The shadow reckoning problem from ancient society as context for learning Trigonometry

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Abstract. In this article, we explore the shadow observations and reckoning from various ancient societies, especially form china and medieval Islam, as inspiration in developing a learning design. We used the evolution of trigonometry from old shadow reckoning techniques to arrange instructional tasks for tenth-grade secondary education students and analyse how its role in supporting students' understanding of trigonometry. We choose Design Research as a method to reach the purpose. However, we restrict our discussion to a part of one phase, namely pilot experiment, from three main phases of design research. The idea of the learning design is involving students in old mathematical activities that concern with concept of shadow ratio as the beginning of basic trigonometry. This research was involving six Indonesian high school students. We developed an instructional design which consists of several activities that begin with solving history-based problem using Chong Cha procedure (double difference) from ancient china, where the calculation method is using shadow assistance from two poles as a gnomon to determine the height or depth and distance of an inaccessible object (e.g. mountains or valleys), and followed by solving a problem by using a tool and method from medieval Islam in the surveying process which is actually also developed from the concept of shadow calculations. The result indicated that the use of ancient problems and method can provide valuable insight for the students and give them an opportunity to learn the trigonometric ratio meaningfully. Furthermore, by using history as a context, students got more opportunities to improve their mathematics literacy performance through formulating various contextual mathematics problems into formal mathematics form.

1. Introduction

Bringing the History of Mathematics (HoM) in the classroom provides wealthy source material for students in gaining the understanding of mathematical concepts and how the origin of the concept evolved. Therefore, developing a learning design with history of mathematics approach becomes one of the important research topics for the development of innovation in teaching mathematics. As an example, Fachrudin, Putri, Kohar and Widadah [1] used the historical perspective of Ancient Babylonian approach developing a design of learning instruction of quadratic equation. Fachrudin et al [2] also used the Ancient China history-based task to support students' geometrical reasoning and mathematical literacy in learning Pythagoras. The result of the studies implied that History of Mathematics provides conditions for gaining the understanding
of concepts by giving students the opportunity to understand how a mathematical concept develops historically. Swetz [3] pointed that shadow reckoning as an important early mathematical activity was used by many ancient societies for a land surveying (determining the height or depth a land or building) then transform into shadow ratio that led to the eventual derivation of the basic trigonometric function we know today. One of the evidences revealed from Ancient China, in Jiuzhang Suanshu or known as “Nine Chapter of Mathematical Art”. From last chapter of Nine Chapter, Gou Gu chapter, there is a problem known as “Sea Island” problem. On this issue, Liu Hui (263 AD) gave a commentary in measuring the height of a mountain by using two gnomons and its shadow. The method used by Liu Hui is known as “Chong Cha” (shadow different method). The Chong Cha method can be expressed in modern interpretation as function of tangent and cotangent. 

In medieval Islam era, the theory of gnomon of the shadow was developed by modifying the concept of chords (Ptolemy’s table of chord), into half chords, following the Indian astronomer who replace the chord with half chord or sine and the sines of the complement as cosine. By defining the concept of turn shadow as tangent and its hypotenuse as secant and straight shadows as cotangent and its hypothenuse as cosecant, the concept of six trigonometry function was firstly introduced in Islamic era. This shows that historically, the development of trigonometry as one of the subjects in mathematics cannot be separated from “shadow reckoning” activity. Kamber & Takaci [4] stressed that in learning trigonometry students should get more opportunities to work on contextual problems. Furthermore, Fachrudin and Putri [5] implied that History of Mathematics plays important role in improving students’ understanding through contextual problems. Aligned with world challenges in Program International Student Assessment (PISA), problem solving from HoM also helps students to improve their mathematics literacy skills through process categories (i.e. formulate, employ, and interpret mathematics in a variety of contexts) of mathematics literacy.

By means of the shadow reckoning activity from ancient societies, we try to develop a set of instructional tasks for learning Trigonometry. The core of idea is by concerning the ancient China shadow reckoning and the development of shadow concept become six trigonometry function in medieval Islamic era, students can build their understanding related to the trigonometry function. In particular, the aim of this study was to describe our preliminary stage in constructing the Hypothetical Learning Trajectory in learning Trigonometry by using Shadow Reckoning Problem from Ancient Societies as a context.

