Particle relabeling symmetry, generalized vorticity, and normal-mode expansion of ideal, incompressible fluids and plasmas in three-dimensional space

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Lagrangian mechanical consideration of the dynamics of ideal, incompressible hydrodynamic, magnetohydrodynamic, and Hall magnetohydrodynamic media, which are formulated as dynamical systems on some appropriate Lie groups equipped with Riemannian metrics, leads to the notion of generalized vorticities, as well as generalized coordinates, velocities, and momenta. The action of each system is conserved against the integral path variation in the direction of the generalized vorticity, and this invariance is associated with the particle relabeling symmetry. The generalized vorticities are formulated by the operation of integro-differential operators upon the generalized velocities. The eigenfunctions of the operators provide sets of orthogonal functions and, expanding by them, we obtain a common formula for these dynamical systems. In particular, we find that the product of the Riemannian metric, $M_{lm}$, and the structure constants of the Lie group, $C_{jk}^m$, is given by the product of the eigenvalue of the operator, $\Lambda(i)$, and a certain totally antisymmetric tensor, $T_{ijk}$: $M_{lm}C_{jk}^m = \Lambda(i)T_{ijk}$.

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I. INTRODUCTION

Though the operation of particle relabeling does not alter the fluid motions visible to us, the associated “symmetry” is well-known to play a crucial role in considerations of the conservation laws of hydrodynamics (HD) and magnetohydrodynamics (MHD). The degrees of freedom for the symmetry come into the description of fluid mechanics when a Lagrangian specification is adopted. Despite this redundancy, the description of fluid mechanics obtains its analytical mechanical foundation.

The Lagrangian specification introduces triplets of functions to indicate the positions of fluid particles at an assigned time, and these triplets work as the generalized coordinates of Lagrangian mechanics of continua in a three-dimensional space. Mathematically, the function space of these triplets, say $G$, is constituted by bijective maps from $M$ onto itself, where $M$ is a three-dimensional manifold or, physically, a container of a fluid. Because their compositions are also the elements of $G$, the function space constitutes a continuous group with respect to the function composition operation, and this is the key to understanding the basics of the mathematical description.

This was strongly recognized by Arnold when he reviewed his studies on dynamical systems on Lie groups equipped with a Riemannian metric and related hydrodynamics topics in a unified form. Since then, many dynamical systems have been recognized as being on some appropriate Lie groups. In particular, extension to the semidirect product groups makes it possible to treat a wide variety of dynamics of continua, for example, ideal, incompressible MHD systems that include two principal components, i.e., the velocity and current fields.

For many cases of physical interest, the second variables were often set to be passive, i.e., assumed to have so-called “frozen-in” natures. Mathematically, many authors have often considered the semidirect product of a Lie group and appropriate vector spaces that do not have group structure, though the notion of a semidirect product itself can be defined between two groups. On the other hand, Vizman developed the theory on the basis of the semidirect product of two groups. This extension gave us the way to formulate the ideal, incompressible Hall magnetohydrodynamics (HMHD) as a dynamical system on a semidirect product group.

The particle-relabeling symmetry in general form is clearly formulated in terms of Lin constraints. Let $q(t)$ and $\delta q(t)$ be a reference path on an appropriate configuration space and small perturbation imposed on it, respectively; the tangent vector along the path $(\dot{q}(t))$ deviates as

$$\delta \dot{q} = \partial_t(\delta q) + [\delta q, \dot{q}],$$

where $[\cdot, \cdot]$ is the Lie bracket of the Lie algebra associated with the Lie group $G$. This formula is derived by approximating small patches of paths using exponential maps and applying the Baker-Campbell-Hausdorff formula to the small patches around them (see, for example, Ref. Appendix A). The second term on the right-hand side is specific to the dynamical systems on Lie groups and yields the quadratic terms of the evolution equation. The zero-velocity perturbation condition $\delta \dot{q} = 0$ leads to the linear differential equation of $\delta q$:

$$\partial_t(\delta q) + [\delta q, \dot{q}] = 0. \quad (1)$$

This formula physically gives the condition that the reference and perturbed paths give the same velocity history,
and thus, corresponds to the operation of particle relabeling. Because the choice of the “initial” condition for Eq. \(1\) is arbitrary, a reference path and small perturbations that satisfy the equation define an ensemble of paths of the same flow history. Thus, if the Lagrangian is solely defined by \(\dot{q}\), the value of the Lagrangian and the action are invariant for any path in the ensemble, leading to the conservation laws, i.e., Noether’s first theorem.

Note that this symmetry is qualitatively different from the symmetry considered, for example, in gauge field theory, wherein group transformation is applicable in principle at any point in the relevant space and time.\(^{23}\)

Though the degrees of freedom for this symmetry are very large, only the helicity-conservation laws have been found for incompressible fluids and plasmas.\(^{13}\) In the present study, the particle-relabeling symmetry is shown to be extended from the fluid component variable to the extended component, i.e., the current field variable for the plasma dynamics case. In the course of these calculations, we derive integro-differential operators that generate the action-preserving perturbation from the generalized velocities. Furthermore, the eigenfunctions of the operators constitute the orthogonal basis for each system, and the spectral representations of physical quantities reveal the relationship among the Riemannian metric, the structure constants of the Lie group, and the eigenvalues of the particle-relabeling operator.

This paper is organized as follows: sections 2 to 4 are devoted to the descriptions of the HMHD, MHD, and HD systems, respectively. Reviews of the mathematical basics and variational calculations are firstly given; then, construction of the generalized vorticity and helicity-based particle-relabeling operator, and normal-mode expansion of the basic quantities follow in each section. In section 5, discussions about the common features of these three systems and a comparison with higher-dimensional cases are given.

II. HALL MAGNETOHYDRODYNAMICS

Let us begin a discussion from the least trivial example: HMHD formulated by the velocity and current fields.

Note that there are two ways of formulating an HMHD system. The difference between them lies in the choice of basic variables; one uses the ion velocity and current fields while the other uses ion and electron velocity fields.\(^{10}\) Here we adopt the former formulation wherein the pair of the ion velocity, \(\vec{V}\), and the current field, \(J\), work as generalized velocities. Hereafter, we use the boldface italic letter with an arrow notation, for example \(\vec{V} = (V_1, -\alpha J_1)\), (where \(\alpha\) is the Hall term strength parameter) to denote an element of the function space of the generalized velocities and call it the “\(\vec{V}\)-variable.”

