A Fast Exact Quantum Algorithm for Solitude Verification

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Abstract
Solitude verification is arguably one of the simplest fundamental problems in distributed computing, where the goal is to verify that there is a unique contender in a network. This paper devises a quantum algorithm that exactly solves the problem on an anonymous network, which is known as a network model with minimal assumptions [Angluin, STOC’80]. The algorithm runs in $O(N)$ rounds if every party initially has the common knowledge of an upper bound $N$ on the number of parties. This implies that all solvable problems can be solved in $O(N)$ rounds on average without error (i.e., with zero-sided error) on the network. As a generalization, a quantum algorithm that works in $O(N \log_2(\max\{k, 2\}))$ rounds is obtained for the problem of exactly computing any symmetric Boolean function, over $n$ distributed input bits, which is constant over all the $n$ bits whose sum is larger than $k$ for $k \in \{0, 1, \ldots, N - 1\}$. All these algorithms work with the bit complexities bounded by a polynomial in $N$.

1 Introduction
1.1 Background
In synchronous distributed models of computation, the number of rounds (also called the round complexity) is one of the most important complexity measures, especially when we want to design fast distributed algorithms. From a complexity-theoretic point of view, seeking low-round complexity leads to clarifying how much parallelism the problem inherently has. This would be reminiscent of the study of shallow circuit classes (e.g., NC), in which the depth of a circuit solving a problem corresponds to the inherent parallelism of the problem. In this paper, we study distributed algorithms with low-round complexity.

The round complexity is closely related to the diameter of the underlying graph of a given network. This is because, when computing global properties of the network, it necessarily takes at least as many rounds for message exchanges as the value of the diameter for some party to get information from the farthest party. Therefore, when every party’s initial knowledge includes an upper bound $\Delta$ on the diameter, the ultimate goal is to achieve a round complexity close to (typically, linear in) $\Delta$. In particular, we are interested in whether the round complexity $O(n)$ (resp., $O(N)$) can be achieved when every party’s initial knowledge includes the number $n$ of parties (resp., an upper bound $N$ on $n$). This can actually be achieved in a straightforward manner if there is a unique party (called the leader) distinguishable from the others or, almost equivalently, if every party has its own identity: The unique leader, which can be elected in $O(\Delta)$ rounds if every party has its own identity, can gather all the distributed inputs, solve the problem, and distribute the solution to every party in $O(\Delta)$ rounds. However, it is not a simple task to bound the achievable round complexity for networks where no party has its own identity (namely, all parties with the same number of communication links are identical). Such a network is called an anonymous network, which was first introduced by Angluin [Ang80] to examine how much each party in a network needs to know about its own identity and other parties’ (e.g., Refs. [IR81, IR90, AM94, KKvdB94, BSV+96, YK96a, YK96b, BV02]).
and thereby understand the fundamental properties of distributed computing. It has been revealed in the literature that anonymous networks make it highly non-trivial or even impossible to exactly solve many distributed problems, including the leader election one, that are easy to solve on non-anonymous networks (i.e., the networks in which every party has its own identity). Here, by “exactly solve”, we mean “solve without error within a bounded time”. The good news is that if the number of parties is provided to each party, all solvable problems can be solved exactly in \( O(n) \) rounds\(^1\) for any unknown underlying graph by constructing tree-shaped data structures, called universal covers [Ang80] or views [YK96a]. Obviously, however, this does not help us deal with the infinitely many instances of fundamental problems that are impossible to solve on anonymous networks. The best known among them is the leader election problem (LE\(_n\)), the problem of electing a unique leader. There are infinitely many \( n \) such that LE\(_n\) cannot be solved exactly for anonymous networks with certain underlying graphs over \( n \) nodes even if \( n \) is provided to each party [Ang80, YK96a, BV02].

The above situation changes drastically if quantum computation and communication are allowed on anonymous networks (called anonymous quantum networks): LE\(_n\) can be solved exactly for any unknown underlying graph, even when only an upper bound \( N \) on \( n \) is provided to every party [TKM12]. This implies that, if a problem is solvable in non-anonymous networks, then it is also solvable in anonymous quantum networks. For the round complexity, however, the known quantum algorithms for electing a unique leader require super-linear rounds in \( N \) (or \( n \) when \( n \) is provided) [TKM05, TKM12].

Motivated by this situation, we study the linear-round exact solvability of another fundamental problem, the solitude verification problem (SV\(_n\)) [AAHK86, AAHK94, HKAA97], in anonymous quantum networks. The goal of SV\(_n\) is to verify that there is a unique contender in a network with an unknown set of contenders (which may be empty) among the \( n \) parties. Although the final target is to clarify whether a unique leader can be elected in linear rounds or not, SV\(_n\) would be a natural choice as the first step. This is because SV\(_n\) is a subproblem of many common problems, including the leader election problem: a unique leader can be elected by repeating attrition and solitude verification as observed in Ref. [AAHK86]. Another reason is that SV\(_n\) is one of the simplest nontrivial problems concerned with the global properties of a given network, as pointed out in Ref. [AAHK94]. Indeed, SV\(_n\) is not always solvable in the classical case: One can easily show, by modifying the proof of Theorem 4.5 in Ref. [Ang80], that it is impossible to exactly solve SV\(_n\) on any anonymous classical network whose underlying graph is not a tree if only an upper bound \( N \) is provided to each party (the problem can be solved exactly in \( O(n) \) rounds if \( n \) is provided). In the quantum setting, the only quantum algorithms for SV\(_n\) are the straightforward ones that first elect a unique leader (with a super-linear round complexity), who then verifies that there is a unique contender.

Recently, Kobayashi et al. [KMT14] proposed an \( O(N) \)-round quantum algorithm when each communication link in the network is bidirectional, i.e., the underlying graph is undirected. However, their algorithm cannot work in the more general case where the underlying graph is directed. This is due to a technicality that is distinctive in quantum computing: Their algorithm uses a standard quantum technique to erase “garbage” information generated in the course of computation. The technique inverts some operations that have been performed and thus involves sending back messages via bidirectional communication links in the distributed computing setting.

\[1.2\] Our Results

Let \( D_n \) be the set of all strongly connected digraphs with \( n \) nodes. Our main result is an \( O(N) \)-round quantum algorithm that exactly solves SV\(_n\), where the input to each party is a binary value indicating whether the party is a contender or not (see Sec. 2.2 for a more formal definition).

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\(^1\)If the diameter \( \delta \) is given, all solvable problems can be solved in \( O(\delta) \) rounds [Hen14].
Let \( D_n \) denote the directed line graph on \( n \) vertices. Consider a collection \( S_n(k) \) of symmetric functions from \( \{ 0, 1 \}^n \rightarrow \{ 0, 1 \} \) for each \( k \in \{ 0, 1, \ldots, n-1 \} \) such that \( f(x) \) is constant on all \( x := (x_1, \ldots, x_n) \in \{ 0, 1 \}^n \) with \( \sum_{i=1}^{n} x_i > k \). Note that \( S_n(k) \subset S_n(k+1) \) for each \( k \in \{ 0, \ldots, n-2 \} \), and \( S_n(k) \) for any \( k \geq n-1 \) represents the set of all symmetric functions over \( n \) bits. In particular, the function corresponding to \( SV_n \) belongs to \( S_n(1) \). We then have the following theorem.

**Theorem 3** Suppose that there are \( n \) parties on an anonymous network with any unknown underlying graph in \( D_n \) in which an upper bound \( N \) on \( n \) is provided to each party. For every \( f \in S_n(k) \) with \( k \in \{ 0, 1, \ldots, N-1 \} \), there exists a quantum algorithm that exactly computes \( f(x) \) over distributed input \( x \in \{ 0, 1 \}^n \) on the network in \( O(N \log_2(\max\{k, 2\})) \) rounds with a bit complexity bounded by some polynomial in \( N \).

Note that an \( O(N) \)-round quantum algorithm for \( LE_n \) would imply that all solvable problems, including computing \( S_n(k) \), can be solved in \( O(N) \) rounds (since the leader can convert the anonymous network into the corresponding non-anonymous one). Computing \( S_n(k) \) is thus something lying between \( SV_n \) and \( LE_n \) with respect to linear-round solvability.

### 1.3 Technical Outline

Recall that the reason the \( O(N) \)-round leader election algorithm in Ref. [KMT14] does not work on directed graphs is that it sends back messages via bidirectional communication links to erase “garbage” information produced in the course of computation. This seems inevitable as it uses (a version of) the quantum amplitude amplification [CK98] (or a special case of the general quantum amplitude amplification [BHMT02]). It is a
critical issue, however, when the underlying graph is directed, since, although strong connectivity ensures at least one directed path on which the message could be sent back, parties cannot identify the path in the anonymous network (since the original sender of the message cannot be identified). In the classical setting, such an issue cannot arise since the message need not be sent back (the sender has only to keep a copy of the message if it needs to).

Our idea for resolving this issue is to employ the symmetry-breaking procedure introduced in [TKM12, Sec. 4]. The procedure was used to solve the leader election problem as follows: Initially, all parties are candidates for the leader and they repeatedly perform a certain distributed procedure that reduces the set \( S \) of the candidates by at least 1. More concretely, the procedure partitions \( S \) into at least two subsets if \(|S| \geq 2\) and removes one of them from \( S \). A simple but effective way of viewing this is that it not only reduces \( S \) but decides whether \(|S|\) is at least two or not, since it can partition \( S \) only when there are at least two candidates. This observation would exactly solve \( SV_n \) by regarding \( S \) as the set of contenders if the procedure outputs the correct answer with certainty. However, the procedure heavily depends on the following unknowns: the cardinality of \( S \) and the number \( n \) of parties. In Ref. [KMT14], a similar problem arises when deciding whether an \( n \)-bit string \( x \) is of Hamming weight at most 1, and it is resolved by running a base algorithm in parallel for all possible guesses at \(|x|\) and making the decision based on the set of all outputs (the “base algorithm” uses amplitude amplification and is totally different from the symmetry-breaking approach). Together with a simple algorithm for testing whether \( x \) is the all-zero string, the parallel execution of the base algorithm is used in the leader election algorithm [KMT14] to verify that a random \( x \) is of Hamming weight exactly one. This verification framework actually works in our case, and it underlies the entire structure of our algorithm. Namely, we replace the base algorithm in the framework with a subroutine constructed by carefully combining the symmetry-breaking procedure introduced in Ref. [TKM12] with classical techniques related to the view [YK96a, Nor95, BV02, Tan12]. This means that all parties collaborate to perform this subroutine in parallel for all possible pairs of guesses at \((n, |S|)\). The round complexity is thus equal to the number of rounds required to perform the subroutine once, i.e., \( O(N) \) rounds. To show the correctness, we prove that the set of the outputs over all possible pairs of the guesses yields the correct answer to any \( SV_n \) instance with certainty. This needs an in-depth and careful analysis of all operations of which our algorithm consists for every pair not necessarily equal to \((n, |S|)\).

Before the present work, it has seemed as if the symmetry-breaking approach introduced in Refs. [TKM05, TKM12] is entirely different from the amplitude amplification approach used in Ref. [KMT14]. Our algorithm first demonstrates that these approaches are quite compatible, and, indeed, the technical core of Refs. [TKM05, TKM12] can effectively function in the algorithmic framework proposed in Ref. [KMT14]. This would contribute to a better understanding of distributed quantum computing and would be very helpful for future studies of quantum algorithms.

Our algorithm can be generalized to the case of computing a family \( S_n(k) \) of more general symmetric functions as follows: All parties collaborate to partition \( S \) into subsets by recursively applying the procedure up to \( \lceil \log_2 \max \{k, 2\} \rceil \) levels. If there is a singleton set among the subsets at a certain recursion level, then the algorithm stops and all parties elect the only member of the subset as a leader, who can compute \(|S|\) and thus compute the value of the given function in \( S_k \). If no singleton set appears even after the\( \lceil \log_2 \max \{k, 2\} \rceil \)-th recursion level, there must be more than \( k \) parties in \( S \), in which case any function in \( S_k \) is constant by the definition.

