ASYMPTOTIC BEHAVIOR OF TRAVELING WAVES FOR A THREE-COMPONENT SYSTEM WITH NONLOCAL DISPERSAL AND ITS APPLICATION

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Abstract. In this paper, we provide a general approach to study the asymptotic behavior of traveling wave solutions for a three-component system with nonlocal dispersal. Then as an important application, we establish a new type of entire solutions which behave as two traveling wave solutions coming from both sides of \( x \)-axis for a three-species Lotka-Volterra competition system.

1. Introduction. In this paper, we are interested in the asymptotic behavior of traveling wave solutions for the following three-component nonlocal dispersal system:

\[
\begin{align*}
\frac{\partial u(x,t)}{\partial t} &= (J_1 \ast u - u)(x,t) + f_1(u,v,w)(x,t), \\
\frac{\partial v(x,t)}{\partial t} &= (J_2 \ast v - v)(x,t) + f_2(u,v,w)(x,t), \\
\frac{\partial w(x,t)}{\partial t} &= (J_3 \ast w - w)(x,t) + f_3(u,v,w)(x,t),
\end{align*}
\]

where \( u(x,t), v(x,t) \) and \( w(x,t) \) measure the density of population at location \( x \in \mathbb{R} \) and time \( t \in \mathbb{R} \), respectively. The reaction field \( (f_1, f_2, f_3) \) describes the population birth, death as well as the competition between individuals and so on. \( J_i \ast \vartheta \) is the standard spatial convolution on \( \mathbb{R} \), that is, \( (J_i \ast \vartheta)(x,t) = \int_{\mathbb{R}} J_i(x-z) \vartheta(z,t)dz \), where \( i = 1, 2, 3 \). Then \( J_i \ast \vartheta - \vartheta (i = 1, 2, 3) \) is called the nonlocal dispersal and represents transportation due to long range dispersion mechanisms. The nonlocal dispersal problem has been proposed in many practical fields, such as population biology \[11\], phase transition \[1,7,8\] and network model \[10\]. Here, the convolution kernels \( J_i (i = 1, 2, 3) \) can be understood as the probability density and satisfy

(J): \( J_i \in C^1(\mathbb{R}), J_i(-x) = J_i(x) \geq 0, \int_{\mathbb{R}} J_i(x)dx = 1 \) and has compact support, \( i = 1, 2, 3 \).

For convenience, we further assume \( J_i(x) = 0 \) if \( |x| > 1, i = 1, 2, 3 \).

Traveling wave solutions play an important role in understanding the nonlinear biological and physical phenomena. Particularly, they usually can be used to model the spreading and invading phenomena in ecology \[12,15,24,29\]. Exactly, a traveling wave solution of system \[1\] is a special translation invariant solution of the form

\[ u(x,t) = \phi^*(\xi), \quad v(x,t) = \psi^*(\xi), \quad w(x,t) = \varphi^*(\xi), \]

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where $\xi = x + ct$ is the wave variable with speed $c$ and $(\phi^*, \psi^*, \varphi^*)$ is the wave profile. Specifically, $(\phi^*, \psi^*, \varphi^*)$ with $(H2)$:

$$
\begin{align*}
&c\phi^* = \int_{R} J_1(z) \phi^*(\xi - z) dz - \phi^* + f_1(\phi^*, \psi^*, \varphi^*), \quad \xi \in \mathbb{R}, \\
&c\psi^* = \int_{R} J_2(z) \psi^*(\xi - z) dz - \psi^* + f_2(\phi^*, \psi^*, \varphi^*), \quad \xi \in \mathbb{R}, \\
&c\varphi^* = \int_{R} J_3(z) \varphi^*(\xi - z) dz - \varphi^* + f_3(\phi^*, \psi^*, \varphi^*), \quad \xi \in \mathbb{R},
\end{align*}
$$

and the following asymptotic boundary condition:

$$(\phi^*, \psi^*, \varphi^*)(-\infty) = (0, 0, 0),$$

where $K$ is a given positive constant and $(\phi^*, \psi^*, \varphi^*)(-\infty) := \lim_{\xi \to -\infty} (\phi^*, \psi^*, \varphi^*)(\xi)$. Here, we only deal with the asymptotic behavior of traveling wave solutions when $\xi$ goes to $-\infty$, then it is not necessary to impose the boundary condition at $+\infty$.

As we all know, the investigation of asymptotic behavior is essential and important in several aspects of traveling wave solutions, such as the stability of traveling wave solutions, monotonicity and uniqueness of wave profiles as well as the estimation of wave speed and existence of front-like entire solutions (see, e.g., [19–21, 24, 35, 36]). Although there are a lot of work devoted to asymptotic behavior, most of them are focused on the scalar equations (see, e.g., [2,3,6,9,30,36]). Notice that the research of asymptotic behavior for some two-component systems largely depends on the particular form of nonlinear term. To the best of our knowledge, there are only a few papers concerning the asymptotic behavior of traveling wave solutions for some $N$-component systems (see, e.g., [14, 23, 27]). Notice that the research of asymptotic behavior for some two-component systems largely depends on the particular form of nonlinear term. To the best of our knowledge, there are only a few papers concerning the asymptotic behavior of traveling wave solutions for some $N$-component systems ([15, 33] ($N \geq 3$). This is the motivation of our current paper. In particular, we will provide a general approach to tackle the asymptotic behavior of traveling wave solutions for a three-component nonlocal dispersal system ([1]). Some ideas come from Wu [33] for a class of three-component lattice dynamical system.

Before stating our main results on the asymptotic behavior of traveling wave solutions, we give some assumptions about the nonlinearity of system ([1]):

(H1): $f_i \in C^1([0, K] \times [0, K] \times [0, K])$ and $f_i(0) = 0$ for $i = 1, 2, 3$.

(H2):

$$
\begin{align*}
&f_1(x) = \alpha_{11}[1 + o(1)]x_1 \text{ as } x \to 0^+; \\
&f_2(x) = \alpha_{21}[1 + o(1)]x_1 + \alpha_{22}[1 + o(1)]x_2 \text{ as } x \to 0^+; \\
&f_3(x) = \alpha_{31}[1 + o(1)]x_1 + \alpha_{32}[1 + o(1)]x_2 + \alpha_{33}[1 + o(1)]x_3 \text{ as } x \to 0^+; \\
\end{align*}
$$

where $x = (x_1, x_2, x_3)$, $x_1 = \phi$, $x_2 = \psi$, $x_3 = \varphi$ and for $i, j = 1, 2, 3$,

$$
\alpha_{ij} = \frac{\partial f_i(0)}{\partial x_j}, \quad \alpha_{ii} \neq 0 \text{ and } \alpha_{ij} \geq 0 \text{ for } i > j.
$$

We emphasize that $\alpha_{ii}$ ($i = 1, 2, 3$) can be positive or negative. As mentioned in [33], we only need to consider the following three different categories associated with (H2):

(H2-1): $\alpha_{21} = 0$, $\alpha_{31} > 0$, $\alpha_{32} > 0$;

(H2-2): $\alpha_{21} > 0$, $\alpha_{31} > 0$, $\alpha_{32} = 0$;

(H2-3): $\alpha_{21} > 0$, $\alpha_{31} > 0$, $\alpha_{32} > 0$.

The way we treat other cases is a whole lot like the above three cases studied.
Hereafter, we assume that (J), (H1) and (H2-1) hold. Then results about the asymptotic behavior of the solutions of (2)-(3) as follows.

Theorem 1.1. Assume that (J), (H1) and (H2-1) hold. Then
\[ \lim_{\xi \to -\infty} \phi^{*r}(\xi) = \Lambda_1(c) \in \{\lambda_1^-(c), \lambda_1^+(c)\}, \]
\[ \lim_{\xi \to -\infty} \psi^{*r}(\xi) = \Lambda_2(c) \in \{\lambda_2^-(c), \lambda_2^+(c)\}, \]
and \( \varphi^* \) satisfies the following dichotomy:
(i)
\[ \lim_{\xi \to -\infty} \phi^*(\xi) = 0 = \lim_{\xi \to -\infty} \psi^*(\xi), \quad \lim_{\xi \to -\infty} \varphi^*(\xi) = \Lambda_3(c) \leq \min\{\Lambda_1(c), \Lambda_2(c)\}, \]
where \( \Lambda_3(c) \in \{\lambda_3^-(c), \lambda_3^+(c)\} \).
(ii) For some \( L > 0 \),
\[ \lim_{\xi \to -\infty} \alpha_{31}\phi^*(\xi) + \alpha_{32}\psi^*(\xi) = L, \quad \lim_{\xi \to -\infty} \varphi^*(\xi) = \min\{\Lambda_1(c), \Lambda_2(c)\}. \]
Further, \( \Lambda_j(c) > 0 \) for \( j = 1, 2, 3 \).

Theorem 1.2. Assume that (J), (H1) and (H2-2) hold. Then \( V_2 = \psi^*, V_3 = \varphi^* \). Then for \( j = 2, 3 \), \( V_j \) satisfy the following dichotomy:
(i)
\[ \lim_{\xi \to -\infty} \phi^*(\xi) = 0, \quad \lim_{\xi \to -\infty} \frac{V_j'(\xi)}{\varphi^*(\xi)} = \Lambda_j(c) \leq \Lambda_1(c), \]
where \( 0 < \Lambda_j(c) \in \{\lambda_j^-(c), \lambda_j^+(c)\} \).
(ii) For some \( L_j > 0 \),
\[ \lim_{\xi \to -\infty} \frac{\phi^*(\xi)}{\varphi^*(\xi)} = L_j, \quad \lim_{\xi \to -\infty} \frac{V_j'(\xi)}{\varphi^*(\xi)} = \Lambda_1(c). \]

Theorem 1.3. Assume that (J), (H1) and (H2-3) hold. Then \( V_1 = \psi^* \). Moreover, the following statements are valid:
(i) \( \psi^* \) satisfies
\[ \lim_{\xi \to -\infty} \phi^*(\xi) = 0, \quad \lim_{\xi \to -\infty} \frac{\psi^*(\xi)}{\varphi^*(\xi)} = \Lambda_2(c) \leq \Lambda_1(c), \]
or for some \( L_1 > 0 \),
\[ \lim_{\xi \to -\infty} \frac{\phi^*(\xi)}{\varphi^*(\xi)} = L_1, \quad \lim_{\xi \to -\infty} \frac{\psi^*(\xi)}{\varphi^*(\xi)} = \Lambda_1(c). \]
(ii) \( \varphi^* \) satisfies
\[ \lim_{\xi \to -\infty} \frac{\phi^*(\xi)}{\varphi^*(\xi)} = 0 = \lim_{\xi \to -\infty} \frac{\psi^*(\xi)}{\varphi^*(\xi)}, \quad \lim_{\xi \to -\infty} \frac{\varphi^*(\xi)}{\varphi^*(\xi)} = \Lambda_3(c) \leq \min\{\Lambda_1(c), \Lambda_2(c)\}, \]
or for some \( L'_1 > 0 \),
\[ \lim_{\xi \to -\infty} \frac{\alpha_{31}\phi^*(\xi) + \alpha_{32}\psi^*(\xi)}{\varphi^*(\xi)} = L'_1, \quad \lim_{\xi \to -\infty} \frac{\varphi^*(\xi)}{\varphi^*(\xi)} = \min\{\Lambda_1(c), \Lambda_2(c)\}. \]
Further, \( 0 < \Lambda_j(c) \in \{\lambda_j^-(c), \lambda_j^+(c)\} \) for \( j = 1, 2, 3 \).
Further, we assume species capacity of each species is assumed to be 1 by some suitable time and space scaling. The rate of species stands for the competition coefficient of species and is the migration rate of species . Besides, the carrying capacity of each species is assumed to be 1 by some suitable time and space scaling. Further, we assume species and have different preference for food resource, that is, there is no competition between species and . However, species can compete with both species and . In , we give an additional hypothesis on the competition coefficients : which indicates that the competition ability of species is stronger than the species and .

Under (A), it is easy to verify that the corresponding kinetic system has an unstable equilibrium and a stable equilibrium . This indicates that we can model the population invasion process between the invader and the residents and . Mathematically, this dynamics could be characterized by a traveling wave solution or an entire solution (see definition below). According to the previous definition, a solution of (10) is called a traveling wave solution connecting (0,1) and (1,0,0) with speed , if

\[
(u(x,t), v(x,t), w(x,t)) = \left(\hat{\phi}(\xi), \hat{\psi}(\xi), \hat{\varphi}(\xi)\right), \quad \xi = x + ct,
\]

for some function \(\left(\hat{\phi}, \hat{\psi}, \hat{\varphi}\right)\) satisfying

\[
\begin{align*}
&c\hat{\phi}' = \int_{\mathbb{R}} J_1(z)\hat{\phi}(\xi - z)dz - \hat{\phi} + r_1 \hat{\phi} \left(1 - \hat{\phi} - b_{12}\hat{\psi} - b_{13}\hat{\varphi}\right), \quad \xi \in \mathbb{R}, \\
&c\hat{\psi}' = \int_{\mathbb{R}} J_2(z)\hat{\psi}(\xi - z)dz - \hat{\psi} + r_2 \hat{\psi} \left(1 - \hat{\psi} - b_{21}\hat{\phi}\right), \quad \xi \in \mathbb{R}, \\
&c\hat{\varphi}' = \int_{\mathbb{R}} J_3(z)\hat{\varphi}(\xi - z)dz - \hat{\varphi} + r_3 \hat{\varphi} \left(1 - \hat{\varphi} - b_{31}\hat{\phi}\right), \quad \xi \in \mathbb{R}, \\
&\left(\hat{\phi}, \hat{\psi}, \hat{\varphi}\right)(-\infty) = (0,1,1), \left(\hat{\phi}, \hat{\psi}, \hat{\varphi}\right)(+\infty) = (1,0,0), \\
&0 \leq \hat{\phi}, \hat{\psi}, \hat{\varphi} \leq 1, \quad \xi \in \mathbb{R}.
\end{align*}
\]

The following theorem gives the existence of solutions of (11).

