Alternating stationary iterative methods based on double splittings

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Abstract

Matrix double splitting iterations are simple in implementation while solving real non-singular (rectangular) linear systems. In this paper, we present two Alternating Double Splitting (ADS) schemes formulated by two double splittings and then alternating the respective iterations. The convergence conditions are then discussed along with comparative analysis. The set of double splittings used in each ADS schemes induce a preconditioned system which helps in showing the convergence of the ADS schemes. We also show that the classes of matrices for which one ADS scheme is better than the other, are mutually exclusive. Numerical experiments confirm the proposed ADS schemes are superior to the existing methods in actual implementation. Though the problems are considered in the rectangular matrix settings, the same problems are even new in non-singular matrix settings.

Keywords: Preconditioners, iterative methods, alternating scheme, double splitting, proper splitting, Moore-Penrose inverse, non-negativity, convergence theorem, comparison theorem

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1. Introduction

Most of the problems in scientific computations, solving a linear system is inevitable. Given a real matrix $A \in \mathbb{R}^{m \times n}$ and a real vector $b \in \mathbb{R}^m$, we consider the following linear system

$$Ax = b,$$

(1.1)

to find an approximate solution $x \in \mathbb{R}^n$. In practice, these systems are large and sparse. So, the iterative methods are more suitable than direct methods. The classical iterative methods are computationally expensive, which attracts the researcher to develop fast iterative solvers. In this context, we formulate two iterative schemes using the notion of proper splittings. A splitting $A = U - V$ of $A \in \mathbb{R}^{m \times n}$ is called a proper splitting [5] if $R(U) = R(A)$ and $N(U) = N(A)$, where $R(U)$ and $N(U)$ denote the range space and the null space of the matrix $U$, respectively. Different methods of construction of proper splittings are shown in Theorem 1, [6] and Theorem 3.3, [32]. In 2018, Mishra and Mishra [31] proved the uniqueness of a proper splitting under some sufficient conditions. In 1974, Berman and Plemmons [5] considered the following classical iterative scheme

$$x^{k+1} = Hx^k + c,$$

(1.2)

as an application of proper splittings where $H = U^\dagger V$ and $c = U^\dagger b$. Here $A^\dagger$ denotes the Moore-Penrose inverse of $A$, and is defined in the next section. It is well-known that an iteration scheme of the form (1.2) is convergent if the spectral radius of the iteration matrix $H$ is less than 1. Corollary 1, [5] assures the convergence of (1.2) to $A^\dagger b$ (the least-squares solution of minimum norm) for any initial vector $x^0$. Several sufficient/equivalent conditions for the convergence of (1.2) are reported in [1] [11], [12], [17] and [29] for different sub-classes of proper splittings. In 2014, Jena et al. [17] introduced two sub-classes of proper splittings known as proper regular splittings and proper weak regular splittings. A proper splitting $A = U - V$ is called as a proper regular splitting if $U^\dagger \geq 0$ and $V \geq 0$ (entry-wise comparison). A proper splitting $A = U - V$ is called as a proper weak regular splitting if $U^\dagger \geq 0$ and $U^\dagger V \geq 0$. Again in [17], the authors showed that the iterations scheme (1.2) converges for a proper weak regular splitting $A = U - V$ if $A^\dagger \geq 0$.

But, if a matrix has two splittings, then a splitting that yields the smaller spectral radius of the iteration matrix is preferred. In this direction, several comparison results are proved in the literature (see [17], [28], [30] and [31]). However, if a matrix has many splittings, then comparison process is time consuming. To avoid this, Mishra [30] in 2018 introduced the alternating iteration scheme using two proper splittings $A = U - V = M - N$, and is recalled below:

$$x^{k+1} = U^\dagger VM^\dagger Nx^k + U^\dagger (VM^\dagger + I)b,$$

(1.3)

motivated by the work of [4]. Convergence theory of (1.3) can be found in [30], [28], [15]. The idea of introducing alternating iteration scheme is inspired from the Alternating
Direction Implicit (ADI) method proposed by Peaceman and Rachford [35] in 1955 to solve higher dimensional Partial Differential Equations (PDEs). The notion of developing different computationally efficient methods like operator splitting method, parallel implementation of algorithms and alternating iteration schemes for linear systems are inspired from the ADI method. In 1959, Birkhoff and Verga [8] first reformulated the ADI scheme as an iteration scheme for solving linear systems derived from the discretization of PDEs, using matrix splittings. Later, the alternating scheme based algorithm is applied to a wide variety of problems, like variational problems [10], optimization problems and statistical learning algorithms [9, 38], alternating two-stage methods for consistent linear systems to obtain the parallel solution of Markov chains [27], saddle-point problems and also for other different type of matrices using Hermitian and Skew-Hermitian Splitting (HSS) [2, 3, 14]. Further, the alternating scheme for the block matrices has been proposed in [43], by using the notion of HSS method. Our aim is to establish the convergence theory for the alternative schemes applied to the block matrices (as shown in (2.2)) with some specific structure and properties such that the convergence is faster than the classical iteration schemes for solving the rectangular system (1.1).

At one hand, different authors in the literature focused on the problem of improving the convergence rate of the iteration scheme (1.2). On the other hand, expanding the convergence theory of the iteration scheme (1.2) for different types of matrix splittings of $A$ is another topic of research interest. In this direction, the notion of double splitting $A = P - R + S$ of a real non-singular matrix $A$ was first introduced by Woźnicki [44] in 1993. Such type of splitting leads to the iterative scheme

$$x^{k+1} = P^{-1}Rx^k - P^{-1}Sx^{k-1} + P^{-1}b, \quad k > 0$$

for solving the non-singular linear system (1.1), when $n = m$. Shen and Huang [36] and Miao et al. [26] studied the convergence and comparison of the above iterative scheme for monotone matrices ($A \in \mathbb{R}^{n \times n}$ is monotone [13] if and only if $A^{-1}$ exists and $A^{-1} \geq 0$). Moreover, several convergence and its comparison results exist in the literature for different types of double splittings (see [19], [20], [21], [25], [30], [37], [39], [41], [45]). In 2019, Li et al. [23] proposed an alternating scheme using double splittings of a matrix to find an approximate solution of a real non-singular linear system of equations.

The present article aims to revisit the theory alternating schemes using double splittings and to extend this idea to a rectangular matrix setting. In particular, we are interested in introducing another alternating scheme which we call as ADS stationary iteration scheme using double splittings like Li et al. [23] and then we show that our scheme performs better in certain cases where the scheme proposed in [23] fails. To this end, this article is organized in the following manner: Section 2 begins with the description
of some useful definitions and preliminary results. Section 3 proposes two ADS schemes and analyzes its convergence criteria. Section 4 shows the performance of the proposed iteration scheme by extensive numerical examples.

2. Prerequisites

In this section, additional notations, definitions and useful results related to non-negative matrices and double proper splittings are presented which are virtually used throughout this article. We denote \( \mathbb{R}^{m \times n} \) the set of all real rectangular matrices of order \( m \times n \) and \( \mathbb{R}^n \) is an \( n \)-dimensional Euclidean space. The rank of a matrix \( A \in \mathbb{R}^{m \times n} \) is denoted by \( r(A) \).

Suppose \( L \) and \( M \) are two complementary subspaces of \( \mathbb{R}^n \). Let \( \tilde{P}_{L,M} \) be the projection on \( L \) along \( M \). Hence \( \tilde{P}_{L,M} A = A \) if and only if \( R(A) \subseteq L \) and \( A \tilde{P}_{L,M} = A \) if and only if \( N(A) \supseteq M \). For \( A \in \mathbb{R}^{m \times n} \), the unique matrix \( X \in \mathbb{R}^{n \times m} \) satisfying the conditions \( AXA = A, XAX = X, (AX)^t = AX \) and \( (XA)^t = XA \) is called the Moore-Penrose inverse of \( A \), where \( A^t \) denotes the transpose of the matrix \( A \). The Moore-Penrose inverse always exists, and is denoted by \( A^\dagger \).

The matrix \( A \in \mathbb{R}^{m \times n} \) is called semi-monotone if \( A^\dagger \geq 0 \). The properties of \( A^\dagger \) which are frequently used in this article: \( R(A^\dagger) = R(A^t); N(A^\dagger) = N(A^t); AA^\dagger = \tilde{P}_{R(A)}; A^\dagger A = \tilde{P}_{R(A^t)} \).

2.1. Spectral radius and non-negative matrices

We denote the set of all eigenvalues of \( A \in \mathbb{R}^{n \times n} \) as \( \sigma(A) \). The spectral radius of \( A \in \mathbb{R}^{n \times n} \), denoted by \( \rho(A) \), is defined as \( \rho(A) = \max_{1 \leq j \leq n} |\lambda_j| \), where \( \lambda_j \in \sigma(A) \). A \( A \in \mathbb{R}^{m \times n} \) is called non-negative if \( A \geq 0 \). Let \( B, C \in \mathbb{R}^{m \times n} \). We write \( B \geq C \) if \( B - C \geq 0 \). The next results deal with non-negativity of a matrix and the spectral radius.

**Theorem 2.1** (Theorem 2.1.11, [7]). Let \( B \in \mathbb{R}^{n \times n}, B \geq 0, x \geq 0 (x \neq 0) \) and \( \alpha \) is a positive scalar. If \( \alpha x \leq Bx \), then \( \alpha \leq \rho(B) \).

**Theorem 2.2** (Lemma 2.2, [36]). Let \( A = \begin{pmatrix} B & C \\ I & 0 \end{pmatrix} \geq 0 \) and \( \rho(B+C) < 1 \). Then, \( \rho(A) < 1 \).

**Theorem 2.3** (Theorem 2.20, [40]). Let \( A \in \mathbb{R}^{n \times n} \) and \( A \geq 0 \). Then
(i) \( A \) has a non-negative real eigenvalue equal to its spectral radius.
(ii) there exists a non-negative eigenvector for its spectral radius.
2.2. Double proper splittings

Motivated by the standard iterative methods like Jacobi, Gauss-Seidel, SOR etc., Woźnicki [44] introduced double splitting theory for finding iteration solution of non-singular linear system $Ax = b$. Neumann [34] extended the non-singular case to singular linear system which he named as 3-part splitting. A double splitting $A = P - R + S$ of $A \in \mathbb{R}^{m \times n}$ is called double proper splitting if $R(P) = R(A)$ and $N(P) = N(A)$. Further, Jena et al. [17] introduced two subclasses of double proper splittings which are recalled below. A double proper splitting $A = P - R + S$ of $A \in \mathbb{R}^{m \times n}$ is called double proper regular splitting if $P^\dagger \geq 0$, $R \geq 0$ and $S \leq 0$. The next subclass contains the above one. A double proper splitting $A = P - R + S$ is called a double proper weak regular splitting [17] if $P^\dagger R \geq 0$ and $P^\dagger S \leq 0$. Mishra [29] again introduced another subclass which contains the above two subclasses. He named it as double proper nonnegative splitting. However, we call the same as double proper weak splitting as the conditions are weaker than the earlier two. A double proper splitting $A = P - R + S$ is called double proper weak splitting if $P^\dagger R \geq 0$ and $P^\dagger S \leq 0$. In the non-singular matrix setting, the above definitions coincide with double regular splitting (or regular double splitting [36]), double weak regular splitting (or weak regular double splitting [36]), and double weak splitting (or double nonnegative splitting [39]), respectively. Analogous to the non-singular case, the following iterative scheme spanned in three iterates (known as double iteration scheme) is proposed by Jena et al. [17] by the help of double proper splitting $A = P - R + S$:

$$x^{k+1} = P^\dagger Rx^k - P^\dagger Sx^{k-1} + P^\dagger b, \quad k > 0. \quad (2.1)$$

The equivalent block-matrix form [17] of (2.1) is

$$x^{k+1} = Tx^k + b, \quad (2.2)$$

where $x^{k+1} = \begin{pmatrix} x_{k+1} \\ x_k \end{pmatrix}$, $x_k = \begin{pmatrix} x^k \\ x^{k-1} \end{pmatrix}$, $T = \begin{pmatrix} P^\dagger R & -P^\dagger S \\ I & 0 \end{pmatrix}$, $b = \begin{pmatrix} P^\dagger b \\ 0 \end{pmatrix}$ and $I$ denotes the identity matrix of order $n$. Then, the iteration scheme (2.2) converges to $A^\dagger b$ of (1.1) if $\rho(T) < 1$. Here the spectral radius of block matrix $T$ is the spectral radius of the full matrix $T$. Rest of the manuscript, we will write $\rho(T)$ instead of $\rho(T)$ for any block matrix $T$. The next two results present the convergence criteria for double proper regular (or weak regular) splittings and double proper weak splittings. But, interested reader may refer [17], [29], [1] and [18] for more detailed convergence theory of (2.2).

