Measure of positive and not completely positive single-qubit Pauli maps

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The time evolution of an initially uncorrelated system is governed by a completely positive (CP) map. More generally, the system may contain initial (quantum) correlations with an environment, in which case the system evolves according to a not-completely positive (NCP) map. It is an interesting question what the relative measure is for these two types of maps within the set of positive maps. After indicating the scope of the full problem of computing the true volume for generic maps acting on a qubit, we study the case of Pauli channels in an abstract space whose elements represent an equivalence class of maps that are identical up to a non-Pauli unitary. In this space, we show that the volume of NCP maps is twice that of CP maps.

I. INTRODUCTION

The elementary units of any quantum information processing task are positive maps acting on the system of interest. The time-evolution of open quantum systems are not unitary as opposed to that for closed systems. The temporal evolution of open quantum systems [1] are described by a dynamical map [2] acting on the set of states of the system of interest. The dynamical map should be linear, hermiticity and trace preserving and preserve the positivity for the states on which it acts on. A stronger version of positivity, named complete-positivity (CP) is often demanded for dynamical maps. Based on the phenomenological theory of relaxation of spins, it was argued that complete positivity was not a necessary requirement [3–5]. A clarification to the issue of complete positivity was brought forward in the series of papers by Pechukas and Alicki based on the concept of linear assignment maps [6–8]. For initially entangled system and environment, it was shown that the reduced dynamics can be not completely positive (NCP) [9, 10] and the notion of positivity and compatibility domains were proposed [9].
An interesting issue here would be the volume of different classes of qubit channels, which would quantitatively depend on specific parametrization schemes and corresponding measures. Some related work here: the volume of Pauli channels (or, generalized depolarizing channels) simulable with a one-qubit environment within the full space of Pauli channels is considered in [11]. A similar calculation for amplitude damping channels is addressed in [12]. Ref. [13] employs the Lebesgue measure to compute the fraction of unital channels in the space of all qubit channels. Ref. [14], using parametrization scheme developed in [15] considers the volume of CP Pauli channels within the space of all CP unital qubit channels. Ref. [16], employing the Choi-Jamiolkowski isomorphism between channels and extended states [17], derives the Hilbert-Schmidt (Euclidean) volumes of the set of positive maps acting on a qudit, and its nested subsets of decomposable maps, CP maps and superpositive maps.

Here, we consider the question of the volume of CP maps within the space of positive maps acting on a qubit. To the best of our knowledge, a volumetric analysis of any class of quantum channels inclusive of NCP maps has not been addressed so far. We are interested in addressing the relative volume of positive and completely positive trace-preserving maps acting on a qubit. After first considering the general problem, we highlight specific technical issues in implementing the task. In particular, these challenges relate to finding a suitable parametrization for channels, and furthermore, characterizing the space of all NCP maps. We show that these constraints require us to consider the more limited case of Pauli CP maps within the space of Pauli positive maps.

The paper is organized as follows. In Sec. II, we discuss the preliminaries and address the issue with the calculation of a volume measure in the space of generic CP maps acting on a qubit. After pointing out the difficulty in handling this general situation, we look into the case of unital maps in the associated Choi matrix representation in Sec. III. The geometrical ideas and a result on the eigenvalue spectra of NCP maps acting on a qubit are given in Sec. IV. We address the reason as to why we further restrict our discussion to the case of Pauli channels. Based on the eigenvalues of the Choi matrix representing the positive maps, we propose a measure of dynamical maps acting on a qubit, motivate it, and prove our main results in Sec. V. We conclude in Sec. VI.
II. CHOI JAMIOLKOWSKI ISOMORPHISM AND VOLUME OF THE SPACE OF QUBIT CHANNELS

Let $\mathcal{E}$ be a positive map acting on a qubit, represented by the state

$$\rho = \frac{1}{2}(1 + a_i \sigma) = \frac{1}{2} \begin{pmatrix} 1 + a_3 & a_1 - ia_2 \\ a_1 + ia_2 & 1 - a_3 \end{pmatrix}, \quad (1)$$

where vector $a = (a_1, a_2, a_3)$, with $|a| \leq 1$, is the Bloch vector. The space of all one qubit states corresponds to the set of all the points on or inside the “Bloch ball” which is the unit ball in the space $\mathbb{R}^3$ parametrized by the axes $a_1, a_2$ and $a_3$.

