ON A LOCALLY COMPACT MONOID OF COFINITE PARTIAL ISOMETRIES OF $\mathbb{N}$ WITH ADJOINED ZERO

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Abstract. Let $\mathcal{C}_\mathbb{N}$ be a monoid which is generated by the partial shift $\alpha : n \mapsto n + 1$ of the set of positive integers $\mathbb{N}$ and its inverse partial shift $\beta : n + 1 \mapsto n$. In this paper we prove that if $S$ is a submonoid of the monoid $\mathbb{N}_\infty$ of all partial cofinite isometries of positive integers which contains $\mathcal{C}_\mathbb{N}$ as a submonoid then every Hausdorff locally compact shift-continuous topology on $S$ with adjoined zero is either compact or discrete. Also we show that the similar statement holds for a locally compact semitopological semigroup $S$ with an adjoined compact ideal.

1. Introduction and preliminaries

In this paper we shall follow the terminology of [12, 13, 15, 16, 18, 38, 43]. The cardinality of a set $X$ is denoted by $|X|$. By $\mathbb{N}$ and $\mathbb{Z}$ we denote the sets of positive integers and the set of all integers. Also we identify $\omega$ with the set $\mathbb{N} \cup \{0\}$.

For any subset $A$ of $\mathbb{N}$ and any $i \in \mathbb{N}$ we denote $i + A = \{i + k : k \in A\}$ when $A \neq \emptyset$ and $i + \emptyset = \emptyset$.

If $S$ is a semigroup, then we shall denote the subset of all idempotents in $S$ by $E(S)$. If $S$ is an inverse semigroup, then $E(S)$ is closed under multiplication and we shall refer to $E(S)$ as a semilattice (or the semilattice of $S$).

A semigroup $S$ is called inverse if for any element $x \in S$ there exists a unique $x^{-1} \in S$ such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$. The element $x^{-1}$ is called the inverse of $x \in S$. If $S$ is an inverse semigroup, then the function $\text{inv} : S \to S$ which assigns to every element $x$ of $S$ its inverse element $x^{-1}$ is called the inversion. On an inverse semigroup $S$ the semigroup operation determines the following partial order $\preceq$: $s \preceq t$ if and only if there exists $e \in E(S)$ such that $s = te$. This order is called the natural partial order on $S$ and it induces the natural partial order on the semilattice $E(S)$ [44]. An inverse subsemigroup $T$ of an inverse semigroup $S$ is called full if $E(T) = E(S)$.

A congruence $\mathcal{C}$ on a semigroup $S$ is called a group congruence if the quotient semigroup $S/\mathcal{C}$ is a group. Any inverse semigroup $S$ admits the minimum group congruence $\mathcal{C}_{\text{mg}}$:

$$a \mathcal{C}_{\text{mg}} b \quad \text{if and only if} \quad \exists e \in E(S) \quad \text{such that} \quad ea = eb.$$ 

Also, we say that a semigroup homomorphism $\mathcal{h} : S \to T$ is a group homomorphism if the image $(S)\mathcal{h}$ is a group, and $\mathcal{h} : S \to T$ is trivial if it is either an isomorphism or annihilating.

The bicyclic monoid $\mathcal{C}(p, q)$ is the semigroup with the identity 1 generated by two elements $p$ and $q$ subjected only to the condition $pq = 1$. The semigroup operation on $\mathcal{C}(p, q)$ is determined as follows:

$$q^kp^lp^m = q^{k + m - \min(l, m)}p^{l + m - \min(l, m)}.$$

It is well known that the bicyclic monoid $\mathcal{C}(p, q)$ is a bisimple (and hence simple) combinatorial $E$-unitary inverse semigroup and every non-trivial congruence on $\mathcal{C}(p, q)$ is a group congruence [15].

If $\alpha : X \to Y$ is a partial map, then we shall denote the domain and the range of $\alpha$ by $\text{dom} \alpha$ and $\text{ran} \alpha$, respectively. A partial map $\alpha : X \to Y$ is called cofinite if both sets $X \setminus \text{dom} \alpha$ and $Y \setminus \text{ran} \alpha$ are finite.

The symmetric inverse (monoid) semigroup $\mathcal{I}_\lambda$ over a cardinal $\lambda$ is the set of all partial one-to-one transformations of a non-zero cardinal $\lambda$ endowed with the semigroup operation of composition of

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Date: December 6, 2022.

2020 Mathematics Subject Classification. 20M18, 20M20, 20M30, 22A15, 54A10, 54D45.

Key words and phrases. Partial isometry, inverse semigroup, partial bijection, bicyclic monoid, discrete, locally compact, topological semigroup, semitopological semigroup.
relations [15]. The symmetric inverse semigroup was introduced by Wagner [44] and it plays a major role in the theory of semigroups. By \( \mathcal{I}^e \) is denoted a subsemigroup of injective partial selfmaps of \( \lambda \) with cofinite domains and ranges. Obviously, \( \mathcal{I}^e \) is an inverse submonoid of the semigroup \( \mathcal{I}_\lambda \). The semigroup \( \mathcal{I}^e \) is called the monoid of injective partial cofinite selfmaps of \( \lambda \) [26].

A partial transformation \( \alpha: (X, d) \rightarrow (X, d) \) of a metric space \((X, d)\) is called isometric or a partial isometry, if \( d((x)\alpha, (y)\alpha) = d(x, y) \) for all \( x, y \in \text{dom} \alpha \). It is obvious that the set of partial cofinite isometries of a metric space \((X, d)\) with the operation of the composition of partial isometries is an inverse submonoid of the monoid of injective partial cofinite selfmaps of the cardinal \(|X|\).

We endow the sets \( \mathbb{N} \) and \( \mathbb{Z} \) with the standard linear order.

The semigroup \( \mathbb{ID}_\infty \) of all partial cofinite isometries of the set of integers \( \mathbb{Z} \) with the usual metric \( d(n, m) = |n - m|, \) \( n, m \in \mathbb{Z} \), was studied in [10, 11, 27].

Let \( \mathbb{IN}_\infty \) be the set of all partial cofinite isometries of the set of positive integers \( \mathbb{N} \) with the usual metric \( d(n, m) = |n - m|, \) \( n, m \in \mathbb{N} \). Then \( \mathbb{IN}_\infty \) with the operation of composition of partial isometries is an inverse submonoid of \( \mathcal{I}_\infty^e \). The semigroup \( \mathbb{IN}_\infty \) of all partial cofinite isometries of positive integers is studied in [28]. There we described the Green relations on the semigroup \( \mathbb{IN}_\infty \), its band and proved that \( \mathbb{IN}_\infty \) is a simple \( E \)-unitary \( F \)-inverse semigroup [38]. Also in [28], the least group congruence \( \mathcal{C}_{mg} \) on \( \mathbb{IN}_\infty \) is described and there it is proved that the quotient-semigroup \( \mathbb{IN}_\infty / \mathcal{C}_{mg} \) is isomorphic to the additive group of integers \( \mathbb{Z}(+) \). In [28] an example of a non-group congruence on the semigroup \( \mathbb{IN}_\infty \) is presented. Also it is proved that a congruence on the semigroup \( \mathbb{IN}_\infty \) is a group congruence if and only if its restriction onto any isomorphic copy of the bicyclic semigroup in \( \mathbb{IN}_\infty \) is a group congruence. In [30] it was shown that every non-annihilating homomorphism \( h: \mathbb{IN}_\infty \rightarrow \mathbb{ID}_\infty \) has the following property: the image \( (\mathbb{IN}_\infty)h \) is isomorphic either to the cyclic group \( \mathbb{Z}_2 \) of order 2 or the additive group of integers \( \mathbb{Z}(+) \). Also it is proved that \( \mathbb{IN}_\infty \) does not have a finite set of generators, and moreover it does not contain a minimal generating set.

Later by \( \mathbb{I} \) we denote the unit element of \( \mathbb{IN}_\infty \).

Remark 1.1. We observe that the bicyclic semigroup is isomorphic to the semigroup \( C_\mathbb{N} \) which is generated by partial transformations \( \alpha \) and \( \beta \) of the set of positive integers \( \mathbb{N} \), defined as follows:

\[
\text{dom} \alpha = \mathbb{N}, \quad \text{ran} \alpha = \mathbb{N} \setminus \{1\}, \quad (n)\alpha = n + 1
\]

and

\[
\text{dom} \beta = \mathbb{N} \setminus \{1\}, \quad \text{ran} \beta = \mathbb{N}, \quad (n)\beta = n - 1
\]

(see Exercise IV.1.11(ii) in [41]). It is obvious that \( \mathbb{I} = \alpha \beta = \alpha^0 = \beta^0 \) and \( C_\mathbb{N} \) is a submonoid of \( \mathbb{IN}_\infty \).

