Quantum energy-mass spectra of relativistic Yang-Mills fields in a functional paradigm

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Abstract

A non-perturbative and mathematically rigorous quantum Yang-Mills theory on 4-dimensional Minkowski spacetime is set up in the functional framework of a complex nuclear Kree-Gelfand triple. It involves a symbolic calculus of operators with variational derivatives and a new kind of infinite-dimensional ellipticity. In the temporal gauge and Schwinger first order formalism, Yang-Mills equations become a semilinear hyperbolic system for which the general Cauchy problem is reduced to initial data with compact supports. For a simple compact Yang-Mills gauge group and the anti-normal quantization of Yang-Mills energy-mass functional of initial data in a box, the quantum energy-mass spectrum is a sequence of non-negative eigenvalues converging to infinity. In particular, it has a positive mass gap. Furthermore, the energy-mass spectrum is self-similar (including the mass gap) in the inverse proportion to an infrared cutoff of the classical energy scale.

Key words: Second quantization; Kree-Gelfand nuclear triples; operators with variational derivatives; infinite-dimensional symbolic calculus; infinite-dimensional ellipticity; Yang-Mills Millennium problem; asymptotic freedom.

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In memoriam I. M. Gelfand on his 100th anniversary.

1 Introduction

As proven in this paper, the quantum energy-mass spectra of relativistic Yang-Mills fields with infrared cutoffs are infinite and discrete. Moreover these spectra are self-similar in an inverse proportion to the classical energy scale, so that the mass gap converges to zero the cutoff goes to infinity.

Presumably this provides a solution for both parts of the 7th Millennium problem of Clay Mathematics Institute:
Prove that for any compact simple global gauge group, a nontrivial quantum Yang-Mills theory exists on Minkowski time-space and has a positive mass gap (cp. JAFFE-WITTEN[1]).

(This formulation is from WITTEN[36 p. 24].)

In accordance with Yukawa principle, a positive mass gap is an experimental fact for weak and strong subnuclear forces: A limited force range suggests its positive mass (cp. Yukawa 1949 Nobel lecture). This is a quantum effect because, when the natural units of the light velocity $c$ and Planck quantum action $\hbar$ are set to 1, the energy-mass component of the relativistic energy-momentum vector has the natural physical dimension of the reciprocal length $[L^{-1}]$. This is in spite of the conformal symmetry of the Yang-Mills action functional on Minkowski spacetime (see, e.g., GLASSEY-STRAUSS[18]).

The infrared cutoffs provide the dimension $[L]$ for the initially dimensionless Yang-Mills coupling constant (cp. FADDEEV[15] discussion of the reciprocal effect of ultraviolet cutoffs.)

When three independent dimension units are chosen, all physical magnitudes become dimensionless, a subject to mathematical calculus.

The well known theorem that elliptic (pseudo)differential operators on compact manifolds have infinite discrete spectra (see, e.g. SHUBIN[30]) may be a hint for the quantum Yang-Mills energy-mass spectrum. However,

Mathematically, quantum field theory involves integration, and elliptic operators, on infinite-dimensional spaces. Naive attempts to formulate such notions in infinite dimensions lead to all sorts of trouble. To get somewhere, one needs the very delicate constructions considered in physics, constructions that at first sight look rather specialized to many mathematicians. For this reason, together with inherent analytical difficulties that the subject presents, rigorous understanding has tended to lag behind development of physics. (WITTEN[35 p.346))

This is in spite of the fact that quantum field theory was born immediately after quantum mechanics that already matured to a rigorous mathematical theory in J. von Neumann’s monograph ”Mathematische Grundlagen der Quantenmechanik” (1932).

The concept of quantum field theory has double meaning itself:

1The official formulation of the Millennium Yang-Mills problem [1] is looking for a quantum Yang-Mills theory with axiomatic properties at least as strong as axioms of Wightman relativistic quantum field theory. However even modified Wightman axioms (see, e.g., BOGOLIUBOV et al [8, chapter 10] are in a serious conflict with the simplest cases of Gupta-Bleuler theory of quantum electromagnetic fields, as well as commonly used local renormalizable gauges (see, e.g. STROCCI[33 Chapter 6 and Appendix A.2])

2Neither mechanism requires an auxiliary Higgs field for providing a mass for Yang-Mills fields (despite of the 2012 LHC discovery of a quasi Higgs particle).
• Classical theory of quantum fields, famously initiated by P. Dirac in 1927, deals with operator-valued solutions of non-linear hyperbolic equations with partial derivatives. Unfortunately, the non-bounded and non-commuting linear operator values seriously aggravate the mathematical meaning of non-linear terms of the equations. Besides, the non-linearity is incompatible with the quantum superposition Dirac postulate.

• Quantum theory of classical fields deals with functional solutions of linear Schrödinger equations with variational derivatives on solution spaces of those hyperbolic equations. There was a vivid discussion among W. Heisenberg, P. Jordan, and W. Pauli of the corresponding "Volterra mathematics". E.g., P. Jordan and W. Pauli considered a 1-dimensional variational Schrödinger equation for eigenfunctionals $\Psi(\phi(x))$ of massless scalar fields $\phi(x), x \in \mathbb{R}$ (Zur Quantumelectrodynamik ladungsfreier Felder. Zeitung für Physik, 47 (1928))

$$-\left(\frac{\hbar}{4\pi}\right)^2 \int dx \left[ \frac{\delta^2}{\delta \phi(x)^2} + c^2 \left( \frac{d \phi(x)}{dx} \right)^2 \right] \Psi(\phi(x)) = \lambda \Psi(\phi(x)).$$

By Bogolubov-Shirkov-Schwinger quantization postulate [8, Chapter II], the variational Schrödinger operator is a quantization of the dynamical energy-mass invariant integral of the initial data for Euler-Lagrange equation for classical fields. [3]

In 1954 GELFAND-MINLOS [16] proposed to solve variational quantum fields equations with variational derivatives via approximations by solutions of partial differential equations with large but finite number of independent variables (cp. BEREZIN [5, Preface]). However the convergence of such approximations so far has not been established until the present paper.

The main content of the paper is split into three sections:

Section 2 provides a mathematically rigorous context for a non-perurbative quantum field theory on nuclear Kree-Gelfand triples. It includes the following items.

• Infinite-dimensional symbolic calculus of operators with variational derivatives as a mathematically rigorous infinite-dimensional extension of formal AGARWAL-WOLF [2] symbolic calculus of partial differential operators.

• Convergence of approximations of operators with variational derivatives by finite-dimensional pseudodifferential operators, along with the convergence of their symbolic calculi. This is a justification for Gelfand-Minlos method for solution [16] of equations with variational derivatives.

