Elementary introduction to discrete soliton equations*

Jarmo Hietarinta†
Department of Physics and Astronomy
University of Turku
FI-20014, Turku, Finland

Abstract
We will give a short introduction to discrete or lattice soliton equations, with the particular example of the Korteweg-de Vries as illustration. We will discuss briefly how Bäcklund transformations lead to equations that can be interpreted as discrete equations on a $\mathbb{Z}^2$ lattice. Hierarchies of equations and commuting flows are shown to be related to multidimensionality in the lattice context, and multidimensional consistency is one of the necessary conditions for integrability. The multidimensional setting also allows one to construct a Lax pair and a Bäcklund transformation, which in turn leads to a method of constructing soliton solutions. The relationship between continuous and discrete equations is discussed from two directions: taking the continuum limit of a discrete equation and discretizing a continuous equation following the method of Hirota.

1 Introduction

We are all familiar with the integrable (continuous) soliton equations that have been studied intensively since their resurrection in late 1960's (see e.g. [1, 5, 6]). Many interesting and useful properties are associated with such systems, such as symmetries, infinite number of conserved quantities, elastic scattering of solitons, and solvability using various methods such as the Inverse Scattering Transform and Hirota's bilinear method. All these nice properties follow from some important underlying mathematical structure, which has been elaborated in many studies (e.g. by Mikio Sato and his collaborators in Kyoto, see e.g. [13]).

With such a beautiful continuous theory of soliton equations, what is the point of a discrete soliton theory? One might say that we need to discretize PDEs in order to do computations with them, or that there cannot be smooth continuity beyond the Planck scale where quantum aspects take over. But the reason proposed here is that discrete soliton equations should be studied because their mathematical properties of are, if possible, even more beautiful than those of the continuum equations.

2 Basic set-up for lattice equations

When one mentions “lattice equations” perhaps the first thing that comes to mind is the ubiquitous Toda lattice equation given by:

$$\ddot{x}_i(t) = e^{-\left(x_i(t) - x_{i-1}(t)\right)} - e^{-\left(x_i(t) - x_{i+1}(t)\right)}, \quad \forall i.$$

Here $x_i$ is the position of the particle having the name $i$ and the equation gives the time evolution $x_i(t)$. Since time is still continuous we would call this a semi-discrete equation.

*to appear in “Nonlinear Systems and Their Remarkable Mathematical Structures”, Norbert Euler (ed.), CRC Press.
†e-mail: jarmo.hietarinta@utu.fi
2.1 Equations on Cartesian lattice

Here we are considering fully discrete soliton equations and therefore the continuous $u(x,t)$ will be replaced by $u_{n,m}$, i.e., both the space and time coordinates are discretized. The most common discrete two-dimensional space is the Cartesian 2D lattice with dependent variables located at the vertices of the lattice, see Figure 1. Other lattices can also be considered, as well as variables not on the vertices but on the links between the vertices.

The soliton equations are typically evolution equations and therefore we must discuss what kind of evolution we can have on the lattice. The simplest equation is the one relating the corners of a lattice square or quadrilateral. This involves four corners and if we give values at three corners we should be able to compute the value on the fourth corner, see Figure 2. For this to be possible we must require that the equation is affine linear in all the corner variables. As examples of such equations we have the lattice potential KdV equation (lpKdV)

\[(u_{n,m} - u_{n+1,m+1})(u_{n,m+1} - u_{n+1,m}) = p^2 - q^2, \quad \forall n,m,\]  

(1)

and the lattice potential modified KdV equation (lpmKdV)

\[p (u_{n,m} u_{n+1,m} - u_{n,m+1} u_{n+1,m+1}) = q (u_{n,m} u_{n,m+1} - u_{n+1,m} u_{n+1,m+1}).\]  

(2)

Figure 1: The Cartesian lattice, $u_{n,m}$ are located at lattice points.

Figure 2: Equation on a quadrilateral: If values at three corners are given, one should be able to compute the value at the fourth corner.
For these examples we can clearly compute, within each quadrilateral, the value at any corner once the other three corner values are given. This is the local situation.

For a global picture it is necessary to define initial values on some curve so that one can proceed to compute values “forward”. One possibility is to give the values on a corner, another is to use staircase initial values, see Figure 3. In the figure the evolution is to the upper-right direction, but there are corresponding initial settings for other directions. Also the staircase can have occasional longer or higher stairs as long as we have uniquely defined evolution.

![Figure 3: a): The Cartesian lattice with corner initial values (black disks) given, one can then compute the values at open circles in the upper right quadrant. b) The same with staircase initial values.](image)

2.2 Discrete structure within continuous soliton equations

The examples above (1) and (2) did not come from thin air. If we use (1) with \( n = 0, m = 0 \) and solve for \( u_{1,1} \) we get

\[
 u_{1,1} = u_{0,0} + \frac{p^2 - q^2}{u_{1,0} - u_{0,1}}.
\]

This may look familiar. Indeed, in 1973 Wahlquist and Estabrook discussed [16] Bäcklund transformation (BT) for KdV solitons and found (translating notation to the present case) that if \( u_{0,0} \) is a “seed” solution and \( u_{1,0} \) is obtained from it by a BT with parameter \( p \), and similarly \( u_{0,1} \) with parameter \( q \), then there is a superposition principle: If one applies BT with \( q \) on \( u_{1,0} \) or with \( p \) on \( u_{0,1} \) then the results can be the same (i.e., the BTs commute) and the unique result is given algebraically according to formula (3).

Similar results were derived even before, within the theory of surfaces. In his studies Bäcklund derived [3] the equation

\[
 \theta_{uv} = 2 \sin(\theta/2),
\]

that is now called the sine-Gordon equation and subsequently Bianchi derived [4] the permutability theorem for the BTs (c.f. (3)):

\[
 \theta_{12} = \theta + 4 \arctan \left[ \frac{\beta_2 + \beta_1}{\beta_2 - \beta_1} \tan \left( \frac{\theta_2 - \theta_1}{4} \right) \right].
\]

If we take tan on both sides and write the result in terms of \( u := \exp(i\theta/2) \) we get

\[
 \beta_1(u u_1 - u_2 u_1) = \beta_2(u u_2 - u_1 u_1).
\]

This can again be elevated to an abstract equation on the \( \mathbb{Z}^2 \) lattice, namely to lpmKdV as given in (2). Here we interpret subscripts as giving directions of steps in the lattice.
One can study various properties of abstract lattice equations, but if they have a connection to continuous soliton equations as noted above, some of the results may have concrete applications for them.

