Design of Event-triggered State-Constrained Stabilizing Controllers for Nonlinear Control Systems

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ABSTRACT This paper addresses the event-triggered stabilization problem for nonlinear control-affine systems under state constraints. First, a strong control Lyapunov barrier function (CLBF) method is applied for constructing continuous state-constrained stabilizing controllers. Additionally, sufficient conditions for the existence of strong CLBFs are derived. With the obtained state-constrained feedback laws, a new event-triggered policy is proposed for reducing the number of communication events without the input-to-state stability (ISS) assumption. It is proved that the Zeno behavior is excluded under this event-triggered policy, that is, the inter-execution times are lower bounded away from zero. Finally, two examples are included to illustrate the theoretical results.

INDEX TERMS Control Lyapunov functions, Control Lyapunov barrier functions, event-triggered, State constraint, State feedback, Nonlinear systems.

I. INTRODUCTION

In the last decade, the study on event-triggered control has attracted considerable attention in the control community [1], [2]. The event-triggered control scheme has been widely applied in networked control systems (NCSs) [3]–[7] for reducing the waste of computation and communication resources. Different from traditional continuous controllers, an event-triggered controller updates the control signal only if the magnitude of the error, which is often defined by the difference between the current state and the last transmitted state in most literature, is larger than a prespecified threshold [1] or a relative threshold [4] while still guarantees the closed-loop stability and the required performance. For the nonlinear case, [1] proposed an event-triggered state feedback stabilizing controller for nonlinear control systems; [2] presented an event-triggered output feedback stabilizing controller for nonlinear control systems; [3] considered the design of distributed and decentralized event-triggered controllers for nonlinear NCSs. For the linear case, [4] presented a framework for stabilizing uncertain dynamical systems by merging the model-based networked control and event-triggered policy, and a relative threshold control strategy was presented to reduce the communication resource; [5], [6] proposed model-based event-triggered stabilizing controllers for NCSs with quantization and packet dropout. In these approaches, the results were derived based on the assumption that the closed-loop system is input-to-state stable (ISS). To relax this ISS assumption, a protocol was provided in [7] to co-design an adaptive controller and an event-triggered mechanism for a particular class of uncertain nonlinear systems.

Although many important results for event-triggered control problems have been proposed, the state-constrained requirement was rarely considered in event-triggered designs. The state trajectories of a practical system may not be allowed to enter some unsafe regions due to mechanical or electrical constraints or other considerations. For example, the rotor angular velocity and the armature current of a motor control system should be constrained by the inherent properties of the motor [8], and the angle and angular velocity of a robot arm also need to be restricted for safety consideration [14]. Considering state constraints in event-triggered control problems is meaningful. In the last two decades, on the basis of (control) Lyapunov functions and barrier functions, several new nonlinear state-constrained design methods have been proposed in the continuous control case. In [8]–[13], the barrier Lyapunov function (BLF) method was used to design state-constrained stabilizing controllers for single-input-
single-output (SISO) nonlinear control systems. In [14]–[16], based on BLFs, state feedback stabilization problems for SISO nonlinear control systems with output constraints were studied. In [17]–[19], the BLF approach was extended to the case of multiple-input-multiple-output (MIMO) systems. With the BLF approach, the nonlinear control systems considered must be in some particular structures [8]–[13]. In [20], a control barrier function (CBF) approach was developed for control syntheses of nonlinear systems to achieve safety control. In [21], for nonlinear control-affine systems, a CBF and a control Lyapunov function (CLF, see [22]–[25]) were integrated as a smooth control Lyapunov-barrier function (CLBF) and then Sontag’s formula was applied to construct continuous controllers to ensure both safety and stability. In [26]–[29], a new type of CBF (called zeroing CBF (ZCBF)) was proposed to provide both safety and robustness of nonlinear control systems. By integrating CLFs and ZCBFs, a control synthesis methodology was developed using quadratic programs (QPs) to guarantee the satisfaction of state constraints. In [30], a new CLBF approach was proposed to solve the state-constrained stabilization problem. The construction of CLBFs and implementation of controllers are easier than those proposed in [21], [26]–[29]. All controllers proposed in [8]–[30] are continuous controllers. Up to date, few results on the state-constrained control have been proposed in the event-triggered scheme. In [31], an event-triggered stabilizing output feedback controller was developed by the BLF approach for nonlinear systems in a particular structure. In [32], a fuzzy adaptive event-triggered control scheme for a class of uncertain nonlinear systems with full state constraints was presented by the BLF approach. In [33], an adaptive neural event-triggered controller was designed for uncertain block pure-feedback nonlinear systems with output constraints. In [34], a new universal-constraint function was proposed to handle both constrained and unconstrained event-triggered control schemes for pure-feedback systems. In [35], a tangent-type nonlinear mapping function was presented to investigate the event-triggered adaptive output feedback control for MIMO uncertain nonlinear systems with time-varying full state constraints. These approaches were developed for nonlinear control systems in particular forms.

In this paper, we consider the design of event-triggered state-constrained stabilizing controllers for general nonlinear control-affine systems. The strong CLBF method, an extension of the new CLBF approach developed in [30], is used to first derive a state-constrained continuous stabilizing feedback law. Sufficient conditions are derived for the existence of strong CLBFs. With the obtained continuous state-constrained feedback law, a new event-triggered policy is designed to reduce the number of communication events while guaranteeing both closed-loop stability and safety. The event-triggered policy is derived by the error between the ideal control signal and the last updated control signal instead of the error between the current state and the last transmitted state, which was used in most event-triggered nonlinear control studies (see, [1], [3], [6]). Under this setting, the traditional ISS assumption is not required in our approach. For practical implementation of event-triggered controllers, infinitely fast transmissions are not allowed. The time intervals between triggered instants must be lower bounded away from zero ( [1], [3], [6]). We show that, under the proposed triggered policy with some additional conditions, there is a minimum amount of time between two consecutive transmissions.

