TWO-PARAMETER QUANTUM GENERAL LINEAR SUPERGROUP AND CENTRAL EXTENSIONS

HUAFENG ZHANG

Abstract. The universal $R$-matrix of two-parameter quantum general linear supergroups is computed explicitly based on the RTT realization of Faddeev–Reshetikhin–Takhtajan. For the two-parameter quantum special linear supergroup associated with $\mathfrak{sl}(2,2)$, a two-fold quasi-central extension is deduced from its Drinfeld–Jimbo realization.

1. Introduction

Let $r, s$ be non-zero complex numbers whose ratio $\frac{r}{s}$ is not a root of unity. Let $M, N$ be positive integers and $\mathfrak{g} := \mathfrak{gl}(M,N)$ be the general linear Lie superalgebra associated to the vector superspace $\mathbb{C}^{M|N}$. The universal enveloping algebra of $\mathfrak{u}(\mathfrak{g})$ as a Hopf superalgebra admits a two-parameter deformation $\mathcal{U}_{r,s}(\mathfrak{g})$ which is neither commutative nor cocommutative. In this paper we compute its universal $R$-matrix, an invertible element in a completed tensor square $R \in \mathcal{U}_{r,s}(\mathfrak{g}) \hat{\otimes}^2$ satisfying among other favorable properties (Theorem 4.1)

$$\Delta^\text{cop}(x) = R \Delta(x) R^{-1} \quad \text{for} \quad x \in \mathcal{U}_{r,s}(\mathfrak{g}).$$

There are explicit formulas of universal $R$-matrices for one-parameter quantum groups associated to simple Lie (super)algebras, quantum affine algebras and double Yangians, and certain quantum Kac–Moody superalgebras (without isotropic roots), to name a few; usually they are realized as the tensor product of orthonormal bases with respect to a bilinear form called the Drinfeld quantum double (or Hopf pairing in the main text). They have important applications in quantum integrable systems, knot invariants, etc.

For the two-parameter quantum group $\mathcal{U}_{r,s}(\mathfrak{gl}(M))$, Benkart–Witherspoon [BW1, BW2] proved the existence of universal $R$-matrix, which leads to a braided structure of a category $\mathcal{O}$ of representations; the exact formula of universal $R$-matrix was unknown. There are two equivalent definitions of $\mathcal{U}_{r,s}(\mathfrak{gl}(M))$: the Drinfeld–Jimbo realization, and the RTT realization of Faddeev–Reshetikhin–Takhtajan [PRT, JL].

In [Z2] for the quantum affine superalgebra associated to $\mathfrak{gl}(1,1)$ the universal $R$-matrix $R$ was computed. The main steps therein were: first relate the quantum double to the RTT realization; then obtain orthonormal bases (and $R$) via a Gauss decomposition in [DF]. Motivated by [BW1, JL, Z2], in the present paper we start with the RTT realization of $\mathcal{U}_{r,s}(\mathfrak{g})$, based on a suitable solution of the quantum Yang–Baxter equation on $\mathbb{C}^{M|N}$.

Our first main result is a factorization formula of $R$ in Equations (4.6–4.8). This is related to the fact that the orthonormal bases are formed of ordered products. We would like to emphasize that: within the framework of the RTT realization of $\mathcal{U}_{r,s}(\mathfrak{g})$, the orthogonality of these ordered products is fairly easy to prove, compared to the usual Drinfeld–Jimbo realization in [BW1]; see Remark 3.5. Making use of the quantum double
we are able to simplify some computations of relations between RTT generators which were done in \cite{JL} almost case by case; see the proof of Corollary 2.8

Our second main result is related to the special linear Lie superalgebra \( \mathfrak{sl}(2, 2) \), which fills in a non-split short exact sequence of Lie superalgebras \( \mathbb{C}^2 \rightarrow \mathfrak{h} \rightarrow \mathfrak{sl}(2, 2) \) with \( \mathbb{C}^2 \) being central in \( \mathfrak{h} \) \cite{IK}. The Lie superalgebra \( \mathfrak{h} \) is a two-fold central extension of \( \mathfrak{sl}(2, 2) \). Its finite-dimensional representation theory plays a special rôle in the integrability structure of the anti-de Sitter/conformal field theory correspondence; see for example \cite{Be, MM}. Not surprisingly, the quantum supergroup \( U_q(\mathfrak{sl}(2, 2)) \) admits a two-fold central extension \( U_q(\mathfrak{h}) \) \cite{Y2}, by forgetting the degree 4 oscillator relations in \( \mathfrak{Y}^2 \).

We propose in Definition 6.1 a two-parameter quantum supergroup \( U_{r,s} \) associated to \( \mathfrak{h} \). Namely we construct a surjective morphism of Hopf superalgebras \( U_{r,s} \rightarrow U_{r,s}(\mathfrak{sl}(2, 2)) \) whose kernel is generated by two elements \( P, Q \). The precise formula of these generators \( P, Q \) is introduced in order to specialize \( U_{r,s}(\mathfrak{sl}(2, 2)) \) to \( U_{r,s}(\mathfrak{g}) \) by modifying the vector representation of \( U_{r,s}(\mathfrak{sl}(2, 2)) \) on \( \mathbb{C}^2| \); the additional parameter \( x \in \mathbb{C}^\times \) arises from the dynamical action of \( P \).

This paper is organized as follows. \( \mathfrak{Y}^2 \) provides the RTT realization of \( U_{r,s}(\mathfrak{g}) \). In \( \mathfrak{Y}^2 \) an orthogonal property of the quantum double is proved. The universal \( R \)-matrix \( \hat{R} \) is written down in \( \mathfrak{Y}^2 \) and a category \( \mathfrak{O} \) of representations is introduced in order to specialize \( \hat{R} \). \( \mathfrak{Y}^2 \) presents another system of generators for \( U_{r,s}(\mathfrak{g}) \), the Drinfeld–Jimbo realization, from which we deduce easily a two-fold quasi-central extension \( U \) in \( \mathfrak{Y}^3 \) whose representation theory is initiated. The final section \( \mathfrak{Y}^7 \) is left to further discussions.

2. Yang–Baxter equation and RTT realization

In this section we provide a solution of the quantum Yang–Baxter equation and use it to define \( U_{r,s}(\mathfrak{gl}(M, N)) \) following Faddeev–Reshetikhin–Takhtajan \cite{FRT}.

Let \( V = \mathbb{C}^{M|N} \) be the vector superspace with basis \( (v_i)_{1 \leq i \leq M+N} \) and parity: \( |v_i| = |i| = \bar{0} \) if \( i \leq M \) and \( |v_i| = |i| = \bar{1} \) if \( i > M \). Since \( r + s \neq 0 \), the tensor square \( V \otimes V \) is easily seen to be a direct sum of the following two sub-vector-superspaces:

\[
S^2 V := \text{Vect}(v_i \otimes v_j \otimes v_k + (-1)^{|j||k|} v_k \otimes v_j \mid i \leq M, j < k),
\]

\[
\wedge^2 V := \text{Vect}(v_i \otimes v_j \otimes v_k - (-1)^{|j||k|} v_k \otimes v_j \mid i > M, j < k).
\]

Let \( P, Q \) be the corresponding projections. Define the two-parameter Perk–Schultz matrices

\[
(2.1) \quad \hat{R} := r P - s Q \in \text{End}(V \otimes V), \quad R := c_{V,V} \hat{R} \in \text{End}(V \otimes V)
\]

where \( c_{V,V} \in \text{End}(V \otimes V) : v_i \otimes v_j \rightarrow (-1)^{|i||j|} v_j \otimes v_i \) is the graded permutation. Let \( E_{ij} \in \text{End} V \) be \( v_k \mapsto \delta_{jk} v_i \). The precise formula of \( R \) is

\[
(2.2) \quad R = (r \sum_{i \leq M} + s \sum_{i > M}) E_{ii} \otimes E_{ii} + \left( \sum_{i > j} + s \sum_{i < j} \right) E_{ii} \otimes E_{jj} + (r - s) \sum_{i < j} (-1)^{|i|} E_{ji} \otimes E_{ij}.
\]

Let \( A, B, C \) be superalgebras and let \( t = \sum_i a_i \otimes b_i \in A \otimes B \). Define the tensors \( t_{12} = t \otimes 1, \ t_{23} = 1 \otimes t, \ t_{13} = \sum_i a_i \otimes 1 \otimes b_i \) with \( 1 \in C \) the identity element. The following quantum Yang–Baxter equation is a consequence of the non-graded case \( N = 0 \) in \cite{BW2}.
Lemma 2.1. \( R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \in \text{EndV}^{\otimes 3} \).

Proof. By Equation (2.1) this is equivalent to the braid relation for \( \hat{R} \):

\[
(*) \quad \hat{R}_{12}\hat{R}_{23}\hat{R}_{12} = \hat{R}_{23}\hat{R}_{12}\hat{R}_{23} \in (\text{EndV})^{\otimes 3}.
\]

Claim 1. \( (*) \) is true if \( N = 0 \) or \( M = 0 \).