The proposed quantum Yang-Mills theory is of Lagrangian, not of Hamiltonian variety. It deals with Schwinger quantization of Noether energy-mass functional, not with Faddeev-Popov quantization of the constraint Hamiltonian dynamics. Even so, by MONCRIEF [25], there is an equivalence between constrained Yang-Mills energy-mass functional and constrained Yang-Mills Hamiltonian (cp. FADDEEV-SLAVNOV [14, Section 2, Chapter III]) by itself.
- New theory of infinite-dimensional elliptic operators including their spectral properties. The ellipticity means the operator domination of a power of the number operator. By Theorem 2.2 the spectrum of an elliptic non-negative operator is a sequence of non-negative eigenvalues converging to infinity. In particular it has positive a gap at the bottom.

Section 3 is an intricate application of the infinite-dimensional ellipticity to a quantum Yang-Mills theory:

- In the temporal gauge the Yang-Mills system of the second order partial differential equations for Yang-Mills connections on four dimensional Minkowski space is equivalent to the Schwinger semi-linear first order hyperbolic with constraint initial data on the Euclidean space $\mathbb{R}^3$.

- By Ladyzhenskaya principle ([20]), the finite speed propagation of solutions of Yang-Mills system implies the reduction of general Cauchy data to Cauchy data with compact supports. By GOGANOV-KAPITANSKII [20], the Schwinger-Yang-Mills evolution system with compactly supported Cauchy data has unique smooth global solutions on Minkowski space subject to necessary constraints. The restriction to compactly supported Cauchy data is equivalent to infrared cutoffs of their frequencies.

- Theorem 3.1 converts the non-linear manifold of constrained initial data on a 3-dimensional euclidian ball into a Gelfand triple of topological vector spaces.

Section 4 presents the anti-normal quantization of the Yang-Mills energy-mass functional in Gelfand-Kree triple over Coulomb quasi gauge. The Main Theorem 3.1 affirms the ellipticity of the non-negative anti-normal quantum energy-mass Yang-Mills operator and, therefore, discreteness of its spectrum. Proposition 3.2 exhibits an asymptotic self-similarity of the energy-mass spectra on the running energy-mass scale.

4Famous 1900 Planck Ansatz that the classical energy of confined electro-magnetic fields is proportional to their frequencies is a historical artifact because of the first appearance of the (non-normalized) Planck constant. That was before advent of both relativity and quantum theories. E. g. it does not imply the existence of photons, another famous Einstein 1905 Ansatz. It is also a mathematical paradox since the classical EM energy is a quadratic functional. Certainly, the discreteness of the confined classical EM energy does not imply massive confined photons.
2 Elliptic operators with variational derivatives

2.1 Review of Kree-Gelfand triples

Consider a Gelfand triple of densely imbedded complex topological spaces with the complex conjugation $^*$ (see, e.g., GELFAND-VILENKIN [17])

$$
\mathcal{H}^\infty \subset \mathcal{H}^0 \subset \mathcal{H}^{-\infty},
$$

(2.1)

where

- The space $\mathcal{H}^0$ is a Hilbert space with a Hermitian sesquilinear form $z^*w$;
- The space $\mathcal{H}^\infty$ of elements $z^*$ is a nuclear countably Hilbert space;
- The space $\mathcal{H}^{-\infty}$ of elements $w$ is the anti-dual space of $\mathcal{H}^\infty$ with respect to the Hermitian form $z^*w$.

Kree-Gelfand nuclear triple $\mathcal{K}$ (KREE [21] and [22]) is a sesqui-holomorphic second quantization of the Gelfand triple $\mathcal{H}$, with the induced conjugation,

$$
\mathcal{K}^\infty \subset \mathcal{K}^0 \subset \mathcal{K}^{-\infty}
$$

(2.2)

where (cp., e.g., [13])

- The space $\mathcal{K}^{-\infty}$ is the countably Hilbert space of all entire holomorphic functionals $\Psi(z^*)$ on $\mathcal{H}^\infty$ with the topology of compact convergence.
- The space $\mathcal{K}^0$ is the Bargmann-Hilbert space of square integrable entire holomorphic functionals on $\mathcal{H}^{-\infty}$ with respect to the Gaussian probability measure (see, e.g., GELFAND-VILENKIN [16]). Bargmann Hermitian form

$$
\langle \Psi | \Phi \rangle := \int dz^* dz e^{-z^*z} \Psi^*(z) \Phi(z^*),
$$

(2.3)

where the $*$-dual $\Psi^*(z)$ is the complex conjugate of $\Psi(z^*)$.
- The space $\mathcal{K}^\infty$ is the anti-dual of $\mathcal{K}^{-\infty}$.
- Borel-Fourier transform (see, e.g., COLOMBEAU [11], Chapter 7, Abstract)

$$
\tilde{\Psi}(\xi) := \langle \Psi | e^{\xi} \rangle, \quad \Psi \in \mathcal{K}^{-\infty}, \quad \tilde{\Psi}(z^*) := \langle \Psi | e^{z^*} \rangle, \quad \Psi \in \mathcal{K}^\infty,
$$

(2.4)

(where $e^{\xi}(z^*) = e^{z^*} = e^z$) is a topological isomorphism between $\mathcal{K}^{-\infty}$ and the nuclear space of entire functionals $\Psi(\xi)$ of exponential type on $\mathcal{H}^{-\infty}$.}

5The notation is bracketless as, e.g., in BEREZIN [5].
6A functional is entire on a locally convex complex vector space if it is continuous and entire on every complex line in that space (see, e.g., COLOMBEAU [11]).
7An entire functional is of exponential type if it has an exponential growth with respect to any continuous seminorm on $\mathcal{H}^{-\infty}$.
• The Borel-Fourier transform intertwines directional differentiation and multiplication

\[(\partial_w \Psi^\dagger) = (w^\ast \zeta) \tilde{\Psi}, \quad (\partial_w \Psi)^\dagger = (\zeta^\ast w) \tilde{\Psi}^\ast.\]  

(2.5)

By Grothendieck kernel theory, the nuclearity of the Kree-Gelfand triples implies that the locally convex vector spaces \(... \rightarrow ...\) of continuous linear operators are topologically isomorphic to the complete sesqui-linear tensor products (both spaces are endowed with the topology of compact uniform convergence).

\[
\left( \mathcal{H}^\infty \rightarrow \mathcal{H}^{-\infty} \right) \simeq \mathcal{H}^{s=\infty} \hat{\otimes} \mathcal{H}^{-\infty},
\]

(2.6)

\[
\left( \mathcal{H}^\infty \rightarrow \mathcal{H}^\infty \right) \simeq \mathcal{H}^{s=\infty} \hat{\otimes} \mathcal{H}^{\infty},
\]

(2.7)

\[
\left( \mathcal{H}^{-\infty} \rightarrow \mathcal{H}^{\infty} \right) \simeq \mathcal{H}^{s=\infty} \hat{\otimes} \mathcal{H}^{\infty},
\]

(2.8)

where the operators are tame in the case of (2.7).

An operator is polynomial if its kernel is a continuous polynomial on \(\mathcal{H}^{s=\infty} \times \mathcal{H}^\infty\), so that polynomial operators are tame.

The formulas present the one-to-one correspondence between operators and the sesqui-holomorphic kernels of their matrix elements.