The main contribution of this paper is to develop a strong CLBF approach to design event-triggered state-constrained stabilizing controllers for nonlinear control-affine systems without the ISS assumption, in a simple and intuitive way. For the continuous control case, in [7] the ISS assumption was also relaxed. However, the systems considered in [7] must have a particular structure. Although the event-triggered state-constrained stabilization problems have been considered in several studies with different approaches [31]–[35], the systems considered in these papers must also be in particular structures. The developed approaches cannot be applied to the general nonlinear control-affine systems that we considered. All results derived in [31]–[35] are based on the BLF approach, while our results are derived based on the CLBF method. The problem we considered are new and our approach can be applied to more practical control systems because the systems need not be in particular structures.

This paper is organized as follows. In Section 2, the problem to be solved is formulated. In Section 3, the concept of strong CLBFs is introduced, and a continuous state-constrained stabilizing feedback law is derived. Moreover, sufficient conditions for the existence of strong CLBFs are derived. In Section 4, a state-constrained event-triggered policy is proposed. Two examples are provided for illustrations in Section 5. Finally, some conclusions are drawn in Section 6.

### NOTATION
\( \partial S, \bar{S} \) and \( S^C \) are the boundary, the closure, and the complement of the set \( S \); \( A \setminus B \) is the set of all elements belonging to the set \( A \) but not belonging to the set \( B \); \( Z^+ \) is the set of all positive integers; \( \emptyset \) denotes an empty set; \( \text{dist}(S, x) := \min \{y - x\} \) denotes the minimal distance between the point \( x \) and the set \( S \); for \( \epsilon > 0 \), \( B_D(S, \epsilon) := \{x \in D | \text{dist}(S, x) < \epsilon\} \); for simplification, let \( B_D(\epsilon) := B_D(0, \epsilon), B(S, \epsilon) := B_R^+(S, \epsilon), B(\epsilon) := B_{R^+}(0, \epsilon) \); \( \nabla V(x) := \frac{\partial V(x)}{\partial x} \).

### II. PROBLEM FORMULATION AND PRELIMINARIES
In this section, we formulate the problem to be solved and introduce the concept of CLFs.

#### A. PROBLEM FORMULATION
Consider a nonlinear control system:
\[
\dot{x}(t) = f(x(t)) + g(x(t))u(t) \tag{1}
\]
where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, and the functions $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^{n \times m}$ are locally Lipschitz. Suppose that $f(0) = 0$, $(f, g)$ is stabilizable.

In the event-triggered framework, the control signal $u$ is updated when the updating event is triggered. Let $t_j, j = 1, 2, \ldots$, be the triggered times. Therefore, the practical control input is

$$
u(t) = p(x(t_j)), t_j \leq t < t_{j+1}, j = 1, 2, \ldots, \tag{2}$$

where $p(x)$ is a state feedback control law to be designed. Define the error signal as

$$e(t) = p(x(t)) - p(x(t_j)), t_j \leq t < t_{j+1}, j = 1, 2, \ldots, \tag{3}$$

Note that $e(t_j) = 0, j = 1, 2, \ldots$.

Let $t_1 = 0$. The triggered time is determined by

$$t_j = \inf\{t > t_{j-1} | \xi(e(t), x(t)) = 0\}, j = 2, 3, \ldots \tag{4}$$

where $\xi(\cdot, \cdot)$, satisfying $\xi(0, x(t)) < 0$ for each $x(t) \neq 0$, is an event-triggered policy to be determined later. The operator $\inf\{\cdot\}$ is used to find the greatest lower bound of the $t$ satisfying $\xi(e(t), x(t)) = 0$ and $t > t_{j-1}$, which is defined as the triggered time [2], [7].

Define

$$D_i \equiv \{x \in \mathbb{R}^n | s_i(x) > 0\}, i = 1, \ldots, N,$$

where $N \in \mathbb{Z}^+$ and functions $s_i(\cdot), i = 1, \ldots, N$, are differentiable. Assume that $\partial D_i \cap \partial D_j = \emptyset$ if $i \neq j$. Let $D \subset \mathbb{R}^n$ be a connected region represented by

$$D \equiv \{x \in \mathbb{R}^n | s_i(x) > 0, i = 1, \ldots, N\}. \tag{5}$$

That is,

$$D = D_1 \cap D_2 \cap \ldots \cap D_N,$$

and

$$\partial D = (\partial D_1 \cup \partial D_2 \cup \ldots \cup \partial D_N) \cap D.$$

The objective of this paper is to find a state feedback law $p(\cdot)$ and an event-triggered policy $\xi(\cdot, \cdot)$ such that the event-triggered controller (2) asymptotically stabilizes the system (1) under the state constraint (suppose that $x(0) \in D$):

$$x(t) \in D, \forall t \geq 0. \tag{6}$$

**B. PRELIMINARIES**

This subsection introduces the CLF and the CLBF.

**Definition 1 [36]:** The system

$$\dot{x} = f(x) \tag{7}$$

is asymptotically stable with $D$ being forward invariant and in the region of attraction if, for each $\epsilon > 0$, there is a $\delta > 0$ such that for each $x(0) \in B_D(\delta)$, the trajectory satisfies

$$x(t) \in B_D(\epsilon) \forall t \geq 0,$$

and

$$\lim_{t \to \infty} x(t) \to 0 \text{ for each } x(0) \in D.$$

For convenience, a function $V : D \to \mathbb{R}$ is said to be positive definite in $D$ if $V(x) > 0$ for all $x \in D \setminus \{0\}$ and $V(0) = 0$. It can be concluded from [30] that the system (7) is asymptotically stable with $D$ being forward invariant and in the region of attraction if there exists a differentiable positive definite function $V : D \to \mathbb{R}$ satisfying $V(x) \to \infty$ as $||x|| \to \infty$ or $x \to \partial D$, such that $\nabla V(x)f(x) < 0 \forall x \in D \setminus \{0\}$.

