Ground State Entropy of Potts Antiferromagnets on Cyclic Polygon Chain Graphs

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We present exact calculations of chromatic polynomials for families of cyclic graphs consisting of linked polygons, where the polygons may be adjacent or separated by a given number of bonds. From these we calculate the (exponential of the) ground state entropy, \(W\), for the \(q\)-state Potts model on these graphs in the limit of infinitely many vertices. A number of properties are proved concerning the continuous locus, \(B\), of nonanalyticities in \(W\). Our results provide further evidence for a general rule concerning the maximal region in the complex \(q\) plane to which one can analytically continue from the physical interval where \(S_0 > 0\).

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I. INTRODUCTION

The $q$-state Potts antiferromagnet (AF) exhibits nonzero ground state entropy, $S_0 > 0$ (without frustration) for sufficiently large $q$ on a given graph or lattice. This is equivalent to a ground state degeneracy per site $W > 1$, since $S_0 = k_B \ln W$. Such nonzero ground state entropy is important as an exception to the third law of thermodynamics. The zero-temperature partition function of the above-mentioned $q$-state Potts antiferromagnet on a graph $G$ satisfies $Z(G, q, T = 0)_{PAF} = P(G, q)$, where $P(G, q)$ is the chromatic polynomial expressing the number of ways of coloring the vertices of the graph $G$ with $q$ colors such that no two adjacent vertices have the same color. Thus $W(G, q) = \lim_{n \to \infty} P(G, q)^{1/n}$ (1.1) where $n = v(G)$ is the number of vertices of $G$ and $\{G\} = \lim_{n \to \infty} G$. The minimum number of colors needed for this coloring of $G$ is called its chromatic number, $\chi(G)$. At certain special points $q_s$ (typically $q_s = 0, 1, ..., \chi(G)$), one has the noncommutativity of limits $
abla\lim_{q \to q_s} \lim_{n \to \infty} P(G, q)^{1/n} \neq \lim_{n \to \infty} \lim_{q \to q_s} P(G, q)^{1/n}$ (1.2) and hence it is necessary to specify the order of the limits in the definition of $W(G, q_s)$. As in [7], we shall use the first order of limits here; this has the advantage of removing certain isolated discontinuities in $W$. In addition to Refs. [2]- [7], some other previous related works include Refs. [8]- [22]. Since $P(G, q)$ is a polynomial, one can generalize $q$ from $\mathbb{Z}_+$ to $\mathbb{R}$ and indeed to $\mathbb{C}$. $W(G, q)$ is a real analytic function for real $q$ down to a minimum value, $q_c(G)$. For a given $\{G\}$, we denote the continuous locus of non-analyticities of $W$ as $B$. This locus $B$ forms as the accumulation set of the zeros of $P(G, q)$ (chromatic zeros of $G$) as $n \to \infty$. $W(G, q)$ satisfies $B(q) = B(q^*)$. A fundamental question concerning the Potts antiferromagnet is the form of this locus for a given graph family or lattice and, in particular, the maximal region in the complex $q$ plane to which one can analytically continue the function $W(G, q)$ from physical values where there is nonzero ground state entropy, i.e., $W > 1$. We denote this region as $R_1$. Further, we denote as $q_c$ the maximal point where $B$ intersects the real axis, which can occur via $B$ crossing this axis or via a line segment of $B$ lying along this axis.

In the present work we present exact calculations of $P(G, q)$ and $W(G, q)$ for families of cyclic graphs consisting of linked polygons, where the polygons may be adjacent or separated by a certain number of bonds. By the term cyclic we mean that these families of graphs contain a global circuit, defined as a route along the graph which has the topology of $S^1$ and a length $\ell_{g.c.}$ that goes to infinity as $n \to \infty$. The results provide new insight into $W$ and $B$.
in an exactly solvable context where one can study their dependence on several parameters characterizing the families of polygon chain graphs. The present work extends our recent studies of cyclic families of graphs [7,22].

From our previous exact calculations of \(P(G, q)\) and \(W(\{G\}, q)\) on a number of families of graphs we have inferred several general results on \(B\): (i) for a graph \(G\) with well-defined lattice directions, a sufficient condition for \(B\) to separate the \(q\) plane into different regions is that \(G\) contains at least one global circuit [19], and (ii) the locus \(B\) for such a graph does not contain any endpoint singularities. Two other general features are that for graphs that (a) contain global circuits, (b) cannot be written in the form \(G = K_p + H\) [15] and (c) have compact \(B\), we have observed that \(B\) (iii) passes through \(q = 0\) and (iv) crosses the positive real axis, thereby always defining a \(q_c\). The families of polygon chain graphs studied in this paper do contain global circuits, and the exact results presented here are in accord with the above general properties.