When \( N = 0 \), the proof of [BW2 Proposition 5.5] indicates that \( P - rs^{-1}Q \) satisfies \( (*) \); so does \( \hat{R} = r(P - rs^{-1}Q)^{-1} \). The case of \( M = 0 \) comes from that of \( N = 0 \) with parameters \( (s, r) \) instead of \( (r, s) \) owing to Equation (2.2); see also Remark 2.3.

Claim 2. \( (*) \) is true when applied to the vector \( v_a \otimes v_b \otimes v_c \) with \( \sharp \{a, b, c\} = 3 \).

Let \( S \) be the matrix \( \hat{R} \) associated to \( C^{M+N|0} \), so that \( S \) satisfies \( (*) \). Let \( S_{ij}^{ij'} \) be the coefficient of \( v_i \otimes v_j \) in \( S(v_i \otimes v_j) \). Then \( S_{ij}^{ij'} \neq 0 \) only if \( \{i, j\} = \{i', j'\} \). When \( i \neq j \),

\[
\hat{R}(v_i \otimes v_j) = S_{ij}^{ij'} v_i \otimes v_j + (-1)^{i+j+1} S_{ji}^{ij'} v_j \otimes v_i.
\]

Now assume \( 1 \leq a, b, c \leq M + N \) two-by-two distinct. Write

\[
S_{12}S_{23}S_{12} = \sum_{a',b',c'} C_{a'b'c'}^{abc} v_{a'} \otimes v_{b'} \otimes v_{c'} = S_{23}S_{12}S_{23}(v_a \otimes v_b \otimes v_c).
\]

Then \( C_{a'b'c'}^{abc} \neq 0 \) only if \( \{a, b, c\} = \{a', b', c'\} \). Furthermore, one can find \( \varepsilon_{a'b'c'}^{abc} = \pm \) depending only on the permutation \( (a', b', c') \) of \( (a, b, c) \) such that

\[
\hat{R}_{12}\hat{R}_{23}\hat{R}_{12}(v_a \otimes v_b \otimes v_c) = \sum_{a',b',c'} \varepsilon_{a'b'c'}^{abc} C_{a'b'c'}^{abc} v_{a'} \otimes v_{b'} \otimes v_{c'} = \hat{R}_{23}\hat{R}_{12}\hat{R}_{23}(v_a \otimes v_b \otimes v_c).
\]

The same arguments can be used to verify the case where: \( (a, b, c) \) is a permutation of \( (i, i, j) \) with \( i \leq M \). We are led to three remaining cases.

Claim 3. For \( M = N = 1 \), \( (*) \) applied to \( v_1 \otimes v_2 \otimes v_2, v_2 \otimes v_1 \otimes v_2, v_2 \otimes v_2 \otimes v_1 \) is true.

Consider the middle vector \( v_2 \otimes v_1 \otimes v_2 =: u \) for example:

\[
\hat{R}_{12}\hat{R}_{23}\hat{R}_{12}(u) = \hat{R}_{12}\hat{R}_{23}(v_1 \otimes v_2 \otimes v_2) = -s\hat{R}_{12}(v_1 \otimes v_2 \otimes v_2) = -s(r-s)v_1 \otimes v_2 \otimes v_2 - rs^2v_2 \otimes v_1 \otimes v_2 = \hat{R}_{23}((r-s)v_1 \otimes v_2 \otimes v_2 - rs^2v_2 \otimes v_2 \otimes v_1) = \hat{R}_{23}\hat{R}_{12}(u)。
\]

The other two vectors can be checked in the same way. \( \square \)

We are ready to associate a two-parameter quantum supergroup to \( \mathfrak{gl}(M, N) \).

Definition 2.2. \( \mathcal{U} := U_{r,s}(\mathfrak{gl}(M, N)) \) is the superalgebra generated by the coefficients of matrices \( T = \sum_{i \leq j} t_{ij} \otimes E_{ij}, S = \sum_{i \leq j} s_{ij} \otimes E_{ij} \in \mathcal{U} \otimes \text{EndV} \) of even parity with relations

\[
R_{23}T_{12}T_{13} = T_{13}T_{12}R_{23}, \quad R_{23}S_{12}S_{13} = S_{13}S_{12}R_{23}, \quad R_{23}T_{12}S_{13} = S_{13}T_{12}R_{23},
\]

and with requirement that the \( s_{ii}, t_{ii} \) be invertible for \( 1 \leq i \leq M + N \).
Remark 2.3. Let $R' := c_{V,V} R^{-1} c_{V,V}$. The above three equations are equivalent to
\[ R'_{23} T_{12} T_{13} = T_{13} T_{12} R'_{23}, \quad R'_{23} S_{12} S_{13} = S_{13} S_{12} R'_{23}, \quad R'_{23} S_{12} T_{13} = T_{13} S_{12} R'_{23}. \]
Let us write $R'^{M|N}_{r,s}, U_{r,s}$ when the dependence on $r, s, M, N$ is relevant. $r s R'^{0|0}_{r,s}$ is exactly the $R$-matrix used in \[ \text{Definition 3.1} \] to define the two-parameter quantum group $U_{r,s}(gl_n)$.
Since $R'^{0|0}_{r,s} = R^{1|0}_{r,s}$, there are algebra morphisms from $U_{r,s}(gl_M)$ and $U_{s,r}(gl_N)$ in \[ \text{Definition 3.1} \] to our $U_{r,s},$ the former being $l^+_{ij} \mapsto s_{ij}, l^-_{ji} \mapsto t_{ji}$ and the latter $l^+_{ij} \mapsto s_{M+i,M+j}, l^-_{ji} \mapsto t_{M+j,M+i}$. (We borrowed the notations $l^+_{ij}, l^-_{ji}$ from that paper.)

$\mathcal{U}$ is a Hopf superalgebra with coproduct $\Delta$ and counit $\varepsilon$:
\[ \Delta(s_{ij}) = \sum_k s_{ik} \otimes s_{kj}, \quad \Delta(t_{ji}) = \sum_k t_{kj} \otimes t_{ki}, \quad \varepsilon(s_{ij}) = \varepsilon(t_{ji}) = \delta_{ij}. \]
The antipode $S : \mathcal{U} \to \mathcal{U}$ is an anti-automorphism of superalgebra defined by equations $(S \otimes \text{Id})(S) = S^{-1}, (S \otimes \text{Id})(T) = T^{-1}$ in $\mathcal{U} \otimes \text{End} V$. Let $\mathcal{U}^+$ (resp. $\mathcal{U}^-$) be the subalgebra of $\mathcal{U}$ generated by the $s_{ij}, s_{kk}^{-1}$ (resp. the $t_{ji}, t_{kk}^{-1}$) for $i \leq j$; these are sub-Hopf-superalgebras.

**Proposition 2.4.** There exists a unique Hopf pairing $\varphi : \mathcal{U}^+ \otimes \mathcal{U}^- \to \mathbb{C}$ such that
\[ \sum_{ijkl} \varphi(s_{ij}, t_{kl}) E_{kl} \otimes E_{ij} = R \in (\text{End} V)^{\otimes 2}. \]
The associated quantum double $D_\varphi = \mathcal{U}^+ \otimes \mathcal{U}^-$ is isomorphic to $\mathcal{U}$ as Hopf superalgebras via the multiplication map $\mathcal{U}^+ \otimes \mathcal{U}^- \to \mathcal{U}, x \otimes y \mapsto xy$. \thmend

**Remark 2.5.** $\varphi$ being a Hopf pairing means that: for $a, a_1, a_2 \in \mathcal{U}^+$ and $b, b_1, b_2 \in \mathcal{U}^-$
\[ \varphi(a, b_1 b_2) = \varphi_2(\Delta(a), b_1 \otimes b_2), \quad \varphi(a_1 a_2, b) = (-1)^{|a_1||a_2|} \varphi_2(a_2 \otimes a_1, \Delta(b)), \]
here $\varphi_2(a_1 \otimes a_2, b_1 \otimes b_2) = (-1)^{|a_2||b_1|} \varphi(a_1, b_1) \varphi(a_2, b_2)$. The quantum double means
\[ ba = (-1)^{|a_{(1)}||b| + |b_{(2)}| + |b_{(3)}|} a_{(2)} b_{(2)} \varphi(a_{(1)}, S(b_{(1)})) a_{(3)} b_{(3)} \varphi(a_{(3)}, b_{(3)}). \]
We refer to \[ \text{[21]} \] for a sketch of proof of the above proposition.

**Proposition 2.6.** The assignment $s_{ij} \mapsto \varepsilon_{ji} t_{ji}, \quad t_{ji} \mapsto \varepsilon_{ij} s_{ij}$ extends uniquely to an isomorphism of Hopf superalgebras $U_{r,s} \to U_{r,s}^{\text{cop}} \rightarrow U_{s^{-1}, r^{-1}}^{\text{cop}}$. Here $\varepsilon_{ij} := (-1)^{|i|+|i||j|}$.

