DYNAMICALLY CONSISTENT NONLINEAR EVALUATIONS
AND EXPECTATIONS

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Abstract. How an agent (or a firm, an investor, a financial market) evaluates a contingent claim, say a European type of derivatives $X$, with maturity $t$? In this paper we study a dynamic evaluation of this problem. We denote by $\{F_t\}_{t \geq 0}$ the information acquired by this agent. The value $X$ is known at the maturity $t$ means that $X$ is an $F_t$–measurable random variable. We denote by $E_{s,t}[\cdot]$ the evaluated value of $X$ at the time $s \leq t$. $E_{s,t}[\cdot]$ is $F_s$–measurable since his evaluation is based on his information at the time $s$. Thus $E_{s,t}[\cdot]$ is an operator that maps an $F_t$–measurable random variable to an $F_s$–measurable one. A system of operators $\{E_{s,t}[\cdot]\}_{0 \leq s \leq t < \infty}$ is called $F_t$–consistent evaluations if it satisfies the following conditions: (A1) $E_{s,t}[X] \geq E_{s,t}[Y]$, if $X \geq Y$; (A2) $E_{t,t}[X] = X$; (A3) $E_{r,s}E_{s,t}[X] = E_{r,t}[X]$, for $r \leq s \leq t$; (A4) $1_AE_{s,t}[X1_A] = 1_AE_{s,t}[X]$, if $A \in F_s$. In the situation where $F_t$ is generated by a Brownian motion, we propose the so-called $g$–evaluation defined by $E_{g,s,t}[X] := y_s$, where $y$ is the solution of the backward stochastic differential equation with generator $g$ and with the terminal condition $y_t = X$. This $g$–evaluation satisfies (A1)–(A4). We also provide examples to determine the function $g = g(y, z)$ by testing.

The main result of this paper is as follows: if a given $F_t$–consistent evaluation is $E_{g_\mu}$–dominated, i.e., $E_{s,t}[X] - E_{s,t}[X'] \leq E_{g_\mu}[X - X']$, for a large enough $\mu > 0$, where $g_\mu = \mu(|y| + |z|)$, then $E_{s,t}[\cdot]$ is a $g$–evaluation.

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1. Introduction

We are interested in the following dynamically consistent evaluation of risky assets: Let \( \eta = (\eta_t)_{t \geq 0} \) be a \( d \)-dimensional process, it may be the prices of stocks in a financial market, the rates of exchanges, the rates of local and global inflations etc. We assume that at each time \( t \geq 0 \), the information for of an agent (a firm, a group of people or even a financial market) is the history of \( \eta \) during the time interval \([0, t]\). Namely, his actual filtration is \( \mathcal{F}_t = \sigma\{\eta_s; s \leq t\} \).

We denote the set of all real valued \( \mathcal{F}_t \)-measurable random variables by \( m_{\mathcal{F}_t} \). Under this notation an \( \eta \)-underlying derivative \( X \), with maturity \( T \in [0, \infty) \), is an \( \mathcal{F}_T \)-measurable random variable, i.e., \( X \in m_{\mathcal{F}_T} \). We denote this evaluated value at the time \( t \) by \( E_{t,T}[X] \). It is reasonable to assume that \( E_{t,T}[X] \) is \( \mathcal{F}_t \)-measurable. In other words,

\[
E_{t,T}[X] : m_{\mathcal{F}_T} \longrightarrow m_{\mathcal{F}_t}.
\]

In particular

\[
E_{0,T}[X] : m_{\mathcal{F}_T} \longrightarrow \mathbb{R}.
\]

We will make the following Axiomatic Conditions for \((E_{t,T}[\cdot])_{0 \leq t \leq T < \infty}\):

(A1) Monotonicity: \( E_{t,T}[X] \geq E_{t,T}[X'] \), if \( X \geq X' \);
(A2) \( E_{T,T}[X] = X, \forall X \in m_{\mathcal{F}_T} \). Particularly \( E_{0,0}[c] = c \);
(A3) Dynamical consistency: \( E_{s,t}[E_{t,T}[X]] = E_{s,T}[X] \), if \( s \leq t \leq T \);
(A4) "Zero–one law": for each \( t \leq T \), \( 1_A E_{t,T}[X] = 1_A E_{t,T}[1_A X], \forall A \in \mathcal{F}_t \).

or, more specially,

(A4') for each \( t \leq T \), \( 1_A E_{t,T}[X] = 1_A E_{t,T}[1_A X], \forall A \in \mathcal{F}_t \).

Remark 1.0.1. The meaning of (A1) and (A2) are obvious. Condition (A3) means that the evaluated value \( E_{t,T}[X] \) can be also treated as a derivative with the maturity \( t \). At a time \( s \leq t \), the "price" of this derivative evaluated by \( E_{s,t}[E_{t,T}[X]] \) is the same as the "price" of the original derivative \( X \) with maturity \( T \), i.e., \( E_{s,t}[X] \).

Remark 1.0.2. The meaning of condition (A4) is: at time \( t \), the agent knows whether \( \eta_{\Lambda t} \) is in \( A \). If \( \eta_{\Lambda t} \) is in \( A \), then the value \( E_{t,T}[X] \) is the same as \( E_{t,T}[1_A X] \) since the two outcomes \( X \) and \( 1_A X \) are exactly the same.
It is clear that, to investigate this abstract evaluation problem, we need to introduce some regulation condition of $\mathcal{E}$. In this paper the information $\mathcal{F}_t$ will be limited to the $\sigma$–filtration of some $d$–dimensional Brownian motion, and $X$ will be assumed to be square–integrable, i.e., $X \in L^2(\mathcal{F}_T)$.

A condition stronger than (A2) is:
\begin{align*}
\langle \text{A2}' \rangle & \quad \mathcal{E}_{s,t}[X] = X, \forall 0 \leq s \leq t, \forall X \in m\mathcal{F}_s.
\end{align*}

The meaning is that the market has zero interest rate for a non–risky asset $X$. In this case we can define $\mathcal{E}[X|\mathcal{F}_t] := \mathcal{E}_{t,T}[X]$, for a sufficiently large $T$, and $\mathcal{E}[X] := \mathcal{E}[X|\mathcal{F}_0]$. It is easy to check that
\begin{align*}
\mathcal{E}[1_A \mathcal{E}[X|\mathcal{F}_t]] = \mathcal{E}[1_A X].
\end{align*}

$\mathcal{E}[X|\mathcal{F}_t]$ is called the $\mathcal{E}$–conditional expectation of $X$ under $\mathcal{F}_t$. It satisfies all properties of a classical expectation, with one exception that it can be a nonlinear operator. $\{\mathcal{E}[X|\mathcal{F}_t]\}_{0 \leq t \leq T}$ is called an $\mathcal{F}_t$–consistent nonlinear expectation.

A typical filtration-consistent nonlinear expectation, called $g$–expectation and denoted by $\{\mathcal{E}_g[X|\mathcal{F}_t]\}_{0 \leq t \leq T}$, was introduced in [28, Peng1997]. A significant feature of this $g$–expectation is that the value of $\mathcal{E}_g[X|\mathcal{F}_t]$ is uniquely determined by a simple function $g(t, y, z)$ with $g(t, y, 0) \equiv 0$. In fact $\{\mathcal{E}_g[X|\mathcal{F}_t]\}_{0 \leq t \leq T}$ is the solution of the backward stochastic differential equation (BSDE in short) with the function $g$ as its generator and $X$ as its terminal condition at the terminal time $T$. It is then not surprising that the behavior of $\mathcal{E}_g[\cdot|\mathcal{F}_t]$ is entirely characterized by this concrete function $g$. For example, $\mathcal{E}_g[\cdot|\mathcal{F}_t]$ is a linear (conditional) expectation if and only if $g$ is independent of $y$ and is a linear function of $z$, i.e., $g$ has a form $g = b_t \cdot z$; $\mathcal{E}_g[X|\mathcal{F}_t]$ is concave (resp. convex) in $X$ if and only if $g$ is concave (resp. convex) in $(y, z)$, etc. For an interesting application of $g$–expectations to the utility in stochastic continuous–time setting with ambiguity (or “model uncertainty” referred by Hansen and Sargent and Anderson, Hansen and [1, Sargent], see [7, Chen and Epstein, 2002]).

$g$–expectations also have very interesting mathematical properties. A nonlinear Doob–Meyer’s decomposition theorem for $g$–supermartingales was obtained by [29, Peng, 1999], for the case of Brownian filtration, and then by Chen and Peng 1998 [11, Chen & Peng, 1998] for a general filtration. In the case where the assumption $g(t, y, 0) \equiv 0$ does not hold, we have to denote the solution of the related BSDE by $\mathcal{E}^g_{s,t}[X]$ instead of $\mathcal{E}_g[X|\mathcal{F}_s]$. $\{\mathcal{E}^g_{s,t}[\cdot]\}_{0 \leq s \leq t \leq T}$ satisfies (A1)–(A4). $\mathcal{E}^g_{s,t}[\cdot]$ can be applied to a wider situation in economics and finance.

The application of BSDE to the pricing of contingent claims in a financial market was studied in [20, El Karoui et al., 1997]. Most of the results in [20] can be interpreted in the language of $\mathcal{E}^g_{s,t}[\cdot]$. Other recent results in $g$–expectations are in [5, 6, 8, 11, 12, 13, 28, 29, 30, 31, 32] where some cases are studied in depth. For nonlinear evaluations, see [30, Peng, 2002], [31, Peng, 2003] and [32, Peng, 2003]).

An interesting problem is: are the notions of $g$–expectations and $g$–evaluations general enough to represent all “enough regular” filtration-consistent nonlinear expectations and evaluations? In this case we can then concentrate ourselves to find the corresponding function $g$ which determine entirely the evaluation.

For the case of filtration–consistent expectations, we have partially solved the problem in [8]: If the assumptions (A1)–(A4) plus (A2’) hold and if for a large enough $\mu > 0$, the nonlinear expectation $\mathcal{E}[\cdot]$ is dominated by the ‘$g^\mu$–expectation’
with $g = \mu |z|$, and furthermore, if $\mathcal{E}[X + \eta |\mathcal{F}_t] = \mathcal{E}[X |\mathcal{F}_t] + \eta$ for all $\mathcal{F}_t$-
measurable $\eta$, then, there exists a unique $g$, independent of $y$, such that $\mathcal{E}[\cdot] = \mathcal{E}_y[\cdot]$.

The main objective of this paper is to prove this problem for the general case of filtration–consistent evaluation: (see Theorem 3.1) if a filtration consistent eval-
uation $\{\mathcal{E}_{s,t}[\cdot]\}_{0 \leq s \leq t \leq T}$ satisfies (A1)–(A4) plus the corresponding $\mathcal{E}^{\eta_0}$–dominated conditions (see (A5) in Section 3), then the mechanism of this seemingly very ab-
stract evaluation $\mathcal{E}_{s,t}[\cdot]$ can be entirely determined by a simple function $g(t,y,z)$.

This means that, there exists a unique function $g$ such that, for each $0 \leq s \leq t \leq T$ and for each $X \in L^2(\mathcal{F}_t)$, the value $\mathcal{E}_{s,t}[X]$ is the solution of the BSDE with generator $g$ and with terminal condition $X$. The result of this paper have non triv-
ially generalized our previous result of [8]: condition (A2') and “$\mathcal{E}_{t,T}[X + \theta] = \theta$, \(\forall X \in m\mathcal{F}_T\text{ and } \eta \in m\mathcal{F}_t\)” are not at all required.

It is worth to point out that the well–known Black–Scholes option pric ing formula is a case where $g$ is a linear function. But in our axiomatic condition (A1)–(A4) as well as the regularity condition (A5), neither the arbitrage free condition, which is a principle argument in Black–Scholes theory, nor utility maximization has been involved. Another point is that the model of the price $\eta$ of the underlying stocks is not specified. This gives us a large freedom to determine the function $g$ in each specific situation. We also explain how the function $g$ can be determined by simply testing the agent’s evaluation. This testing method is very useful to determine an agent’s behavior under risk.

The paper is organized as follows: in section 2 we give a rigorous setting of the notion of $\mathcal{F}_t$-consistent nonlinear evaluation and its special case: $\mathcal{F}_t$–expectations in subsection 2.1. We then give a concrete $\mathcal{F}_t$–evaluation: $\mathcal{E}^{\eta}$–evaluations in subsection 2.2. The main result, Theorem 3.1 will be presented in section 3. We also provide some examples and explain how to find the function $g$ through by testing the input–output data. This main theorem will be proved in Section 9 with several propositions served as lemmas for the proof given in Sections 4–10. Al-
though the whole paper is focused to prove Theorem 3.1, many preparative results of this paper have their own interests, e.g., the existence and uniqueness of BSDE under $\mathcal{E}$ (Theorem 7.1); the new nonlinear supermartingale decomposition theorem of Doob–Meyer’s type (Theorem 8.1), using a new and intrinsic formulation (see Remark 8.1.2 after Theorem 8.1). This decomposition theorem has also the extension of $\mathcal{E}_{s,t}[\cdot]$ to $\mathcal{E}_{\sigma,\tau}[\cdot]$ with stopping times $\sigma$ and $\tau$ and the related optional stopping theorem (Theorem 8.2 and Theorem 10.19). Mathematically, some of them are more fundamental than Theorem 8.1. In particular, the nonlinear decom-
position theorems of Doob–Meyer’s type, i.e., Proposition 10.10 and Theorem 10.2, play crucial roles in the proof of Theorem 3.1. Theorem 10.2 has also an interesting interpretation in finance (see Remark 8.1.1).

Another application of the dynamical expectations and evaluations is to risk measures. Axiomatic conditions for a (one step) coherent risk measure was intro-
duced by Artzner, Delbaen, Eber and Heath 1999 [2] and, for a convex risk measure, by Föllmer and Schied (2002) citeFo-Sc. Rosazza Gianin (2003) studied dynamical risk measures using the notion of $g$–expectations in [Roazza2003] (see also [32]) in which (B1)–(B4) are satisfied. In fact conditions (A1)–(A4), as well as their spe-
cial situation (B1)–(B4) (see Proposition 2.2) provides an ideal characterization of the dynamical behaviors of a the a risk measure. But in this paper we emphasis
the study of the mechanism of the evaluation to a further payoff, for which is, in
general, the translation property in risk measure is not satisfied.

2. Basic setting and \( \mathcal{E}^g \)–evaluations by BSDE

2.1. Basic setting. Let \((\Omega, \mathcal{F}, P)\) be a probability space and let \((B_t)_{t \geq 0}\) be a \(d\)–
dimensional Brownian motion defined in this space. We denote by \((\mathcal{F}_t)_{t \geq 0}\) the
natural filtration generated by \(B\), i.e.,

\[ \mathcal{F}_t := \sigma(\{B_s, s \leq t\} \cap \mathcal{N}). \]

Here \(\mathcal{N}\) is the collection of all \(P\)–null subsets. For each \(t \in [0, \infty)\), we denote by

- \(L^2(\mathcal{F}_t) := \{\text{the space of all real valued } \mathcal{F}_t\text{–measurable random variables such that } \mathbb{E}[|\xi|^2] < \infty\}\).

**Definition 2.1.** A system of operators:

\[ \mathcal{E}_{s,t}[X] : X \in L^2(\mathcal{F}_s) \rightarrow L^2(\mathcal{F}_s), T_0 \leq s \leq t \leq T_1 \]

is called an \(\mathcal{F}_t\)–consistent nonlinear evaluation defined on \([T_0, T_1]\) if it satisfies the
following properties: for each \(T_0 \leq r \leq s \leq t\) and for each \(X, X' \in L^2(\mathcal{F}_t)\),

(A1) \(\mathcal{E}_{s,t}[X] \geq \mathcal{E}_{s,t}[X'], \) a.s., if \(X \geq X', \) a.s.;

(A2) \(\mathcal{E}_{t,t}[X] = X,\) a.s.;

(A3) \(\mathcal{E}_{r,t}[\mathcal{E}_{s,t}[X]] = \mathcal{E}_{r,t}[X],\) a.s.;

(A4) \(1_A \mathcal{E}_{s,t}[X] = 1_A \mathcal{E}_{s,t}[1_A X],\) a.s. \(\forall A \in \mathcal{F}_s.\)

We will often consider (A1)–(A4) plus an additional condition:

(A4) \(\mathcal{E}_{s,t}[0] = 0,\) a.s. \(\forall 0 \leq s \leq t \leq T.\)

**Proposition 2.2.** (A4) plus (A4) is equivalent to

(A4') \(1_A \mathcal{E}_{s,t}[X] = \mathcal{E}_{s,t}[1_A X],\) a.s. \(\forall A \in \mathcal{F}_s.\)

**Proof.** It is clear that (A4') implies (A4). \(\mathcal{E}_{s,t}[0] = 0\) can be derived by putting
\(A = \emptyset\) in (A4'). On the other hand, (A4) plus the additional condition implies

\[ 1_A \mathcal{E}_{s,t}[1_A X] = 1_A \mathcal{E}_{s,t}[1_A X] = 0. \]

We thus have

\[ \mathcal{E}_{s,t}[1_A X] = 1_A \mathcal{E}_{s,t}[X] + 1_A \mathcal{E}_{s,t}[X] = 1_A \mathcal{E}_{s,t}[X]. \]

\(\square\)

**Proposition 2.3.** (A4) is equivalent to, for each \(0 \leq s \leq t\) and \(X, X' \in L^2(\mathcal{F}_t)\),

\[ \mathcal{E}_{s,t}[1_A X + 1_A c X'] = 1_A \mathcal{E}_{s,t}[X] + 1_A c \mathcal{E}_{s,t}[X'], \] a.s. \(\forall A \in \mathcal{F}_s.\)

**Proof.** (A4) \(\Rightarrow (2.1)\): We let \(Y = 1_A X + 1_A c X'.\) Then, by (A4)

\[ 1_A \mathcal{E}_{s,t}[Y] = 1_A \mathcal{E}_{s,t}[1_A Y] = 1_A \mathcal{E}_{s,t}[1_A X] = 1_A \mathcal{E}_{s,t}[X]. \]

Similarly

\[ 1_A c \mathcal{E}_{s,t}[Y] = 1_A c \mathcal{E}_{s,t}[1_A c Y] = 1_A c \mathcal{E}_{s,t}[1_A c X'] = 1_A c \mathcal{E}_{s,t}[X']. \]

Thus \(\Rightarrow (2.1)\) from \(1_A \mathcal{E}_{s,t}[Y] + 1_A c \mathcal{E}_{s,t}[Y] = 1_A \mathcal{E}_{s,t}[X] + 1_A c \mathcal{E}_{s,t}[X']\).
Remark 2.3.1. At time $t$, the agent knows the value of $1_A$. (A4) means that, if $1_A = 1$ then the evaluated value $\mathcal{E}_{s,t}[1_A X]$ should be the same as $\mathcal{E}_{s,t}[X]$ since the two outcomes $X(\omega)$ and $(1_A X)(\omega)$ are exactly the same. (A4) is applied to the evaluation of a final outcome $X$ plus some “dividend” $(D_s)_{s \geq 0}$.

If, instead of (A2), we set

(A2'). $\mathcal{E}_{s,t}[X] = X$, a.s., for each $T_0 \leq s \leq t \leq T$, and $X \in L^2(\mathcal{F}_s)$.

Then we define

$\mathcal{E}[X|\mathcal{F}_t] := \mathcal{E}_{t,T}[X], \ X \in L^2(\mathcal{F}_T)$.

We observe that this notion describes all $\mathcal{E}_{s,t}[X]$ since, when $X \in L^2(\mathcal{F}_t)$, $\mathcal{E}_{s,t}[X] = \mathcal{E}[X|\mathcal{F}_t]$.

Proposition 2.4. With (A2'), the system of operators

$\mathcal{E}[\cdot|\mathcal{F}_t] : L^2(\mathcal{F}_T) \to L^2(\mathcal{F}_t)$

is a $\mathcal{F}_t$-consistent nonlinear expectation, i.e., it satisfies, for each $T_0 \leq s \leq t \leq T$, and $X \in L^2(\mathcal{F}_T)$

(B1) $\mathcal{E}[X|\mathcal{F}_t] \geq \mathcal{E}[X|\mathcal{F}_s], \ a.s., \ if \ X \geq X', \ a.s.$

(B2) $\mathcal{E}[X|\mathcal{F}_t] = X, \ a.s., \ if \ X \in L^2(\mathcal{F}_t)$;

(B3) $\mathcal{E}[\mathcal{E}[X|\mathcal{F}_t]|\mathcal{F}_s] = \mathcal{E}[X|\mathcal{F}_s], \ a.s.;$

(B4) $\mathcal{E}[1_A X|\mathcal{F}_t] = 1_A \mathcal{E}[X|\mathcal{F}_t], \ a.s. \ \forall A \in \mathcal{F}_1$.

Proof. (B1)–(B3) are easy. Since (A2') implies $\mathcal{E}_{s,t}[0] = 0$, thus, by Proposition 2.2 (A4') and then (B4) holds. □

We have the following immediate result

Proposition 2.5. Let $T_0 < T_1 < T_2 < \cdots < T_N$ be given and, for $i = 0, 1, 2, \cdots, N-1$, let

$\mathcal{E}^i_{s,t}[X] : X \in L^2(\mathcal{F}_t) \to L^2(\mathcal{F}_s), \ T_i \leq s \leq t \leq T_{i+1}$

be an $\mathcal{F}_t$-consistent evaluation defined on $[T_i, T_{i+1}]$ in the sense of Definition 2.1. Then there exists a unique $\mathcal{F}_t$-consistent evaluation $\mathcal{E}[\cdot]$ defined on $[T_0, T_N]$ such that, for each $i = 0, 1, \cdots, N-1$, and for each $T_i \leq s \leq t \leq T_{i+1}$,

$\mathcal{E}_{s,t}[X] = \mathcal{E}^i_{s,t}[X], \ \forall X \in L^2(\mathcal{F}_t)$.