The chromatic polynomial \(P(G, q)\) has a general decomposition as

\[
P(G, q) = c_0(q) + \sum_{j=1}^{N_h} c_j(q)(a_j(q))^{t_j n}
\]

where the \(a_j(q)\) and \(c_{j \neq 0}(q)\) are independent of \(n\), while \(c_0(q)\) may contain \(n\)-dependent terms, such as \((-1)^n\), but does not grow with \(n\) like \((\text{const.})^n\) with \(|\text{const.}| > 1\). The expression \(c_0\) may be absent. A term \(a_\ell(q)\) is “leading” if it dominates the \(n \to \infty\) limit of \(P(G, q)\). The locus \(B\) occurs where there is an abrupt nonanalytic change in \(W\) as the leading terms \(a_\ell\) changes; thus this locus \(B\) is the solution to the equation of degeneracy of magnitudes of leading terms. Hence, \(W\) is finite and continuous, although nonanalytic, across \(B\).

It is convenient to define the following polynomial:

\[
D_k(q) = \frac{P(C_k, q)}{q(q - 1)} = a^{k-2} \sum_{j=0}^{k-2} (-a)^{-j} = \sum_{s=0}^{k-2} (-1)^s \binom{k - 1}{s} q^{k-2-s}
\]

1 Some families of graphs that do not have regular lattice directions have noncompact loci \(B\) that separate the \(q\) plane into different regions [16,18,21].

2 The complete graph on \(p\) vertices, denoted \(K_p\), is the graph in which every vertex is adjacent to every other vertex. The “join” of graphs \(G_1\) and \(G_2\), denoted \(G_1 + G_2\), is defined by adding bonds linking each vertex of \(G_1\) to each vertex in \(G_2\).
where

\[ a = q - 1 \]  \hspace{1cm} (1.5)  

and \( P(C_k, q) \) is the chromatic polynomial for the circuit (cyclic) graph \( C_k \) with \( k \) vertices,

\[ P(C_k, q) = a^k + (-1)^k a \]  \hspace{1cm} (1.6)  

This paper is organized as follows. In Sections II and III we present our results for \( P, W, \) and \( B \) for the open and cyclic polygon chain graphs. Some concluding remarks are included in Section IV, and some further results are included in the Appendix.

II. OPEN POLYGON CHAIN GRAPHS

Before giving our results for the cyclic polygon chain graphs, we discuss the simpler case of families of open polygon chain graphs comprised of \( m \) subunits, each subunit consisting of a \( p \)-sided polygon with one of its vertices attached to a line segment of length \( e_g \) edges (bonds). Thus, the members of each successive pair of polygons are separated from each other by a distance (gap) of \( e_g \) bonds along these line segments, with \( e_g = 0 \) representing the case of contiguous polygons. Since each polygon is connected to the rest of the chain at two vertices (taken to be at the same relative positions on the polygons in all cases), the family of graphs depends on two additional parameters, namely the number of edges of the polygons between these two connection vertices, moving in opposite directions along the polygon, \( e_1 \) and \( e_2 \). For \( e_1 \neq e_2 \) we denote the smaller (s) and larger (\( \ell \)) edge lengths as

\[ e_s \equiv \min(e_1, e_2), \quad e_\ell \equiv \max(e_1, e_2) \]  \hspace{1cm} (2.1)  

Clearly,

\[ e_1 + e_2 = p \]  \hspace{1cm} (2.2)  

We denote this family of open (\( o \)) polygon chain graphs as \( G_{e_1, e_2, e_g, m; o} \). Illustrations of these graphs are given in Fig. (II). Since it is immaterial whether one labels the upper and lower routes between the left and right-hand points on each polygon where it connects to the rest of the chain as \( e_1 \) and \( e_2 \) or in the opposite order, as \( e_2 \) and \( e_1 \),

\[ G_{e_1, e_2, e_g, m; o} = G_{e_2, e_1, e_g, m; o} \]  \hspace{1cm} (2.3)  

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FIG. 1. Illustrations of cyclic and open polygon chain graphs $G_{e_1,e_2,e_g,m}$ and $G_{e_1,e_2,e_g,m;o}$ with $(e_1,e_2,e_g,m) = (a) \ (2,2,0,6), \ (b) \ (2,2,1,4), \ (c) \ (2,3,2,3)$. For the cyclic (open) graphs, the rightmost vertex on each graph is identified with (is distinct from) the leftmost vertex at the same level, respectively.

The number of vertices is

$$v(G_{e_1,e_2,e_g,m;o}) = (p + e_g - 1)m + 1 \quad (2.4)$$

and the number of edges or bonds is

$$e(G_{e_1,e_2,e_g,m;o}) = (p + e_g)m \quad (2.5)$$
The chromatic number $\chi = 2$ if $p$ is even and $\chi = 3$ if $p$ is odd. The chromatic polynomial is easily calculated to be

$$P(G_{e_1,e_2,e_g,m;o}, q) = q(a_1)^m$$ \hspace{1cm} (2.6)

where

$$a_1 = (q - 1)^{e_g+1}D_p$$ \hspace{1cm} (2.7)

Evidently, the chromatic polynomial has the form of (1.3) with $c_0 = 0$, $N_a = 1$, $c_1 = q(a_1)^{-t_1}$ and $t_1 = t$, where

$$t = \frac{1}{p + e_g - 1}$$ \hspace{1cm} (2.8)

