Frobenius splitting of thick flag manifolds of Kac-Moody algebras

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Abstract

We explain that the Plücker relations provide the defining equations of the thick flag manifold associated to a Kac-Moody algebra. This naturally transplant the result of Kumar-Mathieu-Schwede about the Frobenius splitting of thin flag varieties to the thick case. As a consequence, we provide a description of the space of global sections of a line bundle of a thick Schubert variety as conjectured in Kashiwara-Shimozono [Duke Math. J. 148 (2009)]. This also yields the existence of a compatible basis of thick Demazure modules, and the projective normality of the thick Schubert varieties.

Introduction

The geometry of flag varieties of a Lie algebra \( g \) is ubiquitous in representation theory. In case \( g \) is a Kac-Moody algebra, we have two versions of flag varieties \( X \) and \( \mathfrak{X} \), that we call the thin flag varieties and thick flag manifolds, respectively (see e.g. [14]). They coincide when \( g \) is of finite type, and in this case we have

\[
X = \mathfrak{X} = \text{Proj} \bigoplus_{\lambda} \mathcal{L}(\lambda)^\vee,
\]

where \( \lambda \) runs over all dominant integral weights and \( \mathcal{L}(\lambda) \) denotes the corresponding integrable highest weight representation of \( g \). The isomorphism (0.1) is less obvious when \( g \) is not finite type since \( \mathcal{L}(\lambda) \) is no longer finite-dimensional. In fact, the both of \( X \) and \( \mathfrak{X} \) are quotients of certain Kac-Moody groups \( G \) associated to \( g \), and we can ask whether we have

\[
G \cong \text{Spec} \ k[G]
\]

as an enhancement of (0.1), where \( k[G] \) is the coordinate ring of \( G \) (cf. Kac-Peterson [8]). However, Kashiwara [9, §6] explains that none of the choice of \( G \) can satisfy (0.2) for any version of a reasonably natural commutative ring \( k[G] \).

The goal of this paper is to explain that despite the above situation, we can still understand the geometry of Kac-Moody flag manifolds as infinite type schemes so that we can deduce some consequences in representation theory.

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To explain what we mean by this, we introduce some more notation: The scheme $X$ admits a natural action of the subgroup $B$ of $G$ that corresponds to the non-negative part of $g$, and the set of $B$-orbits of $X$ is in natural bijection with the Weyl group $W$ of $g$. Hence, we represent a $B$-orbit closure of $X$ by $X^w$ for some $w \in W$. For each integral weight $\lambda$ of $g$, we have an associated line bundle $O_{X}(\lambda)$ and its restriction $O_{X^w}(\lambda)$ to $X^w$.

The main result in this paper is:

**Theorem A** (Theorem 1.23 and Corollary 2.21). For an arbitrary Kac-Moody algebra, the thick flag manifold $X$ admits the presentation $\text{(0.1)}$ as schemes. Similar result holds for each $B$-orbit closure of $X$.

As Kashiwara’s embedding of $X$ into the Grassmannian $\text{[9, \S 4]}$ factors through a highest weight integrable module, Theorem A asserts that it is a closed embedding. Hence, Theorem A affirmatively answers the question in $\text{[9, 4.5.6–4.5.7]}$.

The thin flag variety $X$ forms a Zariski dense subset of $X$ (see e.g. Kashiwara-Tanisaki [14, \S 1.3]). This implies that the projective coordinate ring of $X$ (as an ind-scheme) is the completion of that of $X$ (as a honest scheme). Therefore, we can transplant the Frobenius splitting of $X$ (or its ind-pieces) to that of $X$ provided in Kumar-Schwede [16]:

**Corollary B** (Corollary 2.12). For an arbitrary Kac-Moody algebra over an algebraically closed field $k$ of positive characteristic, the thick flag manifold $X$ admits a Frobenius splitting that is compatible with the $B$-orbits.

From this, we deduce some conclusions on the level of global sections as:

**Theorem C** (Theorem 2.17, 2.19, and Corollary 2.21, 2.22). For each $w \in W$, we have:

1. the natural restriction map
   \[
   \Gamma(X, O_X(\lambda)) \rightarrow \Gamma(X^w, O_{X^w}(\lambda))
   \] (0.3)
   is surjective;

2. the image of the inclusion
   \[
   \Gamma(X^w, O_{X^w}(\lambda)) \subset \Gamma(X, O_X(\lambda)) = L(\lambda)
   \]
   obtained as the dual of $\text{(0.3)}$ is cyclic as a Lie $B$-module;

3. the scheme $X^w$ is projectively normal;

4. the sums of modules in $\{\Gamma(X^w, O_{X^w}(\lambda))\}_{w \in W}$ forms a distributive lattice in terms of intersection.

We remark that Theorem C 4) should be also obtained as a combination of Kashiwara’s crystal basis theory [12] and Littelmann’s path model theory [17] when $g$ is symmetrizable. However, the only reference the author is aware beyond the finite case is the affine case presented in Ariki-Kreimann-Tsuchioka [11, \S 6] (as stated there, the proofs of this part are due to Kashiwara and Sagaki).

We also note that Theorem C 1) and 2) confirms a part of the Kashiwara-Shimozono conjecture [13, Conjecture 8.10] (that originally concerns when $g$ is affine).
1 Defining equations of thick flag manifolds

We work over an algebraically closed field $k$. We employ [15] as a basic reference, and we may refer to [15] also for char $k > 0$ case without a comment (while the book deals only for $k = \mathbb{C}$) when we supply enough (other) results so that its proof carries over based on them.

Let $I$ be a finite set with its cardinality $r$ and let $C = (c_{ij})_{i,j \in I}$ be a generalized Cartan matrix (GCM) in the sense of [7] §1.1. Let $\mathfrak{g}$ be the Kac-Moody algebra associated to $C$, and let $\mathfrak{h}$ be its Cartan subalgebra (we have $\dim_k \mathfrak{h} = 2|I| - \text{rank} C$). Let $Q$ and $Q^\vee$ be the root lattice and the coroot lattice of $\mathfrak{g}$, and $\{\alpha_i \}_{i \in I} \subset Q$ and $\{\alpha_i^\vee\}_{i \in I} \subset Q^\vee$ are the set of simple roots and the set of simple coroots, respectively. Let $X^+$ be a $\mathbb{Z}$-lattice that contains $Q$ and equipped with elements $x_1, \ldots, x_r \in X^+$ so that $X^+ \otimes \mathbb{Z} k \cong \mathfrak{h}^*$, and there exists a pairing

$$\langle \bullet, \bullet \rangle : Q^\vee \times X^+ \longrightarrow \mathbb{Z}$$

that satisfies

$$\langle \alpha_i^\vee, \alpha_j \rangle = c_{ij}, \quad \langle \alpha_i^\vee, x_j \rangle = \delta_{ij}, \quad \text{and} \quad \langle \alpha_i^\vee, X^+ \rangle = \mathbb{Z}.$$

Let $\{E_i, F_i\}_{i \in I}$ be the Kac-Moody generators of $\mathfrak{g}$ so that $[E_i, F_j] = \delta_{ij}\alpha_i^\vee \in \mathfrak{h}$ for $i, j \in I$. Let $n, n^- \subset \mathfrak{g}$ be the Lie subalgebras generated by $\{E_i\}_{i \in I}$ and $\{F_i\}_{i \in I}$, respectively. We set $H := \text{Spec} k[e^\lambda \mid \lambda \in X^+]$. We have Lie $H = \mathfrak{h}$. For each $\alpha \in X^+$, we define

$$\mathfrak{g}_\alpha := \{ \xi \in \mathfrak{g} \mid \text{Ad}(h)\xi = \alpha(h)\xi, \quad \forall h \in H \}, \quad \text{mult} \, \alpha := \dim \mathfrak{g}_\alpha.$$

We set

$$\Delta^+: = \{ \alpha \in X^+ \setminus \{0\} \mid \mathfrak{g}_\alpha \subset n \}, \quad \Delta^- := -\Delta^+.$$

We have reflections $\{s_i\}_{i \in I}$ on $\text{Aut}(X^+)$ that generates a Coxeter group $W$. We denote its length function by $t$, and the Bruhat order by $<$ (see Kumar [15] Definition 1.3.15]). We have a subset

$$\Delta^+_\text{re} := \Delta^+ \cap W\{\alpha_i\}_{i \in I} \subset \Delta^+.$$

Each $\alpha \in \Delta^+_\text{re}$ gives a reflection $s_\alpha \in W$ defined through the conjugation of a simple reflection. We have mult $\alpha = 1$ for $\alpha \in \Delta^+_\text{re}$, and we have $\mathfrak{sl}(2) \cong \mathfrak{g}_\alpha \oplus k\alpha^\vee \oplus \mathfrak{g}_{-\alpha}$ as Lie algebras in this case.

For each $i \in I$, we define $SL(2, i)$ as the connected and simply connected algebraic group with an identification Lie $SL(2, i) = kE_i \oplus k\alpha_i^\vee \oplus kF_i$. For each $n > 0$, we set

$$\Delta^- (n) := \{ \alpha \in \Delta^- \mid \text{ mult } \alpha = n \} = -\sum_{i \in I} m_i \alpha_i, \quad m_i \in \mathbb{Z}_{\geq 0}, \quad \sum m_i \leq n \} \subset \Delta^-.$$

Then, $\bigoplus_{n \in \Delta^- \setminus \Delta^- (n)} \mathfrak{g}_{-\alpha} \subset n$ and $\bigoplus_{n \in \Delta^- \setminus \Delta^- (n)} \mathfrak{g}_\alpha \subset n^-$ are ideals. We denote the quotients by $n(n)$ and $n^-(n)$, respectively. By construction, we have a Lie algebra quotient maps $n(n) \rightarrow n(n')$ and $n^-(n) \rightarrow n^-(n')$ for $n > n'$.

