GLOBAL SOLUTIONS TO A ONE-DIMENSIONAL NON-CONSERVATIVE TWO-PHASE MODEL

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(Communicated by Eduard Feireisl)

Abstract. In this paper we investigate a basic one-dimensional viscous gas-liquid model based on the two-fluid model formulation. The gas is modeled as a polytropic gas whereas liquid is assumed to be incompressible. A main challenge with this model is the appearance of a non-conservative pressure term which possibly also blows up at transition to single-phase liquid flow (due to incompressible liquid). We investigate the model both in a finite domain (initial-boundary value problem) and in the whole space (Cauchy problem). We demonstrate that under appropriate smallness conditions on initial data we can obtain time-independent estimates which allow us to show existence and uniqueness of regular solutions as well as to gain insight into the long-time behavior of the model. These results rely strongly on the fact that we can derive appropriate upper and lower uniform bounds on the gas and liquid mass. In particular, the estimates guarantee that gas does not vanish at any point for any time when initial gas phase has a positive lower limit. The discussion of the Cauchy problem is general enough to take into account the possibility that the liquid phase may vanish at some points at initial time.

1. Introduction. A viscous version of the non-conservative equal-pressure two-fluid model in one dimension can be stated as follows [18]:

\[
\begin{align*}
  m_t + (mu)_x &= 0, \\
  n_t + (nu)_x &= 0, \\
  (mu)_t + (mu^2)_x + \alpha_l P_x &= -f_l u_l + \tilde{C}(u_g - u_l) + (\epsilon_l u_{l,x})_x, \\
  (nu)_t + (nu^2)_x + \alpha_g P_x &= -f_g u_g - \tilde{C}(u_g - u_l) + (\epsilon_g u_{g,x})_x,
\end{align*}
\]

for \((x,t) \in \Omega \times (0, \infty)\) where \(\Omega = (-L, L)\) and \(L \in (0, \infty)\). Here \(m = \alpha_l \rho_l\) represents liquid mass whereas \(n = \alpha_g \rho_g\) represents gas mass. The unknown variables \(\alpha_l, \alpha_g \in [0, 1]\) denote volume fractions satisfying the fundamental relation: \(\alpha_l + \alpha_g = 1\). Furthermore, the other unknown variables \(\rho_l, \rho_g, u_l\) and \(u_g\) denote, respectively, liquid and gas density, liquid and gas velocity; \(P\) is common pressure for both phases; \(\epsilon_l, \epsilon_g\) denote viscosity of the liquid and gas; \(f_l, f_g, \tilde{C}\) represent, respectively, wall frictional forces and interfacial forces.

2010 Mathematics Subject Classification. Primary: 76T10, 76N10; Secondary: 65M12, 35L60.
Key words and phrases. Two-fluid model, Navier-Stokes, existence, uniqueness, long-time behavior.
This kind of two-fluid gas-liquid model plays a crucial role in the design of different flow systems applied by the oil and gas industry. When the model is used to study deepwater wellbore operations there are many challenging phenomena that can take place. Some of them are: (i) dynamic transition zones separating two-phase and single-phase regions; (ii) strong expansion effects related to compressed gas which moves upwardly towards a lower pressure; (iii) complicated friction terms to take into account more realistic flow patterns; (iv) transition from one flow regime to another; (v) fluid flow between the wellbore and surrounding reservoir. A good understanding of mathematical properties of (1.1) is important both for increased insight into physical mechanisms that will dictate the behavior of the flow system as well as for the construction of reliable discrete versions of (1.1).

More generally, the application range of two-fluid based models is wide. Often, they are formulated similar to (1.1), however, without viscous terms in the momentum equations. On the other hand, it is well known that the resulting system may give rise to complex eigenvalues (loss of hyperbolicity), which has motivated researchers to include additional terms to make the model well-defined. For a discussion of different aspects of such two-fluid models and how to compute accurate solutions, we refer to [6, 8, 9] and references therein. We also refer to [14] for related studies of inviscid two-fluid models with incompressible fluids. Another variant of two-fluid models are obtained by assuming a pressure for each of the two phases leading to a two-pressure formulation. For a discussion of this and some results we refer to [12, 19].

A different class of two-fluid based models that are commonly used in the industry is the so-called drift-flux model where the two momentum equations have been replaced by a mixture momentum equation. In order to have a closed system an algebraic equation which relates the two fluid velocities is added. This slip-relation is derived from laboratory experiments and will typically impose a stronger coupling between the two phases which is natural for flow systems where the two phases show a stronger mixing. For some mathematical results on the drift-flux model where viscous effects are included we refer to [10, 20, 11] and references therein.

In this work we shall work with the model (1.1) in a context described as follows: Assuming that liquid is incompressible, i.e., $\rho_l = \text{constant} (> 0)$ whereas gas is polytropic, $P = \rho_g^\gamma$, $\gamma > 1$ is a constant, it follows that

$$P = P(m,n) = \left(\frac{\rho_l n}{\rho_l - m}\right)^\gamma.$$  

Moreover, it is also assumed that $\epsilon_l = \mu_l m$ and $\epsilon_g = \mu_g n$ where we, for simplicity reasons only, set $\mu_l = \mu_g = 1$. Frictional forces between wall and fluids as well interfacial friction are ignored, i.e., $f_l = f_g = 0$, and $\tilde{C} = 0$. Then system (1.1) takes the simplified form:

$$\begin{cases}
  m_t + (mu_l)_x = 0, \\
  n_t + (nu_g)_x = 0, \\
  (mu_l)_t + (mu_l^2)_x + \alpha_l P_x = (mu_l)_x, \\
  (nu_g)_t + (nu_g^2)_x + \alpha_g P_x = (nu_g)_x,
\end{cases}$$  

for $(x,t) \in \Omega \times (0, \infty)$. When $L < \infty$, we consider the following initial-boundary value problem:
\[(m, n, u_l, u_g)(x, 0) = (m_0, n_0, u_{l0}, u_{g0})(x) \text{ in } \Omega, \quad (1.4)\]

and

\[(u_l, u_g)(x, t) = 0 \text{ on } \partial \Omega \times [0, \infty). \quad (1.5)\]

When \(L = \infty\), i.e., \(\Omega = \mathbb{R}\), then we consider the initial value problem:

\[(m, n, mu_l, u_g)(x, 0) = (m_0, n_0, M_0, u_{g0})(x) \text{ in } \mathbb{R}. \quad (1.6)\]

Note that \(M_0\) represents the initial mass flux (momentum) and it is convenient to use this variable in order to take into account the possibility that liquid mass \(m_0\) may vanish at some points, see [15] for a discussion of this in the context of Navier-Stokes equations.

To our knowledge few mathematical results can be found for models similar to (1.3). The most prominent is the recent work by Bresch et al [5]. In the study of the initial-boundary value problem, we employ techniques that first were used in the seminal works by Bresch and collaborators [4, 5] for a two-fluid model and Li et al [15] for Navier-Stokes equations. A main difference between the one-dimensional model studied in [5] and the model we study is the pressure law. The model studied in [5] assumes two pressure-density relations of the form \(p = a \rho^\gamma\). A challenge with the model we study is that the liquid is assumed to be incompressible. Hence, the pressure function becomes different. In particular, we must ensure that pressure remains well-defined by obtaining a strict upper bound on the liquid mass (i.e., the gas phase will not vanish at any point in the spatial domain). Another extension from [5] is that we consider the Cauchy problem.

We are interested in both existence and uniqueness of (1.3) as well as information about the long time behavior, both on a bounded and unbounded domain. A rough summary of the results presented in this work is as follows: To obtain the basic energy equality, especially for the initial value problem (Cauchy problem), we introduce an energy function \(G\) for the pressure function which is designed to fit the model problem we consider. It takes the following form:

\[G(n, \rho_g, \tilde{\rho}_g) = n \int_{\tilde{\rho}_g}^{\rho_n} \frac{s^\gamma - (\tilde{\rho}_g)^\gamma}{s^2} ds.\]

Here \(\tilde{\rho}_g = \frac{\rho_l \tilde{n}}{\rho_l - m_0}\), with \(\rho_l > \tilde{m} > 0\) and \(\tilde{n} > 0\), represents a positive gas reference density. The role of \(\tilde{m}\) and \(\tilde{n}\) is clarified in Theorem 2.1. For the Cauchy problem, \(\tilde{m} = 0\) and \(\tilde{\rho}_g = \tilde{n}\).

Then we introduce the following two quantities in order to characterize the initial state

\[
\begin{align*}
E_0 &= \frac{1}{2} \int_\Omega \left( \frac{M_0^2}{m_0} + n_0 u_{g0}^2 \right) dx + \int_\Omega G(n_0, \rho_{g0}, \tilde{\rho}_g) dx, \\
E_1 &= 3E_0 + 4 \int_\Omega \left[ (\sqrt{m_0})^2 + (\sqrt{n_0})^2 \right] dx,
\end{align*}
\]

(1.7)

where \(\rho_{g0} = \frac{\rho_l n_0}{\rho_l - m_0}\). Appropriate smallness conditions on \(E_0\) and \(E_1\) will be necessary for the result we obtain in this work as summed up by Theorem 2.1–2.4. Note that for the initial condition (1.4), \(\frac{m_0^2}{m_0}\) in (1.7) is replaced by \(m_0 u_{g0}^2\) (since for this case \(m_0\) is assumed to possess a positive lower limit).

More precisely, for the initial-boundary value problem (1.3)–(1.5) we will assume that initial masses \(m_0\) and \(n_0\) have positive lower and upper bounds (which are related to \(\tilde{m}\) and \(\tilde{n}\)) and then show that we obtain uniform lower and upper bounds
for $m$ and $n$ as expressed by Theorem 2.1. Hence, for this case neither gas nor liquid mass will vanish at any point in the spatial domain. We also obtain information about the long-time behavior for this model problem as described by Theorem 2.2. For the Cauchy problem (1.3) and (1.6) we allow the initial liquid mass $n_0$ to vanish whereas $m_0$ does not. We then obtain uniform lower and upper bounds on $m$ and $n$ as expressed by Theorem 2.3. In particular, we must take into account the possibility that the liquid mass may vanish at some points. Finally, a characterization of the long-time behavior for the Cauchy problem is given by Theorem 2.4.

For both model problems, a special challenge is that we must ensure that the gas volume fraction $\alpha_g$ does not vanish at any point, i.e., $\sup_x \alpha_l < 1$. This is necessary in order to ensure that the pressure function $P(m,n)$, given by (1.2), makes sense and is defined through the gas density $\rho_g$. As far as this point is concerned a key estimate is the so-called BD entropy estimate developed by Bresch and Desjardins for Navier-Stokes equations, see for instance [1, 2, 3]. This estimate hangs on the fact that the viscosity coefficients take the form $\epsilon_g = n$ and $\epsilon_l = m$. Combining the BD estimate with the smallness assumption on the initial data represented by $E_0$ and $E_1$, we can obtain the desired uniform bounds on $m$ and $n$, both for the initial-boundary value problem and the Cauchy problem. However, in order to deal with the Cauchy problem where the liquid mass may vanish we must introduce an appropriate approximate system, see (4.2), where a constant viscosity term is added in the liquid momentum equation combined with additional first order terms in both the liquid and gas momentum equation. This approximate system is designed to fit our gas-liquid model problem where the liquid is incompressible.

In order to study the long-time behavior we need more regularities of the solutions, which usually requires that the masses have positive lower bounds. To achieve this, for the initial-boundary value problem we assume that the initial masses are positive and then get a positive bound for both gas and liquid mass for all time with the help of smallness assumptions. Note that the mass-dependent viscosity coefficients $\epsilon_l = m, \epsilon_q = n$ make it difficult to get the positive lower bounds when the initial data can be large. For the Cauchy problem, we still assume that the gas mass has a positive lower bound initially but assume that the initial liquid mass is non-negative, $m_0 \geq 0$. The reason why the initial assumptions on $m_0$ are different is that we have $\int_\Omega m = \int_\Omega m_0$ for the bounded interval $\Omega$ but not for the case $\Omega = \mathbb{R}$ if $\inf m_0 > 0$. Thus for the Cauchy problem, we assume that $m_0 \geq 0$ and $m_0 \in L^1(\mathbb{R})$. Then we still get $\int_\Omega m = \int_\Omega m_0$ which, combined with the estimate of $(\sqrt{m})_x$ (due to the BD type estimate) and Sobolev inequality implies that $m$ is bounded. It’s still unknown if $m$ has a positive lower bound for the Cauchy problem when $\inf m_0 > 0$ even if the initial data is small, since in this case we do not have any information about $m_0$ or $m_0 - \tilde{m}$ in some $L^p$ for some constant $\tilde{m}$. Fortunately, we can obtain an estimate of $n - \tilde{n}$ in $L^2$ thanks to the pressure function, the basic energy estimate and $m_0 \in L^1$. Hence, we can get a positive lower bound of $n$ by combining the estimate of $(\sqrt{n})_x$ and smallness assumptions. We refer to Lemma 4.4 for more details.

