Some Characterizations of Cylindrical Helices in $\mathbb{E}^n$

Ahmad T. Ali
Mathematics Department
Faculty of Science, Al-Azhar University
Nasr City, 11448, Cairo, Egypt
e-mail: atali71@yahoo.com

Rafael López∗
Departamento de Geometría y Topología
Universidad de Granada
18071 Granada, Spain
e-mail: rcamino@ugr.es

January 21, 2009

Abstract

We consider a unit speed curve $\alpha$ in Euclidean $n$-dimensional space $\mathbb{E}^n$ and denote the Frenet frame by $\{V_1, \ldots, V_n\}$. We say that $\alpha$ is a cylindrical helix if its tangent vector $V_1$ makes a constant angle with a fixed direction $U$. In this work we give different characterizations of such curves in terms of their curvatures.

MSC: 53C40, 53C50

Keywords: Euclidean $n$-space; Frenet equations; Cylindrical helices curves.

∗Partially supported by MEC-FEDER grant no. MTM2007-61775 and Junta de Andalucía grant no. P06-FQM-01642.
1 Introduction and statement of results

An helix in Euclidean 3-space $E^3$ is a curve where the tangent lines make a constant angle with a fixed direction. An helix curve is characterized by the fact that the ratio $\kappa/\tau$ is constant along the curve, where $\kappa$ and $\tau$ are the curvature and the torsion of $\alpha$, respectively. Helices are well known curves in classical differential geometry of space curves [4] and we refer to the reader for recent works on this type of curves [2, 7]. Recently, Magden [3] have introduced the concept of cylindrical helix in Euclidean 4-space $E^4$ saying that the tangent lines makes a constant angle with a fixed directions. He characterizes a cylindrical helix in $E^4$ if and only if the function

$$\left(\frac{\kappa_1}{\kappa_2}\right)^2 + \left(\frac{1}{\kappa_3} \left(\frac{\kappa_1}{\kappa_2}\right)\right)^2$$

is constant along the curve, where $\kappa_3$ and $\kappa_4$ are the third and the fourth curvature of the the curve. See also [5].

In this work we consider the generalization of the concept of general helices in Euclidean n-space $E^n$. Let $\alpha : I \subset \mathbb{R} \rightarrow E^n$ be an arbitrary curve in $E^n$. Recall that the curve $\alpha$ is said to be of unit speed (or parameterized by arc-length function $s$) if $\langle \alpha'(s), \alpha'(s) \rangle = 1$, where $\langle , \rangle$ is the standard scalar product in Euclidean space $E^n$ given by

$$\langle X, Y \rangle = \sum_{i=1}^n x_i y_i,$$

for each $X = (x_1, \ldots, x_n)$, $Y = (y_1, \ldots, y_n) \in E^n$.

Let $\{V_1(s), \ldots, V_n(s)\}$ be the moving frame along $\alpha$, where the vectors $V_i$ are mutually orthogonal vectors satisfying $\langle V_i, V_i \rangle = 1$. The Frenet equations for $\alpha$ are given by ([2])

$$\begin{bmatrix} V'_1 \\ V'_2 \\ \vdots \\ V'_{n-1} \\ V'_n \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & 0 & 0 & \cdots & 0 & 0 \\ -\kappa_1 & 0 & \kappa_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \kappa_{n-1} & 0 & \kappa_3 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & \kappa_{n-1} \\ 0 & 0 & 0 & 0 & \cdots & -\kappa_{n-1} & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_{n-1} \\ V_n \end{bmatrix}.$$  

Recall the functions $\kappa_i(s)$ are called the i-th curvatures of $\alpha$. If $\kappa_{n-1}(s) = 0$ for any $s \in I$, then $V_n(s)$ is a constant vector $V$ and the curve $\alpha$ lies in a $(n-1)$-dimensional affine subspace orthogonal to $V$, which is isometric to the Euclidean $(n-1)$-space.
We will assume throughout this work that all the curvatures satisfy $\kappa_i(s) \neq 0$ for any $s \in I$, $1 \leq i \leq n - 1$.

