MDS codes over finite fields

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Abstract

The mds (maximum distance separable) conjecture claims that a nontrivial linear mds \([n,k]\) code over the finite field \(GF(q)\) satisfies \(n \leq (q+1)\), except when \(q\) is even and \(k = 3\) or \(k = q - 1\) in which case it satisfies \(n \leq (q+2)\).

For given field \(GF(q)\) and any given \(k\), series of mds \([q+1,k]\) codes are constructed.

Any \([n,3]\) mds or \([n,n-3]\) mds code over \(GF(q)\) must satisfy \(n \leq (q+1)\) for \(q\) odd and \(n \leq (q+2)\) for \(q\) even. For even \(q\), mds \([q+2,3]\) and mds \([q+2,q-1]\) codes are constructed over \(GF(q)\).

The codes constructed have efficient encoding and decoding algorithms.

1 Introduction

Background on coding theory and related material made be found in [11] or in [4]. Now \(GF(q)\) denotes the finite field of order \(q\) and \(q\) is necessarily a power of a prime. An \([n,k]\) linear code over \(GF(q)\) is a linear code \(C\) of length \(n\) and dimension \(k\) over \(GF(q)\).

The minimum distance \(d\) of \(C\) is bounded by the Singleton bound \(d \leq (n+1-k)\). If \(d = (n+1-k)\), then the code \(C\) is termed a maximum distance separable (mds) code. The mds codes are those with maximum error correcting capability for a given length and dimension. MacWilliams and Sloane refer to mds codes in their book [11] as “one of the most fascinating chapters in all of coding theory”; mds codes are equivalent to geometric objects called \(n\)-arcs and combinatorial objects called orthogonal arrays, [11], and are, quote, “at the heart of combinatorics and finite geometries”.

The mds conjecture is due originally to Segre [12] from 1955.

mds conjecture: If \(C\) is a nontrivial linear mds \([n,k]\) code over \(GF(q)\), then \(n \leq (q+1)\), except when \(q\) is even and \(k = 3\) or \(k = (q-1)\) in which case \(n \leq (q+2)\).

There is a large literature focusing on this problem, for example see [5, 1, 13]. Ball showed [2] that the mds conjecture is true for prime fields. For a list of when the conjecture is known to hold for \(q\) non-prime, see [6, 7]

Here for any given finite field \(GF(q)\) and any given \(k\), series of mds \([q+1,k]\) codes over \(GF(q)\) are constructed.

Methods in [10] can be adopted to give efficient decoding algorithms; the complexity is \(\max\{O(\log n), t^2\}\) where \(n\) is the length and \(t\) is the error-correcting capability which is \(\lfloor \frac{d+1}{2} \rfloor\) where \(d\) is the distance.

For even \(q\), it is shown that any \([n,3]\) mds code and any \([n,n-3]\) mds code over \(GF(q)\) must satisfy \(n \leq (q+2)\) and for odd \(q\) any \([n,3]\) or \([n,n-3]\) mds code over \(GF(q)\) must satisfy \(n \leq (q+1)\). For even \(q\), series of \([q+2,3]\) mds codes and \([q+2,q-1]\) mds codes over \(GF(q)\) are constructed.

The mds codes constructed over prime fields are maximum length for the field. The more general case is dealt with separately.

2 Basics

A primitive \(n^{th}\) root of unity in a field \(F\) is an element \(\omega\) such that \(\omega^n = 1\) but \(\omega^i \neq 1\) for \(1 \leq i < n\).

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In a finite field $GF(q)$, a primitive $(q - 1)$ root of unity always exists; see for example [4] [11] or any book on field theory. Let $\omega$ be a primitive element in $GF(q)$ so that $\omega$ has order $q - 1 = t$. Then $S = \{1, \omega^1, \omega^2, \ldots, \omega^{t-1}\}$ are the distinct elements of $GF(q)/\{0\}$, $\omega' = 1$ and $\omega' \neq 1$ for $1 \leq i < t$.

See for example [4] or [11] for the following result. A $k \times n$ matrix $G$ is the generator matrix of an mds $[n,k]$ code if and only if any $k \times k$ submatrix of $G$ has non-zero determinant. Also a $k \times n$ matrix is a check matrix of a $[n,n-k]$ mds code if and only if any $k \times k$ submatrix has non-zero determinant.

An $[n,k]$ code is an mds code if and only if its dual is an $[n,n-k]$ mds code. [11, 4].

