K-theory and Homotopies of 2-cocycles on Transformation Groups

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February 14, 2014

Abstract

This paper constitutes a first step in the author’s program to investigate the question of when a homotopy of 2-cocycles \( \omega = \{ \omega_t \}_{t \in [0,1]} \) on a locally compact Hausdorff groupoid \( G \) induces an isomorphism of the \( K \)-theory groups of the reduced twisted groupoid \( C^\ast \)-algebras:

\[
K_\ast (C^\ast_r (G, \omega_0)) \cong K_\ast (C^\ast_r (G, \omega_1)).
\]

Generalizing work of Echterhoff et al. from [6], we show that if \( G = G \rtimes X \) is a transformation group, then whenever \( G \) satisfies the Baum-Connes conjecture with coefficients and \( X \) is compact, a homotopy \( \omega = \{ \omega_t \}_{t \in [0,1]} \) of 2-cocycles on \( G \rtimes X \) gives rise to an isomorphism

\[
K_\ast (C^\ast_r (G \rtimes X, \omega_0)) \cong K_\ast (C^\ast_r (G \rtimes X, \omega_1)).
\]

1 Introduction

In this paper, we study the question of when a homotopy of 2-cocycles \( \omega = \{ \omega_t \}_{t \in [0,1]} \) on a transformation group \( G \rtimes X \) induces an isomorphism of the twisted \( K \)-theory groups

\[
K_\ast (C^\ast_r (G \rtimes X, \omega_0)) \cong K_\ast (C^\ast_r (G \rtimes X, \omega_1)).
\]

Variants on this question have been investigated by Packer and Raeburn in [23, 24] and Echterhoff, Lück, Phillips, and Walters in [6], but its origins lie in the results established about the structure and \( K \)-theory of the rotation algebras by Rieffel, Pimsner and Voiculescu, among others, in the early 1980s.

For \( \theta \in \mathbb{R} \), let \( A_\theta \) denote the universal rotation algebra; that is, the algebra generated by two unitaries \( u, v \) satisfying the commutation relation

\[
uv = vue^{2\pi i \theta}.
\]

We can also realize \( A_\theta \) as a twisted group \( C^\ast \)-algebra:

\[
A_\theta = C^\ast (\mathbb{Z}^2, c_\theta),
\]
where the twisting 2-cocycle \( c_\theta : \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{T} \) is given by
\[
c_\theta ((j, k), (m, n)) = e^{2\pi i \theta km}.
\]

Note that the map \( \theta \mapsto c_\theta ((j, k), (m, n)) \) is continuous for each choice of \((j, k), (m, n)\); we say that \( \{c_\theta\}_{\theta \in \mathbb{R}} \) is a homotopy of 2-cocycles on \( \mathbb{Z}^2 \).

Rieffel proved in 1981 [27] that although
\[
A_\theta \sim_{ME} A_{\theta'} \iff [\theta] = \pm [\theta'] \in \mathbb{R}/\mathbb{Z},
\]
the Morita equivalence relation among the rotation algebras is less strict: \( A_\theta \sim A_{\theta'} \iff \theta, \theta' \) are in the same orbit of a natural action of \( GL_2(\mathbb{Z}) \) on \( \mathbb{R} \) (which is defined in Section 2 of [27]). Finally, Pimsner and Voiculescu proved in [25] that for any \( \theta \in \mathbb{R} \), we have
\[
K_0(A_\theta) \cong \mathbb{Z} \oplus \mathbb{Z} \cong K_1(A_\theta).
\]

In other words, \( K \)-theory is the finest invariant that fails to distinguish between the rotation algebras.

This result by Pimsner and Voiculescu was vastly generalized in a 2010 paper [6] by Echterhoff, Lück, Phillips, and Walters. For a locally compact Hausdorff group \( G \) that satisfies the Baum-Connes conjecture with coefficients \( K \), Echterhoff et al. proved in Theorem 1.9 of [6] that if \( \{\omega_t\}_{t \in [0,1]} \) is a homotopy of 2-cocycles on \( G \), then
\[
K_*(C^*_r(G \rtimes X \times [0,1], \omega_t)) \cong K_*(C^*_r(G \rtimes X, \omega_t)).
\]

Our main theorem in this paper is an extension of this result to the case when \( G \) is a transformation group \( G = H \ltimes X \) with \( X \) compact:

**Theorem 5.1** Let \( G \) be a locally compact Hausdorff group acting on a compact space \( X \) such that \( G \) satisfies the Baum-Connes conjecture with coefficients, and let \( \omega \) be a homotopy of continuous 2-cocycles on the transformation group \( G \rtimes X \). For any \( t \in [0,1] \), the \( * \)-homomorphism
\[
q_t : C^*_r(G \rtimes X \times [0,1], \omega) \to C^*_r(G \rtimes X, \omega_t),
\]
given on \( C_c(G \rtimes X \times [0,1]) \) by evaluation at \( t \in [0,1] \), induces an isomorphism
\[
K_*(C^*_r(G \rtimes X \times [0,1], \omega)) \cong K_*(C^*_r(G \rtimes X, \omega_t)).
\]

**Remark 1.1.** Higson and Kasparov proved in [11] that every a-T-menable group satisfies the Baum-Connes conjecture with coefficients. Examples of such groups include all amenable groups (hence all compact, abelian, or solvable groups), all free groups, and the Lie groups \( SO(n,1) \) and \( SU(n,1) \). Moreover, a-T-menability is inherited by closed subgroups, so any closed subgroup of the above groups also satisfies the Baum-Connes conjecture with coefficients.
Remark 1.2. Theorem 5.1 can also be viewed as a generalization of Theorem 4.2 from the 1990 paper [24] of Packer and Raeburn. When translated into the notation of the current paper, Packer and Raeburn’s Theorem 4.2 states that if $G$ is a discrete subgroup of a solvable simply-connected Lie group, and $G$ acts on a locally compact Hausdorff space $X$, then a homotopy of 2-cocycles $\{\omega_t\}_{t\in[0,1]}$ on $G \ltimes X$ induces an isomorphism

$$K_*(\mathcal{C}_r^*(G \ltimes X, \omega_0)) \cong K_*(\mathcal{C}_r^*(G \ltimes X, \omega_1)).$$

Since discrete subgroups of solvable simply-connected Lie groups satisfy the Baum-Connes conjecture with coefficients by Theorem 8.2 of [13], our Theorem 5.1 applies to a much broader class of groups than those covered in Theorem 4.2 of [24], although their result does not require the space $X$ to be compact.

1.1 Context and Future work

A transformation group $G \ltimes X$ is an example of a groupoid, a class of mathematical objects that includes groups, group actions, equivalence relations, and group bundles. The study of the full and reduced $C^*$-algebras $\mathcal{C}^*(G), \mathcal{C}^*_r(G)$ associated to a locally compact groupoid $G$ was initiated by Jean Renault in [26], and has been pursued actively by many researchers. Although Renault also defined the twisted groupoid $C^*$-algebras $\mathcal{C}^*(G, \omega), \mathcal{C}^*_r(G, \omega)$ for a 2-cocycle $\omega \in Z^2(G, T)$ in [26], these objects have received relatively little attention until recently. However, it has now become clear that twisted groupoid $C^*$-algebras can help answer many questions about the structure of untwisted groupoid $C^*$-algebras (cf. [22, 21, 5, 12, 3]), as well as classifying those $C^*$-algebras which admit diagonal subalgebras (also known as Cartan subalgebras) — cf. [16]. In another direction, [29] shows how the $K$-theory of twisted groupoid $C^*$-algebras classifies D-brane charges in many flavors of string theory.

