The Reticulation of a Universal Algebra

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Abstract

The reticulation of an algebra $A$ is a bounded distributive lattice $\mathcal{L}(A)$ whose prime spectrum of filters or ideals is homeomorphic to the prime spectrum of congruences of $A$, endowed with the Stone topologies. We have obtained a construction for the reticulation of any algebra $A$ from a semi-degenerate congruence–modular variety $C$ in the case when the commutator of $A$, applied to compact congruences of $A$, produces compact congruences, in particular when $C$ has principal commutators; furthermore, it turns out that weaker conditions than the fact that $A$ belongs to a congruence–modular variety are sufficient for $A$ to have a reticulation. This construction generalizes the reticulation of a commutative unitary ring, as well as that of a residuated lattice, which in turn generalizes the reticulation of a BL–algebra and that of an MV–algebra.

The purpose of constructing the reticulation for the algebras from $C$ is that of transferring algebraic and topological properties between the variety of bounded distributive lattices and $C$, and a reticulation functor is particularly useful for this transfer. We have defined and studied a reticulation functor for our construction of the reticulation in this context of universal algebra.

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1 Introduction

The reticulation of a commutative unitary ring $R$ is a bounded distributive lattice $\mathcal{L}(R)$ whose prime spectrum of ideals is homeomorphic to the prime spectrum of ideals of $R$. Its construction has appeared in [32], but it has been extensively studied in [52], where it has received the name reticulation. The mapping $R \mapsto \mathcal{L}(R)$ sets a covariant functor from the category of commutative unitary rings to that of bounded distributive lattices, through which properties can be transferred between these categories. In [7], the reticulation has been defined and studied for non–commutative unitary rings and it has been proven that such a ring has a reticulation (with the topological definition above) iff it is quasi–commutative.

Over the past two decades, reticulations have been constructed for ordered algebras related to logic: MV–algebras [8, 9], BL–algebras [37, 20, 38], residuated lattices [11, 12, 13, 15, 16, 19, 20, 35, 60], 0–distributive lattices [19, 60], almost distributive lattices [50], Hilbert algebras [13], hoops [16]. All these algebras possess a “prime spectrum” which is homeomorphic to the prime spectrum of filters or ideals of a bounded distributive lattice; their reticulations consist of such bounded distributive lattices, whose study involves obtaining a construction for them and using that construction to transfer properties between these classes of algebras and bounded distributive lattices.

The purpose of the present paper is to set the problem of constructing a reticulation in a universal algebra framework and providing a solution to this problem in a case as general as possible, that includes the cases of the varieties above and generalizes the constructions which have been obtained in those particular cases. Apart from the novelty of using commutator theory [18, 59] for the study of the reticulation, essentially, the tools needed for obtaining reticulations in this very general setting are quite similar to those which have been put to work for

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the classes of algebras above, and it turns out that many types of results that hold for their reticulations can be
generalized to our setting. In order to obtain strong generalizations, we have worked with hypotheses as weak
as possible; all our results in this paper hold for semi–degenerate congruence–modular varieties whose members
have the sets of compact congruences closed with respect to the commutator, with just a few exceptions that
necessitate, moreover, principal commutators.

The present paper is structured as follows: Section 2 presents the notations and basic results we use in what
follows; Section 3 collects a set of results from commutator theory which we use in the sequel; in Section 4
we present the standard construction of the Stone topologies on prime spectra, specifically the prime spectrum
of ideals of a bounded distributive lattice and the prime spectrum of congruences of a universal algebra whose
commutator fulfills certain conditions. The results in the following sections that are not cited from other papers,
mentioned as being either known or quite simple to obtain, are new and original.

In Section 5, we construct the reticulation for universal algebras whose commutators fulfill certain conditions,
prove that this construction has the desired topological property and obtain some related results.

In Section 6, we provide some examples of reticulations, study particular cases, such as the congruence–
distributive case, show that our construction generalizes constructions for the reticulation which have been
obtained for particular varieties, and prove that our construction preserves finite direct products of algebras
without skew congruences.

In Section 7, we obtain some arithmetical properties on commutators that we need in what follows, as well
as algebraic properties regarding the behaviour of surjections with respect to commutators and to certain types of
congruences.

In Section 8, we study the behaviour of Boolean congruences with respect to the reticulation, in the general
case, but also in particular ones, such as the case of associative commutators or that of semiprime algebras.

In Section 9, we define a reticulation functor; our definition is not ideal, as it only acts on surjections;
extending it to all morphisms remains an open problem. In this final section, we also show that the reticulation
preserves quotients, and that it is a Boolean lattice exactly in the case of hyperarchimedean algebras, which we
also characterize by several other conditions on their reticulation. These characterizations serve as an example
for the transfer of properties to and from the category of bounded distributive lattices which the reticulation
makes possible.

We intend to further pursue the study of the reticulation in this universal algebra setting and use it to
transfer more properties between the variety of bounded distributive lattices and the kinds of varieties that
allow a construction for the reticulation. A theme for a potentially extensive future study is characterizing those
varieties with the property that the reticulations of their members cover the entire class of bounded distributive
lattices.

2 Preliminaries

In this section, we recall some properties on lattices and congruences in universal algebras. For a further study
of the following results on universal algebras, we refer the reader to [1], [12], [27], [34]. For those on lattices, we
recommend [5], [11], [17], [26], [51].

We shall denote by \( \mathbb{N} \) the set of the natural numbers and by \( \mathbb{N}^* = \mathbb{N} \setminus \{0\} \). For any set \( M \), \( \mathcal{P}(M) \) shall be
the set of the subsets of \( M \), \( id_M : M \to M \) shall be the identity map, and we shall denote by \( \Delta_M = \{ (x, x) \mid x \in M \} \)
and \( \nabla_M = M^2 \). For any family \((M_i)_{i \in I}\) of sets and any \( M \subseteq \prod_{i \in I} M_i \), whenever there is no danger of confusion,
by \( a = (a_i)_{i \in I} \in M \) we mean \( a_i \in M_i \) for all \( i \in I \), such that \( a \in M \). For any sets \( M, N \) and any function
\( f : M \to N \), we shall denote by \( \text{Ker}(f) = \{ (x, y) \in M^2 \mid f(x) = f(y) \} \), and the direct and inverse image of \( f \) in
the usual way; we shall denote, simply, \( f = f^2 : \mathcal{P}(M^2) \to \mathcal{P}(N^2) \) and \( f^* = (f^2)^{-1} : \mathcal{P}(N^2) \to \mathcal{P}(M^2) \); so, for
any \( X \subseteq M^2 \) and any \( Y \subseteq N^2 \), \( f(X) = \{ (f(a), f(b)) \mid (a, b) \in X \} \) and \( f^*(Y) = \{ (a, b) \in M^2 \mid (f(a), f(b)) \in Y \} \),
thus \( \text{Ker}(f) = f^*(\Delta_N) \). Also, if \( X_i \subseteq M_i^2 \) for all \( i \in I \), then the direct product of \((X_i)_{i \in I}\) as a family of binary
relations shall be denoted just as the one for sets, because there will be no danger of confusion when using
this notation: \( \prod_{i \in I} X_i = \{ ((a_i)_{i \in I}, (b_i)_{i \in I}) \mid (\forall i \in I) ((a_i, b_i) \in X_i) \} \subseteq M^2 \). Unless mentioned otherwise, the
operations and order relation of a (bounded) lattice shall be denoted in the usual way, and the complementation
of a Boolean algebra shall be denoted by \( \neg \).

Throughout this paper, whenever there is no danger of confusion, any algebra shall be designated by its
support set. All algebras shall be considered non–empty; by trivial algebra we shall mean one–element algebra, and by non–trivial algebra we shall mean algebra with at least two distinct elements. Any direct product of algebras and any quotient algebra shall be considered with the operations defined canonically. For brevity, we shall denote by $A \cong B$ the fact that two algebras $A$ and $B$ of the same type are isomorphic.

Let $L$ be a bounded lattice. By $\Id(L)$ we shall denote the set of the ideals of $L$, that is the non–empty subsets of $L$ which are closed with respect to the join and to lower bounds. By $\Filt(L)$ we shall denote the set of the filters of $L$, that is the ideals of the dual of $L$: the non–empty subsets of $L$ which are closed with respect to the meet and to upper bounds. For any $M \subseteq L$ and any $a \in L$, $(M)$, respectively $|M|$, shall denote the ideal, respectively the filter of $L$ generated by $M$, and the principal ideal, ({a}) = {x \in L | a \geq x}, respectively the principal filter, ({a}) = {x \in L | a \leq x}, generated by $a$ shall also be denoted by $\langle a \rangle$, respectively $\{a\}$; whenever we need to specify the lattice $L$, we shall denote $[M]_L$, $(M)_L$, $\langle a \rangle_L$ and $\{a\}_L$ instead of $[M]$, $(M)$, $\langle a \rangle$ and $\{a\}$, respectively.

It is well known that $(\Id(L), \vee, \cap, \{0\}, L)$ and $(\Filt(L), \lor, \land, \{1\}, L)$ are bounded lattices, with $J \lor K = (J \cup K)$ and $F \lor G = [F \cup G]$ for all $J, K \in \Id(L)$ and all $F, G \in \Filt(L)$, and they are distributive iff $L$ is distributive; moreover, they are complete lattices, with $\bigvee_{i \in I} J_i = \left( \bigcup_{i \in I} J_i \right)$ and $\bigwedge_{i \in I} F_i = \left( \bigcap_{i \in I} F_i \right)$ for any families $(J_i)_{i \in I} \subseteq \Id(L)$ and $(F_i)_{i \in I} \subseteq \Filt(L)$. Obviously, for any $a, b \in L$, $\langle a \rangle \lor \langle b \rangle = \langle a \lor b \rangle$, $\langle a \rangle \land \langle b \rangle = \langle a \land b \rangle$, $\{a\} \lor \{b\} = \{a \lor b\}$ and $\{a\} \land \{b\} = \{a \land b\}$. If $L$ is a complete lattice, then, for any family $(a_i)_{i \in I} \subseteq L$, $\bigvee_{i \in I} (a_i) = (\bigvee_{i \in I} a_i)$ and $\bigwedge_{i \in I} (a_i) = (\bigwedge_{i \in I} a_i)$. By $\PId(L)$, respectively $\PFilt(L)$, we shall denote the set of the principal ideals, respectively the principal filters of $L$. We shall denote by $\MaxId(L)$, respectively $\MaxFilt(L)$, the set of the maximal ideals, respectively the maximal filters of $L$, that is the maximal elements of the set of proper ideals of $L$, $\Id(L) \setminus \{L\}$, respectively that of proper filters of $L$, $\Filt(L) \setminus \{L\}$. By $\SpecId(L)$ we shall denote the set of the prime ideals of $L$, that is the proper ideals $P$ of $L$ such that for any $x, y \in L$, $x \land y \in P$ implies $x \in P$ or $y \in P$. Dually, $\SpecFilt(L)$ shall denote the set of the prime filters of $L$, that is the proper filters $P$ of $L$ such that for any $x, y \in L$, $x \lor y \in P$ implies $x \in P$ or $y \in P$.

For any algebra $A$, $\Con(A)$ shall denote the set of the congruences of $A$, and $\Max(A)$ shall denote the set of the maximal congruences of $A$, that is the maximal elements of the set of proper congruences of $A$: $\Con(A) \setminus \{\nabla_A\}$. Let $\theta \in \Con(A)$, $a \in A, M \subseteq A$ and $X \subseteq A^2$, arbitrary. Then $a/\theta$ shall denote the congruence class of $a$ with respect to $\theta$, $M/\theta = \{x/\theta | x \in M\}$, $p_\theta : A \to A/\theta$ shall be the canonical surjective morphism: $p_\theta(a) = a/\theta$ for all $a \in A, X/\theta = \{(x/\theta, y/\theta) | (x, y) \in X\}$ and $CG_A(X)$ shall be the congruence of $A$ generated by $X$. It is well known that $(\Con(A), \lor, \cap, \Delta_A, \nabla_A)$ is a bounded lattice, ordered by set inclusion, where $\phi \lor \psi = CG_A(\phi \lor \psi)$ for all $\phi, \psi \in \Con(A)$; moreover, this is a complete lattice, in which $\bigvee_{i \in I} \phi_i = CG_A(\bigcup_{i \in I} \phi_i)$ for any family $(\phi_i)_{i \in I} \subseteq \Con(A)$. For any $a, b \in A$, the principal congruence $CG_A(\{a, b\})$ shall also be denoted by $CG_A(a, b)$. The set of the principal congruences of $A$ shall be denoted by $\PCon(A)$. $K(A)$ shall denote the set of the finitely generated congruences of $A$, which coincide to the compact elements of the lattice $\Con(A)$. Clearly, $\PCon(A) \subseteq K(A)$ and $\Delta_A \in \PCon(A)$, because $\Delta_A = CG_A(x, x)$ for any $x \in A$.

Throughout the rest of this paper, $\tau$ shall be a universal algebras signature, $\mathcal{C}$ shall be an equational class of $\tau$–algebras $A$ and $B$ shall be algebras from $\mathcal{C}$ and $f : A \to B$ shall be a morphism in $\mathcal{C}$. Unless mentioned otherwise, by morphism we shall mean $\tau$–morphism. We recall that $A$ is said to be congruence–modular, respectively congruence–distributive, iff the lattice $\Con(A)$ is modular, respectively distributive, and that $\mathcal{C}$ is said to be congruence–modular, respectively congruence–distributive, iff every algebra in $\mathcal{C}$ is congruence–modular, respectively congruence–distributive.

**Remark 2.1.** If $\beta \in \Con(B)$, then $f^*(\beta) \in \Con(A)$; thus $\Ker(f) = f^*(\Delta_B) \in \Con(A)$. Also, $f^*(\beta) \supseteq f^*(\Delta_B) = \Ker(f)$ and $f^*(\beta) \subseteq f^*(\Delta_B^2)$, thus, if $f$ is surjective, then $f^*(\beta) = \beta$.

If $\alpha \in \Con(A)$ such that $\alpha \supseteq \Ker(f)$, then $f(\alpha) \in \Con(f(A))$, so, if $f$ is surjective, then $f(\alpha) \in \Con(B)$. Thus, for any $\alpha \in \Con(A)$, we have $f(\alpha \lor \Ker(f)) \in \Con(f(A))$, so, if $f$ is surjective, then $f(\alpha \lor \Ker(f)) \in \Con(B)$. Moreover, $\alpha \lor f(\alpha)$ is an order isomorphism from $\Ker(f)$ to $\PCon_f(A)$ to $\Con(f(A))$, thus to $\Con(B)$ if $f$ is surjective, having the corresponding restriction of $f^*$ as inverse.

For any $\theta \in \Con(A)$, clearly, $\Ker(p_\theta) = \theta$. By the above, for all $\alpha \in \Con(A)$ such that $\alpha \supseteq \theta, \alpha/\theta = p_\theta(\alpha) = \{(a/\theta, b/\theta) | (a, b) \in \alpha\} \in \Con(A/\theta)$, and $\alpha \lor \theta/\alpha$ is a bijection from $[\theta]$ to $\Con(A/\theta)$. 


3 The Commutator

This section is composed of results on the commutator in arbitrary and in congruence–modular varieties, which are either previously known of very easy to derive from previously known results. For a further study of these results, see [1, 21, 34, 43].

Out of the various definitions for commutator operations on congruence lattices, we have chosen to work with the term condition commutator; from the following definition. Recall that, in algebras from congruence–modular varieties, all definitions for the commutator give the same commutator operation. For any term \( t \) over \( A \), we shall denote by \( t^A \) the derivative operation of \( A \) associated to \( t \).

**Definition 3.1.** [39] Let \( \alpha, \beta \in \text{Con}(A) \). For any \( \mu \in \text{Con}(A) \), by \( C(\alpha; \beta; \mu) \) we denote the fact that the following condition holds: for all \( n, k \in \mathbb{N} \) and any term \( t \) over \( A \) of arity \( n + k \), if \( (a_i, b_i) \in \alpha \) for all \( i \in \{1, n\} \) and \( (c_j, d_j) \in \beta \) for all \( j \in \{1, k\} \), then \( (t^A(a_1, \ldots, a_n, c_1, \ldots, c_k), t^A(a_1, \ldots, a_n, d_1, \ldots, d_k)) \in \mu \) if \( (t^A(b_1, \ldots, b_n, c_1, \ldots, c_k), t^A(b_1, \ldots, b_n, d_1, \ldots, d_k)) \in \mu \). We denote by \( [\alpha, \beta]_A = \bigcap \{ \mu \in \text{Con}(A) : C(\alpha; \beta; \mu) \} \); we call \( [\alpha, \beta]_A \) the commutator of \( \alpha \) and \( \beta \) in \( A \).

**Remark 3.2.** Let \( \alpha, \beta \in \text{Con}(A) \). Clearly, \( C(\alpha; \beta; \nabla_A) \). Since \( \text{Con}(A) \) is a complete lattice, it follows that \( [\alpha, \beta]_A \in \text{Con}(A) \). Furthermore, according to [39] Lemma 4.4(2), for any family \( (\mu_i)_{i \in I} \subseteq \text{Con}(A) \), if \( C(\alpha; \beta; \mu_i) \) for all \( i \in I \), then \( C(\alpha, \beta; \bigcap_{i \in I} \mu_i) \). Hence \( C(\alpha; \beta; [\alpha, \beta]_A) \), and thus \( [\alpha, \beta]_A = \min \{ \mu \in \text{Con}(A) : C(\alpha, \beta; \mu) \} \), which is exactly the definition of the commutator from [40].

**Definition 3.3.** The operation \( [\cdot, \cdot]_A : \text{Con}(A) \times \text{Con}(A) \to \text{Con}(A) \) is called the commutator of \( A \).

**Theorem 3.4.** [21] If \( C \) is congruence–modular, then, for each member \( M \) of \( C \), \( [\cdot, \cdot]_M \) is the unique binary operation on \( \text{Con}(M) \) such that, for all \( \alpha, \beta \in \text{Con}(M) \), \( [\alpha, \beta]_M = \min \{ \mu \in \text{Con}(M) : \mu \subseteq \alpha \cap \beta \} \) and, for any member \( N \) of \( C \) and any surjective morphism \( h : M \to N \) in \( C \), \( h(\nabla_M(h)) = h(\nabla_M(h)) \).

**Theorem 3.5.** [31] If \( C \) is congruence–distributive, then, in each member of \( C \), the commutator coincides to the intersection of congruences.

For brevity, most of the times, we shall use the remarks in this paper without referencing them, and the same goes for the lemmas and propositions that state basic results.

**Proposition 3.6.** [39] Lemma 4.6, Lemma 4.7, Theorem 8.3 The commutator is:

- increasing in both arguments, that is, for all \( \alpha, \beta, \phi, \psi \in \text{Con}(A) \), if \( \alpha \subseteq \beta \) and \( \phi \subseteq \psi \), then \( [\alpha, \phi]_A \subseteq [\beta, \psi]_A \);
- smaller than its arguments, so, for any \( \alpha, \beta \in \text{Con}(A) \), \( [\alpha, \beta]_A \subseteq \alpha \cap \beta \).

If \( C \) is congruence–modular, then the commutator is also:

- commutative, that is \( [\alpha, \beta]_A = [\beta, \alpha]_A \) for all \( \alpha, \beta \in \text{Con}(A) \);
- distributive in both arguments with respect to arbitrary joins, that is, for any families \( (\alpha_i)_{i \in I} \) and \( (\beta_j)_{j \in J} \) of congruences of \( A \), \( \bigvee_{i \in I} \alpha_i, \bigvee_{j \in J} \beta_j = \bigvee_{i \in I, j \in J} [\alpha_i, \beta_j]_A \).

**Remark 3.7.** Assume that \( [\cdot, \cdot]_A \) is commutative. Then the distributivity of \( [\cdot, \cdot]_A \) in both arguments w.r.t. arbitrary joins is equivalent to its distributivity in one argument w.r.t. arbitrary joins, which in turn is equivalent to its distributivity w.r.t. the join in the case when \( \text{Con}(A) \) is finite, in particular when \( A \) is finite.

Obviously, if \( [\cdot, \cdot]_A \) equals the intersection and it is distributive w.r.t. the join (by Proposition 3.6, the latter holds if \( C \) is congruence–modular), then \( A \) is congruence–distributive.

**Lemma 3.8.** [21] If \( C \) is congruence–modular and \( S \) is a subalgebra of \( A \), then, for any \( \alpha, \beta \in \text{Con}(A) \), \( [\alpha \cap S^2, \beta \cap S^2]_A \subseteq [\alpha, \beta]_A \cap S^2 \).

**Proposition 3.9.** [48] Theorem 5.17, p. 48 Assume that \( C \) is congruence–modular, and let \( n \in \mathbb{N}^+ \), \( M_1, \ldots, M_n \) be algebras from \( C \), \( M = \bigprod_{i=1}^n M_i \) and, for all \( i \in \{1, n\}, \alpha_i, \beta_i \in \text{Con}(M_i) \). Then \( \bigprod_{i=1}^n \alpha_i, \bigprod_{i=1}^n \beta_i \big|_M = \bigprod_{i=1}^n \alpha_i, \beta_i \big|_{M_i} \).
Remark 3.10. By Theorem 3.4 and Remark 2.1, if \( C \) is congruence–modular, \( \alpha, \beta, \theta \in \text{Con}(A) \) and \( f \) is surjective, then \( [f(\alpha \lor \text{Ker}(f)), f(\beta \lor \text{Ker}(f))]_B = f([\alpha, \beta]_A \lor \text{Ker}(f)) \), thus \( ([\alpha \lor \theta, (\beta \lor \theta)]_B = ([\alpha, \beta]_A \lor \theta) \), hence, if \( \theta \subseteq [\alpha, \beta]_A \), then \([\alpha/\theta, \beta/\theta]_{A/\theta} = [\alpha, \beta]_A/\theta \).

Definition 3.11. Let \( \phi \) be a proper congruence of \( A \). Then \( \phi \) is called a prime congruence of \( A \) iff, for all \( \alpha, \beta \in \text{Con}(A) \), \( [\alpha, \beta]_A \subseteq \phi \) implies \( \alpha \subseteq \phi \) or \( \beta \subseteq \phi \). \( \phi \) is called a semiprime congruence of \( A \) iff, for all \( \alpha \in \text{Con}(A) \), \( [\alpha, \alpha]_A \subseteq \phi \) implies \( \alpha \subseteq \phi \).

The set of the prime congruences of \( A \) shall be denoted by \( \text{Spec}(A) \). \( \text{Spec}(A) \) is called the (prime) spectrum of \( A \) and \( \text{Max}(A) \) is called the maximal spectrum of \( A \).

Following [34], we say that \( C \) is semi–degenerate iff no non–trivial algebra in \( C \) has one–element subalgebras. For instance, the class of unitary rings and any class of bounded or derred structures is semi–degenerate.

Lemma 3.12. [1] Theorem 5.3] If \( C \) is congruence–modular and semi–degenerate, then:

- any proper congruence of \( A \) is included in a maximal congruence of \( A \);
- any maximal congruence of \( A \) is prime.

Remark 3.13. By Lemma 3.12 if \( A \) is non–trivial and \( C \) is congruence–modular and semi–degenerate, then \( A \) has maximal congruences, thus it has prime congruences.