**Proof.** Let $\tau : \text{End} V \to \text{End} V$ be the super transposition $E_{ij} \mapsto \varepsilon_{ij} E_{ji}$. Then
\[ \tau^{\otimes 2}(R^{-1}_{r,s}) = r^{-1} s^{-1} c_{V,V} R^{-1}_{s^{-1}, r^{-1}} c_{V,V}. \]
The rest is clear from Definition \[ \text{[22]} \] as in the one-parameter case \[ \text{[21]} \]. Here the co-opposite $A^{\text{cop}}$ of a Hopf superalgebra $(A, \Delta, \varepsilon, S)$ is by definition $(A, c_{A,A} \Delta, \varepsilon, S^{-1})$. \thmend

**Lemma 2.7.** There is a vector representation $\rho$ of $\mathcal{U}$ on $V$ defined by
\[ (\rho \otimes 1)(S) = (\tau \otimes 1)(R), \quad (\rho \otimes 1)(T) = rs(\tau \otimes 1)(c_{V,V} R^{-1} c_{V,V}). \]
**Proof.** Straightforward using Lemma \[ \text{[21]} \] and Definition \[ \text{[22]} \].
Corollary 2.8. Let $1 \leq i, j, k \leq M + N$ be such that $j \leq k$. Then
\[
s_{ii}s_{jk} = \varphi(s_{ii}, t_{jj})\varphi(s_{ii}, t_{kk})^{-1}s_{jk}s_{ii}, \quad t_{ii}s_{jk} = \varphi(s_{jj}, t_{ii})^{-1}\varphi(s_{kk}, t_{ii})s_{jk}t_{ii},
\]
\[
t_{ii}t_{kj} = \varphi(s_{jj}, t_{ii})\varphi(s_{kk}, t_{ii})^{-1}t_{kj}t_{ii}, \quad s_{ii}t_{kj} = \varphi(s_{ii}, t_{jj})^{-1}\varphi(s_{ii}, t_{kk})t_{kj}s_{ii}.
\]

Proof. For the second equation, by Proposition 2.4 and Remark 2.5
\[
t_{ii}s_{jk} = \varphi(s_{jj}, s(t_{ii}))s_{jk}t_{ii}\varphi(s_{kk}, t_{ii}) = \varphi(s_{jj}, t_{ii})^{-1}\varphi(s_{kk}, t_{ii})s_{jk}t_{ii}.
\]
Here we have used the three-fold coproduct formula of $t_{ii}, s_{jk}$, and the fact that $\varphi(s_{ab}, t_{ii}) = 0$ if $a < b$. The fourth equation can be proved in the same way.

For the first equation, by comparing the coefficients of $v_i \otimes v_j$ in the identical vectors $R_{23}S_{12}S_{13}(v_i \otimes v_k) = S_{13}S_{12}R_{23}(v_i \otimes v_k) \in \mathcal{U} \otimes V^\otimes 2$ we obtain (let $s_{pq} = 0$ if $p > q$)
\[
x_{ii}x_{kk} + y_{s_ji}x_{ik} = z_{s_jk}s_{ii} + w_{s_ji}s_{ik}
\]
for certain $x, z \in \{1, r, s, r, s\}$ and $y, w \in \{0, r - s, s - r\}$. We prove that $s_{ii}s_{jk} \in \mathbb{C}s_{jk}s_{ii}$. If not, then $j < i < k$, in which case $y = (-1)^{|i|(r - s)} = w$ and $x_{ii}s_{jk} = z_{s_jk}s_{ii}$, a contradiction. Now the first equation is obtained from the representation $\rho$ in Lemma 2.4
\[
\rho(s_{ii}) = \sum_k \varphi(s_{ii}, t_{kk})E_{kk}, \quad \rho(s_{jk}) = \varphi(s_{jk}, t_{kj})e_{kj}E_{jk} \quad \text{for} \ j < k.
\]
The third equation can be proved in the same way. \hfill \Box

Let $P := \bigoplus_{i=1}^{M+N} \mathbb{Z}\epsilon_i$ be the weight lattice. Call a vector $x \in \mathcal{U}$ of weight $\sum_i \lambda_i\epsilon_i \in P$ if
\[
s_{ii}x^{-1}_{ii} = \varphi(s_{ii}, t_{ii})^{-1}(r_s)^{\sum\lambda_i < \lambda_j}x, \quad t_{ii}x^{-1}_{ii} = \varphi(s_{ii}, t_{ii})^{-\lambda_i}(r_s)^{\sum\lambda_i \leq \lambda_j}x.
\]
$s_{ij}$ and $t_{ji}$ are of weight $\pm(\epsilon_i - \epsilon_j)$ respectively. The Hopf superalgebra $\mathcal{U}$ is $P$-graded.

3. Orthogonal property

In this section we prove an orthogonal property of the Hopf pairing $\varphi : \mathcal{U}^+ \times \mathcal{U}^- \to \mathbb{C}$ in Proposition 2.4, it is eventually a consequence of RTT in Definition 2.2.

Let us introduce the following four subalgebras of $\mathcal{U}^\pm$:
\[
\mathcal{U}^0 := \langle s_{ii}^{\pm 1} \mid 1 \leq i \leq M + N \rangle, \quad \mathcal{U}^\geq := \langle a_{ij} := s_{ii}^{-1}s_{ij} \mid 1 \leq i < j \leq M + N \rangle \subseteq \mathcal{U}^+, \\
\mathcal{U}^\leq := \langle b_{ij} := t_{ij}t_{ii}^{-1} \mid 1 \leq i < j \leq M + N \rangle \subseteq \mathcal{U}^-.
\]

$\mathcal{U}^+ = \mathcal{U}^0\mathcal{U}^\geq$ and $\mathcal{U}^- = \mathcal{U}^0\mathcal{U}^\leq$ by Corollary 2.8.

Lemma 3.1. $\varphi(x_+a, x_-b) = \varphi(x_+, x_-)\varphi(a, b)$ for $a \in \mathcal{U}^+, b \in \mathcal{U}^-$ and $x_+ \in \mathcal{U}^{\pm 0}$.

Proof. Observe that: if $x \in \mathcal{U}^+$ and $y \in \mathcal{U}^-$ are of weights $\alpha, \beta$ respectively, then $\varphi(a, b) \neq 0$ only if $\alpha + \beta = 0$. One may assume that $x_+$ is a product of the $s_{ii}^{\pm 1}$. By definition, $\Delta(x_+a) - x_+ \otimes x_+a$ is a sum of $x_i \otimes y_i$ where the weights of the $x_i$ are $\mathbb{Z}_{>0}$-spans of the $\epsilon_k - \epsilon_l$ with $k < l$; which forces $\varphi(x_i, x_-) = 0$. By Remark 2.5
\[
\varphi(x_+a, x_-b) = \varphi_2(\Delta(x_+a), x_- \otimes b) = \varphi_2(x_+ \otimes x_+a, x_- \otimes b) = \varphi(x_+, x_-)\varphi(x_+a, b).
\]
$\Delta(b) - b \otimes 1$ is a sum of $x'_i \otimes y'_i$ with $y'_i$ being of non-zero weight and $\varphi(x_+, y'_i) = 0$. So
\[
\varphi(x_+a, b) = \varphi_2(a \otimes x_+, \Delta(b)) = \varphi_2(a \otimes x_+, b \otimes 1) = \varphi(a, b)\varphi(x_+, 1) = \varphi(a, b).
\]
Here $\varphi(x_+, 1) = 1$ by our assumption on $x_+$. This gives the desired formula. \hfill \Box
Endow the $(1 \leq i < j \leq M + N)$ with the order: $(i, j) < (k, l)$ if $(i < k)$ or $(i = k, j < l)$, in which case write $a_{ij} < a_{kl}$ and $b_{ij} < b_{kl}$. This defines a total order on $X := \{a_{ij} \mid 1 \leq i < j \leq M + N\} \subseteq U^\succ$, $Y := \{b_{ji} \mid 1 \leq i < j \leq M + N\} \subseteq U^\prec$.

**Lemma 3.2.** Fix $1 \leq i < j \leq M + N$ and $p \in \mathbb{Z}_{>0}$. Let $x_1, x_2, \ldots, x_p \in X$ and $y_1, y_2, \ldots, y_p \in Y$ be such that $x_i \succeq a_{ij}$ and $y_i \succeq b_{ji}$ for all $1 \leq l \leq p$.

1. $\varphi(a_{ij}, b_{ji}) = (-1)\left|\begin{array}{l} (s - 1) - r \end{array}\right|$, \hspace{1cm} (1)
2. If $\varphi(a_{ij}, y_1y_2 \cdots y_p) \neq 0$, then $p = 1, y_1 = b_{ji}$.
3. If $\varphi(x_1x_2 \cdots x_p, b_{ji}) \neq 0$, then $p = 1$ and $x_1 = a_{ij}$.

