Abstract
We establish a fixed point theorem for the composition of nonconvex, measurable selection valued correspondences with Banach space valued selections. We show that if the underlying probability space of states is nonatomic and if the selection correspondences in the composition are K-correspondences (meaning correspondences having graphs that contain their Komlos limits), then the induced measurable selection valued composition correspondence takes contractible values and therefore has fixed points. As an application we use our fixed point result to show that all nonatomic uncountable-compact discounted stochastic games have stationary Markov perfect equilibria – thus resolving a long-standing open question in game theory.

Keywords: K-correspondences; USCOs with contractible values; Approximable USCOs; Fixed points of nonconvex measurable selection valued correspondences; Composition correspondences; Komlos limits; Weak∗-compact sets of Banach-space-valued Bochner integrable selections; Stationary Markov perfect equilibria; Discounted stochastic games

1 Introduction
We make two contributions. First, we establish a fixed point theorem for the composition of measurable selection valued correspondences with Banach space valued selections. We show that if the underlying probability space of states is nonatomic and if the selection correspondences in the composition are K-correspondences (meaning correspondences having graphs that contain their Komlos limits), then the induced nonconvex, measurable selection valued composition correspondence is approximable and therefore has fixed points.

Second, we apply our fixed point result to show that all nonatomic, uncountable-compact discounted stochastic games (DSGs) satisfying the assumptions of the Nowak–Raghavan DSG model have stationary Markov perfect equilibria (SMPE) – thus, resolving a long standing open question in game theory (see Nowak and Raghavan [1]).

Regarding our first contribution, the key step allowing us to establish our fixed point result is to show that if the probability measure on the state space is nonatomic and if the
selection correspondences in the composition are $K$-correspondences (or equivalently, are weak* upper semicontinuous correspondences taking nonempty compact values (US-COS), as we show here), then the composition correspondence takes contractible values in its set of selections where contractibility is with respect to the compatibly metrized weak* topologies. It then follows from results in Gorniewicz, Granas, and Kryszewski [2] that the composition correspondence is approximable.

Regarding our second contribution, we first use our fixed point result to show, under mild conditions on primitives, that all nonatomic, parameterized state-contingent games (PSGs) have Blackwell equilibria. An excellent example of a nonatomic PSG is the one-shot game underlying an uncountable-compact nonatomic discounted stochastic game (DSG) satisfying the assumptions of Nowak and Raghavan [1]. By Blackwell’s classical result [3] (extended to games), a DSG has a stationary Markov perfect equilibrium if and only if its underlying one-shot PSG has a Blackwell equilibrium. By our fixed point result any one-shot PSG belonging to a nonatomic Nowak–Raghavan DSG has a Blackwell equilibrium – therefore, by Blackwell [3], all nonatomic Nowak–Raghavan DSG have stationary Markov perfect equilibria.

We note that in a nonatomic DSG, because the underlying Nash selection correspondence is contractibly valued, the key pathology underlying the recent counterexamples to existence due to Levy [4] and Levy and McLennan [5] is ruled out. In particular, because the Nash selection correspondence is contractibly valued, Nash equilibria homeomorphic to the unit circle cannot arise. Also, we note that in the Levy and McLennan [5] discounted stochastic game model, the state space is not nonatomic. However, as noted by Jaskiewicz and Nowak [6], it is easy to modify the Levy–McLennan model so as to allow for a nonatomic state space while preserving the nonapproximability of the Nash payoff selection correspondence – thereby extending the Levy–McLennan counterexamples to the nonatomic case (also see Jaskiewicz and Nowak [7]). Here we will assume that the state space is nonatomic, and under this assumption, together with our result showing that the Nash payoff selection correspondence belonging to any Nowak–Raghavan DSGs is a $K$-correspondence, we will establish that the Nash payoff selection correspondence belonging to any Nowak–Raghavan DSGs is approximable and therefore has fixed points. Finally, we note that if a DSG has a Nash payoff selection correspondence that is approximable, then it will have fixed points, and therefore, by Blackwell’s theorem [3] the DSG will have stationary Markov perfect equilibria (e.g., see Page [8, 9]). By constructing a DSG having no stationary Markov perfect equilibria, and therefore, a DSG having a Nash payoff selection correspondence without fixed points, we can infer that the Nash payoff selection correspondence belonging to a Levy–McLennan DSG model is not approximable. Thus, while not all nonapproximable DSGs have SMPE, as shown by Levy [4] and Levy and McLennan [5], all approximable DSGs do, as shown by Page [8, 9]. Moreover, as we show here, all DSGs satisfying the Nowak–Raghavan assumptions are approximable, and as a consequence, all such DSGs escape the Levy–McLennan counterexample, and possess stationary Markov perfect equilibria in behavioral strategies.

2 Preliminaries

2.1 Spaces

Let $(\Omega, B_\Omega, \mu)$ be a probability space of states $\omega$, where $\Omega$ is a complete, separable metric space with probability measure $\mu$ defined on the Borel $\sigma$-field $B_\Omega$, and let $Y$ be a norm-bounded, weak* closed (i.e., $w^*$-closed), convex subset of $F^*$, the separable norm dual of
a separable Banach space $F$. Equip $Y$ with metric $\rho_Y^*$ compatible with the $w^*$-topology on $Y$ inherited from $F^*$.

We will often assume that the probability space $(\Omega, B_\Omega, \mu)$ is nonatomic. We say that $E \subset \Omega$ is an atom of $\Omega$ relative to $\mu(\cdot)$ if the following implication holds: if $\mu(E) > 0$, then $H \subset E$ implies that $\mu(H) = 0$ or $\mu(E-H) = 0$. If $\Omega$ contains no atoms relative to $\mu(\cdot)$, $\Omega$ is said to be atomless or nonatomic. Because $\Omega$ is a complete, separable metric space, $\mu(\cdot)$ is atomless (or nonatomic) if and only if $\mu(\{\omega\}) = 0$ for all $\omega \in \Omega$ (see Hildenbrand [10], pp. 44–45).

Next, let $L_Y^\infty$ denote the set of $\mu$-equivalence classes of $F^*$-valued, Bochner integrable functions, $v$, with $v(\omega) \in Y$ a.e. $[\mu]^\circ$. Denote by $L_Y^\infty$ the set consisting of all $F^*$-valued, Bochner integrable functions, $v$, with $v(\omega) \in Y$ a.e. $[\mu]$ (i.e., $L_Y^\infty$ is the quotient of $L_Y^\infty$).

We will equip $L_Y^\infty$ with a metric $\rho_Y^{*\infty}$ compatible with the weak* topology inherited from $L_Y^\infty$. We note that the set $L_Y^\infty$ is decomposable, meaning that if the functions $v^0(\cdot)$ and $v^1(\cdot)$ are in $L_Y^\infty$, then for any $E \in B_\Omega$, with indicator function $I_E(\cdot)$,

$$v^0(\cdot)I_E(\cdot) + v^1(\cdot)I_{\Omega\setminus E}(\cdot) \in L_Y^\infty.$$  

(1)

Similarly, let $X$ be a norm-bounded, weak*–closed (i.e., $w^*$-closed), convex subset of $E^*$, the separable norm dual of a separable Banach space $E$, and equip $X$ with metric $\rho_X^*$ compatible with the $w^*$-topology on $X$ inherited from $F^*$. Also, let $L_X^\infty$ denote the set of $\mu$-equivalence classes of $E^*$-valued, Bochner integrable functions $x(\omega)$, with $x(\omega) \in X$ a.e. $[\mu]$. Denote by $L_X^\infty$ the set consisting of all $E^*$-valued, Bochner integrable functions $x(\omega)$ with $x(\omega) \in X$ a.e. $[\mu]$ (i.e., $L_X^\infty$ is the quotient of $L_X^\infty$). We will equip $L_X^\infty$ with a metric $\rho_X^{*\infty}$ compatible with the weak* topology inherited from $L_X^\infty$. We note that the set $L_X^\infty$ is decomposable.

Finally, equip the spaces $Y$ and $X$ with the Borel $\sigma$-fields $B_Y^*$ and $B_X^*$, generated by the $\rho_Y^{*\infty}$ and $\rho_X^{*\infty}$-open sets in $Y$ and $X$, respectively.

### 2.2 Convergences

We will begin by discussing $W^*$-convergence in $L_F^\infty$ and $w^*$-$K$-convergence in its prequotient space, $L_F^\infty$. Then we will present a result clarifying how the two are related.

#### 2.2.1 $W^*$-Convergence in $L_F^\infty$

By Diestel and Uhl (p. 98) [11], because $F^*$ is separable, $L_F^\infty$ is the norm dual of $L_F^1$. We have the following definitions:

**Definition 1 (W*–Convergence in L_F^\infty and W*–limits)** A sequence of $F^*$-valued functions $(v^n(\cdot))_n$ in $L_F^\infty$ converges weak*, or $W^*$-converges to a function $v^*(\cdot) \in L_F^\infty$, if for every function $l(\cdot) \in L_F^1$,

$$\int_{\Omega} \langle v^n(\omega), l(\omega) \rangle d\mu(\omega) \rightarrow \int_{\Omega} \langle v^*(\omega), l(\omega) \rangle d\mu(\omega) := \langle v^*(\cdot), l(\cdot) \rangle. \quad (2)$$

In $L_F^\infty$ a sequence $(v^n(\cdot))_n W^*$-converges to $v^*(\cdot)$ if and only $\rho_F^{*\infty}(v^n, v^*) \rightarrow 0$. 


2.2.2 $w^*-K$-convergence in $L^\infty_F$

For any sequence $\{v^n\}_n \subset L^\infty_F$, let

$$\{\overline{v}^m_k\}_k := \left\{\frac{1}{k} \sum_{r=1}^{k} v^n_r \right\}_k$$

be the corresponding sequence of arithmetic mean functions. Because $F^*$ is convex for any subsequence $\{v^n_k\}_k \subset L^\infty_F$, $\overline{v}^m_k \in L^\infty_F$.

**Definition 2** ($w^*-K$-Sequences, $w^*-K$-Convergence, and $w^*-K$-Limits) We say that a sequence $\{v^n(.)\}_n \subset L^\infty_F$ $w^*-K$-converges to a $w^*-K$-limit function $\overline{v}(.) \in L^\infty_F$ provided every subsequence $\{v^n(.)\}_k$ of $\{v^n(.)\}_n$ has an arithmetic mean sequence $\{\overline{v}^m_k\}_k$ that $w^*$-converges pointwise a.e. $[\mu]$ to $\overline{v}(.)$, that is,

$$\overline{v}^m_k(\omega) \to \overline{v}(\omega) \quad a.e. \ [\mu]. \quad (4)$$

By Theorem 3.2 in Balder [12], if the sequence, $\{v^n(.)\}_n \subset L^\infty_F$, is such that

$$\sup_n \int_{\Omega} \|v^n(\omega)\|_{\infty} \, d\mu(\omega) < +\infty,$$

then there exists a subsequence $\{v^n(.)\}_k$ of $\{v^n(.)\}_n$ and a function $\overline{v}(.) \in L^\infty_F$ such that any subsequence $\{v^n(.)\}_l$ of $\{v^n(.)\}_k$ $w^*-K$-converges to $w^*-K$-limit, $\overline{v}(.)$.

2.2.3 $W^*$-Convergence and $w^*-K$-convergence

Our first result is about the relationship between $W^*$-convergence and $w^*-K$-convergence in $\Sigma^\infty_F$.

**Theorem 1** ($W^*$-Convergence and $w^*-K$-Convergence) Let $\{v^n(.)\}_n$ be any sequence in $L^\infty_F$. Then the following statements are true:

1. If $\{v^n(.)\}_n$ $w^*-K$-converges to $\overline{v}(.) \in L^\infty_F$, then $\{v^n(.)\}_n \subset L^\infty_F$ $W^*$-converges to $\overline{v}(.) \in L^\infty_F$.

2. If $\{v^n(.)\}_n$ $W^*$-converges to $v^*(.) \in L^\infty_F$, then each subsequence $\{v^n(.)\}_k$ has a further subsequence $\{v^n(.)\}_q$, which is $w^*-K$-convergent to some $w^*-K$-limit $\overline{v}(.) \in L^\infty_F$ such that $\overline{v}(\omega) = v^*(\omega)$ a.e. $[\mu]$.

**Proof** We will prove part (2) first. Assume that $\{v^n(.)\}_n \subset L^\infty_F$ $W^*$-converges to $v^*(.) \in L^\infty_F$, and that the subsequence, $\{v^n(.)\}_k$, $w^*-K$-converges to $\overline{v}(.) \in \Sigma^\infty_F$. For each $l \in L^1_F$, we have

$$\frac{1}{k} \sum_{r=1}^{k} [v^n(\omega), l(\omega)] \to [\overline{v}(\omega), l(\omega)] \quad a.e. \ [\mu]$$

and by the dominated convergence theorem we have for each $l \in L^1_F$,

$$\lim_{k \to \infty} \frac{1}{k} \sum_{r=1}^{k} [v^n(\omega), l(\omega)] \, d\mu(\omega) = \lim_{k \to \infty} \frac{1}{k} \sum_{r=1}^{k} \int_{\Omega} [v^n(\omega), l(\omega)] \, d\mu(\omega) = \int_{\Omega} [\overline{v}(\omega), l(\omega)] \, d\mu(\omega).$$
Because \( \{ \nu^n(\cdot) \}_k \) \( W^* \)-converges to \( \nu^* \), we have for each \( l \in \mathcal{L}_1^\infty \),

\[
\lim_{k \to \infty} \int_{\Omega} \langle \nu^k(\omega), l(\omega) \rangle d\mu(\omega) = \int_{\Omega} \langle \nu^*(\omega), l(\omega) \rangle d\mu(\omega),
\]

implying that for each \( l \in \mathcal{L}_1^\infty \),

\[
\lim_{k \to \infty} \frac{1}{k} \sum_{r=1}^{k} \int_{\Omega} \langle \nu^r(\omega), l(\omega) \rangle d\mu(\omega) = \int_{\Omega} \langle \nu^*(\omega), l(\omega) \rangle d\mu(\omega).
\]

Thus, for each \( l \in \mathcal{L}_1^1 \),

\[
\int_{\Omega} \langle \nu(\omega), l(\omega) \rangle d\mu(\omega) = \int_{\Omega} \langle \nu^*(\omega), l(\omega) \rangle d\mu(\omega),
\]

implying that \( \nu(\omega) = \nu^*(\omega) \) a.e. \([\mu]\).

