Introduction to relativistic quantum information

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I discuss the role that relativistic considerations play in quantum information processing. First I describe how the causality requirements limit possible multi-partite measurements. Then the Lorentz transformations of quantum states are introduced, and their implications on physical qubits are described. This is used to describe relativistic effects in communication and entanglement.

To the memory of Asher Peres, teacher and friend

I. INTRODUCTION

Information and physics are closely and fascinatingly intertwined. Their relations become even more interesting when we leave a non-relativistic quantum mechanics for more exiting venues. My notes are planned as a guided tour for the first steps along that road, with open questions and more involved mergers left to the remarks and to the last section.

I start from a brief introduction to causality restrictions on the distributed measurements: the limitations that are imposed by final propagation velocity of the physical interactions. It is followed by the relativistic transformations of the states of massive particles and photons, from which we can deduce what happens to qubits which are realized as the discrete degrees of freedom. Building on this, I discuss the distinguishability of quantum signals, and briefly touch communication channels and the bipartite entanglement.

I do not follow a historical order or give all of the original references. A review [1] is used as the standard reference on quantum information and relativity. The results of the “usual” quantum information are given without any reference: all of them can be found in at least one of the sources [2, 3, 4]. Finally, a word about units: \( \hbar = c = 1 \) are always assumed.

II. CAUSALITY AND DISTRIBUTED MEASUREMENTS

Here I present the causality constraints on quantum measurements. For simplicity, measurements are considered to be point-like interventions. First recall the standard description of the measurement and the induced state transformation. Consider a system in the state \( \rho \) that is subject to measurement that is described by a positive operator-valued measure (POVM) \( \{ E_\mu \} \). The probability of the outcome \( \mu \) is

\[
p_\mu = \text{tr} E_\mu \rho, \tag{1}
\]

while the state transformation is given by some completely positive evolution

\[
\rho \rightarrow \rho'_\mu = \sum_m A_{\mu m} \rho A_{\mu m}^\dagger / p_\mu, \quad \sum_m A_{\mu m} A_{\mu m}^\dagger = E_\mu. \tag{2}
\]

If the outcome is left unknown, the update rule is

\[
\rho \rightarrow \rho = \sum_{\mu m} A_{\mu m} \rho A_{\mu m}^\dagger. \tag{3}
\]

Now consider a bipartite state \( \rho_{AB} \). The operations of Alice and Bob are given by the operators \( A_{\mu m} \) and \( B_{\nu n} \), respectively. It is easy to see that if these operators commute,

\[
[A_{\mu m}, B_{\nu n}] = 0, \tag{4}
\]

then the observation statistics of Bob is independent of Alice’s results and vice versa. Indeed, the probability that Bob gets a result \( \nu \), irrespective of what Alice found, is

\[
p_\nu = \sum_\mu \text{tr} \left( \sum_{m,n} B_{\nu n} A_{\mu m} \rho A_{\mu m}^\dagger B_{\nu n}^\dagger \right). \tag{5}
\]

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Now make use of Eq. (4) to exchange the positions of $A_{\mu n}$ and $B_{\nu n}$, and likewise those of $A_{\mu n}^\dagger$ and $B_{\nu n}^\dagger$, and then we move $A_{\mu n}$ from the first position to the last one in the product of operators in the traced parenthesis. Since the elements of a POVM satisfy $\sum_\mu E_\mu = \mathbb{1}$, Eq. (5) reduces to

$$p_\nu = \text{tr} \left( \sum_n B_{\nu n} \rho B_{\nu n}^\dagger \right),$$

whence all the expressions involving Alice’s operators $A_{\mu n}$ have totally disappeared. The statistics of Bob’s result are not affected at all by what Alice may simultaneously do somewhere else. This proves that Eq. (4) indeed is a sufficient condition for no instantaneous information transfer. In particular, the local operations $A \otimes \mathbb{1}_B$ and $\mathbb{1}_A \otimes B$ are of this form.

Note that any classical communication between distant observers can be considered a kind of long range interaction. The propagation of signals is, of course, bounded by the velocity of light. As a result, there exists a partial time ordering of the various interventions in an experiment, which defines the notions earlier and later. The input parameters of an intervention are deterministic (or possibly stochastic) functions of the parameters of earlier interventions, but not of the stochastic outcomes resulting from later or mutually spacelike interventions.

Even these apparently simple notions lead to non-trivial results. Consider a separable bipartite superoperator $T$,

$$T(\rho) = \sum_k M_k \rho M_k^\dagger, \quad M_k = A_k \otimes B_k,$$

where the operators $A_k$ represent operations of Alice and $B_k$ those of Bob. Not all such superoperators can be implemented by local transformations and classical communication (LOCC) [7]. This is the foundation of the “non-locality without entanglement”.

A classification of bipartite state transformations was introduced in [8]. It consists of the following categories. There are localizable operations that can be implemented locally by Alice and Bob, possibly with the help of prearranged ancillas, but without classical communication. Ideally, local operations are instantaneous, and the whole process can be viewed as performed at a definite time. A final classical output of such distributed intervention will be obtained at some point of the (joint) causal future of Alice’s and Bob’s interventions. For semilocalizable operations, the requirement of no communication is relaxed and one-way classical communication is possible. It is obvious that any tensor-product operation $T_A \otimes T_B$ is localizable, but it is not a necessary condition. For example the Bell measurements, which distinguishes between the four standard bipartite entangled qubit states,

$$|\Psi^\pm \rangle := \frac{1}{\sqrt{2}} (|00 \rangle \pm |11 \rangle), \quad |\Phi^\pm \rangle := \frac{1}{\sqrt{2}} (|01 \rangle \pm |10 \rangle),$$

are localizable.

Other classes of bipartite operators are defined as follows: Bob performs a local operation $T_B$ just before the global operation $T$. If no local operation of Alice can reveal any information about $T_B$, i.e., Bob cannot signal to Alice, the operation $T$ is semicausal. If the operation is semicausal in both directions, it is causal. In many cases it is easier to prove causality than localizability (see Remark 3). There is a necessary and sufficient condition for the semicausality (and therefore, the causality) of operations [8].

These definitions of causal and localizable operators appear equivalent. It is easily proved that localizable operators are causal. It was shown that semicausal operators are always semilocalizable [9]. However, there are causal operations that are not localizable [9].

It is curious that while a complete Bell measurement is causal, the two-outcome incomplete Bell measurement is not. Indeed, consider a two-outcome PVM

$$E_1 = |\Phi^+ \rangle \langle \Phi^+ |, \quad E_2 = \mathbb{1} - E_1.$$  

If the initial state is $|01 \rangle_{AB}$, then the outcome that is associated with $E_2$ always occurs and Alice’s reduced density matrix after the measurement is $\rho_A = |0 \rangle \langle 0 |$. On the other hand, if before the joint measurement Bob performs a unitary operation that transforms the state into $|00 \rangle_{AB}$, then the two outcomes are equiprobable, the resulting states after the measurement are maximally entangled, and Alice’s reduced density matrix is $\rho_A = \frac{1}{2} \mathbb{1}$. A simple calculation shows that after this incomplete Bell measurement two input states $|00 \rangle_{AB}$ and $|01 \rangle_{AB}$ are distinguished by Alice with a probability of 0.75.

