Induced bosonic current by magnetic flux in a higher dimensional compactified cosmic string spacetime

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Abstract

In this paper, we analyse the bosonic current densities induced by a magnetic flux running along an idealized cosmic string in a high-dimensional spacetime, admitting that the coordinate along the string’s axis is compactified. Additionally we admit the presence of a magnetic flux enclosed by the compactification axis. In order to develop this analysis we calculate the complete set of normalized bosonic wave-functions obeying a quasiperiodicity condition, with arbitrary phase $\beta$, along the compactified dimension. In this context, only azimuthal and axial currents densities take place. As to the azimuthal current, two contributions appear. The first contribution corresponds to the standard azimuthal current in a cosmic string spacetime without compactification, while the second contribution is a new one, induced by the compactification itself. The latter is an even function of the magnetic flux enclosed by the string axis and is an odd function of the magnetic flux along its core with period equal to quantum flux, $\Phi_0 = 2\pi/e$. On the other hand, the nonzero axial current density is an even function of the magnetic flux along the core of the string and an odd function of the magnetic flux enclosed by it. We also find that the axial current density vanishes for untwisted and twisted bosonic fields in the absence of the magnetic flux enclosed by the string axis. Some asymptotic expressions for the current density are provided for specific limiting cases of the physical parameter of the model.

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1 Introduction

Cosmic strings are linear gravitational stable topological defects which may have been created as a consequence of phase transitions in the early universe and are predicted in the context of the standard gauge field theory of elementary particle physics with extra symmetries \[1\] \[2\] \[3\]. Observations of anisotropies in the Cosmic Microwave Background Radiation (CMB) by COBE, WMAP and more recently by the Planck Satellite have ruled out cosmic strings as the

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primary source for primordial density perturbations since they can’t explain the acoustic peaks in the angular power spectrum. However, a couple of other effects due to cosmic strings can be investigated through the observations of CMB, such as non-gaussianity signals and temperature anisotropies caused by the gravitational field of moving strings (see [4] for a recent analysis of the possible effects of strings on CMB). Furthermore, the formation of strings can also have astrophysical and cosmological consequences. For instance, emission of gravitational waves and high energy cosmic rays by strings such as neutrinos and gamma-rays, along with observational data, can help to constrain the product of the Newton’s constant, \( G \), and the linear mass density of the string \( \mu_0 \) [2]. All these effects can also be generated by cosmic strings formed in the context of brane inflation [1], which makes the physics of cosmic strings a wide area of worth investigation. In this paper, we will only focus on the effects generated by the spacetime of a infinitely straight static cosmic string as explained below.

The geometry of the spacetime associated with a cosmic string is locally flat, but topologically conical, having a planar angle deficit given by \( \Delta \phi = 8\pi G \mu_0 \) on the two-surface orthogonal to the string. Although this object was first introduced in the literature as being created by a Dirac-delta type distribution of energy and axial stress along a straight infinity and line, it can also be described by classical field theory where the energy-momentum tensor associated with the Maxwell-Higgs system, investigated by Nielsen and Olesen in [7], couples to the Einstein’s equations. This coupled system was further investigated by Garfinkle and Linet in [8] and [9], respectively. These authors have shown that a planar angle deficit, \( \Delta \phi \), arises on the two-surface perpendicular to a string, as well as a magnetic flux running through its core.

Although the geometry of the spacetime produced by an idealized cosmic string is locally flat, its conical structure alters the vacuum fluctuations associated with quantum fields. As a consequence, the vacuum expectation value (VEV) of physical observables like the energy-momentum tensor, \( \langle T_{\mu\nu} \rangle \), gets a nonzero value. The calculation of the VEV of physical observables associated with the scalar and fermionic fields in the cosmic string spacetime has been developed in [10, 11, 12, 13, 14] and [15, 16, 17, 18], respectively. Furthermore, the presence of a magnetic flux running through the core of the string gives additional contributions to the VEVs associated with charged fields [19, 20, 21, 22, 23, 24, 25] as well as induces vacuum current densities, \( \langle j^\mu \rangle \). This phenomenon has been investigated for massless and massive scalar fields in [26] and [27], respectively. In these papers, the authors have shown that induced vacuum current densities along the azimuthal direction arise if the ratio of the magnetic flux by the quantum one has a nonzero fractional part. The calculation of induced fermionic currents in higher-dimensional cosmic string spacetime in the presence of a magnetic flux has been developed in [28].

The induced fermionic current by a magnetic flux in a \((2 + 1)\)-dimensional conical spacetime and in the presence of a circular boundary has also been analyzed in [29].

The presence of compact dimensions also induces topological quantum effects on matter fields. It is well known that the presence of compact dimensions is an important feature in many high-energy theories of fundamental physics, like supergravity and superstring theories. An interesting application of field theoretical models that present compact dimensions can be found in nanophysics. The long-wavelength description of the electronic states in graphene can be formulated in terms of the Dirac-like theory in three-dimensional spacetime, with the Fermi velocity playing the role of the speed of light (see, e.g., [30]). For instance, a single-walled carbon nanotube is generated by rolling up a graphene sheet to form a cylinder. In this case, the background spacetime for the corresponding Dirac-like theory has topology \( R^2 \times S^1 \). The combined effects of the non-trivial topology of the cosmic string spacetime, the compactified dimension along the axis of the string and the presence of a magnetic flux running through its core and enclosed by the compactified axis, on the VEVs of the energy-momentum tensor, \( \langle T_{\mu\nu} \rangle \), and current densities, \( \langle j^\mu \rangle \), associated with charged quantum fermionic fields in a four-dimensional

\footnote{For a review of the observational consequences generated by cosmic strings see [31, 32].}
cosmic string spacetime have recently been investigated in [31] and [32], respectively. Here, in
the present paper, we shall continue along the same line of investigation. We shall calculate
the induced bosonic current in a higher-dimensional cosmic string spacetime, under the same
conditions as considered in these two previous publications.

This paper is organized as follows. The section 2 is devoted to the evaluation of the positive
frequency Wightman function for a massive charged scalar quantum field in a higher-dimensional
space-time. We also consider that the z-axis along the string is compactified to
a circle, by imposing a quasiperiodic boundary condition on the bosonic field with arbitrary
phase. Moreover we assume the presence of magnetic fluxes running through the string’s core and
enclosed by its axis. In section 3, by using the Wightman function, we evaluate the renormalized
vacuum current density induced by the magnetic fluxes and the compactification. As we shall
see, the renormalized charge density and the radial current vanish. For the azimuthal current
density, the compactification induces it to decompose into two parts: one of them coincides with
the corresponding expression in the geometry of a cosmic string without compactification and
the other is the contribution due to the compactification itself. Moreover, as a consequence of
the compactification, a non-vanishing axial current also arises and is a purely topological one.
The most relevant conclusions of the paper are summarized in section 4. We have also dedicated
an Appendix to provide some important expressions used in the development of our calculation
for the induced current densities. Throughout the paper we use natural units $G = \hbar = c = 1.$

2 Wightman function

In this paper we consider a $(D + 1)$-dimensional cosmic string spacetime with $D \geq 3$. By using
the generalized cylindrical coordinates $(x^1, x^2, \ldots, x^D) = (r, \phi, z, x^4, \ldots, x^D)$ with the string on
the $(D - 2)$-dimensional hypersurface $r = 0$, the corresponding geometry is described by the
line element

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = dt^2 - dr^2 - r^2d\phi^2 - dz^2 - \sum_{i=4}^{D}(dx^i)^2.$$  \hspace{1cm} (2.1)

The coordinates take values in the following intervals: $r \geq 0$, $0 \leq \phi \leq 2\pi/q$ and $-\infty < (t, x^i) < +\infty$ for $i = 4, \ldots, D$. The parameter $q \geq 1$ codifies the presence of the cosmic string.
Moreover, we assume that the direction along the z-axis is compactified to a circle with the
length $L$, so $0 \leq z \leq L$. The standard cosmic string space-time is characterized by $D = 3$, with
$z \in (-\infty, \infty)$. In this case $q^{-1} = 1 - 4\mu_0$, being $\mu_0$ the linear mass density of the string. \hspace{1cm} (2.1)

In this paper we are interested in calculating the induced vacuum current density, $\langle j_\mu \rangle$, associated with a charged scalar quantum field, $\varphi(x)$, in the presence of magnetic fluxes running
along the core of the string and enclosed by it, considering that the z-axis is compactified to
a circle. In order to do that we shall calculate the corresponding complete set of normalized
bosonic wavefunction.