The paper is structured as follows. In Section 2, we present our main results which include the well-posedness and large-time behavior for the initial-boundary value problem and Cauchy problem. In Section 3, we give some a priori estimates globally in time and prove Theorems 2.1 and 2.2. In Section 4, based on some useful estimates of Section 3 uniformly for the size of the domain and $T$, the proofs of Theorems 2.3 and 2.4 are completed.
2. Main results. Throughout the rest of the paper, \(C\) \((C(T))\) denotes a generic positive constant depending on the initial data and the known constants \((T)\). For an integer \(n \geq 0\), \(H^n\) denotes the Sobolev space \(H^n(\Omega)\) with the usual norm \(\| \cdot \|_{H^n}\) and \(L^2 = H^0\); \(L^q\) denotes the Lebesgue space \(L^q(\Omega)\) with the usual norm \(\| \cdot \|_{L^q}\) with \(0 \leq q \leq \infty\). We denote \(\int_\Omega f\,dx\) by \(\int f\).

2.1. For \(L < \infty\). For the case \(L < \infty\), the main results are stated as follows.

**Theorem 2.1** (Existence). Assume that \((m_0, n_0) \in H^1(\Omega), (u_{l0}, u_{g0}) \in H^1_0(\Omega), and that for any given positive constants \(\bar{m}, \bar{n}, m_1, n_1, m_2, n_2, \) and \(\bar{m}, \bar{n}, m_1, n_1, m_2, n_2\), where \(m_1 \in (\bar{m}, \rho_1), m_2 \in (0, \bar{m}), n_1 \in (\bar{n}, \infty), and n_2 \in (0, \bar{n})\) such that

\[
\begin{align*}
m_1 &\leq \inf_x m_0 \leq \sup_x m_0 \leq m_1, \\
n_1 &\leq \inf_x n_0 \leq \sup_x n_0 \leq n_1.
\end{align*}
\]

Then there exists a positive constant \(\delta\) depending only on the initial data and the known constants such that (1.3)-(1.5) has a unique global solution \((m, n, u_l, u_g)\) satisfying

\[
\begin{align*}
(m, n) &\in C([0, \infty); H^1(\Omega)), \quad (m_l, n_l) \in C([0, \infty); L^2(\Omega)), \\
(u_l, u_g) &\in C([0, \infty); H^1_0(\Omega) \cap L^2([0, \infty); H^2(\Omega))), \\
(u_{l,t}, u_{g,t}) &\in L^2([0, \infty); L^2(\Omega)),
\end{align*}
\]

provided that

\[
E_2 \triangleq E_1 + \frac{1}{|\Omega|} \int_\Omega |m_0 - \bar{m}| + \frac{1}{|\Omega|} \int_\Omega |n_0 - \bar{n}| \leq \delta. \tag{2.2}
\]

Moreover, we have the following estimates uniformly for time:

\[
m \leq m \leq \bar{m}, \quad (x, t) \in \bar{\Omega} \times [0, \infty),
\]

\[
n \leq n \leq \bar{n}, \quad (x, t) \in \bar{\Omega} \times [0, \infty),
\]

and

\[
\|(m, n)\|_{H^1} + \|(m_l, n_l)\|_{L^2} + \|(u_l, u_g)\|_{H^1_0} \leq C.
\]

where \(m \in (\bar{m}, \rho), m \in (0, \bar{m}), \bar{n} \in (\bar{n}, \infty), and n \in (0, \bar{n})\).

**Remark 2.1.** The positive constant \(\delta\) is given by (3.31).

The following theorem gives the large time asymptotic behavior of the solution \((m, n, u_l, u_g)\) obtained in Theorem 2.1.

**Theorem 2.2** (Large-time behavior). Assume that the conditions in Theorem 2.1 hold. Then we have

\[
\lim_{t \to \infty} \int_\Omega (|u_{l,x}|^2 + |u_{g,x}|^2) = 0, \tag{2.3}
\]

and for any \(q' \geq 1\)

\[
\lim_{t \to \infty} \int_\Omega (|u_l|^{q'} + |u_g|^{q'}) = 0, \tag{2.4}
\]

\[
\lim_{t \to \infty} \|(\rho_g - \bar{\rho}_g)(\cdot, t)\|_{C(\bar{\Omega})} = 0, \tag{2.5}
\]

where \(\bar{\rho}_g = \frac{1}{|\Omega|} \int_\Omega \rho_g\).
Remark 2.2. We only obtain the large-time behavior of the velocities \( u_l, u_g \) and gas density \( \rho_g \). What can be said about the large-time behavior of \( m \) and \( n \)? This seems to be an open question.

2.2. For \( L = \infty \). For the case \( L = \infty \), i.e., \( \Omega = \mathbb{R} \).

Main assumptions.

\[
\begin{align*}
0 & \leq \inf_x m_0 \leq \sup_x m_0 \leq \overline{m}_1 < \rho_l, \ m_0 \in L^1, \ (\sqrt{m_0})_x \in L^2, \\
\var{m_0} & = 0 \ a.e. \mbox{ on } \var{m_0 = 0}, \ \frac{|\var{m_0}|}{\var{m_0}} \in L^1, \ \frac{|\var{m_0}|^2}{\var{m_0}^{2+\sigma}} \in L^1, \\
0 & < \overline{n}_1 \leq \inf_x n_0 \leq \sup_x n_0 \leq \overline{n}_1, \ (\sqrt{\var{n}_0})_x \in L^2, \ (m_0, n_0 - \overline{n}) \in H^1,
\end{align*}
\]

where \( \overline{m}_1, \overline{n}_1, \overline{n} \) and \( \sigma \) are positive constants.

The main results for the Cauchy problem are stated as follows.

Theorem 2.3 (Existence). In addition to (2.6), we assume that \( u_{g0} \in H^1(\mathbb{R}) \). Then there exists a positive constant \( \delta_x \) given by (4.33) such that the Cauchy problem (1.3) and (1.6) has a global solution \((m, n, u_l, u_g)\) for any \( T > 0 \) in the sense that

\[
\begin{align*}
& n - \overline{n} \in C([0, T]; H^1(\mathbb{R})), \ n_t \in C([0, T]; L^2(\mathbb{R})), \\
& m \in L^\infty([0, T]; H^1(\mathbb{R})) \cap L^\infty([0, T]; L^1(\mathbb{R})), \ m_t \in L^\infty([0, T]; H^{-1}), \\
& u_l \in C([0, T]; H^1(\mathbb{R})) \cap L^2([0, T]; H^2(\mathbb{R})), \ u_{g, t} \in L^2([0, T]; L^2(\mathbb{R})), \\
& \sqrt{m}u_l \in L^\infty([0, T]; L^2(\mathbb{R})), \ (\sqrt{m})_x \in L^\infty([0, T]; L^2(\mathbb{R})), \ P_e \in L^\infty([0, T]; L^2(\mathbb{R})),
\end{align*}
\]

and that

- the equations (1.3) and (1.6) are satisfied almost everywhere.
- the liquid mass equation (1.3) and the initial condition (1.6) are satisfied in sense of distribution.
- the liquid momentum equation (1.3) is satisfied as follows:

\[
\begin{align*}
\int_{\mathbb{R}} M_0(x) \varphi(x, 0) \, dx + \int_0^T \int_{\mathbb{R}} \left[ \sqrt{m} \sqrt{m} u_l \varphi_t + (\sqrt{m} u_l)^2 \varphi_x \right] \, dx \, dt \\
- \int_0^T \int_{\mathbb{R}} \alpha_l P_e \varphi \, dt + < m_{t, x}, \varphi_x > = 0,
\end{align*}
\]

for any \( T > 0 \) and any test function \( \varphi \in C_0^\infty(\mathbb{R} \times [0, T]) \), where \( \alpha_l = \frac{n}{m} \) and

\[
< m_{t, x}, \varphi_x > = \int_0^T \int_{\mathbb{R}} \sqrt{m} \sqrt{m} u_{l, x} \varphi_{xx} \, dx \, dt + 2 \int_0^T \int_{\mathbb{R}} (\sqrt{m})_x \sqrt{m} u_l \varphi_x \, dx \, dt,
\]

provided that \( E_0 + \mathring{E}_{30} + \int_{\mathbb{R}} m_0 \, dx \leq \delta_3 \) where \( \delta_3 \in (0, \frac{\delta_1}{16}) \) and

\[
\mathring{E}_{30} = \int_{\mathbb{R}} |(\sqrt{m_0})_x|^2 + |(\sqrt{m_0})_x|^2 \, dx.
\]

Moreover, we have the following estimates uniformly for time.

\[
0 \leq m \leq \overline{m}, \ (x, t) \in \mathbb{R} \times [0, \infty), \quad (2.7)
\]

\[
n \leq n \leq \overline{n}, \ (x, t) \in \mathbb{R} \times [0, \infty), \quad (2.8)
\]
where $\overline{m} \in (\overline{m}_1, \rho_1)$, $\overline{n} \in (\overline{n}_1, \infty)$, and $\underline{n} \in (0, \underline{n}_1)$, and
\[
\left\| n(\cdot, t) - \bar{n} \right\|_{H^1} + \|n_t(\cdot, t)\|_{L^2} + \left\| u_g(\cdot, t) \right\|_{H^1} + \int_0^t \left( \|u_{g,x}(\cdot, s)\|_{H^1}^2 + \|u_{g,t}\|_{L^2}^2 \right) \leq C,
\]
and
\[
\int_{\mathbb{R}} (mu_l^2 + |\sqrt{m}| x|^2) + \int_0^t \int_{\mathbb{R}} \rho_{g,x}^2 \leq C. \tag{2.10}
\]

**Theorem 2.4** (Large time behavior). Assume that the conditions in Theorem 2.3 hold. Then we have
\[
\lim_{t \to \infty} \int_{\mathbb{R}} \|u_{g,x}\|^2 = 0, \tag{2.11}
\]
and for any $q' > 2$
\[
\lim_{t \to \infty} \int_{\mathbb{R}} \|u_g\|_{L^{q'}}^2 = 0, \tag{2.12}
\]
\[
\lim_{t \to \infty} \|\rho_g - \tilde{\rho}_g\|_{L^\infty} = 0, \tag{2.13}
\]
where $\tilde{\rho}_g = \bar{n}$.

3. **Initial-boundary value problem.**

3.1. **Global existence.** For any given $L \in (0, \infty)$, we are going to use the classical strategy, i.e., local existence in combination with global a priori estimates in time, to prove the global existence. The local existence of the solutions as in Theorem 2.1 can be done by using the classical iteration arguments (see for instance [7]). We omit it for brevity. Thus, it suffices to obtain some a priori estimates globally in time.

Let’s begin with an important proposition which is about the upper bound of liquid density $m$.

**Proposition 3.1.** Under the conditions of Theorem 2.1, for any given $\overline{m} \in (\overline{m}_1, \rho_1)$, there exist a constant $\overline{m}_2 \in (\overline{m}_1, \overline{m})$ and a constant $\delta_1 > 0$ which depends only on the initial data and other known constants such that if $(m, n, u_l, u_g)$, a solution to (1.3)-(1.5) on $[-L, L] \times [0, T]$ for $T < T^*$ where $T^* > 0$ is the maximal existence time of the solution as in Theorem 2.1, satisfies
\[
m \leq \overline{m}, \quad (x, t) \in [-L, L] \times [0, T], \tag{3.1}
\]
then the following estimate holds:
\[
m \leq \overline{m}_2, \quad (x, t) \in [-L, L] \times [0, T], \tag{3.2}
\]
provided that $E_2 \leq \delta_1$ where $E_2$ is defined in (2.2).

**Proof.** Proposition 3.1 is an consequence of Lemma 3.3 below. \qed

We start with the following basic energy estimate for $(m, n, u_l, u_g)$.

