TREND TO EQUILIBRIUM OF RENORMALIZED SOLUTIONS TO REACTION-CROSS-DIFFUSION SYSTEMS

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ABSTRACT. The convergence to equilibrium of renormalized solutions to reaction-cross-diffusion systems in a bounded domain under no-flux boundary conditions is studied. The reactions model complex balanced chemical reaction networks coming from mass-action kinetics and thus do not obey any growth condition, while the diffusion matrix is of cross-diffusion type and hence nondiagonal and neither symmetric nor positive semi-definite, but the system admits a formal gradient-flow or entropy structure. The diffusion term generalizes the population model of Shigesada, Kawasaki and Teramoto to an arbitrary number of species. By showing that any renormalized solution satisfies the conservation of masses and a weak entropy-entropy production inequality, it can be proved under the assumption of no boundary equilibria that all renormalized solutions converge exponentially to the complex balanced equilibrium with a rate which is explicit up to a finite dimensional inequality.

1. INTRODUCTION

Multi-species systems appear in many applications in biology, physics and chemistry, and can be modeled by reaction-cross-diffusion systems. We want to study the convergence to equilibrium of reaction-cross-diffusion systems with strongly growing reactions, where the system (without reactions) is of formal gradient-flow structure and thus admits an entropy estimate. But since the reactions do not obey any growth condition, this estimate is not enough to define weak solutions, which motivates the study of renormalized solutions à la J. Fischer [15]. Our goal is to show that any renormalized solution satisfies the conservation of masses and a weak entropy-entropy production inequality, and consequently, under the assumption of no boundary equilibria, all renormalized solutions converge to equilibrium with an exponential rate which is explicit up to a finite dimensional inequality.

The convergence to equilibrium for reaction-diffusion systems with linear diffusion has been studied extensively, see e.g. [1, 9, 11] and references therein, while much less is known for nonlinear diffusion or cross diffusion, see [17] for a porous-medium type diffusion and [8] for Maxwell-Stefan diffusion. In this work, we study the convergence to equilibrium for a cross-diffusion model originally introduced by Shigesada, Kawasaki and Teramoto [21] in population dynamics. The existence of global weak solutions for this class of cross-diffusion models with at most linearly growing reactions has been attracted a lot of attention recently by exploiting its formal gradient-flow structure, see e.g. [2, 3, 4, 12, 13, 19, 20]. Unfortunately, for strongly growing reactions (such as chemical reactions) this does not provide enough regularity to define weak solutions. Hence, the notion of renormalized solutions was introduced in [5] for reaction-cross-diffusion systems in analogy to [15] for reaction-diffusion systems. The standard way for proving convergence to equilibrium via entropy method is to first prove the convergence for an approximate solution, and then by passing to the limit to obtain it also for the constructed weak solution (see e.g. [8]). But since uniqueness for cross diffusion is a very delicate topic (see e.g. [6]), it is desirable to prove convergence to equilibrium for all solutions. This has been recently obtained in [18] for reaction-diffusion systems, and thus in this work, we extend these results to reaction-cross-diffusion systems with strongly growing complex balanced reactions coming from mass-action kinetics.

More precisely, we consider $n$ chemical substances $S_1, \ldots, S_n$ reacting via $R$ reactions of the form

\begin{equation}
\begin{aligned}
y_{r,1}S_1 + \ldots + y_{r,n}S_n \xrightarrow{k_r} y'_{r,1}S_1 + \ldots + y'_{r,n}S_n \quad \text{or shortly} \quad y_r \xrightarrow{k_r} y'_r, \quad r = 1, \ldots, R,
\end{aligned}
\end{equation}

where $y_r = (y_{r,1}, \ldots, y_{r,n}), y'_r = (y'_{r,1}, \ldots, y'_{r,n}) \in (\{0\} \cup [1, \infty))^n$ are the stoichiometric coefficients, and $k_r > 0$ are the reaction rate constants. The corresponding reaction-cross-diffusion system reads for each $i = 1, \ldots, n$ as

