Singular Stochastic Equations on Hilbert Spaces: Harnack Inequalities for their Transition Semigroups

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Abstract

We consider stochastic equations in Hilbert spaces with singular drift in the framework of [7]. We prove a Harnack inequality (in the sense of [13]) for its transition semigroup and exploit its consequences.

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In particular, we prove regularizing and ultraboundedness properties of the transition semigroup as well as that the corresponding Kolmogorov operator has at most one infinitesimally invariant measure \( \mu \) (satisfying some mild integrability conditions). Finally, we prove existence of such a measure \( \mu \) for non-continuous drifts.

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1 Introduction, framework and main results

In this paper we continue our study of stochastic equations in Hilbert spaces with singular drift through its associated Kolmogorov equations started in [7]. The main aim is to prove a Harnack inequality for its transition semigroup in the sense of [18] (see also [1, 16, 19] for further development) and exploit its consequences. See also [14] for an improvement of the main results in [16] concerning generalized Mehler semigroups. To describe our results more precisely, let us first recall the framework from [7].

Consider the stochastic equation

\[
\begin{cases}
    dX(t) = (AX(t) + F(X(t)))dt + \sigma dW(t) \\
    X(0) = x \in H.
\end{cases}
\]

Here \( H \) is a real separable Hilbert space with inner product \( \langle \cdot, \cdot \rangle \) and norm \( |\cdot| \), \( W = W(t), \ t \geq 0, \) is a cylindrical Brownian motion on \( H \) defined on a stochastic basis \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) and the coefficients satisfy the following hypotheses:

(H1) \( (A, D(A)) \) is the generator of a \( C_0 \)-semigroup, \( T_t = e^{tA}, \ t \geq 0, \) on \( H \) and for some \( \omega \in \mathbb{R} \)

\[
\langle Ax, x \rangle \leq \omega |x|^2, \quad \forall x \in D(A).
\]  

(H2) \( \sigma \in L(H) \) (the space of all bounded linear operators on \( H \)) such that \( \sigma \) is positive definite, self-adjoint and
(i) \( \int_0^\infty (1 + t^{-\alpha}) \| T_t \sigma \|^2_{HS} dt < \infty \) for some \( \alpha > 0 \), where \( \| \cdot \|_{HS} \) denotes the norm on the space of all Hilbert–Schmidt operators on \( H \).

(ii) \( \sigma^{-1} \in L(H) \).

(H3) \( F : D(F) \subset H \to 2^H \) is an \( m \)-dissipative map, i.e.,

\[
    \langle u - v, x - y \rangle \leq 0, \quad \forall \ x, y \in D(F), \ u \in F(x), \ v \in F(y),
\]

(“dissipativity”) and

\[
    \text{Range (} I - F \text{)} := \bigcup_{x \in D(F)} (x - F(x)) = H.
\]

Furthermore, \( F_0(x) \in F(x), \ x \in D(F) \), is such that

\[
    |F_0(x)| = \min_{y \in F(x)} |y|.
\]

Here we recall that for \( F \) as in (H3) we have that \( F(x) \) is closed, non empty and convex.

The corresponding Kolmogorov operator is then given as follows: Let \( \mathcal{E}_A(H) \) denote the linear span of all real parts of functions of the form \( \varphi = e^{i \langle h, \cdot \rangle}, \ h \in D(A^*) \), where \( A^* \) denotes the adjoint operator of \( A \), and define for any \( x \in D(F) \),

\[
    L_0 \varphi(x) = \frac{1}{2} \text{Tr} (\sigma^2 D^2 \varphi(x)) + \langle x, A^* D \varphi(x) \rangle + \langle F_0(x), D \varphi(x) \rangle, \quad \varphi \in \mathcal{E}_A(H).
\]

(1.3)

Additionally, we assume:

(H4) There exists a probability measure \( \mu \) on \( H \) (equipped with its Borel \( \sigma \)-algebra \( \mathcal{B}(H) \)) such that

(i) \( \mu(D(F)) = 1 \),

(ii) \( \int_H (1 + |x|^2)(1 + |F_0(x)|) \mu(dx) < \infty \),

(iii) \( \int_H L_0 \varphi d\mu = 0 \) for all \( \varphi \in \mathcal{E}_A(H) \).
Remark 1.1 (i) A measure for which the last equality in (H4) (makes sense and) holds is called *infinitesimally invariant* for \((L_0, \mathcal{E}_A(H))\).

(ii) Since \(\omega\) in (1.2) is an arbitrary real number we can relax (H3) by allowing that for some \(c \in (0, \infty)\)
\[
\langle u - v, x - y \rangle \leq c|x - y|^2, \quad \forall \ x, y \in D(F), \ u \in F(x), \ v \in F(y).
\]
We simply replace \(F\) by \(F - c\) and \(A\) by \(A + c\) to reduce this case to (H3).

(iii) At this point we would like to stress that under the above assumptions (H1)-(H4) (and (H5) below) because \(F_0\) is merely measurable and \(\sigma\) is not Hilbert-Schmidt, it is unknown whether (1.1) has a strong solution.

(iv) Similarly as in [7] (see [7, Remark 4.4] in particular) we expect that (H2)(ii) can be relaxed to the condition that \(\sigma = (-A)^{-\gamma}\) for some \(\gamma \in [0, 1/2]\). However, some of the approximation arguments below become more involved. So, for simplicity we assume (H2)(ii).

The following are the main results of [7] which we shall use below.

**Theorem 1.2 (cf. [6, Theorem 2.3 and Corollary 2.5])** Assume (H1), (H2)(i), (H3) and (H4). Then for any measure \(\mu\) as in (H4) the operator \((L_0, \mathcal{E}_A(H))\) is dissipative on \(L^1(H, \mu)\), hence closable. Its closure \((L_{\mu}, D(L_{\mu}))\) generates a \(C_0\)-semigroup \(P_{\mu}^t\), \(t \geq 0\), on \(L^1(H, \mu)\) which is Markovian, i.e., \(P_{\mu}^t 1 = 1\) and \(P_{\mu}^t f \geq 0\) for all nonnegative \(f \in L^1(H, \mu)\) and all \(t > 0\). Furthermore, \(\mu\) is \(P_{\mu}^t\)-invariant, i.e.,
\[
\int_H P_{\mu}^t f \, d\mu = \int_H f \, d\mu, \quad \forall \ f \in L^1(H, \mu).
\]
Below \(B_b(H), C_b(H)\) denote the bounded Borel-measurable, continuous functions respectively from \(H\) into \(\mathbb{R}\) and \(\| \cdot \|\) denotes the usual norm on \(L(H)\).

**Theorem 1.3 (cf. [6, Proposition 5.7])** Assume (H1)-(H4) hold. Then for any measure \(\mu\) as in (H4) and \(H_0 := \text{supp} \ \mu\) (:=largest closed set of \(H\) whose complement is a \(\mu\)-zero set) there exists a semigroup \(p_{\mu}^t(x, dy), \ x \in H_0, \ t > 0\), of kernels such that \(p_{\mu}^t f\) is a \(\mu\)-version of \(P_{\mu}^t f\) for all \(f \in B_b(H), \ t > 0\), where as usual
\[
p_{\mu}^t f(x) = \int_H f(y)p_{\mu}^t(x, dy), \quad x \in H_0.
\]
Furthermore, for all \( f \in B_b(H) \), \( t > 0 \), \( x, y \in H_0 \)

\[
|p^\mu_t f(x) - p^\mu_t f(y)| \leq \frac{e^{\omega|t|}}{\sqrt{t \wedge 1}} \|f\|_0 \sigma^{-1} \|x - y\| \tag{1.4}
\]

and for all \( f \in \text{Lip}_b(H) \) (:= all bounded Lipschitz functions on \( H \))

\[
|p^\mu_t f(x) - p^\mu_t f(y)| \leq e^{\omega|t|} \|f\|_{\text{Lip}} |x - y|, \quad \forall \ t > 0, \ x, y \in H_0, \tag{1.5}
\]

and

\[
\lim_{t \to 0} p^\mu_t f(x) = f(x), \quad \forall \ x \in H_0. \tag{1.6}
\]

(Here \( \|f\|_0, \|f\|_{\text{Lip}} \) denote the supremum, Lipschitz norm of \( f \) respectively.)

Finally, \( \mu \) is \( p^\mu_t \)-invariant.

**Remark 1.4**

(i) Both results above have been proved in [7] on \( L^2(H, \mu) \) rather than on \( L^1(H, \mu) \), but the proofs for \( L^1(H, \mu) \) are entirely analogous.

(ii) In [7] we assume \( \omega \) in (H1) to be negative, getting a stronger estimate than \( (1.4) \) (cf. [7] (5.11)). But the same proof as in [7] leads to \( (1.4) \) for arbitrary \( \omega \in \mathbb{R} \) (cf. the proof of [7] Proposition 4.3 for \( t \in [0, 1] \)). Then by virtue of the semigroup property and since \( p^\mu_t \) is Markov we get \( (1.4) \) for all \( t > 0 \).

(iii) Theorem 1.3 holds in more general situations since (H2)(ii) can be relaxed (cf. [7] Remark 4.4] and [5] Proposition 8.3.3).

