Poincaré’s theorem for the modular group of real Riemann surfaces

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Abstract. Let \( Mod_g \) denote the modular group of (closed and orientable) surfaces \( S \) of genus \( g \). Each element \([h] \in Mod_g\) induces a symplectic automorphism \( H([h]) = H_1(S, \mathbb{Z}) \). Poincaré showed that \( H : Mod_g \to Sp(2g, \mathbb{Z}) \) is an epimorphism. A real Riemann surface is a Riemann surface \( S \) together with an anticonformal involution \( \sigma \). Let \((S, \sigma)\) be a real Riemann surface, \( \text{Homeo}^\sigma_g \) be the group of orientation preserving homeomorphisms of \( S \) such that \( h \circ \sigma = \sigma \circ h \) and \( \text{Homeo}^{\sigma, 0}_g \) be the subgroup of \( \text{Homeo}^\sigma_g \) consisting of those isotopic to the identity. The group \( Mod^\sigma_g = \text{Homeo}^\sigma_g / \text{Homeo}^{\sigma, 0}_g \) plays the rôle of the modular group in the theory of real Riemann surfaces. In this work we describe the image by \( H \) of \( Mod^\sigma_g \). Such image depends on the topological type of the involution \( \sigma \).

1 Introduction

The modular group, \( Mod_g \), of genus \( g \) closed (compact and without boundary) orientable surfaces (or mapping class group) is the group of orientation preserving autohomeomorphisms of a surface \( S \) of genus \( g \) modulo the homeomorphisms isotopic to the identity. Each element \([h] \in Mod_g\) induces in \( H_1(S, \mathbb{Z}) \) a symplectic automorphism \( H([h]) \), in such a way that \( H : Mod_g \to Sp(2g, \mathbb{Z}) \) is a homomorphism. In the celebrated paper \([\text{P}]\) by Poincaré, one of the principal results is that \( H \) is an epimorphism. This important relation between the modular group of complex curves and the symplectic group has been used, for instance, in the classification of abelian actions on surfaces, see \([\text{CN}]\) and \([\text{E}]\).

The moduli space of complex algebraic curves is a central object in mathematics and recently in mathematical physics. The moduli space is the quotient of the Teichmüller space by the action of the modular group. By this fact, the modular group \( Mod_g \) plays an important rôle in the study of the
geometry and topology of the moduli space of Riemann surfaces of complex algebraic curves, see [N] and [MS].

A real Riemann surface \((S, \sigma)\) is a Riemann surface \(S\) with an anticonformal involution \(\sigma\). Real Riemann surfaces correspond to real algebraic curves and the moduli space of real Riemann surfaces is the moduli of real algebraic curves or “real moduli space”. The study of the moduli space of real algebraic curves is somehow similar to the study of the complex case but it presents some difficulties of its own. The “real moduli space” has importance in its own right and has applications in many areas (see [D]). The modular group of a real Riemann surface (real modular group) is the group defined as follows. Let \((S, \sigma)\) be a real Riemann surface, \(\text{Homeo}^\sigma_{g}\) be the group of orientation preserving homeomorphisms of \(S\) such that \(h \circ \sigma = \sigma \circ h\) and \(\text{Homeo}^\sigma_{g,0}\) be the subgroup of \(\text{Homeo}^\sigma_{g}\) consisting of homeomorphisms isotopic to the identity. The real modular group \(\text{Mod}^\sigma_{g}\) is the quotient \(\text{Homeo}^\sigma_{g}/\text{Homeo}^\sigma_{g,0}\).

The real moduli for real Riemann surfaces with the same topological type than \((S, \sigma)\) is the quotient of a contractible space by the action of \(\text{Mod}^\sigma_{g}\) (see [N]). Our aim is to describe the automorphisms of \(H_1(S, \mathbb{Z})\) defined by \(\text{Mod}^\sigma_{g}\), i.e. to establish the analog of Poincaré theorem for the real modular group.

Of course the involution \(\sigma\) induces a nontrivial involution \(\sigma_H\) on the homology group \(H_1(S, \mathbb{Z})\). The image of the real modular group by the homomorphism \(H\), defined by the action on homology, lies in \(\text{Sp}(2g, \mathbb{Z}) \cap \{h : h \circ \sigma_H = \sigma_H \circ h\} = \text{Aut}^\sigma_{\mu}(H_1(S, \mathbb{Z}))\). A naive conjecture would be that the image by \(H\) of \(\text{Mod}^\sigma_{g}\) is exactly \(\text{Aut}^\sigma_{\mu}(H_1(S, \mathbb{Z}))\). This turn out to be the case when the fixed point set of \(\sigma\) is empty (Theorem 7). When the fixed point set of \(\sigma\) is nonempty and consists of a collection of disjoint simple closed curves, however, the subspace of \(H_1(S, \mathbb{Z})\) spanned by the fixed curves must be preserved. This last condition is not sufficient and somewhat more technical conditions must be imposed. In general \(H(\text{Mod}^\sigma_{g})\) depends on the topological type of \(\sigma\). We now give precise statements.

Orientation reversing involutions \(\sigma\) of \(S\) up topological equivalence are classified by an integer \(\pm k\): the topological type. The set \(\text{Fix}(\sigma)\) is called the real part of \(\sigma\) and it consist of \(k\) disjoint simple closed curves, called ovals. The set \(\text{Fix}(\sigma)\) can be separating if \(S - \text{Fix}(\sigma)\) is disconnected and we say that \(\sigma\) has topological type \(+k\). In this case \(S/\langle \sigma \rangle\) is orientable, \(0 < k \leq g + 1\) and \(k \equiv g + 1 \mod 2\). If the set \(\text{Fix}(\sigma)\) is non-separating, that is to say \(S - \text{Fix}(\sigma)\) is connected, then \(\sigma\) has topological type \(-k\). Now \(S/\langle \sigma \rangle\) is nonorientable and \(0 \leq k \leq g\) (Harnack theorem, see [BEGG]).

As before, let \(H : \text{Mod}^\sigma_{g} \to \text{Sp}(2g, \mathbb{Z})\) denote the homomorphism given by the action on \(H_1(S, \mathbb{Z})\) of the elements of \(\text{Mod}^\sigma_{g}\). Let \((.,.)\) be the intersection form in \(H_1(S, \mathbb{Z})\). The principal results are as follows:

**Case 1** (Theorem 7). If the topological type of \((S, \sigma)\) is 0 then
\[ H(\text{Mod}_g^\sigma) = \{ h \in Sp(2g, \mathbb{Z}) : h \circ \sigma_H = \sigma_H \circ h \}. \]

Case 2 (Theorem 8). If the topological type of \((S, \sigma)\) is \(-k\) then there is a symplectic basis \(\{X_i, Y_i\}_{i=1,...,g}\) in \(H_1(S, \mathbb{Z})\), i.e. \((X_i, X_j) = (Y_i, Y_j) = 0, (X_i, Y_j) = \delta_{ij}\), such that the first \(k\) elements \(X_i\) are represented by the \(k\) ovals (of \(\text{Fix}(\sigma)\)) and \(h \in H(\text{Mod}_g^\sigma)\) iff

1. \(h \in Sp(2g, \mathbb{Z})\),
2. \(h \circ \sigma_H = \sigma_H \circ h\),
3. \(h\{X_1, ..., X_k\} = \{\varepsilon_1 X_1, ..., \varepsilon_k X_k\}\), where \(\varepsilon_i \in \{-1, 1\}, i = 1, ..., k\).
4. if \(i \in \{k+1, ..., g\}\), \((h(X_i), X_j) = 0\), for every \(0 \leq j \leq g\) and \((h(X_i), Y_j)\)
is even for every \(1 \leq j \leq k\).