Theorem 5.1 constitutes a first step in our research program to study the question of when a homotopy $\omega = \{\omega_t\}_{t\in[0,1]}$ of 2-cocycles on a locally compact Hausdorff groupoid $G$ induces an isomorphism of the $K$-theory groups of the reduced twisted groupoid $C^*$-algebras:

$$K_*(\mathcal{C}_r^*(G, \omega_0)) \cong K_*(\mathcal{C}_r^*(G, \omega_1)).$$

Motivated by Theorem 5.4 in [18], we will address this question in a forthcoming paper for the case when $G = \mathcal{G}_\Lambda$ is the groupoid associated to a $k$-graph $\Lambda$, as defined in Section 2 of [17].

We are unaware of any examples of groupoids $G$ and homotopies $\omega$ of cocycles where (1) fails to hold.

1.2 Outline

To tackle the proof of Theorem 5.1 we will first need to understand the reduced twisted groupoid $C^*$-algebra $\mathcal{C}_r^*(G \ltimes X, \omega)$; Section 2 reviews this construction for general groupoids $G$, in order to establish the context and simplify formulas.
In Section 3 we show that a homotopy of 2-cocycles on a compact groupoid $G$ gives rise to a trivial bundle of $C^*$-algebras over $[0,1]$. Section 4 describes carefully the isomorphism between the twisted crossed product $C^*$-algebra defined by Packer and Raeburn in [23], and the twisted groupoid $C^*$-algebra described in Section 2; this material is no doubt well known to experts but we were unable to find a precise reference, so we include it here. Finally, in Section 5 we combine the results of the preceding sections in order to prove Theorem 5.1, following the line of argument presented in Theorem 1.9 of [6]. Since this argument relies on a technical result in equivariant $KK$-theory (namely, Theorem 1.5 of [4]), Section 5 opens with a section reviewing those elements of Kasparov’s equivariant $KK$-theory that we will need to invoke.

Acknowledgments: This research constitutes part of my PhD thesis, and I would like to thank my advisor Erik van Erp for his encouragement and assistance. I would also especially like to thank Jody Trout for his help and support throughout my PhD work. Much of the research in the present paper was completed during a visit to the Universidad Carlos III in Madrid, Spain; thanks are also due to the mathematics department there, in particular Fernando Lledó and Francisco Marcellán, for making that very enjoyable visit possible.

2 Groupoids

In this section we review many of the basic concepts and constructions from the theory of groupoid $C^*$-algebras, focusing on the case where our groupoid $G$ is a transformation group $G \ltimes X$. Many of the definitions are given first for general groupoids, to simplify notation and to contextualize our work in this paper.

**Definition 1.** A **groupoid** is a set $G$ equipped with a subset $G^{(2)} \subseteq G \times G$ (called the **set of composable pairs**), a multiplication map $G^{(2)} \to G$ given by $(x,y) \mapsto xy$, and an inverse map $G \to G$, written $x \mapsto x^{-1}$, such that:

- If $(x,y), (y,z) \in G^{(2)}$, then so are $(x,yz)$ and $(xy,z)$, and $x(yz) = (xy)z$;
- For any $x \in G$, the pair $(x,x^{-1}) \in G^{(2)}$;
- For any $(x,y) \in G^{(2)}$, we have $x^{-1}(xy) = y$ and $(xy)y^{-1} = x$.

Any groupoid comes equipped with range and source maps $r, s : G \to G$, defined by $r(x) = x^{-1}x$ and $s(x) = xx^{-1}$. Observe that $r$ and $s$ have a common image, which we call the **unit space** of $G$, and denote $G^{(0)}$.

If $u \in G^{(0)}$, we will write

$$G_u = \{ x \in G : s(x) = u \}; \quad G^u = \{ x \in G : r(x) = u \}.$$

In this paper we will consider almost exclusively the following class of groupoids.

**Example 2.1.** Suppose $G$ is a group acting (on the left) on a space $X$. We define a groupoid $G \ltimes X$, called the **transformation group**, to be $G \times X$ as a set, with $(G \ltimes X)^{(0)} = X$:

$$s(\gamma, u) = \gamma^{-1} \cdot u, \quad r(\gamma, u) = u.$$
In other words, we can think of an element \( x = (\gamma, u) \in G \ltimes X \) as an arrow (labeled \( \gamma \)) taking us from the point \( v := \gamma^{-1} \cdot u \in X \) to the point \( u \):

\[
\begin{array}{c}
\bullet \rightarrow \bullet \\
\downarrow \gamma \\
\downarrow \downarrow \\
\bullet \rightarrow \bullet \\
\end{array}
\]

It follows that

\[
((\gamma, u), (\eta, v)) \in (G \ltimes X)^{(2)} \Leftrightarrow \gamma^{-1} \cdot u = v
\]

and

\[
(\gamma, u) \cdot (\eta, \gamma^{-1} \cdot u) = (\gamma \eta, u); \quad (\gamma, u)^{-1} = (\gamma^{-1}, \gamma^{-1} \cdot u).
\]

Note that for any \( u \in X \), we have \((G \ltimes X)_u \cong (G \ltimes X)^u \cong G\).

**Remark 2.2.** In this paper, we will use \( x, y, z \) to denote elements of an arbitrary groupoid \( \mathcal{G} \), whereas Greek letters such as \( \gamma, \eta \) will denote elements of a group \( G \). The letters \( u, v \) will denote elements of the unit space of our groupoid – so \( u, v \in X \) if the groupoid \( \mathcal{G} \) under consideration is a transformation group \( G \ltimes X \).

**Definition 2.** We say that a groupoid \( \mathcal{G} \) is a *locally compact Hausdorff groupoid* or *LCH groupoid* if \( \mathcal{G} \) is a locally compact Hausdorff topological space such that multiplication and inversion are continuous (when \( \mathcal{G}^{(2)} \) has the topology induced by the product topology on \( \mathcal{G} \times \mathcal{G} \)).

**Remark 2.3.** Since the range and source maps can be constructed from multiplication and inversion, it follows that \( r, s \) are also continuous in a LCH groupoid.

**Example 2.4.** If \( \mathcal{G} = G \ltimes X \) is a transformation group, where \( G \) and \( X \) are locally compact Hausdorff spaces and the action of \( G \) on \( X \) is continuous, then the product topology on \( G \times X \) makes \( G \ltimes X \) into a LCH groupoid. We will always use this topology on \( G \ltimes X \) in this paper.

**Definition 3.** Let \( \mathcal{G} \) be a LCH groupoid. A (continuous) map \( \omega : \mathcal{G}^{(2)} \rightarrow \mathbb{T} \) is called a (continuous) 2-cocycle if

\[
\omega(x, y)\omega(x y, z) = \omega(x, y z)\omega(y, z)
\]

whenever \((x, y), (y, z) \in \mathcal{G}^{(2)}\), and if

\[
\omega(x, s(x)) = 1 = \omega(r(x), x)
\]

for any \( x \in \mathcal{G} \).