**Definition 1.1.** A unit speed curve $\alpha : I \to \mathbb{E}^n$ is called cylindrical helix if its tangent vector $V_1$ makes a constant angle with a fixed direction $U$.

Our main result in this work is the following characterization of cylindrical helices in Euclidean $n$-space $\mathbb{E}^n$.

**Theorem 1.2.** Let $\alpha : I \to \mathbb{E}^n$ be a unit speed curve in $\mathbb{E}^n$. Define the functions

$$G_1 = 1, \quad G_2 = 0, \quad G_i = \frac{1}{\kappa_{i-1}} \left[ \kappa_{i-2} G_{i-2} + G_{i-1} \right], \quad 3 \leq i \leq n.$$ (3)

Then $\alpha$ is a cylindrical helix if and only if the function

$$\sum_{i=3}^{n} G_i^2 = C$$ (4)

is constant. Moreover, the constant $C = \tan^2 \theta$, being $\theta$ the angle that makes $V_1$ with the fixed direction $U$ that determines $\alpha$.

This theorem generalizes in arbitrary dimensions what happens for $n = 3$ and $n = 4$, namely: if $n = 3$, (4) writes $G_3^2 = \kappa_1/\kappa_2 = \kappa/\tau$ and for $n = 4$, (4) agrees with (1).

## 2 Proof of Theorem 1.2

Let $\alpha$ be a unit speed curve in $\mathbb{E}^n$. Assume that $\alpha$ is a cylindrical helix curve. Let $U$ be the direction with which $V_1$ makes a constant angle $\theta$ and, without loss of generality, we suppose that $\langle U, U \rangle = 1$. Consider the differentiable functions $a_i$, $1 \leq i \leq n$,

$$U = \sum_{i=1}^{n} a_i(s) V_i(s), \quad s \in I,$$ (5)

that is,

$$a_i = \langle V_i, U \rangle, \quad 1 \leq i \leq n.$$

Then the function $a_1(s) = \langle V_1(s), U \rangle$ is constant, and it agrees with $\cos \theta$:

$$a_1(s) = \langle V_1, U \rangle = \cos \theta$$ (6)
for any $s$. By differentiation (6) with respect to $s$ and using the Frenet formula (2) we have
\[ a'_1(s) = \kappa_1 \langle V_2, U \rangle = \kappa_1 a_2 = 0. \]

Then $a_2 = 0$ and therefore $U$ lies in the subspace $Sp(V_1, V_3, \ldots, V_n)$. Because the vector field $U$ is constant, a differentiation in (5) together (2) gives the following ordinary differential equation system
\[
\begin{align*}
\kappa_1 a_1 - \kappa_2 a_3 &= 0 \\
\kappa_3 a_3 - \kappa_4 a_4 &= 0 \\
\kappa_3 a_3 - \kappa_4 a_5 &= 0 \\
\vdots \\
\kappa_{n-2} a_{n-2} - \kappa_{n-1} a_n &= 0 \\
\kappa_{n-1} a_{n-1} &= 0
\end{align*}
\]

Define the functions $G_i = G_i(s) = a_i, 3 \leq i \leq n$. (8)

We point out that $a_1 \neq 0$: on the contrary, (7) gives $a_i = 0$, for $3 \leq i \leq n$ and so, $U = 0$: contradiction. The first $(n - 2)$-equations in (7) lead to
\[
\begin{align*}
G_3 &= \frac{\kappa_1}{\kappa_2} \\
G_4 &= \frac{\kappa_3}{\kappa_4} G'_3 \\
G_5 &= \frac{1}{\kappa_4} \left[ \kappa_3 G_3 + G'_4 \right] \\
\vdots \\
G_{n-1} &= \frac{1}{\kappa_{n-2}} \left[ \kappa_{n-3} G_{n-2} + G'_{n-2} \right] \\
G_n &= \frac{1}{\kappa_{n-1}} \left[ \kappa_{n-2} G_{n-2} + G'_{n-1} \right]
\end{align*}
\]

The last equation of (7) leads to the following condition;
\[ G'_n + \kappa_{n-1} G_{n-1} = 0. \]

