A UNIFORM SOBOLEV INEQUALITY UNDER RICCI FLOW

QI S. ZHANG

CONTENTS

1. Introduction 1
2. Proof of Theorem 1.1 3
3. ADDENDUM 8
References 15

Abstract. Let $M$ be a compact Riemannian manifold and the metrics $g = g(t)$ evolve by the Ricci flow. We prove the following result. The Sobolev imbedding by Aubin or Hebey, perturbed by a scalar curvature term and modulo sharpness of constants, holds uniformly for $(M, g(t))$ for all time if the Ricci flow exists for all time; and if the Ricci flow develops a singularity in finite time, then the same Sobolev imbedding holds uniformly after a standard normalization. As a consequence, long time non-collapsing results are derived, which improve Perelman’s local non-collapsing results. Applications to Ricci flow with surgery are also presented.

1. Introduction

Let $M$ be a compact Riemannian manifold of dimension $n \geq 3$ and $g$ be the metric. It is well known that a Sobolev inequality of the following form holds: there exist positive constants $A, B$ such that, for all $v \in W^{1,2}(M, g)$,

$$\left( \int v^{2n/(n-2)} d\mu(g) \right)^{(n-2)/n} \leq A \int |\nabla v|^2 d\mu(g) + B \int v^2 d\mu(g).$$

This inequality was proven by Aubin [Au] for $A = K^2(n) + \epsilon$ with $\epsilon > 0$ and $B$ depending on bounds on the injectivity radius, sectional curvatures and derivatives. Here $K(n)$ is the best constant in the Sobolev imbedding for $\mathbb{R}^n$. Hebey [H1] showed that $B$ can be chosen to depend only on $\epsilon$, the injectivity radius and the lower bound of the Ricci curvature. Hebey and Vaugon [HV] proved that one can even take $\epsilon = 0$. However the constant $B$ will also depend on the derivatives of the curvature tensor.

Now consider a Ricci flow $(M, g(t))$, $\partial_t g = -2Ric$. Due to the obvious importance of Sobolev inequality in its analysis, it is highly desirable to have a uniform control on the constants $A$ and $B$. For instance Sesum and Tian [ST] showed a uniform Sobolev imbedding for certain Kähler Ricci flow under an additional lower bound of the Ricci curvatures. Chang and Lu [CL] proved that the Yamabe constant has nonnegative derivative at the initial time under Ricci flow and under an additional technical assumption. Also in the paper [HV2], Hebey and Vaugon studied the evolution of Yamabe metrics for short time and proved pinching results.

Date: August 2007.
Unfortunately the controlling geometric quantities for $B$ as stated above are not invariant under the Ricci flow in general. Nevertheless, by using a generalized version Perelman’s W entropy, we are able to prove that the above Sobolev inequality (except for the sharp constants), with a perturbation term involving the scalar curvature, is valid uniformly under the Ricci flow or least after a suitable normalization. As a consequence, we establish long time non-collapsing result which generalizes Perelman’s short time result.

Here is the main result of the paper.

**Theorem 1.1.** Let $M$ be a compact Riemannian manifold with dimension $n \geq 3$ and the metrics $g = g(t)$ evolve by the Ricci flow $\partial_t g = -2\text{Ric}$. Then the following conclusions are true.

(a). Suppose the Ricci flow exists for all time $t \in (0, \infty)$. Then there exist positive functions $A, B$ depending only on the initial metric $g(0)$ and $t$, such that, for all $v \in W^{1,2}(M, g(t))$, $t > 0$, it holds

$$\left( \int v^{2n/(n-2)} d\mu(g(t)) \right)^{(n-2)/n} \leq A \int (|\nabla v|^2 + \frac{1}{4} R v^2) d\mu(g(t)) + B \int v^2 d\mu(g(t)).$$

Here $R$ is the scalar curvature with respect to $g(t)$. The same holds when $\infty$ is replaced by any $T > 0$.

(b). Suppose the Ricci flow is smooth for $t \in (0, 1)$ and is singular at $t = 1$. Let $\tilde{t} = -\ln(1-t)$ and $\tilde{g}(\tilde{t}) = \frac{1}{1-t} g(t)$ which satisfy a normalized Ricci flow

$$\partial_{\tilde{t}} \tilde{g} = -2\tilde{Ric} + \tilde{g}.$$

Then there exist positive constants $A, B$ depending only on the initial metric $g(0)$ such that, for all $v \in W^{1,2}(M, \tilde{g}(\tilde{t}))$, $\tilde{t} > 0$, it holds

$$\left( \int v^{2n/(n-2)} d\mu(\tilde{g}(\tilde{t})) \right)^{(n-2)/n} \leq A \int (|\nabla v|^2 + \frac{1}{4} \tilde{R} v^2) d\mu(\tilde{g}(\tilde{t})) + B \int v^2 d\mu(\tilde{g}(\tilde{t})).$$

Here $\tilde{R}$ is the scalar curvature with respect to $\tilde{g}(\tilde{t})$.

**Remark.** (1). By the work of Hebey [H1] and Brouttelande [Br], one can show that the constants in the theorem depending on $g(0)$ only through $n$, lower bound of the Ricci curvature and injectivity radius. This is explained in Step 2, Case (a) below.

(2). After the first version of the paper was posted on in the arXiv, two interesting related papers [Y] and [HS] have appeared in the arXiv. In one of them [Y], Professor Ye also pointed out an inaccuracy in the statement of the previous Theorem 1.1 (a) and its proof. However, one can see in section 2 below, this can be easily corrected by the same method of using the generalized Perelman entropy. Actually this part of the proof can be done more simply by using just Perelman’s original W-entropy. In the papers [Y] and [HS], the controlling constants of the Sobolev imbedding were computed more precisely and a full log Sobolev inequality were deduced.

In case of Kähler Ricci flow, the theorem takes a particularly simple form due to Perelman’s boundedness result on the scalar curvature, as explained in [ST].

**Corollary 1.** Let $(M, g(t))$ be the normalized Kähler Ricci flow

$$\partial_t g_{i\bar{j}} = g_{i\bar{j}} - R_{i\bar{j}}$$
where $\mathbf{M}$ is a compact, Kähler manifold with complex dimension $n$ and positive first Chern class. Then there exist positive constants $A, B$ depending only on the initial metric $g(0)$ such that, for all $v \in W^{1,2}(\mathbf{M}, g(t))$, $t > 0$, it holds

$$
(\int v^{4n/(2n-2)} d\mu(g(t)))^{(2n-2)/(2n)} \leq A \int |\nabla v|^2 d\mu(g(t)) + B \int v^2 d\mu(g(t)).
$$

By virtue of the work of Carron \cite{Ca} (see also Lemma 2.2 in \cite{H2} and its proof), we know that Sobolev imbedding in the above form implies the following long term non-collapsing result for geodesic balls on manifolds.

