Heterotic – type I superstring duality
and low-energy effective actions

A.A. Tseytlin

Theoretical Physics Group, Blackett Laboratory,
Imperial College, London SW7 2BZ, U.K.

Abstract

We compare order $R^4$ terms in the 10-dimensional effective actions of $SO(32)$ heterotic and type I superstrings from the point of view of duality between the two theories. Some of these terms do not receive higher-loop corrections being related by supersymmetry to ‘anomaly-cancelling’ terms which depend on the antisymmetric 2-tensor. At the same time, the consistency of duality relation implies that the ‘tree-level’ $R^4$ super-invariant (the one which has $\zeta(3)$-coefficient in the sphere part of the action) should appear also at higher orders of loop expansion, i.e. should be multiplied by a non-trivial function of the dilaton.
1. Introduction

It was suggested \[1\] that $SO(32)$ heterotic and type I string theories in ten dimensions are dual to each other in the sense that a strong-coupling region of one of the theories can be described by dynamics of solitonic states which is equivalent to weak-coupling dynamics of elementary states of the other. Further arguments in favour of this duality were given in \[2,3,4,5\]. The $D = 10$ supersymmetry dictates that the leading-order terms in the two effective actions are related by a field redefinition. Somewhat surprisingly, this redefinition involves changing the sign of the dilaton, i.e. it inverts the string coupling \[1\] (we shall use primes to indicate the fields of type I theory)

\[
G'_{\mu\nu} = e^{-\phi} G_{\mu\nu}, \quad \phi' = -\phi, \quad B'_{\mu\nu} = B_{\mu\nu}, \quad A'_{\mu} = A_{\mu}.
\] (1.1)

To understand better how this the duality acts at the string-theory level it would be interesting to compare higher-derivative terms in the two low-energy effective actions.

To be able to test duality using only perturbative string-theory results one is to consider the terms in the effective actions with coefficients that have simple polynomial dependence on string coupling, i.e. receive contributions only from one or few particular orders of string loop expansion \[9\]. Then (1.1) maps them into terms with similar perturbative coefficients.  

There are, indeed, examples of terms in the string effective actions which are either not renormalised by string higher-loop corrections (like second-derivative terms and ‘anomaly-cancelling’ terms, see, e.g., \[12,13,14,15\]) or receive contributions only from specific orders of string perturbative expansion (see \[16,17\]). In \[18\] we considered a term quartic in the gauge field strength $tr F^4$ which, in heterotic string theory, is absent at the tree level \[19,20\] but appears at the one-loop level \[21,22,23\]. We have argued that it does not receive corrections from higher loops since $D = 10$ supersymmetry relates \[24,23\] it to the ‘anomaly-cancelling’ term $Btr F^4$ \[24\] which is not renormalised at higher orders \[27,15\] (see also below). The duality (1.1) maps this term into the tree-level (disc) $tr F^4$ term of the type I effective action \[10,11\]. In the finite $SO(32)$ type I theory \[28\] the $tr F^4$ term does not receive any loop corrections and its coefficient is exactly the same \[18\] as that of the one-loop $tr F^4$ term in the heterotic theory.

---

1. Similar tests of heterotic–type II string duality \[2,1\] in six and four dimensions were done in \[7\] and \[8\].

2. It should be noted that one may ignore possible $\alpha'$-dependent modifications of the duality transformation rule (1.1). Indeed, the effective actions are in any case defined modulo local field redefinitions \[10,1,1\], so it makes sense to compare only ‘irreducible’ terms with coefficients which are not changed under local covariant redefinitions of ‘massless’ fields. Such are the terms which will be discussed below.
As was noted in [18], it seems likely that $D = 10$ superstring effective actions contain infinite series of local terms that receive contributions only from one particular order in string loop expansion. This is necessary for consistency of the duality conjecture and is probably related to the special property of $D = 10$ supersymmetry that certain super-invariants of given dimension may have only specific dilaton dependence [24,9]. Indeed, the dilaton plays a special role in $D = 10$ supergravity, being in the same multiplet with graviton.

Here we shall try to elucidate further the interplay between duality, supersymmetry and the structure of loop expansion in heterotic and type I theories by extending the discussion in [18] to the curvature-dependent $R^4$-terms. We shall see that the relation between the $R^4$ corrections is more complicated than between the $F^4$ terms, reflecting the existence of several super-invariants containing $R^4$ contractions [30,24,25].

2. Duality and effective actions

The general structure of local terms depending on the curvature and gauge field strength in the heterotic and type I string effective actions is

$$S_{het} = \int d^{10}x \sqrt{G} \sum_{n=1}^{\infty} \left[ g_n(\phi) R^n + s_n(\phi) R^{n-2} F^2 + \ldots + f_n(\phi) F^n + \ldots \right], \quad (2.1)$$

$$S_{typeI} = \int d^{10}x \sqrt{G'} \sum_{n=1}^{\infty} \left[ g'_n(\phi') R'^n + s'_n(\phi') R'^{n-2} F'^2 + \ldots + f_n(\phi') F'^n + \ldots \right],$$

where $R^n = (R_{\ldots})^n G^{-n}$ stands for various possible invariants with $n$ factors of the curvature. $F^n$ may also involve different group trace structures which we shall not distinguish at the moment. We assume that the inverse string tensions $\alpha'$ and $\alpha'_I$ are absorbed into the metrics $G_{\mu\nu}$ and $G'_{\mu\nu}$ so that all tensors have geometrical dimension $[T_{\mu_1...\mu_n}] = cm^{-n}$.