The semigroup of cofinite monotone (order preserving) bijective partial transformations of \( \mathbb{N} \) was introduced in [24] and there it was denoted by \( \mathcal{I}_\infty^\mathbb{N} \). Obviously, \( \mathcal{I}_\infty^\mathbb{N} \) is an inverse subsemigroup of the semigroup \( \mathcal{I}_\infty^e \). In [24] Gutik and Repovš studied properties of the semigroup \( \mathcal{I}_\infty^\mathbb{N} \). In particular, they showed that \( \mathcal{I}_\infty^\mathbb{N} \) is an inverse bisimple semigroup and all of its non-trivial group homomorphisms are either isomorphisms or group homomorphisms. It is obvious that \( \mathbb{IN}_\infty \) is an inverse submonoid of \( \mathcal{I}_\infty^\mathbb{N} \) [28].

A partial map \( \alpha: \mathbb{N} \rightarrow \mathbb{N} \) is called almost monotone if there exists a finite subset \( A \) of \( \mathbb{N} \) such that the restriction \( \alpha|_{\mathbb{N} \setminus A} : \mathbb{N} \setminus A \rightarrow \mathbb{N} \) is a monotone partial map. By \( \mathcal{I}_\infty^\mathbb{N} \) we shall denote the semigroup of cofinite almost monotone injective partial transformations of \( \mathbb{N} \). Obviously, \( \mathcal{I}_\infty^\mathbb{N} \) is an inverse subsemigroup of \( \mathcal{I}_\infty^e \) and the semigroup \( \mathcal{I}_\infty^\mathbb{N} \) is an inverse subsemigroup of \( \mathcal{I}_\infty^\mathbb{N} \), too. The semigroup \( \mathcal{I}_\infty^\mathbb{N} \) is studied in [14]. In particular, it is shown that the semigroup \( \mathcal{I}_\infty^\mathbb{N} \) is inverse, bisimple and all of its non-trivial group homomorphisms are either isomorphisms or group homomorphisms. In the paper [29] we showed that every automorphism of a full inverse subsemigroup of \( \mathcal{I}_\infty^\mathbb{N} \) which contains the semigroup \( C_\mathbb{N} \) is the identity map. Also there we constructed a submonoid \( \mathbb{IN}_\infty^{(1)} \) of \( \mathcal{I}_\infty^\mathbb{N} \) with the following property: if \( S \) is an inverse subsemigroup of \( \mathcal{I}_\infty^\mathbb{N} \) such that \( S \) contains \( \mathbb{IN}_\infty^{(1)} \) as a submonoid, then every non-identity congruence \( \mathcal{C} \) on \( S \) is a group congruence. We show that if \( S \) is an inverse submonoid of \( \mathcal{I}_\infty^\mathbb{N} \) such that \( S \) contains \( C_\mathbb{N} \) as a submonoid then \( S \) is simple and the quotient semigroup \( S/\mathcal{C}_{mg} \) is isomorphic to the additive group of
integers. Topologizations of inverse submonoids of $\mathcal{I}_\infty^\omega(\mathbb{N})$ and embeddings of such semigroups into compact-like topological semigroups are investigated in [14,29]. Similar results for semigroups of cofinite almost monotone partial bijections and cofinite almost monotone partial bijections of $\mathbb{Z}$ were obtained in [25].

For an arbitrary positive integer $n_0$ we denote $[n_0) = \{n \in \mathbb{N} : n \geq n_0\}$. Since the set of all positive integers is well-ordered, the definition of the semigroup $\mathcal{I}_\infty^\omega(\mathbb{N})$ implies that for every $\gamma \in \mathcal{I}_\infty^\omega(\mathbb{N})$ there exists the smallest positive integer $n_\gamma^d \in \text{dom } \gamma$ such that the restriction $\gamma|_{[n_\gamma^d)}$ of the partial map $\gamma : \mathbb{N} \to \mathbb{N}$ onto the set $[n_\gamma^d)$ is an element of the semigroup $\mathcal{C}_\mathbb{N}$, i.e., $\gamma|_{[n_\gamma^d)}$ is a shift of $[n_\gamma^d)$. For every $\gamma \in \mathcal{I}_\infty^\omega(\mathbb{N})$ we put $\gamma^\omega = \gamma|_{[n_\gamma^d)}$, i.e.

$$\text{dom } \gamma^\omega = [n_\gamma^d), \quad (x) \gamma^\omega = (x)\gamma \quad \text{for all } x \in \text{dom } \gamma^\omega \quad \text{and} \quad \text{ran } \gamma^\omega = \text{ran } \gamma.$$

Also, we put

$$n_\gamma^d = \min \text{ dom } \gamma \quad \text{for } \gamma \in \mathcal{I}_\infty^\omega(\mathbb{N}).$$

It is obvious that $n_\gamma^d = n_\gamma$ when $\gamma \in \mathcal{C}_\mathbb{N}$, and $n_\gamma^d < n_\gamma$ when $\gamma \in \mathcal{I}_\infty^\omega(\mathbb{N}) \setminus \mathcal{C}_\mathbb{N}$. Also for any $\gamma \in \text{IN}_\infty$ we denote

$$n_\gamma^r = (n_\gamma^d)\gamma = \min \text{ ran } \gamma \quad \text{and} \quad n_\gamma^r = (n_\gamma^d)\gamma.$$

The results of Section 3 of [30] imply that $n_\gamma^r - n_\gamma^d = n_\gamma - n_\gamma^d$ for any $\gamma \in \text{IN}_\infty$, and moreover for any non-negative integer $k$ we have that

$$\text{IN}_\infty^{[k]} = \{ \gamma \in \text{IN}_\infty : n_\gamma^r - n_\gamma^d \leq k \}$$

is a simple inverse subsemigroup of $\text{IN}_\infty$ such that $\text{IN}_\infty$ admits the following infinite semigroup series

$$\mathcal{C}_\mathbb{N} = \text{IN}_\infty^{[0]} \subset \text{IN}_\infty^{[1]} \subset \text{IN}_\infty^{[2]} \subset \text{IN}_\infty^{[3]} \subset \ldots \subset \text{IN}_\infty^{[k]} \subset \ldots \subset \text{IN}_\infty.$$

For any positive integer $k$ the semigroup $\text{IN}_\infty^{[k]}$ is called the monoid of cofinite isometries of positive integers with the noise $k$.

A topological space $X$ is called:

- **Baire** if for each sequence $A_1, A_2, \ldots, A_i, \ldots$ of nowhere dense subsets of $X$ the union $\bigcup_{i=1}^{\infty} A_i$ is a co-dense subset of $X$ [18,31];
- **locally dense** if every point $x$ of $X$ has an open neighbourhood with the compact closure.

A (semi)topological semigroup is a topological space endowed with a (separately) continuous semigroup operation. An inverse topological semigroup with continuous inversion is called a topological inverse semigroup.

A topology $\tau$ on a semigroup $S$ is called:

- a **semigroup** topology if $(S, \tau)$ is a topological semigroup;
- an **inverse semigroup** topology if $(S, \tau)$ is a topological inverse semigroup;
- a **shift-continuous** topology if $(S, \tau)$ is a semitopological semigroup.

The bicyclic monoid admits only the discrete semigroup Hausdorff topology [17]. Bertman and West in [9] extended this result for the case of Hausdorff semitopological semigroups. Stable and $\Gamma$-compact topological semigroups do not contain the bicyclic monoid [1,33,37]. The problem of embedding the bicyclic monoid into compact-like topological semigroups was studied in [3,4,8,23].

In the paper [21] we studied algebraic properties of the monoid $\text{IN}_\infty^{[j]}$ and extend the Eberhard-Selden and Bertman-West results [9,17] to the semigroups $\text{IN}_\infty^{[j]}$, $j \geq 0$. In particular it is shown that for any positive integer $j$ every Hausdorff shift-continuous topology $\tau$ on $\text{IN}_\infty^{[j]}$ is discrete and and if $\text{IN}_\infty^{[j]}$ is a proper dense subsemigroup of a Hausdorff semitopological semigroup $S$, then $S \setminus \text{IN}_\infty^{[j]}$ is a closed ideal of $S$, and moreover if $S$ is a topological inverse semigroup then $S \setminus \text{IN}_\infty^{[j]}$ is a topological group. Also it is described the algebraic and topological structure of the closure of the monoid $\text{IN}_\infty^{[j]}$ in a locally compact topological inverse semigroup.
The well known A. Weil Theorem states that every locally compact monothetic topological group $G$ (i.e., $G$ contains a cyclic dense subgroup) is either compact or discrete (see [45]). A semitopological semigroup $S$ is called monothetic if it contains a cyclic dense subsemigroup. Locally compact and compact monothetic topological semigroups were studied by Hewitt [32], Hofmann [34], Koch [36], Numakura [40] and others (see more information on this topics in the books [13] and [35]). Koch in [36] posed the following problem: “If $S$ is a locally compact monothetic semigroup and $S$ has an identity, must $S$ be compact?” From the other side, Zelenyuk in [46] constructed a countable monothetic locally compact topological semigroup without unit which is neither compact nor discrete and in [47] he constructed a monothetic locally compact topological monoid with the same property. The topological properties of monothetic locally compact (semi)topological semigroups studied in [2, 19, 48, 49].

