INVERSE OF MULTIVECTOR: BEYOND P+Q=5 THRESHOLD

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Abstract. The algorithm of finding inverse multivector (MV) numerically and symbolically is of paramount importance in the applied Clifford geometric algebra (GA) \( \mathcal{Cl}_{p,q} \). The first general MV inversion algorithm was based on matrix representation of MV. The complexity of calculations and size of the answer in a symbolic form grow exponentially with the GA dimension \( n = p + q \). The breakthrough occurred when D. Lundholm and then P. Dadbeh found compact inverse formulas up to dimension \( n \leq 5 \). The formulas were constructed in a form of Clifford product of initial MV and its carefully chosen grade-negation counterparts. In this report we show that the grade-negation self-product method can be extended beyond \( n = 5 \) threshold if, in addition, properly constructed linear combinations of such MV products are used. In particular, we present compact explicit MV inverse formulas for algebras of vector space dimension \( n = 6 \) and show that they embrace all lower dimensional cases as well. For readers convenience, we have also given various MV formulas in a form of grade negations when \( n \leq 5 \).

1. Introduction

The knowledge of how to find inverse multivector (MV) in the Clifford algebra in a symbolic and coordinate-free form is very important both from practical computational and purely theoretical point of views. A universal formula for inverse MV would allow to write down a fast and general algorithm for all occasions rather than to resort to either specific symbolic or numerical cases. The inverse of MV can be used to find explicit solutions of algebraic GA equations with all the ensuing consequences and applications. A closely related problem is the normalization of spinors [1,2]. Invertibility also serves as an important criterion on deciding whether homogeneous versors are the blades [3], which are essential in numerous geometric constructions.

The first attempts of inversion of some specific forms of MVs can be traced back to papers [4,5,6]. However it was not until 2002 when J.P. Fletcher [7] has suggested some general MV inverse formulas for low dimensional algebras. His algorithm was based on decomposition of MV into matrix basis elements, and, as a consequence, resulted into large, inconvenient and signature dependent formulas the size of which grows exponentially with algebra dimension.

The breakthrough occurred after D. Lundholm [8] has presented explicit expressions for \( n \leq 5 \) (determinant) norms and later P. Dadbeh [9] introduced grade-negation operation what has allowed to write down compact and explicit formulas.

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for MV inverse in a coordinate-free form. The heart of algorithm [9] is the geometric product of initial MV and its carefully chosen grade-negated counterpart(s) that after few iterations eventually yields a scalar. As a matter of fact, the product may be related with the determinant of a matrix representation of general MV, from which the inverse multivector can be easily extracted by simply removing the initial MV which is always positioned in either left-most or right-most side of the product. Using the described method P. Dadbeh was able to find explicit inverses for a general MV up to dimension $n \leq 5$. It is important to stress that the obtained formulas are determined by vector space dimension $n = p + q$ only and are independent of a particular GA signature $(p, q)$. The same formulas were also obtained by other authors using different methods (see, for example, [10, 11, 12]). When $n \leq 5$, detailed mathematical proofs are given in [11]. If general multivector is given in expanded form in some orthogonal basis with symbolic coefficients, then the verification of the algorithm can be easily done by a direct substitution of symbolic MV into formula and explicitly computing the inverse $A^{-1}$, and finally checking that the property $AA^{-1} = A^{-1}A = 1$ is satisfied. For $n > 4$, however, calculation of explicit inverse in symbolic form is time consuming and results in lengthy expressions for coefficients at basis elements of $A^{-1}$. Nonetheless, such calculations, in fact, have status of “computer-assisted proof”. Similar formulas for the (determinant) norms of MVs when $n \leq 5$ were also given in [2]. Analysis of general structure of such formulas was presented in [12]. However, until now any attempts to step across the threshold $p + q = 5$ were unsuccessful although there is a need for such formulas in practice.

In this report we show that the grade-negation method can be extended beyond $n = 5$ threshold if, in addition, properly constructed linear combinations of grade-negated MVs are introduced. In particular, we write down explicit MV inverse formulas for algebras with vector space dimension $n = 6$ having all possible $(p, q)$ signatures. We also provide alternative formulas for even MVs and $n = 5$ case. In Sec. 2 the grade-negation method is shortly reviewed and required notation and terminology is introduced. In Sec. 3 the inverse even MVs that follow from higher grade inverse MVs are considered. Finally, in Sec. 4 a general algorithm for construction of inverse MV at $n = 6$ is briefly discussed and the obtained coordinate-free formulas are presented in a form of tables.

2. Grade-negated self-product and inverse of a general MV in $n \leq 5$ case

Following [9] we first introduce a grade-negated self-product that is defined via grade-$r$ negation operation. Applied to the multivector $A$ this operation does what it says, i.e., it changes the sign of grade-$r$ part of $A$. Such grade-$r$ negated MV will be denoted as $A_{\bar{r}}$, with the bar over index designating which of the grades have opposite signs. Formally the grade-$r$ negated MV can be expressed as $A_{\bar{r}} = A - 2\langle A \rangle_r$, or $A_{\bar{r},\bar{s}} = A - 2\langle A \rangle_r - 2\langle A \rangle_s$ for a double negation, where $\langle A \rangle_r$ denotes grade-$r$ projection of multivector $A$. In particular, we have $\langle (A)_{\bar{r}} \rangle_{\bar{r}} = -\langle A \rangle_r$.

The following properties are evident from the definition of grade-negation operation: $(A_{\bar{r}})_{\bar{r}} = A_{\bar{r},\bar{r}} = A_{\bar{r},\bar{r}} \cdot A_{\bar{r}} = A, (A + B)_{\bar{r}} = A_{\bar{r}} + B_{\bar{r}}$. However, $(AB)_{\bar{r}} \neq A_{\bar{r}}B_{\bar{r}}$. If $A$ does not contain grade-$r$ elements then negation returns the same MV, $A_{\bar{r}} = A$. Commutator with grade-negated MV then can be expressed as $AA_{\bar{r}} - A_{\bar{r}}A = 2(\langle A \rangle_r A - A\langle A \rangle_r)$. Multivector $A$ commutes with scalar negated $A_\bar{r}$,
i. e., \([A, A^n_1] = 0\) and, as a consequence, with \(A_{i,\ldots,j}\), where all grades \(i \neq 0, j \neq 0, \ldots\) of \(A\) (except scalar) are negated.

