Ruin probabilities under Sarmanov dependence structure

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Abstract
Our work aims to study the tail behaviour of weighted sums of the form \( \sum_{i=1}^{\infty} X_i \prod_{j=1}^{i} Y_j \), where \((X_i, Y_i)\) are independent and identically distributed, with common joint distribution bivariate Sarmanov. Such quantities naturally arise in financial risk models. Each \(X_i\) has a regularly varying tail. With sufficient conditions similar to those used by [1] imposed on these two sequences, and with certain suitably summable bounds similar to those proposed by [2], we explore the tail distribution of the random variable \(\sup_{n \geq 1} \sum_{i=1}^{n} X_i \prod_{j=1}^{i} Y_j\). The sufficient conditions used will relax the moment conditions on the \(\{Y_i\}\) sequence.

Keywords: Regular variation, product of random variables, ruin probabilities, Sarmanov distribution.

1. Introduction

Regularly varying distributions find several applications in areas of actuarial and financial mathematics, in the analysis of random coefficient linear processes such as ARMA and FARIMA, and in stochastic difference equations. We refer to [3] for the study of the insurance ruin problem. The development of the capital is described as the solution to a stochastic difference equation. The net losses over the years are independent and identically distributed with regularly varying tail. [4] consider a discrete-time risk model with dependent insurance and financial risks. If \(X_n\) denotes the insurance risk and \(Y_n\) the financial risk or the stochastic discount factor in time \(n\), then

\[
S_n = \sum_{i=1}^{n} X_i \prod_{j=1}^{i} Y_j
\]

represents the stochastic discount value of aggregate net losses up to time \(n\). In [4], the finite and infinite time ruin probabilities are analyzed.

A random variable \(X\) with tail distribution \(F\) is said to be regularly varying with index \(-\alpha\), with \(\alpha > 0\), if \(F(xy) \sim y^{-\alpha} F(x)\) as \(x \to \infty\), for all \(y > 0\). This is denoted by \(X \in RV_{-\alpha}\). Let \(\{X_n, n \geq 1\}\) be a sequence of independent and identically distributed random variables with regularly varying tails, and \(\{\Theta_n, n \geq 1\}\) be another sequence of random variables, not necessarily independent of \(\{X_n\}\). The almost sure convergence and tail behaviour of \(\sup_{n \geq 1} \sum_{i=1}^{n} \Theta_i X_i\) has been studied in the literature. Here and later, for two positive functions \(a(x)\) and \(b(x)\), we write \(a(x) \sim b(x)\) as \(x \to \infty\) if \(\lim_{x \to \infty} a(x)/b(x) = 1\).

The study of the almost sure finiteness of the infinite sum \(S_\infty = \sum_{i=1}^{\infty} X_i \prod_{j=1}^{i} Y_j\) has been a topic of sustained interest in the literature. The general problem has been addressed in [5] for the case when the sequences \(\{X_i\}\) and \(\{Y_i\}\) are independent and \(\{X_i\}\) an i.i.d. regularly varying sequence. See also [4] and [7].

We address our problem in two parts: first we analyze the behaviour of the product, and then the sum. The main result in this direction is given in [5], which proves that if \(X \in RV_{-\alpha}\) and \(\Theta\) independent of \(X\) satisfies \(E[\Theta^{\alpha}] < \infty\)
for some \( \epsilon > 0 \), then \( \Theta X \in RV_{-\epsilon} \) with \( P[\Theta X > x] \sim E[\Theta^\alpha]P[X > x] \) as \( x \to \infty \). This result was extended to finite and infinite sums in [3]. They showed that if \( \{X_i\} \) and \( \{\Theta_i\} \) are independent of each other, the \( X_i \)'s are i.i.d \( RV_{-\epsilon} \), and the \( \Theta_i \)'s satisfy some extra moment assumptions, then \( P \left[ \sum_{j=1}^{\infty} \Theta_j X_j > x \right] \sim P[X_1 > x] \sum_{j=1}^{\infty} E[\Theta_j^\alpha] \) as \( x \to \infty \).

\[ \square \] replaced the extra moment assumptions with other sufficient conditions so that \( P[\Theta X > x] \sim E[\Theta^\alpha]P[X > x] \) as \( x \to \infty \). This was again extended to the finite and infinite sum case by [2]. Motivated by the ruin model of [3] above, we restrict ourselves to the setup where \( \Theta_i = \prod_{j=1}^{i} Y_j \), for i.i.d. \( Y_j \).

We consider the finite time ruin probability by time \( n \), given by

\[
\Psi(n, x) = P \left[ \max_{1 \leq k \leq n} S_k > x \right],
\]
and the infinite time ruin probability by

\[
\Psi(x) = P \left[ \sup_{n \geq 1} S_n > x \right].
\]

1.1. Some useful classes of distributions

While classically, the insurance risk \( \{X_i\} \) and discount factor \( \{Y_i\} \) are assumed to be independent, [4] assumed that each \( (X_i, Y_i) \) follows a bivariate Sarmanov distribution, which is defined as follows.