Here is another example of a semicausal and semilocalizable measurement which can be executed with one-way classical communication from Alice to Bob. Consider a PVM measurement, whose complete orthogonal projectors are

$$|0 \rangle \otimes |0 \rangle, \quad |0 \rangle \otimes |1 \rangle, \quad |1 \rangle \otimes +, \quad |1 \rangle \otimes |\rangle,$$

where $|\pm \rangle = (|0 \rangle \pm |1 \rangle) / \sqrt{2}$. The Kraus matrices are

$$A_{\mu j} = E_\mu \delta_{j0},$$

where $\delta_{j0}$ is the Kronecker delta. 

$$\Delta_{\mu j} = E_\mu \delta_{j0},$$

then we move $A_{\mu n}$ from the first position to the last one in the product of operators in the traced parenthesis. Since the elements of a POVM satisfy $\sum_\mu E_\mu = \mathbb{1}$, Eq. (5) reduces to

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$$A_{\mu j} = E_\mu \delta_{j0},$$

where $\delta_{j0}$ is the Kronecker delta.
From the properties of complete orthogonal measurements, it follows that this operation cannot be performed without Alice talking to Bob. A protocol to realize this measurement is the following. Alice measures her qubit in the basis \( \{ |0\rangle, |1\rangle \} \), and tells her result to Bob. If Alice’s outcome was \( |0\rangle \), Bob measures his qubit in the basis \( \{ |0\rangle, |1\rangle \} \), and if it was \( |1\rangle \), in the basis \( \{ |-\rangle, |+\rangle \} \).

If one allows for more complicated conditional state evolution, then more measurements are localizable. In particular, consider a verification measurement, i.e., the measurement yields a \( \mu \)-th result with certainty, if the state prior to the classical interventions was given by \( \rho = E_{\mu} \), but without making any specific demand on the resulting state \( \rho'_{\mu} \).

It is possible to realize a verification measurements by means of a shared entangled ancilla and Bell-type measurements by one of the parties. Verification measurement of Eq. (11) can illustrate this construction. In addition to the state to be tested, Alice and Bob share a Bell state \( |\Psi^-\rangle \). They do not have to coordinate their moves. Alice and Bob perform tasks independently and convey their results to a common center, where a final decision is made.

The procedure is based on the teleportation identity

\[
|\Psi_1\rangle |\Psi_-\rangle_{23} = \frac{1}{2} \left( |\Psi^-\rangle_{12} |\Psi_3\rangle + |\Psi^+\rangle_{12} |\Psi^x\rangle_3 + |\Phi^-\rangle_{12} |\Phi_3\rangle + |\Phi^+\rangle_{12} |\Phi^y\rangle_3 \right),
\]

where \( |\Psi^x\rangle \) means the state \( |\Psi\rangle \) rotated by \( \pi \) around the \( z \)-axis, etc. The first step of this measurement corresponds to the first step of a teleportation of a state of the spin from \( B \) (Bob’s site) to \( A \) (Alice’s site). Bob and Alice do not perform the full teleportation (which requires a classical communication between them). Instead, Bob performs only the Bell measurement at his site which leads to one of the branches of the superposition in the rhs of Eq. (12).

The second step of the verification measurement is taken by Alice. Instead of completing the teleportation protocol, she measures the spin of her particle in the \( z \) direction. According to whether that spin is up or down, she measures the spin of her ancilla in the \( z \) or \( x \) direction, respectively. This completes the measurement and it only remains to combine the local outcomes to get the result of the nonlocal measurement \( |\Psi^-\rangle \). This method can be extended to arbitrary Hilbert space dimensions.

**Remarks**

1. Measurements in quantum field theory are discussed in \([1, 5, 6]\).
2. An algebraic field theory approach to statistical independence and to related topics is presented in \([12]\).
3. To check the causality of an operation \( T \) whose outcomes are the states \( \rho_{\mu} = T_{\mu}(\rho)/p_{\mu} \) with probabilities \( p_{\mu} = \text{tr} T_{\mu}(\rho) \), \( \sum_{\mu} p_{\mu} = 1 \) it is enough to consider the corresponding superoperator

\[
T'(\rho) := \sum_{\mu} T_{\mu}(\rho)
\]

Indeed, assume that Bob’s action prior to the global operation lead to one of the two different states \( |\rho_1\rangle \) and \( |\rho_2\rangle \). Then the states \( T'(|\rho_1\rangle) \) and \( T'(|\rho_2\rangle) \) are distinguishable if and only if some of the pairs of states \( T_{\mu}(|\rho_1\rangle)/p_{\mu1} \) and \( T_{\mu}(|\rho_2\rangle)/p_{\mu2} \) are distinguishable. Such probabilistic distinguishability shows that the operation \( T \) is not semicausal.

3. Absence of the superluminal communication makes possible to evade the theorems on the impossibility of a bit commitment. In particular the protocol RBC2 allows a bit commitment to be indefinitely maintained with unconditionally security against all classical attacks, and at least for some finite amount of time against quantum attacks \([13, 14]\).
4. In these notes I am not going to deal with the relativistic localization POVM. Their properties (and difficulties in their construction) can be found in \([1]\). An exhaustive survey of the spatial localization of photons is given in \([15]\). Here we only note in passing that if \( E(O) \) is an operator that corresponds to the detection of an event in a spacetime region \( O \), since they are not thought to be implemented by physical operations confined to that spacetime area, the condition \( [E(O_1), E(O_2)] = 0 \) is not required \([14, 17]\).

**III. QUANTUM LORENTZ TRANSFORMATIONS**

There is no elementary particle that is called “qubit”. Qubits are realized by particular degrees of freedom of more or less complicated systems. To decide how qubits transform (e.g., under Lorentz transformations) it may be necessary to consider again the entire system. In the following our qubit will be either a spin of a massive particle or a polarization of a photon. A quantum Lorentz transformation connects the description of a quantum state \( |\Psi\rangle \) in two reference frames that are connected by a Lorentz transformation \( \Lambda \) (i.e., their coordinate axes are rotated with respect to each other and the frames have a fixed relative
velocity). Then $|\Psi'\rangle = U(\Lambda)|\Psi\rangle$, and the unitary $U(\Lambda)$ is represented on Fig. 1 below. The purpose of this section is to explain the elements of this quantum circuit.

From the mathematical point of view the single-particle states belong to some irreducible representation of the Poincaré group. An introductory discussion of these representations and their relations with states and quantum fields may be found, e.g., in [18, 19]. Within each particular irreducible representation there are six commuting operators. The eigenvalues of two of them are invariants that label the representation by defining the mass $m$ and the intrinsic spin $j$. The basis states are labelled by three components of the momentum $p$ and the spin operator $\sum_3$. Hence a generic state is given by

$$|\Psi\rangle = \sum_\sigma \int d\mu(p) \psi_\sigma(p)|p, \sigma\rangle.$$  \hspace{1cm} (14)

In this formula $d\mu(p)$ is the Lorentz-invariant measure,

$$d\mu(p) = \frac{1}{(2\pi)^3} \frac{d^3p}{2E(p)},$$  \hspace{1cm} (15)

where the energy $E(p) = p^0 = \sqrt{p^2 + m^2}$. The improper momentum and spin eigenstates are $\delta$-normalized,

$$\langle p, \sigma|q, \sigma'\rangle = (2\pi)^3(2E(p))\delta^{(3)}(p-q)\delta_{\sigma\sigma'},$$  \hspace{1cm} (16)

and are complete on the one-particle space, which is $\mathcal{H} = \mathbb{C}^{2j+1} \otimes L^2(\mathbb{R}^3, d\mu(p))$ for spin-$j$ fields.