Now we will prove part (1). By part (2), we can conclude that \( \mathcal{L}_1^\infty \) is convex and \( W^* \)-compact. Assume that \( \{ \nu^n(\cdot) \}_n \subset \mathcal{L}_1^\infty \) \( w^*-K \)-converges to some \( \nu(\cdot) \in \mathcal{L}_1^\infty \). In order to show that \( \{ \nu^n(\cdot) \}_n \) \( W^* \)-converges to \( \nu(\cdot) \) where \( \nu^*(\omega) = \nu(\omega) \) a.e. \([\mu]\), it suffices to show that the \( \mu \)-equivalence class in \( \mathcal{L}_1^\infty \) containing \( \nu(\cdot) \) is the only limit point of the sequence of \( \mu \)-equivalence classes, \( \{ \nu^n(\cdot) \}_n \subset \mathcal{L}_1^\infty \). Let \( \nu^\infty(\cdot) \) be any \( W^* \)-limit point of the sequence \( \{ \nu^n(\cdot) \}_n \) and let \( \{ \nu^k(\cdot) \}_k \) be a subsequence \( W^* \)-converging to \( \nu^\infty(\cdot) \). By \( w^*-K \)-convergence we know that this subsequence also \( w^*-K \)-converges to \( \nu(\cdot) \), and hence by part (2) we know that \( \nu^\infty(\omega) = \nu(\omega) \) a.e. \([\mu]\). \( \square \)

### 2.3 Contractible continua

Let \((X, \rho_X), (Y, \rho_Y), (Z, \rho_Z)\) be metric spaces. Consider a function, \( f : Z \to X \), from \( Z \) onto \( X \). If \( f \) is continuous and one-to-one, and if its inverse \( f^{-1} \) is also continuous, then we say that \( f \) is a homeomorphism and that the metric spaces \( Z \) and \( X \) are homeomorphic. If \((Z, \rho_Z)\) is compact, then any continuous, one-to-one mapping \( f \) from \( Z \) onto \( X \) is a homeomorphism. A continuous function \( f : Z \to X \) is an embedding if \( f : Z \to f(Z) \) is a homeomorphism.

Given metric space \((Z, \rho_Z)\), a set \( E \subseteq Z \) is connected if \( E \) cannot be written as the union of two disjoint open sets (or two disjoint closed sets). A set \( E \subseteq Z \) is locally connected at \( e \in E \) if each neighborhood \( U_e \) of \( e \) contains a connected neighborhood \( V_e \) of \( e \). Also \( E \) is locally connected if it is locally connected at each \( e \in E \). We note that in any metric space \((Z, \rho_Z)\), the condition of being (i) a locally connected continuum and (ii) the continuous image of an interval are equivalent (this is the Mazurkiewicz–Moore theorem – see Kuratowski [13]).

If the metric space \((Z, \rho_Z)\) is compact and connected, it is called a continuum. A non-empty, closed, connected subset of \( Z \) is called a subcontinuum. We will denote by \( \mathcal{C}(Z) \) the collection of all subcontinua of \( Z \). If in addition, the continuum \((Z, \rho_Z)\) is locally connected, it is called a Peano continuum. Finally, (i) if the Peano continuum \((Z, \rho_Z)\) is unicoherent, meaning that for all subcontinua \( A \) and \( B \) of \( Z \) such that \( Z = A \cup B \), the intersection \( A \cap B \) is connected, and (ii) if all subcontinua of \( Z \) are unicoherent (i.e., if property (i) is hereditary), then \((Z, \rho_Z)\) is a dendrite. Thus, \((Z, \rho_Z)\) is a dendrite if it is a compact metric space that is connected, locally connected, and hereditarily unicoherent (see Charatonik and Charatonik [14] for more details).
A retraction \( r(\cdot) \) is a continuous function from a space \( Z \) into \( Z \) such that \( r(\cdot) \) is the identity on its range (i.e., \( r(r(z)) = r(z) \) for all \( z \in Z \)). A subset \( W \) of \( Z \) is said to be a retract of \( Z \) provided there is a retraction of \( Z \) onto \( W \).

The notion of a homotopy will be important in what follows. A homotopy is a function that essentially provides us with a way to index a set of continuous functions. We have the following formal definition:

**Definition 3** (Homotopies) Let \((Z, \rho_Z)\) and \((X, \rho_X)\) be compact metric spaces and let \( C(Z \times [0,1], X) \) denote the collection of all continuous functions, \( h : Z \times [0,1] \to X \), defined on \( Z \times [0,1] \) taking values in \( X \). A function \( h \in C(Z \times [0,1], X) \) is called a homotopy, and each homotopy \( h \) specifies a continuously indexed set of continuous functions,

\[
C_h(Z,X) := \{ h(\cdot,t) \in C(Z,X) : t \in [0,1] \}.
\]

The indexed collection, \( C_h(Z,X) \), can be thought of as an arc, \( \alpha_h \), in the continuum of continuous functions, \( C(Z,X) \), equipped with the sup metric. The continuous functions \( f \) and \( g \) in \( C(Z,X) \) are homotopically related, or homotopic, if \( f \) and \( g \) are the endpoints of an arc \( \alpha_h \) whose arc type is identified by some function, \( h \in C(Z \times [0,1], X) \). In particular, if \( f, g \in C(Z,X) \) are homotopic, then there is an arc of type \( h \in C(Z \times [0,1], X) \) running from continuous function \( f(\cdot) \) to continuous function \( g(\cdot) \) at \( h(\cdot,1) \). We denote this \( h \)-arc from \( f \) to \( g \) by writing \( g \in [f]_h \) or by writing \( f \hto g \) (and if the orientation is in the opposite direction, then we write \( f \in [g]_h \) or \( g \hto f \)). Constant functions form a special class of homotopy arc end points. Let \( g_\ast \in C(Z,X) \) denote the constant function (i.e., \( g_\ast(z) = \bar{x} \) for all \( z \in Z \)). If \( f \) and \( g_\ast \) are homotopic (i.e., if \( g_\ast \in [f]_h \), that is, if \( f \hto g_\ast \) for some \( \bar{x} \in X \)), then \( f \) is said to be inessential. Moreover, if for some pair of compact metric spaces \((Z, \rho_Z)\) and \((X, \rho_X)\), all pairs of functions \( f, g \in C(Z,X) \) are homotopic, then, in particular, \( f, g_\ast \in C(Z,X) \) are homotopic for some \( h \)-arc and some \( \bar{x} \in X \) – and this means that for this pair of compact metric spaces, \((Z, \rho_Z)\) and \((X, \rho_X)\), all functions \( f \in C(Z,X) \) are inessential (i.e., for each \( f \in C(Z,X) \), there is \((h(\cdot),x) \in (C(Z \times [0,1], X),f, f \hto g_\ast) \).

If \( Z \subseteq X \), then \( Z \) is contractible in \( X \) if for some homotopy \( h \in C(Z \times [0,1], X) \), there is an \( h \)-arc running from the identity (or inclusion) mapping \( f_\text{id} \in C(Z,X) \) to a constant mapping \( g_\ast \in C(Z,X) \), for some \( \bar{x} \in X \). Thus, \( f_\text{id}(z) = h(\cdot,0) \) where \( f_\text{id}(z) = z \) for all \( z \in Z \) is the inclusion mapping (i.e., \( f_\text{id}(z) = z = h(z,0) \) for all \( z \in Z \) and \( h(\cdot, 1) \) is the constant mapping (i.e., \( h(z,1) = \bar{x} \) for all \( z \in Z \) for some \( \bar{x} \in X \)).

We say that \( X \) is contractible if \( X \) is contractible in \( X \). Note that if \( X \) is contractible, then for any \( Z \subseteq X \), \( Z \) is contractible in \( X \). Two useful facts related to the contractibility of continua are the following:

1. If \( X \) is contractible and \( Z \subseteq X \) is a retract of \( X \), then \( Z \) is also contractible. Thus if \( r : X \to Z \), \( r \in C(X,Z) \) where \( r(z) = z \) for all \( z \in Z \), then \( Z \) is also contractible.

2. If \( X \) is contractible, then \( X \) is unicoherent (see Corollary A.12.10 in van Mill [15]) – implying that all pairs of functions, \( f, g \in C(X, S^1) \), are homotopic for the unit circle, \( S^1 := \{x = (x_1,x_2) : (x_1)^2 + (x_2)^2 = 1 \} \). Thus, if \( X \) is contractible, then all continuous functions \( f : X \to S^1 \) are inessential, and we can conclude that \( X \) contains no simple closed curves.

A compact metric space \( K \) is called an absolute retract (AR) provided that whenever \( K \) is embedded in a metric space \( Y \), the embedded copy of \( K \) is a retract of \( Y \). A compact
metric space is called an **absolute extensor (AE)** provided whenever \( B \) is a closed subset of a metric space \( M \) and \( f : B \rightarrow K \) is continuous, then \( f \) can be extended to a continuous function \( F : M \rightarrow K \) (\( F \) being an extension of \( f \) means that \( F|_B = f \)). By Borsuk's theorem, a compact metric space \( K \) is an AR if and only if \( K \) is an AE (see Borsuk [16], see also, 9.1 in Illanes and Nadler [17]).

A compact metric space \( K \) is called an **absolute neighborhood retract (ANR)** provided that whenever \( K \) is embedded in a metric space \( Y \) the embedded copy \( K' \) of \( K \) is a retract of some neighborhood of \( K' \) in \( Y \). A compact metric space \( K \) is called an absolute neighborhood extensor (ANE) provided that whenever \( B \) is a closed subset of a metric space \( M \) and \( f : B \rightarrow K \) is continuous, then there is a neighborhood \( U \) of \( B \) in \( M \) such that \( f \) can be extended to a continuous function \( F : U \rightarrow K \). In parallel to Borsuk's theorem, a compact metric space \( K \) is an ANR if and only if \( K \) is an ANE (see Borsuk [16], see also, 19.5 in Illanes and Nadler [17]).

### 2.4 Correspondences

#### 2.4.1 USCOs and approximability

For compact metric spaces \((X, \rho_X)\) and \((Y, \rho_Y)\), let \( \mathcal{U}_{\rho_Y, \rho_X} \) denote the collection of all upper semicontinuous correspondences \( \Gamma(\cdot) \) defined on \( Y \) taking nonempty, \( \rho_X \)-closed (and hence \( \rho_X \)-compact) values in \( X \). Equip the product space \( Y \times X \) with the sum metric, \( \rho_{Y \times X} := \rho_Y + \rho_X \). Following the literature, all such mappings are USCOs (e.g., see Hola and Holy [18]). We say that the USCO \( \Gamma(\cdot) \in \mathcal{U}_{\rho_Y, \rho_X} \) is \( \rho_Y - \rho_X \)-approximable if for each \( n \), there is a \( \rho_Y - \rho_X \)-continuous function, \( g^n(\cdot) : Y \rightarrow X \), such that for each \((y^n, x^n) \in \text{Gr}^n \subset Y \times X \) (i.e., for each \((y^n, x^n) \in Y \times X \), with \( x^n = g^n(y^n) \in X \)), there exists \((\bar{y}^n, \bar{x}^n) \in \text{Gr} \Gamma(\cdot) \) such that

\[
\rho_Y(y^n, \bar{y}^n) + \rho_X(x^n, \bar{x}^n) < \frac{1}{n}.
\]

Bringing together the relevant parts of (5.6) and (5.12) from Gorniewicz, Granas, and Kryszewski [2], we have the following result concerning the approximability of USCOs:

**Theorem 2** (The Approximability of USCOs) Let \((X, \rho_X)\) and \((Y, \rho_Y)\) be compact metric ANR spaces and let \( \Gamma(\cdot) \in \mathcal{U}_{\rho_Y, \rho_X} \). If for each \( y \in Y \), \( \Gamma(y) \) is contractible, then \( \Gamma(\cdot) \) is \( \rho_Y - \rho_X \)-approximable.

