Abstract

We study an interacting particle system, motivated by a simple random access protocol in wireless systems. There are $\rho n$ particles, $\rho < 1$, moving clockwise, in discrete time, on $n$ cites arranged in a circle. A “free” particle, with no neighbors on either side, certainly moves forward in a slot. A particle with another one immediately in front of it, does not move. A particle which has an empty site in front, but another particle immediately behind, moves forward with probability $0 < p < 1$. (We refer to the latter as a “holdback” property.) From the point of view of holes (empty sites) moving counter-clockwise, this is a zero-range process.

The main question we address is: what is the system flux (or current, or throughput) when $n$ is large, as a function of density $\rho$? The most interesting range of densities is $0 \leq \rho < 1/2$. We define the system typical flux as the limit of the flux in a system with random perturbations, when the perturbations’ rate vanishes. Our main results show that (a) the typical flux is different from the formal flux for the basic system (without perturbations), and (b) the process experiences a phase transition at density $h = p/(1 + p)$. If $\rho < h$, the typical flux is equal to $\rho$, which coincides with the formal flux. If $\rho > h$, a condensation occurs, which results in the formation of large particle clusters; in particular, the typical flux in this case is $p(1 - \rho) < h < \rho$.

Our results include both the steady-state limits (which determine the typical flux) and the transient analysis. In particular, we show that the key “reason” for large cluster formation for densities $\rho > h$ is described by a version of the Ballot Theorem.

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1 Introduction

This paper studies an interacting particle system, motivated by a simple model of packet movement in a wireless communication system under a CSMA (Carrier-Sense Multiple Access) protocol. Suppose, a fixed number of packets move clockwise in a network of nodes arranged in a circle. The time is slotted. There can be at most one packet at a node at any time. Consider a node that in a given time slot has a packet. If both neighbors of the node are “empty,” the packet will certainly move to the next node. If the next (clockwise) node is occupied, the packet will certainly be blocked from moving forward. If the next node is empty, but the previous (clockwise) node is occupied, the access protocol is such that the “competition” from the packet “behind” may prevent the packet from moving forward, with some probability. (This motivating model is very basic.
A more complicated – and more realistic – CSMA model, which exhibits a qualitatively similar behavior, is discussed in Appendix B.)

The corresponding interacting particle system, which is the focus of this paper, is as follows. There are \( n \) sites (or nodes), arranged in a circle and numbered from 0 to \( n - 1 \) in the clockwise order. There is a constant number \( \rho n \) of particles in the system, where the average density \( \rho \in (0, 1] \) is fixed. There is at most one particle at each site at any time. The system evolves in discrete (slotted) time \( t = 0, 1, 2, \ldots \). The state at a given time is described as a sequence of particles (occupied sites) and holes (empty sites). We often refer to the clockwise and counter-clockwise directions as “right” and “left,” respectively. The particles’ movement in the clockwise direction is governed by the following rules:

(a) if a particle has another particle as a right-neighbor, it does not move;
(b) if a particle has holes as neighbors on both sides, it moves to the right-neighbor site;
(c) if a particle has a hole as right-neighbor and a particle as left-neighbor, it moves to the right-neighbor site with probability \( p \in (0, 1] \) which is fixed.

This model may be considered as a version of the discrete-time Totally Asymmetric Simple Exclusion Process (TASEP), with parallel updates [6]. We refer to rule (c) as a “holdback” property and thus call the model discrete-time TASEP-H, where H stands for holdback. In the classical version of discrete-time TASEP a particle cannot move if its right-neighbor is another particle and, otherwise, moves right with a certain fixed probability. (For a general introduction into interacting particle systems see [8]. The class of TASEP models is very rich and has received a lot of attention in literature, because it has many applications, including statistical physics and transportation; see, e.g.,[1, 4, 7]).

Note that from the point of view of holes moving counter-clockwise (left), the rules (a)-(c) are equivalent to the following ones:

(a’) if a hole has another hole immediately to the left, it does not move;
(b’) if a hole has a particle immediately in front of it (to the left), followed by another hole, it moves to the left-neighbor site;
(c’) if a hole has two or more particles immediately in front of it (to the left), it moves to the left-neighbor site with probability \( p \in (0, 1] \) which is fixed.

In the latter interpretation (as holes’ movement), our model is a special case of the model considered in [6] where the probability of a hole movement may be an arbitrary function of the number of holes in front of it. It is shown (see [6, Section 7]) that the stationary distribution of the system may be presented in a product form. (Our model is also related to some other TASEP models considered in literature. We mention \( q \)-TASEP [3] for the continuous-time and [2] for the discrete-time version; both papers however consider movement of a finite number of particles on an infinite line.)

The system flux (or, current) \( \phi \) is defined as the average number of particle movements, per time slot per site. (This corresponds to the throughput in the context of wireless systems.) In this paper we are primarily interested in the dependence of flux \( \phi \) on the density \( \rho \).

It is not hard to observe (following the discussion which will be given shortly), that, as \( \rho \) is increasing from 0 to 1, the steady-state of TASEP-H exhibits a phase transition at \( \rho = 1/2 \) from a completely sparse state, where all particles are “free” (never have neighbors), to a condensed state,
where large particle clusters (contiguously occupied sites) are formed. Correspondingly, the flux $\phi = \rho$ for $\rho < 1/2$, but then “jumps down” at $1/2$ and is continuous decreasing for $\rho > 1/2$. See Figure 1.

The main results of this paper will show that, in fact, if we consider a “typical” steady-state behavior, the condensation threshold is lower than 1/2 and is equal to $h = p/(1 + p)$. We will formally define the “typical” steady-state behavior as that of the TASEP-H system with (low-rate) perturbations.

The term condensation in interacting particle systems is inspired by the famous Bose condensation [5, 6]. (Regarding the terminology, in most of the literature, the condensation occurs at low particle density, as opposed to high. This is consistent with TASEP-H condensation, if we view the process from the point of view of holes’ movement.)

We now discuss the condensation effect and our main results in more detail. Recall that the state of the system at any time slot is described as a sequence of particles and holes. We will refer to a contiguous segment of (at least two) particles as a cluster. Only the particle at the right end of a cluster may move in the given time slot. We will also refer to a contiguous segment of sites (which may include all sites) as a sparse interval, if all particles in it are “free,” i.e. have holes as neighbors. Obviously, each free particle in a given slot moves right with probability 1.

A system state such that all particles are free we will call (interchangeably) as completely sparse, or ideal, or absorbing. Clearly, if $\rho < 1/2$, then for any $n$ and any $p$, the system eventually reaches an absorbing state and, thereafter, this state will only shift right at speed 1. Therefore, when $\rho < 1/2$ the flux $\phi = \rho$. See Figure 1.

However, for the values of $\rho$ close to (but still less than) $1/2$, it may take the system a very long time to reach such an absorbing state. Instead, from some initial states, the system may enter a quasi-stationary regime, which will persist for a long (indeed, exponential in $n$) time. We explain this – heuristically at this point – as follows. Suppose, the system starts in a state such that $\tau^* n$
particles, $0 < \tau^* \leq 1$, form a single cluster, while the the remaining sparse interval, consisting of $(1 - \tau^*)n$ sites, has particles spread out with the density

$$h = p/(1 + p).$$

Note that in this case the overall average density of particles is $\rho = \tau^* + (1 - \tau^*)h > h$, and then

$$\tau^* = \rho(1 + p) - p.$$

Then, the system dynamics is as follows. The particles will leave the cluster from its “right end” at the rate $p$, with the average distance between the released particles being $1 + 1/p = 1/h$. Therefore, the cluster right end, as it moves left at the rate $p$, leaves particle density exactly $h$ in “its wake” on the right. Of course, the free particles (in the sparse interval) move right at speed 1. It is not hard to see that, as free particles “hit” (and thus join) the cluster on its left, the left end moves left at the rate $p$. (This follows because the density in the sparse interval is $h$.) To summarize, the cluster moves left at speed $p$, with its length $\tau^* n$ staying (approximately) constant, and with the density “everywhere” in the sparse interval staying equal to $h$. Therefore, system state remains (almost) invariant, up to the shift of the cluster. We will call such a state a main equilibrium state (MES). (This definition will be made formal later in the paper.) Note that a MES exists and is unique (up to space shift) if and only if $\rho > h$. As long as MES remains the system state, the average flux is $\phi^* = (1 - \tau^*)h = p(1 - \rho) < h < 1/2$.

Further notice that, if the system starts in a MES, it will take a long time, which is exponential in $n$, for the cluster to “dissolve.” Therefore, if the overall density $\rho \in (h, 1/2)$, at least for some initial states, the system may enter a MES and stay in it – as opposed to an absorbing state – for a very long time, during which it will have the flux $\phi^* < h < \rho$, as opposed to the steady-state flux $\rho$.

The graph of the flux $\phi^*$ (when the system stays in a MES) versus $\rho$ is given in Figure 2 which is plotted for the cases $p = 0.1$, $p = 0.5$ and $p = 0.9$. Note that the largest value of $\phi^*$ is $h = p/(1 + p)$ (so for example if $p = 1/2$, the largest value is $1/3$, and the largest value of $\phi^*$ only approaches $1/2$ as $p$ approaches 1), and that $\phi^*$ starts to decrease as $\rho$ becomes larger than $h$.

Note that MES exists for any $\rho > h$, not just $h < \rho < 1/2$. That’s why Figure 2 shows domain $[0, 1]$ for $\rho$. In general, the case when $\rho \geq 1/2$ is a simplified version of the case $h < \rho < 1/2$. For this reason, in this paper we restrict our attention to the most interesting case $\rho < 1/2$.

Due to the observations above, it is not clear what should be considered a “typical” flux of TASEP-H, especially in the case $h < \rho < 1/2$. Is it $\rho$, or $\phi^*$, or maybe something else? In this paper we will define as “typical” the steady-state flux of TASEP-H with perturbations, as we take the limit in $n \to \infty$, and the rate of perturbations vanishing with $n$.

Perturbations in the basic TASEP-H may be defined in different ways. For example, we may replace the probability 1 with which a free particle moves right, with a close to one probability $\pi < 1$. Such a system, which we will refer to as zero-range model, is still within the framework of [6], its stationary distribution has a product form, and the limiting flux $\phi_\pi$, as $n \to \infty$, can be found. We in fact carry out this analysis in Section 3 and demonstrate that $\phi = \lim_{\pi \uparrow 1} \phi_\pi$ is exactly as depicted on Figure 2. Note that here we take the limit $n \to \infty$ first, and limit $\pi \uparrow 1$ second. This means, in particular, that as $n \to \infty$, the perturbation rate remains “high,” namely $O(1)$ per particle per time slot. It is interesting, however, to look at how low a perturbation rate can be so that the flux remains as in Figure 2. This question could be answered by considering $\pi = \pi(n) \uparrow 1$. Unfortunately, analysis of the flux in this latter case (i.e., taking limit $n \to \infty$ with $\pi = \pi(n) \uparrow 1$), appears to be difficult. We also note that the analysis based on the product-form of the steady-state does not shed light into how condensation occurs, i.e. into the questions of the
Figure 2: Flux $\phi$ versus density $\rho$ in the TASEP-H model with three different values of $p$

following type. What is the dynamics of large cluster formation, in particular, what are the most likely trajectories? How fast do the clusters form? An so on.

The main goal of this paper is to demonstrate that even very rare perturbations lead to the typical flux given by Figure 1. Moreover, our analysis sheds light on why, for the densities $\rho > h$, even a very small cluster that appears due to a perturbation has a positive, uniformly bounded away from 0, probability to grow large, bring the system to a MES, which will then persist for a long time. The key “reason” for large cluster formation for densities $\rho > h$ is described by a version of the Ballot Theorem, which we derive in Proposition 5.

The TASEP-H with perturbations model, which is the main focus of this paper, is as follows. We define two versions of the perturbation mechanism, both preventing the system from being absorbed in an ideal (absorbing) state. For both of them, if a perturbation occurs, it consists of the following: a particle is chosen uniformly at random and moved to one of the empty sites, also chosen uniformly at random. The two versions of TASEP-H with perturbations are:

(-) $A$-perturbations ($perturbations$ $of$ $absorbing$ $states$). In every time slot in which the system enters an ideal (absorbing) state, a perturbation happens with probability $\lambda/n$ with some $\lambda > 0$, independently of the process history.

(-) $I$-perturbations ($independent$ $perturbations$). In every time slot a perturbation happens with probability $\lambda/n$ with some $\lambda > 0$, independently of the process history.