\begin{equation}
\begin{aligned}
\partial_t u_i - \text{div} \left( \sum_{j=1}^n A_{ij}(u) \nabla u_j \right) &= f_i(u), \quad \text{for } (x, t) \in \Omega \times (0, T), \quad \text{for } (x, t) \in \partial \Omega \times (0, T),
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
\sum_{j=1}^n A_{ij}(u) \nabla u_j \cdot \nu &= 0, \quad \text{for } x \in \Omega, \quad \text{for } x \in \Omega,
\end{aligned}
\end{equation}

where $u = (u_1, \ldots, u_n)$ are the population densities and $\Omega$ is a bounded domain with smooth boundary $\partial \Omega$, and $\nu$ is the exterior unit normal vector to $\partial \Omega$. The reaction terms represent the reactions in (1), i.e.

\begin{equation}
\begin{aligned}
f_i(u) = \sum_{r=1}^R k_r (y'_{r,i} - y_{r,i})u^{y'_r} \quad \text{with} \quad u^{y'_r} = \prod_{i=1}^n u_i^{y'_{r,i}},
\end{aligned}
\end{equation}

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while the diffusion matrix $A(u) = [A_{ij}(u)]_{i,j=1,\ldots,n}$ is given by

$$A_{ij}(u) = \delta_{ij} \left( a_{i0} + \sum_{k=1}^{n} a_{ik} u_k \right) + a_{ij} u_i,$$

where $a_{i0}, a_{ij} \geq 0$ for all $i, j = 1, \ldots, n$ and $\delta_{ij}$ denotes the Kronecker delta. They are assumed to satisfy (in analogy to [5]) either the weak cross-diffusion condition

$$\alpha := \min_{i=1,\ldots,n} \left( a_{ii} - \frac{1}{4} \sum_{i=1}^{n} (\sqrt{a_{ij}} - \sqrt{a_{ji}})^2 \right) > 0,$$

or the detailed-balance condition\(^1\)

$$a_{ij} = a_{ji} \quad \text{for all } 1 \leq i, j \leq n.$$

Let $m = \text{codim}\{y_r' - y_r\}_{r=1,\ldots,R}$, then if $m > 0$ there exists a matrix $Q \in \mathbb{R}^{m \times n}$ whose rows form a basis of $\text{ker}\{y_r' - y_r\}_{r=1,\ldots,R} \in \mathbb{R}^{n \times R}$. From (2) it follows that $Q[f_1(u), \ldots, f_n(u)]^T = 0$, and therefore (S) formally possesses $m$ conservation laws

$$\dot{\pi}(t) = Q\pi_0 =: M \quad \text{for all } t > 0,$$

where $\pi = (\pi_1, \ldots, \pi_n)$ and $\pi_i = \frac{1}{|\Omega|} \int_{\Omega} u_i dx$. The system (S) is said to satisfy the complex balanced condition if there exists a positive complex balanced equilibrium $u_\infty = (u_{1,\infty}, \ldots, u_{n,\infty}) \in (0, \infty)^n$, such that at $u_\infty$ the total out-flow and in-flow at each complex are balanced, i.e.

$$\sum_{\{r: y_r = y\}} k_r u_{i,i} = \sum_{\{s: y_s = y\}} k_s u_{i,i} \quad \text{for all } y \in \{y_r, y_r'\}_{r=1,\ldots,R}.$$

It was proved in [14] that if $m > 0$ then for each positive initial mass vector $M$ there exists a unique positive complex balanced equilibrium $u_\infty \in (0, \infty)^n$, while when $m = 0$ the system has a unique positive complex balanced equilibrium for any positive initial data. Note that there could possibly exist many boundary equilibria, i.e. $u^* \in \partial(0, \infty)^n$ and $u^*$ satisfies (6).

The main result of this paper reads as follows.