The nuclear Gelfand triple of the sesqui-Hermitian direct products

\[
\mathcal{H}^{s=\infty} \times \mathcal{H}^{s=\infty} \subset \mathcal{H}^{s=0} \times \mathcal{H}^{0} \subset \mathcal{H}^{s=\infty} \times \mathcal{H}^{-\infty}
\]

(2.9)

carries the Hermitian conjugation

\[
(z^\ast, w)^\ast := (w^\ast, z)
\]

(2.10)

The associated Kree-Gelfand triples of sesqui-holomorphic kernels consists of

\[
\mathcal{H}^{s=\infty} \hat{\otimes} \mathcal{H}^{s=\infty} \subset \mathcal{H}^{s=0} \hat{\otimes} \mathcal{H}^{0} \subset \mathcal{H}^{s=\infty} \hat{\otimes} \mathcal{H}^{-\infty}
\]

(2.11)

where \(\mathcal{H}^{s=0} \hat{\otimes} \mathcal{H}^{0}\) is the Hilbert space of Hilbert-Shmidt kernels.

The corresponding exponential functionals are

\[
e^{(\zeta^\ast, \eta)}((z^\ast, w)^\ast) = e^{w^\ast \eta + \zeta^\ast z}.
\]

(2.12)

### 2.2 Operators with variational derivatives

Kree-Gelfand triple (2.7) has the canonical linear representation by continuous linear transformations of \(\zeta \in \mathcal{H}^{-\infty}\) and \(\zeta^\ast \in \mathcal{H}^{\infty}\) into the adjoint operators of creation and annihilation continuous operators of multiplication and directional differentiation

\[
\hat{\zeta} : \mathcal{H}^{\infty} \rightarrow \mathcal{H}^{-\infty}, \quad \hat{\zeta} \Psi(z^\ast) := (z^\ast \zeta) \Psi(z^\ast),
\]

(2.13)

\[
\hat{\zeta}^\ast : \mathcal{H}^{-\infty} \rightarrow \mathcal{H}^{\infty}, \quad \hat{\zeta}^\ast \Psi(z) := (\zeta^\ast z) \Psi(z),
\]

(2.14)

\[
\hat{\zeta}^{\dagger} : \mathcal{H}^{\infty} \rightarrow \mathcal{H}^{\infty}, \quad \hat{\zeta}^{\dagger} \Psi(z^\ast) := \partial_{\zeta^\ast} \Psi(z^\ast),
\]

(2.15)

\[
\hat{\zeta}^{\dagger} : \mathcal{H}^{-\infty} \rightarrow \mathcal{H}^{-\infty}, \quad \hat{\zeta}^{\dagger} \Psi(z) := \partial_{\zeta} \Psi(z),
\]

(2.16)

such that
1. Bosonic commutation relation
\[
[\zeta^+, \eta] = \zeta^+ \eta, \quad [\zeta^+, \eta^+] = \eta^+ \zeta.
\] (2.17)

2. The exponentials \(e^\eta, \eta^+ \in \mathcal{H}^{-\infty}\), and \(e^\eta, \eta \in \mathcal{H}^{\infty}\), are the eigenstates of the annihilation operators
\[
\widehat{\zeta}^+ e^\eta = (\zeta^+ \eta)e^\eta, \quad \zeta^+ e^\eta^+ = (\eta^+ \zeta)e^\eta^+.
\] (2.18)

Creators and annihilators generate strongly continuous abelian operator groups of quantum exponentials in \(\mathcal{H}^{-\infty}\) and \(\mathcal{H}^{-\infty}\) parametrized by \(\zeta\) and \(\zeta^+\):
\[
\begin{align*}
\hat{e}^\zeta & : \mathcal{H}^{\infty} \to \mathcal{H}^{\infty}, & e^\zeta \Psi(z) &= e^{\zeta^+} \Psi(z); \\
\hat{e}^{\zeta^+} & : \mathcal{H}^{-\infty} \to \mathcal{H}^{-\infty}, & e^{\zeta^+} \Psi(z) &= \Psi(z + \zeta); \\
\hat{e}^\zeta & : \mathcal{H}^{-\infty} \to \mathcal{H}^{-\infty}, & e^\zeta \Psi(z) &= e^{\zeta^+} \Psi(z); \\
\hat{e}^{\zeta^+} & : \mathcal{H}^{\infty} \to \mathcal{H}^{\infty}, & e^{\zeta^+} \Psi(z) &= \Psi(z^+ + \zeta^+).
\end{align*}
\] (2.19) - (2.22)

Borel-Fourier transform of kernels \(M(\theta^+, \eta)\)
\[
\hat{M}(z^+, w) = \langle M(\theta^+, \eta) \mid e^{z^+ \eta^+ + \theta^+ \eta} \rangle = \langle M(\theta^+, \eta) \mid e^z \rangle \langle e^\eta \rangle
\] (2.23)

may be quantized as normal, Weyl, and anti-normal pseudovariational operators \(\hat{M}\)
\[
\hat{M} : \mathcal{H}^{-\infty} \to \mathcal{H}^{-\infty}, \quad M \in \mathcal{H}^{\infty} \otimes \mathcal{H}^{\infty}
\] (2.24)

defined by their exponential matrix elements
\[
\begin{align*}
\langle e^x \mid \hat{M}_\nu \mid e^w \rangle &= \langle M_\nu(\theta^+, \eta) \mid e^{x^+} \rangle \langle e^{\hat{\eta}^+} \rangle \langle e^w \rangle, \\
\langle e^x \mid \hat{M}_\omega \mid e^w \rangle &= \langle M_\omega(\theta^+, \eta) \mid e^{x^+} \rangle \langle e^{\hat{\eta}^+} \rangle \langle e^w \rangle, \\
\langle e^x \mid \hat{M}_\alpha \mid e^w \rangle &= \langle M_\alpha(\theta^+, \eta) \mid e^{x^+} \rangle \langle e^{\hat{\eta}^+} \rangle \langle e^w \rangle.
\end{align*}
\] (2.25) - (2.27)

**Proposition 2.1** Any continuous linear operator \(Q\) from \(\mathcal{H}^{-\infty}\) to \(\mathcal{H}^{\infty}\) has a unique normal co-kernel \(\hat{M}_Q(\zeta^+, \eta)\).

**Proof** Since
\[
\langle e^x \mid \hat{M}_\nu \mid e^w \rangle = \langle M_\nu(\theta^+, \eta) \mid e^{x^+ + \theta^+ \zeta} e^w \rangle
\] (2.28)
\[
= \langle e^x \mid e^{\hat{\eta}^+} \mid e^w \rangle = e^{x^+ + \theta^+ \zeta} e^w
\] (2.29)

one has
\[
\langle e^x \mid \hat{M}_\nu \mid e^w \rangle = \langle M_\nu(\theta^+, \eta) \mid e^{x^+ + \theta^+ \zeta} e^w \rangle = \hat{M}_\nu(z^+, w)e^w
\] (2.30)
where $\tilde{M}_v(z^*, w)$ is the sesqui-linear Fourier transform of $M_v(\theta^*, \eta)$ (see (2.12)). Thus $\tilde{M}_v(z^*, w)e^{z^*w}$ is the kernel of $\tilde{M}_v$.