We analyze the behavior of the system as $n \to \infty$, with $\rho$ staying constant. To do that we consider the process under fluid (mean-field) space/time scaling. In addition to the steady-state results – in fact, as a tool for obtaining them – we derive results on the process transient behavior under this scaling. These transient results may be of independent interest.

Our main results are as follows.
• For **TASEP-H with A-perturbations** we prove that the limit (in \( n \to \infty \)) of stationary distributions is such that:

  - If \( \rho < h \), a cluster (if any) contains zero fraction of sites; consequently, the flux is equal to \( \rho \); see Theorem 9.
  - If \( \rho > h \), the system state is the MES; consequently, the flux is equal to \( \phi^* = p(1 - \rho) \); see Theorem 12.

• For **TASEP-H with I-perturbations** we prove that the limit (in \( n \to \infty \)) of stationary distributions is such that:

  - If \( \rho < h \), all clusters (if any) contain zero fraction of sites; consequently, the flux is equal to \( \rho \); see Theorem 17.
  - If \( \rho > h \), the flux \( \phi < \phi^* = p(1 - \rho) \); see Theorem 19.

We conjecture that, if \( \rho > h \), the flux \( \phi = \phi^* \); Conjecture 18.

• For the **zero-range model** we prove that if we first take the limit of stationary distributions as \( n \to \infty \), and then the limit in \( \pi \uparrow 1 \), the flux exactly matches that of TASEP-H with A-perturbations; see Section 5.

• We derive a version of the **Ballot Theorem** (Proposition 8) which serves as a key tool for establishing condensation at high densities \( \rho > h \).

The rest of the paper is organized as follows. Some basic notation, used throughout the paper, is given in Section 1.1. Section 2 defines the basic TASEP-H model (without perturbations), the model with perturbations, and a system flux; it also defines the asymptotic regime (and the corresponding limiting flux), the process under fluid (mean-field) scaling, and the probability space construction. The analysis of TASEP-H with A-perturbations is given in Section 3: we define the process fluid limits and establish their properties (Section 3.1); we state a version of the Ballot Theorem (Proposition 8 in Section 3.2); the main results for TASEP-H with A-perturbations (Theorems 9 and 12) are stated and proved in Section 3.3. Section 4 is devoted to the analysis of TASEP-H with I-perturbations: some preliminary facts are given in Section 4.1; fluid limits are defined and studied in Section 4.2; Sections 4.3 and 4.4 contain our main results and the conjecture (Theorem 17, Theorem 19, Conjecture 18) for TASEP-H with I-perturbations. In Section 5 we formally define the zero-range model and derive its limiting flux (as parameter \( \pi \to 1 \)). In Section 6 we present and discuss further conjectures for both the TASEP-H with perturbations and zero-range models. Appendix A contains the proof of Proposition 8. Appendix B discusses a CSMA model further (indirectly) motivating the holdback property of TASEP-H.

1.1 Basic notation

We denote by \( \mathbb{R} \) and \( \mathbb{Z} \) the sets of real numbers and integers, respectively, and by \( L \) the Lebesgue measure on \( \mathbb{R} \). Abbreviation a.e. means *almost everywhere w.r.t. Lebesgue measure*. Notation \((\partial^-/\partial x)f(x,t)\) means left partial derivative in \( x \). The minimum and maximum of two numbers are denoted \( a \wedge b = \min(a,b) \) and \( a \vee b = \max(a,b) \), respectively. \( \mathbb{I}(A) \) is the indicator of event or condition \( A \). RHS and LHS mean right-hand side and left-hand side, respectively.

Abbreviation w.p.1 means *with probability 1*. Probability distributions are defined on the spaces and corresponding \( \sigma \)-algebras that are clear from the context. We denote by \( \Rightarrow \) the convergence
of random elements in distribution. For a random process \( Y(t), t \geq 0 \), we denote by \( Y(\infty) \) its (random) state in a stationary regime. (In other words, the distribution of \( Y(\infty) \) is a stationary distribution of \( Y(t) \).) RCLL means right-continuous with left limits. We will say that a random variable \( X \) has distribution \( \text{GEOM}(\ell, p) \) [or simply write \( X \sim \text{GEOM}(\ell, p) \)], for integer \( \ell \) and real \( 0 \leq p \leq 1 \), if \( \Pr\{X = i\} = p(1 - p)^{i-\ell}, i \geq \ell; \mathbb{E}X = \ell + (1 - p)/p \).

2 Model

2.1 Basic model

Consider \( n \) sites arranged in a circle. The sites are numbered from 0 to \( n - 1 \) in clockwise order. There is a constant number \( \rho n, 0 < \rho \leq 1 \), of particles in the system, with either 0 or 1 particles located at each site at any time; correspondingly, a site may be empty (0 particles) or occupied (1 particle). We refer to empty sites as “holes.” The clockwise [resp., counterclockwise] direction we often refer to as “right” [resp. “left”] direction.

The system evolves in discrete (slotted) time \( t = 0, 1, 2, \ldots \). The system state at each time is its sites’ occupancy configuration, i.e. the sequence of 1’s and 0’s (particles and holes), indicating the occupancy of each site. Given the state at time \( t \), the (random) state at time \( t + 1 \) is determined by applying the following rules to each particle independently:

(a) if the right-neighbor site of the particle is occupied, the particle does not move;
(b) if both the right-neighbor and left-neighbor sites are empty, the particle moves to the right-neighbor site;
(c) if the right-neighbor site is empty and the left-neighbor site is occupied, the particle moves to the right-neighbor site with probability \( p \in (0, 1] \).

We refer to the rule (c) as a “holdback” property.

To make some definitions that follow later unambiguous, we adopt the following convention about the exact timing of the particles’ movement. Namely, the movement of particles that changes the system state from that at \( t - 1 \) to that at \( t \) is attributed to time \( t \).

We will use the following terminology throughout. We call a particle free if it has holes on both sides. A contiguous set of sites is a cluster if it has holes as neighbors on both sides. A contiguous set of sites is a sparse interval, if it contains only free particles and holes (i.e., does not overlap with any cluster). A system state is called completely sparse, or ideal, if it contains no clusters.

Note that any trajectory of the basic model is such that the number of clusters cannot increase. Indeed, let us assume that as particles “leave” a cluster from the right and “join” it on the left, this cluster retains its “identity.” Then, each initial cluster may “move” to the left, may grow or decrease in length, or may eventually disappear. But no new cluster can ever be created.

2.2 Model with perturbations

We will now introduce the model with perturbations, in fact two different versions of it. For a given system state its (random) perturbation is defined as follows: we pick a particle uniformly at random, remove it, and then place it into one of the empty sites picked uniformly at random. We adopt a convention that the site from which the particle was removed is immediately considered empty, so that the particle may go back to it. (This convention is not essential.)

We also adopt the following convention about the exact timing of a perturbation with respect to time slots. Applying a perturbation in a time slot \( t \) means that it is applied to the state the system enters after the “normal” particles’ movement at time \( t \) (i.e. the movement “between” \( t - 1 \) and \( t \)), and it is done before the next time slot \( t + 1 \). In other words, the perturbation does not
take an extra time slot, and the final system state at time \( t \) is the state after the normal particles’ movement at \( t \) and after a perturbation (if any) is applied.

**Model with I-perturbations.** In this version, in each time slot, one perturbation is applied with probability \( \lambda/n \), \( \lambda > 0 \), independently of the process history up to this slot. (Term I-perturbations is because the perturbations are independent.)

**Model with A-perturbations.** In this version, one perturbation is applied with probability \( \lambda/n \), \( \lambda > 0 \), in each time slot in which the system enters to – or stays at – an ideal state, after the normal particles’ movement. (Term A-perturbations is because the perturbations explicitly prevent absorption.)

Recall that in the basic model (without perturbations) no new cluster can ever be created. Unlike in the basic model, a model with perturbations is such that new clusters may be created (only after a perturbation). Note that, for the A-perturbation model, this necessarily means that, after the system “hits” and ideal state, it can have at most one cluster from that time on. In the I-perturbation model, a cluster may be split into two (only after a perturbation).

For future reference, we summarize these simple observations as the following

**Lemma 1.** Any process trajectory has the following properties, depending on the model type.

(i) The basic model (without perturbations). The number of clusters cannot increase. Each initial cluster may move to the left, may grow or decrease in length, may eventually disappear. But no new cluster can ever be created.

(ii) A-perturbations model. The number of clusters cannot increase until a trajectory “hits” an ideal state. There can be at most one cluster from that time on, which may be created after a perturbation.

(iii) I-perturbations model. The number of clusters cannot increase between perturbations. Upon a perturbation, a new cluster may form and/or a cluster may split into two.

The process without perturbations is trivial in that an ideal state is eventually reached. For both the A-perturbation and I-perturbation models, the corresponding process is a discrete-time finite irreducible aperiodic Markov chain. For A-perturbations, the state space is irreducible if it is a priori restricted to those states reachable from the state where all particles are within one cluster (all other states are transient).

### 2.3 Steady-state flux of a system

We now introduce the steady-state flux of a system which is the main focus of this paper. Denote the state of the system at time \( t \) by \( \vec{Z}(t) = (Z_1(t), \ldots, Z_n(t)) \), where \( Z_i(t) = 1 \) if site \( i \) is occupied by a particle and \( Z_i(t) = 0 \) otherwise. Then, the instantaneous average flux of the system is defined as \( \phi(\rho, n; Z(t)) = \psi(\rho, n; Z(t))/n \), where \( \psi(\rho, n; Z(t)) \) is the expected total distance that will be traveled by all particles at time \( t + 1 \). (This includes particles being relocated due to perturbations – the precise convention will be given below.) The steady-state flux for the system with parameters \( \rho \) and \( n \) is defined as

\[
\phi(\rho, n) = \mathbb{E}\phi(\rho, n; Z(\infty)),
\]

where \( Z(\infty) \) is the (random) value of \( Z(t) \) in a stationary regime. (It is easy to see that \( \phi(\rho, n) \) for a given system does not depend on a steady state chosen, even if steady state is non-unique.)

### 2.4 Asymptotic regime. Limiting steady-state flux

The asymptotic regime that we will consider is such that \( \rho \) and \( \lambda \) remain constant, while \( n \to \infty \). To avoid cumbersome notation, let us assume that \( \rho n \) is integer. This is the total number of particles.
(If \( \rho n \) is non-integer, we could assume that the number of particles is, say, \( [\rho n] \).

The main focus of the paper is on identifying the limiting steady-state flux, or just flux,

\[
\phi = \phi(\rho) = \lim_{n \to \infty} \phi(\rho, n).
\]

2.5 Process definition. Fluid (mean-field) scaling

For a system with parameter \( n \) (the circle length) the process state at time 0 is described by the function

\[
F^n(x,0), \quad x = 0,1,\ldots,n-1,
\]

where \( F^n(x,0) \) is the total number of particles in sites \( 0,\ldots,x \) at time 0.

The (random) movement of particles in the system is described by the flux-function \( \Phi^n(x,t) \), defined for \( x = 0,1,\ldots,n-1 \) and time \( t = 0,1,2,\ldots \) as follows: \( \Phi^n(x,t) \) is the total number of particles that moved (right) from site \( x \) at times \( 1,\ldots,t \), with \( \Phi^n(x,0) = 0 \). By convention, this quantity includes the number of particles that crossed site \( x \) due to perturbations; namely, the convention is that a particle being relocated from site \( x_1 \) to site \( x_2 \) always moves “right,” i.e. \( x_1 \leq x_2 \), and this particle simultaneously “leaves” sites \( x_1, x_1+1,\ldots,x_2-1 \) (or none, if \( x_1 = x_2 \)).

We define

\[
F^n(x,t) = F^n(x,0) - \Phi^n(x,t), \quad x = 0,1,\ldots,n-1, \quad t = 0,1,2,\ldots
\]

It is convenient to extend the definition of \( F^n(x,t) \) to all integer \( x \) by setting \( F^n(n,t) = F^n(0,t) + \rho n \) and assuming that \( F^n(x,t) \) has periodic increments in \( x \) with period \( n \). Clearly, the (random) function \( F^n(x,t) \) completely describes the process evolution. It is non-decreasing in \( x \) and non-increasing in \( t \). (In terms of describing the system states and evolution, only increments of \( F^n(x,t) \) matter. So, for any fixed integer \( C \), \( F^n(\cdot,t) + C \) describes exactly the same system state at \( t \) as \( F^n(\cdot,t) \), and \( F^n(\cdot,\cdot) + C \) describes exactly the same system trajectory as \( F^n(\cdot,\cdot) \). In particular, if \( F^n(t_1) - F^n(t_2) = C \), \( \forall x \), the system states at times \( t_1 \) and \( t_2 \) are equal.)