Proposition 3.14. [31] \( C \) is semi–degenerate iff, for all members \( M \) of \( C \), \( \nabla_M \in \mathcal{K}(M) \).

Proposition 3.15. [21] Theorem 8.5, p. 85] If \( C \) is congruence–modular, then the following are equivalent:

(\(i\)) for any algebra \( M \) from \( C \), \( [\Lambda, \Lambda]_M = \Lambda \);

(\(ii\)) for any algebra \( M \) from \( C \) and any \( \theta \in \text{Con}(M) \), \( [\theta, \Lambda]_M = \theta \);

(\(iii\)) \( C \) has no skew congruences, that is, for any algebras \( M \) and \( N \) from \( C \), \( \text{Con}(M \times N) = \{ \theta \times \zeta \mid \theta \in \text{Con}(M), \zeta \in \text{Con}(N) \} \).

Lemma 3.16. (\(i\)) If \( C \) is congruence–modular and semi–degenerate, then \( C \) fulfills the equivalent conditions from Proposition 3.15.

(\(ii\)) If \( C \) is congruence–distributive, then \( C \) fulfills the equivalent conditions from Proposition 3.15.

Proof. [\(i\)] This is exactly [1] Lemma 5.2.

[\(ii\)] Clear, from Theorem 3.5.

Lemma 3.17. [1] Lemma 1.11], [53] Proposition 1.2] If \( f \) is surjective, then, for any \( a, b \in A \), any \( X \subseteq A^2 \), any \( \theta \in \text{Con}(A) \) and any \( \alpha, \beta \in [\text{Ker}(f)] \):

(\(i\)) \( f(\theta \lor \text{Ker}(f)) = G_{\Lambda}(f(\theta)) \); \( f(\alpha \lor \beta) = f(\alpha) \lor f(\beta) \);

(\(ii\)) \( f(G_{\Lambda}(a, b) \lor \text{Ker}(f)) = G_{\Lambda}(f(a), f(b)) ; f(G_{\Lambda}(X) \lor \text{Ker}(f)) = G_{\Lambda}(f(X)) \);

(\(\theta \lor \beta, \beta/\theta \) \( G_{\Lambda}(\alpha, \beta/\theta) ; (G_{\Lambda}(X) \lor \theta) / \theta = G_{\Lambda}(\alpha, \beta/\theta) \).

We say that \( A \) has principal commutators iff, for all \( \alpha, \beta \in \text{PCon}(A) \), we have \( [\alpha]_A \in \text{PCon}(A) \), that is iff \( \text{PCon}(A) \) is closed with respect to the commutator of \( A \). Following [1], we say that \( C \) has principal commutators iff each member of \( C \) has principal commutators. We say that \( C \) has associative commutators iff, for each member \( M \) of \( C \), the commutator of \( M \) is an associative binary operation on \( \text{Con}(M) \).

Remark 3.18. \( \mathcal{K}(A) = \{ G_{\Lambda}(\theta) \} \cup \{ G_{\Lambda}((a_1, b_1, \ldots, a_n, b_n)) \mid n \in \mathbb{N}^+, a_1, b_1, \ldots, a_n, b_n \in A \} = \{ \Delta_A \} \cup \{ \bigvee_{i=1}^n G_{\Lambda}(a_i, b_i) \mid n \in \mathbb{N}^+, a_1, b_1, \ldots, a_n, b_n \in A \} = \{ \bigvee_{i=1}^n G_{\Lambda}(a_i, b_i) \mid n \in \mathbb{N}^+, a_1, b_1, \ldots, a_n, b_n \in A \} \), since \( \Delta_A \in \text{PCon}(A) \). From this, it is immediate that \( \mathcal{K}(A) \) is closed with respect to finite joins, and, if \( A \) has principal commutators and \( [\cdot, \cdot]_A \) is commutative and distributive w.r.t. the join (for instance if \( C \) is congruence–modular), then \( \mathcal{K}(A) \) is also closed with respect to the commutator of \( A \).
Remark 3.19. If $C$ is congruence–distributive, then, as shown by Theorem 3.5

- $C$ has principal commutators iff $C$ has the principal intersection property (PIP);
- $K(M)$ is closed with respect to the commutator for each member $M$ of $C$ iff $C$ has the compact intersection property (CIP).

As a particular case of Remark 3.18 if $C$ is congruence–distributive and has the PIP, then $C$ has the CIP.

Example 3.20. [11, 10, 22, 31] Theorem 2.8, [33, 36] As shown by Theorem 3.5 any congruence–distributive variety has associative commutators. The variety of commutative unitary rings is semi–degenerate, congruence–modular, with principal commutators and associative commutators, and it is not congruence–distributive. Out of the semi–degenerate congruence–distributive varieties with the CIP, we mention: bounded distributive lattices, residuated lattices (a variety which includes Gödel algebras, product algebras, MTL–algebras, BL–algebras, MV–algebras) and semi–degenerate discriminator varieties (out of which we mention Boolean algebras, $n$–valued Post algebras, $n$–valued Łukasiewicz algebras, $n$–valued MV–algebras, $n$–dimensional cylindric algebras, Gödel residuated lattices).

4 The Stone Topologies on Prime and Maximal Spectra

In what follows, we present the Stone topologies on the prime and maximal spectra of ideals and filters of a bounded distributive lattice and those of congruences of an algebra with the greatest congruence compact from a congruence–modular variety; in particular, the following hold for algebras from semi–degenerate congruence–distributive varieties. The results in this section are either previously known or very easy to derive from previously known results; see, for instance, [30].

Let $L$ be a bounded distributive lattice. For any $I \in \text{Id}(L)$ and any $a \in L$, we shall denote by $V_{\text{Id},L}(I) = \text{Spec}_{\text{Id}}(L) \cap [I] = \{ P \in \text{Spec}_{\text{Id}}(L) \mid I \subseteq P \}$, $D_{\text{Id},L}(I) = \text{Spec}_{\text{Id}}(L) \setminus V_{\text{Id},L}(I) = \{ Q \in \text{Spec}_{\text{Id}}(L) \mid I \not\subseteq Q \}$, $V_{\text{Id},L}(a) = V_{\text{Id},L}((a)) = \{ P \in \text{Spec}_{\text{Id}}(L) \mid a \in P \}$ and $D_{\text{Id},L}(a) = D_{\text{Id},L}((a)) = \text{Spec}_{\text{Id}}(L) \setminus V_{\text{Id},L}(a) = \{ Q \in \text{Spec}_{\text{Id}}(L) \mid a \notin Q \}$. By replacing $\text{Spec}_{\text{Id}}(L)$ with $\text{Spec}_{\text{Filt}}(L)$, in the same way we can define $V_{\text{Filt},L}(F)$, $D_{\text{Filt},L}(F)$, $V_{\text{Filt},L}(a)$ and $D_{\text{Filt},L}(a)$ for any $F \in \text{Filt}(L)$ and any $a \in L$.

Remark 4.1. The following hold, and their duals hold for filters:

- for any $J, K \in \text{Id}(L)$, $V_{\text{Id},L}(J \cap K) = V_{\text{Id},L}(J) \cup V_{\text{Id},L}(K)$ and $D_{\text{Id},L}(J \cap K) = D_{\text{Id},L}(J) \cap D_{\text{Id},L}(K)$;
- for any family $(J_i)_{i \in I} \subseteq \text{Id}(L)$, $V_{\text{Id},L}(\bigvee_{i \in I} J_i) = \bigcap_{i \in I} V_{\text{Id},L}(J_i)$ and $D_{\text{Id},L}(\bigvee_{i \in I} J_i) = \bigcup_{i \in I} D_{\text{Id},L}(J_i)$;
- thus, for any $a, b \in L$, $V_{\text{Id},L}(a \wedge b) = V_{\text{Id},L}(a) \cap V_{\text{Id},L}(b)$, $D_{\text{Id},L}(a \wedge b) = D_{\text{Id},L}(a) \cap D_{\text{Id},L}(b)$, $V_{\text{Id},L}(a \vee b) = V_{\text{Id},L}(a) \cup V_{\text{Id},L}(b)$ and $D_{\text{Id},L}(a \vee b) = D_{\text{Id},L}(a) \cup D_{\text{Id},L}(b)$;
- if $L$ is a complete lattice, then, for any family $(a_i)_{i \in I} \subseteq L$, $V_{\text{Id},L}(\bigvee_{i \in I} a_i) = \bigcap_{i \in I} V_{\text{Id},L}(a_i)$ and $D_{\text{Id},L}(\bigvee_{i \in I} J_i) = \bigcup_{i \in I} D_{\text{Id},L}(J_i)$;
- if $I \in \text{Id}(L)$, then: $D_{\text{Id},L}(I) = \text{Spec}_{\text{Id}}(L)$ iff $V_{\text{Id},L}(I) = \emptyset$ iff $I = L$;
- $D_{\text{Id},L}({\{0\}}) = \emptyset$ and $V_{\text{Id},L}({\{0\}}) = \text{Spec}_{\text{Id}}(L)$;
- if $L$ is distributive (so that the Prime Ideal Theorem holds in $L$ and, hence, any ideal of $L$ equals the intersection of the prime ideals that include it) and $I \in \text{Id}(L)$, then: $D_{\text{Id},L}(I) = \emptyset$ iff $V_{\text{Id},L}(I) = \text{Spec}_{\text{Id}}(L)$ iff $I = \{0\}$.

As shown by Remark 4.1, $\{D_{\text{Id},L}(I) \mid I \in \text{Id}(L)\}$ is a topology on $\text{Spec}_{\text{Id}}(L)$, called the Stone topology, having $\{D_{\text{Id},L}(a) \mid a \in L\}$ as a basis and, obviously, $\{V_{\text{Id},L}(I) \mid I \in \text{Id}(L)\}$ as the family of closed sets and $\{V_{\text{Id},L}(a) \mid a \in L\}$ as a basis of closed sets. Since $\text{Max}_{\text{Id}}(L) \subseteq \text{Spec}_{\text{Id}}(L)$, $\{D_{\text{Id},L}(I) \cap \text{Max}_{\text{Id}}(L) \mid I \in \text{Id}(L)\}$
is a topology on \(\text{Max}_{\text{id}}(L)\), which is also called the **Stone topology**, and it has \(\{D_{\text{id},L}(a) \cap \text{Max}_{\text{id}}(L) \mid a \in L\}\) as a basis, \(\{V_{\text{id},L}(I) \cap \text{Max}_{\text{id}}(L) \mid I \in \text{Id}(L)\}\) as the family of closed sets and \(\{V_{\text{id},L}(a) \cap \text{Max}_{\text{id}}(L) \mid a \in L\}\) as a basis of closed sets. Dually, we have the Stone topologies on \(\text{Spec}_{\text{fin}}(L)\) and \(\text{Max}_{\text{fin}}(L)\). \(\text{Spec}_{\text{id}}(L)\), \(\text{Max}_{\text{id}}(L)\), \(\text{Spec}_{\text{fin}}(L)\) and \(\text{Max}_{\text{fin}}(L)\) are called the *(prime) spectrum of ideals*, maximal spectrum of ideals, *(prime) spectrum of filters* and maximal spectrum of filters of \(L\), respectively.

Throughout the rest of this section, we shall assume that \([\cdot,\cdot]_{A}\) is commutative and distributive w.r.t. arbitrary joins. For each \(\theta \in \text{Con}(A)\), we shall denote by \(V_{A}(\theta) = \text{Spec}(A) \cap [\theta] = \{\phi \in \text{Spec}(A) \mid \theta \subseteq \phi\}\) and by \(D_{A}(\theta) = \text{Spec}(A) \setminus V_{A}(\theta) = \{\psi \in \text{Spec}(A) \mid \theta \not\subseteq \psi\}\). We shall also denote, for any \(a, b \in A\), by \(V_{A}(a, b) = V_{A}(C_{gA}(a, b)) = \{\phi \in \text{Spec}(A) \mid (a, b) \in \phi\}\) and by \(D_{A}(a, b) = D_{A}(C_{gA}(a, b)) = \{\psi \in \text{Spec}(A) \mid (a, b) \notin \psi\}\). The proof of the following result is straightforward.

**Proposition 4.2.** \([1]\) \(\text{Spec}(A), \{D_{A}(\theta) \mid \theta \in \text{Con}(A)\}\) is a topological space, having \(\{D_{A}(a, b) \mid a, b \in A\}\) as a basis and in which, for all \(\alpha, \beta \in \text{Con}(A)\) and any family \((\alpha_{i})_{i \in I} \subseteq \text{Con}(A)\), the following hold:

1. \(D_{A}(\Delta) = \emptyset\) and \(D_{A}(\nabla) = \text{Spec}(A)\); \(V_{A}(\Delta) = \text{Spec}(A)\) and \(V_{A}(\nabla) = \emptyset\);
2. \(D_{A}([\alpha, \beta]_{A}) = D_{A}(\alpha \cap \beta) = D_{A}(\alpha) \cap D_{A}(\beta)\); \(V_{A}([\alpha, \beta]_{A}) = V_{A}(\alpha \cap \beta) = V_{A}(\alpha) \cup V_{A}(\beta)\);
3. \(D_{A}(\bigvee_{i \in I} \alpha_{i}) = \bigcup_{i \in I} D_{A}(\alpha_{i})\); \(V_{A}(\bigvee_{i \in I} \alpha_{i}) = \bigcap_{i \in I} V_{A}(\alpha_{i})\).

\(\{D_{A}(\theta) \mid \theta \in \text{Con}(A)\}\) is called the **Stone topology** on \(\text{Spec}(A)\). Obviously, its family of closed sets is \(\{V_{A}(\theta) \mid \theta \in \text{Con}(A)\}\), and \(\{V_{A}(a, b) \mid a, b \in A\}\) is a basis of closed sets for this topology. The Stone topology on \(\text{Spec}(A)\) induces the **Stone topology** on \(\text{Max}(A)\), namely \(\{D_{A}(\theta) \cap \text{Max}(A) \mid \theta \in \text{Con}(A)\}\).

**Remark 4.3.** Let \(\alpha, \beta \in \text{Con}(A)\). Then, clearly:

- \(V_{A}(\alpha) \subseteq V_{A}(\beta)\) iff \(\text{Spec}(A) \setminus D_{A}(\alpha) \subseteq \text{Spec}(A) \setminus D_{A}(\beta)\) iff \(D_{A}(\beta) \subseteq D_{A}(\alpha)\);
- if \(\alpha \subseteq \beta\), then \(V_{A}(\beta) \subseteq V_{A}(\alpha)\) and \(D_{A}(\alpha) \subseteq D_{A}(\beta)\).

**Proposition 4.4.** If \(C\) is congruence–modular and semi–degenerate, then, for any \(\alpha \in \text{Con}(A)\): \(D_{A}(\alpha) = \text{Spec}(A)\) iff \(V_{A}(\alpha) = \emptyset\) iff \(\alpha \subseteq \nabla_{A}\).

**Proof.** \(D_{A}(\alpha) = \text{Spec}(A)\) iff \(\text{Spec}(A) \setminus D_{A}(\alpha) = \emptyset\) iff \(V_{A}(\alpha) = \emptyset\). Since \(\text{Spec}(A) \subseteq \text{Con}(A) \setminus \{\nabla_{A}\}\), we have \(V_{A}(\nabla_{A}) = \emptyset\), which was also part of Proposition 4.2. If \(\alpha \not\subseteq \nabla_{A}\), then, according to Lemma 3.12 there exists a \(\phi \in \text{Spec}(A)\) such that \(\alpha \subseteq \phi\), that is \(V_{A}(\phi) \neq \emptyset\).

**Remark 4.5.** Recall that, if \(f\) is surjective, then the map \(\alpha \mapsto f(\alpha)\) is a lattice isomorphism from \([\text{Ker}(f)]\) to \(\text{Con}(B)\). Now assume that \(C\) is congruence–modular.

Then this map is an order isomorphism from \(\text{Max}(A) \cap [\text{Ker}(f)]\) to \(\text{Max}(B)\). Furthermore, this map is an order isomorphism from \(\text{Spec}(A) \cap [\text{Ker}(f)]\) to \(\text{Spec}(B)\) (see also [1], [25], [47]). Hence, if \(\text{Ker}(f) \subseteq \alpha \in \text{Con}(A)\), then \(V_{B}(f(\alpha)) = f(V_{A}(\alpha))\) and \([f(\alpha)] \cap \text{Max}(B) = f(\alpha) \cap \text{Max}(A)\).

Therefore, for all \(\theta \in \text{Con}(A)\), the map \(\alpha \mapsto \alpha/\theta\) is a lattice isomorphism from \([\theta]\) to \(\text{Con}(A/\theta)\), an order isomorphism from \(\text{Max}(A) \cap [\theta]\) to \(\text{Max}(A/\theta)\) and an order isomorphism from \(\text{Spec}(A) \cap [\theta]\) to \(\text{Spec}(A/\theta)\); hence, if \(\theta \subseteq \alpha \in \text{Con}(A)\), then \(V_{A/\theta}(\alpha/\theta) = \{\psi/\theta \mid \psi \in V_{A}(\alpha)\}\) and \([\alpha/\theta] \cap \text{Max}(A/\theta) = \{\psi/\theta \mid \psi \in [\alpha] \cap \text{Max}(A)\}\).

### 5 The Construction of the Reticulation of a Universal Algebra and Related Results

Throughout this section, we shall assume that \([\cdot,\cdot]_{A}\) is commutative and distributive w.r.t. arbitrary joins, and that \(\nabla_{A} \in \mathcal{K}(A)\). For every \(\theta \in \text{Con}(A)\), we shall denote by \(\rho_{A}(\theta)\) the **radical of \(\theta\)**, that is the intersection of the prime congruences of \(A\) which include \(\theta\): \(\rho_{A}(\theta) = \bigcap \{\phi \in \text{Spec}(A) \mid \theta \subseteq \phi\} = \bigcap_{\phi \in V_{A}(\theta)} \phi\).

**Remark 5.1.** Let \(\alpha, \beta \in \text{Con}(A)\) and \(\phi \in \text{Spec}(A)\). Then, clearly:

1. \(V_{A}(\nabla_{A}) = \emptyset\) and thus \(\rho_{A}(\nabla_{A}) = \nabla_{A}\).
(ii) \(\rho_A(\phi) = \phi\); moreover, \(\rho_A(\alpha) = \alpha\) iff \(\alpha\) is the intersection of a family of prime congruences of \(A\);

(iii) if \(\alpha \subseteq \beta\), then \(V_A(\alpha) \supseteq V_A(\beta)\), hence \(\rho_A(\alpha) \subseteq \rho_A(\beta)\);

(iv) if \(\alpha \subseteq \phi\), then \(\rho_A(\alpha) \subseteq \phi\), since \(\phi \in V_A(\alpha)\).

Following (ii), for any \(\alpha, \beta \in \text{Con}(A)\) and every \(n \in \mathbb{N}^*\), we denote by \([\alpha, \beta]_A^1 = [\alpha, \beta]_A\) and \([\alpha, \beta]_A^{n+1} = [\alpha, \beta]_A^n\), \(\alpha, \beta\), and by \((\alpha, \beta)_{A}^1 = [\alpha, \beta]_A\) and \((\alpha, \beta)_{A}^{n+1} = (\alpha, \beta)_{A}^n\).

**Lemma 5.2.** For all \(n \in \mathbb{N}^*\), any \(\alpha, \beta \in \text{Con}(A)\) and any family \((\alpha_i)_{i \in I} \in \text{Con}(A)\):

(i) \(\alpha \subseteq \rho_A(\alpha)\);

(ii) \(V_A(\alpha) = V_A(\rho_A(\alpha))\);

(iii) \(V_A(\bigvee_{i \in I} \alpha_i) = V_A(\bigvee_{i \in I} \rho_A(\alpha_i))\);

(iv) \(V_A((\alpha, \beta)_A^1) = V_A((\alpha, \beta)_A) = V_A(\alpha \cap \beta) = V_A(\alpha) \cup V_A(\beta)\);

(v) \(V_A((\alpha, \alpha)_A^1) = V_A((\alpha, \alpha)_A) = V_A(\alpha)\).

**Proof.** (i) Trivial.

(iii) By (i) and Remark 4.3, \(V_A(\rho_A(\alpha)) \subseteq V_A(\alpha)\). If \(\phi \in V_A(\alpha)\), then \(\phi \in V_A(\rho_A(\alpha))\), according to Remark 5.1 (v), thus \(V_A(\alpha) \subseteq V_A(\rho_A(\alpha))\). Hence \(V_A(\alpha) = V_A(\rho_A(\alpha))\).

(iii) By (i) and Proposition 4.2 (iii), \(V_A(\bigvee_{i \in I} \alpha_i) = \bigvee_{i \in I} V_A(\alpha_i) = \bigvee_{i \in I} V_A(\rho_A(\alpha_i)) = V_A(\bigvee_{i \in I} \rho_A(\alpha_i))\).

(iv) By Proposition 4.2 (iv), \(V_A((\alpha, \beta)_A^1) = V_A((\alpha, \beta)_A) = V_A(\alpha \cap \beta) = V_A(\alpha) \cup V_A(\beta)\). Now we prove that \(V_A((\alpha, \beta)_A^n) = V_A(\alpha) \cup V_A(\beta)\) by induction on \(n \in \mathbb{N}^*\). \(V_A((\alpha, \beta)_A^1) = V_A((\alpha, \beta)_A) = V_A(\alpha) \cup V_A(\beta)\). Now let \(n \in \mathbb{N}^*\) such that \(V_A((\theta, \zeta)_A^n) = V_A(\theta) \cup V_A(\zeta)\) for all \(\theta, \zeta \in \text{Con}(A)\). Then \(V_A((\alpha, \beta)_A^{n+1}) = V_A((\alpha, \beta)_A^n) \cup V_A((\alpha, \beta)_A^n) = V_A(\alpha) \cup V_A(\beta)\).

By (v).

**Proposition 5.3.** For all \(\alpha, \beta, \theta \in \text{Con}(A)\), the following hold:

(i) \(\rho_A(\alpha) \subseteq \rho_A(\beta)\) iff \(\alpha \subseteq \rho_A(\beta)\) iff \(V_A(\alpha) \supseteq V_A(\beta)\);

(ii) \(\rho_A(\alpha) = \rho_A(\beta)\) iff \(V_A(\alpha) = V_A(\beta)\);

(iii) if \(\theta \subseteq \alpha\), then \(\rho_A(\alpha/\theta) = \rho_A(\alpha) / \theta\);

(iv) \(\rho_A(\partial A/\theta) = \rho_A(\theta / \theta)\);

(v) \(\rho_A(\partial(\alpha \vee \theta)) = \rho_A(\alpha \vee \theta) / \theta\).

**Proof.** (i) Clearly, if \(V_A(\alpha) \supseteq V_A(\beta)\), then \(\rho_A(\alpha) \subseteq \rho_A(\beta)\). If \(\rho_A(\alpha) \subseteq \rho_A(\beta)\), then, since \(\alpha \subseteq \rho_A(\alpha)\), it follows that \(\alpha \subseteq \rho_A(\beta)\). Finally, if \(\alpha \subseteq \rho_A(\beta)\), then \(V_A(\alpha) \supseteq V_A(\rho_A(\beta)) = V_A(\beta)\), by Remark 5.1 (iii) and Lemma 5.2 (iii).