**Theorem 1.4.** Assume that \((J)\) and \((A)\) hold. Then there exists a positive constant \(c_{\min}\) such that \((11)\) admits a solution \(c, \hat{\phi}, \hat{\psi}, \hat{\varphi}\) satisfying \(\hat{\phi}' > 0, \hat{\psi}' < 0\) and \(\hat{\varphi}' < 0\) in \(\mathbb{R}\) if and only if \(c \geq c_{\min}\).

Although the traveling wave solution is a key object characterizing the dynamics of evolution equations, it is not enough to understand the whole dynamics. In fact, traveling wave solutions are only special examples of the so-called entire solutions which are defined for whole space and all time. The study of entire solutions is crucial and significant. In a nutshell, entire solutions provide some new spread
and invasion ways of the epidemic and species \cite{17,24,27} and can help us for the mathematical understanding of transient dynamics and the structures of global attractor \cite{26}. Accurately speaking, the research of two-front entire solutions can be tracked back to the works of Hamel-Nadirashvili \cite{19,20} and Yagisita \cite{35}. Since then, the investigation of entire solutions for scalar equations or two-component systems with or without delay has been widely concerned (see, e.g., \cite{4,5,13,18,20,23,26,27,31,34}), and (see, e.g., \cite{10,17,22,24,28,32,37}) for nonlocal dispersal equations or lattice differential equations. In this paper, we further study this type of entire solutions for a three-component system with nonlocal dispersal.

We end this part with the last theorem about the entire solutions of system \cite{10}. Unfortunately, we still need the following technical condition on the solution \((c, \phi, \psi, \hat{\varphi})\) of system \cite{11}:

\[ (M): \frac{\hat{\phi}(\xi)}{1 - \psi(\xi)} \geq \eta \text{ for any } \xi \leq 0 \text{ and some } \eta > 0, \]

which has appeared in some papers (see, e.g., \cite{17,24,27,33}). Thankfully, we can verify that (M) is valid under some proper conditions (see Remarks 3 and 4).

**Theorem 1.5.** Assume that (J), (A) and (M) hold. Let \((c_i, \hat{\varphi}_i, \psi_i, \hat{\psi}_i)\) be a solution of system \cite{11} and \(\theta_i\) be some given constants, \(i = 1, 2\). Then there exists an entire solution \((u, v, w)(x, t) \in (0, 1) \times (0, 1) \times (0, 1)\) of system \cite{10} such that

\[ \lim_{t \to -\infty} \sup_{x \geq \frac{2}{\sqrt{t}}} \left\{ |u(x, t) - \hat{\phi}_1(x + c_1 t + \theta_1)| + |v(x, t) - \hat{\psi}_1(x + c_1 t + \theta_1)| \right\} = 0, \quad (12) \]

\[ \lim_{t \to -\infty} \sup_{x \leq -\frac{2}{\sqrt{t}}} \left\{ |u(x, t) - \hat{\phi}_2(-x + c_2 t + \theta_2)| + |v(x, t) - \hat{\psi}_2(-x + c_2 t + \theta_2)| \right\} = 0, \quad (13) \]

and

\[ \lim_{t \to +\infty} \sup_{x \in \mathbb{R}} \{ |1 - u(x, t)| + |v(x, t)| + |w(x, t)| \} = 0. \quad (14) \]

**Remark 1.** We point out that the techniques and theories developed for system \cite{1} under the assumption that \(J_i(x) = 0\), if \(|x| > 1\), \(i = 1, 2, 3\) can be parallel extended to the following situation: \(J_i(x) = 0\), if \(|x| > L_i\), where \(L_i \in \mathbb{R}\) are some given positive constants, \(i = 1, 2, 3\). In fact, let \(J_{iL_i}(\cdot) = 1/L_i J_i(\cdot/L_i)\), then \(J_{iL_i}\) satisfies (J) with the compactly supported set being \([-L_i, L_i]\) if \(J_i\) is compactly supported on \([-1, 1]\). Moreover, all the analysis can be repeated if we replace \(J_i\) by \(J_{iL_i}\), respectively. Therefore, in our paper, we take \(L_i = 1\) \((i = 1, 2, 3)\) directly just for convenience.

The remaining part of this paper is organised as follows. In Section 2, we establish the asymptotic behavior of traveling wave solutions for system \cite{1} and complete the proof of Theorems 1.1-1.3. In Section 3, we first obtain the existence of traveling wave solutions of system \cite{10}, then we apply the results established in Section 2 to construct front-like entire solutions.
2. Asymptotic behavior of traveling wave solutions. In this section, our aim is to study the asymptotic behavior of traveling wave solutions of system \([1]\). For convenience, we denote \((\phi^*, \psi^*, \varphi^*)\) by \((\phi, \psi, \varphi)\) just in the current section. Now we give some preparing works.

Consider the characteristic equations as follows:

\[
k_j(c, \lambda) = c\lambda - \left[ \int_{\mathbb{R}} J_j(x)e^{-\lambda x}dx - 1 + \alpha_j \right] = 0, \quad j = 1, 2, 3. \tag{15}\]

Then we have the following observation.

**Lemma 2.1.** Assume that \((J)\) holds. Then for \(j = 1, 2, 3\), the following assertions hold:

(i) If \(\alpha_{jj} > 0\), then there exists \(c_{j*} > 0\) such that \(k_j(c, \lambda) = 0\) has two real roots \(0 < \lambda_j^-(c) \leq \lambda_j^+(c)\) for \(|c| \geq c_{j*}\), and \(k_j(c, \lambda) = 0\) has no real root for \(|c| < c_{j*}\).

(ii) If \(\alpha_{jj} = 0\), then \(k_j(c, \lambda) = 0\) has two real roots \(\lambda_j^-(c) \leq \lambda_j^+(c)\) (one of them is zero) for any \(c \in \mathbb{R}\).

(iii) If \(\alpha_{jj} < 0\), then \(k_j(c, \lambda) = 0\) has two real roots \(\lambda_j^-(c) < 0 < \lambda_j^+(c)\) for any \(c \in \mathbb{R}\).

We now introduce a lemma coming from [36, Proposition 3.7], which plays an important role in proving Theorem 1.1

**Lemma 2.2.** Assume \(a \neq 0\), \(B(\cdot)\) is a continuous function having finite limits at infinity \(B(\pm \infty) := \lim_{\xi \to \pm \infty} B(\xi)\). Let \(Z(\cdot)\) be a measurable function satisfying

\[
aZ(\xi) = \int_{\mathbb{R}} J(x)e^{\int_{-\xi}^x Z(s)ds}dx + B(\xi), \quad \xi \in \mathbb{R}, \tag{16}\]

where \(J\) is a kernel function and satisfies \((J)\). Then \(Z\) is uniformly continuous and bounded. Moreover, \(z^\pm = Z(\pm \infty)\) exists and are real roots of the characteristic equation \(aw = \int_{\mathbb{R}} J(y)e^{-\omega y}dy + B(\pm \infty)\).

Based on Lemmas 2.1 and 2.2, we can easily obtain the asymptotic behavior of \(\phi\) near \(-\infty\).

**Lemma 2.3.** Assume that \((J)\), \((H1)\) and \((H2)\) hold. Then \([4]\) holds. Moreover,

\[
\sup_{\xi \leq 0, s \in [-1, 1]} \frac{\phi(\xi + s)}{\phi(\xi)} < +\infty. \tag{17}\]

**Proof.** Let \(w(\xi) = \frac{\phi'(\xi)}{\phi(\xi)}\), then

\[
cw(\xi) = \int_{\mathbb{R}} J_1(x)e^{\int_{-\xi}^x w(s)ds}dx + B(\xi),
\]

where \(B(\xi) = -1 + \frac{E(\phi(\xi)))}{\phi(\xi)}\). By \((H2)\), it holds \(B(-\infty) = -1 + \alpha_{11}\). So we have \([4]\) by Lemmas 2.1 and 2.2. In view of \(\phi(-\infty) = 0 < \phi(\cdot)\) in \(\mathbb{R}\), we get \(\Lambda_1(c) > 0\). Moreover, \((17)\) comes from the equality \(\frac{\phi(\xi + s)}{\phi(\xi)} = e^{\int_{-\xi}^x \frac{\phi'(s)}{\phi(s)}ds}\) directly.

The asymptotic behaviors of \(\psi, \varphi\) when \(\xi\) goes to \(-\infty\) are more complicated. We then divide our discussion into several cases according to \((H2-1)-(H2-3)\).
2.1. Asymptotic behavior of solutions under (H2-1). In this subsection, we always assume that (J), (H1) and (H2-1) hold. In this case, similar to Lemma 2.3, the following result holds.

**Lemma 2.4.** \( \phi \) is valid and \( \Lambda_2(c) > 0 \).

Define

\[
P(\xi) := \frac{\alpha_{31} \phi(\xi) + \alpha_{32} \psi(\xi)}{\varphi(\xi)}, \quad \xi \in \mathbb{R}.
\]  

(18)

Then we plan to investigate the behavior of \( P(\xi) \) near \( \xi = -\infty \). In this piece, the following three identities are used frequently:

\[
P'(\xi) = \alpha_{31} \left[ \frac{\phi'(\xi)}{\phi(\xi)} - \frac{\varphi'(\xi)}{\varphi(\xi)} \right] \phi(\xi) + \alpha_{32} \left[ \frac{\psi'(\xi)}{\psi(\xi)} - \frac{\varphi'(\xi)}{\varphi(\xi)} \right] \psi(\xi),
\]  

(19)

\[
\frac{\varphi(\xi - z)}{\varphi(\xi)} = \frac{P(\xi)}{P(\xi - z)} \left[ \alpha_{31} + \alpha_{32} \frac{\psi(\xi - z)}{\phi(\xi - z)} \right] \frac{\phi(\xi - z)}{\varphi(\xi)} \left[ \frac{\alpha_{31} + \alpha_{32} \psi(\xi)}{\phi(\xi)} \right]^{-1},
\]  

(20)

for any \( \xi \in \mathbb{R}, \ z \in [-1, 1] \) and

\[
\frac{\varphi(\xi)}{\varphi(\xi)} = \int_{\mathbb{R}} J_3(z) \frac{\varphi(\xi - z)}{\varphi(\xi)} dz - 1 + \frac{f_3(\phi, \psi, \varphi)(\xi)}{\varphi(\xi)}. \tag{21}
\]

**Lemma 2.5.** Assume that \( c > 0 \) and

\[
f_3(\phi, \psi, \varphi)(\xi) \geq 0 \text{ for all } \xi \text{ near } -\infty. \tag{22}
\]

Then

\[
\sup_{\xi \leq 0} \left| \frac{\varphi'(\xi)}{\varphi(\xi)} \right| + \sup_{\xi \leq 0, s \in [-1, 1]} \frac{\varphi(\xi + s)}{\varphi(\xi)} < +\infty. \tag{23}
\]

**Proof.** By (H2-1) and \( c > 0 \), we can choose \( \mu > \frac{1 + |\alpha_{33}|}{2} \) and \( \xi^0 \ll -1 \) such that \( \varphi'(\xi) + \mu \varphi(\xi) > 0 \) for all \( \xi \leq \xi^0 + 1 \). Thus \( \varphi(\xi - s) < e^{\mu s} \varphi(\xi) \) for any \( s > 0 \) and \( \xi \leq \xi^0 + 1 \). In view of (22), there exists \( \xi^1 \ll -1 \) such that \( f_3(\phi, \psi, \varphi)(\xi) \geq 0 \) for all \( \xi \leq \xi^1 \). Taking \( \xi_0 = \min\{\xi^0, \xi^1\} \). Then for \( \xi \leq \xi_0 \), integrating the third equation of (11) on \( (-\infty, \xi] \), we get

\[
c \varphi(\xi) \geq \int_{\mathbb{R}} J_3(z) \int_{\xi}^{\xi + z} \varphi(s) ds dz
\]

\[
\geq \int_{\frac{1}{2}}^{1} J_3(z) \int_{\xi + \frac{1}{2}}^{\xi + z} \varphi(s) ds dz - \frac{1}{2} \varphi(\xi) e^\mu
\]

\[
\geq \int_{\frac{1}{2}}^{1} J_3(z) \left( z - \frac{1}{2} \right) dz \varphi \left( \xi + \frac{1}{2} \right) e^{-\frac{1}{2} \frac{1}{2} \mu} - \frac{1}{2} \varphi(\xi) e^\mu,
\]

which implies \( \varphi(\xi + 1) \varphi(\xi) \) is bounded uniformly for any \( \xi \leq \xi_0 + 1 \). Therefore

\[
\frac{\varphi(\xi + s)}{\varphi(\xi)} \leq \frac{\varphi(\xi + (\frac{1}{2} - s))}{\varphi(\xi)} \leq \frac{e^{\mu (\frac{1}{2} - s)} \varphi(\xi + \frac{1}{2})}{\varphi(\xi)} < +\infty
\]

for \( s \in [-1, \frac{1}{2}] \). Furthermore, if \( s \in (\frac{1}{2}, 1] \), we have

\[
\frac{\varphi(\xi + s)}{\varphi(\xi)} = \frac{\varphi(\xi + s - \frac{1}{2} + \frac{1}{2}) \varphi(\xi + 1 - (1 - s))}{\varphi(\xi + s - \frac{1}{2}) \varphi(\xi)} \leq \frac{\varphi(\xi + s - \frac{1}{2}) e^{\mu (1 - s)} \varphi(\xi + \frac{1}{2})}{\varphi(\xi + s - \frac{1}{2}) \varphi(\xi)} < +\infty.
\]
Hence $\frac{\varphi'(ξ + s)}{\varphi(ξ)}$ is bounded uniformly for any $ξ \leq ξ_0$ and $|s| \leq 1$. Therefore (23) holds by the equation that $\varphi$ satisfied. This completes the proof.

