Photonic Implementation of Quantum Information Masking

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Recently, it has been shown that the quantum information can be masked by spreading it over nonlocal correlations and hidden from the subsystems. However, most of previous works based on only pure states which greatly limits its application. In this work, we provide the exact geometric structure of the maskable states for qudit mixed states, and experimentally investigate the properties of masking using mixed qubit states in an all optical system. The maskable set is determined to be a disk inscribed in Bloch sphere with infinitesimal thickness, within which we observe that the quantum information in the masked states can be completely transferred into bipartite correlations and faithfully restored. The protocol of masking is further utilized to demonstrate tripartite secret sharing, where all the candidates cooperate together to unlock a secret string. Our experiment provide a clear characterization of qubit masking, and sheds light on its potential application on secure quantum information processing.

Introduction.— The distinctively nonclassical properties of quantum mechanics establish the pronounced discrepancy of quantum information from the classical one [1]. Given the celebrated quantum nonlocality [2], it is natural to consider the possibility of spreading information over the nonlocal correlation to hide it from any observer who only has access to some subsystems, namely, masking the quantum information [3]. However, the postulate of unitarity and linearity [4] of the quantum theory imposes fundamental limitations on our capability of masking quantum information. In the seminal work [3], universal masking of qudit states is deemed impossible. This assertion establishes a novel no-go theorem to paraphrase its conspicuous precedents like the interdicts against universal cloning [5], broadcasting [3, 7], deleting [8, 9] and hiding [10] of an unknown state.

Despite the fundamental handicap in its universal implementation, masking of quantum information admits state-dependent [11] and probabilistic realization [12, 13], which is similar to other generally impossible tasks in quantum information [14, 15]. Extrapolations of masking into multipartite [16], multi-level [17] scenarios and channel states [18] have been proposed. Masking of quantum information unravels profound affiliation with e.g. quantum state discrimination [19], qubit commitment [20, 21], secret sharing [22], and pivotal laws of nature like information conservation [23]. Although significant progress has sprouted regarding the theoretical investigation of masking, to the best of our knowledge, the task of its experimental implementation has not been undertaken.

In this work, we construct a photonic qubit masking machine to investigate the properties of quantum information masking for any qubit state. Assisted by the versatility of the machine, quantum information in a class of nonorthogonal ensembles is masked, which serves as a novel approach to distribute and conceal quantum information. By perturbing the states to be masked, we confirm the zero measure of the maximally maskable set [24] to show that the requirement posed by masking is actually stringent. On the other hand, this property supports effective extraction of the elusive quantum information and its interface with classical information, which is further exploited for tripartite sharing of a secret image. Our experiment deepens the comprehension of the connection between quantum information and nonlocality, provides a systematic method for experimental studies of quantum information masking, which also proves to be useful for novel schemes of quantum secret sharing.

Masking of mixed quantum states.— The formal definition of information masking for mixed quantum states is a direct extrapolation from the case of pure states. First, we define a general qudit “masker”, $\mathcal{U}$, which is a linear isometry correlating a qudit state $\rho_A$ and an $d \times d$ bipartite state $\rho_A^{AB}$:

$$\rho_A^s \rightarrow \rho_A^{s AB} = \mathcal{U} \rho_A^s \otimes |0\rangle \langle 0| \mathcal{U}^\dagger, \quad s \in \{1, 2, \cdots \},$$

where $|0\rangle \langle 0|$ represents the blank state of a qudit. We say that $\mathcal{U}$ “masks” the quantum information contained in a set $\Omega$ of density matrices $\{\rho_A^s \in \Omega\}$ if for all $s$, the marginal states of $\rho_A^{s AB}$ for the two parties are respectively identical [3].

The geometric representation of a quantum state is particularly helpful for the analysis of masking. A qudit
state can be spanned on the $SU(d)$ basis $\{\Lambda_i\}_{i=1}^{d^2-1}$. Explicitly, $\rho = I_d/d + \sum_{i=1}^{d^2-1} x_i \Lambda_i/2$ with $\sum_{i=1}^{d^2-1} x_i^2 \leq r_d^2$, where $x_i$ represents the coefficients, and $r_d$ is determined by the dimension $d$ [1, 2]. Consequently, every quantum state corresponds to a unique point in a hypersphere, which is analogous to the Bloch sphere for qubits. In [3], the authors conjectured that the maskable set corresponds to any linear qudit isometry lays on some hyperdisk. Based on this representation, we are able to prove this conjecture in Supplementary Material (SM) [27] by directly constructing the isometry. Moreover, the converse of this conjecture also holds, that every disk inscribed in the Bloch sphere is maskable. Notably, the noncommuting mixed states cannot be broadcast [3], and commuting mixed states are simultaneously maskable (cf. Section IC of SM [27]). Because the mixed states on the same disk do not necessarily commute, the maskable set strictly contains the broadcastable set.