In the paper [20] it is proved that every Hausdorff locally compact shift-continuous topology on the bicyclic monoid with adjoined zero is either compact or discrete. This result was extended by Bardyla onto the a polycyclic monoid [5] and graph inverse semigroups [6], and by Mokrytskyi onto the monoid of order isomorphisms between principal filters of $\mathbb{N}^n$ with adjoined zero [39]. Also, in [22] it is proved that the extended bicyclic semigroup $\mathcal{E}_2^0$ with adjoined zero admits continuum many different shift-continuous topologies, however every Hausdorff locally compact semigroup topology on $\mathcal{E}_2^0$ is discrete. In [7] Bardyla proved that a Hausdorff locally compact semitopological semigroup McAlister Semigroup $\mathcal{M}_1$ is either compact or discrete. However, this dichotomy does not hold for the McAlister Semigroup $\mathcal{M}_2$ and moreover, $\mathcal{M}_2$ admits continuum many different Hausdorff locally compact inverse semigroup topologies [7].

In this paper we extend the results of paper [20] onto the monoid $\mathbb{I}_\infty$ of all partial cofinite isometries of positive integers with adjoined zero. In particular we prove that if $S$ is a submonoid of $\mathbb{I}_\infty$ which contains $\mathcal{E}_\infty$ as a submonoid then every Hausdorff locally compact shift-continuous topology on $S$ with adjoined zero is either compact or discrete. Also we show that the similar statement holds for a locally compact semitopological semigroup $S$ with an adjoined compact ideal.

2. ON SUBMONOIDS OF THE MONOID $\mathbb{I}_\infty$

Lemma 2.1. Let $S$ be a subsemigroup of $\mathbb{I}_\infty$ which contains $\mathcal{E}_\infty$. Then $S$ is a simple inverse monoid.

Proof. The statement of the lemma is trivial when either $S = \mathbb{I}_\infty$ or $S = \mathcal{E}_\infty$. Hence, we suppose that $S$ is a proper submonoid of $\mathbb{I}_\infty$ which contains $\mathcal{E}_\infty$ as a proper submonoid. To prove that $S$ is an inverse semigroup, it is sufficient to show that every non-idempotent element of $S$ has inverse in $S$.

Let $\gamma$ be an arbitrary non-idempotent element of $S \setminus \mathcal{E}_\infty$. By Lemma 1 of [28], $\gamma$ is a shift of a subset $\gamma_0$ of $\mathbb{N}$. Put $\gamma_0 = \beta_0^{\infty} \alpha_0^{d_0}$. Then $\gamma \subseteq \text{ran} \gamma_0$, $\text{ran} \gamma \subseteq \text{dom} \gamma_0$, and hence

\[
(i)\gamma_0^{-1} = (i)\beta_0^{d_0} \alpha_0^\infty = (i)\gamma
\]

for all $i \in \text{dom} \gamma$. This implies that $\gamma \gamma_0$ is the identity map of the set $\text{dom} \gamma$ and $\gamma_0 \gamma$ is the identity map of the set $\text{ran} \gamma$. Hence we get that the elements $\gamma \gamma_0$ and $\gamma_0 \gamma$ are idempotents in the semigroup $S$, and $\gamma \gamma_0 \gamma = \gamma$. We claim that the element $\gamma_0 \gamma \gamma_0$ is inverse of $\gamma$. Indeed, we have that

\[
\gamma (\gamma_0 \gamma \gamma_0 \gamma) = (\gamma \gamma_0 \gamma \gamma_0) \gamma = \gamma \gamma_0 \gamma = \gamma
\]

and

\[
(\gamma_0 \gamma \gamma_0) \gamma (\gamma_0 \gamma \gamma_0) = \gamma_0 (\gamma_0 \gamma \gamma_0) (\gamma_0 \gamma \gamma_0) = \gamma_0 \gamma (\gamma_0 \gamma) = \gamma_0 (\gamma_0 \gamma) = \gamma_0 \gamma_0 = \gamma_0 \gamma_0.
\]

By Theorem 5 of [29] the monoid $S$ is simple. \qed
Fix an arbitrary non-empty finite subset $A$ of $\mathbb{N}$ such that $1 \in A$. Let $n^0$ be a positive integer such that $n^0 \geq \max A + 2$. We denote $A[n^0] = A \cup \{n^0\}$ and for any $i \in \omega$ we put

$$i + A[n^0] = \{i + k : k \in A[n^0]\}.$$ 

By Lemma 1 of [28] every element of the semigroup $\mathbb{IN}_\infty$ is a partial shift of a co-finite subset of $\mathbb{N}$. Hence in the above presented terminology we get the following descriptions of elements of the semigroup $\mathbb{IN}_\infty$.

**Corollary 2.2.** For every element $\gamma$ of the semigroup $\mathbb{IN}_\infty$ one of the following conditions holds:

(i) $\gamma$ is a shift of the set $\{n^0\}$ for some positive integer $n^0$, i.e., $\text{dom } \gamma = \{n^0\}$;

(ii) $\gamma$ is a shift of the set $i + A[n^0]$ for some $i \in \omega$, $n^0 \in \mathbb{N}$ and a non-empty finite subset $A \subset \mathbb{N}$ such that $\min A = 1$ and $n^0 \geq \max A + 2$, i.e., $\text{dom } \gamma = i + A[n^0]$.

For any non-empty finite subset $A \subset \mathbb{N}$ such that $\min A = 1$, for any nonnegative integer $i$ and any positive integer $n^0 \geq \max A + 2$ by $\varepsilon_A^{n^0}[i]$ we denote the identity map of the set $i + A[n^0]$.

Corollary 2.2 implies

**Proposition 2.3.** Let $\gamma$ be an element of the semigroup $\mathbb{IN}_\infty$ such that $\text{dom } \gamma = i + A[n^0]$ for some non-empty finite subset $A \subset \mathbb{N}$, a nonnegative integer $i$ and a positive integer $n^0 \geq \max A + 2$. Then there exists a nonnegative integer $j$ such that

$$\gamma = \varepsilon_A^{n^0}[i] \cdot \beta^i \alpha^j = \beta^i \alpha^j \cdot \varepsilon_A^{n^0}[j].$$

**Remark 2.4.** We observe that representation (2.1) of a such element $\gamma$ is not unique. Moreover for any non-negative integer $k$ such that $i - k \geq 0$ and $j - k \geq 0$ we have that

$$\gamma = \varepsilon_A^{n^0}[i] \cdot \beta^{i-k} \alpha^{j-k} = \beta^{i-k} \alpha^{j-k} \cdot \varepsilon_A^{n^0}[j].$$

Also, since in formula (2.1) we have that $i = n^d_\gamma$ and $j = n^r_\gamma$, the equalities (2.1) are called the canonical representations of $\gamma \in \mathbb{IN}_\infty$.

On any submonoid $S$ of the semigroup $\mathbb{IN}_\infty$ which contains $\mathcal{C}_\mathbb{N}$ we define a binary relation $\ll$ in the following way:

$$\gamma \ll \delta \quad \text{if and only if there exists a non-negative integer } i \text{ such that } \gamma = \beta^i \delta \alpha^i,$$

for $\gamma, \delta \in S$.

**Proposition 2.5.** The binary relation $\ll$ is a partial order on any submonoid $S$ of the semigroup $\mathbb{IN}_\infty$ which contains $\mathcal{C}_\mathbb{N}$.

**Proof.** Obviously that $\gamma = \beta^0 \gamma^0$ for any $\gamma \in S$, and hence $\ll$ is reflexive.

Suppose that $\gamma \ll \delta$ and $\delta \ll \gamma$ for some $\gamma, \delta \in S$. Then $\gamma = \beta^i \delta \alpha^i$ and $\delta = \beta^j \gamma \alpha^j$ for some $i, j \in \omega$. This implies that $\gamma = \beta^i \beta^j \gamma \alpha^j \alpha^i = \beta^{i+j} \gamma \alpha^{i+j}$. Then the definitions of elements $\alpha$ and $\beta$ imply that $n^d_\gamma \leq n^d_{\beta^{i+j} \gamma \alpha^{i+j}}$, and hence we get that $i + j = 0$. Then $\gamma = \delta$, and hence the binary relation $\ll$ is antisymmetric.