The generalized coordinates that induce the generalized velocities are given by a pair of function triplets, \((\vec{X}(\vec{u}, t), \vec{Y}(\vec{b}, t; s))\). In the present study, an italic letter with an arrow denotes a position vector, i.e., an element of the manifold \(M\), where \(M\) is a container of fluid or plasma. In differential topological terminology, the configuration space is given by the semidirect product of two volume-preserving diffeomorphisms on \(M\), \(SDiff(M)\times SDiff(M)\), say \(G\) in this section. The generalized coordinates and velocities are related by the equations\(^{15}\)

\[
\begin{align*}
\left( V^i(t) \frac{\partial}{\partial x^j} \right) \vec{X}(\bar{x}, t) &= \left( \frac{\partial}{\partial t} \right) \vec{X}(\bar{x}, t), \\
\left( -\alpha J^i(t) \frac{\partial}{\partial s} \right) \vec{Y}(\bar{b}, t; s) &= \left( \frac{\partial}{\partial t} \right) \vec{Y}(\bar{b}, t; s).
\end{align*}
\]

Note that the argument \(t\) provides the line parameter of the integral path of action. A group operation on \(G\), i.e., the composite of the two generalized coordinates is defined by the following formula:\(^{9}\)

\[
(\vec{X}_1, \vec{V}_1) \circ (\vec{X}_2, \vec{V}_2) = (\vec{X}_1(\vec{X}_2(\vec{V}_1(\vec{X}_2^{-1}))))
\]

The Lie bracket associated with this definition is given by

\[
[\vec{V}_1, \vec{V}_2] := \left( \nabla \times (\vec{V}_1 \times \vec{V}_2), -\alpha \nabla \times (\vec{V}_1 \times J_2)
\right.
\left. + J_1 \times \vec{V}_2 - \alpha J_1 \times J_2 \right),
\]

(2)

where \(\nabla \times\) is the curl operator. Divergence-free conditions \(\nabla \cdot \vec{V} = \nabla \cdot \vec{J} = 0\) are used to derive the formula. To derive the HMHD equation, we define the inner product of two \(\vec{V}\)-variables, i.e., the Riemannian metric on \(G\) as follows\(^{16}\):

\[
(\vec{V}_1, \vec{V}_2) := \int_{\mathbb{X} \in M} d^3x \left( \vec{V}_1 \cdot \vec{V}_2 - J_1 \cdot (\triangle^{-1} J_2) \right),
\]

(3)

where \(\triangle^{-1}\) is the inverse of the Laplacian. Using this inner product, we define the Lagrangian as

\[
L = \frac{1}{2} \int_{\mathbb{X} \in M} \left( |\vec{V}|^2 + |\vec{B}|^2 \right) d^3x.
\]

(4)

Hereafter, \(\vec{B}\) is used to denote the magnetic field induced by the current field \(\vec{J}\); i.e., \(\vec{B} = (\nabla \times)^{-1} \vec{J}\), where \((\nabla \times)^{-1}\) is the inverse of the curl operator with “Coulomb gauge” condition \((\nabla \cdot \vec{B} = 0)\). The generalized momenta associated with this Lagrangian, say \(\vec{M} = (\vec{M}_V, \vec{M}_J)\), are

\[
\vec{M}_V := \frac{\partial L}{\partial \vec{V}} = \vec{V}, \quad \vec{M}_J := \frac{\partial L}{\partial (-\alpha \vec{J})} = -\alpha^{-1} \vec{A},
\]

where \(\vec{A}\) is the vector potential of \(\vec{B}\) with a Coulomb gauge condition \((\vec{A} := -\triangle^{-1} \vec{J})\). Introducing the inertia operator, \(\hat{M}\), which is an integral operator given by

\[
\hat{M} = \begin{pmatrix}
I & 0 \\
0 & (\alpha \nabla \times)^{-2}
\end{pmatrix}
\]

(5)

the relationship between the generalized momenta and velocities is expressed as \(\vec{M} = \hat{M} \vec{V}\) (cf. Ref.\(^{17}\) Sect C).
Now we consider the perturbation of the integral path of action. Let \((\vec{X}_t(a, t), \vec{Y}_t(a, t; s))\) be a perturbed path, where \(\epsilon\) is a small parameter and \(\epsilon = 0\) corresponds to the reference path. For sufficiently small \(\epsilon\), the difference between these paths is expressed by an appropriate \(\vec{V}\)-variable, say \(\xi := (\vec{\xi}, -\alpha \vec{\eta})\), as follows

\[
\begin{align*}
\vec{X}_t(a, t) - \vec{X}(a, t) &= \epsilon \vec{X}(\vec{X}_t(a, t), t) + o(\epsilon), \\
\vec{Y}_t(a, t; s) - \vec{Y}(a, t; s &= -\alpha \epsilon \vec{Y}(\vec{Y}_t(a, t; s), t) + o(\epsilon).
\end{align*}
\]

The perturbation parts of the ion velocity and current variables, say \(\dot{V}_i, \dot{J}_i\) for these paths, are related to the perturbed coordinates by the differential equations

\[
\begin{align*}
\partial_t \vec{X}_t^i(a, t) &= (V^i + \epsilon \dot{V}^i + o(\epsilon)) \vec{X}_t^i(a, t), \\
\partial_t \vec{Y}_t^i(a, t; s) &= -\alpha (J^i + \epsilon \dot{J}^i + o(\epsilon)) \vec{Y}_t^i(a, t; s).
\end{align*}
\]

As for the dynamical systems on Lie groups, it is well-known that perturbations of generalized velocities obey the so-called Lin’s constraints \(\vec{V} = \partial_t \vec{\xi} + [\vec{\xi}, \vec{V}]\), where \(\vec{V} := (\dot{V}, -\alpha \dot{J})\). Each field satisfies the relation

\[
\begin{align*}
\vec{V} &= \partial_t \vec{\xi} + \nabla \times (\vec{\xi} \times V), \\
\vec{J} &= \partial_t \vec{\eta} + \nabla \times (\vec{\xi} \times J + \vec{\eta} \times V - \alpha \vec{\eta} \times J).
\end{align*}
\]