For compact metric spaces \((\mathcal{L}^\infty_{\rho_X}, \rho^*_{\mathcal{L}^\infty_{\rho_X}})\) and \((\mathcal{L}^\infty_{\rho_X}, \rho^*_{\mathcal{L}^\infty_{\rho_X}})\), let \( \mathcal{U}^\infty_{\mathcal{L}^\infty_{\rho_X}, \rho^*_{\mathcal{L}^\infty_{\rho_X}}} \) denote the collection of all upper semicontinuous correspondences defined on \( \mathcal{L}^\infty_{\rho_X} \) taking nonempty, \( \rho^*_{\mathcal{L}^\infty_{\rho_X}} \)-closed (and hence \( \rho^*_{\mathcal{L}^\infty_{\rho_X}} \)-compact) values in \( \mathcal{L}^\infty_{\rho_X} \). We will denote this special set of USCOs – USCOs with respect to weak* topologies – by \( \mathcal{U}^\infty_{\mathcal{L}^\infty_{\rho_X}, \rho^*_{\mathcal{L}^\infty_{\rho_X}}} \) by USCO*\). Also, we have the collection of all USCO*’s with convex values (CUSCO*).

#### 2.4.2 K-correspondences and USCO*’s

The formal definition of a **K-correspondence** is the following:

**Definition 4** (K-Limit Property and K-Correspondences) We say that a correspondence \( \mathcal{E}(\cdot) : \mathcal{L}^\infty_{\rho_X} \rightarrow P(\mathcal{L}^\infty_{\rho_X}) \) has the K-limit property, or is a **K-correspondence**, if for any sequence

\[
\{(y^n, x^n)\}_n \subset \text{Gr}\mathcal{E}(\cdot) := \{(v, x(\cdot)) \in \mathcal{L}^\infty_{\rho_X} \times \mathcal{L}^\infty_{\rho_X} : x(\cdot) \in \mathcal{E}(v)\}
\]
with \( v^n \rightarrow \hat{\nu} \in L^\infty_Y \) and \( x^{(i)}_n \rightarrow \tilde{x}_i \in L^\infty_X \), we have \((\hat{\nu}, \tilde{x}_i) \in \text{GrE}(\cdot)\). We will denote the collection of all \( K \)-correspondences defined on \( L^\infty_Y \) with nonempty values in \( L^\infty_X \) by \( UK_{E^Y_L \times E^X_L} \).

As our next result makes clear, because of the near equivalence of \( w^*\)-\( K \)-convergence and \( W^* \)-convergence in \( L^\infty_Y \) and \( L^\infty_X \), \( K \)-correspondences and USCO\(^* \)'s are equivalent. We note that each \( E(\cdot) \in U^*_{L^\infty_L \times L^\infty_X} \) takes decomposable values in \( L^\infty_X \).

**Lemma 1** (Equivalence of \( UK_{E^Y_L \times E^X_L} \) and \( U^*_{E^Y_L \times E^X_L} \)) Let \( E(\cdot) : L^\infty_Y \rightarrow P(L^\infty_X) \) be a correspondence. Then \( E(\cdot) \) is a \( K \)-correspondence if and only if \( E(\cdot) \) is an USCO\(^* \).

**Proof** Given the compactness of \( L^\infty_Y \) and \( L^\infty_X \), \( E(\cdot) \) is an USCO\(^* \) if and only if \( \text{GrE}(\cdot) \) is \( \rho^*_L \times \rho^*_L \)-closed in \( L^\infty_Y \times L^\infty_X \) (\( \rho^*_L \times \rho^*_L := \rho^*_L \times \rho^*_L \)). By Theorem 1 above, if \( E(\cdot) \) is a \( K \)-correspondence, then \( \text{GrE}(\cdot) \) is \( \rho^*_L \times \rho^*_L \)-closed in \( L^\infty_Y \times L^\infty_X \). Moreover, if \( E(\cdot) \) is \( \rho^*_L \times \rho^*_L \)-closed in \( L^\infty_Y \times L^\infty_X \) and \( \{ (v^n, x^{(i)}_n) \} \subset \text{GrE}(\cdot) \) is such that \( v^n \rightarrow \hat{\nu} \in L^\infty_Y \) and \( x^{(i)}_n \rightarrow \tilde{x}_i \in L^\infty_X \), then we have \( v^n \rightarrow \hat{\nu} \) and \( x^{(i)}_n \rightarrow \tilde{x}_i \) and because \( \text{GrE}(\cdot) \) is \( \rho^*_L \times \rho^*_L \)-closed in \( L^\infty_Y \times L^\infty_X \), \((\hat{\nu}, \tilde{x}_i) \in \text{GrE}(\cdot)\), implying that \( E(\cdot) \) is a \( K \)-correspondence. \( \square \)

### 3 Main Results

#### 3.1 Contractibility and approximability

**3.1.1 Contractibility**

Our key result is the following:

**Theorem 3** (The Contractibility Theorem) Let \( E(\cdot) \in UK_{E^Y_L \times E^X_L} \). If \( (\Omega, B_\Omega, \mu) \) is nonatomic, then \( E(\nu) \) is contractible in \( L^\infty_X \) for each \( \nu \in L^\infty_Y \).

**Proof** By Lemma 1 above, \( E(\cdot) \in U^*_{E^Y_L \times E^X_L} \). Given that \( \mu \) is nonatomic, we know by Fryszkowski [19] that Lyapunov’s theorem [20] on the range of a vector measure guarantees the existence of a family of measurable sets \( \{ A_i \}_{i \in [0,1]} \) such that

\[
\begin{align*}
  &t' \leq t \Rightarrow A_{t'} \subseteq A_t, & A_0 = \emptyset & \text{and} & A_1 = \Omega, & \text{and} \\
  &\mu(A_i) = t\mu(\Omega) = t.
\end{align*}
\]

(5)

Using the properties of the system \( \{ A_i \}_{i \in [0,1]} \), and the decomposability of \( E(\nu) \) for each \( \nu \in L^\infty_Y \), we will show that for each \( \nu \) the function \( h_\nu(\cdot, \cdot) \) given by

\[
h_\nu(x_1, t) := x_1|_t{I(A_t)} + x_1|_{t=0}{I(A_0)}
\]

(6)

for all \( (x_1, t) \in E(\nu) \times [0,1] \) is a homotopy (and in particular, a contraction of the selections in \( E(\nu) \) to \( x_1^1 \)). Here \( \nu \in L^\infty_Y \) is fixed, \( I_A(\cdot) \) is the indicator function of set \( A \), and \( x_1^1 \) is any fixed selection in \( E(\nu) \).

It suffices to show that \( h_\nu(\cdot, \cdot) \) is \( \rho^*_L \times [0,1] \)-continuous. Let \( \{ (x^n_1, t^n) \} \) be a sequence such that

\[
x^n_1 \overset{\rho^*_L}{\longrightarrow} x^*_1 \quad \text{and} \quad t^n \overset{1}{\longrightarrow} t^*.
\]
We must show that
\[ h_s(x^n, t^p) \rightarrow h_s(x^n, t^*) \in {\mathcal{E}}(v). \]  

(7)

In particular, we must show that for each \( l(\cdot) \in \mathcal{L}^1 \),
\[ \left| \int_{\Omega} \left( (h_s(x^n, t^p), l(\omega)) - (h_s(x^n, t^*), l(\omega)) \right) d\mu(\omega) \right| \rightarrow 0. \] 

(8)

For each \( l(\cdot) \in \mathcal{L}^1 \), we have
\[ \left| \int_{\Omega} \left( (h_s(x^n, t^p), l(\omega)) - (h_s(x^n, t^*), l(\omega)) \right) d\mu(\omega) \right| \ \leq \ \left| \int_{\Omega} \left( h_s(x^n, l(\omega)) - h_s(x^n, l(\omega)) \right) d\mu(\omega) \right| \\
+ \left| \int_{\Omega} \left( (h_s(x^n, t^p), l(\omega)) - (h_s(x^n, t^*), l(\omega)) \right) d\mu(\omega) \right| \]

(9)

For expressions (9(a)) and (9(b)) above, we have that
\[ \left| \int_{\Omega} \left( (h_s(x^n, l(\omega)) - h_s(x^n, l(\omega)) \right) d\mu(\omega) \right| \ \leq \ \int_{\mathcal{A}} \left| \sum_{r=1}^{\mu(A^\nu \Delta A^\tau)} (x^n, l(\omega)) \right| d\mu(\omega) \]

(10)

and
\[ \left| \int_{\Omega} \left( (h_s(x^n, l(\omega)) - h_s(x^n, l(\omega)) \right) d\mu(\omega) \right| \ \leq \ \int_{\mathcal{A}} \left| \sum_{r=1}^{\mu(A^\nu \Delta A^\tau)} (x^n, l(\omega)) \right| d\mu(\omega) \]

(11)

For expression (9(c)), because \( l(\cdot)|_{\Omega \setminus \{A^\nu \Delta A^\tau\}} \) is also contained in \( \mathcal{L}^1 \) and \( x^n \rightarrow x^n \), we have that
\[ \left| \int_{\Omega} \left( (h_s(x^n, l(\omega)) - h_s(x^n, l(\omega)) \right) d\mu(\omega) \right| \ \rightarrow \ 0. \] 

(12)

Because \( \mu(A^\nu \Delta A^\tau) \rightarrow 0 \), we have that \( M_1 \|l(\cdot)\|_1 \left[ \frac{1}{2} \sum_{q=1}^{\mu} \mu(A^\nu \Delta A^\tau) \right] \rightarrow 0 \), implying that in expression (9) (a) \( \rightarrow 0 \) and (b) \( \rightarrow 0 \), and we have already from expression (12) that (c) \( \rightarrow 0 \). Thus, we have that for each \( l(\cdot) \in \mathcal{L}^1 \),
\[ \left| \int_{\Omega} \left( (h_s(x^n, t^p) - h_s(x^n, t^*)), l(\omega) \right) d\mu(\omega) \right| \rightarrow 0. \] 

(13)
implying that \( h_v(x^n, t^n) \rightarrow h_v(x^*, t^*) \). Because \( \mathcal{E}(\cdot) \) is an USCO* with contractible values (Theorem 3 above), this in turn guarantees the \( \rho^*_{L^\infty Y} - \rho^*_{L^\infty X} \)-approximability of \( \mathcal{E}(\cdot) \), as our next result shows.

**Theorem 4** (The Approximability Theorem) Let \( \mathcal{E}(\cdot) \in \mathcal{UK}(L_Y^\infty, L_X^\infty) \). If \((\Omega, B_\Omega, \mu)\) is nonatomic, then \( \mathcal{E}(\cdot) \) is \( \rho^*_{L^\infty Y} - \rho^*_{L^\infty X} \)-approximable.

**Proof** By Lemma 1 and Theorem 3 above, \( \mathcal{E}(\cdot) \) is an USCO* with contractible values. By Theorem 2 above (see 5.6 and 5.12 in Gorniewicz, Granas, and Kryszewski [2]), because \( \mathcal{E}(\cdot) \) is an USCO* defined on the ANR space of value functions \( L^\infty_X \), taking nonempty, compact, and contractible values in the ANR space, \( L^\infty_Y, \mathcal{E}(\cdot) \) is \( \rho^*_{L^\infty Y} - \rho^*_{L^\infty X} \)-approximable. \( \square \)

3.2 Fixed points for \( K \)-correspondences and their compositions

3.2.1 \( K \)-Correspondences

Consider the correspondence \( \mathcal{U}(\cdot) \) from \( L^\infty_Y \) with nonempty set values in \( L^\infty_Y \).

**Theorem 5** (Fixed Point Theorem for \( K \)-Correspondences) Let \( \mathcal{U}(\cdot) \in \mathcal{UK}(L_Y^\infty, L_X^\infty) \). If \((\Omega, B_\Omega, \mu)\) is nonatomic, then \( \mathcal{U}(\cdot) \) has fixed points (i.e., there exists \( v^* \in L^\infty_Y \) such that \( v^* = \mathcal{U}(v^*) \)).

**Proof** By Theorem 4 above, because \((\Omega, B_\Omega, \mu)\) is nonatomic and \( \mathcal{U}(\cdot) \) is a \( K \)-correspondence, \( \mathcal{U}(\cdot) \) is \( \rho^*_{L^\infty X} - \rho^*_{L^\infty X} \)-approximable. Therefore, we have for each \( n \), a \( \rho^*_{L^\infty Y} - \rho^*_{L^\infty Y} \)-continuous function, \( g^n(\cdot) : L^\infty_Y \rightarrow L^\infty_Y \), such that for each \( (v^a, u^a) \in \text{Gr}g^n \subset L^\infty_Y \times L^\infty_Y \) (i.e., for each \( (v^a, u^a) \in L^\infty_Y \times L^\infty_Y \), with \( u^a = g^n(v^a) \in L^\infty_Y \)) there exists \( (\overline{v}^a, \overline{u}^a) \in \text{Gr}S^n(\Gamma_\cdot) \) such that

\[
\rho^*_{L^\infty Y}(v^a, \overline{v}^a) + \rho^*_{L^\infty Y}(u^a, \overline{u}^a) < \frac{1}{n^2}.
\]

Because each of the functions, \( g^n \), is \( \rho^*_{L^\infty Y} - \rho^*_{L^\infty Y} \)-continuous and defined on the \( \rho^*_{L^\infty Y} \)-compact and convex subset, \( L^\infty_Y \), in \( L^\infty_Y \), taking values in \( L^\infty_Y \), it follows from the Brouwer–Schauder–Tychonoff fixed point theorem (see Aliprantis and Border [22], 17.56), that
each $g^n$ has a fixed point, $v^n \in L_{Y}^{\infty}$ (i.e., for each $n$ there exists some $v^n \in L_{Y}^{\infty}$ such that $v^n = g^n(v^n)$). Let $\{v^n\}_n$ be a fixed point sequence corresponding to the sequence of $\rho_{L_{Y}^{\infty}}^* - \rho_{L_{Y}^{\infty}}$-continuous approximating functions, $\{g^n()\}_n$. Expression (14) can now be rewritten as follows: for each $v^n$ in the fixed point sequence, there is a corresponding pair, $(v^n, \overline{v}^n) \in \text{Gr}U(\cdot)$, such that

$$\rho_{L_{Y}^{\infty}}^*(v^n, \overline{v}^n) < \frac{1}{n^2},$$

and therefore such that

$$\frac{\rho_{L_{Y}^{\infty}}^*(v^n, \overline{v}^n)}{A} + \frac{\rho_{L_{Y}^{\infty}}^*(v^n, \overline{v}^n)}{B} < \frac{1}{n^2}. \quad (15)$$