It is also convenient to extend the definition of \( F^n(x,t) \) to all real \( x \in \mathbb{R} \) and \( t \geq 0 \) by adopting convention

\[
F^n(x,t) = F^n([x],[t]).
\]

Furthermore, we will identify any location \( x \in \mathbb{R} \) with \( x \) (mod \( n \)) \( \in [0,n) \). So, for example, if \( n = 50 \), \( F^{50}(4,t) - F^{50}(4-7,t) \) is the total number of particles, at time \( t \), in the 7 consecutive sites, starting from 4 and going left: 4,3,2,1,0,−1,−2 or, equivalently, 4,3,2,1,0,49,48.

We assume that the system processes, for a given \( n \), are constructed via driving (control) sequences of random variables, described as follows.

We have a countable set, indexed by \( j = 1,2,\ldots \), of the sequences of i.i.d. random variables, \( \xi^j_i \), \( i = 1,2,\ldots \), with distribution \( GEOM(1,p) \), mean \( 1/p \). For each cluster that exists initially, and for each new cluster that forms as the process evolves, a sequence \( \{\xi^j_i\} \) with its “own” \( j \) is assigned. So, \( j \) can be thought of as a cluster index. The indices \( j \) are assigned to the clusters sequentially, “as needed.” If a cluster \( j \) breaks into two (which is possible due to and only due to a perturbation), then, by convention, the right one retains the index \( j \), and the left one gets a new index. If two clusters \( j' \) and \( j \) merge into one larger cluster (which is possible due to and only due to a perturbation), with cluster \( j \) becoming the right part of the new cluster, then, by convention, the new cluster retains the index \( j \), and the index \( j' \) is “eliminated.” A r.v. \( \xi^j_i \) determines the random

Footnote: Note that the contribution of perturbations into the steady-state flux is upper bounded by \( n(\lambda/n)/n \), and therefore vanishes as \( n \to \infty \).
time it takes a particle to “break away” from a cluster, after this particle becomes the right edge of the cluster. (For example, assuming the particle just ahead of ours, that broke away from the same cluster earlier, remains free, then $\xi_i$ gives the number of holes between our particle and the particle ahead, at the time when our particle breaks away.) The random variables $\xi_i^j$, for a given cluster $j$, are “taken” in sequence, as necessary, every time a particle becomes a cluster-right-edge particle.

Perturbations are controlled by an i.i.d. sequence of pairs $(\eta_1^i, \eta_2^i)$, $i = 1, 2, \ldots$, with independent components, each having uniform distribution in $[0, 1)$. Variable $\eta_1^i$ is “responsible” for picking a particle, and $\eta_2^i$ is “responsible” for picking a vacant site where it relocates. Specifically, $\eta_1^i$ is “used” as follows. We label the particles by $1, 2, \ldots, \rho n$, in the order of their locations in the (scaled) interval $[0, 1)$. Then, the particle with label $\lceil \eta_1^i \rho n \rceil$ is picked for a perturbation. A variable $\eta_2^i$ is used analogously for picking a vacant site where a particle relocates. The timing of perturbations is controlled by the i.i.d. sequence $\eta_3^i$, $i = 1, 2, \ldots$, with $\eta_3^i \sim GEOM(0, \lambda/n)$, $\mathbb{E} \eta_3^i = (1 - \lambda/n)/(\lambda/n)$. Namely, $\eta_3^i$ determines the random time from the start of the $i$-th perturbation “clock,” until the actual $i$-th perturbation. For I-perturbations, the clock starts in the time slot immediately after the previous, $(i - 1)$-th, perturbation; for A-perturbations, the clock starts in the first time slot after the previous perturbation, where the system enters an ideal state. Triples $(\eta_1^i, \eta_2^i, \eta_3^i)$ are “used” in sequence, as necessary, when perturbations need to be applied.

For each $n$, we consider a fluid (mean-field) scaled process, where we compress space, time and the number of particles by factor $n$:

$$f^n(x, t) = \frac{1}{n} F^n(nx, nt).$$

For a fixed (scaled) time $t$, $f^n(x, t)$ – the (scaled) system state – has periodic increments with period 1. From now on, we always refer to scaled space and time, unless explicitly stated otherwise.

In this paper we study the asymptotic behavior of the fluid-scaled process $f^n(x, t)$ as $n \to \infty$.

In the rest of the paper, we will often construct the processes for all $n$ on a common probability space. The construction is as follows. We assume that the i.i.d. sequences, $\xi_i^j$, $i = 1, 2, \ldots$, $j = 1, 2, \ldots$, are common for all $n$. They satisfy the following functional strong law of large numbers (FSLLN) condition: w.p.1,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{\lfloor nt \rfloor} \xi_i^j = \frac{1}{p} t, \quad t \geq 0 \ (u.o.c.), \quad \forall j. \quad (1)$$

The i.i.d. sequences $\eta_1^i$ and $\eta_2^i$, $i = 1, 2, \ldots$, will also be common for all $n$. The i.i.d. sequence $\eta_3^i$, $i = 1, 2, \ldots$, will depend on $n$. Using Skorohod representation, we can and will assume that these sequences are such that, w.p.1,

$$\lim_{n \to \infty} n^{-1} \eta_3^i = \hat{\eta}_3^i, \quad \forall i, \quad (2)$$

where $\hat{\eta}_3^i$ are independent (across $i$) exponentially distributed random variables with mean $1/\lambda$. Of course, we also have, by the strong law of large numbers, that w.p.1,

$$\lim_{m \to \infty} \frac{1}{m} \sum_{i=1}^{m} \hat{\eta}_3^i = \frac{1}{\lambda}. \quad (3)$$
3 A-perturbations model

3.1 A-perturbations model: Fluid limits

For any \( n \), \( f^n(\cdot, t) \) is a Markov process, and the main goal of this paper is to study its asymptotic behavior, in particular the asymptotics of its stationary distribution. However, to achieve this goal, we will need to analyze an extended version of this process, which includes additional process variables, explicitly describing clusters, sparse intervals, and the times when clusters exist. Recall that, for the A-perturbations, without loss of generality, we restrict the state space to those states reachable from a state where all particles are within one cluster. Therefore, for A-perturbations model, there is at most one cluster at any time, so the following additional descriptors will suffice.

Denote by \( \ell^n(t) \) and \( r^n(t) \) the (scaled) locations of the particles at the left and right edge of the cluster at time \( t \), if any, and by \( \tau^n(t) = r^n(t) - \ell^n(t) \) its length. (Note that the scaled number of particles within the cluster is \( n\tau^n(t) + 1/n = \tau^n(t) + 1/n \).) By convention, if cluster does not exist at time 0, we assume that \( \ell^n(0) = r^n(0) = \tau^n(0) = 0 \) until the time when a cluster forms. Also by convention, after a cluster dissolves, \( \ell^n(t) = r^n(t) \) remain frozen at the value of \( \ell^n \) at the last time when the cluster existed. We also adopt the convention that when \( \ell^n \) changes as a result of a cluster formation, the new value of \( \ell^n \) is chosen to be within \([0, 1]\), and \( r^n \) is chosen accordingly, to be to the right of (or at) \( \ell^n \).

At any time \( t \), the system state is given by

\[
[f^n(\cdot, t), \ell^n(t), r^n(t), \tau^n(t)].
\]

(Note that \([f^n(\cdot, t), \ell^n(t), r^n(t), \tau^n(t)]\) contains no more information than \( f^n(\cdot, t) \), in the sense that the distribution of \( f^n(\cdot, s), s \geq t \) depends only on \( f^n(\cdot, t) \).) Once again, we use the additional state descriptors for the purposes of analysis.) The metrics on the state space components are as follows. For the \( f \)-component, it is defined by the max-norm:

\[
\|f_1(\cdot) - f_2(\cdot)\| = \max_x |f_1(x) - f_2(x)|.
\]

For the \( \tau \)-component, it is given simply by the distance \( |\tau_1 - \tau_2| \). For the \( \ell \)-component, it is the \( \min_{k \in \mathbb{Z}} |\ell_1 - \ell_2 + k| \).

The metric for the \( r \)-component space is the same as for the \( \ell \)-component. Finally, the metric on the entire system state space is the sum-metric of its components.

For each \( n \), consider a finite or infinite time interval \([0, T^n]\), with \( T^n \) possibly depending on \( n \). (If \( T^n \) does depend on \( n \), it can be though of as a realization of a stopping time.) Trajectories

\[
[f^n(\cdot, t), \ell^n(t), r^n(t), \tau^n(t)], \: t \in [0, T^n],
\]

are considered as elements of the Skorohod space. For the purposes of having the Skorohod metric well-defined, we adopt the convention that, when \( T^n < \infty \), the state remains constant in \([T^n, \infty)\).

Further, let \( \alpha^n_0 \geq 0 \) be the first time when a cluster exists. (If it exists at time 0, then \( \alpha^n_0 = 0 \). If it never exists \( \alpha^n_0 = \infty \).) Let \( \beta^n_0 \geq \alpha^n_0 \) be the first time after \( \alpha^n_0 \) when the system is in an ideal state. \( \beta^n_0 = \infty \) if the system never enters an ideal state after \( \alpha^n_0 \). In other words, \([\alpha^n_0, \beta^n_0] \) is the time interval when the “first” (in time) cluster exists. Similarly, let \( \alpha^n_1 \geq \beta^n_0 \) be the first time after \( \beta^n_0 \) when the “second” cluster appears, and \( \beta^n_1 \geq \alpha^n_1 \) be the first time this cluster disappears. And so on, we define pairs \( (\alpha^n_i, \beta^n_i) \) marking the beginning and end of the \( i \)-th cluster. Note that,
\( \alpha_i^n = \infty \) implies that all consecutive times \( \alpha_i^n \) and \( \beta_i^n \) are also infinite. We also adopt a convention that, if we consider the process on a finite time interval \([0, T^n]\), then any time \( \alpha_i^n \) or \( \beta_i^n \) which is outside \([0, T^n]\) is set to be infinite. Finally, note that all \( \alpha_i^n \), except maybe \( \alpha_0^n \) when it is 0, are the times when perturbations occur. (However, not every perturbation time is one of the \( \alpha_i^n \).)

The following will be called an extended realization (trajectory) of the process on a (finite or infinite) time interval \([0, T^n]\):

\[
\{[f^{n}(\cdot, t), \ell^{n}(t), r^{n}(t), \tau^{n}(t)], \ t \in [0, T^n]; \ [\alpha_i^n, \beta_i^n], i = 0, 1, \ldots \}.
\]

(When this does not cause confusion, which is in most cases, we will call it just a realization.)

**Definition 1** (Fluid sample path (FSP)). Suppose, there is a fixed sequence of the process (extended) realizations on the time intervals \([0, T^n]\)

\[
\{[f^{n}(\cdot, t), \ell^{n}(t), r^{n}(t), \tau^{n}(t)], \ t \in [0, T^n]; \ [\alpha_i^n, \beta_i^n], i = 0, 1, \ldots \},
\]

such that the driving sequences’ realizations satisfy conditions \((i)-(iii)\). Then, a trajectory

\[
\{[f(\cdot, t), \ell(t), r(t), \tau(t)], \ t \in [0, T]; \ [\alpha_i, \beta_i], i = 0, 1, \ldots \},
\]

is called a fluid sample path (FSP) on the interval \([0, T]\) if the following conditions hold.

(i) Points \( 0 \leq \alpha_0 \leq \alpha_1 \leq \ldots \) are such that there is only a finite number of them on any finite interval; \( \alpha_i < \alpha_{i+1} \), as long as \( \alpha_i < \infty \); if \( \alpha_i > T \), then necessarily \( \alpha_i = \infty \). Points \( 0 \leq \beta_0 \leq \beta_1 \leq \ldots \) are such that there is only a finite number of them on any finite interval; \( \alpha_i \leq \beta_i < \alpha_{i+1} \), as long as \( \beta_i < \infty \); if \( \beta_i > T \), then necessarily \( \beta_i = \infty \).

(ii) Functions \( f(\cdot, t) \) and \( \tau(t) \) are continuous in \( t \). Function \( [\ell(t), r(t)] \) is right-continuous with left-limits \( (RCLL) \) in \( t \); its only possible points of discontinuity are those \( \alpha_i, i \geq 1 \), that are finite.