In order for the two actions to be related by the duality transformation (1.1) it should be true that

$$g'_n(\phi') = e^{(n-5)\phi'} g_n(-\phi') , \quad s'_n(\phi') = e^{(n-5)\phi'} f_n(-\phi') , \quad \ldots . \quad (2.2)$$

---

3 In addition to local terms, the massless superstring effective action contains also non-local terms which are non-analytic in momenta. We define the string effective action as the one the tree-level amplitudes of which reproduce the full loop-corrected string amplitudes for massless states. The non-analyticity of the low-energy expansion is due to loops of massless string states which must necessarily be included in order to have a well-defined (finite, anomaly-free) effective action [24].
Our aim is to check whether such relations are consistent with the structure of string perturbative expansions in the two theories.

Since we known how to compute the above coupling functions only using weak-coupling expansions in each theory

$$g_n(\phi) = b_0^{(n)} e^{-2\phi} + b_1^{(n)} e^{2\phi} + ... + b_m^{(n)} e^{2(m-1)\phi} + ... , \quad e_\phi \ll 1 , \quad (2.3)$$

$$g_n'(\phi') = b_0^{(n)} e^{-\phi'} + b_1^{(n)} e^{\phi'} + ... + b_m^{(n)} e^{(m-1)\phi'} + ... , \quad e_{\phi'} \ll 1 , \quad (2.4)$$

to be able to check the duality relations (2.2) one should consider only special invariants which have coupling functions containing only one or few terms in the perturbative series. Indeed, in general, the left and the right sides of the formal relation for $g_n'(\phi')$ in (2.2)

$$b_0^{(n)} e^{-\phi'} + b_1^{(n)} e^{\phi'} + ... + b_m^{(n)} e^{(m-1)\phi'} + ... = b_0^{(n)} e^{(n-3)\phi'} + b_1^{(n)} e^{(n-5)\phi'} + b_2^{(n)} e^{(n-7)\phi'} + ... + b_m^{(n)} e^{(n-3-2m)\phi'} + ... , \quad (2.5)$$

are defined in the different regions of the coupling space, $e_{\phi'} \ll 1$ and $e_{\phi'} \gg 1$, respectively.

Let us start with the $R, R^2, R^3$ terms. Since the 3-graviton amplitude in the heterotic string theory does not receive string loop corrections [31, 19, 20, 13, 21]

$$g_1(\phi) = d_1 e^{-2\phi}, \quad g_2(\phi) = d_2 e^{-2\phi}, \quad g_3(\phi) = d_3 e^{-2\phi}, \quad d_2 = \frac{1}{8} d_1, \quad d_3 = 0 . \quad (2.6)$$

Then (2.2), (2.3) would be satisfied provided

$$g_1'(\phi') = d_1 e^{-2\phi'}, \quad g_2'(\phi') = d_2 e^{-\phi'}, \quad g_3'(\phi') = d_3 . \quad (2.7)$$

These relations can be understood as being simply a consequence of $D = 10$ supersymmetry: the dilaton dependence of the supergravity term is fixed uniquely (up to a field redefinition) as is the structure of the $R^2$ super-invariant; also, there are no super-invariants containing $R^3$ (see [33, 30] and references there). According to [30], the supersymmetric action which starts with the $N = 1, D = 10$ supergravity + Yang-Mills terms with $H_{\mu\nu\lambda}$ modified by the Lorentz Chern-Simons term $\omega_3$ is given by an infinite series of terms (we use heterotic frame)

$$S_{het} = -\frac{1}{8} \int d^{10}x \sqrt{G} \ e^{-2\phi} \left[ R + 4(\partial \phi)^2 - \frac{1}{12} \hat{H}_{\mu\nu\lambda}^2 \right. $$

$$\left. - \frac{1}{2} T + k_2 (3 T_{\mu\nu\lambda\rho}^2 + T_{\mu\nu}^2 + ... + O(T^n) + ... \right] , \quad (2.8)$$

The proof that there is no Gauss-Bonnet-type $R^3$ term involves analysis of the 4-point graviton amplitude [32, 20]. As usual, we ignore terms which can be eliminated by local field redefinitions.
where $R_{\mu\nu}$ stands for $R_{ab}^{\mu\nu} (\omega_-)$. The connection $\omega_-$ has torsion proportional to $\hat{H}_{\mu\nu\lambda}$ which is given by $\hat{H} = dB + k_1 \omega_3 (A) + k_2 \omega_3 (\omega_-)$. The coefficients $k_1$ and $k_2$ are fixed by the condition of anomaly cancellation at the level of low-energy field theory and thus do not contain any particular ‘stringy’ information. They must have the required form in any theory which reduces to supergravity + Yang-Mills at low energies, is supersymmetric and anomaly-free. Since the heterotic and type I effective actions start with the same supergravity + Yang-Mills actions related by the field redefinition (1.1) we conclude that the consistency of duality at this level is automatic.

One may, of course, check the coefficient in the expression for $g'_2$ by directly computing the sum (finite in $SO(32)$ theory [28]) of the 3-graviton amplitudes on the disc and the projective plane. To confirm that $d_3 = 0$, i.e. $g'_3 = 0$ in type I theory one is to show that the $R^3$-contribution which could come from the sum of the annulus, Möbius strip and Klein bottle diagrams does not appear at all.