By (2.11), any Grothendieck kernel has a unique such representation. QED

By the Taylor expansion centered at the origin, the sesqui-entire functionals are uniquely defined by their restrictions to the real diagonal $(z^*, w = z) \in \mathcal{R}(\mathcal{H}^{\times \infty} \times \mathcal{H}^{\infty})$, so that the normal symbol of the operator $Q$

$$\sigma_v^0(z^*, z) := \tilde{M}_v = \langle e^{z^*} | e^z \rangle^0(z^*, z)$$

(2.31)

exists and defines $Q$ uniquely.

By Baker-Campbell-Hausdorff commutator formula and the canonical commutation relations (2.17),

$$e^{\hat{H}}e^{\theta^*} = e^{\hat{H} + \theta^*/2}, \quad e^{\theta^*}e^{\hat{H}} = e^{\hat{H} + \theta^*/2}e^{-\theta^*/2}.$$  (2.32)

Thus any operator $Q$ has Weyl and anti-normal co-kernels $M_{\theta}^0$ and $M_{\theta}^0$. Their restrictions to the real diagonal are Weyl and anti-normal symbols $\sigma_\theta^0$ and $\sigma_\theta^0$ of $Q$.

By (2.32), the symbols of the same operator $Q$ are related via Weierstrass transform (cp. AGARWAL-WOLF[2] formulas (5.29), (5.30), (5.31), page 2173) in a finite dimensional case; DYNIN [13]) in white noise calculus:

$$\sigma_\theta^0(z^*, z) = e^{-\theta/2}\sigma_\theta^0(z^*, z),$$

(2.33)

$$\sigma_\theta^0(z^*, z) = e^\theta\sigma_\theta^0(z^*, z),$$

(2.34)

$$\sigma_\theta^0(z^*, z) = e^\theta\sigma_\theta^0(z^*, z),$$

(2.35)

where the operator $e^{\pm \theta/2}\sigma_{\hat{\mathfrak{g}}}$ is the sesqui-linear Fourier equivalent of the multiplication by $e^{\pm \theta/2}$, i.e.,

$$\frac{\partial}{\partial \xi^*} \frac{\partial}{\partial \zeta} := \text{Trace}(\frac{\partial}{\partial \xi^*} \frac{\partial}{\partial \zeta}).$$  (2.36)

(Note that the multiplication by $e^{\pm \theta}w$ is a continuous operator in $\mathcal{H}^{\times \infty} \otimes \mathcal{H}^{\infty}$, so that the restrictions of $e^{\pm \theta/2}\sigma(\xi^*, \zeta)$ to the real diagonal $\mathcal{R}(\mathcal{H}^{\times \infty} \times \mathcal{H}^{\infty})$ are well defined.)

The following Proposition shows that anti-normal operators are infinite-dimensional Berezin-Toeplitz integral operators from $\mathcal{H}^{\infty}$ to $\mathcal{H}^{\times \infty}$ (cp. BEREZIN[6] Equation (2.5)).

**Proposition 2.2** The kernel of an anti-normal operator $Q = \tilde{M}_\alpha$

$$\langle e^{z^*} | Q | e^w \rangle = e^{z^*w}\sigma_\alpha^0(w^*, w),$$

(2.37)

i.e. $Q$ acts on $\Psi(w^*)$ as the multiplication $\sigma_\alpha^0(w^*, w)\Psi(w^*) \in \text{Ent}(\mathcal{H}^{\times \infty}) \times \mathcal{H}^{\infty}$ followed by the orthogonal projection with the kernel $(e^{z^*w})^* = e^{z^*w}$ onto $\text{Ent}(\mathcal{H}^{\infty})$. 
PROOF Since
\[
\langle e^z | \hat{\theta} e^\eta | e^w \rangle = \langle \hat{\theta} e^z | e^\eta e^w \rangle = \langle e^z e^{\hat{\theta} w + \hat{\theta} w} | e^{\hat{\theta} w} \rangle,
\]
the kernel (2.27)
\[
\langle M_\alpha(\theta^*, \eta) | e^{z^*} e^{\theta^* w} e^{\theta^* w + \theta^* w} \rangle = e^{z^* w} \sigma_\alpha(w, w^*).
\]
QED

Corollary 2.1 The diagonal matrix elements of an anti-normal operator \( Q \) on \( K_\infty \)
\[
\langle e^z | Q | e^z \rangle \geq \inf \sigma_\alpha^Q(z^*, z), \quad (z^*, z) \in H_\infty.
\]
This Corollary is an infinite-dimensional extension of Theorem 7.1 in BEREZIN [6].

2.3 Quantized Galerkin approximations

A Galerkin sequence \( p_n, j = 1, 2, \ldots \), is an increasing sequence of tame orthogonal projections of rank \( n \) strongly convergent to the identity operator in \( H_\infty \). The projectors \( p_n \) are uniquely extended to the Galerkin families in \( H^0 \) and \( H^{-\infty} \). The notation \( p_n \) is kept for all these extensions.

The finite dimensional projectors induce the quantized Galerkin sequence
\[
P_n \Psi(z^*) := \Psi(p_n z^*), \quad P_n \Psi(z) := \Psi(p_n z)
\]
of infinite dimensional projectors in the triple \( H \) onto cylindrical triples isomorphic to the pulled back sesqui-entire triples over the tautological finite dimensional triple \( C^n \subset C^n \subset C^n \).

By Proposition 2.1 the compressions of operators \( Q_n := P_n Q P_n \) of \( Q \) are cylindrical pseudodifferential operators with the exponential kernels,
\[
\langle e^{z^*} | Q_n | e^w \rangle = \langle e^{p_n z^*} | Q | e^{p_n w} \rangle,
\]
i.e. pullbacks from \( C^n \) of finite dimensional pseudodifferential operators of AGARWAL-WOLF [2].

Theorem 2.1 Operator \( Q \) is the strong limit of the cylindrical pseudodifferential operators \( Q_n \) on \( H_\infty \).

PROOF The matrix element \( \langle \Psi^* | Q | \Phi \rangle \) is a separately continuous sesquilinear form on the Frechet space \( H_\infty \). By a Banach theorem (see, e.g., [27, v.1,Theorem V.7]), the sesquilinear form is actually continuous on \( H_\infty \). In particular, operator \( Q \) is
the weak limit of \( Q_n \) in \( \mathcal{H}^{\infty} \). Since \( \mathcal{H}^{\infty} \) is a nuclear space, the weak convergence implies the strong one in the topology of \( \mathcal{H}^{\infty} \).

As \( n \to \infty \), the exponential matrix elements

\[
\langle e^{z^*} | Q_n | e^w \rangle = \langle e^{P_n z^*} | Q | e^{P_n w} \rangle \longrightarrow \langle e^{z^*} | Q | e^w \rangle,
\]

so that symbols of the cylindrical \( Q_\nu \) converge to the corresponding symbols of \( Q \).