Notice that if condition (22) is removed, (23) may be incorrect. Here, we give another estimate of $\frac{\varphi'}{\varphi}$ in a bounded interval.

**Lemma 2.6.** Assume $c > 0$. If $P'(ξ) \geq 0$ for all $ξ \in [a, b]$, then

$$-μ \leq \frac{\varphi'(ξ)}{\varphi(ξ)} \leq \sup_{ξ \in [a, b]} \left[ \frac{\varphi'(ξ)}{\varphi(ξ)} + \frac{\psi'(ξ)}{\varphi(ξ)} \right], \quad ξ \in [a, b],$$

where $μ$ is given in Lemma 2.5.

**Proof.** By virtue of (19) and the choice of $μ$, the conclusion is obvious and we omit the detail here.

To continue our work, we need the following result [33, Lemma 2.6].

**Lemma 2.7.** For given $C^1$ function sequence $u_j : (-∞, 0] \to (0, +∞)$ satisfy

$$\lim_{ξ \to -∞} u'_j(ξ) = a_j > 0 \text{ for each } j = 1, 2, ..., N.$$

Then the following statements are valid:

(i) \begin{align*}
\lim_{ξ \to -∞} \frac{u_j(ξ)}{ω(ξ)} &= 0 \quad \text{if } a_i < a_j, \\
\lim_{ξ \to -∞} \frac{u_j(ξ + s)}{ω(ξ + s)} u_j(ξ) &= 1 \quad \text{for any fixed } s \in \mathbb{R} \text{ if } a_i = a_j.
\end{align*}

(ii) Given a positive function $ω \in C^1((-∞, 0])$ and positive constants $k_j > 0$ for $j = 1, 2, ..., N$. If $\{ξ_n\}$ is a sequence which tends to $-∞$ as $n \to ∞$ such that

$$\sum_{j=1}^{N} k_j \left[ \frac{u'_j(ξ_n)}{ω_j(ξ_n)} - \frac{ω'(ξ_n)}{ω(ξ_n)} \right] \frac{u_j(ξ_n)}{ω(ξ_n)^2} = 0 \text{ for each } n \in \mathbb{N}, \tag{25}$$

then $\lim_{n \to ∞} \frac{ω'(ξ_n)}{ω(ξ_n)^2} = \min\{a_1, a_2, ..., a_N\}$.

Based on Lemma 2.7, the following result can be derived by a parallel discussion to that of [33, Lemma 2.7] and the details are omitted here.

**Lemma 2.8.** It holds that

$$\sup_{ξ \leq 0} P(ξ) < +∞, \tag{26}$$

where $P$ is given by (18).

In the sequel, we say that a function $ϕ(ξ)$ is eventually monotone for $ξ < 0$ ($ξ > 0$), if $ϕ$ has no extreme points on $(-∞, -n]$ (or $[n, +∞)$) for some $n \gg 1$.

**Proposition 1.** $\lim_{ξ \to -∞} P(ξ)$ exists and is finite, where $P$ is given by (18).

**Proof.** If $P(ξ)$ is eventually monotone for $ξ < 0$, then we have the assertion by Lemma 2.8. Next, we continue our discussion for $P(ξ)$ is not eventually monotone for $ξ < 0$, that is, $P$ oscillates as $ξ \to -∞$.

Let $M := \limsup_{ξ \to -∞} P(ξ)$ and $m := \liminf_{ξ \to -∞} P(ξ)$. It then follows from Lemma 2.8 that $0 \leq m \leq M < ∞$. Moreover, we can choose a sequence $\{x_n\}$ (resp., $\{y_n\}$) of local maximal (resp., minimal) points of $P$ such that $x_n \to -∞$ (resp., $y_n \to -∞$)
and \( P(x_n) \to M \) (resp., \( P(y_n) \to m \)) as \( n \to \infty \). If \( M = 0 \), then \( \lim_{\xi \to -\infty} P(\xi) = 0 \) and the conclusion is valid. So we only consider the case \( M > 0 \).

We divide our discussion into two cases. To start with, we assume \( \Lambda_2(c) \geq \Lambda_1(c) \) without loss of generality.

**Case I.** When \( \inf_{\xi \leq 0} P(\xi) > 0 \), then \( m > 0 \). By the choice of \( \{x_n\} \) and Lemma 2.7 (ii), we know

\[
\lim_{n \to \infty} \frac{\varphi'(x_n)}{\varphi(x_n)} = \Lambda_1(c).
\]

Let

\[
U(x_n) := \alpha_{31} \phi(x_n) + \alpha_{32} \psi(x_n) = P(x_n) \varphi(x_n),
\]

then

\[
\lim_{n \to \infty} \frac{U'(x_n)}{U(x_n)} = \lim_{n \to \infty} \frac{P'(x_n) \varphi(x_n) + P(x_n) \varphi'(x_n)}{P(x_n) \varphi(x_n)} = \Lambda_1(c).
\]

From Lemma 2.7 (i), for any fixed \( z \in [-1, 1] \), we have

\[
\lim_{n \to \infty} \frac{\alpha_{31} \phi(x_n - z) + \alpha_{32} \psi(x_n - z)}{\phi(x_n - z)} \frac{\phi(x_n)}{\alpha_{31} \phi(x_n) + \alpha_{32} \psi(x_n)} = 1.
\]

That is,

\[
H(x_n - z) := \left( \alpha_{31} + \alpha_{32} \frac{\psi(x_n - z)}{\phi(x_n - z)} \right) \left( \alpha_{31} + \alpha_{32} \frac{\psi(x_n)}{\phi(x_n)} \right)^{-1} \to 1 \text{ as } n \to \infty,
\]

for any fixed \( z \in [-1, 1] \).

Further, for any given \( \varepsilon > 0 \) and any fixed \( z \in [-1, 1] \), according to the choice of \( \{x_n\} \) and (20), we can get that

\[
\frac{\varphi(x_n - z)}{\varphi(x_n)} \geq \frac{M}{M + \varepsilon} \frac{\phi(x_n - z)}{\phi(x_n)} H(x_n - z) \text{ for all large } n.
\]

In view of (27), (28) and Lemma 2.3, taking \( n \to \infty \) in (21), we have

\[
c \Lambda_1(c) \geq \int_{\mathbb{R}} J_3(z) \frac{M}{M + \varepsilon} e^{-\Lambda_1(c)z} dz - 1 + M + \alpha_{33}.
\]

By the arbitrariness of \( \varepsilon \), we further have

\[
c \Lambda_1(c) \geq \int_{\mathbb{R}} J_3(z) e^{-\Lambda_1(c)z} dz - 1 + M + \alpha_{33}.
\]

On the other hand, replacing \( \{x_n\} \) by \( \{y_n\} \) and using \( m > 0 \), we can obtain that

\[
c \Lambda_1(c) \leq \int_{\mathbb{R}} J_3(z) e^{-\Lambda_1(c)z} dz - 1 + m + \alpha_{33}.
\]

It then follows from (29) and (30) that \( M = m \), that is, the proposition is true.

**Case II.** When \( \inf_{\xi \leq 0} P(\xi) = 0 \). We first choose a local minimal point \( z_0 < 0 \) of \( P \) in \((-\infty, -1)\) such that \( P(z_0) = \min_{\xi \in [z_0, z_0 + 2]} P(\xi) \). Motivated by (14) and (33), we can define \( \{z_n\}_{n=0}^{\infty} \) to be the sequence of local minimal points of \( P \) in \((-\infty, 0)\) such that \( z_n < z_{n-1}, P(z_n) < P(z_{n-1}) \) for all \( n \in \mathbb{N} \), and \( P(z) \geq P(z_{n-1}) \) for any minimal point \( z \) of \( P \) in \((z_n, z_{n-1})\) (if it exists). Notice that this sequence is well defined due to the fact that \( \inf_{\xi \leq 0} P(\xi) = 0 \) and \( P(z_n) \downarrow 0 \) as \( n \to \infty \).
We now give a claim: for any \( \alpha \in [0, 2] \) and \( n \in \mathbb{N} \), there holds
\[
P(z_n) \leq P(z_n + \alpha).
\] (31)
In fact, if not, then there exists \( \alpha_0 \in (0, 2] \) and \( n_0 \in \mathbb{N} \) such that \( P(z_{n_0} + \alpha_0) < P(z_{n_0}) \). If \( z_{n_0} + \alpha_0 \geq z_0 \), it follows the definition of \( z_n \) that \( P(z_{n_0} + \alpha_0) \geq P(z_0) \geq P(z_{n_0}) \), which is a contradiction. Hence, there must hold \( z_{n_0} + \alpha_0 < z_0 \), so there exists \( k_0 \geq 0 \) such that \( z_{n_0} + \alpha_0 \in (z_{n_0-k_0}, z_{n_0-k_0-1}) \). In view of the definition of \( z_n \), we get \( P(z_{n_0} + \alpha_0) \geq P(z_{n_0-k_0-1}) \geq P(z_{n_0-1}) \), hence \( P(z_{n_0}) > P(z_{n_0-1}) \), which contradicts to \( P(z_{n_0}) < P(z_{n_0-1}) \). Thus the claim is true.

According to \( P'(z_n) = 0 \), (19) and Lemma 2.7 (ii), we have \( \lim_{n \to \infty} \frac{\varphi(z_n)}{\varphi(z_{n-1})} = \Lambda_1(c) \). Furthermore, we claim that there exists \( \alpha_1 \in (0, 1) \) and \( N \in \mathbb{N} \), such that for any \( n \geq N \),
\[
P(z_n) > P(z_n - \alpha_1).
\] (32)
If not, then \( P(z_n) \leq P(z_n - \alpha) \) for any \( z \in (0, 1) \) and \( n \in \mathbb{N} \). By (20), (21) and (31),
\[
c \frac{\varphi'(z_n)}{\varphi(z_n)} = \int_\mathbb{R} J_3(z) \frac{P(z_n)}{P(z_n - z)} H(z_n - z) \frac{\phi(z_n - z)}{\phi(z_n)} dz - 1 + \frac{f_3(\phi, \psi, \varphi)(z_n)}{\varphi(z_n)} \leq \int_\mathbb{R} J_3(z) H(z_n - z) \frac{\phi(z_n - z)}{\phi(z_n)} dz - 1 + \frac{f_3(\phi, \psi, \varphi)(z_n)}{\varphi(z_n)},
\]
letting \( n \to \infty \), by Lemma 2.3 and (28), we have
\[
c \Lambda_1(c) \leq \int_\mathbb{R} J_3(z) e^{-\Lambda_1(c)z} dz - 1 + \alpha_{33},
\]
which contradicts to (29) since \( M > 0 \), so (32) holds. Moreover, by the definition of \( z_n \) and (32), if \( z_{n+1} + \alpha_1 < z_n - \alpha_1 \), we have
\[
P'(\xi) \geq 0 \quad \text{for all } \xi \in [z_{n+1} + \alpha_1, z_n - \alpha_1] \quad \text{and all large } n,
\] (33)
which implies that any local maximal point of \( P \) near \( -\infty \) must fall in \([z_n - \alpha_1, z_n + \alpha_1] \) for enough large \( n \in \mathbb{N} \). For convenience, we assume \( x_n \in [z_n - \alpha_1, z_n + \alpha_1] \) for enough large \( n \) (if necessary we take a sub-sequence), where \( x_n \) is defined as above. Note that
\[
\frac{\varphi(z_n)}{\varphi(x_n)} = \frac{\alpha_{31} \phi(z_n) + \alpha_{32} \psi(z_n)}{\alpha_{31} \phi(x_n) + \alpha_{32} \psi(x_n)} \frac{P(x_n)}{P(z_n)}.
\]
Since \( x_n \in [z_n - \alpha_1, z_n + \alpha_1] \) for enough large \( n \in \mathbb{N} \). By Lemma 2.3 and (28), we have
\[
\frac{\alpha_{31} \phi(z_n) + \alpha_{32} \psi(z_n)}{\alpha_{31} \phi(x_n) + \alpha_{32} \psi(x_n)} = \left[ \frac{\alpha_{31} + \alpha_{32} \psi(z_n)}{\phi(z_n)} \right] \left[ \frac{\alpha_{31} + \alpha_{32} \psi(x_n)}{\phi(x_n)} \right]^{-1} \frac{\phi(z_n)}{\phi(x_n)} \geq K_2
\]
for some \( K_2 > 0 \) and for all large \( n \in \mathbb{N} \). Hence
\[
\frac{\varphi(z_n)}{\varphi(x_n)} \geq K_2 \frac{P(x_n)}{P(z_n)} \to +\infty \quad \text{as } n \to \infty.
\] (34)
If we can prove \( \sup_{n \to \infty} \frac{\varphi(z_n)}{\varphi(x_n)} < +\infty \), then \( m = 0 \) can’t occur when \( M > 0 \). Therefore, we next want to show \( \sup_{n \to \infty} \frac{\varphi(z_n)}{\varphi(x_n)} < +\infty \). Specifically, we complete the proof in two ways: \( c > 0 \) and \( c < 0 \).