Another consistency check is provided by an observation that the sphere part of the type I action (which is the same as in type II theory with the NS-NS antisymmetric tensor field set equal to zero) does not contain $R^2$ and $R^3$ terms. The terms $e^{-2\phi} (d'_2 R^2 + d'_3 R^3)$ in the type I action would be related by the duality (1.1), (2.2) to the terms $d'_2 e^{-\phi} R^2 + d'_3 R^3$ in the heterotic action which are certainly absent in the heterotic string perturbative expansion. As mentioned above, the presence of $R^2$ and the absence of $R^3$-term in the two effective actions is, in fact, dictated by the 10-dimensional supersymmetry.

3. $R^4, R^2F^2, F^4$ terms in the $SO(32)$ heterotic string action: super-invariants and non-renormalisation

Let us now consider $R^4$, $R^2F^2$ and $F^4$ terms. We shall start by summarising the known results for the structure of tree-level and 1-loop corrections to the effective action of the heterotic string and then explain how these results can be understood systematically in terms of possible $D = 10$ super-invariants constructed in [24,25].

The relevant tree-level terms in the heterotic string action can be written in the following symbolic form [11,19,20,34] ($\alpha'$ is absorbed into $G_{\mu\nu}$; the trace in the fundamental representation of $SO(32)$ and $\text{tr} R^2 \equiv R^{ab}_{\mu\nu} R^{ba\mu\nu}$

$$S^{(0)}_{het} = -\frac{1}{8} \int d^{10}x \sqrt{G} e^{-2\phi} \left[ R + 4(\partial\phi)^2 - \frac{1}{12} \hat{H}^2_{\mu\nu\lambda} + \frac{1}{3}(\text{tr} F^2 - \text{tr} R^2) \right]$$

5 In addition to the $O(e^{2n\phi})$ contributions from the orientable diagrams of the type II theory (sphere, torus, etc.) the closed string type I scattering amplitudes receive contributions from extra diagrams (with topology of a sphere with disc and crosscap insertions) which should be added together in order to realise the projection onto type I intermediate states [12].
+ b_1 t_8 (\text{tr} F^2 - \text{tr} R^2)^2 + b_2 (t_8 t_8 R^4 - \frac{1}{8} \epsilon_{10} \epsilon_{10} R^4) \right), \quad (3.1)

b_1 = - \frac{1}{28}, \quad b_2 = \frac{\zeta(3)}{3 \cdot 2^9}.

Here $t_8$ is the 10-dimensional extension of the 8-dimensional light-cone gauge ‘zero-mode’ tensor \cite{12} (with $e^{\mu_1 \cdots \mu_8}$-term omitted) built out of $G^{\mu \nu}$. For example,

\begin{align*}
t_8 F^4 & \equiv t^{\mu_1 \nu_1 \cdots \mu_4 \nu_4} F_{\mu_1 \nu_1} F_{\mu_2 \nu_2} F_{\mu_3 \nu_3} F_{\mu_4 \nu_4} \\
& = 16 F^{\mu \nu} F_{\rho \nu} F_{\mu \lambda} F_{\rho \lambda} + 8 F^{\mu \nu} F_{\rho \nu} F_{\mu \lambda} F_{\rho \lambda} - 4 F^{\mu \nu} F_{\mu \nu} F_{\rho \lambda} F_{\rho \lambda} - 2 F^{\mu \nu} F_{\mu \nu} F_{\rho \lambda} F_{\rho \lambda},
\end{align*}

and

\begin{align*}
t_8 t_8 R^4 & \equiv t^{\mu_1 \nu_1 \cdots \mu_4 \nu_4} t_{\mu_1' \nu_1' \cdots \mu_4' \nu_4'} R_{\mu_1 \nu_1} R_{\mu_2 \nu_2} R_{\mu_3 \nu_3} R_{\mu_4 \nu_4} \\
& \equiv \epsilon^{\alpha \beta \mu_1 \nu_1 \cdots \mu_4 \nu_4} \epsilon^{\alpha \beta \mu_1' \nu_1' \cdots \mu_4' \nu_4'} R_{\mu_1 \nu_1} R_{\mu_2 \nu_2} R_{\mu_3 \nu_3} R_{\mu_4 \nu_4}. \quad (3.3)
\end{align*}

We shall use $\epsilon_{10}$ to indicate the totally antisymmetric tensor $e^{\mu_1 \cdots \mu_{10}}$, e.g.,

\begin{align*}
\epsilon_{10} \epsilon_{10} R^4 & \equiv \epsilon^{\alpha \beta \mu_1 \nu_1 \cdots \mu_4 \nu_4} \epsilon^{\alpha \beta \mu_1' \nu_1' \cdots \mu_4' \nu_4'} R_{\mu_1 \nu_1} R_{\mu_2 \nu_2} R_{\mu_3 \nu_3} R_{\mu_4 \nu_4}. \quad (3.4)
\end{align*}

The invariant \cite{3.4} is the Gauss-Bonnet density in 8-dimensions whose presence cannot be detected from the calculation of the 4-point scattering amplitude but can be fixed by comparison with the $\sigma$-model $\beta$-function \cite{14}.

The above terms are consistent with the $D = 10$ supersymmetry. The structures containing $\text{tr} F^2 - \text{tr} R^2$ are ‘anomaly-related’ terms which appear in the super-extension of the supergravity + super Yang-Mills with $H$ modified by the Lorentz Chern-Simons term \cite{30}, i.e. \cite{28} with $k_1 = - k_2 = \frac{1}{4}$. The combination

\begin{align*}
J_0 = t_8 t_8 R^4 - \frac{1}{2} \epsilon_{10} \epsilon_{10} R^4, \quad (3.5)
\end{align*}

is the bosonic part of one of possible super-invariants containing $R^4$-terms \cite{33,24,25}.

