On Certain Diophantine Equations of Diagonal Type

Andrew Bremner and Maciej Ulas

Abstract. In this note we consider Diophantine equations of the form

\[ ax^p - by^q = cz^r - dw^s, \quad \text{where } \frac{1}{p} + \frac{1}{q} + \frac{1}{r} + \frac{1}{s} = 1, \]

with even positive integers \( p, q, r, s \). We show that in each case the set of rational points on the underlying surface is dense in the Zariski topology. For the surface with \((p, q, r, s) = (2, 6, 6, 6)\) we prove density of rational points in the Euclidean topology. Moreover, in this case we construct infinitely many parametric solutions in coprime polynomials. The same result is true for \((p, q, r, s) \in \{(2, 4, 8, 8), (2, 8, 4, 8)\}\). In the case \((p, q, r, s) = (4, 4, 4, 4)\), we present some new parametric solutions of the equation \(x^4 - y^4 = 4(z^4 - w^4)\).

1. Introduction

It is well known that the Diophantine equation \(x^4 + y^4 = z^4 + w^4\) has infinitely many integer solutions (Euler). In fact, as was proved by Swinnerton-Dyer, this diophantine equation has infinitely many rational parametric solutions [13]. This implies that the set of rational points on the surface defined by this equation is dense in all real points. The same is true for the equation \(x^4 + y^4 + z^4 = w^4\) as was proved by Elkies in [4]. It is also classically known that the diophantine equation \(x^6 + y^6 - z^6 = w^2\) has infinitely many integer solutions. According to Dickson [3] this result was obtained by Rignaux by construction of two polynomial solutions. In a recent paper we extended these investigations to equations

\[(1) \quad x^p \pm y^q = \pm z^r \pm w^s, \quad p, q, r, s \in \mathbb{N}_+, \quad \text{where } \frac{1}{p} + \frac{1}{q} + \frac{1}{r} + \frac{1}{s} = 1.\]

We call the above equation an equation of diagonal type. In [1] we considered forty one equations (those not considered by Euler, Rignaux and Elkies) corresponding to configurations of signs and exponents. We showed that for all but one instance the existence of real solutions implies the existence of infinitely many rational solutions. The remaining equation which we were unable to treat is \(w^2 = -x^6 - y^6 + z^6\).

In the range \(x + y + z < 5000\) we know only one numerical solution \((x, y, z, w) = (28, 44, 57, 162967)\) and Noam Elkies [5] has informed us that this is the only solution with \(x, y \leq 2^{15}\).

In this paper we study the natural generalization of the equation (1) in the form

\[(2) \quad a(x^p - y^q) = b(z^r - w^s), \quad p, q, r, s \in \mathbb{N}_+, \quad \text{where } \frac{1}{p} + \frac{1}{q} + \frac{1}{r} + \frac{1}{s} = 1.\]

2010 Mathematics Subject Classification. 11D57, 11D85.

Key words and phrases. rational points, diagonal equations, Zariski density.
Here $a, b$ are fixed nonzero integer numbers. It is clear that we can assume that $gcd(a, b) = 1$. The aim of this paper is to show that for each of the quadruplets $(p, q, r, s)$ with $p, q, r, s$ even, the diophantine equation (2) has infinitely many rational parametric solutions. Of course, there are trivial parametric solutions of (2) corresponding to $x^p = y^q$, $z^r = w^s$, etc., representing lines and curves on the surface in weighted projective space; but these solutions are not of interest to us, and henceforth we discard such possibilities.

In Section 2 we generalize Rignaux’s result by proving that when $(p, q, r, s) = (2, 6, 6, 6)$, then there are infinitely many polynomial solutions to (2) in coprime polynomials $x, y, z, w$. A similar result is proved in Section 3 for $(p, q, r, s) = (2, 4, 8, 8), (2, 8, 4, 8)$. In Section 4 we investigate the three equations (2) that arise from permutations of $(p, q, r, s) = (2, 4, 6, 12)$. We show there are infinitely many coprime polynomial solutions of (2) for $(p, q, r, s) = (2, 4, 6, 12)$, and show there are infinitely many polynomial solutions, not necessarily coprime, for $(p, q, r, s) = (2, 4, 6, 12), (2, 12, 4, 6)$. The basic idea in these three sections is to use an elliptic fibration on the corresponding surface in weighted projective space. In Section 5 we consider the case $(p, q, r, s) = (4, 4, 4, 4)$, which needs different techniques from elementary arithmetic algebraic geometry. It devolves into cases where $a/b$ is not a square; $a/b$ is a square mod fourth powers, not 1 or 4; and $a/b$ mod fourth powers equals 1 or 4. In the last two sections we consider some generalizations of the preceding results.

2. The equation $a(x^2 - y^6) = b(z^6 - w^6)$

In this section we construct polynomial solutions of the equation

$$a(x^2 - y^6) = b(z^6 - w^6),$$

where $a, b$ are coprime integers. The following theorem generalizes the result of Rignaux [10].

**Theorem 2.1.** Fix $a, b \in \mathbb{Z} \setminus \{0\}$. The diophantine equation $a(x^2 - y^6) = b(z^6 - w^6)$ has infinitely many solutions in coprime polynomials $x, y, z, w \in \mathbb{Z}[t]$. Moreover, the set of rational points on the surface $S : a(x^2 - y^6) = b(z^6 - w^6)$ is dense in the Euclidean topology.

**Proof.** Set $x = y^3 + t(z^3 - w^3)$, for an indeterminate $t$ assumed throughout to be non-zero. Then

$$a(x^2 - y^6) - b(z^6 - w^6) = (z^3 - w^3)(2aty^3 + (at^2 - b)z^3 - (at^2 + b)w^3),$$

and the problem reduces to investigation of the smooth cubic curve

$$C : 2aty^3 = -(at^2 - b)z^3 + (at^2 + b)w^3.$$ 

Since the equation defining the curve $C$ is homogeneous, a point $(y : z : w) \in C(\mathbb{Q}(t))$ may be assumed to have coordinates which are polynomials in $t$. Localizing at $t = 2$ results in $4ay^3 = -(4a - b)z^3 + (4a + b)w^3$, which cannot have solutions in $\mathbb{Q}(a, b)$ since at $a = b = 1$ the equation becomes $3z^3 + 4y^3 + 5(-w^3) = 0$ over $\mathbb{Q}$, the famous Selmer cubic with no non-zero rational solution. Thus the set $C(\mathbb{Q}(t))$ is empty for generic $a, b$.

Consider the non-invertible change of variable given by $\varphi : \mathbb{Q} \to \mathbb{Q}$ where $\varphi(t) = t^3$; note that the set $\varphi(\mathbb{Q})$ is dense in $\mathbb{Q}$, which property will be used later. The
corresponding curve $C_\varphi$ takes the form

$$C_\varphi : 2at^3y^3 = -(at^6 - b)z^3 + (at^6 + b)w^3,$$

containing the point $P(y, z, w) = (t, -1, 1)$, and hence is an elliptic curve over $\mathbb{Q}(t)$. The tangent to $C_\varphi$ at $P$ meets the curve again at $Q(y, z, w) = (2bt, 3at^6 + b, 3at^6 - b)$, and

$$\begin{align*}
(x, y, z, w) &= (2bt^3(27a^2t^{12} + 5b^2), 2bt, 3at^6 + b, 3at^6 - b)
\end{align*}$$

shows us that the equation $b$ has infinitely many solutions in coprime integers.

