Charged rotating Kaluza–Klein black holes generated by $G_{2(2)}$ transformation

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Abstract

Applying the $G_{2(2)}$ generating technique for minimal $D = 5$ supergravity to the Rasheed black hole solution, we present a new rotating charged Kaluza–Klein black hole solution to the five-dimensional Einstein–Maxwell–Chern–Simons equations. At infinity, our solution behaves as a four-dimensional flat spacetime with a compact extra dimension and hence describes a Kaluza–Klein black hole. In particular, the extreme solution is non-supersymmetric, which is in contrast to a static case. Our solution has the limits to the asymptotically flat charged rotating black hole solution and a new charged rotating black string solution.

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1. Introduction

In recent years, classical general relativity in more than four spacetime dimensions has been the subject of increasing attention. In particular, higher dimensional black objects have become one of the major subjects in fundamental physics [1–19]. Although recent ideas of braneworld and TeV gravity [20–22] have opened up the possibility of large extra-dimensions, still important and widely believed especially in the context of string theories is the spacetime picture that our macroscopically large four-dimensional spacetime is realized from a higher dimensional spacetime by some mechanism of compactifying extra-dimensions within the size of the fundamental scale of gravity. In this context, it is of great interest and important to study Kaluza–Klein black holes, which are essentially higher dimensional near the event horizon but effectively look like four-dimensional spacetime at infinity. Also, in the context of string theory, the five-dimensional Einstein–Maxwell theory with a Chern–Simons term gathers much attention since it is the bosonic sector of the minimal supergravity. Supersymmetric
black hole solutions to the five-dimensional Einstein–Maxwell equations with a Chern–Simons term have been found by various authors.

In this context of the Kaluza–Klein black holes, one of the recent important works is the discovery of the Ishihara–Matsuno solution [23], which is a static charged Kaluza–Klein black hole solution in the five-dimensional Einstein–Maxwell theory. They found it by using the squashing transformation, i.e., they regarded $S^3$ of a five-dimensional Reissner–Nordström black hole as a twisted $S^1$ bundle over $S^2$ and changed the ratio of the radius of $S^2$ to that of $S^1$ fiber such that the ratio diverges at infinity. A static charged multi Kaluza–Klein black hole solution [24] was constructed immediately from the Ishihara–Matsuno solution. Subsequently the squashing transformation was used to generate many different Kaluza–Klein black hole solutions. In fact, it was demonstrated by Wang [25] that the five-dimensional Kaluza–Klein black hole of Dobiasch and Maison [26, 27] can be reproduced by squashing a five-dimensional Myers–Perry black hole with two equal angular momenta. In [28], the squashing transformation was also applied to Cvetić et al.’s charged rotating black hole solution [13] with two equal angular momenta to the five-dimensional Einstein–Maxwell–Chern–Simons equations. Moreover, the application of the squashing transformation to non-asymptotically flat Kerr–Gödel black hole solutions [29] was considered in [30–32]. In particular, the solution in [28] is a generalization of the Ishihara–Matsuno solution to the rotating black holes in Einstein–Maxwell–Chern–Simons theory, which will be compared with our new solution in this paper. The solution in general describes a non-supersymmetric black hole boosted along the direction of the extra dimension and has the limit to the rotating supersymmetric black hole solution. A similar type of Kaluza–Klein black holes was considered in the context of supersymmetric theories by Gaiotto et al. [33] and Elvang et al. [34]. Furthermore, Kaluza–Klein solutions of this type were generalized to various different solutions by many authors [35–47].

As mentioned in [32], however, the squashing transformation can be applied only to a special class of solutions, i.e., cohomogeneity-1 black hole solutions such as the five-dimensional Myers–Perry black hole solution [1] with two equal angular momenta and Cvetić–Lü–Pope black hole solution [13] with two equal angular momenta. New solutions generated by the squashing transformation also belong to a class of cohomogeneity-1. Therefore, to obtain more general cohomogeneity-2 spacetimes such as Kaluza–Klein black hole with two rotations, or to apply it to cohomogeneity-2 solutions such as black rings [2, 4], black holes with two rotations [1], black lenses [48, 49], one has to consider a generalization of the squashing technique or quietly different methods.

Recently, a new solution-generation technique for $D = 5$ minimal supergravity has been developed by Bouchareb et al. [50]. Under the assumptions of the existence of two commuting Killing vector fields, the five-dimensional Einstein–Maxwell–Chern–Simons equations reduce to equations for eight scalar fields on a three-dimensional space. As first discussed by Mizoguchi et al. [51, 52], the system of the scalar fields is described by a nonlinear sigma model whose action is invariant under $G_{2(2)}$ transformations. In particular, the target space of the three-dimensional sigma model is the coset $G_{2(2)}/[SL(2, \mathbb{R}) \times SL(2, \mathbb{R})]$ in the Euclidean case. Some isometries of the target space construct solution-generation symmetries. They have constructed a simple representation of a coset in terms of $7 \times 7$ matrices [51, 52]. To obtain a new solution from a seed solution, one needs to transform the coset matrix for the seed by one-parametric subgroups of $G_{2(2)}$. In fact, using the formalism, they succeeded in the derivation of a charged black hole solution [13] from the five-dimensional Myers–Perry black hole solution [1].

The purpose of this paper is to construct a new charged rotating Kaluza–Klein black hole solution which belongs to a class of cohomogeneity-2 in the five-dimensional
Einstein–Maxwell–Chern–Simons theory by using the $G_{2(2)}$ transformation from the Rasheed solution [53] and to examine several features of the solution. Our solution can be regarded as the generalization of the Ishihara–Matsuno solution to the rotating case. The black hole is rotating only in the four-dimensional direction, which is in contrast to the charged rotating Kaluza–Klein black hole solution in [28] generated by the squashing transformation. Namely, the black hole is rotating only along the fifth dimensional direction. As shown later, these two black hole solutions have some different properties. In particular, it is interesting to study the difference between two solutions in the extreme cases in the context of supersymmetric theory. In that case, the solution in [28] can become supersymmetric but our solution is non-supersymmetric. Our new solution has many limits to several known solutions and a new electrically charged rotating black string solution.

