The Weil-Petersson geometry of the five-times punctured sphere

Javier Aramayona

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Abstract

We give a new proof that the completion of the Weil-Petersson metric on Teichmüller space is Gromov-hyperbolic if the surface is a five-times punctured sphere or a twice-punctured torus. Our methods make use of the synthetic geometry of the Weil-Petersson metric.

1 Introduction

The large scale geometry of Teichmüller space has been a very important tool in different aspects of the theory of hyperbolic 3-manifolds. Within this context, a natural question to ask is whether Teichmüller space, with a given metric, is hyperbolic in the sense of Gromov (or Gromov hyperbolic, for short). In general, the answer is negative. In their paper [BrFa], the authors prove the following: if $\Sigma$ is a surface of genus $g$ and with $p$ punctures, with $3g - 3 + p > 2$, then the Teichmüller space of $\Sigma$, endowed with the Weil-Petersson metric, is not Gromov hyperbolic. In the case when $3g - 3 + p = 2$ (that is, when the surface is a sphere with five punctures or a twice-punctured torus) they show that the Weil-Petersson Teichmüller space is Gromov hyperbolic. The proof makes reference to very deep results by Masur and Minsky on the Gromov hyperbolicity of the curve complex. The aim of this paper is to give a direct proof of the Gromov hyperbolicity of the Weil-Petersson Teichmüller space in the case $3g - 3 + p = 2$.

The objective of this article is to show the following result:

**Theorem 1** If $\Sigma$ is the five-times punctured sphere or the twice punctured torus, then $T_{WP}(\Sigma)$ is Gromov hyperbolic.

Brock [Br] showed that, for every hyperbolic surface $S$, $T_{WP}(\Sigma)$ is quasiisometric to the pants complex (see [Br] for definitions) $C_P(\Sigma)$ of the surface $\Sigma$. Since Gromov hyperbolicity is a quasi-isometry invariant, we get the following result:

**Corollary 2** If $\Sigma$ is the five-times punctured sphere or the twice punctured torus, then the pants complex $C_P(\Sigma)$ of $\Sigma$ is Gromov hyperbolic.
Remark. Behrstock [Be] has given a direct proof of this last result, using the combinatorial structure of the pants complex.

Let \( \Sigma \) be a hyperbolic surface of genus \( g \) and with \( p \) punctures and let \( \mathcal{T}(\Sigma) \) be the Teichmüller space of \( \Sigma \). The Weil-Petersson metric on \( \mathcal{T}(\Sigma) \) is a non-complete metric of negative sectional curvature.

Remark. We note that Theorem 1 cannot be obtained as a consequence of the negative sectional curvatures of the Weil-Petersson metric, since these curvatures have been shown (see [Hu]) not to be bounded away from zero.

Let \( \mathcal{T}_{WP}(\Sigma) \) denote the Teichmüller space of \( \Sigma \) endowed with the Weil-Petersson metric. The metric completion \( \overline{\mathcal{T}_{WP}(\Sigma)} \) is the augmented Teichmüller space (see [Ma]), i.e. the set of marked metric structures on \( \Sigma \) with nodes on a (possibly empty) collection of different homotopy classes of essential simple closed curves on \( \Sigma \). Here, a curve is essential if it is not null-homotopic nor homotopic to a puncture. From now on we will refer to a homotopy class of essential simple closed curves on \( \Sigma \) simply as a curve, unless otherwise stated.

Let \( \mathcal{C} = \mathcal{C}(\Sigma) \) be the curve complex of \( \Sigma \), as defined in [Har]. Recall that this is a finite-dimensional simplicial complex whose vertices correspond to (homotopy classes of non-trivial, non-peripheral) curves on \( \Sigma \), and that a subset \( A \subseteq V(\mathcal{C}) \) spans a simplex in \( \mathcal{C} \) if the elements of \( A \) can be realised disjointly on \( \Sigma \). Note that inclusion determines a partial order on the set of simplices of \( \mathcal{C} \). Following [Wo], we can define a map \( \Lambda : \overline{\mathcal{T}_{WP}(\Sigma)} \to \mathcal{C} \cup \{\emptyset\} \) that assigns, to a point in \( u \in \mathcal{T}_{WP}(\Sigma) \), the (possibly empty) collection of different curves on \( \Sigma \) on which \( u \) has nodes. The space \( \overline{\mathcal{T}_{WP}(\Sigma)} \) is the union of the level sets of \( \Lambda \). Then, \( \overline{\mathcal{T}_{WP}(\Sigma)} \) has the structure of a stratified space, where the level sets of \( \Lambda \) are the strata (observe that \( \Lambda^{-1}(\{\emptyset\}) = \mathcal{T}_{WP}(\Sigma) \) and that two strata intersect over a stratum if at all). We will refer to \( \Lambda^{-1}(\Lambda(u)) \) as the stratum containing \( u \) and we will say that it has label \( \Lambda(u) \). The strata of \( \overline{\mathcal{T}_{WP}(\Sigma)} \) are isometric embeddings of products of lower dimensional Teichmüller spaces (which come from subsurfaces of \( \Sigma \)) with their corresponding Weil-Petersson metric. It is clear that the stratum with label a collection of curves that determine a pants decomposition on \( \Sigma \) consists of only one point in \( \mathcal{T}_{WP}(\Sigma) \). Let \( \text{Mod}(\Sigma) \) be the mapping class group of \( \Sigma \), i.e. the group of self-homeomorphisms of \( \Sigma \) up to homotopy. It is known (see [Ab]) that \( \text{Mod}(\Sigma) \) acts cocompactly on \( \overline{\mathcal{T}_{WP}(\Sigma)} \). The space \( \overline{\mathcal{T}_{WP}(\Sigma)} \) is not locally compact: a point in \( \overline{\mathcal{T}_{WP}(\Sigma)} \setminus \mathcal{T}_{WP}(\Sigma) \) does not admit a relatively compact neighbourhood. Indeed, let \( u \) be a point in \( \overline{\mathcal{T}_{WP}(\Sigma)} \setminus \mathcal{T}_{WP}(\Sigma) \), which corresponds to a surface with a nodes on the simple closed curve \( \alpha \) (and possibly more), and consider the Dehn twist \( T_\alpha \) along \( \alpha \). Then the \( T_\alpha \) orbit of any point lies in every neighbourhood of \( u \) (see [Wo]). Nevertheless, individual strata are locally compact, since they are (products of) lower dimensional Teichmüller spaces.

