A Shift from Democratic to Tri-bimaximal Neutrino Mixing with Relatively Large $\theta_{13}$

Zhi-zhong Xing *
Institute of High Energy Physics, Chinese Academy of Sciences, Beijing 100049, China

Abstract

Recent neutrino oscillation data hint that the smallest neutrino mixing angle $\theta_{13}$ is possible to lie in the range $5^\circ \lesssim \theta_{13} \lesssim 12^\circ$. We show that reasonable perturbations to the democratic mixing pattern, which is geometrically related to the tri-bimaximal mixing pattern through an equal shift $\theta_* \simeq 9.7^\circ$ of two large mixing angles, can naturally produce a nearly tri-bimaximal neutrino mixing matrix $V$ with sufficiently large $\theta_{13}$. Two especially simple but viable scenarios of $V$ are proposed and their phenomenological consequences are discussed.

PACS number(s): 14.60.Pq, 13.10.+q, 25.30.Pt

*E-mail: xingzz@ihep.ac.cn
Recent solar, atmospheric, reactor and accelerator neutrino oscillation experiments have provided us with very convincing evidence that neutrinos are massive and lepton flavors are mixed [1]. The mixing of lepton flavors is effectively described by a $3 \times 3$ unitary matrix $V$, whose nine elements can be parametrized in terms of three rotation angles and three CP-violating phases. Defining three rotation matrices in the $(1,2)$, $(1,3)$ and $(2,3)$ planes as

$$
R_{12}(\theta_{12}) = \begin{pmatrix}
  c_{12} & s_{12} & 0 \\
  -s_{12} & c_{12} & 0 \\
  0 & 0 & 1 
\end{pmatrix},
$$

$$
R_{13}(\theta_{13}) = \begin{pmatrix}
  c_{13} & 0 & s_{13} \\
  0 & 1 & 0 \\
  -s_{13} & 0 & c_{13} 
\end{pmatrix},
$$

$$
R_{23}(\theta_{23}) = \begin{pmatrix}
  1 & 0 & 0 \\
  0 & c_{23} & s_{23} \\
  0 & -s_{23} & c_{23} 
\end{pmatrix},
$$

(1)

where $c_{ij} \equiv \cos \theta_{ij}$ and $s_{ij} \equiv \sin \theta_{ij}$ (for $1 \leq i < j \leq 3$), one may parametrize $V$ in nine topologically different ways [2]. The so-called standard parametrization takes the form

$$
V = R_{23}(\theta_{23}) \otimes P_\delta \otimes R_{13}(\theta_{13}) \otimes P_\delta^\dagger \otimes R_{12}(\theta_{12}) \otimes P_\nu \\
= \begin{pmatrix}
  c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\
  -s_{12}c_{23} - c_{12}s_{13}s_{23}e^{i\delta} & c_{12}c_{23} - s_{12}s_{13}s_{23}e^{i\delta} & c_{13}s_{23} \\
  s_{12}s_{23} - c_{12}s_{13}c_{23}e^{i\delta} & -c_{12}s_{23} - s_{12}s_{13}c_{23}e^{i\delta} & c_{13}c_{23} 
\end{pmatrix} P_\nu,
$$

(2)

in which $P_\delta = \text{Diag}\{1,1,e^{i\delta}\}$ and $P_\nu = \text{Diag}\{e^{i\rho},e^{i\sigma},1\}$ are two diagonal phase matrices containing three CP-violating phases. A recent global analysis of current neutrino oscillation data yields $\theta_{12} = 34.5^\circ \pm 1.0^\circ$, $\theta_{13} = 5.1_0^{+3.0}_{-3.3}^\circ$ and $\theta_{23} = 42.8_0^{+4.7}_{-2.8}^\circ$ at the 1$\sigma$ level [3], but three phases of $V$ remain entirely unconstrained. The ongoing and forthcoming neutrino oscillation experiments will measure $\theta_{13}$ and $\delta$. On the other hand, the neutrinoless double-beta decay experiments will help to probe or constrain $\rho$ and $\sigma$.