When the results are applied on the qubit case, the complete characterization in [11] can be recovered. For simplicity, we will interchangeably denote a state $\rho = (I_2 + \sigma_x a_x + y_0 \sigma_y + \sigma_z a_z)/2$ using its $SU(2)$ expansion $\rho := (x, y, z)$. A qubit disk containing the reference state $\rho_0 = (x_0, y_0, z_0)$ can be expressed in a parametric form:

$$D_\alpha(\rho_0) = \{\rho : x \sin \alpha \cos \theta + y \sin \alpha \sin \theta + z \cos \alpha = c\},$$

(2)

with $c = x_0 \sin \alpha \cos \theta + y_0 \sin \alpha \sin \theta + z_0 \cos \alpha$, $\alpha \in [0, \pi]$ and $\theta \in [0, 2\pi]$. Our result asserts the existence of isometry $U^{\square}_{\rho_0}(\rho_0)$ which masks the quantum disk $D_\alpha(\rho_0)$, and its exact form can be found in SM [27]. We now implement the qubit masking isometry on a photonic architecture.

A photonic masking machine.— The polarization degree of freedom of the photons is a natural courier of qubit information. Specifically, the correspondences $|H\rangle \leftrightarrow |0\rangle$, $|V\rangle \leftrightarrow |1\rangle$ link the isomorphic Hilbert spaces of a photon’s polarization and a qubit, with $|H\rangle$ and $|V\rangle$ identifying horizontal and vertical polarization, respectively. Here, we exploit the photon fusion gate [4] to construct a class of maskers $U^\square_{\rho}$, which is further promoted to arbitrary parameters $U^\square_{\rho}$ using additional birefringent crystals of wave plates. Using this photonic masking machine, an agent (Alice) can conceal some quantum information into the correlation of her photons with the ones held by another agent (Bob). Discussion about the relation between the fusion gate and our masking machine is deferred to section IIA of [27].

The experimental setup is illustrated in Fig. 1. An ultraviolet laser with a central wavelength of 400nm is used to pump a type-II phase-matched beta-barium borate (BBO) crystal to generate photon pairs with deterministic polarization via the spontaneous parametric down-conversion process. The two photons are collected by single-mode fibers (SMFs), with one of them transmitted to Alice for initial state preparation using a half-wave plate (HWP) and a quarter-wave plate (QWP), and the other directly fed into the masking machine, where its polarization is rotated to $|D\rangle = (|H\rangle + |V\rangle)/\sqrt{2}$. Subsequent to state preparation, Alice inputs her photon to the masking machine, which first undergoes a polarization rotation induced by a HWP sandwiched by two QWPs. This rotation maps the arbitrary disk to be masked to a latitudinal plane in the Bloch sphere [30]. The two photons are then superposed on a polarization beam splitter (PBS) with their trajectory and arrival time carefully aligned to ensure the spatial and temporal wavefunctions of the two photons properly overlap. The photons from one port are kept by Alice and the others belong to Bob. The polarization states of the photons are later analyzed by a PBS preceded by a QWP and a HWP. Finally, the photons are again collected by two SMFs and sent to single-photon avalanche detectors for coincidence counting.

We exemplify the aptitude of the masking machine by masking a disk $D_\theta(\rho_{D})$ with $\theta = \arctan \sqrt{2}$, passing through $\rho_1 = |H\rangle \langle H| = (0, 0, 1)$, $\rho_2 = |D\rangle \langle D| = (1, 0, 0)$.
and $\rho_3 = |L\rangle \langle L| = (0, 1, 0)$ with $|L\rangle = (|H\rangle + i|V\rangle)/\sqrt{2}$. This requires the orientation of the three cascaded wave plates in the masking machine set to 58.18°, 0° and 64.66°. The masker is also applicable for masking the pure state $\rho_4 = (2/3, 2/3, -1/3)$ and the mixed state $\rho_5 = (1/2, 1/2, 0)$, with the latter prepared using temporal-mixing technique [31].

