \|\mu\| = \sup_{u \in C^\infty_0(\mathbb{R}^{n+1}_+), \|u\|_{L^\infty_1} = 1} < \mu, u >. \) Moreover, if \( u = 0 \) and \( \|\mu\| = S_{1,\alpha,\beta}\|\nu\|^{\frac{\beta}{p}}, \) then \( \mu \) and \( \nu \) are concentrated at a single point.

**Proof.** 1). Assume first \( u = 0 \).

1.1. For any \( \phi \in C^\infty_0\left(\mathbb{R}^{n+1}_+\right) \), by inequality (1.5), we have
\[
S_{1,\alpha,\beta}\left( \int_{\mathbb{R}^{n+1}_+} t^\beta|\phi u_m|^p dydt \right)^{\frac{\beta}{p}} \leq \int_{\mathbb{R}^{n+1}_+} t^\alpha|\nabla(\phi u_m)|^2 dydt. \tag{3.1}
\]
By Hölder inequality, we have

\[
\int_{\mathbb{R}_{+}^{n+1}} t^{\alpha} |\nabla (\varphi u_m)|^2 dydt = \int_{\mathbb{R}_{+}^{n+1}} t^{\alpha} |\nabla \varphi u_m + \varphi \nabla u_m|^2 dydt
\]

\[
\leq \int_{\mathbb{R}_{+}^{n+1}} t^{\alpha} |\varphi|^2 |\nabla u_m|^2 dydt + 2 \int_{\mathbb{R}_{+}^{n+1}} t^{\alpha} |\varphi||u_m||\nabla \varphi||\nabla u_m| dydt
\]

\[
+ \int_{\mathbb{R}_{+}^{n+1}} t^{\alpha} |u_m|^2 |\nabla \varphi|^2 dydt.
\]

By the compact embedding (Lemma 7.1), we have \(u_m \to 0\) in \(L^{2}_{\text{loc}}(\mathbb{R}_{+}^{n+1}, t^\alpha dydt)\). Then

\[
\int_{\mathbb{R}_{+}^{n+1}} t^{\alpha} |\varphi||u_m||\nabla \varphi||\nabla u_m| dydt
\]

\[
\leq C(\varphi) \int_{\text{supp} \varphi} t^{\alpha} |u_m||\nabla u_m| dydt
\]

\[
\leq C(\varphi) \left( \int_{\text{supp} \varphi} t^{\alpha} |\nabla u_m|^2 dydt \right)^{\frac{1}{2}} \left( \int_{\text{supp} \varphi} t^{\alpha} |u_m|^2 dydt \right)^{\frac{1}{2}} \to 0,
\]

and

\[
\int_{\mathbb{R}_{+}^{n+1}} t^{\alpha} |u_m|^2 |\nabla \varphi|^2 dydt \leq C(\varphi) \int_{\text{supp} \varphi} t^{\alpha} |u_m|^2 dydt \to 0.
\]

Thus for \(m \to \infty\), we arrive at

\[
S_{1,\alpha,\beta} \left( \int_{\mathbb{R}_{+}^{n+1}} |\varphi|^{p^*} d\nu \right)^{\frac{2}{p^*}} \leq \int_{\mathbb{R}_{+}^{n+1}} |\varphi|^2 d\mu.
\]

A limit process shows

\[
S_{1,\alpha,\beta} \nu(E)^{\frac{2}{p^*}} \leq \mu(E),
\]

for any bounded Borel set \(E \subset \mathbb{R}_{+}^{n+1}\), which implies \(\|\mu\| \geq S_{1,\alpha,\beta} \|\nu\|^{\frac{2}{p^*}}\).

1.2. For any \(R > 1\), choose \(\psi_R \in C^1(\mathbb{R}_{+}^{n+1})\), such that \(0 \leq \psi_R \leq 1\), \(\psi_R(y, t) = 1\) for \(|y| + t \geq R + 1\) and \(\psi_R(y, t) = 0\) for \(|y| + t < R\). By inequality (1.1), we have

\[
\left( \int_{\mathbb{R}_{+}^{n+1}} t^\beta |\psi_R u_m|^{p^*} dydt \right)^{\frac{2}{p^*}} \leq S_{1,\alpha,\beta}^{-1} \int_{\mathbb{R}_{+}^{n+1}} t^{\alpha} |\nabla (\psi_R u_m)|^2 dydt
\]

Similar to the argument in 1.1), we have

\[
\lim_{m \to \infty} \left( \int_{\mathbb{R}_{+}^{n+1}} t^\beta |\psi_R u_m|^{p^*} dydt \right)^{\frac{2}{p^*}} \leq S_{1,\alpha,\beta}^{-1} \lim_{m \to \infty} \int_{\mathbb{R}_{+}^{n+1}} t^{\alpha} \psi_R^2 |\nabla u_m|^2 dydt.
\]
Thus

$$S_{1,\alpha,\beta}^{-1,\mu} = S_{1,\alpha,\beta}^{-1} \lim_{R \to \infty} \lim_{m \to \infty} \int_{B_R(0)} t^\alpha \left| \nabla u_m \right|^2 \, dydt$$

$$\geq S_{1,\alpha,\beta}^{-1} \lim_{R \to \infty} \lim_{m \to \infty} \int_{B_R(0)} t^\alpha \psi_R^2 \left| \nabla u_m \right|^2 \, dydt$$

$$\geq \lim_{R \to \infty} \lim_{m \to \infty} \left( \int_{B_R(0)} t^\beta \left| \psi_R u_m \right|^p \, dydt \right)^{\frac{2}{p}}$$

$$= \left( \lim_{R \to \infty} \lim_{m \to \infty} \int_{B_R(0)} t^\beta \left| \psi_R u_m \right|^p \, dydt \right)^{\frac{2}{p}}$$

$$\geq \left( \lim_{R \to \infty} \lim_{m \to \infty} \int_{B_R(0)} t^\beta \left| u_m \right|^p \, dydt \right)^{\frac{2}{p}}$$

$$= \nu_{\infty}^{\frac{2}{p}}.$$

1.3). Further, if we know that \( \parallel \mu \parallel = S_{1,\alpha,\beta}^{-1} \parallel \nu \parallel^{\frac{2}{p}} \), then, for any \( \nu \in C_0^\infty(\mathbb{R}_+^{n+1}) \),

$$\int_{\mathbb{R}_+^{n+1}} \left| \varphi \right|^p \, d\nu \leq S_{1,\alpha,\beta}^{-1} \left( \int_{\mathbb{R}_+^{n+1}} \left| \varphi \right|^2 \, d\mu \right)^{\frac{p}{2}} \leq S_{1,\alpha,\beta}^{-1,\mu} \parallel \mu \parallel^{\frac{p-2}{2}} \int_{\mathbb{R}_+^{n+1}} \left| \varphi \right|^p \, d\mu,$$

we have

$$\nu = S_{1,\alpha,\beta}^{-1,\mu} \parallel \mu \parallel^{\frac{p-2}{2}} \mu.$$

This means

$$\left( \int_{\mathbb{R}_+^{n+1}} \left| \varphi \right|^p \, d\nu \right)^{\frac{1}{p}} \leq S_{1,\alpha,\beta}^{-\frac{1}{2}} \left( \int_{\mathbb{R}_+^{n+1}} \left| \varphi \right|^2 \, d\mu \right)^{\frac{1}{2}}$$

$$= S_{1,\alpha,\beta}^{-\frac{1}{2}} \parallel \mu \parallel^{\frac{p-2}{2}} \left( \int_{\mathbb{R}_+^{n+1}} \left| \varphi \right|^2 \, d\nu \right)^{\frac{1}{2}}$$

$$= \parallel \nu \parallel^{\frac{p-2}{2p}} \left( \int_{\mathbb{R}_+^{n+1}} \left| \varphi \right|^2 \, d\nu \right)^{\frac{1}{2}}.$$

Then for any open set \( \Omega \),

$$\nu(\Omega)^{\frac{1}{p}} \leq \nu(\mathbb{R}_+^{n+1})^{\frac{p-2}{2p}} \nu(\Omega)^{\frac{1}{2}}.$$

Since \( \alpha < \beta + 2 \), we have that \( p^* > 2 \). If \( \nu(\Omega) > 0 \), we have \( \nu(\mathbb{R}_+^{n+1}) \leq \nu(\Omega) \), which implies that \( \nu \) is centered at a single point, so is \( \mu \).

2). We discuss the general case. We write \( v_m = u_m - u \). Since \( v_m \to 0 \) weakly in \( D_0^{1,2}(\mathbb{R}_+^{n+1}) \), we have for any \( h \in C_0^\infty(\mathbb{R}_+^{n+1}) \),

$$\int_{\mathbb{R}_+^{n+1}} t^\alpha \left| \nabla u_m \right|^2 \, dydt$$

$$= \int_{\mathbb{R}_+^{n+1}} t^\alpha \left| \nabla v_m \right|^2 \, dydt + 2 \int_{\mathbb{R}_+^{n+1}} t^\alpha \nabla v_m \nabla uh \, dydt + \int_{\mathbb{R}_+^{n+1}} t^\alpha \left| \nabla u \right|^2 \, dydt$$

$$\to \int_{\mathbb{R}_+^{n+1}} h \, dy + \int_{\mathbb{R}_+^{n+1}} t^\alpha \left| \nabla u \right|^2 \, dydt.$$

Then we obtain that

$$t^\alpha \left| \nabla u_m \right|^2 \, dydt \to \mu + \left| \nabla u \right|^2 \, dydt \text{ in } M(\mathbb{R}_+^{n+1}).$$
According to Brezis-Lieb Lemma, we have for every nonnegative $h \in C_0^\infty(\mathbb{R}_+^{n+1})$,
\[
\int_{\mathbb{R}_+^{n+1}} t^\beta |u|^p h dy dt = \lim_{m \to \infty} \left( \int_{\mathbb{R}_+^{n+1}} t^\beta |u_m|^p h dy dt - \int_{\mathbb{R}_+^{n+1}} t^\beta |v_m|^p h dy dt \right).
\]
Hence we obtain that
\[
t^\beta |u_m|^p dy dt \to \nu + t^\beta |u|^p dy dt \text{ in } \mathcal{M}(\mathbb{R}_+^{n+1}).
\]
Part (i) follows from the corresponding inequality for \{v_m\}.

Since
\[
\lim_{m \to \infty} \int_{\mathbb{R}_+^{n+1}\setminus B_R(0)} t^\alpha |\nabla v_m|^2 dy dt = \lim_{m \to \infty} \int_{\mathbb{R}_+^{n+1}\setminus B_R(0)} t^\alpha |\nabla u_m|^2 dy dt - \int_{\mathbb{R}_+^{n+1}\setminus B_R(0)} t^\alpha |\nabla u|^2 dy dt,
\]
we obtain that
\[
\lim_{R \to \infty} \lim_{m \to \infty} \int_{\mathbb{R}_+^{n+1}\setminus B_R(0)} t^\alpha |\nabla v_m|^2 dy dt = \lim_{R \to \infty} \lim_{m \to \infty} \int_{\mathbb{R}_+^{n+1}\setminus B_R(0)} t^\alpha |\nabla u_m|^2 dy dt = \mu_\infty.
\]
By Brezis-Lieb Lemma, we have
\[
\lim_{m \to \infty} \left( \int_{\mathbb{R}_+^{n+1}\setminus B_R(0)} t^\beta |u_m|^p dy dt - \int_{\mathbb{R}_+^{n+1}\setminus B_R(0)} t^\beta |v_m|^p dy dt \right) = \int_{\mathbb{R}_+^{n+1}\setminus B_R(0)} t^\beta |u|^p dy dt,
\]
which implies
\[
\lim_{R \to \infty} \lim_{m \to \infty} \int_{\mathbb{R}_+^{n+1}\setminus B_R(0)} t^\beta |v_m|^p dy dt = \nu_\infty.
\]
Part (ii) follows from the corresponding inequality for \{v_m\}.

For every $R > 1$, we have
\[
\lim_{m \to \infty} \int_{\mathbb{R}_+^{n+1}} t^\alpha |\nabla u_m|^2 dy dt
\]
\[
= \lim_{m \to \infty} \int_{\mathbb{R}_+^{n+1}} t^\alpha \psi_R |\nabla u_m|^2 dy dt + \lim_{m \to \infty} \int_{\mathbb{R}_+^{n+1}} t^\alpha (1 - \psi_R) |\nabla u_m|^2 dy dt
\]
\[
= \lim_{m \to \infty} \int_{\mathbb{R}_+^{n+1}} t^\alpha \psi_R |\nabla u_m|^2 dy dt + \int_{\mathbb{R}_+^{n+1}} (1 - \psi_R) d\mu + \int_{\mathbb{R}_+^{n+1}} t^\alpha (1 - \psi_R) |\nabla u|^2 dy dt.
\]
When $R \to \infty$, we get, by Lebesgue dominated convergence theorem, that
\[
\lim_{m \to \infty} \int_{\mathbb{R}_+^{n+1}} t^\alpha |\nabla u_m|^2 dy dt = \mu_\infty + \|\mu\| + \int_{\mathbb{R}_+^{n+1}} t^\alpha |\nabla u|^2 dy dt.
\]
Similarly, we can get
\[
\lim_{m \to \infty} \int_{\mathbb{R}_+^{n+1}} t^\beta |u_m|^p dy dt = \nu_\infty + \|\nu\| + \int_{\mathbb{R}_+^{n+1}} t^\beta |u|^p dy dt.
\]
Lemma \[3.2\] is proved. \(\square\)

**Proof of Proposition 3.1.** Let \(\{u_m\} \subset D_0^{1,2}(\mathbb{R}_+^{n+1})\) be a nonnegative minimizing sequence of functions for \(S_{1,\alpha,\beta}\) with \(\int_{\mathbb{R}_+^{n+1}} t^\beta |u_m|^p dy dt = 1\). Then for any
compact set $K \subset \mathbb{R}^{n+1}_+$, we have

$$\int_K t^\alpha |u_m|^2 \, dydt \leq \left( \int_K t^\beta |u_m|^p \, dydt \right)^{\frac{2}{p}} \left( \int_K t^{\frac{\alpha p^* - 2\beta}{p^* - 2}} \, dydt \right)^{\frac{p^* - 2}{p^*}}.$$  

Since $\frac{\alpha p^* - 2\beta}{p^* - 2} = \frac{(n+1)\alpha - (n-1)\beta}{\beta + 2 - \alpha} \geq 0$ we have

$$\int_K t^\alpha |u_m|^2 \, dydt \leq C(K),$$

i.e. $u \in H^{1,2}_{loc}(\mathbb{R}^{n+1}_+, t^\alpha \, dydt)$. By the compact embedding (Lemma 7.1), we have, after passing to a subsequence, that

$$u_m \rightharpoonup u \text{ in } D^{1,2}_\alpha(\mathbb{R}^{n+1}_+, t^\alpha \, dydt),$$

$$u_m \rightarrow u \text{ in } L^2_{loc}(\mathbb{R}^{n+1}_+, t^\alpha \, dydt),$$

$$u_m \rightarrow u \text{ a.e. in } \mathbb{R}^{n+1}_+,$$

$$\nu_m = t^\beta |u_m - u|^p \, dydt \rightharpoonup \nu \text{ weakly in } \mathcal{M}(\mathbb{R}^{n+1}_+),$$

$$\mu_m = t^\alpha |\nabla (u_m - u)|^2 \, dydt \rightharpoonup \mu \text{ weakly in } \mathcal{M}(\mathbb{R}^{n+1}_+).$$

Define

$$Q_m(\lambda) = \sup_{z \in \partial \mathbb{R}^{n+1}_+} \int_{|y - z| + t < \lambda} t^\beta |u_m|^p \, dydt.$$  

Since for every $m$,

$$\lim_{\lambda \to 0^+} Q_m(\lambda) = 0, \quad \lim_{\lambda \to \infty} Q_m(\lambda) = 1,$$

there exists $\lambda_m > 0$ such that $Q_m(\lambda_m) = \frac{1}{2}$. Moreover, there exists $z_m \in \partial \mathbb{R}^{n+1}_+$ such that

$$\int_{B^+_{\lambda_m}(z_m)} t^\beta |u_m|^p \, dydt = Q_m(\lambda_m) = \frac{1}{2},$$

since

$$\lim_{|z| \to \infty} \int_{B^+_{\lambda_m}(z)} t^\beta |u_m|^p \, dydt = 0.$$  

Due to the translation and dilation invariance for the minimizing sequence, we have (we can replace $u_m$ by $u_m^{\lambda_m, z_m}$ and still denote it as $u_m$) that

$$\frac{1}{2} = \int_{B^+_1(0)} t^\beta |u_m|^p \, dydt = \sup_{z \in \partial \mathbb{R}^{n+1}_+} \int_{B^+_1(z)} t^\beta |u_m|^p \, dydt. \quad (3.3)$$

From Lemma 3.2, we have

$$1 = \lim_{m \to \infty} \int_{\mathbb{R}^{n+1}_+} t^\beta |u_m|^p \, dydt = \int_{\mathbb{R}^{n+1}_+} t^\beta |u|^p \, dydt + \|\nu\| + \nu_{\infty}, \quad (3.4)$$
Moreover, for \( p^* > 2 \), we have

\[
S_{1,\alpha,\beta} = \lim_{m \to \infty} \int_{\mathbb{R}^{n+1}} t^\alpha |\nabla u_m|^2 \, dy \, dt
\]

\[
= \int_{\mathbb{R}_+^{n+1}} t^\alpha |\nabla u|^2 \, dy \, dt + \| \mu \| + \mu_\infty
\]

\[
\geq S_{1,\alpha,\beta} \left[ \left( \int_{\mathbb{R}_+^{n+1}} t^\beta |u|^{p^*} \, dy \, dt \right)^{\frac{2}{p^*}} + \| \nu \|^\frac{2}{p^*} + \nu_\infty \right]
\]

\[
\geq S_{1,\alpha,\beta} \left( \int_{\mathbb{R}_+^{n+1}} t^\beta |u|^{p^*} \, dy \, dt + \| \nu \| + \nu_\infty \right)^{\frac{2}{p^*}}
\]

\[
= S_{1,\alpha,\beta}.
\]

It implies that

\[
\left( \int_{\mathbb{R}^{n+1}_+} |u|^{p^*}_{L^p_{\alpha,\beta}(\mathbb{R}^{n+1}_+)} \right)^{\frac{2}{p^*}} + \| \nu \|^\frac{2}{p^*} + \nu_\infty = \| u \|^{p^*}_{L^p_{\alpha,\beta}(\mathbb{R}^{n+1}_+)} + \| \nu \| + \nu_\infty = 1. \tag{3.5}
\]

Since \( 2/p^* < 1 \), above equality indicates that only one term is equal to 1 and the others must be 0. By \( (3.3) \), \( \nu_\infty \leq \frac{1}{2} \), then \( \nu_\infty = 0 \). If \( \| \nu \| = 1 \), then \( u = 0 \) and \( \| \mu \| = S_{1,\alpha,\beta} \| \nu \|^{\frac{2}{p^*}} \). By the last statement in Lemma 3.2, we have that \( \mu \) and \( \nu \) are concentrated on a single point \( x^* \). We claim \( x^* = (z^*, 0) \in \partial \mathbb{R}^{n+1}_+ \), then by \( (3.3) \)

\[
\frac{1}{2} \int_{B^+_1(x^*)} t^\beta |u_m|^{p^*} \, dy \, dt \to \| \nu \| = 1,
\]

contradiction.