Substituting these relations, we obtain the first variation of action, \(S := \int_0^1 L dt\), and its integration by parts as follows:

\[
\begin{align*}
\frac{dS}{d\epsilon} \bigg|_{\epsilon=0} &= \int_0^1 dt \int_{\vec{x} \in M} d^3\vec{x} \left[ V \cdot [\partial_t \vec{\xi} + \nabla \times (\vec{\xi} \times V)] + A \cdot [\partial_t \vec{\eta} + \nabla \times (\vec{\xi} \times J + \vec{\eta} \times V - \alpha \vec{\eta} \times J) \right]
\end{align*}
\]

\[
\begin{align*}
\int_{\vec{x} \in M} d^3\vec{x} (V \cdot \xi + A \cdot \eta) \bigg|_{t=1}^{t=0} \\
- \int_0^1 dt \int_{\vec{x} \in M} d^3\vec{x} \left[ \partial_t V + \Omega \times V + B \times J \right] \\
- \alpha \eta \cdot \left( \partial_t A + B \times V - \alpha B \times J \right),
\end{align*}
\]

where \(\Omega\) is vorticity, \(\vec{V} = \nabla \times \vec{V}\). For the sake of simplicity, the boundary integrals are assumed to vanish throughout this study. Hamilton’s principle \((\vec{\xi} = \vec{\eta} = 0\) at \(t = 0, 1\)) leads to the Euler-Lagrange equation for the dynamics of an HMHD medium as

\[
\begin{align*}
\partial_t \vec{V} + \Omega \times V + B \times J &= -\nabla P, \\
\partial_t A + B \times V - \alpha B \times J &= -\nabla \phi,
\end{align*}
\]

where the functions \(P, \phi\) are introduced to guarantee the divergence-free conditions and correspond to the generalized pressure and the scalar potential of electro-magnetic field, respectively, in the physical context. Note that Eq. \((\text{14})\) is the nondimensionalized formula of the Hall MHD approximation\(^{18}\).

\[
E + \frac{1}{c} V \times B = \frac{1}{ne} J \times B,
\]

where \(E\) is the electric field \(E := -\partial_t A - \nabla \phi\). Taking the curl of Eqs. \((\text{10})\) and \((\text{11})\), we obtain

\[
\begin{align*}
\partial_t \Omega + \nabla \times (\Omega \times V + B \times J) &= 0, \\
\partial_t B + \nabla \times (B \times V - \alpha B \times J) &= 0.
\end{align*}
\]

The combinations \(\alpha \times (\text{13}) + \text{14}\) and \((\text{13})\) read

\[
\begin{align*}
\partial_t (\alpha \Omega + B) + \nabla \times ((\alpha \Omega + B) \times V) &= 0, \\
\partial_t \Omega + \nabla \times ((\alpha \Omega + B) \times J + \Omega \times V - \alpha \Omega \times J) &= 0.
\end{align*}
\]

On the other hand, we notice that Eq. \((\text{14})\) can be rewritten as the following pair of equations:

\[
\begin{align*}
\partial_t 0 + \nabla \times (0 \times V) &= 0, \\
\partial_t B + \nabla \times (0 \times J + B \times V - \alpha B \times J) &= 0.
\end{align*}
\]

Comparing Eqs. \((\text{15})\) and \((\text{16})\) with Eqs. \((\text{10})\) and \((\text{11})\), we find that the \(\vec{V}\)-variable

\[
\hat{\Omega} = \left( \begin{array}{c}
\xi \Omega \\
-\alpha \eta \Omega
\end{array} \right) = C_C \left( \begin{array}{c}
\alpha \Omega + B \\
-\alpha \Omega
\end{array} \right) - C_M \left( \begin{array}{c}
0 \\
B
\end{array} \right)
\]

(\text{17})

(i.e., the perturbation by \(\hat{\Omega}\) satisfies \(\vec{V} = \vec{J} = 0\). Physically this implies that the value of the Lagrangian along the path is unchanged by the perturbation of the generalized coordinates in the direction of \(\hat{\Omega}\), i.e., the perturbation implies the operation of particle relabeling. Substituting Eqs. \((\text{10})\), \((\text{11})\), and \((\text{13})\) into both sides of the first variation Eqs. \((\text{8})\) and \((\text{9})\), we obtain a conservation law:

\[
\int_{\vec{x} \in M} d^3\vec{x} (V \cdot \xi \Omega + A \cdot \eta \Omega) \bigg|_{t=1}^{t=0} = 0.
\]

Using ordinary variables, the constant of motion, say \(H\), is expressed as

\[
H = C_C \int_{\vec{x} \in M} [\alpha V \cdot \Omega + 2V \cdot B] d^3\vec{x}
\]

\[
+ \frac{C_M}{\alpha} \int_{\vec{x} \in M} A \cdot B d^3\vec{x}
\]

(\text{20})

The constant \(H\) become the magnetic helicity when \((C_C, C_M) = (0, \alpha)\). On the other hand, the parameter value \((C_C, C_M) = (\alpha, \alpha)\) yields the hybrid helicity.\(^{15}\)

Hereafter, we call the \(\vec{V}\)-variable, \(\hat{\Omega}\), the \textit{generalized vorticity} (the reason for so naming will be explained in Sec. \((\text{IV})\)). The generalized vorticity is related to the generalized momenta by \(\hat{\Omega} = D \hat{M}\), where the differential operator \(\hat{D}\) is defined by

\[
\hat{D} := \left( \begin{array}{cc}
C_C \alpha \nabla \times & -C_C \alpha \nabla \times \\
- C_C \alpha \nabla \times & C_M \alpha \nabla \times
\end{array} \right).
\]

Hereafter, we will call \(\hat{D}\) the \textit{generalized curl operator}. Thus, using the inertia operator \(\hat{M}\), we can relate the
generalized vorticities to the generalized velocities by the formula \( \Omega = \hat{W} \hat{V} \), where the integro-differential operator \( \hat{W} \) is defined by

\[
\hat{W} = \hat{D} \hat{M} = \left( \begin{array}{c} C_C \alpha \nabla \times \sigma \\ - C_C \alpha \nabla \times C_M (\alpha \nabla \times)^{-1} \end{array} \right).
\]

In Fig. 1 relations among the generalized velocities, vorticities, and the reference and perturbed paths are presented. It is important that the operator \( \hat{W} \) generates a field that satisfies the relabeling symmetry from the tangent vector of the reference path, though the particle relabeling symmetry itself is realized by an arbitrary time-dependent vector field that satisfies Eq. (18) if the reference path locally satisfies the Euler-Lagrange equation. Thus, we call the operator the helicity-based particle-relabeling operator.