**Theorem 2.4** (Theorem 3.6, [17]). Let $A^\dagger \geq 0$. If $A = P - R + S$ be a double proper regular (or weak regular) splitting of $A \in \mathbb{R}^{m \times n}$, then $\rho(T) < 1$.

**Theorem 2.5** (Theorem 4.5, [29]). Let $A^\dagger P \geq 0$. If $A = P - R + S$ be a double proper weak splitting of $A \in \mathbb{R}^{m \times n}$, then $\rho(T) < 1$. 

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3. Main results

3.1. Formulation of Alternating Double Splitting (ADS) schemes

Motivated by the work of Li et al. [23], where the authors introduced Alternating Double Splitting (ADS) scheme using double splittings to solve a non-singular linear system, and the work of Jena et al. [17], we consider two double iterative schemes with respect to two double proper splittings of \(A \in \mathbb{R}^{m \times n}\), respectively as:

\[
A = P_1^\dagger R_1 + S_1 = P_2^\dagger R_2 + S_2
\]

are

\[
x^{k+1/2} = P_1^\dagger R_1 x^k - P_1^\dagger S_1 x^{k-1/2} + P_1^\dagger b, \quad (3.1)
\]

and

\[
x^{k+1} = P_2^\dagger R_2 x^{k+1/2} - P_2^\dagger S_2 x^k + P_2^\dagger b. \quad (3.2)
\]

The corresponding block iterative schemes can be written in two different ways as mentioned below for \(i = 1, 2\):

\[
T_i = \begin{pmatrix}
P_1^\dagger R_i & -P_1^\dagger S_i \\
I & 0
\end{pmatrix}, \quad G_i = \begin{pmatrix}
I & 0 \\
P_1^\dagger R_i & -P_1^\dagger S_i
\end{pmatrix} \quad \text{and} \quad H_i = \begin{pmatrix}
P_1^\dagger R_i & -P_1^\dagger S_i \\
0 & I
\end{pmatrix}.
\]

From each pair of block forms, we are going to formulate next a ADS scheme.

3.1.1. TG-ADS scheme

\[
\begin{align*}
x^{k+1/2} &= \begin{pmatrix} x^{k+1/2} \\ x^k \end{pmatrix} = \begin{pmatrix} I & 0 \\ P_1^\dagger R_1 & -P_1^\dagger S_1 \end{pmatrix} \begin{pmatrix} x^k \\ x^{k-1/2} \end{pmatrix} + \begin{pmatrix} 0 \\ P_1^\dagger b \end{pmatrix} \\
&= G_1 x^k + b_1
\end{align*}
\]

\[
\begin{align*}
x^{k+1} &= \begin{pmatrix} x^{k+1} \\ x^{k+1/2} \end{pmatrix} = \begin{pmatrix} P_2^\dagger R_2 & -P_2^\dagger S_2 \\ I & 0 \end{pmatrix} \begin{pmatrix} x^{k+1/2} \\ x^k \end{pmatrix} + \begin{pmatrix} P_2^\dagger b \\ 0 \end{pmatrix} \\
&= T_2 x^{k+1/2} + b_2.
\end{align*}
\]

(3.3)

To do the convergence analysis of (3.3), we next formulate a single-step double iteration scheme by composing the half-step double iteration schemes in (3.3),

\[
x^{k+1} = T_2 G_1 x^k + T_2 b_1 + b_2 = W_{12} x^k + b_3.
\]

(3.4)

We call the above scheme as TG-ADS scheme. The iteration matrix \(W_{12}\) and the vector \(b_3\) of the TG-ADS scheme are as follows:

\[
W_{12} = \begin{pmatrix}
P_2^\dagger R_2 - P_1^\dagger S_2 P_1^\dagger R_1 & P_2^\dagger S_2 P_1^\dagger S_1 \\
I & 0
\end{pmatrix} \quad \text{and} \quad b_3 = \begin{pmatrix}
P_2^\dagger (I - S_2 P_1^\dagger) b \\
0
\end{pmatrix}.
\]
3.1.2. HT-ADS scheme

Alike the half-step double iteration schemes used in TG-ADS scheme, we introduce a new ADS scheme by defining another pair of half-step double iteration schemes.

$$
\begin{align*}
\mathbf{x}^{k+1/2} &= \left( \begin{array}{c}
x^{k+1/2} \\
x^k
\end{array} \right) = \left( \begin{array}{c}
P_1^1 R_1 & -P_1^1 S_1 \\
I & 0
\end{array} \right) \left( \begin{array}{c}
x^k \\
x^{k-1/2}
\end{array} \right) + \left( \begin{array}{c}
P_1^1 b \\
0
\end{array} \right) \\
= \mathbf{T}_1 \mathbf{x}^k + \mathbf{b}_1
\end{align*}
$$

$$
\begin{align*}
\mathbf{x}^{k+1} &= \left( \begin{array}{c}
x^{k+1} \\
x^{k+1/2}
\end{array} \right) = \left( \begin{array}{c}
P_2^1 R_2 & -P_2^1 S_2 \\
0 & I
\end{array} \right) \left( \begin{array}{c}
x^{k+1/2} \\
x^k
\end{array} \right) + \left( \begin{array}{c}
P_2^1 b \\
0
\end{array} \right) \\
= \mathbf{H}_2 \mathbf{x}^{k+1/2} + \mathbf{b}_2.
\end{align*}
$$

The corresponding single-step double iteration scheme is derived as follows:

$$
\begin{align*}
\mathbf{x}^{k+1} &= \mathbf{H}_2 \mathbf{T}_1 \mathbf{x}^k + \mathbf{H}_2 \mathbf{b}_1 + \mathbf{b}_2 = \mathbf{W}_{12} \mathbf{x}^k + \mathbf{b}_4. \tag{3.6}
\end{align*}
$$

This scheme is called as _HT-ADS scheme_. The iteration matrix $\mathbf{W}_{12}$ and the vector $\mathbf{b}_4$ of the HT-ADS scheme are

$$
\begin{align*}
\mathbf{W}_{12} &= \left( \begin{array}{cc}
P_2^1 R_2 P_1^1 R_1 - P_2^1 S_2 & -P_2^1 R_2 P_1^1 S_1 \\
I & 0
\end{array} \right) \quad \text{and} \quad \mathbf{b}_4 = \left( \begin{array}{c}
P_2^1 (R_2 P_1^1 + I) b \\
0
\end{array} \right),
\end{align*}
$$

respectively. The iteration schemes (3.4) and (3.6) are called as _ADS alternating iteration schemes_ (ADS schemes) in its block form.

**Remark 3.1.** HT-ADS scheme in its block form yields a three-term recurrence scheme

$$
\begin{align*}
\mathbf{x}^{k+1} &= (P_2^1 R_2 P_1^1 R_1 - P_2^1 S_2) \mathbf{x}^k - P_2^1 R_2 P_1^1 S_1 \mathbf{x}^{k-1} + P_2^1 R_2 P_1^1 b + P_2^1 b, \tag{3.7}
\end{align*}
$$

which is also formed by eliminating $\mathbf{x}^{k+1/2}$ from (3.2). However, one can verify that TG-ADS scheme in its block form which extends the Alternating Double Splitting method proposed by Li et al. [23] does not coincide with (3.7).

The iteration schemes (3.4) and (3.6) converge for any initial guess $\mathbf{x}^0$ to $A^\dagger \mathbf{b}$ if and only if $\rho(\mathbf{W}_{12}) < 1$ and $\rho(\mathbf{W}_{12}) < 1$, respectively [17]. The next section provides different sufficient conditions for the convergence of the above types of ADS schemes.

### 3.2. Convergence analysis

We show the convergence of each ADS scheme by considering the spectral radius of another iteration matrix for solving a new preconditioned system as both the iteration matrices have the same spectral radius. This is shown next.
3.2.1. TG-ADS scheme

Let \( A = P_1 - R_1 + S_1 = P_2 - R_2 + S_2 \) be two double proper splittings of \( A \in \mathbb{R}^{n \times n} \) with \( N(S_2) \subseteq N(P_2), R(S_2) \subseteq R(P_2) \) and \( 1 \notin \sigma(S_2P_1^\dagger) \). So, \( I - S_2P_1^\dagger \) is non-singular. Let us consider the preconditioned linear system

\[
\hat{A}x = \hat{b}, \tag{3.8}
\]

where \( \hat{A} = (I - S_2P_1^\dagger)A \) and \( \hat{b} = (I - S_2P_1^\dagger)b \). Simplifying \( \hat{A} \), we have

\[
\hat{A} = (I - S_2P_1^\dagger)A = A - S_2P_1^\dagger A
\]

\[
= P_2 - R_2 + S_2 - S_2P_1^\dagger(P_1 - R_1 + S_1)
\]

\[
= P_2 - R_2 + S_2 - S_2P_1^\dagger R_1 - S_2P_1^\dagger S_1
\]

\[
= P_2 - (R_2 - S_2P_1^\dagger R_1) + (-S_2P_1^\dagger S_1)
\]

\[
= \hat{P} - \hat{R} + \hat{S}
\]

is a double splitting of \( \hat{A} \). For convenience, we denote \( \hat{P} = P_2, \hat{R} = R_2 - S_2P_1^\dagger R_1 \) and \( \hat{S} = -S_2P_1^\dagger S_1 \). Next, we have to show that \( \hat{A} = \hat{P} - \hat{R} + \hat{S} \) is a double proper splitting of \( \hat{A} \). Let \( x \in N(\hat{A}) \). This implies \( \hat{A}x = 0 \), i.e., \( (I - S_2P_1^\dagger)Ax = 0 \). Pre-multiplying \( (I - S_2P_1^\dagger)^{-1} \) to \( (I - S_2P_1^\dagger)Ax = 0 \) yields \( Ax = 0 \). So \( N(\hat{A}) \subseteq N(\hat{P}) \). Again, suppose \( x \in N(\hat{P}) = N(P_2) = N(A) \). This gives \( Ax = 0 \) which yields \( (I - S_2P_1^\dagger)^{-1}\hat{A}x = 0 \). So, we get \( \hat{A}x = 0 \) which implies \( N(\hat{P}) \subseteq N(\hat{A}) \). Hence \( N(\hat{A}) = N(\hat{P}) \). Next, to show that \( R(\hat{A}) = R(\hat{P}) \). From (3.8), we obtain \( \hat{A} = A - S_2P_1^\dagger A = A - P_2P_1^\dagger S_2P_1^\dagger A = A - AA^\dagger S_2P_1^\dagger A = A(I - A^\dagger S_2P_1^\dagger A) \). This gives \( R(\hat{A}) \subseteq R(A) = R(P_2) = R(\hat{P}) \). Also, \( r(\hat{A}) = r(\hat{P}) \). Hence \( R(\hat{A}) = R(\hat{P}) \). Thus, \( \hat{A} = \hat{P} - \hat{R} + \hat{S} \) is a double proper splitting of \( \hat{A} \) and the corresponding double iterative scheme for the preconditioned system (3.8) can be written as

\[
x^{k+1} = \hat{P}^\dagger \hat{R}x^k - \hat{P}^\dagger \hat{S}x^{k-1} + \hat{P}^\dagger \hat{b}, \quad k > 0 \tag{3.9}
\]

i.e.,

\[
x^{k+1} = \begin{pmatrix} \hat{P}^\dagger \hat{R} \\ \hat{I} \end{pmatrix} x^k - \begin{pmatrix} \hat{P}^\dagger \hat{S} \\ \hat{0} \end{pmatrix} x^{k-1} + \begin{pmatrix} \hat{P}^\dagger \hat{b} \\ \hat{0} \end{pmatrix}.
\]

The iteration matrix is

\[
\hat{T} = \begin{pmatrix} \hat{P}^\dagger \hat{R} & -\hat{P}^\dagger \hat{S} \\ \hat{I} & \hat{0} \end{pmatrix} = \begin{pmatrix} P_2^\dagger R_2 - P_2^\dagger S_2P_1^\dagger R_1 & P_2^\dagger S_2P_1^\dagger S_1 \\ \hat{I} & \hat{0} \end{pmatrix} = \mathbf{W}_{12}
\]

and

\[
\begin{pmatrix} \hat{P}^\dagger \hat{b} \\ \hat{0} \end{pmatrix} = \begin{pmatrix} P_2^\dagger(I - S_2P_1^\dagger) b \\ \hat{0} \end{pmatrix} = \begin{pmatrix} P_2^\dagger R_2 - P_2^\dagger S_2 \\ \hat{I} \end{pmatrix} \begin{pmatrix} 0 \\ P_1^\dagger b \end{pmatrix} + \begin{pmatrix} P_2^\dagger b \\ \hat{0} \end{pmatrix} = \mathbf{T}_2b_1 + b_2.
\]
We remark that if $A = P_1 - R_1 + S_1 = P_2 - R_2 + S_2$ are two double proper regular (or weak regular or weak) splittings of $A \in \mathbb{R}^{m \times n}$ with $N(S_2) \supseteq N(P_2)$, $R(S_2) \subseteq R(P_2)$ and $1 \notin \sigma(S_2 P_1)$, then $\hat{A} = \hat{P} - \hat{R} + \hat{S}$ is also a double proper regular (weak regular or weak) splittings of $A \in \mathbb{R}^{m \times n}$. Hence, an immediate consequence of Theorem 2.4 which generalizes Theorem 2.6, \[23\] is as follows.

