BLOW-UP TIME OF STRONG SOLUTIONS TO A BIOLOGICAL NETWORK FORMATION MODEL IN HIGH SPACE DIMENSIONS

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Abstract. We investigate the possible blow-up of strong solutions to a biological network formation model originally introduced by D. Cai and D. Hu [13]. The model is represented by an initial boundary value problem for an elliptic-parabolic system with cubic non linearity. We obtain an algebraic equation for the possible blow-up time of strong solutions. The equation yields information on how various given data may contribute to the blow-up of solutions. As a by-product of our development, we establish a $W^{1,q}$ estimate for solutions to an elliptic equation which shows the explicit dependence of the upper bound on the elliptic coefficients.

1. Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^N$ with $C^{1,1}$ boundary $\partial \Omega$ and $T$ a positive number. Set $\Omega_T = \Omega \times (0,T)$. We study the possible blow-up of strong solutions to the system

\begin{align}
-\text{div} \left[(I + m \otimes m)\nabla p\right] &= S(x) \quad \text{in } \Omega_T, \\
\partial_t m - D^2 \Delta m + |m|^{2(\gamma-1)}m &= E^2(m \cdot \nabla p)\nabla p \quad \text{in } \Omega_T,
\end{align}

coupled with the initial boundary conditions

\begin{align}
p(x,0) &= 0, \quad m(x,0) = m_0(x), \quad x \in \Omega \\
m(x,t) &= 0, \quad (x,t) \in \Sigma_T = \partial \Omega \times (0,T),
\end{align}

for given functions $S(x)$, $m_0(x)$ and physical parameters $D, E, \gamma$ with properties:

(H1) $m_0(x) \in \left( W^{1,\infty}_0(\Omega) \right)^N$, $N \geq 3$, $S(x) \in L^{\frac{4qN}{4qN+N}}(\Omega)$ for some $q > 1 + \frac{N}{2}$; and

(H2) $D, E \in (0,\infty)$, $\gamma \in (\frac{1}{2},\infty)$.

This system was originally derived in ([12], [13]) as the formal gradient flow of the continuous version of a cost functional describing formation of biological transportation networks on discrete graphs. In this context, the scalar function $p = p(x,t)$ is the pressure due to Darcy’s law, while the vector-valued function $m = m(x,t)$ is the conductance vector. The function $S(x)$ is the time-independent source term. More detailed information on the biological relevance of the problem can be found in [1, 2, 10].

We are interested in the mathematical analysis of the problem. A pair $(m, p)$ is said to be a weak solution to (1.1)-(1.4) if:

\begin{itemize}
\item[(D1)] $m \in L^\infty(0,T; \left(W^{1,2}_0(\Omega) \cap L^{2\gamma}(\Omega)\right)^N)$, $\partial_t m \in L^2(0,T; \left(L^2(\Omega)\right)^N)$, $p \in L^\infty(0,T; W^{1,2}_0(\Omega))$, $(m \cdot \nabla p) \in L^\infty(0,T; L^2(\Omega))$;
\item[(D2)] $m(x,0) = m_0$ in $C \left([0,T]; \left(L^2(\Omega)\right)^N\right)$;
\item[(D3)] Equations (1.1) and (1.2) are satisfied in the sense of distributions.
\end{itemize}
The existence of a weak solution was first established in [9]. It was based upon the following a priori estimates

\[
\frac{1}{2} \int_{\Omega} |\mathbf{m}(x, \tau)|^2 \, dx + D^2 \int_{\Omega} |\nabla \mathbf{m}|^2 \, d\tau + E^2 \int_{\Omega} (\mathbf{m} \cdot \nabla p)^2 \, d\tau \\
+ \int_{\Omega_r} |\mathbf{m}|^{2\gamma} \, d\tau + 2E^2 \int_{\Omega} |\nabla p|^2 \, dx \\
= \frac{1}{2} \int_{\Omega} |\mathbf{m}_0|^2 \, dx + 2E^2 \int_{\Omega} S(x)p \, dx \\
+ \frac{E^2}{2} \int_{\Omega} |\nabla p|^2 \, dx + \frac{1}{2\gamma} \int_{\Omega} |\mathbf{m}|^{2\gamma} \, dx \\
= \frac{D^2}{2} \int_{\Omega} |\nabla \mathbf{m}_0|^2 \, dx + \frac{E^2}{2} \int_{\Omega} (\mathbf{m}_0 \cdot \nabla p_0)^2 \, dx + \frac{1}{2\gamma} \int_{\Omega} |\mathbf{m}_0|^{2\gamma} \, dx \\
+ \frac{E^2}{2} \int_{\Omega} |\nabla p_0|^2 \, dx,
\]

(1.5)

where \( \tau \in (0, T], \Omega_r = \Omega \times (0, \tau) \), and \( p_0 \) is the solution of the boundary value problem

\[
-\text{div}[(I + \mathbf{m}_0 \otimes \mathbf{m}_0)\nabla p] = S(x), \quad \text{in } \Omega, \\
p_0 = 0 \quad \text{on } \partial \Omega.
\]

(1.7)

Partial regularity of weak solutions was addressed in [16, 20] for \( N \leq 3 \). If the space dimension is 2, a classical solution was obtained in [22] for the stationary case and in [19] for the time-dependent case. Finite time extinction or break-down of solutions in the spatially one-dimensional setting for certain ranges of the relaxation exponent \( \gamma \) was carefully studied in [10]. Further modeling analysis and numerical results can be found in [1]. We also mention that the question of existence in the case where \( \gamma = \frac{1}{2} \) is addressed in [10]. In this case the term \( |\mathbf{m}|^{2(\gamma - 1)} \) is not continuous at \( \mathbf{m} = 0 \). However, the general regularity theory remains fundamentally incomplete. In particular, it is not known whether or not weak solutions develop singularities in high space dimensions \( N \geq 3 \) even though a blow-up criterion was obtained in [15] when \( \Omega = \mathbb{R}^3 \).

A strong solution is a weak solution with the additional property

(D4) \( \mathbf{m} \) is Hölder continuous in \( \overline{\Omega_T} \).

If (D4) holds, then equation (1.1) becomes uniformly elliptic. In fact, much better is true. Under (D4) and the assumption

(H3) \( \partial \Omega \) is \( C^{1,1} \),

the result in ([18], p.82) becomes applicable. Upon invoking the result, we conclude that for each \( s > 1 \) there is a positive number \( c \) determined by \( N, s, \Omega \), and the Hölder continuity of \( \mathbf{m} \) such that

\[
\| \nabla p \|_{s, \Omega} \leq c\| S \|_{\frac{s}{s-1}, \Omega}.
\]

(1.9)

This, in turn, will further improve the regularity of \( (\mathbf{m}, p) \). In fact, one can infer from the results in [4, 14] that the system (1.1)-(1.2) is satisfied a.e on \( \Omega_T \). We shall not elaborate.

Our main result is:

**Theorem 1.1.** Let (H1)-(H3) be satisfied. Then there is a positive number \( T_{\text{max}} \) such that problem (1.1)-(1.4) has a strong solution in \( \Omega_T \) for each \( T < T_{\text{max}} \). The number \( T_{\text{max}} \) is the unique solution
of the equation

\[(1.10) \quad \left( \sum_{i=1}^{7} T_{\max}^a ||S||^{b_i}^{\frac{1}{2(N+4q)}} \right)^s \sum_{i=8}^{19} T_{\max}^a ||S||^{b_i}^{\frac{1}{2(N+4q)}} ||\nabla m_0||^{c_i}_{\infty,\Omega} = d_0,\]

where all the exponents \(s > 0, a_i > 0, b_i \geq 0, c_i \geq 0\) for \(i = 1, \cdots, 19\) are determined by \(N, q, \gamma\) only, while \(d_0 > 0\) also depends on \(\Omega\) and the physical parameters \(D, E\) in the problem in addition to \(N, q, \gamma\).

This theorem immediately asserts the local existence of a strong solution. Obviously, the smaller \(||S||^{\frac{1}{2(N+4q)}}\) and \(||\nabla m_0||_{\infty,\Omega}\) are, the longer the life span of such a strong solution is. It is a refinement of Theorem 1.8 in [20]. However, the theorem does not say if the blow-up does occur at \(T_{\max}\). Thus, \(T_{\max}\) only serves as a lower bound for the possible blow-up time.

To describe the mathematical difficulty involved in our problem, first notice the term \((m \cdot \nabla p) \nabla p\) in (1.2) represents a cubic non-linearity, which is still not well-understood. Second, the elliptic coefficients in (1.1) satisfy

\[(1.11) \quad |\xi|^2 \leq (I + m \otimes m) \xi \cdot \xi \leq (1 + |m|^2)|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^N.\]

This allows the possibility that the coefficient matrix oscillate wildly among different eigen-directions unless \(m\) is bounded, which is not known a priori. However, existing results for degenerate and/or singular elliptic equations such as these in [11] often require that the largest eigenvalue \(\lambda_l\) and the smallest eigenvalue \(\lambda_s\) of the coefficient matrix satisfy

\[\lambda_l \leq c \lambda_s.\]

Here and in what follows the letter \(c\) denotes a generic positive number. Thus our problem does not fit into the classical framework.