Thus if operators \( Q_1 \) and \( Q_2 \) are tame then (cp. AGARWAL-WOLF [2, Theorem III.5])

**Corollary 2.2** The symbols of the tame \( Q_3 = Q_2 Q_1 \) are convergent series (polynomials when \( Q_1 \) or \( Q_2 \) is a polynomial operator)

\[
\sigma_{\nu}^{Q_3}(z^*, z) = \sum_{m=0}^{\infty} (m!)^{-1} \partial_z^m \sigma_{\nu}^{Q_2}(z^*, z) \partial_{z^*}^m \sigma_{\nu}^{Q_1}(z^*, z),
\]

\( \sigma_{\alpha}^{Q_3}(z^*, z) = \sum_{m=0}^{\infty} (m!)^{-1} \partial_z^m \sigma_{\alpha}^{Q_2}(z^*, z) \partial_{z^*}^m \sigma_{\alpha}^{Q_1}(z^*, z),\)

\( \sigma_{\omega}^{Q_3}(z^*, z) = \sum_{m=0}^{\infty} (m!)^{-1} \Omega^m (\sigma_{\omega}^{Q_2}(z^*_2, z_2) \sigma_{\omega}^{Q_1}(z^*_1, z_1)),\)

where \( \Omega := (1/2)(\partial_2 \partial_1 - \partial_1 \partial_2), z_2^* = z_1^* = z^*. \)

Another consequence from Theorem 2.1 for essentially selfadjoint

**Corollary 2.3** Let \( Q \) be the Friedrichs extension of a non-negative tame operator in \( \mathcal{H}^{\infty} \) to an (unbounded) selfadjoint operator in \( \mathcal{H}^0 \). Then the spectrum of \( Q \) consists of limits of some spectral values of \( Q_n \) as \( n \) converges to infinity.

**Proof** By Theorem 2.1, the resolvent operators of \( Q_n \) strongly converge to the resolvent operator of \( Q \).

Then [27, v.1,Theorem VIII.24(a)] implies the lower semicontinuity of the spectra convergence.

**QED**

### 2.4 Ellipticity

The tame quadratic number operator \( N \) is defined by its quadratic symbols

\[
\sigma_{\nu}^{N} = z^* z + 1, \quad \sigma_{\alpha}^{N} = z^* z, \quad \sigma_{\omega}^{N} = z^* z + 1/2.
\]

The notation \( N \) is transferred to the Friedrichs extension of the number operator. The eigenspaces \( \mathcal{N}_n, n = 1, 2, \ldots \), of \( N \) with the eigenvalues \( n \) are the spaces spanned by continuous homogeneous polynomials of degree \( n \). The number operator is positive and self-adjoint in the Hilbert space \( \mathcal{H}^0 \).
Thus $N$ has the simple eigenvalue $\lambda_1 = 1$, and all other eigenvalues $\lambda_n = n$ form its essential spectrum.

A non-negative symmetric tame operator $Q$ is elliptic\footnote{The definitions of ellipticity in [4], [24], and [13] are not sufficient for the present paper.} if there exist positive constants $r$ and $c$ such that

$$
\langle \Psi \mid Q \mid \Psi \rangle \geq c \langle \Psi^r \mid N^r \mid \Psi \rangle, \quad \Psi \in \mathcal{H}^\infty.
$$

(2.49)

The notation $Q$ is transferred to the Friedrichs extension of the operator $Q$.

**Theorem 2.2** The spectrum of a positive elliptic operator $Q$ is a sequence $\lambda_n(Q) \to +\infty$. In particular, $Q$ has a mass gap at the bottom of its spectrum.

**Proof** Quantized Galerkin cutoffs $P_j Q P_j$ of elliptic non-negative operators $Q$ in $\mathcal{H}^\infty$ are pullbacks of the finite dimensional elliptic pseudodifferential operators $Q_j$ on $p_jX$. The latter are non-negative and elliptic, so that they have discrete spectra of eigenvalues $\lambda_n(Q_j) \geq 0$ converging to infinity (see, e.g. SHUBIN[30] Theorem 26.3). Then the operators $P_j Q P_j$ have the infinitely degenerated spectra of $Q_j$.

Similarly, for $k \geq j$, the operators $P_j Q_k P_j$ on $p_kX$ are pullbacks of $Q_j$, so that $Q_j$ have trivial extensions $\tilde{Q}_j$ of $Q_j$ (with $\tilde{Q}_j = 0$ on the orthogonal complement $(p_jX)^\perp$ in $p_kX$. Again, both $\tilde{Q}_j$ and $Q_j$ have the same spectra (though with different but finite multiplicities).

Since $Q_k \geq \tilde{Q}_j$, a variational principle (see, e.g. BEREZIN-SHUBIN[7] Appendix 1, Corollary 2 of Proposition 3.1)\footnote{If non-negative operators $A' \leq A''$ are essentially selfadjoint on a common domain $D$ with non-decreasing sequential discrete spectra $\lambda_n'$ and $\lambda_n''$, then $\lambda_n' \leq \lambda_n''$.} implies that the eigenvalues $\lambda_n(Q_j)$ monotonically increase with $j$. Then, by Corollary 2.3 the spectrum of the elliptic operator $Q$ consists of some of the limits of these monotonic sequences.

The cutoffs $P_j N P_j$ of the number operator are pullbacks of the finite dimensional pseudodifferential number operators $N_j = p_j N p_j$ on $p_jX$. All they have the same spectra $\mathbb{Z}_+$, so that $p_j c N^r p_j$ with positive $c$ and $r$ have discrete spectra $cn^r$. Now the same variational principle and ellipticity condition (2.49) imply that the eigenvalues $\lambda_m(Q_j) \geq cn^r$. Therefore, for any given integer $n$ there may be only finitely many limits of monotonic $\lambda_n(Q_j)$ as $j$ increases to infinity.

**QED**

3 Quantum Yang-Mills theory

3.1 Classical Yang-Mills fields

The global gauge group $\mathbb{G}$ of a Yang-Mills theory is a connected semi-simple compact Lie group with the Lie algebra $\text{Ad}(\mathbb{G})$.

The notation $\text{Ad}(\mathbb{G})$ indicates that the Lie algebra carries the adjoint representation $\text{Ad}(g)X = gXg^{-1}, g \in \mathbb{G}, a \in \text{Ad}(\mathbb{G})$, of the group $\mathbb{G}$ and the corresponding
self-representation \( \text{ad}(X)Y = [X,Y] \), \( X,Y \in \text{Ad}(G) \). Then \( \text{Ad}(G) \) is identified with a Lie algebra of skew-symmetric matrices and the matrix commutator as Lie bracket with the positive definite \( \text{Ad} \)-invariant scalar product

\[
X \cdot Y := \text{Trace}(X^T Y),
\]

where \( X^T = -X \) denotes the matrix transposition.