**Corollary 2.** In either case (a) or (b) in Theorem 1.1, suppose also that the scalar curvature is uniformly bounded. Then there exist two positive constants $v_1$ and $v_2$ depending only on $g(0)$, the bound on the scalar curvature and on $t$ in case (a) such that

$$
|B(x, r)|_{g(t)} \geq \min[v_1 r^n, v_2]; \quad |B(x, r)|_{\tilde{g}(\tilde{t})} \geq \min[v_1 r^n, v_2]
$$

for all $t \in [0, \infty)$ and $r > 0$; and respectively for all $\tilde{t} \in [0, \infty)$ and $r > 0$.

Using the property that the distances in a parabolic cube is comparable if the curvature is bounded by the inverse radius square, this result generalizes Perelman’s non-collapsing result from short time to long time case. Applications on local non-collapsing with surgery are also possible. We will present it in the addendum (last section) of the paper.

2. **Proof of Theorem 1.1**

**Step 1.** We show that the monotonous property of $W$, the Perelman $W$ entropy, implies restricted Log Sobolev inequalities (2.5) and (2.6) below.

**Case (a).** We assume the Ricci flow exists for all time.

Clearly, by scaling in space time, we only need to prove the result when $t = 1$. Here is why. For a fixed $t_0 \geq 1$. One can do the scaling $t' = t/t_0$ and $g' = g/t_0$.

Let $t = 1$. For any $\epsilon \in (0, 1)$, we take $\tau(\cdot) = \epsilon^2 + 1 - \epsilon$ so that $\tau_1 = 1 + \epsilon^2$ and $\tau_2 = \epsilon^2$ (by taking $t_1 = 0$ and $t_2 = 1$).

Recall that Perelman’s $W$ entropy is

$$
W(g, f, \tau) = \int_M (\tau (R + |\nabla f|^2) + f - n) u d\mu(g(t))
$$

where $u = e^{-f/(4\pi \tau)^{n/2}}$. Let $u_2$ be a minimizer of the entropy $W(g, f, \tau_2)$ for all $u$ such that $\int u d\mu(g(t_2)) = 1$. We solve the backward heat equation with the final value chosen as $u_2$ at $t = t_2$. Let $u_1$ be the value of the solution of the backward heat equation at $t = t_1$. As usual, we define functions $f_i$ with $i = 1, 2$ by the relation $u_i = e^{-f_i/(4\pi \tau_i)^{n/2}}$, $i = 1, 2$.

Then, by the monotonicity of the $W$ entropy

$$
in \int \int_{u d\mu(g(t_1)) = 1} W(g(t_1), f_0, \tau_1) \leq W(g(t_1), f_1, \tau_1) \leq W(g(t_2), f_2, \tau_2)
$$

(2.1)

$$
= \int \int_{u d\mu(g(t_2)) = 1} W(g(t_2), f, \tau_2).
$$

Here $f_0$ and $f$ are given by the formulas

$$
u_0 = e^{-f_0/(4\pi \tau_1)^{n/2}}, \quad u = e^{-f/(4\pi \tau_2)^{n/2}}.$$
Using these notations we can rewrite (2.1) as
\[
\inf_{\|u\|_1=1} \int_M \left( e^2(R + |\nabla \ln u|^2) - \ln u - \ln(4\pi e^{n/2}) \right) u \, d\mu(g(t)) \\
\geq \inf_{\|u_0\|_1=1} \int_M \left( (1 + e^2(R + |\nabla \ln u_0|^2) - \ln u_0 - \ln(4(1 + e^2))^{n/2} \right) u_0 \, d\mu(g(0)).
\]

Observe that the \(\ln(4\pi)^{n/2}\) terms on both sides of the above inequality can be canceled. Denote \(v = \sqrt{u}\) and \(v_0 = \sqrt{u_0}.\) We obtain, since \(\epsilon \leq 1,\)
\[
\inf_{\|v\|_2=1} \int_M \left( e^2(Rv^2 + 4|\nabla v|^2) - v^2 \ln v^2 \right) \, d\mu(g(t)) - n \ln \epsilon \\
\geq \inf_{\|v_0\|_2=1} \int_M \left( (1 + e^2(Rv_0^2 + 4|\nabla v_0|^2) - v_0^2 \ln v_0^2 \right) \, d\mu(g(0)) - \ln 2^{n/2}.
\]

Since \((M, g(0))\) is a compact Riemannian manifold, it is known that the following log Sobolev inequality holds:

*Given \(\epsilon \in (0, \sqrt{2\pi}]\), for any \(v_0 \in W^{1,2}(M)\) with \(\|v_0\|_2 = 1\), there exists a positive constant \(L_0\) depending only on \(g(0)\) such that\)*
\[
\int_M v_0^2 \ln v_0^2 \, d\mu(g(0)) \leq \frac{\epsilon^2}{\pi} \int_M |\nabla v_0|^2 \, d\mu(g(0)) - n \ln \epsilon + L_0.
\]

Since the parameter \(\epsilon\) is bounded from above by \(\sqrt{2\pi}\) while in the standard log Sobolev inequality \(\epsilon\) is unrestricted, we refer the above as restricted log Sobolev inequality.

Actually even a sharp version of the log Sobolev inequality is valid by [Br] Corollary 1.2. i.e.

*There exist positive constants \(A_0\) and \(B_0\) depending only on the constants of the Sobolev imbedding (1.1) such that for all \(v_0 \in C^\infty(M)\) verifying \(\|v_0\|_2 = 1,\)
\[
\int_M v_0^2 \ln v_0^2 \, d\mu(g(0)) \leq \frac{1}{2} n \ln \left( A_0 \int_M |\nabla v_0|^2 \, d\mu(g(0)) + B_0 \right).
\]

This inequality is obtained by combining the Sobolev imbedding (1.1) with Hölder’s inequality with suitable powers and taking the limit. If one does not care about the precise value of \(B_0,\) one can even take \(A_0 = \frac{2}{\pi \sqrt{2\pi}}\) (c.f. [Br]). For any positive number \(\xi_1\) and \(\xi_2\) and \(\delta \in (0, 1],\) it is easy to see, that there exists \(c > 0\) such that,
\[
\ln(1 + \xi_1 + \xi_2) \leq \delta(\xi_1 + \xi_2) + |\ln \delta| + c \leq \delta \xi_1 + |\ln \delta| + \xi_2 + c.
\]

By choosing \(\xi_1 = \int_M |\nabla v_0|^2 \, d\mu(g(0))\), \(\xi_2 = B_0/A_0\) and taking \(\delta\) appropriately, we know that (2.3) is a direct consequence of this sharp log Sobolev inequality.

