On Shahidi matrix of local coefficients

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To Professor Freydoon Shahidi on his 70th birthday

Abstract

In these notes we study the Shahidi matrices of local coefficients associated with genuine principal series representations of an \( n \)-fold cover of \( p \)-adic \( SL_2(F) \). We show that the entries of these matrices are linear combinations of Tate and Tate type \( \gamma \)-factors and we relate these entries to linear maps defined on the dual of the space of Schwartz functions. We also find new formulas for the Plancherel measure and use it to detect all the reducibility points on the unitary axis and to relate principal series representations of different coverings of \( SL_2(F) \). While we do not assume that the residual characteristic of \( F \) is relatively prime to \( n \) we do assume that \( n \) is not divisible by 4.

0 Introduction

Let \( F \) be a \( p \)-adic field containing the full group of \( n \)-th roots of 1 and let \( SL_2^{(n)}(F) = SL_2(F) \) be the \( n \)-fold cover of \( SL_2(F) \) constructed by Kubota in [10]. In these notes we generalize and extend the study initiated in [8] of the genuine principal series representations of \( SL_2(F) \) and their associated Shahidi matrices of local coefficients and Plancherel measures. While we do not assume that the residual characteristic of \( F \) is relatively prime to \( n \) we do assume for most part of the paper that \( n \) is not divisible by 4. To ease the exposition in this introduction we first discuss our results in the case where \( n \) is odd.

Recall that the \( n \)-th power Hilbert symbol

\[
(\cdot, \cdot) : F^* \times F^* \to \mu_n
\]

plays a fundamental role in the construction of \( SL_2(F) \). It defines a non-degenerate antisymmetric bilinear form on \( F^*/F^{*n} \). The main tool in this paper is the existence of an essentially unique Lagrangian decomposition \( L = (G_0, K_0) \) of \( F^*/F^{*n} \). Namely, the existence of maximal isotropic subgroups \( G_0 \) and \( K_0 \) of \( F^*/F^{*n} \) such \( G_0 \times K_0 = F^*/F^{*n} \) and such that the Hilbert symbol identifies \( K_0 \) with the dual group of \( G_0 \).

A genuine principal series representation of \( SL_2(F) \) is a representation parabolically induced from \( \sigma \), a genuine smooth admissible irreducible representation of \( H(F) \), the inverse image of the diagonal subgroup of \( SL_2(F) \) inside \( SL_2(F) \). The isomorphism class of \( \sigma \) is determined by its central character, \( \chi_\sigma \). Let \( I(\sigma, s) \) be a genuine principal series representation of \( SL_2(F) \) induced from \( \sigma_s \) (here \( s \in \mathbb{C} \) is the usual complex parameter), let \( \psi \) be a
Let \( \chi \) more detail, for \( k \) \( \zeta \) by sending similar to Tate \( \gamma \) appearing in this introduction see the body of this paper). In Theorem 1.1 we show that where \( \gamma \). In Theorem 2.1 we show that \( G \). Here \( \hat{\gamma} \) is a basis for \( \sigma \) with \( \psi \), \( \chi \) be the standard intertwining operator. By composition, \( A_w(\sigma, s) \) induces a map

\[
\hat{A}_w(\sigma, s) : Wh_\psi(I(\sigma^w, -s)) \to Wh_\psi(I(\sigma, s)).
\]

In Corollary 2.1 we show that both \( Wh_\psi(I(\sigma, s)) \) and \( Wh_\psi(I(\sigma^w, -s)) \) have bases associated with the Lagrangian decomposition above. These bases are parameterized by \( K_0 \). Denote by \( \tau_L(a, b, \chi, s, \psi) \) the associated Shahid matrix of local coefficients, namely the matrix representing \( \hat{A}_w(\sigma, s) \) with respect to these bases. Here \( a, b \in K_0 \) and \( \chi \) is a character of \( F^* \) essentially extending \( \chi_\sigma \). We show in Theorems 2.2 and 2.3 that this matrix (along with some simple translations) and its determinant \( D_L(\sigma, s, \psi) \) is an invariant associated with \( \sigma \), \( \psi \) and \( L \). In fact, the trace of a closely related matrix is an invariant associated with \( \sigma \) and \( \psi \) only. In the \( n = 1 \) case, the matrix presented here is the inverse of Shahidi local coefficient, defined in [17], namely \( \gamma(\chi^{-1}, 1 - s, \psi) \).

Recall that Tate \( \gamma \)-factor, \( \gamma(\chi, s, \psi) \), is a matrix representing a linear map defined on a one dimensional space. Precisely, let \( S(F) \) be the space of Schwartz functions on \( F \) and let \( \hat{S}(F)_\chi \) be the space of \( \chi \) eigenfunctionals on \( S(F) \). The last space is one dimensional, see [23] and [12]. \( \hat{S}(F)_\chi \) is spanned by the Mellin transform, \( \phi \mapsto \zeta(s, \chi, \phi) \), where \( \zeta(s, \chi, \phi) \) is the meromorphic continuation of the zeta integral. On \( \hat{S}(F)_\chi \), one defines a linear map by sending \( \zeta(s, \chi, \phi) \) to \( \zeta(1 - s, \chi^{-1}, \tilde{\phi}) \), where \( \tilde{\phi} \in S(F) \) is the \( \psi \)-Fourier transform of \( F \). We have

\[
\zeta(1 - s, \chi^{-1}, \tilde{\phi}) = \gamma(s, \chi, \psi) \zeta(s, \chi, \phi).
\]

In Theorem 2.1 we show that

\[
\tau_L(a, b, \chi, s, \psi) = \gamma_G((\chi^{-1}(a^{-1}b, \cdot), 1 - s, \psi, (ab)^{-1}),
\]

where \( \gamma_G(\chi, s, \psi, k) \) is the partial \( \gamma \)-factor we define in Section 1.3 by

\[
\gamma_G(\chi, s, \psi, k) = (#G_0)^{-1} \sum_{g \in G_0} \gamma(\chi(g, \cdot), s, \psi)(g, k)^{-1} = \int_{Gk^{-1}} \chi^{-1}(y) ||y||^{1-s} \psi(y) d^*_\psi y.
\]

Here \( G \) is the pullback of \( G_0 \) to \( F^* \) and \( k \in K_0 \) (for exact details on integral representations appearing in this introduction see the body of this paper). In Theorem 1.4 we show that similar to Tate \( \gamma \)-factor, \( \gamma_G(\chi, s, \psi, k) \) arises from a functional equation related to \( S(F) \). In more detail, for \( k \in K_0 \) define the partial Mellin transform

\[
\zeta_G(\phi, \chi, s, k) = (#G_0)^{-1} \sum_{g \in G_0} (g, k)^{-1} \zeta(\phi, \chi_{g\psi}, s) = \int_{Gk} \phi(x) \chi(x) ||x||^s d^*_\psi x.
\]

Let \( \chi' \) be the restriction of \( \chi \) to \( G \). In Section 1.2 we show that the set

\[
\mathcal{B}(\chi'_s) = \{ \phi \mapsto \zeta_G(\phi, \chi, s, k) \mid k \in K_0 \}
\]

is a basis for \( \hat{S}(F)_{G, \chi'_s} \), the space of \( \chi'_s \) eigenfunctionals on \( S(F) \) and that

\[
\zeta(\phi, \chi^{-1}, 1 - s, G, k_0^{-1}) = \sum_{k \in K_0} \gamma_G(\chi, s, \psi, k^{-1}k_0) \zeta(\phi, \chi, s, G, k).
\]
In the case where \( n \) is prime to the residual characteristic of \( F \), the matrix of Shahidi local coefficients associated with unramified representations of coverings of \( GL_n(F) \) and a particular Lagrangian decomposition (not available if \( p \) divides \( n \)) was computed by Kazhdan and Patterson in [9]. In this context this matrix is called a scattering matrix. It appears frequently in the study of covering groups. It plays a central role in the study of the distinguished representations in [9]. It also appears in the construction given by Chinta and Offen in [4] of the metaplectic Casselman-Shalika formula. Recently, Brubaker, Buciumas and Bump showed that this matrix is equal to a certain twisted R-matrix, see [3]. McNamara generalized in [14] the computations of this matrix to the context of unramified representations of metaplectic coverings of unramified reductive p-adic groups. Our approach is different from the one used in [9] and [14]. We also use a slightly different normalization for the Whittaker functionals. Our approach originates from the work of Ariturk, [1], who studied unramified genuine principal series representations of \( SL_2^{(3)}(F) \). Thus, the results in these notes gives new and simple interpretation to an object otherwise known to be complicated. For example, as we demonstrate in Section 2.6, the fact that most of the entries of Shahidi matrix of local coefficients appearing in [9] vanish is explained by a simple property of \( \epsilon \)-factors.

In [7] and [27], Gan-Gao and Weissman raised the question whether the Langlands-Shahidi method can be extended to metaplectic groups other than the double cover of \( Sp_{2n}(F) \). Our results given in this paper provide many local clues about the nature of such a theory.

As an application of our approach we give in Theorem 2.4 a new formula for \( \mu_n(\sigma, s) \), the Plancherel measure associated with \( I(\sigma, s) \). This analytic invariant is defined by

\[
A_{w^{-1}}(\sigma^w, -s) \circ A_w(\sigma, s) = \mu_n^{-1}(\sigma, s)Id.
\]

For \( n = 1 \) this invariant is a quotient of \( L \)-functions. Utilizing our formula for the matrix of local coefficients we prove that \( \mu_n(\sigma, s) \) is the harmonic mean of Plancherel measures of principal series representations of the linear group \( SL_2(F) \), namely we prove that

\[
\mu_n^{-1}(\sigma, s) = [F^* : F^*n]^{-1} \sum_{k \in F^*/F^*n} \mu_1^{-1}(\chi(k, \cdot), s).
\]

We use this formula to find all reducible genuine principal series representations of \( SL_2(F) \) induced from a unitary data. We also find a relation between the Plancherel measures of matching representations of \( SL_2^{(n)}(F) \) and \( SL_2^{(m)}(F) \) provided \( m \) divides \( n \).