1.4 Related Work

Pal, Singh, and Kumar [PSK03] and D’Hondt and Panangaden [DP06b] dealt with \( LE_n \) and the GHZ-state sharing problem in a different setting, where pre-shared entanglement is assumed but only classical communication is allowed. The relation between several network models that differ in available quantum
resources has been discussed by Gavoille, Kosowski, and Markiewicz [GKM09]. Recently, Elkin et al. [EKNP14] proved that quantum communication cannot substantially speed up algorithms for some fundamental problems, such as the minimum spanning tree, compared to the classical setting. For fault-tolerant distributed quantum computing, the Byzantine agreement problem and the consensus problem were studied by Ben-Or and Hassidim [BOH05] and Chlebus, Kowalski, and Strojnowski [CKS10], respectively. In the cryptographic context where there are cheating parties, Refs. [AS10, Gan09] devises quantum algorithms that elect a unique leader with a small bias. Some quantum distributed protocols were experimentally demonstrated by Gaertner et al. [GBK08] and Okubo et al. [OWJ08].

See the surveys [BR03, DP06a, BT08] and the references therein for more work on distributed quantum computing.

1.5 Organization

Section 2 defines the network model and the problems considered in this paper. It then mentions several known facts employed in the subsequent sections. Section 3 provides the structure of our algorithm and then proves Theorem 1 assuming several properties of the key subroutine $Q_{h,m}$. Section 4 describes $Q_{h,m}$ step by step and presents numerous claims and propositions to show how each step takes effect. Section 5 proves all the claims and propositions appearing in Section 4 and then completes the proof that $Q_{h,m}$ has the properties assumed in Section 3. Section 6 proves Corollary 2, and then generalizes Theorem 1 for proving Theorem 3.

2 Preliminaries

Let $\mathbb{C}$ be the set of all complex numbers, $\mathbb{N}$ the set of all positive integers, and $\mathbb{Z}^+$ the set of all non-negative integers. For any $m, n \in \mathbb{Z}^+$ with $m < n$, $[m..n]$ denotes the set $\{m, m+1, \ldots, n\}$, and $[n]$ represents $[1..n]$.

2.1 The Distributed Computing Model

We first define a classical model and then adapt it to the quantum model, where every party can perform quantum computation and communication.

A classical distributed network consists of multiple parties and unidirectional communication links, each of which connects a pair of parties. By regarding the parties and links as nodes and edges, respectively, in a graph, the topology of the distributed network can be represented by a strongly connected digraph (i.e., directed graph), which may have multiple edges or self-loops.

A natural assumption is that every party can distinguish one link from another among all communication links incident to the party; namely, it can assign a unique label to every such link. We associate these labels with communication ports. Since every party has incoming and outgoing communication links (although a self-loop is a single communication link, it looks like a pair of incoming and outgoing links for the party), it has two kinds of communication ports accordingly: in-ports and out-ports. For the sake of convenience, we assume that each edge $e := (u, v) \in E$ is labeled with the pair of the associated out-port and in-port (of the two different parties); namely, $(\sigma_{u}^{\text{out}}[e], \sigma_{v}^{\text{in}}[e])$ (each party can know the labels of edges incident to it by only a...
one-round message exchange as described in Example 1). In our model, each party knows the number of its in-ports and out-ports and can choose one of its in-ports or one of its out-ports in any way whenever it sends or receives a message.

In distributed computing, what information each party initially possesses has a great impact on complexity. Let $I_i$ be the information that only party $i$ initially knows, such as its local state and the number of its ports. Let $I_G$ be the information initially shared by all parties. We may call $I_i$ and $I_G$ local and global information, respectively.\(^2\)

Without loss of generality, we assume that every party $l$ runs the same algorithm with $(I_i, I_G)$ as its arguments (or the input to the algorithm), in addition to the instance $x_l$ of a problem to solve. We will not explicitly write $(I_i, I_G)$ as input to algorithms when it is clear from the context. Note that $(I_i, I_G)$ is not part of the problem instance but part of the model. Also note that the algorithm may invoke subroutines with part of $(x_l, I_i, I_G)$ as input to the subroutine. If all parties in a network have the same local information except for the number of their ports, the network is said to be anonymous, and the parties in the anonymous network are said to be anonymous. In this paper, we assume that for each party $l$, $I_i$ consists of a common initial state, a common description of the same algorithm, and the numbers $(d_i^{in}, d_i^{out})$ of in-ports and out-ports. An extreme case of networks is a regular graph, such as a directed ring, in which case each party is identical to any other party; that is, effectively, every party has the same identifier. Obviously, the difficulty of solving a problem depends on the underlying graph. Moreover, it may also depend on port-numberings. This can be intuitively understood from Example 1. When solving problems on distributed networks, we do not assume a particular port-numbering; in other words, we say that a problem can be solved if there is an algorithm that solves the problem for any port-numbering.

This paper deals with only anonymous networks but may refer to a party with its index (e.g., party $i$) only for the purpose of clear descriptions. Our goal is to construct an algorithm that works for any port-numbering on any digraph in $D_n$, where $D_n$ denotes the set of all $n$-node strongly connected digraphs, which may have multiple edges or self-loops, and is used through this paper.

**Example 1** Fig. 1 shows two anonymous networks, (a) and (b), on the same four-node regular graph with different port-numberings $\sigma$ and $\tau$, respectively, where (1) each party has two in-ports and two out-ports, and (2) each directed edge $e := (u, v)$ is labeled with $(\sigma^\text{out}_u(e), \sigma^\text{in}_u(e))$ and $(\tau^\text{out}_u(e), \tau^\text{in}_u(e))$ on networks (a) and (b), respectively, where $\sigma^\text{out}_u(e)$ (resp., $\tau^\text{out}_u(e)$) is put on the source side and $\sigma^\text{in}_u(e)$ (resp., $\tau^\text{in}_u(e)$) is put on the destination side. Observe that each party can know the label of each incoming edge incident to the party by exchanging a message: Each party sends a message “i” out via every out-port $i$, and if another party receives this message via in-port $j$, the receiver concludes that it has an incoming edge with label $(i, j)$. To elect a unique leader on network (a) in Fig. 1, consider the following game: (1) If a party has an incoming edge with label $(i, j)$, then it scores 1 point if $i > j$ (win), 0 points if $i = j$ (draw), -1 point if $i < j$ (lose); (2) each party earns the the sum of points over all its incoming edge; and (3) a party wins the game if it earns the largest sum of points among all parties. It is easy to see that each party can compute the sum of points as we observed. In the case of (a), the upper-left party is the unique winner: it earns 1 point in total, while the others earn 0 or -1 points. This fact can be used to elect a unique leader. In the case of (b), however, all parties earn 0 points. Hence, the above game cannot elect a unique leader. Actually, no deterministic algorithm can elect a unique leader in the case of (b) [YK96a].

A network is either synchronous or asynchronous. In the synchronous case, message passing is performed synchronously. The unit interval of synchronization is called a round, which consists of the following sequential execution of the two (probabilistic) procedures that are defined in the algorithm invoked

\(^2\)In this paper, we do not consider the case where only a subset of parties share some information, since an upper bound on the complexity for that case can be obtained by regarding such information as local information (when dealing with non-cryptographic/non-fault-tolerant problems, which is our case).
by each party [Lyn96]: one procedure changes the local state of the party depending on the current local state and the incoming messages, and then removes the messages from ports; the other procedure then prepares new messages and decides the ports through which the messages should be sent, depending on the current local state, and finally the messages are sent out via the ports. We do not impose any limit on the number of bits in a message sent in each round.

A network that is not synchronous is asynchronous. In asynchronous networks, every party can send messages at any time and the time it takes for a message to go through a communication link is finite but not bounded. This paper deals with synchronous networks for simplicity, but our algorithms can be emulated in asynchronous networks without sacrificing the communication cost by just delaying the local operation that would be done in each round in the synchronous setting until a message arrives at every port.

The only difference between the quantum and classical models is that every party can perform quantum computation and communication in the former model [for the basics of quantum computation and communication, we refer readers to standard textbooks (e.g., Refs. [NC00, KSV02, KLM07])]. More concretely, the two procedures for producing messages and changing local states are replaced with physically realizable super-operators (i.e., a trace-preserving completely positive super-operator) that act on the registers storing the local quantum state and quantum messages received to produce new quantum messages and a new local quantum state and to specify port numbers. Accordingly, we assume that every communication link can transfer quantum messages. For sending quantum messages at the end of each round, each party sends out one of its quantum registers through the specified out-port. The party then receives quantum registers from its neighbors at the beginning of the next round and uses them for local quantum computation.

This paper focuses on the required number of rounds as the primary complexity measure (called round complexity). This is often used as an approximate value of time complexity, which includes the time taken by local operations as well as the time taken by message exchanges. Although our primary goal is to construct algorithms with low round complexities, our algorithms all have bit complexities bounded by certain polynomials in the given upper bound on the number of parties (or polynomial bit complexities for short), where the bit complexity of an algorithm is the number of bits or qubits communicated by the algorithm (a.k.a., communication complexity).

Finally, we assume that there are no faulty parties and no faulty communication links.
2.2 Solitude Verification and Leader Election

Let \( n \in \mathbb{N} \). For any bit string \( x \in \{0,1\}^n \), let \( |x| \) be the Hamming weight of \( x \), i.e., the number of 1's in \( x \). For any \( G := (V,E) \in \mathcal{D}_n \), without loss of generality, we assume that \( V \) is identified with the set \([n]\).

For any \( k \in \mathbb{Z}^+ \), let \( H_k : \{0,1\}^n \rightarrow \{\text{true},\text{false}\} \) be the symmetric Boolean function that is true if and only if \(|x|\) is equal to \( k \) for input \( x \in \{0,1\}^n \) distributed over \( n \) parties (i.e., each party is in possession of one of the \( n \) bits). Note that \( k \) may be larger than \( n \), in which case \( H_k(x) \) is false for all \( x \in \{0,1\}^n \). The function \( H_k \) is hence well-defined even if each party does not know the integer \( n \) (since the value of \( H_k \) depends not on \( n \) but on \( k \) and \(|x|\)). We also define \( T_k : \{0,1\}^n \rightarrow \{\text{true},\text{false}\} \) as the symmetric Boolean function such that \( T_k(x) \) is true if and only if \(|x| \leq k \) for \( x \in \{0,1\}^n \); namely, \( T_k(x) = \bigvee_{i=0}^{k} H_i(x) \). Note again that \( T_k \) is well-defined even if \( n \) is unknown. The solitude verification problem is equivalent to computing \( H_1 \) as can be seen from the following definition.

**Definition 4 (Solitude Verification Problem (**\( SV_n \))** Suppose that there is a distributed network with any underlying graph \( G \in \mathcal{D}_n \), which is unknown to any party. Suppose further that each party \( i \in [n] \) in the network is given as input a Boolean value \( x_i \in \{0,1\} \) and a variable \( y_i \in \{\text{true},\text{false}\} \) initialized to true. The goal is to set \( y_i = H_1(x) \) for every \( i \in [n] \), where \( x := (x_1, \ldots, x_n) \).

If every party has a unique identifier picked from, say, \([n]\), this problem can easily be solved by simply gathering all \( x_i \)'s to the party numbered 1 (although the complexity may not be optimal). On anonymous networks, however, this simple idea can no longer work since the parties do not have a unique identifier. If the global information \( I_G \) includes the exact number \( n \) of parties, \( SV_n \) can be solved deterministically by a non-trivial algorithm, which runs in \( O(n) \) rounds with a polynomial bit complexity [YK96a, BV02, Tan12]. In a more general case, however, this is impossible.

**Fact 5 ([Ang80, IR90])** There are infinitely many \( n \in \mathbb{N} \) such that, if only an upper bound on the number \( n \) of the parties is provided to each party as global information, \( SV_n \) cannot be solved in the zero-error (i.e., Las Vegas) setting as well as in the exact setting on anonymous classical networks with a certain underlying graph \( G \in \mathcal{D}_n \).

