PRESENTATIONS OF PSEUDODISTRIBUTIVE LAWS

CHARLES WALKER

Abstract. By considering the situation in which the involved pseudomonads are presented in no-iteration form, we deduce a number of alternative presentations of pseudodistributive laws including a “decagon” form, a pseudoalgebra form, a no-iteration form, and a warping form. As an application, we show that five coherence axioms suffice in the usual monoidal definition of a pseudodistributive law.

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1. Introduction

Monads are one of the fundamental constructions in category theory, and in recent years have also become more prevalent in computer science [2, 26, 8]. Typically, a monad on a category $\mathcal{C}$ is defined as an endofunctor $T: \mathcal{C} \to \mathcal{C}$ along with natural transformations $u: 1_\mathcal{C} \to T$ and $m: T^2 \to T$ satisfying three coherence conditions.

Distributive laws of monads were introduced by Beck [1] and give a concise description of the data and coherence conditions needed to compose two monads $(T, u, m)$ and $(P, \eta, \mu)$. More precisely, Beck defines a distributive law of monads as a natural transformation $\lambda: TP \to PT$ such that the below two triangles and two pentagons commute:

\[
\begin{array}{ccc}
TP & \xrightarrow{\lambda} & PT \\
\downarrow m & & \downarrow Tm \\
T^2P & \xrightarrow{T\eta} & PT P
\end{array}
\quad
\begin{array}{ccc}
TP & \xrightarrow{\lambda} & PT \\
\downarrow \eta T & & \downarrow \eta T \\
PT & \xrightarrow{T\eta} & PT
\end{array}
\]

\[
\begin{array}{ccc}
TP & \xrightarrow{\lambda} & PT \\
\downarrow \eta T & & \downarrow \eta T \\
PT & \xrightarrow{T\eta} & PT
\end{array}
\quad
\begin{array}{ccc}
TP & \xrightarrow{\lambda} & PT \\
\downarrow \eta T & & \downarrow \eta T \\
PT & \xrightarrow{T\eta} & PT
\end{array}
\]

It is not hard to arrive at this set of four axioms. Indeed, given a $\lambda: TP \to PT$ if one works out what is required to extend the monad $(T, u, m)$ to a monad $(\tilde{T}, \tilde{u}, \tilde{m})$:

\[
\begin{array}{c}
\left(\tilde{T}, \tilde{u}, \tilde{m}\right): \text{Kl}(P, \eta, \mu) \to \text{Kl}(P, \eta, \mu)
\end{array}
\]

on the Kleisli category of $P$, they arrive at two of these axioms from the nullary and binary functoriality conditions of $\tilde{T}$, and the other two from naturality of $\tilde{u}$ and $\tilde{m}$.

It turns out that one may take a different approach to distributive laws, based on the “extensive” (also called “no-iteration” or “Kleisli triple”) presentation of monads as studied by Manes [18], which in fact dates back to early work of Walters [28]. In this extensive form, a monad on a category $\mathcal{C}$ is defined as an assignation on objects $T: \mathcal{C}_{\text{ob}} \to \mathcal{C}_{\text{ob}}$ with a family of arrows $u_X: X \to TX$ and functions $\mathcal{C}(X, TY) \to \mathcal{C}(TX, TY)$. This data is then required to satisfy three different coherence axioms. It is an interesting fact that the functoriality and naturality conditions automatically follow from these three conditions\(^1\). Moreover, this family of functions $\mathcal{C}(X, TY) \to \mathcal{C}(TX, TY)$ comprise what is called a “pasting operator”. Such a pasting operator of a monad is also called the extension operator of an extension system [7, 23].

If one works out what is needed to extend the monad $(T, u, m)$ to the Kleisli category of $P$, with this extension now defined in extensive form, they will naturally arrive at three coherence conditions for distributive laws corresponding to the three axioms for a monad in extensive form. These three axioms are the two triangles:

\[\text{\textbullet\textbullet\textbullet}\]

---

\(^1\)This simplification which happens in $\text{Cat}$ when monads are presented extensively is explained in detail by Marmolejo and Wood [23], as a consequence of any functor having a right adjoint in the bicategory of profunctors $\text{Prof}$. 

from earlier, but with the two pentagons replaced by a single decagon condition

\[
\begin{align*}
TPTPT & \xrightarrow{TPT} TP^2T^2 \xrightarrow{TP^2m} TPT^2 T \xrightarrow{T\mu T} \alpha_{PT} PPT \\
\downarrow_{PT^2} & \downarrow_{PT} \\
P \cdot PmPT & \xrightarrow{TPT} PmPT \\
\downarrow_{PM} & \downarrow \\
PTPT & \xrightarrow{P\alpha} P^2T^2 \xrightarrow{P^2m} P^2T \xrightarrow{\mu T} \alpha_{PT} PPT
\end{align*}
\]

In one dimension, the difference between these two definitions of distributive law is rather trivial. However, in two dimensions the difference becomes significant, as this means pseudodistributive laws can be naturally defined taking three modifications as the basic data rather than the usual four [20]. Moreover, the reduction in the data makes the coherence conditions for pseudodistributive laws much easier to understand conceptually. Interestingly, one recovers a variant of the well known triangle and pentagon axioms for monoidal categories [16] (though it is actually the bicategorical version of these axioms [17] which is relevant here).

It is this understanding, along with known coherence results concerning the triangle and pentagon equations in the context of bicategories [9] and pseudomonads [19, 14], that allow us to deduce that three of Marmolejo and Wood’s eight coherence axioms for pseudodistributive laws [22] are redundant in the sense that they follow from the other five.

However, the goal of this paper is not just to reduce the coherence axioms of pseudodistributive laws, but to give other presentations of them. For instance, the reader will notice the composites \(\lambda T \cdot Pm : TPT \to PT\) appearing in the decagon, so that denoting this composite by \(\alpha\), the decagon may be seen as the hexagon axiom

\[
\begin{align*}
TPTPT & \xrightarrow{TP\alpha} TP^2T \xrightarrow{TP^2T} TPT \\
\downarrow_{\alpha T} & \downarrow \\
PTPT & \xrightarrow{P\alpha} P^2T \xrightarrow{\mu T} PT
\end{align*}
\]

These morphisms \(\alpha : TPT \to PT\) (or morphisms \(PTP \to PT\) in the dual situation\(^2\)) should be familiar to the reader, appearing in the characterization of distributive laws in terms of Kleisli and Eilenberg-Moore objects [21]. Indeed, such characterizations are often useful for considering distributive laws when Kleisli and Eilenberg-Moore objects do not exist [21].

Interestingly, this hexagon axiom leads to a simpler version of these characterizations of [21]. It turns out that distributive laws \(\lambda\) are in bijection with morphisms

\(^2\)One might denote such morphisms by \(\text{res}_{T\eta} : PTP \to PT\) as they exhibit \(T\eta\) as a \(P\)-embedding, using the “admissibility” point of view [27]. In [27] the dual problem of extending to pseudo-algebras extensively was considered, though in the simpler lax-idempotent case [29, 12], and the decagon conditions were not recognized.

When \(P\) is only an endofunctor, one is forced to use this dual version. In particular, a dual version of this hexagon appeared in [4] in the context of wreaths.
\[ \alpha : TPT \to PT \text{ rendering commutative the diagrams} \]

\[
\begin{array}{ccccccc}
TPT & \xrightarrow{\alpha} & PT & T^2 & \xrightarrow{T\eta T} & TPT & TPTPT & \xrightarrow{TP\alpha} & T\mu T & \xrightarrow{\alpha} & TPT \\
\downarrow uPT & & \downarrow m & \downarrow nT & & \downarrow \alpha PT & \downarrow P\alpha & & \downarrow \mu T & & \downarrow \alpha \\
PT & & PT & PT & & PT & PT & & PT & & PT \\
\end{array}
\]

which we refer to as the algebra definition (as the \( T \)-algebra axioms on \( \alpha \) follow from the above conditions). This definition is closely connected to a definition of distributive laws in terms of pasting operators \( (-)^{\lambda} \) due to Marmolejo and Wood \[23, Theorem 6.2\]. Indeed, we get a simplification of their result, finding that with the monad \( P \) defined extensively, a distributive law is a pasting operator \( (-)^{\lambda} : \mathcal{C} (-, PT-) \to \mathcal{C} (T-, PT-) \) such that all \( f : X \to PTY \), \( g : Y \to PTZ \) and \( k : X \to TY \) render commutative the three diagrams

\[
\begin{array}{ccc}
X & \xrightarrow{f} & PTY \\
\downarrow uX & & \downarrow f^\lambda \\
TX & & PTY \\
\end{array}
\quad
\begin{array}{ccc}
TX & \xrightarrow{(\eta TY \cdot k)^{\lambda}} & PTY \\
\downarrow k^T & & \downarrow \eta TY \\
TY & & PTY \\
\end{array}
\quad
\begin{array}{ccc}
TX & \xrightarrow{((\varphi^\lambda)^p f)^{\lambda}} & PTZ \\
\downarrow f^\lambda & & \downarrow (\varphi^\lambda)^p \\
PTY & & PTZ \\
\end{array}
\]

Finally we give the so called “warping” definition, which describes distributive laws in terms of the data of the extended monad’s resulting Kleisli category structure. Of the corresponding three conditions

\[
\begin{array}{ccc}
\mathcal{C} (X, PTY) & \xrightarrow{id} & \mathcal{C} (X, PTY) \\
\xrightarrow{(-)^{\lambda}_{X,Y} (-)^{\lambda}_{X}} & & \xrightarrow{(-)^{\lambda}_{X,Y}} \mathcal{C} (X, PTY) \\
\mathcal{C} (X, TY) & \xrightarrow{(-)^{\lambda}_{X,Y} (-)^{\lambda}_{X,Y}} & \mathcal{C} (X, PTY) \\
\xrightarrow{(-)^{\lambda}_{X,Y} (-)^{\lambda}_{X,Y}} & & \xrightarrow{(-)^{\lambda}_{X,Y}} \mathcal{C} (X, PTY) \\
\end{array}
\]

the first and third conditions are a unit law and associativity law for the Kleisli category of the extended monad \( \tilde{T} \), where the second is slightly stronger in that it gives both the remaining unit law as well as a compatibility condition.

In this paper we will consider the more general 2-dimensional “pseudo" versions of the above to better understand pseudodistributive laws. This allows us to address some problems which have not been practical to solve until now (due to the complicated coherence conditions). Perhaps most important is defining pseudodistributive laws in terms of pasting operators \( (-)^{\lambda} \), which is essential in the setting
of relative pseudomonads [5], where one is forced to use pasting operators. This is perhaps the simplest and most natural presentation\(^3\).

It remains unclear how to generalize these results to distributive laws where both monads are relative. Solving this problem will likely involve some combination of the methods of this paper, the case when one is relative [15], and left and right extension operators [7].