The experimental initial states in a Bloch sphere (blue dots) are shown in Fig. 2a. To retrieve the masked information, we numerically apply the inverse isometry $U^{-1}$ on the reconstructed bipartite density matrix. The final state is also shown in Fig. 2a (orange dots) for comparison. The reconstruction achieved a mean fidelity of 99.87% and the average total absolute spectra error is $3.72 \times 10^{-2}$. The very high fidelity of reconstruction credits to the additional information from the form of the masker, which suppressed some noise present in the bipartite state. The effect of masking can be further reflected by the reduced states for Alice and Bob, which are shown in Fig. 2b. We find that they almost completely overlap, with the average trace distance $T(\rho_i, \rho_j) = \frac{1}{2}||\rho_i - \rho_j||_1$ [32], $(i, j \in [1, 5])$ of Alice’s and Bob’s reduced states being $1.55 \times 10^{-2}$ and $4.06 \times 10^{-2}$, respectively. Although the quantum information retrievals from the local marginal states, joint measurements on two photons show that the average fidelity of the masking-resulted states with respect to the theoretical predictions is 97.70%. Two instances of reconstructed bipartite density matrices are shown in Fig. 2c and 2d, in good accord with their theoretical values.

Zero measure of maskable set.— The maskable set corresponding to any isometry has zero Haar measure [24]. We verify this property on $U^0_0$, which is implemented by setting the orientations of all the three wave plates in the masking machine to 0°. The corresponding maskable disks are every latitudinal plane on the Bloch sphere. Experimentally, the reference states $\rho_0 = (\sin(\phi/2)|H\rangle + \cos(\phi/2)|V\rangle)(\sin(\phi/2)|H\rangle + \cos(\phi/2)|V\rangle)$ on latitude $\phi$, with $\phi$ setting to $0°$, $30°$ and $60°$, are selected. We then prepare some states shifted from $\rho_0$ along a parallel or a meridian, cast $U^0_0$, and take tomography on these states. The reduced states of Bob, $\rho^B_0$, are obtained from the reconstructed bipartite density matrices, and compare with the theoretical value, $\rho^0_B = Tr_A(U^0_B \rho_0 \otimes |0\rangle \langle 0|U^B_0)$ to detect the failure of masking. Our results in Fig. 3 shows that when the shift is along a parallel, $\rho^B_0$ is invariant, which can be revealed by the vanishing $T(\rho^B_0, \rho^0_B)$. Consequently, the state still belongs to the maskable set. However, shifts along a meridian always induce nonzero $T(\rho^B_0, \rho^0_B)$ regardless of $\phi$, indicating failure of qubit masking because extra information is transmitted to Bob. The results coincide with theoretical predictions, and confirm that the maskable disk essentially has zero thickness and thus zero measure.

Masking-based secret sharing.— Besides the theoretical significance, masking of quantum information also has practical merits. To elucidate a potential application, we adopt qubit masking to demonstrate a scheme of secret sharing, in which Alice distributes a colored picture to three recipients (Bob1, Bob2, Bob3). Precisely, Alice utilizes the homomorphism between the Bloch representation of a qubit and the hue-saturation-luminosity color space (cf. section IIb of SM [27]) to encode every pixel of a picture into a quantum state $\rho_0 = (x_0, y_0, z_0)$. She then masks $\rho_0$ with different maskers $U_{(i)}$, sends one of the qubits from the resulted $\rho^1_{IB}$ to Bob1, respectively. With the information from the marginal state, Bob1 can only restrict the masked state onto a disk even they are informed a priori the form of $U_{(i)}$. However, when all Bobs cooperate together and compare their inference of the marginal state (using merely classical communication), as is shown in Fig. 4a, their disks will intersect on one point in the Bloch sphere, which reveals $\rho_0$ as well as the concealed information.

In our experiment, Alice adopts $U_{(1)} = U^0_0/\sqrt{2}, U_{(2)} = U^E_{\pi/2}$ and $U_{(3)} = U^0_0$ to mask the quantum state $\rho_0$ corresponding to the color of each pixel. From the bipartite state resulted by applying $U_{(i)}$, one particle is sent to Bobi and one is kept by Alice. Because the reduced density matrix for Bob after applying the maskers on $\rho_0$ are $(I_2 + x\sigma_x)/2, (I_2 + y\sigma_y)/2,$ and $(I_2 + z\sigma_z)/2$, respectively, this operation effectively grants each Bob access to one of the float numbers, $x_0, y_0$, and $z_0$, which constitutes the original state. By repeatedly applying the
qubit masking also implies the nonexistence of qubit commitments [3], a stronger version of quantum bit commitment [20, 21], the counterpart of bit commitment which is also ruled out by quantum mechanics. Consequently, our work further differentiates the attainable and unattainable tasks in the framework of quantum theory.