**Theorem 1.1.** Let $\Omega$ be a bounded domain with smooth boundary $\partial \Omega$. Assume $a_{i0}, a_{ii} > 0, a_{ij} \geq 0$, and let the diffusion matrix $A(u)$ satisfy either (4) or (5). Assume that (S) satisfies the complex balanced condition (6). Then, for any nonnegative measurable initial data $u_0 \in L^1(\Omega)^n$ such that $u_{i,0} \log u_{i,0} \in L^1(\Omega)$ for all $i = 1, \ldots, n$, there exists a global nonnegative renormalized solution $u = (u_1, \ldots, u_n)$ to (S), that is, for all $T > 0$,

$$u_{i,0} \log u_{i} \in L^\infty(0, T; L^1(\Omega)), \quad \|\sqrt{\pi_r}\|_{L^2(0,T;H^1(\Omega))}, \|u_{i,0}\|_{L^2(0,T;H^1(\Omega))} \leq C(T)$$

and for any smooth function $\xi \in C^\infty([0, \infty)^n)$ with compactly supported $D\xi$, it holds for all test functions $\psi \in C^\infty_0(\overline{\Omega} \times [0, T])$ that

$$-\int_{\Omega} \xi(u_0)\psi(\cdot, 0)dx - \int_0^T \int_{\Omega} \xi(u)\partial_t \psi dx dt = -\sum_{i,k=1}^{n} \int_0^T \int_{\Omega} \partial_i \partial_k \xi(u) \left( \sum_{j=1}^{n} A_{ij}(u) \nabla u_j \right) \nabla u_k \psi dx dt$$

$$-\sum_{i=1}^{n} \int_0^T \int_{\Omega} \partial_i \xi(u) \left( \sum_{j=1}^{n} A_{ij}(u) \nabla u_j \right) \nabla \psi dx dt + \sum_{i=1}^{n} \int_0^T \int_{\Omega} \partial_i \xi(u) f_i(u) \psi dx dt.$$

Assume additionally that (S) does not have any boundary equilibria and fix an initial mass vector $M$. Then, any renormalized solution to (S) with positive initial mass $M$, i.e. $\dot{\pi}_0 = M$, converges exponentially to the equilibrium, i.e.

$$\sum_{i=1}^{n} \|u_i(t) - u_{i,\infty}\|_{L^1(\Omega)} \leq C e^{-\lambda t} \quad \text{for all } t > 0,$$

where $C > 0$ and $\lambda > 0$ are constants which can be computed explicitly up to a finite dimensional inequality.

**Remark 1.2.** The convergence result in Theorem 1.1, in case $m > 0$, depends only on the initial masses but not on the precise initial data. Thus, two solutions with different initial data but same initial masses converge exponentially to the same equilibrium. When $m = 0$, i.e. there are no conservation laws, then all renormalized solutions converge to the unique positive equilibrium for any positive initial data.

\(^1\)This should not be confused with the detailed balance condition occurring in reactions or in even more general micro-reversible processes. Also note that in [5] the detailed balance diffusion condition was $\pi_i a_{ij} = \pi_j a_{ji}$ for some positive constants $\pi_i > 0$. Here we choose $\pi_i = 1$ for $i = 1, \ldots, n$ for the compatibility with the reactions.
The main tool in the proof of Theorem 1.1 is to consider the relative entropy

\[ \mathcal{E}(u|u_x) = \int_{\Omega} E(u|u_x) dx, \]

where \( E(u|u_x) = \sum_{i=1}^{n} (u_i \log(u_i/u_x) - u_i + u_i u_x) \geq 0, \)

for which formally for any solution to (S) the entropy production has the following form

\[ \frac{d}{dt} \mathcal{E}(u|u_x) \leq -\mathcal{D}(u) \quad \text{with} \quad \mathcal{D}(u) = \sum_{i=1}^{n} a_{i0} \int_{\Omega} |\nabla u_i|^2 dx + \sum_{r=1}^{R} k_r u_x^{\beta r} - \int_{\Omega} \frac{\Psi(x,y) u_x^{\beta r}}{u_x^2} dx, \]