The following result summarises some deep and remarkable facts about the geometry of the Weil-Petersson metric on Teichmüller space (see below for the relevant definitions). Part (i) is due to S. Yamada [Ya]; (ii) is due to Wolpert [Wo] and (iii) is due to Daskalopoulos and Wentworth [DW]. Let us note that all these results rely on previous work by Wolpert on the Weil-Petersson metric.
Theorem 3 ([DW], [Wo], [Ya]) Let Σ be a surface of hyperbolic type and let \( T_{WP}(\Sigma) \) be the completion of the Weil-Petersson metric on \( T_{WP}(\Sigma) \). Then,

1. The space \( T_{WP}(\Sigma) \) is a CAT(0) space.
2. The closure of a stratum in \( T_{WP}(\Sigma) \) is convex and complete in the induced metric.
3. The open geodesic segment \([u,v]\) \{u,v\} from \( u \) to \( v \) lies in the stratum with label \( \Lambda(u) \cap \Lambda(v) \).

Let \( X_0 \) be the Teichmüller space of the five-times punctured sphere \( \Sigma_{0,5} \), endowed with the Weil-Petersson metric and let \( X \) be its metric completion. We will write \( X_F = X \setminus X_0 \). Since a pants decomposition of \( \Sigma_{0,5} \) corresponds to two disjoint curves on \( \Sigma_{0,5} \), we get that a stratum in \( X \) has label \( \alpha \) or \( \alpha \beta \), where \( \alpha \) and \( \beta \) are disjoint curves on \( \Sigma_{0,5} \). The closure of a stratum of the type \( S_\alpha \) is given by \( S_\alpha = S_\alpha \cup (\bigcup_{\beta \in B} S_{\alpha \beta}) \), where \( B \) is the set of curves that are disjoint from \( \alpha \). We observe that any curve \( \alpha \) separates \( \Sigma_{0,5} \) into two subsurfaces, namely a four-times punctured sphere and a three-times punctured sphere. Then \( S_\alpha \) corresponds to the Teichmüller space of the four-times punctured sphere, since the Teichmüller space of the other subsurface is trivial. Also, recall that a stratum of the form \( S_{\alpha \beta} \) is a single point in \( X \).

We prove Theorem 1 in the case where the surface is a sphere with five punctures. Using the techniques from Sections 2 and 3 we immediately get the result for the twice-punctured torus. The only difference between the two cases is the nature of the strata in \( T_{WP}(\Sigma) \) which arise from pinching a single curve on the surface. In the first case, these strata correspond to the Teichmüller space of a four-times punctured sphere; in the second, they correspond either to the Teichmüller space of a four-times punctured sphere or to the Teichmüller space of a once-punctured torus, depending on whether the curve giving rise to such a stratum separates the surface. In both cases, \( T_{WP}(\Sigma) \) has the same structure (as a stratified space).

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2 Preliminaries

In order to give a proof of Theorem 1, we will have to make use of some geometric properties of CAT(0) spaces.

Let us begin by recalling the definition of a CAT(0) space. Let \( Y \) be a geodesic metric space, that is, a space in which every two points in the space can be connected by a path which realises their distance; such a path is called a geodesic between the two points. By a triangle in \( Y \) we will mean three points \( x_1, x_2, x_3 \in Y \), the vertices of \( T \), and three geodesics connecting them pairwise. We will write \([x_i, x_j]\) for the geodesic side of \( T \) with endpoints \( x_i \) and \( x_j \). Throughout this article we will denote the euclidean plane by \( \mathbb{E}^2 \) and the euclidean distance in \( \mathbb{E}^2 \) by \( d_e \).