We have demonstrated masking for mixed qubit states with a photonic setup to investigate the connection between nonlocality and quantum information. By spreading the information into correlations between quantum systems, the concealed information is protected by quantum correlations and becomes robust to eavesdropping and untrusted agents. The properties of masking are meticulously characterized. Benefiting from masking’s intrinsic resilience against noise, the entailed method of secret sharing achieves high fidelity. This work provides the first experimental implementation of masking of quantum information with photons, deepens our comprehension of its tie with the basic axioms of the quantum nature, and sheds lights on its future applications in processing of classical as well as quantum information.

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Fig. 3. Observation of the zero measure of maskable sets. The masker is chosen to generate identical marginal states when the initial states are on the same latitude on the Bloch sphere. Each plot shows the trace distance of Bob’s reduced density matrices from the reference state, when Alice slightly shifts the initial state away from a reference point on different latitudes $\phi$. The cases of displacement along a parallel or a meridian on the Bloch sphere are plotted respectively in green and cyan points (experimental data) and curves (theoretical values). The error bars are deduced from Poissonian counting statistics.

above procedure, the entire image is split and transmitted. The Bobs can then assemble their deduced parameters to recover $\rho_0$, which corresponds to a certain color of a pixel. The reconstructed image in Fig. 4c reasonably resembles the original one (cf. Fig. 4b), and the correlation between the two colored images, averaging over red-green-blue channels, is calculated to be 99.35\%.

We mention that this scheme of secret sharing is secure against eavesdropping providing that the masker and the marginal state are not simultaneously divulged: fabrication of either of the two components in masking may eventuate in a non-physical state that locates outside the Bloch sphere being reconstructed, immediately exposing the existence of an eavesdropper.

Discussion.— The protocol of qubit masking in our experiment works on an arbitrary disk in the Bloch sphere. A class of nonorthogonal states can be simultaneously masked. This is drastically different from the situation of classical information, which does not admit superposition of orthogonal states. Moreover, falsification of universal qubit masking also implies the nonexistence of qubit commitments [20, 21], the counterpart of bit commitment which is also ruled out by quantum mechanics. Consequently, our work further differentiates the attainable and unattainable tasks in the framework of quantum theory.

We have demonstrated masking for mixed qubit states with a photonic setup to investigate the connection between nonlocality and quantum information. By spreading the information into correlations between quantum systems, the concealed information is protected by quantum correlations and becomes robust to eavesdropping and untrusted agents. The properties of masking are meticulously characterized. Benefiting from masking’s intrinsic resilience against noise, the entailed method of secret sharing achieves high fidelity. This work provides the first experimental implementation of masking of quantum information with photons, deepens our comprehension of its tie with the basic axioms of the quantum nature, and sheds lights on its future applications in processing of classical as well as quantum information.
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Supplementary Material for
“Photonic Implementation of Quantum Information Masking”

In this Supplementary Material, we give the proofs for the propositions in the main text. We first introduce the geometric representation of a qudit state. Exploiting this representation, we provide a geometric feature of the maximal maskable set in an arbitrary \( n \)-dimensional qudit Hilbert space, and subsequently derive the form for qubit masking, which correspond to \( n = 2 \). Finally, we prove that any commuting, mixed states set can be simultaneously masked, and the set of quantum states that is broadcastable is a proper subset of one that is maskable.

**CONTENTS**

A. Theoretical considerations 1
   1. Geometric representation of qudit 1
   2. General form of maskable set in arbitrary dimensions 1
   3. Strict inclusion of broadcastable set in maskable set 2

B. Experimental implementations 4
   1. The relation between the fusion gate and the masking machine 4
   2. Encoding colored image with qubit states 5

Supplementary Material A: Theoretical considerations

1. Geometric representation of qudit

An arbitrary qudit state \( \rho \) can be expressed as follows using the \( SU(d) \) basis \( \{ \Lambda_i \}_{i=1}^{d^2-1} \), where \( \Lambda_i \) are orthogonal eigenstates satisfying \( \Lambda_i^+ = \Lambda_i \), \( \text{Tr}(\Lambda_i) = 0 \), and \( \text{Tr}(\Lambda_i\Lambda_j) = 2\delta_{ij} \). Namely \([1, 2]\),

\[
\rho = \frac{1}{d} I_d + \frac{1}{2} \sum_{i=1}^{d^2-1} x_i \Lambda_i, \quad \sum_{i=1}^{d^2-1} x_i^2 \leq r_d^2, \quad x_i \in \mathbb{R},
\]

where \( r_d \) satisfies \( \frac{2}{d(d-1)} \leq r_d^2 \leq \frac{2(d-1)}{d} \).

