Lie symmetries of the geodesic equations and projective collineations

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Abstract. We study the Lie symmetries of the geodesic equations in a Riemannian space and show that they coincide with the projective symmetries of the Riemannian metric. We apply the result to the spaces of constant curvature.

1. Motivation
Consider on the Euclidean plane a family of straight lines parallel to the $x$–axis. These curves can be considered either as the integral curves of the ODE $\ddot{y} = 0$ or as the geodesics of the Euclidean metric $dx^2 + dy^2$. Consider a symmetry operation defined by a reshuffling of these lines. According to the first interpretation this symmetry operation is a Lie symmetry of the ODE $\ddot{y} = 0$ and according to the second interpretation it is a projective symmetry of the Euclidean plane. This motivates one to examine to what extend the Lie symmetries of the equation of autoparallels in an affine space endowed with a symmetric connection $\Gamma^i_{jk}$ are related to the projective collineations of that space. In the special case of a Riemannian connection this question reduces to the identification of the Lie symmetries of the geodesic equations with the projective symmetries of the metric of the space.

1.1. Preliminary results
Consider a $C^\infty$ manifold $M$ of dimension $n$, endowed with a symmetric\(^1\) connection.

Let $X^a$ be a vector field on the manifold. The connection defines a family of curves on the manifold, called autoparallels, by the requirement that the covariant derivative of the tangent to these curves is parallel to itself, that is:

$$\nabla_{\dot{x}}(t)\dot{x}(t) = \phi(t)\dot{x}(t),$$

where $t$ is a parameter along the curves. When the parametrization is such that $\phi$ vanishes we say that the autoparallel is affinely parameterized and in this case $t$ is called an affine parameter.

In a local coordinate system $\{x^i | i = 1, \ldots, n\}$ equation (1) is written as the system of ODEs:

$$\ddot{x}^i(t) + \Gamma^i_{jk}(x(t))\dot{x}^j(t)\dot{x}^k(t) = \phi(t)\dot{x}^i(t), \quad i = 1, \ldots, n$$

\(^1\) The coefficients $\Gamma^i_{jk}$ in general are not symmetric. In the autoparallel equation (1) the antisymmetric part of $\Gamma^i_{jk}$ (the torsion) does play a role.
where $\Gamma^i_{jk} \partial_k = \nabla_j \partial_k$. The Lie derivative (of a not-necessarily symmetric connection) $\Gamma^i_{jk}$ with respect to a vector field $X = X^a \partial_a$ is defined as follows (see Yano [1] eqn. (2.16)):

$$\mathcal{L}_X \Gamma^i_{jk} = X^i_{,jk} + \Gamma^i_{jk,l} X^l + X^l_{,k} \Gamma^i_{lj} + X^l_{,j} \Gamma^i_{lk} - X^i_{,l} \Gamma^l_{jk}. \tag{3}$$

We say that the vector field $X$ is an **affine collineation** or **affine motion** iff

$$\mathcal{L}_X \Gamma^i_{jk} = 0. \tag{4}$$

In flat space equation (4) implies the condition:

$$X_{a,bc} = 0, \tag{5}$$

whose solution is:

$$X_a = B_{ab} x^b + C_a, \tag{6}$$

where $B_{ab}, C_a$ are constants.

The geometric property/definition of affine collineations is that they preserve the set of autoparallels of the connection together with their affine parametrization (that is, by an affine symmetry an affinely parameterized autoparallel goes over to an affinely parameterized autoparallel of the same connection). From (6) we infer that in an $n$-dimensional space there are at most $n + n^2 = n(n + 1)$ Affine Collineation Vectors and when this is the case, it can be shown that the space is a space of constant curvature.