Definition. Let \( Y \) be a geodesic metric space and let \( T \) be triangle in \( Y \) with vertices \( x_1, x_2, x_3 \). A comparison triangle for \( T \) in \( \mathbb{E}^2 \) is a geodesic triangle in \( \mathbb{E}^2 \) with vertices \( \overline{x}_1, \overline{x}_2 \) and \( \overline{x}_3 \) such that...
$d(x_i, x_j) = d_e(\overline{x_i}, \overline{x_j})$ for all $i, j = 1, 2, 3$. Given a point $p \in [x_i, x_j]$, for some $i, j = 1, 2, 3$ distinct, a comparison point $\overline{p}$ for $p$ is a point $\overline{p} \in [\overline{x_i}, \overline{x_j}]$ such that $d(x_i, p) = d_e(\overline{x_i}, \overline{p})$ and $d(x_j, p) = d_e(\overline{x_j}, \overline{p})$.

**Definition.** We say that the triangle $T$ satisfies the CAT(0) inequality if for any points $p \in [x_i, x_j]$ and $q \in [x_j, x_k]$, for $i, j, k = 1, 2, 3$ distinct, we have that $d(p, q) \leq d_e(p, q)$, where $p$ and $q$ are comparison points for $p$ and $q$, respectively, in the comparison triangle $\overline{T}$ for $T$. We will say that the space $X$ is a CAT(0) space if every triangle in $X$ satisfies a CAT(0) inequality.

The next three results about CAT(0) spaces are well-known; they will be crucial in our main argument. For a proof see, for instance [BriHa].

**Theorem 4** If $Y$ is a CAT(0) space, then the distance function on $Y$ is convex along geodesics, that is, if $\sigma, \sigma' : [0, 1] \to Y$ are geodesics in $Y$ parametrised proportional to arc-length, then

$$d(\sigma(t), \sigma'(t)) \leq td(\sigma(0), \sigma'(0)) + (1 - t)d(\sigma(1), \sigma'(1)),$$

for all $t \in [0, 1]$.

**Corollary 5** Every CAT(0) space is uniquely geodesic.

**Theorem 6** Let $Y$ be a CAT(0) space and let $C$ be a complete convex subset of $Y$. Given $x \in Y$ there exists a point $\pi(x) \in C$ such that $d(x, \pi(x)) = \inf_{c \in C} d(x, c)$. Moreover, the map $\pi : Y \to C$ is distance non-increasing, that is, for all $x, y \in Y$ we have that $d(x, y) \leq d_C(\pi(x), \pi(y))$, where $d_C$ denotes the subspace metric.

When interested in the large-scale geometry of a CAT(0) space, a natural question to ask is what are the obstructions for such a space to be Gromov hyperbolic. An answer to this question was given by Bowditch [Bo] and Bridson [Bri] separately, generalising a result announced by Gromov. They showed the following.

**Theorem 7** ([Bo], [Bri]) Let $Y$ be complete, locally compact CAT(0) space which admits a co-compact isometric group action. Then, either $Y$ is Gromov hyperbolic or else it contains a totally geodesic embedding of a euclidean plane.

Let $X$ be the completion of the Weil-Petersson metric on the Teichmüller space of the five-times punctured sphere. We are going to show, using some of the techniques in [Bo], that if $X$ is not Gromov hyperbolic then there exists an isometrically embedded euclidean disc in one of the strata of $X$, which is impossible since all the sectional curvatures of $X$ are strictly negative. More specifically, we are going to construct, for each $n \in \mathbb{N}$, a map $\phi_n : D \to X$, where $D$ is the unit disc in $E^2$, such that
\[ \lambda_n d_e(x, y) \leq d(\phi_n(x), \phi_n(y)) \leq d_e(x, y), \]

for all \( x, y \in D \), and where \( \lambda_n \in (0, 1) \) with \( \lambda_n \to 1 \) as \( n \to \infty \). If the space \( X \) were locally compact, one could take the limit of \( \phi_n \) as \( n \) tends to infinity, obtaining in this way an isometric embedding of the euclidean unit disc in \( X \). Even though \( X \) is not locally compact, we are able to use the structure of \( X \) as a stratified space and the fact that individual strata are locally compact to obtain such an isometrically embedded disc. We note that we will not use the cocompact isometric action of the mapping class group on \( X \) to construct these maps; this action will only be used in the arguments in the next section. The construction of the maps \( \phi_n \) is totally analogous to the one in \([Bo]\), where Bowditch shows the following result (we remark that the results in \([Bo]\) are more general than the ones we present here).

**Lemma 8** \([Bo]\) *Let \( Y \) be a CAT(0) space. Given \( n \in \mathbb{N} \) and \( \epsilon > 0 \), there exists a map \( \phi_{n,\epsilon} : ([n, n + 1] \cap \mathbb{Z})^2 \to Y \) such that*

\[ |i - i'| - \epsilon \leq d(\phi_{n,\epsilon}(i, j), \phi_{n,\epsilon}(i', j)) \leq |i - i'| \tag{1} \]

*and*

\[ |j - j'| - \epsilon \leq d(\phi_{n,\epsilon}(i, j), \phi_{n,\epsilon}(i, j')) \leq |j - j'| \tag{2} \]

**Remark.** For the sake of completeness, we now give a brief account on Bowditch's construction. Let \( n \in \mathbb{N} \) and \( \epsilon > 0 \) and let \( q \) be a natural number bigger than \( 2n^2/\epsilon \). Let \( \sigma : [0, qn] \to Y \) be a geodesic segment in \( Y \) and let \( y \) be a point in \( Y \) at distance at least \( n \) from \( \alpha \). Let \( \tau_i : [0, d(y, \sigma(i))] \to Y \) be the unique geodesic from \( y \) to \( \sigma(i) \), for all \( i \in [0, qn] \cap \mathbb{N} \). Bowditch then sets, for all \( i, j = 0, \ldots, n \) and for all \( p = 0, \ldots, q - 1 \), \( \phi_{p,i,j} = \tau_{pm+i}(j) \) and shows that there exists a number \( p = 0, \ldots, n - 1 \) satisfying (1) and (2).

