Contribution from stochastic electrodynamics to the understanding of quantum mechanics

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Abstract

During the last decades there has been a relatively extensive attempt to develop the theory of stochastic electrodynamics (SED) with a view to establishing it as the foundation for quantum mechanics. The theory had several important successes, but failed when applied to the study of particles subject to nonlinear forces. An analysis of the failure showed that its reasons are not to be ascribed to the principles of SED, but to the methods used to construct the theory, particularly the use of a Fokker-Planck approximation and perturbation theory. A new, non-perturbative approach has been developed, called linear stochastic electrodynamics (LSED), of which a clean form is presented here.

After introducing the fundamentals of SED, we discuss in detail the principles on which LSED is constructed. We pay attention to the fundamental issue of the mechanism that leads to the quantum behaviour of field and matter, and demonstrate that indeed LSED is a natural way to the quantum formalism by demanding its solutions to comply with a limited number of principles, each one with a clear physical meaning. As a further application of the principles of LSED we derive also the Planck distribution. In a final section we revisit some of the most tantalizing quandaries of quantum mechanics from the point of view offered by the present theory, and show that it offers a clear physical answer to them.

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1 Introduction

Despite the extraordinary power of quantum mechanics, it is difficult to find in the history of physics another example of a theory that has raised and nourished so many debates and controversies about its meaning. The myriads of papers, books and meetings devoted to the scrutiny of its interpretation testify to the meager progress reached in such disputes since the early stages of quantum

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mechanics. Of course for the physicist who uses the theory in her/his daily undertakings as a tool, there is usually nothing to bother about, so she/he will easily overlook such questions. But there exist also some physicists (not so few as to be negligible, as evidenced by the number of papers) who are deeply interested in the study and solution of these matters and would relish a clear-cut answer to them. Being simultaneously a fundamental theory and an active field of research, quantum theory cannot fully flourish while indefinitely leaving aside the basic conceptual issues that are known to affect it. Thus no wonder that the papers dealing with the fundamentals of quantum theory continue to accumulate, as is easily confirmed by a glance, for example, at the recent monograph by Auletta [1]. Most frequently the problem is tackled from within the quantum theory itself, as is well illustrated with the works of Bohm [2] or Omnès [3]. In the course of time, however, several attempts to find a solution to those problems from a broader framework have been developed, as testified in the old book by Jammer [4] or, of course, the more recent one by Auletta.

Among the varied efforts to construct a theory aimed at contributing to the understanding of quantum mechanics, we shall refer here specifically to stochastic electrodynamics, SED for short. (For its origins see the pioneering works of T. W. Marshall [5], who has contributed also to its optical branch, stochastic optics [6], [7]). We recall that the central premise of SED is that the quantum behavior of the particle is a result of its interaction with the vacuum radiation field, or zero-point field. This field is assumed to pervade the space and, for the purpose of studying atomic or molecular systems, is considered to be in a stationary state with well defined stochastic properties. Its action on the particle impresses upon it in every point of space a stochastic motion, with an intensity characterized by Planck’s constant, which is a measure of the magnitude of the fluctuations of the vacuum field.

Initially put forward by Nernst as a conjecture, the crucial role of vacuum-matter interaction for the stability of the atom and other quantum properties is a fundamental result in SED (see [8] for an almost exhaustive list of references to the end of 1995). Phenomena as diverse and as characteristic of the quantum world as van der Waals and Casimir forces [9], [11], diamagnetism [3], [11], cavity effects on atomic systems [13], thermal effects of acceleration [14]-[16], the quantum harmonic oscillator including its radiative corrections [17]-[19], and several others, have found a consistent, even if in some cases still incomplete physical explanation within the framework of SED, as can be seen in the detailed account given in [8]. This collection of fitting results suggested that the core of the theory is a sound one. However, the difficulties encountered when applying it to systems subject to nonlinear forces [20]-[24] brought the theory almost to a standstill, except for the renewed efforts by Cole and coworkers to advance in the understanding of the hydrogen atom [25], [26], the development of stochastic optics by Marshall, Santos and coworkers [6], [7], and the proposal of an alternative formulation of SED by Cetto and de la Peña [8], [27], [28]. This latter is the subject matter of the present paper.

This alternative theory, termed linear stochastic electrodynamics (LSED) for reasons that will become clear below, shares with SED its basic principle about
the central role played by the zero-point field, but makes a careful review of supplementary hypotheses used along the derivation of the theory. A detailed analysis as the one given in [8] convinced us that the culprit was to be found in the use of a perturbative method to deal with the effect of the field on the mechanical system, and the corresponding use of a generalized Fokker-Planck equation [29]. This has prompted us to propose a nonperturbative approach that makes no recourse to methods associated with the Fokker-Planck treatment of stochastic problems [5, 27]. In the course of time we have been able to refine the arguments that sustain this approach, and we believe that the framework presented here constitutes a more accomplished form of LSED, suitable for those physicists who would like to see quantum mechanics emerge as a physical theory devoid of strange elements or assumptions. So here we show how the usual formalism of quantum mechanics (and quantum electrodynamics, in its nonrelativistic version) ensues from a more general underlying theory, and we use this demonstration to understand the origin of some major quantum peculiarities.

It is not our intention to offer here a full derivation of quantum theory but to provide the fundamental elements that explain how the usual quantum formalism comes about. As will be apparent below, the nature of LSED is such as to expect that it should lead us beyond the framework of present day quantum theory. This certainly constitutes a most attractive feature of the theory that should encourage its further development. For the time being we take the limited step of applying the principles of the theory to understand the physics of today; but already along this restricted way we shall come across some novelties.

The paper is organized as follows. Using as starting point the Abraham-Lorentz equation of motion for a particle subject to an arbitrary external force and the zero-point field, in the first sections we set forth three principles that limit to a considerable extent the class of allowed solutions, by imposing to them clearly defined statistical demands. We explicitly state the approximations introduced to satisfy each one of the principles. We then approach the problem of finding these solutions and show that they are naturally described by the formalism of (nonrelativistic) quantum theory. Having reached that point we take a closer look at the interaction of matter with the radiation field, to distinguish between this equilibrium field and the vacuum field in the absence of matter, i.e., the free vacuum field. As an application we study the equilibrium with a thermal field (including the vacuum, of course) to derive Planck’s distribution along the lines of the old statistical method proposed by Einstein [30]. The final section differs somewhat in nature from the rest of the paper in that we use it to address, in the light of the present theory, some of the conceptual problems of quantum mechanics that have been under discussion for decades.

2 The principles of LSED

We consider the problem of a bound particle, typically an electron, and start from the Abraham-Lorentz equation of sed (usually called Braffort-Marshall
\[ m \ddot{x} = f(x) + m \tau \ddot{x} + eE(t). \]  

(1)

The quantity \( \tau = \frac{2e^2}{3mc^3} \) is of the order of \( 10^{-23} \) s for the electron. The radiation reaction force \( m \tau \ddot{x} \) has the well-known causality problems associated with the third-order time derivative, but these are of no special concern to us here; in [5] the interested reader may find a detailed discussion of this point and an ample list of relevant references. The term \( eE(t) \) stands for the electric force exerted by the vacuum field on the particle; the magnetic term is not included since the discussion is restricted to the nonrelativistic case. Further, as will become evident later, the wavelength of the relevant field modes is assumed to be much larger than the amplitude of the particle’s motion. This allows us to neglect the spatial dependence of \( E \); in other words, we are using the electric dipole approximation.

### 2.1 The quantum regime

Let us now discuss in detail the premises on which LSD is based. The first assumption to be made is the following.

**Principle One.** *The system under study reaches an equilibrium state, at which the average rate of energy radiated by the particle equals the average rate of energy absorbed by it from the field.*

To give a quantitative form to this demand, we multiply Eq.(1) by \( \dot{x} \) and get after some minor transformations

\[ \langle \frac{dH}{dt} \rangle = -m\tau \langle \dot{x}^2 \rangle + e \langle \dot{x} \cdot E \rangle, \]  

(2)

where \( H \) stands for the particle Hamiltonian, including the Schott energy,

\[ H = \frac{1}{2}m \dot{x}^2 + V(x) - m\tau \dot{x} \cdot \ddot{x}, \]  

(3)

and \( V(x) \) is the potential associated to the external force \( f(x) \). The average is being performed over the realizations of the background (zero-point) field. When the system has reached the state of energetic equilibrium, so that

\[ \langle \frac{dH}{dt} \rangle = 0, \]  

(4a)

we have

\[ m\tau \langle \dot{x}^2 \rangle = e \langle \dot{x} \cdot E \rangle. \]  

(4b)

The two sides of this equation are of a very different nature: energy radiation, its average rate being given by the Larmor term \( m\tau \langle \dot{x}^2 \rangle \), is due basically to the orbital motions, whereas energy absorption, whose average rate is \( e \langle \dot{x} \cdot E \rangle \),
comes from the highly irregular motion impressed on the particle by the vacuum field, and more specifically from the radiative (stochastic) corrections to the primary motions, as will be argued below (see Eq. (46)). When this equilibrium condition is satisfied (or nearly satisfied) we say that the system has reached the *quantum regime*. The theory to be developed assumes that this regime has been reached. Below we will show that in LSED an even more stringent condition is satisfied, namely that of detailed energy balance, i.e., balance for each separate frequency. Of course this is to be expected under equilibrium, since otherwise energy could be transferred by the mechanical part of the system from some modes of the field to others, in clear violation of the principles of thermodynamics.

### 2.2 Central role of the vacuum field

Despite the fact that Eq. (4b) is still unfinished (as shown by Eq. (46)) we can draw some initial conclusions from it. One of primary importance is that in equilibrium, \( \langle x^2 \rangle \) is determined by the vacuum field (more specifically, by its energy spectrum, as we will see below). Thus, also the acceleration itself should be determined by the field. The importance of this observation can be recognized by considering a counterexample. Suppose that we examine a state of motion determined *perturbatively* from Eq. (1), taking the field as the perturbation. Then the dominant part of \( x \) comes from the classical equation of motion

\[
m \ddot{x} = f(x)
\]

(along with the field we are neglecting the radiation damping). Under these conditions there is no guarantee that the radiated power

\[
m \tau \langle x^2 \rangle = e \langle x \cdot E \rangle
\]

equal for each possible motion, since \( f(x) \) and \( E(t) \) are entirely independent functions. This was precisely the problem created by the original form of SED, as is discussed in detail in [8] and [27]. From this it follows that it was not SED itself which failed with the nonlinear forces, but the approach developed to study it. We conclude that there is a need to look for a different kind of solutions, such that the acceleration is determined by the vacuum field and Eq. (4b) is guaranteed to hold for all allowed states of motion. We embody this observation in the form of Principle Two:

**Principle Two.** *Once the quantum regime has been attained (and Eq. (4b) holds), the vacuum field has gained control over the motion of the material part of the system.*

To apply this principle to the present problem, we consider the equation of motion (1) in the first place for the free particle,

\[
m \ddot{x} = m \tau \dot{x} + e E(t),
\]

(5)

and to simplify the discussion we consider the one-dimensional case, as there seems to be no problem in generalizing to the multidimensional instance. Now
we express the field as a Fourier transform as follows,

\[ E(t) = \sum_{\beta} \tilde{E}_\beta a_\beta e^{i\omega_\beta t} = \sum_{\omega_\beta > 0} \left( \tilde{E}_\beta^{(+)} a_\beta e^{i\omega_\beta t} + \tilde{E}_\beta^{(-)} a_\beta^* e^{-i\omega_\beta t} \right). \tag{6} \]

The amplitudes \( a_\beta = a(\omega_\beta) \) are stochastic variables with statistical properties that will be fixed from the requirements of the theory itself. In the usual form of SED it was customary to fix a priori these properties by writing them in the form

\[ a(\omega_\beta) = r(\omega_\beta) e^{i\varphi(\omega_\beta)}, \tag{7} \]

with both \( r \) and \( \varphi \) real functions, the amplitude \( r \) following a normal distribution and the phases \( \varphi(\omega_\beta) \) being independent random numbers uniformly distributed in \([0, 2\pi]\), as corresponds to a free field. Here we follow a different path and leave the \( a(\omega_\beta) \) largely unspecified for the time being, since we will find below that the statistical properties of the (near) equilibrium field cannot be freely fixed, but must follow from the principles of the theory. The field amplitudes \( \tilde{E}_\beta \) will be selected so as to assign to each mode of the field the mean energy \( \mathcal{E}_\beta = \frac{1}{2} \hbar \omega_\beta \).