Case 3 (Theorem 10). If the topological type of \((S, \sigma)\) is \(+k\), let the connected components of \(S - \text{Fix}(\sigma)\) be denoted by \(S_1\) and \(S_2\). Let \(H_i \leq H_1(S, \mathbb{Z})\) be spanned by cycles represented by closed curves in \(S_i\), \(i = 1, 2\). Let \(X_1, ..., X_k\) be the \(k\) elements of \(H_1(S, \mathbb{Z})\) represented by the ovals (of \(\text{Fix}(\sigma)\)).

Then \(h \in H(\text{Mod}_g^\sigma)\) iff

1. \(h \in Sp(2g, \mathbb{Z})\),
2. \(h \circ \sigma_H = \sigma_H \circ h\),
3. \(h\{X_1, ..., X_k\} = \{\varepsilon X_1, ..., \varepsilon X_k\}\) where \(\varepsilon = \pm 1\) and \(h(H_i) \subset H_i\) if \(\varepsilon = 1\) and \(h(H_i) \subset H_j, i \neq j\), if \(\varepsilon = -1\).

2 Preliminaries on crystallographic groups

We use 2-dimensional crystallographic groups as a tool for our arguments. The results are attainable using only topology arguments (generators of the mapping class group of surfaces) but the difficulties are comparable.

Let \(\mathbb{U}^2 = S^2, \mathbb{E}^2\) and \(\mathbb{H}^2\) be the sphere, euclidean plane and the hyperbolic plane respectively and let \(\mathcal{G}\) be the group of isometries of \(\mathbb{U}^2\). Let \(\mathcal{G}^+\) be the index two subgroup of \(\mathcal{G}\) consisting of the orientation preserving transformations of \(\mathbb{U}^2\).

A crystallographic group, is a discrete subgroup \(\Gamma\) of the group \(\mathcal{G}\) with compact quotient space. If the group \(\Gamma\) contains orientation reversing transformations then \(\Gamma^+ = \Gamma \cap \mathcal{G}^+\) is an index two subgroup of \(\Gamma\) and \(\Gamma^+\) is the canonical orientation preserving crystallographic subgroup of \(\Gamma\).

If \(\Gamma\) is such a group then its algebraic structure and the geometrical structure of the quotient orbifold \(\mathbb{U}^2/\Gamma\) is given by the signature (see [BEGG]):

\[ s(\Gamma) = (h; \pm; [m_1, ..., m_r]; \{ (n_{11}, ..., n_{1s_1}), ..., (n_{k1}, ..., n_{ks_k}) \}). \] (*)
The orbit space \( U^2/\Gamma \) is an orbifold with underlying surface of genus \( h \), having \( r \) cone points and \( k \) boundary components, each with \( s_j \geq 0 \), \( j = 1, \ldots, k \), corner points. The signs \("+\) and \("-\) correspond to orientable and non-orientable underlying surfaces respectively. The integers \( m_i \) are called the proper periods of \( \Gamma \) and they are the orders of the cone points of \( U^2/\Gamma \). The brackets \( \langle n_{i1}, \ldots, n_{is_i} \rangle \) are the period cycles of \( \Gamma \) and the integers \( n_{ij} \) are the link periods of \( \Gamma \); they are the orders of the corner points of \( U^2/\Gamma \). The group \( \Gamma \) is isomorphic to the orbifold fundamental group \( \pi_1 O(U^2/\Gamma) \).

A group \( \Gamma \) with signature \((*)\) has a canonical presentation with generators:

\[
x_1, \ldots, x_r, e_1, \ldots, e_k, c_{ij}, 1 \leq i \leq k, 0 \leq j \leq s_i \text{ and} \\
a_1, b_1, \ldots, a_h, b_h \text{ if } U^2/\Gamma \text{ is orientable or} \\
d_1, \ldots, d_h \text{ otherwise},
\]

and relators:

\[
x_i^{m_i}, i = 1, \ldots, r, \ c_i^2, (c_{i-1}c_{ij})^{n_{ij}}, c_i^{-1}c_{is_i}, e_i, i = 1, \ldots, k, j = 0, \ldots, s_i \text{ and} \\
x_1 \ldots x_r e_1 \ldots e_k [a_1, b_1] \ldots [a_h, b_h] \text{ or } x_1 \ldots x_r e_1 \ldots e_k d_1^2 \ldots d_h^2 \text{, according as the orbit space is orientable or not. Here } [a_i, b_i] = a_i b_i a_i^{-1} b_i^{-1}.
\]

3 Preliminary results

Let \( S \) be a (closed and orientable) surface of genus \( g \), let \( H_1(S, \mathbb{Z}) \) be the first homology group of \( S \) and \((.,.)\) be the bilinear intersection form in \( H_1(S, \mathbb{Z}) \). Let \( \sigma : S \to S \) be an orientation reversing involution and assume that \( \text{Fix}(\sigma) \) consists in \( k \) ovals, \( k \geq 0 \). We denote by \( \sigma_H : H_1(S, \mathbb{Z}) \to H_1(S, \mathbb{Z}) \) the homomorphism induced by \( \sigma \) in homology.

Lemma 1 If \( \text{Fix}(\sigma) \) is non-separating, i. e. \( S/\langle \sigma \rangle \) is nonorientable, then there is a base \( \{X_i, Y_i\}_{i=1, \ldots, g} \) in \( H_1(S, \mathbb{Z}) \) such that \( (X_i, X_j) = (Y_i, Y_j) = 0 \), \( (X_i, Y_j) = \delta_{ij} \), the cycles \( X_i, i = 1, \ldots, k \) are represented by the ovals of \( \sigma \), \( \sigma_H(X_i) = X_i \), \( i = 1, \ldots, g \), and

\[
\sigma_H(Y_i) = \begin{cases} 
-Y_i - \sum_{j=1}^{g} X_j & i = 1, \ldots, k \\
-Y_i - X_i - \sum_{j=1}^{g} X_j & i = k + 1, \ldots, g 
\end{cases}
\]

Proof. The orbifold \( S/\langle \sigma \rangle \) can be uniformized by a spherical, euclidean or hyperbolic crystallographic group \( \Delta \) of signature

\[
(g - k + 1, -,-, [-]; \{(-), _, _\}),
\]

i. e. \( S/\langle \sigma \rangle = U^2/\Delta \). Let

\[
\langle e_1, \ldots, e_k, c_1, \ldots, c_k, d_{k+1}, \ldots, d_{g+1} : e_1 \ldots e_k d_{k+1}^2 \ldots d_{g+1}^2 = 1, \\
c_i^2 = 1, e_i c_i e_i^{-1} = c_i, i = 1, \ldots, k \rangle
\]
be a canonical presentation of $\Delta$.