(ii) By (i).

(iii) If \(\theta \subseteq \alpha\), then we may write: \(\rho_A(\alpha/\theta) = \bigcap_{\psi \in V_A(\alpha/\theta)} \psi = \bigcap_{\phi \in V_A(\alpha)} \phi \cap \theta = \rho_A(\alpha) / \theta\).

(iv) By (iii), \(\rho_A(\partial A/\theta) = \rho_A(\theta / \theta) = \rho_A(\theta) / \theta\).

By (v).

**Proposition 5.4.** For any \(n \in \mathbb{N}^*\), any \(\alpha \in \text{Con}(A)\) and any family \((\alpha_i)_{i \in I} \in \text{Con}(A)\):

(i) if \(C\) is congruence–modular and semi–degenerate, then: \(\rho_A(\alpha) = \nabla_A\) iff \(\alpha = \nabla_A\);

(ii) \(\rho_A((\alpha, \beta)_A^1) = \rho_A((\alpha, \beta)_A) = \rho_A(\alpha \cap \beta) = \rho_A(\alpha) \cap \rho_A(\beta)\);

(iii) \(\rho_A((\alpha, \alpha)_A^1) = \rho_A((\alpha, \alpha)_A) = \rho_A(\alpha)\).
(iv) $\rho_A(\rho_A(\alpha)) = \rho_A(\alpha)$;
(v) $\rho_A(\bigvee_{i \in I} \rho_A(\alpha_i)) = \rho_A(\bigvee_{i \in I} \alpha_i)$;
(vi) if $C$ is congruence–modular and semi–degenerate, then: $\bigvee_{i \in I} \rho_A(\alpha_i) = \nabla_A$ iff $\bigvee_{i \in I} \alpha_i = \nabla_A$.

**Proof.** Boh results in Remark 5.5, $\nabla_A \subseteq \rho_A(\nabla_A)$, thus $\rho_A(\nabla_A) = \nabla_A$. If $\alpha \neq \nabla_A$, then there exists $\phi \in V_A(\alpha) \subsetneq \nabla_A$.

By Remark 5.1, Lemma 5.2, and Proposition 5.2, $\rho_A([\alpha, \beta]_A) = \rho_A([\alpha, \beta]) = \rho_A(\alpha \cap \beta) = \bigvee_{\phi \in V_A(\alpha \cap \beta)} \bigwedge_{\phi \in V_A(\alpha) \cup V_A(\beta)} \phi \cap \bigwedge_{\phi \in V_A(\beta)} \phi = \rho_A(\alpha) \cap \rho_A(\beta)$.

By Remark 5.1, and Lemma 5.2.

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By (v) and (i), $\bigvee_{i \in I} \rho_A(\alpha_i) = \nabla_A$ iff $\rho_A(\bigvee_{i \in I} \rho_A(\alpha_i)) = \rho_A(\bigvee_{i \in I} \rho_A(\alpha_i)) = \rho_A(\bigvee_{i \in I} \alpha_i) = \nabla_A$ iff $\bigvee_{i \in I} \alpha_i = \nabla_A$.

The radical congruences of $A$ are the congruences $\alpha$ of $A$ such that $\alpha = \rho_A(\alpha)$. Let us denote by $\text{RCon}(A)$ the set of the radical congruences of $A$.

**Remark 5.5.** By Remark 5.1, $\text{Spec}(A) \subseteq \text{RCon}(A)$; moreover, the elements of $\text{RCon}(A)$ are exactly the intersections of prime congruences of $A$.

**Remark 5.6.** $\text{RCon}(A) = \{ \alpha \in \text{Con}(A) \mid \alpha = \rho_A(\alpha) \} = \{ \rho_A(\alpha) \mid \alpha \in \text{Con}(A) \}$. Indeed, the first of these equalities is the definition of $\text{RCon}(A)$ and the second equality follows from Proposition 5.3.

**Proposition 5.7.** If the commutator of $A$ equals the intersection, in particular if $C$ is congruence–distributive, then $\text{RCon}(A) = \text{Con}(A)$.

**Proof.** By Lemma 1.6, the radical congruences of $A$ coincide to its semiprime congruences, that is the congruences $\theta$ of $A$ such that, for all $\alpha \in \text{Con}(A)$, $[\alpha, \alpha]_A \subseteq \theta$ implies $\alpha \subseteq \theta$. Clearly, if $[\cdot, \cdot]_A = \cap$, then every congruence of $A$ is semiprime, and thus radical.

Most of the previous results on the radicals of congruences are known, but, for the sake of completeness, we have provided short proofs for them. For any $\alpha, \beta \in \text{Con}(A)$, let us denote by $\alpha \vee \beta = \rho_A(\alpha \vee \beta)$. For any family $(\alpha_i)_{i \in I} \subseteq \text{Con}(A)$, we shall denote by $\bigvee_{i \in I} \alpha_i = \rho_A(\bigvee_{i \in I} \alpha_i)$.

**Proposition 5.8.** $(\text{RCon}(A), \vee, \cap, \rho_A(\Delta_A), \rho_A(\nabla_A) = \nabla_A)$ is a bounded lattice, ordered by set inclusion. Moreover, it is a complete lattice, in which the arbitrary join is given by the $\bigvee$ defined above.

**Proof.** Of course, $\cap$ is idempotent, commutative and associative, and, clearly, $\vee$ is commutative. Now let $\alpha, \beta, \gamma \in \text{Con}(A)$, and $R = \{ \rho_A(\alpha), \rho_A(\beta), \rho_A(\gamma) \} \subseteq \text{RCon}(A)$; we shall use Proposition 5.3, and (ii) and (iv): $\alpha, \beta, \gamma \in \text{Con}(A)$, $\rho_A(\alpha) \vee \rho_A(\beta) = \rho_A(\rho_A(\alpha) \vee \rho_A(\beta)) = \rho_A(\alpha \vee \beta) = \alpha \vee \beta$, so $\vee$ is idempotent; $\rho_A(\alpha) \vee (\rho_A(\beta) \vee \rho_A(\gamma)) = (\rho_A(\alpha) \vee \rho_A(\beta)) \vee (\rho_A(\alpha) \vee \rho_A(\gamma)) = \rho_A(\alpha) \vee \rho_A(\beta \vee \gamma) = \rho_A(\rho_A(\alpha) \vee \rho_A(\beta) \vee \rho_A(\gamma))$.

By the commutativity of $\vee$, we also have $(\rho_A(\alpha) \vee \rho_A(\beta)) \vee \rho_A(\gamma) = \rho_A(\gamma) \vee (\rho_A(\alpha) \vee \rho_A(\beta)) = \bigvee_{\theta \in R} \theta$, hence $\rho_A(\alpha) \vee (\rho_A(\beta) \vee \rho_A(\gamma)) = (\rho_A(\alpha) \vee \rho_A(\beta)) \vee \rho_A(\gamma)$, so $\vee$ is associative; $\rho_A(\alpha) \vee (\rho_A(\alpha) \vee \rho_A(\beta)) = \rho_A(\alpha) \vee \rho_A(\alpha \vee \beta) = \rho_A(\rho_A(\alpha) \vee \rho_A(\alpha \vee \beta)) = \rho_A(\rho_A(\alpha) \vee \rho_A(\beta \vee \gamma)) = \rho_A(\rho_A(\alpha) \vee \rho_A(\beta) \vee \rho_A(\gamma))$.

By (i) By Lemma 5.2, (i), Proof. Boh results in Remark 5.5, $\nabla_A \subseteq \rho_A(\nabla_A)$, thus $\rho_A(\nabla_A) = \nabla_A$. If $\alpha \neq \nabla_A$, then there exists $\phi \in V_A(\alpha) \subsetneq \nabla_A$.
Remark 5.10. A bounded lattice morphism. Moreover, \(\rho_A(\Delta_A) = \rho_A(\alpha \cap \rho_A(\alpha \lor \beta)) = \rho_A(\alpha \cap \rho_A(\alpha \lor \beta)) = \rho_A(\alpha \cap \rho_A(\alpha \lor \beta)) = \rho_A(\rho_A(\alpha) \cap \rho_A(\rho_A(\alpha \lor \beta))) = \rho_A(\rho_A(\alpha) \cap \rho_A(\alpha \lor \beta)) = \rho_A(\alpha) \lor \rho_A(\alpha \lor \beta) = \rho_A(\alpha \lor (\alpha \lor \beta)) = \rho_A(\alpha),\) so the absorption laws hold. Of course, for all \(\theta, \zeta \in \text{RCon}(A),\) \(\theta \cap \zeta = \theta \iff \theta \subseteq \zeta.\) Therefore \((\text{RCon}(A), \lor, \cap)\) is a lattice, ordered by set inclusion. From Remark 5.1 (iii) and (iv), we obtain that this lattice has \(\rho_A(\Delta_A)\) as first element and \(\rho_A(\lor_A) = \lor_A\) as last element.

Now let us consider a family \((\alpha_i)_{i \in I} \subseteq \text{Con}(A),\) \(M = \{\rho_A(\alpha_i) \mid i \in I\} \subseteq \text{RCon}(A)\) and let us denote by \(\theta = \bigvee_{i \in I} \rho_A(\alpha_i) = \rho_A(\bigvee_{i \in I} \rho_A(\alpha_i)) = \rho_A(\bigvee_{i \in I} \alpha_i),\) by Proposition 5.4 (v). Then \(\theta \in \text{RCon}(A)\) and \(\rho_A(\alpha_i) \subseteq \theta\) for all \(i \in I.\) Now, if \(\zeta \in \text{RCon}(A)\) and \(\rho_A(\alpha_i) \subseteq \zeta\) for all \(i \in I,\) then \(\bigvee_{i \in I} \rho_A(\alpha_i) \subseteq \zeta,\) so, by Remark 5.1 (iii) and Proposition 5.4 (v), \(\zeta = \rho_A(\zeta) \supseteq \rho_A(\bigvee_{i \in I} \rho_A(\alpha_i)) = \bigvee_{i \in I} \rho_A(\alpha_i) = \theta.\) Therefore \(\theta = \sup(M)\) in the bounded lattice \(\text{RCon}(A),\) hence this lattice is complete. □

Let us define a binary relation \(\equiv_A\) on \(\text{Con}(A)\) by: \(\alpha \equiv_A \beta \iff \rho_A(\alpha) = \rho_A(\beta),\) for any \(\alpha, \beta \in \text{Con}(A).\) \(\equiv_A \cap (\mathcal{K}(A))^2\) shall also be denoted by \(\equiv_A.\)

**Remark 5.9.** Clearly, \(\equiv_A\) is an equivalence on \(\text{Con}(A),\) thus also on \(\mathcal{K}(A).\) On \(\text{RCon}(A),\) \(\equiv_A\) coincides to the equality, that is to \(\Delta_{\text{RCon}(A)},\) because, for any \(\alpha, \beta \in \text{Con}(A),\) \(\rho_A(\alpha) \equiv_A \rho_A(\beta) \iff \rho_A(\rho_A(\alpha)) = \rho_A(\rho_A(\beta)) \iff \rho_A(\alpha) = \rho_A(\beta).\) So, trivially, \(\equiv_A\) is a congruence of the lattice \(\text{RCon}(A).\)

On \(\text{Con}(A),\) \(\equiv_A\) preserves the commutator, \(\cap, \lor\) and \(\lor\), even \(\lor\) and \(\lor\) over arbitrary families of congruences, in particular it is a congruence of the lattice \(\text{Con}(A).\) Indeed, if \(\alpha, \alpha', \beta, \beta' \in \text{Con}(A)\) such that \(\alpha \equiv_A \alpha'\) and \(\beta \equiv_A \beta',\) that is \(\rho_A(\alpha) = \rho_A(\alpha')\) and \(\rho_A(\beta) = \rho_A(\beta'),\) then, by Proposition 5.4 (ii), \(\rho_A([\alpha, \beta]_A) = \rho_A(\alpha \lor \beta) = \rho_A(\rho_A(\alpha) \lor \rho_A(\beta)) = \rho_A(\rho_A(\alpha') \lor \rho_A(\beta')) = \rho_A(\rho_A(\alpha') \lor \beta') = \rho_A(\rho_A(\alpha') \lor \beta) = \rho_A(\rho_A(\alpha) \lor \beta) = \rho_A(\rho_A(\alpha') \lor \beta') = \rho_A(\rho_A(\alpha') \lor \beta') = \rho_A(\rho_A(\alpha') \lor \beta'),\) thus \([\alpha, \beta]_A \equiv_A [\alpha', \beta']_A \equiv_A \alpha \land \beta \equiv_A \alpha' \land \beta'.\) Now, if \((\alpha_i)_{i \in I} \subseteq \text{Con}(A)\) and \((\alpha'_i)_{i \in I} \subseteq \text{Con}(A)\) such that, for all \(i \in I, \alpha_i \equiv_A \alpha'_i,\) that is \(\rho_A(\alpha_i) = \rho_A(\alpha'_i),\) then, by Proposition 5.4 (iv), \(\rho_A(\bigvee_{i \in I} \alpha_i) = \rho_A(\rho_A(\bigvee_{i \in I} \alpha_i)) = \rho_A(\bigvee_{i \in I} \rho_A(\alpha_i)) = \rho_A(\bigvee_{i \in I} \rho_A(\alpha'_i) = \rho_A(\bigvee_{i \in I} \alpha'_i),\) hence \(\bigvee_{i \in I} \alpha_i \equiv_A \bigvee_{i \in I} \alpha'_i \equiv_A \bigvee_{i \in I} \alpha_i \equiv_A \bigvee_{i \in I} \alpha'_i.\)

Moreover, as shown by Proposition 5.4 (ii), (iv) and (v), just as in the calculations above, for all \(\alpha, \beta \in \text{Con}(A)\) and all \((\alpha_i)_{i \in I} \subseteq \text{Con}(A),\) \([\alpha, \beta]_A \equiv_A \alpha \land \beta \equiv_A \bigvee_{i \in I} \alpha_i \equiv_A \bigvee_{i \in I} \alpha'_i.\)

Note, also, that, for all \(\alpha \in \text{Con}(A),\) \(\alpha \equiv_A \rho_A(\alpha),\) by Proposition 5.4 (v).

For all \(\alpha \in \text{Con}(A),\) let us denote by \(\hat{\alpha}\) the equivalence class of \(\alpha\) with respect to \(\equiv_A,\) and let \(\hat{\mathcal{K}}(A) = \mathcal{K}(A)/\equiv_A = \{\hat{\theta} \mid \theta \in \mathcal{K}(A)\}.\) Let \(\lambda_A : \text{Con}(A) \to \mathcal{K}(A)/\equiv_A\) be the canonical surjection: \(\lambda_A(\theta) = \hat{\theta}\) for all \(\theta \in \text{Con}(A);\) we denote in the same way its restriction to \(\mathcal{K}(A),\) with its co-domain restricted to \(\Delta_A,\) that is the canonical surjection \(\lambda_A : \mathcal{K}(A) \to \Delta_A.\) Let us define the following operations on \(\text{Con}(A),\) where the second equalities follow from Remark 5.9 as does the fact that these operations are well defined:

- for all \(\alpha, \beta \in \text{Con}(A),\) \(\hat{\alpha} \lor \hat{\beta} = \alpha \lor \beta = \alpha \lor \beta\) and \(\hat{\alpha} \land \hat{\beta} = \alpha \land \beta = [\alpha, \beta]_A;\)
- \(0 = \Delta_A = \rho_A(\Delta_A)\) and \(1 = \lor_A = \rho_A(\lor_A).\)

**Remark 5.10.** By Proposition 5.4 (ii), if \(\mathcal{C}\) is congruence–modular and semi–degenerate, then, for any \(\alpha \in \text{Con}(A),\) \(\hat{\alpha} = 1 \iff \alpha = \lor_A.\)

**Lemma 5.11.** \((\text{Con}(A)/\equiv_A, \lor, \land, 0, 1)\) is a bounded distributive lattice and \(\lambda_A : \text{Con}(A) \to \text{Con}(A)/\equiv_A\) is a bounded lattice morphism. Moreover, \(\text{Con}(A)/\equiv_A\) is a complete lattice, in which \(\bigvee_{i \in I} \hat{\alpha}_i = \hat{\bigvee_{i \in I} \alpha_i}\) and \(\bigwedge_{i \in I} \hat{\alpha}_i = \hat{\bigwedge_{i \in I} \alpha_i}\) for any family \((\alpha_i)_{i \in I} \subseteq \text{Con}(A),\) and the meet is completely distributive with respect to the join, thus \(\text{Con}(A)/\equiv_A\) is a frame.
Proof. By Remark 5.9, $\equiv_A$ is a congruence of the bounded lattice $\text{Con}(A)$, hence $(\text{Con}(A)/\equiv_A, \lor, \land, 0, 1)$ is a bounded lattice and the canonical surjection $\lambda_A : \text{Con}(A) \to \text{Con}(A)/\equiv_A$ is a bounded lattice morphism, in particular it is order–preserving. It is straightforward, from the fact that the lattice $\text{Con}(A)$ is complete and the surjectivity of the lattice morphism $\lambda_A$, that the lattice $\text{Con}(A)/\equiv_A$ is complete and its joins and meets of arbitrary families of elements have the form in the enunciation. By Proposition 3.6, for any families $(\alpha_i)_{i \in I}$ and $(\beta_j)_{j \in J}$ of congruences of $A$, $(\bigvee_{i \in I} \alpha_i) \land (\bigvee_{j \in J} \beta_j) = ((\bigvee_{i \in I} \alpha_i) \land \bigvee_{j \in J} \beta_j) = \bigvee_{i \in I, j \in J} [\alpha_i, \beta_j]$, that is the meet is completely distributive with respect to the join in $\text{Con}(A)/\equiv_A$, thus $\text{Con}(A)/\equiv_A$ is a frame, in particular it is a bounded distributive lattice. \qed

We shall denote by $\leq$ the partial order of the lattice $\text{Con}(A)/\equiv_A$.

**Proposition 5.12.** (\(\text{RCon}(A), \lor, \land, \rho_A(\Delta_A), \rho_A(\land_A) = \land_A\)) is a frame, isomorphic to $\text{Con}(A)/\equiv_A$.

**Proof.** Let $\varphi : \text{Con}(A)/\equiv_A \to \text{RCon}(A)$, for all $\alpha \in \text{Con}(A)$, $\varphi(\alpha) = \rho_A(\alpha)$. If $\alpha, \beta \in \text{Con}(A)$, then the following equivalences hold: $\hat{\alpha} = \hat{\beta}$ if $\alpha \equiv_A \beta$ iff $\rho_A(\alpha) = \rho_A(\beta)$ iff $\varphi(\alpha) = \varphi(\beta)$, hence $\varphi$ is well defined and injective. By Remark 5.6, $\varphi$ is surjective. By Proposition 5.14, $\forall \alpha, \beta \in \text{Con}(A)$, $\varphi(\hat{\alpha} \land \hat{\beta}) = \varphi(\alpha \land \beta) = \rho_A(\alpha \land \beta)$ and $\varphi(\hat{\alpha} \lor \hat{\beta}) = \varphi(\alpha \lor \beta) = \rho_A(\alpha \lor \beta)$ (actually, Proposition 5.4 and Lemma 5.11 show that $\varphi$ preserves arbitrary joins). Therefore $\varphi$ is a lattice isomorphism, thus an order isomorphism, hence it preserves arbitrary joins and meets. From this and Lemma 5.11 we obtain that $\text{RCon}(A)$ is a frame and $\varphi$ is a frame isomorphism. \qed

Throughout the rest of this section, we shall assume that $K(A)$ is closed with respect to the commutator.

**Proposition 5.13.** $\land_A \in K(A)$, hence $(\text{Con}(A)/\equiv_A, \lor, \land, 0, 1)$ is a bounded distributive lattice.

**Proof.** Since $\land_A \in K(A)$, we have $1 = \land_A \in \text{Con}(A)$, thus $0 = 0 = \land_A \in \text{Con}(A)$. If $K(A)$ is closed with respect to the commutator, then, for each $\alpha, \beta \in K(A)$, we have $[\alpha, \beta] = [\alpha \land \beta] \in K(A)$, thus $\alpha \land \beta = [\alpha, \beta] \in K(A)$ again by Remark 5.14 for each $\alpha, \beta \in K(A)$, $\hat{\alpha} \lor \hat{\beta} = \alpha \lor \beta \in K(A)$. Hence $\land_A$ is a bounded sublattice of $\text{Con}(A)/\equiv_A$, which is distributive by Lemma 5.11 thus $\land_A$ is a bounded distributive lattice. \qed

For any $\theta \in \text{Con}(A)$ and any $I \in \text{Id}(\land_A)$, we shall denote by:

- $\theta^* = \{ \hat{\alpha} \mid \alpha \in K(A), \alpha \subseteq \theta \} = \lambda_A(K(A) \cap \theta) \subseteq \land_A$, where $\theta = \theta|_{\text{Con}(A)} \in \text{PId}(\text{Con}(A))$

- $I_\theta = \bigvee\{ \alpha \in K(A) \mid \hat{\alpha} \in I \} = \bigvee_{\alpha \in \lambda_A^{-1}(I)} \alpha \in \text{Con}(A)$; note that $\lambda_A^{-1}(I)$ is non–empty, because $\Delta_A \in K(A)$ and $\Delta_A = \rho_A(\Delta_A) = 0 \in I$.

**Lemma 5.14.** For all $\theta \in \text{Con}(A)$:

- $\theta^* \subseteq (\theta|_{\text{Con}(A)/\equiv_A} \cap \land_A)$ and $\theta^* \in \text{Id}(\land_A)$

- if $\theta \in \text{K}(A)$, then $\theta^* = (\theta|_{\text{Con}(A)/\equiv_A} \cap \land_A) = (\hat{\theta}|_{\land_A}) \in \text{PId}(\land_A)$.

**Proof.** Let $\theta \in \text{Con}(A)$, and, in this proof, let us denote by $\hat{\theta} = \hat{\theta}|_{\text{Con}(A)/\equiv_A}$ and, in the case when $\theta \in \text{K}(A)$, by $\hat{\theta}_\Sigma = \{ \hat{\alpha} \mid \alpha \in \theta \cap \text{K}(A) \}$.

For all $\alpha \in \theta \cap \text{K}(A)$, we have $\hat{\alpha} \in \land_A$ and $\alpha \subseteq \theta$, thus $\hat{\alpha} \in \hat{\theta} \cap \land_A$, hence $\hat{\alpha} \in (\hat{\theta} \cap \land_A)$, therefore $\theta^* \subseteq (\hat{\theta} \cap \land_A)$. $\Delta_A \in \text{K}(A)$ and $\Delta_A \subseteq \theta$, thus $\Delta_A \in \theta^*$, so $\theta^*$ is non–empty. Since $\text{K}(A)$ is closed w.r.t. $[\cdot, \cdot]_A$, $\alpha \lor \beta \in \text{K}(A)$ for any $\alpha, \beta \in \text{K}(A)$. Let $x, y \in \theta^*$, which means that $x = \hat{\alpha}$ and $y = \hat{\beta}$ for some $\alpha, \beta \in \text{K}(A) \cap \theta$. Then $\alpha \lor \beta \in \text{K}(A) \cap \theta$, thus $\alpha \lor \beta \in \theta^*$. Now let $x \in \theta^*$ and $y \in \land_A$ such that $x \leq y$, so that $y = x \land y$. Then $\alpha = \hat{\alpha}$ for $\text{some } \alpha \in \text{K}(A) \cap \theta$ and $y = \hat{\beta}$ for...
some $\beta \in \mathcal{K}(A)$. Thus $[\alpha, \beta]_A \in \mathcal{K}(A)$ and $[\alpha, \beta]_A \subseteq \alpha \cap \beta \subseteq \alpha \subseteq \theta$, hence $[\alpha, \beta]_A \in \mathcal{K}(A) \cap \{\theta\}$, therefore $y = x \wedge y = \widetilde{\alpha} \wedge \widetilde{\beta} = [\alpha, \beta]_A \in \theta^*$. Hence $\theta^* \in \text{Id}(\mathcal{L}(A))$.