where \( \Psi(x,y) = x \log(x/y) - x + y. \) For details we refer to [2] for the cross-diffusion term and to [18] for the reaction term. Moreover, for all nonnegative measurable functions \( u = (u_1, \ldots, u_n) \) satisfying the conservation laws

\[ \mathbb{Q} \mathbf{\Pi} = \mathbf{M}, \]

it was proved (e.g. [18]) that

\[ \mathcal{D}(u) \geq \lambda \mathcal{E}(u|u_x), \]

where \( \lambda \) is an explicit constant up to a finite dimensional inequality. Then, still formally, one obtains the desired exponential decay

\[ \mathcal{E}(u(t)|u_x) \leq e^{-\lambda t} \mathcal{E}(u(0)|u_x). \]

Unfortunately, the notion of renormalized solutions is very weak, so that the entropy-entropy production inequality (9) or even the conservation laws (10) (which only concern the \( L^1 \)-norm of the solution) are not easy to verify. As mentioned before, one can argue via approximating solutions, and thus obtain the convergence to equilibrium for one renormalized solution, see e.g. [8]. However, it is not clear if all renormalized solutions (in the sense of definition in (7)) can be approximated in such a way. Our aim here is to prove that all renormalized solutions with the same initial mass converge to the unique equilibrium. The main idea is to show that the conservation laws (10) and a weaker version of the entropy-entropy production inequality (see Lemma (2.1)) hold for any renormalized solution. Our proof uses the techniques developed in [16].

2. Proof of the main result

Lemma 2.1 (Weak entropy-entropy production inequality). For any renormalized solution \( u \) of (S) it holds that

\[ \mathcal{E}(u(t)|u_x) + \int_{s}^{t} \mathcal{D}(u(\tau)) d\tau \leq \mathcal{E}(u(s)|u_x) \quad \text{for a.e.} \quad t > s > 0, \]

where \( \mathcal{E} \) and \( \mathcal{D} \) are defined in (8) and (9) respectively.

Proof. From this point on, we consider \( C > 0 \) as a generic constant whose value can change from line to line, or even in the same line. For \( M > 0 \), let \( \phi_M : [0, \infty) \to \mathbb{R} \) be a smooth function with

\[ \phi_M(s) = s, \quad \phi_M'(s) = 0, \quad \phi_M''(s) \in [0, 1], \]

\[ |\phi_M''(s)| \leq \frac{C}{1 + s \log(1 + s)} \quad \text{for all} \quad s \geq 0. \]

Moreover, we set

\[ \xi(u) = \phi_M(E(u + \eta|u_x)), \]

where \( u + \eta = (u_1 + \eta, \ldots, u_n + \eta) \) for some \( \eta > 0 \). The regularization \( \eta > 0 \) is needed to deal with the potential singularity of logarithm since renormalized solution is non-negative but in general not strictly positive. For simplicity, we will write \( E(u) \) and \( E(u + \eta) \) instead of \( E(u|u_x) \) and \( E(u + \eta|u_x) \) respectively inside this proof. Then we can compute

\[ \partial_t \xi(u) = \phi_M''(E(u + \eta)) \frac{u_k + \eta}{u_{k|x}} \log \left( \frac{u_k + \eta}{u_{k|x}} \right) \]

\[ \partial_t \partial_k \xi(u) = \phi_M''(E(u + \eta)) \frac{u_k + \eta}{u_{k|x}} \log \left( \frac{u_k + \eta}{u_{k|x}} \right) + \phi_M'(E(u + \eta)) \frac{\delta_{ik}}{u_k + \eta}. \]

