NONPERTURBATIVE SOLUTION OF THE SUPER-VIRASORO CONSTRAINTS

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ABSTRACT

We present the solution of the discrete super-Virasoro constraints to all orders of the genus expansion. Integrating over the fermionic variables we get a representation of the partition function in terms of the one-matrix model. We also obtain the nonperturbative solution of the super-Virasoro constraints in the double scaling limit but do not find agreement between our flows and the known supersymmetric extensions of KdV.

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1. Introduction.

Matrix models provide us with powerful methods to evaluate the sum over geometries in two dimensional quantum gravity. For $c \leq 1$ conformal field theories [1] coupled to gravity, $n$-point correlation functions of gravitationally dressed primary fields have been calculated within this formulation on world sheets with arbitrary topology, a result that is far from being obtained from the continuum Liouville approach [2]. It turns out that matrix models have a close connection to integrable systems [3], which is manifest through the appearance of the Korteweg-de Vries hierarchy in the double scaling limit [4] [5]. The recent discovery of Kontsevich matrix integrals has shown a natural connection to topological gravity [6] [7]. Despite of this important results, the generalization to the supersymmetric case is connected with difficulties. So for example, the role which the known supersymmetric extensions of KdV [8] play in this context remains obscure [9][10]. For $N = 1$ superconformal field theories coupled to two dimensional supergravity correlation functions for spherical topologies have been obtained in [11], using the continuum super-Liouville approach. The torus path integral and the spectrum of physical operators for $\hat{c} \leq 1$ superconformal models coupled to 2D-supergravity have been considered in [9] [12]. However the more powerful counterpart in terms of supermatrix models has not yet been found [13]. To shed some light on this situation, the authors of [14] proposed an “eigenvalue model”, which is described in terms of $N$ even and $N$-Grassmann odd variables. The basic idea for the construction of the model was the generalization of the Virasoro constraints known from the ordinary one matrix model [15] [16] to the $N = 1$ supersymmetric case. Generalizing Kazakov’s loop equations [17] to $N = 1$ superloop equations, the model was evaluated for spherical topologies in [14], obtaining agreement with correlation functions [11] and critical exponents [18] [19] of $(2, 4m)$ superconformal field theories coupled to world-sheet supergravity. Further analysis of the model carried out in [20] [21] showed, that the continuum super-Virasoro constraints are described in terms of a $\hat{c} = 1$ theory, with a $Z_2$-twisted scalar and a Weyl Majorana fermion in the Ramond sector. The solution of the continuum superloop equations in the first
few orders of the genus expansion and a general heuristic argument showed [20],
that there might be a close relation between the bosonic piece of the free energy
and the free energy of the one matrix model.

In this letter we analyze the solution of this model to all orders of the genus
expansion. After a brief review about its construction (section two), we analyze
the solution of the discrete super-Virasoro constraints in section three. Integrating
over the fermionic variables, we get a representation of the partition function in
terms of the one matrix model. In section four we analyze the solution of the
super-Virasoro constraints in the double scaling limit and compare our flows with
the known supersymmetric extensions of KdV [8].

2.Construction of the model.

The bosonic one matrix model is defined as:

\[
Z_B(g_k, N) = \int d^{N^2} \Phi \exp \left[ -\frac{N}{\Lambda_B} \text{tr} \sum_{k \geq 0} g_k \Phi^k \right], \quad \Lambda_B = e^{-\mu} \tag{1}
\]

where \( \Phi \) is a Hermitian \( N \times N \) matrix and \( \mu \) is the bare cosmological constant. It
satisfies the Virasoro constraints [15]:

\[
L_n Z_B = 0, \quad n \geq -1 \; ; \; \; \; [L_n, L_m] = (n - m) L_{n+m}
\]

\[
L_n = \frac{\Lambda_B^2}{N^2} \sum_{k=0}^{n} \frac{\partial^2}{\partial g_{n-k} \partial g_k} + \sum_{k \geq 0} k g_k \frac{\partial}{\partial g_{k+n}}, \tag{2}
\]

which can be derived from (1) by implementing invariance under the shift \( \Phi \to \Phi + \varepsilon \Phi^{n+1} \).