(iii) Trajectory \((5)\) is a limit of trajectories \((4)\), as \( n \to \infty \), in the following sense:

(iii.1) \( T^n \to T \);

(iii.2) \( \alpha_i^n \to \alpha_i \) and \( \beta_i^n \to \beta_i \), for each \( i \);

(iii.3) The following Skorohod space convergence holds:

\[
\{[f^{n}(\cdot, t), \ell^{n}(t), r^{n}(t), \tau^{n}(t)], \ t \in [0, T^n]\} \to \{[f(\cdot, t), \ell(t), r(t), \tau(t)], \ t \in [0, T]\}.
\]

(For the purposes of the Skorohod space metric, by convention, \( [f(\cdot, t), \ell(t), r(t), \tau(t)] \) is defined for all \( t \geq 0 \), with its value being constant for \( t \geq T \).

Note that in Definition 1 convergence

\[
[f^{n}(\cdot, t), \tau^{n}(t)] \to [f(\cdot, t), \tau(t)]
\]

is necessarily uniform on compact sets (since \( [f(\cdot, t), \tau(t)] \) is continuous), and convergence

\[
[\ell^{n}(t), r^{n}(t)] \to [\ell(t), r(t)]
\]

is necessarily uniform on any closed bounded interval, not containing any of the points \( \alpha_i \).

The following lemma describes basic properties of the FSPs, implied by the corresponding basic properties of pre-limit trajectories and FSP definition.
Lemma 2. Any FSP has the following properties.

(i) For any $t$, $f(x,t)$ is Lipschitz with constant 1. Moreover, $f(r(t), t) - f(\ell(t), t) = r(t) - \ell(t) = \tau(t)$ and $f(x,t)$ is Lipschitz with constant 1/2 in $[r(t), \ell(t) + 1]$.

(ii) $f(x + 1, t) - f(x, t) = \rho$ for any $x$ and $t$.

(iii) For $t > 0$, condition $\tau(t) > 0$ necessarily implies that $t \in (\alpha_i, \beta_i \land T]$ for some $i$.

(iv) Function $\tau(t)$ is Lipschitz with constant 1.

(v) Functions $\ell(t)$ and $r(t)$ are Lipschitz non-increasing, with constant 1, on the intervals of the form $(\alpha_i, \beta_i \land T)$. In the sub-intervals of $[0, T)$, not intersecting with any of the intervals $[\beta_i, \alpha_{i+1})$, $\tau(t) = 0$ and $\ell'(t) = r'(t) = 0$.

Proof is rather straightforward. For example, to prove (i), note that for any $n, t$ and $\delta \geq 0$, $f^n(x + \delta, t) - f^n(x, t) \leq 2/n$; moreover, $\delta - 2/n \leq f^n(x + \delta, t) - f^n(x, t) \leq 2/n$ if $[x, x + \delta] \in [\ell^n(t), r^n(t)]$, and $f^n(x + \delta, t) - f^n(x, t) \leq 2/n$ if $[x, x + \delta] \in [r^n(t), \ell^n(t) + 1]$. Given the FSP definition, this easily implies that: $f(x, t)$ is Lipschitz in $x$, with constant 1, everywhere; $(\partial / \partial x)f(x, t) = 1$ if $x \in (\ell(t), r(t))]$; $f(x, t)$ is Lipschitz in $x$, with constant 1/2, in $[r(t), \ell(t) + 1]$. The rest of the properties are easily established as well. We omit further details. □

For an FSP, a time point $t$ is called regular, if the derivatives $\tau'(t), \ell'(t), r'(t)$ exist. In view of Lemma 2(iv)-(v), almost all points (w.r.t. Lebesgue measure) are regular.

We see that an FSP describes the evolution of the distribution of the continuous “particle mass,” or “fluid,” obtained as a limit of rescaled pre-limit particle distributions. For an FSP we will also use the natural notion of a cluster: we say that a cluster $[\ell(t), r(t)]$, of length $\tau(t) = r(t) - \ell(t)$, exists at time $t$, if $t \in (\alpha_i, \beta_i)$ for some $i$; in this case the rest of the unit circle, that is interval $[r(t), \ell(t) + 1]$ is a sparse interval. A sub-interval of a sparse interval we will call a sparse sub-interval. Note that it is possible that a cluster in an FSP exists even when it has zero length, $\tau(t) = 0$. Of course, if $\tau(t) > 0$ then a cluster necessarily exists. If there is no cluster at time $t$, then necessarily $\tau(t) = 0$ and the entire unit circle $[0, 1]$ is a sparse interval. The total particle mass $f(x + 1, t) - f(x, t) = \rho$ is constant at all times. The particle mass density $(\partial / \partial x)f(x, t)$ within a cluster is exactly 1, and the density within a sparse interval is at most 1/2.

The following Lemma 3 describes the basic FSP dynamics. We precede it with an informal description of this dynamics. (See Figure 3 for an illustration.) The particle mass outside a cluster (if any) simply moves to the right at the constant speed 1 until and unless it “hits” the cluster. In a time interval when a cluster $[\ell(t), r(t)]$ exists, its right end $r(t)$ moves left at the constant speed.
p. (Because this is the rate at which particles in a pre-limit system “break-away” from the cluster on the right.) Moreover, as the right end of a cluster moves left, it leaves a sparse sub-interval of the density exactly \( h = p/(1 + p) \) in its “wake.” (Because, in pre-limit process, the average distance between two consecutive breaking away particles is \( 1 + 1/p = 1/h \).) Specifically, in a time interval \([t, t + \delta]\), a sparse sub-interval \([r(t) - \delta p, r(t) + \delta]\) of density \( h \) is created, where \( r(t + \delta) = r(t) - \delta p \) is the right-end location at time \( t + \delta \), and \( r(t) + \delta \) is how far the right edge of the sparse sub-interval moved right by time \( t + \delta \). Now let us consider how the left end \( \ell(t) \) moves. Consider a small interval \([t, t + \delta]\). Let \([\ell(t) - \delta_1, \ell(t)]\) be the sparse sub-interval immediately to the left of the cluster at time \( t \), which hits – and merges into – the cluster in time interval \([t, t + \delta]\). Then the \( \delta_1 \) above is the unique positive solution to equation

\[
\delta = \delta_1 - [f(\ell(t), t) - f(\ell(t) - \delta_1, t)].
\]

Indeed, the time \( \delta \) for the sparse sub-interval \([\ell(t) - \delta_1, \ell(t)]\) (at \( t \)) to join the cluster is exactly equal to the “empty space” in \([\ell(t) - \delta_1, \ell(t)]\) (at \( t \)), which is \( \delta_1 - [f(\ell(t), t) - f(\ell(t) - \delta_1, t)] \); solution \( \delta_1 \) is unique, because the RHS of (6), as function of \( \delta_1 \), is Lipschitz increasing with derivative \( \geq 1/2 \) (recall that the density within a sparse interval is at most \( 1/2 \)).

The following lemma describes the above basic dynamics of an FSP formally. Its proof is also fairly straightforward – most of it uses the corresponding properties of pre-limit trajectories and FSP definition. We will only provide the key points and comments, omitting details.

**Lemma 3.** Any FSP has the following properties.

(i) Consider a time point \( t \) such that \( t \in (\alpha_i, \beta_i \wedge T) \). (Recall that for a \( 0 < t < T \), \( \tau(t) > 0 \) necessarily implies \( t \in (\alpha_i, \beta_i \wedge T) \).) For any sufficiently small \( \delta > 0 \), the following holds. Denote by \( \delta_1 \) the unique positive solution to equation

\[
\delta = \delta_1 - [f(\ell(t), t) - f(\ell(t) - \delta_1, t)]
\]

Then, the FSP state at time \( t + \delta \) is as follows:

\[
\ell(t + \delta) = \ell(t) - (\delta_1 - \delta);
\]

\[
r(t + \delta) = r(t) - p\delta;
\]

for \( x \not\in (\ell(t + \delta), r(t + \delta)) \) (this is the part of sparse interval that just shifted, without any interaction with the cluster) we have

\[
f(x, t + \delta) = f(x - \delta, t);
\]

for \( x \in (\ell(t + \delta), r(t + \delta)) \) (within cluster at time \( t + \delta \))

\[
(\partial/\partial x)f(x, t + \delta) = 1;
\]

for \( x \in (r(t + \delta), r(t) + \delta) \) (the “wake” of right end of cluster)

\[
(\partial/\partial x)f(x, t + \delta) = h.
\]

In addition, if this \( t \) is regular,

\[
\ell'(t) = -\frac{g}{1 - g}, \quad \text{where} \quad g = g(t) = \frac{\partial^-}{\partial x}f(\ell(t), t)
\]

and \( r'(t) = -p \).
(ii) If $t$ is regular, then $\tau'(t) > 0$ if and only if $g > h$.

(iii) If $t$ is regular, $t \in (x_i, \beta_i \wedge T)$ and $\tau(t) = 0$, then, necessarily, $\tau'(t) = 0$ and $g = h$.

(iv) For any interval $[t, t+\delta] \subset [0, T)$ when a cluster does not exist (i.e. not intersecting with any of the intervals $[x_i, \beta_i]$), the particle density moves to the right at speed $1$:

$$f(x, t + \delta) = f(x - \delta, t), \quad \ell(t + \delta) = r(t + \delta) = \ell(t) + \delta = r(t) + \delta, \quad \tau(t) = 0.$$  

(v) Denote by $\mu(t)$ the total Lebesgue measure of all points $x \in [0,1)$ (within the circle) at time $t$, which are outside the cluster $[\ell(t), r(t)]$ (if any) and where the density

$$\frac{\partial}{\partial x} f(x, t) > h.$$  

Function $\mu(t)$ is Lipschitz (and then the derivative exists for almost all $t$), and given the properties (i)-(iv), it is non-increasing. Then, for almost all $t > 0$ (w.r.t. Lebesgue measure), either

$$\mu'(t) = 0,$$

or this time $t$ is regular, $\tau'(t) > 0$ (and then $\tau(t) > 0$, $t \in (x_i, \beta_i \wedge T)$), and

$$g > h \quad \text{and} \quad \mu'(t) = -1 + \ell'(t) = -\frac{1}{1 - g} \leq -\frac{1}{1 - h} = -(1 + p).$$  

(vi) The Lebesgue measure of the time points $t$ where $\tau'(t) > 0$ is upper bounded by $1/(1 + p)$.

From now on we make the definition of a regular point more restrictive by requiring, in addition, that the conclusions of Lemma 3(v) hold. Still, almost all time points $t$ are regular.

Proof. (i) The meaning of (7) (which is same as (6)) is explained in the informal definition above. Consider a fixed $\delta > 0$ and time $t$. For each pre-limit trajectory, as in FSP definition, define $\delta_1^{(n)} \geq 0$ as the smallest number such that all particles located at time $t$ in the interval $[\ell^n(t) - \delta_1^{(n)}, \ell^n(t)]$, will join the cluster by time $t + \delta$. It is easy to observe that

$$\delta = \delta_1^{(n)} - [f(\ell^n(t), t) - f(\ell^n(t) - \delta_1^{(n)}, t)] + O(1/n).$$

It remains to consider the limit in $n$ (as in the FSP definition) and observe that $\delta_1^{(n)}$ must converge exactly to the $\delta$ as defined. All properties before (10) then easily follow. Property (10) is a differential form of (7). Namely, we use the fact that $t$ is regular and therefore $\ell'(t)$ exists; let $\delta \downarrow 0$; from (8) we have

$$\lim_{\delta \to 0} \delta_1/\delta = 1 - \ell'(t);$$

dividing (7) by $\delta$, taking $\delta \downarrow 0$ limit and substituting the above display, we obtain (10). The last property is from (9).

(ii) and (iii) follow from (i). (iv) is obvious.

