1 Introduction

The study of quantum field theory (QFT) in (1 + 1)-dimensions provides valuable insights into many difficult conceptual problems of QFT’s, as well as giving us situations where appropriate use of infinite dimensional symmetries, like conformal invariance and $W_\infty$ symmetry, considerably simplifies the field theoretic computation. One might ask if similar insights could also be obtained by studying QFT’s in (1 + 1)-dimensional noncommutative spacetime. Indeed there is reason for some optimism regarding this query, as it is possible to define notions of conformal invariance, Kac-Moody and Virasoro symmetries [1, 2].

In this article we will address the question of quantum integrability, and investigate the noncommutative analog of the quantum sinh-Gordon model. We will argue that it is integrable, in the sense that there is no particle production, and calculate the exact two-particle scattering matrix.

In Section 2, we will summarize the results on twisted QFT’s based on our earlier work [3, 4], and then review scattering theory for noncommutative QFT’s [5]. In Section 3, we will recall some essential features of the quantum sinh-Gordon model on commutative spacetime, and then go on to construct the two-particle $S$-matrix $S_\Theta$ for its noncommutative counterpart. A discussion of the properties of $S_\Theta$ will be provided in Section 4.

2 Noncommutative (1 + 1)-d spacetime

Two dimensional noncommutative spacetime is generated by operators $\hat{x}_\mu$ satisfying the commutation relations

$$[\hat{x}_\mu, \hat{x}_\nu] = i\Theta_{\mu\nu} \equiv i\Theta\epsilon_{\mu\nu}, \quad \mu, \nu = 0, 1,$$

where $\hat{x}_0, \hat{x}_1$ are hermitian operators, and $\Theta$ is a real constant.

Using the Moyal map, we can map operators built out of (the products of) the $\hat{x}_\mu$’s to functions on Minkowski space $\mathbb{R}^{1,1}$, but with a modified rule for multiplication. If $\hat{f}$ and $\hat{g}$ are two operators that are mapped to functions $f(x)$ and $g(x)$ respectively, then their star-product consistent with (2.1) is

$$(f * g)(x) = f(x)e^{\frac{i}{2}\Theta_{\mu\nu}\overleftarrow{\partial}_\mu \overrightarrow{\partial}_\nu}g(x).$$

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For example, for $e_p(x) = e^{-i p \cdot x}$,

$$e_p(x) \ast e_q(x) = e^{-\frac{i}{2} p \cdot q} e_{p+q}(x), \quad p \wedge q \equiv p^\mu \Theta_{\mu\nu} q^\nu.$$  

Notice that the commutation relation (2.1) is unchanged under translations $\hat{x}_\mu \to \hat{x}_\mu + a_\mu$ and identity-connected Lorentz transformations $\hat{x}_\mu \to \Lambda_\mu^\nu \hat{x}_\nu$. Thus Poincaré transformations, with their usual action on coordinates, belong to the automorphism group of noncommutative $\mathbb{R}^{1,1}$.

Our interest in this noncommutative space is to understand the notion of integrability for quantum theories. As long as the Hamiltonian is (formally) hermitian, the quantum theory on such a space is manifestly unitary, as emphasized in the case of field theories by [7], and demonstrated for single particle quantum mechanics by [8]. We will therefore proceed to write free quantum fields, which we will use to build interactions.

### 2.1 Quantum Fields

A free quantum scalar field $\Phi(x)$ can be expanded as

$$\Phi(x) = \int d\mu(k)(a_k e^{-ik \cdot x} + a_k^\dagger e^{ik \cdot x}), \quad d\mu(k) = \frac{dk}{4 \pi k^0}, \quad k^0 = \sqrt{(k^1)^2 + M^2}.$$  

While it is possible to quantize the field by imposing the standard (or canonical) commutation relations between $a_k$'s and $a_k^\dagger$'s, more general commutation relations are possible as well [3]. These twisted commutation relations are of the form:

$$a_p a_q = G_{p,q} a_q a_p, \quad a_p a_q^\dagger = G_{p,-q} a_q^\dagger a_p^\dagger + 2p_0 \delta(\vec{p} - \vec{q}),$$  
$$a_p^\dagger a_q^\dagger = G_{-p,-q} a_q a_p^\dagger$$

where $G_{p,q}$ is any Lorentz-invariant function of two-momenta $p$ and $q$.