Next, simple reviews of the concept of CLFs [24] and CLBFs [30] will be introduced.

**Definition 2 [24]:** A differentiable, proper, and positive definite function $V_C : \mathbb{R}^n \to \mathbb{R}$ is a CLF of system (1) if for each $x \in \mathbb{R}^n \setminus \{0\}$,

$$\inf_{u \in \mathbb{R}^m} \{\nabla V_C(x)f(x) + \nabla V_C(x)g(x)u\} < 0. \tag{8}$$

**Definition 3 (24):** A CLF $V_C : \mathbb{R}^n \to \mathbb{R}$ of system (1) satisfies the small control property (SCP) if for each $\epsilon > 0$ there is a $\delta > 0$ such that, for each nonzero $x$ satisfying $||x|| < \delta$, there is some $u$ with $||u|| < \epsilon$ such that

$$\nabla V_C(x)f(x) + \nabla V_C(x)g(x)u < 0.$$

Let

$$a_C(x) = \nabla V_C(x)f(x)$$

and

$$b_C(x) = \nabla V_C(x)g(x).$$

Similar to [30], define

$$\eta(\alpha, \beta, \mu) = -\frac{\alpha + \sqrt{\alpha^2 + \mu \beta^2}}{\beta^2} \beta^T,$$

where $\alpha, \mu \in \mathbb{R}, \mu > 0$, and $\beta \in \mathbb{R}^n \setminus \{0\}$. It has been proven in Theorem 1 of [24] that if a CLF $V_C$, satisfying the SCP, exists, a continuous asymptotically stabilizing controller can be constructed by using Sontag’s formula:

$$u = p_s(x) = \left\{ \begin{array}{ll}
\eta(a_C(x), b_C(x), 1), & \text{if } b_C(x) \neq 0 \\
0, & \text{if } b_C(x) = 0. \end{array} \right. \tag{9}$$

**Definition 4 [30]:** A differentiable function $V : D \to \mathbb{R}$ is a CLBF of system (1) if $V$ is positive definite in $D$, $V(x) \to \infty$ as $||x|| \to \infty$ or $x \to \partial D$ and,

$$\inf_{u \in \mathbb{R}^m} \{\nabla V(x)f(x) + \nabla V(x)g(x)u\} < 0, \forall x \in D \setminus \{0\}. \tag{10}$$

In the next section, we first consider how to design a continuous state-constrained stabilizing feedback law for the system (1). An event-triggered policy will be derived based on the obtained feedback law.

**III. CONSTRUCTIONS OF STRONG CLBFs AND STATE-CONSTRAINED STABILIZATION**

For deriving a state-constrained event-triggered policy, we need a strong CLBF instead of the CLBF defined in [30].

**Definition 5:** A smooth function $V : D \to \mathbb{R}$ is a strong CLBF of system (1) if

B1) $V(x) > 0$, for all $x \in D \setminus \{0\}$, and $V(0) = 0$;
B2) $V(x) \to \infty$ as $\|x\| \to \infty$ or $x \to \partial D$.

B3) There is a $\sigma > 0$ such that, for each $x \in D \setminus \{0\}$
\[ \inf_{u \in \mathbb{R}^m} \{ \nabla V(x)f(x) + \nabla V(x)g(x)u + \sigma V(x) \} < 0. \]  
(11)

Here, the term $\sigma V$ is added to simplify the design of the event-triggered policy, which will be provided in the next section. If we only need to derive a continuous state-constrained stabilizing controller, this term can be eliminated. Let
\[ a(x) = \nabla V(x)f(x) + \sigma V(x) \]
and
\[ b(x) = \nabla V(x)g(x) \]

Condition (11) is equivalent to
\[ \forall x \in D \setminus \{0\} \text{ such that } b(x) = 0 \Rightarrow a(x) < 0. \]  
(12)

For a nonlinear control-affine system, a CLBF can be easily constructed by combing a CLF and a barrier function provided that a CLF is available and the conditions in Theorem 1 of [30] hold. The strong CLBFs can be constructed in a similar way with a little modification. To construct strong CLBFs, we need strong CLFs defined below.

**Definition 6:** A differentiable, proper, and positive definite function $V_C : \mathbb{R}^n \to \mathbb{R}$ is a strong CLF of system (1) if for each $x \in \mathbb{R}^n \setminus \{0\}$,
\[ \inf_{u \in \mathbb{R}^m} \{ \nabla V_C(x)f(x) + \nabla V_C(x)g(x)u + \sigma V_C(x) \} < 0. \]  
(13)

One can see that, a strong CLF $V_C(x)$ must satisfy
\[ \forall x \in \mathbb{R}^n \setminus \{0\} \text{ such that } \nabla V_C(x)g(x) = 0 \Rightarrow \nabla V_C(x)f(x) + \sigma V_C(x) < 0. \]  
(14)

The existence of strong CLBFs is sufficient for the existence of state-constrained stabilizing feedback laws. To ensure the continuity of the obtained feedback law, a strong CLBF must satisfy the small control property (SCP).

**Definition 7 (30):** A strong CLBF $V : D \to \mathbb{R}$ of the system (1) satisfies the small control property if for each $\epsilon > 0$ there is a $\delta > 0$ such that, for each $x \in B_D(\delta) \setminus \{0\}$, there is some $u$ with $\|u\| < \epsilon$ such that $a(x) + b(x)u < 0$.