3 Symmetries and hierarchies

3.1 In the continuum

One of the essential concepts of integrable soliton equations is that they do not appear isolated but in hierarchies. For example for the KdV equation we have the hierarchy of equations

\[
\begin{align*}
  u_{t_1} &= \partial_x u \\
  u_{t_2} &= \frac{1}{4} \partial_x [u_{xx} + 3 u^2] \\
  u_{t_3} &= \frac{1}{40} \partial_x [u_{xxx} + 10 u_{xx} u + 5 u_x^2 + 10 u^3]
\end{align*}
\] (4a, 4b, 4c)

Thus in the KdV case we have one space variable \( x \) and multiple time variables \( t_j \), and the flows corresponding to the different times commute. Furthermore, if we assign weight 2 for \( u \), weight 1 for \( \partial_x \) and weight \( j \) for \( \partial_{t_j} \) then all equations are weight homogeneous. There are elegant explanations on why the equations fit together so nicely, e.g. by the Sato theory [13].

3.2 Discrete multidimensionality

For the present discrete case we would also like to have hierarchical and multidimensional structure. To begin with, our (1) is fully symmetric between the \( n \) and \( m \) coordinates of the \( \mathbb{Z}^2 \) lattice, and therefore as we introduce higher dimensionality we would like to keep this symmetry. Thus we introduce a third dimension and the corresponding lattice index \( k \) by \( u_{n,m} \rightarrow u_{n,m,k} \) and rewrite (1) as

\[
(u_{n,m,k} - u_{n+1,m+1,k})(u_{n,m+1,k} - u_{n+1,m,k}) = p^2 - q^2, \quad \forall n, m, k.
\]

This means that we have the same equation on all planes labeled by \( k \). When we look at the situation from this point of view it is natural to propose [14] that we should equally well have equations in which \( m \) labels the plane while \( n, k \) label the corners of the quadrilateral:

\[
(u_{n,m,k} - u_{n+1,m,k+1})(u_{n,m,k+1} - u_{n+1,m,k}) = p^2 - r^2, \quad \forall n, m, k.
\]

Here we have also replaced \( q \) with \( r \), which is the lattice constant for the \( k \) direction. Finally we can have a similar equation in the \( m, k \) plane

\[
(u_{n,m,k} - u_{n,m+1,k+1})(u_{n,m+1,k+1} - u_{n,m,k}) = q^2 - r^2, \quad \forall n, m, k.
\]

As the subscript notation starts to get lengthy it is common in the literature to introduce various kinds of shorthand notations. We sometimes use the notation in which shift in the \( n \)-direction is indicated by a tilde, in the \( m \)-direction by a hat and in the \( k \)-direction by a bar:

\[
\begin{align*}
  u_{n,m,k} &= u, \quad u_{n+1,m,k} = \tilde{u}, \quad u_{n,m,k+1} = \hat{u}, \quad u_{n,m,k+1} = \overline{u}, \\
  u_{n+1,m,k+1} &= \tilde{u}, \quad u_{n+1,m,k+1} = \hat{u}, \quad u_{n,m+1,k+1} = \overline{u}, \quad u_{n+1,m+1,k+1} = \overline{u}.
\end{align*}
\]

Then our equations on the three planes read

\[
\begin{align*}
  Q_{12}(u, \tilde{u}, \hat{u}, \overline{u}; p, q) &:= (u - \tilde{u})(\tilde{u} - \hat{u}) - p^2 + q^2 = 0, \quad (8a) \\
  Q_{23}(u, \tilde{u}, \overline{u}, \overline{u}; q, r) &:= (u - \overline{u})(\overline{u} - \overline{u}) - q^2 + r^2 = 0, \quad (8b) \\
  Q_{31}(u, \overline{u}, \overline{u}, \overline{u}; r, p) &:= (u - \overline{u})(\overline{u} - \overline{u}) - r^2 + p^2 = 0, \quad (8c)
\end{align*}
\]

when written using cyclic changes: tilde \( \rightarrow \) hat \( \rightarrow \) bar, \( p \rightarrow q \rightarrow r \).
3.3 Commuting discrete flows

In the continuum case we know that we cannot introduce arbitrary flows in the different time directions because they would not be compatible, i.e., they would not commute. We have already discussed evolution on the lattice (see Figure 3) and when we assign equations on different planes, the evolution they generate must also satisfy some compatibility conditions. Let us look at this locally. Assuming a common corner \((n, m, k)\) in the \(\mathbb{Z}^3\) lattice we should have a situation as in Figure 2 in each of the three planes intersecting in that corner. If we keep just the elementary plaquettes we get Figure 4.

In terms of equations the situation is as follows: At the various sides of the cube we have the corresponding equations:

\[
\begin{align*}
\text{bottom:} & \quad Q_{12}(u, \bar{u}, \hat{u}, \bar{\hat{u}}; p, q) = 0, & \text{top:} & \quad Q_{12}(\bar{\bar{u}}, \bar{u}, \hat{u}, \bar{\hat{u}}; p, q) = 0, \\
\text{back:} & \quad Q_{23}(u, \bar{u}, \bar{\bar{u}}, \bar{\hat{u}}; q, r) = 0, & \text{front:} & \quad Q_{23}(\bar{u}, \hat{u}, \bar{\bar{u}}, \bar{\hat{u}}; q, r) = 0, \\
\text{left:} & \quad Q_{31}(u, \bar{u}, \bar{\bar{u}}, \bar{\hat{u}}; r, p) = 0, & \text{right:} & \quad Q_{31}(\bar{u}, \hat{u}, \bar{\bar{u}}, \bar{\hat{u}}; r, p) = 0.
\end{align*}
\]

Here we get from the LHS to the RHS by applying on the dependent variables a shift in the direction not yet appearing on the LHS while keeping the equation itself unchanged. We would use this for any candidate equations which are uniform on parallel planes, for lpKdV we have (8).

