Data-Driven Models of Selfish Routing: Why Price of Anarchy Does Depend on Network Topology

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Abstract. We investigate traffic routing both from the perspective of real world data as well as theory. First, we reveal through data analytics a natural but previously uncaptured regularity of real world routing behavior. Agents only consider, in their strategy sets, paths whose free-flow costs (informally their lengths) are within a small multiplicative \((1 + \theta)\) constant of the optimal free-flow cost path connecting their source and destination where \(\theta \geq 0\). In the case of Singapore, \(\theta = 1\) is a good estimate of agents’ route (pre)selection mechanism. In contrast, in Pigou networks the ratio of the free-flow costs of the routes and thus \(\theta\) is infinite, so although such worst case networks are mathematically simple they correspond to artificial routing scenarios with little resemblance to real world conditions, opening the possibility of proving much stronger Price of Anarchy guarantees by explicitly studying their dependency on \(\theta\). We provide an exhaustive analysis of this question by providing provably tight bounds on \(\text{PoA}(\theta)\) for arbitrary classes of cost functions both in the case of general congestion/routing games as well as in the special case of path-disjoint networks. For example, in the case of the standard Bureau of Public Roads (BPR) cost model, \(c_e(x) = a_e x^4 + b\), and more generally quartic cost functions, the standard PoA bound for \(\theta = \infty\) is 2.1505 [23] and it is tight both for general networks as well as path-disjoint and even parallel-edge networks. In comparison, in the case of \(\theta = 1\), the PoA in the case of general networks is only 1.6994, whereas for path-disjoint/parallel-edge networks is even smaller (1.3652), showing that both the route geometries as captured by the parameter \(\theta\) as well as the network topology have significant effects on PoA (Figure 1).

Keywords: Non-atomic Congestion Games · Equilibrium Flow · Data Analytics · Empirics · Primal-Dual Framework.

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Fig. 1: Improved Price of Anarchy bounds in data-driven routing models
1 Introduction

Modern cities are wonders of emergent, largely self-organizing, behavior. Major capitals buzz with the collective hum of millions of people whose lives are intertwined and coupled in myriad and diverse ways. One of the most palpable such phenomena of collective behavior is the emergence and diffusion of traffic throughout the city. A bird’s eye view of any major city would reveal a complex and heterogeneous landscape of thousands upon thousands of cars, buses, trucks, motorcycles, running though the veins of a maze of remarkable complexity and scale consisting of a vast number of streets and highways. As Figure 2 suggests, the full magnitude of the multi-scale complexity of these real-life networks lies outside the perceptive capabilities of any single individual. Nevertheless, as a phenomenon that we get to experience daily, such as the weather, we would like to understand at least some macroscopic, high level characteristics of traffic routing. Quite possibly, one of the most interesting such questions is how efficient is a traffic network?

This question has received a lot of attention within algorithmic game theory. Using the model of congestion games, seminal papers in the area established tight bounds on their Price of Anarchy (PoA), i.e., the worst case inefficiency of traffic routing [15, 25]. For example, the Price of Anarchy of linear non-atomic congestion games is $A/3$, whereas if we apply the standard Bureau of Public Roads (BPR) cost functions that are polynomials of degree four, then the Price of Anarchy is roughly 2.151. On the positive side, these bounds apply to all networks (within the prescribed class of delay/cost functions) regardless of their size or their total demand, or number of agents and are tight even for the simplest possible network instances, i.e., Pigou networks with just two parallel links.

The common interpretation of these bounds is that they are strong and a PoA anywhere in that range (e.g. PoA=2) immediately translates to practical guarantees about real traffic. Some recent purely experimental work, however, has produced new insights that allow us to reexamine these results from a different perspective. For example, [15] showed that the efficiency of real-life traffic networks, as estimated from traffic measurements alone, is really close to optimal even when compared to very optimistic estimates of optimal performance. A Price of Anarchy of 2 implies that the average commuter can increase their mean speed by 100%. Measurements suggest that this level of inefficiencies/improvements is rather unlikely. Since Price of Anarchy is a macroscopic characteristic of a system with countless moving parts, a more useful analogy is that of weather or climate (e.g., average temperature). The differences between 10% and 20% increase to system inefficiency are significant and a 100% increase, i.e., PoA of 2 would have catastrophic consequences.

A natural question emerges: Can we create classes of models, i.e., congestion games, which come closer to representing real world traffic? In this paper we do, by leveraging an intuitive but largely unexplored characteristic of real world traffic routing. Commuters only consider in their strategy sets paths/routes whose free-flow costs (informally their lengths) are approximately equal to each other (within a multiplicative factor of $1 + \theta$).

We call such games $\theta$-free flow games. We generalize the special case of linear congestion $\theta$-free flow games [3] to the case of arbitrary classes of cost functions as well as simultaneously studying both general networks as well as path-disjoint networks. $\theta = 0$ means that all paths considered by each user have exactly equal free-flow cost/length, whereas $\theta = 1$ allows for paths whose lengths are within a factor of 2. Pigou networks may feel intuitively very simple and thus natural due to their small size, but they fail to satisfy this property in the most extreme sense. The ratio of the free-flow costs of the two edges is infinite ($\theta = \infty$). It is like considering two possible paths from home to work, one which is the shortest distance route and one that circumnavigates the globe along the way. Such unnatural paths may indeed be available to us, but we unconsciously and automatically prune them out from the set of alternatives that we consider. Amazingly, enforcing such a natural property on the set of models (routing games) we consider immediately removes from consideration Pigou networks, the worst case examples from a PoA perspective, and thus opens up the possibility of proving stronger Price of Anarchy guarantees. What are the implications of such characteristics to PoA? What other type of attributes can we take advantage of when creating new models? Finally, how well do they match real traffic conditions?

1.1 Our Contribution

In Section 2, we start off by experimentally computing estimates of $\theta$ from real world traffic data. We employ an experimental dataset that contains detailed information (sampled every 13 seconds) on the routing behavior of tens of thousands of commuters in Singapore. Based on this fine-grained information and in combination with a graph representation of the road network of Singapore that we have created we can estimate numerous characteristics of the actual routing behavior at an unprecedented level of accuracy. Using these tools that we believe are of independent interest as well, we find that the $\theta$ values for the vast majority of commuters (close to 80%) are below 1.
Inspired by the above evidence, we introduce a new class of congestion games, that we call free-flow games, parametrized by $\theta$ (Section 3). Building on the primal-dual method introduced in [1], we provide two parametric tight bounds on the Price of Anarchy of free-flow games under general latency functions satisfying mild assumptions, thus largely extending the results given in [3] which are restricted to affine latencies only. The first of these bounds applies to the general case of unrestricted network topologies (indeed, it applies even to congestion games) (Theorem 1), while the second one holds for path-disjoint networks (Theorem 2) which includes the fundamental parallel-links topology. These bounds are never equal as long as $\theta \not\in \{0, \infty\}$. In fact, differently from what happens in the classical setting without the free-flow assumption, where the worst-case situation already arises in a two parallel-links network (the Pigou network), for free-flow games the absence of intersections among paths allows for more efficient equilibria. More precisely, as $\theta$ goes to infinity, both bounds converge to the same limit, but the convergence of the one for parallel-link networks can be significantly slower (see, for instance, Figure 1(a)). We also stress that, with respect to the case of affine latency functions, our findings improve on the results given in [3], as we close the gap between upper and lower bound on the Price of Anarchy for parallel-link networks that was left as an open problem.

One of the most important messages coming from our investigation is that the separation outlined by Theorems 1 and 2 sheds new light on the question of whether the Price of Anarchy is affected by the network topology. In fact, a famous, and perhaps counter-intuitive, result by Roughgarden [23] states that the PoA is independent of the network topology as, in almost all notable cases, worst-case instances are already attained by simple networks, such as parallel-link graphs. Under the free-flow assumption, however, this situation ceases to hold, and the network topology begins to play a critical, if not dominant, role in the efficiency of equilibria. This evidence has major practical implications, as it signifies the fundamental importance of careful road network design and planning for selfish routing. As shown in Figure 1 and in more details in Table 1, in the case of the standard Bureau of Public Roads (BPR) cost model, $c_\theta(x) = a_\theta x^4 + b_\theta$ and more generally quartic cost functions, applying the constraint $\theta = 1$ nearly halves the percentage of inefficiency, and applying the additional constraint of a path-disjoint network halves it once again.

At the technical level, our general formulas depend on whether the free-flow traversing time of some edges is larger than zero, i.e., whether the limit of the edge cost/latency as its load goes to zero is strictly positive. Latency functions for which this does not hold have been termed homogeneous by Roughgarden [23] and they...
represent one of the few exceptions for which he could not prove that the PoA is independent of the network topology. Since under homogeneous latency functions any congestion game is a 0-free flow game, as a by-product of our results, we also obtain that, for (free-flow) games with homogeneous latency functions, the Price of Anarchy is lower than the one attained by non-homogeneous latencies, and it is tight even for parallel-links topologies (Theorem 2), thus answering the open question posed by Roughgarden in 23.

To summarize, we obtain that the Price of Anarchy is independent of the network topology (i.e., the worst-case PoA is attained by parallel-link games) if and only if one of the following cases occurs: (i) \( \theta = 0 \) (which include the case of homogeneous latency functions as a special case) and (ii) \( \theta = \infty \).

For the sake of a more concrete exposition of our results and for empirical purposes, we provide explicitly an instantiation of the PoA bounds in the case of polynomial latency functions (Theorems 3 and 4). The resulting bounds depend on both the maximum and minimum degree of the polynomials and, in the case of non-homogeneous polynomials only, they also depend on \( \theta \). A quantitative representation of our results is partially summarized in Table 1.

| \( (p, q) \) | \( \theta = 0 \) | \( \theta = 1/2 \) | \( \theta = 1 \) | \( \theta = \infty \) |
|---|---|---|---|---|
| | General | Path-disjoint | General | Path-disjoint | General | Path-disjoint | General | Path-disjoint |
| (1, 1) | 1 | 1 | 1.1547 | 1.0909 | 1.2071 | 1.1429 | 1.3333 | 1.3333 |
| (2, 1) | 1.0355 | 1.0355 | 1.2873 | 1.1472 | 1.3852 | 1.2383 | 1.6258 | 1.6258 |
| (2, 2) | 1 | 1 | 1.2873 | 1.1472 | 1.3852 | 1.2383 | 1.6258 | 1.6258 |
| (3, 1) | 1.0982 | 1.0982 | 1.4078 | 1.1869 | 1.5475 | 1.3093 | 1.8956 | 1.8956 |
| (3, 2) | 1.0147 | 1.0147 | 1.4078 | 1.1869 | 1.5475 | 1.3093 | 1.8956 | 1.8956 |
| (3, 3) | 1 | 1 | 1.4078 | 1.1869 | 1.5475 | 1.3093 | 1.8956 | 1.8956 |
| (4, 1) | 1.1676 | 1.1676 | 1.5202 | 1.2170 | 1.6994 | 1.3652 | 2.1505 | 2.1505 |
| (4, 2) | 1.0450 | 1.0450 | 1.5202 | 1.2170 | 1.6994 | 1.3652 | 2.1505 | 2.1505 |
| (4, 3) | 1.0080 | 1.0080 | 1.5202 | 1.2170 | 1.6994 | 1.3652 | 2.1505 | 2.1505 |
| (4, 4) | 1 | 1 | 1.5202 | 1.2170 | 1.6994 | 1.3652 | 2.1505 | 2.1505 |

Table 1: The Price of Anarchy of free-flow games with non-homogeneous (i.e., with constant terms allowed) polynomial latency functions of maximum degree \( p \leq 4 \) and minimum degree \( q \). Unlabelled bounds are proven in this paper. Bounds for homogeneous (i.e., without constant terms) polynomials can be obtained from the case \( \theta = 0 \) (the same upper bounds have been given in 10, but tight lower bounds were only conjectured to exist). As it can be appreciated, the PoA depends on the network topology whenever \( 0 < \theta < \infty \).

1.2 Related Work

Price of anarchy in routing games: Introduced by Koutsoupias and Papadimitriou 15, the ratio between the social cost of the worst equilibrium of a game and its optimum was given the name Price of Anarchy (PoA) in 22. For networks of linear latency and general topology, PoA was bounded tightly by 4/3 23 and 5/2 in the atomic case 6. Following results by Roughgarden 24 studied more general latency functions and atomic routing games and again gave tight bounds on PoA. However, for a large class of natural latency functions, PoA tends to 1 as the demand on the network approaches infinitesimally small or infinitely high levels 7, 8. This casts doubts on the predictive power of PoA on the state of a real system, as noted in Monnot et al. 18.

Strategy sets of routing games are typically exponential in the number of vertices, hence restricting them is a common assumption. The unnatural character of Pigou in real systems was noted by Lu and Yu 17, who assume players have at least one strategy that is not more than \( \lambda \) away from the fastest strategy in congestion games. Restricting the strategy sets to obtain tighter bounds for PoA is also employed in 21 for load balancing games (i.e., congestion games where the strategies of players are singleton sets). Fotakis 11 proved a pure PoA bound for symmetric atomic congestion games on extension-parallel networks, an interesting class of networks with linearly independent paths, that is equal to that of non-atomic congestion games.

Primal-dual techniques for bounding the Price of Anarchy in non-cooperative games have been proposed by Bil 1, Kulkarni and Mirrokni 16, Nadav and Roughgarden 20 and Thang 25. The methods proposed in 1 and 20 operate by explicitly formulating the problem of maximizing the Price of Anarchy of a class of games. Despite using the same formulation, they differ in the choice of the variables. While 20 uses the probability distributions defining the outcomes occurring in the formulation, 11 adopts suitable multipliers for the resource
cost functions. The methods in [16] and [26], instead, build on a formulation for the problem of optimizing the social function, and then implement the equilibria conditions within the choice of the dual variables. We adopt the method proposed in [1] as it appears to be more flexible and powerful in our realm of application. The first advantage is that it generalizes to any type of cost functions, while all the others require some restrictions: the method in [20] can only be applied to affine functions, the one in [16] requires convex functions, while that of [26] needs non-decreasing ones. Secondly, the method (if properly used) always yields tight bounds on the Price of Anarchy, while those in [16] and [26] are limited by the integrality gap of the formulation. Last but not least, it models in a simple, direct and intuitive way any new twist, as the free-flow property considered in this work, one may want to add to the scenario of application.

Transportation research: The seminal work of Wardrop [28] introduces and formalizes one of the first notions of equilibrium in transportation networks. A proof of the equal social costs for equilibria and optimum (i.e., PoA = 1) in parallel links routing games appears in Nagurney and Qiang [21]. Related ideas from sensitivity analysis for edge cost functions are treated in Tobin and Friesz [27]. The Price of Anarchy was estimated for the city of Boston with different means from our study by Zhang et al. [30], where the sensitivity of the social cost at equilibrium with respect to edge parameters is also discussed. The previously cited works rely on the BPR estimation of cost functions [4], which are included in the family of weakly monomial latency functions we define in Section 3. The free-flow property in transportation networks has been first proposed by Jahn et al. [14] with respect to the problem of optimizing a centralized traffic flow without imposing too longer detours to some users.