We define a pro-unipotent group

$$\tilde{N}^- := \lim_{\leftarrow n} N^-(n),$$

where
Lemma 1.2. Let \( Z \) have many elements from \( G \) that is a \((\text{resp. } SL_5) \) as a 6-tuple \((G^-, N(H), N, H, S)\). Applying the Chevalley involution to \( \{N^-(n)\}_{n \geq 1} \), we obtain a pro-unipotent group \( \hat{N} := \lim_{\leftarrow n} N(n) \) corresponding to \( \hat{n} \).

We define \( \hat{B}^+ := H\hat{N} \) and \( \hat{B}^- := H\hat{N}^- \), that are (pro-algebraic) groups (and also a Lie subalgebra \( b := \mathfrak{h} \oplus n \subset \mathfrak{g} \)). For each \( \alpha \in \Delta^{+}_{\mathfrak{h}} \), we have a one-parameter unipotent subgroup \( \rho_{\alpha} : \mathbb{G}_a \rightarrow \hat{B}^+ \) so that \( h\rho_{\alpha}(z)h^{-1} = \rho_{\alpha}(\alpha(h)z) \) for every \( z \in \mathbb{G}_a \) and \( h \in H \). Similarly, we have a one-parameter unipotent subgroup \( \rho_{-\alpha} : \mathbb{G}_a \rightarrow \hat{B}^- \).

We have subgroups \( \hat{N}^+ \subset \hat{N}^+ \) and \( \hat{N}^- \subset \hat{N}^- \) formed by products of finitely many elements from \( \{\rho_{\alpha}(\mathbb{G}_a)\}_{\alpha \in \Delta^{+}_{\mathfrak{h}}} \) and \( \{\rho_{-\alpha}(\mathbb{G}_a)\}_{\alpha \in \Delta^{+}_{\mathfrak{h}}} \), respectively. Let \( N(H) \) denote the group generated by \( H \) and the normalizers of \( H \) inside \( SL(2, i) \) for each \( i \in I \), whose quotient by \( H \) is \( W \). We have a translation of elements of \( \hat{B}^+ \) under the action of \( N(H) \), defined partially (see [15 §6.1]). The positive Kac-Moody group \( G^+ \) is defined as the amalgamated product of \( \hat{B}^+ \) and \( N(H) \), while the negative Kac-Moody group \( G^- \) is defined as the amalgamated product of \( \hat{B}^- \) and \( N(H) \) (see [15 §5.1]). For each \( J \subset I \), we have a partial amalgam \( \hat{B}^J \subset \hat{B}^J \subset G^+ \), that we call the parabolic subgroups corresponding to \( J \).

Let \( U_{\mathfrak{z}}(\mathfrak{g}) \) (resp. \( U_{\mathfrak{z}}(\mathfrak{h}), U_{\mathfrak{z}}(\mathfrak{b}) \) or \( U_{\mathfrak{z}}(\mathfrak{n}^-) \)) be the Chevalley-Kostant \( \mathbb{Z} \)-form of the enveloping algebra of \( \mathfrak{g} \) generated by \( E_i^{(n)}, F_i^{(n)} \) \((i \in I, n \in \mathbb{Z}_{\geq 0})\) and

\[
 h(m) := \frac{h(h-1)\cdots(h-m+1)}{m!} \quad h \in \text{Hom}_{\mathbb{Z}}(X^*, \mathbb{Z}), m \in \mathbb{Z}_{\geq 0}
\]

(resp. \( h(m), E_i^{(n)} \) and \( h(m), \) or \( F_i^{(n)} \)), and let \( U(\mathfrak{g}) \) (resp. \( U(\mathfrak{h}), U(\mathfrak{b}), U(\mathfrak{n}^-) \)) be its specialization to \( \mathbb{K} \) (see e.g. Tits [21] or Mathieu [19] Chapter I).

We understand that a representation of an algebraic group is always algebraic. Note that the complete reducibility of representations always hold for split torus (and we never deal with non-split torus in this paper).

**Definition 1.1** (integrable highest weight modules). A \((U(\mathfrak{h}), H)\)-module \( M \) is said to be a weight module if \( M \) admits a semi-simple action of the above \( h(m)\)'s that integrates to the algebraic \( H\)-action. In this case, we denote by \( M_{\mu} \subset M \) the \( H\)-weight space of weight \( \mu \in X^* \). We call \( M \) restricted if we have \( \dim M_{\lambda} < \infty \) for every \( \lambda \in X^* \).

A \((U(\mathfrak{g}), H)\)-module \( M \) is said to be a highest weight module if \( M \) is a weight module as a \((U(\mathfrak{h}), H)\)-module and \( M \) carries a cyclic \( U(\mathfrak{g})\)-module generator that is a \((U(\mathfrak{b}), H)\)-eigenvector.

A \((U(\mathfrak{g}), H)\)-module \( M \) is said to be an integrable module if it is a restricted weight module, we have \( \dim \text{Span}_{\mathbb{K}}\{E_i^{(n)}F_i^{(m)}v\} < \infty \) for each \( v \in M \) and \( i \in I \), and it integrates to an algebraic \( SL(2, i)\)-action that is compatible with the \( H\)-action.

**Lemma 1.2.** Let \( M \) be an integrable \( U(\mathfrak{g})\)-module. Then, every \( U(\mathfrak{g})\)-submodule of \( M \) is again integrable.
Theorem 1.3 (Mathieu \[19, 20\].) We have a non-dengenerate \(k\)-linear pairing
\[
\langle \bullet, \bullet \rangle : U(\mathfrak{n}^-) \otimes k[\hat{\mathfrak{n}}^-] \ni (P, f) \mapsto (Pf)(1) \in k.
\]

Proof. Note that \(U(\mathfrak{n}^-)\) is equipped with a restricted \((U(\mathfrak{h}), H)\)-module structure arising from the adjoint action of \(H\). Hence, the (restricted) \(k\)-dual of \(U(\mathfrak{n}^-)\) is well-defined. Moreover, the natural Hopf algebra structure of \(U(\mathfrak{n}^-)\) (so that \(\Delta(F_i^{(m)}) = \sum F_i^{(m)} \otimes F_i^{(n-m)}\) for each \(i \in \mathbb{I}\) and \(n \geq 0\)) induces a commutative bialgebra structure on \(U(\mathfrak{n}^-)^{\vee}\). Then, Spec \(U(\mathfrak{n}^-)^{\vee}\) is the pro-algebraic group associated to \(\mathfrak{n}^-\) in \[19, 20\] by \[20\] Lemme 2.

By \[20\] Lemme 3, Spec \(U(\mathfrak{n}^-)^{\vee}\) satisfies the conditions on \(\hat{\mathfrak{n}}^-\) listed above. By replacing the arguments in \[15\] \S 6.1 involving the exponential maps to our pro-unipotent group structures of \(\hat{\mathfrak{n}}^-\) and unipotent one-parameter subgroups \(\{\rho_\alpha\}_\alpha\), we deduce that \((G^-, N(H), \hat{\mathfrak{n}}^-, N, H, S)\) satisfies the conditions in \[15\] Definition 5.2.1 by \[15\] Theorem 6.1.17 and its proof.

We define \(P := \bigoplus_{i \in \mathbb{I}} \mathbb{Z} \omega_i, P_+ := \bigoplus_{i \in \mathbb{I}} \mathbb{Z}_{\geq 0} \omega_i, \) and \(P_{++} := \bigoplus_{i \in \mathbb{I}} \mathbb{Z}_{\geq 1} \omega_i\). For \(J \subset \mathbb{I}\), we set \(P_J := \bigoplus_{i \in J} \mathbb{Z}_{\geq 0} \omega_i\). For each \(\lambda \in P\), we have a Verma module \(M(\lambda)\) defined as:
\[
M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} \mathbb{k}_\lambda.
\]
The Verma modules are restricted weight modules and are generated by a unique vector \(v_\lambda\) with \(H\)-weight \(\lambda\). We define
\[
L(\lambda) := M(\lambda)/\sum_{i \in \mathbb{I}} U(\mathfrak{g})F_i^{(\langle \alpha^\vee_i, \lambda \rangle + 1)}v_\lambda.
\]

Lemma 1.4. For each \(\lambda \in P_+\), the module \(L(\lambda)\) is the maximal integrable quotient of \(M(\lambda)\).

Proof. The assertion is \[15\] Lemma 2.1.7 when \(\text{char} \mathbb{k} = 0\). Its proof also asserts that every integrable module is a quotient of \(L(\lambda)\) when \(\text{char} \mathbb{k} > 0\) (as an effect of our definition of integrality). For each \(i \in \mathbb{I}\), the module
\[
M(\lambda)/U(\mathfrak{g})F_i^{(\langle \alpha^\vee_i, \lambda \rangle + 1)}v_\lambda
\]
is \(SL(2,i)\)-integrable (recall that the pro-unipotent radical of the parabolic subgroup corresponding to \(i \in \mathbb{I}\) is \(SL(2,i)\)-stable by the GCM condition \(\langle \alpha^\vee_i, \alpha_j \rangle \leq 0\) when \(i \neq j\) \[12\] §1.1]), and it is the maximal \(SL(2,i)\)-integrable quotient of \(M(\lambda)\). Hence, we deduce that \(L(\lambda)\) is the maximal integrable quotient of \(M(\lambda)\) as required.

Corollary 1.5. For each \(\lambda \in P_+\), the \(H\)-character of \(L(\lambda)\) obeys the Weyl-Kac character formula.