Note that this chromatic polynomial does not depend on $e_1$ or $e_2$ individually, but instead only on their sum $p$, i.e., the number of sides of each polygon\footnote{One can consider a generalization of this family in which the quantities $e_1$ and $e_2$ can be different for each polygon. The chromatic polynomial and $W$ function for this much larger family of graphs are again given by (2.6) with (2.7) and by (2.10), respectively.}

Let us next consider the limit of an infinitely long chain, i.e., $m \to \infty$. Formally, we denote this limit as

$$L_m : m \to \infty \text{ with } e_1, e_2, e_g \text{ fixed}$$ \hspace{1cm} (2.9)

We find

$$W(\left[ \lim_{m \to \infty} G_{e_1,e_2,e_g,m;o}, q \right], q) = [(q - 1)^{e_g+1}D_p]^{\frac{1}{p + e_g - 1}}$$ \hspace{1cm} (2.10)

The nonanalyticities of this $W$ function consist of discrete branch point singularities at its zeros, including the point $q = 1$ and the $p - 2$ zeros of $D_p$; there is no continuous locus of nonanalyticities of $W$; i.e., $B = \emptyset$. If $p$ is odd, then $D_p$ contains a factor $(q - 2)$, so that one of the zeros of $W$ is at $q = 2$.

Given the relation (2.4), there are also other ways to take the number of vertices to infinity:

$$L_{e_g} : e_g \to \infty \text{ with } e_1, e_2, m \text{ fixed}$$ \hspace{1cm} (2.11)
\[ L_{e_1} : \ e_1 \to \infty \text{ with } e_2, e_g, m \text{ fixed} \] (2.12)

\[ L_{e_2} : \ e_2 \to \infty \text{ with } e_1, e_g, m \text{ fixed} \] (2.13)

and

\[ L_p : \ p \to \infty \text{ with } e_2 - e_1, e_g, m \text{ fixed} \] (2.14)

For the \( L_{e_g} \) limit, we find

\[
W(\lim_{e_g \to \infty} G_{e_1, e_2, e_g, m, o}, q) = q - 1
\] (2.15)

This is analytic for all \( q \) so that, in particular, \( \mathcal{B} = \emptyset \). The \( W \) function in eq. (2.13) is the same as the \( W \) function for linear and, more generally tree, graphs. This is understandable, since as \( e_g \to \infty \), the vertices on the linear connecting segments occupy a fraction approaching unity of the total number of vertices of the graph.

Since the chromatic polynomial (2.6) only depends on \( e_1 \) and \( e_2 \) through the quantity \( p \), it follows that for this family of open polygon chain graphs, the limits \( L_{e_1} \), \( L_{e_2} \), and \( L_p \) all yield the same result. Further, from eq. (1.4), it follows that in these limits, \( W \) is the same as that for a circuit graph \( \overline{G} \), viz.,

\[
W(\lim_{p \to \infty} G_{e_1, e_2, e_g, m, o}, q) = q - 1 \quad \text{for } |q - 1| \geq 1
\] (2.16)

and

\[
|W(\lim_{p \to \infty} G_{e_1, e_2, e_g, m, o}, q)| = 1 \quad \text{for } |q - 1| < 1
\] (2.17)

For this limit, \( \mathcal{B} \) consists of the circle \(|q - 1| = 1\). Again, these results are easily understood, since as \( p \to \infty \), the vertices on the polygonal circuit subgraphs occupy a fraction approaching unity of the total number of vertices.

\footnote{For real \( 0 < q < q_c(\{G\}) \), as well as other regions of the \( q \) plane that cannot be reached by analytic continuation from the real interval \( q > q_c(\{G\}) \), one can only determine the magnitude \(|W(\{G\}, q)|\) unambiguously \( \overline{G} \).}
III. CYCLIC POLYGON CHAIN GRAPHS

We proceed to our main subject, families of graphs consisting of \( m \) \( p \)-sided polygons linked together to form closed circuits. The other parameters, \( e_g, e_1, \) and \( e_2, \) are the same as for the open polygonal chains discussed above. Thus, a given cyclic polygonal chain graph is specified by the four parameters \( (e_1, e_2, e_g, m) \), so we denote it as \( G_{e_1,e_2,e_g,m} \), where the cyclic property is implicitly understood. For the same reason as given above for the open chain, we have

\[
G_{e_1,e_2,e_g,m} = G_{e_2,e_1,e_g,m} \tag{3.1}
\]

The number of vertices in the graph \( G_{e_1,e_2,e_g,m} \) is

\[
n = v(G_{e_1,e_2,e_g,m}) = (p + e_g - 1)m \tag{3.2}
\]

and the number of edges (bonds) is the same as for the open chain, \( e(G_{e_1,e_2,e_g,m}) = (p + e_g)m \). If \( p \) is odd, then the chromatic number \( \chi \) is 3. If \( p \) is even, then \( \chi = 2 \) or 3 depending on the values of \( e_1, e_2, e_g, \) and \( m \). In Fig. (1) we show illustrative examples of families of cyclic polygon chain graphs. From (3.1), it follows that

\[
P(G_{e_1,e_2,e_g,m}, q) = P(G_{e_2,e_1,e_g,m}, q) \tag{3.3}
\]