Suppose that $\gamma \ll \delta$ and $\delta \ll \eta$ for some $\gamma, \delta, \eta \in S$, i.e., $\gamma = \beta^i \delta \alpha^i$ and $\delta = \beta^j \eta \alpha^j$ for some $i, j \in \omega$. This implies that $\gamma = \beta^i \beta^j \eta \alpha^j \alpha^i = \beta^{i+j} \gamma \alpha^{i+j}$. Thus $\gamma \ll \eta$ and hence the binary relation $\ll$ is transitive. \(\square\)

Some interesting property of the relation $\ll$ on the semigroup $\mathbb{IN}_\infty$ presents the following proposition.

**Proposition 2.6.** The restriction of the binary relation $\ll$ on the subsemigroup $\mathcal{C}_\mathbb{N}$ of $\mathbb{IN}_\infty$ is the natural partial order on $\mathcal{C}_\mathbb{N}$. 
Proof. Suppose that $\beta^{i_1} \alpha^{j_1} \preceq \beta^{i_2} \alpha^{j_2}$ for some $\beta^{i_1} \alpha^{j_1}, \beta^{i_2} \alpha^{j_2} \in \mathcal{C}_N$. The definition of the relation $\preceq$ implies that there exists $i \in \omega$ such that $i_1 = i_2 + i$ and $j_1 = j_2 + i$. Then we have that
\[
\beta^{i_2} \alpha^{j_2} \cdot \beta^{j_2+i} \alpha^{j_2+i} = \beta^{i_2} \alpha^{j_2}(\beta^{j_2} \beta^{i}) \alpha^{j_2+i} = \beta^{i_2} \alpha^{j_2} = \beta^{i_2} \alpha^{j_2+i} = \beta^{i_2+i} \alpha^{j_2+i} = \beta^{i_1} \alpha^{j_1},
\]
and hence $\beta^{i_1} \alpha^{j_1} \preceq \beta^{i_2} \alpha^{j_2}$ in $\mathcal{C}_N$.

Suppose that $\beta^{i_1} \alpha^{j_1} \preceq \beta^{i_2} \alpha^{j_2}$ for some $\beta^{i_1} \alpha^{j_1}, \beta^{i_2} \alpha^{j_2} \in \mathcal{C}_N$. By Lemma 1.4.6 of [38] we have that
\[
\beta^{i_1} \alpha^{j_1} = \beta^{i_2} \alpha^{j_2} \cdot (\beta^{i_1} \alpha^{j_1})^{-1} \cdot \beta^{i_1} \alpha^{j_1} = \beta^{i_2} \alpha^{j_2} \cdot \beta^{i_1} \alpha^{j_1} = \beta^{i_2} \alpha^{j_2} \cdot \beta^{i_1} \alpha^{j_1} = \left\{ \begin{array}{ll} \beta^{i_2+j_2-j_1} \alpha^{j_1}, & \text{if } j_2 \leq j_1; \\
 \beta^{i_2} \alpha^{j_2}, & \text{if } j_2 > j_1. \end{array} \right.
\]
This implies $j_2 \leq j_1$ and $i_2 - j_2 + j_1 = i_1$, and hence $k = i_1 - i_2 = j_1 - j_2 \geq 0$. Then
\[
\beta^k \cdot \beta^{i_1} \alpha^{j_1} \cdot \alpha^k = \beta^{i_1-j_2} \cdot \beta^{i_2} \alpha^{j_2} \cdot \alpha^{j_1-j_2} = \beta^{i_1-j_2+i_2+j_2-j_2} = \beta^{i_1} \alpha^{j_2},
\]
which implies that $\beta^{i_1} \alpha^{j_1} \preceq \beta^{i_2} \alpha^{j_2}$ in $\mathcal{C}_N$.

For any $\gamma \in \mathcal{C}_N$ and $M \subseteq \mathcal{C}_N$ we denote
\[
\downarrow \ll \gamma = \{ \delta \in \mathcal{C}_N : \delta \ll \gamma \} \quad \text{and} \quad \downarrow \ll M = \bigcup_{\delta \in M} \downarrow \ll \delta.
\]

A linearly ordered set $(X, \ll)$ is called an $\omega$-chain if it is order isomorphic to $(\omega, \geq)$, where $\geq$ is the dual order to the usual linear order $\leq$ on $\omega$ [42].

**Lemma 2.7.** Let $S$ be a subsemigroup of $\mathcal{C}_N$ which contains $\mathcal{C}_N$. Then for any $\gamma \in S$, $\downarrow \ll \gamma \subseteq S$ and the poset $(\downarrow \ll \gamma, \ll)$ is an $\omega$-chain.

**Proof.** We define a mapping $f : \omega \to \downarrow \ll \gamma$ by the formula $f(i) = \beta^i \cdot \alpha^i$. Simple verifications show $f$ is an order isomorphism of $(\omega, \geq)$ onto $(\downarrow \ll \gamma, \ll)$.

This implies that $\downarrow \ll \gamma = \{ \delta \in \mathcal{C}_N : \delta \ll \gamma \}$ is a subset of $S$. \qed

The proof of Lemma 2.8 is obvious and it follows from the definition of the set $\downarrow \ll \gamma$ for any $\gamma \in \mathcal{C}_N$.

**Lemma 2.8.** Let $S$ be a subsemigroup of $\mathcal{C}_N$ which contains $\mathcal{C}_N$. Then for every $\gamma \in S$ the following statements hold:

1. $\beta^i \cdot \downarrow \ll \gamma \cdot \alpha^i \subseteq \downarrow \ll \gamma$ for any $i \in \omega$;
2. $\downarrow \ll \gamma \subseteq \downarrow \ll (\alpha^i \gamma \beta^i)$ if and only if $n^d_\gamma - i \geq 0$ and $n^r_\gamma - i \geq 0$.

It is easy to check that the elements $\gamma \in \mathcal{C}_N$ such that $n^d_\gamma = 0$ or $n^r_\gamma = 0$ are maximal in the poset $(\mathcal{C}_N, \ll)$.

If $A = \emptyset$ then we put $n^0 = 0$ and $\langle A[n^0] \rangle = \mathcal{C}_N$. Also, for any non-empty finite subset $A \subseteq \mathcal{N}$ such that min $A = 1$ and for any positive integer $n^0 \geq 2 + \max A$ we denote
\[
\langle A[n^0] \rangle = \left\{ \sum_{i, j \in \omega} a_{ij} \cdot \beta^i \cdot \alpha^j : a_{ij} \in \mathcal{C}_N \right\}.
\]

The proof of Lemma 2.1 and Proposition 2.3 imply the following proposition.

**Proposition 2.9.** Let $S$ be a subsemigroup of $\mathcal{C}_N$ which contains $\mathcal{C}_N$. Let $A$ be a non-empty finite subset of $\mathcal{N}$ such that min $A = 1$ and let $n^0$ be a positive integer such that $n^0 \geq 2 + \max A$. If $\langle A[n^0] \rangle \cap S \neq \emptyset$ then $\langle A[n^0] \rangle \subseteq S$. 
Lemma 2.10. For an arbitrary idempotent $\varepsilon^0_A[i] \in \mathbb{IN}_\infty \setminus \mathbb{C}_\infty$ the following conditions hold:

1. $\alpha^p \cdot \varepsilon^0_A[i] = \varepsilon^0_A[i - p] \cdot \alpha^p$ for any non-negative integer $p \leq i$;
2. $\beta^q \cdot \varepsilon^0_A[i] = \varepsilon^0_A[i + q] \cdot \beta^q$ for any non-negative integer $q$;
3. $\varepsilon_A^0[i] \cdot \alpha^p = \alpha^p \cdot \varepsilon_A^0[i + p]$ for any non-negative integer $p$;
4. $\varepsilon_A^0[i] \cdot \beta^q = \beta^q \cdot \varepsilon_A^0[i - q]$ for any non-negative integer $q \leq i$.

Proposition 2.11. Let $A$ be a non-empty finite subset of $\mathbb{N}$ such that $\min A = 1$ and let $n^0$ be a positive integer such that $n^0 \geq 2 + \max A$. Then $\ll A(n^0) = \langle A(n^0) \rangle$.