From a corner we can start evolution and for the configuration of Figure 4 with \(u, \bar{u}, \hat{u}, \bar{\hat{u}}\) as initial values we can compute using LHS equations the values of \(\bar{\bar{u}}, \bar{\hat{u}}, \bar{\bar{u}}\). After this we can compute \(\bar{\bar{u}}\) from each of the three RHS equations and the result should be the same. In the language of the commuting flows we have three different order of flows

- first, independently, (LHS \(Q_{12}\) to get \(\bar{\hat{u}}\), and LHS \(Q_{23}\) to get \(\bar{\bar{u}}\)), after that RHS \(Q_{31}\) to get \(\bar{\bar{u}}\).
- first, independently, (LHS \(Q_{23}\) to get \(\bar{\bar{u}}\), and LHS \(Q_{31}\) to get \(\bar{\hat{u}}\)), after that RHS \(Q_{12}\) to get \(\bar{\hat{u}}\).
- first, independently, (LHS \(Q_{31}\) to get \(\bar{\hat{u}}\), and LHS \(Q_{12}\) to get \(\bar{\bar{u}}\)), after that RHS \(Q_{23}\) to get \(\bar{\bar{u}}\).

Thus we have three flows, which can be arranged in 6 different orders, but since the order in the first pair does not matter we find the condition that the three possibilities listed above should give the
same result, i.e., two consistency conditions. This is also called “Consistency-Around-a-Cube” (CAC) or Multidimensional consistency (MDC). When this is applied to equations (8) we find that in each case
\[
\bar{u} = \frac{\bar{u}}{u} (p^2 - q^2) + \bar{u} (q^2 - r^2) + \bar{u} (r^2 - p^2).
\]
This was already derived by Wahlquist and Estabrook in the context of BT [16].

In the general case the conditions following from CAC are fairly complicated, but under suitable additional assumptions one can obtain a classification of equations, the most interesting being the “ABS list” [2], which contains the above mentioned \(lpKdV\) as “H1” and \(lpmKdV\) as “H3(\(\delta = 0\))”.

4 Lax pairs

4.1 Constructing the Lax pair from CAC

In the discrete case we can use the equations on the consistency cube to generate a Lax pair by taking the bar-variables as the auxiliary linear variables.

Let us take the back and left equations of (9) and solve for \(\hat{u}\) and \(\tilde{u}\). In the case of \(lpKdV\) we get
\[
\begin{align*}
\hat{u} &= u (u - \hat{u}) - q^2 + r^2, \\
\tilde{u} &= u (u - \tilde{u}) - p^2 + r^2.
\end{align*}
\]
Now introducing \(u = \frac{f}{g}\) we can write (10) as
\[
\begin{align*}
\hat{\Phi} &= \Phi, \\
\tilde{\Phi} &= \mathcal{L} \Phi, \\
\Phi &= \begin{pmatrix} f \\ g \end{pmatrix},
\end{align*}
\]
which implies
\[
\tilde{\mathcal{M}} \mathcal{L} = \hat{\mathcal{L}} \mathcal{M}.
\]
Applying the matrices given in (12) (with \(\mu = \lambda = 1\)) to this equation yields
\[
\tilde{\mathcal{M}} \mathcal{L} - \hat{\mathcal{L}} \mathcal{M} = \begin{pmatrix} (u - \hat{u}) (\hat{u} - \tilde{u}) - p^2 + q^2 \\ -1 \\
\end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix},
\]
and thus the Lax pair does generate the equation. However, it should be noted that there can also be “fake Lax pairs”, that is, even if an equation has the CAC property its Lax pair as constructed above might not generate the equation (for example if equation (13) is satisfied automatically).
4.2 Bäcklund transformation for constructing soliton solutions

The Lax pair and the Bäcklund transformation are different ways of interpreting the six equations (9): For a Lax pair we used the back and left equations and wrote them in matrix form. For BT we assume that \(u_{n,m,1}\) solves the top equation and then we use the back and left equations to construct a solution to the bottom equation (which has the same form as the top equation). Since we have the extra lattice parameter \(r\) at our disposal, the solution to the bottom equation should be more general.

The starting point in this construction is a seed solution of the top equation. Usually this is just the null solution, but now we observe that \(u_{n,m,k} = 0\) is not a solution of (5) and the first problem is to find a suitable seed solution. One finds easily that \(u_{n,m,k} = \pm pn \pm q m + ck\) is a simple linear solution. With this in mind let us change dependent variables by

\[
u_{n,m,k} = u_{n,m,k} - pm - qm - rk\tag{14}\]

after which the bottom, back and left equations can be written, respectively, as

\[
egin{align*}
(v_{n+1,m,k} - v_{n,m+1,k} - p + q)(v_{n+1,m+1,k} - v_{n,m,k} - p - q) &= (p^2 - q^2), \\
(v_{n,m+1,k} - v_{n,m,k+1} - q + r)(v_{n,m+1,k+1} - v_{n,m,k} - q - r) &= (q^2 - r^2), \\
(v_{n,m,k+1} - v_{n+1,m,k} - r + p)(v_{n,m,k+1} - v_{n,m,k} - r - p) &= (r^2 - p^2).
\end{align*}
\tag{15a-b-c}
\]

We now use these equations for \(k = 0\), take \(v_{n,m,1} = 0\), \(\forall n, m\), which solves the top equation, and solve for \(\nu_{n,m} := v_{n,m,0}\). We find

\[
\nu_{n,m+1} = \frac{(q - r)\nu_{n,m}}{\nu_{n,m} + q + r},
\]
\[
\nu_{n+1,m} = \frac{(p - r)\nu_{n,m}}{\nu_{n,m} + p + r}.
\tag{16-17}
\]

