Self-similar sequence transformation for critical exponents

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Abstract

Self-similar sequence transformation is an original type of nonlinear sequence transformations allowing for defining effective limits of asymptotic sequences. The method of self-similar factor transformations is shown to be regular. This method is applied for calculating the critical exponents of the $O(N)$-symmetric $\phi^4$ theory in three dimensions by summing asymptotic $\varepsilon$ expansions. It is shown that this method is straightforward and essentially simpler than other summation techniques involving complicated numerical calculations, while enjoying comparable accuracy.

Keywords: Self-similar sequence transformation; asymptotic series; summation methods; critical exponents

1 Introduction

Asymptotic expansions in powers of some parameter are widely used in physics and applied mathematics \cite{1,2}. Since the parameters of interest very rarely are really small, one needs to employ some kind of effective summation of divergent series. The most popular are the method of Padé approximants \cite{3}, Borel summation \cite{4} and its variants, such as Padé-Borel summation and Borel summation with conformal mapping. One also uses the methods of renormalization group, conformal bootstrap, Monte Carlo simulations, and other methods (see review \cite{5}) requiring quite heavy numerical calculations.

In the present paper, we advocate another approach based on self-similar approximation theory \cite{6,7}. The idea of this theory is to represent the transition from one approximation term to another as the motion of a dynamical system, or a renormalization-group equation, with the approximation order playing the role of discrete time. Then the sequence of approximation terms becomes bijective to the dynamical-system trajectory and the limit of the sequence is bijective to the fixed point of the trajectory. In the vicinity of a fixed point, the evolution equation (renormalization-group equation) acquires the form of a self-similar relation, which explains the name of the self-similar approximation theory. Mathematical details can be found in the review articles \cite{10,11}.
The approach has been applied to numerous problems, providing good agreement with the exact results, when these are available, as well as with numerical calculations and other elaborate methods, thus being compatible with other methods of summation of divergent series. Accurate results can be obtained even when just a few terms of an expansion are given, and when other summation methods are not applicable at all (see reviews [10][11]).

The method demonstrates good numerical convergence, which becomes especially evident for the cases, where a large number (of order or larger than ten) of perturbative terms are available. This concerns, e.g., the so-called zero-dimensional model [12], one-dimensional anharmonic oscillator [12], and spin glass [13].

All one needs for the application of the method is an asymptotic expansion centered at a point on the real axis. Its analytical behavior on the whole complex plane is not required. Thus, in the case of the zero-dimensional model and the one-dimensional oscillator, we meet expressions that have on the complex plane a singular point at zero [14]. This does not hinder the use of the method resulting in numerically convergent sequence of approximants for these models [12].

The method allows for the summation of a large class of functions, rational, irrational and transcendental. Moreover, there exists a class of functions that are exactly reproducible by this method [12]. This is the class of functions having the form

$$F_{k_M}(x) = \prod_{i=1}^{M} P_{m_i}^{\alpha_i}(x)$$

of the product of polynomials

$$P_{m_i}(x) = c_{i0} + c_{i1}x + c_{i2}x^2 + \ldots + c_{imi}x^{m_i}$$

where $\alpha_i$ and $c_{ij}$ are either real or the powers $\alpha_i$ and coefficients $c_{ij}$ are complex-valued numbers entering in complex conjugate pairs so that $F_{k_M}$ is real, and

$$k_M = \sum_{i=1}^{M} m_i + M.$$ 

The exponential function also is shown to be reconstructed exactly starting from the second-order approximation [12][15].

Nonlinear differential equations, including singular equations, can be solved by this method, first, by deriving a solution in terms of a series in powers of a variable, and then by summing this series using the self-similar factor transformation. For some nonlinear equations, exact soliton solutions have been obtained [12][16].

