Asymptotic expansions of the Cotton-York tensor on slices of stationary spacetimes

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Abstract

We discuss expansions for the Cotton-York tensor near infinity for arbitrary slices of stationary spacetimes. From these expansions it follows directly that a necessary condition for the existence of conformally flat slices in stationary solutions is the vanishing of a certain quantity of quadrupolar nature (obstruction). The obstruction is nonzero for the Kerr solution. Thus, the Kerr metric admits no conformally flat slices. An analysis of the next order terms in the expansions in the case of solutions such that the obstruction vanishes, suggests that the only stationary solutions admitting conformally flat slices are the Schwarzschild family of solutions.

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1 Introduction

The question whether the Kerr spacetime admits a Cauchy hypersurface that is conformally flat is an issue which has been of particular interest for the numerical relativity community. The reason for this interest lies in the extensive use of conformally flat initial data sets —like the Bowen-York and Brandt-Brügmann data [4, 5]— in the numerical simulations of the collision of, say, two spinning black holes. Therefore, it would be very convenient to have a family of conformally flat data sets which would reduce to a Kerr initial data set under a particular choice of the parameters of the family. Such a family of data would then allow to study the close limit case as perturbations of a Kerr black hole in, for example, a way analogous to the work carried out by Gleiser et al [10]. In their analysis, they considered the close limit for Bowen-York data—which does not reduce to Kerr data under any choice of the parameters. Hence, they were forced to study the close limit as perturbations of a Schwarzschild black hole. Accordingly, they were only able to consider situations with small angular momentum.

The existence or not of such conformally flat slices is also interesting from a more theoretical point of view. More generally, one could ask whether a generic asymptotically flat, stationary spacetime admits conformally flat Cauchy slices or not. For a while now, there has been the suspicion that there are no such slices —unless, of course, the spacetime corresponds to the Minkowski or Schwarzschild solution.

There are a couple of partial negative answers to the aforementioned questions. Garat & Price [8] have shown that there are no axially symmetric, conformally flat slices in a given Kerr spacetime which smoothly reduce to the standard Schwarzschildian $t = constant$ slices in the limit when the angular momentum parameter $a$ goes to zero. Their approach is what one could call the obvious way of dealing with the problem. That is, they calculated the Cotton-York tensor

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of their slices, and found that they could not be chosen so that the Cotton-York tensor vanishes. The Cotton-York tensor is a complicated object involving derivatives of the curvature of the slice. Thus, in order to keep their calculations tractable, they considered expansions in powers of the angular momentum parameter $a$. Remarkably, the obstructions to the existence of conformally flat slices arise at order $a^2$.

More recently, in [16] it was shown that stationary solutions with non-vanishing angular momentum admit no conformally flat, maximal, non-boosted (i.e. with vanishing linear momentum) Cauchy slices. This result was obtained by calculating a certain type of asymptotic expansions near null and spatial infinity. Notably, the only fact about stationary solutions used in this result is that they are known to admit a smooth null infinity.

The results obtained in [8,16] are not fully satisfactory because of the assumptions they make about the nature of the slices. The stationary solutions are well understood, and in a certain sense —via their multipolar expansions—, one could say all is known about them. Thus, it ought to be possible to reduce the assumptions regarding the potential conformally flat slices to the bare minimum. This is the goal of the present work. Our analysis shows that, in order to have conformally flat slices in a given stationary spacetime, certain quantities of quadrupolar nature —to which we shall refer to as obstructions— should vanish. For the particular case of the Kerr solution these obstructions do not vanish, whence this spacetime admits no conformally flat Cauchy hypersurfaces. The only assumption made here about the slices is that they should have an intrinsic 3-metric for which a conformal compactification at infinity is well defined. In order to prove the aforementioned result we have used an approach that can be regarded as an hybrid of the approaches used in [8] and [16]: we have made use of the multipolar expansions for stationary spacetimes to calculate asymptotic expansions of the Cotton-York tensor near infinity. By looking at orders in the expansions higher than the one required to prove our main result, we also obtain further evidence to the suspicion that the only stationary spacetimes admitting conformally flat slices are the Schwarzschild and Minkowski solutions.

In this work we have made extensive use of multipolar expansions similar to those introduced by Simon & Beig in reference [15]. Given the high order in the expansions needed to obtain our results, most of the calculations have been performed using the computer algebra system Maple V. An example, based on the subclass of axially symmetric solutions, of how the computer algebra implementation was carried out can be found in an accompanying Maple V script. Due to the size of some of these expressions, most of them are not given in full. The article is organised as follows: section 2 reviews some of the basic features and formulas concerning multipolar expansions that have been used in our analysis. In particular, we discuss the so-called Geroch-Hansen potentials in both the quotient and rescaled quotient manifolds. The connection of the metrics of these quotient manifolds to the metrics of Cauchy hypersurfaces of the spacetime are also considered. Section 3 is concerned with the asymptotic expansions of the Cotton tensor for some particular slices which we shall call “canonic”. In section 4 the expansions for more generic slices are discussed. Our main result is presented there. Finally in section 5, there are some comments concerning higher order expansions.

**Notation.** Greek indices are spacetime ones, while latin indices are spatial. We have adopted the definition of the Cotton-York tensor as given in the Exact Solutions book [13]. The convention used for spherical harmonics is the one given in Arfken’s book [17]. A quantity $\phi(x')$ will be said to be $O^\infty(f(r))$ if there is a $C^\infty$ function $f(r)$ such that $|\phi| \leq |f|$, $|\partial_r \phi| \leq |\partial f/\partial r|$, $|\partial_r \partial_j \phi| \leq |\partial^2 f/\partial r^2|$, \ldots.