The equation which governs the quantum dynamics of a charged bosonic field with mass $m$, in a curved background and in the presence of an electromagnetic potential vector, $A_\mu$, reads

$$\left(\mathcal{D}^2 + m^2 + \xi R\right) \varphi(x) = 0,$$  \hspace{1cm} (2.2)

where the differential operator above is defined by

$$\mathcal{D}^2 = \frac{1}{\sqrt{|g|}}D_\mu \left(\sqrt{|g|} g^{\mu\nu}D_\nu\right),$$  \hspace{1cm} (2.3)

\footnote{It is interesting to note that the effective metric produced in superfluid $^3$He – A by a radial disgyration is described by $D = 3$ line element \hspace{1cm} (2.1) with a negative planar angle deficit \hspace{1cm} [33].
being $D_\mu = \partial_\mu + ieA_\mu$ and $g = \det(g_{\mu\nu})$. In addition, we have considered the presence of a non-minimal coupling, $\xi$, between the field and the geometry represented by the Ricci scalar, $R$. However, for a thin and infinitely straight cosmic string, $R = 0$ for $r \neq 0$.

In the analysis that we want to develop, it will be assumed that the direction along the $z$-axis is compactified to a circle with length $L$: $0 \leq z \leq L$. The compactification is achieved by imposing the quasiperiodicity condition on the matter field,

$$\varphi(t, r, \phi, z + L, x^4, \ldots, x^D) = e^{2\pi i \beta}(t, r, \phi, z, x^4, \ldots, x^D),$$

with a constant phase $\beta$, $0 \leq \beta \leq 1$. The special cases $\beta = 0$ and $\beta = 1/2$ correspond to the untwisted and twisted fields, respectively, along the $z$-direction. In addition, we shall consider the existence of the following constant vector potential

$$A_\mu = (0, 0, A_\phi, A_z),$$

(2.5)

with $A_\phi = -q\Phi_\phi/(2\pi)$ and $A_z = -\Phi_z/L$, being $\Phi_\phi$ and $\Phi_z$ the corresponding magnetic fluxes.

In quantum field theory the condition (2.4) changes the spectrum of the vacuum fluctuations compared with the case of uncompactified dimension and, as a consequence, the induced vacuum current density changes.

In the spacetime defined by (2.1) and in the presence of the vector potential given above, the equation (2.2) becomes

$$\left[\partial_\tau^2 - \partial_r^2 - \frac{1}{r}\partial_r - \frac{1}{r^2}(\partial_\phi + ieA_\phi)^2 - (\partial_z + ieA_z)^2 - \sum_{i=4}^{D}\partial_i^2 + m^2\right]\varphi(x) = 0 .$$

(2.6)

The positive energy solution of this equation can be obtained by considering the general expression,

$$\varphi(x) = CR(r)e^{-i\omega t + iqn\phi + ik_z z + i\vec{k}_{\parallel} \cdot \vec{r}_{\parallel}},$$

(2.7)

where $\vec{r}_{\parallel}$ represents the coordinates of the extra dimensions, $\vec{k}$ the momentum along these directions and $C$ is a normalization constant. Substituting (2.7) into (2.6) we find that the radial function $R(r)$ must obey the differential equation

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + \lambda^2 - \frac{\nu^2}{r^2}\right)R(r) = 0 ,$$

(2.8)

with

$$\lambda = \sqrt{\omega^2 - \vec{k}_z^2 - k_z^2 - m^2} ,$$

$$\nu = qn + eA_\phi ,$$

$$\vec{k}_z = k_z + eA_z .$$

(2.9)

In the present analysis we shall assume that the wave functions obey the Dirichlet boundary condition on the string’s core. The regular solution at origin is $R(r) = J_\nu(\lambda r)$, where $J_\nu(z)$ represents the Bessel function of order

$$\nu = \nu_n = q|n + \alpha|$$

with

$$\alpha = \frac{eA_\phi}{q} = -\frac{\Phi_\phi}{\Phi_0} ,$$

(2.10)

$\Phi_0 = 2\pi/e$ being the quantum flux. Then, the general solution takes the form

$$\varphi_\sigma(x) = CJ_{q|n+\alpha|}(\lambda r)e^{-i\omega t + iqn\phi + ik_z z + i\vec{k}_{\parallel} \cdot \vec{r}_{\parallel}} .$$

(2.11)
The quasiperiodicity condition (2.3) provides a discretization of the quantum number \( k_z \) as shown below:

\[
k_z = k_l = \frac{2\pi}{L} (l + \beta) \quad \text{with} \quad l = 0, \pm 1, \pm 2, \ldots .
\]  

(2.12)

Under this circumstance the energy takes the form

\[
\omega = \omega_l = \sqrt{m^2 + \lambda^2 + \tilde{k}_l^2 + \tilde{\beta}^2} ,
\]

where

\[
\begin{align*}
\tilde{k}_z &= \tilde{k}_l = \frac{2\pi}{L} (l + \beta) , \\
\tilde{\beta} &= \beta + \frac{eA_z L}{2\pi} = \beta - \frac{\Phi_z}{\Phi_0} .
\end{align*}
\]

(2.14)

The constant \( C \) can be obtained by the normalization condition

\[
- i \int d^D x \sqrt{|g|} [\varphi^*_{\sigma'}(x) \partial_t \varphi_{\sigma}(x) - \varphi_{\sigma}(x) \partial_t \varphi^*_{\sigma'}(x)] = \delta_{\sigma,\sigma'} ,
\]

where \( \delta \) symbol on the right-hand side is understood as Dirac delta function for the continuous quantum number, \( \lambda \) and \( \tilde{k} \), and Kronecker delta for the discrete ones, \( n \) and \( k_l \). From (2.15) one finds

\[
|C|^2 = \frac{q\lambda}{2(2\pi)^{D-2}\omega_l L} .
\]

(2.16)

So, the renormalized bosonic wave-function reads,

\[
\varphi_{\sigma}(x) = \left[ \frac{q\lambda}{2(2\pi)^{D-2}\omega_l L} \right]^\frac{1}{2} J_{|n+\alpha|}(\mu r) e^{-i\omega t + i\mu \phi + ik_l z + i\tilde{\beta} \cdot \tilde{r}} .
\]

(2.17)

The properties of the vacuum state are described by the corresponding positive frequency Wightman function, \( W(x, x') = \langle 0 | \varphi(x) \varphi^*(x') | 0 \rangle \), where \( |0\rangle \) stands for the vacuum state. Having this function we can evaluate the induced bosonic current. For the evaluation of the Wightman function, we use the mode sum formula

\[
W(x, x') = \sum_{\sigma} \varphi_{\sigma}(x) \varphi^*_{\sigma}(x') ,
\]

(2.18)

where we are using the compact notation defined as

\[
\sum_{\sigma} = \sum_{n=-\infty}^{+\infty} \int d\tilde{k} \int_0^{\infty} d\lambda \sum_{l=-\infty}^{+\infty} .
\]

(2.19)

The set \( \{ \varphi_{\sigma}(x), \varphi^*_{\sigma}(x') \} \) represents a complete set of normalized mode functions satisfying the periodicity condition (2.3). In our case, the mode functions in Eq. (2.17) are specified by the set of quantum numbers \( \sigma = (n, \lambda, k_l, \tilde{k}) \), with the values in the ranges \( n = 0, \pm 1, \pm 2, \ldots , -\infty < k^j < +\infty \) with \( j = 4, \ldots D \), \( 0 < \lambda < \infty \) and \( k_l = 2\pi (l + \beta)/L \) with \( l = 0, \pm 1, \pm 2, \ldots \).

Substituting (2.17) into the sum (2.18) we obtain

\[
W(x, x') = \frac{q}{2L(2\pi)^{D-2}} \sum_{\sigma} e^{i\mu \phi} e^{i\tilde{k} \cdot \tilde{r}_l} \lambda J_{|n+\alpha|}(\mu r) J_{|n+\alpha|}(\mu r')
\]

\[
\times e^{-i\omega t + i\mu \phi + ik_l z + i\tilde{\beta} \cdot \tilde{r}_l} ,
\]

(2.20)

where \( \Delta \phi = \phi - \phi' \), \( \Delta \tilde{r}_l = \tilde{r}_l - \tilde{r}_l' \), \( \Delta \mu = \mu - \mu' \) and \( \Delta z = z - z' \).

Having the positive frequency Wightman function above, we are in position to calculate the induced vacuum bosonic current density, \( \langle j_\mu \rangle \). This calculation will be developed in the next section.
3 Bosonic current

The bosonic current density operator is given by,

\[ j_\mu(x) = ie [\phi^*(x)D_\mu \phi(x) - (D_\mu \phi)^* \phi(x)] \]

\[ = ie [\phi^*(x)\partial_\mu \phi(x) - \phi(x)(\partial_\mu \phi(x))^*] - 2e^2 A_\mu(x)|\phi(x)|^2 . \]  

(3.1)

Its vacuum expectation value (VEV) can be evaluated in terms of the positive frequency Wightman function as shown below:

\[ \langle j_\mu(x) \rangle = ie \lim_{x' \to x} \{(\partial_\mu - \partial'_\mu)W(x, x') + 2ieA_\mu W(x, x') \} . \]  

(3.2)

This VEV is a periodic function of the magnetic fluxes \( \Phi_\phi \) and \( \Phi_z \) with period equal to the quantum flux. This can be observed if we write the parameter \( \alpha \) in (2.10) in the form

\[ \alpha = n_0 + \alpha_0 \]  

with \( |\alpha_0| < \frac{1}{2} \),

(3.3)

where \( n_0 \) is an integer number. In this case the VEV of the current density will depend on \( \alpha_0 \) only.