The index two subgroup of $\Delta^+$ containing the orientation transformations of $\Delta$, uniformizes $S$, i.e. $S = \mathbb{U}^2/\Delta^+$. A set of generators of $\Delta^+$ in terms of the above presentation of $\Delta$ is:

$$\{e_1, d_{g+1}c_1, ..., e_k, d_{g+1}c_k, d_{k+1}^2, d_{g+1}d_{k+1}, ..., d_{g}^2, d_{g+1}d_{g}\}$$

The involution $\sigma$ induces on $\Delta^+$ an automorphism given by conjugation by $d_{g+1}$. On the set of generators of $\Delta^+$ produces:

$$\begin{align*}
  e_i & \rightarrow d_{g+1}c_id_{g+1}^{-1} = (d_{g+1}c_i)e_i(d_{g+1}c_i)^{-1} \\
  d_i^2 & \rightarrow d_{g+1}d_i^2d_{g+1}^{-1} = (d_{g+1}d_i)d_i^2(d_{g+1}d_i)^{-1} \\
  d_{g+1}c_i & \rightarrow d_{g+1}d_{g+1}c_id_{g+1}^{-1} = d_{g+1}^2(d_{g+1}c_i)^{-1} \\
  d_{g+1}d_i & \rightarrow d_{g+1}d_{g+1}d_{g+1} = d_{g+1}^2d_{g+1}^2(d_{g+1}d_i)^{-1}
\end{align*}$$

Let $\theta : \Delta^+ \rightarrow H_1(S, \mathbb{Z})$ be the abelianization epimorphism and denote:

$$\begin{align*}
  X_i &= \theta(e_i), \quad i = 1, ..., k \\
  X_{i+k} &= \theta(d_{i+k}^2), \quad i = 1, ..., g - k \\
  Y_i &= \theta(d_{g+1}c_i), \quad i = 1, ..., k \\
  Y_{i+k} &= \theta(d_{g+1}d_{i+k}), \quad i = 1, ..., g - k.
\end{align*}$$

Considering the elements $e_i$ and $d_i$, $c_i$ in $\pi_1 O(\mathbb{U}^2/\Delta)$, the above formula tell us how to represent the homology cycles $X_i$ and $Y_i$ as lifting of paths in $S/\langle \sigma \rangle$ and then to compute the intersection form. We have:

$$(X_i, X_j) = (Y_i, Y_j) = 0, \quad \text{and} \quad (X_i, Y_j) = \delta_{ij}, \quad \text{for} \quad i, j \in \{1, ..., g\}.$$}

The effect of $\sigma_H$ on $X_i$ is:

$\sigma_H(X_i) = X_i$ and since $\theta(d_{g+1}^2) = -\frac{g}{\sum X_i}$ then:

$$\sigma_H(Y_i) = \begin{cases} 
  -Y_i - \sum_{j=1}^{g} X_j & \quad i = 1, ..., k \\
  -Y_i - X_i - \sum_{j=1}^{g} X_j & \quad i = k + 1, ..., g 
\end{cases} \quad \square$$

**Corollary 2** Let $Z \in H_1(S, \mathbb{Z})$ such that $\sigma_H(Z) = Z$, then $Z = \sum \alpha_i X_i$.

**Proof.** Let $Z = \sum \alpha_i X_i + \sum \beta_i Y_i$. If $\sigma_H(Z) = Z$ then $-\beta_i = (Z, X_i) = (\sigma_H(Z), X_i) = \beta_i$. Hence $\beta_i = 0$. $\square$

We denote $V_0 = \langle X_1, ..., X_g \rangle = Fix(\sigma_H)$ and $\text{Aut}^*(H_1(S, \mathbb{Z})) = \{ h \in \text{Aut}(H_1(S, \mathbb{Z})) : h \text{ preserves the symplectic form } (\langle, \rangle \text{ and } h \circ \sigma_H = \sigma_H \circ h\}$. 

5
Corollary 3 If \( g \in Aut^\sigma(H_1(S, \mathbb{Z})) \) then \( g(V_0) = V_0 \) and there is a homomorphism \( \varphi: Aut^\sigma(H_1(S, \mathbb{Z})) \rightarrow Aut(V_0) \) defined by \( \varphi(a) = a|_{V_0} \).

Proof. It is a consequence of the Corollary 2, since \( V_0 = Fix(\sigma_H) \). \( \square \)

Corollary 4 \( \varphi \) is a monomorphism.

Proof. Let \( a \in \ker \varphi \). Hence \( a \in Aut^\sigma(H_1(S, \mathbb{Z})) \) and \( a|_{V_0} = id \).

We know that \( a(X_i) = X_i, i = 1, \ldots, g \) and now we want to show that \( a(Y_i) = Y_i \).

Since \( a \) preserves the symplectic form and \( (X_i, Y_j) = \delta_{ij} \) then \( \delta_{ij} = (X_i, Y_j) = (a(X_i), a(Y_j)) = (X_i, a(Y_j)) \), thus \( a(Y_i) = Y_i + \sum \alpha_{ij}X_j \).

Now we use that \( a \circ \sigma_H = \sigma_H \circ a: \)

\[
\sigma_H \circ a(Y_i) = \sigma_H(Y_i + \sum \alpha_{ij}X_j) = -Y_i - \varepsilon_iX_i - \sum_{i=1}^{g} X_i + \sum \alpha_{ij}X_j,
\]

where \( \varepsilon_i \) is 1 or 0,

\[
a \circ \sigma_H(Y_i) = a \circ ( -Y_i - \varepsilon_iX_i - \sum_{i=1}^{g} X_i ) = -Y_i - \sum \alpha_{ij}X_j - \varepsilon_iX_i - \sum_{i=1}^{g} X_i.
\]

Thus \( \alpha_{ij} = 0 \) and \( a(Y_i) = Y_i \), so \( a = id_{H_1(S, \mathbb{Z})} \). \( \square \)

Lemma 5 If \( Fix(\sigma) \) is separating, i.e. \( S/\langle \sigma \rangle \) is orientable (must be \( k > 0 \)) then there is a base \( \{X_i, Y_i\}_{i=1, \ldots, g} \) in \( H_1(S, \mathbb{Z}) \) such that \( (X_i, X_j) = (Y_i, Y_j) = 0 \), \( (X_i, Y_j) = \delta_{ij} \), the cycles \( X_i, i = 1, \ldots, k - 1 \) are represented by \( k - 1 \) ovals of \( \sigma \) and:

\[
\sigma_H(X_i) = \begin{cases} 
X_i & i = 1, \ldots, k - 1 \\
X_{i+ \frac{g+k+1}{2}} & i = k, \ldots, \frac{g+k+1}{2} \\
X_{i- \frac{g+k+1}{2}} & i = \frac{g+k+1}{2}, \ldots, g 
\end{cases}
\]

\[
\sigma_H(Y_i) = \begin{cases} 
-Y_i & i = 1, \ldots, k - 1 \\
-Y_{i+ \frac{g+k+1}{2}} & i = k, \ldots, \frac{g+k+1}{2} \\
-Y_{i- \frac{g+k+1}{2}} & i = \frac{g+k+1}{2}, \ldots, g 
\end{cases}
\]

Proof. The proof is similar to the proof of Lemma 1 but now using the uniformization of \( S/\langle \sigma \rangle \) by a crystallographic group \( \Delta \) with signature \( ( \frac{g-k+1}{2}, +, [-], \{ (-), k, ( - ) \} ) \).