By the $\rho_{L_{Y}^{\infty}}^*$-compactness of $L_{Y}^{\infty}$, we can assume WLOG that the fixed point sequence, $\{v^n\}_n \subset L_{Y}^{\infty}$, $\rho_{L_{Y}^{\infty}}^*$-converges to a limit $v^* \in L_{Y}^{\infty}$. Thus, by part A of (15), as $n \to \infty$ we have

$$v^n \to v^* \quad \text{and} \quad \overline{v}^n \to v^*,$$

and therefore by part B of (15), as $n \to \infty$ we have $\overline{v}^n \to v^*$. Because $U(\cdot)$ has a $\rho_{L_{Y}^{\infty}}^* - \rho_{L_{Y}^{\infty}}^*$ closed graph in $L_{Y}^{\infty} \times L_{Y}^{\infty}$, $(\{\overline{v}^n, \overline{v}^n\})_n \subset \text{Gr}U(\cdot)$, and $\overline{v}^n \to v^*$ and $\overline{v}^n \to v^*$ imply that $(v^*, v^*) \in \text{Gr}U(\cdot)$. Therefore, $v^* \in U(v^*)$. \hfill \square

3.2.2 Composition correspondences

Finally, let us consider the correspondences,

$$E(\cdot) : L_{X}^{\infty} \to P(L_{X}^{\infty}),$$

$$T(\cdot, \cdot) : L_{X}^{\infty} \times L_{X}^{\infty} \to P(L_{X}^{\infty}),$$

and the composition correspondence,

$$\nu \to U(\nu) := \{ T(\nu, x(\cdot)) : x(\cdot) \in E(\nu) \}.$$ \hfill (17)

Here, the notation $P(L_{X}^{\infty})$ denotes the collections of all nonempty subsets of $L_{X}^{\infty}$ (and similarly for $P(L_{Y}^{\infty})$). We will use the notation $P_f(L_{X}^{\infty})$ to denote the hyperspace of all nonempty, $W^*$-closed (and hence, $W^*$-compact) subsets of $L_{X}^{\infty}$ (and similarly for $P_f(L_{Y}^{\infty})$).

Let

$$T(\cdot, \cdot) : L_{X}^{\infty} \times L_{X}^{\infty} \to P(L_{Y}^{\infty})$$

be a $\rho_{L_{Y}^{\infty}}^* - \rho_{L_{Y}^{\infty}}^*$-upper semicontinuous correspondence. Thus, if $\{(\nu^n, x^n(\cdot), u^n)\}_n$ is a sequence in $\text{Gr}T(\cdot, \cdot)$ such that $\nu^n \to \nu^*$, $x^n(\cdot) \to x^*(\cdot)$, and $u^n \to u^*$, then $(\nu^*, x^*(\cdot), u^*) \in P_f(L_{Y}^{\infty})$.\hfill \square
GrT(·, ·). Let $U^*_X \times L^\infty_Y$ denote the collection of all such upper semicontinuous correspondences. We note that, by Lemma 1 above, $T(·, ·)$ is a $K$-correspondence. Next, let $E(·) \in UK_L^*, L^\infty_X$ and consider the induced composition correspondence,

$$
v \longrightarrow U(v) := T(v, E(v)). \quad (18)
$$

We can now state our main fixed point result.

**Theorem 6** (Fixed Point Theorem for Compositions of $K$-Correspondences) Let $E(·) \in UK_L^*, L^\infty_X$ and $T(·, ·) \in UK_L^*, L^\infty_X$. Consider the composition correspondence $T(·, E(·)) : L^\infty_X \longrightarrow P(L^\infty_Y)$.

The following statements are true:

(1) The correspondence $v \longrightarrow U(v) := T(v, E(v))$ is a $K$-correspondence.

(2) If $(\Omega, B_{\Omega}, \mu)$ is nonatomic, then $U(·)$ has fixed points (i.e., there exists $v^* \in L^\infty_Y$ such that $v^* \in T(v^*, E(v^*))$).

**Proof** (1) Because $E(·) \in UK_L^*, L^\infty_X$ and $T(·, ·) \in UK_L^*, L^\infty_X$, by Lemma 1, $E(·) \in UK_L^*, L^\infty_X$ and $T(·, ·) \in UK_L^*, L^\infty_X$. Thus, the composition $U(·) := T(·, E(·))$ is contained in $UK_L^*, L^\infty_X$ and, by Lemma 1,

$$
U(·) := T(·, E(·)) \in UK_L^*, L^\infty_X.
$$

(2) Because $(\Omega, B_{\Omega}, \mu)$ is nonatomic and $U(·)$ is a $K$-correspondence, we have by Theorem 5 above that $U(·)$ has fixed points. Therefore, there exists $v^* \in L^\infty_Y$ such that $v^* \in U(v^*) := T(v^*, E(v^*))$.

\[\square\]

**4 Applications**

**4.1 Parameterized, state-contingent games (PSGs)**

**4.1.1 Primitives, assumptions, and Blackwell equilibria of PSGs**

We must first expand the number of spaces to take into account that now we have $m$ players indexed by $d = 1, 2, \ldots, m$. For each player $d$, let $Y_d$ be a closed bounded interval $[-M, M]$, $M > 0$, the same for all players and equip $Y_d$ with the absolute value metric – a metric we will continue to denote by $\rho^\infty_d$.

Let $L^\infty_Y$ denote the Banach space of $\mu$-equivalence classes of real-valued, essentially bounded measurable functions equipped with the weak* topology (i.e., the $w^*$-topology), and let $L^\infty_{Y_d}$ be the nonempty, convex, $w^*$-compact, and metrizable subset of $L^\infty_Y$ consisting of functions $v_d$ with $v_d(\omega) \in Y_d$ a.e. $\mu$. Denote by $L^\infty_{Y_d}$ the set consisting of all such functions $v_d$ with $v_d(\omega) \in Y_d$ a.e. $\mu$ (i.e., $L^\infty_{Y_d}$ is the prequotient of $L^\infty_{Y_d}$). We will equip $L^\infty_{Y_d}$ with a metric, $\rho^\infty_{L^\infty_{Y_d}}$, compatible with the $w^*$-topology in $L^\infty_{Y_d}$ inherited from $L^\infty_Y$.

Next let $X_d$ be a norm-bounded, weak* closed (i.e., $w^*$-closed), convex subset of $E^*_d$, the separable norm dual of a separable Banach space $E_d$, and equip $X_d$ with metric $\rho^\infty_{X_d}$ compatible with the $w^*$-topology on $X_d$ inherited from $E^*_d$. Also, let $L^\infty_{E^*_d}$ denote the Banach space of $\mu$-equivalence classes of $E^*_d$-valued, Bochner integrable functions equipped with
the weak* topology, and let \( L_{xd}^{\infty} \) be the nonempty, convex, weak* compact, and metrizable subset of \( \mu \)-equivalence classes of \( E_d^{\infty} \)-valued, Bochner integrable functions \( x_{d(\cdot)} \) with \( x_{d\omega} \in X_d \) a.e. [\( \mu \)]. Denote by \( L_{xd}^{\infty} \) the set consisting of all \( E_d^{\infty} \)-valued, Bochner integrable functions \( x_{d(\cdot)} \) with \( x_{d\omega} \in X_d \) a.e. [\( \mu \)] (i.e., \( L_{xd}^{\infty} \) is the prequotient of \( L_{xd}^{\infty} \)). We will equip \( L_{xd}^{\infty} \) with a metric, \( \rho_{xd}^{\ast}, \) compatible with the weak* topology in \( L_{xd}^{\infty} \) inherited from \( L_{xd}^{\infty} \).

Equip the product spaces \( Y := Y_1 \times \cdots \times Y_m \) and \( X := X_1 \times \cdots \times X_m \) with the sum metrics, denoted by \( \rho_Y^{\ast} := \sum_d \rho_{Y_d}^{\ast} \) and \( \rho_X^{\ast} := \sum_d \rho_{X_d}^{\ast} \), respectively. Also equip the product spaces \( Y \) and \( X \) with the Borel product \( \sigma \)-fields, \( B_Y^{\ast} = B_{Y_1}^{\ast} \times \cdots \times B_{Y_m}^{\ast} \) and \( B_X^{\ast} = B_{X_1}^{\ast} \times \cdots \times B_{X_m}^{\ast} \) generated by the \( \rho_Y^{\ast} \) and \( \rho_X^{\ast} \)-open sets in \( Y \) and \( X \), respectively.

Finally, let \( L_{Y_d}^{\infty} := L_{Y_1}^{\infty} \times \cdots \times L_{Y_m}^{\infty} \) and \( L_{X_d}^{\infty} := L_{X_1}^{\infty} \times \cdots \times L_{X_m}^{\infty} \) with \( Y_d \subset E_d^{\ast} \) for all \( d \) and let \( L_{Y_d}^{\infty} \) and \( L_{X_d}^{\infty} \) denote their prequotients. Equip \( L_{Y_d}^{\infty} \) with the sum metric \( \rho_{L_{Y_d}^{\infty}}^{\ast} := \sum_d \rho_{L_{Y_d}^{\infty}}^{\ast} \), and note that on the prequotient \( L_{Y_d}^{\infty} \), \( \rho_{L_{Y_d}^{\infty}}^{\ast} \) is a semimetric. Similarly, let \( L_{X_d}^{\infty} := L_{X_1}^{\infty} \times \cdots \times L_{X_m}^{\infty} \) and \( L_{X_d}^{\infty} := L_{X_1}^{\infty} \times \cdots \times L_{X_m}^{\infty} \) with \( X_d \subset E_d^{\ast} \) for all \( d \) and let \( L_{X_d}^{\infty} \) and \( L_{Y_d}^{\infty} \) denote their prequotients. Equip \( L_{X_d}^{\infty} \) with the sum metric, \( \rho_{L_{X_d}^{\infty}}^{\ast} := \sum_d \rho_{L_{X_d}^{\infty}}^{\ast} \), and again note that on the prequotient \( L_{X_d}^{\infty} \), \( \rho_{L_{X_d}^{\infty}}^{\ast} \) is a semimetric.

**Label the assumptions above [A-1].**

A parameterized, state-contingent game (denoted by \( \mathcal{PSG} \)) consists of a \( \nu \)-parameterized collection of strategic form games, \( \mathcal{G}_{L_{yd}^{\infty}} := \{ \mathcal{G}_\nu : \nu \in L_{yd}^{\infty} \} \), with each \( \nu \)-game \( \mathcal{G}_\nu \) being specified by primitives

\[
\mathcal{G}_\nu := \left( L_{X_d}^{\infty}, \mathcal{S}^{\infty}(\Phi_{d(\cdot), \nu_d}), U_d(v_{d\cdot}, x_{d(\cdot)}) \right)_{d\nu}
\]  

(19)

Letting \((\Omega, B_\Omega, \mu)\) be the underlying probability space of states \(\omega\), where \(\Omega\) is a complete, separable metric space, \(B_\Omega\) is the Borel \(\sigma\)-field, and \(\mu\) is a nonatomic probability measure, in the \(\nu\)-game \(\mathcal{G}_\nu\), each player \(d \in D\), seeks to choose a feasible state-contingent action (i.e., feasible strategy)

\[
x_{d(\cdot)} \in \mathcal{S}^{\infty}(\Phi_{d(\cdot), \nu_d}) \subset L_{X_d}^{\infty},
\]  

(20)

so as to maximize player \(d\)’s expected payoff

\[
U_d(v_{d\cdot}, x_{d(\cdot)}, x_{-d(\cdot)}) := \int_{\Omega} U_d(\omega, v_{d\cdot}, x_{d\omega}, x_{-d\omega}) \, d\mu(\omega),
\]

(21)

given player \(d\)’s value function \(v_{d\cdot} \in L_{Y_d}^{\infty}\), and the state-contingent actions \(x_{-d(\cdot)} \in L_{X_d}^{\infty}\) of the other players. Also \(\mathcal{S}^{\infty}(\Phi_{d(\cdot), \nu_d})\) denotes the collection of \(\mu\)-equivalence classes of measurable selections of the state-contingent constraint correspondence, \(\Phi_{d(\cdot), \nu_d}\). A typical element of a \(\mu\)-equivalence class of \(\mathcal{S}^{\infty}(\Phi_{d(\cdot), \nu_d})\) is a function \(x_{d(\cdot)} \in L_{X_d}^{\infty}\) such that \(x_{d\omega} \in \Phi_{d(\omega), \nu_d}\) a.e. [\(\mu\)]. Here, \(Y := Y_1 \times \cdots \times Y_m \subset \mathbb{R}^m\) is the set of all possible player payoff profiles, and for each player \(d = 1, 2, \ldots, m\), we have for each possible state-contingent action profile \((x_{d(\cdot)}, x_{-d(\cdot)}) \in L_{X_d}^{\infty} \times L_{X_d}^{\infty}\) that

\[
U_d(v_{d\cdot}, x_{d(\cdot)}, x_{-d(\cdot)}) := \int_{\Omega} U_d(\omega, v_{d\cdot}, x_{d\omega}, x_{-d\omega}) \, d\mu(\omega) \in Y_d = [-M, M], \quad M > 0.
\]

We will assume the following concerning a parameterized game \(\mathcal{G}_{L_{yd}^{\infty}} := \{ \mathcal{G}_\nu : \nu \in L_{yd}^{\infty} \}\). Label these assumptions [\(\mathcal{PSG}-1\)].
We have that

\[
\begin{align*}
(1) \quad (\omega, v_d) \rightarrow \Phi_d(\omega, v_d) & \text{ is player } d \text{'s Caratheodory action constraint correspondence, defined on } \Omega \times X_d^\infty \text{ taking nonempty, convex, and } \rho^*_L \text{-compact values in } X_d, \\
& \text{ with } \rho^*_L = \rho^*_L \text{-continuous in } v_d \text{ for each } \omega \in \Omega \text{ and measurable in } \omega \text{ for each } v_d \in L_\infty^X.
\end{align*}
\]

(2) \quad v_d \rightarrow S^\infty(\Phi_d(v_d)) \text{ is player } d \text{'s } \rho^*_L \text{-continuous strategy constraint correspondence, defined on } L_\infty^X \text{ taking nonempty, convex, and } \rho^*_L \text{-compact values in } L_\infty^X, \text{ with values consisting of } \mu \text{-equivalence classes of strategies } x_d(\cdot) \in L_\infty^X \text{ such that } x_{d(\omega)} \in \Phi_d(\omega, v_d) \text{ a.e. } [\mu].