This is the unique door through which Planck’s constant enters into the theory, fixing the scale of the spectral energy of the zero-point field \([31]\); from here it spreads over the whole theory. Thus we write

\[ \mathcal{E}_\beta = \frac{1}{2} \left( p_\beta^2 + \omega_\beta^2 q_\beta^2 \right), \tag{8} \]

with \( p_\beta = \sqrt{\frac{\mathcal{E}_\beta}{2}} (a_\omega + a_\omega^*) \), \( i\omega_\beta q_\beta = \sqrt{\frac{\mathcal{E}_\beta}{2}} (a_\omega - a_\omega^*) \). \tag{9}

With (6) the solution to Eq.(5) becomes

\[ x(t) = \sum \tilde{x}_\beta a_\beta e^{i\omega_\beta t}, \tag{10a} \]

where

\[ \tilde{x}_\beta = -\frac{e\tilde{E}_\beta}{m\omega_\beta^2 + im\tau\omega_\beta^3}. \tag{10b} \]

Hence,

\[ x(t) = -\sum_{\beta} \frac{e\tilde{E}_\beta a_\beta}{m\omega_\beta^2 + im\tau\omega_\beta^3} e^{i\omega_\beta t}. \tag{10c} \]

In these expressions all quantities except the amplitudes \( a_\beta \) are sure numbers. It is important to note that this includes the amplitudes \( \tilde{x}_\beta \) and the frequencies \( \omega_\beta \). Upon introduction of an external force \( f(x) \), however, these parameters become in principle stochastic variables. Indeed, from the complete equation of motion (11) we get

\[ \sum \left( -m\omega_\beta^2 \tilde{x}_\beta - im\tau\omega_\beta^3 \tilde{x}_\beta + \frac{\tilde{f}_\beta}{a_\beta} \right) a_\beta e^{i\omega_\beta t} = e \sum \tilde{E}_\beta a_\beta e^{i\omega_\beta t}. \tag{11} \]
For a generic force, the Fourier coefficient $\tilde{f}_\beta$ (of the terms that oscillate with frequency $\omega_\beta$) will be a complicated function of both sets, $\{\tilde{x}_\beta\}$ and $\{a_\beta\}$. By writing

\[ \tilde{x}_\beta = -\frac{e \tilde{E}_\beta}{m \omega_\beta^2 + i m \tau \omega_\beta^3 + \frac{f_\beta}{x_\beta a_\beta}} \]  

(12a)

and introducing this into Eq.(10a), we get

\[ x(t) = -\sum \frac{e \tilde{E}_\beta a_\beta}{m \omega_\beta^2 + i m \tau \omega_\beta^3 + \frac{f_\beta}{x_\beta a_\beta}} e^{i\omega_\beta t}. \]  

(12b)

Comparing with (10c) we see that introduction of an external force modifies the response amplitudes $\tilde{x}_\beta$ in a way that may be very important. Firstly, the $\tilde{x}_\beta$ referring to different frequencies become entangled, so that the response to a given frequency depends now on the response to other frequencies; secondly, as already noted they become stochastic parameters, functions of the field amplitudes $a_\beta$, as is seen from Eq.(12a). Also, the $x(t)$ given by Eq.(12b) is determined in an essential way by both the field and the external force, so that it fulfills Principle Two. It should further be noted that one gets a different solution for each realization of the vacuum field, i.e., for each set $\{a_\beta\}$, so that for nonlinear forces, when $\tilde{f}_\beta$ becomes a nonlinear function of the sets $\{\tilde{x}_\beta\}$ and $\{a_\beta\}$, we have a continuous infinity of stochastic solutions.

2.3 Looking for solutions independent of the realization of the field

The problem of determining $x(t)$ in the general case appears impossible to solve. However there is a way to considerably simplify matters, to the extent of transforming the problem into a soluble one, under certain restrictions. Let us consider an ensemble of similarly prepared systems. Owing to differences in the initial conditions, specific realizations of the background field and so on, there would be a whole collection of different states of motion. We are however interested in those that are particularly stable, and which thus become dominant as equilibrium is approached, at least in the mean. One expects that such particularly stable orbits would be those corresponding to a minimum average energy in some appropriate (statistical) sense (to be detailed below, see subsection 7.2). Owing to their greater stability, such motions will result approximately the same (in the statistical sense just mentioned) for a whole family of realizations of the field. Thus, one can characterize them by being to a certain extent independent of the details of the field realization. We propose to stretch this consideration to its limits and consider those motions in the quantum regime that become independent of the realization of the background field.

This request is clearly equivalent to demanding that the near equilibrium vacuum field has adjusted itself to the presence of matter in the given state of motion. A similar situation takes place, for instance, with the equilibrium field
at a temperature $T$, which is not merely the vacuum field, but that corresponding to the Planck distribution at the given temperature. This is the reason we have formerly stated that the field amplitudes $a_\beta$ should be fixed by the demand of equilibrium. The price to be paid for the present major simplification is that the theory becomes unable to describe the detailed behavior of any particular member of the ensemble (or subensemble) considered. It is in this sense that the description has become statistical. This stripped-down description is simply accomplished noticing that the set of solutions described by Eq. (12a) satisfies the stated demand when the amplitudes $a_\beta$ are such that $\tilde{f}_\beta / (\tilde{x}_\beta a_\beta)$ becomes independent of the specific realization.

From the above discussion it follows that the amplitudes $a_\beta$ should be selected so as to guarantee that the following Principle Three holds.

**Principle Three.** There exist states of matter (quantum states) that are unspecific to (or basically independent of) the particular realization of the vacuum field.

The demand that the system of Eqs. (12a)-(12b) possesses solutions that satisfy Principle Three will be considered as the simplest possible approximation to the solutions that satisfy the condition (4b) of (detailed) energy balance, that is, once the quantum regime has been attained. Prior to this, the field may be anyone, possibly closer to the free field. To establish the consequences of Principle Three we expand the Fourier amplitude $\tilde{f}_\beta$ of the external force that corresponds to the frequency $\omega_\beta$ as follows, noting that each factor $\tilde{x}_\beta$ should carry an associated $a_\beta$ factor, as follows from Eq. (10a) (we leave aside the case of a constant force),

$$\tilde{f}_\beta = k_\beta \tilde{x}_\beta a_\beta + k_{\beta'\beta''} \tilde{x}_\beta \tilde{x}_{\beta'} a_{\beta''} a_{\beta'''} + k_{\beta'\beta''\beta'''} \tilde{x}_\beta \tilde{x}_{\beta'} \tilde{x}_{\beta''} a_{\beta'''} a_{\beta''''} a_{\beta'''''} + ... \quad (13a)$$

The point in this expansion is that the nonlinear terms entangle the frequencies, so that there may appear an arbitrary number of terms associated with the same frequency. Each one of the factors $\tilde{x}_{\beta'}$ is accompanied by the factor $a_{\beta'} e^{i\omega_{\beta'} t}$, so that the product of the time functions gives the factor $e^{i(\omega_{\beta'} + \omega_{\beta''} + \omega_{\beta'''} + ... ) t} = e^{i\omega_\beta t}$, a fact that has been already taken into account in writing Eq. (12b). Thus it follows that

$$\frac{\tilde{f}_\beta}{\tilde{x}_\beta a_\beta} = k_\beta + k_{\beta'\beta''} \frac{\tilde{x}_\beta \tilde{x}_{\beta''}}{\tilde{x}_\beta} a_{\beta'} a_{\beta''} + k_{\beta'\beta''\beta'''} \frac{\tilde{x}_\beta \tilde{x}_{\beta''} \tilde{x}_{\beta'''}}{\tilde{x}_\beta} a_{\beta'} a_{\beta''} a_{\beta'''} + ... \quad (13b)$$

Now it is clear that the response functions $\tilde{x}_\beta$ will become sure numbers, independent of the field realization, if the set of conditions

$$a_{\beta'} a_{\beta''} a_{\beta'''} ... a_{\beta^{(n)}} = a_\beta \quad (14)$$

is satisfied for any number of factors, since $\tilde{f}_\beta$ reduces then to

$$\tilde{f}_\beta = \left( k_\beta \tilde{x}_\beta + \sum_{\beta} k_{\beta'\beta''} \tilde{x}_{\beta'} \tilde{x}_{\beta''} + \sum_{\beta} k_{\beta'\beta''\beta'''} \tilde{x}_{\beta'} \tilde{x}_{\beta''} \tilde{x}_{\beta'''} + ... \right) a_\beta \quad (15)$$
so that Eq. (12a) acquires sure values,

\[ \tilde{x}_\beta = -\frac{e\tilde{E}_\beta}{m\omega_\beta^2 + im\tau\omega_\beta^3 + k_\beta + \sum^\beta k_\beta'k_\beta''\frac{\tilde{x}_\beta'\tilde{x}_\beta''}{\tilde{x}_\beta} + \sum^\beta k_\beta'k_\beta''k_\beta'''\frac{\tilde{x}_\beta'}{\tilde{x}_\beta}} \]

(16)

It is to be noted that this expression is exact (although implicit) whenever conditions (14) are satisfied. However, since the amplitudes \(a_\beta\) are stochastic quantities it would be naive to assume that the latter are satisfied exactly. So with the present approximations we are also neglecting the “noise” associated with all such fluctuations. As remarked above, together with requirement (14) a condition on the combination frequencies must be satisfied, namely,

\[ \omega_{\beta'} + \omega_{\beta''} + \omega_{\beta'''} + \cdots + \omega_{\beta(n)} = \omega_\beta. \]

(17)

so that each term in the denominator of Eq. (16) corresponds to the common frequency \(\omega_\beta\). We call relevant frequencies all those frequencies that solve equation (17); a central problem of the theory will be their determination. Note that equation (17) is weaker than Eq. (14): if the latter is met, the former will be automatically satisfied, but not in the opposite sense. That condition (14) is satisfied is the meaning of the superscript \(\beta\) in the sums \(\sum^\beta\) in the above equations.