![Diagram](image)

FIG. 1: Relativistic state transformation as a quantum circuit: the gate $D$ which represents the matrix $D_{\ell\sigma}[W(\Lambda, p)]$ is controlled by both the classical information and the momentum $p$, which is itself subject to the classical information $\Lambda$.

To find the transformation law we have to be more concrete about the spin operator. The operator $\sum_3(p)$ is a function of the generators of the Poincaré group. One popular option is helicity, $\sum_3 = J \cdot \mathbf{P}/|\mathbf{P}|$, which is applicable for both massive and massless particles. For massive particles we use the $z$-component of the rest-frame (or Wigner spin, that we now describe in the next section.

A. Massive particles

The construction involves picking a reference 4-momentum $k$, which for massive particles is taken to be $k_R = (m, 0)$. The Wigner spin $\mathbf{S}(p)$ is defined to coincide with the non-relativistic spin $\mathbf{S}$ in particle’s rest frame. The state of a particle at rest is labelled $|k_R, \sigma\rangle$,

$$\mathbf{S}^2|k_R, \sigma\rangle = j(j+1)|k_R, \sigma\rangle, \hspace{1cm} \sum_3|k_R, \sigma\rangle = \sigma|k_R, \sigma\rangle.$$  \hspace{1cm} (17)

The spin states of arbitrary momenta are defined as follows. The standard rotation-free boost that brings $k_R$ to an arbitrary momentum $p$, $p^\mu = L(p)^\mu_\nu k^\nu$ is given by

$$L(p) = \begin{pmatrix}
\frac{p^1}{m} & 1 + \frac{p_1 p_1}{m(m+E)} & \frac{p_2}{m} & \frac{p_3}{m} \\
\frac{p_2}{m} & \frac{p_1 p_1}{m(m+E)} & \frac{p_2}{m} & \frac{p_3}{m} \\
\frac{p_3}{m} & \frac{p_2 p_2}{m(m+E)} & \frac{p_3}{m} & \frac{p_3}{m} \\
\end{pmatrix}.$$  \hspace{1cm} (18)
The Wigner spin $S(p)$ and the one-particle basis states are defined by

$$|p, \sigma\rangle \equiv U|L(p)||k_R, \sigma\rangle, \quad S_3(p)|p, \sigma\rangle = \sigma|p, \sigma\rangle. \quad (19)$$

In deriving the transformation rules we begin with the momentum eigenstates. Using the group representation property and Eqs. (19) the transformation is written as

$$U(\Lambda) = U[L(\Lambda p)]U[L^{-1}(\Lambda p)\Lambda L(p)]U[L^{-1}(p)] \quad (20)$$

The element of the Lorentz group

$$W(\Lambda, p) \equiv L^{-1}(\Lambda p)\Lambda L(p), \quad (21)$$

leaves $k_R$ invariant, $k_R = Wk_R$. Hence it belongs to the stability subgroup (or Wigner little group) of $k_R$. For $k_R = (m, 0)$ it is a rotation. Pressing on

$$U(\Lambda)|p, \sigma\rangle = U[L(\Lambda p)]U[W(\Lambda, p)]|k_R, \sigma\rangle, \quad (22)$$

and as a result,

$$U(\Lambda)|p, \sigma\rangle = \sum_\xi D_{\xi\sigma}[W(\Lambda, p)]|\Lambda p, \xi\rangle, \quad (23)$$

where $D_{\xi\sigma}$ are the matrix elements of the representation of the Wigner rotation $W(\Lambda, p)$.

We consider only spin-$\frac{1}{2}$ particles, so $\sigma = \pm \frac{1}{2}$. Any $2 \times 2$ unitary matrix can be written as $\tilde{D} = \exp(-i\omega \hat{n} \cdot \sigma)$, where $\omega$ is a rotation angle and $\hat{n}$ is a rotation axis that corresponds to $W(\Lambda, p)$.

The wave functions transform according to $\psi'_\xi(q) = \langle \xi, q|U(\Lambda)|\Psi\rangle$ so the same state in the Lorentz-transformed frame is

$$|\Psi'\rangle = U(\Lambda)|\Psi\rangle = \sum_{\sigma, \xi} \int_{-\infty}^{\infty} D_{\sigma\xi}[W(\Lambda, \Lambda^{-1} p)]\psi(\Lambda^{-1} p)|\sigma, p\rangle d\mu(p). \quad (24)$$

For pure rotation $R$ the three-dimensional (more exactly, 3D block of 4D matrix; here and in the following we use the same letter for a 4D and 3D matrix for $R \in SO(3)$) Wigner rotation matrix is the rotation itself,

$$W(R, p) = R, \quad \forall p = (p^0, \mathbf{p}). \quad (25)$$

As a result, the action of Wigner spin operators on $\mathcal{H}_1$ is given by than halves of Pauli matrices that are tensored with the identity of $L^2$.

\section*{B. Photons}

The single-photon states are labelled by momentum $p$ (the 4-momentum vector is null, $E = p^0 = |p|$) and helicity $\sigma_p = \pm 1$, so the state with a definite momentum is given by $\sum_{\sigma = \pm 1} \alpha_\sigma|p, \sigma_p\rangle$, where $|\alpha_+|^2 + |\alpha_-|^2 = 1$. Polarization states are also labelled by 3-vectors $\epsilon^\alpha_p$, $\mathbf{p} \cdot \epsilon^\alpha_p = 0$, that correspond to the two senses of polarization of classical electromagnetic waves. An alternative labelling of the same state, therefore, is $\sum_{\sigma = \pm 1} \alpha_\sigma|p, \epsilon^\alpha_p\rangle$.

Helicity is invariant under proper Lorentz transformation, but the basis states acquire phases. The little group element $W(\Lambda, p) = L^{-1}(\Lambda p)\Lambda L(p)$ is defined with respect to the standard four-momentum $k_R = (1, 0, 0, 1)$. The standard Lorentz transformation is

$$L(p) = R(\hat{p})B_z(u), \quad (26)$$

where $B_z(u)$ is a pure boost along the $z$-axis with a velocity $u$ that takes $k_R$ to $(|p|, 0, 0, |p|)$ and $R(\hat{p})$ is the standard rotation that carries the $z$-axis into the direction of the unit vector $\hat{p}$. If $\hat{p}$ has polar and azimuthal angles $\theta$ and $\phi$, the standard rotation $R(\hat{p})$ is accomplished by a rotation by $\theta$ around the $y$-axis, that is followed by a rotation by $\phi$ around the $z$-axis. Hence,

$$R(\hat{p}) = \begin{pmatrix}
\cos \theta \cos \phi & -\sin \phi & \cos \phi \sin \theta \\
\cos \theta \sin \phi & \cos \phi & \sin \phi \sin \theta \\
-\sin \theta & 0 & \cos \theta
\end{pmatrix}, \quad (27)$$