We do the change of variables:
\[ t(s) = \int_s^{\kappa_{n-1}(u)} du, \quad \frac{dt}{ds} = \kappa_{n-1}(s). \]
In particular, and from the last equation of (9), we have

\[ G'_{n-1}(t) = G_n(t) - \left( \frac{\kappa_{n-2}(t)}{\kappa_{n-1}(t)} \right) G_{n-2}(t). \]

As a consequence, if \( \alpha \) is a cylindrical helix, substituting the equation (10) in the last equation yields

\[ G''_n(t) + G_n(t) = \frac{\kappa_{n-2}(t)G_{n-2}(t)}{\kappa_{n-1}(t)}. \]

The general solution of this equation is

\[ G_n(t) = \left( A - \int \frac{\kappa_{n-2}(t)G_{n-2}(t)}{\kappa_{n-1}(t)} \sin t \, dt \right) \cos t + \left( B + \int \frac{\kappa_{n-2}(t)G_{n-2}(t)}{\kappa_{n-1}(t)} \cos t \, dt \right) \sin t, \]

where \( A \) and \( B \) are arbitrary constants. Then (11) takes the following form

\[ G_n(s) = \left( A - \int \left[ \frac{\kappa_{n-2}(s)G_{n-2}(s)}{\kappa_{n-1}(s)} \sin \int \kappa_{n-1}(s) \, ds \right] ds \right) \sin \int \kappa_{n-1}(s) \, ds \]

\[ + \left( B + \int \left[ \frac{\kappa_{n-2}(s)G_{n-2}(s)}{\kappa_{n-1}(s)} \cos \int \kappa_{n-1}(s) \, ds \right] ds \right) \cos \int \kappa_{n-1}(s) \, ds. \]  

From (10), the function \( G_{n-1} \) is given by

\[ G_{n-1}(s) = \left( A - \int \left[ \frac{\kappa_{n-2}(s)G_{n-2}(s)}{\kappa_{n-1}(s)} \sin \int \kappa_{n-1}(s) \, ds \right] ds \right) \sin \int \kappa_{n-1}(s) \, ds \]

\[ - \left( B + \int \left[ \frac{\kappa_{n-2}(s)G_{n-2}(s)}{\kappa_{n-1}(s)} \cos \int \kappa_{n-1}(s) \, ds \right] ds \right) \cos \int \kappa_{n-1}(s) \, ds. \]  

From Equation (9) and (13), we have

\[ \sum_{i=3}^{n-2} G_i G'_i = G_3 \kappa_3 G_4 + G_4 \left( \kappa_4 G_5 - \kappa_3 G_3 \right) + \ldots \]

\[ + G_{n-3} \left( \kappa_{n-3} G_{n-2} - \kappa_{n-4} G_{n-4} \right) + G_{n-2} G'_{n-2} \]

\[ = G_{n-2} \left( G'_{n-2} + \kappa_{n-3} G_{n-3} \right) \]

\[ = \kappa_{n-2} G_{n-2} G_{n-1} \]

If we integrate the above equation, we have

\[ \sum_{i=3}^{n-2} G_i^2 = C - \left( A - \int \left[ \kappa_{n-2}(s)G_{n-2}(s) \sin \int \kappa_{n-1} \, ds \right] ds \right)^2 \]

\[ - \left( B + \int \left[ \kappa_{n-2}(s)G_{n-2}(s) \cos \int \kappa_{n-1} \, ds \right] ds \right)^2, \]  

(14)
where $C$ is a constant of integration. From Equations (12) and (13), we have
\[
G_n^2 + G_{n-1}^2 = \left( A - \int \left[ \kappa_{n-2}(s)G_{n-2}(s) \sin \int \kappa_{n-1}ds \right] ds \right)^2 \\
+ \left( B + \int \left[ \kappa_{n-2}(s)G_{n-2}(s) \cos \int \kappa_{n-1}ds \right] ds \right)^2,
\]
(15)

It follows from (14) and (15) that
\[
\sum_{i=3}^{n} G_i^2 = C.
\]
Moreover this constant $C$ calculates as follows. From (8), together the $(n - 2)$-equations (9), we have
\[
C = \sum_{i=3}^{n} G_i^2 = \frac{1}{a_1^2} \sum_{i=3}^{n} a_i^2 = \frac{1 - a_1^2}{a_1^2} = \tan^2 \theta,
\]
where we have used (2) and the fact that $U$ is a unit vector field.