3.1 Charge density and radial current

Let us start the calculation with the charge density. Because \( A_0 = 0 \), we have,

\[ \rho(x) = \langle j^0(x) \rangle = ie \lim_{t' \to t} (\partial_t - \partial'_{t'})W(x, x') . \]  

(3.4)

Substituting (2.20) into the above expression, taking the time derivative and finally the coincidence limit \( x' \to x \), the formal expression for the charge density is:

\[ \rho(x) = qe \frac{(4\pi)^{D-2}}{(2\pi)^{D-2}L} \sum_{n=0}^{\infty} \frac{\lambda}{2} J_{2q|n+\alpha|}^2 (\lambda r) . \]  

(3.5)

Because the above expression is divergent, in order to obtain a finite and well defined result, we have to regularize it by introducing a cutoff function \( e^{-\eta(\lambda^2 + k^2 + k^2)} \), with the cutoff parameter \( \eta > 0 \). At the end of the calculation we shall take the limit \( \eta \to 0 \).

So, using the cutoff function, the integral over the variable \( \lambda \) can be evaluated with the help of [31], and the regularized contribution gives

\[ \int_0^\infty d\lambda \lambda e^{-\eta \lambda^2} J_{2q|n+\alpha|}^2 (\lambda r) = \frac{1}{2\eta} e^{-\frac{2}{2\eta}} I_{|q|n+\alpha|} \left( \frac{r^2}{2\eta} \right) , \]  

(3.6)

with \( I_{\nu}(z) \) being the modified Bessel function. As to the integral over \( \vec{k} \) we have,

\[ \int \frac{d\vec{k}}{4\pi} e^{-\eta k^2} = \left( \frac{\pi}{\eta} \right)^{(D-3)/2} . \]  

(3.7)

Thus, the regularized charge density reads

\[ \rho_{\text{reg}}(x, \eta) = \frac{qe}{(4\pi)^{(D-1)/2}} \frac{e^{-\frac{r^2}{2\eta}}}{L \eta^{(D-1)/2}} \sum_{l=-\infty}^{\infty} e^{-\eta k^2} I(q, \alpha_0, r^2/(2\eta)) , \]  

(3.8)

where

\[ I(q, \alpha_0, w) = \sum_{n=-\infty}^{\infty} I_{q|n+\alpha|}(w) = \sum_{n=-\infty}^{\infty} I_{q|n+\alpha|}(w) . \]  

(3.9)
In Appendix A.1 it is shown that the above summation on the quantum number \( n \) is given by \((A.8)\) and \((A.7)\). Substituting this result into \((3.8)\) we obtain

\[
\rho_{\text{reg}}(x, \eta) = \frac{qe}{(2\pi)^{(D-1)/2} L} \mathcal{W}^{(D-1)}_{\eta/2} e^{-w} \sum_{l=-\infty}^{\infty} e^{-\frac{\pi}{2}l^2} \left[ \frac{e^w}{q} - \frac{1}{\pi} \int_0^{\infty} dy \frac{e^{-w \cosh(y) f(q, \alpha_0, y)}}{\cosh(qy) - \cos(q\pi)} \right],
\]

with \( w = r^2/(2\eta) \). In the above expression \( p = [q/2] \), where \([q/2]\) represents the integer part of \( q/2 \), and the prime on the sign of the summation means that in the case \( q = 2p \) the term \( k = q/2 \) should be taken with the coefficient 1/2.

The first term in the square brackets of \((3.10)\) corresponds to the charge density for \( \alpha_0 = 0 \) and \( q = 1 \). The renormalized value for the charge density is given by subtracting from \((3.10)\) the contribution corresponding to the Minkowski spacetime in the absence of magnetic flux. We can do this in manifest form by discarding the first term inside the bracket. The other contributions contain \( e^{-2w \cosh(y/2)} \) and \( e^{-2w \sin^2(\pi k/q)} \), inside the integral and summation respectively; hence in the limit \( \eta \to 0 \) (\( w \to \infty \)) these terms vanish for \( r > 0 \). Thereby, we conclude that the renormalized value for the charge density is zero, i.e., there is no induced charge density.

The VEV of the current densities along the extra dimensions, \( \langle j^i(x) \rangle \) for \( i = 4, \cdots, D \), are given by

\[
\langle j_i(x) \rangle = i e \lim_{x' \to x} (\partial_{x'} - \partial_{x''}) W(x, x'),
\]

which can be written as

\[
\langle j_i(x) \rangle = \frac{eq}{L(2\pi)^{D-2}} \lim_{x' \to x} \sum_{\sigma} e^{i\sigma \Delta \phi} e^{ik_0 \Delta r_0} k^2 \lambda J_{q(n+\alpha)}(\lambda r) J_{q(n+\alpha)}(\lambda r') \\
\times \frac{e^{-i\omega_1 \Delta t + ik_1 \Delta z}}{\omega_1}.
\]

The integral over the variable \( k^2 \) can be evaluated by using the Eq. \((2.13)\) and the identity

\[
\frac{1}{\sqrt{m^2 + \lambda^2 + k^2}} = \frac{2}{\sqrt{\pi}} \int_0^{\infty} ds \; e^{-s(m^2 + \lambda^2 + k^2)} s^2.
\]

Then, the integral in \( k^2 \) reads

\[
\int_{-\infty}^{\infty} dk^i \; k^i e^{-s^2(k^2)^2} e^{ik^i \Delta x^i} = \frac{i\sqrt{\pi} \Delta x^i}{2s^3} e^{-\frac{(\Delta x^i)^2}{4s^2}}.
\]

This expression goes to zero by taking the limit, \( \Delta x^i \to 0 \). Moreover, at the coincidence limit, \( r' \to r \), the integral over \( \lambda \) provides a result equivalent to \((3.6)\). In addition the integrals over the other components of momentum along the extra dimensions, \( k^r \) with \( r \neq i \), provide finite results similar to the one given by \((3.7)\). Using again the results \((A.8)\) and \((A.7)\) for the sum over the modified Bessel function and identifying the first term with the contribution of the Minkowski space in the absence of magnetic flux, we can renormalize this current density in a manifest form by discarding this term. By doing this, we can see that the terms inside the summation and integral contain factors \( e^{-r^2 \sin^2(\pi k/q)/s^2} \) and \( e^{-r^2 \cosh(y/2)/s^2} \), respectively. Consequently, the integrals over \( s \) of the remaining terms are finite. So, our final conclusion is that, because \((3.14)\) goes to zero at the coincidence limit and the renormalized values for the
other integrals are finite in that limit, there will be no induced vacuum current densities along the extra dimensions. In fact this result is in agreement with the invariance of the system under a boost along the $x^i$ direction.

Now let us analyze the radial current density. Because $A_r = 0$, the VEV of the $r$-component of the current is simply expressed as

$$\langle j_r(x) \rangle = ie \lim_{r' \to r} (\partial_r - \partial_r') W(x, x') . \quad (3.15)$$

Taking the radial derivatives with respect to $r$ and $r'$ in the Wightman function, subtracting both terms and taking the coincidence limit, there appears a cancellation between those terms. Thereby, we also conclude that there is no induced radial current density:

$$\langle j_r(x) \rangle = 0 . \quad (3.16)$$

### 3.2 Azimuthal current

The VEV of the azimuthal current density is given by

$$\langle j_\phi(x) \rangle = ie \lim_{\phi' \to \phi} \{ (\partial_\phi - \partial_\phi') W(x, x') + 2ieA_\phi W(x, x') \} . \quad (3.17)$$

Substituting (2.20) into the above equation we get the formal expression for the azimuthal bosonic current density below:

$$\langle j_\phi(x) \rangle = - \frac{qe}{L(2\pi)^{D-2}} \sum_{n=-\infty}^{\infty} q(n+\alpha) \int d\vec{k} \int_0^{\infty} d\lambda \lambda J_q |n+\alpha|^{2}(\lambda r)$$

$$\times \sum_{l=-\infty}^{\infty} \frac{1}{\sqrt{m^2 + \lambda^2 + \vec{k}^2 + \vec{k}'^2}} . \quad (3.18)$$

In order to develop the summation over the quantum number $l$ we shall apply the Abel-Plana summation formula in the form [35], which is given by

$$\sum_{l=-\infty}^{\infty} g(l + \tilde{\beta}) f(|l + \tilde{\beta}|) = \int_0^{\infty} du \left[ g(u) + g(-u) \right] f(u)$$

$$+ i \int_0^{\infty} du \left[ f(iu) - f(-iu) \right] \sum_{\lambda = \pm 1} \frac{g(i\lambda u)}{e^{2\pi(u+i\lambda\beta)} - 1} . \quad (3.19)$$

Taking $g(u) = 1$ and

$$f(u) = \frac{1}{\sqrt{(2\pi u/L)^2 + \lambda^2 + m^2 + \vec{k}^2}} . \quad (3.20)$$

By using this formula, it is possible to decompose the expression formal to $\langle j_\phi \rangle$, eq. (3.18), as the sum of the two contributions as show below:

$$\langle j_\phi \rangle = \langle j_\phi \rangle_{cs} + \langle j_\phi \rangle_c , \quad (3.21)$$

where the term, $\langle j_\phi \rangle_{cs}$, is the contribution of the first integral on the right hand side of (3.19) and corresponds to the azimuthal current density in the geometry of the higher dimensional cosmic string spacetime without compactification. The term, $\langle j_\phi \rangle_c$, is induced by the compactification of the string along its axis and is provided by the second integral on the right-hand side of (3.19). As we shall see the latter vanishes in the limit $L \to \infty$. 