(iv) \( (1.4) \) above implies that \( p^\mu_t, t > 0, \) is strongly Feller, i.e., \( p^\mu_t(B_0(H)) \subset C(H_0) \) (=all continuous functions on \( H_0 \)). We shall prove below that under the additional condition (H5) we even have \( p^\mu_t(L^p(H, \mu)) \subset C(H_0) \) for all \( p > 1 \) and that \( \mu \) in (H4) is unique. However, so far we have not been able to prove that for this unique \( \mu \) we have \( \text{supp} \ \mu = H \), though we conjecture that this is true. □

For the results on Harnack inequalities, in this paper we need one more condition.

\[(H5)\] (i) \( (1 + \omega - A, D(A)) \) satisfies the weak sector condition (cf. e.g. [12]), i.e., there exists a constant \( K > 0 \) such that

\[
\langle (1 + \omega - A)x, y \rangle \leq K \|(1 + \omega - A)x\|^2 / 2 \langle (1 + \omega - A)y, y \rangle ^{1/2}, \quad \forall \ x, y \in D(A). \tag{1.7}
\]
There exists a sequence of $A$-invariant finite dimensional subspaces $H_n \subset D(A)$ such that $\bigcup_{n=1}^{\infty} H_n$ is dense in $H$.

We note that if $A$ is self-adjoint, then (H2) implies that $A$ has a discrete spectrum which in turn implies that (H5)(ii) holds.

**Remark 1.5** Let $(A, D(A))$ satisfy (H1). Then the following is well known:

(i) (H5) (i) is equivalent to the fact that the semigroup generated by $(1 + \omega - A, D(A))$ on the complexification $H_C$ of $H$ is a holomorphic contraction semigroup on $H_C$ (cf. e.g. [12, Chapter I, Corollary 2.21]).

(ii) (H5) (i) is equivalent to $(1 + \omega - A, D(A))$ being variational. Indeed, let $(\mathcal{E}, D(\mathcal{E}))$ be the coercive closed form generated by $(1 + \omega - A, D(A))$ (cf. [12, Chapter I, Section 2]) and $(\tilde{\mathcal{E}}, D(\tilde{\mathcal{E}}))$ be its symmetric part. Then define

$$ V := D(\mathcal{E}) \text{ with inner product } \tilde{\mathcal{E}} \text{ and } V^* \text{ to be its dual.} \quad (1.8) $$

Then

$$ V \subset H \subset V^* \quad (1.9) $$

and $1 + \omega - A : D(A) \to H$ has a natural unique continuous extension from $V$ to $V^*$ satisfying all the required properties (cf. [12, Chapter I, Section 2, in particular Remark 2.5]).

Now we can formulate the main result of this paper, namely the Harnack inequality for $p_t^\mu$, $t > 0$.

**Theorem 1.6** Suppose (H1) – (H5) hold and let $\mu$ be any measure as in (H4) and $p_t^\mu(x, dy)$ as in Theorem 1.5 above. Let $p \in (1, \infty)$. Then for all $f \in B_0(H)$, $f \geq 0$,

$$ (p_t^\mu f(x))^p \leq p_t^\mu f^p(y) \exp \left[ \|\sigma^{-1}\|^2 \frac{p\omega|x - y|^2}{(p - 1)(1 - e^{-2\omega t})} \right], \quad t > 0, \ x, y \in H_0. \quad (1.10) $$

As consequences in the situation of Theorem 1.6 (i.e. assuming (H1)-(H5)) we obtain:

**Corollary 1.7** For all $t > 0$ and $p \in (1, \infty)$

$$ p_t^\mu(L^p(H, \mu)) \subset C(H_0). $$
Corollary 1.8 \( \mu \) in \((H4)\) is unique.

Because of this result below we write \( p_t(x, dy) \) instead of \( p'_t(x, dy) \).

Finally, we have

Corollary 1.9 (i) For every \( x \in H_0 \), \( p_t(x, dy) \) has a density \( \rho_t(x, y) \) with respect to \( \mu \) and

\[
\| \rho_t(x, \cdot) \|_{L_p}^{p/(p-1)} \leq \frac{1}{\int_H \exp \left[ -\|\sigma^{-1}\|^2 \frac{p\|x-y\|^2}{(1-e^{-2\omega})} \right] \mu(dy)}, \quad x \in H_0, \ p \in (1, \infty).
\]

(ii) If \( \mu(e^{\lambda |\cdot|^2}) < \infty \) for some \( \lambda > 2(\omega \wedge 0)^2 \|\sigma^{-1}\|^2 \), then \( p_t \) is hyperbounded, i.e. \( \| p_t \|_{L^2(H, \mu) \to L^3(H, \mu)} < \infty \) for some \( t > 0 \).

Corollary 1.10 For simplicity, let \( \sigma = I \) and instead of \((H1)\) assume that more strongly \((A, D(A))\) is self-adjoint satisfying \[(1.2)\]. We furthermore assume that \( |F_0| \in L^2(H, \mu) \).

(i) There exists \( M \in \mathcal{B}(H_0) \), \( M \subset D(F) \), \( \mu(M) = 1 \) such that for every \( x \in M \) equation \[(1.11)\] has a pointwise unique continuous strong solution (in the mild sense see \[(4.11)\] below), such that \( X(t) \in M \) for all \( t \geq 0 \) \( \mathbb{P}\)-a.s..

(ii) Suppose there exists \( \Phi \in C([0, \infty)) \) positive and strictly increasing such that \( \lim_{s \to \infty} s^{-1}\Phi(s) = \infty \) and

\[
\Psi(s) := \int_s^\infty \frac{dr}{\Phi(r)} < \infty, \quad \forall \ s > 0.
\]

If there exists a constant \( c > 0 \) such that

\[
\langle F_0(x) - F_0(y), x - y \rangle \leq c - \Phi(|x-y|^2), \quad \forall \ x, y \in D(F),
\]

then \( p_t \) is ultrabounded with

\[
\| p_t \|_{L^2(H, \mu) \to L^\infty(H, \mu)} \leq \exp \left[ \frac{\lambda(1 + \Psi^{-1}(t/4))}{(1 - e^{-\omega t/2})^2} \right], \quad t > 0,
\]

holding for some constant \( \lambda > 0 \).

Remark 1.11 We emphasize that since the nonlinear part \( F_0 \) of our Kolmogorov operator is in general not continuous, it was quite surprising for us that in this infinite dimensional case nevertheless the generated semigroup \( P_t \) maps \( L^1 \)-functions to continuous ones as stated in Corollary \[(1.7)\].
The proof that Corollary 1.9 follows from Theorem 1.6 is completely standard. So, we will omit the proofs and instead refer to [10], [19].

Corollary 1.7 is new and follows whenever a semigroup \( p_t \) satisfies the Harnack inequality (see Proposition 1.1 below).

Corollary 1.8 is new. Since (1.10) implies irreducibility of \( p_t^\mu \) and Corollary 1.7 implies that it is strongly Feller, a well known theorem due to Doob immediately implies that \( \mu \) is the unique invariant measure for \( p_t^\mu \), \( t > 0 \). \( p_t^\mu \), however, depends on \( \mu \), so Corollary 1.8 is a stronger statement. Corollary 1.10 is also new.

Theorem 1.6 as well as Corollaries 1.7, 1.8 and 1.10 will be proved in Section 4. In Section 3 we first prove Theorem 1.6 in case \( F_0 \) is Lipschitz, and in Section 2 we prepare the tools that allow us to reduce the general case to the Lipschitz case. In Section 5 we prove two results (see Theorems 5.2 and 5.4) on the existence of a measure satisfying (H4) under some additional conditions and present an application to an example where \( F_0 \) is not continuous. For a discussion of a number of other explicit examples satisfying our conditions see [7, Section 9].

2 Reduction to regular \( F_0 \)

Let \( F \) be as in (H3). As in [7] we may consider the Yosida approximation of \( F \), i.e., for any \( \alpha > 0 \) we set

\[
F_\alpha(x) := \frac{1}{\alpha} (J_\alpha(x) - x), \quad x \in H,
\]

where for \( x \in H \)

\[
J_\alpha(x) := (I - \alpha F)^{-1}(x), \quad \alpha > 0,
\]

and \( I(x) := x \). Then each \( F_\alpha \) is single valued, dissipative and it is well known that

\[
\lim_{\alpha \to 0} F_\alpha(x) = F_0(x), \quad \forall x \in D(F), \tag{2.2}
\]

\[
|F_\alpha(x)| \leq |F_0(x)|, \quad \forall x \in D(F). \tag{2.3}
\]

Moreover, \( F_\alpha \) is Lipschitz continuous, so \( F_0 \) is \( \mathcal{B}(H) \)-measurable. Since \( F_\alpha \) is not differentiable in general, as in [7] we introduce a further regularization by setting

\[
F_{\alpha, \beta}(x) := \int_H e^{\beta y} F_\alpha(e^{\beta B} x + y) B^{-1}_{\beta B - 1}(dy), \quad \alpha, \beta > 0, \tag{2.4}
\]
where $B : D(B) \subset H \to H$ is a self-adjoint, negative definite linear operator such that $B^{-1}$ is of trace class and as usual for a trace class operator $Q$ the measure $N_Q$ is just the standard centered Gaussian measure with covariance given by $Q$.

$F_{\alpha,\beta}$ is dissipative, of class $C^\infty$, has bounded derivatives of all the orders and $F_{\alpha,\beta} \to F_\alpha$ pointwise as $\beta \to 0$.

Furthermore, for $\alpha > 0$

$$c_\alpha := \sup \left\{ \frac{|F_{\alpha,\beta}(x)|}{1 + |x|} : x \in H, \beta \in (0, 1] \right\} < \infty. \quad (2.5)$$

We refer to [10, Theorem 9.19] for details.