In the present paper, we consider two important points, one technical and the other of great interest in physics. The first point is the proof of the method regularity. We consider the method of self-similar factor transformation and show that this method is regular, which implies that it sums every convergent series to the same sum as that to which the series converges. This point is of high importance for the justification of the approach. It is the standard way of dealing with summation methods, when one, first, shows the regularity of the method and then extrapolates it to divergent sequences, by demonstrating its compatibility with other reliable methods and observing numerical convergence for some test problems [17][18]. Then we apply the method to the transformation of $\varepsilon$ expansions for critical exponents. As a concrete example, we study the $O(N)$ -symmetric $\varphi^4$ theory in three dimensions. We show that the method of self-similar factor
transformation provides the accuracy comparable with other elaborate techniques involving heavy numerical calculations, while being essentially simpler.

In Sec. 2, we formulate the method of self-similar factor transformation. We do not plunge into the foundations of the whole theory, but will just give the receipt of the method usage. In Sec. 3, we prove the regularity of the method. The application to defining the critical exponents is given in Sec. 4, where we compare our results with the most accurate values obtained by other methods summarized in Refs. [5,19,21]. We compare our results with Monte Carlo simulations, conformal bootstrap, hypergeometric Meijer summation, Borel summation, Borel summation with conformal mapping, and the method of nonperturbative renormalization group. This comparison shows good agreement of the self-similar factor approximants with the calculations by other methods. The last Sec. 5 concludes.

2 Self-similar factor transformation

In this section, we describe the method that can be used for the summation of arbitrary asymptotic series. Suppose we have got an asymptotic expansion for a real function

\[ f_k(x) = f_0(x) \left(1 + \sum_{n=1}^{k} a_n x^n\right) \]  

in powers of a real parameter \( x \) assumed to be asymptotically small. However, we need to find the value of the function at a finite value of the parameter. The extrapolation of the expansion \( f_k(x) \) to arbitrary values of the parameter \( x \) can be done by means of the self-similar factor transformation \[10,22,23\]. This method transforms the truncated expansion (1) to the factor form

\[ f_k^*(x) = f_0(x) \prod_{j=1}^{N_k} (1 + A_j x)^{n_j}, \]  

where the number of factors is

\[ N_k = \begin{cases} \frac{k}{2}, & k = 2, 4, 6, \ldots \\ \frac{(k+1)}{2}, & k = 3, 5, 7, \ldots \end{cases} \] (3)

The parameters \( A_j \) and \( n_j \) are uniquely determined by the accuracy-through-order procedure, by equating the like-order terms in the expansions at small \( x \),

\[ f_k^*(x) \simeq f_k(x) \quad (x \to 0). \] (4)

This procedure gives the equations

\[ \sum_{j=1}^{N_k} n_j A_j^n = D_n \quad (n = 1, 2, \ldots, k), \] (5)

in which

\[ D_n \equiv \frac{(-1)^{n-1}}{(n-1)!} \lim_{x \to 0} \frac{d^n}{dx^n} \ln \left(1 + \sum_{m=1}^{n} a_m x^m\right). \] (6)
In the case of an even order $k$, Eq. (5) consists of $k$ equations uniquely defining the $k/2$ parameters $A_j$ and $k/2$ parameters $n_j$. For odd orders $k$, the system of $k$ equations (5) contains $k + 1$ unknowns, where one of the parameters $A_j$, say $A_1$, is arbitrary. Normalizing $A_j$ in units of $A_1$ implies $A_1 = 1$, which makes the system of equations (5) self-consistent and all parameters uniquely defined $[10, 15]$. If the found parameters lead to a complex-valued approximant, it is replaced by the nearest real-valued approximant. The final result is given by the average between the last two approximants $[f_k(x) + f_{k-1}(x)]/2$ and the error bar is defined as the half-difference between the last two different approximants $[f_k(x) - f_{k-1}(x)]/2$. As is seen, the scheme is very simple and straightforward.

3 Method regularity

In this section, we show that the self-similar factor transformation is a regular method of summation.