In [4], we argued that in spacetime dimension greater than 2, compatibility with quantum statistics of identical particles requires $G_{p,q}$ to be of the form

$$G_{p,q} = e^{ip \wedge q}.$$  

We will work henceforth with this choice of $G_{p,q}$, and argue in Section 3 why this leads to integrability of the $S$-matrix for the noncommutative sinh-Gordon model.

Even though the commutation relations (2.5) are twisted, the operators $a_p, a_p^\dagger$ act on the usual (bosonic) Fock space. To see this, let $c_p^\dagger, c_q$ be the ordinary or untwisted creation and annihilation operators:

$$[c_p, c_q] = 0, \quad [c_p, c_q^\dagger] = 2p_0 \delta(\vec{p} - \vec{q}).$$  

There is a simple relation between the $c_p$'s and the $a_p$'s (for $G_{p,q} = e^{ip \wedge q}$). To this end, consider the Fock space momentum operator $P_\mu$ associated with $\Phi(x)$:

$$P_\mu = \int d\mu(p)p_\mu a_p^\dagger a_p,$$  
$$[P_\mu, \Phi(x)] = -i \partial_\mu \Phi(x)$$  

(2.11)
Using $P_\mu$, we can realize the twisted creation-annihilation operators in terms of the untwisted ones:

$$a_p = c_p e^{-\frac{1}{2}p^\mu P_\mu}, \quad (2.12)$$

$$a_p^\dagger = c_p^\dagger e^{\frac{1}{2}p^\mu P_\mu}, \quad (2.13)$$

It is easy to check that (2.12, 2.13) reproduce the twisted commutation relations (2.5–2.7). Thus the $a_p$’s act on the same Fock space as that of the untwisted creation and annihilation operators. The map (2.12,2.13) is a “dressing transformation” (first discussed in [9]), and the commutation relations for the $a_p$’s and $a_p^\dagger$’s that follow from it are simple examples of the algebra discussed in [10, 11].

A twisted $n$-particle state is

$$|p_1, p_2, \cdots p_n\rangle_\Theta = a_{p_1}^\dagger \cdots a_{p_n}^\dagger |0\rangle \quad (2.14)$$

This is related very simply to the ordinary $n$-particle state. Using (2.13),

$$|p_1, p_2, \cdots p_n\rangle_\Theta = e^{\frac{i}{2} \sum \langle i < j p_i \wedge P \rangle} |p_1, p_2, \cdots p_n\rangle_0 \quad (2.15)$$

In (1+1) dimensions, it is often convenient to work in light-cone coordinates, with the two-momentum characterized by rapidity $\eta$: $(p^0, p^1) = (M \cosh \eta, M \sinh \eta)$. The twisted commutation relation (2.7) then becomes

$$a^\dagger (\eta_1) a^\dagger (\eta_2) = e^{-i \Theta M^2 \sinh(\eta_1 - \eta_2)} a^\dagger (\eta_2) a^\dagger (\eta_1) \quad (2.16)$$

Finally, the quantum field $\Phi(x)$ even though free, is not local: the commutator $[\Phi(x), \Phi(y)]$ is non-zero for $x$ and $y$ space-like separated [4].

### 2.2 Noncommutative Scattering Theory

Consider a theory with interaction Hamiltonian

$$H_I = g \int dx \Phi^n_\ast, \quad \Phi^n_\ast = \Phi(x) \ast \Phi(x) \cdots \Phi(x) \quad (2.17)$$

The scattering operator for this theory is

$$S_\Theta = T e^{-i \int H_I dt} \quad (2.18)$$

The first non-trivial term in the perturbative expansion of the above is

$$S^{(1)}_\Theta = -i g \int d^2 x \Phi^n_\ast \quad (2.19)$$

Using the mode expansion for $\Phi(x)$, let us look at a typical term in the above, which is of the form $-i g \int d^2 x a_{p_1} \cdots a_{p_n} e_{p_1}(x) \cdots e_{p_n}(x)$. Using (2.13) and (2.3),

$$-i g \int d^2 x a_{p_1} \cdots a_{p_n} e_{p_1}(x) \cdots e_{p_n}(x) \quad (2.20)$$

$$= -i g \int d^2 x c_{p_1} \cdots c_{p_n} e^{-\frac{1}{2} \sum \langle i < j p_i \wedge P \rangle} e^{-\frac{1}{2} \sum \langle i < j p_i \wedge p_j \rangle} e_{p_1+p_2+\cdots p_n}(x) e^{-\frac{1}{2} \sum \langle i < j p_i \wedge p_j \rangle} \quad (2.21)$$

$$= -i g \int d^2 x c_{p_1} \cdots c_{p_n} e_{p_1+p_2+\cdots p_n}(x) e^{\frac{1}{2} \sum \langle i < j p_i \wedge P \rangle} \quad (2.22)$$

$$= -i g \int d^2 x c_{p_1} \cdots c_{p_n} e_{p_1}(x) \cdots e_{p_n}(x) \quad (2.23)$$
where we have integrated by parts and discarded surface terms to obtain the last expression. But (2.23) is just the same as the corresponding term from the commutative scattering theory. Hence to order $g$, the noncommutative scattering operator is the same as in the commutative theory.