Proof. It suffices to prove the case $N = 2$. Because after we then can apply this result to prove the cases $[T_0, T_3] = [T_0, T_2] \cup [T_2, T_3], \cdots, [T_0, T_N] = [T_0, T_{N-1}] \cup [T_{N-1}, T_N]$. We define

$\mathcal{E}_{s,t}[X] = \left\{ \begin{array}{ll}
(i) & \mathcal{E}^1_{s,t}[X], \quad T_0 \leq s \leq t \leq T_1; \\
(ii) & \mathcal{E}^2_{s,t}[X], \quad T_1 \leq s \leq t \leq T_2; \\
(iii) & \mathcal{E}^1_{T_1,T}[\mathcal{E}^2_{T_1,t}[X]] \quad T_1 \leq s < T_1 < t \leq T_2.
\end{array} \right.$
It is clear that, on $[T_0, T_2]$, $\mathcal{E}_{s,t}[:]=\mathcal{E}_{s,t}[X]$ satisfies (A1) and (A2). To prove (A3) we only need to check the relation

$$\mathcal{E}_{r,s}[\mathcal{E}_{s,t}[X]] = \mathcal{E}_{r,t}[X], \quad T_0 \leq r \leq s \leq t \leq T_1$$

for two cases: $T_0 \leq r \leq s \leq T_1 \leq t \leq T_2$ and $T_0 \leq r \leq T_1 \leq s \leq t \leq T_2$. For the first case

$$\mathcal{E}_{r,s}[\mathcal{E}_{s,t}[X]] = \mathcal{E}_{r,T_1}[\mathcal{E}_{T_1,t}[X]] = \mathcal{E}_{r,t}[X].$$

For the second case

$$\mathcal{E}_{r,s}[\mathcal{E}_{s,t}[X]] = \mathcal{E}_{r,T_1}[\mathcal{E}_{T_1,t}[X]] = \mathcal{E}_{r,t}[X].$$

We now prove (A4). Again it suffices to check the case $T_0 \leq s \leq T_1 \leq t \leq T_2$. In this case, for each $A \in \mathcal{F}_s \subset \mathcal{F}_{T_1}$, (A4) is derived from

$$1_A \mathcal{E}_{s,t}[X] = 1_A \mathcal{E}_{s,T_1}[\mathcal{E}_{T_1,t}[X]] = 1_A \mathcal{E}_{s,T_1}[\lambda \mathcal{E}_{T_1,t}[X]] = 1_A \mathcal{E}_{s,t}[\lambda X].$$

It remains to prove the uniqueness of $\mathcal{E}[:]=\mathcal{E}_{s,t}[.]$. Let $\mathcal{E}^a[:]=\mathcal{E}^a_{s,t}[.]$ be an $\mathcal{F}_t$–consistent evaluation such that,

$$\mathcal{E}^a_{s,T_1}[X] = \mathcal{E}_{s,T_1}[X], \quad \forall X \in L^2(\mathcal{F}_t), \quad i = 1, 2.$$

We then have, when $T_0 \leq s \leq t \leq T_1$ and $T_1 \leq s \leq t \leq T_2$, $\mathcal{E}^a_{s,t}[X] = \mathcal{E}_{s,t}[X], \forall X \in L^2(\mathcal{F}_t)$. For the remaining case, i.e., $T_0 \leq s < T_1 < t \leq T_1$, since $\mathcal{E}^a$ satisfies (A3),

$$\mathcal{E}^a_{s,t}[X] = \mathcal{E}^a_{s,T_1}[\mathcal{E}^a_{T_1,t}[X]] = \mathcal{E}_{s,T_1}[\mathcal{E}_{T_1,t}[X]] = \mathcal{E}_{s,t}[X], \forall X \in L^2(\mathcal{F}_t).$$

Thus $\mathcal{E}^a_{s,t}[.] = \mathcal{E}_{s,t}[.]$. This completes the proof. \[\square\]

Remark 2.5.1. (i) In the remaining of this paper, we mainly consider the situation $t \in [0, T]$ for a fixed $T$. The conclusions can be extended to $[0, \infty)$, using the above Proposition. (ii) The argument of the above Proposition can be also applied to a filtration different from $\{\mathcal{F}_t\}_{t \geq 0}$, e.g., $\{\mathcal{F}_{t\wedge \tau}\}_{t \geq 0}$, where $\tau$ is an $\mathcal{F}_t$–stopping time.

2.2. $\mathcal{E}^q$–evaluations induced by BSDE. In the remaining of this paper, we limited ourselves within the time interval $[0, T]$ for some fixed $T > 0$. The results of this paper can be extended to the whole interval $[0, \infty)$, using Proposition. We need the following notations. Let $p \geq 1$ and $\tau \leq T$ be a given $\mathcal{F}_t$–stopping time.

- $L^p(\mathcal{F}_t; \mathbb{R}^m):=\{\text{the space of all } \mathbb{R}^m–\text{valued } \mathcal{F}_t–\text{measurable random variables such that } E[|\xi|^p] < \infty\};$
- $L^p_\tau(0, \tau; \mathbb{R}^m):=\{\mathbb{R}^m–\text{valued and } \mathcal{F}_t–\text{predictable stochastic processes such that } E \int_0^\tau |\phi_t|^p \, dt < \infty\};$
We recall that all elements in

\[ L^p \] (2.4)  

\[ g \]

where the unknown is the pair of the adapted processes \((Y, Z)\)

\[ \{ \text{all RCLL processes in } L^p(\mathcal{F}_t; R^m) \} \]

\[ \{ \text{all continuous processes in } D^p(\mathcal{F}_t; R^m) \} \]

\[ \{ \text{the collection of all } \mathcal{F}_t \text{-stopping times bounded by } T \} \]

\[ \{ \tau \in \mathcal{S}_T \text{ and } \bigcup_{i=1}^{t_1} \tau = \Omega, \text{ with some deterministic } 0 \leq t_1 < \cdots < t_N \} \]

In the case \( m = 1 \), we denote them by \( L^p(\mathcal{F}_t), L^p(\mathcal{F}_t; R), D^p(\mathcal{F}_t), \) and \( S^p(\mathcal{F}_t) \).

We will prove that \((E, F, t)\) is \( \mathcal{F}_t \)-predictable.

For each given \( t \in [0, T] \) and \( X \in L^2(\mathcal{F}_t) \), we solve the following BSDE

\[
Y_s = X + \int_s^t g(r, Y_r, Z_r)dr - \int_s^t Z_r dB_r, \quad s \in [0, t],
\]

where the unknown is the pair of the adapted processes \((Y, Z)\). Here the function

\[
g : (\omega, t, y, z) \in \Omega \times [0, T] \times R \times R^d \rightarrow R
\]

satisfies the following basic assumptions for each \( \forall y, y' \in R, z, z' \in R^d \)

\[
(2.5)
\begin{align*}
(i) & \quad g(\cdot, y, z) \in L^2(0, T), \\
(ii) & \quad |g(t, y, z) - g(t, y', z')| \leq \mu(|y - y'| + |z - z'|).
\end{align*}
\]

In some cases it is interesting to consider the following situation:

\[
(2.6)
\begin{align*}
(a) & \quad g(\cdot, 0, 0) \equiv 0, \\
(b) & \quad g(\cdot, y, 0) \equiv 0, \quad \forall y \in R.
\end{align*}
\]

Obviously (b) implies (a). This kind of BSDE was introduced by Bismut \[3, 4\] for the case where \( g \) is a linear function of \((y, z)\). Pardoux and Peng \[25\] obtained the following result (see Theorem 4.1 for a more general situation): for each \( X \in L^2(\mathcal{F}_t) \), there exists a unique solution \((Y, Z) \in S^2(0, t) \times L^2(0, t; R^d)\) of the BSDE \(\mathcal{F}_t\).

**Definition 2.6.** We denote by \( E^{g}_{s,t}[X] := Y_s, 0 \leq s \leq t \).

We thus define a system of operators

\[
E^{g}_{s,t} : L^2(\mathcal{F}_t) \rightarrow L^2(\mathcal{F}_s), \quad 0 \leq s \leq t \leq T.
\]

We will prove that \(\{E^{g}_{s,t}[\cdot]\}_{0 \leq s \leq t \leq T}\) forms an \(\mathcal{F}_t\)-consistent evaluation on \([0, T]\). This evaluation is entirely determined by the simple function \(g\).

**3. Main Result:** \(E^{g}_{s,t}[\cdot]\) is determined by a function \(g\)

From now on the system \(E^{g}_{s,t} : L^2(\mathcal{F}_t) \rightarrow L^2(\mathcal{F}_s), 0 \leq s \leq t \leq T\), is always a fixed \(\mathcal{F}_t\)-consistent nonlinear evaluation, i.e., satisfying (A1)–(A4), with additional assumptions (A4b) and the following \(E^{g_{mu}}\)-domination assumption:

\[ g_{mu}(y, z) := \mu |y| + \mu |z|, \quad (y, z) \in R \times R^d. \]

The main theorem of this paper is:
Theorem 3.1. Let $\mathbb{E}_{s,t}[: ] : L^2(\mathcal{F}_t) \to L^2(\mathcal{F}_s)$, $0 \leq s \leq t \leq T$, satisfy (A1)–(A4), (A40) and (A5). Then there exists a function $g(\omega, t, y, z)$ satisfying (3.3) with $g(s,0,0,0) \equiv 0$, such that, for each $0 \leq s \leq t \leq T$,

\begin{equation}
\mathbb{E}_{s,t}[X] = \mathbb{E}_{s,t}^g[X], \forall X \in L^2(\mathcal{F}_t).
\end{equation}

Remark 3.1.1. The case where $\mathbb{E}_{s,t}[: ]$ satisfy (A1)–(A5), without (A40), can be obtained as corollaries of the this main theorem. This will be given in Corollaries 5.11 and 5.12. In this more general situation the condition $g(s,0,0,0) \equiv 0$ is not imposed.

We consider some special situations of our theorem.

Example 3.2. If moreover, $g(s,y,0,0) \equiv 0$. Then, by [28], (A2') holds. Thus, according to Proposition 2.4, $\mathbb{E}_{s,t}^g[: ]$ becomes an $\mathcal{F}_t$–consistent nonlinear expectation:

$$\mathbb{E}[X|\mathcal{F}_t] = \mathbb{E}_g[X|\mathcal{F}_t] := \mathbb{E}_{s,t}^g[X] = \mathbb{E}_{s,T}^g[X].$$

This is the so called $g$–expectation introduced in [28].

This extends non trivially the result obtained in [8], (see also [32] for a more systematical presentation and explanations in finance), where we needed a more strict domination condition plus the following assumption

$$\mathbb{E}[X + \eta|\mathcal{F}_t] = \mathbb{E}[X|\mathcal{F}_t] + \eta, \forall \eta \in \mathcal{F}_t.$$

Under these assumptions we have proved in [8] that there exists a unique function $g = g(s,z)$, with $g(s,0) \equiv 0$, such that $\mathbb{E}_g[X] \equiv \mathbb{E}[X] = \mathbb{E}[X|\mathcal{F}_0]$.

Example 3.3. Consider a financial market consisting of $d + 1$ assets: one bond and $d$ stocks. We denote by $P_0(t)$ the price of the bond and by $P_i(t)$ the price of the $i$-th stock at time $t$. We assume that $P_0$ is the solution of the ordinary differential equation: $dP_0(t) = r(t)P_0(t)dt$, and $\{P_i\}_{i=1}^d$ is the solution of the following SDE

\begin{align*}
\frac{dP_i(t)}{P_i(t)} &= \sigma_i(t)dB_i(t), \\
P_i(0) &= P_{i0}, \quad i = 1, \ldots, d.
\end{align*}

Here $r$ is the interest rate of the bond; $\{b_i\}_{i=1}^d$ is the rate of the expected return, $\{\sigma_i\}_{i,j=1}^d$ the volatility of the stocks. We assume that $r, b, \sigma$ and $\sigma^{-1}$ are all $\mathcal{F}_t$–adapted and uniformly bounded processes on $[0,\infty)$. Black and Scholes have solved the problem of the market evaluation of an European type of derivative $X \in L^2(\mathcal{F}_T)$ with maturity $T$. In the point of view of BSDE, the problem can be treated as follows: consider an investor who has, at a time $t \leq T$, $n_0(t)$ bonds and $n_i(t)$ $i$-stocks, $i = 1, \ldots, d$, i.e., he invests $n_0(t)P_0(t)$ in bond and $n_i(t)P_i(t)$ in the $i$-th stock. $\pi(t) = (\pi_1(t), \ldots, \pi_d(t))$, $0 \leq t \leq T$ is an $R^d$ valued, square-integrable and adapted process. We define by $y(t)$ the investor’s wealth invested in the market at time $t$:

$$y(t) = n_0(t)P_0(t) + \sum_{i=1}^d n_i(t).$$

We make the so called self–financing assumption: in the period $[0, T]$, the investor does not withdraw his money from, or put his money in his account $y_t$. Under this condition, his wealth $y(t)$ evolves according to

$$dy(t) = n_0(t)dP_0(t) + \sum_{i=1}^d n_i(t)dP_i(t).$$
or
\[dy(t) = [r(t)y(t) + \sum_{i=1}^{d} (b_i(t) - r(t))\pi_i(t)]dt + \sum_{i,j=1}^{d} \sigma_{ij}(t)\pi_i(t)dB_i^t.\]

We denote \(g(t, y, z) := -r(t)y - \sum_{i,j=1}^{d} (b_i(t) - r(t))\sigma_{ij}^{-1}(t)z_j\). Then, by the variable change \(z_j(t) = \sum_{i=1}^{d} \sigma_{ij}(t)\pi_i(t)\), the above equation is
\[-dy(t) = g(t, y(t), z(t))dt - z(t)dB_t.\]

We observe that the function \(g\) satisfies (2.5). It follows from the existence and uniqueness theorem of BSDE (Theorem 4.1) that for each derivative \(X \in L^2(F_T)\), there exists a unique solution \((y(\cdot), z(\cdot)) \in L^2_T(0, T; R^{1+d})\) with the terminal condition \(y_T = X\). This meaning is significant: in order to replicate the derivative \(X\), the investor needs and only needs to invest \(y(t)\) at the present time \(t\) and then, during the time interval \([t, T]\) and then to perform the portfolio strategy \(\pi_i(s) = \sigma_{ij}^{-1}(s)z_j(s)\).

Furthermore, by Comparison Theorem of BSDE, if he wants to replicate a \(X'\) which is bigger than \(X\), i.e., \(X' \geq X\), a.s., \(P(X' \geq X) = 0\), then he must pay more, i.e., there is no arbitrage opportunity. This \(y(t)\) is called the Black–Scholes price, or Black–Scholes evaluation, of \(X\) at the time \(t\). We define, as in (4.8), \(\mathcal{E}^g_{t,T}[X] = y_t\).

We observe that the function \(g\) satisfies (b) of condition (4.4). It follows from Proposition 4.6 that \(\mathcal{E}^g_{t,T}[\cdot]\) satisfies properties (A1)–(A4) for \(F_t\)-consistent evaluation.

**Example 3.4.** An very important problem is: if we know that the evaluation of an investigated agent is a \(g\)-evaluation \(\mathcal{E}^g\), how to find this function \(g\). We now consider a case where \(g\) depends only on \(z\), i.e., \(g = g(z) : R^d \rightarrow R\). In this case we can find such \(g\) by the following testing method. Let \(\tilde{z} \in R^d\) be given. We denote \(Y_s := \mathcal{E}^g_{t,T}[\tilde{z}(B_T - B_t)], s \in [t, T]\), where \(t\) is the present time. It is the solution of the following BSDE
\[Y_s = \tilde{z}(B_T - B_t) + \int_s^T g(Z_u)du - \int_s^T Z_u dB_u, s \in [t, T].\]

It is seen that the solution is \(Y_s = \tilde{z}(B_s - B_t) + \int_s^T g(\tilde{z})ds, Z_s \equiv \tilde{z}\). Thus
\[\mathcal{E}^g_{t,T}[\tilde{z}(B_T - B_t)] = Y_t = g(\tilde{z})(T - t),\]

or
\[g(\tilde{z}) = (T - t)^{-1} \mathcal{E}^g_{t,T}[\tilde{z}(B_T - B_t)].\]

Thus the function \(g\) can be tested as follows: at the present time \(t\), we ask the investigated agent to evaluate \(\tilde{z}(B_T - B_t)\). We thus get \(\mathcal{E}^g_{t,T}[\tilde{z}(B_T - B_t)]\). Then \(g(\tilde{z})\) is obtained by (3.4).

**Remark 3.4.1.** The above test works also for the case \(g : [0, \infty) \times R^d \rightarrow R\), or for a more general situation \(g = \gamma y + g_0(t, z)\).

An interesting problem is, in general, how to find the function \(g\) through a testing of the input–output behaviour of \(\mathcal{E}^g[\cdot]\)? Let \(b : R^n \rightarrow R^n, \sigma : R^n \rightarrow R^{n \times d}\) be two Lipschitz functions.

\[X^{x,x}_s = x + \int_t^s b(X^{t,x}_r)dr + \int_t^s \sigma(X^{t,x}_r)dB_r, \ s \geq t.\]

The following result was obtained in Proposition 2.3 of [5].
Proposition 3.5. We assume that the generator $g$ satisfies (2.6). We also assume that, for each fixed $(y, z), \ g(\cdot, y, z) \in D^2_T(0, T)$. Then for each $(t, x, p, y) \in [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$, we have
\[
L^2 - \lim_{\epsilon \to 0} \frac{1}{\epsilon} \mathbb{E}^\mathbb{Q}_{t\epsilon}(y + p \cdot (X_{t\epsilon}^x - x) - y) = g(t, y, \sigma^T(x)p + p \cdot b(x)).
\]

4. A more general formulation: $\mathcal{E}_{\eta}^\mathbb{Q}([\cdot]; K]$-evaluation

To prove our main result, we need to introduce a more general type of $\mathcal{F}_t$-consistent evaluations $\mathcal{E}_{\eta}^\mathbb{Q}([\cdot]; K]$ induced by $\mathcal{E}_{\eta}^\mathbb{Q}([\cdot])$, for each given process $K \in D^2_T(0, T)$. In finance, $K$ often represents a dividend and/or a consumption process. We will firstly consider $\mathcal{E}_{\eta}^\mathbb{Q}([\cdot]; K]$. For technical convenience, we will directly consider stopping times $\sigma \leq \tau \leq T$ in the place of deterministic times $s$ and $t$.

Let $\tau \in \mathcal{S}_T$ be a given stopping time. We consider the following backward stochastic differential equation:
\[
Y_s = X + K_\tau - K_{s \wedge \tau} + \int_{s \wedge \tau}^\tau g(r, Y_r, Z_r)dr - \int_{s \wedge \tau}^\tau Z_rdB_r, \ s \in [0, T].
\]
Here the pair $(Y, Z)$ is the unknown process to be solved. $X \in L^2(\mathcal{F}_T), \ K \in D^2_\mathcal{F}(0, T)$ are given.

We recall the following basic results of BSDE.

Theorem 4.1. (25, 27) We assume (2.3). Then there exists a unique solution $(Y, Z) \in L^2(0, \tau; \mathbb{R} \times \mathbb{R}^d)$ of BSDE (4.1). We denote it by
\[
(Y_{s, t}^{\tau, X, K}, Z_{s, t}^{\tau, X, K}) = (Y_s, Z_s), \ s \in [0, \tau].
\]

We have
\[
Y_{\tau, X, K} + K \in S^2_\mathcal{F}(0, \tau).
\]

and the estimate
\[
E \int_0^\tau |Z_{s}^{\tau, X, K}|^2 ds + E[\sup_{s \in [0, \tau]} |Y_{s, t}^{\tau, X, K} - K_s|^2] 
\leq CE((X + K_\tau)^2) + CE \int_0^\tau (K_s^2 + |g(s, 0, 0)|^2) ds,
\]
where the constant $C$ depends only on $\mu$ and $T$. Furthermore, let $X' \in L^2(\mathcal{F}_T), \ K' \in D^2_\mathcal{F}(0, T)$ be also given. Then we have
\[
E[\sup_{s \in [0, \tau]} |Y_{s, t}^{\tau, X, K} - Y_{s, t}^{\tau, X', K'} + K_s - K'_s|^2] + E \int_0^\tau |Z_{s}^{\tau, X, K} - Z_{s}^{\tau, X', K'}|^2 ds 
\leq CE((X - X' + K_\tau - K'_\tau)^2) + CE \int_0^\tau |K_s - K'_s|^2 ds,
\]
where the constant $C$ depends only on $\mu$ and $T$.

Proof. In (25) (see also 20), the result of BSDE is for $\tau = T$ and $K_t = \int_0^t \phi ds$ for some $\phi \in L^2(0, T)$. The present situation can be treated by defining (see 28)
\[
\hat{Y}_s := Y_s + K_s,
\hat{g}(s, y, z) := g(s, y - K_s, z)1_{[0, \tau]}(s)
\]
and considering the following equivalent BSDE
\begin{equation}
\bar{Y}_s = X + K_\tau + \int_s^T \bar{g}(r, \bar{Y}_r, Z_r)dr - \int_s^T Z_rdB_r, \ s \in [0, T].
\end{equation}

It is clear that \(\bar{Y}_s \equiv X + K_s, \ Z_s \equiv 0\) on \([\tau, T]\). Since \(\bar{g}\) is a Lipschitz function with the same Lipschitz constant \(\mu\) and

\[\bar{g}(\cdot, 0, 0) = g(\cdot, -K(\cdot, 0))1_{[0, \tau]} \in L^2_x(0, T),\]

thus, by [25, 27], the BSDE (4.6) has a unique solution \((\bar{Y}, Z)\). We also have
\[E \int_0^T |Z|^2ds + E \sup_{s \in [0, \tau]} |\bar{Y}_s|^2 \leq CE[(X + K_\tau)^2] + CE \int_0^\tau (K_s^2 + |g(s, 0, 0)|^2)ds,\]

where the constant \(C\) depends only on \(\mu\) and \(T\). We thus have estimate (4.8). Moreover, let \((\bar{Y}', Z')\) denotes the solution of the (4.6) with \(X\) and \(K\) in the place of \(X'\) and \(K'\) in Definition 2.6. Clearly when (2.6)–(a) is satisfied, we have
\[E \sup_{s \in [0, \tau]} |\bar{Y}_s - \bar{Y}'_s|^2 \leq CE[(X - X' + K - K')^2] + CE \int_0^\tau (K_s - K'_s)^2ds.\]

where \(C\) is the same as in (4.8). We then have (4.4). The proof is complete. \(\square\)

We introduce a new notation.

**Definition 4.2.** We denote, for \(\sigma, \tau \in S_T, \ \sigma \leq \tau,\)
\begin{align}
E_{\sigma, \tau}^g[X; K] &:= Y_{\sigma}^{\tau, X, K} \\
E_{\sigma, \tau}^g[X] &:= E_{\sigma, \tau}^g[X; 0].
\end{align}

This notion generalizes that of \(E_{t,t}^g[\cdot]\) in Definition 2.6. Clearly when \(E_{t,t}^g[\cdot]\)–(a) is satisfied, we have \(E_{\sigma, \tau}^g[0] = E_{\sigma, \tau}^g[0; 0] = 0\). In particular \(E_{s,t}^g[0] \equiv 0, \ 0 \leq s \leq t \leq T.\)

**Remark 4.2.1.** About the notations \(E^g[\cdot].\) This notation was firstly introduced in [25] in the case where \(g\) satisfies (2.6)–(b). In this situation it is easy to check that
\[E_{s,t}^g[X] = E_{s,t}^{g'}[X], \ \forall 0 \leq s \leq t \leq T.\]

In other words, \(E^g\) is a nonlinear expectation, called \(g\)-expectation. The general situation, i.e., without (2.6) was introduced in [27] and [12].