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1The Cotton-York tensor, sometimes also called the Bach tensor, locally characterises the conformal flatness of 3-dimensional manifolds. The Cotton-York vanishes if and only if the the 3-dimensional manifold is locally conformally flat. It can be thought of as the 3-dimensional analogue of the Weyl tensor. A classical discussion of this fact can be found in Eisenhart’s book, chapter II, section 28 [7]. A discussion in terms of more modern language can be found in [9,12]. As an historical remark, it is noted that the first use of this object in General Relativity seems to be found in [1], chapter 7, section 7-5.4. I thank S. Deser for this observation.
2 Stationary spacetimes and their multipolar expansions

Let \((\tilde{M}, \tilde{g}_{\mu\nu})\) be a stationary vacuum spacetime, and let \(\xi^\mu\) be the corresponding timelike Killing vector. The collection of the orbits of the Killing vector \(\xi^\mu\) defines the so-called quotient manifold, \(\tilde{X}\). In terms of quantities defined on the quotient manifold, the metric \(\tilde{g}_{\mu\nu}\) locally takes the form
\[
\tilde{g} = \lambda \left(dt + \beta_i dx^i\right)^2 - \lambda^{-1} \tilde{\gamma}_{ij} dx^i dx^j, \quad i, j = 1, 2, 3,
\]
where \(\lambda, \beta_i\) and \(\tilde{\gamma}_{ij}\) (the metric of the quotient manifold) depend only on the spatial coordinates \(x^k\). For the purposes of our later discussion, we will also need to have the metric \(\tilde{g}\) written in terms of a 3+1 decomposition with respect to the \(t = \text{constant}\) hypersurfaces. One has
\[
\tilde{g} = \tilde{N}^2 dt^2 - \tilde{h}_{ij} \left(N^i dt + dx^i\right) \left(N^j dt + dx^j\right).
\]

The relation between the two sets of quantities is given by —see e.g. \(5\)—
\[
\tilde{N}^2 = \frac{\lambda}{1 - \lambda^2 \beta_i \beta_i},
\]
\[
N_j = \lambda \beta_j, \quad N^j = -\tilde{N}^2 \lambda \tilde{\beta}^j,
\]
\[
\tilde{h}_{ij} = \lambda^{-1} \tilde{\gamma}_{ij} - \lambda \beta_i \beta_j, \quad \tilde{h}^{ij} = \lambda \tilde{\gamma}^{ij} + \lambda^2 \tilde{N}^2 \beta^i \beta^j,
\]
where
\[
\tilde{N}_j = \tilde{h}_{ij} N^i, \quad \tilde{\beta}^j = \tilde{\gamma}^{jk} \beta_k.
\]

The stationary field equations can be conveniently be written in terms of certain quantities defined in the quotient manifold: the metric \(\tilde{\gamma}_{ij}\), and 3 potentials, \(\tilde{\phi}_M, \tilde{\phi}_S, \tilde{\phi}_K\). These potentials are certain algebraic combinations of \(\lambda = \tilde{g}_{\mu\nu} \xi^\mu \xi^\nu\) (the square of the norm of the Killing vector field) and of a scalar \(\omega\) (the twist of \(\xi^\mu\)) such that
\[
\tilde{D}_i \omega_i = \omega_i = -\lambda^2 \sqrt{\det \tilde{\gamma}_{ijk}} \tilde{D}_j \beta^k.
\]

More precisely, one defines
\[
\tilde{\phi}_M = \frac{1}{4\lambda} (\lambda^2 + \omega^2 - 1), \quad \tilde{\phi}_S = \frac{1}{2\lambda} \omega, \quad \tilde{\phi}_K = \frac{1}{4\lambda} (\lambda^2 + \omega^2 + 1).
\]

These potentials are not independent, but satisfy the relation \(\tilde{\phi}_M^2 + \tilde{\phi}_S^2 - \tilde{\phi}_K^2 = -1/4\). The stationary vacuum field equations read:
\[
\tilde{D}_i \tilde{D}_i \tilde{\phi} = 2 \tilde{R} \tilde{\phi}, \quad (9a)
\]
\[
\tilde{R}_{ij} = 2 \left( \tilde{D}_i \tilde{\phi}_M \tilde{D}_j \tilde{\phi}_M + \tilde{D}_i \tilde{\phi}_S \tilde{D}_j \tilde{\phi}_S - \tilde{D}_i \tilde{\phi}_K \tilde{D}_j \tilde{\phi}_K \right), \quad (9b)
\]
where \(\tilde{\phi}\) denotes any of the potentials \(\tilde{\phi}_M, \tilde{\phi}_S, \tilde{\phi}_K\) and \(\tilde{R}_{ij}\) is the Ricci tensor of \(\tilde{\gamma}_{ij}\), and \(\tilde{R}\) its Ricci scalar.

2.1 Expansions in the quotient manifold \(\tilde{X}\)

Simon & Beig [15] have studied certain (asymptotic) expansions of the potentials \(\tilde{\phi}_M, \tilde{\phi}_S, \tilde{\phi}_K\) which are closely related to the Geroch-Hansen multipoles [11]. More precisely, they have shown that:

Theorem 1 (Simon & Beig, 1983). For all stationary vacuum solutions there is a (Cartesian) coordinate system \(\tilde{x}^k (\tilde{r} = \sqrt{\delta_{ij} \tilde{x}^i \tilde{x}^j})\) on the quotient manifold \(\tilde{X}\) and there are sets of constants
\[ \tilde{\phi}_M = \sum_{l=0}^{m-1} \frac{E_{a_1 \cdots a_l} \tilde{x}^{a_1} \cdots \tilde{x}^{a_l}}{l! \tilde{r}^{2l+1}} + \mathcal{O}^\infty(\tilde{r}^{-(m+1)}), \]
\[ \tilde{\phi}_S = \sum_{l=0}^{m-1} \frac{F_{a_1 \cdots a_l} \tilde{x}^{a_1} \cdots \tilde{x}^{a_l}}{l! \tilde{r}^{2l+1}} + \mathcal{O}^\infty(\tilde{r}^{-(m+1)}), \]
\[ \tilde{\phi}_K = \frac{1}{2} + \sum_{l=0}^{m-1} \frac{G_{a_1 \cdots a_l} \tilde{x}^{a_1} \cdots \tilde{x}^{a_{l-1}}}{l! \tilde{r}^{2l+1}} + \mathcal{O}^\infty(\tilde{r}^{-(m+1)}), \]
\[ \tilde{\tau}_{ij} = \delta_{ij} + \sum_{l=0}^{m-1} \left( \frac{\tilde{x}_i \tilde{x}_j A_{a_1 \cdots a_{l-1} \cdots a_l} \tilde{x}^{a_1} \cdots \tilde{x}^{a_{l-1}}}{\tilde{r}^{2l}} + \frac{\delta_{ij} B_{a_1 \cdots a_{l-1} \cdots a_l} \tilde{x}^{a_1} \cdots \tilde{x}^{a_{l-1}}}{\tilde{r}^{2l-2}} \right) + \mathcal{O}^\infty(\tilde{r}^{-(m+1)}). \]