Let

\[
< e_1 \ldots e_k, c_1, \ldots, c_k, a_1, b_1, \ldots, a_{ \frac{g+k+1}{2}}, b_{ \frac{g-k+1}{2}} : e_1 \ldots e_k [a_1, b_1] \ldots [a_{ \frac{g+k+1}{2}}, b_{ \frac{g-k+1}{2}}] = 1, c_i^2 = 1, e_i c_i e_i^{-1} = c_i, i = 1, \ldots, k >
\]
Lemma 6

There is an isomorphism

\[
\text{formizing } S/\sigma \text{morphism that produces an automorphism } \alpha \text{ orientation preserving transformations, then } \sigma \text{U}
\]

and there is a 2-fold orbifold covering \( S \) ing transformations of \( \Delta \) uniformizes \( S \)ing transformations of \( \Delta \) is:

\[
\text{tallographic groups, for spherical and euclidean groups is a classical fact).}
\]

is as given in the statement of the Lemma.

\[
\text{Let } \theta : \Delta^+ \to H_1(S, \mathbb{Z}) \text{ be the abelianization epimorphism and denote:}
\]

\[
X_i = \theta(e_i), \quad i = 1, ..., k - 1, \\
Y_i = \theta(c_k c_i), \quad i = 1, ..., k - 1, \\
X_i = \theta(a_i), \quad i = k, ..., g - \frac{k - 1}{2}, \\
Y_i = \theta(b_i), \quad i = k, ..., g - \frac{k - 1}{2}, \\
X_i = \theta(c_i a_i c_k), \quad i = \frac{g + k + 1}{2}, ..., g, \\
Y_i = \theta(c_i b_i c_k^{-1}), \quad i = \frac{g + k + 1}{2}, ..., g.
\]

Hence \((X_i, X_j) = (Y_i, Y_j) = 0\) and \((X_i, Y_j) = \delta_{ij}\). The effect of \( \sigma_H \) on \( X_i \) is as given in the statement of the Lemma.

\( \square \)

Let \( \Delta \) be a crystallographic group with signature

\[
\left( \frac{g - k + 1}{\eta}, \pm, [-], \{-\}, (\cdot), (-) \right),
\]

where \( \eta \) is 2 or 1 depending if there is a + or a − in the signature, uniformizing \( S/\langle \sigma \rangle \). Let \( \Delta^+ \) be the index two subgroup of \( \Delta \) containing the orientation preserving transformations, then \( \sigma : S = \mathbb{U}^2/\Delta^+ \to \mathbb{U}^2/\Delta^+ = S \) and there is a 2-fold orbifold covering \( S = \mathbb{U}^2/\Delta^+ \to \mathbb{U}^2/\Delta = S/\langle \sigma \rangle \).

Lemma 6 There is an isomorphism \( \psi : \text{Aut}(\Delta) \to \text{Mod}_g^\prime \).

Proof. Let \( \alpha : \Delta \to \Delta \) be an automorphism, there is a homeomorphism \( h : \mathbb{U}^2/\Delta \to \mathbb{U}^2/\Delta \) inducing \( \alpha \) (see [MS] Corollary 7.4 for hyperbolic crystallographic groups, for spherical and euclidean groups is a classical fact).

Let \( h^+ : \mathbb{U}^2/\Delta^+ \to \mathbb{U}^2/\Delta^+ \) be the orientation preserving lifting of \( h \) to \( \mathbb{U}^2/\Delta^+ \). Thus \( h^+ \circ \alpha = \alpha \circ h^+ \) and we define \( \psi(\alpha) = h^+ \).

Conversely, if \( h^+ : S = \mathbb{U}^2/\Delta^+ \to \mathbb{U}^2/\Delta^+ \) and \( h^+ \circ \alpha = \alpha \circ h^+ \) then there is \( h : S/\langle \sigma \rangle = \mathbb{U}^2/\Delta \to \mathbb{U}^2/\Delta \). The homomorphism \( h \) is an orbifold automorphism that produces an automorphism \( \alpha \) of the orbifold fundamental group \( \pi_1 O(S/\langle \sigma \rangle) = \Delta \).

\( \square \)
4 Poincaré’s theorem for real algebraic curves with empty real part.

Let $S$ be a (closed and orientable) surface of genus $g$. Let $\sigma : S \to S$ be an orientation reversing involution and assume that $\text{Fix}(\sigma) = \emptyset$. We denote by $\sigma_H : H_1(S, \mathbb{Z}) \to H_1(S, \mathbb{Z})$ the homomorphism induced by $\sigma$ in homology.

By Lemma 1 there is a base $\{X_i, Y_i\}_{i=1,\ldots,g}$ in $H_1(S, \mathbb{Z})$ such that $(X_i, Y_j) = (Y_i, Y_j) = 0$, $(X_i, Y_j) = \delta_{ij}$ and

$$\sigma_H(X_i) = X_i, \sigma_H(Y_i) = -Y_i - X_i - \sum_{j=1}^{g} X_j, \quad i = 1, \ldots, g.$$ 

By Lemma 2 if $Z \in H_1(S, \mathbb{Z})$ such that $\sigma_H(Z) = Z$, then $Z = \sum \alpha_i X_i$.

Hence $V_0 = \langle X_1, \ldots, X_g \rangle = \text{Fix}(\sigma_H)$ and we denote $\text{Aut}^{\sigma_H}(H_1(S, \mathbb{Z})) = \{h \in \text{Aut}(H_1(S, \mathbb{Z})) : h \text{ preserves the symplectic form } (\ldots) \text{ and } h \circ \sigma_H = \sigma_H \circ h\}$.

By Corollaries 3 and 4 there is a monomorphism $\varphi : \text{Aut}^{\sigma_H}(H_1(S, \mathbb{Z})) \to \text{Aut}(V_0)$ defined by $\varphi(a) = a|_{V_0}$.

Let $H : \text{Mod}_g^\sigma \to \text{Aut}^{\sigma_H}(H_1(S, \mathbb{Z}))$ be the homomorphism given by $H(h) = h_H$, where $h_H$ is the homomorphism induced in the homology by $h$, then our main result in this Section is:

**Theorem 7** If $\sigma : S \to S$ is an orientation reversing involution such that $\text{Fix}(\sigma) = \emptyset$ then $H(\text{Mod}_g^\sigma) = \text{Aut}^{\sigma_H}(H_1(S, \mathbb{Z}))$ is an epimorphism.