(here only the non-trivial 3D block is shown).
An arbitrary little group element for a massless particle is decomposed according to
\[ W(A, p) = S(\beta, \gamma)R_z(\xi), \] (28)
where the elements \( S(\beta, \gamma) \) form a subgroup that is isomorphic to the translations of the Euclidean plane and \( R_z(\xi) \) is a rotation around the \( z \)-axis. We are interested only in the angle \( \xi \), since \( \beta \) and \( \gamma \) do not correspond to the physical degrees of freedom. However, they are important for gauge transformations. Finally, the little group elements are represented by
\[ D_{\sigma, \sigma'} = \exp(i\xi_\sigma \delta_{\sigma, \sigma'}). \] (29)

It is worthwhile to derive more explicit expressions for \( \xi \). I begin with rotations, \( \Lambda = \mathcal{R} \). Since rotations form a subgroup of a Lorentz group, \( R^{-1}(\mathcal{R}\hat{p})\mathcal{R}R(\hat{p}) \) is a rotation that leaves \( \hat{z} \) invariant and thus is of the form \( R_z(\omega) \) for some \( \omega \). A boost in \((t, z)\) plane and a rotation around \( z \)-axis commute, \([R_z, B_z] = 0\), so
\[ W(\mathcal{R}, p) = R^{-1}(\mathcal{R}\hat{p})\mathcal{R}R(\hat{p}) = R_z(\xi). \] (30)

Any rotation can be described by two angles that give a direction of the axis and the third angle that gives the amount of rotation around that axis. If \( \mathcal{R} \hat{p} = \hat{q} \), we decompose the rotation matrix as
\[ \mathcal{R} = R_q(\omega)R(\hat{q})R^{-1}(\hat{p}), \] (31)
where \( R_q(\omega) \) characterizes a rotation around \( \hat{q} \), and \( R(\hat{q}) \) and \( R(\hat{p}) \) are the standard rotations that carry the \( z \)-axis to \( \hat{q} \) and \( \hat{p} \), respectively. Using Eq. (30) we find that \( S = 1 \) and the two rotations are of the same conjugacy class,
\[ R_z(\xi) = R^{-1}(\mathcal{R}\hat{p})R_{\mathcal{R}\hat{p}}(\omega)R(\mathcal{R}\hat{p}), \] (32)
so we conclude that \( \xi = \omega \).

A practical description of polarization states is given by spatial vectors that correspond to the classical polarization directions. Taking again \( k_R \) as the reference momentum, two basis vectors of linear polarization are \( e_{k_R}^1 = (1, 0, 0) \) and \( e_{k_R}^2 = (0, 1, 0) \), while to the right and left circular polarizations correspond \( e_{k_R}^\pm = (\epsilon_{k_R}^1 \pm i\epsilon_{k_R}^2)/\sqrt{2} \).

Phases of the states obtained by the standard Lorentz transformations \( L(p) \) are set to 1. Since the standard boost \( B_z(u) \) leaves the four-vector \((0, e_{k_R}^\pm)\) invariant, we define a polarization basis for any \( p \) as
\[ \epsilon_p^\pm = e_p^\pm \equiv R(\hat{p})e_{k_R}^\pm, \] (33)
while the transformation of polarization vectors under an arbitrary rotation \( \mathcal{R} \) is given by the rotation itself. To see the agreement between transformations of spatial vectors and states, consider a generic state with a momentum \( p \). Its polarization is described by the polarization vector \( \alpha(p) = \alpha_+e_p^+ + \alpha_-e_p^- \), or by the state vector \( \alpha_+|p, +\rangle + \alpha_-|p, -\rangle \). Using Eq. (33) we see that the transformation of \( \alpha(p) \) is given by
\[ \mathcal{R}\alpha(p) = R_{\mathcal{R}\hat{p}}(\omega)R(\mathcal{R}\hat{p})R^{-1}(\hat{p})\alpha(p) = R_{\mathcal{R}\hat{p}}(\omega)R(\mathcal{R}\hat{p})\alpha(k_R) = R_{\mathcal{R}\hat{p}}(\omega)\alpha(\mathcal{R}\hat{p}). \] (34)

If \( q = \mathcal{R}p \) the transformation results in \( \alpha_+e^{i\omega}e_q^+ + \alpha_-e^{-i\omega}e_q^- \), and since \( \omega = \xi \), it is equivalent to the state transformation
\[ U(\mathcal{R})(\alpha_+|p, +\rangle + \alpha_-|p, -\rangle) = \alpha_+e^{i\xi}|q, +\rangle + \alpha_-e^{-i\xi}|q, -\rangle. \] (35)

For a general Lorentz transformations the triad \((e_p^1, e_p^2, \hat{p})\) is rigidly rotated, but in a more complicated fashion. To obtain the phase for a general Lorentz transformation, we decompose the latter into two rotations and a standard boost \( B_z \) along the \( z \)-axis:
\[ \Lambda = \mathcal{R}_2B_z(u)\mathcal{R}_1. \] (36)
It can be shown that \( B_z \) alone does not lead to a phase rotation. Therefore,
\[ \xi = \omega_1 + \omega_2, \] (37)
where both \( \omega_1 \) and \( \omega_2 \) are due to the rotations and are given by Eq. (31). Note that although \( B_z(u) \) alone does not lead to a phase rotation, it can affect the value of \( \omega_2 \), since it indirectly appears in the definition of \( \mathcal{R}_2 \).
Remarks

1. A comprehensive discussion of the Poincaré group in physics can be found in [20,21]. Useful expressions for Wigner rotations and their applications for massive particles are given in [22,23,24].

2. In this transformation I do not assume any additional normalization factors. A condition of unitarity is $UU^\dagger = U^\dagger U = 1$, but there also other conventions in the literature.

3. A double infinity of the positive energy solutions of the Dirac equation (functions $u_p^{(1/2)}$ and $u_p^{(-1/2)}$) spans an improper basis of this space. There is a one-to-one correspondence between Wigner and Dirac wave functions. Basis vectors of Wigner and Dirac Hilbert spaces are in the one-to-one correspondence [21].

$$u_p^{(1/2)} \leftrightarrow |\frac{1}{2}, p\rangle, \quad u_p^{(-1/2)} \leftrightarrow |\frac{1}{2}, p\rangle,$$

while the wave functions are related by

$$\Psi^\alpha(p) = \psi_{1/2}(p)u_p^{(1/2)\alpha} + \psi_{-1/2}(p)u_p^{(-1/2)\alpha}$$

$$2m\psi_\sigma(p) = \sum_{\alpha=1}^{4} \theta^{(-\sigma)}_{p\alpha} \Psi^\alpha(p)$$

4. Another approach to the construction of the Wigner rotation $\hat{D}$ is based on the homomorphism between Lorentz group and $SL(2, \mathbb{C})$ [21].

5. When not restricted to a single-particle space the Wigner spin operator is given by

$$S = \frac{i}{2} \sum_{\eta, \zeta} \sigma_{\eta\zeta} \int d\mu(p) (\hat{a}^\dagger_{\eta p} \hat{a}_{\zeta p} + \hat{b}^\dagger_{\eta p} \hat{b}_{\zeta p}),$$

where $\hat{a}^\dagger_p$ creates a mode with a momentum $p$ and spin $\eta$ along the $z$-axis, etc. A comparison of different spin operators can be found in [25].