Note that the equation of the autoparallels for affine parametrization takes the form:

$$\ddot{x}^i(t) + \Gamma^i_{jk}(x(t)) \dot{x}^j(t) \dot{x}^k(t) = 0, \quad i = 1, \ldots, n. \tag{7}$$

We say that a vector field $X$ is a projective collineation of the connection if there exists a function $f(x^i)$ so that the following condition holds:

$$\mathcal{L}_X \Gamma^i_{jk} = f_{,j} \delta^i_k + f_{,k} \delta^i_j. \tag{8}$$

In flat space condition (8) implies that:

$$X_{a,b} = B_{ab} + (A_c x^c) g_{ab} + C_b x_a, \tag{9}$$

which has the solution:

$$X_a = B_{ab} x^b + (A_b x^b) x_a + C_a, \tag{10}$$

where again the various coefficients are constants. In an $n$-dimensional space there are at most $n^2 + n + n = n(n + 2)$ projective collineations of the connection and when this is the case, it can be shown that the space is a space of constant curvature. This holds in any space irrespective of the signature of the metric and the (finite) dimension of space. The dimension of the space fixes the range of values of the indices $a, b, c, \ldots$.

The geometric property/definition of the (proper) projective collineations is that they preserve the set of autoparallels, but they do not preserve the affine parametrization. Obviously the affine collineations are not proper projective collineations.

The affine collineations form a group which is an invariant subgroup of the projective collineations. Concerning the algebras we have that the algebra of affine collineations contains the proper subalgebras of the Killing vectors, the homothetic Killing vector and does not contain the subalgebra of conformal motions.

The issue we wish to address in this work is:
Consider the autoparallel equation (2) as a system of second order ODEs and study its Lie symmetries. Then consider the quantities \( \Gamma^i_{jk} \) to be the components of a connection on the manifold and establish a relation between the Lie symmetries of the geodesic equations and the projective collineations of this connection. If such a relation exists then one has established a dual perspective for the symmetries of an ODE, that is, either as an non-geometric property (Lie symmetry) and as a geometric collineation (geometric approach). This double approach can be used to establish an interrelation between the study of Lie symmetries through Differential Geometry (where many aspects of the collineations have been addressed) and conversely the study of geometric problems, using the rather formal approach of the Lie method\(^2\).

2. Lie point symmetries of the geodesic equations
We consider the system of ODEs (2) and determine its Lie point symmetries generated by the vector field \( X \). Because the ODEs are of second order, we must prolong \( X \) up to the second prolongation. Let \( G^{[1]} \) be the coefficients of the first prolongation. From the prolongation formula we have
\[
G^{[1]} = \frac{d}{dt}\eta^i - \dot{x}^i \frac{d}{dt}\xi = \eta^i_1 + \eta^i_j \dot{x}^j - \xi_i \dot{x}^i - \xi_j \dot{x}^i \dot{x}^j.
\]

For the second prolongation coefficient \( G^{[2]} \) of \( X \) we have:
\[
G^{[2]} = \frac{d}{dt}G^{[1]} - \ddot{x}^i \frac{d}{dt}\xi = \eta^i_1 + 2\eta^i_j \dot{x}^j + \eta^i_{jk} \dot{x}^k \dot{x}^j - \xi_i \dot{x}^j - 2\xi_j \dot{x}^j \dot{x}^i - \xi_{jk} \dot{x}^j \dot{x}^i \dot{x}^k \dot{x}^j + \eta^i_j \ddot{x}^j - 2\xi_i \ddot{x}^i - \xi_j \ddot{x}^j \dot{x}^i - 2\xi_j \dot{x}^j \ddot{x}^i .
\]

The computation of Lie symmetries can be done in two different ways: Either by using the linear operator \( A \) associated with the system of ODEs and demanding the symmetry condition \([X^{[1]}, A] = \lambda A\), or by computing directly the action of the prolonged generator on the equation and demanding that \( X^{[1]}(ODE) = 0 \mod (ODE = 0) \) where ODE is given by (2).