**Definition 1.1.** The pair of random variables \((X, Y)\) is said to follow a bivariate Sarmanov distribution, if

\[
P(x, dy) = (1 + \theta \phi_1(x)\phi_2(y))F(dx)G(dy), \quad x \in \mathbb{R}, y \geq 0,
\]

where the kernels \( \phi_1 \) and \( \phi_2 \) are two real valued functions and the parameter \( \theta \) is a real constant satisfying

\[
E[\phi_1(X)] = E[\phi_2(Y)] = 0 \quad \text{and} \quad 1 + \theta \phi_1(x)\phi_2(y) \geq 0, \quad x \in D_X, y \in D_Y,
\]

where \( D_X \subset \mathbb{R} \) and \( D_Y \subset \mathbb{R}^+ \) are the supports of \( X \) and \( Y \), with marginals \( F \) and \( G \) respectively.

This class of bivariate distributions is quite wide, covering a large number of well-known copulas such as the Farlie-Gumbel-Morgenstern (FGM) copula, which is recovered by taking \( \phi_1(x) = 1 - 2F(x) \) and \( \phi_2(y) = 1 - 2G(y) \). We refer the reader to [10] for further discussion. A bivariate Sarmanov distribution is called proper if \( \theta \neq 0 \) and none of \( \phi_1 \) and \( \phi_2 \) vanishes identically. To study the dependence structure of Sarmanov distribution, we need to define the class of dominatedly tail varying distributions.

**Definition 1.2.** A random variable \( X \) with distribution function \( F \) is called dominatedly-tail-varying, denoted by \( X \in \mathcal{D} \), if for all \( 0 < y < 1, \lim \sup_{x \to \infty} \frac{F(xy)}{F(x)} < \infty \).

It is traditional to study the tail of the product of random variables in terms of the Breiman’s condition, which we strive to weaken. For that we need to state definitions of certain useful classes of distributions.

**Definition 1.3.** A random variable \( X \) is said to be long tailed and denoted by \( X \in \mathcal{L} \) if \( P[X > x] \sim P[X > x + y] \) as \( x \to \infty \), for any \( y \).

**Definition 1.4.** A non-negative function \( f \) is in the class \( S_\alpha \) and called a subexponential density if

\[
\lim_{x \to \infty} \int_0^x \frac{f(x-y)}{f(x)} f(y) dy = 2 \int_0^\infty f(u) du < \infty.
\]

If \( f \in S_\alpha \) is such that \( f(x) = P[U > x] \) for some random variable \( U \), we say that \( U \in \mathcal{S}' \).

**Definition 1.5.** A non-negative random variable \( T \) is in class \( S(\gamma) \), \( \gamma \geq 0 \), if as \( x \to \infty \), we have

\[
P[T > x + y] \to e^{-\gamma y} \quad \text{and} \quad \frac{P[T + T' > x]}{P[T > x]} \to 2E[e^{\gamma T}] < \infty,
\]

where \( T' \) is an i.i.d. copy of \( T \). For \( \gamma = 0 \), we get the class \( S \) of subexponential distributions.
The crucial property used by [4] is that the bivariate Sarmanov dependence is not very strong. For this, they further assumed that the generic bivariate Sarmanov random vector $(X, Y)$ satisfies

$$X \in RV_{-\alpha} \quad \text{and} \quad \lim_{x \to \infty} \phi_1(x) = d_1. \quad (4)$$

These assumptions will also be made throughout this paper. If $(X, Y)$ is bivariate Sarmanov, then asymptotically, the product $XY$ has the same tail distribution as the product $XY_0$ where $X$ and $Y_0$ are independent and $Y_0$ is obtained through a change of measure. It has the distribution function $G_0$ with

$$G_0(dy) = P[Y_0^* \in dy] = (1 + \theta d_1 \phi_2(y))G(dy). \quad (5)$$

This is formalized in Lemma 3.1 of [4], but we need a less generalized version given in Theorem 1.6.

**Theorem 1.6.** Assume that $(X, Y)$ follows a bivariate Sarmanov distribution and (4) holds. Let $X^*$ and $Y^*$ be two independent random variables identically distributed as $X$ and $Y$ respectively, i.e. having marginals $F$ and $G$ respectively. Let $\overline{F}(x) = P[X^*Y^* > x]$. If now $H^* \in \mathcal{D}$ and $\overline{G}(x) = o(\overline{F}(x))$, then $P[XY > x] \sim P[X^*Y^*_0 > x]$, where $X^*, Y^*_0$ mutually independent and $Y^*_0$ has distribution $G_0$ as defined in (4).

[4] considered one of the conditions proposed by [1] on $(X, Y)$, and showed that

$$P[XY > x] \sim (E[Y^\alpha] + \theta d_1 E[\phi_2(Y)Y^\alpha])\overline{F}(x), \quad (6)$$

In Section 2 we show that [4] still holds under the remaining three conditions assumed by [1].

Under the same condition as used in establishing (6), [4] showed that the finite time ruin probability

$$\Psi(x, n) \sim \frac{1 - E[Y^\alpha]^n}{1 - E[Y^\alpha]}(E[Y^\alpha] + \theta d_1 E[\phi_2(Y)Y^\alpha])\overline{F}(x), \quad (7)$$

where they used the convention that $(1 - E[Y^\alpha]^n)/(1 - E[Y^\alpha]) = n$ when $E[Y^\alpha] = 1$. In section 3 we again extend (7) under the remaining three conditions of [1].

[4] showed that the infinite time ruin probability, assuming extra moments of $Y_j$s as in [2], satisfies

$$\Psi(x) \sim \frac{E[Y^\alpha] + \theta d_1 E[\phi_2(Y)Y^\alpha]}{1 - E[Y^\alpha]}\overline{F}(x). \quad (8)$$

In Section 4 we prove (8) assuming only the conditions in [1], and some uniform integrability assumptions.