Using the deletion-contraction theorem to get recursion relations which we then solve, we calculate the chromatic polynomial

\[
P(G_{e_1,e_2,e_g,m}, q) = (a_1)^m + (q - 1)(a_2)^m \tag{3.4}
\]

where the term \( a_1 \) is the same as the term appearing in the chromatic polynomial for the open polygonal chain, (2.7), and

\[
a_2 = (-1)^{p + e_g}q^{-1}\left[q - 2 + (1 - q)^e_1 + (1 - q)^e_2\right] \tag{3.5}
\]

\[
= (-1)^{p + e_g}\left[1 - p - \sum_{s=2}^{e_1} \binom{e_1}{s}(-q)^{s-1} - \sum_{s=2}^{e_2} \binom{e_2}{s}(-q)^{s-1}\right] \tag{3.6}
\]

The chromatic polynomial (3.4) has the form of (1.3) with \( c_0 = 0, N_a = 2, c_1 = 1, c_2 = (q-1), \) and \( t_1 = t_2 = t \), where \( t \) was given in eq. (2.8). Note that, owing to the second term, \( a_2 \), the chromatic polynomial (3.4) depends on the values of \( e_1 \) and \( e_2 \) individually, in contrast to the result (2.6) for the open polygon chain, which only depends on \( e_1 \) and \( e_2 \) through their sum, \( p \).
If the smaller distance \( e_s = 1 \), then the chromatic polynomial (3.4) reduces to the factorized form

\[
P(G_{e_s=1,e_1,e_2,m}, q) = (D_p)^m P(C_{nc}, q) = q(q - 1)(D_p)^m D_{nc}
\]

(3.7)

where

\[
n_c = m(1 + e_g)
\]

(3.8)

Having calculated the chromatic polynomial (3.4), we can next obtain the \( W \) function and its continuous locus of nonanalyticities \( \mathcal{B} \) for the various limits (2.9) and (2.11)-(2.14). This locus is determined as the solution of the degeneracy equation of (nonzero) leading terms

\[
|a_1| = |a_2|
\]

(3.9)

It will be useful to reexpress this in the variable \( a = q - 1 \) of eq. (1.3). Multiplying both sides of (3.9) by \( |a + 1| \), we obtain

\[
|a^{e_g}(a^p + (-1)^p a)| = |a - 1 + (-a)^{e_1} + (-a)^{e_2}|
\]

(3.10)

(Since the value \( q = 0 \), i.e., \( a + 1 = 0 \), is already a solution of (3.9), multiplying both sides of this equation by \( |a + 1| \) to get (3.10) does not introduce a distinct spurious zero.)

We first discuss the limits that yield the simplest results. For the \( L_{e_g} \) limit, \( \mathcal{B} \) consists of the unit circle \( |q - 1| = 1 \) and divides the \( q \) plane into two regions, the exterior and interior of this circle, which regions are denoted as \( R_1 \) and \( R_2 \), respectively. If \( |q - 1| > 1 \), then \( a_1 \) is the dominant term and

\[
W([\lim_{e_g \to \infty} G_{e_1,e_2,e_g,m}], q) = q - 1 \quad \text{for} \quad |q - 1| > 1
\]

(3.11)

If \( |q - 1| < 1 \), then \( a_2 \) is dominant and

\[
|W([\lim_{e_g \to \infty} G_{e_1,e_2,e_g,m}], q)| = 1 \quad \text{for} \quad |q - 1| < 1
\]

(3.12)

That is, in the \( L_0 \) limit, the \( W \) function is the same as for the infinite-vertex limit of the circuit graph. This is expected, since as \( e_g \) increases, more and more of the vertices of the graph lie on a circuit graph where, e.g., the route of the circuit follows either the \( e_1 \) or \( e_2 \) bonds of a given polygon, then traverses the \( e_g \) connecting bonds to the next polygon, and so forth around the cyclic chain. As a result of the symmetry (3.3), the \( L_{e_1}, L_{e_2}, \) and \( L_p \) limits are equivalent; in this case, we find that for \( |q - 1| > 1 \), the \( a_1 \) term is dominant and \( W \) is again given by the right-hand side of (3.11) while for \( |q - 1| < 1 \), neither term
dominates over the other, and \(|W| = 1\). The results for \(W\) are thus the same as for the \(L_{eg}\) limit, and the reason is similar: as \(p \to \infty\), the vertices located on the polygons occupy a fraction approaching unity of all of the vertices.