1.1. Structure of the paper. In Section 3 we recall the monoidal and extensive (no-iteration) definitions of pseudomonads, and their coherence axioms. Then in Section 4 we give five new presentations of pseudodistributive laws \(\lambda: TP \to PT\) of pseudomonads; namely:

1. the “pseudomonoidal” definition. This is the pseudo version of the usual definition due to Beck [1], involving a pseudonatural transformation \(\lambda: TP \to PT\) and four invertible modifications comprising two triangles and two pentagons satisfying five coherence axioms;

2. the “Kleisli-decagon” definition. This involves the decagonal conditions one finds for distributive laws when the involved monads are presented in extensive form and the usual \(\lambda: TP \to PT\) is taken as the data. This version comprises three modifications satisfying a version of the triangle and pentagon equations, plus a third compatibility axiom\(^4\);

3. the “pseudoalgebra” definition in terms of maps \(\alpha: TPT \to PT\). This is a reduced version of the above in which a change of variables leads to a simplification in the axioms. Moreover, this definition may be regarded as a “base case” for definitions of pseudodistributive laws in terms of pasting operators \((-)^\lambda\) as one may apply such pasting operators to identities to recover \(\alpha\);

4. the “no-iteration” definition in terms of pseudo-pasting operators

\[ (-)^\lambda : \mathcal{C}(-, PT-) \to \mathcal{C}(T-, PT-). \]

This no-iteration definition is intended to avoid any iteration of the involved pseudomonads \(T\) and \(P\), which is important in the “relative” case [5];

5. the “warping” definition in terms of the data of the Kleisli bicategory of the extended pseudomonad \(\tilde{T}\). This formulation allows for applications of MacLane and Paré’s coherence theorem [17], and hence applications to the corresponding pseudodistributive law data in the earlier formulations. Moreover, this generalizes Street and Lack’s equivalence of monads and warpings [14] to one of distributive laws and “distributivity warpings”.

The third and forth presentations above give an improvement of a result of Marmolejo and Wood [23, Theorem 6.2], both simplifying and generalizing from one to two dimensions. In fact, understanding this result was the original motivation for this paper.

In Section 5 we justify our five definitions of pseudodistributive law by proving they are in equivalence with compatible extensions of \(T\) to the Kleisli bicategory of the pseudomonad \(P\). In the case of the pseudomonoidal definition, we will

\(^3\)One might argue the warping definition is the most natural, since it is based entirely on (pseudo)pasting operators. However, this presentation is more complicated to state.

\(^4\)The “Kleisli” prefix refers to the fact the decagon starts with \(TPTPT\), as happens when one extends to the Kleisli category extensively. There is also a dual version starting from \(PTPTP\).
also explain in Subsection 5.3 how one deduces the redundancy of three of the usual pseudodistributive law axioms from the decagon formulation’s version of the triangle and pentagon equations.

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3. PSEUDOMONADS

We start this section by recalling two equivalent definitions of pseudomonad (namely the monoidal and no-iteration forms), including the three axioms which are known to be redundant in monoidal form by results of Marmolejo [19], and in no-iteration form (also called extensive form) by results of Street and Lack [14].

3.1. Pseudomonads in pseudomonoidal and no-iteration form. In order to define pseudomonads, we first need the notions of pseudonatural transformations and modifications. The notion of pseudonatural transformation is the (weak) 2-categorical version of natural transformation. Modifications, also defined below, take the place of morphisms between pseudonatural transformations.

**Definition 3.1.1.** A pseudonatural transformation of pseudofunctors \( t: F \Rightarrow G: \mathcal{A} \to \mathcal{B} \) where \( \mathcal{A} \) and \( \mathcal{B} \) are bicategories provides for each 1-cell \( f: A \to B \) in \( \mathcal{A} \), 1-cells \( t_A \) and \( t_B \) and an invertible 2-cell \( t_f \) in \( \mathcal{B} \) as below

\[
\begin{array}{ccc}
FA & \xrightarrow{Ff} & FB \\
\downarrow t_A & & \downarrow t_B \\
GA & \xrightarrow{GFf} & GB
\end{array}
\]

coherent with composition as in [11, Definition 2.2]. Given two pseudonatural transformations \( s, t: F \Rightarrow G: \mathcal{A} \to \mathcal{B} \) as above, a modification \( \alpha: s \Rightarrow t \) consists of, for every object \( A \in \mathcal{A} \), a 2-cell \( \alpha_A: s_A \Rightarrow t_A \) such that for each 1-cell \( f: A \to B \) in \( \mathcal{A} \) we have the equality \((\alpha_B \cdot Ff) \circ t_f = s_f \circ (Gf \cdot \alpha_A)\).

By considering pseudomonoids in a Gray-monoid of endo-pseudofunctors one naturally arrives at the following definition.

**Definition 3.1.2.** A pseudomonad (in pseudomonoidal form) on a bicategory \( \mathcal{C} \) consists of a pseudofunctor equipped with pseudonatural transformations as below

\[
\begin{array}{cccc}
T: \mathcal{C} \to \mathcal{C}, & u: 1\mathcal{C} \to T, & m: T^2 \to T
\end{array}
\]

along with three invertible modifications

\[
\begin{array}{ccc}
T \xrightarrow{uT} T^2 & \xleftarrow{T^u} & T \\
\downarrow \alpha & \cong & \downarrow \beta & \cong & \downarrow m \\
\text{id} & \circlearrowleft & \circlearrowright & \text{id} & \circlearrowleft & \circlearrowright \end{array}
\]

\[
\begin{array}{ccc}
T^3 \xrightarrow{Tm} T^2 & \xleftarrow{T^3} & T \\
\downarrow \gamma & \cong & \downarrow m & \cong & \downarrow m \\
T^2 & \xrightarrow{mT} & T
\end{array}
\]
subject to the two coherence axioms

\[
\begin{align*}
T^2 \xrightarrow{T \mu T} T^3 & \quad \text{and} \quad T^2 \xrightarrow{T \nu T} T^3 \\
T^3 \xrightarrow{T \nu T} T^2 & \quad \text{and} \quad T^3 \xrightarrow{T \mu T} T^2 \\
mT \xrightarrow{T \mu T} T^3 & \quad \text{and} \quad mT \xrightarrow{T \mu T} T^2 \\
mT \xrightarrow{T \nu T} T^3 & \quad \text{and} \quad mT \xrightarrow{T \nu T} T^2 \\
\end{align*}
\]

\(T^2 \xrightarrow{T \mu T} T^3 \xrightarrow{T \nu T} T^2\)

\(T^3 \xrightarrow{T \mu T} T^2 \xrightarrow{T \nu T} T^3\)

\(mT \xrightarrow{T \mu T} T^3 \xrightarrow{T \nu T} T^2\)

\(mT \xrightarrow{T \mu T} T^2 \xrightarrow{T \nu T} T^3\)

Remark 3.1.3. One should note here that there are three useful consequences of these pseudomonad axioms [19, Proposition 8.1] motivated by results of Kelly [9]. These are as follows:

\[
\begin{align*}
1 \otimes u & \xrightarrow{T \alpha} T^2 \xrightarrow{T \beta} uT \\
\end{align*}
\]

\[
\begin{align*}
T^3 & \xrightarrow{T \mu T} T^2 \\
\end{align*}
\]

\[
\begin{align*}
mT & \xrightarrow{T \mu T} T^3 \\
\end{align*}
\]

\[
\begin{align*}
mT & \xrightarrow{T \mu T} T^2 \\
\end{align*}
\]

\[
\begin{align*}
\end{align*}
\]

We only mention these redundant axioms as they will be important later on. Indeed, a version of these three redundant axioms appear in the coherence conditions for a pseudodistributive law, albeit in a very convoluted way which is why it was not noticed earlier. As we will later see in Section 5.3, the first appears most directly, whilst the appearance of the other two only becomes apparent when one combines the two pentagons into a decagon.

The definition of a pseudomonad in no-iteration form is due to Marmolejo and Wood [24]. However, it will be more convenient to use the presentation given by Fiore, Gambino, Hyland and Winskel [5] for relative pseudomonads (with the “relative” part taken to be an identity).

Definition 3.1.4. [24, 5] A pseudomonad (in no-iteration form) on a bicategory \(C\) consists of:

\[
\begin{align*}
deck
\end{align*}
\]
• an assignation on objects $\mathcal{C}_{ob} \rightarrow \mathcal{C}_{ob}: X \mapsto TX$;
• for each $X \in \mathcal{C}$, a 1-cell $u_X: X \rightarrow TX$;
• for each $X, Y \in \mathcal{C}$ a functor $(-)^T_{X,Y}: \mathcal{C}(X, TY) \rightarrow \mathcal{C}(TX, TY)$;
• for each $f: X \rightarrow TY$, an isomorphism $\phi_f: f \Rightarrow f^T \cdot u_X$ natural in $f$;
• for each $X \in \mathcal{C}$, an isomorphism $\theta_X: u_X^T \Rightarrow \text{id}_{TX}$;
• for each $f: X \rightarrow TY$ and $g: Y \rightarrow TZ$, an isomorphism $\delta_{g,f}: (g^T \cdot f)^T \Rightarrow g^T \cdot f^T$ natural in $f$ and $g$;

satisfying the two coherence conditions involving the unitors and associators of $\mathcal{C}$:

1. each $f: X \rightarrow TY$ renders commutative

\[
\begin{array}{ccc}
  f^T & \xrightarrow{\phi_f^T} & (f^T u_X)^T \\
  \downarrow \text{unitor} & & \downarrow \text{id} \\
  f^T \theta_X & & \end{array}
\]

2. each $f: X \rightarrow TY$, $g: Y \rightarrow TZ$, and $h: Z \rightarrow TV$ renders commutative

\[
\begin{array}{ccc}
  (h^T g^T f)^T & \xrightarrow{\delta_{h,g,f}^T} & (h^T g)^T f^T \\
  \downarrow \text{assoc.} & & \downarrow \text{assoc.} \\
  (h^T (g^T f))^T & \xrightarrow{\delta_{h,g}^T} & h^T (g^T f)^T \\
  \downarrow \text{assoc.} & & \end{array}
\]

Remark 3.1.5. The three useful consequences of the pseudomonad axioms listed earlier in Remark 3.1.3 now become the assertion [5, Lemma 3.2] which states that any morphisms $f: X \rightarrow TY$ and $g: Y \rightarrow TZ$ render commutative

\[
\begin{array}{ccc}
  u_X^T & \xrightarrow{\theta_{u_X}^T} & u_X^T u_X \\
  \downarrow \text{id} & & \downarrow \text{id} \\
  u_X & \xrightarrow{\theta_X u_X} & \end{array}
\]

The redundancy of these pseudomonad axioms in no-iteration form was first noticed by Street and Lack [14].

Given that these two versions of pseudomonads are in equivalence, it is not at all surprising that there are three corresponding redundant axioms in the two definitions. What is surprising (and is shown later) is that this causes three of the usual pseudodistributive law axioms to be redundant. It is not at all expected that the redundant three of five pseudomonad axioms should correspond to three of the ten pseudodistributive law axioms (as these are a very different looking sets of axioms). In fact, this is the best situation one might hope for.