By choosing \( \psi = 1 \), or more precisely a smooth version of 1 with compact support in \([0, T - \delta]\) then let \( \delta \to 0 \) (see [5, Lemma 11] for more details) in the definition of the renormalized solutions, we get

\[ I_1(\eta, M) := \int_{\Omega} \phi_M(E(u + \eta)) dx \bigg|_{s}^{t}, \]

\[ = - \sum_{i,k=1}^{n} \int_{s}^{t} \left( \phi_M''(E(u + \eta)) \frac{u_k + \eta}{u_{k|x}} \log \left( \frac{u_k + \eta}{u_{k|x}} \right) + \phi_M'(E(u + \eta)) \frac{\delta_{ik}}{u_k + \eta} \right) \left( \sum_{j=1}^{n} A_{ij}(u) \nabla u_j \right) \nabla u_k dx dt \]

\[ + \sum_{i=1}^{n} \int_{s}^{t} \phi_M(E(u + \eta)) \frac{u_i + \eta}{u_{i|x}} f_i(u) dx dt \]

\[ =: I_2(\eta, M) + I_3(\eta, M). \]
Our first goal now is to pass to the limit $\eta \to 0$ in (12). Clearly, due to the dominated convergence theorem, we have for the left-hand side of (12) that

$$\lim_{\eta \to 0} I_1(\eta, M) = \int_\Omega \phi_M(E(u)) \, dx \bigg|_{t_s}^t.$$  

Next, since $\phi_M'$ has compact support the integrand of $I_3(\eta, M)$ vanishes when $|u|$ is large. Now for $|u| \leq C(M)$ we can use the property $f_i(u) \geq 0$ when $u_i = 0$ and the local Lipschitz continuity of $f_i(u)$ to estimate $f_i(u) \geq -C(M)u_i$. Hence by considering the signs of $f_i(u)$ and $\log \left( \frac{u_i + \eta}{u_i} \right)$ one obtains easily

$$f_i(u) \log \left( \frac{u_i + \eta}{u_i} \right) \leq C(M)u_i \left| \log \frac{u_i + \eta}{u_i} \right|.$$  

Thus, Fatou’s lemma yields

$$\limsup_{\eta \to 0} I_3(\eta, M) \leq \sum_{i=1}^n \int_{\Omega} \phi'_M(E(u)) \log \left( \frac{u_i}{u_i} \right) f_i(u) \, dx \, d\tau.$$  

Next, we split $I_2(\eta, M)$ in (12) into

$$I_2(\eta, M) = - \sum_{i,k=1}^n \int_{\Omega} \left( \phi'_M(E(u + \eta)) \log \left( \frac{u_k + \eta}{u_k} \right) \log \left( \frac{u_i + \eta}{u_i} \right) \left( \sum_{j=1}^n A_{ij}(u) \nabla u_j \right) \nabla u_k \, dx \, d\tau \right.$$

$$- \sum_{i=1}^{n} \int_{\Omega} \phi'_M(E(u + \eta)) \frac{1}{u_i} \left( \sum_{j=1}^n A_{ij}(u) \nabla u_j \right) \nabla u_i \, dx \, d\tau$$

$$=: I_4(\eta, M) + I_5(\eta, M).$$  

In order to show the convergence of $I_4$, we use that $|A_{ij}(u)| \leq C(1 + \sum_{k=1}^n |u_k|)$ and $\|\nabla u_j\|_{L^2(\Omega \times (0, T))} \leq C(T)$ thanks to the regularity of renormalized solutions. Then, recalling $\phi'_M$ has a compact support, we obtain by dominated convergence theorem that

$$\lim_{\eta \to 0} I_4(\eta, M) = - \sum_{i=1}^n \int_{\Omega} \phi'_M(E(u)) \log \left( \frac{1}{u_i} \right) \log \left( \frac{u_i}{u_i} \right) \left( \sum_{j=1}^n A_{ij}(u) \nabla u_j \right) \nabla u_i \, dx \, d\tau.$$  

In a similar way, we obtain

$$\lim_{\eta \to 0} I_5(\eta, M) = - \sum_{i=1}^n \int_{\Omega} \phi'_M(E(u)) \log \left( \frac{1}{u_i} \right) \log \left( \frac{u_i}{u_i} \right) \left( \sum_{j=1}^n A_{ij}(u) \nabla u_j \right) \nabla u_i \, dx \, d\tau.$$  