All constants are symmetric in all their indices. \(A_{a_1 \cdots a_{l-1} \cdots a_l}, B_{a_1 \cdots a_{l-1} \cdots a_l}, C_{a_1 \cdots a_{l-1} \cdots a_l}, D_{a_1 \cdots a_{l-1} \cdots a_l} \), \(G_{a_1 \cdots a_{l-1} \cdots a_l} \), and the trace parts of \(E_{a_1 \cdots a_{l-1} \cdots a_l} \) and \(F_{a_1 \cdots a_{l-1} \cdots a_l} \) for \(0 \leq l \leq m \), depend on the trace free parts of \(E_{a_1 \cdots a_{l-1} \cdots a_l} \) and \(F_{a_1 \cdots a_{l-1} \cdots a_l} \), with \(1 \leq l \leq m \).

In [13] the explicit form of these expansions for \(m = 3 \) has been calculated. Unfortunately, it turns out that our analysis actually requires knowing the expansions up to \(m = 5 \). For computational purposes—as it is more amenable to a computer algebra implementation—it is more convenient to work not in the Cartesian coordinates discussed in the aforementioned theorem, but in the associated spherical coordinates given by
\[ \tilde{x}^1 = \tilde{r} \sin \theta \cos \varphi, \quad \tilde{x}^2 = \tilde{r} \sin \theta \sin \varphi, \quad \tilde{x}^3 = \tilde{r} \cos \theta. \]

Based on theorem [11] we make the following Ansatz for the expansions (in spherical coordinates) of the Hansen potentials and the metric of the quotient manifold:

\[ \tilde{\phi}_M = \frac{M}{\tilde{r}} + \sum_{k=3}^{5} \sum_{l=0}^{k-1} \sum_{m=-l}^{l} \frac{M_{klm} Y_{lm}}{\tilde{r}^k} + \mathcal{O} \left( \frac{1}{\tilde{r}^6} \right), \] (10a)
\[ \tilde{\phi}_S = \frac{5Y_{00} + S_{210} Y_{10}}{\tilde{r}^2} + \sum_{k=3}^{5} \sum_{l=0}^{k-1} \sum_{m=-l}^{l} \frac{S_{klm} Y_{lm}}{\tilde{r}^k} + \mathcal{O} \left( \frac{1}{\tilde{r}^6} \right), \] (10b)
\[ \tilde{\phi}_K = \frac{1}{2} + \sum_{k=2}^{5} \sum_{l=0}^{k-1} \sum_{m=-l}^{l} \frac{K_{klm} Y_{lm}}{\tilde{r}^k} + \mathcal{O} \left( \frac{1}{\tilde{r}^6} \right), \] (10c)

where \(M (\neq 0) \) denotes the mass of the stationary spacetime, and \(Y_{lm} \) are standard spherical harmonics expressed in terms of spherical coordinates. If the stationary spacetime is also axially symmetric the above expressions contain only \(Y_{00} \) harmonics. In the above expansions the angular momentum monopole has been set to zero in order to guarantee asymptotic flatness. Furthermore, a translation and a rotation have been used —without loss of generality—to set the mass dipolar terms equal to zero and to “align the angular momentum along the z axis”, that is, to set equal to zero the coefficients coming with the spherical harmonics \(Y_{11} \) and \(Y_{1-1} \) at order \(1/\tilde{r}^2 \) in \( \tilde{\phi}_S \).

The potentials \(\tilde{\phi}_M, \tilde{\phi}_S \) and \(\tilde{\phi}_K \) are real. Hence, the diverse coefficients in the expansions have to satisfy the reality conditions
\[ M_{klm} = (-1)^m M_{klm}, \quad S_{klm} = (-1)^m S_{klm}, \quad K_{klm} = (-1)^m K_{klm}. \] (11)
Similarly, we write for the components of the metric $\tilde{\gamma}_{ij}$,

\begin{align}
\tilde{\gamma}_{rr} &= 1 + \sum_{k=2}^{4} \sum_{l=0}^{k-1} \sum_{m=-l}^{l} \frac{A_{klm}Y_{lm}}{r^k} + O\left(\frac{1}{r^2}\right), \\
\tilde{\gamma}_{\theta \theta} &= \sum_{k=1}^{3} \sum_{l=0}^{k-1} \sum_{m=-l}^{l} \frac{B_{klm}Y_{lm}}{r^k} + O\left(\frac{1}{r^4}\right), \\
\tilde{\gamma}_{\varphi \varphi} &= \sum_{k=1}^{3} \sum_{l=0}^{k-1} \sum_{m=-l}^{l} \frac{C_{klm}\sin \theta Y_{lm}}{r^k} + O\left(\frac{1}{r^4}\right), \\
\tilde{\gamma}_{\theta \varphi} &= \sum_{k=0}^{2} \sum_{l=0}^{k-1} \sum_{m=-l}^{l} \frac{D_{klm}Y_{lm}}{r^k} + O\left(\frac{1}{r^5}\right), \\
\tilde{\gamma}_{\varphi \theta} &= \sum_{k=0}^{2} \sum_{l=0}^{k-1} \sum_{m=-l}^{l} \frac{E_{klm} \sin \theta Y_{lm}}{r^k} + O\left(\frac{1}{r^5}\right), \\
\tilde{\gamma}_{\varphi \varphi} &= \tilde{\gamma}_{\theta \theta} + \sum_{k=0}^{2} \sum_{l=0}^{k-1} \sum_{m=-l}^{l} \frac{F_{klm} \sin \theta Y_{lm}}{r^k} + O\left(\frac{1}{r^5}\right). 
\end{align}

Again, the different coefficients in the latter expansions are subject to reality conditions analogous to those for the potentials.