To prove the claim, we argue by contradiction. Assume \( x^* = (z^*, t^*) \) for some \( t^* > 0 \). For \( n > 1 \), since \( \beta < \frac{n+1}{n-1} \alpha \), we know \( p^* < 2^* := \frac{2(n+1)}{n-1} \). For every \( 0 < \varepsilon < t^* \), we have

\[
\lim_{m \to \infty} \int_{B_\varepsilon(x^*)} t^\beta |u_m|^{p^*} \, dy \, dt = 1.
\]

But

\[
\left( \int_{B_\varepsilon(x^*)} t^\beta |u_m|^{p^*} \, dy \, dt \right)^{\frac{2}{p^*}} \leq C \left( \int_{B_\varepsilon(x^*)} |u_m|^{2^*} \, dy \, dt \right)^{\frac{2}{2^*}}
\]

\[
\leq C \left( \int_{B_\varepsilon(x^*)} |u_m|^{2^*} \, dy \, dt \right)^{\frac{2}{2^*}} \varepsilon^\frac{2(2^*-p^*)}{2(p^*-1)(n+1)}
\]

\[
\leq C \varepsilon^\frac{2(2^*-p^*)}{2(p^*-1)(n+1)} \int_{B_\varepsilon(x^*)} |\nabla u_m|^2 \, dy \, dt
\]

\[
\leq C \varepsilon^\frac{2(2^*-p^*)}{2(p^*-1)(n+1)} \int_{B_\varepsilon(x^*)} t^\alpha |\nabla u_m|^2 \, dy \, dt
\]

\[
\leq C \varepsilon^\frac{2(2^*-p^*)}{2(p^*-1)(n+1)} \to 0,
\]

as \( \varepsilon \to 0 \).

contradiction. For \( n = 1 \), we replace \( 2^* \) by a power \( q > p^* \) in the above calculation. Similarly, we can get the same contradiction. \( \square \)

**Remark 3.3.** For \( \beta = \frac{n+1}{n-1} \alpha \), the minimizer may not exist. In \( \[4\] \) combined with the analysis in \( \[39\] \), we know that for \( n = 2, \alpha = 1, \beta = 3 \), the minimizer doesn’t exist.
4. Regularity of extremal functions

Throughout this section, we always assume

\[-1 < \beta, \quad \alpha + \beta \geq 0, \quad \frac{n-1}{n+1} \beta \leq \alpha < \beta + 2. \quad (4.1)\]

Since \(p^* = \frac{2(n+\beta+1)}{n+\alpha-1}\), the above condition indicates \(p^* > 2\).

In this section, we shall prove Theorem 1.6: under condition (4.1) on \(\alpha\) and \(\beta\), the weak positive solutions to (1.8) are Hölder continuous up to the boundary.

**Proposition 4.1.** Suppose \(0 \leq u \in \mathcal{D}_{1,2}^{1,2}(\mathbb{R}^{n+1})\) is a weak solution to equation (1.8) and \(\alpha, \beta\) satisfy condition (4.1). Then, for any \(1 \leq q < \infty\), we have

\[u^q \in \mathcal{D}_{1,2}^{1,2}(\mathbb{R}^{n+1})\]

and

\[u^q \in L^2_{loc}(\mathbb{R}^{n+1}, t^\beta \, dy \, dt).\]

**Proof.** We shall prove this by iteration. Suppose \(\eta \in C_0^\infty(\mathbb{R}^{n+1})\) and \(0 \leq \beta \geq 0\) and \(K \geq 0\). Denote

\[\phi = \eta^2 u \cdot \min\{u_\theta^2, K^2\},\]

where \(\theta\) is to be chosen. Notice that \(\phi \in \mathcal{D}_{1,2}^{1,2}(\mathbb{R}^{n+1})\). Testing (1.8) with \(\phi\), we have

\[\int_{\mathbb{R}^{n+1}_+} t^\alpha |\nabla u| \cdot \nabla \phi \, dy \, dt = \int_{\mathbb{R}^{n+1}_+} t^\beta u^{p^*-1} \phi \, dy \, dt. \quad (4.2)\]

While the LHS can be calculated

\[LHS = \int_{\mathbb{R}^{n+1}_+} t^\alpha |\nabla u|^2 \min\{u_\theta^2, K^2\} \eta^2 \, dy \, dt + 2\theta \int_{\{u_\theta \leq K\}} t^\alpha |\nabla u|^2 u_\theta^2 \eta^2 \, dy \, dt\]

\[+ \int_{\mathbb{R}^{n+1}_+} t^\alpha |\nabla u| \cdot \nabla \eta^2 \cdot u \min\{u_\theta^2, K^2\} \, dy \, dt\]

and for the last term, one can use Hölder’s inequality

\[\int_{\mathbb{R}^{n+1}_+} t^\alpha |\nabla u| \cdot \nabla \eta^2 \cdot u \min\{u_\theta^2, K^2\} \, dy \, dt \geq -\frac{1}{2} \int_{\mathbb{R}^{n+1}_+} t^\alpha |\nabla u|^2 \min\{u_\theta^2, K^2\} \eta^2 \, dy \, dt - 2 \int_{\mathbb{R}^{n+1}_+} t^\alpha |u|^2 \min\{u_\theta^2, K^2\} |\nabla \eta|^2 \, dy \, dt.\]

Putting these inequalities back to (4.2),

\[\frac{1}{2} \int_{\mathbb{R}^{n+1}_+} t^\alpha |\nabla u|^2 \min\{u_\theta^2, K^2\} \eta^2 \, dy \, dt + 2\theta \int_{\{u_\theta \leq K\}} t^\alpha u_\theta^2 |\nabla \theta|^2 \eta^2 \, dy \, dt \leq 2 \int_{\mathbb{R}^{n+1}_+} t^\beta u^{p^*} \min\{u_\theta^2, K^2\} |\nabla \eta|^2 \, dy \, dt + \int_{\mathbb{R}^{n+1}_+} t^\beta u^{p^*} \min\{u_\theta^2, K^2\} \eta^2 \, dy \, dt\]
where \( \hat{\theta} = \theta^{-1} \) if \( \theta > 0 \) and \( \hat{\theta} = 0 \) if \( \theta = 0 \). Denote \( w = u \cdot \min\{u^\theta, K\} \eta \). The above inequality implies

\[
\int_{\mathbb{R}^n_+} t^\alpha |\nabla w|^2 \, dy dt \leq C \int_{\mathbb{R}^n_+} t^\alpha u^2 \min\{u^{2\theta}, K^2\} |\nabla \eta|^2 \, dy dt + C \int_{\mathbb{R}^n_+} t^\beta u^{p^* - 2} u^2 \, dy dt
\]

\[
\leq C(\eta) \int_{\mathbb{R}^n_+} t^\alpha w^2 \, dy dt + C L p^* - 2 \int_{\mathbb{R}^n_+} t^\beta w^2 \, dy dt
\]

\[
+ C \left( \int_{\{|u| \leq L) \cap \text{supp}(\eta)\}} t^\beta u^{p^*} \, dy dt \right)^{1 - 2/p^*} \left( \int_{\mathbb{R}^n_+} t^\beta w^{p^*} \, dy dt \right)^{2/p^*}.
\]

Since \( \int_{\{|u| > L) \cap \text{supp}(\eta)\}} t^\beta u^{p^*} \, dy dt \to 0 \) as \( L \to \infty \), then one can fix \( L \) large enough such that

\[
\int_{\mathbb{R}^n_+} t^\alpha |\nabla w|^2 \, dy dt \leq C(\eta, L) \int_{\mathbb{R}^n_+} t^\alpha w^2 \, dy dt + C(L) \int_{\mathbb{R}^n_+} t^\beta w^2 \, dy dt. \tag{4.3}
\]

We claim that for \( \alpha \) and \( \beta \) satisfy [1.1] and \( w \) with compact support

\[
\int_{\mathbb{R}^n_+} t^\alpha w^2 \, dy dt \leq \begin{cases} C \int_{\mathbb{R}^n_+} t^\beta w^2 \, dy dt & \text{if } \alpha \geq \beta, \\ C \left( \int_{\mathbb{R}^n_+} t^\beta w^2 \, dy dt \right)^{1 - \lambda} \left( \int_{\mathbb{R}^n_+} t^\alpha |\nabla w|^2 \, dy dt \right) & \text{if } \alpha > \beta,
\end{cases}
\]

for some constant \( C > 0 \) and \( \lambda \in (0, 1) \). In fact, if \( \alpha \geq \beta \), the above inequality is obvious, so does \( \alpha = 0 \). Now consider \( 0 < \alpha < \beta \) and [1.1] holds. Choose some \( \gamma \in (0, \alpha) \) such that \( \alpha < \gamma + 2 \),

\[
\int_{\mathbb{R}^n_+} t^\alpha w^2 \, dy dt \leq \left( \int_{\mathbb{R}^n_+} t^\beta w^2 \, dy dt \right)^{1 - \lambda} \left( \int_{\mathbb{R}^n_+} t^\gamma w^2 \, dy dt \right) \lambda
\]

where \( \lambda \in (0, 1) \) satisfies \( \alpha = (1 - \lambda) \beta + \lambda \gamma \). Applying Hölder’s inequality and [1.5] for \( \gamma \) and \( \alpha, \beta \) one gets

\[
\int_{\mathbb{R}^n_+} t^\alpha w^2 \, dy dt \leq C S_{\alpha, \gamma}^{-1} \left( \int_{\mathbb{R}^n_+} t^\beta w^2 \, dy dt \right)^{1 - \lambda} \left( \int_{\mathbb{R}^n_+} t^\alpha |\nabla w|^2 \, dy dt \right) \lambda.
\]

So the claim is proved. Inserting the above claim to [4.3] gives

\[
\int_{\mathbb{R}^n_+} t^\alpha |\nabla w|^2 \, dy dt \leq C \int_{\mathbb{R}^n_+} t^\beta w^2 \, dy dt + C \left( \int_{\mathbb{R}^n_+} t^\beta w^2 \, dy dt \right)^{1 - \lambda} \left( \int_{\mathbb{R}^n_+} t^\alpha |\nabla w|^2 \, dy dt \right) \lambda
\]

where \( C \) depends on \( \alpha, \beta, \eta, L \). Now letting \( K \to \infty \) in the above inequality, if we can find \( \theta \) such that \( w = \eta u^{\theta + 1} \in L^2(\mathbb{R}^n_+), \) then \( u \in D^{1,2}(\mathbb{R}^n_+) \). Hence by [1.5]

\[
w \hat{w} \in L^2(\mathbb{R}^n_+). \tag{4.5}
\]

Now we let \( \theta_0 = 0 \), it is easy to see that \( \eta u \in L^2(\mathbb{R}^n_+) \). The assumption of our first step of iteration is satisfied. Hence by letting \( \theta_i + 1 = (\theta_{i-1} + 1) L^2_T \) if \( i \geq 1 \), one can iterate the above process to get the conclusion. \( \square \)

**Proposition 4.2.** Suppose \( 0 \leq u \in D^{1,2}(\mathbb{R}^n_+) \) is a weak solution to equation [1.8] and \( \alpha, \beta \) satisfy condition [4.1], then \( u \in L^\infty(\mathbb{R}^n_+) \).
Proof. Testing (1.7) by \( \phi = \eta^2 u^{2\theta+1} \) for some \( \theta > 0 \) and \( \text{supp}(\eta) \subset B^+_2 \),

\[
2 \int_{B^+_2} t^\alpha \eta u^{2\theta+1} \nabla \eta \cdot \nabla u dy dt + (2\theta + 1) \int_{B^+_2} t^\alpha \eta^2 u^{2\theta} |\nabla u|^2 dy dt = \int_{B^+_2} t^\beta \eta^2 u^{p^*+2\theta} dy dt.
\]

It follows that

\[
\int_{B^+_2} t^\alpha \eta^2 u^{2\theta} |\nabla u|^2 dy dt \leq C(\theta) \int_{B^+_2} t^\alpha |\nabla \eta|^2 u^{2\theta+2} + t^\beta \eta^2 u^{p^*+2\theta} dy dt
\]

for some constant \( C(\theta) > 0 \). Write \( w = u^{\theta+1} \). We have

\[
\int_{B^+_2} t^\alpha |\nabla (\eta w)|^2 dy dt \leq C(\theta)(\theta + 1)^2 \int_{B^+_2} t^\alpha |\nabla \eta|^2 w^2 + t^\beta \eta^2 u^{p^*-2} w^2 dy dt. \quad (4.6)
\]

Using Hölder’s inequality, we obtain

\[
\int_{B^+_2} t^\beta \eta^2 u^{p^*-2} w^2 dy dt \leq \left( \int_{B^+_2} t^\beta u^{(p^*-2)\frac{2}{p^*-1}} dy dt \right)^{\frac{1}{q}} \left( \int_{B^+_2} t^\beta (\eta w)^{2q} dy dt \right)^{\frac{1}{q}}
\]

for some \( q \) fixed such that \( p^* > 2q > 2 \). By Theorem 4.1, we know

\[
V := \left( \int_{B^+_2} t^\beta u^{(p^*-2)\frac{2}{p^*-1}} dy dt \right)^{1-\frac{1}{q}} < \infty;
\]

Also, by Young’s inequality, we have

\[
\left( \int_{B^+_2} t^\beta (\eta w)^{2q} dy dt \right)^{\frac{1}{q}} \leq \delta \left( \int_{B^+_2} t^\beta (\eta w)^{p^*} dy dt \right)^{\frac{1}{p^*}} + \delta^{-\sigma} \left( \int_{B^+_2} t^\beta \eta^2 w^2 dy dt \right)^{\frac{1}{\sigma}}
\]

where \( \sigma = \frac{p^*(q-1)}{p^*-2q} \). Putting these back to (1.6), one gets

\[
\int_{B^+_2} t^\alpha |\nabla (\eta w)|^2 dy dt \leq C(1 + \theta)^2 \left[ \int_{B^+_2} t^\alpha |\nabla \eta|^2 w^2 dy dt + V \delta^{2} \left( \int_{B^+_2} t^\beta (\eta w)^{p^*} dy dt \right)^{\frac{1}{p^*}} \right. \\
\left. + V \delta^{-2\sigma} \int_{B^+_2} t^\beta \eta^2 w^2 dy dt \right].
\]

Using inequality (4.4) and (1.5) and choosing \( \delta \) small enough, we have

\[
\left( \int_{B^+_2} t^\beta (\eta w)^{p^*} dy dt \right)^{\frac{1}{p^*}} \leq C(1 + \theta)^2 \int_{B^+_2} t^\alpha |\nabla \eta|^2 w^2 dy dt + C(1 + \theta)^{2\sigma+2} V^{\sigma+1} \int_{B^+_2} t^\beta \eta^{2} w^2 dy dt
\]

\[
\leq C(1 + \theta)^2 \int_{B^+_2} t^\alpha |\nabla \eta|^2 w^2 dy dt + C(1 + \theta)^{2\sigma+2} V^{\sigma+1} \int_{B^+_2} t^\beta \eta^{2} w^2 dy dt. \quad (4.7)
\]

For \( r < 2 \) and \( p > 2 \), define

\[
\Phi(p, r) = \left( \int_{B^+_2} t^\beta u^p dy dt \right)^{\frac{1}{\beta}}.
\]
Set $\gamma = 2(1 + \theta)$, choose $\eta = 1$ in $B^+_1$ and $\eta = 0$ in $B^+_2 - B^+_1$. Then (4.7) shows that for any $\gamma > 2$

$$\Phi \left( \frac{p^*}{2}, r_1 \right) \leq \left[ \frac{C(\gamma \sqrt{V})^{\sigma + 1}}{r_2 - r_1} \right] \Phi(\gamma, r_2).$$

By iterating the above inequality: set $r_m = 1 + 2^{-m}$ and $\gamma_0 = p > 2$ and $\gamma_m = \gamma_{m-1} \frac{\sqrt{V}}{2}$, $m = 1, 2, \ldots$, one gets

$$\Phi(\gamma_m, r_m) \leq \left( C \cdot \sqrt{V} p^* \right)^{(1+\sigma)\sum k(p^*/2)^{-k}} \Phi(p, 2).$$

Since $p^* > 2$, then $\sum k(p^*/2)^{-k} < \infty$. Letting $m \to \infty$, we have $\sup_{B^+_1} u < \infty$. $\square$

The $L^\infty$ bound yields that $u$ is actually smooth in $\mathbb{R}^{n+1}$ by the standard elliptic estimates. Next, we shall show that $u$ is Hölder continuous up to the boundary. To that end, we firstly need to establish some lemmas.

We need the following weak Poincaré inequality. Let $Q_r(X)$ denote the cube in $\mathbb{R}^{n+1}$ with length of sides equal $r$ and centered at $X$.

**Lemma 4.3.** Suppose that $\alpha \geq 0$. There exists $C$ depends only on $n$ and $\alpha$ (does not depend on $X$) such that

$$\int_{Q_r} |t|^{\alpha}|u - u_{Q_r}|^2 dydt \leq C r \int_{Q_{2r}} |t|^{\alpha}u^2 dydt,$$

holds for any $r > 0$ and $u \in \mathcal{D}^{1,2}_{\alpha, loc}(2Q_r(X))$ which is even with respect to $t$. Here we write $Q_r = Q_r(X)$ for short and

$$u_{Q_r} = \frac{\int_{Q_r} |t|^{\alpha}u(y,t) dydt}{\int_{Q_r} |t|^{\alpha} dydt}.$$

**Proof.** It suffices to prove the above inequality for $r = 1$ and $u \in C^1(\overline{Q_1})$, the general case follows from scaling and approximation. Since $u$ is even, without loss of generality, assume $X = (0, t_c)$ lies in the upper half plane or its boundary.