From [6] we know that if $A(u)$ satisfies (4), then

$$\sum_{i=1}^n \frac{1}{u_i} \left( \sum_{j=1}^n A_{ij}(u) \nabla u_j \right) \nabla u_i \geq 4 \sum_{i=1}^n a_{i0} |\nabla u_i|^2 + \alpha \sum_{i=1}^n |\nabla u_i|^2,$$

and if $A(u)$ satisfies (5), then

$$\sum_{i=1}^n \frac{1}{u_i} \left( \sum_{j=1}^n A_{ij}(u) \nabla u_j \right) \nabla u_i \geq 4 \sum_{i=1}^n a_{i0} |\nabla u_i|^2 + 2 \sum_{i=1}^n a_{i} |\nabla u_i|^2 + 2 \sum_{i \neq j} a_{ij} |\nabla u_i u_j|^2.$$  

From both cases we infer, by noticing that $a_{i0} > 0$ and $4|\nabla u_i|^2 = |\nabla u_i|^2/\sqrt{u_i}$,

$$\lim_{\eta \to 0} I_5(\eta, M) \leq - \sum_{i=1}^n \int_{\Omega} \phi'_M(E(u)) a_{i0} \frac{|\nabla u_i|^2}{u_i} \, dx \, d\tau.$$  

Putting everything together yields from (12)

$$\int_{\Omega} \phi_M(E(u)) \, dx \bigg|_{t_s}^t \leq - \sum_{i,k=1}^n \int_{\Omega} \phi'_M(E(u)) \log \left( \frac{1}{u_k} \right) \log \left( \frac{u_i}{u_i} \right) \left( \sum_{j=1}^n A_{ij}(u) \nabla u_j \right) \nabla u_k \, dx \, d\tau$$

$$- \sum_{i=1}^n a_{i0} \int_{\Omega} \phi'_M(E(u)) \nabla u_i \, dx \, d\tau + \sum_{i=1}^n \int_{\Omega} \phi'_M(E(u)) \log \left( \frac{u_i}{u_i} \right) f_i(u) \, dx \, d\tau$$

$$=: I_6(M) + I_7(M) + I_8(M).$$  

Our goal now is to pass to the limit $M \to \infty$ in (13). For the left-hand side of (13), the convergence is clear due the dominated convergence theorem. For $I_7$ we can use $\sqrt{u_i} \in L^2(0, T; H^1(\Omega))$ and the dominated convergence theorem to obtain