The map $\mathcal{E}$ can be represented by a 4 dimensional Hermitian matrix, usually referred to as the Choi matrix $[18]$. If the Choi matrix is positive, the map is CP and else NCP. The trace of the Choi matrix acting on a qubit is 2. Apart from the trace, the Choi matrix is equivalent to a valid state in 4 dimensions. This is the isomorphism between maps and states, usually referred to as the Choi-Jamiolkowski (CJ) isomorphism in literature. The required measure of one-qubit channels will then be the measure of two-qubit states. The two main technical issues we are confronted with is (1) to identify a proper parametrization for the volume of two-qubit states; (2) finding the proper generalization of the parameters to include NCP maps, which correspond to non-positive two-qubit states.

The density matrix in 4 dimensions can be expressed in terms of the generators of SU(4) Lie algebra. Following $[19]$, any $4 \times 4$ density matrix, $\tilde{\rho}$ can be parametrized using the generators of SU(4) Lie algebra, $\{\Lambda_i\}$ with 12 Euler angles $\alpha_i$ and three rotation angles $\theta_i$ as follows

$$\tilde{\rho} = e^{i\Lambda_3 \alpha_1} e^{i\Lambda_2 \alpha_2} e^{i\Lambda_3 \alpha_3} e^{i\Lambda_5 \alpha_5} e^{i\Lambda_7 \alpha_7} e^{i\Lambda_9 \alpha_9} e^{i\Lambda_{10} \alpha_{10}} e^{i\Lambda_{12} \alpha_{12}}$$

$$\times \left( \frac{1}{4} 1 + \frac{1}{2} (-1 + 2a^2)b^2 e \Lambda_3 + \frac{1}{2\sqrt{3}} (-2 + 3b^2)e^2 \Lambda_8 + \frac{1}{2\sqrt{6}} (-3 + 4c^2) \Lambda_{15} \right)$$

$$\times e^{-i\Lambda_2 \alpha_{12}} e^{-i\Lambda_3 \alpha_{11}} e^{-i\Lambda_5 \alpha_{10}} e^{-i\Lambda_3 \alpha_9} e^{-i\Lambda_4 \alpha_8} e^{-i\Lambda_5 \alpha_7} e^{-i\Lambda_6 \alpha_6} e^{-i\Lambda_4 \alpha_5} e^{-i\Lambda_3 \alpha_4} e^{-i\Lambda_4 \alpha_3} e^{-i\Lambda_2 \alpha_2} \times e^{-i\Lambda} (2)$$

where

$$a^2 = \sin^2(\theta_1),$$

$$b^2 = \sin^2(\theta_2),$$

$$c^2 = \sin^2(\theta_3), \quad (3)$$
with the ranges for the 12 $\alpha$ parameters and the three $\theta$ parameters given by

$$0 \leq \alpha_1, \alpha_3, \alpha_5, \alpha_7, \alpha_9, \alpha_{11} \leq \pi,$$

$$0 \leq \alpha_2, \alpha_4, \alpha_6, \alpha_8, \alpha_{10}, \alpha_{12} \leq \frac{\pi}{2},$$

$$\frac{\pi}{4} \leq \theta_1 \leq \frac{\pi}{2},$$

$$\cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \leq \theta_2 \leq \frac{\pi}{2},$$

$$\frac{\pi}{3} \leq \theta_3 \leq \frac{\pi}{2}.$$  \hspace{1cm} (4)

Based on this parametrization, a volumetric analysis could be performed based on the Haar measure of the SU(4) states. In other words, via the CJ isomorphism the volume of the space of CP/NCP maps can be evaluated in terms of positive/negative Choi matrices, in principle. But, as we can see, a calculation of volume in terms of the above mentioned Euler-angle parametrization is tedious and an analytical expression is hard to be obtained, let alone for NCP maps.

Therefore, we ask the question whether a volume issue could be addressed for a sub-class of maps acting on a qubit. A rather limited, and yet very important and large set of maps is the unital maps (which map a unit matrix back to itself).