Again we would like to use matrix notation to write these results, and for that purpose we introduce

\[
\psi_{n,m} = \begin{pmatrix} a_{n,m} \\ b_{n,m} \end{pmatrix}, \quad M := \mu \begin{pmatrix} q - r & 0 \\ 1 & q + r \end{pmatrix}, \quad L := \lambda \begin{pmatrix} p - r & 0 \\ 1 & p + r \end{pmatrix},
\]

so that the equations to solve are

\[
\psi_{n,m+1} = M \psi_{n,m}, \quad \psi_{n+1,m} = L \psi_{n,m}.
\]

Since \(M, L\) are commuting constant matrices and

\[
M^m = \begin{pmatrix} F^m \\ (1 - F^m)/(2r) \end{pmatrix}, \quad L^n = \begin{pmatrix} G^n \\ (1 - G^n)/(2r) \end{pmatrix},
\]

where \(F := (q - r)/(q + r), G := (p - r)/(p + r)\), we find

\[
\psi_{n,m} = M^m L^n \psi_{0,0}.
\]

Putting everything together yields the result

\[
\nu_{n,m} = \frac{a_{n,m}}{b_{n,m}} = 2r \frac{\rho_{n,m}}{1 - \rho_{n,m}}, \quad \text{where} \quad \rho_{n,m} = \left(\frac{q - r}{q + r}\right)^m \left(\frac{p - r}{p + r}\right)^n \frac{v_{0,0}}{2r + v_{0,0}}.
\tag{18}
\]
Figure 5: There are two ways to squeeze the quadrilateral to obtain continuum limits. \( \dot{u} \) and \( u' \) are the corresponding derivatives.

5 Continuum limits

When we compare discrete and continuous spaces we will match the origins and then for a generic point we have \( (x, y) = (\epsilon n, \delta m) \), where \( \epsilon \) and \( \delta \) measure the lattice distances. For a quad equation there are two ways to take a continuum limit as illustrated in Figure 5: In the top figure (straight limit) the square is squeezed in the \( m \)-direction, in the bottom figure (skew limit) it is first rotated 45° and then squeezed.

5.1 Skew limit

Here we will only consider the skew limit. For that purpose we rotate the coordinates by \( (n, m) \rightarrow (n' + m - 1, m' - n) \), furthermore let us denote \( n + m = n', \ m - n = m' \) and then equation (15a) reads

\[
(v_{n', m'} - v_{n', m' + 1} - p + q)(v_{n' + 1, m} - v_{n' - 1, m'} - p - q) = p^2 - q^2.
\] (19)

Since we take the limit in the \( m' \) direction we set

\[
v_{n' + \nu, m' + \mu} = V_{n' + \mu}(t + \delta \mu),
\] (20)

where \( \delta \) is the lattice distance in the \( m' \) direction. Thus we will take

\[
m' \rightarrow \infty, \quad \delta \rightarrow 0, \quad \text{while} \quad m' \delta = t \quad \text{stays fixed.}
\] (21)

We still have the question of how \( \delta \) and \( p, q \) are related. Some help can be obtained from the form of \( \rho \) in (18). We know that the soliton solutions are constructed using exponential functions and \( \rho \) can be interpreted as a discrete form of the exponential, due to the well known limit formula

\[
\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = e^x.
\] (22)

In our case we have to consider the combination with \( m - n \) power, i.e.

\[
\left(\frac{q - r}{q + r} \cdot \frac{p + r}{p - r}\right)^{(m-n)} = \left(1 + \frac{2r(q-p)}{(p-r)(q+r)}\right)^{m'}
\]
Since \( r \) is a soliton parameter it will stay finite and nonzero and therefore we take \( q - p = \delta \rightarrow 0 \). Substituting \( q = p + \delta \) and using (21) we get

\[
\left(1 + \frac{2r\delta}{(p - r)(p + \delta + r)}\right)^{m'} = \left(1 + \frac{t}{m'} \frac{2r}{(p + t/m' + r)}\right)^{m'} \rightarrow \exp \left(\frac{2rt}{p^2 - r^2}\right).
\]

Thus the limit works and produces a reasonable plane wave factor. We then proceed to insert \( q = p + \delta \) and (20) into (19) and expand in \( \delta \). This yields

\[
\partial_t V_n(t) = 1 - \frac{2p}{V_{n-1}(t) - V_{n+1}(t) + 2p} \tag{23}
\]

where we have dropped the primes in \( n \). This equation is therefore the skew semi-discrete limit of the \((\text{translated})\) lpKdV equation given in (15a). It is a bona-fide integrable equation, having a Lax pair etc.

We can next take a continuum limit in the remaining \( n \) variable. For this purpose we take \( p = 1/\epsilon \) and write

\[
V_{n,n}(t) = U(t, \xi + \nu \epsilon) \tag{26}
\]

and expand in epsilon. The result is

\[
\partial_t U = \epsilon^2 U_x + \epsilon^4 \partial_x + \epsilon^4 \partial_{\tau} \tag{24}
\]

then the leading term \( \epsilon^4 \) yields

\[
U_{\tau} = \frac{1}{6} [U_{xxx} + 6U_x^2], \tag{25}
\]

which is a potential form of the KdV equation (pKdV). The need for some sort of new “squeezed” variables as in (24) is obvious: the starting discrete equation is very symmetric while the continuum target equation is asymmetric, with \( x \) playing a different role in comparison to the \( t_i \).

