Tiling Vertices and the Spacing Distribution of their Radial Projection

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Abstract

The Fourier-based diffraction approach is an established method to extract order and symmetry properties from a given point set. We want to investigate a different method for planar sets which works in direct space and relies on reduction of the point set information to its angular component relative to a chosen reference frame. The object of interest is the distribution of the spacings of these angular components, which can for instance be encoded as a density function on $\mathbb{R}^+$. In fact, this radial projection method is not entirely new, and the most natural choice of a point set, the integer lattice $\mathbb{Z}^2$, is already well understood.

We focus on the radial projection of aperiodic point sets and study the relation between the resulting distribution and properties of the underlying tiling, like symmetry, order and the algebraic type of the inflation multiplier.

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1 Radial Projection Method

Given a locally finite point set $\Lambda \subset \mathbb{R}^2$, the following procedure is applied (see Fig. 1 for an example):

(a) For a reference point $x_0 \in \Lambda$ determine the subset $\Lambda'$ of points visible from this $x_0$. We call a point $p$ invisible if there exists a $p_0 \in \Lambda$ such that

$$\exists \ t \in (0,1) : p_0 = x_0 + t \cdot (p - x_0) . \tag{1}$$

(b) Select a radius $R > 0$ and project all $x \in \Lambda' \cap B_R(x_0)$ to $\partial B_R(x_0)$. If $x$ is given as $(a, b) \in \mathbb{R}^2$, then this amounts to mapping $x$ to $\arctan(b/a)$.

(c) The step in (b) produces a list of distinct (because of the visibility condition) angles $\Phi(R)$. Since $\Lambda$ is locally finite, we can sort the list

$$\Phi(R) = \{ \varphi_1, \ldots, \varphi_n \}$$

in ascending order. We also renormalise the $\varphi_i$ with the factor $\frac{1}{2\pi}$ such that the mean distance between consecutive entries becomes 1.

(d) Define $d_i := \varphi_{i+1} - \varphi_i$ and consider the discrete probability measure

$$\nu_R := \frac{1}{n-1} \sum_{i=1}^{n-1} \delta_{d_i} .$$

(e) Assuming that it exists, the spacing distribution is now obtained by taking the limit $R \to \infty$ in the sense of weak convergence of measures.

The choice of the point $x_0$ can be arbitrary and, in general the limit measure $\nu := \lim_{R \to \infty} \nu_R$ depends on it. For now, we restrict ourselves to reference points with high symmetry (see Fig. 2 and 3). Further investigations are needed to decide whether an averaging over multiple $x_0$ makes more sense here.

Figure 1: Radial projection using the example of the $\mathbb{Z}^2$ lattice.

All results (except for the reference cases) are currently strictly numerical and therefore rely on the computation of large circular patches of the point set and the determination of visibility. For an introduction to the topic of aperiodic tilings we refer to [1].

2 Integer Lattice $\mathbb{Z}^2$

The lattice $\mathbb{Z}^2$ provides the most ordered case of a planar point set. In this regard, it represents one reference point set for a potential classification of order. The set of visible points from the origin is given by

$$V_{\mathbb{Z}^2} = \{(a, b) \in \mathbb{Z}^2 : \gcd(a, b) = 1 \} . \tag{2}$$
In 2000, a closed expression \[2\] was derived for the limiting measure. With our setup, the explicit density function reads
\[
g(t) = \begin{cases} 
0, & 0 < t < \frac{3}{\pi},
\frac{12}{\tau t} \cdot \log \left( 2 \left/ \left( 1 + \sqrt{1 - \frac{12}{\pi^2 t}} \right) \right. \right), & \frac{12}{\tau t} < t < \frac{12}{\pi},
\frac{12}{\pi t} \cdot \log \left( 2 \left/ \left( 2 + \sqrt{1 - \frac{12}{\pi^2 t}} \right) \right. \right), & t > \frac{12}{\pi},
\end{cases}
\]
but in fact the existence also holds for more general expanding regions (which is a circle here). A Taylor expansion of the tail of \( g(t) \) gives
\[
g(1/t) = \frac{36}{\pi^4} t^3 + \frac{162}{\pi^6} t^4 + O(t^5) \text{ for } t \to 0^+,
\]
making it obvious that the moments of order \( k \geq 2 \) do not exist. A plot of \( g(t) \) is overlayed over most histograms (see e.g. Fig. 4 and 5).

For all other cases (apart from the next), only histograms were computed.

3 Poisson Distributed Points

In contrast to the \( \mathbb{Z}^2 \) case, the point set generated by a homogeneous spatial Poisson process gives us the other reference point for our classification. In terms of order, it represents the case of total disorder. The model is also known as complete spatial randomness (CSR) or ideal gas in terms of physics.