3. Calculation of Lie symmetries using the associated linear operator
We write the system of ODEs in the form \( \ddot{x}^i = \omega^i(x, \dot{x}, t) \) where:
\[
\omega^i(x, \dot{x}, t) = -\Gamma^i_{jk}(x) \dot{x}^j \dot{x}^k - \phi(x) \dot{x}^i .
\]

The associated linear operator defined by this system of ODEs is:
\[
A = \frac{\partial}{\partial t} + \dot{x}^i \frac{\partial}{\partial x^i} + \omega^i(t, x, \dot{x}) \frac{\partial}{\partial \dot{x}^i}.
\]

The condition for a Lie symmetry for the system of equations is [3]:
\[
[X^{[1]}, A] = \lambda(x^a)A,
\]
where \( X^{[1]} \) is the first prolongation of the symmetry vector \( X = \xi(t, x) \partial_t + \eta^i(t, x) \partial_{x^i} \) defined as follows:
\[
X^{[1]} = \xi(t, x, \dot{x}) \partial_t + \eta^i(t, x, \dot{x}) \partial_{x^i} + G^{[1]} \partial_{\dot{x}^i}.
\]

\(^2\) A question, which could be addressed further, is if one can define a specific set of connection coefficients for a given ODE so that the solution curves of the ODE coincide with the autoparallels of that connection. If this is done then we have geometrized the ODE.
It is a standard result [3] that (14) leads to the following three conditions:

\[ -A\xi = \lambda, \quad G^{[1]} = A\eta^i - \dot{x}^i A\xi, \quad X^{[1]}(\omega^i) - A(G^{[1]}) = -\omega^i A\xi. \]

(16) \hspace{1cm} (17) \hspace{1cm} (18)

For any function \( f(t, x^i) \) the \( A f = \frac{df}{dt} - \xi \frac{df}{dx^i} \frac{dx^i}{dt} \) is the total derivative of \( f \). Using this result we write the symmetry conditions as follows:

\[ \lambda = -\frac{d\xi}{dt}, \quad G^{[1]} = \frac{d\eta^i}{dt} - \dot{x}^i \frac{d\xi}{dt}, \quad X^{[1]}(\omega^i) - A(G^{[1]}) = -\omega^i \frac{d\xi}{dt}. \]

(19) \hspace{1cm} (20) \hspace{1cm} (21)

We note that the second condition (20) defines the first extension \( G^{[1]} \). The first equation (19) gives the factor \( \lambda \), therefore the essential symmetry condition is equation (21).

To compute the symmetry condition first we have to compute the quantities \( X^{[1]}(\omega^i) \) and \( A(G^{[1]}) - \omega^i \frac{df}{dt} \) taking into consideration (20) and (12). The result of this formal calculation is:

\[ X^{[1]}(\omega^i) = (\xi \partial_t + \eta^i \partial_{x^i} + G^{[1]} \partial_{x^i}) (-\Gamma^i_{jk}(x) \dot{x}^j \dot{x}^k - \phi(x) \dot{x}^i) \]

\[ = -\eta^i \phi \]

\[ + (-\xi \phi \gamma^i_{\dot{j}j} - \phi \xi \gamma^i_{\dot{k}k} - \eta^i \Gamma^j_{jk} - \eta^i \Gamma^i_{jk} - \phi \eta^i_{\dot{j}j} + \phi \xi \eta^i_{\dot{k}k}) \dot{x}^j \]

\[ + (-\xi \Gamma^i_{jk}(t) - \eta^i \Gamma^i_{jk}(t) - \eta^i \Gamma^i_{jk}(t) + \phi \xi \eta^i_{\dot{j}j} + 2\xi \xi \eta^i_{\dot{k}k} + \phi \xi \eta^i_{\dot{k}k}) \dot{x}^j \dot{x}^k \]

\[ + \xi \Gamma^i_{jk}(\dot{t}) \dot{x}^j \dot{x}^k \dot{x}^l. \]

