Local vertical measurements and violation of Bell inequality

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(Dated: February 2, 2008)

For two qubits belonging to Alice and Bob, we derive an approach to setup the bound of Bell operator in the condition that Alice and Bob continue to perform local vertical measurements. For pure states we find that if the entanglement of the two qubits is less than 0.2644 (measured with von Neumann entropy) the violation of the Bell inequality will never be realized, and only when the entanglement is equal to 1 the maximal violation \((2\sqrt{2})\) can occur. For specific form of mixed states, we prove that the bound of the Bell inequality depends on the concurrence. Only when the concurrence is greater than 0.6 the violation of the Bell inequality can occur, and the maximal violation can never be achieved. We suggest that the bound of the Bell operator in the condition of local vertical measurements may be used as a measure of the entanglement.

PACS numbers: 03.65.Ud, 03.65.Ta

The local realism theory (LRT) [1] states that physical systems can be described by local objective properties (physical reality) which are independent of observation. The Bell inequality [2], however, sets bound for correlations of local observables within any LRT. The violation of the Bell inequality means that the quantum mechanics cannot be regarded as a LRT. There is profound relation between the violation of the Bell inequality and the quantum entanglement, and this relation has been formulated as the entanglement witness [3]. As early as 1991, Gisin et al. [4] pointed out that the Bell inequality is satisfied for any separable quantum state, but may be violated by any purely entangled state if one chooses a proper measurement setting.

The original Bell inequality has been extended to a more general inequality by Clauser, Horne, Shimony, and Holt (CHSH inequality) [5]. Consider a bipartite quantum system including qubit \(a\) belonging to Alice and qubit \(b\) belonging to Bob. Alice and Bob are at distant sites and choose to measure one of two dichotomous observables: \(A\) or \(A'\) at qubit \(a\) and \(B\) or \(B'\) at qubit \(b\). All observables have the spectrum in \([-1, 1]\). In this Letter we only consider traceless spin observables, e.g., \(A = \mathbf{a} \cdot \sigma\) and analogously for \(A', B, B'\). There is a so-called Bell operator [6]:

\[
W = A \otimes (B + B') + A' \otimes (B - B').
\]

The CHSH inequality is

\[
|\langle W \rangle_{\rho}| \leq 2,
\]

where \(\langle W \rangle_{\rho}\) is the expected value of \(W\) in state \(\rho\). For any quantum states, a bound of \(W\) is given by the Tsirelson inequality [7, 8]

\[
|\langle W \rangle_{\rho}| \leq \sqrt{4 + |\langle [A, A'] \otimes [B, B'] \rangle_{\rho}|}.
\]

Landau [8] has pointed out that the Tsirelson inequality is tight, i.e., for any choices of the observables, there exists a state \(\rho\) which can make

\[
\max_{\rho \in D} |\langle W \rangle_{\rho}| = \sqrt{4 + |\langle [A, A'] \otimes [B, B'] \rangle_{\rho}|},
\]

where \(D\) is the set of all quantum states. Tsirelson [6] has proved that for spin observables \(\max_{\rho \in D} |\langle W \rangle_{\rho}|\) can be obtained in a pure two-qubit state. From the Tsirelson inequality it is clear that if one wants to produce a violation of the CHSH inequality he must carry out measurements on pairs of non-commuting spin observables for both particles, and if one wants to achieve the maximal violation \((2\sqrt{2})\) allowed by the quantum theory he has to choose both pairs of local observables to be anti-commuting. The latter corresponds to the case that both Alice and Bob carry out vertical measurements, \(a \cdot a' = \mathbf{b} \cdot \mathbf{b'} = 0\). In a recent work [9], Seevinck and Uffink have proved that for entangled state if both the local angles, \(\theta_a = \arccos(\mathbf{a} \cdot \mathbf{a'})\) and \(\theta_b = \arccos(\mathbf{b} \cdot \mathbf{b'})\), increase from zero to \(\pi/2\) the maximal violation of the CHSH inequality increases.

In this Letter we will investigate the following question: What is the bound of \(|\langle W \rangle_{\rho}|\) for a given state \(\rho\) in the condition of local vertical measurements? For this purpose we derive the analytical expression of the tight upper bound of \(|\langle W \rangle_{\rho}|\) for any given pure state \(\rho\), and show that if Alice and Bob both perform vertical measurements they would never find violation of CHSH inequality if \(\rho\) is not an “enough” entangled state. We also derive an approach which can be used to deal with the case of mixed states. For mixed states of a specific form we calculate the bound and find that it depends on the concurrence. We argue that this bound of the Bell operator in the condition of local vertical measurements can be used as a measure of the quantum entanglement for any states.