Convention: Throughout the remainder of this paper we will suppress the modification data when describing pseudomonads. Thus instead of $(T, u, m, \alpha, \beta, \gamma)$
we simply write \((T, u, m)\) or \(T\). We will also suppress the modification data for the pseudomonad \(P\), and the isomorphisms in naturality squares. The reason for this is the proof carries through the same whether \(T\) and \(P\) are pseudomonads or 2-monads. It is only the modification data on the extended pseudomonad that is important for proving the results of this paper. Moreover, labeling all of this data may be distracting or overwhelming for the reader, especially in the larger diagrams which involve higher order expressions such as \(TPTPTP\).

We will also follow the convention of Street and Lack [13], where a distributive law of type \(\lambda: TP \to PT\) is one of \(T\) over \(P\).

**Definition 3.1.6.** The Kleisli bicategory of a pseudomonad \((T, u, m)\) on a 2-category \(\mathcal{C}\), denoted \(\text{Kl}(T)\), is the bicategory with:

- the same objects as \(\mathcal{C}\);
- a morphism \(f: X \rightsquigarrow Y\) in \(\text{Kl}(T)\) is a morphism \(f: X \to TY\) in \(\mathcal{C}\);
- the identity \(\text{id}_X: X \rightsquigarrow X\) on an object \(X\) is the unit \(uX: X \to TX\);
- for each \(f: X \rightsquigarrow Y\) and \(g: Y \rightsquigarrow Z\) the composite \(g \cdot f: X \rightsquigarrow Z\) is given by

\[
\begin{array}{ccc}
X & \xrightarrow{f} & TY \\
& \xrightarrow{Tg} & T^2Z \\
& \xrightarrow{mZ} & TZ
\end{array}
\]

The unitality and associativity laws only hold up to coherent isomorphism, thus giving only a bicategory. The coherence data for these laws is constructed using the modifications comprising the pseudomonad. We will omit the details here, as the reader can refer to [3, Definition 4.1].

**Remark 3.1.7.** The reader will notice the composite above may be written as \(g^T \cdot f\) when the pseudomonad is presented in no-iteration form. As this formulation of pseudomonad naturally lends itself to describing the Kleisli bicategory, it is also sometimes called the Kleisli presentation.

Following this line of reasoning, we observe that the basic structure required to form the Kleisli bicategory is a (pseudo) pasting operator \((-)^T: \mathcal{C}(X, TY) \to \mathcal{C}(TX, TY)\) along with some data and axioms. If the reader is being consistent, they will express all of the data and axioms in terms of pasting operators. This gives the (unsimplified) warping definition of a pseudomonad.

4. **Presentations of pseudodistributive laws**

4.1. **Pseudomonoidal definition of pseudodistributive laws.** Even when dealing with strict 2-monads, it is often the case that one has no strict distributive law between them, but only a pseudodistributive law where the usual diagrams (two triangles and two pentagons) only commute up to invertible modifications [3]. Work on these “pseudo” versions of distributive laws started with Kelly [10], who considered the case where the usual axioms held strictly with the exception of one of the pentagons.

Later, pseudodistributive laws were considered in the general case (where all four axioms only hold up to isomorphism) by Marmolejo [20], who imposed nine coherence conditions on the four invertible modifications. It was then later shown by Marmolejo and Wood [22], that one of the original nine axioms, in addition to a tenth axiom introduced by Tanaka [25], are redundant, thus reducing the number of coherence axioms to eight.
We now give another reduction in the coherence axioms, using just five to define a pseudodistributive law. We again follow the convention of suppressing the modification data of the pseudomonads $T$ and $P$.

**Remark 4.1.1.** Note that the usual ten coherence axioms for a pseudodistributive law come from understanding the structure of a pseudomonad in the Gray category of pseudomonads [6]. From there, it is a matter of working out which are redundant in that they follow from the others$^5$.

**Remark 4.1.2.** To explain why the “pseudo” version of distributive laws is interesting to consider, we point the reader to the trivial fact that a strict law is merely a strict monad morphism and opmorphism. This fact is somewhat misleading as it fails in the weak (pseudo) setting where an extra coherence axiom is required, as detailed in Theorem 4.1.8. Hence this weak case motivates us to stop viewing distributive laws as a monad morphism and opmorphism, but rather to think of them as their own independent structure; namely, as a generalization of monads. This is especially evident in the no-iteration setting, where the data and axioms for a distributive law are similar to that of a monad. This also follows Beck [1] who generalized the concept of algebras from monads to distributive laws. These are ultimately the algebras of the composite monad but with a simpler description in terms of the distributive law data.

**Definition 4.1.3.** A pseudodistributive law (in pseudomonoidal form) of pseudomonads $(T, u, m)$ over $(P, \eta, \mu)$ is a pseudonatural transformation $\lambda: TP \to PT$ and four invertible modifications as below$^6$

\[
\begin{align*}
TP & \xrightarrow{\lambda} PT \\
\xleftarrow{uP} P & \xrightarrow{P\mu} PT \\
\xleftarrow{T\eta} T & \xrightarrow{\eta T} T
\end{align*}
\]

satisfying the following five coherence axioms. The first two coherence axioms are the unitality axioms of a pseudomonad morphism and pseudomonad opmorphism.

$^5$An exception to this is in the lax idempotent setting, where one can define pseudodistributive laws using different sets of coherence axioms. However, it appears unlikely any choice would be better than the five axioms given here in the general case.

$^6$The directions of the modifications below are chosen such that they will naturally compose into decagons later on, and such that the directions of the induced pseudomonad modifications will match with that of a no-iteration pseudomonad as in Definition 3.1.4 (which is defined as in [5]). Though these choices of directions do not matter in the sense that these modifications are invertible, it will make the later proofs easier to follow.
The next two axioms are the associativity axioms of a pseudomonad morphism and pseudomonad opmorphism
The last axiom ensures that the pentagons $\omega_3$ and $\omega_4$ are compatible, and asks

is equal to

Remark 4.1.4. Since the above axiom (W5) is the most complicated, it may be helpful to note that it may be seen as an instance of a pseudomodule transformation axiom. This is explained by Gambino and Lobbio [6, Remark 4.2] who denote the axiom (C5).

For convenience and easy reference, we also list the five redundant coherence conditions of a pseudodistributive law.
Theorem 4.1.5. Given a pseudodistributive law \((\lambda, \omega_1, \omega_2, \omega_3, \omega_4) : TP \to PT\) in pseudomonoidal form, the following five conditions are derivable.

(W6)

(W7)

(W8)

(W9)

(W10)
Remark 4.1.6. The redundancy of (W6) and (W7) is shown directly by Marmolejo and Wood [22]. However, this result can be seen more easily by noting that pseudomonad morphisms can be seen as instances of pseudoalgebras (as is well known in one dimension [21, 23]), and that one of the unitality axioms for a pseudoalgebra is redundant [19, Lemma 9.1]. Curiously the methods of this paper give another proof of the redundancy, though this proof would be less strong as it uses additional pseudodistributive law axioms.

Remark 4.1.7. Note that this is the best that one might hope for, in that only one compatibly axiom is needed relating the pseudomonad morphism and opmorphism data. This is why the set of five axioms given here is expected to be minimal. Stated more precisely, this becomes the following result.

**Theorem 4.1.8.** A pseudodistributive law \((\lambda, \omega_1, \omega_2, \omega_3, \omega_4) : TP \to PT\) is equivalently a pseudomonad morphism \((\lambda, \omega_1, \omega_3) : T \to T\) along \(P\), and a pseudomonad opmorphism \((\lambda, \omega_2, \omega_4) : P \to P\) along \(T\), such that \(\omega_3\) and \(\omega_4\) satisfy axiom (W5).

4.2. **Decagon definition of pseudodistributive laws.** The following is the definition of pseudodistributive law one finds after working out the conditions on a pseudonatural transformation \(\lambda : TP \to PT\) needed for extending a pseudomonad \((T, u, m)\) to the Kleisli bicategory of a pseudomonad \((P, \eta, \mu)\) in pseudoextensive form. In practice one would likely not use this definition, but it will be needed for the later proofs and explanation of redundant coherence axioms.

**Definition 4.2.1.** A pseudodistributive law (in Kleisli-decagon form) of pseudomonads \((T, u, m)\) over \((P, \eta, \mu)\) is a pseudonatural transformation \(\lambda : TP \to PT\) and three invertible modifications comprising the two triangles

\[
\begin{array}{ccc}
TP & \xrightarrow{\lambda} & PT \\
\downarrow uP & \searrow & \downarrow Pu \\
\downarrow T\eta & & \downarrow T\eta \\
P & \underset{\omega_1}{\leftarrow} & T \\
& \underset{\omega_2}{\leftarrow} & PT \\
\end{array}
\]

and the decagon

\[
\begin{array}{ccccccc}
TP & \xrightarrow{\lambda} & PT & \xrightarrow{TP\lambda T} & TP^2T^2 & \xrightarrow{TP^2m} & TP^2T & \xrightarrow{T\mu T} & TP \\
\downarrow XPT & & \downarrow \Psi \Omega & & \downarrow TP^2 & & \downarrow TPT \\
PT^2PT & & & & PT^2 & & PT \\
\downarrow PmPT & & \downarrow P\lambda T & & \downarrow P\lambda T & & \downarrow Pm \\
PTPT & \xrightarrow{P\lambda T} & P^2T^2 & \xrightarrow{P^2m} & P^2T & \xrightarrow{\mu T} & PT \\
\end{array}
\]
satisfying the triangle equation:

(D1)

the pentagon equation:

(D2)

is equal to
and compatibility of $\Omega$ with $\mathcal{C} \rightarrow \mathcal{C}_P$.

\[ \text{(DC2)} \]

\[ \begin{array}{ccccccccc}
TPT^2 & \xrightarrow{TPT \eta T} & TPTPT & \xrightarrow{TPT \lambda T} & TPT^3 & \xrightarrow{TP \mu T} & TPT \\
PT^3 & \xrightarrow{PT \eta T} & PTPT & \xrightarrow{PT \lambda T} & PT^2 & \xrightarrow{PT \omega} & PT \\
PT^2 & \xrightarrow{PT \eta T} & PT^2 & \xrightarrow{PT \omega} & PT & & \\
\end{array} \]

is equal to

\[ \begin{array}{ccccccccc}
TPT^2 & \xrightarrow{TPT \eta T} & TPTPT & \xrightarrow{TPT \lambda T} & TPT^3 & \xrightarrow{TP \mu T} & TPT \\
PT^3 & \xrightarrow{PT \eta T} & PTPT & \xrightarrow{PT \lambda T} & PT^2 & \xrightarrow{PT \omega} & PT \\
PT^2 & \xrightarrow{PT \eta T} & PT^2 & \xrightarrow{PT \omega} & PT & & \\
\end{array} \]

\[ \text{Remark 4.2.2.} \] When the data for enforcing compatibility is not part of the data for forming the extension, the compatibility axioms are necessary. This is why all of the later presentations will require both compatibility axioms.