Using the variables \( V \) and \( B \), the eigenvalue problem of the operator \( \hat{W} \) reads

\[
\alpha \nabla \times V + B = \frac{\Lambda}{C_C} V,
\]

where \( \Lambda \) is the eigenvalue. The eigenfunctions are known as the double Beltrami flows (DBF) and provide a series of force-free solutions of the two-fluid plasma model. Because the operator \( \hat{W} \) is constituted by the curl operator and its inverse, the eigenfunction is expressed in terms of the Beltrami functions, say \( \phi = \phi(\vec{k}, \sigma; \vec{x}) \), where \( k \), \( \sigma = \pm 1 \) are respectively the wavenumber and the helicity. Using the relation \( \nabla \times \phi = \sigma |\vec{k}| \phi \), we obtain the eigenvalue and corresponding eigenfunction (DBF) as

\[
\Lambda = \Lambda(\vec{k}, \sigma, s) = \frac{1}{2} \left[ C_C K + \frac{C_M}{K} + s \sqrt{(C_C K - \frac{C_M}{K})^2 + 4C_C^2} \right],
\]

where \( K = \alpha s |\vec{k}| \), \( s = \pm 1 \). Because the eigenvalues are real and non-degenerate with respect to \( s \), and the curl operator is Hermitian, the eigenfunctions are orthogonal with respect to the inner product Eq. (3).

Using the relation

\[
-\alpha \nabla \times (V_1 \times J_2 + J_1 \times V_2 - \alpha J_1 \times J_2) = \nabla \times \left[ (V_1 - \alpha J_1) \times (V_2 - \alpha J_1) \right] - \nabla \times (V_1 \times V_2)
\]

and integrating by parts, we obtain the inner product of the \( \hat{V} \)-variable and the Lie bracket as

\[
\langle \hat{V}_1 | [\hat{V}_2, \hat{V}_3] \rangle = \frac{1}{\alpha} \int_{\vec{x} \in M} d^3x \left\{ \left( \alpha \Omega_1 + B_1 \right) \cdot (V_2 \times V_3) - B_1 \cdot \left[ (V_2 - \alpha J_2) \times (V_3 - \alpha J_3) \right] \right\}.
\]

If \( (V_1 - \alpha J_1) \) is the eigenfunction of \( \hat{W} \) with the eigenvalue \( \Lambda \), substituting Eqs. (29) and (30) into the right hand side of Eq. (27), we obtain the formula

\[
\langle \hat{V}_1 | [\hat{V}_2, \hat{V}_3] \rangle = \Lambda T_{123}.
\]

where \( T_{123} \) is a 3rd-order totally antisymmetric tensor given by

\[
T_{123} = \frac{1}{\alpha} \int_{\vec{x} \in M} d^3x \left\{ \frac{1}{C_C} \frac{1}{V_1 \cdot (V_2 \times V_3)} + \frac{1}{C_M} \left( \frac{1}{C_C} \frac{1}{V_1 \cdot (V_2 \times V_3)} \right) \times (V_1 - \alpha J_1) \cdot \left[ (V_2 - \alpha J_2) \times (V_3 - \alpha J_3) \right] \right\}.
\]

The value of each component of the Riemannian metric tensor and the structure constant of the Lie group are defined by

\[
M_{\alpha \beta \gamma} = \langle \hat{\Phi}_\alpha | \hat{\Phi}_\beta \rangle,
\]

\[
[\hat{\Phi}_\alpha, \hat{\Phi}_\beta] = C^{\alpha \beta \gamma}_{\mu \nu \delta} \hat{\Phi}_\mu \hat{\Phi}_\nu \hat{\Phi}_\delta,
\]

where \( \alpha, \beta, \) and \( \gamma \) stand for the mode indices of the eigenfunctions and Einstein’s summation convention is being used for repeated Greek-letter indices. Substituting Eqs. (30) and (31) into Eq. (28), we obtain

\[
M_{\alpha \beta \gamma} C^{\alpha \beta \gamma}_{\mu \nu \delta} = \Lambda(\alpha) T_{\alpha \beta \gamma},
\]

where the symbols are defined by

\[
T_{\alpha \beta \gamma} = \left[ \frac{1}{C_C} \left( \Lambda(\alpha) - \frac{C_M}{K(\alpha)} \right) \left( \Lambda(\beta) - \frac{C_M}{K(\beta)} \right) \times \left( \Lambda(\gamma) - \frac{C_M}{K(\gamma)} \right) + \frac{1}{C_C} \frac{1}{C_M} \Lambda(\alpha) \delta(\beta) \Lambda(\gamma) \right] \times \int_{\vec{x} \in M} d^3x \left\{ \phi(\vec{k}_a, \sigma_a) \cdot \left( \phi(\vec{k}_b, \sigma_b) \times \phi(\vec{k}_c, \sigma_c) \right) \right\},
\]

\[
M_{\alpha \beta} = \left[ \frac{\Lambda(\alpha)}{C_C K(\alpha)} \right]^2 + C^2 C \delta_{\alpha \beta},
\]

\[
\Lambda(\alpha) = \Lambda(\vec{k}_a, \sigma_a, s_a),
\]

\[
\Lambda(\alpha) = \Lambda(\vec{k}_a, \sigma_{-a}, -s_a),
\]