2 Experimental Evidence for \( \theta \)-Free-Flow Time in Singapore

We look for experimental evidence that commuters use the heuristic presented in the introduction to guide their routing decisions. Namely, we make the conjecture that commuters consider only paths with “length” at most a multiplicative factor \( 1 + \theta \) away from the shortest path taking them to their destination (where “length” is measured as a latency, or travel time). Does this conjecture hold in practice?

To answer, we must obtain data on the routing behavior of a sampled population. Knowing the route taken by individuals in the sample, we must be able to infer what their travel time would have been in free-flow road conditions, i.e., without anyone else on the road (the data free-flow time). Finally, we must compute the shortest free-flow travel time on any path connecting their origin to their destination (the best free-flow time). By comparing data and best free-flow time for each individual in the sample, we arrive at a distribution of \( \theta \) over our set of trips.

In this section, we make use of the Singapore National Science Experiment dataset to understand the routing behavior of its participants. First, we introduce the dataset and our data processing methods. Second, we provide the methodology for estimating the data free-flow trip duration of the subjects’ chosen morning route, computed from the collected data. Third, we compare this measure with the best free-flow time, optimized over all commuting paths.

2.1 The National Science Experiment

As part of the Smart Nation programme, the National Science Experiment (NSE) is a nationwide project in Singapore in which over 90,000 students from primary, secondary and junior college wore a sensor, called SENSg, for up to one week per student in 2015 and 2016. The SENSg sensors collect ambient temperature, relative humidity, atmospheric pressure, light intensity, sound pressure level, and 9-degree of freedom motion data. The NSE initiative led up to the mass-production of 50,000 sensor nodes. The SENSg scans the Wi-Fi hotspots which are used to localize the sensor nodes as well as to move sensor data to a back-end server. All environment and motion values are sampled every 13 seconds using the Wi-Fi based localization system. The raw collected datapoints are then post-processed to obtain semantic data, employing state-of-the-art methods described in [18, 19, 29] and employed to study the relationship between efficiency and inequality in [12]. The semantic data covers the identification of individual trips within the discrete stream of locations, inference of the activity performed at each endpoint and transportation mode classification.

The NSE 2016 dataset contains data from 49,526 students who wore the SENSg sensor. This work uses the mode identification algorithm developed in [31] where five different modes can be identified, namely: (a) stationary; (b) walking; (c) riding a train; (d) riding a bus; and (e) riding a car. With additional information

\(^5\) Modelling assumptions and a formal definition of \( \theta \) are presented in Section 3.
from Singapore’s Land Transportation Authority, the algorithm detection covers 8 rail lines, 106 train stations, 260 bus services and 4,684 bus stops. Similarly, the 164 km of expressways and the 698 km of arterial roads in Singapore feed the algorithm to distinguish whether a subject is traveling in a car.

To ensure the quality of our empirical results, we perform a strict data cleaning process over the complete dataset. A total of 34,121 clean trips are considered, with 16,563 unique students and 89 different schools. This work focuses on morning travels of students who get to their schools from their homes. Two main reasons were considered for this choice.

First, in the following analyses, the latency, or duration of the trip, is considered as the primary “cost” of the subjects, discounting any other monetary cost. Morning trips typically feature subjects optimizing to minimize their latency. Evening trips are more sparse since the battery of sensor is expected to be charged at night while the subject is home. By the end of the day, if it has run out due to not being charged properly, the evening trip is not recorded. We have however in the dataset 21,065 samples for which both morning and evening trips are recorded. For these pairs, the average duration of the morning trip is 29 minutes and 6 seconds, while it is 33 minutes and 33 seconds for the evening trip.

Second, the data source—students of Singapore—may not constitute a fully representative sample of Singapore’s population. However, their exposure to traffic during the morning hours—which are effectively the most congested conditions—allows us to infer properties of the system as a whole. The geographical distribution of their homes broadly correspond to the population density of Singapore, and thus provides additional confidence on the representativeness. Additionally, the number of students by school type is approximately equally distributed, hence capturing the routing behavior of subjects over a large space in Singapore.

Our dataset contains highly granular information concerning the routing decisions of the subjects. With the help of the onboard sensors in the device and the mode identification algorithm, we are able to obtain for each trip an accurate representation of its segments and their endpoints. For instance, typical segments making up a trip may be “Walk - Car - Walk”, or “Walk - Bus - Train - Bus - Walk”. The following study focuses on car trip segments. In this dataset \cite{18}, looking at the population of public transport users only, Price of Anarchy was upper bounded by 1.18. Conversely, Price of Anarchy for car users only was bounded by 1.86. Putting both populations together, Price of Anarchy was bounded by 1.34.

2.2 Estimation of Free-Flow Time for Selected Route

We compute a graph representation from a road map of Singapore, where each vertex is located at an intersection or a bend in the road. An edge connecting two vertices indicates the presence of a segment of road going from one vertex to the other. Edges also possess additional metadata: their physical length (in meters) as well as the road type—such as expressway, local street, arterial road, and so on.

Every edge is assigned with a cost representing how much time is needed to traverse it. This latency is obtained from edge features such as the road type and the posted speed limit on the road. For each private transportation trip segment in the dataset, we associate its origin and destination with the closest vertex in the graph. We run a shortest path algorithm to estimate the free-flow travel time of the trip segment, referred to in the following as the best free-flow time. This best free-flow time is compared with the data free-flow time, or the time it would take the subject to travel the same trip segment if no one was on the road. We describe how the data free-flow time is estimated in the following paragraph.

A segment measured by the sensor consists of a stream of geographical locations. For each datapoint, we associate the closest edge in the graph. The size of the graph (61,151 vertices and 65,596 edges) implies a lengthy lookup phase to associate the point to its closest edge. For this reason, we consider a smaller dataset of 449 car segments out of the 17,897 segments in the larger dataset. These selected segments are well distributed across Singapore as depicted by Figure 2.

The direction in which the subject traversed the edge is assigned by a heuristic based on the distance of each endpoint to the endpoints of edges preceding and following the edge under consideration. In other words, the heuristic attempts to minimize the amount of back and forth, selecting the direction that least creates deviations.

Information on the origin and destination of the trip as well as the list of directed edges traversed by the subject does not suffice. Where the sensor does not record a datapoint we must provide a best guess on which edges were crossed during the trip.

\footnote{Geographical location is obtained by scanning surrounding WiFi access points. The method does not always yield accurate enough measurements, but the issue can be mitigated with proper data processing \cite{19}.}
We assume that latency functions are non-decreasing, positive, and continuous $\sum$ and $\mathbb{R}$. 

Non-atomic Congestion Games.

is a tuple $CG = ([n],\{r_i\}_{i \in [n]}, E, (\ell_e)_{e \in E}, (\Sigma_i)_{i \in [n]})$, where $[n]$ is a set of types, $E$ is a set of resources, $\ell_e : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is the latency function of resource $e \in E$, and, for each $i \in [n]$, $r_i \in \mathbb{R}_{\geq 0}$ is the amount of players of type $i$ and $\Sigma_i \subseteq 2^E \setminus \emptyset$ is the set of strategies for players of type $i$ (i.e. a strategy is a non-empty subset of resources).

We assume that latency functions are non-decreasing, positive, and continuous. Anyway, our theoretical results hold even with the weaker assumption of right-continuity.

### 2.3 Deviation and Estimate of $\theta$

For each trip segment, two estimates are obtained: the best free-flow time and the data free-flow time. We call deviation the ratio between these two estimates. The deviation is strongly related to the parameter $\theta$ we introduce in Section 1. It measures the free-flow time difference between the best route the subject could have chosen and the route actually selected, both in a situation of no congestion. The distribution of the deviation among subjects provides a clue to estimating $\theta$ for the routing game of Singapore. A small value of $\theta$ yields support to the hypothesis that agents only consider routes which connect origin and destination in a straightforward manner (under no congestion) as part of their strategy set, see Figure 3.

Fig. 3: The deviation is measured by the ratio of the selected route free-flow time to the minimum free-flow time among all routes between the origin and the destination. Close to 80% of the $\theta$ values are below 1, implying that the free-flow time of the selected route is rarely twice as long as the best free-flow time.

This experimental result provides justification for the upper bound of PoA estimated from the same dataset in previous work [19]. This benchmark is meaningful for real road networks, as latency functions are typically estimated using affine quartic monomials [4]. As noted in our introduction as well as in more details in the next section, our model is based on the assumption of a uniform $\theta$ bound over the whole population. We should note that this assumption is consistent with our experimental measurements, since these measurements provide us with estimates on the lower bounds of the agents’ $\theta$’s. More detailed models with a heterogeneous population/distribution of $\theta$’s is an interesting direction for future work.

### 3 Model and Definitions

For a positive integer $i$, let $[i] := \{1, 2, \ldots, i\}$. Given a set $A$ and a set $B \supseteq A$, let $\chi_A : B \rightarrow \{0, 1\}$ denote the indicator function, i.e., $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ if $x \notin A$. Given a tuple of numbers $(\alpha_1, \alpha_2, \ldots, \alpha_k)$, we write $(\alpha_1, \alpha_2, \ldots, \alpha_k) > 0$ if $\alpha_i > 0$ for any $i \in [k]$ and $\alpha_i > 0$ for some $i \in [k]$.

Non-atomic Congestion Games. A non-atomic congestion game (from now on, simply a congestion game) is a tuple $CG = ([n],\{r_i\}_{i \in [n]}, E, (\ell_e)_{e \in E}, (\Sigma_i)_{i \in [n]})$, where $[n]$ is a set of types, $E$ is a set of resources, $\ell_e : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is the latency function of resource $e \in E$, and, for each $i \in [n]$, $r_i \in \mathbb{R}_{\geq 0}$ is the amount of players of type $i$ and $\Sigma_i \subseteq 2^E \setminus \emptyset$ is the set of strategies for players of type $i$ (i.e. a strategy is a non-empty subset of resources).

7 The property of continuity is well-motivated by most of the real-life scenarios modelled by non-atomic congestion games. Anyway, our theoretical results hold even with the weaker assumption of right-continuity.
Classes of Congestion Games. A network congestion game is a congestion game based on a graph \( G = (V, E) \), where the set of resources coincides with \( E \), each type \( i \) is associated with a pair of nodes \((u_i, v_i) \in V \times V \), so that the set of strategies of players of type \( i \) is the set of paths from \( u_i \) to \( v_i \) in graph \( G \). If there exists \( u^* \in V \) such that \( u^* = u_i \) for any \( i \in [n] \), the game is called single-source network congestion game. Let \( \mathcal{P} \) be the set of all the paths \( P \) connecting source \( u_i \) with destination \( v_i \), for any pair source-destination \((u_i, v_i)\). The game is called path-disjoint network congestion game if all the paths in \( \mathcal{P} \) are pair-wise node-disjoint.

A load balancing game is a congestion game in which each strategy is a singleton, i.e., \( S = \{e\} \) for some \( e \in E \), for any strategy \( S \in \Sigma_i \) and type \( i \in [n] \). A parallel-link game (or symmetric load balancing game) is a load balancing game in which all players have the same set of strategies. It is well-known that each load balancing game (resp. parallel-link game) can be modelled as a single-source congestion game (resp. path-disjoint network congestion game).

Latency Functions. For the sake of simplicity, we extend the domain of each latency function \( \ell(x) \) to \( x = 0 \) in such a way that \( \ell(0) = \lim_{x \to 0^+} \ell(x) \). Given a class of latency functions \( \mathcal{F} \), let \( \mathcal{F}[H] := \{f : f(x) = g(x) - g(0), \, g \in \mathcal{F}\} \). Observe that \( f(0) = 0 \) for any \( f \in \mathcal{F}[H] \) by definition. In the following, we use similar definitions as in [23]. \( \mathcal{F} \) is homogeneous if \( \mathcal{F} = \mathcal{F}[H] \). \( \mathcal{F} \) is weakly diverse if \( \mathcal{F}[H] \subseteq \mathcal{F} \) and it contains at least one constant function (i.e., a function \( f \) such that \( f(x) = \beta \) for any \( x > 0 \), for some \( \beta > 0 \)). \( \mathcal{F} \) is scale-closed if it contains all the functions \( f \) such that \( f(x) = \alpha g(x) \), for any \( g \in \mathcal{F} \) and \( \alpha > 0 \). \( \mathcal{F} \) is strongly diverse if it contains all the functions \( f \) such that \( f(x) = \alpha g(x) + \beta \), for any \( g \in \mathcal{F}[H] \) and \( (\alpha, \beta) > 0 \).

A polynomial latency function of maximum degree \( p \) and minimum degree \( q \) (with \( p \geq q \geq 1 \)) is defined as \( \ell_p(x) := \sum_{d=0}^{q} a_d x^d + \beta_\ell \), where \( (a_0, a_1, \ldots, a_p, \beta_\ell) > 0 \). Let \( \mathcal{P}_{p,q} \) denote the class of polynomial latency functions of maximum degree \( p \) and minimum (non-zero) degree \( q \). A weakly monomial latency function of degree \( p \) is defined as \( \ell_p(x) := \alpha_p x^p + \beta_\ell \), with \( \alpha_p > 0 \), and \( \beta_\ell \geq 0 \). In the previous definition, \( \ell_\ell \) is called monomial latency function of degree \( p \) if \( \beta_\ell = 0 \). Let \( WM_p \) (resp. \( WM_p \)) denote the class of of weakly monomial latency functions (resp. monomial latency functions). Observe that \( WM_p \subseteq WM_p \) for any integer \( p \geq 1 \).

A latency function \( \ell_\ell \) is affine if \( \ell_\ell \in WM_1 \), and it is linear if \( \ell_\ell \in M_1 \).

Strategy Profiles and Pure Nash Equilibria. A strategy profile is a tuple \( \sigma := (\sigma_i, S) \in [n], S \subseteq \Sigma_i \) with \( \sum_{S \subseteq \Sigma_i} \sigma_i, S = r_i \) for any \( i \in [n] \), that is a state of the game where \( \sigma_i, S \geq 0 \) is the total amount of players of type \( i \) selecting strategy \( S \) for any \( i \in [n] \) and \( S \subseteq \Sigma_i \). Given a strategy profile \( \sigma, k_\sigma := \sum_{i \in [n], S \subseteq \Sigma_i; e \in S} \sigma_i, S \) is the congestion of \( e \) in \( \sigma \), i.e., the total amount of players selecting \( e \) in \( \sigma \), and given a strategy \( S \), \( c_i(S) := \sum_{e \in S} \ell_\ell(k_\sigma(e)) \) is the cost of players selecting \( S \) in \( \sigma \). A strategy profile \( \sigma \) is a pure Nash equilibrium (or Wardrop equilibrium, or equilibrium flow) if and only if, for each \( i \in [n] \), \( S \subseteq \Sigma_i : \sigma_i, S > 0 \) and \( S' \subseteq \Sigma_i \), it holds that \( c_i(S) \leq c_{i'}(S') \).

Quality of Equilibria. A social function that is usually used as a measure of the quality of a strategy profile in congestion games is the total latency, defined as \( \text{SUM}(\sigma) := \sum_{e \in E} k_\ell(\sigma) \ell_\ell(k_\sigma(e)) = \sum_{i \in [n]} r_i c_i(\sigma) \) at equilibrium \( \sigma \). A social profile is a strategy profile \( \sigma^* \) minimizing \( \text{SUM} \).