The one-loop terms in the effective action which are local and have explicitly computable coefficients are related to the special amplitudes on the torus which have ‘nearly holomorphic’ integrands \cite{14} so that they receive contributions only from a boundary ($\text{Im} \tau \to \infty$) of the moduli space. They can be reconstructed from string scattering amplitudes \cite{14,33,21,22} or computed directly \cite{23} using the partition function representation for the string effective action \cite{36}. The parity-odd ‘anomaly-cancelling’ terms in the heterotic string theory have the following structure \cite{14}:

\begin{align*}
S^{(1)}_{\text{het}} = \int d^{10} x \sqrt{G} \left( - \frac{1}{16} \epsilon_{10} B X_8 + \ldots \right), \quad (3.6)
\end{align*}

\footnote{We quote the 1-loop coefficients relative to the tree-level term \cite{3.1}. In the notation we are using here $\kappa = 2 \alpha'^2 g$, $g_{10}^2 = 2 \kappa^2 / \alpha'$, $g = e^{\phi}$, i.e. $S^{(0)}_{\text{het}} = \int d^{10} x \sqrt{G} e^{-2 \phi} \left( - \frac{1}{2 \pi^2} \dddot{R} + \ldots \right)$.}
where \( \epsilon_{10} \) is multiplied by \( B_{\mu\nu} \) and the 8-rank tensor \( X_8 \) which is related to the anomaly 12-form (and elliptic genus \( \mathcal{A} \)) by \( I_{12} = [\mathcal{A}_q]_{12} = \frac{1}{2\pi} (\text{tr} F^2 - \text{tr} R^2) X_8 \),

\[
X_8 = \beta (32\text{tr} F^4 - 4\text{tr} F^2 \text{tr} R^2 + 4\text{tr} R^4 + \text{tr} R^2 \text{tr} R^2), \quad \beta \equiv -\frac{1}{3} \cdot \frac{1}{2^{14} \pi^5} .
\] (3.7)

The parity-even local one-loop terms are given by \([21,22,23]\)

\[
S_{\text{het}}^{(1)} = \int d^{10} x \sqrt{G} \left( \frac{1}{4} t_8 X_8 + \ldots \right),
\] (3.8)

where \( t_8 X_8 \equiv t_{\mu_1 \ldots \mu_8} X_8^{\mu_1 \ldots \mu_8} \). The similarity of the expressions (3.6) and (3.8) suggests that they are related by supersymmetry (which was indeed anticipated in \([23]\)). A heuristic argument explaining the connection between \( t_{\mu_1 \ldots \mu_8} X_8^{\mu_1 \ldots \mu_8} \) and \( \epsilon_{10} B X_8 \) terms is the following. The 8-dimensional light-cone gauge ‘zero-mode’ tensor which appears in the 4-point amplitude \([12]\)

\[
(t_{\mu_1 \ldots \mu_8})_8 = t_{\mu_1 \ldots \mu_8} - \frac{1}{2} \epsilon_{\mu_1 \ldots \mu_8}
\]

may be given a 10-dimensional generalisation:

\[
\hat{t}_{\mu_1 \ldots \mu_8} \equiv t_{\mu_1 \ldots \mu_8} - \frac{1}{4} B_{\lambda \rho} \epsilon^{\lambda \rho \mu_1 \ldots \mu_8}
\] (3.9)

(assuming that in the light-cone gauge \( B_{uv} = 1, \ F_{uv} = R_{uv} = 0 \)). This is also consistent with the structure of the corresponding string amplitudes \([21,25]\) which, of course, should satisfy the requirements of linearised supersymmetry. Then the combinations \( t_8 X_8 \) and \( \epsilon_{10} B X_8 \) in (3.6) and (3.8) should be parts of the \( D = 10 \) super-invariant

\[
J_1 \equiv \hat{t}_8 X_8 = t_8 X_8 - \frac{1}{4} \epsilon_{10} B X_8 ,
\] (3.10)

\[
S_{\text{het}}^{(1)} = \int d^{10} x \sqrt{G} \left( \frac{1}{4} J_1 + \ldots \right) .
\] (3.11)

Indeed, \( J_1 \) is a combination of \( R^4, F^4, R^2 F^2 \) type \( D = 10 \) super-invariants recently constructed in \([24,25]\). According to \([24,25]\) the bosonic parts of a set of 6 independent super-invariants are given by (an invariant action should start with \( R + \text{tr} F^2 \)-terms)

\[
I_1 = t_8 \text{tr} F^4 - \frac{1}{4} \epsilon_{10} B \text{tr} F^4 , \quad I_2 = t_8 \text{tr} F^2 \text{tr} F^2 - \frac{1}{4} \epsilon_{10} B \text{tr} F^2 \text{tr} F^2 ,
\] (3.12)

\[
I_3 = t_8 \text{tr} R^4 - \frac{1}{4} \epsilon_{10} B \text{tr} R^4 , \quad I_4 = t_8 \text{tr} R^2 \text{tr} R^2 - \frac{1}{4} \epsilon_{10} B \text{tr} R^2 \text{tr} R^2 ,
\] (3.13)

\[
I_5 = t_8 \text{tr} R^2 \text{tr} F^2 - \frac{1}{4} \epsilon_{10} B \text{tr} R^2 \text{tr} F^2 ,
\] (3.14)

which all contain parity-odd terms, and also by the parity-even combination \( J_0 \) in (3.5). Note also that

\[
t_8 t_8 R^4 = 24 t_8 \text{tr} R^4 - 6 t_8 \text{tr} R^2 \text{tr} R^2 .
\] (3.15)
As a result, $J_1$ (3.11) which appears in the 1-loop heterotic string action (3.11) is a linear combination of the super-invariants (see (3.7))

$$J_1 = \beta (32I_1 + 4I_3 + I_4 - 4I_5) .$$ (3.16)