If $t_c > 1$, then $t/t_c$ is uniformly bounded above and below in $Q_1$, (4.8) can be reduced to the Poincaré inequality in Euclidean space without weight, which is obviously true.

If $t_c \in [0, \frac{1}{2}]$, $Q_1$ will overlap with the lower half space. We claim it suffices to find some $c$ such that

$$\int_{Q_1} |t|^{\alpha}|u(y,t) - c|^2 dydt \leq C \int_{Q_1} |t|^{\alpha} |\nabla u|^2 dydt. \tag{4.9}$$

Indeed, the reason follows from the following interpolation argument

$$\int_{Q_1} |t|^{\alpha}|u(y,t) - u_{Q_1}|^2 dydt \leq 2 \int_{Q_1} |t|^{\alpha}|u - c|^2 dydt + 2 \int_{Q_1} |t|^{\alpha}|c - u_{Q_1}|^2 dydt \leq 4 \int_{Q_1} |t|^{\alpha}|u - c|^2 dydt.$$ 

Now, we need to divide into three cases according to the value of $\alpha$.

1) $\alpha \in (0, 1)$. Since a general fact that $|t|^{\alpha}$ is an $A_p$-weight for $p$ satisfying $p - 1 > \alpha$, then in this case one can take $p = 2$. For $A_2$-weight, [13] theorem 1.5 implies

$$\int_{Q_1} |t|^{\alpha}|u(y,t) - u_{Q_1}|^2 dydt \leq C \int_{Q_1} |t|^{\alpha} |\nabla u|^2 dydt.$$
which implies (4.8).

2) \( \alpha = m \) for some positive integer \( m \). Define \( \tilde{u} \) on \( \mathbb{R}^{n+m+1} \) by \( \tilde{u}(y,z) = u(y,t) \) with \( z \in \mathbb{R}^{n+1} \) and \( |z| = t \). Suppose \( Q^+_1 = Q_1 \cap \{ t > 0 \} = [a_1,b_1] \times \cdots \times [0,t_c + \frac{1}{2}], \)
and define \( \tilde{Q}^+_1 = [a_1,b_1] \times \cdots \times B^m_{t_c + \frac{1}{2}}(0) \) where \( B^m_r(0) \) denotes the ball in \( \mathbb{R}^{n+1} \)
centered at 0 with radius \( r \). Then \( \tilde{u}(y,z) \) is defined on \( \tilde{Q}^+_1 \). Since \( \tilde{Q}^+_1 \) is a convex
domain, it holds a Poincaré inequality as the following

\[
\left( \int_{\tilde{Q}^+_1} |\tilde{u} - \tilde{u}_{\tilde{Q}^+_1}|^p dydz \right)^{\frac{1}{p}} \leq C(p,n) \int_{\tilde{Q}^+_1} |\nabla \tilde{u}|^2 dydz \tag{4.10}
\]

for any \( p \geq 1 \). However, changing coordinates back to \((y,t)\), the above inequality is equivalent to

\[
\left( \int_{\tilde{Q}^+_1} t^m |u - \tilde{u}_{\tilde{Q}^+_1}|^p dydt \right)^{\frac{1}{p}} \leq C \int_{\tilde{Q}^+_1} t^m |\nabla u|^2 dydt.
\]

Taking \( p = 2 \), since \( u \) is even, \( \int_{Q_1} |t^m|u - c|^2 dydt \leq 2 \int_{Q_1^+} t^m|u - c|^2 dydt \), (4.8) is proved.

3) \( \alpha \in (m - 1, m) \) for some integer \( m \geq 2 \). Note that

\[
\int_{Q_1} |t|^{\alpha}|u - u_{Q_1}|^2 dydt \leq \left( \int_{Q_1} |t|^m|u - u_{Q_1}|^4 dydt \right)^{\frac{1}{4}} \left( \int_{Q_1} |t|^{2\alpha - m} dydt \right)^{\frac{1}{2}}
\]

\[
\leq C \left( \int_{Q_1} |t|^m|u - u_{Q_1}|^4 dydt \right)^{\frac{1}{4}} \left( \int_{Q_1} |t|^{2\alpha - m} dydt \right)^{\frac{1}{2}}
\]

because \( 2\alpha - m \geq m - 2 \geq 0 \). Now it follows from the previous case that

\[
\int_{Q_1} |t|^{\alpha}|u - u_{Q_1}|^2 dydt \leq C \int_{Q_1} |t|^m|\nabla u|^2 dydt \leq C \int_{Q_1} |t|^{\alpha}|\nabla u|^2 dydt.
\]

Hence (4.8) is established for this case.

We are left with the case \( t_c \in \left( \frac{1}{2}, 1 \right) \). Notice in this case \( Q_1 \) lies entirely in the
upper half plan but not far from \( \{ t = 0 \} \). Then \( Q_2 = Q_2(X) \) will intersect \( \{ t = 0 \} \).
Suppose \( \eta \) is a cut off function whose support contained in \( Q_2 \) and \( \eta \equiv 1 \) in \( Q_1 \).

By Hölder’s and Young’s inequality and (4.3)

\[
\int_{Q_2} |t|^{\alpha}|\eta(u - c)|^2 dydt \leq C \left( \int_{Q_2} |t|^{\alpha}|\eta(u - c)|^{\frac{2(\alpha + 2\alpha + 1)}{\alpha + 1}} dydt \right)^{\frac{\alpha + 1 - \alpha}{\alpha + 1}}
\]

\[
\leq C \int_{Q_2} |t|^{\alpha}|u - c|^2 dydt + C \int_{Q_2} |t|^{\alpha}|\nabla u|^2 dydt.
\]

Since \( Q_2 \cap \{ t = 0 \} \neq \emptyset \), the previous proof shows

\[
\int_{Q_2} |t|^{\alpha}|u - c|^2 dydt \leq C \int_{Q_2} |t|^{\alpha}|\nabla u|^2 dydt,
\]

for some constant \( c \). Then (4.8) is established. \( \square \)

Now let us deal with general \( \alpha \) and \( \beta \) satisfying (4.1).
Lemma 4.4. Suppose $\alpha$ and $\beta$ satisfy \((1,1)\), then there exists $C$ depends upon $n, \alpha, \beta$ such that
\[ \frac{1}{\|Q_r\|} \int_{Q_r} |t|^{\beta} |u(y,t) - w| \, dy \, dt \leq C \left( \frac{1}{\|Q_r\|} \int_{Q_r} |t|^\alpha |\nabla u|^2 \, dy \, dt \right)^{\frac{1}{2}} \]
holds for any $r > 0$ and $u \in D^{1,2}_{\alpha,\text{loc}}(Q_r(X))$ and $u$ is even with respect to $t$. Here we write $Q_r = Q_r(X)$ for short and
\[ u_{Q_r} = \frac{\int_{Q_r} |t|^{\beta} u(y,t) \, dy \, dt}{\|Q_r\|} \]

Proof. It also suffices to prove the above inequality for $r = 1$ and $u \in C^1(Q)$. For the same reason in the last lemma, if we can find a constant such that
\[ \int_{Q_1} |t|^{\beta} |u(y,t) - c| \, dy \, dt \leq C \left( \int_{Q_1} |t|^\alpha |\nabla u|^2 \, dy \, dt \right)^{\frac{1}{2}} \] (4.11)
then the conclusion is verified.

Suppose $\eta$ is a cut off function whose support contained in $Q_2 = Q_2(X)$ and $\eta = 1$ in $Q_1$. By Hölder’s inequality and \((1,3)\), we have
\[ \int_{Q_2} |t|^{\beta} |\eta(u-c)| \, dy \, dt \leq C \left( \int_{Q_2} |t|^\beta |\eta(u-c)^{p^*} \, dy \, dt \right)^{\frac{1}{p}} \]
\[ \leq C \left( \int_{Q_2} |t|^\alpha |u-c|^2 \, dy \, dt \right)^{\frac{1}{2}} + C \left( \int_{Q_2} |t|^\alpha |\nabla u|^2 \, dy \, dt \right)^{\frac{1}{2}} \]
Taking $c = \int_{Q_2} |t|^\alpha u \, dy \, dt / \int_{Q_2} |t|^\alpha \, dy \, dt$ and using Lemma \((4.3)\) we get the conclusion. $\square$

Suppose $d\mu$ is a doubling measure on some domain $\Omega \subset \mathbb{R}^{n+1}$, that is $\mu(2B) \leq C(\mu)\mu(B)$ for any $2B \subset \Omega$. A function $w \in L^1_{\text{loc}}(\Omega, d\mu)$ is said to be in $BMO(\Omega, \mu)$ if there is a constant $C > 0$ such that for every ball $B$ satisfying $2B \subset \Omega$ it holds that
\[ \frac{1}{\mu(B)} \int_B |w - w_B| \, d\mu \leq C. \]
Here $w_B = \frac{1}{\mu(B)} \int_B w \, d\mu$ is the average on $B$. One can also use cubes instead of balls to define $BMO$. The two definitions are equivalent. The least $C$ such that the above inequality holds is called the $BMO(\Omega, \mu)$--norm of $w$. Similar to the classical result of BMO space on Euclidean space, we have the following result from Corollary 19.10 in \cite{27}.

Lemma 4.5 (John-Nirenberg lemma for doubling measures). Suppose $\mu$ is a doubling measure. A function $w$ is in $BMO(\Omega, d\mu)$ if and only if there exist constant $c$ and $C$ such that
\[ \frac{1}{\mu(B)} \int_B e^{c|w-w_B|} \, d\mu \leq C \]
for every ball $B$ such that $2B \subset \Omega$.

One consequence of this lemma is that
\[ \int_B e^{cw} \, d\mu \int_B e^{-cw} \, d\mu \leq C[\mu(B)]^2. \] (4.12)
Proposition 4.6. Suppose (4.13) holds and $0 \leq u \in D^{1,2}_{\alpha,loc}(\mathbb{R}^{n+1})$ satisfying
\[
\int_{\mathbb{R}^{n+1}} t^\alpha \nabla u \cdot \nabla \phi \, dydt \geq \int_{\mathbb{R}^{n+1}} t^\beta g \phi \, dydt
\] (4.13)
for some $g \in L^\infty_{loc}$ and any $0 \leq \phi \in C_c^\infty(\mathbb{R}^{n+1})$. Then there exist $C > 0$ depend upon on $n, \alpha, \beta$ such that for any $r > 0$
\[
C \left( \inf_{B^+_2} u + r^{\beta+2-\alpha} |g|_{L^\infty(B_r)} \right) \geq \frac{1}{r^{n+1+\beta}} \int_{B^+_2} t^\beta u \, dydt.
\] (4.14)

Proof. We just prove the result for $r = 1$, the general case follows from rescaling. Let $k = |g|_{L^\infty(B_r)} + \varepsilon$ for some $\varepsilon > 0$. Define $\bar{u} = u + k$. Plugging $\phi = \eta^2 \bar{u}^{2\gamma+1}$ in (1.7) for some $\theta < -\frac{1}{2}$ and $\text{supp}(\eta) \subset B^+_2$ leads to
\[
2 \int_{B^+_2} t^\alpha \eta^2 \bar{u}^{2\gamma+1} \nabla \eta \cdot \nabla \bar{u} \, dydt + (2\theta + 1) \int_{B^+_2} t^\alpha \eta^2 \bar{u}^{2\gamma+1} |\nabla \bar{u}|^2 \, dydt \geq \int_{\mathbb{R}^{n+1}} t^\beta \eta^2 \bar{u}^{2\gamma+1} g \, dydt.
\]
Thus
\[
\int_{B^+_2} t^\alpha \eta^2 \bar{u}^{2\gamma+1} |\nabla \bar{u}|^2 \, dydt \leq C(\theta) \int_{B^+_2} t^\alpha |\nabla \eta|^2 \bar{u}^{2\gamma+2} + t^\beta \eta^2 \frac{|g|}{k} \bar{u}^{2\gamma+2} \, dydt
\]
where $C(\theta)$ is bounded when $\theta$ is away from $-\frac{1}{2}$. Define $w = \bar{u}^{\theta+1}$ if $\theta \neq -1$, $w = \log \bar{u}$ if $\theta = -1$. Inserting it to the above equation. One gets
\[
\int_{B^+_2} t^\alpha |\nabla (\eta w)|^2 \, dydt \leq \begin{cases} C(\theta) \int_{B^+_2} t^\alpha |\nabla \eta|^2 w^2 + t^\beta \eta^2 w^2 \, dydt & \text{if } \theta \neq -1 \\ C \int_{B^+_2} t^\alpha |\nabla \eta|^2 + t^\beta \eta^2 \, dydt & \text{if } \theta = -1. \end{cases}
\] (4.15)

Combining with inequality (1.5), we have
\[
\left( \int_{B^+_2} t^\alpha (\eta w)^p \, dydt \right)^{\frac{1}{p}} \leq C \int_{B^+_2} t^\alpha |\nabla \eta|^2 w^2 \, dydt.
\] (4.16)

For $p \neq 0$, define
\[
\Phi(p,r) = \left( \int_{B^+_2} t^\beta \bar{u}^p \, dydt \right)^{\frac{1}{p}}
\]
Set $\gamma = 2(\theta + 1)$ for some $\theta < -\frac{1}{2}$, choose $\eta = 1$ in $B^+_1$ and $\eta = 0$ in $B^+_2 - B^+_r$. Then (4.16) implies
\[
\Phi(\gamma, r_2) \leq \left[ \frac{C}{r_2 - r_1} \right]^{\frac{1}{\gamma}} \Phi(\gamma^\frac{p^*}{2}, r_1) \quad \text{if } \gamma < 0,
\] (4.17)
\[
\Phi(\gamma^\frac{p^*}{2}, r_1) \leq \left[ \frac{C}{r_2 - r_1} \right]^{\frac{p^*}{\gamma p^*}} \Phi(\gamma, r_2) \quad \text{if } \gamma > 0.
\] (4.18)

Iterate inequality (4.17). Setting $r_m = 2 + 2^{-m}$ and $\gamma_0 = -p_0$ for some $p_0 \in (0,1)$ and $\gamma_m = \gamma_{m-1} - p_0$, $m = 1, 2, \ldots$, we have
\[
C \inf_{B^+_1} \bar{u} \geq \Phi(-p_0, 3)
\] (4.19)
where $C > 0$ depends upon $n, \alpha, p_0$.

Iterating the inequality (4.18). Setting $r_m = 2 + 2^{-m}$ and $\gamma_0 = p_0 \in (0,1)$ and $\gamma_m = \gamma_{m-1} - p_0$, $m = 1, 2, \ldots$, after some finite steps, one gets
\[
\Phi(1,2) \leq C \Phi(p_0, 3).
\]
Next, we want to show for some $p_0$ small enough that
\[
\Phi(p_0, 3) \leq C \Phi(-p_0, 3). \tag{4.20}
\]

To prove such an inequality, one can extend $u$ evenly to the whole $\mathbb{R}^{n+1}$, that is $u(y, t) = u(y, -t)$ for $t < 0$. We can show that $u$ is a well-defined weak solution to \((1.7)\) in the whole space (with $t$ replaced by $|t|$). Letting $w = \log \bar{u}$, one can get the following inequality from \((4.15)\)
\[
\int_{B_6} |t|^\alpha |\nabla (\eta w)|^2 dydt \leq C \int_{B_6} |t|^\alpha |\nabla \eta|^2 + |t|^\beta \eta^2 dydt \tag{4.21}
\]
for any $\eta$ which is some cut-off function with $\text{supp}(\eta) \subset B_6$. Taking any ball $B_r(X)$ for some $X \in B_6$ such that $B_{2r}(X) \subset B_6$, one can choose a cut-off function $\eta$ such that $\eta = 1$ on $B_r(X)$, $\text{supp}(\eta) \subset B_2(X)$ and $|\nabla \eta| \leq 2/r$. Then the above inequality implies
\[
\int_{B_{r}(X)} |t|^\alpha |\nabla w|^2 dydt \leq Cr^{-2} \int_{B_{2r}(X)} |t|^\alpha dydt.
\]

It follows from Lemma 4.4 that
\[
\int_{B_{r}(X)} |t|^\beta |w - w_{B_r(X)}| dydt \leq C \int_{B_{r}(X)} |t|^\beta dydt.
\]
This shows $w \in BMO(B_3, |t|^\beta dydt)$. Since $|t|^\beta$ is a weight with doubling property, that is
\[
\int_{B_{2r}(X)} |t|^\beta dydt \leq C(\beta) \int_{B_{r}(X)} |t|^\beta dydt.
\]
Using the above Lemma 4.5 there exist some $p_0 > 0$ small such that
\[
\int_{B_2} e^{p_0 w |t|^\beta} \int_{B_2} e^{-p_0 w |t|^\beta} dydt \leq C.
\]
Notice that $w = \log \bar{u}$, and $u$ is even with respect to $t$, the above inequality exactly means $\Phi(p_0, 3) \leq \Phi(-p_0, 3)$. Combining \((4.19)\) with \((4.20)\) and letting $\varepsilon \to 0$, we get our conclusion. \hfill \Box

**Corollary 4.7.** Suppose $0 \leq u \in D^{1,2}_{\alpha, \text{loc}}(\mathbb{R}^{n+1})$ is a weak solution to equation \((1.8)\) and $\alpha, \beta$ satisfy condition \((4.1)\), then $u$ is H"older continuous up to the boundary $\partial \mathbb{R}^{n+1}$.

