Group-valued invariant of knots in the full torus

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1 Introduction

Knot concordance plays a crucial role in the low dimensional topology [3].

The better understanding of knot sliceness and the concordance group of classical knots led to significant results [4, 5].

In the last decades, detecting new $\mathbb{Z}^\infty$ summands in the group $\Theta_2^3$ were considered as significant results [1].

To tackle these problems, a very elaborated techniques was developed. Say, Heegaard-Floer Homology originally used some count of holomorphic discs and $spin^c$-structures on manifolds.

The present paper deals with the sliceness problems for knots in the full torus $S^1 \times D^1$.

We propose a very elementary techniques which allows one to construct a lot of sliceness obstructions for such knots.

Our approach deals with group theoretical techniques; it is completely combinatorial, and the groups are very easy to deal with.

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1 Note that once forever we deal only with homologically trivial knots (having winding number 0 with respect to $S^1$.)
2 Definitions

Let \( K \) be a knot in the full torus \( S^1 \times \mathbb{R}^2 \). The full torus can be considered as a thickening of the cylinder \( S^1 \times \mathbb{R} \). Hence, the knot \( K \) is presented by its diagram \( D \) in the cylinder.

Consider the group

\[ G' = \{ a, b, b', B, B' \mid a^2 = 1, ab = (b')^{-1}a, aB = (B')^{-1}a, bB^{-1} = B'(b')^{-1}, b^{-1}B = (B')^{-1}b' \} . \tag{1} \]

There is a bijection \( \phi \) on \( G' \) given by the formula

\[ \phi(x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}) = x_2^{-\alpha_2}\cdots x_n^{-\alpha_n}x_1^{-\alpha_1} \]

where \( x_i = a, b, b', B \) or \( B' \), and \( \alpha_i = \pm 1 \). Let \( \hat{G}' = G'/\phi \).

With an oriented knot \( K \) in the full torus \( S^1 \times \mathbb{R}^2 \) having winding number 0 we associate an element \( f'(K) \in \hat{G}' \) as follows.

We fix a reference point \( P \) (not a crossing) on the Gauss diagram \( G(D) \) of the knot. We enumerate the chord endpoints in the core circle as they appear according to the orientation. The endpoints are called to be in odd position if the number of endpoints containing itself from the starting point of \( G(D) \) is odd. Otherwise the endpoint is called to be in even position.

We start walking from the reference point along the core circle of \( G(D) \) and write the letters according to the following rule:

1. We associate \( a \) with an even chord end;

2. We associate \( b, b', B \) or \( B' \) to chord ends in odd positions and we associate \( b^{-1}, (b')^{-1}, B^{-1} \) or \( (B')^{-1} \) to chord ends in even position;

3. We choose \( b, b^{-1}, B \) or \( B^{-1} \) if the chord considered is linked with evenly many even chords and \( b', (b')^{-1}, B', (B')^{-1} \), otherwise;

4. We associate \( b, b^{-1}, b' \) or \( (b')^{-1} \) to undercrossings, and \( B, B^{-1}, B', (B')^{-1} \) to overcrossings.

The product of the letters is a word \( f'(K) \) considered as an element of \( \hat{G}' \).
Theorem 1. The element $f'(K)$ is an invariant of oriented knots $K$ in the full torus.

The theorem is proved by standard checking Reidemeister moves. The bijection $\phi$ appears whenever we change the reference point.

It turns out that the invariant sees much more than just knot invariance:

Theorem 2. Let $K$ be a knot in the full torus. If $f'(K) = \tilde{1} \in \tilde{G}'$ then $K$ is not slice.

The theorem follows from a more general statement.

Let $G$ be a group and $\Phi$ be a family of bijections $\phi: G \to G$ such that $\phi(1) = 1$. For example,

$$
\Phi = \{Ad_g \mid g \in G\} \text{ where } Ad_g(h) = ghg^{-1}, h \in G.
$$

Let $\tilde{G} = G/\Phi$ be the set of equivalence classes of elements of $G$ modulo the relations $g \sim \phi(g)$, $g \in G$, $\phi \in \Phi$. We will denote the equivalence class of an element $g \in G$ by $\tilde{g}$. Note that by definition $\tilde{1} = \{1\}$.

Let $f$ be an invariant of oriented links with ordered components whose value on a link $L = K_1 \cup \cdots \cup K_n$ is an element $f(L) = (f_i(L))_{i=1}^n \in \prod_{i=1}^n \tilde{G} = \tilde{G}^m$. Assume $f$ obeys the following conditions:

1. For any permutation $\sigma \in \Sigma_n \sigma \circ f(L) = f \circ \sigma(L)$. The permutation interchanges the components of a link $L$ and interchanges the components of the value $f(L)$;

2. If the component $K_i$ is trivial then $f_i(L) = \tilde{1}$;

3. Let $L = K_1 \cup \cdots \cup K_n$, $L' = K_1 \cup \cdots \cup K_{n-2} \cup (K_{n-1} \# K_n)$ and $f(L) = (f_i(L))_{i=1}^n \in \tilde{G}^m$. Then $f(L') = (f_i(L'))_{i=1}^{n-1} \in \tilde{G}^{m-1}$ where $f_i(L') = f_i(L)$ for $1 \leq i \leq n - 2$ and $f(L')_{n-1} = g_{n-1}g_n$ for some representatives $g_{n-1}, g_n \in G$ of the classes $f_{n-1}(L)$ and $f_n(L)$ respectively.