(22)

\[ A(G^{[1]}) - \omega^i \frac{d\xi}{dt} = \eta^i \frac{\partial t}{\partial t} \]

\[ + (\eta^i \Gamma^i_{jk} + \xi \eta^i \Gamma^i_{jk} - \eta^i \Gamma^i_{jk} + \phi \xi \eta^i \Gamma^i_{jk} - 2\phi \xi \eta^i \Gamma^i_{jk} - \phi \xi \eta^i \Gamma^i_{jk} - 2\phi \xi \eta^i \Gamma^i_{jk} - \phi \xi \eta^i \Gamma^i_{jk}) \dot{x}^j \dot{x}^k \]

\[ + (\xi \eta^i \Gamma^i_{jk} + \xi \eta^i \eta^i \eta^i \Gamma^i_{jk} - \xi \eta^i \eta^i \eta^i \Gamma^i_{jk} - \xi \eta^i \eta^i \eta^i \Gamma^i_{jk}) \dot{x}^j \dot{x}^k \dot{x}^l. \]

(23)

Substituting in the symmetry condition (21) and collecting terms of the same order in \( \dot{x}^i \) we find the following equations: \((i = 1, \ldots, n)\):

(\(\dot{x}\))^0 terms:

\[ \eta^i_{\dot{t}t} + \eta^i_{\dot{t}l} = 0, \]

(24)

(\(\dot{x}\))^1 terms:

\[ \xi_{\dot{t}i} \delta^i_{\dot{j}j} - \xi \phi \delta^i_{\dot{j}j} - 2[\eta^i_{\dot{t}j} + \eta^i_{\dot{t}l} \Gamma^l_{jk} - [\phi \xi \phi + \phi \xi \phi] \delta^i_{\dot{j}j} = 0, \]

(\(\dot{x}\))^2 terms:

\[ (-\eta^i_{\dot{j}j} - \eta^i_{\dot{t}j} \Gamma^i_{jk} - \eta^i_{\dot{t}l} \Gamma^l_{jk} - \eta^i_{\dot{t}l} \Gamma^i_{jk} + 2\xi_{\dot{t}l} \delta^i_{\dot{j}j} - 2\phi \xi_{\dot{t}l} \delta^i_{\dot{j}j} - \xi \Gamma^i_{jk}, t) = 0 \]

\[ \Rightarrow \]

\[ 2\phi \xi_{\dot{t}l} \delta^i_{\dot{j}j} + \xi \Gamma^i_{jk}, t + 2\xi \Gamma^i_{jk}, t, \]

(25)
(\dot{x})^3 \text{ terms:} \\
\left( \xi_{(jk)} - \xi_{(e)} \Gamma_i^{(jk)} \delta_i^e \right) = 0. \tag{26}

Define the quantity:
\Phi = \xi,_{\tau} - \phi \xi. \tag{27}

Then condition (25) is written:

\mathcal{L}_\eta \Gamma_i^{(jk)} = 2 \Phi (\delta_i^j) - \xi \Gamma_i^{(kj),\tau}. \tag{28}

If we consider the vector \( \xi = \xi \partial_t \) (which does not have components along \( \partial_i \)) we compute that:

\mathcal{L}_\xi \Gamma_i^{(jk)} = \xi \Gamma_i^{(kj),\tau}, \tag{29}

hence (28) is written:

\mathcal{L}_X \Gamma_i^{(jk)} = 2 \Phi (\delta_i^j), \tag{30}

where \( X = \xi + \eta = \xi \partial_t + \eta^i (t, x) \partial_x \). We observe that this condition is precisely condition (8) for a projective collineation of the connection \( \Gamma_i^{(jk)} \) along the symmetry vector \( X \) and with projecting function \( \Phi \). Concerning the other conditions we note that (24) can be written in covariant form (relevant to the indices \( a = 1, 2, ..., n \)) as follows:

\Phi_t \delta_i^j - 2 \eta^i_t \delta_i^j = 0, \tag{31}

where \( \eta^i_t \delta_i^j = \eta^i_t + \eta^k_t \Gamma_i^{(kj)} \) is the covariant derivative with respect to \( \Gamma_i^{(kj)} \) of the vector \( \eta^i_t \).

Similarly condition (26) can be written:

\xi_{(jk)} \delta_i^e = 0.