The Price of Anarchy of a congestion game \( CG \) (with respect to the social function \( \text{SUM} \)), denoted as \( \text{PoA}(CG) \), is the supremum of the ratio \( \text{SUM}(\sigma)/\text{SUM}(\sigma^*) \), where \( \sigma^* \) is a pure Nash equilibrium for \( CG \) and \( \sigma^* \) is a social optimum for \( CG \). As shown in [23], all pure Nash equilibria of any congestion game have the same total latency. Thus, the Price of Anarchy can be redefined as the ratio \( \text{SUM}(\sigma)/\text{SUM}(\sigma^*) \), where \( \sigma \) is an arbitrary pure Nash equilibrium for \( CG \) and \( \sigma^* \) is a social optimum for \( CG \).

Free-Flow Congestion Games Given \( \theta \in [0, \infty) \), a \( \theta \)-free-flow congestion game \( CG_\theta \) is a congestion game in which, for each \( i \in [n] \) and \( S, S' \subseteq \Sigma_i \), it holds that \( \sum_{\ell \in S} \ell_\ell(0) \leq (1 + \theta) \sum_{\ell \in S'} \ell_\ell(0) \), i.e., all the strategies available to players of type \( i \) are kept. When evaluated in absence of congestion, are within a factor \( 1 + \theta \) one from the other. Observe that free-flow congestion games are congestion games obeying some special properties. Thus, all positive results holding for congestion games carries over to \( \theta \)-free-flow congestion games for any value of \( \theta \). Moreover, for \( \theta = \infty \), any congestion game is a \( \theta \)-free-flow congestion game.

4 Price of Anarchy of Free-Flow Congestion Games

In this section, we give tight bounds on the Price of Anarchy of free-flow congestion games. Before going into details, we sketch the high level building blocks of the proofs of the upper bounds.
For the general case, by adapting [11], we formulate the problem of bounding the Price of Anarchy of $\theta$-free-flow congestion games by means of a factor-revealing pair of primal-dual linear programs. The techniques work as follows.

Given a $\theta$-free-flow congestion game $CG_\theta$ and a family of latency functions $F$, we know that we can model the latency of every resource $e \in E$ as $f_e(x) = \alpha_e x + \beta_e$, with $f_e \in [F]_H$, $\alpha_e \in [0, 1]$ and $\beta_e \geq 0$. We fix a Nash equilibrium $\sigma$ and a social optimum $\sigma^*$ for $CG_\theta$. Hence, for every $e \in E$, the congestions $k_e(\sigma)$ and $k_e(\sigma^*)$ of $e$ in $\sigma$ and $\sigma^*$, respectively, become fixed constants. As the Price of Anarchy measures the worst-case ratio of $\sum(\sigma)$ over $\sum(\sigma^*)$, our goal becomes that of choosing suitable values for $\alpha_e$ and $\beta_e$, for every $e \in E$, so as to maximize $\sum(\sigma)$ under the assumption that $\sum(\sigma^*) = 1$, $\sigma$ is a Nash equilibrium and $CG_\theta$ is a $\theta$-free-flow game. In particular, constraint $\sum(\sigma^*) = 1$ can be assumed without loss of generality by a simple scaling argument, provided we relax the condition $\alpha_e \in [0, 1]$ with $\alpha_e \geq 0$. Thus, an optimal solution to the resulting linear program, call it $LP$, provides an upper bound to the Price of Anarchy of $CG_\theta$.

Next step is to compute and analyze the dual of $LP$, that we call $DLP$. $DLP$ has three variables, namely $x$, $y$ and $\gamma$, with $x \geq 0$, $y \geq 0$ and $\gamma$ defining its objective value. Thus, by the Weak Duality Theorem, any feasible solution $(x^*, y^*, \gamma^*)$ for $DLP$ yields an upper bound of $\gamma^*$ to the optimal solution of $LP$ and so an upper bound to the Price of Anarchy of $CG_\theta$. For each function $f_e \in F_H$, $DLP$ has two constraints, namely $c_1(f_e, k_e(\sigma), k_e(\sigma^*), x, \gamma)$ and $c_2(f_e, k_e(\sigma), k_e(\sigma^*), y, \gamma)$, providing lower bounds on $\gamma$. Thus, the hard work becomes that of determining an optimal dual solution, i.e., a triple $(x^*, y^*, \gamma^*)$ satisfying both constraints and minimizing the value of $\gamma^*$. In fact, the lower is the value of $\gamma^*$, the more accurate becomes the estimation of the Price of Anarchy. As we shall determine a feasible solution that is independent of the values of $k_e(\sigma)$ and $k_e(\sigma^*)$, the final derived upper bound is independent of the particular choice of both $\sigma$ and $\sigma^*$ and, by the arbitrariness of $CG_\theta$, we get an upper bound on the Price of Anarchy of $\theta$-free-flow congestion games with latency functions in $F$.

By analyzing the structure of the dual constraints and by the arbitrariness of $k_e(\sigma)$ and $k_e(\sigma^*)$, it is possible to show that, for any fixed function $f_e$, $\gamma$ is minimized for $y = \frac{l f_k}{f_l}$ and $x \geq 1$. After applying these simplifications, we finally obtain two significant lower bounds for $\gamma$, where we can assume without loss of generality that $k_e(\sigma^*) > 0$. Both bounds depend on a structural property of $f_e$; moreover, the first is also influenced by the choice of $x$, while the second exhibits a dependence from $\theta$. For any class of latency functions $G$, by using $k$ and $l$ as a shorthand for $k_e(\sigma)$ and $k_e(\sigma^*)$, respectively, these bounds are denoted as $\gamma(G)$ and $\gamma_\theta(G)$, with

$$\gamma(G) := \inf_{x \geq 1, \theta > 0} \inf_{f_e \in G} \left( \frac{k + x(-k + l)}{l} \right) \frac{f(k)}{f(l)}$$

and

$$\gamma_\theta(G) := \sup_{k > 0, \theta > 0} \sup_{f_e \in G} \frac{f(k)(1 + \theta) - l}{f(k)(k - l)(1 + \theta) + lf(l)\theta}.$$
4.1 The Main Theorems

Theorem 1. Let $CG_\theta$ be a $\theta$-free-flow congestion game with latency functions in $F$ and $\theta \in [0, \infty]$. We have

$$\text{PoA}(CG_\theta) \leq \begin{cases} \gamma([F]_H) & \text{if } \theta = 0, \\ \gamma([F]) & \text{if } \theta = \infty, \\ \max\{\gamma([F]_H), \gamma_\theta([F]_H)\} & \text{if } \theta \in (0, \infty). \end{cases}$$

This bound is tight for single-source network games if $F$ is weakly diverse and even for load balancing games if $F$ is strongly diverse.

Proof. We first show the upper bound for $\theta \in [0, \infty)$. Let $\sigma = (\sigma_i, S)_{i \in [n], S \in \Sigma_i}$ and $\sigma^* = (\sigma^*_i, S)_{i \in [n], S \in \Sigma_i}$ be a pure Nash equilibrium and a social optimum for $CG_\theta$, respectively. Let $k_e := k_e(\sigma)$ and $l_e := k_e(\sigma^*)$ for any $e \in E$. Let $\ell_e(x) := \alpha_e f_e(x) + \beta_e$ be the latency function of each resource $e \in E$, with $\alpha_e \in \{0, 1\}$, $\beta_e \geq 0$, and $f_e \in [F]_H$. By applying the primal-dual method [1], we have that the optimal solution of the following linear program in variables $(\alpha_e)_{e \in E}$ and $(\beta_e)_{e \in E}$ is an upper bound on $\text{PoA}(CG_\theta)$:

$$\text{LP : max } \sum_{e \in E} (\alpha_e k_e f_e(k_e) + \beta_e)$$

s.t. \begin{align*}
\sum_{e \in E} (\alpha_e k_e f_e(k_e) + \beta_e) & \leq \sum_{e \in E} (\alpha_e l_e f_e(k_e) + \beta_e l_e) \quad (1) \\
\sum_{e \in E} (\beta_e l_e) & \leq \sum_{e \in E} (1 + \theta) \beta_e k_e \quad (2) \\
\sum_{e \in E} (\alpha_e l_e f_e(l_e) + \beta_e l_e) & = 1 \quad (3) \\
\alpha_e, \beta_e & \geq 0 \quad \forall e \in E.
\end{align*}

Indeed:

- Each latency function $\ell_e$ can be expressed as $\ell_e(k_e) = \alpha_e f_e(k_e) + \beta_e$ with $\alpha_e \in \{0, 1\}$.
- For any $i \in [n]$, and any two strategies $S, S^* \in \Sigma_i$, let $\sigma_{i,S,S^*} \geq 0$ denote the amount of players of type $i$ selecting $S$ in $\sigma$ and selecting $S^*$ in $\sigma^*$. By the pure Nash equilibrium conditions, we have that $\sum_{e \in S} (\alpha_e f_e(k_e) + \beta_e) \leq \sum_{e \in S^*} (\alpha_e f_e(k_e) + \beta_e)$, for any $i \in [n]$, and for any two strategies $S, S^* \in \Sigma_i$ such that $\sigma_{i,S,S^*} > 0$. Then, we have that

$$0 \geq \sum_{i \in [n]} \sum_{S^* \in \Sigma_i} \sigma_{i,S,S^*} \left( \sum_{e \in S} (\alpha_e f_e(k_e) + \beta_e) - \sum_{e \in S^*} (\alpha_e f_e(k_e) + \beta_e) \right)$$

$$= \sum_{e \in E} \left( \sum_{i \in [n], S \in \Sigma_i : e \in S} \sigma_{i,S,S^*} \right) (\alpha_e f_e(k_e) + \beta_e) - \sum_{e \in E} \left( \sum_{i \in [n], S^* \in \Sigma_i : e \in S^*} \sigma_{i,S,S^*} \right) (\alpha_e f_e(k_e) + \beta_e)$$

$$= \sum_{e \in E} (\alpha_e k_e f_e(k_e) + \beta_e k_e) - \sum_{e \in E} (\alpha_e l_e f_e(l_e) + \beta_e l_e),$$

and this implies constraint [1].
- By using the definition of $\theta$-free-flow congestion games, we have that $\sum_{e \in S^*} \beta_e \leq (1 + \theta) \sum_{e \in S} \beta_e$, for any $i \in [n]$, and for any strategies $S, S^* \in \Sigma_i$ such that $\sigma_{i,S,S^*} > 0$. Thus

$$0 \geq \sum_{i \in [n]} \sum_{S^* \in \Sigma_i} \sigma_{i,S,S^*} \left( \beta_e - (1 + \theta) \sum_{e \in S} \beta_e \right)$$

$$= \sum_{e \in E} \left( \sum_{i \in [n], S^* \in \Sigma_i : e \in S^*} \sigma_{i,S,S^*} \right) \beta_e - (1 + \theta) \sum_{e \in E} \left( \sum_{i \in [n], S \in \Sigma_i : e \in S} \sigma_{i,S,S^*} \right) \beta_e$$

$$= \sum_{e \in E} \ell_e \beta_e - (1 + \theta) \sum_{e \in E} \ell_e \beta_e,$$

and this implies constraint [2].
The Price of Anarchy of CGθ is
\[
\frac{\text{SUM}(x)}{\sum e \in \gamma} = \sum_{e \in S} (\alpha_k f_e + \beta_k),
\]
which, by maximizing such value over all the possible \(\alpha_k, \beta_k \geq 0\) subject to constraints \(1\) and \(2\), we get an upper bound on the Price of Anarchy of CGθ. If we restrict the values of \(\alpha_k\) and \(\beta_k\) in such a way that the further normalization constraint \(3\) holds, we do not affect the maximum value considered above, thus finding such maximum value is equivalent to find the optimal solution of LP. This can be achieved by relaxing the condition \(\alpha_k \in \{0, 1\}\) to \(\alpha_k \geq 0\).

We call generating set a generic finite set \(S \subseteq \{(k, l, f) : k, l \geq 0, f \in [F]_H\}\). For any generating set \(S\), we define a linear program in variables \(\gamma, x, y\):

\[
\text{DLP}(S) \quad \min \gamma \quad \text{s.t.} \quad l\gamma \geq k + x(-k + l) + y(0 + (1 + \theta)k) \quad \forall (k, l, f) \in S
\]

\[
x \geq 0.
\]

Let \(S_E := \{(k_e, l_e, f_e) : e \in E\}\). Observe that DLP\((S_E)\) is the dual of LP. Indeed, each dual constraint of type \(1\) (resp. \(5\)) is associated to some primal variable \(\alpha_k\) (resp. \(\beta_k\)), and \(x\) (resp. \(y\), resp. \(\gamma\)) is the dual variable associated to the primal constraint defined in \(1\) (resp. 2, resp. 3).

For any generating set \(S\), we can assume without loss of generality that \(S\) contains some sufficiently large \(k\) (indeed, by adding new triples the optimal \(\gamma\) cannot decrease). By exploiting constraint \(5\), we can set \(y := \frac{1}{1 + \theta}\) to get an optimal solution. Indeed, constraint \(5\) can be rewritten as \(l\gamma \geq (1 - x + (1 + \theta) y) k + (x - y l)\). Observe that the coefficient of \(k\) in the previous inequality, i.e., term \(1 - x + (1 + \theta) y\), must be non-positive, otherwise, we get an arbitrarily large \(k\) by considering a constraint of type \(5\) with a triple \((k', 1, f')\) having a sufficiently large \(k'\). Thus, by imposing \(1 - x + (1 + \theta) y \leq 0\), we have that the worst-case constraint of type \(5\) is obtained by triple \((0, 0, f')\).

The coefficient of \(l\) becomes \((x - y l)\), the optimal value for \(\gamma\) is obtained by setting \(y\) as high as possible. Since \(1 - x + (1 + \theta) y \leq 0\) is equivalent to \(y \leq \frac{1}{1 + \theta}\), it is sufficient setting \(y := \frac{1}{1 + \theta}\).

Now, we can assume without loss of generality that any optimal solution of DLP\((S)\) is such that \(x \geq 1\), otherwise we get an arbitrarily large \(\gamma\) by considering a constraint of type \(1\) for triple \((k', 1, f')\) with a sufficiently large \(k'\). Furthermore, we can avoid constraints of type \(1\) with \(l = 0\), as DLP\((S)\) is always satisfied for \(l = 0\), considering that we have assumed \(x \geq 1\). We conclude that, by setting \(y := \frac{1}{1 + \theta}\) in DLP\((S)\), we get the following linear program which is equivalent to DLP\((S)\):

\[
\text{DLP2}(S) : \quad \min \gamma \quad \text{s.t.} \quad \gamma \geq \frac{k f(k) + x(-k f(k) + l f(k))}{l f(l)}, \quad \forall (k, l, f) \in S : l > 0
\]

\[
\gamma \geq \frac{x \theta + 1}{1 + \theta}
\]

\[
x \geq 1.
\]

By constraint \(7\), we get that any feasible solution \((\gamma, x)\) of DLP2\((S)\) verifies \(\gamma \geq 1\), thus DLP2\((S)\) admits an optimal solution. Let \((\gamma^*, x^*)\) be the optimal solution DLP2\((S)\), chosen as extreme point of the feasible region of DLP2\((S)\). Thus \((\gamma^*, x^*)\) verifies one of the following cases: (a) two constraints of type \(6\) are tight; (b) exactly one constraint of type \(6\) and constraint \(7\) are tight. If case (a) holds, we conclude that

\[
\text{PoA}(CGθ) \leq \gamma^* = \max_{(k, l, f) \in S : l > 0} \frac{k f(k) + x^*(-k f(k) + l f(k))}{l f(l)} \leq \gamma([F]_H).
\]

and the claim follows.