Proof. This is \[15\] Theorem 8.3.1 when \(\text{char} \mathbb{k} = 0\). When \(\text{char} \mathbb{k} > 0\), the arguments in \[19\] asserts that some integrable submodule of \(L(\lambda)\) obtained as a successive application of Demazure-Joseph functors obeys the Weyl-Kac character formula. As such a submodule contains \(v_\lambda\), it must be the whole \(L(\lambda)\).
For each \( w \in W \) and \( \lambda \in P^+ \), we have a unique non-zero vector \( v_{w\lambda} \in L(\lambda) \) of weight \( w\lambda \) up to scalar. We define the thin Demazure module and thick Demazure module as:

\[
L_w(\lambda) := U(n)v_{w\lambda}, \quad L^w(\lambda) := U(n^-)v_{w\lambda} \subset L(\lambda).
\]

These admit \( H \)-eigenspace decompositions.

We define the tensor product of two restricted weight modules \( M, N \) as:

\[
M \otimes N := \bigoplus_{\lambda, \mu \in X^*} M_\lambda \otimes N_\mu.
\]

We define the dual of a restricted weight module \( M \) as:

\[
M^\vee := \bigoplus_{\lambda \in X^*} M^*_\lambda,
\]

for which the natural inclusion \( M^\vee \subset M^* \) defines a \( H \)-submodule. The completion of a restricted weight module \( M \) is defined as:

\[
M^\wedge := \prod_{\lambda \in X^*} M_\lambda.
\]

It is straightforward to see that if \( M \) admits a Lie algebra action that contains \( \mathfrak{h} \) whose action prolongs to the \( H \)-action, then so are \( M^\vee \) and \( M^\wedge \). Note that the \( H \)-action on \( M^\vee \) is \( H \)-finite.

**Definition 1.6** (Thin flag varieties; [15] §7.1). The thin flag variety \( X \) is defined set-theoretically as \( G^+(k)/\tilde{B}^+(k) \), and the generalized thin flag variety \( X_J \) for \( J \subset \mathfrak{l} \) is defined set-theoretically as \( G^+(k)/\tilde{B}^+_J(k) \). In particular, we have \( X = X_\emptyset \). Their indscheme structures are given through an embedding into \( \bigcup_{w \in W} \mathbb{P}(L_w(\lambda)) = \mathbb{P}(L(\lambda)) \) for \( \lambda \in P^+_+ \) (see [15] §7). For each \( w \in W \), we set \( X_w := X \cap \mathbb{P}(L_w(\lambda)) \) and \( X_{w,J} := X_J \cap \mathbb{P}(L_w(\lambda)) \), and call them the thin Schubert variety and the generalized thin Schubert variety, respectively.

**Remark 1.7.** The (ind-)scheme structures of \( X, X_J, X_w, X_{w,J} \) are independent of the choice of \( \lambda \) (see [15] Theorem 7.1.15 and Remark 7.1.16] and Mathieu [20] Corollaire 2).

By [15] Proposition 7.1.15], we know that \( X = \bigcup_w X_w \) and \( X_J = \bigcup_w X_{w,J} \).

We have an embedding \( L(\lambda) \subset L(\lambda)^\wedge \) for \( \lambda \in P^+_+ \). The group \( G^- \) acts on \( L(\lambda)^\wedge \), while the group \( G^+ \) acts on \( L(\lambda) \subset L(\lambda)^\wedge \). Let \( \mathcal{O}^\wedge \) be the \( \tilde{N}^- \)-orbit of \([v_\lambda]\) in \( \mathbb{P}(L(\lambda)^\wedge) \), whose scheme structure is independent of the choice of \( \lambda \in P^+_+ \).

**Definition 1.8** (Thin flag manifolds; [9] §5.8). The thin flag manifold \( \mathcal{X}' \) is defined set-theoretically as the union of \( \mathcal{N}(H)(k) \)-translates of \( \mathcal{O}^\wedge(k) \).

**Remark 1.9.** In the following, we only need to use the fact that the scheme structure of the thin flag manifold \( \mathcal{X}' \) given in [9] has \( \mathcal{X}'(k) = \mathcal{N}(H)(k) \cdot \mathcal{O}^\wedge(k) \) as its set of \( k \)-valued points, it admits the \( G^- \)-action, and it has \( \mathcal{O}^\wedge \) as its \( \tilde{B}^- \)-stable (affine open) subscheme. Note that we have \( \mathcal{O}^\wedge \cong \tilde{N}^- \), and its scheme structure is the same as these transported from \( \mathbb{P}(L(\lambda)^\wedge) \) (for every choice of \( \lambda \in P^+_+ \)) or the Grassmannian employed in [9].
Remark 1.10. Assume \( g \) not to be of finite type. By construction, we easily find an inclusion \( X \subseteq \mathcal{X}' \). This inclusion cannot be an equality as the dimension of \( X \) is countable, while the dimension of \( \mathcal{X}' \) is uncountable. In general, \( X \) is not smooth \([3]\), but can be formally smooth \([22]\), while \( \mathcal{X}' \) is always smooth (in the sense it is a union of affine spaces) by construction.

Theorem 1.13 identifies \( M(\lambda)^\vee \) with a rank one \( k[\widehat{N}^-] \)-module. This also induces an inclusion

\[
L(\lambda)^\vee \hookrightarrow M(\lambda)^\vee \cong k[\widehat{N}^-] \otimes_k k_{-\lambda} \quad \lambda \in P_+.
\]

Note that \( M(\lambda)^\vee \) naturally admits an action of \( U(g) \), with a unique cocyclic \( H \)-eigenvector of weight \(-\lambda\). Hence, we have an inclusion

\[
\bigoplus_{\lambda \in P_+} L(\lambda)^\vee \subset \bigoplus_{\lambda \in P_+} M(\lambda)^\vee \cong k[\widehat{N}^-] \otimes \bigoplus_{\lambda \in P_+} k_{-\lambda} \subset k[\widehat{B}^-],
\]

where the RHS is a commutative ring.

Lemma 1.11. For each \( \lambda, \mu \in P_+ \), we have a unique \( U(g) \)-module morphism (up to a scalar)

\[
m_{\lambda, \mu} : L(\lambda)^\vee \otimes L(\mu)^\vee \longrightarrow L(\lambda + \mu)^\vee
\]

that makes \( \bigoplus_{\lambda \in P_+} L(\lambda)^\vee \) into an integral commutative subring of \( k[\widehat{N}^-] \). Moreover, the map \( m_{\lambda, \mu} \) is surjective for every \( \lambda, \mu \in P_+ \), and the ring \( \bigoplus_{\lambda \in P_+} L(\lambda)^\vee \) is generated by \( \bigoplus_{i \in I} L(\varpi_i)^\vee \).

Proof. By the comparison of the defining equation, we have a unique \( U(g) \)-module map (up to scalar)

\[
m^*_{\lambda, \mu} : L(\lambda + \mu) \rightarrow L(\lambda) \otimes L(\mu)
\]

that respects the \( H \)-weight decomposition. By taking the dual, we obtain the desired map. Each \( L(\lambda) \) is a quotient of \( M(\lambda) \), and we have an isomorphism

\[
M(\lambda)^\vee \cong k[\widehat{N}^-] \otimes_k k_{-\lambda}
\]

as \( U(n^-) \)-modules. The \( H'N^- \)-equivariant multiplication of \( k[\widehat{B}^-] \) is uniquely determined by that of the \( N^- \)-fixed elements, that is \( k[X^*] \). This forces \( L(\lambda)^\vee \cdot L(\mu)^\vee \subset M(\lambda + \mu)^\vee \) inside \( k[\widehat{B}^-] \). Since the tensor product of integrable modules is integrable, we deduce that \( L(\lambda)^\vee \cdot L(\mu)^\vee \subset L(\lambda + \mu)^\vee \) inside \( k[\widehat{B}^-] \). Therefore, the inclusion \( \bigoplus_{\lambda \in P_+} L(\lambda)^\vee \) respects the product structure (uniquely) induced by \( m_{\lambda, \mu} \). The resulting ring is commutative and integral by \([20]\) Lemme 2, and its multiplication maps are surjective by \([20]\) Corollaire 2.

The commutativity of the product and the integrality of \( \bigoplus_{\lambda \in P_+} L(\lambda)^\vee \) can be also deduced from these of \( k[\widehat{B}^-] \) (though our Theorem 1.13 depends on these facts through \([20]\) Lemme 2) unless we employ the theory of global base \([11][12]\) to prove it by additionally assuming \( g \) is symmetrizable).

Definition 1.12. Let \( J \subseteq I \). For a \( P^2 \)-graded ring \( R = \bigoplus_{\lambda \in P^2_+} R_\lambda \) with \( R_0 = k \) that is generated by \( \bigoplus_{i \in I \setminus J} R_{\varpi_i} \), we define \( \text{Proj}_J R \) to be

\[
\text{Proj}_J R := \left( \text{Spec } R \setminus \{ x \in \text{Spec } R \mid x \neq 0 \text{ on } R_{\varpi_i}, \forall i \in I \setminus J \} \right) / H,
\]

where \( H \) acts on \( R_{\varpi_i} \) through the character \( \varpi_i \) for each \( i \in I \). We might drop subscript \( J \) when the meaning is clear from the context.
Remark 1.13. We note that our condition guarantees \( \text{Proj}_J R \subset \prod_{i \in I \setminus J} \mathbb{P}(R_{\lambda_i}^*) \), that in turn implies that \( P^J_+ \) is in the closure of the ample cone of \( \text{Proj}_J R \).