We obtain a relation to a free massless boson, if we introduce an infinite set of
creation and annihilation operators related to the coupling constants

\[
\alpha_{-n} = -\frac{N}{\Lambda_B \sqrt{2}} n g_n, \quad n > 0 \; ; \; \; \; \alpha_n = -\frac{\Lambda_B \sqrt{2}}{N} \frac{\partial}{\partial g_n}, \quad n \geq 0 \tag{3}
\]

The Virasoro generators \( L_n = \frac{1}{2} \sum : \alpha_{-k} \alpha_{k+n} : \) are the modes of the energy
momentum tensor \( T(p) = \sum L_n p^{-n-2} \) of the scalar field \( \partial X(p) = \sum \alpha_n p^{-n-1} = \)
\[ \partial X^+(p) + \partial X^-(p). \] From (1) and (3) we see that the potential of the model is related to the negative modes of the scalar field

\[ \lim_{N \to \infty} \frac{1}{N} Z_B^{-1} \partial X^- Z_B \sim V'(p) = \sum_{k>0} k g_k p^{k-1}, \] (4)

As a generalization to the \( N = 1 \) supersymmetric case, we take a \( \hat{c} = 1 \) free massless superfield \( X(p, \Pi) = x(p) + \Pi \psi(p) \) with energy momentum tensor \( T(p, \Pi) = DX \, \partial X = \psi \partial_p x + \Pi : (\partial_p x \partial_p x + \partial \psi) : \) and the mode expansion \( \partial X(p) = \sum \alpha_n p^{-n-1}, \psi(p) = \sum b_r p^{-r-1/2}. \) The relation to the \( \hat{c} = 1 \) super-Virasoro algebra:

\[ [L_m, L_n] = (n - m) L_{m+n} + \frac{(m^3 - m)}{8} \delta_{n+m,0}, \] (5)

\[ [L_m, G_r] = (\frac{m}{2} - r) G_{m+r}, \] (6)

\[ \{ G_r, G_s \} = 2 L_{r+s} + \frac{1}{2} (r^2 - \frac{1}{4}) \delta_{r+s,0}, \] (7)

and the coupling constants of the model is introduced, if we define the modes as:

\[ \alpha_p = -\frac{\Lambda_S}{N} \frac{\partial}{\partial g_p} \quad ; \quad \alpha_{-p} = -\frac{N}{\Lambda_S} p g_p, \quad p = 0, 1, 2, \ldots \]

\[ b_{p + \frac{1}{2}} = -\frac{\Lambda_S}{N} \frac{\partial}{\partial \xi_{p + \frac{1}{2}}} \quad ; \quad b_{-p - \frac{1}{2}} = -\frac{N}{\Lambda_S} \xi_{p + \frac{1}{2}}, \quad p = 0, 1, 2, \ldots . \] (8)

In terms of the coupling constants the Virasoro generators then become:
\[ G_{n-\frac{1}{2}} = \sum_{k=0}^{\infty} \xi_{k+\frac{1}{2}} \frac{\partial}{\partial g_{k+n}} + \sum_{k=0}^{\infty} kg_k \frac{\partial}{\partial \xi_{k+n-\frac{1}{2}}} + \frac{\Lambda_S^2}{N^2} \sum_{k=0}^{n-1} \frac{\partial}{\partial \xi_{k+\frac{1}{2}}} \frac{\partial}{\partial g_{n-1-k}} \quad n \geq 0 \quad , \quad (9) \]

\[ L_n = \sum_{k=1}^{\infty} kg_k \frac{\partial}{\partial g_{k+n}} + \sum_{r=\frac{1}{2}}^{\infty} \left( \frac{n}{2} + r \right) \xi_r \frac{\partial}{\partial \xi_{n+r}} + \frac{\Lambda_S^2}{2N^2} \sum_{k=0}^{n} \frac{\partial^2}{\partial g_k \partial g_{n-k}} + \frac{\Lambda_S^2}{2N^2} \sum_{r=\frac{1}{2}}^{n-\frac{1}{2}} \left( \frac{n}{2} + r \right) \frac{\partial^2}{\partial \xi_r \partial \xi_{n-r}} \quad n \geq -1 \quad . \quad (10) \]