Theorem. Let us consider the sequence of the terms

$$f_k(x) = \sum_{n=0}^{k} a_n x^n \quad (x \in \mathbb{D})$$

that are defined on a domain $\mathbb{D} \subset \mathbb{R}$ including the point $x = 0$. The self-similar factor transformation reduces the term $f_k(x)$ to the form

$$f^*_k(x) \equiv a_0 \prod_{j=1}^{N_k} (1 + A_j x)^{n_j}$$

that is a smooth function on $\mathbb{D}$, hence being infinitely differentiable, whose parameters $A_j$ and $n_j$ are prescribed by the accuracy-through-order procedure, such that

$$\frac{f_k^{(n)}(0)}{n!} = a_n \quad (n = 0, 1, 2, \ldots),$$

where $f_k^{(n)}(x)$ is the $n$-th derivative of $f_k^*(x)$ over $x$.

If the sequence of terms (7) converges to a smooth function

$$f(x) = \sum_{n=0}^{\infty} a_n x^n,$$

then the sequence of the factor approximants (8) converges to the function

$$f^*(x) \equiv a_0 \prod_{j=1}^{\infty} (1 + A_j x)^{n_j}$$

coinciding with $f(x)$:

$$f^*(x) = f(x).$$
Proof. The convergence of the sequence of terms (7) to a smooth function \( f(x) \) implies that the latter can be represented as the Taylor series

\[
f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n,
\]

with

\[
\frac{f^{(n)}(0)}{n!} = a_n \quad (n = 0, 1, 2, \ldots)
\]

and with the remainder

\[
R_k[f(x)] = \frac{f^{(k+1)}(c_1)}{(k+1)!} x^{k+1},
\]

where \( c_1 \in (0, x) \), this remainder tending to zero when \( k \to \infty \),

\[
\lim_{k \to \infty} R_k[f(x)] = 0.
\]

Comparing expressions (9) and (13), we see that

\[
f^{(n)}(0) - f^{(n)}_* (0) = 0 \quad (n = 0, 1, 2, \ldots).
\]

Because of this, and since the derivatives are continuous, there exists a finite value \( \varepsilon \) such that

\[
| f^{(n)}(c_1) - f^{(n)}_* (c_2) | < \varepsilon < \infty,
\]

where \( 0 \leq c_2 \leq x \) and \( n = 0, 1, 2, \ldots \). Considering the difference between the remainder (14) and the remainder

\[
R_k[f_*(x)] = \frac{f^{(k+1)}_* (c_2)}{(k+1)!} x^{k+1},
\]

we have

\[
| R_k[f_*(x)] - R_k[f(x)] | < \frac{\varepsilon}{(k+1)!} | x |^{k+1}.
\]

Using the Stirling formula \( n! \simeq \sqrt{2\pi n}(n/e)^n \), we find that

\[
\lim_{n \to \infty} \frac{\varepsilon}{(n+1)!} | x |^{n+1} = \frac{\varepsilon}{\sqrt{2\pi}} \lim_{n \to \infty} \left( \frac{| x | e}{n} \right)^n = 0
\]

for any fixed \( x \). From here, and taking into account the limit (15), we get

\[
\lim_{k \to \infty} R_k[f_*(x)] = 0.
\]

Then, by the Taylor theorem, the factor approximant (11) can be represented in the form of the Taylor series

\[
f_*(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}_* (0)}{n!} x^n,
\]

with

\[
\frac{f^{(n)}_* (0)}{n!} = a_n \quad (n = 0, 1, 2, \ldots).
\]

Comparing series (12) and (21), under equalities (13) and (22), we come to the conclusion that the functions \( f_*(x) \) and \( f(x) \) coincide. \( \square \)
4 Critical exponents

Critical exponents can be presented in the form of \( \varepsilon \)-expansions in powers of \( \varepsilon = 4 - d \), where \( d \) is space dimensionality. For \( O(N) \)-symmetric \( \varphi^4 \) theory in three dimensions, the five-loop expansions can be found in the book [24]. The summation of the five-loop expansions by means of self-similar approximants was considered in Ref. [13] for all \( N \). It was shown that for \( N = -2 \) and \( N \to \infty \) self-similar approximations yield the exact values for the exponents. The results are very accurate for large \( N \gg 1 \), with the errors decreasing as \( 1/N \) with increasing \( N \). However the accuracy for the lower \( N \) was not sufficient.