Now we consider the following regularized stochastic equation

$$\begin{cases}
    dX_{\alpha,\beta}(t) = (AX_{\alpha,\beta}(t) + F_{\alpha,\beta}(X_{\alpha,\beta}(t)))dt + \sigma dW(t) \\
    X_{\alpha,\beta}(0) = x \in H.
\end{cases} \quad (2.6)$$

It is well known that (2.6) has a unique mild solution $X_{\alpha,\beta}(t, x)$, $t \geq 0$. Its associated transition semigroup is given by

$$P_t^{\alpha,\beta}f(x) = \mathbb{E}[f(X_{\alpha,\beta}(t, x))], \quad t > 0, \quad x \in H,$$

for any $f \in B_b(H)$. Here $\mathbb{E}$ denotes expectation with respect to $\mathbb{P}$.

**Proposition 2.1** Assume (H1) – (H4). Then there exists a $K_\sigma$-set $K \subset H$ such that $\mu(K) = 1$ and for all $f \in B_b(H)$, $T > 0$ there exist subsequences $(\alpha_n)$, $(\beta_n)$ → 0 such that for all $x \in K$

$$\lim_{n \to \infty} \lim_{m \to \infty} P_{t}^{\alpha_n,\beta_m}f(x) = \mu^*f(x) \quad \text{weakly in } L^2(0, T; dt). \quad (2.7)$$

**Proof.** This follows immediately from the proof of [7, Proposition 5.7]. (A closer look at the proof even shows that (2.7) holds for all $x \in H_0 = \text{supp } \mu$.) \&

As we shall see in Section 4, the proof of Theorem 1.6 follows from Proposition 2.1 if we can prove the corresponding Harnack inequality for each $P_t^{\alpha,\beta}$. Hence in the next section we confine ourselves to the case when $F_0$ is dissipative and Lipschitz.

### 3 The Lipschitz case

In this section we assume that (H1)-(H3) and (H5) hold and that $F_0$ in (H3) is in addition Lipschitz continuous. The aim of this section is to prove Theorem
We first note that since $F_0$ is Lipschitz (1.1) has a unique mild solution $X(t, x)$, $t \geq 0$, for every initial condition $x \in H$ (cf. [10]) and we denote the corresponding transition semigroup by $P_t$, $t > 0$, i.e.

$$P_t f(x) := \mathbb{E}[f(X(t, x))], \quad t > 0, \ x \in X,$$

where $f \in B_b(H)$.

Now we need to consider an appropriate Galerkin approximation. To this end let $e_k \in D(A)$, $k \in \mathbb{N}$, be orthonormal such that $H_n = \text{linear span} \{e_1, ..., e_n\}$, $n \in \mathbb{N}$. Hence $\{e_k : k \in \mathbb{N}\}$ is an orthonormal basis of $(H, \langle \cdot, \cdot \rangle)$. Let $\pi_n : H \to H_n$ be the orthogonal projection with respect to $(H, \langle \cdot, \cdot \rangle)$. So, we can define

$$A_n := \pi_n A_{|H_n} (= A_{|H_n} \text{ by (H5)(ii)})$$

and, furthermore

$$F_n := \pi_n F_{0|H_n}, \quad \sigma_n := \pi_n \sigma_{|H_n}.$$  

Obviously, $\sigma_n : H_n \to H_n$ is a self-adjoint, positive definite linear operator on $H_n$. Furthermore, $\sigma_n$ is bijective, since it is one-to-one. To see the latter, one simply picks an orthonormal basis $\{e_\sigma^1, ..., e_\sigma^n\}$ of $H_n$ with respect to the inner product $\langle \cdot, \cdot \rangle_\sigma$ defined by $\langle x, y \rangle_\sigma := \langle \sigma x, y \rangle$. Then if $x \in H_n$ is such that $\sigma_n x = \pi_n \sigma x = 0$, it follows that

$$\langle x, e_\sigma^i \rangle_\sigma = \langle \sigma x, e_\sigma^i \rangle_\sigma = 0, \quad \forall \ 1 \leq i \leq n.$$  

But $x = \sum_{i=1}^n \langle x, e_\sigma^i \rangle_\sigma e_\sigma^i$, hence $x = 0$.

Now fix $n \in \mathbb{N}$ and on $H_n$ consider the stochastic equation

$$\begin{cases}
  dX_n(t) = (A_n X_n(t) + F_n(X_n(t)))dt + \sigma_n dW_n(t) \\
  X_n(0) = x \in H_n,
\end{cases}$$

where $W_n(t) = \pi_n W(t) = \sum_{i=1}^n \langle e_k, W(t) \rangle e_k$.

(3.2) has a unique strong solution $X_n(t, x)$, $t \geq 0$, for every initial condition $x \in H_n$ which is pathwise continuous $\mathbb{P}$-a.s.. Consider the associated transition semigroup defined as before by

$$P_t^n f(x) = \mathbb{E}[f(X_n(t, x))], \quad t > 0, \ x \in H_n,$$

where $f \in B_b(H_n)$.

Below we shall prove the following:
Proposition 3.1 Assume that \((H1) - (H5)\) hold. Then:

(i) For all \(f \in C_b(H)\) and all \(t > 0\)

\[
\lim_{n \to \infty} P_t^n f(x) = P_t f(x), \quad \forall x \in H_{n_0}, \quad n_0 \in \mathbb{N}.
\]

(ii) For all nonnegative \(f \in B_b(H)\) and all \(n \in \mathbb{N}, p \in (1, \infty)\)

\[
(P_t^n f(x))^p \leq P_t^n f^p(y) \exp \left[ \frac{p \sigma |x - y|^2}{(p - 1)(1 - e^{-2\omega})} \right], \quad t > 0, \quad x, y \in H_n.
\]

Proof. (i): Define

\[
W_{A,\sigma}(t) := \int_0^t e^{(t-s)A} \sigma dW(s), \quad t \geq 0.
\]

Note that by \((H2)(i)\) we have that \(W_{A,\sigma}(t), \ t \geq 0,\) is well defined and pathwise continuous. For \(x \in H_{n_0}, \ n_0 \in \mathbb{N}\) fixed, let \(Z(t), \ t \geq 0,\) be the unique variational solution (with triple \(V \subset H \subset V^*\) as in Remark 1.5(ii), see e.g. [15]) to

\[
\begin{cases}
  dZ(t) = [AZ(t) + F_0(Z(t) + W_{A,\sigma}(t))]dt \\
  Z(0) = x,
\end{cases}
\]

which then automatically satisfies

\[
\mathbb{E} \sup_{t \in [0,T]} |Z(t)|^2 < +\infty.
\]

Then we have (see [10]) that \(Z(t) + W_{A,\sigma}(t), \ t \geq 0,\) is a mild solution to \((1.1)\) (with \(F_0\) Lipschitz), hence by uniqueness

\[
X(t, x) = Z(t) + W_{A,\sigma}(t), \quad t \geq 0.
\]

Clearly, since

\[
\mathbb{E} \sup_{t \in [0,T]} |W_{A,\sigma}(t)|^2 < +\infty,
\]

we have

\[
\pi_n W_{A,\sigma}(t) \to W_{A,\sigma}(t) \quad \text{as} \quad n \to \infty \quad \text{in} \quad L^2(\Omega, \mathcal{F}, \mathbb{P}), \quad \forall t \geq 0.
\]

We set \(X_n(t) := X_n(t, x) \ (= \text{solution of (3.2)})\). Defining

\[
W_{A_n,\sigma_n}(t) = \int_0^t e^{(t-s)A_n} \sigma_n dW_n(t), \quad t \geq 0,
\]

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and
\[ Z_n(t) := X_n(t) - W_{A_n,\sigma_n}(t), \quad n \in \mathbb{N}, \ t \geq 0, \]
it is enough to show that
\[ Z_n(t) \rightarrow Z(t) \quad \text{as} \ n \rightarrow \infty \quad \text{in} \ L^2(\Omega, \mathcal{F}, \mathbb{P}), \ \forall \ t \geq 0, \tag{3.9} \]
because then by (3.7)
\[ X_n(t) \rightarrow X(t) \quad \text{as} \ n \rightarrow \infty \quad \text{in} \ L^2(\Omega, \mathcal{F}, \mathbb{P}), \ \forall \ t \geq 0, \]
and the assertion follows by Lebesgue’s dominated convergence theorem. To show (3.9) we first note that by the same argument as above
\[ dZ_n(t) = [A_n Z_n(t) + F_n(Z_n(t) + W_{A_n,\sigma_n}(t))]dt \]
and thus (in the variational sense), since \( A = A_n \) on \( H_n \) by (3.1)
\[ d(Z(t) - Z_n(t)) = [A(Z(t) - Z_n(t)) + F_0(X(t)) - F_n(X_n(t))]dt. \]
Applying Itô’s formula we obtain that for some constant \( c > 0 \)
\[ \frac{1}{2} |Z(t) - Z_n(t)|^2 \leq \int_0^t \left[ (\omega + 1/2)|Z(s) - Z_n(s)|^2 \\
+ |F_0(X(s)) - F_0(X_n(s))|^2 + |(1 - \pi_n)F_0(X(s))|^2 \right] ds \\
\leq c \int_0^t |Z(s) - Z_n(s)|^2 ds + c \int_0^t |W_{A,\sigma}(s) - W_{A_n,\sigma_n}(s)|^2 ds \\
+ \int_0^t |(1 - \pi_n)F_0(X(s))|^2 ds. \]
Now (3.9) follows by the linear growth of \( F_0 \), (3.6)-(3.8) and Gronwall’s lemma, if we can show that
\[ \int_0^T \mathbb{E}|W_{A,\sigma}(s) - W_{A_n,\sigma_n}(s)|^2 ds \rightarrow 0 \quad \text{as} \ n \rightarrow \infty. \tag{3.10} \]
To this end we first note that a straightforward application of Duhamel’s formula yields that
\[ e^{tA}|_{H_n} = e^{tA_n} \quad \forall \ t \geq 0. \]
Therefore
\[ W_{A,\sigma}(s) - W_{A_n,\sigma_n}(s) = \int_0^s e^{(t-r)A}(\sigma - \pi_n \sigma \pi_n)dW(r), \]
and thus
\[
\mathbb{E}|W_{A,\sigma}(s) - W_{A_n,\sigma_n}(s)|^2 = \int_0^s \|e^{(t-r)A}(\sigma - \pi_n\sigma\pi_n)\|_{HS}^2 dr \\
= \sum_{i=1}^{\infty} \int_0^s |e^{rA}(\sigma - \pi_n\sigma\pi_n)e_i|^2 dr.
\]

Since for any \( i \in \mathbb{N}, r \in [0, s] \), the integrands converge to 0, Lebesgue’s dominated convergence theorem implies (3.10).