Of course Eq. (16) correctly contains the particular case of the harmonic oscillator, for which \(k_\beta = -m\omega_\beta^2, k_\beta'k_\beta'' = k_\beta'k_\beta''k_\beta''' = \cdots = 0\), just as a particular instance of the general description. This is to be remarked because in the original (conventional) SED theory, the harmonic oscillator, being a linear system, was dealt with directly with a Fourier development, just as is done here, so that the answer is the same in both theories, except for those features that depend on the different statistical properties of the \(a_\beta\). However, in conventional SED, non-linear problems are treated using perturbation theory around the corresponding classical motion, so that nothing equivalent to Eq. (16) for the general case has place in that theory. In the present theory, we have extended the treatment of the harmonic oscillator to the generic case. There remains however an important difference between the linear oscillator and the more general problem; this comes from the fact that for the oscillator we get

\[ \bar{x}_\beta = -\frac{e\tilde{E}_\beta/m}{\omega_\beta^2 + i\tau\omega_\beta^3 - \omega_\beta^2}, \]

(18)

so there is no explicit need to impose the conditions (14), which means that for all stochastic fields (or all realizations of a given field) one obtains the same set \(\bar{x}_\beta\). This includes the free vacuum field (the one assumed in conventional SED), as well as several other representations of the quantized radiation field. This is but a manifestation of a well known result, namely, that the harmonic oscillator can reach an equilibrium state with any background field. We could say that what we have achieved here is equivalent to extending the property of the linear harmonic oscillator of being independent of the specific realization of the field, to all dynamical systems in the quantum regime.
3 Solutions in the quantum regime

Let us now attempt to give a precise meaning to the above equations and find their solution. We start by considering conditions (14)\[a_{\beta'}a_{\beta''}a_{\beta'''}...a_{\beta(n)} = a_{\beta},\]
which we rewrite using the polar representation (7), 
\[a_{\beta} = r_\beta e^{i\varphi_\beta},\]
to obtain
\[r_{\beta'}r_{\beta''}r_{\beta'''}...r_{\beta(n)}e^{i(\varphi_{\beta'}+\varphi_{\beta''}+\varphi_{\beta'''}+...+\varphi_{\beta(n)})} = r_\beta e^{i\varphi_\beta}.\] (19)

Since the number of factors \(r_\beta\) in the left hand side is arbitrary and their product should equal \(r_\beta\) in all cases, this equation requires that we take \(r_\beta = 1\), and so on.

Thus the stochastic amplitudes simplify to
\[a_{\beta} = e^{i\varphi_\beta},\] (20)
with \(\varphi_\beta\) a random phase uniformly distributed in \([-\pi, \pi]\). With this, equation (19) reduces to
\[\varphi_{\beta'} + \varphi_{\beta''} + \varphi_{\beta'''} + ... + \varphi_{\beta(n)} = \varphi_\beta.\] (21)

Thus the conditions on the phases and on the (relevant) frequencies, Eq. (17) become similar. It is clear that Eqs. (17) and (21) relate only relevant frequencies or phases among themselves; in other words, not any phase (or frequency) enters into the conditions, so that our problem is just the specification of the relevant ones. Let us consider first the case of only two phases, so that
\[\varphi_{\beta'} + \varphi_{\beta''} = \varphi_\beta.\] (22a)

It is clear that even if the phases entering in this equation are random, they have become correlated one with another. So we may write for instance
\[\varphi_{\beta'} = \varphi_\beta + \phi_{\beta'}\], \quad \varphi_{\beta''} = \varphi_\beta + \phi_{\beta''}, \quad \varphi_{\beta'''} = \varphi_\beta + \phi_{\beta'''} + \phi_{\beta''},\] (22b)
or
\[\phi_{\beta'} = \varphi_{\beta'} - \varphi_\beta, \quad \text{and so on,}\] (23)
with each \(\phi_{\beta'}\) a random phase. Substituting in Eq. (22a) one obtains
\[\varphi_\beta + \phi_{\beta'} + \phi_{\beta''} - \phi_{\beta'\beta'} = \varphi_{\beta''} - \phi_{\beta''} + \phi_{\beta''} + \varphi_{\beta'''} - \phi_{\beta'''} = \varphi_\beta,\]
or, simplifying with the help of Eq. (22a),
\[\phi_{\beta'} = \phi_{\beta'} + \phi_{\beta''}.\] (24)

Since, according to Eq. (22b), each phase \(\phi_{\beta'}\) can be written as the difference of two random phases, and the latter are uniformly distributed in \([-\pi, \pi]\), also the \(\phi_{\beta'}\) are uniformly distributed in the same interval, modulo \(\pi\). This result establishes the condition that the \(\phi_{\beta'}\) should obey to guarantee that Eq. (22a) is satisfied. We thus find that the indices of the new phases must follow a chain rule as shown in Eq. (24), which is easily generalized to any number of terms, so for the general case we have
\[\phi_{\beta'} = \phi_{\beta'} + \phi_{\beta''} + \phi_{\beta'''} + \phi_{\beta''} + \phi_{\beta'''} + ... + \phi_{\beta(n-1)} + \phi_{\beta(n)}.\] (25)
In terms of the original phases $\phi_\beta$ the mechanism that leads to the fulfilment of Eq. (21) is the successive cancellation of pairs of phases. Thus, for instance Eq. (25) is equivalent to

$$\phi_{\beta'} - \phi_\beta = \phi_{\beta''} - \phi_{\beta'} + \phi_{\beta'''} - \phi_{\beta''} + \phi_{\beta''''} - \phi_{\beta'''} + \ldots - \phi_{\beta(n)} + \phi_{\beta(n-1)} - \phi_\beta,$$

which is automatically satisfied. In summary, this means that Eq. (21) should be written in terms of the phases $\phi_{\beta'}$ instead of the original phases $\phi_\beta$, and that in that doing the original single index should be replaced by a pair of indices that fulfill the chain rule. This also means that Eq. (20) should be rewritten in the form

$$a_{\alpha\beta} = e^{i\phi_{\alpha\beta}} = e^{i(\phi_\alpha - \phi_\beta)},$$

(26)

and, more generally, that the original single index should be replaced throughout by a pair of indices that combine themselves according to the chain rule made explicit in Eq. (25). In particular, we must apply this rule to Eq. (17), $\omega_\beta = \omega_{\beta'} + \omega_{\beta''} + \omega_{\beta'''} + \ldots + \omega_{\beta(n)}$, which now reads

$$\omega_{\beta\beta(n)} = \omega_{\beta\beta'} + \omega_{\beta\beta''} + \omega_{\beta\beta'''} + \ldots + \omega_{\beta\beta(n-1)\beta(n)}.$$

(27)

This is the precise meaning that one should ascribe to the symbol $\sum^\beta$ used in previous equations, as (15) and (16). The frequencies that enter into all these relations are just the relevant frequencies defined above, and the $a_{\alpha\beta}$ are the relevant stochastic amplitudes.

It is clear that the demand (17) has implied a drastic reduction of “useful” frequencies and stochastic amplitudes to those that qualify as relevant, leaving the rest aside from the present consideration. This is a direct and most important consequence of Principle Three. Although the “non-relevant” frequencies still exist and operate, their combined action reduces merely to a noise that we have been systematically neglecting and that adds to the motions described by the present approximation. Indeed they are part of the source of the (nonrelativistic) radiative corrections, as will become clear below. This explains also a most tantalizing feature of the present theory, namely that the behavior of a mechanical (atomic) system controlled by the random vacuum field may be described in terms of response functions and characteristic (relevant) frequencies that are sure numbers. As it is clear from the present discussion, this occurs only insofar as the demand (16), which now should be written as

$$a_{\beta\beta'}a_{\beta\beta''}a_{\beta\beta'''}\ldots a_{\beta\beta(n-1)\beta(n)} = a_{\beta\beta(n)},$$

(28)

is fulfilled. As has been already said, this can occur only approximately, and taking it as an exact relation as a result of the application of Principle Three means neglecting the residual noise.

Eqs. (25) for the random phases and (27) for the relevant frequencies have exactly the same structure, so that the mechanism that solves the former also solves the latter. Thus, to solve Eq. (27) we write

$$\omega_{\alpha\beta} = \Omega_\alpha - \Omega_\beta.$$

(29)
where the parameters $\Omega_\lambda$ are sure numbers to be determined later (see Eq. (45b)). It can be easily seen that this form satisfies Eq. (27) identically.

The solution (26) is precisely the one arrived at during the foundations of matrix mechanics [33]. There is however an important difference between that original derivation and the present one. Although in both cases Eq. (29) is used to ensure that each term in a Fourier development corresponds to the appropriate frequency, in the present case it is not a formal device required to get the correct Fourier development but a result of the three principles we have used to construct LSED, i.e., it is a consequence of fundamental postulates and has a physical meaning over and above its mathematical necessity.

3.1 The response amplitudes

Let us now turn to the equations that determine the response functions and the relevant frequencies. These are the set of equations (16), which in the new notation become

$$\bar{x}_{\alpha\beta} = -\frac{e\bar{F}_{\alpha\beta}}{m\omega_{\alpha\beta}^2 + im\tau\omega_{\alpha\beta}^3},$$  

(30a)

where $\bar{f}_{\alpha\beta}$ is the Fourier component of frequency $\omega_{\alpha\beta}$ of the external force. In its turn, Eq. (12b) reads now

$$x_{\alpha}(t) = \sum_{\beta} \frac{-e\bar{F}_{\alpha\beta}a_{\alpha\beta}}{m\omega_{\alpha\beta}^2 + im\tau\omega_{\alpha\beta}^3} e^{i\omega_{\alpha\beta}t}.$$  

(30b)

This equation shows that now we have a whole set of solutions, labeled by the index $\alpha$ that we have been forced to add by following the above rules. These solutions are obtained by solving the complete set of simultaneous equations (30a), where $\bar{f}_{\alpha\beta}$ is expressed as a function of the response functions themselves and the relevant frequencies. For example, for an external force expressed as a power series we would have something like

$$\bar{f}_{\alpha\beta} = \left[ k_1\bar{x}_{\alpha\beta} + k_2 \sum_{\beta'} \bar{x}_{\alpha\beta'}\bar{x}_{\beta'\beta} + k_3 \sum_{\beta',\beta''} \bar{x}_{\alpha\beta'}\bar{x}_{\beta'\beta''}\bar{x}_{\beta''\beta} + \ldots \right] a_{\alpha\beta}.$$  

(30c)

Taken as an implicit equation for the response function $\bar{x}_{\alpha\beta}$, Eq. (30a) has in the general case solutions dominated by its poles. We should expect that at each of the corresponding frequencies there is a strong response of the mechanical system, and since $\tau$ is a very small parameter (normally $\tau\omega_{\alpha\beta} \ll 1$, so that in the quantum mechanical case the term $im\tau\omega_{\alpha\beta}^3$ coming from the radiation damping is neglected ) this response is extremely sharp. This suggests to take them as resonances that occur at the corresponding frequencies, which are just the relevant frequencies.

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Eq. (30b) shows that the function $x_\alpha(t)$ is expressed as a linear function of the stochastic amplitudes $a_{\alpha\beta}$; the same is true for $p_\alpha(t) = m\dot{x}_\alpha(t)$ (we are here again neglecting a small radiative correction). Since the product of any number of relevant stochastic amplitudes can be expressed as a single amplitude by applying Eq. (28), any dynamic variable, taken as a function of the $x_\alpha(t)$ and $p_\alpha(t)$, can in principle be expressed as a linear function of the amplitudes $a_{\alpha\beta}$ (or, in exceptional cases, independent of them). This is the reason for having named the present theory linear stochastic electrodynamics, LSED. It is a fundamental and distinctive feature of the theory; it explains, for example, why all systems described by it (and hence by quantum mechanics) behave as if they consisted of a set of linear oscillators.