One expects that the parity-odd ‘anomaly-cancelling’ terms $\epsilon_{10} B \text{tr} F^4$, etc., should appear only at the 1-loop order, since, in particular, there should be no higher-loop contributions to the anomalies $\delta S$. The absence of corrections to these terms was shown directly at the level of higher-loop heterotic string amplitudes $[27,15]$. The relation between the string coupling and the dilaton $\phi$, and also the gauge nature of the 2-form field $B_{\mu \nu}$, suggest another simple explanation for that. If one would get, e.g., $f(g)\epsilon_{10} B \text{tr} F^4$ with $f(g) = a_1 + a_2 g^2 + \ldots$, this would imply the presence of the $e^{2n\phi}\epsilon_{10} B \text{tr} F^4$ terms, which, however, are not consistent with the gauge invariance $B \to B + d\lambda$ unless $n = 0$.

Combining the non-renormalisation of the ‘anomaly-cancelling’ terms with their relation to $R^4, F^4, R^2 F^2$-terms in (3.12),(3.13),(3.14) by $D = 10$ supersymmetry we are led to the important conclusion that the coefficients of the latter terms also do not receive two and higher loop corrections. This is an interesting new example of a $D = 10$ non-renormalisation theorem’ which applies to terms originating from certain 4-point string amplitudes.

This non-renormalisation allows one to extend the duality relation between the leading terms in the type I and heterotic string effective actions to those higher-derivative terms.

---

7 A close connection between the calculation of the anomaly index and the one-loop $O(R^4, R^2 F^2, F^4)$ term represented as a torus partition function in a background was suggested as an indication that this term should not receive higher heterotic string loop corrections $[23]$. It should be emphasised that the condition of preservation of supersymmetry is crucial for this non-renormalisation. One may also argue for non-renormalisation of, e.g., $F^4$ term by modifying the proof $[15]$ of the absence of renormalisation of $\epsilon_{10} B F^4$ term. The $g$-loop parity-conserving 4-vector $(V = \zeta \cdot (\partial x + i k \psi \psi) e^{i k x})$ amplitude in RNS approach has $2g - 2$ supermoduli integrals and thus contains $2g - 2$ supercurrent $(T_f = \psi \partial x + \ldots)$ insertions. Since we need only 4 powers of momenta $k$ to get $F^4$-structure and at the same time are to saturate the integral over 10 fermionic zero modes, the two (additional to 8) $\psi$-factors should come from the supercurrent insertions. Then the remaining free $\partial x$ from $T_f$ should be contracted with $e^{i k x}$ giving extra power of momenta. This implies that though there may be higher-derivative corrections, $F^4$ will not be renormalised.

8 Since the field strengths contain non-linear commutator terms and the curvature terms expanded near flat space contain terms of all orders in $h_{\mu \nu}$, the gauge invariance implies similar non-renormalisation of certain low-momentum terms in infinite sets of vector and graviton amplitudes.
Note, however, that we are unable to make a similar non-renormalisation claim for the parity-even ‘tree-level’ super-invariant $J_0 = t_8 t_8 R^4 - \frac{1}{8} \epsilon_{10} \epsilon_{10} R^4$ in (3.5). As we shall see below, this invariant must, in fact, receive higher loop corrections in order to avoid contradiction with the duality conjecture.

The $t_8 t_8 R^4$ term is present both in the tree [34,11] and the 1-loop [38,22] parts of the type II superstring effective action. The $D = 10$ supersymmetry implies that it should actually appear in the combination $J_0$ (3.5), i.e.

$$S_{\text{type}II} = \int d^{10}x \sqrt{G} \left[ g(\phi) J_0 + \ldots \right],$$

(3.17)

$$g(\phi) = c_0 e^{-2\phi} + c_1 + \ldots, \quad c_0 = -\frac{1}{8} b_2, \quad c_1 = -\frac{1}{3} \cdot 2^{18} \pi^5,$$

where $b_2$ is the same as in (3.1). In contrast to the heterotic string case, here the needed $R^4$ kinematic structure is already produced by the light cone Green-Schwarz fermionic zero-mode contribution so that its coefficient is simply given by the volume of the moduli space. As a result, there are no 1-loop contributions proportional to $I_3$ and $I_4$ in (3.13) (in agreement with the absence of the ‘anomaly-cancelling’ terms).

4. Comparison with type I theory

One may argue that the comparison of the coefficients of the above terms ($R^4$, etc.) in the heterotic and type I actions does not represent a non-trivial check of the duality as such: the values of these coefficients are determined just by the requirement of anomaly cancellation in low-energy field theory. Indeed, the coefficients of the ‘anomaly-cancelling’ terms ($\epsilon_{10} BR^4$, etc.) are directly related to the anomalous contributions in massless fermionic amplitudes of low-energy field theory and thus are fixed once one demands the anomaly cancellation via Green-Schwarz mechanism [26]. Since the supersymmetry relates them to the coefficients of the $R^4$, etc. terms, the latter coefficients are also uniquely determined by the low-energy field theory. Since both type I and heterotic string theories are consistent (supersymmetric, anomaly-free) extensions of the same low-energy field theory, the values of the coefficients in the two theories which are uniquely determined by the limiting field theory must be the same.