The smallness of $\theta_{13}$ and the largeness of $\theta_{12}$ and $\theta_{23}$ have motivated some speculations about a constant neutrino mixing matrix with $\theta_{13} = 0^\circ$, such as the “democratic” pattern

$$
U_0 = \begin{pmatrix}
  \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} & 0 \\
  -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} \\
  \sqrt{\frac{1}{3}} & -\sqrt{\frac{1}{3}} & \sqrt{\frac{1}{3}} 
\end{pmatrix}
$$

(3)

with $\theta_{12}^{(0)} = 45^\circ$, $\theta_{13}^{(0)} = 0^\circ$ and $\theta_{23}^{(0)} = \arctan(\sqrt{2}) \approx 54.7^\circ$ [4] or the “tri-bimaximal” pattern

$$
V_0 = \begin{pmatrix}
  \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{3}} & 0 \\
  -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{6}} & \sqrt{\frac{1}{2}} \\
  \sqrt{\frac{1}{6}} & -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{2}} 
\end{pmatrix}
$$

(4)

with $\theta_{12}^{(0)} = \arctan(1/\sqrt{2}) \approx 35.3^\circ$, $\theta_{13}^{(0)} = 0^\circ$ and $\theta_{23}^{(0)} = 45^\circ$ [5]. Either of them can be obtained in the limit of a certain flavor symmetry (e.g., the discrete $S(3)$ flavor symmetry.
for $U_0$ or the discrete $A_4$ symmetry for $V_0$ [6]), and the latter has to be broken in order to generate nonzero $\theta_{13}$ and CP violation. Since $V_0$ is much closer to the best-fit values of current data on three neutrino mixing angles, it has recently attracted much more interest.

Note that the entries of $V_0$ are actually the same as those of $U_0$, although their positions are essentially different. Hence it is interesting to explore not only an intrinsic relationship between $U_0$ and $V_0$ but also how to link them to the realistic neutrino mixing matrix $V$ via reasonable perturbations. Note also that there are some preliminary hints that the smallest neutrino mixing angle $\theta_{13}$ might not be very small. For example, $\theta_{13} \simeq 5.1^{+3.9}_{-3.3}^\circ$ (1$\sigma$) by Gonzalez-Garcia et al [3], $\theta_{13} \simeq 7.3^{+2.9}_{-2.2}^\circ$ (1$\sigma$) by Fogli et al [7], and $\theta_{13} \simeq 8.1^{+2.8}_{-4.5}^\circ$ as the best-fit value by the KamLAND Collaboration [8]. Although the statistical significance of these results remains quite low, they do imply that $\theta_{13}$ is possible to lie in the range $5^\circ \lesssim \theta_{13} \lesssim 12^\circ$. On the theoretical side, it is certainly likely that $\theta_{13}$ may take a value in the above range [9]. So it makes sense to discuss how to confront a constant neutrino mixing pattern with a relatively large value of $\theta_{13}$. On the other hand, a definite determination of $\theta_{13}$ may serve as a crucial turning-point of experimental neutrino physics to the era of precision measurements, in which the detection of leptonic CP violation and the search for new physics will become feasible [10].

Let us pose two immediate and interesting questions: (1) what is the geometric relation between the democratic and tri-bimaximal neutrino mixing patterns? (2) which of them is more natural to receive relatively significant perturbations in order to accommodate relatively large $\theta_{13}$? In this paper we shall point out that two large mixing angles predicted in the democratic mixing pattern $U_0$ are intrinsically related to their counterparts in the tri-bimaximal mixing pattern $V_0$ through an equal shift

$$
\theta_* \equiv \theta_{12}^{(0)} - \vartheta_{12}^{(0)} = \theta_{23}^{(0)} - \vartheta_{23}^{(0)} \simeq 9.7^\circ .
$$

(5)

This geometric relation keeps unchanged if a universal perturbation is imposed onto three mixing angles of $U_0$ or $V_0$. Although it is likely to generate relatively large $\theta_{13}$ by perturbing $V_0$, the perturbation term has to be adjusted in such a way that two large mixing angles of $V_0$ are slightly modified but the smallest angle of $V_0$ is significantly modified. In contrast, it is more natural to produce a nearly tri-bimaximal neutrino mixing matrix $V$ with sufficiently large $\theta_{13}$ by introducing comparable perturbations to three mixing angles of $U_0$. We shall propose two especially simple but viable scenarios of $V$ — scenario A is based on the standard parametrization of $U_0$ and scenario B relies on a very useful parametrization proposed by Fritzsch and Xing (FX) [11]. Scenario A predicts $\theta_{12} \simeq 35.3^\circ$, $\theta_{13} \simeq 9.7^\circ$ and $\theta_{23} = 45^\circ$ together with the maximal strength of leptonic CP violation $J_{\text{max}} = (\sqrt{2} + 1)/(36\sqrt{3}) \simeq 3.9\%$; and scenario B predicts $28.3^\circ \lesssim \theta_{12} \lesssim 42.2^\circ$, $\theta_{13} \simeq 6.9^\circ$ and $\theta_{23} \simeq 44.6^\circ$ together with $J_{\text{max}} = 1/36 \simeq 2.8\%$. Both scenarios are in good agreement with current data, and they can soon be tested in a variety of neutrino oscillation experiments.