Contracting the indices \( i, j \) we find the final form:

\xi_{(jk)} = 0. \tag{32}

This implies \( \xi_{\tau} = f(t) \) (where \( f(t) \) is an arbitrary function of its argument) is a gradient KV of the metric of the space \( x^i \).

Condition (23) is obviously in covariant form with respect to the indices \( a \).

We arrive at the following conclusion:

(i) The Lie symmetries of the autoparallel equations (12) (not necessarily affinely parameterized) for a general connection defined on a \( C^\infty \) manifold are the following:

\eta^i_t + \eta^i_\tau \phi = 0, \tag{33}

\xi_{(jk)} = 0, \tag{34}

\Phi_t \delta_i^j - 2 \eta^i_t \delta_i^j = 0, \tag{35}

\mathcal{L}_X \Gamma_i^{(jk)} = 2 \Phi (\delta_i^j). \tag{36}

These are covariant equations because if we consider the connection in the augmented \( 1 + n \) space \( \{x^i, t\} \), all components of \( \Gamma \) which contain an index along the direction of \( t \) vanish, therefore the partial derivatives with respect to \( t \) can be replaced with covariant derivative with respect to \( t \).
(ii) Equation (33) gives the functional dependence of $\eta^i$ on $t$ and the parametrization

(iii) Equation (34) gives that the vector $\xi_{,i}$ is a gradient Killing vector of the $n-$dimensional space $\{x^i\}$.

(iv) Equation (35) relates the functional dependence of $\eta^i$ and $\xi$ in terms of $t$.

(v) Equation (36) is the most important equation to our purpose, because it states that the symmetry vector $X$ is an affine collineation in the jet space $\{t, x^i\}$ because it preserves both the geodesic and its parametrization. In the space $\{x^i\}$ the vectors $\eta^i(t, x) \partial_{,i}$ are projective collineations because they preserve the geodesics and not necessarily their parametrization.

In the following we restrict our considerations to the case of Riemannian connections that is the $\Gamma^i_{jk}$ are symmetric and the covariant derivative of the metric vanishes.

4. Calculation of the symmetry vector for a Riemannian connection

We compute the Lie symmetry vectors for the case of affine parametrization ($\Leftrightarrow \phi = 0$) and the assumption $\Gamma^i_{jk,t} = 0$ i.e. the $\Gamma^i_{jk}$ are independent of the parameter $t$. The later is a logical assumption because the $\Gamma^i_{jk}$ are computed in terms of the metric which does not depend on the parameter $t$. Under these assumptions the symmetry conditions (33) - (36) read:

$$\eta_{,tt}^i = 0,$$  
(37)

$$\xi_{(jk)} = 0,$$  
(38)

$$\xi_{,tt}^i \delta_j^i - 2 \eta_{,i}^i = 0,$$  
(39)

$$\mathcal{L}_\eta \Gamma^i_{jk} = 2 \xi_{,t(j \delta_k^i).}$$  
(40)

We proceed with the solution of this system of equations. Equation (37) implies:

$$\eta^i(t, x) = A^i(x)t + B^i(x),$$  
(41)

where $A^i(x), B^i(x)$ are arbitrary differentiable vector fields.

The solution of equation (38) is:

$$\xi(t, x) = C_J(t)S^J(x) + D(t),$$  
(42)

where $C_J(t), D(t)$ are arbitrary functions of the affine parameter $t$ and $S^J(x)$ is a function whose gradient is a gradient KV i.e.:

$$S(x)_{(i,j)} = 0.$$  
(43)

The index $J$ runs through the number of gradient KVs of the metric.