Whilst this suggests axiom (DC2) might be redundant (because the compatibility data $\omega_2$ is part of the extension data)\(^7\), the fact it does not involve the unit $u$ (whereas (D1) does) suggests it may be necessary. The issue is that MacLane and Paré’s coherence theorem applies to $\omega_1$, $\Omega$ and $\mathcal{C}_P$ instead of $\omega_2$ (even though this simplifies to $\omega_2$ when composed with appropriate pseudomonad data)\(^8\). Hence the coherence theorem only applies to pasting diagrams constructed in a certain way. This problem creates a persistent issue where the redundancy of certain coherence conditions only becomes apparent when restricted by a unit map $u$.

4.3. Pseudoalgebra definition of pseudodistributive laws. The following is intended to provide a definition of pseudodistributive laws involving a pseudoalgebra structure map $\alpha$ (with the pseudoalgebra data and axioms being derivable from the law). This is in the spirit of Marmolejo, Rosebrugh and Wood [21, Prop. 3.5] who in one dimension considered taking algebra structure maps as the data for a distributive law, though directly assuming the algebra axioms.

We point out that as the usual monoidal definition of monads is natural, defining a distributive law in monoidal form requires no extra “change of variables” axioms. If the reader agrees the no-iteration definition of monad is also natural, then the no-iteration (and by extension pseudoalgebra) versions of a distributive law should

\[^7\text{Another reason to expect (DC2) to be redundant is that the other compatibility axiom is.}\]

\[^8\text{Note this data directly appears in axiom (D1).}\]
also avoid any such axioms. This is the conceptual reason as to why a simplification of [21, Prop. 3.5] is possible, reducing five axioms to three in dimension one.\footnote{In the skew setting the monoidal and no-iteration definitions of skew monad are distinct [14] and so the reader is forced to decide which one is natural. The same issue applies to distributive laws in the skew setting (as the bijectivity of the change of variables $\lambda \mapsto \alpha$ fails), forcing the reader to decide which formulation is the natural one.}

**Definition 4.3.1.** A pseudodistributive law (in pseudoalgebra form) of pseudomonads $(T, u, m)$ over $(P, \eta, \mu)$ is a pseudonatural transformation $\alpha : TPT \to PT$ and three invertible modifications

![Diagram of pseudodistributive law](image)

satisfying the triangle equation;

\[(M1)\]

\begin{align*}
TPT & \xrightarrow{TPT} PT \\
\eta PT & \xrightarrow{id} TPT \\
PT & \xrightarrow{TPT} \mu PT
\end{align*}

the pentagon equation;

\[(M2)\]

\begin{align*}
TPT & \xrightarrow{TPT} TPT \\
\eta PT & \xrightarrow{\alpha PT} TPT \\
PT & \xrightarrow{TPT} \mu PT
\end{align*}
is equal to

\[
\begin{array}{c}
\text{TPTPTPT} \xrightarrow{TPT\alpha} \text{TPTP}^2 \xrightarrow{TPT\mu} \text{TPTP} \\
\text{PTPTPT} \xrightarrow{T\alpha PT} \text{TPT^2} \xrightarrow{T\mu PT} \text{TPTP} \\
\text{PTPTPT} \xrightarrow{P_{\alpha PT}} \text{P^2TP} \xrightarrow{\mu PT} \text{PTP} \\
\end{array}
\]

compatibility of \( \psi \) with \( \mathcal{C} \to \mathcal{C}_P \);

(MC1)

and compatibility of \( \Psi \) with \( \mathcal{C} \to \mathcal{C}_P \);

(MC2)

is equal to

\[
\begin{array}{c}
\text{T^3} \xrightarrow{T\eta^2} \text{TPT^2} \xrightarrow{TPT\eta T} \text{TPTP} \xrightarrow{T\alpha} \text{TPT}^2 \xrightarrow{T\mu} \text{TPT} \\
\text{T^2} \xrightarrow{m T} \text{TPT} \xrightarrow{\xi T} \text{TPT} \xrightarrow{\alpha T} \text{TPT} \\
\text{T} \xrightarrow{\eta T} \text{PT} \xrightarrow{\Psi} \text{PT} \\
\end{array}
\]

Remark 4.3.2. Most of the technicalities in changing to this formulation arise from replacing the data \( \omega_2 \) with the new data \( \xi \), which now serves two separate roles. Firstly, the data \( \xi \) is what is required to provide compatibility with the Kleisli functor \( \mathcal{C} \to \mathcal{C}_P \). Secondly, it yields the pasting*
which is the data required to construct an extension (without enforcing compatibility). In this way one may view $\xi$ as two pieces of coherence data combined into one. In fact, asking that these two pieces of data are related this way is the condition for compatibility of $\xi$ with $\mathcal{C} \to \mathcal{C}_P$.

**Remark 4.3.3.** The restriction of (MC1) along the unit $u: 1 \to T$ and the restriction of (MC2) along $T^2u: T^2 \to T^3$ are both redundant in that they follow from the pair of axioms (M1) and (M2)\(^{10}\).

**4.4. No-iteration definition of pseudodistributive laws.** The following defines a pseudodistributive law in terms of pseudo-pasting operators $(-)^\lambda_XY$, which in our case arise as families of functors $C(C,X,PTY) \to C(TX,PTY)$ induced by pasting with a 2-cell $\alpha: TPT \Rightarrow PT$ as in the diagram

\[
\begin{array}{c}
X \\
\downarrow f \\
\B \\
\downarrow \alpha \\
T \\
\downarrow \eta T \\
Y \\
\downarrow P T \\
\end{array}
\]

This definition (which generalizes [23, Definition 2.1] to two dimensions) should be useful in that it more readily generalizes to the relative case [5] (in which one may no longer have such a 2-cell $\alpha$).

**Definition 4.4.1.** A **pseudo-pasting operator** in a tricategory $\mathcal{K}$

$$(-)^\#: \mathcal{K}(C,D)(-,s-) \to \mathcal{K}(C,E)(t-,u-)$$

is a family of functors indexed by $f: C \to D$ and $g: C \to \text{doms} = \text{dom} u$

$$(-)^\#_{f,g}: \mathcal{K}(C,D)(f,sg) \to \mathcal{K}(C,E)(tf,ug)$$

equipped with:

1. (whiskering data) for all $\vartheta: f \Rightarrow sg: C \to D$ and $h: A \to C$ equalities $\vartheta^\# h = (\vartheta h)^\#\(^{11}\);
2. (blistering data) for all $\vartheta: f \Rightarrow sg: C \to D$ and $\xi: p \Rightarrow f: C \to D$ a 3-isomorphism $\text{bl}^\#(\vartheta,\xi): \vartheta^\#(t\xi) \cong (\vartheta\xi)^\#$;

such that using $\circ$ for vertical composition and $\cdot$ for horizontal whiskering:

1. (respecting naturality) all $\nu: \vartheta \Rightarrow \vartheta': f \Rightarrow sg: C \to D$ render commutative

\[
\begin{array}{c}
\vartheta^\# \circ (t \cdot \xi) \\
\downarrow \text{bl}^\#(\vartheta,\xi) \\
(\vartheta \circ \xi)^\#
\end{array}
\]

\[
\begin{array}{c}
\vartheta'^\# \circ (t \cdot \xi) \\
\downarrow \text{bl}^\#(\vartheta',\xi) \\
(\vartheta' \circ \xi)^\#
\end{array}
\]

\[^{10}\text{The former is straightforward to check and the latter is a pseudoalgebra version of (D4).}\]

\[^{11}\text{We use an equality here since it holds strictly in the cases of interest. In particular for a pseudonatural transformation $\alpha$ and whiskering by a pseudofunctor $F$ we have $(\alpha F)_X := \alpha_{FX}$.}\]
Remark 4.4.2. When $\mathcal{K}$ is a 2-category each $\text{bl}_\# (\vartheta, \xi)$ is an equality and we recover the usual one dimensional version of pasting operators.

Remark 4.4.3. In our case of interest we will take $\mathcal{K}$ to be the tricategory of 2-categories, pseudofunctors, pseudonatural transformations and modifications, $D = E = \mathcal{C}, t = T$ and $s = u = PT$. To replace $f$, we denote by $X \in \mathcal{C}$ a generalized element $X: \mathcal{F} \to \mathcal{C}$ of $\mathcal{C}$ so that $C = \mathcal{F}$ and a pseudo-pasting operator consists of functors $\mathcal{K}(\mathcal{F}, \mathcal{C})(X, PTY) \to \mathcal{K}(\mathcal{F}, \mathcal{C})(TX, PTY)$. This may be further simplified since it is equivalent to a coherent family in which each $\mathcal{F}$ is the terminal 2-category.\(^\text{12}\)

Remark 4.4.4. A basic example of a pseudo-pasting operator is $(-)^u$ induced by a pseudonatural transformation $u: 1_{\mathcal{C}} \to T$.

We can now give the no-iteration definition of a pseudodistributive law, which involves no iteration of the pseudomonads $T$ and $P$.

**Definition 4.4.5.** A **pseudodistributive law (in no-iteration form)** of pseudomonads $(T, u, (-)^T)$ over $(P, \eta, (-)^P)$ on a 2-category $\mathcal{C}$ is a pseudo-pasting operator

$$(-)^X_{X,Y}: \mathcal{C}(X, PTY) \to \mathcal{C}(TX, PTY), \quad X, Y \in \mathcal{C}$$

along with for all $f: X \to PTY$, $k: X \to TY$, and $g: Y \to PTZ$ a family of invertible 2-cells

$$\begin{array}{c}
\begin{tikzpicture}[scale=0.6]
\node (A) at (0,0) {$TX$};
\node (B) at (2,0) {$TY$};
\node (C) at (0,-2) {$TX$};
\node (D) at (2,-2) {$TY$};
\node (E) at (4,0) {$TX$};
\node (F) at (4,-2) {$TY$};
\node (G) at (6,0) {$TX$};
\node (H) at (6,-2) {$TY$};
\node (I) at (8,0) {$TX$};
\node (J) at (8,-2) {$TY$};
\node (K) at (10,0) {$TX$};
\node (L) at (10,-2) {$TY$};
\node (M) at (12,0) {$TX$};
\node (N) at (12,-2) {$TY$};
\node (O) at (14,0) {$TX$};
\node (P) at (14,-2) {$TY$};
\node (Q) at (16,0) {$TX$};
\node (R) at (16,-2) {$TY$};
\node (S) at (18,0) {$TX$};
\node (T) at (18,-2) {$TY$};
\node (U) at (20,0) {$TX$};
\node (V) at (20,-2) {$TY$};
\node (W) at (22,0) {$TX$};
\node (X) at (22,-2) {$TY$};
\node (Y) at (24,0) {$TX$};
\node (Z) at (24,-2) {$TY$};
\node (AA) at (26,0) {$TX$};
\node (BB) at (26,-2) {$TY$};
\node (CC) at (28,0) {$TX$};
\node (DD) at (28,-2) {$TY$};
\node (EE) at (30,0) {$TX$};
\node (FF) at (30,-2) {$TY$};
\node (GG) at (32,0) {$TX$};
\node (HH) at (32,-2) {$TY$};
\node (II) at (34,0) {$TX$};
\node (JJ) at (34,-2) {$TY$};
\node (KK) at (36,0) {$TX$};
\node (LL) at (36,-2) {$TY$};
\node (MM) at (38,0) {$TX$};
\node (NN) at (38,-2) {$TY$};
\node (OO) at (40,0) {$TX$};
\node (PP) at (40,-2) {$TY$};
\node (QQ) at (42,0) {$TX$};
\node (RR) at (42,-2) {$TY$};
\node (SS) at (44,0) {$TX$};
\node (TT) at (44,-2) {$TY$};
\node (UU) at (46,0) {$TX$};
\node (VV) at (46,-2) {$TY$};
\node (WW) at (48,0) {$TX$};
\node (XX) at (48,-2) {$TY$};
\node (YY) at (50,0) {$TX$};
\node (ZZ) at (50,-2) {$TY$};
\node (AAA) at (52,0) {$TX$};
\node (BBB) at (52,-2) {$TY$};
\node (CCC) at (54,0) {$TX$};
\node (DDD) at (54,-2) {$TY$};\end{tikzpicture}\\
\end{array}$$