Bowditch then shows that if the space \( Y \) is, in addition, not Gromov hyperbolic then the maps \( \phi_n \) satisfy the following non-degeneracy condition.

**Lemma 9** *Let \( Y \) be a CAT(0) space and suppose \( Y \) is not Gromov hyperbolic. Then the maps \( \phi_n \) in the result above satisfy, in addition, that*

\[ d(\phi_{n,\epsilon}(i, j), \phi_{n,\epsilon}(i + 1, j + 1)) \geq 1/2 \tag{3} \]

*and*

\[ d(\phi_{n,\epsilon}(i + 1, j), \phi_{n,\epsilon}(i + 1, j)) \geq 1/2 \tag{4} \]

*for all \( n \in \mathbb{N} \).*

We now give an extension, using standard arguments about CAT(0) spaces, to Bowditch's construction. It will play a central role in the proof of Theorem [**].
Lemma 10 Let $Y$ be a CAT(0) space and suppose that $Y$ is not Gromov hyperbolic. Fix a number $N \in \mathbb{N}$ and consider $K = [-N, N] \times [-N, N] \subseteq \mathbb{R}^2$, endowed with the euclidean metric. Then there exists a sequence $(\phi_n)_{n \in \mathbb{N}}$ of maps $\phi_n : K \to Y$ such that

$$
\lambda_n d_e(x, y) \leq d(\phi_n(x), \phi_n(y)) \leq d_e(x, y),
$$

where $\lambda_n \in (0, 1)$ for all $n \in \mathbb{N}$ and $\lambda_n \to 1$ as $n \to \infty$.

Proof Let $(\epsilon_n)_{n \in \mathbb{N}}$ be a sequence of positive real numbers tending to zero and let $\phi_n = \phi_{n, \epsilon_n}$ be the map described above. First, we are going to extend the map $\phi_n$ to $K$ as follows:

Let $\sigma_n : [0, 1] \to Y$ be the unique geodesic segment in $Y$ from $\phi_n(0, 0)$ to $\phi_n(1, 0)$. Similarly, let $\sigma'_n : [0, 1] \to Y$ be the geodesic segment in $Y$ from $\phi_n(0, 1)$ to $\phi_n(1, 1)$. Here, and from now on, we assume that all the geodesics are parametrised proportional to arc-length. For $t \in [0, 1]$, let $\tau^n(t) : [0, 1] \to Y$ be the unique geodesic in $Y$ connecting $\sigma(t)$ and $\sigma'(t)$. We set $\phi_n(t, s) := \tau^n(s)$, for all $t, s \in [0, 1]$. We can extend $\phi_n$ to $K$ by performing this construction on each square $[i, i+1] \times [j, j+1]$ for all $i, j \in [-N, N] \cap \mathbb{Z}$, provided $n$ is large enough so that $\phi_n$ is defined on the whole of $K$. Note that, up to extracting a subsequence, we can assume that this is the case.

Since the distance function on $Y$ (Theorem 4) is convex along geodesics we get, from inequality (2), that

$$
d(\phi_n(t, j), \phi_n(t, j + 1)) \leq 1,
$$

for all $t \in [-N, N]$, $j \in [-N, N] \cap \mathbb{Z}$ and $n \in \mathbb{N}$. The convexity of the distance function on $Y$ yields that the real function $[t \to d(\phi_n(t, j), \phi_n(t, j + 1))]$ tends, as $n$ grows, to a real function which is convex (and also bounded, from (2)). But a real function which is convex and bounded must be constant and therefore $d(\phi_n(t, j), \phi_n(t, j + 1)) \to 1$ as $n \to \infty$, from (1).

Note that, from (2) and the properties of the extension $\phi_n$ to $K$, we can deduce that the images of a vertical segment of the form $\{i\} \times [-N, N]$ under $\phi_n$, where $i \in [-N, N] \cap \mathbb{Z}$, get arbitrarily close to being geodesic in $Y$ as $m$ grows; by this we mean that the Hausdorff distance between $\phi_n(\{i\} \times [-N, N])$ and the unique geodesic in $Y$ with the same endpoints tends to 0 as $n$ tends to infinity. By a totally analogous convexity argument we obtain that the images of any two vertical lines in $K$ under $\phi_n$ get arbitrarily close to being parallel geodesics as $n \to \infty$. Note that this implies that, up to taking a subsequence, the maps $\phi_n$ are injective on $K$.