**Theorem 3.1.** Let $A = P_1 - R_1 + S_1 = P_2 - R_2 + S_2$ be two double proper regular (weak regular) splittings of a semi-monotone matrix $A \in \mathbb{R}^{m \times n}$. If $N(S_2) \supseteq N(P_2)$, $R(S_2) \subseteq R(P_2)$, $1 \notin \sigma(S_2 P_1)$ and $\hat{A}^\dagger \geq 0$, then $\rho(W_{12}) < 1$.

The next example shows that the converse of the above theorem is not true.

**Example 3.1.** Let $A = \begin{bmatrix} \frac{3}{27} & -\frac{5}{54} \\ -\frac{5}{54} & \frac{3}{27} \end{bmatrix} = P_1 - R_1 + S_1 = P_2 - R_2 + S_2$ be two double regular splittings of a monotone matrix $A$, where

\[
P_1 = \begin{bmatrix} \frac{4}{27} & -\frac{27}{2} \\ -\frac{27}{2} & \frac{4}{27} \end{bmatrix}, \quad R_1 = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{27} \end{bmatrix}, \quad S_1 = \begin{bmatrix} 0 & -\frac{1}{54} \\ -\frac{1}{54} & 0 \end{bmatrix},
\]
\[
P_2 = \begin{bmatrix} \frac{27}{2} & \frac{5}{27} \\ \frac{5}{27} & \frac{27}{2} \end{bmatrix}, \quad R_2 = \begin{bmatrix} 0 & 0 \\ 0 & \frac{3}{27} \end{bmatrix}, \quad S_2 = \begin{bmatrix} 0 & -\frac{2}{9} \\ -\frac{2}{9} & 0 \end{bmatrix}.
\]

Here $S_2 P_1^{-1} = \begin{bmatrix} -1 & -2 \\ -2 & -1 \end{bmatrix}$. We have $\rho(W_{12}) = 0.8306 < 1$ but $1 \in \sigma(S_2 P_1^{-1})$.

Next result discusses the case when $A$ has two double proper weak splittings. This extends Theorem 2.4, \[23\] to rectangular matrices.

**Theorem 3.2.** Let $A = P_1 - R_1 + S_1 = P_2 - R_2 + S_2$ be two double proper weak splitting of $A \in \mathbb{R}^{m \times n}$. If $\hat{A}^\dagger \hat{P} \geq 0$, $N(S_2) \supseteq N(P_2)$, $R(S_2) \subseteq R(P_2)$ and $1 \notin \sigma(S_2 P_1)$, then $\rho(W_{12}) < 1$.

### 3.2.2. HT-ADS scheme

Let $A = P_1 - R_1 + S_1 = P_2 - R_2 + S_2$ be two double proper regular (weak regular) splittings of $A \in \mathbb{R}^{m \times n}$ such that $N(R_2) \supseteq N(P_2)$, $R(R_2) \subseteq R(P_2)$ and $-1 \notin \sigma(R_2 P_1^\dagger)$. We then get another preconditioned linear system

$$
\hat{A} x = \hat{b},
$$

where $\hat{A} = (I + R_2 P_1^\dagger)A$. Proceeding similarly as in the convergence analysis discussed in the subsection 3.2.1, we thus have $\hat{A} = \hat{P} - \hat{R} + \hat{S}$ is a double proper regular (weak regular) splitting of $\hat{A}$ where $\hat{P} = P_2$, $\hat{R} = R_2 P_1^\dagger R_1 - S_2$ and $\hat{S} = R_2 P_1^\dagger S_1$. The iteration
matrix of the double iteration scheme \((3.9)\) with respect to the double proper splitting \(\hat{A} = \hat{P} - \hat{R} + \hat{S}\) is

\[
\hat{\tau} = \begin{pmatrix}
\hat{P}^\dagger \hat{R} & -\hat{P}^\dagger \hat{S} \\
I & 0
\end{pmatrix}
= \begin{pmatrix}
P_1^\dagger R_2 P_1^\dagger R_1 - P_2^\dagger S_2 & -P_2^\dagger R_2 P_1^\dagger S_1 \\
I & 0
\end{pmatrix} = \mathcal{W}_{12}.
\]

We therefore have the following convergence theorem for the HT-ADS scheme by using Theorem 2.4.

**Theorem 3.3.** If \(A = P_1 - R_1 + S_1 = P_2 - R_2 + S_2\) be two double proper regular (weak regular) splittings such that \(N(R_2) \supseteq N(P_2), R(R_2) \subseteq R(P_2), -1 \notin \sigma(R_2 P_1^\dagger)\) and \(\hat{A}^\dagger \geq 0\), then \(\rho(\mathcal{W}_{12}) < 1\).

Note that the condition \(\hat{A}^\dagger \geq 0\) will be replaced by \(\hat{A}^\dagger \hat{P} \geq 0\) in the case of \(A\) having double proper weak splittings.

### 3.3. Comparison Results: TG-ADS scheme

Convergence theory of ADS schemes will be meaningful if the proposed ADS schemes \((3.4)\) and \((3.6)\) converge faster than the two individual double iteration schemes of the form \((2.1)\). This is discussed first in Theorem 3.4 before moving into other problems. In this context, the following question arises now which is highly useful in practice, i.e., how to choose the second double splitting \(A = P_2 - R_2 + S_2\) if \(A = P_1 - R_1 + S_1\) is given such that the TG-ADS scheme converges faster than the double iteration scheme arising out of the splitting \(A = P_1 - R_1 + S_1\). This is addressed in the next result.

**Theorem 3.4.** Let \(A = P_1 - R_1 + S_1\) be a double proper weak regular splitting and \(A = P_2 - R_2 + S_2\) be a double proper regular splitting of a semi-monotone matrix \(A \in \mathbb{R}^{m \times n}\). Suppose that \(N(S_2) \supseteq N(P_2), R(S_2) \subseteq R(P_2)\), \(-1 \notin \sigma(S_2 P_1^\dagger)\) and \(\hat{A}^\dagger \geq 0\). If \(P_1^\dagger R_1 \geq P_2^\dagger R_2\) and \(P_2^\dagger S_2 \geq P_1^\dagger S_1\), then \(\rho(\mathcal{W}_{12}) \leq \rho(T_1) < 1\).

**Proof.** By Theorem 2.3 and Theorem 3.1 we have \(\rho(T_1) < 1\) and \(\rho(\mathcal{W}_{12}) < 1\), respectively. Case (i): \(\rho(\mathcal{W}_{12}) = 0\). The proof is obvious.

Case (ii): \(\rho(\mathcal{W}_{12}) \neq 0\). Since \(\mathcal{W}_{12} \geq 0\), there exists a non-negative eigenvector \(\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\) such that \(\mathcal{W}_{12} \mathbf{x} = \rho(\mathcal{W}_{12}) \mathbf{x}\) by Theorem 2.3. This gives

\[
(P_2^\dagger R_2 - P_2^\dagger S_2 P_1^\dagger R_1) x_1 + P_2^\dagger S_2 P_1^\dagger S_1 x_2 = \rho(\mathcal{W}_{12}) x_1
\]

\[
x_1 = \rho(\mathcal{W}_{12}) x_2.
\]

We next have

\[
(P_2^\dagger P_2 R_2 - P_2^\dagger P_2 S_2 P_1^\dagger R_1) x_1 + \frac{1}{\rho(\mathcal{W}_{12})} P_2^\dagger P_2 S_2 P_1^\dagger S_1 x_1 = \rho(\mathcal{W}_{12}) P_2 x_1
\]
by pre-multiplying $P_2$ in (3.11). Also, $P_2P_2^\dagger R_2 = R_2$ as $R(R_2) \subseteq R(P_2)$ which follows from $R(S_2) \subseteq R(P_2)$ and $R(A) = R(P_2)$, and $P_2P_2^\dagger S_2 = S_2$ as $R(S_2) \subseteq R(P_2)$. Using (3.12), it then yields

$$
\rho(W_{12})^2 P_2 x_1 = \rho(W_{12})(R_2 - S_2 P_1^\dagger R_1)x_1 + S_2 P_1^\dagger S_1 x_1,
$$

(3.13)

which results $P_2 x_1 \geq 0$. Also, we have

$$
0 = \rho(W_{12})^2 P_2 x_1 - \rho(W_{12})(R_2 - S_2 P_1^\dagger R_1)x_1 - S_2 P_1^\dagger S_1 x_1 \\
\leq \rho(W_{12}) P_2 x_1 - \rho(W_{12})(R_2 - S_2 P_1^\dagger R_1)x_1 - \rho(W_{12}) S_2 P_1^\dagger S_1 x_1 \\
= \rho(W_{12}) \left( P_2 - R_2 + S_2 P_1^\dagger (R_1 - S_1) \right) x_1 \\
= \rho(W_{12}) \left( A - S_2 + S_2 P_1^\dagger (P_1 - A) \right) x_1 \\
= \rho(W_{12}) \left( A - S_2 + S_2 P_1^\dagger P_2 - S_2 P_1^\dagger A \right) x_1 \quad \text{by } P_1^\dagger P_1 = P_2^\dagger P_2 \text{ as } R(P_1) = R(A) = R(P_2) \\
= \rho(W_{12}) \left( A - S_2 P_1^\dagger A \right) x_1 \quad \because S_2 P_1^\dagger P_2 = S_2 \text{ as } N(S_2) \supseteq N(P_2) \\
= \rho(W_{12}) \left( I + (-S_2 P_1^\dagger) \right) Ax_1.
$$

Thus, $Ax_1 \geq 0$. Now

$$
\mathbf{T}_1 \mathbf{x} - \rho(W_{12}) \mathbf{x} = \begin{pmatrix} P_1^\dagger R_1 x_1 - P_1^\dagger S_1 x_2 - \rho(W_{12}) x_1 \\ x_1 - \rho(W_{12}) x_2 \end{pmatrix} \\
= \begin{pmatrix} \Delta \\ 0 \end{pmatrix},
$$

where

$$
\Delta = (P_1^\dagger R_1 - P_2^\dagger R_2)x_1 + P_2^\dagger S_2 P_1^\dagger R_1 x_1 - \frac{1}{\rho(W_{12})} P_1^\dagger S_1 x_1 - \frac{1}{\rho(W_{12})} P_2^\dagger S_2 P_1^\dagger S_1 x_1 \\
\geq P_2^\dagger S_2 P_1^\dagger R_1 x_1 - \frac{1}{\rho(W_{12})} P_1^\dagger S_1 x_1 - \frac{1}{\rho(W_{12})} P_2^\dagger S_2 P_1^\dagger S_1 x_1 \\
\geq \frac{1}{\rho(W_{12})} \left( P_2^\dagger S_2 P_1^\dagger (R_1 - S_1) \right) x_1 - \frac{1}{\rho(W_{12})} P_1^\dagger S_1 x_1 \\
= \frac{1}{\rho(W_{12})} \left( P_2^\dagger S_2 P_1^\dagger (P_1 - A) - P_1^\dagger S_1 \right) x_1 \\
= \frac{1}{\rho(W_{12})} (P_2^\dagger S_2 - P_1^\dagger S_1)x_1 + \frac{1}{\rho(W_{12})} (-P_2^\dagger S_2) P_1^\dagger A x_1 \geq 0.
$$

Therefore, $\mathbf{T}_1 \mathbf{x} - \rho(W_{12}) \mathbf{x} \geq 0$. By Theorem 2.4 we thus have $\rho(W_{12}) \leq \rho(T_1) < 1$. 