We do the converse of Theorem. Assume that the condition (9) is satisfied for a curve $\alpha$. Let $\theta \in \mathbb{R}$ be so that $C = \tan^2 \theta$. Define the unit vector $U$ by
\[
U = \cos \theta \left[ V_1 + \sum_{i=3}^{n} G_i V_i \right].
\]

By taking account (9), a differentiation of $U$ gives that $\frac{dU}{ds} = 0$, which it means that $U$ is a constant vector field. On the other hand, the scalar product between the unit tangent vector field $V_1$ with $U$ is
\[
\langle V_1(s), U \rangle = \cos \theta.
\]
Thus $\alpha$ is a cylindrical helix curve. This finishes the proof of Theorem 1.2.

As a direct consequence of the proof, we generalize Theorem 1.2 in Minkowski space and for timelike curves.

**Theorem 2.1.** Let $E_1^n$ be the Minkowski $n$-dimensional space and let $\alpha : I \rightarrow E_1^n$ be a unit speed timelike curve. Then $\alpha$ is a cylindrical helix if and only if the function $\sum_{i=3}^{n} G_i^2$ is constant, where the functions $G_i$ are defined as in (3).

**Proof.** The proof carries the same steps as above and we omit the details. We only point out that the fact that $\alpha$ is timelike means that $V_1(s) = \alpha'(s)$ is a timelike vector field. The other $V_i$ in the Frenet frame, $2 \leq i \leq n$, are unit spacelike vectors and so, the second equation in (2) changes to $V_2' = \kappa_1 V_1 + \kappa_2 V_3$ ([1, 6]).
3 Further characterizations of cylindrical helices

In this section we present new characterizations of cylindrical helix in $\mathbb{E}^n$. The first one is a consequence of Theorem 1.2.

**Theorem 3.1.** Let $\alpha : I \subset \mathbb{R} \rightarrow \mathbb{E}^n$ be a unit speed curve in Euclidean space $\mathbb{E}^n$. Then $\alpha$ is a cylindrical helix if and only if there exists a $C^2$-function $G_n(s)$ such that

$$G_n = \frac{1}{\kappa_{n-1}} \left[ \kappa_{n-2} G_{n-2} + G_{n-1}' \right], \quad \frac{dG_n}{ds} = -\kappa_{n-1}(s) G_{n-1}(s),$$

where

$$G_1 = 1, G_2 = 0, G_i = \frac{1}{\kappa_{i-1}} \left[ \kappa_{i-2} G_{i-2} + G_{i-1}' \right], \quad 3 \leq i \leq n - 1.$$  

**Proof.** Let now assume that $\alpha$ is a cylindrical helix. By using Theorem 1.2 and by differentiation the (constant) function given in (4), we obtain

$$0 = \sum_{i=3}^{n} G_i G_i'$$

$$= G_3 \kappa_3 G_4 + G_4 \left( \kappa_4 G_5 - \kappa_3 G_3 \right) + ... + G_{n-1} \left( \kappa_{n-1} G_n - \kappa_{n-2} G_{n-2} \right) + G_n G_n'$$

$$= G_n \left( G_n' + \kappa_{n-1} G_{n-1} \right).$$

This shows (16). Conversely, if (16) holds, we define a vector field $U$ by

$$U = \cos \theta \left[ V_1 + \sum_{i=3}^{n} G_i V_i \right].$$

By the Frenet equations (2), $\frac{dU}{ds} = 0$, and so, $U$ is constant. On the other hand, $\langle V_1(s), U \rangle = \cos \theta$ is constant, and this means that $\alpha$ is a cylindrical helix. \qed

We end giving an integral characterization of a cylindrical helix.