As a consequence of this and the work [H1], we know that \(L_0\) can be chosen to depend on \(n,\) the injectivity radius and the lower bound of the Ricci curvature of \((M, g(0))\) only.

Substituting the log Sobolev inequality (2.3) to (2.2), we deduce
\[
\inf_{\|v\|_2=1} \int_M \left( e^2(Rv^2 + 4|\nabla v|^2) - v^2 \ln v^2 \right) \, d\mu(g(t)) - n \ln \epsilon \\
\geq - \max R^- (\cdot, 0) - L_0 - c.
\]
Therefore, by renaming $\epsilon$ and $c$, we reach the uniform restricted log Sobolev inequality:

\[
(2.5) \int_M v^2 \ln v^2 \, d\mu(g(t)) \leq \frac{\epsilon^2}{2\pi} \int_M (|\nabla v|^2 + \frac{1}{4} Rv^2) \, d\mu(g(t)) - n \ln \epsilon + L + \max R^-(\cdot, 0).
\]

Here $L$ depends only on $n$ and $g(0)$ through the lower bound of the Ricci curvature and injectivity radius.

Case (b). We assume the Ricci flow exists for $t \in (0, 1)$ and becomes singular at $t = 1$. We have

\[
(2.6) \int_M v^2 \ln v^2 \, d\mu(\tilde{g}(\tilde{t})) \leq \frac{\epsilon^2}{2\pi} \int_M (|\nabla v|^2 + \frac{1}{4} \tilde{R}v^2) \, d\mu(\tilde{g}(\tilde{t})) - n \ln \epsilon + L + \max R^-(\cdot, 0).
\]

Here $L$ depends only on $n$ and $g(0)$ through the lower bound of the Ricci curvature and injectivity radius.

This is an immediate consequence of Case (a) by scaling.

Step 2. Fix a time $t_0$ during the Ricci flow or $\tilde{t}_0$ during the normalized one. Suppose on $(M, g(t_0))$ or $(\tilde{M}, \tilde{g}(\tilde{t}_0))$, the restricted Log Sobolev inequalities (2.5) and (2.6) hold respectively. We show that they imply short time heat kernel upper bound for the fundamental solution of

\[
(2.7) \Delta u(x, t) - \frac{1}{4} R(x, t_0)u(x, t) - \partial_t u(x, t) = 0
\]

under the fixed metric $g(t_0)$ or $\tilde{g}(\tilde{t}_0)$. The proof, which does not distinguish between the Ricci flow or the normalized Ricci flow case, follows the original ideas of Davies [Da]. There are only two extra issues to deal with here. One is that the range of $\epsilon$ is restricted. The other is that the negative part of the scalar curvature may make the semigroup generated by $\Delta - \frac{1}{4} R$ not contractive. However the modification in the proof is moderate since we are dealing with short time upper bound only. We also benefit from the fact that the most negative value of the scalar curvature does not decrease under either the Ricci flow or the normalized one in the theorem. For this reason, we will be brief in the presentation.

Let $u$ be a positive solution to (2.7). Given $T \in (0, 1]$ and $t \in (0, T)$, we take $p(t) = T/(T - t)$ so that $p(0) = 1$ and $p(T) = \infty$. By direct computation

\[
\partial_t \|u\|_{p(t)} = \partial_t \left( \int_M u^{p(t)}(x, t) \, dx \right)^{1/p(t)} = -\frac{p'(t)}{p^2(t)} \|u\|_{p(t)} \ln \int_M u^{p(t)}(x, t) \, dx + \frac{1}{p(t)} \left( \int_M u^{p(t)}(x, t) \, dx \right)^{(1/p(t)) - 1} \times \left[ \int_M u^{p(t)}(\ln u)p'(t) \, dx + p(t) \int_M u^{p(t) - 1}(\Delta u - \frac{1}{4} Ru) \, dx \right].
\]

Here $dx$ means the integral element with respect to $g(t_0)$. We adopt this notation to symbolize the property that $g(t_0)$ is not evolving with respect to $t$. Using integration by
parts on the term containing $\Delta u$ and multiplying both sides by $p^2(t)\|u\|_{p(t)}^p$, we reach
\[
p^2(t)\|u\|_{p(t)}^p \partial_t \ln \|u\|_{p(t)} = -p'(t)\|u\|_{p(t)}^{p(t)+1} \ln \int_M u^{p(t)}(x,t)dx + p(t)p'(t)\int_M u^{p(t)} \ln u(x,t)dx \\
- p^2(t)(p(t)-1)\|u\|_{p(t)} \int_M u^{p(t)-2} |\nabla u|^2(x,t)dx \\
- p^2(t)\|u\|_{p(t)} \int_M \frac{1}{4} R(x,t_0)u^{p(t)}(x,t)dx.
\]
Dividing both sides by $\|u\|_{p(t)}$, we obtain
\[
p^2(t)\|u\|_{p(t)}^p \partial_t \ln \|u\|_{p(t)} = -p'(t)\|u\|_{p(t)}^{p(t)+1} \ln \int_M u^{p(t)}(x,t)dx + p(t)p'(t)\int_M u^{p(t)} \ln u(x,t)dx \\
- 4(p(t)-1)\int_M |\nabla (u^{p(t)/2})|^2(x,t)dx - p^2(t)\int_M \frac{1}{4} R(x,t_0)(u^{p(t)/2})^2(x,t)dx.
\]
Merging the first two terms on the righthand side of the above equality and making the substitution $v = u^{p(t)/2}/\|u^{p(t)/2}\|_2$, we arrive at, after dividing by $\|u\|_{p(t)}^p$,
\[
p^2(t)\partial_t \ln \|u\|_{p(t)} = p'(t)\int_M v^2 \ln v^2(x,t)dx - 4(p(t)-1)\int_M |\nabla v|^2(x,t)dx - p^2(t)\int_M \frac{1}{4} R(x,t_0)v^2(x,t)dx \\
= p'(t)\int_M v^2 \ln v^2(x,t)dx - 4(p(t)-1)\int_M (|\nabla v|^2(x,t) + \frac{1}{4} R(x,t_0)v^2)dx \\
+ (4(p(t)-1) - p^2(t))\int_M \frac{1}{4} R(x,t_0)v^2(x,t)dx.
\]
It is easy to check $\|v\|_2 = 1$ and also
\[
\frac{4(p(t)-1)}{p'(t)} = \frac{4t(T-t)}{T} \leq T \leq 1,
\]
\[
-T \leq \frac{4(p(t)-1) - p^2(t)}{p'(t)} = \frac{4t(T-t) - T^2}{T} \leq 0.
\]
Hence
\[
p^2(t)\partial_t \ln \|u\|_{p(t)} \leq p'(t)\left(\int_M v^2 \ln v^2(x,t)dx - \frac{4(p-1)}{p'(t)} \int_M (|\nabla v|^2(x,t) + \frac{1}{4} R(x,t_0)v^2)dx + T \sup R^-(x,t_0)\right).
\]
Take $\epsilon$ so that
\[
\frac{\epsilon^2}{\pi} = \frac{4(p(t)-1)}{p'(t)} \leq T \leq 1
\]
in the restricted log Sobolev inequality \[25\], we deduce
\[
p^2(t)\partial_t \ln \|u\|_{p(t)} \leq p'(t)\left(- n \ln \sqrt{\pi 4(p(t)-1)/p'(t)} + L + T \sup R^-(x,0)\right).
\]
Here we also used the fact that sup $R^{-}(x, t_0) \leq \sup \, R^{-}(x, 0)$ as remarked earlier. 