The most interesting and complicated results are for the \(L_m\) limit. We have proved a number of general properties of \(\mathcal{B}\). We show calculations of \(\mathcal{B}\) and comparison with chromatic zeros for long finite chains in Figs. 2-5 (and, in addition, Figs. 6-9 in the Appendix). Because of the factorization [3.7], the families with \(e_s = 1\) yield very simple results (see below) so that in the figures we concentrate on the cases with \(e_s \geq 2\): \((e_1, e_2) = (2, 2), (2, 3), (2, 4), \) and \((3, 3)\) and, for each pair \((e_1, e_2)\), the values \(0 \leq e_g \leq 3\). The general properties of \(\mathcal{B}\) are

- (B1) \(\mathcal{B}\) is compact.
- (B2) \(\mathcal{B}\) passes through \(q = 0\).
- (B3) \(\mathcal{B}\) encloses regions in the \(q\) plane.
- (B4) if \(e_s = 1\), then \(\mathcal{B}\) is the unit circle \(|q - 1| = 1\) independent of the values of \(e_\ell\) and \(e_g\); thus, in this case, \(q_c = 2\). The chromatic zeros of \(G_{e_1, e_2, e_g, m}\) lie exactly on this locus, independently of the values of \(e_\ell\), \(e_g\), and \(m\).
- (B5) if \(p\) is even, then \(q_c = 2\).
- (B6) if \(p\) is odd and \(e_s \geq 2\), then \(q_c < 2\) and for fixed \((e_1, e_2)\), \(q_c\) increases monotonically as \(e_g\) increases, approaching 2 from below as \(e_g \to \infty\).

To prove (B1), we re-express eq. (3.9) in terms of the variable

\[ y = \frac{1}{a} = \frac{1}{q - 1} \]  

Then eq. (3.10) becomes

\[ |1 + (-1)^p y^{p-1}| = |y^{e_g+e_s}(y^{e_\ell-1} - y^{e_\ell} + (-1)^{e_\ell} + (-1)^{e_s} y^{e_\ell-e_s})| \]  

where we have divided (3.9) by \(|a^{e_s+p}|\), which factor is nonzero for \(q \neq 1\) and thus, in particular, for the large-\(|q|\) (small \(|y|\)) region of interest here. The necessary and sufficient condition that \(\mathcal{B}\) is noncompact, i.e. unbounded is that the equation (3.14) has a solution for \(y = 0\). Clearly, it does not have such a solution, which proves property (B1).

For (B2), we use the property that

\[ D_p(q = 0) = (-1)^p(p - 1) \]  

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so that $|a_1| = |p - 1|$ at $q = 0$. From the second equivalent form of $a_2$ in eq. (3.6), it follows that $|a_2| = |p - 1|$ at $q = 0$ also, so that the point $q = 0$ is a solution to eq. (3.9) and hence is on the locus $\mathcal{B}$, which proves (B2).

Property (B3) can be proved by explicit solution of (3.9). One also observes from the results presented in Figs. 2-9 that in cases where $\mathcal{B}$ does not contain any multiple points\footnote{In the technical terminology of algebraic geometry \cite{23}, a multiple point on an algebraic curve is a point where several branches of the curve intersect. If all $n_i$ of the branches have different tangents, the multiple point is said to have index $n_i$.}, the number of regions, $N_{\text{reg.}}$, and the number of connected components on $\mathcal{B}$ satisfy the relation $N_{\text{reg.}} = N_{\text{comp.}} + 1$. As was the case in our earlier work \cite{21}, one can get a general upper bound on the number $N_{\text{comp.}}$ using the Harnack theorem \cite{23} from algebraic geometry, but it is not very restrictive. To do this, we write the equation (3.9) whose solution set is $\mathcal{B}$ out in terms of the real and imaginary components of $q$ or, more conveniently, of $a = q - 1$. This yields a polynomial equation of general degree $d = 2(p + e_g)$ in these components. In the cases without singular (multiple) points on $\mathcal{B}$, the Harnack theorem then gives the upper bound $N_{\text{comp.}} \leq h + 1$, where $h$, the genus of the algebraic curve comprising $\mathcal{B}$, is $h = (d - 1)(d - 2)/2$ \cite{23}. Thus, for example, for the lowest case $(e_1, e_2, e_g) = (2, 2, 1)$, the Harnack theorem gives the upper bound $N_{\text{comp.}} \leq 37$. Clearly, this is a very weak upper bound compared with the actual result $N_{\text{comp.}} = 2$ for all families of the form $(e_1, e_2, e_g) = (2, 2, e_g)$. Similar remarks apply for the Harnack bound applied to other cases $(e_1, e_2, e_g)$.

To prove (B4), we observe that for families where $e_s = 1$, because of the factorization (3.7), for any values of $e_\ell$ and $e_g$, the locus $\mathcal{B}$ is the unit circle $|q - 1| = 1$. The result on chromatic zeros follows immediately from (3.7) and the fact that the zeros of the chromatic polynomial of the circuit graph lie exactly on the circle $|q - 1| = 1$ \cite{15}.