Our results for \( n \equiv 2 \pmod{4} \) are similar. In this case the \( \gamma \)-factor is replaced by the metaplectic \( \tilde{\gamma} \)-factor defined in [20], arising from another linear map on \( S\hat{(F)}_{\chi s} \). This distinction between even and odd fold covers of \( SL_2(F) \) is compatible with the theory given by Weissman in [27] which states that the dual group of \( SL_2^{(n)}(F) \) is \( PGL_2 \) if \( n \) is odd and \( SL_2 \) if \( n \) is even.

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1 Functional equations

1.1 A Lagrangian decomposition

Let $F$ be a finite extension of $\mathbb{Q}_p$. Denote by $q$ the cardinality of its residue field. Denote by $O_F$ its ring of integers. Fix $\varpi$, a generator of $\mathcal{P}_F$, the maximal ideal of $O_F$. We normalize the absolute value on $F$ such that $|\varpi| = q^{-1}$. Fix an integer $n \geq 1$ (starting at Section 2.2 we shall assume that $n$ is not divisible by 4). We shall assume that $F^*$ contains the full group of $n^{th}$ roots of 1. Denote this cyclic group by $\mu_n$. We identify $\mu_n$ with the group of $n^{th}$ roots of unity in $\mathbb{C}^*$ and suppress this identification.

For $m \in \mathbb{N}$ which divides $n$ let

$$(\cdot, \cdot)_m : F^* \times F^* \to \mu_m$$

be the $m^{th}$ power Hilbert symbol. The kernel of this anti-symmetric bilinear form is $F^{*m} \times F^{*m}$. Hence, it gives rise to a non-degenerate bilinear form on $F^*/F^{*m} \times F^*/F^{*m}$. In particular, it identifies $F^*/F^{*m}$ with its dual, $\widetilde{F^*/F^{*m}}$, which may also be identified with the group of characters of $F^*$ whose order divides $m$. Note that since $F^*$ contains $\mu_m$

$$[F^* : F^{*m}] = m^2 |m|^{-1},$$

see Page 48 of [13] for example. For all $x \in F^*$,

$$(x, x)_m = (-1, x)_m. \quad (1.2)$$

Also, if $n = ml$ then

$$(x, y)_n = (x, y)_l.$$  

See Section 5 in Chapter IV of [6] for example. Denote

$$d = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even;} \\ n & \text{if } n \text{ is odd.} \end{cases}$$

Observe that $F^*$ contains $\mu_{2d}$. Indeed, if $n$ is even there is nothing to explain while if $n$ is odd we note that $\mu_n \cup -\mu_n = \mu_{2n}$. Thus, $-1 \in F^{*d}$. Using (1.2) we conclude that for all $x \in F^*$

$$(x, x)_d = 1. \quad (1.3)$$

For $x \in F^*$ let $\eta_x$ be the character of $F^*$ defined by

$$y \mapsto \eta_x(y) = (x, y)_d.$$ 

Note that the map $x \mapsto \eta_x$ factors through $F^{*d}$. When convenient we shall also think of $\eta_x$ as an element in $\widetilde{F^*/F^{*d}}$.

A subgroup $G_0$ of $F^*/F^{*d}$ (of $F^*$) is called a maximal isotropic subgroup or a Lagrangian subgroup if

$$G_0 = \bigcap_{x \in G_0} \ker(\eta_x).$$
Let $G_0$ and $K_0$ be two Lagrangian subgroups of $F^*/F^{*d}$. We say that $L = (G_0, K_0)$ is a Lagrangian decomposition of $F^*/F^{*d}$ if $F^*/F^{*d}$ is a direct product of $G_0$ and $K_0$ and the map

$$k \mapsto \eta_k|_{G_0}$$

is an isomorphism from $K_0$ to the dual group of $G_0$ (in particular $G_0 \simeq K_0$). Note that if $(G_0, K_0)$ is a Lagrangian decomposition of $F^*/F^{*d}$ then by (1.1) we have

$$\# G_0 = \# K_0 = \sqrt{|F^*: F^{*d}|} = d |d|^{\frac{1}{2}}.$$

**Lemma 1.1.** A Lagrangian decomposition of $F^*/F^{*d}$ exists. Furthermore, if $(G_0, K_0)$ and $(G'_0, K'_0)$ are two Lagrangian decompositions of $F^*/F^{*d}$ then there exists an automorphism $\theta$ of $F^*/F^{*d}$ preserving $(\cdot, \cdot)_d$ such that $\theta(G_0) = G'_0$ and $\theta(K_0) = K'_0$.

**Proof.** Let $A$ be a finite abelian group and let $k$ be a field. In [5], Davydov defines a non-degenerate bilinear form $[\cdot, \cdot] : A \times A \to k^*$ to be alternative if

$$[x, x] = 1, \forall x \in A$$

and proves in Lemma 4.2 of [5] that a Lagrangian decomposition exists. By (1.3), this proof applies to our case. We shall now prove the second assertion in the context studied in [5]. Suppose $(G_0, K_0)$ is a Lagrangian decomposition of $A$. If we write

$$G_0 \simeq C_{p_1}^{k_1} \times C_{p_2}^{k_2} \times \ldots \times C_{p_j}^{k_j}$$

where $p_1, p_2, \ldots, p_j$ are prime integers, $k_1, k_2, \ldots, k_j$ are positive integers and $C_m$ is the cyclic group of order $m$, then

$$A \simeq C_{p_1}^{k_1} \times C_{p_2}^{k_2} \times \ldots \times C_{p_j}^{k_j} \times C_{p_j}^{k_j} \times \ldots$$

From the general theory of abelian groups it now follows that if $A \simeq G \times G$ then $G \simeq G_0$. Let $(G'_0, K'_0)$ be another Lagrangian decomposition of $A$. By the last remark there exists an isomorphism

$$\alpha : G_0 \to G'_0.$$

This map induces an isomorphism

$$\beta : K_0 \to K'_0$$

such that for all $g \in G_0$, $k \in K_0$

$$[g, k] = [\alpha(g), \beta(k)].$$

Clearly $\theta = \alpha \times \beta$ satisfies the conditions given in the lemma.

**Example 1.1.** In the case where $\gcd(d, p) = 1$ set

$$G_0 = \mathbb{O}_F^* F^{*d} / F^{*d}, K_0 = < \omega > F^{*d} / F^{*d}.$$

It follows from Chap. XIII, Sec. 5 of [24] that $L = (G_0, K_0)$ is a Lagrangian decomposition of $F^*/F^{*d}$. Both $G_0$ and $K_0$ are isomorphic to $C_d$. Suppose in addition that $d = m^2$ for
some integer. Then $H_0 = (F^*/F^{*d})^m$ is a maximal isotropic subgroup. But since $H_0 \not\cong C_d$ it follows from the proof of Lemma 1.1 that $H_0$ does not fit into a Lagrangian decomposition. In example 4.4 of [2] the author shows directly that the extension

$$H_0 \to F^*/F^{*d} \to (F^*/F^{*d})/H_0$$

does not split.

We shall now fix once and for all a Lagrangian decomposition $L = (G_0, K_0)$ of $F^*/F^{*d}$. Let $G$ and $K$ be the pullbacks of $G_0$ and $K_0$ respectively to $F^*/F^{*d}$. Both $G$ and $K$ are maximal isotropic subgroups of $F^*$, $GK = F^*$ and $G \cap K = F^{*d}$.

**Lemma 1.2.** The following hold.
1. $F^*/G$ is isomorphic to the dual group of $G_0$. The pairing is given by the Hilbert symbol.
2. For $k \in F^*$, the map
   $$x \mapsto (#G_0)^{-1} \sum_{g \in G_0} \eta_g(xk^{-1})$$

defined on $F^*$ is the characteristic function of $Gk$.
3. The map
   $$kG \mapsto kF^{*d}$$

is an isomorphism from $F^*/G$ and $K_0$.

**Proof.** Fix $x \in G_0$. Since $G \subseteq \ker(\eta_x)$, $\eta_x$ is a well defined character of $F^*/G$. From the non-degeneracy of the Hilbert symbol it follows that $x \mapsto \eta_x$ is an embedding of $G_0$ in the dual group of $F^*/G$. Also, $xG \mapsto \eta_x|_{G_0}$ is an embedding of $F^*/G$ into the dual group of $G_0$. Since both $G_0$ and $F/G$ are finite abelian groups it follows that these two embeddings are isomorphisms. This proves the first assertion. The second assertion follows from the first. The last assertion follows simply from the third isomorphism theorem along with the fact the $K_0 \cong (F^*/F^{*d})/G_0$.

**Corollary 1.1.** By picking representatives for $F^*/G$ in $K$ one identifies $F^*/G$ with $K_0$.

**Lemma 1.3.** Fix $x, y \in G$.

$$(x, y)_n = \begin{cases} 1 & \text{n is odd;} \\ (x, y)_2 & \text{n \equiv 2 (mod 4).} \end{cases}$$

**Proof.** This lemma is trivial if $n$ is odd. If $n \equiv 2 (mod 4)$ then $d$ is odd. Also $(x, y)_n = \pm 1$ for $x, y \in G$ since $(x, y)_n^2 = (x, y)_d = 1$. These two observations imply that for all $x, y \in G$

$$(x, y)_n = (x, y)_n^d = (x, y)_2.$$
1.2 Eigen functionals

Let $\chi_0$ be a character of $F^{*d}$. From the general theory of locally compact abelian groups it follows that $\chi_0$ can be extended to $F^*$. Let $\chi$ be one these extensions and let $\chi'$ be the restriction of $\chi$ to $G$. This notation will be fixed throughout this paper. Note that

$$\{\chi_0 y \mid y \in F^*/F^{*d}\}$$

is the set of all extensions of $\chi_0$ to $F^*$,

$$\{\chi' y \mid y \in K_0\}$$

is the set of all extensions of $\chi_0$ to $G$ and

$$\{\chi y \mid y \in G_0\}$$

is the set of all extensions of $\chi'$ to $F^*$.

Let $S(F)$ be the space of Schwartz functions on $F$ and let $\widehat{S(F)}$ be the space of linear functionals on $S(F)$. $F^*$ acts on $S(F)$ by right translations. Denote this action by $\rho$. $F^*$ also acts on $\widehat{S(F)}$ by

$$\lambda(g) \phi = \xi(\rho(g^{-1}) \phi).$$

Let $\widehat{S(F)}_{\chi}$ be the space of $\chi$ eigenfunctionals on $S(F)$.