\(^{12}\text{We remind the reader that in the case of a no-iteration monad (or an extension system) the naturality of the unit and multiplication may be shown as a consequence of the definition. Indeed, a similar calculation may be done for the distributive law version concerning the resulting $\alpha: TPT \to PT$. Here $(PTf)^\lambda \cong \alpha_Y \cdot TPTf$ by blistering, and the isomorphism $(PTf)^\lambda \cong PTf \cdot \alpha_X$ is $\Psi^{XY} u_X f \cdot \alpha_X$ simplified using $\xi^\lambda f$. This means one may use global elements in place of generalized elements.}
natural in \( f,k, g \) and respecting whiskering and blistering\(^{13} \), such that we have for all \( g: Y \to PTZ \) the triangle equation;

\[
\begin{align*}
\text{(I1)} & \\
TY & \xrightarrow{g^\lambda} & PTZ \\
\downarrow & & \\
\downarrow & & \\
TY & \xrightarrow{(g^\lambda u_Y)^\lambda} & PTZ \\
\downarrow & & \\
\downarrow & & \\
TY & \xrightarrow{((g^\lambda)^p \cdot \eta TY \cdot u_Y)^\lambda} & PTZ \\
\downarrow & & \\
\downarrow & & \\
TY & \xrightarrow{\xi^\lambda} & PTZ \\
\downarrow & & \\
\downarrow & & \\
TY & \xrightarrow{\eta TY} & PTZ \\
\downarrow & & \\
\downarrow & & \\
TY & \xrightarrow{g^\lambda} & PTZ
\end{align*}
\]

for all \( f: X \to PTY, g: Y \to PTZ \) and \( h: Z \to PTW \) the pentagon equation;

\[
\begin{align*}
\text{(I2)} & \\
TX & \xrightarrow{((h^\lambda)^p)^{g^{\lambda}}} & PTW \\
\uparrow & & \\
\uparrow & & \\
TX & \xrightarrow{\psi f^K} & PTW \\
\downarrow & & \\
\downarrow & & \\
TX & \xrightarrow{(h^\lambda)^p} & PTW \\
\downarrow & & \\
\downarrow & & \\
TX & \xrightarrow{\eta TY} & PTZ \\
\downarrow & & \\
\downarrow & & \\
TX & \xrightarrow{g^\lambda} & PTZ \\
\downarrow & & \\
\downarrow & & \\
TX & \xrightarrow{(g^\lambda)^p} & PTW
\end{align*}
\]

\(^{13}\)The whiskering axiom ensures \( f^\lambda K = (fK)^\lambda \) and the blistering axiom ensures \( g^\lambda Tf = (gf)^\lambda \). That the family of \( \psi_f \) respects these axioms ensures the following: with \( X,Y: \mathcal{B} \to \mathcal{C} \) and \( K: \mathcal{A} \to \mathcal{B} \) we have \( \psi f^K = \psi fK \) (whiskering), and \( \psi_f \) may be composed by a naturality square of \( u \) at \( p: W \to X \) to give \( \psi f_p \) (blistering). Both families of \( \xi^\lambda \) and \( \Psi g,f \) satisfy similar conditions. Like naturality, we will avoid detailing each whiskering and blistering condition as they are uninteresting. If the reader requires them, they may be derived from the warping formulation as instances of the axioms of morphisms of pseudo-pasting operators.
is equal to

\[
\begin{array}{c}
TX \xrightarrow{\left(\left((h^\lambda)^p g^\lambda\right)^\lambda\right)} PTW \\
\Psi\left((\psi^h g^\lambda)^\lambda\right) \xrightarrow{} \Psi\left((h^\lambda)^p g^\lambda\right)^\lambda \\
TX \xrightarrow{\left((h^\lambda)^p g^\lambda\right)^\lambda} PTW \\
\Psi\left((\psi^h g^\lambda)^\lambda\right) \xrightarrow{} \Psi\left((h^\lambda)^p g^\lambda\right)^\lambda \\
TX \xrightarrow{(g^\lambda)^p f} PTZ \xrightarrow{(h^\lambda)^p} PTW \\
\Psi\left((\psi^h g^\lambda)^\lambda\right) \xrightarrow{} \Psi\left((h^\lambda)^p g^\lambda\right)^\lambda \\
TX \xrightarrow{f^\lambda} PTY \xrightarrow{(g^\lambda)^p} PTZ \xrightarrow{(h^\lambda)^p} PTW \\
\end{array}
\]

for all \(k: X \rightarrow TY\) compatibility of \(\psi\) with \(C \rightarrow C_p\);

\[
\begin{array}{c}
X \xrightarrow{k} TY \\
\eta_Y \xrightarrow{\xi^k} \eta_Y \\
\xi^k \xrightarrow{\eta_Y \cdot f} TY \\
\end{array}
\]

(\text{IC1})

and for all \(f: X \rightarrow TY\) and \(g: Y \rightarrow TZ\) compatibility of \(\Psi\) with \(C \rightarrow C_p\);

\[
\begin{array}{c}
TX \xrightarrow{\left((\eta_Z g^\lambda)^p\right)^\lambda} PTZ \\
\Psi\left((\psi^h g^\lambda)^\lambda\right) \xrightarrow{} \Psi\left((\eta_Z g^\lambda)^p\right)^\lambda \\
TX \xrightarrow{\left((\eta_Z g^\lambda)^p\right)^\lambda} PTZ \\
\Psi\left((\psi^h g^\lambda)^\lambda\right) \xrightarrow{} \Psi\left((\eta_Z g^\lambda)^p\right)^\lambda \\
TX \xrightarrow{(g^\lambda)^p f^\lambda} PTZ \xrightarrow{(\xi^g)^p} PTZ \\
\Psi\left((\psi^h g^\lambda)^\lambda\right) \xrightarrow{} \Psi\left((\eta_Z g^\lambda)^p\right)^\lambda \\
TX \xrightarrow{g^\lambda f^\lambda} PTZ \xrightarrow{(\xi^g)^p} PTZ \\
\end{array}
\]

Remark 4.4.6. The data \(\xi^u_Y\) suffices for constructing the extension (the first two axioms) whereas the more general data \(\xi^k\) is required for compatibility (the remaining two axioms).
4.5. Warping definition of pseudodistributive laws. In [14] Street and Lack showed that pseudomonads presented in extensive form are in equivalence with warpings. These (pseudomonad) warpings correspond to Kleisli bicategory structures, with the two unitors and associator of the pseudomonad being used to construct the two unitors and associator of the Kleisli bicategory. Conversely the two unitors and associator of the pseudomonad can be recovered from that of the Kleisli bicategory structure. It is by this correspondence, that MacLane and Paré’s coherence theorem [17] which applies to the Kleisli bicategory, must apply to the pseudomonad data also. This is how three of the no-iteration pseudomonad axioms were shown to be redundant [14].

Here we extend this correspondence from pseudomonads to pseudodistributive laws, using the fact that these laws correspond to extended pseudomonads on the Kleisli bicategory \( C \). The technicality here is that pseudodistributive laws are more than just a \( T \), as we also have compatibility data and axioms. Hence the coherence theorem only applies to the Kleisli bicategory of \( T \), constructed from the pseudodistributive law data \( \xi^aX \) and arbitrary \( \psi \) and \( \Psi \), but not the more general compatibility data \( \xi^k \), explaining the need for two additional compatibility axioms.

The definition of warping used by Street and Lack [14] does not directly mention pasting operators due to simplifications which are possible in that reduced case. In our more general setting however we will use both pseudo-pasting operators and their morphisms, as it is unclear if one can give a natural definition without them.

**Definition 4.5.1.** Given two pseudo-pasting operators in a tricategory \( \mathcal{X} \)

\[
(-)^\#, (-)^\#: \mathcal{X}(C,D)(-,sg) \to \mathcal{X}(C,E)(tf,ug)
\]

a morphism of pseudo-pasting operators \((-)\# : (-)^\# \Rightarrow (-)^\#'\) is a family of natural transformations indexed by \( f: C \to D \) and \( g: C \to doms = domu \)

\[
(-)_{f,g} : (-)^# \Rightarrow (-)^#' : \mathcal{X}(C,D)(f,sg) \to \mathcal{X}(C,E)(tf,ug)
\]

such that:

1. (whiskering) for all \( \vartheta: f \Rightarrow sg: C \to D \) and \( h: A \to C \) we have \( \vartheta \# h = (\vartheta h)^\# \);  
2. (blistering) all \( \vartheta: f \Rightarrow sg: C \to D \) and \( \xi: p \Rightarrow f: C \to D \) render commutative

\[
\begin{array}{c}
\vartheta (t\xi) \\
\downarrow \vartheta (t\xi) \\
\vartheta (t\xi') \\
\downarrow \vartheta (t\xi')
\end{array} 
\begin{array}{c}
\bl_{\vartheta,\xi}(\vartheta (t\xi)) \\
\bl_{\vartheta,\xi} (\vartheta (t\xi)) \\
\bl_{\vartheta,\xi} (\vartheta (t\xi))
\end{array} 
\begin{array}{c}
(\vartheta \xi)^\# \\
(\vartheta \xi)^\# \\
(\vartheta \xi)^\#'
\end{array}
\]

This formulation of pseudodistributive laws is designed to look as similar as possible to that of warpings for pseudomonads, and thus might be called a “distributivity warping”. Whilst this warping form may appear to be a complicated formulation, it provides the machinery which simplifies the previous formulations and also gives the data of the resulting compatible Kleisli bicategory structure. It may be possible to more easily derive this formulation by iterating the notion of warping (and enforcing compatibility).