9 Note also that while the tree-level terms in the heterotic (3.1) and type II (3.17) actions coincide for the special background $R = F$, this does not apply to the 1-loop corrections (cf. (3.3), (3.7), (3.15)). Though the two world-sheet actions become formally equivalent, the structure of the two 1-loop path integrals remains different.

10 It is thus should not be surprising that computed directly from string theory they are given by the contribution of the ‘infra-red’ boundary of the moduli space.
Still, given that the structures of the loop expansion in the two string theories are very different, it is instructive to see how the duality transformation relates the terms which appear at different loop orders in a way consistent with $D = 10$ supersymmetry. A test of the duality is actually the equality of these coefficients combined with their non-renormalisation by higher-loop corrections.

Let us start with the tree-level terms in the heterotic string (3.1) and discuss their type I images under the duality transformation (2.2). Since the ‘anomaly-related’ terms like $\sqrt{G}e^{-2\phi}t^8(trF^2 - trR^2)^2$ in (3.1) are parts of the supersymmetric completion of the low-energy supergravity action (with Chern-Simons modified $H$) these terms should not be renormalised by higher loop corrections for the same reason as the leading-order supergravity terms. Then according to (2.2), (2.5) the counterpart of the above term under the duality transformation (1.1) is $\sqrt{G'}e^\phi t^8(trF'^2 - trR'^2)^2$. This term should originate from the combination of type I diagrams with the Euler number $-1$ (sphere with 3 holes, etc). These diagrams are much more complicated than the sphere one which led to (3.1) in the heterotic string. This illustrates the non-triviality of the duality relation between the two theories.

The same reasoning does not automatically apply to $J_0$ in (3.5), (3.1) since it is not clear how one could argue that it does not receive corrections from higher loops. This term thus needs a special discussion (see below).

Since the 1-loop torus terms in (3.6), (3.16) do not appear at the tree level and two and higher loop levels, the corresponding function $g(\phi)$ in (2.3) is constant. Then under the duality (2.2), (2.3) they are mapped into ‘tree-level’ disc terms $e^{-\phi'}t^8 (trF'^4, e^{-\phi'}R^4, e^{-\phi'}R^2trF^2)$. Thus the absence of the $trF^2trF^2$ term in the torus correction in $SO(32)$ heterotic string is crucial for the consistency of the duality: in type I theory such double-trace term could not originate from the disc diagram which has only one boundary. As was pointed out in [18], the coefficient of the disc $t^8 trF'^4$ term in type I theory [10,11] is indeed the same as the 1-loop coefficient of this term (see (3.8), (3.7)) the heterotic string [21,22].

The ‘anomaly-cancelling’ terms in (3.6) do not depend on the dilaton and constant part of the metric and thus do not change their form (for $\phi =$ const) under the duality transformation (1.1). Their coefficients (determined, as mentioned above, uniquely by the same low-energy field theory) must indeed agree in the two string theories even though they originate from two different string diagrams – torus in the heterotic string [14] and disc (with $B_{\mu\nu}$ R-R vertex operator insertion adding extra power of $e^\phi$) in type I theory [38]. Since the field redefinition (1.4) implies the corresponding change in the supersymmetry transformation laws, the type I theory form of the super-invariants (3.12) is

$$I_1 = e^{-\phi'}t^8 trF'^4 - \frac{1}{4}t^8 trF'^4, \quad I_3 = e^{-\phi'}t^8 trR'^4 - \frac{1}{4}t^8 trR'^4, ..., \quad (4.1)$$

where the prime is used to indicate that the corresponding tensors are constructed using $G'_{\mu\nu}$. As in the heterotic case, the non-renormalisation of ‘anomaly-cancelling’ terms
combined with supersymmetry implies the non-renormalisation of the $\text{tr}F^4, R^4, \text{tr}F^2 R^2$-terms appearing in (3.12), (3.13), (3.14) in type I theory.\footnote{A direct argument relating the non-renormalisation of $\text{tr}F^4$ to finiteness of type I theory was sketched in \cite{18}.}

Let us now discuss whether there are other $R^4$, etc. terms in the type I action in addition to the ones which are dual images of the terms in the heterotic action considered above. The ‘anomaly-related’ terms and the terms which are connected by supersymmetry to the unique ‘anomaly-cancelling’ terms are also unique; the duality transformation (1.1) just relates the corresponding super-invariants. This does not, however, apply to the $R^4$-terms which form the super-invariant $J_0$ (3.3) appearing in the tree-level heterotic string action (3.1). Its direct type I counterpart under (1.1)

$$\sqrt{G}e^{-2\phi} J_0 = \sqrt{G'}e^{\phi'} J'_0, \quad J'_0 = t'_s t'_8 R'^4 - \frac{1}{8} \epsilon'_{10} \epsilon'_{10} R'^4,$$  \hspace{1em} (4.2)

is the bosonic part of only one of the super-invariants of that structure which are present in the effective action of type I theory. Indeed, the type I theory contains $J_0$ terms coming from the sphere and the torus since the contributions of these diagrams are the same as in the type II theory. The latter terms were already given in (3.17). The corresponding type I contributions are obtained by adding primes on $G$ and $\phi$, i.e.\footnote{The value of the coefficient $c_1$ may, in principle, be different from its value in the type II theory: in addition to the torus, the Euler number zero diagrams in the type I theory include also the Klein bottle, the annulus and the Möbius strip which may also produce contributions proportional to $J_0$.}

$$S_{\text{type} I} = \int d^{10}x \sqrt{G'} \left[ g'(\phi') J'_0 + \ldots \right], \quad g'(\phi') = c_0 e^{-2\phi'} + c'_1 + \ldots. \quad (4.3)$$