\[
K(\alpha) = \alpha \sigma_a |\vec{k}_a|.
\]
Furthermore, introducing the symbol
\[ D^{ab} = \Lambda(a) \left[ \left( \Lambda(a) - \frac{C_M}{K(a)} \right)^2 + C_C^2 \right]^{-1} \delta^{ab}, \] (38)
which corresponds to the operator \( \tilde{D} \) (see Eq. (21)), the structure constants are expressed as
\[ C^i_{bc} = D^{ab} T_{bce}. \] (39)
In terms of these quantities, the Euler-Lagrange equations \( \Phi^{10} \) and \( \Phi^{11} \) are synthesized into a single expression as
\[ M_{i\alpha} \ddot{V}^\alpha + T_{i\alpha\beta} D^{\alpha\gamma} M_{\gamma\beta} \dot{V}^\alpha \dot{V}^\beta = 0, \] (40)
where the \( V^a \)s are the expansion coefficients: \( \bar{V} = V^a \tilde{\Phi}_a \), the derivation of which is summarized in Sect. \( \forall \). Operating on this equation with \( D^{ji} \), introducing the notation \( \Omega^i := D^{ij} M_{i\alpha} V^\alpha \), which corresponds to the relation \( D \bar{V} = \tilde{D} M \bar{V} \), and using Eq. (39), we obtain the expansion coefficient expression of Eq. (18) as
\[ \tilde{\Omega}^i + C^i_{\alpha\beta} \Omega^\alpha \dot{V}^\beta = 0, \] (41)
where \( \tilde{\Omega} = \Omega^a \tilde{\Phi}_a \). In the following sections, it is shown that the same mathematical structures are found in the MHD and HD systems.

III. MAGNETOHYDRODYNAMICS

The configuration space of the MHD system is given by a semidirect product of a volume-preserving diffeomorphism and the function space of the vector fields on \( M \), \( \text{SDiff}(M) \times \mathfrak{X}(M) \), say \( G \) in the present section. The generalized velocities of this formulation are given by the pair of the ion velocity, \( \bar{V} \), and the current fields, \( \bar{J} \), say \( \bar{V} = (V, J) \). The generalized coordinates that induce the generalized velocities are given by the pair of a function triplet and a vector field, \( (\tilde{X}(\tilde{a}, t), J(\tilde{x}, t)) \). The variables \( V \) and \( \tilde{x} \) are related by
\[ V^i(\tilde{X}(\tilde{a}, t), t) = \frac{\partial X^i}{\partial t} \bigg|_{(\tilde{x}, t)}. \]

Group operation on \( G \) is defined by the following formula:
\[ \left( \tilde{X}_1, J_1 \right) \circ \left( \tilde{X}_2, J_2 \right) = \left( \tilde{X}_1(\tilde{X}_2), J_1 + \text{Ad}_{\tilde{X}_1} J_2 \right), \]
where Ad denotes the adjoint representation of \( \text{SDiff}(M) \) on \( \mathfrak{X}(M) \), and this definition yields the following Lie bracket:
\[ [\bar{V}_1, \bar{V}_2] := \left( \nabla \times (V_1 \times V_2), \right. \]
\[ \left. \nabla \times (V_1 \times J_2 + J_1 \times V_2) \right). \] (42)

To derive the MHD equation, we define the inner product of two \( \tilde{V} \)-variables, i.e., the Riemannian metric on \( G \) as follows:
\[ \langle \bar{V}_1 | \bar{V}_2 \rangle := \int_M d^3\tilde{x} \left( V_1 \cdot V_2 - J_1 \cdot (\Delta^{-1} J_2) \right). \] (43)
The Lagrangian \( L = \frac{1}{2} \langle \bar{V} | \bar{V} \rangle \) yields the generalized momenta,
\[ \bar{M}_V := \frac{\partial L}{\partial \dot{V}} = V, \quad \bar{M}_J := \frac{\partial L}{\partial \dot{J}} = A, \] (44)
and the corresponding inertia operator, \( \tilde{M} \):
\[ \tilde{M} := \begin{pmatrix} I & 0 \\ 0 & (\alpha \nabla \times)^{-2} \end{pmatrix}. \] (45)
Let \( \bar{\xi} := (\xi, \eta) \), \( \bar{V} := (\tilde{V}, \bar{J}) \) be the perturbation direction of the integral path of action and the associated deviation of the generalized velocities, respectively. Lin’s constraints are given by
\[ \bar{V} = \partial_t \xi + \nabla \times (\xi \times V), \] (46)
\[ \bar{J} = \partial_t \eta + \nabla \times (\xi \times J + \eta \times V). \] (47)
Substituting these relations, we obtain the first variation of action, \( S := \int_0^1 L dt \), and its integration by parts as follows:
\[ \frac{\partial S}{\partial \bar{\xi}} \bigg|_{\bar{\xi}=0} = \int_0^1 dt \int_{\tilde{x} \in M} d^3\tilde{x} \left( V \cdot [\partial_t \xi + \nabla \times (\xi \times V)] \right. \]
\[ + A \cdot [\partial_t \eta + \nabla \times (\xi \times J + \eta \times V)] \bigg\}, \] (48)
\[ = \int_{\tilde{x} \in M} d^3\tilde{x} \left( V \cdot \xi + A \cdot \eta \right) \bigg|_{t=0}^{t=1} \]
\[ - \int_0^1 dt \int_{\tilde{x} \in M} d^3\tilde{x} \left[ \xi \cdot (\partial_t V + \Omega \times V + B \times J) \right. \]
\[ \left. - \alpha \eta \cdot (\partial_t A + B \times V) \right]. \] (49)
Hamilton’s principle \( (\xi = \eta = 0 \text{ at } t = 0, 1) \) leads to the Euler-Lagrange equation for the dynamics of an MHD medium as
\[ \partial_t V + \Omega \times V + B \times J = -\nabla P, \] (50)
\[ \partial_t A + B \times V = -\nabla \phi, \] (51)
where \( P, \phi \) are the generalized pressure and the scalar potential of an electro-magnetic field, respectively. Taking the curl of Eqs. (50) and (51), we obtain
\[ \partial_t \Omega + \nabla \times (\Omega \times V + B \times J) = 0, \] (52)
\[ \partial_t B + \nabla \times (B \times V) = 0. \] (53)
Rewriting Eq. (53) as the following pair of equations:
\[ \partial_t 0 + \nabla \times (0 \times V) = 0, \]
\[ \partial_t B + \nabla \times (0 \times J + B \times V) = 0. \] (54)
and comparing Eqs. [52], [53], and [54] with Eqs. [46] and [47], we find that the $\hat{V}$-variable,

$$\hat{\Omega} = \left( \begin{array}{c} \xi_\Omega \\ \eta_\Omega \end{array} \right) = C_C \left( \begin{array}{c} B \\ \Omega \end{array} \right) + C_M \left( \begin{array}{c} 0 \\ B \end{array} \right),$$  