The rest of the paper is organized as follows. In section 2, we briefly review the solution-generation techniques based on the results of the paper [50]. Then, in section 3, we review the Rasheed solution, which is considered as a seed solution in our work. In section 4, following the results in [50], we generate a new solution from the Rasheed solution under a special choice of the parameters and in section 5, we study their basic properties. Section 6 is devoted to summary and discussions.

2. $G_{2(2)}$ generation technique

Here, following [50], we review the results of the $G_{2(2)}$ transformation which generates charged solutions to the five-dimensional Einstein–Maxwell–Chern–Simons equations from solutions to the five-dimensional vacuum Einstein equations.

Consider the five-dimensional Einstein–Maxwell theory with a Chern–Simons term, whose action\(^4\) is given by

\[
S = \frac{1}{16\pi G_5} \int d^5x \left[ \sqrt{-g} \left( R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) + \frac{1}{12\sqrt{3}} \epsilon^{\mu\nu\rho\sigma\lambda} A_\mu F_{\nu\rho} F_{\sigma\lambda} \right],
\]

where $R$ is the five-dimensional scalar curvature, $F = dA$ is the 2-form of the five-dimensional gauge field associated with the gauge potential 1-form $A$ and $G_5$ is the five-dimensional Newton constant. Varying the action (1), one derives the Einstein equation

\[
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{1}{2} \left( F_{\mu\lambda} F^{\lambda}_{ \nu} - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \right)
\]

and the Maxwell equation

\[
F^{\mu\nu} + \frac{1}{4\sqrt{3}} (\sqrt{-g})^{-1} \epsilon^{\mu\nu\rho\sigma\lambda} F_{\rho\sigma} F_{\nu\lambda} = 0.
\]

We assume that the spacetime admits two commuting Killing vector fields $\partial_a$ ($a = 0, 1$). The metric and the gauge potential 1-form can be written as follows, respectively,

\[
d^2 = \lambda_{ab} (dz^a + a^a_i \, dx^i) (dz^b + a^b_j \, dx^j) + \tau^{-1} h_{ij} \, dx^i \, dx^j,
\]

\[
A = \sqrt{3} \psi_a \, dz^a + A_i \, dx^i,
\]

where $\tau = -\det(\lambda_{ab})$ and $i = 2, 3, 4$. Dualization of $F_{ij} = \partial_i A_j - \partial_j A_i$ gives the scalar field $\mu$ by the equation

\[
\frac{1}{\sqrt{3}} F^{ij} = a^{ij} \partial^i \psi_a - a^{ai} \partial^j \psi_a + \frac{1}{\tau \sqrt{h}} \epsilon^{ijkl} (\partial_k \mu + \epsilon_{ab} \psi_a \partial_k \psi_b),
\]

\(^4\) The actions written here and in [23, 30–32] are different in the coefficients of $F_{\mu\nu}$ and $A_\mu F_{\nu\rho} F_{\sigma\lambda}$ by the multiplier of $1/4$. To avoid confusion, in this paper we follow [50].
and dualization of $G_{ij}^b = \partial_i a^b_j - \partial_j a^b_i$ gives the two vectors $\omega^a$ by the equation

$$
\tau \lambda_{ab} G^{bij} = \frac{1}{\sqrt{h}} e^{ijk} [\partial_k \omega_a - \psi_a (3 \partial_k \mu + \epsilon^{bc} \psi_b \partial_k \psi_c)],
$$

(7)

where $h = \det(h_{ij})$.

Introduce the $G_2(2)/(SL(2, \mathbb{R}) \times SL(2, \mathbb{R}))$ coset matrix $M$ constructed by

$$
M = \begin{pmatrix}
A & B & \sqrt{2} U \\
B^T & C & \sqrt{2} V \\
\sqrt{2} U^T & \sqrt{2} V^T & S
\end{pmatrix},
$$

(8)

with

$$
A = \begin{pmatrix}
(1 - y) \lambda + (2 + x) \psi \psi^T - \tau^{-1} \tilde{\omega} \tilde{\omega}^T + \mu (\psi \psi^T \lambda^{-1} J - J \lambda^{-1} \psi \psi^T) \tau^{-1} \tilde{\omega} \\
\tau^{-1} \tilde{\omega}^T - \tau^{-1}
\end{pmatrix},
$$

$$
B = \begin{pmatrix}
(\psi \psi^T - \mu J) \lambda^{-1} - \tau^{-1} \tilde{\omega} \psi J \\
\tau^{-1} \psi J
\end{pmatrix},
$$

$$
C = \begin{pmatrix}
(1 + x) \lambda^{-1} - \lambda \psi \psi^T \lambda^{-1} J - J (z - \mu J \lambda^{-1}) \psi \\
\tilde{\omega} \psi J + \psi \psi^T (z + \mu \lambda^{-1} J) J - J \lambda^{-1} \psi - \mu (1 + x - 2y - xy + z^2)
\end{pmatrix},
$$

$$
U = \begin{pmatrix}
(1 + x - \mu J \lambda^{-1}) \psi - \mu \tau^{-1} \tilde{\omega} \\
\mu \tau^{-1}
\end{pmatrix},
$$

$$
V = \begin{pmatrix}
(\lambda^{-1} + \mu \tau^{-1}) J \psi \\
\psi \lambda^{-1} \tilde{\omega} - \mu (1 + x - z)
\end{pmatrix},
$$

$$
S = 1 + 2(x - y),
$$

where

$$
\tilde{\omega} = \omega - \mu \psi,
$$

$$
x = \psi \psi^T \lambda^{-1} \psi, \quad y = \tau^{-1} \mu^2, \quad z = y - \tau^{-1} \psi J \tilde{\omega},
$$

(9)

and the matrix $J$ is

$$
J = \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}.
$$

(11)

$M$ is symmetric ($M^T = M$). $M$ transforms as a symmetric, second-rank tensor field under global $G_2(2)$ transformations. The action of $G_2(2)$ on the coset matrix can be written as $M'(\Phi^A) = C^T M(\Phi^A) C$, where $C$ is an exponential of the $g_{2(2)}$ Lie algebra generator and $\Phi^A = (\lambda_{ab}, \omega_a, \psi_a, \mu)$ expresses the coordinates in the target space. First, one needs to read off the target space variables $\Phi^A$ for a seed solution via equations (6) and (7). The new target space variables $\Phi'^A = (\lambda'_{ab}, \omega'_a, \psi'_a, \mu')$ can be written in terms of the target space variables $\Phi^A$ of a seed from $M'(\Phi'^A) = M(\Phi^A)$. At the final step, by solving the set of equations (6) and (7) for the new variables $\Phi'^A$, one can obtain the explicit forms of the metric and the gauge potential 1-form of a new solution.