Lemma 1.4. $\widehat{S(F)}_{\chi}$ is a one dimensional space.

This fundamental uniqueness result is implicit in Tate’s thesis, [21]. It is proven in [23]. See also Theorem 3.4 of [12].

Lemma 1.5.

$$\widehat{S(F)}_{\chi} = \bigoplus_{y \in G_0} \widehat{S(F)}_{\chi y}. $$

In particular,

$$\dim \widehat{S(F)}_{\chi} = d|d|^{-\frac{1}{2}}.$$ 

Proof. Clearly,

$$\widehat{S(F)}_{\chi} \supset \bigoplus_{y \in G_0} \widehat{S(F)}_{\chi y}.$$ 

We prove the converse inclusion. Fix $\xi \in \widehat{S(F)}_{\chi}$. For $g \in G_0$ define

$$\xi_g = \sum_{k \in K_0} (\chi y)^{-1}(k) \lambda(k) \xi.$$ 

It is easy to verify that $\xi_g$ is well defined. We shall now show that $\xi_g \in \widehat{S(F)}_{\chi y}$. Since $\widehat{S(F)}_{\chi'}$ is an $F^*$ space it follows from the definition of $\xi_g$ that for $g' \in G$

$$\lambda(g') \xi_g = \chi(g') \xi_g.$$
Recalling that \( F^* = GK \) it is sufficient to check that for \( k' \in K \), \( \lambda(k')\xi_g = (\chi\eta_g)(k')\xi_g \). Indeed
\[
\lambda(k')\xi_g = \sum_{k \in K_0} (\chi\eta_g)^{-1}(k)\lambda(kk')\xi = (\chi\eta_g)(k')\sum_{k \in K_0} (\chi\eta_g)^{-1}(k)\lambda(k)\xi.
\]

We finally note that
\[
(\#G_0)^{-1} \sum_{g \in G_0} \xi_g = (\#G_0)^{-1} \sum_{g \in G_0} (\chi\eta_g)^{-1}(k)\lambda(k)\xi = \sum_{k \in K_0} \chi^{-1}(k)\lambda(k)\xi (\#G_0)^{-1} \sum_{g \in G_0} \eta_g^{-1}(k) = \xi.
\]

Thus, \( \xi \in \bigoplus_{g \in G} \hat{S}(F)_{\chi\eta_g} \).

For \( s \in \mathbb{C} \) let \( \chi_s \) be the character of \( F^* \) given by
\[
x \mapsto \chi(x)\|x\|^s.
\]

Same definition applies for characters of subgroups of \( F^* \). A non-zero element in \( \hat{S}(F)_{\chi_s} \) is given by
\[
\phi \mapsto L^{-1}(s, \chi)\zeta(s, \chi, \phi)
\]
where \( \zeta(s, \chi, \phi) \) is the rational function in \( q^{-s} \) given by the meromorphic continuation of
\[
\int_{F^*} \phi(x)\chi_s(x) d^*_\psi x.
\]

Here \( d^*_\psi x \) is a Haar measure on \( F^* \) normalized as explained in Section 1.3 below. This integral is absolutely convergent for \( Re(s) >> 0 \). From Lemma 1.5 it follows that away from the poles
\[
B(\chi_s) = \{ \phi \mapsto \zeta(s, \chi, \phi) \mid g \in G_0 \}
\]
is a basis for \( \hat{S}(F)_{G, \chi_s} \). For \( k \in K_0 \) we now define
\[
\zeta_G(\phi, \chi, s, k) = (\#G_0)^{-1} \sum_{g \in G_0} \eta_g(k^{-1})\zeta(\phi, \chi\eta_g, s)
\]
and we set
\[
\overline{B}(\chi_s) = \{ \phi \mapsto \zeta_G(\phi, \chi, s, k) \mid k \in K_0 \}.
\]

**Lemma 1.6.** Away from the poles, \( \overline{B}(\chi_s) \) is a basis for \( \hat{S}(F)_{G, \chi_s} \). Also, \( \zeta_G(\phi, \chi, s, k) \) is the meromorphic continuation of
\[
\int_{Gk} \phi(x)\chi_s(x) d^*_\psi x.
\]
This integral converges absolutely for \( Re(s) >> 0 \).

**Proof.** From the duality of \( G_0 \) and \( K_0 \) it follows that
\[
\zeta(\phi, \chi\eta_g, s) = \sum_{k \in K_0} \eta_g(k)\zeta_G(\phi, \chi, s, k).
\]
Hence, $B(\chi_s') \subseteq \overline{B}(\chi_s')$. The first assertion is now proven since

$$\# B(\chi_s') = \dim \widehat{S}(F)_{G,\chi'}.$$ 

We now prove the second assertion. Suppose that $Re(s) >> 0$ so that all the integrals below are absolutely convergent. We have

$$\zeta_G(\phi, \chi, s, k) = (\# G_0)^{-1} \sum_{g \in G_0} \eta_g(k^{-1}) \zeta(\phi, \chi \eta_g, s) = (\# G_0)^{-1} \sum_{g \in G_0} \int_{F^*} \phi(x) \chi_s(x) \eta_g(xk^{-1}) d^* x$$

$$= \int_{F^*} (\# G_0)^{-1} \sum_{g \in G_0} \eta_g(xk^{-1}) \phi(x) \chi_s(x) d^* x.$$ 

By Lemma 1.2 the proof is done. \(\square\)

The following Lemma will not be used later. It is included here for the sake of completeness.

**Lemma 1.7.** For $k \in F^*$ let $\widehat{S}(F)_{G,\chi',k}$ be the subspace of $\widehat{S}(F)_{G,\chi'}$ which consists of functionals supported on $Gk$.

$$\dim \widehat{S}(F)_{G,\chi',k} = 1$$

and $\zeta_G(\phi, \chi, s, k) \in \widehat{S}(F)_{G,\chi',k}$.

**Proof.** The second assertion follows immediately from the integral formula given in Lemma 1.6. In particular, for any $k \in F^*$,

$$\dim \widehat{S}(F)_{G,\chi',k} \geq 1.$$ 

The first assertion now follows since

$$\bigoplus_{k \in K_0} \widehat{S}(F)_{G,\chi',k} \subseteq \widehat{S}(F)_{\chi s}$$

and

$$\# K_0 = \dim \widehat{S}(F)_{G,\chi'}.$$ 

\(\square\)

### 1.3 Tate $\gamma$-factor and related factors

We fix a non-trivial character $\psi$ of $F$. Given $\phi \in S(F)$ we define $\hat{\phi} \in S(F)$ to be its $\psi$-Fourier transform, i.e.,

$$\hat{\phi}(x) = \int_F \phi(y) \psi(xy) d\psi y.$$ 

Here $d\psi y$ is the self dual measure with respect to $\psi$. By Tate’s thesis, [21],

$$\phi \leftrightarrow L^{-1}(1-s, \chi^s) \zeta(1-s, \chi^{-1}, \hat{\phi}) \in \widehat{S}(F)_{\chi s}.$$
This observation along with the uniqueness result in Lemma 1.4 give rise to Tate $\gamma$-factor

$$\gamma(s, \chi, \psi) = \epsilon(s, \chi, \psi) \frac{L(1-s, \chi)}{L(s, \chi)}$$

This rational function in $q^{-s}$ is defined via the functional equation.

$$\zeta(1-s, \chi^{-1}, \hat{\phi}) = \gamma(s, \chi, \psi)\zeta(s, \chi, \phi). \quad (1.4)$$

It is well known that $\gamma(\chi^{-1}, 1-s, \psi)$ is given by the meromorphic continuation of

$$\lim_{r \to \infty} \int_{P_{x}^{r}} \chi_{s}(x)\psi(x) d^{*}_{\psi}x.$$

Here $d^{*}_{\psi}x = \|x\|^{-1}d_{\psi}x$. This limit exists for $Re(s) >> 0$. We now define a map

$$M_{\psi} \in GL\left(S(F)_{G, \chi'}\right)$$

by setting

$$M_{\psi}\left(\phi \mapsto \zeta(\phi, \chi\eta_{g}, s)\right) = \phi \mapsto \zeta(\hat{\phi}, (\chi\eta_{g})^{-1}, 1-s)$$

and then by linear extension. Clearly

$$M_{\psi}\left(\phi \mapsto \zeta_{G}(\phi, \chi, s, k_{0})\right) = \phi \mapsto \zeta_{G}(\hat{\phi}, \chi^{-1}, 1-s, k_{0}^{-1}). \quad (1.5)$$

By (1.4) the matrix representing $M_{\psi}$ with respect to $B(\chi_{s}')$ is diagonal. We now describe the matrix representing $M_{\psi}$ with respect to $B(\chi_{s}')$. For $k \in K_{0}$ we set

$$\gamma_{G}(\chi, s, \psi, k) = (\#G_{0})^{-1} \sum_{g \in G_{0}} \gamma(\chi\eta_{g}, s, \psi)\eta_{g}(k^{-1}). \quad (1.6)$$

It is a rational function in $q^{-s}$.