(v) Obviously, $\mu'(t) = 0$ for almost all time points $t$ such that a cluster does not exist. Consider now a $t$ which is strictly within a time interval when a cluster exists. Once again, consider the movement of the left end $\ell$ of the cluster in a small interval $[t, t+\delta]$. Then, using (10), (7) can be written as

$$-\delta_1 = \int_t^{t+\delta} [-1 + \ell'(s)] ds = \int_t^{t+\delta} \frac{-1}{1 - g(s)} ds, \quad (11)$$

where $-1 + \ell'(s)$ is the instantaneous (negative) rate at which the sparse sub-interval $[\ell(t) - \delta, \ell(t)]$ of length $\delta_1$, which existed at time $t$, “shrinks” due to its right end $\ell(s)$ moving at speed $\ell'(s)$ left
and its left end $\ell(t) - \delta_1 + (s-t)$ moving at speed 1 right. The integrand in the RHS exists for almost all $t$ (and is unique up the time subsets of zero Lebesgue measure). Note that, for a given $t$, the density $(\partial / \partial x)f(x,t)$ exists a.e. in $x$, and therefore $(\partial^- / \partial x)f(x,t) = (\partial / \partial x)f(x,t)$ a.e. (Recall that $g(t)$ is defined as left derivative.) Analogously to (11), it is easy to obtain its generalization, which gives the (negative) increment of $\mu$ in the interval $[t,t+\delta]$, by integrating over only those times $s$, where $(\partial^- / \partial x)f(\ell(s), s) = g(s) > h$:

$$\mu(t+\delta) - \mu(t) = \int_t^{t+\delta} \frac{-1}{1-g(s)}1\{g(s) > h\} ds. \quad (12)$$

Therefore, for almost all $s \in [t,t+\delta]$,

$$\mu'(s) = \frac{-1}{1-g(s)}1\{g(s) > h\}. $$

From here, along the fact that almost all times $s$ are regular, all properties stated in (v) follow.

(vi) Follows from (v). $\square$

**Lemma 4.** Suppose, $\rho < h$. Then any FSP is such that the following properties hold.
(i) For any $i$ such that $\alpha_i < \infty$, we have $\beta_i \wedge T - \alpha_i < 1$.
Consequently, for any $t_0 \leq T - 1$ there exist $t_0 < t_1 \leq t_0 + 1$ such that $\tau(t_1) = 0$.
(ii) $\int_0^T \tau(t) dt \leq \bar{C} = 1 + \frac{1}{2(1+p)} \quad (13)$

**Proof.** (i) Suppose not, i.e. $(t_0, t_0 + 1]$ is entirely within an interval of the form $[\alpha_i, \beta_i \wedge T)$. Then, at time $t_0 + 1$ the FSP state is such that everywhere outside the cluster the density is equal to $h$. (This follows from Lemma 3(i), density $h_i$.) This is impossible, because it would imply that $\rho \geq h$.

(ii) Consider the set of points $t \in [0,T]$, where $\tau(t) > 0$. This set consists of possibly an interval $[0,s)$, whose length $s$ cannot exceed 1 (by (i)), and a countable number of open intervals $(t_1, t_2)$ with lengths also not exceeding 1 (by (i)). Obviously,

$$\int_0^s \tau(t) dt \leq s \leq 1.$$

Consider any of the open intervals $(t_1, t_2)$. We have

$$\max_{t \in (t_1, t_2)} \tau(t) \leq \int_{t \in (t_1, t_2)} \tau'(t) dt \leq \int_{t \in (t_1, t_2)} |\tau'(t)| dt \leq \frac{1}{2} \mathcal{L}\{ t \in (t_1, t_2) : \tau'(t) > 0 \},$$

and therefore (recall that $t_2 - t_1 \leq 1$)

$$\int_{t_1}^{t_2} \tau(t) dt \leq \frac{1}{2} \mathcal{L}\{ t \in (t_1, t_2) : \tau'(t) > 0 \}.$$

Summing up over all intervals where $\tau(t) > 0$, we obtain

$$\int_0^T \tau(t) dt \leq 1 + \frac{1}{2} \mathcal{L}\{ t \in (0,T) : \tau'(t) > 0 \} \leq 1 + \frac{1}{2(1+p)} \leq 1 + \frac{1}{2(1+p)},$$

where in the last inequality we used Lemma 3(vi). $\square$
Suppose $h < \rho < 1$. Recall $h = p/(1 + p)$. State $[f^*(\cdot), \ell^*, r^*, \tau^*]$ is called a main equilibrium state (MES), if it satisfies the following conditions: $\tau^* = \frac{\rho - h}{1 - \rho}$, $r^* = \ell^* + \tau^*$, $f'(x) = 1$ for $x \in (\ell^*, r^*)$, $f'(x) = h$ for $x \in (r^*, \ell^* + 1)$. The definition shows that a MES is essentially unique, up to a space shift of the cluster; therefore, we will often refer to any MES as the MES. Note that if the steady-state of a system with large $n$ is close to MES, then the flux is close to $\phi^* = \phi^*(\rho) = (1 - \tau^*)h = p(1 - \rho)$.

**Lemma 5.** Suppose, $\rho > h$. Then any FSP is such that the following properties hold.

(i) If FSP initial state is a MES, the FSP is unique and is stationary, i.e. it stays in this state (up to shift of the cluster).

(ii) If $\alpha_0 = 0$ and $\beta_0 \geq 1 - \rho$, then the FSP state at time $1 - \rho$ is a MES, and the trajectory is stationary (staying in a MES) in the interval $[1 - \rho, T]$.

**Proof.** (i) According to derivatives’ expressions for a MES, it cannot change with time, up to the cluster moving left at the constant speed $p$.

(ii) The entire particle mass, which is originally outside the cluster is at most $\rho$. We have $\beta_0 \geq 1 - \rho$, so the cluster exists for the time at least $1 - \rho$. The time interval $[0, 1 - \rho]$ is long enough for all the mass which was originally outside the cluster to join the cluster. Therefore, at some time point in $t_0 \in [0, 1 - \rho]$, a state is reached, consisting of a cluster and a sparse interval with constant density $h$ (by Lemma 3(i).) This is equivalent to the state being a MES. The same argument as in the proof of (i) shows that the trajectory must stay in MES in $[t_0, T]$. □

**Lemma 6.** Consider a sequence of systems, with $n \to \infty$, with arbitrary random initial states $[f^n(\cdot, 0), \ell^n(0), r^n(0), \tau^n(0)]$. For each $n$ consider some stopping time $T^n$ (finite or infinite). Then, with prob. 1, any subsequence of (extended) realizations has a further subsequence, along which the convergence to an FSP holds in the sense of Definition 7(iii).

**Proof.** This type of a fluid-limit result is standard. If the sequence of processes is constructed on the common probability space, as specified above, the driving sequences satisfy the properties required in the FSP definition w.p.1. Then, w.p.1., from any subsequence we can choose a further subsequence, such that there is a convergence to some trajectory. The latter trajectory must be an FSP, by the FSP definition. We omit further details. □

**Lemma 7.** Suppose $\rho > h$. Consider a sequence of systems, with $n \to \infty$, with arbitrary random initial states $[f^n(\cdot, 0), \ell^n(0), r^n(0), \tau^n(0)]$. For a fixed constant $B \geq 1 - \rho$ and each $n$ consider a stopping time $T^n$, which is the minimum of constant $B$ and the first time after 0, when the process is in an ideal state. Then, w.p.1, any subsequence of (extended) realizations has a further subsequence, along which it converges to an FSP of duration $T \leq B$. This FSP satisfies one of the following two properties:

(a) $T < 1 - \rho$ and $\tau(T) = 0$;

(b) $T = B$ and the FSP state in $[1 - \rho, B]$ is a MES.

**Proof.** The probability 1 subsequential convergence to FSPs is due to Lemma 6. Then the required FSP properties follow from Lemma 5(ii). □

### 3.2 A version of the ballot theorem

The following combinatorial fact plays a key role in the proofs of our main results for the high density, $h < \rho < 1/2$. We will derive it from Theorem 1.2.5 – a ballot theorem – in [12].
Proposition 8. Suppose an integer \( n \geq 1 \) and a sequence of real non-negative numbers, \( k_1, k_2, \ldots, k_n \) are fixed. Let us extend the definition of the sequence \( \{k_r\} \) to all integer \( r \geq 1 \) by periodicity, \( k_{r+n} = k_r \), and denote \( \psi(j) = \sum_{\ell=1}^{j} k_{\ell} \) for \( j \geq 0 \), where \( \psi(0) = 0 \) by convention. Suppose a real number \( m > \psi(n) \) is fixed. Let \( N \) be the number of those indices \( j \in \{0, 1, \ldots, n-1\} \), for which \( \psi(j+r) - \psi(j) < \frac{m}{n} r, \quad r = 1, \ldots, n \). Then,

\[
N \geq \left\lceil n \left(1 - \frac{\psi(n)}{m}\right)\right\rceil.
\]

The proof is in Appendix A.

3.3 **A-perturbations model: Main results**

**Theorem 9.** Consider the model with A-perturbations. Assume \( \rho < h \). Then, as \( n \to \infty \), the sequence of the stationary distributions of \( \{f^n(\cdot, t), \ell^n(t), \tau^n(t)\} \) is such that its any subsequential weak limit is concentrated on the states with \( \tau = 0 \). In other words, as \( n \to \infty \), \( \tau^n(\infty) \Rightarrow 0 \) [equivalently, \( \mathbb{E} \tau^n(\infty) \to 0 \)] and, consequently, the limiting steady-state flux \( \phi = \phi(\rho) = \rho \).

**Proof of Theorem 9.** It suffices to show that any distributional limit is such that \( \mathbb{E} \tau = 0 \). Suppose not. Consider a fixed interval \([0, B]\), and stationary versions of the processes in this interval. Then there exists \( \epsilon > 0 \) such that, for any \( B > 0 \),

\[
\liminf_{n \to \infty} \frac{1}{B} \mathbb{E} \int_{0}^{B} \tau^n(t)dt \geq \epsilon > 0.
\]

However, using our construction above, we can construct all these (stationary) processes on a common probability space, so that, w.p.1., we have a subsequential convergence to an FSP. Convergence to an FSP, in particular, means that \( \tau^n(\cdot) \to \tau(\cdot) \) u.o.c.. Then, for any \( B > 0 \), from Lemma 4(ii) we obtain

\[
\limsup_{n \to \infty} \mathbb{E} \int_{0}^{B} \tau^n(t)dt \leq \bar{C}.
\]

This contradicts the fact that (16) must hold for all \( B > 0 \). □

**Lemma 10.** Consider the model with A-perturbations. Assume \( h < \rho < 1/2 \). Consider the system without time/space rescaling. Consider a (random) initial system state, at initial time 0, which is formed as follows: we pick an arbitrary ideal state and apply a perturbation to it. (In other words, a particle is picked uniformly at random, and it is placed into one of the empty sites, chosen uniformly at random.) Then, there exists \( \delta = \delta(p, \rho) > 0 \) such that, uniformly in all the original (pre-perturbation) ideal states and uniformly in all sufficiently large \( n \) (and then in all \( n \)), the following event occurs with probability at least \( \delta \): The perturbation creates a cluster and this cluster will not disappear within (unscaled) time interval \([0, (1 - \rho)n - 4]\).

**Proof.** It will be convenient to relabel time, so that the initial time is 1 (instead of 0). Consider the initial state after the perturbation. Consider the relocated particle, which we will refer to as a “seed” particle; the site to which it relocates we will call the “seed” site. If the perturbation happens to form a cluster, let us consider the dynamics of this cluster, starting the initial time 1. (If the perturbation did not form a cluster, let us view such event as a cluster “disappearance” immediately at time 1.) Obviously, as long as the cluster exists, the particles with leave it on the
right as a Bernoulli process, with “success” (leaving) probability \( p \), starting time 2. (By convention, a particle does not leave the cluster at initial time 1.) Of course, the process of “failures” (non-departures) is Bernoulli with probability \( 1 - p \). Fix \( \epsilon = (\hat{p} - p)/3 \), where \( \hat{p} = \rho/(1 - \rho) > p \), because \( p = h/(1 - h) \). (The meaning of \( \hat{p} \) will be explained shortly.) Let \( D(t) \), \( t = 1, 2, \ldots \), be the cumulative number of “successes” of the Bernoulli process described above (with \( D(1) = 0 \) by convention), by and including time \( t \), and let \( \bar{D}(t) = t - D(t) \) be the corresponding cumulative number of “failures” (with \( \bar{D}(1) = 1 \) by convention). We observe that

\[
\delta_1 = \mathbb{P}\{\bar{D}(t) \geq [(1 - \rho) - \epsilon]t, \ \forall t \geq 1\} > 0. \tag{17}
\]

Now consider how the cluster (if any) “grows” on the left due to particles joining it on the left. (By convention, if the seed particle finds another one immediately to the left of the seed site, we assume that the latter particle joined the cluster immediately at the initial time 1.) Note that the initial cluster (if any) has at most 1 particle to the right of the seed particle and, therefore, at time 1, at least \( \rho n - 3 \) particles (i.e., “almost all”) form a sparse interval to the left of the cluster. Therefore, at least \( \rho n - 2 \) particle which are initially to the left of the seed particle will move unobstructed, at speed 1, until and unless they join the cluster.