In [50], they have succeeded in constructing the Harrison transformation to preserve asymptotic flatness, which can generate charged solutions to the five-dimensional Einstein–Maxwell–Chern–Simons equations from vacuum solutions to five-dimensional Einstein
equations. Following [50], the transformed coset matrix can be expressed in terms of the seed matrix as
\[ M' = C^T M C, \]  
(12)
where the one-parameter subgroup is given by
\[ C = \begin{pmatrix} 
  c^2 & 0 & 0 & s^2 & 0 & 0 & \sqrt{2}sc \\
  0 & c & 0 & 0 & 0 & s & 0 \\
  0 & 0 & c & 0 & -s & 0 & 0 \\
  s^2 & 0 & 0 & c^2 & 0 & 0 & \sqrt{2}sc \\
  0 & 0 & -s & 0 & c & 0 & 0 \\
  0 & s & 0 & 0 & 0 & c & 0 \\
  \sqrt{2}sc & 0 & 0 & \sqrt{2}sc & 0 & 0 & c^2 + s^2 
\end{pmatrix}, \]  
(13)
where \( c := \cosh \gamma \) and \( s := \sinh \gamma \), and \( \gamma \) is a parameter. Note that when \( \gamma = 0 \), \( C \) is a \( 7 \times 7 \) unit matrix. From this one can read off transformed potentials as follows:
\[ \lambda'_{00} = D^{-2} \lambda_{00}, \]  
(14)
\[ \lambda'_{01} = D^{-2} (c^3 \lambda_{01} + s^3 \lambda_{00} \omega_0), \]  
(15)
\[ \tau' = D^{-1} \tau, \]  
(16)
\[ \omega'_0 = D^{-2} \left[ c^3 (c^2 + s^2 \lambda_{00}) \omega_0 - s^3 (2c^2 + c^2 + s^2) \lambda_{00} \lambda_{01} \right], \]  
(17)
\[ \omega'_1 = \omega_1 + D^{-2} s^3 \left[ -c^3 \lambda_{01}^2 + s (2c^2 - \lambda_{00}) \lambda_{01} \omega_0 - c^3 \omega_0^2 \right], \]  
(18)
\[ \psi'_0 = D^{-1} s c (1 + \lambda_{00}), \]  
(19)
\[ \psi'_1 = D^{-1} s c (c \lambda_{01} - s \omega_0), \]  
(20)
\[ \mu' = D^{-1} s c (c \omega_0 - s \lambda_{01}), \]  
(21)
where
\[ D = 1 + s^2 (1 + \lambda_{00}). \]  
(22)

3. Rasheed solutions

In the following section, using the \( G_{2(2)} \) transformation mentioned in the previous section, we will generate a new charged rotating black hole solution from the Rasheed solution [53], which is the most general Kaluza–Klein black hole solution to the five-dimensional vacuum Einstein equations. Hence, in this section, we briefly explain the Rasheed solution.

3.1. Metric

As shown by Maison [54], assuming symmetry of a spacetime, the five-dimensional pure Einstein equations reduce to equations for five scalar fields on a three-dimensional space. The system of the scalar fields is described by a nonlinear \( \sigma \) model which is invariant under the global \( SL(3, R) \) transformation. In [53], Rasheed derived the black hole solution by applying the \( SL(3, R) \) transformation to a trivial \( S^1 \) bundle over the Kerr spacetime with the mass parameter \( M_k \) and the rotation parameter \( a \). Actually to assure the asymptotic flatness in the context of four dimensions, the \( SL(3, R) \) transformation is restricted to the specific \( SO(1, 2) \) transformation which are labeled by two boost parameters \( (\alpha, \beta) \). The generated
new Kaluza–Klein black hole solution is specified by boost parameters \((\alpha, \beta)\) in addition to old parameters \((M_k, a)\). These parameters, \(\alpha\) and \(\beta\), are related to an electric charge and a magnetic charge under dimensional reduction, restrictively. See [55] about another derivation of the Rasheed solution.

The metric is given by
\[
ds^2 = \frac{B}{A} \left( dx^5 + 2A_{\mu} dx^\mu \right)^2 - \frac{f^2}{B} \left( dr + \omega^0_\phi d\phi \right)^2 + \frac{A}{\Delta} dr^2 + A d\theta^2 + \frac{A\Delta}{f^2} \sin^2 \theta d\phi^2,
\]
where
\[
A = \left( r - \frac{\Sigma}{\sqrt{3}} \right)^2 - \frac{2P^2}{\Sigma - \sqrt{3}M} + a^2 \cos^2 \theta + \frac{2JPQ \cos \theta}{(M + \Sigma/\sqrt{3})^2 - Q^2},
\]
\[
B = \left( r + \frac{\Sigma}{\sqrt{3}} \right)^2 - \frac{2Q^2}{\Sigma + \sqrt{3}M} + a^2 \cos^2 \theta - \frac{2JPQ \cos \theta}{(M - \Sigma/\sqrt{3})^2 - P^2},
\]
\[
C = 2Q \left( r - \frac{\Sigma}{\sqrt{3}} \right) - \frac{2JPQ \cos \theta(M + \Sigma/\sqrt{3})}{(M - \Sigma/\sqrt{3})^2 - P^2},
\]
\[
\omega^0_\phi = \frac{2J \sin^2 \theta}{f^2} \left[ r - M + \frac{(M^2 + \Sigma^2 - P^2 - Q^2)(M + \Sigma/\sqrt{3})}{(M + \Sigma/\sqrt{3})^2 - Q^2} \right],
\]
\[
\omega^5_\phi = \frac{2P \Delta \cos \theta}{f^2} - \frac{2QJ \sin^2 \theta [r(M - \Sigma/\sqrt{3}) + M \Sigma/\sqrt{3} + \Sigma^2 - P^2 - Q^2]}{f^2[(M + \Sigma/\sqrt{3})^2 - Q^2]},
\]
\[
\Delta = r^2 - 2Mr + P^2 + Q^2 - \Sigma^2 + a^2,
\]
\[
f^2 = r^2 - 2Mr + P^2 + Q^2 - \Sigma^2 + a^2 \cos^2 \theta,
\]
\[
2A_{\mu} dx^\mu = \frac{C}{B} dr + \left( \omega^5_\phi + \frac{C}{B} \omega^0_\phi \right) d\phi,
\]
where \(A_{\mu}\) is the electromagnetic vector potential derived by dimensional reduction. Here \(M, P, Q, J, \Sigma\) are the mass, the magnetic charge, the electric charge, the angular momentum and dilaton charge, respectively. They are parameterized by the boost parameters \((\alpha, \beta)\) as
\[
M = \frac{(1 + \cosh^2 \alpha \cosh^2 \beta) \cosh \alpha}{2\sqrt{1 + \sinh^2 \alpha \cosh^2 \beta}} M_k,
\]
\[
\Sigma = \sqrt{3} \cosh \alpha (1 - \cosh^2 \beta + \sinh^2 \alpha \cosh^2 \beta) \frac{M_k}{2\sqrt{1 + \sinh^2 \alpha \cosh^2 \beta}},
\]
\[
Q = \sinh \alpha \sqrt{1 + \sinh^2 \alpha \cosh^2 \beta} M_k,
\]
\[
P = \frac{\sinh \beta \cosh \beta}{\sqrt{1 + \sinh^2 \alpha \cosh^2 \beta}} M_k,
\]
\[
J = \cosh \beta \sqrt{1 + \sinh^2 \alpha \cosh^2 \beta} a M_k.
\]
It should be noted that these parameters are not independent since they are related to the equation
\[
\frac{Q^2}{\Sigma + \sqrt{3}M} + \frac{P^2}{\Sigma - \sqrt{3}M} = \frac{2\Sigma}{3},
\]
and $M_k$ is written in terms of these parameters as
\[ M_k^2 = M^2 + \Sigma^2 - P^2 - Q^2. \]  
(38)