Note that $p'(t)/p^2(t) = 1/T$ and $4(p(t) - 1)/p'(t) = 4t(T - t)/T$. Hence 

$$\partial_t \ln \|u\|_{p(t)} \leq \frac{1}{T} \left(-\frac{n}{2}\ln \pi 4t(T - t) + L + T \sup \, R^{-}(x, 0)\right).$$

This yields, after integration from $t = 0$ to $t = T$, 

$$\ln \frac{\|u(\cdot, T)\|_{\infty}}{\|u(\cdot, 0)\|_1} \leq -\frac{n}{2} \ln (4\pi T) + L + T \sup \, R^{-}(x, 0).$$

Since 

$$u(x, T) = \int_M P(x, y, T)u(y, 0)dy,$$

this shows 

$$P(x, y, T) \leq \exp \left(L + T \sup \, R^{-}(x, 0)\right)/(4\pi T)^{n/2}.$$ 

Here $P$ is heat kernel of $[2.7]$. 

**Step 3.** We show that the short term heat kernel upper bound implies the Sobolev imbedding in Theorem 1.1. 

This is more or less standard. Let $t_0$ be a fixed time during Ricci flow. Let $F = \sup \, R^{-}(x, 0)$ and $P_F$ be the heat kernel of the operator $\Delta - \frac{1}{4}R(x, t_0) - F - 1$. Since $R^{-}(x, t_0) \leq F$, from the short time upper bound for $P$, we know that $P_F$ obeys the global upper bound 

$$P_F(x, t, y) \leq \frac{\Lambda}{t^{n/2}}, \quad t > 0.$$ 

Here $\Lambda$ depends only on $L$ and $F$. Moreover $P_F$ is a contraction. By Hölder inequality, for any $f \in L^2(M)$, we have 

$$|\int_M P_F(x, t, y)f(y)dy| \leq \left(\int_M P_F^2(x, t, y)dy\right)^{1/2} \|f\|_2 \leq \Lambda^{1/2}t^{-n/4}\|f\|_2.$$ 

The Sobolev inequality in Theorem 1.1 now follows from Theorem 2.4.2 in [Da]. i.e. there exist positive constants $A, B$ depending only on the initial metric through $\Lambda$ such that, for all $v \in W^{1,2}(M, g(t_0))$, it holds 

$$\left(\int v^{2n/(n-2)}d\mu(g(t_0))\right)^{(n-2)/n} \leq A \int (|\nabla v|^2 + \frac{1}{4}Rv^2)d\mu(g(t_0)) + B \int v^2d\mu(g(t_0)).$$ 

The same also holds for the normalized Ricci flow. Since $t_0$ is arbitrary, the proof is done. 

One can also prove it by establishing a Nash type inequality first and using an argument in [BCLS].

**Acknowledgement.** We would like to thank Professors T. Coulhon, L. F. Tam, G. Tian and X. P. Zhu for their helpful communications.
3. ADDENDUM

In this section, we generalize the previous result to Ricci flow with surgery. As an application we prove a noncollapsing result with surgery without using the concepts of reduced distance and reduced volume by Perelman.

Here is the main result. For detailed information and related terminology such as $(r, \delta)$ surgery, $\delta$ neck etc on Ricci flow with surgery we refer the reader to [CZ], [KL] and [MT].

**Theorem A.1.** Given real numbers $T_1 < T_2 < T_3$, let $(M, g(t))$ be a 3 dimensional Ricci flow with normalized initial condition defined on the time interval containing $[T_1, T_3]$. Suppose the following conditions are met.

(a). $g(t)$ is smooth except when $t = T_2$.
(b). A $(r, \delta)$ surgery occurs at $t = T_2$ with parameter $h$, i.e. the surgery occurs in a $\delta$ neck of radius $h$. Here $\delta$ is sufficiently small.
(c). For $t \in [T_1, T_2)$ and $A_1 > 0$, the Sobolev imbedding

$$\left( \int v^{2n/(n-2)} d\mu(g(t)) \right)^{(n-2)/n} \leq A_1 \int (|\nabla v|^2 + \frac{1}{4} R v^2) d\mu(g(t)) + A_1 \int v^2 d\mu(g(t)).$$

holds for all $v \in W^{1,2}(M, g(t))$. Here $R$ is the scalar curvature.

Then for all $t \in (T_2, T_3]$, the Sobolev imbedding below holds for all $v \in W^{1,2}(M, g(t))$.

$$\left( \int v^{2n/(n-2)} d\mu(g(t)) \right)^{(n-2)/n} \leq A_2 \int (|\nabla v|^2 + \frac{1}{4} R v^2) d\mu(g(t)) + A_2 \int v^2 d\mu(g(t))$$

Here

$$A_2 \leq a_1 e^{a_2(T_3-T_2+R_0)}(A_1 + R_0^- + a_3)$$

with $a_i, i = 1, ..., 3$ being positive numerical constants; and $R_0^- = \sup R^-(x, 0)$.

Moreover, the Ricci flow is $\kappa$ noncollapsing in the whole interval $[T_1, T_3]$ under a scale $\epsilon$ where $\kappa$ and $\epsilon$ are independent of the surgery radius $h$, or the canonical neighborhood scale $r(t)$ in $(T_2, T_3)$ or the smallness of $\delta$.

**Remark.** 1. It is well known that any compact Riemannian manifold with a given metric supports a Sobolev imbedding ([Au], [H1] e.g). Therefore the theorem shows that under a Ricci flow with finite number of surgeries in finite time, the Sobolev imbedding is preserved. By Lemma A.2 below, $\kappa$ noncollapsing property is also preserved in this case. The constant $\kappa$ will depend on time and number of surgeries, but it is independent of the surgery scale or the canonical neighborhood scale $r(t)$.

2. In this paper we use the following definition of $\kappa$ non-collapsing by Perelman [P2], as elucidated in Definition 77.9 of [KL].