To prove that there are infinitely many polynomial solutions, note that with $P$ as the origin of the group law, then $C_\varphi$ is birationally equivalent to the elliptic cubic with Weierstrass equation

$$E : y^2 = x^3 - 27a^2(at^6 - b)^2(2at^6 + b)^2.$$ 

The image of $Q$ on $E$ is the point

$$\begin{align*}
R &= ((3a^2t^{12} + b^2)/t^4, b(-9a^2t^{12} + b^2)/t^6);
\end{align*}$$

and by characterization of the torsion points on curves of type $y^2 = x^3 + D$ (see, for example, Silverman [11, p. 323], we have that $R$ is of infinite order. The points $mR$, for $m = 2, 3, 4, \ldots$, pull back to points $(y_m, z_m, w_m)$ on $C_\varphi$, with $\gcd(y_m, z_m, w_m) = 1$, and then $(x_m, y_m, z_m, w_m) = (y_m^3 + t^3(z_m^3 - w_m^3), y_m, z_m, w_m)$ gives infinitely many coprime polynomial solutions of $b$.

We will prove that the set of rational points on the surface $S : a(x^2 - y^6) = b(z^6 - w^6)$ is dense in the Euclidean topology. However, we first prove Zariski density of the set of rational points. It is clear that in order to prove this result it is enough to prove that the set of rational points is dense in the Zariski topology on $C_\varphi$. Because the curve $E$ is of positive rank over $\mathbb{Q}(t)$, the set of multiples of the point $R$, i.e. $mR = (X_m(t), Y_m(t))$ for $m = 1, 2, \ldots$, gives infinitely many $\mathbb{Q}(t)$-rational points on the curve $E$. Now, regarding $E$ as an elliptic surface in the space with coordinates $(X, Y, t)$ we see that each rational curve $(X_m, Y_m, t)$ is included in the Zariski closure, say $\mathcal{R}$, of the set of rational points on $E$. Because this closure consists of only finitely many components, it has dimension two, and as the surface $E$ is irreducible, $\mathcal{R}$ is the whole surface. Thus the set of rational points on $E$ is dense in the Zariski topology and the same is true for $C_\varphi$ and thus for $C$ and $S$. This follows from the fact that $C_\varphi$ comes from $C \simeq S$ after a non-invertible change of variable.

To obtain the density of the set $E(\mathbb{Q})$ in the Euclidean topology, we use two results: a theorem of Hurwitz [6] (see also [12, p. 78]) and a theorem of Silverman [11, p. 368]. The theorem of Hurwitz states that if an elliptic curve $E$ defined over $\mathbb{Q}$ has positive rank and at most one torsion point of order two (defined over $\mathbb{Q}$) then the set $E(\mathbb{Q})$ is dense in $E(\mathbb{R})$. The same result holds if $E$ has three torsion points of order two under the assumption that we have a rational point of infinite order on the bounded branch of the set $E(\mathbb{R})$.

Silverman’s theorem states that if $E$ is an elliptic curve defined over $\mathbb{Q}(t)$ with positive rank, then for all but finitely many $t_0 \in \mathbb{Q}$, the curve $E_{t_0}$ obtained from the curve $E$ by specialization at $t = t_0$ has positive rank. From this result we see that for all but finitely many $t \in \mathbb{Q}$ the elliptic curve $E_t$ is of positive rank. Let us denote by $\mathcal{G}$ the set of $t \in \mathbb{Q}$ such that the specialization $R_t$ of the point $R$ at $t$ is of finite order on the curve $E_t$. From the remark at the beginning of this
section we know that the order of a torsion point on the curve $E_t$ is at most six. Thus, in order to find $G$ it is enough to find all $t \in \mathbb{Q}$ such that $R_t$ has finite order $\leq 6$, that is, we need to characterize all $t \in \mathbb{Q}$ such that $mR_t = \mathcal{O}$ for some $m \in \{1, 2, 3, 4, 6\}$. The assumption on $a, b$ immediately implies that there are at most two specializations of $t \in \mathbb{Q}$ such that the $Y$ coordinate of the point $R$ is equal to zero, i.e. the order of the point $R$ is two, and these specializations correspond to the rational roots of the equation $(3at^6 - b)(3at^6 + b) = 0$. The vanishing of the 3-division polynomial of $E$ evaluated at $X(R_t)$ determines a polynomial of degree 4 in $t^{12}$, so there can be at most 8 rational roots $t$, and at most 8 values of $t \in \mathbb{Q}$ leading to $R_t$ of order three. Similarly, the vanishing of the 4-division polynomial at $X(R_t)$ determines a polynomial of degree 8 in $t^{12}$, so there are at most 16 rational values of $t$ leading to $R_t$ of order four. Finally, a point of order six can occur if and only if $-27a^2(at^6 - b)^2(at^6 + b)^2$ equals a sixth power, which is clearly impossible. To sum up, there are at most $2 + 8 + 16 = 26$ rational specializations of $t$ such that the point $R_t$ has finite order, and thus $|G| \leq 26$ (note also that $t = 0$ is forbidden).

Using the Silverman theorem we deduce that for all $t \in \mathbb{Q} \setminus G$ the curve $E_t$ is of positive rank. The Hurwitz theorem now implies that the set $E_t(\mathbb{Q})$ is dense in $E_t(\mathbb{R})$. It follows that $C_\infty(t)(\mathbb{Q})$ is dense in $C_\infty(t)(\mathbb{R})$ because the image of $\mathbb{Q}$ by the function $\varphi(t) = t^3$ is dense in the Euclidean topology of $\mathbb{R}$ (since $\varphi(\mathbb{Q}) = \mathbb{R}$). If we put now $H = \varphi(\mathbb{Q}) \subset \mathbb{R}$ we get that for all but finitely many $t \in H$ the set $C_t(\mathbb{Q})$ is dense in $C_t(\mathbb{R})$. Because $H$ is dense in $\mathbb{R}$, it follows that the set $S(\mathbb{Q})$ of rational points on $S$ is dense in the Euclidean topology in $S(\mathbb{R})$. The theorem is proved.

**Corollary 2.2.** Fix $a, b \in \mathbb{Z} \setminus \{0\}$ and $n \in \mathbb{N}_+$. The diophantine equation

$$a(x^2 - y^{6n}) = b(z^6 - w^6)$$

has infinitely many solutions in integers. Moreover, if $b = 1$, then the equation has infinitely many solutions in coprime integers.

**Proof.** Substituting $t = (2b)^{n-1}T^n$ into the expression for $x, y, z, w$ in (4) we get a polynomial solution of (4). Unfortunately this solution has the common factor $b$. However, if $b = 1$ then we get a solution of the equation $a(x^2 - y^{6n}) = z^6 - w^6$ in the following form:

$$x = 2^{3n-2}T^{3n}(27a^22^{12(n-1)}T^{12n} + 5), \quad y = 2T,$$

$$z = 3a2^{6(n-1)}T^{6n} + 1, \quad w = 3a2^{6(n-1)}T^{6n} - 1.$$  

This solution clearly satisfies the condition $1 = \gcd(z, w) = \gcd(x, y, z, w)$. □

3. The equation $a(x^p - y^q) = b(z^r - w^s)$ with $(p, q, r, s) = (2, 4, 8, 8), (2, 8, 4, 8)$

In this section we construct polynomial solutions of the equation

$$a(x^p - y^q) = b(z^r - w^s),$$

where $a, b$ are coprime integers and $(p, q, r, s) \in \{(2, 4, 8, 8), (2, 8, 4, 8)\}$.