Our aim in the present paper is to demonstrate that the accuracy of self-similar factor approximants for the \( O(N) \)-symmetric \( \varphi^4 \) theory in three dimensions can be drastically improved by employing the available seven-loop \( \varepsilon \)-expansions that are known for \( N = 1 \) [25] and are derived, using the seven-loop coupling parameter expansions [26], in Ref. [27] for \( N = 0, 1, 2, 3, 4 \). These expansions are as follows.

(i) \( N = 0 \).

For \( N = 0 \), we have

\[
\nu^{-1} = 2 - 0.25 \varepsilon - 0.08594 \varepsilon^2 + 0.11443 \varepsilon^3 - 0.28751 \varepsilon^4 + 0.95613 \varepsilon^5 - 3.8558 \varepsilon^6 + 17.784 \varepsilon^7, \tag{23}
\]

\[
\eta = 0.015625 \varepsilon^2 + 0.016602 \varepsilon^3 - 0.0083267 \varepsilon^4 + 0.026505 \varepsilon^5 - 0.09073 \varepsilon^6 + 0.37851 \varepsilon^7, \tag{24}
\]

and

\[
\omega = \varepsilon - 0.65625 \varepsilon^2 + 1.8236 \varepsilon^3 - 6.2854 \varepsilon^4 + 26.873 \varepsilon^5 - 130.01 \varepsilon^6 + 692.1 \varepsilon^7. \tag{25}
\]

We calculate the corresponding self-similar factor approximants \( f^*_k(\varepsilon) \), as is explained in the previous section, and set \( \varepsilon = 1 \). The results of this calculation for the exponent \( \nu \) are illustrated in Table 1. Following the same procedure for the exponents \( \eta \) and \( \omega \), we obtain the values shown in Table 2, where we compare the results obtained by means of factor approximants (FA) with those of other methods: Monte Carlo simulations (MC) [28–33], Conformal bootstrap (CB) [34–38], Hypergeometric Meijer summation (HGM) [27], Borel summation complemented by additional conjectures on the behavior of coefficients (BAC) [39], Borel summation with conformal mapping (BCM) [40], and Nonperturbative renormalization group (NPRG) [5, 41, 43].

(ii) \( N = 1 \).

The \( \varepsilon \) expansions for the critical exponents read as

\[
\nu^{-1} = 2 - 0.333333 \varepsilon - 0.11728 \varepsilon^2 + 0.12453 \varepsilon^3 - 0.30685 \varepsilon^4 + 0.95124 \varepsilon^5 - 3.5726 \varepsilon^6 + 15.287 \varepsilon^7, \tag{26}
\]

\[
\eta = 0.018519 \varepsilon^2 + 0.01869 \varepsilon^3 - 0.0083288 \varepsilon^4 + 0.025656 \varepsilon^5 - 0.081273 \varepsilon^6 + 0.31475 \varepsilon^7, \tag{27}
\]

and

\[
\omega = \varepsilon - 0.62963 \varepsilon^2 + 1.6182 \varepsilon^3 - 5.2351 \varepsilon^4 + 20.75 \varepsilon^5 - 93.111 \varepsilon^6 + 458.74 \varepsilon^7. \tag{28}
\]

The results for the factor approximants are summarized in Table 3, where they are compared with the values obtained by other methods listed above.

(iii) \( N = 2 \).