4 The equations of motion

The algebraic relationships just obtained between dynamic quantities strongly suggest to adopt a matrix language to frame the whole theory. Take Eq. (30c) as an example, with the series of terms within the square brackets contributing to the force term $\tilde{f}_{\alpha\beta}$. Each one of these is easily recognizable as a matrix element, e.g. for the third-order term we have

$$\sum_{\beta',\beta''} \tilde{x}_{\alpha\beta'} \tilde{x}_{\beta'\beta''} = (\tilde{x}^3)_{\alpha\beta}$$

(31)

a.s.o., so that also the quantity $\tilde{f}_{\alpha\beta}/a_{\alpha\beta}$ can be recognized as a matrix element. Going now back to Eq. (11) in the new notation, i.e. with the second subindex introduced, we have for every Fourier component:

$$-\left( m\omega_{\alpha\beta}^2 \tilde{x}_{\alpha\beta} + im\tau\omega_{\alpha\beta}^3 \tilde{x}_{\alpha\beta} + \tilde{f}_{\alpha\beta}/a_{\alpha\beta} \right) e^{i\omega_{\alpha\beta} t} = e^{i\omega_{\alpha\beta} t},$$

(32a)

which is itself an equation relating matrix elements. Associating the elementary oscillator $e^{i\omega_{\alpha\beta} t}$ to each one of the matrix elements we can rewrite this as a dynamic equation:

$$m \frac{d^2 \tilde{x}_{\alpha\beta}(t)}{dt^2} = \tilde{f}_{\alpha\beta}(t) + m\tau \frac{d^3 \tilde{x}_{\alpha\beta}(t)}{dt^3} + e \tilde{E}_{\alpha\beta}(t).$$

(32b)

with

$$\tilde{x}_{\alpha\beta}(t) = \tilde{x}_{\alpha\beta} e^{i\omega_{\alpha\beta} t}, \quad \tilde{E}_{\alpha\beta}(t) = \tilde{E}_{\alpha\beta} e^{i\omega_{\alpha\beta} t}, \quad \tilde{f}_{\alpha\beta}(t) = \tilde{f}_{\alpha\beta}/a_{\alpha\beta} e^{i\omega_{\alpha\beta} t}. $$

(32c)

In closed matrix notation, Eq. (32b) reads

$$m \frac{d^2 \tilde{x}}{dt^2} = \tilde{f} + m\tau \frac{d^3 \tilde{x}}{dt^3} + e \tilde{E}. $$

(32d)
The field operator written in full is

$$\hat{E}(\omega) = iN\sqrt{\mathcal{E}} (\hat{a} e^{i\omega t} - \hat{a}^\dagger e^{-i\omega t}) ,$$  \hspace{1cm} (32e)

with

$$\hat{a} = -\frac{i\hat{E}^+(\omega)}{N\sqrt{\mathcal{E}}} , \quad \hat{a}^\dagger = \frac{i\hat{E}^-(\omega)}{N\sqrt{\mathcal{E}}}.$$  \hspace{1cm} (32f)

$\mathcal{E} = \frac{1}{2}\hbar \omega$ and $N$ a suitable normalizing factor, as explained below ($\hat{E}(\pm)$ are now matrices). In writing these equations we have separated positive and negative frequencies for clarity (in Eq. (32e) $\omega > 0$).

Eq. (32b) is the law of motion for the mechanical subsystem in the quantum regime according to LSED; it agrees with the corresponding equation of non-relativistic quantum electrodynamics [34]. If now we neglect the field and the radiation reaction terms to get a purely mechanical description, which can be done since owing to the principles under which the theory has been constructed, the field has already played its central role in stabilizing the atomic subsystem and driving it to the quantum regime, we get the usual Heisenberg equations of motion of nonrelativistic quantum mechanics,

$$\frac{d\hat{p}}{dt} = \hat{f},$$  \hspace{1cm} (32g)

$$\hat{p} = \hbar \frac{d\hat{x}}{dt}.$$  \hspace{1cm} (32h)

The neglected terms, when reintroduced, lead to the (well-known) radiative corrections arising both from the background field and radiation reaction.

4.1 Completing the description

Establishing the full equivalence between quantum mechanics and LSED still requires some additional results, to which we now pay attention. We start by considering the Poisson brackets of dynamical variables in the LSED description. Originally the configuration and momentum coordinates are the $x, p$ of the particle (or particles) and the $q_{\alpha\beta}, p_{\alpha\beta}$ of the field. However, once the quantum regime is established, the particle variables $x_{\alpha}, p_{\alpha}$ corresponding to a stationary state are not any more independent, as they have become functions of the field variables $q_{\alpha\beta}, p_{\alpha\beta}$ or, equivalently, of the random amplitudes $a_{\alpha\beta}, a_{\alpha\beta}^*$, as follows from Eq. (33), which in full notation reads

$$p_{\alpha\beta} = \sqrt{\mathcal{E}_{\alpha\beta}/2} (a_{\alpha\beta} + a_{\alpha\beta}^*) , \quad i\omega_{\alpha\beta} q_{\alpha\beta} = \sqrt{\mathcal{E}_{\alpha\beta}/2} (a_{\alpha\beta} - a_{\alpha\beta}^*) .$$  \hspace{1cm} (33)

with $\mathcal{E}_{\alpha\beta} = \frac{1}{2}\hbar \omega_{\alpha\beta}$. This means that the Poisson bracket of the couple of dynamical variables $A_{\alpha}, B_{\alpha}$ should be written as

$$[A, B]_{PB} = \sum_{\lambda} \left[ \frac{\partial A}{\partial q_{\alpha\lambda}} \frac{\partial B}{\partial p_{\alpha\lambda}} - \frac{\partial A}{\partial p_{\alpha\lambda}} \frac{\partial B}{\partial q_{\alpha\lambda}} \right]$$
\[
\sum_{\lambda} - \frac{i \omega_{\alpha \lambda}}{2 \xi_{\alpha \lambda}} \left[ \frac{\partial A}{\partial a_{\alpha \lambda}} \frac{\partial B}{\partial a_{\alpha \lambda}^*} - \frac{\partial A}{\partial a_{\alpha \lambda}^*} \frac{\partial B}{\partial a_{\alpha \lambda}} \right]
\]
\[
= \frac{1}{i \hbar} \sum_{\lambda} \left[ \frac{\partial A}{\partial a_{\alpha \lambda}} \frac{\partial B}{\partial a_{\alpha \lambda}^*} - \frac{\partial A}{\partial a_{\alpha \lambda}^*} \frac{\partial B}{\partial a_{\alpha \lambda}} \right],
\]

(34a)

where both variables \(A\) and \(B\) are understood to refer to the state \(\alpha\). The quantity appearing in the last equality within brackets was introduced in a similar context in [35] with the name Poissonian, and in a different context under the name of commutator in [36]; denoting it with \(\langle A; B \rangle\) (the index \(\alpha\) is implied as above) we get

\[
[A, B]_{PB} = \frac{1}{i \hbar} \langle A; B \rangle.
\]

(34b)

It is easy to generalize Eq. (34a) to include nondiagonal elements, by writing

\[
\langle A; B \rangle_{\alpha \beta} = \sum_{\lambda} \left[ \frac{\partial A}{\partial a_{\alpha \lambda}} \frac{\partial B}{\partial a_{\beta \lambda}^*} - \frac{\partial A}{\partial a_{\beta \lambda}^*} \frac{\partial B}{\partial a_{\alpha \lambda}} \right] a_{\alpha \beta}
\]
\[
= \sum_{\lambda} \left[ \frac{\partial A}{\partial a_{\alpha \lambda}} \frac{\partial B}{\partial a_{\alpha \lambda}^*} - \frac{\partial B}{\partial a_{\alpha \lambda}^*} \frac{\partial A}{\partial a_{\alpha \lambda}} \right] a_{\alpha \beta}.
\]

(35)

In writing this equation we have taken into account that from Eq. (26) it follows that \(a_{\alpha \beta}^* = a_{\beta \alpha}\); similarly, \(\omega_{\alpha \beta} = - \omega_{\beta \alpha}\). This result can be recast immediately in terms of the matrix elements of the variables \(A\) and \(B\). Indeed, by writing

\[
A = \sum A_{\alpha \lambda} a_{\alpha \lambda} e^{i \omega_{\alpha \lambda} t}, \quad B = \sum B_{\alpha \lambda} a_{\alpha \lambda} e^{i \omega_{\alpha \lambda} t},
\]

(36)

we get successively

\[
\langle A; B \rangle_{\alpha \beta} = \sum_{\lambda} \left[ \tilde{A}_{\alpha \lambda} \tilde{B}_{\lambda \beta} - \tilde{B}_{\alpha \lambda} \tilde{A}_{\lambda \beta} \right] a_{\alpha \beta} = \left[ \hat{A}, \hat{B} \right]_{\alpha \beta} a_{\alpha \beta}.
\]

(37)

In the last expression we have introduced the commutator of the matrices \(\hat{A}\) and \(\hat{B}\) with matrix elements

\[
\left[ \hat{A}, \hat{B} \right]_{\alpha \beta} = \sum_{\lambda} \left[ \tilde{A}_{\alpha \lambda} \tilde{B}_{\lambda \beta} - \tilde{B}_{\alpha \lambda} \tilde{A}_{\lambda \beta} \right].
\]

(38)

We have thus found the correspondences

\[
[A, B]_{PB} \leftrightarrow \frac{1}{i \hbar} \langle A; B \rangle \leftrightarrow \frac{1}{i \hbar} \left[ \hat{A}, \hat{B} \right].
\]

(39)

Two important applications are the following:

The identity \([x, p]_{PB} = 1\) leads to the fundamental commutator \([\hat{x}, \hat{p}] = i \hbar \hat{1}\).
The eq. of motion $d\hat{A}/dt = [A, H]_{PB}$ leads to the Heisenberg eq. $i\hbar d\hat{A}/dt = [\hat{A}, \hat{H}]$.

(41)

It must be noted that whereas the classical equation $[x, p]_{PB} = 1$ is an identity, the corresponding commutator $[\hat{x}, \hat{p}] = i\hbar$ is a derived equation that holds only in the quantum regime. It has a dynamic meaning, precisely because it implies that the mechanical system has already reached the quantum regime, in which the mechanical variables are driven by the field variables; in other words, it is a physical law. From Eq. (33) it is easy to see that with the definition given above, Eq. (40) holds also for the field variables (because they describe quantum oscillators), so that we recover the usual rule

$$[\hat{a}, \hat{a}^\dagger] = -i\omega \frac{2\hbar}{\tilde{E}} [\hat{q}, \hat{p}] = \frac{1}{i\hbar} [\hat{q}, \hat{p}] = \hat{1}. \tag{42}$$

Another related point that merits some attention is the following. The matrix elements of any dynamical variable are given by equations as (30a) or (32a), which contain no arbitrary elements in principle. In quantum mechanics, however, as there is no explicit reference to the vacuum field components $\tilde{E}$, the scale of the matrix elements is lost. This problem is solved by normalizing the state vectors to unity (and so $\langle \alpha | \beta \rangle = \delta_{\alpha \beta}$). In the present theory, such scale can be introduced by means of Eq. (42), which fixes the normalization factor $N$ that was left undetermined in Eq. (32c).