2. Literature review

2.1. Shadow Reckoning from Ancient China

The Jiuzhang suanshu or known as “Nine Chapter on Mathematical Art” is the ancient Chinese work that consists of nine different parts. The last chapter, or also known as Gou Gu, discussed about twenty-four problem solving concerning about right triangle’s properties [4]. In Gou Gu, there are 24 problems regarding right triangles properties, specifically in Pythagoras, where the 24th problem involve surveying problems which the principle of “shadow reckoning” technique of gnomon is used to solve the problem. Liu Hui, specifically elaborating and adding nine more the surveying problems and explained the solution using shadow reckoning technique or known as “Double Differences Technique” or “Chong Cha” method. Further, in Tang Dynasty (618-906), this additional problem was discussed separately from Jiuzhang and was called Haidao Suanjing (Sea Island Mathematical Manual). The following is an example of one problem in Haidao Suanjing and the solution technique using Chong Cha method which in principle used the “Shadow Reckoning” concept.

There is a sea island that is to be measured. Two rods that are 30 chih high are erected at a distance of 1,000 pu from each other, so that the rear rod aligns with the first rod and the island. When a man walks 123 pu back from the nearer rod, the top of the island is just visible through the end of that rod, if he tries to see with his eye brought
on the ground. The summit of the island's peak is also seen to align with the end of the rear rod, when seen bringing the end in contact with the ground from a point 127 pu to the rear of that rod. It is required to know the height and distance of this island. (See Figure 1.) [6]

1 chi = 1 feet
1 pu= 6 chi

To solve the problem, Swetz [7] explained that Liu Hui illustrate the problem on a form of a triangular problem as follows (Figure 2). If we suppose the height of the gnomon or $RB = h$, while $GE = a_1$ is the length of the shadow of the first gnomon, while $CI = a_2$ is the length of the second gnomon’s shadow. The Sea Island height is $AB = y$, distance of first gnomon to the base of the peak of sea island is $EB = z$, while the distance between two sticks is $IE = x$.

\[KO \] is made such that parallel to $GM$, consequently $MN=OP$.
$PL=HN$ and $MN=OP$ consequently (area of) $POJL$= area of $NMFH$.
Since $MNHF = EBRM$ then $EBRM = POJL$ (shaded area) ...(1)
Since $IBRO = QOJD$, and \(1\), then $IEMO = QPLD$ ...(2)
$IE \cdot EM = QP \cdot PL$ then we got $AR = \frac{IE \cdot EM}{QP}$ or $AR = \frac{hx}{a_3-a_5}$
$EB \cdot EM = OP \cdot \frac{hx}{a_4-a_5}$ then $EB = \frac{a_4x}{a_4-a_5}$

The height and the distance of sea Island can be found:
and

$$y = \frac{h_x}{a_2-a_1} + h \quad \text{and} \quad z = \frac{a_1x}{a_2-a_1}$$

We called the quantity \((a_2 - a_1)\) as the “shadow difference”. By imposing the concept of angle on the condition, \(m(\angle AGB) = \alpha\) and \(m(\angle ACB) = \beta\), the formula can be expressed in terms of the trigonometric form,

$$y = \frac{\tan \beta}{\tan \beta - \tan \alpha} \cdot (a_2 - a_1 - x)$$

Under the modern interpretation, the Chong Cha procedure could refer to trigonometric (tangent) function. This is a proof that the primitive form of the trigonometric equation comes from the shadow reckoning activity.

2.2. Shadow Concept and The Beginning of Six Trigonometry Function from Medieval Islam

For various studies on astronomy and the determination of location coordinates during the Islamic Era, the shadow reckoning concept began to be developed. Mohammed ibn Jabir al-Battani al-Harrani (c. 858–929), was one of the Islamic astronomers who continued and developed the concept of “half chord” or Sine and the complement of Sine or known as Cosines developed by scientists from Hinduism [8].