Let $M$ be a Ricci flow with surgery defined on $[a, b]$. Suppose that $(x_0, t_0) \in M$ and $r > 0$ are such that $t_0 - r^2 \geq a$, $B(x_0, t_0, r) \subset M_{t_0}^-$ is a proper ball and the parabolic ball $P(x_0, t_0, r, -r^2)$ is unscathed. Then $M$ is $\kappa$-collapsed at $(x_0, t_0)$ at scale $r$ if $|Rm| \leq r^{-2}$ on $P(x_0, t_0, r, -r^2)$ and $vol(B(x_0, t_0, r)) < \kappa r^3$; otherwise it is $\kappa$-noncollapsed.

In order to prove the theorem, we need the following two lemmas. The first lemma extends Theorem 1.1 in that a more accurate upper bound for the evolving Sobolev constants are derived.

**Lemma A.1.** Let $(M, g(t))$ be a smooth Ricci flow for $t \in [0, T_0]$. Suppose for a constant $A(0) > 0$, the following Sobolev imbedding holds for the initial metric:
for all \( v \in W^{1,2}(M, g(0)) \),

\[
\left( \int v^{2n/(n-2)} d\mu(g(0)) \right)^{(n-2)/n} \leq A(0) \int (|\nabla v|^2 + \frac{1}{4} R v^2) d\mu(g(0)) + A(0) \int v^2 d\mu(g(0)).
\]

Then there exists a positive constant \( A(T_0) \) such that the following Sobolev imbedding holds for the metric \( g(T_0) \):

\[
\left( \int v^{2n/(n-2)} d\mu(g(T_0)) \right)^{(n-2)/n} \leq A(T_0) \int (|\nabla v|^2 + \frac{1}{4} R v^2) d\mu(g(T_0)) + A(T_0) \int v^2 d\mu(g(T_0)).
\]

Moreover, there exists numerical constants \( a_1, a_2 > 0 \) such that

\[
A(T_0) \leq a_1 \exp(a_2(T_0 + 1 + R_{\infty}^0)) A(0),
\]

where \( R_{\infty}^0 = \sup R^{-}(x, 0) \).

In the above and later, \( R \) stands for the scalar curvature under the given metrics.

**Proof.**

We begin by deriving a log Sobolev inequality from the Sobolev inequality (A.1) using the argument in [Br], p.119-120.

Using (A.1) and interpolation (Hölder) inequality, we have (see (1.3) in [Br])

\[
\left( \int v^r d\mu(g(0)) \right)^{2/(r\theta)} \leq \left( A(0) \int (|\nabla v|^2 + \frac{1}{4} R v^2) d\mu(g(0)) + A(0) \int v^2 d\mu(g(0)) \right)^{\frac{2}{n \theta}} \times \left( \int v^s d\mu(g(0)) \right)^{\frac{2(1-\theta)/(s\theta)}{}}
\]

where

\[
\frac{1}{r} = \frac{\theta(n - 2)}{2n} + \frac{1 - \theta}{s}.
\]

Next we take \( r = 2 \) and \( v \) such that \( \|v\|_2 = 1 \), divide the last integral term on the right hand side to the left, and take log on the resulting inequality. Then we take \( \theta \to 0 \), i.e. \( s \to 2 \). It is straightforward to check that we will arrive at the subsequent log Sobolev inequality:

for those \( v \in W^{1,2}(M, g(0)) \) such that \( \|v\|_2 = 1 \), it holds

\[
\int v^2 \ln v^2 d\mu(g(0)) \leq \frac{n}{2} \ln \left( A(0) \int (|\nabla v|^2 + \frac{1}{4} R v^2) d\mu(g(0)) + A(0) \right).
\]

Recall the elementary inequality: for all \( z, \sigma > 0 \),

\[
\ln z \leq \sigma z - \ln \sigma - 1.
\]

Its proof goes as follows. Write

\[ f(z) = \ln z - \sigma z + \ln \sigma + 1. \]

Then \( f'(z) = (1/z) - \sigma \) and \( f''(z) = -1/z^2 < 0 \). So \( f \) is concave down. It is absolute maximum is reached when \( f' = 0 \), i.e. when \( z = 1/\sigma \). But \( f(1/\sigma) = 0 \). Hence \( f(z) \leq 0 \) for all \( z > 0 \), yielding (A.4).
Using (A.4) on the right hand side of (A.3), we deduce

\[(A.5) \int v^2 \ln v^2 d\mu(g(0)) \leq \frac{n}{2} \sigma \left( A(0) \int (|\nabla v|^2 + \frac{1}{4} Rv^2) d\mu(g(0)) + A(0) \right) - \frac{n}{2} \ln \sigma - \frac{n}{2}. \]

We write

\[\frac{n}{2} \sigma A(0) \equiv \frac{\alpha^2}{2\pi}, \quad \text{i.e.} \quad \sigma = \frac{\alpha^2}{2\pi nA(0)}.\]

Then (A.5) becomes

\[(A.6) \int v^2 \ln v^2 d\mu(g(0)) \leq \frac{\alpha^2}{2\pi} \int (|\nabla v|^2 + \frac{1}{4} Rv^2) d\mu(g(0)) + \frac{\alpha^2}{2\pi} - \frac{n}{2} \ln \frac{\alpha^2}{2\pi} + \frac{n}{2} \ln \frac{nA(0)}{2} - \frac{n}{2}. \]

Our next task is to use the monotonicity of Perelman’s \(W\) entropy to extend (A.6) to a log Sobolev inequality for \(g(T_0)\).

Let \(t \in (0, T_0]\) and \(\epsilon \in (0, 1]\). We define

\[\tau = \tau(t) = \epsilon^2 + T_0 - t\]

so that \(\tau_1 = \epsilon^2 + T_0\) and \(\tau_2 = \epsilon^2\) (by taking \(t_1 = 0\) and \(t_2 = T_0\)).

Let \(u_2\) be a minimizer of the entropy \(W(g, f, \tau_2)\) for all \(u\) such that \(\int ud\mu(g(t_2)) = 1\). We solve the backward heat equation with the final value chosen as \(u_2\) at \(t = t_2\). Let \(u_1\) be the value of the solution of the backward heat equation at \(t = t_1\). As usual, we define functions \(f_i\) with \(i = 1, 2\) by the relation \(u_i = e^{-f_i/(4\pi\tau_i)^{n/2}}, \ i = 1, 2\). Then, by the monotonicity of the \(W\) entropy

\[\inf \int W(g(t_1), f_0, \tau_1) \leq W(g(t_1), f_1, \tau_1) \leq W(g(t_2), f_2, \tau_2) = \inf \int W(g(t_2), f, \tau_2).\]

Here \(f_0\) and \(f\) are given by the formulas

\[u_0 = e^{-f_0/(4\pi\tau_1)^{n/2}}, \quad u = e^{-f/(4\pi\tau_2)^{n/2}}.\]

