Family of solutions for axisymmetric electrovacuum Einstein-Maxwell field equations

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Abstract

We present a family of solutions for the axisymmetric Plebanski-Demianski metric and other corresponding reduced metrics. We also present the black hole characteristics using a new set of parameters for Kerr-Newman metric.

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I. INTRODUCTION

In the last four decades there has been an extraordinary progress in solving the Einstein-Maxwell equations with two Killing vectors \[1–6\]. The most important solutions have been related to charged black holes, i.e., Reissner-Nordström static black hole and Kerr-Newman rotating black hole. In addition, it is well known the solution of Plebanski-Demianski (PD) \[1\] metric, containing the last solutions as particular cases; the PD solution is the most general solution in the Petrov classification type D. Besides, recently, these solutions were generalized in the metric affine gravity \[7–9\].

The present work is the result of an investigation that we did using computer algebra methods of the family of solutions of Einstein-Maxwell equations with cosmological constant \(\lambda\)

\[
G_{\mu\nu} + \lambda g_{\mu\nu} - \kappa T_{\mu\nu} = 0. \tag{1}
\]

In particular, we have extensively used REDUCE 3.6 \[10\].

The Maxwell equations can be written in tensorial form as

\[
\partial_\beta F^{\alpha\beta} = j^\alpha, \tag{2}
\]

\[
\partial_\alpha F_{\beta\theta} + \partial_\theta F_{\alpha\beta} + \partial_\beta F_{\alpha\theta} = 0, \tag{3}
\]

where \(F^{\alpha\beta}\) is the Maxwell tensor

\[
F_{ab} = \partial_b A_a - \partial_a A_b, \tag{4}
\]

with the quadripotential \(A^\alpha\), and \(j^\alpha\) the current density quadrivector (in this work we take \(j^\alpha = 0\)).

II. STRUCTURE OF THE PD METRIC

Following reference \[1\], we consider the class of space-times with signature \((-,+,+,+\)) in real generalized coordinates \(\chi^\alpha(\tau, q, p, \sigma)\) with the line element given by

\[
ds^2 = g_{00}(\chi^\alpha) d\tau^2 + 2g_{03}(\chi^\alpha) d\tau d\sigma + g_{11}(\chi^\alpha) dq^2 + g_{22}(\chi^\alpha) dp^2 + g_{33}(\chi^\alpha) d\sigma^2, \tag{5}
\]

where we take only the following two-coordinate dependence of \(\chi^\alpha = \chi^\alpha(q, p)\). The PD metric has a similar structure \[1\]

\[
ds^2 = \frac{1}{H^2} \left\{ -\frac{P}{\Delta} (d\tau - p^2 d\sigma)^2 + \frac{\Delta}{Q} dq^2 + \frac{\Delta}{P} dp^2 + \frac{\Delta}{\Delta} (d\tau + q^2 d\sigma)^2 \right\}, \tag{6}
\]

or

\[
ds^2 = \frac{P - Q}{H^2 \Delta} d\tau^2 + \frac{2(p^2 Q + q^2 P)}{H^2 \Delta} d\tau d\sigma + \frac{\Delta}{H^2 Q} dq^2 + \frac{\Delta}{H^2 P} dp^2 + \frac{q^4 P - p^4 Q}{H^2 \Delta} d\sigma^2. \tag{7}
\]

The structure functions \(P\) and \(Q\) depend on the coordinates \(p\) and \(q\), respectively. The functions \(H\) and \(\Delta\) are defined as \(H = 1 - pq\) and \(\Delta = p^2 + q^2\), respectively. This space-time
has two Killing vectors \( \partial/\partial \tau \) and \( \partial/\partial \sigma \) that commute between them. The vector \( \partial/\partial \tau \) is timelike in the region where \( Q - P > 0 \), and the space-time is stationary there.

The real electromagnetic potential is given by

\[
A^\alpha = \frac{1}{\Delta} ((e_0 q + g_0 p), 0, 0, (g_0 p - e_0 p)pq),
\]

(8)

where \( e_0 \) and \( g_0 \) are the electric- and magnetic-like charges, respectively.