Similar to [37], a strong CLBF $V$ of the system (1) satisfies the SCP if and only if
\[ \lim_{\delta \to 0} \sup_{x \in B_D(\delta)} \frac{a(x)}{\|b(x)\|} \leq 0. \]  
(15)

Let $k : D \to \mathbb{R}$ be a differentiable function with $k(x) > 0$ for all $x \in D$. We have the following result.

**Lemma 1:** [30] If there exists a CLBF $V : D \to \mathbb{R}$, satisfying the SCP, for the system (1), then there exists a continuous state feedback controller $u = p(x)$, such that the closed-loop system is asymptotically stable with $D$ being forward invariant and in the region of attraction. In this case,
\[
\begin{aligned}
u &= p(x) \\
&= \left\{ \begin{array}{ll}
\eta(a(x), b(x), \frac{b(x)}{k(x) + \|b(x)\|}), & \text{if } b(x) \neq 0 \\
0, & \text{if } b(x) = 0
\end{array} \right.
\tag{16}
\end{aligned}
\]
is one such controller.

Because the definition of strong CLBFs is a little different from CLBFs in [30], the conditions for the existence of strong CLBFs are different from those for CLBFs provided in [30]. For convenience, define
\[ Z_{sgi} = \{ x \in \bar{D}_i \cap \bar{D} \mid \nabla s_i(x)g(x) = 0 \}, \]
\[ C_{Mi} = \{ x \in Z_{sgi} \mid \nabla V(x)g(x) = 0 \}, \]
\[ C_{Li} = \{ x \in \bar{D}_i \cap \bar{D} \setminus Z_{sgi} \mid \text{there is a } \gamma > 0 \text{ such that } \nabla V_C(x)g(x) = \gamma \nabla s_i(x)g(x) \}. \]

**Lemma 2:** Suppose that $C_{Mi} \cap \partial D_i$ and $C_{Li} \cap \partial D_i$ are bounded for each $i = 1, 2, \ldots, N$. Assume $\{ 0 \} \notin \partial D_i(\epsilon) \cup D^c_i$ and $\partial D_i(\epsilon) \cap \partial D_j(\epsilon) = \emptyset$ if $i \neq j$ for some $\epsilon > 0$. If $V_C : \mathbb{R}^n \to \mathbb{R}$ is a strong CLF for the system (1) that satisfies the SCP such that for $i = 1, 2, \ldots, N$,
\[ a). \quad \nabla s_i(x)f(x) > 0, \quad \forall x \in (Z_{sgi} \cup C_{Li}) \cap \partial D_i, \]
\[ b). \quad \nabla V_C(x)f(x) < -\sigma V(x), \quad \forall x \in C_{Li} \cap \partial D_i \]
then there exists a strong CLBF, satisfying the SCP, for system (1).

**Proof:** As in [30], define
\[ V(x; \epsilon) = \left\{ \begin{array}{ll}
V_C(x) + \frac{(\epsilon - s_i(x))^2}{s_i(x)}, & \text{if } x \in \partial D_i(\epsilon) \cap D_i \\
V_C(x), & \text{if } x \in D \setminus \cup_{i=1}^N \partial D_i(\epsilon).
\end{array} \right. \]  
(17)

Clearly, $V(x; \epsilon)$ is differentiable and positive definite in $D$. We first show that, for sufficiently small $\epsilon$,
\[ \inf_{u \in \mathbb{R}^m} \nabla V(x; \epsilon)(f(x) + g(x)u) < -\sigma V(x; \epsilon) \]
for each $x \in D \setminus \{0\}$.

**Case 1:** for $x \in D \setminus \cup_{i=1}^N \partial D_i(\epsilon)$. In this region, $V(x; \epsilon) = V_C(x)$. Since $V_C(x)$ is a strong CLF of system (1), one can see that, for each nonzero $x \in D \setminus \cup_{i=1}^N \partial D_i(\epsilon)$,
\[ \inf_{u \in \mathbb{R}^m} \nabla V(x; \epsilon)(f(x) + g(x)u) = \inf_{u \in \mathbb{R}^m} \nabla V_C(x)(f(x) + g(x)u) \]
\[ < -\sigma V_C(x) = -\sigma V(x; \epsilon). \]

**Case 2:** for $x \in \partial D_i(\epsilon) \cap D$ for $i \in \{1, 2, \ldots, N\}$. In this region, we have
\[
\begin{aligned}
\nabla V(x; \epsilon)(f(x) + g(x)u) &= \frac{1}{s_i^2(x)}(a_{V_C}(x) - a_{s_i}(x; \epsilon)) \\
&\quad + \frac{1}{s_i^2(x)}(b_{V_C}(x) - b_{s_i}(x; \epsilon))u.
\end{aligned}
\tag{18}
\]
where
\[ a_V(x) = s(x) \nabla V(x) f(x) \]
\[ a_s(x; \epsilon) = (\epsilon^2 - s^2(x)) \nabla s(x) f(x) \]
\[ b_V(x) = s(x) \nabla V(x) g(x) \]
\[ b_s(x; \epsilon) = (\epsilon^2 - s^2(x)) \nabla s(x) g(x) \]

Let
\[ C_{Z \epsilon}(\epsilon) = \{ x \in \partial D_i(\epsilon) \cap D | \nabla V(x; \epsilon) g(x) = 0 \}. \]