$$\lim_{M \to \infty} I_7(M) = - \sum_{i=1}^n a_{i0} \int_{\Omega} \frac{|\nabla u_i|^2}{u_i} \, dx \, d\tau.$$
Since $\sum_{i=1}^n \log \left( \frac{u_i}{u_{i,x}} \right) f_i(u) \leq 0$, we get by Fatou’s lemma that
\[
\limsup_{M \to \infty} I_6(M) \leq \sum_{i=1}^n \int_0^T \int_\Omega \log \left( \frac{u_i}{u_{i,x}} \right) f_i(u) \, dx \, dt.
\]
For $I_6(M)$ we first use the identity $\sum_{j=1}^n A_{ij}(u) \nabla u_j = a_{i0} \nabla u_i + \sum_{j=1}^n a_{ij} (u_j \nabla u_i + u_i \nabla u_j)$ to estimate $I_6(M) \leq I_{61}(M) + I_{62}(M) + I_{63}(M)$ where
\[
I_{61}(M) = C \sum_{i,j,k=1}^n \int_s^t \int_\Omega \phi''_{M}(E(u)) \log \left( \frac{u_k}{u_{k,x}} \right) \frac{u_i}{u_{i,x}} \nabla u_i \nabla u_k \, dx \, dt,
\]
\[
I_{62}(M) = C \sum_{i,j,k=1}^n \int_s^t \int_\Omega \left| \phi''_{M}(E(u)) \right| |u_j| \log \left( \frac{u_k}{u_{k,x}} \right) \nabla u_k \, \left( \frac{u_i}{u_{i,x}} \right) \nabla u_i \, dx \, dt,
\]
\[
I_{63}(M) = C \sum_{i,j,k=1}^n \int_s^t \int_\Omega \left| \phi''_{M}(E(u)) \right| \log \left( \frac{u_k}{u_{k,x}} \right) \frac{u_i}{u_{i,x}} \nabla u_i \nabla u_k \, dx \, dt.
\]
For $I_{61}(M)$ we write $\nabla u_i \nabla u_k = 4 \sqrt{u_i \sqrt{u_k}} (\nabla u_i \sqrt{u_k})$, then we use the property of $\phi''_M$ in (11) to estimate
\[
\left| \phi''_{M}(E(u)) \right| \log \left( \frac{u_k}{u_{k,x}} \right) \frac{u_i}{u_{i,x}} \nabla u_i \nabla u_k \leq C \left| \log \left( \frac{u_k}{u_{k,x}} \right) \frac{u_i}{u_{i,x}} \right| \left( \frac{u_k}{u_{k,x}} \right) \frac{u_i}{u_{i,x}} \leq C.
\]
Hence, from the bound $|\nabla \sqrt{u_k}|_{L^2(\Omega \times (0,T))} \leq C(T)$ we obtain by dominated convergence that $\lim_{M \to +\infty} I_{61}(M) = 0$. To estimate $I_{62}(M)$ we have first
\[
\left| \phi''_{M}(E(u)) \right| \log \left( \frac{u_k}{u_{k,x}} \right) \nabla u_k \leq \chi(u_{k,x} > 1) \left| \phi''_{M}(E(u)) \right| \log \left( \frac{u_k}{u_{k,x}} \right) \nabla u_k \leq C|\nabla \sqrt{u_k}|,
\]
and similarly $\left| \phi''_{M}(E(u)) \right| \nabla u_i \leq \chi(u_{i,x} > 1) \left| \phi''_{M}(E(u)) \right| \nabla u_i \leq C|\nabla \sqrt{u_i}|$. Therefore
\[
I_{62}(M) \leq C \sum_{i,j,k=1}^n \int_s^t \int_\Omega \left( J_1(M) |\nabla u_k| |\nabla u_i| + J_2(M) |\nabla u_k| |\nabla \sqrt{u_i}| + J_3(M) |\nabla \sqrt{u_k}| |\nabla u_i| + J_4(M) |\nabla \sqrt{u_k}| |\nabla \sqrt{u_i}| \right) dx \, dt
\]
with
\[
J_1(M) = \left| \phi''_{M}(E(u)) \right| |u_j| \chi(u_{k,x} > 1) \chi(u_{i,x} > 1) \left| \phi''_{M}(E(u)) \right| \left( \frac{u_i}{u_{i,x}} \right) \left( \frac{u_k}{u_{k,x}} \right), \quad J_4(M) = \left| \phi''_{M}(E(u)) \right| |u_j| \chi(u_{i,x} > 1) \left| \phi''_{M}(E(u)) \right| \left( \frac{u_k}{u_{k,x}} \right)
\]
\[
J_2(M) = \left| \phi''_{M}(E(u)) \right| |u_j| \chi(u_{i,x} > 1) \left| \phi''_{M}(E(u)) \right| \left( \frac{u_k}{u_{k,x}} \right), \quad J_3(M) = \left| \phi''_{M}(E(u)) \right| |u_j| \chi(u_{i,x} > 1) \left| \phi''_{M}(E(u)) \right| \left( \frac{u_i}{u_{i,x}} \right).
\]
Using (11) we see that $|J_i(M)| \leq C$ for all $i = 1, \ldots, 4$. Taking into account that $|\nabla u_i|_{L^2(\Omega \times (0,T))}, |\nabla \sqrt{u_i}|_{L^2(\Omega \times (0,T))} \leq C(T)$ we conclude by the dominated convergence theorem that $\lim_{M \to +\infty} I_{62}(M) = 0$. The proof of $\lim_{M \to +\infty} I_{63}(M) = 0$ is similar so we omit it. Consequently, by collecting all results together and using the fact that
\[
\sum_{i=1}^n f_i(u)(\log u_i - \log u_{i,x}) = - \sum_{r=1}^R k_r u_r \nabla \phi''_{M}(E(u)) \nabla u_r \leq 0,
\]
(see the computations in [10, Proposition 2.1]), we obtain the desired result.