Using these notations we can rewrite the above as

\[\inf \int \mathcal{M} \left( e^2(R + |\nabla \ln u|^2) - \ln u - \ln(4\pi e^2)^{n/2} \right) u d\mu(g(T_0))\]

\[\geq \inf \int \mathcal{M} \left( (\epsilon^2 + T_0)(R + |\nabla \ln u_0|^2) - \ln u_0 - \ln(4\pi(\epsilon^2 + T_0))^{n/2} \right) u_0 d\mu(g(0)).\]

Denote \(v = \sqrt{u}\) and \(v_0 = \sqrt{u_0}\). This inequality is converted to

\[\inf \int \mathcal{M} \left( e^2(Rv^2 + 4|\nabla v|^2) - v^2 \ln v^2 \right) d\mu(g(T_0)) - \ln(4\pi e^2)^{n/2}\]

\[\geq \inf \int \mathcal{M} \left( 4(\epsilon^2 + T_0)\left(\frac{1}{4} Rv_0^2 + |\nabla v_0|^2\right) - v_0^2 \ln v_0^2 \right) d\mu(g(0)) - \ln(4\pi(\epsilon^2 + T_0))^{n/2}.\]

Apply the log Sobolev inequality (A.6) on the right hand side of the above inequality with the choice

\[\frac{\alpha^2}{2\pi} = 4(T_0 + \epsilon^2).\]
Proof. Then \((t \in \int \text{imbedding for section 2 of the paper. It is well known that (A.8) implies the desired Sobolev inequality for } g(T_0):\)

\[
\int_M v^2 \ln v^2 \, d\mu(g(T_0)) \leq \frac{\epsilon^2}{2\pi} \int_M (|\nabla v|^2 + \frac{1}{4} Rv^2) \, d\mu(g(T_0)) - n \ln \epsilon + 4(T_0 + 1) + \frac{n}{2} \ln A(0) + c.
\]

Let \(p(x, y, t)\) be the heat kernel of \(\Delta - \frac{1}{4} R\) in \((M, g(T_0))\). Then (A.7) implies, for \(t \in (0, 1],\)

\[
(A.8) \quad p(x, y, t) \leq \exp(4(T_0 + 1) + \frac{n}{2} \ln A(0) + c + R_0^{-}) \frac{1}{(4\pi t)^n/2} \equiv \frac{A}{t^n/2}.
\]

This can be proven by adapting the standard method in heat kernel estimate as demonstrated in section 2 of the paper. It is well known that (A.8) implies the desired Sobolev imbedding for \(g(T_0), \) i.e.

for all \(v \in W^{1,2}(M, g(T_0)),\)

\[
\left( \int v^{2n/(n-2)} \, d\mu(g(T_0)) \right)^{(n-2)/n} \leq A(T_0) \int (|\nabla v|^2 + \frac{1}{4} Rv^2) \, d\mu(g(T_0)) + A(T_0) \int v^2 \, d\mu(g(T_0)).
\]

By keeping track of the constants (see also Theorem 2.2 in [S-C]), we know that

\[A(T_0) \leq c_1 A^{2/n}\]

for some numerical constant \(c_1.\) Therefore, there exists numerical constants \(a_1, a_2 > 0\) such that

\[A(T_0) \leq a_1 \exp(a_2(T_0 + 1 + R_0^{-})) A(0).\]

\[\square\]

The second Lemma of the addendum relates the Sobolev imbedding in Lemma A.1 to local noncollapsing of volume of geodesic balls. We follow the idea in [Ca].

**Lemma A.2.** Let \((M, g)\) be a Riemannian manifold. Given \(x_0 \in M\) and \(r \in (0, 1].\) Let \(B(x_0, r)\) be a proper geodesic ball, i.e. \(M - B(x_0, r)\) is non empty. Suppose the scalar curvature \(R\) satisfies \(|R(x)| \leq 1/r^2\) in \(B(x_0, r)\) and the following Sobolev imbedding holds: for all \(v \in W^{1,2}_0(B(x_0, r)),\) and a constant \(A \geq 1,\)

\[
\left( \int v^{2n/(n-2)} \, d\mu(g) \right)^{(n-2)/n} \leq A \int (|\nabla v|^2 + \frac{1}{4} Rv^2) \, d\mu(g) + A \int v^2 \, d\mu(g).
\]

Then \(|B(x_0, r)| \geq 2^{-(n+5)n/2} A^{-n/2} r^n.\)

**Proof.** Since \(R \leq 1/r^2,\) \(r \leq 1\) and \(A \geq 1\) by assumption, the Sobolev imbedding can be simplified to

\[
\left( \int v^{2n/(n-2)} \, d\mu(g) \right)^{(n-2)/n} \leq A \int |\nabla v|^2 \, d\mu(g) + \frac{2A}{r^2} \int v^2 \, d\mu(g).
\]
Under the scaled metric $g_1 = g/r^2$, we have, for all $v \in W^{1,2}_0(B(x_0, 1, g_1))$, 
\[
\left( \int v^{2n/(n-2)} d\mu(g_1) \right)^{(n-2)/n} \leq A \int |\nabla v|^2 d\mu(g_1) + 2A \int v^2 d\mu(g_1).
\]
Now, by [C] (see p33, line 4 of [H2]), it holds 
\[
|B(x_0, 1, g_1)|_{g_1} \geq \min\left\{ \frac{1}{2\sqrt{2A}}, \frac{1}{2(\pi/2)^{n/2}} \right\}^n.
\]
Therefore 
\[
|B(x_0, r, g)|_g \geq 2^{-(n+5)n/2} A^{-n/2} r^n.
\]

Now we are in a position to give a

**Proof of Theorem A.1.**

We begin by finding an upper bound for the Sobolev constant of for the metric right after the surgery. Recall that surgery occurs at a $\delta$ neck, called $N$, of radius $h$ such that $(N, h^{-2}g)$ is $\delta$ close in the $C^{0,\frac{\alpha}{2}}$ topology to the standard round neck $S^2 \times (-\delta^{-1}, \delta^{-1})$ of scalar curvature 1. Let $\Pi$ be the diffeomorphism from the standard round neck to $N$ in the definition of $\delta$ closeness. Denote by $z$ for a number in $(-\delta^{-1}, \delta^{-1})$. For $\theta \in S^2$, $(\theta, z)$ is a parametrization of $N$ via the diffeomorphism $\Pi$. We can identify the metric on $N$ with its pull back on the round neck by $\Pi$ in this manner.