(55)

(where $C_C$, and $C_M$ are arbitrary constants) satisfies the particle-relabeling condition ($\hat{V} = \hat{J} = 0$), and thus, yields a conservation law

$$\int_{\xi \in M} d^3\xi (\hat{V} \cdot \hat{\xi}_\Omega + A \cdot \eta_\Omega) \bigg|_{t=0}^{t=1} = 0.$$  

(56)

Using the ordinary variables, the constant of motion is found to be given by the linear combination of the cross and magnetic helicities:

$$H = 2C_C \int_{\xi \in M} V \cdot B d^3\xi + C_M \int_{\xi \in M} A \cdot B d^3\xi.$$  

(57)

The relations between the generalized vorticities, momenta, and velocities are given by $\hat{\Omega} = \hat{D} \hat{M} = \hat{W} \hat{V}$, where the differential operators $\hat{D}$ and $\hat{W}$ are defined by

$$\hat{D} := \left( \begin{array}{cc} 0 & \frac{C_c}{C_C} \frac{\nabla \times}{\nabla \times} \frac{C_m}{C_C} \frac{\nabla \times}{\nabla \times} \right),$$  

(58)

$$\hat{W} = \hat{D} \hat{M} = \left( \begin{array}{cc} \frac{C_c}{C_C} \frac{\nabla \times}{\nabla \times} \frac{C_m}{C_C} \frac{\nabla \times}{\nabla \times} \right).$$  

(59)

Using the variables $V$ and $B$ the eigenvalue problem of the operator $\hat{W}$ reads

$$C_C B = \Lambda V,$$  

(60)

$$C_C \nabla \times V + C_M B = \Lambda \nabla \times B.$$  

(61)

Due to Eq. (60), the eigenvalue problem reduces to a single Beltrami flow problem. Using the Beltrami function, $\phi$, we obtain the eigenvalue and corresponding eigenfunction as

$$\Lambda = \Lambda(\vec{\kappa}, \sigma, s) = \frac{1}{2} \left[ C_M \frac{K}{K} + s \sqrt{ \left( \frac{C_M}{K} \right)^2 + 4 C_C^2 } \right],$$  

(62)

$$\Phi = \Phi(\vec{\kappa}, \sigma, s) = \left( \frac{C_C}{\Lambda K} \phi \right),$$  

(63)

where $K = \sigma |\vec{\kappa}|$, $s = \pm 1$. Because the eigenvalues are real and non-degenerate with respect to $s$ and the curl operator is Hermitian, the eigenfunctions are orthogonal with respect to the inner product Eq. (13).

The inner product of the $\hat{V}$-variable and Lie bracket is given by

$$\langle \hat{V}_1 | [\hat{V}_2, \hat{V}_3] \rangle = \int_{\xi \in M} d^3\xi \left[ \Omega_1 \cdot (V_2 \times V_3) + B_1 \cdot (V_2 \times J_3 + J_2 \times V_3) \right].$$  

(64)

If $(\hat{V}_1, J_1)$ is the eigenfunction of $\hat{W}$ with the eigenvalue $\Lambda$, noticing that Eqs. (60) and (61) are rewritten as

$$B = \frac{\Lambda}{C_C} V, \quad \Omega = -\frac{C_M \Lambda}{C_C^2} V + \frac{\Lambda}{C_C} J,$$

and substituting them into the right hand side of Eq. (64), we obtain the formula

$$\langle \hat{V}_1 | [\hat{V}_2, \hat{V}_3] \rangle = \Lambda T_{123},$$  

(65)

where $T_{123}$ is a 3rd-order totally antisymmetric tensor given by

$$T_{123} = \int_{\xi \in M} d^3\xi \left\{ - \frac{C_M}{C_C} V_1 \cdot (V_2 \times V_3) + \frac{1}{C_C} [J_1 \cdot (V_2 \times V_3) + V_1 \cdot (J_2 \times V_3) + V_1 \cdot (V_2 \times J_3)] \right\}.$$  

(66)

Thus, substituting the eigenfunctions into Eq. (65), we obtain a relation among the Riemannian metric, the Lie bracket, and the eigenvalue of the helicity-based particle-relabeling operator that agrees with Eq. (32):

$$M_{a \beta} C_{bc}^\beta = \Lambda(a) T_{abc},$$

where the symbols are defined by

$$M_{a \beta} = \left( C_C^2 + \Lambda(a)^2 K(a)^2 \right) \delta_{ab},$$  

(67)

$$\Lambda(a) = \Lambda(\vec{\kappa}_a, \sigma_a, s_a),$$  

(68)

$$K(a) = \sigma_a |\vec{\kappa}_a|,$$  

(69)

$$T_{abc} = C_C (\Lambda(a) K(a) + \Lambda(b) K(b) + \Lambda(c) K(c) - C_M) \times \int_{\xi \in M} d^3\xi \left[ \phi(\vec{\kappa}_a, \sigma_a) \cdot (\phi(\vec{\kappa}_b, \sigma_b) \times \phi(\vec{\kappa}_c, \sigma_c)) \right].$$  

(70)

As was done for Eq. (21), introducing the symbol

$$D^{a \beta} = \Lambda(a) \left( C_C^2 + \Lambda(a)^2 K(a)^2 \right)^{-1} \delta^{a \beta},$$  

(71)

which corresponds to the operator $\hat{D}$ (see Eq. (68)), the structure constants are expressed in the same way as Eq. (39).

$$C_{bc}^a = D^{a \beta} T_{\beta bc}.$$  

Consequently, we recognize that the MHD system is formulated by the same equations as Eqs. (40) and (41).