To gain some insights into our problem, we take a look at the one-dimensional case

\[(1.12) \quad -(1 + m^2)p_x |_{x} = S(x) \text{ in } (0, 1) \times (0, \infty),\]

\[(1.13) \quad m_t - D^2 m_{xx} + |m|^{2(\gamma - 1)} m = E^2 mp_x^2 \text{ in } (0, 1) \times (0, \infty),\]

\[(1.14) \quad p(0, t) = p(1, t) = 0 \text{ on } (0, \infty),\]

\[(1.15) \quad m(0, t) = m(1, t) = 0 \text{ on } (0, \infty),\]

\[(1.16) \quad m(x, 0) = m_0(x) \text{ on } (0, 1).\]

According to Rolle’s theorem, for each \(t \in (0, \infty)\) there is an \(x^*(t) \in (0, 1)\) such that

\[p_x(x^*(t), t) = 0.\]

For \(x \in (0, 1)\) we integrate (1.12) over \((x^*(t), x)\) to derive

\[(1.17) \quad p_x = -\frac{1}{1 + m^2} \int_{x^*(t)}^{x} S(y)dy.\]

Subsequently,

\[||p_x||_{\infty} \leq ||S||_1.\]

Substitute (1.17) into (1.13) to obtain

\[m_t - D^2 m_{xx} + |m|^{2(\gamma - 1)} m = \frac{E^2 m}{(1 + m^2)^2} \left( \int_{x^*(t)}^{x} S(y)dy \right)^2.\]
We multiply through the above equation by \( m_t \) and integrate to obtain

\[
\int_0^1 m_t^2 \, dt + \frac{D^2}{2} \int_0^1 m_t^2 \, dt + \frac{1}{2\gamma} \int_0^1 m_t^2 \, dt = E^2 \int_0^1 \left( \frac{mm_t}{1 + m^2} \right)^2 \left( \int_{x^*(t)}^x S(y) \, dy \right)^2 \, dx
\]

\[
\leq E^2 \|S\|_1^2 \int_0^1 \frac{m^2}{(1 + m^2)^2} \, dx.
\]

This implies that blow-up in \( m \) does not occur. The key here is (1.17), which is not available in high space dimensions. To seek a substitute, we have developed the following theorem which seems to be of interest on its own right.

**Theorem 1.2.** Assume that (H3) holds and

\[(1.18) \quad w \in \left( W^{1,\ell}(\Omega) \right)^N \text{ for some } \ell > N.\]

Let \( p \) be the solution of the boundary value problem

\[
-\text{div} \left[ (I + w(x) \otimes w(x)) \nabla p \right] = S(x) \text{ in } \Omega, \tag{1.19}
\]

\[
p = 0 \text{ on } \partial \Omega, \tag{1.20}
\]

where \( S \) is a given function in a suitable function space. For each

\[(1.21) \quad q > \frac{N}{N-1},\]

there is a positive number \( c = (N, \Omega, q, \ell) \) such that

\[
\|\nabla p\|_{q, \Omega} \leq c \left( 1 + \|w\|_{W^{1,\ell}(\Omega)}^N \right)^{s_1} \left( \|\nabla p\|_{1, \Omega} + \|S\|_{\frac{Nq}{N+q}, \Omega} \right), \tag{1.22}
\]

where \( s_1 = \frac{5N(2\ell-N+N\ell)(Nq-N)}{q(\ell-N)}.\)

The advantage of this theorem over the classical result (1.9) is that it gives the explicit dependence of the upper bound on the coefficient matrix. This is very crucial to our applications. Condition (1.21) is to ensure that \( \frac{Nq}{N+q} > 1 \). When \( N = 2 \), a version of (1.22) was obtained in [19] by deriving an equation for the term \( (I + w \otimes w) \nabla p \cdot \nabla p \). Unfortunately, this technique only works for \( N = 2 \).

Our approach here is largely inspired by the papers [3, 4, 6].

The rest of the paper is organized as follows. A refinement of a classical uniform bound for solutions to a linear parabolic equation is given in Section 2. The proof of Theorem 1.2 is presented in Section 3. Our main theorem is established in Section 4.

Finally, we remark that unless stated otherwise our generic constant \( c \) depends only on \( N, q, \Omega \) and the three physical parameters \( D, E, \gamma \) in the problem. In particular, it does not depend on \( T, \mathbf{m}_0(x), \) or \( S(x) \). The following two inequalities are frequently used without acknowledgment:

\[
(1.23) \quad (a + b)^\alpha \leq a^\alpha + b^\alpha \text{ for } a \geq 0, b \geq 0, \alpha \in (0, 1),
\]

\[
(1.24) \quad (a + b)^\alpha \leq 2^{\alpha-1} (a^\alpha + b^\alpha) \text{ for } a \geq 0, b \geq 0, \alpha \geq 1.
\]

2. Preliminary results

In this section, we collect a few relevant known results. The first lemma contains some elementary inequalities whose proof can be found in ([17], p. 146-148).

**Lemma 2.1.** Let \( x, y \) be any two vectors in \( \mathbb{R}^N \). Then:
(i) For $\gamma \geq 1$,
\[
(\langle |x|^{2\gamma-2} - |y|^{2\gamma-2} \rangle \cdot (x-y)) \geq \frac{1}{2^{2\gamma-1}} |x-y|^{2\gamma};
\]
(ii) For $\frac{1}{2} < \gamma \leq 1$,
\[
(|x| + |y|)^{2-2\gamma} \left(\langle |x|^{2\gamma-2} - |y|^{2\gamma-2} \rangle \cdot (x-y)\right) \geq (2\gamma - 1)|x-y|^2.
\]

The next lemma plays a central role in our main result.

**Lemma 2.2.** Let $h(\tau)$ be a continuous non-negative function defined on $[0, T_0]$ for some $T_0 > 0$. Suppose that there exist three positive numbers $\varepsilon, \delta, b$ such that
\[
(2.1) \quad h(\tau) \leq \varepsilon h^{1+\delta}(\tau) + b \quad \text{for each } \tau \in [0, T_0].
\]
Then
\[
(2.2) \quad h(\tau) \leq \frac{1}{\varepsilon(1+\delta)} h_0 \quad \text{for each } \tau \in [0, T_0]
\]
whenever
\[
(2.3) \quad \varepsilon \leq \frac{\delta^\delta}{(b + \delta)^\delta (1+\delta)^{1+\delta}} \quad \text{and} \quad h(0) \leq h_0.
\]

We will use the proof of this lemma which is given in [21].

The following lemma can be found in ([5], p.12).

**Lemma 2.3.** Let $\{y_n\}, n = 0, 1, 2, \cdots$, be a sequence of positive numbers satisfying the recursive inequalities
\[
y_{n+1} \leq cb^n y_n^{1+\alpha} \quad \text{for some } b > 1, c, \alpha \in (0, \infty).
\]
If
\[
y_0 \leq c^{-\frac{1}{\alpha}} b^{-\frac{1}{\alpha}},
\]
then $\lim_{n \to \infty} y_n = 0$.

**Lemma 2.4.** Let $u$ be a sub-solution of the problem
\[
(2.3) \quad \partial_t u - D^2 \Delta u = gu + f + \text{div} g \quad \text{in } \Omega_T,
\]
\[
(2.4) \quad u = 0 \quad \text{on } \Sigma_T,
\]
\[
(2.5) \quad u = u_0 \quad \text{on } \Omega,
\]
where $u_0, g, f, g$ are given functions in suitable function spaces. Then for each
\[
(2.6) \quad q > 1 + \frac{N}{2}
\]
there exists a positive number $c = c(D, N, \Omega, q)$ such that
\[
(2.7) \quad \sup_{\Omega_T} u \leq 2 \sup_{\Omega} u_0 + c \left(\left\|g\right\|_{q, \Omega_T}^{\left(\frac{q-1}{2q}\right)n} + 1\right) \left\|u_0^+\right\|_{\frac{2q}{q-1}, \Omega_T}^{\frac{2q}{q-1}} + \left\|f\right\|_{q, \Omega_T}^{\frac{1}{2q}} + \left\|g\right\|_{2q, \Omega_T},
\]
where
\[
(2.8) \quad s_0 = \frac{q(N+2)}{2q - N - 2}.
\]

This lemma is essentially known. The interest here lies in the fact that it gives the precise dependence of the uniform up bound on $T$ and the given functions. This is very important to our late development.
Then we easily see from Young’s inequality (2.9)
\[ \frac{1}{2} \sup_{0 \leq t \leq T} \int_{\Omega} [(u - k_{n+1})^+]^2 \, dx + D^2 \int_{\Omega_T} |\nabla (u - k_{n+1})^+]^2 \, dx \, dt \]
be selected as below. Define
\[ k_n = k - \frac{k}{2^{n+1}}, \quad n = 0, 1, 2, \ldots. \]
Use \((u - k_{n+1})^+\) as a test function in (2.3) and use the fact that \((u - k_{n+1})^+|_{t=0} = 0\) to get
\[ \frac{1}{2} \sup_{0 \leq t \leq T} \int_{\Omega} [(u - k_{n+1})^+]^2 \, dx + D^2 \int_{\Omega_T} |\nabla (u - k_{n+1})^+]^2 \, dx \, dt \]
(2.10) \leq 2 \int_{\Omega_T} gu(u - k_{n+1})^+ \, dx \, dt + 2 \int_{\Omega_T} f(u - k_{n+1})^+ \, dx \, dt - 2 \int_{\Omega_T} g \cdot \nabla (u - k_{n+1})^+ \, dx \, dt.
Set
\[ Q_n = \{(x, t) \in \Omega_T : u(x, t) \geq k_n\}. \]
Then we easily see from Young’s inequality ([8], p.145) that
\[ \left| \int_{\Omega_T} g \cdot \nabla (u - k_{n+1})^+ \, dx \, dt \right| \leq \varepsilon \int_{\Omega_T} |\nabla (u - k_{n+1})^+]^2 \, dx \, dt + \frac{c}{\varepsilon} \|g\|_{2, Q_n+1}^2, \quad \varepsilon > 0. \]
By Hölder’s inequality,
\[ \left| \int_{\Omega_T} f(u - k_{n+1})^+ \, dx \, dt \right| \leq \|f\|_{\frac{2(N+2)}{N+4}, \Omega_n+1} \|(u - k_{n+1})^+\|_{\frac{2(N+2)}{N}, \Omega_T}. \]
Note that
\[ [(u - k_n)^+]^2 \geq (u - k_n)^+(u - k_{n+1})^+ \geq u \left(1 - \frac{k_n}{k_{n+1}}\right) (u - k_{n+1})^+ \geq \frac{1}{2^{n+2}} u(u - k_{n+1})^+. \]
Consequently,
\[ \left| \int_{\Omega_T} gu(u - k_{n+1})^+ \, dx \, dt \right| \leq 2^{n+2} \int_{\Omega_T} |g| [(u - k_n)^+]^2 \, dx \, dt \]
(2.13)
\[ \leq 2^{n+2} \|g\|_{q, \Omega_T} y_n^{\frac{q-1}{2}}, \]
where
\[ y_n = \int_{\Omega_T} [(u - k_n)^+]^{\frac{2q}{q-1}} \, dx \, dt. \]
Substitute (2.11), (2.12), and (2.13) into (2.10) and choose \(\varepsilon\) suitably small in the resulting inequality to derive
\[ \sup_{0 \leq t \leq T} \int_{\Omega} [(u - k_{n+1})^+]^2 \, dx + \int_{\Omega_T} |\nabla (u - k_{n+1})^+]^2 \, dx \, dt \]
(2.14)
\[ \leq c2^n\|g\|_{q, \Omega_T} y_n^{\frac{q-1}{2}} + c\|f\|_{\frac{2(N+2)}{N+4}, Q_n+1} \|(u - k_{n+1})^+\|_{\frac{2(N+2)}{N}, \Omega_T} + c\|g\|_{2, Q_n+1}^2. \]
Consequently, the last two terms in the above inequality can be estimated as follows:

\[
\int_{\Omega_T} \left[ (u - k_{n+1})^+ \right]^{2 + \frac{4}{N}} \, dt \\
\leq \int_0^T \left( \int_\Omega \left[ (u - k_{n+1})^+ \right]^{2} \, dx \right)^{\frac{2}{N}} \left( \int_\Omega \left[ (u - k_{n+1})^+ \right]^{\frac{2N}{N-2}} \, dx \right)^{\frac{N-2}{2N}} \, \frac{dt}{T}
\]

\[
\leq c \left( \sup_{0 \leq t \leq T} \int_\Omega \left[ (u - k_{n+1})^+ \right]^{2} \, dx \right)^{\frac{2}{N}} \int_{\Omega_T} \left| \nabla (u - k_{n+1})^+ \right|^2 \, dx dt
\]

\[
\leq c \left(2^n \|g\|_{q, \Omega_T, y_n}^q + \left( \int_{\Omega_T} \left[ (u - k_{n+1})^+ \right]^{\frac{2N}{N-2}} \, dx \right)^{\frac{N-2}{2N}} \|g\|_{2, q, \Omega_T}^q \right)^{\frac{N+2}{N}}
\]

\[
\leq c 2^{\frac{n(N+2)}{N}} \|g\|_{q, \Omega_T, y_n}^q \left[ \left( \frac{q-1}{q} \right)^{\frac{N+2}{N}} \right]^{\frac{N+2}{N}} \int_{\Omega_T} \left[ (u - k_{n+1})^+ \right]^{2} \, dx dt
\]

\[
\leq \varepsilon \int_{\Omega_T} \left[ (u - k_{n+1})^+ \right]^{2 + \frac{4}{N}} \, dx dt + c \varepsilon \left( \int_{\Omega_T} \left[ (u - k_{n+1})^+ \right]^{\frac{2(N+2)}{N-2}} \, dx \right)^{\frac{N-2}{2N}} \|g\|_{2, q, \Omega_T}^2
\]

\[
+ c 2^{\frac{n(N+2)}{N}} \|g\|_{q, \Omega_T, y_n}^q \left[ \left( \frac{q-1}{q} \right)^{\frac{N+2}{N}} \right]^{\frac{N+2}{N}}
\]

(2.15) \quad \varepsilon > 0.

Consequently,

(2.16) \quad \int_{\Omega_T} \left[ (u - k_{n+1})^+ \right]^{2 + \frac{4}{N}} \, dx dt \leq c 2^{\frac{n(N+2)}{N}} \|g\|_{q, \Omega_T, y_n}^q \left[ \left( \frac{q-1}{q} \right)^{\frac{N+2}{N}} \right]^{\frac{N+2}{N}} + c \varepsilon \left( \int_{\Omega_T} \left[ (u - k_{n+1})^+ \right]^{\frac{2(N+2)}{N-2}} \, dx \right)^{\frac{N-2}{2N}} \|g\|_{2, q, \Omega_T}^2.

The last two terms in the above inequality can be estimated as follows:

\[
\|f\|_{2, q, \Omega_T}^{2(N+2)/N} = \left( \int_{Q_{n+1}} \left| f \right|^{2(N+2)/N} \, dx dt \right)^{N+2}/N \leq \|f\|_{q, \Omega_T}^{2(N+2)/N} |Q_{n+1}|^{N+2}/N, \quad N, q, \Omega_T
\]

(2.17) \quad \|g\|_{2, q, \Omega_T}^{2(N+2)/N} \leq \|g\|_{2, q, \Omega_T}^{2(N+2)/N} |Q_{n+1}|^{N+2}/N^q, \quad N, q, \Omega_T

Use this in (2.16) to obtain

\[
\int_{\Omega_T} \left[ (u - k_{n+1})^+ \right]^{2 + \frac{4}{N}} \, dx dt \leq \quad c 2^{\frac{n(N+2)}{N}} \|g\|_{q, \Omega_T, y_n}^q \left[ \left( \frac{q-1}{q} \right)^{\frac{N+2}{N}} \right]^{\frac{N+2}{N}} + c \varepsilon \left( \int_{\Omega_T} \left[ (u - k_{n+1})^+ \right]^{\frac{2(N+2)}{N-2}} \, dx \right)^{\frac{N-2}{2N}} \|g\|_{2, q, \Omega_T}^2.
\]

(2.18) \quad + c \|g\|_{2, q, \Omega_T}^{2(N+2)/N} \left( \frac{N+2}{N} \right)^{N+2}/N^q.

By (2.6),

\[
\frac{q}{q - 1} < \frac{N + 2}{N}.
\]
This together with (2.18) implies
\[
y_{n+1} = \int_{\Omega_T} \left[ (u - k_{n+1})^+ \right]^{2q} \frac{2q}{q-1} \, dx \, dt
\]
\[
\leq \left( \int_{\Omega_T} \left[ (u - k_{n+1})^+ \right]^{2q} \frac{2q}{q-1} \, dx \, dt \right)^{\frac{qN}{(q-1)(N+2)}} |Q_{n+1}|^{1-\frac{qN}{(q-1)(N+2)}}
\]
\[
\leq c 2^{\frac{2q}{q-1}} \|g\|_{\frac{q}{q-1} \Omega_T}^q y_n |Q_{n+1}|^{1-\frac{qN}{(q-1)(N+2)}}
\]
\[
+ c \|f\|_{\frac{2q}{q-1} \Omega_T}^q |Q_{n+1}|^{(N+2)q-2(N+2)} (q-1)(N+2)^{+1-\frac{qN}{(q-1)(N+2)}}
\]
\[
+ c \|g\|_{\frac{2q}{q-1} \Omega_T}^q |Q_{n+1}|^{(N+2)(q-N-2)} (q-1)(N+2)^{+1-\frac{qN}{(q-1)(N+2)}}
\]
\[
= c 2^{\frac{2q}{q-1}} \|g\|_{\frac{q}{q-1} \Omega_T}^q |Q_{n+1}|^{\alpha} + c \|f\|_{\frac{2q}{q-1} \Omega_T}^q |Q_{n+1}|^{1+2\alpha} + c \|g\|_{\frac{2q}{q-1} \Omega_T}^q |Q_{n+1}|^{1+\alpha},
\]
where
\[
\alpha = \frac{2q - N - 2}{(q-1)(N+2)} = \frac{q}{(q-1)s_0} > 0.
\]
We easily see that
\[
y_n \geq \int_{Q_{n+1}} (k_{n+1} - k_n)^{\frac{2q}{q-1}} \, dx = \frac{k_{n+1}^{\frac{2q}{q-1}}}{2^{\frac{2q}{q-1}}} |Q_{n+1}|.
\]
Therefore,
\[
|Q_{n+1}|^\alpha \leq \frac{2^{\frac{2q}{q-1}(n+2)\alpha}}{k_{n+1}^{\frac{2q}{q-1}}} y_n^\alpha,
\]
\[
|Q_{n+1}|^{1+\alpha} \leq \frac{2^{\frac{2q}{q-1}(n+2)(1+\alpha)}}{k_{n+1}^{\frac{2q(1+\alpha)}{q-1}}} y_n^{1+\alpha},
\]
\[
|Q_{n+1}|^{1+2\alpha} \leq |Q_{n+1}|^{1+\alpha} |\Omega_T|^\alpha \leq \frac{2^{\frac{2q}{q-1}(n+2)(1+\alpha)}}{k_{n+1}^{\frac{2q}{q-1}(1+\alpha)}} |\Omega_T|^\alpha y_n^{1+\alpha}.
\]
Collecting the preceding three estimates in (2.19) to get
\[
y_{n+1} \leq \frac{c 2^{\frac{2q}{q-1}(n+1)\alpha}}{k_{n+1}^{\frac{2q}{q-1}}} \|g\|_{\frac{q}{q-1} \Omega_T}^q y_n^{1+\alpha}
\]
\[
+ \frac{c 2^{\frac{2q}{q-1}(n+1)\alpha}}{k_{n+1}^{\frac{2q}{q-1}(1+\alpha)}} \left( \|f\|_{\frac{2q}{q-1} \Omega_T}^q T^{\alpha} + \|g\|_{\frac{2q}{q-1} \Omega_T}^q \right) y_n^{1+\alpha}.
\]
We choose \(k\) so large that
\[
\frac{\|f\|_{\frac{2q}{q-1} \Omega_T}^q T^{\alpha} + \|g\|_{\frac{2q}{q-1} \Omega_T}^q}{k_{n+1}^{\frac{2q}{q-1}}} \leq 1.
\]
Use this in (2.24) to get
\[
y_{n+1} \leq \frac{c 2^{\frac{2q}{q-1}(n+1)\alpha}}{k_{n+1}^{\frac{2q}{q-1}}} \left( \|g\|_{\frac{q}{q-1} \Omega_T}^q + 1 \right) y_n^{1+\alpha}
\]
According to Lemma 2.3, if we further require $k$ to satisfy

$$y_0 \leq c \left( \frac{\frac{2q}{q-1}}{c \left( \|g\|_{q,\Omega_T} + 1 \right)} \right)^{\frac{1}{\alpha}},$$

then

$$\text{(2.27)} \sup_{\Omega_T} u \leq k.$$

In view of (2.25) and (2.9), it is enough for us to take

$$\text{(2.28)} \quad k = 2 \sup_{\Omega} u_0 + c \left( \|g\|_{\frac{q}{q-1},\Omega_T} + 1 \right) \frac{\frac{q-1}{q}}{y_0} + \|f\|_{q,\Omega_T} T^{\frac{1}{q-1}} + \|g\|_{2q,\Omega_T}.$$