As 2-cocycles are the only flavor of cocycle we will discuss in this paper, we will usually drop the 2 and refer to them simply as cocycles.

**Example 2.5.** For any groupoid \( \mathcal{G} \), the function \( \omega : \mathcal{G}^2 \rightarrow \mathbb{T} \) given by \( \omega(x, y) = 1 \ \forall (x, y) \in \mathcal{G}^{(2)} \) is a 2-cocycle, called the trivial 2-cocycle.
Example 2.6. Let $G = \mathbb{Z}^2$ and fix $t \in \mathbb{R}$. Then the function $c_t : \mathbb{Z}^2 \times \mathbb{Z}^2 \rightarrow T$ given by

$$c_t ((j,k), (m,n)) := e^{2\pi i t (k m)}$$

is a 2-cocycle.

Remark 2.7. If $\omega_0, \omega_1$ are two 2-cocycles on $G$, then the pointwise product $\omega_0 \omega_1$ also satisfies the cocycle condition, as does the pointwise inverse $\omega_0^{-1}$ for any cocycle $\omega_0$.

Definition 4. Let $G$ be a LCH groupoid. We consider $G \times [0,1]$ to be a LCH groupoid by equipping it with the product topology (using the standard topology on $[0,1]$). Set

$$(G \times [0,1])^{(2)} = G^{(2)} \times [0,1].$$

In other words, as a groupoid, $G \times [0,1]$ is a bundle of groupoids over $[0,1]$, and all groupoid operations preserve the fibers.

We say that $\omega$ is a homotopy of 2-cocycles on $G$ if $(G \times [0,1])^{(2)}$ is a continuous 2-cocycle.

Remark 2.8. Every homotopy of cocycles on $G$ corresponds to a family $\{\omega_t\}_{t \in [0,1]}$ of continuous 2-cocycles on $G$ which varies continuously in $t$: for each fixed $(x,y) \in G^{(2)}$, the map $t \mapsto \omega_t (x,y) := \omega_t ((x,t), (y,t))$ is continuous, and for each fixed $t \in [0,1]$, $\omega_t : G^{(2)} \rightarrow T$ is continuous.

In order to associate $C^*$-algebras to LCH groupoids $G$ and continuous cocycles $\omega$, we will start by turning the continuous compactly supported functions $C_c(G)$ into a convolution algebra, and then taking an appropriate completion. To do so, we need to integrate over $G$, which requires a Haar system on $G$.

Definition 5. Let $G$ be a locally compact Hausdorff groupoid. A collection $\{\lambda^u\}_{u \in G^{(0)}}$ of non-negative Radon measures is a Haar system if

- $\text{supp}(\lambda^u) = G^u$ for any $u \in G^{(0)}$,
- For any $f \in C_c(G)$, the function $u \mapsto \int_{G^u} f(x)d\lambda^u (x)$ is in $C_0(G^{(0)})$,
- The system of measures is left-invariant: For any $f \in C_c(G)$ and any $x \in G$,

$$\int_{G^r(x)} f(xy)d\lambda^s(x)(y) = \int_{G^r(x)} f(y)d\lambda^r(x)(y).$$

Example 2.9. If $G$ is a LCH group, then $G^{(0)}$ contains only one point, namely, the unit $e$ of $G$, and so the first two conditions of the definition are irrelevant. The third condition tells us that any Haar measure on $G$ constitutes a Haar system.

Unlike for groups, Haar systems for groupoids need not exist or be unique. One generally assumes from the beginning the existence of a fixed Haar system on the groupoid in question, a precedent we will follow in this paper.
Example 2.10. If \( \mathcal{G} = G \ltimes X \) for a LCH group \( G \) and a LCH space \( X \), then fix a Haar measure \( \lambda \) on \( G \). Setting \( \lambda^u = \lambda \) for every \( u \in X \) makes \( \{ \lambda^u \}_{u \in X} \) into a Haar system on \( G \ltimes X \). We will always use this Haar system on a transformation group in this paper.

Given a continuous cocycle \( \omega \) on a LCH groupoid \( \mathcal{G} \), and a Haar system \( \{ \lambda^u \}_{u \in \mathcal{G}^{(0)}} \) on \( \mathcal{G} \), we can turn \( C_c(\mathcal{G}) \) into a convolution algebra \( C^*_c(\mathcal{G}, \omega) \):

\[
\begin{align*}
\hat{f} * \hat{g}(x) & := \int_{\mathcal{G}\setminus(\gamma)} f(y)g(y^{-1}x) \, \omega((y, y^{-1}x)) \, d\lambda^\gamma(y) \\
\hat{f}(x) & := \int (x^{-1}) \, \omega(x, x^{-1}).
\end{align*}
\]

In the case when \( \mathcal{G} = G \ltimes X \) is a transformation group, the formula for convolution becomes

\[
f * \omega g(\gamma, u) := \int_G f(\eta, u)g(\eta^{-1}\gamma, \eta^{-1}u) \omega((\eta, u), (\eta^{-1}\gamma, \eta^{-1}u)) \, d\lambda(\eta). \tag{4}
\]

Convolution multiplication is evidently linear, but also is easily checked to be associative and to satisfy \( f^* * \omega g^* = (g * \omega f)^* \) (cf. [20] II.1). One needs to invoke the cocycle condition (2) in order to show associativity.

To make \( C^*_c(\mathcal{G}, \omega) \) into a \( C^* \)-algebra, we need to complete it with respect to a suitable norm. There are two \( C^* \)-algebras canonically associated to \( C^*_c(\mathcal{G}, \omega) \); however, in this paper we will only be concerned with the reduced twisted \( C^* \)-algebra of a transformation group \( \mathcal{G} = G \ltimes X \), so we present only this construction here. The reader who wishes to understand groupoid \( C^* \)-algebras in more generality is referred to [20] or [26].

Definition 6. Given a groupoid \( \mathcal{G} \), let \( \mu \) be a measure on \( \mathcal{G}^{(0)} \) with full support. As in [20] Definition 2.45, we construct a measure \( \nu^{-1} \) on \( \mathcal{G} \), which is the pullback under the map \( x \mapsto x^{-1} \) of the measure induced by \( \mu \) on \( G \). To be precise, a function \( \xi \) on \( \mathcal{G} \) is in \( L^2(\mathcal{G}, \nu^{-1}) \) iff

\[
\|\xi\|_2^2 := \int_{\mathcal{G}^{(0)}} \int_{\mathcal{G}^{u}} |\xi(x^{-1})|^2 d\lambda^u(x) d\mu(u) < \infty.
\]

In the case when \( \mathcal{G} = G \ltimes X, \) so \( \mathcal{G}^{(0)} = X \) and \( \lambda^u \) is Haar measure on \( G \), the formula above becomes

\[
\|\xi\|_2^2 = \int_X \int_G |\xi(\gamma^{-1}, \gamma^{-1} \cdot u)|^2 d\lambda(\gamma) d\mu(u).
\]

Convolution multiplication defines a \(*\)-representation of \( C^*_c(\mathcal{G}, \omega) \) on \( L^2(\mathcal{G}, \nu^{-1}) \) which we will denote \( \text{Ind} \mu \):

Definition 7. For \( f \in C^*_c(\mathcal{G}, \omega) \), we define the operator \( \text{Ind} \mu(f) \) on \( L^2(\mathcal{G}, \nu^{-1}) \) by

\[
\text{Ind} \mu(f)\xi(x) := (f * \omega \xi)(x) = \int_{\mathcal{G}} f(xy)\xi(y^{-1}) \omega(xy, y^{-1}) \, d\lambda^x(y)
\]

for \( \xi \in L^2(\mathcal{G}, \nu^{-1}) \).
Remark 2.11. The notation \( \text{Ind} \mu \) indicates that the representation of \( C_c(\mathcal{G}, \omega) \) is induced from the multiplication representation of \( C_0(\mathcal{G}^{(0)}) \) on \( L^2(\mathcal{G}^{(0)}, \mu) \).