Recall the mds codes constructed in [10].

A Fourier matrix is a special type of Vandermonde matrix. Let $\omega$ be a primitive $n^{th}$ root of unity in a field $F$; primitive here means that $\omega^n = 1$ and $\omega^i \neq 1$ for $1 \leq i < n$. The Fourier matrix $F_n$, relative to $\omega$ and $F$, is the $n \times n$ matrix

$$
F_n = \begin{pmatrix}
1 & 1 & 1 & \ldots & 1 \\
1 & \omega & \omega^2 & \ldots & \omega^{n-1} \\
1 & \omega^2 & \omega^4 & \ldots & \omega^{2(n-1)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \omega^{n-1} & \omega^{2(n-1)} & \ldots & \omega^{(n-1)(n-1)}
\end{pmatrix}
$$

Simplifications can be made to some of the powers from $\omega^n = 1$. An $n^{th}$ root of unity can only exist in a field provided the characteristic of the field does not divide $n$ and in this case $n^{-1}$ exists.

Then

$$
\begin{pmatrix}
1 & 1 & 1 & \ldots & 1 \\
1 & \omega & \omega^2 & \ldots & \omega^{n-1} \\
1 & \omega^2 & \omega^4 & \ldots & \omega^{2(n-1)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \omega^{n-1} & \omega^{2(n-1)} & \ldots & \omega^{(n-1)(n-1)}
\end{pmatrix}
\begin{pmatrix}
1 & 1 & 1 & \ldots & 1 \\
1 & \omega^{n-1} & \omega^{2(n-1)} & \ldots & \omega^{(n-1)(n-1)} \\
1 & \omega^{n-2} & \omega^{2(n-1)} & \ldots & \omega^{(n-1)(n-2)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \omega^2 & \omega^{2(n-1)} & \ldots & \omega^{(n-1)}
\end{pmatrix} = nI_n
$$

The inverse of $F_n$ can be obtained from the above by multiplying through by $n^{-1}$ when it exists.

Recall the following from [10]:

**Theorem 2.1** [11]

(i) Let $F_n$ be a Fourier $n \times n$ matrix over a field $F$. Let $C$ be a code obtained by choosing in order $r$ rows of $F_n$ in arithmetic sequence with arithmetic difference $k$ satisfying $gcd(n,k) = 1$. Then $C$ is an mds $[n, r, n - r + 1]$ code.

In particular this is true when $k = 1$, that is, when the $r$ rows are chosen in succession.

(ii) Let $C$ be as in part (i). Then there exist explicit efficient encoding and decoding algorithms for $C$.

Thus series of mds codes are formed from rows of a Fourier matrix using this unit-derived method developed initially in [9].

It is possible to choose rows which ‘wrap over’ and still get an mds code as long as the arithmetic difference $k$ between the rows is the same and satisfies $gcd(k,n) = 1$ – consider row $n + i$ the same as row $i$.

Some of the methods of [10] are generalizations of those of [8] but the papers are independent.

In particular the following mds codes are formed from a finite field $GF(q)$ by using the primitive $(q - 1)$ root.

**Theorem 2.2** (See [11]) Let $GF(q)$ be a finite field and $\omega$ a primitive $(q - 1)$ root of unity in $GF(q)$.
Form the Fourier \((q - 1) \times (q - 1)\) matrix relative to \(\omega:\)

\[
F_{q-1} = \begin{pmatrix}
1 & 1 & 1 & \cdots & 1 \\
1 & \omega & \omega^2 & \cdots & \omega^{q-2} \\
1 & \omega^2 & \omega^4 & \cdots & \omega^{2(q-2)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \omega^{q-2} & \omega^{2(q-2)} & \cdots & \omega^{(q-2)(q-2)}
\end{pmatrix}
\]

Then choosing any \(r\) rows of \(F_{q-1}\) in arithmetic sequence with difference \(k\) satisfying \(\gcd(q - 1, k) = 1\) gives a generator matrix for an \([q - 1, r]\) mds code. In particular taking consecutive rows gives an mds code.

Further there exist explicit efficient encoding and decoding algorithms for the codes.

