PARABOLIC RAYNAUD BUNDLES

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Abstract. Let \( X \) be an irreducible smooth projective curve defined over complex numbers, \( S = \{ p_1, p_2, \ldots, p_n \} \subset X \) a finite set of closed points and \( N \geq 2 \) a fixed integer. For any pair \( (r, d) \in \mathbb{N} \times \frac{1}{N} \mathbb{Z} \), there exists a parabolic vector bundle \( R_{r,d,*} \) on \( X \), with parabolic structure over \( S \) and all parabolic weights in \( \frac{1}{N} \mathbb{Z} \), that has the following property: Take any parabolic vector bundle \( E_* \) of rank \( r \) on \( X \) whose parabolic points are contained in \( S \), all the parabolic weights are in \( \frac{1}{N} \mathbb{Z} \) and the parabolic degree is \( d \). Then \( E_* \) is parabolic semistable if and only if there is no nonzero parabolic homomorphism from \( R_{r,d,*} \) to \( E_* \).

1. Introduction

Let \( X \) be an irreducible smooth projective curve defined over \( \mathbb{C} \). Faltings showed that a vector bundle \( E \) over \( X \) is semistable if and only if there exists a vector bundle \( F \) on \( X \) such that \( H^i(X, \mathcal{H}om(F, E)) = 0 \) for \( i = 0, 1 \) [4]. This result was developed further by Popa (see [9]) to obtain estimates for the rank of \( F \) which depend only on the rank of \( E \). Given \( r \) and \( d \), a vector bundle \( R_{r,d} \) on \( X \) is called a Raynaud bundle if the following holds: A vector bundle \( E \) on \( X \) of rank \( r \) and degree \( d \) is semistable if and only if \( \text{Hom}(R_{r,d}, E) = 0 \) [6]. We note that these are called Raynaud bundles because the vector bundles considered by Raynaud in [10] are Raynaud bundles for \( r = 2 \) and \( d = 2(\text{genus}(X) - 1) \) (see Theorem 1.1 in [6]).

In [6], building on Popa’s result it was shown that Raynaud bundles exist. In [3], an analog of Faltings’ semistability criterion is given for parabolic vector bundles. Our aim here is to prove the existence of the Raynaud bundles in the parabolic context.

Given two parabolic vector bundles \( E_* \) and \( F_* \) on \( X \), the global parabolic homomorphisms from \( F_* \) to \( E_* \) will be denoted by \( \text{Hom}_{\text{par}}(F_*, E_*) \) (see [7] p. 212, Definition 1.5(II)) for the definition of a parabolic homomorphism).

We prove the following theorem:

**Theorem 1.1.** Let \( S = \{ p_1, p_2, \ldots, p_n \} \subset X \) a finite set of closed points of \( X \), and \( N \geq 2 \) a fixed integer. For any pair \( (r, d) \in \mathbb{N} \times \frac{1}{N} \mathbb{Z} \), there exists a parabolic vector bundle \( R_{r,d,*} \) on \( X \), with parabolic structure over \( S \) and all parabolic weights in \( \frac{1}{N} \mathbb{Z} \), that has the following property: Take any parabolic vector bundle \( E_* \) on \( X \) such that
(1) the parabolic points of $E_*$ are contained in $S$,
(2) all the parabolic weights of $E_*$ are in $\frac{1}{N}\mathbb{Z}$, and
(3) the rank of $E_*$ is $r$, and the parabolic degree of $E_*$ is $d$.
Then $E_*$ is parabolic semistable $\iff \text{Hom}_\text{par}(R_{r,d,*}, E_*) = 0$.

2. Preliminaries

2.1. The equivalence of equivariant bundles and parabolic bundles. We will recall a correspondence between parabolic vector bundles and equivariant vector bundles which will be used in the proof of Theorem 1.1.

We assume that at least one of the following two conditions hold:

- genus($X$) ≥ 1
- $|S| \neq 1$.

Fix $S$ and $d$ as in Theorem 1.1. Fix a Galois algebraic covering

(1) $f : Y \rightarrow X$

such that

- $f$ is ramified exactly over $S$, and
- the ramification index of each point in $f^{-1}(S)$ is $N - 1$.