As it is clear from (4.2), (4.3), the $J_0$-term in type I theory is likely to receive corrections from all orders of perturbation theory, $g'(\phi') = c_0 e^{-2\phi'} + c'_1 + c'_2 e^{\phi'} + \ldots$. This then avoids contradiction with duality making it simply non-applicable term-by-term in the two weak-coupling expansions. If the duality (1.1) would apply to the terms in (4.3), they would be mapped into

$$S_{\text{type} I}(G, \phi) = \int d^{10}x \sqrt{G} \left[ (c_0 e^{\phi} + c'_1 e^{-\phi} + \ldots) J_0(G) + \ldots \right]. \quad (4.4)$$

Such terms are certainly absent in the heterotic string perturbative expansion being of odd power in the string coupling constant.

The absence of contradiction with duality implies that $J_0$ must receive higher-loop corrections also in the heterotic theory (otherwise duality would map the tree-level heterotic string $J_0$ term in (3.1) into the unique type I term (4.2)). This indicates that there
should exist a super-extension of the term $g(\phi)J_0$ with an arbitrary dilaton function. Such a possibility does not seem to be ruled out by a scarce information available about the structure of ‘anomaly-unrelated’ super-invariants in $D = 10$ supergravity. In particular, the (on-shell or only linear) superspace constructions of $R^4$ superinvariants [38] are not sufficient in order to fix uniquely the structure of possible dilaton prefactor. Though the fact that the dilaton is in the same multiplet with gravitons seems to suggest that that the form of the dilaton dependence can not be arbitrary, this expectation seems to be in conflict with the above discussion of the $g(\phi)J_0$-terms in the type I superstring.

While one is thus unable to check duality by comparing $J_0$-terms in the two theories, one can use the duality conjecture to predict the strong-coupling behaviour of the corresponding $g(\phi)$ function in each theory: it should be the same as its weak coupling behaviour in the dual theory. Thus, for example, in the heterotic string theory (see (2.2),(4.4))

$$g(\phi)_{\phi \to -\infty} = c_0 e^{-2\phi} + ..., \quad g(\phi)_{\phi \to \infty} = c_0 e^\phi + c'_1 e^{-\phi} ... .$$

(4.5)

5. Concluding remarks

Similar conclusion about a non-trivial modification to all loop orders applies to all terms which appear in the low-energy expansion of the sphere contribution to the type I effective action. Indeed, the $R^n$ terms there have the dilaton factor $g'_n = b_n' e^{-2\phi'}$. If they were not renormalised, the duality (1.1),(2.2) would map them into $b_n' e^{(n-3)\phi} R^n$ terms in the heterotic action. Since the supersymmetry seems to rule out higher-order terms with odd powers of $R$ this means that all $R^n$ ($n > 1$) terms in the sphere part of the type I action would be mapped into the heterotic string terms multiplied by odd powers of string coupling. Such terms, though presumably consistent with $D = 10$ supersymmetry, cannot appear in the perturbative loop expansion of the heterotic string. Thus all of them should be ‘dressed’ by non-trivial functions of dilaton.

At the same time, all $\text{tr}F^n$ terms in the disc (‘Born-Infeld’) part of the type I action should not receive higher loop corrections [18]. This implies, by duality, that the $\text{tr}F^n$ terms in the heterotic string action should appear only at specific orders of the loop expansion, i.e. should have the following exact dilaton dependence: $a_n e^{(n-4)\phi} \text{tr}F^n$. Like $a_4$, other $a_n$ coefficients probably be given by the contributions of boundaries of moduli spaces, in agreement with their ‘tree-level’ (disc) origin in the type I theory. Each of these terms should be consistent with supersymmetry since the $F^n$-cominations which originate from the expansion of the (abelian) Born-Infeld action are indeed bosonic parts of (global $D = 10$) super-invariants [40].

Thus one expects that $D = 10$ superstring effective actions should contain local higher-derivative terms with coefficients which receive contributions only from certain loop orders and only from boundaries of moduli spaces so that they are explicitly computable. A
A low-dimensional example of such ‘non-renormalisation’ is provided by $N = 2, D = 4$ supersymmetric terms $\sim R^2 F^{g-2}$ in type II theory compactified on a Calabi-Yau space which are generated only at $g$-loop order \cite{16,17}. This follows from the $N = 2$ supersymmetry (the dilaton is a member of a hypermultiplet) and is also related to the fact that the gravi-photon $F$ is a R-R field (so that its power should be correlated with the power of $e^{\phi}$).

It is likely that analogous ‘non-renormalisation theorems’ may apply also directly to type II theory in 10 dimensions. Given that superspace non-renormalisation arguments are currently not applicable in the $D = 10$ case, one may try to use indirect arguments, demanding consistency between string loop expansion and duality combined also with some information about $D = 10$ supersymmetry. In particular, it would be interesting to apply self-duality of the type IIB theory \cite{6,1} to try to determine the dependence of its effective action on the two antisymmetric 2-tensors (NS-NS and R-R ones) which are interchanged under the special duality transformation $\phi' = -\phi$, etc. Since the dilaton dependence of the terms with the R-R field is likely to be fixed, this may also constrain possible dependence on the NS-NS field.