**Proof.** We compute $\varphi(a_{ij}, b_{ji})$ by using Proposition 2.4, Corollary 2.8 and Lemma 3.1

\[
\varphi(a_{ij}, b_{ji}) = \varphi(s_{ij}, s_{ji}) = \varphi(s_{ij}a_{ij}, b_{ji}t_{ii}) = \varphi(s_{ij}a_{ij}, \varphi(s_{ij}, t_{ii})^{-1} \varphi(s_{jj}, t_{ii})t_{ij}b_{ji}) = \varphi(s_{jj}, t_{ii})\varphi(a_{ij}, b_{ji}) = rs\varphi(a_{ij}, b_{ji}) \implies (1).
\]

For (2), notice that the first tensor factors in $\Delta(a_{ij}) - a_{ij} \otimes s_{ii}^{-1}s_{jj}$, being either 1 or $x \in X$ with $x < a_{ij}$, are orthogonal to $y_i \succeq b_{ji}$. So $\varphi(a_{ij}, y_1y_2 \cdots y_p) = (a_{ij}, y_1)\varphi(s_{ij}^{-1}s_{jj}, y_2 \cdots y_p)$. In view of the weight grading, $p = 1$ and $y_1 = b_{ji}$. (3) is proved in the same way. □

**Corollary 3.3.** Fix $1 \leq i < j \leq M + N$. Let $x_1, x_2, \ldots, x_p \in X$ and $y_1, y_2, \ldots, y_q \in Y$ be such that $x_i \succeq a_{ij}$ and $y_i \succeq b_{ji}$ for all $l$. Let $m, n \in \mathbb{Z}_{>0}$. Then

\[
\varphi(x_1x_2 \cdots x_p a_{ij}^m, y_1y_2 \cdots y_q b_{ji}^n) = \varphi_2(x_1x_2 \cdots x_p \otimes a_{ij}^m, y_1y_2 \cdots y_q \otimes b_{ji}^n),
\]

\[
\varphi(x_1^m, b_{ji}^n) = \delta^m_n (m)_{ij}^l \varphi(a_{ij}, b_{ji})^m.
\]

Here $(m)_a = \frac{m!}{a!}, (m)_a = \prod_{k=1}^m (k)_a$ and $\tau_{ij} = (-1)^i+j(r^2)^{-1}\varphi(s_{ii}, t_{ii})\varphi(s_{jj}, t_{jj})$.

**Proof.** Use induction on $\max(m, n)$: the case $m = n = 0$ is trivial. Assume $m > 0$ (the case $n > 0$ can be treated similarly). The left hand side of the first formula becomes

\[
\text{lhs}_1 = (-1)^{\theta_1} \varphi_2(a_{ij} \otimes x_1x_2 \cdots x_p a_{ij}^{m-1}, \Delta(y_1y_2 \cdots y_q b_{ji}^n)), \hspace{1cm} \theta_1 = |a_{ij}|a_{ij}^{m-1}x_1x_2 \cdots xp.
\]

Notice that the first tensor factors of $\Delta(b_{ji}) - b_{ji} \otimes 1 - t_{ij}t_{ij}^{-1} \otimes b_{ji}$ are of the form $xy$ where $x \in U^{-r}$, $y \in Y$ such that $y \succeq b_{ji}$ and $\varphi(1, x) = 1$. By Lemmas 3.1 3.2

\[
\text{lhs}_1 = (-1)^{\theta_1} \varphi_2(a_{ij} \otimes x_1x_2 \cdots x_p a_{ij}^{m-1}, \prod_{k=1}^q (z \otimes y_k) \sum_{l=1}^n (z \otimes b_{ji})t_{ij}^{-l}(b_{ji} \otimes 1)(z \otimes b_{ji})^{n-l}.
\]

Here $z = t_{ij}t_{ij}^{-1}$, $z_k \in U^{-r}$ and $\varphi(1, z_k) = 1$ for $1 \leq k \leq q$. By Corollary 2.8

\[
b_{ji}z = zb_{ji}\varphi(s_{ii}, t_{ij})^{-1}\varphi(s_{jj}, t_{jj})^{-1}\varphi(s_{ii}, t_{ii})\varphi(s_{jj}, t_{jj}) = zb_{ji}t_{ij}(-1)^{|b_{ji}|}.
\]

Thus $(z \otimes b_{ji})t_{ij}^{-l}(b_{ji} \otimes 1)(z \otimes b_{ji})^{n-l} = (-1)^{|b_{ji}|}t_{ij}^{-l}z^{n-l}b_{ji} \otimes b_{ji}^{n-l}$ and

\[
\text{lhs}_1 = (-1)^{\theta_1+\theta_2} (n)_{ij} \varphi_2(a_{ij} \otimes x_1x_2 \cdots x_p a_{ij}^{m-1}, z_1z_2 \cdots z_{n-l}b_{ji} \otimes y_1y_2 \cdots y_q b_{ji}^{n-1})
\]

\[
= (-1)^{\theta_1+\theta_2} (n)_{ij} \varphi_2(a_{ij} \otimes x_1x_2 \cdots x_p a_{ij}^{m-1}, b_{ji} \otimes y_1y_2 \cdots y_q b_{ji}^{n-1})
\]

\[
= (n)_{ij} \varphi_2(x_1x_2 \cdots x_p a_{ij}^{m-1} \otimes a_{ij}, y_1y_2 \cdots y_q b_{ji}^{n-1} \otimes b_{ji}).
\]

Here $\theta_2 = |b_{ji}|b_{ji}^{n-1}y_1y_2 \cdots y_q$. In the last identity $\varphi_2(a \otimes b, c \otimes d) = \varphi_2(b \otimes a, d \otimes c) \times (-1)^{|a||b|+|c||d|}$ as $\varphi$ respects the parity. The rest is clear from the induction hypothesis. □
Let $\Gamma$ be the set of functions $f : X \to \mathbb{Z}_{\geq 0}$ such that $f(x) \leq 1$ if $|x| = 1$. Such an $f$ induces, by abuse of language, another function $f : Y \to \mathbb{Z}_{\geq 0}$ with $f(b_{ji}) := f(a_{ij})$. Set
\begin{equation}
(3.4) \quad a[f] := \prod_{x \in X} x^{f(x)} \in \mathcal{U}^>, \quad b[f] := \prod_{y \in Y} y^{f(y)} \in \mathcal{U}^<.
\end{equation}
Here $\prod$ means the product with descending order. If $i \leq M < j$, then $\tau_{ij} = -1$ and $\varphi(a_{ij}^m b_{ji}^n) = 0$ for $m > 1$, which is the reason for $f(a_{ij}) \leq 1$.

**Corollary 3.4.** For $f, g \in \Gamma$, $\varphi(a[f], b[g]) \neq 0$ if and only if $f = g$. Moreover, the $a[f]$ and the $b[f]$ form a basis of $\mathcal{U}^>$ and $\mathcal{U}^<$ respectively.

**Proof.** The first statement comes from Corollary [MRS] and Lemma [Y1] notably the $a[f]$ (resp. the $b[f]$) are linearly independent. For the second statement, consider $\mathcal{U}^>$ for example. A slight modification of the arguments in the proof of [MRS] Lemma 2.1 by using $R_{23}S_{12}S_{13} = S_{13}S_{12}R_{23}$ shows that $\mathcal{U}^>$ is spanned by ordered monomials of the $a_{ij}$. It suffices to check that $s_{ij}^2 = 0$ (and so $a_{ij}^2 = 0$) for $i \leq M < j$; this comes from a comparison of coefficients of $v_i \otimes v_i$ in the identical vectors $R_{23}S_{12}S_{13}(v_j \otimes v_j) = S_{13}S_{12}R_{23}(v_j \otimes v_j) \in \mathcal{U} \otimes \mathcal{V}^{\otimes 2}$. □

**Remark 3.5.** For the one-parameter quantum supergroup $U_q(\mathfrak{gl}(M, N))$, there is a Hopf pairing with an orthogonal property [Y1] §10.2 similar to Corollary 3.4. The proof in loc. cit. was a lengthy calculation on coproduct estimation and commuting relations of root vectors defined as proper quantum brackets of Drinfeld–Jimbo generators; see also [R] §II. In our situation, the RTT generators can be viewed as root vectors; their coproduct and commuting relations are given for free in the definition.