**Proof.** For $r < \frac{1}{2}$, define $M(r) = \sup_{B_2^r} u$ and $m(r) = \inf_{B_2^r} u$, and $\omega(r) = M(r) - m(r)$, then
\[
\int_{B_2^r} t^\alpha \nabla [M(4r) - u] \cdot \nabla \phi dydt = - \int_{B_2^r} t^\beta u^{r-1} \phi dydt \geq -M(2)^{r-1} \int_{B_2^r} t^\beta \phi dydt, \\
\int_{B_2^r} t^\alpha \nabla [u - m(4r)] \cdot \nabla \phi dydt = \int_{B_2^r} t^\beta u^{r-1} \phi dydt \geq 0.
\]
for any $\phi \geq 0$ with compact support in $B_{2r}^+$. One can mimic the proof of theorem \cite[1.6]{4.6} to get
\[
\frac{1}{r^{n+1+\beta}} \int_{B_{2r}^+} t^\beta [M(4r) - u] dy dt \leq C \inf_{B_t^+} [M(4r) - u] + Cr^{\beta+2-\alpha} M(2)^{p^*-1} = C [M(4r) - M(r)] + Cr^{\beta+2-\alpha} M(2)^{p^*-1},
\]
\[
\frac{1}{r^{n+1+\beta}} \int_{B_{2r}^+} t^\beta [u - m(4r)] dy dt \leq C \inf_{B_t^+} [u - m(4r)] = C [m(r) - m(4r)].
\]
Summing the above two equations leads to
\[
\omega(r) \leq \frac{C - 1}{2} \omega(4r) + r^{\beta+2-\alpha} M(2)^{p^*-1}.
\]
Using this inequality, it is standard, for example see \cite[Lemma 8.23]{24}, to conclude that $u$ is Hölder continuous up to the boundary. \hfill \Box

5. Classification results

Though, in certain cases (see, for example, Obata \cite{35}, Escobar \cite{11}, Beckner \cite{43}, Jerison and Lee \cite{29}) one can use conformal invariant property to obtain the best constant for the sharp Sobolev type inequalities, the more powerful way is to classify all positive solutions to the Euler-Lagrange equations satisfied by the best constant for the sharp Sobolev type inequalities. In this section, we shall prove Theorem 1.7 through the proof of the following theorem.

**Theorem 5.1.** Let $\beta > -1$, $\alpha + \beta \geq 0$ and $\frac{\alpha - 1}{n+1} \beta < \alpha < \beta + 2$. If $n = 1$, in addition assume
\[
\frac{1 - (1 - \alpha)^2}{4} \leq \frac{\alpha(2 + \beta)}{(\alpha + \beta + 2)^2}.
\]
Assume that $u \in D^{1,2}_{\alpha,loc}(\mathbb{R}^{n+1})$ is a positive weak solution to equation (1.8). Then,
\[
u(y, t) = \left(\frac{1}{|y - y^o|^2 + (t + A)^2}\right)^{\frac{n-1}{2}} \psi\left(\frac{(y - y^o, t + A)}{|y - y^o|^2 + (t + A)^2} - (y^o, A)\right)
\]
for some $y^o \in \mathbb{R}^n$, $A > 0$, and $\psi(r) > 0$ satisfies an ordinary differential equation
\[
\begin{align*}
\psi'' + \left(\frac{n}{r} - \frac{2nA}{r^2}\right) \psi' - \frac{(\alpha(\alpha + \alpha + 1)A}{2n^2 - r^2} \psi &= -C \left(\frac{1}{4\pi} - r^2\right)^{\beta-\alpha} \psi^{\frac{n+2\beta+\alpha-1}{2n+2}}, \quad 0 < r < \frac{1}{4\pi},
\end{align*}
\]
for some constant $C > 0$ independent of $A$. Further, there is only one solution to equation (5.3).
(1) If $\beta = \alpha - 1$, $\alpha \geq \frac{1}{2}$ for $n > 1$ or $\alpha \in \left\{\frac{1}{2}\right\} \cup \left[\frac{1}{2}, \frac{1}{4}\right], \infty$ for $n = 1$, then up to some constant $u(y, t)$ must be the form of
\[
u(y, t) = \left(\frac{A}{(A + t)^2 + |y - y^o|^2}\right)^{\frac{n+\alpha-1}{2}},
\]
where $A > 0$, $y^o \in \mathbb{R}^n$, and
\[
S_{1, \alpha, \alpha-1} = \alpha(n + \alpha - 1) \left[\frac{\Gamma(\alpha)}{\Gamma(n + 2\alpha)}\right]^{\frac{1}{n+\alpha}}.
\]
If $\beta = \alpha$, $\alpha > 0$ for $n > 1$ or $\alpha \geq \sqrt{2}$ for $n = 1$, then up to some constant $u(y,t)$ must be the form of
\[
u(y,t) = \left(\frac{A}{A^2 + t^2 + |y - y^0|^2}\right)^{n+\alpha-1},
\] where $A > 0$, $y^0 \in \mathbb{R}^n$, and
\[
S_{1,\alpha,\alpha} = (n + \alpha - 1)(n + \alpha + 1)\left[\frac{n}{2} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+\alpha+1}{2}\right)}\right]^{\frac{2}{n+\alpha+1}}.
\]

Remark 5.2. By Theorem 1.6 we know that $u \in C^2(\mathbb{R}^{n+1}_+ \cap C^1(\mathbb{R}^{n+1}_+))$ for some $\gamma \in (0, 1)$.

Remark 5.3. For $\alpha > 0$, if $u \in C^2(\mathbb{R}^{n+1}_+ \cap C^1(\mathbb{R}^{n+1}_+))$, then $u(y,t)$ is a classical solution to equation (1.8).

Remark 5.4. Formula (5.2) indicates that $u(y,t)$ is “almost” a radially symmetric function in the sense that equation (1.8) can be reduced into the ODE (5.3).

First, we use the method of moving spheres to determine the boundary value $u(y,0)$.

**Proposition 5.5.** Assume that $\beta > -1$ and $\alpha + \beta \geq 0$ with $\frac{n-1}{n+1} \beta < \alpha < \beta + 2$. Let $u(y,t) \in D^{1,2}_{\alpha,\alpha}(\mathbb{R}^{n+1}_+)$ be a positive weak solution of (1.8). Then on $\partial \mathbb{R}^{n+1}_+$, $u(y,t)$ takes the form of
\[
u(y,0) = k\left(\frac{A}{A^2 + |y - y^0|^2}\right)^{\frac{n}{n+1}},
\] for some $k, A > 0$ and $y^0 \in \mathbb{R}^n$.

For any fixed $b \in \partial \mathbb{R}^{n+1}_+$, set
\[
u_b(y,t) = u((y,t) + b),
\]
\[
v_b(y,t) = \frac{1}{|(y,t)|^{n+\alpha-1}}u_b\left(\frac{(y,t)}{|(y,t)|}\right).
\]
Then we know $v_b$ is a weak solution in $\mathbb{R}^{n+1}_+ \setminus \{0\}$ (see the proof of Lemma 7.2 in Appendix), and
\[
\lim_{|(y,t)| \to \infty} |(y,t)|^{n+\alpha-1}v_b(y,t) = u_b(0,0) = u(b) > 0. \tag{5.6}
\]

For $\lambda > 0$, set
\[
v_{\lambda,b}(y,t) = \frac{\lambda^{n+\alpha-1}}{|(y,t)|^{n+\alpha-1}}v_b\left(\frac{(y,t)}{|(y,t)|}\right)
\]
and
\[
w_{\lambda,b}(y,t) = v_b(y,t) - v_{\lambda,b}(y,t).
\]
Then $w_{\lambda,b}$ satisfies
\[
\int_{\mathbb{R}^{n+1}_+} t^\alpha \nabla w_{\lambda,b} \cdot \nabla \phi dydt = \int_{\mathbb{R}^{n+1}_+} t^\beta \varphi^{2(\beta - \alpha - 1)} w_{\lambda,b} \phi dydt \tag{5.7}
\]
for $\forall \phi(y,t) \in D^{1,2}_{\alpha}(\mathbb{R}^{n+1}_+)$ that vanishes near $\{0\}$, where $\varphi(y,t) = s(y,t)v_b(y,t) + (1 - s(y,t))v_{\lambda,b}(y,t)$ for some $s(y,t) \in [0, 1]$.

**Claim 1.** When $\lambda$ is large enough, $w_{\lambda,b} \leq 0$ in $\mathbb{R}^{n+1}_+ \setminus B_\lambda(0)$. 

**Proof.** Define $\Sigma_{\lambda,b} = \{(y,t) \in \mathbb{R}^{n+1}_+ \setminus B_{\lambda}(0) : w_{\lambda,b}(y,t) > 0\}$ and $w^+_{\lambda,b} = \max\{w_{\lambda,b}, 0\}$ in $\mathbb{R}^{n+1}_+ \setminus B_{\lambda}(0)$, and trivially extend it to the whole space. Easy to see that $w^+_{\lambda,b} \in D^{1,2}_{\alpha}(\mathbb{R}^{n+1}_+)$ (see Lemma 7.3 in the Appendix). Taking $w^+_{\lambda,b}$ as the test function to (5.7), we have

$$\int_{\Sigma_{\lambda,b}} t^\alpha |\nabla w^+_{\lambda,b}|^2 dydt = \int_{\Sigma_{\lambda,b}} t^\beta \frac{2(\beta-\alpha+2)}{n+\alpha-1} |w^+_{\lambda,b}|^2 dydt.$$ 

Since in $\Sigma_{\lambda,b}$, $0 < v_{\lambda,b} \leq \varphi \leq v_b$ and $\alpha < \beta + 2$, we have

$$\int_{\Sigma_{\lambda,b}} t^\alpha |\nabla w^+_{\lambda,b}|^2 dydt \leq \int_{\Sigma_{\lambda,b}} t^\beta v_b \frac{2(\alpha+\beta+1)}{n+\alpha+1} |w^+_{\lambda,b}|^2 dydt \leq \frac{1}{2},$$

which implies

$$\int_{\Sigma_{\lambda,b}} t^\alpha |\nabla w^+_{\lambda,b}|^2 = 0.$$ 

Then $w_{\lambda,b} \leq 0$ in $\mathbb{R}^{n+1}_+ \setminus B_{\lambda}(0).$ \hfill $\square$

**Claim 2.** There exists $b \in \partial \mathbb{R}^{n+1}_+$, such that $\lambda_b > 0$.

**Proof.** If for all $b \in \partial \mathbb{R}^{n+1}_+$, $\lambda_b = 0$, we have for all $b \in \partial \mathbb{R}^{n+1}_+$ and $\lambda > 0,$

$$v_b(y,t) \leq \frac{\lambda^{n+\alpha-1}}{|(y,t)|^{n+\alpha-1}} v_b\left(\frac{\lambda^2(y,t)}{|(y,t)|^2}\right) \text{ in } \mathbb{R}^{n+1}_+ \setminus B_{\lambda}(0).$$

It follows that for all $b \in \partial \mathbb{R}^{n+1}_+$ and $\lambda > 0,$

$$u_b(y,t) \geq \frac{\lambda^{-(n+\alpha-1)}}{|(y,t)|^{n+\alpha-1}} u_b\left(\frac{\lambda^{-2}(y,t)}{|(y,t)|^2}\right) \text{ in } \mathbb{R}^{n+1}_+ \setminus B_{\lambda}(0).$$

It follows from the first Li-Zhu lemma (see, for example, Dou and Zhu [10, Lemma 3.7]) that $u$ only depends on $t$. Writing $v(t) = u(y,t)$, we know that $v \in C^2(0,\infty) \cap C^0[0,\infty)$ satisfies

$$\begin{cases}
(t^\alpha v'(t))' = -t^\beta v \frac{n+2-\alpha}{n+\alpha-1}, & 0 < t < \infty, \\
v(t) > 0, & 0 \leq t < \infty.
\end{cases}$$

An elementary phase-plane argument shows that $v(t) < 0$ for large $t$, contradicting with $v > 0.$ We have verified Claim 2. \hfill $\square$

**Claim 3.** Suppose $\lambda_b > 0$ for some $b \in \partial \mathbb{R}^{n+1}_+$, then we have $w_{\lambda,b,b} \equiv 0$ in $\mathbb{R}^{n+1}_+.$

**Proof.** Firstly, by continuity, $w_{\lambda,b,b} \leq 0$ in $\mathbb{R}^{n+1}_+ \setminus B_{\lambda_b}(0).$ If $w_{\lambda,b,b} < 0$ somewhere in $\mathbb{R}^{n+1}_+ \setminus B_{\lambda_b}(0)$, then by the maximum principle, we have $w_{\lambda,b,b} < 0$ in $\mathbb{R}^{n+1}_+ \setminus B_{\lambda_b}(0).$
Take $\delta_1$ small enough, such that $0 < \delta_1 < \lambda_b$ and for any $\lambda \in [\lambda_b - \delta_1, \lambda_b]$, there exists $R$ large enough, such that

$$S_{1,\alpha,\beta}^{-1}\left(\int_{\Sigma,\lambda, b} t^\beta v_b^{2(\alpha + \beta + 1)} \right)^{\frac{2 - \alpha + 2}{2n + \alpha + 1}} < \frac{1}{4}.$$ 

Take $\delta_2$ small enough, such that $0 < \delta_2 < \delta_1$ and

$$S_{1,\alpha,\beta}^{-1}\left(\int_{\Omega,\delta_2} t^\beta v_b^{2(\alpha + \beta + 1)} \right)^{\frac{2 - \alpha + 2}{2n + \alpha + 1}} < \frac{1}{4},$$

where $\Omega,\delta_2 = \left( (B^+_R(0) \setminus B^+_{\lambda_b + \delta_2}(0)) \cap \{(y, t) : 0 < t < \delta_2\} \right) \cup (B^+_R(0) \setminus B^+_{\lambda_b - \delta_2}(0))$ (See Figure 1).

![Figure 1: Domain of $\Omega,\delta_2$](image)

Since $w_{\lambda_b, b} < 0$ in compact set $(B^+_R(0) \setminus B^+_{\lambda_b + \delta_2}) \cap \{(y, t) : t > \delta_2\}$, we have $w_{\lambda_b, b} < -K < 0$ in $(B^+_R(0) \setminus B^+_{\lambda_b + \delta_2}) \cap \{(y, t) : t > \delta_2\}$.

By continuity, there exists $\delta_3$ small enough, such that $0 < \delta_3 < \delta_2$ and for any $\lambda \in [\lambda_b - \delta_3, \lambda_b]$

$$w_{\lambda, b} < -\frac{K}{2} < 0 \text{ in } (B^+_R(0) \setminus B^+_{\lambda_b + \delta_2}) \cap \{(y, t) : t > \delta_2\}.$$ 

Then for $\lambda \in [\lambda_b - \delta_3, \lambda_b]$, we have

$$S_{1,\alpha,\beta}^{-1}\left(\int_{\Sigma,\lambda, b} t^\beta v_b^{2(\alpha + \beta + 1)} \right)^{\frac{2 - \alpha + 2}{2n + \alpha + 1}} < \frac{1}{2}.$$ 

Similar to the proof of Claim 1, we have for $\lambda \in [\lambda_b - \delta_3, \lambda_b]$, $w_{\lambda, b} \leq 0$ in $\mathbb{R}^{n+1}_+ \setminus B_\lambda(0)$, contradicting with the definition of $\lambda_b$. Then $w_{\lambda_b, b} \equiv 0$ in $\mathbb{R}^{n+1}_+$.

\textbf{Claim 4.} For all $b \in \partial \mathbb{R}^{n+1}_+$, $\lambda_b > 0$. 

□
Suppose the contrary to Claim 4 for some \( \bar{b} \in \partial \mathbb{R}^{n+1} \) such that \( \lambda_b > 0 \) and \( w_{\lambda_b, \bar{b}}(y, t) = 0, \forall (y, t) \in \mathbb{R}^{n+1}_+ \). It follows that

\[
 u_{\bar{b}}(y, t) = \frac{1}{\lambda_b^{n+\alpha-1} |(y, t)|^{n+\alpha-1}} u_{\bar{b}}(\frac{(y, t)}{\lambda_b^2 |(y, t)|^2}), \quad \forall (y, t) \in \mathbb{R}^{n+1}_+.
\]

Clearly,

\[
 \lim_{|(y, t)| \to \infty} |(y, t)|^{n+\alpha-1} u_{\bar{b}}(y, t) = \frac{u_{\bar{b}}(0, 0)}{\lambda_b^{n+\alpha-1}}.
\]

Namely,

\[
 \lim_{|y| \to \infty} |(y, t)|^{n+\alpha-1} u(y, t) = \frac{u(\bar{b})}{\lambda_b^{n+\alpha-1}}. \tag{5.8}
\]

Suppose the contrary to Claim 4 for some \( b \in \partial \mathbb{R}^{n+1}_+ \), namely,

\[
 v_b(y, t) - \frac{\lambda_b^{n+\alpha-1}}{|(y, t)|^{n+\alpha-1}} v_b \left( \frac{\lambda_b^2 (y, t)}{|(y, t)|^2} \right) \leq 0, \quad \forall \lambda > 0, (y, t) \in \mathbb{R}^{n+1}_+ \setminus B_\lambda(0).
\]

Then

\[
 u_b \left( \frac{(y, t)}{|(y, t)|^2} \right) \leq \frac{|(y, t)|^{n+\alpha-1}}{\lambda_b^{n+\alpha-1}} u_b \left( \frac{(y, t)}{\lambda_b^2} \right), \quad \forall \lambda > 0, (y, t) \in \mathbb{R}^{n+1}_+ \setminus B_\lambda(0).
\]

Fixing \( \lambda > 0 \) in the above and sending \( |(y, t)| \) to \( \infty \), by (5.8), we have

\[
 u_b(0) \leq \frac{\lambda_b^{n+\alpha-1}}{\lambda_b^{n+\alpha-1}} u(\bar{b}).
\]

Sending \( \lambda \) to 0, we have

\[
 u(b) = u_b(0) \leq 0.
\]

Contradiction. \( \square \)

Using the second Li-Zhu Lemma in [23, Lemma 2.5] and its generalization for continuous functions due to Li and Nirenberg [29, Lemma 5.8], we have

**Lemma 5.6.** Suppose \( \alpha \in \mathbb{R} \) and \( f \in C(\mathbb{R}^n) \) \((n \geq 1)\) satisfying: \( \forall b \in \mathbb{R}^n \), there exists \( \mu_b \in \mathbb{R} \) such that

\[
 f(x' + b) = \left( \frac{\mu_b}{|x'|^{\alpha+1}} \right)^{n+\alpha-1} f \left( \frac{\mu_b^2 |x'|^2 + b}{|x'|^2} \right), \quad \forall x' \in \mathbb{R}^n \setminus \{0\}. \tag{5.9}
\]

Then for some \( a \geq 0, d > 0, x_0' \in \mathbb{R}^n \),

\[
 f(x') = \frac{a}{|x' - x_0'|^2 + d} \left( \frac{a}{|x' - x_0'|^2 + d} \right)^{\alpha+1}, \quad \forall x' \in \mathbb{R}^n,
\]

or

\[
 f(x') = -\frac{a}{|x' - x_0'|^2 + d} \left( \frac{a}{|x' - x_0'|^2 + d} \right)^{\alpha+1}, \quad \forall x' \in \mathbb{R}^n.
\]

**Proof of Proposition 5.5.** From Claim 4, we know \( \lambda_b > 0 \) for all \( b \in \partial \mathbb{R}^{n+1}_+ \). Then if follows from Claim 3 that

\[
 v_b(y, t) = \frac{\lambda_b^{n+\alpha-1}}{|(y, t)|^{n+\alpha-1}} v_b \left( \frac{\lambda_b^2 (y, t)}{|(y, t)|^2} \right).
\]

That is,

\[
 u_b(y, t) = \frac{\mu_b^{n+\alpha-1}}{|(y, t)|^{n+\alpha-1}} u_b \left( \frac{\mu_b^2 (y, t)}{|(y, t)|^2} \right). \tag{5.10}
\]
where \( \mu_b = \lambda_b^{-1} \). By Lemma 5.6 and \( u > 0 \), we have
\[
u(y) = k \left( \frac{A}{A^2 + |y - y^0|^2} \right)^{\frac{n + \alpha - 1}{2}}, \quad y \in \mathbb{R}^n,
\]
for some \( k, A > 0 \) and \( y^0 \in \mathbb{R}^n \).