A variation on Theorem 6.18 from [8] tells us that \( \text{Ind} \mu(f) \xi \in L^2(\mathcal{G}, \nu^{-1}) \) whenever \( f \in C_c(\mathcal{G}, \omega) \) and \( \xi \in L^2(\mathcal{G}, \nu^{-1}) \), and moreover that the operator norm \( \|\text{Ind} \mu(f)\| \) is bounded by the \( L^1 \) norm \( \|f\|_{\nu^{-1}} \) of \( f \) with respect to the measure \( \nu^{-1} \). Since convolution multiplication is linear, associative, and \( \ast \)-preserving, it follows that \( \text{Ind} \mu \) is a \( \ast \)-representation of \( C_c(\mathcal{G}, \omega) \).

Definition 8. The \emph{reduced norm} \( \| \cdot \|_r \) on \( C_c(\mathcal{G}, \omega) \) is given by

\[
\|f\|_r := \|\text{Ind} \mu(f)\|.
\]

The completion of \( C_c(\mathcal{G}, \omega) \) with respect to the norm \( \| \cdot \|_r \) is the \emph{reduced twisted groupoid C*-algebra} \( C^*_r(\mathcal{G}, \omega) \).

Remark 2.12. One can check that any full measure \( \mu \) on \( \mathcal{G}^{(0)} \) will give rise to an equivalent norm to \( \| \cdot \|_r \), so the C*-algebra \( C^*_r(\mathcal{G}, \omega) \) is independent of \( \mu \). In fact, there are multiple equivalent definitions of the reduced norm (cf. [19] Definition 6.3 and [26] Definition III.2.8).

3 The Compact Case

As mentioned in the Introduction, our proof of Theorem 5.1 was inspired by a 2010 paper [6] by Echterhoff, Lück, Phillips, and Walters. In particular, Theorem 1.9 in that paper proves a version of our Theorem 5.1 for groups, rather than transformation groups. The idea of the proof of Theorem 1.9 in [6] is to use a technical theorem from [4] to reduce the question to the case of \( G \) a compact group, which is much more tractable thanks to results of Echterhoff and Williams in [7] on \( C_0(X) \)-linear actions on continuous trace algebras. The same technique of reduction to the compact case works in the case of a transformation group as well, as we shall see, so we will begin by examining the case when our transformation group is compact.

The main result in this section is the following:

**Proposition 3.1.** Let \( \omega \) be a homotopy of cocycles on a compact Hausdorff groupoid \( \mathcal{G} \). Then the map

\[
q_t : C^*_r(\mathcal{G} \times [0, 1], \omega) \to C^*_r(\mathcal{G} \times [0, 1], \omega_t)
\]

given on \( C_c(\mathcal{G} \times [0, 1]) \) by \( q_t(f)(x) = f(x, t) \) is a homotopy equivalence, and thus induces an isomorphism

\[
K_*(C^*_r(\mathcal{G} \times [0, 1], \omega)) \to K_*(C^*_r(\mathcal{G} \times [0, 1], \omega_t)).
\]

The proof will proceed through a series of lemmas, some of which have slightly more general hypotheses than those in the statement of Proposition 3.1. We begin with a definition.
**Definition 9.** Let \( \omega : G^{(2)} \to \mathbb{T} \) be a 2-cocycle on a groupoid \( G \). A function \( \theta : G^{(2)} \to \mathbb{R} \) is a cocycle logarithm for \( \omega \) if
\[
\omega(x, y) = \exp(\theta(x, y)) := e^{2\pi i \theta(x, y)}
\]
for all \((x, y) \in G^{(2)}\).

**Lemma 3.2.** Let \( G \) be a compact groupoid, and let \( \omega \) be a homotopy of cocycles on \( G \). Given \( t \in [0, 1] \), we define a cocycle \( h_t \) on \( G \times [0, 1] \) by
\[
h_t(x, y, s) = \omega(x, y, t).
\]
For each \( t \in [0, 1] \), we can find \( \epsilon_t > 0 \) such that the cocycle \( H_t \) on \( G \times ((t - \epsilon_t, t + \epsilon_t) \cap [0, 1]) \) defined by
\[
H_t(x, y, s) := \omega(x, y, s)h_t(x, y, s)^{-1}
\]
admits a continuous cocycle logarithm.

**Proof:** Observe that \( \omega \) and \( h_t \) are continuous cocycles on \( G \times [0, 1] \) by definition. Moreover, Proposition 1.10 in [10] tells us that \( G^{(2)} \times [0, 1] \) is a closed subset of a compact space, and hence is compact. Thus, for each \( t \in [0, 1] \) we can find \( \epsilon_t > 0 \) such that \(|s-t| < \epsilon_t \) implies that \( H_t(x, y, s) \) lies in the right-hand half of the circle, strictly between \(-i\) and \(i\), for all \((x, y) \in G^{(2)}\). This implies that the principal branch of the logarithm is a cocycle logarithm for \( H_t \), for any \( t \). Let \( \theta_t(x, y, s) \) to be the argument of \( H_t(x, y, s) \) that lies in the interval \((-\pi, \pi)\). Then since the product of cocycles is a cocycle by Remark 2.7, the cocycle identity for \( H_t \) implies that for each fixed \((x, y) \in G^{(2)} \) and \( s \in [0, 1] \),
\[
\theta_t(x, y, s) + \theta_t(xy, z, s) = \theta_t(x, yz, s) + \theta_t(y, z, s) + 2\pi k
\]
for some \( k \in \mathbb{Z} \). If \( k = 0 \) then the principal branch of the logarithm is a cocycle logarithm as claimed.

In order to have \( k \neq 0 \) we must have that at least one of the pairs
\[
\{\theta_t(x, y, s), \theta_t(x, yz, s)\}, \{\theta_t(xy, z, s), \theta_t(y, z, s)\}
\]
has a difference of at least \( \pi \). However, since \(-\pi/2 < \theta_t(x, y, s) < \pi/2 \) for all \((x, y) \in G^{(2)}\), this is impossible, so the principal branch of the logarithm is a cocycle logarithm as claimed. \( \square \)

When a cocycle \( \omega \) admits a logarithm, we can use left invariance of the Haar system to show that in many cases, \( \omega \) is cohomologous to the trivial cocycle. This will be an important ingredient in our proof, because cohomologous cocycles give rise to isomorphic \( C^* \)-algebras.