If case (b) holds, we have that there exists a triple \((k, l, f) \in S\) such that a constraint of type \(6\) is satisfied, and constraint \(7\) is satisfied. We have that \(k > l\). Indeed, if any tight constraint of type \(6\) is associated to a triple \((k, l, f)\) with \(k \leq l\), the optimal value does not decrease if we reduce the value of \(x^*\), and this contradicts the fact that \((\gamma^*, x^*)\) is an extreme point of the feasible region. Thus, there exist \(k, l \geq 0\) with \(k > l > 0\), \(f \in [F]_H\), and \(x \geq 1\), such that \(\gamma(S) = \frac{k f(k) + x(-k f(k) + l f(k))}{l f(l)}\) and \(\gamma(S) = \frac{x \theta + 1}{1 + \theta}\), that is, \(\gamma(S) = \frac{f(k)(k + 1 + \theta - l)}{f(k)(1 - \theta) + f(l)}\).
We conclude that
\[
\text{PoA}(\mathcal{CG}_\theta) \leq \gamma(S) = \frac{f(k)(k(1+\theta)-l)}{f(k)(k-l)(1+\theta) + l(f(l)-l)\theta} \leq \gamma_{\theta}(\mathcal{F}|_H). \tag{9}
\]

As either (8) or (9) holds, we get \(\text{PoA}(\mathcal{CG}_\theta) \leq \max\{\gamma(\mathcal{F}|_H), \gamma_{\theta}(\mathcal{F}|_H)\}\), and the claim follows. For the particular case of \(\theta = 0\), as \(\gamma(\mathcal{F}|_H) = 1\), we get \(\text{PoA}(\mathcal{CG}_0) \leq \gamma(\mathcal{F}|_H)\).

Now, we move to the case of \(\theta = \infty\). Here, we assume that, for each resource \(e \in E\), the latency function is of the form \(f_e(x) = \alpha_e f_e(x) + \beta_e\), with \(f_e \in F\). By using similar arguments as before, we derive a new linear program LP, deprived of constraint (7) and of variables \((\beta_e)_{e \in E}\), whose optimal value is an upper bound to \(\text{PoA}(\mathcal{CG}_\theta)\). Then, the dual of LP is equal to DLP(S), but without constraint (7) and with \(f_e \in F\). Let \((\gamma^*, x^*)\) be the optimal solution of this dual program. As constraint (7) has been removed, we have that case (a) considered before is the unique possible case that can occur. Thus, we get \(\text{PoA}(\mathcal{CG}_\theta) \leq \gamma^* = \max\{k(l_1 + \theta f(l_1))/l(1+\theta), f(l_1)/l\} \leq \gamma(\mathcal{F})\).

The construction of the matching lower bounding instances is deferred to the appendix. \(\square\)

We now show that, when considering either parallel-links games or path-disjoint network congestion games, a better bound on the Price of Anarchy can be achieved. To this aim, given a class of latency functions \(\mathcal{G}\), let us define
\[
\eta_\theta(\mathcal{G}) := \sup_{k > l > 0, f \in F} \frac{k f(k)(1+\theta)}{l f(l)(1+\theta) + (lf(l)-l f(k))\theta}.
\]

**Theorem 2.** Fix a value \(\theta \in (0, \infty)\) and a class of latency functions \(\mathcal{F}\). Let \(\text{PLG}_\theta\) be a \(\theta\)-free-flow path-disjoint network congestion game with latency functions in \(\mathcal{F}\). Then, \(\text{PoA}(\text{PLG}_\theta) \leq \max\{\gamma(\mathcal{F}|_H), \eta_\theta(\mathcal{F}|_H)\}\). The bound is tight in general and even for parallel-links networks if \(\mathcal{F}\) is scale-closed.

**Proof.** Here, we show the claim for the restricted case \(\theta\)-free-flow parallel-links games only. For the case of path-disjoint network games, see the Appendix. Let \(\text{PLG}_\theta\) be a \(\theta\)-free-flow parallel-link game with latency functions in \(\mathcal{F}\). Let \(\ell_e(x) := \alpha_e f_e(x) + \beta_e\) be the latency function of each resource \(e \in E\), with \(\alpha_e \in \{0, 1\}\), \(\beta_e \geq 0\), and \(f_e \in \mathcal{F}\). Let \(\sigma\) and \(\sigma^*\) be a pure Nash equilibrium and a social optimum for \(\text{PLG}_\theta\), respectively. Let \(k_e := k_e(\sigma)\) and \(l_e := k_e(\sigma^*)\) for any \(e \in E\). Let \(E^+ := \{e \in E : k_e > l_e\}\) and \(E^- := \{e \in E : k_e < l_e\}\). If one set among \(E^+\) and \(E^-\) is empty, then \(E^+ = E^- = \emptyset\) necessarily, and we have that the Price of Anarchy of \(\text{PLG}_\theta\) is 1. Then, as \(\eta_\theta(\mathcal{F}) \geq 1\), the claim holds. Thus, we assume that \(E^+ \neq \emptyset\) and then \(E^- \neq \emptyset\). Furthermore, we assume without loss of generality that there are no resources \(e \in E\) such that \(k_e = l_e\), otherwise, by removing these resources and their users from the game, the Price of Anarchy does not decrease. Observe that, for any pair of resources \(u \in E^+\) and \(v \in E^-\), we can assign an amount of flow \(w_{u,v} \in [0, 1]\) to pair \((u, v)\) in such a way that \(k_u - l_u = \sum_{e \in E^-} w_{u,e}\) and \(l_v - k_v = \sum_{e \in E^+} w_{u,e}\). For any two resources \(u \in E^+\) and \(v \in E^-\), let \(\xi_{u,v} := w_{u,v}/(k_u - l_u)\) and \(\psi_{u,v} := w_{u,v}/(l_v - k_v)\). Observe that \(\sum_{v \in E^-} \xi_{u,v} = 1\) for any \(u \in E^+\) and \(\sum_{u \in E^+} \xi_{u,v} = 1\) for any \(v \in E^-\). We have that
\[
\text{SUM}(\sigma) = \sum_{u \in E^+} k_u \xi_u(k_u) + \sum_{v \in E^-} \sum_{u \in E^+} k_u \psi_{u,v} \xi_v(k_v).
\]

and
\[
\text{SUM}(\sigma^*) = \sum_{u \in E^+} l_u \xi_u(l_u) + \sum_{v \in E^-} \sum_{u \in E^+} l_u \psi_{u,v} \xi_v(l_v).
\]
Theorem 4. Fix a value \( u \) and \( \ell \) latency functions of maximum degree \( \theta \). For the general case, we have the following theorem.

As consequence of the previous results, we can determine the exact Price of Anarchy of free-flow congestion games with polynomial latency functions. For parallel-links games and free-flow path-disjoint network games, we get the following result.

By exploiting (10) and (11) we get:

\[
\text{SUM}(\sigma) = \sum_{u \in E^+, v \in E^-} w_{u,v} \left( \frac{k_u}{k_u - l_u} \ell_u(k_u) + \frac{k_v}{l_v - k_v} \ell_v(k_v) \right) \leq \max_{(u,v): u \in E^+, v \in E^-} \frac{k_u}{k_u - l_u} \ell_u(k_u) + \frac{k_v}{l_v - k_v} \ell_v(k_v),
\]

\[
= \max_{(u,v): u \in E^+, v \in E^-} \frac{\alpha}{k_u - l_u} (\alpha f_u(k_u) + \beta_u) + \frac{\gamma_v}{l_v - k_v} (\alpha f_v(k_v) + \beta_v),
\]

(12)

Let \( u \in E^+, v \in E^- \) be the resources maximizing (12), so that (12) is at most

\[
F(\alpha_u, \beta_u, \alpha_v, \beta_v) := \frac{\alpha}{k_u - l_u} (\alpha f_u(k_u) + \beta_u) + \frac{\gamma_v}{l_v - k_v} (\alpha f_v(k_v) + \beta_v).
\]

In the following, we show that \( \max \{ \gamma([F]H), \eta_\theta([F]H) \} \) is an upper bound to \( F(\alpha_u, \beta_u, \alpha_v, \beta_v) \). By the equilibrium conditions, we have that \( \alpha f_u(k_u) + \beta_u \leq \alpha f_v(k_v) + \beta_v \). Only if \( k_v = 0 \), it might be the case that \( \alpha f_u(k_u) + \beta_u < \alpha f_v(k_v) + \beta_v \). In such a case, one can reduce the values of \( \alpha_v \) and \( \beta_v \) as much as possible so as to meet the \( \theta \)-free-flow conditions, while guaranteeing that \( \alpha f_u(k_u) + \beta_u = \alpha f_v(k_v) + \beta_v \) holds and the value of \( F(\alpha_u, \beta_u, \alpha_v, \beta_v) \) does not decrease. Thus, we assume without loss of generality that the equilibrium conditions are tight, i.e., \( \alpha f_u(k_u) + \beta_u = \alpha f_v(k_v) + \beta_v \) holds. As \( \beta_v \leq (1 + \theta) \beta_u \), and since \( k_u < l_u \), we have that, by increasing \( \beta_v \) and decreasing \( \alpha_v \) and as much as possible so that the \( \theta \)-free-flow conditions are preserved, the equilibrium conditions are satisfied, and the value \( \alpha f_u(k_u) + \beta_u \) does not change, we get that \( \alpha f_v(k_v) + \beta_v \) does not increase, and then \( F(\alpha_u, \beta_u, \alpha_v, \beta_v) \) does not decrease. Thus, we can assume without loss of generality that \( \beta_u := \beta_v/(1 + \theta) \) (by the \( \theta \)-free-flow conditions) and that \( \alpha_u := (\alpha f_v(k_v) + (1 - 1/(1 + \theta)) \beta_v)/f_u(k_u) \) (by the equilibrium conditions). By using these values of \( \alpha_v \) and \( \beta_u \), we can prove the following result which yields the claim:

**Lemma 1.** It holds that \( F(\alpha_u, \beta_u, \alpha_v, \beta_v) \leq \max \{ \gamma([F]H), \eta_\theta([F]H) \} \).

Also in this case, the construction of the matching lower bounding instances is deferred to the appendix. □

### 4.2 Polynomial Latency Functions

As consequence of the previous results, we can determine the exact Price of Anarchy of free-flow congestion games with polynomial latency functions. For the general case, we have the following theorem.

**Theorem 3.** Fix a value \( \theta \in [0, \infty] \). The Price of Anarchy of \( \theta \)-free-flow congestion game with polynomial latency functions of maximum degree \( p \) and minimum degree \( q \) is \( \max \{ \gamma([P_{p,q}]H), \eta_\theta([P_{p,q}]H) \} \), with

\[
\gamma(\eta_{\theta}[P_{p,q}]H) = \sup_{t \in \mathbb{N}} \frac{t^p (t(1 + \theta) - 1)}{t^p-t(1+\theta) + \eta_\theta}, \quad \text{and}
\]

(13)

\[
\gamma([P_{p,q}]H) = \frac{p^p (p-1) + 1}{(p+1)^{p+1}} \frac{\chi_{(p-1)}(q) + \chi_{(p)}(q)}{q^p}. \quad \text{(14)}
\]

For parallel-links games and free-flow path-disjoint network games, we get the following result.

**Theorem 4.** Fix a value \( \theta \in [0, \infty) \). The Price of Anarchy of both \( \theta \)-free-flow parallel-links games and \( \theta \)-free-flow path-disjoint network congestion games with polynomial latency functions of maximum degree \( p \) and minimum degree \( q \) is \( \max \{ \gamma([P_{p,q}]H), \eta_\theta([P_{p,q}]H) \} \), where

\[
\eta_\theta([P_{p,q}]H) = \sup_{t \in \mathbb{N}} \frac{t^{p+1} (1 + \theta)}{t^{p+1}(1 + \theta) + (1 - t^\theta)}.
\]

and \( \gamma([P_{p,q}]H) \) is defined as in Theorem 3.
5 Interpretation and Discussion of the Results

In this section, we provide a detailed discussion of the implications of our theoretical results and how they relate to previous work.

**Congestion Games with Homogeneous Latency Functions.** Consider the case of $\theta = \infty$, i.e., general congestion games without the free-flow hypothesis. From [9], we know that $\lambda(\mathcal{F}) := \sup_{k>l>0, f \in \mathcal{F}} \frac{k f(k)}{k f(k)(1+\theta)}$ is an upper bound on the Price of Anarchy of congestion games with latency function in $\mathcal{F}$, and such bound is tight, even for parallel-links games, if $\mathcal{F}$ contains at least one constant function. However, if $\mathcal{F}$ is homogeneous, this upper bound is not guaranteed to be tight.

As in the case of homogeneous latency functions any congestion game is a 0-free-flow game, by Theorems [1] and [2] and by the fact that $\gamma_0([\mathcal{F}]_H) = \eta_0([\mathcal{F}]_H) = 1$, we get that the price of anarchy is equal to $\gamma(\mathcal{F}) = \max\{\gamma([\mathcal{F}]_H), \eta_0([\mathcal{F}]_H)\} = \max\{\gamma([\mathcal{F}]_H), \gamma_0([\mathcal{F}]_H)\}$ and it is attained even for parallel-links networks. Hence, as a byproduct of our analysis, we close an open problem posed by Roughgarden [23], in which he asked if there exists a simple parallel-links game matching the worst-case Price of Anarchy of arbitrary classes of homogeneous latency functions.

Moreover, by exploiting Theorem [3] with $\mathcal{F} := [\mathcal{P}_{p,q}]_H$, one can show by direct computation that $\gamma(\mathcal{F}) < \lambda(\mathcal{F})$. Thus, we also obtain that the PoA for homogeneous functions may be strictly lower than the one for functions admitting constant terms.

**Free-Flow Games with Polynomial Latency Functions.** By Theorem [3] the Price of Anarchy of $\theta$-free-flow congestion games with monomial latency functions of degree $p \geq 1$ is

$$\max\{\gamma([\mathcal{M}_p]_H), \gamma_0([\mathcal{M}_p]_H)\} = \max\{\gamma([\mathcal{M}_p]_H)\} = \gamma([\mathcal{M}_p]_H) = \gamma([\mathcal{W}_M]_H) = \gamma([\mathcal{P}_{p,p}]_H) = 1,$$

(where the first equality comes from the fact that, for homogeneous latency functions, a game is $\theta$-free-flow if and only if it is 0-free-flow, and the last equality comes from [14]), thus reobtaining a well-known result in the literature (see, for instance, [13]). For weakly-monomial latency functions of degree $p \geq 1$, instead, by Theorem [3] we get a bound of

$$\max\{\gamma([\mathcal{W}_M]_H), \gamma_0([\mathcal{W}_M]_H)\} = \max\{\gamma_0([\mathcal{W}_M]_H), 1\} = \gamma([\mathcal{W}_M]_H) = \gamma_0([\mathcal{P}_{p,p}]_H),$$

i.e., the value defined in [13]. For the particular case of affine functions, i.e., class $\mathcal{W}_M$, we reobtain the same bounds of [9]. However, we give an improved result, as our lower bounds hold even for load balancing and single-source network congestion games (in [3], tight lower bounds are given for general congestion games only).