First, a simple lemma.

**Lemma 3.1.** Consider the elliptic quartic $E : y^2 = ax^4 + \beta, \alpha + \beta = \gamma^2, \alpha, \beta, \gamma \in \mathbb{Z}$. If $\alpha \beta \neq \square$ and $\alpha \beta \neq -1$ mod fourth powers, then the curve has torsion group of order 2; and since $E$ contains the four points $(\pm1, \pm\gamma)$, then $E$ has positive rank.
The curve $\mathcal{C}$ induction using the addition formula that $mQ$ in terms of $(\ )$ for $m$ so that $x$ order. We have

Let $t$ for an indeterminate $Y$.

Proof. Consider first the case $(p, q, r, s) = (2, 4, 8, 8)$. Set $x = -y^2 + t^2(z^4 + w^4)$, for an indeterminate $t$. Then

$$a(x^2 - y^4) - b(z^8 - w^8) = -(z^4 + w^4)(2at^2y^2 - (at^4 - b)z^4 - (at^4 + b)w^4).$$

We prove the desired result by showing that the curve

$$C : 2at^2y^2 = (at^4 - b)z^4 + (at^4 + b)w^4$$

has infinitely many solutions in the ring of polynomials $\mathbb{Z}[t]$. Note that if $y, z, w \in \mathbb{Z}[t]$ satisfy this equation and $t \not| \gcd(z, w)$ then $\gcd(z, w)^2$ divides $y$. This implies that we can assume $\gcd(y, z, w) = 1$ in $\mathbb{Z}[t]$. Thus the solution will be in coprime polynomials provided $t \not| \gcd(z, w)$.

Now $C$ contains the points $(y, z, w) = (\pm t, \pm 1, 1)$. If we take $(t, 1, 1)$ as the point at infinity, then by the remarks of Lemma 3.1, the point $Q = (-t, 1, 1)$ is of infinite order. We have

$$2Q(y, z, w) = (t(-3b^4 + 4a^4t^{16}), b^2 - 2abt^4 - 2a^2t^8, b^2 + 2abt^4 - 2a^2t^8),$$

so that

$$x = t^2(-7b^8 + 32a^2b^6t^8 - 88a^4b^4t^{16} + 128a^6b^2t^{24} + 16a^8t^{32}),$$

$$y = t(-3b^4 + 4a^4t^{16}),$$

$$z = b^2 - 2abt^4 - 2a^2t^8,$$

$$w = b^2 + 2abt^4 - 2a^2t^8,$$

gives infinitely many solutions of our equation in coprime integers.

Let $mQ = (y_m, z_m, w_m), m = 1, 2, 3, \ldots$. The recurrence formulae that determine $mQ$ in terms of $(m - 1)Q$ are necessarily complicated, but it may be checked by induction using the addition formula that

$$z_m \equiv w_m \equiv b^{m^2-m} \mod t, \quad y_m \equiv 0 \mod t,$$

for $m = 1, 2, 3, \ldots$. Hence $t \not| \gcd(z_m, w_m)$, and the points $mQ$, together with $x_m = -y^2 + t^2(z^4 + w^4)$, lead to infinitely many coprime polynomial solutions.

The proof in the case $(p, q, r, s) = (2, 8, 4, 8)$ is similar. This time set $x = -y^4 + t^4(z^2 + w^4)$ and get

$$a(x^2 - y^8) - b(z^4 - w^8) = -(z^2 + w^4)(bz^2 - bw^4 + 2ay^4t^4 - az^2t^8 - aw^4t^8).$$

The curve

$$C : (b - at^8)z^2 = (b + at^8)w^4 - 2at^4y^4$$
has points \((\pm y, \pm z, w) = (t, 1, 1)\), and so by Lemma \([5.1]\) on taking \((t, 1, 1)\) as the origin, the point \(Q = (-t, 1, 1)\) is of infinite order. Then \(2Q\) gives
\[
x = t^4 (-79b^6 + 600a^2 t^8 + 3068a^2 b^6 t^{16} + 18984a^3 b^5 t^{24} + 101126a^4 b^4 t^{32} \\
+ 155112a^5 b^3 t^{40} + 172604a^6 b^2 t^{48} + 109848a^7 b t^{56} + 28561a^8 t^{64}),
\]
\[
y = t (3b^2 + 6ab t^8 + 13a^2 t^{16}),
\]
\[
z = b^4 - 52ab^3 t^8 - 138a^2 b^2 t^{16} - 340a^3 b t^{24} - 239a^4 t^{32},
\]
\[
w = b^2 + 14ab t^8 + a^2 t^{16}.
\]
With similar reasoning to the previous case, we easily deduce the existence of infinitely many coprime polynomial solutions to the equation \(a(x^2 - y^8) = b(z^4 - w^8)\).

There is an immediate consequence from the above results.

**Corollary 3.3.** Consider the surfaces
\[
S_1 : a(x^2 - y^4) = b(z^8 - 1), \quad S_2 : a(x^2 - y^8) = b(z^4 - 1).
\]
The set of rational points on \(S_i\) is Zariski dense for \(i = 1, 2\).

**Proof.** This follows from the existence of infinitely many rational curves lying on \(S_i, i = 1, 2\), and the reasoning given in the proof of Theorem 2.1. \(\square\)

4. The equation \(a(x^p - y^q) = b(z^r - w^s)\) with 
\((p, q, r, s) \in \{(2, 4, 6, 12), (2, 6, 4, 12), (2, 12, 4, 6)\}\n
In this section we are interested in constructing polynomial solutions of the equation
\[
a(x^p - y^q) = b(z^r - w^s),
\]
where \(a, b\) are coprime integers and \((p, q, r, s) \in \{(2, 4, 6, 12), (2, 6, 4, 12), (2, 12, 4, 6)\}\). We prove the following:

**Theorem 4.1.** Let \(a, b \in \mathbb{Z} \setminus \{0\}\). The diophantine equation \(a(x^2 - y^4) = b(z^6 - w^{12})\) has infinitely many coprime solutions in the ring of polynomials \(\mathbb{Z}[t]\).