(3) \quad x_d \rightarrow u_d(\omega, v_d, x_d, x_{-d}) \text{ is player } d \text{'s Caratheodory payoff function, defined on } \Omega \times X_d^\infty \times X \text{ taking values in } Y_d \text{ measurable in } \omega \text{ and weak* continuous in } (v_d, x) \text{ on } L_\infty^X \times X, \text{ with } u_d(\omega, v_d, x_{-d}) \text{ quasiconcave in } x_d \text{ given } \omega \text{ value function } v_d \in L_\infty^X, \text{ and other players' actions } x_{-d} \in X_{-d}, \text{ where for each } (v_d, x_d) \in L_\infty^X \times L_\infty^X \text{ player } d \text{'s induced state-contingent payoff function, } u_d(\cdot, v_d, x_d), \text{ is contained in } L_\infty^X, \text{ and is } \rho^*_L \times \rho^*_L \text{-continuous (i.e., } v^*_d \rightarrow v^*_d \text{ and } x^*_d \rightarrow x^*_d, \text{ then } u(\cdot, v^*_d, x^*_d) \rightarrow u(\cdot, v^*_d, x^*_d)) \text{.}

An excellent example of a PSG satisfying assumptions [PSG-1] as well as assumptions [A-1] is the one-shot game underlying any nonatomic discounted stochastic game satisfying the Nowak–Raghavan assumptions (see Nowak and Raghavan [1]). We will provide just such an example below.

A Blackwell equilibrium of a parameterized game \( G_{L_X} \) is defined as follows:

**Definition 5** (Blackwell Equilibrium – Selection Form) Let \( G_{L_X} \) be a PSG satisfying assumptions [A-1] and [PSG-1]. A value function-strategy profile \((v^*, x^*_d) \in L_\infty^X \times L_\infty^X\) is a Blackwell equilibrium of \( G_{L_X} \) provided for each player \( d \),

\[
\begin{align*}
v_d^* &= u_d(\cdot, v_d^*, x_d^*(\cdot), x_{-d}^*(\cdot)) \\
& \text{ and } \\
U_d(v_d^*, x_d^*(\cdot), x_{-d}^*(\cdot)) &= \max_{x_d(\cdot) \in S^\infty(\Phi_d(v_d^*))} U_d(v_d^*, x_d(\cdot), x_{-d}^*(\cdot)).
\end{align*}
\]

By Theorem 2.2 in Hiai and Umegaki [23], we have that

\[
\begin{align*}
\max_{x_d(\cdot) \in S^\infty(\Phi_d(v_d^*))} U_d(v_d^*, x_d(\cdot), x_{-d}^*(\cdot)) \\
& := \max_{x_d(\cdot) \in S^\infty(\Phi_d(v_d^*))} \int\Omega U_d(\omega, v_d^*, x_d(\cdot), x_{-d}^*) \, d\mu(\omega) \\
& = \int\Omega \max_{x_d(\cdot) \in S^\infty(\Phi_d(v_d^*))} U_d(\omega, v_d^*, x_d(\cdot), x_{-d}^*) \, d\mu(\omega).
\end{align*}
\]

Thus, in state-contingent form, an equivalent way of writing down the definition of a Blackwell equilibrium of a parameterized game \( G_{L_X} \) is the following:

**Definition 6** (Blackwell Equilibrium – State-Contingent Form) Let \( G_{L_X} \) be a PSG satisfying assumptions [A-1] and [PSG-1]. A value function-strategy profile \((v^*, x^*_d) \in L_\infty^X \times L_\infty^X\) is a Blackwell equilibrium provided for each player \( d \) and a.e. \([\mu]\) in \( \omega \),

\[
\begin{align*}
v_d^*(\omega) &= u_d(\omega, v_d^*, x_d^*(\omega), x_{-d}^*(\omega)) \\
& \text{ and } \\
U_d(\omega, v_d^*, x_d^*(\omega), x_{-d}^*(\omega)) &= \max_{x_d(\cdot) \in S^\infty(\Phi_d(\omega, v_d^*))} U_d(\omega, v_d^*, x_d(\cdot), x_{-d}^*(\omega)).
\end{align*}
\]
The first condition in expression (24) above requires that each player’s value function $v_d^*(\cdot)$ satisfies state by state a.e. $[\mu]$ the Bellman equation

$$v_d^*(\omega) = u_d(\omega, v_d^*, x_d^*),$$ \hspace{1cm} (25)

given players’ Nash equilibrium strategy profile

$$x_d^* := (x_d^*_{11}, \ldots, x_d^*_{m_1}) \in S^\infty(\Phi_1(\cdot, v_1)) \times \cdots \times S^\infty(\Phi_m(\cdot, v_m)) \subset L^\infty_\Phi.$$

The second condition in expression (24) above requires that each player’s strategy $x_{d(\cdot)}^* \in S^\infty(\Phi_d(\cdot, v_d)) \subset L^\infty_\Phi$ satisfies state by state a.e. $[\mu]$ the Nash condition

$$u_d(\omega, v_d^*_{d(\cdot)}, x_{d_{d(\cdot)}}^*, x_{d_{\neq d(\cdot)}}^*) = \max_{x_d \in G_d(\omega, v_d^*{d(\cdot)})} u_d(\omega, v_d^*_{d(\cdot)}, x_d, x_{d_{\neq d(\cdot)}}),$$ \hspace{1cm} (26)

given player $d$’s value function $v_d \in L^\infty_\Phi$, and the strategies of the other players $x_{d_{\neq d(\cdot)}}^* \in S^\infty(\Phi_{d_{\neq d(\cdot)}}(\cdot, v_{d_{\neq d(\cdot)}})) \subset L^\infty_{\Phi_{d_{\neq d(\cdot)}}}$.

Under assumptions [A-1] and [PSG-1], we know that each $(\omega, v)$-game,

$$G_{(\omega, v)} := (X_d, \Phi_d(\omega, v_d), u_d(\omega, v_d^*_{d(\cdot)}, x_{d_{d(\cdot)}}))_d,$$

has a nonempty, $\rho_\infty^\times$-compact set of Nash equilibria, denoted by $\mathcal{N}(\omega, v)$. Moreover, by the Berge maximum theorem and the measurable maximum theorem (see 17.31 and 18.19 in Aliprantis and Border [22]), we know that the collection of $(\omega, v)$-games $G_{(\omega, v)} : (\omega, v) \in \Omega \times L^\infty_\Omega$ has a Nash correspondence $(\omega, v) \mapsto \mathcal{N}(\omega, v)$ and an associated Nash payoff correspondence $(\omega, v) \mapsto \mathcal{P}(\omega, v)$ that are upper Carathéodory (i.e., jointly measurable in $(\omega, v)$ and upper semicontinuous in $v$ with nonempty compact values). Here, for each $(\omega, v) \in \Omega \times L^\infty_\Omega$, we have that

$$\mathcal{P}(\omega, v) := u(\omega, v, \mathcal{N}(\omega, v))$$
$$:= \left\{ u \in Y : u = (u_1(\omega, v_1, x), \ldots, u_m(\omega, v_m, x)) : x \in \mathcal{N}(\omega, v) \right\}.$$ \hspace{1cm} (27)

Let $\mathcal{G}_{\Omega \times L^\infty_\Omega}$ denote the collection of $(\omega, v)$-games underlying the parameterized game $G_{L^\infty_\omega \times L^\infty_\Phi}$. We have that $(v^*, x^*_{(\cdot)}) \in L^\infty_\Omega \times L^\infty_\Phi$ is a Blackwell equilibrium of parameterized game $G_{L^\infty_\omega \times L^\infty_\Phi}$ if and only if

$$(v^*(\omega), x^*_\omega) \in \mathcal{P}(\omega, v^*) \times \mathcal{N}(\omega, v^*) \text{ a.e. }[\mu].$$ \hspace{1cm} (28)

Moreover, let

$$\mathcal{S}^\infty(\mathcal{P}_v) := \left\{ u \in L^\infty_\Omega : u(\omega) \in \mathcal{P}(\omega, v) \text{ a.e. } [\mu] \right\} \hspace{1cm} (29)$$

and

$$\mathcal{S}^\infty(\mathcal{N}_v) := \left\{ x_{(\cdot)} \in L^\infty_\Phi : x_\omega \in \mathcal{N}(\omega, v) \text{ a.e. } [\mu] \right\}, \hspace{1cm} (30)$$
be the Nash selection correspondences, we have that \((v^*, x^*_d) \in \mathcal{L}^\infty_Y \times \mathcal{L}^\infty_X\) is a Blackwell equilibrium of parameterized game \(G_{C^\infty_Y}\) if and only if

\[
(v^*, x^*_d) \in S^\infty(P_v) \times S^\infty(N_v),
\]

where \(S^\infty(P_v)\) is the Nash payoff selection correspondence and \(S^\infty(N_v)\) is the Nash selection correspondence.

Finally, we note that for any \(u \in S^\infty(P_v)\), we can deduce the existence of Nash strategy profile \(x^*_d \in S^\infty(N_v)\) such that \(u(\omega) = u(\omega, v, x^*_d)\) a.e. [\(\mu\)] using implicit measurable selection (see Theorem 7.1 in Himmelberg [24]). Thus, a necessary and sufficient condition for parameterized game \(G_{C^\infty_Y}\) to have a Blackwell equilibrium is that there exist a \(v^* \in \mathcal{L}^\infty_Y\) such that \(v^* \in S^\infty(P_v^*)\).

### 4.1.2 Best strategy response correspondences

Given a Blackwell equilibrium \((v^*, x^*_d)\), belonging to a parameterized \(G_{C^\infty_Y}\), we have by the Nash condition above (26) and by Theorem 2.2 in Hiai and Umegaki [23] that for each player \(d\) and a.e. [\(\mu\)] in \(\omega\),

\[
x^*_d \in \arg \max_{x^*_d \in \Phi_d(\omega, v^*)} U_d(\omega, v^*, x^*_d),
\]

or equivalently,

\[
x^*_d \in \arg \max_{x^*_d \in S^\infty(\Phi_d(v^*) \omega)} \int \mathcal{D} U_d(\omega, v^*, x^*_d) d\mu(\omega) = \arg \max_{x^*_d \in S^\infty(\Phi_d(v^*) \omega)} U_d(\omega, v^*, x^*_d).
\]

Under assumptions [A-1] and \([\mathcal{P} \mathcal{S} \mathcal{G}-1]\), we have for each player \(d\) that the strategy constraint correspondence

\[
v_d \rightarrow S^\infty(\Phi_d(\cdot, v_d))
\]

is continuous and because \(v^*_d \rightarrow v_d\) and \(x^*_d \rightarrow x^*_d\) imply (via assumption \([\mathcal{P} \mathcal{S} \mathcal{G}-1]\)(3) above) that \(U_d(v^*_d, x^*_d) \rightarrow U_d(v_d, x_d)\), i.e., because the real-valued function \(U_d(\cdot, \cdot)\) is \(\rho_{L^\infty_{L(c)}}\)-continuous on \(\mathcal{L}^\infty_{X_d} \times \mathcal{L}^\infty_X\), we have by Berge’s maximum theorem (see 17.31 in Aliprantis and Border [22]) that player \(d\)’s best strategy response correspondence (i.e., the argmax correspondence)

\[
(v_d, x^*_d) \rightarrow B_d(v_d, x^*_d) := \max_{x^*_d \in S^\infty(\Phi_d(\cdot, v_d))} U_d(v_d, x^*_d)
\]

is upper semicontinuous and takes nonempty and \(\rho_{L^\infty_{L(c)}}\)-compact values. Moreover, because \(U_d(v_d, x^*_d)\) is quasiconcave on \(\mathcal{L}^\infty_{X_d}\) and because \(S^\infty(\Phi_d(\cdot, v_d))\) is convex-valued, we have that player \(d\)’s best strategy response correspondence \(B_d(\cdot, \cdot)\) is a CUSCO* (i.e., is a convex valued USCO*; see Hola and Holy [18]). Thus, the best strategy response correspondence \(B(\cdot, \cdot) : \mathcal{L}^\infty_Y \times \mathcal{L}^\infty_X \rightarrow P_{\mathcal{C}}(\mathcal{L}^\infty_X)\) belonging to the parameterized \(G_{C^\infty_Y}\), given
by

\[ B(v, x_i) := B(v_1, x_{i(1)}) \times \cdots \times B(v_m, x_{i(m)}) \]

where for each player \( d = 1, \ldots, m, \)

\[ B_d(v_d, x_{d,i}) := \{ x_{d,i} \in L^\infty_X : x_{d,i} \in \arg \max x_{d,i} \in S^\infty(\Phi_d(v_d)) U_d(v_d, x'_{d,i}, x_{-d,i}) \} \]  