6. If one works with the 4-vectors, then in the helicity gauge the polarization vector is given by $\epsilon_p = (0, \epsilon_p)$. A formal connection between helicity states and polarization vectors is made by first observing that three spin-1 basis states can be constructed from the components of a symmetric spinor of rank 2. Unitary transformations of this spinor that are induced by $\mathcal{R}$ are in one-to-one correspondence with transformations by $\mathcal{R}$ of certain linear combinations of a spatial vector. In particular, transformations of the helicity ±1 states induced by rotations are equivalent to the rotations of $(\epsilon^1_{k\sigma} \pm i\epsilon^2_{k\sigma})/\sqrt{2}$ (the $z$-axis is the initial quantization direction). While $p_\mu \epsilon^\mu_p = 0$ gauge condition is Lorentz-invariant, the spatial orthogonality is not. The role of gauge transformations in preserving the helicity gauge and some useful expressions for the phase that photons acquire can be found in [26,27,28].

IV. IMPLICATIONS OF QUANTUM LORENTZ TRANSFORMATIONS

A. Reduced density matrices

In a relativistic system whatever is outside the past light cone of the observer is unknown to him, but also cannot affect his system, therefore does not lead to decoherence (here, I assume that no particle emitted by from the outside the past cone penetrates into the future cone). Since different observers have different past light cones, by tracing out they exclude from their descriptions different parts of spacetime. Therefore any transformation law between them must tacitly assume that the part excluded by one observer is irrelevant to the system of another.

Another consequence of relativity is that there is a hierarchy of dynamical variables: primary variables have relativistic transformation laws that depend only on the Lorentz transformation matrix $\Lambda$ that acts on the spacetime coordinates. For example, momentum components are primary variables. On the other hand, secondary variables such as spin and polarization have transformation laws that depend not only on $\Lambda$, but also on the momentum of the particle. As a consequence, the reduced density matrix for secondary variables, which may be well defined in any coordinate system, has no transformation law relating its values in different Lorentz frames.

Moreover, an unambiguous definition of the reduced density matrix is possible only if the secondary degrees of freedom are unconstrained, and photons are the simplest example when this definition fails. In the absence of a general prescription, a case-by-case treatment is required. I describe a particular construction, valid with respect to a certain class of tests.
**B. Massive particles**

For a massive qubit the usual definition of quantum entropy has no invariant meaning. The reason is that under a Lorentz boost, the spin undergoes a Wigner rotation, that as shown on Fig. 1 is controlled both by the classical data and the corresponding momentum. Even if the initial state is a direct product of a function of momentum and a function of spin, the transformed state is not a direct product. Spin and momentum become entangled.

Let us define a reduced density matrix,

$$\rho = \int d\mu(p) \psi(p) \psi^\dagger(p).$$

(42)

It gives statistical predictions for the results of measurements of spin components by an ideal apparatus which is not affected by the momentum of the particle. Note that I tacitly assumed that the relevant observable is the Wigner spin. The spin entropy is

$$S = -\text{tr} (\rho \log \rho) = -\sum \lambda_j \log \lambda_j,$$

(43)

where \(\lambda_j\) are the eigenvalues of \(\rho\).

As usual, ignoring some degrees of freedom leaves the others in a mixed state. What is not obvious is that in the present case the amount of mixing depends on the Lorentz frame used by the observer. Indeed consider another observer (Bob) who moves with a constant velocity with respect to Alice who prepared that state. In the Lorentz frame where Bob is at rest, the state is given by Eq. (24).

As an example, take a particle prepared by Alice to be

$$|\Psi\rangle = \chi \int \psi(p)|p\rangle d\mu(p), \quad \chi = \begin{pmatrix} \zeta \\ \eta \end{pmatrix},$$

(44)

where \(\psi\) is concentrated near zero momentum and has a characteristic spread \(\Delta\). Spin density matrices of all the states that are given by Eq. (44) are

$$\rho = \begin{pmatrix} |\zeta|^2 & \zeta \eta^* \\
\zeta^* \eta & |\eta|^2 \end{pmatrix},$$

(45)

and are independent of the specific form of \(\psi(p)\). To make calculations explicit (and simpler) I take the wave function to be Gaussian, \(\psi(p) = N \exp(\frac{p^2}{2\Delta^2})\), where \(N\) is a normalization factor. Spin and momentum are not entangled, and the spin entropy is zero. When that particle is described in Bob’s Lorentz frame, moving with velocity \(v\) at the angle \(\theta\) with Alice’s \(z\)-axis, a detailed calculation shows that the the spin entropy is positive [1]. This phenomenon is illustrated in Fig. 2. A relevant parameter, apart from the angle \(\theta\), is in the leading order in momentum spread,

$$\Gamma = \frac{\Delta}{m} \frac{1 - \sqrt{1 - v^2}}{v},$$

(46)

where \(\Delta\) is the momentum spread in Alice’s frame. The entropy has no invariant meaning, because the reduced density matrix \(\tau\) has no covariant transformation law, except in the limiting case of sharp momenta. Only the complete density matrix transforms covariantly.

I outline some of the steps in this derivation. First, we calculate the rotation parameters \((\omega, \hat{n})\) of the orthogonal matrix \(W(\Lambda, p)\) for a general momentum. The rotation axis and angle are given by

$$\hat{n} = \hat{v} \times \hat{p}, \quad \cos \theta = \hat{v} \cdot \hat{p}, \quad 0 \leq \theta \leq \pi$$

(47)

where \(\hat{v}\) is boost’s direction, while the leading order term for the angle is

$$\omega = \frac{1 - \sqrt{1 - v^2}}{v} \frac{p^2}{m^2} \sin \theta - O\left(\frac{p^2}{m^2}\right)$$

(48)

Without a loss of generality we can make another simplification. We can choose our coordinate frame in such a way that both \(\zeta\) and \(\eta\) are real. The matrix \(D[W(\Lambda, \Lambda^{-1}p)]\) takes the form

$$D[W(\Lambda, p')] = \sigma_0 \cos \frac{\omega}{2} - i \sin \frac{\omega}{2}(- \sin \phi \sigma_x + \cos \phi \sigma_y),$$

(49)
FIG. 2: Dependence of the spin entropy \( S \), in Bob’s frame, on the values of the angle \( \theta \) and a parameter \( \Gamma = [1 - (1 - v^2)^{1/2}] \Delta/mv \).

where \((\theta, \phi)\) are the spherical angle of \( p' \) (to be consistent with Eq. (24) momentum in Alice frame carries a prime, \( p' = \Lambda^{-1}p \)).