The introduction of the hyperdisk may also benefits the comprehension of qudit masking. A hyperdisk \( D \) is defined as the intersection of a hypersphere and a hyperplanes in the Euclidean space. Namely,

\[
D = \{ (x_1, x_2, ..., x_d) : \sum_{i=1}^d x_i^2 \leq r_d^2, \sum_{i=1}^d a_i x_i + b = 0, a_i \text{ not all be zero} \}.
\]

2. General form of maskable set in arbitrary dimensions

**Theorem 1.** the maximal maskable set for an arbitrary \( d \)-dimensional qudit state lays on a hyperdisk in \( d^2 - 1 \) dimensional Euclidean space.

**Proof.** — Suppose \( \rho \) is the density matrix of the mixed states on \( H_A \), and \( U \) is unitary operator, \( U \rho \otimes |0\rangle\langle 0| U^\dagger = \rho^{AB} \). Let \( \rho_A = \text{Tr}_B(\rho^{AB}) \), then

\[
\rho_A(k,l) = \sum_{i=1}^{n^2-1} a_i^{(k,l)} x_i + b^{(k,l)}, k, l = 1, 2, ..., d.
\]

where \( \rho_A(k,l) \) is the \( k \)-th row and \( l \)-th column element of matrix \( \rho_A, a_i^{(k,l)}, b^{(k,l)} \in \mathbb{C} \). A given subset which is maskable iff. \( \rho_A(k,l) \) for \( \rho_A \), and similarly \( \rho_B(k,l) \) for \( \rho_B \), are constant, namely, \( \exists h^{(k,l)}_X \in \mathbb{C}, \rho_X(k,l) = h^{(k,l)}_X, X = A \) or \( B \).
Obviously, for a masker $\mathcal{U}$, if all $\rho_X(k, l)$ are constant for all $\rho$, then masking of all quantum qudit pure states is possible, which contradicts the known results, so $a_i^{(kl)}, i = 1, 2, ..., n$ can’t all be zero, namely, there exists $\text{Re}\rho_X(k, l) = \text{Re}h_X^{(kl)}$ or $\text{Im}\rho_X(k, l) = \text{Im}h_X^{(kl)}$, which is a nontrivial hyperplane. So the maximal maskable set of states with respect to $\mathcal{U}$ always lays on a hyperdisk in $n^2 - 1$ dimensional space.

Note that when $d = 2, r_d \equiv 1$, in other words, the map between the set of points in the Bloch sphere and the set of qubit states is a bijection, so the maximal maskable set of qubit states with respect to $\mathcal{U}$ is on a disk in the Bloch sphere. Moreover, an arbitrary disk passing through the point $\rho_0 = (x_0, y_0, z_0)$ in the Bloch sphere can always be represented as $D^\rho_0(\rho_0)$. We now prove that $D^\rho_0(\rho_0)$ can be masked by a masker $\mathcal{U}_\alpha^0$ with

$$
\mathcal{U}_\alpha^0 = \mathcal{U}_0^0(\mathcal{R}(\alpha, \theta) \otimes I_2)(\mathcal{R}(0, 0) \otimes I_2)^{-1} = \begin{pmatrix}
\cos(\alpha/2) & 0 & e^{-i\theta} \sin(\alpha/2) & 0 \\
0 & \cos(\alpha/2) & 0 & e^{-i\theta} \sin(\alpha/2) \\
0 & \sin(\alpha/2) & 0 & -e^{-i\theta} \cos(\alpha/2) \\
\sin(\alpha/2) & 0 & e^{-i\theta} \cos(\alpha/2) & 0
\end{pmatrix}.
$$

(S1)

In fact, let $\mathcal{U}_\alpha^0 \otimes |0\rangle\langle 0|\mathcal{U}_\alpha^0 \dagger = \rho^{AB}$, then it immediately follows that the reduced density matrices are

$$
\rho_A = \rho_B = \frac{1}{2}(1 + c)|0\rangle\langle 0| + \frac{1}{2}(1 - c)|1\rangle\langle 1|,
$$

therefore, we recover the following Proposition, which is asserted without proof in the main text.