### 2. Product results

We start this section with collecting the main product results of [1]. We first recall the complete characterization of slowly varying functions from Lemma 2.1 of [1] in our Lemma 2.1. We then state, in Theorem 2.2, the four sufficient conditions given in Propositions 2.1 through 2.3 of [1].

**Lemma 2.1.** Let $X$ be nonnegative with tail distribution $\overline{F} \in RV_{-\alpha}$. We write $\overline{F}(x) = x^{-\alpha}L(x)$, where $L$ is slowly varying. In this case, $L$ must be one of the following forms:

(i) $L(x) = c(x)$;

(ii) $L(x) = c(x)/P(V > \log x)$;

(iii) $L(x) = c(x)P(U > \log x)$;

(iv) $L(x) = c(x)P(U > \log x)/P(V > \log x)$;

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where $U$ and $V$ are long tailed random variables and $\lim_{x \to \infty} c(x) = c \in (0, \infty)$.

introduced the following four conditions, referred here as (DZ) conditions, enough to ensure Breiman-type results.

**Theorem 2.2.** Let $X$ be nonnegative with tail distribution $F \in RV_{-\alpha}$, and $Y$ be independent of $X$, satisfying $E(Y^\alpha) < \infty$ and $P(Y > x) = o(P(X > x))$ as $x \to \infty$. We write $\overline{F}(x) = x^{-\alpha}L(x)$, $L$ slowly varying. Consider the following conditions:

1. $\lim_{x \to \infty} \sup_{y \in [1, x]} L(y)/L(x) < \infty$;
2. $L$ is of type (ii) or (iii) and $L(OZ3)$.
3. $DZ1$. $\lim_{x \to \infty} \overline{F}(x) = \infty$.
4. $DZ2$. $\limsup_{x \to \infty} \frac{\overline{L}(x)}{L(x)} < \infty$.
5. $DZ3$. $\limsup_{x \to \infty} \frac{\overline{L}(x)}{L(x)} < \infty$.
6. $DZ4$. When $E[Y] = \infty$ or equivalently $E[X^\alpha] = \infty$, we define \( m(x) = \int_0^x v^\alpha F(sv) \to \infty \), and assume $P(Y > x) = o(P(X > x)/m(x))$ and $\limsup_{x \to \infty} \sup_{x \leq y \leq \alpha} L(y)/L(x) < \infty$.

If any one of the above conditions holds, then $P(XY > x) \sim E(Y^\alpha)P(X > x)$.

We need one more property of bivariate Sarmanov distribution, which is from Proposition 1.1 of [4].

**Lemma 2.3.** Assume that $(X, Y)$ follows a proper bivariate Sarmanov distribution. Then there exist two positive constants $b_1$ and $b_2$ such that $|\phi_1(x)| \leq b_1$ for all $x \in D_X$ and $|\phi_2(y)| \leq b_2$ for all $y \in D_Y$.

In the rest of the section, $X$ and $Y$ jointly follow bivariate Sarmanov, [4] holds and we additionally have

$$E[Y^\alpha] < \infty \quad \text{and} \quad P(Y > x) = o(P(X > x)) \Rightarrow \overline{G}(x) = o(\overline{F}(x)).$$

(9)

We further assume that any one of the last three (DZ) conditions $DZ2$, $DZ3$ and $DZ4$ holds, and investigate the behaviour of the product $XY$.

Let $X^*$ and $Y^*$ be two mutually independent random variables with distribution functions $F$ and $G$ respectively. Let $H_\alpha(x) = P[X^*Y^* > x]$. Let $Y_\alpha$ be the twisted version of $Y$ as given by [5]. Observe that by Lemma 2.3

$$\overline{G}(x) = \int_x^{\infty} (1 + \theta d_1|\phi_2(y)|)dG(y) \leq (1 + |\theta d_1|b_2)\overline{G}(x) = o(\overline{F}(x)), \quad \text{(10)}$$

and

$$E[(Y^*)^\alpha] = \int_0^{\infty} y^\alpha (1 + \theta d_2\phi_1(y))dG(y) \leq (1 + |\theta d_2|b_2)E[(Y^\alpha)] < \infty.$$

In Lemma 2.4 we show how any (DZ) condition that holds for $(X, Y)$, also extends to $(X^*, Y_\alpha)$. As a result, using Theorem 2.2 we are able to conclude that

$$P(XY > x) \sim [E(Y^\alpha) + \theta d_1E(\phi_2(Y)Y^\alpha)]/\overline{F}(x).$$

**Lemma 2.4.** Let any one of the four (DZ) conditions hold for $(X, Y)$. Then it also holds for $(X^*, Y_\alpha)$.

**Proof.** Because each of the four (DZ) conditions involves only the properties of the marginal distributions of $X$ and $Y$, hence if they hold for $(X, Y)$, they also hold for $(X^*, Y_\alpha)$. For this same reason, $DZ1$ and $DZ2$ extend to $(X^*, Y_\alpha)$, and because of (10), (DZ3) also extends to $(X^*, Y_\alpha)$.