The reduced density matrix in Bob’s frame is

\[
\rho_B^{\sigma \xi} = \int d\mu(p') D_{\sigma \nu}^* \psi(p') \psi^*(p').
\]

The symmetry of \( \psi(\Lambda^{-1}p) \) is cylindrical. Hence the partial trace is taken by performing a momentum integration in cylindrical coordinates. This simplification is a result of the spherical symmetry of the original \( \psi \). The two remaining integrations are performed by first expanding in powers of \( p/\Delta \) and taking Gaussian integrals. Finally,

\[
\rho' = \begin{pmatrix}
\zeta^2(1 - \Gamma^2/4) + \eta^2\Gamma^2/4 & \zeta\eta^*(1 - \Gamma^2/4) \\
\zeta^*\eta(1 - \Gamma^2/4) & \zeta^2\Gamma^2/4 + \eta^2(1 - \Gamma^2/4)
\end{pmatrix}.
\]

Fidelity can be used to estimate the difference between the two density matrices. It is defined as

\[
f = \chi^\dagger \rho' \chi,
\]

and it is easy to get an analytical result for this quantity. Set \( \zeta = \cos \theta \) and \( \eta = \sin \theta \). Then

\[
f = 1 - \frac{\Gamma^2}{2} \left( 3 + \frac{\cos 4\theta}{8} \right).
\]

Consider now a pair of orthogonal states that were prepared by Alice, e.g. the above state with \( \chi_1 = (1, 0) \) and \( \chi_2 = (0, 1) \). How well can moving Bob distinguish them? I use the simplest criterion, namely the probability of error \( P_E \), defined as follows: an observer receives a single copy of one of the two known states and performs any operation permitted by quantum theory in order to decide which state was supplied. The probability of a wrong answer for an optimal measurement is

\[
P_E(\rho_1, \rho_2) = \frac{1}{4} - \frac{1}{4} \text{tr} \sqrt{(\rho_1 - \rho_2)^2}.
\]

In Alice’s frame \( P_E = 0 \). In Bob’s frame the reduced density matrices are

\[
\rho_1^B = \begin{pmatrix}
1 - \Gamma^2/4 & 0 \\
0 & \Gamma^2/4
\end{pmatrix}, \quad \rho_2^B = \begin{pmatrix}
\Gamma^2/4 & 0 \\
0 & 1 - \Gamma^2/4,
\end{pmatrix}
\]

respectively. Hence the probability of error is \( P_E(\rho_1, \rho_2) = \Gamma^2/4 \).
C. Photons

The relativistic effects in photons are essentially different from those for massive particles that were discussed above. This is because photons have only two linearly independent polarization states. As we know, polarization is a secondary variable: states that correspond to different momenta belong to distinct Hilbert spaces and cannot be superposed (an expression such as $|\epsilon_k^±\rangle + |\epsilon_q^±\rangle$ is meaningless if $k \neq q$). The complete basis $|p, \epsilon_p^±\rangle$ does not violate this superselection rule, owing to the orthogonality of the momentum basis. The reduced density matrix, according to the usual rules, should be

$$\rho = \int d\mu(p)|\psi(p)|^2|p, \alpha(p)\rangle\langle p, \alpha(p)|.$$

(56)

However, since $\xi$ in Eq. 29 depends on the photon’s momentum even for ordinary rotations, this object will have no transformation law at all. It is still possible define an “effective” density matrix adapted to a specific method of measuring polarization 29, 30. I describe one such scheme.

The labelling of polarization states by Euclidean vectors $\epsilon\pm_k$ suggests the use of a $3 \times 3$ matrix with entries labelled $x, y$ and $z$. Classically, they correspond to different directions of the electric field. For example, a component $\rho_{xx}$ would give the expectation values of operators representing the polarization in the $x$ direction, seemingly irrespective of the particle’s momentum.

To have a momentum-independent polarization is to admit longitudinal photons. Momentum-independent polarization states thus consist of physical (transverse) and unphysical (longitudinal) parts, the latter corresponding to a polarization vector $\epsilon_x^l = \hat{p}$. For example, a generalized polarization state along the $x$-axis is

$$|\hat{x}\rangle = x_+(p)|\epsilon_p^+\rangle + x_-(p)|\epsilon_p^-\rangle + x_\ell(p)|\epsilon_p^\ell\rangle,$$

(57)

where $x_\pm(p) = \hat{x} \cdot \epsilon_p^\pm$, and $x_\ell(p) = \hat{x} \cdot \hat{p} = \sin \theta \cos \phi$. It follows that $|x_+|^2 + |x_-|^2 + |x_\ell|^2 = 1$, and we thus define

$$\epsilon_x(p) = \frac{x_+(p)|\epsilon_p^+\rangle + x_-(p)|\epsilon_p^-\rangle}{\sqrt{x_+^2 + x_-^2}},$$

(58)

as the polarization vector associated with the $x$ direction. It follows from 57 that $\langle \hat{x}|\hat{x}\rangle = 1$ and $\langle \hat{x}|\hat{y}\rangle = \hat{x} \cdot \hat{y} = 0$, and likewise for the other directions, so that

$$|\hat{x}\rangle\langle \hat{x}| + |\hat{y}\rangle\langle \hat{y}| + |\hat{z}\rangle\langle \hat{z}| = \mathbb{1}_p,$$

(59)

where $\mathbb{1}_p$ is the unit operator in momentum space.

To the direction $\hat{x}$ there corresponds a projection operator

$$P_{xx} = |\hat{x}\rangle\langle \hat{x}| \otimes \mathbb{1}_p = |\hat{x}\rangle\langle \hat{x}| \otimes \int d\mu(k)|p\rangle\langle p|,$$

(60)

The action of $P_{xx}$ on $|\Psi\rangle$ follows from Eq. 57 and $\langle \epsilon_p^±|\epsilon_p^\ell\rangle = 0$. Only the transverse part of $|\hat{x}\rangle$ appears in the expectation value:

$$\langle \Psi|P_{xx}|\Psi\rangle = \int d\mu(p)|\psi(p)|^2|x_+(p)\alpha_+(p) + x_-(p)\alpha_-(p)|^2.$$

(61)

It is convenient to write the transverse part of $|\hat{x}\rangle$ as

$$|b_x(p)\rangle \equiv (|\epsilon_p^+\rangle\langle \epsilon_p^+| + |\epsilon_p^-\rangle\langle \epsilon_p^-|)|\hat{x}\rangle = x_+(p)|\epsilon_p^+\rangle + x_-(p)|\epsilon_p^-\rangle.$$

(62)

Likewise define $|b_y(p)\rangle$ and $|b_z(p)\rangle$. These three state vectors are neither of unit length nor mutually orthogonal.

Finally, a POVM element $E_{xx}$ which is the physical part of $P_{xx}$, namely is equivalent to $P_{xx}$ for physical states (without longitudinal photons) is

$$E_{xx} = \int d\mu(k)|p, b_x(p)\rangle\langle p, b_x(p)|,$$

(63)

and likewise for the other directions. The operators $E_{xx}, E_{yy}$ and $E_{zz}$ indeed form a POVM in the space of physical states, owing to Eq. 59.