The concept of the shadow of a gnomon (miqyas in Arabic) is Tangent and Cotangent (figure. 3). Mohammed abu'l-Wafa al-Buzjani (940–998) introduced the “gnomon circle” with the length of gnomon as its radius (figure. 2) and the length of turned shadow as its Tangent.

Abu'l-Wafa provides definitions of several terms by combining the concept of “Chord” (denoted by “crd”) in a circle (which was previously introduced by Hipparchus and Ptolemy) as shown in Fig. 4. But the difference is that if Ptolemy uses a standard size of 60 units of radius, during the Islamic era it was introduced that the standard radius of a circle is 1 unit. For the angle on the circle, it uses a concept introduced by Ptolemy which divides the perimeter of a circle into 360 parts. In Islamic times, trigonometry was originally a unit of "Length", not "ratio" as it is known today.
Figure 4. The relation between chord and sine, also definition six trigonometric “length”

Muhammad ibn Ahmad al-Biruni (973-1055) in his work “Exhaustive Treatise on Shadows” defines that when there is a gnomon parallel to the horizon (OB), the length of the shadow (reversed shadow) is tangent. Meanwhile, the hypotenuse length of the shadow (OD) is Secant. Whereas when a gnomon is perpendicular to the horizon (OE), the shadow length (direct shadow) of the gnomon is called the Cotangent (EK), and the hypotenuse length of the direct shadow (OK) is Cosecant (see Figure 4).

3. Method
Design research is the method used in this research. The aim of this type of research is developing a Local Instruction Theory (LIT) [9] by testing and comparing the Hypothetical Learning Trajectory (HLT) as initial design with the actual learning through pilot experiment and teaching experiment. However, we restricted our discussion on a pilot experiment, from three main phases of design research. This research was involving 6 students. The data collected were field notes and students’ work. After doing the HLT testing, we conducted retrospective analysis to compare the actual learning and the HLT. The results of the analysis are used to revise the HLT.

4. Result and Discussion
In this section we will explain briefly the learning activities and example of student answers based on the hypothetical learning trajectory we have developed. We developed a series of learning activities based on the shadow reckoning activity from ancient china, the used of Chong Cha method for surveying, and medieval Islam era, the shadow concept as the beginning of six trigonometry functions. Students are asked to construct the trigonometric table by connecting the Islamic shadow concept and the Ptolemy’s concept of Chord. Furthermore, we asked the students solving and completing the Al Biruni’s work in determining the earth’s radius using the concept of Trigonometry.

Based on the experiment, the following is the quantitative data analysis of comparison between HLT and Actual Learning.

Table 1. Actual Learning result compared with HLT conjectures for the tasks for each activity

| Conjecture | C.1 | C.2.1 | C.2.2 | C.3.1 | C.3.2 | C.4.1 | C.4.2 |
|------------|-----|-------|-------|-------|-------|-------|-------|
| +          | x   | x     |       |       | x     | x     |       |
| ±          |     | x     |       |       | x     |       | x     |

Note: an x means how well the conjecture matched to the observed learning (− refers to confirmation for up to 1/3 of the total students, and + to at least 2/3 of the total students)

Based on the table above, we concluded that in general the actual learning trajectory has been running in accordance with the conjectures in HLT (i.e. around 2/3 of the total students can complete the task in accordance with the conjecture made).

The following is a brief explanation and analysis of our set of instructional tasks developed. We also presented the example of student work to show how the actual learning trajectory and the student’s understanding in solving the tasks’ problem.