We now consider $DZ4$. Because $X^*$ has the same distribution $F$ as $X$, hence $E[U] = \infty$ so that $m(x) \to \infty$, and

$$\lim_{x \to \infty} \sup_{y \in [1, x]} \frac{L(y)}{L(x)} < \infty.$$

From (10),

$$\lim_{x \to \infty} \frac{P[Y_\alpha > x]}{P[X^* > x]}m(x) \leq \lim_{x \to \infty} \left(1 + |\theta d_1|b_2\overline{G}(x)\right)m(x) = (1 + |\theta d_1|b_2) \lim_{x \to \infty} \frac{\overline{G}(x)}{\overline{F}(x)}m(x) = 0.$$

Thus all aspects of condition (DZ4) are satisfied by $(X^*, Y_\alpha)$.


Summarizing everything, we have the following theorem.

Theorem 2.5. The pair of random variables $(X, Y)$ jointly follow bivariate Sarmanov as given in Definition 1.1. Let $X$ be nonnegative and $X \in RV_{\alpha}$. We also assume $E[Y^\alpha] < \infty$, $P[Y > x] = o(P[X > x])$ and $\lim_{x \to \infty} \phi_1(x) = d_1$ exists. If any one of the three conditions (DZ2), (DZ3), (DZ4) holds, then $P[XY > x] \sim (E[Y^\alpha] + \theta d_1 E[\phi_2(Y)^{\alpha}])F(x)$.

Note that the similar result under the condition (DZ1) has already been proved in [4].

3. Finite Sum

We consider a sequence $\{(X_i, Y_i)\}$ of independent and identically distributed random vectors, with the generic random vector $(X, Y)$ following bivariate Sarmanov and satisfying (3). The finite time ruin probability under (DZ1) has already been studied by [4]. We now show that if (3) and any one of the three sufficient conditions (DZ2), (DZ3) and (DZ4) is satisfied, then the same conclusion as in (7) will hold.

Recall that the finite time ruin probability $\Psi(x, n) = P[\max_{1 \leq k \leq n} S_k > x]$. The first step is to prove

Lemma 3.1. Assume that $\{(X_i, Y_i) : i \in \mathbb{N}\}$ is an i.i.d. sequence of random vectors with the generic random vector $(X, Y)$ following bivariate Sarmanov distribution as in Definition 1.1. Each $X_i$ is regularly varying with index $-\alpha$, and $\phi_1(x)$ holds. If $\bar{H}(x) = P[X, \prod_{j=1}^{\infty} Y_{j-1}] > x$ and any of the conditions (DZ2), (DZ3) and (DZ4) holds, then

$$\Psi(x, n) \sim \sum_{i=1}^{n} \bar{H}(x).$$

The proof of (11) is similar to that of Theorem 4.1 in [4], and hence we omit it. The crucial step is then to establish that

$$\bar{H}(x) \sim (E[Y^\alpha])^{i-1} \bar{H}(x) \sim (E[Y^\alpha])^{i-1} [E(Y^\alpha) + \theta d_1 E(\phi_2(Y)^{\alpha})]F(x) = (E[Y^\alpha])^{i-1} E[Y^\alpha] \bar{F}(x)$$

where $\bar{H}(x) = \bar{H}(x) = P[X, Y_1 > x]$.

We proceed using induction on $i$. It holds for $i = 1$ using Theorem 2.5. Assume that (12) holds for some $i \geq 1$ which implies that $\bar{H}_i(x) \in RV_{\alpha}$ since $\bar{F} \in RV_{\alpha}$. Hence we can write $\bar{H}_i(x) = x^{-\alpha} L_i(x)$ where $L_i$ is a positive slowly varying function. Clearly this means that, by our induction hypothesis, $L_i(x) \sim (E[Y^\alpha])^{-1} E[Y^\alpha] L(x)$, where $\bar{F}(x) = x^{-\alpha} L(x)$. Hence it is immediate that $L_i$ will have the same form as $L_i$, that is, the appropriate one from (6) through (8) of Lemma 2.1 holds. Since (DZ2) and (DZ3) involve only the asymptotic tail properties of $L_i$, they carry over to $L_i$ as well. We separately check the same extension of the result for (DZ4).

Lemma 3.2. If $(X, Y)$, or equivalently, $(F, G)$, satisfies (DZ2), and (12) holds for some $i \geq 1$, then the joint distribution $(H_i, G)$ also satisfies (DZ4).

Proof. Since, by induction hypothesis, we have $L_i(x)/L_i(x) \rightarrow (E[Y^\alpha])^{-1} E[Y^\alpha]$, and (DZ4) holds for $L_i$, we have $\lim \sup_{x \to \infty} \sup_{1 \leq k \leq n} L_i(y)/L_i(x) < \infty$.

Let us define $m_i(x) = \int_0^x s^{-\alpha} \bar{H}_i(s) ds - x^{-\alpha} \bar{H}_i(x), \quad \text{and} \quad m(x) = \alpha \int_0^x s^{-\alpha} \bar{F}(s) ds - x^{-\alpha} \bar{F}(x).