**Lemma 3.1.** Under the assumptions of Proposition 3.1, it holds that for any $t \in [0, T]$
\[
\frac{1}{2} \int_{-L}^{L} (mu_l^2 + nu_g^2) + \int_{-L}^{L} G(n, \rho_g, \tilde{\rho}_g) + \int_0^t \int_{-L}^{L} (mu_{l.x}^2 + nu_{g.x}^2)
\]
\[ E_0 = \frac{1}{2} \int_{-L}^{L} (m_0 u_0^2 + n u_0^2) + \int_{-L}^{L} G(n_0, \rho_{g0}, \tilde{\rho}_g) := E_0. \] (3.3)

**Proof.** Multiplying (1.3)3 and (1.3)4 by \( u_t \) and \( u_g \), respectively, then integrating the resulting equations with respect to \( x \) over \((-L, L)\), and adding them up, we have

\[
\frac{1}{2} \frac{d}{dt} \int_{-L}^{L} (m u_t^2 + n u_g^2) + \int_{-L}^{L} [m u_{t,x}^2 + n u_{g,x}^2] = - \int_{-L}^{L} (\alpha_g u_g + \alpha_l u_l) P_x := I_1. \] (3.4)

In order to handle \( I_1 \) we introduce a new energy function designed for the pressure function in question, which is crucial, especially for the Cauchy problem:

\[ G(n, \rho_g, \tilde{\rho}_g) = n \int_{\tilde{\rho}_g}^{\rho_g} \frac{s^\gamma - (\tilde{\rho}_g)^\gamma}{s^2} ds, \quad \tilde{\rho}_g := \frac{\rho_g \tilde{n}}{\rho_g - \tilde{m}}. \] (3.5)

Then we have

\[
G_t = n_t \int_{\tilde{\rho}_g}^{\rho_g} \frac{s^\gamma - (\tilde{\rho}_g)^\gamma}{s^2} ds + n(\rho_g) \frac{(\rho_g)^\gamma - (\tilde{\rho}_g)^\gamma}{(\rho_g)^2} - \int_{\tilde{\rho}_g}^{\rho_g} \frac{\gamma s^{\gamma-1} (\tilde{\rho}_g)^\gamma - (\tilde{\rho}_g)^\gamma}{s^2} ds \bigg|_{x} + \alpha_g u_g (\rho_g) x + (\rho_g) \frac{(\rho_g)^\gamma - (\tilde{\rho}_g)^\gamma}{(\rho_g)^2} \bigg|_{x} + \alpha_l \int_{-L}^{L} G(n, \rho_g, \tilde{\rho}_g) = - \int_{-L}^{L} (\alpha_g u_g + \alpha_l u_l) P_x = I_1. \] (3.6)

Substituting (3.6) into (3.4), we have

\[
\frac{1}{2} \frac{d}{dt} \int_{-L}^{L} (m u_t^2 + n u_g^2) + \frac{d}{dt} \int_{-L}^{L} G(n, \rho_g, \tilde{\rho}_g) + \int_{-L}^{L} (m u_{t,x}^2 + n u_{g,x}^2) = 0. \] (3.7)

Integrating (3.7) with respect to \( t \) over \((0, t)\), we complete the proof of Lemma 3.1. \( \square \)

Based on ideas employed in [5] and a treatment of the pressure function similar to that used in Lemma 3.1, we get the next BD type of entropy estimate. The BD entropy estimate was first proposed by Bresch and Desjardins [1, 2, 3] in the context of compressible Navier-Stokes equations. This provides us with some useful information on higher regularity of the liquid and gas mass \((m \text{ and } n)\).
Lemma 3.2. Under the assumptions of Proposition 3.1, it holds that for any \( t \in [0, T] \)

\[
\frac{1}{2} \int_{-L}^{L} [m(u_t + \frac{m_x}{m})^2 + n(u_g + \frac{n_x}{n})^2] + \int_{-L}^{L} G(n, \rho_g, \rho_g) + \gamma \int_{0}^{t} \int_{-L}^{L} \alpha_g \rho_g^2 \rho_g^2 = \leq 2E_0 + 4 \int_{-L}^{L} [(\sqrt{m_0})^2 + (\sqrt{n_0})^2].
\]

(3.8)

Proof. From equation (1.3)\(_1\), we obtain

\[
m_{xt} + (m_x u_t)_x + (mu_t)_x = 0.
\]

(3.9)

Summing (3.9) and the equation (1.3)\(_3\) up yields

\[
[m(u_t + \frac{m_x}{m})]_t + [mu_t(u_t + \frac{m_x}{m})]_x = -\alpha_l P_x.
\]

(3.10)

Similarly, we have

\[
[n(u_g + \frac{n_x}{n})]_t + [nu_g(u_g + \frac{n_x}{n})]_x = -\alpha_g P_x.
\]

(3.11)

Multiplying (3.10) and (3.11) by \( u_t + \frac{m_x}{m} \) and \( u_g + \frac{n_x}{n} \) respectively, integrating with respect to \( x \) over \((-L, L)\), and adding the resulting equalities, we have

\[
\frac{1}{2} \frac{d}{dt} \int_{-L}^{L} [m(u_t + \frac{m_x}{m})^2 + n(u_g + \frac{n_x}{n})^2]
\]

\[= - \int_{-L}^{L} \alpha_l P_x(u_t + \frac{m_x}{m}) - \int_{-L}^{L} \alpha_g P_x(u_g + \frac{n_x}{n}) = I_2,
\]

(3.12)

where we have used (1.3)\(_1\) and (1.3)\(_2\).

Using (1.3)\(_1\), (1.3)\(_2\), (3.6) and the fact \( \alpha_l + \alpha_g = 1 \), \( I_2 \) can be handled as follows:

\[
I_2 = I_1 - \int_{-L}^{L} P_x[\alpha_l \frac{m_x}{m} + \alpha_g \frac{n_x}{n}]
\]

\[= - \frac{d}{dt} \int_{-L}^{L} P_x[\alpha_l \frac{m_x}{m} + \alpha_g \frac{n_x}{n} + \alpha_g \rho_g \rho_g] - \int_{-L}^{L} P_x[\alpha_l \frac{m_x}{m} + \alpha_g \rho_g \rho_g] + \gamma \int_{-L}^{L} \alpha_g \rho_g^2 \rho_g^2.
\]

(3.13)

Substituting the above equality into (3.12), we have

\[
\frac{1}{2} \frac{d}{dt} \int_{-L}^{L} [m(u_t + \frac{m_x}{m})^2 + n(u_g + \frac{n_x}{n})^2] + \frac{d}{dt} \int_{-L}^{L} G(n, \rho_g, \rho_g)
\]

\[+ \gamma \int_{-L}^{L} \alpha_g \rho_g^2 \rho_g^2 = 0.
\]

(3.14)

Integrating (3.14) over \((0, t)\), we get

\[
\int_{-L}^{L} \left[ \frac{m}{2} (u_t + \frac{m_x}{m})^2 + \frac{n}{2} (u_g + \frac{n_x}{n})^2 + G(n, \rho_g, \rho_g) \right] + \gamma \int_{0}^{t} \int_{-L}^{L} \alpha_g \rho_g^2 \rho_g^2
\]

\[= \int_{-L}^{L} \left[ \frac{m_0}{2} (u_0 + \frac{m_0 x}{m_0})^2 + \frac{n_0}{2} (u_g + \frac{n_0 x}{n_0})^2 + G(n_0, \rho_g, \rho_g) \right]
\]
Here is a corollary of Lemmas 3.1 and 3.2.

**Corollary 3.1.** Under the assumptions of Proposition 3.1, it holds that

$$
\int_{-L}^{L} [ (\sqrt{m})^2 + (\sqrt{n})^2 ] \leq E_1.
$$

Note that from (3.16) we directly obtain an upper bound of $\sqrt{m}$ by using that $\sqrt{m} \leq \frac{1}{2T} \int_{-L}^{L} \sqrt{m} + \int_{-L}^{L} |(\sqrt{m})_x|$ combined with Hölder’s inequality. However, this may not guarantee that the pressure $P(m, n)$ remains bounded. Hence, a finer upper bound of $m$, as expressed by Proposition 3.1, must be derived.

**Lemma 3.3.** There exist a constant $\overline{m}_2 \in (\overline{m}_1, \overline{m})$ and a constant $\delta_1 > 0$ which depends only on the initial data and other known constants, such that if the solution $(m, n, u_l, u_g)$ of (1.3)-(1.5) on $[-L, L] \times [0, T]$ satisfies (3.1), then

$$
m \leq \overline{m}_2 \text{ for } (x, t) \in [-L, L] \times [0, T],
$$

provided that $E_2 \leq \delta_1$.

**Proof.** From Corollary 3.1 and (3.1), we have

$$
|m(x, t) - \overline{m}| = \left| m(x, t) - \overline{m} - \frac{1}{2L} \int_{-L}^{L} (m - \overline{m}) + \frac{1}{2L} \int_{-L}^{L} (m - \overline{m}) \right|
$$

$$
= \left| m(x, t) - \frac{1}{2L} \int_{-L}^{L} m + \frac{1}{2L} \int_{-L}^{L} (m_0 - \overline{m}) \right|
$$

$$
= \frac{1}{2L} \int_{-L}^{L} \int_{y}^{x} m_\xi d\xi dy + \frac{1}{2L} \int_{-L}^{L} (m_0 - \overline{m}).
$$

This combined with Hölder inequality and the liquid mass equation deduces

$$
|m(x, t) - \overline{m}| \leq 2 \int_{-L}^{L} \sqrt{m} |(\sqrt{m})_x| + \frac{1}{2L} \int_{-L}^{L} |m_0 - \overline{m}|
$$

$$
\leq 2(\int_{-L}^{L} m) \frac{1}{2} E_1^\frac{1}{2} + \frac{1}{2L} \int_{-L}^{L} |m_0 - \overline{m}| \ (3.18)
$$

$$
= 2(\int_{-L}^{L} m_0) \frac{1}{2} E_1^\frac{1}{2} + \frac{1}{2L} \int_{-L}^{L} |m_0 - \overline{m}|.
$$

Let $\delta_1 > 0$ sufficiently small such that

$$
2\delta_1^\frac{1}{2} (\int_{-L}^{L} m_0) \frac{1}{2} + \delta_1 \leq \overline{m}_2 - \overline{m}.
$$

(3.19)

If we choose $E_2 \leq \delta_1$, we obtain from (3.18)

$$
m \leq 2\delta_1^\frac{1}{2} (\int_{-L}^{L} m_0) \frac{1}{2} + \delta_1 + \overline{m} \leq \overline{m}_2.
$$

(3.20)
**Remark 3.1.** With Lemma 3.3, the proof of Proposition 3.1 is complete. Based on Proposition 3.1 and the classical continuity arguments, (3.1) holds in \([-L, L] \times [0, T^*)\), i.e.,

\[ m \leq \underline{m} \quad \text{for} \quad (x, t) \in [-L, L] \times [0, T^*), \]

provided that \( E_2 \leq \delta_1 \), where \( T^* \) is the maximal existence time of the solution.

From (3.21), we have got the upper bound of \( m \). The next lemma is mainly about the upper and lower bounds of \( n \) and the lower bound of \( m \).

**Lemma 3.4.** There exists a positive constant \( \delta \) depending only on the initial data and other known constants, such that for any \( T \in (0, T^*) \)

\[ m \leq m \leq m < \rho, \quad (x, t) \in [-L, L] \times [0, T], \]

\[ n \leq n \leq n, \quad (x, t) \in [-L, L] \times [0, T], \]

provided that \( E_2 \leq \delta \), where \( m \in (0, m_1) \subseteq (0, \bar{m}) \), \( n \in (0, n_1) \subseteq (0, \bar{n}) \), and \( \pi \in (\bar{n}, \infty) \subseteq (\bar{n}, \infty) \).

**Proof.** Let \( \delta_2 > 0 \) satisfy

\[ 2\delta_2^\frac{1}{2} (\int_{-L}^{L} m_0)^\frac{1}{2} + \delta_2 \leq \bar{m} - \underline{m}. \]  

(3.24)

If we choose \( E_2 \leq \delta_2 \), we obtain from (3.18)

\[ m \geq \bar{m} - 2\delta_2^\frac{1}{2} (\int_{-L}^{L} m_0)^\frac{1}{2} - \delta_2 \geq \underline{m}. \]  

(3.25)

Using the similar arguments as that in (3.18), we have

\[ |n(x, t) - \bar{n}| \leq 2 \int_{-L}^{L} \sqrt{n} (|n|)_x + \frac{1}{2L} \int_{-L}^{L} |n_0 - \bar{n}| \]

\[ \leq 2(\int_{-L}^{L} n_0)^\frac{1}{2} E_1^\frac{1}{2} + \frac{1}{2L} \int_{-L}^{L} |n_0 - \bar{n}|. \]  

(3.26)

Let \( \delta_3 > 0 \) satisfy

\[ 2\delta_3^\frac{1}{2} (\int_{-L}^{L} n_0)^\frac{1}{2} + \delta_3 \leq \bar{n} - \underline{n}. \]  

(3.27)

If we choose \( E_2 \leq \delta_3 \), we obtain from (3.26)

\[ n \leq 2\delta_3^\frac{1}{2} (\int_{-L}^{L} n_0)^\frac{1}{2} + \delta_3 + \bar{n} \leq \bar{n}. \]  

(3.28)

Similarly, if we choose \( E_2 \leq \delta_4 \) where \( \delta_4 \) satisfies

\[ 2\delta_4^\frac{1}{2} (\int_{-L}^{L} n_0)^\frac{1}{2} + \delta_4 \leq \bar{n} \]

(3.29)

then

\[ n \geq \bar{n} - 2\delta_4^\frac{1}{2} (\int_{-L}^{L} n_0)^\frac{1}{2} - \delta_4 \geq \underline{n}. \]  

(3.30)

From (3.21), (3.25), (3.28) and (3.30), we complete the proof of Lemma 3.4, if we choose

\[ E_2 \leq \delta = \min \{ \delta_1, \delta_2, \delta_3, \delta_4 \}, \]

(3.31)

where \( \delta_1, \delta_2, \delta_3, \delta_4 \) are determined by (3.19), (3.24), (3.27), and (3.29), respectively.
We are going to obtain $H^1$ estimates of $(u_t, u_g)$.