**Proof.** Let $\Delta$ be a crystallographic group with signature $(g+1, -1, \{-\}, \{-\})$ uniformizing $S/\langle \sigma \rangle$. We have $\text{Aut}(\Delta) \xrightarrow{\psi} \text{Mod}_g^\sigma \xrightarrow{H} \text{Aut}^{\sigma_H}(H_1(S, \mathbb{Z})) \xrightarrow{\varphi} \text{Aut}(V_0)$. By Corollary 4, $\varphi$ is a monomorphism then we need only to prove that $\varphi \circ H \circ \psi$ is an epimorphism. If $g \leq 1$ the result is obvious since $V_0$ in such cases has dimension $\leq 1$. Let now assume that $g \geq 2$.

To prove the Theorem we need to show that $\varphi \circ H \circ \psi$ is an epimorphism. Let $\langle d_1, \ldots, d_{g+1} : d_1^2 \ldots d_{g+1}^2 = 1 \rangle$ be a canonical presentation of $\Delta$. We define the automorphisms $\alpha_i, i = 1, \ldots, g, \beta$ and $\gamma$ by:

$$\alpha_i : \begin{cases} 
  d_i &\to d_i^2 d_{i+1} - d_i^2 \\
  d_{i+1} &\to d_i \\
  d_j &\to d_j \quad j \neq i, i + 1
\end{cases}$$

$$\beta : \begin{cases} 
  d_1 &\to d_1^{-1} \\
  d_2 &\to d_2^{-1} \\
  d_j &\to d_j^{-1} \quad j \geq 3
\end{cases}$$

$$\gamma : \begin{cases} 
  d_1 &\to d_1 d_2 - d_1 \\
  d_2 &\to d_2 d_3 - d_2 \\
  d_j &\to d_j \quad j \geq 3
\end{cases}$$

Now we shall compute $\varphi \circ H \circ \psi$ of the above automorphisms. Let $\{X_i\}_{i=1,\ldots,g}$ be the basis of $V_0$ obtained from the above canonical presentation.
of $\Delta$ following the proof of Lemma 1. If $A_i = \varphi \circ H \circ \psi(\alpha_i)$ and $X_i = \theta(d_i^2)$ then $A_i(X_i) = X_{i+1}$, $A_i(X_{i+1}) = X_i$, $A_i(X_j) = X_j$, $j \neq i, i + 1$.

The matrix of $A_i$ in the basis $\{X_i\}_{i=1,\ldots,g}$ is:

$$
\begin{bmatrix}
1 & 0 & \ldots & 0 \\
0 & 1 \\
& \ddots & \ddots & \ddots \\
0 & \ldots & 0 & 1 \\
\end{bmatrix}
$$

If $B = \varphi \circ H \circ \psi(\beta)$, then $B(X_1) = \sum_{i=1}^{g} X_i$ and $B(X_j) = -X_{g-j+2}$. The matrix of $B$ in the basis $\{X_i\}_{i=1,\ldots,g}$ is:

$$
\begin{bmatrix}
1 & 0 & \ldots & 0 & 0 \\
1 & 0 & \ldots & 0 & -1 \\
1 & 0 & \ldots & -1 & 0 \\
1 & 0 & \ldots & 0 & 0 \\
1 & -1 & \ldots & 0 & 0 \\
\end{bmatrix}
$$

And $C = \varphi \circ H \circ \psi(\gamma)$, $C(X_1) = -X_2$, $C(X_2) = 2X_2 + X_1$, $C(X_j) = X_j$, $j \neq 1, 2$, with matrix in the basis $\{X_i\}_{i=1,\ldots,g}$:

$$
\begin{bmatrix}
0 & 1 & 0 & \ldots & 0 \\
-1 & 2 & 0 & \ldots & 0 \\
0 & 0 & 1 & 0 & \ldots \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & \ldots & 0 \\
\end{bmatrix}
$$

Note that multiplying by the left or the right by $A_i$ we can interchange the rows and columns of a given matrix. The following matrix obtained from $B$ permuting rows corresponds to an element in the image of $\varphi \circ H \circ \psi$:

$$
B' = \begin{bmatrix}
1 & 0 & 0 & \ldots & 0 \\
1 & -1 & 0 & \ldots & 0 \\
1 & 0 & -1 & 0 & \ldots \\
1 & 0 & 0 & -1 & 0 \\
1 & 0 & 0 & \ldots & 0 \\
\end{bmatrix}
$$
Now the following matrices also corresponds to elements in the image of \( \varphi \circ H \circ \psi \):

\[
G_1 = CA_1B' = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
1 & 1 & 0 & & 0 \\
0 & 0 & -1 & 0 & \cdots & 0 \\
0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & & \cdots & 0 & -1
\end{bmatrix}
\]

and:

\[
G_2 = \begin{bmatrix}
1 & 1 & 0 & \cdots & 0 \\
0 & 1 & 0 & & 0 \\
0 & 0 & -1 & 0 & \cdots & 0 \\
0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & & \cdots & 0 & -1
\end{bmatrix}
\]

We have also:

\( G_3 = B'CA_1B' \) with matrix:

\[
\begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & -1 & 0 & & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & & \cdots & 0 & 1
\end{bmatrix}
\]

Using \( A_i \) and \( G_3 \) we can obtain from \( G_2 \) and \( G_3 \) the matrices:

\[
G'_1 = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
1 & 1 & 0 & & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & & \cdots & 0 & 1
\end{bmatrix}
\]

and

\[
G'_2 = \begin{bmatrix}
1 & 1 & 0 & \cdots & 0 \\
0 & 1 & 0 & & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & & \cdots & 0 & 1
\end{bmatrix}
\]

To finish the proof it is enough to show that given an integer \( g \times g \) matrix \( M \) with \( \det M = \pm 1 \) then \( M \) is a product of the matrices \( A_i, G'_1 \) and \( G'_2 \). Equivalently:

\[
w_1(A_i, G'_1, G'_2)Mw_2(A_i, G'_1, G'_2) = I
\]

where \( w_1(A_i, G'_1, G'_2) \) and \( w_2(A_i, G'_1, G'_2) \) are words in \( A_i, B, C \) and this fact follows from [V], vol. 2 pg. 107.
5 Poincaré’s theorem for real Riemann surfaces with non-separating fixed point set.