(ii) Fix \( T > 0, n \in \mathbb{N} \) and \( x, y \in H_n \). Let \( \xi_T \in C^1([0, \infty)) \) be defined by
\[
\xi_T(t) := \frac{2\omega e^{-\omega t}|x - y|}{1 - e^{-2\omega T}}, \quad t \geq 0.
\]
Consider for \( X_n(t) = X_n(t, x), \ t \geq 0 \), see the proof of (i), the stochastic equation
\[
\begin{cases}
\frac{dX_n(t)}{dt} = \left[ A_nX_n(t) + F_n(X_n(t)) + \xi_T(t) \frac{X_n(t) - Y_n(t)}{|X_n(t) - Y_n(t)|} \mathbb{1}_{X_n(t) \neq Y_n(t)} \right] dt \\
+\sigma_n dW_n(t), \\
Y_n(0) = y.
\end{cases}
\]
(3.11)

Since
\[
z \rightarrow \frac{X_n(t) - z}{|X_n(t) - z|} \mathbb{1}_{X_n(t) \neq z}
\]
is dissipative on \( H_n \) for all \( t \geq 0 \) (cf [19]), (3.11) has a unique strong solution \( Y_n(t) = Y_n(t, y), \ t \geq 0 \), which is pathwise continuous \( P \)-a.s.

Define the first coupling time
\[
\tau_n := \inf\{t \geq 0 : X_n(t) = Y_n(t)\}.
\]
(3.12)
Writing the equation for \( X_n(t) - Y_n(t), \ t \geq 0 \), applying the chain rule to \( \phi_\epsilon(z) := \sqrt{z + \epsilon^2}, \ z \in (-\epsilon^2, \infty), \ \epsilon > 0 \), and letting \( \epsilon \to 0 \) subsequently, we obtain
\[
\frac{d}{dt} |X_n(t) - Y_n(t)| \leq \omega |X_n(t) - Y_n(t)| - \xi_T(t) \mathbb{1}_{X_n(t) \neq Y_n(t)} \quad t \geq 0,
\]
which yields
\[
d(e^{-\omega t}|X_n(t) - Y_n(t)|) \leq -e^{-\omega t}\xi_T(t) \mathbb{1}_{X_n(t) \neq Y_n(t)} dt, \quad t \geq 0.
\]
(3.13)
In particular, \( t \mapsto e^{-\omega t}|X_n(t) - Y_n(t)| \) is decreasing, hence \( X_n(T) = Y_n(T) \) for all \( T \geq \tau_n \). But by (3.13) if \( T \leq \tau_n \) then

\[
|X_n(T) - Y_n(T)| e^{-\omega T} \leq |x - y| - |x - y| \int_0^T \frac{2\omega e^{-2\omega t}}{1 - e^{-2\omega T}} dt = 0.
\]

So, in any case

\[
X_n(T) = Y_n(T), \quad \mathbb{P}\text{-a.s.} \tag{3.14}
\]

Let

\[
R := \exp \left[ - \int_0^{T \wedge \tau_n} \frac{\xi^T(t)}{|X_n(t) - Y_n(t)|} \langle X_n(t) - Y_n(t), \sigma^{-1} dW_n(t) \rangle - \frac{1}{2} \int_0^{T \wedge \tau_n} \frac{(\xi^T(t))^2 |\sigma^{-1}(X_n(t) - Y_n(t))|^2}{|X_n(t) - Y_n(t)|^2} dt \right]
\]

By (3.14) and Girsanov’s theorem for \( p > 1 \),

\[
(P_T^nf(y))^p = \mathbb{E}[f(Y_n(T))]^p = \mathbb{E}[Rf(X_n(T)))]^p \leq (P_T^n f^p(x))^{(\mathbb{E}[R^{p/(p-1)}])^{p-1}}. \tag{3.15}
\]

Let

\[
M_p = \exp \left[ - \frac{p}{p-1} \int_0^{T \wedge \tau_n} \frac{\xi^T(t)}{|X_n(t) - Y_n(t)|} \langle X_n(t) - Y_n(t), \sigma^{-1} dW_n(t) \rangle - \frac{p^2}{2(p-1)^2} \int_0^{T \wedge \tau_n} \frac{(\xi^T(t))^2 |\sigma^{-1}(X_n(t) - Y_n(t))|^2}{|X_n(t) - Y_n(t)|^2} dt \right]
\]

We have \( \mathbb{E}M_p = 1 \) and hence,

\[
\mathbb{E}R^{p/(p-1)} = \mathbb{E}\left\{ M_p \exp \left[ \frac{p}{2(p-1)^2} \int_0^{T \wedge \tau_n} \frac{(\xi^T(t))^2 |\sigma^{-1}(X_n(t) - Y_n(t))|^2}{|X_n(t) - Y_n(t)|^2} dt \right] \right\}
\]

\[
\leq \sup_\Omega \exp \left[ \frac{p}{2(p-1)^2} \int_0^{T \wedge \tau_n} (\xi^T(t))^2 |\sigma^{-1}|^2 dt \right] \leq \exp \left[ ||\sigma^{-1}||^2 \frac{p\omega|x - y|^2}{(p-1)^2(1 - e^{-2\omega T})} \right].
\]

Combining this with (3.15) we get the assertion (with \( T \) replacing \( t \)). \( \square \)
Proof and consequences of Theorem 1.6

On the basis of Propositions 3.1 and 2.1 we can now easily prove Theorem 1.6.

**Proof of Theorem 1.6** Let \( f \in Lip_b(H), f \geq 0 \). By Proposition 3.1(i) it then follows that (3.4) holds with \( P_t f \) replacing \( P_n t f \) provided \( F \) is Lipschitz. Using that \( \bigcup_{n \in \mathbb{N}} H_n \) is dense in \( H \) and that \( P_t f(x) \) is continuous on \( x \) (cf. [10]) we obtain (3.4) for all \( x, y \in H \). In particular, this is true for \( P_t^{\alpha_n, \beta_n} f \) from Proposition 2.1.

Now fix \( t > 0 \) and \( k \in \mathbb{N} \), let

\[
\chi_k(s) := \frac{1}{k} \mathbb{1}_{[t,t+1/k]}(s), \quad s \geq 0.
\]

Using (3.4) for \( P_t^{\alpha_n, \beta_m} f \), (1.6), Proposition 2.1 and Jensen’s inequality, we obtain for \( x, y \in K_p \)

\[
p_t^\mu f(x) = \lim_{k \to \infty} \frac{1}{k} \int_t^{t+1/k} p_s^\mu f(x) ds
\]

\[
= \lim_{k \to \infty} \lim_{n \to \infty} \lim_{m \to \infty} \int_0^{t+1} \chi_k(s) P_s^{\alpha_n, \beta_m} f(x) dx
\]

\[
\leq \lim_{k \to \infty} \lim_{n \to \infty} \lim_{m \to \infty} \int_0^{t+1} \chi_k(s)(P_s^{\alpha_n, \beta_m} f^p(y))^{1/p} \exp \left[ \|\sigma^{-1}\|^2 \frac{\omega |x - y|^2}{(p - 1)(1 - e^{-2\omega s})} \right] ds
\]

\[
\leq \lim_{k \to \infty} \lim_{n \to \infty} \lim_{m \to \infty} \left( \int_0^{t+1} \chi_k(s) P_s^{\alpha_n, \beta_m} f^p(y) \exp \left[ \|\sigma^{-1}\|^2 \frac{p \omega |x - y|^2}{(p - 1)(1 - e^{-2\omega s})} \right] ds \right)^{1/p}
\]

\[
= (p_t^\mu f^p(y))^{1/p} \exp \left[ \|\sigma^{-1}\|^2 \frac{\omega |x - y|^2}{(p - 1)(1 - e^{-2\omega t})} \right],
\]

where we note that we have to choose the sequences \((\alpha_n), (\beta_n)\) such that (2.7) holds both for \( f \) and \( f^p \) instead of \( f \). Since \( K \) is dense in \( H_0 \), (1.10) follows for \( f \in C_b(H) \), for all \( x, y \in H_0 \), since \( p_t^\mu f \) is continuous on \( H_0 \) by (1.4).