5.2 Double limit

On the basis of the above we could try to take a limit in \( n, m \) directions simultaneously, but at the same time we should somehow introduce suitable scaling. Thus we try

\[
v_{n,m} = V(x + \epsilon(na_1 + mb_1), t + \epsilon^3(na_3 + mb_3)), \tag{26}
\]

where we have chosen the powers of \( \epsilon \) following the expected relative scaling of \( x \) and \( t_3 \). Inserting this with \( p = \alpha/\epsilon \), \( q = \beta/\epsilon \) into (15a) and expanding in \( \epsilon \) we find that if we choose

\[
a_j = \frac{2j}{j\alpha}, \quad b_j = \frac{2j}{j\beta}, \quad \alpha^2 \neq \beta^2, \tag{27}
\]

we get, as the leading term, the pKdV equation in the form

\[
V_t = \frac{1}{4} [V_{xxx} + 3V_x^2]. \tag{28}
\]

But there is more: If we look at the next order in \( \epsilon \) we find an \( x \) derivative of the above equation, and then at the next order, after using (28) to eliminate time derivatives, the expression

\[
V_{xxxxx} + 10V_{xxx} V_x + 5V_x^2 + 10V_x^3,
\]
which appears in the square brackets on the RHS of the fifth order KdV (4c). Thus it seems that the
discrete equation contains inside it the whole hierarchy of continuum equations! In order to explore this
further, let us use multiple time variables as follows:

\[ v_{n,m} = V(x + \epsilon (na_1 + mb_1), t_3 + \epsilon^3 (na_3 + mb_3), t_5 + \epsilon^5 (na_5 + mb_5), \cdots) \]

When we expand (15a) using this multi-time expression with parameters (27) and in the results eliminate
lower order times using lower order equations, and change \( V = u \), we get the sequence of higher order
members of the KdV hierarchy, some of which were given in (4).

The above observations can be made into precise statements using more refined mathematics, for
example by using the Sato theory. In that formalism infinite number of time variables are used at
the outset and one can find a simple correspondence between the discrete and continuum hierarchies.
The main observation to take away from this is that the nicely symmetric and simple looking discrete
formalism is in effect as rich as the corresponding continuum theory. And this statement holds also for
the more general equations such as KP.

## 6 Discretizing a continuous equation

One approach to discrete equations is to take a known continuous integrable equation and try to construct
a discrete version with as many as possible integrability properties. One important object that we
would like to preserve is the class of solutions, perhaps in the sense that the discrete solutions approach the
continuous ones in a smooth fashion.

### 6.1 A simple 1D example

Consider the nonlinear ODE (Verhulst’s population model)

\[ \dot{x}(t) = \alpha x(t)(1 - x(t)). \]  

(29)

We would like to discretize this so that the solutions of the discrete version follow closely the continuous
solution, which can be derived easily:

\[ x(t) = \frac{1}{1 + e^{\alpha (t-t_0)}}. \]

(30)

How should this equation be discretized? A naive discretization would be to replace the derivative by
a forward difference:

\[ h^{-1}(x(t+h) - x(t)) = \alpha x(t)(1 - x(t)). \]

(31)

This is the logistic equation which is well known to lead to chaotic behavior for most values of the
parameter \( \alpha \), while the solution (30) is always smooth. We need a different discretization.

In order to proceed we note that equation (29) can be linearized:

\[ x(t) = \frac{1}{1 + y(t)} \quad \Rightarrow \quad \dot{y}(t) + \alpha y(t) = 0. \]

(32)

For the linear equation the naive discretization works: The solution to the continuous \( y \) equation (32) is
given by

\[ y(t) = \exp[-\alpha (t - t_0)] \]

(33)

while the solution to the discretized version of (32)

\[ h^{-1}(y(t+h) - y(t)) + \alpha y(t) = 0 \]

is given by

\[ y(t + nh) = (1 - \alpha h)^{n+(t-t_0)/h}. \]

(34)
This solution approximates the solution (33), due to the limit formula (22):

\[(1 - \alpha h)^{(t-t_0)/h} \rightarrow e^{-\alpha(t-t_0)} \text{ as } h \rightarrow 0^+.
\]

Let us denote \(y(t + nh) = y_n\), \((t - t_0)/h = -n_0\) and reverse the steps above. We find

\[y_n := (1 - \alpha h)^{n-n_0}\]

solves \(y_{n+1} = (1 - \alpha h)y_n\)

and since \(x = 1/(1 + y)\),

\[x_n := \frac{1}{1 + (1 - \alpha h)^{n-n_0}}, \tag{35a}\]

solves

\[x_{n+1} = \frac{x_n}{1 - \alpha h + \alpha h x_n}. \tag{35b}\]

The solution for \(x_n\) (35a) is a good approximation to (30) but the equation it solves (35b) is not at all like the one obtained by naive discretization (31).

### 6.2 Hirota’s method of discretization

For PDE’s the situation is much more complicated. This is the case in particular because we do not know all solutions, or rather, the general solution is too complicated to work with. One approach is to make sure that at least the soliton solutions carry over from continuous to discrete. For this we follow R. Hirota, who in a series of papers in 1977 discretized many soliton equations while preserving their \(N\)-soliton solutions \([9, 10, 11]\). The culmination of this work was the “DAGTE” equation \([12]\) from which many other soliton equations follow.

#### 6.2.1 Bilinear form of continuous KdV

Hirota’s method is based on a transformation of the dependent variable so that in terms of the new dependent variable the soliton solutions are simply polynomials of exponentials with linear exponents. Instead of the standard form of the KdV equation

\[u_t + 6u u_x + u_{xxx} = 0. \tag{36}\]

it is better to introduce the variable \(v\) by \(u = v_x\), and integrate (36) into the pKdV equation

\[v_t + 3v^2 + v_{xxx} = 0. \tag{37}\]

The new dependent variable \(f\) is defined by

\[v = 2\partial_x \log(f), \quad \text{or} \quad u = 2\partial_x^2 \log(f), \tag{38}\]

and when this is used in (37) we get a fourth order equation

\[D_x(D_t + D_x^3) f \cdot f = 0, \tag{39}\]

which is written in terms of Hirota’s bilinear derivatives, defined by

\[D_x^m D_t^n f \cdot g = (\partial_x - \partial_{x'})^n (\partial_t - \partial_{t'})^m f(x, t) g(x', t') \big|_{x' = x, y' = y}.
\]
6.2.2 Discretizing KdV

In order to continue we need a discrete version of the bilinear derivative. For usual derivatives we have

\[ e^{ax} f(x) = f(x + a) \]

and therefore we have, for example,

\[ e^{aD_x} f \cdot g = f(x + a) g(x - a), \]
\[ \sinh(aD_x) f \cdot g = \frac{1}{2} [f(x + a) g(x - a) - f(x - a) g(x + a)]. \]