For $\theta$-free-flow path-disjoint games with weakly-monomial latency functions of degree $p \geq 1$, by Theorem [4] the Price of Anarchy gets equal to

$$\max\{\gamma([\mathcal{W}_M]_H), \eta_0([\mathcal{W}_M]_H)\} = \eta_0([\mathcal{W}_M]_H) = \eta_0([\mathcal{P}_{p,p}]_H) = \frac{(1 + \theta)(p + 1) + 1}{(1 + \theta)(p + 1) + \theta},$$

and it is tight even for parallel-links games with two resources only. Also in this case, with respect to affine functions, we improve on the lower bounds given in [3].

**Simpler Upper Bounds for Path-Disjoint Free-Flow Games.** We observe that

$$\eta_0([\mathcal{F}]_H) = \sup_{k>l>0, f \in [\mathcal{F}]_H} \frac{k f(k)(1+\theta)}{k f(k)(1+\theta) + (lf(l) - lf(k))\theta} \leq \sup_{k>l>0, f \in [\mathcal{F}]_H} \frac{k f(k)(1+\theta)}{k f(k)(1+\theta) - k f(k)\theta} = 1 + \theta,$$

thus, by Theorem [2] the Price of Anarchy of path-disjoint free-flow games with latency functions in $\mathcal{F}$ is at most $\max\{1 + \theta, \gamma([\mathcal{F}]_H)\}$. Such upper bound is not tight in general as that considered in Theorem [2] but it does not require the computation of $\eta_0([\mathcal{F}]_H)$ for any $\theta > 0$. Furthermore, if $\gamma([\mathcal{F}]_H) = 1$ (as for weakly-monomial latency functions), we have that the Price of Anarchy is at most $1 + \theta$, thus getting a simple and good upper bound for small values of $\theta$. 


General vs Path-Disjoint Free-Flow Games. Let $F$ be a class of latency functions. By standard arguments of calculus, one can show that $\max\{\gamma(F_H), \eta_\theta(F_H)\} \leq \max\{\gamma(F_H), \gamma_\theta(F_H)\}$ for any $\theta \in [0, \infty)$, and $\lim_{\theta \to \infty} \max\{\gamma(F_H), \eta_\theta(F_H)\} = \lambda(F) = \lim_{\theta \to \infty} \max\{\gamma(F_H), \gamma_\theta(F_H)\}$. Thus, the Price of Anarchy of $\theta$-free-flow path-disjoint games and that of general $\theta$-free-flow congestion games converge to the same value. However, the rate of convergence can be significantly lower in path-disjoint games. In fact, if $F := WM_p$, by Theorems 3 and 4, we have that $\max\{\gamma(F_H), \eta_\theta(WM_p)\} < \gamma_\theta(WM_p) = \max\{\gamma(F_H), \gamma_\theta(F_H)\}$.

For instance, for $p = 4$ and $\theta = 1$, we get $\eta_\theta(WM_p) = 1.3652$ and $\gamma_\theta(WM_p) = 1.6994$. This is an important difference with the classical setting with $\theta = \infty$, where, given an arbitrary strongly diverse class of latency functions $F$, the Price of Anarchy of general congestion games with latency functions in $F$ is matched by a parallel-links game. Instead, in our case the Price of Anarchy of general $\theta$-free-flow can be higher even than the one for $\theta$-free-flow path-disjoint games.

6 Concluding Remarks

In this paper, we introduce the class of $\theta$-free-flow routing games, aiming to capture the behavior of real-world networks with a stronger assumption on edge costs than typical PoA analysis. This assumption is supported by granular data of commuters’ car trips in Singapore. Indeed, the data shows evidence that agents only evaluate a small subset of their entire strategy sets to solve the routing problem. Specifically, 75% of the agents would consider paths that are at most 88% longer that the shortest path at free-flow. Price of anarchy analysis in $\theta$-free-flow routing games (and variants thereof) provides much tighter Price of Anarchy guarantees that can be significantly smaller that the vanilla PoA bounds and which themselves are in better agreement with experimental investigations of Price of Anarchy [18]. Furthermore, we show that the Price of Anarchy in $\theta$-free-flow routing games, in general, is not independent on the network topology, differently from what happens in classical non-atomic congestion games [23].

As a by-product of our analysis, we also determine the structure of a parallel-links game that matches the Price of Anarchy of games with homogeneous latency functions, thus solving an open problem posed by Roughgarden [23], and we tighten several bounds on the Price of Anarchy shown in [3] for the case of affine functions, which are extended to more general latency functions, too.

We hope that this paper opens up a new direction for tighter coupling between data analytics, modelling and theory in congestion games and beyond. Analyzing different cities as well as introducing models that take into account the difference between public and private transport seems like an exciting direction for future work.
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A Appendix

In the appendix, we use the following further definitions:

\[ \hat{\gamma}_\theta([F]_H) := \max\{\gamma_\theta([F]_H), \gamma([F]_H)\}, \quad \hat{\eta}_\theta([F]_H) := \max\{\eta_\theta([F]_H), \gamma([F]_H)\}. \]

A.1 Tightness of the Upper Bound Shown in Theorem 1

In the following theorem, we show that the upper bound provided in Theorem 1 is tight for single-source network games if \( F \) is weakly diverse and even for load balancing games if \( F \) is strongly diverse.

**Theorem 5.** Fix a value \( \theta \in [0, \infty] \), a class of latency functions \( F \) and a value \( M < \hat{\gamma}_\theta([F]_H) \). (i) If \( \theta < \infty \) and \( F \) is strongly diverse, there exists a (non-symmetric) \( \theta \)-free-flow load balancing game \( LBG_\theta \) with latency functions in \( F \) such that \( \text{PoA}(LBG_\theta) > M \). (ii) If \( \theta < \infty \) and \( F \) is weakly diverse only, there exists a \( \theta \)-free-flow single-source network congestion game \( NCG_\theta \) such that \( \text{PoA}(NCG_\theta) > M \). (iii) If \( \theta = \infty \), there exists a load-balancing game \( \text{PoA}(NCG_\theta) > M \). (iv) If \( \theta = \infty \) and \( F \) is scale-closed, there exists a parallel-link game \( PLG_\theta \) with latency functions in \( F \) such that \( \text{PoA}(PLG_\theta) > M \).

The proof of Theorem 5 consists into two further theorems (Theorem 6 and 10) and several preliminary results. First of all, we show in Theorem 6 that the upper bound \( \hat{\gamma}_\theta([F]_H) \) is tight.

**Theorem 6.** Fix a value \( \theta \in [0, \infty] \), a class of latency functions \( F \) and a value \( M < \gamma_\theta([F]_H) \). (i) If \( F \) is strongly diverse, then there exists a (non-symmetric) load balancing \( \theta \)-free-flow game \( LBG_\theta \) with latency functions in \( F \) such that \( \text{PoA}(LBG_\theta) > M \). (ii) If \( F \) is weakly diverse only, then there exists a single-source network \( \theta \)-free-flow congestion game \( NCG_\theta \) such that \( \text{PoA}(NCG_\theta) > M \).

**Proof.** We first show part (i). We do not consider the case \( \theta = 0 \), since in such case \( \gamma_\theta([F]_H) = 1 \), and any congestion game has a Price of Anarchy of at least 1. Thus, we assume that \( \theta > 0 \). Let \( k, l > 0 \), with \( k > l \), and \( f \in [F]_H \) be such that \( \gamma_\theta(k, l, f) := \frac{f(kk^{\theta-1})}{\min\{\theta-1, k\}} > M \) (such a triple \( (k, l, f) \) exists by the definition of supremum and since \( M < \gamma_\theta([F]_H) \)). Observe that \( k \) and \( l \) can be chosen in such a way that \( \frac{k}{T} \) is integer for some \( n \in N \). Indeed, if \( \frac{k}{T} \) is not integer, we proceed as follows. First of all, observe that function \( \gamma_\theta(k, l, f) \) is right-continuous in \( t \in (0, k) \) for any fixed \( k > 0 \) and \( f \in [F]_H \). Since \( \gamma_\theta(k, l, f) > M \), by exploiting the right-continuity of \( \gamma_\theta(k, l, f) \), we have that there exists a value \( n' \) sufficiently close to \( n \) such that \( \frac{n'}{T} \) is an integer and \( \gamma_\theta(k, n', f) > M \).

To construct the lower bounding instance, we resort to a representation called load balancing graph: (a) the nodes are the resources, (b) each edge \( (u, v) \) is a player having two strategies \( \{u\} \) and \( \{v\} \), where \( u \) (resp. \( v \)) is called the first resource (resp. the second resource) of the considered player; (c) the weight \( w_e \) of any edge \( e \in E \) denotes the total amount of players associated to edge \( (u, v) \). Given an integer \( m \geq 2 \), let \( LBG_\theta(m) \) be the load balancing game associated to a load balancing graph \( G_\theta = (V, E) \) defined as follows: (a)
the nodes of $V$ are partitioned into $m$ levels, where each level $s \in [m]$ has $n^{m-1} \left(\frac{1}{k}\right)^{m-s}$ nodes (observe that such a number is an integer as $\frac{n}{k} \in \mathbb{N}$); (b) for any level $s \in [m-1]$, there are $n^m \left(\frac{1}{k}\right)^{m-s}$ edges going from $s$ to $s+1$ in such a way that the out-degree of each node $u$ at level $s$ is $n$, and the in-degree of each node $v$ at level $s+1$ is $\frac{n}{k}$; (c) $w_e = \frac{k}{n}$ for any edge $e \in E$, and the latency function of any node/resource at level $s \in [m]$ is defined as $\ell_s(x) = \alpha_s f(x) + \beta_s$, where $\alpha_s = 1 - (1 + \theta)^{s-m}$, and $\beta_s = (1 + \theta)^{s-m} f(k)$. Observe that $LBG_\theta(m)$ is a $\theta$-free-flow game (as $\beta_s(1 + \theta) = \beta_{s+1}$ for any $s \in [m-1]$) with latency functions in $\mathcal{F}$.

Let $\sigma$ and $\sigma^*$ be the strategy profiles in which each player selects her first and her second resource, respectively. Observe that $k_u(\sigma) = k$ (resp. $k_u(\sigma^*) = l$) for any resource $u$ at level $s \in [m-1]$ (resp. $s \in [m] \setminus \{1\}$), and $k_u(\sigma) = 0$ for resources at level $m$ (resp. 1). We show that $\sigma$ is a pure Nash equilibrium. Given $s \in [m-1]$ and a player $(u, v)$ such that $u$ is at level $s$, we get $\ell_s(k_u(\sigma)) = \alpha_s f(k_u(\sigma)) + \beta_s = (1 - (1 + \theta)^{s-m}) f(k) + (1 + \theta)^{s-m} f(k) = f(k) = (1 - (1 + \theta)^{s+1-m}) f(k) + (1 + \theta)^{s+1-m} f(k) = \alpha_{s+1} f(k_u(\sigma)) + \beta_{s+1} = \ell_{s+1}(k_u(\sigma))$, and this shows that $\sigma$ is a pure Nash equilibrium. We have that

\[
\begin{align*}
SUM(\sigma) &= \sum_{s=1}^{m-1} n^{m-1} \left(\frac{1}{k}\right)^{m-s} k \ell_s(k) = \sum_{s=1}^{m} n^{m-1} \left(\frac{1}{k}\right)^{m-s} k f(k), \\
SUM(\sigma^*) &= \sum_{s=2}^{m} n^{m-1} \left(\frac{1}{k}\right)^{m-s} l \ell_s(l) = \sum_{s=2}^{m} n^{m-1} \left(\frac{1}{k}\right)^{m-s} l \left(1 - (1 + \theta)^{s-m}\right) f(l) + (1 + \theta)^{s-m} f(k). \quad (15)
\end{align*}
\]

Given a sufficiently small $\epsilon > 0$ such that $\gamma_\theta(k, l, f) > M + \epsilon$ and a sufficiently large $m$, by using (15) and (16) we get

\[
\begin{align*}
\text{PoA}(LBG(m)_\theta) &\geq \frac{SUM(\sigma)}{SUM(\sigma^*)} \\
&= \frac{\sum_{s=1}^{m} n^{m-1} \left(\frac{1}{k}\right)^{m-s} k f(k)}{\sum_{s=2}^{m} n^{m-1} \left(\frac{1}{k}\right)^{m-s} l \left(1 - (1 + \theta)^{s-m}\right) f(l) + (1 + \theta)^{s-m} f(k) + (1 + \theta)^{s+1-m} f(k) + (1 + \theta)^{s+1-m} f(k)} \\
&= \frac{\left(\frac{1}{k}\right)^{m-1} f(k)}{\sum_{s=0}^{m-2} \left(\frac{1}{k}\right)^{s} l \left(1 - (1 + \theta)^{s+2-m}\right) f(l) + \sum_{s=0}^{m-2} \left(\frac{1}{k}\right)^{s} \left(1 + \theta\right)^{s+2-m} f(k) + (1 + \theta)^{s+2-m} f(k) + (1 + \theta)^{s+2-m} f(k)} \\
&= \frac{\left(\frac{1}{k}\right)^{m-1} f(k)}{\sum_{s=0}^{m-2} \left(\frac{1}{k}\right)^{s} f(l) + \sum_{s=0}^{m-2} \left(\frac{1}{k}\right)^{s} \left(1 + \theta\right)^{s+2-m} f(l) + (1 + \theta)^{s+2-m} f(k) - f(l)} \\
&= \lim_{m \to \infty} \frac{\left(\frac{1}{k}\right)^{m-1} f(k)}{\sum_{s=0}^{m-2} \left(\frac{1}{k}\right)^{s} f(l) + \sum_{s=0}^{m-2} \left(\frac{1}{k}\right)^{s} \left(1 + \theta\right)^{s+2-m} f(l) + (1 + \theta)^{s+2-m} f(k) - f(l)} \\
&= \frac{\left(\frac{1}{k}\right)^{m-1} f(k)}{f(l) + \left(\frac{1}{k}\right)^{m-1} \left(1 + \theta\right)^{m+2} f(k) - f(l)} \\
&= \frac{f(k) \left(1 + \theta\right)^{m+2} - f(l) - \epsilon}{f(k) \left(1 + \theta\right)^{m+2} - f(l) - \epsilon} \\
&= \frac{f(k) \left(1 + \theta\right)^{m+2} - f(l) - \epsilon}{f(k) \left(1 + \theta\right)^{m+2} - f(l) - \epsilon} \\
&= \gamma_\theta(k, l, f) - \epsilon
\end{align*}
\]
and this shows part (i) of the claim.

