A study of large field configurations in MC simulations
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We discuss a new approach of scalar field theory where the small field contributions are treated perturbatively and the large field configurations (which are responsible for the asymptotic behavior of the perturbative series) are neglected. In two Borel summable $\lambda \phi^4$ problems improved perturbative series can be obtained by this procedure. The modified series converge towards values exponentially close to the exact ones. For $\lambda$ larger than some critical value, the method outperforms Padé’s approximants and Borel summations. The method can also be used for series which are not Borel summable such as the double-well potential series and provide a perturbative approach of the instanton contribution. Semi-classical methods can be used to calculate the modified Feynman rules, estimate the error and optimize the field cutoff. We discuss Monte Carlo simulations in one and two dimensions which support the hypothesis of dilution of large field configurations used in these semi-classical calculations. We show that Monte Carlo methods can be used to calculate the modified perturbative series.

In a recent publication\textsuperscript{[1]}, we have shown with three nontrivial $\phi^4$ problems that cutting off the integration range of $\phi$ at each site modifies the large order behavior of these series. The modified series converge toward values which are exponentially close to the exact ones. The examples were the anharmonic oscillator and the Landau-Ginzburg hierarchical model which are Borel summable, but also the double-well which is not Borel summable, and in this case, the method incorporates instanton effects. More generally, we are interested in developing approximate treatments where the small field configurations are treated perturbatively and the large field configurations semi-classically.

All the examples discussed above can be solved accurately with numerical methods\textsuperscript{[2],[3]}. In the following, we discuss the possibility of using the Monte Carlo method to perform such calculations. We consider scalar models with one component at each site. We define the norm of a configuration $C$ as $|C| = \max_x \{ |\phi_x| \}$. The norm distributions of MC configurations for $\lambda \phi^4$ models in 1 and 2 dimensions are shown in Fig.

1. One sees that the distributions have rapidly falling tails and that the number of large norm configurations is exponentially suppressed. One would thus expect that the error on correlation functions caused by cutting-off the configurations with $|C| > \phi_{\max}$ has the generic form

$$\text{Error} \sim \phi_{\max}^{A_1} e^{-A_3 \phi_{\max}^{A_2}}.$$ \hspace{1cm} (1)

In addition, the sites with large fields are clustered, and the configuration has a typical shape in these regions as shown in Fig. 2. This suggests that the large field configurations contribute directly to the functional integral. This is just what is needed for a semi-classical calculation.

The similarity of the (Borel summable) results found in Ref.\textsuperscript{[1]} suggests that in general the corrections due to the field cutoffs can be expressed as simple one-dimensional integrals. The correction to the zeroth order of the ground state of the anharmonic oscillator (in other words, the correction to the ground state of the harmonic oscillator) has the semi-empirical form\textsuperscript{[4]}.

$$\delta E^{(0)}_n \simeq 4\pi^{-1/2} \phi_{\max}^{2} \int_{\phi_{\max}}^{+\infty} d\phi \phi^{-1/2} e^{-\phi^2}.$$ \hspace{1cm} (2)
Fig. 3 shows that this is a good approximation which becomes better as $\phi_{\text{max}}$ increases.

When $\phi_{\text{max}}$ is large, the semi-empirical formula has the asymptotic form

$$\delta E^{(0)}_0 \simeq 2\pi^{-1/2}\phi_{\text{max}} e^{-\phi_{\text{max}}^2}$$

This correction has the same functional dependence as the semi-classical splitting for the first two levels of the double-well [4]:

$$\delta E^{(0)}_0 \propto \left(\frac{S_0}{2\pi}\right)^{1/2} e^{-S_0}$$

provided that we take $S_0 = \phi_{\text{max}}^2$, the classical action from the classical configuration $\phi_{\text{max}} e^{-|\tau-\tau_0|}$. This confirms the validity of the dilute gas approximation. Calculations of higher order perturbative coefficients and in higher dimensions are in progress.

It is possible to use the MC method to calculate the coefficients of the modified perturbative series. This will be particularly useful to check the validity of semi-classical methods in higher dimensions. In the case of the anharmonic oscillator $H = p^2/2 + \phi^2/2 + \lambda \phi^4$, the first order coefficient is

$$<\sum_x \phi_x^4>_{\text{Gaussian}} / \sum_x = 3/4 .$$

We have calculated the modified averages corresponding to various field cuts. The results are shown in Fig. 4 and are in good agreement with
At a fixed value of $\phi_{max}$, the error due to the field cut decreases when the coupling increases. At a fixed order in the modified perturbative series corresponding to the same fixed $\phi_{max}$, the error due to the omission of the higher orders decreases when the coupling decreases. Consequently, at fixed $\phi_{max}$, there is a value of the coupling where the the accuracy is optimal. This value is approximately where the two error curves meet. Conversely, at fixed coupling, one can find an optimal value of $\phi_{max}$. This value can be related to the optimal values of variational parameters used in Refs. [3,4]. In these calculations, an intermediate mass term is introduced. The effect of this term is at the same time to reduce the contributions of the large fields and to reduce the argument of the exponential which is expanded. Simple power law relations appear [4] in the strong and weak coupling limits for the problem of the one dimensional integral [5].

The coupling dependence of the optimal $\phi_{max}$ is a question which needs to be understood if we want to apply similar methods to gauge theories. In Wilson’s formulation, at fixed lattice spacing, a field cut inversely proportional to the gauge coupling is imposed when one uses the compact group integration. Consequently, the possibility for scalar models of having an optimal $\phi_{max}$ varying like the inverse square root of the quartic coupling in some regime would be quite interesting. We have started investigating the possibility of introducing field cuts in gauge theories by using the public OSU quenched configurations [9]. These configurations are in the Landau gauge, and the distributions of values of a fixed matrix element have similarities with the scalar case.

Finally, we would like to mention that, in the large-$N$ limit, the critical potentials of the $O(N)$ models in 3-dimensions have a finite radius of convergence when expanded in the $O(N)$ invariant quantity $\phi^2$ [10]. This is due to two conjugated branch cuts in the complex $\phi^2$ plane. It is tempting to consider the model with a field cut which corresponds to this radius of convergence. However, this procedure generates sizable errors compared to the reliable procedure which consists in defining the critical potential using Padé approximants.

REFERENCES

1. Y. Meurice, Phys. Rev. Lett. 88 (2002) 141601.
2. B. Bacus, Y. Meurice, and A. Soemadi, J. Phys. A 28 (1995) L381; Y. Meurice, quant-ph/0202047, to appear in J. Phys. A.
3. J. J. Godina, Y. Meurice, and M. Oktay, Phys. Rev. D 57 (1998) R6581 and D 59 (1999) 096002.
4. S. Coleman, *Aspects of Symmetry* (Cambridge University Press, Cambridge, 1985).
5. L. Li and Y. Meurice, U. of Iowa preprint, in preparation.
6. I. Buckley, A. Duncan and H. Jones, Phys. Rev. D 47 (1993) 2554.
7. W. Janke and H. Kleinert, Phys. Rev. Lett. 75 (1995) 2787.
8. B. Kessler and Y. Meurice, work in progress.
9. G. Kilcup, D. Pekurovsky, L. Venkataraman, Nucl. Phys. (Proc. Suppl.) 53 (1997) 345.
10. Y. Meurice, hep-th/0208181.