Let $0 \leq t, t', s, s' \leq 1$. Again from the convexity of the distance function we obtain that

$$
d(\tau^n_i(s), \tau^n_j(s)) \leq sd(\tau^n_i(0), \tau^n_j(0)) + (1 - s)d(\tau^n_i(1), \tau^n_j(1)) = sd(\sigma_n(t), \sigma_n(t')) + (1 - s)d(\sigma'_n(t), \sigma'_n(t')) = |t - t'|.
$$
From this inequality and the fact that \( \tau^n_t : [0, 1] \to Y \) is a geodesic parametrised proportional to arc length for every \( t \in [0, 1] \), we deduce that

\[
d(\phi_n(t, s), \phi_n(t', s')) = d(\tau^n_t(s), \tau^n_t(s')) \\
= d(\tau^n_t(s), \tau^n_{t'}(s')) + d(\tau^n_{t'}(s'), \tau^n_{t'}(s')) \\
\leq |s - s'| + |t - t'|,
\]

and thus \( \phi_n \) is continuous on \( K \) for all \( n \in \mathbb{N} \).

Recall that the images of vertical lines in \( K \) under the maps \( \phi_n \) get arbitrarily close to being parallel geodesics in \( Y \) as \( n \) grows. Consider now a horizontal line \([-N, N] \times \{s_0\} \) in \( K \) and let \( \rho_n : [0, 1] \to Y \) be the geodesic in \( Y \) between \( \phi_n(0, s_0) \) and \( \phi_n(1, s_0) \). We know, from the discussion above, that \( d(\sigma_n(t), \sigma'_n(t)) \to 1 \) as \( n \to \infty \) for all \( t \in [0, 1] \). From the convexity of the distance function we get that \( d(\sigma_n(t), \rho_n(t)) \to s_0 \) and \( d(\sigma'_n(t), \rho_n(t)) \to 1 - s_0 \) as \( n \to \infty \). But \( d(\phi_n(t, s_0), \sigma_n(t)) \to s_0 \) and \( d(\phi_n(t, s_0), \sigma'_n(t)) \to 1 - s_0 \) as \( n \to \infty \), since \([s \to \phi_n(t, s)]\) is geodesic for a fixed \( t \in [0, 1] \). Thus the images of a horizontal line in \( K \) under \( \phi_n \) are paths that get arbitrarily close to being geodesics \( n \to \infty \). By a similar argument we get that the same holds for the images of a horizontal line in \( K \) under \( \phi_n \).

Let \( d_n \) be the pull-back metric on \( K \) determined by \( \phi_n \). Since \( d_n((-N, -N), (N, N)) \) is bounded above and below (note that, in particular, it is bounded below away from 0, from (1)) we can assume that \( d_n((-N, -N), (N, N)) \to a > 0 \), up to extracting a subsequence. Also recall, from inequalities (1) and (2), that \( d_n((-N, -N), (-N, N)) \to 2N \) and \( d_n((-N, -N), (N, -N)) \to 2N \) as \( n \to \infty \).

Let \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) be an affine map such that \( f(t, 0) = (t, 0) \), for all \( t \in \mathbb{R} \), and \( d_e(f(-N, -N), f(N, N)) = a \). By considering \( \phi_n \circ f^{-1} \), which we denote again by \( \phi_n \) abusing notation, we deduce that the maps \( \phi_n \), restricted to \( K \), satisfy that

\[
d_e(x, y) - \epsilon_n \leq d(\phi_n(x), \phi_n(y)) \leq d_e(x, y),
\]

for all \( x, y \in K \). Consider the restriction of \( \phi_n \) to the unit disc \( D \) in \( \mathbb{R}^2 \). It follows immediately from the last inequality that there exists a sequence \( (\lambda_n)_{n \in \mathbb{N}} \) of real numbers, with \( \lambda_n = \lambda_n(\epsilon_n) \) and \( \lambda_n \in (0, 1) \), such that

\[
\lambda_n d_e(x, y) \leq d(\phi_n(x), \phi_n(y)) \leq d_e(x, y),
\]

as desired. \( \diamond \)
3 Proof of the Theorem

Let us begin with some technical results that will be important to prove Theorem 2.

Remark. Let $\Sigma$ be a hyperbolic surface. In [Wo], Corollary 21, Wolpert shows that for a stratum $S$ defined by the vanishing of the length sum $l = l_1 + \ldots + l_i$, where $l_i$ corresponds to the length of some curve $\alpha_i$ on the surface for $i = 1, \ldots, n$, the distance to the stratum is given locally as $d(p, S) = (2\pi l)^{1/2} + O(l^2)$. In particular, if $(u_m)_{m \in \mathbb{N}}$ is a sequence in $X$ such that $d(u_m, S) \to 0$ as $m \to \infty$ we get that $l_{u_m}(\alpha_i) \to 0$ as $m \to \infty$, for all $i = 1, \ldots, n$, where $l_{u_m}(\alpha)$ denotes the length of $\alpha$ in $u_m$.

Lemma 11. Let $S_\alpha$ and $S_\beta$ be two different strata in $X_F$ such that $\overline{S_\alpha} \cap \overline{S_\beta} \neq \emptyset$. Let $(u_n)_{n \in \mathbb{N}}$ and $(v_n)_{n \in \mathbb{N}}$ be sequences in $\overline{S_\alpha}$ and $\overline{S_\beta}$, respectively, such that $d(u_n, v_n) \to 0$ as $n \to \infty$. Then $d(u_n, S_{\alpha\beta}) \to 0$ and $d(v_n, S_{\alpha\beta}) \to 0$ as $n \to \infty$.