By the above existence and uniqueness theorem, we have for each stopping times \(0 \leq \rho \leq \sigma \leq \tau \leq T\) and for each \(X \in L^2(F_\tau)\) and \(K \in D^2_\tau(0, T),\)
\begin{equation}
E_{\rho, \sigma}^g[E_{\sigma, \tau}^g[X; K]; K] = E_{\rho, \tau}^g[X; K], \ \text{a.s.}
\end{equation}

It is also easy to check that, with the notation \(g_-(t, y, z) := -g(t, -y, -z)\)
\begin{equation}
-E_{\sigma, \tau}^g[X; K] = E_{\sigma, \tau}^{g_-}[-X; -K].
\end{equation}

We will see that \(\{E_{t,T}^g[X]\}_{0 \leq t \leq T}, \ X \in L^2(F_T)\) form an \(F_t\)-consistent nonlinear evaluation. The following monotonicity property is the comparison theorem of BSDE.
Theorem 4.3. We assume (2.5). If we assume that the elements in the above theorem satisfy $X \geq X'$, a.s., and that $K - K'$ is an increasing process. Then, for each stopping times $0 \leq \sigma \leq \tau \leq T$, we have

$$\mathcal{E}_{\sigma}^g[X; K] \geq \mathcal{E}_{\sigma}^g[X'; K'], \text{ a.s.} \quad (4.11)$$

In particular,

$$\mathcal{E}_{\sigma}^g[X] \geq \mathcal{E}_{\sigma}^g[X'], \text{ a.s.} \quad (4.12)$$

If $A \in D^2_T(0, T)$ is an increasing process, then

$$\mathcal{E}_{\sigma}^g[X; A] \geq \mathcal{E}_{\sigma}^g[X]. \quad (4.13)$$

Proof. The case $K \equiv K' \equiv 0$ is the classical comparison theorem of BSDE. The present general situation, see [27] or [32].

We recall the special function $g_\mu(y, z)$ defined in [42].

Corollary 4.4. $\mathcal{E}^g$ is dominated by $\mathcal{E}^{g_\mu}$ in the following sense: for each stopping times $0 \leq \sigma \leq \tau \leq T$ and $X, X' \in L^2(F_T)$, $K, K' \in D^2_T(0, T)$, we have

$$\mathcal{E}^{g_\mu}[X; K] - \mathcal{E}^{g_\mu}[X'; K'] \leq \mathcal{E}^{g_\mu}[X - X'; (K - K')], \text{ a.s.} \quad (4.14)$$

where $\mu$ is the Lipschitz constant of $g$ given in [42].

Proof. By the definition of $\mathcal{E}^g[$], The processes defined by $Y_s = \mathcal{E}_{s\wedge \tau}^g[X; K]$ and $Y'_s = \mathcal{E}_{s\wedge \tau}^g[X'; K']$ solve respectively the following BSDEs on $[0, \tau]$:

$$Y_s = X + K_T - K_{s \wedge \tau} + \int_{s \wedge \tau}^\tau g(r, Y_r, Z_r) dr - \int_{s \wedge \tau}^\tau Z_r dB_r, \quad Y'_s = X' + K'_T - K'_{s \wedge \tau} + \int_{s \wedge \tau}^\tau g(r, Y'_r, Z'_r) dr - \int_{s \wedge \tau}^\tau Z'_r dB_r.$$ 

We denote $\hat{Y} = Y - Y'$, $\hat{Z} = Z - Z'$ and

$$\hat{K}_t = K_t - K'_t + \int_0^t [-g_\mu(\hat{Y}_s, \hat{Z}_s) + g(s, Y_s, Z_s) - g(s, Y'_s, Z'_s)] ds.$$

$(\hat{Y}, \hat{Z})$ solves BSDE

$$\hat{Y}_s = X - X' + \hat{K}_T - \hat{K}_{s \wedge \tau} + \int_{s \wedge \tau}^\tau g_\mu(r, \hat{Y}_r, \hat{Z}_r) dr - \int_{s \wedge \tau}^\tau \hat{Z}_r dB_r.$$

We compare it to the BSDE

$$\tilde{Y}_s = X - X' + (K - K')_T - (K - K')_{s \wedge \tau} + \int_{s \wedge \tau}^\tau g_\mu(r, \tilde{Y}_r, \tilde{Z}_r) dr - \int_{s \wedge \tau}^\tau \tilde{Z}_r dB_r.$$

Since $d(K - K' - \hat{K}) \geq 0$, thus, by comparison theorem, i.e., Theorem 4.3, $\tilde{Y}_s \geq \hat{Y}_s = Y_s - Y'_s$. We thus have \[ \square \]

Proposition 4.5. We have the following uniform estimate: for each $X \in L^2(F_T)$ and $g_0(\cdot) \in L^2_T(0, T)$

$$E[(\mathcal{E}_{1,T}^{g_0}[X; \int_0 g_0(s) ds])^2] \leq E[|X|^2]e^{\beta(T-t)} + E[\int_t^T e^{\beta(s-t)} |g_0(s)|^2 ds]. \quad (4.15)$$

and

$$E[\sup_{t \in [0, T]} (\mathcal{E}_{1,T}^{g_0}[X; \int_0 g_0(s) ds])^2] \leq CE[|X|^2 + \int_0^T |g_0(s)|^2 ds]. \quad (4.16)$$
where $\beta = 2\mu^2 + 2\mu + 2$. The constant $C$ depends only on $T$ and the Lipschitz constant $\mu$ in (2.5).

**Proof.** $Y_t = e^{\beta t}\mathcal{E}^g_{t,T}[X; \int_0^T g_0(s)ds]$ satisfies the following BSDE on $[0, T]$:

\begin{equation}
Y_t = X + \int_t^T g_0(s)ds + \mu \int_t^T (|Y_s| + |Z_s|)ds - \int_t^T Z_s dB_s .
\end{equation}

We apply Itô’s formula to $|Y_t|^2 e^{\beta t}$:

\begin{equation}
E[|Y_t|^2 e^{\beta t} + \int_t^T e^{\beta s}(|Z_s|^2 + \beta |Y_s|^2)ds
\end{equation}

\begin{equation}
= E[|X|^2 e^{\beta T} + \int_t^T e^{\beta s}2Y_s (g_0(s) + \mu |Y_s| + \mu |Z_s|)ds
\end{equation}

\begin{equation}
\leq E[|X|^2 e^{\beta T} + \int_t^T e^{\beta s} (2\mu^2 + 2\mu + 1)|Y_s|^2 + \frac{1}{2}|Z_s|^2]ds.
\end{equation}

We thus have (4.14) and

\begin{equation}
E \int_0^T e^{\beta s}(|Z_s|^2 + |Y_s|^2)ds \leq E[|X|^2 e^{\beta T} + E \int_0^T e^{\beta s} |g_0(s)|^2 ds.
\end{equation}

With (4.17), we now apply BDG’s inequality to $Y_t^2$. Then (4.16) follows.

We now can assert that

**Proposition 4.6.** Let $g$ satisfies (2.5). Then, for each fixed $K \in L^2(\mathcal{F}_T)$, the system of operators

\begin{equation}
\mathcal{E}^g_{s,T}[X; K] : L^2(\mathcal{F}_s) \rightarrow L^2(\mathcal{F}_s), 0 \leq s \leq t \leq T.
\end{equation}

defined in (4.7) is an $\mathcal{F}_t$-consistent nonlinear evaluation, i.e., it satisfies (A1)–(A4) of Definition 2.1.

**Proof.** (A1) is given by (4.11). (A2) is clearly true by the definition. (A3) is proved by (4.10). We now consider (A4). In fact we can prove stronger results: for each stopping times $0 \leq \sigma \leq \tau \leq T$ and $X \in L^2(\mathcal{F}_\tau)$, we have

\begin{equation}
1_A \mathcal{E}^g_{\sigma,T}[X; K] = 1_A \mathcal{E}^g_{\sigma,T}[1_A X; K], \ \forall A \in \mathcal{F}_\tau,
\end{equation}

as well as

\begin{equation}
1_A \mathcal{E}^g_{\sigma,T}[X; K] = \mathcal{E}^g_{\sigma,T} [1_A X; K^\sigma,A], \ \forall A \in \mathcal{F}_\tau,
\end{equation}

where we set

\begin{equation}
g_{\sigma,A}(t, y, z) := 1\mathbb{1}_{[0, \sigma)}(t)g(t, y, z) + 1\mathbb{1}_{[\sigma, \tau]}(t)1_A g(t, y, z),
\end{equation}

\begin{equation}K^{\sigma,A}_t := 1\mathbb{1}_{[0, \sigma)}(t)K_t + 1\mathbb{1}_{[\sigma, \tau]}(t)1_A (K_t - K_\sigma).
\end{equation}

We will give the proof of (4.19). The proof of (4.20) is similar. According to BSDE (4.11) for each stopping time $\rho \in [\sigma, \tau]$, $Y_\rho := \mathcal{E}^g_{\rho,T}[X; K]$ and $\bar{Y}_\rho := \mathcal{E}^g_{\rho,T}[1_A X; K]$ solve respectively

\begin{equation}
Y_\rho = X + K_\tau - K_\rho + \int_\rho^\tau g(r, Y_r, Z_r)dr - \int_\rho^\tau Z_r dB_r,
\end{equation}

and

\begin{equation}\bar{Y}_\rho = 1_A X + K_\tau - K_\rho + \int_\rho^\tau g(r, \bar{Y}_r, \bar{Z}_r)dr - \int_\rho^\tau \bar{Z}_r dB_r.
\end{equation}
We multiply $1_A$, $A \in \mathcal{F}_\pi$ on both sides of the above two BSDEs. Since $1_A g(r, Y_r, Z_r) = 1_A g(r, Y_t, Z_r, 1_A)$, we have

$$1_A Y_\rho = 1_A X + 1_A K_\tau - 1_A K_\rho + \int_\rho^\tau 1_A g(r, 1_A Y_r, 1_A Z_r) dr - \int_\rho^\tau 1_A Z_r dB_r,$$

and

$$1_A \bar{Y}_\rho = 1_A X + 1_A K_\tau - 1_A K_\rho + \int_\rho^\tau 1_A g(r, 1_A \bar{Y}_r, 1_A \bar{Z}_r) dr - \int_\rho^\tau 1_A \bar{Z}_r dB_r.$$

It is clear that $1_A Y_\rho$ and $1_A \bar{Y}_\rho$ satisfy exactly the same BSDE with the same terminal condition on $[\sigma, \tau]$, and $1_A E^g_t, \bar{Y}_\rho$ on $[\sigma, \tau]$, i.e., $1_A E^g_{\sigma, \tau}[X; K] = 1_A E^g_{\sigma, \tau}[1_A X; K]$. The proof is complete. \qed

We now consider nonlinear martingales induced by $E^g$.

**Definition 4.7.** Let $K \in D^2_{\mathbb{F}}(0, T)$ be given. A process $Y \in D^2_{\mathbb{F}}(0, T)$ is said to be an $E^g[\cdot; K]$–martingale (resp. $E^g[\cdot; K]$–supermartingale, $E^g[\cdot; K]$–submartingale) if for each $0 \leq s \leq t \leq T$,

$$E^g_{s,t}[Y_t; K] = Y_s, \text{ (resp. } \leq Y_s, \geq Y_s).\quad(4.23)$$

**Remark 4.7.1.** If $(y, z) \in L^2_{\mathbb{F}}(0, T; R \times R^d)$ solves the BSDE

$$y_s = y_t + K_t - K_s + \int_s^t g(r, y_r, z_r) dr - \int_s^t z_r dB_r, \quad s \leq t.$$

It is clear that $(-y, -z)$ solves

$$-y_s = -y_t + (K_t - K_s) + \int_s^t [-g(r, -y_r, -z_r)] dr - \int_s^t (-z_r) dB_r.$$

Thus, if $y$ is an $E^g[\cdot; K]$–martingale (resp. $E^g[\cdot; K]$–supermartingale, $E^g[\cdot; K]$–submartingale), then $-y$ is an $E^g[\cdot; -K]$–martingale (resp. $E^g[\cdot; -K]$–supermartingale, $E^g[\cdot; -K]$–submartingale), where we denote

$$g_s(t, y, z) := -g(t, -y, -z).$$

Therefore many results concerning $E^g[\cdot; K]$–supermartingales can be also applied to situations of submartingales.

**Example 4.8.** Let $X \in L^2(\mathcal{F}_T)$ and $A \in D^2_{\mathbb{F}}(0, T)$ be given such that $A$ is an increasing process. By the monotonicity of $E^g$, i.e., Theorem 4.4.3, we have, for $t \in [0, T]$,

$$Y_t := E^g_{t,T}[X] = E^g_{t,T}[X; 0] \text{ is a } E^g\text–martingale, \quad Y^+_t := E^g_{t,T}[X; A] \text{ is a } E^g\text–supermartingale, \quad Y^-_t := E^g_{t,T}[X; -A] \text{ is a } E^g\text–submartingale.\quad(4.24)$$

As in classical situation, an interesting and hard problem is the inverse one: if $Y$ is an $E^g$–supermartingale, can we find an increasing and predictable process $A$ such that $Y_t \equiv E^g_{t,T}[X; A]$? This nonlinear version of Doob–Meyer’s decomposition theorem will be stated as follows. It plays a crucially important role in this paper.

The following result is a nonlinear version of optional sampling theorem for $g$–supermartingale. See [32] (also Theorem 10.1.14 for a more general situation).
Proposition 4.9. Let $g$ satisfy (2.5)–(i) and (ii) and let $Y \in D_2^F(0, T)$ be an $\mathcal{E}^g$–martingale (resp. $\mathcal{E}^g$–supermartingale, $\mathcal{E}^g$–submartingale). Then for each stopping times $0 \leq \sigma \leq \tau \leq T$, we have
\begin{equation}
\mathcal{E}_{\sigma, \tau}^g[Y_\tau] = Y_\sigma, \quad (\text{resp. } \leq Y_\sigma, \geq Y_\sigma).
\end{equation}

We have the following $\mathcal{E}^g$–supermartingale decomposition theorem of Doob–Meyer’s type. This nonlinear decomposition theorem was obtained in [29]. But the formulation using the new notation $\mathcal{E}_{\sigma, \tau}^g[\cdot; A]$ is new. In fact we think this is the intrinsic formulation since it becomes necessary in the more abstract situation of the $\mathcal{E}$–supermartingale decomposition theorem, i.e., Theorem 8.1 which can considered as a generalization of the following result.

Proposition 4.10. We assume (2.5)–(i) and (ii). Let $Y \in D_2^F(0, T)$ be an $\mathcal{E}^g$–supermartingale. Then there exists a unique increasing process $A \in D_2^F(0, T)$ (thus predictable) with $A_0 = 0$, such that
\begin{equation}
Y_t = \mathcal{E}_{t, T}^g[Y_T; A], \quad \forall 0 \leq t \leq T.
\end{equation}

Corollary 4.11. Let $K \in D_2^F(0, T)$ be given and let $Y \in D_2^F(0, T)$ be an $\mathcal{E}^g[\cdot; K]$–supermartingale in the following sense
\begin{equation}
\mathcal{E}_{s, t}^g[Y_t; K] \leq Y_s, \quad \forall 0 \leq s \leq t \leq T.
\end{equation}
Then there exists a unique increasing process $A \in D_2^F(0, T)$ with $A_0 = 0$, such that
\begin{equation}
Y_t = \mathcal{E}_{t, T}^g[Y_T; K + A], \quad \forall 0 \leq t \leq T.
\end{equation}

Proof. By the notations of (2.5) with $\tau = T$, we have
\begin{equation}
\mathcal{E}_{s, t}^g[Y_t; K] + K_s = \mathcal{E}_{s, t}^g[Y_t + K_t].
\end{equation}
It follows that (2.20) is equivalent to
\begin{equation}
\mathcal{E}_{s, s}^g[Y_t + K_t] \leq Y_s + K_s, \quad \forall 0 \leq s \leq t \leq T.
\end{equation}
In other words, $Y + K$ is an $\mathcal{E}^g$–supermartingale in the sense of (1.23). By the above supermartingale decomposition theorem, Proposition 4.10, there exists an increasing process $A \in D_2^F(0, T)$ with $A_0 = 0$, such that
\begin{equation}
Y_t + K_t = \mathcal{E}_{t, T}^g[Y_T + K_T; A], \quad \forall 0 \leq t \leq T,
\end{equation}
or, equivalently (4.27). □

5. $\mathcal{E}_{s, t}[\cdot; K]$ AND RELATED PROPERTIES

5.1. $\mathcal{E}_{s, t}[\cdot; K]$ AND IT’S MAIN PROPERTIES. In this section, $\{\mathcal{E}_{s, t}[\cdot]\}_{0 \leq s \leq t \leq T}$ is a fixed $\mathcal{F}_t$–consistent evaluation satisfying (A1)–(A5) (without (A40)) as well as (5.1). To prove our main theorem, we shall use $\mathcal{E}_{s, t}[\cdot]$ to generate an operator $\mathcal{E}_{s, t}[\cdot; K]$, $K \in D_2^F(0, T)$ which plays the same role as $\mathcal{E}_{s, t}^g[\cdot; K]$ defined in (4.27).

To this end we first define such operator on the space of step processes defined by
\begin{equation}
D_2^F(0, T) := \{K \in D_2^F(0, T), K_t = \sum_{i=0}^N \xi_i 1_{[t_i, t_{i+1})}(t), \text{ for some } 0 = t_0 < t_1 \ldots < t_N = T\}.
\end{equation}
We have

\[ K \in D^{2,0}_F(0, T) \] in the form

\[ K_t = \sum_{i=0}^{N} \xi_{1[t_i, t_{i+1})}(t), \quad t \in [0, T]. \]

For each \( i = 0, 1, 2, \ldots, N - 1 \), we define, for \( T_i \leq s \leq t \leq T_{i+1} \) and \( X \in L^2(F_t) \),

\[ \mathcal{E}_{s,t}^i[X; K] := \mathcal{E}_{s,t}[X + K_t - K_s]. \]

We have

**Lemma 5.1.** For each \( i = 0, 1, 2, \ldots, N - 1 \), \( \mathcal{E}_{s,t}[\cdot; K] \), \( t_i \leq s \leq t \leq t_{i+1} \) is an \( \mathcal{F}_t \)-consistent evaluation.

**Proof.** It is easy to check that (A1), (A2) and (A3) holds. We now prove (A4), i.e., for each \( t_i \leq s \leq t \leq t_{i+1} \) and \( X \in L^2(F_t) \),

\[ \mathcal{E}_{s,t}^i[X; K] = 1_A \mathcal{E}_{s,t}^i[A X; K], \quad \forall A \in \mathcal{F}_s. \]

We have

\[ 1_A \mathcal{E}_{s,t}^i[X; K] = 1_A \mathcal{E}_{s,t}[X + K_t - K_s] = 1_A \mathcal{E}_{s,t}[1_A X + K_t - K_s] = 1_A \mathcal{E}_{s,t}[1_A X + K_t - K_s] = 1_A \mathcal{E}_{s,t}[1_A X; K]. \]

Thus (A4) holds. \( \square \)

By Proposition 2.6 there exists a unique \( \mathcal{F}_t \)-consistent evaluation that coincides with \( \mathcal{E}^i[\cdot; K] \) for each interval \( [T_i, T_{i+1}] \).

**Definition 5.2.** We denote this unique \( \mathcal{F}_t \)-consistent evaluation that coincides with \( \mathcal{E}^i[\cdot; K] \) by \( \mathcal{E}_{s,t}[\cdot; K] \):

\[ \mathcal{E}_{s,t}[X; K] : X \in L^2(F_t) \rightarrow L^2(F_s). \]

**Lemma 5.3.** If there is some function \( g \) satisfying (A5) such that \( \mathcal{E} \) coincides with \( \mathcal{E}^g \), i.e., for each \( 0 \leq s \leq t \leq T \) and \( X \in L^2(F_T) \) we have \( \mathcal{E}_{s,t}[X] = \mathcal{E}_{s,t}^g[X] \), then, for each \( K \in D^{2,0}_F(0, T) \), \( \mathcal{E}_{s,t}[\cdot; K] \) also coincides with \( \mathcal{E}_{s,t}^g[\cdot; K] \).

**Proof.** It is easy to check that \( \mathcal{E}_{s,t}[X; K] = \mathcal{E}_{s,t}^g[X; K] = \mathcal{E}_{s,t}^g[X; K], t_i \leq s \leq t \leq t_{i+1} \). Thus we can apply Proposition 2.6 to prove this lemma for \( 0 \leq s \leq t \leq T \). \( \square \)

**Lemma 5.4.** \( \mathcal{E} \) is dominated by \( \mathcal{E}^{g_n} \) in the following sense: for each \( K, K' \in D^{2,0}_F(0, T) \) and for each \( 0 \leq s \leq t \leq T \), \( X, X' \in L^2(F_t) \), we have

\[ \mathcal{E}^{g_n}_{s,t}[X - X'; (K - K')] \leq \mathcal{E}_{s,t}[X; K] - \mathcal{E}_{s,t}^i[X'; K'] \leq \mathcal{E}^{g_n}_{s,t}[X - X'; (K - K')], \quad a.s. \]

(5.3)

(5.4)

**Proof.** We only prove (5.3). The proof of (5.4) is similar. Without loss of generality, we can set \( K_t = \sum_{i=0}^{N} \xi_{1[t_i, t_{i+1})}(t) \) and \( K'_t = \sum_{i=0}^{N} \xi'_{1[t_i, t_{i+1})}(t) \) for some \( 0 = t_0 < t_1 \ldots < t_N = T \). When \( t_i \leq s \leq t \leq t_{i+1} \), we have, since \( \mathcal{E}[\cdot] \) satisfies (A5)

\[ \mathcal{E}_{s,t}[X; K] - \mathcal{E}_{s,t}[X'; K'] = \mathcal{E}_{s,t}[X + K_t - K_s] - \mathcal{E}_{s,t}[X' + K'_t - K'_s] \leq \mathcal{E}^{g_n}_{s,t}[X - X'] + (K_t - K_s) - (K'_t - K'_s) \]

\[ = \mathcal{E}^{g_n}_{s,t}[X - X'; (K - K')]. \]
Now let \( t_{i-1} \leq s \leq t_i \leq t_{i+1} \), for some \( 1 \leq i \leq N - 1 \), we have

\[
\mathcal{E}_{s,t}[X; K] - \mathcal{E}_{s,t}[X'; K'] = \mathcal{E}_{s,t_i}[X; K] - \mathcal{E}_{s,t_i}[X'; K'] - \mathcal{E}_{s,t_{i+1}}[X; K] + \mathcal{E}_{s,t_{i+1}}[X'; K']
\]

\[
\leq \mathcal{E}_{s,t_i}^{g_{\mu}}[X; K] - \mathcal{E}_{s,t_i}[X'; K'] - \mathcal{E}_{s,t_{i+1}}[X; K] + \mathcal{E}_{s,t_{i+1}}^{g_{\mu}}[X'; K']
\]

\[
\leq \mathcal{E}_{s,t_i}^{g_{\mu}}[X - X'; (K - K')] + \mathcal{E}_{s,t_{i+1}}^{g_{\mu}}[X - X'; (K - K')].
\]

We can repeat this procedure to prove that

\[
\mathcal{E}_{s,t}[X; K] - \mathcal{E}_{s,t}[X'; K'] \leq \mathcal{E}_{s,t_s}^{g_{\mu}}[X - X'; (K - K')], \quad \forall 0 \leq s \leq t \leq T.
\]

We then have obtained the second inequality of (5.3). The first inequality is obtained by changing the positions of \((X, K)\) and \((X', K')\) in the second inequality of (5.3) and by observing that

\[
-(\mathcal{E}_{s,t}^{g_{\mu}}[X - X; (K' - K)] = \mathcal{E}_{s,t}^{-g_{\mu}}[X - X'; (K - K')].
\]

The proof is complete. \(\square\)

**Corollary 5.5.** We have the following estimate

\[
E[ \sup_{s \in [0,t]} |\mathcal{E}_{s,t}[X; K] + K_s - (\mathcal{E}_{s,t}[X'; K'] + K'_s)|^2]
\]

\[
\leq CE[(X - X' + K_s - K'_s)^2] + CE \int_0^t (K_s - K'_s)^2 ds.
\]

where C only depends on \(T\) and \(\mu\).