The substitution of the Ansatz for the Hansen potentials and the metric of the quotient manifold into the stationary vacuum field equations (12a) and (12b) yields —consistently with theorem 1— that:

(i) the coefficients $S_{200}$, $M_{klm}$, $S_{klm}$ with $k = 3, \ldots, 5$, $l = 0, \ldots, k - 2$, $m = -l, \ldots, l$ in $\phi_M$ and $\phi_S$;

(ii) all the coefficients $K_{klm}$ in $\phi_K$;

(iii) and all the coefficients $A_{klm}$, $B_{klm}$, $C_{klm}$, $D_{klm}$, $E_{klm}$, $F_{klm}$ in the components of $\tilde{\gamma}_{ij}$,

can be written in terms of the coefficients $M$, $M_{k,k-1,m}$ and $S_{k,k-1,m}$, $m = 1 - k, \ldots, k - 1$. These coefficients are essentially the multipole moments of Geroch & Hansen. The explicit form dependence in these expansions for the axially symmetric case can be found in the accompanying Maple V script.

**The Kerr solution.** The Kerr spacetime is, arguably, the most important stationary solution. Part of its relevance lies in the fact that stationary solutions are Kerrian at first order in the angular momentum —see e.g. [2]. A quick calculation with the Kerr initial data in Bowen-York coordinates reveals that

\begin{equation}
S_{110} = 2\sqrt{\frac{\pi}{3}}Ma, \quad Q_{200} = -86\sqrt{\frac{\pi}{3}}Ma^2, 
\end{equation}

where $a$ is the Kerr parameter —which has dimensions of angular momentum per unit mass. Furthermore, because of the axial symmetry

\begin{equation}
M_{222} = M_{221} = 0.
\end{equation}

### 2.2 Expansions in the conformally rescaled quotient manifold $X$

On calculational grounds, it will be much more convenient for us to work not in $\tilde{X}$ but in a conveniently conformally rescaled version thereof. The asymptotic flatness of the stationary spacetime ensures that there exists a manifold $X$ consisting of $\tilde{X}$ plus an additional point $i$ such that the metric in $X$ is given by

\begin{equation}
\gamma_{ij} = \Omega^{\frac{1}{2}}\tilde{\gamma}_{ij},
\end{equation}

where $\Omega$ is a function of $\tilde{r}$.
We define the rescaled potentials

$$\Omega = \frac{1}{2B^2} \left( \sqrt{1 + 4(\hat{\phi}_M^2 + \hat{\phi}_S^2)} - 1 \right),$$

(16)

for some real constant $B^2 > 0$. Furthermore, one has that

$$\Omega(i) = 0, \quad D_i \Omega(i) = 0 \quad D_j D_k \Omega(i) = 2\gamma_{jk}(i).$$

(17)

We define the rescaled potentials

$$\phi_M = \tilde{\phi}_M / \sqrt{\Omega}, \quad \phi_S = \tilde{\phi}_S / \sqrt{\Omega}, \quad \phi_K = \tilde{\phi}_K / \sqrt{\Omega}.$$  

(18)

The expansions discussed in section 2.1 then imply that

$$\Omega = r^2 + \left( \left( M^2 + \frac{1}{4\pi M^2} S_{110}^2 \right) + \sum_{m=-2}^{2} \left( \frac{2}{M} M_{22m} + \frac{1}{\sqrt{5\pi M^2}} S_{110}^2 \delta_{0m} \right) Y_{2m} \right) + O(r^5),$$

(19)

where we have introduced the new coordinate

$$r = 1/\tilde{r},$$

(20)

and have set $B = M$. In order to perform the calculations to be described in the following sections, it is actually necessary to know the expansions of $\Omega$ up to order $r^6$ —that is, two orders more! The leading terms of the rescaled potentials read

$$\phi_M = M + \left( \frac{M^3}{2} - \frac{1}{8\pi M} S_{110}^2 \right) - \frac{1}{2M \sqrt{5\pi}} S_{110}^2 Y_{20} \right) r^2 + O(r^3),$$

(21a)

$$\phi_S = S_{110} Y_{10} r - \left( \sum_{m=-2}^{2} S_{22m} Y_{2m} \right) r^2 + O(r^3),$$

(21b)

$$\phi_K = \frac{1}{2r} + \left( \frac{3}{4} M^2 - \frac{1}{16\pi M^2} S_{110}^2 \right) - \sum_{m=-2}^{2} \left( \frac{1}{2M} M_{22m} - \frac{1}{4\pi M^2} S_{110}^2 \delta_{0m} \right) Y_{2m} \right) r + O(r^3).$$

(21c)