To check (DZ4) for $(H_i, G)$ it is enough to check that $m_i(x)/m(x)$ is bounded. Observe that

$$\limsup_{x \to \infty} \frac{m_i(x)}{m(x)} = \limsup_{x \to \infty} \frac{1 - \frac{x^{-\alpha} \bar{H}_i(x)}{\alpha \int_0^x s^{-\alpha} \bar{H}_i(s) ds}}{1 - \frac{x^{-\alpha} \bar{F}(x)}{\alpha \int_0^x s^{-\alpha} \bar{F}(s) ds}}$$

$$= \limsup_{x \to \infty} \frac{\int_0^x s^{-\alpha} \bar{H}_i(s) ds}{\int_0^x s^{-\alpha} \bar{F}(s) ds} \leq \sup_{x \to \infty} \frac{\bar{H}_i(x)}{\bar{F}(x)} < \infty,$$
where the second equality follows from Karamata’s theorem. Hence, by the (DZ) condition on \( L \),
\[
\lim_{x \to \infty} \frac{G(x)}{H(x)} m_i(x) = \lim_{x \to \infty} \frac{G(x)}{F(x)} F(x) m_i(x) m(x) \leq \lim_{x \to \infty} \frac{G(x)}{F(x)} m(x) m(x) \frac{F(x)}{H(x)} \lim_{x \to \infty} m_i(x) m(x) = 0.
\]

Lastly, \( Y_1 \) is independent of \( X_{i+1} Y_1 + \ldots Y_2 \) with distribution \( H_i \). The appropriate (DZ) condition for \( (H_i, G) \) gives
\[
H_i + 1(X) = P[(X_{i+1} Y_1 + \ldots Y_2) X_1 > x] \sim E(Y^a) H_i(x) \sim \{E(Y^a)\} H_i(x).
\]

This shows that the result (12) holds for \( i + 1 \) as well, and the induction is completed.

Summarizing, we now have the following theorem.

**Theorem 3.3.** Let \( [(X_i, Y_i)] \) be a sequence of independent and identically distributed random vectors, with the generic random vector \( (X, Y) \) following bivariate Sarmanov as in Definition (7), with \( X \in \text{RV}_\phi \). Suppose \( E[Y^a] < \infty \), \( P[Y > x] = 0 \) \( \lim_{x \to \infty} \phi_1(x) = C \). Let \( \Psi(x, n) \) be as defined in (7). If any one of the conditions (DZ), (DZ2) and (DZ3) holds, then we have
\[
\Psi(x, n) \sim \frac{(1 - E[Y^a]) \{E[Y^a] + \theta d_1 \} \{E[Y^a] E[Y^a] \}}{(1 - E[Y^a])} F(x),
\]
with the convention that \((1 - E[Y^a])/(1 - E[Y^a]) = 1 \) when \( E[Y^a] = 1 \).

4. Infinite sum

In this section, we consider again a sequence \( [(X_i, Y_i)] \) of i.i.d. random vectors with the generic random vector \( (X, Y) \) jointly bivariate Sarmanov, with both (3) and (9) satisfied. Additionally, we assume that \( E[Y^a] < 1 \). Now we show that, if any of the four (DZ) conditions is also satisfied, along with some uniform integrability condition, then the same conclusion as (9) holds, that is
\[
\lim_{x \to \infty} \frac{\Psi(x)}{E[Y^a]^x} = \frac{E[Y^a] + \theta d_1 E[Y^a] E[Y^a]}{1 - E[Y^a]} = \frac{E[Y^a]}{1 - E[Y^a]},
\]
where \( Y_\theta \) is the twisted version of \( Y \) given in (5). The lower bound for \( \Psi(x)/F(x) \) follows immediately from a common argument for all the four (DZ) conditions.

For any \( m \in \mathbb{N} \), using Theorem 3.3 or Theorem 4.1 of (4), we get
\[
\frac{\Psi(x)}{E[Y^a]^x} \leq \frac{\Psi(x, m)}{E[Y^a]^x} \leq \frac{1 - E[Y^a]^m}{E[Y^a]} \cdot E[Y_\theta^a].
\]

and the desired lower bound now follows by letting \( m \to \infty \).

For the upper bound we proceed as follows. Let \( \zeta_1 = \prod_{i=1}^{m} Y_i \) and \( Z_i = X_i Y_i \). Observe that \( Z_i \) and \( \zeta_i \) are mutually independent. Then for any natural number \( m \), any constant \( 0 < \delta < 1 \) and any \( x \geq 0 \),
\[
P \left[ \sup_{1 \leq m < \infty} \sum_{i=1}^{n} Z_i \zeta > x \right] \leq P \left[ \max_{1 \leq k \leq m} \sum_{i=1}^{k} Z_i \zeta > (1 - \delta) x \right] + P \left[ \sum_{i=k+1}^{\infty} Z_i \zeta > \delta x \right].
\]

Using Theorem 4.1 of (4) for (DZ1), and Theorem 3.3 for (DZ2), (DZ3) or (DZ4), we have
\[
P \left[ \max_{1 \leq k \leq m} \sum_{i=1}^{k} Z_i \zeta > (1 - \delta) x \right] = \Psi((1 - \delta)x, m) \sim \frac{1 - E[Y^a]^m}{E[Y^a]} E[Y_\theta^a]\Phi(1 - \delta)x).
\]

Since \( \Phi \in \text{RV}_{\phi} \), we have \( \lim_{x \to \infty} P[\max_{1 \leq k \leq m} \sum_{i=1}^{k} Z_i \zeta > (1 - \delta)x]/\Phi(1 - \delta)x] \leq \frac{E[Y^a]}{1 - E(Y^a)} (1 - \delta)^{-a} \).
We obtain the desired upper bound by making the second term of (13) arbitrarily small for suitably large $m$ and for all sufficiently large values of $x$, and finally letting $\delta \to 0$.