Condition (39) gives:

$$2A(x)^i_{\mid j} = [C_J(t)_{,tt} S^J(x) + D(t)_{,tt}] \delta_j^i.$$  
(44)

Because the left hand side is a function of $x$ only we must have:

$$D(t)_{,tt} = M \Rightarrow D(t) = \frac{1}{2} Mt^2 + kt + L, \text{ where } M, K, L \text{ constants}$$  
(45)

$$C_J(t)_{,tt} = G_J = \text{constant} \Rightarrow C_J(t) = \frac{1}{2} G_J t^2 + E_J t + F_J, \text{ where } G_J, E_J, F_J \text{ constants}.$$  
(46)

Replacing in (44) we find:

$$2A(x)^i_{\mid j} = (G_J S^J(x) + M) \delta_j^i \Rightarrow A(x)_{i,ij} = \frac{1}{2} (G_J S^J(x) + M) g_{ij},$$  
(47)
where we have lowered the index because the connection is metric (i.e. \( g_{ijk} = 0 \)). The last equation implies that the vector \( A(x)^i \) is a conformal Killing vector with conformal factor \( \psi = \frac{1}{2} (G_J S^J(x) + M) \). Because \( A(x)|_{ij} = 0 \) this vector is a gradient CKV.

We continue with condition (40) and replace \( \eta^i(t, x) \) from (41):

\[
\mathcal{L}_A \Gamma^i_{jk} + \mathcal{L}_B \Gamma^i_{jk} = 2 \xi_{i, (j} \delta^i_k) = 2 \left[ (G_J t + E_J) S^J(x) + M t + K \right]_{(j} \delta^i_k) = 2 (G_J t + E_J) S^J(x)_{(j} \delta^i_k) \Rightarrow 
\]

\[
\mathcal{L}_A \Gamma^i_{jk} = 2 G_J S^J(x)_{(j} \delta^i_k), 
\]

(48)

\[
\mathcal{L}_B \Gamma^i_{jk} = 2 E_J S^J(x)_{(j} \delta^i_k), 
\]

(49)

The last two equations imply that the vectors \( A^i(x), B^i(x) \) are projective collineations of the metric - or one of their specializations - with projective functions \( G_J S^J(x) \) and \( E_J S^J(x) \) respectively. Note that relations (48), (49) remain true if we add a KV to the vectors \( A^i(x), B^i(x) \), therefore these vectors are determined up to a KV.

Now it is well known that in a Riemannian space a CKV \( K^i \) with conformal factor \( \psi(x^i) \) satisfies the following identity:

\[
\mathcal{L}_K \Gamma^i_{jk} = g^{is} [\psi_j g_{ks} + \psi_k g_{js} - \psi_s g_{jk}] . 
\]

(50)

Applying this identity to the CKV \( A^i \) we find:

\[
G_J S^J(x)_{,k} = 0 \Rightarrow G_J S^J(x) = 2 \rho = \text{constant}. 
\]

(51)

This implies that \( A^i \) is a gradient HKV with homothetic factor \( \rho + \frac{1}{2} M \). Furthermore (47) implies:

\[
2 A^i = (2 \rho + M) x^i + 2 L^i \Rightarrow 
\]

\[
A^i = (\rho + \frac{1}{2} M) x^i + L^i , 
\]

(52)

where \( L^i \) is a KV.

We continue with the projective collineation vector \( B^i \). For this vector we use the property that for a symmetric connection the following identity holds:

\[
\mathcal{L}_B \Gamma^i_{jk} = B^i_{jk} - R^i_{jkl} B^l . 
\]

Replacing the left hand side from (49) we find:

\[
B^i_{jk} - R^i_{jkl} B^l = 2 E_J S^J(x)_{(j} \delta^i_k) . 
\]

(53)

Contracting the indices \( i, j \) we find:

\[
(B^i_{ji} - (n + 1) E_J S^J(x))_{,k} = 0, 
\]

(54)

which implies that:

\[
B^i_{ji} = (n + 1) E_J S^J(x) + 2 b , 
\]

(55)

where \( b = \text{constant} \).