The \( \varepsilon \) expansions are

\[
\nu^{-1} = 2 - 0.4 \varepsilon - 0.14 \varepsilon^2 + 0.12244 \varepsilon^3 - 0.30473 \varepsilon^4 + 0.87924 \varepsilon^5 - 3.103 \varepsilon^6 + 12.419 \varepsilon^7, \tag{29}
\]
\[ \eta = 0.02 \varepsilon^2 + 0.019 \varepsilon^3 - 0.0078936 \varepsilon^4 + 0.023209 \varepsilon^5 - 0.068627\varepsilon^6 + 0.24861 \varepsilon^7 , \tag{30} \]

and

\[ \omega = \varepsilon - 0.6 \varepsilon^2 + 1.4372 \varepsilon^3 - 4.4203 \varepsilon^4 + 16.374 \varepsilon^5 - 68.777 \varepsilon^6 + 316.48 \varepsilon^7 . \tag{31} \]

The calculated factor approximants for the critical exponents are presented in Table 4, where they are compared with the exponents found by other methods listed above.

(iv) \( N = 3 \).

The \( \varepsilon \) expansions read as

\[ \nu^{-1} = 2 - 0.45455 \varepsilon - 0.1559 \varepsilon^2 + 0.11507 \varepsilon^3 - 0.2936 \varepsilon^4 + 0.78994 \varepsilon^5 - 2.6392 \varepsilon^6 + 9.9452 \varepsilon^7 , \tag{32} \]

\[ \eta = 0.02066 \varepsilon^2 + 0.018399 \varepsilon^3 - 0.0074495 \varepsilon^4 + 0.020833 \varepsilon^5 - 0.057024 \varepsilon^6 + 0.19422 \varepsilon^7 , \tag{33} \]

and

\[ \omega = \varepsilon - 0.57025 \varepsilon^2 + 1.2829 \varepsilon^3 - 3.7811 \varepsilon^4 + 13.182 \varepsilon^5 - 52.204 \varepsilon^6 + 226.02 \varepsilon^7 . \tag{34} \]

The corresponding exponents calculated by means of the factor approximants are shown in Table 5, together with the results of calculation by other methods listed above.

(v) \( N = 4 \).

The \( \varepsilon \) expansions take the form

\[ \nu^{-1} = 2 - 0.5 \varepsilon - 0.16667 \varepsilon^2 + 0.10586 \varepsilon^3 - 0.27866 \varepsilon^4 + 0.70217 \varepsilon^5 - 2.2337 \varepsilon^6 + 7.9701 \varepsilon^7 , \tag{35} \]

\[ \eta = 0.020833 \varepsilon^2 + 0.017361 \varepsilon^3 - 0.0070852 \varepsilon^4 + 0.017631 \varepsilon^5 - 0.047363 \varepsilon^6 + 0.15219 \varepsilon^7 , \tag{36} \]

and

\[ \omega = \varepsilon - 0.54167 \varepsilon^2 + 1.1526 \varepsilon^3 - 3.2719 \varepsilon^4 + 10.802 \varepsilon^5 - 40.567 \varepsilon^6 + 166.26 \varepsilon^7 . \tag{37} \]

The resulting values of the factor approximants and the values of the exponents found by other methods listed above are given in Table 6.

As has been explained above, the construction of the factor approximants for each given expansion is straightforward. In order that the reader would grasp the overall structure of these approximants, we adduce below, as an example, the explicit expressions for \( \nu^{-1} \) in the case of \( N = 0 \). The self-similar approximant of order \( k = 2 \) is

\[ f_2^* (\varepsilon) = 2(1 - 0.81252\varepsilon)^{0.153842} . \]

For the approximant of order \( k = 3 \), we have

\[ f_3^* (\varepsilon) = \frac{2(1 + \varepsilon)^{0.169077}}{(1 + 0.229573 \varepsilon)^{1.28098}} . \]

In the fourth order \( (k = 4) \), we get

\[ f_4^* (\varepsilon) = 2(1 - 0.343467 \varepsilon)^{0.411916}(1 + 3.21435 \varepsilon)^{0.0051269} . \]