For the purposes of this article, the expansions for $\phi_M$ and $\phi_S$ need to be known up to order $r^5$, while that of $\phi_K$ only up to order $r^4$. Similarly, the leading terms of the rescaled metric $\gamma_{ij}$ read

$$\gamma_{11} = 1 + \left( \left( 2M^2 + \frac{1}{2\pi M^2} S_{110}^2 \right) + \sum_{m=-2}^{2} \left( \frac{4}{M} M_{22m} + \frac{2}{\sqrt{5\pi M^2}} S_{110}^2 \delta_{0m} \right) Y_{2m} \right) r^2$$

$$+ O(r^3),$$

(22a)

$$\gamma_{12} = O(r^6),$$

(22b)

$$\gamma_{13} = O(r^6),$$

(22c)

$$\gamma_{22} = r^2 + \left( \left( M^2 + \frac{1}{2\pi M^2} S_{110}^2 \right) + \sum_{m=-2}^{2} \left( \frac{4}{M} M_{22m} + \frac{2}{\sqrt{5\pi M^2}} S_{110}^2 \delta_{0m} \right) Y_{2m} \right) r^4$$

$$+ O(r^5),$$

(22d)

$$\gamma_{23} = O(r^6),$$

(22e)

$$\gamma_{33} = \sin^2 \theta r^2 + \sin^2 \theta \left( \left( M^2 + \frac{1}{2\pi M^2} S_{110}^2 \right) + \sum_{m=-2}^{2} \left( \frac{4}{M} M_{22m} + \frac{2}{\sqrt{5\pi M^2}} S_{110}^2 \delta_{0m} \right) Y_{2m} \right) r^4$$

$$+ O(r^5),$$

(22f)

Later computations will require knowing of the expansions of $\gamma_{11}$ up to order $r^4$, and up to order $r^6$ for $\gamma_{22}$ and $\gamma_{33}$. At this point it is worth making the following remark: due to the work of Beig & Simon, we know that for any triplet $(\bar{\gamma}_{ij}, \bar{\phi}_M, \bar{\phi}_S)$ solution of the stationary field
equations (9a) and (9b), there is a (Cartesian) coordinate chart around $i$ in $X$ such that $\tilde{\gamma}_{ij}$, $\tilde{\phi}_M$, $\tilde{\phi}_S$ and $\Omega$ are analytic. Thus, by a standard coordinate transformation there is a chart in spherical coordinates (not necessarily the one we are using) for which expansions as the ones used above —equations (21a), (21b) and (22a)-(22f)— are well defined. Hence, we shall not be concerned with questions of convergence of our expansions.

Starting from these expansions, it is not too hard to obtain those of the scalar $\lambda$ appearing in the metric given by equation (1) —the square of the norm of the stationary Killing vector field. Its leading terms read

$$\lambda = \frac{1}{2\sqrt{\Omega}(\phi_K - \phi_M)} = 1 + 2Mr + 2M^2r^2 + 2\left(M^3 + \sum_{m=-2}^{2} M_{2,2,m}Y_{2,m}\right)r^3 + O(r^4).$$

The calculations in this article require the computation of the above expansions up to a further order. More complicated is the calculation of expansions for the shift $\beta_i$ appearing in the metric (1). In order to obtain an equation for $\beta_i$ we proceed as follows. We begin by considering the identity

$$\left(D^iD_i - \frac{1}{8}R\right)\tilde{\phi} = \Omega^{5/2}\left(D^iD_i - \frac{1}{8}R\right)\phi,$$

where again $\tilde{\phi}$ is any of the Hansen potentials. Thus, the stationary equation (23a) can be written as:

$$\left(D^iD_i - \frac{1}{8}R\right)\phi = \frac{15}{8}\Omega^{-2}\tilde{R}\phi.$$  

Now, consider the equation (25) for $\phi_M$ and for $\phi_S$, multiply them by $\phi_S$ and $\phi_M$ respectively and take their difference to obtain $\phi_S D^iD_i\phi_M - \phi_M D^iD_i\phi_S = 0$. From here it follows that the expression $\phi_SD_i\phi_M - \phi_M D_i\phi_S$ is divergence free. An analogous argument can be used to extract the same conclusion from $\phi_M D_i\phi_K - \phi_K D_i\phi_M$ and $\phi_K D_i\phi_K - \phi_K D_i\phi_S$. Thus it follows from equation (7) that $\beta^i$ satisfies the “curl-like” equation

$$\bar{\epsilon}_{ijk}D^j\beta^k = \frac{4}{\sqrt{|\gamma|}}\left(\phi_SD_i\phi_K + \phi_M D_i\phi_M - \phi_K D_i\phi_S - \phi_S D_i\phi_M\right).$$

Note that because of its “curl-like”, its solutions are defined up to a gradient. The properties of the solutions to this equation have been discussed in lemma 2.5 in reference [6].

**Lemma 1** (Dain, 2001). There exists a solution $\beta_i$ of equation (26) which in normal (Cartesian) coordinates has the form

$$\beta_i = \beta^1_i + \frac{1}{r}\beta^2_i,$$

where $\beta^1_i$ and $\beta^2_i$ are analytic and $O(x^3)$.

Based on the latter result, we shall consider solutions to equation (26) whose leading terms are given by

$$\beta_r = 2\sqrt{5}(S_{221}Y_{11} - \overline{S}_{221}Y_{1,-1})r + O(r^2), \quad \beta_\theta = O(r^4) \quad \beta_\phi = -\sin^2 \theta \sqrt{\frac{3}{\pi}}S_{110}r + O(r^2).$$

These last expansions need to be calculated up to order $r^4$.  

\[7\]
2.3 The 3-metric of arbitrary slices

Consistently with our overall strategy, we shall not be interested in the metric (1), but in a conveniently conformally rescaled version thereof. Following [6], we define the (time independent) conformal factor for the spacetime

\[ \hat{\Omega} = \sqrt{\lambda} \Omega, \]

so that

\[ g_{\mu\nu} = \hat{\Omega}^2 \tilde{g}_{\mu\nu} \quad \text{and} \quad h_{ij} = \hat{\Omega}^2 \tilde{h}_{ij}. \]

In terms of a 3+1 decomposition one would have

\[ g = N^2 dt^2 - h_{ij} (N^i dt + dx^i)(N^j dt + dx^j). \]