Using the above results we find for \( \xi(t, x) \):

\[
\xi(t, x) = C_J(t) S^J + D(t), 
\]

\[
= \left( \frac{1}{2} G_J t^2 + E_J t + F_J \right) S^J + \frac{1}{2} M t^2 + K t + L , 
\]

\[
= \frac{1}{2} (G_J S^J + M) t^2 + (E_J S^J + K) t + F_J S^J + L . 
\]

We summarize the above results in the following Theorem
Theorem 1. The Lie symmetry vector $X = \xi + \eta = \xi(t, x) \partial_t + \eta^j(t, x) \partial_x^j$ of the equation of geodesics (7) in a Riemannian space involves all symmetry vectors, that is, (gradient) KVs, (gradient) HKVs, PCs and their degeneracies HKVs and ACs as follows:

a. The function:
$$\xi(t, x) = \left( \rho + \frac{1}{2} M \right) t^2 + \left[ E_J S^J + K \right] t + F_J S^J + L,$$
where $\rho, M, b, K, F_J, L$ are constants, the index $J$ running along the number of gradient KVs.

b. The vector:
$$\eta^j(t, x) = A^i(x) t + B^i(x) + D^i(x),$$
where the vector $A^i(x)$ is a gradient HKV with conformal factor $\psi = \rho + \frac{1}{2} M$, $D^i(x)$ is a KV of the metric and $B^i(x)$ is a projective collineation with projection function $E_J S^J(x)$ whose divergence $B^i_{;i}$ must satisfy condition (55).

5. The Lie symmetries of geodesic equations of spaces of constant curvature

The metric of a space of constant curvature $K = 0, \pm 1$ and dimension $n$ in Cartesian (stereographic) coordinates is:
$$ds^2 = U(x) \eta_{ij} dx^i dx^j,$$
where $U(x) = \frac{1}{1 + \frac{K}{4} x^k x_k}$, $\eta_{ij} = diag(\pm 1, \pm 1, \ldots, \pm 1)$. It is easy to show that the connection coefficients of this metric are:
$$\Gamma^i_{jk} = -\frac{K U}{2} (x_k \delta^i_j + x_j \delta^i_k - x^i \eta_{jk}),$$

hence the geodesics equations for affine parametrization are:
$$\ddot{x}^i - \frac{K U}{2} (2 x_k \dot{x}^k \dot{x}^i - (\eta_{jk} \dot{x}^j \dot{x}^k)) \dot{x}^i = 0,$$
where a dot indicates derivation with respect to the affine parameter $s$ (say). We shall apply Theorem 1 to determine the Lie symmetries of the geodesic equations (58). Feroze et al [9] have computed the Lie symmetries of these equations for some well known spaces of constant curvature and essentially they considered the Killing vectors of the corresponding metrics. In the following we shall extend and complete their results to all spaces of constant curvature. Furthermore we clarify the general and important results of Aminova [6], [7].

In order to apply Theorem 1 we need to know the conformal and the projective collineations of spaces of constant curvature. From the literature we have the following results.

A. Flat spaces

a. Conformal algebra

\[ n \binom{n(n-1)}{2} \]-Non-gradient Killing vectors:
$$X_{IJ} = \delta^i_I \delta^j_J x_j \partial_i,$$

$n$ Gradient Killing vectors:
$$\delta^i_I \partial_i,$$

1 Gradient Homothetic Killing vector:
$$x^i \partial_i \text{ with homothety factor } \psi = 1,$$
n non-gradient special CKVs:

\[
\left[ 2x_I x^i - \delta^i_J (x^k x_k) \right] \partial_i \text{ with conformal factor } \psi = 2x^i,
\]

b. \(n(n+1)\) Proper Affine Collineations [1]:

\[
\left[ b^j_i x^j + c^i \right] \partial_i,
\]

c. \(n(n+2)\) Proper Projective Collineations [2]:

\[
\left[ (c_j x^j) x^i + b^j_i x^j + a^i \right] \partial_i.
\]

B. Spaces of constant non-vanishing curvature (the indices \(I, J\) count vectors):

a. Conformal algebra [10]:

\(n\) Non-gradient Killing vectors:

\[
\frac{1}{U} \left[ (2U - 1) \delta^i_I + \frac{KU}{2} x_I x^i \right] \partial_i,
\]

\(\frac{n(n-1)}{2}\) Non-gradient Killing vectors:

\[X_{IJ} = \delta^i_I \delta^j_J \partial_i,\]