Next result shows that the ADS scheme performs better than the other double iteration scheme formed by $A = P_2 - R_2 + S_2$. This extends Theorem 3.5 to rectangular matrix case and can be proved proceeding similarly as in the non-singular case.
Theorem 3.5. Let \( A = P_1 - R_1 + S_1 \) be a double proper weak regular splitting and \( A = P_2 - R_2 + S_2 \) be a double proper regular splitting of a semi-monotone matrix \( A \in \mathbb{R}^{m \times n} \). If \( N(S_2) \supseteq N(P_2), R(S_2) \subseteq R(P_2), 1 \notin \sigma(S_2P_1^t) \) and \( \hat{A} \geq 0 \), then \( \rho(W_{12}) \leq \rho(T_2) < 1 \).

The importance of the TG-ADS scheme is discussed in the next result which is a combination of Theorem 3.4 and Theorem 3.5. The result says that the proposed ADS scheme (3.4) converges faster than usual double iteration scheme (2.2) under suitable assumptions.

Theorem 3.6. Let \( A = P_1 - R_1 + S_1 \) be a double proper weak regular splitting and \( A = P_2 - R_2 + S_2 \) be a double proper regular splitting of a semi-monotone matrix \( A \in \mathbb{R}^{m \times n} \). If \( N(S_2) \supseteq N(P_2), R(S_2) \subseteq R(P_2), 1 \notin \sigma(S_2P_1^t), \hat{A} \geq 0, P_1^tR_1 \geq P_2^tR_2 \) and \( P_2^tS_2 \geq P_1^tS_1 \), then
\[
\rho(W_{12}) \leq \min\{\rho(T_1), \rho(T_2)\} < 1.
\]

The corollary obtained below is even new in the non-singular matrix setting.

Corollary 3.7. Let \( A = P_1 - R_1 + S_1 \) be a double weak regular splitting and \( A = P_2 - R_2 + S_2 \) be a double regular splitting of a monotone matrix \( A \in \mathbb{R}^{n \times n} \). If \( 1 \notin \sigma(S_2P_1^{-1}), \hat{A}^{-1} \geq 0, P_1^{-1}R_1 \geq P_2^{-1}R_2 \) and \( P_2^{-1}S_2 \geq P_1^{-1}S_1 \), then
\[
\rho(W_{12}) \leq \min\{\rho(T_1), \rho(T_2)\} < 1.
\]

The next result provides different sufficient conditions to draw the conclusion of Theorem 3.4.

Theorem 3.8. Let \( A = P_1 - R_1 + S_1 \) be a double proper weak regular splitting and \( A = P_2 - R_2 - S_2 \) be a double proper regular splitting of a semi-monotone matrix \( A \in \mathbb{R}^{m \times n} \). Suppose that \( N(S_2) \supseteq N(P_2), R(S_2) \subseteq R(P_2), 1 \notin \sigma(S_2P_1^t) \) and \( \hat{A} \geq 0 \). If \( P_1^t \leq P_2^t \) and \( P_1^tR_1 \leq P_2^tR_2 \), then \( \rho(W_{12}) \leq \rho(T_1) < 1 \).

Proof. By Theorem 2.4 and Theorem 3.1, we have \( \rho(T_1) < 1 \) and \( \rho(W_{12}) < 1 \), respectively. Case (i): \( \rho(W_{12}) = 0 \). The proof is obvious.

Case (ii): \( \rho(W_{12}) \neq 0 \). Since \( W_{12} \geq 0 \), there exists a non-negative eigenvector \( \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \) such that \( W_{12}\mathbf{x} = \rho(W_{12})\mathbf{x} \), i.e., \( \hat{T}\mathbf{x} = \rho(W_{12})\mathbf{x} \) by Theorem 2.3. This gives
\[
\hat{P}_1^t\hat{R}_1\mathbf{x}_1 - \hat{P}_1^t\hat{S}_2\mathbf{x}_2 = \rho(W_{12})\mathbf{x}_1, \quad \mathbf{x}_1 = \rho(W_{12})\mathbf{x}_2.
\]

Now
\[
T_1\mathbf{x} - \rho(W_{12})\mathbf{x} = \begin{pmatrix} P_1^tR_1\mathbf{x}_1 - P_1^tS_1\mathbf{x}_2 - \rho(W_{12})\mathbf{x}_1 \\ x_1 - \rho(W_{12})x_2 \end{pmatrix}.
\]
The condition $P_1^t R_1 \leq P_2^t R_2$ yields

$$P_1^t R_1 - P_2^t R_2 + P_2^t S_2 P_1^t R_1 \leq 0 \quad \text{i.e.,} \quad P_1^t R_1 - P_2^t (R_2 - S_2 P_1^t R_1) \leq 0.$$ 

Hence $P_1^t R_1 - \hat{P}^t \hat{R} \leq 0$. Now

$$P_1^t R_1 x_1 - P_1^t S_1 x_2 - \rho(W_{12})x_1 - \frac{1}{\rho(W_{12})}(P_1^t R_1 - P_1^t S_1)x_1 - \frac{1}{\rho(W_{12})}(\hat{P}^t \hat{S} - \hat{P}^t \hat{R})x_1$$

$$= P_1^t R_1 x_1 - \frac{1}{\rho(W_{12})} P_1^t S_1 x_2 - \frac{1}{\rho(W_{12})} \hat{P}^t \hat{S} x_1$$

$$= \left(1 - \frac{1}{\rho(W_{12})}\right) P_1^t R_1 x_1 + \frac{1}{\rho(W_{12})} \hat{P}^t \hat{S} x_1$$

$$= \left(1 - \frac{1}{\rho(W_{12})}\right) (P_1^t R_1 - \hat{P}^t \hat{R}) x_1 \geq 0.$$ 

Therefore,

$$P_1^t R_1 x_1 - P_1^t S_1 x_2 - \rho(W_{12})x_1 \geq \frac{1}{\rho(W_{12})}(P_1^t R_1 - P_1^t S_1 + \hat{P}^t \hat{S} - \hat{P}^t \hat{R})x_1$$

$$= \frac{1}{\rho(W_{12})} \left(P_1^t (R_1 - S_1) + \hat{P}^t (\hat{S} - \hat{R})\right) x_1$$

$$= \frac{1}{\rho(W_{12})} \left(P_1^t (P_1 - A) + \hat{P}^t (\hat{A} - \hat{P})\right) x_1$$

$$= \frac{1}{\rho(W_{12})} (P_2^t P_1 - \hat{P}^t \hat{A} - P_1^t A) x_1$$

$$= \frac{1}{\rho(W_{12})} (P_2^t A - P_1^t S_2 P_1^t A - P_1^t A) x_1$$

$$= \frac{1}{\rho(W_{12})} (P_2^t - P_1^t) A x_1 + \frac{1}{\rho(W_{12})} (P_2^t S_2 P_1^t A) x_1 \geq 0,$$

as $A x_1 \geq 0$ can be shown as in the previous proof. We thus have $T_1 x - \rho(W_{12}) x \geq 0$ resulting $\rho(W_{12}) \leq \rho(T_1) < 1$ by Theorem 2.4. 

**Corollary 3.9.** Let $A = P_1 - R_1 + S_1$ be a double weak regular splitting and $A = P_2 - R_2 - S_2$ be a double regular splitting of a monotone matrix $A \in \mathbb{R}^{n \times n}$. Suppose that $1 \notin \sigma(S_2 P_1^{-1})$ and $\hat{A}^{-1} \geq 0$. If $P_1^{-1} \leq P_2^{-1}$ and $P_1^{-1} R_1 \leq P_2^{-1} R_2$, then

$$\rho(W_{12}) \leq \rho(T_1) < 1.$$ 

Note that the conclusion of Theorem 3.0 also follows if the conditions $P_1^t R_1 \geq P_2^t R_2$ and $P_2^t S_2 \geq P_1^t S_1$ are replaced by $P_1^t \leq P_2^t$ and $P_1^t R_1 \leq P_2^t R_2$. Based on the above-discussed results, it is confirmed that the TG-ADS scheme is a better choice for a certain class of matrices. However, if a matrix $A$ has many pairs of double proper splittings satisfying the desired convergence criteria for the TG-ADS scheme, we face another problem,
i.e., if \( A \) has three or more double proper splittings of \( A \in \mathbb{R}^{m \times n} \), the problem is to choose which pair of double proper splittings to frame the TG hybrid scheme. And to do this, we present a few comparison results next.

Let \( A = P_1 - R_1 + S_1 = P_2 - R_2 + S_2 = P_3 - R_3 + S_3 \) be three double proper splittings of \( A \in \mathbb{R}^{m \times n} \). Then, the iteration matrices for framing ADS schemes as in (3.3) and (3.5) are

\[
G_1 = \begin{pmatrix} I & 0 \\ P_1^tR_1 & -P_1^tS_1 \end{pmatrix}, \quad G_2 = \begin{pmatrix} I & 0 \\ P_2^tR_2 & -P_2^tS_2 \end{pmatrix}
\]

and

\[
T_3 = \begin{pmatrix} P_3^tR_3 & -P_3^tS_3 \\ I & 0 \end{pmatrix}.
\]

If \( N(S_3) \supseteq N(P_3) \), \( R(S_3) \subseteq R(P_3) \) and \( 1 \notin \sigma(S_3P_1^t) \), then \( A = P_1 - R_1 + S_1 = P_3 - R_3 + S_3 \) induce a double proper splitting \( \hat{A}_1 = \hat{P}_1 - \hat{R}_1 + \hat{S}_1 \) to solve the preconditioned linear system \( \hat{A}_1x = \hat{b}_1 \). The iteration matrix corresponding the double iterative scheme (3.9) is

\[
W_{13} = T_3G_1 = \begin{pmatrix} P_3^tR_3 - P_3^tS_3P_1^tR_1 & P_3^tS_3P_1^tS_1 \\ I & 0 \end{pmatrix}.
\]

Similarly, assuming \( N(S_3) \supseteq N(P_3) \), \( R(S_3) \subseteq R(P_3) \) and \( 1 \notin \sigma(S_3P_2^t) \), the other pair of double proper splittings induces another double proper splitting \( \hat{A}_2 = \hat{P}_2 - \hat{R}_2 + \hat{S}_2 \) to solve \( \hat{A}_2x = \hat{b}_2 \). The corresponding iteration matrix is

\[
W_{23} = T_3G_2 = \begin{pmatrix} P_3^tR_3 - P_3^tS_3P_2^tR_2 & P_3^tS_3P_2^tS_2 \\ I & 0 \end{pmatrix}.
\]

Next result presents a comparison result between the spectral radii of \( W_{13} \) and \( W_{23} \) which will help to know which pair of double splittings yields a better ADS scheme.

**Theorem 3.10.** Let \( A = P_1 - R_1 + S_1 = P_2 - R_2 + S_2 \) be two double proper weak regular splittings of \( A \in \mathbb{R}^{m \times n} \). Suppose that \( A = P_3 - R_3 + S_3 \) is a double proper regular splitting with \( N(S_3) \supseteq N(P_3) \), \( R(S_3) \subseteq R(P_3) \), \( 1 \notin \sigma(S_3P_i^t) \) and \( \hat{A}_i^t\hat{P}_i \geq 0 \) for \( i = 1, 2 \). If \( P_1^t \geq P_2^t \) and one of the following conditions

1. \( P_1^tR_1 \geq P_2^tR_2 \)
2. \( P_1^tS_1 \geq P_2^tS_2 \)

holds, then \( \rho(W_{13}) \leq \rho(W_{23}) < 1 \).

**Proof.** Clearly, we have \( \rho(W_{13}) < 1 \) and \( \rho(W_{23}) < 1 \), by Theorem 3.2.

Case (i): \( \rho(W_{13}) = 0 \). The proof is obvious.