For \( c > 0 \). If \( \frac{\varphi}{\varphi} \) is unbounded for \( \xi \leq 0 \), then we can choose a sequence \( \{ \tau_n \}_{n=1}^{\infty} \) such that \( \frac{\varphi(\tau_n)}{\varphi(z_{\tau_n})} \to +\infty \), \( \tau_n \downarrow -\infty \) as \( n \to \infty \). In view of the proof of Lemma 2.5
there holds $\varphi(x - s) < e^{\mu x} \varphi(x)$ for $s > 0$ and $\xi \leq \xi_0 + 1$ with $\xi_0 \ll -1$. By (21), we have

$$\int_{-1}^{0} J_3(z) \frac{\varphi(\tau_n - z)}{\varphi(\tau_n)} dz \to +\infty \text{ and } \tau_n \downarrow -\infty \text{ as } n \to \infty. \quad (35)$$

We now claim: for all large $n$, $\tau_n \in [\xi_k - \alpha_1, \xi_k + \alpha_1]$ for some $k = k(n) \gg 1$, where $\alpha_1$ is given by (32). If not, then for enough large $n$, there exists enough large $k$, such that $\tau_n \in [\xi_k + 1 + \alpha_1, \xi_k - \alpha_1]$. By (32) and Lemma 2.6, we get $-\mu \leq \frac{\varphi'(\tau_n)}{\varphi(\tau_n)} < +\infty$, which contradicts to $\frac{\varphi'(\tau_n)}{\varphi(\tau_n)} \to +\infty$ as $n \to -\infty$. Thus the claim holds.

From (31) and (32), we have $P(z_n - \alpha_1) \leq P(z_n) \leq P(z_n + 2)$ for enough large $n$. Let $G(\phi; z_k) = \exp \left\{ \int_{z_n - \alpha_1}^{z_n + 2} \frac{\varphi'(s)}{\varphi(s)} ds \right\}$. Then

$$\alpha_3 \phi(z_k - \alpha_1) G(\phi; z_k) + \alpha_3 \psi(z_k - \alpha_1) G(\psi; z_k)$$

$$= P(z_n + 2) \geq P(z_n - \alpha_1)$$

$$= \alpha_3 \phi(z_k - \alpha_1) + \alpha_3 \psi(z_k - \alpha_1).$$

So $G(\phi; z_k) \geq 1$ or $G(\psi; z_k) \geq 1$. Without loss of generality, we assume the former occurs. Then using $\tau_n \in [\xi_k - \alpha_1, \xi_k + \alpha_1]$ for all large $n$ and Lemma 2.5, we have

$$\int_{-1}^{0} \frac{\varphi'(s)ds}{\varphi(\tau_n)} \geq \int_{-1}^{0} \frac{\varphi'(s)ds}{\varphi(\tau_n)} \geq \int_{-1}^{0} \frac{\varphi'(s)ds}{3\mu}.$$}

By Lemma 2.3, we further obtain that

$$\frac{\varphi(\tau_n + 1)}{\varphi(\tau_n)} \leq \exp \{ 3\mu + 3\sup_{\xi \leq 0} \frac{\varphi'(\xi)}{\varphi(\xi)} \} < +\infty \text{ for all enough large } n.$$

Thus for any $z \in [0, 1]$, $\frac{\varphi(\tau_n + z)}{\varphi(\tau_n)} = \frac{\varphi(\tau_n + 1 - (1 - z))}{\varphi(\tau_n)} \leq e^{\mu \frac{\varphi(\tau_n + 1)}{\varphi(\tau_n)}} < +\infty$, which contradicts to (35). Hence, $\frac{\varphi'(\xi)}{\varphi(\xi)}$ is bounded for $\xi \leq 0$.

For $c < 0$, by a similar argument as Lemma 2.5, we have

$$\frac{\varphi(\xi + s)}{\varphi(\xi)} \leq L_0 < +\infty, \text{ for } \xi \leq \xi_0 \ll -1 \text{ and } 0 \leq s \leq 1.$$

Thus the boundedness of $\frac{\varphi'}{\varphi}$ for $\xi \leq 0$ and $\int_{0}^{1} J_3(z) \frac{\varphi(\xi - z)}{\varphi(\xi)} dz \leq 0$ for $\xi \leq 0$ is equivalent.

Assuming $\frac{\varphi'}{\varphi}$ is unbounded for $\xi \leq 0$. In view of $c < 0$ and (20), we get that $M + \alpha_3 < 0$. So there exists $\xi_1 < \xi_0$ such that for all $\xi \leq \xi_1 + 1$, $J_3(\phi, \psi, \varphi)(\xi) < 0$.

According to the unboundedness of $\int_{0}^{1} J_3(z) \frac{\varphi(\xi - z)}{\varphi(\xi)} dz$, we can choose $\xi_2 < \xi_1$ such that $\frac{\varphi(\xi_2 - 1)}{\varphi(\xi_2)} > L_0$. Also, since $\varphi(\xi) = 0$, there exists $\xi_3 \leq \xi_2$ such that

$$\varphi(\xi_3) = \max \{ \varphi(\xi) | \xi \leq \xi_2 \}. \quad (36)$$

For the above $\xi_3$, we claim that

$$\varphi(\xi_3 + z) \leq \varphi(\xi_3) \text{ for any } z \in [0, 1]. \quad (37)$$

If not, there exists $z_0 \in (0, 1]$ such that

$$\varphi(\xi_3 + z_0) > \varphi(\xi_3). \quad (38)$$

Then $\xi_3 + z_0 > \xi_2$ by the definition of $\xi_3$. Thus there exists $\theta_1 \in (0, 1]$ such that

$$\xi_3 = \theta_1(\xi_2 - z_0) + (1 - \theta_1)\xi_2 = \xi_2 - \theta_1 z_0.$$
Note that
\[ \varphi(\xi_3) \geq \varphi(\xi_3 - 1) > L_0 \varphi(\xi_3) \geq \varphi(\xi_3 + \theta) = \varphi(\xi_3 + \theta_1 z_0 + \theta) \]
for any \( \theta \in [0, 1] \). When \( \theta_1 = 1 \), taking \( \theta = 0 \), then \( \xi_3 + \theta_1 z_0 + \theta = \xi_3 + z_0 \).
When \( \theta_1 \in (0, 1) \), taking \( \theta = z_0 (1 - \theta_1) \), then \( \xi_3 + \theta_1 z_0 + \theta = \xi_3 + z_0 \). Hence, \( \varphi(\xi_3) > \varphi(\xi_3 + z_0) \), which contradicts to (38), so (37) holds. Consequently,
\[
0 = c \varphi'(\xi_3) = \int_{\mathbb{R}} J_3(z) \varphi(\xi_3 - z) dz - \varphi(\xi_3) + f_3(\phi, \psi, \varphi)(\xi_3)
\]
\[
= \int_{0}^{1} J_3(z) \varphi(\xi_3 - z) dz + \int_{0}^{1} J_3(z) \varphi(\xi_3 + z) dz - \varphi(\xi_3)
\]
\[
+ f_3(\phi, \psi, \varphi)(\xi_3)
\]
\[
= \int_{0}^{1} J_3(z)(\varphi(\xi_3 - z) - \varphi(\xi_3)) dz + \int_{0}^{1} J_3(z)(\varphi(\xi_3 + z) - \varphi(\xi_3)) dz
\]
\[
+ f_3(\phi, \psi, \varphi)(\xi_3)
\]
\[
< 0,
\]
which is a contradiction. So whether \( c > 0 \) or \( c < 0 \), we obtain that \( \frac{\varphi'}{\varphi} \) is bounded for \( \xi \leq 0 \), which contradicts to (34). The above discussion indicates that \( m = 0 \) can not occur when \( M > 0 \). This completes the proof of Proposition 1. □

We are ready to prove Theorem 1.1

Proof of Theorem 1.1 By Lemmas 2.3 and 2.4 we have (4) and (5). Note that \( P(-\infty) \) exists and is finite by Proposition 1, where \( P \) is given by (18).

Let \( P(-\infty) = L_1 \in [0, +\infty) \). By Lemma 2.2 there exists \( \omega \in \mathbb{R} \) such that
\[
c \omega = \int_{\mathbb{R}} J_3(z) e^{-\omega z} dz - 1 + L_1 + \alpha_{31}.
\]

If \( L_1 = 0 \), then \( \omega \in \{ \lambda_{3}^{\pm}(c), \lambda_{3}^{\pm}(c) \} \). It then follows from \( \alpha_{31} \neq 0 \) that \( \lambda_{3}^{\pm}(c) \neq 0 \). Further, because \( \varphi(-\infty) = 0 < \varphi(\cdot) \) in \( \mathbb{R} \), it holds \( \omega > 0 \). Without loss of generality, we assume \( \min\{\Lambda_1(c), \Lambda_2(c)\} = \Lambda_1(c) \). We then want to show \( \omega \leq \Lambda_1(c) \). If not, that is, \( \omega > \Lambda_1(c) \). Then, by Lemma 2.7 we have \( P(-\infty) = +\infty \), which contradicts to Lemma 2.8. Thus \( \omega \leq \Lambda_1(c) = \min\{\Lambda_1(c), \Lambda_2(c)\} \).

If \( L_1 > 0 \), we further show that \( \omega = \Lambda_1(c) = \min\{\Lambda_1(c), \Lambda_2(c)\} \). If \( \omega > \Lambda_1(c) \), repeating the discussion as above, we still get a contradiction. Therefore \( \omega \leq \Lambda_1(c) \).

If \( \omega < \Lambda_1(c) \), by Lemma 2.7 \( P(-\infty) = 0 \), which contradicts to \( L_1 > 0 \), thus \( \omega = \min\{\Lambda_1(c), \Lambda_2(c)\} \). □

2.2. Asymptotic behavior of solutions under (H2-2) and (H2-3). Under (H1) and (H2-2), the asymptotic behavior of \( \psi, \varphi \) depends on \( \phi \), but they are independent of each other. Our proof is similar to Subsection 2.1 we first study the behavior of the ratio function:

\[
P_1(\xi) := \frac{\alpha_{31}}{\phi(\xi)} \frac{\phi(\xi)}{\varphi(\xi)}
\]

near \( \xi = -\infty \). Clearly, by taking \( \alpha_{32} = 0 \) in (18), we then obtain (39). By a similar argument as Proposition 1 we have the following result.

Lemma 2.9. \( \lim_{\xi \to -\infty} P_1(\xi) \) exists and is finite, where \( P_1 \) is given by (39).
Proof of Theorem 1.3. The proof is similar to Theorem 1.1 and we do not repeat it here.

Proof of Theorem 1.3. Under (H1) and (H2-3), the asymptotic behavior of $\psi$ as $\xi \to -\infty$ only depends on $\phi$, thus it can be obtained by the same argument as the assumptions (H1) and (H2-2) are hold. Therefore, the behavior of $\psi$ satisfies either (8) or (9). In addition, the behavior of $\varphi$ can be derived as in Section 2.1. Thus Theorem 1.3 holds.

3. Application. In this section, we focus on a three-species Lotka-Volterra competition system (10). We first study the existence of traveling wave solutions and then obtain the asymptotic behaviour of them by using the abstract results in Section 2. Finally, we construct a class of front-like entire solutions.

3.1. Existence of traveling wave solutions of system (10). Let $\phi(\xi) := \hat{\phi}(\xi)$, $\psi(\xi) := 1 - \hat{\psi}(\xi)$ and $\varphi(\xi) := 1 - \hat{\varphi}(\xi)$. Then (11) becomes

\begin{equation}
\begin{aligned}
c\dot{\phi}' &= \int_{\mathbb{R}} J_1(z)\phi(\xi - z)dz - \phi + r_1\phi(1 - b_{12} - b_{13} - \phi + b_{12}\psi + b_{13}\varphi), \quad \xi \in \mathbb{R}, \\
c\dot{\psi}' &= \int_{\mathbb{R}} J_2(z)\psi(\xi - z)dz - \psi + r_2(1 - \psi)(b_{21}\phi - \psi), \quad \xi \in \mathbb{R}, \\
c\dot{\varphi}' &= \int_{\mathbb{R}} J_3(z)\varphi(\xi - z)dz - \varphi + r_3(1 - \varphi)(b_{31}\phi - \varphi), \quad \xi \in \mathbb{R},
\end{aligned}
\end{equation}

(40)

For any $c, \mu > 0$ and any $\xi \in \mathbb{R}$, denote

\begin{equation}
\begin{aligned}
H_1(\phi, \psi, \varphi)(\xi) &= \frac{1}{c} \left\{ c\mu\phi - \phi + r_1\phi(1 - b_{12} - b_{13} - \phi + b_{12}\psi + b_{13}\varphi) \right. \\
&\quad + \left. \int_{\mathbb{R}} J_1(z)\phi(\xi - z)dz \right\}, \\
H_2(\phi, \psi, \varphi)(\xi) &= \frac{1}{c} \left\{ c\mu\psi - \psi + r_2(1 - \psi)(b_{21}\phi - \psi) + \int_{\mathbb{R}} J_2(z)\psi(\xi - z)dz \right\}, \\
H_3(\phi, \psi, \varphi)(\xi) &= \frac{1}{c} \left\{ c\mu\varphi - \varphi + r_3(1 - \varphi)(b_{31}\phi - \varphi) + \int_{\mathbb{R}} J_3(z)\varphi(\xi - z)dz \right\}.
\end{aligned}
\end{equation}

Define an operator $T = (T_1, T_2, T_3)$ by

\begin{equation}
T_i(\phi, \psi, \varphi)(\xi) := \int_{-\infty}^{\xi} e^{-\mu(\xi - s)} H_i(\phi, \psi, \varphi)(s)ds
\end{equation}

for $i = 1, 2, 3$. Note that we can choose $\mu$ large enough, such that $T$ is monotone increasing on $C([0, 1]) \times C([0, 1]) \times C([0, 1])$. It is clear that $(c, \phi, \psi, \varphi)$ satisfies (40) if and only if $(\phi, \psi, \varphi) = T(\phi, \psi, \varphi), (\phi, \psi, \varphi)(-\infty) = (0, 0, 0)$ and $(\phi, \psi, \varphi)(+\infty) = (1, 1, 1)$.

In order to establish the existence of solutions for system (40), we introduce the notion of upper-solution as follows.