Proof. Fix an arbitrary $\delta \in \langle A(n^0) \rangle$. Then $\delta = \varepsilon_A^0[i] \cdot \beta^j \alpha^j$ for some $i, j \in \omega$. Lemma 2.10 implies that $\beta^j \cdot \varepsilon_A^0[i] \cdot \alpha^k = \varepsilon_A^0[i + k]$ for any $k \in \omega$. This implies that for any $k \in \omega$ we have that

$$
\beta^k \cdot \delta \cdot \alpha^k = \beta^k \cdot \varepsilon_A^0[i] \cdot \beta^j \alpha^j \cdot \alpha^k = \\
\beta^k \cdot \varepsilon_A^0[i] \cdot \alpha^k \beta^j \cdot \beta^j \alpha^j \cdot \alpha^k = \\
\varepsilon_A^0[i + k] \cdot \beta^j \alpha^j \cdot \alpha^k \in \langle A(n^0) \rangle,
$$

and hence $\ll \delta \in \langle A(n^0) \rangle$. This implies the equality $\ll \langle A(n^0) \rangle = \langle A(n^0) \rangle$. \qed

Lemma 2.12. Let $\varepsilon^0_{A_1}[i_1]$ and $\varepsilon^0_{A_2}[i_2]$ be idempotents of the monoid $\mathbb{IN}_\infty$. Then:

1. $\varepsilon^0_{A_1}[i_1] \cdot \beta^i \alpha^i$ is the canonical representation of $\varepsilon^0_{A_1}[i_1]$;
2. $\varepsilon^0_{A_1}[i_1] \ll \varepsilon^0_{A_2}[i_2]$ if and only if $A_1 = A_2$, $n^0_1 = n^0_2$ and $i_1 \geq i_2$;
3. if $\delta_1 = \varepsilon^0_{A_1}[i_1] \cdot \beta^i \alpha^i$ and $\delta_2 = \varepsilon^0_{A_2}[i_2] \cdot \beta^j \alpha^j$ are canonical representations, then $\delta_1 \ll \delta_2$ if and only if $A_1 = A_2$, $n^0_1 = n^0_2$ and $i_1 - i_2 = j_1 - j_2 = k$ for some $k \in \omega$.

Proof. Statement (1) is trivial because $i_1 = \min \text{dom} \varepsilon^0_{A_1}[i_1] = \min \text{ran} \varepsilon^0_{A_1}[i_1]$.

2. By (1), $\varepsilon^0_{A_1}[i_1] \cdot \beta^i \alpha^i$ and $\varepsilon^0_{A_2}[i_2] \cdot \beta^j \alpha^j$ are the canonical representations of idempotents $\varepsilon^0_{A_1}[i_1]$ and $\varepsilon^0_{A_2}[i_2]$, respectively. The relation $\varepsilon^0_{A_1}[i_1] \ll \varepsilon^0_{A_2}[i_2]$ holds if and only if $\varepsilon^0_{A_1}[i_1] = \beta^k \cdot \varepsilon^0_{A_2}[i_2] \cdot \alpha^k$ for some $k \in \omega$. By Lemma 2.10(2) we have that

$$
\varepsilon^0_{A_1}[i_1] = \varepsilon^0_{A_1}[i_1] \cdot \beta^i \alpha^i = \\
\beta^k \cdot \varepsilon^0_{A_2}[i_2] \cdot \beta^j \alpha^j \cdot \alpha^k = \\
\varepsilon^0_{A_2}[i_2 + k] \cdot \beta^j \alpha^j \cdot \alpha^k,
$$

and hence by (1), $\varepsilon^0_{A_1}[i_1] \ll \varepsilon^0_{A_2}[i_2]$ if and only if $A_1 = A_2$, $n^0_1 = n^0_2$ and $i_1 \geq i_2$.

3. Implication (⇐) is obvious.

(⇒) If $\delta_1 \ll \delta_2$ then there exists $k \in \omega$ such that $\delta_1 = \beta^k \delta_2 \alpha^k$ for some $k \in \omega$. By Lemma 2.10(4) we have that

$$
\delta_1 = \varepsilon^0_{A_1}[i_1] \cdot \beta^i \alpha^i = \beta^i \varepsilon^0_{A_1}[i_1] \cdot \alpha^i; \\
\delta_2 = \varepsilon^0_{A_2}[i_2] \cdot \beta^j \alpha^j = \beta^j \varepsilon^0_{A_2}[i_2] \cdot \alpha^j,
$$

and hence $\varepsilon^0_{A_1}[i_1] \ll \varepsilon^0_{A_2}[i_2]$ if and only if $A_1 = A_2$, $n^0_1 = n^0_2$ and $i_1 \geq i_2$. The proof is complete.
and since every element $\gamma$ of the monoid $\mathbb{N}_\infty$ has the unique canonical representation $\gamma = \varepsilon_A^n [i] \beta^i \alpha^j$, the equalities
\[
\beta^{i_1} \varepsilon_{A_1}^0 [0] \alpha^{j_1} = \delta_1 = \\
= \beta^k \delta_2 \alpha^k = \\
= \beta^k \beta^{i_2} \varepsilon_{A_2}^0 [0] \alpha^{j_2} \alpha^k = \\
= \beta^{i_2 + k} \varepsilon_{A_2}^0 [0] \alpha^{j_2 + k},
\]
implies that $A_1 = A_2, n_1^0 = n_2^0$ and $i_1 - i_2 = j_1 - j_2 = k$.

**Corollary 2.13.** If $\gamma$ and $\delta$ are comparable elements of the monoid $\mathbb{N}_\infty$ with respect to the partial order $\ll$ then either there exist a non-empty finite subset $A \subset \mathbb{N}$ such that $\min A = 1$ and a positive integer $n^0 \geq 2 + \max A$ such that $\gamma, \delta \in \langle A[n^0] \rangle$ or $\gamma, \delta \in \mathbb{C}_N$.

**Lemma 2.14.** Let $S$ be a subsemigroup of $\mathbb{N}_\infty$ which contains $\mathbb{C}_N$. Let $\gamma_0 \in S$ and $\beta^{i_1} \alpha^{j_1}, \beta^{i_2} \alpha^{j_2} \in S$ be such that $\gamma_0, \beta^{i_1} \alpha^{j_1}, \gamma_0, \beta^{i_2} \alpha^{j_2} \in \langle A[n^0] \rangle$, where $A$ is a non-empty finite subset of $\mathbb{N}$ such that $\min A = 1$ and a positive integer $n^0 \geq 2 + \max A$ or $\langle A[n^0] \rangle = \mathbb{C}_N$. Then
\[
\beta^{i_1} \alpha^{j_1} \cdot \eta \cdot \beta^{i_2} \alpha^{j_2} \in \downarrow \gamma_0 (\beta^{i_1} \alpha^{j_1} \cdot \gamma_0 \cdot \beta^{i_2} \alpha^{j_2}) \cap \langle A[n^0] \rangle
\]
for all $\eta \in \downarrow \gamma_0$.

**Proof.** By Proposition 2.9 if $\langle A[n^0] \rangle \cap \mathbb{C}_N \neq \emptyset$ then $\langle A[n^0] \rangle = \mathbb{C}_N$.

In the case when $\langle A[n^0] \rangle = \mathbb{C}_N$ by Proposition 2.6 the restriction of the partial order $\ll$ onto $\mathbb{C}_N$ is the natural partial order on $\mathbb{C}_N$, and next we apply Proposition 1.4.7 of [38].

Suppose that $\langle A[n^0] \rangle \subseteq S \setminus \mathbb{C}_N$ and let $\gamma_0 = \varepsilon_A^0 [i_0] \beta^{j_0} \alpha^{j_0}$ be the canonical representation of $\gamma_0$. Since $\gamma_0, \beta^{i_1} \alpha^{j_1}, \gamma_0, \beta^{i_2} \alpha^{j_2} \in \langle A[n^0] \rangle$ we have that $j_1 \leq i_0$ and $i_2 \leq j_0$. Lemma 2.10 implies that
\[
\beta^{i_1} \alpha^{j_1} \cdot \gamma_0 \cdot \beta^{i_2} \alpha^{j_2} = \beta^{i_1} \alpha^{j_1} \cdot \varepsilon_A^0 [i_0] \beta^{j_0} \alpha^{j_0} \cdot \beta^{i_2} \alpha^{j_2} \\
= \varepsilon_A^0 [i_0 - j_1] \beta^{j_0} \alpha^{j_0} \beta^{i_2} \alpha^{j_2} = \\
= \varepsilon_A^0 [i_0 - j_1 + i_1] \beta^{j_0} \alpha^{j_0} \beta^{i_2} \alpha^{j_2} = \\
= \varepsilon_A^0 [i_0 - j_1 + i_1] \beta^{j_0 - j_1 + i_1} \alpha^{j_0 - i_2 + j_2}.
\]