Note that

$$\frac{\frac{q-1}{q}}{y_0} \leq \left( \int_{\Omega_T} (u^+) \frac{q-1}{q} \right)^{\frac{1}{\alpha}} = \|u^+\|_{\frac{2q}{q-1},\Omega_T}.$$

This combined with (2.27) and (2.20) gives (2.7). The proof is complete. \(\square\)

If

$$\text{sup}_{\Omega_T} u = \|u\|_{\infty,\Omega_T},$$

we can apply the interpolation inequality ([8], p.146) to obtain

$$\|u^+\|_{\frac{2q}{q-1},\Omega_T} \leq \|u\|_{\frac{2q}{q-1},\Omega_T} \leq \varepsilon \|u\|_{\infty,\Omega_T} + \frac{1}{\varepsilon^{\frac{q-1}{q}}} \|u\|_{1,\Omega_T}, \quad \varepsilon > 0.$$

Plug this into (2.7) to get

$$\|u\|_{\infty,\Omega_T} \leq c \left( \|g\|_{\frac{q}{q-1},\Omega_T} + 1 \right) \left( \varepsilon \|u\|_{\infty,\Omega_T} + \frac{1}{\varepsilon^{\frac{q-1}{q}}} \|u\|_{1,\Omega_T} \right)$$

$$+ 2 \sup_{\Omega} u_0 + \|f\|_{q,\Omega_T} T^{\frac{1}{q-1}} + \|g\|_{2q,\Omega_T}.$$

Take $\varepsilon$ so that the coefficient of the term $\|u\|_{\infty,\Omega_T}$ on the right hand side of the above inequality

$$c \left( \|g\|_{\frac{q}{q-1},\Omega_T} + 1 \right) \varepsilon = \frac{1}{2}$$

to drive

$$\|u\|_{\infty,\Omega_T} \leq 4 \sup_{\Omega} u_0 + c \left( \|g\|_{\frac{q}{q-1},\Omega_T} + 1 \right) \|u\|_{1,\Omega_T} + 2 \|f\|_{q,\Omega_T} T^{\frac{1}{q-1}} + 2 \|g\|_{2q,\Omega_T}.$$

Here we have used (2.20).

**Lemma 2.5.** Assume that (H3) holds. Let $u$ be the solution of the problem

$$\text{(2.30)} \quad \partial_t u - D^2 \Delta u = f \quad \text{in} \ \Omega_T,$$

$$\text{(2.31)} \quad u = 0 \quad \text{on} \ \Sigma_T,$$

$$\text{(2.32)} \quad u = u_0 \quad \text{on} \ \Omega,$$

where

$$u_0 \in W^{1,\infty}_0(\Omega), \quad f \in L^{2q}(\Omega_T) \quad \text{for some} \ q > 1 + \frac{N}{2}.$$

Then there is a positive number $c = c(\Omega, N, q)$ such that

$$\text{(2.33)} \quad \|\nabla u\|_{\infty,\Omega_T} \leq c \|\nabla u_0\|_{\infty,\Omega} + c \|f\|_{2q,\Omega_T}.$$
This lemma is known. In fact, it is not difficult for us to see (2.34). Indeed, \( u \) satisfies (2.3) with \( f \) being replaced by \( f_x \) and \( g, g \) being 0. A full proof can be inferred from Proposition 2.3 in [21].

3. \( W^{1,q} \) estimates for elliptic equations

Before we prove Theorem 1.2, we recall some results from [3].

**Definition 3.1.** A function \( k(x) \) on \( \mathbb{R}^N \setminus \{0\} \) is called a Calderón-Zygmund kernel (in short, C-Z kernel) if:

(i) \( k \in C^\infty(\mathbb{R}^N \setminus \{0\}) \);
(ii) \( k(x) \) is homogeneous of degree \(-N\), i.e., \( k(tx) = t^{-N}k(x) \);
(iii) \( \int_{\partial B_1(0)} k(x) d\mathcal{H}^{N-1} = 0 \).

The most fundamental result concerning C-Z kernels [3] is the following

**Lemma 3.2.** Given a C-Z kernel \( k(x) \), we define

\[
K_\varepsilon f(x) = \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} k(x - y)f(y)dy \quad \text{for } \varepsilon > 0 \quad \text{and} \quad f \in L^q(\mathbb{R}^N) \quad \text{with} \quad q \in (1, \infty).
\]

Then:

(CZ1) For each \( f \in L^q(\mathbb{R}^N) \) there exists a function \( Kf \in L^q(\mathbb{R}^N) \) such that

\[
\lim_{\varepsilon \to 0} \|K_\varepsilon f - Kf\|_{q,\mathbb{R}^N} = 0.
\]

In this case we use the notation

\[
Kf(x) = \text{P.V.} k \ast f(x) = \text{P.V.} \int_{\mathbb{R}^N} k(x - y)f(y)dy.
\]

(CZ2) \( K \) is a bounded operator on \( L^q(\mathbb{R}^N) \). More precisely, we have

\[
\|Kf\|_{q,\mathbb{R}^N} \leq c \left( \int_{\partial B_1(0)} k^2(x) d\mathcal{H}^{N-1} \right)^{\frac{1}{2}} \|f\|_{q,\mathbb{R}^N},
\]

where the positive number \( c \) depends only on \( N, q \).

We are ready to prove Theorem 1.2.

**Proof of Theorem 1.2.** As in [3, 6], the proof comprises a local interior estimate and a boundary estimate. To establish the former, we fix \( x_0 \in \Omega \). Let \( 0 < \delta < r \) with \( B_r(x_0) \subseteq \Omega \). Pick a smooth cutoff function \( \xi \in C^\infty_0(\mathbb{R}^N) \) such that

\[
\xi = 1 \quad \text{on} \quad B_\delta(x_0), \quad \xi = 0 \quad \text{outside} \quad B_r(x_0), \quad 0 \leq \xi \leq 1 \quad \text{on} \quad B_r(x_0), \quad |\nabla \xi| \leq \frac{c}{r - \delta} \quad \text{on} \quad B_r(x_0).
\]

Set

\[
u = p\xi.
\]

We can easily verify that \( u \) satisfies the equation

\[
-\text{div} [(I + w(x) \otimes w(x))\nabla u] = -\text{div} [p(I + w(x) \otimes w(x))\nabla \xi] + F \quad \text{in} \quad \mathbb{R}^N,
\]

where

\[
F = \xi S(x) - \nabla \xi (I + w(x) \otimes w(x))\nabla p.
\]
Set

\[ A(x_0) = I + \int_{B_r(x_0)} w(x) dx \otimes \int_{B_r(x_0)} w(x) dx. \]

Then we can write the above equation in the form

\[ -\text{div}[A(x_0) \nabla u(x)] = \text{div} \left( (w(x) - \int_{B_r(x_0)} w(x) dx) \cdot \nabla u(x) \right) \nabla u(x) \]

\[ + \text{div} \left( \int_{B_r(x_0)} w(x) dx \cdot \nabla u(x) \right) (w(x) - \int_{B_r(x_0)} w(x) dx) \]

\[ - \text{div} [p(x)(I + w(x) \otimes w(x)) \nabla \xi(x)] + F(x), \]

(3.4)

Recall that the fundamental solution of the equation \(-\text{div}[A(x_0) \nabla v(x)] = 0\) is given by

\[ \Gamma(x_0, x) = \frac{1}{(N-2)\omega_N \sqrt{\det A(x_0)}} \left( \sum_{i,j=1}^{N} A_{ij}(x) x_i x_j \right)^{\frac{2-N}{2}}, \]

where \(A_{ij}(x_0)\) is the co-factor of the entry that lies in the \(i\)th row and the \(j\)th column in the matrix \(A(x_0)\) and \(\omega_N\) is the surface area of the unit sphere. Then we have the frequently used representation formula

\[ u(y) = -\int_{B_r(x_0)} \Gamma(x_0, x-y) \text{div}[A(x_0) \nabla u(x)] dx \]

whenever \(u\) is compactly supported in \(B_r(x_0)\). We can easily verify that \(\frac{\partial^2}{\partial x_i \partial x_j} \Gamma(x_0, x), i,j = 1, \cdots, N\) are C-Z kernels [3, 6]. Our key observation is the following

**Lemma 3.3.** There is a positive number \(c = c(N)\) such that

\[ \left| \frac{\partial^2 \Gamma(x_0, x)}{\partial x_i \partial x_j} \right| \leq c \left( 1 + \left| \int_{B_r(x_0)} w(x) dx \right|^2 \right)^{\frac{2-N}{2}} \leq c(1 + \|w\|_{\infty, \Omega})^{5N-2} \text{ for } x \in \partial B_1(0). \]