**Lemma 3.5.** Under the assumptions of Theorem 2.1, it holds that

$$\int_{-L}^{L} (u_{t,x}^2 + u_{g,x}^2) + \int_{0}^{T} \int_{-L}^{L} (u_{t,t}^2 + u_{g,t}^2) \leq C \quad (3.32)$$

for a.e. $T \in (0, T^*)$.

**Proof.** Multiplying $(1.3)_4$ by $u_{g,t}$, integrating the resulting equality with respect to $x$ over $(-L, L)$, and using integration by parts, Cauchy inequality, $(1.3)_2$ and $(3.23)$, we have

$$\frac{1}{2} \frac{d}{dt} \int_{-L}^{L} n_t u_{g,x}^2 + \int_{-L}^{L} n_{u_{g,t}}^2$$

$$\leq \frac{1}{2} \int_{-L}^{L} n_t u_{g,x}^2 - \int_{-L}^{L} n_u u_{g,x} u_{g,t} - \frac{1}{\rho_t} \int_{-L}^{L} (\rho_t - m) u_{g,t} P_{x}$$

$$\leq -\frac{1}{2} \int_{-L}^{L} n_{u_{g,x}}^2 - \frac{1}{2} \int_{-L}^{L} n_u u_{g,x} u_{g,t} + \frac{1}{4} \int_{-L}^{L} n_{u_{g,t}}^2 + C \int_{-L}^{L} n_{u_{g,x}} u_{g,x}^2 + C \int_{-L}^{L} P_{x}^2$$

$$\leq C \|n_{u_{g,x}}\|_{L^\infty} \int_{-L}^{L} n_{u_{g,x}}^2 + \frac{1}{2} \int_{-L}^{L} \left( \frac{1}{n} \right) u_{g}(n_{u_{g,x}})^2 + \frac{1}{4} \int_{-L}^{L} n_{u_{g,t}}^2$$

$$+ C \int_{-L}^{L} n_{u_{g,x}} u_{g,x}^2 + C \int_{-L}^{L} P_{x}^2,$$

which, combined with integration by parts, $(1.3)_4$, $(3.23)$ and Cauchy inequality, deduces

$$\frac{1}{2} \frac{d}{dt} \int_{-L}^{L} n_{u_{g,x}}^2 + \frac{3}{4} \int_{-L}^{L} n_{u_{g,t}}^2$$

$$\leq C \|n_{u_{g,x}}\|_{L^\infty} \int_{-L}^{L} n_{u_{g,x}}^2 - \frac{1}{2} \int_{-L}^{L} \left( \frac{1}{n} \right) u_{g,x}(n_{u_{g,x}})^2 - \int_{-L}^{L} \frac{1}{n} u_{g}(n_{u_{g,x}}) u_{g,t} u_{g,x}$$

$$+ C \|u_{g}\|^2_{L^\infty} \int_{-L}^{L} n_{u_{g,x}}^2 + C \int_{-L}^{L} P_{x}^2$$

$$\leq C \|n_{u_{g,x}}\|_{L^\infty} \int_{-L}^{L} n_{u_{g,x}}^2 - \int_{-L}^{L} u_{g} u_{g,t} (n_{u_{g,x}} + \alpha_{g} P_{x})$$

$$+ C \|u_{g}\|^2_{L^\infty} \int_{-L}^{L} n_{u_{g,x}}^2 + C \int_{-L}^{L} P_{x}^2$$

$$\leq C \|n_{u_{g,x}}\|_{L^\infty} \int_{-L}^{L} n_{u_{g,x}}^2 + \frac{1}{4} \int_{-L}^{L} n_{u_{g,t}}^2 + C \|u_{g}\|^2_{L^\infty} \int_{-L}^{L} n_{u_{g,x}}^2 + C \int_{-L}^{L} P_{x}^2.$$

By Sobolev inequality, $(1.3)_4$, $(3.23)$ and Lemma 3.1, we get

$$\|n_{u_{g,x}}\|_{L^\infty} \leq C \|n_{u_{g,x}}\|_{L^2} + C \|(n_{u_{g,x}})_{x}\|_{L^2}$$

$$\leq C \|\sqrt{n} u_{g,x}\|_{L^2} + C \|u_{g,t} + n_{u_{g,t}} + \alpha_{g} P_{x}\|_{L^2} \quad (3.34)$$

$$\leq C \left( 1 + \|u_{g}\|_{L^\infty} \right) \|\sqrt{n} u_{g,x}\|_{L^2} + C \|\sqrt{n} u_{g,t}\|_{L^2} + C \|P_{x}\|_{L^2},$$

and

$$\|u_{g}\|_{L^\infty} \leq C \left( \|u_{g}\|_{L^2} + \|u_{g,x}\|_{L^2} \right) \leq C \left( 1 + \|\sqrt{n} u_{g,x}\|_{L^2} \right). \quad (3.35)$$
Substituting (3.34) and (3.35) into (3.33), we obtain
\[
\frac{1}{2} \frac{d}{dt} \int_{-L}^{L} nu_{g,x}^2 + \frac{1}{4} \int_{-L}^{L} nu_{g,t}^2 \leq C \left( \int_{-L}^{L} nu_{g,x}^2 \right)^2 + C \int_{-L}^{L} P_x^2 + \int_{-L}^{L} nu_{g,x}^2, \tag{3.36}
\]
where we have used Young’s inequality.

Then by (3.22), (3.23), Lemmas 3.1-3.2 and Gronwall’s inequality, we obtain
\[
\int_{-L}^{L} nu_{g,x}^2 \leq \exp \left( C \int_{0}^{T} \int_{-L}^{L} nu_{g,x}^2 \right) \left\{ \int_{-L}^{L} nu_{0,x}^2 + C \int_{0}^{T} \left( \int_{-L}^{L} nu_{g,x}^2 + \int_{-L}^{L} P_x^2 \right) \right\}, \tag{3.37}
\]
where we have used
\[
\int_{0}^{T} \int_{-L}^{L} P_x^2 \leq C \int_{0}^{T} \int_{-L}^{L} \alpha \rho \gamma \rho_x^2 \leq C. \tag{3.38}
\]

By (3.36) and (3.37), we have
\[
\int_{-L}^{L} nu_{g,x}^2 + \int_{0}^{T} \int_{-L}^{L} nu_{g,t}^2 \leq C. \tag{3.39}
\]

This combined with (3.23) concludes that
\[
\int_{-L}^{L} u_{g,x}^2 + \int_{0}^{T} \int_{-L}^{L} u_{g,t}^2 \leq C. \tag{3.40}
\]

**Lemma 3.6.** Under the assumptions of Theorem 2.1, it holds that
\[
\| (u_l, u_g) \|_{L^\infty} \leq C, \tag{3.41}
\]
and
\[
\int_{-L}^{L} (m_l^2 + n_l^2 + P_l^2) \leq C, \tag{3.42}
\]
for a.e. \( t \in (0, T^*) \).

**Proof.** (3.41) is a consequence of Lemma 3.1, Lemmas 3.4-3.5 and (3.35). From Corollary 3.1, (3.22), (3.23) and the expression of \( P \), we can get
\[
\int_{-L}^{L} (m_x^2 + n_x^2 + P_x^2) \leq C. \tag{3.43}
\]
(3.43) together with (1.3)\(_1\), (1.3)\(_2\) and Lemma 3.5 gives (3.42).

**Lemma 3.7.** Under the assumptions of Theorem 2.1, it holds that
\[
\int_{0}^{T} \int_{-L}^{L} (u_{l,xx}^2 + u_{g,xx}^2) \leq C, \tag{3.44}
\]
\[
\int_{0}^{T} (\| u_l \|_{L^\infty}^2 + \| u_g \|_{L^\infty}^2) \leq C, \tag{3.45}
\]
for a.e. $T \in (0, T^*)$.

**Proof.** It follows from (1.3)\(_4\) that

$$n u_{g,xx} = n u_{g,t} + n u_{g,x} + \alpha_g P_x - n x u_{g,x}. \quad (3.46)$$

Then by (3.23), (3.34), (3.41), (3.43), (3.38) and Lemmas 3.1, 3.5, we have

\[
\int_{0}^{T} \int_{-L}^{L} u_{g,xx}^{2} \leq C \int_{0}^{T} \int_{-L}^{L} u_{g,t}^{2} + C \int_{0}^{T} \left( \|u_{g}\|_{L^{\infty}}^{2} \int_{-L}^{L} u_{g,x}^{2} \right) + C \int_{0}^{T} \int_{-L}^{L} P_{x}^{2} \\
+ C \int_{0}^{T} \|u_{g,x}\|_{L^{\infty}}^{2} \int_{-L}^{L} n_{x}^{2} \\
\leq C + C \int_{0}^{T} \|u_{g,x}\|_{L^{\infty}}^{2} \int_{-L}^{L} n_{x}^{2} \\
\leq C + C \int_{0}^{T} \|u_{g,x}\|_{L^{\infty}}^{2} \leq C + C \int_{0}^{T} \int_{-L}^{L} u_{g,t}^{2} \leq C.
\]

In the meantime (in view of (3.34), (3.41), (3.38), and (3.32)), we also obtain

$$\int_{0}^{T} \|u_{g,x}\|_{L^{\infty}}^{2} \leq C.$$  

Similarly, we have

$$\int_{0}^{T} \int_{-L}^{L} u_{l,xx}^{2} \leq C,$$

and

$$\int_{0}^{T} \|u_{l,x}\|_{L^{\infty}}^{2} \leq C.$$

\[\square\]

**Conclusion.** Based on the *a priori* estimates globally in time as obtained in Section 3.1, it can be concluded that the maximal existence time of the solution $T^* = \infty$. Thus, the proof of the global existence part of Theorem 2.1 is complete.

### 3.2. Uniqueness

Now we are in a position to prove the uniqueness of the solutions as stated in Theorem 2.1. Let $(m, n, u_{l}, u_{g})$ and $(\hat{m}, \hat{n}, \hat{u}_{l}, \hat{u}_{g})$ be the two solutions to (1.3)-(1.5) with the same initial data and the same boundary conditions. At first, using the two liquid mass equations

$$m_{t} + (mu_{l})_{x} = 0,$$

and

$$\hat{m}_{t} + (\hat{m}\hat{u}_{l})_{x} = 0,$$

we can obtain

$$(m - \hat{m})_{t} = -(m - \hat{m})_{x}u_{l} - \hat{m}_{x}(u_{l} - \hat{u}_{l}) - (m - \hat{m})\hat{u}_{l,x} - m(u_{l} - \hat{u}_{l})_{x}. \quad (3.47)$$

Then, using the two liquid momentum equations

$$(mu_{l})_{t} + (mu_{l}^{2})_{x} + \frac{m}{\rho_{l}} P_{x} = (mu_{l,x})_{x},$$

and

$$(\hat{m}\hat{u}_{l})_{t} + (\hat{m}\hat{u}_{l}^{2})_{x} + \frac{\hat{m}}{\rho_{l}} P_{x} = (\hat{m}\hat{u}_{l,x})_{x},$$
we can obtain
\[
m(u_t - \dot{u}_t)_t + mu_t(u_t - \dot{u}_t)_x = -(m - \dot{m})\dot{u}_t(x) - m(u_t - \dot{u}_t)_x \rho_t
\]
\[
-\frac{1}{\rho_t}(m - \dot{m})\dot{P}_x + ((m - \dot{m})\dot{u}_l(x))_x. \quad (3.48)
\]

Similarly, we have
\[
(n - \dot{n})_t = -(n - \dot{n})_x u_g - \dot{n}_x(u_g - \dot{u}_g) - (n - \dot{n})\dot{u}_g(x) - n(u_g - \dot{u}_g)_x, \quad (3.49)
\]
and
\[
n(u_g - \dot{u}_g)_t + nu_g(u_g - \dot{u}_g)_x = -(n - \dot{n})\dot{u}_g(x) - n(u_g - \dot{u}_g)_x
\]
\[
-\frac{1}{\rho_t}(m - \dot{m})\dot{P}_x + ((n - \dot{n})\dot{u}_g(x))_x. \quad (3.50)
\]

Next, multiplying (3.47) and (3.49) by \((m - \dot{m})\) and \((n - \dot{n})\), respectively, then integrating the resulting equations with respect to \(x\) over \((-L, L)\), and adding them up, we can easily get
\[
\frac{1}{2} \frac{d}{dt} \int_{-L}^{L} [(m - \dot{m})^2 + (n - \dot{n})^2]
\]
\[
\leq C(\|u_t\|_{L^\infty} + \|u_g\|_{L^\infty} + \|\dot{u}_t\|_{L^2}^2 + \|\dot{u}_g\|_{L^2}^2)
\]
\[
	imes (\|m - \dot{m}\|_{L^2}^2 + \|n - \dot{n}\|_{L^2}^2 + \|\sqrt{m}(u_t - \dot{u}_t)\|_{L^2}^2 + \|\sqrt{n}(u_g - \dot{u}_g)\|_{L^2}^2)
\]
\[
+ \frac{1}{4}(\|\sqrt{m}(u_t - \dot{u}_t)\|_{L^2}^2 + \|\sqrt{n}(u_g - \dot{u}_g)\|_{L^2}^2). \quad (3.51)
\]