4. Universal $R$-matrix

In this section we compute the universal $R$-matrix of $\mathcal{U}_{r,s}$. For this purpose, we first work with a topological version of quantum supergroups and view $r, s$ as formal variables:
\[
r = e^h \in \mathbb{C}[[h, \nu]], \quad s = e^\nu \in \mathbb{C}[[h, \nu]].
\]

**Step 1.** Extend $\mathcal{U}^\pm, \mathcal{U}$ to topological Hopf superalgebras over $\mathbb{C}[[h, \nu]]$ according to Equation (2.3); first add commutative primitive elements $(\epsilon^i_j)^1 \leq j \leq M+N$ of even parity such that $[\epsilon^i_j, x] = \lambda_i x$ for $x \in \mathcal{U}^\pm$, $\mathcal{U}$ of weight $\lambda = \sum_i \lambda_i \epsilon_i \in \mathcal{P}$; then identify
\begin{equation}
(4.5) \quad s_{ii} = e^{(h+\nu)\sum_{j<i} \epsilon^*_j} \times \begin{cases} e^{h_1^*} & (i \leq M), \\
e^{h_2^*} & (i > M), \end{cases} \quad t_{ii} = e^{(h+\nu)\sum_{j<i} \epsilon^*_j} \times \begin{cases} e^{\nu_1^*} & (i \leq M), \\
e^{\nu_2^*} & (i > M). \end{cases}
\end{equation}

Denote by $\mathcal{U}_{h,\nu}^\pm, \mathcal{U}_{h,\nu}$ the resulting topological Hopf superalgebras. They contain
\[
H_i := (h + \nu) \sum_{j<i} \epsilon^*_j + \epsilon^*_i \begin{cases} \nu & (i \leq M), \\
h & (i > M). \end{cases}
\]

Extend $\varphi$ to a Hopf pairing $\overline{\varphi} : \mathcal{U}_{h,\nu}^+ \times \mathcal{U}_{h,\nu}^- \to \mathbb{C}((h, \nu))$ by $\overline{\varphi}(\epsilon^i_j, H_j) = \delta_{ij}$. Observe that $\overline{\varphi}(s_{ii}, t_{jj}) = \varphi(s_{ii}, t_{jj})$, which shows in turn that $\overline{\varphi}$ exists uniquely. As in Proposition 2.4, the multiplication map induces a surjective morphism of topological Hopf superalgebras from the quantum double $\mathcal{D}_\varphi$ to $\mathcal{U}_{h,\nu}$ with kernel generated by the $\epsilon^*_i \otimes 1 - 1 \otimes \epsilon^*_i$. 

\]
Step 2. Let $\mathcal{U}^0$ be the topological subalgebra of $\mathcal{U}^+_{h,p}$ generated by the $\epsilon_i^*$. Then $\mathcal{U}^+_{h,p} = \mathcal{U}^0\mathcal{U}^>$ and $\mathcal{U}^-_{h,p} = \mathcal{U}^0\mathcal{U}^\prec$. Lemma 3.4 holds true for $x_\pm \in \mathcal{U}^0$. Together with Corollaries 3.3 and 3.4 we obtain orthonormal bases of $\mathfrak{g}$ and the universal $R$-matrix of $\mathcal{U}^+_{h,p}$:

\[(4.6) \quad \mathcal{R} = \mathcal{R}_0\mathcal{R}_+, \quad \mathcal{R}_+ := \sum_{f \in \Gamma} (-1)^{|a[f]|} a[f] \otimes b[f] = \prod_{1 \leq i < j \leq M+N} \mathcal{R}_{ij}, \]

\[(4.7) \quad \mathcal{R}_0 := \prod_{i=1}^{M+N} e_i^\otimes H_i = \left( \prod_{i=1}^{M+N} s_i^\otimes e_i^* \right) \times \left( \prod_{j=M+1}^{M+N} r_j^\otimes e_j^* \right) \times \left( \prod_{1 \leq i < k \leq M+N} (r_k^\otimes s_i^*), (r_k^\otimes e_i^*), (r_k^\otimes e_i^*), \right), \]

\[(4.8) \quad \mathcal{R}_{ij} = \begin{cases} \sum_{n=0}^{\infty} \left( a_{ij}^n \otimes b_{ji}^n \right) \text{ if } (i < j \leq M), \\ \sum_{n=0}^{\infty} \left( a_{ij}^n \otimes b_{ji}^n \right) \text{ if } (M < i < j), \\ 1 - a_{ij} \otimes b_{ji} \text{ if } (i \leq M < j). \end{cases} \]

The formula of $\mathcal{R}$ is similar to that for $U_q(\mathfrak{gl}(M,N))$ in [Y1] §10.6 when $r = q = s^{-1}$. We shall evaluate $\mathcal{R}$ in a certain class of representations (defined over $\mathbb{C}$). From now on assume that $r,s \in \mathbb{C}^\times$ and $\frac{r}{s}$ is not a root of unity. We work with $\mathcal{U}_{r,s} = \mathcal{U}$ instead of $\mathcal{U}_{h,p}$.

Step 3. Let $V$ be a $\mathcal{U}$-module and $0 \neq v \in V$. Call $v$ a weight vector of weight $\lambda \in \mathfrak{p}$ if

\[(4.9) \quad s_i v = \varphi(s_{ii}, t_{ii})^\lambda (r_s)^{\sum_{j<i} \lambda_j} v, \quad t_i v = \varphi(s_{ii}, t_{ii})^{-\lambda} (r_s)^{\sum_{i<j} \lambda_j} v. \]

Call $V$ a weight module if it is spanned by weight vectors. Define $\mathcal{Q}$ (resp. $\mathcal{Q}^\dagger$) to be the $\mathbb{Z}$-span (resp. the $\mathbb{Z}_{\geq 0}$-span) of the $\epsilon_i - \epsilon_j$ for $1 \leq i < j \leq M + N$. As in [BW1], $V$ is said to be in category $\mathcal{O}$ if it is a weight module with finite-dimensional weight spaces and (BGG) the weights of $V$ are contained in $\cup_{\lambda \in \mathfrak{p}} (\lambda - \mathcal{Q}^\dagger)$ for some finite subset $F \subset \mathfrak{p}$.

Let $V,W$ be in category $\mathcal{O}$. Firstly $(\mathcal{R}_0)_{V,W} \in \text{End}(V \otimes W)$ is well-defined by

\[v \otimes w \mapsto v \otimes w \times s^\sum_{i=1}^{M} \lambda_i \mu_i r^\sum_{j>M} \lambda_j \mu_j (r_s)^{\sum_{k<i} \lambda_k \mu_k} \]

for $v \in V$ and $w \in W$ being of weight $\lambda = \sum_i \lambda_i \epsilon_i$ and $\mu = \sum_i \mu_i \epsilon_i$ respectively. (In the case $(M,N) = (n,0)$, this is exactly the operator $s \times f_{V,W}$ in [BW1] §4.) Secondly, let $v \in V$ is of weight $\lambda \in \mathfrak{p}$. For $f \in \Gamma$, by Equations (2.3) and (3.4), the vector $a[f] v \in V$ is of weight $\lambda + \sum_{i<j} f(a_{ij})(\epsilon_i - \epsilon_j)$. By (BGG), $a[f] v = 0$ for all but finite $f$. This implies that $(\mathcal{R}_+)_{V,W} \in \text{End}(V \otimes W)$ is also well-defined. Let $\mathcal{R}_{V,W} := (\mathcal{R}_0)_{V,W}(\mathcal{R}_+)_{V,W}$. From the quantum double construction of $\mathcal{U}$, we obtain (compare [BW1] Theorems 4.11, 5.4)

Theorem 4.1. Let $V_1, V_2, V_3$ be in category $\mathcal{O}$. Then

\[(\mathcal{R}_{V_1,V_2})_{12}(\mathcal{R}_{V_1,V_3})_{13}(\mathcal{R}_{V_2,V_3})_{23} = (\mathcal{R}_{V_2,V_3})_{23}(\mathcal{R}_{V_1,V_3})_{13}(\mathcal{R}_{V_1,V_2})_{12} \in \text{End}(V_1 \otimes V_2 \otimes V_3) \]

and $c_{V_1,V_2} \cdot \mathcal{R}_{V_1,V_2} : V_1 \otimes V_2 \rightarrow V_2 \otimes V_1$ is an isomorphism of $\mathcal{U}$-modules.

Example 4.2. Consider the vector representation of $\mathcal{U}$ on $V$ in Lemma 2.7. From the proof of Corollary 2.8 we see that $v_i$ is of weight $\epsilon_i$ and $V$ is in category $\mathcal{O}$. Moreover,

\[\rho(s_{jk}) = E_{jk} (r - s) (-1)^{|k|}, \quad \rho(t_{kj}) = E_{kj} (s - r) (-1)^{|k|} \quad (j < k). \]

This gives $\mathcal{R}_{V,V} = c_{V,V} R_s^{-1}$. Compare [Y1] §10.7.
5. Drinfeld–Jimbo realization

We present the Drinfeld–Jimbo realization of \( U_{r,s}(\mathfrak{gl}(M,N)) \). As in [JL, §4], let us introduce Drinfeld–Jimbo generators: for \( 1 \leq i < M + N \)
\[
E_i := a_{i,i+1} = s_{ii}^{-1}s_{i,i+1}, \quad F_i := b_{i+1,i} = t_{i+1,i}t_{ii}^{-1}, \quad K_i := s_{ii}^{-1}s_{i+1,i+1}, \quad L_i := t_{i+1,i+1}t_{ii}^{-1}
\]
It follows that \( K_i, L_i \) are grouplike and \( E_i, F_i \) are skew-primitive:
\[
\Delta(E_i) = 1 \otimes E_i + E_i \otimes K_i, \quad \Delta(F_j) = L_j \otimes F_j + F_j \otimes 1, \quad \Delta(g) = g \otimes g \text{ for } g = K_i, L_i.
\]