Case (ii): \( \rho(W_{13}) \neq 0 \). Since \( W_{13} \geq 0 \), there exists a non-negative eigenvector \( x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \)
such that $W_{13}x = \rho(W_{13})x$ by Theorem 2.3. This implies

$$(P_{i}^1 R_3 - P_{i}^1 S_3 P_{i}^1 R_1)x_1 + P_{i}^1 S_3 P_{i}^1 S_1 x_2 = \rho(W_{13})x_1$$

$$x_1 = \rho(W_{13})x_2.$$ 

As an immediate consequence, we have

$$W_{23}x - \rho(W_{13})x = \begin{pmatrix} (P_{i}^1 R_3 - P_{i}^1 S_3 P_{i}^1 R_2)x_1 + P_{i}^1 S_3 P_{i}^1 S_2 x_2 - \rho(W_{13})x_1 \\ x_1 - \rho(W_{13})x_2 \end{pmatrix}$$

$$= \begin{pmatrix} -P_{i}^1 S_3 P_{i}^1 R_2 x_1 + \frac{1}{\rho(W_{13})} P_{i}^1 S_3 P_{i}^1 S_2 x_1 + P_{i}^1 S_3 P_{i}^1 R_1 x_1 - \frac{1}{\rho(W_{13})} P_{i}^1 S_3 P_{i}^1 S_1 x_1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \nabla \\ 0 \end{pmatrix},$$

where

$$\nabla = P_{i}^1 S_3(P_{i}^1 R_1 - P_{i}^1 R_2)x_1 + \frac{1}{\rho(W_{13})} P_{i}^1 S_3(P_{i}^1 S_2 - P_{i}^1 S_1)x_1.$$ 

If the first condition $P_{i}^1 R_1 \geq P_{i}^1 R_2$ holds, we then have

$$\nabla - \frac{1}{\rho(W_{13})} P_{i}^1 S_3(P_{i}^1 R_1 - P_{i}^1 R_2)x_1 - \frac{1}{\rho(W_{13})} P_{i}^1 S_3(P_{i}^1 S_2 - P_{i}^1 S_1)x_1$$

$$= P_{i}^1 S_3(P_{i}^1 R_1 - P_{i}^1 R_2)x_1 - \frac{1}{\rho(W_{13})} P_{i}^1 S_3(P_{i}^1 R_1 - P_{i}^1 R_2)x_1$$

$$= \begin{pmatrix} 1 - \frac{1}{\rho(W_{13})} \end{pmatrix} P_{i}^1 S_3(P_{i}^1 R_1 - P_{i}^1 R_2)x_1 \geq 0.$$ 

Therefore,

$$\nabla \geq \frac{1}{\rho(W_{13})} P_{i}^1 S_3(P_{i}^1 R_1 - P_{i}^1 R_2)x_1 + \frac{1}{\rho(W_{13})} P_{i}^1 S_3(P_{i}^1 S_2 - P_{i}^1 S_1)x_1$$

$$= \frac{1}{\rho(W_{13})} P_{i}^1 S_3 \left( P_{i}^1 R_1 - P_{i}^1 S_1 + P_{i}^1 S_2 - P_{i}^1 R_2 \right) x_1$$

$$= \frac{1}{\rho(W_{13})} P_{i}^1 S_3 \left( P_{i}^1 (R_1 - S_1) + P_{i}^1 (S_2 - R_2) \right) x_1$$

$$= \frac{1}{\rho(W_{13})} P_{i}^1 S_3 \left( P_{i}^1 (P_1 - A) + P_{i}^1 (A - R_2) \right) x_1$$

$$= \frac{1}{\rho(W_{13})} P_{i}^1 S_3(P_{i}^1 A - P_{i}^1 A)x_1 \quad (\because P_{i}^1 P_2 = P_{i}^1 P_1)$$

$$= \frac{1}{\rho(W_{13})} P_{i}^1 S_3(P_{i}^1 - P_{i}^1) A x_1 \geq 0,$$

using the fact $Ax_1 \geq 0$ as shown in Theorem 3.3. Hence $W_{23}x - \rho(W_{13})x \geq 0$. By Theorem 2.1, $\rho(W_{13}) \leq \rho(W_{23})$.

Similarly, if $P_{i}^1 S_1 \geq P_{i}^1 S_2$, one can easily show that

$$\nabla - P_{i}^1 S_3(P_{i}^1 R_1 - P_{i}^1 R_2)x_1 - P_{i}^1 S_3(P_{i}^1 S_2 - P_{i}^1 S_1)x_1 \geq 0$$

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i.e.,\[
\nabla \geq P_3^i S_3 (P_1^i R_1 - P_2^i R_2) x_1 + P_3^i S_3 (P_2^i S_2 - P_1^i S_1) x_1 \geq 0.
\]
Thus, \(W_{23}x - \rho(W_{13})x \geq 0\). By Theorem 2.1, it follows that \(\rho(W_{13}) \leq \rho(W_{23})\).

Note that the condition \(\hat{A}_i^i \hat{P}_i \geq 0\) for \(i = 1, 2\) is assumed in the above theorem as the class of matrices \((\hat{A}_i^i \hat{P}_i \geq 0)\) is bigger than the class \(\hat{A}_i^i \geq 0\) for \(i = 1, 2\) and each double proper regular (weak) splitting is also a double proper weak splitting. The above theorem is also true if we replace the condition \(P_1^i \geq P_2^i\) by \(P_1^i A \geq P_2^i A\). We have the following corollary to the above result in the case of non-singular matrix setting.

**Corollary 3.11** (Theorem 3.2, [17]). Let \(A = P_1 - R_1 + S_1 = P_2 - R_2 + S_2\) be two double weak regular splittings and \(A = P_3 - R_3 + S_3\) be a double regular splitting of a real non-singular matrix \(A\). Suppose that \(1 \notin \sigma(S_3 P_1^{-1})\) and \(\hat{A}_i^{-1} \hat{P}_i \geq 0\) for \(i = 1, 2\). If \(P_1^{-1} \geq P_2^{-1}\) and one of the following conditions

1. \(P_1^{-1} R_1 \geq P_2^{-1} R_2\)
2. \(P_1^{-1} S_1 \geq P_2^{-1} S_2\)

holds, then \(\rho(W_{13}) \leq \rho(W_{23}) < 1\).

We have the following comparison result between the spectral radii of the iteration matrices \(W_{13}\) and \(W_{23}\) which is motivated by the proof of Theorem 3.7, [23].

**Theorem 3.12.** Let \(A = P_1 - R_1 + S_1 = P_2 - R_2 + S_2\) be two double proper weak regular splittings of a semi-monotone matrix \(A \in \mathbb{R}^{m \times n}\). Suppose that \(A = P_3 - R_3 + S_3\) is a double proper regular splitting with \(N(S_3) \supseteq N(P_3), R(S_3) \subseteq R(P_3), 1 \notin \sigma(S_3 P_1^i)\) and \(\hat{A}_i^i \geq 0\) for \(i = 1, 2\). If \(\hat{P}_1^i \hat{A}_1 \geq \hat{P}_2^i \hat{A}_2\) and \(\hat{P}_1^i \hat{R}_1 \geq \hat{P}_2^i \hat{R}_2\), then \(\rho(W_{13}) \leq \rho(W_{23}) < 1\).

**Proof.** We have \(\hat{A}_1 = (I - S_3 P_1) A, \hat{P}_1 = P_3, \hat{R}_1 = R_3 - S_3 P_1 R_1\) and \(\hat{S}_1 = -S_3 P_1 S_1\) corresponding to the two double proper weak regular splittings \(A = P_1 - R_1 + S_1 = P_3 - R_3 + S_3\). Again, the double proper weak regular splittings \(A = P_2 - R_2 + S_2 = P_3 - R_3 + S_3\) give \(\hat{A}_2 = (I - S_3 P_1) A, \hat{P}_2 = P_3, \hat{R}_2 = R_3 - S_3 P_2 R_2\) and \(\hat{S}_2 = -S_3 P_2 S_2\). The respective iteration matrices are

\[
\hat{T}_1 = \begin{pmatrix}
\hat{P}_1^i \hat{R}_1 & -\hat{P}_1^i \hat{S}_1 \\
I & 0
\end{pmatrix} = W_{13}, \quad \hat{T}_2 = \begin{pmatrix}
\hat{P}_2^i \hat{R}_2 & -\hat{P}_2^i \hat{S}_2 \\
I & 0
\end{pmatrix} = W_{23}.
\]

By Theorem 3.11 we have \(\rho(W_{13}) < 1\) and \(\rho(W_{23}) < 1\).

Case (i): \(\rho(W_{13}) = 0\). The proof is obvious.

Case (ii): \(\rho(W_{13}) \neq 0\). Since \(W_{13} \geq 0\), there exists a non-negative eigenvector \(x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\) such that \(W_{13} x = \rho(W_{13}) x\) by Theorem 2.3. This yields

\[
\hat{P}_1^i \hat{R}_1 x_1 - \hat{P}_1^i \hat{S}_1 x_2 = \rho(W_{13}) x_1
\]

\[
x_1 = \rho(W_{13}) x_2.
\]
By using the above two equations, we have

\[
\mathbf{W}_{23} \mathbf{x} - \rho(\mathbf{W}_{13}) \mathbf{x} = \begin{pmatrix}
\hat{P}_2^1 \hat{R}_2 x_1 - \hat{P}_2^1 \hat{S}_2 x_2 - \rho(\mathbf{W}_{13}) x_1 \\
\hat{P}_2^1 \hat{R}_2 x_1 - \hat{P}_1^1 \hat{R}_1 x_1 + \hat{P}_1^1 \hat{S}_1 x_2 \\
0
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\hat{P}_2^1 \hat{R}_2 x_1 - \hat{P}_2^1 \hat{S}_2 x_2 - \rho(\mathbf{W}_{13}) x_1 \\
0
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\hat{P}_2^1 \hat{R}_2 x_1 - \hat{P}_1^1 \hat{R}_1 x_1 - \frac{1}{\rho(\mathbf{W}_{13})} (\hat{P}_2^1 \hat{S}_2 - \hat{P}_1^1 \hat{S}_1) x_1
\end{pmatrix}
\]

\[
\geq \begin{pmatrix}
\frac{1}{\rho(\mathbf{W}_{13})} (\hat{P}_2^1 \hat{R}_2 - \hat{P}_1^1 \hat{R}_1 - \hat{P}_2^1 \hat{S}_2 + \hat{P}_1^1 \hat{S}_1) x_1
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\frac{1}{\rho(\mathbf{W}_{13})} (\hat{P}_2^1 (\hat{R}_2 - \hat{S}_2) - \hat{P}_1^1 (\hat{R}_1 - \hat{S}_1)) x_1
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\frac{1}{\rho(\mathbf{W}_{13})} (\hat{P}_1^1 \hat{A}_1 - \hat{P}_2^1 \hat{A}_2) x_1
\end{pmatrix}
\]

Hence, the conclusion follows by Theorem 2.13.

**Corollary 3.13.** Let \( A = P_1 - R_1 + S_1 = P_2 - R_2 + S_2 \) be two double weak regular splittings of a monotone matrix \( A \in \mathbb{R}^{n \times n} \). Suppose that \( A = P_3 - R_3 + S_3 \) be a double regular splitting with \( 1 \notin \sigma(S_3 P_i^{-1}) \) and \( \hat{A}_i^{-1} \geq 0 \) for \( i = 1, 2 \). If \( \hat{P}_1^{-1} \hat{A}_1 \geq \hat{P}_2^{-1} \hat{A}_2 \) and \( \hat{P}_1^{-1} \hat{R}_1 \geq \hat{P}_2^{-1} \hat{R}_2 \), then \( \rho(\mathbf{W}_{13}) \leq \rho(\mathbf{W}_{23}) < 1 \).

Observe that the above comparisons are made between two pairs of double proper splittings with one common double proper splitting out of three independent double proper splittings. Further, we are interested to reveal the comparison of two independent pairs of double proper splittings tailored by the TG-ADS scheme (3.4). To this end, let us consider the first pair of double proper splittings \( A = P_1 - R_1 + S_1 = P_2 - R_2 + S_2 \) such that \( N(S_2) \supseteq N(P_2) \), \( R(S_2) \subseteq R(P_2) \) and \( 1 \notin \sigma(S_2 P_1) \). As per the convergence analysis described in the subsection 3.2, the corresponding induced splitting \( \hat{A}_1 = \hat{P}_1 - \hat{R}_1 + \hat{S}_1 \) is a double proper splitting, where \( \hat{A}_1 = (I - S_2 P_1^1) A \), \( \hat{P}_1 = P_2 \), \( \hat{R}_1 = R_2 - S_2 P_1^1 R_1 \) and \( \hat{S}_1 = -S_2 P_1^1 S_1 \). In this case, the iteration matrix of the TG-ADS scheme is

\[
\mathbf{W}_{12} = \begin{pmatrix}
P_2^1 R_2 & P_1^1 S_2 P_1^1 R_1 & P_1^1 S_2 P_1^1 S_1 \\
I & 0
\end{pmatrix}
\]

which is also the iteration matrix of double iteration scheme (3.9) to solve the system \( \hat{A}_1 x = \hat{b}_1 \). Similarly, the other pair of double proper splittings \( A = P_3 - R_3 + S_3 = P_4 - R_4 + S_4 \), satisfying \( N(S_4) \supseteq N(P_4) \), \( R(S_4) \subseteq R(P_4) \) and \( 1 \notin \sigma(S_4 P_1) \) yields
The following theorem presents different sufficient conditions for choosing a better ADS scheme, and its proof is analogous to the proof of Theorem 3.12

**Theorem 3.14.** Let \( A = P_1 - R_1 + S_1 = P_2 - R_2 + S_2 = P_3 - R_3 + S_3 = P_4 - R_4 + S_4 \) be four double proper weak regular splittings of a semi-monotone matrix \( A \in \mathbb{R}^{m \times n} \). Suppose that \( \hat{A}_i \geq 0 \) for \( i = 1, 2 \), \( N(S_2) \supseteq N(P_2) \), \( R(S_2) \subseteq R(P_2) \), \( N(S_4) \supseteq N(P_4) \), \( R(S_4) \subseteq R(P_4) \), \( 1 \notin \sigma(S_2 P_1^\dagger) \) and \( 1 \notin \sigma(S_4 P_3^\dagger) \). If \( \hat{P}_1^\dagger \hat{A}_1 \geq \hat{P}_2^\dagger \hat{A}_2 \) and \( \hat{P}_1^\dagger \hat{R}_1 \geq \hat{P}_2^\dagger \hat{R}_2 \), then \( \rho(W_{12}) \leq \rho(W_{34}) < 1 \).