Assume that \( \sigma \) is an orientation reversing autohomeomorphism that fixes \( k \) closed curves and such that \( S/\langle \sigma \rangle \) is non-orientable. If \( h \) is an autohomeomorphism of \( S \) such that \( h \circ \sigma = \sigma \circ h \), then induces a homeomorphism \( h' : S/\langle \sigma \rangle \to S/\langle \sigma \rangle \). Hence \( h' \) sends boundary components to boundary components and \( h \) sends fixed curves by \( \sigma \) in fixed curves, may be changing the orientation. Let \( X_1, \ldots, X_k \) be the cycles in \( H_1(S,\mathbb{Z}) \) represented by the ovals of \( \sigma \). Thus \( h_H \) must permute the elements in the set \( \{ \pm X_1, \ldots, \pm X_k \} \), i.e. \( h(X_i) = \pm X_j, i, j \in \{1, \ldots, k\} \). Let \( \{X_i, Y_i\}_{i=1,\ldots,g} \) be a base of \( H_1(S,\mathbb{Z}) \) given by the Lemma 1. We shall denote:

\[
\begin{align*}
\text{Aut}^\sigma &_{\mu-k}(H_1(S,\mathbb{Z})) = \{ h \in \text{Aut}(H_1(S,\mathbb{Z})) : h \text{ preserves the symplectic form } \langle ., . \rangle , h \circ \sigma_H = \sigma_H \circ h , h(X_i) = \pm X_j, i, j \in \{1, \ldots, k\} , \\
h(X_i) &= \sum_{j=1}^{k} 2\alpha_{ij} X_j + \sum_{l=k+1}^{g} \beta_{il} X_l, i \in \{k+1, \ldots, g\}, \alpha_{ij}, \beta_{il} \in \mathbb{Z} \\
\text{Aut}^{-k}(V_0) &= \{ h \in \text{Aut}(V_0) : h(X_i) = \pm X_j, i, j \in \{1, \ldots, k\} , \\
h(X_i) &= \sum_{j=1}^{k} 2\alpha_{ij} X_j + \sum_{l=k+1}^{g} \beta_{il} X_l, i \in \{k+1, \ldots, g\}, \alpha_{ij}, \beta_{il} \in \mathbb{Z} \}.
\end{align*}
\]

If \( H : \text{Mod}_g^\sigma \to \text{Aut}(H_1(S,\mathbb{Z})) \) is the homomorphism given by \( H(h) = h_H \), where \( h_H \) is the homomorphism induced in the homology by \( h \) then \( H(\text{Mod}_g^\sigma) \subset \text{Aut}^\sigma_{\mu-k}(H_1(S,\mathbb{Z})) \). Following the notation in Section 3, note that \( X_i = \theta(d_i e_i) \) with \( i \in \{k+1, \ldots, g\} \). If \( H(h) = h_H \), then \( h_H(X_i) = \theta(h_*(d_i) h_+(d_i)) \), where \( h_* \) is the homomorphism induced by \( h \) on \( \Delta \). Now \( h_+(d_i) \) is a word in the \( d_i \) and \( e_i \) and then \( \theta(h_+(d_i) h_+(d_i)) \) is a linear combination of \( 2X_j = \theta(e_j^2), j \in \{1, \ldots, k\} \) and \( X_l = \theta(d_l^2), l \in \{k+1, \ldots, g\} \). Hence \( H(\text{Mod}_g^\sigma) \subset \text{Aut}^\sigma_{\mu-k}(H_1(S,\mathbb{Z})) \).

The main result of this Section is:

**Theorem 8** If \( \sigma \) is an orientation reversing autohomeomorphism of a surface \( S \) such that \( \sigma \) fixes \( k \) closed curves and \( S/\langle \sigma \rangle \) is non-orientable then \( H(\text{Mod}_g^\sigma) = \text{Aut}^\sigma_{\mu-k}(H_1(S,\mathbb{Z})) \).

**Proof.** Assume that \( S/\langle \sigma \rangle \) is non-orientable and has \( k \neq 0 \) boundary components.

Let \( \{X_1, Y_1, \ldots, X_g, Y_g\} \) be a basis of \( H_1(S,\mathbb{Z}) \) given by Lemma 1 and let \( h_H \in \text{Aut}^\sigma_{\mu-k}(H_1(S,\mathbb{Z})) \).

**Lemma 9** There are \( h'_H \in \text{Aut}^\sigma_{\mu-k}(H_1(S,\mathbb{Z})) \) and \( f \in \text{Mod}_g^\sigma \), such that \( H(f) \circ h_H = h'_H \) and \( h'_H(X_i) = X_i, i = 1, \ldots, k \) and the subspace \( \langle X_{k+1}, \ldots, X_g \rangle \) is invariant by the automorphism \( h'_H \).
Proof of the Lemma. We define the automorphisms $\delta, \rho_i$ and $\mu_i$ of $\Delta$ by:

$$
\delta : \begin{cases} 
   e_k & \rightarrow e_k d_{k+1} e_k^{-1} d_{k+1}^{-1} e_k^{-1} \\
   d_{k+1} & \rightarrow e_k d_{k+1} \\
   c_k & \rightarrow e_k d_{k+1} c_k d_{k+1} e_k^{-1} \\
   y & \rightarrow y 
\end{cases}
$$

for every canonical generator different from $e_k, d_{k+1}$ and $c_k$

$$
\rho_i : \begin{cases} 
   e_i & \rightarrow e_i e_{i+1} e_i \\
   e_{i+1} & \rightarrow e_i \\
   c_i & \rightarrow e_i c_{i+1} e_i^{-1} \\
   y & \rightarrow y 
\end{cases}
$$

for every canonical generator different from $e_i, e_{i+1}$ and $c_i$

$$
\mu_i : \begin{cases} 
   d_i & \rightarrow d_i^2 d_{i+1} d_i^{-2} \\
   d_{i+1} & \rightarrow d_i \\
   y & \rightarrow y 
\end{cases}
$$

for every canonical generator different from $d_i$ and $d_{i+1}$

The above automorphisms produce $D = \phi \circ H \circ \varphi(\delta), R_i = \phi \circ H \circ \varphi(\rho_i), M_i = \phi \circ H \circ \varphi(\mu_i)$. In the basis $\{X_l\}_{l=1,...,g}$ of $V_0$ the automorphisms $D, R_i$ and $M_i$ give:

- $D(X_k) = -X_k$
- $D(X_{k+1}) = X_{k+1} + 2X_k$
- $D(X_l) = X_l, l \neq k, k + 1$

- $R_i(X_i) = X_{i+1}$
- $R_i(X_{i+1}) = X_i$
- $R_i(X_l) = X_l, l \neq i, i + 1, i \in \{1, ..., k - 1\}$

- $M_i(X_i) = X_{i+1}$
- $M_i(X_{i+1}) = X_i$
- $M_i(X_l) = X_l, l \neq i, i + 1, i \in \{k + 1, ..., g - 1\}$

We can construct a word $w(D, R_i, M_i)$ in $D, R_i, M_i$ such that:

- $w(D, R_i, M_i)(h(X_j)) = X_j, j = 1, ..., k$ and
- $w(D, R_i, M_i)(h(X_l)) \in \langle X_{k+1}, ..., X_g \rangle, l \in \{k + 1, ..., g\}$.

Hence the element $f \in Mod_g^\sigma$ that we are looking for is $\psi(w(\delta, \rho_i, \mu_i))$.

We shall denote by $h_{H,1}$ the restriction of $h_H'$ to $\langle X_{k+1}, ..., X_g \rangle$.