It is well known that the structure functions \( P \) and \( Q \) are polynomials of fourth degree, whose coefficients, depending on seven parameters, are classified as mass \( m \), NUT parameter \( n \), angular momentum \( j_0 \), acceleration parameter \( b \), cosmological constant \( \lambda \), and the charges \( e_0, g_0 \).

\[
P = (b - g_0^2) + 2np - \epsilon p^2 + 2mp^3 - \left( b + e_0^2 + \frac{\lambda}{3} \right) p^4,
\]

(9)

\[
Q = (b + e_0^2) - 2mq + \epsilon q^2 - 2nq^3 - \left( b - g_0^2 + \frac{\lambda}{3} \right) q^4,
\]

(10)

where \( b \to \gamma - \frac{\lambda}{6} \).

We mention that recently the following generalization of this metric has been found by García and Macías [3], including a new parameter \( \mu \) (new acceleration parameter)

\[
P = (b - g_0^2) + 2np - \epsilon p^2 + 2m\mu p^3 - \left[ \mu^2 (b + e_0^2) + \frac{\lambda}{3} \right] p^4,
\]

(11)

\[
Q = (b + e_0^2) - 2mq + \epsilon q^2 - 2n\mu q^3 - \left[ \mu^2 (b - g_0^2) + \frac{\lambda}{3} \right] q^4,
\]

(12)

\[
H = 1 - \mu p q,
\]

\[
\Delta = p^2 + q^2.
\]

(13)

(14)

III. FAMILY OF SOLUTION FOR PD METRIC

We present now the family of solutions for Einstein-Maxwell equations with cosmological constant \( \lambda \)

\[
G_{\mu \nu} + \lambda g_{\mu \nu} - \kappa T_{\mu \nu} = 0.
\]

(15)

Using the program for Excalc package [11] in REDUCE 3.6, we can introduce the following ansatz for the polynomials \( P(p) \) and \( Q(q) \)

\[
P(p) = p_0 + p_1 p + p_2 p^2 + p_3 p^3 + p_4 p^4,
\]

(16)

\[
Q(q) = q_0 + q_1 q + q_2 q^2 + q_3 q^3 + q_4 q^4.
\]

(17)

Via computer algebra we obtained the following relations
\[ q_0 - p_0 = e_0^2 + g_0^2, \]  
(18)  
\[ p_3 = -\mu q_1, \]  
(19)  
\[ q_3 = -\mu p_1, \]  
(20)  
\[ p_2 = -q_2, \]  
(21)  
\[ p_4 = -\frac{\lambda}{3} - \mu^2 q_0, \]  
(22)  
\[ q_4 = p_4 + \mu^2(e_0^2 + g_0^2). \]  
(23)

At this point we generalize the relation (18) by introducing two new real parameters, \( \alpha \) and \( \beta \), as follows

\[ q_0 - p_0 = \left[ b + (\alpha + 1)e_0^2 + \beta g_0^2 \right] - \left[ b + \alpha e_0^2 + (\beta - 1)g_0^2 \right] = e_0^2 + g_0^2. \]  
(24)

Finally, the family of solutions given via the structure functions \( P \) and \( Q \), can be written by taking into account the structure function for the PD metric

\[ P(p) = \left[ b + \alpha e_0^2 + (\beta - 1)g_0^2 \right] + 2np - \epsilon p^2 + 2m\mu p^3 \]
\[ - \left[ \mu^2(b + (\alpha + 1)e_0^2 + \beta g_0^2) + \frac{\lambda}{3} \right] p^4, \]  
(25)  
\[ Q(q) = \left[ b + (\alpha + 1)e_0^2 + \beta g_0^2 \right] - 2mq + \epsilon q^2 - 2n\mu q^3 \]
\[ - \left[ \mu^2(b + \alpha e_0^2 + (\beta - 1)g_0^2) + \frac{\lambda}{3} \right] q^4, \]  
(26)  
\[ H = 1 - \mu pq, \]  
(27)  
\[ \Delta = p^2 + q^2. \]  
(28)

The PD metric corresponds to the following values in the new parameters: \( \mu = 1, \alpha = 0 \) and \( \beta = 0 \).