\[
P \left[ \sum_{i=m+1}^{\infty} Z_i \xi_i > x \right] \leq \sum_{i=m+1}^{\infty} P[Z_i \xi_i > x] + P \left[ \sum_{i=m+1}^{\infty} Z_i \xi_i 1_{[Z_i \xi_i \leq x]} > x \right].
\] (14)

We bound the second term of (14), separately for $\alpha < 1$ and $\alpha \geq 1$, arguing as in the proof of Theorem 4.2 in [4]. For $\alpha < 1$, we use Markov’s inequality and for $\alpha \geq 1$ we use Minkowski’s inequality. In both cases, using Karamata’s theorem, we get a constant $C$ such that

\[
P \left[ \sum_{i=m+1}^{\infty} Z_i \xi_i 1_{[Z_i \xi_i \leq x]} > x \right] \leq \begin{cases} C \sum_{i=m+1}^{\infty} \frac{P[Z_i \xi_i > x]}{p_i} & \text{if } \alpha < 1, \\ \sum_{i=m+1}^{\infty} \frac{P[Z_i \xi_i > x]}{p_i} & \text{if } \alpha \geq 1. \end{cases}
\]

Then the upper bound will be established by showing that $P[Z_i \xi_i > x]/p_i$ uniformly for all large values of $x$, that is, there exists $x_0$ such that for all $x > x_0$, we have $P[Z_i \xi_i > x] \leq B_i p_i(x)$ for all $i$. Here $B_i$ is a finite positive constant such that

\[
\sum_{i=1}^{\infty} B_i < \infty \quad \text{for } \alpha < 1 \quad \text{and} \quad \sum_{i=1}^{\infty} B_i \frac{1}{p_i} = \infty \quad \text{for } \alpha \geq 1.
\] (15)

For this, it will be sufficient to produce an upper bound for $P[Z_i \xi_i > x]/p_i$ which satisfies (15). We split the ratio as follows:

\[
P[Z_i \xi_i > x]/p_i = \int_{(0,1)} + \int_{(1,\infty)} \frac{P[Z_i \xi_i > x]}{p_i} G_i(dv),
\]

where $G_i$ is the distribution function of $\xi_i$. As $x \to \infty$, the integrand converges to $v^\alpha$ uniformly in $v$ over the first interval and hence, for all large enough $x$, $\int_{[0,1]} \frac{P[Z_i \xi_i > x]}{p_i} G_i(dv) \leq 2E(\xi_i^\alpha)$. The bound for the other integral is provided separately for the four (DZ) conditions. Recall that the (DZ) conditions are given in terms of the slowly varying function $L(x) = x^\alpha F(x)$.

Lemma 4.1. Let $(X_n, Y_n)$ be i.i.d. random vectors, with the generic random vector $(X, Y)$ jointly distributed as bivariate Sarmanov, and satisfying (4) and (9). Also, the (DZ1) condition holds and $E[Y^\alpha] < 1$. Then

\[
\int_{(1,\infty)} \frac{P[Z_i \xi_i > x]}{p_i} G_i(dv) \leq C'E(\xi_i^\alpha),
\] (16)

for some constant $C'$ independent of $i$, and for all sufficiently large $x$ uniformly in $i$.

Proof. We have $p_i(x) = P(Z_i > x) = P(X_i Y_i > x) = x^{-\alpha} L_i(x)$ where $L_i$ is slowly varying. Then $\lim_{x \to \infty} L_i(x)/L(x) = E[Y_i^\alpha] \in (0, \infty)$. Thus $\lim_{x \to \infty} \sup_{1 \leq x \leq x} L_i(x)/L(x) = \sup_{1 \leq x \leq x} L_i(x)/L(x)$ is finite. We split the integral in (16) over two intervals, $(1, x]$ and $(x, \infty)$. For the integral over $(x, \infty)$, for all $x$ large enough, uniformly in $i$, we have:

\[
\int_{(1,\infty)} \frac{P[Z_i \xi_i > x]}{p_i} G_i(dv) \leq \frac{E(\xi_i^\alpha)}{L_i(x)},
\]

which is bounded above by a constant multiple of $E(\xi_i^\alpha)$, the constant being independent of $i$. For the integral over the range $(1, x]$, for all sufficiently large $x$ uniformly in $i$, we have:

\[
\int_{(1,\infty)} \frac{P[Z_i \xi_i > x]}{p_i} G_i(dv) \leq \sup_{1 \leq x \leq x} \frac{L_i(y)}{L_i(x)} \int_{(1,\infty)} v^\alpha G_i(dv),
\]

which is once again bounded above by a constant multiple of $E(\xi_i^\alpha)$, the constant free of $i$. \hfill \Box
**Lemma 4.2.** Assume that \((X_i, Y_i), i \geq 1\) are i.i.d. random vectors with the generic random vector \((X, Y)\) following a bivariate Sarmanov distribution, satisfying (4) and (9). Also (DZ2) holds and \(E(Y^{\alpha}) < 1\). Further \(C_i = \sup_{x \geq 1} P[\xi > x] = \infty\) when \(\alpha < 1\) and \(\sum_{i=2}^{\infty} C_i < \infty\) when \(\alpha \geq 1\) for some \(\varepsilon > 0\). Then, for all sufficiently large \(x\) uniformly in \(i\), some constant \(\eta\) independent of \(i\),

\[
\int_{(1, \infty)} \frac{P[Z_i > x/v]}{P[Z_i > x]} G_i(dv) < \eta C_i + E[\zeta_i^\alpha].
\]  