Our main contribution is a quantum algorithm that exactly computes the function \( T_1 \) in rounds linear in \( N \) even if only an upper bound \( N \) on \( n \) is provided. This implies that there exists a quantum algorithm that exactly computes \( H_1 = T_1 \land \neg T_0 \) (and thus \( SV_n \)) in rounds linear in \( N \), since there is a simple (classical) deterministic algorithm for computing \( T_0 \) (i.e., the negation of OR over all input bits).

We next define the leader election problem, which is closely related to \( SV_n \).

**Definition 6 (Leader Election problem (**\( LE_n \))** Suppose that there is a distributed network with any underlying graph \( G \in \mathcal{D}_n \), which is unknown to any party. Suppose further that each party \( i \in [n] \) is given a variable \( y_i \) initialized to 1. The goal is to set \( y_k = 1 \) for arbitrary but unique \( k \in [n] \) and \( y_i = 0 \) for every remaining \( i \in [n] \setminus \{k\} \).

\( LE_n \) is a fundamental problem with a long history of research starting from the dawn of distributed computing; there are a lot of studies on efficient algorithms for solving it on non-anonymous networks. On anonymous networks, however, it is impossible to exactly solve \( LE_n \).

**Fact 7 ([Ang80, IR90, YK96a, BV02])** There are infinitely many \( n \in \mathbb{N} \) such that, even if the number \( n \) is provided to each party as global information, \( LE_n \) cannot be solved exactly on anonymous classical networks with a certain underlying graph \( G \in \mathcal{D}_n \). Moreover, if only an upper bound on \( n \) is provided to each party, it is impossible to solve \( LE_n \) even with zero-error.
Actually, the former part of this fact is a corollary of a more general theorem proved in Ref. [YK96a, BV02], which provides a necessary and sufficient condition on underlying graphs and port-numbering for exactly solving $\text{LE}_n$ when every party knows the number $n$ as its global information. The latter part of Fact 7 (i.e., the zero-error unsolvability of $\text{LE}_n$) follows from Fact 5 and the fact\(^3\) that $\text{SV}_n$ is reducible to $\text{LE}_n$.

In contrast, it is possible to solve the problems on anonymous quantum networks.

**Fact 8 ([TKM05])** There exists a quantum algorithm that, for every $n \in \mathbb{N}$ with $n \geq 2$, if an upper bound $N$ on $n$ is provided to each party as global information, exactly solves $\text{LE}_n$ in $\Theta(N \log N)$ rounds with a polynomially bounded bit complexity $O(N \cdot p(N))$ on an anonymous network with any unknown underlying graph $G \in \mathcal{D}_n$, where $p(N)$ is the bit complexity of constructing the view [YK96a, BV02], a tree-like data-structure, of depth $O(N)$. The best known bound on $p(N)$ is $\tilde{O}(N^6)$ [Tan12].

**Remark 9** When an upper bound $\Delta$ on the diameter of the graph is provided to each party, the round complexity becomes $O(\Delta(\log N)^2)$ by using the recent result on the view in Ref. [Hen14].

We next provide a powerful primitive in classical distributed computing: a linear-round algorithm that deterministically computes any symmetric function on anonymous networks. This algorithm is actually obtained from a more generic one that effectively uses the full-power of deterministic computation in the anonymous classical network: during the execution of the generic algorithm, every party constructs a tree-like data structure, called view [YK96a, BV02], which contains as much information as it can gather in the anonymous network. In terms of graph theory, the view of depth $k$ is defined for each node $v$, and it is a labeled tree rooted at $v$ that is obtained by sharing the maximal common suffix of every pair of $k$-length directed paths to the node $v$. It is not difficult to see that the view of depth $k$ can be constructed by exchanging messages $k$ times [YK96a, BV02] as follows: (1) every party creates a 0-depth view, which is nothing but a single node, $r$, labeled by the input to the party; (2) for each $j = 1, \ldots, k$ in this order, each party sends its $(j - 1)$-depth view to every neighbor, receives a $(j - 1)$-depth view from every neighbor, and then makes a $j$-depth view by connecting the node $r$ with the roots of received views by edges labeled with port numbers. Since it is proved in Ref. [Nor95] that setting $k = 2n - 1$ is necessary and sufficient to gather all the information in the network, $(2n - 1)$-depth views need to be constructed in general for solving problems. For $k = 2n - 1$, the above naive construction algorithm obviously has an exponential bit complexity in $n$, but it is actually possible to compress each message so that the total bit complexity is polynomially bounded in $n$ [Tan12]. Since the Hamming weight $|x|$ of $n$ distributed input bits can be locally computed as a rational function of the number $n$ and the number of non-isomorphic subtrees of depth $n - 1$ in the view [YK96a, Nor95, BV02], every party can compute any symmetric function on $x$ from its view whenever $n$ is given. This algorithm is summarized as follows.

**Fact 10 (Computing Symmetric Functions [YK96a, Nor95, BV02, Tan12])** For any $n \in \mathbb{N}$ with $n \geq 2$, suppose an anonymous network with any unknown underlying graph $G \in \mathcal{D}_n$, where the number $n$ of parties is given to each party as global information. Then, there exists a deterministic algorithm that computes any symmetric function over $n$ distributed input bits in $O(n)$ rounds with bit complexity $O(p(n))$, where $p(\cdot)$ is the function defined in Fact 8.

To exactly compute the symmetric function with this algorithm, every party needs to know the exact number $n$ of the parties. Nevertheless, even when a wrong $m \in \mathbb{N}$ is provided instead of $n$, the algorithm can still run through and output some value. Namely, it constructs the view of depth $2m - 1$ for every party and outputs the value of the rational function over the number $m$ and the number of isomorphic subtrees in the view, as can be seen from the above sketch of the algorithm. This requires $O(m)$ rounds and $O(\max\{p(m), p(n)\})$.

\(^3\)Once elected, the leader can verify that $|x|$ is one: The leader first assigns a unique identifier to each party. It then gathers all $x_i$ together with the identifier of the owner of $x_i$ (along a spanning tree after setting it up).
bits of communication. In fact, we will use this algorithm for a guess \( m \) at \( n \) (this \( m \) is not necessarily equal to \( n \)). Although the output may be wrong, the set of the outputs over all possible guesses \( m' \)'s contains useful information as will be described in the following sections.

Finally, we define some terms. Suppose that each party \( l \) has a \( c \)-bit string \( z_l \in \{0,1\}^c \) (i.e., the \( n \) parties share a \( cn \)-bit string \( z := (z_1, z_2, \ldots, z_n) \)). Given a set \( S \subseteq [n] \), the string \( z \) is said to be consistent over \( S \) if \( z_l \) has the same value for all \( l \) in \( S \). Otherwise, \( z \) is said to be inconsistent over \( S \). In particular, if \( S \) is the empty set, then any string \( z \) is consistent over \( S \). We also say that a \( cn \)-qubit pure state \( |\psi\rangle := \sum_{z \in \{0,1\}^n} \alpha_{\! z} |z\rangle \) shared by the \( n \) parties is consistent (inconsistent) over \( S \) if \( \alpha_{\! z} \neq 0 \) only for \( z \)'s that are consistent (inconsistent) over \( S \). Note that there are pure states that are neither consistent nor inconsistent over \( S \) (i.e., superpositions of both consistent string(s) and inconsistent string(s) over \( S \)). We may simply say “consistent/inconsistent strings/states” if the associated set \( S \) is clear from the context. We say that a quantum state \( |\psi\rangle \) is an \( m \)-partite GHZ-state if \( |\psi\rangle \) is of the form \( \frac{1}{\sqrt{2}} (|0\rangle^\otimes m + |1\rangle^\otimes m) \) for some natural number \( m \). When \( m \) is clear from the context, we may simply call the state \( |\psi\rangle \) a GHZ-state.

3 Solitude Verification (Proof of Theorem 1)

This section proves Theorem 1 by showing a quantum algorithm that exactly computes \( H_1 \) on \( n \) bits distributed over an anonymous quantum network with \( n \) parties when an upper bound \( N \) on \( n \) is provided as global information. Suppose that a bit \( x_i \) is provided to each party \( i \in [n] \) as input. We say that any party \( i \) with \( x_i = 1 \) is active and any party \( j \) with \( x_j = 0 \) is inactive. Let \( x := (x_1, \ldots, x_n) \).

The algorithm actually computes the functions \( T_1 \) and \( T_0 \) on input \( x \) in parallel and then outputs \( H_1(x) = \neg T_0(x) \wedge T_1(x) \). The function \( T_0 \) can be computed in any anonymous network by a simple and standard deterministic algorithm in \( O(N) \) rounds if any upper bound \( N \) on \( n \) is given, as stated in Proposition 11 (the proof is provided in Sec. 5 for completeness).

**Proposition 11 (Computing \( T_0 \))** Suppose that there are \( n \) parties on an anonymous network with any underlying graph \( G := (V, E) \in D_n \), in which an upper bound \( \Delta \) on \( n \) is given as global information. Then, there exists a deterministic algorithm that computes \( \neg \bigvee_{i \in [n]} x_i \) on the network in \( O(\Delta) \) rounds with bit complexity \( O(m\Delta) \), where \( m := |E| \), if every party \( i \in V \) gives a bit \( x_i \in \{0,1\} \) as the input to the algorithm.

The difficult part is computing \( T_1 \). It runs another quantum algorithm \( Q_{h,m} \) as a subroutine for all pairs of \( (h, m) \in [0..m] \times [2..N] \). The integers \( h \) and \( m \) mean guesses at the number of active parties (i.e., \( |x| \)) and the number of parties (i.e., \( n \)), respectively, which are unknown to any party. The following lemma states the properties of \( Q_{h,m} \).

**Lemma 12 (Main)** There exists a set of distributed quantum algorithms \( \{Q_{h,m}: (h, m) \in [0..m] \times [2..N]\} \) such that, if every party \( i \) (\( i = 1, \ldots, n \)) performs the algorithm \( Q_{h,m} \) with a bit \( x_i \in \{0,1\} \) and an upper bound \( N \) on the number \( n \) of parties, then \( Q_{h,m} \) always reaches a halting state\(^4\) in \( O(N) \) rounds with \( O(N^3) \) qubits and \( O(N^6) \) classical bits of communication for every \( m \in [2..N] \) and every \( h \in [0..m] \) and satisfies the following properties:

1. For each \( (h, m) \), \( Q_{h,m} \) outputs “true” or “false” at each party, where these outputs agree over all parties;
2. for \( (h, m) = (|x|, n) \), \( Q_{h,m} \) computes \( T_1(x) \) with certainty for every \( x \); and

\(^4\)This property is crucial when one emulates our algorithm in asynchronous (anonymous) networks, since deadlocks that parties cannot detect may occur.
3. for every \( m \in [2..N] \) and every \( h \in [0..m] \), \( Q_{h,m} \) outputs “true” with certainty whenever \( |x| \in \{0,1\} \),

where \( x := (x_1, \ldots, x_n) \).

Section 4 demonstrates how \( Q_{h,m} \) works, and Sec. 5 proves that it works as stated in the lemma.

After all parties collaborate to run \( Q_{h,m} \) (in parallel) for all pairs \( (h, m) \), every party obtains \( (N - 1)(m + 1) \) classical results:

\[
\{(h, m, q_{h,m}) : (h, m) \in [0..m] \times [2..N], q_{h,m} \in \{true, false\}\},
\]

where \( q_{h,m} \) is the output of \( Q_{h,m} \) for input \( x \). Note that this set of results agree over all parties by property 1 in Lemma 12. From properties 2 and 3, we can observe that there is at least one false in \( \{q_{h,m}\} \) if and only if \( T_1(x) \) is “false” (since \( q_{h,m} \) is equal to \( T_1(x) \) whenever \( (h, m) \) equals \( (|x|, n) \)). Thus, every party can locally compute \( T_1(x) \), once it obtains all the classical results \( \{q_{h,m}\} \). Therefore, together with the algorithm in Proposition 11, \( H_1(x) = \neg T_0(x) \land T_1(x) \) can be computed exactly. Moreover, Proposition 11 and Lemma 12 imply that the entire solitude verification algorithm runs in \( O(N) \) rounds and with \( O(N^5) \) qubits and \( O(N^8) \) classical bits of communication. The pseudo code, Algorithm QSV, provided below summarizes the above operations.