Fix an arbitrary $k \in \omega$ and put $\eta = b^k \gamma_0 \alpha^k$. Then by Lemma 2.10 for any $k \geq k_0$ we have that
\[
\beta^k \gamma_0 \alpha^k = \beta^k \varepsilon_A^0 [i_0] \beta^{j_0} \alpha^{j_0} \alpha^k = \\
= \varepsilon_A^0 [i_0 + k] \beta^k \beta^{j_0} \alpha^{j_0} \alpha^k = \\
= \varepsilon_A^0 [i_0 + k] \beta^{j_0 + k} \alpha^{j_0 + k}
\]
and since $j_1 \leq i_0$ and $i_2 \leq j_0$ we get that
\[
\beta^{i_1} \alpha^{j_1} \cdot \beta^k \gamma_0 \alpha^k \cdot \beta^{i_2} \alpha^{j_2} = \beta^{i_1} \alpha^{j_1} \cdot \varepsilon_A^0 [i_0 + k] \beta^{j_0 + k} \alpha^{j_0 + k} \cdot \beta^{i_2} \alpha^{j_2} = \\
= \beta^{i_1} \cdot \varepsilon_A^0 [i_0 - j_1 + k] \beta^{j_0 + k} \alpha^{j_0 + k} \beta^{i_2} \alpha^{j_2} = \\
= \varepsilon_A^0 [i_0 - j_1 + i_1 + k] \beta^{j_0 - j_1 + i_1 + k} \alpha^{j_0 - i_2 + j_2 + k}.
\]

Lemma 2.12(3) implies that $\beta^{i_1} \alpha^{j_1} \cdot \eta \cdot \beta^{i_2} \alpha^{j_2} \ll \beta^{i_1} \alpha^{j_1} \cdot \gamma_0 \cdot \beta^{i_2} \alpha^{j_2}$, and hence
\[
\beta^{i_1} \alpha^{j_1} \cdot \eta \cdot \beta^{i_2} \alpha^{j_2} \in \downarrow \gamma_0 (\beta^{i_1} \alpha^{j_1} \cdot \gamma_0 \cdot \beta^{i_2} \alpha^{j_2}) \cap \langle A[n^0] \rangle
\]
for all $\eta \in \downarrow \gamma_0$. \qed
3. On locally compact submonoids of $\mathbb{IN}_\infty$ with adjoined zero

In this section we assume that $S$ is a submonoid of the semigroup $\mathbb{IN}_\infty$ which contains $\mathbb{C}_N$. By $S^0$ we denote $S$ with the adjoined zero $0$.

**Definition 3.1 ([14]).** We shall say that a semigroup $S$ has:
- an $F$-property if for every $a, b, c, d \in S^1$ the sets $\{x \in S \mid a \cdot x = b\}$ and $\{x \in S \mid x \cdot c = d\}$ are finite;
- an $FS$-property if $S$ is simple and has $F$-property.

**Proposition 3.2.** If $S^0$ is a Hausdorff Baire semitopological semigroup, then $S$ is a discrete subspace of $S^0$.

**Proof.** Since $S^0$ is a Hausdorff space, $S$ is an open subspace of $S^0$ and hence the space $S$ is Baire.

By Proposition 2.2 of [24] the monoid $\mathcal{J}_\infty(\mathbb{N})$ has $F$-property. This implies that $\mathbb{IN}_\infty$ has $F$-property, and hence $S$ has $F$-property, too. By Theorem 5 of [29] the monoid $S$ is simple. Then by Theorem 5 of [14] every shift-continuous topology on $S$ is discrete.

**Corollary 3.3.** If $S^0$ is a Hausdorff locally compact semitopological semigroup, then $S$ is a discrete subspace of $S^0$.

Corollary 3.3 implies that every open neighbourhood $U(0)$ of $0$ in Hausdorff locally compact semitopological semigroup $S^0$ is a closed subset, i.e., the closure of an open neighbourhood $U(0)$ of $0$ coincides with $U(0)$. These arguments and Corollary 3.3 imply the following lemma.

**Lemma 3.4.** Let $\tau$ be a non-discrete Hausdorff locally compact shift-continuous topology on the semigroup $S^0$. Then for any compact-and-open neighbourhoods $U(0)$ and $V(0)$ of $0$ in $(S^0, \tau)$ both sets $U(0) \setminus V(0)$ and $V(0) \setminus U(0)$ are finite.

Later in all statements by $U(0)$ we denote any compact-and-open neighbourhood of zero in $(S^0, \tau)$.

**Lemma 3.5.** Let $\tau$ be a non-discrete Hausdorff locally compact shift-continuous topology on the semigroup $S^0$. Then there exists $\gamma^* \in S$ such that $\downarrow \gamma^* \cap U(0)$ is infinite.

**Proof.** Suppose to the contrary that the set $\downarrow \gamma \cap U(0)$ is finite for any $\gamma \in S$ and any neighbourhood $U(0)$ of $0$. This implies that the set $\downarrow \gamma \cap U(0)$ is non-empty for some infinitely many $\gamma \in S$. Hence for such elements $\gamma$ the set $\downarrow \gamma \cap U(0)$ contains the minimum and the maximum elements with the respect to the order $\ll$ on $S$. By separate continuity of the semigroup operation in $S^0$ there exists a compact-and-open neighbourhood $V(0) \subseteq U(0)$ of $0$ in $(S^0, \tau)$ such that $\beta \cdot V(0) \cdot \alpha \subseteq U(0)$. Then the above arguments and Lemma 2.8(1) imply that the set $U(0) \setminus V(0)$ is infinite, which contradicts Lemma 3.4. Hence there exists $\gamma_0 \in S$ such that the set $\downarrow \gamma_0 \cap U(0)$ is infinite.

**Proposition 3.6.** Let $\tau$ be a non-discrete Hausdorff locally compact shift-continuous topology on the semigroup $S^0$. Then there exists $\gamma^* \in S$ such that $\downarrow \gamma^* \subseteq U(0)$.

**Proof.** By Lemma 3.5 there exists $\gamma_0 \in S$ such that the set $\downarrow \gamma_0 \cap U(0)$ is infinite. We claim that the set $\downarrow \gamma_0 \setminus U(0)$ is finite. Suppose to the contrary that there exists a compact-and-open neighbourhood $V(0)$ of $0$ in $(S^0, \tau)$ such that the set $\downarrow \gamma_0 \setminus V(0)$ is infinite. By the separate continuity of the semigroup operation in $S^0$ there exists a compact-and-open neighbourhood $W(0) \subseteq V(0)$ of $0$ in $(S^0, \tau)$ such that $\beta \cdot W(0) \cdot \alpha \subseteq V(0)$. Then infiniteness of $\downarrow \gamma_0 \setminus V(0)$ and Lemma 2.8(1) imply that the set $V(0) \setminus W(0)$ is infinite, which contradicts Lemma 3.4. Hence the set $\downarrow \gamma_0 \setminus V(0)$ is finite, and by Lemma 3.4 the set $\downarrow \gamma_0 \setminus U(0)$ is finite, as well. By Lemma 2.8 there exists $\gamma^* \in \downarrow \gamma_0$ such that $\downarrow \gamma^* \subseteq U(0)$.

**Proposition 3.7.** Let $\tau$ be a non-discrete Hausdorff locally compact shift-continuous topology on the semigroup $S^0$. Then there exists a subset of the form $\langle A[n^0] \rangle$ in $S$ such that the set $\langle A[n^0] \rangle \setminus U(0)$ is finite.
Proof. By Proposition 3.6 there exists $\gamma_0 \in S$ such that $\downarrow \ll \gamma_0 \subset U(0)$. By Propositions 2.3 and 2.9 there exists a subset of the form $\langle A[n^0]\rangle$ in $S$ such that $\gamma_0 \in \langle A[n^0]\rangle$, and moreover Proposition 2.11 implies that $\downarrow \ll \gamma_0 \subset \langle A[n^0]\rangle$.

Suppose that $\gamma_0 \in \mathcal{C}_N$. Then we have that $\langle A[n^0]\rangle = \mathcal{C}_N$. By Corollary 3.3 every point of $S$ is isolated in $S^0$, and hence $\mathcal{C}_N^0 = \mathcal{C}_N \cup \{0\}$ is a closed subspace of $S^0$. Theorem 3.3.8 of [18] implies that the space $\mathcal{C}_N^0$ with the induced topology from $S^0$ is locally compact. Proposition 3.6 and Theorem 1 of [20] imply that the space $S^0$ is compact. This implies that the set $\langle A[n^0]\rangle \setminus U(0)$ is finite, because $\langle A[n^0]\rangle = \mathcal{C}_N$ in the case when an element $\gamma_0$ of the set $\langle A[n^0]\rangle$ belongs to $\mathcal{C}_N$.