Proof Suppose, for contradiction, that the result is not true. Let $\Gamma$ be the mapping class group of $\Sigma_{0,5}$. Since the action of $\Gamma$ on $X$ is cocompact there is a compact subset $Z$ of $X$ and a sequence $(\gamma_n)_{n \in \mathbb{N}}$ in $\Gamma$ such that $\gamma_n(u_n) \in Z$ for all $n \in \mathbb{N}$. Note that $d(u_n, v_n) \to 0$ as $n \to \infty$ if and only if $d(\gamma_n(u_n), \gamma_n(v_n)) \to 0$ as $n \to \infty$, since $\Gamma$ acts isometrically on $X$ and therefore $d(\gamma_n(u_n), \gamma_n(v_n)) = d(u_n, v_n)$ for all $n \in \mathbb{N}$. We replace $u_n$ and $v_n$ by $\gamma_n(u_n)$ and $\gamma_n(v_n)$, respectively; abusing notation we denote the new points by $u_n$ and $v_n$ again. Then, up to extracting a subsequence, we get that the sequence $(u_n)_{n \in \mathbb{N}}$ converges to a point $u_0 \in X$, since $Z$ is compact. Since $u_n \in \overline{S_\alpha}$, then $l_{u_n}(\alpha) = 0$ for all $n \in \mathbb{N}$. Similarly, we have that $l_{v_n}(\beta) = 0$ for all $n$. From the remark above and since $d(u_n, S_{\alpha\beta}) > K$ for infinitely many $n$, it follows that there must exist a constant $L = L(K) > 0$ such that $l_{u_n}(\beta) > L$, for infinitely many $n \in \mathbb{N}$. Therefore, for the simple closed curve $\beta$ we have that $l_{u_n}(\beta) > L > 0$ and $l_{v_n}(\beta) = 0$, for infinitely many $n$, which is impossible since we know from the hypotheses that $d(u_n, v_n) \to 0$ as $n \to \infty$.

As a trivial consequence, we get the following corollary:

Corollary 12. Given $r > 0$ and a stratum of the form $S_{\alpha\beta}$, let $B(r) = B(S_{\alpha\beta}, r)$ be the ball of radius $r$ around $S_{\alpha\beta}$. Then, there exists $D = D(r) > 0$ such that $d(u, v) > D$, for all $u \in \overline{S_\alpha} \setminus (B(r) \cap \overline{S_\alpha})$ and $v \in \overline{S_\beta} \setminus (B(r) \cap \overline{S_\beta})$.

Proof [of Theorem 2] Suppose, for contradiction, that $X$ is not Gromov hyperbolic. Since $X$ is a CAT(0) space, we can construct a sequence of maps $(\phi_n : D \to X)_{n \in \mathbb{N}}$ as described in the previous section. Using the structure of $X$ as a stratified space, there is a stratum $S \subseteq X$ such that $\phi_n(D) \subseteq \overline{S}$ (note that $S$ may have label $\emptyset$, so that $\overline{S} = X$). Also, $S$ cannot have a pants decomposition as label, since recall that such a stratum consists of only one point). The stratum $S$ corresponds to the Weil-Petersson Teichmüller space of a properly embedded hyperbolic subsurface $\Sigma_S \subseteq \Sigma_{0,5}$ (possibly
with \( \Sigma_S = \Sigma_{0,5} \). Let \( \Gamma_0 \) be the mapping class group of \( \Sigma_S \), and recall that \( \Gamma_0 \) acts cocompactly on \( \overline{S} \). We will denote \( \overline{S} \setminus S \) by \( S_F \). We now have two possibilities:

(a) There exists a point \( x_0 \in D \) and a constant \( \delta = \delta(x_0) \) such that \( d(\phi_n(x_0), S_F) > \delta \), for infinitely many \( n \in \mathbb{N} \). We can assume, up to passing to \( X/\Gamma_0 \) and lifting back, that the sequence \( (\phi_n(x_0))_{n \in \mathbb{N}} \) converges to a point \( w \in S \), since the action of \( \Gamma_0 \) on \( \overline{S} \) is cocompact.

**Notation.** We will write \( D_e(a, r) \) to denote the euclidean disc in \( E^2 \) with centre \( a \) and radius \( r \).

Consider \( D_e(x_0, \eta) \), where \( \eta = \min(\delta/4, 1 - d_e(0, x_0)) \). Then,
\[
d(\phi_n(x), S_F) \geq d(\phi_n(x_0), S_F) - d(\phi_n(x), \phi_n(x_0)) \geq \delta - 2\eta \geq \delta/2 > 0,
\]
for all \( x \in D_e(x_0, \eta) \). So, \( d(\phi_n(D_e(x_0, \eta), S_F) > \delta/2 \), for all \( n \in \mathbb{N} \). Since \( S \) is locally compact, the maps \( \phi_n \) converge (maybe after passing to a subsequence) on \( D_e(x_0, \eta) \) to a map \( \phi \). Therefore, \( \phi(D_e(x_0, \eta)) \) is a copy of a euclidean disc in \( S \), which is impossible since all its sectional curvatures are strictly negative.

(b) Otherwise, for all \( x \in D_e, d(\phi_n(x), S_F) \to 0 \) as \( n \to \infty \). Again we can assume (up to the action of \( \Gamma \)) that \( \phi_n(0) \to w' \) as \( n \to \infty \), for some \( w' \in S_F \).