Definition 3.1. Given a constant $c > 0$, a continuous function $(\phi_+, \psi_+, \varphi_+)$ from $\mathbb{R}$ to $(0, 1] \times (0, 1] \times (0, 1]$ is called an upper-solution of (40), if the following hold:

(i) There exists some $\xi_+ \in \mathbb{R}$ such that $\psi_+(\xi_+) < 1, \varphi_+(\xi_+) < 1$;
(ii) $\phi_+(+\infty) = \psi_+(+\infty) = \varphi_+(+\infty) = 1$;
(iii) $\phi_+(\xi), \psi_+(\xi)$ and $\varphi_+(\xi)$ are differentiable a.e. in $\mathbb{R}$ such that
Proposition 3. Let 
\[
\begin{align*}
R & a similar argument as that for [15, Proposition 1] and the details are omitted here. 
R & then system \[(40)\] satisfying \[c\phi'_+ \geq \int_\mathbb{R} J_1(z)\phi_+(\xi-z)dz - \phi_+ + r_1\phi_+(1-b_{12} - b_{13} - \phi_+ + b_{12}\psi_+ + b_{13}\varphi_+),
\]
\[c\psi'_+ \geq \int_\mathbb{R} J_2(z)\psi_+(\xi-z)dz - \psi_+ + r_2(1-\psi_+)(b_{21}\phi_+ - \psi_+),
\]
\[c\varphi'_+ \geq \int_\mathbb{R} J_3(z)\varphi_+(\xi-z)dz - \varphi_+ + r_3(1-\varphi_+)(b_{31}\phi_+ - \varphi_+)
\]
hold a.e. in \(\mathbb{R}\).

Proposition 2. If there exists an upper-solution \((\phi_+, \psi_+, \varphi_+)\) of system \[(40)\] satisfying
\[
\phi_+(\cdot) = \psi_+(\cdot) = \varphi_+(\cdot) = 1 \text{ on } [0, +\infty).
\]
Then system \[(40)\] admits a solution \((\phi, \psi, \varphi)\) with \(\phi' > 0, \psi' > 0\) and \(\varphi' > 0\) in \(\mathbb{R}\).

Inspired by [15], we introduce the following truncated problem. Let
\[
(\phi, \psi, \varphi)(\xi) = (T^n_1(\phi, \psi, \varphi), T^n_2(\phi, \psi, \varphi), T^n_3(\phi, \psi, \varphi))(\xi) \quad \text{for } \xi \in [-n, 0],
\]
with the boundary conditions:
\[
\phi(\xi) = \psi(\xi) = \varphi(\xi) = 1, \quad \text{for } \xi \in (0, +\infty),
\]
\[
\phi(\xi) = \psi(\xi) = \varphi(\xi) = \varepsilon, \quad \text{for } \xi \in (-\infty, -n),
\]
where \(\varepsilon \in [0, 1], n \in \mathbb{N}\) and
\[
T^n_i(\phi, \psi, \varphi)(\xi) = \int_{-\infty}^{-n} e^{-\mu(\xi-s)} \mu \varepsilon ds + \int_{-n}^{\xi} e^{-\mu(\xi-s)} H_i(\phi, \psi, \varphi)(s) ds, \quad i = 1, 2, 3.
\]
Note that \((\phi, \psi, \varphi)\) satisfies \[(40)\] on \((-n, 0)\) if it satisfies \[(41)\].

The following lemma can be easily obtained due to the monotonicity of operator \(T^n_i, i = 1, 2, 3\). We will not repeat the proof here. For the details, one can refer to [14 Lemma 2.2].

Lemma 3.2. Assume that \((J)\) and \((A)\) hold. Then for each \(n \in \mathbb{N}\) and \(\varepsilon \in [0, 1]\), there exists a unique function \((\phi^{n, \varepsilon}, \psi^{n, \varepsilon}, \varphi^{n, \varepsilon})\) from \(\mathbb{R}\) to \([\varepsilon, 1] \times [\varepsilon, 1] \times [\varepsilon, 1]\) satisfying \[(41)-(43)\] and having the following properties:
(i) \(\phi^{n, \varepsilon}, \psi^{n, \varepsilon}, \varphi^{n, \varepsilon} \in C^1((-\infty, 0)) \cap C((-\infty, 0]).\n\]
(ii) \((\phi^{n, \varepsilon}), (\psi^{n, \varepsilon}), (\varphi^{n, \varepsilon})' > 0\) on \((-n, 0)\) for any \(\varepsilon \in [0, 1]\).
(iii) \(\frac{d\phi^{n, \varepsilon}}{dx}, \frac{d\psi^{n, \varepsilon}}{dx}, \frac{d\varphi^{n, \varepsilon}}{dx} \geq e^{-\mu(\varepsilon+n)}\) for \(\xi \in [-n, 0]\).

To proceed further, we also recall the Helly’s lemma (see, e.g. [15 Proposition 2]).

Proposition 3. (Helly’s Lemma) Let \(\{U_n\}_{n \in \mathbb{N}}\) be a sequence of uniformly bounded and non-decreasing functions defined in \(\mathbb{R}\), then there exists a subsequence \(\{U_{n_i}\}\) of \(\{U_n\}\) and a non-decreasing function \(U\) such that \(U_{n_i} \to U\) as \(i \to \infty\) point-wise in \(\mathbb{R}\).

Thanks to Lemma 3.2 and Proposition 3 we can obtain Proposition 2 by applying a similar argument as that for [15 Proposition 1] and the details are omitted here.

Now we are ready to prove Theorem 1.4.
Proof of Theorem 1.4. Let

\[ c_0 := \max \left\{ \int_{\mathbb{R}} J_1(z)e^{-z}dz - 1 + r_1, \int_{\mathbb{R}} J_2(z)e^{-z}dz - 1 + r_2(b_{21} - 1), \int_{\mathbb{R}} J_3(z)e^{-z}dz - 1 + r_3(b_{31} - 1) \right\} \]

and

\[ \phi_+ (\xi) = \psi_+ (\xi) = \varphi_+ (\xi) = \min \{ e^\xi, 1 \}. \]

Then it is not hard to verify that \((\phi_+, \psi_+, \varphi_+)\) is an upper-solution of \((40)\) for \(c \geq c_0\). By Proposition \(2\) \((40)\) admits a solution \((\phi, \psi, \varphi)\) satisfying \(\phi' > 0, \psi' > 0\) and \(\varphi' > 0\). Therefore, the constant

\[ c_{\min} = \inf\{c > 0 \mid (40) \text{ has a solution } (c, \phi, \psi, \varphi) \text{ with } \phi' > 0, \psi' > 0 \text{ and } \varphi' > 0 \text{ in } \mathbb{R} \} \]

is well-defined and \(c_{\min} \geq 0\). Moreover, by Helly’s Lemma and a similar argument as that for \(15\) Theorem 2, we can prove the existence of traveling waves with \(c \geq c_{\min}\) and \(c_{\min} > 0\). Then the proof is complete.

Remark 2. We further estimate the minimal speed \(c_{\min}\) for system \((40)\). Define

\[ c_* := \inf_{\lambda > 0} \int_{\mathbb{R}} J_1(x)e^{-\lambda x}dx - 1 + r_1(1 - b_{12} - b_{13}) \]

By Theorem 1.4 \((40)\) admits a solution \((c, \phi, \psi, \varphi)\) for any \(c \geq c_{\min} > 0\). Since

\[ c \frac{\phi'(\xi)}{\phi(\xi)} = \int_{\mathbb{R}} J_1(x) \frac{\phi(\xi - x)}{\phi(\xi)} dx - 1 + r_1(1 - b_{12} - b_{13} - \phi(\xi) + b_{12} \psi(\xi) + b_{13} \varphi(\xi)). \]

Let \(Z(\xi) = \frac{\phi'(\xi)}{\phi(\xi)}\), then above equation becomes

\[ cZ(\xi) = \int_{\mathbb{R}} J_1(x)e^{\xi - \lambda x}Z(s)ds dx + B(\xi), \]

where \(B(\xi) = -1 + r_1(1 - b_{12} - b_{13}) - \phi(\xi) + b_{12} \psi(\xi) + b_{13} \varphi(\xi))\). In view of \(B(-\infty) = -1 + r_1(1 - b_{12} - b_{13})\), by Lemma 2.2 there exists \(\lambda > 0\) such that

\[ c\lambda = \int_{\mathbb{R}} J_1(x)e^{-\lambda x}dx - 1 + r_1(1 - b_{12} - b_{13}). \]

Therefore, \(c \geq c_*\) and hence \(c_{\min} \geq c_*\).

3.2. Asymptotic behavior of traveling wave solutions of system \((40)\). Consider the following characteristic equations:

\[ k_1(c, \lambda) = c\lambda - \int_{\mathbb{R}} J_1(x)e^{-\lambda x}dx + 1 - r_1(1 - b_{12} - b_{13}), \]

\[ k_j(c, \lambda) = c\lambda - \int_{\mathbb{R}} J_j(x)e^{-\lambda x}dx + 1 + r_j, \quad j = 2, 3. \]

Then under (A), it follows from Lemma 2.1 that for \(c \geq c_*\) \((c_*)\) is given in Remark 2\), \(k_1(c, \lambda) = 0\) admits two real roots \(0 < \lambda^- (c) \leq \lambda^+ (c)\) and \(k_j(c, \lambda) = 0\) has a real root \(\delta_j (c) > 0\) with \(j = 2, 3\).
Moreover, if δ > 6306, FANG-DI DONG, WAN-TONG LI AND JIA-BING WANG

Remark 4. If l holds, then
\( (i) \lim_{\xi \to -\infty} \phi' = \Lambda(c) = \{\lambda^- (c), \lambda^+ (c)\}, \)
\( (ii) \lim_{\xi \to -\infty} \psi' = \delta_2 (c), \text{ if } \delta_2 (c) \leq \Lambda(c), \)
\( (iii) \lim_{\xi \to -\infty} \varphi' = \delta_3 (c), \text{ if } \delta_3 (c) \leq \Lambda(c), \)

Moreover, if \( \delta_2 (c) > \Lambda(c) \) (resp., \( \delta_3 (c) > \Lambda(c) \)), then
\( \lim_{\xi \to -\infty} \phi = L_{11} > 0 \) (resp., \( \lim_{\xi \to -\infty} \phi = L_{12} > 0 \))
for some \( L_{11}, L_{12} > 0 \).

According to Corollary 1 we give some remarks about the assumption (M).

Remark 3. By the relationship between \( \hat{\phi}, \hat{\psi}, \hat{\varphi} \) and \( \phi, \psi, \varphi \), we have
\( l_1 := \lim_{\xi \to -\infty} \frac{\hat{\phi}}{1 - \psi}, \quad l_2 := \lim_{\xi \to -\infty} \frac{\hat{\phi}}{1 - \varphi} \)
exists and is equal to 0 or a positive number. Moreover, by Corollary 1 when either
(i) \( \lim_{\xi \to -\infty} \psi' = \delta_2 (c) \) and \( \lim_{\xi \to -\infty} \varphi' = \delta_3 (c) \) or
(ii) \( \Lambda(c) < \delta_2 (c) \) and \( \Lambda(c) < \delta_3 (c) \) holds, then \( l_1, l_2 > 0 \). Therefore (M) holds.

Remark 4. If \( J_1 (x) = J_2 (x) = J_3 (x) \) for all \( x \in \mathbb{R} \), or \( J_1 (x) \geq \max \{J_2 (x), J_3 (x)\} \) for all \( x \in \mathbb{R} \), we must have \( \Lambda(c) < \delta_2 (c) \) and \( \Lambda(c) < \delta_3 (c) \). Then (M) holds.

So far, we have obtained the asymptotic behavior of the solutions for (40) at \( -\infty \). Next, we want to investigate the behavior of wave tails near \( +\infty \). To start with, we study the following characteristic equations:
\( \kappa_j (c, \lambda) = c\lambda - \int_\mathbb{R} J_j (x) e^{-\lambda x} dx + 1 + b_j (b_j - 1), \quad j = 2, 3, \)
\( \kappa_1 (c, \lambda) = c\lambda - \int_\mathbb{R} J_1 (x) e^{-\lambda x} dx + 1 + r_1. \)

By (A) and Lemma 2.1 we can verify that for each \( c < 0 \), \( \kappa_j (c, \lambda) = 0 \) has one real root \( \lambda_j (c) > 0, j = 2, 3 \) and \( \kappa_1 (c, \lambda) = 0 \) admits a real root \( \lambda_1 (c) > 0 \).
and

Similarly, we can verify that

\[ (40) \]

Assume that (J) and (A) hold, and let

\[ \phi' = \int_R J_1(z) \phi(\xi - z) dz - \phi + r_1 \left( 1 - \phi \right) \left( b_{12} \psi + b_{13} \bar{\psi} - \phi \right), \quad \xi \in \mathbb{R}, \]

\[ -c \psi' = \int_R J_2(z) \psi(\xi - z) dz - \psi + r_2 \psi \left( 1 - \psi - b_{21} + b_{23} \bar{\psi} \right), \quad \xi \in \mathbb{R}, \]

\[ -c \bar{\psi}' = \int_R J_3(z) \bar{\psi}(\xi - z) dz - \bar{\psi} + r_3 \bar{\psi} \left( 1 - \bar{\psi} - b_{31} + b_{33} \phi \right), \quad \xi \in \mathbb{R}, \]

\[ \left( \phi, \psi, \bar{\psi} \right)(\xi) = \left( \phi(\xi), \psi(\xi), \bar{\psi}(\xi) \right), \]

\[ 0 \leq \phi, \psi, \bar{\psi} \leq 1, \quad \xi \in \mathbb{R}. \]