Let $C$ be a closed curve in $S/\langle \sigma \rangle$ such that $S/\langle \sigma \rangle - C$ has two connected components $F_1$ and $F_2$, such that $F_1$ is homeomorphic to a non-orientable
surface of genus $g+1$ with connected boundary and $\overline{F_2}$ is homeomorphic to a planar surface with $k+1$ boundary components. Note that $C = \overline{F_1} \cap \overline{F_2}$ and that the preimage by $\pi: S \to S/\langle \sigma \rangle$ of $C$ is a set of two closed and disjoint curves $C_1$ and $C_2$.

Let $\widetilde{F_1}$ be the surface obtained from $\overline{F_1}$ closing the boundary component with a disc and $\widetilde{F_1}$ be the surface obtained from $\pi^{-1}(F_1)$ closing with two discs the boundaries $C_1$ and $C_2$. Let $\sigma_1$ be the orientation reversing automorphism of $\widetilde{F_1}$ giving as quotient $\overline{F_1}$. Hence we can identify $\text{Fix}(\sigma_1) \subset H_1(\overline{F_1},\mathbb{Z})$ with $\langle X_{k+1},...,X_g \rangle$. By Theorem 7 we have a homeomorphism $h_1 : \widetilde{F_1} \to \widetilde{F_1}$ such that $h_1 \circ \sigma_1 = \sigma_1 \circ h_1$, $(h_1)_H = h_{H,1}$ and we can choose $h_1$ in such a way that $h_1$ fixes $C_1$ and $C_2$. The extension of $h_1|_{\pi^{-1}(F_1)}$ to $S$ by the identity give us an element $h'$ of $\text{Mod}^p_g$ such that $H(h') = h'_H$.

\[ \square \]

**Example.** Let $S$ be a surface of genus 2 and $\sigma$ be an orientation reversing autohomeomorphism of $S$ with one (non-separating) oval. Let $\{X_1,Y_1,X_2,Y_2\}$ be a base given by the Lemma 1. Hence: $\sigma_H(X_i) = X_i$ and $\sigma_H(Y_1) = -X_1 - X_2$, $\sigma_H(Y_2) = -Y_2 - X_1 - 2X_2$.

Let us consider the automorphism of $H_1(S,\mathbb{Z})$:

$h(X_1) = X_1$, $h(X_2) = X_1 + X_2$

$h(Y_1) = -X_1 - X_2 + Y_1 - Y_2$, $h(Y_2) = Y_2 - X_1$.

It is easy to verify that $h$ preserves the symplectic form $(\,,\,)$ and $h \circ \sigma_H = \sigma_H \circ h$, but, by Theorem 8, $h$ is not induced by an autohomeomorphism of $S$ that commutes with $\sigma$.

### 6 Poincaré’s theorem for real Riemann surfaces with separating fixed point set.

Assume that $\sigma$ is an orientation reversing autohomeomorphism that fixes $k$ closed curves and such that $S/\langle \sigma \rangle$ is orientable, then $S - \text{Fix}(\sigma)$ has two connected components $S_1$ and $S_2$. If $h$ is an orientation preserving autohomeomorphism of $S$ such that $h \circ \sigma = \sigma \circ h$, $h$ induces an orientation preserving homeomorphism $h' : S_i \to S_i$ where $i = j$ or $i \neq j$ (depending on the nature of $h$). Hence $h'$ sends boundary components to boundary components and $h$ sends fixed curves by $\sigma$ in fixed curves. Let $X_1,...,X_{k-1}$, $\sum_{i=1}^{k-1} X_i$ be the cycles in $H_1(S,\mathbb{Z})$ represented by the ovals of $\sigma$. Thus $h_H$ satisfy

$h(X_i) = \varepsilon X_j$ or $\sum_{i=1}^{k-1} X_i$, $i,j \in \{1,...,k-1\}$, $\varepsilon = 1$ if $h' : S_i \to S_i$ and $\varepsilon = -1$ if $h' : S_i \to S_j$, $i \neq j$. We can identify

$H_1(S_1,\mathbb{Z}) = \langle X_1,...,X_{k-1},X_k,Y_k,...,X_{2-(k-1)/2},Y_{2-(k-1)/2} \rangle$ and

$H_1(S_2,\mathbb{Z}) = \langle X_1,...,X_{k-1},X_{2-(k+1)/2},Y_{2-(k+1)/2},...,X_g,Y_g \rangle$. 

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If we denote by
\[ H_1 = \langle X_k, Y_k, \ldots, X_{2^{i-1}}, Y_{2^{i-1}} \rangle \text{ and } H_2 = \langle X_{2^{k-1}}, Y_{2^{k-1}}, \ldots, X_{y}, Y_y, \rangle, \]
then the homology classes in \( H_i \) are represented by curves contained in one of the connected components \( S_1 \) or \( S_2 \). Then \( h_H(H_1) \subset H_i \) if \( i, j \in \{1, 2\} \) if \( \varepsilon = 1 \) and \( h_H(H_1) \subset H_j \) if \( i, j \in \{1, 2\} \) if \( \varepsilon = -1 \). We shall denote:

\[ \text{Aut}^{\sigma H, +k}(H_1(S, \mathbb{Z})) = \{ h \in \text{Aut}(H_1(S, \mathbb{Z})) : h \text{ preserves the symplectic form } (.,.) , h \circ \sigma_H = \sigma_H \circ h, (X_i) = \varepsilon X_j \text{ or } \varepsilon \sum_{i=1}^{k-1} X_i, i, j \in \{1, ..., k-1\}, \]

\[ \varepsilon = \pm 1, h(H_i) \subset H_i \text{ if } \varepsilon = 1 \text{ and } h(H_i) \subset H_j \text{ if } \varepsilon = -1 \}. \]

If \( H : \text{Mod}_g^\sigma \to \text{Aut}(H_1(S, \mathbb{Z})) \) is the homomorphism given by \( H(h) = h_H \), we have observed above that \( H(\text{Mod}_g^\sigma) \subset \text{Aut}^{\sigma H, +k}(H_1(S, \mathbb{Z})) \).

**Theorem 10** If \( \sigma \) is an orientation reversing autohomeomorphism such that \( \sigma \) fixes \( k \) closed curves and such that \( S/\langle \sigma \rangle \) is orientable then \( H(\text{Mod}_g^\sigma) = \text{Aut}^{\sigma H, +k}(H_1(S, \mathbb{Z})) \).

**Proof.** Assume that \( S/\langle \sigma \rangle \) is orientable and has \( k \neq 0 \) boundary components.

Let \( \{X_1, Y_1, \ldots, X_y, Y_y\} \) be the basis of \( H_1(S, \mathbb{Z}) \) given in Lemma 4 and let \( h_H \in \text{Aut}^{\sigma H, +k}(H_1(S, \mathbb{Z})) \).

**Lemma 11** There are \( h'_H \in \text{Aut}^{\sigma H, +k}(H_1(S, \mathbb{Z})) \) and \( f \in \text{Mod}_g^\sigma \), such that \( H(f) \circ h_H = h'_H, h'_H(X_i) = X_i, i = 1, \ldots, k \) and \( h'_H(H_i) = H_j, i, j \in \{1, 2\} \).