Letting

\[ N_j = h_{ij} N^i, \quad \beta^i = \gamma^{ik} \beta_k, \]

one finds the following relations between the two decompositions:

\[ N = \hat{\Omega} \tilde{N}, \quad N = \frac{\Omega^2 \lambda^2}{1 - \hat{\Omega}^2 \lambda^2 \beta_j \beta^j}, \quad \beta^j = -\frac{N^j}{N}, \]

and most importantly,

\[ h_{ij} = \gamma_{ij} - \Omega^2 \lambda^2 \beta_i \beta_j. \]

The aforementioned 3+1 decomposition together with a 1-form \( \beta_i \) whose leading terms are given by equations (27a)-(27c) render a foliation of the stationary spacetime that in a sense can be regarded as “canonical” —see the discussion after theorem 2.6 in [6]. For example, the slices \( t = \text{constant} \) in the Kerr solution given in Boyer-Lindquist coordinates are an example of such a canonical foliation. Any other foliation can be obtained by introducing a new time coordinate \( \bar{t} \) such that

\[ t = \bar{t} - F(r, \theta, \phi). \]

The assumptions being made on \( F = F(r, \theta, \phi) \) will be described later, but for the time being we shall assume it is at least of class \( C^4 \) in the coordinates \( (r, \theta, \phi) \). Substitution of the latter into (34) renders

\[ g = \Omega^2 \lambda^2 (dt + [\beta_i - \partial_i F] dx^i)^2 - \gamma_{ij} dx^i dx^j. \]

Thus, writing \( \bar{\gamma}_{ij} = \beta_i - \partial_i F \) one finds that the corresponding 3-metric, \( \bar{h}_{ij} \), associated with the new foliation is given by

\[ \bar{h}_{ij} = \gamma_{ij} - \Omega^2 \lambda^2 \bar{\gamma}_i \bar{\gamma}_j, \]

\[ = \gamma_{ij} - \Omega^2 \lambda^2 (\beta_i - \partial_i F)(\beta_j - \partial_j F) . \]

3 The Cotton-York tensor in the “canonical” slices

As mentioned in the introduction, our strategy is to discuss the existence of conformally flat slices in stationary spacetimes by calculating the Cotton-York tensor of the prospective slices. Now, the Cotton-York tensor is a complicated object. However, on the other hand, there is some evidence suggesting that generic stationary solutions do not admit conformally flat slices. The Cotton-York tensor is a conformal invariant, thus, if it vanishes for a certain 3-dimensional manifold, then it also vanishes for any other manifold which is conformally related to the original one. Whence, in order to establish a no-go result, we just need, for example, to show the non-vanishing of the Cotton-York tensor in a neighbourhood of infinity, \( i \) of the slice. The simplest way of doing the latter is by means of asymptotic expansions around \( i \). As a warming up, we firstly consider the existence of conformally flat slices in the “canonic” foliation —i.e. that for which \( F = 0 \). Given the 3-metric \( h_{ij} \), its Cotton-York tensor is given by

\[ B^{ijkl} = 2\epsilon^{ijkl} D_t \left( R^i_k - \frac{1}{4} \delta^i_k R \right), \]
where $R^j_k$, $R$, and $D_i$ denote respectively the mixed Ricci tensor, the Ricci scalar and the covariant derivative with respect to $h_{ij}$. For the case of the so-called canonical slicing the leading terms of the initial 3-metric read,

\[
h_{rr} = 1 + \left(2M^2 + \frac{1}{2\pi M^2}S^2_{110} + \sum_{m=-2}^{2} \left(\frac{4}{M}M_{22m} + \frac{2}{\sqrt{3}\pi M^2}S^2_{110}\delta_{0m}\right)Y_{2m}\right)r^2
\]

\[+ O(r^3), \tag{38a}\]

\[h_{r\theta} = O(r^6), \tag{38b}\]

\[h_{r\varphi} = O(r^6), \tag{38c}\]

\[h_{\theta\theta} = r^2 + \left(M^2 + \frac{1}{2\pi M^2}S^2_{110} + \sum_{m=-2}^{2} \left(\frac{4}{M}M_{22m} + \frac{2}{\sqrt{3}\pi M^2}S^2_{110}\delta_{0m}\right)Y_{2m}\right)r^4
\]

\[+ O(r^5), \tag{38d}\]

\[h_{\theta\varphi} = O(r^6), \tag{38e}\]

\[h_{\varphi\varphi} = \sin^2 \theta r^2 + \sin^2 \theta \left(M^2 + \frac{1}{2\pi M^2}S^2_{110} + \sum_{m=-2}^{2} \left(\frac{4}{M}M_{22m} + \frac{2}{\sqrt{3}\pi M^2}S^2_{110}\delta_{0m}\right)Y_{2m}\right)r^4
\]

\[+ O(r^5). \tag{38f}\]

The expansions for $h_{rr}$ have been calculated up to order $r^4$, while those for $h_{\theta\theta}$ and $h_{\varphi\varphi}$ up to order $r^6$. From here a lengthy calculation with Maple yields

\[B_{rr} = O(r^4), \tag{39a}\]

\[B_{r\theta} = iM \left(2\overline{M}Y_{2,-2} - 2M_{222}Y_{2,-1} - M_{221}Y_{2,1} + M_{221}Y_{21}\right)r^4 + O(r^5), \tag{39b}\]

\[B_{r\varphi} = \sin \theta \left(\frac{1}{\sqrt{\pi}} \left(45S^2_{110} + 2\sqrt{3}\pi MM_{220}\right) \left(\frac{3\sqrt{7}}{35}Y_{30} - \frac{3\sqrt{3}}{5}Y_{10}\right) + M\overline{M}_{221} \left(\frac{3\sqrt{5}}{5}Y_{1,-1} - 4\sqrt{\frac{2}{35}}Y_{3,-1}\right) + MM_{221} \left(-\frac{3\sqrt{5}}{5}Y_{1,1} + 4\sqrt{\frac{2}{35}}Y_{3,1}\right)
\]

\[+ \frac{2\sqrt{7}}{7}MM\overline{M}_{222}Y_{3,-2} + \frac{2\sqrt{7}}{7}M^2MM_{222}Y_{32}\right)r^4 + O(r^5), \tag{39c}\]

\[B_{\theta\theta} = O(r^5), \tag{39d}\]

\[B_{\theta\varphi} = O(r^5), \tag{39e}\]

\[B_{\varphi\varphi} = O(r^5). \tag{39f}\]