It remains still to determine the meaning of the parameter $\Omega_{\lambda}$ introduced in Eq. (29). To achieve this we combine this equation with (32c) to write

$$\tilde{x}_{\alpha\beta} = -i\omega \frac{\hbar}{2} \tilde{x}_{\alpha\beta} = i (\Omega_{\alpha} - \Omega_{\beta}) \tilde{x}_{\alpha\beta}$$

$$= i \sum_{\lambda} (\Omega_{\alpha} \delta_{\alpha\lambda} \tilde{x}_{\lambda\beta} - \tilde{x}_{\alpha\lambda} \Omega_{\beta} \delta_{\lambda\beta}). \tag{43}$$

On the other hand, from the equation of motion (41) it follows that

$$i\hbar \tilde{x}_{\alpha\beta} = \sum_{\lambda} \left( \tilde{x}_{\alpha\lambda} \tilde{H}_{\lambda\beta} - \tilde{H}_{\alpha\lambda} \tilde{x}_{\lambda\beta} \right). \tag{44}$$

A comparison gives

$$\tilde{H}_{\alpha\beta} = \hbar \Omega_{\alpha} \delta_{\alpha\beta} + c \tilde{x}_{\alpha\beta} + d \delta_{\alpha\beta}, \tag{45a}$$

with $c$ and $d$ arbitrary. By using $\tilde{p}_{\alpha\beta} = i\omega_{\alpha\beta} \tilde{p}_{\alpha\beta}$ and applying a similar procedure we conclude that necessarily $c = 0$. The constant $d$ simply shifts the overall reference level of $H$, and can therefore be dropped. Therefore, the matrix representing the Hamiltonian in state $\alpha$ is diagonal and has sure values,

$$H_{\alpha} = \sum_{\lambda} H_{\alpha\lambda} a_{\alpha\lambda} = h\Omega_{\alpha} a_{\alpha\alpha} = h\Omega_{\alpha} \equiv E_{\alpha}, \tag{45b}$$
as was to be expected, since the stationary states were defined from the very start (Eq. 4a) as those for which energy equilibrium had been reached. In terms of the energy $E_\alpha$ of state $\alpha$, Eq. (29) becomes Bohr’s rule for the transition frequencies,

$$\hbar \omega_{\alpha\beta} = E_\alpha - E_\beta.$$  

(45c)

One can thus identify the relevant frequencies with the corresponding quantum transition frequencies. Since these coincide with the spectroscopic frequencies, Eq. (45c) corresponds to the old Ritz principle, stating that each spectroscopic frequency can be written as the difference of two terms. In the present theory, however (as in quantum theory) this is a prediction. It is important to observe that Principle Three, by assigning a sure value to $\omega_{\alpha\beta}$, concurrently assigns sure values to the energy, which correspond to the eigenvalues of the Hamiltonian, as stated in Eq. (45b) (although in a different language). Principle Three can therefore be considered as the quantization principle, a point on which we elaborate below. In this form we have verified that LSED is formally equivalent to (nonrelativistic) QED and to quantum mechanics in the radiationless approximation.

Recently, Cole and Zou [26] obtained a series of appealing numerical results for the ground state of the hydrogen atom directly from the principles of SED, having a strong resemblance with the corresponding predictions of quantum mechanics. Their computations coincide in spirit with the present theory, since both approaches are based on the principles of SED but are stripped from the old methodological assumptions, so neither perturbative nor Fokker-Planck methods are being used. Thus the present work gives theoretical backing to their results, while at the same time it is at least in part underpinned by their numerical experiments.

Still the conceptual differences between LSED and quantum mechanics are momentous. A brief discussion of these matters is given in the Discussion section at the end. For the time being let us just briefly remark that from the present point of view, quantum mechanics furnishes an approximate, time-asymptotic statistical description of the mechanical (atomic) part of the system under study, valid once the quantum regime has set in. The passage to QED improves the description by adding part of the lost noise and by leading to matter and field quantization, which calls for new phenomena. But even then, the description continues to be approximate, statistical and time asymptotic. Only a return to the initial, complete description could lead to a qualitative improvement of the account. This is a task that pertains to the future.

## 5 Detailed energy balance

Now we come back to Eq. (1b)

$$m\tau \langle \mathbf{x}^2 \rangle = e \langle \mathbf{x} \cdot \mathbf{E} \rangle$$
describing the average power balance. Our first undertaking will be to give to this equation a more finished form. The quantum mechanical solutions follow from Eq. (32g), but they must be amended using Eq. (32d) to take into account the radiative corrections and other phenomena. Treating the corrections as a perturbation, the solution will read (once more in one dimension) \( x = x_0 + x_1 \), where \( x_0 \) represents the unperturbed solution and \( x_1 \) its correction. The average absorbed and radiated power are now \( m\tau \langle \cdot \rangle_{x_0} \) and \( e \langle (\dot{x}_0 + \dot{x}_1) E \rangle \), respectively, where we have neglected the small corrections to the radiation.

Let us now assume that \( x_0 \) is proportional to \( e^r \), whereas \( x_1 \) is proportional to \( e^s \). Here \( e \) stands for the coupling constant to the radiation field, which in the present context appears as merely incidental. Thus, the charge \( e \) that appears here is foreign to quantum mechanics, as it is linked to the background field \( E \). Further, since \( m\tau = 2e^2/3c^3 \), the power radiated is proportional to \( e^{2r+2} \), whereas the power absorbed due to \( e\dot{x}_0 \) is proportional to \( e^{r+1} \). For these two quantities to be equal, we must have \( r = -1 \). But this is contrary to quantum mechanics, since with the normalization used there, one should have \( r = 0 \), as has just been argued. Thus, the term \( e \langle \dot{x}_0 E \rangle \) cannot contribute to the energy absorption. However, the term proportional to \( x_1 \) requires that \( 2r + 2 = s + 1 \), or \( s = 2r + 1 \). Putting here \( r = 0 \) we get \( s = 1 \).

As will be shown, this corresponds exactly to the correction \( x_1 \) (see Eq. (47)), so we conclude that the equation for the energy balance, omitting the spurious term, reads

\[
m\tau \langle \cdot \rangle_{x_0} = e \langle x_1 \cdot E \rangle \tag{46}
\]

A more formal argument to arrive at this equation goes as follows. The quantity \( \dot{x}_0 E \) belongs to QED, where the vacuum field is a natural element of the theory and the term accounts for the effect of the radiative correction. However, \( \dot{x}_0 \) and \( E \) belong to different Hilbert spaces, so that \( \langle \dot{x}_0 E \rangle \) is proportional to \( \langle 0 | E | 0 \rangle = 0 \), and the contribution effectively cancels out.

Incidentally, we note that Eq. (46) is independent not just from the charge, that is, from the strength of the particle’s coupling to the vacuum field (we stress that \( x_1 = \delta x \) in Eq. (47) is proportional to \( e \)), but also from the mass of the particle. This strongly suggests a principle of universality, according to which the variance of the acceleration is largely independent from the specific details of the particle and perhaps, of the interaction. This principle has already been advocated from different considerations within SED [37], and to the extent that it holds, SED would be but a particular version of a more general theory, in which different kinds of vacuum field may participate (of course, all of them with the same average energy per mode and hence in equilibrium among them). In its turn, this points to the possibility that a more general theory could be formulated not in terms of random vacua, but of a fluctuating metric, which would be a truly universal theory [3].

The next step is the determination of \( x_1 \), which constitutes a conventional
problem readily solved using perturbation theory. The result to first order is (we return to our previous notation, so that $x_0$ is denoted by $x$, and instead of $x_1$ we write $\delta x$)

$$\delta x_\alpha(t) = \frac{2e}{\hbar} \sum_\beta |\tilde{\alpha}_\beta|^2 \int_0^\infty E(t - s) \sin \omega_{\alpha\beta} s \, ds.$$ (47)

We now take into account that

$$\langle E_i(x, t) E_j(x', t') \rangle = \delta_{ij} \int_0^\infty S(\omega) \cos \omega(t - t') \, d\omega,$$ (48a)

where $S(\omega)$ is the power spectrum of the vacuum field and $\rho(\omega)$ is its (energy) spectral density. Inserting these results into the expression for the average power absorbed and performing the integration, we get

$$e \langle \delta x E \rangle_\alpha = -\frac{4\pi^2 e^2}{3\hbar} \sum_\beta \omega_{\alpha\beta} \rho(\omega_{\alpha\beta}) |\tilde{x}_{\alpha\beta}|^2.$$ (49)

In its turn, in the quantum regime the average power radiated is $(2e^2/3c^3) \langle \dot{x}^2 \rangle = (2e^2/3c^3) \omega_{\alpha\beta}^2 |\tilde{x}_{\alpha\beta}|^2$. Thus Eq. (49) transforms into

$$\sum_\beta \frac{2e^2}{3c^3} \left[ -|\omega_{\alpha\beta}|^3 + \frac{2\pi^2 e^3}{\hbar} \rho(\omega_{\alpha\beta}) \right] |\omega_{\alpha\beta}| |\tilde{x}_{\alpha\beta}|^2 = 0.$$ (50)

In writing this equation we assumed that $\omega_{\alpha\beta}$ is negative, as is the case for the ground state. We assumed also that the frequencies $\omega_{\alpha\beta}$ are nondegenerate. This equation is satisfied irrespective of the coefficients $\tilde{x}_{\alpha\beta}$, which means without regard to the specific system under study, if the expression within brackets vanishes for every $\omega_{\alpha\beta}$, or if

$$\rho(\omega) = \rho_0(\omega) \equiv \frac{\hbar \omega^3}{2\pi^2 e^3} \quad (\omega > 0).$$ (51)

This is just the spectral density of a vacuum with average energy per mode $\hbar \omega/2$, so it corresponds to that of the vacuum field of SED (and QED). This result means that indeed the balance equation (40) is satisfied by each frequency separately (whether or not there is degeneracy), or that detailed energy balance holds for any bounded system described by LSED. Alternatively, the argument can be seen as a derivation of the zero-point spectrum from the requirement of detailed balance. The result is important, not only because it shows the internal consistency of the theory, but also on the account that it stands in sharp contrast with the corresponding classical result for a general system with harmonics, where detailed balance holds only for the Rayleigh-Jeans spectrum $\rho(\omega) \sim \omega^2$ and, perhaps worse, only for a Laplacian distribution of energy (Maxwell-Boltzmann statistics) [38].
6 Some generalizations. Planck’s distribution

With the purpose of providing a more general perspective of the theory we give some generalizations of the above results, without however entering into their detailed derivations, which can be found elsewhere [8]. Let us first consider a dynamic variable \( \xi \) that represents an integral of motion of the unperturbed system. It is possible to demonstrate that to first order in perturbation theory the equilibrium condition for this variable reads

\[
\sum_{\beta} \left[ -\frac{\hbar}{2\pi^2 c^3} \omega_{\alpha\beta}^3 + \rho(\omega_{\alpha\beta}) \right] (\xi_\alpha - \xi_\beta) |\tilde{x}_{\alpha\beta}|^2 = 0. \tag{52}
\]

For \( \xi = H \) this result reduces to Eq.(50), as it should. For any other integral of motion for which \( \xi_\alpha \neq \xi_\beta \) Eq.(52) leads to the same equilibrium spectral energy density \( \rho_0(\omega) \), as should be expected in advance.