**Remark 4.5.2.** The definition of pasting operators is not suggestive of closure under composition, though those we are considering certainly are composable as they
correspond to 2-cells. This is why a number of composition functors appear in the following definition.

**Definition 4.5.3.** A pseudodistributive law (in warping form) of pseudomonads \(\left(T, u, (-)^T\right)\) over \(\left(P, \eta, (-)^P\right)\) on a 2-category \(\mathcal{C}\) consists of:

- a pseudo-pasting operator \((-)^{\lambda}_{X,Y} : \mathcal{C}(X, PTY) \to \mathcal{C}(TX, PTY)\);
- morphisms of pseudo-pasting operators

\[
\begin{align*}
\mathcal{C}(X, PTY) & \xrightarrow{(-)^{\lambda}_{X,Y}} \mathcal{C}(TX, PTY) \xrightarrow{\mathcal{C}(X, TX)} \mathcal{C}(X, PTY) \\
\mathcal{C}(X, TY) & \xrightarrow{(-)^{\lambda}_{X,Y}} \mathcal{C}(TX, PTY) \xrightarrow{\mathcal{C}(TX, TX)} \mathcal{C}(TX, PTY)
\end{align*}
\]

satisfying (labeling only the cells due to space constraints)\(^{14}\) the triangle equation;

\[
\begin{align*}
\mathcal{C}(Y, PTZ) & \xrightarrow{((-)^{\lambda}_{Y,Z})_{TX, TX}^{P}} \mathcal{C}(TX, PTZ) \xrightarrow{\mathcal{C}(TX, TX)} \mathcal{C}(TX, PTZ) \\
\mathcal{C}(Y, PTZ) & \xrightarrow{((-)^{\lambda}_{X,Y})_{TX, TX}^{P}} \mathcal{C}(TX, PTZ) \xrightarrow{\mathcal{C}(TX, TX)} \mathcal{C}(TX, PTZ)
\end{align*}
\]

\(^{14}\)On components these axioms reduce to the earlier no-iteration versions. Moreover, it should be straightforward to recover the full diagrams if one substitutes the three morphisms of pasting operators in full detail, though some composites of pasting operators must be decomposed.
the pentagon equation:

\[
\begin{array}{c}
\Psi(Y,Z,W) \cdot C(X,PTY) \\
\downarrow \\
\Psi(TX,PTY) \cdot C(X,PTY) \\
\downarrow \\
\Psi(X,Y,Z) \\
\downarrow \\
\Psi(X,Z,W) \\
\end{array}
\]

is equal to

\[
\begin{array}{c}
\Psi(Y,Z,W) \cdot C(X,PTY) \\
\downarrow \\
\Psi(X,Y,W) \\
\downarrow \\
\Psi(X,Z,W) \\
\downarrow \\
\Psi(X,Z,W) \\
\end{array}
\]

compatibility of \((\_\psi)\) with \(\mathcal{C} \to \mathcal{C}_P\):

\[
\begin{array}{c}
\Psi(Y,Z,W) \cdot C(X,PTY) \\
\downarrow \\
\Psi(X,Y,W) \\
\downarrow \\
\Psi(X,Z,W) \\
\downarrow \\
\Psi(x,z) \\
\end{array}
\]

Remark 4.5.4. The reason for giving the coherence axioms here (as opposed to just relying on the no-iteration form) is to remind the reader how the conditions of pseudodistributive laws correspond to bicategorical coherence axioms.
Remark 4.5.5. Warpings in the context of wreaths have been considered by Chikhladze [4]. This should give a realistic approach to finding the coherence axioms of pseudo-wreaths. Furthermore, there are structures between distributive laws and wreaths which may be of interest. For instance, the above formulation with the compatibility data\textsuperscript{15} and axioms omitted is more general than a distributive law, less general than a wreath, and has a simpler presentation than both.

5. Equivalence of presentations of pseudodistributive laws

As all five of our definitions of pseudodistributive laws are new, we must justify them by showing they are equivalent to a pseudodistributive law in the sense of Marmolejo [20]. This is the reason for proving the following theorem, which makes use of the equivalence between Marmolejo’s definition of pseudodistributive law and compatible extensions of a pseudomonad to the Kleisli bicategory shown in [3].

Theorem 5.0.1. Given two pseudomonads \((T, u, m)\) and \((P, \eta, \mu)\) on a 2-category \(\mathcal{C}\), the following are in equivalence:

1. a pseudodistributive law \(\lambda: TP \to PT\) in pseudomonoidal form;
2. a pseudodistributive law \(\lambda: TP \to PT\) in Kleisli-decagon form;
3. a pseudodistributive law \(\alpha: TPT \to PT\) in pseudoalgebra form;
4. a pseudodistributive law \((-)^\lambda: \mathcal{C} (-, PT-) \to \mathcal{C} (T-, PT-)\) in no-iteration form;
5. a pseudodistributive law \((-)^\lambda: \mathcal{C} (-, PT-) \to \mathcal{C} (T-, PT-)\) in warping form;
6. an extension of \((T, u, m)\) to a pseudomonad on the Kleisli bicategory of \((P, \eta, \mu)\) which is compatible with the Kleisli pseudofunctor \(\mathcal{C} \to \mathcal{C}_P\).

We will give a sketch proof of this theorem by constructing functors as in the below diagram

\[
\begin{array}{ccc}
\lambda-8 & \text{equiv} & \lambda-5 \\
\text{monoidal} & \alpha \text{ps-alg} & \text{no-it} \\
\lambda-dec & \text{warp} & \\
\tilde{T} & \text{equiv} & \tilde{T} \text{no-iteration}
\end{array}
\]

and explaining why these are all equivalences. The top row lists Marmolejo and Wood’s 8-axiom presentation, our 5-axiom reduction and the decagon presentation. The middle row lists the pseudoalgebra, no-iteration and warping presentations. The bottom row references compatible extensions of \(T\) to the Kleisli bicategory of \(P\), presented in monoidal or no-iteration form. We label by “equiv” those functors which are well known to be equivalences by results of Marmolejo and Wood [22, 24], and (i) to (vi) the remaining functors we must define and show are equivalences.

Remark 5.0.2. We will not burden this paper with the definitions of morphisms of pseudodistributive laws, as these are simply modifications \(\lambda \Rightarrow \lambda'\) or \(\alpha \Rightarrow \alpha'\) such that the obvious pasting diagrams agree.

\textsuperscript{15}The loss of compatibility data may break the bijectivity of the change of variables \(\lambda \Rightarrow \alpha\).
Remark 5.0.3. The pseudoalgebra, no-iteration and warping formulations are in a sense all the same definition but with different notations, and so all have a similar set of four axioms. It is only a non-trivial change of variables which causes a non-trivial change of axioms.

Remark 5.0.4. Given the equivalence of our five formulations of pseudodistributive law, one can show any reasonable coherence axiom must hold. This is since any such condition may be rewritten in terms of the standard pseudomonoidal form, and then shown directly. For example, to verify one has a pseudoalgebra structure on \( \alpha \), meaning we have coherent isomorphisms \( \alpha \cdot mPT \cong \alpha \cdot T\alpha \), one may rewrite \( \alpha \) in terms of \( \lambda \) and use the standard axioms of the pseudomonoidal form.

In proving coherence conditions concerning pseudomonads and their data, it is worth remembering that MacLane and Paré’s coherence theorem for bicategories [17] applies. This is a direct consequence of the warping formulation of pseudomonads [14], which shows that pseudomonads may be expressed in terms of the data of the resulting Kleisli bicategory. In the case of interest, the coherence theorem reduces to the following.

**Proposition 5.0.5.** Suppose that \( T \) is a pseudomonad on any 2-category \( \mathcal{C} \). Then any two pasting diagrams (which are formal in the sense of the following Remark) constructed from the pseudomonad data with the same outside must be equal.

Remark 5.0.6. Just as in MacLane’s theorem, we will need an analogue of a formal path:

- (pseudomonoidal form) here formal means it is constructed entirely from pastings of naturality squares of \( u \) and \( m \) and the three given modifications. Any unexpected equalities involving \( u, m \) and the remaining data (such as the \( mT = Tm \) in Isbell’s counterexample) cannot be used in the pasting diagram, even if they are true;
- (no-iteration form) here formal means it is constructed entirely from pastings of the given unit and multiplication cells. Any rewriting of \( f \) as \( f' \) is not allowed, even if \( f = f' \);

We also note that if this proposition was not true, then the definition of pseudomonad would need extra coherence axioms.

5.1. A restricted equivalence of the pseudomonoidal and decagon forms. The following lemma sets the stage for proving the equivalence (ii) of the monoidal and decagon definitions by directly showing a restricted version of this equivalence (meaning a number of axioms are left out of both formulations).

Remark 5.1.1. For checking equalities of pasting diagrams as below, it is helpful to note that given two diagrams constructed entirely from the same data (for example if we are given two pasting diagrams both constructed from two \( \omega_2 \) cells and one \( \Omega \) cell), then showing they are equal is typically a matter of applying naturality (middle four interchange).

**Lemma 5.1.2.** For a given \( \lambda \), the data \( (\omega_1, \omega_2, \omega_3, \omega_4) \) with axioms (W1) and (W2) is in bijection with the data \( (\omega_1, \omega_2, \Omega) \) with axiom (D1).

**Proof.** From the modifications comprising the pentagons \( \omega_3 \) and \( \omega_4 \), the decagon \( \Omega \) is constructed as the pasting diagram
Conversely, given the decagon \( \Omega \) one recovers the pentagon \( \omega_4 \) as

and the pentagon \( \omega_3 \) as
Now given \((\omega_1, \omega_2, \omega_3, \omega_4)\) with axioms \((W1)\) and \((W2)\), we may start with the left hand side of \((D1)\) and substitute the formula for the decagon \(\Omega\) in terms of \(\omega_3\) and \(\omega_4\). The \(\omega_2\) cell is moved by naturality to the \(\omega_4\) cell and both are eliminated by \((W2)\), and similarly the \(\omega_1\) cell and \(\omega_3\) cell are eliminated by \((W1)\), leaving the identity or right hand side of \((D1)\). Replacing \(\Omega\) by its formula in terms of \(\omega_3\) and \(\omega_4\) in the above two diagrams, we recover \(\omega_3\) and \(\omega_4\) by the same process of moving cells by naturality and eliminating.

Conversely, given \((\omega_1, \omega_2, \Omega)\) with axiom \((D1)\). We may start with the left hand side of \((W1)\) and substitute the formula for \(\omega_3\) in terms of \(\Omega\). This gives a decagon \(\Omega\) with a \(\omega_2\) cell on the left, and a \(\omega_1\) cell on the top. This should already look similar to axiom \((D1)\), which can be applied with a minor rearrangement. The proof of \((W2)\) is similar. Writing the formula for \(\Omega\) in terms of \(\omega_3\) and \(\omega_4\) and then substituting the above formulas for \(\omega_3\) and \(\omega_4\) in terms of \(\Omega\), one may move the \(\omega_1\) cell by naturality to apply \((D1)\) leaving only \(\Omega\) after some simplification. \(\square\)

5.2. Decagons to pseudoalgebras to no-iteration forms. We now define \((iii)\) and \((v)\), and show these are equivalences. The construction \((iii)\) is the following simple rewriting.