**Proof.** Set \(x = -y^2 + t^6(z^3 + w^6)\), where \(t\) is an indeterminate, and get
\[
a(x^2 - y^4) - b(z^6 - w^{12}) = -(z^3 + w^6)(2at^6y^2 - (at^{12} - b)z^3 - (at^{12} + b)w^6).
\]
The problem reduces to investigating the curve
\[
C : 2at^6y^2 = (at^{12} - b)z^3 + (at^{12} + b)w^6,
\]
with Weierstrass form
\[
\mathcal{E} : Y^2 = X^3 + 8a^3(at^{12} - b)^2(at^{12} + b)
\]
under the mapping
\[
(X, Y) = \left(\frac{2a(at^{12} - b)z}{w^2}, \frac{4a^2 t^3(at^{12} - b)y}{w^3}\right).
\]
There is the obvious point \( P(z, w, y) = (1, 1, t^3) \) mapping to \( Q = (2a(at^{12} - b), -4a^2t^6(at^{12} - b)) \) on \( E \). Just as for the point \( R \) at \( 1 \), we can show that \( Q \) is of infinite order. The point \( 2Q \) leads to the following solution of \( a(x^2 - y^4) = b(z^6 - w^{12}) \):

\[
\begin{align*}
x &= -4a^2(729b^4 + 2430ab^3t^{12} + 3024a^2b^2t^{24} + 2178a^3bt^{36} - 169a^4t^{48}), \\
y &= 2a(-27b^2 - 18abt^{12} + 13a^2t^{24}), \\
z &= -2at^2(9b + 7at^{12}), \\
w &= 4at^7,
\end{align*}
\]

and in general, we construct infinitely many polynomial solutions by pulling back multiples of \( Q \). If \( z, w \) are both divisible by an irreducible polynomial \( f(t) \) coprime to \( t(at^{12} - b) \), then necessarily \( f^2 | z, f^3 | y \), and we have the polynomial solution \( (z/f^2, w/f, y/f^3) \). To give infinitely many polynomial solutions with coprime \( (x, y, z, w) \) it therefore suffices to give an infinite family where \( y \) is coprime to \( t(at^{12} - b) \). Inductively, we claim that the pull-backs of points \( 2^nQ, n = 1, 2, 3, ..., \) satisfy this condition.

Let \( (z_n, w_n, y_n) = (2(z_{n-1}, w_{n-1}, y_{n-1}) \) on \( C \), with \( (z_0, w_0, y_0) = (1, 1, t^3) \). The duplication formula gives

\[
\begin{align*}
z_n &= 2at^2z_{n-1} - (at^{12} - b)z_{n-1}^3 - 8(at^{12} + b)w_{n-1}^6, \\
w_n &= 4at^4w_{n-1}y_{n-1}, \\
y_n &= 2a(-at^{12} - b)^2z_{n-1}^6 - 20(at^{12} + b)(at^{12} - b)z_{n-1}^3w_{n-1}^6 + 8(at^{12} + b)^2w_{n-1}^6
\end{align*}
\]

Modulo \( at^{12} - b \), it follows that

\[y_n \equiv 64ab^{12}w_{n-1}^4 \equiv 64a^2by_{n-1}^4.
\]

Since \( y_0 = t^3 \), it immediately follows that \( y_n \) is coprime to \( at^{12} - b \) for \( n \geq 0 \). Assume now that \( t^{21} || z_{n-1}, t^{12} || w_{n-1}, \) and \( t \not| y_{n-1} \); this is certainly true in the case \( n = 2 \) from the above particular solution. The recurrence relations above show that \( t^{10} || z_n, t^{11} || w_n, t^{12} || y_n \). Replacing \( (z, w, y) \) by \( (z/f^2, w/t^4, y/t^{12}) \) gives a reduced polynomial solution in which \( t^2 || z_n, t^7 || w_n, \) as required.

**Remark 4.2.** Unfortunately, we are unable to find coprime polynomial solutions of the equation \( a(x^p - y^q) = b(z^r - w^s) \) with \( (p, q, r, s) = (2, 6, 4, 12), (2, 12, 4, 6) \). However, we do construct polynomial solutions of these equations. First, consider the equation \( a(x^p - y^q) = b(z^r - w^s) \) with \( (p, q, r, s) = (2, 6, 4, 12) \). Set \( x = y^3 + t^6(z^2 + w^6) \) to get

\[a(x^2 - y^6) - b(z^4 - w^{12}) = (z^2 + w^6)(2at^6y^3 + (at^{12} - b)z^2 + (at^{12} + b)w^6),
\]

and our problem is reduced to investigation of the curve

\[C: (at^{12} - b)z^2 = -2at^6y^3 - (at^{12} + b)w^6.
\]

The curve \( C \) is birationally equivalent with the Weierstrass cubic:

\[E: Y^2 = X^3 - 4a^2(at^{12} - b)^3(at^{12} + b),
\]

under the mapping

\[X, Y = \left(\frac{-2at^2(at^{12} - b)y}{w^2}, \frac{2a(at^{12} - b)^2z}{w^3}\right).
\]
There is an obvious point \((y, w, z) = (-t^2, 1, 1)\) on \(C\) mapping to the point 
\[ Q = (2at^4(at^{12} - b), 2a(at^{12} - b)^2), \]
on \(E\), which as before is of infinite order. The point \(2Q\) leads to the following solution of the diophantine equation \(a(x^2 - y^6) = b(z^4 - w^{12})\):
\[
\begin{align*}
  x &= 8t^6(at^{12} - b)^2(125a^4t^{48} + 409a^3bt^{36} + 588a^2b^2t^{24} + 256ab^3t^{12} + 80b^4), \\
  y &= -2t^2(at^{12} - b)(5at^{12} + 4b), \\
  z &= -4(at^{12} - b)(11a^2t^{24} + 14abt^{12} + 2b^2), \\
  w &= 2(at^{12} - b).
\end{align*}
\]

Again, we can write down infinitely many polynomial solutions (not necessarily coprime).

Second, consider the equation \(a(x^2 - y^{12}) = b(z^4 - w^6)\). Set \(x = y^6 - t^6(z^2 - w^3)\), where \(t\) is an indeterminate, and get
\[
a(x^2 - y^{12}) - b(z^4 - w^6) = (w^3 - z^2)((at^{12} + b)w^3 + 2at^6y^6 - (at^{12} - b)z^2),
\]
and our problem is reduced to investigating the curve
\[ C : \ (at^{12} - b)z^2 = (at^{12} + b)w^3 + 2at^6y^6. \]
The curve \(C\) is birationally equivalent with the Weierstrass cubic
\[ E : \ Y^2 = X^3 + 2a(at^{12} - b)^3(at^{12} + b)^2 \]
under the mapping
\[
(X, Y) = \left( \frac{(at^{12} + b)(at^{12} - b)w}{t^2y^2}, \frac{(at^{12} - b)^2(at^{12} + b)z}{t^3y^3} \right).
\]
The obvious point \((w, y, z) = (-1, t, 1)\) on \(C\) maps to
\[
Q = \left( \frac{-(at^{12} - b)(at^{12} + b)}{t^4}, \frac{-(at^{12} - b)^2(at^{12} + b)}{t^6} \right)
\]
on \(E\), and as before is of infinite order. The point \(2Q\) leads to the following solution of the diophantine equation \(a(x^2 - y^{12}) = b(z^4 - w^6)\):
\[
\begin{align*}
  x &= -2t^6(at^{12} - b)^2(32a^4t^{48} + 4849a^3bt^{36} + 867a^2b^2t^{24} + 115ab^3t^{12} - 31b^4), \\
  y &= 2t(at^{12} - b), \\
  z &= (at^{12} - b)(-71a^2t^{24} - 38abt^{12} + b^2), \\
  w &= (at^{12} - b)(17at^{12} + b).
\end{align*}
\]
Infinitely many (not necessarily coprime) polynomial solutions are constructed by taking the pullbacks of \(mQ\), \(m = 2, 3, 4, \ldots\).