We assume that

\[
\begin{align*}
A &= \sigma_z^a, & A' &= \sigma_z^a, \\
B &= \frac{-\sigma_z^b - \sigma_z^b}{\sqrt{2}}, & B' &= \frac{\sigma_z^b - \sigma_z^b}{\sqrt{2}}.
\end{align*}
\]
thus $W$ can be written as

$$W \equiv A \otimes (B + B') + A' \otimes (B - B') \quad \quad \text{Eq. (6)}$$

So an arbitrary local vertical measurement scheme can be written as $(U^a \otimes U^b)W(U^a \otimes U^b)$, where $U^{a(b)}$ is an arbitrary unitary operation on $a(b)$. For a given state $\rho$, we have to find some $U^a$ and $U^b$ such that $|(U^a \otimes U^b)W(U^a \otimes U^b)\rangle_{\rho}$ takes its maximum value.

We first discuss the case of $\rho$ being a pure state. The eigenvalues and eigenvectors of $W$ are

$$\begin{align*}
-2\sqrt{2} & \leftrightarrow \eta_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \\
2\sqrt{2} & \leftrightarrow \eta_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \\
0 & \leftrightarrow \eta_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \\
0 & \leftrightarrow \eta_4 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}.
\end{align*}$$

An arbitrary unitary operation on a single qubit can be written as $U^a \otimes U^b$ as follows:

$$U^a \otimes U^b = 
\begin{pmatrix}
\xi_{11} \cos(\gamma/2) & \xi_{12} \sin(\gamma/2) & -\xi_{13} \cos(\gamma/2) & \xi_{14} \sin(\gamma/2) \\
\xi_{21} \sin(\gamma/2) & \xi_{22} \cos(\gamma/2) & \xi_{23} \sin(\gamma/2) & \xi_{24} \cos(\gamma/2) \\
-\xi_{31} \sin(\gamma/2) & -\xi_{32} \cos(\gamma/2) & -\xi_{33} \sin(\gamma/2) & -\xi_{34} \cos(\gamma/2) \\
-\xi_{41} \cos(\gamma/2) & -\xi_{42} \sin(\gamma/2) & -\xi_{43} \cos(\gamma/2) & -\xi_{44} \sin(\gamma/2)
\end{pmatrix}$$

where all $\xi_{ij}$ are related to tunable parameters $\alpha, \alpha', \beta, \beta', \delta, \delta'$ from Eq. (10). Now we map $|\psi\rangle$ into
subspace $\mathcal{H}_2$ by choosing suitable $U^a \otimes U^b$. We denote $|\psi\rangle = U^a \otimes U^b |\psi\rangle$. It is found that we have to choose $\gamma$ and $\gamma'$ in such a way that $\cos(\gamma/2) = \sin(\gamma/2) = 0$ or $\sin(\gamma'/2) = \cos(\gamma/2) = 0$, because otherwise $\psi'$ will have component state which is in $\mathcal{H}_2$ and this will reduce $|\langle \psi' | W |\psi\rangle|$. When $\cos(\gamma'/2) = \sin(\gamma/2) = 0$, we can obtain

$$\psi = U^a \otimes U^b |\psi\rangle = \left( -e^{i\xi_{12}} \cos(\theta/2) \quad 0 \quad 0 \quad e^{i(\xi_{13}+\chi)} \sin(\theta/2) \right)^T. \quad (11)$$

Then $|\langle (U^a \otimes U^b)^\dagger W (U^a \otimes U^b) \rangle_\rho|$ can be obtained as follows

$$|\langle (U^a \otimes U^b)^\dagger W (U^a \otimes U^b) \rangle_\psi| = |\langle \psi' | W |\psi'\rangle| = \left| -2\sqrt{2} \cdot |\langle \eta_1 |\psi'\rangle|^2 \right| = \sqrt{2} \cdot \left| -e^{i\xi_{12}} \cos(\theta/2) + e^{i(\xi_{13}+\chi)} \sin(\theta/2) \right|^2. \quad (12)$$