**IV. HYDRODYNAMICS**

The generalized velocity is the velocity field, $\vec{V}$, itself. The generalized coordinates that induce the generalized velocity are given by a function triplet, $\vec{X}(\vec{a}, t)$, which is an element of a volume-preserving diffeomorphism on $M$, $SDiff(M)$. The variables $\vec{V}$ and $\vec{X}$ are related by

$$V^i(\vec{X}(\vec{a}, t), t) = \frac{\partial X^i}{\partial t} |_{(\vec{a}, t)}.$$  

Group operation is simply given by $\vec{X}_1 \circ \vec{X}_2 = \vec{X}_1(\vec{X}_2)$, and the Lie bracket associated with this definition reads

$$[\vec{V}_1, \vec{V}_2] := \nabla \times (\vec{V}_1 \times \vec{V}_2).$$  

(72)
To derive the HD equation, we define the inner product, i.e., the Riemannian metric on \( G \) as follows:

\[
\langle V_1 | V_2 \rangle := \int_{x \in M} V_1 \cdot V_2 d^3 \tilde{x}.
\]  

The Lagrangian \( L = \frac{1}{2} \langle V | V \rangle \) yields the generalized momenta,

\[
\mathcal{M}_V := \frac{\partial L}{\partial \dot{V}} = V.
\]  

The corresponding inertia operator, \( \hat{M} \), is the identity map:

\[
\hat{M} = I.
\]  

Let \( \xi \) and \( \tilde{V} \) be the perturbation direction of the integral path of action and the associated deviation of the generalized velocities, respectively. Lin’s constraints are given by

\[
\tilde{V} = \partial_t \xi + \nabla \times (\xi \times V),
\]  

Substituting this relation, we obtain the first variation of action, \( S := \int_0^1 L dt \), and its integration by parts as follows:

\[
\frac{\partial S}{\partial c} \bigg|_{c=0} = \int_0^1 dt \int_{x \in M} V \cdot \left[ \partial_t \xi + \nabla \times (\xi \times V) \right] d^3 \tilde{x},
\]  

Hamilton’s principle \( (\xi = 0 \text{ at } t = 0, 1) \) leads to the Euler-Lagrange equation for the dynamics of an HD medium as

\[
\partial_t V + \Omega \times V = -\nabla P,
\]  

where \( P \) is the pressure. Taking the curl of Eq. \((70)\), we obtain the vorticity equation:

\[
\partial_t \Omega + \nabla \times (\Omega \times V) = 0.
\]  

The equation obviously satisfies the particle-relabeling condition \( (\tilde{V} = 0 \text{ for Eq. } (76)) \), and thus, yields the helicity conservation law

\[
\int_{x \in M} V \cdot \Omega d^3 \tilde{x} \bigg|_{t=0}^{t=1} = 0.
\]  

The relations between the generalized vorticities, momenta, and velocities are given by \( \Omega = \hat{D} \mathcal{M} = \hat{W} \tilde{V} \), where the differential operators \( \hat{D}, \hat{W} \) are defined by

\[
\hat{D} := \nabla \times, \quad \hat{W} := \hat{D} \hat{M} = \nabla \times.
\]  

Thus, the helicity-based particle relabeling operator \( \hat{W} \) for an HD system is the curl operator. In summary, the consideration of the hydrodynamic system reveals that the theoretical counterpart of the variable \( \Omega \) is the vorticity, and this is the reason for which we named it “generalized vorticity”.

The eigenfunction is given by the Beltrami function, \( \phi \), and its eigenvalue is \( \Lambda = \sigma |\tilde{k}| \). Because the curl operator is Hermitian, the eigenfunctions are orthogonal with respect to the inner product, Eq. \((73)\).

The inner product of \( V \) and a Lie bracket is given by

\[
\langle V_1 | [V_2, V_3] \rangle = \int_{x \in M} \Omega_1 \cdot (V_2 \times V_3) d^3 \tilde{x}.
\]  

Substituting the eigenequation \( \Omega_1 = \Lambda V_1 \) into the right hand side of Eq. \((84)\), we obtain the formula

\[
\langle V_1 | [V_2, V_3] \rangle = \Lambda T_{123},
\]  

Thus, substituting the Beltrami functions into Eq. \((85)\), we again obtain the relation among the Riemannian metric, the Lie bracket, and the eigenvalue of the helicity-based particle-relabeling operator Eq. \((82)\):

\[
M_{a\beta}C^\alpha_{bc} = \Lambda(a)T_{abc},
\]

where the symbols are defined by

\[
M_{ab} = \delta_{ab}, \quad \Lambda(a) = \sigma_a |\tilde{k}_a|, \quad T_{abc} = \int_{x \in M} \left[ \phi(\tilde{k}_a, \sigma_a) \times (\phi(\tilde{k}_b, \sigma_b) \times \phi(\tilde{k}_c, \sigma_c)) \right] d^3 \tilde{x}.
\]  

As was done for Eq. \((21)\), introducing the symbol

\[
D^{ab} = \sigma_a |\tilde{k}_a| \delta^{ab},
\]

which corresponds to the operator \( \hat{D} = \nabla \times \), the structure constants are expressed in the same way as Eq. \((88)\),

\[
C^a_{bc} = D^{ab}T_{bca}.
\]

Consequently, we recognize that the HD system is formulated by the same equations as Eqs. \((40)\) and \((41)\).