**Theorem 1.1.**

$$\zeta_{G}(\hat{\phi}, \chi^{-1}, 1-s, k_{0}^{-1}) = \sum_{k \in K_{0}} \gamma_{G}(\chi, s, \psi, k^{-1}k_{0})\zeta_{G}(\phi, \chi, s, k).$$

Also, $\gamma_{G}(\chi^{-1}, 1-s, \psi, k)$ is the meromorphic continuation of

$$\lim_{r \to \infty} \int_{P_{x}^{r} \cap Gk^{-1}} \chi_{s}(x)\psi(x) d^{*}_{\psi}x.$$

**Proof.**

$$M_{\psi}\left(\zeta_{G}(\phi, \chi, s, k_{0})\right) = (\#G_{0})^{-1} \sum_{g \in G_{0}} \eta_{g}(k_{0}^{-1})M_{\psi}(\zeta(\phi, \chi\eta_{g}, s))$$

$$= (\#G_{0})^{-1} \sum_{g \in G_{0}} \eta_{g}(k_{0}^{-1})\gamma(\chi\eta_{g}, s, \psi)\zeta(\phi, \chi\eta_{g}, s).$$
Plugging (1.6) we obtain
\[ M_\psi(\zeta_G(\phi, \chi, s, k_0)) = (\#G_0)^{-1} \sum_{g \in G_0} \eta_g(k_0^{-1}) \gamma(\chi \eta_g, s, \psi) \sum_{k \in K_0} \eta_g(k) \zeta_G(\phi, \chi, s, k) \]
\[ = \sum_{k \in K_0} \left( (\#G_0)^{-1} \sum_{g \in G_0} \gamma(\chi \eta_g, s, \psi) \eta_g(kk_0^{-1}) \right) \zeta_G(\phi, \chi, s, k). \]

This proves the first assertion. We now prove the second assertion. We may assume that \( \text{Re}(s) \gg 0 \) is such that all the limits below exist.
\[ \gamma_G(\chi^{-1}, 1 - s, \psi, k) = (\#G_0)^{-1} \sum_{g \in G_0} \eta_g(k^{-1}) \gamma(\chi^{-1} \eta_g, 1 - s, \psi) \]
\[ = (\#G_0)^{-1} \sum_{g \in G_0} \eta_g(k^{-1}) \lim_{r \to \infty} \int_{F^r} \psi(x)(\chi \eta_g^{-1})_s(x) d^* x = \]
\[ = (\#G_0)^{-1} \lim_{r \to \infty} \int_{F^r} \left( \sum_{g \in G_0} \eta_g(kx) \right) \psi(x) \chi_s(x) d^* x. \]

Using Lemma 1.2 again we are done.

**Remark 1.2.** We have been using the notation \( \zeta_G(\phi, \chi, s, k_0) \) and \( \gamma_G(\chi, s, \psi, k) \) rather than \( \zeta_L(\phi, \chi, s, k_0) \) and \( \gamma_L(\chi, s, \psi, k) \) since the results in Sections 1.2, 1.3 and Section 1.4 below do not depend on the existence of a Lagrangian decomposition. These results only uses the fact that \( G \) is a maximal Abelian subgroup. \( K_0 \) can be replaced by \( F/G \).

\[ \square \]

### 1.4 Metaplectic Tate \( \tilde{\gamma} \)-factor and related factors

Let
\[ \gamma_\psi : F^* \mapsto \mu_4 \subseteq \mathbb{C} \]
be the normalized Weil index associated with \( \psi \). It is well known that \( \gamma_\psi(F^*^2) = 1 \) and that for all \( x, y \in F^* \)
\[ \gamma_\psi(xy) = \gamma_\psi(x) \gamma_\psi(x,y), \quad (1.7) \]
For \( \phi \in S(F) \) we define \( \tilde{\phi} : F^* \to \mathbb{C} \) by
\[ \tilde{\phi}(x) = \int_{F^*} \phi(y) \gamma_\psi^{-1}(xy) \psi(xy) d^* y. \]

Although \( \tilde{\phi}(x) \) is typically not an element of \( S(F) \) it was proven in [20] that
\[ \int_{F^*} \tilde{\phi}(x) \chi_s(x) d^* x \]
converges absolutely for \( a < \text{Re}(s) < a + 1 \), for some \( a \in \mathbb{R} \), to a rational function in \( q^{-s} \). This enables the natural definition of \( \zeta(s, \chi, \tilde{\phi}) \) as the meromorphic continuation of the integral above. Furthermore, by [20] there exists a rational function in \( q^{-s} \), \( \tilde{\gamma}(s, \chi, \psi) \), such that
\[ \zeta(1 - s, \chi^{-1}, \tilde{\phi}) = \zeta(s, \chi, \phi) \tilde{\gamma}(s, \chi, \psi) \]
for all $\phi \in S(F)$. In particular

$$\phi \mapsto \zeta(1 - s, \chi, \tilde{\phi}) \in \hat{S}(F)_{\chi s}.$$ 

It was also proven in [20] that $\tilde{\gamma}(\chi^{-1}, 1 - s, \psi)$ is the meromorphic continuation of

$$\lim_{r \to \infty} \int_{F_F^{-r}} \chi_s(x) \gamma_{\psi}^{-1}(x) \psi(x) d_{\psi}^* x.$$ 

This limit exists for $Re(s) >> 0$. Its computation is contained in an unpublished notes of W. Jay Sweet, [19], see also the appendix of [8]:

$$\tilde{\gamma}(1 - s, \chi, \psi) = \gamma_{F}^{-1}(\psi_{-1})(2s, \chi^2, \psi) \gamma(s + \frac{1}{2}, \chi, \psi).$$ (1.8)

Here, for $a \in F$, $\psi_a$ is the character of $F$ defined by $x \mapsto \psi(ax)$ and $\gamma_{F}^{-1}(\psi_{-1}) \in \mu_8$ is the unnormalized Weil index associated with $\psi^{-1}$. From Lemma 4.1 in [11] and from the fact that $\gamma_{\psi} = \gamma_{\psi_{-1}}^{-1}$ it follows that

$$\gamma_{F}^{-2}(\psi_{-1}) = \gamma_{\psi}^{-1}(-1).$$ (1.9)

Similar to Section 1.3, for $k \in K_0$ we define

$$\zeta_G(\tilde{\phi}, \chi, s, k) = (\#G_0) \sum_{g \in G_0} \eta_g(k^{-1}) \zeta(\phi, \chi \eta_g, s).$$

$\zeta_G(\tilde{\phi}, \chi, s, k)$ is the meromorphic continuation of

$$\int_{G_{k_{0}^{-1}}} \tilde{\phi}(x) \chi_s(x) d^* x.$$ 

For $k \in K_0$ we set

$$\tilde{\gamma}_G(\chi, s, \psi, k) = (\#G_0)^{-1} \sum_{g \in G_0} \tilde{\gamma}(\chi \eta_g, s, \psi) \eta_g(k^{-1}).$$ (1.10)

We have

$$\zeta(\tilde{\phi}, \chi^{-1}, 1 - s, G, k_0^{-1}) = \sum_{k \in K_0} \tilde{\gamma}_G(\chi, s, \psi, k^{-1} k_0) \zeta(\phi, \chi, s, G, k).$$

Also, $\tilde{\gamma}_G(\chi^{-1}, 1 - s, \psi, k)$ is the meromorphic continuation of

$$\lim_{r \to \infty} \int_{F_F^{-r} \cap G_{k_{0}^{-1}}} \chi_s(x) \gamma_{\psi}^{-1}(x) \psi(x) d_{\psi}^* x.$$ (1.11)

This last limit exists for $Re(s) >> 0$.

**Remark 1.3.** Let $\theta$ be an automorphism of $\hat{F^*/F^*d}$. The notion of a maximal isotropic subgroup and the results in Sections 1.1-1.4 are unchanged if we replace $\eta_x$ by $\theta(\eta_x)$ since $x \mapsto \theta(\eta_x)$ is another pairing of $F^*/F^*d$ with itself. In particular, if $n \equiv 2 (mod 4)$ then $d$ is odd and $m = \frac{d+1}{2} \in \mathbb{N}$ is relatively prime to $d$. Thus, if we define

$$x \mapsto \eta_x' = \eta_x^m$$ (1.12)

then $\eta_x \mapsto \eta_x'$ is an automorphism of the dual group of $F^*/F^*d$. 

12
2 Genuine principal series representations of $\widetilde{SL}_2(F)$.

2.1 An $n$ fold cover of $SL_2(F)$.

Let $SL_2(F)$ be the group of two by two matrices with entries in $F$ whose determinant is 1. Let $N(F) \simeq F$ be the group of upper triangular unipotent matrices. Let $H(F) \simeq F^*$ be the group of diagonal elements inside $SL_2(F)$. Denote $B(F) = H(F) \rtimes N(F)$. For $x \in F$, and $a \in F^*$ we shall write $n(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$, $h(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$, $w_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Let $\widetilde{SL}_2(F)$ be the topological central extension of $SL_2(F)$ by $\mu_n$ constructed by Kubota in [10]. More precisely, we realize $\widetilde{SL}_2(F)$ as the set $SL_2(F) \times \mu_n$ along with the multiplication

$$(g, \epsilon)(g', \epsilon') = (gg', c(g, g')\epsilon \epsilon'),$$

where

$$c(g, g') = (x(gg')x^{-1}(g), x(gg')x^{-1}(g'))_n.$$  \hspace{1cm} (2.1)

Here

$$x \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} c & c \neq 0; \\ d & c = 0. \end{cases}$$

We shall denote by $s$ the map from $SL_2(F)$ to $\widetilde{SL}_2(F)$ given by

$$s(g) = (g, 1)$$

(generally, it is not a group homomorphism). We set $w = s(w_0)$. For a subset $A$ of $SL_2(F)$ we shall denote by $A^{(n)}$ its inverse image in $\widetilde{SL}_2(F)$. For most part of this paper, $n$ is fixed. Thus, when convenient we shall drop the index $n$ when discussing $\widetilde{SL}_2^{(n)}(F)$ and its subgroups. It is only toward the end of Section 2.7 that $n$ varies.

For $l$ sufficiently large, $\widetilde{SL}_2(F)$ splits over

$$K_l = \{ g \in SL_2(F) \mid g = I_{2n} \pmod{P_F^l} \}$$

via the trivial section. The topology on $\widetilde{SL}_2(F)$ is defined so that $\{s(K_l)\}_{l>0}$ form a basis of the neighborhoods of the identity.