Following the notations on p424 of [CZ], the metric $\tilde{g} = \tilde{g}(T_2)$ right after the surgery is given by 
\[
\tilde{g} = \begin{cases} 
\tilde{g}, & z \leq 0, \\
e^{-2f} \tilde{g}, & z \in [0, 2], \\
\phi e^{-2f} \tilde{g} + (1 - \phi)e^{-2f} h^2 g_0, & z \in [2, 3], \\
e^{-2f} h^2 g_0, & z \in [3, 4].
\end{cases}
\]
Here $\tilde{g}$ is the nonsingular part of the limit $\lim_{t \to T_2^-} g(t)$; $g_0$ is the standard metric on the round neck; and $f = f(z)$ is a smooth function given by (c.f. p424 [CZ])
\[
f(z) = 0, & z \leq 0; \\
f(z) = ce^{-P/z}, & z \in (0, 3]; f''(z) > 0, & z \in [3, 3.9]; \\
f(z) = -\frac{1}{2} \ln(16 - z^2), & z \in [3.9, 4].
\]
Here a small $c > 0$ and a large $P > 0$ are suitably chosen to ensure that the Hamilton-Ivey pinching condition remains valid. $\phi$ is a smooth bump function with $\phi = 1$ for $z \leq 2$ and $\phi = 0$ for $z \geq 3$.

Now we define another cut-off function $\eta$ on $N$ such that, $0 \leq \eta \leq 1$ and 
\[
\eta(x) = \begin{cases} 
1, & -100 \leq z \leq -10, \\
0, & z \geq 0.
\end{cases}
\]
Here $x \in \Pi(S^2 \times \{z\})$. i.e. $\eta(x) \equiv \xi(z(x))$ where $\xi$ is a suitable cut-off function on the real line. We then extend $\eta$ to be a Lipschitz cut-off function on the whole manifold $(M, \tilde{g}(T_2))$ by taking $\eta(x) = 1$ when $x \in M - \Pi(S^2 \times (-100, 4))$. 

\[\]
By \( \delta \) closeness, there exists a constant \( c \), independent of the smallness of \( \delta \), such that 
\[
|\nabla \eta(x)| \leq c h^{-1}.
\]
Here \( \nabla \) is with respect to the metric \( \tilde{g}(T_2) \) which is identical to the unscathed metric \( \bar{g}(T_2) \) in the support of \( \nabla \eta \).

Notice that \( \nabla \eta(x) = 0 \) when \( \eta(x) = 1 \); and when \( \eta(x) < 1 \), then \( x \) is in the \( \delta \) neck where the scalar curvature is close to \( h^{-2} \) when \( \delta \) is small. Therefore there exists a constant \( c \), independent of the smallness of \( \delta \), such that

\[
(A.9) \quad |\nabla \eta(x)|^2 \leq c(R(x, T_2) + R_0), \quad x \in M, \text{ with metric } \tilde{g}(T_2).
\]

Here, as before, \( R_0^\sim = \sup R^-(x, 0) \geq \sup R^-(x, t), \ t \geq 0 \).

Now we pick \( v \in C^\infty(M, \tilde{g}(T_2)) \). We compute

\[
(A.10) \quad \left( \int v^{2n/(n-2)} d\mu(\tilde{g}(T_2)) \right)^{(n-2)/n} = \left( \int [(\eta + 1 - \eta)v]^{2n/(n-2)} d\mu(\tilde{g}(T_2)) \right)^{(n-2)/n}
\]

\[
\leq 2 \left( \int (\eta v)^{2n/(n-2)} d\mu(\tilde{g}(T_2)) \right)^{(n-2)/n} + 2 \left( \int ((1 - \eta)v)^{2n/(n-2)} d\mu(\tilde{g}(T_2)) \right)^{(n-2)/n}
\]

\[
= 2I_1 + 2I_2.
\]

Observe that \( \text{supp}(\eta v) \) is in the unscathed portion of \( M \) so that \( \tilde{g}(T_2) = \bar{g}(T_2) \) here. Hence condition (c) in the statement of the theorem is applicable to \( \eta v \). Therefore

\[
(A.11) \quad I_1 \leq A_1 \int (|\nabla (\eta v)|^2 + \frac{1}{4} R(\eta v)^2) d\mu(\tilde{g}(T_2)) + A_1 \int (\eta v)^2 d\mu(\tilde{g}(T_2))
\]

\[
= A_1 \int (|\nabla (\eta v)|^2 + \frac{1}{4} R(\eta v)^2) d\mu(\tilde{g}(T_2)) + A_1 \int (\eta v)^2 d\mu(\tilde{g}(T_2)).
\]

Next we estimate \( I_2 \). From the construction, it is easy to see that the region \( \{x \in M, \tilde{g}(T_2) \mid z \geq -10\} \) is, after scaling with factor \( h^{-2}, \delta^{1/2} \) close (in \( C^{(n-1/2)} \) topology) to a region in the standard capped infinite cylinder, called \( C \), whose scalar curvature is comparable to 1 everywhere (see Lemma 7.3.4 [CZ] e.g.). It is well known that the Yamabe constant of \( C \) is strictly positive, i.e.

\[
0 < Y_0 \equiv \inf_{0 < v \in C^\infty(C)} \frac{4n-1}{2n} \int_C |\nabla v|^2 d\mu(g) + \int_C R v^2 d\mu(g)}{(\int_C v^{2n/(n-2)} d\mu(g))^{(n-2)/n}}.
\]

Here all geometric quantities are with respect to the standard metric of \( C \).

It is clear that the Yamabe constant is invariant under the scaling of the metric. It is also known that the Yamabe constant is stable under perturbation of metric (c.f. [BB]). Hence, when \( \delta \) is sufficiently small, we have, for a constant \( c > 0 \),

\[
(A.12) \quad I_2 = \left( \int_M ((1 - \eta)v)^{2n/(n-2)} d\mu(\tilde{g}(T_2)) \right)^{(n-2)/n}
\]