(33)

is also a CUSCO* but one defined on \( L^\infty_X \times L^\infty_X \) with nonempty, convex, and \( \rho^\infty \)-compact values in \( S^\infty(\Phi(\cdot, v_d)) \subset L^\infty_X \), implying that players' best strategy response correspondence \( B(v, \cdot) \) is a CUSCO* for each possible profile of player value functions \( v \). Moreover, the players' best strategy response CUSCO* has associated with it a fixed point correspondence

\[ \nu \mapsto \mathcal{E}_B(v) = \{ x_i \in L^\infty_X : x_i \in B(v, x_i) \} \]  

(34)

(see 17.28 in Aliprantis and Border [22]). By the Kakutani–Fan–Glicksberg fixed point theorem (see 17.55 in Aliprantis and Border [22]), we know that each \( \nu \)-BSR CUSCO* \( B(v, \cdot) : L^\infty_X \rightarrow P_{gf}(L^\infty_X) \) has a nonempty, \( \rho^\infty \)-compact set of fixed points. Thus, \( \mathcal{E}_B(\cdot) \) takes nonempty values, and because \( B(\cdot, \cdot) \) is a CUSCO*, and therefore because \( B(\cdot, \cdot) \) has a weak* closed graph in \( L^\infty_Y \times L^\infty_X \times L^\infty_X \), it is easy to show that \( \mathcal{E}_B(\cdot) \) has a weak* closed graph in \( L^\infty_Y \times L^\infty_X \), implying that \( \mathcal{E}_B(\cdot) \) is an USCO* (see Hola and Holy [18]) and takes decomposable values in \( L^\infty_X \). By Lemma 1 above \( \mathcal{E}_B(\cdot) \) is a K-correspondence. It then follows from Theorems 1–6 above that the correspondence \( \nu \mapsto \mathcal{U}(\nu) = T(\nu, \mathcal{E}_B(\nu)) \) has fixed points in \( L^\infty_Y \).

Moreover, because \( \mathcal{E}_B(\nu) = S^\infty(N_\nu) \) and \( T(\nu, \mathcal{E}_B(\nu)) = S^\infty(P_\nu) \) for all \( \nu \in L^\infty_Y \), where \( S^\infty(N_\nu) \) is the game's Nash selection correspondence and where \( S^\infty(P_\nu) \) is the game's Nash payoff selection correspondence, we know that the game has *Blackwell equilibria*, provided there exists \( (\nu^*, x^*_\nu) \in L^\infty_Y \times L^\infty_X \) with

\[ (\nu^*, x^*_\nu) \in S^\infty(P_\nu^*) \times S^\infty(N_\nu^*). \]  

(35)

Thus, for any Blackwell equilibrium \( (\nu^*, x^*_\nu) \), we have, for each player \( d \) and for \( \mu \)-almost every state \( \omega \), that

\[ v^*_d(\omega) = u_d(\omega, \nu^*_d, x^*_d, x^*_{d\omega}), \]

\[ u_d(\omega, \nu^*_d, x^*_d, x^*_{d\omega}) = \max_{x_{d\omega} \in \Phi_d(\omega, \nu^*_d)} u_d(\omega, \nu^*_d, x^*_d, x^*_{d\omega}). \]  

(36)

As we will see in the next subsection, the key facts about a one-shot \( P\mathcal{S}\mathcal{G} \) belonging to a discounted stochastic game \( D\mathcal{S}\mathcal{G} \) is that it has an equilibrium strategy correspondence \( \mathcal{E}_B(\cdot) \) and an equilibrium payoff function correspondence \( T(\cdot, \mathcal{E}_B(\cdot)) \), that are K-correspondences with decomposable values. These facts are particularly important because they guarantee that equilibrium payoff correspondence \( T(\cdot, \mathcal{E}_B(\cdot)) \) is an USCO*, and if \( \mu \) is nonatomic then \( T(\cdot, \mathcal{E}_B(\cdot)) \) is an USCO* with contractible values – guaranteeing that \( T(\cdot, \mathcal{E}_B(\cdot)) \) is approximable – and finally, that \( T(\cdot, \mathcal{E}_B(\cdot)) \) has fixed points in \( L^\infty_Y \). Moreover, given that for all \( \nu \in L^\infty_Y \),

\[ S^\infty(N_\nu) = \mathcal{E}_B(\nu) \quad \text{and} \quad S^\infty(P_\nu) = T(\nu, \mathcal{E}_B(\nu)), \]  

(37)
for the Nash selection correspondences $S^\infty(N \mu)$ and $S^\infty(P \mu)$, belonging to the parameterized game $G^\infty_{\gamma}$, we have that the Nash payoff selection correspondence $S^\infty(P \mu)$ is approximable and has fixed points.

### 4.2 Discounted stochastic games (DSGs)

For the one-shot $P \sigma$ game

\[
G^\sigma \subseteq \llbracket G^\sigma : \nu \in L^\infty_{\gamma} \rrbracket,
\]  

underlying any discounted stochastic game $D \sigma$, satisfying the Nowak–Raghavan assumptions, each $\nu$-game $G^\nu$ is specified by the primitives

\[
G^\nu := (L^\infty_{\Delta(A_d)}, S^\infty(\Delta(\Phi_d(\cdot))), U_d(\nu_d, \sigma_{-d}(\cdot)))_{d}.
\]  

The following list of Nowak–Raghavan assumptions specializes and extends assumptions [A-1] and [PSG-1] above:

1. $(\Omega, B_\omega, \mu)$ is a probability space of states $\omega$, where $\Omega$ is a complete, separable metric space with probability measure $\mu$ defined on the Borel $\sigma$-field $B_\omega$.

2. $A$ is a compact metric space of actions available to player $d$. We will equip $A_d$ with metric $\rho_{A_d}$ and we will equip $A := A_1 \times \cdots \times A_m$ with the sum metric, $\rho_A = \sum_d \rho_{A_d}$.

3. $\Phi_d(\cdot)$ is player $d$'s measurable, state-contingent action constraint correspondence (not dependent on $\nu_d \in L^\infty_{\gamma}$).

4. $\Delta(\Phi_d(\cdot))$ is player $d$'s measurable, state-contingent behavioral action constraint correspondence with

\[
\Delta(\Phi_d(\omega)) := \{ \sigma_d \in \Delta(\Lambda_d) : \sigma_d(\Phi_d(\omega)) = 1 \ a.e. [\mu] \}.
\]  

5. $L^\infty_{\Delta(A_d)}$ is the collection of $\mu$-equivalences classes of measurable functions $\sigma_d(\cdot) : \Omega \rightarrow \Delta(\Lambda_d)$ (i.e., for $\omega$ a.e. $[\mu]$), $\sigma_d(\cdot|\omega) \in \Delta(\Lambda_d)$) and is the norm dual of $L^\infty_{\Delta(A_d)}$, where $C(\Lambda_d)$ is the Banach space of continuous functions on compact metric space $\Lambda_d$ with the sup-norm (see Theorem V-2 in Castaing and Valadier [25]).

6. $S^\infty(\Delta(\Phi_d(\cdot)))$ is player $d$’s behavioral strategy set given by the collection of $\mu$-equivalences classes of measurable functions $\sigma_d(\cdot) : \Omega \rightarrow \Delta(\Lambda_d)$ such that for $\omega$ a.e. $[\mu]$, $\sigma_d(\cdot|\omega) \in \Delta(\Lambda_d)$ and $\sigma_d(\Phi_d(\omega)|\omega) = 1$ (i.e., $\sigma_d(\cdot|\omega) \in \Delta(\Phi_d(\omega))$ a.e. $[\mu]$). We will equip $S^\infty(\Delta(\Phi_d(\cdot)))$ with a metric $\rho_{W_\sigma}$ compatible with its $W_\sigma$-topology inherited from $L^\infty_{\Delta(A_d)}$.

7. $S^\infty(\Delta(\Phi(\cdot))) := S^\infty(\Delta(\Phi_1(\cdot))) \times \cdots \times S^\infty(\Delta(\Phi_m(\cdot)))$ is the set of players’ strategy profiles. We will equip $S^\infty(\Delta(\Phi(\cdot)))$ with the sum metric, $\rho_{W_\sigma} := \sum_d \rho_{W_\sigma}$, and we will denote a typical element of $S^\infty(\Delta(\Phi(\cdot)))$ by

\[
\sigma(da_{1|}) := (\sigma_1(da_{1|}), \ldots, \sigma_m(da_{m|})).
\]
Also, given \( \sigma(da|\omega) := (\sigma_1(da_1|\omega), \ldots, \sigma_m(da_m|\omega)) \in S^\infty(\Delta(\Phi(\cdot))) \), for each state \( \omega \) we will denote by

\[
\pi \sigma(da|\omega) := \pi(\sigma_1(da_1|\omega), \ldots, \sigma_m(da_m|\omega)) := \bigotimes_{d=1}^m \sigma_d(da_d|\omega) \in \Delta(\Phi(\omega))
\]

the product probability measure determined by the \( m \)-tuple of probability measures, 
\( \sigma(1(da_1|\omega), \ldots, \sigma_m(da_m|\omega)) \) (i.e., behavioral actions) chosen by the players in state \( \omega \).

(8) \( L^\infty_{\Omega_1} := L^\infty_{\Omega_1,M,M} \subset L^\infty_R \) and \( L^\infty_Y := L^\infty_{\Omega_1,M,M} \subset L^\infty_{\Omega_1} \) is the set of players’ state-contingent payoff profiles.

(9) \( r_d(\cdot, \cdot) : \Omega \times A \rightarrow Y_d \) is player \( d \)’s Carathéodory stage payoff function (i.e., \( r_d(\omega, a) \) is \( \rho_\Phi \)-continuous on \( A \) and \( r_d(\cdot, a) \) is \( (B_\Omega, B_{Y_d}) \)-measurable on \( \Omega \) for each \( a \)).

(10) \( q(\cdot|\cdot, \cdot) : \Omega \times A \rightarrow \Delta(\Omega) \) is the law of motion defined on \( \Omega \times A \) taking values in the space of probability measures on \( \Omega \), having the following properties: (i) \( q(\cdot|\omega, a) \ll \mu \) for all \( (\omega, a) \in \Omega \times A \) (i.e., each probability measure \( q(\cdot|\omega, a) \) is absolutely continuous with respect to the probability measure \( \mu \)), (ii) for each \( E \in B_\Omega, q(E|\cdot, \cdot) \) is measurable on \( \Omega \times A \), and (iii) the collection of probability density functions

\[
H^d_\mu := \{ h(\cdot|\omega, a) : (\omega, a) \in \Omega \times A \}
\]

of \( q(\cdot|\omega, a) \) with respect to \( \mu \) is such that for each state \( \omega \), the function

\[
a \rightarrow h(a|\omega, a)
\]

is continuous in \( a \) e. [\( \mu \)] in \( \omega \).

(11) \( \sigma_d(\cdot) \rightarrow U_d(\nu_d, \sigma_d(\cdot), \sigma_{-d}(\cdot)) \) is player \( d \)’s expected stage payoff function given 
\( (\nu_d, \sigma_{-d}(\cdot)) \in L^\infty_{\Omega_1} \times L^\infty_{\Delta(\Omega_1)} \), where

\[
U_d(\nu_d, \sigma(\cdot)) := \int_\Omega (1 - \beta) \int_a r_d(\omega, a) \pi(\sigma(da|\omega)) + \beta_d \int_a [\int_{\Omega_1} \nu_d(\omega') h(\omega'|\omega, a) d\mu(\omega')] \pi(\sigma(da|\omega)) d\mu(\omega)
\]

\[
= \int_\Omega \int_a u_d(\omega, \nu_d, \sigma(\omega)) d\mu(\omega)
\]

\[
= \int_\Omega \int_A u_d(\omega, \nu_d, a) \pi(\sigma(da|\omega)) d\mu(\omega),
\]

where for each \( \omega \), \( (\nu_d, a) \rightarrow u_d(\omega, \nu_d, a) \) is jointly continuous in \( (\nu_d, a) \) on \( L^\infty_{\Omega_1} \times A \) (see Appendix 2).

Label the assumptions above \( \{DSG^{NR}\} \).