**Lemma 2.1.** The center of $\widetilde{H}(F)$, is $\widetilde{H_0}(F)$ where

$$\widetilde{H_0}(F) = H^d(F).$$

Also, if we set

$$H(G) = \{ h(a) \mid a \in G \}$$

Then $\widetilde{H}(G)$ is a maximal abelian subgroup of $\widetilde{H}(F)$. 
Proof. From (2.1) it follows that
\[ c(h(a), h(b)) = (b, a)_n. \]
Hence, inverse images of \( h(a) \) and \( h(b) \) in \( \widetilde{SL_2(F)} \) commute if and only if \((b, a)^2 = 1\). This is equivalent to \((a, b)_d = 1\). Both assertions now follow. \( \square \)

2.2 Representations

A (complex) representation of \( \widetilde{SL_2(F)} \) or any of its subgroups is called genuine if the central subgroup
\[ \{ (I_2, \epsilon) \mid \epsilon \in \mu_n \} \]
acts by the identity character. By Stone-Von Neumann Theorem, see Theorem 3.1 in [25] for example, we have

**Lemma 2.2.** The isomorphism class of a genuine smooth admissible irreducible representation \( \sigma \) of \( H(F) \) is determined by its central character \( \chi_\sigma \). Moreover, a realization of \( \sigma \) is given by
\[ i(\chi'_\sigma) = \text{Ind}_{\widetilde{H(G)}}^{\widetilde{H(F)}} \chi'_\sigma \]
where \( \chi'_\sigma \) is a character of \( \widetilde{H(G)} \) which extends \( \chi_\sigma \). In particular, the dimension of \( \sigma \) is
\[ [\widetilde{H(F)} : \widetilde{H(G)}] = d\|d\|^{-\frac{1}{2}}. \]

**Remark 2.1.** In [26] Weissman refers to
\[ [\widetilde{H(F)} : \widetilde{H(G)}] = \sqrt{[\widetilde{H(F)} : \widetilde{H_0(F)}]} \]
as the central index of \( \widetilde{H(F)} \). It arises in the context of Lagrangian decompositions of \( \widetilde{T/Z(T)} \), where \( \widetilde{T} \) is a cover of a torus \( T \) defined over a local field and \( Z(\widetilde{T}) \) is its center.
In the case of \( \widetilde{SL_2(F)} \) these Lagrangian decompositions of \( \widetilde{H(F)}/\widetilde{H_0(F)} \) are in a natural bijection with the Lagrangian decompositions of \( F^* / F^*d \) discussed in Section L.7

From this point we assume that \( n \) is not divisible by 4. This assumption implies that \([F^* : F^*d]\) is odd. We define
\[ \chi_\psi : \widetilde{H(F)} \to \mathbb{C} \]
by
\[ \chi_\psi(h(a), \epsilon) = \epsilon \chi(a) \begin{cases} 1 & n \text{ is odd;} \\ \gamma^{-1}_\psi(a) & n \equiv 2 \pmod{4}. \end{cases} \]
Denote its restriction to \( \widetilde{H_0(F)} \) and \( \widetilde{H(G)} \) by \( (\chi_0)_\psi \) and \( \chi'_\psi \) respectively. We continue to denote by \( \chi_\psi \), the function on \( F^* \) defined by \( a \mapsto \chi_\psi(s(h(a))) \). Observe that by Lemma L.3 and by (L.7), \( (\chi_0)_\psi \) and \( \chi'_\psi \) are genuine characters of \( \widetilde{H_0(F)} \) and \( \widetilde{H(G)} \) respectively. Moreover, since the quotient of two genuine characters factors via the projection map it
follows that any genuine character of $\widehat{H_0(F)} (\widehat{H(G)})$ has this form. In other words, the set of genuine characters of $\widehat{H_0(F)} (\widehat{H(G)})$ is parameterized by the group of characters of $H_0(F) (\widehat{H(G)})$. This parametrization is canonical if and only if $n$ is odd.

$SL_2(F)$ splits over $N(F)$ via the trivial section and $H(F)$ normalizes $s(N(F))$. Therefore, any representation of $H(F)$ can be extended to a representations of $B(F)$ by defining it to be trivial on $s(N(F))$. Thus, as in the linear case, we shall not distinguish between representations of $H(F)$ and those of $B(F)$. A genuine principal series representations $I(\sigma)$ of $SL_2(F)$ is a representation induced from a genuine smooth admissible irreducible representation $\sigma$ of $B(F)$. Using induction by stages, it follows from the above that any genuine principal series representation $I(\sigma)$ of $SL_2(F)$ may be realized as

$$I(\chi^\prime_\psi) = \text{Ind}_{B(G)}^{SL_2(F)} \chi^\prime_\psi,$$

where $B(G) = H(G)N(F)$ and where $\chi^\prime_\psi$ extends $\chi_\sigma$. We assume that the induction is normalized in the usual way, namely that for any $f \in I(\chi^\prime_\psi)$

$$f(s(nh)g) = \chi^\prime_\psi(s(h))|h||f(g).$$

Here $h \in H(G)$, $n \in N(F)$, $g \in SL_2(F)$.

Given a genuine smooth admissible irreducible representation $\sigma$ of $H(F)$ and $s \in \mathbb{C}$ we define $\sigma_s$ to be the genuine smooth admissible irreducible representation of $H(F)$ whose central character is

$$(a, \epsilon) \mapsto \chi_\sigma(a, \epsilon)||a||^s.$$  

Note that $\sigma_0 \simeq \sigma$. We shall denote by $I(\sigma, s)$ the principle series representation induced from $\sigma_s$. A realization for $I(\sigma, s)$ is given by inducing from $(\chi^\prime_\sigma)_\psi$. We shall denote this realization by $I(\chi^\prime_\psi, s)$.

### 2.3 Whittaker functionals

A $\psi$-Whittaker functional $\lambda$ on a representation $(\pi, V)$ of $SL_2(F)$ is a functional on $V$ which satisfies

$$\lambda \circ \pi(s(n(x))) = \psi(x)\lambda.$$  

Let $Wh_\psi(\pi)$ be the space of $\psi$-Whittaker functionals on $(\pi, V)$.

**Lemma 2.3.** Let $\sigma$ be a genuine smooth admissible irreducible representation of $\widehat{B(F)}$.

1. $\dim Wh_\psi(I(\sigma)) = d|d|^{-\frac{1}{2}}$

2. If $|\chi(\varpi^d)| < q^{Re(s)d}$ then for any $h \in \widehat{H(F)}$ and $f_s \in I(\chi^\prime_\psi, s)$ the integral

$$\int_F f_s(h\beta s(n(x))))\psi^{-1}(x) \, d\beta x$$
converges absolutely to a polynomial in $q^{-s}$. Moreover, for all $s$,

$$
\lim_{r \to \infty} \int_{F^r} f_s(hwn(n(x))) \psi^{-1}(x) \, d\psi x
$$

exists.

3. Let $\Delta_{h,\chi,\psi,s}(f_s)$ denote the analytic continuation of

$$
\int_{F} f_s(hs(wn(x))) \psi^{-1}(x) \, d\psi x
$$

The map $f_s \mapsto \Delta_{h,\chi,\psi,s}(f_s)$ is a $\psi$ Whittaker functional on $I(\chi_{\omega},\psi,s)$.

4. Let $R$ be a set of representatives of $\widetilde{H}(F)/\widetilde{H}(G)$. The set

$$
\{ \Delta_{h,\chi,\psi,s} \mid h \in R \}
$$

is a basis for $Wh_\psi(I(\chi',s))$.

Proof. This is well known. Part 1 of this Lemma was first proven in [9], Lemma 1.32, for unramified genuine series representations. See also Theorem 7 in [14] for a more general case containing ours. This part of the lemma is a Heredity of Rodier type, [15], namely, $\dim Wh_\psi(I(\sigma))$ equals $\dim Wh_\psi(\sigma)$. Since the unipotent radical of $H(F)$ is trivial we conclude that $\dim Wh_\psi(\sigma) = \dim \sigma$. The convergence and the analytic continuation of the integral in Part 2 are proven similar to the linear case. Part 3 is obvious and Part 4 follows since the functionals in discussion are linearly independent.

Note that the map

$$
h \mapsto \Delta_{h,\chi,\psi,s}
$$

is not a well defined map on $\widetilde{H}(F)/\widetilde{H}(G)$. Our novelty in this section is the following. Note that

$$(a,e)\widetilde{H}(G) \mapsto aG$$

is a natural isomorphism from $\widetilde{H}(F)/\widetilde{H}(G)$ to $F^*/G$. Recall that in Corollary 1.1 we have proven that choosing representatives for $F^*/G$ one identifies this group with $K_0$. Also note that the map

$$
x \mapsto \left( (\chi(1+s))_\psi \right)^{-1}(x) \Delta_{s(x),\chi,\psi,s}
$$

defined on $K$ factors through $F^{*d}$. Thus, For $a = kF^{*d} \in K_0$ we may normalize the Whittaker functionals and set

$$
\lambda_{a,\chi,\psi,s} = \left( (\chi(1+s))_\psi \right)^{-1}(k) \Delta_{s(k),\chi,\psi,s}.
$$

From the 4th assertion in Lemma 2.3 we conclude the following.

Corollary 2.1.

$$
\{ \lambda_{a,\chi,\psi,s} \mid a \in K_0 \}
$$

is a basis for $Wh_\psi(I(\chi',s))$. 
2.4 Metaplectic Shahidi local coefficients

Given two representations $\pi$ and $\varsigma$ of $\widetilde{SL}_2(F)$ and

$$A \in \text{Hom}_{\widetilde{SL}_2(F)}(\pi, \varsigma)$$

one defines

$$\widehat{A} : Wh_\psi(\varsigma) \to Wh_\psi(\pi)$$

by setting

$$\widehat{A}(\xi) = \xi \circ A.$$

**Remark 2.2.** If $\pi = \varsigma$ and $A$ is a scalar map corresponding to $\mu$ then $\widehat{A}$ is also a scalar map corresponding to $\mu$.

For $t \in \widetilde{H}(F)$ set $t^w = wtw^{-1}$. Let $\sigma^w$ is the smooth admissible irreducible representation of $\widetilde{H}(F)$ defined by

$$t \mapsto \sigma(t^w).$$

The following is proven as in the linear case. See Section 4 in [14] for example.

**Lemma 2.4.** For any $f_s \in I(\sigma, s)$ the integral

$$\int_F f_s(ws(n(x))g)d\psi x$$

converges absolutely to a rational function in $q^{-s}$ provided that $|\chi(\varpi^d)| < q^{\text{Re}(s)d}$. We shall denote its meromorphic continuation by $A_w(\sigma_s)(f_s)$. Away from its poles,

$$A_w(\sigma_s) \in \text{Hom}_{\widetilde{SL}_2(F)}(I(\sigma, s), (\sigma^w, s)).$$

Observe that $I(\chi'^{-1}_\psi, -s)$ is a realization of $I(\sigma^w, -s)$. When convenient we shall think of $A_w(\sigma_s)$ as an element of

$$\text{Hom}_{\widetilde{SL}_2(F)}(I(\chi'_\psi, s), I(\chi'^{-1}_\psi, -s))$$

and denote it by $A_w(\chi'_\psi, s)$. For $a \in K_0$ define now

$$\lambda^w_{a, \chi, \psi, s} = \widehat{A}_w(\chi'_\psi, s)(\lambda^w_{a, \chi^{-1}, \psi, -s}).$$

From Corollary [23] it follows that

$$\lambda^w_{a, \chi, \psi, s} = \sum_{b \in K_0} \tau_L(a, b, \chi, s, \psi) \lambda_b(\chi, \psi, s),$$

where the functions $\tau_L(a, b, \chi, s, \psi)$ are rational functions in $q^{-s}$.