As a consequence of Theorem 3.1, Remark 3.2, and the reasoning as in Corollary 3.4, we have the following.

**Corollary 4.3.** Consider the surfaces
\[ S_i : \ a(x^2 - y^4) = b(z^6 - 1), \quad S_4 : \ a(x^2 - y^6) = b(z^4 - 1), \quad S_5 : \ a(x^2 - 1) = b(z^4 - w^6) \]
The set of rational points on \(S_i\) is Zariski dense for \(i = 3, 4, 5\).
5. The equation \( a(x^4 - y^4) = b(z^4 - w^4) \)

We take the equation in the form
\[
V : x^4 - y^4 = h(z^4 - w^4),
\]
representing a surface in projective three-space. Choudhry [2] has some elementary results showing how to derive new points from known points, essentially arising from elliptic fibrations of the surface of type
\[
x^2 - y^2 = t(z^2 - w^2), \quad t(x^2 + y^2) = h(z^2 + w^2),
\]
where the known point means the intersection of the two quadrics is an elliptic curve.

We ask whether there exist parametrizable curves on \( V \), that is, curves of geometric genus 0. We are only able to treat the case of curves that have arithmetic genus 0, hence geometric genus 0, and hence parametrizable. The arguments we use are those of Swinnerton-Dyer [13].

It is known (see for example Pinch & Swinnerton-Dyer [5]) that there are precisely 48 straight lines on the surface \( V \), given as follows:
\[
\begin{align*}
(x = \alpha_1 y, z = \beta_1 w), & \quad \alpha_1^4 = \beta_1^4 = 1, \\
(x = \alpha_2 y, y = \beta_2 w), & \quad \alpha_2^4 = \beta_2^4 = h, \\
(x = \alpha_3 y, y = \beta_3 w), & \quad \alpha_3^4 = \beta_3^4 = -h.
\end{align*}
\]
further, the Néron-Severi group of \( V \) over \( \mathbb{C} \) is generated by the classes of these lines.

Write \( h = \theta^4 \). The hyperplane \( x - y - \theta (z - w) = 0 \) cuts the surface \( V \) in the lines \( (x = y, z = w), (x = \theta z, y = \theta w) \), with residual intersection the irreducible conic
\[
x^2 - xy + 2y^2 - \theta xz + 3\theta yz + 2\theta^2 z^2 = 0.
\]
In this way we generate 128 distinct irreducible conics on \( V \), typified by the above.

In the first instance, we suppose \( h \) is not a perfect rational square. Pinch & Swinnerton-Dyer show that the Néron-Severi group of \( V \) over \( \mathbb{Q} \) is of rank 6. It is straightforward to show that the group has a \( \mathbb{Z} \)-basis given by (the classes of) the following four lines and two line pairs:
\[
\begin{align*}
\Delta_1 &= (x = y, z = w), & \Delta_2 &= (x = y, z = -w), \\
\Delta_3 &= (x = -y, z = w), & \Delta_4 &= (x = -y, z = -w), \\
\Delta_5 &= (x = y, z = iw) + (x = y, z = -iw), & \Delta_6 &= (x = iy, z = iw) + (x = -iy, z = -iw).
\end{align*}
\]
The corresponding intersection matrix is given in Table 1.

For a curve \( \Gamma \) defined over \( \mathbb{Q} \), set
\[
\Gamma \sim n_1 \Delta_1 + n_2 \Delta_2 + n_3 \Delta_3 + n_4 \Delta_4 + n_5 \Delta_5 + n_6 \Delta_6.
\]
Defining \( g(\Gamma) \) to be the arithmetic genus of \( \Gamma \), we have
\[
2 g(\Gamma) - 2 = (\Gamma \cdot \Gamma) = -2n_1^2 + 2n_1 n_2 + 2n_1 n_3 + 4n_1 n_5 - 2n_2^2 + 2n_2 n_4 + 4n_2 n_5 \\
- 2n_3^2 + 2n_3 n_4 - 2n_4^2 - 2n_5^2 + 4n_5 n_6 - 4n_6^2.
\]
Table 1. Intersection matrix for a $\mathbb{Z}$-basis of $V$ with $h$ not a square

| $\Delta_1$ | $\Delta_2$ | $\Delta_3$ | $\Delta_4$ | $\Delta_5$ | $\Delta_6$ |
|------------|------------|------------|------------|------------|------------|
| $\Delta_1$ | $-2$       | $1$        | $1$        | $0$        | $2$        | $0$        |
| $\Delta_2$ | $1$        | $-2$       | $0$        | $1$        | $2$        | $0$        |
| $\Delta_3$ | $1$        | $0$        | $-2$       | $1$        | $0$        | $0$        |
| $\Delta_4$ | $0$        | $1$        | $1$        | $-2$       | $0$        | $0$        |
| $\Delta_5$ | $2$        | $2$        | $0$        | $0$        | $-2$       | $2$        |
| $\Delta_6$ | $0$        | $0$        | $0$        | $0$        | $2$        | $-4$       |

Writing $\text{deg}(\Gamma) = d = n_1 + n_2 + n_3 + n_4 + 2n_5 + 2n_6$, we have

\[
d^2 - 4(\Gamma \cdot \Gamma) = (-d + 4n_5)^2 + 4(-d + n_2 + 3n_5 + n_3 + 2n_4 + 2n_6)^2 + 4(d - n_5 - n_3)^2 + 16n_6^2,
\]

allowing efficient computation of divisors over $\mathbb{Q}$ of given degree and genus.

Suppose that $\Gamma$ is irreducible and is distinct from any of the known lines or conics. Then $\Gamma$ will have non-negative intersection number with all the straight lines and conics, and this determines linear inequalities on the coefficients $n_i$, $i = 1, \ldots, 6$. In this way, we obtain 31 linear constraints to be satisfied by the $n_i$, and accordingly $(n_1, n_2, n_3, n_4, n_5, n_6)$ is a point of the corresponding convex cone. Supporting hyperplanes of this cone are 22 in number (we used the routine MinimalInequalities in Magma \[7\]). Now $g(\Gamma) - 1 = \frac{1}{2}(\Gamma \cdot \Gamma) = \frac{1}{8}d^2 - (\text{positive definite form in the } n_i)$, and we shall see that $\frac{1}{8}d^2 - (\text{pos. def. form in the } n_i)$ is non-negative on this reduced cone, forcing $g(\Gamma) \geq 1$. On the hyperplane $d = \text{const}$, this latter form takes its minimum at a vertex of the resulting convex polytope, and so it is enough to see that the form, equivalently, $(\Gamma \cdot \Gamma)$, is non-negative on the extremal rays of the cone (that is, the lines joining the origin to the vertices of the convex polytope in which any hyperplane meets the cone). In this particular instance, there are 41 such rays, and the minimum value taken by $(\Gamma \cdot \Gamma)$ is 0. It follows that there are no curves $\Gamma$ on $V$ of (arithmetic) genus 0 other than lines and conics.