Regarding part (ii), we resort to similar arguments as in [23]: we reconsider the lower bound instance of part (i), and, by replacing each resource $e$ with a path $P$ simulating the latency function of $e$, we transform the load balancing instance in a lower bounding instance having the structure of a single-source network congestion game. Let us consider a load balancing game $\text{LBG}_\theta$ defined as in part (i), such that $\text{PoA}(\text{LBG}_\theta) > M$. Let $\theta, k, l, f, n, m$ be the parameters characterizing $\text{LBG}_\theta$, and $g \in \mathcal{F}$ be a constant latency function defined as $g(x) := \beta$ for some $\beta > 0$ (such a function exists as $\mathcal{F}$ is weakly-diverse). We can assume without loss of generality that $\theta$ and $f(k)/\beta$ are rational numbers. Indeed, if it is not the case, as $\gamma_\theta(k, l, f) > M$ is continuous with respect to variables $\theta, k, l, f$, and $f(x)$ is right-continuous in $x$, there exist $\theta' < \theta$, $k' > k$, $l' > l$, and an integer $n' \geq 1$, such that $\gamma_\theta(k', l', f) > M$, $k' > l' > 0$, and $\frac{\ell'}{\ell'}$ is integer, and $f(k')/\beta$ is rational. Thus, a load balancing instance $\text{LBG}_\theta'$ based on values $\theta', k', l', f, n', m$ is a $\theta$-free-flow game (as $\theta' < \theta$), and verifies $\text{PoA}(\text{LBG}_\theta') > M$, and we can choose $\text{LBG}_\theta'$ in place of $\text{LBG}_\theta$.

Now, let $\text{NCG}_\theta$ be a single-source network congestion game constructed from $\text{LBG}_\theta$ as follows. Consider an undirected graph $G = (V', E')$ initially empty. Let $g$ be a constant function, with $g(x) = \beta$ for some $\beta > 0$. For any resource $e$ at level $s \in [m]$, we add in $G$ a source-node $u^*$, and two consecutive directed paths $P_e := (u^*_e, 0, u^*_e, 1, \ldots, u^*_e, a_e)$ and $Q_e = (v^*_e, 0, v^*_e, 1, \ldots, v^*_e, b_e)$ with $u^*_e, 0 = u^*$ and $u^*_e, a_e = v^*_e, 0$, in such a way that there exists an integer $h$ (not depending on the level $s$) such that $a_e = a_e h$ and $\beta_e = b_e h/\beta$, where $a_e$ and $\beta_e$ are defined as in part (i). Such an integer exists since, by our choice of $\theta$ and $k$, quantities $a_e$ and $\beta_e$ are rational numbers. The latency function of each edge of paths of type $P_e$ (resp. $Q_e$) is $f$ (resp. $g$). For any amount of players of $\text{LBG}_\theta$ having $u$ and $v$ as first and second resource respectively, we include the same amount of players in $\text{NCG}_\theta$, whose strategies are all the simple paths from $u^*$ to the last node of $Q_u$, denoted as $z_u$. The unique simple path from $v$ to the last node of $Q_u$ is $(P_u, Q_u)$ and $(P_v, Q_v)$ only, denoted respectively as first and second path of player of type $(u^*, z_v)$.

As shown for $\text{LBG}_\theta$ in part (i), $\text{NCG}_\theta$ is a $\theta$-free-flow game as well. Let $\phi$ and $\phi^*$ be the strategy profiles in which each player selects her first path and her second path, respectively. By adapting the proof-arguments used in part (i) to game $\text{NCG}_\theta$, one can easily show that the strategy profile $\phi$ is a pure Nash equilibrium, and that $\text{PoA}(\text{NCG}_\theta) \geq \frac{\sum_{s=1}^{\infty} \phi_{s}}{\sum_{s=1}^{\infty} \phi_{s}^*} \geq \gamma_\theta(k, l, f) - \epsilon > M + \epsilon - \epsilon = M$. \hfill \Box

In Theorem [10] we address the lower bound $\gamma(\mathcal{F})$ which, does not depend on $\theta$. For such a reason, the lower bound is proven with respect to classical congestion games (i.e., as if $\theta = \infty$). In the proof of Theorem [5] it will be clarified how to use this result to give tight lower bounds holding also for general free-flow games. We first give some preliminary notations and results.

Given a class $\mathcal{F}$ of latency functions and a value $x \in [1, \infty)$, define

\begin{align*}
\gamma_>(\mathcal{F}, x) &:= \sup_{k > l > 0, f \in \mathcal{F}} \left( k + x(-k + l) \right) \frac{f(k)}{f(l)}, \\
\gamma_\leq(\mathcal{F}, x) &:= \sup_{l > 0, 0 \leq k \leq l, f \in \mathcal{F}} \left( k + x(-k + l) \right) \frac{f(k)}{f(l)},
\end{align*}

Observe that $\gamma_>(\mathcal{F}, x)$ and $\gamma_\leq(\mathcal{F}, x)$, which are functions from $[1, \infty)$ to $\mathbb{R}_{\geq 0} \cup \{\infty\}$, are, respectively, non-increasing and non-decreasing in $x$. Moreover, $\gamma_>(\mathcal{F}, 1) \geq \gamma_\leq(\mathcal{F}, 1)$. In the following lemma, we show that $\gamma_\leq(\mathcal{F}, x)$ and $\gamma_>(\mathcal{F}, x)$ are continuous in $x \geq 1$.

**Lemma 2.** Both $\gamma_\leq(\mathcal{F}, x)$ and $\gamma_>(\mathcal{F}, x)$ are continuous in $x \geq 1$.

**Proof.** When showing the continuity of $\gamma_\leq(\mathcal{F}, x)$ and $\gamma_>(\mathcal{F}, x)$, we implicitly consider the topological space $\mathbb{R}_{\geq 0} \cup \{\infty\}$ as codomain.

Let us start with $\gamma_\leq(\mathcal{F}, x)$. First of all, we show that $\gamma_\leq(\mathcal{F}, x)$ is right-continuous in $x \geq 1$. Fix a value $x \geq 1$. If $\gamma_\leq(\mathcal{F}, x) = \infty$, as $\gamma_\leq(\mathcal{F}, x)$ is non-decreasing, the right-continuity follows. Now, we assume that $\gamma_\leq(\mathcal{F}, x) < \infty$. We have that

\[
\sup_{0 \leq k \leq l \in \mathcal{F}, l > 0} \frac{f(k)}{f(l)} \leq \sup_{0 \leq k \leq l \in \mathcal{F}, l > 0} \left( \frac{k + (x + \delta)(-k + l)}{l} \right) \frac{f(k)}{f(l)} = \gamma_\leq(\mathcal{F}, x) < \infty.
\]
Thus, there exists a constant $c > 0$ such that $\sup_{0 \leq k \leq l, f \in F > 0} \frac{f(k)}{f(l)} \leq c$. Let $\delta := \epsilon/c$. Given $\xi \in [x, x + \delta)$, we get

$$
\gamma_{\leq}(F, \xi) \leq \gamma_{\leq}(F, x + \delta)
$$

$$
= \sup_{0 \leq k \leq l, f \in F} \left( \frac{k + \delta}{l} \right) \cdot \sup_{0 \leq k \leq l, f \in F} \left( \frac{-k + l}{f(k)} \cdot \delta \frac{f(k)}{f(l)} \right)
$$

$$
\leq \sup_{0 \leq k \leq l, f \in F > 0} \left( \frac{k + x - k}{l} \right) \cdot \sup_{0 \leq k \leq l, f \in F > 0} \left( \frac{f(k)}{f(l)} \right) + \sup_{0 \leq k \leq l, f \in F > 0} \left( \frac{\delta \frac{f(k)}{f(l)}}{f(k)} \right)
$$

$$
\leq \sup_{0 \leq k \leq l, f \in F > 0} \left( \frac{k + x - k}{l} \right) \cdot \sup_{0 \leq k \leq l, f \in F > 0} \left( \frac{f(k)}{f(l)} \right) + c \delta
$$

$$
= \gamma_{\leq}(F, x) + \epsilon,
$$

thus, by the arbitrariness of $\epsilon$, $\gamma_{\leq}(F, x)$ is right-continuous in $x$.

Now, we show that $\gamma_{>}(F, x)$ is left-continuous in $x > 1$ (for our aim, the left-continuity does not make sense in $x = 1$). Fix two values $x > 1$ and $M < \gamma_{>}(F, x)$, and let $\epsilon$ be an arbitrary number such that $0 < \epsilon < \gamma_{>}(F, x) - M$. By the definition of supremum, there exists a triple $(k, l, f)$ with $l \leq k \leq l$, and $f \in F$ such that $f_{k, l, f}(x) := \frac{k + x - k + l}{l} \cdot \frac{f(k)}{f(l)} M + \epsilon$. By continuity of $f_{k, l, f}(x)$ in $x > 1$, we have that there exists $\delta > 0$ such that $f_{k, l, f}(\xi) > (M + \epsilon) - \epsilon = M$ for any $\xi \in (x - \delta, x]$, and by the definition of supremum and the monotonicity of $\gamma_{>}(F, \xi)$, we get $\gamma_{>}(F, x) \geq \gamma_{>}(F, \xi) = \sup_{0 \leq k \leq l, f \in F > 0} f_{k, l, f}(\xi) > M$ for any $\xi \in (x - \delta, x]$. Thus, by the arbitrariness of $M$, the left-continuity follows.

Now, we proceed with $\gamma_{>}(F, x)$. First of all, we show that $\gamma_{>}(F, x)$ is left-continuous in $x > 1$ (observe that left-continuity in $x = 1$ does not make sense). Fix a value $x > 1$. If $\gamma_{>}(F, x) = \infty$, the left-continuity follows, as $\gamma_{>}(F, x)$ is non-increasing in $x > 1$.

Now, assume that $\gamma_{>}(F, x) < \infty$. Let $y$ be an arbitrary number such that $\frac{x - 1}{2} < y < 1$. We necessarily have that $\sup_{0 \leq k \leq l, f \in F} \frac{f(k)}{f(l)} < \infty$. Indeed, if this is not the case, by setting $k^* := \frac{xy}{(x - 1)}$ we have that $x(1 - y) = \sup_{0 \leq k \leq l, f \in F} \frac{f(k)}{f(l)} = \sup_{0 \leq k \leq l, f \in F} \left( \frac{k + (1 - y)}{l} \right) \cdot \frac{f(k)}{f(l)} \leq \gamma_{>}(F, x)$, but this contradicts the hypothesis $\gamma_{>}(F, x) < \infty$. Thus, we get the following fact:

**Fact 7** There exists a constant $b$ such that $b > \frac{f(k)}{f(l)}$ for any $l > 0$ and $f \in F$.

Let $\psi$ be an arbitrary number such that $1 < \psi < x$. Observe that $k + (\xi - k + l) < 0$ for any $\xi \in [\psi, x]$ and for any $k > \frac{\psi}{\psi - 1}$. Thus, when evaluating the supremum of $\left( \frac{k + (\xi - k + l)}{l} \right) \cdot \frac{f(k)}{f(l)}$, we can avoid considering all values $k > \frac{\psi}{\psi - 1}$, and we get the following fact:

**Fact 8** $\gamma_{>}(F, x) = \sup_{k > l > 0, f \in F, k \leq \frac{\psi}{\psi - 1} \frac{\psi}{\psi - 1}} \left( \frac{k + (\xi - k + l)}{l} \right) \cdot \frac{f(k)}{f(l)}$ for any $\xi \in [\psi, x]$.

Let $h$ be the first integer such that $\left( \frac{\psi}{\psi - 1} \right)^{h} \geq \frac{\psi}{\psi - 1}$, and let $c := \frac{\psi}{\psi - 1}$. We have that

$$
\sup_{k > l > 0, f \in F, k \leq \frac{\psi}{\psi - 1}} \left( \frac{k - l}{l} \right) \cdot \frac{f(k)}{f(l)} \leq \sup_{k > l > 0, f \in F, k \leq \frac{\psi}{\psi - 1}} \left( \frac{k - l}{l} \right) \cdot \sup_{k > l > 0, f \in F, k \leq \frac{\psi}{\psi - 1}} \frac{f(k)}{f(l)}
$$

$$
\leq \sup_{k > l > 0, f \in F} \left( \frac{k - l}{l} \right) \cdot \frac{f(k)}{f(l)}
$$

$$
\leq \frac{1}{\psi - 1} \sup_{l > 0, f \in F} \frac{f}{f} \left( \frac{xy}{x - 1} \right)^{l}
$$

$$
\leq \frac{1}{\psi - 1} \sup_{l > 0, f \in F} \prod_{i=1}^{h} \frac{f}{f} \left( \frac{xy}{x - 1} \right)^{i-1}
$$

$$
\leq \frac{1}{\psi - 1} \sup_{f \in F} \prod_{i=1}^{h} \left( \frac{f}{f} \left( \frac{xy}{x - 1} \right)^{i-1} \right)
$$
Lemma 3. We have that there exists \( \gamma \) such that the equality follows from the monotonicity of \( \gamma \).

Proof. Now, given an arbitrary \( \gamma \), we separately consider the case

\[
\gamma(F, \xi) \leq \gamma(F, x - \delta)
\]

(20)

where (18) comes from Fact 7. Thus, by (19), we get the following fact:

Fact 9. There exists a constant \( c > 0 \) such that

\[
\sup_{k > l > 0, f \in F, k \leq \psi^2 \varepsilon l} \frac{(k - l)}{l} \frac{f(k)}{f(l)} \leq c.
\]

(19)

Now, given an arbitrary \( \epsilon > 0 \), let \( \delta := \min\{x - \psi, \epsilon/c\} \). For any \( \xi \in (x - \delta, x] \) we have that:

\[
\gamma(F, \xi) \leq \gamma(F, x - \delta)
\]

(21)

where (20) comes from Fact 8 and (21) comes from Fact 9. Thus, by the arbitrariness of \( \epsilon \), \( \gamma(F, x) \) is left-continuous in \( x \).

Now, we show that \( \gamma(F, x) \) is right-continuous in \( x \geq 1 \). Fix two values \( x \geq 1 \) and \( M < \gamma(F, x) \), and let \( \epsilon \) be an arbitrary number such that \( 0 < \epsilon < \gamma(F, x) - M \). By the definition of supremum, there exists a triple \( (k, l, f) \) with \( 0 < l < k \) and \( f \in F \) such that \( f_{k,l,f}(x) := \frac{k + x(-k + l)}{l} \frac{f(k)}{f(l)} > M + \epsilon \). By continuity of \( f_{k,l,f}(x) \) in \( x \geq 1 \), we have that there exists \( \delta > 0 \) such that \( f_{k,l,f}(\xi) > (M + \epsilon) - \epsilon = M \) for any \( \xi \in [x - \delta, x] \), and by the definition of supremum and the monotonicity of \( \gamma(F, \xi) \), we get \( \gamma(F, x) \geq \gamma(F, \xi) = \sup_{0 < l < k, f \in F} f_{k,l,f}(\xi) > M \) for any \( \xi \in [x, x + \delta] \). Thus, by the arbitrariness of \( M \), the right-continuity follows.

By using the previous results, we get the following lemma.

Lemma 3. Let \( 1 \leq x_0 \leq x_1 \) be such that \( \gamma(F, x_1) \leq \gamma(F, x_0) \), and let \( x_0 := 1 \) or such that \( \gamma(F, x_0) \geq \gamma(F, x_0) \). Then there exists \( \hat{x} \in [x_0, x_1] \) such that \( \gamma(F, x) = \gamma(F, \hat{x}) = \gamma(F, \hat{x}) \). If such \( x_1 \) does not exist, then \( \gamma(F) = \lim_{x \to \infty} \gamma(F, x) = \lim_{x \to -\infty} \gamma(F, x) = \infty \) and \( \gamma(F, x) > \gamma(F, x) \) for any \( x \geq 1 \).