On the other hand, multiplying (3.48) and (3.50) by \((u_t - \dot{u}_t)\) and \((u_g - \ddot{u}_g)\), respectively, then integrating the resulting equations with respect to \(x\) over \((-L, L)\), and adding them up, we can obtain
\[
\frac{1}{2} \frac{d}{dt} \int_{-L}^{L} [m(u_t - \dot{u}_t)^2 + n(u_g - \dot{u}_g)^2] + \int_{-L}^{L} [m(u_t - \dot{u}_t)_x^2 + n(u_g - \dot{u}_g)_x^2]
\]
\[
\leq C(\|u_t\|_{L^\infty}^2 + \|u_g\|_{L^\infty}^2 + \|\dot{u}_t\|_{L^2}^2 + \|\dot{u}_g\|_{L^2}^2)
\]
\[
\times (\|m - \dot{m}\|_{L^2}^2 + \|n - \dot{n}\|_{L^2}^2 + \|\sqrt{m}(u_t - \dot{u}_t)\|_{L^2}^2 + \|\sqrt{n}(u_g - \dot{u}_g)\|_{L^2}^2)
\]
\[
+ \frac{1}{4}(\|\sqrt{m}(u_t - \dot{u}_t)\|_{L^2}^2 + \|\sqrt{n}(u_g - \dot{u}_g)\|_{L^2}^2). \quad (3.52)
\]

Combining (3.51) and (3.52) yields
\[
\frac{d}{dt}(\|m - \dot{m}\|_{L^2}^2 + \|n - \dot{n}\|_{L^2}^2 + \|\sqrt{m}(u_t - \dot{u}_t)\|_{L^2}^2 + \|\sqrt{n}(u_g - \dot{u}_g)\|_{L^2}^2)
\]
\[
\leq CA(t)(\|m - \dot{m}\|_{L^2}^2 + \|n - \dot{n}\|_{L^2}^2 + \|\sqrt{m}(u_t - \dot{u}_t)\|_{L^2}^2 + \|\sqrt{n}(u_g - \dot{u}_g)\|_{L^2}^2),
\]
where
\[
A(t) = \|u_t\|_{L^\infty} + \|u_g\|_{L^\infty} + \|\dot{u}_t\|_{L^2}^2 + \|\dot{u}_g\|_{L^2}^2 + \|P_t\|_{L^2}^2 + \|\dot{P}_x\|_{L^2}^2
\]
\[
+ \|u_t\|_{L^\infty} + \|u_g\|_{L^\infty} + \|\dot{u}_t\|_{L^2}^2 + \|\dot{u}_g\|_{L^2}^2,
\]
which belongs to \(L^1(0, T)\) for any \(T < \infty\). In view of Gronwall’s inequality and the fact that the initial data of the two solutions is the same, we have
\[
\|m - \dot{m}\|_{L^2}^2 + \|n - \dot{n}\|_{L^2}^2 + \|\sqrt{m}(u_t - \dot{u}_t)\|_{L^2}^2 + \|\sqrt{n}(u_g - \dot{u}_g)\|_{L^2}^2 = 0.
\]
Thus, \((m, n, u_l, u_g) = (\hat{m}, \hat{n}, \hat{u}_l, \hat{u}_g)\) in \([-L, L] \times [0, \infty)\) where we have used \(m > 0\) and \(n > 0\). The proof of the uniqueness is complete.

### 3.3. Large-time behavior.

In this subsection, we will explore the large time asymptotic behavior of the solution \((m, n, u_l, u_g)\) as defined in Theorem 2.1. The precise result is stated in Theorem 2.2.

**Proof of Theorem 2.2.** The equation (1.3) can be rewritten as follows:

\[
\dot{u}_g + \alpha \rho_{g,x} = (n\dot{u}_g)_x \quad (3.53)
\]

where \(\dot{u}_g = u_{g,t} + u_g u_{g,x}\).

Multiplying (3.53) by \(\dot{u}_g\), and integrating the resulting equality with respect to \(x\) over \([-L, L]\), we have

\[
\int_{-L}^L n(\dot{u}_g)^2 + \frac{1}{2} I'(t) = -\int_{-L}^L \alpha \rho_{g,x} \dot{u}_g - \int_{-L}^L n u_g^3, \quad (3.54)
\]

where \(I(t) = \int_{-L}^L n u_g^2\).

Then from (3.54), we have

\[
\int_0^T |I'(t)| \leq C \int_0^T \int_{-L}^L n(\dot{u}_g)^2 + C \int_0^T \int_{-L}^L \rho_{g,x}^2 + C \int_0^T \int_{-L}^L u_g^4, \quad (3.55)
\]

where we have used (3.3), (3.22), (3.23), (3.38), (3.41), and Lemmas 3.5 and 3.7.

From (3.55) and Lemma 3.1 (which implies \(\int_0^T I(t) \leq C\)), we have

\[
\lim_{t \to \infty} I(t) = 0,
\]

which combined with (3.23) implies

\[
\lim_{t \to \infty} \int_{-L}^L |u_{g,x}|^2 = 0.
\]

Similarly, we have

\[
\lim_{t \to \infty} \int_{-L}^L |u_{l,x}|^2 = 0.
\]

Then we obtain (2.3). (2.4) is an easy consequence of (2.3) combined with the boundary condition and Sobolev inequality.

From (3.22), (3.23), Corollary 3.1 and Lemma 3.2, we have

\[
\sup_{0 \leq t \leq T} (\|\rho_g\|_{L^\infty} + \|\rho_g^{-1}\|_{L^\infty} + \|\rho_{g,x}\|_{L^2}) + \int_0^T \|\rho_{g,x}\|_{L^2}^2 \leq C. \quad (3.56)
\]

Define

\[
g(t) = \int_{-L}^L |\rho_g - \bar{\rho}_g|^4, \quad (3.57)
\]
where
\[ \tilde{\rho}_g = \frac{1}{2L} \int_{-L}^{L} \rho_g. \]

**Claim.**
\[ g(t) \to 0, \text{ as } t \to \infty. \] (3.58)

In fact, from (3.56) and the Poincaré–Sobolev inequality, we have
\[ \int_0^T g(t) dt \leq C \sup_{0 \leq t \leq T} \| \rho_g - \tilde{\rho}_g \|_{L^\infty} \int_0^T \int_{-L}^L (\rho_g - \tilde{\rho}_g)^2 \leq C \epsilon^2 \int_0^T \int_{-L}^L \rho_{g,x}^2 \leq C \epsilon^2. \] (3.59)

On the other hand,
\[ |g'(t)| = \left| 4 \int_{-L}^{L} (\rho_g - \tilde{\rho}_g)^3 \rho_g, t - 4(\tilde{\rho}_g)_t \int_{-L}^{L} (\rho_g - \tilde{\rho}_g)^3 \right| \]
\[ \leq C(L) \int_{-L}^{L} |\rho_g - \tilde{\rho}_g|^3 (|m_t| + |n_t|) + C(L) (\|m_t\|_{L^2} + \|n_t\|_{L^2}) \| \rho_g - \tilde{\rho}_g \|_{L^\infty}^2 \]
\[ \leq C(L) \| \rho_g - \tilde{\rho}_g \|_{L^\infty}^2 \leq C(L) \| \rho_g - \tilde{\rho}_g \|_{L^2}^2 + C(L) \| \rho_{g,x} \|_{L^2}^2 \]
\[ \leq C(L) \| \rho_{g,x} \|_{L^2}^2, \]
where we have used (3.22), (3.23), Hölder inequality, Poincaré–Sobolev inequality and (3.42). This combined with (3.56) implies
\[ \int_0^T |g'(t)| \leq C(L). \] (3.60)

Since the upper bounds in (3.59) and (3.60) are independent of \( T \), we get (3.58).

From (3.56) and the Gagliardo–Nirenberg interpolation inequality, we have
\[ \| (\rho_g - \tilde{\rho}_g)(\cdot, t) \|_{C([-L,L])} \leq C(L) \| (\rho_g - \tilde{\rho}_g)(\cdot, t) \|_{L^2}^{\frac{1}{2}} \| \rho_{g,x} \|_{L^2}^{\frac{1}{2}} \]
\[ \leq C(L) \| (\rho_g - \tilde{\rho}_g)(\cdot, t) \|_{L^2}^{\frac{1}{2}} \to 0, \text{ as } t \to \infty. \] (3.61)

The proof of Theorem 2.2 is complete. \( \square \)

4. **Cauchy problem.** In this section we focus on the Cauchy problem. Note that we will allow for the possibility that the initial liquid mass \( m_0(x) \) may vanish at some points, as described by the assumptions stated in (2.6). In order to deal with this new situation we define an appropriate approximate system on the interval \([-L,L]\), see (4.2)–(4.4). This is defined to fit the model problem we study where the pressure \( P(m,n) \) reflects that liquid is incompressible. In particular, since the initial gas \( n_0 \) has a positive lower limit, no additional constant viscosity term is added to the gas momentum equation, in contrast to what is done for the liquid momentum equation. However, an approximate first order term of the form \( \epsilon^{1/4}[\rho^{2\gamma}]_x \) is still included (in the gas momentum equation) and seems necessary in order to derive a BD type of estimate, see Lemma 4.2. Note that, clearly, an additional challenge with the Cauchy problem is to derive estimates that are independent of both \( L, \epsilon, \) and \( T \).

4.1. **Existence.**

4.1.1. **An auxiliary theorem.** In order to apply the conclusion of Theorem 2.1, we need to mollify the initial data as follows.
For sufficiently large $L \in (1, \infty)$ and any fixed $\epsilon \equiv \epsilon(L) \in (0, 1) \cap (0, \rho_L - \overline{m}_1) \cap (0, \frac{1}{L^2})$. Assume that $\eta(\cdot) = \frac{1}{\sqrt{2\pi}} \psi(\cdot)$ where $\psi$ is the usual mollifier. Denote

$$
\begin{align*}
& m^L_0 = m_0 + \epsilon(L), \quad n^L_0 = n_0, \\
& u^L_{0t} = \psi_L \frac{1}{m_0^{\frac{1}{2}+\sigma}} \eta \ast (M_0 m_0^{-\frac{1}{2}+\sigma}), \\
& u^L_{0x} = \psi_L \eta \ast u_{0x},
\end{align*}
$$

where $\psi_L \in C_0^\infty(-L,L)$, $\psi_L = 1$ in $[-\frac{L}{2}, \frac{L}{2}]$, $\psi_L = 0$ in $[-L,L]/[-\frac{2L}{3}, \frac{2L}{3}]$ and $|\psi_L'| \leq \frac{\epsilon}{L}$. Then $m^L_0 \geq \epsilon(L) > 0$, $(u^L_{0t}, u^L_{0x}) \in C_0^\infty(-L,L)$, and

$$
\begin{align*}
m^L_0 \rightarrow m_0 \text{ in } L^1_{\text{loc}}(\mathbb{R}) \cap H^1_{\text{loc}}(\mathbb{R}), \quad (\sqrt{m^L_0}) \rightarrow (\sqrt{m_0}) \text{ in } L^2(\mathbb{R}), \\
\left(\frac{|m^L_0 u^L_{0x}|^2}{m^L_0}, \frac{|m^L_0 u^L_{0x}|^{2+\sigma}}{m^L_0^{1+\sigma}}\right) \rightarrow \left(\frac{|m_0^2}{m_0}, \frac{|m_0^{2+\sigma}}{m_0^{1+\sigma}}\right) \text{ in } L^1(\mathbb{R}), \quad u^L_{0x} \rightarrow u_{0x} \text{ in } H^1(\mathbb{R})
\end{align*}
$$

as $L \rightarrow \infty$.

We consider an approximate system of the Cauchy problem of (1.3) and (1.6):

$$
\begin{align*}
& m_t + (mu)_t = 0, \\
& n_t + (nu)_t = 0, \\
& (mu)_t + [m(u)_2] + \alpha_t([\rho]_2) + \epsilon^2 m_x = [(m + \epsilon)u]_x, \\
& (nu)_t + [n(u)_2] + \alpha_t([\rho]_2) + \epsilon^2 [n^{2\gamma}] = (nu)_{x},
\end{align*}
$$

with the initial condition:

$$
(m,n,u_l,u_g)(x,0) = (m^L_0, n^L_0, u^L_{0t}, u^L_{0x})(x) \quad \text{for } x \in (-L,L),
$$

and the boundary condition:

$$
(u_l,u_g)(-L,t) = (u_l,u_g)(L,t) = 0 \quad \text{for } t \in [0, \infty),
$$

where $\alpha_t = \frac{m}{\rho_l}, \rho_l = \frac{\rho_l - m}{\rho_l - \overline{m}_1}$, and $\alpha_g = \frac{\rho_l - m}{\rho_l}$.