Now assume that $\theta \in \mathcal{K}(A)$, so that $\widetilde{\theta} \in \mathcal{L}(A)$. By the above, $\theta^* \subseteq \widetilde{\theta} \cap \mathcal{L}(A) = \widetilde{\theta}$. Let $x \in \widetilde{\theta}$, so that there exists an $\alpha \in \mathcal{K}(A)$ with $\mu = x \leq \mu$, thus $[\alpha, \theta]_A = \mu \cap \widetilde{\theta} = \mu = x$. But $[\alpha, \theta]_A \in \mathcal{K}(A) \cap \{\theta\}$, so $x = [\alpha, \theta]_A \in \theta^*$. Therefore we also have $\theta^* \subseteq \widetilde{\theta}$, hence $\theta^* = \widetilde{\theta} \in \text{Id}(\mathcal{L}(A))$. \hfill $\square$

By the above, we have two functions:

- $\theta \in \text{Con}(A) \rightarrow \theta^* \in \text{Id}(\mathcal{L}(A))$;

- $I \in \text{Id}(\mathcal{L}(A)) \rightarrow I_* \in \text{Con}(A)$.

**Lemma 5.15.** The two functions above are order-preserving.

**Proof.** For any $\theta, \zeta \in \text{Con}(A)$ such that $\theta \subseteq \zeta$, we have $[\theta] \subseteq [\zeta]$, hence $\theta^* \subseteq \zeta^*$. For any $I, J \in \text{Id}(\mathcal{L}(A))$ such that $I \subseteq J$, we have $\lambda^{-1}_A(I) \subseteq \lambda^{-1}_A(J)$, thus $I_* \subseteq J_*$.

**Lemma 5.16.** Let $\alpha \in \mathcal{K}(A)$ and $I \in \text{Id}(\mathcal{L}(A))$. Then: $\alpha \subseteq I_*$ iff $\widetilde{\alpha} \in I$.

**Proof.** “$\Rightarrow$” If $\widetilde{\alpha} \in I$, then $\alpha \subseteq \lambda^{-1}_A(I)$, thus $\alpha \subseteq I_*$.

“$\Leftarrow$” If $\alpha \subseteq I_*$, then $\alpha \subseteq \bigvee \{\beta \in \mathcal{K}(A) \mid \beta \in I\}$, then, since $\alpha \in \mathcal{K}(A)$, it follows that there exist an $n \in \mathbb{N}$ and $\beta_1, \ldots, \beta_n \in \mathcal{K}(A)$ such that $\widetilde{\beta}_1, \ldots, \widetilde{\beta}_n \in I$ and $\alpha \subseteq \bigvee_{i=1}^n \beta_i$, hence $\widetilde{\alpha} \subseteq \bigvee_{i=1}^n \widetilde{\beta}_i = \bigvee_{i=1}^n \beta_i \in I$, thus $\widetilde{\alpha} \in I$.

**Lemma 5.17.** (i) For any $\theta \in \text{Con}(A)$, $\theta \subseteq (\theta^*)_*$.

(ii) For any $I \in \text{Id}(\mathcal{L}(A))$, $I = (I_*)^*$.

**Proof.** (i) Let $\theta \in \text{Con}(A)$. For any $(a, b) \in \theta$, $Cg_A(a, b) \in P\text{Con}(A) \subseteq \mathcal{K}(A)$ and $Cg_A(a, b) \subseteq \theta$, thus $Cg_A(a, b) \in \mathcal{K}(A) \cap \{\theta\}$, hence $Cg_A(a, b) \in \theta^*$, therefore $Cg_A(a, b) \subseteq (\theta^*)_*$ by Lemmas 5.14 and 5.16, so $(a, b) \in (\theta^*)_*$. Hence $\theta \subseteq (\theta^*)_*$.

(ii) For any $x \in \mathcal{L}(A)$, by Lemma 5.10 the following equivalences hold: $x \in (I_*)^*$ iff there exists an $\alpha \in \mathcal{K}(A)$ such that $\alpha \subseteq I_*$ and $x = \mu$ iff there exists an $\alpha \in \mathcal{K}(A)$ such that $\mu \subseteq \alpha \subseteq I$. Therefore $(I_*)^* = I$.

**Proposition 5.18.** (i) The map $I \in \text{Id}(\mathcal{L}(A)) \mapsto I_* \in \text{Con}(A)$ is injective.

(ii) The map $\theta \in \text{Con}(A) \mapsto \theta^* \in \text{Id}(\mathcal{L}(A))$ is surjective.

**Proof.** (i) Let $I, J \in \text{Id}(\mathcal{L}(A))$ such that $I_* = J_*$. Then $(I_*)^* = (J_*)^*$, so $I = J$ by Lemma 5.17. (ii) Let $I \in \text{Id}(\mathcal{L}(A))$, and denote $\theta = I_* \in \text{Con}(A)$. Then $\theta^* = (I_*)^* = I$ by Lemma 5.17.

**Lemma 5.19.** For any $\phi \in \text{Spec}(A)$, $\phi = (\phi^*)_*$.

**Proof.** Let $\phi \in \text{Spec}(A)$. Then $\phi \subseteq (\phi^*)_*$ by Lemma 5.17. Now let $\beta \in \mathcal{K}(A)$ such that $\beta \subseteq \phi$, which means that $\beta = \mu$ for some $\alpha \in \mathcal{K}(A)$ with $\alpha \subseteq \phi$. Since $\beta = \mu$, we have $\rho_A(\beta) = \rho_A(\alpha)$, while $\alpha \subseteq \phi$ gives us $\rho_A(\alpha) \subseteq \rho_A(\phi) = \phi$, where the last equality follows from the fact that $\phi \in \text{Spec}(A)$. Hence $\beta \subseteq \rho_A(\beta) \subseteq \phi$. Therefore $(\phi^*)_* = \bigvee \{\gamma \in \mathcal{K}(A) \mid \gamma \subseteq \phi \} \subseteq \phi$.

Hence $\phi = (\phi^*)_*$.

**Lemma 5.20.** For any $\phi \in \text{Spec}(A)$, we have $\phi^* \in \text{Spec}_{\text{Id}}(\mathcal{L}(A))$.

**Proof.** Let $\phi \in \text{Spec}(A)$. Then $\phi^* \in \text{Id}(\mathcal{L}(A)) = \text{Id}(\mathcal{K}(A)/\{\theta\})$. Let $\alpha, \beta \in \mathcal{K}(A)$ such that $[\alpha, \beta]_A = \mu \wedge \beta = \phi^* = \bigvee \{\gamma \mid \gamma \in \mathcal{K}(A), \gamma \subseteq \phi\}$. Then there exists a $\gamma \in \mathcal{K}(A)$ such that $\gamma \subseteq \phi$ and $\mu = [\alpha, \beta]_A$, thus $\rho_A(\gamma) = \rho_A([\alpha, \beta]_A)$ and $\rho_A(\gamma) \subseteq \rho_A(\phi) = \phi$ since $\phi \in \text{Spec}(A)$. Hence $[\alpha, \beta]_A \subseteq \rho_A([\alpha, \beta]_A) \subseteq \phi$, hence $\alpha \subseteq \phi$ or $\beta \subseteq \phi$ since $\phi \in \text{Spec}(A)$. But this means that $\phi \subseteq \phi^*$. Therefore $\phi^* \in \text{Spec}_{\text{Id}}(\mathcal{L}(A))$.

**Lemma 5.21.** For any $P \in \text{Spec}_{\text{Id}}(\mathcal{L}(A))$, we have $P_* \in \text{Spec}(A)$.
Proof. Let $P \in \text{Spec}_{\text{Id}}(\mathcal{L}(A))$. Then $P_\ast \in \text{Con}(A)$. Let $\alpha, \beta \in \text{PCon}(A)$ such that $[\alpha, \beta]_A \subseteq P_\ast$. Then $\alpha, \beta \in \mathcal{K}(A)$, so that $[\alpha, \beta]_A \in \mathcal{K}(A)$, and $[\alpha, \beta]_A \subseteq [\gamma \in \mathcal{K}(A) \mid \hat{\gamma} \in P]$, hence there exist an $n \in \mathbb{N}^+$ and $\gamma_1, \ldots, \gamma_n \in \mathcal{K}(A)$ such that $\hat{\gamma}_1, \ldots, \hat{\gamma}_n \in P$ and $[\alpha, \beta]_A \subseteq \bigvee_{i=1}^n \gamma_i$. But then $\bigvee_{i=1}^n \gamma_i = \bigvee_{i=1}^n \hat{\gamma}_i \in P$, hence $\hat{\alpha} \wedge \hat{\beta} = \hat{[\alpha, \beta]_A} \in P$, thus $\hat{\alpha} \in P$ or $\hat{\beta} \in P$ since $P \in \text{Spec}_{\text{Id}}(\mathcal{L}(A))$. By Lemma 5.16 it follows that $\alpha \subseteq P_\ast$ or $\beta \subseteq P_\ast$. Therefore $P_\ast \in \text{Spec}(A)$.

By Lemmas 5.20 and 5.24 we have these restrictions of the functions defined above:

- $u : \text{Spec}(A) \to \text{Spec}_{\text{Id}}(\mathcal{L}(A))$, for all $\phi \in \text{Spec}(A)$, $u(\phi) = \phi^\ast$;
- $v : \text{Spec}_{\text{Id}}(\mathcal{L}(A)) \to \text{Spec}(A)$, for all $P \in \text{Spec}_{\text{Id}}(\mathcal{L}(A))$, $v(P) = P_\ast$.

Proposition 5.22. $u$ and $v$ are homeomorphisms, inverses of each other, between the prime spectrum of $A$ and the prime spectrum of ideals of $\mathcal{L}(A)$, endowed with the Stone topologies.

Proof. By Lemma 5.17, for all $P \in \text{Spec}_{\text{Id}}(\mathcal{L}(A))$, we have $v(u(P)) = P$. By Lemma 5.19 for all $\phi \in \text{Spec}(A)$, we have $u(v(\phi)) = \phi$. Thus $u$ and $v$ are bijections and they are inverses of each other.

Let $\theta \in \text{Con}(A)$ and $\phi \in V_A(\theta)$, that is $\phi \in \text{Spec}(A)$ and $\theta \subseteq \phi$. Then, by Lemmas 5.21 and 5.15 $\phi^\ast \in \text{Spec}_{\text{Id}}(\mathcal{L}(A))$ and $\theta^\ast \subseteq \phi^\ast$, so $\phi^\ast \in \mathcal{L}(A)(\theta^\ast)$, and we have $u(\phi) = \phi^\ast$. Hence $u(V_A(\theta)) \subseteq \mathcal{L}(A)(\theta^\ast)$. Now let $P \in \mathcal{L}(A)(\theta^\ast)$, that is $P \in \text{Spec}_{\text{Id}}(\mathcal{L}(A))$ and $\theta^\ast \subseteq P$. Then, by Lemma 5.17, for all $P \in \text{Spec}(A)$, thus $P_\ast \in \text{Spec}(A)$, and we have $u(P_\ast) = u(v(P)) = P$. Hence $\mathcal{L}(A)(\theta^\ast) \subseteq u(V_A(\theta))$. Therefore $u(V_A(\theta)) = \mathcal{L}(A)(\theta^\ast)$, thus $u$ is closed, hence $u$ is open, so $v$ is continuous.

Now let $I \in \text{Id}(\mathcal{L}(A))$. Then, according to Proposition 5.18, (ii), $I = \theta^\ast$ for some $\theta \in \text{Con}(A)$. By the above, $u(V_A(\theta)) = \mathcal{L}(A)(\theta^\ast) = \mathcal{L}(A)(I)$, hence $v(V_A(\theta)) = v(u(V_A(\theta))) = V_A(\theta)$, therefore $v$ is closed, hence $v$ is open, thus $u$ and $v$ are homeomorphisms.

Corollary 5.23 (existence of the reticulation). $\mathcal{L}(A)$ is a reticulation for the algebra $A$.

Proposition 5.24. [5, 20] If $L$ and $M$ are bounded distributive lattices whose prime spectra of ideals, endowed with the Stone topologies, are homeomorphic, then $L$ and $M$ are isomorphic.

Corollary 5.25 (uniqueness of the reticulation). The reticulation of $A$ is unique up to a lattice isomorphism.

Corollary 5.26. If $\mathcal{C}$ is congruence–modular and semi–degenerate, then $u$ and $v$ induce homeomorphisms, inverses of each other, between the maximal spectrum of $A$ and the maximal spectrum of ideals of $\mathcal{L}(A)$, endowed with the Stone topologies.

Proof. By Lemma 3.12 Proposition 5.22 and the fact that, as Lemma 5.15 ensures us, $u$ and $v$ are order–preserving, and hence they are order isomorphisms between the posets $\text{Spec}(A)$ and $\text{Spec}_{\text{Id}}(\mathcal{L}(A))$.

Proposition 5.27. [1] Proposition 4.1] For any $\theta \in \text{Con}(A)$, $\rho_A(\theta) = \{ (a,b) \in A^2 \mid (\exists n \in \mathbb{N}^+)[Cg_A(a,b), Cg_A(a,b)]_A^{n} \subseteq \theta) \}$, so $\rho_A(\Delta_A) = \{ (a,b) \in A^2 \mid (\exists n \in \mathbb{N}^+)[Cg_A(a,b), Cg_A(a,b)]_A^{n} = \Delta_A) \}$.

Proposition 5.28. For any $\theta \in \text{Con}(A)$, $\theta^\ast = \rho_A(\theta)$.

Proof. For every $\beta \in \mathcal{K}(A)$ such that $\hat{\beta} \in \theta^\ast = \{ \hat{\gamma} \mid \gamma \in \mathcal{K}(A), \gamma \subseteq \theta \}$, there exists an $\alpha \in \mathcal{K}(A)$ such that $\alpha \subseteq \theta$ and $\hat{\alpha} = \hat{\beta}$, thus $\beta \subseteq \rho_A(\beta) = \rho_A(\alpha) \subseteq \rho_A(\theta)$. Therefore $\theta^\ast = \bigvee \{ \hat{\gamma} \in \mathcal{K}(A) \mid \hat{\gamma} \in \theta^\ast \} \subseteq \rho_A(\theta)$. Now let $(a,b) \in \rho_A(\theta)$, so that, according to Proposition 5.27, Lemma 5.17, and Lemma 5.16 for some $n \in \mathbb{N}^+$, $[Cg_A(a,b), Cg_A(a,b)]_A^n \subseteq \theta \subseteq \theta^\ast$, hence $[Cg_A(a,b), Cg_A(a,b)]_A^n \in \theta^\ast$. But $\rho_A([Cg_A(a,b), Cg_A(a,b)]_A^n) = \rho_A(Cg_A(a,b))$, thus $Cg_A(a,b) \in \theta^\ast$, hence $(a,b) \in Cg_A(a,b) \subseteq \theta^\ast$, by Lemma 5.16 Therefore $\rho_A(\theta) \subseteq \theta^\ast$. Hence $\theta^\ast = \rho_A(\theta)$.

Corollary 5.29. (i) For all $\theta \in \text{Con}(A)$, $\rho_A(\theta^\ast) = \theta^\ast$.

(ii) For all $I \in \text{Id}(\mathcal{L}(A))$, $\rho_A(I_\ast) = I_\ast$. 
we shall denote by

Remark 6.1. If Spec(\

commutator of

following examples is finite, we have

Hence

Proof. Assume that \([\cdot,\cdot]_A = \cap \Delta_A \in \text{PCon}(A) \subseteq K(A)\). By Remark 6.18, \(K(A)\) is closed w.r.t. the join, and we are under the assumptions that \(\nabla_A \in K(A)\) and \(K(A)\) is closed w.r.t. the commutator, so w.r.t. the intersection. Hence \(K(A)\) is a bounded sublattice of \(\text{Con}(A)\). By Proposition 6.24, \(\equiv_A = \Delta_{K(A)}\), thus \(L(A) = K(A)/\Delta_{K(A)} \cong K(A)\) and the canonical surjection \(\lambda_A : K(A) \to L(A)\) is a lattice isomorphism.

Remark 6.3. If \(\text{Con}(A) = K(A)\), in particular if \(A\) is finite, then \(L(A) = \text{Con}(A)/\equiv_A\), so, if, furthermore, the commutator of \(A\) equals the intersection, in particular if \(C\) is congruence–distributive, then \(L(A) \cong \text{Con}(A)\) by Proposition 6.22, thus we may take \(L(A) = \text{Con}(A)\).

As a fact that may be interesting by its symmetry, if \(A\) is finite and its commutator equals the intersection, so that \(\text{Con}(A)\) is a finite distributive lattice, then \(L(\text{Con}(A)) = \text{Con}(\text{Con}(A)) = \text{Con}(L(A))\). It might also be interesting to find weaker conditions on \(A\) under which \(L(\text{Con}(A)) \cong \text{Con}(L(A))\).
Remark 6.4. By Proposition 6.2, if \( A \) is a residuated lattice, then \( \mathcal{L}(A) = \mathcal{K}(A) \). If we denote by \( \text{Filt}(A) \) the set of the filters of \( A \) and by \( \text{PFilt}(A) \) the set of the principal filters of \( A \), then, since \( \text{Con}(A) \cong \text{Filt}(A) \) and the finitely generated filters of \( A \) are principal filters \([22], [28]\), it follows that \( \mathcal{L}(A) = \mathcal{K}(A) \cong \text{PFilt}(A) \), which is the dual of the reticulation of a residuated lattice obtained in \([11], [22], [23]\), where the reticulation has the prime spectrum of filters homeomorphic to the prime spectrum of filters, thus to that of congruences of \( A \) by the above, so this duality to the construction of \( \mathcal{L}(A) \) from Section 5 was to be expected.

Remark 6.5. If \( A \) is a commutative unitary ring and \( \text{Id}(A) \) is its lattice of ideals, then it is well known that \( \text{Id}(A) \cong \text{Con}(A) \). If, for all \( I \in \text{Id}(A) \), we denote by \( \sqrt{I} \) the intersection of the prime filters of \( A \) which include \( I \), then \([7]\) Lemma, p. 1861] shows that, for any \( J \in \text{Id}(A) \), there exists a finitely generated ideal \( K \) of \( A \) such that \( \sqrt{I} = \sqrt{K} \). From this, it immediately follows that the lattice \( \mathcal{L}(A) \) is isomorphic to the reticulation of \( A \) constructed in \([7]\).

Remark 6.6. Let \( n, k \in \mathbb{N}^* \) and assume that \( C \) is congruence–modular, \( S \) is a subalgebra of \( A, \alpha, \beta \in \text{Con}(A) \), \( M_1, \ldots, M_n \) are algebras from \( C \), \( M = \prod_{i=1}^n M_i \) and, for all \( i \in \overline{1, n} \), \( \alpha_i, \beta_i \in \text{Con}(M_i) \).

From Lemma 6.8 it is immediate that \( [\alpha \cap S^2, \beta \cap S^2]_k \subseteq [\alpha_k, \beta_k]_A \cap S^2 \) and \( [\alpha \cap S^2, \beta \cap S^2]_k \subseteq [\alpha_k, \beta_k]_A \cap S^2 \). Hence, if \( A \) is Abelian or solvable or nilpotent, then \( S \) is Abelian or solvable or nilpotent, respectively.

From Proposition 6.9 it is immediate that \( \prod_{i=1}^n [\alpha_i, \beta_i]_{M_i} = \prod_{i=1}^n [\alpha_i, \beta_i]_{M_i} \) and \( \prod_{i=1}^n [\alpha_i, \beta_i]_{M_i} = \prod_{i=1}^n [\alpha_i, \beta_i]_{M_i} \).

From this, it is easy to prove that: \( M \) is Abelian or solvable or nilpotent iff \( M_1, \ldots, M_n \) are Abelian or solvable or nilpotent, respectively.

Example 6.7. For any group \((G, \cdot)\), any \( x \in G \) and any normal subgroup \( H \) of \( G \), let us denote by \( \langle x \rangle \) the subgroup of \( G \) generated by \( x \) and by \( \equiv_H \) the congruence of \( G \) associated to \( H \): \( \equiv_H = \{(y, z) \in G^2 \mid yz^{-1} \in H\} \).

As shown by the following commutator calculations, the quaternion group, \( C_8 = \{1, -1, i, -i, j, -j, k, -k\} \), is a solvable algebra which is not Abelian, while the group \( S_3 = \{1, t, u, v, c, d\} \) of the permutations of the set \( \overline{1, 3} \), where \( t = \text{id}(1, 3), u = (1, 3), v = (2, 3), c = (1, 2, 3) \) and \( d = c \circ c \), has \( \text{Spec}(S_3) = \emptyset \), without being solvable or nilpotent. The following are the subgroups of \( C_8 \), respectively \( S_3 \), all of which are normal, and the proper ones are cyclic, thus Abelian: \( \langle 1 \rangle, \langle -1 \rangle, \langle i \rangle, \langle j \rangle, \langle k \rangle \) and \( C_8 \), respectively \( \langle 1 \rangle, \langle i \rangle, \langle u \rangle, \langle c \rangle \) and \( S_3 \), so \( C_8 \) and \( S_3 \) have the following congruence lattices and commutators, which suffice to conclude that \( \text{Spec}(C_8) = \text{Spec}(S_3) = \emptyset \), since we are in a congruence–modular variety, and thus \( \mathcal{L}(C_8) \cong \mathcal{L}(S_3) \cong \mathcal{L}_1 \), by Remark 6.1.

Notice, also, that \( C_8 \) is solvable, as we have announced, thus, according to Remark 6.6, so is any finite direct product whose factors are subgroups of \( C_8 \), which, of course, is Abelian if all those subgroups are proper.