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As we have already mentioned in the beginning of this paper, the calculation of the induced azimuthal bosonic current density by a magnetic flux in the geometry of an idealized cosmic string has been developed in [26] and [27] for massless and massive quantum fields, respectively. In [27] the calculation of the azimuthal vacuum current density was developed for a higher dimensional cosmic string spacetime considering the case where the parameter $1 \leq q < 2$. However, to our knowledge, a closed expression for the induced azimuthal current considering general values of $q$ is missed. In this sense, here we generalize the results in [27] considering general values of the parameter, $q$, as well as consider the case where there exists a magnetic flux running through the core of a string whose the axis is compactified to a circle. Thus, in the present paper, we want to investigate the induced bosonic current as general as possible, combining all the above effects.

From the first integral on the right hand side of (3.19), Eq. (3.18) gives

$$\langle j_\phi(x)\rangle_{cs} = -\frac{2eq}{(2\pi)^{D-1}} \sum_{n=-\infty}^{\infty} q(n+\alpha) \int d\vec{k} \int_{0}^{\infty} d\lambda \lambda J^2_{q|n+\alpha|}(\lambda r) \int_{0}^{\infty} dy \frac{dy}{\sqrt{y^2 + \lambda^2 + \vec{k}^2 + m^2}},$$

where we have introduced a new variable $y = 2\pi u/L$.

In order to provide a more workable expression, we use the identity (3.13). This allows us to integrate over $\lambda$ by using (3.14). Also the integral over the momentum on the extra dimensions is easily evaluated. Finally, writing $\alpha$ in the form given in (3.3) we obtain

$$\langle j_\phi(x)\rangle_{cs} = -\frac{eq}{(2\pi)^{(D+1)/2} r^{D-1}} \int_{0}^{\infty} dw \frac{w^{(D-3)/2}}{2} e^{-w - w^{2} / 2w} \sum_{n=-\infty}^{\infty} q(n+\alpha_0) I_{q|n+\alpha_0|}(w),$$

where we have defined $w = r^2 / 2s^2$.

In Appendix A.2 it is shown that the above summation in the quantum number $n$ is given by (A.19) and (A.18). Substituting this result into (3.23), we obtain

$$\langle j_\phi(x)\rangle_{cs} = \frac{4em^{D+1}}{(2\pi)^{(D+1)/2}} \left[ \sum_{k=1}^{p} \sin(2k\pi / q) \sin(2k\pi \alpha_0) F_{(D+1)/2} [2mr \sin(k\pi / q)] \right]$$

$$+ \frac{q}{\pi} \int_{0}^{\infty} dy \frac{g(q, \alpha_0, 2y) \sinh(2y)}{\cosh(2qy) - \cos(q\pi)} F_{(D+1)/2} [2mr \cosh(y)],$$

where we use the notation

$$F_\nu(x) = \frac{K_\nu(x)}{x^\nu},$$

being $K_\nu(x)$ the modified Bessel function. We can see that $\langle j_\phi(x)\rangle_{cs}$ vanishes for the case $\alpha_0 = 0$. For $1 \leq q < 2$ the first term on the right hand side of (3.24) is absent and the result coincides with the one found for the azimuthal induced bosonic current in [27]. From Eq. (3.24), we can see that $\langle j_\phi(x)\rangle_{cs}$ is an odd function of $\alpha_0$, with period equal to the quantum flux $\Phi_0$. In Fig.1 we plot the behavior of the azimuthal current density as a function of $\alpha_0$ for the case where $D = 3$, considering $mr = 0.5$ and different values of $q$. One can see that the effect of the cosmic string parameter, $q$, is to amplify the oscillatory nature of the azimuthal current with respect to the parameter $\alpha_0$.

For a massless field, from Eq. (3.24), we obtain the following expression

$$\langle j_\phi(x)\rangle_{cs} = \frac{4e \Gamma \left(\frac{D+1}{2}\right)}{\pi \left(\frac{D+1}{2} \right)^2 (2\pi)^{D+1}} \left[ \sum_{k=1}^{p} \cos(k\pi / q) \sin(2k\pi \alpha_0) \sin^D(k\pi / q) \right]$$

$$+ \frac{q}{\pi} \int_{0}^{\infty} dy \frac{\sinh(y)}{\cosh(2qy) - \cos(q\pi)} \frac{g(q, \alpha_0, 2y)}{\cosh^D(y)},$$

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Figure 1: The azimuthal current density without compactification for \( D = 3 \) is plotted, in units of \( "m^4e"\), in terms of \( \alpha_0 \) for \( mr = 0.5 \) and \( q = 1.5, 2.5 \) and \( 3.5 \).

Taking for the above expression \( D = 3 \), we have

\[
\langle j^\phi(x) \rangle_{cs} = \frac{e}{4\pi^2 r^4} \left[ \sum_{k=1}^{p} \frac{\cot(k\pi/q) \sin(2k\pi\alpha_0)}{\sin^2(k\pi/q)} \right. \\
\left. + \frac{q}{\pi} \int_0^\infty dy \frac{\tanh(y)}{\cosh(2qy) - \cos(q\pi)} \frac{g(q, \alpha_0, 2y)}{\cosh^2(y)} \right].
\] (3.27)

Considering \( D = 3 \) and \( q > 2 \), the behavior of the \( \langle j^\phi(x) \rangle_{cs} \) at large distance from string, \( mr \gg 1 \), is dominated by the first term of (3.24) with \( k = 1 \). It reads,

\[
\langle j^\phi(x) \rangle_{cs} \approx \frac{e m}{(2\pi)^2} \left( \frac{m}{\pi r \sin(\pi/q)} \right)^{\frac{1}{2}} \sin(2\pi\alpha_0) \cot(\pi/q) e^{-2mr \sin(\pi/q)}.
\] (3.28)

Now let us develop the calculation of the contribution to the azimuthal current induced by the compactification. So, we substitute the second term of (3.19) into (3.18). Doing this we obtain:

\[
\langle j^\phi(x) \rangle_c = -\frac{4eq}{(2\pi)^{D-1}} \sum_{n=-\infty}^{\infty} q(n + \alpha) \int d\vec{k} \int_0^\infty d\lambda \lambda j^2_{q|n+\alpha|}(\lambda r) \\
\times \int\frac{1}{\sqrt{\lambda^2 + k^2 + m^2}} \frac{dy}{\sqrt{y^2 - \lambda^2 - k^2 - m^2}} \sum_{\lambda = \pm 1} e^{Ly - 2\pi\gamma k^2 + \lambda - 1}.
\] (3.29)

To continue our analysis, we shall use the series expansion \((e^u - 1)^{-1} = \sum_{l=1}^{\infty} e^{-lu}\) in the above expression, and with the help of [34] we get,

\[
\langle j^\phi(x) \rangle_c = -\frac{4eq}{(2\pi)^{D-1}} \sum_{l=1}^{\infty} \cos(2\pi l \beta) \sum_{n=-\infty}^{\infty} q(n + \alpha) \int d\vec{k} \int_0^\infty d\lambda \lambda j^2_{q|n+\alpha|}(\lambda r) \times \\
\times K_0 \left( 1L \sqrt{\lambda^2 + k^2 + m^2} \right).
\] (3.30)

At this point, we shall use the integral representation below for the Macdonald function [34]

\[
K_\nu(x) = \frac{1}{2} \left( \frac{x}{2} \right)^\nu \int_0^{\infty} dt \frac{e^{-\frac{t-x^2}{4}}}{t^{\nu+1}}.
\] (3.31)
By using this representation, it is possible to integrate over the variable \( \lambda \) and over the momentum along the extra dimensions \( k \). So, we obtain

\[
\langle j_\phi(x) \rangle_c = -\frac{eq}{2} \sum_{l=1}^{\infty} \cos(2\pi l \beta) \int_0^\infty dw \left\{ F_{(D+1)} \left[ -\frac{w}{2} e^{-w \left[ 1 + \frac{1}{2} l^2 \right]} - \frac{m^2}{2w} \right] \right\} \times \sum_{n=-\infty}^{\infty} q(n + \alpha_0) I_{n+\alpha_0}(w),
\]

where we have used \( \vec{k} \) momentum along the extra dimensions.