**Proof of Theorem 5.1.** Without loss of generality, we assume that \( k = 1, A = 1 \) and \( y^0 = 0 \). By (5.8), we have
\[
1 = \lim_{|y| \to \infty} |y|^n \alpha - 1 u(y, t) = \mu_b^{n + \alpha - 1} u(b) = \mu_b^{n + \alpha - 1} \left( \frac{1}{1 + |b|^2} \right)^{\frac{n + \alpha - 1}{2}},
\]
which implies
\[
\mu_b = \sqrt{1 + |b|^2}.
\]
Then by (5.10), for any \( (y, t) \in \mathbb{R}^{n+1}_+ \),
\[
u(y, t) = \frac{(1 + |b|^2)^{\frac{n + \alpha - 1}{2}}}{(y, t) - b|^{n + \alpha - 1} u(b + \frac{(1 + |b|^2)((y, t) - b)}{|(y, t) - b|^2})}.
\]
Set \( \epsilon_{n+1} = (0, 1) \), and
\[
x := (x', x_{n+1}) = -\epsilon_{n+1} + \frac{(y, t) + \epsilon_{n+1}}{|(y, t) + \epsilon_{n+1}|^2},
\]
\[
\psi(x) = \frac{1}{|x + \epsilon_{n+1}|^{n+1}} u(-\epsilon_{n+1} + \frac{x + \epsilon_{n+1}}{|x + \epsilon_{n+1}|^2}),
\]
\[
B = B_\frac{\epsilon_{n+1}}{2}.
\]
Proposition 5.6 implies that pointwise \( \psi \) satisfies
\[
\begin{cases}
\Delta \psi - \frac{2 \alpha \nabla \psi \cdot (x + \epsilon_{n+1})}{|x + \epsilon_{n+1}|^2} - \frac{\alpha (n + \alpha - 1)}{4 - \alpha |x + \epsilon_{n+1}|^2} \psi = -C \left( \frac{1}{4} - |x + \epsilon_{n+1}|^2 \right)^{\frac{3 - \alpha}{2}} \psi \left( \frac{n + 2\beta - \alpha + 3}{n + \alpha - 1} \right), & \text{on } B, \\
\psi = 1, & \text{on } \partial B.
\end{cases}
\]
for some unknown \( C > 0 \). Next, we will show that \( \psi \) is radially symmetric about the center \( -\frac{\epsilon_{n+1}}{2} \).

Combining (5.11) with (5.13), we have
\[
\frac{1}{|(y, t) + \epsilon_{n+1}|^{n+1}} \psi(x) = \frac{(1 + |b|^2)^{\frac{n + \alpha - 1}{2}}}{|(y, t) - b|^{n + \alpha - 1} b + \epsilon_{n+1} + \frac{1}{|((y, t) - b)|^2} (\frac{1 + |b|^2)}{|(y, t) - b|^2} - 1) \psi(x^b),
\]
where
\[
x^b = -\epsilon_{n+1} + \frac{\frac{1 + |b|^2)}{|(y, t) - b|^2} - 1 + \epsilon_{n+1}}{1 + \frac{|b|^2)}{|(y, t) - b|^2} - 1 + \epsilon_{n+1}^2}.
\]
By simple calculation, we have that
\[
|\frac{x + \epsilon_{n+1}}{2}|^2 = |x^b + \epsilon_{n+1}|^2 = \frac{1}{4} - \frac{t}{|(y, t) + \epsilon_{n+1}|^2},
\]
and
\[
\frac{\psi(x)}{\psi(x^b)} = \left( \frac{(1 + |b|^2)}{|(y, t) - b|^2} b + \frac{1 + |b|^2)}{|(y, t) - b|^2} \right)^{\frac{n + \alpha - 1}{2}} = 1.
\]
Since $x^b$ runs all $\partial B \sqrt{\frac{1}{4} (|x|^2 + 4x_1^2)} (-\frac{2n+a}{2})$ as $b$ runs all $\partial \mathbb{R}^{n+1}$, we have $\psi$ is radial symmetric about the center $-\frac{e_n}{2}$. Multiplied by a suitable positive constant (still denote it $\psi$), $\psi$ satisfies the following ODE

$$\begin{align*}
\psi'' + \left(\frac{\beta}{1 - r^2}\right)\psi' - \frac{\alpha(n+1-\alpha)}{4 - r^2}\psi &= -(\frac{1}{4} - r^2)^{\beta - \alpha}\psi, \quad r \in (0, \frac{1}{2}), \\
\psi(\frac{1}{2}) &= K
\end{align*}$$

(5.15)

for some unknown constant $K > 0$. Summarizing the above analysis, we shall consider $0 < \psi \in C^2[0, \frac{1}{2}) \cap C^0[0, \frac{1}{2})$ satisfying the following ODE in $(0, \frac{1}{2})$

$$\begin{align*}
\psi''(r) + \left(\frac{\beta}{1 - r^2}\right)\psi'(r) - \frac{\alpha(n+1-\alpha)}{4 - r^2}\psi &= -(\frac{1}{4} - r^2)^{\beta - \alpha}\psi^{p-1}, \\
\psi(\frac{1}{2}) &= K, \quad \psi'(0) = 0
\end{align*}$$

(5.16)

for some $K > 0$.

**Proposition 5.7.** Suppose $\frac{2n-1}{2} \beta < \alpha < \beta + 2$, $\alpha + \beta \geq 0$ and $\beta > -1$. If $n = 1$, in addition assume

$$1 - \frac{(1 - \alpha)^2}{4} \leq \frac{\alpha(2 + \beta)}{(\alpha + \beta + 2)^2}.$$

Then there exists at most one $K$ such that (5.16) has a solution $0 < \psi \in C^2[0, \frac{1}{2}) \cap C^0[0, \frac{1}{2})$.

Let $w(r) = \left(\frac{1 - r^2}{4}\right)^{\frac{n+1-\alpha}{2}} \psi(\frac{1}{2})$. Then, for $r \in [0, 1)$, $w(r)$ satisfies

$$\begin{align*}
\left[\frac{1 - r^2}{2}\right] (w'' + \frac{n}{r} w') + (n - 1) \frac{1 - r^2}{2} rw' + \frac{n^2 - (1 - \alpha)^2}{4} w &= -w^{p-1}. \\
(5.17)
\end{align*}$$

We also have $w'(0) = 0$.

Now we view $w$ as a positive radial function lives on the unit disc $B^{n+1}$. The above equation actually can be interpreted in hyperbolic space. That is: if $B^{n+1} = \mathbb{H}^{n+1}$ is equipped with standard metric $4/(1 - |x|^2)|dx|^2$, the above equation is equivalent to (for example, see [32 pg. 666]):

$$\Delta_{\mathbb{H}} w + \frac{n^2 - (1 - \alpha)^2}{4} w = -w^{p-1}. \\
(5.18)$$

Such an equation is already studied by [32]. We will borrow some of their arguments to establish our uniqueness result.

Let $v(t) := w(\tanh \frac{t}{\alpha})$ and $q(t) = (\sinh t)^n$, then equation (5.18) can be written as

$$\begin{align*}
v'' + \frac{n}{\tanh t} v' + \frac{n^2 - (1 - \alpha)^2}{4} v + v^{p-1} &= 0, \quad v'(0) = 0.
\end{align*}$$

(5.19)

Noting $v(r)$ is bounded, we know the asymptotic behavior of $v(t)$ as $t \to \infty$:

$$\lim_{t \to \infty} v(t) \cdot e^{\frac{(n+1-\alpha)n}{2}} = L_1,$$

(5.20)

for some positive number $L_1$.

Proposition 5.7 follows from the following lemma.
Lemma 5.8. Suppose \( \frac{\alpha-1}{n+1} \beta < \alpha < \beta + 2, \alpha + \beta \geq 0 \) and \( \beta > -1 \). If \( n = 1 \), in addition assume

\[
1 - (1 - \alpha)^2 \leq \frac{2p^s}{(p^s + 2)^2} = \frac{\alpha(2 + \beta)}{(\alpha + \beta + 2)^2}.
\]

Then there is at most one positive solution to equation (5.19) which satisfies asymptotic condition (5.20).

Lemma 5.8 can be proved along the line of the proof of Theorem 1.3 in [3]. Even though, \( u(r) \) may not be in \( H^1(\mathbb{R}^{n+1}) \), but we do have the asymptotic behavior (5.20) for \( v(t) \), which yields

\[
\int_0^\infty q v^p \, dt < \infty.
\]

For example, considering \( E_v(t) = \frac{v'^2}{2} + \frac{n^2 - (1 - \alpha)^2}{8} v^2 + \frac{v^p}{p^s} \), we know

\[
\frac{d}{dt} E_v(t) = -\frac{1}{\tanh t} v'^2 \leq 0.
\]

Thus, we know that \( E_v(t) \) is decreasing to a nonnegative limit. One can easily see that the limit must be zero, and show that \( \lim_{t \to \infty} v'(t) = 0 \) and \( v'(t) < 0 \). We skip the other details here.

We now continue the proof of Theorem 5.1. First we observe

(i) If \( \beta = \alpha - 1 \), (5.10) has an obvious solution, \( \psi = [\alpha(n + \alpha - 1)]/(p^s - 2) \).

By Proposition 5.7, it is the unique solution provided \( n \geq 2 \) or \( n = 1 \) and \( \alpha \in (0, \frac{1}{2}) \cup \left(\frac{1}{2}(1 + \sqrt{17}), \infty \right) \). Since \( u \) is related to \( \psi \) by (5.13), thus \( u \) takes the form of (5.4).

(ii) If \( \beta = \alpha \), it is easy to verify (5.10) has an solution \( \psi(r) = C_{n,\alpha} (r^2 + \frac{1}{4})^{-\frac{n+\alpha-1}{2}} \)

for some suitable \( C_{n,\alpha} \). By Proposition 5.7, it is the unique solution provided \( n \geq 2 \) or \( n = 1 \) and \( \alpha \geq \sqrt{2} \). Using (5.13), we know that \( u \) takes the form of (5.3).

Then we need to compute the best constants. Using rearrangement (see, for example, [5] Proposition 4.2)) and the strong Maximum Principle for \( t > 0 \), we know that there are positive extremal functions. We first compute the best constant \( S_{1,\alpha,\alpha} \). From observation(ii), we know the extremal functions of \( S_{1,\alpha,\alpha} \) have the following form

\[
U_\alpha(y, t) = \left( \frac{A}{A^2 + t^2 + |y - y^o|^2} \right)^{\frac{n+\alpha-1}{2}}
\]

for any \( (y, t) \in \mathbb{R}_+^{n+1}, y^o \in \mathbb{R}^n \).

With loss of generality, we assume that \( A = 1 \) and \( y^o = 0 \), (since \( u(y, t) \) is translation invariant with respect to \( y \) direction). It is easy to verify

\[
\frac{\partial U_\alpha}{\partial y_i}(y, t) = -(n + \alpha - 1)(1 + t^2 + |y|^2)^{-\frac{n+\alpha-1}{2}-1} y_i, \quad i = 1, 2, \ldots, n,
\]

\[
\frac{\partial U_\alpha}{\partial t}(y, t) = -(n + \alpha - 1)(1 + t^2 + |y|^2)^{-\frac{n+\alpha-1}{2}-1} t,
\]

and then

\[
|\nabla U_\alpha(y, t)|^2 = (n + \alpha - 1)^2 \left( \frac{1}{1 + t^2 + |y|^2} \right)^{n+\alpha+1} (t^2 + |y|^2)
\]

\[
= (n + \alpha - 1)^2 \left[ \left( \frac{1}{1 + t^2 + |y|^2} \right)^{n+\alpha} - \left( \frac{1}{1 + t^2 + |y|^2} \right)^{n+\alpha+1} \right].
\]
Moreover, by the change of variable, we have

\[
\int_{\mathbb{R}^n_+} t^\alpha |U_\alpha(y, t)| \frac{2(n+\alpha+1)}{n+\alpha} dy dt = \int_0^\infty \int_{\mathbb{R}^n} t^\alpha \left(\frac{1}{1+t^2+\rho^2}\right)^{n+\alpha} \rho^{-1} d\rho d\mu
\]

\[
= n\omega_n \int_0^\infty \int_0^\infty t^\alpha \left(\frac{1}{1+t^2+\rho^2}\right)^{n+\alpha} \rho^{-1} d\rho d\mu
\]

\[
= n\omega_n \int_0^\infty t^\alpha \int_0^\infty (1+t^2)^{-(n+\alpha+1)+\frac{n+\alpha}{2}} \left(\frac{1}{1+(\sqrt{1+t^2})^2}\right)^{n+\alpha} \left(\frac{\rho}{\sqrt{1+t^2}}\right)^{n-1} d\rho d\mu
\]

\[
= n\omega_n \int_0^\infty \frac{t^\alpha}{(1+t^2)^{\frac{n+\alpha}{2}+\alpha+1}} dt \int_0^\infty \frac{r^{n-1}}{(1+r^2)^{n+\alpha}} dr \quad (r = \frac{\rho}{\sqrt{1+t^2}})
\]

\[
= n\omega_n \frac{\Gamma(\frac{\alpha+1}{2})\Gamma(\frac{n+\alpha+1}{2})}{\Gamma(\frac{n}{2}+\alpha+1)} \frac{\Gamma(\frac{n}{2}+\alpha+1)}{\Gamma(n+\alpha+1)}
\]

\[
= n\omega_n \frac{\Gamma(\frac{\alpha+1}{2})\Gamma(\frac{n+\alpha+1}{2})\Gamma(\frac{n}{2}+\alpha+1)}{\Gamma(n+\alpha+1)}
\]

where \(\omega_n = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)}\) is the volume of the \(n\)-dimensional unit ball in \(\mathbb{R}^n\).

Similarly,

\[
\int_0^\infty \int_{\mathbb{R}^n} t^\alpha \left(\frac{1}{1+t^2+\rho^2}\right)^{n+\alpha} \rho^{-1} d\rho d\mu
\]

\[
= n\omega_n \int_0^\infty \int_0^\infty t^\alpha \left(\frac{1}{1+t^2+\rho^2}\right)^{n+\alpha} \rho^{-1} d\rho d\mu
\]

\[
= n\omega_n \int_0^\infty \frac{t^\alpha}{(1+t^2)^{\frac{n+\alpha}{2}+\alpha+1}} dt \int_0^\infty \frac{r^{n-1}}{(1+r^2)^{n+\alpha}} dr \quad (r = \frac{\rho}{\sqrt{1+t^2}})
\]

\[
= n\omega_n \frac{\Gamma(\frac{\alpha+1}{2})\Gamma(\frac{n+\alpha+1}{2})}{\Gamma(\frac{n}{2}+\alpha+1)} \frac{\Gamma(\frac{n}{2}+\alpha+1)}{\Gamma(n+\alpha+1)}
\]

\[
= n\omega_n \frac{\Gamma(\frac{\alpha+1}{2})\Gamma(\frac{n+\alpha+1}{2})\Gamma(\frac{n}{2}+\alpha+1)}{\Gamma(n+\alpha+1)}
\]

Combining the above into \(1.0\) with \(\beta = \alpha\), we have

\[
S_{1,\alpha,\alpha} = \frac{\int_{\mathbb{R}^n_+} t^\alpha |\nabla U_\alpha|^2 dy dt}{(\int_{\mathbb{R}^n_+} t^\alpha |U_\alpha|^2 dy dt)^{\frac{n+\alpha}{n+\alpha+1}}}
\]

\[
= (n+\alpha-1)^2 \frac{\int_{\mathbb{R}^n_+} t^\alpha \left[\left(\frac{1}{1+r^2+|y|^2}\right)^{n+\alpha} - \left(\frac{1}{1+r^2+|y|^2}\right)^{n+\alpha+1}\right] dy dt}{(\int_{\mathbb{R}^n_+} t^\alpha \left(\frac{1}{1+r^2+|y|^2}\right)^{n+\alpha+1} dy dt)^{\frac{n+\alpha}{n+\alpha+1}}}
\]
By the change of variable, we have

\[
\int_{\mathbb{R}^n_+} t^\alpha \left( \frac{1}{1 + t^2 + |y|^2} \right)^{n+\alpha} \, dy \, dt
\]

\[
- \int_{\mathbb{R}^n_+} t^\alpha \left( \frac{1}{1 + t^2 + |y|^2} \right)^{n+\alpha+1} \, dy \, dt
\]

\[
= (n + \alpha - 1)^2 \left\{ \left[ \frac{n\omega_n}{4} \frac{\Gamma(\frac{\alpha+1}{2})\Gamma(\frac{n+\alpha+1}{2})\Gamma(\frac{n}{2})}{\Gamma(n + \alpha + 1)} \right]^{\frac{2}{n+\alpha+1}} \cdot 2(n + \alpha) \right\}
\]

\[
- \frac{n\omega_n}{4} \frac{\Gamma(\frac{\alpha+1}{2})\Gamma(\frac{n+\alpha+1}{2})\Gamma(\frac{n}{2})}{\Gamma(n + \alpha + 1)}
\]

\[
= (n + \alpha - 1)(n + \alpha + 1) \left[ \frac{n\pi^\frac{n}{2} \frac{\Gamma(\frac{\alpha+1}{2})\Gamma(\frac{n+\alpha+1}{2})\Gamma(\frac{n}{2})}{\Gamma(n + \alpha + 1)} \right]^{\frac{2}{n+\alpha+1}}
\]

We next compute the best constant \( S_{1,\alpha,\alpha-1} \). From observation(i), we know the extremal functions of \( S_{1,\alpha,\alpha-1} \) have the following form

\[
U_{\alpha-1}(y, t) = u(y, t) = \left( \frac{A}{(A + t)^2 + |y - y_0|^2} \right)^{\frac{2+n+\alpha-1}{2}}
\]

for any \((y, t) \in \mathbb{R}^{n+1}_+, y_0 \in \mathbb{R}^n\).