**Theorem 3.2.** Let $\alpha : I \subset \mathbb{R} \rightarrow \mathbb{E}^n$ be a unit speed curve in Euclidean space $\mathbb{E}^n$. Then $\alpha$ is a cylindrical helix if and only if the following condition is satisfied

$$G_{n-1}(s) = \left( A - \int \left[ \kappa_{n-2} G_{n-2} \sin \int \kappa_{n-1} ds \right] ds \right) \sin \int^s \kappa_{n-1}(u) du$$

$$- \left( B + \int \left[ \kappa_{n-2} G_{n-2} \cos \int \kappa_{n-1} ds \right] ds \right) \cos \int^s \kappa_{n-1}(u) du.$$ 

for some constants $A$ and $B$. 

7
Proof. Suppose that \( \alpha \) is a cylindrical helix. By using Theorem 3.1, let define \( m(s) \) and \( n(s) \) by
\[
\phi(s) = \int s_0^s \kappa_n(u) du, \\
m(s) = G_n(s) \cos \phi + G_{n-1}(s) \sin \phi + \int \kappa_{n-2} G_{n-2} \sin \phi ds, \\
n(s) = G_n(s) \sin \phi - G_{n-1}(s) \cos \phi - \int \kappa_{n-2} G_{n-2} \cos \phi ds.
\]
(18)

If we differentiate equations (18) with respect to \( s \) and taking into account of (17) and (16), we obtain
\[
\frac{dm}{ds} = 0 \quad \text{and} \quad \frac{dn}{ds} = 0.
\]
Therefore, there exist constants \( A \) and \( B \) such that \( m(s) = A \) and \( n(s) = B \). By substituting into (18) and solving the resulting equations for \( G_{n-1}(s) \), we get
\[
G_{n-1}(s) = \left( A - \int \kappa_{n-2} G_{n-2} \sin \phi ds \right) \sin \phi - \left( B + \int \kappa_{n-2} G_{n-2} \cos \phi ds \right) \cos \phi.
\]

Conversely, suppose that (17) holds. In order to apply Theorem 3.1, we define \( G_n(s) \) by
\[
G_n(s) = \left( A - \int \kappa_{n-2} G_{n-2} \sin \phi ds \right) \cos \phi + \left( B + \int \kappa_{n-2} G_{n-2} \cos \phi ds \right) \sin \phi.
\]
with \( \phi(s) = \int s_0^s \kappa_{n-1}(u) du \). A direct differentiation of (17) gives
\[
G'_{n-1} = \kappa_{n-1} G_n - \kappa_{n-2} G_{n-2}.
\]
This shows the left condition in (16). Moreover, a straightforward computation leads to \( G'_n(s) = -\kappa_{n-1} G_{n-1} \), which finishes the proof.

We end this section with a characterization of cylindrical helices only in terms of the curvatures of \( \alpha \). From the definitions of \( G_i \) in (3), one can express the functions \( G_i \) in terms of \( G_3 \) and the curvatures of \( \alpha \) as follows:
\[
G_j = \sum_{i=0}^{j-3} A_{ji} G_3^{(i)}, \quad 3 \leq j \leq n, \quad (19)
\]
where
\[
G_3^{(i)} = \frac{d^{(i)} G_3}{ds^i}, \quad G_3^{(0)} = G_3 = \frac{\kappa_1}{\kappa_2}.
\]

Then
\[
G_4 = \kappa_3^{-1} G_3' = A_{41} G_3' + A_{40} G_3, \quad A_{41} = \kappa_3^{-1}, A_{40} = 0, \\
G_5 = A_{52} G_3 + A_{51} G_3' + A_{50} G_3, \quad A_{52} = \kappa_4^{-1} A_{41}, A_{51} = \kappa_4^{-1} A_{41}' A_{41}, A_{50} = \kappa_4^{-1} \kappa_3
\]
and so on. Define the following functions:

\[ A_{30} = 1, \quad A_{40} = 0 \]

\[ A_{j0} = \kappa_{j-1}^{-1}\kappa_{j-2}^{-1}A_{(j-2)0} + \kappa_{j-1}^{-1}A'_{(j-1)0}, \quad 5 \leq j \leq n \]

\[ A_{j(3)} = \kappa_{j-1}^{-1}\kappa_{j-2}^{-1}\kappa_{j-3}^{-1}\kappa_{j-4}^{-1}\kappa_3^{-1}, \quad 4 \leq j \leq n \]