Theorem 3. The map $f$ is a concordance invariant of links.

Corollary 1. If $K$ is a slice knot then $f(K) = \tilde{1}$.
2.1 Concordance of links in the thickened torus

Let $L = L_1 \cup L_2 \cup \cdots \cup L_n$ be a link in $S^3$ such that $L_1 \cup L_2$ forms the Hopf link. Then $S^3 \setminus (L_1 \cup L_2)$ is diffeomorphic to the thickening $M = T^2 \times (0,1)$ of the torus. Hence, the link $\bar{L} = L_3 \cup \cdots \cup L_n$ can be considered as a link in the thickened torus $M$.

**Theorem 4.** Two links $L = L_1 \cup L_2 \cup \cdots \cup L_n$ and $L' = L_1' \cup L_2' \cup \cdots \cup L_n'$ in $S^3$ such that $L_1 \cup L_2$ and $L_1' \cup L_2'$ are Hopf links, are concordant if and only if the links $\bar{L}$ and $\bar{L}'$ are concordant in $M$, i.e. there exists a smooth embedding $\bar{W} : \cup_{i=3}^n S^1 \times [0,1] \to M \times [0,1]$ such that $\bar{W} \cap M \times \{0\} = \partial_0 \bar{W} = \bar{L} \times \{0\}$ and $\bar{W} \cap M \times \{1\} = \partial_1 \bar{W} = \bar{L}' \times \{1\}$.

**Proof.** After some isotopy of $L'$ we can assume that $L_1 = L_1'$ and $L_2 = L_2'$.

Let $\bar{L}$ and $\bar{L}'$ are concordant in $M$, i.e. exists a set of cylinders $\bar{W} : \bar{W} : \bar{W} : \cup_{i=3}^n S^1 \times [0,1] \to M \times [0,1]$ such that $\partial_0 \bar{W} = \bar{L} \times \{0\}$ and $\bar{W} \cap M \times \{1\} = \partial_1 \bar{W} = \bar{L}' \times \{1\}$.

Assume that $\bar{L}_i \times \{0\}$ and $\bar{L}_i \times \{1\}$ belong to one component of $W$, $i = 1, 2$. Assume also that there exists an isotopy of $S^3 \times [0,1]$ which is fixed on the boundary and transforms the components of $W$ containing $\bar{L}_i \times \{0\}$, $i = 1, 2$, to $\bar{L}_i \times [0,1]$. Denote the $\bar{W} \cap M \times \{0\}$ to $\bar{W} \cap M \times \{1\}$ is a concordance of $\bar{L}$ and $\bar{L}'$ in $S^3 \setminus (L_1 \cup L_2) = M$.

\[ \square \]

3 Proof of Theorem 3

**Proof.** Consider first the case of knots. Let $K_0$ and $K_1$ be two concordant knots. Then there exists an annulus $W \subset \mathbb{R}^3 \times [0,1]$ such that $W \cap \mathbb{R}^3 \times \{i\} = K_i$, $i = 0, 1$.

Let $t$ be the coordinate on the interval $[0,1]$. We can assume that $t$ is a simple Morse function on $W$. Consider the Reeb graph $T$ of the function $t$ on $W$. That means the graph $T$ is the quotient space $W/ \sim$ where $x \sim y$ iff $x, y$ belong to the same connected component of the level $t^{-1}(c) \subset W$ for some $c \in [0,1]$. The graph $T$ is a trivalent graph and its vertices correspond to components of $t^{-1}(c)$ which contains singular points of the map $t$ except the vertices $v_0, v_1$ which corresponds to the knots $K_0$ and $K_1$. 
The graph $T$ is a tree because $W$ is an annulus. Then there is a unique path $P$ in $T$ with ends $v_0$ and $v_1$.

For any edge $e$ in the graph $T$ choose a point $z \in e$. Let $c = t(z)$, $L_c = t^{-1}(c) \subset W$ and $K_i$ be the component of the link $L_c$ that corresponds to $z$. Denote the element $f(L_c)_i \in \tilde{G}$ by $f(z)$. Since $f$ is an invariant, the value $f(z)$ is the same for all internal points $z \in e$. Hence, there is a well defined map $f : E(T) \to \tilde{G}$ where $E(T)$ is the set of edges of $T$.