Let us define the following process over the time interval \( 1, 2, \ldots, n - (\rho n - 1) = (1 - \rho) n + 1 \). (It is such that it exactly describes the process \( A(t) \) of the cumulative number of particles joining the cluster from the left, in the (a bit shorter) time interval \( 1, 2, \ldots, n - (\rho n - 1) = 2 \cdot (1 - \rho) n - 3 \).) We will illustrate the definition by an example. Let \( n = 10 \) and the total number of particles is \( \rho n = 4 \). Consider the configuration of the \( \rho n - 1 = 3 \) particles on the circle, excluding the seed particle, with respect to the seed site. Starting from the site immediately to the left from the seed site, and moving left, the configuration of the particles (excluding seed particle) is, for example, this sequence of length \( n = 10 \):

\[
1, 0, 0, 1, 0, 0, 0, 0, 1, 0 \tag{18}
\]

(Note that the last element of this sequence is necessarily 0, because it corresponds to the seed site, and we do not include the seed particle into this sequence.) Consider now the sequence of length \( n - (\rho n - 1) = 10 - 3 = 7 \), obtained from \( \hat{A}(5) \) by removing each 0, which immediately follows a 1. We obtain

\[
1, 0, 1, 0, 0, 0, 1 \tag{19}
\]

The cumulative number of 1’s by time (= place in the sequence) \( t \) is the function \( A(t) \), \( t = 1, 2, \ldots, (1 - \rho) n + 1 \). It is easy to see that it would exactly describe the number of particles joining the cluster on the left, assuming the cluster would continue to exist, and assuming all particles except the seed one would move freely at rate 1. It is also easy to observe that, in a little shorter interval \( t = 1, 2, \ldots, (1 - \rho) n - 3 \), \( A(t) \) is exactly the number of particles joining the cluster on the left, assuming the cluster would continue to exist.

Clearly, the process \( A(t) \) satisfies the following conservation law: \( A((1 - \rho) n + 1) = \rho n - 1 \). Let us denote by \( \hat{A}(t) = t - A(t) \) the corresponding cumulative process of “failures” (zeros). The conservation law for \( \hat{A}(t) \) is: \( \hat{A}((1 - \rho) n + 1) = ((1 - \rho) n + 1) - (\rho n - 1) = (1 - 2\rho) n + 2 \). Note that the average slope of \( \hat{A}(t) \) is

\[
\frac{(1 - 2\rho) n + 2}{(1 - \rho) n + 1} = \frac{(1 - 2\rho) + 2/n}{(1 - \rho) + 1/n}, \tag{20}
\]

and is close to

\[
\frac{1 - 2\rho}{1 - \rho} = 1 - \hat{p}, \quad \hat{p} = \rho/(1 - \rho).
\]
Therefore, for all large \( n \), the average slope \( \mathbb{E} \) is at most \( (1 - \hat{p}) + \epsilon < (1 - \hat{p}) + 2\epsilon = (1 - p) - \epsilon \).

Notice that, given locations of all particles except the seed particle on the circle, since the seed particle chooses one of the empty sites (to become the seed site) uniformly at random, all cyclical permutations of a sequence \( \sigma(1, 1) \) appear with equal probabilities. Using the latter observation, we can apply Proposition 8 to obtain the following estimate: for all sufficiently large \( n \),

\[
P\{ \hat{A}(t) < [(1 - \hat{p}) + 2\epsilon]t, \forall 1 \leq t \leq (1 - \rho)n + 1 \} \geq \delta_2 = \frac{\epsilon}{(1 - \hat{p}) + 2\epsilon} > 0. \tag{21}
\]

Combining estimates (17) and (21), we conclude that, uniformly on all initial (pre-perturbation) ideal states and in all sufficiently large \( n \), with probability at least \( \delta = \delta_1 \delta_2 \), a perturbation will create a cluster at time 1 and this cluster will not disappear by and including time \( (1 - \rho)n - 3 \). \( \square \)

As a corollary of Lemmas 7 and 10, we obtain the following

**Lemma 11.** Consider the model with \( A \)-perturbations. Assume \( h < \rho < 1/2 \). For each \( n \) consider the scaled process, with (random) initial state formed by a perturbation of an arbitrary ideal state. Fix \( B \geq 1 - \rho \). Consider the stopping time \( T^n \) which is the minimum of \( B \) and the first time the process hits an ideal state. Then any subsequential (in \( n \to \infty \)) weak limit of the state distributions at time \( T^n \) is such that, with probability at least \( \delta \) (defined in Lemma 10) the state is a MES. (The subsequential weak limits exist, because the entire state space of \( T \) at time \( n \) process hits an ideal state. Then any subsequential (in \( n \to \infty \)) weak limit of the sequence of the stationary distributions of \( [f^n(\cdot,t), \ell^n(t), r^n(t), \tau^n(t)] \) is compact, if we consider the unique “standard” version of each state, satisfying \( f^n(0,t) = 0 \) and \( \ell^n(t) \in [0,1] \).)

**Theorem 12.** Consider the model with \( A \)-perturbations. Assume \( h < \rho < 1/2 \). Then, as \( n \to \infty \), any subsequential weak limit of the sequence of the stationary distributions of \( [f^n(\cdot,t), \ell^n(t), r^n(t), \tau^n(t)] \) is a distribution concentrated on the main equilibrium states. Consequently, the limiting steady-state flux \( \phi = \phi(\rho) = \phi^* = (1 - \tau^*)h = p(1 - \rho) \).

**Remark 13.** Theorem 12 actually holds for \( h < \rho < 1 \). The proof for \( 1/2 < \rho \leq 1 \) is a simplified version of that for \( h < \rho < 1/2 \), because when \( \rho > 1/2 \) a cluster always exists and \( A \)-perturbations never occur. If \( \rho = 1/2 \), the corresponding sequence of pre-limit systems is such that an ideal state either exists infinitely often, or does not exist infinitely often, or both. In any case, if we consider the corresponding subsequences of systems, the proof is the same as for either \( h < \rho < 1/2 \) or \( h > 1/2 \).

**Proof of Theorem 12**. For each \( n \), consider the process with any fixed initial state at 0. Consider stopping time \( \theta^p_1 \), which is the minimum of 2 and the first time (after 0) the system is in an ideal state. Consider the random system state at \( \theta^p_1 \). Consider its any subsequential limit in distribution. By Lemma 7, this distributional limit is such that, almost surely, the state is either a main equilibrium state (MES) or it is an ideal state. (A small subtlety here: we use the fact that, as \( n \to \infty \), the probability that a perturbation occur immediately at a time of hitting an ideal state, vanishes.)

Consider stopping time \( \theta^p_2 \), which is the minimum of \( \theta^p_1 + 2 \) and the first time after \( \theta^p_1 \) the system is in an ideal state. Note that, as \( n \to \infty \), the time to wait for a perturbation converges to exponentially distributed one with mean \( 1/\lambda \); in particular, uniformly in all large \( n \), the probability that a time to wait for a perturbation is less than 1, is at least \( \epsilon = (1 - e^{-\lambda})/2 \). If the state at \( \theta^p_1 \) was ideal, then with probability \( \geq \epsilon \), there will be a perturbation in \( [\theta^p_1, \theta^p_1 + 1] \). Using Lemma 11, we conclude that any subsequential limit in distribution of the system state at \( \theta^p_2 \) is either a MES with probability at least \( \zeta = \epsilon \delta > 0 \), or it is an ideal state. (The \( \delta \) is as defined in Lemmas 10 and 11)
We define stopping times \( \theta^n_k \) for all \( k \geq 2 \) analogously to the definition of \( \theta^n_2 \). By induction, we conclude that any distributional limit of the system state at \( \theta^n_k \) is such that it is either a MES with probability at least \( 1 - (1 - \zeta)^{k-1} \), or it is an ideal state. Note that \( \theta^n_k \leq 2k \). Finally, using Lemma 5(i), we establish that any subsequential limit in distribution of the system state at time \( 2k \) is such that with probability at least \( 1 - (1 - \zeta)^{k-1} \), the state is a MES. This is true for any fixed initial states for each \( n \), and any \( k \). This implies that the limit of stationary distributions of the process is concentrated on MESes. \( \square \)

4 I-perturbations model

Unless specified otherwise, in this section we consider the model with I-perturbations.

4.1 Some preliminary facts.

As a corollary of Lemma 1, we obtain the following fact.

**Lemma 14.** Suppose \( 0 < \rho < h \). Assume there are no perturbations in the (scaled) time interval \([0,1]\). Then, uniformly on the initial states at time 0, which may contain one or multiple clusters,

\[
\lim_{n \to \infty} \mathbb{P}\{\text{At (scaled) time 1 the system is in an ideal state}\} = 1.
\]

*Proof of Lemma 14.* For each \( n \), consider an arbitrary system state at time 0, and pick any cluster (if any). Let us show that, uniformly on the initial states and the choice of the cluster,

\[
\lim_{n \to \infty} \mathbb{P}\{\text{The cluster will disappear before (scaled) time 1}\} = 1. \tag{22}
\]

WLOG, let the right edge of the cluster be located at (scaled) coordinate 0. Then, the “best case,” in terms of the cluster not disappearing for as long as possible, is when this cluster contains all particles in the system. From the law of large numbers, we easily verify that, for a fixed \( \epsilon > 0 \) such that \( \rho + \epsilon < 1 \), as \( n \to \infty \),

\[
\lim_{n \to \infty} \mathbb{P}\{\text{The cluster “dissolves” before the particle initially at 0 reaches point } -\rho - \epsilon\} = 1.
\]

This implies (22).

We are not done, because the total number of initial clusters may increase to infinity with \( n \). However, the following observation resolves this potential issue. Fix a small \( \epsilon > 0 \), such that \( \rho + \epsilon < 1 \). The argument we used above for a single cluster easily generalizes to show the following property:

\[
\lim_{n \to \infty} \mathbb{P}\{\text{All clusters, which at time 0 overlap with } [-\epsilon,0], \text{ will disappear before (scaled) time 1}\} = 1. \tag{23}
\]

Indeed, if at time 0 we “fill in” all holes within the scaled interval \([-\epsilon,0]\) with additional particles, we produce a single cluster (completely covering \([-\epsilon,0]\)), then the original and modified processes can be coupled so that the disappearance of the cluster in the modified system implies the disappearance of all clusters initially overlapping with \([-\epsilon,0]\) in the original system. Note that the total (scaled) number of particles in the modified system is at most \( \rho + \epsilon \). Therefore, (22) applies to the modified system, which proves (23). Obviously, (23) implies the lemma statement, because the circle can be covered by a finite number of \( \epsilon \)-long intervals. \( \square \)

From Lemma 14 we obtain the following corollary.
Lemma 15. Consider the system with I-perturbations. Suppose \( \rho < h \). Then, uniformly on all sufficiently large \( n \) and all initial process states, the probability that the process reaches an ideal state at some time \( t_1 < 1 \) is at least \( e^{-\lambda}/2 \).

Proof. The probability that there are no perturbations in \([0, 1]\) converges to \( e^{-\lambda} \). It remains to apply Lemma 14. \( \square \)

Lemma 16. Suppose \( \rho < h \). Then, uniformly in \( n \), the steady-state number of clusters is stochastically upper bounded by a proper (finite w.p.1) random variable.

Proof. It suffices to show that the lemma statement holds for all sufficiently large \( n \). Fix an arbitrary integer \( k \). By Lemma 15, uniformly on all initial states at time 0, the probability that the process reaches an ideal state in \([0, k]\) is at least \( 1 - (1 - e^{-\lambda}/2)^k \). Considering the time interval from hitting an ideal state within \([0, k]\) until \( k \), and taking into account the (Bernoulli) structure of the perturbation process, we easily obtain the following. (Recall also that a perturbation can increase the number of clusters by at most 2.) Uniformly on the initial states and in all large \( n \), with probability at least \( 1 - (1 - e^{-\lambda}/2)^k \), the number of clusters in the system is stochastically dominated by a random variable \( 2(1 + H_{2\lambda k}) \), where \( H_{2\lambda k} \) has Poisson distribution with mean \( 2\lambda k \). Since this is true for all integers \( k \), we obtain a uniform (in \( n \)) stochastic upper bound on the number of clusters in steady-state. \( \square \)

4.2 FSPs and their properties

In this subsection we define FSPs for I-perturbation process, describe their properties, and give corresponding fluid limit results. For our purposes, it will suffice to consider FSPs with a finite number of initial clusters. The definitions and results are described informally, because, in essence, they are straightforward generalizations of those for the A-perturbation model. We believe this informal description is sufficient – an interested reader can easily fill in all formalities.