**Theorem 2.1.**

$$\tau_L(a, b, \chi, s, \psi) = \begin{cases} \gamma_G((\chi \eta_{ab^{-1}})^{-1}, 1 - s, \psi, (ab)^{-1}) & n \text{ is odd;} \\ \overline{\gamma}_G((\chi \eta'^{-1}_{ab^{-1}})^{-1}, 1 - s, \psi, (ab)^{-1}) & n \equiv 2 \pmod{4}. \end{cases}$$
Proof. For $b \in K_0$ and $l >> 0$ let $f_{l,b,\chi,\psi, s} \in I(\chi'_\psi, s)$ be the function supported in
\[ \widetilde{B(G)}s(h(b)K_lw_0) \]
normalized such that
\[ f_{l,b,\chi,\psi, s}(hs(h(b))wk) = \text{Vol}^{-1}_\psi(\mathbb{P}^d_F)(\chi_{s+1, \psi, b})^{-1}(h) \]
for all $h \in \widetilde{B(G)}$, $k \in K_l^+ \cap s(N(F))$. Arguing as in Lemma 1.31 of [9] we obtain
\[ \lambda_{c, \chi, \psi, s}(f_{l,b,\chi,\psi, s}) = \begin{cases} ((\chi_{s+1, \psi, b})^{-1}) b = c; \\ 0 \quad b \neq c. \end{cases} \]
This implies that
\[ \tau_L(a, b, \chi, s, \psi) = (\chi_{1+s, \psi, b})\lambda_{s(a), \chi, \psi, s}(f_{l,b,\chi,\psi, s}). \]
We argue that for $\text{Re}(s) >> 0$
\[ \lambda_{a,\chi,\psi, s}(f_{l,b,\chi,\psi, s}) = (((\chi^{-1})_{1-s, \psi, b})^{-1}) \times (b, -a)_n|ab^{-1}| \lim_{r \to \infty} \int_{G_{ab}\mathbb{P}^r F} (\chi_s)(z a^{-1} b^{-1})(ab^{-1}, z)_n \psi(z) d^*_\psi z. \]
Indeed, this was proven in Lemma 4.2 of [8] in the case where $n$ is relatively prime to the residual characteristic of $F$ and $L$ is as in Example 1.1. The reader can verify that the argument there applies to the general case as well.

If $n$ is odd then $\chi_\psi = \chi$, $d = n$ and $(b, -a)_n = 1$ as both $a, b \in K_0$. Thus,
\[ \tau_L(a, b, \chi, s, \psi) = \lim_{r \to \infty} \int_{G_{ab}\mathbb{P}^r F} \chi_s(z) \eta_{ab^{-1}, s}(z) \psi(z) d^*_\psi z. \]
The theorem for the odd case now follows from the second assertion in Lemma 1.1. Suppose now that $n \equiv 2 \pmod{4}$. In this case
\[ \chi_\psi(x) = \gamma_{\psi}^{-1}(x)\chi(x) = \chi(x)\gamma_{\psi}(x)(x, -1)_2. \]
This gives
\[ \tau_L(a, b, \chi, s, \psi) = (b, -1)_2(a, b)_2(b, -a)_n \lim_{r \to \infty} \int_{G_{ab}\mathbb{P}^r F} \gamma_{\psi}^{-1}(z)(ab, z)_2(\chi_s)(z)(ab^{-1}, z)_n \psi(z) d^*_\psi(z). \]
Note now that since $n = 2d$ it follows that
\[ (ab, z)_2(ab^{-1}, z)_n = (ab^{-1}, z)_2(ab^{-1}, z)_n = (ab^{-1}, z)^d_n(ab^{-1}, z)_n = (ab^{-1}, z)^{d+1}_n. \]
Recall that in Remark 1.3 we have denoted $m = \frac{d+1}{2} \in \mathbb{N}$. With the notation introduced in 1.12 we have
\[ (ab^{-1}, z)_n^{d+1} = ((ab^{-1}, z)^2)_n^m = (ab^{-1}, z)^m_d = \eta'_{ab^{-1}}(z). \]
Using similar arguments one shows that
\[ (b, -1)_2(a, b)_2(b, -a)_n = \eta'_b(a). \]
As in the odd case, $\eta'_b(a) = 1$. The theorem for the even case now follows from 1.11. \qed
The inducing data for $I(\sigma)$ is $\chi_{\sigma} = (\chi_0)_\psi$ rather than $\chi_\psi$. Recall that all the extensions of $\chi_0$ to $F^*$ are given by $\chi_{\eta x}$ for some $x \in F^*/F^{*d}$. Any $x \in F^*/F^{*d}$ has the form $x = gkF^{*d}$ for some $g \in G_0$, $k \in K_0$. We argue that

$$\tau_L(a, b, \chi_{\eta x}, s, \psi) = \eta_g^{-1}(ab)\tau_L(a\sqrt{k}, b\sqrt{k^{-1}}, \chi, s, \psi).$$

(2.2)

Here $k \mapsto \sqrt{k}$ is the inverse of the automorphism $k \mapsto k^2$ of $K_0$. Indeed, the $K_0$ equivalence property follows at once from (2.1) while the $G_0$ equivalence property follows either from the definitions of $\gamma_G$ and $\wt{\gamma}_G$ or simply by noting that modifying $\chi$ by $\eta_g$ changes the normalization of $\lambda_{a, \chi, \psi, s}$. We have proven the following.

**Theorem 2.2.** Shahidi matrix of local coefficients $\tau_L(a, b, \chi, s, \psi)$ along with its translations given in the right hand side of (2.2) is an invariant associated with a genuine principal series representation $I(\sigma, s)$ of $\SL_2(F)$, an additive character $\psi$ of $F$ and a Lagrangian decomposition $L = (G_0, K_0)$ of $F^*/F^{*d}$.

Note that

$$\tau_L((ab, ab^{-1}, \chi, s, \psi) = \begin{cases} 
\gamma_G((\chi_{\eta a})^{-1}, 1 - s, \psi, b) & n \text{ is odd;} \\
\wt{\gamma}_G((\chi_{\eta a})^{-1}, 1 - s, \psi, b) & n \equiv 2 \pmod{4}.
\end{cases}$$

Hence, from (1.6) and (1.10) along with their inversions one deduce the following property of $\tau_L(a, b, \chi, s, \psi)$ which is independent of the choice of the Lagrangian decomposition.

**Proposition 1.**

$$\text{span}\{s \mapsto \tau(a, b, \chi, s, \psi) \mid a, b \in K_0\}
\begin{cases} 
\text{span}\{s \mapsto \gamma((\chi_{\eta x})^{-1}, 1 - s, \psi) \mid x \in F^*/F^{*d}\} & n \text{ is odd;} \\
\text{span}\{s \mapsto \wt{\gamma}((\chi_{\eta x})^{-1}, 1 - s, \psi) \mid x \in F^*/F^{*d}\} & n \equiv 2 \pmod{4}.
\end{cases}$$

2.5 $T(\sigma, s, \psi)$ and $D_L(\sigma, s, \psi)$.

$\tau_L(a, b, \chi, s, \psi)$ represents a linear map from $Wh_\psi(I(\chi^{-1}_\psi, s))$ to $Wh_\psi(I(\chi'_\psi, s))$ with respect to two compatible ordered bases. Taking (1.5) into account it seems natural to invert the ordering of one of the bases. We set

$$\tau_L(a, b, \chi_{\eta x}, s, \psi) = \tau_L(a, b^{-1}, \chi, s, \psi).$$

Observe that this inversion reflects the action of $w$ on $H(F)$. Namely,

$$\lambda_{a, \chi, \psi, s}^w = \sum_{b \in K_0} \tau_L(a, b, \chi, s, \psi)\lambda_{b\psi, \chi, \psi, s}.$$

Define

$$T(\sigma, s, \psi) = \text{trace}(\tau_L(a, b, \chi, s, \psi))$$

and

$$D_L(\sigma, s, \psi) = \text{det}(\tau_L(a, b, \chi, s, \psi)).$$

(2.3)
Theorem 2.3. $D_L(\sigma, s, \psi)$ is an invariant of $\sigma$, $\psi$ and $L$. $T(\sigma, s, \psi)$ is an invariant of $\sigma$ and $\psi$ only. Moreover

\[
T(\sigma, s, \psi) = \begin{cases} 
\sum_{\eta \in \hat{F}^*/F^*} \gamma(\chi \eta, s, \psi) & \text{if } n \text{ is odd;} \\
\sum_{\eta \in \hat{F}^*/F^*} \widetilde{\gamma}(\chi \eta, s, \psi) & \text{if } n \equiv 2 \pmod{4}
\end{cases}
\]

and for $\text{Re}(s) >> 0$

\[
T(\sigma, s, \psi) = d^{-1}\left|d\right|^{\frac{1}{2}} \lim_{r \to \infty} \left\{ \int_{F^* \cap F^d} \chi_{\sigma}(s(x)) \gamma^{-1}(x) \psi(x) d^*_\psi x \right\} \quad \text{is odd;}
\]

\[
\int_{F^* \cap F^d} \chi_{\sigma}(s(x)) \gamma^{-1}(x) \psi(x) d^*_\psi x \equiv 2 \pmod{4}.
\]

Proof. The translations in (2.2) are mapped to conjugations of $\tau_L(a, b, \chi, s, \psi)$. Thus, its trace and determinant are invariants of $L$, $\sigma$ and $\psi$. By computing the sign of the permutation $g \mapsto g^{-1}$ defined on $G_0$ one concludes that

\[
\det(\tau_L(\sigma, a, \chi, s, \psi)) = (-1)^{\frac{\#G_0 - 1}{2}} \det(\tau_L(a, b, \chi, s, \psi)).
\]

This proves the first assertion. The rest follows from a straightforward computation and from arguments we have already used.

\[\square\]

2.6 An Unramified computation

In this section only we assume that $n > 1$ is relatively prime to the residual characteristic of $F^*$ and that $L$ is the lagrangian decomposition given in Example 1.1. $\gamma_G(\chi, s, \psi, a)$, $\widetilde{\gamma}_G(\chi, s, \psi, a)$ and $\tau_L(a, b, \chi, s, \psi)$ were computed in \[8\] using slightly different definitions and normalization. We give the computations of $\gamma_G(\chi, s, \psi, a)$ and $\tau_L(a, b, 1, s, \psi)$ here in the case where $n$ is odd and $\psi$ is normalized to demonstrate the simplicity of our definitions and use these to compute $T(1_H, s, \psi)$ and $D_L(1_H, s, \psi)$. Here $1_H$ is the genuine smooth admissible irreducible representation of $H(F)$ whose central character is trivial on $s(H(F))$. It should be noted that similar type of computations can be used for the matrix of local coefficients in \[9\] for both odd and even fold covers of $GL_n(F)$. Namely, one does not need to use $\widetilde{\gamma}_G(\chi, s, \psi, a)$ for these covering groups. An analogous formula for $D_L(1_H, s, \psi)$ in the context of coverings of $GL_2(F)$ can be found in \[2\].