**Lemma 5.1.** \( s_{ab}s_{bc} - s_{bc}s_{ab} = (-1)^{|b|(r - s)}s_{ac}s_{cb} \) for \( 1 \leq a < b < c \leq M + N \).

**Proof.** Straightforward by comparing the coefficients of \( v_b \otimes v_a \) in the two identical vectors
\[
R_{23}S_{12}S_{13}(v_c \otimes v_b) = S_{13}S_{12}R_{23}(v_c \otimes v_b) \in \mathcal{U} \otimes V \otimes V^2.
\]

**Proposition 5.2.** \( \mathcal{U} \) is generated as a superalgebra by the \( E_i, F_i, s_{jj}^{\pm 1}, t_{jj}^{\pm 1} \). Moreover:
\[
\begin{align}
E_i^2 E_{i+1} - (r + s)E_i E_{i+1}E_i + rsE_i E_{i+1}E_i^2 &= 0 \text{ if } (1 \leq i < M + N - 1, i \neq M), \\
E_{i-1} E_i^2 - (r + s)E_{i-1} E_i E_i + rsE_{i-1} E_i E_i^2 &= 0 \text{ if } (1 \leq i < M + N, i \neq M), \\
rs F_i F_{i+1} - (r + s)F_i F_{i+1}F_i + F_i F_{i+1}F_i^2 &= 0 \text{ if } (1 \leq i < M + N - 1, i \neq M), \\
rs F_{i-1} F_i^2 - (r + s)F_{i-1} F_i F_i + F_{i-1} F_i F_i^2 &= 0 \text{ if } (1 \leq i < M + N, i \neq M), \\
E_i E_j = E_j E_i, \quad F_i F_j = F_j F_i, \quad E_M^2 = F_M^2 = 0 \text{ if } (|i - j| > 1),
\end{align}
\]
When \( M, N > 1 \) the following relations also hold true:
\[
\begin{align}
E_{M-1} E_M E_{M+1} E_M + rsE_{M+1} E_M E_{M-1} E_M + E_M E_{M-1} E_M E_{M+1} + rsE_M E_{M+1} E_M E_{M-1} - (r + s)E_M E_{M+1} E_M E_{M-1} &= 0, \\
r s F_M F_{M+1} F_M F_M + F_M F_{M+1} F_M F_M + rsF_M F_{M-1} F_M F_M + F_M F_{M-1} F_M F_M &= 0.
\end{align}
\]

**Proof.** (Sketch) This is based on lengthy but straightforward calculations as done in the non-graded case in [JL, §4]. Equations (5.11)–(5.16) agree with [JL, Equations (4.17), (4.19)–(4.21), (4.24)–(4.26)] by Remark 2.3. We only consider Equations (5.10) and (5.11).

From Remark 2.3, Lemma 3.1 and Equation (5.10) we obtain
\[
F_j E_i = (-1)^{|E_i||F_j|}(L_j \varphi(E_i, F_j) + E_i F_j + \varphi(E_i, S(F_j)K_i)) = (-1)^{|E_i||F_j|} \varphi(E_i, F_j)(L_j - K_i) + (-1)^{|E_i||F_j|}E_i F_j,
\]
\[
[E_i, F_j] = \varphi(E_i, F_j)(K_i - L_j) = \delta_{ij}(-1)^{|i|}(s^{-1} - r^{-1})(K_i - L_i). \quad \text{Equation (5.16).}
\]

To prove Equation (5.17), we can assume \( M = N = 2 \). By comparing coefficients of \( v_1 \otimes v_2 \) in the identical vectors \( R_{23}S_{12}S_{13}(v_1 \otimes v_2) = S_{13}S_{12}R_{23}(v_1 \otimes v_2) \) we obtain \( s_{23}s_{14} + rs \times s_{14}s_{23} = 0 \). After left multiplication by \( s_{23}^{-1}s_{14}^{-1} \) this relation becomes
\[
E_{23} s_{14}^{-1} + s_{14}^{-1} E_2 = 0.
\]
Let us express $s_{11}^{-1}s_{13}, s_{11}^{-1}s_{14}$ in terms of $E_1, E_2, E_3$ by using Lemma 5.1:

$$s_{11}^{-1}s_{13} = (rs)s_{11}^{-1}s_{22}s_{22} = \frac{rs}{r-s}s_{11}^{-1}s_{22}^{-1}(s_{12}s_{23} - s_{23}s_{12}) = \frac{rs}{r-s}(s^{-1}E_1E_2 - E_2E_1),$$

$$s_{11}^{-1}s_{14} = \frac{rs}{s-r}s_{11}^{-1}s_{33}^{-1}(s_{13}s_{34} - s_{34}s_{13}) = \frac{rs}{s-r}(r^{-1}s_{11}s_{13}E_3 - E_3s_{11}s_{13})$$

$$= \frac{-(rs)^2}{(r-s)^2}(r^{-1}(s^{-1}E_1E_2 - E_2E_1)E_3 - E_3(s^{-1}E_1E_2 - E_2E_1)).$$

Equation (5.17) comes from Equation (5.19) and $E_1E_3 = E_3E_1, E_2^2 = 0$. □

Let $U_{r,s}(gl(M,N))$ be the subalgebra of $U$ generated by the $E_i, F_i, K_i^{\pm 1}, L_i^{\pm 1}$ with $1 \leq i < M + N$; this is a sub-Hopf-superalgebra. We believe that Proposition 5.2 and Corollary 2.8 exhaust the defining relations of $U_{r,s}(gl(M,N))$ as in [DF, Theorem 2.1]. A proof (Proposition 6.3) will be given for $M = N = 2$, in which case we have: for $1 \leq i, j \leq 3$

$$K_jE_iK_j^{-1} = c_{ij}E_i, \quad K_jF_iK_j^{-1} = c_{ij}^{-1}F_i, \quad L_jF_iL_j^{-1} = d_{ij}F_i, \quad L_jE_iL_j^{-1} = d_{ij}^{-1}E_i,$$

where $(c_{ij}) = \begin{pmatrix} r^{-1}s & s^{-1} & 1 \\ r & 1 & r^{-1} \\ 1 & s & rs^{-1} \end{pmatrix}$ and $(d_{ij}) = \begin{pmatrix} r^{-1}s & r & 1 \\ s^{-1} & 1 & s \\ 1 & r^{-1} & rs^{-1} \end{pmatrix}$ are three-by-three matrices with non-zero entries; furthermore $\varphi(K_i, L_j) = c_{ij}^{-1}$.

6. TWO-FOLD QUASI-CENTRAL EXTENSION

In this section we show that Relations (5.17)–(5.18) can be dropped, leading to a two-fold quasi-central extension of $U_{r,s}(gl(2,2)) =: U^{sl}$. Namely, we will construct a sextuple $(U, \theta, \tau, f, P, Q)$ where $U$ is a Hopf superalgebra, $f : U \rightarrow U^{sl}$ is a morphism of Hopf superalgebras, $\theta, \tau \in Aut(U)$ are superalgebra automorphisms and $P, Q \in U$ such that:

(QC1) $f$ is surjective with kernel generated as an ideal by $P, Q$;
(QC2) $Px = \theta(x)P$ and $Qx = \tau(x)Q$ for $x \in U$.

Definition 6.1. $U = U_{r,s}$ is the superalgebra generated by $e_i, f_i, k_i^{\pm 1}, l_i^{\pm 1}$ for $1 \leq i \leq 3$ with $e_2, f_2$ being odd and other generators even, and subject to (view $E_i, F_i, K_i, L_i$ as $e_i, f_i, k_i, l_i$)

Equations (5.20) and (5.11)–(5.16) where $M = N = 2$ and $|1| = |2| = 0, |3| = 1$.

$U$ is a Hopf superalgebra with the same coproduct formula as Equation (5.10). Let $U^+$ (resp. $U^-$) be the subalgebra of $U$ generated by the $e_i, k_i^{\pm 1}$ (resp. the $f_i, l_i^{\pm 1}$); these are sub-Hopf-superalgebras. By definition $e_i, f_i, k_i, l_i \mapsto E_i, F_i, K_i, L_i$ extends to a surjective morphism of Hopf superalgebras $f : U \rightarrow U^{sl}$. The following lemma is motivated by Propositions 2.2 and 2.6 whose proof is omitted.