Let the Minkowski space \( \mathbb{M} \) be oriented and time oriented with the Minkowski metric signature \((-1,1,1,1)\). In a Minkowski coordinate systems \( x^\mu, \mu = 0,1,2,3 \), the metric tensor is diagonal. In the natural unit system, the time coordinate \( x^0 = t \). Thus \( (x^\mu) = (t,x^i), \ i = 1,2,3 \).

The local gauge group \( \mathcal{G} \) is the group of infinitely differentiable \( G \)-valued functions \( g(x) \) on \( \mathbb{M} \) with the pointwise group multiplication. The local gauge Lie algebra \( \text{Ad}(\mathcal{G}) \) consists of infinitely differentiable \( \text{Ad}(G) \)-valued functions on \( \mathbb{M} \) with the pointwise Lie bracket.

\( \mathcal{G} \) acts via the pointwise adjoint action on \( \text{Ad}(\mathcal{G}) \) and correspondingly on \( \mathcal{A} \), the real vector space of gauge fields \( A = A_\mu(x) \in \text{Ad}(\mathcal{G}) \).

Gauge fields \( A \) define the covariant partial derivatives

\[
\partial_\mu X := \partial_\mu X - \text{ad}(A_\mu)X, \quad X \in \text{Ad}(\mathcal{G}).
\]

This definition shows that in the natural units gauge connections have the mass dimension \( 1/[L] \).

Any \( g \in \mathcal{G} \) defines the affine gauge transformation

\[
A_\mu \mapsto A^g_\mu := \text{Ad}(g)A_\mu - (\partial_\mu g)^{-1}, \quad A \in \mathcal{A},
\]

so that \( A^{g_1}A^{g_2} = A^{g_1g_2} \).

Yang-Mills curvature tensor \( F(A) \) is the antisymmetric tensor

\[
F(A)_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu].
\]

The curvature is gauge invariant:

\[
\text{Ad}(g)F(A) = F(A^g),
\]

as well as Yang-Mills Lagrangian

\[
(1/4)F(A)^{\mu\nu} \cdot F(A)_{\mu\nu}.
\]

The corresponding gauge invariant Euler-Lagrange equation is a 2nd order non-linear partial differential equation \( \partial_\mu F(A)^{\mu\nu} = 0 \), called the Yang-Mills equation

\[
\partial_\mu F^{\mu\nu} - [A_\mu, F^{\mu\nu}] = 0.
\]

Yang-Mills fields are solutions of Yang-Mills equation.

\( ^{\text{10}} \)The dimensionless Yang-Mills coupling \( g_{YM}^2 \) is set to 1.
3.2 First order formalism

In the temporal gauge $A_0(t,x^k) = 0$ the 2nd order Yang-Mills equation (3.7) is equivalent to the 1st order Schwinger hyperbolic system for the time-dependent $A_j(t,x^k)$, $E_j(t,x^k)$ on $\mathbb{B}$ (see, e.g., GOGANOV-KAPITANSKII [20, Equation (1.3)])

$$\partial_t A_k = E_k, \quad \partial_t E_k = \partial_j F^j_k - [A_j, F^j_k], \quad F^j_k = \partial^j A_k - \partial_k A^j - [A^j, A_k]. \quad (3.8)$$

and the constraint equations

$$[A_k, E_k] = \partial_k E_k, \quad i.e. \quad \partial_k A E = 0. \quad (3.9)$$

By GOGANOV-KAPITANSKII [20], the evolution system is a semilinear first order partial differential system with finite speed propagation of the initial data, and the Cauchy problem for it with constrained initial data at $t = 0$

$$a_k(x) := A(0,x_k), \quad e_k(x) := E(0,x_k), \quad \partial^k e_k = [a_k, e_k] \quad (3.10)$$

is globally and uniquely solvable in local Sobolev spaces on the whole Minkowski space $\mathbb{M}$ (with no restrictions at the space infinity.)

This fundamental theorem has been derived via Ladyzhenskaya 1949 method (see [20]) by a reduction to the case of Cauchy data on 3-dimensional balls $\mathbb{B} = \mathbb{B}(R) : |x| \leq R$.

If the constraint equations are satisfied at $t = 0$, then, in view of the evolution system, they are satisfied for all $t$ automatically. Thus the 1st order evolution system along with the constraint equations for Cauchy data is equivalent to the 2nd order Yang-Mills system. Moreover the constraint equations are invariant under time independent gauge transformations. As the bottom line, we have

**Proposition 3.1** In the temporal gauge Yang-Mills fields $A$ are in one-one correspondence with their gauge transversal Cauchy data $(a,e)$ satisfying the equation $\partial_a e = 0$.

Consider the chain of Sobolev-Hilbert spaces $\mathcal{A}^s$, $-\infty < s < \infty$, of (generalized) connections $a(x)$ on a $\mathbb{B}$ of radius $R$ with respect to the norms

$$|a|_s^2 := \int_{\mathbb{B}} d^3x (a \cdot (1 - \Delta)^s a) < \infty. \quad (3.11)$$

They define the real Gelfand nuclear triple (cp., e.g., [16])

$$\mathcal{A} : \mathcal{A}^{-\infty} := \bigcap \mathcal{A}^s \subset \mathcal{A}^0 \subset \mathcal{A}^{\infty} := \bigcup \mathcal{A}^s, \quad (3.12)$$

\[11\] Segal theory [29] of infinite-dimensional Sobolev Lie groups implies that for any infinitely differentiable gauge field on $\mathbb{M}$ there is a unique infinitely differentiable gauge transformation to the temporal gauge.
where $\mathcal{H}^\infty$ is a nuclear countably Hilbert space with the dual $A^{-\infty}$.

Similarly we define the chain of Sobolev-Hilbert spaces $\mathcal{S}^s, -\infty < s < \infty$, of (generalized) Lorentz scalar fields $u(x)$ on $\mathbb{B}$ with values in $\text{Ad} \mathbb{G}$ and the Hilbert norms $|u|_s$. Let

$$\mathcal{S} : \mathcal{S}^\infty := \bigcap \mathcal{S}^s \subset \mathcal{S}^0 \subset \mathcal{S}^{-\infty} := \bigcup \mathcal{S}^s$$

be the corresponding Gelfand triple.