Along the lines of the proof of Theorem 2.1, one can get the following theorem about the global well posedness of (4.2)-(4.4) (the global solution $(m^L, n^L, u^L_t, u^L_{0x})$ is still denoted by $(m, n, u_l, u_g)$):

**Theorem 4.1.** Under the assumptions of Theorem 2.3, there exists a positive constant $\delta_2$ depending only on the initial data and some other known constants but independent of $L$ and $T$, such that for given $L$, (4.2)-(4.4) has a global solution $(m, n, u_l, u_g)$ satisfying

$$
\begin{align*}
& (m, n) \in C([0, \infty); H^1), \quad (m_t, n_t) \in C([0, \infty); L^2), \\
& (u_l, u_g) \in C([0, \infty); H^2) \cap L^2([0, \infty); H^2), \\
& (u_{lt}, u_{tx}) \in L^2([0, \infty); L^2),
\end{align*}
$$

provided that $E_0^L + E_{3,0}^L + \int_{-L}^L m_0 \leq \delta_2$ where $\delta_2$ is given by (4.33) and $E_0^L$ and $E_{3,0}^L$ are given by (4.7) and (4.9).

Moreover, we have the following estimates uniformly for time.

$$
0 < C(L) \leq m \leq \overline{m}, \quad (x,t) \in [-L,L] \times [0, \infty),
$$
Lemma 4.1. Under the assumptions of Proposition 4.1, it holds that

\[ \tilde{m} \leq m \leq \bar{m}, \quad (x, t) \in [-L, L] \times [0, \infty), \]

where \( \bar{m} \in (\bar{m}_1, \rho_1), \) \( \bar{n} \in (\bar{n}_1, \infty), \) and \( m \in (0, m_1), \) and

\[ \| n(\cdot, t) - \bar{n} \|_{H^1} + \| n_u(\cdot, t) \|_{L^2} + \| u_g(\cdot, t) \|_{H^1} + \int_{0}^{t} (\| u_{g,x}(\cdot, s) \|_{H^1} + \| u_{g,t} \|_{L^2}^2) \, ds \leq C, \]

and

\[ \int_{-L}^{L} (m u_t^2 + m + |(\sqrt{m})_x|^2) + \int_{0}^{t} \int_{-L}^{L} [m + \epsilon(L)] u_{t,x}^2 \leq C, \]

and

\[ \int_{0}^{t} \int_{-L}^{L} \left[ \rho_{g,x}^2 + [\epsilon(L)]^{\gamma} m_x^2 + [\epsilon(L)]^{\gamma} n_x^2 \right] \leq C, \]

where \( C \) is independent of \( t \) and \( L. \)

Proof of Theorem 4.1. The local existence can be obtained by using similar arguments as in [7], we omit it here for brevity. Then we will obtain some *a priori* estimates (global in time) of the solution to the approximate system uniformly for \( L. \) In fact, some of them can be obtained by using the same arguments as in Section 3.1. Based on the local existence and these *a priori* estimates global in time, we can prove Theorem 4.1. \( \square \)

More precisely, we begin with an important Proposition which is used to get the upper bound of \( m. \)

**Proposition 4.1.** Under the conditions of Theorem 4.1, for any given \( \bar{m} \in (\bar{m}_1, \rho_1), \) there exist a constant \( \bar{m}_2 \in (\bar{m}_1, \bar{m}) \) and a constant \( \bar{\delta}_1 > 0 \) which depends only on the initial data and other known constants such that if \( (m, n, u_1, u_g), \) a solution to (4.2)-(4.4) on \([-L, L] \times [0, T] \) for \( T < T^* \), where \( T^* > 0 \) is the maximal existence time of the solution as in Theorem 4.1, satisfies

\[ m \leq \bar{m}, \quad (x, t) \in [-L, L] \times [0, T], \]

then the following estimate holds:

\[ m \leq \bar{m}_2, \quad (x, t) \in [-L, L] \times [0, T], \]

provided that \( E_0^L + E_{3,0}^L \leq \bar{\delta}_1. \)

The proof of this proposition is based on the next two lemmas.

**Lemma 4.1.** Under the assumptions of Proposition 4.1, it holds that

\[ E^L(t) + \int_{0}^{t} \int_{-L}^{L} [(m + \epsilon(L)) u_{t,x}^2 + n u_{g,x}^2] = E^L(0) = E_0^L, \]

where

\[ E^L = \int_{-L}^{L} \left[ \frac{1}{2} m u_t^2 + \frac{1}{2} n u_g^2 + G(n, \rho_g, \tilde{\rho}_g^L) \right. \]

\[ + [\epsilon(L)]^{\gamma} \left( m \log \frac{m}{\epsilon(L)} - m + \epsilon(L) \right) + \left[ \epsilon(L) \right]^{\gamma} \frac{n^{2\gamma}}{2\gamma - 1} \],

Here \( \tilde{\rho}_g^L = \frac{\rho \bar{\alpha}}{\rho_1 - \epsilon(L)}. \)
Proof. Multiplying $(4.2)_3$ and $(4.2)_4$ by $u_l$ and $u_g$, respectively, integrating the resulting equations with respect to $x$ over $(-L,L)$, and then using some similar arguments as in Lemma 3.1, we have

$$\frac{d}{dt} \int_{-L}^{L} \left[ \frac{1}{2} m u_l^2 + \frac{1}{2} m u_g^2 + G(n, \rho_g, \tilde{\rho}_g L) \right] + \int_{-L}^{L} [(m + \epsilon) u_{l,x} + n u_{g,x}^2]$$

$$= \frac{\epsilon}{2} \int_{-L}^{L} (m - \epsilon) u_{l,x} + \frac{\epsilon}{2} \int_{-L}^{L} n^{2\gamma} u_{g,x}$$

$$= -\frac{\epsilon}{2} \frac{d}{dt} \int_{-L}^{L} (m \log \frac{m}{\epsilon} - m + \epsilon) - \frac{\epsilon}{2} \frac{d}{dt} \int_{-L}^{L} \frac{n^{2\gamma}}{2\gamma - 1}.$$ 

Thus, we have

$$\frac{d}{dt} \int_{-L}^{L} \left[ \frac{1}{2} m u_l^2 + \frac{1}{2} m u_g^2 + G(n, \rho_g, \tilde{\rho}_g L) + \epsilon \frac{1}{2} (m \log \frac{m}{\epsilon} - m + \epsilon) + \epsilon \frac{1}{2} \frac{n^{2\gamma}}{2\gamma - 1} \right]$$

$$+ \int_{-L}^{L} [(m + \epsilon) u_{l,x} + n u_{g,x}^2] = 0. \tag{4.8}$$

Integrating $(4.8)$ over $(0,t)$, we complete the proof of Lemma 4.1. \hfill \Box

Lemma 4.2. Under the assumptions of Proposition 4.1, it holds that

$$E_3^L(t) + \int_{0}^{t} \int_{-L}^{L} \left[ \gamma \alpha_g \rho_g - 2 \rho_{g,x} [\epsilon(L)] \frac{4}{3} \frac{m_x^2}{m} + \frac{[\epsilon(L)]^2}{2} \frac{m_x^2}{m^2} + \gamma [\epsilon(L)] \frac{4}{3} n^{2\gamma - 2} \frac{n_x^2}{m} \right]$$

$$= E_3^L(0) \equiv E_{3,0}^L, \tag{4.9}$$

where

$$E_3^L = \int_{-L}^{L} \left[ \frac{1}{2} m (u_l + \frac{m_x}{m})^2 + \frac{1}{2} m (u_g + \frac{n_x}{m})^2 + G(n, \rho_g, \tilde{\rho}_g L) + \frac{[\epsilon(L)]^2}{3} (m \log \frac{m}{\epsilon} - m + \epsilon(L)) + \frac{[\epsilon(L)]^2}{2} \frac{n^{2\gamma}}{2\gamma - 1} \right].$$

Proof. From $(4.2)_1$, we obtain

$$[m_x (1 + \frac{\epsilon}{m})]_t + [m_x (1 + \frac{\epsilon}{m}) u_l]_x + [(m + \epsilon) u_{l,x}]_x = 0. \tag{4.10}$$

Summing $(4.10)$ and $(4.2)_3$ up yields

$$[m (u_l + \frac{m_x}{m})]_t + [m_{ul} (u_l + \frac{m_x}{m}) + \frac{cm_{x}}{m^2}]_x = -\alpha_l P_x - \epsilon \frac{1}{2} m_x. \tag{4.11}$$

Similarly, we have

$$[n (u_g + \frac{n_x}{n})]_t + [n u_g (u_g + \frac{n_x}{n})]_x = -\alpha_g P_x - \epsilon \frac{1}{2} (n^{2\gamma})_x. \tag{4.12}$$

Multiplying $(4.11)$ and $(4.12)$ by $u_l + \frac{m_x}{m} + \frac{cm_{x}}{m^2}$ and $u_g + \frac{n_x}{n}$ respectively, integrating the result over $(-L,L)$, and using arguments similar to those of Lemmas 3.2 and
For the second term on the right hand side of (4.13), we have
\[ \frac{1}{2} \frac{d}{dt} \int_{-L}^{L} \left[ m(u_t + \frac{m_x}{m} + \frac{em_{xx}}{m^2})^2 + n(u_g + \frac{n_x}{n})^2 \right] \\
+ \frac{d}{dt} \int_{-L}^{L} G(n, \rho_g, \rho_g^L) + \gamma \int_{-L}^{L} \alpha_g \rho_g^{\gamma - 2} \rho_g^x \\
= - \epsilon \int_{-L}^{L} \alpha_l P_x \frac{m_x}{m^2} - \epsilon \int_{-L}^{L} m_x u_t - \epsilon \int_{-L}^{L} \frac{m_x^2}{m} - \epsilon \int_{-L}^{L} \frac{m_x^2}{m^2} \\
- \epsilon \int_{-L}^{L} (n^{2\gamma})_x u_g - 2\gamma \epsilon \int_{-L}^{L} n^{2\gamma - 2} n_x^2 \\
= - \frac{d}{dt} \int_{-L}^{L} \left[ \epsilon \frac{1}{\epsilon} (m \log \frac{m}{\epsilon} - m + \epsilon) + \frac{4\gamma}{2\gamma - 1} \right] - \epsilon \int_{-L}^{L} \alpha_l P_x \frac{m_x}{m^2} - \epsilon \int_{-L}^{L} \frac{m_x^2}{m} \\
- \epsilon \int_{-L}^{L} \frac{m_x^2}{m^2} - 2\gamma \epsilon \int_{-L}^{L} n^{2\gamma - 2} n_x^2. \] (4.13)

For the second term on the right hand side of (4.13), we have
\[ -\epsilon \int_{-L}^{L} \alpha_l P_x \frac{m_x}{m^2} = -\epsilon \int_{-L}^{L} m m_x \gamma(\frac{\rho_n}{\rho_l - \rho_n})^{\gamma - 1} \left( \frac{\rho_l n_x}{\rho_l - \rho_n} + \frac{\rho_l \rho_m n_x}{(\rho_l - \rho_n)^2} \right) \]
\[ \leq -\epsilon \int_{-L}^{L} \gamma(\frac{\rho_n}{\rho_l - \rho_n})^{\gamma - 1} m_x \frac{n_x}{m} \frac{n_x}{\rho_l - \rho_n} \] (4.14)
\[ \leq \epsilon \frac{\gamma}{2} \int_{-L}^{L} \frac{m_x^2}{m} + \frac{\gamma}{2} \frac{\rho_l^{\gamma - 2}}{(\rho_l - \rho_n)^{2\gamma}} \int_{-L}^{L} n^{2\gamma - 2} n_x^2. \]

Let \( \epsilon > 0 \) be sufficiently small such that
\[ \frac{\epsilon}{2} \frac{\gamma}{\rho_l - \rho_n} \leq 1. \] (4.15)

Then substituting (4.14) into (4.13), we have
\[ \frac{1}{2} \frac{d}{dt} \int_{-L}^{L} \left[ m(u_t + \frac{m_x}{m} + \frac{em_{xx}}{m^2})^2 + n(u_g + \frac{n_x}{n})^2 \right] \\
+ \frac{d}{dt} \int_{-L}^{L} G(n, \rho_g, \rho_g^L) + \gamma \int_{-L}^{L} \alpha_g \rho_g^{\gamma - 2} \rho_g^x \\
= - \frac{d}{dt} \int_{-L}^{L} \left[ \epsilon \frac{1}{\epsilon} (m \log \frac{m}{\epsilon} - m + \epsilon) + \frac{4\gamma}{2\gamma - 1} \right] - \epsilon \int_{-L}^{L} \frac{m_x^2}{m} \\
- \frac{3}{2} \int_{-L}^{L} \frac{m_x^2}{m^2} - \gamma \epsilon \int_{-L}^{L} n^{2\gamma - 2} n_x^2. \] (4.16)

Integrating (4.16) over \((0, t)\), we get (4.9).

**Corollary 4.1.** Under the assumptions of Proposition 4.1, it holds that
\[ \int_{-L}^{L} \left[ m_0 \frac{m_x^2}{m} + [\epsilon(L)]^2 \frac{m_x^2}{m^3} + \frac{n_x^2}{n} \right] \leq 4(E^L_0 + E^L_{\delta_1}). \] (4.17)

**Lemma 4.3.** There exist a constant \( \overline{m} \in (\overline{m}_1, \overline{m}) \) and a constant \( \tilde{\delta}_1 > 0 \) which depends only on the initial data and other known constants such that if the solution
(m, n, u_1, u_2) of (4.2)-(4.4) on \([-L, L] \times [0, T]\) satisfies (4.5), then
\[ m \leq m_2 \quad \text{for } (x, t) \in [-L, L] \times [0, T], \]
provided that \(E_{L0} + E_{L3,0} \leq \tilde{\delta}_1\).