**Proof.** Obviously,

\[ |y|^2 \leq \left( I + \int_{B_r(x_0)} w(x) dx \otimes \int_{B_r(x_0)} w(x) dx \right) y \cdot y \leq \left( 1 + \left| \int_{B_r(x_0)} w(x) dx \right|^2 \right) |y|^2 \text{ for all } y \in \mathbb{R}^N. \]

This implies

\[ \text{the largest eigenvalue of } A(x_0) \leq 1 + \left| \int_{B_r(x_0)} w(x) dx \right|^2, \]

(3.7)

\[ \text{the smallest eigenvalue of } A(x_0) \geq 1. \]

(3.8)

Hence,

\[ \det A(x_0) = \text{the product of the eigenvalues, counted with multiplicity} \]

\[ \in \left( 1, \left( 1 + \left| \int_{B_r(x_0)} w(x) dx \right|^2 \right)^2 \right)^{N}. \]

(3.9)
We calculate
\[
\frac{\partial^2_{x_kx_x} \Gamma(x_0, x)}{\partial x_k \Gamma(x_0, x)} = -\frac{\sum_{i=1}^{N} (A_{ik}(x_0) + A_{kj}(x_0)) x_i}{2w_N \sqrt{\det A(x_0)} \left( \sum_{i,j=1}^{N} A_{ij}(x_0)x_ix_j \right)^{\frac{3}{2}}},
\]
\[
\frac{\partial^2_{x_kx_x} \Gamma(x_0, x)}{\partial x_k \Gamma(x_0, x)} = -\frac{A_{ik}(x_0) + A_{kj}(x_0)}{2w_N \sqrt{\det A(x_0)} \left( \sum_{i,j=1}^{N} A_{ij}(x_0)x_ix_j \right)^{\frac{3}{2}}}
\]
(3.10)
\[
\sum_{i,j=1}^{N} A_{ij}(x_0) x_i x_j = \det A(x_0) A^{-1}(x_0) x \cdot x.
\]
(3.11)

Note that
\[
A^{-1}(x_0) = \frac{1}{\det A(x_0)} \begin{pmatrix} A_{11} & \cdots & A_{N1} \\ \vdots & \ddots & \vdots \\ A_{1N} & \cdots & A_{NN} \end{pmatrix}.
\]

It is easy to see that
\[
\sum_{i,j=1}^{N} A_{ij}(x_0) x_i x_j = \det A(x_0) A^{-1}(x_0) x \cdot x.
\]

Recall that \( \lambda \) is an eigenvalue of an invertible matrix \( A \) if and only if \( \frac{1}{\lambda} \) is an eigenvalue of \( A^{-1} \). With this in mind, we derive from (3.7), (3.8), and (3.9) that
\[
\sum_{i,j=1}^{N} A_{ij}(x_0) x_i x_j \leq \det A(x_0) |x|^2 \leq \left( 1 + \left| \int_{B_r(x_0)} w(x) \ dx \right| \right)^2 |x|^2,
\]
(3.12)
\[
\sum_{i,j=1}^{N} A_{ij}(x_0) x_i x_j \geq \frac{\det A(x_0)}{1 + \left| \int_{B_r(x_0)} w(x) \ dx \right|^2} |x|^2 \geq \frac{1}{1 + \left| \int_{B_r(x_0)} w(x) \ dx \right|^2} |x|^2.
\]

Remember that the determinant of an \( n \times n \) matrix \( A \) is the signed sum over all possible products of \( n \) entries of \( A \) with exactly one entry being selected from each row and from each column of \( A \). Thus,
\[
|A_{ij}(x_0)| \leq (N - 1)! \left( 1 + \left| \int_{B_r(x_0)} w(x) \ dx \right| \right)^2 |x|^{N-1}.
\]
(3.13)

Here we have used the fact that each entry in \( A(x_0) \) is bounded by \( 1 + \left| \int_{B_r(x_0)} w(x) \ dx \right|^2 \). We are ready to estimate for \( x \in \partial B_1(0) \) that
\[
|\frac{\partial^2_{x_kx_x} \Gamma(x_0, x)}{|\partial x_k \Gamma(x_0, x)}| \leq c \left( 1 + \left| \int_{B_r(x_0)} w(x) \ dx \right| \right)^2 \left( \frac{N+2}{2} \right)^{N-1}.
\]
(3.14)

The proof is complete. \( \square \)
Return to the proof of Lemma 1.2. Since \( u \) is compactly supported in \( B_r(x_0) \), we have from (3.5) that
\[
\begin{align*}
\mathcal{L}(y) &= \int_{B_r(x_0)} \left( (w(x) - \frac{f}{B_{r_0}} w(x) dx \right) \cdot \nabla u(x) w(x) \cdot \nabla \Delta \Gamma(x_0, y - x) dx \\
&\quad + \int_{B_r(x_0)} \left( f w(x) dx \cdot \nabla u(x) \right) w(x) \cdot \nabla \Delta \Gamma(x_0, y - x) dx \\
&\quad + \int_{B_r(x_0)} p(x) (I + \Delta w(x) \otimes w(x)) \nabla \xi(x) \cdot \nabla \Delta \Gamma(x_0, y - x) dx \\
&\quad + \int_{B_r(x_0)} F(x) \Delta \Gamma(x_0, y - x) dx.
\end{align*}
\]
(3.15)

Differentiate the above equation with respect to \( y_i \) to obtain
\[
\begin{align*}
u_{y_i}(y) &= -\int_{B_r(x_0)} ((h(x) - m(x)) u_{x_i}(x)) m_j(x) \frac{\partial^2}{\partial x_i \partial x_j} \Gamma(x_0, y - x) dx \\
&\quad - \int_{B_r(x_0)} (m(x) u_{x_i}(x)) (m_j(x) - m_j(x)) \frac{\partial^2}{\partial x_i \partial x_j} \Gamma(x_0, y - x) dx \\
&\quad - \int_{B_r(x_0)} p(x) (I + \Delta w(x) \otimes w(x)) \nabla \xi(x) \cdot \nabla \Delta \Gamma(x_0, y - x) dx \\
&\quad - \int_{B_r(x_0)} F(x) \Delta \Gamma(x_0, y - x) dx.
\end{align*}
\]
(3.16)

According to Morrey’s inequality ([7], p.143), for \( \ell > N \) there is a positive number \( c = c(N, \ell) \) such that
\[
\left| w(x) - \frac{f}{B_{r_0}} w(x) dx \right| \leq cr^{1-N} \norm{\nabla w}_{\ell, B_r(x_0)} \text{ on } B_r(x_0).
\]
(3.17)

With this in mind, we apply (CZ2) to (3.16) to deduce
\[
\begin{align*}
\norm{\nabla u}_{q, B_r(x_0)} &\leq cr^{1-N} \sum_{i,j=1}^N \left\| \frac{\partial^2}{\partial x_i \partial x_j} \Gamma(x_0, x) \right\|_{2, \partial B_1(0)} \left\| w \right\|_{\infty, B_r(x_0)} \left\| \nabla w \right\|_{\infty, B_r(x_0)} \left\| \nabla u \right\|_{q, B_r(x_0)} \\
&\quad + c \sum_{i,j=1}^N \left\| \frac{\partial^2}{\partial x_i \partial x_j} \Gamma(x_0, x) \right\|_{2, \partial B_1(0)} \left( 1 + \left\| w \right\|_{\infty, B_r(x_0)}^2 \right) \left\| p \right\|_{q, B_r(x_0)} \\
&\quad + c \left\| \int_{B_r(x_0)} F(x) \Delta \Gamma(x_0, y - x) dx \right\|_{q, B_r(x_0)} \\
&\leq cr^{1-N} \left( 1 + \left\| w \right\|_{\infty, B_r(x_0)} \right)^{5N-1} \left\| \nabla w \right\|_{\infty, B_r(x_0)} \left\| \nabla u \right\|_{q, B_r(x_0)} \\
&\quad + c \left\| p \right\|_{q, B_r(x_0)} \left( 1 + \left\| w \right\|_{\infty, B_r(x_0)} \right)^{5N} + \left\| \int_{B_r(x_0)} F(x) \Delta \Gamma(x_0, y - x) dx \right\|_{q, B_r(x_0)}.
\end{align*}
\]
(3.18)

The last step is due to (3.14). To estimate the last term in the above inequality, we derive from (3.10), (3.12), and (3.13) that
\[
\left| \Gamma(x_0, y - x) \right| \leq \frac{c \left( 1 + \left| \int_{B_r(x_0)} w(x) dx \right|^2 \right)^{3N-2}}{|x - y|^{N-1}}.
\]
(3.19)
Consequently,
\[
\left| \int_{B_r(x_0)} F(x) \Gamma_{x_i}(x_0, y-x) \, dx \right| 
\leq c \left( 1 + \|w\|_{\infty, B_r(x_0)} \right)^{3N-2} \int_{B_r(x_0)} \frac{|\xi S(x)| + |\nabla \xi| \left( 1 + \|w\|_{\infty, B_r(x_0)}^2 \right) |\nabla p|}{|x-y|^{N-1}} \, dx.
\]

By the remark following Lemma 7.12 in [8], we obtain
\[
\left\| \int_{B_r(x_0)} F(x) \Gamma_{x_i}(x_0, y-x) \, dx \right\|_{q, B_r(x_0)} \leq c \left( 1 + \|w\|_{\infty, B_r(x_0)} \right)^{3N-2} \|S\|_{N+q, B_r(x_0)}^{N_q} + \frac{c \left( 1 + \|w\|_{\infty, B_r(x_0)} \right)^{3N}}{r-\delta} \|\nabla p\|_{N+q, B_r(x_0)}^{N_q}.
\]