To prove property (B5), we first observe that since $p$ is even, the left-hand side of eq. (3.10) has the value 2 for $q = 2$, i.e., for $a = 1$. The condition that $p$ is even means that either both $e_1$ and $e_2$ are even or both $e_1$ and $e_2$ are odd. In both of these cases, the right-hand side of eq. (3.10) is also equal to 2 for $q = 2$, so that $q = 2$ is a solution of this equation. Furthermore one easily checks that (3.10) has no solution for larger real $q$. This yields (B5).

For property (B6) pertaining to odd $p$, we recall that the case $e_s = 1$ is already covered by (B4); for $e_s \geq 2$, we first note that since $p$ is odd, one of $(e_1, e_2)$ is odd and the other is
even; these are denoted $e_{\text{even}}$ and $e_{\text{odd}}$. We next observe that for odd $p = 2k+1$, $D_p$ contains a factor $(q - 2)$ times a polynomial:

$$D_{2k+1} = (q - 2)F_{2k+1}$$  \hfill (3.16)

where

$$F_{2k+1} = k \quad \text{at} \quad q = 2$$  \hfill (3.17)

Now let us consider the limit $q \to 2$. From eqs. (3.16) and (3.17), we have

$$a_1 \to \frac{(q - 2)(p - 1)}{2} \quad \text{as} \quad q \to 2$$  \hfill (3.18)

Furthermore,

$$a_2 \to \frac{1}{2}(-1)^{e_g+1}(q - 2)(1 + e_{\text{even}} - e_{\text{odd}}) \quad \text{as} \quad q \to 2$$  \hfill (3.19)

Hence for the chromatic polynomial (3.4),

$$P(G_{e_{\text{even}},e_{\text{odd}},e_g,m}, q \to 2) \to \left[ \frac{(q - 2)(p - 1)}{2} \right]^m \left[ 1 + \left( \frac{(-1)^{e_g+1}(1 + e_{\text{even}} - e_{\text{odd}})}{p - 1} \right)^m \right]$$  \hfill (3.20)

Since

$$\frac{|1 + e_{\text{even}} - e_{\text{odd}}|}{p - 1} < 1$$  \hfill (3.21)

for the relevant case $e_s \geq 2$ considered here, it follows that, taking the $m \to \infty$ limit first, before the $q \to 2$ limit, the contribution of the second term in the square brackets, i.e., the $a_2$ term, in eq. (3.20), relative to that of the first, i.e., $a_1$ term, goes to zero. Recalling our order of limits (1.2), it follows that at $q = 2$, the function $W$ is determined completely by the $a_1$ term. Hence, although $a_1$ and $a_2$ both vanish at $q = 2$, this point is not on the locus $B$ since $W$ is determined only by $a_1$ at this point. This shows that the condition of the degeneracy of terms $|a_1(q)| = |a_2(q)|$ is necessary in order that the point $q$ lie on the locus $B$, but, as noted in connection with eq. (3.9), a further necessary (and sufficient) condition is that this equality in magnitudes holds where $a_1$ and $a_2$ are both nonzero leading terms. Since $W$ is determined by $a_1$ at $q = 2$, the same as in region $R_1$ (see below), this shows that $q_c < 2$, which was to be proved.

The other property in (B6) follows in a similar manner from the solution to eq. (3.10). The fact that $q_c \to 2$, i.e., $a_c \to 1$, where $a_c = q_c - 1$, is clear from (3.10) since if $a > 1$ ($a < 1$), the left-hand side diverges (vanishes) as $e_g \to \infty$, precluding equality with the right-hand side.
We observe several other interesting features of the locus \( B \). For \( e_g = 0 \) and \( e_s \geq 2 \) and for certain values of \((e_1, e_2)\), this locus, as an algebraic curve, may contain one or more multiple point(s), which are of index 2, i.e., they involve a crossing of four curves forming two branches of \( B \). A particularly simple set of families for which this occurs is the set where \( e_1 = e_2 = p/2 \equiv e_{12} \). By an analysis of eq. (3.10), we find that for this set there are

\[
N_{\text{m.p.}} = e_{12} - 1
\]

multiple points, which occur at

\[
q_j = 1 + e^{i\theta_j}
\]

where

\[
\theta_j = \pm \frac{2j\pi}{e_{12}}, \quad j = 0, 1, ..., \frac{e_{12} - 2}{2} \quad \text{for} \quad e_{12} \text{ even}
\]

\[
\theta_j = \pm \frac{(2j + 1)\pi}{e_{12}}, \quad j = 0, 1, ..., \frac{e_{12} - 3}{2} \quad \text{for} \quad e_{12} \text{ odd}
\]

For example, in Fig. (2)(a) for \((e_1, e_2, e_g) = (2, 2, 0)\) one sees that there is one such multiple point, located at \( q = q_c = 2 \), while in Fig. (8)(a) for \((e_1, e_2, e_g) = (3, 3, 0)\) there are 2 complex-conjugate multiple points, located at \( q = 1 + e^{\pm i\pi/3} \). Fig. (8) illustrates another case with a multiple point, namely \((e_1, e_2, e_g) = (2, 4, 0)\), for which the multiple point is at \( q = q_c = 2 \). In Figs. 2-3 and 6-9 one sees that the multiple points that are present for \( e_g = 0 \) disappear when \( e_g \geq 1 \). Figs. 4-5 illustrate a family with \( p \) odd, viz., \((e_1, e_2) = (2, 3)\) for which \( B \) has no multiple points.