We denote the ring \( \bigoplus_{\lambda \in P^J_+} L(\lambda)^\vee \) in Lemma 1.11 by \( R \). We define

\( \mathbb{X} := \text{Proj} R. \)

Note that each \( SL(2, i) \) \((i \in I)\) and \( H \) acts on \( L(\lambda)^\vee \), and hence on \( \mathbb{X} \). Hence, we derive an action of \( N(H) \) on \( \mathbb{X} \). By construction, we have a line bundle \( \mathcal{O}_\mathbb{X}(\lambda) \) on \( \mathbb{X} \) for each \( \lambda \in P^J_+ \).

**Corollary 1.14.** For each \( w \in W \) and \( \lambda, \mu \in P_+ \), the multiplication map \( m_{\lambda, \mu} \)

\[ m_{\lambda, \mu} : L_w(\lambda)^\vee \otimes L_w(\mu)^\vee \rightarrow L_w(\lambda + \mu)^\vee , \quad m_{\lambda, \mu}^w : L_w^\vee(\lambda)^\vee \otimes L_w^\vee(\mu)^\vee \rightarrow L_w^\vee(\lambda + \mu)^\vee \]

that define quotient rings of \( R \) (and hence they are associative).

**Proof.** By the dual of Lemma 1.11 we have \( L(\lambda + \mu) \subset L(\lambda) \otimes L(\mu) \).

By \( m_{\lambda, \mu}^w(v_{w(\lambda + \mu)}) = v_{w\lambda} \otimes v_{w\mu} \), we deduce that the inclusion \( L(\lambda + \mu) \subset L(\lambda) \otimes L(\mu) \) yields inclusions \( L_w(\lambda + \mu) \subset L_w(\lambda) \otimes L_w(\mu) \) and \( L^w(\lambda + \mu) \subset L^w(\lambda) \otimes L^w(\mu) \). Hence, the multiplication map \( m_{\lambda, \mu} \) induce well-defined surjective maps

\[ m_{\lambda, \mu}^w : L_w^\vee(\lambda)^\vee \otimes L_w^\vee(\mu)^\vee \rightarrow L_w^\vee(\lambda + \mu)^\vee , \quad m_{\lambda, \mu}^w : L_w^\vee(\lambda)^\vee \otimes L_w^\vee(\mu)^\vee \rightarrow L_w^\vee(\lambda + \mu)^\vee \]

that define quotient rings of \( R \) (and hence they are associative). \( \square \)

For each \( w \in W \), we have two commutative algebras:

\[ R^w := \bigoplus_{\lambda \in P_+} L^w(\lambda)^\vee , \quad \text{and} \quad R_w := \bigoplus_{\lambda \in P_+} L_w(\lambda)^\vee , \]

whose multiplications are given in Corollary 1.14.

We have a natural \( G^+ \)-equivariant line bundle \( \mathcal{O}_{X_w}(\lambda) \) for each \( w \in W \) and \( \lambda \in P^J_+ \), and we have a natural \( G^+ \)-equivariant line bundle \( \mathcal{O}_{X_w, J}(\lambda) \) for each \( w \in W \) and \( \lambda \in P^J_+ \) (cf. [13 §7.2]).

**Theorem 1.15** (Mathieu [19 Théorème 3, cf. [15 Theorem 8.2.2]). For each \( \lambda \in P^J_+ \), we have

\[ H^i(X_w, \mathcal{O}_{X_w}(\lambda)) \cong \begin{cases} L_w(\lambda)^\vee & (i = 0) \\ \{0\} & (i > 0) \end{cases}. \]

The analogous assertion holds for generalized thin Schubert varieties corresponding to \( J \subset I \) for every \( \lambda \in P^J_+ \). \( \square \)

**Corollary 1.16.** For each \( w \in W \), we have \( X_w = \text{Proj} R_w \). The analogous assertion holds for generalized thin Schubert varieties corresponding to \( J \subset I \) by setting

\[ R_w, J := \bigoplus_{\lambda \in P^J_+} L_w(\lambda)^\vee . \]

**Proof.** Combine Theorem 1.15 and the fact that \( X_{w, J} \subset \mathbb{P}(L_w(\lambda)) \) is a closed immersion for each \( \lambda \in P^J_+ \) so that \( \langle \alpha_i^\vee, \lambda \rangle > 0 \) for each \( i \in I \setminus J \). \( \square \)
Thanks to Corollary 1.16, we have an embedding $X_w \subset X$ for each $w \in W$. This particularly implies $\bigcup_w X_w = X \subset X$.

**Lemma 1.17.** The set of $H$-fixed points of $X$ is in bijection with $W$.

**Proof.** A $H$-fixed point $x$ of $X$ gives a collection of non-zero $H$-eigenvectors $\{v^\lambda_w\}_{\lambda \in \mathcal{P}_+} \in \prod_{\lambda \in \mathcal{P}_+} L(\lambda)$ so that $m_{\lambda, \mu}(v^\lambda_w, v^\mu_w) = v^\lambda_w \otimes v^\mu_w$ for $\lambda, \mu \in P_+$ by Lemma 1.11. By Theorem 1.15 there exists $w \in W$ so that $x = X_w$. It follows that

$$\bigcup_{w \in W} X^H_w = X^H.$$

The set of $H$-fixed points of $X^H_w$ is in common among all characteristic and is a subset of the translation of $\{[v_\lambda]\}_{\lambda \in \mathcal{P}_+}$ by $N(H)$ that descends to $W$ (see [15, §7.1]). Therefore, we conclude that $X^H$ is in bijection with $W$.

Let $x_w$ denote the $H$-fixed point of $X^H_w \subset X$ corresponding to the cyclic $H$-eigenvectors of $\{L_w(\lambda)\}_{\lambda \in \mathcal{P}_+}$ for each $w \in W$. By examining the stabilizer, we deduce an isomorphism

$$\hat{B}^- x_w \cong \hat{A}^\infty$$

for each $w \in W$

inside $\prod_{\lambda \in \mathcal{P}_+} \mathbb{P}(L(\lambda)^\vee) = \prod_{\lambda \in \mathcal{P}_+} \mathbb{P}(L(\lambda)^{\vee,*})$. We set $\mathcal{O}^w := \hat{B}^- x_w (= \hat{N}^- x_w)$. It is easy to see that $\mathcal{O}^w$ here is isomorphic to $\mathcal{O}^c$ employed in the definition of $X$ as a $\hat{B}^-$-homogeneous space.

We denote $\hat{N}^+ x_w = N^+ x_w \subset X$ by $\mathcal{O}_w$.

**Proposition 1.18.** We have an inclusion $\mathcal{O}^c \subset X$ obtained by inverting finitely many rational functions on $X$. In other words, $\mathcal{O}^c$ is a standard open set of $X$ in the terminology of [1].

**Proof.** By [11], inverting the unique $H$-weight $-\lambda$ vector $v^\lambda_x \in L(\lambda)^\vee$ (up to scalar) yields

$$\sum_{\lambda \in \mathcal{P}_+} (v^\lambda_x)^{-1} L(\lambda)^\vee \cong U(n^-)^\vee \cong \mathbb{C}[\hat{N}^-]$$

as algebras, where the second isomorphism is through the Hopf algebra structure of $U(n^-)$. We can rearrange $\{v^\lambda_x\}_{\lambda \in \mathcal{P}_+}$ so that it is closed under the multiplication. It follows that

$$\mathcal{O}^c = X \setminus \{v^\lambda_x = 0\}_{i \in I}$$

as required.

Proposition 1.18 asserts that we have an inclusion $\mathcal{O}^c \subset X$ with a $\hat{B}^-$-action extending the $N^-$-action on $X$. By using the $SL(2, i)$-actions for every $i \in I$, we deduce an action of $\hat{B}^-$ (and hence the $G^-$-action) on $X$ extending the $N^-$-action. We set $X^w := \mathcal{O}^w \subset X$ and call it the thick Schubert variety corresponding to $w \in W$.

**Lemma 1.19.** The ind-scheme $X$ is Zariski dense in $X$.

**Proof.** Since we have $L(\lambda) = \bigcup_{w \in W} L_w(\lambda)$ for each $\lambda \in P_+$ ([15 Lemma 8.3.3]), the regular functions on $X$ can be distinguished on $X$.  

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Theorem 1.20. For each $\lambda \in P_+$, we have
\[ H^0(\mathcal{X}, \mathcal{O}_X(\lambda)) \cong L(\lambda)^\vee. \]

Proof. We first prove the first assertion. By Lemma 1.19 we have
\[ H^0(\mathcal{X}, \mathcal{O}_X(\lambda)) \cong \Gamma(X, \mathcal{O}_X(\lambda)). \]
This induces an injective map
\[ H^0(\mathcal{X}, \mathcal{O}_X(\lambda)) \subset \lim_{\leftarrow w} H^0(X_w, \mathcal{O}_{X_w}(\lambda)). \]
By Theorem 1.15 (or directly from [15, Corollary 8.3.12]; see also the proof of Lemma 2.10), we have \[ \lim_{\leftarrow w} H^0(X_w, \mathcal{O}_{X_w}(\lambda)) \cong L(\lambda)^\vee. \] Therefore, we conclude
\[ H^0(\mathcal{X}, \mathcal{O}_X(\lambda)) \subset L(\lambda)^\vee \] as $g$-modules. Here we have
\[ H^0(\mathcal{X}, \mathcal{O}_X(\lambda)) \hookrightarrow H^0(\mathcal{O}_e, \mathcal{O}_X(\lambda)) \cong M(\lambda)^\vee. \]
In particular, \[ H^0(\mathcal{X}, \mathcal{O}_X(\lambda)) \] is $H$-semisimple, and hence we deduce
\[ H^0(\mathcal{X}, \mathcal{O}_X(\lambda)) \subset L(\lambda)^\vee = L(\lambda)^\vee \cap M(\lambda)^\vee \subset M(\lambda)^\vee. \]
By examining the ring $R$, we deduce that \[ L(\lambda)^\vee \subset H^0(\mathcal{X}, \mathcal{O}_X(\lambda)). \] This forces
\[ H^0(\mathcal{X}, \mathcal{O}_X(\lambda)) \cong L(\lambda)^\vee \] as required.