Example 6.8. This is the algebra from \([2]\) Example 6.3] and \([3]\) Example 4.2\): \( U = \{(0, a, b, c, d), +\} \), with + defined by the following table, which has the congruence lattice represented below, where \( U/\alpha = \{0, a, \{b, c, d\}\} \), \( U/\beta = \{0, b, \{a, c, d\}\} \), \( U/\gamma = \{0, c, d, \{a, b\}\} \) and \( U/\delta = \{0, \{a\}, \{b\}, \{c, d\}\} \):

\[
\begin{array}{ccccc}
+ & 0 & a & b & c & d \\
0 & 0 & a & b & c & d \\
a & a & 0 & c & b & b \\
b & b & c & 0 & a & a \\
x & c & b & a & 0 & 0 \\
y & d & b & a & 0 & 0 \\
z & & & & & \\
\end{array}
\]

\[
\begin{array}{ccccc}
\Delta_U & \alpha & \beta & \gamma & \delta \\
\alpha & \Delta_U & \delta & \delta & \delta \\
\beta & \delta & \Delta_U & \delta & \delta \\
\gamma & \delta & \delta & \Delta_U & \delta \\
\delta & \delta & \delta & \delta & \Delta_U \\
\end{array}
\]

\[
\begin{array}{c}
\mathcal{L}(U) \cong \mathcal{L}_1 \text{ by Remark 6.1} \\
\end{array}
\]
Example 6.9. Let $M = \{a, b, x, y, z\}$ and $N = \{a, b, c, x, y, z\}$, with $+$ defined by the following tables. Then $\text{Con}(M)$ and $\text{Con}(N)$ have the Hasse diagrams below, where:

- $M/\alpha = \{(a, b), \{x, y, z\}\}, M/\beta = \{(a, b), \{x, y\}, \{z\}\}, M/\gamma = \{(a, b), \{x, z\}, \{y\}\}, M/\delta = \{(a, b), \{x\}, \{y, z\}\}$ and $M/\varepsilon = \{(a, b), \{x\}, \{y\}, \{z\}\}$;
- $N/\chi = \{(a, b, c), \{x, y\}\}, N/\xi = \{(a, b, c), \{x\}\}, N/\eta = \{(a, b, c), \{x, y\}\}, N/\zeta = \{(a, b), \{c\}, \{x, y\}\}$, $N/\psi = \{(a, b, c), \{x, y\}\}, N/\varphi = \{(a, b), \{c\}, \{x, y\}\}$.

\[
\begin{array}{c|cccccc}
+ & a & b & x & y & z \\
\hline
a & a & b & a & a & a \\
b & b & b & b & b & b \\
x & x & x & x & x \\
y & y & y & y & y \\
z & z & z & z & z \\
\end{array}
\]  
\[
\begin{array}{c|cccc}
\alpha & a & b & c & x \\
\hline
a & a & b & c & c \\
b & b & b & c & c \\
x & x & x & x & x \\
y & y & y & y & y \\
z & z & z & z & z \\
\end{array}
\]

Note that, despite the fact that $M$ is congruence–modular and $N$ is congruence–distributive, neither $\mathcal{HSP}(M)$, nor $\mathcal{HSP}(N)$ is congruence–modular, because $S = \{(a, b), +\} \cong (\mathbb{Z}_2, \text{max}) \cong (\mathbb{Z}_2, \cdot)$ is a subalgebra of both $M$ and $N$, and it can be easily checked that $S^2$ is not congruence–modular. Thus neither $\mathcal{HSP}(M)$, nor $\mathcal{HSP}(N)$ is semidegenerate, which is also obvious from the fact that $\{(a, b), +\}$ is a subalgebra of both $M$ and $N$.

We have: $[\theta, \xi]_M = \xi$ for all $\theta, \xi \in \mathcal{E}$ and, of course, $[\Delta_M, \theta]_M = [\theta, \Delta_M]_M = \Delta_M$ for all $\theta \in \text{Con}(M)$, hence $\text{Spec}(M) = \{\Delta_M\}$ and thus $\mathcal{L}(M) \cong \mathbb{L}_2$, while $\mathcal{L}(N)$ is given by the following table, thus $\text{Spec}(N) = \{\psi, \xi\}$, so $\rho_N$ is defined as follows and hence $\mathcal{L}(M) \cong \mathbb{L}_2$:

| $[\cdot, \cdot]_N$ | $\Delta_N$ | $\psi$ | $\psi_1$ | $\phi$ | $\xi$ | $\chi$ | $\chi_1$ | $\nabla_N$ | $\theta$ | $\rho_N(\theta)$ |
|-----------------|------------|---------|---------|-------|-------|-------|-------|---------|-------|---------|
| $\Delta_N$     | $\Delta_N$ | $\Delta_N$ | $\Delta_N$ | $\Delta_N$ | $\Delta_N$ | $\Delta_N$ | $\Delta_N$ | $\Delta_N$ | $\Delta_N$ | $\psi$ |
| $\psi$          | $\Delta_N$ | $\psi_1$ | $\psi_1$ | $\psi_1$ | $\psi_1$ | $\psi_1$ | $\psi_1$ | $\psi_1$ | $\psi_1$ | $\psi_1$ |
| $\psi_1$        | $\Delta_N$ | $\psi_1$ | $\psi_1$ | $\psi_1$ | $\psi_1$ | $\psi_1$ | $\psi_1$ | $\psi_1$ | $\psi_1$ | $\psi_1$ |
| $\phi$          | $\Delta_N$ | $\Delta_N$ | $\Delta_N$ | $\Delta_N$ | $\Delta_N$ | $\Delta_N$ | $\Delta_N$ | $\Delta_N$ | $\Delta_N$ | $\phi$ |
| $\xi$           | $\Delta_N$ | $\Delta_N$ | $\Delta_N$ | $\Delta_N$ | $\Delta_N$ | $\Delta_N$ | $\Delta_N$ | $\Delta_N$ | $\Delta_N$ | $\xi$ |
| $\xi_1$         | $\Delta_N$ | $\Delta_N$ | $\Delta_N$ | $\Delta_N$ | $\Delta_N$ | $\Delta_N$ | $\Delta_N$ | $\Delta_N$ | $\Delta_N$ | $\xi$ |
| $\chi$          | $\Delta_N$ | $\Delta_N$ | $\Delta_N$ | $\Delta_N$ | $\Delta_N$ | $\Delta_N$ | $\Delta_N$ | $\Delta_N$ | $\Delta_N$ | $\chi$ |
| $\chi_1$        | $\Delta_N$ | $\Delta_N$ | $\Delta_N$ | $\Delta_N$ | $\Delta_N$ | $\Delta_N$ | $\Delta_N$ | $\Delta_N$ | $\Delta_N$ | $\chi_1$ |
| $\nabla_N$      | $\Delta_N$ | $\Delta_N$ | $\Delta_N$ | $\Delta_N$ | $\Delta_N$ | $\Delta_N$ | $\Delta_N$ | $\Delta_N$ | $\Delta_N$ | $\nabla_N$ |

Example 6.10. Here are some finite congruence–distributive examples, thus in which the reticulations are isomorphic to the congruence lattices. Regarding the preservation properties fulfilled by the reticulation, these examples show that there is no embedding relation between the reticulation of an algebra and those of its subalgebras: if $\mathcal{E}$ is the following bounded lattice, then, for instance, $\{0, x, y, 1\} = \mathbb{L}_4 = \mathbb{L}_2 \oplus \mathbb{L}_2 \oplus \mathbb{L}_2$, $\mathcal{D} = \{0, a, x, y, 1\}$ and $\mathcal{P} = \{0, a, x, y, 1\}$ are bounded sublattices of $\mathcal{E}$. We have: $\mathcal{L}(\mathcal{E}) \cong \text{Con}(\mathcal{E}) = \{\Delta_\mathcal{E}, \mu, \nabla_\mathcal{E}\} \cong \mathbb{L}_3$, where $\mathcal{E}/\mu = \{\{0\}, \{a\}, \{x, y\}, \{b\}, \{1\}\}$, $\mathcal{L}(\mathcal{L}_4) \cong \text{Con}(\mathcal{L}_4) = \text{Con}(\mathbb{L}_2 \oplus \mathbb{L}_2 \oplus \mathbb{L}_2) \cong \text{Con}(\mathbb{L}_2)^3 \cong \mathbb{L}_3^3$, $\mathcal{L}(\mathcal{D}) \cong \text{Con}(\mathcal{D}) = \{\Delta_\mathcal{D}, \nabla_\mathcal{D}\} \cong \mathbb{L}_2$ and $\mathcal{L}(\mathcal{P}) \cong \text{Con}(\mathcal{P}) = \{\Delta_\mathcal{P}, \alpha, \beta, \gamma, \nabla_\mathcal{P}\} \cong \mathbb{L}_2 \oplus \mathbb{L}_2$, where $\mathcal{P}/\alpha = \{\{0, x, y\}, \{a, 1\}\}$, $\mathcal{P}/\beta = \{\{0, a\}, \{x, y, 1\}\}$ and $\mathcal{P}/\gamma = \{\{0\}, \{a\}, \{x, y, 1\}\}$.

Remark 6.11. By [24] Lemma 3.3, in any variety, arbitrary intersections commute with arbitrary direct products of congruences. If $\mathcal{C}$ is congruence–modular and $M$ is an algebra from $\mathcal{C}$ such that $A \times M$ has no skew congruences, then $\text{Spec}(A \times M) = \{\phi \times \nabla_M \mid \phi \in \text{Spec}(A)\} \cup \{\nabla_A \times \psi \mid \psi \in \text{Spec}(M)\}$. This follows from Proposition 5.9 in the same way as in the congruence–distributive case, treated in [24] Proposition 3.5.(ii).

Proposition 6.12 (the reticulation preserves finite direct products without skew congruences). Let $M$ be an algebra from $\mathcal{C}$ such that the direct product $A \times M$ has no skew congruences. Then:
(ii) By (i), for all defined and surjective and fulfills:
\[ \zeta \in Cg \]
\[ \alpha \in Con(\mathcal{A} \times M) \]
This also shows that the map \((\alpha, \mu) \rightarrow \alpha \times \mu\) is a lattice isomorphism from \(Con(\mathcal{A}) \times Con(\mathcal{M})\) to \(Con(\mathcal{A} \times M)\).

If \(C\) is congruence–modular and \(\nabla_M \in K(\mathcal{M})\), then:
- for all \(\alpha \in Con(\mathcal{A})\) and all \(\mu \in Con(\mathcal{M})\), \(\rho_{\mathcal{A} \times \mathcal{M}}(\alpha \times \mu) = \rho A(\alpha) \times \rho M(\mu)\);
- \(\equiv_{\mathcal{A} \times \mathcal{M}} \equiv_{\mathcal{A}} \times \equiv_{\mathcal{M}}\) and \(\mathcal{L}(\mathcal{A} \times M) \cong \mathcal{L}(\mathcal{A}) \times \mathcal{L}(\mathcal{M})\).

**Proof.** \(\mathcal{A} \times M\) has no skew congruences, that is \(Con(\mathcal{A} \times M) = \{\alpha \times \mu \mid \alpha \in Con(\mathcal{A}), \mu \in Con(\mathcal{M})\}\).

By Remark 6.11, \(Cg_{\mathcal{A} \times \mathcal{M}}(\mathcal{A} \times M) = \bigcap \{\theta \in Con(\mathcal{A} \times M) \mid \mathcal{X} \times \mathcal{Y} \subseteq \theta\} = \bigcap \{\alpha \times \mu \mid \alpha \in Con(\mathcal{A}), \mu \in Con(\mathcal{M}), \mathcal{X} \times \mathcal{Y} \subseteq \alpha \times \mu\} = (\bigcap \{\alpha \in Con(\mathcal{A}) \mid \mathcal{X} \subseteq \alpha\}) \times (\bigcap \{\mu \in Con(\mathcal{M}) \mid \mathcal{Y} \subseteq \mu\}) = Cg_{\mathcal{A}}(\mathcal{X}) \times Cg_{\mathcal{M}}(\mathcal{Y})\).

Thus, we obtain: \(\mathcal{K}(\mathcal{A} \times M) = \{\mathcal{C}_{\mathcal{A} \times \mathcal{M}}(\{(a_1, u_1), \ldots, (a_n, u_n)\}) \mid n \in \mathbb{N}^*, a_1, \ldots, a_n \in A, u_1, \ldots, u_n \in M\} = \bigcap_{i=1}^n Cg_{\mathcal{A} \times \mathcal{M}}(a_i, u_i)\), hence the expression of \(PCon(\mathcal{A} \times M)\) in the emmunciation. From this and the second statement in 6.11, we obtain: \(\mathcal{K}(\mathcal{A} \times M) = \{\mathcal{C}_{\mathcal{A} \times \mathcal{M}}(\{(a_1, u_1), \ldots, (a_n, u_n)\}) \mid n \in \mathbb{N}^*, a_1, \ldots, a_n \in A, u_1, \ldots, u_n \in M\} = \bigcap_{i=1}^n Cg_{\mathcal{A}}(a_i) \times Cg_{\mathcal{M}}(u_i)\), hence the expression of \(PCon(\mathcal{A} \times M)\) in the emmunciation.
7 Further Results on The Commutator

Throughout this section, we shall assume that $[\cdot, \cdot]_A$ is commutative and distributive w.r.t. arbitrary joins and $\nabla_A \in K(A)$.

**Lemma 7.1.** For all $n \in \mathbb{N}^*$ and all $\alpha, \beta \in A$ we have $[\alpha, [\alpha, \beta]_{A, A}]_{A, A} = [[\alpha, \beta]_A, [\alpha, \beta]_A]_{A, A}$.

**Proof.** Let $\alpha, \beta \in A$. We proceed by induction on $n$. By its definition, $[\alpha, \beta]_A = ([\alpha, \beta]_A, [\alpha, \beta]_A)_A$. Now let $n \in \mathbb{N}^*$ such that $[\alpha, \beta]_{n+1} =([\alpha, \beta]_A, [\alpha, \beta]_A)_{A}$. Then, by the induction hypothesis, $[\alpha, \beta]_{n+2} = ([\alpha, \beta]_{n+1}, [\alpha, \beta]_{n+1})_{A}$. Thus, by the definition of the commutator, $[\alpha, \beta]_{n+1} = ([\alpha, \beta]_{n+1}, [\alpha, \beta]_{n+1})_{A}$. We shall denote by $\beta, \alpha, \alpha \in A$.

**Lemma 7.2.** If the commutator of $A$ is associative, then, for any $n \in \mathbb{N}^*$ and all $\alpha, \beta \in A$, $[\alpha, \beta]_{n+1} = ([\alpha, \beta]_{n+1}, [\alpha, \beta]_{n+1})_{A}$.

**Proof.** Assume that the commutator of $A$ is associative, and let us also use its commutativity, along with Lemma 7.1. Let $\alpha, \beta \in A$. We apply induction on $n$. For $n = 1$, $[\alpha, \beta]_{n+2} = ([\alpha, \beta]_{n+1}, [\alpha, \beta]_{n+1})_{A} = [([\alpha, \beta]_{n+1}, [\alpha, \beta]_{n+1})_{A}]_{A, A} = ([\alpha, \beta]_{n+2}, [\alpha, \beta]_{n+2})_{A, A}$. Then, by the induction hypothesis, $[\alpha, \beta]_{n+2} = ([\alpha, \beta]_{n+1}, [\alpha, \beta]_{n+1})_{A}$.

**Lemma 7.3.** For all $n, k \in \mathbb{N}$ and all $\alpha, \beta, \phi, \psi, \alpha_1, \alpha_2, \ldots, \alpha_k \in A$:

(i) if $\alpha \subseteq \beta$ and $\phi \subseteq \psi$, then $[\alpha, \phi]_{A} \subseteq [\beta, \psi]_{A}$;

(ii) if $\alpha \subseteq \beta$ and $\phi \subseteq \psi$, then $[\alpha, \phi]_{A} \subseteq [\beta, \psi]_{A}$;

(iii) if $\beta \subseteq \alpha$ and $\psi \subseteq \phi$, then $[\alpha, \phi]_{A} \subseteq [\beta, \psi]_{A}$;

(iv) if $\omega \subseteq \alpha$ and $\phi \subseteq \psi$, then $[\alpha, \phi]_{A} \subseteq [\beta, \psi]_{A}$;

(v) if $\omega \subseteq \alpha$ and $\phi \subseteq \psi$, then $[\alpha, \phi]_{A} \subseteq [\beta, \psi]_{A}$;

(vi) if $\omega \subseteq \alpha$ and $\phi \subseteq \psi$, then $[\alpha, \phi]_{A} \subseteq [\beta, \psi]_{A}$;

(vii) if $\omega \subseteq \alpha$ and $\phi \subseteq \psi$, then $[\alpha, \phi]_{A} \subseteq [\beta, \psi]_{A}$.

**Proof.**

(i) By Proposition 3.4, for all $\alpha, \beta \in \mathbb{N}$, $[\alpha, \beta]_{n+1} = ([\alpha, \beta]_{n+1}, [\alpha, \beta]_{n+1})_{A}$, hence the inclusion in the formula.

(ii) By Proposition 3.4, for all $\alpha, \beta \in \mathbb{N}$, $[\alpha, \beta]_{n+1} = ([\alpha, \beta]_{n+1}, [\alpha, \beta]_{n+1})_{A}$, hence the inclusion in the formula.

(iii) By Proposition 3.4, for all $\alpha, \beta \in \mathbb{N}$, $[\alpha, \beta]_{n+1} = ([\alpha, \beta]_{n+1}, [\alpha, \beta]_{n+1})_{A}$, hence the inclusion in the formula.

(iv) By Proposition 3.4, for all $\alpha, \beta \in \mathbb{N}$, $[\alpha, \beta]_{n+1} = ([\alpha, \beta]_{n+1}, [\alpha, \beta]_{n+1})_{A}$, hence the inclusion in the formula.

(v) By Proposition 3.4, for all $\alpha, \beta \in \mathbb{N}$, $[\alpha, \beta]_{n+1} = ([\alpha, \beta]_{n+1}, [\alpha, \beta]_{n+1})_{A}$, hence the inclusion in the formula.

(vi) By Proposition 3.4, for all $\alpha, \beta \in \mathbb{N}$, $[\alpha, \beta]_{n+1} = ([\alpha, \beta]_{n+1}, [\alpha, \beta]_{n+1})_{A}$, hence the inclusion in the formula.

(vii) By Proposition 3.4, for all $\alpha, \beta \in \mathbb{N}$, $[\alpha, \beta]_{n+1} = ([\alpha, \beta]_{n+1}, [\alpha, \beta]_{n+1})_{A}$, hence the inclusion in the formula.

For all $\phi, \psi \in \mathbb{N}$, $[\alpha, \phi]_{A} \subseteq [\beta, \psi]_{A}$ and by $\delta \subseteq \theta \Delta_{A} = \wedge{\alpha \in \mathbb{N}}{[\alpha, \beta]_{A} \subseteq \delta}$.
Remark 7.4. For all \( \theta, \zeta \in \text{Con}(A) \), \( \theta \rightarrow \zeta = \max\{\alpha \in \text{Con}(A) \mid [\theta, \alpha]_A \subseteq \zeta\} \), because, if we denote by 
\( M = \{\alpha \in \text{Con}(A) \mid [\theta, \alpha]_A \subseteq \zeta\} \), then 
\( [\theta, \theta \rightarrow \zeta]_A = [\theta, \bigvee_{\alpha \in M} \alpha]_A = \bigvee_{\alpha \in M} [\theta, \alpha]_A \subseteq \zeta \), hence \( \theta \rightarrow \zeta \in M \).

Lemma 7.5. For all \( \alpha, \beta, \gamma \in \text{Con}(A) \), \( [\alpha, \beta]_A \subseteq \gamma \iff \alpha \subseteq \beta \rightarrow \gamma \).

Proof. \( \Rightarrow \): Let \( \beta \rightarrow \gamma = \bigvee\{\theta \in \text{Con}(A) \mid [\beta, \theta]_A \subseteq \gamma\} \). Since \( [\beta, \alpha]_A = [\alpha, \beta]_A \subseteq \gamma \), it follows that \( \alpha \subseteq \beta \rightarrow \gamma \).

\( \Leftarrow \): Suppose \( \alpha \subseteq \beta \rightarrow \gamma \). For all \( \alpha, \beta, \gamma \in \text{Con}(A) \), then \( [\beta, \theta]_A \subseteq \gamma \), hence \( [\beta, \alpha]_A = [\alpha, \beta]_A \subseteq [\beta, \beta \rightarrow \gamma]_A = [\beta, \bigvee\{\theta \in \text{Con}(A) \mid [\beta, \theta]_A \subseteq \gamma\}]_A = \bigvee\{[\beta, \theta]_A \mid \theta \in \text{Con}(A), [\beta, \theta]_A \subseteq \gamma\} \subseteq \gamma \).

For the following results, recall, also, the equivalences in Proposition 7.6.

Lemma 7.6. For all \( \alpha, \beta \in \text{Con}(A) \) such that \( [\alpha, \nabla A]_A = \alpha: \alpha \rightarrow \beta = \nabla A \iff \alpha \subseteq \beta \).

Proof. \( \alpha \rightarrow \beta = \nabla A \iff \nabla A \subseteq \alpha \rightarrow \beta \iff [\nabla A, \alpha]_A \subseteq \beta \), according to Lemma 7.5.

Remark 7.7. By the above, if \([\cdot, \cdot]_A\) is associative, then \((\text{Con}(A) \cup \nabla, \cap, [\cdot, \cdot]_A, \rightarrow, \Delta_A, \nabla A)\) is a residuated lattice, and, if \( A \) is a congruence–distributive variety, then \((\text{Con}(A) \cup \nabla, \cap, [\cdot, \cdot]_A, \rightarrow, \Delta_A, \nabla A)\) is, moreover, a Gödel algebra.

Proposition 7.8. If \( [\theta, \nabla A]_A = \theta \) for all \( \theta \in \text{Con}(A) \), for any \( \alpha, \beta, \gamma \in \text{Con}(A) \):

(i) if \( \alpha \lor \beta = \nabla A \), then \( [\alpha, \beta]_A = \alpha \lor \beta \); 

(ii) if \( \alpha \lor \beta = \alpha \lor \nabla A = \nabla A \), then \( [\alpha, [\beta, \gamma]_A] = \alpha \lor ([\beta, \gamma]_A) = \nabla A \); 

(iii) if \( \alpha \lor \beta = \nabla A \), then \( [\alpha, \alpha]_A \lor [\beta, \beta]_A = \nabla A \) for all \( n \in \mathbb{N}^* \).

Proof. 

(i) Assume that \( \alpha \lor \beta = \nabla A \). Since \( \alpha, \beta \in \text{Con}(A) \) and \( [\alpha, \beta]_A = \bigvee\{\theta \in \text{Con}(A) \mid [\alpha \lor \beta, \theta]_A \subseteq [\alpha, \beta]_A\} \) and \( [\alpha \lor \beta, \theta, \gamma]_A \subseteq [\alpha, \beta]_A \) and \( [\alpha \lor \beta, \beta]_A \subseteq [\alpha, \beta]_A \), it follows that \( \alpha \subseteq [\alpha \lor \beta] \) and \( \beta \subseteq [\alpha \lor \beta] \), hence \( \nabla A = \alpha \lor \beta \subseteq [\alpha \lor \beta] \), therefore \( \alpha \lor \beta = [\alpha, \beta]_A = \nabla A \), thus \( \alpha \lor \beta \subseteq [\alpha, \beta]_A \) by Lemma 7.6. Since the converse inclusion always holds, it follows that \( \alpha \lor \beta = [\alpha, \beta]_A \).