This family of solutions have the property that they remain solutions under the inversion transformation

\[ q \rightarrow -\frac{1}{q} \]  
(29)

performed in both the structure functions and the electromagnetic potential. In addition, we can find the cylindrically symmetric counterpart of the metric ([8]) in similar form to García et. al, [8].

**IV. ROOTS AND SINGULARITIES IN THE PD GENERALIZED**

When we consider the problem of obtaining the roots and singularities in this generalized metric, via calculating the invariants \( R, R_{\alpha\beta}, R^\alpha_{\beta\delta}, R^\alpha_{\beta\delta\gamma} \), we found an intrinsic singularity for \( \Delta = p^2 + q^2 = 0 \). Under the coordinate transformation \( (q \rightarrow r, p \rightarrow -j_0 \cos \theta) \), the function \( \Delta = 0 \) gives the corresponding ring Kerr singularity, where the new parameters \( \alpha \) and \( \beta \) do not modify this property.
The roots in the polynomials $P = 0$ (25) and $Q = 0$ (26) represent singularities in the coordinates, and therefore they are removable via an adequate coordinate transformations. The singularities in the coordinates, show us some deficiencies in the use of these coordinate systems to describe the space-time under consideration.

The roots in the polynomial $P = 0$ do not have any physical interest, and we omit them. However, the real roots in the polynomial $Q = 0$, represent null hypersuperfaces of event horizon type with $q = \text{constant}$.

The solution for $Q = 0$ is obtained via conventional methods [12]. Consider the following form

$$[b + (\alpha + 1)e_0^2 + \beta g_0^2] - 2mq + \epsilon q^2 - 2n\mu q^3 - \left[\mu^2(b + \alpha e_0^2 + (\beta - 1)g_0^2) + \frac{\lambda}{3}\right]q^4 = 0, \quad (30)$$

or,

$$q^4 + a_3q^3 + a_2q^2 + a_1q + a_0 = 0, \quad (31)$$

where

$$a_3 = \frac{2n\mu}{\mu^2(b + \alpha e_0^2 + (\beta - 1)g_0^2) + \frac{\lambda}{3}},$$

$$a_2 = 2m \frac{1}{\mu^2(b + \alpha e_0^2 + (\beta - 1)g_0^2) + \frac{\lambda}{3}},$$

$$a_1 = \frac{\mu^2(b + \alpha e_0^2 + (\beta - 1)g_0^2) + \frac{\lambda}{3}}{\mu^2(b + \alpha e_0^2 + \beta g_0^2)}.$$  

Using the Ferrari method for equations of fourth degree and the Cardano method for those of third degree [12], we can get the following roots

$$q_{a,b} = \frac{1}{2}(-A \pm B^{1/2}), \quad q_{c,d} = \frac{1}{2}(-C \pm D^{1/2}), \quad (32)$$

where

$$A = \left[\frac{a_3}{2} - \frac{a_3^2}{3A} + a_2 - (t_1)^{1/3} - (t_2)^{1/3}\right],$$

$$B = \left\{A^2 - 2\left(\frac{t_1}{3} + (t_2)^{1/3} - \frac{1}{2}a_3[(t_1)^{1/3} + (t_2)^{1/3} - a_1] \left(\frac{a_3}{2} - A\right)^{1/2}\right)\right\},$$

$$C = \left[\frac{a_3}{2} + \frac{s_3^2}{4} - a_2 + (t_1)^{1/3} + (t_2)^{1/3}\right],$$

$$D = \left\{C^2 - 2\left(\frac{t_1}{3} + (t_2)^{1/3} + \frac{1}{2}a_3[(t_1)^{1/3} + (t_2)^{1/3} - a_1] \left(\frac{a_3}{2} - A\right)^{1/2}\right)\right\},$$

$$t_1 = -\frac{u}{2} + \left(\frac{u^2}{4} + \frac{v^3}{27}\right)^{1/2}, \quad t_2 = u - t_1,$$
\[ u = b_1 - \frac{b_2^2}{3}, \quad v = b_0 - \frac{b_1 b_2}{3} + \frac{2b_2^2}{27}, \]