The fifth order \( (k = 5) \) yields

\[ f_5^* (\varepsilon) = 2(1 + \varepsilon)^{0.0459476}(1 - 0.172432 \varepsilon)^{1.02395}(1 + 4.48424 \varepsilon)^{0.00125184} . \]
The sixth order \( k = 6 \) gives
\[
 f^*_6(\varepsilon) = 2(1 - 0.261332 \varepsilon)^{0.58267}(1 + 1.98816 \varepsilon)^{0.0126183}(1 + 5.44898 \varepsilon)^{0.00040002}.
\]
And the approximant of seventh order \((k = 7)\) is given by the expression
\[
 f^*_7(\varepsilon) = 2(1 + \varepsilon)^{0.0275211}(1 - 0.222915 \varepsilon)^{0.734195}(1 + 3.13825 \varepsilon)^{0.0032879}(1 + 6.2879 \varepsilon)^{0.000131004}.
\]
Substituting here \( \varepsilon = 1 \), we come to the corresponding values of the exponent \( \nu \).

All other factor approximants for the related expansions are obtained in the same way. In addition to the exponents \( \nu, \eta, \) and \( \omega \), we find other exponents \( \alpha, \beta, \gamma, \) and \( \delta \) through the relations
\[
 \alpha = 2 - 3\nu, \quad \beta = \frac{\nu}{2}(1 + \eta), \quad \gamma = \nu(2 - \eta), \quad \delta = \frac{5 - \eta}{1 + \eta}.
\] (38)

Table 7 summarizes these results.

As is seen, the self-similar factor approximants are in good agreement with the results of other methods, while the approach using factor approximants is very simple and allowing for the explicit analytical construction of these approximants.

It is worth mentioning one point, where the situation remains not completely settled. This is the case of the exponent \( \alpha \) for \( N = 2 \). The case of \( N = 2 \) is of special interest representing the superfluid helium and magnetic \( XY \) models \[44, 45\]. The situation concerns the fact that practically all calculational methods yield the values of \( \alpha \) that are in good accordance with each other as well as with the experimental values, but that are a bit lower than the result of Monte Carlo simulations and the value following from the conformal bootstrap conjecture.

Table 8 illustrates the situation for \( N = 2 \), comparing the exponent \( \alpha \) obtained by different methods listed in Table 2 with experimental data, including the extremely precise results of the specific heat measurements for liquid helium in zero gravity \[46\] and the series of measurements for several magnetic materials of the \( XY \) class \[47–49\].

5 Discussion

The method of self-similar transformations is a very simple and convenient tool for the summation of asymptotic series. The basis of the method is the consideration of the transfer from one approximation to another as a motion in the space of approximations, with the approximation order playing the role of time. The motion in the vicinity of a fixed point is described by an equation having the form of a self-similar relation, which is equivalent to a renormalization-group equation. The fixed point of the evolution equation defines the sought effective limit of the transformed sequence. A representation for the effective limit acquires the form of self-similar factor approximants. Following the usual way of dealing with sequence transformations, we show that the method is regular and then extrapolate it to the case of divergent sequences.

We apply the method of self-similar factor transformations for the summation of \( \varepsilon \) expansions for the \( O(N) \)-symmetric theory in three dimensions. The series of seventh order in \( \varepsilon \) are used. The method is shown to provide accurate approximations, at the same time being very simple and allowing for the construction of explicit analytical expressions. The results are compatible with other known methods of summation.