Let now \( f \in B_0(H), f \geq 0 \). Let \( f_n \in C_b(H), n \in \mathbb{N} \), such that \( f_n \to f \) in \( L^p(H, \mu) \) as \( n \to \infty \), \( p \in (1, \infty) \) fixed. Then, since \( \mu \) is invariant for \( p_t^\mu \), \( t > 0 \), selecting a subsequence if necessary, it follows that there exists \( K_1 \in \mathcal{B}(H) \), \( \mu(K_1) = 1 \), such that

\[
p_t^\mu f_n(x) \to p_t^\mu f(x) \quad \text{as} \ n \to \infty, \ \forall \ x \in K_1.
\]
Taking this limit in (1.10) we obtain (1.10) for all \(x, y \in K_1\). Taking into account that \(\mu_t\) is continuous and that \(K_1\) is dense in \(H_0 = \text{supp } \mu\), (1.10) follows for all \(x, y \in H_0\). \(\square\)

Corollary 1.7 immediately follows from Theorem 1.6 and the following general result:

**Proposition 4.1** Let \(E\) be a topological space and \(P\) a Markov operator on \(B_b(E)\). Assume that for any \(p > 1\) there exists a continuous function \(\eta_p\) on \(E \times E\) such that

\[
    |Pf|(x) \leq (P|f|^p(y))^{1/p}e^{\eta_p(x,y)} \quad \forall \ x, y \in E, \ f \in B_b(E).
\]

(4.1)

Then \(P\) is strong Feller, i.e. maps \(B_b(E)\) into \(C_b(E)\). Furthermore, for any \(\sigma\)-finite measure \(\mu\) on \((E, \mathcal{B}(E))\) such that

\[
    \int_E |Pf|d\mu \leq C \int_E |f|d\mu, \quad \forall \ f \in B_b(E),
\]

(4.2)

for some \(C > 0\), \(P\) uniquely extends to \(L^p(E, \mu)\) with \(PL^p(E, \mu) \subset C(E)\) for any \(p > 1\).

**Proof.** Since \(P\) is linear, we only need to consider \(f \geq 0\). Let \(f \in B_b(E)\) be nonnegative. By (4.1) and the property of \(\eta_p\) we have

\[
    \limsup_{x \to y} Pf(x) \leq (Pf^p(y))^{1/p}, \quad p > 1.
\]

Letting \(p \downarrow 1\) we obtain \(\limsup_{x \to y} Pf(x) \leq Pf(y)\). Similarly, using \(f^{1/p}\) to replace \(f\) and replacing \(x\) with \(y\), we obtain

\[
    (Pf^{1/p}(y))^p \leq (Pf(x))e^{p\eta_p(y,x)}, \quad \forall \ x, y \in E, \ p > 1.
\]

First letting \(x \to y\) then \(p \to 1\), we obtain \(\liminf_{x \to y} Pf(x) \geq Pf(y)\). So \(Pf \in C_b(E)\). Next, for any nonnegative \(f \in L^p(E, \mu)\), let \(f_n = f \wedge n, \ n \geq 1\).

By (4.2) and \(f_n \to f\) in \(L^p(E, \mu)\) we have \(P|f_n - f_m|^p \to 0\) in \(L^1(E, \mu)\) as \(n, m \to \infty\). In particular, there exists \(y \in E\) such that

\[
    \lim_{n, m \to \infty} P|f_n - f_m|^p(y) = 0.
\]

(4.3)

Moreover, by (4.1), for \(B_N := \{x \in E : \eta_p(x, y) < N\}\)

\[
    \sup_{x \in B_N} |Pf_n(x) - Pf_m(x)|^p \leq \sup_{x \in B_N} (P|f_n - f_m|(x))^p \leq (P|f_n - f_m|^p(y))e^{pN}.
\]
Since by the strong Feller property $P f_n \in C_b(E)$ for any $n \geq 1$ and noting that $C_b(B_N)$ is complete under the uniform norm, we conclude from (1.3) that $P f$ is continuous on $B_N$ for any $N \geq 1$, and hence, $P f \in C(H)$. □

**Proof of Corollary 1.8** Let $\mu_1, \mu_2$ be probability measures on $(H, \mathcal{B}(H))$ satisfying (H4). Define $\mu := \frac{1}{2} \mu_1 + \frac{1}{2} \mu_2$. Then $\mu$ satisfies (H4) and $\mu_i = \rho_i \mu$, $i = 1, 2$, for some $\mathcal{B}(H)$-measurable $\rho_i : H \to [0, 2]$. Let $i \in \{1, 2\}$.

Since $\rho_i$ is bounded, by (H4)(iii) and Theorem 1.2 it follows that

\[
\int_H L^\mu u \, d\mu_i = 0, \quad \forall \, u \in D(L^\mu).
\]

Hence

\[
\frac{d}{dt} \int_H e^{tL^\mu} u \, d\mu_i = \int_H L^\mu(e^{tL^\mu} u) \, d\mu_i = 0, \quad \forall \, u \in D(L^\mu),
\]

i.e.

\[
\int_H p_t^\mu u \, d\mu_i = \int_H u \, d\mu_i \quad \forall \, u \in \mathcal{E}_A(H).
\]

Since $\mathcal{E}_A(H)$ is dense in $L^1(H, \mu_i)$, $\mu_i$ is ($p_t^\mu$)-invariant. But as mentioned before, by Theorem 1.6 it follows that ($p_t^\mu$) is irreducible on $H_0$ (see [11]) and it is strong Feller on $H_0$ by Corollary 1.7. So, since $\mu_i(H_0) = 1$, $\mu_i = \mu$. □

**Proof of Corollary 1.10** Let

\[
\tilde{A} := A - \omega I, \quad D(\tilde{A}) := D(A)
\]

\[
\tilde{F}_0 := F_0 + \omega I.
\]

By (H2), $\tilde{A}$ has discrete spectrum. Let $e_k \in H$, $-\lambda_k \in (-\infty, 0]$, be the corresponding orthonormal eigenvectors, eigenvalues respectively.

For $k \in \mathbb{N}$ define

\[
\varphi_k(x) := \langle e_k, x \rangle, \quad x \in H.
\]

We note that by a simple approximation (1.5) also holds for any Lipschitz function on $H$ and thus (cf. the proof of [7, Proposition 5.7(iii)]) also (1.6) holds for such functions, i.e. in particular, for all $k \in \mathbb{N}$

\[
[0, \infty) \ni t \mapsto p_t \varphi_k(x) \quad \text{is continuous for all } x \in H_0. \quad (4.4)
\]

Since any compactly supported smooth function on $\mathbb{R}^N$ is the Fourier transform of a Schwartz test function, by approximation it easily follows that setting

\[
\mathcal{F} C_b^\infty(\{e_k\}) := \{g(\langle e_1, \cdot \rangle, \ldots, \langle e_N, \cdot \rangle) : N \in \mathbb{N}, \, g \in C_b^\infty(\mathbb{R}^N)\},
\]
we have $\mathcal{F}C^\infty_b(\{e_k\}) \subset D(L_\mu)$ and for $\varphi \in \mathcal{F}C^\infty_b(\{e_k\})$

$$L_\mu \varphi(x) = \frac{1}{2} \text{Tr} [D^2 \varphi(x)] + \langle x, AD \varphi(x) \rangle + \langle F_0(x), D \varphi(x) \rangle \quad x \in H.$$ 

Then by approximation it is easy to show that

$$\varphi_k, \varphi^2_k \in D(L_\mu) \text{ and } L_\mu \varphi_k = -\lambda_k \varphi_k + \langle e_k, \tilde{F}_0 \rangle,$$

$$L_\mu \varphi^2_k = -2\lambda_k \varphi^2_k + 2\varphi_k \langle e_k, \tilde{F}_0 \rangle + 2 \quad \forall \ k \in \mathbb{N}. \quad (4.5)$$

Since we assume that $|F_0|$ is in $L^2(H, \mu)$, by [3, Theorem 1.1] we are in the situation of [17, Chapter II]. So, we conclude that by [17, Chapter II, Theorem 1.9] there exists a normal (that is $P^x[X(0) = x] = 1$) Markov process $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, (X(t))_{t \geq 0}, (P^x)_{x \in H_0})$ with state space $H_0$ and $M \in \mathcal{B}(H_0)$, $\mu(M) = 1$, such that $X(t) \in M$ for all $t \geq 0 \ P^x$-a.s. for all $x \in M$ and which has continuous sample paths $P_x$-a.s for all $x \in M$ and for which by the proof of [7, Proposition 8.2] and (4.4), (4.5) we have that for all $k \in \mathbb{N}$

$$\beta^x_k(t) := \varphi_k(X(t)) - \varphi_k(x) - \int_0^t L_\mu \varphi_k(X(s)) ds, \quad t \geq 0,$$

$$M^x_k(t) := \varphi^2_k(X(t)) - \varphi^2_k(x) - \int_0^t L_\mu \varphi^2_k(X(s)) ds, \quad t \geq 0,$$

are continuous local $(\mathcal{F}_t)$-martingales with $\beta^x_k(0) = M_k(0) = 0$ under $P^x$ for all $x \in M$. Fix $x \in M$. Below $\mathbb{E}_x$ denotes expectation with respect to $P^x$. Since for $T > 0$

$$\int_H \int_0^T \mathbb{E}_x(1 + |X(s)|^2)(1 + |F_0(X(s))|) ds \mu(dx)$$

$$= T \int_H (1 + |x|^2)(1 + |F_0(x)|) \mu(dx) < \infty,$$

making $M$ smaller if necessary, by (H4)(ii) we may assume that

$$\mathbb{E}_x \int_0^T (1 + |X(s)|^2)(1 + |F_0(X(s))|) ds < \infty. \quad (4.7)$$

By standard Markov process theory we have for their covariation processes under $P^x$,

$$\langle \beta^x_k, \beta^x_{k'} \rangle_t = \int_0^t \langle D \varphi_k(X(s)), D \varphi_{k'}(X(s)) \rangle ds = t \delta_{k,k'}, \quad t \geq 0. \quad (4.8)$$
Indeed, an elementary calculation shows that for all \( k \in \mathbb{N}, \ t \geq 0 \),
\[
\beta_k^x(t)^2 - \int_0^t |D\varphi_k(X(s))|^2 \, ds
\]
\[
= M_k^x(t) - 2\varphi_k(x)\beta_k^x(t) - \int_0^t (\beta_k^x(t) - \beta_k^x(s)) L_{\mu, \varphi_k}(X(s)) \, ds,
\]
where all three summands on the right hand side are martingales. Since we have a similar formula for finite linear combinations of \( \varphi_k' \)s replacing a single \( \varphi_k \), by polarization we get (4.8). Note that by (4.5) and (4.7) all integrals in (4.6), (4.9) are well defined.