Theorem 1.21 (cf. [10]). For each $\lambda \in P_+$, we have
\[ H^0(\mathcal{X}', \mathcal{O}_{X'}(\lambda)) \cong L(\lambda)^\vee. \]
If we assume \( \text{char} \, k = 0 \) in addition, then we have
\[ H^{>0}(\mathcal{X}', \mathcal{O}_{X'}(\lambda)) \cong \{0\}. \]

Proof. Since $\mathcal{X}'$ is the $G^-$-translate of $\mathcal{O}_e$, we have
\[ H^0(\mathcal{X}', \mathcal{O}_{X'}(\lambda)) \subset H^0(\mathcal{O}_e, \mathcal{O}_{X'}(\lambda)) \cong M(\lambda)^\vee. \]
Let $U \subset \mathcal{X}'$ be a $\hat{B}^-$-stable open subset. By $SL(2)$-consideration, imposing the regularity conditions on a section of $H^0(U, \mathcal{O}_{X'}(\lambda))$ along $SL(2, i)U$ is equivalent to impose the $SL(2, i)$-finiteness. We know that $G^-$ is topologically generated by $SL(2, i)$ for all $i \in I$. Therefore, the maximal integrable submodule of $M(\lambda)^\vee$ is exactly the space of global sections of $\mathcal{O}_{X'}(\lambda)$. This proves the first assertion by Lemma 1.4.

Now we assume \( \text{char} \, k = 0 \) to consider the latter assertion. The case of symmetrizable $g$ is [10, Theorem 5.2.1]. The Kempf resolution presented in [13] (8.6) is valid for arbitrary Kac-Moody algebras, as the differential between terms can be interpreted as a $SL(2)$-calculation if one removes unnecessary strata. We have the BGG resolution for arbitrary Kac-Moody algebras [10] [3] not by changing the construction (see e.g. [13] [9.2]) but by proving that the resulting homology group is integrable. Therefore, their comparison yields the second assertion in general. 

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Corollary 1.22. We have an embedding $\mathcal{X}' \hookrightarrow \prod_{\lambda \in P_+} \mathbb{P}(L(\lambda)^\vee)$ of schemes.

Proof. The morphism exists by Theorem 1.21. Since the morphism is an embedding on $\mathbb{O}^e$ and equivariant with respect to the $N(H)$-action, we conclude that it is an embedding.

Theorem 1.23. The scheme $\mathcal{X}$ is isomorphic to the thick flag manifold $\mathcal{X}'$.

Proof. We borrow some notation from the proof of Proposition 1.18. By Corollary 1.22, we have $\mathcal{X}' := \bigcup_{w \in W} O_w \subset \mathcal{X}$. We set $E := \mathcal{X}\setminus \mathcal{X}'$.

It suffices to show $E = \emptyset$. Thanks to Proposition 1.18, the set $E$ is contained in the locus that $v_{\lambda}^* = 0$ for some $\lambda \in P_+$. Note that $E$ admits natural $SL(2,i)$-action for each $i \in I$ as $R$ and $X'$ do. It follows that $E \subset \bigcap_{w \in W} \{ v_{w,\lambda}^* = 0 \}$.

For each $\lambda \in P_+$, we have a natural map $\psi_\lambda : \mathcal{X} \to \mathbb{P}(H^0(\mathcal{X}, \mathbb{O}_\mathcal{X}(\lambda))^*) = \mathbb{P}(L(\lambda)^\vee)$ by Theorem 1.20.

Claim A. The map $\psi_\lambda$ sends $E$ to $\mathbb{P}(M^\vee)$, where $M \subset L(\lambda)$ is a $U(\mathfrak{g})$-stable $H$-submodule that does not contain $H$-weight $\{ w\lambda \}_{w \in W}$-part for each $\lambda \in P_+$.

Proof. Assume to the contrary to deduce contradiction. Then, we have some $x \in E$ so that $\psi_\lambda(x) \notin \mathbb{P}(M^\vee)$ for every $U(\mathfrak{g})$-stable $H$-submodule that does not contain $H$-weight $\{ w\lambda \}_{w \in W}$-part. Then, applying $SL(2,i)$-action repeatedly, we obtain a point $y \in E$ so that $\psi_\lambda(y) \in \{ v_{\lambda}^* \neq 0 \}$. This is a contradiction and we conclude the result.

We return to the proof of Theorem 1.23. By taking the fixed point of a $G_m$-action that shrinks $\mathcal{X}'$, we deduce that $E^H \cap \mathcal{X}^H = \emptyset$.

This forces $E = \emptyset$ (our $G_m$-action always send a point to a limit point as the set of $H$-weight of $L(\lambda)$ in contained in $\lambda - Z_{\geq 0} \Delta^+$), and we conclude the assertion.

Corollary 1.24 (of the proof of Theorem 1.23). We have $\mathcal{X} = \bigcup_{w \in W} O_w$. □

Corollary 1.25. We have $X_w = O_w$, and the thin flag variety $X$ of $\mathfrak{g}$ is obtained as $\bigcup_{w \in W} X_w$ inside $\mathcal{X}$. □

Theorem 1.26 (Kashiwara [9] §4 and Kashiwara-Tanisaki [14] §1.3). For each $w, v \in W$, we have:

1. $O_w \subset X_v$ if and only if $w \leq v$;
2. $O^w \subset X^v$ if and only if $w \geq v$.

Moreover, we have $dim X_w = \ell(w)$ and $\text{codim}_X X^w = \ell(w)$. □
2 Frobenius splitting of thick flag manifolds

We retain the setting of the previous section. Let $B := N^+ H \subset \tilde{B}^+$. For each $i \in I$, we have an overgroup $B_{i} \subset B_{i+1}$ so that $\text{Lie} B_{i} \cong k F_{i} \oplus \text{Lie} B$. We similarly define $B_{i} : = N^{-} H$ and $B_{i}^{-}$ for each $i \in I$. Let $i = (i_{1}, i_{2}, \ldots, i_{k}) \in \mathbb{I}^{k}$ be a sequence. We have a Bott-Samelson-Demazure-Hansen variety

\[ Z(i) := B_{i_{1}} \times B_{i_{2}} \times B_{i_{3}} \times \cdots \times B_{i_{k}}/B. \]

In case $w = s_{i_{1}} s_{i_{2}} \cdots s_{i_{k}}$ satisfies $\ell(w) = \ell$ (i.e. $i$ is a reduced expression of $w$), we have the BSDH resolution (see e.g. [15, Chapter VIII])

\[ \sigma_{1} : Z(i) \ni (g_{1}, g_{2}, \ldots, g_{k}) \mapsto g_{1}g_{2}\cdots g_{k}B/B \in X_{w}. \]

The variety $Z(i)$ admits a left $B$-action, that makes $\sigma_{1}$ into a $B$-equivariant morphism. For each $1 \leq k \leq \ell$, we define a $B$-stable divisor $H_{k} \subset Z(i)$ by requiring $g_{k}B$ for $(g_{1}, g_{2}, \ldots, g_{k}) \in Z(i)$. Note that $H_{k}$ is naturally isomorphic to $Z(i^k)$, where $i^k \in \mathbb{I}^{k-1}$ is obtained from $i$ by omitting the $k$-th entry. In addition, every subword $i' = (i_{j_{1}}, \ldots, i_{j_{\nu}}) \in \mathbb{I}^{\nu}$ of $i$ (so that $1 \leq j_{1} < j_{2} < \cdots < j_{\nu} < \ell$) gives us a $B$-equivariant embedding described as

\[ Z(i') \ni (g_{1}, \ldots, g_{\nu}) \mapsto (\underbrace{1, \ldots, 1}_{j_{1}-1}, g_{j_{1}}, \ldots, \underbrace{1, \ldots, 1}_{j_{\nu}-j_{1}-1}, g_{j_{\nu}}, \ldots) \in Z(i). \]

We follow the generality on Frobenius splitting in [2], that considers separated schemes of finite type. We sometimes use the assertions from [2] without finite type assumption when the assertion is independent of that, whose typical disguises are properness, finite generation, and the Serre vanishing theorem. Note that a closed subscheme of a projective space is separated.

**Definition 2.1** (Frobenius splitting of a ring). Let $R$ be a commutative ring over $k$ with characteristic $p > 0$, and let $R^{(1)}$ denote the set $R$ equipped with the map

\[ R \times R^{(1)} \ni (r, m) \mapsto r^{p^{m}} \in R^{(1)}. \]

This equips $R^{(1)}$ an $R$-module structure over $k$ (the $k$-vector space structure on $R^{(1)}$ is also twisted by the $p$-th power operation), together with an inclusion $\iota : R, 1 \subset R^{(1)}$. An $R$-module map $\phi : R^{(1)} \to R$ is said to be a Frobenius splitting if $\phi \circ \iota$ is an identity.

**Definition 2.2** (Frobenius splitting of a scheme). Let $\mathcal{X}$ be a separated scheme defined over field $k$ with positive characteristic, Let $\text{Fr}$ be the (relative) Frobenius endomorphism of $\mathcal{X}$ (that induces a $k$-linear endomorphism). We have a natural inclusion $\iota : \mathcal{O}_{X} \to \text{Fr}_{\ast} \mathcal{O}_{X}$. A Frobenius splitting of $\mathcal{X}$ is a $\mathcal{O}_{X}$-linear morphism $\phi : \text{Fr}_{\ast} \mathcal{O}_{X} \to \mathcal{O}_{X}$ so that the composition $\phi \circ \iota$ is the identity.