Let $u$ and $k$ be primitive elements in $G_0$ and $K_0$ respectively. We shall assume that $k$ is the image of $\varpi$ in $F^*/F^d$. Observe that $\eta_u$ is unramified. Denote

\[
\eta_u(x) = |x|^c.
\]

Let $\xi$ be the primitive $n^{th}$ root of 1 defined by

\[
\xi = \eta_q(k) = q^{-c}.
\]

Proposition 2. Let $h = k^t$. If $\chi$ is ramified then

\[
\gamma_G(\chi, s, \psi, h) = \begin{cases} 
e(\chi, s, \psi) & t \equiv e(\chi) \pmod{n}; \\
0 & \text{otherwise.}
\end{cases}
\]
Here $e(\chi)$ is the conductor of \(\chi\). Also,

\[
\gamma_G(1, s, \psi, h) = L(n(1 - s), 1) q^{(n-t)(s-1)} \times \begin{cases} L^{-1}(1 - (n(1 - s), 1) & t = 1; \\ (1 - q^{-1}) & 2 \leq t \leq n. \end{cases}
\]

Let \(a = k^i, b = k^j\) where \(0 \leq i, j < n, i \neq j\).

\[
\tau_L(a, b, 1, s, \psi) = \begin{cases} \epsilon(nh^a, 1 - s, \psi) & j + i = n - 1; \\ 0 & \text{otherwise.} \end{cases}
\]

\[
\tau_L(a, a, 1, s, \psi) = L(ns, 1) \times \begin{cases} q^{-2is}(1 - q^{-1}) & 0 \leq i < \frac{n-1}{2}; \\ q^{(1-n)sL^{-1}(1 - ns, 1)} & i = \frac{n-1}{2}; \\ q^{(n-2i)s}(1 - q^{-1}) & \frac{n-1}{2} < i \leq n - 1. \end{cases}
\]

**Proof.**

\[
\gamma_G(\chi, s, \psi, h) = \frac{1}{n} \sum_{g \in G_0} \gamma(\chi n g, s, \psi) n g^{-1}(h) = \frac{1}{n} \sum_{t=0}^{n-1} \gamma(\chi, s + lc, \psi) \xi^{-t}. 
\]

Suppose first that \(\chi\) is ramified. Then, from the known properties of \(\gamma\) and \(\epsilon\)-factors, see [22], we obtain

\[
\gamma(\chi, s + lc, \psi) = \epsilon(\chi, s + lc, \psi) = q^{-e(\chi)lc} \epsilon(\chi, s + lc, \psi).
\]

Thus,

\[
\gamma_G(\chi, s, \psi, h) = \epsilon(\chi, s, \psi) \frac{1}{n} \sum_{t=0}^{n-1} \xi^{e(\chi) - t}.
\]

The computation for the ramified case follows. We move to the unramified case.

\[
\gamma_G(1, s, \psi, h) = \frac{1}{n} \sum_{t=0}^{n-1} L(1 - s - lc, 1) L^{-1}(s + lc, 1) \xi^{-t} 
\]

\[
= \frac{1}{n} \prod_{t=0}^{n-1} (1 - q^{s-1} \xi^{-t})^{-1} \sum_{t=0}^{n-1} (1 - q^{-s} \xi^t) \xi^{-t} \prod_{0 \leq m \leq n-1, m \neq t} (1 - q^{s-1} \xi^{-m}).
\]

By using the elementary identities

\[
\prod_{m=0}^{n-1} (1 - x \xi^{-m}) = 1 - x^n, \quad \prod_{0 \leq m \leq n-1, m \neq t} (1 - x \xi^{-m}) = \sum_{m=0}^{n-1} x^m \xi^{-tm}.
\]

we deduce that

\[
\gamma_G(1, s, \psi, a, t) = L(n(1-s), 1) \sum_{m=0}^{n-1} q^{m(s-1)} \left( \frac{1}{n} \sum_{t=0}^{n-1} \xi^{-t(m+t)} - \sum_{m=0}^{n-1} q^{m(s-1)-s} \left( \frac{1}{n} \sum_{t=0}^{n-1} \xi^{-t(m+t-1)} \right) \right).
\]
The formula for the unramified case now follows. The formulas for \( \tau_L(a, b, 1, s, \psi) \) and \( \tau_L(a, a, 1, s, \psi) \) follow from the formulas for \( \gamma_G(\chi, s, \psi, a) \) and from the fact that since \( n \) is relatively prime to the residual characteristic of \( F^* \), the conductor of any ramified charter of \( F^* \) which is trivial on \( F^{\pi} \) is 1.

**Corollary 2.2.**

\[
T(1_H, s, \psi) = L(ns, 1)(1 - q^{-1}), \\
D_L(1_H, s, \psi) = (-1)^{\frac{n-1}{2}} q^{2s(1-n)-1}(\frac{a_1}{2}) L^n(1, ns)L^{-\frac{n+1}{2}}(1 - ns) L^{-\frac{n-1}{2}}(-1 - ns).
\]

**Proof.** From the formulas for \( \tau_L(a, b, 1, s, \psi) \) and \( \tau_L(a, a, 1, s, \psi) \) it follows that

\[
T(1_H, s, \psi) = \tau_L(1, 1, 1, s, \psi)
\]

and that

\[
D_L(1_H, s, \psi) = \tau_L(k^{\frac{n-1}{2}}, k^{\frac{n+1}{2}}, 1, s, \psi) \prod_{i=0}^{n-3} D_i.
\]

where

\[
D_i = \tau_L(k^i, k^i, 1, s, \psi) - \tau_L(k^{n-1-i}, k^{n-1-i}, 1, s, \psi) = \tau_L(k^i, k^{n-1-i}, 1, s, \psi) - \tau_L(k^{n-1-i}, k^i, 1, s, \psi).
\]

The first assertion is now clear. To prove the second assertion is is sufficient to show that for any \( 0 \leq i \leq \frac{n-3}{2} \),

\[
D_i = -q^{2s(1-n)-1} L^2(1, ns)L^{-1}(1 - ns)L^{-1}(-ns - 1).
\]

This follows from a direct computation along with the fact that for \( 1 \neq h \in K_0 \),

\[
e(\eta_h, 1 - s, \psi) = q^{2s-1}.
\]



**2.7 Plancherel measure**

Let \( \mu_n(\sigma, s) \) be the rational function defined by

\[
A_{w^{-1}}((\sigma^w)_s) \circ A_w(\sigma_s) = \mu_n^{-1}(\sigma, s) Id.
\]

Since we realize \( I(\sigma_s) \) as \( I(\chi_{\psi}', s) \) it is convenient to use the notation

\[
\mu(\chi_{\psi}', s) = \mu_n(\sigma, s).
\]

However, it is important to note that \( \mu(\chi_{\psi}', s) \) is an invariant of \( I(\sigma, s) \). It is independent of the choice of a maximal abelian subgroup and of the chosen extension of \( \chi_{\sigma} \). In particular

\[
\mu_n^{-1}(\chi_{\psi}', s) = (\# K_0)^{-1} \sum_{k \in K_0} \mu_n^{-1}(\chi_{\psi}', s).
\]

In the non metaplastic case, \( n = 1 \), it is well known that

\[
\mu_1^{-1}(\chi, s) = (\chi)_s \gamma(\chi^{-1}, 1 - s, \psi) = q^{\epsilon(\psi) - \epsilon(\chi)} \frac{L(s, \chi)L(-s, \chi^{-1})}{L(1 - s, \chi^{-1})L(1 + s, \chi)}. 
\]
Here \(e(\psi)\) is the conductor of \(\psi\). In the \(n = 2\) case, which is the only metaplectic case where \(\widetilde{H}(F)\) is abelian, we have

\[
\mu_2^{-1}(\chi \psi, s) = (-1,-1) \, 2 \chi \psi(-1) \, \overline{\gamma}(\chi, 1 - s, \psi) \, \overline{\gamma}(\chi, 1 + s, \psi) \quad (2.6)
\]

\[
= q^{e(\psi_2) - e(\psi)} \, L(2s, \chi^2) L(-2s, \chi^{-2}) \, L(1 - 2s, \chi^{-2}) L(1 + 2s, \chi^2).
\]

This formula follows from the \(n=2\) case in Theorem 2.1 along with (1.8) and (1.9) (in fact both (2.5) and (2.6) are proven below as particular cases).