Since $0 \leq \theta \leq \pi$, we can take $-e^{i\xi_{12}} = e^{i(\xi_{13}+\chi)} = 0$, and get the maximum value of $|\langle (U^a \otimes U^b)^\dagger W (U^a \otimes U^b) \rangle_\rho|$ as

$$|\langle (U^a \otimes U^b)^\dagger W (U^a \otimes U^b) \rangle_\psi|_{max} = \sqrt{2} \cdot (\sin \theta + 1). \quad (13)$$

When $\sin(\gamma'/2) = \cos(\gamma/2) = 0$, we have

$$\psi = U^a \otimes U^b |\psi\rangle = \left( -e^{i(\xi_{13}+\chi)} \sin(\theta/2) \quad 0 \quad 0 \quad e^{i\xi_{12}} \cos(\theta/2) \right)^T, \quad (14)$$

and $|\langle (U^a \otimes U^b)^\dagger W (U^a \otimes U^b) \rangle_\rho|$ can be calculated as

$$|\langle (U^a \otimes U^b)^\dagger W (U^a \otimes U^b) \rangle_\psi| = |\langle \psi' | W |\psi'\rangle| = \left| -2\sqrt{2} \cdot |\langle \eta_1 |\psi'\rangle|^2 \right| = \sqrt{2} \cdot \left| -e^{i(\xi_{13}+\chi)} \sin(\theta/2) + e^{i\xi_{12}} \cos(\theta/2) \right|^2. \quad (15)$$

Since $0 \leq \theta \leq \pi$, we take $-e^{i(\xi_{13}+\chi)} = e^{i\xi_{12}} = 0$ and obtain the same maximum value of $|\langle (U^a \otimes U^b)^\dagger W (U^a \otimes U^b) \rangle_\rho|$.

We can also map $\psi$ into subspace $\mathcal{H}_2$ and obtain the same maximum of $|\langle (U^a \otimes U^b)^\dagger W (U^a \otimes U^b) \rangle_\psi|$ in a similar way. In Fig. 1 we plot the maximum of $|\langle (U^a \otimes U^b)^\dagger W (U^a \otimes U^b) \rangle_\psi|$, which we call as “bound”, and the entanglement of $\psi$, which is calculated using the von Neumann entropy, as functions of $\theta$. We find that when $\sin(\theta) \leq \sqrt{2} - 1$ and Alice and Bob continue to perform local vertical measurements, they will never reach the violation of the CHSH inequality. According to the Tsirelson inequality, if one wants to achieve the maximal violation allowed by the quantum theory he has to properly choose both pairs of local vertical measurements. So from Fig. 1 we can see that one cannot get the maximal violation ($2\sqrt{2}$) of the CHSH inequality unless $\psi$ is a maximally entangled state.

In [11], Braunstein et al. showed that mixed states can produce maximal violations of the CHSH inequality, and the necessary and sufficient condition for violating the CHSH inequality in an arbitrary mixed spin-$1/2$ state is presented in [12]. Here we present a numerical method which can be used to calculate the maximum of $|\langle W \rangle_\rho|$ for any mixed state $\rho$ in the condition of local measurement setting.

By using the eigenvectors of $W$ in Eq. (7), we can rewrite $|\langle U^\dagger W U \rangle_\rho|_{max}$ ($U = U^a \otimes U^b$) as

$$|\langle U^\dagger W U \rangle_\rho|_{max} = \max_U |\langle \eta_1 | U \rho U^\dagger | \eta_1 \rangle \langle \eta_1 | W |\eta_1 \rangle + \langle \eta_3 | U \rho U^\dagger | \eta_3 \rangle \langle \eta_3 | W |\eta_3 \rangle | = \max_U \left| 2\sqrt{2} \cdot \left[ \text{Tr} (U^\dagger | \eta_3 \rangle \langle \eta_3 | U \rho) - \text{Tr} (U^\dagger | \eta_1 \rangle \langle \eta_1 | U \rho) \right] \right|. \quad (16)$$

FIG. 2: The numerically calculated bound of $|\langle U^\dagger W U \rangle_\psi|$ for the pure state $\psi$. It is independent of the azimuthal angle $\chi$. FIG. 3: The bound of $|\langle U^\dagger W U \rangle_\rho|$ (solid line) as a function of $\lambda$ for the mixed state. When it is greater than the classical bound (dashed line), we say that Alice and Bob can achieve the violation of the CHSH inequality. The concurrence (dot-dashed line) of $\rho$ is also shown.
Substituting \( U^a \otimes U^b, \eta_1, \eta_3 \) and \( \rho \) into Eq. (16), we can numerically calculate the bound of \( |\langle U^\dagger W U \rangle |^\rho \). In the case of pure state, \( \rho = |\psi \rangle \langle \psi | \) where \( |\psi \rangle \) is the state in Eq. (9), we calculate the bound by using the numerical scheme. The results are shown in Fig. 2. They are the same as those calculated from Eq. (13).