Acknowledgements

I am grateful to E. Bergshoeff, J. Louis, M. de Roo, K. Stelle and E. Witten for important remarks and suggestions. I would like to thank the organizers of the CERN workshop on duality for hospitality and invitation to present a talk. I acknowledge also the support of PPARC, ECC grant SC1*-CT92-0789 and NATO grant CRG 940870.
References

[1] E. Witten, Nucl. Phys. B443 (1995) 85, hep-th/9503124.
[2] A. Dabholkar, Phys. Lett. B357 (1995) 307, hep-th/9506160.
[3] C.M. Hull, Phys. Lett. B357 (1995) 545, hep-th/9506194.
[4] J. Polchinski and E. Witten, “Evidence for heterotic – type I string duality”, IASSNS-HEP-95-81, NSF-ITP-95-135, hep-th/9510169.
[5] P. Horava and E. Witten, “Heterotic and type I string dynamics from eleven dimensions”, IASSNS-HEP-95-86, hep-th/9510209.
[6] C.M. Hull and P.K. Townsend, Nucl. Phys. B438 (1995) 109, hep-th/9410167.
[7] C. Vafa and E. Witten, Nucl. Phys. B447 (1995) 261.
[8] I. Antoniadis, E. Gava, K.S. Narain and T.R. Taylor, Nucl. Phys. B455 (1995) 109, hep-th/9507115.
[9] E. Witten, private communication.
[10] A.A. Tseytlin, Nucl. Phys. B276 (1986) 391.
[11] D.J. Gross and E. Witten, Nucl. Phys. B277 (1986) 1.
[12] M.B. Green, J.H. Schwarz and E. Witten, Superstring Theory (Cambridge U.P., 1988).
[13] J. Ellis and L. Mizrachi, Nucl. Phys. B327 (1989) 595; R. Kallosh and A. Morozov, Phys. Lett. B207 (1988) 164.
[14] W. Lerche, B.E.W. Nilsson and A.N. Schellekens, Nucl. Phys. B289 (1987) 609; W. Lerche, B.E.W. Nilsson, A.N. Schellekens and N.P. Warner, Nucl. Phys. B299 (1988) 91.
[15] O. Yasuda, Phys. Lett. B218 (1989) 455.
[16] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, Commun. Math. Phys. 165 (1994) 31.
[17] I.Antoniadis, E. Gava, K.S. Narain and T.R. Taylor, Nucl. Phys. B413 (1994) 162.
[18] A.A. Tseytlin, “On SO(32) heterotic - type I superstring duality in ten dimensions”, Imperial/TP/95-96/6, hep-th/9510173.
[19] Y. Cai and C. Nunez, Nucl. Phys. B287 (1987) 279; Y. Kikuchi and C. Marzban, Phys. Rev. D35 (1987) 1400.
[20] D.J. Gross and J.H. Sloan, Nucl. Phys. B291 (1987) 41.
[21] J. Ellis, P. Jetzer and L. Mizrachi, Nucl. Phys. B303 (1988) 1.
[22] M. Abe, H. Kubota and N. Sakai, Phys. Lett. B200 (1988) 461; Nucl. Phys. B306 (1988) 405.
[23] W. Lerche, Nucl. Phys. B308 (1988) 102.
[24] M. de Roo, H. Suemmann and A. Wiedemann, Phys. Lett. B280 (1992) 39; Nucl. Phys. B405 (1993) 326.
[25] H. Suemmann, “Supersymmetry and string effective actions”, Ph.D. thesis, Groningen, 1994.
[26] M.B. Green and J.H. Schwarz, Phys. Lett. B149 (1984) 117; Nucl. Phys. B255 (1985) 93.
[27] L. Mizrachi, Nucl. Phys. B338 (1990) 209.
[28] M.B. Green and J.H. Schwarz, Phys. Lett. B151 (1985) 21.
[29] A.A. Tseytlin, Int. J. Mod. Phys. A5 (1990) 589.
[30] E. Bergshoeff and M. de Roo, Nucl. Phys. B328 (1989) 439.
[31] D.J. Gross, J.A. Harvey, E. Martinec and R. Rohm, Nucl. Phys. B256 (1985) 253; B267 (1986) 75.
[32] R.R. Metsaev and A.A. Tseytlin, Phys. Lett. B185 (1987) 52.
[33] R.E. Kallosh, Phys. Scr. T15 (1987) 118; B.E.W. Nilsson and A.K. Tollsten, Phys. Lett. B181 (1986) 63.
[34] M.T. Grisaru, A.E.M. van de Ven and D. Zanon, Nucl. Phys. B277 (1986) 388, 409; M.T. Grisaru and D. Zanon, Phys. Lett. B177 (1986) 347; M.D. Freeman, C.N. Pope, M.F. Sohnius and K.S. Stelle, Phys. Lett. B178 (1986) 199.
[35] J. Ellis and L. Mizrachi, Nucl. Phys. B302 (1988) 65.
[36] E.S. Fradkin and A.A. Tseytlin, Phys. Lett. B158 (1985) 316; Phys. Lett. B160 (1985) 69.
[37] O. Yasuda, Phys. Lett. B215 (1988) 306.
[38] N. Sakai and Y. Tani, Nucl. Phys. B287 (1987) 457.
[39] C.G. Callan, C. Lovelace, C.R. Nappi and S.A. Yost, Phys. Lett. B206 (1988) 41; Nucl. Phys. B308 (1988) 221.
[40] E. Bergshoeff, M. Rakowski and E. Sezgin, Phys. Lett. B185 (1987) 371; R.R. Metsaev and M.A. Rahmanov, Phys. Lett. B193 (1987) 202; R.R. Metsaev, M.A. Rahmanov and A.A. Tseytlin, Phys. Lett. B193 (1987) 207.