The standard involutions such as MV reversion \(\tilde{A}\), grade inversion \(\hat{A}\), and Clifford conjugate \(\bar{A} = \tilde{A} = \hat{A}\) in terms of negation can be written as \(\tilde{A} = A_{1,3,5,7,9,11}, \hat{A} = A_{1,3,5,7,9,11,\ldots}\) and \(\bar{A} = A_{1,2,5,6,9,10,\ldots}\), respectively. For finite algebras the index series should be chopped off when negation index becomes larger than that of the pseudoscalar. It is worth mentioning that the grades 0, 4, 8, 12, \ldots are absent in the above list of standard GA involutions. From this follows that the inverse of a general MV cannot be expressed using just standard involutions, or their combinations. As we shall see this observation also is directly related to the MV inversion problem for \(n \geq 6\).

By grade-negated self-product we identify the geometric product of general MV \(A\) with any number of its grade-negated counterparts, for example, geometric product \(AA_{r,\ldots,t}A_{s,\ldots,t}\) is the left grade-negated self-product and \(A_{r,\ldots,t}A_{s,\ldots,t}\) is the right grade-negated self-product. These self-products, where the initial MV stands in the left-most or right-most position, will be of special importance in finding the inverse MV \(A^{-1}\).

The explicit inverse formulas which we will construct rely on our ability to get a real scalar using just grade-negated self-products (or, as we shall see, some linear combinations of them for \(n > 5\)), where initial MV \(A\) is mandatory and is located either in the left-most or right-most position. We shall call the real scalar

\[
(1) \quad s_m = \overbrace{AAA_{r,\ldots,t}A_{s,\ldots,t}\cdots}^{m} = A f(A) = f(A) A
\]

obtained in this way the determinant norm\(^1\). If we succeed in constructing such a scalar \(s_m\) then it is easy to see that inverse MV formula for \(A\) (correspondingly, for \(\tilde{A}\)) can be obtained simply by removing either extreme left initial MV \(A\) or extreme right initial MV \(A\) from the scalar scalar \(s_m\) in (1) (correspondingly, from the \(s'_{m} = \cdots \tilde{A}_{s,\ldots,t}A_{r,\ldots,t}\tilde{A}\)), and then dividing the result by \(s_m\) or \(s'_m = \tilde{A} f(\tilde{A}) = \tilde{f}(\tilde{A}) \tilde{A}\).

\(^1\) In literature, the GA terminology on MV norms is quite confusing. The book [13] and most of introductory GA lecture courses, for example [14], typically define the MV norm \(|A|^2\) as a scalar part of product \((A\tilde{A})_{1}^{0}\), which can be negative. (A better name would be the pseudonorm). If sign is positive this kind of norm is also called the magnitude. The same term sometimes is used to define a positive square root of absolute value of \(\sqrt{\text{abs}(|A|^2)}\). On the other hand, in [8] a different scalar (named MV norm) is defined which coincides with our determinant norm \(s_n\) for \(n \leq 5\), whereas in [12] a similar construction is called “the discriminant”. In [9] [10] this construction has been named “the determinant”, because a determinant of matrix representation of \(A\) always coincides with the scalar. Other authors [11] bypass naming confusion calling it just “a real scalar”. In order to maintain the analogy with MV norm defined in [8] and to distinguish it from the norm definition of [13], we will use prefixed form “determinant (pseudo)norm” or shorter form “determinant norm”, or just a “determinant” if it is clear from the context what is meant. In general, the last term should not be confused with a determinant of matrix that represents the MV \(A\). It should be noted, however, that the term “determinant” in GA is usually addressed in the context of linear transformations (see, for example, [13], p. 108, the subsection “The determinant”), the definition and meaning of which is related to the exterior product of basis vectors and has nothing to do neither with the above determinant norm nor with a determinant of matrix that represents \(A\).
The inverse formulas for a MV $A$, thus, become
\[
(2) \quad A^{-1} = \frac{A_{r,i} \cdots A_{s,i} \cdots}{s_m} = \frac{f(A)}{s_m}, \quad \overline{A^{-1}} = \frac{\cdots \overline{A_{s,i}} \cdots}{s_m} = \frac{\overline{f(A)}}{s_m}.
\]

All known cases for $n \leq 5$ (for detailed proofs see [11]) show that inverse formulas in (2) can be applied to arbitrary signature algebra at a fixed vector space dimension $n$. Furthermore, it is easy to check explicitly, for example, that the inverse formula for $n = 5$ also yields inverses for $n = 4, 3, 2, 1$ as well, i.e., each of formula for larger $n$ automatically contains inverses of all lower algebras $m \leq n$ with all possible signatures. Therefore, we conjecture that explicit formulas for determinant norm (1) are signature independent, and that, when restricted to lower dimension vector spaces, they yield determinant norms of lower dimensional vector spaces, generally raised in some power, which can be easily determined from the dimension of matrix representation (see Example 5 below). The known cases also suggest that formulas (2) always ensure that the inverse commutes with all three main involutions: reversion, grade inversion and Clifford conjugate, i.e., $\overline{A^{-1}} = (\overline{A})^{-1}$, $\overline{\overline{A^{-1}}} = (\overline{A})^{-1}$ and $\overline{\overline{A^{-1}}} = (\overline{A})^{-1}$. This is not generally true for an arbitrary involution, for example, for an arbitrary combination of grade negations: $(A^{-1})_{i,\ldots} \neq (A_{i,\ldots})^{-1}$.

It is well known that Clifford algebras are isomorphic to algebras of square matrices, the left and right inverses of which coincide. The Clifford MV, therefore, has only one inverse which, depending on our needs, can be written using either of the two determinant norm forms (1). As a result we have that $AA^{-1} = A^{-1}A = 1$ which can also serve as a test of correctness of GA inverse algorithm.

**Two-dimensional quadratic space.** Since the one-dimensional case is trivial we start with the algebras $Cl_{2,0}$, $Cl_{1,1}$ and $Cl_{0,2}$. For $n = 2$ let it be $Cl_{2,0}$. Writing $A = \sum_{J=0}^{2^n-1} a_J e_J$, where the multi-index $J$ covers all orthonormal base elements arranged in the increasing order of the degrees followed by the lexicographic order, i.e., the lowest grade elements appear first while elements of the same grade are ordered lexicographically. In the sum the multi-index takes the following explicit values $J = [\{\}, \{1\}, \{2\}, \{1,2\}]$, which illustrate inverse degree lexicographic ordering [10]. We can check that self-product $AA_{1,2}$ immediately gives the required scalar $s_2 = a_1^2 - a_2^2 - a_3^2 + a_4^2$. In the standard notation it is more convenient to rewrite the coefficient indices as in $s_2 = a_0^2 - a_1^2 - a_2^2 + a_3^2$, where the scalar coefficient was indexed by zero. The indices of coefficients for vector components run from 1 to $n$, while numerical indices for higher grades increase monotonically up to $2^n$. In general, when programming, for a coefficient in front of grade-$r$ base it is convenient to start the element index enumeration by $\sum_{k=0}^{r-1} (\frac{n}{2})$ and to end by

2The informal explanation is very simple. Geometric product of two vectors splits into anti-symmetric and symmetric parts: $ab = a \wedge b + a \cdot b$. Because the commutators vanish, $[e_i, e_i] = 0$ for $i = 1, 2, \ldots$, only the diagonal part (i.e. signature) is not fixed in the totally antisymmetric expression (whatever that might mean), when the geometric product of vectors is extended to the whole algebra (whatever that might mean). Since the result (the determinant norm) is a real scalar obtained by the same fixed antisymmetrization construction (i.e. the formula), it can only be a function of signature.