Since

\[ \sinh(aD_x) = aD_x + \text{higher order terms in } aD_x \]

it seems reasonable to use discretization rules like \( D \to \sinh(aD) \). The precise replacement proposed by Hirota was (ref [9], equation (2.3))

\[ \sinh[\frac{1}{4} (D_x + \delta D_t)] \{ 2 \delta^{-1} \sinh[\frac{1}{4} \delta D_t] + 2 \sinh[\frac{1}{4} D_x] \} \ f(x, t) \cdot f(x, t) = 0, \tag{40} \]

which can also be written as

\[ \{ \cosh[\frac{1}{4} \delta D_t + \frac{3}{4} D_x] + \delta^{-1} \cosh[\frac{3}{4} \delta D_t + \frac{1}{4} D_x] - (1 + \delta^{-1}) \cosh[\frac{3}{4} \delta D_t - \frac{1}{4} D_x] \} \ f(x, t) \cdot f(x, t) = 0. \]

In order to write this as shifts we note that

\[ \cosh(\alpha D_x + \beta D_t) f(x, t) \cdot f(x, t) = f(x + \alpha, t + \beta) f(x - \alpha, t - \beta) \]

and if we convert shifts to discrete subscript notation

\[ f(x + \frac{1}{4} \nu, t + \frac{1}{4} \mu) = f_{n, \mu} + \frac{1}{4} \nu, m + \frac{1}{4} \mu, \]

we can write (40) as

\[ f_{n+\frac{1}{4}, m+\frac{1}{4}} f_{n-\frac{1}{4}, m-\frac{1}{4}} + \delta^{-1} f_{n+\frac{1}{4}, m+\frac{1}{4}} f_{n-\frac{1}{4}, m-\frac{1}{4}} + (1 + \delta^{-1}) f_{n-\frac{1}{4}, m+\frac{1}{4}} f_{n+\frac{1}{4}, m-\frac{1}{4}} = 0. \tag{41} \]

This does not sit at the points of the \( \mathbb{Z}^2 \) lattice but if we make a 45° rotation and a shift according to

\[ (n + \nu, m + \mu) \to (n + m + \nu + \mu, n - m + \nu - \mu + 1) = (n' + \nu + \mu, m' + \nu - \mu) \]

we get

\[ f_{n'+1, m'+1} f_{n'-1, m'} + \delta^{-1} f_{n'+1, m'} f_{n'-1, m'+1} - (1 + \delta^{-1}) f_{n', m'} f_{n'+1, m'+1} = 0. \tag{42} \]

The dependent variables are now on the points of the \( \mathbb{Z}^2 \) lattice, but the equation connects points on two quadrilaterals. (This is typical for Hirota bilinear equations, in fact the only one-component equation that can exist on a single quadrilateral is trivial.)

Equations (41) and (42) have the main properties essential in Hirota’s approach to constructing soliton solutions: a) \( f_{n, m} \equiv 1 \) is a solution, and b) in each product the sum of indices is the same. This last property implies gauge invariance: if \( f_{n, m} \) is a solution, so is \( f_{n, m} := A^n B^m f_{n, m} \) for any constants \( A, B \).

6.2.3 Soliton solutions

In Hirota’s approach soliton solutions are constructed perturbatively:

**Background solution:** The bilinear form (42) obviously has \( f_{n, m} \equiv 1 \) as the vacuum or background solution.
**One-soliton solution:** The ansatz for the one-soliton solution of (42) is
\[ f_{n,m} = 1 + cA(p, k_1)^n B(q, k_1)^m, \]
where \( k_1 \) is the parameter of the soliton. We have also noted possible dependence on lattice parameters: the plane wave factor \( A \) may depend on \( p \) because it is associated with the \( n \) direction, similarly \( B \) may depend on \( q \). The constant \( c \) is constant only in \( n,m \) but may depend on \( p,q,k_1 \). This ansatz leads to the dispersion relation
\[ A(p, k_1) - B(q, k_1) = \delta \left[ 1 - A(p, k_1)B(q, k_1) \right], \]
and evidently \( \delta \) should also depend on \( p,q \). This relation is resolved by
\[ A(p, k_1) = \frac{p - k_1}{p + k_1}, \quad B(q, k_1) = \frac{q - k_1}{q + k_1}, \quad \delta(p, q) = \frac{p - q}{p + q}. \]

Note the beautiful symmetry which even encompasses the parameter \( \delta \).

**Two-soliton solution:** Following Hirota’s perturbative approach the 2SS ansatz is
\[ f_{n,m} = 1 + c_1 A(p, k_1)^n B(q, k_1)^m + c_2 A(p, k_2)^n B(q, k_2)^m + A(k_1, k_2) c_1 c_2 A(p, k_1)^n B(q, k_1)^m A(p, k_2)^n B(q, k_2)^m. \]

This form is dictated by the condition that when solitons are far apart they look like 1SS. There is a new parameter \( A(k_1, k_2) \) called the phase factor. When this ansatz is substituted into (42) with (44), we find that it is a solution, provided that the phase factor is given by
\[ A(k_1, k_2) = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}. \]

This is exactly same as for continuous KdV equation.