Lemma 6.2. There exists a unique Hopf pairing $\widehat{\varphi} : U^+ \times U^- \rightarrow \mathbb{C}$ such that

$$\widehat{\varphi}(e_i, f_j) = \delta_{ij}(-1)^{|i|}(s^{-1} - r^{-1}), \quad \widehat{\varphi}(k_i, l_j) = c_{ij}^{-1}.$$

As Hopf superalgebras the quantum double is isomorphic to $U$ via multiplication. The assignment $e_i \mapsto (-1)^{|e_i|}r^{-1}s^{-1}f_i, \quad f_i \mapsto e_i, \quad k_i \mapsto l_i, \quad l_i \mapsto k_i$ extends uniquely to a Hopf superalgebra isomorphism $U_{r,s} \rightarrow U_{r,s}^{cop}$. □
The automorphisms $\theta, \tau$ are defined by: $\theta(w) = \tau(w) = w$ for $w \in \{e_2, f_2, k_2, l_2\}$, 
$\theta(x) = x, \theta(y) = rsy, \theta(z) = (rs)^{-1}z$ for $x \in \{f_1, f_3\}, y \in \{e_3, k_3, l_3\}, z \in \{e_1, k_1, l_1\}$, 
$\tau(a) = a, \tau(b) = rsb, \tau(c) = (rs)^{-1}c$ for $a \in \{e_1, e_3\}, b \in \{f_1, k_1, l_1\}, c \in \{f_3, k_3, l_3\}$.

$P, Q$ are the left hand side of Equations (5.17)–(5.18) respectively:

\begin{align*}
(6.21) \quad P &:= e_1e_2e_3e_2 + rs e_3 e_2 e_1 e_2 + e_2 e_1 e_2 e_3 + rs e_2 e_3 e_2 e_1 - (r + s) e_2 e_1 e_3 e_2, \\
(6.22) \quad Q &:= rs f_1 f_2 f_3 f_2 + f_3 f_2 f_1 f_2 + rs f_2 f_1 f_3 f_2 + f_2 f_3 f_2 f_1 - (r + s) f_2 f_1 f_3 f_2.
\end{align*}

**Proposition 6.3.** (QC1)–(QC2) hold for $(\mathbf{U}, \theta, \tau, f, P, Q)$.

**Proof.** Motivated by the proof of Proposition 5.2 we introduce $e_{i,i+1} := e_i$ for $i = 1, 2, 3$ and 
$e_{13} := \frac{rs(s^{-1}e_1e_2 - e_2e_1)}{r - s}, \quad e_{24} := \frac{rs(r^{-1}e_2e_3 - e_3e_2)}{s - r}, \quad e_{14} := \frac{rs(r^{-1}e_3e_1 - e_1e_3)}{s - r}$.

Then $f(e_{ij}) = s_{ii}^{-1}s_{ij} = a_{ij}$ (see [3]) and 
$P = \frac{-rs}{(r - s)^2}(e_{14}e_2 + e_2e_{14}), \quad \Delta(P) = 1 \otimes P + P \otimes k_1k_2k_3$.

It follows from $e_2^2 = 0$ that $Pe_2 = e_2P$. Next $Pe_1 = r^{-1}s^{-1}e_1P$ follows from 
$e_1^2e_2 - (r + s)e_1e_1e_2 + rs e_2 e_1^2 = 0, \quad e_1 e_2 e_1 e_2 = rs e_2 e_1 e_2 e_1, \quad e_1 e_3 = e_3 e_1$.

Similarly we obtain $Pe_3 = rs e_3 P$. The relations between $P$ and the $k_i, l_i$ come from Equation (5.20). From $\hat{\Delta}(P, f_i) = 0$, the coproduct formula $\Delta(P)$ and the quantum double construction in Remark 2.5, it follows that $P, f_i$ commute. Thus $Px = \theta(x)P$ for $x \in \mathbf{U}$. This implies $Qx = \tau(x)Q$ by using Lemma 6.2 (QC2) is proved.

We can show directly that $\mathbf{U}^+$ is spanned by ordered products of $k_i^{\pm 1} > e_{34} > e_{24} > e_{23} > P > e_{14} > e_{13} > e_{12}$; for example $e_{12}e_{13} = re_{13}e_{12}$ and $e_{12}e_{14} = re_{14}e_{12}$. By Corollary 3.4 the kernel of the restriction $f|_{\mathbf{U}^+}$ is the ideal generated by $P$. By Lemma 6.2 the same holds true for $(f|_{\mathbf{U}^-}, Q)$ and $(f, \mathbf{U})$. This proves (QC1).

**Corollary 6.4.** Let $(\pi, V)$ be a finite-dimensional simple representation of $\mathbf{U}$ on a vector superspace $V$. If $\pi(P) \neq 0$ or $\pi(Q) \neq 0$, then $(r s)^{\dim(V)} = 1$.

**Proof.** Assume $\pi(P) \neq 0$ (see Lemma 6.2 for $Q$). Let $v \in V$ be an eigenvector of $P$ of eigenvalue $x \in \mathbb{C}$. Since $V = \mathbf{U}v$, by (QC2) the non-zero action of $P$ on $V$ is semi-simple with eigenvalues in $x(rs)^2$. This implies $x \neq 0$. For $y \in x(rs)^2$, let $V_y \subseteq V$ be the eigenspace of $P$ of eigenvalue $y$. Then the action $k_3 : V \rightarrow V$ induces an isomorphism $V_y \cong V_{yrs}$. Since $V$ is a direct sum of the $V_y$, we must have $(r s)^{\dim(V)} = 1$.

As a direct consequence, if $rs$ is not a root of unity, then any finite-dimensional simple $\mathbf{U}$-module factorizes through $f : \mathbf{U} \rightarrow \mathbf{U}^{sl}$. In the following we will construct $\mathbf{U}$-modules such that the action of $P$ is non-zero. Let us first recall the vector representation $(\rho, V)$ of $\mathbf{U}^{sl}$ in Lemma 2.2 and Example 5.2 ($M = N = 2$):

\begin{align*}
\rho(K_1) &= sE_{11} + rE_{22} + E_{33} + E_{44}, \quad \rho(K_2) = E_{11} + sE_{22} + sE_{33} + E_{44}, \\
\rho(K_3) &= E_{11} + E_{22} + rE_{33} + sE_{44}, \quad \rho(L_i) = \rho(K_i)|_{(r,s) \rightarrow (s,r)}, \\
\rho(E_1) &= (1 - r^{-1}s)E_{12}, \quad \rho(F_1) = (1 - rs^{-1})E_{21}, \quad \rho(E_2) = (r^{-1}s - 1)E_{23},
\end{align*}
\[ \rho(F_2) = (rs^{-1} - 1)E_{32}, \quad \rho(E_3) = (1 - rs^{-1})E_{34}, \quad \rho(F_3) = (1 - r^{-1}s)E_{43}. \]

Motivated by the construction in [^2] §II, let us define the vector superspace

\[ V^\infty := V \otimes \mathbb{C}[X, X^{-1}], \quad |X| = \overline{0}, \quad y_n := v_y \otimes X^n \text{ for } n \in \mathbb{Z}, y = 1, 2, 3, 4. \]

Note that the \( y_n \) form a basis of \( V^\infty \). As usual, let \( E_{y_m} \in \text{End}(V^\infty) \) be the operator which sends \( z_m \) to \( y_n \) and kills other basis vectors.

**Proposition 6.5.** Let \( x \in \mathbb{C}^X \). There is a representation \((\pi_x, V^\infty)\) of \( U \):

\[
\pi_x(k_1) = \sum_n (sE_{1n+1} + rE_{2n+2} + E_{3n+3} + E_{4n+4}), \\
\pi_x(k_3) = \sum_n (E_{1n-1} + E_{2n-2} + rE_{3n-3} + sE_{4n-4}), \\
\pi_x(k_2) = \sum_n (E_{1n+1} + E_{2n+2} + sE_{3n+3} + E_{4n+4}), \\
\pi_x(e_1) = (1 - r^{-1}s) \sum_n E_{1n+2}, \quad \pi_x(f_1) = (1 - rs^{-1}) \sum_n E_{2n+1}, \\
\pi_x(e_3) = (1 - rs^{-1}) \sum_n E_{3n-3}, \quad \pi_x(f_3) = (1 - r^{-1}s) \sum_n E_{4n-4}, \\
\pi_x(e_2) = (r^{-1}s - 1) \sum_n (E_{2n+3} + x(rs)^{-n}E_{4n+1}), \quad \pi_x(f_2) = (rs^{-1} - 1) \sum_n E_{3n+2}. 
\]

In particular, \( P_{y_n} = x(rs)^{-n}(1 - r^{-1}s)(1 - rs^{-1})(r^{-1}s - 1)^2y_n \) and \( Qy_n = 0 \). If \( rs \) is not a root of unity, then \( \pi_x \) is a simple representation.

**Proof.** \( \pi_x \) comes from the vector representation \((\rho, V)\) of \( U_{sl^2} \); essentially it suffices to check that \( \pi_x \) respects the relations between \( e_2 \) and \( k_1, k_3 \), which is straightforward.

Assume that \( rs \) is not a root of unity. Let \( W \) be a non-zero \( \rho \)-directed \( V \)-module of \( V^\infty \). Then \( W \) contains an eigenvector \( v \otimes X^n \) of \( P \). Applying the \( f_l \) leads to \( 4_m \in W \). Acting \((e_3, e_2, e_1)\) on \( 4_m \) gives \( 3_{m-1}, 2_{m-1}, 1_m \in W \). By applying \( k_1^{-1} \), we see that \( W \) contains all the \( y_n \) and so \( W = V^\infty \). This proves that \( V^\infty \) is a simple \( U \)-module. □

For \( \ell \) a positive integer, let \( V^\ell \) be the quotient of \( V^\infty \) by the relations \( y_n = y_{n+\ell} \).