Suppose second that $h$ is a perfect rational square, but not equal to 1 or 4 modulo fourth powers. In this case the Néron-Severi group of $V$ over $\mathbb{Q}$ is of rank 7, with basis as above together with the extra divisor

\[
\Delta_7 = (x = \theta z, y = \theta w) + (x = -\theta z, y = -\theta w).
\]

Arguing as before, an irreducible curve $\sum_{i=1}^7 n_i \Delta_i$, distinct from the straight lines and known conics, determines a point $(n_1, \ldots, n_7)$ that lies within a polytope defined by the 49 half-planes arising from demanding non-negative intersection of $\Gamma$ with the known lines and conics. The reduced cone is defined by 30 half-planes, with 113 extremal rays. Again the minimum value of $(\Gamma \cdot \Gamma)$ on the extremal rays is equal to
0, and it follows that there are no rational curves \( \Gamma \) on \( V \) of (arithmetic) genus 0 other than lines and conics.

Third, if \( h \) is 4 modulo fourth powers, then without loss of generality, \( h = 4 \). This surface appears in the literature (Choudhry\([2]\)), who presents parameterizations of degrees 3 and 13, but a full treatment seems not to have been given. The surface has Néron-Severi rank 9, with basis as above, together with the two additional line degrees 3 and 13, but a full treatment seems not to have been given. The surface has been

\[
\Delta_8 = (x = (1 + i)w, y = (1 + i)z) + (x = (1 - i)w, y = (1 - i)z),
\]

\[
\Delta_9 = (x = (1 + i)w, y = -(1 + i)z) + (x = (1 - i)w, y = -(1 - i)z).
\]

Using similar analysis as before, we can show that there is a unique curve of degree 3, parameterized by:

\[
x : y : z : w = 4 - 2t + 4t^2 + t^3 : 2 + 2t + 4t^2 - t^3 : 2 + 4t - t^2 + t^3.
\]

There are no curves of degree 5, and up to symmetry a unique curve of degree 7:

\[
x : y : z : w = -64 - 24t - 24t^2 + 132t^3 - 144t^4 + 138t^5 - 22t^6 + 9t^7 :
- 96 + 8t - 24t^2 - 252t^3 + 168t^4 - 78t^5 + 42t^6 - 7t^7 :
- 56 + 168t - 156t^2 + 168t^3 - 126t^4 - 6t^5 + t^6 :
- 72 + 88t - 276t^2 + 144t^3 - 66t^4 + 6t^5 + 3t^6 + 4t^7.
\]

It seems plausible that there are no rational curves of even degree, and curves of every odd degree at least 7: (with some effort) we have written down such for all odd degrees up to 25. The techniques of Swinnerton-Dyer should resolve this question, in that all curves should be realisable by repeated application to the straight lines of a finite set of automorphisms of the surface. We have not taken this further here.

Finally, if \( h \) is a perfect fourth power, then without loss of generality, \( h = 1 \). The surface was known by Euler to contain parametrizable curves, and has been fully treated by Swinnerton-Dyer \([13]\).

6. The equation \( a(y_1^4 - f_1(X))^2 = b(y_2^4 - f_2(X))^2 \)

The same ideas as used in the preceding sections allow treatment of slightly more general equations. In a recent paper \([14]\), \( \textit{inter alia} \), it is proved that for any pair of non-zero integers \( a, b \) and any positive odd \( n \) the diophantine equation

\[
a(y_1^4 - x_1^{2n}) = b(y_2^4 - x_2^{2n}),
\]

has infinitely many rational parametric solutions. In order to get this result the variety defined by the equation \((10)\) is treated as an intersection of two rational hypersurfaces defined over the field \( \mathbb{Q}(t) \). Using a suitable (non-invertible) change of variables the study of the intersection is reduced to the problem of constructing \( \mathbb{Q}(t) \)-rational points on a certain hyperelliptic quartic curve, say \( C \). An important feature of this construction is that the curve \( C \) has a \( \mathbb{Q}(t) \)-rational point at infinity. Thus, essentially we can transform \( C \) into an elliptic curve, say \( E \), with Weierstrass equation defined over the field \( \mathbb{Q}(t) \). The existence of another \( \mathbb{Q}(t) \)-point on \( C \) guarantees that \( E \) has positive \( \mathbb{Q}(t) \)-rank, implying that the set of \( \mathbb{Q}(t) \)-rational points on \( E \), and thus on \( C \), is infinite. From this it follows that the set of parametric solutions of \((10)\) is infinite. This method is very simple and in essence is very
similar to the method used in previous sections here. A natural question arises as to whether we can use a corresponding approach to other diophantine equations of similar nature.

In this section we are interested in the following generalization of a result related to the solvability of the diophantine equation \( \textup{11} \).

**Theorem 6.1.** Let \( a, b \) be non-zero integers and let \( f_1(\overline{x}), f_2(\overline{x}) \) be homogenous forms with integer coefficients, where \( \overline{x} = (X_1, X_2, \ldots, X_n) \) is a vector of variables. Suppose that \( \text{deg} f_1 = \text{deg} f_2 = 2m + 1, m \in \mathbb{Z} \). Moreover, suppose that the set of rational points on the variety

\[
\mathcal{H} : Y^2 = -f_1(\overline{x})f_2(\overline{x})
\]

is infinite. Then there are infinitely many rational points lying on the hypersurface defined by the equation

\[
\mathcal{V} : a(y_1^4 - f_1(\overline{x})^2) = b(y_2^4 - f_2(\overline{x})^2).
\]

If \( \mathcal{H} \) contains rational points, then so does \( \mathcal{V} \).

**Proof.** Without loss of generality we can assume that \( \text{gcd}(a, b) = 1 \). Instead of considering \( \mathcal{V} \), consider the variety defined by the intersection

\[
\text{12} \quad a(y_1^4 - f_1(\overline{x})) = bU(y_2^4 - f_2(\overline{x})), \quad U(y_1^4 + f_1(\overline{x})) = y_2^4 + f_2(\overline{x}),
\]

where \( U \) is an indeterminate parameter. In order to solve the above system, take

\[
\text{13} \quad y_1 = T^m, \quad y_2 = vT^m, \quad X_i = u_iT \quad \text{for} \quad i = 1, 2, \ldots, n.
\]

From the first equation in \( \text{12} \) we get (after clearing the common factor \( T^{2m} \))

\[
\text{14} \quad T = \frac{a - bUv^2}{af_1 - bf_2U},
\]

where to shorten the notation put \( f_i = f_i(\overline{\pi}) \), with \( \overline{\pi} = (u_1, \ldots, u_n) \) and \( i = 1, 2 \).

From the second equation in \( \text{12} \):

\[
(bf_1U^2 - 2bf_2U + af_1)uv^2 = -bf_2U^2 + 2af_1U - af_2.
\]

Thus we get the equation of a hyperelliptic quartic curve in the form

\[
\mathcal{C}_{a,b} : V^2 = (btU^2 + 2aU + at)(bU^2 + 2btU + a),
\]

where \( V = v(bf_1U^2 - 2bf_2U + af_1)/f_1 \) and \( t = t(\overline{\pi}) = -f_2(\overline{\pi})/f_1(\overline{\pi}) \). From the assumption which says that the variety \( Y^2 = -f_1(\overline{\pi})f_2(\overline{\pi}) \) has infinitely many rational points we know that for infinitely many \( n \)-tuples \( \overline{\pi} \) of rational numbers the value of \( t(\overline{\pi}) \) is a square. We can treat the curve \( \mathcal{C}_{a,b} \) as the curve defined over the function field \( \mathbb{Q}(\mathcal{H}) \) of the variety \( \mathcal{H} \).