3As known, the summation is orderless with respect to base elements. Nevertheless, it is convenient in advance to settle some order which is required if we want to enumerate coefficients $a_i$ in front of base elements in a unique way.
\[
\left( \sum_{k=0}^{n-1} \binom{\frac{n}{2}}{k} \right) + \binom{n}{\frac{n}{2}} - 1. \text{ Here } \binom{n}{k} \text{ denotes the binomial coefficient. For example, for } Cl_{3,0}, \text{ which has three vectors and three bivectors, these expressions give 1 and 3 for first/last vector indices, and 4 and 6 for first/last bivector indices. Taking into account that binomial with negative index vanishes, this convention enumerates all for first/last vector indices, and 4 and 6 for first/last bivector indices. Taking into account that binomial with negative index vanishes, this convention enumerates all coefficients of MV and allows an easy transition to more standard notation in a consistent way, for example, for initial MV we write } A = a_0 + a_1 e_1 + a_2 e_2 + a_3 e_{12} \text{ and for grade-negated MV } A_{\bar{1},\bar{2}} = a_0 - a_1 e_1 - a_2 e_2 - a_3 e_{12}. \text{ This notation will be used throughout the paper. Note, that indices of base elements will never be renamed and always be referred to as multi-indices. Before going to higher dimensional algebras a few comments are in place here. The comments are of common character and can be easily generalized to higher dimensional algebras.}

1. The explicit form of determinant } s_m \text{ comprises all coefficients } a_i \text{ of a multivector } A.

2. The condition for a MV to have inverse is determined by determinant which must be nonzero, } s_m \neq 0. \text{ If some of intermediate results in } s_m \text{ after negation reduce to zero the entire determinant turns out to zero automatically. This explicit statement was given in order to resolve indeterminate cases like zero division by zero.}

3. Because reversion operation leaves the scalar (determinant norm) invariant, the above formula for } n = 2 \text{ can be rewritten as } s_2 = AA_{1,2} = A_{1,2}A = \tilde{A}_{1,2} \text{ for an arbitrary multivector } A. \text{ The right hand side of equality, therefore, can be understood as the determinant norm of a multivector } \tilde{A} \text{ written in the right hand side form (where } \tilde{A} \text{ now stands in the right most position). Due to this arbitrariness, inversions can always be written in two forms: } A^{-1} = A_{1,2}/s_2 = A_{1,2}/(AA_{1,2}) = A_{1,2}/(A_{1,2}A) \text{ and, correspondingly, } \tilde{A}^{-1} = \tilde{A}_{1,2}/(\tilde{A}_{1,2}\tilde{A}) = \tilde{A}_{1,2}/(\tilde{A}\tilde{A}_{1,2}) = \tilde{A}_{1,2}/(A_{1,2}A) = \tilde{A}^{-1}, \text{ where we explicitly had used } \tilde{A} = A_{2,3}. \text{ Determinant norms of } A \text{ and } \tilde{A} \text{ are equal } [10], \text{ because the matrix representation of the reversed MV yields the same determinant. Construction of matrix operation itself, which corresponds to MV } \tilde{A} \text{ reversion for any signature is described in [17]. Since } A_{1,\bar{2}} = \tilde{A}, \text{ the negated MV for } n = 2 \text{ algebras can be expressed through standard involutions.}

4. The determinant expression for } n = 2 \text{ contains only two MVs. From 8-periodicity table } [18] \text{ it follows that the algebras } Cl_{2,0}, Cl_{1,1} \text{ and } Cl_{0,2} \text{ are isomorphic to the algebra } \mathbb{R}(2) \text{ of real } 2 \times 2 \text{ matrices or to the algebra } \mathbb{H} \text{ of quaternions. The determinant of these matrices is a quadratic polynomial in the coefficients } a_i \text{ of the MVs; therefore, this polynomial can be constructed by multiplying just two MVs. Below we shall see that in all cases the total polynomial degree (where coefficients } a_i \text{ play the role of variables) of matrix determinant always matches the number of MVs in the determinant product. The determinant of matrix representation with quaternionic elements can be calculated using isomorphism } \mathbb{H} \cong \mathbb{C}(2), i.e. \text{ we first replace quaternions by } 2 \times 2 \text{ block matrices and then calculate the determinant (the real scalar) of the resulting matrix. This practical procedure allows us to avoid considering numerous definitions of the determinant of } \mathbb{H}(n) \text{ due to element non-commutativity.}

**Three-dimensional quadratic space.** Our goal is to eliminate as many grades as possible by forming suitable self-products until finally the grade-0 element (determinant norm) is left. As another example, let us calculate the inverse of } Cl_{2,1}
algebra. It is easy to check that geometric product of \( A = a_0 + a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4 + a_5e_5 + a_6e_6 + a_7e_{123} \) with \( A_{1,2} \) lacks grades 1 and 2. The result is a new multivector \( B = AA_{1,2} = b_0 + b_7e_{123} \) which consists of scalar and grade-3 element with coefficients \( b_0 = a_0^2 - a_1^2 - a_2^2 + a_3^2 - a_4^2 - a_5^2 + a_6^2 + a_7^2 \) and \( b_7 = -2a_3a_4 + 2a_2a_5 - 2a_1a_6 + 2a_0a_7. \) Repeating grade negation procedure one finds that grade-3 part can be removed too, and we obtain the determinant norm \( s_4 = BB_3 = AA_{1,2}(AA_{1,2})_3 = b_0^2 - b_7^2. \) Reversion of \( s_4 \) then immediately yields an alternative form of the norm \( s'_4 = (A_{1,2}A)^3A_{1,2}A. \) Of course, for different 3D algebras the same formula will give distinct real expressions, which differ in signs at individual coefficients \( a_i. \) The important thing is that despite the fact that the scalar \( s_4' \) was calculated in \( Cl_{2,1} \) algebra, exactly the same sequence of products and grade negations will produce the scalar (generally different) in all other algebras with \( n = 3. \) Also note, that the determinant norm in this case is determined by two terms, \( b_0^2 \) and \( b_7^2, \) the difference of which should not be equal to zero for an invertible multivector \( A \) to exist. The condition \( b_0^2 - b_7^2 \neq 0 \) exactly matches the MV invertibility condition, which according to 8-periodicity table may be obtained by calculating the determinant of a matrix representation of MV.