**Proof.** Since both \(g_{\mu}\) and \(-g_{\mu}\) satisfies conditions (2.5) for \(g\). We set \(g_1 := \mathcal{E}_{s,t}^{g_{\mu}}[X - X; (K' - K)]\) and \(g_2 := \mathcal{E}_{s,t}^{-g_{\mu}}[X - X; (K' - K)]\). We observe that

\[
\mathcal{E}_{s,t}^{g_{\mu}}[0; 0] = \mathcal{E}_{s,t}^{-g_{\mu}}[0; 0] = 0.
\]

We then can apply estimate (4.1) with \(\tau = t\), to get, for \(i = 1, 2\),

\[
E[ \sup_{s \in [0,t]} |g_i + K_s - K'_s|^2]
\]

\[
\leq CE[(X - X' + K_s - K'_s)^2] + CE \int_0^t (K_s - K'_s)^2 ds.
\]

This with (5.6) derives (5.6). The proof is complete. \(\square\)

For each \(K \in \mathcal{D}_x^2(0, T)\) and for each \(0 \leq s \leq t \leq T\), \(X \in L^2(\mathcal{F}_s)\), we take a sequence \(\{K_i\}_{i=1}^{\infty}\) in \(\mathcal{D}_x^2(0, T)\) such that \(\{K_i\}_{i=1}^{\infty}\) converges in \(L^2(\mathcal{F}_s)\) to \(K\) and such that \(K_i^s = K_s\). \(K_i^t = K_t\). It follows that \(\{\mathcal{E}_{s,t}[X; K_i] + K_i^s\}_{i=1}^{\infty}\) is a Cauchy sequence in \(L^2(\mathcal{F}_s)\). Consequently, \(\{\mathcal{E}_{s,t}[X; K_i]\}_{i=1}^{\infty}\) is a Cauchy sequence in \(L^2(\mathcal{F}_s)\).

**Remark 5.5.1.** A sequence \(\{K_i\}_{i=1}^{\infty}\) satisfying the above condition can be realized by, for example, taking \(0 = t_0^i < t_1^i < \cdots < t_j^i = T\), \(\max_j(t_{j+1}^i - t_j^i) \to 0\), with \(s = t_{j_1}^i\) and \(t = t_{j_2}^i\) for some \(j_1 \leq j_2 \leq i\), and then define

\[
K_i^t := \sum_{j=0}^i K_{t_{j+1}}^i 1(t_{j+1}^i - t_j^i)(t), \quad t \in [0, T].
\]
Definition 5.6. We denote the limit of the Cauchy sequence \( \{ \mathcal{E}_{s,t}[X; K^i]\}_{i=1}^{\infty} \) in \( L^2(\mathcal{F}_s) \) by \( \mathcal{E}_{s,t}[X; K] \).

The following property still holds true for \( K \in D^2_{F}(0, T) \).

Proposition 5.7. We assume (A1)–(A5) as well as (5.11). Then \( \mathcal{E}[: K] \) is dominated by \( \mathcal{E}^g \) in the following sense, for each \( K, K' \in D^2_{F}(0, T) \) and for each \( 0 \leq s \leq t \leq T, X, X' \in L^2(\mathcal{F}_t) \), we have

\[
\mathcal{E}_{s,t}^g[X - X'; (K - K')] \leq \mathcal{E}_{s,t}[X; K] - \mathcal{E}_{s,t}[X'; K'] \\
\leq \mathcal{E}_{s,t}^g[X - X'; (K - K')], \text{ a.s.}
\]

(5.8)

\[
\mathcal{E}_{s,t}^g[0; K + K^0] \leq \mathcal{E}_{s,t}[0; K] \leq \mathcal{E}_{s,t}^g[0; K + K^0]
\]

Proof. Let \( \{K^i\}_{i=1}^{\infty} \) and \( \{K'^i\}_{i=1}^{\infty} \) be sequences in \( D^2_{F}(0, T) \) satisfying the conditions of Definition 5.6 for \( K \) and \( K' \), respectively. By Lemma 5.4 we have

\[
\mathcal{E}_{s,t}^g[X - X'; (K^i - K'^i)] \leq \mathcal{E}_{s,t}[X; K^i] - \mathcal{E}_{s,t}[X'; K'^i] \\
\leq \mathcal{E}_{s,t}^g[X - X'; (K^i - K'^j)], \text{ a.s.}
\]

We then pass to the limit to get (5.7). The proof of (5.8) is similar. □

Corollary 5.8. For \( K, K' \in D^2_{F}(0, T) \), we have

\[
E[\sup_{s \in [0,t]}|\mathcal{E}_{s,t}[X; K] + K_s - (\mathcal{E}_{s,t}[X'; K'] + K'_s)|^2] \\
\leq CE[(X - X' + K_t - K'_t)^2] + CE \int_0^t (K_s - K'_s)^2 ds.
\]

The following properties come immediately from Lemma 5.1 and Lemma 5.4.

Proposition 5.9. For a given \( K \in D^2_{F}(0, T) \), \( \mathcal{E}_{s,t}[; K] \) is an \( \mathcal{F}_t \)-consistent evaluation: (A1)–(A5), i.e.,

(A1) \( \mathcal{E}_{s,t}[X; K] \geq \mathcal{E}_{s,t}[X'; K], \text{ a.s.; if } X \geq X', \text{ a.s. ;}
\)

(A2) \( \mathcal{E}_{s,t}[X; K] = X; \)

(A3) \( \mathcal{E}_{s,t}[\mathcal{E}_{r,s}[X; K]]; K] = \mathcal{E}_{r,s}[X; K], \forall 0 \leq r \leq s \leq t; \)

(A4) \( 1_A \mathcal{E}_{s,t}[X; K] = 1_A \mathcal{E}_{s,t}[1_A X; K], \forall A \in \mathcal{F}_s; \)

(A5) for each \( K, K' \in D^2_{F}(0, T) \), inequalities (5.8) and (5.4) hold true.

Proposition 5.10. If there is some function \( g \) satisfying (2.5) such that \( \mathcal{E} \) coincides with \( \mathcal{E}^g \), i.e., for each \( 0 \leq s \leq t \leq T \) and \( X \in L^2(\mathcal{F}_T) \) we have \( \mathcal{E}_{s,t}[X] = \mathcal{E}^g_{s,t}[X] \), then, for each \( K \in D^2_{F}(0, T) \), \( \mathcal{E}_{s,t}[; K] \) also coincides with \( \mathcal{E}^g_{s,t}[; K] \).

5.2. Two corollaries from Theorem 3.11 The situation without assumption (A4.0) can be derived by Theorem 3.11.

Corollary 5.11. Let \( \mathcal{E}_{s,t}[; L^2(\mathcal{F}_s) \rightarrow L^2(\mathcal{F}_s), 0 \leq s \leq t \leq T, \text{ satisfy (A1)–(A5) and}
\]

\[
\mathcal{E}_{s,t}^g[0; K^0] \leq \mathcal{E}_{s,t}[0] \leq \mathcal{E}_{s,t}^g[0; K^0],
\]

with a given \( K^0 \in D^2_{F}(0, T) \). Then there exists a function \( g(\omega, t, y, z) \) satisfying (2.5) with \( g(s, 0, 0) \equiv 0 \), such that

\[
\mathcal{E}_{s,t}[X] = \mathcal{E}^g_{s,t}[X; K^0],
\]
We define \( \mathcal{E}^0_{s,t}[X] := \mathcal{E}_{s,t}[X; -K^0] \). The above proposition ensures that \( \mathcal{E}^0[\cdot] \) satisfies all conditions (A1)–(A5). Moreover, by (5.13)

\[
0 = \mathcal{E}^{-g^0}_{s,t}[0; K^0 - K^0] \leq \mathcal{E}^0_{s,t}[0] \leq \mathcal{E}^{g^0}_{s,t}[0; K^0 - K^0] = 0, \text{ a.s.}
\]

Thus \( \mathcal{E}^0_{s,t}[\cdot] \) also satisfies (A4). It follows from Theorem 3.1 that there exists \( g \) with \( g(s, 0, 0) \equiv 0 \), such that

\[
\mathcal{E}^0_{s,t}[X] = \mathcal{E}^g_{s,t}[X]
\]

or

\[
\mathcal{E}_{s,t}[X; -K^0] = \mathcal{E}^g_{s,t}[X].
\]



\( \square \)

**Corollary 5.12.** Let \( \mathcal{E}_{s,t}[\cdot] : L^2(\mathcal{F}_t) \rightarrow L^2(\mathcal{F}_s), 0 \leq s \leq t \leq T, \text{ satisfy (A1)–(A5)} \) and

\[
(5.13) \quad \mathcal{E}^{-g^0}_{s,t}[0] \leq \mathcal{E}_{s,t}[0] \leq \mathcal{E}^{g^0}_{s,t}[0],
\]

with a given \( g^0 \in L^2_{\mathcal{F}}(0, T) \). Then there exists a function \( g(\omega, t, y, z) \) satisfying (2.5) with \( g(s, 0, 0) \equiv g^0_s \), such that, for each \( 0 \leq s \leq t \leq T \),

\[
(5.14) \quad \mathcal{E}_{s,t}[X] = \mathcal{E}^{g}_{s,t}[X], \quad \forall X \in L^2(\mathcal{F}_t).
\]

**Proof.** We set \( K^0_t := \int_0^t g^0_s ds, t \in [0, T] \). By the definition of \( \mathcal{E}^g[\cdot; K] \), condition (5.13) reads as

\[
\mathcal{E}^{-g^0}_{s,t}[0; K^0] \leq \mathcal{E}_{s,t}[0] \leq \mathcal{E}^{g^0}_{s,t}[0; K^0].
\]

It follows from Corollary 5.11 that there exists a function \( \bar{g}(\omega, t, y, z) \) satisfying (2.5) with \( \bar{g}(s, 0, 0) \equiv 0 \), such that, for each \( 0 \leq s \leq t \leq T \) and \( X \in L^2(\mathcal{F}_t) \) we have \( \mathcal{E}_{s,t}[X] = \mathcal{E}^{g}_{s,t}[X; K^0], \) or equivalently, \( \mathcal{E}_{s,t}[X] = \mathcal{E}^{g}_{s,t}[X] \), where we set \( g(s, y, z) := \bar{g}(s, y, z) + g^0_s \). The proof is complete. \( \square \)

6. \( \mathcal{E}[:; K] \)-**Martingales**

Hereinafter, \( \mathcal{E}[:;] \) will be a fixed \( \mathcal{F}_t \)-consistent evaluation satisfying (A1)–(A5) and (A4). We introduce the notion of \( \mathcal{E}[:; K] \)-martingale:

**Definition 6.1.** Let \( K \in D^2_{\mathcal{F}}(0, T) \) be given. A process \( Y \in L^2_{\mathcal{F}}(t_0, t_1) \) satisfying \( E[\text{ess sup}_{s \in [t_0, t_1]} |Y_t|^2] < \infty \), is said to be an \( \mathcal{E}[:; K] \)-martingale (resp. \( \mathcal{E}[:; K] \)-supermartingale, \( \mathcal{E}[:; K] \)-submartingale) on \([t_0, t_1]\) if for each \( t_0 \leq s \leq t \leq t_1 \), we have

\[
(6.1) \quad \mathcal{E}_{s,t}[Y_t; K] = Y_s, \text{ (resp. } \leq Y_s, \geq Y_s\text{), a.s.}
\]

**Proposition 6.2.** We assume (A1)–(A5) and (A4). Let \( K \) and \( K' \in D^2_{\mathcal{F}}(0, T) \) be given. Then for each fixed \( t_1 \in [0, T] \) and \( X, X' \in L^2(\mathcal{F}_{t_1}) \), the process defined by

\[
(6.2) \quad Y_{s,t_1}^{X,K} := \mathcal{E}_{s,t_1}[X; K], s \in [0, t_1]
\]

is an \( \mathcal{E}[:; K] \)-martingale, an \( \mathcal{E}^{g_\nu}[; K] \)-submartingale as well as an \( \mathcal{E}^{-g_\nu}[; K] \)-supermartingale. The difference of the processes

\[
(6.3) \quad Y_s = \mathcal{E}_{s,t_1}[X; K] - \mathcal{E}_{s,t_1}[X'; K'], s \in [0, t_1]
\]
is also an $\mathcal{E}^{g^0}[\cdot; K - K']$-submartingale and an $\mathcal{E}^{-g^0}[\cdot; K - K']$-supermartingale.

**Proof.** The first assertion comes directly from (A3) of Proposition 5.9. Now, for each $0 \leq s \leq t \leq t_1$,

\[
Y_s = \mathcal{E}_{s,t_1}[X; K] - \mathcal{E}_{s,t_1}[X'; K']
\]

\[
= \mathcal{E}_{s,t}[\mathcal{E}_{t,t_1}[X; K]] - \mathcal{E}_{s,t}[\mathcal{E}_{t,t_1}[X'; K']]
\]

\[
\leq \mathcal{E}_{s,t}[\mathcal{E}_{t,t_1}[X; K] - \mathcal{E}_{t,t_1}[X'; K'”; K - K']
\]

\[
\leq \mathcal{E}_{s,t}[Y_t; K - K']
\]

Thus $Y$ is an $\mathcal{E}^{g^0}[\cdot; K - K']$ submartingale on $[0, t_1]$. Similarly we can prove that it is an $\mathcal{E}^{-g^0}[\cdot; K - K']$–supermartingale. □

We will prove that $\mathcal{E}_{s,t_1}[X; K]$, $s \in [0, t_1]$ have an RCLL modification. The following upcrossing inequality can be found in [32]. We denote $\mathcal{E}^\mu[\cdot] := \mathcal{E}_{0,T}^\mu[\cdot]$, $\mathcal{E}^{-\mu}[\cdot] := \mathcal{E}_{0,T}^{-\mu}[\cdot]$ with $g^0_\mu(z) = \mu|z|$, $z \in \mathbb{R}^d$.

**Theorem 6.3.** We assume that $g$ satisfies (i) and (ii) of (2.5). Let $Y = (Y_t)_{t \in [0,T]}$ be a $\mathcal{E}^\mu$–supermartingale, $D$ be a denumerable dense subset of $[0,T]$. Then for each $a, b \in R$, $r, s \in [0,T]$ such that $a < b$ and $r < s$, we have

\[
\mathcal{E}^{-\mu}[\mathcal{E}_{0,T}^b(Y, D \cap [r; s]) \leq \frac{e^{2\mu(s-r)}}{b-a}\{\mathcal{E}^\mu[(Y_s-a)^-] + \mathcal{E}^\mu[\int_r^s e^{\mu t}g^0_\mu|dt] + a\mu(s-r)\},
\]

where $\mu$ is the Lipschitz constant of $g$ and $g^0_\mu = g(s, 0, 0)$. In particular

\[
\mathcal{E}^{-\mu}[\mathcal{E}_{0,T}^b(Y, D)] \leq \frac{e^{2\mu T}}{b-a}\{\mathcal{E}^\mu[(Y_T-a)^-] + \mathcal{E}^\mu[\int_0^T e^{\mu t}g^0_\mu|dt] + a\mu T\}.
\]

Moreover, $U_{a,T}^b(Y, D) < \infty$, a.s.

**Proposition 6.4.** For each $X \in L^2(\mathcal{F}_{t_1})$, the process $(\mathcal{E}_{s,t_1}[X])_{0 \leq s \leq t_1}$ has an RCLL modification.

**Proof.** Without loss of generality, we set $t_1 = T$. Since $\mathcal{E}_{s,T}[X]$ is an $\mathcal{E}^{g^0}$–supermartingale, by upcrossing inequality, it is classical that the $\mathcal{F}_t$–adapted process $\bar{Y}_t$ defined by

\[
\bar{Y}_t := \lim_{s \in \mathbb{Q} \cap [0,T], s \searrow t} \mathcal{E}_{s,T}[X], \quad t \in [0,T).
\]

is RCLL. Thus it suffices to prove that, for each $t \in [0, T)$, $\bar{Y}_t = \mathcal{E}_{t,T}[X]$, a.s. Indeed, let $\{s_n\}_{n=1}^\infty \subset \cap(t, T]$ be such that $s_n \searrow t$. By $E[\sup_{0 \leq s \leq T} |\mathcal{E}_{s,T}[X]|^2] < \infty$ and $\mathcal{E}_{s,T}[X; K] \rightarrow \bar{Y}_t$ it follows that

$$
\mathcal{E}_{s,T}[X] \rightarrow \bar{Y}_t, \quad t \in [0,T).
\]

Thus for each $A \in \mathcal{F}_t$, $\mathcal{E}_{s,T}[X_1A] \rightarrow \bar{Y}_t 1_A$, in $L^2(\mathcal{F}_T)$. Or

$$
\mathcal{E}_{s,T}[X_1A] \rightarrow \bar{Y}_t 1_A, \quad \text{in } L^2(\mathcal{F}_T).
\]

Thus, by (4.4) with $K = K' = 0$, $\tau = T$, $X = \mathcal{E}_{s,T}[X_1A]$ and $X' = \bar{Y}_t 1_A$, we have

\[
E[|\mathcal{E}_{0,s_n}[\mathcal{E}_{s,T}[X_1A]] - \mathcal{E}_{0,s_n}[\bar{Y}_t 1_A]|^2] \leq E[|\mathcal{E}_{s,T}[X_1A] - \bar{Y}_t 1_A|^2] \rightarrow 0.
\]
We also have
\[ |E_{0,s_n}[\tilde{Y}_t 1_A] - E_{0,t}[\tilde{Y}_t 1_A]|^2 \leq cE[|E_{0,s_n}[\tilde{Y}_t 1_A] - \tilde{Y}_t 1_A|^2] \]

It follows from (10.34) that \( E_{0,s_n}[\tilde{Y}_t 1_A] \to E_{0,t}[^t\tilde{Y}_t 1_A] \). The above two convergences imply
\[ E_{0,s_n}[E_{s,n,T}[X 1_A]] = E_{0,t}[E_{t,T}[X 1_A]]. \]

But on the other hand,\( E_{0,s,n}[E_{s,n,T}[X 1_A]] = E_{0,t}[E_{t,T}[X 1_A]]. \) We thus have
\[ E_{0,s_n}[E_{s,n,T}[X 1_A]] = E_{0,t}[E_{t,T}[X 1_A]], \quad \forall A \in F_t. \]

From which it follows that \( E_{t,T}[X] = Y, \ a.s. \)

We will always take an RCLL version of \( E_{.,T}[^t\cdot] \).

**Proposition 6.5.** We assume (A1)–(A5) and (A40). Then, for each \( X \in L^2(F_t) \) and \( K \in D^2_T(0,T) \) the process \( (E_{t,T}[X; K])_{0 \leq t \leq T} \) belongs to \( D^2_T(0,T) \).

**Proof.** In the case where \( K \in D^2_T(0,T) \), from the definition and the above lemma it follows that \( E_{.,T}[X; K] \) is also RCLL. This with (6.10) (by setting \( Y' = 0 \)) we deduce that \( E_{.,t_i}[X; K_i] \in D^2_T(0,t_i) \). Now let \( K \in D^2_T(0,T) \) and let \( \{K_i\}_{i=1}^\infty \) be a sequence in \( D^2_T(0,T) \) such that \( K_i \to K \) in \( L^2_T(0,T) \) and \( K_{i_n} \to K_{i_1} \) in \( L^2(F_T) \). From (6.10) it follows that
\[ E[\sup_{s \in [0,t_i]} |E_{s,t_i}[X; K_i^t] + K_i^t - (E_{s,t_i}[X; K] + K_s)|^2] \leq CE[(K_{i_n}^t - K_t)^2] + CE \int_{0}^{t_1} (K_s - K_s') ds \to 0. \]

Since \( (E_{s,t_i}[X; K_i] - K_i)_{0 \leq t \leq t_i}, i = 1, 2, \ldots \) are in \( D^2_T(0,t_i) \), \( (E_{s,t_i}[X; K] - K_i)_{0 \leq t \leq t_i} \) and \( (E_{s,t_i}[X; K])_{0 \leq t \leq t_i} \) are also in \( D^2_T(0,t_i) \).

We then can apply \( E^g \)-supermartingale decomposition theorem, i.e., Proposition 4.10 to get the following result.

**Proposition 6.6.** We assume (A1)–(A5) and (A40). Let \( K \in D^2_T(0,T) \) be given. For fixed \( t \in [0,T] \) and \( X \in L^2(F_t) \), the process \( Y_{t,X,K} := E_{s,t}[X; K], s \in [0,t], \) has the following expression: there exist processes \( \{g_{t,X,K}, z_{t,X,K}\} \in L^2_T(0,T; R \times R^2) \) such that
\[ Y_{s,X,K}^t = X + K_t - K_s + \int_s^t g_{s,r,X,K} dr - \int_s^t z_{s,r,X,K} dB_r, \quad s \in [0,t], \]

such that
\[ |g_{s,r,X,K}^t| \leq \mu(|Y_{s,r,X,K}^t| + |z_{s,r,X,K}^t|), \forall s \in [0,t]. \]

Moreover let \( Y_{s,X',K'} := E_{s,t}[X'; K'], s \in [0,t], \) for some other \( K' \in D^2_T(0,T), X' \in L^2(F_t) \) and let \( (g_{s,r,X',K'}, z_{s,r,X',K'}) \) be the corresponding expression in \( [0,t], \) then we have
\[ |g_{s,r,X,K}^t - g_{s,r,X',K'}^t| \leq \mu(|Y_{s,r,X,K}^t - Y_{s,r,X',K'}^t| + |z_{s,r,X,K}^t - z_{s,r,X',K'}^t|), \forall s \in [0,t]. \]

**Proof.** Since \( (Y_{s,X,K})_{s \in [0,t]} \), is an \( E_{g_{t,X,K}} \)-submartingale and \( E_{g_{t,X,K}} \)-supermartingale, by Proposition 4.10 and Corollary 4.11 there exists an increasing process \( A^+ \in D^2_T(0,t) \) and \( A^- \in D^2_T(0,t) \) with \( A_0^+ = A_0^- = 0, \) such that
\[ Y_{s,X,K}^t = g_{s,t|[X;(K-A^+)]} = E_{s,t|[X;(K-A^-)]}, s \in [0,t]. \]
According to the notion of $\mathcal{E}^g$ defined in (6.7), $Y_{s}^{t,X,K}$ is the solution of the following BSDE on $[0,t]$:

\begin{equation}
Y_{s}^{t,X,K} = X + (K - A^{+})_t - (K - A^{+})_s + \int_s^t \mu(|Y_{r}^{t,X,K}| + |Z_{r}^{+}|)dr - \int_s^t Z_{r}^{+}dB_r
\end{equation}

and

\begin{equation}
Y_{1}^{t,X,K} = X + (K + A^{-})_t - (K + A^{-})_s - \int_s^t \mu(|Y_{r}^{t,X,K}| + |Z_{r}^{-}|)dr - \int_s^t Z_{r}^{-}dB_r.
\end{equation}

It then follows that $Z_{s}^{t,X,K} := Z_{s}^{+} \equiv Z_{s}^{-}$, $s \in [0,t]$ and thus

\begin{equation}
-dA_{s}^{+} + \mu(|Y_{s}^{t,X,K}| + |Z_{s}^{+}|)ds \equiv dA_{s}^{+} - \mu(|Y_{s}^{t,X,K}| + |Z_{s}^{+}|)ds, \quad s \in [0,t]
\end{equation}

or

\begin{equation}
dA_{s}^{-} + dA_{s}^{+} \equiv 2\mu(|Y_{s}^{t,X,K}| + |Z_{s}^{+}|)ds, \quad s \in [0,t]
\end{equation}

Thus $dA^{+}$ and $dA^{-}$ are absolutely continuous with respect to $ds$. We denote $a_{s}^{+}ds = dA_{s}^{+}$ and $a_{s}^{-}ds = dA_{s}^{-}$. It is clear that

\begin{align*}
0 & \leq a_{s}^{+} \leq 2\mu(|Y_{s}^{t,X,K}| + |Z_{s}^{+}|), \\
0 & \leq a_{s}^{-} \leq 2\mu(|Y_{s}^{t,X,K}| + |Z_{s}^{+}|), \quad dp \times dt - \text{a.e.}
\end{align*}

We then can rewrite (6.10) as

\begin{equation}
Y_{s}^{t,X,K} = X + K_t - K_s + \int_s^t [-a_{r}^{+} + \mu(|Y_{r}^{t,X,K}| + |Z_{r}^{+}|)]dr - \int_s^t Z_{r}^{+}dB_r.
\end{equation}

Thus, by setting $g_{s}^{t,X,K} := -a_{s}^{+} + \mu(|Y_{s}^{t,X,K}| + |Z_{s}^{+}|)$, we have the expression (6.6) as well as the estimate (6.7).