**Remark.** One of the consequences of the Collar Lemma is that if \( \alpha \) and \( \beta \) are intersecting curves on a hyperbolic surface \( \Sigma \), then their lengths cannot be very small simultaneously. In the light of this result, it is possible to show (see [Wo], Corollary 22) that there exists a constant \( k_0 = k_0(\Sigma) \) such that two strata \( S_1, S_2 \subseteq \overline{T_{WP}(\Sigma)} \setminus T_{WP}(\Sigma) \) either have intersecting closures or they satisfy \( d(S_1, S_2) \geq k_0 \).

Let \( k_0 = k_0(\Sigma_S) \) be the constant given in the remark above and consider the maps \( \phi_n \) restricted to \( D_e(0, k_0/3) \). We may as well assume that \( k_0 \leq 1 \) (if not, take \( k_0' = \min\{1, k_0\} \)). Also from the remark above, we deduce that \( S \) must have label \( \emptyset \). Otherwise, if \( S \) had label \( \alpha \), for some simple closed curve \( \alpha \), the fact that \( d(\phi_n(x), S_F) \to 0 \) as \( n \to \infty \), for all \( x \in D_e(0, k_0/3) \) would imply that there exists a curve \( \beta \) in \( \Sigma_{0,5} \), disjoint from \( \alpha \), such that \( d(\phi_n(x), S_{0,\beta}) \to 0 \) as \( n \to \infty \), for all \( x \in D_e(0, k_0/3) \). But we know that a stratum of the form \( S_{0,\beta} \) consists of only one point, which contradicts the construction of the maps \( \phi_n \) since we know that the maps \( \phi_n \) contract distances by a factor \( \lambda_n \) at most. Thus we assume, from now on, that \( S \) has label \( \emptyset \) and so \( \overline{S} = X \) and \( S_F = X_F \).

We observe that, given \( u \in X_F \), there are at most two non-trivial strata, say \( S_\alpha \) and \( S_\beta \), in \( X_F \) such that \( u \in \overline{S_\alpha} \cap \overline{S_\beta} \). These strata correspond to two (possibly equal) simple closed curves on the surface on which \( u \) has nodes. Note that in the case when there are exactly two strata, we get that \( \{u\} = \overline{S_\alpha} \cap \overline{S_\beta} \). So we deduce that \( d(\phi_n(x), \overline{S_\alpha} \cap \overline{S_\beta}) \to 0 \) as \( n \to \infty \), for all \( x \in D_e(0, k_0/3) \).

Our next aim is to show that there exists \( x_1 \in D_e(0, k_0/3) \) and \( k_1 = k_1(x_1) > 0 \), with \( k_1 \leq k_0/3 \), so that the images under \( \phi_n \) of the points in \( D_e(x_1, k_1) \) get uniformly arbitrarily close to the same stratum as \( n \to \infty \). Using this result, we will be able to define a distance non-decreasing projection from \( D_e(x_1, k_1) \) to the closure of that particular stratum.

Suppose that \( \phi_n(0) \to w' \in \overline{S_\alpha} \setminus S_{0,\beta} \) as \( n \to \infty \). In particular, there exists \( r > 0 \) such that, maybe considering a subsequence, \( d(\phi_n(0), S_{0,\beta}) > r \), say. Let \( D = D(r) \) be the constant given in Corollary 4 and consider \( D_e(0, k) \), where \( k = \min(D/3, k_0/3) \). We claim that we can take
Suppose, for contradiction, that the images of \( D_e(0, k) \) do not get uniformly arbitrarily close to \( \overline{S}_\alpha \); that is, there exists \( K_0 > 0 \), a subsequence \((\phi_m)_{m \in \mathbb{N}} \subseteq (\phi_n)_{n \in \mathbb{N}}\) and points \( u_m \in \phi_m(D_e(0, k)) \) such that \( d(u_m, \overline{S}_\alpha) \geq K_0 \) for all \( m \in \mathbb{N} \). We know that, for all \( m \in \mathbb{N} \), \( u_m = \phi_m(x_m) \) for some \( x_m \in D_e(0, k) \) and thus, up to a subsequence, \( x_m \to y \in D_e(0, k) \).

From the construction of the maps \( \phi_n \) we have that \( d(u_m, \phi_m(y)) \to 0 \) as \( n \to \infty \). Therefore, \( d(u_m, \overline{S}_\alpha \cup \overline{S}_\beta) \to 0 \) as \( m \to \infty \) and thus \( d(u_m, \overline{S}_\alpha) \to 0 \), since we know that \( d(u_m, \overline{S}_\alpha) \geq K_0 \) for all \( m \). This represents a contradiction since \( d(u_m, \phi_m(0)) \leq D/3 \), the points \( u_m \), and \( \phi_m(0) \) lie in \( X \setminus B(r) \), for all \( m \in \mathbb{N} \), but \( d(\overline{S}_\alpha \setminus (B(r) \cap \overline{S}_\alpha), \overline{S}_\beta \setminus (B(r) \cap \overline{S}_\alpha)) > D \).