More generally, this is true to any order in perturbation theory (see [5] for details of the proof). In particular, this means that
\[ S_\Theta = S_0. \] (2.24)

The scattering operator on noncommutative space is the same as that for the commutative counterpart. In particular, if a commutative theory is renormalizable, so is its noncommutative counterpart, because the number of counterterms is the same.

As far as scattering is concerned, the main difference between a commutative theory and its noncommutative counterpart is in the nature of asymptotic states. But since we know the explicit map (2.15) between these two kinds of asymptotic states, the matrix elements of $S_\Theta$ can be calculated in terms of those of $S_0$.

The result (2.24) is true for theories without interacting non-Abelian gauge fields only [12]. For example, in noncommutative QCD there are effects that violate Lorentz invariance. However, this caveat is not applicable here, as we will only consider theories where the interaction Hamiltonian is made up of matter fields only.

### 3 Noncommutative sinh-Gordon Model

One of the simplest non-trivial integrable model in (1 + 1)-dimensional commutative spacetime is the sinh-Gordon model. This is the theory with interaction Hamiltonian of the form

\[ H^{(sG)}_I = : \int dx \left( \frac{M^2}{2} \Phi^2 + \frac{1}{4!} \Phi^4 + \frac{g^2}{6!} \Phi^6 + \cdots \right) : = : \int dx \frac{M^2}{g^2} (\cosh g\Phi - 1) : \] (3.1)

where the double dots above stand for normal-ordering. For this theory, there is no particle production: the amplitude $S_{m \rightarrow n}$ for producing $n$ outgoing particles by colliding $m$ incoming particles in zero if $m \neq n$. In addition, the “elastic” amplitude $S_{m \rightarrow m}$ factorizes into products of two-particle scattering amplitudes $S_{2 \rightarrow 2}$. The exact expression of the two-particle $S$-matrix on commutative spacetime has been given in [6]:

\[ S_0(\eta) = \frac{\tanh \left( \frac{1}{2} \left( \eta - i \frac{\pi}{2} B(g) \right) \right)}{\tanh \left( \frac{1}{2} \left( \eta + i \frac{\pi}{2} B(g) \right) \right)}, \quad \text{where} \quad B(g) = \frac{2g^2}{8\pi + g^2} \] (3.3)

and $\eta$ the relative rapidity.

A very nice argument motivating quantum integrability for the commutative case has been given by Dorey in [13]. We will use this argument, adapting it appropriately to the noncommutative case.

Consider the theory based on the free field $\Phi(x)$ as in (2.4), with the free Hamiltonian $H_0 = \int d\mu(k) a_k^\dagger a_k$, and an interaction $H^{(1)}_{int} = : \frac{1}{4!} \int dx \Phi^4(x) :$, where by $\Phi^4(x)$ we mean $\Phi(x) * \Phi(x) * \Phi(x) * \Phi(x)$. 
There are two diagrams (Figure 1) that contribute to the $2 \to 4$ scattering amplitude at tree level. The amplitude for this process is

$$S_\Theta(p_1, p_2; k_1, k_2, k_3, k_4) = \langle 0 | a_k a_k a_k a_k | (-i \lambda) \left( \int d^2 x H_1(x, t) \right) | a_{p_1} a_{p_2} | 0 \rangle$$  \hspace{1cm} (3.4)

$$= S_0(p_1, p_2; k_1, k_2, k_3, k_4) e^{\frac{i}{\hbar} p_{1} \wedge p_{2} - \sum_{i<j} k_i \wedge k_j}$$  \hspace{1cm} (3.5)

$$= i \lambda^2 M^2 e^{\frac{i}{\hbar} p_{1} \wedge p_{2} - \sum_{i<j} k_i \wedge k_j}$$  \hspace{1cm} (3.6)

where we have used (2.15, 2.24), and the fact that $S_0 = i \frac{\lambda^2}{\hbar}$.  