See [8, p. 26, Proposition 1.2.12] for the existence of $f$ satisfying these conditions. We note that the assumption (genus($X$), $|S|$) $\neq (0, 1)$ is needed for the existence of $f$.

Let

$\Gamma := \text{Gal}(f)$

be the Galois group for the covering $f$. A $\Gamma$–linearized vector bundle on $Y$ is an algebraic vector bundle $E$ equipped with a lift of the action of $\Gamma$ as vector bundle automorphisms. This means that $\Gamma$ acts on the total space of $E$ as algebraic automorphisms, and the action of each $\gamma \in \Gamma$ on $E$ is an isomorphism of the vector bundle $E$ with $(\gamma^{-1})^*E$.

In [1], a natural bijective correspondence between the following two classes was established:

(1) the $\Gamma$–linearized vector bundles $W$ on $Y$, and
(2) the parabolic vector bundles $E_*$ over $X$ for which the parabolic divisor in contained in $S$ and all the parabolic weights are in $\frac{1}{N}\mathbb{Z}$.

This bijective correspondence takes the usual tensor product (respectively, direct sum) of $\Gamma$–linearized vector bundles to the tensor product (respectively, direct sum) of the corresponding parabolic vector bundles. Similarly, for any $\Gamma$–linearized vector bundle $W$, this bijective correspondence takes $W^*$ to the parabolic dual of the parabolic vector bundle corresponding to $W$. (See [2], [11] for the above mentioned operations on parabolic vector bundles.)
For any parabolic vector bundle $E_*$ of the above type, if $\hat{E}$ is the corresponding $\Gamma$–linearized vector bundle, then
\begin{equation}
\text{deg}(\hat{E}) = |\Gamma| \cdot \text{deg}_{\text{par}}(E_*),
\end{equation}
where $\text{deg}_{\text{par}}(E_*)$ is the parabolic degree of $E_*$. \[1\] p. 318, (3.12)]. Furthermore,
\begin{equation}
E_* \text{ is parabolic semistable } \iff \hat{E} \text{ is semistable}
(\text{see } [1, \text{ p. 318, Lemma 3.13}]).
\end{equation}

2.2. Raynaud bundles on smooth algebraic curves. In [6] an irreducible smooth projective curve $Y$ is considered. It is shown that for any pair of integers $(r, d)$, there exists a vector bundle $R_{r,d}$ on $Y$ (which is called a Raynaud bundle) with the following property:

For a vector bundle $E$ of rank $r$ and degree $d$ on $Y$,
\begin{equation}
E \text{ is semistable } \iff \text{Hom}(R_{r,d}, E) = 0
(\text{see (i) } \iff (v) \text{ in Theorem 2.12 of [6]}).
\end{equation}
This way we obtain a short cohomological criterion which enables us to check semistability. We can compute the rank and degree of $R_{r,d}$ in terms of the genus $g_Y$ of $Y$ and the integers $r$ and $d$ (see Proposition 2.2 and Corollary 3.4 in [6]).

3. Proof of Theorem 1.1

As in Section 2.1, we will assume that at least one of the following two conditions hold:
\begin{itemize}
    \item genus($X$) $\geq$ 1
    \item $|S| \neq 1$.
\end{itemize}
The remaining case where $(\text{genus}(X), |S|) = (0, 1)$ will be treated separately.

Let $E_*$ be a parabolic vector bundle on $X$ satisfying conditions (1)–(3) of Theorem 1.1. Denoting by $\hat{E}$ the associated $\Gamma$–linearized vector bundle on $Y$ in [1], we have that $\hat{E}$ is of rank $r$ and degree $d|\Gamma|$, and the parabolic semistability of $E_*$ is equivalent to the semistability of $\hat{E}$ (see (3)).