Let $a \in \mathcal{S}^{s+3}, s \geq 0$. Then, by Sobolev embedding theorem $a$ is continuously $s+2$-differentiable on $\mathbb{B}$ and, therefore, the following gauged vector calculus operators are continuous:

- **Gauged gradient** $\text{grad}^a : \mathcal{S}^{s+1} \rightarrow \mathcal{S}^s$,
  $$\text{grad}^a u := \partial_k u - [a_k, u].$$  
  \hspace{10cm} (3.14)

- **Gauged divergence** $\text{div}^a : \mathcal{S}^{s+1} \rightarrow \mathcal{S}^s$,
  $$\text{div}^a b := \text{div} b - [a; b], \quad [a; b] := a_k b_k.$$  
  \hspace{10cm} (3.15)

- **Gauged curl** $\text{curl}^a : \mathcal{S}^{s+1} \rightarrow \mathcal{S}^s$,
  $$\text{curl}^a b := \text{curl} b - [a \times b], \quad [a \times b]_i := \varepsilon_{ijk} [a_j b_k].$$  
  \hspace{10cm} (3.16)

- **Gauged Laplacian** $\triangle^a : \mathcal{S}^{s+2} \rightarrow \mathcal{S}^s$,
  $$\triangle^a u := \text{div}^a (\text{grad}^a u).$$  
  \hspace{10cm} (3.17)

The adjoints of the gauged operators are

$$(\text{grad}^a)^* = -\text{div}^a, \quad \text{curl}^a = -\text{curl}^a.$$  
  \hspace{10cm} (3.18)

**Lemma 3.1** If $a \in \mathcal{S}^{s+3}, s \geq 0$, then the operator $\text{div}^a : \mathcal{S}^{s+1} \rightarrow \mathcal{S}^s$ is surjective.

**Proof** Let $\mathcal{S}^{s+2}, s \geq 0$, denote the closure in $\mathcal{S}^{s+2}$ of the space of $a$’s with compact support in the interior of $\mathbb{B}$. The conventional Laplacian $\triangle^0 : \mathcal{S}^{s+2} \rightarrow \mathcal{S}^s$ is an isomorphism (see e.g. AGRANOVICE ET AL [3]).

The gauged Laplacian $\triangle^a$ differs from the usual Laplacian $\triangle^0$ by first order differential operators, and, therefore is a Fredholm operator of zero index from $\mathcal{S}^{s+2}$ to $\mathcal{S}^s, s \geq 0$.

If $\triangle^a u = 0$ then $\triangle^a u^* u = (\text{grad}^a u)^*(\text{grad}^a u)$, so that $\text{grad} u = [a, u]$. The computation

$$\left(1/2\right) \partial_h (u \cdot u) = (\partial_k u \cdot u) = [a_k, u] \cdot u = -\text{Trace} (a_k uu - u a_k u) = 0$$  
  \hspace{10cm} (3.19)

shows that the solutions $u \in \mathcal{S}^{s+2}$ are constant. Because they vanish on the ball boundary, they vanish on the whole ball. Since the index of the Fredholm operator $\triangle^a$ is zero, its range is a closed subspace with the codimension equal to the dimension of its null space. Thus the operator $\text{div}^a \text{grad}^a$ is surjective , and so is $\text{div}^a$.  

QED
3.3 Coulomb quasi gauge

Consider the bundles $\mathcal{C}^s, s \geq 0$ of constraint Cauchy data with the base $\mathcal{A}^\infty$ and the null spaces $\mathcal{E}^{s+1}_a$ of the operators $\text{div}^a : \mathcal{E}^{s+1} \to \mathcal{E}^s$ as fibers over $a \in \mathcal{A}^\infty$.

Their intersection $\mathcal{C}^\infty$ is a bundle of nuclear countably Hilbert spaces over the nuclear countably Hilbert base $\mathcal{A}^\infty$. Together with the unions of the dual spaces $\mathcal{C}^{-s}$ they form a bundle of nuclear Gelfand triples $\mathcal{C}$ over the same base.

**Theorem 3.1** The bundle $\mathcal{C}^\infty$ is smoothly trivial, so that the total space of $\mathcal{C}^\infty$ is smoothly isomorphic to the direct product of its base $\mathcal{A}^\infty$ and the fiber $\mathcal{C}^\infty_{a=0}$, the nullspace of the operator $\text{div}$ in $\mathcal{E}^\infty$.

**Proof** For $0 \leq s \leq \infty$ consider the mapping

$$f : \mathcal{A}^{s+2} \times \mathcal{E}^{s+1} \to \mathcal{A}^s, \quad f(a, e) := \text{div}_a(e)$$

Sobolev imbedding theorem shows that the mapping is continuous. Lemma 3.1 implies that the continuous partial Frechet derivatives $\partial_e f(a, e)$ are bounded linear operators onto a fixed Hilbert space $T(s)$, the orthogonal complement of constant $a$’s. The restrictions of $\partial_e f(a, e)$ to the orthogonal complements of the null spaces of $\text{div}_a$ are one-to-one. By the implicit function theorem on Hilbert spaces (see, e.g., [23]), this implies that the explicit solutions $e = e(a)$ of the equation $f(a, e) = 0$ provide infinitely smooth local trivializations of Hilbert bundles $\mathcal{C}^s$.

Their intersection $\mathcal{C}^\infty = \cap \mathcal{C}^s$ is a locally trivial $\mathcal{C}^\infty$-bundle over $\mathcal{A}^\infty$ with the associated locally trivial bundle of smooth $\ast$-orthonormal frames in the fibers.

Since $\mathcal{A}^\infty$ is a Frechet space, its smooth homothety retraction to the origin $a = 0$ has a homotopy lifting to the frame space. Thus the bundle $\mathcal{C}^\infty$ is trivial, so that the total set of constraint Cauchy data carries the global chart $\mathcal{A}^\infty \times \mathcal{C}^\infty_{a=0}$. QED

Let $\mathcal{A}^s$ and $\mathcal{E}^s$ denote the nullspaces of the operator $\text{div}$ in $\mathcal{A}^s$ and $\mathcal{E}^s$.

By DELL’ANTONIO-ZWANZIGER[12], the closures of smooth gauge orbits in $\mathcal{H}^0 := \mathcal{A}^0 \times \mathcal{E}^0$ intersect $\mathcal{A}^0$. These closures are the orbits of the Sobolev group, the closure in Sobolev space $W^{1,2}(\mathbb{B})$ of the group of smooth gauge transformations. (The Sobolev group is a topological group of continuous transformations in $\mathcal{A}^0$.) Thus $\mathcal{H}^0 := \mathcal{A}^0 \times \mathcal{E}^0$ is a quasi-gauge for the orbifold of the direct product of the parallel transports (i.e. every $(a, e) \in \mathcal{H}^0$ is on an orbit but some orbits may intersect $\mathcal{H}^0$ more than once (cp. SINGER[32] and NARASIMHAN-RAMADAS[26]).

The Gelfand triple

$$\mathcal{H} : \mathcal{H}^\infty := \mathcal{A}^\infty \times \mathcal{E}^\infty \subset \mathcal{H}^0 := \mathcal{A}^0 \times \mathcal{E}^0 \subset \mathcal{H}^{-\infty} := \mathcal{A}^{-\infty} \times \mathcal{E}^{-\infty}$$

(3.21)

is the direct product of the Gelfand triples $\mathcal{A}$ and $\mathcal{E}$.