**Corollary 6.6.** Let \( x \in \mathbb{C}^X, \ell \in \mathbb{Z}_{\geq 0} \) and \( rs \) be a primitive \( \ell \)-th root of unity. \( \pi_x \) factorizes through \( V^\infty \rightarrow V^\ell \) and induces a \( 4\ell \)-dimensional simple representation \((\pi_x^\ell, V^\ell)\) of \( U \).

**Proof.** That \( \pi_x \) factorizes comes directly from \((rs)^{-\ell} = 1 \). Set \( j_{[m]} := \sum_{p=1}^{\ell} (rs)^{mp}j_p \in V^\ell \) for \( 1 \leq j \leq 4, m \in \mathbb{Z} \). Since \( rs \) is a primitive \( \ell \)-th root of unity, the \( j_{[m]} \) with \( 1 \leq m \leq \ell, 1 \leq j \leq 4 \) form a basis of \( V^\ell \), and they are common eigenvectors of the \( k_j \) with eigenvalues distinct two-by-two. Let \( W \) be a non-zero \( \rho \)-directed \( V \)-module of \( V^\ell \). Then \( W \) contains certain \( y_{[n]} \). Applying the \( f_i \) leads to \( 4_{[n]} \in W \). Next,

\[
e_34_{[n]} = (rs - r^2)3_{[n-1]}, \quad e_23_{[n-1]} = (r^{-1}s - 1)2_{[n-1]}, \quad e_12_{[n-1]} = (r^{-1}s - 1 - r^{-2})1_{[n]}, \\
e_21_{[n]} = (r^{-1}s - 1) \sum_{p=1}^{\ell} x(rs)^{-p}(rs)^{mp}4_p = (r^{-1}s - 1)x4_{[n-1]}.
\]
Since $rs^{-1}$ is not a root of unity, $4_{[n-1]} \in W$. The same arguments imply that $j_{[m]} \in W$ for all $1 \leq m \leq \ell, 1 \leq j \leq 4$. This proves $W = V^\ell$ and the simplicity of $V^\ell$.

**Remark 6.7.** Let $(r, s) = (q, q^{-1})$ with $q \in \mathbb{C}^\times$ not being root of unity. $U = U_{q, q^{-1}}$ has a quotient isomorphic to the superalgebra $\tilde{U}^0$ and $(\pi_x, V_x)$ can be identified with the representation $(\rho_x, V_x)$ restricted to $U^0$ in [Y2] §II after renormalization.

In loc. cit. other finite-dimensional simple representations $(\rho, W)$ of $\tilde{U}^0$ were constructed from $\rho_x$ via a fusion procedure. We expect that each of them induces a family $(\rho^\ell, W^\ell)$ of irreducible representations of $U_{r,s}$ where $r = q$ and $s = q^{-1}\exp(\frac{2\pi i}{r})$.

7. Further discussions

In the present paper, we have studied the quasi-triangular structure of $U_{r,s}(\mathfrak{gl}(M, N))$ a Hopf superalgebra extension $U_{r,s}$ of $U_{r,s}(\mathfrak{sl}(2, 2))$. Let us make some further remarks not used in the proof of the main results.

1. The condition (QC2) in [E] seems to define an operator algebra structure on $U$ in the sense of [FV] §3. In this way, one might view $(\pi_x, V^\infty)$ in Proposition 6.5 as a four-dimensional dynamical representation, with $P, Q$ playing the role of $h$ in loc. cit. In [Y2] by considering a one-parameter representation $\rho_x$ of $U_{q, q^{-1}}$ on $\mathbb{C}^2[1]$ (see Remark 6.7), Yamane obtained an $R$-matrix which satisfies the dynamical Yang–Baxter equation in [FV] instead of the quantum Yang–Baxter equation in [2]. It might be useful to look at dynamical quantum groups associated to centrally extended $\mathfrak{sl}(2, 2)$ and their dynamical representations.

2. Following Remark 2.3, set $R(z, w) := zR - wR'$; it satisfies the quantum Yang–Baxter equation with spectral parameters and leads to an RTT realization of the two-parameter quantum affine superalgebra $U_{r,s}(\mathfrak{gl}(M, N))$. In [CWW] $R(1, z)\frac{R(z, s)}{z - s}$ with $(r, s) = (p^{-1}, q)$ and $M = N = 1$ was used to deduce the Drinfeld loop realization of $U_{r,s}(\mathfrak{gl}(1, 1))$.

3. Our computation of the universal $R$-matrix in [2,4] might shed light on the second remark in [O] §7 where the RTT realizations for the quantum queer superalgebra $U_q(\mathfrak{q}_n)$ and for its dual were obtained. The universal $R$-matrix, presumably being in $U_q(\mathfrak{q}_n)\widehat{\otimes} U_q(\mathfrak{q}_n)^*$ rather than in $U_q(\mathfrak{q}_n)\widehat{\otimes} 2$, is still missing to the best of the author’s knowledge.

**Acknowledgments.** The author thanks Niklas Beisert, Giovanni Felder and Reimar Hecht for discussions. This work is supported by the National Center of Competence in Research SwissMAP–The Mathematics of Physics of the Swiss National Science Foundation.

**References**

[Be] N. Beisert, *The analytic Bethe ansatz for a chain with centrally extended su(2|2) symmetry*, J. Stat. Mech. (2007), P01017.

[BW1] G. Benkart and S. Witherspoon, *Two-parameter quantum groups and Drinfel’d doubles*, Algeb. Represent. Theory 7, no. 3 (2004): 261–286.

[BW2] G. Benkart and S. Witherspoon, *Representations of two-parameter quantum groups and Schur–Weyl duality*, Hopf algebras, Lecture Notes in Pure and Appl. Math., vol. 237, Dekker, New York, 2004, 65–92. [arXiv:math/0108038].

[CWW] J. Cai, S. Wang and K. Wu, *Two-parameter quantum affine superalgebra $U_{p,q}(\mathfrak{gl}(1|1))$ and its Drinfel’d realization*, Preprint [arXiv:q-alg/9703030].
[DF] J. Ding and I. Frenkel, *Isomorphism of two realizations of quantum affine algebras*, Commun. Math. Phys. 156, no. 2 (1993): 277–300.

[FRT] L. Faddeev, N. Reshetikhin and L. Takhtajan, *Quantization of Lie groups and Lie algebras*, Leningrad Math. J. 1 (1990): 193–225.

[FV] G. Felder and A. Varchenko, *On representations of the elliptic quantum group $E_{r,s}(\mathfrak{sl}_2)$*, Commun. Math. Phys. 181, no. 3 (1996): 741–761.

[IK] K. Iohara and Y. Koga, *Central extensions of Lie superalgebras*, Comment. Math. Helv. 76, no. 1 (2001): 110–154.

[JL] N. Jing and M. Liu, *$R$-matrix realization of two-parameter quantum group $U_{r,s}(\mathfrak{gl}_n)$*, Commun. Math. Stat. 2, no. 3 (2014): 211–230.

[MM] T. Matsumoto and A. Molev, *Representations of centrally extended Lie superalgebra $\mathfrak{psl}(2|2)$*, J. Math. Phys. 55 (2014), 091704.

[MRS] A. Molev, E. Ragoucy and P. Sorba, *Coideal subalgebras in quantum affine algebras*, Rev. Math. Phys. 15, no. 8 (2003): 789–822.

[O] G. Olshanski, *Quantized universal enveloping superalgebra of type Q and a super-extension of the Hecke algebra*, Lett. Math. Phys. 24, no. 2 (1992): 93–102.

[R] M. Rosso, *An analogue of P.B.W. theorem and the universal $R$-matrix of $U_q\mathfrak{sl}(N + 1)$*, Commun. Math. Phys. 124, no. 2 (1989): 307–318.

[Y1] H. Yamane, *Quantized enveloping algebras associated with simple Lie superalgebras and their universal $R$-matrices*, Publ. RIMS, Kyoto Univ. 30, no. 1 (1994): 15–87.

[Y2] H. Yamane, *A central extension of $U_q\mathfrak{sl}(2|2)^{(1)}$ and $R$-matrices with a new parameter*, J. Math. Phys. 44, no. 11 (2003): 5450–5455.

[Z1] H. Zhang, *RTT realization of quantum affine superalgebras and tensor products*, (to appear) Intern. Math. Res. Not. doi:10.1093/imrn/rnu167. arXiv:1407.7001

[Z2] H. Zhang, *Universal $R$-matrix of quantum affine $\mathfrak{gl}(1,1)$*, Lett. Math. Phys. 105, no. 11 (2015): 1587–1603.

Departement Matematik, ETH Zürich, CH-8092 Zürich, Switzerland.

Institut für Theoretische Physik, ETH Zürich, CH-8093 Zürich, Switzerland.

E-mail address: huafeng.zhang@math.ethz.ch