Proof. Let \( 1 \leq x_0 \leq x_1 \) be such that \( \gamma(F, x_1) \leq \gamma(F, x_1) \) and \( \gamma(F, x_0) \geq \gamma(F, x_0) \). We do not consider separately the case \( x_0 = 1 \), since \( \gamma(F, 1) = \gamma(F, 1) \). Since \( \gamma(F, x) \) and \( \gamma(F, x) \) are continuous in \( x \geq 1 \) (as shown in Lemma 2), we necessarily have that \( \gamma(F, \hat{x}) = \gamma(F, \hat{x}) \) (by the Intermediate Zero Theorem), and the claim follows. If there is no \( x_1 \geq 1 \) such that \( \gamma(F, \hat{x}) \leq \gamma(F, \hat{x}) \), we necessarily have that \( \gamma(F) = \lim_{x \to -\infty} \gamma(F, x) \geq \lim_{x \to -\infty} \gamma(F, x) \), where the first equality follows from the monotonicity of \( \gamma(F, x) \). Thus, as \( \lim_{x \to -\infty} \gamma(F, x) = \infty \), the claim follows.

The following lemma is the main ingredient to show the desired lower bound, and the proof is based on Lemma 3.

Lemma 4. Fix a value \( M < \gamma(F) \). Then, there exist \( k_1, l_1, f_1, k_2, l_2, f_2 \) with \( f_1, f_2 \in F \), \( k_1 > l_1 > 0 \), \( 0 \leq k_2 \leq l_2 \), and \( l_2 > 0 \), such that

\[
\gamma(k_1, l_1, f_1, k_2, l_2, f_2) := \frac{(l_2 - k_2)f_2(k_2)}{(l_2 - k_2)f_2(k_2)l_1f_2(l_1) + (k_1 - l_1)f_1(k_1)l_2f_2(l_2)} > M.
\]
Proof. First of all, assume that there exists $x_1 > 1$ such that $\gamma_>(\mathcal{F}, x_1) < \gamma_≤(\mathcal{F}, x_1)$. By Lemma 3 we have that $\gamma(\mathcal{F}) = \gamma_>(\mathcal{F}, \hat{x}) = \gamma_≤(\mathcal{F}, \hat{x})$ for some $\hat{x} \in [1, x_1]$. Thus, by the definition of supremum, there exist two triples $(k_1, l_1, f_1)$ and $(k_2, l_2, f_2)$, with $k_1 > l_1 > 0$, $0 \leq k_2 \leq l_2$, $f_1, f_2 \in \mathcal{F}$, and $l_2 > 0$, such that

$$
\frac{k_1 + \hat{x}(-k_1 + l_1)}{l_1} f_1(k_1) > M, \quad \frac{k_2 + \hat{x}(-k_2 + l_2)}{l_2} f_2(k_2) > M.
$$

(22)

As $k_2 \leq l_2$ and $l_1 < k_1$, we have that there exists $x^* > 0$ such that $\gamma^* := \left(\frac{k_1 + x^*(-k_1 + l_1)}{l_1} f_1(k_1)\right) = \left(\frac{k_2 + x^*(-k_2 + l_2)}{l_2} f_2(k_2)\right)$. Then, by (22), $(x^*, \gamma^*)$ is the optimal solution of the following linear program in variables $\hat{x}, \gamma$:

$$
\text{DLP : } \min \gamma
$$

$$
s.t. \ \gamma \geq \frac{k_1 f_1(k_1) + x(-k_1 f_1(k_1) + l_1 f_1(k_1))}{l_1 f_1(l_1)},
$$

(23)

$$
\gamma \geq \frac{k_2 f_2(k_2) + x(-k_2 f_2(k_2) + l_2 f_2(k_2))}{l_2 f_2(l_2)},
$$

(24)

$x \geq 0,$

and constraints (23) and (24) are tight for $x = x^*$. By considering the dual of the linear program considered above we get the following linear program in variables $\alpha_1, \alpha_2$:

$$
\hat{\text{LP}} : \max \alpha_1 k_1 f_1(k_1) + \alpha_2 k_2 f_2(k_2)
$$

$$
s.t. \ \alpha_1 (k_1 - l_1) f_1(k_1) + \alpha_2 (k_2 - l_2) f_2(k_2) \leq 0
$$

(25)

$$
\alpha_1 l_1 f_1(l_1) + \alpha_2 l_2 f_2(l_2) = 1
$$

(26)

By the Strong Duality Theorem, the optimal solution of $\hat{\text{LP}}$ has value $\gamma^* > M$. Furthermore, since $x^* > 0$, by the complementary slackness conditions, we have that the optimal solution $(\alpha^1, \alpha^2)$ of $\hat{\text{LP}}$ is such that constraint (25) is tight, that, together with constraint (26), gives

$$
\alpha^1 = \frac{(l_2 - k_2) f_2(k_2)}{(l_2 - k_2) f_2(k_2) l_1 f_1(l_1) + (k_1 - l_1) f_1(k_1) l_2 f_2(l_2)} \geq 0
$$

$$
\alpha^2 = \frac{(k_1 - l_1) f_1(k_1)}{(l_2 - k_2) f_2(k_2) l_1 f_1(l_1) + (k_1 - l_1) f_1(k_1) l_2 f_2(l_2)} \geq 0.
$$

By putting $\alpha^1$ and $\alpha^2$ in the objective function of $\hat{\text{LP}}$, we get the optimal value $\gamma^*$ of $\hat{\text{LP}}$, that is

$$
M < \gamma^* = \alpha^1 k_1 f_1(k_1) + \alpha^2 k_2 f_2(k_2) = \frac{(l_2 - k_2) f_2(k_2) k_1 f_1(k_1) + (k_1 - l_1) f_1(k_1) l_2 f_2(l_2)}{(l_2 - k_2) f_2(k_2) l_1 f_1(l_1) + (k_1 - l_1) f_1(k_1) l_2 f_2(l_2)} \leq \gamma(\mathcal{F}),
$$

and this shows the claim.

Finally, if there is no $\hat{x} \geq 1$ such that $\gamma_>(\mathcal{F}, \hat{x}) = \gamma_≤(\mathcal{F}, \hat{x})$, by Lemma 3 we have that $\gamma(\mathcal{F}) = \lim_{x \to \infty} \gamma_>(\mathcal{F}, x) = \lim_{x \to \infty} \gamma_≤(\mathcal{F}, x) = \gamma_>(\mathcal{F}, x)$ for any $\gamma \geq 1$. Thus, by the definition of supremum, there exist a sufficiently large $\hat{x} \geq 1$, two triples $(k_1, l_1, f_1)$ and $(k_2, l_2, f_2)$, with $k_1 > l_1 > 0$, $0 < k_2 \leq l_2$, $f_1, f_2 \in \mathcal{F}$, such that

$$
\frac{k_1 + \hat{x}(-k_1 + l_1)}{l_1} f_1(k_1) > \frac{k_2 + \hat{x}(-k_2 + l_2)}{l_2} f_2(k_2) > M.
$$

Thus, using the same arguments as in the previous case, one can show the claim as well. □

Armed with the above lemma, we are ready to show Theorem 10.

**Theorem 10.** Fix a class of latency functions $\mathcal{F}$ and a value $M < \gamma(\mathcal{F})$. (i) There exists a path-disjoint network congestion game $\text{PNCG}$ with latency functions in $\mathcal{F}$ such that $\text{PoA}(\text{PNCG}) > M$. (ii) If $\mathcal{F}$ is scale-closed, then there exists a parallel-link game $\text{PLG}$ with latency functions in $\mathcal{F}$ such that $\text{PoA}(\text{PLG}) > M$. 


Proof. We first show part (ii). Let \( k_1, l_1, f_1, k_2, l_2, f_2 \) be the quantities specified in the claim of Lemma 4, i.e., such that \( f_1, f_2 \in F, k_1 > l_1 > 0, \) and \( 0 \leq k_2 \leq l_2 \), and \( \gamma(k_1, l_1, f_1, k_2, l_2, f_2) > M \). Observe that \( k_1, l_1, k_2, l_2 \) can be chosen in such a way that \( \frac{k_2 - l_1}{l_2 - k_2} \) is an integer for some \( n \in \mathbb{N} \) and in such a way that \( l_2 > k_2 \). Indeed, assume that such property is not satisfied. As \( \gamma(k_1, l_1, f_1, k_2, l_2, f_2) \) is continuous in \( l_2 \), by using similar arguments as in the proof of Theorem 6 one can consider a value \( l_2' \) in place of \( l_2 \), in such a way that \( 0 \leq k_2 < l_2' \), \( \gamma(k_1, l_1, f_1, k_2, l_2', f_2) > M \), and \( \frac{k_2 - l_1}{l_2' - k_2} \) is an integer for some \( n \in \mathbb{N} \).

Let PLG be a parallel-link game defined as follows: (a) the set of resources \( E \) is partitioned into two subsets \( E^+ \) and \( E^- \); (b) \( E^+ \) contains \( n \) resources having latency function defined as \( \ell_e(x) := f_2(k_2)f_1(x) \), and \( E^- \) contains \( \frac{k_2 - l_1}{l_2 - k_2} n \) resources having latency function defined as \( \ell_e(x) := f_1(k_1)f_2(x) \); (c) the total amount of players is \( \frac{(k_1 l_1 - k_2 l_2)}{l_2 - k_2} n \). Let \( \sigma \) (resp. \( \sigma^* \)) be the strategy profile in which each resource of \( E^+ \) is selected by \( k_1 \) (resp. \( l_1 \)) players and each resource of \( E^- \) is selected by \( k_2 \) (resp. \( l_2 \)) players. One can easily observe that all resources have the same latency in \( \sigma \), thus \( \sigma \) is a pure Nash equilibrium. We have that

\[
\begin{align*}
\text{SUM}(\sigma) &= |E^+|k_1 \ell_1(k_1) + |E^-|k_2 \ell_2(k_2) = \left( k_1 f_2(k_2) f_1(k_1) + \frac{(k_1 - l_1) f_1(k_1) f_2(k_2)}{l_2 - k_2} \right) n \\
\text{SUM}(\sigma^*) &= |E^+|l_1 \ell_1(l_1) + |E^-|l_2 \ell_2(l_2) = \left( l_1 f_2(k_2) f_1(l_1) + \frac{(k_1 - l_1) f_1(k_1) f_2(l_2)}{l_2 - k_2} \right) n,
\end{align*}
\]

thus

\[
\text{PoA}(\text{PLG}) \geq \frac{\text{SUM}(\sigma)}{\text{SUM}(\sigma^*)} = \frac{\left( k_1 f_2(k_2) f_1(k_1) + \frac{(k_1 - l_1) f_1(k_1) f_2(k_2)}{l_2 - k_2} \right) n}{\left( l_1 f_2(k_2) f_1(l_1) + \frac{(k_1 - l_1) f_1(k_1) f_2(l_2)}{l_2 - k_2} \right) n} = \gamma(k_1, l_1, k_2, l_2, f_2) > M,
\]

and this shows the claim.

Regarding part (i), we resort to a similar proof as in Theorem 6. We reconsider the lower bounding instance of part (ii), and transform it in a lower bounding instance having the structure of a path-disjoint network congestion game. Let us consider a parallel-link game defined as in part (ii) such that \( \text{PoA}(\text{PLG}) > M \). Let \( k_1, l_1, f_1, k_2, l_2, f_2, n \) be the parameters characterizing PLG. By using similar arguments as in the proof of Theorem 6, we can assume without loss of generality that \( f_1(k_1) \) and \( f_2(k_2) \) are rational numbers.

Let PNCG be a path-disjoint network congestion game constructed from PLG as follows. Consider an undirected graph \( G = (V', E') \) initially empty. We add in \( G \) a source-node \( u^* \) and a sink-node \( v^* \). Then, for any resource \( e \) of group \( E^+ \) (resp. \( E^- \) ), we add a path \( P_e := (u^*, u'_{e,1}, \ldots, u'_{e,n-1}, v^*) \) (resp. \( Q_e = (u^*, v'_{e,1}, \ldots, v'_{e,h-1}, v^*) \) with \( h := \frac{(k_1 - l_1)n}{l_2 - k_2} \). The latency function of each edge of paths of type \( P_e \) (resp. \( Q_e \)) is \( f_1 \) (resp. \( f_2 \)). The amount of players is the same as in PLG, and their possible strategies are all the simple paths from \( u^* \) to \( v^* \).

Let \( \phi \) (resp. \( \phi^* \)) be the strategy profile in which each path of type \( P_e \) (resp. \( Q_e \)) is selected by the same amount of player selecting resource \( e \) in \( \sigma \) (resp. in \( \sigma^* \)). By adapting the proof-arguments used in part (ii) to game PNCG, one can easily show that the strategy profile \( \phi \) is a pure Nash equilibrium, and that \( \text{PoA}(\text{PNCG}) \geq \frac{\text{SUM}(\phi)}{\text{SUM}(\phi^*)} = \gamma(k_1, l_1, k_2, l_2, f_2) > M \), thus showing the claim.

By using Theorem 6 and 10, we can show Theorem 5.

Proof (Proof of Theorem 5). Recall that \( \gamma_\theta(|F|_H) = \max\{\gamma_\theta(|F|_H), \gamma(|F|_H)\} \) and observe that, when considering latency functions in \( |F|_H \) (i.e., homogeneous functions), any congestion game is also a \( \theta \)-free-flow game for any \( \theta \in [0, \infty] \). Thus, we can apply Theorem 10 with respect to the value \( \gamma(|F|_H) \).

First of all, we show parts (i) and (ii). If \( |F|_H \) is strongly diverse, then it is also scale-closed, and so the claim follows from Theorems 6 (part (ii)) and 10 (part (ii)). If \( |F|_H \) is weakly diverse, then the claim follows from Theorems 6 (part (ii)) and 10 (part (i)).