Let us add an extra interaction of the form $H_{(2)}^{\text{int}} = \frac{\lambda'}{\hbar} \int dx \Phi_6^\ast$. We see that the $2 \to 4$ process receives an additional contribution from a third diagram (Figure 2), which of the form

$$S'_\Theta = -i \lambda' e^{\frac{i}{\hbar} p_{1} \wedge p_{2} - \sum_{i<j} k_i \wedge k_j}$$  \hspace{1cm} (3.7)

By choosing $\lambda' = \lambda^2 / M^2$, we can make the total tree-level amplitude $S_\Theta + S'_\Theta$ for the $2 \to 4$ process to vanish.

For the new interaction Hamiltonian $H_1^{(1)} + H_1^{(2)}$, the amplitude for the $2 \to 6$ process is now a non-zero constant, which can be made to vanish by judiciously choosing an extra interaction piece of the form $\int dx \Phi_8^\ast$. Continuing in this manner, one finds that the theory with the interaction Hamiltonian of the form

$$H_I = \frac{M^2}{g^2} \int dx (\cosh (g \Phi) - 1)$$  \hspace{1cm} (3.8)
has no particle production at tree-level: all processes of the form $2 \to n \ (n > 2)$ are forbidden (We have moved the mass term $M^2 \Phi^2 / 2$ from the free Hamiltonian to the interaction Hamiltonian, so that it can be presented in the convenient form (3.8)).

A crucial ingredient in the argument above is the specific choice (2.8) of the twist function $G_{p,q}$ for the commutation relations of the free field creation and annihilation operators. Had we chosen it to be of some other form, it is easy to see that we would lose the no particle production condition, and hence quantum integrability. In particular, choosing the twist function to be identity (i.e. using conventional commutation relations (2.9) for the free field creation/annihilation operators) leads to particle production at tree-level itself, as shown explicitly by [14] (however, see also [15–17] for a discussion of the absence of tree-level particle production in the noncommutative sine-Gordon model).

Our argument also extends to higher loops. For a scalar field theory (with, say, polynomial interactions) in commutative $(1 + 1)$ dimensions, the only source of ultraviolet divergence in perturbation theory is from single closed loops (see for example [18]). These can be absorbed by renormalizing the mass of the particle, or equivalently by working with the normal-ordering the interaction Hamiltonian to start with, as we have done.

As we argued earlier, although the scattering operator for the noncommutative theory and its commutative counterpart is the same, the asymptotic states are different. Using the map (2.15), we find the two-particle amplitude for the noncommutative case to be

$$S_{\Theta}(\eta) = \frac{\tanh \left[ \frac{1}{2} \left( \eta - i \frac{\pi}{2} B(g) \right) \right]}{\tanh \left[ \frac{1}{2} \left( \eta + i \frac{\pi}{2} B(g) \right) \right]} e^{-i \Theta M^2 \sinh \eta}$$

where $M$ is the (physical) mass of the sinh-Gordon particle.

### 4 Discussion

It is obvious that $S_{\Theta}(\eta)$ satisfies the following conditions:

Real analyticity: $S_{\Theta}(\eta)$ is real for $\eta$ purely imaginary.

Unitarity: $S_{\Theta}(\eta)S_{\Theta}(-\eta) = 1$.

Crossing: $S_{\Theta}(\eta) = S_{\Theta}(i\pi - \eta)$

Yang-Baxter equation: $S_{\Theta}(\eta_{12})S_{\Theta}(\eta_{13})S_{\Theta}(\eta_{23}) = S_{\Theta}(\eta_{23})S_{\Theta}(\eta_{13})S_{\Theta}(\eta_{12})$

It also possesses the strong-weak duality symmetry: $S_{\Theta}$ is invariant under $B \to 2 - B$, or equivalently under

$$g \to \frac{8\pi}{g} \ .$$

(4.1)

The fact the noncommutative scattering matrix for the sinh-Gordon model differs from commutative one only by a phase may seem surprising at first. It was pointed out by Mitra [19] that overall phases of the form $e^{i \sum_{\ell=0}^{\infty} b_{\ell} \sinh(2\ell+1)\eta}$ are allowed, over and above the form of the $S$-matrix dictated by dynamics. For local fields, the $b_{\ell}$ are required to vanish. However, since our field is non-local, we have no such restriction. We find, in fact, that $b_{\ell} = -\Theta M^2$. 


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