**Four-dimensional quadratic space.** In \( n = 4 \) case, in trying to eliminate as much grades as possible we can proceed in two alternative ways. Firstly, we can negate simultaneously the grades 1 and 2 and then in the next step the grades 3 and 4. Alternatively, we can eliminate 2 and 3, and then 1 and 4 grades. Both choices are valid. However, if we choose 1 and 3, and then 2 and 4 grade combinations, neither one will do the job. Thus, we find the following formulas for determinant norm \( s_4 = AA_{1,2}(AA_{1,2})_3 = AA_{2,3}(AA_{2,3})_{1,4} \) and \( s'_4 = (A_{1,2}A)^{3,4}A_{1,2}A = (A_{2,3}A)^{1,4}A_{2,3}A. \) It is easy to check that both expressions indeed give determinant norms. The occurrence of symbol \( A \) as often as four times in the products \( s_4 \) and \( s'_4 \) is again what we expect from the matrix representations for \( n = 4. \) The total degree of the determinant, considered as a polynomial function of the coefficients of MV, is 4 for all algebras in the case \( n = 4. \) Despite different forms, the expanded explicit expressions for \( s_m \) and \( s'_m \) were found to be equal, \( s_m = s'_m, \) as it should be [11].

Our computations show that formulas \( s_m \) and \( s'_m \) are the only possible equivalent ways to get determinant norm in \( n = 4 \) case using geometric product and negation operations.

**Five-dimensional quadratic space.** This is the largest dimension, \( n = 5, \) when consecutive grade elimination works by factorizing the determinant norm into product of the initial MV and negated ones. The grade elimination sequence is similar [11]: (1) Eliminate simultaneously grades 2 and 3; (2) Then eliminate grades 1 and 4; (3) Finally eliminate grade 5. Apart from the last step this sequence is exactly the same as the second alternative of \( n = 4 \) case. The final result is \( s'_5 = (FA)_{3,4}F\bar{A} \) with \( F = (A_{2,3}A)_{1,4}A_{2,3}A_{1,2}A = (A_{2,3}A)^{1,4}A_{2,3}A. \)

The total degree of determinant polynomial in this case is 8, which again exactly matches the number of MVs in the determinant norm product. With our program [19] we have found that \( s_8 \) determinant norm can be written in 52 different ways as presented in Table [1].

*Example 1.* Let’s take \( A = 3 + e_2 + e_5 - e_{12} - e_{15} + 3e_{125} \) in \( Cl_{4,1}. \) It is easy to check that \( AA_{2,3} = 0. \) In Table 1 the formulas with \( H = AA_{2,3} \) immediately allow to conclude that the determinant norm of this MV is zero and therefore the
| N | Abbreviation → | $H = AA\hat{\sim} = AA_{2,3}$ | $H' = AA\hat{\sim} = AA_{1,2,5}$ |
|---|----------------|-------------------------|-------------------------|
| 13 | $H(H(H_{1,4}H_5)_{1,4})$ | $H(H(HH_{1,4}H_5)_{1,4})$ | $H(H(HH_{1,5})_{1,5})$ |
| 14 | $H(H_{1,4}H_{4,5})_{1,4}$ | $H(H_{1,4}H_{4,5})_{1,4}$ | $H(H_{1,5}H_{4,5})_{1,5}$ |
| 15 | $H(H_{1,4}H_{5})_{1,4}$ | $H(H_{1,4}H_{5})_{1,4}$ | $H(H_{1,5}H_{4,5})_{1,5}$ |
| 17 | $H(H_{1,4})_{1,4}$ | $H(H_{1,4})_{1,4}$ | $H(H_{1,5})_{1,5}$ |
| 18 | $H(H_{1,4})_{1,4}$ | $H(H_{1,4})_{1,4}$ | $H(H_{1,5})_{1,5}$ |
| 15 | $H'(H'(H'H')_{1,4})_{1,4}$ | $H'(H'H')_{1,4}$ | $H'(H'H')_{1,4}$ |
| 16 | $H'(H'(H'H')_{1,4})_{1,4}$ | $H'(H'H')_{1,4}$ | $H'(H'H')_{1,4}$ |

| Table 1. Alternative formulas for MV determinant norm for GAs of vector space dimension $n = 5$ listed by increasing number of negations. For example, the number of negations $N$ in the first line is determined by four $H = AA_{2,3}$, each including 2 negations, (2, 3), and five explicit negations $(1, 4)+5+(1, 4)$ in the formula, resulting in $2*4+5 = 13$ total negations. Computationally preferred forms that contain the largest number of repeating pieces are underlined. |

The inverse of $A$ does not exist. Now let’s try to find the determinant of $A$ with formula that does not contain $AA_{2,3} = 0$, for example with $H' = AA_{1,2,5}$ (see the first line in Table I), from which the final result is not so obvious. First, we calculate $H' = 18 + 18 \epsilon_{125}$, which is not zero. However, computing the next step $H'(H')_{1,4}$ and $(H')_{1,5}H'$ in the last two lines in Table I we get zero again.

**Example 2.** Given $A = 1 + 2e_1 + 3e_2 + 4e_{2345}$ in $Cl_{5,0}$ let us find $A^{-1}$ using a couple of alternative formulas. First, we shall use new computationally efficient formula $D_1 = HH_{1,5}(HH_{1,5})_{1,5}$ (underlined in Table I). Computation of $H$ yields $H = AA_{2,3} = 30 + 4e_1 + 8e_{2345} + 16e_{12345}$. Then $HH_{1,5} = 692 + 352e_{2345}$. And $HH_{1,5} = 354960$. Then the inverse is $A^{-1} = \frac{AA_{2,3}(HH_{1,5})_{1,5}}{D_1} = \frac{354960(3576 + 96e_1 + 5383e_{235} - 1507e_{235} - 8592e_{123} - 8592e_{123} + 47424e_{2345} - 8256e_{12345})}$. Now let’s compute the determinant norm of $A$ using $D_2 = HH_{1,5}HH_{1,4}$, which computationally is less efficient, because it contains smaller number of repeating parts. Computation of $HH_{1,5}$ yields $596 - 16e_1$. Then, $HH_{1,5}HH_{1,4}$ gives $17944$ --
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\[ Cl_{p,q} \left( A^+ \right)^{-1} \]

\[
\begin{align*}
p + q = 2 & \quad \frac{B(A^* B)}{(A^* B)^2}, \quad B = (v)\bar{1} = -v \\
p + q = 3 & \quad \frac{C(A^* C)}{(A^* C)^2}, \quad C = (A^+ v)\bar{1} \\
p + q = 4 & \quad \frac{C(A^* C)}{(A^* C)^2}, \quad C = (A^+ v)\bar{1} \\
p + q = 5 & \quad \frac{D(A^* D)}{(A^* D)^2}, \quad D = (A^+ v)\bar{3}(A^+ (A^+ v)\bar{3})\bar{1} \\
p + q = 6 & \quad \frac{D(A^* D)\bar{6}}{(A^* D)(A^* D)\bar{6}}, \quad D = (A^+ v)\bar{3}(A^+ (A^+ v)\bar{3})\bar{1}
\end{align*}
\]