**Algorithm QSV:** Every party \( i \) performs the following operations.

- **Input:** a classical variable \( x_i \in \{0, 1\} \) and \( N \in \mathbb{N} \).
- **Output:** \( y \in \{true, false\} \).

begin
  foreach \( (h, m) \in [0..m] \times [2..N] \) do
    perform in parallel \( Q_{h,m} \) with \( (x_i, N) \) to obtain an output \( q_{h,m} \);
  end
  if \( q_{h,m} = false \) for at least one of the pairs \( (h, m) \) then set \( y_1 := false \)
  else set \( y_1 := true \);
  compute \( y_0 := T_0(x) \) with input \( x_i \) and \( N \);
  return \( y := \neg y_0 \land y_1 \).
end

4 Algorithm \( Q_{h,m} \)

To demonstrate simply how \( Q_{h,m} \) works, we mainly consider the case where \( (h, m) \) equals \( (|x|, n) \) (and defer the analysis in the other cases until the following sections). In fact, \( Q_{h,m} \) originates from the following simple idea. Every active party flips a coin, broadcasts the outcome, and receives the outcomes of all parties. Obviously, those outcomes cannot include both heads and tails if there is at most one active party. Otherwise, they include both with high probability. This implies that, with high probability, one can tell whether the number of active parties is at most one or not, based on whether the outcomes include both heads and tails. One can thus obtain the correct answer to \( SV_n \) with high probability, since one can easily check whether there exists at least one active party. However, this still yields the wrong answer with positive probability. Since we want to solve the problem exactly, we need to suppress this error. For this purpose, we carefully combine several quantum tools with classical techniques based on the idea sketched in [TKM12, Sec. 4].

Suppose that every party \( i \) has a two-qubit register \( R_i \). Let \( \{|0\}, \{|0\}, \{|1\}, \{|0,1\} \} \) be an orthonormal basis of \( \mathbb{C}^4 \) ("\( \emptyset \)" represents the empty set). For notational simplicity, we also use \( 0, 1, \times \) to denote \( \{0\}, \{1\}, \{0,1\} \), respectively ("\( \times \)" intuitively means inconsistency since we have both 0 and 1 in the
corresponding set). Without loss of generality, we identify \(|\hat{0}, \hat{1}\rangle\), \(|0\rangle\), \(|\times\rangle\) with \(|00\rangle, |01\rangle, |10\rangle, |11\rangle\), respectively. We now start to describe \(Q_{h,m}\) step by step and explain the effect of each step by showing claims and propositions. We defer their proofs to the next section.

**STEP 1:** Every active party \(i\) prepares \((|\hat{0}\rangle + |\hat{1}\rangle)/\sqrt{2}\) in register \(R_i\), while every inactive party \(j\) prepares \(|0\rangle\) in \(R_j\).

Let \(R_S\) be the set of \(R_i\) over all \(i \in S\), where \(S\) is the set of active parties, and let \(R_{\overline{S}}\) be the set of \(R_i\) over all \(i \in [n] \setminus S\). We then denote by \(R\) the pair \((R_S, R_{\overline{S}})\). The quantum state \(|\psi_x(0)\rangle\) over \(R\) can be written as

\[
|\psi_x(0)\rangle_R = \left[|0\rangle^{\otimes n} - |x\rangle\right]_{R_S} \otimes \left[\frac{(|\hat{0}\rangle + |\hat{1}\rangle)}{\sqrt{2}}\right]^{\otimes |S|}_{R_{\overline{S}}}.
\]

**STEP 2:** All parties attempt to project \(|\psi_x(0)\rangle\) onto the space spanned by \(|z\rangle: z \in A\) or \(|z\rangle: z \in A^c\), where \(A\) is the set of all strings \(z\) in \([0, 1, \emptyset]\)^n such that \(z\) does not contain both 0 and 1 simultaneously, i.e., \(A := \{0, \emptyset\}^n \cup \{1, \emptyset\}^n\), and \(A^c := \{0, 0, 1, \times\}^n \setminus A\) is the complement of \(A\).

For **STEP 2**, every party applies the operator \(\Phi_\Delta\) with \(\Delta := \frac{N}{4}\) to \(|\psi_x(0)\rangle\), where \(\Phi_\Delta\) is defined in the following claim. The resulting state \(|\psi_x(1)\rangle_{(R,Y,G)} := \Phi_\Delta |\psi_x(0)\rangle_R\) is is \(\frac{1}{\sqrt{2^{|S|}}}\) times

\[
\sum_{z \in A} |z\rangle_R \otimes (|\text{consistent}\rangle^{\otimes n})_Y \otimes |g_N(z)\rangle_G + \sum_{z \in A^c} |z\rangle_R \otimes (|\text{inconsistent}\rangle^{\otimes n})_Y \otimes |g_N(z)\rangle_G. \tag{1}
\]

**Claim 13** There exists a distributed quantum algorithm \(Q_{\text{CONSISTENCY}}\) that, for given upper bound \(\Delta\) on the diameter of the underlying graph \(G := (V, E) \in \mathcal{D}_n\), implements the following operator\(^5\):

\[
\Phi_\Delta := \sum_{z \in A} |z\rangle_R \otimes (|\text{consistent}\rangle^{\otimes n})_Y \otimes |g_\Delta(z)\rangle_G + \sum_{z \in A^c} |z\rangle_R \otimes (|\text{inconsistent}\rangle^{\otimes n})_Y \otimes |g_\Delta(z)\rangle_G, \tag{2}
\]

where \(|\text{consistent}\rangle, |\text{inconsistent}\rangle\) is an orthonormal basis of \(\mathbb{C}^2\), \(Y := (Y_1, \ldots, Y_n)\) denotes a set of ancillary single-qubit registers \(Y_i\) possessed by party \(i\) for all \(i \in [n]\), and \(G\) denotes a set of all the other ancillary registers, the content \(g_\Delta(z)\) of which is a bit string uniquely determined\(^6\) by \(\Delta\) and \(z\). Moreover, \(Q_{\text{CONSISTENCY}}\) runs in \(O(\Delta)\) rounds and communicates \(O(\Delta |E|)\) qubits. In particular, any upper bound \(N\) of \(n\) can be used as \(\Delta\).

Note that the operator \(\Phi_\Delta\) has an effect similar to the measurement \(\{\Pi_0, \Pi_1\}\) on \(|\psi_x(0)\rangle\), where \(\Pi_0 := \sum_{z \in A} |z\rangle\langle z|\) and \(\Pi_1 := I_{4^n} - \Pi_0\) for the identity operator \(I_{4^n}\) over the space \(\mathbb{C}^{4^n}\). The difference is that \(\Phi_\Delta\) leaves the garbage part \(|g_\Delta(z)\rangle\), which is due to the “distributed execution” of the measurement.

**STEP 3:** Every party \(i\) measures the register \(Y_i\) of \(|\psi_x(1)\rangle_{(R,Y,G)}\) in the basis \(|\text{consistent}\rangle, |\text{inconsistent}\rangle\). If the outcome is “inconsistent”, then \(Q_{h,m}\) halts and returns “false”.

To understand the consequence of each possible outcome, we first provide an easy proposition.

**Claim 14** \(|\psi_x(0)\rangle_R\) is in the space spanned by \(|z\rangle: z \in A\) if and only if \(|x| \leq 1\).

---

\(^5\)More formally, this operator should be written as a unitary one acting on ancillary registers initialized to \(|0\rangle\) as well as \(R_i\).
\(^6\)Actually, \(g_\Delta(z)\) also depends on the underlying graph, but we assume without loss of generality that the graph is fixed.
If the outcome is “inconsistent”, there must exist \( z \in A^c \) such that \( |z\rangle \) has a non-zero amplitude in \( |\psi_x^{(0)}\rangle_R \), implying that \( |\psi_x^{(0)}\rangle_R \) does not lie in the space spanned by \( \{|z\rangle : z \in A\} \) and thus \(|x| \geq 2\) by Claim 14. Therefore, every party can conclude that \( T_1(x) \) is “false” without error if the outcome is “inconsistent”.

If the outcome is “consistent”, we have the resulting state (from Eq. (1)):

\[
|\psi_x^{(2)}\rangle_{(R,G)} = c \sum_{z \in A} \alpha_{z(x)}|z\rangle_R \otimes |g_{\Delta}(z)\rangle_G,
\]

where \( c := 1/\sqrt{\sum_{z \in A}\alpha_z(x)^2} \) is the normalizing factor. In this case, the parties cannot determine the value of \( T_1 \) on \( x \) since \( |\psi_x^{(0)}\rangle_R \) may or may not be in the space \( \{|z\rangle : z \in A\} \). Hence, we need a few more steps.

For proceeding to the next step, the following observation is crucial, which is obtained from the definition of anonymous networks, however, every (active) party has count the number of outcomes \( \sum_{i \leq n} \alpha_{z(x)} \) and return a correct answer. In general cases, however, the value of \( h \) is not necessarily equal to \( n \). To rotate the state by the angle \( \pi/|x| \), every active party applies to its share of the registers the rotation operator for the angle \( \pi/|x| \), so that the sum of the angles over all active parties is \( \pi \). This works correctly if the number of active parties is given (as \( h \)). With the assumption that \( h \) equals \( |x| \), the state should be rotated by the correct angle \( \pi \). In general cases, however, the value of \( h \) is not necessarily equal to \( |x| \).

In the case where \( |x| \) equals 0, no active parties exist and thus no operations are performed in this step.

The following claim summarizes the effect of GHZ-SCALEDOWN.
Claim 16 There exists a distributed quantum algorithm GHZ-SCALEDOWN such that, for given \((h, m)\),

- if \(|x|\) equals zero, then for any \((h, m)\) it halts with the output “true” at every party or applies the identity operator to \(|\psi_x^{(2)}\rangle_{R, G}\);
- else if \((h, m)\) equals \((|x|, n)\), it transforms \(|\psi_x^{(2)}\rangle_{R, G}\) to \(|\psi_x^{(3)}\rangle_{R_S} := \frac{1}{\sqrt{2}}(|\hat{0}\rangle_{|x|} + |\hat{1}\rangle_{|x|})_{R_S}\);
- else it halts with the output “true” at every party or transforms \(|\psi_x^{(2)}\rangle_{R, G}\) to the state \(|\tilde{\psi}_x^{(3)}\rangle_{R_S} := \frac{1}{\sqrt{2}}(|\hat{0}\rangle_{|x|} + e^{i\theta_{h,m}}|\hat{1}\rangle_{|x|})_{R_S}\) for some real \(\theta_{h,m}\).

Moreover, GHZ-SCALEDOWN runs in \(O(m)\) rounds and communicates \(O(\max\{p(m), p(n)\})\) classical bits, where \(p(\cdot)\) is the function defined in Fact 8.

Then, \(Q_{h,m}\) proceeds to STEP 5. By Claim 16, STEP 5 is performed with \(|\tilde{\psi}_x^{(3)}\rangle_{R_S}\) only if both \((h, m) \neq (|x|, n)\) and \(x \geq 1\) hold. In this case, to meet item 3 in Lemma 12, we only need to examine the output of \(Q_{h,m}\) on \(x\) with \(|x| = 1\) (we defer until the next section easy proofs that item 1 holds and that \(Q_{h,m}\) reaches a halting state for all \(x\)). For \(x\) with \(|x| = 1\), the behavior of \(Q_{h,m}\) is almost the same as in the case of \((h, m) = (|x|, n)\). Thus, in the following part of this section, we assume that \(|x|\) is equal to 0 or STEP 5 starts with \(|\psi_x^{(3)}\rangle_{R_S}\) for simplicity (the proof of Lemma 12 provided in the next section will rigorously analyze all cases).