**Definition 10.** Let \( \omega_0, \omega_1 \) be two cocycles on a groupoid \( G \). We say that \( \omega_0, \omega_1 \) are cohomologous if there exists a function \( b : G \to \mathbb{T} \) such that for any \((x, y) \in G^{(2)}\), we have
\[
\omega_0(x, y) = (\delta b)(x, y)\omega_1(x, y) := b(x)b(y)b(xy)^{-1}\omega_1(x, y).
\]
Remark 3.3. When \( \mathcal{G} \) is a LCH groupoid and \( \omega_0, \omega_1, b \) are continuous, then a straightforward check shows that the map \( f \mapsto f \cdot b \) on \( C_c(\mathcal{G}) \) induces an isomorphism

\[ C^*_r(\mathcal{G}, \omega_0) \cong C^*_r(\mathcal{G}, \omega_1). \]

The class of groupoids alluded to above, for which the existence of a cocycle logarithm implies that the cocycle is cohomologous to the trivial cocycle, is the class of proper groupoids.

Definition 11. A groupoid \( \mathcal{G} \) is proper if the map \((r, s) : \mathcal{G} \to \mathcal{G}^{(0)} \times \mathcal{G}^{(0)}\) is proper.

Remark 3.4. Proposition 1.10 of [10] tells us that \( \mathcal{G}^{(0)} \subseteq \mathcal{G} \) is closed in any LCH groupoid. Consequently, if \( \mathcal{G} \) is compact then \( \mathcal{G} \) is proper.

Example 3.5. A transformation group \( G \ltimes X \) is proper iff \( G \) acts properly on \( X \) – that is, iff for any \( K_1, K_2 \subseteq X \) compact, \( \{ \gamma \in G : \gamma K_1 \cap K_2 \neq \emptyset \} \) is compact.

If \( \mathcal{G} \) is a proper groupoid, then Tu describes in Section 6.2 of [28] how to construct a cutoff function \( c : \mathcal{G}^{(0)} \to \mathbb{R}^+ \) such that for any \( u \in \mathcal{G}^{(0)} \),

\[ \int_{\mathcal{G}^{(0)}} c(s(x))d\lambda^u(x) = 1 \quad (5) \]

We will use this cutoff function to show that a cocycle with a continuous logarithm is cohomologous to the trivial cocycle.

Proposition 3.6. Let \( \mathcal{G} \) be a proper groupoid. Suppose that a 2-cocycle \( \omega \) on \( \mathcal{G} \) admits a continuous cocycle logarithm \( \theta : \mathcal{G}^{(2)} \to \mathbb{R} \). Then \( \omega \) is cohomologous to the trivial cocycle on \( \mathcal{G} \).

Proof: Define \( b(x) = \int \theta(x, y)c(s(y))d\lambda^u(y) \). Using the left-invariance of the Haar system, and [13], we see that whenever \((x, y) \in \mathcal{G}^{(2)}\),

\[ b(x) + b(y) - b(xy) = \theta(x, y). \]

Thus,

\[ \omega(x, y) = \exp(\theta(x, y)) = \exp(b(x)) \exp(b(y)) \exp(b(xy))^{-1} \]

\[ = \delta(\exp \circ b)(x, y), \]

so \( \omega \) is a coboundary, as claimed. □

Proof of Proposition 3.1. The two previous Lemmas, combined with our earlier observation that cohomologous cocycles induce isomorphic twisted \( C^* \)-algebras, imply that if \( \omega \) is a homotopy of continuous cocycles on a compact transformation group \( \mathcal{G} = H \ltimes X \), then for any \( t \in [0, 1] \),

\[ C^*_r(\mathcal{G} \times (t - \epsilon_t, t + \epsilon_t), \omega) \cong C^*_r(\mathcal{G} \times (t - \epsilon_t, t + \epsilon_t), h_t). \]

1This isomorphism in fact holds for both the full and the reduced \( C^* \)-algebras.
We claim that $C^*_r(\mathcal{G} \times (t - \epsilon_t, t + \epsilon_t), h_t) \cong C_0((t - \epsilon_t, t + \epsilon_t), C^*_r(\mathcal{G}, \omega_t))$. To see this, let $\epsilon = (t - \epsilon_t, t + \epsilon_t)$ and observe that $C_c(\epsilon \times \mathcal{G})$ is dense in both algebras. Since the cocycle $h_t$ doesn’t depend on $s \in \epsilon$, the multiplication and involution operations on $C_c(\epsilon \times \mathcal{G})$ are the same in both algebras.

To see that the norms agree on the two algebras, recall that in the groupoid $\mathcal{G} \times \epsilon$, we have $(x, s)^{-1} = (x^{-1}, s)$. If we write $\nu^{-1}$ for the measure on $\mathcal{G}$ induced from a full measure $\mu$ on $\mathcal{G}^{(0)}$ as in Definition 6, and $\tilde{\nu}^{-1}$ for the measure on $\mathcal{G} \times \epsilon$ induced from $\mu$ and from Lebesgue measure on $\epsilon$, then

$$\int_{\mathcal{G} \times \epsilon} f(x, s)d\tilde{\nu}^{-1}(x, s) = \int_{\epsilon} \int_{\mathcal{G}^{(0)}} \int_{\mathcal{G}^u} f(x^{-1}, s)d\lambda^n(x)d\mu(u)ds = \int_{\epsilon} \int_{\mathcal{G}} f(x, s)d\nu^{-1}(x)ds.$$

In other words, $L^2(\tilde{\nu}^{-1}) = L^2(\epsilon, L^2(\nu^{-1})) \cong L^2(\epsilon) \otimes L^2(\nu^{-1})$. Moreover, a quick examination of the formula for the representation $\text{Ind} \mu$ of $C_c(\mathcal{G} \times \epsilon)$ on $L^2(\tilde{\nu}^{-1}) \cong L^2(\epsilon) \otimes L^2(\nu^{-1})$ will show that this representation decomposes into pointwise multiplication on $L^2(\epsilon)$ and convolution on $L^2(\nu^{-1})$. Thus, the representation $\text{Ind} \mu$ of the groupoid convolution algebra $C_c(\mathcal{G} \times \epsilon, h_t)$ is the same as the representation of $C_c(\epsilon) \otimes C^*_r(\mathcal{G}, \omega_t)$ on $L^2(\epsilon) \otimes L^2(\nu^{-1})$ by multiplication in the first component, and twisted convolution multiplication in the second. Since this latter representation gives rise to $C_0(\epsilon) \otimes C^*_r(\mathcal{G}, \omega_t) \cong C_0(\epsilon, C^*_r(\mathcal{G}, \omega_t))$, we’re done.