**Definition 2.3** (Compatible splitting). Let $\mathcal{Y} \subset \mathcal{X}$ be an inclusion of separated schemes defined over $k$. A Frobenius splitting $\phi$ of $\mathcal{X}$ is said to be compatible with $\mathcal{Y}$ if $\phi(\text{Fr}_{\ast} \mathcal{I}_{\mathcal{Y}}) \subset \mathcal{I}_{\mathcal{Y}}$.

**Remark 2.4.** A Frobenius splitting of $\mathcal{X}$ compatible with $\mathcal{Y}$ induces a Frobenius splitting of $\mathcal{Y}$ (see e.g. [2] Remark 1.1.4 (ii))).
Theorem 2.5 ([2] Lemma 1.1.11 and Exercise 1.1.E). Let \( X \) be a separated scheme of finite type over \( k \) with semiample line bundles \( L_1, \ldots, L_r \). If \( X \) admits a Frobenius splitting, then the multi-section ring
\[
\bigoplus_{n_1, \ldots, n_r \geq 0} \Gamma(X, L_1^{\otimes n_1} \otimes \cdots \otimes L_r^{\otimes n_r})
\]
admits a Frobenius splitting \( \phi \). Moreover, a closed subscheme \( Y \subset X = \Proj S \) admits a compatible Frobenius splitting if and only if the homogeneous ideal \( I_Y \subset S \) that defines \( Y \) satisfies \( \phi(I_Y) \subset I_Y \).

\[ \square \]

Definition 2.6 (\( B \)-canonical splitting). Let \( X \) be a separated scheme equipped with a \( B \)-action. A Frobenius splitting \( \phi \) is said to be \( B \)-canonical if it is \( H \)-fixed, and each \( i \in I \) yields
\[
\rho_{\alpha_i}(z)\phi(\rho_{\alpha_i}(-z)f) = \sum_{j=0}^{p-1} \phi_{i,j}(f),
\]
where \( \phi_{i,j} \in \Hom_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X) \). We similarly define the notion of \( B^- \)-canonical splitting by using \( \{\rho_{-\alpha_i}\}_{i \in I} \) instead. The \( B \)-canonical splitting of a commutative ring \( S \) over \( k \) is defined through its spectrum.

Theorem 2.7 ([2] Exercise 4.1.E.2). Assume that \( \text{char } k > 0 \). For each \( i \in I \), there exists a unique \( B \)-canonical splitting of \( Z(i) \) that is compatible with the subvarieties \( Z(i') \) obtained by subwords \( i' \) of \( i \).

\[ \square \]

Corollary 2.8. In the setting of Theorem 2.7, the restriction of the \( B \)-canonical splitting to \( Z(i') \) is \( B \)-canonical.

Proof. The condition of \( B \)-canonical splitting is preserved by the restriction to a \( B \)-stable compatibly split subset.

\[ \square \]

Lemma 2.9. For each \( w \in W \), the ind-scheme \( (X \cap X^w) \) is Zariski dense in \( X^w \).

Proof. Assume to the contrary to deduce contradiction. Let \( w = s_{i_1}s_{i_2} \cdots s_{i_l} \) be a reduced expression. For a \( B^- \)-stable subset \( Y \subset X^v \) that is not Zariski dense in \( X^w \) and \( i \in I \) so that \( s_i < v \), the inclusion
\[
SL(2, i)Y \subset \rho_{\alpha_i}(\mathcal{O}_X)\mathcal{O}_X \subset SL(2, i)X^v = X^v \cup X^{s_i^v} = X^{s_i^v}
\]
cannot be Zariski dense. Moreover, \( SL(2, i)Y \) is again \( B^- \)-stable by the Bruhat decomposition (of \( SL(2, i) \)). As \( (X \cap X^w) \) is stable under the action of \( B^- \), we repeatedly apply the above estimate to conclude
\[
SL(2, i_2) \cdots SL(2, i_1)(X \cap X^w) \subset X
\]
is not Zariski dense. By the Bruhat decomposition, we have \( X^{s_i^v} \subset SL(2, i)X^v \) for each \( i \in I \) and \( v \in W \). Each rational point \( x \) of \( X \) satisfies
\[
SL(2, i_1) \cdots SL(2, i_1)x \cap X^w \neq \emptyset
\]
by its repeated application. It follows that
\[
X = SL(2, i) \cdots SL(2, i_1)(X \cap X^w) \subset X
\]
is also not Zariski dense. This gives a contradiction to Lemma 1.19 and we conclude the result.

\[ \square \]
Lemma 2.10. Assume that char $k > 0$. For each $w \in W$, the ring $R$ and $R_w$ admits a $B$-canonical splitting.

Proof. Let $i \in I^t$ be a sequence so that $\text{Im } \pi_i = X_w$. Then, we have

$$H^0(Z(i), \pi_i^* \mathcal{O}_X(\lambda)) \cong H^0(X_w, \mathcal{O}_X(\lambda)) \cong L_w(\lambda)^* \quad \lambda \in P_+$$

by [13, Théorème 3] (cf. [15, Theorem 8.2.2]).

Applying Theorem 2.5 we deduce that the ring $R_w$ admits a $B$-canonical splitting. Choose a series of sequences $i_k \in I^k \ (k \geq 1)$ so that

1. $i_k$ is obtained from $i_{k+1}$ by omitting the first entry:

2. $\bigcup_{k \geq 1} \pi_{i_k}(Z(i_k)) = X$,

(whose existence is guaranteed by the subword property of the Bruhat order [15, Lemma 1.3.16]). Let $w_k \in W$ be so that $X_{w_k} = \pi_{i_k}(Z(i_k))$ (that exists as $Z(i_k)$ is irreducible). Then, we have

$$L(\lambda) = \lim_{k \to \infty} L_{w_k}(\lambda).$$

This induces a dense inclusion of algebras

$$R \subset \lim_{k \to \infty} R_{w_k},$$

where the LHS is the $H$-finite part of the RHS. The system \{R_{w_k}\}_{k \geq 1} is an inverse system with surjective transition maps. Therefore, Corollary 2.8 induces a Frobenius splitting of $\lim_{k \to \infty} R_{w_k}$ from the $B$-canonical splittings of \{R_{w_k}\}_{k \geq 1}. Since our splitting preserves the $H$-weights, it descends to the $H$-finite part $R$ as required.

Corollary 2.11. Assume that char $k > 0$. The ring $R$ admits a $B^-$-canonical splitting, and hence $X$ is $B^-$-canonically Frobenius split.

Proof. We retain the setting of the proof of Lemma 2.10. Our ring $R$ is a $H$-finite graded algebra that admits a $B$-canonical splitting. Note that $R$ admits a rational action of $SL(2, i)$ for each $i \in I$ as each $L(\lambda)$ is integrable. Hence, [2, Excercise 4.1 (1)] forces a $B$-canonical splitting of $R$ to induce a $B^-$-canonical splitting as desired.

Corollary 2.12. Assume that char $k > 0$. For each $w \in W$, the $B^-$-canonical splitting of $X$ (constructed above) is compatible with $X_w$.

Proof. We argue along the line of [16, Proposition 5.3], that was stated with the symmetrizability assumption (that we drop here).

We already know that the scheme $X$ (or rather its projective coordinate ring) admits a $B^-$-canonical splitting by Corollary 2.11.

We show that our splitting splits the $H$-fixed points as in [16, Proof of Proposition 5.3 Assertion II]. The $H$-fixed point $x_w$ of $X$ corresponding to $w \in W$ is contained in $X_w$. Hence, we have $H$-algebra morphisms

$$\bigoplus_{\lambda \in P_+} \mathbb{k}_{-w\lambda} \hookrightarrow R_w \twoheadrightarrow \bigoplus_{\lambda \in P_+} \mathbb{k}_{-w\lambda}$$
corresponding to \(x_w \in X_w\), whose composition is the identity. As our Frobenius splitting induces that of \(R_w\) and preserves \(H\)-weight spaces, we conclude that our splitting splits the \(H\)-fixed points of \(X\) by Lemma 1.17.

We show that our splitting splits each \(X_w\) compatibly as in [16, Proof of Proposition 5.3 Assertion III] to complete the proof. Let \(I_w\) be the ideal of \(x_w\). The ideal \(I_w\) is preserved by our Frobenius splitting. Therefore, the ideal \(I^w := \bigcap_{b \in B} b \cdot I_w \subset R\) is preserved by our \(B^-\)-canonical splitting thanks to [2, Proposition 4.1.8]. By Lemma 2.9 the ideal \(I^w\) defines the Zariski closure of \(\hat{B}^{-} \cdot x_w\) (as that is the same as \(B^{-} \cdot x_w\)) inside \(X\), that is \(X^w\). It follows that \(X\) splits compatibly with \(X^w\) through our splitting as required.

**Remark 2.13.** According to Kumar-Schwede [16], the essential part of our proof of Corollary 2.12 traces back to a result of Olivier Mathieu. As the author has no access to it, he cites it from [16].

**Corollary 2.14.** For each \(w \in W\), the scheme \(X^w\) is integral.

**Proof.** Apply [2, Proposition 1.2.1] to Corollary 2.12 if \(\text{char } k > 0\). As the integrality of \(X^w\) follows by the integrality of \(R^w\), we apply [2, Proposition 1.6.5] to subalgebras of \(R^w\) generated by finitely many \(H\)-weight spaces (so that it is finitely generated) to deduce the integrality in \(\text{char } k = 0\).