**Proposition 1.** the maximal maskable set of states in the Bloch sphere with respect to $\mathcal{U}$ is on a disk and any disk is a maskable set.

Furthermore, we consider the measure of the maskable set. Because the volume of the Bloch sphere is $\frac{4\pi}{3}$ and the maximal maskable set is a disk with a volume of zero, the measure of the maskable set corresponding to all qubit states is zero. This conclusion also holds for qudit states, because the volume of all qudit states is greater than the maximal maskable set is a disk with a volume of zero, the measure of the maskable set corresponding to all qubit states is zero. This conclusion also holds for qudit states, because the volume of all qudit states is greater than

3. **Strict inclusion of broadcastable set in maskable set**

In this section, we consider the relationship between the set of the maskable and broadcastable qudit states. Here, we show that the latter set is a proper subset of the former one. Our start point is that the density matrices of the broadcastable states commute [3], and the proof is based on the following two observations.

**Theorem 2.** Commuting mixed states set can be simultaneously masked, and the set of quantum states that is broadcastable is a proper subset of one that is maskable.

**Proof.** Suppose the spectrum decomposition of $\rho_x^A$ is:

$$
\rho_x^A = \sum_{i=1}^{d} p_i |i\rangle\langle i|, \text{ for all } \sum_{i=1}^{d} p_i = 1.
$$

Let’s construct a Vandermonde matrix $(a_{kl})_{d \times d}$ of unit roots, namely

$$
(a_{kl})_{d \times d} = \begin{pmatrix}
1 & 1 & \cdots & 1 \\
x_1 & x_2 & \cdots & x_d \\
x_1^2 & x_2^2 & \cdots & x_d^2 \\
\cdots & \cdots & \cdots & \cdots \\
x_1^{d-1} & x_2^{d-1} & \cdots & x_d^{d-1}
\end{pmatrix},
$$

(S2)
where \( x_i \) are the roots of \( x^d = 1 \). One may verify that the inner product of the two different row vectors of this matrix \((a_{kl})_{d \times d}\) is zero. Let us denote \( |\Psi_k\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} a_{ki} |i\rangle |l\rangle \), then \( \{ |\Psi_k\rangle \}_{k=1}^{d} \) is a set of orthonormal bases. Also, we choose the isometric linear operator \( \mathcal{M} \) acting on the base \( \{ |k\rangle \}_{k=1}^{d} \) in the following way,

\[
|k\rangle \rightarrow |k\rangle |1\rangle \rightarrow \Psi_k, \quad k = 1, \ldots, d.
\]

From the definition of the isometry, \( \mathcal{M} \) can always be dilated to an unitary operator \( \mathcal{U} \) on the space \( \mathcal{H}_A \otimes \mathcal{H}_B \), that is, there exist a unitary operator \( \mathcal{U}, \mathcal{U}|k\rangle |0\rangle = \mathcal{M}|k\rangle = |\Psi_k\rangle \). Note that \( \mathcal{U} \rho_A^k \otimes |0\rangle \langle 0| \mathcal{U}^\dagger = \rho_{AB}^k \). Direct calculation of \( \rho_A = Tr_B(\rho_{AB}^k) \) yields the reduced density matrices

\[
\rho_A = \rho_B = \frac{1}{d} \sum_{k=1}^{d} \sum_{i=1}^{d} p_i \| a_{ik} \| |k\rangle \langle k| = \frac{1}{d} \sum_{k=1}^{d} |k\rangle \langle k|.
\]

Where \( \rho_A = \rho_B \) is guaranteed by the symmetric form of the isometry. We see that the eigenvalues \( p_i \) cease to appear in the final reduced density matrices, so the masking succeed.

Further, let

\[
|\psi_0\rangle = \left( \frac{d-1}{d} + \frac{1}{d} i \right) |1\rangle + \sum_{k=2}^{d} \left( \frac{-1}{d} + \frac{1}{d} i \right) |k\rangle,
\]

where \( i \) is the unit imaginary number. Applying the isometry \( \mathcal{M} \) defined in (S3) on \( |\psi_0\rangle \), we find

\[
Tr_A(\mathcal{M}|\psi_0\rangle \langle \psi_0| \mathcal{M}^\dagger) = Tr_B(\mathcal{M}|\psi_0\rangle \langle \psi_0| \mathcal{M}^\dagger) = \frac{1}{d} \sum_{k=1}^{d} |k\rangle \langle k|,
\]

and thus \( |\psi_0\rangle \langle \psi_0| \) and \( \rho_A^k \) is simultaneously maskable by \( \mathcal{M} \). By the linearity of the trace operation, the set of quantum states \( \{ \rho \lambda_0 |\psi_0\rangle \langle \psi_0| + \sum_{i=1}^{d} \lambda_k |k\rangle \langle k|, \sum_{k=0}^{d} \lambda_k = 1 \} \) is maskable by \( \mathcal{M} \). However, it is straightforward to check that for \( \prod_{i=0}^{d} \lambda_i \neq 0 \), these states are predominantly not broadcastable, so the dimension of the set of the maskable quantum states is at least 1 plus which of the broadcastable ones.