Now let us consider a mixed state which has a single positive parameter

\[
\rho = \frac{1}{9} \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 4 & \lambda & 0 \\
0 & \lambda & 4 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\]

(17)

where we take \( 0 \leq \lambda \leq 4 \) to ensure the positivity of \( \rho \). This state is usually used to investigate the evolution of the entanglement of a pair of qubits exposed to local noisy environments [13, 14]. We substitute Eq. (17) into Eq. (16) and calculate the bound of the Bell operator. The obtained results are shown in Fig. 3. There is a turning point of the curve near \( \lambda = 3.52 \). From Fig. 3 we can see that the bound is highly consistent with the concurrence except for this turning point. Due to the restriction of the local vertical measurement scheme, only when the concurrence is greater than 0.6 Alice and Bob can achieve the violation of the CHSH inequality. In this form of mixed states the maximal violation of the CHSH inequality cannot be realized. It is expected that only when the concurrence of a mixed state is equal to 1 the maximal violation of the CHSH inequality could be achieved.

In summary, for any pure state we present an analytical expression of the bound of the Bell operator in the condition that Alice and Bob both perform local vertical measurements, and for a general state we derive an numerical method for the calculation of the bound. The results show intimate relationship between the bound and the concurrence. We suggest that the bound of the Bell operator in the condition of local vertical measurements may be used as a measure of the entanglement.

Acknowledgments This work was supported by the State Key Programs for Basic Research of China (Grant Nos.2005CB623605 and 2006CB921803), and by National Foundation of Natural Science in China Grant Nos. 10474033 and 60676056.

[1] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. 47, 777 (1935).
[2] J. S. Bell, Physics (Long Island City, N.Y.) 1, 195 (1964); J. S. Bell, Speakable and Unspeakable in Quantum Mechanics (Cambridge University Press, Cambridge, England, 1988).
[3] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Lett. A 223, 1 (1996); B. M. Terhal, ibid. 271, 319 (1996); M. Lewenstein, B. Kraus, J. I. Cirac, and P. Horodecki, Phys. Rev. A 62, 052310 (2000); D. Bruß, J. I. Cirac, P. Horodecki, F. Hulpke, B. Kraus, M. lewenstein, and A. Sanpera, J. Mod. Opt. 49, 1399 (2002); Philipp Hyllus, Otfried Gühne, Dagmar Bruß, and Maciej Lewenstein, Phys. Rev. A 72, 012321 (2005).
[4] N. Gisin, Phys. Lett. A 154, 201 (1991); N. Gisin and A. Peres, ibid. 162, 15 (1992); S. Popescu and D. Rohrlich, ibid. 166, 293 (1992).
[5] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Phys. Rev. Lett. 23, 880 (1969); 24, 549 (E) (1970).
[6] S. L. Braunstein, A. Mann, and M. Revzen, Phys. Rev. Lett. 68, 3259 (1992).
[7] B. S. Cirel’son, Lett. Math. Phys. 4, 93 (1980).
[8] L. J. Landau, Phys. Lett. A 120, 54 (1987).
[9] Michael Seevinck and Jos Uffink, Phys. Rev. A 76, 042105 (2007).
[10] See for example, M.A. Nielsen and L.L. Chuang, “Quantum Computation and Quantum Information”, CUP, Cambridge (2000).
[11] Samuel L. Braunstein, A. Mann, and M. Revzen, Phys. Rev. Lett. 68, 3259 (1992).
[12] R. Horodecki, P. Horodecki, and M. Horodecki, Phys. Lett. A 200, 340 (1995).
[13] Ting Yu and J. H. Eberly, Phys. Rev. Lett. 93, 140404 (2007).
[14] Ting Yu and J. H. Eberly, Phys. Rev. Lett. 97, 140403 (2007).