As a consequence, we have the next result.

**Corollary 3.15.** Let \( A = P_1 - R_1 + S_1 = P_2 - R_2 + S_2 = P_3 - R_3 + S_3 = P_4 - R_4 + S_4 \) be four double weak regular splittings of a monotone matrix \( A \in \mathbb{R}^{m \times n} \). Suppose that \( \hat{A}_i^{-1} \geq 0 \) for \( i = 1, 2 \), \( 1 \notin \sigma(S_2 P_1^{-1}) \) and \( 1 \notin \sigma(S_4 P_3^{-1}) \). If \( \hat{P}_1^{-1} \hat{A}_1 \geq \hat{P}_2^{-1} \hat{A}_2 \) and \( \hat{P}_1^{-1} \hat{R}_1 \geq \hat{P}_2^{-1} \hat{R}_2 \), then \( \rho(W_{12}) \leq \rho(W_{34}) < 1 \).

### 3.4. Comparison Results: HT-ADS scheme

In this sub-section, we establish that the HT-ADS scheme converges faster than the classical double splitting schemes. Further, we answer the natural question that under what condition the HT-ADS scheme performs better than the TG-ADS scheme and vice versa. The comparisons among \( \rho(W_{12}), \rho(W_{23}) \) and \( \rho(W_{34}) \), like the TG-ADS scheme are omitted as they are very similar.

**Theorem 3.16.** Let \( A = P_1 - R_1 + S_1 = P_2 - R_2 + S_2 \) be a double proper weak regular splitting and \( A = P_2 - R_2 + S_2 \) be a double proper regular splitting of a semi-monotone matrix \( A \in \mathbb{R}^{m \times n} \). Suppose that \( N(R_2) \supseteq N(P_2) \), \( R(R_2) \subseteq R(P_2) \), \(-1 \notin \sigma(R_2 P_1^\dagger) \) and \( \hat{A}_1 \geq 0 \). If \( P_1^\dagger R_1 + P_2^\dagger S_2 \leq 0 \), then \( \rho(W_{12}) \leq \rho(T_2) < 1 \).

**Proof.** By Theorem 3.3 and Theorem 2.4, we have \( \rho(W_{12}) < 1 \) and \( \rho(T_2) < 1 \), respectively. Case (i): \( \rho(W_{12}) = 0 \), the proof is trivial.

Case (ii): \( \rho(W_{12}) \neq 0 \), i.e., \( 0 < \rho(W_{12}) < 1 \). Applying Theorem 2.3 to \( W_{12} \), i.e., there exists a vector

\[
\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \geq 0, \quad \mathbf{x} \neq 0
\]

such that \( W_{12} \mathbf{x} = \rho(W_{12}) \mathbf{x} \). This gives

\[
(P_1^\dagger R_1 + P_2^\dagger S_2)x_1 - P_2^\dagger R_2 P_1^\dagger S_1 x_2 = \rho(W_{12})x_1
\]

\[
x_1 = \rho(W_{12})x_2.
\]
Hence
\[ T_2x - \rho(W_{12})x = \left( P_{21}^t R_2 x_1 - P_{21}^t S_2 x_2 - \rho(W_{12})x_1 \right). \]

By suitable substitutions in the first component of the above expression, we have
\[
\begin{align*}
P_{21}^t R_2 x_1 & - \frac{1}{\rho(W_{12})} P_{21}^t S_2 x_1 - P_{21}^t R_2 P_{11}^t R_1 x_1 + P_{21}^t S_2 x_1 + \frac{1}{\rho(W_{12})} P_{21}^t R_2 P_{11}^t S_1 x_1 \\
& \geq P_{21}^t R_2 x_1 + \frac{1}{\rho(W_{12})} P_{21}^t S_2 x_1 - \frac{1}{\rho(W_{12})} P_{21}^t R_2 P_{11}^t R_1 x_1 + \frac{1}{\rho(W_{12})} P_{21}^t R_2 P_{11}^t S_1 x_1 \\
& = P_{21}^t R_2 x_1 + \left( 1 - \frac{1}{\rho(W_{12})} \right) P_{21}^t S_2 x_1 + \frac{1}{\rho(W_{12})} P_{21}^t R_2 P_{11}^t (-R_1 + S_1) x_1 \\
& = P_{21}^t R_2 x_1 + \left( 1 - \frac{1}{\rho(W_{12})} \right) P_{21}^t S_2 x_1 + \frac{1}{\rho(W_{12})} P_{21}^t R_2 P_{11}^t (A - P_1) x_1 \\
& \geq \left( 1 - \frac{1}{\rho(W_{12})} \right) P_{21}^t R_2 x_1 + \left( 1 - \frac{1}{\rho(W_{12})} \right) P_{21}^t S_2 x_1 \\
& \geq (1 - \frac{1}{\rho(W_{12})})(P_{21}^t R_2 + P_{21}^t S_2) \geq 0.
\end{align*}
\]

Hence \( T_2x - \rho(W_{12})x \geq 0 \). We thus have \( \rho(W_{12}) \leq \rho(T_2) < 1 \), by Theorem 2.4. \( \square \)

**Theorem 3.17.** Let \( A = P_1 - R_1 + S_1 \) be a double proper weak regular splitting and \( A = P_2 - R_2 + S_2 \) be a double proper regular splitting of a semi-monotone matrix \( A \in \mathbb{R}^{m \times n} \). Suppose that \( N(R_2) \supseteq N(P_2) \), \( R(R_2) \subseteq R(P_2) \), \(-1 \notin \sigma(R_2 P_{11}^t) \) and \( \hat{A} \geq 0 \). If \( P_{11}^t R_1 \geq P_{21}^t R_2 \), \( P_{21}^t S_2 \geq P_{11}^t S_1 \) and \( P_{21}^t R_2 + P_{21}^t S_2 \leq 0 \), then \( \rho(W_{12}) \leq \rho(T_1) < 1 \).

**Proof.** By Theorem 3.3 and Theorem 2.4, we get \( \rho(W_{12}) < 1 \) and \( \rho(T_1) < 1 \), respectively.

Case (i): \( \rho(W_{12}) = 0 \), the proof is trivial.

Case (ii): \( \rho(W_{12}) \neq 0 \), i.e., \( 0 < \rho(W_{12}) < 1 \). Applying Theorem 2.3 to \( W_{12} \), i.e., there exists a vector
\[
x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \geq 0, \ x \neq 0
\]
such that \( W_{12}x = \rho(W_{12})x \). This gives
\[
(P_{21}^t R_2 P_{11}^t R_1 - P_{21}^t R_2 P_{11}^t S_2) x_1 - P_{21}^t R_2 P_{11}^t S_1 x_2 = \rho(W_{12})x_1 \\
x_1 = \rho(W_{12})x_2.
\]

Now
\[
T_1x - \rho(W_{12})x = \begin{pmatrix} P_{11}^t R_1 x_1 - P_{11}^t S_1 x_2 - \rho(W_{12})x_1 \\ x_1 - \rho(W_{12})x_2 \end{pmatrix}.
\]
By suitable substitutions in the first component of the above expression, we have

\[
P_1^\dagger R_1 x_1 - \frac{1}{\rho(W_1)} P_1^\dagger S_1 x_1 - P_2^\dagger R_2 P_1^\dagger R_1 x_1 + P_2^\dagger S_2 x_1 + \frac{1}{\rho(W_1)} P_2^\dagger R_2 S_1 x_1
\]

\[
\geq P_2^\dagger R_2 x_1 - \frac{1}{\rho(W_1)} P_1^\dagger S_1 x_1 - \frac{1}{\rho(W_1)} P_2^\dagger R_2 P_1^\dagger R_1 x_1 + \frac{1}{\rho(W_1)} P_2^\dagger R_2 P_1^\dagger S_1 x_1 + P_2^\dagger S_2 x_1
\]

\[
\geq P_2^\dagger R_2 x_1 + \left( 1 - \frac{1}{\rho(W_1)} \right) P_1^\dagger S_2 x_1 + \frac{1}{\rho(W_1)} P_2^\dagger R_2 P_1^\dagger (-R_1 + S_1)x_1
\]

\[
= P_2^\dagger R_2 x_1 + \left( 1 - \frac{1}{\rho(W_1)} \right) P_2^\dagger S_2 x_1 + \frac{1}{\rho(W_1)} P_2^\dagger R_2 P_1^\dagger (A - P_1)x_1
\]

\[
= P_2^\dagger R_2 x_1 + \left( 1 - \frac{1}{\rho(W_1)} \right) P_2^\dagger S_2 x_1 + \frac{1}{\rho(W_1)} P_2^\dagger R_2 P_1^\dagger A x_1 - \frac{1}{\rho(W_1)} P_2^\dagger R_2 x_1
\]

\[
\geq \left( 1 - \frac{1}{\rho(W_1)} \right) P_2^\dagger R_2 x_1 + \left( 1 - \frac{1}{\rho(W_1)} \right) P_2^\dagger S_2 x_1 \quad (\because P_2^\dagger R_2 P_1^\dagger A x_1 \geq 0)
\]

\[
= \left( 1 - \frac{1}{\rho(W_1)} \right) (P_2^\dagger R_2 + P_2^\dagger S_2) \geq 0.
\]

Hence \( T_1 x - \rho(W_1) x \geq 0 \). By Theorem 2.1 we have \( \rho(W_1) \leq \rho(T_1) < 1 \).

Combining the above two results, we have the following one.

**Theorem 3.18.** Let \( A = P_1 - R_1 + S_1 \) be a double proper weak regular splitting and \( A = P_2 - R_2 + S_2 \) be a double proper regular splitting of a semi-monotone matrix \( A \in \mathbb{R}^{m \times n} \).

Suppose that \( N(R_2) \supseteq N(P_1) \), \( R(R_2) \subseteq R(P_2) \), \(-1 \notin \sigma(R_2 P_1^\dagger)\) and \( \hat{A}^{-1} \geq 0 \). If \( P_1^\dagger R_1 \geq P_2^\dagger R_2 \), \( P_2^\dagger S_2 \geq P_1^\dagger S_1 \), and \( P_2^\dagger R_2 + P_2^\dagger S_2 \leq 0 \), then \( \rho(W_1) \leq \min\{\rho(T_1), \rho(T_2)\} < 1 \).

In the case of non-singular \( A \), we obtain the following result as a corollary.

**Corollary 3.19.** Let \( A = P_1 - R_1 + S_1 \) be a double weak regular splitting and \( A = P_2 - R_2 + S_2 \) be a double regular splitting of a monotone matrix \( A \). Suppose that \(-1 \notin \sigma(R_2 P_1^{-1})\) and \( \hat{A}^{-1} \geq 0 \). If \( P_1^{-1} R_1 \geq P_2^{-1} R_2 \), \( P_2^{-1} S_2 \geq P_1^{-1} S_1 \), and \( P_2^{-1} R_2 + P_2^{-1} S_2 \leq 0 \), then \( \rho(W_1) \leq \min\{\rho(T_1), \rho(T_2)\} < 1 \).

In support of Theorem 3.18, the following example is illustrated.

**Example 3.2.** Suppose \( A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = P_1 - R_1 + S_1 = P_2 - R_2 + S_2 \)

\[
\begin{bmatrix} 5 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} + \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}
\]

\[
\begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}
\]

are two double proper splittings of \( A \) such that the first one is double proper weak regular splitting and the second one is double proper regular splitting. Also, it satisfy all conditions of Theorem 3.18 Therefore, \( \rho(W_1) = 0.6667 \leq \min\{\rho(T_1) = 0.8633, \rho(T_2) = 0.7675\} < 1 \).