$J$ is also related to $a$ by
\[ J^2 = a^2 \left[ (M + \Sigma/\sqrt{3})^2 - Q^2 \right] \left[ (M - \Sigma/\sqrt{3})^2 - P^2 \right]. \]  
(39)

The locations of the horizon are given by the zeros of $\Delta_1$ when the parameters satisfy the inequality
\[ M^2 \geq P^2 + Q^2 + a^2 - \Sigma^2. \]  
(40)

3.2. Potentials

First, one has to identify the target space variables $\Phi^A = (\lambda_{ab}, \omega_a, \psi_a, \mu)$ which correspond to the seed solution. Choose the coordinates as $z^0 = t, z^1 = x^5, x^2 = \phi, x^3 = r$ and $x^4 = \theta$.

In this paper, we consider the case $\alpha = 0$ as a seed solution since the complex form of the metric makes it impossible for us to read off the new metric from the transformed target space variables $\Phi^A$. Under this identification, we can read off the target space variables $\Phi^A$ from the metric (23) as follows:

\[ \lambda_{00} = g_{rr} = -f_2^2 + 4 \frac{B}{A} A_t^2, \]  
(41)

\[ \lambda_{01} = g_{r5} = 2 \frac{B}{A} A_t, \]  
(42)

\[ \lambda_{11} = g_{55} = \frac{B}{A}, \]  
(43)

\[ \omega_0 = \frac{2aM_k \cos \theta \cosh \beta}{A}, \]  
(44)

\[ \omega_1 = \frac{M_k (2r - M_k \sinh^2 \beta) \sinh \beta \cosh \beta}{A}, \]  
(45)

\[ \tau = -\det(\lambda_{ab}) = \frac{f^2}{A}, \]  
(46)

\[ \psi_0 = 0, \]  
(47)

\[ \psi_1 = 0, \]  
(48)

\[ \mu = 0, \]  
(49)

where in order to obtain $\omega_a (a = 0, 1)$, we substituted the equations

\[ a_0^0 \phi = \frac{\lambda_{01} g_{\phi 5} - \lambda_{11} g_{\phi 1}}{\tau} = a_0^0 \phi, \]  
(50)

\[ a_1^0 \phi = \frac{-\lambda_{00} g_{\phi 5} - \lambda_{01} g_{\phi 1}}{\tau} = a_1^0 \phi, \]  
(51)

into equations (6) and (7) and integrated them. The three-dimensional metric $h_{ij}$ and the square root of the determinant $h$ are given by, respectively,

\[ h_{rr} = \tau g_{rr} = \frac{f^2}{\Delta}. \]  
(52)
\[ h_{\theta\theta} = \tau g_{\theta\theta} = f^2, \]
\[ h_{\phi\phi} = \tau \left( g_{\phi\phi} - 2\lambda_{01} a^0_{\phi} a^1_{\phi} - \lambda_{11} (a^1_{\phi})^2 - \lambda_{00} (a^0_{\phi})^2 \right) = \Delta \sin^2 \theta, \]
\[ \sqrt{h} = f^2 \sin \theta. \]

4. New charged solutions

4.1. New solution

To obtain the metric and the gauge potential 1-form of the Maxwell field of the new solution, one must solve equations (6) and (7) for the transformed target space variables \( \Phi' = (\lambda'_{ab}, \omega'_{a}, \psi'_{a}, \mu') \). Integrating them, we can obtain the explicit forms \( a^0_{\phi} \) and \( A_{\phi} \) for the new solution, respectively, as follows:
\[ a^0_{\phi} = c^3 a^0_{\phi} + s^3 y, \]
\[ a^1_{\phi} = a^1_{\phi}, \]
\[ A_{\phi} = -2\sqrt{3} c a M_k [\sin^2 \theta \cosh(\beta + \gamma)(r + 3M_k \sinh^2 \beta/2) + \cos^2 \theta \sinh(\beta + \gamma) M_k \sinh 2\beta] d\phi, \]
where the function \( y \) is given by
\[ y = a M_k \left( 2r - M_k \sinh^2 \beta - 4M_k \right) \sin^2 \theta \sinh \beta. \]