In sum, when $\mathcal{G}$ is a compact groupoid, $C^*_r(\mathcal{G} \times [0, 1], \omega)$ is a locally trivial continuous field of $C^*$-algebras over $[0, 1]$, with fiber algebra $C^*_r(\mathcal{G}, \omega_t)$ over $t \in [0, 1]$. Consequently, the fiber algebras $C^*_r(\mathcal{G}, \omega_t)$ are all isomorphic. Since any locally trivial fiber bundle over a contractible space is trivializable, we have

$$C^*_r(\mathcal{G} \times [0, 1], \omega) \cong C([0, 1], C^*_r(\mathcal{G}, \omega_t)) \cong C([0, 1]) \otimes C^*_r(\mathcal{G}, \omega_t).$$

Moreover, since $[0, 1]$ is compact and contractible, the natural map

$$q_t : C^*_r(\mathcal{G} \times [0, 1], \omega) \to C^*_r(\mathcal{G}, \omega_t)$$

is a homotopy equivalence, and induces an isomorphism

$$K_*(C^*_r(\mathcal{G} \times [0, 1], \omega)) \cong K_*(C^*_r(\mathcal{G}, \omega_t))$$

as claimed. □

4 Twisted crossed products

In this section, we connect our construction of the reduced twisted transformation-group $C^*$-algebra from Section 2 with that of Packer and Raeburn in [23, 24]. Our eventual goal is to invoke the Packer-Raeburn “stabilization trick” (Theorem 3.4 in [23]) to show that

$$C^*_r(G \times X, \omega) \otimes \mathcal{K} \cong C_0(X, \mathcal{K}) \rtimes_r G.$$
In order to do this we need to exhibit an isomorphism between the reduced twisted groupoid \( C^* \)-algebra \( C^*_r(G \ltimes X, \omega) \) as defined in Section 2 and the reduced twisted crossed product \( C^* \)-algebra \( \hat{\pi} \times R(C_0(X) \rtimes \alpha, \omega) \) described in Definition 3.10 and Remark 3.12 of [23].

To that end, suppose that \( \omega \) is a 2-cocycle on the transformation group \( G \ltimes X \), and let \( A = C_0(X) \). The action of \( G \) on \( X \) that underlies the transformation-group structure gives rise to an action \( \alpha \) of \( G \) on \( A \) by

\[
\alpha_\gamma(f)(v) := f(\gamma^{-1} \cdot v).
\]

Moreover, if we define \( \mu : G \times G \to C(X, \mathbb{T}) = UM(A) \) by

\[
\mu(\gamma, \eta)(v) := \omega((\gamma, v), (\eta, \eta^{-1} \cdot v)),
\]

then the fact that \( \omega \) satisfies the cocycle condition (2) tells us that for any \( \gamma, \eta, \rho \in G \) we have

\[
\alpha_\gamma(\mu(\eta, \rho))\mu(\gamma, \eta \rho) = \mu(\gamma, \eta)\mu(\gamma \eta, \rho).
\]

In other words, \((A, G, \alpha, \mu)\) is a twisted dynamical system in the sense of [23] Definition 2.1.

Conversely, if \((A, G, \alpha, \mu)\) is any twisted dynamical system with \( A = C_0(X) \) abelian, the action \( \alpha \) of \( G \) on \( A \) must arise from an action of \( G \) on \( X \). Then if we define \( \omega : (G \ltimes X)^{(2)} \to \mathbb{T} \) via

\[
\omega((\gamma, v), (\eta, \gamma^{-1} \cdot v)) := \mu(\gamma, \eta)(v),
\]

an easy check will show that \( \omega \) satisfies the cocycle condition (2). In other words, twisted dynamical systems \((A, G, \alpha, \mu)\) as defined in [23] with \( A = C_0(X) \) abelian are in bijection with cocycles \( \omega \) on the transformation group \( G \ltimes X \).

We include the following proposition here because we have not found a satisfactory reference to it in the literature.

**Proposition 4.1.** Let \( \mu \) be a full measure on \( X \). Under the above identifications, \( C^*_r(G \ltimes X, \omega) \) is isomorphic to the reduced twisted crossed product \( \hat{\pi} \times R(C_0(X) \rtimes \alpha, \omega) \) described in Remark 3.12 of [23], where \( \pi : C_0(X) \to B(L^2(X, \mu)) \) is the representation of \( C_0(X) \) on \( L^2(X, \mu) \) by multiplication operators.

**Proof:** From Definitions 2.4 and 3.10, we see that \( C_c(G, \omega) \) is a dense subalgebra of \( \hat{\pi} \times R(C_0(X) \rtimes \alpha, \omega) \), where the norm is given by the representation \( \hat{\pi} \times R \) of \( C_c(G, \omega) \) on \( L^2(G, L^2(X, \mu)) \). If \( f \in C_c(G, \omega) \), \( \xi \in L^2(G, L^2(X, \mu)) \), then

\[
\hat{\pi} \times R(f)\xi(\gamma, v) := \int_G f(\eta, \gamma^{-1} v)\Delta(\eta)^{1/2} \omega((\gamma, v), (\eta, \gamma^{-1} v))\xi(\gamma \eta, v) d\lambda(\eta),
\]

where \( \Delta : G \to \mathbb{R}^+ \) is the modular function of \( G \). We will define a \(*\)-homomorphism \( \phi \) from the dense \(*\)-subalgebra \( C_c(G \ltimes X, \omega) \) of \( C^*_r(G \ltimes X, \omega) \) onto the dense \(*\)-subalgebra \( C_c(G, \omega) \) of \( \hat{\pi} \times R(C_0(X) \rtimes \alpha, \omega) \), and a unitary

\[
U : L^2(G \ltimes X, \nu^{-1}) \to L^2(G, L^2(X)),
\]
that intertwine the representations $\tilde{\pi} \times R$ and $\operatorname{Ind} \mu$. That is, for any $f \in C_c(G \times X, \omega), \xi \in L^2(G, L^2(X))$, we will show that

$$
\tilde{\pi} \times R(\phi(f))(\xi) = U \operatorname{Ind} \mu(f)(U^* \xi).
$$

(6)

It follows that $\phi$ is norm-preserving, and therefore extends to a $*$-homomorphism $C^*_\omega(G \times X, \omega) \to \tilde{\pi} \times R(C_0(X) \rtimes_{\alpha,u} G)$ that implements the desired isomorphism.

Using the definition of the representation $\tilde{\pi} \times R$ given by Definition 3.10 and Definition 2.4 of [23], we see that the algebraic operations on $C_c(G, C_0(X)) \subseteq \tilde{\pi} \times R(C_0(X) \rtimes_{\alpha,u} G)$ are given by

$$
f^* g(\gamma, v) := \int f(\eta, v)g(\eta^{-1} \gamma, \eta^{-1} \cdot v)\omega((\eta, v), (\eta^{-1} \gamma, \eta^{-1} \cdot v)) d\lambda(\gamma);
$$

$$
f^*(\gamma, v) := \overline{f(\gamma^{-1}, \gamma^{-1} \cdot v)\Delta(\gamma^{-1})\omega((\gamma^{-1}, \gamma^{-1} \cdot v), (\gamma, v))}.
$$

The appearance of the modular function in the involution above means that the standard inclusion $C_c(G \times X) \hookrightarrow C_c(G, C_0(X))$ will not be a $*$-homomorphism. Instead, we define $\phi : C_c(G \times X, \omega) \to C_c(G, C_0(X))$ by

$$
\phi(f)(\gamma, v) := f(\gamma, v)\Delta(\gamma^{-1})^{1/2};
$$

a straightforward check shows that $\phi$ is multiplicative and $*$-preserving.