\[
\leq c Y_0^{-1} \int_M (|\nabla((1 - \eta)v)|^2 + \frac{1}{4} R((1 - \eta)v)^2) d\mu(\tilde{g}(T_2)).
\]
Taking these inequalities to the right hand side of (A.14), we get
\[
\left( \int_{M} v^{2n/(n-2)} d\mu(\tilde{g}(T_2)) \right)^{(n-2)/n} \leq 2A_1 \int (|\nabla (\eta v)|^2 + \frac{1}{4}R(\eta v)^2) d\mu(\tilde{g}(T_2)) + 2A_1 \int (\eta v)^2 d\mu(\tilde{g}(T_2)) \\
+ 2cY_0^{-1} \int_{M} (|\nabla ((1-\eta)v)|^2 + \frac{1}{4}R((1-\eta)v)^2) d\mu(\tilde{g}(T_2)).
\]
Combining the like terms on the right hand side of the above inequality, we deduce
\[
\left( \int_{M} v^{2n/(n-2)} d\mu(\tilde{g}(T_2)) \right)^{(n-2)/n} \leq (4A_1 + 4cY_0^{-1}) \int (\eta^2 + (1-\eta)^2)|\nabla v|^2 d\mu(\tilde{g}(T_2)) \\
+ (4A_1 + 4cY_0^{-1}) \int (|\nabla \eta|^2 + |\nabla (1-\eta)|^2)v^2 d\mu(\tilde{g}(T_2)) \\
+ 2A_1 \int \frac{1}{4}\eta^2 Rv^2 d\mu(\tilde{g}(T_2)) + 2cY_0^{-1} \int \frac{1}{4}(1-\eta)^2 Rv^2 d\mu(\tilde{g}(T_2)) + 2A_1 \int v^2 d\mu(\tilde{g}(T_2)).
\]
Note that \(\eta^2 + (1-\eta)^2 \leq 1\). More importantly, by (A.9),
\[|\nabla \eta|^2 + |\nabla (1-\eta)|^2 \leq 2c(R + R_0^-).\]
Also
\[(\eta^2 + (1-\eta)^2)R \leq R + R_0^-.
\]
Taking these inequalities to the right hand side of (A.14), we get
\[
\left( \int_{M} v^{2n/(n-2)} d\mu(\tilde{g}(T_2)) \right)^{(n-2)/n} \leq (4A_1 + 4cY_0^{-1}) \int |\nabla v|^2 d\mu(\tilde{g}(T_2)) \\
+ (4A_1 + 4cY_0^{-1})8c \int \frac{1}{4}(R + R_0^-)v^2 d\mu(\tilde{g}(T_2)) \\
+ (2A_1 + 2cY_0^{-1}) \int \frac{1}{4}(R + R_0^-)v^2 d\mu(\tilde{g}(T_2)) + 2A_1 \int v^2 d\mu(\tilde{g}(T_2)).
\]
\[ \left( \int_M v^{2n/(n-2)} d\mu(\tilde{g}(T_2)) \right)^{(n-2)/n} \leq c(A_1 + R_0^- + Y_0^-) \int (|\nabla v|^2 + \frac{1}{4} R v^2 + v^2) d\mu(\tilde{g}(T_2)). \]

Theorem A.1 is now proven by applying Lemma A.1 and A.2. \[ \square \]

References

[Au] Aubin, Thierry, *Problèmes isopérimétriques et espaces de Sobolev.* (French) J. Differential Geometry 11 (1976), no. 4, 573–598.

[BB] Bérard-Bergery, Lionel, *La courbure scalaire des variétés riemanniennes.* (French) [The scalar curvature of Riemannian manifolds] Bourbaki Seminar, Vol. 1979/80, pp. 225–245, Lecture Notes in Math., 842

[Br] BrouutELANDE, Christophe, *The best-constant problem for a family of Gagliardo-Nirenberg inequalities on a compact Riemannian manifold.* Proc. Edinb. Math. Soc. (2) 46 (2003), no. 1, 117–146.

[BCLS] Bakry, D.; Coulhon, T.; Ledoux, M.; Saloff-Coste, L, *Sobolev inequalities in disguise.* Indiana Univ. Math. J. 44 (1995), no. 4, 1033–1074.

[Ca] Carron, Gilles, *Inégalités isopérimétriques de Faber-Krahn et conséquences.* (French) Actes de la Table Ronde de Géométrie Différentielle (Luminy, 1992), 205–232, Sémin. Congr., 1, Soc. Math. France, Paris, 1996

[CCGGIIKLLN] Bennett Chow, Sun-Chin Chu, David Glickenste in, Christine Guenther, Jim Isenberg, Tom Ivey, Dan Knopf, Peng Lu, Feng Luo, Lei Ni. *The Ricci flow: Techniques and Applications.* AMS 2007

[CL] Shu-Cheng Chang and Peng Lu, *Evolution of Yamabe constants under Ricci flow,* Ann. of Global Analysis and Geometry, to appear.

[CZ] Huai-Dong Cao and Xi-Ping Zhu, *A Complete Proof of Poincare and Geometrization Conjectures-Application of the Hamilton-Perelman Theory of the Ricci Flow,* Asian J. Math. International Press Vol. 10, No. 2, pp. 165-492, June 2006

[Da] Davies, E. B. *Heat Kernel and Spectral Theory,* Cambridge University Press, 1989.

[H1] Hebey, Emmanuel, *Optimal Sobolev inequalities on complete Riemannian manifolds with Ricci curvature bounded below and positive injectivity radius.* Amer. J. Math. 118 (1996), no. 2, 291–300.

[H2] Hebey, E., *Nonlinear Analysis on Manifolds: Sobolev Spaces and Inequalities,* Courant Lecture Notes 1999.

[Hs] Hsu, Shu-yu., *Uniform Sobolev inequalities for manifolds evolving by Ricci flow,* arXiv: 0708.0803v1, August 2007

[HV] Hebey, Emmanuel and Vaugon, Michel, *Meilleures constantes dans le theoreme d’inclusion de Sobolev.* (French) Ann. Inst. H. Poincar Anal. Non Lineaire 13 (1996), no. 1, 57–93.

[HV2] Hebey, E., and Vaugon, M., *Effective Lp pinching for the concircular curvature* Journal of Geometric Analysis Vol. 6, Number 4, 1996, 531-553

[KL] Bruce Kleiner and John Lott, *Notes on Perelman’s papers,* http://arXiv.org/math.DG/0605067 v1(May 25, 2006)

[KZ] Shilong Kuang and Qi S. Zhang, *A gradient estimate for all positive solutions of the conjugate heat equation under Ricci flow,* preprint 2006, available in arxiv.

[Li] Jun-Fang Li, *Eigenvalues and entropy functionals with monotonicity formula,* Math. Annalen, to appear.

[MT] John W. Morgan and Gang Tian, *Ricci Flow and the Poincare Conjecture,* 25 July, 2006, http://arXiv.org/math.DG/0607007 v1

[P] Grisha Perelman, *The Entropy formula for the Ricci flow and its geometric applications,* 11 Nov. 2002, http://arXiv.org/math.DG/0211159

[P2] Grisha Perelman, *Ricci flow with surgery on three manifolds,* http://arXiv.org/math.DG/0303109.
[S-C] Saloff-Coste, L. *A note on Poincaré, Sobolev, and Harnack inequalities*. Internat. Math. Res. Notices 1992, no. 2, 27–38.

[ST] Natasa Sesum and Gang Tian, *Bounding scalar curvature and diameter along the Kähler Ricci flow (after Perelman) and some applications*, preprint

[T] Peter Topping, *Lectures on the Ricci Flow*, L.M.S. Cambridge University Press (12 Oct 2006), ISBN: 0521689473.

[Y] Ye, Rugang, *The Logarithmic Sobolev inequality along the Ricci flow*, arXiv: 0707.2424v2. July 2007

e-mail: qizhang@math.ucr.edu

Department of Mathematics, University of California, Riverside, CA 92521, USA