Let us now extend the result (50) to cover the case of excited states and a more general external random electromagnetic field, to study the equilibrium conditions. We write the spectral energy density of the field in the form

\[
\rho = \rho_e + \rho_0,
\]

where \( \rho_0 \) is given by Eq.(51) and \( \rho_e \) represents the spectral density of the field above the zero-point. We can write now

\[
\langle \frac{dH}{dt} \rangle = -\frac{4\pi^2 e^2}{3\hbar} \sum_{\beta} \omega_{\alpha\beta} \left[ \rho + \text{sign}(\omega_{\alpha\beta}) \rho_0 \right] |\tilde{x}_{\alpha\beta}|^2. \tag{53}
\]

We now separate positive and negative frequencies, adding a superindex \( \pm \) to \( \tilde{x}_{\alpha\beta} \) to keep track of this sign,

\[
\langle \frac{dH}{dt} \rangle = -\frac{4\pi^2 e^2}{3\hbar} \sum_{\beta} \omega_{\alpha\beta} \left[ (\rho - \rho_0) |\tilde{x}_{\alpha\beta}^{(-)}|^2 + (\rho + \rho_0) |\tilde{x}_{\alpha\beta}^{(+)}|^2 \right], \tag{54}
\]

and recast the result into the form

\[
\langle \frac{dH}{dt} \rangle = W_{ab} - W_{em}, \tag{55a}
\]

where

\[
W_{ab} = \frac{4\pi^2 e^2}{3\hbar} \sum_{\omega_{\alpha\beta} < 0} |\omega_{\alpha\beta}| (\rho - \rho_0) |\tilde{x}_{\alpha\beta}^{(-)}|^2, \tag{55b}
\]

\[
W_{em} = \frac{4\pi^2 e^2}{3\hbar} \sum_{\omega_{\alpha\beta} > 0} \omega_{\alpha\beta} (\rho + \rho_0) |\tilde{x}_{\alpha\beta}^{(+)}|^2 \tag{55c}
\]

are the contributions of the absorptions and emissions to the energy change, respectively. Eq. (55b) clearly shows that for absorptions to occur, necessarily \( \rho > \rho_0 \), i.e., \( \rho_e \) must be present. Thus, there are no ‘spontaneous absorptions’, \( W_{ab}^{\text{spont}} = 0 \) in the present theory, just as happens in QED and in nature, of course. This behavior is due to the fact, clearly shown in Eq. (55b), that the
ground state is just that supported by the vacuum field; to get into a higher state the atomic system should be immersed in a field with $\rho > \rho_0$. It is also interesting to have a closer look at Eq. (55c) for the probability of an emission to take place. As follows from Eq. (50) the contribution containing $\rho_0$ comes from the effects of radiation reaction, whereas the term that involves the whole spectral density $\rho$ is due to the fluctuating motions. For a pure vacuum $\rho = \rho_0$ both contributions become alike and contribute with equal amounts to the emissions, whereas they exactly cancel out for absorptions. Of course this latter result is but another form to express the fact that the system has reached the quantum regime with the vacuum [39]. This is an important point because SED (and presumably lSED) has been charged of being a semiclassical theory and thus necessarily predicting spontaneous absorptions [40]. In fact all absorptions are induced with probability

$$W_{\text{ind}}^{\text{ab}} = \frac{4\pi^2e^2}{3\hbar} \sum_{\omega_{\alpha\beta}<0} \omega_{\alpha\beta} \rho_e |\tilde{x}_{\alpha\beta}^{(-)}|^2. \quad (55\text{d})$$

On the other hand, writing $\rho + \rho_0 = \rho_e + 2\rho_0$ in Eq. (55c) we obtain induced and spontaneous emissions, the latter being due solely to the action of the vacuum field,

$$W_{\text{ind}}^{\text{em}} = \frac{4\pi^2e^2}{3\hbar} \sum_{\omega_{\alpha\beta}>0} \omega_{\alpha\beta} \rho_e |\tilde{x}_{\alpha\beta}^{(+)}|^2, \quad (55\text{e})$$

$$W_{\text{spont}}^{\text{em}} = \frac{8\pi^2e^2}{3\hbar} \sum_{\omega_{\alpha\beta}>0} \omega_{\alpha\beta} \rho_0 |\tilde{x}_{\alpha\beta}^{(+)}|^2. \quad (55\text{f})$$

From these results (or their generalization to any other integral of motion) it is easy to obtain the Planck distribution as the equilibrium solution for $\rho_e$ by following the well known statistical method introduced by Einstein. For this purpose, let us consider a system with only two active levels, so that a single frequency, which we call $\omega_{\alpha\beta}$, is relevant. To support the state of thermodynamic equilibrium the system must be embedded in an appropriate field that allows for upward and downward transitions to occur at the same constant rate. Since the system is in thermodynamic equilibrium, the populations of the levels $\alpha$ and $\beta$ should be proportional to $e^{-\beta E_\alpha}$ and $e^{-\beta E_\beta}$, respectively, with $\beta = 1/(k_B T)$ (we are neglecting the possibility of degeneracies, as they would add nothing but complications to the argument). Thus from the equilibrium condition $W_{\text{ab}} = W_{\text{em}}$ applied to Eqs. (55c)-(55f), we get

$$e^{-\beta E_\alpha} \rho_e |\tilde{x}_{\alpha\beta}^{(-)}|^2 = e^{-\beta E_\beta} (\rho_e + 2\rho_0) |\tilde{x}_{\alpha\beta}^{(+)}|^2, \quad (56)$$

and since $|\tilde{x}_{\alpha\beta}^{(-)}|^2 = |\tilde{x}_{\alpha\beta}^{(+)}|^2 = |\tilde{x}_{\alpha\beta}|^2$, this gives for the equilibrium condition

$$e^{-\beta E_\alpha} \rho_e = e^{-\beta E_\beta} (\rho_e + 2\rho_0), \quad (57)$$
which in its turn leads to the blackbody distribution (with zero-point field, of course)

\[ \rho = \rho_0 + \rho_e = \rho_0 \cosh \left( \frac{\hbar \omega}{2} \right). \]  

(58)

A nice point of this derivation is that it clearly exhibits the Planck distribution as a universal result, independent of the nature and specific properties of the material system, since the only elements in Eq.(56) referring to such system, the \(|\tilde{x}_{\alpha\beta}|^2\), cancel out to lead to Eq.(57). Also, from Eq.(54) we observe that in the present theory this result comes from the quantum properties of matter, not those of the field. This is an interesting point, since it is traditional to consider the Planck distribution as the first known illustration of the quantum properties of the radiation field. A similar argument is known in other instances, as is the case with the photoelectric effect. This effect was explained by Einstein as arising from the quantum properties of the radiation field, and has been since then taken as such. However there have been solid arguments \[11\] to show that this effect can equally well be interpreted as arising from the quantum properties of matter. Since as we have seen (and is well known), quantization of matter and of the radiation field imply one another so they go together, the coexistence of both possible points of view is understandable.

Of course the equivalent calculation is well known in quantum theory. What we are trying to stress with the present reckoning is that \textsc{lsed} furnishes the results of quantum mechanics and (nonrelativistic) \textsc{qed} in a quite direct and transparent way. It is possible to go even further with the calculation of the radiative corrections (the Lamb shift and others \[8, 13\]), but for the purposes of illustration the above examples should suffice. We thus conclude that the principles used to construct \textsc{lsed} are sufficient to transform an apparently classical theory into a sound quantum theory. The reason of this seemingly miraculous transformation is twofold. Firstly, the theory contains a crucial ingredient, the vacuum field, foreign to classical physics and with statistical properties specified by \(\hbar\), which makes the theory stricto sensu a nonclassical one from the start—or a quantum one, as substantiated by the end results. Secondly, the set of principles used to develop the theory, particularly Principle Three, is strong enough as to select a class of (approximate) solutions to the equations of motion that corresponds just to the quantum behavior. These reasons explain our proviso apparently classical, used to stress the fact that if it were a plain classical theory, it would be impossible to derive quantum results from it.

7 Discussion

We have arrived at quantum mechanics from a fresh point of view. Even if for utilitarian purposes the present derivation may seem to be of limited interest, on the conceptual level it has the benefit of providing a new and valuable perspective to the foundations of quantum theory. For example, the theory furnishes a physical explanation on the origin of quantization as due to the selection of allowed solutions as the robust ones against fluctuations, \(i.e.,\) the ones that are immune to small changes in the particular realization of the vacuum field, once
the system has entered the quantum regime. Further, the new picture elucidates the mechanism leading to atomic stability, a problem that has puzzled physicists for nearly a century (we elaborate further on these and related points below and in the Appendix). But in addition, there appear some points where the predictions of the theory may permit someday to explore as yet unknown territories. For the time being let us make a brief tour from the perspective afforded by the present theory onto some of those traits of quantum mechanics that have been the subject of endless discussions and controversies on the matter.

In the usual perspective, quantum mechanics constitutes both the point of departure and the final reference for our inquiries about the meaning of the theory itself. Its conceptual problems must therefore be looked at from the inside, which creates a kind of circular reasoning leading almost nowhere, as is amply testified by endless discussions on such subjects. Since the point of departure for LSED is a wider physical theory, it offers a qualitatively different opportunity. This fact allows in principle to answer such conceptual questions with a fresh and deeper understanding from an ‘external’ perspective, without the need to resort to philosophical or ideological preconceptions. We now attempt to exploit these possibilities to address, albeit very succinctly and in a schematic fashion, some of the most abiding quantum questions.

7.1 Atomic stability

One point of the proposed theory that surely catches the reader’s attention is the one related to atomic stability. In usual quantum theory the stationary atomic levels are well predicted by the equations, but the physical reason for their stability remains undisclosed. In LSED they appear as those states which comply with the requirement of belonging to the quantum regime, that is, those for which the rate of radiated and absorbed energy is the same in the mean. That the levels belong normally to a discrete spectrum comes from the fact that only for certain orbital motions such equilibrium can be attained, as follows clearly from Eq.(46). In quantum mechanics such an explanation is impossible, since there is no field from which to absorb energy, and thus the point remains as a mystery, and can find only a formal answer.

7.2 Energy eigenvalues and the origin of quantization

A related point is that of the energy (and other) eigenvalues. That in an essentially stochastic theory the dynamical variables may attain sure values, seems to be a contradiction, or at least a very obscure property. The answer that can be derived from the present development goes as follows. In the first place, the quantum description is approximate, since Principle Three cannot be satisfied exactly by natural systems; and it is just this principle that leads to the existence of eigenvalues. Thus, nature is noisier than what the present theoretical description asserts (which, indeed, must be corrected by adding at least the field that gives rise to the radiative corrections, a very special kind of noise, a correction leading to the QED description as shown above). Secondly, the quantum
description (according to LSED) refers to a kind of average behavior, as follows
from the principles of the theory and is amplified here, and thus its dynamical
variables are normally partially averaged quantities describing the behavior of
subensembles that comply locally with the statistical requirements. A further
but fundamental reason for the appearance of sharp values for some dynamical
variables is that they correspond to stable stationary motions, which makes
them emerge through the statistics as preferred motions, selected by Principle
Three. This is why there can be eigenvalues at all in the theory. In the next
subsections we expand on these considerations.