Set

\[ \tilde{f}_1(\psi, \bar{\psi}, \phi)(\xi) = r_2 \psi(\xi) \left( 1 - \psi(\xi) - b_{21} + b_{23} \bar{\psi}(\xi) \right), \]

\[ \tilde{f}_2(\psi, \bar{\psi}, \phi)(\xi) = r_3 \bar{\psi}(\xi) \left( 1 - \bar{\psi}(\xi) - b_{31} + b_{33} \phi(\xi) \right), \]

\[ \tilde{f}_3(\psi, \bar{\psi}, \phi)(\xi) = r_1 \left( 1 - \phi(\xi) \right) \left( b_{12} \psi(\xi) + b_{13} \bar{\psi}(\xi) - \phi(\xi) \right). \]

Similarly, we can verify that \( \tilde{f} = (\tilde{f}_1, \tilde{f}_2, \tilde{f}_3) \) satisfies (H1) and (H2-1). Then the following statements come from Theorem 1.1. Specifically,

\[ \lim_{\xi \to -\infty} \tilde{\psi}'(\xi) = \lambda_2(c), \quad \lim_{\xi \to -\infty} \tilde{\bar{\psi}}'(\xi) = \lambda_3(c), \quad \text{for any } \xi \in \mathbb{R} \quad (44) \]

\[ \lim_{\xi \to -\infty} \frac{\tilde{\phi}(\xi)}{\phi(\xi)} = \begin{cases} \lambda_1(c), & \text{if } \lambda_1(c) \leq \min\{\lambda_2(c), \lambda_3(c)\}, \\ \min\{\lambda_2(c), \lambda_3(c)\}, & \text{if } \lambda_1(c) \geq \min\{\lambda_2(c), \lambda_3(c)\}. \end{cases} \quad (45) \]

Furthermore,

\[ \frac{\tilde{\psi}'(\xi) + \tilde{\bar{\psi}}'(\xi)}{\phi(\xi)} \leq K_3, \quad \text{for any } \xi \in \mathbb{R} \quad \text{and some } K_3 > 0. \quad (46) \]

By the earlier rules, we have for any \( \xi \in \mathbb{R} \),

\[ \begin{cases} \phi(\xi) = \phi(\xi) = 1 - \phi(-\xi), \\ 1 - \psi(\xi) = \psi(\xi) = \psi(-\xi), \\ 1 - \varphi(\xi) = \varphi(\xi) = \varphi(-\xi), \end{cases} \quad (47) \]

which together with (44)-(46) lead to the following results.

**Corollary 2.** Assume that (J) and (A) hold, and let \( (c, \phi, \psi, \varphi) \) be a solution of (40). Then we have

\[ \lim_{\xi \to +\infty} \tilde{\psi}'(\xi) = \lambda_2(c), \quad \lim_{\xi \to +\infty} \frac{\psi'(\xi)}{1 - \psi(\xi)} = \lambda_3(c) \]

and

\[ \lim_{\xi \to +\infty} \frac{\phi'(\xi)}{1 - \phi(\xi)} = \begin{cases} \lambda_1(c), & \text{if } \lambda_1(c) \leq \min\{\lambda_2(c), \lambda_3(c)\}, \\ \min\{\lambda_2(c), \lambda_3(c)\}, & \text{if } \lambda_1(c) \geq \min\{\lambda_2(c), \lambda_3(c)\}. \end{cases} \]

Moreover, there exists \( K_3 > 0 \) such that

\[ \frac{(1 - \psi(\xi)) + (1 - \varphi(\xi))}{1 - \phi(\xi)} \leq K_3, \quad \text{for any } \xi \in \mathbb{R}. \]
3.3. Existence of entire solutions of system \(^{(10)}\). This subsection is devoted to the existence of front-like entire solutions for \(^{(10)}\). In particular, the limiting argument of a sequence of Cauchy problems combining comparison principle and upper/lower-solutions method are used in solving this issue.

3.3.1. Preliminaries. Let \( u^*(x, t) := u(x, t), v^*(x, t) := 1 - v(x, t) \) and \( w^*(x, t) := 1 - w(x, t) \). Then \(^{(10)}\) yields

\[
\begin{align*}
  u_t^* &= (J_1 * u^* - u^*) + r_1 u^*(1 - b_{12} - b_{13} - u^* + b_{12}v^* + b_{13}w^*), \\
  v_t^* &= (J_2 * v^* - v^*) + r_2(1 - v^*)(b_{21}u^* - v^*), \\
  w_t^* &= (J_3 * w^* - w^*) + r_3(1 - w^*)(b_{31}u^* - w^*). \\
\end{align*}
\]

\( (48) \)

We just study the entire solutions of \(^{(48)}\), since \(^{(48)}\) and \(^{(10)}\) are equivalent. Consider the following initial value problem:

\[
\begin{align*}
  u_t^* &= (J_1 * u^* - u^*) + r_1 u^*(1 - b_{12} - b_{13} - u^* + b_{12}v^* + b_{13}w^*), \\
  v_t^* &= (J_2 * v^* - v^*) + r_2(1 - v^*)(b_{21}u^* - v^*), \\
  w_t^* &= (J_3 * w^* - w^*) + r_3(1 - w^*)(b_{31}u^* - w^*), \\
  u^*(x, 0) &= u_0^*(x), v^*(x, 0) = v_0^*(x), w^*(x, 0) = w_0^*(x),
\end{align*}
\]

where \( x \in \mathbb{R}, t > 0 \).

In what follows, we use the usual notations for the standard ordering in \( \mathbb{R}^3 \). That is, for \( u = (u_1, u_2, u_3) \) and \( v = (v_1, v_2, v_3) \), we define \( u \leq v \) if \( u_i \leq v_i \), \( i = 1, 2, 3 \), and \( u < v \) if \( u_i \leq v_i \), \( i = 1, 2, 3 \), but \( u \neq v \). If \( u \leq v \), we denote \( (u, v) = \{ w \in \mathbb{R}^3 : u < w < v \} \); \( [u, v) = \{ w \in \mathbb{R}^3 : u < w \leq v \} \); and \( [u, v] = \{ w \in \mathbb{R}^3 : u \leq w \leq v \} \). Let \( || \cdot || \) denote the Euclidean norm in \( \mathbb{R}^3 \).

Let \( X = \text{BUC}(\mathbb{R}, \mathbb{R}^3) \) be the Banach space of all bounded and uniformly continuous functions from \( \mathbb{R} \) to \( \mathbb{R}^3 \) with the supremum norm \( || \cdot ||_X \). Let \( X^+ = \{ V = (u, v, w) \in X : u(x) \geq 0, v(x) \geq 0, w(x) \geq 0, x \in \mathbb{R} \} \). It’s easy to see that \( X^+ \) is a closed cone of \( X \). For any \( V_1, V_2 \in X \), we write \( V_1 \leq_X V_2 \) if \( V_2 - V_1 \in X^+ \). For \( V_1, V_2 \in X \) with \( V_1 \leq_X V_2 \), we denote \( [V_1, V_2]_X = \{ V \in X : V_1 \leq_X V \leq_X V_2 \} \).

For \( V = (u, v, w) \in X \), we define \( T_1(t)u = e^{-\mu_1 t}u, T_2(t)v = e^{-\mu_2 t}v \) and \( T_3(t)w = e^{-\mu_3 t}w \), where \( \mu_1 = 1 + r_1(1 + b_{12} + b_{13}), \mu_2 = 1 + r_2(1 + b_{21}) \) and \( \mu_3 = 1 + r_3(1 + b_{31}) \).

We rewrite \(^{(49)}\) as the following integral form:

\[
\begin{align*}
  u^*(x, t) &= T_1(t)u_0^*(x) + \int_0^t T_1(t-s)[(J_1 * u^*)(x, s) + \mu_1 u^*(x, s) - u^*(x, s)]ds, \\
  v^*(x, t) &= T_2(t)\tilde{v}_0^*(x) + \int_0^t T_2(t-s)[(J_2 * v^*)(x, s) + \mu_2 v^*(x, s) - v^*(x, s)]ds, \\
  w^*(x, t) &= T_3(t)\tilde{w}_0^*(x) + \int_0^t T_3(t-s)[(J_3 * w^*)(x, s) + \mu_3 w^*(x, s) - w^*(x, s)]ds,
\end{align*}
\]

which is equivalent to

\[
\Psi(t) = T(t)Z + \int_0^t T(t-s)B(\Psi(s))ds,
\]
where

\[
\Psi(t) = \begin{pmatrix}
u^*(t) \\
v^*(t) \\
w^*(t)
\end{pmatrix}, \quad T(t) = \begin{pmatrix}
T_1(t) & 0 & 0 \\
0 & T_2(t) & 0 \\
0 & 0 & T_3(t)
\end{pmatrix}, \quad Z = \begin{pmatrix}
u_0^* \\
v_0^* \\
w_0^*
\end{pmatrix}
\]

and

\[
B(\Psi) = \begin{pmatrix}
J_1 \ast u^* + \mu_1 u^* - u^* + r_1 u^* (1 - b_{12} - b_{13} - u^* + b_{12} v^* + b_{13} w^*) \\
J_2 \ast v^* + \mu_2 v^* - v^* + r_2 (1 - v^*) (b_{21} u^* - v^*) \\
J_3 \ast w^* + \mu_3 w^* - w^* + r_3 (1 - w^*) (b_{31} u^* - w^*)
\end{pmatrix}.
\]

**Definition 3.3.** A continuous function \( \Psi = (u, v, w) : [\tau, T) \to X, \tau < T \), is called an upper-solution (a lower-solution) of (48) on \((\tau, T)\) if for any \( \tau < T, \Psi \) is an upper-solution (a lower-solution) of (48) on \([\tau, T)\).

**Lemma 3.4.** For any \( Z \in [0, 1]^3 \) with \( 0 := (0, 0, 0) \) and \( 1 := (1, 1, 1) \), (49) has a unique classical solution \( \Psi(x, t; Z) = (u^*(x, t; Z), v^*(x, t; Z), w^*(x, t; Z)) \) on \((x, t) \in \mathbb{R} \times [0, +\infty) \) with \( \Psi(x, 0; Z) = Z \) and \( 0 \leq \Psi(x; t; Z) \leq 1 \) for \( x \in \mathbb{R}, t \geq 0 \).

Furthermore, for any pair of upper-solution \( \Psi^+(x, t) \) and lower-solution \( \Psi^-(x, t) \) of (48) on \([0, +\infty) \) with \( \Psi^+(x, 0) \geq \Psi^-(x, 0) \) and \( 0 \leq \Psi^-(x, t), \Psi^+(x, t) \leq 1 \) for \( x, t \in \mathbb{R} \times [0, +\infty) \), there holds \( 0 \leq \Psi^-(x, t) \leq \Psi^+(x, t) \leq 1 \) for all \( (x, t) \in \mathbb{R} \times [0, +\infty) \).

3.3.2. Upper-lower solutions. As a consequence of Corollaries 1 and 2 we have following lemma which is essential in constructing lower/upper-solutions of (48).

**Lemma 3.5.** Assume that \((J), (A)\) and \((M)\) hold, and let \((\epsilon, \phi, \psi, \varphi)\) be a solution of (49). Then there exists \( \mu_1 > 0, m > 0 \) and \( M > 0 \) such that

\[
0 < u(\xi) \leq Me^{\mu_1 \xi}, \quad \text{for any } \xi \leq 1,
\]

\[
m \leq \frac{u'(\xi)}{u(\xi)} \leq M, \quad \text{for any } \xi \leq 1,
\]

\[
m \leq \frac{u'(-\xi)}{1 - u(\xi)} \leq M, \quad \text{for any } \xi \geq -1,
\]

\[
m \leq \frac{1 - u(\xi)}{1 - \phi(\xi)} \leq M, \quad \text{for any } \xi \in \mathbb{R},
\]

and

\[
\frac{1 - u(\xi + s)}{1 - u(\xi)} \leq M, \quad \text{for any } \xi \in \mathbb{R}, s \in [-1, 1],
\]

\[
\frac{u(\xi + s)}{u(\xi)} \leq M, \quad \text{for any } \xi \in \mathbb{R}, s \in [-1, 1],
\]

where \( u = \phi, \psi, \varphi. \)
Proof. The proofs of (50)-(53) follow from Corollaries 1 and 2 directly and we omit the details here.

Note that
\[
1 - u(\xi + s) = e^{-\int_\xi^{\xi+s} \frac{u'(t)}{u(t)} dt}, \quad u(\xi + s) = e^{\int_\xi^{\xi+s} \frac{u'(t)}{u(t)} dt}
\]
and \(u(-\infty) = 0\). Then we can obtain (54) and (55) by (52) and (51), respectively. This completes the proof.

Lemma 3.6. Assume that \((J), (A)\) and \((M)\) hold, and let \((c_i, \phi_i, \psi_i, \varphi_i)\) with \(i = 1, 2\) be the solutions of (40). Define

\[
B_1(y, p) := \phi_1(y + p)\psi_2(-y + p)(1 - \phi_2(-y + p))(1 - \psi_1(y + p)),
B_2(y, p) := \phi_2(-y + p)\psi_1(y + p)(1 - \phi_1(y + p))(1 - \psi_2(-y + p)),
D_1(y, p) := \phi_1(y + p)\psi_2(-y + p)(1 - \phi_2(-y + p))(1 - \varphi_1(y + p)),
D_2(y, p) := \phi_2(-y + p)\psi_1(y + p)(1 - \phi_1(y + p))(1 - \varphi_2(-y + p)),
E_1(y, p, z) := (\psi_2(-y + p + z) - \psi_2(-y + p))(\phi_1(y + p) - \varphi_1(y + p - z)),
E_2(y, p, z) := (\varphi_2(-y + p) - \varphi_2(-y + p - z))(\phi_1(y + p + z) - \varphi_1(y + p)),
A(y, p) := \phi'_1(y + p)(1 - \phi_2(-y + p)) + \phi'_2(-y + p)(1 - \phi_1(y + p)),
\]

where \(z \in [0, 1]\). Then there exists \(M_1 > 0\) such that for any given \(p < 0\), we have

\[
\frac{B_1(y, p)}{A(y, p)} \leq M_1 e^{\mu_1 p}, \quad y \in \mathbb{R}, \quad (56)
\]
\[
\frac{B_2(y, p)}{A(y, p)} \leq M_1 e^{\mu_1 p}, \quad y \in \mathbb{R}, \quad (57)
\]
\[
\frac{D_1(y, p)}{A(y, p)} \leq M_1 e^{\mu_1 p}, \quad y \in \mathbb{R}, \quad (58)
\]
\[
\frac{D_2(y, p)}{A(y, p)} \leq M_1 e^{\mu_1 p}, \quad y \in \mathbb{R}, \quad (59)
\]
\[
\frac{E_1(y, p, z)}{A(y, p)} \leq M_1 e^{\mu_1 p}, \quad y \in \mathbb{R}, \quad z \in [0, 1], \quad (60)
\]
\[
\frac{E_2(y, p, z)}{A(y, p)} \leq M_1 e^{\mu_1 p}, \quad y \in \mathbb{R}, \quad z \in [0, 1]. \quad (61)
\]

Proof. We only prove (56), (58) and (60), since the others can be obtained similarly.