In analogy to the one matrix model the potential is determined from the negative modes of the superfield \( DV(p, \Pi) \sim DX^- \):

\[ V(p, \Pi) = \sum_{k \geq 0} \left( g_k p^k + \Pi \xi_{k+1/2} p^k \right) \quad . \quad (11) \]

This suggests the introduction of \( N \) even-and \( N \) Grassmann-odd eigenvalues, obtaining for the partition function:

\[ Z_S(g_k, \xi_{k+\frac{1}{2}}, N) = \int \prod_{i=1}^{N} d\lambda_i d\theta_i \Delta(\lambda, \theta) \exp \left( -\frac{N}{\Lambda_S} \sum_{k \geq 0} \sum_{i=1}^{N} (g_k \lambda_i^k + \xi_{k+\frac{1}{2}} \theta_i \lambda_i^k) \right) \quad . \quad (12) \]

The generalization of the usual Vandermonde determinant, \( \Delta(\lambda, \theta) \), is determined by implementing invariance of (12) under the super-Virasoro constraints (9) (10), which yields a differential equation for the measure:

\[ \sum_i \lambda_i^n \left( -\frac{\partial}{\partial \theta_i} + \theta_i \frac{\partial}{\partial \lambda_i} \right) \Delta = \Delta \sum_{i \neq j} \theta_i \frac{\lambda_i^n - \lambda_j^n}{\lambda_i - \lambda_j} \quad , \quad (13) \]

whose unique solution up to an irrelevant constant is \( \Delta = \prod_{i<j} (\lambda_i - \lambda_j - \theta_i \theta_j) \).

The model that we want to solve in the large \( N \) limit is thus:
\[ Z_S = \int \prod_{i=1}^{N} d\lambda_i d\theta_i \prod_{i<j} (\lambda_i - \lambda_j - \theta_i \theta_j) e^{x p \left( -\frac{N}{\Lambda_S} \sum_{k \geq 0} \sum_{i=1}^{N} (g_k \lambda_i^k + \xi_{k+\frac{1}{2}} \theta_i \lambda_i^k) \right)} . \]  

(14)

In (14) \( N \) is taken to be even in order to get a Grassmann even partition function.

3. Solution of the discrete model.

We analyze the solution of the discrete superloop equations [14] [20] or equivalently the discrete super-Virasoro constraints through the representation (14) of the partition function. We first consider the case without fermionic couplings \( \xi_{k+\frac{1}{2}} = 0 \). The partition function can be written as a function of the Pfaffian of the antisymmetric matrix \( M_{ij} = \lambda_{ij}^{-1} \), where then \( \lambda_{ij} = \lambda_i - \lambda_j, i \neq j \), and the usual Vandermonde \( \Delta(\lambda) = \prod_{i<j} (\lambda_i - \lambda_j) \) [14]:

\[ Z_S(g_k, \xi_{k+\frac{1}{2}} = 0, 2N) = \int \prod_{i=1}^{2N} d\lambda_i \Delta(\lambda) Pf^{2N}(\lambda_{ij}^{-1}) e^{x p \left( -\frac{2N}{\Lambda_S} \sum_{k} \sum_{i=1}^{2N} g_k \lambda_i^k \right)} . \]  

(15)