In particular, if one has an axially symmetric stationary spacetime then $M_{221} = M_{222} = 0$ and one obtains the obstruction:

\[\Upsilon = 45S^2_{110} + 2\sqrt{3}\pi MM_{220}, \tag{40}\]

which can be shown to be different from zero for the Kerr solution. Indeed,

\[\Upsilon_{\text{Kerr}} = -112\pi M^2a^2. \tag{41}\]

Note that $\Upsilon$ is a quantity of quadrupolar nature which closely resembles the structure of the Newman-Penrose constants of stationary spacetimes —see [14]. However, at this point it is not possible to exhibit a connection —if any.

Remark. Guided by the suspicion that the Kerr spacetime admits no conformally flat slices, it would be natural to reckon that the angular momentum is responsible for this feature. However, Garat & Price’s perturbative analysis already hints that the responsible of the non-existence of conformally flat slices in stationary solutions has to be an object of quadrupolar nature. The quantity $M^2a^2$ is the only quadrupolar object one can form out of the parameters of the Kerr solution. Due to the specialness of the latter spacetime, the form of the obstructions for more general stationary spacetimes cannot be inferred from merely looking at the Kerrian case.
4 The Cotton-York tensor in an arbitrary slice

Before calculating the Cotton-York tensor of the $\mathcal{I} = \text{constant}$ slices, one has to make some assumptions on the function $F = F(r, \theta, \varphi)$ defining the coordinate transformation. Our Ansatz will be

$$F(r, \theta, \varphi) = F_1(\theta, \varphi)r + \mathcal{O}(r^2),$$

(42)

where the symbol $\mathcal{O}(r^2)$ means we assume that $\partial_r F = F_1 + \mathcal{O}(r)$, $\partial_\theta F = O(1)$, $\partial_\varphi F = O(1)$ and $\partial_r \partial_\theta \partial_\varphi F = O(1)$. Furthermore, we will require the metric $\overline{h}_{ij}$, given by equation (38), to have a well defined conformal compactification at infinity. This requires the existence of a conformal factor $\overline{\Omega}$ which is $C^2$ near $i$, such that

$$\overline{\Omega}(i) = 0, \quad \overline{D}_i \overline{\Omega}(i) = 0, \quad \overline{D}_i \overline{D}_j \overline{\Omega}(i) = 2h_{ij}(i),$$

(43)

where $\overline{D}$ is the covariant derivative with respect to the metric $\overline{h}_{ij}$. The latter forces $\overline{h}_{ij}$ to be at least $C^3$ in a neighbourhood of $i$. Now, a short calculation using Cartesian coordinates and arguments similar to that used in the remarks to theorem 2.6 in reference [6] shows that if the coefficient $F_1(\theta, \varphi)$ contains $l = 2$ harmonics then the metric $\overline{h}_{ij}$ is at best of class $C^{0,\alpha}$ in a neighbourhood of $i$. Indeed, the spherical harmonics $Y_{2m}$ are in Cartesian coordinates given by

$$Y_{20} = \frac{1}{4} \sqrt{\frac{5}{\pi}} \left( \frac{3z^2}{r^2} - 1 \right), \quad Y_{21} = -\frac{1}{2} \sqrt{\frac{15}{2\pi}} \frac{z(x + iy)}{r^2}, \quad Y_{22} = \frac{1}{4} \sqrt{\frac{15}{2\pi}} \frac{(x + iy)^2}{r^2}.$$}

Thus, a function $F_1$ made up of these harmonics would be such that $rF_1 = f_1/r + f_2r$ where $f_1$ and $f_2$ are analytic. From here a “book-keeping” argument shows that $\overline{h}_{ij} = \overline{h}_{ij}^1 + \overline{r} \overline{h}_{ij}^2$, where $\overline{h}_{ij}^1$ and $\overline{h}_{ij}^2$ are analytic and $r \in C^{0,\alpha}$. The situation is even worse for higher harmonics. On the other hand, if $F_1(\theta, \varphi)$ contains only $l = 0$ and $l = 1$ harmonics then $\overline{h}_{ij}$ is at least of class $C^{2,\alpha}$.

Under our assumptions the leading terms of the 3-metric $\overline{h}_{ij}$ read:

$$\overline{h}_{rr} = 1 + \mathcal{O}(r^2),$$

(44a)

$$\overline{h}_{r\theta} = -F_1 \partial_\theta F_1 r^5 + \mathcal{O}(r^6),$$

(44b)

$$\overline{h}_{r\varphi} = -F_1 \left( \partial_\varphi F_1 - S_{110} \frac{3}{\pi} \sin^2 \theta \right) r^5 + \mathcal{O}(r^6),$$

(44c)

$$\overline{h}_{\theta\theta} = r^2 + \mathcal{O}(r^4),$$

(44d)

$$\overline{h}_{\theta\varphi} = \mathcal{O}(r^6),$$

(44e)

$$\overline{h}_{\varphi\varphi} = \sin^2 \theta r^2 + \mathcal{O}(r^4).$$

(44f)

The Cotton-York tensor in this case is such that

$$\overline{\mathcal{T}}_{rr} = -\frac{1}{2\sqrt{\pi} \sin^4 \theta} \left( A \partial_{\theta\theta} F_1 + B \partial_{\varphi\varphi} F_1 + C \partial_{\theta\varphi} F_1 + D \partial_{\theta\varphi} F_1 + E \partial_{\theta\theta} F_1 + G \partial_{\varphi\varphi} F_1 + H \partial_{\varphi\varphi} F_1 + I \partial_{\theta\varphi} F_1 \right) r^3 + \mathcal{O}(r^4),$$

(45)

\text{2} f \in C^{p,\alpha} \text{ means that the function has } p\text{-th order derivatives } f \text{ which are Hölder continuous with exponent } \alpha. \text{ A function } f \text{ is said to be Hölder continuous with exponent } \alpha \text{ at a point } x_0 \text{ if there is a constant } C \text{ such that } |f(x) - f(x_0)| \leq |x - x_0|^\alpha, \text{ } 0 < \alpha < 1 \text{ for } x \text{ in a neighborhood of } x_0.