**Proof of the Lemma.** We define the automorphisms \( \omega, \xi, \nu, \mu, \rho \) of \( \Delta \) by:

\[
\omega : \begin{cases}
  a_1 \to a_1 b_1 \\
  y \to y
\end{cases}
\text{for every canonical generator different from } a_1
\]

\[
\tau : \begin{cases}
  a_1 \to a_1 b_1 \\
  b_1 \to a_1^{-1} \\
  y \to y
\end{cases}
\text{for every canonical generator different from } a_1, b_1
\]

\[
\nu_i : \begin{cases}
  a_i \to a_i+1 \\
  b_i \to b_i+1 \\
  a_{i+1} \to w_{i+1}^{-1} a_i w_{i+1} \\
  b_{i+1} \to w_{i+1}^{-1} b_i w_{i+1} \\
  y \to y
\end{cases}
\text{where } w_{i+1} = [a_{i+1}, b_{i+1}]
\text{for every canonical generator different from } a_i, a_{i+1}, b_i, b_{i+1}
\]

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\[ \mu : \begin{cases} 
  e_k & \rightarrow & a_k^{-1}e_ka_1 \\
  a_1 & \rightarrow & [a_1^{-1}, e_k^{-1}]a_1 \\
  b_1 & \rightarrow & b_1a_1^{-1}e_ka_1 \\
  y & \rightarrow & y 
\end{cases} \]

for every canonical generator different from \( e_k, a_1 \) and \( b_1 \)

\[ \rho_i : \begin{cases} 
  e_i & \rightarrow & e_ie_{i+1}e_i \\
  e_{i+1} & \rightarrow & e_i \\
  c_i & \rightarrow & e_ic_{i+1}e_i^{-1} \\
  c_{i+1} & \rightarrow & c_i \\
  y & \rightarrow & y 
\end{cases} \]

for every canonical generator different from \( e_i, e_{i+1} \) and \( c_i \)

Having in consideration that \( \theta(e_k) = -\sum_{i=1}^{k-1} X_i \), the above automorphisms produce \( Z = \varphi \circ H \circ \varphi(\omega) \), \( T = \varphi \circ H \circ \varphi(\tau) \), \( N_i = \varphi \circ H \circ \varphi(\nu_i) \), \( M = \varphi \circ H \circ \varphi(\mu) \), \( R_i = \varphi \circ H \circ \varphi(\rho_i) \) described by the formulae:

\[ Z(X_k) = X_k + Y_k, \quad Z(X_{k+2+\frac{k+1}{2}}) = X_{k+2+\frac{k+1}{2}} - Y_{k+2+\frac{k+1}{2}} \]

\[ T(X_k) = X_k + Y_k, \quad T(Y_k) = -X_k \]

\[ T(X_{k+2+\frac{k+1}{2}}) = X_{k+2+\frac{k+1}{2}} - Y_{k+2+\frac{k+1}{2}}, \quad T(Y_{k+2+\frac{k+1}{2}}) = X_{k+2+\frac{k+1}{2}} \]

\[ N_i(X_i) = X_{i+1}, \quad N_i(Y_i) = Y_{i+1} \]

\[ N_i(X_{i+1}) = X_i, \quad N_i(Y_{i+1}) = Y_i \]

\[ N_i(X_{i+2+\frac{k+1}{2}}) = X_{i+2+\frac{k+1}{2}}, \quad N_i(Y_{i+2+\frac{k+1}{2}}) = Y_{i+2+\frac{k+1}{2}} \]

\[ M(Y_k) = Y_k - \sum_{i=1}^{k-1} X_i, \quad M(Y_{k+2+\frac{k+1}{2}}) = Y_{k+2+\frac{k+1}{2}} - \sum_{i=1}^{k-1} X_i \]

\[ R_i(X_i) = X_{i+1}, \quad R_i(Y_i) = Y_{i+1} \]

\[ R_i(X_{i+1}) = X_i, \quad R_i(Y_{i+1}) = Y_i \]

\( i \in \{1, \ldots, k-2\} \),

\[ R_{k-1}(X_{k-1}) = -\sum_{i=1}^{k-1} X_i, \quad R_{k-1}(Y_{k-1}) = -Y_{k-1} \]

\[ R_{k-1}(Y_i) = Y_i - Y_{k-1}, \quad i = 1, \ldots, k-2. \]

In the above formulae we have just written the image of the elements that change by the automorphisms.

Finally we consider \( U \) induced by an orientation preserving involution \( u \) in \( S \) with \( \sigma \circ u = u \circ \sigma \) such that \( U(S_1) = S_2 \), \( u \) has two fixed points on each oval of \( \sigma \) and

\[ U(X_i) = -X_i, \quad U(Y_i) = -Y_i, \quad i = 1, \ldots, k-1. \]
\[ U(X_i) = X_{i + \frac{g+k+1}{2}}, \quad U(Y_i) = Y_{i + \frac{g+k+1}{2}}, \quad i = k, \ldots, \frac{g+k-1}{2}. \]
\[ U(X_i) = X_{i - \frac{g+k+1}{2}}, \quad U(Y_i) \rightarrow Y_{i - \frac{g+k+1}{2}}, \quad i = \frac{g+k+1}{2}, \ldots, g. \]

Now, as in the proof of Lemma 9, we can construct a word \( w(Z, T, N_i, M, R_i, U) \) in \( Z, T, N_i, M, R_i, U \) such that:

\[ w(Z, T, N_i, M, R_i, U)(h(X_j)) = X_j, \quad j = 1, \ldots, k - 1 \text{ and } \]
\[ w(Z, T, N_i, M, R_i, U)(H_i) \subset H_i, \quad i, j \in \{1, 2\}. \]

Hence the element \( f \in Mod_\sigma^g \) that we are looking for is \( \psi(w(\omega, \tau, \nu_i, \mu, \rho_i)). \)

We shall denote by \( h_{H,1} \) the restriction of \( h'_H \) to \( H_1 \). Let \( C \) be a closed curve in \( S_1 \) such that \( S_1 - C \) has two connected components \( F_1 \) and \( F_2 \), such that \( \overline{F_1} \) is homeomorphic to an orientable surface of genus \( \frac{g-k+1}{2} \) with connected boundary and \( \overline{F_2} \) is homeomorphic to a planar surface with \( k+1 \) boundary components.

Let \( \widetilde{F_1} \) be the surface obtained from \( \overline{F_1} \) closing the boundary component with a disc. Hence we can identify \( H_1(\widetilde{F_1}, Z) \) with \( H_1 \). By Poincaré theorem there is \( h_1 : \widetilde{F_1} \rightarrow \widetilde{F_1} \) such that \( (h_1)_H = h_{H,1} \) and \( h_1 \) fixes \( C_1 \). We construct now

\[ h'(z) = h_1(z) \text{ for } z \in \widetilde{F_1} \cap S_1, \]
\[ h'(z) = z \text{ for } z \in S_1 - \widetilde{F_1}, \]
\[ h'(z) = \sigma \circ h' \circ \sigma (z) \text{ for } z \in S_2. \]

Thus we have an element \( h' \) of \( Mod_\sigma^g \) such that \( H(h') = h'_H \).

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