III. UNITAL MAPS ACTING ON A QUBIT

The general form of the Choi matrix $\tilde{B}$ acting on a qubit which is unital and trace-preserving can be parametrized as follows. This is obtained by considering a 4 dimensional Hermitian matrix such that $\text{tr}_1\tilde{B} = 1 = \text{tr}_2\tilde{B}$. Note that $a$ is real and $x, y, z$ and $w$ are in general complex.

$$\tilde{B} = \begin{pmatrix} a & x & y & z \\ x^* & 1-a & w & -y \\ y^* & w^* & 1-a & -x \\ z^* & -y^* & -x^* & a \end{pmatrix}.$$ \hspace{1cm} (5)

Unfortunately, even in this case, although computing the volume of maps is comparatively easier than in the general case, there seems to be no way yet to bound the full volume of maps including the NCP case. We shall thus require a further restriction of the sub-class of Pauli channels, discussed below, for which we can use an existing result to compute the volume of both CP and NCP maps in an abstract, albeit well-motivated, space. The reason for not using the general form as in Eq. (5) is explained later in the manuscript.
Let us consider the following form of the Choi matrix $B$ representing a trace-preserving $E$ on a qubit,

$$B = \frac{1}{2} \begin{pmatrix} 1 + t_3 + x_3 & t_1 - t_2 & 0 & x_1 + x_2 \\ t_1 + t_2 & 1 - t_3 - x_3 & x_1 - x_2 & 0 \\ x_1 - x_2 & t_1 + t_2 & 1 - t_3 + x_3 \\ x_1 + x_2 & 0 & t_1 + t_2 & 1 - t_3 + x_3 \end{pmatrix}.$$  

(6)

This is obtained by considering the action of the map $E$ on the basis $\{\mathbb{1}, \sigma_i = 1, 2, 3\}$ as

$$E(\mathbb{1}) = \mathbb{1} + \sum_{i=1}^{3} t_i \sigma_i,$$

$$E(\sigma_i) = x_i \sigma_i,$$

(7)

where $\mathbb{1}$ is the identity matrix in two dimensions and $\sigma_i$ are the familiar Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

This means that the map scales each of the three independent directions with scaling factors $x_i$ and translates $t_i$ along the three directions. The dynamical matrix Eq. (6) can be easily written down by inspection, by using the definition of the action of the map as in Eq. (7) on Eq. (1). Eq. (6) represents a dynamical map acting on a qubit with affine shifts. Such a map is referred to as non-unital. Note that $\text{tr}_2 B = \mathbb{1}$, which means that the map is trace-preserving.

Restricting ourselves to the case of Pauli channels, we set $t_j \equiv 0$ in Eq. (6), so that

$$B = \frac{1}{2} \begin{pmatrix} 1 + x_3 & 0 & 0 & x_1 + x_2 \\ 0 & 1 - x_3 & x_1 - x_2 & 0 \\ x_1 - x_2 & 1 - x_3 & 0 \\ x_1 + x_2 & 0 & 0 & 1 + x_3 \end{pmatrix}.$$  

(8)

The eigenvalues of $B$ are

$$\lambda_1 = \frac{1}{2}(1 + x_1 - x_2 - x_3),$$

$$\lambda_2 = \frac{1}{2}(1 - x_1 + x_2 - x_3),$$

$$\lambda_3 = \frac{1}{2}(1 - x_1 - x_2 + x_3),$$

$$\lambda_4 = \frac{1}{2}(1 + x_1 + x_2 + x_3).$$  

(9)
IV. PAULI UNITAL MAPS, POSITIVITY AND GEOMETRY

The set of maps parametrized as in Eq. (8) is usually referred to as the Pauli channels. A Pauli channel has a Kraus representation as the convex combination of application of the four Pauli operators. However, the considerations of this and the next section would equally apply to any other sub-class of unital maps where the operators are a set of four orthogonal unitary operator basis elements. With the assumption of unitality, the map can be represented as an action on the Bloch vector as $a' = Ta$ where $T$ is a $3 \times 3$ matrix. $T$ can be diagonalized and the eigenvalues of $T$ are invariant under the unitary transformation. The matrix of eigenvalues is notated as $D$ in the subsequent discussions. For the unital map $B$ in Eq. (8), the diagonal matrix, $D$ has entries $\bar{x} = \{x_1, x_2, x_3\}$ which is equivalent to the fact that the $T$ matrix itself is diagonal.