Table 2. Explicit formulas of the inverse of even MVs in a coordinate-free form for Clifford algebras of dimension \( n = p + q \leq 6 \). The quantities \( (A^+ B)^2/v^2 \), \( (A^+ C)^2/v^2 \), \( (A^+ D)^2/v^4 \) and \( (A^+ D)\bar{6}/v^4 \) are the determinant norms of respective MVs. Here, nonisotropic and unnormalized vector \( v \) also can be replaced by one of orthonormal base vector \( e_i \) for computational efficiency.

\[ 2864e_1 + 5024e_{2345} - 9664e_{12345}. \] Finally, \( D_2 = 354960 \). Of course, these formulas give the same explicit expressions for inverse MV in symbolic form as well.

Can the above described determinant computation procedure be extended beyond \( n = 5 \)? The short answer is ‘yes’ if, as shown below, we allow linear combinations of grade-negated self-products with properly chosen numerical coefficients. Before describing this case let us derive some useful formulas for even subalgebras when \( n \leq 6 \). The even subalgebras are directly related with the spinor groups that are very important in the quantum mechanics [1].

3. INVERSE OF EVEN MULTIVECTORS

When MV consists of even grade elements only, i.e. \( A^+ = \langle A \rangle_0 + \langle A \rangle_2 + \langle A \rangle_4 + \cdots \), simpler formulas for inverse MVs can be constructed as shown in Table 2. The nonisotropic unnormalized vector \( v \) in these formulas play the role of dummy variable.

It is interesting to observe that in contrast to general case considered in the next section there exists a single self-negated product for inverse of even MV (see Table 2) that is not a linear combination. This property is to be expected (compare with a general MV form for \( n = 5 \) in Table 3), because there exists the well-known isomorphism between the even subalgebra of one vector space dimension larger algebra and the full lower dimensional Clifford algebra,

\[ Cl_{p,q+1}^+ \cong Cl_{p,q}, \quad Cl_{p+1,q}^+ \cong Cl_{q,p}. \]

Because we can write inverse MV as a single self-negated product for dimension \( n = 5 \) (see Table 3), it is not surprising that according to isomorphisms (3) we can do this for even subalgebra of dimension \( n = 6 \). Unfortunately, as we shall see in section 4 this property does not extend to the full six dimensional Clifford algebra.
4. Inverse of general MV in 6-dimensional quadratic space

4.1. Insufficiency of single self-negated product in \( n = 6 \) case. Let us take \( Cl_{6,0} \) algebra. Using Mathematica GA package [19], after some experimentation with a pair of self-negated product formed from a general MV it is not difficult to ascertain that we can eliminate simultaneously either grades 1, 2, 5 and 6 (then 0, 3 and 4 grades survive) or, alternatively, the grades 2, 3 and 6 (then the grades 0, 1, 4 and 5 survive). In both cases the grade 4 remains and, therefore, cannot be eliminated by the method of simple self-negated product used till now. This conclusion strictly follows from an attempt to simultaneously nullify all coefficients of grade-4 base elements using all \( 2^6 = 64 \) possible combinations of grade negations in the two term product. The grade-4 therefore is distinct from the other grades and deserves special examination. From experiments by computer it turns out that any self product of multivector \( B \) there exists a subalgebra with base formed by elements \[ \{1, e_{1256}, e_{1346}, e_{2345}\} \] such that any self product of multivector \( B = a_0 + a_{47}e_{1256} + a_{49}e_{1346} + a_{52}e_{2345} \) by grade-negated \( B \) will yield new MV having at least one grade-4 base element present. This is the reason why such a single self-product fails in eliminating grade-4 part and, as we shall see, one has to use a combination of at least a pair of self-products.

4.2. Linear combination of self-products. Before considering general case, we shall note that the determinant of a matrix representation (computed using symbolic coefficients) of restricted MV, which consists of just scalar and general grade-4 element, can be written as a square of some polynomial of MV coefficients \( s_4 \), i.e. as \( \det ([A]_0 + [A]_4) = s_8 = (s_4)^2 \). Thus, we assume that one can always extract the square root from \( s_8 \). From this follows that in search of inverse of \( [A]_0 + [A]_4 \), as a first step one can try to test only linear combination of product of four negated MVs instead of eight as required in general case. Due to above arguments we assume that square root of determinant norm may be written as a linear combination of the following form

\[
s_{4+4} = s_{4f} + s_{4g} = b_1 BF_5\left(f_4(B)f_3(f_2(B)f_1(B))\right) + b_2 BG_5\left(g_4(B)g_3(g_2(B)g_1(B))\right),
\]

where each of \( f_j \) and \( g_j \) is either the identity mapping or the grade-4 negation, and \( b_1, b_2 \) are unknown scalar coefficients of the linear combination.

Once a pattern of linear combination of square root of the determinant norm was fixed we can calculate explicit symbolic form of matrix representation of the above mentioned MV \( B = a_0 + a_{47}e_{1256} + a_{49}e_{1346} + a_{52}e_{2345} \), then compute the matrix determinant, take square root and compare it with the GA expression (4) after the same GA multivector \( B \) was inserted.