FSP on a (finite or infinite) interval \([0, T]\) has the following structure:

\[
\{f(\cdot, t), \quad t \in [0, T]; \quad [\ell_i(t), r_i(t), \tau_i(t)], \quad t \in [0, T], \quad [\alpha_i, \beta_i], \quad i = 0, 1, \ldots \}. \tag{24}
\]

Function \( f(\cdot, t) \) describes the FSP state at \( t \). The times \( \alpha_i \leq \beta_i \leq \infty \) are the beginning and end times of the existence of \( i \)-cluster, and \( \ell_i(t), r_i(t), \tau_i(t) = r_i(t) - \ell_i(t) \) are the left-end, right-end and the length of the \( i \)-th cluster at time \( t \). There is at most a finite number of initial clusters (at time 0) – they are indexed by \( i = 0, 1, \ldots \) in an arbitrary order; for those initial clusters, \( \alpha_i = 0 \). FSP has at most a finite number of clusters by any finite time \( t \) (i.e., those with \( \alpha_i \leq t \)). At any given time \( t \), the clusters present at \( t \) do not overlap. New clusters are indexed in the order of their appearance, i.e. in the order of increasing \( \alpha_i \). We adopt the following convention: if in a pre-limit process a cluster divides into two due to a perturbation, we “ignore” this fact and still treat the divided cluster as one; this does not cause problems, because the “holes” created within a cluster remain “invisible” in the corresponding FSP on any finite time interval, in the sense that their “size” remains 0 and their existence does not affect the dynamics (derivatives) of the left and right ends, \( \ell_i(t) \) and \( r_i(t) \), of the original cluster in the FSP. Given this convention, except possibly time 0, no more than one new cluster may appear (due to a perturbation) at any time \( t > 0 \). By another convention, before cluster \( i \) exists, i.e. for \( t < \alpha_i \), \( \ell_i(t) = r_i(t) = \tau_i(t) = 0 \); and after it disappears, i.e. for \( t \geq \beta_i \), \( \tau_i(t) = 0 \) and \( \ell_i(t) = r_i(t) \) remain frozen at their value at \( \beta_i \). Convergence in the FSP definition is understood as convergence to \([\alpha_i, \beta_i]\) and Skorohod convergence to \([\ell_i(t), r_i(t), \tau_i(t)], t \in [0, T] \), for each \( i \), and the Skorohod convergence to \( f(\cdot, t), \quad t \in [0, T] \).
For each cluster $i$, we have the (same as in the case of A-perturbations) Lipschitz properties of $\ell_i(t), r_i(t), \tau_i(t)$ within $[\alpha_i, \beta_i]$. If we denote $\tau(t) = \sum_i \tau_i$, it is still Lipschitz. We have exactly same properties describing the dynamics of each $\tau_i(t)$ in terms of the density at its left edge $\ell_i(t)$.

Finally, the following properties hold for $\rho < h$, and are proved analogously to the way it is done for A-perturbations:

(i) Denote by $U_i$ the set of those time points in $[0, \infty)$, where $\tau_i'(t) > 0$. Then,

$$\sum_i \mathcal{L}(U_i) \leq 1/(1 + p).$$

(ii) For each cluster $i$, we have that $\alpha_i < \infty$ implies $\beta_i \wedge T - \alpha_i < 1$, i.e., the duration of any cluster is less than 1.

(iii) We have (13):

$$\int_0^T \tau(t) dt \leq \bar{C} = 1 + \frac{1}{2(1 + p)}.$$

4.3 I-perturbations, low-density result.

**Theorem 17.** Consider the model with I-perturbations and $\rho < h$. Then, as $n \to \infty$, the sequence of the stationary distributions of $[f^n(\cdot, t), \tau^n(t)]$ is such that its any subsequential limit is concentrated on the states with $\tau = 0$ and finite number of clusters. In other words, $\tau^n(\infty) \Rightarrow 0$ [equivalently, $\mathbb{E}\tau^n(\infty) \to 0$] and, consequently, the steady-state average flux $\phi = \phi(\rho) \to \rho$.

Proof of Theorem 17 is essentially same as that of Theorem 9. We do use the fact that both the pre-limit process and the FSPs have a finite number of clusters on any finite interval – this means that, just like for A-perturbations, convergence of pre-limit trajectories to an FSP implies $\tau^n(\cdot) \to \tau(\cdot)$ u.o.c.. □

4.4 I-perturbations, high density conjecture.

It is natural to conjecture that for I-perturbations, $h < \rho < 1/2$, the same result as Theorem 12 for A-perturbations holds.

**Conjecture 18.** Consider the model with I-perturbations and $h < \rho < 1/2$. Then,

$$\phi = \phi(\rho) = \lim_{n \to \infty} \phi(\rho; n) = \phi^*.$$

The proof of Theorem 12 however does not completely go through. The key difficulty is that, for I-perturbations with $h < \rho < 1/2$, the number of clusters in steady-state is not uniformly stochastically bounded. The key ideas of the proof of Theorem 12 can be applied to obtain the following partial result, which shows that, when $\rho > h$, the flux equal to $\rho$ is not achievable.

**Theorem 19.** For any $\epsilon > 0$, there exists $\delta > 0$, such that for all large $n$ and the density $\rho \in (h + \epsilon, 1/2)$, the steady-state flux $\phi(\rho, n) < \rho - \delta$.

We do not give a proof of this partial result, because it is a much more involved version of the proof of Theorem 12 and does not provide new insights.

Conjecture 18 is supported by extensive simulations. The simulations confirm that when $\rho > h$ the flux is indeed $\phi^*$. Furthermore, they show that the system “typical” state is “close to” MES in that it contain essentially a single cluster, “punctured” by holes due to perturbations. Figure 4 contains snapshots of the states of two systems (with $n = 400$ sites in the left plot.
Figure 4: The states of the system of $n = 400$ (left) and $n = 900$ sites after $10^7$ time slots

and $n = 900$ sites in the right plot) with the values $\lambda = 1$, $p = 1/2$ and $\rho = 0.4$ (so indeed $1/2 > \rho > \eta = p/(1 + p)$). The systems are started from a random state, where the initially occupied sites are a random sample of $\rho n$ sites out of $n$ without replacement, and the snapshots are taken after $10^7$ time slots. For each site $i$ the plots show $x(i) \ast x(i + 1)$, where $x(i)$ is 1 or 0 if site $i$ is a particle or a hole, respectively. This is a convenient way to observe clusters in a large system. One can see that there is a single cluster on the left and essentially one cluster on the right, except it is “punctured” by two holes.

5 A related zero-range model

Consider the following version of the model we study in this paper. As before, there are $\rho n$ particles moving clockwise on $n$ sites forming a circle. There are no perturbations. A particle that has holes as both neighbors, jumps forward with probability $\pi \in (0, 1]$. A particle that has another particle immediately behind and a hole immediately ahead jumps forward with probability $p \in (0, 1)$. This model, as we will see shortly, can be viewed as a zero-range model, considered in [6]. Denote by $v_\pi(\rho; n)$ and $\phi_\pi(\rho; n)$ the steady-state particle velocity and flux, respectively, of this system. (Subscript $\pi$ will indicate that we refer to a flux for this model, as opposed to TASEP-H.) It is shown in [6] that the stationary distribution of this process has a product-form; moreover, the expressions are derived, which allow to determine the limiting steady-state particle velocity and limiting steady-state flux,

$$v_\pi = v_\pi(\rho) = \lim_{n \to \infty} v_\pi(\rho; n), \quad \phi_\pi = \phi_\pi(\rho) = \lim_{n \to \infty} \phi_\pi(\rho; n),$$

in the same asymptotic regime as in this paper, with $n \to \infty$ and particle density $\rho$ staying constant. If $\pi = 1$, we obtain exactly the basic model of this paper (without perturbations), so that $\phi_1 = \rho$ for any $\rho < 1/2$. If $\pi = 1 - \epsilon < 1$, but close to 1, then the zero-range model can be viewed as our basic model with “perturbations” of a different kind; namely, at each time, with small probability $\epsilon > 0$, a free particle stays in place instead of moving forward. Note, however, that in the zero-range model (with $\pi$ independent of $n$) the “perturbation” rate is much higher – $O(1)$ per particle per unit time – than for our TASEP-H with perturbations model, where the perturbation rate is $O(1/n^2)$ per particle per unit time.

Using the results of [6], we will now show that $\phi_\pi$ is discontinuous at $\pi = 1$, with the limit $\lim_{\pi \to 1} \phi_\pi$ being exactly equal to limiting flux of TASEP-H with perturbations model, namely
\[ \lim_{\pi \to 1} \phi_\pi = \begin{cases} ho, & \rho < h, \\ \phi^* = p(1 - \rho), & \rho > h, \end{cases} \tag{25} \]

Indeed, consider an equivalent model of holes moving left: a hole with exactly one consecutive particle immediately to the left, jumps left with probability \( \pi \); a hole with two or more consecutive particles immediately to the left, jumps left with probability \( p \). For convenience, denote by \( \gamma = 1 - \rho \) the density of holes.

Let \( \eta_\pi = \eta_\pi(\rho) \) denote the limiting (in \( n \to \infty \)) steady-state velocity of a hole. The relation to the velocity of a particle \( v_\pi \) is via the equality of fluxes: \((1 - \rho)\eta_\pi = \rho v_\pi\).

It is clear that (25) it equivalent to

\[ \lim_{\pi \to 1} \phi_\pi = \begin{cases} 1 - \gamma, & \gamma > 1 - h, \\ p\gamma, & \gamma < 1 - h. \end{cases} \]

This, in turn, is equivalent to

\[ \lim_{\pi \to 1} \eta_\pi = \begin{cases} \frac{1 - \gamma}{\gamma}, & \gamma > 1 - h, \\ p, & \gamma < 1 - h. \end{cases} \]

Let us use notation \( q = \frac{1 - \gamma}{\gamma} \). Note that as \( h = \frac{p}{1 + p} \), the condition \( \gamma > 1 - h \) is equivalent to \( q < p \). So finally, we conclude that in order to show (25), it is sufficient to show that

\[ \lim_{\pi \to 1} \eta_\pi = \begin{cases} q, & q < p, \\ p, & q > p, \end{cases} \tag{26} \]

or simply \( \lim_{\pi \to 1} \eta_\pi = q \land p \).

Results of [6] are directly applicable to the model seen as movement of holes. In the notation of [6, Section 5], \( p(1) = \pi, \ p(k) = p \) for \( k \geq 2 \). The function \( f(k) \) (as in [6, Section 5]) in our case is then as follows (we can get rid of the common factor \( 1 - \pi \), because this function is defined up to a positive factor):

\[ f(0) = 1, \quad f(1) = \frac{1}{\pi}, \]

\[ f(k) = \frac{1 - \pi}{\pi p} \left( \frac{1 - p}{p} \right)^{k-2}, \quad k \geq 2. \]

Note that the generating function of \( \{f(k)\} \) is

\[ G(z) = \sum_{k=0}^{\infty} f(k)z^k = 1 + \frac{z}{\pi} + \frac{1 - \pi}{\pi p} \frac{z^2}{1 - (\frac{1 - p}{p})z}; \tag{27} \]

clearly, for real \( z \in [0, p/(1 - p)] \), \( G(z) \) is increasing and \( G(z) \uparrow \infty, \ z \uparrow p/(1 - p) \), and it is easy to check that same is true for the function \( zG'(z)/G(z) \).