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Theorem 3.20. Let $A = P_1 - R_1 + S_1$ be a double proper weak regular splitting and $A = P_2 - R_2 + S_2$ be a double proper regular splitting of a semi-monotone matrix $A \in \mathbb{R}^{m \times n}$. Suppose that $N(S_2) = N(P_2)$, $R(S_2) = R(P_2)$, $1 \notin \sigma(S_2 P_1^\dagger)$, $-1 \notin \sigma(R_2 P_1^\dagger)$, $\hat{A}^\dagger \geq 0$ and $\hat{\tilde{A}}^\dagger \geq 0$. If $\hat{\tilde{P}}^\dagger \hat{\tilde{R}} \geq \hat{\tilde{P}}^\dagger \hat{\tilde{R}}$ and $\hat{\tilde{P}}^\dagger \hat{\tilde{A}} \geq \hat{\tilde{P}}^\dagger \hat{\tilde{A}}$, then $\rho(W_{12}) \leq \rho(W_{12}) < 1$.

Proof. As in the earlier proof, we get $\hat{A} = \hat{P} - \hat{\tilde{R}} + \hat{\tilde{S}}$ and $\hat{\tilde{A}} = \hat{P} - \hat{\tilde{R}} + \hat{\tilde{S}}$ as double proper weak regular splittings. By Theorem 3.11 and Theorem 3.33 we have $\rho(W_{12}) < 1$ and $\rho(W_{12}) < 1$, respectively.

Case (i): $\rho(W_{12}) = 0$. The proof is obvious.

Case (ii): $\rho(W_{12}) \neq 0$. Since $W_{12} \geq 0$, there exists a non-negative eigenvector $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ such that $W_{12}x = \rho(W_{12})x$ by Theorem 2.23.

\[
\hat{\tilde{P}}^\dagger \hat{\tilde{R}} x_1 - \hat{\tilde{P}}^\dagger \hat{\tilde{S}} x_2 = \rho(W_{12})x_1,
\]
\[
x_1 = \rho(W_{12})x_2.
\]

Now
\[
W_{12}x - \rho(W_{12})x = \begin{pmatrix} \hat{\tilde{P}}^\dagger \hat{\tilde{R}} x_1 - \hat{\tilde{P}}^\dagger \hat{\tilde{S}} x_2 - \rho(W_{12})x_1 \\ x_1 - \rho(W_{12})x_2 \end{pmatrix}
= \begin{pmatrix} \hat{\tilde{P}}^\dagger \hat{\tilde{R}} x_1 - \frac{1}{\rho(W_{12})} \hat{\tilde{P}}^\dagger \hat{\tilde{S}} x_1 - \hat{\tilde{P}}^\dagger \hat{\tilde{R}} x_1 + \frac{1}{\rho(W_{12})} \hat{\tilde{P}}^\dagger \hat{\tilde{S}} x_1 \\ 0 \end{pmatrix}
\geq \begin{pmatrix} \frac{1}{\rho(W_{12})} (\hat{\tilde{P}}^\dagger \hat{\tilde{R}} - \hat{\tilde{P}}^\dagger \hat{\tilde{R}}) x_1 - \frac{1}{\rho(W_{12})} (\hat{\tilde{P}}^\dagger \hat{\tilde{S}} - \hat{\tilde{P}}^\dagger \hat{\tilde{S}}) x_1 \\ 0 \end{pmatrix}
= \begin{pmatrix} \frac{1}{\rho(W_{12})} (\hat{\tilde{P}}^\dagger (\hat{P} - \hat{\tilde{A}}) - \hat{\tilde{P}}^\dagger (\hat{P} - \hat{\tilde{A}})) x_1 \\ 0 \end{pmatrix}
\geq \begin{pmatrix} \frac{1}{\rho(W_{12})} (\hat{\tilde{P}}^\dagger \hat{\tilde{A}} - \hat{\tilde{P}}^\dagger \hat{\tilde{A}}) x_1 \\ 0 \end{pmatrix} \geq 0.
\]

Hence $\rho(W_{12})x \leq W_{12}x$. By Theorem 2.21 we therefore have $\rho(W_{12}) \leq \rho(W_{12}) < 1$.

Remark 3.2. If the conditions are again replaced by $\hat{\tilde{P}}^\dagger \hat{\tilde{R}} \geq \hat{\tilde{P}}^\dagger \hat{\tilde{R}}$ and $\hat{\tilde{P}}^\dagger \hat{\tilde{A}} \geq \hat{\tilde{P}}^\dagger \hat{\tilde{A}}$, then the HT-ADS scheme performs better than the GT-ADS scheme.

As a consequence of Theorem 3.20 we have the the following corollary.

Corollary 3.21. Let $A = P_1 - R_1 + S_1$ be a double weak regular splitting and $A = P_2 - R_2 + S_2$ be a double regular splitting of a monotone matrix $A$. Suppose that $1 \notin \sigma(S_2 P_1^{-1})$, $-1 \notin \sigma(R_2 P_1^{-1})$, $\hat{A}^{-1} \geq 0$ and $\hat{\tilde{A}}^{-1} \geq 0$. If $\hat{P}^{-1} \hat{\tilde{R}} \geq \hat{P}^{-1} \hat{\tilde{R}}$ and $\hat{P}^{-1} \hat{\tilde{A}} \geq \hat{P}^{-1} \hat{\tilde{A}}$, then $\rho(W_{12}) \leq \rho(W_{12}) < 1$. 21
4. Numerical results

In this section, numerical results are given to demonstrate the accuracy and effectiveness of the proposed ADS schemes. The computations are carried out using Mathematica 10.0 and MATLAB R2018a on an intel(R) Core(TM)i5, 2.5GHz, 16GB RAM. The stopping criteria is \( \|x_k - x_{k-1}\| \leq \epsilon = 10^{-7} \). We have considered two different examples: one for the case of non-singular matrices and the other for rectangular matrices.

**Example 4.1 (Example 4.1, [23])**. Applying second order five-point central difference scheme for the following two-dimensional convection-diffusion equation:

\[
-\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = \sin x, \quad (x, y) \in \Omega = [0, 1] \times [0, 1],
\]

we obtain a system of linear equations \( Ax = b \), where \( A \) is non-singular matrix. The discretization is made using uniform grids with \( N_x \times N_y \) interior nodes, where the solution is known at the boundary. Therefore, the coefficient matrix \( A \) is of the form

\[
A = I_y \otimes J_x + J_y \otimes I_x.
\]

Here \( \otimes \) is the Kronecker product, and the matrices \( J_x \) and \( J_y \) are tridiagonal matrices of order \( N_x \) and \( N_y \) respectively, i.e.,

\[
J_x = \text{tridiagonal} (-2 - h_x, 8, h_x - 2) \quad \text{and} \quad J_y = \text{tridiagonal} (-2 - 2h_y, 0, 2h_y - 2),
\]

where \( h_x \) and \( h_y \) are the uniform step size along \( x \) and \( y \) directions, respectively. Similarly, the identity matrices \( I_x \) and \( I_y \) are of the dimension \( N_x \) and \( N_y \), respectively. We can observe \( A \) is not a symmetric matrix but its diagonally dominant block tridiagonal matrix hence irreducible. This properties of matrices implies they are monotonic, which is very useful while investigating our theoretical findings by numerical experiments. The proposed TG-ADS scheme is compared with the iterative methods of [22], [24], [36], [46] and [47]. The Table 1 compares the residual norm (\( \|r_k\| = \|b - Ax_k\| \)), error norm (\( \|e_k\| = \|A^\dagger b - x_k\| \)) and Mean Time(MT). The symbol (\( - \)) represents that the TG-ADS scheme does not converge within the maximum allowed iteration (4000). Figure 1 presents the computational time of the present ADS scheme which outperforms the iteration schemes used in Table 1. The same figure shows that the computational time for the increasing size of the discretization matrices (by reducing the step length \( h \)). The computational time of the TG-ADS scheme is consistently lesser than the existing schemes for all size of matrices.
Figure 1: Comparison of existing methods with the TG-ADS scheme

Table 1: Comparison analysis of different schemes for $\epsilon = 10^{-7}$

| Order of $A$ | Method       | $n$ | $\|r_k\|_2$        | $\|e_k\|_2$         | MT        |
|--------------|--------------|-----|-------------------|---------------------|-----------|
| 15 $\times$ 15 | Method of [22] | 453 | $4.6057e^{-8}$    | $2.3197e^{-6}$      | 0.00955   |
|               | Method of [24] | 568 | $5.9575e^{-8}$    | $2.9791e^{-6}$      | 0.01525   |
|               | Method of [36] | 161 | $2.6862e^{-8}$    | $6.4999e^{-7}$      | 0.00798   |
|               | Method of [46] | 701 | $7.6163e^{-8}$    | $3.7715e^{-6}$      | 0.01704   |
|               | TG-ADS        | 93  | $1.1769e^{-8}$    | $2.8399e^{-7}$      | 0.00544   |
| 25 $\times$ 25 | Method of [22] | 1230| $4.7809e^{-8}$    | $6.3031e^{-6}$      | 0.16492   |
|               | Method of [24] | 1514| $6.1655e^{-8}$    | $8.0690e^{-6}$      | 0.18487   |
|               | Method of [36] | 283 | $1.7583e^{-8}$    | $1.0791e^{-6}$      | 0.11376   |
|               | Method of [46] | 1904| $7.7942e^{-8}$    | $1.0127e^{-5}$      | 0.25060   |
|               | TG-ADS        | 162 | $8.9456e^{-9}$    | $5.4889e^{-7}$      | 0.07400   |
| 35 $\times$ 35 | Method of [22] | 2399| $4.8831e^{-8}$    | $1.2277e^{-5}$      | 1.94247   |
|               | Method of [24] | 3007| $6.2089e^{-8}$    | $1.5513e^{-5}$      | 2.39230   |
|               | Method of [36] | 408 | $1.3870e^{-8}$    | $1.6075e^{-6}$      | 0.62750   |
|               | Method of [46] | 3716| $7.8517e^{-8}$    | $1.9509e^{-5}$      | 2.97910   |
|               | TG-ADS        | 233 | $7.2425e^{-9}$    | $8.4203e^{-7}$      | 0.37433   |
| 50 $\times$ 50 | Method of [22] | –   | –                 | –                   | –         |
|               | Method of [24] | –   | –                 | –                   | –         |
|               | Method of [36] | 602 | $9.8102e^{-9}$    | $2.2771e^{-6}$      | 3.92001   |
|               | Method of [46] | –   | –                 | –                   | –         |
|               | TG-ADS        | 343 | $5.3138e^{-9}$    | $1.2329e^{-6}$      | 2.18799   |
Next, we will perform a few computational experiments to understand the efficiency of the preconditioners induced by the ADS schemes. The preconditioning matrix which will modify the original matrix such that the new matrix will be closer to the identity matrix or at least that the eigenvalues of the new matrix are clustered together, see [42] by Wathen in 2015. Hence, we can compute \( \| I - L A \| \) with respect to different preconditioning matrix \( L \) and compare with the \( \| I - A \| \) to identify the efficient preconditioning matrix. In Table 2 and 3 we have compared the efficiency of the preconditioners along with that we have observed the decrease in condition number of the coefficient matrix with respect to the increase in efficiency of the preconditioners induced by the ADS schemes.

| Order System n | Time | Condition number | Efficiency |
|---------------|------|-----------------|------------|
| \((A, b)\) 178 | 0.22885 | 100.3994 | 0.9808 |
| \((\hat{A}, \hat{b})\) 93 | 0.13691 | 53.0487 | 0.9657 |
| \((A, b)\) 312 | 5.34683 | 266.0636 | 0.9927 |
| \((\hat{A}, \hat{b})\) 186 | 3.98814 | 242.2502 | 0.9920 |
| \((A, b)\) 451 | 43.16318 | 510.6155 | 0.9962 |
| \((\hat{A}, \hat{b})\) 268 | 37.29997 | 464.6237 | 0.9958 |
| \((A, b)\) 665 | 412.50718 | 1025.400 | 0.9981 |
| \((\hat{A}, \hat{b})\) 395 | 359.45053 | 932.6497 | 0.9979 |
| \((A, b)\) 546 | 336.15932 | 542.8390 | 0.9897 |

For the computations in Table 2 we have selected a second splitting \( A = P_2 - R_2 + S_2 \) such that that HT-ADS scheme converges faster than TG-ADS scheme. As a result, it shows that the preconditioned system (3.10) is better than the earlier one (3.8). The comparison theorem (i.e., Theorem 3.20) served the sufficient conditions under which the faster convergence of the HT-ADS scheme is guaranteed. In particular, one can observe that the condition number of \( A \) reduces from 1025.400 to 542.839 when matrix size is 2500 for the preconditioned system (3.10) induced by the HT-ADS scheme. The purpose of the last column of the table is crucial in order to measure the efficiency of the preconditioning matrix by computing the norm of the difference of the matrix or the preconditioned matrices from the identity matrix. The minimum norm will assure that the preconditioned matrix is the closest to identity matrix and confirm the corresponding preconditioner is the most efficient and its resulting system have the least condition number. For all sizes of matrices, considered in the table, the HT-ADS scheme preconditioner is consistently efficient and the condition number is less. Due to this effect, HT-ADS scheme converges
with the least number of iterations and computational time.