The negation functions \( f_j \) and \( g_j \) that control signs of grades can be modelled as a multiplication by unknown coefficient \( p_{4jk} \), where the index \( j \) denotes the involution number \( f_j, g_j \) in the self-product and \( k \) is the term number in linear combination (when \( n = 6 \), \( k = 1 \) for \( f \) and \( k = 2 \) for \( g \)). Later, when we shall deal with negations of other grades the first index \( i \) in \( p_{ijk} \) will indicate possibly negated grade-\( i \). For the moment the index \( i \) is fixed to 4, which corresponds to current nontrivial grade of \( B \) (we will ignore negations of scalar, because it is equivalent to negation of all other remaining MV grades). The coefficients \( p_{ijk} \) acquire values \( \pm 1 \) only, where \(-1\) means that involution which changes sign of grade \( i \) is to be applied, while \(+1\) means the identity map.
In the considered $n = 6$ case, comparison with the square root of the determinant of a matrix representation of $B$ yields the system of four equations for each of base element (including the scalar) of $B$. The system is too long to be fully presented here, therefore a small characteristic part of it is written down in a truncated form below,

$$\begin{cases}
  a_1^4 - 2a_0^2a_1^2 + b_2a_0^2a_1^2p_{412}p_{422} + <72 \text{ monomials}> = 0 \\
  b_1a_0^2a_{52} + 2b_2a_0^2a_{52}p_{412}p_{422}p_{432}p_{442}p_{452} + <72 \text{ monomials}> = 0 \\
  <74 \text{ monomials}> = 0 \\
  <74 \text{ monomials}> = 0
\end{cases}$$

(5)

We see that even for a simple $B$ which contains only 4 base elements (the scalar, $e_{1256}, e_{1346}$ and $e_{2345}$) the system is highly nonlinear in $p_{4ij}$. We can, however, try to substitute concrete values for $p_{4ij}$ one by one to get much simpler systems that contain only variables $b_1, b_2$ and $a_i$. Then we can try to solve each of simple systems separately with respect to $b_1, b_2$ for arbitrary coefficients $a_i$. Only few of them have solutions. In fact, we have solved the systems with Mathematica command `SolveAlways[]`. After testing all $2^{10} = 1024$ possible values of $p_{4ij}$ we have found two sets of solutions, $\{b_1 = -2/3, b_2 = -1/3, p_{411} = -1, p_{412} = 1, p_{421} = -1, p_{422} = 1, p_{431} = -1, p_{432} = -1, p_{441} = -1, p_{442} = 1, p_{451} = -1, p_{452} = 1\}$ and $\{b_1 = -1/3, b_2 = -2/3, p_{411} = 1, p_{412} = -1, p_{421} = 1, p_{422} = -1, p_{431} = -1, p_{432} = -1, p_{441} = 1, p_{442} = -1, p_{451} = 1, p_{452} = -1\}$. The obtained solution is unique up to the permutation of two terms. The common sign of coefficients $b_1$ and $b_2$ in general is not fixed as yet, because square root of determinant norm was calculated at this stage.

It is easy to check that the above solution (though computed from highly simplified MV, namely, the scalar plus three grade-4 base elements) works flawlessly for a more general MV (scalar plus any number of grade-4 elements) $C = a_0 + a_4e_{1234} + a_{43}e_{1235} + a_{44}e_{1236} + a_{45}e_{1245} + a_{46}e_{1246} + a_{47}e_{1256} + a_{48}e_{1345} + a_{49}e_{1346} + a_{50}e_{1356} + a_{51}e_{1456} + a_{52}e_{2345} + a_{53}e_{2346} + a_{54}e_{2356} + a_{55}e_{2456} + a_{56}e_{2456}$. This is what one expects, because grade involution done on the same grade elements acts in exactly identical way. Starting with a simple MV and then augmenting it till the number of solutions cannot be further decreased allows one to keep the whole calculation size manageable. This considerably speeds up search procedure for involutions and balances calculation complexity against the speed.

Once $f_j, g_j$ and coefficients $b_j$ in equations (5) and (4) have been determined, we can renew the search for other negation involutions in exactly the same way. Our next task is to double the number of multivectors in the products (4), because we know that determinant of general MV requires 8 multipliers in order to match the total degree polynomial of determinant of a MV matrix representation.

It was already mentioned that in the product $AA_{i,j,...}$ one can simultaneously eliminate either grades 1, 2, 5 and 6 (then grades 0, 3 and 4 remain), or, alternatively, the grades 2, 3 and 6 (then grades 0, 1, 4 and 5 survive). All in all there are $\frac{6!}{2!2!1!} + \frac{6!}{3!1!1!} + 1 = 36$ elements of grades 2, 3 and 6 which can be eliminated simultaneously. This is more than 28 elements of grades 1, 2, 5 and 6. Therefore, if we replace $B$ in equation (4) by self-negated product $AA_{2,3,6}$ we are left to deal with self-negated product of four multivectors of grades 0, 1, 4, 5 only, from which we can ignore the grades 0 and 4 for which $f_j$ and $g_j$ already have been established. Thus, repeating the same procedure we can find a number of valid solutions for coefficients $p_{1jk}, p_{2jk}$ and $p_{5jk}$. In order to speed up the derivation we had used
the multivector \( a_0 + a_1 e_1 + a_22 e_{123} + a_{47} e_{1256} \) first. The obtained result then was explicitly verified \( (i.e. \) proved by explicit expansion in orthogonal basis) using the most general \( Cl_{6,0} \) multivector with symbolic coefficients. Finally we found 320 valid forms of determinant norm. After removing all superfluous involutions a number of possible determinant forms was reduced to 16+4=20. All these forms are immediately written using equation (2). The results for inverse MVs for algebras are summarized in Table 3.

The denominators, which are real scalars, in these expressions are the determinant norms of respective MVs. At the same time they give the condition for existence of the inverse MV. It was found that determinant norms exactly match expressions for determinants calculated from matrix representation of respective MVs (in fact, the whole derivation of the determinant norm formulas relied on this match). All formulas were proved by explicit calculation of symbolic inverses of the most general MV for concrete Clifford algebras with all possible signatures \((p,q)\). For this task we wrote the Mathematica package for symbolic calculations in Clifford algebras [19], which also can be found in Mathematica package data base [http://packagedata.net/]. The specific property of our GA program is that it can simultaneously work with a number of Clifford algebras having different signatures \((p,q)\) what was a great help in computer-assisted verifications.

### 5. Classification and examples

In Table 4 we have collected all formulas obtained by the described method that represent the determinant norm at \(n = 6\) in the pattern [14]. All expressions for determinant norm naturally split into two types. The first four formulas at the

| \(Cl_{p,q}\) | \(A^{-1}\) |
|----------------|----------------|
| \(p + q = 0\) | \(B = 1\) |
| \(p + q = 1\) | \(B = 1\) |
| \(p + q = 2\) | \(C = A_{1,2}\) |
| \(p + q = 3\) | \(C = A_{1,2}\) |
| \(p + q = 4\) | \(D = A_{2,3}(AA_{2,3})_{1,4}\) or \(D = A_{1,2}(AA_{1,2})_{3,4}\) |
| \(p + q = 5\) | \(D = A_{2,3}(AA_{2,3})_{1,4}\) |
| \(p + q = 6\) | \(G = \frac{1}{3}A_{2,3,6} \left( H(HH_{1,5})_{1,5} + 2(H_{4}(H_{1,4})_{1,4})_{1,4} \right) \) or \(G = \frac{1}{4}A_{2,3,6} \left( (H(HH_{1,5})_{1,5} + 2(H_{4,5}(H_{1,4})_{1,4})_{1,4} \right) \) with \(H = AA_{2,3,6} = AA\) |