First of all, the following relation between the democratic mixing matrix $U_0$ and the tri-bimaximal mixing matrix $V_0$ comes into our notice:

$$
V_0 = R_{23}^T(\theta_*) \otimes U_0 \otimes R_{12}^T(\theta_*) ,
$$

(6)

where “$T$” means a transpose, and $\theta_*$ has been defined in Eq. (5). As a matter of fact, $U_0$ and $V_0$ can be decomposed into
We proceed to discuss reasonable perturbations to the democratic mixing pattern $U_0$ so as to produce a realistic neutrino mixing matrix $V$ with naturally large $\theta_{13}$. To illustrate, we propose two simple but viable scenarios of $V$ in two different parametrizations of $U_0$. They are based on the standard and FX parametrizations of $U_0$, leading respectively to the predictions $\theta_{13} \simeq 9.7^\circ$ and $\theta_{13} \simeq 6.9^\circ$ in the standard parametrization of $V$.

\[ U_0 = R_{23}(45^\circ + \theta_\ast) \otimes R_{12}(45^\circ), \]
\[ V_0 = R_{23}(45^\circ) \otimes R_{12}(45^\circ - \theta_\ast). \]  

(7)

Two nonzero mixing angles of $U_0$ turn out to be $\theta_{12}^{(0)} = 45^\circ$ and $\theta_{23}^{(0)} = 45^\circ + \theta_\ast$, and those of $V_0$ are $\vartheta_{12}^{(0)} = 45^\circ - \theta_\ast$ and $\vartheta_{23}^{(0)} = 45^\circ$. Their geometrical relations in the real plane are shown in FIG. 1. So $\theta_{12}^{(0)}$ and $\theta_{23}^{(0)}$ are intrinsically related to $\vartheta_{12}^{(0)}$ and $\vartheta_{23}^{(0)}$ via an equal shift $\theta_\ast \simeq 9.7^\circ$. Although the size of $\theta_\ast$ is not small, it is smaller than the Cabibbo angle of quark mixing (i.e., $\theta_C \simeq 13^\circ$ [1]). In this sense we argue that $V_0$ can be regarded as a consequence of $U_0$ whose $(1,2)$ and $(2,3)$ mixing angles are corrected by $\theta_\ast$ in a destructive way.

Since $V_0$ itself is very close to the best-fit result obtained from current experimental data on three neutrino mixing angles [3], any possible perturbations to $V_0$ must be small enough (see, e.g., Refs. [12–14]). In the standard parametrization the overall perturbation matrix can be expressed as

\[ \Omega_\varepsilon = R_{23}^T(\varepsilon_{23}) \otimes P_\delta \otimes R_{13}(\varepsilon_{13}) \otimes P_\delta^\dagger \otimes R_{12}^T(\varepsilon_{12}) \]  

(8)

with $|\varepsilon_{ij}| \ll 1$ (for $ij = 12, 13, 23$), and thus the overall neutrino mixing matrix is given by

\[
V = R_{23}(45^\circ) \otimes \Omega_\varepsilon \otimes R_{12}(45^\circ - \theta_\ast) \otimes P_\nu \\
= R_{23}(45^\circ - \varepsilon_{23}) \otimes P_\delta \otimes R_{13}(\varepsilon_{13}) \otimes P_\delta^\dagger \otimes R_{12}(45^\circ - \theta_\ast - \varepsilon_{12}) \otimes P_\nu
\]  