Hence, by (4.8) \( \beta_k^x, \ k \in \mathbb{N} \), are independent standard \((\mathcal{F}_t)\)-Brownian motions under \( P_x \). Now it follows by [13, Theorem 13] that, with \( W^x = (W^x(t))_{t \geq 0} \), being the cylindrical Wiener process on \( H \) given by \( W^x = (\beta_k^x e_k)_{k \in \mathbb{N}} \), we have for every \( t \geq 0 \),
\[
X(t) = e^{tA}x + \int_0^t e^{(t-s)A} F_0(X(s)) \, ds + \int_0^t e^{(t-s)A} dW^x(s), \ \mathbb{P}\text{-a.s.}, \ (4.10)
\]
that is, the tuple \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}_x, W^x, X)\) is a solution to
\[
\begin{cases}
Y(t) = e^{tA}Y(0) + \int_0^t e^{(t-s)A} F_0(Y(s)) \, ds + \int_0^t e^{(t-s)A} dW(s), \ \mathbb{P}\text{-a.s.}, \ \forall \ t \geq 0, \\
\text{law } Y(0) = \delta_x (: = \text{Dirac measure in } x),
\end{cases}
\]
(4.11)
in the sense of [13, page 4].

We note that the zero set in (4.10) is indeed independent of \( t \), since all terms are continuous in \( t \) \( \mathbb{P}_x \)-a.s. because of (H2)(ii) and (4.7).

**Claim** We have \( X \)-pathwise uniqueness for equation (4.11) (in the sense of [13, page 98]).

For any given cylindrical \((\mathcal{F}_t')\)-Wiener process \( W \) on a stochastic basis \((\Omega', \mathcal{F}', (\mathcal{F}_t')_{t \geq 0}, \mathbb{P}')\) let \( Y = Y(t), \ Z = Z(t), \ t \geq 0, \) be two solutions of (4.11) such that \( \text{law}(Z) = \text{law}(Y) = \text{law}(X) \) and \( Y(0) = Z(0) \) \( \mathbb{P}'\)-a.s.. Then by (4.7)
\[
\mathbb{E}' \int_0^T |F_0(Y(s))| \, ds = \mathbb{E}' \int_0^T |F_0(Z(s))| \, ds = \mathbb{E}_x \int_0^T |F_0(X(s))| \, ds < \infty.
\]
(4.12)
(which, in particular implies by (4.11) and by (H2)(i) that both $Y$ and $Z$ have $\mathbb{P}'$-a.s. continuous sample paths). Hence applying [13, Theorem 13] again (but this time using the dual implication) we obtain for all $k \in \mathbb{N}$

$$
\langle e_k, Y(t) - Z(t) \rangle = -\lambda_k \int_0^t \langle e_k, Y(s) - Z(s) \rangle ds
$$

$$
+ \int_0^t \langle e_k, \tilde{F}_0(Y(s)) - \tilde{F}_0(Z(s)) \rangle ds, \quad t \geq 0, \mathbb{P}'\text{-a.s.}
$$

Therefore, by the chain rule for all $k \in \mathbb{N}$

$$
\langle e_k, Y(t) - Z(t) \rangle^2 = -2\lambda_k \int_0^t \langle e_k, Y(s) - Z(s) \rangle^2 ds
$$

$$
+ 2 \int_0^t \langle e_k, Y(s) - Z(s) \rangle \langle e_k, \tilde{F}_0(Y(s)) - \tilde{F}_0(Z(s)) \rangle ds, \quad t \geq 0, \mathbb{P}'\text{-a.s.}
$$

Dropping the first term on the right hand side and summing up over $k \in \mathbb{N}$ (which is justified by (4.11) and the continuity of $Y$ and $Z$), we obtain from (H3) that

$$
|Y(t) - Z(t)|^2 \leq 2 \int_0^t \langle Y(s) - Z(s), \tilde{F}_0(Y(s)) - \tilde{F}_0(Z(s)) \rangle ds
$$

$$
\leq 2\omega \int_0^t |Y(s) - Z(s)|^2 ds, \quad t \geq 0, \mathbb{P}'\text{-a.s.}
$$

Hence, by Gronwall’s lemma $Y = Z$ $\mathbb{P}'$-a.s. and the Claim is proved.

By the Claim we can apply [13, Theorem 10, (1) $\iff$ (3)] and then [13, Theorem 1] to conclude that equation (4.11) has a strong solution (see [13, Definition 1]) and that there is one strong solution with the same law as $X$, which hence by (4.7) has continuous sample paths a.s. Now all conditions in [13, Theorem 13.2] are fulfilled and, therefore, we deduce from it that on any stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ with $(\mathcal{F}_t)$-cylindrical Wiener process $W$ on $H$ and for $x, y \in M$ there exist pathwise unique continuous strong solutions $X(t, x), X(t, y), t \geq 0$, to (4.11) such that

$$
\mathbb{P} \circ X(\cdot, x)^{-1} = \mathbb{P}_x \circ X^{-1}
$$

and

$$
\mathbb{P} \circ X(\cdot, y)^{-1} = \mathbb{P}_y \circ X^{-1},
$$

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in particular, $X(0, x) = x$ and $X(0, y) = y$ and

\[
\mathbb{P} \circ X(t, x)^{-1}(dz) = p_t(x, dz), \quad t \geq 0,
\]
\[
\mathbb{P} \circ X(t, y)^{-1}(dz) = p_t(y, dz), \quad t \geq 0.
\]

(4.13)

In particular, we have proved (i). To prove (ii), below for brevity we set

$X := X(\cdot, x)$, $X' := X(\cdot, y)$. Then proceeding as in the proof of the Claim, by (1.13) and noting that $s^{-1}\Phi(s) \to \infty$ as $s \to \infty$, we obtain

\[
\frac{d}{dt} |X(t) - X'(t)|^2 \leq a - \Phi_0(|X(t) - X'(t)|^2)
\]

(4.14)

for some constant $a > 0$, only depending on $\omega$ and $\Phi$, where $\Phi_0 = \frac{1}{2} \Phi$.

Now we consider two cases.

**Case 1.** $|x - y|^2 \leq \Phi_0^{-1}(2a)$.

Define $f(t) := |X(t) - X'(t)|^2$, $t \geq 0$, and suppose there exists $t_0 \in (0, \infty)$ such that

$f(t_0) > \Phi_0^{-1}(a)$.

Then we can choose $\delta \in [0, t_0]$ maximal such that

$f(t) > \Phi_0^{-1}(a), \quad \forall \ t \in (t_0 - \delta, t_0]$.

Hence, because by (4.14) $f$ is decreasing on every interval where it is larger than $\Phi_0^{-1}(a)$, we obtain that

$f(t_0 - \delta) \geq f(t_0) > \Phi_0^{-1}(a)$.

Suppose $t_0 - \delta > 0$. Then $f(t_0 - \delta) \leq \Phi_0^{-1}(a)$ by the continuity of $f$ and the maximality of $\delta$. So, we must have $t_0 - \delta = 0$, hence

$f(t_0) \leq f(t_0 - \delta) = f(0) = |x - y|^2 \leq \Phi_0^{-1}(2a)$.

So,

$|X(t) - X'(t)|^2 \leq \Phi_0^{-1}(2a), \quad \forall \ t > 0$.

**Case 2.** $|x - y|^2 > \Phi_0^{-1}(2a)$.

Define $t_0 = \inf\{t \geq 0 : |X(t) - X'(t)|^2 \leq \Phi_0^{-1}(2a)\}$. Then by Case 1, starting at $t = t_0$ rather than $t = 0$ we know that

$|X(t) - X'(t)|^2 \leq \Phi_0^{-1}(2a), \quad \forall \ t \geq t_0$.