From [6] the (limiting in \( n \to \infty \)) fugacity \( z^* = z^*(\gamma, \pi) \) satisfies the equation

\[ 1 - \gamma = \gamma z^* \frac{\partial \log G(z^*)}{\partial z^*} = \gamma z^* G'(z^*)/G(z^*). \tag{28} \]

Note that (28) defines the fugacity \( z^* \) uniquely, and \( z^* \) is also clearly continuous w.r.t. \( \pi \in (0, 1) \) and \( (1 - \gamma)/\gamma \in (0, \infty) \), viewed as parameters. (We will only use the continuity in \( \pi \).)
The expression for the velocity \( \eta_\pi \) in terms of fugacity \( z^* \), is as follows (analogous to (11) in \cite{6}, holds for \( z = z^* \)):

\[
\eta_\pi = [\pi z f(1) + p \sum_{k=2}^{\infty} z^k f(k)]G(z)^{-1}.
\]

The equation above gives the dependence of \( \eta_\pi \) on \( z^* \). For the zero-range model that we consider, this dependence is very simple, and it holds for a quite general function \( p(k) \) (with \( p(0) = 0 \) and \( 0 < p(k) \leq \delta < 1 \) for \( k \geq 1 \)). We have

\[
f(0) = 1, \\
f(k) = \frac{1}{1 - p(k)} \prod_{m=1}^{k-1} \frac{1 - p(m)}{p(m)}, \quad k \geq 1.
\]

Note that

\[
p(k) f(k) = \frac{p(k)}{1 - p(k)} \prod_{m=1}^{k} \frac{1 - p(m)}{p(m)} = \prod_{m=1}^{k-1} \frac{1 - p(m)}{p(m)} = (1 - p(k - 1)) f(k - 1).
\]

Hence the expression for \( \eta_\pi \) is (we write \( z \) instead of \( z^* \) for simplicity below)

\[
\eta_\pi = \sum_{k=1}^{\infty} \frac{p(k) f(k) z^k}{G(z)} = \sum_{k=1}^{\infty} \frac{(1 - p(k - 1)) f(k - 1) z^k}{G(z)}
\]

\[
= \frac{z \sum_{k=1}^{\infty} f(k - 1) z^{k-1} - z \sum_{k=1}^{\infty} f(k - 1) p(k - 1) z^{k-1}}{G(z)}
\]

\[
= \frac{z G(z) - z v G(z)}{G(z)} = z - z \eta_\pi.
\]

Therefore,

\[
\eta_\pi = \frac{z^*}{z^* + 1}, \quad \text{or} \quad z^* = \frac{\eta_\pi}{1 - \eta_\pi}.
\] (29)

Note that the dependence of \( \eta_\pi \) on \( \pi \in (0, 1) \) (and \( \gamma \in (0, 1) \)) is continuous.

Using (28) and (29), for a fixed \( \gamma \), we will now study the limit of \( \eta_\pi \) as \( \pi \uparrow 1 \), and will prove (26).

Substituting (27) into (28), and then \( z^* = \eta_\pi / (1 - \eta_\pi) \), we obtain

\[
(q - \eta_\pi)(p - \eta_\pi)(\eta_\pi(p - \pi - \pi p) + \pi p) = p^2(1 - \pi) \eta_\pi(1 - \eta_\pi).
\] (30)

Note that, as \( \pi \uparrow 1 \), the RHS of (30) converges to 0, while staying positive, because for \( \pi < 1 \) velocity cannot be 1, and cannot be 0 (in view of the LHS of (30)).

Note that \( \eta_\pi \leq q \) must hold for any \( \pi \), because \( q \) is the max velocity achievable for the holes, achieved when all particles move at speed 1. Therefore, to show that \( \eta_\pi \leq q \wedge p \), it suffices to show that \( \eta_\pi \leq p \) in the case \( p < q \). If \( \pi = p \), we have \( \eta_\pi \leq p \) simply because the velocity of a hole in this case cannot possibly exceed \( p \). It is also easy to see that \( \eta_\pi < p \). (In fact, in this case the velocity is known explicitly: \( \eta_\pi = (1 - \sqrt{1 - 4 \pi \gamma (1 - \gamma) / 2 \rho}) \), cf. [6].) As we continuously increase \( \pi \) in the interval \( [p, 1] \), the RHS of (30) must stay positive, so for all those \( \pi \) we must have \( \eta_\pi < p \). This completes the proof of the fact that \( \eta_\pi \leq q \wedge p \) for all \( \pi < 1 \).

Finally, again, as we continuously increase \( \pi \) in the interval \( [p, 1] \), velocity \( \eta_\pi \) changes continuously, the RHS of (30) must stay positive and converge to 0. The only option is that \( \eta_\pi \rightarrow q \wedge p \) as \( \pi \uparrow 1 \). □
6 Discussion and further conjectures

Our analysis of the TASEP-H model with perturbations suggests the following generalizations. Indeed, consider the A-perturbation model. As $n \to \infty$, the average time it takes the cluster in a MES to dissolve, grows exponentially fast with $n$. (A standard large-deviations analysis should apply.) Therefore, if the inter-perturbation times grow sub-exponentially in $n$, then the fraction of time when the system spends in MES will dominate.

We will say that a positive, monotone function $g(k)$ is sub-exponential in $k$ (where either $k \to \infty$ or $k \to -\infty$), or write $g(k) = \text{SUBEXP}(k)$, if

$$\lim_{|k| \to \infty} \log(g(k))/|k| = 0.$$  

Conjecture 20. Consider the model with either A-perturbations or I-perturbations. Suppose, the perturbation rate per-particle, per-time-slot is $g(n) = \text{SUBEXP}(n)$. Then the limiting steady-state fluxes are as follows.

(i) If $0 < \rho < h$,

$$\phi(\rho) = \lim_{n \to \infty} \phi(\rho; n) = \rho.$$ 

(ii) If $h < \rho < 1/2$,

$$\phi(\rho) = \lim_{n \to \infty} \phi(\rho; n) = \phi^*.$$  

The intuition that leads to Conjecture 20 for TASEP-H with perturbations, as well as the informal connection between TASEP-H and the zero-range model, also suggests that the following conjecture is very plausible.

Conjecture 21. Consider the zero-range model. Suppose, parameter $\pi$ depends on $n$ so that $1 - \pi(n) = \text{SUBEXP}(n)$ Then the limiting steady-state fluxes are as follows.

(i) If $0 < \rho < h$,

$$\lim_{n \to \infty} \phi_{\pi(n)}(\rho; n) = \rho.$$ 

(ii) If $h < \rho < 1/2$,

$$\lim_{n \to \infty} \phi_{\pi(n)}(\rho; n) = \phi^*.$$  

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A Proof of Proposition [8]

The result is derived from Theorem 1.2.5 – a ballot theorem – in [12]. Without loss of generality, we can assume that \( m = n \). (Otherwise, \( m \) and all \( k_r \) can be rescaled by the same factor, and this obviously does not change \( N \).) Then, conditions (14) and (15) become, respectively,

\[
\psi(j + r) - \psi(j) < r, \quad r = 1, \ldots, n,
\]

and

\[
N \geq \lceil n - \psi(n) \rceil.
\]

For each integer \( j \geq 0 \), let \( \eta(j) = 1 \) if (31) holds, and \( \eta(j) = 0 \) otherwise. (So that, \( N = \sum_{0}^{n-1} \eta(j) \).) Let us extend the definition of \( \psi(j) \) to all real \( u \geq 0 \) by letting \( \psi(u) = \psi(\lfloor u \rfloor) \). For each \( u \geq 0 \), let \( \delta(u) = 1 \) if

\[
v - u \geq \psi(v) - \psi(u) \quad \text{for all} \quad v \geq u,
\]

and \( \delta(u) = 0 \) otherwise. Denote by \( D \) the set of those \( u \in [0, n) \), for which \( \delta(u) = 1 \). By Theorem 1.2.5 in [12], \( \mathcal{L}(D) = n - \psi(n) \), where \( \mathcal{L} \) denotes the Lebesgue measure. For any interval \( [j, j+1) \), \( j = 0, 1, \ldots, n - 1 \), observe the following: if \( \delta(u) = 1 \) for some \( j < u < j + 1 \), then necessarily \( \eta(j) = 1 \). Indeed, in this case, for any \( r = 1, \ldots, n \),

\[
r - [\psi(j + r) - \psi(j)] = (u - j) + (j + r - u) - [\psi(j + r) - \psi(u)] \geq u - j > 0,
\]

where the first inequality is by (33). This implies that for each interval \( [j, j+1) \) such that \( \mathcal{L}(D \cap [j, j+1)) > 0 \) we have \( \eta(j) = 1 \). Since, obviously, \( \mathcal{L}(D \cap [j, j+1)) \leq 1 \) for any \( j \), and

\[
\sum_{j=0}^{n-1} \mathcal{L}(D \cap [j, j+1)) = \mathcal{L}(D) = n - \psi(n),
\]

we see that the (integer) number \( N \) of those \( j \in \{0, 1, \ldots, n - 1\} \) for which \( \eta(j) = 1 \) cannot be less than \( n - \psi(n) \). This proves (15). \( \square \)
B A CSMA model indirectly motivating the holdback property of TASEP-H

In this section we describe an interacting particle system, which is a model of a fairly realistic wireless network under a CSMA protocol. This system is far more complicated than TASEP-H, but exhibits qualitatively similar behavior. We will formulate some hypotheses and discuss the difficulties one faces in analyzing such a system. We briefly introduce the model here and refer to [10] for an account of known results on the topic, in particular stability analysis.

As in the TASEP-H setting, there are \( n \) nodes arranged in a circle, the overall density \( \rho \) of particles is fixed, the time is slotted. There is however no restriction on the number of particles present at a node at a given time slot. Moreover, a particle may move to another node even if that node is non-empty (the particles that do move in a time slot are chosen according to a randomised competition described below). At the beginning of each time slot the nodes are given access priorities, forming a permutation of numbers 1, \ldots, \( n \), picked independently (across time slots), uniformly at random from all possible permutations. From the node with the highest priority, if it is not empty, exactly one particle (chosen at random, say) moves to its neighbor on the right. From the node with the second-highest priority, exactly one particle moves to the neighbor on the right, as long as the node is not empty and that no particle has already moved from any of its neighbor nodes. And so on until all nodes are checked in their priority order. The procedure described above is repeated independently over time slots.

The restriction that a particle can only move from a node if no particle moved from any of the neighbors of the node models the interference in wireless networks, and it is the most important characteristic of models of these networks.

Thus, there are \( \rho n \) particles moving on a circle of \( n \) sites, where \( \rho \in (0, \infty) \) (not necessarily less than 1). We are interested in the flux of this system. As in the TASEP-H model, if \( \rho < 1/2 \) an ideal (absorbing) state will be eventually reached and the system will stay in it, thus the steady-state flux is equal to \( \rho \) as long as \( \rho < 1/2 \) (see Figure 5).

A typical flux (defined via a system with perturbations) that we observe in simulations is shown in Figure 5 (The typical flux plot in Figure 5 is just a qualitative illustration – it does not describe a specific simulation experiment.) We observe that the flux is monotone increasing, asymptotically converging to the parking constant \( c = (1/2)(1 - e^{-2}) \approx 0.43 \) as \( \rho \to \infty \). The asymptotic limit \( c \) is intuitive. Indeed, as \( \rho \to \infty \), almost all nodes will be typically occupied. If so, when \( n \) is large, the fraction of nodes that will move a particle to the next node will be equal to the expected number of cars parked per slot in the so-called discrete parking system (we refer to [10] for an explanation of the connections between the models). We also conjecture that there is a phase transition at some density level \( h^* \). The conjectured phase transition should be qualitatively similar to that for TASEP-H model. Namely, when the density \( \rho < h^* \), the clusters that may emerge will not have a tendency to grow very large, because, if/when they become large they “lose” particles on the right faster than “acquire” new particle on the left (due to free particles hitting it). As a result, when \( \rho < h^* \) almost all (in the \( n \to \infty \) limit – all) particles remain free. If \( \rho > h^* \), very large clusters form and contain a non-zero fraction of particles, while the density of particles in the sparse intervals is \( h^* \). (The value of \( h^* \) is not known to us. A simple heuristic argument leads us to believe that \( h^* \geq 1/9 \). Indeed, the rate at which particles “break away” on the right from a cluster, followed by a sparse interval, is at least 1/8; within any two time slots, a particle breaks away with probability at least \((1/2)^2 \). And if the density of the sparse interval on the left of the cluster is at most 1/9, then particles join the cluster on the left at the rate at most \((1/9)/(1 - 1/9) = 1/8 \). Thus, if the overall density \( \rho < 1/9 \), clusters should have a tendency to dissolve.)
We further conjecture that for $\rho > 1/2$ the steady-state flux (of the system without perturbations) coincides with the typical flux. That is, the steady-state flux jumps “down” at $\rho = 1/2$, and then is equal to the typical flux (see Figure 5).

We note that an analysis of the system in this section is much more difficult than that of the TASEP-H system due to a number of reasons, including, but not limited to, the following: in any time slot, any cluster may in principle break into any number of sub-clusters even without perturbations; similarly, a large number of sub-clusters may join and form a much larger cluster; even if the right-most site of a cluster moves a particle to the right, the latter particle does not necessarily enter a sparse interval (because the site may have had other particles in it).

![Figure 5: Stationary and typical flux in the CSMA model](image-url)