Plug this into (3.18) and then use the definition of \(\xi\) to obtain
\[
\|\nabla p\|_{q, B_r(x_0)} \leq c r^{1-\frac{N}{2}}\left( 1 + \|w\|_{\infty, B_r(x_0)} \right)^{5N-1} \|\nabla m\|_{q, B_r(x_0)} \|\nabla p\|_{q, B_r(x_0)}
\]
\[
+ \frac{c r^{1-\frac{N}{2}}}{r-\delta} \left( 1 + \|w\|_{\infty, B_r(x_0)} \right)^{5N-1} \|\nabla m\|_{q, B_r(x_0)} \|p\|_{q, B_r(x_0)}
\]
\[
+ \frac{c}{r-\delta} \left( 1 + \|w\|_{\infty, B_r(x_0)} \right)^{5N} \|p\|_{q, B_r(x_0)} + c \left( 1 + \|w\|_{\infty, B_r(x_0)} \right)^{3N-2} \|S\|_{N+q, B_r(x_0)}^{N_q}.
\]

Set
\[
K_1 = r^{1-\frac{N}{2}}\left( 1 + \|w\|_{\infty, B_r(x_0)} \right)^{5N-1} \|\nabla m\|_{q, B_r(x_0)},
\]
\[
K_2 = \frac{c r^{1-\frac{N}{2}}}{r-\delta} \left( 1 + \|w\|_{\infty, B_r(x_0)} \right)^{5N-1} \|\nabla m\|_{q, B_r(x_0)} \|p\|_{q, B_r(x_0)}
\]
\[
+ \left( 1 + \|w\|_{\infty, B_r(x_0)} \right)^{5N} \|p\|_{q, B_r(x_0)} + \left( 1 + \|w\|_{\infty, B_r(x_0)} \right)^{3N} \|\nabla p\|_{N+q, B_r(x_0)}^{N_q},
\]
\[
K_3 = \left( 1 + \|w\|_{\infty, B_r(x_0)} \right)^{3N-2} \|S\|_{N+q, B_r(x_0)}^{N_q}.
\]

We can write (3.22) as
\[
\|\nabla p\|_{q, B_r(x_0)} \leq c K_1 \|\nabla p\|_{q, B_r(x_0)} + \frac{c}{r-\delta} K_2 + c K_3.
\]

Define
\[
r_n = r - \frac{r}{2^{n+1}}, \quad n = 0, 1, 2, \ldots .
\]

Take \((\delta, r) = (r_n, r_{n+1})\) in (3.26) and keep in mind the fact that \(K_1, K_2,\) and \(K_3\) are all increasing with \(r\) to get
\[
\|\nabla p\|_{q, B_r(x_0)} \leq c K_1 \|\nabla p\|_{q, B_{r+1}(x_0)} + \frac{c}{r_{n+1} - r_n} K_2 + c K_3
\]
\[
\leq c K_1 \|\nabla p\|_{q, B_{r+1}(x_0)} + \frac{c 2^{n+2}}{r} K_2 + c K_3.
\]

By iteration,
\[
\|\nabla p\|_{q, B_r(x_0)} \leq (c K_1)^n \|\nabla p\|_{q, B_{r_n}(x_0)} + \frac{c}{r} K_2 \sum_{i=0}^{n-1} (2c K_1)^i + c K_3 \sum_{i=0}^{n-1} (c K_1)^i.
\]
In view of (3.23), we can take \( r \) so that
(3.29) \[ c2K_1 \leq \frac{1}{2}. \]

Then let \( n \to \infty \) in (3.28) to get
(3.30) \[
\|\nabla p\|_{q,B^2_r(x_0)} \leq \frac{c}{r} K_2 + c K_3 \\
= c r^{-\frac{N}{3}} (1 + \|w\|_{\infty,B^2_r(x_0)}^{5N-1}\|\nabla m\|_{q,B^2_r(x_0)}\|p\|_{q,B^2_r(x_0)} \\
+ \frac{c}{r} (1 + \|w\|_{\infty,B^2_r(x_0)})^{5N}\|p\|_{q,B^2_r(x_0)} + c (1 + \|w\|_{\infty,B^2_r(x_0)})^{3N-2}\|\nabla p\|_{\frac{Nq}{N+q},B^2_r(x_0)}.\]

By virtue of (3.23), for (3.29) to hold, it is enough for us to take
(3.31) \[ c r^{1-\frac{N}{r}} \left(1 + \|w\|_{W^{1,\ell}(\Omega)}\right)^{5N} = \frac{1}{4}. \]

This combined with (3.30) yields
(3.32) \[
\|\nabla p\|_{q,B^2_r(x_0)} \leq c \left(1 + \|w\|_{W^{1,\ell}(\Omega)}\right)^{\frac{5N(2\ell-N)}{t-N}}\|p\|_{q,\Omega} + c \left(1 + \|w\|_{W^{1,\ell}(\Omega)}\right)^{\frac{N(6\ell-3N)}{t-N}}\|\nabla p\|_{\frac{Nq}{N+q},\Omega} \\
+ c \left(1 + \|w\|_{\infty,\Omega}\right)^{3N-2}\|S\|_{\frac{Nq}{N+q},\Omega} + c\left(1 + \|w\|_{W^{1,\ell}(\Omega)}\right)^{\frac{5N(2\ell-N)}{t-N}}\left(\|\nabla p\|_{\frac{Nq}{N+q},\Omega} + \|S\|_{\frac{Nq}{N+q},\Omega}\right)\].

Here we have applied Poincaré’s inequality to \( p \) and the Sobolev embedding theorem to \( w \).

If \( x_0 \in \partial\Omega \), the same estimate still holds with \( B_r(x_0) \) (resp. \( B^2_r(x_0) \)) being replaced by \( B_r(x_0) \cap \Omega \) (resp. \( B^2_r(x_0) \cap \Omega \)). This can be achieved by the classical technique of flattening \( B_r(x_0) \cap \partial\Omega \) and then turning \( x_0 \) into an interior point ([8], p.300). Also see [21] for a rather detailed implementation of the technique. We shall omit it here.

Finally, let \( r \) be determined by (3.31). There is an integer \( j \) with the property
(3.33) \[ j - 1 \leq \frac{\text{diam}(\Omega)}{r} \leq j. \]

We can find at most \((2j)^N\) balls \( \{B_r(x_0)\} \) with
\[ \Omega \subset \bigcup_{i=1}^{(2j)^N} B^2_r(x_0^{(i)}). \]

Consequently,
(3.34) \[
\|\nabla p\|_{q,\Omega} \leq \sum_{i=1}^{(2j)^N} \|\nabla p\|_{q,B^2_r(x_0^{(i)})} \\
\leq c j^N \left(1 + \|w\|_{W^{1,\ell}(\Omega)}\right)^{\frac{5N(2\ell-N)}{t-N}}\left(\|\nabla p\|_{\frac{Nq}{N+q},\Omega} + \|S\|_{\frac{Nq}{N+q},\Omega}\right).\]

Observe from (3.33) and (3.31) that
\[ j^N \leq \frac{c}{r^N} + c \]
\[ \leq c \left(1 + \|\nabla w\|_{W^{1,\ell}(\Omega)}\right)^{\frac{5N^2}{t-N}} + c. \]
Substitute this into (3.34) to obtain
\[ \|\nabla p\|_{q, \Omega} \leq c(1 + \|\nabla w\|_{W^{1, \ell}(\Omega)}) \frac{5N(2\ell - N + N)}{q(\ell - N)} \left( \|\nabla p\|_{Nq, \Omega} + \|S\|_{Nq, \Omega} \right). \]

On account of (1.21) and the interpolation inequality ([8], p.146), we have
\[ \|\nabla p\|_{Nq, \Omega} \leq c \|\nabla p\|_{q, \Omega} + \frac{1}{\varepsilon} \|\nabla p\|^{1_\Omega}, \varepsilon > 0. \]

Plug this into (3.35) and choose \( \varepsilon \) appropriately in the resulting inequality to get
\[ \|\nabla p\|_{q, \Omega} \leq c \left( 1 + \|\nabla w\|_{W^{1, \ell}(\Omega)} \right) \frac{5N(2\ell - N + N)}{q(\ell - N)} \left( \|\nabla p\|_{1, \Omega} + \|S\|_{Nq, \Omega} \right). \]

The proof is complete. \( \square \)

Note that if we wish to further weaken \( w \) to a VMO function as did in [3, 4, 6] we will run into a technical problem. That is, we do not know how the constant \( c \) in inequality (2.3) of [3] depends on \( \|k\|_{2, \partial B_1(0)} \).

4. Blow-up time

In this section we offer the proof of the main theorem.

Assume that \((m, p)\) is a strong solution to (1.1)-(1.4). The existence of such an “approximate” solution will be made clear later.

By virtue of the boundary condition (1.3), we have
\[ \|m\|_{W^{1, \ell}(\Omega)} \leq c \|\nabla m\|_{\infty, \Omega}. \]

Use \( p \) as a test function in (1.19) to get
\[ \int_{\Omega} |\nabla p|^2 \, dx \leq \int_{\Omega} S(x)p \, dx \leq \|S\|_{2, \Omega} \|p\|_{2, \Omega} \leq c \|S\|_{2, \Omega} \|\nabla p\|_{2, \Omega}. \]

Thus,
\[ \|\nabla p\|_{2, \Omega} \leq c \|S\|_{2, \Omega}. \]

Thus we can write (1.22) as
\[ (4.1) \quad \|\nabla p\|_{4q, \Omega} \leq c \left( 1 + \|\nabla m\|_{\infty, \Omega} \right)^{s_1} \|S\|_{Nq, \Omega}. \]

Here we have replaced \( q \) by \( 4q \). Take the \((4q)^{th}\) power, integrate over \((0, T)\), and then take the \((4q)^{th}\) root to derive
\[ (4.2) \quad \|\nabla p\|_{4q, \Omega_T} \leq c T^{\frac{1}{4q}} \left( 1 + \|\nabla m\|_{\infty, \Omega} \right)^{s_1} \|S\|_{Nq, \Omega}. \]

The rest of the proof of Theorem 1.1 is divided into several lemmas.