Regarding parts (iii) and (iv), as \( \gamma_\theta(|F|_H) = \gamma(|F|_H) \), the lower bound given in Theorem 10 exactly matches the upper bound provided by Theorem 1. \( \square \)

### A.2 Proof of Theorem 3

To show the claim, by Theorems 1 and 5 and since \( \mathcal{P}_{p,q} \) is strongly diverse, it suffices computing the values of \( \gamma_\theta(|\mathcal{P}_{p,q}|_H) \) and \( \gamma(|\mathcal{P}_{p,q}|_H) \). We get

\[
\gamma_\theta(|\mathcal{P}_{p,q}|_H) = \sup_{k > t \geq 0, f \in |\mathcal{P}_{p,q}|_H} \frac{f(k)(k(1 + \theta) - l)}{f(k)(k - l)(1 + \theta) + l f(l) \theta}
\]
exploited in (27) and (28), we have that 

\[ \hat{x} \]

and thus obtaining (13). To compute \( \gamma \) in (13), we use the characterization given in Lemma 4. Given \( x \geq 1 \), we have that:

\[
\gamma_{\gamma}(\mathcal{P}_{p,q}\mid H, x) = \sup_{k > 0, f \in [\mathcal{P}_{p,q}\mid H]} \left( \frac{k + x(-k + l)}{l} \right) f(k) f(l)
\]

and

\[
\gamma_{\gamma}(\mathcal{P}_{p,q}\mid H, x) = \sup_{0 < k \leq l, f \in [\mathcal{P}_{p,q}\mid H], l > 0} \left( \frac{k + x(-k + l)}{l} \right) f(k) f(l)
\]

Let \( x_0 := q + 1 \) and \( x_1 := p + 1 \). Observe that (27) (resp. (28)) is decreasing (resp. increasing) in \( x \) in \( [1, p + 1] \) (resp. \( x \geq q + 1 \)) and equal to 1 if \( x \geq p + 1 \) (resp. \( x \in [q + 1] \)). Thus, we have that \( \gamma_{\gamma}(\mathcal{P}_{p,q}\mid H, x_0) = 1 \leq \gamma_{\gamma}(\mathcal{P}_{p,q}\mid H, x_1) \geq 1 = \gamma_{\gamma}(\mathcal{P}_{p,q}\mid H, x_1). \) By Lemma 5, there exists \( \hat{x} \in [q + 1, p + 1] \) such that \( \gamma(\mathcal{P}_{p,q}\mid H, \hat{x}) = \gamma(\mathcal{P}_{p,q}\mid H, \hat{x}) \). If \( p = q \), we necessarily have that \( \hat{x} = p + 1 \), and then \( \gamma(\mathcal{P}_{p,q}\mid H, \hat{x}) = \gamma(\mathcal{P}_{p,q}\mid H, \hat{x}) = \gamma_{\gamma}(\mathcal{P}_{p,q}\mid H, \hat{x}) = 1 \), thus (14) holds. If \( p > q \), by using the characterizations exploited in (27) and (28), we have that \( \hat{x} \) necessarily verifies \( (q + 1)^{p+1} \lambda_{q+1}(x-1)^{p+1} \). Finally, by substituting the value of \( \hat{x} \) in \( \gamma_{\gamma}(\mathcal{P}_{p,q}\mid H, \hat{x}), \) we get (14).

A.3 Proof of Lemma 1

By substituting \( \beta_u := \beta_u/(1 + \theta) \) and \( \alpha_u := (\alpha_u f_u(k_u) + (1 - 1/(1 + \theta)) \beta_u)/f_u(k_u) \) in the definition of \( F(\alpha_u, \beta_u, \alpha_u, \beta_u) \), we get

\[
F(\alpha_u, \beta_u, \alpha_u, \beta_u)
\]
Proof. For any strategy/path functions containing \( \hat{\eta} = (\hat{\eta}_1, \hat{\eta}_2) \) is the optimal value of DLP (where L\textsc{P} is based on \( k_1, l_1, f_1, k_2, l_2, f_2 \)). As the dual of DLP is LP and the optimal value of LP is upper bounded by \( \gamma(|F|_H) \), inequality \( \gamma_k(1, l_1, f_1, k_2, l_2, f_2) \leq \gamma(|F|_H) \) follows by the Weak Duality Theorem.

Fact 11 We have that:

\[
\frac{(l_2 - k_2) f_2(k_2) k_1 f_1(k_1) + (k_1 - l_1) f_1(k_1) k_2 f_2(k_2)}{(l_2 - k_2) f_2(k_2) l_1 f_1(l_1) + (k_1 - l_1) f_1(k_1) l_2 f_2(l_2)} \leq \gamma(|F|_H)
\]

for any \( k_1, l_1, f_1, k_2, l_2, f_2 \) such that \( k_1 > l_1 \geq 0, l_2 > k_2 \geq 0 \), and \( f_1, f_2 \in |F|_H \).

Proof. Reconsider the proof of Lemma 7. We have that \( \gamma_k(1, l_1, f_1, k_2, l_2, f_2) \) is the optimal value of DLP (where L\textsc{P} is based on \( k_1, l_1, f_1, k_2, l_2, f_2 \)). As the dual of DLP is LP and the optimal value of LP is upper bounded by \( \gamma(|F|_H) \), inequality \( \gamma_k(1, l_1, f_1, k_2, l_2, f_2) \leq \gamma(|F|_H) \) follows by the Weak Duality Theorem.

Fact 12 We have that:

\[
\frac{k_u}{k_u - l_u} \left( \frac{\theta}{f_u(k_u) + f_u(k_u)} \right) + \frac{k_v}{l_v - k_v} \leq \eta_0(|F|_H).
\]

Proof. Let \( a := \frac{k_u}{k_u - l_u} \), \( b := \frac{l_v}{k_u - l_u} \left( \frac{\theta}{f_u(k_u) + f_u(k_u)} \right) \), and \( t := \frac{k_v}{k_v - l_v} \in [0, 1] \), so that the left-hand side of (30) is equal to \( \frac{a + t}{b + t} \). As \( 1 \leq \text{PoA}(\text{PLG}_e) \leq \frac{\gamma + t}{b + t} \), we necessarily have that \( a \geq b \). Thus, by standard arguments of calculus, one can show that \( \frac{a + t}{b + t} \) is maximized by \( t = 0 \), i.e., \( \frac{a + t}{b + t} \leq \frac{a}{b} \). Thus,

\[
\frac{a + t}{b + t} \leq \frac{a}{b} = \frac{f_u(k_u) k_u (1 + \theta)}{f_u(k_u) k_u (1 + \theta) + (l_u f_u(l_u) - l_u f_u(k_u)) \theta} \leq \eta_0(|F|_H).
\]

By using the previous facts, we get that (29) is upper bounded by max \( \{\gamma(|F|_H), \eta_0(|F|_H)\} \).

A.4 Case of Path-disjoint Network Games in Theorem 2

Let \( F' := \{g : g(x) = \sum_{r=1}^{s} f_r(x) \forall x > 0, f_1, \ldots, f_s \in F, s \geq 1\} \), i.e., \( F' \) is the smallest class of latency functions containing \( F \) and closed under sums of latency functions. Let PNCG\textsc{g} be a \( \theta \)-free-flow path-disjoint network congestion game with latency functions in \( F \). As \( F \subseteq F' \), PNCG\textsc{g} has also latency functions in \( F' \).

For any strategy/path \( P \) of PNCG\textsc{g}, we replace \( P \) with a unique resource \( e_P \) having latency function defined as \( \ell_{\text{res}}(x) := \sum_{e \in P} \ell_e(x) \), \( \ell_e \) is the latency function of edge \( e \) in game PNCG\textsc{g}. By construction of \( F' \), we have that \( \ell_{\text{res}} \in F' \) for any path \( P \) of PNCG\textsc{g}.

Thus, the resulting game is a \( \theta \)-free-flow parallel-link game with latency function in \( F' \) whose Price of Anarchy is at most \( \hat{\eta}_0(|F'|_H) = \max\{\eta_0(|F'|_H), \gamma(|F'|_H)\} \). We observe that:

\[
\eta_0(|F'|_H) := \sup_{k > 0, f_1, \ldots, f_s \in |F'|_H, s \geq 1} \sum_{r=1}^{s} k f_r(k) \theta = \sup_{k > 0, f \in |F'|_H} k f(k) \theta = \eta_0(|F|_H),
\]

and with analogue proof arguments, we get \( \gamma(|F'|_H) = \gamma(|F|_H) \). We conclude that \( \text{PoA}(\text{PNCG}_e) \leq \hat{\eta}_0(|F'|_H) \leq \max\{\eta_0(|F'|_H), \gamma(|F'|_H)\} \).
A.5 Tightness of the Upper Bound Shown in Theorem 2

Theorem 13. Fix a value $\theta \in [0, \infty)$, a class of latency functions $\mathcal{F}$ and a value $M < \eta_\theta([\mathcal{F}]_H)$. (i) If $\mathcal{F}$ is strongly diverse, there exists a $\theta$-free-flow parallel-link game $\mathcal{PLG}_\theta$ with latency functions in $\mathcal{F}$ such that $\text{PoA}(\mathcal{PLG}_\theta) > M$. (ii) If $\mathcal{F}$ is weakly-diverse only, there exists a $\theta$-free-flow path-disjoint network congestion game $\mathcal{PNCG}_\theta$ with latency functions in $\mathcal{F}$ such that $\text{PoA}(\mathcal{PNCG}_\theta) > M$.

Proof. As Theorem 10 guarantees the existence of parallel-link games and symmetric network congestion games whose Price of Anarchy is at least $M$ for $M < \gamma([\mathcal{F}]_H)$ and when considering homogeneous latency functions any congestion game is $\theta$-free-flow for any value of $\theta$, it is sufficient to show the claim for $M < \eta_\theta([\mathcal{F}]_H)$.

We first show part (i). Fix $M < \eta_\theta([\mathcal{F}]_H)$, and let $k, l, f$ such that $f \in [\mathcal{F}]_H$, $k > l > 0$, and $\eta_\theta(k, l, f) := \frac{k f(k)(1 + \theta)}{k f(k)(1 + \theta) + (l f(l) - l f(k)) \theta} > M$. Consider a parallel-link game $\mathcal{PLG}_\theta$ with two resources $u$ and $v$ and an amount of players equal to $k$, where $u$ and $v$ have latency function defined as $\ell_u(x) = \theta f(x) + f(k)$ and $\ell_v(x) = (1 + \theta) f(k)$, respectively. Observe that $\ell_u(0)(1 + \theta) = \ell_v(0)$, thus $\mathcal{PLG}_\theta$ is with a $\theta$-free-flow game.

Let $\sigma$ (resp. $\sigma'$) be the strategy profile such that $k$ (resp. $l$) players select resource $u$, and 0 (resp. $k - l$) players select resource $v$. One can easily observe that $\ell_u(k_u(\sigma)) = (1 + \theta) f(k) = \ell_v(k_v(\sigma))$, thus $\sigma$ is a pure Nash equilibrium. We have that

$$
\text{SUM}(\sigma) = k \ell_u(k) = k f(k)(1 + \theta),
$$

$$
\text{SUM}(\sigma') = l \ell_u(l) + (k - l) \ell_v(k - l) = l f(l) + (k - l) f(k)(1 + \theta) = k f(k)(1 + \theta) + (l f(l) - l f(k)) \theta,
$$

thus $\text{PoA}(\mathcal{PLG}_\theta) \geq \frac{\text{SUM}(\sigma)}{\text{SUM}(\sigma')} = \frac{k f(k)(1 + \theta)}{k f(k)(1 + \theta) + (l f(l) - l f(k)) \theta} = \eta_\theta(k, l, f) > M$, and this shows the claim.

Regarding part (ii), it is sufficient considering a similar argument as in Theorem 6 and 10. Let $k, l$ and $f$ be defined as in part (i), and let $g \in \mathcal{F}$ be a constant latency function defined as $g(x) := \beta$. As shown in Theorems 6 and 10, one can assume without loss of generality that $f(k)/\beta$ and $\theta$ are rational numbers, and $(k - l)n$ is an integer for some $n \in \mathbb{N}$.

Let $\mathcal{PNCG}_\theta$ be a path-disjoint network congestion game constructed from $\mathcal{PLG}_\theta$ as follows. Consider an undirected graph $G = (V', E')$ initially empty. We add in $G$ a source-node $u^*$ and a sink-node $v^*$. Let $h \in \mathbb{N}$ be such that $a := \theta h, b := f(h)/\beta$, and $c := (1 + \theta)f(h)/\beta$ are integers. As $f(k)/\beta$ and $\theta$ are rational numbers, such an integer $h$ exists. Then, we add two consecutive paths $P := (u_1', u_2', \ldots, u_h')$ and $Q := (v_1', v_2', \ldots, v_h')$ with $u_1' = u^*$, $u_h' = v^*$, and $v_h' = v^*$, such that the latency function of each edge in $P$ (resp. $Q$) is defined as $\ell'(x) := f(x)$ (resp. $\ell'(x) := g(x) := \beta$). Let $P + Q$ be the path obtained by concatenating paths $P$ and $Q$. Finally, we add a path $R := (r_0', r_1', \ldots, r_{h'}')$, with $r_0' = u^*$ and $r_{h'}' = v^*$ such that the latency function of each edge in $R$ is defined as $\ell'(x) := g(x)$.

As shown for $\mathcal{PLG}_\theta$ in part (i), $\mathcal{PNCG}_\theta$ is a $\theta$-free-flow game as well. Let $\phi$ (resp. $\phi'$) be the strategy profile in which path $P + Q$ is selected by an amount of $k$ (resp. $l$) players and path $R$ is selected by an amount of 0 (resp. $k - l$) players. By adapting the proof-arguments used in part (i) for game $\mathcal{PNCG}_\theta$, one can easily show that the strategy profile $\phi$ is a pure Nash equilibrium, and that $\text{PoA}(\mathcal{PNCG}_\theta) \geq \frac{\text{SUM}(\phi)}{\text{SUM}(\phi')} = \eta_\theta(k, l, f) > M$, thus showing the claim. \qed

A.6 Proof of Theorem 4

To show the claim, by Theorem 2 and 13 and since $\mathcal{P}_{p,q}$ is strongly diverse, it suffices computing the values of $\eta_\theta([\mathcal{P}_{p,q}]_H)$ and $\gamma([\mathcal{P}_{p,q}]_H)$. $\gamma([\mathcal{P}_{p,q}]_H)$ has been computed in Theorem 3 thus it is sufficient computing the value of $\eta_\theta([\mathcal{F}]_H)$. We have that:

$$
\eta_\theta([\mathcal{P}_{p,q}]_H) = \sup_{k > l > 0, f \in [\mathcal{P}_{p,q}]_H} \frac{f(k)(1 + \theta)}{k f(k)(1 + \theta) + (l f(l) - l f(k)) \theta}
$$

$$
= \sup_{k > l > 0, (\alpha_0, \alpha_1, \ldots, \alpha_p) > 0} \frac{\sum_{d=0}^{p} \alpha_d k^{d+1}(1 + \theta)}{\sum_{d=0}^{p} \alpha_d k^{d+1}(1 + \theta) + (\sum_{d=0}^{p} \alpha_d k^{d+1} - \sum_{d=0}^{p} \alpha_d k^{d}) \theta}
$$

$$
= \sup_{k > l > 0, (\alpha_0, \alpha_1, \ldots, \alpha_p) > 0} \frac{\sum_{d=0}^{p} \alpha_d (k^{d+1}(1 + \theta) + (l^{d+1} - l k^{d}) \theta)}{\sum_{d=0}^{p} \alpha_d (k^{d+1}(1 + \theta) + (l^{d+1} - l k^{d}) \theta)}
$$
\[
\begin{align*}
\text{max} & \sup_{d \in [p] \setminus [q-1]} \sup_{k>1 \theta} k^{d+1}(1 + \theta) \\
& \text{max} \sup_{d \in [p] \setminus [q-1]} \sup_{t>1} t^{d+1}(1 + \theta) + (1 - t^d)\theta \\
& \text{max} \sup_{d \in [p] \setminus [q-1]} \sup_{t>1} t^{p+1}(1 + \theta) + (1 - t^p)\theta \\
& \sup_{t>1} t^{p+1}(1 + \theta) + (1 - t^p)\theta
\end{align*}
\]