Consider cases where the first \(r\) rows are chosen; other cases are similar.

\[
A = \begin{pmatrix}
1 & 1 & 1 & \cdots & 1 \\
1 & \omega & \omega^2 & \cdots & \omega^{q-2} \\
1 & \omega^2 & \omega^4 & \cdots & \omega^{2(q-2)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \omega^{r-1} & \omega^{2(r-1)} & \cdots & \omega^{(r-1)(q-2)}
\end{pmatrix}
\]

This is a generator matrix for an \([q - 1, r]\) mds code and \(A\) is an \(r \times (q - 1)\) matrix.

Any \(r \times r\) submatrix of \(A\) has non-zero determinant as \(A\) generates an mds code.

Now extend the length \((q - 1)\) to \((q - 1 + 2) = (q + 1)\) as follows to obtain an mds code \([q + 1, r]\) code. Extend \(A\) by adding on two further \(r \times 1\) columns \(v = (1, 0, \ldots, 0)^T, w = (0, 0, \ldots, 0, 1)^T\) to obtain the \(r \times (q + 1)\) matrix \(B = (v, w, A)\).

**Theorem 2.3** The matrix \(B\) generates an mds \([q + 1, r]\) code.

**Proof:** This is proved by showing that any \(r \times r\) submatrix of \(B\) has non-zero determinant.

If the \(r \times r\) submatrix is from \(A\), only, then it has non-zero determinant since \(A\) generates an mds \([q - 1, r]\) code.

Consider the case where the \(r \times r\) submatrix \(P\) is formed by taking the first column of \(B\) together with \((r - 1)\) columns of \(A\).

Then \(P = (v, A')\) where \(A'\) is an \(r \times (r - 1)\) of \(A\) with rows of \(F_{q-1}\) in sequence.

Evaluate the determinant of \(P\) by expanding via the first column and get \(\det(P) = \det(A'')\) where \(A''\) is an \((r - 1) \times (r - 1)\) submatrix of \(F_{q-1}\) with rows in sequence. Then \(\det(A'') \neq 0\) and so \(\det(P) \neq 0\).

Similarly the case where an \(r \times r\) submatrix formed by taking the second column of \(B\) with \((r - 1)\) columns of \(A\) can be shown to have non-zero determinant.

Now form \(Q = (v, w, A''')\) where \(A'''\) consists of \((r - 2)\) columns of \(A\). Expand by first column and get that \(\det(Q) = \det(w', A''')\) where \(w'\) is \(w\) with first zero omitted and \(A'''\) is \((r - 1) \times (r - 2)\) submatrix of \(A\) with rows in sequence from \(F_n\). Now expand by first column and get that \(\det(Q) = \pm \det(A''')\) where \(A'''\) is from \(F_n\) with rows in sequence and so \(\det(A''') \neq 0\). Hence \(\det(Q) \neq 0\) as required.

The result depends on the fact that any (square) \(y \times y\) submatrix of \(y\) rows in sequence of the Fourier matrix have non-zero determinant [8][10]; this requires the \(\{v, w\}\) to have the forms given – with one starting with 1 and the other ending with 1 and all other entries equal to zero.

It is clear from Theorem 2.1 that the \(A\) may be chosen by taking \(r\) rows of \(F_{q-1}\) in arithmetic sequence with difference \(k\) satisfying \(\gcd(k, q - 1) = 1\).

Encoding and decoding is obtained by adapting the methods in [10] to the present situation.

More generally get the following result:
**Theorem 2.4** Given the finite field $GF(q)$, form the Fourier $(q - 1) \times (q - 1)$ matrix $F_{q-1}$ using a primitive $(q - 1)$ element in $GF(q)$. Form the $r \times (q - 1)$ matrix $A$ by choosing $r$ rows of $F_{q-1}$ in arithmetic sequence with arithmetic difference $k$ satisfying $\gcd(k, q - 1) = 1$. Let $v = (1, 0, \ldots, 0)^T$, $w = (0, 0, \ldots, 1)^T$ where these are of size $r \times 1$. Let $B$ be the $r \times (q + 1)$ matrix obtained by adding $\{v, w\}$ as columns to $A$. Then $B$ is the generator matrix of an $[q + 1, r]$ mds code.

Moreover the methods in [10] may be adopted to give efficient encoding and decoding algorithms for the code generated by $B$. The complexity of this is $\max\{O(n \log n), t^2\}$ where $t = \lfloor \frac{d - 1}{d} \rfloor$ with $d(= q - r + 2)$ is the distance of the mds code $[q+1, r]$.