The sum over the quantum number \( n \) in (3.32) can be developed by using the compact result (A.19). This allows us to perform the integrals over \( x \). Our final expression is:

\[
\langle j_\phi(x) \rangle_c = \frac{8em^{D+1}}{(2\pi)^{\frac{D+1}{2}}} \sum_{l=1}^{\infty} \cos(2\pi l \beta) \left\{ \sum_{k=1}^{P} \sin(2k\pi/q) \sin(2k\pi\alpha_0) \frac{F_{(D+1)}}{\sqrt{m^2 + \rho_k^2}} \right\} \times \sum_{n=-\infty}^{\infty} q(n + \alpha_0) I_{n+\alpha_0}(w),
\]

where we have defined

\[
\rho_k = \frac{2r \sin(\pi k/q)}{L}, \quad \eta(y) = \frac{2r \cosh(y)}{L}.
\]

From the above expression we can see that the contribution due to the compactification on the bosonic current density is an even function of the parameter \( \beta \) and is an odd function of the magnetic flux along the core of the string, with period equal to \( \Phi_0 \). In particular, in the case of an untwisted bosonic field, \( \langle j_\phi(x) \rangle_c \) is an even function of the magnetic flux enclosed by the string’s axis. Also we can see that for \( \alpha_0 = 0 \) the induced current above vanishes. Moreover, \( \langle j_\phi(x) \rangle_c = 0 \) for \( r = 0 \). This result is in contrast with the fermionic case where the azimuthal current induced by the compactification does not vanish at \( r = 0 \), as shown in [31]. In Fig.2 we plot the behavior of the compactified azimuthal current density as a function of \( \alpha_0 \) for \( D = 3 \), considering \( mr = 0.5 \), \( mL = 1 \), \( \beta = 0.1, 0.7 \) and different values of the parameter \( q \). On the graphs below we can see that the effect of the cosmic string parameter, \( q \), is to amplify the oscillatory nature of the azimuthal current with respect to the parameter \( \alpha_0 \) while the effect of the parameter, \( \beta \), is to change the direction of oscillation as well as diminish the absolute values of \( \langle j_\phi(x) \rangle_c \).

For large values of the length of the compact dimension, \( mL \gg 1 \), assuming that \( mr \) is fixed and considering \( D = 3 \), the main contribution comes from the \( l = 1 \) term and to leading order we find

\[
\langle j_\phi(x) \rangle_c \approx \frac{\sqrt{2em^3}}{\pi^2 L^2} \left\{ \sum_{k=1}^{P} \sin(2k\pi/q) \sin(2k\pi\alpha_0) \right\} \times \frac{q}{\pi} \int_0^\infty dy \frac{g(q, \alpha_0, 2y) \sinh(2y)}{\cosh(2qy) - \cos(q\pi)} \right\},
\]

where we can see that there appear an exponential decay. So in this limit, the contribution to the total current density is dominated by \( \langle j_\phi(x) \rangle_{cs} \).

For a massless field and also considering \( D = 3 \), we obtain

\[
\langle j_\phi(x) \rangle_c = \frac{4e}{\pi^2 L^2} \left\{ \sum_{k=1}^{P} \sin(2k\pi/q) \sin(2k\pi\alpha_0) G_c(\beta, \rho_k) \right\} \times \frac{q}{\pi} \int_0^\infty dy \frac{g(q, \alpha_0, 2y) \sinh(2y)}{\cosh(2qy) - \cos(q\pi)} G_c(\beta, \eta(y)) \right\},
\]

where we have used the compact result (A.19). This allows us to perform the integrals over \( x \). Our final expression is:

\[
\langle j_\phi(x) \rangle_c = \frac{8em^{D+1}}{(2\pi)^{\frac{D+1}{2}}} \sum_{l=1}^{\infty} \cos(2\pi l \beta) \left\{ \sum_{k=1}^{P} \sin(2k\pi/q) \sin(2k\pi\alpha_0) \frac{F_{(D+1)}}{\sqrt{m^2 + \rho_k^2}} \right\} \times \sum_{n=-\infty}^{\infty} q(n + \alpha_0) I_{n+\alpha_0}(w),
\]

where we have defined

\[
\rho_k = \frac{2r \sin(\pi k/q)}{L}, \quad \eta(y) = \frac{2r \cosh(y)}{L}.
\]
Figure 2: The azimuthal current density induced by the compactification for $D = 3$ is plotted, in units of $"m^4e"$, in terms of $a_0$ for $mr = 0.5$, $mL = 1$ and $q = 1.5, 2.5$ and $3.5$. The plot on the left is for $\tilde{\beta} = 0.1$ while the plot on the right is for $\tilde{\beta} = 0.7$.

where we have defined

$$G_c(\tilde{\beta}, x) = \sum_{l=1}^{\infty} \frac{\cos(2\pi l \tilde{\beta})}{(l^2 + x^2)^2}.$$  \hspace{1cm} (3.37)

The summation above can be developed with the help of [34]. So, after some elementary steps we obtain:

$$G_c(\tilde{\beta}, x) = -\frac{1}{2x^4} + \frac{\pi^2 \cosh(2\pi \tilde{\beta} x)}{4x^2 \sinh^2(\pi x)} + \frac{\pi \cosh[\pi(1-2\tilde{\beta})x] + 2\pi \tilde{\beta} x \sinh[\pi(1-2\tilde{\beta})x]}{4x^3 \sinh(\pi x)},$$  \hspace{1cm} (3.38)

for $0 \leq \tilde{\beta} \leq 1$.

With (3.38) we can also obtain the dominant behavior of $\langle j^\phi(x) \rangle_c$ in the region $r << L$. It reads,

$$\langle j^\phi(x) \rangle_c \approx -\frac{2e}{45L^4} [30\tilde{\beta}^2(1-\tilde{\beta})^2 - 1] \left[ \sum_{k=1}^{p} \sin(2k\pi/q) \sin(2k\pi a_0) + \frac{q}{\pi} \int_{0}^{\infty} dy g(q, a_0, 2y) \sinh(2y) \right].$$  \hspace{1cm} (3.39)

Also with (3.38) it is possible to obtain the dominant behavior of $\langle j^\phi(x) \rangle_c$ in the region $r >> L$. Considering $x >> 1$ and $\tilde{\beta} = 0$ or $\tilde{\beta} = 1$, $G_c(\tilde{\beta}, x) \approx \pi/(4x^3)$. On the other hand taking $x >> 1$ and for $0 < \tilde{\beta} < 1$, $G_c(\tilde{\beta}, x) \approx -1/(2x^4)$. So we conclude that the dominant behavior of $\langle j^\phi(x) \rangle_c$ depends on the value assumed for the parameter $\tilde{\beta}$.

Combining Eqs. (3.24) and (3.33) we can write the total azimuthal current as

$$\langle j^\phi(x) \rangle = \frac{8cm^{D+1}}{(2\pi)^{D+1}/2} \sum_{l=0}^{\infty} \cos(2\pi l \tilde{\beta}) \left( \sum_{k=1}^{p} \sin(2k\pi/q) \sin(2k\pi a_0) F_{(D+1)/2} \left[ mL \sqrt{t^2 + \rho_k^2} \right] \right) \left[ mL \sqrt{t^2 + \eta^2(y)} \right] + \frac{q}{\pi} \int_{0}^{\infty} dy g(q, a_0, 2y) \sinh(2y) \right].$$  \hspace{1cm} (3.40)
where the prime on the sum over $l$ means that the term with $l = 0$ should be taken with the weight $1/2$. In fig. 3 we plot the total azimuthal current density as a function of $mr$ for $D = 3$, considering $q = 2.5$, $\alpha_0 = 0.25$ and different values of $mL$. In the left plot we consider $\tilde{\beta} = 0$ while in the right one $\tilde{\beta} = 0.5$. For $A_z = 0$, one has $\tilde{\beta} = \beta$, and the cases $\beta = 0$ and $\beta = 1/2$ are related with the untwisted and twisted bosonic fields, respectively. One can see that the effect of the parameter $\tilde{\beta}$ is to alter the curves for the finite values of $mL$ compared with the solid curve for $mL \to \infty$.

### 3.3 Axial current

Here we shall analyze the bosonic current density along the string. As we shall see, due to the compactification of the direction along the string axis, there appear a non-vanishing current.

The VEV of the axial current is given by

$$
\langle j_z(x) \rangle = ie \lim_{z' \to z} \left\{ (\partial_z - \partial_{z'})W(x, x') + 2ieA_zW(x, x') \right\} .
$$

(3.41)

Using (2.20) and the fact that $A_z = -\Phi_z/L$, a formal expression for this current can be provided. It reads,

$$
\langle j_z(x) \rangle = -\frac{eq}{L(2\pi)^{D-2}} \sum_{n=-\infty}^\infty \int d\tilde{k} \int_0^\infty \lambda J_{q|n+\alpha|}(\lambda r) \sum_{l=-\infty}^\infty \frac{\tilde{k}_l}{\sqrt{m^2 + \lambda^2 + \tilde{k}_l + k}} ,
$$

(3.42)

where $\tilde{k}_l$ is given by (2.14).