Suppose that $\gamma_0 \in S \setminus \mathcal{C}_N$ and let $\gamma_0 = \varepsilon_A^0[i_0]\beta^0\alpha_0$ be the canonical representation of $\gamma_0$. Fix an arbitrary $\gamma \in \langle A[n^0]\rangle$ with the canonical representation $\delta = \varepsilon_A^0[i]\beta^\alpha$. By Lemma 2.10(4) we have that

$$\beta^\alpha \cdot \gamma_0 \cdot \beta^0 \cdot \gamma = \beta^\alpha \cdot \varepsilon_A^0[i_0] \cdot \beta^0 \cdot \gamma = \beta^\alpha \cdot \varepsilon_A^0[i_0 - i_0] \cdot \beta^0 \cdot \gamma = \beta^\alpha \cdot \varepsilon_A^0[0] \cdot (\alpha \beta^0) \cdot \gamma = \beta^i \cdot \varepsilon_A^0[i] \cdot \gamma = \varepsilon_A^0[i] \cdot \gamma$$

By the separate continuity of the semigroup operation in $(S^0, \tau)$ there exists a compact-and-open neighbourhood $V_\gamma(0)$ of $0$ such that $\beta^\alpha \cdot V_\gamma(0) \cdot \beta^0 \cdot \gamma \subseteq U(0)$. Lemma 3.4 implies that the set $U(0) \setminus V(0)$ is finite. Since $\downarrow \ll \gamma_0 \subset U(0)$ there exists $\gamma_1 \ll \gamma_0$ such that $\downarrow \ll \gamma_1 \subset V(0)$. By Lemma 2.12(3) there exists $k \in \omega$ such that $\gamma_1 = \varepsilon_A^0[i_0 + k] \beta^0 \cdot \alpha^0 \cdot k$ is the canonical representation of $\gamma_1$. Then we get that

$$\beta^\alpha \cdot \gamma_1 \cdot \beta^0 \cdot \gamma = \beta^\alpha \cdot \varepsilon_A^0[i_0 + k] \cdot \beta^0 \cdot \gamma = \beta^\alpha \cdot \varepsilon_A^0[0] \cdot (\alpha \beta^0) \cdot \gamma = \beta^i \cdot \varepsilon_A^0[0] \cdot \gamma = \varepsilon_A^0[i] \cdot \gamma$$

and hence $\beta^\alpha \cdot \downarrow \ll \gamma_1 \cdot \beta^0 \cdot \gamma = \downarrow \ll (\beta^\alpha \cdot \gamma_1 \cdot \beta^0 \cdot \gamma)$. This implies that for any neighbourhood $U(0)$ of the zero $0$ in $(S^0, \tau)$ and any $\gamma \in \langle A[n^0]\rangle$ there exists $\delta \in \langle A[n^0]\rangle$ such that $\downarrow \ll \gamma \ll \delta \subseteq U(0)$.

Suppose to the contrary that the set $\langle A[n^0]\rangle \setminus U(0)$ is infinite. Then the above arguments imply that there exists an infinite sequence $\{\gamma_p\}_{p \in \omega} \subseteq \langle A[n^0]\rangle \setminus U(0)$ such that all elements of the sequence $\{\gamma_p\}_{p \in \omega}$ are incomparable with respect to the partial order $\ll$ on $S^0$ and $\downarrow \ll \gamma_p \subseteq U(0)$.

By the separate continuity of the semigroup operation in $(S^0, \tau)$ there exists a compact-and-open neighbourhood $W(0)$ of $0$ such that $\alpha \cdot W(0) \cdot \beta \subseteq U(0)$. The above arguments imply that $\{\gamma_p\}_{p \in \omega} \cap U(0) = \emptyset$ and $\{\beta^0 \cdot \alpha\}_{p \in \omega} \cap U(0) = \emptyset$. But $\{\beta^0 \cdot \alpha\}_{p \in \omega} \cap W(0) = \emptyset$ and hence the set $U(0) \setminus W(0)$ is infinite, which contradicts Lemma 3.4. The obtained contradiction completes the proof of the proposition.

Proposition 3.8. Let $\tau$ be a non-discrete Hausdorff locally compact shift-continuous topology on the semigroup $S^0$. If there exists a subset of the form $\langle A[n^0]\rangle$ in $S$ such that the set $\langle A[n^0]\rangle \cap U(0)$ is infinite then the set $\langle A[n^0]\rangle \setminus U(0)$ is finite.

Proof. We claim that there exists $\gamma_0 \in \langle A[n^0]\rangle$ such that $\downarrow \ll \gamma_0 \subset U(0)$. Suppose the contrary: the set $\downarrow \ll \gamma_0 \setminus U(0)$ is infinite for any $\gamma_0 \in S$. By the separate continuity of the semigroup operation in $S^0$ there exists a neighbourhood $W(0) \subseteq V(0)$ of $0$ in $(S^0, \tau)$ such that $\beta \cdot W(0) \cdot \alpha \subseteq V(0)$. Then the
set $U(0) \setminus V(0)$ is infinite, which contradicts Lemma 3.4. Hence $\subseteq \gamma_0 \subseteq U(0)$ for some $\gamma_0 \in \langle A[n^0] \rangle$.

Next, in a similar way as in the proof of Proposition 3.7 it can be show that the set $\langle A[n^0] \rangle \setminus U(0)$ is finite.

**Proposition 3.9.** Let $\tau$ be a non-discrete Hausdorff locally compact shift-continuous topology on the semigroup $S^0$. Then for any subset of the form $\langle A[n^0] \rangle$ in $S$ the set $\langle A[n^0] \rangle \setminus U(0)$ is finite.

**Proof.** Fix any arbitrary open set $U(0)$ of the zero $0$ in $(S^0, \tau)$. First we shall show that the set $\langle A[n^0] \rangle \setminus U(0)$ is finite for $\langle A[n^0] \rangle = \mathcal{C}_N$. By Proposition 3.7 there exists a subset of the form $\langle A[n^0] \rangle$ in $S$ such that the set $\langle A[n^0] \rangle \setminus U(0)$ is finite. Then there exists a positive integer $k_V$ such that the element $\gamma_l = \epsilon_A^n[0] \beta^n \alpha^n \in \langle A[n^0] \rangle$ belongs to $U(0)$ for any $l \geq k_V$. Since the set $U(0) \setminus W(0)$ finite for any compact-and-open neighbourhood $W(0)$ of $0$, there exists a positive integer $k_W \geq k_V$ such that the element $\gamma_l = \epsilon_A^n[0] \beta^n \alpha^n \in \langle A[n^0] \rangle$ belongs to $W(0)$ for any $l \geq k_V$.

By the separate continuity of the semigroup operation in $S^0$ there exists a compact-and-open neighbourhood $W(0) \subseteq U(0)$ of the zero $0$ in $(S^0, \tau)$ such that $\alpha^n \cdot W(0) \subseteq U(0)$. Since $\operatorname{ran}(\alpha^n) \subseteq \operatorname{dom}(\epsilon_A^n[0])$ and $\epsilon_A^n[0]$ is an idempotent of $S$, we get that $\alpha^n \cdot \epsilon_A^n[0] = \alpha^n$, and hence $\alpha^n \cdot \gamma_l = \epsilon_A^n[0] \beta^n \alpha^n \alpha^n = \epsilon_A^n[0] \beta^n \alpha^n = \alpha^n \cdot \gamma_{l+1} \in U(0)$, which implies that the set $\mathcal{C}_N \cap U(0)$ is finite. Then by Proposition 3.8 the set $\mathcal{C}_N \cap U(0)$ is finite.

Fix any subset of the form $\langle A[n^0] \rangle$ in $S$ distinct from $\mathcal{C}_N$. By the separate continuity of the semigroup operation in $S^0$ there exists a neighbourhood $V(0) \subseteq U(0)$ of the zero $0$ in $(S^0, \tau)$ such that $\epsilon_A^n[0] \cdot V(0) \subseteq U(0)$. By the previous part of the proof the set $\mathcal{C}_N \setminus V(0)$ is finite, and hence there exists a positive integer $k_V$ such that the element $\gamma_l = \beta^n \in \mathcal{C}_N$ belongs to $V(0)$ for any $l \geq k_V$. This implies that the neighbourhood $U(0)$ contains the infinite set $\{ \epsilon_A^n[0] \cdot \beta^n : l \geq k_V \}$. By Proposition 3.8 the set $\langle A[n^0] \rangle \setminus U(0)$ is finite.

**Lemma 3.10** shows that on a semigroup $S^0$, where $S$ be a some subsemigroup of $\mathbb{IN}_\infty$ which contains $\mathcal{C}_N$, there exists a Hausdorff topology $\tau_{\mathbb{A}}$ such that $(S^0, \tau_{\mathbb{A}})$ is a compact semitopological semigroup.

**Lemma 3.10.** Let $T$ be a semigroup with $F$-property and $T^0$ be the semigroup $T$ with adjoined zero. Let $\tau_{\mathbb{A}}$ be the topology on $T^0$ such that

(i) every element of $T$ is an isolated point in the space $(T^0, \tau_{\mathbb{A}})$;

(ii) the family $\mathcal{B}(0) = \{ U \subseteq T^0 : U \ni 0 \text{ and } T^0 \setminus U \text{ is finite} \}$ determines a base of the topology $\tau_{\mathbb{A}}$ at zero $0 \in T^0$.