(ii) Assume that \( \alpha \lor \beta = \alpha \lor \nabla A, \) so that \( [\alpha \lor \beta, \alpha \lor \nabla A] = [\alpha \lor \beta, \alpha \lor \gamma]_A = [\alpha, \alpha]_A \lor [\beta, \gamma]_A \lor [\alpha, [\beta, \gamma]_A] \subseteq [\alpha \lor \beta, \beta \lor \gamma] \subseteq \nabla A \), hence \( [\alpha \lor \beta, \gamma]_A = \alpha \lor ([\beta, \gamma]_A) = \nabla A \).

(iii) We apply induction on \( n \). Assume that \( \alpha \lor \beta = \nabla A \), so that, by (ii), \( \alpha \lor \beta = \nabla A \), thus \( [\alpha, \alpha]_A \lor [\beta, \beta]_A = \nabla A \), hence the implication holds in the case \( n = 1 \). Now, if \( n \in \mathbb{N}^* \) fulfills the implication in the enunciation for all \( \alpha, \beta \in \text{Con}(A) \), and assume that \( \alpha \lor \beta = \nabla A \), so that \( [\alpha, \alpha]_A \lor [\beta, \beta]_A = \nabla A \). Then, by the case \( n = 1 \), it follows that \( [\alpha, \alpha]_A \lor [\beta, \beta]_A = \nabla A \), and \( [\alpha, \alpha]_A \lor [\beta, \beta]_A = [\alpha, \alpha]_A \lor [\beta, \beta]_A = \nabla A \).

Lemma 7.9. If \( [\gamma, \nabla A]_A = \gamma \) for all \( \gamma \in \text{Con}(A) \), then, for all \( \alpha \in \mathcal{B}(\text{Con}(A)) \) and all \( \theta \in \text{Con}(A) \), \( [\alpha, \theta]_A = \alpha \lor \theta \).

Proof. Let \( \theta \in \text{Con}(A) \) and \( \alpha \in \mathcal{B}(\text{Con}(A)) \), so there exists a \( \beta \in \text{Con}(A) \) with \( \alpha \lor \beta = \nabla A \) and \( \alpha \lor \beta = \Delta_A \). Then the following holds: \( [\alpha, \theta]_A \subseteq \alpha \lor \theta = [\nabla A, \alpha \lor \theta]_A = [\alpha \lor \beta, \alpha \lor \theta]_A = [\alpha \lor \beta \lor [\beta, \alpha \lor \theta]_A \subseteq [\alpha, \alpha \lor \theta]_A \lor [\beta, \alpha \lor \theta]_A \subseteq [\alpha, \theta]_A \lor [\beta, \alpha \lor \theta]_A = [\alpha, \theta]_A \lor \Delta_A = [\alpha, \theta]_A \), hence \( [\alpha, \theta]_A = \alpha \lor \theta \). We have followed the argument from Lemma 4.

Remark 7.10. By Lemma 7.9, if \( [\gamma, \nabla A]_A = \gamma \) for all \( \gamma \in \text{Con}(A) \), then, in \( \mathcal{B}(\text{Con}(A)) \), the commutator of \( A \) equals the intersection in particular the intersection in \( \mathcal{B}(\text{Con}(A)) \) is distributive with respect to the join.

Lemma 7.11. (i) If \( f \) is surjective, then:

- \( f(\text{PCon}(A) \cap [\text{Ker}(f)]) \subseteq f(\{\alpha \lor \text{Ker}(f) \mid \alpha \in \text{PCon}(A)\}) = \text{PCon}(B) \);
- \( f(\text{K}(A) \cap [\text{Ker}(f)]) \subseteq f(\{\alpha \lor \text{Ker}(f) \mid \alpha \in \text{K}(A)\}) = \text{K}(B) \);
- If \( C \) is congruence–modular and semi–degenerate, then \( f(\mathcal{B}(\text{Con}(A)) \cap [\text{Ker}(f)]) \subseteq f(\{\alpha \lor \text{Ker}(f) \mid \alpha \in \mathcal{B}(\text{Con}(A))\}) \subseteq \mathcal{B}(\text{Con}(B)) \).

(ii) For all \( \theta \in \text{Con}(A) \):

- \( \{\alpha / \theta \mid \alpha \in \text{PCon}(A) \cap \{\theta\}\} \subseteq \{\alpha / \theta \mid \alpha \in \text{PCon}(A)\} = \text{PCon}(A/\theta) \);
\[
\{\alpha/\theta \mid \alpha \in \mathcal{K}(A) \cap \{\theta\}\} \subseteq \{(\alpha \lor \theta)/\theta \mid \alpha \in \mathcal{K}(A)\} = \mathcal{K}(A/\theta);
\]

if \(C\) is congruence–modular and semi–degenerate, then \(\{\alpha/\theta \mid \alpha \in B(\mathcal{K}(A)) \cap \{\theta\}\} \subseteq \{(\alpha \lor \theta)/\theta \mid \alpha \in B(\mathcal{K}(A))\} \subseteq B(\mathcal{K}(A/\theta))\).

**Proof.** The first inclusion in each statement is trivial. By Lemma 5.12(iii), for the statements on principal and on compact congruences. Now let \(\alpha \in B(\mathcal{K}(A))\), so that \(\alpha \lor \beta = \nabla A\) and \([\alpha,\beta]_A = \Delta A\) for some \(\beta \in \mathcal{K}(A)\), hence, by Lemma 5.17(i), and Remark 5.10, \(f(\alpha \lor \text{Ker}(f)) \lor f(\beta \lor \text{Ker}(f)) = f(\alpha \lor \text{Ker}(f) \lor \beta \lor \text{Ker}(f)) = f(\nabla A) = \nabla B\) and \(f(\alpha \lor \text{Ker}(f)) \lor f(\beta \lor \text{Ker}(f)) = f([\alpha,\beta]_A \lor \text{Ker}(f)) = f(\Delta A) = \Delta B\), therefore \(f(\alpha \lor \text{Ker}(f)) \in B(\mathcal{K}(B))\).

**Proposition 7.12.** (i) Assume that \(f\) is surjective. Then: if \(\nabla A \in \text{PCon}(A)\), then \(\nabla B \in \text{PCon}(B)\), while, if \(\nabla A \in \mathcal{K}(A)\), then \(\nabla B \in \mathcal{K}(B)\).

(ii) \(\nabla A \in \text{PCon}(A)\) iff \(\nabla A/\theta \in \text{PCon}(A/\theta)\) for all \(\theta \in \text{Con}(A)\). \(\nabla A \in \mathcal{K}(A)\) iff \(\nabla A/\theta \in \mathcal{K}(A/\theta)\) for all \(\theta \in \text{Con}(A)\).

**Proof.** (i) By Lemma 7.11(ii). (ii) By (i) for the direct implications, and the fact that \(A/\Delta A\) is isomorphic to \(A\), for the converse implications.

**Lemma 7.13.** If \(C\) is congruence–modular, then, for all \(n \in \mathbb{N}^*\) and any \(\alpha,\beta \in \mathcal{K}(A)\):

(i) if \(f\) is surjective, then \([f(\alpha \lor \text{Ker}(f)), f(\beta \lor \text{Ker}(f))]_B^n = f([\alpha,\beta]_A^n \lor \text{Ker}(f))\);

(ii) for any \(\theta \in \text{Con}(A)\), \([([\alpha \lor \theta]/\theta, ([\beta \lor \theta]/\theta)]_A/\theta^n = ([\alpha,\beta]_A^n \lor \theta)/\theta^n\);

(iii) for any \(\theta \in \text{Con}(A)\) and any \(X,Y \in \mathcal{P}(A^2)\), \([Cg_{A/\theta}(X/\theta), Cg_{A/\theta}(Y/\theta)]_A/\theta^n = ([Cg_A(X), Cg_A(Y)]_A^n \lor \theta)/\theta^n\).

**Proof.** We proceed by induction on \(n\). For \(n = 1\), this holds by Remark 5.10. Now take an \(n \in \mathbb{N}^*\) such that \([f(\alpha \lor \text{Ker}(f)), f(\beta \lor \text{Ker}(f))]_B^n = f([\alpha,\beta]_A^n \lor \text{Ker}(f))\). Then, by the induction hypothesis and Remark 5.10 \([f(\alpha \lor \text{Ker}(f)), f(\beta \lor \text{Ker}(f))]_B^n = f([[\alpha,\beta]_A^n \lor \text{Ker}(f))_B^n = f([\alpha,\beta]_A^n \lor \text{Ker}(f))\) for all \(\theta \in \text{Con}(A)\), hence, if \(C\) is congruence–distibutive, then every member of \(C\) is semiprime.

**Proposition 8.2.** \(A/\rho_A(\Delta A)\) is semiprime.

**Proof.** By Proposition 5.7(iv), and Proposition 5.4(iv), \(\rho_A(\Delta A) = \rho_A(\Delta A)/\rho_A(\Delta A)\).

**Lemma 8.3.** If \(A\) is semiprime, then, for all \(\alpha,\beta \in \text{Con}(A)\):

- \(\lambda_A(\alpha) = 0\) iff \(\alpha = \Delta A\);
- \([\alpha,\beta]_A = \Delta A\) iff \(\alpha \cap \beta = \Delta A\).

**Proof.** Let \(\alpha,\beta \in \text{Con}(A)\). Since \(\lambda_A(\Delta A) = 0\) and \([\alpha,\beta]_A \subseteq \alpha \cap \beta\), the converse implications always hold. Now assume that \(A\) is semiprime. If \(\lambda_A(\alpha) = 0 = \lambda_A(\Delta A)\), then \(\alpha \subseteq \rho_A(\alpha) = \rho_A(\Delta A) = \Delta A\), thus \(\alpha = \Delta A\). If \([\alpha,\beta]_A = \Delta A\), then \(\lambda_A(\alpha \cap \beta) = \lambda_A([\alpha,\beta]_A) = \lambda_A(\Delta A) = 0\), hence \(\alpha \cap \beta = \Delta A\) by the above.

**Lemma 8.4.** For any \(\theta \in \text{Con}(A)\), the following hold:
(i) \( \rho_A(\theta) = \bigvee \{ \alpha \in \text{Con}(A) \mid (\exists k \in \mathbb{N}^+) \{ [\alpha, \alpha]^k_A \subseteq \theta \} \} = \bigvee \{ \alpha \in \mathcal{K}(A) \mid (\exists k \in \mathbb{N}^+) \{ [\alpha, \alpha]^k_A \subseteq \theta \} \} = \bigvee \{ \alpha \in \text{PCon}(A) \mid (\exists k \in \mathbb{N}^+) \{ [\alpha, \alpha]^k_A \subseteq \theta \} \} ; \)

(ii) for any \( \alpha \in \mathcal{K}(A) \), \( \alpha \subseteq \rho_A(\theta) \) iff there exists a \( k \in \mathbb{N}^* \) such that \( [\alpha, \alpha]^k_A \subseteq \theta. \)

Proof. \( \Box \) By Proposition 8.5 and the fact that \( \text{PCon}(A) \subseteq \mathcal{K}(A) \subseteq \text{Con}(A) \), \( \rho_A(\theta) = \bigvee \{ \text{CG}_A(a, b) \mid (a, b) \in A^2, (\exists k \in \mathbb{N}^+) \{ [\text{CG}_A(a, b), \text{CG}_A(a, b)]^k_A \subseteq \theta \} \} = \bigvee \{ \alpha \in \text{PCon}(A) \mid (\exists k \in \mathbb{N}^+) \{ [\alpha, \alpha]^k_A \subseteq \theta \} \} \subseteq \bigvee \{ \alpha \in \mathcal{K}(A) \mid (\exists k \in \mathbb{N}^+) \{ [\alpha, \alpha]^k_A \subseteq \theta \} \} \subseteq \bigvee \{ \alpha \in \text{Con}(A) \mid (\exists k \in \mathbb{N}^+) \{ [\alpha, \alpha]^k_A \subseteq \theta \} \} \subseteq \bigvee \{ \alpha \in \text{PCon}(A) \mid (\exists k \in \mathbb{N}^+) \{ [\alpha, \alpha]^k_A \subseteq \theta \} \} \), where the last inclusion holds because, for any \( \alpha \in \text{Con}(A) \), if \( k \in \mathbb{N}^* \) is such that \( [\alpha, \alpha]^k_A \subseteq \theta \), then, for any \( (a, b) \in \alpha \), \( [\text{CG}_A(a, b), \text{CG}_A(a, b)]^k_A \subseteq [\alpha, \alpha]^k_A \subseteq \theta \), so that

\[ \rho_A(\theta) = \bigvee \{ \alpha \in \text{Con}(A) \mid (\exists k \in \mathbb{N}^+) \{ [\alpha, \alpha]^k_A \subseteq \theta \} \} = \bigvee \{ \alpha \in \mathcal{K}(A) \mid (\exists k \in \mathbb{N}^+) \{ [\alpha, \alpha]^k_A \subseteq \theta \} \} = \bigvee \{ \alpha \in \text{PCon}(A) \mid (\exists k \in \mathbb{N}^+) \{ [\alpha, \alpha]^k_A \subseteq \theta \} \} . \]

The converse implication follows directly from \( \Box \).

For the direct implication, from \( \Box \) it follows that, for any \( \alpha \in \mathcal{K}(A) \) such that \( \alpha \subseteq \rho_A(\theta) \), there exist non–empty families \( (\beta_j)_{j \in J} \subseteq \mathcal{K}(A) \) and \( (k_j)_{j \in J} \subseteq \mathbb{N}^* \) such that \( \alpha \subseteq \bigvee_{j \in J} \beta_j \) and \( [\beta_j, \beta_j]^k_A \subseteq \theta \) for all \( j \in J \). Since \( \alpha \in \mathcal{K}(A) \), it follows that there exist an \( n \in \mathbb{N}^* \) and \( j_1, \ldots, j_n \in J \) such that \( \alpha \subseteq \bigwedge_{i=1}^n \beta_{j_i} \). Let \( j = \max\{j_1, \ldots, j_n\} \in \mathbb{N}^* \). Then \( [\beta_{j_i}, \beta_{j_i}[^k_A \subseteq \theta \) for each \( i \in \{1, n\} \), thus, by Lemma 8.4, \( \alpha \subseteq [\bigwedge_{i=1}^n \beta_{j_i}]^k_A \subseteq \theta. \)

**Proposition 8.5.** (i) \( \rho_A(\Delta_A) = \bigvee \{ \alpha \in \text{Con}(A) \mid (\exists k \in \mathbb{N}^+) \{ [\alpha, \alpha]^k_A = \Delta_A \} \} = \bigvee \{ \alpha \in \mathcal{K}(A) \mid (\exists k \in \mathbb{N}^+) \{ [\alpha, \alpha]^k_A = \Delta_A \} \} = \bigvee \{ \alpha \in \text{PCon}(A) \mid (\exists k \in \mathbb{N}^+) \{ [\alpha, \alpha]^k_A = \Delta_A \} \} ; \)

(ii) for any \( \alpha \in \mathcal{K}(A) \), \( \alpha \subseteq \rho_A(\Delta_A) \) iff there exists a \( k \in \mathbb{N}^* \) such that \( [\alpha, \alpha]^k_A = \Delta_A \).

Proof. By Lemma 8.4 \( \Box \)

**Corollary 8.6.** \( A \) is semiprime iff, for any \( \alpha \in \mathcal{K}(A) \) and any \( k \in \mathbb{N}^* \), if \( [\alpha, \alpha]^k_A = \Delta_A \), then \( \alpha = \Delta_A. \)

Throughout the rest of this section, we shall assume that \( \mathcal{K}(A) \) is closed w.r.t. the commutator of \( A \) and \( [\theta, \nabla_A] = \theta \) for all \( \theta \in \text{Con}(A) \); see also Proposition 8.15.

For any bounded lattice \( L \), we shall denote by \( B(L) \) the set of the complemented elements of \( L \). If \( L \) is distributive, then \( B(L) \) is the Boolean center of \( L \). Although \( \text{Con}(A) \) is not necessarily distributive, we shall call \( B(\text{Con}(A)) \) the **Boolean Center** of \( \text{Con}(A) \). So \( B(\text{Con}(A)) \) is the set of the \( \alpha \in \text{Con}(A) \) such that there exists a \( \beta \in \text{Con}(A) \) which fulfills \( \alpha \vee \beta = \nabla_A \) and \( \alpha \wedge \beta = \Delta_A \), thus also \( [\alpha, \beta] = \Delta_A \).

**Remark 8.7.** Obviously, \( \Delta_A, \nabla_A \in B(\text{Con}(A)) \).

**Lemma 8.8.** \( B(\text{Con}(A)) \subseteq \mathcal{K}(A) \).

Proof. Let \( \alpha \in B(\text{Con}(A)) \), so that \( \alpha \vee \beta = \nabla_A \) and \( \alpha \wedge \beta = \Delta_A \) for some \( \beta \in \text{Con}(A) \). Now let \( \emptyset \neq (\alpha_i)_{i \in I} \subseteq \text{Con}(A) \) such that \( \alpha \subseteq \bigvee_{i \in I} \alpha_i \), so that \( \beta \vee \bigvee_{i \in I} \alpha_i = \nabla_A \in \mathcal{K}(A) \), thus \( \nabla_A = \beta \vee \bigvee_{j=1}^n \alpha_{i_j} \) for some \( n \in \mathbb{N}^* \) and some \( i_1, \ldots, i_n \subseteq I \), hence, by Proposition 8.8, \( \alpha = [\alpha, \nabla_A]_A = [\alpha, \beta \vee \bigvee_{j=1}^n \alpha_{i_j}]_A = [\alpha, \beta]_A \vee [\alpha, \bigvee_{j=1}^n \alpha_{i_j}]_A = \Delta_A \vee [\alpha, \bigvee_{j=1}^n \alpha_{i_j}]_A = [\alpha, \bigvee_{j=1}^n \alpha_{i_j}]_A \subseteq \bigvee_{j=1}^n \alpha_{i_j} \), hence \( \alpha \in \mathcal{K}(A) \).

**Proposition 8.9.** If \( \mathcal{K}(A) = B(\text{Con}(A)) \), then \( \mathcal{L}(A) = B(\mathcal{L}(A)) \).
Lemma 8.10. For any $\sigma, \theta \in \text{Con}(A)$: $\theta^\perp = \bigvee \{ \alpha \in \text{PCon}(A) \mid [\alpha, \theta]_A = \Delta_A \} = \bigvee \{ \alpha \in \text{Con}(A) \mid [\alpha, \theta]_A = \Delta_A \} = \max\{ \alpha \in \text{Con}(A) \mid [\alpha, \theta]_A = \Delta_A \}$, thus: $\sigma \leq \theta^\perp$ iff $[\sigma, \theta]_A = \Delta_A$.

Proof. Let $M = \{ \alpha \in \text{Con}(A) \mid [\alpha, \theta]_A = \Delta_A \}$. For all $\alpha, \beta \in \text{Con}(A)$, $[Cg_A(a, b), \theta]_A \subseteq [\alpha, \theta]_A = \Delta_A$, thus $Cg_A(a, b) \in M \cap \text{PCon}(A)$. Hence $\theta^\perp = \bigvee_{\alpha \in M} \alpha = \bigvee_{\alpha \in M \cap \text{K}(A)} \alpha$. Note, also, that $[\theta^\perp, \theta]_A = \bigvee_{\alpha \in M} [\alpha, \theta]_A = \bigvee_{\alpha \in M} \Delta_A = \Delta_A$, hence $\theta^\perp \in M$, thus $\theta^\perp = \max(M)$. If $\sigma \leq \theta^\perp$, then $[\sigma, \theta]_A \subseteq [\theta^\perp, \theta]_A = \Delta_A$, thus $[\sigma, \theta]_A = \Delta_A$, and conversely: if $[\sigma, \theta]_A = \Delta_A$, then $\sigma \in M$, thus $\sigma \leq \max(M) = \theta^\perp$.

Lemma 8.11. $\lambda_A(\text{B(Con}(A))) = \text{B(Con}(A))/\equiv_A \subseteq \text{B}(\text{L}(A)) \subseteq \text{B}(\text{Con}(A)/\equiv_A)$ and $\lambda_A |_{\text{B(Con}(A))}: \text{B(Con}(A)) \to \text{B}(\text{L}(A))$ is a Boolean morphism.

Proof. $\lambda_A(\text{B(Con}(A))) = \text{B(Con}(A))/\equiv_A$. Now we use Lemma 8.8. Let $\alpha \in \text{B(Con}(A)) \subseteq \text{K}(A)$, so that $\lambda_A(\alpha) \in \text{L}(A)$ and, for some $\beta \in \text{B(Con}(A)) \subseteq \text{K}(A)$, we have $\alpha \vee \beta = \nabla_A$ and $\alpha \wedge \beta = \Delta_A$. Then $\lambda_A(\beta) \in \text{L}(A)$. $1 = \lambda_A(\nabla_A) = \lambda_A(\alpha \vee \beta) = \lambda_A(\alpha) \vee \lambda_A(\beta)$. $0 = \lambda_A(\Delta_A) = \lambda_A(\alpha \wedge \beta) = \lambda_A(\alpha) \wedge \lambda_A(\beta)$, hence $\lambda_A(\beta) \in \text{B}(\text{L}(A))$. Therefore $\lambda_A(\text{B(Con}(A))) \subseteq \text{B}(\text{L}(A))$. Since $\text{L}(A)$ is a bounded sublattice of the bounded distributive lattice $\text{Con}(A)/\equiv_A$, it follows that $\theta^\perp = \text{B}(\text{L}(A))$ is a Boolean subalgebra of $\text{B}(\text{Con}(A)/\equiv_A)$. Hence $\lambda_A(\text{B(Con}(A))) = \text{B}(\text{Con}(A)) = \subseteq \text{B}(\text{L}(A)) \subseteq \text{B}(\text{Con}(A)/\equiv_A)$. $\lambda_A : \text{Con}(A) \to \text{Con}(A)/\equiv_A$ is a (surjective) bounded lattice morphism. Hence $\lambda_A |_{\text{B(Con}(A))}: \text{B(Con}(A)) \to \text{B}(\text{L}(A))$ is well defined and it is a bounded lattice morphism, thus it is a Boolean morphism.

Throughout the rest of this section, $C$ shall be congruence–modular and semi–degenerate.

Proposition 8.12. (i) The Boolean morphism $\lambda_A |_{\text{B(Con}(A))}: \text{B(Con}(A)) \to \text{B}(\text{L}(A))$ is injective.

(ii) If the commutator of $A$ is associative, then $\lambda_A(\text{B(Con}(A))) = \text{B}(\text{L}(A)) = \text{B(Con}(A))/\equiv_A \subseteq \text{B}(\text{Con}(A))/\equiv_A)$ and $\lambda_A |_{\text{B(Con}(A))}: \text{B(Con}(A)) \to \text{B}(\text{L}(A))$ is a Boolean isomorphism.

(iii) If $A$ is semiprime, then $\lambda_A(\text{B(Con}(A))) = \text{B}(\text{L}(A)) = \text{B}(\text{Con}(A))/\equiv_A = \text{B}(\text{Con}(A))/\equiv_A)$ and $\lambda_A |_{\text{B(Con}(A))}: \text{B(Con}(A)) \to \text{B}(\text{L}(A))$ is a Boolean isomorphism.

Proof. By Lemma 8.11 $\lambda_A |_{\text{B(Con}(A))}: \text{B(Con}(A)) \to \text{B}(\text{L}(A))$ is a Boolean morphism. By Remark 5.10 $\lambda_A(\alpha) = 1$ iff $\alpha = \nabla_A$, hence this Boolean morphism is injective.