With loss of generality, we assume that \( A = 1 \) and \( y_0 = 0 \), (since \( u(y, t) \) is translation invariant the with respect to \( y \) direction). Note that

\[
\frac{\partial U_{\alpha-1}}{\partial y_i}(y, t) = -(n + \alpha - 1)((1 + t)^2 + |y|^2)^{-\frac{n+\alpha-1}{2}}y_i, \quad i = 1, 2, \ldots, n,
\]

\[
\frac{\partial U_{\alpha-1}}{\partial t}(y, t) = -(n + \alpha - 1)((1 + t)^2 + |y|^2)^{-\frac{n+\alpha-1}{2}}(1 + t),
\]

and

\[
|\nabla U_{\alpha-1}(y, t)|^2 = (n + \alpha - 1)^2 \left( \frac{1}{(1 + t)^2 + |y|^2} \right)^{n+\alpha+1}((1 + t)^2 + |y|^2)
\]

\[
= (n + \alpha - 1)^2 \left( \frac{1}{(1 + t)^2 + |y|^2} \right)^{n+\alpha}.
\]

By the change of variable, we have

\[
\int_{\mathbb{R}^n_+} t^\alpha \left| U_{\alpha-1}(y, t) \right|^{\frac{2(n+\alpha)}{n+\alpha+1}} \, dy \, dt = \int_0^\infty \int_{\mathbb{R}^n} t^\alpha \left( \frac{1}{(1 + t)^2 + |y|^2} \right)^{n+\alpha} \, dy \, dt
\]

\[
= n\omega_n \int_0^\infty \int_0^\infty t^\alpha \left( \frac{1}{(1 + t)^2 + \rho^2} \right)^{n+\alpha} \rho^{n-1} \, d\rho \, dt
\]
Combining the above into (1.6) with \( \beta = -\alpha - 1 \), we have

\[
\begin{align*}
S_{1,\alpha-1} &= \frac{1}{\Gamma(n+2\alpha)} \int_{\mathbb{R}^{n+1}} |\nabla U_{\alpha-1}| \, dydt \nonumber \\
&= \frac{1}{\Gamma(n+2\alpha)} \int_{\mathbb{R}^{n+1}} t^\alpha \left( \frac{1}{(1+t^2+|y|^2)^{n+\alpha}} \right)^{\frac{n+\alpha}{n+\alpha-1}} \left( \frac{1}{1+(1+t^2+|y|^2)^{n+\alpha}} \right)^{\frac{n+\alpha}{n+\alpha-1}} \, dydt \\
&= (n+\alpha-1)^2 \frac{1}{\Gamma(n+2\alpha)} \int_{\mathbb{R}^{n+1}} t^\alpha \left( \frac{1}{(1+t^2+|y|^2)^{n+\alpha}} \right)^{\frac{n+\alpha}{n+\alpha-1}} \left( \frac{1}{1+(1+t^2+|y|^2)^{n+\alpha}} \right)^{\frac{n+\alpha}{n+\alpha-1}} \, dydt \\
&= (n+\alpha-1)^2 \frac{\alpha}{n+\alpha-1} \left[ \frac{n\omega_n \Gamma(\alpha) \Gamma\left(\frac{n+\alpha}{2}\right) \Gamma\left(\frac{n+\alpha}{2} + \alpha\right)}{2 \Gamma(n+2\alpha)} \right]^{\frac{n+\alpha}{n+\alpha-1}} \\
&= \alpha(n+\alpha-1) \left[ \frac{n\omega_n \Gamma(\alpha) \Gamma\left(\frac{n+\alpha}{2}\right) \Gamma\left(\frac{n+\alpha}{2} + \alpha\right)}{2 \Gamma(n+2\alpha)} \right]^{\frac{n+\alpha}{n+\alpha-1}}.
\end{align*}
\]

The proof of Theorem 5.1 is completed. \( \square \)

**Remark 5.9.** It is not clear to us whether any nonnegative weak solutions to equation (1.8) must be positive.
6. Baouendi-Grushin operator and inequality

As an application of the sharp Gagliardo-Nirenberg inequality, we shall derive the best constants for the sharp form of inequality (1.15). We first prove Proposition 1.8.

**Proof of Proposition 1.8** Due to the rearrangement argument, we only need to prove inequality (1.15) for \( u(x, z) = u(|x|, |y|) \in C^\infty_0(\mathbb{R}^{n+m}) \).

**Case 1:** \( m = 1 \). For given \( \alpha \in [0, 1) \) and \( z > 0 \), let \( z = (1 - \alpha)z + \frac{1}{1 - \alpha} \) (that is: \( t = (1 - \alpha)z + \frac{1}{1 - \alpha} \)), \( x = \frac{u}{1 - \alpha} \) (that is: \( y = (1 - \alpha)x + \frac{1}{1 - \alpha} + 1)z \)), thus \( dz = (1 - \alpha)^{-1}dy \). Writing \( u(y, t) = u(x, z) \), we can check that \( u(y, t) \in D^{1,2}_\alpha(\mathbb{R}^{n+1}) \). We have

\[
\int_{\mathbb{R}^{n+1}} t^\alpha |\nabla u|^2 dydt = \int_{\mathbb{R}^{n+1}} t^\alpha (|\nabla y|^2 + u_1^2)dydt = (1 - \alpha)^{n+\alpha} \int_{\mathbb{R}^{n+1}} (z\alpha z (1 - \alpha)^{-1} |\nabla x|^2 + u_1^2)dx dz ,
\]

and

\[
\int_{\mathbb{R}^{n+1}} t^\beta |u|^{\frac{2n + 2\beta + 2}{\alpha}} dydt = \int_{\mathbb{R}^{n+1}} t^\beta \cdot (\frac{t}{1 - \alpha})^\alpha |u|^{\frac{2n + 2\beta + 2}{\alpha}} dydz = (1 - \alpha)^{n+\beta} \int_{\mathbb{R}^{n+1}} z^{-\alpha} |u|^{\frac{2n + 2\beta + 2}{\alpha}} dx dz .
\]

Thus, if we choose \( \alpha, \beta \) satisfy conditions in Corollary 1.2, inequality (1.15) implies: there is a positive constant \( C_{2,\alpha,\beta} = (1 - \alpha)^{\frac{2n + 2\beta + 2}{\alpha}} C_{1,\alpha,\beta} \), such that for any \( u(x, z) \in C^0_\alpha(\mathbb{R}^{n+m}) \),

\[
\left( \int_{\mathbb{R}^{n+1}} z^{-\alpha} |u|^{\frac{2n + 2\beta + 2}{\alpha}} dx dz \right)^{\frac{n+\alpha-1}{\alpha}} \leq C_{2,\alpha,\beta} \int_{\mathbb{R}^{n+1}} ((1 - \alpha)^{-2} z^{-\alpha} |\nabla x|^2 + u_1^2)dx dz .
\]

In particular, we choose \( \alpha \in (0, 1) \), \( \beta = -\alpha \), and using the density argument, we have

\[
\left( \int_{\mathbb{R}^{n+1}} |u|^{\frac{2n + 2\beta + 2}{\alpha}} dx dz \right)^{\frac{n+\alpha-1}{\alpha}} \leq C_{2,\alpha,\beta} \int_{\mathbb{R}^{n+1}} ((1 - \alpha)^{-2} z^{-\alpha} |\nabla x|^2 + u_1^2)dx dz .
\]

(6.1)

Changing \( z \) to \( -z \), we know that inequality (6.1) still holds in the lower half space \( \mathbb{R}^{n+1}_- \). Writing \( \tau = \frac{n-1}{1 - \alpha} \in [0, \infty) \), we obtain inequality (1.15) with

\[
S_{\tau}(n, \alpha) = (2^{-\frac{2(1 - \alpha)}{n - \alpha - 1}} C_{2,\alpha,\beta})^{-1} = 2^{\frac{2(1 - \alpha)}{n - \alpha - 1}} (1 - \alpha)^{-\frac{2n}{n - \alpha - 1}} S_{1,\alpha,\beta} .
\]

**Case 2:** \( m > 2 \). For \( \alpha \in [1, m - 1] \), let \( \tau = \frac{m - \alpha - 1}{\alpha - 1} \). Thus \( \tau \geq 0 \). Let \( t = r^{\tau+1}, z = r\xi, |\xi| = 1 \) on \( \mathbb{R}^m \), and \( y = x \). Writing \( u(y, t) = u(x, z) \), then \( dt = (\tau + 1) r^{\tau+1} dr, u_t = \)
Now choose Case 3: 

$$m = 1$$, we know from Theorem 1.7 part 1), that

$$\frac{u_0}{(n+1)\tau}.$$ We can check that $$u(y, t) \in D^{1,2}(\mathbb{R}_+^{n+1})$$, and

$$\int_{\mathbb{R}_+^{n+1}} t^\alpha |\nabla u|^2 dydt = \int_{\mathbb{R}_+^{n+1}} t^\alpha [|\nabla_y u|^2 + u_1^2] dydt

= \int_0^\infty \int_{\mathbb{R}^n} r^{(\tau+1)\alpha} [|\nabla_y u|^2 + \frac{1}{((\tau+1)r^\tau)^{2}}} u_1^2] (\tau+1)r^\tau dydr

= \frac{1}{\tau+1} \int_{\mathbb{R}^n} \int_0^\infty [(\tau+1)r^\tau|\nabla_y u|^2 + u_1^2] r^{(\tau+1)\alpha} dydr

= \frac{1}{\mu \omega_m(\tau+1)} \int_{\mathbb{R}_{n+m}} [(\tau+1)|z|^2|\nabla_x u|^2 + |\nabla_z u|^2] dx dz \quad (6.2)$$

and

$$\int_{\mathbb{R}_+^{n+1}} t^\beta |u|^{\frac{2n+2\beta+2}{m\tau+1}} dydt = \int_0^\infty \int_{\mathbb{R}^n} r^{(\tau+1)\beta} |u|^{\frac{2n+2\beta+2}{m\tau+1}} (\tau+1)r^\tau dydr

= \frac{\tau+1}{\mu \omega_m} \int_{\mathbb{R}_{n+m}} |u|^{\frac{2n+2\beta+2}{m\tau+1}} r^{(\tau+1)(\beta+1)-m} dx dz. \quad (6.3)$$

Now choose $$\beta = \frac{m}{\tau+1} - 1 = \frac{m}{\alpha-1} - 1 > -1$$. Easy to check: 

$$\frac{2n+2\beta+2}{m\tau+1} = \frac{2(n+m)}{(n+\alpha-1)\tau+1} = \frac{2Q}{\tau+1}. \quad \text{Clearly, } \alpha + \beta > 0 \text{ and } \alpha < \beta + 2. \text{ And } \frac{\alpha-1}{\tau+1} \beta \leq \alpha \text{ implies } \tau \geq \frac{1}{\beta-1}. \text{Bringing \ref{6.2} and \ref{6.3} into \ref{1.5}, we have: for } \tau \geq 0,$$

$$\left( \int_{\mathbb{R}_{n+m}} |u|^{\frac{2Q}{(\tau+1)\beta}} dx dz \right)^{\frac{Q-2}{2}} \leq S^{-1}_{\tau} (n, m) \int_{\mathbb{R}_{n+m}} [(\tau+1)^2 |z|^2 |\nabla_x u|^2 + |\nabla_z u|^2] dydz, \quad (6.4)$$

where 

$$S_{\tau}(n, m) = \left( \frac{2Q}{(\tau+1)\beta} \right)^{\frac{Q-2}{2}} S^{1,\alpha,\beta}_1, \text{ with } \alpha = \frac{m+\tau-1}{\tau+1} \text{ and } \beta = \frac{m}{\tau+1} - 1.$$

**Case 3:** 

$$m = 2$$.

For $$\tau \geq 0$$, we use the same substitution for variables in Case 2, and choose $$\alpha = 1$$, $$\beta = \frac{\tau+1}{\tau+1}$$. $$\tau \geq 0$$ implies $$\beta > 0$$, thus $$\alpha + \beta > 0$$. Clearly, $$\alpha = 1 \leq \beta + 2$$, and $$\frac{\alpha-1}{\tau+1} \beta \leq \alpha$$. Thus, bringing \ref{6.2} and \ref{6.3} into \ref{1.5}, we obtain \ref{6.4}.

The existence of extremal functions follow from Theorem 1.4. \qed

Now we prove Theorem 1.9 from Theorem 1.7.

**Proof of Theorem 1.9** We first prove part 1). Similar to the proof of Proposition 1.8 for $$m \neq 2$$ or $$m = 2, n \neq 1$$, we choose $$\alpha = m/2, \beta = m/2 - 1$$. Thus, for 

$$\tau = 1$$, we know from Theorem 1.7 part 1), that 

$$u(x, z) = \frac{1}{(1 + |z|^2)^2 + |x|^2}^{2n+m-2} \quad \forall(x, z) \in \mathbb{R}_{n+m}^{n+m}$$
is an extremal function to the sharp inequality (1.13). Thus

\[ S_1(n, m) = \frac{\int_{\mathbb{R}^{n+m}} |\nabla_x u|^2 \, dx \, dz + 4 |\nabla_x u|^2 \, dx \, dz}{\int_{\mathbb{R}^{n+m}} |u|^{2(m+\tau)} \, dx \, dz}^{\frac{2\tau}{2m+\tau}} \]
\[ = 2m\omega_m \int_{\mathbb{R}^{n+1}} t^{m-1} |u|^{\frac{2(m+\tau)}{2m+\tau}} \, dy \, dt \]
\[ = 2^{2-\frac{2}{m+\tau}} (m\omega_m)^{\frac{2}{m+\tau}} S_1(n, m) \]
\[ = 2^{2-\frac{2}{m+\tau}} \left( \frac{2\pi^\tau}{\Gamma(\frac{m}{2})} \right)^{\frac{m}{m+\tau}} m^2 n + m - 2 \left[ \pi^\tau \frac{\Gamma(m)}{\Gamma(n+1)} \right]^{\frac{2}{m+\tau}} \]
\[ = m(2n + m - 2) \left[ \pi^{\frac{n+m}{m+\tau}} \frac{\Gamma(m)}{\Gamma(n+1)} \right]^{\frac{2}{m+\tau}}. \]

Now assume that \( u(x, z) \in C^2(\mathbb{R}^{n+m}) \) is a positive solution to the equation (1.14) for \( \tau = 1 \), and assume that \( u(x, z) \) is rotationally symmetric about \( z \) variable. For simplicity, we only consider the case of \( m = 1 \). The case of \( m > 1 \) can be proved in the same way.

We first obtain the value for \( u(x, z) \) for \( z \geq 0 \).

Define \( w(x, t) = u(x, t^\frac{1+\tau}{\tau}) \) for \( t \geq 0 \), and for \( z \geq 0 \), let \( z = t^\frac{1+\tau}{\tau} \) for \( m = 1 \).

We have \( \frac{\partial w}{\partial t} = 2z \), \( u_z = 2zu_t \), \( u_{zz} = 4z^2 u_{tt} + 2u_t \), thus, for \( t > 0 \),

\[ w^{\frac{1+\tau}{\tau}}(x, t) = -w^{\frac{1+\tau}{\tau}}(x, z) = 4|z|^2 \Delta_x u + \Delta_z u \]
\[ = 4t \Delta_x w + 4tw_t + 2w_t \]
\[ = 4t^\frac{\tau}{1+\tau} \text{div}(t^\frac{1+\tau}{\tau} \nabla w)(x, t). \]

Since \( u(x, z) \in C^2(\mathbb{R}^{n+m}) \) and \( \partial u/\partial z = 0 \) at \( z = 0 \), using above equation one can check that \( v(x, t) := 2^{-n+1/2} w(x, t) \in C^2(\mathbb{R}^{n+1}+) \cap C^{1/2}(\mathbb{R}^{n+1}+) \) is a weak solution to

\[ \begin{cases} \text{div}(t^{1/2} \nabla v) = -t^{-1/2} w^{\frac{1+\tau}{\tau}}, & v(x, t) > 0, \quad \text{in } \mathbb{R}^{n+1}, \\ t^{1/2} \frac{\partial w}{\partial t} = 0, & \text{on } \partial \mathbb{R}^{n+1}. \end{cases} \] (6.5)

For \( m = 1 \) and \( \tau = 1 \), we first obtain the value for \( u(x, z) \) for \( z \geq 0 \). Choose \( z = t^\frac{\tau}{1+\tau} \) with \( t > 0 \). It follow from Theorem 1.7 that up to the multiple of some constant,

\[ v(x, t) = \left( \frac{1}{(\lambda_+ + t)^2 + |x - x_+|^2} \right)^{\frac{2n-1}{2}} \]

for some \( \lambda_+ > 0 \) and \( x_+^0 \in \partial \mathbb{R}^{n+1}_+ \). This yields:

\[ u(x, z) = 2^{2n-1} \left( \frac{1}{(\lambda_+ + |z|^2)^2 + |x - x_+|^2} \right)^{\frac{2n-1}{2}}, \quad \forall z \geq 0. \]

Similar argument applying to \( u(x, z) \) for \( z \leq 0 \) yields

\[ u(x, z) = 2^{2n-1} \left( \frac{1}{(\lambda_+ + |z|^2)^2 + |x - x_+|^2} \right)^{\frac{2n-1}{2}}, \quad \forall z \leq 0, \]
for some $\lambda_+ > 0$, and $x_0^0 \in \partial \mathbb{R}^{n+1}_+$. Since $u(x, z)$ is continuous, we know that $\lambda_+ = \lambda_-$ and $x_0^0 = x_0^0$. We thus have

$$u(x, z) = 2^{\frac{2n-1}{m}} \left( \lambda_+ + |z|^2 + |x - x_0^0|^2 \right)^{\frac{2n-1}{2m}}, \quad \forall (x, z) \in \mathbb{R}^{n+m}.$$ 