Table 3. Summary of formulas for inverse multivectors \(A^{-1}\) in Clifford algebras of dimension \(n \leq 6\).
beginning of Table 4 \((N = 38)\) constitute the first type or class. All four expressions from this class can be obtained by forming pairs from two sets

\[
\begin{align*}
\{ & \frac{1}{3} H(H(H H_{1,5})_4)_{1,5}, \quad \frac{1}{3} H H_{1,5}(H_{1,5} H)_4, \\
& \frac{2}{3} H(H_{1,5} H_{1,5} H_{1,5})_{4,5}, \quad \frac{2}{3} H H_{1,4,5}(H_{1,5} H_{1,5})_{4,5}\}
\]

in any combination. This gives the first four, \(2 \times 2 = 4\), formulas. The remaining 16 formulas, which constitute the second class can be obtained similarly by forming all possible pairs from sets

\[
\begin{align*}
\{ & \frac{1}{3} H H(H H)_{1,4,5}, \quad \frac{1}{3} H(H_{1,5} H_{1,5})_{4,5}, \quad \frac{1}{3} H H(H_{1,5} H_{1,5})_{4,5}, \\
& \frac{1}{3} H H(H_{1,5} H_{1,5})_{4,5}, \quad \frac{1}{3} H H_{1,5}(H_{1,5} H)_{4,5}, \quad \frac{2}{3} H(H_{1,5} H_{1,5} H_{1,5})_{4,5}\}
\]

\(\text{Table 4. The formulas for determinant norm of MV A for Clifford algebras of vector space dimension } n = 6. N \text{ counts the total number of negations in the norm (see Table 1 for explanation).}\)
The representatives of both classes were included in Table 3. In the first class only terms in a pair always coincide within the class but differ between different classes. Examples are pretty valid choices.

In the next step, however, we get zero since reversed forms.

The inverse MV formula can be constructed by taking any three expressions either from all weight coefficients being the same and equal to 1 or from sets either from Table 4 can also be rewritten as a sum of three different terms with formulas of Table 4.

By isotropic vector (e3, 1) from the left.

This yields remaining 4×4 = 16 formulas.

Both classes are well defined, because symbolic expressions for each of separate terms in a pair always coincide within the class but differ between different classes. The representatives of both classes were included in Table[3]. In the first class only grade-3 and grade-4 cancellation can occur between both terms in a pair. In the second class, the MV in each pair generally is made up of grades 1, 4 and 5, which cancel out in the final result (see Example 4). It is also interesting to note that formulas of Table[4] can also be rewritten as a sum of three different terms with all weight coefficients being the same and equal to 1/3 as shown in Table[5]. The inverse MV formula can be constructed by taking any three expressions either from sets S1, S2 and S3, or from sets T1, T2 and T3 listed in Table[5]. For example, the formulas

\[ \frac{1}{3} H(H_{45}(H_{14}H_{4}H_{4}))_{1,4,5}, \frac{1}{3} H(H_{45}(H_{14}H_{14}H_{14}))_{1,4,5}, \frac{1}{3} H(H_{145}(H_{145}H_{145}))_{1,4,5} \]

are pretty valid choices.

Generally the Table[5] allows to make 4^3 = 64 different triplets of weight n = 6 algebras without taking into account possible reversed forms.

**Example 3.** In Cl4,2 let’s take A = 2 + e1 + e5 − 2e15 + 3e26 + 3e1256 and find the inverse with formula from Table[5] AG = \( \frac{1}{3} HH(HH)_{1,4,5} + \frac{1}{3} HH(H_{14}H_{14})_{1,4,5} \), which has a minimal number of negations. We obtain \( H = AA' \) = 8e1 + 8e5. In the next step, however, we get zero since \( HH = 0 \) as well as \( H_{45}H_{1} = 0 \). In fact, the MV in the considered example was constructed multiplying the MV \( 1 + 2e_{1} + 3e_{126} \) by isotropic vector (e1, e5) from the left.

It is easy to check that the last mentioned MV \( A' = 1 + 2e_{1} + 3e_{126} \) is non-invertible as well. Indeed, \( H = A'A'_{123,4} = -4 + 4e_{1} \), then \( HH = 32 - 32e_{1} \), and
finally $HH(HH)_{1,1,5} = 0$. In a similar way $H_1(H_1H_4)_{1,1,5} = 0$. One can check that all formulas in Table 4 will yield the same result, namely, zero. In [20] one can find more information on MVs that contain isotropic multipliers.

**Example 4.** In $Cl_{1,5}$ let’s find inverse of $A = 2 + e_1 + 4e_4 + e_{15} + 3e_{126}$ using the same generic formula with minimal number of negations. Computation steps are:

1. $H = AA_{2,3,6} = -3 + 4e_1 + 16e_3 - 2e_5 - 24e_{1236}$,
2. $HH = -811 - 24e_1 - 96 e_3 + 12 e_5 + 144 e_{1236} - 96e_{1256}$,
3. $HH(HH)_{1,1,5} = 1/3(678025 + 27648e_1 + 2304e_{1236} + 18432e_{1256} - 4608e_{2356} + 3456e_{1256})$,
4. $(H_1H_4)_{1,1,5} = -811 + 24e_1 + 96e_3 - 12e_5 + 144e_{1236} - 96e_{1256}$,
5. $(H_1(H_1H_4)_{1,1,5})_1 = -2487 - 3316e_1 - 13264e_3 - 646e_5 + 19704e_{1236} - 1536e_{1256} + 384e_{2356} + 864e_{1256}$,
6. $\frac{1}{2}H((H_1H_4)_{1,1,5})_1 = 2/3(678025 - 13824e_1 - 1152e_{1236} - 9216e_{1256} + 2304e_{2356} - 1728e_{1256})$.

From 3) and 5) we get that determinant norm of $A$ is equal to 678025. Doing calculations in a similar way we find the inverse:

$$A^{-1} = \frac{1}{678025} (44766 - 9765e_1 - 95588e_3 + 1841e_{15} + 8412e_{26} - 5176e_{35} - 71355e_{126} - 12112e_{135} + 20568e_{236} - 1554e_{1236} + 1960e_{2356} - 7488e_{1256} + 47152e_{456} + 21336e_{2356} - 4032e_{1256}) = \frac{1}{678025} (44766 - 9765e_1 - 95588e_3 + 1841e_{15} + 8412e_{26} - 1720e_{35} - 71355e_{126} - 12112e_{135} + 19416e_{236} - 6162e_{1236} + 20760e_{2356} - 5184e_{1256})$$

If different formula from Table 4 is employed in finding the inverse, for example, using the first line $\frac{1}{2}H(H(HH)_{1,1,5})_{1,1,5} + \frac{1}{3}H(H_1H_4)_{1,1,5})_{1,1,5}$, then intermediate results will be different. In particular, for the first term $\frac{1}{2}H(H(HH)_{1,1,5})_{1,1,5}$ we find 678025/3, and for the second term $\frac{1}{2}H(H(H_1H_4))_{1,1,5})_{1,1,5}$ we find 1356050/3. Thus, in this case no cancellation between nonzero grades of both terms in the pair occurs.