**STEP 5**: Every active party \(i\) applies the local unitary operator \(W_h\) to its register \(R_i\), where \(W_h\) is defined as follows:

\[
W_h := \begin{cases} 
I_2 \otimes U_h & h \text{ is even and at least two} \\
V_h \cdot \text{CNOT}_{2\rightarrow1} & h \text{ is odd and at least three} \\
I_2 \otimes I_2 & h \text{ is zero or one},
\end{cases}
\]

where \(U_h\) and \(V_h\) are defined in Sec. 3.3 of Ref. [TKM12], CNOT\(_{2\rightarrow1}\) acts on the first qubit, controlled by the second qubit, and \(I_2\) is the identity over \(\mathbb{C}_2\).

To see how the operator \(W_h\) works, we provide the following claim.

Claim 17 (Adaptation from Lemmas 3.3 and 3.5 in Ref. [TKM12]) **STEP 5** satisfies the following:

- If \(|x|\) is zero, then **STEP 5** is effectively skipped.
- If \(|x|\) is one, then \(|\psi_x^{(4)}\rangle_{R_S} := W_h|\psi_x^{(3)}\rangle_{R_S} = (\alpha|\hat{0}\rangle + \beta|\hat{1}\rangle + \gamma|\emptyset\rangle + \delta|x\rangle)_{R_S}\) for some \(\alpha, \beta, \gamma, \delta \in \mathbb{C}\).
- If \(|x|\) is at least two and \(h\) equals \(|x|\), then it holds that

\[
|\psi_x^{(4)}\rangle_{R_S} := W_h^\otimes |\psi_x^{(3)}\rangle_{R_S} = \sum_{y \in B} \beta_y |y\rangle_{R_S},
\]

where \(B := \{|z| : z \in \{\emptyset, \hat{0}, \hat{1}, \emptyset, x\}^{\|x\|} \setminus \{\hat{0}|x\rangle, \hat{1}|x\rangle, \emptyset|x\rangle, x|x\rangle\}\) and \(\beta_y \in \mathbb{C}\).

That is, if \(|x|\) is zero, then the state over \((R, G)\) is a tensor product of \(|\emptyset\rangle\) by Claims 15 and 16.

If \(|x|\) is exactly one, i.e., \(S = \{i^*\}\) for some \(i^*\), then, whatever unitary operator \(W_h\) has acted on \(R_{i^*}\), the resulting state \(|\psi_x^{(4)}\rangle_{R_S} := W_h|\psi_x^{(3)}\rangle_{R_S}\) is a quantum state over the register \(R_{i^*}\). This state is obviously consistent over \(S\), since there is only one active party.

If \(|x|\) is at least two and \(h\) equals \(|x|\), **STEP 5** transforms the state \(|\psi_x^{(3)}\rangle_{R_S}\) into \(|\psi_x^{(4)}\rangle_{R_S}\), which is an inconsistent state over \(S\). Intuitively, the state \(|\psi_x^{(4)}\rangle_{R_S}\) is a superposition of only the basis states that
correspond to the situations where at least two active parties have different contents chosen from \{\hat{0}, \hat{1}, 0, \times\} in their \(R_i\). This fact exhibits a striking difference from the case of \(|x| \leq 1\). We should emphasize that the amplitude \(\beta_y\) vanishes for each \(y \in \{\hat{0}^{|x|}, \hat{1}^{|x|}, 0^{|x|}, \times^{|x|}\}\) only when \(h\) equals \(|x|\). In general, however, \(h\) may not be equal to \(|x|\), in which case the amplitude \(\beta_y\) is nonzero for some \(y \in \{\hat{0}^{|x|}, \hat{1}^{|x|}, 0^{|x|}, \times^{|x|}\}\).

**STEP 6:** Every active party measures \(R_i\) in the basis \{\(|\hat{0}\rangle, |\hat{1}\rangle, |0\rangle, |\times\rangle\}\) and obtains a two-bit classical outcome \(r_i\).

Suppose that \(|x|\) is at least one and that the quantum state over \(R_S\) just before STEP 6 is \(|\psi_x^{(4)}\rangle\rangle R_S\). If \(|x|\) is at least two and \(h\) equals \(|x|\), Claim 17 implies that there are at least two distinct outcomes among those obtained by active parties. If \(|x|\) equals one, then there is obviously a single outcome in the network.

Finally, suppose that \(|x|\) is zero. Since there are no active parties, there are no outcomes that would be obtained by the measurement.

The number of distinct outcomes is hence different between these three cases.

**STEP 7:** All parties collaborate to decide whether the number of distinct elements among the outcomes \(\{r_i: i \in S\}\) is at most one by running the distributed deterministic algorithm provided in Proposition 18. If the number is at most one, return true; otherwise, return false.

To realize STEP 7, it suffices to decide whether the string \((r_1, \ldots, r_n)\) is consistent over \(S\), where \(r_j\) for \(j \notin S\) is set at an appropriate value that is distinguishable from any possible outcome \(r_i\) for \(i \in S\) (technically, \(r_i\) is a three-bit value). For this purpose, we use the distributed algorithm CONSISTENCY given in Proposition 18 (a special form of a more general statement [KKvdB94]). The algorithm yields the correct output if an upper bound of the diameter of the underlying graph is known. In our setting, \(N\) can be used to upper-bound the diameter. Every party can therefore decide whether \(T_1(x)\) is true or false.

**Proposition 18 (Color Counting [KKvdB94])** Let \(G := (V, E) \in \mathcal{D}_n\) be the the underlying graph of an anonymous network with a diameter upper-bounded by \(\Delta\). Let \(S \subseteq [n]\) be the set of active parties and let \(C\) be the set of a constant number of colors. Then, there is a deterministic algorithm COLORCOUNT that, if the upper bound \(\Delta\), a color \(c_i \in C\), and a bit \(S_i\) indicating whether \(i \in S\) or not are given at each party \(i \in [n]\), decides which is true among the following three cases and informs every party of the decision:

\[
(0) : |\bigcup_{i \in S} \{c_i\}| = 0, \quad (1) : |\bigcup_{i \in S} \{c_i\}| = 1, \quad (2) : |\bigcup_{i \in S} \{c_i\}| \geq 2
\]

in \(O(\Delta)\) rounds with \(O(\Delta |E|)\) classical bits of communication [the case (0) occurs if and only if \(S\) is the empty set]. As a special case, the algorithm decides whether all active parties are assigned a certain single color [called the “consistent” case, corresponding to (0) or (1)] or at least two colors [called the “inconsistent” case, corresponding to (2)]. When the algorithm is used for this purpose, it is denoted by CONSISTENCY.

The operations performed by each party \(i\) in executing \(Q_{h,m}\) are described in Algorithm \(Q_{h,m}\) shown below. We should emphasize that these operations are independent of the index \(i\) (recall that this is the requirement of computing on anonymous networks). Indeed, when executing \(Q_{h,m}\), each party need not tell its own registers from the other parties’; it just needs to distinguish between its local registers and perform operators on them in a way independent of its index (but dependent on input). Thus, the subscript \(i\) of each register/variable in Algorithm \(Q_{h,m}\) can safely be dropped without introducing any ambiguity from the viewpoint of the party \(i\).

5 Proof of Lemma 12

This section first provides all the proofs of the propositions and claims in Sec. 4 (except those appearing in previous works). Then it proves Lemma 12.
Algorithm \textit{Q}_{h,m}: Every party $i$ performs the following operations.

\begin{algorithm}
\begin{tabular}{l}
\textbf{Input}: a classical variable $x_i$, and $N \in \mathbb{N}$.
\textbf{Output}: true or false.
\end{tabular}

\textbf{Notation} Let $R_i$, $Y_i$, and $G_i$ be (a set of) quantum registers, and let $y_i$, $r_i$, and $\bar{r}_i$ be classical variables.
Assume that $|0\rangle$, $|1\rangle$, $|\emptyset\rangle$, $|\times\rangle$ are $|00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$, respectively.

\begin{description}
\item[begin]
\item[STEP1] if $x_i = 1$ then prepare $\frac{1}{\sqrt{2}}(|\emptyset\rangle + |\bar{1}\rangle)$ in $R_i$ else prepare $|\emptyset\rangle$ in $R_i$.
\item[STEP2] run QCONSISTENCY on $R_i$ with $(S_i, \Delta) \leftarrow (x_i, N)$ to output $(R_i, Y_i, G_i)$
\item[STEP3] $(G_i$ is the party’s share of $G$, so that $G = (G_1, \ldots, G_n))$
\item[STEP4] measure $Y_i$ in the \{“consistent”\}, \{“inconsistent”\} basis to obtain an outcome $y_i$;
\item[STEP5] if $y_i$ is “inconsistent” then return false.
\item[STEP6] run GHZ-SCALEDOWN on $(R_i, G_i)$ with $(h, m)$;
\item[STEP7] if the output is “true” then return true.
\item[STEP8] run \textit{COLORCOUNT} on $(S_i, \Delta) \leftarrow (x_i, N)$ to output $\bar{r}_i$;
\item[STEP9] if $\bar{r}_i$ is “consistent” then return true else return false.
\end{description}
\end{algorithm}

5.1 Proofs of Claims and Propositions in Sec. 3

All proofs except the one for Claim 16 essentially follow from the ideas in Refs. [KKvdB94, TKM12]. Readers familiar with these references can safely skip the proofs. The proof for Claim 16 is based on Ref. [TKM12] for the special case, but it also discusses the other cases that the reference does not deal with.

Proposition 18 is simply an adaptation of Theorem 1 in Ref. [KKvdB94]. For completeness, we provide its proof.

\textit{Proof of Proposition 18.} Suppose that every party $i$ prepares a variable $Y$. Each active party initializes $Y$ to $\{c_i\}$, while each inactive party initializes $Y$ to “$\emptyset$”, representing the empty set. Every party then sends a copy of $Y$ via every out-port while keeping a copy of $Y$ and receives a message as a variable $Z_k$ via every in-port $k$. The party then updates $Y$ by setting it at the union of $Y$ and $Z_k$ over all $k$. Every party repeats the above sending/receiving at most $\Delta$ times. It is easy to see that every party’s final $Y$ is the set of all colors in active parties’ possession. Since each message is of constant size, the bit complexity is $O(\Delta|E|)$. \hfill \Box

As a corollary of Proposition 18, we obtain Proposition 11.

\textit{Proof of Proposition 11.} Let $C := \{1\}$ and $S := \{i \in [n]: x_i = 1\}$. Every party runs \textit{COLORCOUNT} in Proposition 18 and decides which of the two cases, (0) or (1), holds (event (2) never occurs since $|C| = 1$). \hfill \Box

We henceforth provide proofs of the claims in order.

\textit{Proof of Claim 13.} The claimed quantum algorithm essentially follows from “CONSISTENCY$_d$” on page 21 of Ref. [TKM12], where $n$ is replaced with $\Delta + 1$ and the binary operation “$\circ$” is interpreted as the union operation over sets in our case. Note that “CONSISTENCY$_d$” is a simple quantization of the classical
algorithm in Proposition 18.

The ancillary quantum registers left at each party after CONSISTENCY\(_d\) runs in \(\Delta\) rounds are (1) the master copy \(Y_0^{(t)}\) whose copies have been sent out to neighbors in each round \(t \in [\Delta]\), (2) all registers \((Z_0^{(t)}, \ldots, Z_d^{(t)})\) received in each round \(t \in [\Delta]\), and (3) the final output register \(Y_0^{(\Delta+1)}\). Let \(Y\) be the collection \((Y_1, \ldots, Y_n)\), where \(Y_i\) for \(i \in [n]\) denotes the register \(Y_0^{(\Delta+1)}\) in the possession of party \(i\), and let \(G\) be all the ancillary registers in the network except \(Y\), namely, the collection of \((Y_0^{(1)}, \ldots, Y_0^{(\Delta)}), (Z_0^{(1)}, \ldots, Z_d^{(1)}), \ldots, (Z_1^{(\Delta)}, \ldots, Z_d^{(\Delta)})\) over all parties \(i\). It is easy to see that the whole action of the quantum algorithm can be written as the operator \(\Phi_\Delta\) in the claim.