Now we prove part 2). We consider $m \neq 2$ or $m = 2, n \neq 1$. For $\tau \geq 0$, let $|z| = t^{\frac{1}{\alpha}}$ for $t > 0$, we have $\frac{d}{dt} = (\tau + 1)|z|^{\tau} z_j, u_{z_j} = (\tau + 1)|z|^{\tau} z_j w_t, u_{z_j z_j} = (\tau + 1)^2 |z|^{2(\tau - 1)} z_j^2 w_{tt} + (\tau + 1)|z|^{\tau - 1} w_t \left( (\tau - 1) \frac{z^2}{|z|^2} + 1 \right)$, for $j = 1, 2, \ldots, m$. Thus

$$\Delta_z u = (\tau + 1)^2 |z|^{2\tau} w_{tt} + (\tau + 1)(\tau - 1 + m)|z|^{\tau - 1} w_t$$

$$= (\tau + 1)^2 |z|^{2\tau} w_{tt} + (\tau + 1)(\tau - 1 + m)t^{\frac{\tau - 1}{\alpha}} w_t,$$

and then

$$-w^\frac{\alpha}{\alpha - 2}(x, t) = -u^\frac{\alpha}{\alpha - 2}(x, z) = (\tau + 1)^2 |z|^{2\tau} \Delta_x u + \Delta_z u$$

$$= (\tau + 1)^2 t^{\frac{\alpha}{\alpha - 2}} \Delta_x w + (\tau + 1)^2 t^{\frac{\alpha}{\alpha - 2}} w_{tt} + (\tau + 1)(\tau - 1 + m)t^{\frac{\tau - 1}{\alpha}} w_t$$

$$= (\tau + 1)^2 t^{\frac{\alpha}{\alpha - 2}} \left[ \Delta_x w + w_{tt} + \frac{\tau - 1 + m}{(\tau + 1)t} w_t \right]$$

$$= (\tau + 1)^2 t^{\frac{\alpha}{\alpha - 2}} \text{div}(t^{\frac{\alpha}{\alpha - 2}} \nabla w)(x, t).$$

Since $u(x, z) \in C^2(\mathbb{R}^{n+m})$ and $\partial u / \partial z = 0$ at $z = 0$, using above equation one can check that $v(x, t) := (\tau + 1)^{-\frac{\alpha}{\alpha - 2}} w(x, t) \in C^2(\mathbb{R}^{n+1}_+) \cap C^{1/2}(\mathbb{R}^{n+1}_+)$ is a weak solution to

$$\left\{ \begin{array}{ll}
\text{div}(t^{\frac{\alpha}{\alpha - 2}} \nabla v) = -t^{\frac{\alpha}{\alpha - 2}} \Delta_x v \quad & \text{in } \mathbb{R}^{n+1}_+, \\
\frac{\partial v}{\partial t} = 0, & \text{on } \partial \mathbb{R}^{n+1}_+. 
\end{array} \right. \quad (6.6)$$

Using (1.10) with $\alpha = \frac{\alpha - 1 + m}{\alpha + 1}$, we have

$$u(x, z) = \frac{1}{|x - x_0^0|^2 + (|z|^{\tau - 1} + A)^2} \psi(|x - x_0^0|^{\tau - 1} + A) \left( \frac{1}{|x - x_0^0|^2 + (|z|^{\tau - 1} + A)^2} - (x^0, A) \right).$$

The proof is hereby completed. $\Box$

**Remark** 6.1. If $m = 1$, we can prove the same classification result in Theorem [1.9] without the assumption that $u(x, z)$ is symmetric in $z$ variable. We only need to show that $v(x, z)$ satisfies equation (6.5) in a slight weak sense, and use the method of moving spheres to classify all positive weak solutions. Unfortunately, this argument does not work for $m \geq 2$. It remains as an open problem to prove that any positive $C^2$ solution to equation (1.14) is symmetric in variable $z$.

7. Appendix

We provide proofs for some technical lemmas in this appendix. For $\Omega \subset \mathbb{R}^{n+1}_+$, define the $\|\cdot\|_{H^{1,2}((\alpha, \infty), dy)}$ by

$$\|u\|_{H^{1,2}((\alpha, \infty), dy)} := \left( \int_{\Omega} t^{\alpha} |\nabla u|^2 dy + \int_{\Omega} t^{\alpha} |u|^2 dy \right)^{\frac{1}{2}}.$$

**Lemma 7.1.** For $\alpha > -1$, the embedding from $H^{1,2}_{loc}(\mathbb{R}^{n+1}_+, t^\alpha dy)$ to $L^2_{loc}(\mathbb{R}^{n+1}_+, t^\alpha dy)$ is compact.
Proof. Suppose \( \{u_i(y,t)\} \) is bounded in \( H^{1,2}(U \times T, t^\alpha dydt) \), where \( U \subset \mathbb{R}^n \) is compact and \( T \subset [0,\infty) \) is a compact interval.

1). If \( \alpha = m \) is a nonnegative integer. Define

\[
v_i(y,z) = u_i(y,t),
\]

where \( z \in \mathbb{R}^{m+1} \) and \( |z| = t \). Set \( B_T = \{ z \in \mathbb{R}^{m+1}, |z| \in T \} \), then we have

\[
\int_{U \times B_T} |v_i(y,z)|^2 dydz = \int_U \int_{B_T} |v_i(y,z)|^2 dzdy
\]

\[
= (m+1)\omega_{m+1} \int_U \int_T |u_i(y,t)|^2 t^m dt dy
\]

\[
= (m+1)\omega_{m+1} \int_{U \times T} t^m |u_i(y,t)|^2 dydt
\]

and

\[
\int_{U \times B_T} |\nabla v_i(y,z)|^2 dydz = \int_U \int_{B_T} (|\nabla_y v_i(y,z)|^2 + |\nabla_z v_i(y,z)|^2 ) dzdy
\]

\[
= (m+1)\omega_{m+1} \int_U \int_T (|\nabla_y u_i(y,t)|^2 + |\nabla_z u_i(y,t)|^2 ) t^m dt dy
\]

\[
= (m+1)\omega_{m+1} \int_{U \times T} t^m |\nabla u_i(y,t)|^2 dydt,
\]

where \( \omega_{m+1} \) is the volume of unit ball in \( \mathbb{R}^{m+1} \). It follows that \( \{v_i\} \) is bounded in \( H^{1,2}(U \times B_T) \). By compact Sobolev embedding, there is a subsequence, still denoting it by \( \{v_i\} \), converges in \( L^2(U \times B_T) \), i.e.

\[
\int_{U \times B_T} |v_i(y,z) - v_j(y,z)|^2 dydz \to 0 \quad \text{as } i,j \to \infty.
\]

By similarly calculation,

\[
\int_{U \times T} t^m |u_i(y,t) - u_j(y,t)|^2 dydt \to 0 \quad \text{as } i,j \to \infty.
\]

Then \( \{u_i(y,t)\} \) converges in \( L^2(U \times T, t^\alpha dydt) \).

2). For \( \alpha > 1 \), there is a positive integer \( m \), such that \( m - 1 \leq \alpha < m \). Without loss of generality, we assume \( \{u_i(y,t)\} \) is bounded in \( H^{1,2}(\tilde{\Omega}, t^\alpha dydt) \), where \( \tilde{\Omega} \cap \mathbb{R}^{n+1}_+ \) is open and \( U \times T \subset \tilde{\Omega} \). Take a smooth cut-off function \( 0 \leq \eta \leq 1 \) such that \( \eta = 1 \) in \( U \times T \) and \( \eta = 0 \) in \( \mathbb{R}^{n+1}_+ \setminus \tilde{\Omega} \).

We first assume \( T \subset [0,1] \), then

\[
\|u_i\|_{H^{1,2}(U \times T, t^\alpha dydt)} \leq \|u_i\|_{H^{1,2}(U \times T, t^\alpha dydt)} \leq C.
\]

By 1), we know that there is a subsequence, still denoting it by \( \{u_i\} \), converges in \( L^2(U \times T, t^m dydt) \), i.e.

\[
\int_{U \times T} t^m |u_i(y,t) - u_j(y,t)|^2 dydt \to 0 \quad \text{as } i,j \to \infty.
\]
Using the cut-off function, we have (noting $\alpha > 1$)

$$
\int_{U \times T} t^{\alpha - 2} |u_i(y, t) - u_j(y, t)|^2 dydt \\
\leq \int_{\mathbb{R}^n_+} t^{\alpha - 2} |\eta(u_i(y, t) - u_j(y, t))|^2 dydt \\
\leq S_{1, \alpha, \alpha - 2} \int_{\mathbb{R}^n_+} t^{\alpha} |\nabla(\eta(u_i(y, t) - u_j(y, t)))|^2 dydt \\
\leq 2S_{1, \alpha, \alpha - 2} \int_{\mathbb{R}^n_+} t^{\alpha} (|\nabla \eta|^2 |(u_i(y, t) - u_j(y, t))|^2 + \eta^2 |\nabla u_i(y, t) - \nabla u_j(y, t)|^2) dydt \\
\leq C \|u_i - u_j\|_{{\mathcal H}^{1, 2}(\tilde{\Omega}, t^{\alpha} dydt)}^2 \\
\leq C. 
$$

(7.1)

Then

$$
\int_{U \times T} t^{\alpha} |u_i(y, t) - u_j(y, t)|^2 dydt \\
\leq (\int_{U \times T} t^{\alpha - 2} |u_i(y, t) - u_j(y, t)|^2 dydt)^{\frac{m-\alpha}{m-\alpha+2}} (\int_{U \times T} t^m |u_i(y, t) - u_j(y, t)|^2 dydt)^{\frac{2}{m-\alpha+2}} \\
\leq C (\int_{U \times T} t^m |u_i(y, t) - u_j(y, t)|^2 dydt)^{\frac{2}{m-\alpha+2}} 
$$

as $i, j \to \infty$.

Thus $\{u_i(y, t)\}$ converges in $L^2(U \times T, t^\alpha dydt)$.

If $T \subset (1, \infty)$, similarly, we have

$$
\int_{U \times T} t^{m-1} |u_i(y, t) - u_j(y, t)|^2 dydt \to 0 \quad \text{as } i, j \to \infty.
$$

Then

$$
\int_{U \times T} t^{\alpha} |u_i(y, t) - u_j(y, t)|^2 dydt \\
\leq (\max_T t^{\alpha-m+1}) \int_{U \times T} t^{m-1} |u_i(y, t) - u_j(y, t)|^2 dydt \to 0 \quad \text{as } i, j \to \infty.
$$

For general $T$, we consider $T \cap [0, 1]$ and $T \cap (1, \infty)$ separately, then we can get that $\{u_i(y, t)\}$ converges in $L^2(U \times T, t^\alpha dydt)$.

3. For $\alpha \in (-1, 1)$, we use the same argument in Section 5.7 of Evans [12]. We assume that $\{u_i\}$ is bounded in $H_{\text{loc}}^{1, 2}(\mathbb{R}^{n+1}, |t|^{\alpha} dydt)$. By extension theorem, without loss of generality, we can assume $\text{supp} u_i$ are compact and $\text{supp} u_i \subset V$, where $V$ is an open and bounded set in $\mathbb{R}^{n+1}$. Let us first consider the smooth functions

$$
u_i = \eta \ast u_i,
$$

where $\eta$ denotes the usual mollifier. We may assume $\{u_i\}$ all have support in $V$ as well.

Claim 1. $\nu_i \to u_i$ in $L^2(V, t^\alpha dydt)$ as $\varepsilon \to 0$, uniformly in $i$. 
To prove this, we first note that if \( u_i \) is smooth, then
\[
u_i^\varepsilon(y,t) - u_i(y,t) = \int_{B_1(0)} \eta(x,s)[u_i((y,t) - \varepsilon(x,s)) - u_i(y,t)]dxds\]
\[
= \int_{B_1(0)} \eta(x,s)[\int_0^1 \frac{d}{dr} u_i((y,t) - \varepsilon r(x,s))dr]dxds\]
\[
= -\varepsilon \int_{B_1(0)} \eta(x,s)[\int_0^1 \nabla u_i((y,t) - \varepsilon r(x,s)) \cdot (x,s)dr]dxds.\]

Then
\[
\int_V |t|^{\alpha}|\nu_i^\varepsilon(y,t) - u_i(y,t)|dydt\]
\[
\leq \varepsilon \int_{B_1(0)} \eta(x,s)[\int_0^1 (\int_V |t|^{\alpha} |\nabla u_i((y,t) - \varepsilon r(x,s))|dydt)dr]dxds\]
\[
\leq \varepsilon \int_V |t|^{\alpha} |\nabla u_i(y,t)|dydt\]
\[
\leq \varepsilon \int_V |t|^{\alpha} |\nabla u_i(y,t)|^2 dydt\frac{1}{2} (\int_V |t|^{\alpha} dydt)^{\frac{1}{2}}\]
\[
\leq C\varepsilon \|\nabla u_i\|_{L^2(V,|t|^{\alpha}dydt)}.\]

By approximation, this estimate also holds for \( u_i \in H^{1,2}(V,|t|^{\alpha}dydt) \), i.e.\[
\|u_i^\varepsilon - u_i\|_{L^1(V,|t|^{\alpha}dydt)} \leq \varepsilon \|\nabla u_i\|_{L^1(V,|t|^{\alpha}dydt)} \leq \varepsilon \|\nabla u_i\|_{L^2(V,|t|^{\alpha}dydt)}.\]

Then we get
\[
u_i^\varepsilon \to u_i \quad \text{in} \quad L^1(V,|t|^{\alpha}dydt), \text{uniformly in} \ i.\]

By Hölder inequality and inequality (1.5), we get that\[
\|u_i^\varepsilon - u_i\|_{L^2(V,|t|^{\alpha}dydt)} \leq \|u_i^\varepsilon - u_i\|^\theta_{L^1(V,|t|^{\alpha}dydt)} \|u_i^\varepsilon - u_i\|^{1-\theta}_{L^\infty(\mathbb{R}^{n+1})} \|u_i\|_{L^{1,2}(V,|t|^{\alpha}dydt)} \]
\[
\leq C\|u_i^\varepsilon - u_i\|^\theta_{L^1(V,|t|^{\alpha}dydt)} \|u_i\|_{L^{1,2}(V,|t|^{\alpha}dydt)} \]
\[
\leq C\|u_i^\varepsilon - u_i\|^\theta_{L^1(V,|t|^{\alpha}dydt)},\]

where \( \theta \in (0,1) \) satisfies \( \frac{1}{2} = \theta + (1-\theta) \frac{n+\alpha-1}{2(n+\alpha+1)} \) and the second inequality is similar with (7.1). Then we get that
\[
u_i^\varepsilon \to u_i \quad \text{in} \quad L^2(V,|t|^{\alpha}dydt), \text{uniformly in} \ i.\]

Claim 2. For each fixed \( \varepsilon > 0 \), \( \{u_i^\varepsilon\} \) is uniformly bounded and equicontinuous. In fact, for every \((y,t) \in \mathbb{R}^{n+1} \) and \( i = 1, 2, \cdots, \) since \(-1 < \alpha < 1 \) we have \[
|u_i^\varepsilon(y,t)| \leq \int_{B_\varepsilon(x,s)} \eta_\varepsilon((y,t) - (x,s))u_i(x,s)dxds\]
\[
\leq \|\eta_\varepsilon\|_{L^\infty(\mathbb{R}^{n+1})} (\int_V |t|^{\alpha} |u_i|^2 dxds)^\frac{1}{2} (\int_V t^{-\alpha} dxds)^\frac{1}{2}\]
\[
\leq \frac{C}{\varepsilon^{n+1}} < \infty,\]
and
\[
|\nabla u^\varepsilon_i(y,t)| \leq \int_{B^\varepsilon_{\varepsilon}(y,t)} \left|\nabla \eta_i(x,t) - (x,s)\right| |u_i(x,s)| \, dx \, ds
\]
\[
\leq \left\|\nabla \eta_i\right\|_{L^\infty(\mathbb{R}^{n+1})} \left(\int_{\Omega} |t|^\alpha |u_i|^2 \, dy \, dt\right)^{\frac{1}{2}} \left(\int_{\Omega} t^{-\alpha} \, dy \, dt\right)^{\frac{1}{2}}
\]
\[
\leq \frac{C}{\varepsilon^{n+2}} < \infty.
\]

Claim 2 follows from these two estimates.

**Claim 3.** For any fixed \(\delta > 0\), there exists a subsequence \(\{u_{i,j}\}\), such that
\[
\limsup_{j,k \to \infty} \left\|u_{i,j} - u_{i,k}\right\|_{L^2(V,|t|^{\alpha} \, dy \, dt)} \leq \delta.
\]  
(7.2)

First, by Claim 1, there is \(\varepsilon > 0\) small enough, such that
\[
\left\|u_i - u^\varepsilon_i\right\|_{L^2(V,|t|^{\alpha} \, dy \, dt)} \leq \delta
\]  
(7.3)

for \(i = 1, 2, \cdots\). Next, by Claim 2 and Arzela-Ascoli Theorem, there is subsequence \(\{u_{i,j}\}\), converges uniformly on \(V\), then

\[
\limsup_{j,k \to \infty} \left\|u_{i,j} - u_{i,k}\right\|_{L^2(V,|t|^{\alpha} \, dy \, dt)} \leq \limsup_{j,k \to \infty} \left\|u_{i,j} - u_{i,k}\right\|_{L^\infty(V)} \left(\int_{\Omega} |t|^\alpha \, dy \, dt\right)^{\frac{1}{2}} = 0. (7.4)
\]

By (7.3) and (7.4), we get (7.2).