(4.15)
Furthermore, it follows from (4.14) that
\[
d|X(t) - X'(t)|^2 \leq -\frac{1}{2} \Phi_0(|X(t) - X'(t)|^2) dt, \quad \forall \ t \leq t_0.
\]
This implies
\[
\Psi(|X(t) - X'(t)|^2) \geq \frac{1}{2} \int_{|X(t) - X'(t)|^2}^{|x-y|^2} \frac{dr}{\Phi_0(r)} \geq \frac{t}{4}, \quad \forall \ t \leq t_0.
\]
Therefore,
\[
|X(t) - X'(t)|^2 \leq \Psi^{-1}(t/4), \quad \forall \ t \leq t_0. \tag{4.16}
\]
Combining Case 1, (4.15) and (4.16) we conclude that
\[
|X(t) - X'(t)|^2 \leq \Psi^{-1}(t/4) + \Phi_0^{-1}(2a), \quad \forall \ t > 0. \tag{4.17}
\]
Combining (4.17) with Theorem 1.6 for all \( f \in B_b(H) \) we obtain
\[
(p_{t/2}|f|(X(t/2)))^2 \leq (p_{t/2}f^2(X'(t/2))) \exp \left[ \frac{\lambda(1 + \Psi^{-1}(t/8))}{(1 - e^{-\omega t/2})^2} \right], \quad \forall \ t > 0
\]
for some constant \( \lambda > 0 \). By Jensen’s inequality and approximation it follows that for all \( f \in L^2(H, \mu) \)
\[
(p_t|f|(x))^2 \leq \mathbb{E}(p_{t/2}|f|(X(t/2)))^2 \leq (p_t f^2(y)) \exp \left[ \frac{\lambda(1 + \Psi^{-1}(t/8))}{(1 - e^{-\omega t/2})^2} \right], \quad \forall \ t > 0, \ \forall \ x, y \in M. \tag{4.18}
\]
But since \( H_0 = \text{supp} \ \mu, \ M \) is dense in \( H_0 \), hence by the continuity of \( p_t f \) (cf. Corollary 1.7) (4.18) holds for all \( x \in H_0, y \in M \). Since \( \mu(M) = 1 \) this completes the proof by integrating both sides with respect to \( \mu(dy) \). □

**Remark 4.2** We would like to mention that by using [2] instead of [17] we can drop the assumption that \( |F_0| \in L^2(H, \mu) \). So, by (4.9) and the proof above we can derive (4.8) avoiding to assume the usually energy condition
\[
\int_0^t |F_0(X(s))|^2 ds < \infty, \quad \mathbb{P}_x\text{-a.s.}
\]
Details will be included in a forthcoming paper. We would like to thank Tobias Kuna at this point from whom we learnt identity (4.9) by private communication.
5 Existence of measures satisfying \((H4)\)

To prove existence of invariant measures we need to strengthen some of our assumptions. So, let us introduce the following conditions.

\((H1)'\) \((A, D(A))\) is self-adjoint satisfying \((1.2)\).

\((H6)\) There exists \(\eta \in (\omega, \infty)\) such that
\[
\langle F_0(x) - F_0(y), x - y \rangle \leq -\eta|x - y|^2, \quad \forall \ x, y \in D(F).
\]

**Remark 5.1** (i) Clearly, \((H1)'\) implies \((H1)\) and \((H5)\). \((H1)'\) and \((H2)(i)\) imply that \((A, D(A))\) and thus also \((1+\omega-A, D(A))\) has a discrete spectrum. Let \(\lambda_i \in (0, \infty), \ i \in \mathbb{N},\) be the eigenvalues of the latter operator. Then by \((H2)\)
\[
\sum_{i=1}^{\infty} \lambda_i^{-1} < \infty. \quad (5.1)
\]

(ii) If we assume \((5.1)\), i.e. that \((1 + \omega - A)^{-1}\) is trace class, then all what follows holds with \((H2)\) replaced by \((H2)(i)\). So, \(\sigma^{-1} \in L(H)\) is not needed in this case.

Let \(F_\alpha, \alpha < 0,\) be as in Section 2. Then e.g. by \([6, \text{Theorem 3.2}]\) equation \((1.1)\) with \(F_\alpha\) replacing \(F_0\) has a unique mild solution \(X_\alpha(t, x), \ t \geq 0.\) Since there exist \(\tilde{\eta} \in (\omega, \infty)\) and \(\alpha_0 > 0\) such that each \(F_\alpha, \alpha \in (0, \alpha_0),\) satisfies \((H6)\) with \(\tilde{\eta}\) replacing \(\eta,\) by \([6, \text{Section 3.4}]\) \(X_\alpha\) has a unique invariant measure \(\mu_\alpha\) on \((H, \mathcal{B}(H))\) such that for each \(m \in \mathbb{N}\)
\[
\sup_{\alpha \in (0,\alpha_0)} \int_H |x|^m \mu_\alpha(dx) < \infty. \quad (5.2)
\]

That these moments are indeed uniformly bounded in \(\alpha,\) follows from the proof of \([6, \text{Proposition 3.18}]\) and the fact that \(\tilde{\eta} \in (\omega, \infty).\)

Let \(N_Q\) denote the centered Gaussian measure on \((H, \mathcal{B}(H))\) with covariance operator \(Q\) defined by
\[
Qx := \int_0^{\infty} e^{tA} \sigma e^{tA} x dt, \quad x \in H,
\]
which by \((H2)(ii)\) is trace class.
Let $W^{1,2}(H, N_Q)$ be defined as usual, that is as the completion of $E_A(H)$ with respect to the norm
\[
\|\varphi\|_{W^{1,2}} := \left( \int_H (\varphi^2 + |D\varphi|^2) dN_Q \right)^{1/2}, \quad \varphi \in E_A(H),
\]
where $D$ denotes first Fréchet derivative. By [9] we know that
\[
W^{1,2}(H, N_Q) \subset L^2(H, N_Q), \quad \text{compactly. (5.3)}
\]

**Theorem 5.2** Assume that (H1)', (H2), (H3) and (H6) hold and let $\mu_\alpha, \alpha \in (0, \alpha_0)$ be as above. Suppose that there exists a lower semi-continuous function $G : H \to [0, \infty]$ such that
\[
\{G < \infty\} \subset D(F), \quad |F_0| \leq G \text{ on } D(F) \quad \text{and} \quad \sup_{\alpha \in (0, \alpha_0)} \int_H G^2 d\mu_\alpha < \infty. \quad (5.4)
\]
Then $\{\mu_\alpha : \alpha \in (0, \alpha_0)\}$ is tight and any limit point $\mu$ satisfies (H4) and hence by Corollary 1.8 all of these limit points coincide. Furthermore, for all $m \in \mathbb{N}$
\[
\int_H (|F_0(x)|^2 + |x|^m) \mu(dx) < \infty \quad (5.5)
\]
and there exists $\rho : H \to [0, \infty), \mathcal{B}(H)$-measurable, such that $\mu = \rho N_Q$ and $\sqrt{\rho} \in W^{1,2}(H, \mu)$.

**Proof.** We recall that by [3] Theorem 1.1] for each $\alpha \in (0, \alpha_0)$
\[
\mu_\alpha = \rho_\alpha N_Q; \quad \sqrt{\rho_\alpha} \in W^{1,2}(H, N_Q) \quad (5.6)
\]
and as is easily seen from its proof, that
\[
\int_H |D\sqrt{\rho_\alpha}|^2 dN_Q \leq \frac{1}{4} \int_H |F_\alpha|^2 d\mu_\alpha. \quad (5.7)
\]
But by (2.3) and (5.4) the right hand side of (5.7) is uniformly bounded in $\alpha$. Hence by (5.3) there exists a zero sequence $\{\alpha_n\}$ such that
\[
\sqrt{\rho_{\alpha_n}} \to \sqrt{\rho} \quad \text{in } L^2(H, N_Q) \text{ as } n \to \infty,
\]
for some $\sqrt{\rho} \in W^{1,2}(H, N_Q)$ and therefore, in particular,
\[
\rho_{\alpha_n} \to \rho \quad \text{in } L^1(H, N_Q) \text{ as } n \to \infty. \quad (5.8)
\]
Define $\mu := \rho N_Q$ and $\rho_n := \rho_{\alpha_n}$, $n \in \mathbb{N}$. Since $G$ is lower semi-continuous and $\mu_{\alpha_n} \to \mu$ as $n \to \infty$ weakly, (5.2) and (5.4) imply
\[
\int_H (G^2(x) + |x|^m)\mu(dx) < \infty \quad \forall \ m \in \mathbb{N}.
\] (5.9)

Hence by (5.4) both (H4)(i) and (H4)(ii) follow. So, it remains to prove (H4)(iii).

Since $\sigma$ is independent of $\alpha$, to show (5.9) it is enough to prove that for all $\varphi \in C_b(H)$, $h \in D(A)$
\[
\lim_{n \to \infty} \int_H F^h_{\alpha_n}(x) \varphi(x) \mu_{\alpha_n}(dx) = \int_H F^h_0(x) \varphi(x) \mu(dx),
\] (5.10)

where $F^h_\alpha := \langle h, F_{\alpha} \rangle$, $\alpha \in [0, \alpha_0)$. We have
\[
\left| \int_H F^h_{\alpha_n} \varphi d\mu_{\alpha_n} - \int_H F^h_0 \varphi d\mu \right| \\
\leq \|\varphi\| \int_H \left| F^h_{\alpha_n} - F^h_0 \right| \rho_n dN_Q + \int_H |F^h_0 \varphi| |\rho_n - \rho| dN_Q.
\] (5.11)

But by (2.3) and (5.4) we have
\[
\int_H |F^h_{\alpha_n} - F^h_0| \rho_n dN_Q \leq \int_{|G| \leq M} |F^h_{\alpha_n} - F^h_0| \rho_n dN_Q + \frac{2|h|}{M} \sup_{\alpha \in (0, \alpha_0)} \int_H G^2 d\mu_{\alpha}.
\]

Hence first letting $n \to \infty$ then $M \to \infty$ by (2.2), (5.4) and (5.8) Lebesgue’s generalized dominated convergence theorem implies that the first term on the right hand side of (5.11) converges to 0. Furthermore, for every $\delta \in (0, 1)$
\[
\left| \int_H F^h_0 \varphi d\mu_{\alpha_n} - \int_H F^h_0 \varphi d\mu \right| \\
\leq \left| \int_H \frac{F^h_0}{1 + \delta |F^h_0|} \varphi(\rho_n - \rho) dN_Q \right| \\
+ \delta \|\varphi\| \left( \int_H |F^h_0|^2 d\mu_{\alpha_n} + \int_H |F^h_0|^2 d\mu \right).
\] (5.12)

Since by (2.3) and (5.4)
\[
\sup_{\alpha \in (0, \alpha_0)} \int_H |F^h_0|^2 d\mu_{\alpha} < \infty,
\] 25
(H4)(iii) follows from (5.12) by letting first $n \to \infty$ and then $\delta \to 0$, since for fixed $\delta > 0$ the first term in the right hand side converges to zero by (5.8).