\[ A_{j(4)} = \kappa_{j-1}^{-1}\left(\kappa_{j-2}^{-1}\kappa_{j-3}^{-1}\kappa_{j-4}^{-1}\kappa_3^{-1}\right) + \kappa_{j-1}^{-1}\kappa_{j-2}^{-1}\left(\kappa_{j-3}^{-1}\kappa_{j-4}^{-1}\kappa_3^{-1}\right) \]

\[ + \ldots + \kappa_{j-1}^{-1}\kappa_{j-2}^{-1}\kappa_{j-3}^{-1}\kappa_{j-4}^{-1}\kappa_3^{-1}\left(\kappa_3^{-1}\right)^\prime, \quad 5 \leq j \leq n \]

\[ A_{ji} = \kappa_{j-1}^{-1}\kappa_{j-2}A_{(j-2)i} + \kappa_{j-1}^{-1}\left(A'_{(j-1)i} + A_{(j-1)(i-1)}\right) \quad 1 \leq i \leq j - 5, \quad 6 \leq j \leq n \]

and \( A_{ji} = 0 \) otherwise.

The second equation of (16), leads the following condition:

\[ A_{n(n-3)}G_3^{(n-2)} + \left(A'_{n(n-3)} + A_{n(n-4)}\right)G_3^{(n-3)} \]

\[ + \sum_{i=1}^{n-4} \left[A'_{ni} + A_{n(i-1)} + \kappa_{n-1}A_{(n-1)i}\right]G_3^{(i)} \]

\[ + \left(A'_{n0} + \kappa_{n-1}A_{(n-1)0}\right)G_3 = 0, \quad n \geq 3. \quad (20) \]

As a consequence of (20) and Theorem 1.2, we have the following

**Corollary 3.3.** Let \( \alpha : I \to E^n \) be a unit speed curve in \( E^n \). The next statements are equivalent:

1. \( \alpha \) is a cylindrical helix.

2. \[
0 = A_{n(n-3)} \left(\frac{\kappa_1}{\kappa_2}\right)^{(n-2)} + \left(A'_{n(n-3)} + A_{n(n-4)}\right) \left(\frac{\kappa_1}{\kappa_2}\right)^{(n-3)} \\
+ \sum_{i=1}^{n-4} \left[A'_{ni} + A_{n(i-1)} + \kappa_{n-1}A_{(n-1)i}\right] \left(\frac{\kappa_1}{\kappa_2}\right)^{(i)} \\
+ \left(A'_{n0} + \kappa_{n-1}A_{(n-1)0}\right) \left(\frac{\kappa_1}{\kappa_2}\right), \quad n \geq 3.
\]

3. The function

\[
\sum_{j=3}^{n} \sum_{i=0}^{j-3} \sum_{k=0}^{j-3} A_{ji}A_{jk} \left(\frac{\kappa_1}{\kappa_2}\right)^{(i)} \left(\frac{\kappa_1}{\kappa_2}\right)^{(k)} = C
\]

is constant, \( j - i \geq 3, \quad j - k \geq 3. \)
References

[1] A. Ali, R. López, Slant helices in Minkowski space $E^3_1$, preprint 2008: arXiv:0810.1464v1 [math.DG].

[2] H. Gluck, Higher curvatures of curves in Euclidean space, Amer. Math. Monthly, 73 (1996), 699–704.

[3] A. Magden, On the curves of constant slope, YYÜ Fen Bilimleri Dergisi, 4 (1993), 103–109.

[4] R. S. Milman, G. D. Parker, Elements of Differential Geometry, Prentice-Hall Inc., Englewood Cliffs, New Jersey, 1977.

[5] E. Özdamar, H. H. Hacisalihoglu, A characterization of inclined curves in Euclidean n-space, Comm Fac Sci Univ Ankara, series A1, 24A (1975), 15–23.

[6] M. Petrovic-Torgasev, E. Sucurovic, W-curves in Minkowski spacetime, Novi. Sad. J. Math. 32 (2002), 55–65.

[7] P. D. Scofield, Curves of constant precession, Amer. Math. Monthly, 102 (1995), 531–537.