**N-soliton solution in determinant form:** We could follow this perturbative route and construct an ansatz for 3SS, with only \( k_3 \) as a new parameter, and verify that it is a solution. But we can do better and construct a general determinant formula for the \( N \)-soliton solution. For that purpose let us define
\[ \psi_{n,m}(j,l) := \rho_j^+ (p + k_j)^n (q + k_j)^m k_j^l + \rho_j^- (p - k_j)^n (q - k_j)^m (-k_j)^l. \]

It is easy to verify that \( \psi_{n,m}(1,0) \) is gauge equivalent to \( f_{n,m} \) of (43) for \( c = \frac{\rho_1^-}{\rho_1^+} \). With \( \psi \) given we write the 2SS as
\[ f_{n,m}^{[2ss]} = \begin{vmatrix} \psi_{n,m}(1,0) & \psi_{n,m}(1,1) \\ \psi_{n,m}(2,0) & \psi_{n,m}(2,1) \end{vmatrix}, \]
and this is gauge equivalent to (45) if we connect parameters \( c_j, \rho_j^+, \rho_j^- \) by
\[ \frac{\rho_j^-}{\rho_j^+} = c_1 \frac{k_2 - k_1}{k_1 + k_2}, \quad \frac{\rho_j^-}{\rho_j^+} = c_2 \frac{k_1 - k_2}{k_1 + k_2}. \]

The \( N \)-soliton solution is given by the natural extension
\[ f_{n,m}^{[Nss]} = \begin{vmatrix} \psi_{n,m}(1,0) & \psi_{n,m}(1,1) & \cdots & \psi_{n,m}(1,N-1) \\ \psi_{n,m}(2,0) & \psi_{n,m}(2,1) & \cdots & \psi_{n,m}(2,N-1) \\ \vdots & \vdots & \ddots & \vdots \\ \psi_{n,m}(N,0) & \psi_{n,m}(N,1) & \cdots & \psi_{n,m}(N,N-1) \end{vmatrix}. \]

That this is a solution of (42) for \( \delta \) as given above can be shown by determinantal manipulations as in [8] (cf. Equations (2.20), (5.17b) with bar \( \rightarrow \) tilde, and (5.18)).
6.2.4 From bilinear to nonlinear

Recall that the change of dependent variables from continuous nonlinear to continuous bilinear by (38) involved derivatives, which are easy. The reverse operation involves integration and is more involved, especially for the discrete case.

We start with (41) and shift it by \((n, m) \rightarrow (n - \frac{1}{2}, m + \frac{1}{2})\) and by \((n, m) \rightarrow (n + \frac{1}{2}, m - \frac{1}{2})\) and get

\[
\begin{align*}
&f_{n+\frac{1}{2},m+\frac{1}{2}} f_{n-\frac{1}{2},m-\frac{1}{2}} + \delta^{-1} f_{n+1,m+1} f_{n-1,m-1} - (1 + \delta^{-1}) f_{n+\frac{1}{2},m+\frac{1}{2}} f_{n,m} = 0, \\
&f_{n+\frac{1}{2},m-\frac{1}{2}} f_{n-\frac{1}{2},m+\frac{1}{2}} + \delta^{-1} f_{n+\frac{1}{2},m+\frac{1}{2}} f_{n,m-1} - (1 + \delta^{-1}) f_{n,m} f_{n+\frac{1}{2},m-\frac{1}{2}} = 0,
\end{align*}
\]

respectively. Multiplying the first equation by \(f_{n+\frac{1}{2},m-\frac{1}{2}}\) the second by \(f_{n-\frac{1}{2},m+\frac{1}{2}}\) and subtracting them yields a four term equation and after multiplying it by \(f_{n,m}/(f_{n+\frac{1}{2},m+\frac{1}{2}} f_{n+\frac{1}{2},m-\frac{1}{2}} f_{n-\frac{1}{2},m+\frac{1}{2}} f_{n-\frac{1}{2},m-\frac{1}{2}})\) we can write the result as

\[
\frac{f_{n-1,m} f_{n,m}}{f_{n+\frac{1}{2},m-\frac{1}{2}} f_{n-\frac{1}{2},m+\frac{1}{2}}} - \frac{f_{n+1,m} f_{n,m}}{f_{n+\frac{1}{2},m+\frac{1}{2}} f_{n-\frac{1}{2},m-\frac{1}{2}}} = \frac{1}{\delta} \left( \frac{f_{n,m+1} f_{n,m}}{f_{n+\frac{1}{2},m+\frac{1}{2}} f_{n-\frac{1}{2},m+\frac{1}{2}}} - \frac{f_{n,m-1} f_{n,m}}{f_{n+\frac{1}{2},m-\frac{1}{2}} f_{n-\frac{1}{2},m-\frac{1}{2}}} \right)
\]

Now introduce the quantity

\[
W := \frac{f_{n+\frac{1}{2},m} f_{n-\frac{1}{2},m}}{f_{n,m+\frac{1}{2}} f_{n,m-\frac{1}{2}}}, \quad \text{(48)}
\]

in terms of which the above equation can be written as

\[
W_{n+\frac{1}{2},m} - W_{n-\frac{1}{2},m} = \frac{1}{\delta} \left( \frac{1}{W_{n,m+\frac{1}{2}}} - \frac{1}{W_{n,m-\frac{1}{2}}} \right). \quad \text{(49)}
\]

In order to get a convenient form we still make a change in the discrete variables by

\[
(n + \nu, m + \mu) \rightarrow (n + m + (\nu + \mu + \frac{1}{2}), n - m + (\nu - \mu + \frac{1}{2})),
\]

after which equation (49), when written in terms of \(n' := n + m, m' := n - m\), reads

\[
W_{n',m'} - W_{n'+1,m'+1} = \frac{1}{\delta} \left( \frac{1}{W_{n'+1,m'}} - \frac{1}{W_{n',m'+1}} \right). \quad \text{(50)}
\]

This now has the standard quad form as it depends on the corner variables of the quadrilateral as in Figure 2.

If we apply the double continuous limit (26) on the relation (48) we get

\[
\lim_{\epsilon \to 0} \frac{1}{\epsilon^2} (W - 1) = \partial_x^2 \log(f(x,t)).
\]

Comparing this to (38) we see that equation for \(W - 1\) should be taken as the discrete KdV equation.