To evaluate the summation over the quantum number $l$ we shall use the generalized Abel-Plana summation formula, Eq. (3.19). For this case we have $g(u) = 2\pi u/L$ and $f(u)$ is given by (3.20). Taking these expressions into consideration, we can see that the first term on the right hand side of Eq. (3.19) vanishes due to the fact that $g(u)$ is an odd function. Thus, it remains only a contribution due to the second term on the right hand side of Eq. (3.19). This contribution is a consequence of the compactification assumed for the direction along the cosmic string’s axis.
The axial current density induced by the compactification can be written as

\[
\langle j_z(x) \rangle_c = -\frac{2ieq}{(2\pi)^{D-1}} \sum_{n=-\infty}^{\infty} \int \frac{dk}{k} \int_0^\infty d\lambda \lambda J_{q|n+\alpha|}(\lambda r) \left\langle j \right\rangle_w y e^{-lLy} \frac{y e^{-lLy}}{\sqrt{\lambda^2+k^2+m^2}} \left( \frac{\lambda}{\lambda^2-k^2-m^2} \right)^j. \quad (3.43)
\]

Using again the series expansion, \((e^u - 1)^{-1} = \sum_{l=1}^\infty e^{-lu}\), for the summation in \(j\) present in Eq. (3.43), we find

\[
\langle j_z(x) \rangle_c = \frac{4eq}{(2\pi)^{D-1}} \sum_{l=1}^\infty \sin(2\pi l \beta) \int \frac{dk}{k} \int_0^\infty d\lambda \lambda J_{q|n+\alpha|}(\lambda r) \left\langle j \right\rangle_w y e^{-lLy} \frac{y e^{-lLy}}{\sqrt{\lambda^2+k^2+m^2}} \left( \frac{\lambda}{\lambda^2-k^2-m^2} \right)^j. \quad (3.44)
\]

The integral over \(y\) can be evaluated with help of [34]. The result is given in terms of the modified Bessel function of first order, \(K_1(z)\). On the other hand, using the integral representation \((9.31)\) for this function, and the fact that \(K_1(z) = K_{-1}(z)\) we can show that

\[
\int_0^\infty \frac{dy}{\sqrt{\lambda^2+k^2+m^2}} y e^{-lLy} = \frac{1}{IL} \int_0^\infty dt \ e^{-t^2 \frac{l^2(\lambda^2+k^2+m^2)}{4L^2}}. \quad (3.45)
\]

Substituting the right hand side of the above identity into (3.44), it is possible to develop the integral over the variable \(\lambda\) and also the integral over the extra momenta. Moreover, defining a new variable \(w = 2tr^2/L^2\), we obtain

\[
\langle j_z(x) \rangle_c = \frac{2qeL}{(2\pi)^{D+1}} \sum_{l=1}^\infty l \sin(2\pi l \beta) \int_0^\infty dw \ w^{(D-1)/2} e^{-w \left[ 1 + \frac{l^2}{2r^2} \right] - \frac{2r^2}{2w}} \left[ e^w q \right]^w - \frac{1}{\pi} \int_0^\infty dy \ f(q, \alpha_0, y) \ e^{-w \cosh y} + \frac{2}{q} \sum_{k=1}^{p} \cos(2k\pi \alpha_0) e^{w \cos(2k\pi/q)} \right], \quad (3.47)
\]

where we have used \(\alpha\) in the form \((8.4)\) and \(f(q, |\alpha_0|, y)\) is given by \((1.7)\). Integrating over the variable \(w\), the above expression can be written as

\[
\langle j_z(x) \rangle_c = \langle j_z(x) \rangle_c^{(0)} + \langle j_z(x) \rangle_c^{(q, \alpha_0)} \quad (3.48)
\]

The first term inside the bracket in (3.47), provides a contribution that does not depend on \(\alpha_0\) and \(q\). It is a pure topological term, a consequence of the compactification only. For this contribution one has

\[
\langle j^z(x) \rangle_c^{(0)} = -\frac{2em}{(2L)^{D+1}} \sum_{l=1}^{D+1} \sin(2\pi l \beta) \frac{1}{l^{(D+1)/2}} K_{(D+1)/2}(lL). \quad (3.49)
\]
Notice that this term is independent on the radial distance, r. We can say that the above equation corresponds to the current density in the \((D+1)\) Minkowski spacetime with the spatial topology \(R^{D-1} \times S\). The current density in Minkowski spacetime with the topology \(R^{p} \times (S^{1})^{q}\) is recently investigated in [36] (for the corresponding problem in de Sitter spacetime see [37]). Moreover, from (3.47) we can see that the axial current density vanishes for integer and half-integer values of \(\tilde{\beta}\).

For \(D = 3\), Eq. (3.49) reads

\[
\langle j^z(x) \rangle^{(0)}_c = -\frac{em^2}{\pi^2 L} \sum_{l=1}^{\infty} \frac{\sin(2\pi l \tilde{\beta})}{l} K_2(2mL) .
\]

This value is exactly half of the corresponding value for the fermionic case found in [32].

The second contribution to the axial current coming from the magnetic flux and planar angle deficit is:

\[
\langle j^z(x) \rangle^{(q,\alpha_0)}_c = -\frac{8em^{D+1} L}{(2\pi)^{D+1}} \sum_{l=1}^{\infty} l \sin(2\pi l \tilde{\beta}) \left\{ \sum_{k=1}^{p} \cos(2k\pi \alpha_0) F_{(D+1)} \left[ mL \sqrt{l^2 + \rho_k^2} \right] 
- \frac{q}{\pi} \int_0^{\infty} dy f(q, \alpha_0, 2y) \left[ mL \sqrt{l^2 + \eta^2(y)} \right] \right\} ,
\]

where \(\rho_k\) and \(\eta(y)\) are given by (3.34). We can see that Eq. (3.51) is an odd function of the parameter \(\tilde{\beta}\) and is an even function of \(\alpha_0\), with period equal to the quantum flux \(\Phi_0\). In particular, in the case of an untwisted bosonic field, Eq. (3.51) is an odd function of the magnetic flux enclosed by the string’s axis. Moreover, this contribution vanishes for \(q = 1\) and \(\alpha_0 = 0\). In fig. 4 we plot the behavior of \(\langle j^z(x) \rangle^{(q,\alpha_0)}_c\) as function of \(\tilde{\beta}\) for \(D = 3\), considering \(mr = 0.4, mL = 1\) and \(q = 1.5, 2.5, 3.5\). The left plot is for \(\alpha_0 = 0\) and the right plot is for \(\alpha_0 = 0.25\). By these plots we can see that the amplitude of the current increases with \(q\) and the effect of \(\alpha_0\) is to change the orientation of the current.

![Figure 4](image-url)

Figure 4: The axial current density for \(D = 3\) in Eq. (3.51) is plotted, in units of “\(m^3 e\)”, in terms of \(\tilde{\beta}\) for the values \(mr = 0.4, mL = 1\) and \(q = 1.5, 2.5, 3.5\). The left plot is for \(\alpha_0 = 0\) while the right plot is for \(\alpha_0 = 0.25\).
For \( r = 0 \) and \( D = 3 \), Eq. (3.51) becomes

\[
\langle j^z(x) \rangle_c^{(q,\alpha_0)} = -\frac{2e_m^2}{\pi^2 L} \sum_{l=1}^{D} K_2(lmL) \sin(2l\pi \tilde{\beta}) \left[ \sum_{k=1}^{p} \cos(2k\pi \alpha_0) \right] - \frac{q}{\pi} \int_{0}^{\infty} dy \frac{f(q, \alpha_0, 2y)}{\cosh(2qy) - \cos(q\pi)} - \frac{q}{\pi} \int_{0}^{\infty} dy \frac{f(q, \alpha_0, 2y)}{\cosh(2qy) - \cos(q\pi)}, \quad (3.52)
\]

where we can see that this contribution is finite.

Now, considering \( mL \gg 1 \) and \( mr \) fixed, the main contribution to (3.51) comes from the \( l = 1 \) term. So, for \( D = 3 \) we have

\[
\langle j^z(x) \rangle_c^{(q,\alpha_0)} \approx -\frac{8e_m^2 \sin(2\pi \tilde{\beta}) e^{-mL}}{(2\pi L)^2} \left\{ \sum_{k=1}^{p} \cos(2k\pi \alpha_0) \right\} - \frac{q}{\pi} \int_{0}^{\infty} dy \frac{f(q, \alpha_0, 2y)}{\cosh(2qy) - \cos(q\pi)}, \quad (3.53)
\]

where there appears an exponential decay.