Using the result (1) together with (2) and (3) we obtain that
\begin{equation}
E_* \text{ is parabolic semistable } \iff \text{Hom}(R_{r,d|\Gamma|}, \hat{E}) = 0.
\end{equation}

**Step 1:** Set now
\begin{equation}
\tilde{R}_{r,d} = \bigoplus_{\gamma \in \Gamma} \gamma^* R_{r,d}
\end{equation}
to be the direct sum. Since $\hat{E}$ is $\Gamma$–linearized,
\[
\text{Hom}(\tilde{R}_{r,d|\Gamma|}, \hat{E}) \cong \text{Hom}(R_{r,d|\Gamma|}, \hat{E})^{\oplus |\Gamma|}.
\]
Therefore,
\[
\text{Hom}(\tilde{R}_{r,d|\Gamma|}, \hat{E}) = 0 \iff \text{Hom}(R_{r,d|\Gamma|}, \hat{E}) = 0.
\]
Combining this with (5) we conclude that

$$E_\ast \text{ is parabolic semistable } \iff \text{Hom}(\tilde{R}_{r,d|\Gamma}, \hat{E}) = 0.$$  

**Step 2:** The vector bundle $\tilde{R}_{r,d}$ in (6) admits a canonical $\Gamma$–linearization. Let $R'_{r,d,\ast}$ be the parabolic vector bundle over $X$ corresponding to this $\Gamma$–linearized vector bundle $\tilde{R}_{r,d}$.

Consider the trivial vector bundle over $Y$

$$\hat{W} := \mathcal{O}_Y \otimes_{\mathbb{C}} \mathbb{C}(\Gamma),$$

where $\mathbb{C}(\Gamma)$ is the group algebra of $\Gamma$. The action of $\Gamma$ on $Y$ lifts to an action of $\Gamma$ on $\mathcal{O}_Y$. The natural action of $\Gamma$ on $\mathbb{C}(\Gamma)$ and the action of $\Gamma$ on $\mathcal{O}_Y$ together define a $\Gamma$–linearization on the vector bundle $\hat{W}$ in (3).

Let $W_\ast$ denote the parabolic vector bundle over $X$ corresponding to the $\Gamma$–linearized vector bundle $\hat{W}$.

**Lemma 3.1.** Let $F_\ast$ be a parabolic vector bundle over $X$ satisfying condition (1) and condition (2) in Theorem 1.1. Let $\hat{F}$ be the $\Gamma$–linearized vector bundle on $Y$ corresponding to $F_\ast$. Then

$$\text{Hom}(\tilde{R}_{r,d|\Gamma}, \hat{F}) = \text{Hom}_{\text{par}}(R'_{r,d,\ast} \otimes W_\ast, F_\ast),$$

where $R'_{r,d,\ast} \otimes W_\ast$ is the parabolic tensor product of the parabolic vector bundles $R'_{r,d,\ast}$ and $W_\ast$ constructed above.

**Proof.** The parabolic vector bundle $W_\ast$ is constructed as follows. Consider the direct image

$$W = f_* \mathcal{O}_Y,$$

where $f$ is the covering map in (1). We have a filtration of subsheaves

$$W_1 \subset \cdots \subset W_i \subset \cdots \subset W_{N-1} \subset W_N = W,$$

where $W_j := f_* \mathcal{O}_Y(-(N-j)f^{-1}(S)_{\text{red}})$. For any point $x \in S$, let

$$0 \subset W^1_x \subset \cdots \subset W^j_x \subset W^{j+1}_x \subset \cdots \subset W^{N-1}_x \subset W^N_x = W_x$$

be the filtration of subspaces given by the above filtration of subsheaves. The dimension of each successive quotient in (10) is $|\Gamma|/N$.

The vector bundle underlying the parabolic vector bundle $W_\ast$ is $W$ (defined in (9)), its parabolic divisor is $S$, its quasiparabolic filtration on each point $x \in S$ is the one in (10), and the parabolic weight of the subspace $W^j_x \subset W_x$ in (10) is $(N-j)/N$.