It remains to prove (6.8). By (A5) of Proposition 4.10, $\hat{Y}_{s} = Y_{s}^{t,X,K} - Y_{s}^{t,X',K'}$ is an $\mathcal{E}^{\hat{g}_{s}[::K - K']}$-submartingale and an $\mathcal{E}^{-\hat{g}_{s}[::K - K']}$-supermartingale on $[0,t]$.

Thus we can repeat the above procedure to prove that there exist processes $(\hat{g}_{s}, \hat{Z}_{s}) \in L_{F_{s}}^{2}(0,t;R \times R^{d})$ such that

\begin{equation}
\hat{Y}_{s} = X - X' + (K - K')_t - (K - K')_s + \int_s^t \hat{g}_{r}dr - \int_s^t \hat{Z}_{r}dB_r, \quad s \in [0,t],
\end{equation}

such that

\begin{equation}
|\hat{g}_{s}| \leq \mu(|\hat{Y}_{s}| + |\hat{Z}_{s}|), \quad \forall s \in [0,t].
\end{equation}

But by (6.6) and $\hat{Y}_{s} = Y_{s}^{t,X,K} - Y_{s}^{t,X',K'}$, we immediately have

\begin{equation}
\hat{g}_{s} \equiv g_{s}^{t,X,K} - g_{s}^{t,X',K'}, \quad \hat{Z}_{s} \equiv z_{s}^{t,X,K} - z_{s}^{t,X',K'}.
\end{equation}

This with (6.15) yields (6.8). The proof is complete. $\square$

**Corollary 6.7.** Let $K^{1}$ and $K^{2} \in D_{F_{2}}^{2}(0,T)$ and $X^{1} \in L^{2}(\mathcal{F}_{1})$, $X^{2} \in L^{2}(\mathcal{F}_{2})$ be given for some fixed $0 \leq t_{1} \leq t_{2} \leq T$ and let $(g_{s}^{t,X,K}), (z_{s}^{t,X,K})_{s \in [0,t]}$, $i = 1,2$, be the pair in (6.6) for $Y_{s}^{t,X',K}$, $\mathcal{E}_{s,t_{1}}[X^{i};(K^{i})]$, $i = 1,2$, respectively. Then we have

\begin{equation}
|g_{s}^{t,X',K} - g_{s}^{t,X,K}| \leq \mu(|Y_{s}^{t,X',K^{2}} - Y_{s}^{t,X,K^{2}}| + |z_{s}^{t,X,K^{1}} - z_{s}^{t,X,K^{2}}|), \quad \forall s \in [0,t_{1}].
\end{equation}
Proof. With the observation
\[ Y_t^{X,K} = \mathcal{E}_{s,t}[Y_t^{X,K}], \quad s \in [0,t], \]
it is an immediate consequence of Proposition 6.6. \(
\)

**Corollary 6.8.** For each \( t \in [0,T] \) and \( X \in L^2(\mathcal{F}_t) \), \( K \in D^2_T(0,T) \), the process \( (\mathcal{E}_{s,t}[X;K])_{s \in [0,t]} \) is also in \( D^2_T(0,t) \). If moreover, \( K \in S^2_T(0,T) \) (resp. Itô’s process), then \( (\mathcal{E}_{s,t}[X;K])_{s \in [0,t]} \) is also in \( S^2_T(0,t) \) (resp. Itô’s process).

7. BSDE UNDER \( \mathcal{E}[: \cdot] \)

We now consider the following kind of backward stochastic differential equations: Let \( X \in L^2(\mathcal{F}_T) \) be given and let
\[
(7.1) \quad f : (\omega, t, y) \in \Omega \times [0,T] \times R \rightarrow R
\]
be a given function. We assume that \( f \) satisfies
\[
(7.2) \quad \begin{cases} 
(i) & f(\cdot, y) \in L^2_T(0,T), \text{ for each } y \in R, \\
(ii) & |f(t, y) - f(t, y')| \leq c(|y - y'|), \forall y, y' \in R.
\end{cases}
\]
We consider the following kind of BSDE
\[
(7.3) \quad Y_t = \mathcal{E}_{t,T}[X; \int_0^T f(s, Y_s)ds], \quad t \in [0,T].
\]
We have the following existence and uniqueness result

**Theorem 7.1.** Let \( \mathcal{E}[: \cdot] \) satisfy (A1)-(A5) and (A4) and let \( f \) satisfies (7.2). Then for each \( X \in L^2(\mathcal{F}_T) \), there exists a unique solution \( Y \in S^2_T(0,T) \) of BSDE (7.3).

**Proof.** With Corollary 6.8 we only need to prove that BSDE (7.3) has a unique solution \( Y \in L^2_T(0,T) \). To this end we define a mapping \( \Lambda[:\cdot] : L^2_T(0,T) \rightarrow L^2_T(0,T) \) by
\[
(7.4) \quad \Lambda_t[y] := Y_t = \mathcal{E}_{t,T}[X; \int_0^T f(s, Y_s)ds], \quad y \in L^2_T(0,T).
\]
By Corollary 6.8 \( \Lambda[:\cdot] \) belongs to \( S^2_T(0,T) \). From Proposition 7.7 we have
\[
(7.5) \quad \Lambda_t[y^1] - \Lambda_t[y^2] \leq \mathcal{E}^{y^1}_{t,T}[0; \int_0^T (f(s, y^1_s) - f(s, y^2_s))ds], \forall y^1, y^2 \in L^2_T(0,T).
\]
By Proposition 6.3 we have, with \( \beta = \mu^2 + 2\mu + 1 \),
\[
E[(\Lambda_t[y^1] - \Lambda_t[y^2])^2] \leq E[(\mathcal{E}^{y^1}_{t,T}[0; \int_0^T (f(s, y^1_s) - f(s, y^2_s))ds)^2]] \leq E[\int_t^T e^{\beta(s-t)}(f(s, y^1_s) - f(s, y^2_s))^2ds] \leq CE[\int_t^T |y^1_s - y^2_s|^2ds],
\]
where $C = c^2 e^{\beta T}$ and $c$ is the Lipschitz constant of $f$. We multiple $e^{2Ct}$ on both sides and integrate on $[0, T]$,

$$E[\int_{0}^{T} e^{2Ct}(\Lambda_t[y^1] - \Lambda_t[y^2])^2 dt] \leq CE[\int_{0}^{T} \int_{t}^{T} e^{2Ct}|y^1_s - y^2_s|^2 dsdt]$$

$$= CE[\int_{0}^{T} (\int_{0}^{s} e^{2Ct} ds)|y^1_s - y^2_s|^2 ds]$$

$$= C(2C)^{-1} E[\int_{0}^{T} (e^{2Ct} - 1)|y^1_s - y^2_s|^2 ds].$$

We thus have

$$E[\int_{0}^{T} e^{2Ct}(\Lambda_t[y^1] - \Lambda_t[y^2])^2 dt] \leq \frac{1}{2} E[\int_{0}^{T} e^{2Ct}|y^1_t - y^2_t|^2 dt].$$

We observe that the following two norms are equivalent to each others in $L^2_{Y}(0, T)$:

$$[E \int_{0}^{T} |\phi_t|^2 dt]^{1/2} \sim [E \int_{0}^{T} e^{2Ct}|\phi_t|^2 dt]^{1/2}.$$

It follows that $\Lambda_t[\cdot]$ is a contraction mapping. Thus there exists a unique fixed point $Y \in L^2_{Y}(0, T)$, such that

$$Y_t = \Lambda_t[Y] = E_{t,T}[X; \int_{0}^{T} f(s, Y_s)ds].$$

The prove is complete. □

**Proposition 7.2.** Let $Y \in S_{F}^{2}(0, T)$ be the solution of BSDE (7.3). Then, for each $t \in [0, T)$ we have

$$Y_s = E_{s,t}[Y_t; \int_{0}^{T} f(s, Y_s)ds], \ s \in [0, t].$$

**Proof.** We set $K_t := \int_{0}^{t} f(s, Y_s)ds$. By proposition 0.2 $Y_s = E_{s,T}[X; K_t]$ is an $E_{s,T}[\cdot; K_t]$-martingale. Thus (7.7) holds. □

**Proposition 7.3.** Let $X, X' \in L^2(\mathcal{F}_T)$ and $\phi \in L^2_{Y}(0, T)$ be given. Let $Y \in S_{F}^{2}(0, T)$ be the solution of BSDE (7.3), $Y' \in S_{Y}^{2}(0, T)$ be the solution of

$$Y'_t = E_{t,T}[X'; \int_{0}^{T} (f(s, Y'_s) + \phi_s)ds].$$

Then $Y' - Y$ is an $\mathcal{E}^{g_{\mu+c, \mu} + \phi[\cdot]}$-submartingale and an $\mathcal{E}^{-g_{\mu+c, \mu} + \phi[\cdot]}$-supermartingale on $[0, T]$, where $c \geq 0$ is the Lipschitz constant of $f$ with respect to $y$ and $g_{\mu+c, \mu}(y, z) := (c + \mu)|y| + \mu|z|$.

**Proof.** By Proposition 5.7 and the above proposition, we have, for each $0 \leq s \leq t \leq T$,

$$Y'_s - Y_s = E_{s,t}[Y'_t; \int_{0}^{T} (f(s, Y'_s) + \phi_s)ds] - E_{s,t}[Y_t; \int_{0}^{T} f(s, Y_s)ds]$$

$$\geq E_{s,t}^{-g_{\mu}}[Y'_t - Y_t; \int_{0}^{T} ((f(s, Y'_s) - f(s, Y_s)) + \phi_s)ds].$$
Thus $Y' - Y$ is an $\mathcal{E}^{-\theta_s} \cdot K$ supermartingale, where $K_t := \int_0^t ((f(s, Y'_s) - f(s, Y_s) + \phi_s)ds$. By $\mathcal{E}^\theta$-supermartingale decomposition theorem (Corollary 4.11), there exists an increasing process $A \in D^2_T(0, T)$ with $A_0 = 0$, such that

$$Y_t - Y'_t = \mathcal{E}^{g_t}_{t,T}[X' - X; (K + A)].$$

Define $\hat{Y}_t := Y_t - Y'_t$ solves the following BSDE

$$\hat{Y}_t = X' - X + A_T - A_t + \int_t^T ((f(s, Y'_s) - f(s, Y_s) + \phi_s)ds$$

$$- \int_t^T \mu(|\hat{Z}_s| + |\hat{Y}_s|)ds - \int_t^T \hat{Z}_s dB_s$$

or, equivalently

$$\hat{Y}_t = X' - X + (A + \hat{A})_T - (A + \hat{A})_t$$

$$- \int_t^T [(\mu + \sigma)(|\hat{Y}_s| + |\hat{Z}_s|) + \phi_s]ds - \int_t^T \hat{Z}_s dB_s.$$

Here we set

$$\hat{A}_t = \int_0^t c|\hat{Y}_s| + f(s, Y'_s) - f(s, Y_s) ds.$$

It is clear that $\hat{A}$ and thus $A + \hat{A}$ is an increasing process. Thus $\hat{Y}$ is an $\mathcal{E}^{g_t + \mu + \phi} - \text{supermartingale}$. Analogously, we can prove that it is an $\mathcal{E}^{g_t + \mu + \phi}$-submartingale. □

**Corollary 7.4.** If $X' \geq X$ and $\phi_s \geq 0$, $dt \times dP$-a.e., then we have $Y'_t - Y_t \geq 0$, $dt \times dP$-a.e..

8. $\mathcal{E}$–SUPERMARTINGALE DECOMPOSITION THEOREM: INTRINSIC FORMULATION

Our objective of this section is to prove the following $\mathcal{E}$–supermartingale decomposition theorem of Doob–Meyer’s type. Since $(\mathcal{E}_{s,t}[,]_{s \leq t})$ is abstract and nonlinear, it is necessary to introduce the intrinsic form $\mathbb{E}$. This theorem plays an important role in the proof of the main theorem of this paper. It can be also considered as a generalization of Proposition A.10.

**Theorem 8.1.** We assume $(A1)$–(A5) as well as $(A4)$. Let $Y \in S^2_T(0, T)$ be an $\mathcal{E}[-] - \text{supermartingale}$. Then there exists an increasing process $A \in S^2_T(0, T)$ with $A_0 = 0$, such that $Y$ is an $\mathcal{E}[-; A]$-martingale, i.e.,

$$Y_t = \mathcal{E}_{t,T}[Y_T; A], \quad t \in [0, T]. \quad (8.1)$$

**Remark 8.1.1.** This theorem has an interesting interpretation in finance: the fact that $Y \in S^2_T(0, T)$ is an $\mathcal{E}[-]$–supermartingale is equivalent to that there exists an increasing process $A$ such that $Y$ is the flow of the dynamical evaluation of the sum of the final payoff $Y_T$ at $T$ plus the flow of the “dividend” $A$ during the whole period $[0, T]$.

**Remark 8.1.2.** In the case where $(\mathcal{E}_{s,t}[,])_{0 \leq s \leq t \leq T}$ is a system of linear mappings, $(8.1)$ becomes

$$Y_t + A_t = \mathcal{E}_{t,T}[Y_T + A_T], \quad t \in [0, T],$$

i.e., as in classical situation, $Y + A$ is an $\mathcal{E}[-]$–martingale. But, the intrinsic formulation that can be applied to nonlinear situation is that $Y$ is an $\mathcal{E}[-; A]$–martingale.
In order to prove this theorem, we need to extend the definition of $\mathcal{E}_{s,t}[: \cdot]$, $\mathcal{E}_{\sigma,\tau}[: \cdot]$ from deterministic times $s, t \in [0, T]$ to $\mathcal{E}_{\sigma,\tau}[: \cdot]$ of stopping times $\sigma$ and $\tau$.

**Theorem 8.2.** There exists a unique extension of $(\mathcal{E}_{s,t}[: \cdot])_{0 \leq s \leq t \leq T}$ to the $\mathcal{F}_\tau$-consistent nonlinear evaluation:

$$\mathcal{E}_{\sigma,\tau}[: \cdot] : L^2(\mathcal{F}_\tau) \to L^2(\mathcal{F}_\sigma), \sigma, \tau \in \mathcal{S}_T, \sigma \leq \tau,$$

for each $X, X' \in L^2(\mathcal{F}_\tau),$

(A1) $\mathcal{E}_{\sigma,\tau}[X] \geq \mathcal{E}_{\sigma,\tau}[X']$, a.s., if $X \geq X'$, a.s.;

(A2) $\mathcal{E}_{\sigma,\tau}[X] = X$, a.s.;

(A3) $\mathcal{E}_{\sigma,\tau}[^{\rho}[X]] = \mathcal{E}_{\sigma,\tau}[X]$, $\forall 0 \leq \sigma \leq \rho \leq \tau, \rho \in \mathcal{S}_\tau$;

(A4') $1_{A} \mathcal{E}_{\sigma,\tau}[X] = \mathcal{E}_{\sigma,\tau}[1_{A} X], \forall A \in \mathcal{F}_\sigma$;

(A5) $\mathcal{E}^{\rho}$–domination: $\forall X, X' \in L^2(\mathcal{F}_\tau), \forall K, K' \in \mathcal{S}_2^2(0, T)$.

(8.2) $\mathcal{E}_{\sigma,\tau}[X; K] - \mathcal{E}_{\sigma,\tau}[X'; K'] \leq \mathcal{E}^{\rho}_{\sigma,\tau}[X - X'; K - K'].$

**Remark 8.2.1.** The “unique extension” is in the following sense: if the system $\mathcal{E}_{\sigma,\tau}[: \cdot] : L^2(\mathcal{F}_\tau) \to L^2(\mathcal{F}_\sigma), \sigma, \tau \in \mathcal{S}_T, \sigma \leq \tau$, satisfies also the above (A1)–(A3), (A4) and (A5) such that $\mathcal{E}_{s,t}[X] = \mathcal{E}_{s,t}[X]$, a.s., for each deterministic times $0 \leq s \leq t \leq T$ and for each $X \in L^2(\mathcal{F}_t)$, then we have $\mathcal{E}_{s,t}[X] = \mathcal{E}_{s,t}[X]$, a.s., for each $\sigma, \tau \in \mathcal{S}_T, \sigma \leq \tau$, and for each $X \in L^2(\mathcal{F}_\tau)$.

We will give the proof of this Theorem in the last section.

To prove Theorem 8.2, we need to introduce a sequence of BSDEs of the following form: for $n = 1, 2, \cdots,$

(8.3) $y^n_t = \mathcal{E}_{t,T}[Y_T; n \int_0^t (Y_s - y^n_s)ds].$

The solution $y^n \in \mathcal{S}_2^2(0, T)$ has the following interesting property.

**Lemma 8.3.** For each $n = 1, 2, \cdots, \exists$ such that $y^n_t \leq Y_t, dt \times dP$–a.e..<br>

**Proof.** For each fixed $n$, and any $\delta > 0$, we define

$$\sigma^n := \inf\{t \geq 0; y^n_t \geq Y_t + \delta\} \land T.$$

If for all $\delta > 0$ we always have $P\{\sigma^n = T\} = 1$, then we have our conclusion. Otherwise, there exists a $\delta > 0$ such that

$$P(\sigma^n > T) > 0.$$

We then define

$$\tau := \inf\{t \geq \sigma^n; y^n_t \leq Y_t\}.$$

Since $y^n_T = Y_T$, we have $P(\tau \leq T) = 1$. Since $\mathcal{E}_{s,t}[: n \int_0^t (Y_s - y^n_s)ds], 0 \leq s \leq t \leq T,$ is an $\mathcal{F}_t$–consistent evaluation satisfying (A1)–(A5), by Theorem 8.2, it can be uniquely extended to $\mathcal{E}_{\rho,\sigma}[: n \int_0^t (Y_s - y^n_s)ds], \rho \leq \sigma, \rho \in \mathcal{S}_T.$ This with the fact that $(y^n_t)_{t \in [0, T]}$ is an martingale under $\mathcal{E}_{s,t}[: n \int_0^t (Y_s - y^n_s)ds]$, it follows from the Optional Stopping Theorem that

$$y^n_\rho = \mathcal{E}_{\rho,\sigma}[y^n_\rho; n \int_0^t (Y_s - y^n_s)ds], \mathcal{E}_{\rho,\sigma}[Y_\rho] \leq Y_\rho, \text{ a.s.}$$
We then have
\[ y^n_{\sigma^+, \tau} = \mathcal{E}_{\sigma^+, \tau}[y^n_{\tau}; \int_0^\tau n(Y_s - y^n_s)ds] \]
\[ \leq \mathcal{E}_{\sigma^+, \tau}[y^n_{\tau}] + \mathcal{E}_{\sigma^+, \tau}^{\mu, \delta}[0; \int_0^\tau n(Y_s - y^n_s)ds]. \]
The inequality is from (8.2). But since \( n(Y_s - y^n_s) \leq 0 \), on \([\sigma^+, \tau]\), by the definition of \( \mathcal{E}^\mu[\cdot; \mathcal{K}] \) (see (4.7)) and the Comparison Theorem of BSDE we have \( \mathcal{E}_{\sigma^+, \tau}^{\mu, \delta}[0; \int_0^\tau n(Y_s - y^n_s)ds] \leq 0 \). Thus
\[ y^n_{\sigma^+, \tau} \leq \mathcal{E}_{\sigma^+, \tau}[y^n_{\tau}] = \mathcal{E}_{\sigma^+, \tau}[Y_{\tau}] \leq Y_{\sigma^+}. \]
But this contradicts with \( y^n_{\tau} \geq Y_{\tau} + \delta \) on \( \{\sigma^+ < T\} \) and \( P(\sigma^+ < T) > 0 \). The proof is complete. \( \square \)

By Lemma 8.4, we can rewrite BSDE (8.6) as:
(8.4) \[ y^n_t = \mathcal{E}_{\tau, T}[Y_T; n \int_0^T (Y_s - y^n_s)^+ ds]. \]
By comparison theorem (Corollary 7.4), we have
(8.5) \[ y^1_t \leq y^2_t \leq \cdots \leq Y_t. \]
Since \( y^1 \) and \( Y \) are both in \( S^2_{\mathbb{F}}(0, T) \) it follows from \( |y^n_{\tau}| \leq |y^1_{\tau}| + |Y_{\tau}| \) that, there exists a constant \( C > 0 \) which is independent of \( n \) such that
(8.6) \[ E[\sup_{t \in [0, T]} |y^n_t|^2] \leq C. \]
We define
(8.7) \[ A^n_t := n \int_0^t (Y_s - y^n_s)^+ ds = n \int_0^t (Y_s - y^n_s)ds, \quad t \in [0, T]. \]
It follows from Proposition 6.6 that \( y^n \) has the expression
(8.8) \[ y^n_t = Y_T + A^n_T - A^n_t + \int_t^T g^n_s ds - \int_0^T z^n_s dB_s, \quad t \in [0, T], \]
where \((g^n, z^n) \in L^2(0, T; \mathbb{R} \times \mathbb{R}^d)\) satisfies, for each \( m, n = 1, 2, \cdots \),
(8.9) \[ |g^n_t| \leq \mu|y^n_t| + |z^n_t|, \]
(8.10) \[ |g^n_t - g^m_t| \leq \mu|y^n_t - y^m_t| + \mu|z^n_t - z^m_t|, \quad \forall t \in [0, T], \ dt \times dP, \text{ a.e.} \]
We have the following estimates.