For a massless field and \( D = 3 \), the equation (3.51) reads,

\[
\langle j^z(x) \rangle_c^{(q,\alpha_0)} = - \frac{4e}{\pi L^3} \left\{ \sum_{k=1}^{p} \cos(2k\pi \alpha_0) V_c(\tilde{\beta}, \rho_k) \right\} - \frac{q}{\pi} \int_{0}^{\infty} dy \frac{f(q, \alpha_0, 2y)}{\cosh(2qy) - \cos(q\pi)} V_c(\tilde{\beta}, \eta(y)), \quad (3.54)
\]

where we have defined

\[
V_c(\tilde{\beta}, x) = \sum_{l=1}^{\infty} \frac{l \sin(2\pi l \tilde{\beta})}{(l^2 + x^2)^2} \cdot (3.55)
\]

Once more with the help of [34], we were able to develop the summation (3.55). The result is:

\[
V_c(\tilde{\beta}, x) = -\frac{\pi^2}{4x} \frac{\sinh(2\pi \tilde{\beta} x) - 2\tilde{\beta} \sinh(\pi x) \cosh[\pi x (1 - 2\tilde{\beta})]}{\sinh^2(\pi x)}, \quad (3.56)
\]

for \( 0 \leq \tilde{\beta} \leq 1 \). Clearly we can see that \( V_c(\tilde{\beta}, x) \) vanishes for \( \tilde{\beta} = 0, 1/2, 1 \).

With (3.56) we can obtain the dominant behavior of \( \langle j^z(x) \rangle_c^{q \neq 1} \) for \( r << L \). It reads,

\[
\langle j^z(x) \rangle_c^{(q,\alpha_0)} \approx - \frac{4e}{L^3} \frac{\tilde{\beta}(1 - \tilde{\beta})(1 - 2\tilde{\beta})}{3} \left\{ \sum_{k=1}^{p} \cos(2k\pi \alpha_0) \right\} - \frac{q}{\pi} \int_{0}^{\infty} dy \frac{f(q, \alpha_0, 2y)}{\cosh(2qy) - \cos(q\pi)}, \quad (3.57)
\]

In the opposite limit, we can obtain the behavior of \( \langle j^z(x) \rangle_c^{q \neq 1} \) for \( r >> L \). Taking the limit \( x >> 1 \) in (3.56), we can verify that \( V_c(\tilde{\beta}, x) \approx \pi^2 \tilde{\beta}/(2x) e^{-2\pi \tilde{\beta} x} \) for \( 0 < \tilde{\beta} < 1/2 \), and \( V_c(\tilde{\beta}, x) \approx -\pi^2/(2x) e^{-2\pi(1-\tilde{\beta}) x} \) for \( 1/2 < \tilde{\beta} < 1 \). In both cases the axial current presents an exponential decay, however there appears a different signal.
4 Conclusion

In this paper we have investigated the bosonic current density in a higher dimensional compactified cosmic string spacetime induced by magnetic fluxes, one of them enclosed by the compactified direction and the other running along the string’s core. The calculations were performed by imposing the quasiperiodicity condition, with arbitrary phase \( \beta \), on the solution of the Klein-Gordon equation. The general solution was obtained by considering a constant vector potential in Eq. (2.5) and, after imposing the quasiperiodicity condition and calculating the normalization constant (2.16), was presented in its final form in Eq. (2.17).

The positive frequency Wightman function (2.20), which is necessary to calculate the VEV of the bosonic current density in Eq. (3.2), was constructed by the complete set of normalized wave function (2.17). In this context we were able to show that the renormalized charge and radial current densities vanish. Moreover, we have seen that the compactification induces the azimuthal current density to decompose into two parts. The first one corresponds to the expression in the geometry of a cosmic string without compactification and is presented in Eq. (3.24). The second contribution is due to the compactification and is presented in Eq. (3.33). The former is an odd function of \( \beta \) and is an even function of the parameter \( \tilde{\alpha} \). In Fig.1, we plotted Eq. (3.33) for \( \alpha_0 = 0 \). For a massless field and considering \( mL \leq 1 \), assuming \( D = 3 \) and \( m \) fixed. A plot of the total azimuthal current density, in units of “\( m^4 e^5 \)”, with respect to \( \alpha_0 \) as it can be seen in Fig.1. By this graph we can see that the intensity of the current increases with the parameter \( q \).

Furthermore, we have seen that the azimuthal current density induced by the compactification is an even function of the parameter \( \tilde{\beta} \) and is an odd function of the magnetic flux along the core of the string, with period equal to \( \Phi_0 \). We have checked that when \( \beta = 0 \) (untwisted bosonic field), Eq. (3.33) becomes an even function of the magnetic flux enclosed by the strings axis. We have also checked that this induced current vanishes for the case \( \alpha_0 = 0 \). In addition, in contrast with the fermionic case investigated in Ref. [32], the azimuthal current density, \( \langle j^\phi(x) \rangle_c \), vanishes for \( r = 0 \). For a massless field and considering \( D = 3 \), Eq. (3.33), is further simplified and is given by Eq. (3.36). In this case, the dominant behavior of \( \langle j^\phi(x) \rangle_c \) depends on the values assumed for the parameter \( \tilde{\beta} \). In Fig.2, we plotted Eq. (3.33) for \( D = 3 \), in units of “\( m^4 e^5 \)”, with respect to \( \alpha_0 \). Also by this graph we can see that the intensity of the current increases with \( q \), and the effect of the parameter \( \tilde{\beta} \) plays an important role on the sign of direction.

For the total azimuthal current density, that is, the sum of Eqs. (3.24) and (3.33), we have seen that it is dominated by \( \langle j^\phi(x) \rangle_{cs} \), for large values of the length of the compact dimension, \( mL \gg 1 \), assuming \( D = 3 \) and \( mr \) fixed. A plot of the total azimuthal current density, in units of “\( m^4 e^5 \)”, with respect to \( mr \) is presented in Fig.3 for \( D = 3 \) and for two different values of \( \tilde{\beta} \). From this graph we can see that the relevance of the compactified part of the current depends on the product \( mL \), decreasing when \( mL \) becomes larger. Moreover, the relative intensity of the total current, compared with the \( \langle j^\phi(x) \rangle_{cs} \), depends on \( \tilde{\beta} \).

We have also shown that the VEV of the axial current density in Eq. (3.47) has a purely topological origin and vanishes when \( \tilde{\beta} = 0, 1/2 \) and 1. This VEV can be expressed as the sum of two terms. One of them is given by Eq. (3.49) and independ of the radial distance \( r \), the cosmic string parameter \( q \) and \( \alpha_0 \). This contribution corresponds to the current density in \( (D + 1) \) Minkowski spacetime with the spatial topology \( R^{D-1} \times S \). The other contribution is given by Eq. (3.51) and is due to the magnetic fluxes and the planar angle deficit. We verified that this contribution is an odd function of the parameter \( \tilde{\beta} \) and is an even function of \( \alpha_0 \), with period equal to the quantum flux \( \Phi_0 \). For the particular case when \( \beta = 0 \), Eq. (3.51) becomes an odd function of the magnetic flux enclosed by the strings axis. A plot of the azimuthal current as function \( \tilde{\beta} \) is presented in Fig.4 for two different values of \( \alpha_0 \) and considering \( D = 3 \). By this graph we can see that the amplitude of the current increases with the parameter \( q \) and the effect of \( \alpha_0 \) is to change the orientation of the current.
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A Summation formulas

Here, we shall develop the summation involving the modified Bessel functions in Eqs. (3.9) and (3.23).

A.1 Summation formula involving the modified Bessel function $I_{\beta_n}(w)$

We start first with the expression (3.9). Let us consider the sum

$$I(w, \alpha, q) = \sum_{n=-\infty}^{\infty} I_{q|\alpha_0|}(w) + \sum_{n=1}^{\infty} [I_{q(n+\alpha_0)}(w) + I_{q(n-\alpha_0)}(w)].$$

(A.1)

A very useful integral representation for $I_{\beta_n}(w)$ has previously been considered in [29] and will also be used here. This representation is given by

$$I_{\beta_n}(w) = \frac{\sin(\pi \beta_n)}{\pi \beta_n} e^{-w} + \frac{w}{\pi} \int_0^\pi dy \sin y \frac{\sin(y \beta_n)}{\beta_n} e^{wy \cos y} - \frac{\sin(\pi \beta_n)}{\pi} \int_0^\infty dy e^{-w \cosh y - \beta_n y},$$

(A.2)

where in our case $\beta_n = q|\alpha_0|$. Upon substituting Eq. (A.2) into Eq. (A.1), one can work out each term separately. Thereby, using Eq. (06) (subsection 5.4.3) from [38], the summation in $n$ of the first and second terms on the right hand side of (A.2) is found to be

$$\sum_{n=-\infty}^{\infty} \sin(\beta_n \theta) \frac{\pi}{\beta_n} \sin[(2k+1)\pi \alpha_0].$$

(A.3)

which is only valid for $2k\pi/q < \theta < (2k+2)\pi/q$.