As \(p < 0\), we divide \(\mathbb{R}\) into four intervals: \((-\infty, p], [p, 0], [0, -p] \) and \([-p, \infty)\). For simplicity, we use \(\tilde{C}_i\) and \(C_i\) with \(i = 1, 2, \cdots, 8\) to denote some positive constants.

For \(y \in (-\infty, p]\), by (52), (53) and (50), we have

\[
\frac{B_1(y, p)}{A(y, p)} \leq \frac{\phi_1(y + p)\psi_2(-y + p)(1 - \phi_2(-y + p))(1 - \psi_1(y + p))}{\phi'_2(-y + p)(1 - \phi_1(y + p))} \\
\leq \frac{\phi_1(y + p)\psi_2(-y + p)}{\phi'_2(-y + p)} \frac{1 - \phi_2(-y + p)}{1 - \phi_1(y + p)} \\
\leq \tilde{C}_1 \phi_1(y + p) \\
\leq C_1 e^{\mu_1 p}.
\]
For $y \in [p, 0]$, by (53), (51), (50) and (M), we have
\[
\frac{B_1(y, p)}{A(y, p)} \leq \frac{\phi_1(y + p)\psi_2(-y + p)(1 - \phi_2(-y + p))(1 - \psi_1(y + p))}{\phi'_1(y + p)(1 - \phi_1(y + p))}
\]
\[\leq \phi_1(y + p)(1 - \phi_2(-y + p))\frac{\psi_2(-y + p) 1 - \psi_1(y + p)}{\phi'_2(-y + p)(1 - \phi_1(y + p))},\]
\[\leq \bar{C}_2\phi_1(y + p)\frac{\psi_2(-y + p) \phi_2(-y + p)}{\phi_2(-y + p) \phi'_2(-y + p)},\]
\[\leq C_2e^{\mu_1 p}.
\]
For $y \in [0, -p]$, by (51) and (50), we have
\[
\frac{B_1(y, p)}{A(y, p)} \leq \frac{\phi_1(y + p)\psi_2(-y + p)(1 - \phi_2(-y + p))}{\phi'_1(y + p)(1 - \phi_2(-y + p))}
\]
\[= \psi_2(-y + p)\frac{\phi_1(y + p)}{\phi'_1(y + p)}\]
\[\leq \bar{C}_3\psi_2(-y + p)\]
\[\leq C_3e^{\mu_1 p}.
\]
For $y \in [-p, \infty)$, by (52), (50) and (53), we have
\[
\frac{B_1(y, p)}{A(y, p)} \leq \frac{\psi_2(-y + p)(1 - \phi_2(-y + p))(1 - \psi_1(y + p))}{\phi'_1(y + p)(1 - \phi_2(-y + p))}
\]
\[= \psi_2(-y + p)\frac{1 - \psi_1(y + p)}{\phi'_1(y + p)}\]
\[= \psi_2(-y + p)\frac{1 - \psi_1(y + p) 1 - \phi_1(y + p)}{\phi_1(y + p) \phi'_1(y + p)}\]
\[\leq \bar{C}_4\psi_2(-y + p)\]
\[\leq C_4e^{\mu_1 p}.
\]
Thus (56) is valid when we take $M_1 \geq \max\{C_i | i = 1, 2, 3, 4\}$.
In fact, (58) can be proved by replacing $\psi_2$ with $\phi_2$ in the proof of (56) and we omit the details here.

We now prove (60). For $y \in (-\infty, p]$, by (52), (50), (51) and (54), we have
\[
\frac{E_1(y, p, z)}{A(y, p)} = \frac{(\phi_2(-y + p + z) - \phi_2(-y + p))(\phi_1(y + p) - \phi_1(y + p - z))}{\phi'_1(y + p)(1 - \phi_2(-y + p)) + \phi'_2(y + p)(1 - \phi_1(y + p))}
\]
\[\leq \frac{\phi'_2(-y + p + \eta_1)\phi'_1(y + p - \eta_1)}{\phi'_2(-y + p)(1 - \phi_1(y + p))}
\]
\[\leq \frac{\phi'_1(y + p - \eta_1)}{1 - \phi_1(0)} \frac{\phi'_2(-y + p + \eta_1)}{1 - \phi_2(-y + p + \eta_1)}
\]
\[\times \frac{1 - \phi_2(-y + p + \eta_1)}{1 - \phi_2(-y + p)} \frac{1 - \phi_2(-y + p)}{\phi'_2(-y + p)}\]
\[\leq \bar{C}_5\phi'_1(y + p - \eta_1)\]
\[\leq C_5e^{\mu_1 p} \text{ for some } 0 \leq \eta, \eta_1 \leq 1.
\]
For $y \in [p, 0]$, by (51), (50) and (55), we get
\[
\frac{E_1(y, p, z)}{A(y, p)} = \frac{(\phi_2(y + p + z) - \phi_2(y + p))(\phi_1(y + p) - \phi_1(y + p - z))}{\phi_1(y + p)(1 - \phi_2(y + p)) + \phi_2(y + p)(1 - \phi_1(y + p))}
\leq \frac{\phi_2(y + p + \eta_{21})\phi_1(y + p - \eta_{22})}{\phi_2(y + p)(1 - \phi_1(y + p))}
\leq \frac{\phi_1(y + p - \eta_{22})}{1 - \phi_1(0)} \frac{\phi_2(-y + p + \eta_{21})\phi_2(-y + p + \eta_{21})\phi_2(-y + p)}{\phi_2(-y + p + \eta_{22})\phi_2(-y + p)}
\leq C_6\phi_1(y + p - \eta_{22})
\leq C_6 e^{\mu_1 p}
\text{ for some } 0 \leq \eta_{21}, \eta_{22} \leq 1.
\]

For $y \in [0, -p]$, by (51), (50), we get
\[
\frac{E_1(y, p, z)}{A(y, p)} = \frac{(\phi_2(-y + p + z) - \phi_2(-y + p))(\phi_1(y + p) - \phi_1(y + p - z))}{\phi_1(y + p)(1 - \phi_2(-y + p)) + \phi_2(-y + p)(1 - \phi_1(y + p))}
\leq \frac{\phi_2(-y + p + \eta_{31})\phi_1(y + p - \eta_{32})}{(1 - \phi_2(-y + p))\phi_1(y + p)}
\leq \frac{\phi_2(-y + p + \eta_{31})\phi_1(y + p - \eta_{32})}{1 - \phi_2(0)} \frac{\phi_1(y + p - \eta_{32})\phi_1(y + p)}{\phi_1(y + p)}
\leq C_7\phi_1(-y + p + \eta_{31})
\leq C_7 e^{\mu_1 p}
\text{ for some } 0 \leq \eta_{31}, \eta_{32} \leq 1.
\]

For $y \in [-p, \infty)$, by (52), (50) and (54), we have
\[
\frac{E_1(y, p, z)}{A(y, p)} = \frac{(\phi_2(y + p + z) - \phi_2(y + p))(\phi_1(y + p) - \phi_1(y + p - z))}{\phi_1(y + p)(1 - \phi_2(y + p)) + \phi_2(y + p)(1 - \phi_1(y + p))}
\leq \frac{\phi_2(-y + p + \eta_{41})\phi_1(y + p - \eta_{42})}{(1 - \phi_2(-y + p))\phi_1(y + p)}
\leq \frac{\phi_2(-y + p + \eta_{41})\phi_1(y + p - \eta_{42})}{1 - \phi_2(0)} \frac{\phi_1(y + p - \eta_{42})}{1 - \phi_1(y + p)}
\leq C_8\phi_2(-y + p + \eta_{41})
\leq C_8 e^{\mu_1 p}
\text{ for some } 0 \leq \eta_{41}, \eta_{42} \leq 1.
\]

Then (60) holds by taking $M_1 \geq \max\{C_i|i = 5, 6, 7, 8\}$. Now we complete the proof. \hfill \square

**Lemma 3.7.** Assume that (J), (A) and (M) hold, and let $(c_i, \phi_i, \psi_i, \varphi_i)$ be a solution of (40), $i = 1, 2$. Define
\[
F(y, p, z; \zeta) := [c_1(y + p + z) - c_1(y + p)]|\zeta_2(y + p) - \zeta_2(y + p - z)|,
G(y, p, z; \zeta) := [c_1(y + p + z) - c_1(y + p - z)]|\zeta_2(y + p + z) - \zeta_2(y + p)|,
H(y, p, \zeta) := c_1(y + p)|\zeta_2(y + p) - \zeta_2(y + p - z)|[1 - c_1(y + p)]\zeta_2(y + p),
I(y, p, \zeta) := c_1(y + p)[1 - \zeta_2(y + p)] + c_2(-y + p)[1 - (1 - \zeta_2(y + p))],
\]
where $z \in [0, 1]$, $\zeta = \psi, \varphi$. Then there exists $M_2 > 0$ such that for any given $p < 0$, we have
\[
\frac{F(y, p; \zeta)}{I(y, p; \zeta)} \frac{G(y, p; \zeta)}{I(y, p; \zeta)} \frac{H(y, p; \zeta)}{I(y, p; \zeta)} \leq M_2 e^{\mu_1 p}, \quad y \in \mathbb{R}.
\]
Proof. The proof is similar to that of Lemma 3.6 and then we omit the details here.

Before constructing upper/lower-solutions, we transform (10) into a new system by letting \( y = x + \frac{c_1 + c_2}{2} t \), and \( (u, v, w)(y, t) = (u, 1 - v, 1 - w)(x, t) \),

\[
\begin{cases}
L_1(u, v, w) := u_t + \frac{c_1 - c_2}{2} u_y - J_1 * u + u - f_1(u, v, w) = 0, (y, t) \in \mathbb{R}^2, \\
L_2(u, v, w) := v_t + \frac{c_1 - c_2}{2} v_y - J_2 * v + v - f_2(u, v, w) = 0, (y, t) \in \mathbb{R}^2, \\
L_3(u, v, w) := w_t + \frac{c_1 - c_2}{2} w_y - J_3 * w + w - f_3(u, v, w) = 0, (y, t) \in \mathbb{R}^2,
\end{cases}
\tag{62}
\]

where

\[
\begin{align*}
f_1(u, v, w) &= r_1 u (1 - u - b_12(1 - v) - b_{13}(1 - w)), \\
f_2(u, v, w) &= r_2 (1 - v)(b_{21} u - v), \\
f_3(u, v, w) &= r_3 (1 - w)(b_{31} u - w).
\end{align*}
\]

We call \( (\pi, \varphi, \psi) \) an upper-solution of (62) on \( \mathbb{R} \times [T_1, T_2] \) if \( L_j(\pi, \varphi, \psi)(y, t) \geq 0 \) for any \( (y, t) \in \mathbb{R} \times [T_1, T_2] \) and \( j = 1, 2, 3 \). Similarly, a lower-solution \( (\underline{u}, \underline{v}, \underline{w}) \) is defined by reversing the above inequalities.