Assume that $A$ is semiprime, and let $x \in \text{B(Con}(A))/\equiv_A$, so that $x \vee y = 1$ and $x \wedge y = 0$ for some $y \in \text{B(Con}(A))/\equiv_A$. Hence there exist $\alpha, \beta \in \text{Con}(A)$ such that $x = \lambda_A(\alpha)$ and $y = \lambda_A(\beta)$, thus $1 = x \vee y = \lambda_A(\alpha) \vee \lambda_A(\beta) = \lambda_A(\alpha \vee \beta)$. $0 = x \wedge y = \lambda_A(\alpha) \wedge \lambda_A(\beta) = \lambda_A(\alpha \wedge \beta)$, therefore $\alpha \vee \beta = \nabla_A$ and $\alpha \wedge \beta = \Delta_A$, by Remark 5.10 and Lemma 8.3. Hence $\alpha \in \text{B}(\text{L}(A))$, thus $x = \lambda_A(\alpha) = \lambda_A(\text{B(Con}(A))) = \text{B}(\text{Con}(A))/\equiv_A$, therefore, by Lemma 8.11 $\lambda_A(\text{B(Con}(A))) \subseteq \lambda_A(\text{B}(\text{Con}(A))) = \text{B}(\text{Con}(A))/\equiv_A \subseteq \text{B}(\text{L}(A)) \subseteq \text{B}(\text{Con}(A))/\equiv_A)$, hence $\lambda_A(\text{B(Con}(A))) = \text{B}(\text{Con}(A))/\equiv_A = \text{B}(\text{Con}(A))/\equiv_A)$. Therefore $\lambda_A |_{\text{B(Con}(A))}: \text{B(Con}(A)) \to \text{B}(\text{L}(A))$ is surjective, so, by (i), it is a Boolean isomorphism.

Assume that $A$ is semiprime, and let $x \in \text{B}(\text{L}(A)) \subseteq \text{L}(A) = \lambda_A(\text{Con}(A))$, so that $x \vee y = 1$ and $x \wedge y = 0$ for some $y \in \text{B}(\text{Con}(A))/\equiv_A$. Then $\lambda_A(\alpha) \vee \beta = \lambda_A(\alpha) \vee \lambda_A(\beta) = x \vee y = 1 = \lambda_A(\nabla_A)$, hence $\alpha \vee \beta = \nabla_A$. We also have $\lambda_A(\alpha, \beta)_A = \lambda_A(\alpha) \wedge \lambda_A(\beta) = x \wedge y = 0 = \lambda_A(\Delta_A)$, thus $[\alpha, \beta]_A \subseteq \rho_A(\alpha \wedge \beta) = \rho_A(\Delta_A)$, and, since $K$ is closed with respect to the commutator, we have $[\alpha, \beta]_A \subseteq \lambda_A(\text{Con}(A))$, thus, according to Proposition 8.9. Therefore $[\alpha, \beta]_A = \lambda_A([\alpha, \beta]_A) = \Delta_A$ for some $k \in \mathbb{N}^*$; we have applied Lemmas 7.1 and 7.2. Thus $\alpha \vee \beta = \nabla_A$, and, since $\alpha \wedge \beta = \Delta_A$, hence $\alpha \wedge \beta = \Delta_A$, and, since $\alpha \wedge \beta = \Delta_A$, hence $\lambda_A(\text{B(Con}(A))) = \text{B}(\text{Con}(A))/\equiv_A \subseteq \text{B}(\text{L}(A)) \subseteq \text{B}(\text{Con}(A))/\equiv_A)$ by Lemma 8.11. Therefore $\lambda_A |_{\text{B(Con}(A))}: \text{B(Con}(A)) \to \text{B}(\text{L}(A))$ is surjective, so, by (i), it is a Boolean isomorphism.
Lemma 8.13. If A is semiprime and \( \alpha \in \text{Con}(A) \), then: \( \alpha \in \mathcal{B}(\text{Con}(A)) \) iff \( \lambda_A(\alpha) \in \mathcal{B}(\mathcal{L}(A)) \).

Proof. We apply Lemma 8.11 which, first of all, gives us the direct implication. For the converse, assume that \( \lambda_A(\alpha) \in \mathcal{B}(\mathcal{L}(A)) = \mathcal{B}(\text{Con}(A)/\equiv_A) \), so that there exists a \( \beta \in \text{Con}(A) \) with \( \lambda_A(\alpha \lor \beta) = \lambda_A(\alpha) \lor \lambda_A(\beta) = 1 = \lambda_A(\neg \alpha) \) and \( \lambda_A(\alpha \land \beta) = \lambda_A(\alpha) \land \lambda_A(\beta) = 0 \), thus \( \alpha \lor \beta = \neg \alpha \) and \( \alpha \land \beta = \Delta_A \) by Remark 5.10 and Lemma 8.3. Therefore \( \alpha \in \mathcal{B}(\text{Con}(A)) \).

For any \( \Omega \subseteq \text{Con}(A) \), let us consider the property:

\[(A, \Omega) \] for all \( \alpha, \beta \in \Omega \) and all \( n \in \mathbb{N}^* \), there exists a \( k \in \mathbb{N}^* \) such that \( [\alpha, \alpha]_\Omega, [\beta, \beta]_\Omega \subseteq [\alpha, \beta]_\Omega \)

Remark 8.14. By Lemma 7.22 if the commutator of A is associative, then \((A, \text{Con}(A))\) holds.

Notice, from the proof of statement (ii) from Proposition 8.12, that this statement, and thus the fact that \( \lambda_A(\mathcal{B}(\text{Con}(A))) : \mathcal{B}(\text{Con}(A)) \to \mathcal{B}(\mathcal{L}(A)) \) is a Boolean isomorphism, also hold if property \((A, \mathcal{K}(A))\) is fulfilled, instead of the associativity of the commutator of A.

Open problem 8.15. Under the current context, determine whether \((A, \mathcal{K}(A))\) always holds; if it doesn’t, then determine whether \((A, \mathcal{K}(A))\) is equivalent to the associativity of the commutator of A.

Lemma 8.16. \((\mathcal{B}(\text{Con}(A)), \lor, \lceil \cdot \rceil_A) = \lor, \Delta_A, \nabla_A)\) is a Boolean algebra.

Proof. We follow, in part, the argument from Lemma 4. Let \( \alpha, \beta \in \mathcal{B}(\text{Con}(A)) \), so that there exist \( \overline{\alpha}, \overline{\beta} \in \mathcal{B}(\text{Con}(A)) \) such that \( \alpha \lor \beta = \overline{\alpha} \lor \overline{\beta} = \nabla_A \) and \( \alpha \land \beta = \overline{\alpha} \land \overline{\beta} = \Delta_A \). Then, by Remark 7.10, the following holds: \( (\alpha \lor \beta) \lor \overline{\alpha} \lor \overline{\beta} = (\alpha \lor \overline{\alpha}) \lor (\beta \lor \overline{\beta}) = \Delta_A \lor \Delta_A = \Delta_A \) and, since \( \overline{\alpha} \lor \overline{\beta} \subseteq \overline{\Delta_A} \), it follows that \( \alpha \lor \beta \lor \overline{\alpha} \lor \overline{\beta} \lor \overline{\Delta_A} = \Delta_A \lor \Delta_A \lor \overline{\Delta_A} = \Delta_A \lor \Delta_A \). An analogously, \( (\overline{\alpha} \lor \overline{\beta}) \lor \overline{\alpha} \lor \overline{\beta} = \Delta_A \lor \Delta_A \lor \overline{\Delta_A} = \Delta_A \lor \Delta_A \). Hence \( \alpha \lor \beta, \alpha \land \beta \in \mathcal{B}(\text{Con}(A)) \). Clearly, \( \Delta_A, \nabla_A \in \mathcal{B}(\text{Con}(A)) \). Therefore \( \mathcal{B}(\text{Con}(A)) \) is a bounded distributive sublattice of \( \mathcal{B}(\text{Con}(A)) \). By Remark 7.10 it follows that \((\mathcal{B}(\text{Con}(A)), \lor, \lceil \cdot \rceil_A) = \lor, \Delta_A, \nabla_A)\) is a bounded distributive lattice, and, by its definition, it is also complemented, thus it is a Boolean lattice. By a well-known characterization of the complement in a Boolean lattice, for any \( \theta \in \mathcal{B}(\text{Con}(A)) \), the complement of \( \theta \) in \( \mathcal{B}(\text{Con}(A)) \) is \( \overline{\theta} = \max(\alpha \in \mathcal{B}(\text{Con}(A)) | \alpha \land \theta = \Delta_A) \) or \( \max(\alpha \in \mathcal{B}(\text{Con}(A)) | \alpha \lor \theta = \Delta_A) \) or \( \theta \lor \theta^\perp = \nabla_A \). Again by Lemma 8.11 \( \Delta_A = [\theta, \theta^\perp]_\mathcal{A} = \theta \land \theta^\perp \). Therefore \( \theta^\perp \in \mathcal{B}(\text{Con}(A)) \) and \( \theta^\perp \) is the complement of \( \theta \) in \( \mathcal{B}(\text{Con}(A)) \).

For any bounded lattice \( L \) and any \( I \in \text{Id}(L) \), we shall denote by \( \text{Ann}(I) \) the annihilator of \( I \) in \( L \): \( \text{Ann}(I) = \{a \in L | (\forall x \in I) (a \land x = 0)\} \). It is immediate that, if \( L \) is distributive, then \( \text{Ann}(I) \in \text{Id}(L) \). Throughout the rest of this paper, all annihilators shall be considered in the bounded distributive lattice \( \mathcal{L}(A) \), so they shall be ideals of the lattice \( \mathcal{L}(A) \). Recall that \( \mathcal{L}(A) = \mathcal{L}(\mathcal{K}(A)) \).

Lemma 8.17. For any \( \alpha \in \mathcal{K}(A) \):

- \( \text{Ann}(\alpha^*) = \{\lambda_A(\beta) | \beta \in \mathcal{K}(A), \lambda_A(\alpha, \beta) = 0\} \)
- If \( A \) is semiprime, then \( \text{Ann}(\alpha^*) = \{\lambda_A(\beta) | \beta \in \mathcal{K}(A), [\alpha, \beta]_A = \Delta_A\} \).

Proof. By Lemma 8.14 \( \text{Ann}(\alpha^*) = \text{Ann}(\lambda_A(\alpha)) = \{\lambda_A(\beta) | \beta \in \mathcal{K}(A) \land (\forall x \in (\lambda_A(\alpha)))(x \lor \lambda_A(\beta) = 0)\} = \{\lambda_A(\beta) | \beta \in \mathcal{K}(A), \lambda_A(\alpha) \land \lambda_A(\beta) = 0\} = \{\lambda_A(\beta) | \beta \in \mathcal{K}(A), \lambda_A(\alpha, \beta) = 0\} = \{\lambda_A(\beta) | \beta \in \mathcal{K}(A), \lambda_A(\alpha, \beta) = 0\} \). By Lemma 8.3, if \( A \) is semiprime, then, for any \( \beta \in \mathcal{K}(A) \), \( \lambda_A(\alpha, \beta) = 0 \) iff \( [\alpha, \beta]_A = \Delta_A \), hence the second equality in the enumeration.

Lemma 8.18. For any \( \alpha \in \text{Con}(A) \) and any \( I \in \text{Id}(\mathcal{L}(A)) \), if \( \text{Ann}(\alpha^*) \subseteq I \), then \( \alpha^+ \subseteq I^* \). If \( A \) is semiprime and \( \alpha \in \mathcal{K}(A) \), then the converse implication holds, as well.

Proof. For the direct implication, assume that \( \text{Ann}(\alpha^*) \subseteq I \) and let \( \beta \in \mathcal{K}(A) \) such that \( [\alpha, \beta]_A = \Delta_A \), hence \( \lambda_A(\alpha) \land \lambda_A(\beta) = \lambda_A(\alpha, \beta) = \lambda_A(\alpha, \beta) = 0 \). Now let \( x \in \alpha^* \), so that \( x = \lambda_A(\gamma) \) for some \( \gamma \in \mathcal{K}(A) \) with \( \gamma \subseteq \alpha \). Then \( x = \lambda_A(\gamma) \subseteq \lambda_A(\alpha) \lor \lambda_A(\beta) \lor \lambda_A(\gamma) \lor \lambda_A(\beta) \leq \lambda_A(\alpha) \land \lambda_A(\beta) = 0 \), so \( x \lor \lambda_A(\beta) = 0 \), thus \( \lambda_A(\beta) \in \text{Ann}(\alpha^*) \subseteq I \), therefore \( \beta \subseteq I^* \) by Lemma 5.10. According to Lemma 8.10 \( \alpha^+ = \lor \beta \in \mathcal{K}(A) | [\alpha, \beta]_A = \Delta_A \subseteq \mathcal{L}(A) \).

For the converse implication, assume that \( A \) is semiprime, \( \alpha \in \mathcal{K}(A) \) and \( \alpha^+ \subseteq I^* \), and let \( x \in \text{Ann}(\alpha^*) \), which means that \( x = \lambda_A(\beta) \) for some \( \beta \in \mathcal{K}(A) \) with \( [\alpha, \beta]_A = \Delta_A \), according to Lemma 8.17. Hence, by Lemmas 8.10 and 5.10 \( \beta \subseteq \alpha^+ \subseteq I^* \), thus \( x = \lambda_A(\beta) \in I \), therefore \( \text{Ann}(\alpha^+) \subseteq I \).
Proposition 8.24.

For any \( \theta \in \text{Con}(A) \):

(i) \( (\theta^\perp)^* \subseteq \text{Ann}(\theta^*) \);

(ii) if \( A \) is semiprime, then \( (\theta^\perp)^* = \text{Ann}(\theta^*) \).

Proof. \( (\theta^\perp)^* = \{ \lambda_A(\alpha) \mid \alpha \in K(A), \alpha \subseteq \theta^\perp \} = \{ \lambda_A(\alpha) \mid \alpha \in K(A), [\alpha, \theta]_A = \Delta_A \} \), by Lemma 8.10. \( \text{Ann}(\theta^*) = \{ \lambda_A(\alpha) \mid \alpha \in K(A), (\forall x \in \theta^*) (\lambda_A(\alpha) \wedge x = \lambda_A(\Delta_A)) \} = \{ \lambda_A(\alpha) \mid \alpha \in K(A), (\forall \beta \in K(A)) (\beta \subseteq \theta \Rightarrow \lambda_A([\alpha, \beta]_A) = \lambda_A(\alpha) \wedge \lambda_A(\beta) = \lambda_A(\Delta_A)) \} = \{ \lambda_A(\alpha) \mid \alpha \in K(A), (\forall \beta \in K(A)) (\beta \subseteq \theta \Rightarrow \rho_A([\alpha, \beta]_A) = \rho_A(\Delta_A)) \} \).

(i) Let \( \alpha \in K(A) \) such that \( \lambda_A(\alpha) \in (\theta^\perp)^* \), which means that \( [\alpha, \theta]_A = \Delta_A \). Then, for any \( \beta \in K(A) \) fulfilling \( \beta \subseteq \theta \), we have \( [\alpha, \beta]_A \subseteq [\alpha, \theta]_A = \Delta_A \), so \( [\alpha, \beta]_A = \Delta_A \), thus \( \rho_A([\alpha, \beta]_A) = \rho_A(\Delta_A) \), hence \( \lambda_A(\alpha) \in \text{Ann}(\theta^*) \). Therefore \( (\theta^\perp)^* \subseteq \text{Ann}(\theta^*) \).

(ii) Assume that \( A \) is semiprime and \( \alpha \in K(A) \) such that \( \lambda_A(\alpha) \in \text{Ann}(\theta^*) \), which means that, for all \( \beta \in K(A) \) such that \( \beta \subseteq \theta \), \( [\alpha, \beta]_A \subseteq \rho_A([\alpha, \beta]_A) = \rho_A(\Delta_A) = \Delta_A \), so \( [\alpha, \beta]_A = \Delta_A \). Thus, \( \theta = \bigvee_{(a,b) \in \theta} Cg_A(a, b) \subseteq \bigvee_{\beta \in K(A) \mid \beta \subseteq \theta} \{ \alpha, [\beta \in K(A) \mid \beta \subseteq \theta]_A = \bigvee_{\beta \in K(A), [\beta]_A \subseteq \theta} \{ \lambda_A(\alpha) \mid \beta \in K(A), \beta \subseteq \theta \} = \bigvee_{\Delta_A \mid \beta \in K(A), \beta \subseteq \theta} \Delta_A = \Delta_A \), therefore \( \lambda_A(\alpha) \in (\theta^\perp)^* \), hence \( \text{Ann}(\theta^* \subseteq (\theta^\perp)^* \).

\( \Box \)

Proposition 8.20.

For any \( I \in \text{Id}(\mathcal{L}(A)) \):

(i) \( (I_s)^\perp \subseteq \text{Ann}(I)^* \);

(ii) if \( A \) is semiprime, then \( (I_s)^\perp = \text{Ann}(I)^* \).

Proof. \( (I_s)^\perp = \bigvee \{ \alpha \in K(A) \mid [\alpha, I_s]_A = \Delta_A \} = \bigvee \{ \alpha \in K(A) \mid [\alpha, \bigvee \{ \beta \in K(A) \mid \lambda_A(\beta) \in I \}]_A = \Delta_A \} = \bigvee \{ \alpha \in K(A) \mid \bigwedge \{ [\alpha, \beta]_A \in K(A) \mid [\beta]_A \subseteq \theta \} = \Delta_A \} = \bigvee \{ \alpha \in K(A) \mid (\forall \beta \in K(A), \lambda_A(\alpha) \in \text{Ann}(I)) (\lambda_A(\beta) \in [\alpha, \beta]_A = \Delta_A) \} \). \( \text{Ann}(I)^* = \{ \alpha \in K(A) \mid \lambda_A(\alpha) \in \text{Ann}(I) \} = \{ \alpha \in K(A) \mid (\forall \beta \in K(A)) (\lambda_A(\alpha) \in \text{Ann}(I) \Rightarrow \lambda_A(\beta) \in [\alpha, \beta]_A = \Delta_A) \} \).

For all \( \alpha, \beta \in \text{Con}(A) \), if \( [\alpha, \beta]_A = \Delta_A \), then \( \lambda_A([\alpha, \beta]_A) = \lambda_A(\Delta_A) \), hence \( (I_s)^\perp \subseteq \text{Ann}(I)^* \).

(ii) If \( A \) is semiprime, then, for every \( \alpha \in \text{Con}(A) \), \( \lambda_A([\alpha, \beta]_A) = \lambda_A(\Delta_A) \) implies \( [\alpha, \beta]_A \subseteq \rho_A([\alpha, \beta]_A) = \rho_A(\Delta_A) = \Delta_A \), thus \( [\alpha, \beta]_A = \Delta_A \), hence \( \text{Ann}(I_s) \subseteq (I_s)^\perp \). By (i), it follows that \( (I_s)^\perp = \text{Ann}(I_s)^* \).

We call \( A \) a hyperarchimedean algebra iff, for all \( \alpha \in \text{PCon}(A) \), there exists an \( n \in \mathbb{N}^* \) such that \( [\alpha, \alpha]_A^n \in B(\text{Con}(A)) \).

Remark 8.21.

If \( \alpha \in \text{Con}(A) \) and \( n \in \mathbb{N}^* \) are such that \( [\alpha, \alpha]_A^n \in B(\text{Con}(A)) \), then, by Remark 7.10, \( [\alpha, \alpha]_A^{n+1} = [\alpha, \alpha]_A^n \cap [\alpha, \alpha]_A^n = [\alpha, \alpha]_A^n \), for all \( k \in \mathbb{N} \) such that \( k \geq n \).

Remark 8.22.

If \( [\alpha, \alpha]_A \in B(\text{Con}(A)) \) for all \( \alpha \in \text{PCon}(A) \), then \( A \) is hyperarchimedean. Thus, if \( \text{PCon}(A) \subseteq B(\text{Con}(A)) \) and \( A \) has principal commutators, then \( A \) is hyperarchimedean. If the commutator of \( A \) equals the intersection, for instance if \( \mathcal{L} \) is congruence–distributive, then: \( A \) is hyperarchimedean iff \( \text{PCon}(A) \subseteq B(\text{Con}(A)) \).

By Lemmas 8.10 and 8.8 the following equivalences hold: \( \text{PCon}(A) \subseteq B(\text{Con}(A)) \iff \text{K}(A) \subseteq B(\text{Con}(A)) \iff \mathcal{L}(A) = B(\mathcal{L}(A)) \).

Remark 8.23.

By Lemma 8.10, the lattice \( \text{Con}(A) \) is Boolean iff \( \text{Con}(A) = B(\text{Con}(A)) \), which implies that the commutator of \( A \) equals the intersection, according to Remark 7.10 and thus, since \( \text{PCon}(A) \subseteq \text{Con}(A) = B(\text{Con}(A)) \), \( A \) is hyperarchimedean, while Remark 8.1 ensures us that \( A \) is semiprime. From Lemma 8.8, we obtain the following equivalences: \( \text{Con}(A) \) is a Boolean lattice iff \( B(\text{Con}(A)) = \text{Con}(A) \iff \text{B}(\text{Con}(A)) = \text{K}(A) = \text{Con}(A) \). Of course, since \( \mathcal{L}(A) \) is a bounded distributive lattice, \( \mathcal{L}(A) \) is a Boolean algebra iff \( \mathcal{L}(A) = B(\mathcal{L}(A)) \).

Proposition 8.24.

(i) If \( A \) is semiprime, then \( \mathcal{K}(A) = B(\text{Con}(A)) \iff \mathcal{L}(A) = B(\mathcal{L}(A)) \).

(ii) If \( \text{Con}(A) \) is a Boolean lattice, then \( A \) is hyperarchimedean and semiprime and \( \mathcal{L}(A) \) is isomorphic to \( \text{Con}(A) \), in particular \( \mathcal{L}(A) \) is a Boolean lattice, as well.
Proof. by Proposition \[8.9\] the direct implication is Proposition \[8.9\]. For the converse, let \(\alpha \in \mathcal{K}(A)\), so that \(\lambda_A(\alpha) \in \mathcal{L}(A) = \mathcal{B}(\mathcal{L}(A))\), thus \(\alpha \in \mathcal{B}((\mathcal{C}_{\mathcal{K}(A)})\) by Lemma \[8.8\]. Hence \(\mathcal{K}(A) \subseteq \mathcal{B}((\mathcal{C}_{\mathcal{K}(A)})\), thus \(\mathcal{K}(A) = \mathcal{B}((\mathcal{C}_{\mathcal{K}(A)})\) by Lemma \[8.8\].

By Lemma \[8.28\] we obtain that \(A\) is hyperarchimedean and semi-prime, and \(\mathcal{B}((\mathcal{C}_{\mathcal{K}(A)}) = \mathcal{K}(A) = \mathcal{C}(A)\), hence \(\mathcal{L}(A) = \mathcal{B}(\mathcal{L}(A))\) by Proposition \[8.9\] and thus \(\lambda_A : \mathcal{C}(A) = \mathcal{B}((\mathcal{C}_{\mathcal{K}(A)}) \to \mathcal{B}(\mathcal{L}(A)) = \mathcal{L}(A)\) is a Boolean isomorphism, according to Proposition \[8.12\].

\[8.25\] Lemma. If \(A\) is hyperarchimedean, then \(A/\theta\) is hyperarchimedean for all \(\theta \in \mathcal{K}(A)\).