For the complexity, CONSISTENCY\(_d\) requires \(O(\Delta)\) rounds. In addition, CONSISTENCY\(_d\) communicates \(O(\Delta|E|)\) qubits, since a single register representing a subset of \(\{0, 1, \emptyset, x\}\) is sent through each communication link at each round. □

**Proof of Claim 14.** The claim is very easy. We omit the proof. □

**Proof of Claim 15.** Since the state \(|\psi_x^{(2)}\rangle_{(R,G)}\) is a superposition of \(|z\rangle_R \otimes |g_\Delta(z)\rangle_G\) with \(z \in A\), the contents of the registers \(R_i\) of all active parties \(i\) are identical by the definition of \(A\). More precisely, they are either all “\(0\)’” or all “\(1\)’” with probability 1/2. Meanwhile, the contents of \(R_j\) of all inactive parties \(j\) are “\(\emptyset\)’”. The state \(|\psi_x^{(2)}\rangle_{(R,G)}\) is hence of the form

\[
|\psi_x^{(2)}\rangle_{(R,G)} = \frac{1}{\sqrt{2}} \left( |\hat{0}\rangle \otimes |\chi\rangle \otimes |g(z_0)\rangle + |\hat{1}\rangle \otimes |\chi\rangle \otimes |g(z_1)\rangle \right)_{(R_S,G)} \otimes \left( |\emptyset\rangle \otimes (\mathbb{I} - |\chi\rangle \langle \chi|) \right)_{R_S}. 
\]

This fact and items 1 to 3 of the following Claim 19 imply

\[
|\psi_x^{(2)}\rangle_{(R,G)} = \frac{1}{\sqrt{2}} \left( |\hat{0}\rangle \otimes |\emptyset\rangle + |\hat{1}\rangle \otimes |\emptyset\rangle \right)_{(R_S,G')} \otimes \left( |\emptyset\rangle \otimes |\emptyset\rangle \right)_{(R_S,G'')}, 
\]

where \(G''\) is the set of (untangled) registers in \(G\) whose contents are “\(0\)’”, and \(G'\) is the remaining registers in \(G\). It is obvious that \(p + q \geq |S| + |\bar{S}| = n\). It also holds that \(p \geq |S|\), since the content of \(R_i\) for \(i \in S\) is either “\(0\)’” and “\(1\)’”. In particular, when \(|S| = 0\), item 4 of Claim 19 implies that \(p\) equals 0. This completes the proof of Claim 15. □

**Claim 19** For a fixed set \(S\) of active parties, suppose that the state of \((R, G)\) is \(|\psi_x^{(2)}\rangle_{(R,G)}\). Then the following hold.

1. If the content of \(R_i\) is “\(0\)’” for every \(i \in S\), then the content of every register in \(G\) is either “\(0\)’” or “\(\emptyset\)’”.
2. If the content of \(R_i\) is “\(1\)’” for every \(i \in S\), then the content of every register in \(G\) is either “\(1\)’” or “\(\emptyset\)’”.
3. If a register in \(G\) has the content “\(\emptyset\)’”, then the register is unentangled.
4. If there are no active parties, then the content of every register in \((R, G)\) is “\(\emptyset\)’”.

**Proof** Claim 19 essentially follows from the fact that the operator \(\Phi_\Delta\), i.e., CONSISTENCY\(_d\), consists of idempotent operations, copy operations via CNOT, and register exchanges. We thus omit the proof. This claim is (implicitly) observed in Sec. 4 of Ref. [TKM12]. □
Proof of Claim 16. The algorithm GHZ-SCALEDDOWN is a modification of the algorithm sketched in Sec. 4 of Ref. [TKM12], which is obtained by combining standard quantum and classical techniques. That is, every party \(i\) performs the following procedure on \((R, G)\), which is supposed to be in the state \(|\psi^{(2)}_x\rangle_{(R,G)}\). Let \(|\hat{+}\rangle := (|0\rangle + |1\rangle)/\sqrt{2}\) and \(|\hat{-}\rangle := (|0\rangle - |1\rangle)/\sqrt{2}\).

(i) Measure every local register in \(G\) in the basis \(\{|\hat{+}\rangle, |\hat{-}\rangle, |0\rangle, |\times\rangle\}\).

(ii) Locally count the number \(s_i \mod 2\) of the measurement outcomes “−” that party \(i\) has obtained.

(iii) Attempt to compute the sum of \(s_i \mod 2\) over all parties by invoking the algorithm in Fact 10 with \(s_i\) and \(m\). Let \(\chi\) be the output of the algorithm.

(iv) Decide with CONSISTENCY given in Proposition 18 whether the string induced by \(\chi\)'s is consistent over all parties. If either the result is “inconsistent” or \(\chi\) is not a non-negative integer, then output “true” and halt.

(v) If the party \(i\) is active (i.e., \(i \in S\)), the output \(\chi \mod 2\) equals one, and \(h\) is at least 1, then perform

\[
R(\frac{\pi}{h}) := \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/h} \end{bmatrix}
\]

on the subspace spanned by \(|\hat{0}\rangle, |\hat{1}\rangle\) on \(R_i\) and output \(R_i\).

The reason Step (iv) is needed is as follows: For \(m \neq n\), the algorithm involved in Step (iii) may not work correctly, and some party could output \(\chi\) that is different from other parties’ \(\chi\) and/or is even a non-integer value (actually, the algorithm never outputs negative values, which follows from the details of the algorithm, which we do not touch on in this paper). Step (iv) is for the purpose of detecting that the guess at \(m\) is wrong and precluding this wrong guess from leading to the wrong final decision when the number of active parties is at most one.

First assume that \((h, m)\) equals \((|x|, n)\) and \(|x|\) is at least one. In this case, Steps (i)-(v) transform the state \(|\psi^{(2)}_x\rangle_{RS} = (|\hat{0}\rangle^{|x|} + |\hat{1}\rangle^{|x|})_{RS}/\sqrt{2}\) by following Sec. 4 in Ref. [TKM12].

Next suppose that \(|x|\) is zero. For any pair \((h, m)\), every party can still perform Steps (i) and (ii) since these steps are independent of \((h, m)\). Every party then performs Step (iii). Note that this is possible even when \(m\) is not equal to \(n\) (see the description just after Fact 10). If \(m\) is not equal to \(n\), however, the value \(\chi\) obtained in Step (iii) may be wrong or the string induced by \(\chi\)'s could be inconsistent over all parties.\(^7\) Since CONSISTENCY can run and make a common decision for each party by setting \(\Delta\) at \(N\) (Proposition 18), either every party halts with output “true” at Step (iv) or every party proceeds to Step (v) with a non-negative integer \(\chi\) that agrees with any other party’s \(\chi\). In the latter case, no operations are performed in Step (v) since no active parties exist (i.e., \(|x| = |S| = 0\)). Thus, the procedure effectively applies the identity operator to \(|\psi^{(2)}_x\rangle_{(R,G)}\), i.e., a tensor product of \(|\hat{0}\rangle\).

Finally, suppose that \((h, m)\) is not equal to \((|x|, n)\) and \(|x|\) is at least one. For any pair \((h, m)\), every party can still perform Steps (i), (ii), and (iii) as in the case of \(|x| = 0\), either every party halts with output “true” at Step (iv) or every party proceeds to Step (v) with a non-negative integer \(\chi\) that agrees with any other party’s \(\chi\). Suppose that it proceeds to Step (v). If \(h\) is at least 1 and the value \(\chi \mod 2\) happens to be one, then every active party applies \(R(\frac{\pi}{h})\). This effectively multiplies the state \(|\hat{1}\rangle^{|x|}_{RS}\) by a factor \(e^{i\pi/|x|}\); namely, it transforms \(|\psi^{(2)}_x\rangle_{RS}\) to \(|\tilde{\psi}^{(3)}_x\rangle_{RS} := (1/\sqrt{2})(|0\rangle^{|x|} \pm e^{i\pi/|x|}|1\rangle^{|x|})_{RS}\). If \(h\) or \(\chi \mod 2\) equals zero, then Step (iv) performs no operations, and thus the entire procedure effectively transforms \(|\psi^{(2)}_x\rangle_{RS}\) to \(|\tilde{\psi}^{(3)}_x\rangle_{RS} := (1/\sqrt{2})(|0\rangle^{|x|} \pm |1\rangle^{|x|})_{RS}\).

For the communication cost, observe that only Steps (iii) and (iv) involve (classical) communication. Step (iii) just runs the algorithm provided in Fact 10 with \(m\) instead of (unknown) \(n\). In this case, the

\(^7\)When \(|x|\) is zero, \(s_i\) is zero for all \(i\). This actually implies that \(\chi\) is zero for all parties regardless of \(m\), which follows from the details found in Ref. [YK96a]. However, we consider the possibility of halting at Step (iv) to avoid getting into the details of the algorithm in Fact 10.
algorithm runs in $O(m)$ rounds and communicates $O(\max\{p(m), p(n)\})$ classical bits, as described just after Fact 10. The cost of Step (iv) is shown in Proposition 18 and dominated by the cost of Step (iii). □

**Proof of Claim 17.** If $|x|$ is zero, there exist no active parties. STEP 5 is hence effectively skipped. If $|x|$ is one, the register $R_S$ consists of two qubits. Since $\{|0\rangle, |1\rangle, |\emptyset\rangle, |\times\rangle\}$ is an orthonormal basis of $\mathbb{C}^4$, the statement follows. Now suppose that $|x|$ is at least two. In addition, assume that $h$ equals $|x|$. Recall that $(|0\rangle, |1\rangle, |\emptyset\rangle, |\times\rangle)$ denotes $(|00\rangle, |01\rangle, |10\rangle, |11\rangle)$. We thus have $|\psi^{(3)}_x\rangle_{R_S} = (1/\sqrt{2})(|00\rangle \otimes |x\rangle + |01\rangle \otimes |\times\rangle)$. Every active party applies the unitary operator $W_h$ to its $R_i$, where $W_h$ is either $I_2 \otimes U_h$ for even $h$ or $V_h \cdot \text{CNOT}_{2\rightarrow 1}$ for odd $h$. Lemmas 3.3 and 3.5 in Ref. [TKM12] imply that $|\psi^{(3)}_x\rangle_{R_S}$ is transformed into an inconsistent state over $S$ whenever $h \geq 2$: $|\psi^{(3)}_x\rangle_{R_S} \mapsto |\psi^{(4)}_x\rangle_{R_S} := \sum_{y \in B} \beta_y |y\rangle_{R_S}$ for certain amplitudes $\{\beta_y\}$. □

### 5.2 Proof of Lemma 12

We first show that, for any $(h, m)$, STEPs 1 and 2 can run and yield $|\psi^{(1)}_x\rangle_{(R,Y,G)}$. Since STEP 1 is independent of $(h, m)$, it always yields $|\psi^{(0)}_x\rangle_{R}$. STEP 2 runs the algorithm provided in Claim 13, which only requires an upper bound $\Delta$ on the diameter of the underlying graph $G$. By setting $\Delta$ at $N$, Claim 13 implies that STEP 2 always yields $|\psi^{(1)}_x\rangle_{(R,Y,G)}$.

For STEP 3 and latter steps, we consider three cases separately: (A) $|x| = 0$; (B) $|x| \geq 1$ and $(h, m) = (|x|, n)$; (C) $|x| \geq 1$ and $(h, m) \neq (|x|, n)$.

Suppose case (A): $|x| = 0$. STEP 3 can be performed, since it consists of only a measurement that is independent of $(h, m)$. Moreover, every party can obtain the same outcome after STEP 3, since $|\psi^{(1)}_x\rangle_{(R,Y,G)}$ is of the form shown in Eq. (1). The common outcome is “consistent” with certainty by Claim 14. Thus, the state resulting from the measurement is always $|\psi^{(2)}_x\rangle_{(R,G)}$, which is a tensor product of $|\emptyset\rangle$ by Claim 15. STEP 4 consists of a distributed algorithm GHZ-SCALEDOWN, which either returns “true” or keeps the tensor product in $(R, G)$, as asserted by Claim 16. STEPs 5 and 6 are effectively skipped, since there are no active parties. Hence, there are no measurement outcomes in the network, which STEP 6 would yield if it were not skipped. This forces STEP 7 to determine by using Proposition 18 that the number of distinct elements among the outcomes is zero with certainty, which leads to returning “true”. Note that STEP 7 consists of the distributed algorithm CONSISTENCY provided in Proposition 18, which makes the decision of “true” at every party for any $(h, m)$. Therefore, items 1 through 3 in Lemma 12 hold when $|x|$ is zero.