As in [18], consider the following initial value problem

\[
p'(t) = \frac{c_1 + c_2}{2} + Le^{\mu_1 p(t)}, \quad t \leq 0 \quad \text{and} \quad p(0) = p_0 < 0,
\]

where \( \mu_1 \) is given by Lemma 3.5 and \( L \gg 1 \) is to be determined. Moreover, we have

\[
p(t) = p_0 + \frac{c_1 + c_2}{2} t - \frac{1}{\mu_1} \ln \left( 1 + \frac{2L}{c_1 + c_2} e^{\mu_1 p_0} \left( 1 - e^{-\frac{c_1 + c_2}{2} \mu_1 t} \right) \right) < 0, \quad t \leq 0,
\]

and

\[
\lim_{t \to -\infty} \left( p(t) - \frac{c_1 + c_2}{2} t \right) = -\frac{1}{\mu_1} \ln \left( e^{-\mu_1 p_0} + \frac{2L}{c_1 + c_2} \right) < 0. \tag{63}
\]

In the following, we give some useful inequalities on the reaction terms:

\[
\begin{align*}
f_1(\phi_1 + \phi_2 - \phi_1 \phi_2, \psi_1 + \psi_2 - \psi_1 \psi_2, \varphi_1 + \varphi_2 - \varphi_1 \varphi_2) &= (1 - \phi_1) f_1(\phi_1, \psi_1, \varphi_1) \\
- (1 - \phi_1) f_1(\phi_2, \psi_2, \varphi_2) \\
= & r_1 \left[ -\phi_1 \phi_2 (1 - \phi_1)(1 - \phi_2) - (\phi_1 + \phi_2 - \phi_1 \phi_2)[b_{12}(1 - \psi_1)(1 - \psi_2)
+ b_{13}(1 - \varphi_1)(1 - \varphi_2) + b_{12}(1 - \varphi_1)(1 - \psi_1) + b_{13}(1 - \varphi_2)(1 - \psi_1)
+ b_{13}(1 - \varphi_2)(1 - \psi_1) + b_{13}(1 - \varphi_1)(1 - \psi_2)]
\right] \\
\leq & r_1 b_{12}[\phi_1 \psi_2(1 - \phi_2)(1 - \psi_1) + \phi_2 \psi_1(1 - \phi_1)(1 - \psi_2)] \\
+ r_1 b_{13}[\phi_1 \varphi_2 (1 - \phi_2)(1 - \varphi_1) + \phi_2 \varphi_1(1 - \phi_1)(1 - \varphi_2)], \tag{64}
\end{align*}
\]

\[
\begin{align*}
f_2(\phi_1 + \phi_2 - \phi_1 \phi_2, \psi_1 + \psi_2 - \psi_1 \psi_2) &= (1 - \psi_1)f_2(\phi_1, \psi_1, \varphi_1) - (1 - \psi_1)f_2(\phi_2, \psi_2) \\
= & r_2 \psi_1 \psi_2(1 - \psi_1)(1 - \psi_2) - b_{31} \phi_1 \phi_2(1 - \psi_1)(1 - \psi_2) \\
\leq & r_2 \psi_1 \psi_2(1 - \psi_1)(1 - \psi_2). \tag{65}
\end{align*}
\]

and

\[
\begin{align*}
f_3(\phi_1 + \phi_2 - \phi_1 \phi_2, \varphi_1 + \varphi_2 - \varphi_1 \varphi_2) &= (1 - \varphi_1)f_3(\phi_1, \varphi_1) - (1 - \varphi_1)f_3(\phi_2, \varphi_2) \\
= & r_3 \varphi_1 \varphi_2(1 - \varphi_1)(1 - \varphi_2) - b_{31} \phi_1 \phi_2(1 - \varphi_1)(1 - \varphi_2) \\
\leq & r_3 \varphi_1 \varphi_2(1 - \varphi_1)(1 - \varphi_2). \tag{66}
\end{align*}
\]

Define

\[
U(v_1, v_2)(y, t) = v_1(y + p(t)) + v_2(-y + p(t)) - v_1(y + p(t))v_2(-y + p(t)).
\]
Lemma 3.8. Assume that \((J), (A)\) and \((M)\) hold and \((c_i, \phi_i, \psi_i, \varphi_i)\) are solutions of \([40]\), \(i = 1, 2\). Let \((\overline{u}, \overline{v}, \overline{w})(y, t) = (U(\phi_1, \phi_2), U(\psi_1, \psi_2), U(\varphi_1, \varphi_2))(y, t)\) and \(L \geq \max\{M_1(1 + r_1b_{12} + r_1b_{13}), 2M_2r_2, 2M_3r_3\}\). Then \((\overline{u}, \overline{v}, \overline{w})(y, t)\) is an upper-solution of \([42]\) for any \(y \in \mathbb{R}\) and \(t \leq 0\).

Proof. Denote \(u_1 = u_1(y + p(t)), u_2 = u_2(-y + p(t))\) and \(u = \phi, \psi, \varphi\). By a direct calculation, we get

\[
L_1(\overline{u}, \overline{v}, \overline{w})(y, t)
= \left[p'(t) - \frac{c_1 + c_2}{2}\right] A(y, p(t)) - \int_{\mathbb{R}} J_1(z)[(\phi_2(\phi_2 - y + p(t)) - \phi_2(\phi_2 - y + p(t) + z))(\phi_1(y + p(t) - z) - \phi_1(y + p(t)))]dz
= \int_{0}^{1} J_1(z)\left[(\phi_2(\phi_2 - y + p(t)) - \phi_2(\phi_2 - y + p(t) + z))(\phi_1(y + p(t) - z) - \phi_1(y + p(t))\right)
\times (\phi_1(y + p(t) + z) - \phi_1(y + p(t)))\right)dz
= \int_{0}^{1} J_1(z)E_1(y, p(t), z)dz + \int_{0}^{1} J_1(z)E_2(y, p(t), z)dz
\leq \frac{1}{2} M_1 e^{p(t)} A(y, t) + \frac{1}{2} M_1 e^{p(t)} A(y, t) \quad \text{by} \quad [60] \quad \text{and} \quad [61]
= A(y, p(t)) M_1 e^{p(t)}
\]

and

\[
f_1(U(\phi_1, \phi_2), U(\psi_1, \psi_2), U(\varphi_1, \varphi_2)) - (1 - \phi_2)f_1(\phi_1, \psi_1, \varphi_1) - (1 - \phi_1)
\times f_1(\phi_2, \psi_2, \varphi_2)
= f_1(\phi_1, \phi_2 - \phi_1, \psi_1 + \psi_2 - \psi_1, \varphi_1 + \varphi_2 - \varphi_1, \varphi_2) - (1 - \phi_2)f_1(\phi_1, \psi_1, \varphi_1)
\times (1 - \phi_1)f_1(\phi_2, \psi_2, \varphi_2)
\leq r_1b_{12}[\phi_1\psi_2(1 - \phi_2)(1 - \psi_1) + \phi_2\psi_1(1 - \phi_1)(1 - \psi_2)]
\times r_1b_{13}[\phi_1\varphi_2(1 - \phi_2)(1 - \varphi_1) + \phi_2\varphi_1(1 - \phi_1)(1 - \varphi_2)]
= r_1b_{12}[B_1(y, p(t)) + B_2(y, p(t)) + r_1b_{13}[D_1(y, p(t)) + D_2(y, p(t))]
\leq A(y, p(t)) \left[r_1b_{12} M_1 e^{p(t)} + r_1b_{13} M_1 e^{p(t)}\right] \quad \text{by} \quad [64] \quad \text{and} \quad [50] - [59].
\]

Thus

\[
L_1(\overline{u}, \overline{v}, \overline{w})(y, t) \geq \left[p'(t) - \frac{c_1 + c_2}{2}\right] A(y, p(t)) - A(y, p(t)) M_1 e^{p(t)}
\times A(y, p(t)) \left[r_1b_{12} M_1 e^{p(t)} + r_1b_{13} M_1 e^{p(t)}\right]
= A(y, p(t)) \left[Le^{p(t)} - M_1 e^{p(t)} - r_1 M_1 e^{p(t)}(b_{12} + b_{13})\right]
\geq 0
\]
for \((y, t) \in \mathbb{R} \times (-\infty, 0]\) provided that \(L \geq M_1(1 + r_1 b_{12} + r_1 b_{13}).\)

Similarly, by (65), (66) and Lemma 3.7, we have
\[
\mathcal{L}_2(\overline{u}, \overline{v}, \overline{w})(y, t) \geq \left[ p'(t) - \frac{c_1 + c_2}{2} \right] I(y, p(t); \psi) - \int_0^1 J_2(z) G(y, p(t), z; \psi) dz\]
\[+ \int_0^1 J_2(z) F(y, p(t), z; \psi) dz - r_2 H(y, p(t); \psi) \geq I(y, p(t); \psi) \left[ L e^{\mu_1 p(t)} - 2 r_2 M_2 e^{\mu_1 p(t)} \right] \geq 0 \]
for \((y, t) \in \mathbb{R} \times (-\infty, 0]\) when \(L \geq 2 r_2 M_2.\) On the other hand,
\[
\mathcal{L}_3(\overline{u}, \overline{v}, \overline{w})(y, t) \geq \left[ p'(t) - \frac{c_1 + c_2}{2} \right] I(y, p(t); \varphi) - \int_0^1 J_2(z) G(y, p(t), z; \varphi) dz\]
\[+ \int_0^1 J_2(z) F(y, p(t), z; \varphi) dz - r_3 H(y, p(t); \varphi) \geq I(y, p(t); \varphi) \left[ L e^{\mu_1 p(t)} - 2 r_3 M_2 e^{\mu_1 p(t)} \right] \geq 0 \]
for \((y, t) \in \mathbb{R} \times (-\infty, 0]\) provided that \(L \geq 2 r_3 M_2.\)

The above progress shows that if \(L \geq \max\{M_1(1 + r_1 b_{12} + r_1 b_{13}), 2 M_2 r_2, 2 M_2 r_3\},\)
\((\overline{u}, \overline{v}, \overline{w})(y, t)\) is an upper-solution of (62) for any \(y \in \mathbb{R}\) and \(t \leq 0.\) Now the proof is complete.

Denote the solution of system (48), (10) and (62) by \((u^*, v^*, w^*)(x, t),\)
\((u, v, w)(x, t)\) and \((u, v, w)(y, t)\) with \(y = x + \frac{c_1 + c_2}{2} t,\) respectively. Then
\[
\begin{cases}
  u^*(x, t) = u(x, t) = u(y, t), \\
v^*(x, t) = 1 - v(x, t) = v(y, t), \\
w^*(x, t) = 1 - w(x, t) = w(y, t).
\end{cases}
\]
Hence, we further obtain that \((u^+, v^+, w^+)(x, t)\) with the following expressions:
\[
u^+(x, t) = \varphi_1 \left( x + \frac{c_1 + c_2}{2} t + p(t) \right) + \varphi_2 \left( -x - \frac{c_1 + c_2}{2} t + p(t) \right)
- \varphi_1 \left( x + \frac{c_1 + c_2}{2} t + p(t) \right) \varphi_2 \left( -x - \frac{c_1 + c_2}{2} t + p(t) \right),
\]
\[
v^+(x, t) = \psi_1 \left( x + \frac{c_1 + c_2}{2} t + p(t) \right) + \psi_2 \left( -x - \frac{c_1 + c_2}{2} t + p(t) \right)
- \psi_1 \left( x + \frac{c_1 + c_2}{2} t + p(t) \right) \psi_2 \left( -x - \frac{c_1 + c_2}{2} t + p(t) \right),
\]
\[
w^+(x, t) = \phi_1 \left( x + \frac{c_1 + c_2}{2} t + p(t) \right) + \phi_2 \left( -x - \frac{c_1 + c_2}{2} t + p(t) \right)
- \phi_1 \left( x + \frac{c_1 + c_2}{2} t + p(t) \right) \phi_2 \left( -x - \frac{c_1 + c_2}{2} t + p(t) \right),
\]
is an upper-solution of (48) for all \(x \in \mathbb{R}\) and \(t \leq 0.\)
Define
\[ U^-(v_1, v_2)(x, t) = \max\{v_1(x + c_1 t + \tau), v_2(-x + c_2 t + \tau)\} \]
with \( \tau = -\frac{1}{\mu_1} \ln \left\{ e^{-\mu_1 p_0} + \frac{2L}{c_1 + c_2} \right\} < 0 \) and set
\[ (u_-, v_-, w-) = (U^-(\phi_1, \phi_2), U^-(\psi_1, \psi_2), U^-(\varphi_1, \varphi_2)). \]
Then it is easy to verify that \((u_-, v_-, w-)\) is a lower-solution of (48) for all \(x \in \mathbb{R}\) and \(t \in \mathbb{R}\).

3.3.3. Existence. Based on the above preparing works, we are ready to show the existence of entire solutions for (10) and complete the proof of Theorem 1.5.

Proof of Theorem 1.5. We only prove the case of \(\theta_1 = \theta_2 = \tau\) since the general case can be obtained by shifting a suitable space and time such that \(\theta_1 = \theta_2 = \tau\) (see, e.g., [18]). Moreover, since (48) is a monotone system, then applying a standard method (see, e.g., [4, 18, 24]) and Lemma 3.4, we conclude that system (48) admits an entire solution \((u, v, w)(x, t)\) for all \(x \in \mathbb{R}\) and \(t \in \mathbb{R}\), which is continuous and differentiable with respect to \(t\).

Furthermore, we have
\[ (u_-, v_-, w_-)(x, t) \leq (u, v, w)(x, t) \leq (u^+, v^+, w^+)(x, t), \]
for all \(x \in \mathbb{R}, t \leq 0\),
\[ (u_-, v_-, w_-)(x, t) \leq (u, v, w)(x, t) \leq (1, 1, 1), \]
for all \(x \in \mathbb{R}, t \in \mathbb{R}\).

By (63), we have
\[ \lim_{t \to -\infty} \sup_{x \in \mathbb{R}} [\phi^+(x, t) - \phi^-(x, t)] = 0, \quad \phi = u, v, w. \]
Thus, \((u, v, w)(x, t)\) satisfy
\[ \lim_{t \to -\infty} \sup_{x \geq \frac{2c_1 - \tau}{c_1}} \{ |u(x, t) - \phi_1(x + c_1 t + \tau)| + |v(x, t) - \psi_1(x + c_1 t + \tau)| + |w(x, t) - \varphi_1(x + c_1 t + \tau)| \} = 0 \]
and
\[ \lim_{t \to -\infty} \sup_{x \leq \frac{2c_1 - \tau}{c_1}} \{ |u(x, t) - \phi_2(-x + c_2 t + \tau)| + |v(x, t) - \psi_2(-x + c_2 t + \tau)| + |w(x, t) - \varphi_2(-x + c_2 t + \tau)| \} = 0, \]
which combining with (67) and (47) imply that both (12) and (45) hold. Further, since the lower-solution is defined for all \(t \in \mathbb{R}\) and satisfies
\[ \lim_{t \to -\infty} \sup_{x \in \mathbb{R}} [1 - \phi^-(x, t)] = 0, \quad \phi = u, v, w, \]
we get that
\[ \lim_{t \to -\infty} \sup_{x \in \mathbb{R}} [\|1 - u(x, t)\| + \|1 - v(x, t)\| + \|1 - w(x, t)\|] = 0, \]
which together with (67) yield (44). Now we complete all the proof of the theorem.

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