**Samples**

1. Let the field be $GF(3^2)$. The examples from this small field may be obtained directly but are chosen to illustrate the general methods. Let $\omega$ be a primitive $8^{th}$ root of unity in $GF(9)$. Here $q = 9, q + 1 = 10$ by reference to Theorem 2.3 or 2.4. It is required to construct a $[10, r]$ mds code over $GF(9)$.

Consider $r = 4$ as an illustration; the construction for a general $r$ is similar.

From the general construction above, the following is an $[10, 4]$ mds code.

$$
\begin{pmatrix}
1 & 0 & 1 & 1 & 1 & \cdots & 1 \\
0 & 0 & 1 & \omega & \omega^2 & \cdots & \omega^7 \\
0 & 0 & 1 & \omega^2 & \omega^4 & \cdots & \omega^{14} \\
0 & 1 & 1 & \omega^3 & \omega^6 & \cdots & \omega^{21}
\end{pmatrix}
$$

Note that $\omega^8 = 1$ and some of the powers may be simplified.

There are other possibilities by varying the matrix $A$ obtained from the Fourier matrix, Theorem 2.3. For example choose $2^{nd}, 5^{th}, 8^{th}$ rows (with arithmetic difference 3 and $\gcd(3, 8) = 1$) of Fourier $F_8$ over $GF(9)$ using $\omega$ to get

$$A = \begin{pmatrix}
1 & \omega & \omega^2 & \cdots & \omega^7 \\
1 & \omega^4 & \omega^5 & \cdots & \omega^{20} \\
1 & \omega^7 & \omega^6 & \cdots & \omega^1
\end{pmatrix}.
$$

Then add the two columns $(1, 0, 0)^T, (0, 0, 1)^T$ to the front of $A$ to get a $3 \times 10$ matrix $B$ which is then a generator matrix for a $[10, 3]$ mds code over $GF(9)$.

Choosing $\{2^{nd}, 5^{th}, 8^{th}, 11^{th} = 3^{rd}\}$ rows, by wrapping, to construct $A$ and then add the two columns as before to get $B$ which is then a $[10, 4]$ mds code over $GF(3^2)$.

There are many choices.

2. Consider $GF(3^3)$. Here the $q = 27$ and $q + 1 = 28$ from general considerations. Construct $[28, r]$ mds codes over $GF(27)$.

Let $\omega$ be a primitive $26^{th}$ root of unity in $GF(27)$. Form the Fourier $F_{26 \times 26}$ matrix over $GF(27)$ using $\omega$.

Say $r = 4$ for illustration; the more general $r$ is similar.

$$B = \begin{pmatrix}
1 & 0 & 1 & 1 & 1 & \cdots & 1 \\
0 & 0 & 1 & \omega & \omega^2 & \cdots & \omega^{25} \\
0 & 0 & 1 & \omega^2 & \omega^4 & \cdots & \omega^{50} \\
0 & 1 & 1 & \omega^3 & \omega^6 & \cdots & \omega^{78}
\end{pmatrix}
$$

(Some of the powers may be simplified on noting $\omega^{26} = 1$.) $B$ is formed using the first 4 rows of $F_{26 \times 26}$ together with $v = (1, 0, 0, 0)^T, w = (0, 0, 0, 1)^T$.

Then $B$ is an $[28, 4]$ mds code over $GF(27)$.

To get a $[28, 24]$ mds code over $GF(27)$, take $B$ as the check matrix of a code. Alternatively take an $[26, 24]$ mds code from the Fourier $26 \times 26$ matrix and add on the two columns $\{v, w\}$ as before. There are many other ways that the $A$ could be formed as noted previously and the $B$ is obtained by adding on the two extra columns, one beginning with 1 and the other ending with 1 and all other entries zero.
3. Consider the prime field $GF(257)$. Now the order of $(3 \mod 257)$ is 256. Thus $\omega = (3 \mod 257)$ is a primitive element in $GF(257)$. Here $q = 257, q + 1 = 258$ from general considerations. Construct $[258, r]$ mds codes over $GF(257)$ as follows.

Form the Fourier $256 \times 256$ matrix $F_{256}$ over $GF(257)$ using $\omega = (3 \mod 256)$ as the primitive element. Choose $r$ rows of $F_{256}$ chosen is arithmetic sequence with difference $k$ satisfying $gcd(k, 256) = 1$ to form a $r \times 256$ matrix $A$. Now add the two columns $u = (1, 0, 0, \ldots, 0)^T, w = (0, 0, 0, \ldots, 0, 1)^T$ of length $r$ to the front of $A$ to form a matrix $B$. Then $B$ generates an mds $[258, r]$ code.