The present formulation is based on a Fourier development of the field on the
frequency \( \omega \), which means that each term of the development corresponds to
an infinity of field components with all possible values for the wave vector \( k \) and
polarization, with \( k = \omega/c \). In other words, in each case we are considering the
combined effect of all such stochastic components, which vary from realization
to realization, as a single, unique instance, a simplification that is equivalent to
perform a partial averaging over the corresponding field modes \( \mathcal{S} \). This is one
of the reasons we stated above that the dynamic variables are frequently par-
tially averaged quantities. A second obvious reason of the said implicit partial
averaging is the neglect of the effects of the noisy (“nonrelevant”) components
of the field.

Since once the quantum regime is reached and Principle Three holds, the de-
tailed motions do not depend on the specific realization of the field, it becomes
impossible to trace back the trajectory followed by a given particle that reached
the corresponding stationary state. In this sense, the description becomes inde-
pendent of the initial conditions and refers only to subensembles, i.e., the set of
those particles that reached the final state, whatever the trajectory they may
have followed.

It seems convenient to further elaborate on the matter. We have just seen
that Principle Three, by selecting the reduced set of solutions that are insen-
sitive to the specific realization of the field and thus particularly stable, is the
source of quantization in the present theory. From a more physical point of view
it becomes intuitively clear that the demand of detailed energy balance can be
satisfied only by a reduced (and frequently discrete) set of motions. As we have
seen, this latter requirement is the outcome of the very stringent conditions
imposed by the simultaneous demand of energy balance and independence of
the response functions (and the relevant frequencies) from the specific realiza-
tion of the field (Principle Three). It is the mutual reinforcement of these two
requirements which leads to the selection of a well defined class of stationary
allowed solutions, and thus to quantization. By its physical content, the present
explanation of quantization stands in sharp contrast to the usual one related
with the properties of the wave function, although it is formally similar to the
one afforded by matrix mechanics, where quantization arises from solving a set
of simultaneous algebraic equations similar to those given by Eq. (30a) (with due
allowance for the normalization to avoid the numerator \( cE_{\alpha\beta} \), and neglecting
the radiation reaction force). However, this explanation lacks the transparency

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provided by Principle Three, besides being purely formal. In the Appendix it is shown that the allowed stationary solutions correspond to extremum (indeed, minimum) values for the energy.

7.3 Are there trajectories?

This is a major point in the interpretations of quantum mechanics. Since the notion of trajectory does not enter into the axioms of quantum mechanics, the dominant point of view is that in this theory (as it is) there are no trajectories. However, this correct conclusion is frequently amplified to mean that in nature there are no trajectories, when one refers to the systems dealt with in quantum mechanics. This conclusion is tightly bound to the origins of quantum mechanics, for instance, to the foundational work of Heisenberg [42], and has found its way to almost every textbook on the subject. The argument is founded on the Heisenberg inequalities related to noncommuting variables, such as $x$ and $p$. Now in the present theory, the trajectories exist, as follows from the starting premises out of which the quantum description emerges. However, as discussed above, the theory does not strictly describe the motion of individual particles, but of subensembles of particles that satisfy (approximately) the statistical demands on which it is constructed. From such a description, the individual trajectory becomes unrecoverable. Therefore, the trajectories exist in nature (as accounted in the initial description), but they do not belong to the set of ingredients that comprise the final (partially averaged, approximate and time-asymptotic) description. Hence quantum mechanics cannot legitimately be used as a weapon against realism, as is frequently done.

The absence of trajectories in the quantum description is a serious obstacle for the description of fast events, as are the transitions between states, the quantum jumps. In the usual description there is some magic in these, since the jumping electron must ‘know’ in advance the energy of the orbit to which it will be landing, to decide the frequency of the photon to be radiated or absorbed. According to the present theory, transitions occur due to resonant interactions with the background field. Given the state of the atomic electron, there is a defined set of resonant frequencies to which it may respond. Which will be the one selected in each instance is a matter of chance, but there is no more guessing by the part of the electron. Of course, ‘chance’ should be understood here to mean that the end result depends, among other things, on the specific realization of the stochastic vacuum field, upon which we have no control.

There are attempts to introduce hidden variables into quantum theory to recover the hidden trajectories, the best known one being Bohm’s causal theory [2]. According to the present view such attempts are doomed to failure, since the individual behavior of a particle becomes irretrievable once its stochastic motion has been smoothed out, either by averaging or by approximations. The only sensible way to follow the real trajectories is to go back to the original equation of motion [1], but even then we have the intrinsic problem of any stochastic description, namely the specific realization of the field is unknown and with it the specific trajectory. The best we can do in any real situation is to resort to
a statistical treatment of the problem. In plain words, this means that to the extent that LSED is a sensible theory, the mere addition of hidden variables to the usual quantum mechanical description to recover determinism or realism is a very poor course. Even if one attempts to complete the theory by adding the background field (as is usually done in QED), the trajectory of a specific particle remains undetermined; this simply means that an indeterministic description of the quantum system is unavoidable. It is interesting to compare this with the old eagerness, expressed so many times by Einstein as his most tenacious devotee, for a final description free of statistical elements. Unfortunately (at least for some) that seems to be untenable. It is convenient to stress once more that this does not mean a noncausal behavior of the particle: the full theory is both causal and realist, since it is a branch of electrodynamics.

This latter remark is conveniently supported by the work of Cole and Zou already cited [26], where all calculations are performed by following the particles along their trajectories and computing the relevant probability distributions. As already noted, with such procedure the authors recover results that are close to the quantum mechanical ones for the ground state of the hydrogen atom. We have thereat an explicit numerical example of the possibility of interpreting the quantum results in terms of trajectories [43].

7.4 Single particle versus ensemble interpretation

Here we are at the core of the problems of interpretation of quantum mechanics, since the answer to the present dilemma in one or the other sense defines the person as an orthodox or unorthodox (and thus heretic). The present theory gives an answer to this quandary and just on the iconoclastic side. A theory that satisfies Principle Three (and detailed balance as is here defined) cannot lead to the detailed description of the motions of a single particle, but gives in a natural way a statistical rendering as discussed formerly. A single particle will almost never satisfy the principles of the present theory, although a subensemble of a big enough collection of similarly prepared systems can satisfy them statistically. It cannot be excluded that under certain conditions a single particle follows closely enough the principles of the theory; under such circumstance of course the theory describes approximately a single particle. But one swallow does not make a summer, so we must adhere to the ensemble interpretation when trying to extract the general implications of the present theory. This point of view is certainly reinforced by the remaining considerations in this section and the whole of the paper.

The above conclusion allows us to remove the need for the observer and the collapse of the state vector, thus avoiding the paradoxes that they entail [44]. The observer becomes unnecessary because different results in a series of a given measurement performed on the same ensemble are the direct result of measuring on different members of the ensemble. The reduction (or collapse) of the state vector becomes dispensable because the realization of a measurement on one of the partners of an entangled system means changing the ensemble to adjust it to the new knowledge afforded by the result of the measurement, which is just
equivalent to a reduction of the state vector \[45\].

7.5 Quantum Fluctuations and Uncertainty

Quantum fluctuations are usually considered irreducible, on the basis of relationships of the kind \(\Delta x \Delta p \geq \hbar / 2\). These inequalities are often interpreted in terms of unavoidable perturbations attributed to measurements or, in a hazier language, to observations. In the present theory these fluctuations reappear, but now as a result of the interaction of matter with the background stochastic field—which in quantum mechanics remains hidden until we appeal to QED—and they attain their full force only when the system reaches the quantum regime. Thus, from the perspective of LSED the quantum fluctuations are not intrinsic to matter, but induced upon it by its interaction with the vacuum field. Since the vacuum fluctuations are measured by \(\hbar\), as shown by Eq. (51), also the induced equilibrium fluctuations, as measured by relations such as \(\Delta x \Delta p \geq \hbar / 2\), are determined by \(\hbar\). This not only fixes the general scale of the (minimum) fluctuations, but gives a causal meaning to them. Thus LSED implies that one should consider the quantum properties of matter not as intrinsic (hence irreducible), but as acquired properties. It is important to observe that, according to the present theory, the Heisenberg inequalities hold only once the quantum regime is attained. So for extremely short time intervals after the particle gets connected to the vacuum, they could be violated, and although this possibility is extremely difficult to verify for the moment, it remains open in principle. For example, for an oscillator of frequency \(\omega_0\) the relaxation time is of order \((\tau \omega_0^2)^{-1}\), which, for optical frequencies, is about \(10^{-10}\) s.

7.6 Quantum non-causality and indeterminism

As noted above, for LSED an entirely deterministic description [46] of the behavior of a quantum system seems to be an impossible task. However, this is neither the result of the perturbations of the system by our observations nor even less is it due to an intrinsic, ontological indeterminism of the electron, as is usual to assume. Such behavior is simply the result of the electron being in constant contact with a stochastic—thus unknown—field. Had we strictly adhered to the detailed original description (instead of developing an approximate statistical formulation), and assumed the field to be known, everything would remain causal and determined. But the kind of system we are considering and the approximations and restrictions made along the derivation of the main equations leading to LSED contravene these requirements, and so the ensuing theory violates in principle both causality and determinism. Causality is lost, since the agent responsible for the quantum (fluctuating) behavior of matter, the vacuum field, is neglected in the quantum mechanical description. As already said, a partial restitution of causality is achieved in the transition from quantum mechanics to QED, but it is introduced too late to recover a fully causal theory.
There are other instances in theoretical physics where approximations transform an otherwise causal theory into one that violates causality. Perhaps one of the best known examples is the Abraham-Lorentz equation of motion (also used here). This equation is derived from a perfectly causal combination of Maxwell’s theory and classical mechanics. The end result, the Abraham-Lorentz equation, can however give rise to noncausal phenomena as preacceleration, the anticipated response to a future force. Again in this case, the root of such noncausal behavior is to be found in the approximations leading from the original causal and full description to the final simplified (and noncausal) one. Approximate physical theories are not bound to satisfy the same rigorous requirements that fundamental theories are supposed to fulfil; this is particularly true with regard to consistency with first principles [8].

7.7 Wavelike behavior of matter

LSED contains a physical field in interaction with matter, and thus it should be able to explain the appearance of the wave behavior of matter as something not intrinsic, but impressed by the field and revealed by the particles. This idea has been a guiding element of SED [10, 47, 48] for a long time and is worth closer attention, because it helps to develop a heuristic picture of some of the most puzzling properties of quantum systems [49]. Our point of departure here is that the linear response to the field characteristic of LSED, means that where several fields superpose, the quantum response functions will add. Thus, the degree of coherence of the underlying superposed fields will be reflected under appropriate conditions in the coherence of the ‘guided’ matter, so to speak.