\[ b_0 = \frac{4}{q_4^2} \left( e q_0 q_4 + n^2 \mu^2 q_0 - 4 m^2 q_3 \right), \]

\[ b_1 = \frac{4}{q_4^2} \left( m n \mu + 4 q_0 q_4 \right), \quad b_2 = \frac{\epsilon}{q_4}, \]

\[ q_0 = \left[ b + \alpha e_0^2 + (\beta - 1) g_0^2 \right], \]

\[ q_4 = \left[ \mu^2 (b + (\alpha + 1) e_0^2 + \beta g_0^2) + \frac{\lambda}{3} \right]. \]

The polynomial \( Q \) (30) can be rewritten as

\[ q_4 \left\{ q^4 + a_3 q^3 + a_2 q^2 + a_1 q + a_0 \right\} = q_4 (q - q_I)(q - q_{II})(q - q_{III})(q - q_{IV}), \] (33)

where \( q_I, q_{II}, q_{III} \) and \( q_{IV} \) are the roots (32) in ascendent order of magnitude, satisfying the following relations

\[ \sum_{i=1}^{IV} q_i = -a_3 = -\frac{2n\mu}{[\mu^2 (b + \alpha e_0^2 + (\beta - 1) g_0^2) + \frac{\lambda}{3}]}, \] (34)

\[ \sum_{i>j} q_i q_j = a_2 = -\frac{\epsilon}{\mu^2 (b + \alpha e_0^2 + (\beta - 1) g_0^2) + \frac{\lambda}{3}}, \] (35)

\[ \sum_{i>j>k} q_i q_j q_k = -a_1 = -\frac{2m}{\mu^2 (b + \alpha e_0^2 + (\beta - 1) g_0^2) + \frac{\lambda}{3}}, \] (36)

\[ \Pi_i q_i = a_0 = -\frac{b + (\alpha + 1) e_0^2 + \beta g_0^2}{\mu^2 (b + \alpha e_0^2 + (\beta - 1) g_0^2) + \frac{\lambda}{3}}. \] (37)

where \( a_i, i = 0, 1, 2, 3 \) are the coefficients of \( Q \).

Also, we can obtain red-shifted hypersurfaces for this generalized solution, which are given by the equation (13)

\[ g_{00} = \frac{P - Q}{H^2 \Delta} = 0. \] (38)

The roots of (38) are analogue in form to those in equations (32), except that the coefficient \( a_0 \) is now a function of the coordinate \( p \).

V. REDUCED FAMILY OF SOLUTIONS FROM THE FAMILY OF SOLUTIONS TO PD METRIC

Certain family of reduced solutions can be obtained from the family of solutions of the PD metric when we drop appropriately the new parameters and perform the coordinate transformations from \( (\tau, p, q, \sigma) \rightarrow (other \ coordinates) \) as used in García [2]. We give these reduced generalized metrics in table I. All of them correspond to stationary axisymmetric space-times with Maxwell field as a matter source.
\[ ds^2 = \frac{1}{r^2} \left\{ -\frac{\Delta}{\lambda^2} (d\tau - p^2 d\sigma)^2 + \frac{\Delta}{\lambda} dq^2 + \frac{\Delta}{\lambda} dp^2 + \frac{\Delta}{\lambda} (d\tau + q^2 d\sigma)^2 \right\} \]

Electromagnetic potential
\[ A = \frac{1}{\lambda} ((e_0 q + g_0 p), 0, 0, (g_0 p - e_0 q)pq) \]