Finally, we iterate (7.2) with \(\delta = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\) and use diagonal argument to extract a subsequence \(\{u_{i,l}\}\) which satisfies
\[
\limsup_{l,k \to \infty} \left\|u_{i,l} - u_{i,k}\right\|_{L^2(V,|t|^{\alpha} \, dy \, dt)} = 0.
\]

The proof is hereby completed. \(\Box\)

**Lemma 7.2.** For \(\alpha > 0\), \(v_b\) is a weak solution in \(\mathbb{R}^{n+1}_+\setminus\{0\}\).

**Proof.** For any \(\psi \in C_0^\infty(\mathbb{R}^{n+1}_+\setminus\{0\})\), we suppose \(\text{supp} \psi \subset \Omega \subset \subset \mathbb{R}^{n+1}_+\setminus\{0\}\). Then by a simple computation, we have
\[
\int_{\Omega} t^\alpha \nabla v_b(y,t) \nabla \psi(y,t) \, dy \, dt
\]
\[
= -(n-1+\alpha) \int_{\Omega} \frac{t^\alpha}{|(y,t)|^{n+1+\alpha}} v_b(\frac{y,t}{|y,t|^2}) \, dy \, dt + \int_{\Omega} \frac{t^\alpha}{|(y,t)|^{n+1+\alpha}} |\nabla v_b(\frac{y,t}{|y,t|^2})| \cdot \nabla \psi(y,t) \, dy \, dt
\]
\[
- 2 \int_{\Omega} \frac{t^\alpha}{|(y,t)|^{n+1+\alpha}} |(\nabla v_b(\frac{y,t}{|y,t|^2})| \cdot \frac{y,t}{|y,t|^2})| \cdot \nabla \psi(y,t) \cdot (y,t) \, dy \, dt.
\]
Let \((x, s) = \frac{(y, t)}{||y, t||}\in \hat{\Omega} \subset \mathbb{R}_+^{n+1}\). Then
\[
\int_\Omega t^\alpha \nabla v_b(y, t) \nabla \psi(y, t) dydt
= -(n + \alpha - 1) \int_\Omega \frac{s^\alpha}{||x, s||^{n+\alpha+1}} u_b(x, s)(\nabla \psi(\frac{(x, s)}{||x, s||^2}) \cdot \frac{(x, s)}{||x, s||^2}) dxds
+ \int_\Omega \frac{s^\alpha}{||x, s||^{n+\alpha+1}}[\nabla u_b(x, s) \cdot \nabla \psi(\frac{(x, s)}{||x, s||^2})] dxds
- 2 \int_\Omega \frac{s^\alpha}{||x, s||^{n+\alpha+1}}[\nabla u_b(x, s) \cdot (x, s)](\nabla \psi(\frac{(x, s)}{||x, s||^2}) \cdot \frac{(x, s)}{||x, s||^2}) dxds
= -(n + \alpha - 1) \int_\Omega \frac{s^\alpha}{||x, s||^{n+\alpha+1}} u_b(x, s)(\nabla \psi(\frac{(x, s)}{||x, s||^2}) \cdot \frac{(x, s)}{||x, s||^2}) dxds
+ \int_\Omega \frac{s^\alpha}{||x, s||^{n+\alpha+1}} \nabla u_b(x, s) \cdot \nabla \psi(\frac{1}{||x, s||^{n+\alpha-1}}) dxds
+(n + \alpha - 1) \int_\Omega \frac{s^\alpha}{||x, s||^{n+\alpha-1}} \psi(\frac{(x, s)}{||x, s||^2}) \nabla u_b(x, s) \cdot (x, s) dxds
= \int_\Omega s^\beta u_b(x, s) \frac{s^\alpha}{||x, s||^{n+\alpha+1}} \psi(\frac{(x, s)}{||x, s||^2}) dxds
+(n + \alpha - 1) \int_\Omega \frac{s^\alpha}{||x, s||^{n+\alpha+1}} \nabla u_b(x, s) \psi(\frac{(x, s)}{||x, s||^2}) \cdot (x, s) dxds
= \int_\Omega t^\alpha v_b(y, t) \frac{s^\alpha}{||x, s||^{n+\alpha+1}} \psi(y, t) dydt.
\]
The last equality is due to
\[
\int_\hat{\Omega} \frac{s^\alpha}{||x, s||^{n+\alpha+1}}[\nabla (u_b(x, s) \psi(\frac{(x, s)}{||x, s||^2})) \cdot (x, s)] dxds
= \sum_{i=1}^n \int_{\partial \hat{\Omega}} \frac{s^\alpha}{||x, s||^{n+\alpha+1}} u_b(x, s) \psi(\frac{(x, s)}{||x, s||^2}) dS
+ \int_{\partial \hat{\Omega}} \frac{s^\alpha}{||x, s||^{n+\alpha+1}} u_b(x, s) \psi(\frac{(x, s)}{||x, s||^2}) dS
= 0.
\]
By approximation, for any \(\varphi \in D^1_\alpha(\mathbb{R}_+^{n+1})\) that vanishes near \(\{0\}\), we have
\[
\int_{\mathbb{R}_+^{n+1}} t^\alpha \nabla v_b(y, t) \nabla \psi(y, t) dydt = \int_{\mathbb{R}_+^{n+1}} t^\alpha v_b(y, t) \frac{s^\alpha}{||x, s||^{n+\alpha+1}} \psi(y, t) dydt.
\]
The proof is completed. 

Lemma 7.3. \(w^+_{\lambda, b} \in D^2_\alpha(\mathbb{R}_+^{n+1})\)

Proof. One can check:
\[
|\nabla v_b|^2(y, t) = \frac{(n + \alpha - 1)^2}{||(y, t)|^{2n+2\alpha}} u_b^2(\frac{(y, t)}{||(y, t)||^2}) + \frac{1}{||(y, t)|^{2n+2\alpha}} (\nabla u_b)^2(\frac{(y, t)}{||(y, t)||^2})
+ \frac{2(n + \alpha - 1)}{||(y, t)|^{2n+2\alpha}} u_b(\frac{(y, t)}{||(y, t)||^2}) (\nabla u_b)(\frac{(y, t)}{||(y, t)||^2})
\]
For any compact set \(K \subset \mathbb{R}_+^{n+1}\)\(\{0\}\), set \(\hat{K} := \{(\hat{y}, \hat{t}) : (\hat{y}, \hat{t}) = \frac{(y, t)}{||(y, t)||^2}, (y, t) \in K\}\), then \(\hat{K}\) is also a compact set in \(\mathbb{R}_+^{n+1}\)\(\{0\}\).
\[
\int_K t^\alpha |\nabla v_b|^2 (y, t) dy dt \\
= \int_K \frac{(n + \alpha - 1)^2 t^\alpha}{|y|^2} u_b \frac{\partial^2}{\partial n y^2} \eta (y, t) \, dy dt + \int_K \frac{t^\alpha}{|y|^2} |\nabla u_b|^2 \frac{d}{dy} \frac{\partial^2}{\partial n y^2} \eta (y, t) \, dy dt \\
+ \int_K \frac{2(n + \alpha - 1) t^\alpha}{|y|^2} u_b \eta \frac{\partial^2}{\partial n y^2} \eta (y, t) \, dy dt \\
\leq 2 \int_K \frac{(n + \alpha - 1)^2 t^\alpha}{|y|^2} u_b \eta \frac{\partial^2}{\partial n y^2} \eta (y, t) \, dy dt + 2 \int_K t^\alpha |\nabla u_b|^2 \eta \frac{\partial^2}{\partial n y^2} \eta (y, t) \, dy dt \\
\leq 2(n + \alpha - 1)^2 \int_K \frac{t^\alpha}{|y|^2} |\nabla u_b|^2 \eta \frac{\partial^2}{\partial n y^2} \eta (y, t) \, dy dt \\
+ 2 \int_K t^\alpha |\nabla u_b|^2 \eta \frac{\partial^2}{\partial n y^2} \eta (y, t) \, dy dt \\
\leq C(K) \int_K t^\alpha |\nabla u_b|^2 \eta \frac{\partial^2}{\partial n y^2} \eta (y, t) \, dy dt.
\]

Then \( v_b \in D^{1,2}_{\alpha, loc} (\mathbb{R}^{n+1}_+) \). Since \( v_{\lambda, b} (y, t) = \frac{1}{\lambda^{n+1}} u_b (\frac{y}{\lambda}, t) \), we have \( v_{\lambda, b} \in D^{1,2}_{\alpha, loc} (\mathbb{R}^{n+1}_+) \). We thus know that \( w_{\lambda, b}^+ \in D^{1,2}_{\alpha, loc} (\mathbb{R}^{n+1}_+) \).

For any \( R > \lambda \), choose a cut-off function \( \eta_R \in C^\infty_0 (\mathbb{R}^{n+1}_+) \), satisfying \( 0 \leq \eta_R \leq 1 \), \( \eta_R \equiv 1 \) in \( B_R (0) \), \( \eta_R \equiv 0 \) in \( \mathbb{R}^{n+1}_+ \setminus B_{2R} (0) \) and \( |\nabla \eta_R| \leq \frac{C}{R} \) for some constant \( C \). It is easy to check that \( \eta_R w_{\lambda, b}^+ \in D^{1,2}_{\alpha, loc} (\mathbb{R}^{n+1}_+) \). Then we can take \( \eta_R w_{\lambda, b}^+ \) as a test function in \( \mathcal{E} \).

\[
\int_{\mathbb{R}^{n+1}_+} t^\alpha \nabla w_{\lambda, b} \cdot \nabla (\eta_R w_{\lambda, b}^+) \, dy dt = C \int_{\mathbb{R}^{n+1}_+} t^\beta \frac{d}{dy} \frac{\partial^2}{\partial n y^2} w_{\lambda, b} \eta_R w_{\lambda, b}^+ \, dy dt.
\]

LHS \[= \int_{\Sigma, b} t^\alpha \eta_R^2 |\nabla w_{\lambda, b}^+|^2 \, dy dt + 2 \int_{\Sigma, b} t^\alpha \eta_R w_{\lambda, b}^+ \nabla w_{\lambda, b}^+ \cdot \nabla \eta_R \, dy dt \]
\[\geq \frac{1}{2} \int_{\Sigma, b} t^\alpha \eta_R^2 |\nabla w_{\lambda, b}^+|^2 \, dy dt - 2 \int_{\Sigma, b} t^\alpha |\nabla \eta_R|^2 |w_{\lambda, b}^+|^2 \, dy dt.\]

Then we have
\[
\int_{\Sigma, b} t^\alpha \eta_R^2 |\nabla w_{\lambda, b}^+|^2 \, dy dt \\
\leq 4 \int_{\Sigma, b} t^\alpha |\nabla \eta_R|^2 |w_{\lambda, b}^+|^2 \, dy dt + 2C \int_{\Sigma, b} t^\alpha \eta_R^2 \frac{d}{dy} \frac{\partial^2}{\partial n y^2} |w_{\lambda, b}^+|^2 \, dy dt \\
\leq 4 \int_{\Sigma, b} t^\alpha |\nabla \eta_R|^2 |v_b|^2 \, dy dt + 2C \int_{\Sigma, b} t^\beta v_b \frac{d}{dy} \frac{\partial^2}{\partial n y^2} |v_b|^2 \, dy dt.
\]
In the last step we use the fact that in $\Sigma_{\lambda,b}$, $0 \leq \varphi \leq v_b$ and $0 \leq w^+_{\alpha,b} \leq v_b$. Since $v_b(y,t) = O\left(\frac{1}{|y,t|^{n+\alpha-1}}\right)$ as $|(y,t)| \to \infty$, we have
\[
\int_{\Sigma_{\lambda,b}} t^{\beta} v_b^{2\frac{(n+2\alpha+1)}{n+\alpha-1}} dydt \leq C_1.
\]
And there is $R_0 > \lambda$ large enough, such that for $|(y,t)| \geq R_0$, we have $|v_b| \leq C |(y,t)|^{n+\alpha-1}$. Then, for $R \geq R_0$, we have
\[
\int_{\Sigma_{\lambda,b}} t^{\alpha} |\nabla \eta_R|^2 |v_b|^2 dydt = \int_{B^+_R(0) \setminus B^+_R(0)} t^{\alpha} |\nabla \eta_R|^2 |v_b|^2 dydt 
\leq CR^\alpha \cdot \frac{C}{R^2} \cdot \frac{C}{R^{2(n+\alpha-1)}} \cdot R^{n+1} 
\leq C_2,
\]
where $C_2$ is independent of $R$. On the other hand, for $R \in (\lambda, R_0)$,
\[
\int_{\Sigma_{\lambda,b}} t^{\alpha} |\nabla \eta_R|^2 |v_b|^2 dydt \leq \int_{\Sigma_{\lambda,b} \cap B^+_R(0) \setminus B^+_R(0)} t^{\alpha} |\nabla \eta_R|^2 |v_b|^2 dydt 
\leq C(\lambda, R_0).
\]
Then we get that for $R > \lambda$,
\[
\int_{\Sigma_{\lambda,b}} t^{\alpha} |\nabla \eta_R|^2 |w^+_{\alpha,b}|^2 dydt \leq C,
\]
where $C$ is independent of $R$. Letting $R \to \infty$, we get that $w^+_{\alpha,b} \in D^{1,2}_{\alpha}(\mathbb{R}^{n+1})$. \(\square\)

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References
[1] M. S. Baouendi, Sur une classe d’Opérateurs elliptiques dégénérés. Bull. Soc. Math. France, 95 (1967), 45-87.
[2] D. Bakry, I. Gentil and M. Ledoux, Analysis and geometry of Markov diffusion operators, Grundlehren der Mathematischen Wissenschaften 348, Springer, Berlin, 2013.
[3] W. Beckner, On the Grushin operator and hyperbolic symmetry. Proc. Amer. Math. Soc., 129 (2001), 1233-1246.
[4] R. Benguria, R. Frank, M. Loss, The sharp constant in the Hardy-Sobolev-Maz’ya inequality in the three dimensional upper half-space, Math. Res. Lett., 15 (2008), 613-622.
[5] X. Cabre, X. Ros-Oton, Sobolev and isoperimetric inequalities with monomial weights, J. Differ. Equ., 255 (2013), 4312-4336.
[6] L. Caffarelli, B. Gidas and J. Spruck, Asymptotic symmetry and local behavior of semilinear elliptic with critical Sobolev growth, Comm. Pure Appl. Math., 42 (1989), 271-297.
[7] L. Caffarelli, L. Silvestre, An extension problem related to the fractional Laplacian, Comm. PDE, 32 (2007), 1245-1260.
[8] S. Chen, A new family of sharp conformally invariant integral inequalities, Int. Math. Res. Not., 2014 (2014), 1205-1220.
[9] J. Dou, Q. Guo, M. Zhu, Subcritical approach to sharp Hardy-Littlewood-Sobolev type inequalities on the upper half space, Adv. Math., 312 (2017), 1-45.
[10] J. Dou, M. Zhu, Sharp Hardy-Littlewood-Sobolev inequality on the upper half space, Int. Math. Res. Not., 2015 (2015), 651-687.
[11] J. F. Escobar, Sharp constant in a Sobolev trace inequality, Indiana Univ. Math. J., 37, (1988), 687-698.
[12] C. Evans, Partial differential equations, Amer. Math. Soc., second printing, 2010.
[13] E.B. Fabes, C.E. Kenig, R.P. Serapioni, The local regularity of solutions of degenerate elliptic equations, Comm. PDE, 7 (1) (1982), 77-116.
[14] G. Folland, E. Stein, Estimates for the $\bar{\partial}$ complex and analysis on the Heisenberg group, Comm. Pure Appl. Math., 27 (1974), 429-522.
[15] B. Franchi, C. E.Gutiérrez, R. L. Wheeden, Weighted Sobolev-Poincaré inequalities for Grushin type operators. Comm. PDE, 19 (1994), 523-604.
[16] B. Franchi, E. Lanconelli, An embedding theorem for Sobolev spaces related to non-smooth vector fields and Harnack inequality, Comm. PDE, 9 (1984), 1237-1264.
[17] E. Gagliardo, Ulteriori proprietà di alcune classi di funzioni in più variabili. Ricerche Mat., 8(1959), 24-51.
[18] N. Garofalo, D. Vassilev, Symmetry properties of positive entire solutions of Yamabe-type equations on groups of Heisenberg type, Duke Math. J., 106 (2001), 411-448.
[19] B. Gidas, W. M. Ni, L. Nirenberg, Symmetry of positive solutions of nonlinear elliptic equations, in: $\mathbb{R}^n$, Mathematical Analysis and Applications, Part A, Advances in Mathematics Supplementary Studies, vol. 7a, Academic Press, New York, London, 1981, 39-402.
[20] M. Gluck, Subcritical approach to conformally invariant extension operators on the upper half space, J. Func. Anal., online.
[21] M. Gluck, M. Zhu, An extension operator on bounded domains and applications, Calc. Var. (2019) 58: 79.
[22] V. V. Grushin, On a class of hypoelliptic operators. Math. USSR-Sb 12(1970), 458-476.
[23] V. V. Grushin, A certain class of elliptic pseudodifferential operators that are degenerate on a submanifold, (Russian) Mat. Sb. (N.S.), 84 (126) (1971), 163-195.
[24] D. Gilbarg, N. S. Trudinger, Elliptic partial differential equations of second order, Springer, Berlin-Heidelberg-New York, 1983.
[25] D. Jerison, J.M. Lee, The Yamabe problem on CR manifolds, J. Diff. Geom., 25 (1987), 167-197.
[26] D. Jerison, Lee D. Jerison, J.M. Lee, Extremals for the Sobolev inequality on the Heisenberg group and the CR Yamabe problem, J. Amer. Math. Soc., 1 (1988) 1-13.
[27] J. Heinonen, T. Kilpeläinen, O. Martio, Nonlinear potential theory of degenerate elliptic equations, Dover version, New York: Clarendon Press, 1993.
[28] Y. Y. Li, M. Zhu, Uniqueness theorems through the method of moving spheres, Duke Math. J., 80 (1995), 383-417.
[29] Y. Y. Li, Remark on some conformally invariant integral equations: the method of moving spheres, J. Eur. Math. Soc., 6 (2004), 153-180.

[30] P. L. Lions, The concentration-compactness principle in the calculus of variations. The limit case I, Rev. Mat. Iberoamericana, 1 (1985), 145-201.

[31] P. L. Lions, The concentration-compactness principle in the calculus of variations. The limit case II, Rev. Mat. Iberoamericana, 1 (1985), 45-121.

[32] G. Mancini, K. Sandeep, On a semilinear elliptic equation in $\mathbb{R}^n$, Ann. Scuola Norm. Sup. Pisa Cl. Sci, (5) 2008, 635–671.

[33] V. Maz’ya, Sobolev spaces, Springer, second edition, 2011.

[34] R. Monti, D. Morbidelli, Kelvin transform for Grushin operators and critical semilinear equation, Duke Math. J., 131(1)(2006),167-202.

[35] R. Monti, Sobolev inequalities for weighted gradients, Comm. PDE, 31 (2006), 1479-1504.

[36] L. Nirenberg, On elliptic partial differential equations. Ann. Scuola Norm. Sup. Pisa., 13 (1959), 115-162.

[37] Van H. Nguyen, Sharp weighted Sobolev and Gagliardo-Nirenberg inequalities on half-spaces via mass transport and consequences, Proc. London Math. Soc., 111 (2015), 127-148.

[38] M. Obata, The conjecture on conformal transformations of Riemannian manifolds, J. Diff. Geom., 6 (1971), 247-258.

[39] A. Tertikas, K. Tintarev, On existence of minimizers for the Hardy-Sobolev-Maz’ya inequality, Ann. Mat. Pura Appl., 186 (2007), 645-662.

[40] L. Wang, M. Zhu, Liouville theorems on the upper half space, submitted, preprint: arxiv.org 1902.05187.

[41] L. Wang, Hölder estimates for subelliptic operators. J. Funct. Anal., 199 (2003), 228-242.

[42] M. Zhu, Thesis, Rutgers, 1996.