Here the product of the Vandermonde and the Pfaffian is totally symmetric under exchange of two eigenvalues. We want to show, that the relation of the partition function (15) to the bosonic partition function of the one matrix model (1) is given by:

\[ Z_S(g_k, \xi_{k+\frac{1}{2}} = 0, 2N) = \frac{1}{2^N} \binom{2N}{N} Z_B^2(g_k, N) . \]  

(16)

To prove (16) we show, that the following equivalence for the measures holds:
\[
P f^{2N} (\lambda_i^{-1}) \prod_{i<j}^{2N} (\lambda_i - \lambda_j) = \frac{1}{2^N} \sum_{I,J} S^{I, J} \Delta^2(I) \Delta^2(J), \tag{17}
\]

Here we have introduced the notation \( I = (\lambda_{i_1}, \ldots, \lambda_{i_N}), \) \( i_1 < \ldots < i_N \) (same for \( J \)), and \( S^{I, J} \) means symmetric permutations between the elements of \( I \) and \( J \). The equality (16) is a consequence of (17), since the partition function (15) with 2N eigenvalues will then factorize into \( \binom{2N}{N} \) products of two independent one matrix models with \( N \) eigenvalues.

Obviously (16) holds for the case of two eigenvalues, so that to illustrate the situation we start with the first nontrivial case of four eigenvalues, \( N = 2 \). We will consider the right and left hand side of (17) as polynomials \( Q(\lambda), P(\lambda) \) respectively in \( \lambda = \lambda_4 \):

\[
P(\lambda) = (\lambda - \lambda_2)(\lambda - \lambda_3)\lambda_12\lambda_13 - (\lambda - \lambda_1)(\lambda - \lambda_3)\lambda_12\lambda_23 + (\lambda - \lambda_1)(\lambda - \lambda_2)\lambda_13\lambda_23 \tag{18}
\]

\[
Q(\lambda) = \frac{1}{2}[ (\lambda - \lambda_3)^2\lambda_1^2 + (\lambda - \lambda_2)^2\lambda_1^2 + (\lambda - \lambda_1)^2\lambda_2^2]. \tag{19}
\]

From \( P(\lambda_i) = Q(\lambda_i) \) for \( i = 1, 2, 3 \) it follows \( P(\lambda) = Q(\lambda), \forall \lambda \).

We show that (17) is correct by induction over \( N \). For \( N > 2 \) we set \( \lambda = \lambda_{2N+2} \) so that \( P(\lambda), Q(\lambda) \) become polynomials of degree \( 2N - 2 \) in \( \lambda \). It is enough to show that they coincide for \( \lambda = \lambda_{2N+1} \) since the symmetry under the interchange of \( \lambda_i \) and \( \lambda_j \) guarantees \( P(\lambda_i) = Q(\lambda_j) \) for \( i = 1, \ldots, 2N \) and thus \( Q(\lambda) = P(\lambda) \) \( \forall N \). If we consider both polynomials of (17) for \( N \rightarrow N + 1 \) and \( \lambda = \lambda_{2N+1} \) we obtain the desired result by applying the induction hypothesis.

\[
Q(\lambda_{2N+1}) = \frac{1}{2^N} \sum_{I,J} S^{I, J} \Delta^2(I) \Delta^2(J) \prod_{i_1 < \ldots < i_N}^{i_1 < \ldots < i_N} (\lambda_i - \lambda_{2N+1})^2(\lambda_j - \lambda_{2N+1})^2
\]
\[ P f^{2N}(\lambda_{ij}^{-1}) \prod_{i<j} (\lambda_i - \lambda_j) \prod_{i_1<...<i_N} (\lambda_i - \lambda_{2N+1})^2 \prod_{j_1<...<j_N} (\lambda_j - \lambda_{2N+1})^2 = P(\lambda_{2N+1}) \]  

(20)

We conclude that the bosonic part of the supersymmetric \((2N \times 2N)\) model is proportional to the square of the corresponding \((N \times N)\) bosonic one matrix model (16). With this result we can already find the form of the discrete string equation for arbitrary genus.