| Metric | structure functions |
|--------|---------------------|
| PD generalized (10 parameters) | \[ \mathcal{P} = \left[ b + c_0 e_0^2 + (\beta - 1) g_0^2 \right] + 2np - \epsilon^2 p^2 + 2m \mu p^4 \] |
| | \[ \mathcal{Q} = \left[ b + (\alpha + 1) c_0^2 + \beta g_0^2 \right] - 2mq + \epsilon q^2 - 2n \mu q^3 \] |
| | \[ \Delta = p^2 + q^2 \] |
| | \[ H = 1 - \mu pq \] |
| PD standard (\( \mu = 1, \alpha = 0, \beta = 0 \)) | \[ \mathcal{P} = (b - g_0^2) + 2np - \epsilon^2 p^2 + 2m \mu p^4 \] |
| | \[ \mathcal{Q} = (b + e_0^2) - 2mq + \epsilon q^2 - 2n \mu q^3 \] |
| | \[ \Delta = p^2 + q^2 \] |
| | \[ H = 1 - pq \] |
| PC generalized (\( \mu = 0 \)) | \[ \mathcal{P} = \left[ b + c_0 e_0^2 + (\beta - 1) g_0^2 \right] + 2np - \epsilon^2 p^2 - \frac{4}{3} p^4 \] |
| | \[ \mathcal{Q} = \left[ b + (\alpha + 1) c_0^2 + \beta g_0^2 \right] - 2mq + \epsilon q^2 - \frac{4}{3} q^4 \] |
| | \[ \Delta = p^2 + q^2 \] |
| | \[ H = 1 \] |
| KN generalized (\( \mu = 0, n = 0, \lambda \neq 0 \) \( b \rightarrow j_0^2, \epsilon \rightarrow (1 - \lambda j_0^2) \)) | \[ \mathcal{P} = \left[ j_0^2 + c_0 e_0^2 + (\beta - 1) g_0^2 \right] - 1 - \frac{\lambda j_0^2}{3} \] |
| | \[ \mathcal{Q} = \left[ j_0^2 + (\alpha + 1) c_0^2 + \beta g_0^2 \right] - 2mq + \epsilon q^2 \] |
| | \[ \Delta = p^2 + q^2 \] |
| | \[ H = 1 \] |
| KN generalized (\( \mu = 0, n = 0, \lambda = 0, b \rightarrow j_0^2, \epsilon \rightarrow 1 \)) | \[ \mathcal{P} = j_0^2 - p^2 \] |
| | \[ \mathcal{Q} = \left( j_0^2 + c_0^2 + g_0^2 \right) - 2mq + q^2 \] |
| | \[ \Delta = p^2 + q^2 \] |
| | \[ H = 1 \] |
| KN standard \( \mu = 0, n = 0, \lambda = 0, \alpha = 0, \beta = 1, b \rightarrow j_0^2, \epsilon \rightarrow 1 \) | \[ \mathcal{P} = j_0^2 - p^2 \] |
| | \[ \mathcal{Q} = (j_0^2 + c_0^2 + g_0^2) - 2mq + q^2 \] |
| | \[ \Delta = p^2 + q^2 \] |
| | \[ H = 1 \] |

Table I

Generalization and standard cases contained in the PD solutions

VI. WHAT TO DO THE NEW PARAMETERS \( \alpha \) AND \( \beta \)?

In order to study the features of the new parameters \( \alpha \) and \( \beta \), we take the KN metric as an example. As it is known, the exterior gravitational field of a black hole bounded by an event horizon is characterized by three parameters: mass \( m \), angular momentum \( j_0 \), and the electric charge \( e_0 \).

When we take the Boyer-Lindquist coordinate transformation in the generalized PD metric, we obtain the generalized KN metric (with \( \lambda = 0 \)) through the transformation
\[ T : \{ \tau \rightarrow t - j_0 \phi, q \rightarrow r, p \rightarrow -j_0 \cos \theta, \sigma \rightarrow -\phi/j_0 \}, \]  

where now the space-time have the following coordinate \( \chi^\mu = (t, r, \theta, \phi) \), and the explicit form of the metric is

\[
ds^2 = \frac{P_{KN} - Q_{KN}}{\Delta_{KN}} dt^2 - \frac{2}{a \Delta_{KN}} \left\{ (j_0^2 + r^2)P_{KN} - j_0^2 \sin^2 \theta Q_{KN} \right\} dt d\phi + \frac{\Delta_{KN}}{Q_{KN}} dr^2
\]

\[ + \frac{j_0^2 \sin^2 \theta \Delta_{KN}}{P_{KN}} d\theta^2 + \frac{1}{j_0^2 \Delta_{KN}} \left\{ (j_0^2 + r^2)^2P_{KN} - j_0^4 \sin^4 \theta Q_{KN} \right\} d\phi^2, \]