In the cases where \( B \) contains multiple point(s) for \( e_g = 0 \), which disappear when \( e_g \geq 1 \), each such disappearance is accompanied by the appearance of a new disconnected component of \( B \). As \( e_g \) increases, the sizes of the regions bounded by these components of \( B \) decrease, and as \( e_g \to \infty \), the regions shrink to points. For example, in Figs. (2) and (3) one sees that for the case \((2, 2, e_g)\), as \( e_g \) increases from \( e_g = 0 \) to \( e_g \geq 1 \), the multiple point at \( q = q_c = 2 \) that was present for \( e_g = 0 \) disappears and there appears a disconnected component of \( B \) defining a new region, so that the number \( N_{\text{comp.}} \) of (separate, disconnected) components of \( B \) increases from 1 to 2. As \( e_g \to \infty \), this region shrinks to a point at \( q = 3/2 \). Similarly, in Figs. (8) and (1) one sees that for the case \((e_1, e_2) = (3, 3)\), as \( e_g \) increases from 0 to \( e_g \geq 1 \), the complex-conjugate multiple points that were present for \( e_g = 0 \) at \( q = 1 + e^{\pm i\pi/3} \) disappear and \( N_{\text{comp.}} \) increases from 1 to 3. As \( e_g \to \infty \), the inner regions shrink to points at \( q \approx 3/2 \pm i/2 \).
Another feature that one observes in the results shown in the figures is that these loci \( B \) do not have any support for \( \text{Re}(q) < 0 \). This is in accord with our conjecture \([19,22]\) that global circuits are a necessary but not sufficient condition for \( B \) to have support for \( \text{Re}(q) < 0 \).

For \( W(\lim_{m \to \infty} G_{e_1,e_2,e_3,m}, q) \), which henceforth will be abbreviated simply as \( W \), we have the following general results. In region \( R_1 \),

\[
W = (a_1)^{\frac{1}{p+e_g-1}} = [(q - 1)^{e_g + 1}D_p]^{\frac{1}{p+e_g-1}} \quad \text{for} \quad q \in R_1
\]  

We note that the function on the right-hand side of eq. (3.26) is the same as that for the open chain, (2.10), although in the latter case, the result holds throughout the full \( q \) plane whereas for the cyclic chain, the result holds only in region \( R_1 \). One can approach the origin, \( q = 0 \) from the left staying within the region \( R_1 \), and in this limit, using the property (3.15), we have

\[
W = (p - 1)^{\frac{1}{p+e_g-1}} \quad \text{for} \quad q \to 0 , \quad q \in R_1
\]  

Let us denote region \( R_2 \) as the one containing the point \( q = 1 \); this region is contiguous to \( R_1 \) at the origin, \( q = 0 \). In region \( R_2 \), \( a_2 \) is the dominant term, so that

\[
|W| = |a_2|^{\frac{1}{p+e_g-1}} \quad \text{for} \quad q \in R_2
\]  

In particular, for families of graphs with \( e_s = 1 \), where the factorization (3.7) holds,

\[
|W| = |D_p|^{\frac{1}{p+e_g-1}} \quad \text{for} \quad e_s = 1 \quad \text{and} \quad q \in R_2
\]  

For cases where there are additional regions \( R_j \) enclosed within \( R_2 \),

\[
|W| = |a_1|^{\frac{1}{p+e_g-1}} \quad \text{for} \quad q \in R_j \quad \text{enclosed within} \quad R_2
\]  

For \( e_g = 0 \) these additional regions are contiguous with \( R_1 \) at multiple points on \( B \).

Concerning values of \( W \) at special points, we observe that since \( q_c \leq 2 \), the expression (3.26) always holds at \( q = 2 \), so that, as a consequence of the property (3.16),

\[
W(q = 2) = 0 \quad \text{for} \quad \text{odd} \quad p
\]  

Note that this does not follow just from the property that the chromatic polynomial \( P \) vanishes at \( q = 2 \) for odd \( p \), but requires the stronger property that \( P \) contains the factor \( (q - 2)^\nu \) with \( \nu \propto n \) as \( n \to \infty \). In contrast, for even \( p = 2k \), using the property

\[
D_{2k} = 1 \quad \text{for} \quad q = 2
\]  

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we have

$$W(q = 2) = 1 \quad \text{for even } p$$

(3.33)

As one approaches \( q = 0 \) from within region \( R_2 \), using (3.13), it follows that

$$|W| = |p - 1|^{\frac{1}{p+e_g-1}} \quad \text{for } q \to 0, \quad q \in R_2$$

(3.34)

in accord with (3.27) and the equality of the magnitude of \(|W|\) across any point on \( B \).