Employing this method, it is even possible to get reasonable estimates for two-dimensional systems, for which one has to set \( \varepsilon = 2 \). For the two-dimensional systems, the symmetry can be
broken only for \( N = 0 \) and \( N = 1 \) \[50\]. The estimates for the corresponding critical exponents can be compared with the values conjectured in Ref. \[51\] for \( N = 0 \) and with the known exact values \[52\] for \( N = 1 \). Thus the self-similar factor approximants for \( N = 0 \) give \( \nu = 0.748 \), \( \eta = 0.18 \), and \( \omega = 1.62 \), as compared with the exact values \( \nu = 0.75 \), \( \eta = 5/24 \), and \( \omega = 2 \). And for \( N = 1 \), we obtain \( \nu = 0.997 \), \( \eta = 0.21 \), and \( \omega = 1.6 \), as compared with the exact values \( \nu = 1 \), \( \eta = 0.25 \), and \( \omega = 1.75 \).

**Author Contributions**
Both the authors, V.I. Yukalov and E.P. Yukalova, equally contributed to this work.

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Table 1: Self-similar factor approximants of order $k = 2, 3, 4, 5, 6, 7$ for the exponent $\nu$ and different number of components $N$.

| $k$ | $N = 0$ | $N = 1$ | $N = 2$ | $N = 3$ | $N = 4$ |
|-----|---------|---------|---------|---------|---------|
| 2   | 0.64688 | 0.73938 | 0.83405 | 0.91870 | 0.98471 |
| 3   | 0.57949 | 0.61666 | 0.65176 | 0.68457 | 0.71492 |
| 4   | 0.59026 | 0.63394 | 0.67562 | 0.71452 | 0.75006 |
| 5   | 0.58665 | 0.62845 | 0.66875 | 0.70688 | 0.74221 |
| 6   | 0.58789 | 0.63030 | 0.67127 | 0.71000 | 0.74573 |
| 7   | 0.58744 | 0.62968 | 0.67073 | 0.70985 | 0.74614 |

Table 2: Critical exponents for $N = 0$, found by different methods: Self-similar factor approximants (FA), Monte Carlo simulations (MC), Conformal bootstrap (CB), Hypergeometric Meijer summation (HGM), Borel summation with additional conjectures on the behaviour of coefficients (BAC), Borel summation with conformal mapping (BCM), and Nonperturbative renormalization group (NPRG).

| Method | $\nu$     | $\eta$   | $\omega$ |
|--------|-----------|----------|----------|
| FA     | 0.5877 (2)| 0.0301 (2)| 0.821 (15) |
| MC     | 0.587597 (4)| 0.031043 (3)| 0.899 (12) |
| CB     | 0.5877 (12)| 0.0282 (4) | $-$ |
| HGM    | 0.5877 (2)| 0.0312 (7)| 0.8484 (17) |
| BAC    | 0.5874 (2)| 0.0304 (2)| 0.846 (15) |
| BCM    | 0.5874 (3)| 0.0310 (7)| 0.841 (13) |
| NPRG   | 0.5876 (2)| 0.0312 (9)| 0.901 (24) |
Table 3: Critical exponents for $N = 1$, found by different methods listed in Table 2.

| Method | $\nu$         | $\eta$         | $\omega$         |
|--------|---------------|----------------|------------------|
| FA     | 0.6300 (3)    | 0.0353 (3)     | 0.808 (9)        |
| MC     | 0.63002 (10)  | 0.03627 (10)   | 0.832 (6)        |
| CB     | 0.62999 (5)   | 0.03631 (3)    | 0.830 (2)        |
| HGM    | 0.6298 (2)    | 0.0365 (7)     | 0.8231 (5)       |
| BAC    | 0.6296 (3)    | 0.0355 (3)     | 0.827 (13)       |
| BCM    | 0.6292 (5)    | 0.0362 (6)     | 0.820 (7)        |
| NPRG   | 0.63012 (16)  | 0.0361 (11)    | 0.832 (14)       |

Table 4: Critical exponents for $N = 2$, found by different methods listed in Table 2.