This code is the maximum length code that can be formed from $GF(257)$. Note also that the arithmetic for the codes is modular arithmetic performed in $GF(257)$ using powers of $(3 \mod 257)$ only. Note that efficient encoding and decoding algorithms exist of complexity $O(n \log n, t^2)$ where $n = 256, t = \lfloor \frac{d}{2}\rfloor$ where $d$ is the distance which equals $257 - r$.

3 Even $q$ and dimension 3

Consider $GF(q)$ where $q$ is even.

3.1 Sample

Consider $GF(2^3)$ initially. Let $\omega$ be a primitive 7th root of unity in $GF(8)$. Let $A$ be the first three rows of the Fourier $7 \times 7$ matrix formed using $\omega$.

Define $B = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & \omega & \omega^2 & \omega^3 \\ 0 & 0 & 1 & 1 & \omega^2 & \omega^4 & \omega^6 \end{pmatrix}$

This is $B = (u, v, w, A)$ where $u = (1, 0, 0)^T, v = (0, 1, 0)^T, w = (0, 0, 1)^T$ and $A$ is the first three rows of the Fourier $7 \times 7$ matrix formed using $\omega$ as the primitive 7th root of unity.

Show that $B$ is an mds $[10, 3]$ code over $GF(8)$ as follows.

Now any $3 \times 3$ submatrix of $B$ involving columns of $A$, only, has non-zero determinant as $A$ generates an mds code, Theorem 2.1, [10].

If any of $\{u, v, w\}$ with two columns of $A$ are used to form a $3 \times 3$ submatrix then for it to have a zero determinant it must be that $A$ has a $2 \times 2$ submatrix with zero determinant. It may be verified that no $2 \times 2$ submatrix of $A$ has zero determinant directly using Lemma 3.1 below; a direct proof of a more general result which includes this is given in Proposition 3.1.

If a $3 \times 3$ matrix formed with two of $\{u, v, w\}$ together with a column of $A$ has a zero determinant then this means that $A$ has a zero element which it doesn’t.

Thus $B$ is a generator of an mds $[10, 3]$ code.

To form an $[10, 7]$ mds code we may take $B$ as the check matrix of a $[10, 3]$ code.

There are many choices for a $3 \times 10$ matrix $A$ from the Fourier matrix and then add on the three columns as noted to get a $B$ which is then a $[10, 3]$ mds code.

3.2 General case

Consider $GF(2^n)$. To construct $[2^n + 2, 3]$ mds codes in $GF(2^n)$ consider the Fourier $F_{(2^n+1) \times (2^n+1)}$ matrix formed using the primitive $(2^n - 1)^{th}$ root of unity $\omega$ in $GF(2^n)$. Take the first three rows, or any three rows in arithmetic sequence with difference $k$ satisfying $gcd(k, 2^n - 1) = 1$, of this Fourier matrix to form a $3 \times (2^n - 1)$ matrix $A$, which generates an mds code. Now form the matrix $B$ by adding the three columns of $I_3$ to the front (or anywhere indeed) of $A$.

The proof that this $B$ is an $[2^n + 2, 3]$ mds code then reduces to showing that this $3 \times (2^n - 1)$ matrix $A$ from the Fourier matrix has no $2 \times 2$ submatrix with determinant equal to zero.

Using this $3 \times (2^n + 2)$ matrix $B$ as a check matrix gives a $[2^n + 2, 2^n - 1]$ mds code.
Lemma 3.1 Let $\omega$ be a primitive $n^{\text{th}}$ root of unity. Then $\det\left(\begin{array}{cc}
\omega^i & \omega^j \\
\omega^k & \omega^l
\end{array}\right) = 0$ if and only if $i - k \equiv j - l \mod n$.

Proof: $\det\left(\begin{array}{cc}
\omega^i & \omega^j \\
\omega^k & \omega^l
\end{array}\right) = 0$ if and only if $\omega^\ell \omega^j - \omega^i \omega^k = 0$ if and only if $\omega^{i + l} = \omega^{j + k}$ if and only if $i + l \equiv j + k \mod n$ if and only if $i - k \equiv j - l \mod n$.