**Lemma 4.1.** We have
\[ \sup_{\Omega_T} |m| \leq c T^{\frac{s_0}{4q}} \left( T^{\frac{1}{4q}} \|m_0\|_{2, \Omega} + T \|S\|_{2, \Omega} \right) \|\nabla p\|_{4q, \Omega_T}^{s_0} \]
\[ + c \left( \|m_0\|_{\infty, \Omega} + T^{\frac{1}{4q}} \|m_0\|_{2, \Omega} + T \|S\|_{2, \Omega} \right), \]
where \( s_0 \) is given as (2.8).
Proof. Take the dot product of (1.2) with \( \mathbf{m} \) to derive

\[
\frac{1}{2} \partial_t |\mathbf{m}|^2 - \frac{D^2}{2} \Delta |\mathbf{m}|^2 + D^2 |\nabla \mathbf{m}|^2 + |\mathbf{m}|^{2\gamma} = E^2 (\mathbf{m} \cdot \nabla p)^2 \leq E^2 |\nabla p|^2 |\mathbf{m}|^2 \text{ in } \Omega_T.
\]

Drop the two non-negative terms on the left-hand side and then apply (2.29) to derive

\[
\sup_{\Omega_T} |\mathbf{m}|^2 \leq c \sup_{\Omega} |\mathbf{m}_0|^2 + c \left( \|\nabla p\|_{2q, \Omega_T}^2 + 1 \right) \|\mathbf{m}\|^2_{2, \Omega_T}.
\]

We can deduce from (1.5) that

\[
\sup_{0 \leq t \leq T} \int_{\Omega} |\mathbf{m}(x, t)|^2 dx \leq c \int_{\Omega} |\mathbf{m}_0(x)|^2 dx + c T |S(x)|_2^2.
\]

Use this in (4.5) to get (4.3). \( \square \)

**Lemma 4.2.** We have

\[
\|\nabla \mathbf{m}\|_{\infty, \Omega_T} \leq c G(T) + c F(T) \|\nabla p\|_{4q, \Omega_T}^{s_4},
\]

where \( s_4 \) is a positive number determined by \( N, q, \gamma \),

\[
F(T) = T^{\frac{s_0}{4q}} \left( T^{\frac{1}{2}} \|\mathbf{m}_0\|_{2, \Omega} + T \|S(x)\|_{2, \Omega} \right)
+ \left( \|\mathbf{m}_0\|_{\infty, \Omega} + T^{\frac{1}{2}} \|\mathbf{m}_0\|_{2, \Omega} + T \|S(x)\|_{2, \Omega} \right)
+ T^\frac{s_0(2\gamma - 1) + 2}{4q} \left( T^{\frac{2\gamma - 1}{2}} \|\mathbf{m}_0\|_{2, \Omega}^{2\gamma - 1} + T^{2\gamma - 1} \|S(x)\|_{2, \Omega}^{2\gamma - 1} \right), \text{ and}
\]

\[
G(T) = \|\nabla \mathbf{m}_0\|_{\infty, \Omega} + F(T) + T^\frac{s_0}{2q} \left( \|\mathbf{m}_0\|_{\infty, \Omega}^{2\gamma - 1} + T^{\frac{2\gamma - 1}{2}} \|\mathbf{m}_0\|_{2, \Omega}^{2\gamma - 1} + T^{2\gamma - 1} \|S(x)\|_{2, \Omega}^{2\gamma - 1} \right).
\]

Proof. We can write (1.2) in the form

\[
\partial_t m_i - D^2 \Delta m_i = E^2 (\mathbf{m} \cdot \nabla p)_{px_i} - |\mathbf{m}|^{2(\gamma - 1)} m_i \text{ in } \Omega_T, \quad i = 1, \ldots, N.
\]

This puts us in a position to apply Lemma 2.34. Upon doing so, we arrive at

\[
\|\nabla m_i\|_{\infty, \Omega_T} \leq c \|\nabla \mathbf{m}_0\|_{\infty, \Omega} + c \left( \|E^2 (\mathbf{m} \cdot \nabla p)_{px_i} - |\mathbf{m}|^{2(\gamma - 1)} m_i\|_{2q, \Omega_T} \right).
\]

We estimate from (4.3) that

\[
\left\| E^2 (\mathbf{m} \cdot \nabla p)_{px_i} - |\mathbf{m}|^{2(\gamma - 1)} m_i \right\|_{2q, \Omega_T}
\leq c \|\mathbf{m}\|_{\infty, \Omega_T} \|\nabla p\|_{4q, \Omega_T}^2 + c T^\frac{1}{2q} \|\mathbf{m}\|_{\infty, \Omega_T}^{2\gamma - 1}
\leq c T^\frac{s_0}{2q} \left( T^{\frac{1}{2}} \|\mathbf{m}_0\|_{2, \Omega} + T \|S(x)\|_{2, \Omega} \right) \|\nabla p\|_{4q, \Omega_T}^{s_0 + 2}
+ c \left( \|\mathbf{m}_0\|_{\infty, \Omega} + T^{\frac{1}{2}} \|\mathbf{m}_0\|_{2, \Omega} + T \|S(x)\|_{2, \Omega} \right) \|\nabla p\|_{4q, \Omega_T}^2
+ c T^\frac{s_0(2\gamma - 1) + 2}{4q} \left( T^{\frac{2\gamma - 1}{2}} \|\mathbf{m}_0\|_{2, \Omega}^{2\gamma - 1} + T^{2\gamma - 1} \|S(x)\|_{2, \Omega}^{2\gamma - 1} \right) \|\nabla p\|_{4q, \Omega_T}^{s_0(2\gamma - 1)}
+ c T^\frac{1}{2q} \left( \|\mathbf{m}_0\|_{\infty, \Omega}^{2\gamma - 1} + T^{\frac{2\gamma - 1}{2}} \|\mathbf{m}_0\|_{2, \Omega}^{2\gamma - 1} + T^{2\gamma - 1} \|S(x)\|_{2, \Omega}^{2\gamma - 1} \right).
\]

Set

\[
s_4 = \max\{s_0 + 2, s_0(2\gamma - 1)\}.
\]
Then Young’s inequality asserts
\begin{align}
(4.12) \quad \|\nabla p\|_{4q, \Omega_T}^{s_0+2} & \leq c\|\nabla p\|_{4q, \Omega_T}^{s_4} + c, \\
(4.13) \quad \|\nabla p\|_{4q, \Omega_T}^{2} & \leq c\|\nabla p\|_{4q, \Omega_T}^{s_4} + c, \\
(4.14) \quad \|\nabla p\|_{4q, \Omega_T}^{s_0(2\gamma-1)} & \leq c\|\nabla p\|_{4q, \Omega_T}^{s_4} + c. 
\end{align}

Combining this with (4.10) and (4.9) yields (4.6).

Use (4.6) in (4.2) to obtain
\begin{align}
(4.15) \quad \|\nabla p\|_{4q, \Omega_T} & \leq cG_1(T) + cF_1(T)\|\nabla p\|_{4q, \Omega_T}^{s_5},
\end{align}

where
\begin{align}
(4.16) \quad G_1(T) = T^{\frac{1}{s_5}} (G^{s_1}(T) + 1) \|S\|_{\frac{4Nq}{N+4q}, \Omega}, \\
F_1(T) = T^{\frac{1}{s_5}} F^{s_1}(T) \|S\|_{\frac{4Nq}{N+4q}, \Omega}, \quad s_5 = s_1 s_4.
\end{align}

Note that we can represent \( G_1(T) \) as the sum of 12 terms, each of which is of the form
\[ T^a |S|^b \|\nabla \mathbf{m}_0\|_{C^0, \Omega} \]
with \( a > 0, b \geq 0, c \geq 0 \) being determined by \( N, q, \gamma \) only. The same can be done for \( F_1(T) \) except that there are only seven terms in \( F_1(T) \). It follows from (4.15) that
\begin{align}
(4.17) \quad \|\nabla p\|_{4q, \Omega_T} & \leq cF_1(T)\|\nabla p\|_{4q, \Omega_T}^{s_5} + cG_1(T) \quad \text{for each } \tau \in [0, T].
\end{align}

Now we are in the situation of Lemma 2.2 where \( h(\tau) = \|\nabla p\|_{4q, \Omega_T} \). Following the proof of the lemma in [21], we consider the function \( g(h) = cF_1(T)h^{s_5} - h + cG_1(T) \) on \([0, \infty)\). Then (4.17) implies
\begin{align}
(4.18) \quad g(h(\tau)) & \geq 0 \quad \text{for each } \tau \in [0, T].
\end{align}

We compute
\[ g'(h) = cs_5 F_1(T)h^{s_5-1} - 1. \]
Thus, \( g(h) \) is decreasing on \( \left(0, \frac{1}{(cs_5 F_1(T))^{s_5-1}}\right) \) and increasing on \( \left(\frac{1}{(cs_5 F_1(T))^{s_5-1}}, \infty\right) \). The minimum value of \( g \) is given by
\begin{align}
(4.19) \quad m_g & = g\left(\frac{1}{(cs_5 F_1(T))^{s_5-1}}\right) = -\frac{s_5 - 1}{s_5(cs_5 F_1(T))^{s_5-1}} + cG_1(T).
\end{align}
It is easy to see that there is a unique solution \( T_{max} \) to the equation
\begin{align}
(4.20) \quad \frac{s_5 - 1}{s_5(cs_5 F_1(T_{max}))^{s_5-1}} cG_1(T_{max}) = 1.
\end{align}
We have
\[ m_g < 0 \quad \text{for each } T < T_{max}. \]
Fix \( T < T_{max} \). Observe that \( \|\nabla p\|_{4q, \Omega_T} \) is a continuous function of \( \tau \) and \( \lim_{\tau \to 0} \|\nabla p\|_{4q, \Omega_T} = 0 \). We can infer from (4.18) that
\begin{align}
(4.21) \quad \|\nabla p\|_{4q, \Omega_T} < \frac{1}{(cs_5 F_1(T))^{s_5-1}} \quad \text{for each } \tau \leq T.
\end{align}
In particular,
\begin{align}
(4.22) \quad \|\nabla p\|_{4q, \Omega_T} < \frac{1}{(cs_5 F_1(T))^{s_5-1}}.
\end{align}