(17)

**Proof.** We split the integral in (17) over \((1, x)\) and \((x, \infty)\). The integral over \((x, \infty)\) is bounded as:

\[
\int_{(x, \infty)} \frac{P[Z_i > x/v]}{P[Z_i > x]} G_i(dv) \leq \frac{P[\xi_i > x]}{P[Z_i > x]} \leq C_i \overline{F}(x).
\]

(18)

Since, from Theorem 2.3 we know that \(\overline{F}(x)/P[Z_i > x] \to \{E[Y_i^\alpha]\}^{-1}\), hence the right hand side of (18) is bounded by a constant.

We perform integration by parts on the integral over the interval \((1, x)\) to get

\[
\int_{(1, x)} \frac{P[Z_i > x/v]}{P[Z_i > x]} G_i(dv) \leq \int_{(1, x)} \frac{P[\xi_i > v]}{P[Z_i > x]} dv P[Z_i > x/v].
\]

The first term gets bounded by \(E(\zeta_i^\alpha)\) by Markov inequality. Substituting \(u = \log v\) the second term is bounded by

\[
C_i \gamma \int_{(0, \log x)} \frac{P[\log Z_i > u]}{P[\log Z_i > \log x]} du P[\log Z_i > \log x - u].
\]

Recall that \(\overline{H}(x) = P[X, Y_i > x] = x^{-\alpha} L_1(x)\), where \(L_1\) has the same representation out of (iii) or (iv) of Lemma 2.1 as \(L\). Also, \(L_1(e^x) \in \mathcal{S}_L\). From Theorem 2.1 of [1] this implies that \((\log Z_i)^\gamma \in \mathcal{S}(\alpha)\). Therefore, there exists some \(x_2\) large enough, independent of \(t\), such that for all \(x \geq x_2\)

\[
\int_{(0, \log x)} \frac{P[\log Z_i > u]}{P[\log Z_i > \log x]} du P[\log Z_i > \log x - u] \leq 3E[\exp(\alpha(\log Z_i)^\gamma)] \leq 3E[Z_i^\alpha] + 1).
\]

Hence the result follows. 

**Lemma 4.3.** Assume that \((X_i, Y_i), i \geq 1\) are i.i.d. random vectors with the generic random vector \((X, Y)\) following a bivariate Sarmanov distribution, satisfying (4) and (9). Also the condition (DZ2) holds and \(E(Y^{\alpha}) < 1\). We further have

\[
\sup_{x \geq 1} \frac{P[\xi_i > x]}{x^{-\alpha} P[U > \log x]} = C_i,
\]

where \(\sum_{i=2}^{\infty} C_i = \infty\) when \(\alpha < 1\) and \(\sum_{i=2}^{\infty} C_i < \infty\) when \(\alpha \geq 1\) for some \(\varepsilon > 0\). Then, we have, for all sufficiently large \(x\) uniformly in \(i\), and for two constants \(\gamma, \eta\) independent of \(i\),

\[
\int_{(1, \infty)} \frac{P[Z_i > x/v]}{P[Z_i > x]} G_i(dv) \leq \gamma E[\zeta_i^\alpha] + \eta C_i.
\]  

(19)

**Proof.** We split the integral in (19) over two intervals, \((1, x)\) and \((x, \infty)\). For the integral over \((x, \infty)\), we have

\[
\int_{(x, \infty)} \frac{P[Z_i > x/v]}{P[Z_i > x]} G_i(dv) = \frac{P[\zeta_i > x]}{P[Z_i > x]} \leq \frac{P[\zeta_i > x]}{c_1(x) x^{-\alpha} P[U > \log x]} \leq \frac{2C_i}{c_1}.
\]

Now \(\overline{F}(x) = P[X, Y_i > x] = x^{-\alpha} L_1(x)\). Accordingly as \(L\) is of the form (iii) or (iv) of Lemma 2.1, \(L_1\) will have an analogous form with \(c(x)\) replaced by \(c_1(x)\). Thus we have, for all sufficiently large \(x\) uniformly in \(i\),