The elements of $D$ specify a tetrahedron $T$ in the parameter space $\{x_1, x_2, x_3\}$. $T$ is a simplex, given by the convex hull of the points representing $\mathbb{1}$ and the Pauli matrices. The points forming the vertices of $T$ correspond to unitary maps, the edges and faces represent two and three operator maps respectively. Points inside $T$ need all four operators. Thus, $T$ represents the set of all Pauli channels. If $E$ has to be completely positive, then all $\Lambda_i$ in Eq. (9) must be positive semi-definite. This means that the scaling parameters $x_i$ have to be such that

$$|1 \pm x_3| \geq |x_1 \pm x_2| \Leftrightarrow \text{CP map},$$

$$|1 \pm x_3| \leq |x_1 \pm x_2| \Leftrightarrow \text{NCP map}.$$ (10)

Applying the general map on a general density matrix for a qubit, i.e, applying the map Eq. (6) in Eq. (1), one can see that for positivity, $|x_i| \leq 1$. Therefore, the space of positive and trace preserving (PTP) maps form a unit cube, irrespective of whether the map is unital or not. This is physically meaningful, as positivity is a statement about the action of the map, unlike complete positivity which is a statement about the map itself. If the eigenvalues of $\mathcal{B}$ are positive, then the map is complete positive (CP). If there is one eigenvalue which is negative, the map is not completely positive (NCP). We now state the relevant details and prove our results subsequently.

**Theorem 1.** (Wolf and Cirac [20]), For any positive map which is trace-preserving, the determinant of $T$ is contained in $[-1, 1]$.

Based on Theorem 1, we state our result on the bound on the eigenvalues of the Choi matrix associated with a Pauli channel.

**Lemma 1.** The eigenvalues of the Choi matrix, $B$ corresponding to any positive Pauli channel on a qubit which is trace preserving are bounded and cannot be greater than 2 in absolute value.
Proof: From Theorem 1, we have that $\prod x_i \in [-1, 1]$, from which it follows that

$$\forall i x_i \in [-1, 1].$$

(11)

Consider $\lambda_1$: in view of Eq. (9), it assumes the largest absolute value by setting $x_1 := \pm 1$, $x_2 := \mp 1$ and $x_3 := \mp 1$, with this value being 2. Repeating this for the other eigenvalues $\lambda_i$, we find that:

$$\forall i |\lambda_i| \leq 2,$$

(12)

meaning that any eigenvalue is bounded.

If the map is CP, given that the eigenvalues must sum to 2, it is very clear that no eigenvalue can be greater than 2, since all are positive. But, for NCP maps (wherein the dynamical matrix can take negative eigenvalues), this result is non-trivial. ■

For the form of $\tilde{B}$ as in Eq. (5), the map in the affine form gives the $T$ matrix as

$$T = \begin{pmatrix}
\text{Re}[w + z] & -\text{Im}[w + z] & 2\text{Re}[x] \\
\text{Im}[z - w] & \text{Re}[z - w] & 2\text{Im}[x] \\
2\text{Re}[y] & -2\text{Im}[y] & 2a - 1
\end{pmatrix}.
$$

(13)

From the determinant of the $T$ matrix as in Eq. (13) and the eigenvalues of $\tilde{B}$ of Eq. (5), one can easily note that for certain choices of the parameters of $\tilde{B}$, the eigenvalues can blow up to very high values. It should be noted that the problem arises only for NCP maps, where the eigenvalues can take negative values. For CPTP maps, the eigenvalues are still bounded. This makes the characterization of the set of positive maps corresponding to general qubit unital maps difficult and hence we consider only the special class of Pauli unital channels as parametrized by Eq. (8).

V. MEASURE OF THE MAP

We now parametrize all Pauli unital qubit maps in terms of the eigenvalues of their dynamical matrix, i.e., the set of three real numbers corresponding to the scaling factors $(x_1, x_2, x_3)$. We define the measure $\mu$ of a set of maps as the volume of the set in the above parameter space $C$ of eigenvalues of the dynamical matrix. That is, given set $S \in C$, the measure of $S$ is its volume in Cartesian coordinates up to a normalization factor $\kappa$

$$\mu(S) = \kappa \int \int \int_S dx dy dz,$$

(14)

where $\kappa$ is fixed so that $\mu(C) = 1$, i.e., the measure of the space of all positive trace-preserving (PTP) maps is unity. Later, we shall find that $\kappa = \frac{1}{8}$. Note that this space is not the physical
space. Operationally, the measure \( \mu(S) \) of a set \( S \) of maps in this space is the probability with which we would pick an element of \( S \) in the set of all PTP unital maps of a qubit.

Some properties of this parameter space are noted below.

**Lemma 2.** In the space \( \mathcal{C} \) of all Pauli PTP unital qubit maps, the measure \( \mu \) of unitaries is 0.

**Proof:** The measure of the tetrahedron \( \mathcal{T} \) is, by definition, \( \frac{8}{3} \). As noted earlier, the set \( \mathcal{U} \) of unitaries correspond to points forming the vertices of \( \mathcal{T} \). As their Cartesian volume is 0, it follows that \( \mu(\mathcal{U}) = 0 \).