Sometimes we will write \( \sigma_1(\omega) \otimes \cdots \otimes \sigma_m(\omega) \) rather than \( \sigma_1(da_1|\omega) \otimes \cdots \otimes \sigma_m(da_m|\omega) \) to denote the product probability measure determined by \( \sigma(da|\cdot) := (\sigma_1(da_1|\cdot), \ldots, \sigma_m(da_m|\cdot)) \) \( \in S^\infty(\Delta(\Phi(\cdot))) \). Also, we will write

\[
\pi(\sigma_d(\cdot), \sigma_{-d}(\cdot)) = \pi(\sigma_{-d}(\cdot)) \otimes \sigma_d(\cdot) = \pi(\sigma_d(\cdot)) \otimes \sigma_{-d}(da_d|\cdot),
\]

where \( \pi(\sigma_d(\cdot)) := \bigotimes_{d \in D_{\Omega} \setminus \{d\}} \sigma_d(\cdot) \). Note that \( U_d(\nu_d, \sigma_d(\cdot), \sigma_{-d}(\cdot)) \) is linear in \( \sigma_d(\cdot) \in L^\infty_{\Delta(\Omega_1)} \).

In order to show that the parameterized, one-shot game \( G^{NR}_{\Omega_1} \), underlying any Nowak–Raghavan discounted stochastic game \( DSG^{NR} \), has Blackwell equilibria, it suffices to verify
that each player \( d \) has a state-contingent, expected stage payoff function

\[
u_d(\cdot, v_d, \sigma(\cdot)) := \int_A \left( (1 - \beta_d) r_d(\cdot, a) + \beta_d \int_{\Omega} v_d(\omega') h(\omega'|\cdot, a) \, d\mu(\omega') \right) \pi(d\sigma(\cdot))
\]

\[(42)\]

satisfying assumption \([\mathcal{PSG}-1]\)(3) above.

**Theorem 7** (Continuity of Players’ State-Contingent Payoff Functions) Consider a \( \mathcal{DSG}^{NR} \) having a one-shot game \( \mathcal{G}_v^{NR} \) satisfying assumptions \([\mathcal{DSG}^{NR}]\) with players’ state-contingent payoff function profile

\[
(v, \sigma(\cdot)) \rightarrow u(\cdot, v, \sigma(\cdot)) := (u_1(\cdot, v_1, \sigma(\cdot)), \ldots, u_m(\cdot, v_m, \sigma(\cdot)) \in L^\infty_Y.
\]

If \( \{(v^n, \sigma^n(\cdot))\}_n \) is a sequence of value function-strategy profile pairs in \( L^\infty_Y \times S^\infty(\Delta(\Phi(\cdot))) \) such that \( v^n \rightarrow v^* \) and \( \sigma^n(\cdot) \Rightarrow \sigma^*(\cdot) \), then the sequence of state-contingent, expected stage payoff functions in \( (42) \),

\[
\rho^{\sigma^n(\cdot)} \text{-converges to } u(\cdot, v^*, \sigma^*(\cdot)) \in L^\infty_Y.
\]

**Proof** Let \( \{(v^n, \sigma^n(\cdot))\}_n \subset L^\infty_Y \times S^\infty(\Delta(\Phi(\cdot))) \) be a sequence such that

\[
\begin{aligned}
&\text{\( v^n \rightarrow v^* \in L^\infty_Y \) } \quad \text{ and } \quad \text{\( \rho^{\sigma^n(\cdot)} \Rightarrow \rho^{\sigma^*(\cdot)} \in S^\infty(\Delta(\Phi(\cdot))) \).} \\
&\text{By Balder [26], we have for every subsequence \( \{(v^{n_k}, \sigma^{n_k}(\cdot))\}_k \) of \( \{v^n, \sigma^n(\cdot)\}_n \), a further subsequence \( \{v^{n_{k_r}}, \sigma^{n_{k_r}}(\cdot)\}_r \), such that} \\
&\text{\( v^{n_{k_r}} \rightarrow v^* \) and \( \sigma^{n_{k_r}}(\cdot) \rightarrow \sigma^*(\cdot). \) (44)}
\end{aligned}
\]

First, consider the sequence \( \{R_d(\cdot, \sigma^n(\cdot))\}_n \subset L^\infty_{Y_d} \) where

\[
R_d(\cdot, \sigma^n(\cdot)) := \int_A r_d(\omega, a) \pi (d\sigma^n(\cdot)) \omega.
\]

Because \( \sigma^n(\cdot) \Rightarrow \sigma^*(\cdot) \), by the linearity of the integral in \( (45) \), we have for any subsequence \( \{\sigma^{n_i}(\cdot)\}_i \) of \( \{\sigma^n(\cdot)\}_n \) that

\[
R_d \left( \omega, \frac{1}{k} \sum_{r=1}^k \sigma^{n_{k_r}}(\cdot) \right) = \frac{1}{k} \sum_{r=1}^k R_d \left( \omega, \sigma^{n_{k_r}}(\cdot) \right).
\]
Thus, because $\sigma^n(\cdot) \Rightarrow \sigma^*(\cdot)$, we have by the linearity of the integral in (45), that for every subsequence $\{R_d(\cdot, \sigma^n(\cdot))\}_k$ of $\{R_d(\cdot, \sigma^*(\cdot))\}_n$, there is a further subsequence $\{R_d(\cdot, \sigma^n(\cdot))\}_k$, such that $R_d^* = R_d(\cdot, \sigma^*(\cdot)) \in \mathcal{L}_{Yd}^\infty$. Thus, by (1) $\iff$ (2) of Balder [26], we have that for each player $d$, 

$$R_d(\cdot, \sigma^n(\cdot)) \rightarrow R_d(\cdot, \sigma^*(\cdot)).$$

Next, consider

$$V(\omega, \nu_d, a) := \int_{\Omega} v_d(\omega') h(\omega'|\omega, a) \, d\mu(\omega').$$

For $v_d' \in \mathcal{L}_{Yd}^\infty$, by arguments similar to those immediately above, we have for each player $d$ that

$$V(\cdot, v_d', \sigma^n(\cdot)) := \int_{\mathcal{A}} V(\cdot, v_d', a) \pi \sigma^n(da|\cdot)$$

$$\rightarrow \int_{\rho_{w_d}} V(\cdot, v_d', a) \pi \sigma^*(da|\cdot) := V(\cdot, v_d', \sigma^*(\cdot)). \quad (47)$$

Finally, because $(\mathcal{L}_{Yd}^\infty \times A, \rho_{w_d} \times A)$ is a compact metric space and because for each $\omega \in \Omega$, $V(\omega, \cdot, \cdot)$ is continuous on $\mathcal{L}_{Yd}^\infty \times A$, we have for each $\omega \in \Omega$, that $V(\omega, \cdot, \cdot)$ is uniformly continuous on $\mathcal{L}_{Yd}^\infty \times A$ for each $d$ (see proof of Theorem A.1 in Appendix 2). This implies that for any $\varepsilon > 0$, there exists a $\delta_\varepsilon > 0$ such that for any $d$ and any pair, $(v_d, a)$ and $(v_d', a')$ in $\mathcal{L}_{Yd}^\infty \times A$ with

$$\rho_{w_d}(v_d, a), (v_d', a') := \rho_{w_d}(v_d, v_d') + \rho_A(a, a') < \delta_\varepsilon,$$

we have $|V(\omega, v_d, a) - V(\omega, v_d', a')| < \varepsilon$. Because for each player $d$, $v_d' \rightarrow v_d$ (i.e., $\rho_{w_d}(v_d', v_d') \rightarrow 0$), we can choose $N_\varepsilon$ sufficiently large so that for $n > N_\varepsilon$,

$$|V(\omega, v_d', a) - V(\omega, v_d, a)| < \varepsilon \quad \text{for each } d \text{ and for all } a \in A.$$
implying that for each \(d\),

\[
\int_{\mathcal{A}} V(\cdot, v_d^\alpha, a) \pi_{\alpha}(da|\cdot) \rightarrow_{\rho_{U^2}} \int_{\mathcal{A}} V(\cdot, v_d^\alpha, a) \pi_{\alpha}(da|\cdot).
\]  

(48)

Together expressions (47) and (48) imply that for each \(d\),

\[
V(\cdot, v_d^\alpha, a) := \int_{\mathcal{A}} V(\cdot, v_d^\alpha, a) \pi_{\alpha}(da|\cdot) := \int_{\Omega} \int_{\Omega} v_d^\alpha(\omega^1|\cdot, a) d\mu(\omega^1) \pi_{\alpha}(da|\cdot)
\]

\[
\int_{\Omega} \int_{\Omega} v_d^\alpha(\omega^1|\cdot, a) d\mu(\omega^1) \pi_{\alpha}(da|\cdot) := \int_{\mathcal{A}} V(\cdot, v_d^\alpha, a) \pi_{\alpha}(da|\cdot) := V(\cdot, v_d^\alpha, a) \pi_{\alpha}(da|\cdot).
\]

We have, therefore, for each \(d\) that \(v_d^\alpha \rightarrow v_d^\alpha\) and \(\sigma^n(\cdot) \Rightarrow \sigma^*(\cdot)\) imply that

\[
\{ u_d(\cdot, v_d^\alpha, \sigma^n(\cdot)) := (1 - \beta_d)R_d(\cdot, \sigma^n(\cdot)) + \beta_d V(\cdot, v_d^\alpha, \sigma^n(\cdot)) \} \rightarrow_{\rho_{U^2}} V(\cdot, v_d^\alpha, a) \pi_{\alpha}(da|\cdot).
\]

(49)

Because \(Y\) is bounded and because \(\nu^n \rightarrow \nu^*\) and \(\sigma^n(\cdot) \Rightarrow \sigma^*(\cdot)\) imply that

\[
\{ u(\cdot, v^n, \sigma^n(\cdot)) \} \rightarrow_{\rho_{U^2}} u(\cdot, v^*, \sigma^*(\cdot)),
\]

we have that

\[
U(v^n, \sigma^n(\cdot)) = \int_{\Omega} u(\cdot, v^n, \sigma^n(\cdot)) d\mu(\omega) \rightarrow_{\rho_Y} \int_{\Omega} u(\cdot, v^*, \sigma^*(\cdot)) d\mu(\omega) = U(v^*, \sigma^*(\cdot)).
\]

Thus, by Theorems 1–6 above, the correspondence

\[
\nu \rightarrow T(\nu, E_\mathcal{B}(\nu)) := \{ u(\cdot, \nu, \sigma(\cdot)) \in \mathcal{L}_Y^\infty : \sigma(\cdot) \in E_\mathcal{B}(\nu) \}
\]

has fixed points in \(\mathcal{L}_Y^\infty\), and because \(T(\nu, E_\mathcal{B}(\nu)) = S^\infty(P_{\nu})\) for all \(\nu \in \mathcal{L}_Y^\infty\), where \(S^\infty(P_{\nu})\) is the one-shot game’s (i.e., \(G_{\mathcal{L}_Y^\infty, NR}\)) Nash payoff selection correspondence, we know that the game has a Blackwell equilibrium – a pair \((v^*, \sigma^*(\cdot))\) \(\in \mathcal{L}_Y^\infty \times \mathcal{L}_A^\infty\) such that

\[
(v^*, \sigma^*(\cdot)) \in S^\infty(P_{\nu^*}) \times S^\infty(\mathcal{N}^{\nu^*}).
\]

(50)

Thus, we have for each player \(d\) and for \(\mu\)-almost every state \(\omega\) that

\[
\nu_d^\alpha(\omega) = u_d(\omega, v_d^\alpha(\cdot), \sigma_d^\alpha(\omega)), \quad \sigma_d^\alpha(\omega))
\]

\[
\text{and} \quad u_d(\omega, v_d^\alpha(\cdot), \sigma_d^\alpha(\omega)) = \max_{\sigma_d \in \Delta(\sigma_d(\omega))} u_d(\omega, v_d^\alpha(\cdot), \sigma_d(\omega)).
\]

(51)

The following theorem, our main result on the existence of stationary Markov perfect equilibria for uncountable-compact \(\mathcal{D}SG\)s satisfying the Nowak–Raghavan assumptions, summarizes the discussion above.
Therefore, if the Nash selection correspondence v ∋ (T(v, x_1)) : x_1 ∈ E(v) = T(v, E(v)),

(52)

where E(·) : L_Ω^∞ → P(L_Ω^∞) and T(·, ·) : L_Ω^∞ × L_Ω^∞ → P(L_Ω^∞) are K-correspondences, we show that if the probability space of states (Ω, B_Ω, μ), underlying the function spaces L_Ω^∞ and L_Ω^∞, is equipped with a nonatomic probability measure μ, then for each v ∈ L_Ω^∞, T(v, E(v)) is contractible with respect to the compatibly metrized weak* topologies, and, as a consequence, the composition correspondence T(·, E(·)) is approximable and has fixed points.

Next, we consider a discounted stochastic game DSG, satisfying the Nowak–Raghavan assumptions, with underlying parameterized, state-contingent, one-shot game G^{NR}_{C^2}, with Nash selection correspondence v ↦ S^∞(N_v), and Nash payoff selection correspondence v ↦ S^∞(P_v). We show that if the correspondence E(·) in expression (52) is given by

v ↦ E_B(v) := \{x_1 ∈ L_Ω^∞ : x_1 ∈ B(v, x_1)\},

where B(·, ·) : L_Ω^∞ × L_Ω^∞ → P_β(L_Ω^∞) is the best strategy response correspondence (a CUSCO*) belonging to the DSG’s parameterized, state-contingent, one-shot game G^{NR}_{C^2}, then E_B(·) and T(·, E_B(·)) are K-correspondences with

S^∞(N_v) = E_B(v) and S^∞(P_v) = T(v, E_B(v)).