1 Gradient Homothetic Killing vector:

\[x^i \partial_i,\]

\(n\) Gradient Conformal Killing vectors:

\[
X_I = \frac{1}{U} \left[ \delta^i_I - \frac{1}{2} KU x_I x^i \right] \partial_i.
\]

b. Affine Collineations [11]:

Not admitted.

c. \(n(n+2)\) Proper Projective Collineations [11]:

\[\phi, i + D_i (n \geq 2),\]

where \(\phi = c + \frac{(1-Kr^2/4)a_i x^i + b_i x^i x^j}{(1-Kr^2/4)^2} a_i, b_{ij} = b_{ji}, c\) and \(D_i\) is a Killing vector. The constant \(c\) is irrelevant because it does not give a proper projective collineation.

5.1. The case of a flat space

In this case the equation of geodesics when affinely parameterized is:

\[\ddot{x}^i = 0 \quad i = 1, 2, \ldots, n.\]

We know that these equations admit the maximal number of Lie symmetries. We determine these symmetries using Theorem 1. The gradient KV \(\delta^i_I \partial_i\) implies:

\[S^I(x) = \delta^I_i x^i,\]
(plus an irrelevant constant). The gradient homothetic vector \( x^i \partial_i \) implies:
\[
A^i = x^i, L^i = 0.
\]
and the homothetic factor \( \psi = 1 \):
\[
\rho + \frac{1}{2}M = 0.
\]
Equation (47) gives:
\[
2\delta^i_j = (G_j S^i(x) + M) \delta^i_j \Rightarrow G_j = 0, M = 2.
\]
The condition (55) on the proper projective collineations gives:
\[
b^i_{\ i} + (n + 1)(c_i x^i) = (n + 1)E_i x^i + 2b \Rightarrow E_i = c_i, b = \frac{1}{2}b^i_i.
\]
We have now for the symmetry function \( \xi(t, x) \):
\[
\xi(t, x) = t^2 + (c_i x^i + K) t + F_i x^i + L
\]
where \( K, F_i, L \) are constants. Concerning the symmetry vector \( n^i \) we have:
\[
n^i = x^i t + [(c_j x^j) x^i + b^i_j x^j + a^i] + \delta^i_j \delta^j_i x^i + \delta^i_i.
\]
These symmetry quantities must be a solution of the symmetry conditions (37) - (40). It is an easy exercise to show that (37) - (39) are satisfied. Concerning (40) we find:
\[
\mathcal{L}_n \Gamma^i_{jk} = 2c_j \delta^i_k = n^i_{jk},
\]
which is correct. Therefore the solution we have found satisfies the Lie symmetry conditions.

5.2. The case of a space of constant non-vanishing curvature
The system of equations we want to find the Lie symmetries are (58). This time we do not have gradient KVFs therefore \( S^i(x) = 0 \) and the index \( J \) is suppressed. The space does not admit a gradient homothetic vector therefore we set \( \psi(A) = 0 \). This implies \( \rho + \frac{1}{2}M = 0 \). Condition (55) cannot be satisfied therefore the proper projective collineations \( \phi_{\ i} \) are not admitted. The answer is then:
\[
\xi(t, x) = K t + L,
\]
\[
n^i = \delta^i_j \delta^j_i x^i + \frac{1}{U} \left( (2U - 1) \delta^i_l + \frac{KU}{2} x^l x^i \right).
\]
It is easy to show that these quantities satisfy all the conditions (37) - (40).

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We define the sign of the curvature tensor form the identity $A^a_{;bc} - A^a_{;cb} = R^a_{bcd}A^d$ or $A^a_{;bc} - A^a_{;cb} = R^a_{dabc}A^d$. In terms of the connection coefficients $R^a_{bcd} = \Gamma^a_{db,c} - \Gamma^a_{cb,d} + \Gamma^e_{cb} \Gamma^a_{de} - \Gamma^e_{cd} \Gamma^a_{eb}$.

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