Clearly, \( C_{Z \epsilon}(\epsilon) \subset (C_{M \epsilon}(\epsilon) \cup C_{L \epsilon}(\epsilon)) \cap \partial D_i(\epsilon). \) For \( V(x; \epsilon) \) to be a strong CLBF, \( \nabla V(x; \epsilon) f(x) \) must be smaller than \( -\sigma V(x; \epsilon) \) for each \( x \in C_{Z \epsilon}(\epsilon). \) Note that \( \partial D_i(\epsilon) \) and thus \( C_{Z \epsilon}(\epsilon) \) shrinks as \( \epsilon \) becomes smaller. By condition (a), as \( C_{Z \epsilon}(\epsilon) \) is bounded (notice that \( C_{M \epsilon}(\epsilon) \cap \partial D_i(\epsilon) \) and \( C_{L \epsilon}(\epsilon) \partial D_i(\epsilon) \) are bounded), an \( \epsilon \geq 0 \) exists such that, for each \( \epsilon \) satisfying \( 0 < \epsilon < \epsilon_i \), if \( x \in C_{Z \epsilon}(\epsilon) \),
\[ \nabla s(x) f(x) > \sigma s(x) > \frac{\epsilon - s_i(x)}{\epsilon + s_i(x)}. \]

Then,
\[ -\frac{(\epsilon^2 - s_i^2(x)) \nabla s(x) f(x)}{s_i^2(x)} < -\sigma \left( \frac{\epsilon - s_i(x)}{s_i(x)} \right)^2. \]

Because \( V_C(x) \) is a strong CLF and by condition (b), there exists an \( \epsilon_2 > 0 \) such that, for each \( \epsilon \) satisfying \( 0 < \epsilon < \epsilon_2 \), we have
\[ \nabla V_C(x) f(x) < -\sigma V_C(x), \forall x \in C_{Z \epsilon}(\epsilon). \]

As a result, from (18),
\[ \nabla V(x; \epsilon)(f(x) + g(x) u) < -\sigma V(x; \epsilon) - \frac{\sigma (\epsilon - s_i(x))^2}{s_i(x)} = -\sigma V(x; \epsilon). \]

The previous discussion shows that for \( \epsilon < \min \{ \epsilon_1, \epsilon_2 \} \),
\[ \inf \nabla V(x; \epsilon)(f(x) + g(x) u) < -\sigma V(x; \epsilon) \]
for each \( x \in D \setminus \{ 0 \} \) and therefore \( V(x; \epsilon) \) is a strong CLBF of the system (1).

On the other hand, that \( V(x; \epsilon) \) satisfies the SCP is obvious since \( V_C(x) \) satisfies the SCP and \( \{ 0 \} \in D \setminus \partial D(\epsilon). \)

**IV. EVENT-TRIGGERED POLICY**

Based on the feedback law obtained in the previous section, a new event-triggered policy for system (1) will be proposed.

From (1) and (2), where \( p(\cdot) \) is defined in (16) and \( V(\cdot) \) is a strong CLBF, the event-triggered closed-loop system is given by
\[ \dot{x}(t) = f(x(t)) + g(x(t)) p(x(t_j)), t_j \leq t < t_{j+1}. \]

Further, according to (3), we can rewrite the closed-loop system as follows:
\[ \dot{x}(t) = f(x(t)) + g(x(t)) p(x(t)) - g(x(t)) e(t). \]

Along with the solutions of the closed-loop system (19) yields
\[ \dot{V}(x) = \nabla V(x)(f(x) + g(x)p(x) - g(x)e) = a(x) + b(x)p(x) - \sigma V(x) - b(x)e. \]

Since \( V \) is a strong CLBF, we have
\[ \dot{V}(x) < -\sigma V(x) - b(x)e, \forall x \neq 0. \]

Hence, for nonzero \( x \), if \( -b(x)e < \sigma V(x) \), we have \( \dot{V}(x) < 0 \).

As a result, define the triggered times as
\[ t_j = \inf \{ t > t_{j-1} | -b(x(t)) \epsilon(t) = \sigma V(x(t)) \}, j = 2, 3, \ldots \]

That is, \( \zeta(\epsilon, x) = -b(x)e - \sigma V(x) \) in (4). At each \( t_j \), the control signal is updated according to (2).

Similar to the Sontag’s formula [24], the feedback law (16) is differentiable at each \( x \neq 0 \) and continuous at \( 0 \). To ensure that the inter-execution times will not become arbitrarily close under this triggered policy, we show that they are lower bounded away from zero provided that some additional conditions hold.

**Theorem 1:** Suppose that there exists a strong CLBF \( V : D \rightarrow R \) for the system (1). With the control law (2) and the event-triggered policy (20), where \( p(x) \) is defined in (16), if

a. \( \lim_{x \to 0} \| \nabla p(x) \| \leq \infty \),

b. \( \lim_{x \to 0} \| \nabla p(x)(f(x) + g(x)p(x)) \| / |V(x)| \leq \infty \),

c. there exists a \( \lambda > 0 \) such that \( \nabla V(x)(f(x) + g(x)p(x)) > -\lambda V(x) \),

then, for any compact set \( S \subset D \) containing the origin, there exists a \( \tau > 0 \) such that, for any initial condition in \( S \), the inter-execution times \( \{ t_{j+1} - t_j \} \) implicitly defined by the policy (20) are lower bounded by \( \tau \).