Note that when \( t(\overline{\pi}) \) is square, then the curve \( \mathcal{C}_{a,b} \) possesses the \( \mathbb{Q}(\mathcal{H}) \)-rational point \( P = (0, a\sqrt{t(\overline{\pi})}) \). Taking the point \( P \) as a point at infinity on the curve \( \mathcal{C}_{a,b} \), then \( \mathcal{C}_{a,b} \) is birationally equivalent to the elliptic curve with Weierstrass equation

\[
\mathcal{E}_{a,b} : Y^2 = X^3 + 4ab(a - bt^4)^2X.
\]

The curve \( \mathcal{E}_{a,b} \) contains the rational point \( Q = (X, Y) \), where

\[
X = \frac{(a^2 - 6abt^4 + b^2t^8)^2}{4t^2(a + bt^4)^2},
\]

\[
Y = \frac{(a^2 - 6abt^4 + b^2t^8)(a^4 + 20a^3bt^4 - 20a^2b^2t^8 + 20ab^3t^{12} + b^4t^{16})}{8t^3(a + bt^4)^3}.
\]
As before, it is easy to see that $Q$ is of infinite order in the group $\mathcal{E}_{a,b}(Q(H))$. Indeed, from the infinitude of the rational points on the variety $H$ one can find a $n$-tuple $\mathfrak{r}$ such that $t$ is finite and non-zero. Moreover the point $Q$ is non-trivial, i.e., $XY \neq 0$, and $4ab(a - bt^2)^2 \neq 4$ and it is not a fourth power. This implies that the corresponding $Q$ is of infinite order. Summing up, we see that for any $k \in \mathbb{N}_+$ the point $kQ = (X_k, Y_k)$ allows the computation of values of $v$ and $T$ and thus leads to a rational point on the hypersurface $V$ from the expressions (13). Because $Q$ is of infinite order we get infinitely many rational points on $V$. The argument is analogous for when $H$ contains rational curves. This finishes the proof of the theorem. ∎

We note an interesting result that follows from this theorem.

**Corollary 6.2.** Consider $n$-forms $f_i(X) = L_i(X)F_i(X)^2$, where $L_i$ is a linear $n$-form for $i = 1, 2$ and $L_1, L_2$ are independent. Then the hypersurface $V$ given by the equation (17) has infinitely many rational parametric solutions depending on $n - 1$ variables.

**Proof.** It is clear that the result will follow if we show that the hypersurface $H$ given in the preceding theorem contains a rational hypersurface parameterized by rational functions depending on $n - 1$ parameters. In our situation $H$ takes the form

$$H : Y^2 = -L_1(X)L_2(X)(F_1(X)F_2(X))^2.$$ 

$H$ is clearly rational. Indeed, a parameterization can be obtained easily by solving the system $L_1(X) = U_1^2$, $L_2(X) = -U_2^2$ (which clearly has solution by independence of $L_1$ and $L_2$) and putting $Y = U_1U_2F_1(X)F_2(X)$. ∎

We also prove the following result of independent interest.

**Theorem 6.3.** Let $a, b$ be non-zero fixed integers, $n \in \mathbb{N}_+$ and put $X = (x_1, \ldots, x_n)$. Let $f_1, f_2$ be homogenous forms with integer coefficients and suppose that $\deg f_2 = 2\deg f_1 + 1$. Then the set of rational points on the hypersurface defined by the equation

$$\mathcal{W} : a(y_1^4 - f_1(X)^4) = b(y_2^4 - f_2(X)^2)$$

is dense in the Zariski topology. Moreover, there are infinitely many rational curves which lie on $\mathcal{W}$.

**Proof.** Put $\deg f_1 = m$ and $\deg f_2 = 2m + 1$. Without loss of generality, $\gcd(a, b) = 1$. In order to construct rational points on the hypersurface $\mathcal{W}$ we make the following change of variables

\[
(15) \quad x_1 = \frac{f_1(1, w)^2}{f_2(1, w)}r, \quad x_i = w_ix_1, \quad y_1 = pf_1(1, w)x_1^m, \quad y_2 = qf_1(1, w)x_1^m,
\]

for $i = 2, \ldots, n$, where $w = (w_2, \ldots, w_n)$. The inverse mapping is given by

$$w_i = \frac{x_i}{x_1}, \quad p = \frac{y_1}{f_1(X)}, \quad q = \frac{y_2}{f_1(X)}r = \frac{f_2(X)}{f_1(X)^2},$$

for $i = 2, \ldots, n$. The substitution given by (15) leads to the surface $\mathcal{W}'$ given by the equation (after clearing the common factor $f_1(1, w)^{4m}$)

$$\mathcal{W}' : a(p^4 - 1) = b(q^4 - r^2).$$
The transformation shows that \( W \) is just a cone over the surface \( W' \). It is well known that \( W' \) is a del Pezzo surface of degree two with known rational point \((1,1,1)\) and is unirational over \( \mathbb{Q} \). In particular, this implies the existence of a rational map of the following form

\[
\varphi : \mathbb{Q}^2 \ni (u,v) \mapsto (p,q,r) = (g_1(u,v), g_2(u,v), g_3(u,v)) \in W'.
\]

The coordinates of \( \varphi \) are given by

\[
g_1(u,v) = \frac{bu^2(u^2 - 4uv + 2v^2) + a}{bu^2(u^2 - 2v^2) + a}, \quad g_2(u,v) = \frac{bu^2(u^3 - 2u^2v + 2uv^2) + a(u - 2v)}{bu^2(u^2 - 2v^2) + a} \]

and \( g_3(u,v) = u^2(g_1(u,v)^2 + 1) - g_2(u,v)^2 \). In particular \( \varphi(\mathbb{Q}) = W'(\mathbb{R}) \) in the Zariski topology and this property immediately implies the density (in the Zariski topology) of rational points on the hypersurface \( W \). The existence of a map \( \varphi \) implies also the existence of a rational two-parametric solution of the equation defining \( W \).

\[\square\]

7. Some additional remarks on \( ax^2 + by^2p = cz^2q + dw^2r \) with \( \frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1 \) and \( abcd \) square

The equations of Sections 2 to 4 are special cases of the following:

\[ S : ax^2 + by^2p = cz^2q + dw^2r, \quad \frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1, \]

where \( a, b, c, d \in \mathbb{N} \) and \( abcd = \Box \). The triples of positive integers \((p,q,r)\) satisfying the condition \( \frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1 \), \( p \leq q \leq r \), are precisely \((2,3,6), (2,4,4), (3,3,3)\). In this section we make some remarks concerning the existence of rational points on \( S \). The main observation is the following.

**Theorem 7.1.** The surface \( S \) is birationally equivalent with a genus one curve defined over the field of rational functions \( \mathbb{Q}(t) \). In the case \((p,q,r) = (2,3,6)\) the curve can be given in Weierstrass form.