Then, in terms of old variables, the metric generated by \( G_{2(2)} \) transformation can be expressed in the form
\[ ds^2 = D^{-2\lambda_{00}} [dt + \Omega']^2 + D \left[ \tau^{-1} h_{\phi\phi} d\phi^2 - \frac{\tau}{\lambda_{00}} (dx^5 + a^1_{\phi} d\phi)^2 + \tau^{-1} (h_{rr} dr^2 + h_{\theta\theta} d\theta^2) \right], \]
where the 1-form \( \Omega' \) can be written as
\[ \Omega' = \Omega'_\phi d\phi + \Omega'_5 dx^5 \]
with
\[ \Omega'_\phi = c^3 \Omega_{\phi} + s^3 (y + \omega_0 a^1_{\phi}), \]
\[ \Omega'_5 = c^3 \Omega_5 + s^3 \omega_0, \]
\[ \Omega_\phi = a^0_{\phi} + \lambda_{01} \lambda_{00} a^1_{\phi}, \]
\[ \Omega_5 = \frac{\lambda_{01}}{\lambda_{00}}. \]

Here we introduce a new radial coordinate \( \rho := r + 2s^2 M_k - \frac{1}{2} \sinh^2 \beta M_k \). Then in terms of the coordinates \( (t, x^5, \phi, \rho, \theta) \) and parameters including in the seed solution, the metric can be explicitly rewritten as
\[ ds^2 = -\frac{X Y}{Z^2} \left[ dt + \Omega \right]^2 + Z \left[ \frac{W}{X} \left( dx^5 + \frac{2P \Delta \cos \theta}{W} d\phi \right)^2 + Y \left( \frac{\Delta \sin^2 \theta}{W} d\phi^2 + \frac{d\rho^2}{\Delta} + d\theta^2 \right) \right]. \]
where the functions $X, Y, Z, W$ and the 1-form $\Omega'$ are given by
\begin{align*}
X &= (\rho - 2c^2 M_k)(\rho - 2s^2 M_k + 2M_k \sinh^2 \beta) + a^2 \cos^2 \theta, \\
Y &= (\rho - 2s^2 M_k)(\rho - 2s^2 M_k + 2M_k \sinh^2 \beta) + a^2 \cos^2 \theta, \\
Z &= \rho(\rho - 2s^2 M_k + 2M_k \sinh^2 \beta) + a^2 \cos^2 \theta, \\
W &= (\rho - 2c^2 M_k)(\rho - 2s^2 M_k) + a^2 \cos^2 \theta, \\
\Delta &= (\rho - 2e^2 M_k)(\rho - 2s^2 M_k) + a^2
\end{align*}
and
\begin{align*}
\Omega' &= \left( \frac{c^3}{X} X_\phi + \frac{s^3}{Y} Y_\phi \right) d\phi + \left( \frac{c^3}{X} X_5 + \frac{s^3}{Y} Y_5 \right) dx^5
\end{align*}
with
\begin{align*}
X_\phi &= 2a M_k \cosh \beta [\sin^2 \theta(\rho - 2M_k s^2) + 2M_k \sinh^2 \beta], \\
X_5 &= 2a M_k \cos \theta \sinh \beta, \\
Y_\phi &= 2a M_k \sinh \beta [\sin^2 \theta(\rho - 2M_k e^2) + 2M_k \cosh^2 \beta], \\
Y_5 &= 2a M_k \cos \theta \cosh \beta.
\end{align*}

The gauge potential 1-form of the Maxwell field is given by
\begin{align*}
A &= \frac{2\sqrt{3cM_k}}{Z} \left[ (\rho + 2M_k(\sinh^2 \beta - s^2)) dt + a \cos \theta \sinh(\beta + \gamma) dx^5 \\
&\quad - a(\sin^2 \theta \cosh(\beta + \gamma)(\rho + 2M_k(\sin^2 \beta - s^2)) \\
&\quad + \cos^2 \theta \sinh(\beta + \gamma) \sinh 2\beta M_k) d\phi \right].
\end{align*}

This new solution has four independent parameters $(M_k, a, \beta, \gamma)$. The parameter region such that there are two black hole horizons is given by
\begin{align*}
M_k > 0, \quad M_k^2 > a^2
\end{align*}
As shown later, in the sense of five dimensions, the four parameters are physically related to the mass, the angular momentum in the direction of four dimensions, the charge and the size of an extra dimension at infinity. Instead of $(M_k, \gamma)$, it is better to use the physical parameters $(m, q)$ which are proportional to the mass and the charge as follows:
\begin{align*}
m = (s^2 + c^2)M_k, \quad q = 2scM_k
\end{align*}
but in this paper we do not use these to avoid the complicated form of the metric.

5. Basic properties

5.1. Asymptotic structure and asymptotic charges

At the infinity, $\rho \to \infty$, the metric behaves as
\begin{align*}
\text{d}s^2 \simeq -\text{d}t^2 + \rho^2 (\text{d}\theta^2 + \sin^2 \theta \text{d}\phi^2) + (\text{d}x^5 + 2P \cos \theta \text{d}\phi)^2
\end{align*}
The spacetime is asymptotically locally flat, in particular, in the case of $P \neq 0$, the spacetime asymptotically approaches a twisted $S^1$ bundle over a four-dimensional Minkowski spacetime. We set the periodicity of $x^5$ to $8\pi P$. Then the spatial infinity is a squashed $S^3$. In the case
of $P = 0$, the asymptotic structure is a trivial $S^1$ bundle over a four-dimensional Minkowski spacetime. From equation (57), asymptotic structure is preserved since $D \to 1$ and $\Omega \to 0$ at infinity.

We give the conserved charges of the new solution. The charge is

$$Q_c = \frac{1}{4\pi G_5} \int \left( * F - \frac{1}{\sqrt{3}} A \wedge F \right) = -\frac{16\sqrt{3}\pi}{G_5} \sinh \beta \cosh \beta M_k^2 \mathrm{sc}. \quad (78)$$

The Komar mass and the Komar angular momenta at infinity, which are associated with Killing vector fields $t^\mu = (\partial_t)^\mu$, $\phi^\mu = (\partial_\phi)^\mu$ and $\psi^\mu = (\partial_\psi)^\mu$, are

$$M_{\text{Komar}} = -\frac{3}{32\pi G_5} \int_{S_\infty} dS_{\mu\nu} \nabla^\mu t^\nu = \frac{6\pi}{G_5} \sinh \beta \cosh \beta (c^2 + s^2) M_k^2, \quad (79)$$

$$J_{\phi}^{\text{Komar}} = -\frac{1}{16\pi G_5} \int_{S_\infty} dS_{\mu\nu} \nabla^\mu \phi^\nu = -\frac{8\pi}{3G_5} \sinh \beta \cosh \beta (c^3 \cosh \beta + s^3 \sinh \beta) M_k^2 a, \quad (80)$$

$$J_{\psi}^{\text{Komar}} = -\frac{1}{16\pi G_5} \int_{S_\infty} dS_{\mu\nu} \nabla^\mu \psi^\nu = 0, \quad (81)$$

respectively. Here the angular coordinate $\psi$ with periodicity of $4\pi$ is defined by $\psi := x^5 / 2P_5$. Therefore, the spacetime has only an angular momentum in the direction of four dimensions.