a. Activity I: “Shadow reckoning” from Ancient China for surveying
In the first activity, students are introduced to the sea island problems from Haidao Siunjang and the ancient method, Chong Cha, to solve the problem. The purpose of this activity is to introduce the students that “shadow reckoning” was used to solve surveying problems, in this case determining the height of an inaccessible mountain. The problem we have presented in second part of this paper, the “Shadow Reckoning from Ancient China” part. The following is the example of student work in solving first problem using Chong Cha method.
b. Activity II: “Shadow reckoning” from medieval Islam as the origin of trigonometry
In the second activity, students are presented the development of shadow concept as the elaboration of Ptolemy’s Chord concept. In this activity, the learning process are teacher-centered, because students need preliminary knowledge about the concept of the chord of certain angle on a circle and its relation to sine and cosine values. It is intended that in the third activity students can construct the table of sine by using chord approach. Moreover, this activity’s aim is presenting a historical approach such that students get a chance to learn trigonometry meaningfully. Concept and relation among Sine, Cosine, Tangent, Secant, Cosecant, and Cotangent of the “Gnomon Circle” are taught in this activity. On the other hand, another important concept which is trigonometry as a length also taught in this activity. Students also learn the relationship between gnomon and the shadow. If a gnomon parallel to the horizon, the length of the shadow (reversed shadow) is called tangent. Meanwhile, the hypotenuse length of the shadow is called Secant. Whereas when a gnomon is perpendicular to the horizon, the shadow length (direct shadow) of the gnomon is called Cotangent, and the length of the hypotenuse of the direct shadow is Cosecant.

Some ideas obtained by students in this activity are:

\[
\sin a = \frac{1}{2} \text{crd} 2a, \cos a = \sin(90 - a), \tan a = \frac{\sin a}{\cos a}, \sin^2 a + \cos^2 a = 1
\]

b. Activity III: Construct the table of Sin (Trigonometric table) using Ptolemy’s Approach
In this part, students are asked to construct the table of Sine using Chord approach and their knowledge gained previously. Some sine values that must be determined by students are for the angle of 30°, 45°, 60°, 90°, 36°, 72°, 12°, 6° and 3°. The following is the example of student answers.
Figure 6. student’s work in determining the Sine 30°, 45°, 60° and 90°

Figure 7. student’s work in determining the Sine 36° and 72°

Based on Figure 7, students can determine grades of sine 36° and 72° using the Chord approach, not rely on a calculator.

d. Activity IV: Use Trigonometric Function to solve problem

The main tasks of this activity are completing the work of al Biruni in measuring the earth’s radius and determining the height of the certain building by using trigonometry and digital clinometer. The following is first task of the fourth activity.

Al-Biruni devised a method for determining the radius r of the earth by sighting the horizon from the top of a mountain of known height h. That is, al Biruni assumed that one could measure α, the angle of depression from
the horizontal at which one sights the apparent horizon. Al Biruni performed this measurement was 0.5667° and the height of the mountain was 305.1 m. Calculate the radius of the earth and its circumference!

The following is an example of the student’s answer.

![Image of student's answer]

**Figure 8.** Student’s Answer in Completing Al Biruni Work

The main purpose of this activity is to give students the opportunity to be able to explore the students’ ability in solving problems with the concept of trigonometry. In addition, by using contextual problems, students will get the opportunity to improve their mathematics.

5. **Conclusion**

Based on the HLT developed, the general view of the design of the learning trigonometry using the shadow reckoning from ancient societies can be done through several activities including:

a. The Shadow Reckoning from ancient China; b. Combining Chord Concept of Ptolemy and the shadow concept in Medieval Islam to understand the origin of trigonometry; c. Constructing The trigonometry table (Sine) (inspire by Ptolemy approach in constructing the table of Chord) d. Use Trigonometric to solve problem (completing the al biruni work of earth measurement and measure certain building).

From the result regarding the implementation of pilot experiment, we conclude that shadow reckoning problem from the history of mathematics can support students’ understanding of the concept of trigonometry. Moreover, using shadow reckoning problem as context give students a chance to learn trigonometry meaningfully. This is aligning with Freudenthal [10] statements that learning will occur when it is meaningful for the students. Moreover, HoM plays an important role in improving students' mathematical literacy skills through contextual problems in HoM that are aligned with real-world challenges in PISA.

The shadow reckoning from the history from ancient society supplies an illumination link how the trigonometry evolved, and help student to improve their understanding about trigonometric function and how its relation. In general, this study revealed that HoM, in this term we use the shadow reckoning from china and medieval Islamic era, have a plentiful material and didactical source to improve the quality of mathematics pedagogy and design of mathematics instruction. On conclusion, we strongly recommend other researchers to develop history-based learning designs on other topics.

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