To complete the construction of the density matrix, we introduce additional directions. Following the standard practice of state reconstruction, we consider $P_{x+z,x+z}, P_{x+iz,x+iz}$ and similar combinations. For example,

$$P_{x+z,x+z} = \frac{1}{2}(|\hat{x}| + |\hat{z}|)(|\hat{x}| + |\hat{z}|) \otimes \mathbb{1}_p.$$
The diagonal elements of the new polarization density matrix are defined as
\[ \rho_{mm} = \langle \Psi | E_{mm} | \Psi \rangle, \quad m = x, y, z, \] (65)
and the off-diagonal elements are recovered by combinations such as
\[ \rho_{xz} = \langle \Psi | (| \hat{z} \rangle \otimes I_p) | \Psi \rangle = \langle \Psi | E_{xx+z,x+z} - i E_{x-xz,x-z} + (1 - i)(E_{xx} - E_{zz})/2 | \Psi \rangle. \] (66)
Denote \(| \hat{x} \rangle \otimes I_p\) as \(P_{xz}\), and its “physical” part by \(E_{xz}\). Then the effective polarization density matrix is
\[ \rho_{mm} = \langle \Psi | E_{mm} | \Psi \rangle = \int d\mu(k) | f(p) |^2 \langle \alpha(p) | b_m(p) \rangle \langle b_n(p) | \alpha(p) \rangle, \quad m, n = x, y, z. \] (67)

It is interesting to note that this derivation gives a direct physical meaning to the naive definition of a reduced density matrix,
\[ \rho_{\text{naive}} = \int d\mu(p) | \phi(p) |^2 \alpha_m(p) \alpha_n^*(p) = \rho_{mn} \] (68)

It is possible to show that this POVM actually corresponds to a simple photodetection model [31].

The basis states \(| p, \epsilon_p \rangle\) are direct products of momentum and polarization. Owing to the transversality requirement \(\epsilon_p \cdot p = 0\), they remain direct products under Lorentz transformations. All the other states have their polarization and momentum degrees of freedom entangled. As a result, if one is restricted to polarization measurements as described by the above POVM, there do not exist two orthogonal polarization states. In general, any measurement procedure with finite momentum sensitivity will lead to the errors in identification, as demonstrated as follows:

Let two states \(| \Phi \rangle\) and \(| \Psi \rangle\) be two orthogonal single-photon states. Their reduced polarization density matrices, \(\rho_\Phi\) and \(\rho_\Psi\), respectively, are calculated using Eq. (67). Since the states are entangled, the von Neumann entropies of the reduced density matrices, \(S = -\text{tr}(\rho \ln \rho)\), are positive. Therefore, both matrices are at least of rank two. Since the overall dimension is 3, it follows that \(\text{tr}(\rho_\Phi \rho_\Psi) > 0\) and these states are not perfectly distinguishable. An immediate corollary is that photon polarization states cannot be cloned perfectly, because the no-cloning theorem forbids exact copying of unknown non-orthogonal states.

In general, any measurement procedure with finite momentum sensitivity will lead to the errors in identification. First I present some general considerations and then illustrate them with a simple example. Let us take the z-axis to coincide with the average direction of propagation so that the mean photon momentum is \(k_A \hat{z}\). Typically, the spread in momentum is small, but not necessarily equal in all directions. Usually the intensity profile of laser beams has cylindrical symmetry, and we may assume that \(\Delta_x \sim \Delta_y \sim \Delta_r\) where the index \(r\) means radial. We may also assume that \(\Delta_r \gg \Delta_z\). We then have
\[ f(p) \propto f_1[(p_z - k_A)/\Delta_z] f_2(p_r/\Delta_r). \] (69)
We approximate
\[ \theta \approx \tan \theta \equiv p_r/p_z \approx p_r/k_A. \] (70)
In pictorial language, polarization planes for different momenta are tilted by angles up to \(\sim \Delta_r/k_A\), so that we expect an error probability of the order \(\Delta_z^2/k_A^2\). In the density matrix \(\rho_{mn}\), all the elements of the form \(\rho_{mz}\) should vanish when \(\Delta_r \to 0\). Moreover, if \(\Delta_z \to 0\), the non-vanishing \(xy\) block goes to the usual (monochromatic) polarization density matrix,
\[ \rho_{\text{pure}} = \begin{pmatrix} |\alpha|^2 & \beta & 0 \\ \beta^* & 1 - |\alpha|^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \] (71)
As an example, consider two states which, if the momentum spread could be ignored, would be \(| k_A \hat{z}, \epsilon_{k_A \hat{z}}^\pm \rangle\). To simplify the calculations we assume a Gaussian distribution:
\[ f(p) = Ne^{-(p_x-k_A)^2/2\Delta_z^2}e^{-p_r^2/2\Delta_r^2}, \] (72)
where \(N\) is a normalization factor and \(\Delta_z, \Delta_r \ll k_A\). In general the spread in \(p_z\) may introduce an additional incoherence into density matrices, in addition to the effect caused by the transversal spread. However, when all momentum components carry the same helicity, this spread results in corrections of the higher order. In the example below we take the polarization components to be \(\epsilon_p \equiv R(\hat{p})\epsilon_{kS}^\pm\). That means we have to analyze the states
\[ |\Psi_\pm\rangle = \int d\mu(p) f(p) |p, \epsilon_p^\pm\rangle, \] (73)
where \( f(p) \) is given above.

It is enough to expand \( R(\hat{p}) \) up to second order in \( \theta \). The reduced density matrices are calculated by techniques similar to those for massive particles, using rotational symmetry around the \( z \)-axis and normalization requirements. At the leading order in \( \Omega = \Delta r / k_A \)

\[
\rho_+ = \frac{1}{2}(1 - \frac{1}{2}\Omega^2) \begin{pmatrix} 1 & -i & 0 \\ i & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{2}\Omega^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]

(74)

and \( \rho_- = \rho_+^* \). At the same level of precision,

\[
P_E(\rho_+ \rho_-) = \Delta^2 / 4k^2_A.
\]

(75)

It is interesting to note that the optimal strategy for distinguishing between these two states is a polarization measurement in the \( xy \)-plane. Since everything else in the integral remains the same, the effect of relative motion is given by a substitution

\[
\Omega_B = \sqrt{\frac{1 + v}{1 - v}} \Omega_A, \quad P_E^B = \frac{1 + v}{1 - v} P_E^A,
\]

(78)

so Bob can distinguish the signals either better or worse than Alice [29].

Remarks

1. A modification of the spin operator [32] will allow for a momentum-independent transformation of the spin density matrix between two frames that are related by a fixed Lorentz transformation \( \Lambda_{12} \). Its relation to our scheme is discussed in [33].

2. An additional motivation for introduction of effective polarization density matrices comes from the analysis of one-photon scattering [30].

3. I have discussed only discrete variables. To explore the relativistic effects with continuous variables [34] it is convenient to express the quantum Lorentz transformations in terms of mode creation and annihilation operators [35].

V. COMMUNICATION CHANNELS

What happens when Alice and Bob that are in a relative motion try to communicate? Assume that they use qubits that were described above. Under a general Lorentz transformation \( \Lambda \) that relates Alice’s and Bob’s frames, the state of this qubit will be transformed due to three distinct effects, which are:
(i) A Wigner rotation due to the Lorentz boost $\Lambda$, which occurs even for momentum eigenstates. If $\Lambda$ is known, then to the extent that the wave-packet spread can be ignored, this is inconsequential.