Lemma 8.4. There exists a constant \( C > 0 \) which is independent of \( n \) such that
(8.11) \[ E[\int_0^T |z^n_t|^2 dt] \leq C, \quad E[|A^n_T|^2] \leq C. \]

Proof. From (8.8), we have
(8.12) \[ A^n_T = y^n(0) - y^n_T - \int_0^T g^n_s ds + \int_0^T z^n_s dB_s \]
\[ \leq |y^n(0)| + |y^n_T| + \int_0^T (|y^n_s| + |z^n_s|)ds + |T| \int_0^T z^n_s dB_s|. \]
With (8.6), it follows that there are two constants \( c_1 \) and \( c_2 \), independent of \( n \), such that

\[
E|A_T^n|^2 \leq c_1 + c_2 E \int_0^T |z_s^n|^2 ds.
\]

On the other hand, Itô’s formula applied to \(|y^n(\cdot)|^2\) gives:

\[
E[|y^n(0)|^2] = E|Y_T|^2 + E \int_0^T [2y^n_s \cdot g^n_s - |z_s^n|^2]ds
\]

\[
+ 2E \int_0^T y^n_s dA^n_s
\]

\[
\leq E|Y_T|^2 + E \int_0^T [2\mu|y^n_s|(|z^n_s| + |y^n_s|) - |z^n_s|^2]ds
\]

\[
+ E[2A_T^n \sup_{0 \leq s \leq T} |y^n_s|].
\]

Thus, by \( 2\mu|y^n||z^n| \leq 2\mu^2|y^n|^2 + 1/2|z^n|^2 \) and \( 2A_T^n \sup_{0 \leq s \leq T} |y^n_s| \leq \frac{1}{2c_2}|A_T^n|^2 + 4c_2 \sup_{0 \leq s \leq T} |y^n_s|^2 \),

\[
E \int_0^T |z^n_s|^2 ds \leq 2E|Y_T|^2 + E \int_0^T (4\mu^2 + 4\mu + 1)|y^n_s|^2 ds
\]

\[
+ 8c_2E[ \sup_{0 \leq s \leq T} |y^n_s|^2] + \frac{1}{2c_2}E[|A_T^n|^2]
\]

\[
\leq 2E|Y_T|^2 + E \int_0^T (4\mu^2 + 4\mu + 1)|y^n_s|^2 ds
\]

\[
+ 8c_2E[ \sup_{0 \leq s \leq T} |y^n_s|^2] + \frac{c_1}{2c_2} + \frac{1}{2}E \int_0^T |z^n_s|^2 ds.
\]

The last inequality is due to (8.12). Then the first estimate of (8.11) yields immediately from (8.6). From which and (8.12) we obtain the second one. The proof is complete. \( \square \)

We rewrite (8.8) in the following forward version:

\[
y_t^n = y_0^n - A_t^n - \int_0^t g_s^n ds + \int_0^t z_s^n dB_s, \quad t \in [0, T],
\]

in order to apply the following monotonicity limit theorem (see [23, Peng 1999], Theorem 2.1).

**Proposition 8.5.** Let \( \{A^n\}_{n=1}^\infty \) be a sequence of increasing processes in \( S^2_T(0, T) \) with \( A_0^n = 0 \), and let \( \{g^n, z^n\}_{n=1}^\infty \) be uniformly bounded in \( L^2_T(0, T) \):

\[
E \int_0^T [\|g^n_s\|^2 + |z^n_s|^2] \leq C.
\]

If \( \{(y^n_t)_{t \in [0, T]}\}_{n=1}^\infty \) increasingly converges to \( \{y_t\}_{t \in [0, T]} \) with \( E[\sup_{t \in [0, T]} |y_t|^2] < \infty \), then this limit process has the following form

\[
y_t = y_0 - A_t - \int_0^t g_s ds + \int_0^t z_s dB_s, \quad t \in [0, T],
\]
where $A \in D^2_T(0, T)$ is an increasing process such that $A_0 = 0$, $(g, z) \in L^2_T(0, T; R \times R^d)$, and

$$A^n_t \rightarrow A_t \text{ weakly in } L^2(F_T), \forall t \in [0, T],$$

(8.16) $$ \lim_{n \rightarrow \infty} E \int_0^T |z^n_t - z_t|^pdt = 0, \text{ for each fixed } p \in [1, 2).$$

Moreover, if $y$ is continuous, i.e., $y \in S^2_T(0, T)$, then we have

$$ \lim_{n \rightarrow \infty} E \int_0^T |z^n_t - z_t|^2dt = 0.$$  

We now can proceed to give

**Proof of Theorem 8.7** Since $A^n$ defined by (8.7) is bounded by the second estimate of (8.11), it follows that $y^n \rightarrow Y_t dt \times dP$–a.e.. On the other hand, by (8.6), (8.9) and the first estimate of (8.11), \{(g^n, z^n)\}_{n=1}^\infty is also uniformly bounded in $L^2_T(0, T)$. We then can apply Proposition 8.5 to derive that

$$ \lim_{n \rightarrow \infty} \int_0^T |E \left[ (A_T - A^n_T)^2 \right] \leq CE \int_0^T (A_t - A^n_t)^2 dt \rightarrow 0.$$  

We then have

$$ \lim_{n \rightarrow \infty} y^n_t = \lim_{n \rightarrow \infty} E_{t,T}[Y_T; A^n] = E_{t,T}[Y_T; A].$$

The proof is complete. □

9. **Proof of Theorem 8.1**

For each fixed $(t, y, z) \in [0, T] \times R \times R^d$, we consider the solution $Y^{t,y,z} \in S^2_T(0, T)$ of an Itô’s equation on $[t, T]$, and a BSDE on $[0, t]$:  

(9.1) $$ dY^{t,y,z}_s = -\mu(|Y^{t,y,z}_s| + |z|)ds + zdB_s, \quad s \in (t, T],$$

(9.2) $$ Y^{t,y,z}_t = y.$$  

It is easy to check that $Y^{t,y,z}$ is an $\mathcal{F}_s[\cdot]$–martingale. Thus, by (8.1), it is also an $\mathcal{E}[\cdot]$–supermartingale. By Decomposition Theorem 8.1 there exists an increasing process $A^{t,y,z} \in S^2_T(0, T)$ with $A^{0,y,z}_0 = 0$, such that

$$ Y^{t,y,z}_s = \mathcal{E}_{s,T}[Y^{t,y,z}_T; A^{t,y,z}].$$

By Proposition 8.1 and Corollary 8.7 there exists $(g^{t,y,z}, Z^{t,y,z}) \in L^2_T(0, T)$ such that

(9.3) $$ -dY^{t,y,z}_s = dA^{t,y,z}_s + g^{t,y,z}_s ds - Z^{t,y,z}_s dB_s, \quad s \in [0, T],$$

and such that, for each different $(t, y, z), (t', y', z') \in [0, T] \times R \times R^d$

(9.4) $$ |g^{t,y,z}_s - g^{t',y',z'}_s| \leq \mu|Y^{t,y,z}_s - Y^{t',y',z'}_s| + \mu|Z^{t,y,z}_s - Z^{t',y',z'}_s|, \quad s \in [t \vee t', T],$$
Thus (9.4), (9.5) and (9.8) become, respectively,

\[ g(s,t,y,z) = \mu|Y^{t',y,z}| + \mu|Z^{t',y,z}|, \quad s \in [t,T], \quad ds \times dP - \text{a.e.} \]

Now for each \( X \in L^2(F_\tau) \), we set

\[ \tilde{Y}_s^{t',X} := \mathcal{E}_{s,t'}[X] = \mathcal{E}_{s,t'}[X;0]. \]

We use once more Proposition 6.6 and Corollary 6.7, there exists \( \tilde{g}^{t',X} \in L^2(0,t') \) such that, for \( s \in [0,t'] \),

\[(9.11) \quad |\tilde{g}^{t',X} - \tilde{g}^{t',X'}| \leq \mu|\tilde{Y}^{t',X} - \tilde{Y}^{t',X'}| + \mu|\tilde{Z}^{t',X} - \tilde{Z}^{t',X'}|, \quad s \in [0,t'], \quad ds \times dP - \text{a.e..} \]

On the other hand, comparing to (9.1) and (9.3), we have

\[ Y^{t',X} = \tilde{Y}^{t',X}d\tilde{B}_s, \quad \tilde{Y}_\tau = X, \]

such that

\[ |\tilde{g}^{t',X} - \tilde{g}^{t',X'}| \leq \mu|\tilde{Y}^{t',X} - \tilde{Y}^{t',X'}| + \mu|\tilde{Z}^{t',X} - \tilde{Z}^{t',X'}|, \quad s \in [0,t'], \quad ds \times dP - \text{a.e..} \]

Thus (9.11) and (9.13) become, respectively,

\[ (9.11) \quad |g^{t,y,z} - g^{t',y',z'}| \leq \mu|Y^{t,y,z} - Y^{t',y',z'}| + \mu|z - z'|, \quad s \in [t \vee t', T], \quad ds \times dP - \text{a.e..} \]

\[ (9.10) \quad |g^{t,y,z}| \leq \mu|Y^{t,y,z}| + \mu|z|, \]

and

\[ (9.11) \quad |g^{t,y,z} - g^{t',y,z'}| \leq \mu|Y^{t,y,z} - Y^{t',y,z'}| + \mu|z - z'|, \quad s \in [t,t'], \quad ds \times dP - \text{a.e..} \]

Now, for each \( n = 1, 2, 3, \ldots \), we set \( t^n_i = i2^{-n}T, \quad i = 0, 1, 2, \ldots, 2^n \), and define

\[ (9.12) \quad g^n(s,y,z) := \sum_{i=0}^{2^n-1} g^{t^n_i,y,z}1_{[t^n_i,t^n_{i+1})}(s), \quad s \in [0,T]. \]

It is clear that \( g^n \) is an \( F_\tau \)-adapted process. We also have

**Lemma 9.1.** For each fixed \((y,z) \in \mathbb{R} \times \mathbb{R}^d\), \( \{g^n(\cdot,y,z)\}_{n=1}^{\infty} \) is a Cauchy sequence in \( L^2_\mathcal{F}(0,T) \).

To prove this lemma, we need the following classical result of Itô’s SDE. The proof is classic.

**Lemma 9.2.** We have the following estimate: there exist a constant depending only on \( \mu \) and \( T \) such that, for each \((t,y,z) \in [0,T] \times \mathbb{R} \times \mathbb{R}^d\),

\[ (9.13) \quad E[|Y^{t,y,z} - y|^2] \leq C(|y|^2 + |z|^2 + 1)(s-t), \quad \forall s \in [t,T]. \]

We can give
Proof of Lemma 9.1. Let 0 < m < n be two integers. For each \( s \in [0, T) \), there are some integers \( i \leq 2^m - 1 \) and \( j \leq 2^n - 1 \) with \( t_i^m \leq t_j^m \), such that \( s \in (t_i^m, t_{i+1}^m) \), \( s \in [t_{j}^n, t_{j+1}^n) \). We have, by (9.9)

\[
|g^m(s, y, z) - g^n(s, y, z)| = |g_{t_i^m, y, z} - g_{t_j^n, y, z}| \\
\leq |Y_{t_i^m, y, z} - Y_{t_j^n, y, z}| \\
\leq |Y_{t_i^m, y, z} - y| + |Y_{t_j^n, y, z} - y|.
\]

By (9.13)

\[
E[|g^m(s, y, z) - g^n(s, y, z)|^2] \leq 2\mu^2 E[|Y_{s, t_i^m, y, z} - y|^2 + |Y_{s, t_j^n, y, z} - y|^2] \\
\leq 2\mu^2 E[|Y_{s, t_i^m, y, z} - y|^2 + |Y_{s, t_j^n, y, z} - y|^2] \\
\leq 2\mu^2 C(|y|^2 + |z|^2 + 1)(2^{-m} + 2^{-n})T.
\]

Thus

\[
\sup_{s \in [0, T)} E[|g^m(s, y, z) - g^n(s, y, z)|^2] \leq 2\mu^2 C(|y|^2 + |z|^2 + 1)(2^{-m} + 2^{-n})T.
\]

Thus \( \{g^n(\cdot, y, z)\}_{n=1}^\infty \) is a Cauchy sequence in \( L_2^T(0, T) \). \( \square \)

Definition 9.3. For each \((y, z) \in R \times R^d\), we denote \( g(\cdot, y, z) \in L_2^T(0, T) \), the Cauchy limit of \( \{g^n(\cdot, y, z)\}_{n=1}^\infty \) in \( L_2^T(0, T) \).

We will prove that this function is just what we are looking for in Theorem 5.1. We still need to investigate some important properties of \( g \). We have the following estimates for the function \( g \).

Lemma 9.4. The limit \( g : \Omega \times [0, T] \times R \times R^d \rightarrow R^d \) satisfies the following properties:

\[
\text{(i) } g(\cdot, y, z) \in L_2^T(0, T), \text{ for each } (y, z) \in R \times R^d; \\
\text{(ii) } |g(s, y, z) - g(s, y', z')| \leq \mu(|y - y'| + |z - z'|), \forall y, y' \in R, z, z' \in R^d; \\
\text{(iii) } g(s, 0, 0) \equiv 0; \\
\text{(iv) } |g(s, y, z) - g^t X| \leq \mu|y - Y^t X| + \mu|z - Z^t X|, s \in [0, t].
\]

where, in (iv), \( t \in [0, T) \) and \( X \in L_2^T(F_t) \) are arbitrarily given. \((Y^t X, Z^t X)\) is the process defined in (9.6) and (9.7).

Proof. (i) is clear. To prove (ii), we choose \( t_i^m = i2^{-n}T, i = 0, 1, 2, \ldots, 2^n \) as in (9.12). For each \( s \in [0, T) \). We have, once more by (9.9),

\[
|g(s, y, z) - g^n(s, y', z')| = \sum_{j=0}^{2^n-1} 1_{[t_j^n, t_{j+1}^n)}(s)|g_{s, t_j^n, y, z} - g_{s, t_j^n, y', z'}|
\leq \mu \sum_{j=0}^{2^n-1} 1_{[t_j^n, t_{j+1}^n)}(s)(|Y_{s, t_j^n, y, z} - Y_{s, t_j^n, y', z'}| + |z - z'|)
\leq \mu \sum_{j=0}^{2^n-1} 1_{[t_j^n, t_{j+1}^n)}(s)(|Y_{s, t_j^n, y, z} - y| + |Y_{s, t_j^n, y', z'} - y'|)
+ \mu(|y - y'| + |z - z'|)
\]
For the first term $I^n(s)$ of the left hand, we have, by (9.13),

$$E[|I^n(s)|^2] \leq 2\mu^2 \sum_{i=0}^{2^n-1} 1[t^n_j, t^n_{j+1}) (s) E[|Y_{s}^{t^n_j, y, z} - y|^2 + |Y_{s}^{t^n_j, y', z'} - y'|^2]$$

Thus $I^n(\cdot) \to 0$ in $L^2_\mathcal{F}(0, T)$ as $n \to \infty$. (ii) is obtained by passing to the limit in both sides of (9.15). (iii) is proved similarly by using (9.10) and (9.13).

To prove (iv), We apply (9.11),

$$|g^n(s, y, z) - \bar{g}^{t, X} s| = \sum_{i=0}^{2^n-1} 1[t^n_j, t^n_{j+1}) (s) |g_{s}^{t^n_j, y, z} - \bar{g}^{t, X} s|$$

$$\leq \sum_{i=0}^{2^n-1} 1[t^n_j, t^n_{j+1}) (s) \mu |Y_{s}^{t^n_j, y, z} - \bar{Y}^{t, X} s| + \mu |z - \bar{Z}^{t, X} s|$$

Then we pass to the limit on both sides. □

Finally, We give

**Proof of Theorem 3.1.** For each fixed $t \in [0, T]$ and $X \in L^2(\mathcal{F}_t)$, we consider $Y^{t, X}_s := \mathcal{E}_{s,t}[X]$, $s \in [0, T]$. By Proposition 6.6 and Corollary 6.7, we can write

$$Y^{t, X}_s = X + \int_s^t \bar{g}^{t, X}_r dr - \int_s^t \bar{Z}^{t, X}_r dB_r, \ s \in [0, t].$$

On the other hand, let $(Y^{t, X}, Z^{t, X})$ be the solution of the following BSDE

$$Y_s = X + \int_s^t g(r, Y_r, Z_r) dr - \int_s^t Z_r dB_r, \ s \in [0, t].$$

By Lemma 9.4 (i) and (ii), this BSDE is well-posed. We then apply Itô’s formula to $|Y^{t, X} - Y|^2$ in the interval $[0, t]$, take expectation and then apply (iv) of Lemma 9.4. Exactly as the classical proof of the uniqueness of BSDE, we have
It then follows by using Gronwall’s inequality that \( \bar{Y}_{s}^{t,X} \equiv Y_{s}, \ s \in [0,t] \). Recall that \( Y_{s} = \mathcal{E}_{g,\cdot}^{s}[X] \). We thus have the desired result. The proof is complete. \( \square \)

10. Proof of Theorem 8.2 and optional stopping theorem for \( \mathcal{E}_{\sigma,\tau}[\cdot] \)

We now consider the situation where the time indices \( s \) and \( t \) in \( \mathcal{E}_{s,t}[\cdot] \) is replaced by stopping times \( \sigma, \tau \in \mathcal{S}_{T}, \sigma \leq \tau \leq T \). We will extend \( \mathcal{E}_{s,t}[\cdot] \) to \( \mathcal{E}_{\sigma,\tau}[\cdot] \) and prove Theorem 8.2. We will also obtain a generalized version of the optional stopping theorem for \( \mathcal{E} \)–super and \( \mathcal{E} \)–sub–martingale. We note that it is not at all a trivial task to define \( \mathcal{E}_{\sigma,\tau}[\cdot] \), especially for the second parameter \( \tau \). We will first consider the situation of discrete–valued stopping times, i.e., \( \sigma, \tau \in \mathcal{S}_{0}^{\text{d}} \). Then we will pass to the limit to treat the \( \mathcal{S}_{T} \) case.

10.1. Simple case: \( \mathcal{E}_{\sigma,\tau}^{g}[\cdot] \) with \( \sigma, \tau \in \mathcal{S}_{0}^{\text{d}} \). \( \mathcal{E}_{\sigma,\tau}^{g}[\cdot], \ \sigma, \tau \in \mathcal{S}_{T} \) will provide us a concrete example. In this situation. For a given \( X \in \mathcal{F}_{\tau}, \) we can directly solve the BSDE

\[
E[\hat{Y}_{t}^{t,X} - Y_{t}|^2 + E \int_{s}^{t} |\hat{Z}_{r}^{t,X} - Z_{r}|^2 dE_{r} \]

\[
= 2E \int_{s}^{t} (\hat{Y}_{r}^{t,X} - Y_{r})(\hat{g}_{r}^{t,X} - g(r,Y_{r},Z_{r}))dr \]

\[
\leq 2E \int_{s}^{t} (|\hat{Y}_{r}^{t,X} - Y_{r}| \cdot |\hat{g}_{r}^{t,X} - g(r,Y_{r},Z_{r})|)dr \]

\[
\leq 2E \int_{s}^{t} |\hat{Y}_{r}^{t,X} - Y_{r}| \cdot (|\hat{Y}_{r}^{t,X} - Y_{r}| + |\hat{Z}_{r}^{t,X} - Z_{r}|)dr \]

\[
\leq E \int_{s}^{t} (2(\mu + \mu^2)|\hat{Y}_{r}^{t,X} - Y_{r}|^2 + \frac{1}{2}|\hat{Z}_{r}^{t,X} - Z_{r}|^2)dr. \]

It then follows by using Gronwall’s inequality that \( \bar{Y}_{s}^{t,X} \equiv Y_{s}, \ s \in [0,t] \). Recall that \( Y_{s} = \mathcal{E}_{g,\cdot}^{s}[X] \). We thus have the desired result. The proof is complete. \( \square \)

Proposition 10.1. The system of operators

\[
\mathcal{E}_{\sigma,\tau}^{g}[\cdot] : L^{2}(\mathcal{F}_{\tau}) \to L^{2}(\mathcal{F}_{\sigma}), \ \sigma \leq \tau, \ \sigma, \tau \in \mathcal{S}_{T}, \]

is an \( \mathcal{F}_{\tau} \)–consistent nonlinear evaluation, i.e., it satisfies (A1)–(A5) in the following sense: for each \( X, X' \in L^{2}(\mathcal{F}_{\tau}), \)
(a1) \( \mathcal{E}^{g}_{\sigma,\tau}[X] \geq \mathcal{E}^{g}_{\sigma,\tau}[X'], \) a.s., if \( X \geq X', \) a.s.
(a2) \( \mathcal{E}^{g}_{\tau,\tau}[X] = X; \)
(a3) \( \mathcal{E}^{g}_{\rho,\sigma}[\mathcal{E}^{g}_{\sigma,\tau}[X]] = \mathcal{E}^{g}_{\rho,\tau}[X], \forall 0 \leq \rho \leq \sigma \leq \tau; \)
\((a4')\) \(1_A \mathcal{E}_\tau^g[X] = \mathcal{E}_\tau^g[1_A X]\), \(\forall A \in \mathcal{F}_\tau;\)
\((a5)\) for each \(0 \leq \sigma \leq \tau \leq T,\)
\[
\mathcal{E}_\sigma^g[X] - \mathcal{E}_\tau^g[X'] \leq \mathcal{E}_\sigma^g[X - X'], \quad \forall X, X' \in L^2(\mathcal{F}_\tau).
\]

**Proof.** The proof is analogous to the situation of \(\mathcal{E}_s^g[]\). We omit it. \(\square\)

To define \(\mathcal{E}_{s,t}^g[]\), we first consider the situation \(\mathcal{E}_{s \wedge \tau, t \wedge \tau}^g[]\), where \(0 \leq s \leq t \leq T\) and \(\tau \in S_T^r\). We often let \(\tau\) be characterized by
\[
\bigcup_{i=1}^n \{\tau = t_i\} = \Omega, \quad 0 = t_0 < t_1 < \cdots < t_n = t_{n+1} = T.
\]

We consider a more special case where
\[
t_i \leq s < t \leq t_{i+1}, \text{ for some } i \in \{1, 2, \ldots, n\}.
\]

**Lemma 10.2.** In the situation \(10.5\) with \(s\) and \(t\) limited to \(10.6\), we have, for each \(X \in \mathcal{F}_{t \wedge \tau}\)
\[
\{ \begin{array}{l}
(i) \quad \mathcal{E}_{t \wedge \tau, t \wedge \tau}^g[X] = X; \\
(ii) \quad \mathcal{E}_{s \wedge \tau, t \wedge \tau}^g[X] = 1_{\{t \wedge \tau \leq s\}} X + \mathcal{E}_{s,t}^g[X].
\end{array}
\]

**Proof.** (i) is easy. To prove (ii), we first observe that
\[
\{t \wedge \tau \leq s\}^C = \{t \wedge \tau = t\}
\]
and \(\{t \wedge \tau \leq s\} = \{t \wedge \tau \leq t_i\}\). Thus \(1_{\{t \wedge \tau \leq s\}} X \in \mathcal{F}_{t_i}\). We also have \(1_{\{t \wedge \tau = t\}} X \in \mathcal{F}_t\).