For the purpose of illustration, let us recall the typical example of the double slit setup, with the detector far away from the two slits. Quantum mechanics tells us that the passage of an electron through one slit is affected by the existence of the neighboring slit, but it gives no physical explanation to this fact. Of course once more we know the formal answer, that such behavior is a consequence of the superposition of probability amplitudes. We have at hand also the popular ‘explanation’ that the phenomenon is due to the self-interference of the electron. But strictly speaking, this explains nothing, it merely describes what we observe. Neither the formal answer nor the popular one solves the puzzle, which in his famous lectures [50] Feynman considered as the real mystery of quantum mechanics. How is it that the mere existence of a second slit affects the passage of an electron through the other nearby slit?

A qualitative explanation to this question can be offered from the perspective of LSED—to provide a quantitative answer remains an open task, although some work on it is in course. Any nearby body modifies the background field, so that in the neighborhood of a periodic structure the components of the field that fall onto it are enhanced in some preferred directions due to diffraction, and curtailed in others. Under the knowledge that the electron responds more strongly to the relevant waves of the zero-point field, the main effect of diffraction on the particle will be to reinforce the angular deviations specific to such waves, thus giving shape to an interference pattern superimposed to the noisy background. Hence
it is the background field that carries the required information and operates accordingly on the particle. The picture that emerges reminds us of the image suggested by J. Clauser some time ago: “If a bunch of surfers pass through a breakwater with two entrances, you’ll see the two-slit pattern later on the beach in surfer flesh!” (quoted in [51], p. 116). And indeed, we have been observing for over 70 years many-slit patterns in electron flesh.

7.8 Final remark

We have found that LSED explains in a most natural way some basic properties that distinguish quantum systems from the corresponding classical ones, including the wave-like properties of matter, atomic stability, quantization, indeterminism, and so on, in addition to leading to the correct quantitative description. Despite these most favourable traits that substantiate its postulates, the theory here disclosed contains several insufficiencies, the most important among them being the lack of an assessment of the probability with which the trajectories can meet Principle Three within reasonable limits. This is equivalent to an evaluation of the probabilities with which the original free field amplitudes would evolve towards the amplitudes $a_{\alpha\beta}$ that fulfil Eq.(28) with reasonable accuracy and within acceptable time intervals. This is a primary problem that requires close scrutiny to confirm the soundness of the present theory.

8 Appendix

In this Appendix we show that the mean value of the energies associated with the solutions that comply with the principles of LSED correspond to an extremum, in fact a minimum. We start by analyzing the mean kinetic energy as follows from the solutions given by Eq. (10a), which may be written in the following form, introducing all the required indexes, but neglecting the Larmor term,

$$\langle T \rangle = \frac{1}{2m} \langle p^2 \rangle = \frac{m}{2} \sum_{\beta'\beta} \omega_{\alpha\beta} \omega_{\alpha\beta'} \bar{x}_{\alpha\beta}^* x_{\alpha\beta'} \left( a_{\alpha\beta}^* a_{\alpha\beta'} + a_{\alpha\beta'}^* a_{\alpha\beta} \right) e^{-i(\omega_{\alpha\beta} - \omega_{\alpha\beta'}) t}. \quad (A1)$$

We consider a small variation due the independent variation of the amplitudes $a_{\alpha\beta}^*, a_{\alpha\beta'}^*$:

$$\delta \langle T \rangle = \frac{m}{2} \sum_{\beta'\beta} \omega_{\alpha\beta} \omega_{\alpha\beta'} \bar{x}_{\alpha\beta}^* x_{\alpha\beta'} \left( a_{\alpha\beta}^* \delta a_{\alpha\beta'} + a_{\alpha\beta'}^* \delta a_{\alpha\beta}^* \right) e^{-i(\omega_{\alpha\beta} - \omega_{\alpha\beta'}) t}. \quad (A2)$$

We are considering that Principle Three holds, whence neither $\bar{x}_{\alpha\beta}$ nor $\omega_{\alpha\beta}$ depends any more on the amplitudes $a_{\alpha\beta}$. Now from Eq. (28) it follows that

$$\delta a_{\lambda \mu} = ia_{\lambda \mu} \delta \phi_{\lambda \mu} = i a_{\lambda \mu} \left( \delta \phi_{\lambda} - \delta \phi_{\mu} \right). \quad (A3)$$

Since $\phi_{\alpha}$ is common to both $a_{\alpha\beta}$ and $a_{\alpha\beta'}$ the independence of their variation means that only $\phi_{\beta}$ and $\phi_{\beta'}$ change, so that $\delta a_{\alpha\beta} = -i a_{\alpha\beta'} \delta \phi_{\beta'}, \delta a_{\alpha\beta'}^* = -i a_{\alpha\beta} \delta \phi_{\beta'}$.
\[ i a_{\alpha\beta}^* \delta \varphi_\beta. \] Therefore,

\[ \delta \langle T \rangle = i \frac{m}{2} \sum_{\beta' \beta} \omega_{\alpha\beta} \omega_{\alpha\beta'} \overline{x}_{\alpha\beta}^* \overline{x}_{\alpha\beta'} \left( a_{\alpha\beta}^* a_{\alpha\beta'} \right) \left( \delta \varphi_\beta - \delta \varphi_\beta' \right) e^{-i(\omega_{\alpha\beta} - \omega_{\alpha\beta'})t} \]

\[ = i \frac{m}{2} \sum_{\beta' \beta} \omega_{\alpha\beta} \omega_{\alpha\beta'} \overline{x}_{\alpha\beta}^* \overline{x}_{\alpha\beta'} \left( a_{\alpha\beta}^* a_{\alpha\beta'} \right) \left( \delta \varphi_\beta - \delta \varphi_\beta' \right) e^{-i(\omega_{\alpha\beta} - \omega_{\alpha\beta'})t} \]

\[ = i \frac{m}{2} \sum_{\beta' \beta} \omega_{\alpha\beta} \omega_{\alpha\beta'} \overline{x}_{\alpha\beta}^* \overline{x}_{\alpha\beta'} \delta \beta' \left( \delta \varphi_\beta - \delta \varphi_\beta' \right) e^{-i(\omega_{\alpha\beta} - \omega_{\alpha\beta'})t} = 0. \quad (A4) \]

Due to the properties of the amplitudes \( a_{\lambda \nu} \) under Principle Three, a similar result holds for \( \langle V(x) \rangle \), as follows from a power series expansion. Thus we conclude that \( \langle \delta E \rangle = 0 \) to first order under the principles of \( \text{LS6E} \) for arbitrary independent variations of the phases of the stochastic amplitudes. This verifies that the mean energy of the stationary solutions corresponds to an extremum. Moreover, it is clear that these extrema are indeed minima, since otherwise the states would become unstable. To this last observation we can arrive in a simpler way by recalling that the answers afforded by the theory are just the quantum mechanical ones, which very often correspond to the solutions of an eigenvalue problem. It is well known that the eigenvalues of hermitian operators are local minima determined by a variational principle. This is of course the case of the energy eigenvalues, which shows that indeed the extremum values of the energies \( E_\alpha \) correspond to minima.

The observation that Eq. \( (A4) \) is identical to that at each relevant frequency with a vacuum density \( \rho \sim \omega^3 \), as given in Eq. \( (51) \) (for \( T = 0 \); for higher temperatures the same will hold for the Planck spectrum), although obvious from the present point of view is however highly nontrivial, since for classical separable systems, for example, equilibrium occurs only with the Rayleigh-Jeans spectrum, proportional to \( \omega^2 \) \( [38] \).

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[31] That the energy of each field mode should be proportional to its frequency is a relativistic demand, equivalent to asserting that the spectral energy density of the vacuum field is proportional to $\omega^3$. This spectrum is the only one consistent with the demand, among others, of being isotropic in all inertial systems. See, e. g., Ref. ([30]), T. H. Boyer, Phys. Rev. **182**, 1374 (1969) or the discussion and ample list of references in ([30]).

[32] A. Papoulis, *Probability, Random Variables, and Stochastic Processes* (McGraw-Hill, Tokyo, 1965).

[33] M. Born, W. Heisenberg and P. Jordan, *Zeitschr. f. Phys.* **35**, 557 (1926). Reprinted in *Sources of Quantum Mechanics*, B. L. van der Waerden, ed. (Dover, New York, 1968).

[34] J. Dalibard, J. Dupont-Roc and C. Cohen-Tannoudji, J. Physique **43**, 1617 (1982).

[35] L. de la Peña and A. M. Cetto, Nuovo Cim. **B 92**, 189 (1986).

[36] E. Santos, *A definition of commutator of two stationary processes. Application to stochastic electrodynamics*, preprint 1983, University of Santander (Spain), unpublished.

[37] See e.g. E. Santos, J. Math. Phys. **15**, 1954 (1974); also in *Proceedings of the Einstein Centennial Symposium on Fundamental Physics*, S. M. Moore, A. M. Rodríguez-Vargas, A. Rueda and G. Violini (eds.) (Universidad de los Andes, Bogotá, 1981).
See e.g. J. H. van Vleck and D. L. Huber, Rev. Mod. Phys. 49, 939 (1977).

Due to the method of calculation used here there is no difficulty in identifying the source and meaning of each of the terms involved in Eqs. (55b) and (55c). This is not the case in the usual QED treatments of the problem, because their interpretation depends on the order of the creation and annihilation operators describing the quantized field, and there exist a continuous number of combinations of both possible orders. However, a detailed analysis of the problem leads just to the same conclusion as in the text. A detailed account can be find in Ref. [40].

See e.g. P. W. Milonni, The Quantum Vacuum. An Introduction to Quantum Electrodynamics (Academic Press, Inc., San Diego, 1994).

W. E. Lamb and M. O. Scully in Polarisation, Matière et Rayonnement (Presses Universitaires de France, Paris, 1969).

See e.g. M. Jammer, The Conceptual Development of Quantum Mechanics (McGraw-Hill, New York, 1966).

In Bohm’s causal interpretation there are also trajectories. However, there is a fundamental difference between Bohm’s description and the trajectories of the present and Cole and Zou theories. In the former theory the trajectories are constructed ad hoc from the quantum results, whereas in both latter cases quantum mechanics follows from the trajectories.

We have here in mind the usual (and nonlocal) sense of the term ‘collapse’ in quantum theory, according to which a measurement performed on one particle that belongs to an entangled state collapses the state vector, so that both partners acquire well defined values for the measured observable, independently of the distance between them. On occasion the term is used to refer to real (and local) physical processes produced by real interactions, as when photons from an entangled state pass through a polarizer, so that those that pass become polarized and their state vector is reduced by the physical interaction. We are excluding this second meaning from our discussion.

See e.g. D. Home, Conceptual Foundations of Quantum Physics. An Overview from Modern Perspectives (Plenum Press, New York, 1997).

We are using the term deterministic to refer to the description, not to an ontological property, as is done and explained in Ref. [41] or in T. Brody, The Philosophy Behind Physics, L. de la Peña and P. Hodgson, eds. (Springer-Verlag, Berlin, 1993).

S. C. Tiwari, Proc. Einstein Found. Intern. 3, 63 (1986).

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[49] For a complementary point of view see A. F. Kracklauer, Phys. Essays 5, 226 (1992); L. de la Peña and A. M. Cetto, Found. Phys. 24, 917 (1994), 25, 573 (1995) or Ref. (3).

[50] R. P. Feynman, R. B. Leighton and M. Sands, The Feynman Lectures in Physics, Vol. III (Addison-Wesley, Reading, MA, 1965).

[51] D. Wick, The Infamous Boundary. Seven Deades of Heresy in Quantum Physics (Copernicus, New York, 1995).