In Figs. (2)-(9) we have also shown the chromatic zeros of \( G_{e_1,e_2,e_g,m} \) for large finite values of \( m \) to compare with the \( m \to \infty \) accumulation set \( B \). We observe first that the chromatic zeros lie quite close to the asymptotic locus \( B \). Second, the density of chromatic zeros appears to vanish at multiple points, just as was the case for multiple points on complex-temperature phase boundaries for which we carried out exact calculations previously [24]. As one increases \( e_g \) above 0, so that new components of \( B \) pinch off, one still sees a remnant of the reduction of density on the sides facing the location of the multiple point that had existed for \( e_g = 0 \).
FIG. 2. Boundary $B$ in the $q$ plane for $W$ function for $\lim_{m \to \infty} G_{e_1, e_2, e_g, m}$ with $(e_1, e_2, e_g) = (a) (2,2,0)$, 
(b) $(2,2,1)$. Chromatic zeros for $m = 14$ (i.e., $n = 42$ and $n = 56$ for (a) and (b)) are shown for comparison.
FIG. 3. Boundary $B$ in the $q$ plane for $W$ function for $\lim_{m \to \infty} G_{e_1, e_2, e_g, m}$ with $(e_1, e_2, e_g) = (a)\ (2,2,2), 
(b)\ (2,2,3)$. Chromatic zeros for $m = 10$ (i.e., $n = 50$ and $n = 60$ for (a) and (b)) are shown for comparison.
FIG. 4. Boundary $B$ in the $q$ plane for $W$ function for $\lim_{m \to \infty} G_{e_1, e_2, e_g, m}$ with $(e_1, e_2, e_g) = (a) (2,3,0)$, (b) (2,3,1). Chromatic zeros for $m = 14$ (i.e., $n = 56$ and $n = 70$ for (a) and (b)) are shown for comparison.
FIG. 5. Boundary $\mathcal{B}$ in the $q$ plane for $W$ function for $\lim_{m \to \infty} G_{e_1, e_2, e_g, m}$ with $(e_1, e_2, e_g) = (a) (2,3,2), (b) (2,3,3)$. Chromatic zeros for $m = 10$ (i.e., $n = 60$ and $n = 70$ for (a) and (b)) are shown for comparison.
IV. CONCLUSIONS

In this paper we have given exact expressions for the chromatic polynomial $P(G, q)$ and the resultant exponent of the ground state entropy for the Potts antiferromagnet in the $n \to \infty$ limit, $W({\{G\}}, q)$ for families of cyclic polygon chain graphs $G = G_{e_1, e_2, e_g, m}$. We have studied several types of limits yielding $n \to \infty$, viz., $L_{eg}$, $L_p$, and $L_m$, the last of which yielded the most interesting results. From these calculations we have derived the continuous locus, $B$, of nonanalyticities of $W$, which is also the accumulation set of the zeros of the chromatic polynomial in the $n \to \infty$ limit. Various properties of this locus were proved. A comparison with the results for the open polygon chain graph shows the important effect of global circuits. The results of the present study agree with and extend our earlier inferences concerning the locus $B$, in particular, that a sufficient condition that in the $n \to \infty$ limit the locus $B$ separates the $q$ plane into two or more regions is that the graph has a global circuit with $\lim_{n \to \infty} \ell_{g.c.} = \infty$.

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V. APPENDIX

In this appendix we present results for the families of cyclic polygon chain graphs with $(e_1, e_2) = (2, 4)$ and $(3,3)$, and, for each case, $0 \leq e_g \leq 3$. 
FIG. 6. Boundary $B$ in the $q$ plane for $W$ function for $\lim_{m \to \infty} G_{e_1, e_2, e_g, m}$ with $(e_1, e_2, e_g) = (a) (2, 4, 0)$, (b) $(2, 4, 1)$. Chromatic zeros for $m = 10$ (i.e., $n = 50$ and $n = 60$ for (a) and (b)) are shown for comparison.
FIG. 7. Boundary $B$ in the $q$ plane for $W$ function for $\lim_{m \to \infty} G_{e_1,e_2,e_g,m}$ with $(e_1,e_2,e_g) = (a)\ (2,4,2), (b)\ (2,4,3)$. Chromatic zeros for (a) $m = 8$ (i.e., $n = 56$) and (b) $m = 6$ (i.e., $n = 48$) are shown for comparison.
FIG. 8. Boundary $\mathcal{B}$ in the $q$ plane for $W$ function for $\lim_{m \to \infty} G_{e_1, e_2, e_g, m}$ with $(e_1, e_2, e_g) = (a) (3,3,0), (b) (3,3,1)$. Chromatic zeros for $m = 10$ (i.e., $n = 50$ and $n = 60$ for (a) and (b)) are shown for comparison.
FIG. 9. Boundary $\mathcal{B}$ in the $q$ plane for $W$ function for $\lim_{m \to \infty} G_{e_1,e_2,e_g,m}$ with $(e_1, e_2, e_g) = (a) (3,3,2), (b) (3,3,3)$. Chromatic zeros for $m = 8$ (i.e., $n = 56$) and $n = 64$ for (a) and (b) are shown for comparison.
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