with the structure functions \( P_{KN}, Q_{KN} \) and \( \Delta_{KN} \) given by

\[
P_{KN} = \left[ \alpha e_0^2 + (\beta - 1)g_0^2 \right] + j_0^2 \sin^2 \theta, \]

\[
Q_{KN} = \left[ j_0^2 + (\alpha + 1)e_0^2 + \beta g_0^2 \right] - 2mr + r^2, \]

\[
\Delta_{KN} = r^2 + j_0^2 \cos^2 \theta. \]

The hypersurfaces of infinite shift would satisfy the condition

\[ g_{tt} = \frac{P_{KN} - Q_{KN}}{\Delta_{KN}} = 0, \]

implying that there exist two such hypersurfaces at the radial distance

\[ r = m \pm \left[ m^2 - (e_0^2 + g_0^2 + j_0^2 \cos^2 \theta) \right]^{1/2}, \]

where the equation

\[ 0 < e_0^2 + g_0^2 + j_0^2 \leq m^2, \]


gives us the condition for non existence of imaginary values and of negative radius.

Examining the quadratic invariants, we find an intrinsic ring singularity for \( \Delta_{KN} = 0 \) that is similar to Kerr’s singularity

\[ \Delta_{KN} = r^2 + j_0^2 \cos^2 \theta = 0. \]

As is usual, the event horizon can be calculated from the equation \( Q_{KN} = 0 \), leading to

\[ r = m \pm \left[ m^2 - (\alpha + 1)e_0^2 + \beta g_0^2 + j_0^2 \right]^{1/2}. \]

We can conclude that the parameters \( \alpha \) and \( \beta \) modify the radius of the event horizon of the generalized solution of KN metric. We investigate this modification in the following.

**A. Parametrization of the event horizon in the KN space time**

The ideal case under consideration corresponds to the equation (48), where there are no event horizon with imaginary or negative radius. Therefore, we have the condition

\[ 0 < (\alpha + 1)e_0^2 + \beta g_0^2 + j_0^2 \leq m^2 \]
or

$$- (e_0^2 + j_0^2) < \alpha e_0^2 + \beta g_0^2 \leq m^2 - (e_0^2 + j_0^2).$$  \hfill (50)

- (e_0^2 + j_0^2) < \alpha e_0^2 + \beta g_0^2 \leq m^2 - (e_0^2 + j_0^2).

Fig. 1.: Family of solutions to the equation (50).

The set of solutions shown in Figure 1 represents a family of straight lines of slope \( \gamma = -1 \). If the values for \( \alpha \) and \( \beta \) given at the points \( P_2 \) or \( P_3 \) are introduced in the metric (40), the standard KN solution and the corresponding event horizons are recovered. In fact, any value for \( \alpha \) and \( \beta \) coinciding with a straight line crossing through the points \( P_2 \) and \( P_3 \) will lead to the standard KN metric. Then, the family of parallel straight lines given by the equation (50) represents different KN space-times for each case, each with its own event horizon.

Any straight line that crosses the solution range of \( \alpha \) and \( \beta \), intersecting the family of parallel lines of slope \( \gamma \), will lead to all spacetimes found in this range of solutions. Then we suggest that there is a simplification in the set of parameters if we take the straight line as the axis \( \alpha (\beta = 0) \) and introduce a new parameter \( \nu \) that reruns this set of solutions. We propose the following form of this straight line

$$e_0^2 \alpha = m^2 \nu - (e_0^2 + j_0^2), \quad 0 < \nu \leq 1.$$  \hfill (51)

Thus, the new structure functions are

- \( \mathcal{P}_{KN} = [m^2 \nu - (e_0^2 + g_0^2 + j_0^2)] + j_0^2 \sin^2 \theta, \hfill (52) \)
- \( \mathcal{Q}_{KN} = m^2 \nu - 2m r + r^2, \hfill (53) \)
- \( \Delta_{KN} = r^2 + j_0^2 \cos^2 \theta, \hfill (54) \)

where \( \nu \) satisfies equation (51).