6.2.5 Relation to the lpKdV version of KdV

Equation (50) was obtained from the potential KdV equation (37) by discretizing its bilinear form (39) as (42). The discrete bilinear form was then nonlinearized into (50). But this equation is different from the lpKdV equation (1) which we announced in the beginning as being the discrete form of KdV. The reason for the difference is in that lpKdV (1) is discrete potential KdV while (50) is discrete KdV or lattice KdV (IKdV).
The explicit relation is obtained as follows: In (1) let us introduce new variables as follows:

\[ W_{n,m} := u_{n,m+1} - u_{n+1,m}, \quad Z_{n,m} := u_{n,m} - u_{n+1,m+1}, \]

(51)
after which (1) can be written as

\[ W_{n,m} Z_{n,m} = p^2 - q^2. \]

(52)
According to the definitions (51) \( W, Z \) are related by

\[ W_{n,m} - W_{n+1,m+1} = Z_{n,m+1} - Z_{n+1,m} \]

(53)
and if we solve for \( Z_{n,m} \) from (52) and substitute it into (53) we get

\[ W_{n,m} - W_{n+1,m+1} = (p^2 - q^2) \left( \frac{1}{W_{n,m+1}} - \frac{1}{W_{n+1,m}} \right), \]

(54)
which is (50) up to the constant coefficient. The reason for calling this the lattice KdV and (1) the lattice potential KdV is seen from the relation \( W_{n,m} := u_{n,m+1} - u_{n+1,m} \), which is analogous to \( u = v_x \) in the continuous case.

7 Integrability test

As usual we are mainly concerned with integrable equations, but when faced with a new equation how can we tell whether it is potentially integrable? A method that only requires direct computation would be desirable.

In the continuous case we have the Painlevé property and for 1D discrete equations the singularity confinement (SC) idea has been proposed as an analogous property. SC has turned out to be a very useful concept as an indicator, although it is only a necessary condition.

For discrete equations there is also the concept of “algebraic entropy” which states that the complexity of the iterates should not grow exponentially, but only polynomially. Complexity is here measured by the degree of an iterate with respect to the initial values. This is computationally demanding but can be automatized.

As an example consider the corner initial values as in Figure 3 a) and a quadratic equation (such as (1)). If we define \( u_{n,0} = x_n, u_{0,m} = y_n (y_0 = x_0) \) we find that generically \( u_{1,1} \) is quadratic over linear in the initial values, \( u_{1,2} \) and \( u_{2,1} \) are cubic over quadratic, and \( u_{2,2} \) is order six over order five. However, for the particular case of (1), which is integrable, the numerator and denominator of \( u_{2,2} \) have a common factor which can be canceled and the degrees are just order five over order four. Such cancellations continue for higher orders. According to [15], for (1) the degree of the numerator is \( d_{n,m} = nm + 1 \), that is the growth is polynomial. For a generic quadratic equation the asymptotic growth of degrees is exponential.

8 Summary

In this brief introduction to the discrete or lattice soliton equations we have looked at some of their important features, and as an example we have given explicit details for the Korteweg-de Vries equation. We have discussed how the permutation property of Bäcklund transformations can be interpreted as discrete equations on a \( \mathbb{Z}^2 \) lattice and how lattice evolutions are typically defined. The fundamental idea of hierarchy of equations is in the lattice setting provided by multidimensionality. The multidimensional consistency condition was discussed in detail, along with its consequence of providing Lax pair and Bäcklund transformation, which was used to construct a one-soliton solution. We have also discussed continuum limits and discretization, in particular Hirota’s discretization and the ensuing soliton solutions.

For further introductory material we refer the reader to the book [7].
Acknowledgements

I would like to thank Da-jun Zhang for comments on the manuscript.

References

[1] Ablowitz M J and Clarkson P A. *Solitons, nonlinear evolution equations and inverse scattering*, volume 149 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1991.

[2] Adler V, Bobenko A, and Suris Yu. Classification of integrable equations on quad-graphs. The consistency approach. *Comm. Math. Phys.*, 233(3):513–543, 2003.

[3] Bäcklund A V, Om ytor med konstant negative krökning, *Lund Univ. Årsskrift*, 19 (1883), 1-48.

[4] Bianchi L, Sulla trasformazione di Bäcklund per le superficie pseudoferiche, *Rend. Lincei*, 5 (1892), 3-12.

[5] Drazin P G and Johnson R S. *Solitons: an introduction*. Cambridge Texts in Applied Mathematics. Cambridge University Press, Cambridge, 1989.

[6] Faddeev L and Takhtajan L. *Hamiltonian methods in the theory of solitons*. Classics in Mathematics. Springer, Berlin, English edition, 2007. Translated from the 1986 Russian original by Alexey G. Reyman.

[7] Hietarinta J, Joshi N, Nijhoff F, *Discrete Systems and Integrability*. Cambridge University Press, Cambridge, 2016.

[8] Hietarinta J and Zhang D-j. Soliton solutions for ABS lattice equations. II. Casoratians and bilinearization. *J. Phys. A*, 42(40):404006, 30, 2009.

[9] Hirota R. Nonlinear partial difference equations. I. A difference analogue of the Korteweg-de Vries equation. *J. Phys. Soc. Japan*, 43(4):1424–1433, 1977.

[10] Hirota R. Nonlinear partial difference equations. II. Discrete-time Toda equation. *J. Phys. Soc. Japan*, 43(6):2074–2078, 1977.

[11] Hirota R. Nonlinear partial difference equations. III. Discrete sine-Gordon equation. *J. Phys. Soc. Japan*, 43(6):2079–2086, 1977.

[12] Hirota R. Discrete analogue of a generalized Toda equation. *J. Phys. Soc. Japan*, 50(11):3785–3791, 1981.

[13] Miwa T, Jimbo M, and Date E. *Solitons*, volume 135 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2000. Differential equations, symmetries and infinite-dimensional algebras, Translated from the 1993 Japanese original by Miles Reid.

[14] Nijhoff F, Ramani A, Grammaticos B, and Ohta Y. On discrete Painlevé equations associated with the lattice KdV systems and the Painlevé VI equation. *Stud. Appl. Math.*, 106(3):261–314, 2001.

[15] Tremblay S, Grammaticos B, and Ramani A. Integrable lattice equations and their growth properties. *Phys. Lett. A*, 278(6):319–324, 2001.

[16] Wahlquist H, and Estabrook F, Bäcklund transformation for solutions of the Korteweg-de Vries equation, *Phys. Rev. Lett.*, 31 (1973), 1386–1390.