\[
\frac{P[\zeta_i > x]}{P[Z_i > x]} \leq \frac{P[\zeta_i > x]}{c_1(x) x^{-\alpha} P[U > \log x]} \leq \frac{2C_i}{c_1}.
\]
For the integral over \((1, x]\), when \(L\) is of the form \((\text{iii})\) or \((\text{iv})\),
\[
\int_{(1, x]} \frac{P[Z > x/v]}{P[Z_i > x]} G_i(dv) \leq \sup_{v \in (1, x]} \frac{c_i(x/v)}{c_i(x)} \int_{(1, x]} \frac{P[U > \log x - \log v]}{P[U > \log x]} \nu^d G_i(dv)
\]
\[
\leq a \int_{(1, x]} \frac{P[U > \log x - \log v]}{P[U > \log x]} \nu^d G_i(dv),
\]
(20)
since \(\lim_{x \to \infty} c_i(x) = c_i\) and hence \(\sup_{v \in (1, x]} c_i(x/v) < \infty\). We bound the integral in (20) by integration by parts, which gives the bound
\[
\overline{G}_i(1) + a \int_{(1, x]} \frac{\nu^a - 1}{P[U > \log x]} \nu^d G_i(dv) + \int_{(1, x]} \frac{\overline{G}_i(v)^a}{P[U > \log x]} d_s P[U > \log x - \log v].
\]
The first term is bounded by \(E(\nu)^a\) by Markov inequality. The second term can be dealt with as follows:
\[
a \int_{(1, x]} \frac{\nu^a - 1}{P[U > \log x]} \nu^d G_i(dv) \leq aC_i \int_{(1, x]} \frac{P[U > \log x - \log v]}{P[U > \log x]} \nu^d dv
\]
\[
< 3aC_i \int_0^\infty P[U > u] du
\]
for all sufficiently large \(x\) uniformly in \(i\). The last inequality follows from the substitution \(w = \log v\) and noting that \(U \in \mathcal{S}\) implies \(U\) is subexponential. For the third term, we have, again for all sufficiently large \(x\) uniformly in \(i\),
\[
\int_{(1, x]} \frac{\overline{G}_i(v)^a}{P[U > \log x]} d_s P[U > \log x - \log v] \leq C_i \int_{(1, x]} \frac{P[U > \log v]}{P[U > \log x]} d_s P[U > \log x - \log v]
\]
\[
= C_i \frac{P[U + U'] > \log x]}{P[U > \log x]} < 3C_i,
\]
where in the last step we use subexponentiality of \(U\) and \(U'\). Combining everything, the result follows.

\[\square\]

**Lemma 4.4.** Assume that \([(X_i, Y_i), i \geq 1]\) are i.i.d. random vectors with the generic random vector \((X, Y)\) following a bivariate Sarmanov distribution, satisfying \((\text{ii})\) and \((\text{iv})\). The condition \((\text{ii})\) also holds and \(E(\nu) < 1\). We also have
\[
\sup_{x \geq 1} \frac{P[\zeta_i > x]}{F(x)} = C_i \in (0, \infty)
\]
with \(\sum_{i=1}^\infty C_i < \infty\) when \(a < 1\) and \(\sum_{i=1}^\infty C_i^{1+a} < \infty\) when \(a \geq 1\) for some \(\varepsilon > 0\). Then for all sufficiently large \(x\) uniformly in \(i\), and constants \(\gamma, \eta\) independent of \(i\),
\[
\int_{(1, \infty)} \frac{P[Z > x/v]}{P[Z_i > x]} G_i(dv) \leq \gamma E(\nu) + \eta C_i.
\]
(21)

**Proof.** We split the integral in (21) over two intervals, \((1, x]\) and \((x, \infty)\). We bound the integral over \((x, \infty)\) as follows:
\[
\int_{(x, \infty)} \frac{P[Z > x/v]}{P[Z_i > x]} G_i(dv) \leq \frac{P[\zeta_i > x]}{F(x)} \cdot \frac{\overline{F}(x)}{P[Z_i > x]}
\]
Since \(\overline{F}(x)/P[Z_i > x]\) converges and hence bounded, and \(m(x) \to \infty\), leading to \(P[\zeta_i > x]/\overline{F}(x) \leq C_i\), for all sufficiently large \(x\) uniformly in \(i\), thus we have, again for all sufficiently large \(x\) uniformly in \(i\),
\[
\int_{(x, \infty)} \frac{P[Z > x/v]}{P[Z_i > x]} G_i(dv)
\]
is bounded above by a constant multiple of \(C_i\), the constant independent of \(i\).

We now consider the integral over the interval \((1, x]\) and further split it into two sub-intervals: \((1, \sqrt{x}]\) and \((\sqrt{x}, x]\) and bound them separately.
Theorem 4.5. Assume that \((X_i, Y_i), i \geq 1\) are i.i.d. random vectors with the generic random vector \((X, Y)\) following a bivariate Sarmanov distribution, as defined in Definition 1.1, with \(X \in RV_{-\infty}\). Let \(E[Y^\alpha] < 1\) and \(G(x) = o(F(x))\) and \(\lim_{x \to \infty} \phi_1(x) = d_1\). Assume that one of the four (DZ) conditions holds. If one of (DZ3) and (DZ4) is satisfied, then define

\[
C_i = \begin{cases} 
\sup_{x} \frac{P(X > x)}{P(X > y)}, & \text{when (DZ2) holds,} \\
\sup_{y \sim P(Y = y)} \frac{P(X > x)}{m(x)}, & \text{when (DZ3) holds,} \\
\sup_{x \sim P(Y = x)} \frac{P(X > x)}{m(x)}, & \text{when (DZ4) holds,}
\end{cases}
\]

and further assume that

\[
\sum_{i=2}^{\infty} C_i < \infty \quad \text{when } \alpha < 1 \quad \text{and} \quad \sum_{i=2}^{\infty} C_i^{\alpha+1} < \infty \quad \text{when } \alpha \geq 1
\]

for some \(\epsilon > 0\). Then

\[
P \left[ \sup_{\eta \geq 1} \sum_{i=1}^{n} X_i \prod_{j=1}^{i} Y_j > x \right] \leq \frac{E[Y^\alpha] + \theta d_1 E[\phi_2(Y)^\alpha]}{1 - E[Y^\alpha]} P[X_1 > x].
\]
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