Lemma 3.2 Suppose in a field the order of $\omega$ is $t$ where $t = 2j + 1$ is odd. Then the matrix

\[
\begin{pmatrix}
1 & 1 & 1 & \ldots & 1 \\
1 & \omega & \omega^2 & \ldots & \omega^{t-1} \\
1 & \omega^2 & \omega^4 & \ldots & \omega^{2t-2}
\end{pmatrix}
\]

has no $2 \times 2$ submatrix with determinant equal to zero.

Proof: Note that $\omega^{2i-j} = \omega^i$ and $\omega^{2i} = \omega^{2j}$ if and only if $\omega^{2(i-j)} = 1$ if and only if $\omega^{j-i} = 1$ as $\omega$ has odd order; for $i, j < t$ this implies $i = j$.

It is clear that there are no $2 \times 2$ submatrices formed from the first row and either of the second or third rows with determinant zero as (i) $\{\omega, \omega^2, \ldots, \omega^{t-1}\}$ are distinct and (ii) $\{\omega^2, \omega^4, \ldots, \omega^{2t-2}\}$ are distinct.

It remains to show that a $2 \times 2$ submatrix formed from second and third row cannot have determinant equal to zero.

Work out the differences between the powers in the third row with those immediately above in the second row: $\{0, \omega, \omega^2, \ldots, \omega^{1}, \omega^{1} + 1, \omega^{1} + 2, \ldots, \omega^{2}\}$.

These are all different and so by Lemma 3.1 there is no $2 \times 2$ submatrix from second and third rows with determinant equal to zero.

Corollary 3.1 Suppose $\omega$ has odd order $t = 2j + 1$ in a field. Then if $A = \begin{pmatrix}
1 & 1 & 1 & \ldots & 1 \\
1 & \omega & \omega^2 & \ldots & \omega^{t-1} \\
1 & \omega^2 & \omega^4 & \ldots & \omega^{2t-2}
\end{pmatrix}$

has no $3 \times 3$ submatrix with determinant zero then $B = \begin{pmatrix}
1 & 0 & 0 & 1 & 1 & \ldots & 1 \\
0 & 1 & 0 & 1 & \omega & \omega^2 & \ldots & \omega^{t-1} \\
0 & 0 & 1 & 1 & \omega^2 & \omega^4 & \ldots & \omega^{2t-2}
\end{pmatrix}$ has no $3 \times 3$ submatrix with determinant equal to zero.

Proof: If the $3 \times 3$ submatrix is taken from $A$, only, then know it has non-zero determinant as $A$ generates an mds code. If the submatrix involves one of the first three columns of $B$ then by expanding along this column its determinant is zero if and only if $A$ has a $2 \times 2$ submatrix whose determinant is zero; by Lemma 3.2 this cannot happen. If it involves two of the first three columns of $B$ then by expansion by these columns in order its determinant is non-zero as $A$ has no zero entry. If it involves all three of the first columns of $B$ then obviously the determinant is 1 and is non-zero.

Proposition 3.1 Form the Fourier matrix $F_{2^n-1}$ from a primitive $2^n - 1$ root of unity in $GF(2^n)$. Let $A$ be the matrix of the first three rows of $F_{2^n-1}$ and $B$ the matrix formed by adding the columns of $I_3$ to $A$. Then $B$ generates an mds $[2^n + 2, 3]$ code.

Proof: This follows from Lemma 3.2 and Corollary 3.1.

The matrix $A$ in the construction may be formed by taking three rows of the Fourier matrix, as in Proposition 3.1, in arithmetic sequence with difference $k$ satisfying $\gcd(k, 2^n - 1) = 1$. Then adding in the columns of the matrix $I_3$ gives an mds $[2^n + 2, 3]$ code over $GF(2^n)$.

By taking the matrix $B$ as the check matrix of a code, an $[2^n + 2, 2^n - 1]$ mds code is obtained.

For even $q$ any $[n, 3]$ mds code over $GF(q)$ must satisfy $n \leq q + 2$; this is shown in Proposition 3.2. Thus the $[q + 2, 3]$ mds codes and $[q + 2, q - 1]$ produced here are best possible length over the field $GF(q)$ for those dimensions and even $q$. 

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That efficient encoding and decoding methods are available by adapting the methods of [10] is a great advantage. The complexity of encoding and decoding is \( \max\{n \log n, t^2\} \) where \( n \) is the length and \( t = \left\lfloor \frac{d-1}{2} \right\rfloor \) with distance \( d \).