□

Example 5.3 Let $H = L^2(0,1), Ax = \Delta x, x \in D(A) := H^2(0,1) \cap H_0^1(0,1)$. Let $f : \mathbb{R} \to \mathbb{R}$ be decreasing such that for some $c_3 > 0, m \in \mathbb{N}$,

$$|f(s)| \leq c_3(1 + |s|^m), \quad \forall s \in \mathbb{R}. \quad (5.13)$$

Let $s_i \in \mathbb{R}, i \in \mathbb{N}$, be the set of all arguments where $f$ is not continuous and define

$$\tilde{f}(s) = \begin{cases} [f(s_{i+}), f(s_{i-})], & \text{if } s = s_i \text{ for some } i \in \mathbb{N}, \\ f(s), & \text{else} \end{cases}$$

Define

$$F : D(F) \subset H \to 2^H, \; x \mapsto \tilde{f} \circ x,$$

where

$$D(F) = \{x \in H : \tilde{f} \circ x \subset H\}.$$ 

Then $F$ is $m$-dissipative. Let $F_0$ be defined as in Section 2.

Since $A \leq \omega$ for some $\omega < 0$, it is easy to check that all conditions (H1)', (H2), (H3), (H6) with $\eta = 0$ hold for any $\sigma \in L(H)$ such that $\sigma^{-1} \in L(H)$. Define

$$G(x) := \begin{cases} \left( \int_0^1 |x(\xi)|^{2m} d\xi \right)^{1/2} & \text{if } x \in L^{2m}(0,1) \\ +\infty & \text{if } x \notin L^{2m}(0,1). \end{cases}$$

Then $\{G < \infty\} \subset D(F)$ and $|F_0| = |F_0|_{L^2(0,1)} \leq G$ on $D(F)$. Furthermore, by [7, (9.3)]

$$\sup_{\alpha \in (0,\alpha_0)} \int_H G^2 d\mu_\alpha < \infty. \quad (5.14)$$

Note that from [7, Hypothesis 9.5] only the first inequality, which clearly holds by (5.13) in our case, was used to prove [7, (9.3)]. Hence all assumptions of Theorem 5.2 above hold and we obtain the existence of the desired unique probability measure $\mu$ satisfying (H4) in this case. We emphasize that no continuity properties of $f$ and $F_0$ are required. In particular, then all results stated in Section 1 except for Corollary 1.10(ii) hold in this case.

If moreover there exists an increasing positive convex function $\Phi$ on $[0, \infty)$ satisfying (1.12) such that

$$(f(s) - f(t))(s - t) \leq c - \Phi(|s - t|^2), \quad s, t \in \mathbb{R},$$

then...
then by Jensen’s inequality (1.13) holds. Hence, by Corollary 1.10 one obtains an explicit upper bound for \( \| p_t \|_{L^2(H, \mu) \rightarrow L^\infty(H, \mu)} \). A natural and simple choice of \( \Phi \) is \( \Phi(s) = s^m \) for \( m > 1 \).

One can extend these results to the case, where \((0, 1)\) above is replaced by a bounded open set in \(\mathbb{R}^d\), \(d = 2\) or \(3\) for \(\sigma = (-\Delta)^\gamma, \gamma \in (\frac{d-2}{4}, \frac{1}{2})\), based on Remark 1.1(iv).

Before to conclude we want to present a condition in the general case (i.e for any Hilbert space \(H\) as above) that implies (5.4), hence by Theorem 5.2 ensures the existence of a probability measure satisfying (H4) so that all results of Section 1 apply also to this case. As will become clear from the arguments below, such condition is satisfied if the eigenvalues of \(A\) grow fast enough in comparison with \( |F_0| \). To this end we first note that by (5.1) for \( i \in \mathbb{N} \) we can find \( q_i \in (0, \lambda_i) \), \( q_i \uparrow \infty \) such that \( \sum_{i=1}^{\infty} q_i^{-1} < \infty \) and \( \frac{q_i}{\lambda_i} \to 0 \) as \( i \to \infty \). Define \( \Theta : H \to [0, \infty] \) by

\[
\Theta(x) := \sum_{i=1}^{\infty} \frac{\lambda_i}{q_i} \langle x, e_i \rangle^2, \quad x \in H, \tag{5.15}
\]

where \( \{e_i\}_{i \in \mathbb{N}} \) is an eigenbasis of \( (1 + \omega - A, D(A)) \) such that \( e_i \) has eigenvalue \( \lambda_i \). Then \( \Theta \) has compact level sets and \( | \cdot |^2 \leq \Theta \).

Below we set

\[
H_n := \text{lin span } \{e_1, \ldots, e_n\}, \quad \pi_n := \text{projection onto } H_n,
\]

\[
\tilde{A} := A - (1 + \omega)I, \quad D(\tilde{A}) := D(A), \tag{5.16}
\]

\[
\tilde{F}_0 := F_0 + (1 + \omega)I. \tag{5.17}
\]

We note that obviously \( H_n \subset \{ \Theta < +\infty \} \) for all \( n \in \mathbb{N} \).

**Theorem 5.4** Assume that (H1)', (H2), (H3) and (H6) hold and let \( \mu_\alpha, \alpha \in (0, \alpha_0) \), be as above. Suppose that \( \{ \Theta < +\infty \} \subset D(F) \) and that for some \( C \in (0, \infty), m \in \mathbb{N} \)

\[
|F_0(x)| \leq C(1 + |x|^m + \Theta^{1/2}(x)), \quad \forall \ x \in D(F). \tag{5.18}
\]

Then

\[
\sup_{\alpha \in (0, \alpha_0)} \int_H \Theta d\mu_\alpha < \infty \tag{5.19}
\]

and (5.4) holds, so Theorem 5.2 applies.
Proof. Consider the Kolmogorov operator $L_\alpha$ corresponding to $X_\alpha(t, x)$, $t \geq 0$, $x \in H$, which for $\varphi \in \mathcal{F}C^2_b(\{e_n\})$, i.e., $\varphi = g(\langle e_1, \cdot \rangle, ..., \langle e_N, \cdot \rangle)$ for some $N \in \mathbb{N}$, $g \in C^2_b(\mathbb{R}^N)$, is given by

$$L_\alpha \varphi(x) := \frac{1}{2} \text{Tr} \left[ \sigma^2 D^2 \varphi(x) \right] + \langle x, AD \varphi(x) \rangle + \langle F_\alpha(x), D \varphi(x) \rangle, \quad x \in H, \quad (5.20)$$

where $D^2$ denotes the second Fréchet derivative. Then, an easy application of Itô’s formula shows that the $L^1(H, \mu_\alpha)$-generator of $(P_\alpha t)$ (given as before by $P_\alpha t f(x) = \mathbb{E}[f(X_\alpha(t, x))]$) is given on $\mathcal{F}C^2_b(\{e_n\})$ by $L_\alpha$. In particular,

$$\int_H L_\alpha \varphi \, d\mu_\alpha = 0, \quad \forall \varphi \in \mathcal{F}C^2_b(\{e_n\}).$$

By a simple approximation argument and (5.2) we get for $\alpha \in (0, \alpha_0)$ and

$$\varphi_n(x) := \sum_{i=1}^n q_i^{-1} \langle x, e_i \rangle^2, \quad x \in H, \quad n \in \mathbb{N},$$

that also

$$\int_H L_\alpha \varphi_n \, d\mu_\alpha = 0. \quad (5.21)$$

But for all $x \in H$, with $\tilde{F}_\alpha$ defined as $\tilde{F}_0$ in (5.17), we have

$$L_\alpha \varphi_n(x) = -2 \sum_{i=1}^n \frac{\lambda_i}{q_i} \langle x, e_i \rangle^2 + 2 \sum_{i=1}^n q_i^{-1} \langle \tilde{F}_\alpha(x), e_i \rangle \langle x, e_i \rangle$$

$$+ \sum_{i,j=1}^n q_i^{-1} \langle \sigma_n e_i, \sigma_n e_j \rangle$$

$$\leq -2\Theta(\pi_n x) + 2 \left( \sum_{i=1}^n q_i^{-1} \langle \tilde{F}_\alpha(x), e_i \rangle^2 \right)^{1/2} \left( \sum_{i=1}^n q_i^{-1} \langle x, e_i \rangle^2 \right)^{1/2}$$

$$+ \sum_{i=1}^n q_i^{-1} |\sigma_n e_i|^2$$

$$\leq -2\Theta(\pi_n x) + c_1 (1 + |x|^{m+1} + \Theta^{1/2}(x)|x|) + \|\sigma\|^2 \sum_{i=1}^\infty q_i^{-1}, \quad (5.22)$$

for some constant $c_1$ independent of $n$ and $\alpha$. Here we used (2.3) and (5.18).

Now (5.21), (5.2) and (5.22) immediately imply that for some constant $\tilde{c}_1$

$$\sup_{\alpha \in (0, \alpha_0)} \int_H \Theta(x) \mu_\alpha(dx) \leq \sup_{\alpha \in (0, \alpha_0)} \tilde{c}_1 \left( 1 + \int_H |x|^{m+2} \mu_\alpha(dx) \right) + \|\sigma\|^2 \sum_{i=1}^\infty q_i^{-1} < \infty.$$
So, (5.19) is proved, which by (5.18) implies (5.4) and the proof is complete.

□

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