**Theorem 2.4.** Let \(\sigma\) and \(\chi\) be as above.

\[
\mu_n^{-1}(\sigma, s) = \chi_\sigma(-I_2, (-1, -1)_n) D_L(\sigma, s, \psi) D_L(\sigma^w, -s, \psi) \quad (2.7)
\]

\[
\mu_n^{-1}(\sigma, s) = d^{-2} \left[ d \right] \sum_{\eta \in F^*/F^d} \mu_{n/d}^{-1}((\chi\eta)_\psi, s). \quad (2.8)
\]

**Proof.** First note that by Remark (2.2)

\[
\widehat{A}_w(\chi \psi, s) \circ \widehat{A}_w^{-1}(\chi^{-1}_\psi, -s) = \mu_1^{-1}(\chi'_\psi, s).
\]

Next note that since

\[w^{-1} = (-I_2, (-1, -1)_n) w\]

and since \((-I_2, (-1, -1)_n) \in \widetilde{H}_0(F)\) we obtain

\[
A_{w^{-1}}(\chi_{\psi}^{-1}, -s) = \chi_{\sigma}(-I_2, (-1, -1)_n) A_w(\chi_{\psi}^{-1}, -s). \quad (2.9)
\]

(2.7) now follows from the definition of \(D_L(\sigma, s, \psi)\) given in (2.3). (2.9) also implies that for any \(a \in K_0\)

\[
\mu_n^{-1}(\chi_{\psi}, s)(\chi_{\sigma})^{-1}(-I_2, (-1, -1)_n) = \sum_{k \in K_0} \tau_L(a, k, \chi, s, \psi) \tau_L(k, a, \chi^{-1}, -s, \psi). \quad (2.10)
\]

We shall pick \(a = 1\). Observe also that for any \(g \in G\)

\[
\chi_{\sigma}(-I_2, (-1, -1)_n) = (-1, -1)_n(\chi\eta)_\psi(-1) \quad (2.11)
\]

We first prove (2.8) under the assumption that \(n\) is odd. By (2.5), (2.10) and (2.11) it is sufficient to show that

\[
\sum_{k \in K_0} \tau_L(1, k, \chi, s, \psi) \tau_L(k, 1, \chi^{-1}, -s, \psi) \quad (2.12)
\]

\[
= [F^*: F^{*d}]^{-1} \sum_{x \in F^*/F^{*d}} \gamma(\chi^{-1} \eta_x^{-1}, 1 - s, \psi) \gamma(\chi\eta_x, 1 + s, \psi).
\]

By Theorem (2.1) and by (1.6)

\[
\sum_{k \in K_0} \tau_L(1, k, \chi, s, \psi) \tau_L(k, 1, \chi^{-1}, -s, \psi)
\]

\[
= [F^*: F^{*d}]^{-1} \sum_{k \in K_0} \sum_{h \in G_0} \sum_{g \in G_0} \gamma(\chi^{-1} \eta_{gk}, 1 - s, \psi) \gamma(\chi\eta_{hk^{-1}}, 1 + s, \psi) \eta_{gh}(k).
\]
We now change a summation index, \( g \mapsto h^{-1}g \). This gives

\[
\sum_{k \in K_0} \tau_L(1, k, \chi, s, \psi) \tau_L(k, 1, \chi^{-1}, -s, \psi) = [F^* : F^{*d}]^{-1} \sum_{k \in K_0} \sum_{h \in G_0} \sum_{g \in G_0} \gamma((\chi^{-1} \eta_{hk^{-1}})^{-1}\eta_g, 1 - s, \psi) \gamma(\chi \eta_{hk^{-1}}, 1 + s, \psi) \eta_g(k)
\]

\[
= [F^* : F^{*d}]^{-1} \sum_{x \in F^*/F^{*d}} \sum_{g \in G_0} \gamma((\chi \eta_x)^{-1}\eta_g, 1 - s, \psi) \gamma(\chi \eta_x, 1 + s, \psi) \eta_g(x^{-1}).
\]

By (2.10) along with (2.10) we now conclude that

\[
\sum_{k \in K_0} \tau_L(1, k, \chi, s, \psi) \tau_L(k, 1, \chi^{-1}, -s, \psi) = [F^* : F^{*d}]^{-1} \sum_{y \in K_0} \sum_{x \in F^*/F^{*d}} \sum_{g \in G_0} \gamma((\chi \eta_x)^{-1}\eta_g, 1 - s, \psi) \gamma(\chi \eta_x, 1 + s, \psi) \eta_g(x^{-1}).
\]

We finally change another summation index, \( x \mapsto y^{-1}x \) and change the order of summation.

\[
\sum_{k \in K_0} \tau_L(1, k, \chi, s, \psi) \tau_L(k, 1, \chi^{-1}, -s, \psi) = [F^* : F^{*d}]^{-1} \sum_{y \in K_0} \sum_{x \in F^*/F^{*d}} \sum_{g \in G_0} \gamma((\chi \eta_x)^{-1}\eta_g, 1 - s, \psi) \gamma(\chi \eta_x, 1 + s, \psi) \eta_g(x^{-1}y)
\]

\[
= [F^* : F^{*d}]^{-1} \sum_{x \in F^*/F^{*d}} \sum_{g \in G_0} \gamma((\chi \eta_x)^{-1}\eta_g, 1 - s, \psi) \gamma(\chi \eta_x, 1 + s, \psi) \eta_g(x^{-1}) \sum_{y \in K_0} \eta_g(y).
\]

(2.12) now follows. The proof for the case \( n \equiv 2 \pmod{4} \) follows in the same way, replacing \( \gamma \) by \( \widetilde{\gamma} \) and \( \eta \) by \( \eta' \).

\[\Box\]

**Corollary 2.3.** Suppose that \( \sigma \) is unitary. Then \( I(\sigma) \) is reducible if and only if \( n \) is odd and \( \chi_0 = \chi_0 \psi \), where \( \chi_0 \) is a non-trivial quadratic character of \( F^{*n} \).

**Proof.** From the Knapp-Stein dimension Theorem extended by Savin in [16] to a maximal parabolic induction on metaplectic groups, it follows that given that \( \sigma \) is unitary then \( I(\sigma) \) is reducible if and only if \( \sigma \simeq \sigma^w \) and \( \mu_n^{-1}(\sigma, s) \) is analytic at \( s = 0 \). See Section 5 of [8] for details. Denote \( \chi_\sigma = (\chi_0)^{\psi} \). Since \( \chi_{\sigma^w} = (\chi_0^{-1})^\psi \) it follows that \( \sigma \simeq \sigma^w \) is equivalent to \( \chi_0^2 = 1 \). Let \( \chi \) be any extension of \( \chi \) to \( F^* \). Since \( \chi_0^2 = 1 \) one concludes that the order of \( \chi \) divides \( 2d \). Suppose now that \( n \) is even. In this case \( 2d = n \). From (2.6) it follows that there exist \( [F^* : F^{*2}] \) identical summands in the right hand side of (2.8) which have a pole of order 2 at \( s = 0 \). The rest of the summands are analytic. Thus, \( \sigma \simeq \sigma^w \) implies that \( \mu_n^{-1}(\sigma, s) \) has a pole at \( s = 0 \). Suppose now that \( n \) is odd. If \( \chi \) is of order dividing \( n \) then \( \chi_\sigma \) is trivial and and from (2.5) it follows that exactly one of the summands in the right hand side of (2.8) has a pole of order 2 at \( s = 0 \). Thus, \( \mu_n^{-1}(\sigma, s) \) has a pole at \( s = 0 \) in this case. However if \( \chi_0^2 = 1 \) and \( \chi_0 \) is non-trivial then \( \chi \) must be of order \( 2n \). In this case all the summands in the right hand side of (2.8) are analytic. This implies that \( \mu_n^{-1}(\sigma, s) \) is analytic at \( s = 0 \).

\[\Box\]
Recall that $\widetilde{H}_0^{(n)}(F)$ is the center of $\widetilde{H}^{(n)}(F)$, the inverse image of $H(F)$ inside $\widetilde{SL}_2^{(n)}(F)$. From Lemma 2.1 it follows that if $m \mid n$ and $gcd(n,2) = gcd(m,2)$ then $\widetilde{H}_0^{(n)}(F)$ is a subgroup of $\widetilde{H}_0^{(m)}(F)$ although $\widetilde{H}^{(n)}(F)$ is not a subgroup of $\widetilde{H}^{(m)}(F)$. We say that a genuine smooth admissible irreducible representation $\sigma$ of $\widetilde{H}^{(n)}(F)$ and a genuine smooth admissible irreducible representation $\pi$ of $\widetilde{H}^{(m)}(F)$ are related if $\chi_\sigma$ is the restriction of $\chi_\pi$. From the description of the genuine characters of $\widetilde{H}_0^{(n)}(F)$ given in Section 2.2 one concludes that the set $E_m(\sigma)$ of genuine smooth admissible irreducible representations $\pi$ of $\widetilde{H}_0^{(m)}(F)$ related to $\sigma$ is parameterized by $H(m,n)$ which is defined to be the dual of $(F^{\ast n}/gcd(n,2)/F^{\ast})/(F^{\ast m}/gcd(m,2)/F^{\ast})$.

Thus, by writing the right hand side of (2.8) as

$$(\#E_m(\sigma))^{-1} \sum_{\pi \in H(m,n)} \left(\frac{m/gcd(m,2)}{m/gcd(m,2)}\right)^{1/2} \sum_{\eta \in F^{\ast m}/gcd(m,2)/F^{\ast}} \mu_{m/gcd(m,2)}((\chi_\eta \pi) \psi, s)$$

we have proven

**Corollary 2.4.**

$$\mu_{n}^{-1}(\sigma, s) = (\#E_m(\sigma))^{-1} \sum_{\pi \in E_m(\sigma)} \mu_{m}^{-1}(\pi, s).$$

In the case where $gcd(n, p) = 1$ it was proven in Theorem 5.1 of [8] that

$$\mu_{n}^{-1}(\sigma, s) = q^{e(\chi_n^{-1})-e(\psi)} \frac{L(ns, \chi^{-n})L(-ns, \chi^{-n})}{L(1-ns, \chi^{-n})L(1+ns, \chi^{-n})}.$$ 

This formula holds also when $n \equiv 0 \pmod{4}$ (in which case $SL_2^{(n)}(F)$ splits over $H_0^{(n)}(F)$ via the trivial section). By similar computations to those presented in Section 2.6, one shows that under the assumption $gcd(n, p) = 1$, (2.8) is equivalent to this last explicit formula. Thus, (2.8) also holds in the case $n \equiv 0 \pmod{4}$ provided that $gcd(n, p) = 1$. Same is true for Corollary 2.4.

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