\text{3} The spherical coordinates are singular at } r = 0. \text{ Consequently, the discussion regarding the regularity has to be carried out in Cartesian coordinates.
where

\[
A = -\sin^4 \theta \left(4\sqrt{\pi} \partial_{\varphi} F_1 + 3\sqrt{3} S_{110}\right),
\]
\[
B = 4\sqrt{\pi} \sin^2 \theta \partial_\varphi F_1,
\]
\[
C = 4\sqrt{\pi} \sin^2 \theta \partial_\theta F_1,
\]
\[
D = -\sin^2 \theta \left(4\sqrt{\pi} \partial_{\varphi} F_1 + 3\sqrt{3} S_{110}\right),
\]
\[
E = -\cos \theta \sin^3 \theta \left(4\sqrt{\pi} \partial_{\varphi} F_1 + 15\sqrt{3} S_{110}\right),
\]
\[
G = -2\sin \theta \cos \theta \left(3\sqrt{3} S_{110} \sin^2 \theta - 4\sqrt{\pi} \partial_{\varphi} F_1\right),
\]
\[
H = 4\sqrt{\pi} \cos \theta \sin^3 \theta \partial_\theta F_1,
\]
\[
I = \sin^2 \theta \partial_\varphi F_1 \left(18\sqrt{3} S_{110} \cos^4 \theta - 27\sqrt{3} \cos^2 \theta + 9\sqrt{3} S_{110} + 4\sqrt{\pi} \partial_{\varphi} F_1\right).
\]

Furthermore, one also has that

\[
\bar{B}_{r\theta} = \mathcal{O}(r^4), \quad \bar{B}_{r\varphi} = \mathcal{O}(r^4), \quad \bar{B}_{\theta\theta} = \mathcal{O}(r^5), \quad \bar{B}_{\theta\varphi} = \mathcal{O}(r^5), \quad \bar{B}_{\varphi\varphi} = \mathcal{O}(r^5). \quad (46)
\]

In order to analyse the condition imposed by the leading term of \(\bar{B}_{rr}\), and consistently with our Ansatz for \(F\), we write

\[
F_1 = f_{00} Y_{00} + f_{1,-1} Y_{1,-1} + f_{1,0} Y_{10} + f_{11} Y_{11}, \quad (47)
\]

where \(f_{00}, f_{1,-1}, f_{1,0}\) and \(f_{11}\) are complex numbers such that the coefficient \(F_1\) is a real function. The expression one obtains from substituting the latter into the leading term of \(45\), which we shall denote by \(\Upsilon_F\) is quite complicated. In order to extract its content, we shall calculate its inner product with different spherical harmonics. A computation using Maple V reveals that

\[
\int_0^{2\pi} \int_0^\pi \Upsilon_F Y_{00} \sin \theta d\theta d\varphi = -\frac{225}{64} \pi f_{1,0} S_{110}. \quad (48)
\]

Similarly, one has,

\[
\int_0^{2\pi} \int_0^\pi \Upsilon_F Y_{2,1} \sin \theta d\theta d\varphi = -\frac{189}{256} \sqrt{15} \pi f_{1,-1} S_{110}. \quad (49)
\]

Thus, if \(\Upsilon_F\) is to vanish then necessarily

\[
f_{1,-1} = f_{1,0} = f_{1,1} = 0, \quad (50)
\]

unless, of course, \(S_{110} = 0\). Now, if \(F_1 = f_{00} Y_{00}\) then it follows that the leading terms of the expansions of the Cotton-York tensor near infinity are identical to those in the expansions given in the preceding section —equations \(43\) and \(44\). Thus, the (real) coefficient \(f_{00}\) cannot be used to enforce conformal flatness.

We summarise the results of this and the previous sections in the following

**Theorem.** For an asymptotically flat stationary spacetime, necessary conditions for the existence of a conformally flat slice which in a neighbourhood of spatial infinity is given by

\[
\tilde{t} = t + F(r, \theta, \varphi) = \text{constant},
\]

where \(F(r, \theta, \varphi)\) is at least \(C^4\) in a neighbourhood of \(r = 0\), and can be expanded as

\[
F(r, \theta, \varphi) = r F_1(\theta, \varphi) + \mathcal{O}(r^2),
\]

are

\[
M_{222} = M_{221} = 45 S_{110}^2 + 2\sqrt{5} \pi M M_{220} = 0.
\]

In particular, the Kerr spacetime (with nonvanishing angular momentum) admits none of such slices.
5 On expansions at higher orders

In the light of our main theorem it is natural to ask what happens in the expansions at higher orders if the conditions given in the main theorem are fulfilled. In order to keep the complexity of the calculations at bay, we shall assume that the stationary spacetime is also axially symmetric.

If \( 45S_{110}^2 + 2\sqrt{\pi \tau} M_{220} = 0 \) then for the canonical foliation, the leading term of \( B_{r\varphi} \) —which is of order \( r^5 \)— implies that in order to have a conformal flatness one needs

\[
\sqrt{7\pi} M_{330} \cos^3 \theta + 21S_{110}^2 \cos^2 \theta - \sqrt{7\pi} M_{330} \cos \theta - 21S_{110}^2 = 0. \tag{51}
\]

From the last condition it readily follows that

\[
M_{330} = 0, \quad S_{110} = 0. \tag{52}
\]

This last result brings further support to the opinion that the only stationary solutions admitting conformally flat slices are the Schwarzschild and Minkowski spacetimes.

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