By restricting \(O_X(\lambda) (\lambda \in P)\), we obtain a line bundle \(O_{X^w}(\lambda)\) on \(X^w\) for each \(w \in W\).

Let \(J \subset I\). Consider the subring

\[ R_J := \bigoplus_{\lambda \in P^+_J} L(\lambda)^\vee \subset R. \]

We set \(X_J := \text{Proj } R_J\). This also defines a line bundle \(O_{X_J}(\lambda)\) for each \(J \subset I\) and \(\lambda \in P^+_J\). We have natural map

\[ \pi_J : X \to X_J. \]

**Lemma 2.15.** Let \(J \subset I\). The morphism \(\pi_J\) is \(G^-\)-equivariant and surjective. We have a \(B^-\)-canonical splitting of \(X_J\) that is compatible with the \(\hat{B}^-\)-orbits.

**Proof.** Since the dual of the homogeneous coordinate rings of \(X\) and \(X_J\) admits the \(B^-\)-action and \(N(H)\)-action, we conclude that \(\pi_J\) is equivariant with respect to the group generated by \(\hat{B}^-\) and \(N(H)\), that is \(G^-\).

The \(B^-\)-canonical splitting of \(X\) induces that of \(X_J\) through the description of its projective coordinate ring. This must be compatible with the Zariski closure of the image of \(\hat{B}^-\)-orbits. Hence, it remains to show that \(\pi_J\) is surjective.

Fix \(w \in W\). The analogous map to \(\pi_J\) defined for \(X_w\) is surjective (see [15 Proposition 7.1.14]). The same proof as Lemma 2.17 (relying on [15]) implies \(X_J^H \subset \pi_J(\hat{X}^H)\). Hence, the same argument as in Theorem 1.23 yields that every \(\hat{B}^-\)-orbit of \(X_J\) is the image of a \(\hat{B}^-\)-orbit of \(X\) as required.

**Lemma 2.16.** Let \(J \subset I\). The fiber of \(\pi_J\) is isomorphic to the thick flag manifold of the Kac-Moody subalgebra of \(\mathfrak{g}\) corresponding to \(J\). Moreover, we have \((\pi_J)_* O_X(\lambda) \cong O_{X_J}(\lambda)\) for \(\lambda \in P^+_J\).
Proof. Let \( \mathfrak{g}' \) denote the Kac-Moody algebra that is a subalgebra of \( \mathfrak{g} \) corresponding to \( J \), and let \( W' \) denote its Weyl group that is a subgroup of \( W \). Let \( R^1 \) be the minimal homogeneous coordinate ring of \( \pi_1^{-1}(B_2/B_1) \) so that we have an algebra map \( \phi : R \to R^1 \) corresponding to \( \pi_1^{-1}(B_2/B_1) \subset X \). Since \( \mathcal{X}_J \) is \( G^- \)-homogeneous, we find that the scheme \( \pi_1^{-1}(B_2/B_1) \) is reduced.

Let \( \tilde{N}_J \subset \tilde{B}^- \) be the pro-unipotent radical of \( \tilde{B}_J^- \). We find a \( H \)-stable complementary pro-unipotent group \( \tilde{U} \subset \tilde{B}^- \) so that \( \tilde{N}^- = \tilde{U}\tilde{N}_J \), \( \tilde{U} \) normalizes \( \tilde{N}_J \), and \( \tilde{U} \cap \tilde{N}_J = \{ \text{id} \} \).

A point \( x \in \mathcal{X} \) is written as \( x = gwv \) for some \( g \in \tilde{N} \) and a lift \( \tilde{w} \in N(H) \) of \( w \in W \), so that \( x \) gives a point \([gwv_w] \in \mathbb{P}(L(\pi_1)^\wedge)\) for each \( i \in \mathbb{I} \). If \( g \notin \tilde{U} \), then we have

\[
gwv_w \in \{ kwv_w \} \notin \{ kwv \} \quad \text{for some } i \notin J \text{ and every } w \in W'.
\]

Note that \( w \in W' \) belongs to \( W' \) if and only if \( w\alpha_i = \alpha_i \) for every \( i \notin J \). Therefore, we find that every point in \( \pi_1^{-1}(B_2/B_1) \) is of the form \( x = gwv \) for \( g \in \tilde{U} \) and \( w \in W' \) (by Corollary 1.24 and Lemma 1.15).

Therefore, if we represent a point \( x \in \pi_1^{-1}(B_2/B_1) \) as a point \( \{ [x_i] \} \in \prod_{i \in J} \mathbb{P}(L(\pi_i)^\wedge) \) (using Definition 1.12), then the vector \( x_i \) does not contain \( H \)-weights except for \( \alpha_i - \mathbb{Z}_{\geq 0} \{ \alpha_i \mid i \in J \} \). By the cocyclicity of the dual Verma modules, it follows that \( x_i \) belongs to the (maximal) integrable highest weight module \( L'(\lambda) \) of \( \mathfrak{g}' \) spanned by \( v_w \). Moreover, the \( H \)-weight comparison implies that \( L'(\lambda)^\wedge \subset L(\lambda)^\wedge \) is precisely the \( \tilde{N}_J^- \)-invariant part.

Since \( \pi_1^{-1}(B_2/B_1) \) is \( \tilde{N}^- \)-invariant and reduced, it follows that the map \( \phi \) factors through

\[
R^1 := \bigoplus_{i \in J} L'(\alpha_i)^\wedge,
\]

that is the homogeneous coordinate ring of the thick flag manifold of \( \mathfrak{g}' \). As \( \pi_1^{-1}(B_1/B_1) \) is a closed subscheme of \( \mathcal{X} \), we conclude the \( \phi \) must be in fact an equality. Hence, the fibers of \( \pi_2 \) are isomorphic to the thick flag manifold of \( \mathfrak{g}' \).

By examining the sections on the fibers of \( \pi_2 \), we conclude that \( (\pi_1)_* \mathcal{O}_X(\lambda) \) is a line bundle. Since \( \mathcal{X} \) is homogeneous and \( (\pi_1)_* \mathcal{O}_X(\lambda) \) is \( G^- \)-equivariant, we conclude the assertion by the comparison (of characters) on fibers. \( \square \)

**Theorem 2.17** ([13] second part of Conjecture 8.10). For each \( \lambda \in P_+ \) and \( w, v \in W \) so that \( v < w \), the natural restriction map

\[
H^0(\mathcal{X}^w, \mathcal{O}_{\mathcal{X}^w}(\lambda)) \longrightarrow H^0(\mathcal{X}^w, \mathcal{O}_{\mathcal{X}^w}(\lambda))
\]

is surjective.

**Proof.** We set \( J := \{ i \in \mathbb{I} \mid \langle \alpha_i^\vee, \lambda \rangle = 0 \} \). By Lemma 2.16, the assertion reduces to the surjectivity of

\[
H^0(\mathcal{X}_J^w, \mathcal{O}_{\mathcal{X}_J^w}(\lambda)) \longrightarrow H^0(\mathcal{X}_J^w, \mathcal{O}_{\mathcal{X}_J^w}(\lambda)),
\]

where \( \mathcal{X}_J^w := \pi_1(\mathcal{X}^w) \) for each \( w \in W \). We might omit the subscript \( J \) in the below for simplicity. Note that \( \mathcal{O}_{\mathcal{X}_J}(\lambda) = \mathcal{O}_{\mathcal{X}}(\lambda) \) is ample. By the associativity of the restriction maps (and Theorem 1.20), we can assume \( v = e \).

For each \( v \in W \) and \( w \in W' \), we have a restriction map

\[
\varphi_v^w : H^0(\mathcal{X}^w, \mathcal{O}_{\mathcal{X}^w}(\lambda)) \longrightarrow H^0(\mathcal{X}^w \cap X_w, \mathcal{O}_{\mathcal{X}^w \cap X_w}(\lambda)).
\]
By Lemma 2.9, the inverse limit of \( \{ \varphi_w^i \}_w \) yields an inclusion

\[
\varphi^v : H^0(X^v, \mathcal{O}_{X^v}(\lambda)) \rightarrow H^0(\mathcal{X}^v \cap X, \mathcal{O}_{\mathcal{X}^v \cap X}(\lambda)).
\]  

(2.2)

Let \( \Psi \subset \Delta^+ \) be a finite set. Let us consider a linear functional \( h \) on \( \Delta^+ \subset X^*(H) \otimes \mathbb{R} \) so that \( 0 < h(\alpha_i) \) for each \( i \in I \) and \( h(\Psi) < 1 \). Then, the subset

\[ \Delta^+(h) := \{ \beta \in \Delta^+ \mid h(\beta) < 1 \} \subset \Delta^+ \]

is finite, and every \( \mathbb{Z}_{\geq 0} \)-linear combination of elements of \( \Delta^+ \setminus \Delta^+(h) \) does not belong to \( \Delta^+(h) \).

For each \( w \in W \), the set of \( H \)-weights of \( \bigoplus_{i \in I} L_w(\omega_i) \) is finite, and hence so is the set \( \Psi_w \) of positive roots obtained by the difference of two \( H \)-weights of \( \bigoplus_{i \in I} L_w(\omega_i) \). Applying the above construction, we can find a partition \( \Delta^+ = \Delta^+_1 \sqcup \Delta^+_2 \) (\( \Delta^+_1 \) is \( \Delta^+(h) \) obtained by setting \( \Psi = \Psi_w \)) so that every \( x \in \tilde{N}^- \) factors into \( x = x_1x_2 \), where \( x_2 \) is the product of one-parameter subgroup corresponding to \( \Delta^+_2 \), and

\[ L_w(\omega_i) \cap x_2L_w(\omega_i) = \{ v \in L_w(\omega_i) \mid x_2v = v \}. \]

This implies

\[ X^u \cap X_w = B^u x_v \cap Bx_w = B^u x_v \cap Bx_v, \]

where the most RHS is the definition of the Richardson variety in [16] (when \( J = \emptyset \)).