We now solve \(Y_{s \wedge \tau} = \mathcal{E}_{s \wedge \tau, t \wedge \tau}^g[X]\) by, as in \(10.2\),
\[
Y_{s \wedge \tau} = X + \int_s^T 1_{[0,t \wedge \tau]}(r)g(r, Y_r, Z_r)dr - \int_s^T 1_{[0,t \wedge \tau]}(r)Z_r dB_r.
\]

Since \(1_{[0,t \wedge \tau]} = 1_{\{t \wedge \tau \leq t_i\}} 1_{[0,t]} + 1_{\{t \wedge \tau = t\}} 1_{[0,t]}\). By respectively multiplying \(1_{\{t \wedge \tau \leq t_i\}}\) and \(1_{\{t \wedge \tau = t\}}\) on both sides of \(10.9\), we have, on \(s \in [t_i, t)\),
\[
Y_{s \wedge \tau} 1_{\{t \wedge \tau \leq t_i\]} = X 1_{\{t \wedge \tau \leq t_i\}},
\]
and
\[
Y_{s \wedge \tau} 1_{\{t \wedge \tau = t\}} = 1_{\{t \wedge \tau = t\}} X + \int_s^T 1_{[0,t]}(r) 1_{\{t \wedge \tau = t\}} g(r, Y_r, Z_r)dr
\]
\[
- \int_s^T 1_{[0,t]} 1_{\{t \wedge \tau = t\}}(r)Z_r dB_r
\]
\[
= 1_{\{t \wedge \tau = t\}} X + \int_s^t g(r, 1_{\{t \wedge \tau = t\}} Y_r, 1_{\{t \wedge \tau = t\}} Z_r)dr - \int_s^t 1_{[0,t]} Z_r dB_r.
\]

We observe that, the last relation solves a BSDE on \([t_i, t]\). Thus
\[
Y_{s \wedge \tau} 1_{\{t \wedge \tau = t\}} = 1_{\{t \wedge \tau = t\}} \mathcal{E}_{s,t}^g[1_{\{t \wedge \tau = t\}} X] = 1_{\{t \wedge \tau = t\}} \mathcal{E}_{s,t}^g[X].
\]

This with \(10.10\) and \(10.8\), we then have (ii). \(\square\)
10.2. $E_{\sigma, \tau}[-]$ with $\sigma, \tau \in S^0_T$. Let a stopping time $\tau \in S^0_T$ be characterized by $\{1\}$. For each $i = 0, 1, \ldots, n$, for each $t_i \leq s < t \leq t_{i+1}$, $X \in \mathcal{F}_{t \wedge \tau}$, we define

$$E_{t, t}^i[X] = E_{t \wedge \tau, t \wedge \tau}[X] := X;$$

$$E_{s, t}^i[X] = E_{s \wedge \tau, t \wedge \tau}[X] := 1_{\{t \wedge \tau \leq s\}}X + 1_{\{t \wedge \tau = t\}}E_{s, t}[X].$$

The reason that we set $E_{\sigma, \tau}[-]$ satisfying (ii) is as follows

**Lemma 10.3.** Let $E_{\sigma, \tau}^i[-] : L^2(\mathcal{F}_\tau) \to L^2(\mathcal{F}_\sigma)$, $\sigma, \tau \in S^0_T$, $\sigma \leq \tau$, be a system of operators satisfying the following ($\mathcal{F}_\tau$-consistent) conditions: for each $X \in \mathcal{F}_\tau$

(a) $E_{\sigma, \tau}^i[X] = X$;
(b) $E_{\rho, \sigma}^i[E_{\sigma, \tau}^i[X]] = E_{\rho, \tau}^i[X]$, $\forall \rho \leq \sigma \leq \tau$;
(c) $1 \mathcal{A}_\sigma^i E_{\sigma, \tau}^i[X] = 1 \mathcal{A}_\tau^i[1 \mathcal{A}_\tau^i X]$, $\forall A \in \mathcal{F}_\tau$;
(d) $E_{\sigma, \tau}^i[X] = \sum_{i=1}^{m} 1_{\{s = i\}}E_{s \wedge \tau, t \wedge \tau}^i[X]$, if $\sum_{i=1}^{m} \{s = i\} = \Omega$.

We assume that $E'$ coincides with $E$ in the sense that $E_{s, t}^i[X] = E_{s, t}[X]$, for all (deterministic) $0 \leq s \leq t \leq T$, and $X \in \mathcal{F}_\tau$. Then, necessarily, $E'$ satisfies (i) and (ii) of (10.1).

**Proof.** (i) comes directly from (a). We now prove (ii). Since $\{s \wedge \tau = t \wedge \tau\} = \{t \wedge \tau \leq s\}$, we have, by (d) and (a),

$$1_{\{t \wedge \tau \leq s\}}E_{s \wedge \tau, t \wedge \tau}^i[X] = 1_{\{t \wedge \tau = t\}}1_{\{s \wedge \tau = t \wedge \tau\}}E_{s \wedge \tau, t \wedge \tau}^i[X] = 1_{\{t \wedge \tau \leq s\}}X.$$

By (d) we have

$$1_{\{t \wedge \tau = t\}}E_{s \wedge \tau, t \wedge \tau}^i[X] = 1_{\{t \wedge \tau = t\}}E_{s, t}^i[X] = 1_{\{t \wedge \tau = t\}}X.$$
It satisfies (A1)–(A5) in the following sense: for each \(0 \leq s \leq t \leq T\) and \(X, X' \in L^2(\mathcal{F}_{t \vee \tau})\),

(\textbf{a1}) \(\mathcal{E}_{s \wedge \tau, t \vee \tau}[X] \geq \mathcal{E}_{s \wedge \tau, t \vee \tau}[X'],\) a.s., if \(X \geq X',\) a.s.

(\textbf{a2}) \(\mathcal{E}_{s \wedge \tau, t \vee \tau}[X] = X;\)

(\textbf{a3}) \(\mathcal{E}_{r \wedge \tau, s \wedge \tau}[\mathcal{E}_{s \wedge \tau, t \vee \tau}[X]] = \mathcal{E}_{r \wedge \tau, s \wedge \tau}[X],\) \(\forall 0 \leq r \leq s \leq t;\)

(\textbf{a4'}) \(1_A \mathcal{E}_{s \wedge \tau, t \vee \tau}[X] = \mathcal{E}_{s \wedge \tau, t \vee \tau}[1_A X],\) \(\forall A \in \mathcal{F}_{s \wedge \tau}.

(\textbf{a5}) For each \(\tau \in S^0_T,\) \(X, X' \in L^2(\mathcal{F}_{t \vee \tau})\) and \(K, K' \in S^2_T(0, T)\)

\[
(10.15) \quad \mathcal{E}_{s \wedge \tau, t \vee \tau}[X; K] - \mathcal{E}_{s \wedge \tau, t \vee \tau}[X'; K'] \leq \mathcal{E}^{g}_{s \wedge \tau, t \vee \tau}[X - X'; K - K'].
\]

\textbf{Proof.} We first prove that, for each \(i, \mathcal{E}^{i}_{s,t}[\cdot] = \mathcal{E}^{i}_{s \wedge \tau, t \vee \tau}[\cdot],\) \(t_i \leq s \leq t \leq t_{i+1}\) satisfies (a1)–(a3), (a4') and (a5). (a1)–(a3) are easy to check. To prove (a4'), we observe that, for each \(A \in \mathcal{F}_{s \wedge \tau},\)

\[
A \cap \{ t \wedge \tau = i \} = A \cap \{ s \wedge \tau \leq s \} \cap \{ t \wedge \tau = i \} \in \mathcal{F}_s,
\]

\[
A \cap \{ t \wedge \tau \leq s \} = A \cap \{ s \wedge \tau \leq s \} \cap \{ t \wedge \tau \leq s \} \in \mathcal{F}_s.
\]

By (10.11)–(ii), (a4) follows from

\[
1_A \mathcal{E}_{s \wedge \tau, t \vee \tau}[X] = 1_{\{ t \wedge \tau \leq s \}} 1_A X + 1_{\{ t \wedge \tau = i \}} 1_A \mathcal{E}_{s,t}[X]
\]

\[
= 1_{\{ t \wedge \tau \leq s \}} 1_A X + 1_{\{ t \wedge \tau = i \}} 1_A \mathcal{E}_{s,t}[1_{\{ t \wedge \tau = i \}} 1_A X]
\]

\[
= 1_{\{ t \wedge \tau \leq s \}} 1_A X + 1_{\{ t \wedge \tau = i \}} 1_A \mathcal{E}_{s,t}[1_A X]
\]

\[
= 1_A \mathcal{E}_{s \wedge \tau, t \vee \tau}[1_A X].
\]

This with \(\mathcal{E}_{s \wedge \tau, t \vee \tau}[0] = 0\) yields (a4'). It then follows from Proposition 2.5 and Remark 2.3 that, there exists a unique \(\mathcal{F}_{t \vee \tau}\)-consistent evaluation satisfying (a1)–(a4) that coincides to \(\mathcal{E}^i\) on each \([t_i, t_{i+1}].\) (a4) plus \(\mathcal{E}_{s \wedge \tau, t \vee \tau}[0] = 0\) implies (a4').

We now prove (a5). We only prove the second relation. The proof of the first one is similar. We still let \(\tau\) be characterized by (10.14). We already have (a5) when \(t_i \leq s \leq t \leq t_{i+1},\) for each \(i = 0, 1, 2, \ldots.\) Now if \(s \in [t_i, t_{i+1}], t \in (t_{i+1}, t_{i+2}],\) for some \(i = 0, 1, 2, \ldots,\) we have

\[
\mathcal{E}_{s \wedge \tau, t \vee \tau}[X; K] - \mathcal{E}_{s \wedge \tau, t \vee \tau}[X'; K']
\]

\[
\leq \mathcal{E}^{g}_{s \wedge \tau, t \vee \tau}[\mathcal{E}_{s \wedge \tau, t \vee \tau}[X; K] - \mathcal{E}_{s \wedge \tau, t \vee \tau}[\mathcal{E}_{s \wedge \tau, t \vee \tau}[X'; K']]
\]

\[
\leq \mathcal{E}^{g}_{s \wedge \tau, t \vee \tau}[\mathcal{E}_{s \wedge \tau, t \vee \tau}[X - X'; K - K']].
\]

Thus the inequality is still true. We can repeat this procedure to conclude that, for all \(0 \leq s \leq t \leq T,\) the inequality holds. The proof of the first inequality of (10.15) is analogous. □
Lemma 10.6. Let $\tau \in \mathcal{S}^0_T$ be characterized by (10.4). Then, for each $X \in \mathcal{F}_\tau$, $s$, $t \in [0, T]$ and $i = 1, 2, \cdots, n$, we have

\begin{equation}
1_{\{t \wedge \tau = s\}} \mathcal{E}_{s \wedge \tau, \tau} [X] = 1_{\{t \wedge \tau = s\}} \mathcal{E}_{s \wedge \tau, \tau} [X].
\end{equation}

Proof. This problem can be divided into three cases: case 1: $t < s$. In this case (10.17) holds since $1_{\{t \wedge \tau = s\}} \equiv 0$. Case 2: $t = s$. (10.17) is clearly true. We now consider the last case: $t > s$. We assume that $t \in (t_k, t_{k+1}]$, with $i \leq k \leq n$. In this case we have, by Lemma 10.6 and (10.11)–(ii),

\begin{align}
1_{\{t \wedge \tau = s\}} \mathcal{E}_{s \wedge \tau, s} [X] & = 1_{\{t \wedge \tau = s\}} \mathcal{E}_{s \wedge \tau, \tau} \left[ \cdots \mathcal{E}_{t_k \wedge \tau, \tau} [X] \cdots \right] \\
& = 1_{\{t \wedge \tau = s\}} 1_{\{t \wedge \tau \leq s\}} \mathcal{E}_{s \wedge \tau, t_{k+1} \wedge \tau} \left[ \cdots \mathcal{E}_{t_k \wedge \tau, \tau} [X] \cdots \right] \\
& = \cdots \\
& = 1_{\{t \wedge \tau = s\}} 1_{\{t \wedge \tau \leq t_k\}} \mathcal{E}_{t_k \wedge \tau, \tau} [X] \\
& = 1_{\{t \wedge \tau = s\}} \mathcal{E}_{s \wedge \tau, \tau} [X].
\end{align}

\[\square\]

We now consider the general case of $\mathcal{E}_{s, \tau}[\cdot]$ for $\sigma, \tau \in \mathcal{S}^0_T$ with $\sigma \leq \tau$. Let $\sigma$ be characterized by

\begin{equation}
\bigcup_{i=1}^m \{\sigma = s_i\} = \Omega, \ 0 \leq s_1 < \cdots < s_m \leq T.
\end{equation}

Definition 10.7. Let $\sigma$ be characterized by (10.18). $\mathcal{E}_{s, \tau}[\cdot]$ is defined by, as in classical situations,

\begin{equation}
\mathcal{E}_{s, \tau} [X] := \mathcal{E}_{t \wedge \tau, \tau} [X] |_{t = \sigma} = \sum_{i=1}^m 1_{\{\sigma = s_i\}} \mathcal{E}_{s_i \wedge \tau, \tau} [X], \ X \in \mathcal{F}_\tau.
\end{equation}

Remark 10.7.1. By Lemma 10.6, it is clear that (10.18) is satisfied in the case $\sigma = t \wedge \tau$.

Lemma 10.8. Let $\sigma$, $\tau \in \mathcal{S}^0_T$ be such that $\sigma \leq \tau$. Then for each $0 \leq s \leq t \leq T$ and $X \in L^2(\mathcal{F}_\tau)$ we have

\begin{equation}
\mathcal{E}_{s \wedge \sigma, t \wedge \sigma} [\mathcal{E}_{t \wedge \sigma, \tau} [X]] = \mathcal{E}_{s \wedge \sigma, \tau} [X].
\end{equation}

Proof. Without loss of generality we let both $\sigma$ and $\tau$ be valued in $\{t_0, t_1, \cdots, t_n\}$ with

\begin{equation}
0 = t_0 < t_1 < \cdots < t_n \leq t_{n+1} = T.
\end{equation}

($\{\sigma = t_i\}$ or $\{\tau = t_i\}$ may be an empty set for some $i$). For a fixed $i \in \{0, 1, \cdots, n\}$, we consider, the case $t_i \leq s < t \leq t_{i+1}$ (the case $s = t$ is clearly true). By (10.11)–(ii), we have

\begin{align}
\mathcal{E}_{t \wedge \sigma, \tau} [\mathcal{E}_{s \wedge \sigma, \tau} [X]] & = 1_{\{t \wedge \sigma \leq s\}} \mathcal{E}_{t \wedge \sigma, \tau} [X] + 1_{\{t \wedge \sigma = t\}} \mathcal{E}_{s, t} [\mathcal{E}_{t \wedge \sigma, \tau} [X]] \\
& = \mathcal{E}_{s \wedge \sigma, \tau} [X] + 1_{\{t \wedge \sigma = t\}} \mathcal{E}_{s, t} [\mathcal{E}_{t \wedge \sigma, \tau} [X]].
\end{align}

The second term is due to the assumption (A4) of $\mathcal{E}$, with the observation that $1_{\{t \wedge \sigma = t\}}$ is $\mathcal{F}_{t_i}$ (and thus $\mathcal{F}_s$) measurable. We repeatedly use (10.19) and (A4) to
the second term

\[(10.23) \{t \land \sigma = t\} \mathcal{E}_{x,t}[1_{\{t \land \sigma = t\}} \mathcal{E}_{t \land \sigma, \tau}[X]] = 1_{\{t \land \sigma = t\}} \mathcal{E}_{x,t}[1_{\{t \land \sigma = t\}} \mathcal{E}_{t \land \sigma, \tau}[X]]
\]

\[= 1_{\{t \land \sigma = t\}} \mathcal{E}_{x,t}[\mathcal{E}_{t \land \sigma, \tau}[X]]
\]

\[= 1_{\{t \land \sigma = t\}} \mathcal{E}_{x,t}[1_{\{t \land \tau = t\}} \mathcal{E}_{t \land \tau, \tau}[X]]
\]

\[= 1_{\{t \land \sigma = t\}} \mathcal{E}_{x,t}[1_{\{t \land \tau = t\}} \mathcal{E}_{t \land \tau, t \land \tau}[\mathcal{E}_{t \land \tau, \tau}[X]]]
\]

\[= 1_{\{t \land \sigma = t\}} \mathcal{E}_{x,t}[1_{\{s \land \sigma = s\}} \mathcal{E}_{s \land \tau, \tau}[X]].
\]

Here we use the fact that \{t \land \sigma = t\} \subset \{t \land \tau = t\}, since \tau \geq \sigma, and \{s \land \sigma = t\} \subset \{s \land \sigma = \tau\}. By \[(10.12)\], \[1_{\{s \land \sigma = s\}} \mathcal{E}_{s \land \tau, \tau}[X] = 1_{\{s \land \sigma = s\}} \mathcal{E}_{s \land \sigma, \tau}[X]. \]
Thus \[(10.22)\] finally becomes

\[\mathcal{E}_{s \land \sigma, t \land \sigma}[\mathcal{E}_{t \land \sigma, \tau}[X]] = 1_{\{t \land \sigma \leq s\}} \mathcal{E}_{s \land \sigma, \tau}[X] + 1_{\{t \land \sigma = t\}} \mathcal{E}_{s \land \sigma, \tau}[X]
\]

\[= \mathcal{E}_{s \land \sigma, \tau}[X],
\]

since \{t \land \sigma \leq s\} + \{t \land \sigma = t\} = \Omega.

We have proved \[(10.20)\] for the case \[t_i \leq s \leq t \leq t_{i+1}, i \in \{0, \ldots, n\}. \] For the general situation, say \[s \in [t_i, t_{i+1}], t \in [t_j, t_{j+1}], 0 \leq i < j \leq n, \]
\[(10.20)\] can be deduced by

\[\mathcal{E}_{s \land \sigma, t \land \sigma}[\mathcal{E}_{t \land \sigma, \tau}[X]]
\]

\[= \mathcal{E}_{s \land \sigma, t \land \sigma}[\mathcal{E}_{t \land \sigma, t \land t \land \sigma}[\mathcal{E}_{t \land \sigma, \tau}[X]]]
\]

\[= \mathcal{E}_{s \land \sigma, t \land \sigma}[\mathcal{E}_{t \land \sigma, t \land t \land \sigma}[\mathcal{E}_{t \land \sigma, \tau}[X]]]
\]

\[= \cdots
\]

\[= \mathcal{E}_{s \land \sigma, t \land \sigma}[\mathcal{E}_{t \land \sigma, \tau}[X]]
\]

\[= \mathcal{E}_{s \land \sigma, \tau}[X].
\]

The proof is complete. \(\square\)

By this result and the definition of \[\mathcal{E}_{\sigma, \tau}[\cdot]\] in \[(10.19)\], we immediately have

**Lemma 10.9.** Let \(\rho, \sigma \) and \(\tau \in S^\mathcal{T}_F \) be such that \(\rho \leq \sigma \leq \tau. \) Then for each \(X \in L^2(\mathcal{F}_\tau)\) we have

\[(10.24) \mathcal{E}_{\rho, \sigma}[\mathcal{E}_{\sigma, \tau}[X]] = \mathcal{E}_{\sigma, \tau}[X].\]

We now consider an \(\mathcal{E}-\)supermartingale. We will prove the following optional stopping theorem

**Lemma 10.10.** Let \(Y \in D^2_\mathcal{T}(0, T)\) be an \(\mathcal{E}-\)martingale (respectively \(\mathcal{E}-\)supermartingale, \(\mathcal{E}-\)submartingale). Then for each \(\sigma, \tau \in S^\mathcal{T}_F \) such that \(\sigma \leq \tau, \) we have

\[(10.25) \mathcal{E}_{\sigma, \tau}[Y_\tau] = \mathcal{E}_{\sigma, \tau}[Y_\sigma], \text{ (resp. } \leq Y_\sigma, \text{  \geq Y}_\sigma \text{) a.s.}\]

**Proof.** We only prove the case for \(\mathcal{E}-\)supermartingale. It is clear that, once we have

\[(10.26) \mathcal{E}_{t \land \tau, \tau}[Y_\tau] \leq \mathcal{E}_{t \land \tau, \tau}[Y_\sigma], \forall t \in [0, T],\]

then, by \[(10.19)\], we can also prove \[(10.25)\]. We still let \(\tau\) be characterized by \[(10.25)\]. We will prove this inequality by induction. Firstly, when \(t \geq t_n, \)
\[(10.26)\] holds since \(\mathcal{E}_{t \land \tau, \tau}[Y_\tau] = \mathcal{E}_{t \land \tau, \tau}[Y_\sigma] = \mathcal{E}_{t \land \tau, \tau}[Y_\sigma]. \) Now suppose that for a fixed \(i \in \{1, \ldots, n\}, \)
\[(10.26)\] holds for \(t \geq t_i, \) we shall prove that it also holds for \(t \geq t_{i-1}. \) We need to
check the case \( t \in [t_{i-1}, t_i) \). By (10.11)–(ii) and applying (A4) (since \( 1_{\{t_i \wedge \tau = t_i\}} \) is \( \mathcal{F}_t \)-measurable), we have

\[
\mathcal{E}_{t \wedge \tau, t \wedge \tau}[Y_{t \wedge \tau}] = 1_{\{t_i \wedge \tau \leq t\}} Y_{t \wedge \tau} + 1_{\{t_i \wedge \tau = t_i\}} \mathcal{E}_{t \wedge \tau}[Y_{t \wedge \tau}]
\]

\[
= 1_{\{t_i \wedge \tau \leq t\}} Y_{t \wedge \tau} + 1_{\{t_i \wedge \tau = t_i\}} \mathcal{E}_{t \wedge \tau}[Y_{t_i}]
\]

\[
= 1_{\{t_i \wedge \tau \leq t\}} Y_{t \wedge \tau} + 1_{\{t_i \wedge \tau = t_i\}} Y_{t_i}
\]

\[
\leq 1_{\{t_i \wedge \tau \leq t\}} Y_{t \wedge \tau} + 1_{\{t_i \wedge \tau = t_i\}} Y_{t_i}
\]

\[
= Y_{t \wedge \tau}.
\]

The last step is from \( \{ t_i \wedge \tau \leq t \} + \{ t_i \wedge \tau = t_i \} = \Omega \) and then \( t \wedge \tau = t_i \wedge \tau 1_{\{ t_i \wedge \tau \leq t \}} + t 1_{\{ t_i \wedge \tau = t_i \}} \). From this result we derive

\[
\mathcal{E}_{t \wedge \tau, t \wedge \tau}[Y_{t \wedge \tau}] = \mathcal{E}_{t \wedge \tau, t \wedge \tau}[\mathcal{E}_{t \wedge \tau, t \wedge \tau}[Y_{t \wedge \tau}]]
\]

\[
\leq \mathcal{E}_{t \wedge \tau, t \wedge \tau}[Y_{t \wedge \tau}]
\]

\[
\leq Y_{t \wedge \tau}.
\]

Thus (10.26) holds for \( t \geq t_{i-1} \). It follows by induction that (10.26) holds for \( t \in [0, T] \). The proof is complete. \( \square \)

We conclude

**Lemma 10.11.** The system of operators

\[
\mathcal{E}_{\sigma, \tau}[\cdot] : L^2(\mathcal{F}_\tau) \to L^2(\mathcal{F}_\sigma), \ \sigma \leq \tau, \ \sigma, \tau \in \mathcal{S}_T^0
\]

is an \( \mathcal{F}_t \)-consistent nonlinear evaluation in the following sense: for each \( X, X' \in L^2(\mathcal{F}_\tau) \),

(a1) \( \mathcal{E}_{\sigma, \tau}[X] \geq \mathcal{E}_{\sigma, \tau}[X'] \), a.s., if \( X \geq X' \), a.s.