Next, suppose case (B): $|x| \geq 1$ and $(h, m) = (|x|, n)$. Claim 14 implies that, if $|x|$ is one, then each party obtains with certainty the outcome “consistent” of the measurement made in STEP 3. Note that, with the same argument as in case (A), the outcomes of all parties agree. STEP 3 can thus measure “inconsistent” and output “false” only if $|x|$ is at least two. Hence, whenever STEP 3 outputs “false”, this output agrees with $T_1(x)$. Assume henceforth that the outcome is “consistent”. The resulting state is then $|\psi^{(2)}_x\rangle_{(R,G)}$, which is of the form in Eq. (3) in Claim 15. STEP 4 then transforms $|\psi^{(2)}_x\rangle_{(R,G)}$ to $|\psi^{(3)}_x\rangle_{R_S}$ with certainty as asserted by Claim 16. STEP 5 further transforms $|\psi^{(3)}_x\rangle_{R_S}$ to $|\psi^{(4)}_x\rangle_{R_S}$ as implied by Claim 17. If $|x| = (|S|)$ is one, then $|\psi^{(4)}_x\rangle_{R_S}$ is exactly the state of $R_i$ of the only active party $i$, and thus only a single outcome of the measurement is obtained in the whole network in STEP 6. In this case, STEP 7 returns “true”, which matches $T_1(x)$. If $|x|$ is at least two, then the state $|\psi^{(4)}_x\rangle_{R_S}$ is inconsistent over $S$ by Claim 17, so that the string induced by the measurement outcomes obtained in STEP 6 is also inconsistent over $S$. Thus, there are two or more distinct outcomes in the whole network, and STEP 7 returns “false”, matching $T_1(x)$. Therefore, items 1 through 3 in Lemma 12 hold if $|x| \geq 1$ and $(h, m) = (|x|, n)$. It suffices to show that items 1 and 3 in
Lemma 12 hold in this case; namely, that STEPs 3 through 7 can be performed. \( Q_{h,m} \) returns a common decision (i.e., true or false) to every party for any \((h, m)\), and \( Q_{h,m} \) always returns “true” if \(|x|\) is one. As described in case (A), every party obtains the same outcome of the measurement for any \((h, m)\) in STEP 3. With the same argument used in case (B), if \(|x|\) is one, then Claim 14 implies that STEP 3 never returns “false” and every party proceeds to STEP 4 with the state \(|\psi_x^{(2)}\rangle_{(R,G)}\). Claim 16 then implies that, for any \(|x| \geq 1\) and for any \((h, m) \neq (|x|, n)\), STEP 4 either returns “true” to every party or transforms \(|\psi_x^{(2)}\rangle_{(R,G)}\) to \(|\tilde{\psi}_x^{(3)}\rangle_{R_S}\). Assume the latter case. In STEP 5, every active party applies the local unitary operator \(W_h\) to its share of \(|\tilde{\psi}_x^{(3)}\rangle_{R_S}\). This is possible for any \((h, m)\), since \(W_h\) is defined for every possible \(h\). STEP 6 can obviously be performed, since it consists of a measurement that is independent of \((h, m)\). For any \((h, m)\), STEP 7 makes a common decision at every party as stated in Proposition 18. If \(|x|\) is one, the register \(R_S\) is exactly \(R_i\) of the only active party \(i\). Therefore, whatever state in \(R_S\) results from STEP 5, STEP 6 yields only a single outcome in the whole network. STEP 7 thus returns “true”. This shows that items 1 and 3 in Lemma 12 hold.

To bound the complexity, observe that all the communication performed by \(Q_{h,m}\) is devoted to STEPs 2, 4, and 7. The complexities of these steps are shown in Claims 13 and 16 and Proposition 18 with \(N\) as \(\Delta\). Summing them up shows that \(Q_{h,m}\) runs in \(O(N)\) rounds and communicates \(O(N^3)\) qubits and \(O(p(N)) \leq \hat{O}(N^0)\) classical bits.

6. Applications

This section provides some applications of the solitude verification algorithm.

6.1 Zero-Error Leader Election (Proof of Corollary 2)

The algorithm in Theorem 1, called QSV, leads to a simple zero-error algorithm for the leader election problem (a pseudo-code is given as Algorithm ZQLE). This application is somewhat standard, but we will sketch how it works for completeness.

For every \(s \in [2..N]\), every party \(i\) sets \(x_i^{(s)}\) to a random bit, which is 1 with probability \(1/s\) or 0 with probability \(1 - 1/s\). The party then performs QSV with \(x_i^{(s)}\) and \(N\) over all \(s\) in parallel. Let QSV[\(x^{(s)}, N\)] be the (common) output of QSV that every party obtains, where \(x^{(s)} := (x_1^{(s)}, \ldots, x_n^{(s)})\). If there exists at least one \(s\) such that QSV[\(x^{(s)}, N\)] is “true”, then every party \(i\) outputs \(z_i := x_i^{(s_{\text{max}})}\), where \(s_{\text{max}}\) is the maximum of \(s\) such that QSV[\(x^{(s)}, N\)] is “true”; Otherwise, it gives up. The party with \(z_i = 1\) is elected as a unique leader. Note that this elects a unique leader without error whenever \(s_{\text{max}}\) exists. The probability of successfully electing a unique leader is at least some constant, since for \(s = n\), the probability that there is exactly a single \(i \in [n]\) with \(x_i^{(s)} = 1\) is \(\binom{n}{1} \frac{1}{n} (1 - \frac{1}{n})^{n-1} > 1/e\). By the standard argument, this probability can be amplified to a constant arbitrarily close to one by simply repeating QZLE sufficiently many but constant times. Since all communication in QZLE is devoted to QSV, which runs for all \(s \in [2..N]\) in parallel, the overall round complexity is still \(O(N)\) and the overall bit complexity is \(\hat{O}(N^8) \times (N - 1) = \hat{O}(N^9)\).

6.2 Computing General Symmetric Functions (Proof Sketch of Theorem 3)

The simple idea in the formal proof is likely to be hidden under complicated notations. We thus only sketch the proof and relegate its formal description to Appendix 6.2.

Recall that if there are at least two active parties, then during the execution of Algorithm QSV (on page 11), Algorithm \(Q_{h,m}\) (on page 16) for some \((h, m)\) outputs “false”. Let \((h^*, m^*)\) be the
process recursively, the active parties will eventually be partitioned into equivalence classes $V$ such that at least one of them is a singleton. This can be verified as follows: For each equivalence class thus partitions the set of active parties into equivalence classes naturally defined by $r$. The parties can thus tell the value of $x_i$ in $O(N)$ rounds with a polynomially bounded bit complexity, which is more formally stated as Claim 20.

Claim 20 Suppose that there are $n$ parties on an anonymous (classical) network with any underlying graph $G := (V, E)$ in $D_n$ in which an upper bound $N$ on $n$ is given as global information. Suppose further that each party $i$ in the network has a variable $S_i$ such that $S_i$ = “leader” for a certain $l \in [n]$ and $S_i$ = “follower” for all $i \in [n] \setminus \{l\}$. If every party $i$ is given $x_i \in \{0, 1\}$, then every party can compute $|x|$ in $O(N)$ rounds with the bit complexity $O(N|V|^2|E| \log |V|)$.

The parties can thus tell the value of $f(x)$ for any fixed $f \in S_n(k)$. Observe that such a singleton class appears within $[\log_2 |x|]$ levels of recursion. Then, it suffices for the following reason to continue the process up to $[\log_2 k]$-th recursion level: If there are no singleton sets at $[\log_2 k]$-th recursion, then $|x|$ must be larger than $k$, and thus the parties can determine the value of $f(x)$; otherwise, the parties can compute $f(x)$ as we have already shown. The total number of rounds is thus $O(N \log(\max\{k, 2\})$).

Remark 21 For readers familiar with Algorithm II in Ref. [TKM05] (which works even for any strongly connected directed graph), an alternative algorithm for computing $f \in S(k)$ can be considered as follows: First start Algorithm II and stop after the first $[\log_2 k]$ phases have finished. Then, verify with Algorithm QSV that a unique leader is elected. If this is the case, the leader can compute $|x|$ as in Claim 20.

Algorithm ZQLE: Every party $i$ performs the following operations.

| Input: an upper bound $N$ on the number of parties. |
| Output: $z_i \in \{0, 1, \text{give-up}\}$. |

Notation: Let $P_s$ be the distribution over $\{0, 1\}$ for which $P_{Z \in P_s(0,1)}[Z = 1] = 1/s$.

begin
forall the $s \in [2..N]$ do
perform in parallel Algorithm QSV with input $(x_i(s), N)$ to obtain output $y_i(s)$, where $x_i(s) \in P_s \{0, 1\}$. end
if $y_i(s) = \text{true}$ for some $s$ then
return $z_i := x_i(s_{\max})$, where $s_{\max} := \max\{s: y_s = \text{true}\}$; else
return $z_i := \text{give-up}$. end
end

The string thus partitions the set of active parties into equivalence classes naturally defined by the outcomes. Similarly, if STEP 7 outputs “false”, then the measurement outcomes $r_i (i \in [n])$ defined in $Q_{h, m^*}$, the string $r_1 \ldots r_n$ is inconsistent over the set $S$ of active parties in this case, the string induced by the set of all the measurement outcomes is inconsistent over $S$. The string thus partitions the set of active parties into equivalence classes naturally defined by the outcomes. Similarly, if STEP 7 outputs “false”, then the measurement outcomes $r_i (i \in [n])$ defined in $Q_{h, m^*}$, the string $r_1 \ldots r_n$ is inconsistent over the set $S$ of active parties. The string thus partitions the set of active parties into equivalence classes naturally defined by $r_i$'s. By repeating this process recursively, the active parties will eventually be partitioned into equivalence classes $(V_1, \ldots, V_l)$ such that at least one of them is a singleton. This can be verified as follows: For each equivalence class $V_j$, run QSV with the members of $V_j$ as active parties. If two or more singleton classes are found, then all parties agree on one of the singleton classes in an arbitrary way. The parties then decide that the unique member of the class be a leader. It is not difficult to show that once a unique leader is elected, the leader can compute $|x|$ in $O(N)$ rounds with a polynomially bounded bit complexity, which is more formally stated as Claim 20.
If it fails, then this implies that $|x|$ is more than $2^\lceil \log k \rceil \geq k$ and thus determines the value of $f(x)$. Since each phase consists of $O(N)$ rounds, the whole algorithm runs in $O(N \log(\max\{k, 2\}))$ rounds (with a polynomially bounded bit complexity).

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References

[AAHK86] Karl R. Abrahamson, Andrew Adler, Lisa Higham, and David G. Kirkpatrick. Probabilistic solitude verification on a ring. In Proceedings of the Fifth Annual ACM Symposium on Principles of Distributed Computing (PODC ’86), pages 161–173, 1986.

[AAHK94] Karl R. Abrahamson, Andrew Adler, Lisa Higham, and David G. Kirkpatrick. Tight lower bounds for probabilistic solitude verification on anonymous rings. Journal of the ACM, 41(2):277–310, 1994.

[AM94] Yehuda Afek and Yossi Matias. Elections in anonymous networks. Information and Computation, 113(2):312–330, 1994.

[Ang80] Dana Angluin. Local and global properties in networks of processors (extended abstract). In Proceedings of the 12th Annual ACM Symposium on Theory of Computing, pages 82–93, 1980.

[AS10] N. Aharon and J. Silman. Quantum dice rolling: a multi-outcome generalization of quantum coin flipping. New Journal of Physics, 12(033027), 2010. Also available in arXiv:0909.4186.

[BHMT02] Gilles Brassard, Peter Høyer, Michele Mosca, and Alain Tapp. Quantum amplitude amplification and estimation. In Quantum Computation and Information, volume 305 of Contemporary Mathematics, pages 53–74. American Mathematical Society, 2002.