(ii) A decoherence due to the entangling of spin and momentum under the Lorentz transformation $\Lambda$ because the particle is not in a momentum eigenstate. Although reduced or effective density matrices have no general transformation rule, such rules can be established for particular classes of experimental procedures. We can then ask how these effective transformation rules, $\rho' = T(\rho)$, fit into the framework of general state transformations. E.g., for the massive qubit of Sec. 4.2 the effective transformation is given by

$$\rho' = \rho(1 - \frac{\Gamma^2}{4}) + (\sigma_x \rho \sigma_x + \sigma_y \rho \sigma_y) \frac{\Gamma^2}{8}.$$

(79)

If $\Lambda$ is known and it is possible to implement the operators that were mentioned in the Remark 1 above, then this effect is absent. Otherwise, this noise is unavoidable. Still, it is worth to keep in mind that the motion can improve the message fidelity, as in Eq. (80).

(iii) Another kind of decoherence arises due to Bob’s lack of knowledge about the transformation relating his reference frame to Alice’s frame. Using the techniques of the decoherence-free subspaces, it is possible to eliminate this effect completely. E.g., for massive particles four physical qubits may be used to encode a logical qubit, while for photons $2 \rightarrow 1$ encoding is sufficient. In both cases using the block encoding it is possible to reach an asymptotically unit efficiency.

Entanglement between the “qubit” and spatial degrees of freedom leads to an interesting complication of the analysis. It is known [1], that the dynamics of a subsystem may be not completely positive, there is a prior entanglement with another system and the dynamics is not factorizable. Since in Eq. (73) and in the discussion following Eq. (54) we have seen that distinguishability can be improved, we conclude that these transformations are not completely positive. The reason is that the Lorentz transformation acts not only on the “interesting” discrete variables, but also on the primary momentum variables that we elected to ignore and to trace out, and its action on the interesting degrees of freedom depends on the “hidden” primary ones. Of course, the complete state, with all the variables, transforms unitarily and distinguishability is preserved.

This technicality has one important consequence. In quantum information theory quantum channels are described by completely positive maps that act on qubit states. Qubits themselves are realized as discrete degrees of freedom of various particles. If relativistic motion is important, then not only does the vacuum behave as a noisy quantum channel, but the very representation of a channel by a CP map fails.

### VI. ENTANGLEMENT AND DIFFERENT LORENTZ OBSERVERS

In this section I consider only two-particle states. Even in this simple setting there are several possible answers to the question what happens to the entanglement, depending on the details of the question. Since the quantum Lorentz transformation is given by a tensor product $U_1(\Lambda) \otimes U_2(\Lambda)$, the overall entanglement between the states is Lorentz-invariant.

Let us assume that the states can be approximated by momentum eigenstates. Then, the same conclusion applies to the spin-spin (or polarization-polarization) entanglement between the particles, and it is possible to write an appropriate entanglement measures that capture the effects of particle statistics and Lorentz-invariance of the entanglement [23]. However, it does not mean that this invariance will be observed in an experiment, or that the violation of Bell-type inequalities that is observed in an experiment that is performed in Alice’s frame will be observed if the same equipment is placed in Bob’s frame.

While a field-theoretical analysis shows that violations of Bell-type inequalities are generic, there are conditions that are imposed on the experimental procedures that are used to detect them. Consider the CHSH inequality. For any two spacelike separated regions and any pairs of of operators, $a, b$, there is a state $\rho$ such that the CHSH inequality is violated, i.e., $\zeta(a, b, \rho) > 1$. With additional technical assumptions the existence of a maximally violating state $\rho_m$ can be proved:

$$\zeta(a, b, \rho_m) = \sqrt{2},$$

(80)

for any spacelike separated regions $O_L$ and $O_R$. It follows from convexity arguments that states that maximally violate Bell inequalities are pure. What are then the operators that lead to the maximal violation? It was shown [37] that the operators $A_j$ and $B_k$ that give $\zeta = \sqrt{2}$ satisfy $A_j^2 = 1$ and $A_1 A_2 + A_2 A_1 = 0$, and likewise for $B_k$. If we define $A_3 := -i[A_1, A_2]/2$, then these three operators have the same algebra as Pauli spin matrices.

In principle the vacuum state may lead to the maximal violation of Bell-type inequalities. Their observability was discussed in [39].

The operators [38]

$$A_i = 2 \frac{m}{p^0} a_i + \left(1 - \frac{m}{p^0}\right) (a \cdot n)n \cdot S \equiv 2 \alpha(a, p) \cdot S,$$

(81)
where $S$ is the Wigner spin operator and $n = p/|p|$ appear quite naturally as the candidates for the measurement description. The length of the auxiliary vector $\alpha$ is

$$|\alpha| = \frac{\sqrt{(p \cdot a)^2 + m^2}}{p^0},$$

(82)

so generically $A_1^2 = \alpha^2 \mathbb{1} < \mathbb{1}$, and indeed, the degree of violation decreases with the velocity of the observer. Nevertheless, it is always possible to compensate for a Wigner rotation by an appropriate choice of the operators \[1\].

Realistic situations involve wave packets. For example, a general spin-$\frac{1}{2}$ two-particle state may be written as

$$|\Upsilon_{12} \rangle = \sum_{\sigma_1, \sigma_2} \int d\mu(p_1)d\mu(p_2)g(\sigma_1\sigma_2, p_1, p_2)|p_1, \sigma_1; p_2, \sigma_2 \rangle.$$  

(83)

For particles with well defined momenta, $g$ sharply peaks at some values $p_{10}$, $p_{20}$. Again, a boost to any Lorentz frame $S'$ will result in a unitary $U(\Lambda) \otimes U(\Lambda)$, acting on each particle separately, thus preserving the entanglement. Nevertheless, since they can change entanglement between different degrees of freedom of a single particle, the spin-spin entanglement is frame-dependent as well. Having investigated the reduced density matrix for $|\Upsilon_{12} \rangle$ and made explicit calculations for the case where $g$ is a Gaussian, as in the Sec. 4.2 above, it is possible to show that if two particles are maximally entangled in a common (approximate) rest frame (Alice’s frame), then the concurrence, as seen by a Lorentz-boosted Bob, decreases when $v \to 1$. Of course, the inverse transformation from Bob to Alice will increase the concurrence \[40\]. Thus, we see that that spin-spin entanglement is not a Lorentz invariant quantity, exactly as spin entropy is not a Lorentz scalar. Relativistic properties of the polarization entanglement are even more interesting \[28\], since there is no frame where polarization and momentum are unentangled.

VII. THE OMISSIONS & PERSPECTIVES

Because of the lack of space I am only going to mention the various fascinating areas of the interplay between quantum information theory and relativistic physics. Quantum field theory provides us with new situations that should be investigated. For example, it is possible to ask all the usual questions about entanglement, distillability, etc and their invariance\[1, 41, 42, 43\]. So far we discussed only observers that move with constant velocity. An accelerated observer sees Unruh radiation. It leads to a host of interesting effects if we consider a teleportation between a stationary and accelerated observer \[44, 45, 46\]. Dynamical entanglement — the one appearing in the scattering processes or between the decay products also have been investigated \[24, 47, 48\].

Going to more exotic settings, I just mention that black hole physics, cosmology, loop quantum gravity and string theory provide extremely interesting scenarios where the questions of information can and should be asked\[1, 43, 44, 50, 51, 52, 53, 54\].

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