Then (D4) follows from the classical regularity result for linear parabolic equations in ([14], p.204). We can easily transform (4.20) into (1.10).
The existence of a solution in the preceding calculations can be established via the Leray-Schauder fixed point theorem ([8], p.280). To this end, let \( q \) be given as in (H1) and set

\[
\mathcal{B} = L^{4q} \left( 0, T; W^{1,4q}_0(\Omega) \right).
\]

Then define an operator \( \mathcal{T} \) from \( \mathcal{B} \) into itself as follows: Let \( p \in \mathcal{B} \). We say \( w = \mathcal{T}(p) \) if \( w \) is the unique solution of the problem

\[
-\operatorname{div} \left( (I + n \otimes n) \nabla w \right) = S(x) \quad \text{in } \Omega_T, \\
w = 0 \quad \text{on } \Sigma_T,
\]

where \( n \) solves the problem

\[
\partial_t n - D^2 \Delta n + |n|^{2(\gamma-1)} n = E^2(n \cdot \nabla p) \nabla p \quad \text{in } \Omega_T, \\
n = 0 \quad \text{on } \Sigma_T, \\
n(x,0) = m_0(x) \quad \text{on } \Omega.
\]

We have:

**Lemma 4.3.** Let (H1)-(H3) hold. Then there is a unique weak solution \( n \) to (4.25)-(4.27) in the space \( C(\Omega_T) \cap L^2(0,T;W^{1,2}_0(\Omega))N \). Furthermore,

\[
\| \nabla w \|_{4q,\Omega} \leq c \| S(x) \|_{\frac{4Nq}{N+4q},\Omega}.
\]

From whence follows

\[
\| \nabla w \|_{4q,\Omega_T} \leq c T^{\frac{1}{4q}} \| S(x) \|_{\frac{4Nq}{N+4q},\Omega}.
\]

For \( \mathcal{T} \) to have a fixed point, we must verify

(C1) \( \mathcal{T} \) is continuous;
(C2) \( \mathcal{T} \) maps bounded sets into precompact ones;
(C3) There is a constant \( c \) such that

\[
\| p \|_{\mathcal{B}} \leq c
\]

for all \( p \in \mathcal{B} \) and \( \sigma \in [0,1] \) satisfying

\[
p = \sigma \mathcal{T}(p).
\]

To see (C2), suppose

\[
p_n \to p \quad \text{weakly in } L^{4q} \left( 0, T; W^{1,4q}_0(\Omega) \right).
\]

Denote by \( n_n \) the solution to (4.25)-(4.27) with \( p \) being replaced by \( p_n \). In view of (4.28), we can extract a sub-sequence of \( \{n_n\} \), not relabeled, such that

\[
n_n \to n \quad \text{uniformly in } (C(\Omega_T))^N.
\]

Note that we have

\[
- \operatorname{div} \left( (I + n_n \otimes n_n) \nabla p_n \right) = S(x) \quad \text{in } \Omega_T.
\]

Thus, we can pass to the limit in the above equation to get

\[
- \operatorname{div} \left( (I + n \otimes n) \nabla p \right) = S(x) \quad \text{in } \Omega_T.
\]
Subtract this equation from (4.34) to derive

\[-\text{div} \left[ (I + n_n \otimes n_n) \nabla (p_n - p) \right] = \text{div} \left[ (n_n \otimes n_n - n \otimes n) \nabla p \right] \quad \text{in } \Omega_T.\]

Once again, we can use (1.9) to get

\[\|\nabla (p_n - p)\|_{4q, \Omega_T} \leq c \|n_n \otimes n_n - n \otimes n\|_{4q, \Omega_T} \leq c \|n_n \otimes n_n - n \otimes n\|_{\infty, \Omega_T} \to 0.\]

That is, (C2) holds. Each problem in the definition of \(\mathcal{T}\) has a unique solution. This together with (C2) implies (C1).

We easily see that (4.32) is equivalent to the problem

\[
\begin{align*}
-\text{div} \left( (I + m \otimes m) \nabla p \right) &= \sigma_S(x) \quad \text{in } \Omega_T, \\
\partial_t m - D^2 \Delta m + |m|^{2(\gamma - 1)} m &= E^2 (m \cdot \nabla p) \nabla p \quad \text{in } \Omega_T, \\
p &= 0 \quad \text{on } \Sigma_T, \\
m &= 0 \quad \text{on } \Sigma_T, \\
m(x,0) &= m_0(x) \quad \text{on } \Omega.
\end{align*}
\] 

(4.36) (4.37) (4.38) (4.39) (4.40)

Now we are in a position to employ the earlier argument. Upon doing so, we obtain (C3) for each \(T < T_{\text{max}}\).

**Proof of Lemma 4.3.** We first establish the uniqueness assertion. Suppose that (4.25)-(4.27) has two solutions, say, \(n_1, n_2\). Then \(n \equiv n_1 - n_2\) satisfies

\[
\begin{align*}
\partial_t n - D^2 \Delta n + |n|^{2(\gamma - 1)} n_1 - |n_2|^{2(\gamma - 1)} n_2 &= E^2 (n \cdot \nabla p) \nabla p \quad \text{in } \Omega_T, \\
n &= 0 \quad \text{on } \Sigma_T, \\
m(x,0) &= m_0(x) \quad \text{on } \Omega.
\end{align*}
\] 

(4.41) (4.42) (4.43)

Recall from Lemma 2.1 that

\[
\left( |n_1|^{2(\gamma - 1)} n_1 - |n_2|^{2(\gamma - 1)} n_2 \right) \cdot n \geq 0.
\]

With this in mind, we take the dot product of (4.41) with \(n\) to derive

\[
\frac{1}{2} \sup_{0 \leq t \leq T} \int_{\Omega_T} |n|^2 \, dx + D \int_{\Omega_T} |\nabla n|^2 \, dx \, dt \leq 2E^2 \int_{\Omega_T} (n \cdot \nabla p)^2 \, dx \, dt.
\] 

(4.44)

By Poincaré’s inequality,

\[
\begin{align*}
\int_{\Omega_T} |n|^{2 + \frac{4}{N}} \, dx \, dt &\leq \int_{\Omega_T} \left( \int_{\Omega} |n|^{2N} \, dx \right)^{\frac{N-2}{N}} \left( \int_{\Omega} |n|^2 \, dx \right)^{\frac{2}{N}} \, dt \\
&\leq c \sup_{0 \leq t \leq T} \left( \int_{\Omega} |n|^2 \, dx \right)^{\frac{N}{N+2}} \int_{\Omega_T} |\nabla n|^2 \, dx \, dt \\
&\leq c \left( \int_{\Omega_T} (n \cdot \nabla p)^2 \, dx \, dt \right)^{\frac{N+2}{N}} \\
&\leq c \int_{\Omega_T} |n|^{2 + \frac{4}{N}} \, dx \, dt \left( \int_{\Omega_T} |\nabla p|^{N+2} \, dx \, dt \right)^{\frac{2}{N}}.
\end{align*}
\] 

(4.45)

Here \(c\) depends only on \(N, D, E\). Obviously, we can pick a positive number \(\tau \leq T\) such that

\[
c \left( \int_{\Omega_T} |\nabla p|^{N+2} \, dx \, dt \right)^{\frac{2}{N}} < 1.
\]

Then (4.45) implies

\[
n = 0 \quad \text{in } \Omega \times [0, \tau].
\]
If \( \tau < T \), then we apply the preceding proof to the problem on \( \Omega \times (\tau, T) \). In a finite number of steps, we can achieve
\[
    n = 0 \quad \text{in } \Omega_T. 
\]
To establish (4.28), it is enough for us to show that
\[
    n \in (L^\infty(\Omega_T))^N. 
\]
To see this, we write (4.25) as
\[
    \partial_t n - D^2 \Delta n = E^2 (n \cdot \nabla p) \nabla p - |n|^{2(\gamma - 1)} n \in L^q(\Omega_T). 
\]
Since \( q > 1 + \frac{N}{2} \), we can invoke the classical result in (\cite{14}, p.204) and Lemma 2.5 to conclude (4.28). As for (4.46), we take the dot product of (4.25) with \( n \) to deduce
\[
    \partial_t |n|^2 - D^2 \Delta |n|^2 \leq 2E^2 |\nabla p|^2 |n|^2 \quad \text{in } \Omega_T. 
\]
We can infer (4.46) from Lemma 2.4.

The existence of a weak solution to (4.25)-(4.27) can also be established via the Leray-Schauder fixed point theorem. In this case, we define an operator \( B \) from \( (L^\infty(\Omega_T))^N \) into itself as follows: For each \( m \in (L^\infty(\Omega_T))^N \) we let \( w = B(m) \) be the unique solution of the problem
\[
    \begin{align*}
        \partial_t w - D^2 \Delta w &= E^2 (m \cdot \nabla p) \nabla p - |m|^{2(\gamma - 1)} m \quad \text{in } \Omega_T, \\
        w &= 0 \quad \text{on } \Sigma_T, \\
        w &= m_0(x) \quad \text{on } \Omega. 
    \end{align*}
\]
The term on the right-hand side of (4.48) lies in \( L^{2q}(\Omega_T) \). Thus \( w \) is Hölder continuous on \( \overline{\Omega_T} \). With this in mind, we can easily verify that (C1)-(C3) are all satisfied by \( B \). This completes the proof.

\[ \square \]

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