Writing \(Z = e^{N^2 F_S}\), equation (16) implies, that the fermionic independent piece of the supersymmetric free energy \(F_S^{(0)}\) is related to the free energy of the one matrix model \(F_B\) (up to an irrelevant additive constant):

\[ F_S^{(0)} = \frac{F_B}{2} \]  

(21)

Since the dependence of the free energy on \(g_0\) is trivial \(\frac{\partial F}{\partial g_0} = -1/\Lambda\), (21) implies a rescaling of the cosmological constant \(\Lambda_S = 2\Lambda_B\). This relation can also be obtained, if we evaluate the bosonic piece of the \(L_0\) constraint (10):

\[ \Lambda_S = 2 \sum_{k=1}^{\infty} k g_k \partial_{\Lambda_B} \left( -\Lambda_B^2 \frac{\partial F_B}{\partial g_k} \right) \]  

(22)

Up to a factor of two the cosmological constant satisfies the same string equation as in the one matrix model. Including the fermionic couplings, the following equality for the second order in fermions holds:

\[ \frac{\partial^2 Z_S(2N)}{\partial \xi_{k+2} \partial \xi_{n+1}} = \frac{1}{2^{N-1}(2N)} \left( \frac{\partial^2 Z_B(N)}{\partial g_{k+1} \partial g_n} Z_B(N) - \frac{\partial Z_B(N)}{\partial g_{k+1}} \frac{\partial Z_B(N)}{\partial g_n} \right) - (k \leftrightarrow n) \]  

(23)

The proof of this relation is straightforward, using the relation:
\[
\sum_{\alpha,\beta=1}^{2N} \lambda^k_\alpha \lambda^n_\beta \int \prod_{i=1}^{2N} d\theta_i \Delta(\lambda, \theta) \theta_\alpha \theta_\beta =
\]

\[
\frac{1}{2^{N-1}} \sum_{I,J} S^{I,J} \Delta^2(I) \Delta^2(J) \sum_{\alpha,\beta=1}^{N} (\lambda^{k+1}_i \lambda^n_j - \lambda^{k+1}_i \lambda^n_j) - (k \leftrightarrow n), \tag{24}
\]

which can be shown using equation (17). Formula (23) is purely bosonic in the sense, that after derivation we set the fermionic couplings to zero.

Equations (21) and (23) imply that the free energy until the second order in fermionic couplings is given by:

\[
F_S = \frac{F_B}{2} - \frac{1}{2} \sum_{k,n} \xi_n k+\frac{1}{2} \zeta_{n+\frac{1}{2}} \frac{\partial^2 F_B}{\partial g_{k+1} \partial g_n}. \tag{25}
\]

Since the solution to the continuum super-Virasoro constraints will be analyzed in a moment, it will then become clear that (25) is the complete expansion of the free energy and no dependence on more than two fermionic couplings appears.

Equation (25) gives a representation of the model (12) in terms of the one matrix model since the fermionic variables \( \theta_i \) have been integrated out. We think that further analysis of this interesting relation to the one matrix model for the discrete theory could lead to the geometrical interpretation of the model and the derivation of Feynman rules. We hope to report on this results elsewhere and turn now to the solution of the super-Virasoro constraints in the double scaling limit.

4. Solution to the super-Virasoro constraints in the double scaling limit.

The continuum super-Virasoro constraints \([20][21]\) are described by the super-energy momentum tensor of a \( \hat{c} = 1 \) superconformal field theory.
\[ T_F = \frac{1}{2} \partial \varphi(z) \psi(z) \quad , \quad T_B = \frac{1}{2} : \partial \varphi(z) \partial \varphi(z) : + \frac{1}{2} : \partial \psi(z) \psi(z) : + \frac{1}{8z^2} \]