**Proof.** Following the calculations of Lemma 3.3 with \(m_0\) replaced by \(m_0 L\), we have
\[
m(x, t) \leq 2 \int_{-L}^{L} \sqrt{m(|\sqrt{m}|_x)} + \frac{1}{2L} \int_{-L}^{L} m_0^L
\leq 2 \left( \int_{-L}^{L} m \right)^{\frac{1}{2}} \left( \int_{-L}^{L} \left(|\sqrt{m}|_x\right)^2 \right)^{\frac{1}{2}} + \frac{1}{2L} \int_{-L}^{L} m_0 + \epsilon
\leq 2 \left( \int_{-L}^{L} m_0 \right)^{\frac{1}{2}} (E_{L0}^L + E_{L3,0}^L)^{\frac{1}{2}} + \bar{m}_1 + \epsilon
\leq 2 (2L \epsilon + \int_{-L}^{L} m_0)^{\frac{1}{2}} (E_{L0}^L + E_{L3,0}^L)^{\frac{1}{2}} + \bar{m}_1 + \epsilon.
\]
Let \(\tilde{\delta}_1 > 0\) and
\[
\epsilon \triangleq \epsilon(L) \leq \frac{1}{L},
\]
and
\[
2 (2L \epsilon + \int_{-L}^{L} m_0)^{\frac{1}{2}} \tilde{\delta}_1^{\frac{1}{2}} + \epsilon \leq \bar{m}_2 - \bar{m}_1.
\]
Then we obtain from (4.19) and (4.21)
\[
m \leq 2 (2L \epsilon + \int_{-L}^{L} m_0)^{\frac{1}{2}} \tilde{\delta}_1^{\frac{1}{2}} + \epsilon + \bar{m}_1 \leq \bar{m}_2,
\]
provided that \(E_{L0}^L + E_{L3,0}^L \leq \tilde{\delta}_1\).

The proof of Proposition 4.1 is complete. Based on it, we have the following corollary.

**Corollary 4.2.** Under the conditions of Theorem 4.1, it holds that
\[
m \leq \overline{m} \quad \text{for } (x, t) \in [-L, L] \times [0, T^*),
\]
provided that \(E_{L0}^L + E_{L3,0}^L \leq \tilde{\delta}_1\), where \(T^*\) is the maximal existence time of the solution.

**Lemma 4.4.** There exists a positive constant \(\tilde{\delta}_2\) depending only on the initial data and other known constants, such that for any \(T \in (0, T^*)\)
\[
m \geq C(L) > 0, \quad (x, t) \in [-L, L] \times [0, T],
\]
\[
\underline{n} \leq n \leq \overline{n}, \quad (x, t) \in [-L, L] \times [0, T],
\]
provided that \(E_{L0}^L + E_{L3,0}^L + \int_{-L}^{L} m_0 \leq \tilde{\delta}_2\), where \(C(L)\) and \(\tilde{\delta}_2\) is given by (4.25) and (4.33) respectively, and \(\underline{n} \in (0, \overline{n}_1) \subseteq (0, \bar{n})\), and \(n \in (\overline{n}_1, \infty) \subseteq (\bar{n}, \infty)\).

**Proof. Step 1. lower bound of \(m\).** Using (4.2)_1 and the boundary conditions, we have
\[
\int_{-L}^{L} m = \int_{-L}^{L} m_0^L.
\]
Then for any $t \in [0, T]$, there exist $x_1(t), x_2 \in (-L, L)$ such that

$$m(x_1(t), t) = \frac{1}{2L} \int_{-L}^{L} m = \frac{1}{2L} \int_{-L}^{L} m_0 = m_0(x_2) \geq \epsilon.$$  

This combined with Hölder inequality deduces

$$\log m(x, t) = \log m(x_1, t) + \int_{x_1}^x \frac{m(y, t)}{m} dy = \log m_0(x_2) + \int_{x_1}^x \frac{m(y, t)}{m} dy \geq \log \epsilon - \left( \int_{-L}^{L} \frac{m^3}{m^2} \right)^{\frac{1}{2}} \left( \int_{-L}^{L} m \right)^{\frac{1}{2}}$$

$$\geq \log \epsilon - \frac{2}{\epsilon} \left( E_0^L + E_{x,0}^L \right)^{\frac{1}{2}} \left( \int_{-L}^{L} m_0 \right)^{\frac{1}{2}}$$

$$\geq \log \epsilon - \frac{2\delta_1^2}{\epsilon} \left( \int_{-L}^{L} m_0 + 2L \epsilon \right)^{\frac{1}{2}}.$$

This implies that

$$m(x, t) \geq \epsilon(L) \exp \left\{ -\frac{2\delta_1^2}{\epsilon(L)} \left( \int_{-L}^{L} m_0 + 2L \epsilon(L) \right)^{\frac{1}{2}} \right\} \triangleq C(L). \quad (4.25)$$

Step 2. upper bound of $n$. Since $n = \frac{\rho_2 (\rho_1 - m)}{\rho_1}$ and

$$G(n, \rho_2, \rho_2^L) = \frac{n}{\rho_2^L} s^{\gamma - (\rho_2^L)^{\gamma}} ds = n \left( \frac{\rho_2^{\gamma - 1}}{\gamma - 1} - (\rho_2^L)^{\gamma - 1} \right) \left( \frac{1}{\rho_2^L} - \frac{1}{\rho_2^L} \right)$$

$$\geq n \left( \frac{\rho_2^{\gamma - 1}}{\gamma - 1} - (\rho_2^L)^{\gamma - 1} \right)$$

$$\geq \frac{\rho_1 - \bar{m}}{\rho_1} \left( \frac{\rho_2^{\gamma}}{\gamma - 1} - \rho_2^L (\rho_2^L)^{\gamma - 1} - \rho_2^L (\rho_2^L)^{\gamma - 1} \right),$$

one can easily get

$$\rho_2^L \leq C + CG(n, \rho_2, \rho_2^L), \quad (4.26)$$

where $C = C(\rho_2, \rho_1, \bar{m}, \gamma)$ and we have used Young inequality.

Denote $\Omega_x = (-L, L) \cap (x - \frac{1}{2}, x + \frac{1}{2})$. Using Sobolev inequality, (4.17), (4.26), Hölder inequality and the fact that $n \leq \rho_2$, we have

$$n(x, t) = \sqrt{\bar{m}(x, t)} \leq C \sqrt{\bar{m}}_{L^2(\Omega_x)} + C \sqrt{\bar{m}}_{L^2(\Omega_x)} \leq C + C \int_{\Omega_x} n \leq C + C \int_{\Omega_x} \bar{m} \leq C + C \left( \int_{\Omega_x} \rho_2^L \right)^{\frac{1}{2}} \quad (4.27)$$

$$\leq C + C \left( \int_{\Omega_x} G(n, \rho_2, \rho_2^L) \right)^{\frac{1}{2}} \leq \pi,$$

for any $(x, t) \in (-L, L) \times [0, T]$ and some positive constant $\pi$. 
Step 3. lower bound of \( n \). (4.27) and (4.22) implies that
\[
\rho_g \leq C. \tag{4.28}
\]
From (4.28), (4.22) and the fact that \( \rho_g > 0 \) before the first possible singular time \( T^* \), one can easily get
\[
C|\rho_g - \tilde{\rho}_g| \leq G(n, \rho_g, \tilde{\rho}_g). \tag{4.29}
\]
We are going to deduce the lower bound of \( n \). First, note that \( \tilde{\rho}_g L = \rho_l \tilde{n} \rho_l - \epsilon(L) \), and that
\[
|n - \tilde{n}| \leq |(\rho_g - \tilde{\rho}_g) \rho_l - m| - \tilde{n}|^2 
\leq 2|\rho_g - \tilde{\rho}_g| L + 4|\tilde{\rho}_g - \tilde{n}| + 4|\tilde{\rho}_g|^2 m^2 
\leq C\bar{C}(n, \rho_g, \tilde{\rho}_g) + C\epsilon^2 + Cm, \tag{4.30}
\]
where we have used (4.29) and (4.22).

Using Sobolev inequality, (4.17), (4.30), (4.7) and the mass conservation, we have
\[
\sqrt{n} - \sqrt{\tilde{n}}(x, t) \leq \left( \int_{-L}^{L} |(\sqrt{n})_x|^2 \right)^{\frac{1}{2}} + \left( \int_{-L}^{L} |\sqrt{n} - \sqrt{\tilde{n}}|^2 \right)^{\frac{1}{2}} 
\leq C(E_0^L + E_{3,0}^L)^{\frac{1}{2}} + \left( \int_{-L}^{L} \frac{1}{n} |n - \tilde{n}|^2 \right)^{\frac{1}{2}} 
\leq C(E_0^L + E_{3,0}^L)^{\frac{1}{2}} + C\epsilon^2 L^2 + C\left( \int_{-L}^{L} m_0^L \right)^{\frac{1}{2}} 
\leq C(E_0^L + E_{3,0}^L)^{\frac{1}{2}} + C\epsilon L + C\left( \int_{-L}^{L} m_0 \right)^{\frac{1}{2}}. \tag{4.31}
\]

For any given \( n \in (0, n_1) \), let
\[
\epsilon \equiv \epsilon(L) \leq \min\left\{ \frac{1}{L^2}, \frac{(\tilde{n} - \sqrt{n})^4}{2C^4} \right\}, \tag{4.32}
\]
and \( E_0^L + E_{3,0}^L + \int_{-L}^{L} m_0 \leq \delta_2 \) where \( \delta_2 > 0 \) sufficiently small such that \( \delta_2 \leq \delta_1 \) and
\[
C\delta_2^{\frac{1}{2}} + C\delta_2^{\frac{1}{2}} \leq \sqrt{n} - \sqrt{\tilde{n}} - C\epsilon. \tag{4.33}
\]
By (4.31) and (4.33), we get
\[
n \geq \bar{n}. \tag{4.34}
\]

The next estimate is concerned about \( H^1 \) estimates of \( (u_l, u_g) \).

**Lemma 4.5.** Under the assumptions of Theorem 4.1, it holds that
\[
\int_{-L}^{L} u_{g,x}^2 \, dx + \int_{0}^{T} \int_{-L}^{L} u_{g,t}^2 \, dx \, dt \leq C. \tag{4.34}
\]
and
\[ \int_{-L}^{L} v_{t,x}^2 + \int_{0}^{T} \int_{-L}^{L} u_{t}^2 \leq C(\epsilon). \quad (4.35) \]

**Proof.** Multiplying (1.3)_4 by \( u_{g,t} \), integrating by parts over \((-L, L)\), and using arguments similar to those of Lemma 3.5, we have
\[ \frac{1}{2} \frac{d}{dt} \int_{-L}^{L} n_{g,x}^2 + \frac{1}{4} \int_{-L}^{L} n_{g,t}^2 \leq C \left( \int_{-L}^{L} n_{g,x}^2 \right)^2 + C \int_{-L}^{L} P_x^2 + \int_{-L}^{L} n_{g,x}^2 - \epsilon \frac{1}{4} \int_{-L}^{L} [n^{2\gamma}]_x u_{g,t}. \quad (4.36) \]

The next step is to handle the last term on the right hand side of (4.36).
\[ -\epsilon \frac{1}{4} \int_{-L}^{L} [n^{2\gamma}]_x u_{g,t} = -2\gamma \epsilon \frac{1}{4} \int_{-L}^{L} n^{2\gamma-1}n_x u_{g,t} \leq \frac{1}{8} \int_{-L}^{L} n_{g,t}^2 + C \epsilon \frac{1}{4} \int_{-L}^{L} n^{2\gamma-2} n_x^2, \quad (4.37) \]
where we have used (4.24) and Cauchy inequality.

Substituting (4.37) into (4.36), and then using (4.7), (4.9), (4.24) and Gronwall inequality, we have
\[ \int_{-L}^{L} n_{g,x}^2 + \int_{0}^{T} \int_{-L}^{L} n_{g,t}^2 \leq C, \]
which combined with (4.24) gives (4.34). Similarly, we obtain (4.35).

Similar to Lemmas 3.6 and 3.7, we get the following estimates for the approximate system (4.2)–(4.4):

**Lemma 4.6.** Under the assumptions of Theorem 4.1, it holds that
\[ \int_{-L}^{L} (m_x^2 + n_x^2 + P_x^2) \leq C, \quad (4.38) \]
\[ \|u_g\|_{L^\infty} \leq C, \quad \|u_l\|_{L^\infty} \leq C(\epsilon), \quad (4.39) \]
\[ \int_{-L}^{L} n_t^2 \leq C, \quad \int_{-L}^{L} (m_t^2 + P_t^2) \leq C(\epsilon), \quad (4.40) \]
\[ \int_{0}^{T} \|u_{g,x}\|_{L^\infty}^2 \leq C, \quad \int_{0}^{T} \|u_{l,x}\|_{L^\infty}^2 \leq C(\epsilon), \quad (4.41) \]
and
\[ \int_{0}^{T} \int_{-L}^{L} u_{g,xx}^2 \leq C, \quad \int_{0}^{T} \int_{-L}^{L} u_{l,xx}^2 \leq C(\epsilon). \quad (4.42) \]

Note that some of the estimates related to the liquid phase depend on \( \epsilon \). This is due to the fact that the lower bound on \( m \) has been used.