5.2. Horizon and regularity

Two black hole horizons, an outer horizon and an inner horizon, are located at $\rho = \rho_{\pm}(\rho_+ < \rho_+)$, where $\rho_{\pm}$ are roots of the quadratic equation $\Delta = 0$, with respect to $\rho$ and they are explicitly written in terms of the parameters $M_k, a, \gamma$ as

$$\rho_{\pm} = M_k + 2s^2 M_k \pm \sqrt{M_k^2 - a^2}. \quad (82)$$

The functions $X, Y, Z, W, \Omega_1'$ and $\Omega_5'$ do not vanish at $\rho = \rho_{\pm}$ and only the metric component $g_{\rho\rho}$ apparently diverges there. In order to remove the divergence, we introduce the coordinates $(t', \phi')$ defined as

$$dt' = dt - \frac{a}{2\Omega_H \sqrt{M_k^2 - a^2} \rho - \rho_+} d\rho, \quad (83)$$

$$d\phi' = d\phi - \frac{a}{2\sqrt{M_k^2 - a^2} \rho - \rho_+} d\rho, \quad (84)$$

where $\Omega_H$ denotes the angular velocity of the horizon and is given by

$$\Omega_H = -\frac{1}{\Omega_5'(\rho = \rho_+)} = \frac{a}{2M_k \left[ c^3 \left( M_k + \sqrt{M_k^2 - a^2} \right) \cosh \beta + s^3 \left( -M_k + \sqrt{M_k^2 - a^2} \right) \sinh \beta \right]}. \quad (85)$$

The black hole is rotating only in the direction of $\partial_\theta$. Then near $\rho = \rho_+$, the metric behaves as

$$ds^2 \simeq -\frac{X_+ Y_+}{Z_+^2} \left[ dt' + \Omega_{\phi+} d\phi' + \Omega_{5+} dx^5 \right]^2 + \frac{Z_+}{Y_+} \left[ \frac{W_+}{X_+} \left( dx_5^2 + \frac{2Pa \cos \theta}{W_+} d\rho \right)^2 + 2aY_+ \sin^2 \theta \frac{W_+}{W_+} d\phi' d\rho + Y_+ d\theta^2 \right] + \mathcal{O}(\rho - \rho_+). \quad (86)$$
where \((X_\star, Y_\star, Z_\star, W_\star, \Omega_\star, \Omega_\phi)\) denote the values on \(\rho = \rho_\star\) of the functions \((X, Y, Z, W, \Omega_\star, \Omega_\phi)\). The metric is well defined on \(\rho = \rho_\star\). Furthermore, to show that \(\rho = \rho_\star\) corresponds to the horizon, introduce new coordinates \((u, \phi'')\) defined as

\[
    du = dt', \quad d\phi'' = d\phi'' - \Omega_H dt' .
\]

Then, near \(\rho = \rho_\star\), the metric behaves as

\[
ds^2 \simeq -\frac{X_\star Y_\star}{Z_\star^2} \left[ \Omega_\phi' \sinh^2 \frac{\theta}{W_\star} \left( \frac{dx^5}{X_\star} + \frac{2Pa \cos \theta}{W_\star} \right)^2 + \frac{Z_\star}{Y_\star} \left( \frac{dx^5}{X_\star} + \frac{2Pa \cos \theta}{W_\star} \right)^2 + \frac{2a Z_\star \Omega_H \sin^2 \theta}{W_\star} \right] \left( d\rho + \mathcal{O}(\rho - \rho_\star) \right) .
\]

The Killing vector field \(\mathbf{V} := \partial_u\) is null on \(\rho = \rho_\star\). Since \(V_\mu \, dx^\mu = g_{\mu \nu} \, d\rho\) there, the vector field is orthogonal and tangent to the null surface \(\rho = \rho_\star\). Hence the null hypersurface \(\rho = \rho_\star\) is a Killing horizon. In addition, in the coordinate system \((u, x^5, \phi'', \rho, \theta)\), the metric is smooth everywhere in the region \(\rho \geq \rho_\star\). So there is no conical singularity on the base space. Moreover, in the case of \(\beta \neq 0\), the first Chern number is computed as

\[
c_1 = \frac{1}{8\pi P} \int_{S^2} \mathcal{F} = \frac{A_\theta(\theta = \pi) - A_\theta(\theta = 0)}{4P} = -1 ,
\]

where \(\mathcal{F} = dA\). Therefore we have a non-trivial bundle, i.e., equation (90) is a metric on \(S^3\).

### 5.3. Ergorregion

An ergoregion is the region where \(g_{tt} = -XY/Z^2\) becomes positive. We see that \(Y > 0\) outside the outer horizon. Therefore, the ergosurface is located at the zero of the function \(X\), i.e.,

\[
    \rho = M_\star (1 + 2x^2 - \sinh^2 \beta) + \sqrt{M_\star^2 \cosh^4 \beta - a^2 \cos^2 \theta} .
\]

As far as \(a \neq 0\), \(\rho > \rho_\star\) for arbitrary values of \(\theta\). The ergosurface is located outside the outer horizon.
5.4. Various limits

5.4.1. $a \to 0$. Taking the limit of $a \to 0$ and defining the parameters as

\[ r_\infty := 4P, \quad \rho_+ := 2c^2 M_k, \quad \rho_- := 2s^2 M_k, \quad \rho_0 := 2M_k (\sinh^2 \beta - s^2), \]

we can obtain the following metric:

\[ ds^2 = -V(\rho) \, dt^2 + \frac{K^2(\rho)}{V(\rho)} \, d\rho^2 + \rho^2 K^2(\rho) (d\theta^2 + \sin^2 \theta \, d\phi^2) + W^2(\rho) \chi^2, \]

where the functions $V, K, W$ are

\[ V(\rho) = \frac{(\rho - \rho_+)(\rho - \rho_-)}{\rho^2}, \]
\[ K^2(\rho) = \frac{\rho + \rho_0}{\rho}, \]
\[ W^2(\rho) = \frac{r^2}{4} K^{-2}(\rho). \]

This solution coincides with the static charged Ishihara–Matsumoto solution to the five-dimensional Einstein–Maxwell equations. In particular, in the case of $r_\infty \to \infty$, the solution becomes the five-dimensional Reissner–Nordström solution, which is asymptotically flat in the standard five-dimensional sense. In the case of $\rho_+ = \rho_-$, it has a degenerate horizon and is supersymmetric because it is included in a class of solutions on Taub-NUT base space in [56], in which all purely bosonic supersymmetric solutions of minimal supergravity in five dimensions are classified.