Proof: Let \( C \) be the compact set defined by all points \( x \in D \) satisfying \( V(x) \leq 0 \), where \( \mu > 0 \) is large enough so that \( S \subset C \). Such \( \mu \) always exists since \( V \) is a strong CLBF. Set \( C \) is forward invariant for the closed-loop system since the triggered policy (20) guarantees
\[ \dot{V} < 0, \forall x \neq 0. \]

Define
\[ L = \max_{x \in C} \frac{\| b(x) \|}{\sigma}. \]

By enforcing a more conservative inequality \( L \| e \| < V(x) \), we have
\[ \| b(x) \| \| e \| < \sigma V(x), \]
which guarantees
\[ -b(x)e \leq \| b(x) \| \| e \| \leq L \| e \| < \sigma V(x). \]

Under conditions (a) and (b), we can define
\[ K = \sup_{x \in C \setminus \{ 0 \}} \| \nabla p(x)(g(x)) \|, \]
and
\[ M = \sup_{x \in C \setminus \{ 0 \}} \| \nabla p(x)(f(x) + g(x)p(x)) \| / V(x). \]
Because the triggered policy (20) is looser than the virtual triggered policy $L_v\|e\| = V(x)$, it suffices to show that the inter-execution times are bounded under the virtual triggered policy $L_v\|e\| = V(x)$. We can start by looking at the dynamics of $V(x)$. For $x \neq 0$, by conditions (a), (b), and (c), the triggered policy (20), we have

$$\frac{d}{dt} \|e\| = (e^T)\frac{\dot{x}}{2} V(x) = \frac{1}{2} \|e\| \dot{V}(x) + \lambda \|e\| \sigma V(x)$$

According to (21),

$$\frac{d}{dt} \|e\| \leq \|e\| \dot{V}(x) + \lambda \|e\| \sigma V(x)$$

If we denote $\|e\|_y$ by $y$ and let $\phi(t, \phi_0)$ be the solution of

$$\dot{\phi} = M + (K + \lambda + \sigma) \phi$$

satisfying $\phi(0, \phi_0) = \phi_0$, then

$$\dot{y} \leq M + (K + \lambda + \sigma)y,$$

and therefore, $y(t) \leq \phi(t, \phi_0)$. Hence the inter-execution times are bounded by the solution of $\phi(t, 0) = \frac{1}{L}$, where $\tau$ can be derived by solving the above differential equation. In fact, we can get

$$\phi(t, 0) = \frac{M}{K + \lambda + \sigma} e^{(K + \lambda + \sigma)t} - \frac{M}{K + \lambda + \sigma}.$$

As a result, the solution of $\phi(t, 0) = \frac{1}{L}$ is

$$\tau = \frac{1}{(K + \lambda + \sigma) \ln(1 + \frac{K + \lambda + \sigma}{LM})} > 0.$$
The release time interval

![State trajectories of the closed-loop system](image1)

**FIGURE 1.** State trajectories of the closed-loop system (blue: $\epsilon = 0.5$, $\sigma = 1.6$; red: $\epsilon = 0.1$, $\sigma = 1.8$; green: controlled by (9)).

![The release time interval of the closed-loop system](image2)

**FIGURE 2.** The release time interval of the closed-loop system starting at the initial state $[9 - 3]^T$ with $\epsilon = 0.5$, $\sigma = 1.6$.

Table 1 presents the number of triggered events in the first 1.5s for different initial conditions in the case of $(\epsilon, \sigma) = (0.5, 1.6)$. The result for initial state $x(0) = [9 - 3]^T$ is the worst case in this simulation because the number of triggered events is 905 in the first 1.5s. In all other cases, the number of triggered events is small. The interval release times of each event defined by the event-triggered policy (28) with initial state $x(0) = [9 - 3]^T$ are presented in Fig. 2. It is observed that triggered events occur frequently as state trajectories are close to $\partial D$. This is because, with the strong CLBF $V(x; \epsilon)$ defined in (17), the ideal control signal may change rapidly as the state trajectory approaches the boundary. However, the number of triggered events decreases as the state trajectory leaves the boundary. The maximum time interval between successive triggered events is 0.416s. According to Table 1, obviously, the number of communication events was successfully reduced.

Fig. 3 presents the responses of the closed-loop systems controlled by the controller (9) and the event-triggered controller (2) with the feedback law (16) and the event-triggered policy (28), starting at the same initial state $x(0) = [9 - 3]^T$, respectively. It can be seen that the values of $s_1(x)$ and $s_2(x)$ are always positive for the system controlled by the event-triggered controller (2), while the value of $s_1(x)$ is negative in some time interval for the system controlled by (9).

**B. EXAMPLE 2:**

In this example, we show that even the sufficient conditions in Lemma 2 do not hold, finding a CLBF that satisfies the conditions in Definition 5 is possible.

Consider the following single-link robot arm system [14]:

$$
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{mgl}{J} \sin x_1 - \frac{d}{J} x_2 + \frac{1}{J} u
\end{align*}
$$

(29)

where $x_1$ and $x_2$ denote the angle and angular velocity of the
arm, $u \in R$ denotes the control input, $g$ is the gravitational constant, $l$, $m$, $J$, and $d$ denote the length, mass, inertia, and damping of the arm, respectively. The parameter values are chosen as those in [14]: $m = 1$, $J = 1$, $d = 2$, $l = 0.5$, and $g = 9.8$. Define $s_1(x) = 0.5 + x_2$ and $s_2(x) = 0.5 - x_2$. Similar to the previous example, a continuous state feedback law $u = p(x)$ will be firstly derived to stabilize the system (29) under the state constraint:

$$x(t) \in D \equiv D_1 \cap D_2,$$

where $D_i = \{x \in R^2 | s_i(x) > 0\}, i = 1, 2$.