The case \( \phi_n(0) \to w' \in S_{\alpha\beta} \) as \( n \to \infty \) is dealt with in complete analogy, considering \( x_1 \) to be any point in \( D_e(0, k_0/3) \setminus \{0\} \) and and defining \( k_1 \) in a similar way as we did above.

Since \( X \) is a CAT(0) space and \( \overline{S}_\alpha \) is complete and convex we can consider the orthogonal projection \( \pi : X \to \overline{S}_\alpha \) as defined in Theorem 6. Recall that this projection is distance non-increasing; in particular, for all \( x, y \in D_e(x_1, k_1) \) we have that

\[
d(\pi(\phi_n(x)), \pi(\phi_n(y))) \leq d(\phi_n(x), \phi_n(y)) \leq d_e(x, y).
\]

Choose a sequence \((\delta_m)_{m \in \mathbb{N}}\) of positive reals such that \( \delta_m \to 0 \) as \( m \to \infty \). Up to a subsequence we can assume that, given \( m \in \mathbb{N} \), \( d(\phi_n(x), \overline{S}_\alpha) < \delta_m \), for \( n \geq m \) and for all \( x \in D_e(x_1, k_1) \). We have that

\[
d(\phi_n(x), \phi_n(y)) \leq d(\phi_n(x), \pi(\phi_n(x))) + d(\phi_n(y), \pi(\phi_n(y))) + d(\pi(\phi_n(x)), \pi(\phi_n(y)));
\]

and thus, for all \( m \in \mathbb{N} \),

\[
d(\pi(\phi_m(x)), \pi(\phi_m(y))) \geq d(\phi_m(x), \phi_m(y)) - 2\delta_m \geq \lambda_m d_e(x, y) - 2\delta_m.
\]

Let \( \psi_n = \pi \circ \phi_n : D_e(x_1, k_1) \to \overline{S}_\alpha \). Then exists a sequence \((\lambda'_n)_{n \in \mathbb{N}}\) of positive real numbers tending to 1 (we could simply take \( \lambda'_n = \lambda_n - 2\delta_n \), since \( k_1 < 1 \) and therefore \( 2\delta_n < 2\delta_n d_e(x, y) \) for all \( x, y \in D_e(x_1, k_1) \)) so that

\[
\lambda'_nd_e(x, y) \leq d(\psi_n(x), \psi_n(y)) \leq d_e(x, y).
\]

We are now back to the situation described at the beginning of Section 2, this time in a stratum of the form \( \overline{S}_\alpha \). Reasoning in a totally analogous way to cases (a) and (b) we get that either we can find an isometrically embedded copy of a euclidean disc in \( X \) or else the maps \( \phi_n \) collapse \( D_e(x_1, k_1) \) to a point in \( X \), which is impossible since we know that \( \phi_n \) decreases distances by a factor \( \lambda_n \). In any case, we get a contradiction.

Therefore \( X \) is Gromov hyperbolic, as desired. \( \diamond \)
References

[Ab] W. Abikoff, *The real analytic theory of Teichmüller space*, Lecture Notes in Mathematics, 820. Springer, Berlin, 1980.

[Be] J. Behrstock, *Asymptotic geometry of the mapping class group and Teichmüller space*, arxiv:math.GT/0502367.

[Bo] B.H. Bowditch, *Minkowskian subspaces of non-positively curved metric spaces*: Bull. London Math. Soc. 27 (1995), no. 6, 575–584.

[Br] J. Brock, *The Weil-Petersson metric and volumes of 3-dimensional hyperbolic convex cores*: J. Amer. Math. Soc., 16 (2003), 495–535.

[BrFa] J. Brock, B. Farb, *Curvature and rank of Teichmüller space*: preprint (2001).

[Bri] M.R. Bridson, *On the existence of flat planes in spaces of nonpositive curvature*, Proc. Amer. Math. Soc. 123 (1995), no. 1, 223–235.

[BriHa] M.R. Bridson, A. Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 319. Springer-Verlag, Berlin, 1999.

[DW] G. Daskalopoulos, R. Wentworth, *Classification of Weil-Petersson isometries*, Amer. J. Math. 125 (2003), no. 4, 941–975.

[Har] W.J. Harvey, *Boundary structure of the modular group*, Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook conference. Ann. of Math. Stud. 97, Princeton, 1981.

[Hu] Z. Huang, *Asymptotic flatness of the Weil-Petersson metric on Teichmüller Space*, 2001. arxiv:math.DG/0312419.

[Ma] H. Masur, *Extension of the Weil-Petersson metric to the boundary of Teichmüller space*, Duke Math. J., 43(3):623-635, 1976.

[Wo] S.A. Wolpert, *Geometry of the Weil-Petersson completion of Teichmüller space*, Surveys in Differential Geometry, VIII: Papers in Honor of Calabi, Lawson, Siu and Uhlenbeck, editor S. T. Yau, International Press, Nov. 2003.

[Ya] S. Yamada, *Weil-Petersson Completion of Teichmüller Spaces and Mapping Class Group Actions*, 2001. arxiv:math.DG/0112001 W. Abikoff, *The real analytic theory of Teichmüller space*, Lecture Notes in Mathematics, 820. Springer, Berlin, 1980.

Javier Aramayona
Mathematics Institute, University of Warwick, Coventry CV4 7AL, U.K.
jaram@maths.warwick.ac.uk