Defining $U : L^2(G \times X, \omega^{-1}) \to L^2(G, L^2(X))$ by

$$
U\xi(x) = \xi(x^{-1})\omega(x, x^{-1})
$$

$$
= \xi(\gamma^{-1}, \gamma^{-1} \cdot v)\omega((\gamma, v), (\gamma^{-1}, \gamma^{-1} \cdot v))
$$

if $x = (\gamma, v) \in G \times X$, a similarly straightforward check (invoking the cocycle condition (2)) shows that (6) holds, and moreover that $U$ is a unitary operator. It follows (as remarked above) that

$$
C^*_\omega(G \times X, \omega) \cong \tilde{\pi} \times R(C_0(X) \rtimes_{\alpha,u} G)
$$

as desired. □

5 The Main Theorem

We are now ready to tackle the proof of Theorem 5.1. Our proof closely parallels the proof given in the group case (Theorem 1.9 in [6]) by Echterhoff, Lück, Phillips, and Walters; thus, our first step is to find an element

$$
x \in KK^G(C(X \times [0, 1], K), C(X, K))
$$

in the $G$-equivariant $KK$-theory of $A := C(X \times [0, 1], K)$ and $B := C(X, K)$, such that if $H \leq G$ is compact, and we write $x^H$ for the element $x$ thought of as merely an $H$-equivariant $KK$-element, the map

$$
\overline{\#}x^H : KK^H(\mathbb{C}, A) \to KK^H(\mathbb{C}, B)
$$

13
given by taking the Kasparov product with $x^H$ is an isomorphism for any $H \leq G$ compact. We will do this by showing that $x$ corresponds to the element

$$[q_t] \in KK(C^*_r(G \rtimes X \times [0,1], \omega), C^*_r(G \rtimes X, \omega_t))$$

arising from the $*$-homomorphism $q_t : C^*_r(G \rtimes X \times [0,1], \omega) \to C^*_r(G \rtimes X, \omega_t)$ of “evaluation at $t \in [0,1]$,” since we know from Proposition 3.1 that $q_t$ induces a homotopy equivalence, and hence a $KK$-equivalence, when the group is compact. Knowing that $x^H_H$ is an isomorphism for any compact subgroup $H \leq G$ will tell us that $x$ satisfies the hypotheses of Proposition 1.6 of [6], from which we can deduce that $x$ (and thus $[q_t]$) gives rise to an isomorphism on $K$-theory, as claimed.

5.1 $KK$-theory

We begin by reviewing a few fundamental constructions in equivariant $KK$-theory. First, recall that if two $C^*$-algebras $A, B$ admit actions by a locally compact Hausdorff group $G$, then an element of $KK^G(A, B)$ is given by an equivalence class of $KK$-triples $(E, T, F)$, where $E$ is a $\mathbb{Z}/2\mathbb{Z}$-graded right Hilbert $B$-module admitting an action of $G$; $T : A \to L(E)$ is a graded $*$-homomorphism from $A$ into the bounded adjointable operators on $E$; $F \in L(E)$ is a degree-1 operator; and the module operations on $E$ and the operators $T, F$ are all $G$-equivariant.

In particular, [2] Examples 17.1.2(a) tells us that if $\psi : A \to B$ is a $G$-equivariant $*$-homomorphism, then $\psi$ gives rise to a $KK$-triple $(B, \psi, 0)$ and hence to an element $[(B, \psi, 0)] \in KK^G(A, B)$. Almost all of the $KK$-elements that will concern us in this paper will be of this form. We will often denote $[(B, \psi, 0)]$ simply by $[\psi]$.

Given a $G$-algebra $A$, we can also use $KK$-theory to define the topological $K$-theory of $G$ with coefficients in $A$ as the abelian group

$$K^\text{top}_*(G; A) = \lim_{L \subseteq EG} KK^G_*(C_0(L), A),$$

where the limit is taken over all $G$-compact subspaces $L$ of a universal proper $G$-space $EG$. There is a natural map (see [1] Section 9) from the topological $K$-theory of $G$ with coefficients in $A$ to the usual $K$-theory group of the reduced crossed product $A \rtimes_r G$:

$$\mu : K^\text{top}_*(G; A) \to K_*(A \rtimes_r G).$$

The map $\mu$ is called the assembly map, and $G$ is said to satisfy the Baum-Connes conjecture with coefficients if $\mu$ is an isomorphism for any $G$-algebra $A$. As mentioned in Remark [1] every a-T-menable group satisfies the Baum-Connes conjecture with coefficients.

Following [30] Sections 2.2 and 7.2, given a $G$-algebra $A$ and a faithful representation $\pi$ of $A$ on a Hilbert space $\mathcal{H}$, we define the reduced crossed product
Having reviewed the preliminaries, we now begin the process of finding the

5.2 Proof of Theorem 5.1

is an isomorphism.

Finally, Theorem 5.4 in \[15\] tells us how to construct an element $\Lambda_{pt} \in KK_*(\mathbb{C}, C^*_r(G))$ such that if $G$ is compact and $B$ is a $C^*$-algebra with a $G$-action, then the map $KS : KK_*(\mathbb{C}, B) \to KK_*(\mathbb{C}, B \rtimes_r G)$ given by

$$KS(x) = \Lambda_{pt} # j^G(x)$$

is an isomorphism.

5.2 Proof of Theorem 5.1

Having reviewed the preliminaries, we now begin the process of finding the element $x \in KK_*(A, B)$ alluded to at the beginning of this section, where $A = C(X \times [0, 1], \mathcal{K})$, $B = C(X, \mathcal{K})$ for a compact $G$-space $X$.

Let $\omega$ be a homotopy of cocycles on $G \rtimes X$, and write $ev_t : A \to B$ for the $*$-homomorphism $ev_t(f)(x) = f(x, t)$. Proposition \[4.1\] tells us how to construct twisted dynamical systems

$$(C_0(X \times [0, 1]), G, \alpha, u), (C_0(X), G, \alpha, u_t)$$

such that

$$C^*_r(G \rtimes X \times [0, 1], \omega) \cong \hat{\pi} \times R(C_0(X \times [0, 1]) \rtimes_{\alpha, u} G);$$

$$C^*_r(G \rtimes X, \omega_1) \cong \hat{\pi} \times R(C_0(X) \rtimes_{\alpha, u} G).$$

Now, Theorems 3.4 and 3.11 of \[23\] tell us how to construct, from the above twisted dynamical systems, actions $\beta, \beta_\tau$ of $G$ on $A, B$ respectively such that

$$\hat{\pi} \times R(C_0(X \times [0, 1]) \rtimes_{\alpha, u} G) \otimes \mathcal{K} \cong A \rtimes_{\beta, \tau} G,$$

$$\hat{\pi} \times R(C_0(X) \rtimes_{\alpha, u} G) \otimes \mathcal{K} \cong B \rtimes_{\beta, \tau} G.$$
Moreover, examining the formula for $\beta$ given by Equation 3.1 in [23] in this particular case reveals immediately that $\beta$ preserves the fibers over $[0,1]$. In other words, $ev_t$ is equivariant, and consequently induces a $*\text{-homomorphism}$

$$ev^G_t : A \rtimes_{r,\beta} G \to B \rtimes_{r,\beta_t} G,$$

which is given on the dense $*$-subalgebra $C_c(G \times X \times [0,1], K)$ by

$$ev^G_t(f)(\gamma,u,t) := f(\gamma,u,t).$$

Since the descent map $j : KK^G(A,B) \to KK^G(A \rtimes_r G, B \rtimes_r G)$ generalizes the map $\text{Hom}_G(A,B) \to \text{Hom}(A \rtimes_r G, B \rtimes_r G)$, it follows that $j^G([ev_t]) = [ev^G_t]$.