**Proof.** Recall (see for example Richmond [3]) that a quadratic form \( aX^2 + bY^2 - cZ^2 - dW^2 \), with \( abcd = \Box \), and possessing a non-zero rational point, can be written in the form \( L_1L_2 - L_3L_4 \) where \( L_i \) is a linear form in the variables \( X,Y,Z,W \) with rational coefficients, for \( i = 1,2,3,4 \). This immediately implies that the equation defining the surface \( S \) can be written in the form

\[ S : L_1(x,y^p,z^q,w^r)L_2(x,y^p,z^q,w^r) = L_3(x,y^p,z^q,w^r)L_4(x,y^p,z^q,w^r), \]

Instead, we view \( S \) as a curve \( C \) defined over \( \mathbb{Q}(t) \) by the following intersection:

\[ C : L_1(x,y^p,z^q,w^r) = tL_3(x,y^p,z^q,w^r), \quad tL_2(x,y^p,z^q,w^r) = L_4(x,y^p,z^q,w^r). \]

Eliminating \( x \) (these are linear equations) there results an equation of the form:

\[ C : A(t)y^p + B(t)z^q + C(t)w^r = 0, \]

where \( A, B, C \in \mathbb{Z}[t] \) depend on \( a,b,c,d \). Because \((p,q,r) \in \{(2,3,6), (2,4,4), (3,3,3)\}\), it follows immediately that \( C \) is of genus 1. If \((p,q,r) = (2,3,6)\) then \( C \) is birationally equivalent to the elliptic curve in Weierstrass form \( Y^2 = X^3 + A(t)^3B(t)^2C(t) \), where \( Y = A(t)^2B(t)\frac{w^r}{w} \) and \( X = A(t)B(t)\frac{z^q}{z} \).

\[\square\]
Example 7.2. Consider the equation \( x^2 + y^6 = 2(z^6 + w^6) \). It is readily checked that
\[
(3X - Y - 2Z - 4W)(7X - Y - 10Z) - (3X + Y - 4Z - 2W)(5X - 5Y - 8Z - 6W) = 6(X^2 + Y^2 - 2Z^2 - 2W^2).
\]
Thus \( x^2 + y^6 = 2(z^6 + w^6) \) if and only if there exists a rational number \( t \) such that
\[
(3x + y^3 - 4z^3 - 2w^3) = t(3x - y^3 - 2z^3 - 4w^3),
\]
\[
t(5x - 5y^3 - 8z^3 - 6w^3) = 7x - y^3 - 10z^3.
\]
It follows that
\[
x = \frac{(t + 1)y^3 + 2(t - 2)z^3 + 2(2t - 1)w^3}{3(t - 1)},
\]
and
\[
C_t : \ (5t^2 - 8t + 5)y^3 + (7t^2 - 10t + 1)z^3 + (-t^2 + 10t - 7)w^3 = 0.
\]
A small numerical search reveals that when \( t = 1/13 \), the cubic curve has rational point \( P(y, z, w) = (5, 18, 7) \) corresponding to \( x = 8261 \). It is easily checked that \( P \) is of infinite order on \( C_{1/13} \), and thus the set of rational points on the surface \( x^2 + y^6 = 2(z^6 + w^6) \) is infinite.

Remark 7.3. We undertook a small numerical search for rational points on the surface \( ax^2 + by^6 = cz^6 + dw^6 \) with \( 1 \leq a, b, c, d \leq 5 \) and \( c \leq d \). For all but one case in this range we found that the equation is insolvable modulo 3, or has a solution with small height \((\leq 100)\). The only case resisting this attack is the equation \( 2x^2 + y^6 = 2z^6 + 4w^6 \). This equation can be written in the alternative form \( 2(x^2 - z^6) = 4w^6 - y^6 \) and thus contains the pencil of cubic curves
\[
C_t : \ 2(t^2 - 2)w^3 - (t^2 + 2)y^3 + 4tw^3 = 0.
\]
We used the Magma procedure \texttt{PointsCubicModel} for \( t \) in the range \( H(t) \leq 100 \) in order to find curves \( C_t \) containing rational points with \( \text{max}\{|x|, |y|, |z|\} \leq 10^6 \). However, no solution was found in this range.

Acknowledgments. The authors thank the referee for a careful reading of the paper, and for suggesting numerous improvements. The first author acknowledges with gratitude the hospitality of the Jagiellonian University, Kraków, for a short visit when the results presented in this paper were finalized; research of the second author was supported by Polish Government funds for science, grant IP 2011 057671 for the years 2012–2013.

References

[1] A. Bremner, M. Ulas, On \( x^6 \pm y^6 \pm z^6 \pm w^6 = 0 \), 1/a + 1/b + 1/c + 1/d = 1 \, , Int. J. Number Theory, 7 (8) (2011) 2081-2090.
[2] A. Choudhry, On the Diophantine equation \( A^4 + hB^4 = C^4 + hD^4 \), Indian J. Pure Appl. Math. 26 (1995), no. 11, 1057–1061.
[3] L. E. Dickson, History of the Theory of Numbers, Vol. II: Diophantine Analysis, Dover Publications, 2005.
[4] N.D. Elkies, On \( A^4 + B^4 + C^4 = D^4 \), Math. Comp. 51 (1988), no. 184, 825–835.
[5] N. D. Elkies, Personal communication.
[6] A. Hurwitz, Über ternäre diophantische Gleichungen dritten Grades, Vierteljahrschrift d. Naturforsch. Ges. Zürich 62 (1917), 207–229.
[7] Wieb Bosma, John Cannon, and Catherine Playoust, The Magma algebra system. I. The user language, J. Symbolic Comput., 24 (1997), 235265.
[8] R.G.E. Pinch and H.P.F. Swinnerton-Dyer, Arithmetic of diagonal quartic surfaces I. L-functions and arithmetic (Durham, 1989), 317338, London Math. Soc. Lecture Note Ser., 153, Cambridge Univ. Press, Cambridge, 1991.

[9] H.W. Richmond, On the Diophantine equation \( F = ax^4 + by^4 + cz^4 + dw^4 = 0 \), the Product \( abcd \) Being a Square Number, J. London Math. Soc. 19, (1944), 193–194.

[10] M. Rignaux, L’intermédiaire des math., 25, 1918, 7 (Dickson Vol II, page 699).

[11] J. Silverman, The Arithmetic of Elliptic Curves, Springer-Verlag, New York, 1986.

[12] Th. Skolem, Diophantische Gleichungen, New York : Chelsea Publishing Company, 1950.

[13] H.P.F. Swinnerton-Dyer, Applications of algebraic geometry to number theory. 1969 Number Theory Institute (Proc. Sympos. Pure Math., Vol. XX, State Univ. New York, Stony Brook, N.Y., 1969), pp. 1–52. Amer. Math. Soc., Providence, R.I., 1971.

[14] M. Ulas, On certain diophantine systems with infinitely many parametric solutions and applications, Acta Arith. 145 (3) (2010) 305–313.

Andrew Bremner, School of Mathematical and Statistical Sciences, Arizona State University, Tempe AZ 85287-1804, USA; e-mail: bremner@asu.edu

Maciej Ulas, Jagiellonian University, Faculty of Mathematics and Computer Science, Institute of Mathematics, Łojasiewicza 6, 30-348 Kraków, Poland; email: Maciej.Ulas@im.uj.edu.pl