(27)

where \( \varphi(z) \) is a bosonic scalar field with antiperiodic boundary conditions and \( \psi(z) \) is a Weyl-Majorana Fermion in the Ramond sector. The modes of the fermionic part of the energy momentum tensor \( T_F = \frac{1}{2} \sum G_{n+\frac{1}{2}} z^{-n-2} \) are described in terms of the coupling constants of the model for \( n \geq -1 \):

\[ G_{n+\frac{1}{2}} = \frac{t_0 \tau_0}{4\kappa^2} \delta_{n,-1} + \sum_{k \geq 0} (k + \frac{1}{2}) t_k \frac{\partial}{\partial \tau_{n+k+1}} + \frac{\tau_0}{2} \frac{\partial}{\partial \tau_n} + \sum_{k \geq 0} \tau_{k+1} \frac{\partial}{\partial \tau_{n+k+1}} + \kappa^2 \sum_{k=0}^n \frac{\partial^2}{\partial t_k \partial \tau_{n-k}} \]

(28)

and the bosonic modes are obtained by the anticommutation relations (7). The constraints imposed on the partition function \( Z_S(t_k, \tau_k) = \exp(F_S) \) are:

\[ G_{n+\frac{1}{2}} Z_S(t_k, \tau_k) = 0 \quad , \quad n \geq -1 \]

(29)

The equations which describe the model (14) in the double scaling limit are determined by (28) (29):

\[ \kappa^2 D F_S = D^{-1} u - 2 \tau D \tau \]

(30)

\[ t = - \sum_{k \geq 1} (2k + 1) t_k R_k[u] \quad , \quad \tau = - \sum_{k \geq 0} \tau_k R_k[u] \]

(31)

where \( D = \frac{\partial}{\partial t_0}, \tau = \tau + \frac{\tau_0}{2} = \frac{\partial F_S}{\partial t_0} \) is the first fermionic scaling variable, \( t \) is the renormalized cosmological constant and \( u \) is the bosonic piece of the two point function of the puncture operator \( \langle \sigma_0 \sigma_0 \rangle \), satisfying the KdV-flow equation.
\[
\frac{\partial u}{\partial t_n} = DR_{n+1}[u]
\]

(32)

\(R_k[u]\) are the Gel’fand-Dikii polynomials defined through the recursion relations:

\[
DR_{n+1}[u] = (\kappa^2 D^3 + 4u D + 2(Du))R_n[u],
\]

(33)

\[R_0 = \frac{1}{2}, \quad R_1 = u, \quad R_2 = (3u^2 + \kappa^2 D^2 u), \ldots\]

To show that (30) (31) is the solution to (29) it is enough to show that \(G_{-\frac{1}{2}}\) and \(G_{\frac{3}{2}}\) are satisfied, since the algebra (5)-(7) guarantees for all the remaining constraints. That this is indeed the case, can be shown by simple calculations, where we have used well known relations of the one matrix model theory like (33).

The equations (30) (31) coincide with the solution presented in [14][20] for the first few orders of the genus expansion. The free energy (30) can be written as a function of the free energy of the one matrix model, which is very similar to the relation (26) of the discrete model and has an expansion at most bilinear in the fermionic couplings

\[
F_S = F_B - \sum_{n \geq 0} \tau_n \tau_k \frac{\partial^2 F_B}{\partial t_n \partial t_{k-1}}.
\]

(34)

The form of the solution obtained (30) (31) implies, that the bosonic piece of the model is equivalent to an ordinary one matrix model, where the constraints act on the square root of the partition function \(L_n \sqrt{Z_B} = 0, \quad n \geq -1\) [16]. The fermionic dependence of the free energy determines, that correlation functions involving more than two fermionic operators vanish, since the fermionic scaling variable \(\hat{\tau}\) is linear in fermionic couplings. These properties were known to hold for the first orders of the genus expansion [14][20] and are in agreement with the previous analysis of the discrete model.
The correlation functions in the double scaling limit which follow from (30) (31) as a function of the two point function of the puncture operator $U = u - 2\tau D^2 \tau$ are given by:

$$\frac{\partial U}{\partial \tau_k} = 2D(R_k[u]D\tau) \quad , \quad \frac{\partial U}{\partial t_k} = DR_{k+1}[u] - 2\frac{\partial}{\partial t_k}(\tau D^2 \tau) \quad .$$  \hspace{1cm} (35)

The scaling dimensions of the scaling operators (35) and the string susceptibility (31) are in agreement with the results of (2,4m)-minimal superconformal models coupled to two dimensional supergravity [14] [18] [19].