### 3.3 General \([n, 3], [n, n - 3]\)

Suppose \( G \) is a generator matrix for an mds \([n, r]\) code. Then the row-reduced echelon form of \( G \) is \((I_r, A)\). Now all the entries in \( A \) must be non-zero for otherwise an \( r \times r \) submatrix exists with zero determinant.

**Theorem 3.1** Let \( G = (I_r, A) \) be the generator matrix of an \([n, r]\) mds code. Then no \( j \times j \) submatrix of \( A \) has \( \det = 0 \) for \( j \leq r \).

In fact:

**Theorem 3.2** Let \( G = (I_r, A) \) be the generator matrix of an \([n, r]\) mds code. The code is an mds code if and only no \( j \times j \) submatrix of \( A \) has \( \det = 0 \) for \( j \leq r \).

The proof is not included.

**Proposition 3.2** Suppose \([n, 3]\) is an mds code over \( GF(q) \). Then \( n \leq (q + 1) \) when \( q \) is odd and \( n \leq (q + 2) \) when \( q \) is even.

**Proof:** Let \( \omega \) be a primitive element in \( GF(q) \) and thus \( \omega \) is a primitive \((q - 1)\) root of unity. By row operations a generator matrix for the \([n, 3]\) mds code has the form \( G = (I_3, A) \) where \( A \) is a \( 3 \times (n - 3) \) matrix. If any zero appears in \( A \) then there exists a \( 3 \times 3 \) submatrix with determinant equal to zero and then the code would not be an mds code. Hence

\[
G = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\omega^0 & \omega^1 & \omega^2 \\
\omega^0 & \omega^1 & \omega^2 \\
\omega^0 & \omega^1 & \omega^2
\end{pmatrix}
\]

where the \( \omega^{i,j} \) are powers of the primitive element \( \omega \). We are considering \( 3 \times 3 \) submatrices and their determinants so we can consider that the first row of the \( A \) part of \( G \) consists of 1’s.

\[
G = \begin{pmatrix}
1 & 0 & 0 & 1 & 1 & \ldots & 1 \\
0 & 1 & 0 & \omega^1 & \omega^1 & \ldots & \omega^{n-3} \\
0 & 0 & 1 & \omega^2 & \omega^2 & \ldots & \omega^{2,n-3}
\end{pmatrix}
\]

If there is a repeat in any of the second or third rows then together with a \((1, 1)\) from the first row this gives a \( 2 \times 2 \) submatrix of \( A \) with determinant 0. Thus the second and third rows of the \( A \) part of \( G \) contain all the elements \( \{1, \omega, \omega^2, \ldots, \omega^{q-2}\} \) once only.

Thus \( n \leq q + 2 \) in all cases.

No element is repeated in second row and no element is repeated in third row and all powers of \( \omega \) appear in second and third rows of the \( A \) part of \( G \).

Suppose now \( q \) is odd and \( n = q + 2 \). Then the sum of the powers of \( \omega \) in rows 2, 3 is congruent to \( \frac{q-1}{2} \mod q \). When subtracting the powers of third row from the powers of the second row above then all the powers must appear or otherwise a \( 2 \times 2 \) submatrix from second and third rows with determinant 0 is obtained. Thus the subtraction must result in all the powers appearing and hence the sum of these powers is \( \equiv \frac{q-1}{2} \mod q \). But since each of the second and third row sums is \( \equiv \frac{q-1}{2} \mod q \), the subtraction results in powers summing to \( \equiv 0 \mod q \). Therefore in all cases there exists a \( 2 \times 2 \) submatrix of \( A \) part with determinant 0. Then \( G \) has a \( 3 \times 3 \) submatrix with determinant 0.

Thus \( n \leq q + 1 \) when \( q \) is odd.

\( \square \)

In summary then we get the following for dimension 3. Let \( C \) be an \([n, 3]\) or an \([n, n - 3]\) mds code over a finite field \( GF(q) \). Then
1. If \( q \) is odd then \( n \leq (q + 1) \). For each such odd \( q \), series of examples of \([q + 1, 3]\) and \([q + 1, q - 2]\) mds codes with efficient decoding algorithms are constructed using Theorem 2.4 of Section 2.

2. If \( q \) is even then \( n \leq q + 2 \). For each such even \( q \), series of examples of \([q + 2, 3]\) and \([q + 2, q - 1]\) mds codes with efficient decoding algorithms are constructed using Proposition 3.1.

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