Then $(T^0, \tau_{\mathbb{A}})$ is a Hausdorff compact semitopological semigroup.

**Proof.** Since all points of the semigroup $T$ are isolated in $(T^0, \tau_{\mathbb{A}})$ it is sufficient to show that the semigroup operation in $(T^0, \tau_{\mathbb{A}})$ is separately continuous in the following two cases:

$\gamma \cdot 0 = 0$ and $0 \cdot \gamma = 0$, for $\gamma \in T$.

Fix an arbitrary open set $U(0) \in \mathcal{B}(0)$. Then $K = T^0 \setminus U(0)$ is finite. Let

$K_\gamma = \{ \delta \in T : \gamma \cdot \delta \in K \} \cup \{ \delta \in T : \delta \cdot \gamma \in K \}$.

Since the semigroup $T$ has $F$-property, the set $K_\gamma$ is finite. Then

$\gamma \cdot V(0) \subseteq U(0)$ and $V(0) \cdot \gamma \subseteq U(0)$

for $V(0) = U(0) \setminus K_\gamma \in \mathcal{B}(0)$.

**Remark 3.11.** By Corollary 3.3 the discrete topology is a unique Hausdorff locally compact shift-continuous topology on a subsemigroup $S$ of $\mathbb{IN}_\infty$ which contains $\mathcal{C}_N$. So $\tau_{\mathbb{A}}$ is the unique compact shift-continuous topology on $S^0$.

**Theorem 3.12.** Let $S$ be a some susmigroup of $\mathbb{IN}_\infty$ which contains $\mathcal{C}_N$. Then every Hausdorff locally compact shift-continuous topology $\tau$ on the semigroup $S^0$ is either compact or discrete.
Proof. If the zero \(0\) of the semigroup \(S^0\) is an isolated point in \((S^0, \tau)\) then the space \(S\) is locally compact. By Corollary 3.3, \(S\) is discrete and hence \((S^0, \tau)\) is discrete, too.

Suppose that \(0\) is a nonisolated point in \((S^0, \tau)\). Let \(U(0)\) be any neighbourhood of \(0\) in \((S^0, \tau)\). By Proposition 3.9 for any subset of the form \(\langle A[n^0]\rangle\) in \(S\) the set \(\langle A[n^0]\rangle \setminus U(0)\) is finite. If the semigroup \(S\) is the union of a finite family \(\{\mathcal{C}_N, \langle A_1[n^0]\rangle, \ldots, \langle A_k[n^0]\rangle\}\) of subsets of \(S\), such that \(\min A_i = 1\) and \(n_i^0 \geq 2 + \max A_i\) for all positive integer \(i \leq k\), \(k \in \mathbb{N}\), then Proposition 3.9 implies that \(S^0 \setminus U(0)\) is finite, and hence the space \((S^0, \tau)\) is compact.

Next, we suppose that the semigroup \(S\) is the union of an infinite family of distinct sets
\[
\{\mathcal{C}_N, \langle A_1[n^0]\rangle, \ldots, \langle A_k[n^0]\rangle, \ldots\}
\]
of \(S\), such that \(\min A_i = 1\) and \(n_i^0 \geq 2 + \max A_i\) for \(i \in \mathbb{N}\). Suppose to the contrary that there exists a neighbourhood \(U(0)\) of \(0\) in \((S^0, \tau)\) such that \(S^0 \setminus U(0)\) is infinite. By Proposition 3.9 without loss of generality we may assume that there exists an infinite sequence \(\{\gamma_i\}_{i \in \mathbb{N}}\) of distinct elements of \(S^0 \setminus U(0)\) which satisfies the following properties:

1. \(\gamma_i \in \langle A_i[n^0]\rangle\) for any \(i \in \mathbb{N}\);
2. \(\beta \cdot \gamma_i \cdot \alpha \in U(0)\) for any \(i \in \mathbb{N}\);
3. if \(i \neq j\) then \(\langle A_i[n^0]\rangle \cap \langle A_j[n^0]\rangle = \emptyset\).

By the separate continuity of the semigroup operation in \((S^0, \tau)\) there exists an open neighbourhood \(W(0) \subset U(0)\) of \(0\) such that \(\alpha \cdot W(0) \cdot \beta \subset U(0)\). Then our assumption implies that \(\beta \cdot \gamma_i \cdot \alpha \notin W(0)\) for any \(i \in \mathbb{N}\). Hence the set \(U(0) \setminus W(0)\) is infinite which contradicts Lemma 3.4. The obtained contradiction implies that the space \((S^0, \tau)\) is compact.

Since the bicyclic monoid does not embed into any Hausdorff compact topological semigroup [1], Theorem 3.12 implies the following corollary.

Corollary 3.13. Let \(S\) be a some subsemigroup of \(\mathbb{IN}_\infty\) which contains \(\mathcal{C}_N\). Then every Hausdorff locally compact semigroup topology on \(S^0\) is discrete.

Remark 3.14. In the paper [20] the example that a counterpart of the statement of Corollary 3.13 (and hence the statement of Theorem 3.12) does not hold when \(\mathcal{C}_N\) is a \(\check{C}\)ech-complete metrizable topological inverse semigroup is constructed.

Later we need the following trivial lemma, which follows from separate continuity of the semigroup operation in semitopological semigroups.

Lemma 3.15. Let \(S\) be a Hausdorff semitopological semigroup and \(I\) be a compact ideal in \(S\). Then the Rees-quotient semigroup \(S/I\) with the quotient topology is a Hausdorff semitopological semigroup.

Lemma 3.16. Let \(X\) be a Hausdorff locally compact space and \(I\) be a compact subset of \(X\). Then there exists an open neighbourhood \(U(I)\) of \(I\) with the compact closure \(\overline{U(I)}\).

Proof. Fix for any \(x \in I\) an open neighbourhood \(U(x)\) of \(x\) in \(X\) such that the closure \(\overline{U(x)}\) is compact. Then \(\{U(x) : x \in I\}\) is an open cover of \(I\). Since \(I\) is compact, \(I \subseteq U(I) = U(x_1) \cup U(x_2) \cup \cdots \cup U(x_k)\) for some finitely many \(x_1, x_2, \ldots, x_k \in I\). Then the set \(\overline{U(I)} = \overline{U(x_1)} \cup \overline{U(x_2)} \cup \cdots \cup \overline{U(x_k)}\) is compact. \(\square\)

Theorem 3.17. Let \(S\) be a some subsemigroup of \(\mathbb{IN}_\infty\) which contains \(\mathcal{C}_N\). Let \((S_I, \tau)\) be a Hausdorff locally compact semitopological semigroup, where \(S_I = S \cap I\) and \(I\) is a compact ideal of \(S_I\). Then either \((S_I, \tau)\) is a compact semitopological semigroup or the ideal \(I\) open.

Proof. Suppose that \(I\) is not open. By Lemma 3.15 the Rees-quotient semigroup \(S_I/I\) with the quotient topology \(\tau_q\) is a semitopological semigroup. Let \(\pi : S_I \to S_I/I\) be the natural homomorphism which is a quotient map. It is obvious that the Rees-quotient semigroup \(S_I/I\) is isomorphic to the semigroup \(S^0\) and without loss of generality we can assume that \(\pi(S_I) = S^0\) and the image \(\pi(I)\) is zero of \(S^0\).

By Lemma 3.16 there exists an open neighbourhood \(U(I)\) of \(I\) with the compact closure \(\overline{U(I)}\). Since by Corollary 3.3 every point of \(S\) is isolated in \((S_I, \tau)\) we have that \(\overline{U(I)} = U(I)\) and its image \(\pi(U(I))\) is
compact-and-open neighbourhood of zero in $S^0$. Since for any open neighbourhood $V(I)$ of $I$ in $(S_I, \tau)$ the set $\bar{U}(I) \cap V(I)$ is compact, Theorem 3.12 implies that $S^0 \setminus \pi(U(I))$ is finite for any compact-and-open neighbourhood $U(I)$ of $I$ in $(S_I, \tau)$. Then compactness of $I$ implies that $(S_I, \tau)$ is compact as well.

**Corollary 3.18.** Let $S$ be a some submonoid of the semigroup $\mathbb{IN}_\infty$ which contains $\mathbb{C}_\mathbb{N}$. If $(S_I, \tau)$ is a locally compact topology topological semigroup, where $S_I = S \sqcup I$ and $I$ is a compact ideal of $S_I$, then the ideal $I$ is open.

**Acknowledgements**

The authors acknowledge Alex Ravsky and the referees for useful important comments and suggestions.

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