5.4.2. $\gamma \to 0$. In the limit of $\gamma \to 0$, it is clear that $D \to 1$ and $\Omega' \to \Omega$. Therefore, this solution coincides with an uncharged Rasheed solution with $Q = 0 (\alpha = 0)$.

5.4.3. $\beta \to 0$. In the limit of $\beta \to 0$, $X = W$ and $X_5 = 0$ and $Y_\phi = 0$. Then the metric takes the form of

\[ ds^2 = -\frac{[(\rho - 2c^2 M_k)(\rho - 2s^2 M_k) + a^2 \cos^2 \theta][(\rho - 2s^2 M_k)^2 + a^2 \cos^2 \theta]}{[\rho(\rho - 2s^2 M_k) + a^2 \cos^2 \theta]^2} (dt + \Omega')^2 + \frac{\Delta \sin^2 \theta \, d\phi^2}{(\rho - 2s^2 M_k)(\rho - 2c^2 M_k) + a^2 \cos^2 \theta} + \frac{\Delta}{\Delta} + \frac{d\rho^2}{\rho(\rho - 2s^2 M_k) + a^2 \cos^2 \theta} + \frac{d\rho^2}{(\rho - 2s^2 M_k)^2 + a^2 \cos^2 \theta} (dx^5)^2, \]

where the 1-form $\Omega'$ is given by

\[ \Omega' = 2a M_k \left( \frac{c^3 (\rho - 2s^2 M_k) \sin^2 \theta}{(\rho - 2s^2 M_k)(\rho - 2c^2 M_k) + a^2 \cos^2 \theta} \, d\phi + \frac{s^3 \cos \theta}{(\rho - 2s^2 M_k)^2 + a^2 \cos^2 \theta} \, dx^5 \right). \]

The gauge potential 1-form is given by

\[ A = \frac{2\sqrt{3} c M_k [(\rho - 2s^2 M_k) \, dt + a \cos \theta \, dx^5 - a \sin^2 \theta (\rho - 2s^2 M_k) \, d\phi]}{\rho(\rho - 2s^2 M_k) + a^2 \cos^2 \theta}. \]

At infinity, $\rho \to \infty$, the metric asymptotically behaves as

\[ ds^2 = -dr^2 + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2) + (dx^5)^2, \]

which is the trivial $S^1$ bundle over a four-dimensional Minkowski spacetime.
Note that \( A_\phi(\theta = \pi) = A_\phi(\theta = 0) = 0 \) on the horizon \( \rho = \rho_+ \). So the first Chern number is
\[
c_1 = \frac{1}{\Delta x^2} \oint_{S^2} F = \frac{2\pi A_\phi(\theta = \pi) - A_\phi(\theta = 0)}{\Delta x^2} = 0. \tag{105}
\]
Therefore the horizon is also a trivial bundle, i.e., it is topologically \( S^2 \times S^1 \). This solution corresponds to an electrically charged rotating black string solution.

5.4.4. \( \beta \to \infty \). We take the limit of \( \beta \to \infty \) with the following quantities kept fixed:
\[
\begin{align*}
\tilde{m} &= 8(s^2 + c^2) \sinh^2 \beta M_k^2, \tag{106} \\
\tilde{q} &= 16cs \sinh^2 \beta M_k^2, \tag{107} \\
\tilde{a} &= -2 \sinh \beta (c - s)^{-1} a. \tag{108}
\end{align*}
\]
The functions \( X, Y, Z \) behave as the following functions, respectively,
\[
\begin{align*}
X &\to 2M_k \sinh^2 \beta (\rho - 2c^2 M_k), \tag{109} \\
Y &\to 2M_k \sinh^2 \beta (\rho - 2s^2 M_k), \tag{110} \\
Z &\to 2M_k \sinh^2 \beta \rho. \tag{111}
\end{align*}
\]
Define a new radial coordinate \( r \), three new angular coordinates \((/Theta_1, \phi_1, \phi_2)\) as
\[
\begin{align*}
r^2 &= \tilde{\rho}^2 - a^2 (\tilde{\rho}^2 = (8M_k \sinh^2 \beta) \rho), \\
/Theta_1 &= \theta/2, \\
\phi_1 &= (\psi - \phi)/2 \text{ and } \phi_2 = (\psi + \phi)/2, \tag{112}
\end{align*}
\]
we obtain the solution
\[
\begin{align*}
ds^2 &= -dt^2 - \frac{2\tilde{q}}{\tilde{p}^2} \tilde{\nu}(dt - \tilde{\omega}) + \frac{\tilde{f}}{\tilde{p}^2}(dt - \tilde{\omega})^2 + \frac{\tilde{p}^2 r^2}{\Delta} dr^2 \\
&\quad + \tilde{p}^2 [d\Theta^2 + \sin^2 \Theta d\phi_1^2 + \cos^2 \Theta d\phi_2^2], \tag{113} \\
A &= \frac{\sqrt{3}\tilde{q}}{\tilde{p}^2}(dt - \tilde{\omega}), \tag{114}
\end{align*}
\]
where
\[
\begin{align*}
\tilde{\omega} &= -\tilde{\nu} = \tilde{a}(\sin^2 \Theta d\phi_1 - \cos^2 \Theta d\phi_2), \\
\tilde{f} &= 2\tilde{m}\tilde{\rho}^2 - \tilde{q}^2, \tag{115} \\
\Delta &= (r^2 + \tilde{a}^2)^2 + \tilde{q}^2 - 2\tilde{a}^2 \tilde{q} - 2\tilde{m}r^2. \tag{116}
\end{align*}
\]
This solution coincides with the Cveti\v{c}–L"{u}–Pope solution in [13] for \( g = 0, b = -a \), which is the asymptotically flat charged black hole solution with equal angular momenta with an opposite sign \( J_{\phi_1} = -J_{\phi_2} \).
5.4.5. $M_k \rightarrow a$. According to [56], all supersymmetric solutions of the five-dimensional minimal supergravity have a non-spacelike Killing vector field, and when the Killing vector field $\partial_t$ is timelike, the metric and the gauge potential are given, respectively, by