For system (29), with a CLF $V_C(x) = \frac{1}{2} x_1^2 + \frac{1}{2} (x_2 + 3x_1)^2$ obtained by the traditional backstepping approach, sufficient conditions in Lemma 2 do not hold. Nevertheless, it is easy to verify that

$$V(x) = \frac{1}{2} x_1^2 + \frac{1}{2} (x_2 + 3x_1)^2 + \frac{1}{2} 0.25x_2$$

is a strong CLBF of the system (29) because the conditions B1), B2), and B3) in Definition 5 hold with $\sigma = 1.6$. Therefore, a continuous state-constrained stabilizing controller can be constructed by (16) with $k(x) = 0.1 + 2\|x\|$. The event-triggered policy is

$$-b(x)e - 1.6V(x) = 0.$$  

Fig. 4 shows the state trajectories of the closed-loop system controlled by the continuous state-constrained controller (16), the event-triggered state-constrained controller (2), and the controller implemented with the Songtag’s formula (9). The trajectories controlled by (9) enter the unsafe region. The trajectories controlled by the event-triggered controller (2) and the continuous controller (16) all satisfy the state constraint (30). Fig. 5 presents the responses of the closed-loop systems controlled by the continuous controller (16) and the event-triggered controller (2) starting at the same initial state $x(0) = [0.6 \quad -0.48]^T$, respectively. As expected, the response controlled by the continuous controller (16) is smoother than the one controlled by the event-triggered controller (2). Fig. 6 presents the release time interval of the closed-loop system controlled by the event-triggered controller (2) with initial state $x(0) = [0.6 \quad 0.48]^T$. Similar to Example 1, the triggered events occur frequently in the first 0.5s as the state trajectory is close to $\partial D$ and then become less and less. In the first three seconds, the maximum time interval between successive triggered events is 0.598s.

### VI. CONCLUSIONS

In this paper, a strong CLBF method was developed for designing event-triggered asymptotically stabilizing controllers for nonlinear control-affine systems under multiple state constraints. In this approach, an event-triggered policy can be derived without the input-to-state stability (ISS) assumption. The inter-execution times were shown to be lower bounded away from zero. Sufficient conditions for the existence of strong CLBFs were derived. Further extensions of the proposed approach to the state-constrained stabilization problem with communication time delay, possible packet dropout, etc., are possible.

### REFERENCES

[1] P. Tabuada, “Event-triggered real-time scheduling of stabilizing control tasks,” IEEE Trans. Autom. Control, vol. 52, no. 9, pp. 1680-1685, 2007.

[2] M. A. Davó, C. Prieur, and M. Fiaccchini, “Stability analysis of output feedback control systems with a memory-based event-triggering mechanism,” IEEE Trans. Autom. Control, vol. 62, no. 12, pp. 6625-6632, 2017.

[3] X. Wang and M. D. Lemmon, “Event-triggering in distributed networked control systems,” IEEE Trans. Autom. Control, vol. 56, no. 3, pp. 586-601, 2011.

[4] E. García and P. J. Antsaklis, “Model-based event-triggered control with time-varying network delays,” in Proc. 50th IEEE Conf. Decis. Control Eur. Control Conf., Orlando, FL, USA, Dec. 2011, pp. 1650-1655.

[5] X. Yin, D. Yue, S. Hu, C. Peng, and Y. Xue, “Model-based event-triggered predictive control for networked systems with data dropout,” SIAM J. Control Optim., vol. 54, no. 2, pp. 567-586, 2016.

[6] E. Garcia and P. J. Antsaklis, “Model-based event-triggered control for systems with quantization and time-varying network delays,” IEEE Trans. Autom. Control, vol. 58, no. 2, pp. 422-434, 2013.
The release time interval

0.1
0.2
0.3
0.4
0.5
0.6

J.-Y. Jhang

at the initial state

[20] P. Wieland and F. Allgower, “Constructive safety using control barrier

[19] X. Jin, “Adaptive fixed-time control for MIMO nonlinear systems with

[18] K. Sachan and R. Padhi, “Output-constrained robust adaptive control for

[17] K. Zhao, Y. Song, and Z. Zhang, “Tracking control of MIMO nonlinear

[16] B. Niu and J. Zhao, “Barrier Lyapunov functions for the output tracking

[15] Y. J. Liu and S. Tong, “Barrier Lyapunov functions-based command filtered output feedback control for full-state constrained nonlinear systems,” Automatica, vol. 105, pp. 71-79, 2019.

[14] Y. J. Liu and S. Tong, “Barrier Lyapunov functions-based adaptive control for a class of nonlinear pure-feedback systems with full state constraints,” Automatica, vol. 64, pp. 70-75, 2016.

[13] Z. Liu, B. Chen, C. Lin, and Y. Shang, “Adaptive neural constraint output control for a class of quantized input switched nonlinear system,” IEEE Access, vol. 7, pp. 121493-121500, 2019.

[12] Y. J. Liu and Y. Zhao, “Barrier Lyapunov functions for the output tracking control of constrained nonlinear switched systems,” Syst. Control Lett., vol. 62, no. 10, pp. 963-971, 2013.

[11] K. Zhao, Y. Song, and Z. Zhang, “Tracking control of MIMO nonlinear systems under full state constraints: A single-parameter adaptation approach free from feasibility conditions,” Automatica, vol. 107, pp. 52-60, 2019.

[10] J. Yu, L. Zhao, H. Yu, and C. Lin, “Barrier Lyapunov functions-based command filtered output feedback control for full-state constrained nonlinear systems,” IEEE Access, vol. 7, pp. 83-93, 2018.

[9] K. B. Ngo, R. Mahony, and Z. P. Jiang, “Integrator backstepping using barrier functions for systems with multiple state constraints,” in Proc. 44th IEEE Conf. Decis. Control Eur. Control Conf., Seville, Spain, Dec. 2005, pp. 8306-8312.

[8] Y. J. Liu and S. Tong, “Barrier Lyapunov functions for nussbaum gain adaptive control of full state constrained nonlinear systems,” Automatica, vol. 76, pp. 143-152, 2017.