Therefore, if the DSG has a probability space of states (Ω, B_Ω, μ), equipped with a nonatomic probability measure μ, then, by our Theorem 6, the Nash payoff selection correspondence v ↦ S^∞(P_v) = T(v, E_B(v)) has fixed points. By Blackwell’s theorem [3], it then follows that the nonatomic, Nowak–Raghavan DSG to which S^∞(P_v) belongs has stationary Markov perfect equilibria. All of this is true provided the DSG satisfies the
Nowak–Raghavan assumptions. It then suffices to show that each player’s state-contingent payoff function

\[ u_d(v_d, \sigma) = (1 - \beta_d) \int_A r_d(v, a) \pi \sigma(da) \]

is sequentially weak* continuous. Thus, we show in Theorem 7 that if \( v^n \to v^* \) and \( \sigma^n(\cdot) \to \sigma(\cdot) \), then \( u(\cdot, v^n, \sigma^n(\cdot)) \to u(\cdot, v^*, \sigma(\cdot)) \).

**Appendix 1: Measurability and integrability of Banach space valued functions**

Recall the set-up: \( (\Omega, B_\Omega, \mu) \) is a probability space of states (not necessarily nonatomic), where \( \Omega \) is Polish with probability measure \( \mu \) defined on the Borel \( \sigma \)-field \( B_\Omega \), and \( Y \) is a norm-bounded, weak* -closed (i.e., \( w^* \)-closed), convex subset of \( F^* \), the norm dual of a separable Banach space \( F \), with \( F^* \) having the Radon–Nikodym property.

1. Because \( F \) is separable, \( F^* \) is (norm-) separable if and only if \( F^* \) has the Radon–Nikodym property (Bourgin [27], Theorem 5.2.12). Moreover, \( F^* \) has the Radon–Nikodym property if and only if for each \( \mu \)-continuous vector measure \( G : B_\Omega \to F^* \) of bounded variation, there exists an integrable function \( g \in L^1_\mu \), such that for all \( A \in B_\Omega \),

\[ G(A) = \int_A g(\omega) \, d\mu(\omega) \]

(Diestel and Uhl [11]).

2. A function \( v : \Omega \to F^* \) is (a) strongly measurable if there exists a sequence \( \{\varphi^n\}_n \) of \( F^* \)-valued, \( (B_\Omega, \mu) \)-measurable) simple functions such that

\[ \|v(\omega) - \varphi^n(\omega)\| \to 0 \quad \text{a.e.} \; [\mu], \]

(b) scalarly or weakly measurable if \( \omega \to \langle l, v(\omega) \rangle \) is \( (B_\Omega, B_B) \)-measurable for all \( l \in E \), where \( B_B \) is the Borel \( \sigma \)-field in \( R \) (the real numbers), and (c) \( (B_\Omega, B_{w^*}) \)-measurable if for all Borel sets \( B \in B_{w^*} \),

\[ v^{-1}(B) = \{ \omega \in \Omega : v(\omega) \in B \} \in B_\Omega, \]

where \( B_{w^*} \) is the Borel \( \sigma \)-field generated by the \( w^* \)-topology in \( F^* \). By Lemma 11.37 in Aliprantis and Border [22], if \( v(\cdot) \) is strongly measurable, then \( v(\cdot) \) is \( (B_\Omega, B_{w^*}) \)-measurable.

By the Pettis measurability theorem (Diestel and Uhl [11], p. 42) if \( v(\Omega \setminus N) \) is norm separable for \( N \in B_\Omega \) with \( \mu(N) = 0 \) (i.e., off a set of \( \mu \)-measure zero), then the range of \( v(\cdot) \) is norm separable and if \( v(\cdot) \) is \( (B_\Omega, B_{w^*}) \)-measurable, then \( v(\cdot) \) is strongly measurable. In addition, by Proposition A.1 in Cornet and Martin-da-Rocha [28], \( v(\cdot) \) is \( (B_\Omega, B_{w^*}) \)-measurable if and only if \( v(\cdot) \) is scalar measurable. Thus, letting \( L^\infty_{w^*} \) (the prequotient of \( L^\infty_{w^*} \)) be the set of all \( (B_\Omega, B_{w^*}) \)-measurable functions defined on \( \Omega \) taking values a.e. \( [\mu] \) in the \( w^* \)-closed and \( \| \cdot \|_* \)-bounded subset \( Y \) of the norm dual \( F^* \), we have for each \( v(\cdot) \in L^\infty_{w^*} \) that \( v(\cdot) \) is strongly measurable because \( v(\Omega \setminus N) \subseteq Y \) for \( N \in B_\Omega \) with \( \mu(N) = 0 \) and, by
Theorem 7.7 in Kahn [29], \( Y \) is \( \| \cdot \|^* \)-separable. Thus, each function \( \nu(\cdot) \) in the prequotient space \( L_1^\infty \) of \( (\Omega, B, \mu) \)-measurable functions defined on \( \Omega \) and taking values a.e. \([\mu]\) in the \( \omega^* \)-compact subset \( Y \) of the norm dual \( F^* \) is not only \( (\Omega, B, \mu) \)-measurable, but also scalarly measurable, as well as strongly measurable.

(3) Given probability space \((\Omega, B, \mu)\), a strongly measurable \( F^* \)-valued function \( f \) is Bochner integrable if there is a sequence of simple functions \( \{\varphi^n\}_n \) such that

\[
\int_{\Omega} \| f(\omega) - \varphi^n(\omega) \|^* \, d\mu(\omega) = 0.
\]

If \( f \) is Bochner integrable, then for each \( A \in B \), \( \int_A f \, d\mu \) is defined by

\[
\int_A f(\omega) := \lim_n \int_A \varphi^n(\omega) \, d\mu(\omega),
\]

where the last limit is in the norm topology on \( F^* \) (i.e., the \( \| \cdot \|^* \)-topology). By Theorem 2 of Diestel and Uhl [11] (p. 45), a strongly measurable, \( F^* \)-valued function \( f \) is Bochner integrable if and only if its real-valued norm function \( \| f(\cdot) \|^* \) is integrable, i.e.,

\[
\int_{\Omega} \| f(\omega) \|^* \, d\mu(\omega) < \infty.
\]

Let \( L_1^\infty \) denote the Banach space of \( \mu \)-equivalence classes of Bochner integrable functions \( v: \Omega \to F^* \) such that

\[
\| v \| := \text{ess sup} \{ \| v(\omega) \|^* : \omega \in \Omega \} < \infty,
\]

and let \( L_1^1 \) denote the Banach space of \( \mu \)-equivalence classes of Bochner integrable functions (with prequotient \( L_1^1 \)) \( l: \Omega \to F \) such that

\[
\| l(\omega) \|_1 := \int_{\Omega} \| l(\omega) \| \, d\mu(\omega) < \infty.
\]

Because \( F^* \) has the Radon–Nikodym property, it follows from Theorem 1 of Diestel and Uhl [11] (p. 98) that \( L_1^\infty \) is the norm dual of \( L_1^1 \).

Appendix 2: Continuity of payoff functions

**Theorem A.1** (Continuity of Payoff Functions) Suppose assumptions \([DSG^{NR}]\) hold. Consider a \( DSG \) with underlying state-contingent game \( G_{\Omega \times L_1^\infty} \), given by

\[
G_{\Omega \times L_1^\infty} := \{(A_d, \Phi_d(\omega), u_d(\omega, v_d, \cdot))_d : (\omega, v) \in \Omega \times L_1^\infty \},
\]

where

\[
u_d(\omega, v_d, a) := (1 - \beta_d)r_d(\omega, a) + \beta_d \int_{\Omega} \nu_d(\omega') H(\omega'|\omega, a) \, d\mu(\omega'),
\]

and

\[
(v, a) \mapsto u(\omega, v, a) := (u_1(\omega, v_1, a), \ldots, u_m(\omega, v_m, a)).
\]
If \( \{(v^n, a^n)\}_n \) is a sequence in \( \mathcal{L}_2^{\infty} \times A \) such that \( v^n \to v^* \) and \( a^n \to a^* \), then in each state \( \omega \in \Omega \),
\[
u(\omega, v^n, a^n) \to \nu(\omega, v^*, a^*).
\]

**Proof.** Let \( \{(v^n, a^n)\}_n \) be a sequence such that \( v^n \to v^* \) and \( a^n \to a^* \). Let \( \omega \) be given and fixed, and observe that for each player \( d \):
\[
\begin{align*}
|\nu_d(\omega, v^n_d, a^n) - \nu_d(\omega, v^*_d, a^*)|_R &\leq (|\nu_d(\omega, v^n_d, a^n) - \nu_d(\omega, v^*_d, a^*)|_R + |\nu_d(\omega, v^n_d, a^n) - \nu_d(\omega, v^*_d, a^*)|_R) \\
&\leq \left\{ \begin{array}{l}
\left( \int_{\Omega} \nu_d(\omega, v^n_d, a^n) d\mu(\omega) \right) - \int_{\Omega} \nu_d(\omega, v^*_d, a^*) d\mu(\omega)
\end{array} \right\}.
\end{align*}
\]

We will carry out our proof for one player \( d \), keeping in mind that the argument holds for all players simultaneously. Consider \( B^n \) first. We have
\[
B^n = \beta_d \left( \int_{\Omega} \nu_d(\omega, v^n_d, a^n) d\mu(\omega) \right) - \int_{\Omega} \nu_d(\omega, v^*_d, a^*) d\mu(\omega).
\]

Let \( h(\cdot | \omega, a^*) \) be a density of \( q(\cdot | \omega, a^*) \) with respect to \( \mu \). Given that \( h(\cdot | \omega, a^*) \in \mathcal{L}_R^1 \) and \( v^n_d \to v^*_d \), where \( \{v^n_d\}_n \subset \mathcal{L}_R^{\infty} \subset \mathcal{L}_2^{\infty} \), we have
\[
\begin{align*}
\int_{\Omega} \nu_d(\omega, v^n_d, a^n) d\mu(\omega) &= \int_{\Omega} \nu_d(\omega, v^*_d, a^*) d\mu(\omega)
\end{align*}
\]

Thus, \( B^n \to 0 \).

Next, consider \( A^n \). We have
\[
A^n \leq (1 - \beta_d) \left( \int_{\Omega} r_d(\omega, a^n) d\mu(\omega) \right) - \int_{\Omega} r_d(\omega, a^*) d\mu(\omega).
\]

Continuity of \( r_d(\omega, \cdot) \) and \( a^n \to a^* \) imply that \( A^n \to 0 \). To see that \( A^n \to 0 \), observe that
\[
\begin{align*}
&\left\{ \int_{\Omega} \nu_d(\omega, v^n_d, a^n) d\mu(\omega) \right\} - \int_{\Omega} \nu_d(\omega, v^*_d, a^*) d\mu(\omega)
\end{align*}
\]

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Abbreviations

\( \mathcal{DSG} \): discounted stochastic games; SMPE, stationary Markov perfect equilibria; USCOs, upper semicontinuous correspondences taking nonempty compact values; USCO*, USCO where upper semicontinuity is with respect to weak* topologies; \( \mathcal{PSG} \): parameterized state-contingent games; AR, absolute retract; AE, absolute extensor; ANR, absolute neighborhood retract; ANE, absolute neighborhood extensor; CUSCO*, the collection of all USCO* with convex values.

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Authors' contributions

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Endnotes

a See Appendix 1 for a definition and a discussion of Banach space valued Bochner integrable functions.
b A correspondence is approximable if it has a graph about which, for any \( \varepsilon > 0 \), an open \( \varepsilon \)-ball can be placed containing the graph of a continuous function (see De Blasi and Myjak [30], and Cellina [31]).
c We will often refer to an \( m \)-player \( \mathcal{DSG} \) having an uncountable state space and compact metric action spaces as an uncountable-compact \( \mathcal{DSG} \).
d An USCO is an upper semicontinuous correspondence taking nonempty compact values (e.g., see Hola and Holy [18]). A weak* USCO is an USCO where upper semicontinuity is with respect to weak* topologies (USCO*).
e See Appendix 1 for a discussion of measurability and Bochner integrability.
f We will assume that the subset \( X \) of the dual Banach space \( \mathcal{E}^* \) is norm bounded by \( M_\varepsilon \) i.e., each \( x \in X, \|x\| < M_\varepsilon \).
g Local connectedness differs from connectedness. To see this, note, for example, that the set \( E \) in \( R \) given by \( E = [0,1) \cup (1,2] \) is locally connected but not connected (because \( E \) is equal to the union of two disjoint, half-open intervals). While the set \( G \) in \( R^* \) given by

\[
G = \left\{ (x,0), \left( \frac{1}{n}, 0 \right) : 0 < x \leq 1 \text{ and } n = \pm 1, \pm 2, \ldots \right\} \bigcup \left\{ (0,y), (1,y) : y \in R \right\}
\]

is connected but not locally connected (because only the points \( (0,0) \) and \( (1,0) \) in \( G \) possess a collection of connected neighborhoods). These examples are taken from Willard [32], Chap 8.
h Unicoherent continua are particularly important to game theory because unicoherent continua are not homeomorphic to the unit circle and therefore contain no simple closed curves. In game theory, the presence of simple closed curves can create nonexistence-of-equilibrium problems (see Levy [40]).
i In a complete separable metric space, \( \mu \) is nonatomic if and only if \( \mu(\emptyset) = 0 \).
j Here, \( \mathcal{P}_c^{\mathcal{L}_N} \) denotes the collection of all nonempty, convex, \( \mathcal{L}_N\)-closed (and hence \( \mathcal{L}_N \)-compact) subsets of \( \mathcal{L}_N \).

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