[BOH05] Michael Ben-Or and Avinatan Hassidim. Fast quantum Byzantine agreement. In Proceedings of the 37th Annual ACM Symposium on Theory of Computing, pages 481–485, 2005.

[BR03] Harry Buhrman and Hein Röhrig. Distributed quantum computing. In Proceedings of the 28th International Symposium Mathematical Foundations of Computer Science (MFCS 2003), volume 2747 of Lecture Notes in Computer Science, pages 1–20. Springer, 2003.

[BSV+96] Paolo Boldi, Shella Shammah, Sebastiano Vigna, Bruno Codenotti, Peter Gemmell, and Janos Simon. Symmetry breaking in anonymous networks: Characterizations. In Proceedings of the Fourth Israel Symposium on Theory of Computing and Systems, pages 16–26. IEEE Computer Society, 1996.

[BT08] Anne Broadbent and Alain Tapp. Can quantum mechanics help distributed computing? SIGACT News, 39(3):67–76, 2008.

[BV02] Paolo Boldi and Sebastiano Vigna. Fibrations of graphs. Discrete Mathematics, 243(1-3):21–66, 2002.
[CK98] Dong Pyo Chi and Jinsoo Kim. Quantum database search by a single query. In Proceedings of the First NASA International Conference Quantum Computing and Quantum Communications, volume 1509 of Lecture Notes in Computer Science, pages 148–151. Springer, 1998.

[CKS10] Bogdan S. Chlebus, Dariusz R. Kowalski, and Michal Stojnowski. Scalable quantum consensus for crash failures. In Proceedings of the 24th International Symposium Distributed Computing (DISC 2010), volume 6343 of Lecture Notes in Computer Science, pages 236–250. Springer, 2010.

[DP06a] Vasil S. Denchev and Gopal Pandurangan. Distributed quantum computing: A new frontier in distributed systems or science fiction? ACM SIGACT News, 39(3):77–95, 2006.

[DP06b] Ellie D’Hondt and Prakash Panangaden. The computational power of the W and GHZ states. Quantum Information and Computation, 6(2):173–183, 2006.

[EKNP14] Michael Elkin, Hartmut Klauck, Danupon Nanongkai, and Gopal Pandurangan. Can quantum communication speed up distributed computation? In Proceedings of the 33rd ACM SIGACT-SIGOPS Symposium on Principles of Distributed Computing (PODC ’14), pages 166–175, 2014.

[Gan09] Maor Ganz. Quantum leader election. arXiv:0910.4952, 2009.

[GBK+08] Sascha Gaertner, Mohamed Bourennane, Christian Kurtsiefer, Adán Cabello, and Harald Weinfurter. Experimental demonstration of a quantum protocol for byzantine agreement and liar detection. Physical Review Letters, 100(070504), 2008.

[GKM09] Cyril Gavoille, Adrian Kosowski, and Marcin Markiewicz. What can be observed locally? In Idit Keidar, editor, Proceedings of the 23rd International Symposium on Distributed Computing (DISC 2009), volume 5805 of Lecture Notes in Computer Science, pages 243–257. Springer, 2009.

[Hen14] Julien M. Hendrickx. Views in a graph: To which depth must equality be checked? IEEE Transactions on Parallel and Distributed Systems, 25(7):1907–1912, 2014.

[HKAA97] Lisa Higham, David G. Kirkpatrick, Karl R. Abrahamson, and Andrew Adler. Optimal algorithms for probabilistic solitude detection on anonymous rings. Journal of Algorithms, 23(2):291–328, 1997.

[IR81] Alon Itai and Michael Rodeh. Symmetry breaking in distributive networks. In Proceedings of the 22nd Annual IEEE Symposium on Foundations of Computer Science, pages 150–158, 1981.

[IR90] Alon Itai and Michael Rodeh. Symmetry breaking in distributed networks. Information and Computation, 88(1):60–87, 1990.

[KKvdB94] Evangelos Kranakis, Danny Krizanc, and Jacob van den Berg. Computing boolean functions on anonymous networks. Information and Computation, 114(2):214–236, 1994.

[KLM07] Phillip Kaye, Raymond Laflamme, and Michele Mosca. An Introduction to Quantum Computing. Oxford University Press, 2007.

[KMT14] Hirotada Kobayashi, Keiji Matsumoto, and Seiichiro Tani. Simpler exact leader election via quantum reduction. Chicago Journal of Theoretical Computer Science, 2014(10), 2014.
Appendix

Proof of Claim 20. To compute the value $|x|$, the leader first assigns a unique identifier $id_i$ to each party $i$ and then every party collects all pairs $(id_i, x_i)$ for $i \in [n]$ by using Proposition 18, from which every party can compute the value $|x|$ locally.

To assign unique identifiers, the leader first sends a message “$j$” of $O(\log |V|)$ bits via every out-port $j$. The leader ignores any message it has received. Suppose then that a follower $i$ has received a message $m$. If this message is the second one that the follower has received, it ignores the message; otherwise, it sets $id_i := m$ and sends a message $m \circ j$ via every out-port $j$, where ‘$\circ$’ means concatenation (when the follower $i$ receives multiple messages at once, the follower arbitrarily breaks the tie and chooses one of them as the first message). Since each message is a sequence of out-port numbers, this chain of messages
uniquely determines a directed path starting from the leader to each party that receives one of the messages without ignoring it. Hence, id_i is not equal to id_j whenever i \neq j.

The message-passings stop in at most N rounds, since the number of required rounds is equal to one plus the length of the longest path among those determined by the chains of messages, and any such path includes each party at most once. Every party thus moves to the next procedure after N rounds. The bit complexity is \(O(\|V\|E \log|V|)\), since the size of each message is \(O(\|V\| \log|V|)\) and each communication link is used for exactly one message.

To collect all pairs \((id_i, x_i)\) for \(i \in [n]\), all parties run (a slight modification of) COLORCOUNT in Proposition 18 for \(C := \{(id_i, x_i): \, i \in [n]\}\). Each party then obtains \(C\) in \(O(N)\) rounds. Notice that, unlike the statement of Proposition 18, the size of \(C\) is not constant in this case. Hence, the bit complexity should be multiplied by at most the size of a message: \(O(|V|^2 \log|V|)\) (since the set \(C\) has \(|V|\) pairs of \(O(|V| \log|V|)\) bits). Thus, the bit complexity is \(O(N|V|^2 |E| \log|V|)\).

**Proof of Theorem 3.** For each recursion level \(t \in [\lceil \log_2 k \rceil]\), let \(\Xi^{(t)}\) be the collection of all possible equivalence classes of active parties such that each class \(\xi^{(t)}\) in \(\Xi^{(t)}\) is the subset of active parties that have obtained the same sequence \(r^{(1)}, \ldots, r^{(t)}\) of outcomes in the first through \(t\)th levels of recursion [recall that each outcome is obtained by measuring \(R_t\) at (modified) STEP 3 or STEP 6]. Note that some \(\xi^{(t)} \in \Xi^{(t)}\) may be the empty set. Define \(\Xi^{(0)} := \{\{i: \, x_i = 1\}\}\). Since each outcome is a two-bit value, we have \(|\Xi^{(0)}| = 4^t\). More concretely, \(\Xi^{(1)}\) is the finer collection obtained by partitioning \(\Xi^{(0)}\) into four possible equivalence classes associated with four possible outcomes of \(r^{(1)}\): \(\{00, 01, 10, 11\}\). For each \(t \in [\lceil \log_2 k \rceil]\), let \(\varphi^{(t)}\) be a bijection that maps each element in \(\Xi^{(t)}\) to the corresponding sequence of outcomes \((r^{(1)}, \ldots, r^{(t)})\) \(\in \{00, 01, 10, 11\}^t\). We also define \(\varphi^{(0)}: \Xi^{(0)} \mapsto \text{null}\) for the unique element \(\xi^{(0)}\) in \(\Xi^{(0)}\). For simplicity, we identify each element \(\xi^{(t)} \in \Xi^{(t)}\) with \(\varphi^{(t)}(\xi^{(t)})\).

Next, we make a slight modification to Algorithm QSV as follows (let QSV’ be the modified version): If \(|x| \geq 2\) or \(|x| = 0\), then QSV’ outputs \(r_i\) at each party \(i\), where \(r_i\) is the outcome of measurement on \(R_i\) made at (modified) STEP 3 or STEP 6 in \(Q_{h^{*}, m}\); if \(|x| = 1\), QSV’ outputs “true” as the original QSV does.

Now we are ready to present Algorithm QSYM for exactly computing a given \(f \in S_n(k)\).
QSYM consists of \(\lceil \log_2 k \rceil\) stages defined as follows: At stage 1, every party \(i\) performs QSV' with input \((x_i(\xi^{(0)})), N)\), where \(x_i(\xi^{(0)})\) means the input bit \(x_i\). Let \(y_i(\xi^{(0)})\) be the output of QSV'. If \(|x|\) is one, then \(y_i(\xi^{(0)})\) is “true” by the definition of QSV'. In this case, only the party \(i\) with \(x_i = 1\) sets \(S_i := “\text{leader}”\), and then every party can compute \(|x|\) by the algorithm in Claim 20, from which it can compute \(f(x)\) locally. If \(|x| \geq 2\) or \(|x| = 0\), then QSV' returns the measurement outcome \(y_i(\xi^{(0)})\) to every party \(i\). Every party \(i\) then decides which class in \(\Xi(1)\) it belongs to by using the value of \(y_i(\xi^{(0)})\). Moreover, for each class \(\xi^{(1)} \in \Xi(1)\), the party sets the input \(x_i(\xi^{(1)})\) for the second stage to 1 if it is a member of \(\xi^{(1)}\) (i.e., \(\varphi(\xi^{(1)}) = y_i(\xi^{(1)})\)), and to 0 otherwise. The algorithm then proceeds to the second stage to further partition the equivalence classes (actually, all parties can check whether \(|x|\) is zero or not by computing \(T_0\) at the beginning of QSYM, but we design the algorithm as above just to simplify the descriptions).

More generally, at each stage \(t\), every party \(i\) performs QSV' with \((x_i(\xi^{(t-1)})), N)\) for each \(\xi^{(t-1)} \in \Xi^{(t-1)}\). If QSV' returns \(y_i(\xi^{(t-1)}) = \text{true}\), then every party computes \(|x|\) by Claim 20 and outputs \(f(x)\). Otherwise, QSV' returns \(y_i(\xi^{(t-1)}) \in \{0, 1\}^2\). For each \(z \in \{0, 1\}^2\) and each \(\xi^{(t-1)} \in \Xi^{t-1}\), a unique \(\xi^{(t)} \in \Xi_t\) satisfies \(\xi^{(t)} = \xi^{(t-1)}z\). Every party \(i\) then sets the input for stage \(t + 1\) as follows:
\[
x_i(\xi^{(t)}) = x_i(\xi^{(t-1)}z) := x_i(\xi^{(t-1)}) \land [z = y_i(\xi^{(t-1)})],
\]
where \([z = y_i(\xi^{(t-1)})]\) is the predicate, which is 1 if and only if \(z = y_i(\xi^{(t-1)})\); in other worlds, for each \(\xi^{(t-1)}z \in \Xi_t\), \(x_i(\xi^{(t-1)}z)\) is 1 if the party \(i\) is a member of \(\xi^{(t-1)}z\) and 0 otherwise.

If the algorithm runs up to the \(\lceil \log_2 k \rceil\)th stage and does not output “true” for any \(\xi^{(\lceil \log_2 k \rceil)} \in \Xi^{(\lceil \log_2 k \rceil)}\), then \(|x|\) should be larger than \(k\). Every party thus chooses an arbitrary \(x' \in \{0, 1\}^N\) with \(|x'| > k\), and computes the value of \(f\) on \(x'\).

For each \(t\), all the communication is devoted to running QSV' and the algorithm in Claim 20, both of which require \(O(N)\) rounds and a polynomially bounded number of (qu)bits for communication. \(\square\)