Proof. Let \(\theta \in \mathcal{K}(A)\). For any \(a, b \in A\), there exists an \(n \in \mathbb{N}^+\) such that \([C_G(a, b), C_G(a, b)]^n_A \in \mathcal{B}((\mathcal{C}_{\mathcal{K}(A)})\). Then, according to Lemma \[8.13\] (iii), and Lemma \[8.11\] (iv), \([C_G(\alpha, \beta) , C_G(\alpha, \beta)]^n_A / \theta = \langle ([C_G(a, b), C_G(a, b)]^n_A \lor \theta) / \theta \in \mathcal{B}((\mathcal{C}_{\mathcal{K}(A)})\rangle\), therefore \(A/\theta\) is hyperarchimedean.

\[8.26\] Lemma. If \(A\) is hyperarchimedean, then \(\mathcal{L}(A)\) is a Boolean lattice.

Proof. Let \(\theta \in \mathcal{K}(A)\), so that \(\theta = \alpha_1 \lor \ldots \lor \alpha_n\) for some \(n \in \mathbb{N}^+\) and \(\alpha_1, \ldots, \alpha_n \in \mathcal{P}(\mathcal{C}_{\mathcal{K}(A)})\). Since \(A\) is hyperarchimedean, there exists a \(k \in \mathbb{N}^+\) such that, for all \(i \in \mathbb{N}^+\), \([\alpha_i, \alpha_i]^k_A \in \mathcal{B}((\mathcal{C}_{\mathcal{K}(A)})\), thus \(\lambda_A(\alpha_i) = \lambda_A([\alpha_i, \alpha_i]^k_A) \in \lambda_A(\mathcal{B}((\mathcal{C}_{\mathcal{K}(A)})\) \subseteq \mathcal{B}(\mathcal{L}(A))\) by Lemma \[8.11\] so that \(\lambda_A(\theta) = \lambda_A(\alpha_1) \lor \ldots \lor \lambda_A(\alpha_n) \in \mathcal{B}(\mathcal{L}(A))\). Hence \(\lambda_A(\mathcal{K}(A)) = \mathcal{L}(A) \subseteq \mathcal{B}(\mathcal{L}(A))\), thus \(\mathcal{L}(A) = \mathcal{B}(\mathcal{L}(A))\), so \(\mathcal{L}(A)\) is a Boolean lattice.

9 A Reticulation Functor

Throughout this section, \(\mathcal{C}\) shall be congruence-modular and semi-degenerate and such that, in each of its members, the set of the compact congruences is closed w.r.t. the commutator. Also, the morphism \(f : A \to B\) shall be surjective, so that the map \(\varphi_f : \mathcal{C}(A) \to \mathcal{C}(B), \varphi_f(\alpha) = f(\alpha \lor \ker(f))\) for all \(\alpha \in \mathcal{C}(A)\), is well defined.

Remark 9.1. By Lemma \[8.11\] (i), \(\varphi_f(\mathcal{K}(A)) = \mathcal{K}(B)\).

For any algebra \(M\) from \(\mathcal{C}\) and any \(X \subseteq M^2\), let us denote \(V_M(X) = V_M(C_G(M)(X))\). Then, by the proof of Proposition \[2.1\] and Lemma \[8.14\] (ii), for all \(\alpha \in \mathcal{C}(A)\), \(\{f(\phi) \mid \phi \in \mathcal{V}_A(\alpha)\} = f(\mathcal{V}_A(\alpha)) = \mathcal{V}_B(C_G(f(\alpha))) = \mathcal{V}_B(\varphi_f(\alpha))\).

\[
\begin{array}{ccc}
\mathcal{C}(A) & \xrightarrow{\varphi_f} & \mathcal{C}(B) \\
\uparrow \quad \uparrow & & \uparrow \quad \uparrow \\
\mathcal{K}(A) & \xrightarrow{\lambda_A} & \mathcal{K}(B) \\
\downarrow \quad \downarrow & & \downarrow \quad \downarrow \\
\mathcal{L}(A) & \xrightarrow{\mathcal{L}(f)} & \mathcal{L}(B) \\
\end{array}
\]

Let us define \(\mathcal{L}(f) : \mathcal{L}(A) \to \mathcal{L}(B)\), for all \(\alpha \in \mathcal{K}(A)\), \(\mathcal{L}(f)(\hat{\alpha}) = \varphi_f(\alpha)\), that is \(\mathcal{L}(f)(\lambda_A(\alpha)) = \lambda_B(f(\alpha \lor \ker(f)))\).

Proposition 9.2. \(\mathcal{L}(f)\) is well defined and it is a surjective lattice morphism.

Proof. By Remark \[8.1\] the restriction \(\varphi_f \mid_{\mathcal{K}(A)} : \mathcal{K}(A) \to \mathcal{K}(B)\) is well defined and surjective. Let \(\alpha, \beta \in \mathcal{K}(A)\) such that \(\lambda_A(\alpha) = \lambda_B(\beta)\), so that \(\rho_A(\alpha) = \rho_B(\beta)\), thus \(V_A(\alpha) = V_B(\beta)\), hence \(V_B(\varphi_f(\alpha)) = f(V_A(\alpha)) = V_B(\varphi_f(\beta))\), thus \(\rho_B(\varphi_f(\alpha)) = \rho_B(\varphi_f(\beta))\), so \(\lambda_B(\varphi_f(\alpha)) = \lambda_B(\varphi_f(\beta))\), that is \(\mathcal{L}(f)(\lambda_A(\alpha)) = \mathcal{L}(f)(\lambda_A(\beta))\); we have used Proposition \[8.3\] (iii), and Remark \[8.1\]. Hence \(\mathcal{L}(f)\) is well defined. \(\lambda_B : \mathcal{K}(B) \to \mathcal{K}(B)\) \(\mathcal{L}(f)\mid_{\mathcal{K}(A)} : \mathcal{K}(A) \to \mathcal{K}(B)\) are surjective, thus so is their composition, and, since \(\mathcal{L}(f) \circ \lambda_A = \lambda_B \circ \varphi_f\), it follows that \(\mathcal{L}(f)\) is surjective.

By Remark \[8.1\] (ii), \(\mathcal{L}(f)\mid_{\mathcal{K}(A)} : \mathcal{K}(A) \to \mathcal{K}(B)\) is a lattice morphism.
Remark 9.3. Clearly, if $C$ is an algebra from $C$ and $g: B \to C$ is a surjective morphism in $C$, then $\mathcal{L}(g \circ f) = \mathcal{L}(g) \circ \mathcal{L}(f)$. Hence we have defined a covariant functor $\mathcal{L}$ from the partial category of $C$ whose morphisms are exactly the surjective morphisms from $C$ to the partial category of the category $\mathcal{D}01$ of bounded distributive lattices whose morphisms are exactly the surjective morphisms from $\mathcal{D}01$.

Open problem 9.4. Extend the definition of $\mathcal{L}$ to the whole category $C$, with the image in $\mathcal{D}01$, of course.

Remark 9.5. By Proposition 6.2, if $C$ is congruence–distributive, then we may take $\mathcal{L}(f) = \varphi_f |_{K(A)}: K(A) \to K(B)$, with $K(A)$ and $K(B)$ bounded sublattices of $\text{Con}(A)$ and $\text{Con}(B)$, respectively.

For any bounded lattice morphism $h: L \to M$, let us denote by $\text{Ker}_{\text{Id}}(h) = h^{-1}(\{0\}) = \{x \in L \mid h(x) = 0\} \in \text{Id}(L)$, so that $L/\text{Ker}_{\text{Id}}(h) \cong h(L)$ by the Main Isomorphism Theorem (for lattices and lattice ideals).

Proposition 9.6 (the reticulation preserves quotients). For any $\theta \in \text{Con}(A)$, the lattices $\mathcal{L}(A/\theta)$ and $\mathcal{L}(A)/\theta^*$ are isomorphic.

Proof. Recall that $\theta^* = \lambda_A(K(A) \cap \langle \theta \rangle) = \{\hat{\alpha} \mid \alpha \in K(A), \alpha \subseteq \theta\} \in \text{Id}(\mathcal{L}(A))$. $p_\theta: A \to A/\theta$ is a surjective morphism in $C$, so we can apply the construction above:

$$
\begin{array}{ccc}
\mathcal{L}(A) & \xrightarrow{\varphi_{p_\theta}} & \mathcal{L}(A/\theta) \\
\mathcal{K}(A) & \xrightarrow{\varphi_{p_\theta}} & \mathcal{K}(A/\theta) \\
\lambda_A & \xrightarrow{\mathcal{L}(p_\theta)} & \lambda_{A/\theta} \\
\end{array}
$$

For all $\alpha \in \text{Con}(A)$, $\varphi_{p_\theta}(\alpha) = p_\theta(\alpha \vee \text{Ker}(p_\theta)) = (\alpha \vee \theta)/\theta$, so, for all $\alpha \in K(A)$, $\mathcal{L}(p_\theta)(\hat{\alpha}) = (\alpha \vee \theta)/\theta \in \mathcal{L}(A/\theta)$. Thus, for any $\alpha \in \mathcal{K}(A)$, $\hat{\alpha} \in \text{Ker}_{\text{Id}}(\mathcal{L}(p_\theta))$ iff $\mathcal{L}(p_\theta)(\hat{\alpha}) = \hat{\Delta}_{A/\theta}$ iff $(\alpha \vee \theta)/\theta = \hat{\theta}/\theta$, that is $\lambda_{A/\theta}(\alpha \vee \theta)/\theta = \lambda_A(\alpha \vee \theta)/\theta$, iff $\rho_{A/\theta}(\alpha \vee \theta)/\theta = \rho_A(\alpha \vee \theta)/\theta$ iff $\rho_A(\alpha \vee \theta)/\theta \in \text{Ker}_{\text{Id}}(\mathcal{L}(p_\theta))$. Proposition 9.2 ensures us that the lattice morphism $\mathcal{L}(p_\theta)$ is surjective, so, from the Main Isomorphism Theorem, we obtain: $\mathcal{L}(A/\theta) \cong \mathcal{L}(A)/\theta^*$.

Proposition 9.7. The lattices $\mathcal{L}(A)$ and $\mathcal{L}(A/\rho_A(\Delta_A))$ are isomorphic.

Proof. By Corollary 5.29, and Proposition 4.10, $\rho_A(\Delta_A)^* = \Delta_A^*$, hence the lattice $\mathcal{L}(A/\rho_A(\Delta_A))$ is isomorphic to $\mathcal{L}(A)/\rho_A(\Delta_A)^* = \mathcal{L}(A)/\Delta_A$, which, in turn, is isomorphic to $\mathcal{L}(A/\Delta_A)$, and thus to $\mathcal{L}(A)$, since the algebras $A/\Delta_A$ and $A$ are isomorphic.

Remark 9.8. Propositions 8.2 and 9.7 show that the reticulation of any algebra $M$ from a semi–degenerate congruence–modular variety, such that $K(M)$ is closed with respect to the commutator of $M$ and $\nabla_M \in K(M)$, is isomorphic to the reticulation of a semiprime algebra from the same variety.

Corollary 9.9. $\mathcal{B}(\mathcal{L}(A))$ and $\mathcal{B}(\text{Con}(A/\rho_A(\Delta_A)))$ are isomorphic Boolean algebras.

Proof. By Propositions 8.2, 8.12 and 9.7, $A/\rho_A(\Delta_A)$ is semiprime, thus the Boolean algebra $\mathcal{B}(\text{Con}(A/\rho_A(\Delta_A)))$ is isomorphic to $\mathcal{B}(\mathcal{L}(A/\rho_A(\Delta_A)))$, which in turn is isomorphic to $\mathcal{B}(\mathcal{L}(A))$.

Recall the well–known Nachbin's Theorem, which states that, given a bounded distributive lattice $L$, we have: $L$ is a Boolean algebra if $\text{Max}_{\text{Id}}(L) = \text{Spec}_{\text{Id}}(L)$ if $\text{Max}_{\text{Filt}}(L) = \text{Spec}_{\text{Filt}}(L)$.

Proposition 10.10. The following are equivalent:

(i) $A$ is hyperarchimedean;
(ii) $A/\rho_A(\Delta_A)$ is hyperarchimedean;
(iii) $\text{Max}(A) = \text{Spec}(A)$;
(iv) $\mathcal{L}(A)$ is a Boolean lattice;
(v) the lattice $\mathcal{L}(A)$ is isomorphic to $\mathcal{B}(\text{Con}(A))$;
(vi) the lattice $\mathcal{L}(A)$ is isomorphic to $\mathcal{B}({\text{Con}}(A/\rho_A(\Delta_A)))$.

Proof. By Nachbin’s Theorem, Proposition 5.22 and Corollary 5.26 (iii) is equivalent to (iv). Trivially, (vi) implies (iv), while the converse holds by Corollary 9.9.

If $A$ is semiprime, that is $\rho_A(\Delta_A) = \Delta_A$, so that $A/\rho_A(\Delta_A) = A/\Delta_A$ is isomorphic to $A$, then (i) is equivalent to (iii) and (vi) is equivalent to (vi). Now let us drop the condition that $A$ is semiprime. But $A/\rho_A(\Delta_A)$ is semiprime, according to Proposition 8.2, hence, by the above, (ii) is equivalent to $\text{Max}(A/\rho_A(\Delta_A)) = \text{Spec}(A/\rho_A(\Delta_A))$ and to the fact that $\mathcal{L}(A/\rho_A(\Delta_A))$ is a Boolean lattice, which, in turn, is equivalent to (iv) by Proposition 9.7. But, as shown by Lemma 3.12 and Remark 4.5, $\text{Max}(A/\rho_A(\Delta_A)) = \text{Spec}(A/\rho_A(\Delta_A))$ iff $\text{Max}(A) \cap [\rho_A(\Delta_A)] = \text{Spec}(A) \cap [\rho_A(\Delta_A)]$ iff $\text{Max}(A) = \text{Spec}(A)$, since $\text{Max}(A) \subseteq \text{Spec}(A) \subseteq [\rho_A(\Delta_A)]$.

References

[1] P. Agliano, Prime Spectra in Modular Varieties, *Algebra Universalis* 30 (1993), 581–597.
[2] P. Agliano, A. Ursini, On Subtractive Varieties, II: General Properties, *Algebra Universalis* 36, Issue 2 (June 1996), 222–259.
[3] P. Agliano, A. Ursini, On Subtractive Varieties, III: from Ideals to Congruences, *Algebra Universalis* 37 (1997), 296–333.
[4] K. A. Baker, Primitive Satisfaction and Equational Problems for Lattices and Other Algebras, *Trans. Amer. Math. Soc.* 190 (1974), 125–150.
[5] R. Balbes, P. Dwinger, *Distributive Lattices*, University of Missouri Press, Columbia, Missouri, 1974.
[6] L. P. Belluce, Semisimple Algebras of Infinite Valued Logic and Bold Fuzzy Set Theory, *Can. J. Math.* 38, No. 6 (1986), 1356–1379.
[7] L. P. Belluce, Spectral Spaces and Non–commutative Rings, *Communications in Algebra* 19, Issue 7 (1991), 1855–1865.
[8] L. P. Belluce, Semisimple Algebras of Infinite Valued Logic and Bold Fuzzy Set Theory, *Canadian Journal of Mathematics* 38 (1986), 1356–1379.
[9] L. P. Belluce, A. DiNola, A. Lettieri, Subalgebras, Direct Products and Associated Lattices of MV–algebras, *Glasgow Math. J.* 34 (1992), 301–307.
[10] W. J. Blok, D. Pigozzi, On the Structure of Varieties with Equationally Definable Principal Congruences I, *Algebra Universalis* 15 (1982), 195–227.
[11] T. S. Blyth, *Lattices and Ordered Algebraic Structures*, Springer–Verlag London Limited, 2005.
[12] S. Burris, H. P. Sankappanavar, *A Course in Universal Algebra*, Graduate Texts in Mathematics, 78, Springer–Verlag, New York–Berlin (1981).
[13] D. Bușneag, D. Piciu, The Belluce–lattice Associated with a Bounded Hilbert Algebra, *Soft Computing* 19, Issue 11 (November 2015), 3031–3042.
[14] J. L. Castiglioni, M. Menni, W. J. Zuluaga Botero, A Representation Theorem for Integral Rings and Its Applications to Residuated Lattices, *Journal of Pure and Applied Algebra* 220 (2016), 3533–3566.
[15] I. Chajda, A congruence Modular Variety that Is Neither Congruence Distributive Nor 3-permutable, *Soft Computing* 17 (2013), 1467–1469.
[16] D. Cheptea, Reticulation of the Hoops, in preparation.
[17] P. Crawley, R. P. Dilworth, *Algebraic Theory of Lattices*, Prentice Hall, Englewood Cliffs (1973).
[18] J. Czelakowski, *The Equationally-defined Commutator. A Study in Equational Logic and Algebra*, Birkhäuser Mathematics, 2015.
[19] J. Czelakowski, Additivity of the Commutator and Residuation, *Reports on Mathematical Logic* 43 (2008), 109–132.

[20] A. DiNola, G. Georgescu, L. Leuștean, Boolean Products of BL–algebras, *Journal of Mathematical Analysis and Applications* 251, Issue 1 (November 2000), 106–131.

[21] R. Freese, R. McKenzie, *Commutator Theory for Congruence–modular Varieties*, London Mathematical Society Lecture Note Series 125, Cambridge University Press, 1987.

[22] N. Galatos, P. Jipsen, T. Kowalski, H. Ono, *Residuated Lattices: An Algebraic Glimpse at Substructural Logics, Studies in Logic and The Foundations of Mathematics* 151, Elsevier, Amsterdam/ Boston /Heidelberg /London /New York /Oxford /Paris /San Diego / San Francisco /Singapore /Sydney /Tokyo, 2007.

[23] G. Georgescu, I. Voiculescu, Some Abstract Maximal Ideal–like Spaces, *Algebra Universalis* 26 (1989), 90–102.

[24] G. Georgescu, C. Mureșan, Congruence Boolean Lifting Property, to appear in *Journal of Multiple–valued Logic and Soft Computing*, [arXiv:1502.06907](http://arxiv.org/abs/1502.06907) [math.LO].

[25] G. Georgescu, C. Mureșan, Going Up and Lying Over in Congruence–modular Algebras, [arXiv:1608.04985](http://arxiv.org/abs/1608.04985) [math.RA].

[26] G. Grätzer, *General Lattice Theory*, Birkhäuser Akademie–Verlag, Basel–Boston–Berlin (1978).

[27] G. Grätzer, *Universal Algebra*, Second Edition, Springer Science+Business Media, LLC, New York, 2008.

[28] P. Jipsen, C. Tsinakis, A Survey of Residuated Lattices, *Ordered Algebraic Structures*, Kluwer Academic Publishers, Dordrecht, 2002, 19–56.

[29] P. Jipsen, Generalization of Boolean Products for Lattice–orderered Algebras, *Annals of Pure and Applied Logics* 161, Issue 2 (November 2009), 224–234.

[30] P. T. Johnstone, *Stone Spaces*, Cambridge Studies in Advanced Mathematics 3, Cambridge University Press, Cambridge/London/New York/New Rochelle/Melbourne/Sydney, 1982.

[31] B. Jónsson, Congruence–distributive Varieties, *Math. Japonica* 42, No. 2 (1995), 353–401.

[32] A. Joyal, Le Théorème de Chevalley – Tarski et Remarques sur l'algèbre Constructive, *Cahiers Topol. Géom. Différ.*, 16 (1975), 256–258.

[33] J. Kaplansky, *Commutative Rings*, First Edition: University of Chicago Press, 1974; Second Edition: Polygonal Publishing House, 2006.

[34] J. Kollár, Congruences and One–element Subalgebras, *Algebra Universalis* 9, Issue 1 (December 1979), 266–267.

[35] T. Kowalski, A. Ledda, F. Paoli, On Independent Varieties and Some Related Notions, *Algebra Universalis* 70, Issue 2 (October 2013), 107–136.

[36] A. Ledda, F. Paoli, C. Tsinakis, Lattice–theoretic Properties of Algebras of Logic, *J. Pure Appl. Algebra* 218, No. 10 (2014), 1932–1952.

[37] L. Leuștean, The Prime and Maximal Spectra and the Reticulation of BL–algebras, *Central European Journal of Mathematics* 1 (2003), No. 3, 382–397.

[38] L. Leuștean, *Representations of Many–valued Algebras*, Editura Universitară, Bucharest, 2010.

[39] R. McKenzie and J. Snow, *Congruence Modular Varieties: Commutator Theory and Its Uses*, in *Structural Theory of Automata, Semigroups, and Universal Algebra*, Springer, Dordrecht, 2005.

[40] W. DeMeo, The Commutator as Least Fixed Point of a Closure Operator, [arXiv:1703.02764](http://arxiv.org/abs/1703.02764) [math.LO].
[41] C. Mureșan, The Reticulation of a Residuated Lattice, *Bull. Math. Soc. Sci. Math. Roumanie* 51 (99), No. 1 (2008), 47–65.

[42] C. Mureșan, *Algebras of Many–valued Logic. Contributions to the Theory of Residuated Lattices*, Ph. D. Thesis, 2009.

[43] C. Mureșan, Characterization of the Reticulation of a Residuated Lattice, *Journal of Multiple–valued Logic and Soft Computing* 16, No. 3–5 (2010), Special Issue: *Multiple–valued Logic and Its Algebras*, 427–447.

[44] C. Mureșan, Dense Elements and Classes of Residuated Lattices, *Bull. Math. Soc. Sci. Math. Roumanie* 53 (101), No. 1 (2010), 11–24.

[45] C. Mureșan, Further Functorial Properties of the Reticulation, *Journal of Multiple-valued Logic and Soft Computing* 16, No. 1–2 (2010), 177–187.

[46] C. Mureșan, Co–Stone Residuated Lattices, *Annals of the University of Craiova, Mathematics and Computer Science Series* 40 (2013), 52–75.

[47] C. Mureșan, Taking Prime, Maximal and Two–class Congruences Through Morphisms, submitted, arXiv:1607.06901 [math.RA].

[48] P. Ouwehand, *Commutator Theory and Abelian Algebras*, arXiv:1309.0662 [math.RA].

[49] Y. S. Pawar, Reticulation of a 0-distributive Lattice, *Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica* 54, Issue 1 (2015), 121–128.

[50] Y. S. Pawar, I. A. Shaikh, Reticulation of an Almost Distributive Lattice, *Asian–European Journal of Mathematics* 08, Issue 04 (December 2015).

[51] E. T. Schmidt, *A Survey on Congruence Lattice Representations*, Teubner–Texte zur Mathematik, Leipzig (1982).

[52] H. Simmons, Reticulated Rings, *Journal of Algebra* 66, Issue 1 (September 1980), 169–192.

[53] A. Ursini, On Subtractive Varieties, V: Congruence Modularity and the Commutator, *Algebra Universalis* 43 (2000), 51–78.