From (35) we obtain for the first nontrivial bosonic flow, a set of two equations:

$$\frac{\partial u}{\partial t_1} = D(3u^2 + \kappa^2 D^2 u) \quad , \quad \frac{\partial \tau}{\partial t_1} = 6uD\tau + \kappa^2 D^3 \tau \quad ,$$  \hspace{1cm} (36)

where the first one is the ordinary KdV equation. Equations (36) are invariant under the global supersymmetric transformations $\delta u = \epsilon D\tau$, $\delta \tau = \epsilon u$, where $\epsilon$ is a constant anticommuting parameter, as it is known to hold for the supersymmetric extensions of KdV of Manin-Radul or Mulase-Rabin [8]:

$$\frac{\partial W_0}{\partial t} = \frac{1}{2} \partial_x^3 W_0 + 3W_0 \partial_x W_0 + \frac{3}{2} \partial_x^2 W_1 W_1 \quad , \quad \frac{\partial W_1}{\partial t} = \frac{1}{2} \partial_x^2 W_1 + \frac{3}{2} \partial_x (W_0 W_1) \quad .$$  \hspace{1cm} (37)

Here $W_0$ is a bosonic variable, while $W_1$ is fermionic. We can write the form of our generalized KdV-equation in terms of $U$, the perhaps more convenient variable:

$$\frac{\partial U}{\partial t_1} = D(3U^2 + \kappa^2 D^2 U - 12DU\tau D\tau + 6\kappa^2 D\tau D^3 \tau) \quad ,$$  \hspace{1cm} (38)

$$\frac{\partial \tau}{\partial t_1} = 6U D\tau - 12\tau D\tau D^2 \tau + \kappa^2 D^3 \tau \quad .$$  \hspace{1cm} (39)

Equations (38) and (39) can independently be derived from the first orders of
the genus expansion, with the method proposed in [20]. From (36) or (38) (39) we have not been able to see any relation to the known supersymmetric extensions of the KdV-equation [8], so for example, the perhaps most natural identification $W_0 \sim U$ is not consistent with (37) nor the other extensions of KdV [8]. We may conclude that our correlation functions (35) describe a new supersymmetric extension of the KdV-hierarchy.

The bosonic flows satisfy the recursion relations which can be obtained from the recursion relations of the $R_k[u]$’s:

$$
\begin{pmatrix}
\frac{\partial u}{\partial \tau_{k+1}} \\
\frac{\partial \tau}{\partial \tau_{k+1}}
\end{pmatrix}
= 
\begin{pmatrix}
\kappa^2 D^2 + 2u + 2DuD^{-1} & 0 \\
2D\tau D^{-1} + 2D^{-1} D\tau & \kappa^2 D^2 + 2u + 2D^{-1}uD
\end{pmatrix}
\begin{pmatrix}
\frac{\partial u}{\partial \tau_k} \\
\frac{\partial \tau}{\partial \tau_k}
\end{pmatrix}
$$

(40)

While for the fermionic flows we have simply:

$$
\frac{\partial u}{\partial \tau_k} = 0, \quad \frac{\partial \tau}{\partial \tau_k} = (\kappa^2 D^2 + 2u + 2D^{-1}uD) \frac{\partial \tau}{\partial \tau_k}
$$

(41)

It would be important to study further properties of the differential equations which describe our correlation functions to see whether they share the standard properties of completely super-integrable systems. This will be left for work in the future.

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