**Proposition 5.2.1.** A pseudodistributive law \((\lambda, \omega_1, \omega_2, \Omega)\) in decagon form with axiom \((W10)\) gives rise to a pseudodistributive law \((\alpha, \psi, \xi, \Psi)\) in pseudoalgebra form. This also holds with the compatibility axioms \((W10), (DC2)\) and \((MC1), (MC2)\) removed respectively.

**Proof.** Given a \(\lambda: TP \rightarrow PT\) we define \(\alpha: TPT \rightarrow PT\) as the composite

\[
TPT \xrightarrow{\lambda T} PT^2 \xrightarrow{Pm} PT
\]

Fortunately, the decagon \(\Omega\) and axiom \((D2)\) is constructed from such composites. Thus we can simply take \(\Psi = \Omega\), and \((M2)\) is just \((D2)\) with these composites relabeled as \(\alpha\). The modifications \(\psi\) and \(\xi\) are given by

\[
\begin{align*}
TPT & \xrightarrow{\lambda T} PT^2 \xrightarrow{Pm} PT \\
PT & \xrightarrow{id} PT
\end{align*}
\]

and

\[
\begin{align*}
T^2 & \xrightarrow{T\eta T} PT \\
PT & \xrightarrow{Pm} PT
\end{align*}
\]

respectively. The reader will notice these both appearing in axiom \((D1)\), thus by relabeling we recover axiom \((M1)\).

For compatibility, we note the left of \((MC1)\) expressed in terms of the data \((\lambda, \omega_1, \omega_2, \Omega)\) is

\[
\begin{align*}
T & \xrightarrow{\eta T} PT \\
PT & \xrightarrow{Pm} PT
\end{align*}
\]
and it is a clear consequence of \((W10)\) that this simplifies to \(\text{id}_\eta T\). Similarly, by substitution the left of \((MC2)\) becomes

\[
\begin{array}{c}
\xymatrix{
T^2 \ar[r]^{TPT^2} & TPT^2 \ar[r]^{TP\eta T} & TPT \ar[r]^{TP^2 T} & TP^2 T \ar[r]^{TP^2 m} & TP^2 T \ar[r]^{T \mu T} & TPT \\
mT \ar[r]_{\eta T} & TPT \ar[r]_{TP^2 T} & TP^2 T \ar[r]_{TP^2 m} & TP^2 T \ar[r]_{TP^2 m} & TP^2 T \ar[r]_{TP^2 m} & TP^2 T \\
t \ar[r]_{\eta T} & T \ar[r]_{\eta T} & T \ar[r]_{\eta T} & T \ar[r]_{\eta T} & T \\
T \ar[r]_{\eta T} & T \ar[r]_{\eta T} & T \ar[r]_{\eta T} & T \ar[r]_{\eta T} & T \\
\end{array}
\]

which by \((DC2)\) reduces to

\[
\begin{array}{c}
\xymatrix{
T^3 \ar[r]^{TPT^2} & TPT^2 \ar[r]^{TP\eta T} & TPT \ar[r]^{TP^2 T} & TP^2 T \ar[r]^{TP^2 m} & TP^2 T \ar[r]^{TP^2 m} & TP^2 T \\
mT \ar[r]_{\eta T} & TPT \ar[r]_{TP^2 T} & TP^2 T \ar[r]_{TP^2 m} & TP^2 T \ar[r]_{TP^2 m} & TP^2 T \ar[r]_{TP^2 m} & TP^2 T \\
t \ar[r]_{\eta T} & T \ar[r]_{\eta T} & T \ar[r]_{\eta T} & T \ar[r]_{\eta T} & T \\
T \ar[r]_{\eta T} & T \ar[r]_{\eta T} & T \ar[r]_{\eta T} & T \ar[r]_{\eta T} & T \\
\end{array}
\]

and naturality of \(m\) then gives

\[
\begin{array}{c}
\xymatrix{
T^3 \ar[r]^{TPT^2} & TPT^2 \ar[r]^{TP\eta T} & TPT \ar[r]^{TP^2 T} & TP^2 T \ar[r]^{TP^2 m} & TP^2 T \ar[r]^{TP^2 m} & TP^2 T \\
mT \ar[r]_{\eta T} & TPT \ar[r]_{TP^2 T} & TP^2 T \ar[r]_{TP^2 m} & TP^2 T \ar[r]_{TP^2 m} & TP^2 T \ar[r]_{TP^2 m} & TP^2 T \\
t \ar[r]_{\eta T} & T \ar[r]_{\eta T} & T \ar[r]_{\eta T} & T \ar[r]_{\eta T} & T \\
T \ar[r]_{\eta T} & T \ar[r]_{\eta T} & T \ar[r]_{\eta T} & T \ar[r]_{\eta T} & T \\
\end{array}
\]

which is the right hand side of \((MC2)\). \(\square\)

We now see how a pseudodistributive law \((\alpha, \psi, \xi, \Psi)\) in pseudoalgebra form gives rise to a pseudomonad on the Kleisli bicategory, defining \((v)\).

**Proposition 5.2.2.** A pseudodistributive law \((\alpha, \psi, \xi, \Psi)\) in pseudoalgebra form gives rise to a compatible pseudomonad \(\tilde{T}\) extending \(T\) to the Kleisli bicategory of \(P\). With the compatibility axioms \((MC1), (MC2)\) removed we have a possibly non-compatible extension.

**Proof.** Suppose we are given a pseudodistributive law \(\alpha: TPT \to PT\) in pseudoalgebra form. We will define a pseudomonad \(\tilde{T}\) in pseudoextensive form (as in Definition 3.1.4) on the Kleisli bicategory of \((P, \eta, \mu)\). We define \(\tilde{T}\) to have the same action on objects as \(T\). For each \(X \in \text{Kl}(P)\), we take our unit \(\tilde{u}_X: X \simeq TX\) to be the composite

\[
X \xrightarrow{\eta_X} TX \xrightarrow{\eta_T X} PTX
\]
Each functor $K_1(P) (X, TY) \to K_1(P) (TX, TY)$, or more simply $\mathcal{C} (X, PTY) \to \mathcal{C} (TX, PTY)$, is defined by sending an $f : X \to PTY$ to $\alpha_Y \cdot Tf : TX \to PTY$.

For each $f : X \to PTY$ we take the 2-cell $\phi_f : f \Rightarrow \tilde{f} \cdot \tilde{u}_X$ as the pasting

![Diagram]

for each $X$ we take the 2-cell $\theta_X : (\tilde{u}_X) \Rightarrow id_{TX}$ to be

![Diagram]

and for all $f : X \to PTY$ and $g : Y \to PTZ$ we take $\delta_{g,f} : (g \tilde{T} \cdot f) \Rightarrow g \tilde{T} \cdot f \tilde{T}$ as

![Diagram]

Note that technically there is no real choice of the pseudomonad data of $\tilde{T}$ here, as it is forced by the compatibility conditions required for an extension to the Kleisli bicategory. Naturality is clear in the above definitions. Moreover, the two axioms ($M1$) and ($M2$) ensure the two coherence conditions of a pseudomonad in no-iteration form are satisfied.

To ensure compatibility with the Kleisli pseudofunctor $\mathcal{C} \to \mathcal{C}_P$ we must provide an isomorphism in the square where both $T$ and $\tilde{T}$ are defined extensively

![Diagram]

This is the identity on objects, and for a map $k : X \to TY$ in $\mathcal{C}$ it is the isomorphism $(\eta_{TY} \cdot k) \tilde{T} \cong \eta_{TY} \cdot k \tilde{T}$ denoted $\xi^k$ and constructed as

![Diagram]

That $\xi^k$ respects the unit $\phi$ is ($MC1$), that $\xi^k$ respects the unit $\theta$ is that $\theta_X$ must be $\xi^{u_X}$, and finally that $\xi^k$ respects the associator $\delta$ is ($MC2$). \[\Box\]

\[16\] This operation may also be denoted by the pseudo-pasting operator $(-)^\lambda$. 
5.3. Redundancy of pseudomonoidal coherence axioms. From axioms (W1) through to (W5) follow the axioms (D1) and (D2), from which follow three redundant axioms (D3), (D4) and (D5). This restricted version of the decagon formulation (without the condition (DC2) for compatibility) is enough to construct the extension and show redundancy of the three axioms (W8), (W9) and (W10).

The three redundant axioms (D3), (D4) and (D5) come from considering the redundant axioms of a pseudomonad in no-iteration form as in Remark 3.1.5 (being careful to understand these axioms in the context of the extended pseudomonad \( \tilde{T} \) not \( T \)), where the pseudomonad \( \tilde{T} \) is constructed from the decagon formulation of a pseudodistributive law. The middle axiom of Remark 3.1.5 (in the base case with \( f \) an identity) expressed in decagon form is then that

\[
(D4) \quad TPT \xrightarrow{\lambda_T} TPT^2 \xrightarrow{\mu_T} TPT
\]

must be equal to

\[
TPT \xrightarrow{TPT_u} TPT^2 \xrightarrow{TPT \eta_T} TPT \xrightarrow{TPT \eta_T} TPT^2 \xrightarrow{TPT \eta_T} TPT \xrightarrow{\lambda_T} TPT
\]

Following a similar calculation, the leftmost redundant axiom of Remark 3.1.5 works out to be (W10) which we also call (D5), and the rightmost axiom becomes (D3) which we have not stated since it is not strictly necessary for the proof. In particular, this gives the following.

**Proposition 5.3.1.** In the decagon presentation the axiom (W10) is redundant.

The axiom (W10) must also be redundant in the pseudomonoidal presentation, since the decagon formulation may be constructed from it. The proof of the redundancy of (W8) and (W9) is slightly more work, so we give it below.

**Proposition 5.3.2.** In the pseudomonoidal presentation the axioms (W8), (W9) and (W10) are redundant and (i) defines an isomorphism.
Proof. Instead of giving the argument for both remaining axioms, we will show the redundancy of (W8) in some detail. The argument for (W9) is then similar.

Given the left hand side of (W8), we start by substituting the formula for the pentagon \( \omega_3 \) in terms of the decagon \( \Omega \) as in Lemma 5.1.2 giving

We move the bottom \( \omega_3 \) along naturality squares of \( u \) and \( \eta \), so that we are more ready to apply axiom (D4), giving

and we also restrict the \( \omega_2TPT \) to a \( \omega_2T \) to look more like the right hand side of (W8), giving
Now applying axiom (D4) the above reduces to

From here, it is just a matter of restricting the cell $TP\omega_2 T$ to a $T\omega_2$ by naturality. Simplifying, this is the right hand side of (W8).

\[\square\]

Remark 5.3.3. Whilst one can certainly use a similar version of the above to show the axiom (W9) is redundant, doing so is not necessary. This is because we may construct a compatible extension to the Kleisli without ever using it.