In case \( J = \emptyset \), [16] Proposition 5.3] equips \( (X^v \cap X_w) \) a Frobenius splitting compatible with \( (X^v \cap X_w)^{\prime} \)'s in its closure.

In case \( J \neq \emptyset \), the pullback of \( (X^v \cap X_w)^{\prime} \) to \( \mathcal{X} \) is a (possible infinite) union of Richardson varieties of \( X \subset \mathcal{X} \). Therefore, we can transplant the Frobenius splitting \( \phi \) (that we have constructed through the Richardson varieties) on \( \mathcal{X} \) to a Frobenius splitting of \( (X^v \cap X_w)^{\prime} \) compatible with \( (X^v \cap X_w)^{\prime} \)'s in its closure through

\[ \text{Fr}_* \mathcal{O}_{\mathcal{X}_j} \rightarrow (\pi_j)_* \text{Fr}_* \mathcal{O}_{\mathcal{X}} \xrightarrow{\pi_n \phi} (\pi_j)_* \mathcal{O}_{\mathcal{X}} \cong \mathcal{O}_{\mathcal{X}_j}. \]

Therefore, [2 Theorem 1.2.8] yields that the map

\[ H^0(X^v \cap X_w, \mathcal{O}_{X^v \cap X_w}(\lambda)) \rightarrow H^0(\mathcal{X}^v \cap X, \mathcal{O}_{\mathcal{X}^v \cap X}(\lambda)) \]

is surjective for every \( w, v, v' \in W \) so that \( v \leq v' \) when \( \text{char} k > 0 \). Since the both of \( (X^v \cap X_w) \) and \( (X^v \cap X_w) \) are finite type schemes, [2 Corollary 1.6.3] lifts this surjection to the case of \( \text{char} k = 0 \). Hence, we deduce a surjection

\[ H^0(X^v \cap X, \mathcal{O}_{X^v \cap X}(\lambda)) \rightarrow H^0(\mathcal{X}^v \cap X, \mathcal{O}_{\mathcal{X}^v \cap X}(\lambda)) \]

for every \( v, v' \in W \) so that \( v \leq v' \) by taking the inverse limits with respect to surjective inverse systems (so that they satisfies the Mittag-Leffler condition), regardless of the characteristic.

The space \( H^0(X^v, \mathcal{O}_{X^v}(\lambda)) \) is \( H \)-finite since

\[ H^0(X^v, \mathcal{O}_{X^v}(\lambda)) \subset H^0(\mathcal{X}^v, \mathcal{O}_{\mathcal{X}^v}(\lambda)) \cong k[\mathcal{O}^v] \otimes_k k_{\lambda}. \]
In case \( v = e \), the LHS of (2.2) is given in Theorem 1.20, and the RHS is given in [19] (cf. Theorem 1.15). In particular, the LHS is the \( H \)-finite part of the RHS. Therefore, the commutative diagram

\[
\begin{array}{c}
H^0(\mathcal{X}, \mathcal{O}_\mathcal{X}(\lambda)) \to H^0(\mathcal{X} \cap \mathcal{X}, \mathcal{O}_{\mathcal{X} \cap \mathcal{X}}(\lambda)) \\
\downarrow \downarrow \downarrow \downarrow \downarrow \\
H^0(\mathcal{X}^v, \mathcal{O}_{\mathcal{X}^v}(\lambda)) \to H^0(\mathcal{X}^v \cap \mathcal{X}, \mathcal{O}_{\mathcal{X}^v \cap \mathcal{X}}(\lambda))
\end{array}
\] (2.3)

yields the surjectivity of the left vertical arrow, that implies our assertion. 

**Corollary 2.18.** Let \( \mathcal{Y}, \mathcal{Y}' \) be reduced unions of thick Schubert varieties so that \( \mathcal{Y}' \subset \mathcal{Y} \). Then, the natural restriction map

\[
H^0(\mathcal{Y}, \mathcal{O}_\mathcal{Y}(\lambda)) \to H^0(\mathcal{Y}', \mathcal{O}_{\mathcal{Y}'}(\lambda))
\]

is surjective for each \( \lambda \in P_+ \).

**Proof.** We can formally replace \( \mathcal{X}^v \cap \mathcal{X}^w \) with reduced unions of them (that are compatibly split along the intersections by our canonical splitting) in the proof of Theorem 2.17 to deduce the assertion. 

The following is a consequence of Theorem 2.17 as described in [13, §8] after Conjecture 8.10 (when \( g \) is of affine type).

**Theorem 2.19.** For each \( \lambda \in P_+ \), we have

\[
H^0(\mathcal{X}^w, \mathcal{O}_{\mathcal{X}^w}(\lambda)) \cong L^w(\lambda)^\vee.
\]

**Proof.** Combining Theorem 1.20 and Theorem 2.17, we have

\[
H^0(\mathcal{X}^w, \mathcal{O}_{\mathcal{X}^w}(\lambda))^\vee \subset L(\lambda).
\]

In addition, the integrality of \( \mathcal{X}^w \) implies that \( H^0(\mathcal{X}^w, \mathcal{O}_{\mathcal{X}^w}(\lambda))^\vee \) is cyclic as its covering module \( H^0(\mathcal{O}_{\mathcal{X}^w}, \mathcal{O}_{\mathcal{X}^w}(\lambda))^\vee \) is a \( U(n^-) \)-module with cyclic \( H \)-eigenvector \( v_{w,\lambda} \) by [20, Lemme 4]. These imply our result.

**Theorem 2.20** (Kashiwara-Shimozono [13] Proposition 3.2). For each \( w \in W \), the scheme \( \mathcal{X}^w \) is normal.

**Proof.** The argument in [13] Proposition 3.2] is stated for symmetrizable \( g \) and \( \text{char } k = 0 \), but there are no place this assumption is used until [13] Proposition 3.2] in the main body of [13].

**Corollary 2.21.** For each \( w \in W \), we have an isomorphism

\[
\mathcal{X}^w \cong \text{Proj } R^w.
\]

In particular, \( \mathcal{X}^w \) is projectively normal.

**Proof.** The first assertion is the direct consequence of Theorem 2.19 since \( \mathcal{X}^w \) is a closed subscheme of \( \mathcal{X} \).

We prove the second assertion. Corollary 1.14 and Lemma 1.11 asserts that the ring \( R^w \) is generated by \( \bigoplus_{i \in I} L^w(\varpi_i)^\vee \). This verifies a sufficient condition of projective normality (see e.g. Hartshorne [5] Chapter II, Exercise 5.14] for singly graded case) in the presence of the normality of \( \mathcal{X}^w \). Therefore, we conclude the assertion. 

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The following result implies that \( \{L_w(\lambda)\}_{w \in W} \) forms a filtration of \( L(\lambda) \) for each \( \lambda \in P_+ \), that is previously recorded when \( g \) is of affine type (see [1] Theorem 6.23). An analogous result is known for \( \{L_w(\lambda)\}_{w \in W} \) by the works of many people (cf. Littelmann [15 §8] and Kumar [15 VIII]).

**Corollary 2.22.** For each finite subset \( S \subset W \), there exists another subset \( S' \subset W \) so that

\[
\bigcap_{w \in S} L_w(\lambda) = \sum_{v \in S'} L_v(\lambda).
\]

**Proof.** Let \( T \subset W \) and set \( X(T) := \bigcup_{w \in T} X^w \) (here the union is understood to be the reduced union). We have a sequence of maps

\[
O_X(\lambda) \to O_X(T)(\lambda) \to \bigoplus_{w \in T} O_X(\lambda).
\]

Thanks to Corollary 2.18 we deduce

\[
\bigoplus_{w \in T} L_w(\lambda) = \bigoplus_{w \in T} \Gamma(X, O_X(\lambda))^w \to \Gamma(X, O_X(T)(\lambda))^w \to \Gamma(X, O_X(\lambda))^w = L(\lambda).
\]

Moreover, the restriction of the composition maps to a direct summand \( L_w(\lambda) \) yields the standard embedding. Thus, we conclude

\[
\Gamma(X, O_X(T)(\lambda))^w = \sum_{w \in T} L_w(\lambda) \subset L(\lambda). \quad (2.4)
\]

Let us divide \( T = T_1 \sqcup T_2 \), and we set \( Y_i := \bigcup_{w \in T_i} X^w \) for \( i = 1, 2 \). By [2] Proposition 1.2.1], the scheme \( Y_i \) \((i = 1, 2) \) and the scheme-theoretic intersection \( Y := Y_1 \cap Y_2 \) are reduced.

Since \( Y \) is \( \hat{B}^- \)-stable, we have \( Y = X(T') \) for some \( T' \subset W \). We have a short exact sequence

\[
0 \to O_X(T')(\lambda) \to O_{Y_1}(\lambda) \oplus O_{Y_2}(\lambda) \to O_{Y_1 \cup Y_2}(\lambda) \to 0.
\]

Thanks to (2.4) and Corollary 2.18 we conclude a short exact sequence

\[
0 \to \sum_{w \in T} L_w(\lambda) \to \left( \sum_{w \in T_1} L_w(\lambda) \right) \oplus \left( \sum_{w \in T_2} L_w(\lambda) \right) \to \sum_{w \in T'} L_w(\lambda) \to 0
\]

of \( b^- \)-modules. In particular, the third term can be identified with the intersection of the direct summands of the second term inside \( L(\lambda) \). This proves the assertion by induction on \( |S| \) (since the case \( |S| = 1 \) is apparent from \( S = S' \)).

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