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3.4 Quantum Yang-Mills energy-mass spectrum

The Noether energy-mass functional of smooth Yang-Mills Cauchy data (cp. GLASSEY-STRAUSS [18, Section 3]) on \( B \) is

\[
M(a, e) := (1/2) \int_B d^3x \left( (\text{curl} a - [a \times a]) \cdot (\text{curl} a - [a \times a]) + e \cdot e \right).
\] (3.22)

The density \((\text{curl} a - [a \times a]) \cdot (\text{curl} a - [a \times a])\) is the scalar gauge curvature of \( a \) and, as such, is invariant under the gauge parallel transport but the density \( e \cdot e \) is not.

At the same time the density \( e \cdot e \) is invariant under the flat isometric parallel transport provided by Theorem 3.1. Thus the energy-mass functional \( M \) is constant on smooth orbits of the direct product of both parallel transports.

Convert the Coulomb quasi-gauge triple \( 3.21 \) into the complex Gelfand triple \( \mathcal{H} \) with conjugation where the real and imaginary parts are the direct factors

\[
\mathbb{R}\mathcal{H} := \mathbb{R} \hat{\mathcal{H}}, \quad \mathbb{I}\mathcal{H} := \mathbb{I} \hat{\mathcal{H}}.
\] (3.23)

Let the polynomial energy-mass functional \( M(z, z^*) := M(a, e) \) (ref: eq:Noether) be the anti-normal symbol of the tame quantum Yang-Mills energy-mass operator \( \hat{M}_\alpha : \mathcal{H}^\text{sm} \to \mathcal{H}^\infty \).

**Theorem 3.2** The (unique) non-negative selfadjoint Friedrichs extension of \( \hat{M}_\alpha \) in \( \mathcal{M}^0 \) is elliptic. Its spectrum is an infinite sequence of non-negative eigenvalues. In particular, the spectrum has a positive mass gap.

**PROOF** (A)

\[
\int_B d^3x \left( \text{curl} a - [a \times a] \right) \cdot \left( \text{curl} a - [a \times a] \right)
= \int_B d^3x \left( \text{curl} a \cdot \text{curl} a + ([a \times a]) \cdot ([a \times a]) \right)
- 2 \int_B d^3x \text{curl} a \cdot ([a \times a]).
\] (3.24)

The vector identity

\[
\text{curl}([a \times b]) = (\text{div} a)b - a(\text{div} b) + (b \cdot \nabla)a - (a \cdot \nabla)b
\]

(where \( \nabla \) is the gradient vector) converts the last integral after integration by parts into

\[
- \int_B d^3x a \cdot \text{curl}([a \times a]) = - \int_B d^3x \text{div} a \cdot a.
\] (3.24)

Therefore, by (3.1), the energy-mass functional (3.22) on \( \mathcal{H}^\text{sm} \) (so that \( \text{div} a = 0 \)) becomes

\[
M(a, e) = (1/2) \int_B d^3x \left( \text{curl} a \cdot \text{curl} a + [a \times a] \cdot [a \times a] + e \cdot e \right).
\] (3.25)
Let $b_i$ be a basis for $\text{Ad}(G)$ with $b_i \cdot b_j = \delta_{ij}$. Then, since the gauge group $G$ is a simple Lie group, the structure constants $c_{ij}^k = [b_i, b_j] \cdot b_k$ are anti-symmetric under interchanges of all $i, j, k$. Thus if $a = \alpha^i b_i$ then (see Simon[31, page 217])

\[
([a \times a]) \cdot ([a \times a]) = \alpha^i \alpha^j \alpha^m e^{i j k} ([b_i, b_j] \cdot b_k) e^{lmk} ([b_l, b_m] \cdot b_k) \quad (3.26)
\]

\[
= \sum_k \alpha^i \alpha^j \alpha^m c_{ij}^k c_{lm}^k = \sum_k (\alpha^i \alpha^j c_{ij}^k)^2. \quad (3.27)
\]

Because $i \neq j$, the Laplacian $\partial^2 a := \sum_j \partial^2 / \partial \alpha_j \partial \alpha_j$ applied to $[a \times a] \cdot ([a \times a])$ produces

\[
2 \sum_k (\alpha^i \alpha^j c_{ij}^k) (\alpha^i \alpha^j c_{ij}^k) = -2 K(a, a) = c(a \cdot a) > 0 \quad (3.28)
\]

where $K(a, a)$ is the negative definite Killing quadratic form on $\text{Ad}(G)$ (see Simon[31, Equation (13)]) and $c$ is a (positive) constant, by simplicity of the gauge group $G$.

Since $\partial_\alpha a = \partial^2 a + \partial_\alpha^2$,

\[
e^{i \alpha^j \partial / 2} (([a \times a]) \cdot ([a \times a])) \underset{3.28}{=} ([a \times a]) \cdot ([a \times a]) + \frac{c}{2} a \cdot a + c. \quad (3.29)
\]

Therefore the Weyl symbol of the operator $H := \tilde{M}_\alpha$

\[
\sigma^H_\alpha (a, e) \underset{2.35, 3.29}{=} \int_B d^3 x (([a \times a]) \cdot ([a \times a]) + e \cdot e + \frac{c}{2} a \cdot a + c + \frac{1}{2}). \quad (3.30)
\]

The Weyl quantization of $([a \times a]) \cdot ([a \times a])$ is the non-negative operator of multiplication with $([a \times a]) \cdot ([a \times a]) \geq 0$ in the "$(a, e)$-representation" of the canonical commutation relations (cp. Agarwal-Wolf[2, Section VII, page 2177]).

By (2.48),

\[
\int_B d^3 x (a \cdot a + e \cdot e + 1/2)
\]

is the anti-normal symbol of the number operator $N$.

Altogether we get the operator inequality of non-negative operators

\[
\tilde{M}_\alpha \geq C \tilde{N}_\alpha \quad (3.31)
\]

where $C$ is a positive constant.

Now the spectral Theorem 2.2 implies the Theorem 3.2

QED

The restriction to the periodic Yang-Mills fields is equivalent to infrared cutoffs.

**Proposition 3.2** The spectra of quantum Yang-Mills energy-mass operators are self-similar in the inverse proportion to the radius of the ball $B(R)$. 

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The scaling transformation \((x, a, e) \mapsto (x/R, a/R, e/R^2)\) converts the energy-mass functional (3.22) over \(B(R)\) of the radius \(R\)

\[
\frac{1}{2} \int_{B(R)} d^3 x \left( (\text{curl} a - [a \times a]) \cdot (\text{curl} a - [a \times a]) + e \cdot e \right) \tag{3.32}
\]

into the scaled energy-mass functional over the unit ball \(B(1)\)

\[
\frac{1}{2R} \int_{B(1)} d^3 x \left( (\text{curl} a - [a \times a]) \cdot (\text{curl} a - [a \times a]) + e \cdot e \right) \tag{3.33}
\]

The modified scalar product

\[
\int_{B(1)} d^3 x \left( a \cdot a/R + Re \cdot e \right) \tag{3.34}
\]

is invariant under the scaling, so that the quantum canonical relations are conserved under the scaling. QED

**Remark 3.1** The quantum Yang-Mills energy-mass excitations above the mass gap may indicate the existence of infinitely many massive glueballs.

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