(a2) \( \mathcal{E}_{\sigma, \tau}[X] = X \);

(a3) \( \mathcal{E}_{\rho, \sigma}[\mathcal{E}_{\sigma, \tau}[X]] = \mathcal{E}_{\rho, \tau}[X], \ \forall 0 \leq \rho \leq \sigma \leq \tau; \)

(a4') \( 1_A \mathcal{E}_{\sigma, \tau}[X] = \mathcal{E}_{\sigma, \tau}[1_A X], \ \forall A \in \mathcal{F}_\sigma \).

We also have

(a5) For each \( 0 \leq \sigma \leq \tau \leq T \),

\[
(10.27) \quad \mathcal{E}_{\sigma, \tau}[X; K] - \mathcal{E}_{\sigma, \tau}[X'; K'] \leq \mathcal{E}_{\sigma, \tau}^0[X - X'; K - K'], \ \forall X, X' \in L^2(\mathcal{F}_\tau).
\]

Moreover, \( \mathcal{E}_{\sigma, \tau}[\cdot] \) is the unique extension of \( \mathcal{E}_{s, t}[\cdot] \) in the following sense: for each system of operator

\[
\mathcal{E}_{s, t}'[\cdot] : L^2(\mathcal{F}_\tau) \to L^2(\mathcal{F}_\sigma), \ \sigma \leq \tau, \ \sigma, \tau \in \mathcal{S}_T^0
\]

satisfying satisfies (a1)–(a4') such \( \mathcal{E}_{s, t}'[X] = \mathcal{E}_{s, t}[X] \), for each (deterministic) \( 0 \leq s \leq t \leq T \) and for each \( X \in \mathcal{F}_s \), then, for each \( \sigma \leq \tau, \ \sigma, \tau \in \mathcal{S}_T^0 \) and for each \( X \in \mathcal{F}_\tau \), we have \( \mathcal{E}_{s, \tau}'[X] = \mathcal{E}_{s, \tau}'[X] \).

**Proof.** (a1) and (a2) are easily checked from Definition (10.14) of \( \mathcal{E}_{s, t}[\cdot] \) and Lemma 10.4 (a3) is given in Lemma 10.4. (a5) can be proved by using (10.14) and the (a5) part of Lemma 10.4. It remains to prove (a4'). Let \( A \in \mathcal{F}_\sigma \) and let \( \sigma \) be characterized by (10.18). From (10.18) and the fact that \( A \cap \{ \sigma = s_i \} \in \mathcal{F}_{s_i \wedge \sigma} \), we
derive, using the \((a4')\) part of Lemma \textbf{10.3}

\[ 1_A\mathcal{E}_{\sigma,\tau}[X] = \sum_{i=1}^{m} 1_A(\sigma=s_i)\mathcal{E}_{s_i,\tau,\tau}[X] \]

\[ = \sum_{i=1}^{m} 1_A(\sigma=s_i)\mathcal{E}_{s_i,\tau,\tau}[1_A(\sigma=s_i)X] \]

\[ = \sum_{i=1}^{m} 1_A(\sigma=s_i)\mathcal{E}_{s_i,\tau,\tau}[1_AX] \]

\[ = 1_A\mathcal{E}_{\sigma,\tau}[1_AX]. \]

This with \(\mathcal{E}_{\sigma,\tau}[0] = 0\) implies \((a4')\). The uniqueness is a direct consequence of Lemma \textbf{10.3} and the uniqueness part of Lemma \textbf{10.4}. \(\square\)

\textbf{10.3. \(S_T\) case: Proof of Theorem \textbf{8.2} and optional stopping theorem.} We now extend \(\mathcal{E}_{\sigma,\tau}[]\) from \(\sigma, \tau \in S_T^0\) to the general case, i.e., \(\sigma, \tau \in S_T\). Firstly, by Remark \textbf{10.4.1}, for each \(\sigma \in S_T\), we have \(\mathcal{E}_{X,\tau}[X] \in D_2^T(0,T)\). This with the definition of of \(\mathcal{E}_{\sigma,\tau}[X]\) it follows that

\textbf{Lemma 10.12.} Let \(\sigma \in S_T\), \(\tau \in S_T^0\) be such that \(\sigma \leq \tau\) and let \(X \in L^2(\mathcal{F}_T)\). Then for each sequence \(\{\sigma_n\}_{n=1}^\infty\) of \(S_T^0\) such that \(\sigma \leq \sigma_n \leq \tau\) and \(\lim_{n \to \infty} \sigma_n = \sigma\), a.s., \(\{\mathcal{E}_{\sigma_n,\tau}[X]\}_{n=1}^\infty\) is a Cauchy sequence in \(L^2(\mathcal{F}_T)\). Moreover the limit of this sequence is in \(L^2(\mathcal{F}_\sigma)\). We denote it by \(\mathcal{E}_{\sigma,\tau}[X]_{0 \leq \sigma \leq \tau \leq T}\):

\[ \mathcal{E}_{\sigma,\tau}[] : L^2(\mathcal{F}_T) \to L^2(\mathcal{F}_\sigma), \sigma \in S_T, \tau \in S_T^0. \]

With the above convergence result and Lemma \textbf{10.11} we can easily have

\textbf{Lemma 10.13.} The system of operators \(\mathcal{E}_{\sigma,\tau}[]\), defined in \textbf{(10.28)} satisfies the following properties: for each \(\sigma \in S_T\), \(\tau \in S_T^0\), \(\sigma \leq \tau\), and \(X, X' \in L^2(\mathcal{F}_T)\), we have

\(a1\) \(\mathcal{E}_{\sigma,\tau}[X] \geq \mathcal{E}_{\sigma,\tau}[X'],\) a.s., if \(X \geq X',\) a.s.

\(a3\) \(\mathcal{E}_{\sigma,\rho}[\mathcal{E}_{\rho,\tau}[X]] = \mathcal{E}_{\sigma,\tau}[X],\) \(\forall 0 \leq \sigma \leq \rho \leq \tau,\rho \in S_T^0;\)

\(a4'\) \(1_A\mathcal{E}_{\sigma,\tau}[X] = \mathcal{E}_{\sigma,\tau}[1_AX],\) \(\forall A \in \mathcal{F}_\tau.\)

\(a5\) For each \(0 \leq \sigma \leq \tau \leq T\) and \(X, X' \in L^2(\mathcal{F}_T)\),

\[ \mathcal{E}_{\sigma,\tau}[X;K] - \mathcal{E}_{\sigma,\tau}[X';K] \leq \mathcal{E}_{\sigma,\tau}^{0^+}[X - X'; K - K']. \]

Consequently, the estimates in Lemma \textbf{10.12} still hold for \(\sigma \in S_T\) and \(\tau \in S_T^0\).

To proceed, we need the following estimates

\textbf{Lemma 10.14.} For each \(\sigma, \tau \in S_T,\sigma \leq \tau\) and \(X \in L^2(\mathcal{F}_\sigma)\), we have the following estimate

\[ E[|\mathcal{E}_{\sigma,\tau}^{\sigma}[X] - X|^2] \leq CE \int_\sigma^\tau |g(s,0,0)|^2ds + CE[(\tau - \sigma)|X|^2] \]

as well as

\[ E[\sup_{s \in [0,T]} |\mathcal{E}_{\tau,\tau}^{\sigma}[X]|^2] \leq CE[|X|^2] + CE \int_0^\tau |g(s,0,0)|^2ds, \]

where the constant \(C\) depends only on \(\mu\) and \(T\).
Proof. (10.31) is a special case of (4.3) with $K_t \equiv 0$. In order to prove (10.30), we set $\tilde{y}_t \equiv X$ on $[\sigma, \tau]$. Observe that $\mathcal{E}_{\sigma,\tau}^g[X] = y_\sigma$, where $(y_t)_{t \in [0, T]}$ is the solution of the BSDE

$$y_t = X + \int_t^T 1_{[\sigma, \tau]}(s)g(s, y_s, z_s)ds - \int_t^T z_s dB_s.$$ 

We apply Itô’s formula to $|y_t - \tilde{y}_t|^2 e^{\beta t}$ on $[\sigma, \tau]$, with $\beta = 2\mu + 3\mu^2$. By $|g(s, y_s, z_s)| \leq \mu(|y_s - \tilde{y}_s| + |\tilde{y}_s| + |z_s|) + |g^0_s|$ with $g^0_s = g(s, 0, 0, 0)$, we have

$$E[|y_\sigma - \tilde{y}_\sigma|^2 e^{\beta \sigma}] + E \int_\sigma^\tau (\beta |y_s - \tilde{y}_s|^2 + |z_s|^2) e^{\beta s} ds.$$

Thus we have (10.30) with $C = e^{2\beta T}$. \(\square\)

**Corollary 10.15.** For each $\sigma, \tau \in \mathcal{S}_T^0$, $\sigma \leq \tau$ and $X, X' \in L^2(\mathcal{F}_\tau)$, we have the following estimates

\begin{equation}
E[\sup_{s \in [0, T]}|\mathcal{E}_{\tau,\tau}^s[X]|^2] \leq C \mathbb{E}[|X|^2],
\end{equation}

\begin{equation}|\mathcal{E}_{\sigma,\tau}[X] - \mathcal{E}_{\sigma,\tau}[X']|^2 \leq C \mathbb{E}[|X - X'|^2]
\end{equation}

and, for each $X \in L^2(\mathcal{F}_\sigma)$,

\begin{equation}
E[|\mathcal{E}_{\sigma,\tau}[X] - X|^2] \leq C \mathbb{E}[(\tau - \sigma)|X|^2],
\end{equation}

where the constant $C$ depends only on $\mu$ and $\tau$.

**Proof.** By (a5) of Lemma 10.11

$$\mathcal{E}_{\sigma,\tau}^{2g_\mu}[X] - X \leq \mathcal{E}_{\sigma,\tau}[X] - X \leq \mathcal{E}_{\sigma,\tau}^{2g_\mu}[X] - X.$$ 

Thus

$$|\mathcal{E}_{\sigma,\tau}[X] - X| \leq |\mathcal{E}_{\sigma,\tau}^{2g_\mu}[X] - X| + |\mathcal{E}_{\sigma,\tau}^{2g_\mu}[X] - X|.$$ 

It then follows from Lemma 10.14 that (10.34) holds for $X \in L^2(\mathcal{F}_\sigma)$. Using (a5), the proofs of (10.32) and (10.33) are similar. Here we need the estimates (4.16). \(\square\)

To extend $\tau$ to $\mathcal{S}_T$ we need

**Lemma 10.16.** Let $\sigma \in \mathcal{S}_T$, $\tau, \tau' \in \mathcal{S}_T^0$ be such that $\sigma \leq \tau \vee \tau'$, and let $X \in L^2(\mathcal{F}_{\tau \wedge \tau'})$. Then we have

\begin{equation}E[|\mathcal{E}_{\sigma,\tau}[X] - \mathcal{E}_{\sigma,\tau'}[X]|^2] \leq c \mathbb{E}[(\tau \vee \tau' - \tau \wedge \tau')|X|^2],
\end{equation}

where $c$ depends only on $\mu$ and $T$. 


Proof. We have
\[
|\mathcal{E}_{\sigma,\tau}[X] - \mathcal{E}_{\sigma,\tau'}[X]| \leq |\mathcal{E}_{\sigma,\tau \wedge \tau'}[\mathcal{E}_{\sigma,\tau \wedge \tau'}[X]] - \mathcal{E}_{\sigma,\tau \wedge \tau'}[X]| + |\mathcal{E}_{\sigma,\tau \wedge \tau'}[X] - \mathcal{E}_{\sigma,\tau \wedge \tau'}[\mathcal{E}_{\sigma,\tau \wedge \tau'}[X]]|.
\]
For the first term, by (10.33) and then (10.34),
\[
E[|\mathcal{E}_{\sigma,\tau \wedge \tau'}[\mathcal{E}_{\sigma,\tau \wedge \tau'}[X]] - \mathcal{E}_{\sigma,\tau \wedge \tau'}[X]|^2] \leq C E[|\mathcal{E}_{\tau \wedge \tau'}[X] - X|^2] \leq C^2 E[(\tau - \tau')^2|X|^2].
\]
Similarly
\[
E[|\mathcal{E}_{\sigma,\tau \wedge \tau'}[X] - \mathcal{E}_{\sigma,\tau \wedge \tau'}[\mathcal{E}_{\sigma,\tau \wedge \tau'}[X]]|^2] \leq C E[|\mathcal{E}_{\tau \wedge \tau'}[X] - X|^2] \leq C^2 E[(\tau' - \tau \wedge \tau')|X|^2].
\]
From the above three inequalities we have (10.35). □

By this estimate we have

**Lemma 10.17.** Let \( \sigma, \tau \in S_T \), be such that \( \sigma \leq \tau \) and let \( X \in L^2(\mathcal{F}_{\tau \wedge \tau'}). \) Then for each sequence \( \{\tau_n\}_{n=1}^\infty \) in \( S_T^0 \) such that \( \tau \leq \tau_n \) and \( \lim_{n \to \infty} \tau_n = \tau \), a.s., the sequence \( \{\mathcal{E}_{\sigma,\tau_n}[X]\}_{n=1}^\infty \) is a Cauchy sequence in \( L^2(\mathcal{F}_\sigma) \).

**Proof.** By (10.36)
\[
E[|\mathcal{E}_{\sigma,\tau_n}[X] - \mathcal{E}_{\sigma,\tau_n}[X]|^2] \leq c E[(\tau_m \lor \tau_n - \tau)|X|^2].
\]
We then have that \( \{\mathcal{E}_{\sigma,\tau_n}[X]\}_{n=1}^\infty \) is a Cauchy sequence in \( L^2(\mathcal{F}_\sigma) \). □

**Definition 10.18.** We denote the limit of the sequence \( \{\mathcal{E}_{\sigma,\tau_n}[X]\}_{n=1}^\infty \) in \( L^2(\mathcal{F}_\sigma) \) of the above Lemma \( \mathcal{E}_{\sigma,\tau}[X] \):
\[
\mathcal{E}_{\sigma,\tau}[\cdot] : L^2(\mathcal{F}_\tau) \to L^2(\mathcal{F}_\sigma), \sigma, \tau \in S_T.
\]

**Proof of Theorem 8.2** We have already defined \( \mathcal{E}_{\sigma,\tau} \) in (10.37). With which (A1), (A4') and (A5) are proved by simply using Lemma 10.18 and Lemma 10.17 and by passing to the limit. (A2) is proved by
\[
0 = \mathcal{E}_{\sigma,\tau}^0[X] - X \leq \mathcal{E}_{\sigma,\tau}[X] - X \leq \mathcal{E}_{\tau,\tau'}^0[X] - X = 0, \text{ a.s.}
\]
Once we have these properties, it is easy to check that the estimates (10.34)–(10.33) still hold for \( \sigma, \tau \in S_T \).

We now prove (A3), i.e.,
\[
\mathcal{E}_{\rho,\sigma}[\mathcal{E}_{\sigma,\tau}[X]] = \mathcal{E}_{\rho,\tau}[X], \forall 0 \leq \rho \leq \sigma \leq \tau.
\]
We first prove this relation for the case \( \rho, \sigma \in S_T \) and \( \sigma \in S_T^0 \). Let \( \{\sigma_n\}_{n=1}^\infty \) be a sequence in \( S_T^0 \) such that \( \sigma \leq \sigma_n \leq \tau, \ n = 1, 2, \cdots \) and \( \lim_{n \to \infty} \sigma_n = \sigma \), a.s. By Lemma 10.12 and Lemma 10.17 we have
\[
\lim_{n \to \infty} E[|\mathcal{E}_{\rho,\sigma_n}[\mathcal{E}_{\sigma,\tau}[X]] - \mathcal{E}_{\rho,\sigma}[\mathcal{E}_{\sigma,\tau}[X]]|^2] = 0.
\]
and
\[
\lim_{n \to \infty} E[|\mathcal{E}_{\sigma_n,\tau}[X] - \mathcal{E}_{\sigma,\tau}[X]|^2] = 0.
\]
On the other hand, by (a3) of Lemma 10.13 we have

\begin{equation}
(10.41) \quad E_{\rho,\tau}[X] - E_{\rho,\sigma_n}[E_{\sigma,\tau}[X]] = E_{\rho,\sigma_n}[E_{\sigma,\tau}[X]] - E_{\rho,\sigma_n}[E_{\sigma,\tau}[X]].
\end{equation}

It follows from (10.38) and (10.40) that, as $n$ tends to infinity,

\[ E[|E_{\rho,\tau}[X] - E_{\rho,\sigma_n}[E_{\sigma,\tau}[X]]|^2] \leq CE[|E_{\rho,\tau}[X] - E_{\sigma,\tau}[X]|^2] \rightarrow 0. \]

From this and (10.38), it follows that (10.38) holds for $\rho$, $\sigma$, and $\tau \in S_T^0$.

We now prove this relation for the general case: $\rho$, $\sigma$, and $\tau \in S_T$. Let $\{\tau_n\}_{n=1}^\infty$ be a sequence in $S_T$ such that $\tau \leq \tau_n$, $n = 1, 2, \cdots$ and $\lim_{n \to \infty} \tau_n = \tau$, a.s.. We have

\begin{equation}
(10.42) \quad E_{\rho,\sigma}[E_{\sigma,\tau_n}[X]] = E_{\rho,\sigma}[X].
\end{equation}

From (10.38), we have

\[ |E_{\rho,\sigma}[E_{\sigma,\tau_n}[X]] - E_{\rho,\sigma}[E_{\sigma,\tau}[X]]| \leq CE[|E_{\sigma,\tau_n}[X] - E_{\sigma,\tau}[X]|^2].\]

But by Lemma 10.17 both \{\sigma_n\}_{n=1}^\infty and \{\rho_n\}_{n=1}^\infty are Cauchy sequences in $L^2(F_T)$. We then can pass to the limit on both sides of (10.42) to obtain (10.38). It is easy to check that, once we have (A1)–(A2), (A3) and (A5), for $\rho$, $\sigma$, $\tau \in S_T$, the estimate (10.38) still holds for $\sigma, \sigma' \in S_T$ and $\tau, \tau' \in S_T$. From these estimates we have the continuity of $E_{\sigma,\tau}[\cdot]$ in the following sense: for each $\sigma, \tau \in S_T$, $X \in L^2(F_T)$ and sequences $\{\sigma_n\}_{n=1}^\infty$, $\{\tau_n\}_{n=1}^\infty$ in $S_T$ such that $\sigma \leq \sigma_n \leq \sigma, \tau_n \leq \tau$, $\tau \leq \tau_n$, $\lim_{n \to \infty} \sigma_n = \sigma, \lim_{n \to \infty} \tau_n = \tau$, we have

\[ E_{\sigma,\tau}[X] = \lim_{n \to \infty} E_{\sigma_n,\tau_n}[X] = \lim_{n \to \infty} E_{\sigma,\tau_n}[X], \quad \text{in } L^2(F_T). \]

The uniqueness of $E_{\sigma,\tau}[\cdot]$ is due to the uniqueness part of Lemma 10.11 and the continuity of $E_{\sigma,\tau}[\cdot]$ in $\sigma$ and $\tau$. The proof is complete. $\square$

We also have the following optional stopping theorem.

**Theorem 10.19.** Let $Y \in D^2_T(0, T)$ be an $E$–supermartingale (resp. $E$–submartingale). Then for each $\sigma, \tau \in S_T$ such that $\sigma \leq \tau$, we have

\begin{equation}
(10.43) \quad E_{\sigma,\tau}[Y_\tau] \leq Y_\sigma \quad \text{resp. } \geq Y_\sigma, \quad \text{a.s. .}
\end{equation}

**Proof.** We only prove the supermartingale part. We first consider the case $\sigma \in S_T$ and $\tau \in S_T^0$. Let $\{\sigma_n\}_{n=1}^\infty$ be a sequences in $S_T^0$ such that $\sigma \leq \sigma_n \leq \tau$, $n = 1, 2, \cdots$ and $\lim_{n \to \infty} \sigma_n = \sigma$, a.s.. By Lemma 10.13

\[ E_{\sigma_n,\tau}[Y_\tau] \leq Y_\sigma, \quad \text{a.s. .} \]

From the convergence result Lemma 10.12 the left hand side converges to $E_{\sigma,\tau}[Y_\tau]$ in $L^2(F_T)$. Since $Y \in D^2_T(0, T)$, $Y_{\sigma_n} \to Y_\sigma$, a.s.. We then have proved (10.43) for the case $\sigma \in S_T$ and $\tau \in S_T^0$.

Now let $\sigma, \tau \in S_T$ and let $\{\tau_n\}_{n=1}^\infty$ be a sequences in $S_T^0$ such that $\tau \leq \tau_n$, $n = 1, 2, \cdots$ and $\lim_{n \to \infty} \tau_n = \tau$, a.s.. We have proved that

\[ E_{\sigma,\tau_n}[Y_{\tau_n}] \leq Y_\sigma, \quad \text{a.s. .} \]

It is clear that $Y_{\tau_n} \to Y_\tau$ a.s. We also have $|Y_{\tau_n}| \leq \sup_{0 \leq t \leq T} |Y_t|$. Since the right hand side is in $L^2(F_T)$, it follows by Lebesgue’s dominated convergence theorem that $Y_{\tau_n} \to Y_\tau$ in $L^2(F_T)$. Since we have

\[ |E_{\sigma,\tau_n}[Y_{\tau_n}] - E_{\sigma,\tau}[Y_\tau]| \leq |E_{\sigma,\tau_n}[Y_{\tau_n}] - E_{\sigma,\tau_n}[Y_\tau]| + |E_{\sigma,\tau_n}[Y_\tau] - E_{\sigma,\tau}[Y_\tau]|. \]
The second term converges to zero in $L^2(\mathcal{F}_T)$. For the first term, we apply (10.33). It follows that

$$E[|E_{\sigma,\tau_n}[Y_{\tau_n}]-E_{\sigma,\tau_n}[Y_{\tau}]|^2] \leq CE[|Y_{\tau_n}-Y_{\tau}|^2] \to 0.$$  

We finally have (10.43) for the general situation. □

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