Let $V_\ast$ be a parabolic vector bundle over $X$ satisfying condition (1) and condition (2) in Theorem 1.1. The vector bundle underlying $V_\ast$ will be denoted by $V_0$. Let $\hat{V}$ be the $\Gamma$–linearized vector bundle over $Y$ corresponding to $V_\ast$. Then

$$H^0(Y, \hat{V})^\Gamma = H^i(X, V_0)$$
Given any finite dimensional complex left $\Gamma$–module $M$, there is a canonical $\mathbb{C}$–linear isomorphism

$$M \rightarrow \text{Hom}_{\mathbb{C}(\Gamma)}(\mathbb{C}(\Gamma), M)^\Gamma$$

that sends any $v \in M$ to the homomorphism of $\Gamma$–modules $\rho_v : \mathbb{C}(\Gamma) \rightarrow M$ uniquely determined by the condition that $\rho_v(\gamma) = \gamma \cdot v$ for all $\gamma \in \Gamma$. Using this canonical isomorphism, from (11) it follows that

$$H^0(Y, \hat{V}) = \text{Hom}_{\text{par}}(W_*, V_*) .$$

Since the bijective correspondence between the parabolic vector bundles and the $\Gamma$–linearized vector bundles is compatible with tensor product, dual and homomorphism, the parabolic vector bundle corresponding to the $\Gamma$–linearized vector bundle $\tilde{R}^{\ast}_{r,d|\Gamma} \otimes \hat{F}$ is the parabolic tensor product $(R'_{r,d,s})^* \otimes F_*$, where $(R'_{r,d,s})^*$ is the parabolic dual of $R'_{r,d,s}$.

Now the lemma follows by substituting the $\Gamma$–linearized vector bundle $\tilde{R}^{\ast}_{r,d|\Gamma} \otimes \hat{F}$ in place of $\hat{V}$ in (12).

Let

$$(13) \quad R_{r,d,*} := W_* \otimes R'_{r,d,*}$$

be the parabolic tensor product. In view of (7) and Lemma 3.1 we conclude that the parabolic vector bundle $R_{r,d,*}$ constructed in (13) satisfies the condition in Theorem 1.1.

**Step 3:** To complete the proof of Theorem 1.1 we now consider the remaining case where $(\text{genus}(X), |S|) = (0,1)$. So $X = \mathbb{CP}^1$ and $S = \{x\}$ is a singleton set.

Any vector bundle over $\mathbb{CP}^1$ decomposes into a direct sum of line bundles [5, p. 126, Théorème 2.1]. Using this it follows immediately that a parabolic vector bundle $E_*$ of rank $r$ on $\mathbb{CP}^1$ with parabolic structure over $x$ is parabolic semistable if and only if the vector bundle $E_0$ underlying $E_*$ is a direct sum

$$E_0 = L^{\oplus r},$$

and the quasiparabolic filtration is $0 \subset (E_0)_x$.

Given $r$ and $d$ as in Theorem 1.1 let

$$d_0 = \lfloor d/r \rfloor$$

be the integral part of $d/r$. Set

$$\alpha := d/r - d_0 \in [0,1).$$

If $\alpha < (N-1)/N$, then set $R_{r,d,*}$ to be the line bundle $\mathcal{O}_{\mathbb{CP}^1}(d_0)$ equipped with parabolic weight $\alpha + 1/N$ at $x$.

If $\alpha = (N-1)/N$, then set $R_{r,d,*}$ to be the line bundle $\mathcal{O}_{\mathbb{CP}^1}(d_0 + 1)$ with the trivial parabolic structure.
Take any parabolic vector bundle $E_\ast$ of rank $r$ and parabolic degree $d$ on $\mathbb{CP}^1$ with parabolic structure on $x$. If $E_\ast$ is parabolic semistable, then from the above observation that the underlying vector bundle $E_0$ is a direct sum of $r$ copies of a line bundle and the quasiparabolic flag of $E_\ast$ is trivial it follows immediately that $\text{Hom}_{\text{par}}(R_{r,d,\ast}, E_\ast) = 0$.

Now assume that
\begin{equation}
\text{Hom}_{\text{par}}(R_{r,d,\ast}, E_\ast) \neq 0.
\end{equation}
Let
\[ E_0 = \bigoplus_{i=1}^{r} L_i \]
be a decomposition into a direct sum of line bundles of the underlying vector bundle $E_0$. From (15) it follows immediately that $\text{degree}(L_i)$ is independent of $i$. Therefore, $E_0$ is a direct sum of $r$ copies of a line bundle. Again from (15) it follows that all the parabolic weights are at most $\alpha$ defined in (14). Therefore, the parabolic weight must be $\alpha$ with multiplicity $r$. Consequently, $E_\ast$ is parabolic semistable. This completes the proof of Theorem 1.1.

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