Application of the radial projection procedure, with an arbitrary choice of reference point, yields a 1-dimensional CSR with intensity \( \lambda = 1 \) (due to the normalisation) in step\[6\]. From there, it is not hard to see that the density of the radial projection measure is given by
\[
f_\lambda(x) = \begin{cases} 
\lambda \exp(-\lambda x), & x \geq 0,
0, & x < 0,
\end{cases}
\]
since step\[6\] just asks the question what the distribution of the waiting time between jumps of the process is. The density \( f_\lambda(x) \) (a plot can be seen in Fig. 1) therefore provides the second explicit function we can test histograms against.

4 Cyclotomic Model Sets and Visibility

Many aperiodic tilings can not only be realised by inflation of a set of prototiles, but also by projecting a higher-dimensional lattice into \( \mathbb{R}^2 \). This is often described as the cut and project method or model set description.

Our general interest is in cyclotomic model sets, since these can be used to construct \( n \)-fold symmetric tilings. Also, this description is more suitable for radial projection since it only gives the tiling vertices and allows precise control over the patch size.

The Ammann–Beenker (AB) and the Tübingen triangle (TT) tiling can both be described as a cyclotomic...
with \( \lambda_{\text{sm}} = 1 + \sqrt{2} \), the silver mean. The \( \mathbb{Z} \)-module in this case can be decomposed into \( \mathbb{Z}[\zeta_8] = \mathbb{Z}[\sqrt{2}] \oplus \mathbb{Z}[\sqrt{2}] \cdot \zeta_8 \) and coprime in this context means that for \( \mathbb{Z}[\zeta_8] \ni x = x_1 + x_2 \cdot \zeta_8 \) the \( \gcd(x_1, x_2) \) is a unit in \( \mathbb{Z}[\sqrt{2}] \).

The function \( n \) decomposes a \( x \in \mathbb{Z}[\zeta_{12}] \) into the direct-sum representation

\[
\mathbb{Z}[\sqrt{3}] \oplus \mathbb{Z}[\sqrt{3}] \cdot \zeta_{12}
\]

with \( \lambda_{12} := 2 + \sqrt{3} \), and then computes \( |N(\gcd(x_1, x_2))| \), the absolute value of the algebraic norm of the gcd of the components. The two \textit{rescaling factors} for the visibility test are given by

\[
\lambda_1 := \sqrt{\lambda_{12} / 2} \quad \text{and} \quad \lambda_2 := \sqrt{\lambda_{12} / 2} .
\]

The first part of the set \( V_{\text{GS}} \) (indicated in grey in Fig. 7) comprises the already known coprime elements.

Even though Fig. 8 deviates a lot more from the \( \mathbb{Z}^2 \) distribution (compared to Fig. 4 and 5), all considered cyclotomic cases exhibit the special threefold structure (gap, bulk and tail). Let us consider a different tiling.
6 A Non-Pisot Case / Lançon–Billard

The Lançon–Billard (LB) tiling is an example of a chiral (the tiles only appear in one chirality) inflation tiling with a multiplier \( \lambda_{LB} = \sqrt{(5 + \sqrt{5})/2} \), which is a non-Pisot number. The inflation rules (a stone inflation is only possible with a fractal tile boundary) generate a very irregular tiling.

![Figure 10](image1.png)

**Figure 10:** Tile A maps to 3×A and 1×B.

![Figure 11](image2.png)

**Figure 11:** Tile B maps to 1×A and 2×B.

The resulting distribution shows that the radial projection is sensitive to this irregularity.

![Figure 12](image3.png)

**Figure 12:** LB 5-fold symmetric patch.

In fact, on this level, the LB tiling is almost indistinguishable from the Poisson case.

7 Some Additional Examples

We have already seen in Secs. 4 and 5 that the radial projection reacts to the order of symmetry of the tiling. The vertices of the chair tiling in Fig. 14 are a subset of \( \mathbb{Z}^2 \), but a different type of visibility condition has to be applied. The bulk section notices these changes, and shows a lot of structure in general.

![Figure 13](image4.png)

**Figure 13:** Spacing distribution of a large LB patch.

8 Tail Decay Behaviour

The expansion in Eq. (6) gives us an idea how the tail decay of the distribution behaves. A power law fit (\( c_k \) the coefficient of \( t^k \)) applied to the numerical data indicates that at least the CMS cases also show a similar kind of decay.

![Figure 14](image5.png)

**Figure 14:** Spacing distribution of the chair (left) and rhombic Penrose (right) tiling.

![Figure 15](image6.png)

**Figure 15:** Spacing distribution of the Penrose-Robinson tiling and zoom into the bulk.

| Tiling | Gap Size | \( c_3 \) | \( c_4 \) | \( e \) |
|--------|----------|---------|---------|------|
| \( \mathbb{Z}^2 \) | 0.304 | 0.369 | 0.168 | — |
| AB | 0.222 | 0.248 | 0.496 | 2.79 |
| TT | 0.182 | 0.239 | 0.513 | 2.60 |
| GS | 0.152 | 0.232 | 0.547 | 4.75 |

Table 1: Statistical data generated from the radial projection (\( e = \text{error} \cdot 10^{-10} \)).

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