All black hole properties mentioned above are maintained with the inclusion of this new parameter and the important point is that this solution, it not violate no-hair theorem.

The equation \( \mathcal{Q}_{KN} = 0 \) for the structure function (54) determines the event horizons whose radius is given by

$$r_{\pm} = m [1 \pm \sqrt{1 - \nu}], \quad 0 < \nu \leq 1.$$  \hfill (55)
Therefore, from the range of $\nu$ we get the following values for the interior $r_−$ and exterior $r_+$ radii

$$ 0 < r_− \leq m, \quad m \leq r_+ < 2m. \quad (56) $$

One can see that the corresponding solutions are proportional to the geometric mass. The results given in equations (55) - (56) represent the parametrization for the event horizon of KN type black hole, where each value of $\nu$ gives a different space-time with the interior and exterior horizon.

**B. Quantitative analysis to the KN space-times and its event horizons**

It is easy to see that with $\nu = (e_0^2 + g_0^2 + j_0^2)/m^2$ we can recover the standard KN space-time.

Equation (46) gives us two ideal cases

$$ 0 < e_0^2 + g_0^2 + j_0^2 < m^2, \quad (57) $$

and

$$ e_0^2 + g_0^2 + j_0^2 = m^2. \quad (58) $$

The set of solutions for the KN spacetimes taking into account the parameter $\nu$, are shown in figure 2.

To analyse this figure, we can divide it in two regions

$$ I : 0 < \nu \leq (e_0^2 + g_0^2 + j_0^2)/m^2 $$

$$ II : (e_0^2 + g_0^2 + j_0^2)/m^2 \leq \nu \leq 1. \quad (59) $$

The first region I is not physical, because the event horizons cross the infinite redshift surface in different points of the rotation axis, and the corresponding angles of intersection for both surfaces violate the range for the function $\sin^2 x$ ($0 \leq \sin^2 x \leq 1$).

The region II describes the physical parametrization, because it contains the family of KN space-times with their event horizons of radii.
\[ r_{KN} = m[1 \pm \sqrt{1 - \nu}] . \] (60)

The set of spacetimes corresponding to the value \( \nu = 1 \), represents non-extreme black holes with only one event horizon at \( r_{KN} = m \).

**VII. CONCLUSIONS**

Although, we dealt with a difficult algebra we have been able to obtain a generalization of the PD metric by extensive use of the REDUCE 3.6 system. The latter could be used in any available computer resources.

We introduced a PD metric structure with 10 parameters and an appropriate reduction of parameters in the generalized PD metric leads to obtain two new families of solutions to the Plebanski-Carter and Kerr-Newman metrics, containing the new parameters \( \alpha \) and \( \beta \). We point out that the introduction of the parameters \( \alpha \) and \( \beta \) does not modify the generalized solutions when the electromagnetic charge parameters \( e_0 \) and \( g_0 \) are interchanged, as it does in the structure functions and the electromagnetic potential considering all their possible combinations.

Interpretation of the generalized KN metric as the exterior field of a collapsed star surrounded by an event horizon, i.e., as a black hole, has lead to a new equivalent generalized metric, but now simplified with only one dimensionless parameter called \( \nu \). In addition, we have determined the way \( \nu \) modifies the radius of event horizon such that the corresponding solutions are proportional to the geometric mass. On the other hand, the redshift surfaces remain constant and independent of \( \nu \).

The minimum value of the parameter \( \nu \) corresponds to the space-time of a standard KN type black hole. Increments in the value of \( \nu \) increase the region between the redshift surfaces and the event horizons (the ergospheres) until the maximum value \( \nu = 1 \). This maximum value corresponds to the space-time of a KN type black hole with only one event horizon which is not necessarily an extreme black hole. The space-time of a standard extreme KN type black hole can also be recovered.

Dropping the electromagnetic charge parameters allows to get a generalized Kerr solution that corresponds to a vacuum stationary axisymmetric solution of Einstein equations. The analysis and interpretation as a black hole and parametrization of the event horizons is completely analogous to the generalized KN case.

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