Now, let us consider the summation in $n$ of the last term on the right hand side of (A.2). This summation can be rewritten as

$$\sum_{n=-\infty}^{\infty} \sin(\pi \beta_n) e^{-\beta_n y} = \sin(\pi q \alpha_0) e^{-q\alpha_0 y} + \sum_{n=1}^{\infty} \left[ \sin[(n + \alpha_0)q\pi] e^{-(n+\alpha_0)qy} + \sin[(n - \alpha_0)q\pi] e^{-(n-\alpha_0)qy} \right].$$

(A.4)

In addition, one can further consider the sum on the right hand side of (A.4) in the following form:

$$e^{\pi \alpha_0 qy} \sum_{n=1}^{\infty} \sin[(n \pm \alpha_0)q\pi] e^{-nqy} = e^{\pi \alpha_0 qy} \left[ \cos(\alpha_0 \pi q) \sum_{n=1}^{\infty} \sin(n \pi q) e^{-nqy} \pm \sin(\alpha_0 \pi q) \sum_{n=1}^{\infty} \cos(n \pi q) e^{-nqy} \right].$$

(A.5)

Thus, we can use Eqs. (01) and (02) (subsection 5.4.12) from [38] to perform the sums on the right hand side of Eq. (A.5). After doing so, and substituting the result in Eq. (A.4), we obtain

$$\sum_{n=-\infty}^{\infty} \sin(\pi \beta_n) e^{-\beta_n y} = \frac{f(q, \alpha_0, y)}{\cosh(qy) - \cos(\pi q)},$$

(A.6)
where

\[ f(q, \alpha_0, y) = \sin[(1 - |\alpha_0|)qY] \cosh(|\alpha_0|y) + \sin(|\alpha_0|qY) \cosh[(1 - |\alpha_0|)qy]. \]  

(A.7)

As it can be seen in Eq. (A.7), we are considering only absolute values of the parameter \( \alpha_0 \). The absolute value \(|\alpha_0|\) is necessary in order to make Eq. (A.7) an even function of \( \alpha_0 \) and, therefore, compatible with Eq. (A.1) which is also an even function of the same parameter.

Finally, combining Eqs. (A.1), (A.2), (A.3) and (A.6), we are able to show that

\[ I(w, \alpha_0, q) = \frac{e^{wq}}{q} - \frac{1}{\pi} \int_0^\infty dy \frac{e^{-wq} y f(q, \alpha_0, y)}{\cosh(qy) - \cos(\pi q)} \]

\[ + \frac{2}{q} \sum_{k=1}^{[q/2]} \cos(2k\pi \alpha_0) e^{w \cos(2k\pi/q)}, \]  

(A.8)

where \([q/2]\) represents the integer part of \(q/2\), and the prime on the sign of the summation means that in the case \( q = 2p \) the term \( k = q/2 \) should be taken with the coefficient \(1/2\). Note that by taking the summation in \( k \) from \(-p\) to \(+p\) in (A.3) it provides the last term on the right hand side of (A.8) after combining Eqs. (A.1), (A.2), (A.3) and (A.6). We can see now that Eq. (A.8) is in perfect agreement with Eq. (A.1), i.e., both are even functions of \( \alpha_0 \). This is only possible by considering the absolute values of \( \alpha_0 \) in Eq. (A.7).

For integer values of \( q \) and \( \alpha_0 = 0 \), we have:

\[ I(w, q) = \frac{e^{wq}}{q} + \frac{1}{q} \sum_{k=1}^{q-1} e^{w \cos(2k\pi/q)}. \]  

(A.9)

The special case in Eq. (A.9) has been considered in a number of other contexts (see, e.g., [21]-[25]).

A.2 A second summation formula involving the modified Bessel function

\( I_{\beta_n}(z) \)

Let us now turn to the proof of the summation in \( n \) presented in Eq. (3.23), which is considered here in the following form:

\[ S = \sum_{n=-\infty}^\infty (n + \alpha_0) I_{q(n+\alpha_0)}(w) = \alpha_0 I_{q|\alpha_0|}(w) + \sum_{n \geq 1} \left[ (n + \alpha_0) I_{q(n+\alpha_0)}(w) - (n - \alpha_0) I_{q(n-\alpha_0)}(w) \right], \]

(A.10)

which is an odd function of the parameter \( \alpha_0 \). Thus, taking firstly only positive values of \( \alpha_0 \) and using the relations [31]:

\[ (n + \alpha_0) I_{q(n+\alpha_0)}(w) = -w \frac{d}{dw} I_{q(n+\alpha_0)}(w) + \frac{w}{q} I_{q(n+\alpha_0)-1}(w), \]

\[ (n - \alpha_0) I_{q(n-\alpha_0)}(w) = w \frac{d}{dw} I_{q(n-\alpha_0)}(w) - \frac{w}{q} I_{q(n-\alpha_0)+1}(w), \]  

(A.11)

in Eq. (A.10), we get

\[ S = -\frac{w}{q} \frac{d}{dw} \left[ I_{q|\alpha_0|}(w) + \sum_{n \geq 1} I_{q(n+\alpha_0)}(w) + \sum_{n \geq 1} I_{q(n-\alpha_0)}(w) \right] \]

\[ + \frac{w}{q} \left[ I_{q|\alpha_0|-1}(w) + \sum_{n \geq 1} I_{q(n+\alpha_0)-1}(w) + \sum_{n \geq 1} I_{q(n-\alpha_0)+1}(w) \right]. \]

(A.12)
Upon defining, $\tilde{\alpha}_0 = \alpha_0 - 1/q$, we are able to rewrite (A.12) as

$$S = -\frac{w}{q} \int_{-\infty}^{\infty} I_{q(n+\alpha_0)}(w) + \frac{w}{q} \sum_{n=-\infty}^{\infty} I_{q(n+\tilde{\alpha}_0)}(w).$$  \hspace{1cm} (A.13)

On the other hand, we can also exchange the factors $(n \pm \alpha)$ between the relations presented in Eq. (A.11). This provides

$$S = \frac{w}{q} \int_{-\infty}^{\infty} I_{q(n+\alpha_0)}(w) - \frac{w}{q} \sum_{n=-\infty}^{\infty} I_{q(n+\tilde{\alpha}_0)}(w),$$  \hspace{1cm} (A.14)

where now, $\tilde{\alpha}_0 = \alpha_0 + 1/q$. Substituting (A.8) into (A.14), one can separate the latter in two terms, i.e

$$S = S_{\alpha_0} + S_{\tilde{\alpha}_0}$$  \hspace{1cm} (A.15)

The first term is given by

$$S_{\alpha_0} = -\frac{2w}{q^2} \sum_{k=0}^{[q/2]} \cos(2k\pi/q) \cos(2k\pi\alpha_0)e^{w \cos(2k\pi/q)}$$

$$- \frac{w}{q\pi} \int_{0}^{\infty} dy \cosh y f(q, \alpha_0, y),$$  \hspace{1cm} (A.16)

and the second term, $S_{\tilde{\alpha}_0}$, is given in terms of $S_{\alpha_0}$ as shown below:

$$S_{\tilde{\alpha}_0} = -S_{\alpha_0} + \frac{2w}{q^2} \sum_{k=1}^{[q/2]} \sin(2k\pi/q) \sin(2k\pi\alpha_0)e^{w \cos(2k\pi/q)}$$

$$+ \frac{w}{q\pi} \int_{0}^{\infty} dy \sinh y g(q, \alpha_0, y),$$  \hspace{1cm} (A.17)

where the function, $g(q, \alpha_0, y)$, is defined as

$$g(q, \alpha_0, y) = \sin(q\pi\alpha_0) \sinh[(1 - |\alpha_0|)qy] - \sinh(yq\alpha_0) \sin[(1 - |\alpha_0|)\pi q].$$  \hspace{1cm} (A.18)

As we took only positive values of $\alpha_0$, Eq. (A.15) does not need to depend on $|\alpha_0|$. However, if we took negative values of $\alpha_0$ in Eq. (A.10) we would get a similar expression as in Eq. (A.18) but with the opposite sign. This suggests that in order to satisfy both possibilities one has to include absolute values of $\alpha_0$ as showed in Eq. (A.18).

Therefore, substituting (A.17) into (A.15) we finally arrive at

$$S = \frac{2w}{q^2} \sum_{k=1}^{[q/2]} \sin(2k\pi/q) \sin(2k\pi\alpha_0)e^{w \cos(2k\pi/q)} + \frac{w}{q\pi} \int_{0}^{\infty} dy \sinh y g(q, \alpha_0, y).$$  \hspace{1cm} (A.19)

Note that both expression in Eqs. (A.10) and Eq. (A.19) are odd function of $\alpha_0$, as it should be. This is only possible if we take absolute values of $\alpha_0$ as showed in Eq. (A.18). Note also that by using Eq. (A.13), instead of Eq. (A.14), we would obtain the same expression (A.19).
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