As a final remark, we give an intuitive explanation of the proof in the 2-dimensional case. The geometric form of a set of commuting mixed states of qubit states in the Bloch sphere is a line segment passing through the center of the sphere. However, any disk in the Bloch sphere is a maskable set, so any broadcastable set is a proper subset of some maskable sets, and the resource of maskable states is far more abundant than the resource of broadcastable states.

\[\square\]
Supplementary Material B: Experimental implementations

1. The relation between the fusion gate and the masking machine

Here, a detailed analysis of the relation between the fusion gate [4] and the masking operator \( \mathcal{U}_0 \) is given. The schematic illustration of the photon fusion gate is shown in Fig. S1, where the wavefunction of two photons interfere on a polarizing beam splitter (PBS). The effective surface of the PBS is denoted by the anti-diagonal line crossing the center of the PBS. Regardless of the original direction of propagation, a photon will be reflected by the PBS if it has vertical polarization (denoted by \( |V\rangle \)), and will pass through the PBS if its direction of polarization is horizontal (denoted by \( |H\rangle \)).

A general polarization state \( |\psi^A\rangle \) of a single photon can be spanned on the horizontal-vertical basis, namely,
\[
|\psi^A\rangle = \beta |H\rangle + \gamma e^{i\theta} |V\rangle,
\]
with \( \beta, \gamma \) and \( \theta \) being real numbers and \( \beta^2 + \gamma^2 = 1 \). For simplicity, in this section we only consider the case of pure states—the situation for mixed states can be proved immediately using the linearity of quantum mechanics. In the masking machine (cf. Fig. 1 in the main text), this photon will interfere with another photon with fixed polarization of \( |D\rangle = (|H\rangle + |V\rangle)/\sqrt{2} \) on the surface of the PBS. After the PBS gate, the wavefunction propagating from down to is sent to Alice, and the one propagating from left to right is sent to Bob. Moreover, only the photon detection events that result in one photon detected in Alice’s site (A) and one in Bob’s site (B) are registered as coincidence counting, and the other events are discarded.

To address the quantum interference on the PBS, we introduce the annihilation operator \( a_{\mu}^\dagger \) of the photon, which denotes a detection event of a photon with polarization of \( \mu \in \{H, V\} \) happening at site \( \nu \in \{A, B\} \). The conversion of the wavefunctions of the two input photons writes [5]:
\[
|\psi^A\rangle = (\beta |H\rangle + \gamma e^{i\theta} |V\rangle),
\]

\[
|\psi^A\rangle \otimes |D\rangle \xrightarrow{\mathcal{U}} (\beta a_H^\dagger + i\gamma e^{i\theta} a_V^\dagger) \frac{1}{\sqrt{2}} (a_H^\dagger + i a_V^\dagger) |0^A0^B\rangle
\]
\[
= \left( \frac{\beta a_H^\dagger a_H^\dagger - \gamma e^{i\theta} a_V^\dagger a_V^\dagger}{\sqrt{2}} + i \frac{\beta a_H^\dagger a_V^\dagger + \gamma e^{i\theta} a_H^\dagger a_V^\dagger}{\sqrt{2}} \right) |0^A0^B\rangle,
\]
where \( |0^A0^B\rangle \) denotes the vacuum state. The second fraction in the parenthesis, however, does not result in coincidence counting. Consequently, the effective wavefunction after the PBS that reaches coincidence counting, after normalization, reads:
\[
|\psi^{AB}\rangle = \beta |H^A H^B\rangle - \gamma e^{i\theta} |V^A V^B\rangle.
\]