V. DISCUSSION

A. Common mathematical structure specific to the incompressible HMHD, MHD, and HD systems

It is well-known that a dynamical system can be defined on a Lie group equipped with a Riemannian metric. The basic features of each dynamical system on a Lie group are determined by the Riemannian metric tensor, \( M_{ab} \), and the structure constants of the Lie group, \( C^a_{bc} \). In the present study, we reviewed the fact that the ideal, incompressible HMHD, MHD, and HD systems
have this structure, where the coefficients are obtained using an appropriate function set, \( M_{ab} = \langle \Phi_t, \Phi_b \rangle \), \([\Phi_b, \Phi_c]\) = \( C^a_{bc} \Phi_a \). In terms of the function expansion, the first variation of the action is summarized as follows:

\[
\frac{\partial S}{\partial \xi^\no} = \int_0^1 dt \langle \dot{\Phi}^\no \dot{\Phi} \rangle + \int_0^1 dt M_{ab} V^a \left[ \dot{\xi}^b + C^\gamma_{b\delta} \dot{\Phi}^\gamma \right]
\]

\[
= \int_0^1 dt M_{ab} V^a \left[ \dot{\xi}^b + C^\gamma_{b\delta} \dot{\Phi}^\gamma \right] = \left( M_{ab} \right)_{t=0} \dot{\Phi}^\no \dot{\Phi} + \int_0^1 dt \dot{\xi}^b \left[ M_{ab} V^a - M_{\alpha\gamma} C^\gamma_{b\delta} V^\delta \right].
\]

Thus, we obtain the spectral representation of the Euler-Lagrange equation: \( M_{\alpha\beta} V^\alpha = C^\gamma_{\beta\delta} V^\gamma \). This equation is well-known as the Euler-Poincare equation, \( L \) is well-known as the Euler-Poincare equation, and \( -L \) is the associated generalized vorticity, \( \Omega \). The helicity-based particle-relabeling operator, \( \tilde{W} \), is defined as an associated linear integro-differential operator based on the generalized vorticity and inertia operators. \( \tilde{W} = \tilde{D} \tilde{M} \). This feature yields the decomposition of the structure constants into the product of the generalized curl operator and the antisymmetric tensor (Eq. (69)), \( C^a_{bc} = \tilde{T}_{abc} \tilde{D} \tilde{c} \); thus, the Euler-Lagrange equation acquires the form given in Eq. (69).

The last relation leads to the commutation relation between \( D^{ab} \) and \( C^a_{bc} \). Their product satisfies the following relations:

\[
D^{ab} C^d_{bc} = D^{ab} \tilde{T}_{bc} \tilde{D} \tilde{d} = C^a_{cd} \tilde{D} \tilde{d}.
\]

This relation is the key to deriving the spectral representation of the evolution equation of the particle-relabeling field (see the derivation of Eq. (11)). It should be remarked that, for the HD case, this relation corresponds to the commutability of the exterior differentiation and the Lie derivatives for differential forms \( \langle dL \rangle = L_u d \), which is a key identity for the proof of Lagrangian invariants. For the HMHD and MHD cases, we found the triple product integrals Eqs. (29) and (66), which were analogous to Eq. (56). This led us to conjecture that there are counterparts of the Lie derivative and the exterior differentiation for the \( \tilde{V} \)-variable function spaces. Note that the divergence-free condition of the vector fields seems crucial for the derivation process of these triple products, because we fully used the fact that the commutator is given by the curl of the vector product of the vector fields \( (\tilde{B} \cdot \nabla) \tilde{A} - (\tilde{A} \cdot \nabla) \tilde{B} = \nabla \times (\tilde{A} \times \tilde{B}) \), which holds when \( \nabla \cdot \tilde{A} = \nabla \cdot \tilde{B} = 0 \).

B. Remark upon the particle-relabeling field in \( n \)-dimensional space

In the derivation process of the helicity-based particle-relabeling operator, three-dimensionality seems crucial for the linearity of the operator.

Noticing the general helicity conservation laws of ideal fluids in \( n \)-dimensional space, which were proved by Khesin and Chekanov, gives us a clue to consider the particle relabeling procedure. The equation of motion is described in terms of differential 1-forms by \( (\partial_t + L_u) u = -dp \), where \( u \) is the Lie and exterior differentiation operators, respectively. As was shown by, for example, Tur and Yanovsky, the substantial derivative, \( \partial_t + L_u \), is commutative with exterior differentiation and becomes a derivation with respect to the wedge product, \( \wedge \). Using this property, we recognize that the \( k \)-times wedge product of the differential 2-form, \( du \), has the frozen-in feature for \( 2k < n \):

\[
(\partial_t + L_u) (du \wedge \cdots \wedge du) = 0.
\]

Thus, for the odd-dimensional \( n = 2k + 1 \) case, the integral of the differential \( n \)-form,

\[
H = \int u \wedge du \wedge \cdots \wedge du,
\]
becomes a constant of motion.

Now we introduce the correspondence between a vector field, say $\xi$, and a differential $(n-1)$-form, say $\omega^{n-1}_\xi$, via the following integral of a differential $n$-form:

$$\int w(\xi)dx^1 \wedge \cdots \wedge dx^n = \int w \wedge \omega^{n-1}_\xi,$$

where $w$ is an arbitrary differential 1-form. By the bracket, $[\cdot]$, we tentatively denote the correspondence: $[\cdot] : \Omega^{n-1} \rightarrow \mathfrak{X}$, $\xi := [\omega^{n-1}_\xi]$. If the velocity field, $u$, and the differential form are divergence-free ($du^{n-1} = 0$), the Lie derivative and $[\cdot]$ are commutative: $[L_u \omega^{n-1}_\xi] = L_u [\omega^{n-1}_\xi]$. Thus, the vector field $\xi_\Omega = [du \wedge \cdots \wedge du]$ satisfies the evolution equation $(\partial_t + L_u)\xi_\Omega = 0$, i.e., $\xi_\Omega$ provides a particle-relabeling field.

The mapping from $u$ to $\xi_\Omega$ is, in general, nonlinear. Only when there are three spatial dimensions ($n = 3$) does the mapping become linear: $\xi_\Omega = du$. This is conjectured that a similar restriction on the space dimensionality occurs for the dynamics of the HMHD and MHD plasmas, i.e., that the helicity-based particle-relabeling operator is simply given by an integro-differential operator only when $n = 3$. This feature is partly supported by the fact that, in the ion and electron velocity-formulation of the HMHD system, the helicity conservation law is given by the integral

$$H = C_i \int M_i \wedge dM_i + C_e \int M_e \wedge dM_e,$$

where $M_i$ and $M_e$ are the generalized momenta, which are given as the dual of the ion and electron velocities.

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15As is obvious from these equations that, because the arguments of the coefficients $(\partial X_i / \partial t)$ and the base vector field $(\partial / \partial x^i)$ do not agree each other, the time derivative of the Lagrangian coordinate itself is not a proper mathematical object from differential topological viewpoint.

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