Note that as the edges and faces correspond to maps with two and three Kraus operators, even these have zero measure. Only the set of points inside \( \mathcal{T} \), which require all four Kraus operators, have finite measure \( \mu \).

**Lemma 3.** Each point in the space \( \mathcal{C} \) corresponds to a map up to a non-Pauli unitary.

**Proof:** Each point in \( \mathcal{C} \) represents a map parametrized in terms of the eigenvalue spectrum of its dynamical matrix. Given a density matrix \( \rho \), let its spectral decomposition be \( \sum_j \Lambda_j |j\rangle \langle j| \) and let \( U \) be any unitary acting on it. Then, \( U \rho U^\dagger = \sum_j \Lambda_j |(U)j\rangle \langle (U)j| \), where \( |(U)j\rangle \equiv U |j\rangle \).

Clearly, the eigenvalue spectrum doesn’t change. It thus follows that the eigenvalues of the dynamical matrix are invariant when rotated by any given unitary acting on the qubit. This unitary equivalence can be used to establish an equivalence class of noise. However, note that when the unitary is a Pauli operator, it has the effect of permuting the eigenvalues. Excluding the Pauli from the unitaries that define our equivalence would entail that two channels are equivalent iff they differ by a unitary other than a Pauli operation. Thus, the set of unitary channels is represented by four vertices, corresponding to the Pauli qubit operations.

In the scope of this definition of parameter space \( \mathcal{C} \), we show that the ratios of the volume of the space of CPTP and NCPTP unital maps on a qubit represent the relative volume (measure).

**Lemma 4.** The volume of Pauli unital NCPTP maps is twice as large as that of CPTP maps.

**Proof:** Let \( V_{PTP}, V_{CPTP}, V_{NCPTP} \) denote the volumes of positive, completely-positive and non-completely positive trace preserving maps respectively in the above parameter space. Obviously, \( V_{PTP} = V_{CPTP} + V_{NCPTP} \). The volume of space of PTP maps is that of the cube with each side ranging in \([-1, 1]\) and is 8 units. By normalization, the measure of this cube is set to 1, so that \( \kappa \) in Eq. (14) is set to \( \frac{1}{8} \). The CPTP maps, determined by the condition (10), are contained in the
tetrahedron $T$, which is of volume $\kappa T = \frac{1}{3}$. The difference in volume $\kappa(8 - V_{CPTP}) = 16\kappa/3 = \frac{2}{3}$ corresponds to that of the complementary space of NCPTP maps. It thus follows that the volume of NCPTP maps is twice as large as that of CPTP maps.

In addition to giving a compact comparison of CP and NCP maps, this result can be useful for simulating PTP, CPTP or NCPTP maps, in light of the bounds given in Eq. (10). That is, a program may randomly pick three reals $x_j$ subject to these constraints, and this would define a PTP map up to a unitary.

VI. CONCLUSIONS

For long, the study of open systems remained confined to CP maps. With the recognition of the importance of initial correlations, non-markovian dynamics [21, 22] and improved experimental techniques, NCP maps are now studied ever more actively [23]. This motivates the question of how likely an arbitrary trace-preserving dynamical map is likely to be unitary, CP or NCP. Exploiting the CJ isomorphism, we transform this to the question of comparing the volume of two-qubit positive and non-positive states. This turns out to be rather hard to handle for two reasons: (1) dealing with the 15-parameter space required to characterize this set; (2) fixing the exact limits when NCP maps are included. The first problem becomes considerably simplified when we restrict to the class of unital channels, but the second problem persists. We make a further restriction, to the class of Pauli unital channels. The resulting volumetric analysis should be applicable to any situation where the Pauli set is replaced by another unitary qubit operator basis. In the context of qubit dynamics we defined a measure of dynamical maps based on the eigenvalues of the dynamical matrix. Physically, this corresponds to an abstract space where each element represents an equivalence of class of noisy channels up to a non-Pauli unitary. In this framework, we have investigated the relative measure of CP and NCP Pauli unital maps and showed that the volume of NCP maps is twice that of CP maps. Future work may consider issues related to bounding the space of NCP maps. Another direction would be to explore the possibility for two-qubit parametrization other than Eq. (2), possibly one based on Bloch ball representation.

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