In Table 3, we have a second splitting \( A = P_2 - R_2 + S_2 \) (in the complementary class of the splitting considered in Table 2) such that the TG-ADS scheme converges faster than the HT-ADS scheme. On the contrary to the results in Table 2, the TG-ADS scheme induces the efficient preconditioner and the preconditioned linear system has the least condition number. For this case, the guaranteed conditions on the splittings are reported in Remark 3.2, which are the sufficient conditions. Simultaneously, the iteration numbers and computational times are the least for the most efficient preconditioner, which has been consistently observed for the matrices of sizes 225, 625, 1225 and 2500 derived form the discretized PDE.

### Table 3: Comparison of preconditioners: complementary case of Table 2

| Order | Systems | n   | Time   | Condition number | Efficiency |
|-------|---------|-----|--------|------------------|------------|
| 15 × 15 | \((A, \hat{b})\) | 178 | 0.21937 | 100.3994 | 0.9808 |
|       | \((\hat{A}, \hat{b})\) | 96  | 0.18625 | 60.3614  | 0.9696 |
|       | \((\hat{A}, \hat{b})\) | 101 | 0.15694 | 75.7562  | 0.9752 |
| 25 × 25 | \((A, \hat{b})\) | 312 | 4.79311 | 266.0636 | 0.9927 |
|       | \((\hat{A}, \hat{b})\) | 168 | 3.68142 | 159.2569 | 0.9882 |
|       | \((\hat{A}, \hat{b})\) | 177 | 3.87634 | 199.9162 | 0.9905 |
| 35 × 35 | \((A, \hat{b})\) | 451 | 39.98544 | 510.6155 | 0.9962 |
|       | \((\hat{A}, \hat{b})\) | 242 | 32.91968 | 305.4747 | 0.9938 |
|       | \((\hat{A}, \hat{b})\) | 255 | 32.91633 | 383.2356 | 0.9950 |
| 50 × 50 | \((A, \hat{b})\) | 665 | 433.69228 | 1025.400 | 0.9981 |
|       | \((\hat{A}, \hat{b})\) | 356 | 333.72665 | 613.5452 | 0.9969 |
|       | \((\hat{A}, \hat{b})\) | 375 | 360.78457 | 769.1473 | 0.9975 |

The following example demonstrates Theorem 3.5, and also used to generate large rectangular matrices that are used for the computation in Table 4.

**Example 4.2.** Let \( A = \)

\[
\begin{bmatrix}
1 & 18 & -\frac{1}{2} & -\frac{1}{4} & -\frac{1}{8} & -\frac{1}{16} & -\frac{1}{32} & -\frac{1}{64} & -\frac{1}{128} & 0 \\
0 & -\frac{1}{2} & 14 & -\frac{1}{2} & -\frac{1}{4} & -\frac{1}{8} & -\frac{1}{16} & -\frac{1}{32} & -\frac{1}{64} & 0 \\
0 & -\frac{1}{4} & -\frac{1}{2} & 20 & -\frac{1}{2} & -\frac{1}{4} & -\frac{1}{8} & -\frac{1}{16} & -\frac{1}{32} & 0 \\
0 & -\frac{1}{8} & -\frac{1}{4} & -\frac{1}{2} & 11 & -\frac{1}{2} & -\frac{1}{4} & -\frac{1}{8} & -\frac{1}{16} & 0 \\
0 & -\frac{1}{16} & -\frac{1}{8} & -\frac{1}{4} & -\frac{1}{2} & 14 & -\frac{1}{2} & -\frac{1}{4} & -\frac{1}{8} & 0 \\
0 & -\frac{1}{32} & -\frac{1}{16} & -\frac{1}{8} & -\frac{1}{4} & -\frac{1}{2} & 19 & -\frac{1}{2} & -\frac{1}{4} & 0 \\
0 & -\frac{1}{64} & -\frac{1}{32} & -\frac{1}{16} & -\frac{1}{8} & -\frac{1}{4} & -\frac{1}{2} & 19 & -\frac{1}{2} & 0 \\
0 & -\frac{1}{128} & -\frac{1}{64} & -\frac{1}{32} & -\frac{1}{16} & -\frac{1}{8} & -\frac{1}{4} & -\frac{1}{2} & 19 & 1
\end{bmatrix}
\]
\( P_1 = \begin{bmatrix}
871705488637 & 470 & -1 & -1 & -1 & -1 & -1 & -1 & 128 & 72320000 \\
334182018661 & -1 & 479 & -1 & -1 & -1 & -1 & -1 & -1 & 181040000 \\
334182018661 & -1 & 479 & -1 & -1 & -1 & -1 & -1 & -1 & 4763145523 \\
315318939509 & -1 & -1 & 430 & -1 & -1 & -1 & -1 & -1 & 4763145523 \\
31762951685 & -1 & -1 & -1 & 315 & -1 & -1 & -1 & -1 & 642800000 \\
12894356669 & -1 & -1 & -1 & -1 & 315 & -1 & -1 & -1 & 14298436559 \\
18512985050 & -1 & -1 & -1 & -1 & -1 & 429 & -1 & -1 & 3192346000 \\
12894356669 & 16 & -1 & -1 & -1 & -1 & -1 & 128 & 1109889120 \\
180755200 & 16 & -1 & -1 & -1 & -1 & -1 & -1 & 4763145523 \\
4763145523 & 32 & 16 & -1 & -1 & -1 & -1 & -1 & -1 & 338973990 \\
152460000 & 16 & -1 & -1 & -1 & -1 & -1 & -1 & 4763145523 \\
4763145523 & 64 & -1 & -1 & -1 & -1 & -1 & -1 & 4763145523 \\
4763145523 & 128 & 64 & -1 & -1 & -1 & -1 & -1 & 4763145523 \\
\end{bmatrix} \\
\]

and

\( R_1 = \begin{bmatrix}
628772602482 & 339 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 54240000 \\
334182018661 & 0 & 1395 & 0 & 0 & 0 & 0 & 0 & 0 & 4763145523 \\
944509299661 & 0 & 0 & 615 & 0 & 0 & 0 & 0 & 0 & 4763145523 \\
13338639074644 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 4763145523 \\
15775591525 & 0 & 0 & 0 & 0 & 1245 & 0 & 0 & 0 & 4763145523 \\
66604373922 & 0 & 0 & 0 & 0 & 0 & 429 & 0 & 0 & 0 \\
7920700564 & 0 & 0 & 0 & 0 & 0 & 0 & 4763145523 \\
4763145523 & 0 & 0 & 0 & 0 & 0 & 0 & 4763145523 \\
4763145523 & 0 & 0 & 0 & 0 & 0 & 0 & 4763145523 \\
952091046 & 0 & 0 & 0 & 0 & 0 & 0 & 5083894959 \\
135579900 & 0 & 0 & 0 & 0 & 0 & 0 & 4763145523 \\
4763145523 & 0 & 0 & 0 & 0 & 0 & 0 & 4763145523 \\
4763145523 & 0 & 0 & 0 & 0 & 0 & 0 & 4763145523 \\
51240000 & 0 & 0 & 0 & 0 & 0 & 0 & 80363610579 \\
4763145523 & 0 & 0 & 0 & 0 & 0 & 0 & 4763145523 \\
\end{bmatrix} \\
\]

Again, \( A = P_2 - R_2 + S_2 \) is a double proper regular splitting of \( A \) which satisfies
\( N(S_2) \supseteq N(P_2), \; R(S_2) \subseteq R(P_2), \; ||S_2P_1^t|| < 1 \) and \( \tilde{A}^t \geq 0 \). Here, we have

\( P_2 = \begin{bmatrix}
775256538861 & 418 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 64000000 \\
334182018661 & -1 & 414 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 467200000 \\
81253413200 & -1 & 414 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 4763145523 \\
100026055983 & -1 & -1 & 420 & -1 & -1 & -1 & -1 & -1 & -1 & 47289436559 \\
100026055983 & -1 & -1 & -1 & 420 & -1 & -1 & -1 & -1 & -1 & 47289436559 \\
15775591525 & -1 & -1 & -1 & -1 & 414 & -1 & -1 & -1 & -1 & 47289436559 \\
4197316000 & -1 & -1 & -1 & -1 & -1 & 414 & -1 & -1 & -1 & 47289436559 \\
12894356669 & 16 & -1 & -1 & -1 & -1 & -1 & 419 & -1 & -1 & 14298436559 \\
4763145523 & 32 & -1 & -1 & -1 & -1 & -1 & -1 & 4763145523 \\
123200000 & 32 & -1 & -1 & -1 & -1 & -1 & -1 & 4763145523 \\
4763145523 & 128 & 64 & -1 & -1 & -1 & -1 & -1 & 4763145523 \\
\end{bmatrix} \\
\]

and
Therefore, \(0.9720 = \rho(W_{12}) \leq 0.9752 = \rho(T_2) < 1\). The computational performance of the TG-ADS scheme with the double iteration scheme (2.1) is summarized in Table 4.

### Table 4: Comparison analysis for rectangular matrices

| Order | Method | n     | \(\|r_n\|\)     | \(\|e_n\|\)     | \(\rho\) | MT      |
|-------|--------|-------|----------------|----------------|--------|---------|
| 8 × 10| TG-ADS  | 626   | 1.0675e-6      | 9.8514e-8      | 0.9720 | 0.00469 |
|       | Method of \[17\] | 707   | 1.0592e-6      | 9.7765e-8      | 0.9752 | 0.00625 |
| 18 × 20| TG-ADS | 675   | 8.9624e-7      | 9.7691e-8      | 0.9751 | 0.01250 |
|       | Method of \[17\] | 897   | 9.0380e-7      | 9.8409e-8      | 0.9812 | 0.01406 |
| 28 × 30| TG-ADS | 678   | 9.8211e-7      | 9.9963e-8      | 0.9754 | 0.02609 |
|       | Method of \[17\] | 983   | 9.8307e-7      | 9.8492e-8      | 0.9826 | 0.03403 |
| 48 × 50| TG-ADS | 727   | 9.9325e-7      | 9.9956e-8      | 0.9812 | 0.04712 |
|       | Method of \[17\] | 1188  | 9.9534e-7      | 9.9867e-8      | 0.9963 | 0.06548 |

We have selected four rectangular matrices by the column extension of the diagonally dominant matrices of sizes 8, 18, 28 and 48, respectively. Example 4.2 is explained for a diagonally dominant matrix of size 8, whose columns are extended to 10. Explicitly, we have obtained two double splittings, which satisfy the necessary conditions such that the preconditioned matrix induced by the TG-ADS scheme has a convergent double proper regular(weak) splitting. This rectangular matrix \(A\) of size 8 × 10 is semi-monotone. Similarly, the rest of the three matrices can be shown as semi-monotone matrices. We have computed the error norm to make sure that the approximate solution is achieved within the required digit accuracy before the stopping criteria meet the tolerance.

We next generate a 38 × 40 semi-monotone matrix as in Example 4.2 to illustrate the residual and error of different iterative schemes. Residual & error norms are plotted against the iteration number in figure [2].
5. Conclusions

In this paper, we have proposed a new alternating scheme using double splittings (HT-ADS scheme) like the one introduced by Li et al. [23] in 2019, and studied the extension of both the schemes to rectangular matrix setting. The important findings are summarized as follows:

- Formulation of the proposed schemes: TG-ADS scheme and HT-ADS scheme, are shown in Section 3.1 in the rectangular matrix setting. Then, the convergence analysis is carried out in 3.2 for the class of double proper weak regular splittings. This is done by considering another preconditioned linear system which is induced by the ADS scheme.

- The significance of introducing ADS schemes are studied next. In this context, we have established several analytical results which justifies the importance by showing the faster convergence of the ADS schemes. We have also presented a few results which will guide us to choose a particular ADS scheme in case we have more than one same type of ADS schemes. More importantly, we have shown that one ADS scheme outperforms the other for a certain case. This is proved in Theorem 3.20.

- As illustrated in Example 4.1, there are substantial examples of linear systems of PDEs. Our numerical experiments with test matrices from different applications suggest that the ADS schemes are fairly robust. These computations also show that the ADS scheme performs better than some other existing schemes in the literature. Residual and error norms of the ADS scheme are monotonically convergent and faster than the double iteration scheme.
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