Remark 5.3.4. The decagon formulation includes two coherence conditions which ensure the extension is compatible with the Kleisli functor $C \to C$. The first property (DC1) is simply (W10) or (D5) which we have already explained is redundant. The second property (DC2) is almost (D4) but without the restriction along the unit $TPTu : TPT \to TPT^2$. 
Remark 5.3.5. The redundancy of (W8) almost gives a proof of the redundancy of axiom (DC2). In particular, the left of (DC2) is

\[
\begin{array}{ccccccccccc}
\lambda T^2 & \xrightarrow{\lambda T^3} & P^2 T & \xrightarrow{P^2 T \eta T} & PT & \xrightarrow{TP \eta T} & TP & \xrightarrow{T P^2 T} & TP^2 T_m & \xrightarrow{T P^2 T} & TP^2 T & \xrightarrow{T \mu T} & TPT \\
& & & & & & & & & & & \\
& & & & & & & & & & & \\
& & & & & & & & & & & \\
& & & & & & & & & & & \\
& & & & & & & & & & & \\
& & & & & & & & & & & \\
& & & & & & & & & & & \end{array}
\]

and one can immediately simplify using (W8). The issue is (W7) would then be required to simplify to the right of (DC2), which likely cannot be shown here.

Corollary 5.3.6. The assignment (ii) is a functor and defines an isomorphism.

Proof. Since all remaining axioms (W6) to (W10) may be deduced from (W1) to (W5), we know that the remaining condition (DC2) must also follow, as we may simplify the diagram of Remark 5.3.5. It follows that (ii) is a well defined functor.

We also point out that from the full decagon form we get the pseudoalgebra and then no-iteration form above which is a compatible extension to the Kleisli, and thus axioms (W3) to (W5) may be recovered (because we necessarily get a full pseudodistributive law). Hence Lemma 5.1.2 extends to an isomorphism (ii). □

5.4. Equivalence of decagon and pseudoalgebra forms.

Corollary 5.4.1. The functor (iii) and thus also (v) define equivalences.

Proof. Note that the diagram (5.1) pseudo-commutes, and so the composite

\[
\begin{array}{ccc}
\lambda \text{ dec} & \xrightarrow{(iii)} & \alpha \text{ ps-alg} \\
& \xrightarrow{(iii)} & \lambda \text{ dec}
\end{array}
\]

is isomorphic to the identity, where the second arrow denotes (v) composed with the other equivalences. The more complex part is checking that

\[
\begin{array}{ccc}
\alpha \text{ ps-alg} & \xrightarrow{(iii)} & \lambda \text{ dec} \\
& \xrightarrow{(iii)} & \alpha \text{ ps-alg} \\
\alpha & \xrightarrow{(iii)} & \alpha \cdot TPu \\
& \xrightarrow{(iii)} & Pm \cdot \alpha T \cdot TPuT
\end{array}
\]
is also isomorphic to the identity\(^{20}\). This coherent isomorphism may be constructed directly as the pasting

\[(G1)\]

\[
\begin{array}{ccccccccc}
TPT & \xrightarrow{TP\eta T} & TP^2 T & \xrightarrow{id} & TPT \\
TP\alpha T & \xleftarrow{TPuPT} & TP\alpha T & \xleftarrow{TP\phi} & TP\phi T & \xrightarrow{id} & TP\phi T & \xleftarrow{TP\psi} & TP\psi T \\
TPT^2 & \xrightarrow{TPT\eta T} & TPTPT & \xrightarrow{\alpha T} & TP\alpha T & \xrightarrow{TP\phi} & TPT & \xrightarrow{\alpha} & TPT \\
PT^2 & \xrightarrow{PT\eta T} & PPTPT & \xrightarrow{P\alpha} & P^2 T & \xrightarrow{\mu T} & PT \\
Pm & \xrightarrow{P\xi} & P^2 T & \xrightarrow{\mu T} & PT \\
\end{array}
\]

which shows that (iii) is an equivalence, and as (v) belongs to a pseudo-commuting square where everything else is an equivalence it must be one itself. \(\square\)

**Remark 5.4.2.** By a similar argument one may show \(\alpha\) respects the multiplication of \(T\). More explicitly, we have an isomorphism \(Pm \cdot \alpha T \cong \alpha \cdot TPm\) constructed as the pasting

\[(G2)\]

\[
\begin{array}{ccccccccc}
TPT^2 & \xrightarrow{TPm} & TPT & \xrightarrow{id} & TPT \\
TPT\eta T & \xrightarrow{TP^{-1}} & TPTPT & \xrightarrow{T P\phi} & TP\phi T & \xrightarrow{id} & TP\phi T & \xrightarrow{TP\psi} & TP\psi T \\
Pm & \xrightarrow{P\xi} & P^2 T & \xrightarrow{\mu T} & PT \\
\end{array}
\]

It is an interesting observation that this construction requires \(\xi\) to be invertible, whereas the simplified restriction along \(TPuT\) given by (G1) does not. This means whilst Corollary 5.4.1 may generalize from an equivalence to an adjunction in the skew setting, the invertibility of \(\xi\) would still be required for much of the theory.

5.5. **Equivalence of pseudoalgebra, no-iteration and warping forms.** The equivalence of the pseudoalgebra and no-iteration forms relies on the equivalence of 2-cells in a tricategory and pseudo-pasting operators, generalizing the result [23, Lemma 2.2] in dimension one.

**Lemma 5.5.1.** For \(s, t\) and \(u\) configured such that \(st\) exists and has the same domain and codomain as \(u\), the pseudo-pasting operators of Definition 4.4.1\(^{21}\) are in equivalence with 2-cells \(ts \Rightarrow u\).

**Proof.** As in [23, Lemma 2.2] we use the same method of writing a general \(\vartheta^\#\) in standard form as \(\text{id}^\# \cdot g \cdot t\vartheta\), so that pasting operators are again determined by their operation on identities. The only issue here is that the rewriting is only the same up to isomorphism (since blistering is only an isomorphism), and to ensure

\(^{20}\)We omit the level of modifications since it is mostly trivial pastings constructed from the isomorphism data.

\(^{21}\)It is possible the general definition of pseudo-pasting operator (where one lacks this configuration) would have extra axioms. Our formulation is based on what is required in the proof of this Lemma.
this is an isomorphism of pasting operators we require the blistering data of \((-)^\#\) to be natural and respect whiskering and blistering. This is why Definition 4.4.1 contains these three respective conditions.

We now finish the proof of Theorem 5.0.1, by proving the equivalences (iv) and (vi) of the pseudoalgebra, no-iteration and warping formulations of a pseudodistributive law.

**Proposition 5.5.2.** The pseudoalgebra and no-iteration formulations of a pseudodistributive law are in equivalence.

**Proof.** Firstly, note that given a no-iteration pseudodistributive law \((-)^\lambda\) we may take \(f, g, h\) and \(k\) to be identities (and even take \(X, Y, Z\) to be the identity on \(\mathcal{C}\)). From this the data of the pseudoalgebra version is recovered with \(\psi, \xi\) and \(\Psi\) respectively given by (where \(\alpha\) is the action of \((-)^\lambda\) on an identity)

\[
\begin{align*}
PT & \xrightarrow{id} TPT \\
TPT & \xrightarrow{id^T = \alpha} PT \\
\xi & \xrightarrow{\psi \cdot \xi \cdot \Psi} PT
\end{align*}
\]

and

\[
\begin{align*}
PTPT & \xrightarrow{id^T_{PTT} = \alpha PT} PT \\
PTPT & \xrightarrow{\xi^T \cdot \alpha \cdot \alpha'} PT \\
\Psi & \xrightarrow{\psi \cdot \xi \cdot \Psi} PT
\end{align*}
\]

We have used here the fact \((-)^\lambda\) must be defined by sending an \(f: X \to PTY\) to the composite

\[
TX \xrightarrow{Tf} TPTY \xrightarrow{\alpha Y} PTY
\]

and a 2-cell \(\theta: f \Rightarrow g: X \to PTY\) to the whiskering \(\alpha Y \cdot T\theta\), since \((-)^\lambda\) respects whiskering and blistering. Note also that \(\psi, \xi\) and \(\Psi\) become modifications by a similar argument to [24, Prop. 3.4 and 3.5]. Axioms (I1) and (I2) respectively become (M1) and (M2) by similarly substituting identities. In this calculation, it is worth noting that the application of \((-)^\lambda\) to \(\psi\) is what gives the \(T\psi\) in (M1). Moreover, the non-identity cases such as the \(\xi^\alpha Y\) in (I1) can be reduced to identity cases, by identifying the two diagrams

\[
\begin{align*}
TY & \xrightarrow{(\eta TY \cdot uY)^\lambda} PTY \\
TY & \xrightarrow{\eta TY} PTY
\end{align*}
\]

using that the family \(\xi^k\) respects blistering\(^{22}\) and is natural in \(k\).

\(^{22}\)Both \(\xi^\alpha Y\) and \(\xi^\text{id}\) play a central role in this paper, as the unit law data \(\theta\) in the extended pseudomonad, and as the data \(\xi\) in pseudoalgebra form respectively. These two pieces of data are combined into a single family \(\xi^k\) which cohere via the blistering operation.
Conversely, given the pseudoalgebra version one will recover the no-iteration version by defining the pasting operator $(\cdot)^\lambda : C (\cdot, PT\cdot) \to C (T\cdot, PT\cdot)$ by the same formula $f \mapsto \alpha Y \cdot Tf$. It is then easy to construct $\psi^f$ from $\psi$, $\xi^k$ from $\xi$, and $\Psi^{g,f}$ from $\Psi$. This is done by taking blisterings to upgrade from identities to general maps; for example $\xi^\psi Y$ is constructed from $\xi^{id} = \xi$ by $uY$-blistering as in the above diagram. In much the same way, the coherence axioms (M1) and (M2) directly give (I1) and (I2) for the identity case, and are upgraded to general maps (such as the $g$ in (I1)) by blistering. By the same method we see the coherence axioms (MC1) and (MC2) give (IC1) and (IC2). □

**Proposition 5.5.3.** The no-iteration and warping formulations of a pseudodistributive law are in isomorphism.

**Proof.** This is simply a matter of restating the axioms on the morphisms of pseudopasting operators $(-)^\psi, (-)^\xi$ and $(-)^\Psi$ given in Definition 4.5.3 in terms of the components. The four coherence axioms correspond respectively to the four of the no-iteration form of Definition 4.4.5. □

### 6. Declaration Statement

**6.1. Author’s contribution.** This was entirely the work of the only author.

**6.2. Conflict of interest.** The author declares he has no conflict of interest.

**6.3. Availability of data and materials.** Not applicable.

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Department of Mathematics and Statistics, Masaryk University, Kotlářská 2, Brno 61137, Czech Republic

*Email address: charles.walker.math@gmail.com*