$$ds^2 = -H^{-2}(dt + \omega)^2 + H \, dx^2_B,$$

$$A = \frac{\sqrt{3}}{2} [H^{-1}(dt + \omega) - \beta],$$

(119)

where $dx^2_B$ is a metric of a hyper-Kähler space $B$. The scalar function $H$, 1-forms $\omega$ and $\beta$ on $B$ are given by

$$\Delta H = \frac{4}{9} (G^+)^2,$$

$$dG^+ = 0,$$

$$d\beta = \frac{2}{3} G^+, \quad (120)$$

Here, $\Delta$ is the Laplacian on $B$ and the 2-form $G^+$ is the self-dual part of the 2-form $H^{-1}d\omega$, given by

$$dG^+ := \frac{1}{2} H^{-1}(d\omega + * d\omega),$$

(121)

where $(G^+)^2 := \frac{1}{2} G_{mn} G^{mn}$ and $*$ is the Hodge dual operator on $B$. Since $\partial_t$ is a Killing vector field associated with time translation, all components are independent of the time coordinate $t$.

In the limit of $M_k \rightarrow a$ in our solution, two horizons degenerate. However, there is no timelike killing vector field which globally exists outside the outer horizon except for $a = 0$. Therefore this is non-supersymmetric. In the case of $a = 0, \rho_+ = \rho_-$ corresponds to the extreme and supersymmetric case. Introducing the coordinate $R := \rho - \rho_+$ and the parameter $N := \rho_0 + \rho_+$, we obtain the metric

$$ds^2 = H^{-2} \, dt^2 + H \, ds^2_{TN},$$

(122)

$$H = 1 + \frac{\rho_+}{R},$$

(123)

$$ds^2_{TN} = \left(1 + \frac{N}{R}\right)^{-1} (dx^5 + \cos \theta \, d\phi)^2 + \left(1 + \frac{N}{R}\right) (dR^2 + R^2 \, d\theta^2 + R^2 \sin^2 \theta \, d\phi^2),$$

(124)

where $ds^2_{TN}$ is the metric of the self-dual Euclidean Taub-NUT space. Hence this solution is a supersymmetric black hole solution on the Taub-NUT space [56].

6. Summary and discussions

In this paper, applying $G_{32}$ generating techniques for minimal $D = 5$ supergravity to the Rasheed Kaluza–Klein black hole solution with $\alpha = 0$, we have obtained a new charged rotating Kaluza–Klein black hole solution to the five-dimensional Einstein–Maxwell–Chern–Simons equations. The solution has four independent parameters $(M_k, a, \beta, \gamma)$. It is a generalization of the Ishihara–Matsuno static charged solution to the rotating solution. The spacetime has two black hole horizons, an outer horizon and an inner horizon. The event horizon on the cross sections with timeslice is topologically $S^3$. Thus, near the horizon, the spacetime behaves as five dimensions but at infinity the spacetime effectively behaves as a four-dimensional spacetime, which has a twisted $S^1$ bundle over a four-dimensional Minkowski spacetime. Our new solution has many limits to several known solutions, e.g., (i) $\beta \rightarrow \infty$ case: the asymptotically flat charged rotating black hole with two equal angular momenta [13], (ii) $a = 0$ case: the Ishihara–Matsuno static charged Kaluza–Klein black hole solution [23] and (iii) $\beta = 0$ case: a new electrically charged rotating black string solution which asymptotes to a trivial bundle over a four-dimensional Minkowski spacetime.

It is interesting to compare our solution with the rotating solution in [28]. These two solutions are the generalization of the Ishihara–Matsuno static charged black hole solution
[23] to rotating black hole solutions. As mentioned previously, the extreme case (i.e., the $m = \pm q$ case) of the Ishihara–Matsuno solution corresponds to a supersymmetric solution, which is a black hole solution on the Euclidean self-dual Taub-NUT space [56]. However, in the solution in [28], the two cases, $m = -q$ and $m = q$, describe different solutions due to the existence of a Chern–Simons term. Only the case of $m = -q$ corresponds to a supersymmetric Kaluza–Klein black hole solution which was found by Gaiotto et al [33]. In our solution, two extreme cases ($M_2^k = a^2$, $\gamma > 0$) and ($M_2^k = a^2$, $\gamma < 0$) are also different since a Chern–Simons term exists. Furthermore, in contrast to the solution in [28], neither ($M_2^k = a^2$, $\gamma > 0$) nor ($M_2^k = a^2$, $\gamma < 0$) is supersymmetric.

In this paper, we considered the Rasheed solution with $\alpha = 0$. The solution generated from the general seed solution with $\alpha \neq 0$ and $\beta \neq 0$ by the $G_{2(2)}$ transformation will include both our solution and the solution in [28], i.e., in such a solution, a black hole will be rotating in the extra dimension $\partial_5$ as well as in the direction of $\partial_\phi$. However, it turns out that the complex form of the metric with $\alpha \neq 0$ and $\beta \neq 0$ makes it impossible for us to read off the metric from the transformed target space variables.

Finally, we would like to comment on the squashing transformation. The squashing transformation is a very powerful tool to generate Kaluza–Klein black hole solutions from asymptotically flat and non-asymptotically flat black hole solutions. The transformation can be applied only to a cohomogeneity-1 black hole solution such as the five-dimensional Myers–Perry black hole solution with two equal angular momenta and the Cvetiˇc–Lü–Pope black hole solution with two equal angular momenta. The solutions generated by the transformation also belong to a class of cohomogeneity-1, though our solution has the limit to the cohomogeneity-1 asymptotically flat Cvetiˇc-Lü–Pope black hole solution with two equal angular momenta.

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