We will prove that $f(e) = \tilde{1}$ for any $e \not\in P$ and $f(e) = f(K_0)$ for any $e \in P$.

Let $e \in E(T) \setminus P$. Let $T(e)$ be the component of the graph $T \setminus e$ which does not contain the path $P$ and $v(e)$ be the end of the edge $e$ in $T_e$. The height $h(e)$ of the edge $e$ is the number of the edges in $T(e)$.

Let us prove that $f(e)$ for any $e \in E(T) \setminus P$ by induction on the height $h(e)$.

Let $h(e) = 0$. Then $e$ is a leaf and it corresponds to a trivial component $K_i$. Then $f(e) = \tilde{1}$ by the second property of $f$.

Let $h(e) > 0$. Denote the other edges incident to $v(e)$ by $e'$ and $e''$. Then $h(e') < h(e)$ and $h(e'') < h(e)$, hence, $f(e') = f(e'') = \tilde{1}$. With respect to the function $t$ there are two cases.

1. Let $e'$ and $e''$ merge to the edge or the edge $e$ split into $e'$ and $e''$. Then the edges $e, e', e''$ correspond to link components $K, K'$ and $K''$ such that $K = K' \# K''$. By the third property of the invariant $f$, there exist elements $g \in f(e), g' \in f(e'), g'' \in f(e'')$ such that $g = g'g''$. Since $g' = g'' = 1$, we have $g = 1$, hence $f(e) = \tilde{1}$.

2. Let $e$ and $e'$ merge to the edge $e''$ or the edge $e''$ split into $e$ and $e'$. Then there exist $g \in f(e), g' \in f(e'), g'' \in f(e'')$ such that $g'' = gg'$. Since $g' = g'' = 1$ by the induction, $g = 1$ and $f(e) = \tilde{1}$.

Now, let us prove that $f(e) = f(K_0)$ for any $e \in P$. The path $P$ is a sequence of edges $e_0, e_1, \ldots, e_n$ where $v_0$ is incident to $e_0$ and $v_1$ is incident to $e_n$. We prove that $f(e_i) = f(K_0)$ by induction on $i$. By definition, $f(e_0) = f(v_0) = f(K_0)$.

Assume that $f(e_i) = f(K_0)$. There can be two cases. Let the edge $e_i$ split into the edge $e_{i+1}$ and some edge $e \in E(T) \setminus P$. By the third property of the invariant $f$, there exist elements $g_i \in f(e_i), g_{i+1} \in f(e_{i+1}), g \in f(e)$ such that $g_i = g_{i+1}g$. Since $e \in E(T) \setminus P$, $f(e) = \tilde{1}$ and $g = 1$. Then $g_i = g_{i+1}$ and $f(e_{i+1}) = f(e_i) = f(K_0)$.

If the edge $e_i$ merge with some edge $e \in E(T) \setminus P$ to the edge $e_{i+1}$ then $g_{i+1} = g_i g$ for some elements $g_i \in f(e_i), g_{i+1} \in f(e_{i+1}), g \in f(e)$. Since $e \in P$,
$E(T) \setminus P$, $f(e) = \tilde{1}$ and $g = 1$. Then $g_i = g_{i+1}$ and $f(e_{i+1}) = f(e_i) = f(K_0)$.

Thus, we have $f(K_1) = f(v_1) = f(e_n) = f(K_0)$. That means, $f$ is a concordance invariant.

The concordance invariance for links can be proved analogously.

**Example 1.** Consider the RGB-link from [5] with parameters $(0, 1, 0, 1, 2, 1)$, see Fig. 1.

The red and the green components of the link form the Hopf link. Then the green and the blue components can be considered as a 2-component link in the thickened cylinder (see Fig. 2 left), and the blue component can be considered as a knot $K$ in the thickened torus (see Fig. 2 right).

Let us calculate the invariant $f'$ of the knot $K$:

$$f'(K) = bb^{-1}B'(B')^{-1}BB^{-1}B'B'abB^{-1}bab'a'(b')^{-1}.$$ 

$$\cdot ab^{-1}aB^{-1}b'aaeB^{-1}Bb^{-1}ab^{-1} = B'B^{-1}B'B^{-1}b'b^{-1}b^{-1}b^{-1}.$$ 

Note that the group $G'$ contains a subgroup isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ which consists of the elements $(B'B^{-1})^k(b'b^{-1})^l$, $k, l \in \mathbb{Z}$. The bijection $\phi$ maps the element $(B'B^{-1})^k(b'b^{-1})^l$ to the element $(B'B^{-1})^{-k}(b'b^{-1})^{-l}$. This means that the invariant $f'(K)$ is not trivial in $\tilde{G}'$.

Thus, the knot $K$ in the thickened torus is not slice.
Figure 2: The RGB-link defines a GB-link in the cylinder and a B-knot in the torus

Aknowlegements

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