[7] L. Xing, C. Wen, Z. Liu, H. Su, and J. Cai, “Event-triggered adaptive control for a class of uncertain nonlinear systems,” IEEE Trans. Autom. Control, vol. 62, no. 4, pp. 2071-2076, 2016.

[6] Y. Liu, J. Yu, H. Yu, C. Lin, and L. Zhao, “Barrier Lyapunov functions-based adaptive neural control for permanent magnet synchronous motors with full-state constraints,” IEEE Access, vol. 5, pp. 10382-10389, 2017.

[5] K. Artstein, “Stabilization with relaxed control,” in Proc. 36th IEEE Conf. Decision and Control, San Diego, CA, USA, Dec. 1997, pp. 1287-1292.

[4] Y. Hua, and T. Zhang, “Adaptive neural event-triggered control of MIMO pure-feedback systems with asymmetric output constraints and unmodeled dynamics,” IEEE Access, vol. 8, pp. 37684-37696, 2020.

[3] Z. Cao, C. Wen, and Y. Song, “A unified event-triggered control approach for uncertain pure-feedback systems with or without state constraints,” IEEE Trans. Cybern., vol. 51, no. 3, pp. 1267-1271, 2021.

[2] Y. Wei, P. Zhou, W. Xie, J. Tang, D. Duan, “Event-triggered adaptive output-feedback control for nonlinear state-constrained systems using tangent-type nonlinear mapping,” Asian J. Control, vol. 13, pp. 1582-1594, 2021.

[1] Z. Artstein, “Stabilization with relaxed control,” SIAM J. Control Optim., vol. 20, no. 3, pp. 655-687, 1982.

**References**

[23] E. D. Sontag, “A Lyapunov-like characterization of asymptotic controllability,” SIAM J. Control Optim., vol. 21, no. 3, pp. 462-471, 1983.

[22] Z. Artstein, “Stabilization with relaxed control,” Nonlinear Anal. Theory, Methods Appl., vol. 7, no. 11, pp. 1163-1173, 1983.

[21] W. Xiang, “A ‘Universal’ construction of Artstein’s theorem on nonlinear stabilization,” Syst. Control Lett., vol. 13, no. 2, pp. 117-123, 1989.

[20] R. A. Freeman, and P. V. Kokotovic, Robust Nonlinear Control Design. Boston, MA, USA: Birkhäuser, 1996.

[19] J. W. Grizzle, and P. Tabuada, “Control barrier function based quadratic programs with application to adaptive cruise control,” in Proc. 53rd IEEE Conf. Decision. Control, Los Angeles, CA, USA, Dec. 2014, pp. 6271-6278.

[18] A. D. Ames, J. W. Grizzle, and P. Tabuada, “Control barrier function based quadratic programs with application to adaptive cruise control,” in Proc. 53rd IEEE Conf. Decision. Control, Los Angeles, CA, USA, Dec. 2014, pp. 6271-6278.

[17] A. D. Ames, “Stabilization of Nonlinear Control-Affine Systems with Multiple State Constraints,” IEEE Access, vol. 8, pp. 179735-179744, 2020.

[16] H. Changchun, L. Shiyi, L. Yafeng, and G. Xinka, “Output feedback control based on event triggered for nonlinear systems with full state constraints,” in Proc. 36th Chinese Control Conf. (CCC), Dalian, China, Jul. 2017, pp. 1091-1096.

[15] L. B. Wu, J. H. Park, X. P. Xie, C. Gao, and N. N. Zhao, “Fuzzy adaptive event-triggered control for a class of uncertain nonlinear systems with full state constraints,” IEEE Trans. Fuzzy Syst., vol. 29, no. 4, pp. 904-916, 2021.

[14] J. Y. Jhang, J. L. Wu, and C. F. Yung, “Stabilization of Nonlinear Control- Affine Systems with Multiple State Constraints,” IEEE Access, vol. 8, pp. 179735-179744, 2020.

[13] J. Yu, L. Zhao, H. Yu, and C. Lin, “Barrier Lyapunov functions-based command filtered output feedback control for full-state constrained nonlinear systems,” Automatica, vol. 105, pp. 71-79, 2019.

[12] Y. J. Liu and Y. Zhao, “Barrier Lyapunov functions for the output tracking control of constrained nonlinear switched systems,” IEEE Access, vol. 7, pp. 121493-121500, 2019.

[11] K. P. Tee, S. S. Ge, and E. H. Tay, “Barrier Lyapunov functions for the control of output-constrained nonlinear systems,” Automatica, vol. 45, no. 4, pp. 918-927, 2009.

[10] B. Niu and J. Zhao, “Barrier Lyapunov functions for the output tracking control of constrained nonlinear switched systems,” Syst. Control Lett., vol. 62, no. 10, pp. 963-971, 2013.

[9] K. Zhao, Y. Song, and Z. Zhang, “Tracking control of MIMO nonlinear systems under full state constraints: A single-parameter adaptation approach free from feasibility conditions,” Automatica, vol. 107, pp. 52-60, 2019.

[8] X. Jin, “Adaptive fixed-time control for MIMO nonlinear systems with asymmetric output constraints using universal barrier functions,” IEEE Trans. Autom. Control, vol. 64, no. 7, pp. 3046-3053, 2019.

[7] P. Wieland and F. Allgower, “Constructive safety using control barrier functions,” in Proc. 7th IFAC Symp. Nonlin. Control Syst., Pretoria, South Africa, Aug. 2007, pp. 462-467.

[6] Z. M. Romdhony and B. Jayawardhana, “Stabilization with guaranteed safety using control Lyapunov-barrier function,” Automatica, vol. 66, pp. 39-47, 2016.