We can now complete the proof of our main theorem.

**Theorem 5.1.** Let $G$ be a locally compact Hausdorff group acting on a compact space $X$ such that $G$ satisfies the Baum-Connes conjecture with coefficients, and let $\omega$ be a homotopy of continuous cocycles on $G \rtimes X$. For any $t \in [0,1]$, the $*\text{-homomorphism}$

$$q_t : C^*_r(G \rtimes X \times [0,1], \omega) \to C^*_r(G \rtimes X, \omega_t),$$

given on $C_c(G \rtimes X \times [0,1])$ by evaluation at $t \in [0,1]$, induces an isomorphism

$$K_* C^*_r(G \rtimes X \times [0,1], \omega) \cong K_* C^*_r(G \rtimes X, \omega_t)).$$

**Proof:** We begin by examining the case when $H \leq G$ is a compact subgroup. We will show that the diagram

$$
\begin{array}{c}
\begin{array}{ccc}
KK_*^H(\mathbb{C}, A) & \xrightarrow{[ev_t]} & KK_*^H(\mathbb{C}, B) \\
\downarrow{KS} & & \downarrow{KS} \\
KK_*(\mathbb{C}, A \rtimes_{\beta, r} H) & \xrightarrow{[ev^H_t]} & KK_*(\mathbb{C}, B \rtimes_{\beta_t, r} H) \\
\downarrow{\phi} & & \downarrow{\phi_t} \\
KK_*(\mathbb{C}, C^*_r(H \rtimes X \times [0,1], \omega)) & \xrightarrow{[q_t]} & KK_*(\mathbb{C}, C^*_r(H \rtimes X, \omega_t))
\end{array}
\end{array}
$$

commutes, where $A = C(X \times [0,1], K), B = C(X, K)$ as above. Here $\Phi, \Phi_t$ denote the $KK$-equivalences induced by the $C^*$-algebraic isomorphisms

$$\phi : A \rtimes_{\beta, r} H \cong C^*_r(H \rtimes X \times [0,1], \omega) \otimes K, \quad \phi_t : B \rtimes_{\beta_t, r} H \cong C^*_r(H \rtimes X, \omega_t) \otimes K$$

provided by Theorems 3.4 and 3.11 of [23] and Proposition 4.1 of the present paper.
The functoriality of the descent map \( j^H \) tells us that if \( x \in KK^H(C, A) \), we have

\[
j^H(x) \# j^H([ev_t]) = j^H(x \# [ev_t]) \in KK_*(C^*_r(H), B \rtimes_{\beta_t,r} H);
\]

consequently,

\[
KS(x \# [ev_t]) = \Lambda_{pt} \# j^H(x \# [ev_t]) = \Lambda_{pt} \# j^H([ev_t])
\]

\[
= KS(x) \# [ev_t^H].
\]

In other words, the top square of the diagram commutes.

To see that the bottom square of the diagram commutes, since all of the maps in question come from \(*\)-isomorphisms, it suffices to check that

\[
(q_t \otimes \text{id}) \circ \phi = \phi_t \circ ev_t^H.
\]

Examining the formula for the actions \( \beta, \beta_t \) given in Equation 3.1 of [23], and the isomorphisms of our Proposition 4.1 and Corollary 3.7 in [23], we see that these all preserve the fiber over \( t \in [0, 1] \). Consequently, we have the desired equality (10), and the bottom square of (7) commutes as well.

Furthermore, we know by Proposition 3.1 that \( q_t : C^*_r(G \rtimes X \times [0, 1], \omega) \to C^*_r(H \rtimes X, \omega_t) \) is a homotopy equivalence if \( H \) and \( X \) are compact, and hence induces an isomorphism on \( KK \)-theory. It follows from the commutativity of (7) that taking the Kasparov product with \([ev_t] \) induces an isomorphism

\[
KK_*^H(C, A) \cong KK_*^H(C, B)
\]

whenever \( H \leq G \) is compact. In other words, the element \([ev_t] \in KK_*^G(A, B)\) satisfies the hypotheses of Proposition 1.6 in [6]. The conclusion of that proposition tells us that \( ev_t \) induces an isomorphism

\[
K_*^{\text{top}}(G; A) \to K_*^{\text{top}}(G; B),
\]

and since we hypothesized that \( G \) satisfies the Baum-Connes conjecture with coefficients, we moreover have

\[
K_*(A \rtimes_{\beta,r} G) \cong K_*(B \rtimes_{\beta_t,r} G).
\]

The isomorphism of (11) now tells us that for any \( t \in [0, 1] \), we have

\[
K_*(C^*_r(G \rtimes X \times [0, 1], \omega)) \cong K_*(C^*_r(G \rtimes X, \omega_t)).
\]

It remains to check that this isomorphism is implemented by \( q_t \), as claimed.

As described in Section 9 of [1], the Baum-Connes assembly map \( \mu \) is a modification of the descent map \( j^G \). Since we know that \( j^G([ev_t]) = ev_t^G \), equation (10) tells us that the isomorphism induced by \( ev_t \) on

\[
K_*(C^*_r(G \rtimes X \times [0, 1], \omega)) \cong K_*(C^*_r(G \rtimes X, \omega_t))
\]

is also given by evaluation at \( t \) on \( C_*(G \rtimes X \times [0, 1]) \subseteq C^*_r(G \rtimes X \times [0, 1], \omega) \).

This finishes the proof of Theorem 5.1. \( \square \)

---

\(^2\)This equation holds regardless of whether \( H \) and \( X \) are compact.
6 Future Work

We suspect that Theorem 5.1 also holds with slightly weaker hypotheses — if $X$ is merely locally compact but the action of $G$ on $X$ is proper, for example. It might also be possible to remove the requirement that each cocycle $\omega_t$ in the homotopy be continuous by using the framework of groupoid twists (introduced in [16]) instead of cocycles. However, in order to apply the proof techniques we have employed here, following [6], to a broader class of groupoids, we will need to work harder. Proposition 1.6 in [6], which is the crucial technical lemma for Theorem 1.9 in [6] as well as for our Theorem 5.1, allows us to study a homotopy of cocycles $\omega$ on $G \rtimes X$ by simply studying the restriction of $\omega$ to $H \rtimes X$ for all compact subgroups $H$ of $G$. Since compact groups are very friendly, well-understood objects, this latter question is much easier to solve. As of this writing, we are unaware of any analogous simplification results for more general groupoids.

As mentioned in the Introduction, Kumjian, Pask, and Sims have used completely different techniques in [18] to show that if a cocycle $\omega$ on a higher-rank graph $\Lambda$ admits a cocycle logarithm, so that $\omega(\lambda, \mu)$ can be written as $\exp(c(\lambda, \mu))$ for some $\mathbb{R}$-valued 2-cocycle $c$ on $\Lambda$, then

$$K_*(C^*(\Lambda_\omega)) \cong K_*(C^*(\Lambda)).$$

In a forthcoming paper [9], we extend the techniques of [18] to show that any homotopy of cocycles on $\Lambda$ gives rise to $K$-equivalent twisted higher-rank graph $C^*$-algebras; it has been suggested to us that these techniques might also apply to the case of Deaconu-Renault groupoids.

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