Calculating the two reduced density operators of the bipartite system yields:
\[
\text{Tr}_A \left( |\psi^{AB}\rangle \langle \psi^{AB}| \right) = \text{Tr}_B \left( |\psi^{AB}\rangle \langle \psi^{AB}| \right) = \begin{pmatrix} \beta^2 & 0 \\ 0 & \gamma^2 \end{pmatrix}.
\]
We see that the parameter \( \theta \) has ceased to appear in (S4), which means that the states with the same \( \beta, \gamma \) and different \( \theta \)s are masked. These states lays on a latitudinal parallel on the Bloch sphere. Following the definition of the masker, we conclude that the fusion gate realizes \( \mathcal{U}_0 \).
Furthermore, the exact form of the masking isometry $\mathcal{U}_\theta^0$ can also be derived from the form of mode conversion taking place in the fusion gate. Because the fusion gate maps $\begin{pmatrix} \beta \\ \gamma \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ to $\begin{pmatrix} \beta \\ 0 \\ 0 \\ -\gamma \end{pmatrix}$ up to a renormalization of the final state, we can write

$$U_0^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \quad (S5)$$

Note that the from of the second and the fourth columns of the $U_0^0$ can be freely chosen as long as the value assignments preserve unitarity of the evolution. Finally, by noticing that a unitary evolution $\mathcal{R}_\alpha^\theta = \begin{pmatrix} \cos(\alpha/2) & e^{-i\theta} \sin(\alpha/2) \\ \sin(\alpha/2) & -e^{-i\theta} \cos(\alpha/2) \end{pmatrix}$ rotates the edge of the disk $D^\theta_\alpha(p_0)$ to a latitudinal circle on Bloch sphere, we can construct the masking isometry $\mathcal{U}_\alpha^0$ as is given in (S1).

2. Encoding colored image with qubit states

In our protocol of secret sharing of an image, the color of each pixel is determined by three float numbers interpreted by the receivers, “Bobs”. For this course, a correspondence between the string of numbers decoded from Bobs’ quantum states and the reconstructed color has to be established. Here, this is achieved by resorting to the hue-saturation-luminosity (HSL) representation of a color.

We recall the geometric representation for any mixed qubit state in the Bloch sphere, that is,

$$\rho(\vec{r}) = \rho(x, y, z) = \frac{1}{2} (I_2 + x\sigma_x + y\sigma_y + z\sigma_z).$$

For every point in the Bloch sphere, we define the three color parameters, viz., hue, saturation and luminosity, as

$$h = \frac{1}{2} + \frac{\arctan(x, y)}{2\pi},$$
$$s = \frac{x^2 + y^2}{1 - z^2},$$
$$l = \frac{1 + z}{2},$$

respectively, where $\arctan(x, y) \in [-\pi, \pi)$ is the arc tangent of $y/x$, taking into account which quadrant the point $(x, y)$ is in. The value of hue is undefined at the points $(x, y, z)$ with $x = y = 0$, and the saturation is undefined at $z = \pm 1$. These singularities will not cause confusion to the final rendition of colors, which is described using the red, green and blue (RGB) values of the colors. The RGB and HSL values of a color is linked by

$$\{r, g, b\} = \{f(0), f(8), f(4)\},$$
with $f(n) = l - s \min(l, 1 - l) \max[-1, \min(k - 3, 9 - k, 1)]$,

and $k = \left( n + \frac{6h}{\pi} \right) \mod 12$.

The bijection of the RGB colors and the quantum state’s location in the Bloch sphere is illustrated in FigS2. The singularities on the $z$-axis resolve because they represent grayscale colors can be solely described by the luminosity. In our experiment, for every pixel to be transmitted, Alice encodes the message $(x_0, y_0, z_0)$ into the quantum state $\rho_0$. By applying the set of maskers $\mathcal{U}_\pi^0, \mathcal{U}_\pi^0, \mathcal{U}_\pi^0$ on $\rho_0$, and sending one of the resulted qubits to each Bob, she effectively prepares the marginal states for the three Bobs to be $\begin{pmatrix} (1 + x_0)/2 & 0 \\ 0 & (1 - x_0)/2 \end{pmatrix}, \begin{pmatrix} (1 + y_0)/2 & 0 \\ 0 & (1 - y_0)/2 \end{pmatrix}$.
Fig. S2. The one-on-one correspondence between possible colors and quantum state’s location in the Bloch sphere.

and \( \begin{pmatrix} (1 + z_0)/2 & 0 \\ 0 & (1 - z_0)/2 \end{pmatrix} \), respectively. From the form of the masker and the marginal state, each of the three Bobs can determine a disk in the Bloch sphere that the original state must belong to, and the three disks lays on three mutually orthogonal planes so they necessarily intercept at one point in the Bloch sphere. The intercepting point in turn represents the color information of the transmitted pixel.

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