\begin{lemma}[Conservation laws]
When $m > 0$, for any renormalized solution $u$ to (S) it holds that
\[
Q(\infty) = Q_0 \quad \text{for all} \quad t > 0.
\]
\end{lemma}

\begin{proof}
Our proof follows from [16, Proposition 6] where Fischer proved the conservation laws for reaction-diffusion systems. We denote by $q = (q_1, \ldots, q_n)$ an arbitrary row of $Q$. Then, we have that $\sum_{i=1}^n q_i f_i(u) = 0$. Let $\phi_M$ be chosen in the same way as in the proof Lemma 2.1. By choosing $\xi(u) = \phi_M(\beta \sum_{i=1}^n q_i u_i + E(u + \eta u_{i,x}))$ where $\beta \in \mathbb{R}$ and $\psi = 1$ in the definition of renormalized solutions, we can pass to the limits $\eta \to 0$ and $M \to +\infty$ like in the proof of Lemma 2.1 to obtain
\[
\left( \beta \sum_{i=1}^n q_i u_i \, dx + \delta'(u|u_{i,x}) \right) \bigg|_0^T \leq \int_0^T \mathcal{P}(u(\tau)) \, d\tau.
\]
By dividing both sides by $\beta > 0$ and letting $\beta \to +\infty$, we get that
\[
\sum_{i=1}^n q_i u_i(T) \, dx \leq \sum_{i=1}^n q_i u_{i,0}(x) \, dx.
\]
Repeating the arguments with $\beta < 0$ and letting $\beta \to -\infty$, we obtain that $\sum_{i=1}^{n} \int_{\Omega} q_i u_i(T) \, dx \geq \sum_{i=1}^{n} \int_{\Omega} q_i u_{i0}(x) \, dx$, which finishes the proof of the conservation laws.

We are now ready to give the proof of the main result.

**Proof of Theorem 1.1.** The existence of a global renormalized solution follows from [5, Theorem 1] since under the complex balanced condition the reactions satisfy (14), which is (H4) in [5] with $\pi_i = 1$ and $\lambda_i = -\log w_{i,x}$ for all $i = 1, \ldots, n$.

We now turn to the convergence to equilibrium. Since the system possesses no boundary equilibria, it follows from [18, Theorem 1.1] that $\mathcal{P}(u) \geq \lambda \mathcal{E}(u|u_x)$ for all measurable nonnegative functions $u$ satisfying $Q u = Q u_x$, where $\lambda > 0$ is an explicit constant up to a finite dimensional inequality ([18, inequality (11)]). Note that this inequality does not require any other higher regularity of $u$. Therefore, thanks to Lemma 2.2, for any renormalized solution to (S) it holds

$$\mathcal{P}(u(s)) \geq \lambda \mathcal{E}(u(s)|u_x) \quad \text{for a.e.} \quad s > 0.$$ 

Using this and Lemma 2.1 it follows that

$$\mathcal{E}(u(t)|u_x) + \lambda \int_{s}^{t} \mathcal{E}(u(\tau)|u_x) \, d\tau \leq \mathcal{E}(u(s)|u_x) \quad \text{for a.e.} \quad t > s.$$ 

By Gronwall’s inequality we get

$$\mathcal{E}(u(t)|u_x) \leq e^{-\lambda(t-s)} \mathcal{E}(u_0|u_x),$$

and a Csiszár-Kullback-Pinsker type inequality (see e.g. [18, Lemma 2.2]) completes the proof of Theorem 1.1. \qed

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