| Method | $\nu$         | $\eta$         | $\omega$         |
|--------|---------------|----------------|------------------|
| FA     | 0.6710 (3)    | 0.0372 (4)     | 0.809 (11)       |
| MC     | 0.67169 (7)   | 0.03810 (8)    | 0.789 (4)        |
| CB     | 0.67175 (10)  | 0.0385 (6)     | 0.811 (10)       |
| HGM    | 0.6708 (4)    | 0.0381 (6)     | 0.789 (13)       |
| BAC    | 0.6706 (2)    | 0.0374 (3)     | 0.808 (7)        |
| BCM    | 0.6690 (10)   | 0.0380 (6)     | 0.804 (3)        |
| NPRG   | 0.6716 (6)    | 0.0380 (13)    | 0.791 (8)        |

Table 5: Critical exponents for $N = 3$, found by different methods listed in Table 2.

| Method | $\nu$         | $\eta$         | $\omega$         |
|--------|---------------|----------------|------------------|
| FA     | 0.7099 (1)    | 0.0372 (4)     | 0.7919 (3)       |
| MC     | 0.7116 (10)   | 0.0378 (3)     | 0.773            |
| CB     | 0.7121 (28)   | 0.0386 (12)    | 0.791 (22)       |
| HGM    | 0.7091 (2)    | 0.0381 (6)     | 0.764 (18)       |
| BAC    | 0.70944 (2)   | 0.0373 (3)     | 0.794 (4)        |
| BCM    | 0.7059 (20)   | 0.0378 (5)     | 0.795 (7)        |
| NPRG   | 0.7114 (9)    | 0.0376 (13)    | 0.796 (11)       |
Table 6: Critical exponents for $N = 4$, found by different methods listed in Table 2.

| Method | $\nu$       | $\eta$     | $\omega$     |
|--------|-------------|------------|--------------|
| FA     | 0.7459 (2)  | 0.0361 (4) | 0.7913 (8)   |
| MC     | 0.750 (2)   | 0.0360 (3) | 0.765 (30)   |
| CB     | 0.751 (3)   | 0.0378 (32)| 0.817 (30)   |
| HGM    | 0.7443 (3)  | 0.0367 (4) | 0.7519 (13)  |
| BAC    | 0.7449 (4)  | 0.0363 (2) | 0.7863 (9)   |
| BCM    | 0.7397 (35) | 0.0366 (4) | 0.794 (9)    |
| NPRG   | 0.7478 (9)  | 0.0360 (12)| 0.761 (12)   |

Table 7: Critical exponents for the three-dimensional $O(N)$-symmetric $\varphi^4$ field theory, calculated by means of self-similar factor approximants

| $N$ | $\alpha$ | $\beta$      | $\gamma$ | $\delta$ | $\nu$    | $\eta$    | $\omega$   |
|-----|----------|--------------|----------|----------|----------|----------|------------|
| 0   | 0.2369   | 0.3027       | 1.15771  | 4.8247   | 0.5877   | 0.0301   | 0.821      |
| 1   | 0.1100   | 0.3261       | 1.23776  | 4.7954   | 0.6300   | 0.0353   | 0.808      |
| 2   | −0.0130  | 0.3480       | 1.31704  | 4.7848   | 0.6710   | 0.0372   | 0.809      |
| 3   | −0.1297  | 0.3682       | 1.39339  | 4.7848   | 0.7099   | 0.0372   | 0.792      |
| 4   | −0.2377  | 0.3864       | 1.46487  | 4.7910   | 0.7459   | 0.0361   | 0.791      |

Table 8: Critical exponents $\alpha$ for $N = 2$, found by different methods listed in Table 2, and experimental data.

| Method       | $\alpha$      |
|--------------|----------------|
| FA           | −0.0130 (8)    |
| MC           | −0.0151 (2)    |
| CB           | −0.0152 (3)    |
| HGM          | −0.0124 (12)   |
| BAC          | −0.0118 (6)    |
| BCM          | −0.0070 (30)   |
| NPRG         | −0.0148 (18)   |
| experiment [46] | −0.0127 (3)   |
| experiment [47] | −0.0130 (30)  |
| experiment [48] | −0.0130 (10)  |
| experiment [49] | −0.0130 (20)  |