**Example 5.** The example shows that, in principle, one can use $n = 6$ formula to find the determinant norm and inverse MVs of all smaller algebras, $n < 6$, as well. At first sight this may be a bit unexpected. The following example explains how does it happen. Let’s take $Cl_{2,2}$ MV, $A = 45 + 55e_1 + 84e_{12} + 39e_{134} + 93e_{234} + 15e_{1234}$, and use the determinant norm formula of $n = 6$ instead of $n = 4$:

$$AG = \frac{1}{2}H(H(HH)_{1,1,5} + \frac{1}{3}H(H_1H_1)_{1,1,5})_{1,1,5} = 6716445910339801/3, \quad (H_1H_4)_{1,1,5} = 6716445910339801/3, \quad AG = 6716445910339801/3 = (259164901)^2,$$

where $\sqrt{AG} = 259164901$ is exactly the determinant norm of the MV calculated by $n = 4$ formula. So, calculation using the $n = 6$ norm formula yields the squared determinant of matrix representation of $Cl_{2,2}$ multivector. It is easy then to check that the $n = 6$ formula yields the inverse of $Cl_{2,2}$ MV as well. Indeed,
7) \([HH]_{1,4,5}+2(H(H(H_4H_4))_{1,4,5})_1=17494408312203−6017809001220e_1+8093719858230e_2+6904152962640e_{1234}\)

8) Finally, the inverse of \(Cl_{2,2}\) MV calculated using \(n=6\) formula, is

\[A^{-1} = \frac{1}{259164901}(559831173421635−630567641500575e_1
+127983403060830e_2−1024375706022402e_{12}+61927453093950e_{34}
−56087612602467e_{134}−115698425071379e_{234}−30220830582585e_{1234})\]

which does coincide with the inverse MV of \(Cl_{2,2}\) computed using the inverse formula for \(n=4\).

And this is not a coincidence. Explicit symbolic computations confirm that the \(n=6\) determinant norm formula when is applied to the MV of any algebra with vector space dimension \(n<6\) indeed yields either the determinant of matrix representation of MV (for \(n=5\)), or the determinant raised in power of 2 for algebras with the vector space dimension \(n=3\) and \(n=4\), or in power of 4 for \(n=2\) and \(n=1\). Exactly the same happens, when formula of determinant norm for \(n=5\) is applied to \(n=4,3,2\) and \(n=1\) cases. Similarly, when the determinant norm formula for \(n=4\) is used to \(n=3\) MV, we obtain the determinant of matrix representation of the MV, whereas the same procedure for \(n=2\) and \(n=1\) yields the determinant raised in power of 2, etc. So, the determinant norms disclose very interesting onion-like structure, when each higher dimensional formula embraces all lower dimensional ones. We think that this property may be important in further investigations.

6. Conclusions

From this computer-aided study we conjecture that the inverse of a general MV of arbitrary Clifford algebra can be always expressed as a linear combination of a proper number of specially constructed self-negated products, where the initial MV may stay in the left-most or right-most position. For the first time we have explicitly derived compact formulas for inverse MV that consist of sums of two grade-negated products, Tables 3 and 4, for all Clifford algebras of dimension \(n=6\). The formulas are independent of a particular algebra signature \((p, q)\) and may be useful in numerical and especially in symbolic programming. They reduce to known expressions of inverses for vectors, bivectors etc, or their combinations.

We have also presented different formulas for inverse of even Clifford subalgebras for dimensions \(n \leq 6\) (Table 2), which are important for spinor algebras. A number of previously unknown explicit expressions were provided for \(n=5\) dimensional algebras, Table 4, some of which may be interesting from computational point of view as well. We believe, that formulas in Table 4 exhaust all possible single term forms of writing determinant for \(n=5\) algebras using geometric product and involutions only. We have shown that determinant norm formulas for \(n=6\) split into two classes, Tables 4 and 5. We also presented inverses of \(n=6\) in a more symmetric three term linear combination with all weights equal to 1/3 (Table 5). Though this form is not preferred from computational point of view it may be interesting when searching similar formulas for higher dimension Clifford algebras. The computation of inverse MV by formulas in Table 3 requires considerably smaller
number of multiplications since many pieces in the formulas are iterated many times. The speed of calculation can be improved further if MV multiplication algorithm is realized in such a way, that it can skip calculation of some specific grades, because we know in advance that some of grades should vanish from the product. In this case a large number of multiplications can be avoided.

The number \( m \) of multipliers in a self-negated product of the determinant norm is determined by total degree of polynomial of representation of matrix in 8-periodicity table \([18]\). For algebra with \( n = 2 \) the determinant norm consists just of two multipliers, whereas for \( n = 3 \) as well as for \( n = 4 \) one has to make products of four MV multipliers in accord with dimension of a respective matrix in 8-periodicity table. The Clifford algebras of vector space dimensions 5 or 6, require eight multivectors to construct the determinant norm. Then, looking at 8-periodicity table one may expect that the determinant norm of algebras of vector space dimension 7 and 8 can be constructed by multilinear combinations from products of 16 multivectors, etc.

From the considerations above it should be clear that the complexity of finding explicit expression of general MV inverse grows rapidly with the increase of algebra dimension. For example, a direct head-on symbolic multiplication of eight general arbitrary multivectors in \( n = 6 \) dimensional vector space in the worst case requires \((2^6)^8 = 281474976710656\) geometric multiplications of base elements, which fortunately can be bypassed due to significant grade reduction in the determinant formula as discussed in this article. However, in practice it does not help much when searching for correct shape of determinant norm formula. This is why reduced (shorter) MVs are always worth to try first which, under good choice, may greatly reduce the number of candidates for possible linear combinations. It should be stressed that the determinant norms and inverse multivectors obtained in the paper make computations self-contained within the geometric product and negation structure without any need to resort to matrix representations.

Finally, we demonstrated that \( n = 6 \) formula alone can generate MV determinant norms and respective inverses of all algebras of vector space dimension \( n \leq 6 \) and of any signature, albeit using more geometric multiplications. The identified onion-like structure of determinant norms and numerous shapes of inverse multivectors presented in this article incline us to believe that similar formulas (which include only operations of grade negation, geometric product and summation) may exist for Clifford algebras of all dimensions.

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