**Conclusion.** Based on these global a priori estimates in time in Section 4, it concludes that the maximal existence time \( T^* = \infty \). Thus, the proof of Theorem 4.1 is complete.
4.1.2. Proof of Theorem 2.3. Assume that
\[ E_0 + \tilde{E}_{30} + \int_R m_0 \, dx \leq \tilde{\delta}_3, \]
where
\[ E_0 = \frac{1}{2} \int_R \left( \frac{M_0^2}{m_0} + n_0 \rho_{0g}^2 \right) \, dx + \int_R G(n_0, \rho_{0g}, \tilde{\rho}_g) \, dx \]
and
\[ \tilde{E}_{30} = \int_R \left[ |(\sqrt{m_0})_x|^2 + |(\sqrt{n_0})_x|^2 \right] \, dx, \]
for some constant $\tilde{\delta}_3 \in (0, \frac{\delta_3}{16})$. Then for sufficiently large $L \in (1, \infty)$, there exists a constant $\epsilon_1 > 0$ such that if $\epsilon \triangleq \epsilon(L) \in (0, \epsilon_1) \cap (0, \frac{1}{16})$, then
\[ E_0^L + E_{3,0}^L + \int_{-L}^L m_0 \leq \tilde{\delta}_2. \]
Using Theorem 4.1, for any given $L$ sufficiently large and $\epsilon \in (0, \epsilon_1) \cap (0, \frac{1}{16})$, we get a global solution $(m^L, n^L, u^L, u^L_g)$ to the approximate system (4.2)-(4.4) with the following estimates:
\[ 0 < C(L) \leq m^L \leq m, \quad \langle x, t \rangle \in [-L, L] \times [0, \infty), \]
where $m \in (\bar{m}_1, m)$, $\bar{m} \in (\bar{m}_1, \infty)$, and $\bar{m} \in (0, \bar{m}_1)$, and
\[ n^L(t) \leq \bar{n} \leq \bar{n} \leq \bar{n}, \quad \langle x, t \rangle \in [-L, L] \times [0, \infty), \]
where $\bar{m} \in (\bar{m}_1, \rho_1)$, $\bar{n} \in (\bar{m}_1, \infty)$, and $\bar{n} \in (0, \bar{m}_1)$, and
\[ \|n^L(t) - \bar{n}\|_{\mathcal{M}^1} + \|n^L(t)\|_{L^2} + \|u^L_{g,x}(t)\|_{\mathcal{M}^1} + \int_0^t \left( \|u^L_{g,x}(s)\|_{\mathcal{M}^1} + \|u^L_{g,x}\|_{L^2} \right) \, ds \leq C, \]
and
\[ \int_{-L}^L \left( m^L|u^L_{t}|^2 + m^L + |(\sqrt{m^L})_x|^2 \right) + \int_0^t \int_{-L}^L \left( m^L + \epsilon \right)|u^L_{t,x}|^2 \leq C, \]
and
\[ \int_0^t \int_{-L}^L \left[ |\rho^L_{g,x}|^2 + \epsilon \frac{1}{2} |m^L_x|^2 + \epsilon \frac{1}{2} |n^L_x|^2 \right] \leq C, \]
where $C$ is independent of $t$ and $L$.
In order to get a strong convergence of $\sqrt{m^L} u^L_t$, following the arguments of [13, 16, 17], we need one more estimate:
\[ \int_{-L}^L m^L|u^L_t|^{2+\sigma} \leq C(T), \quad (4.43) \]
for some $\sigma \in (0, \frac{1}{2})$. In fact, (4.43) can be obtained by multiplying (4.2) by $|u^L_t|^\sigma u^L_t$ and using some straightforward calculations.
Then we use compactness arguments similar to those found in [13, 16, 17], and get a solution $(m, n, u_l, u_g)$ in the sense of Theorem 2.3 to the Cauchy problem (1.3) and (1.6). More precisely, denote the zero-extension of $(m^L, u^L_t, u^L_g)$ and $\bar{n}$—extension of $n^L$ outside $[-L,L]$ still by themselves. If $L \to \infty$ (take subsequence if necessary), we have the following arguments which will ensure that $(m, n, u_l, u_g)$ is the solution over $\mathbb{R} \times [0, T]$ for any $T > 0$. 


Global Solutions for a Two-Phase Model

- **Weak (\(\ast\)) Convergence**
  
  \( (m^L, n^L - \hat{n}) \rightarrow (m, n - \hat{n}) \) in \([L^\infty(0, T; L^\infty(\mathbb{R}) \cap L^2(\mathbb{R}))]^2 \)
  
  \( (m^L_x, n^L_x) \rightarrow (m_x, n_x) \) in \([L^\infty(0, T; L^2_{loc}(\mathbb{R}))]^2 \)
  
  \( n^L_t \rightarrow n_t \) in \(L^\infty(0, T; L^2(\mathbb{R})) \)
  
  \( (u^L_g, u^L_{g,x}) \rightarrow (u_g, u_{g,x}) \) in \(L^\infty(0, T; L^2(\mathbb{R})) \times L^\infty(0, T; L^2_{loc}(\mathbb{R})) \)
  
  \( (u^L_{g,t}, u^L_{g,xx}) \rightarrow (u_{g,t}, u_{g,xx}) \) in \(L^2(0, T; L^2(\mathbb{R})) \times L^2(0, T; L^2_{loc}(\mathbb{R})) \)
  
  and
  
  \( m^L u^L_t \rightarrow M \) in \(L^\infty(0, T; L^2(\mathbb{R})) \).

- **Strong Convergence**
  
  Since \((m^L, n^L, u^L_g)\) is bounded in \(L^\infty(0, T; H^1(K)) \times L^\infty(0, T; H^1(K)) \times L^\infty(0, T; H^1(K))\) and \((m^L_t, n^L_t, [u^L_g]_t)\) is bounded in \(L^\infty(0, T; H^{-1}(K)) \times L^\infty(0, T; L^2(K)) \times L^2(0, T; L^2(K))\) for every compact set \(K \subset \mathbb{R}\), using Aubin-Lions compactness Lemma, we have
  
  \( (m^L, n^L, u^L_g) \rightarrow (m, n, u_g) \) in \(C([0, T]; C(K)) \). \(\text{(4.44)}\)
  
  It is easy to verify that \(m^L u^L_t\) is bounded in \(L^2(0, T; W^{1,1}(K)) \cap L^\infty(0, T; L^2(K))\) and that \((m^L u^L_g)_t\) is bounded in \(L^2(0, T; H^{-2}(K))\) for every compact set \(K \subset \mathbb{R}\). This combined with Aubin-Lions compactness Lemma yields that
  
  \( m^L u^L_t \rightarrow M \) in \(L^2(0, T; L^p(K)) \cap C([0, T]; H^{-1}(K))\) for any \(p \in (1, \infty)\).
  
  Thus
  
  \( m^L u^L_t \rightarrow M \) almost everywhere in \(\mathbb{R} \times (0, T) \). \(\text{(4.45)}\)
  
  With \((4.43)\), \((4.44)\) and \((4.45)\), following the same arguments as in \([16]\), we get
  
  \( \sqrt{m} u^L_t \rightarrow M/\sqrt{m} \) (defined to be zero when \(M = 0\)) in \(L^2(0, T; L^2(K))\).
  
  In particular, we have \(M = 0\) a.e. on \(\{m = 0\}\). Then \(M = m u_t\) and
  
  \[ \frac{M}{\sqrt{m}} = \sqrt{m} u_t \ \text{where} \ u_t = \begin{cases} \frac{\sqrt{m}}{m} u_t & \text{if} \ m \neq 0, \\ 0 & \text{if} \ m = 0. \end{cases} \]

4.2. **Large-time Behavior.** From \((2.7)-(2.10)\), we obtain

\[
\sup_{0 \leq t \leq T} (\|\rho_g\|_{L^\infty} + \|\rho_g^{-1}\|_{L^\infty} + \|\rho_{g,x}\|_{L^2} + \|\rho_g - \hat{\rho}_g\|_{L^2}) + \int_0^T \|\rho_{g,x}\|_{L^2}^2 \leq C \quad (4.46)
\]

where \(C\) is independent of time. Define

\[ \tilde{G}(t) = \int_\mathbb{R} |\rho_g - \hat{\rho}_g|^6. \]

**Claim.**

\[ \tilde{G}(t) \rightarrow 0, \text{ as } t \rightarrow \infty. \quad (4.47) \]

In fact, using \((4.46)\) and the Gagliardo-Nirenberg inequality, we have

\[
\int_0^T \tilde{G}(t) \, dt \leq C \int_0^T \|\rho_g - \hat{\rho}_g\|_{L^2} \|\rho_{g,x}\|_{L^2}^2 \leq C \int_0^T \int_\mathbb{R} \rho_{g,x}^2 \leq C. \quad (4.48)
\]
Moreover,
\[
G'(t) = 6 < (\rho_g - \tilde{\rho}_g)^5, \rho_{g,t} >_{H^1(\mathbb{R}) \times H^{-1}(\mathbb{R})}
\]
\[
= 6\rho_t \int_{\mathbb{R}} \left[ \frac{(\rho_g - \tilde{\rho}_g)^5}{\rho_l - m} \right] x n u_g + 6\rho_t \int_{\mathbb{R}} \left[ \frac{n(\rho_g - \tilde{\rho}_g)^5}{(\rho_l - m)^2} \right] x n u_l
\]
\[
= 6\rho_t \int_{\mathbb{R}} \left[ \frac{5(\rho_g - \tilde{\rho}_g)^4 \rho_{g,x}}{\rho_l - m} + \frac{(\rho_g - \tilde{\rho}_g)^5 m_x}{(\rho_l - m)^2} \right] n u_g
\]
\[
+ 6\rho_t \int_{\mathbb{R}} \left[ \frac{(\rho_g - \tilde{\rho}_g)^5 n_x}{(\rho_l - m)^2} + \frac{5(\rho_g - \tilde{\rho}_g)^4 \rho_{g,x} n}{(\rho_l - m)^2} + 2(\rho_g - \tilde{\rho}_g)^5 m x \right] n u_l.
\]  
(4.49) combined with (2.7)-(2.10), (4.46), the Hölder inequality and the Gagliardo-Nirenberg inequality, we have
\[
\int_0^T |G'(t)| \, dt 
\]
\[
\leq C \int_0^T \left( \|\sqrt{\mu u_l}\|_{L^2} + \|\sqrt{n u_g}\|_{L^2} \right) \left( \|\rho_{g,x}\|_{L^2} + \|n_x\|_{L^2} + \|m_x\|_{L^2} \right) \|\rho_g - \tilde{\rho}_g\|_{L^\infty}
\]
\[
\leq C \int_0^T \|\rho_g - \tilde{\rho}_g\|_{L^2}^2 \leq C \int_0^T \|\rho_{g,x}\|_{L^2}^2 \|\rho_{g,x}\|_{L^2} \leq C, 
\]
which together with (4.48) implies (4.47).

Then, (4.46) and (4.47) combined with the Gagliardo-Nirenberg inequality yield
\[
\|\rho_g - \tilde{\rho}_g\|_{L^\infty(\mathbb{R})} \leq C \|\rho_{g,x}\|_{L^2} \|\rho_{g,\cdot,t}\|_{L^\infty} \|\rho_{g,\cdot,t}\|_{L^4} \leq C G(t)^{\frac{2}{3}} \rightarrow 0, \text{ as } t \rightarrow \infty.
\]  
(4.50)

Using the similar arguments as in Section 3.3, we get
\[
\lim_{t \rightarrow \infty} \int_{\mathbb{R}} |u_{g,x}|^2 = 0
\]
which combined with \(u_g \in L^\infty([0, \infty); L^2)\) and the Gagliardo-Nirenberg inequality yields
\[
\lim_{t \rightarrow \infty} \int_{\mathbb{R}} |u_g|^q' = 0.
\]
for any \(q' > 2\).

Acknowledgments. The authors would like to thank the anonymous referees for their instructive comments. This work was done when Yao visited IPT, University of Stavanger, Norway and when Wen was a Postdoctoral researcher at IPT, University of Stavanger, Norway. They would like to thank the department for its hospitality. Wen was partially supported by the National Natural Science Foundation of China #11301205 and by the Fundamental Research Funds for the Central Universities #D2154560. Yao was supported by the National Natural Science Foundation of China #11101331, 11571280, 11331005, FANEDD #201315, Science and Technology Program of Shaanxi Province #2013KJXX-23.

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Received March 2014; revised November 2014.

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