(9)

with $\theta_{12} = 45^\circ - \theta_\ast - \varepsilon_{12}$, $\theta_{13} = \varepsilon_{13}$ and $\theta_{23} = 45^\circ - \varepsilon_{23}$. In view of $\theta_{12} = 34.5^\circ \pm 1.0^\circ$, $\theta_{13} = 5.1^{+3.0}_{-3.9}^\circ$ and $\theta_{23} = 42.8^{+4.7}_{-3.9}^\circ$ (1$\sigma$) extracted from a global analysis of current neutrino oscillation data [3], one immediately obtains $\varepsilon_{12} = 0.8^\circ \pm 1^\circ$, $\varepsilon_{13} = 5.1^{+3.0}_{-3.3}^\circ$ and $\varepsilon_{23} = 2.2^{+4.7}_{-3.9}^\circ$. One might argue that it would be somewhat unnatural if a perturbation to the smallest mixing angle of $V_0$ were much larger than the ones to two large angles of $V_0$. In this sense $|\varepsilon_{13}| \lesssim |\varepsilon_{12}| \lesssim |\varepsilon_{23}|$ seems to be a natural choice of three perturbation parameters, just corresponding to the fact $\theta_{13} < \theta_{12} < \theta_{23}$. Then $\theta_{13} = \varepsilon_{13}$ is expected to be very small, and it is most likely to lie in the range $0^\circ \lesssim \theta_{13} < 5^\circ$. In other words, it seems rather unlikely to obtain $\theta_{13} \gtrsim 5^\circ$ by introducing *natural* perturbations to three mixing angles of $V_0$. Note, however, that such arguments might not work when the neutrino mixing matrix is derived from a lepton mass model. From the point of view of model building, one is not subject to the assumption of $|\varepsilon_{13}| \lesssim |\varepsilon_{12}| \lesssim |\varepsilon_{23}|$ because three mixing angles may receive contributions from both the charged-lepton and neutrino sectors at the tree level [15] and they can also receive appreciable quantum corrections at the loop level [16].

Different from $V_0$, $U_0$ is not so close to the best-fit neutrino mixing pattern extracted from a global analysis of current neutrino oscillation data. Hence large perturbations to three mixing angles of $U_0$ can naturally be allowed, in order to bring $U_0$ to a phenomenologically favored form. In this case even $\theta_{13} \sim \theta_\ast$ may be achieved from reasonable perturbations to $U_0$, as we shall see later on.
Scenario A

The standard parametrization of $U_0$ has been given in Eq. (7). Here we assume a universal angle $\varepsilon_{12} = \varepsilon_{13} = \varepsilon_{23} = \theta_*$ to perturb three mixing angles of $U_0$, in order to obtain a nearly tri-bimaximal neutrino mixing matrix $V$ with sufficiently large $\theta_{13}$. Such a perturbation term is reasonable in the sense that its magnitude is actually smaller than the Cabibbo angle $\theta_C$. Let us recall that the realistic Cabibbo-Kobayashi-Maskawa quark mixing matrix [1] can be regarded as a result obtained from small perturbations to the identity matrix, in which the maximal perturbation term is just characterized by $\sin \theta_C$. Now a realistic neutrino mixing matrix $V$ arises from a universal perturbation of $O(\theta_*)$ to three mixing angles of $U_0$. In this case the overall perturbation matrix reads

$$\Omega_* = R_{23}^T(\theta_*) \otimes P_\delta \otimes R_{13}(\theta_*) \otimes P_\delta^\dagger \otimes R_{12}^T(\theta_*) . \quad (10)$$

The resultant neutrino mixing matrix is

$$V = R_{23}(45^\circ + \theta_*) \otimes \Omega_* \otimes R_{12}(45^\circ) \otimes P_\nu$$

$$= R_{23}(45^\circ) \otimes P_\delta \otimes R_{13}(\theta_*) \otimes P_\delta^\dagger \otimes R_{12}(45^\circ - \theta_*) \otimes P_\nu$$

$$= \left( \begin{array}{ccc}
\sqrt{\frac{1}{3}} c_* & \sqrt{\frac{1}{3}} c_* & s_* e^{-i\delta} \\
-\sqrt{\frac{1}{6}} c_* - \sqrt{\frac{1}{6}} s_* e^{i\delta} & \sqrt{\frac{1}{6}} c_* - \sqrt{\frac{1}{6}} s_* e^{i\delta} & \sqrt{\frac{1}{2}} c_* \\
-\sqrt{\frac{1}{6}} c_* - \sqrt{\frac{1}{6}} s_* e^{i\delta} & -\sqrt{\frac{1}{6}} c_* - \sqrt{\frac{1}{6}} s_* e^{i\delta} & \sqrt{\frac{1}{2}} c_* 
\end{array} \right) P_\nu , \quad (11)$$

where $c_* \equiv \cos \theta_* = (\sqrt{2} + 1)/\sqrt{6}$ and $s_* \equiv \sin \theta_* = (\sqrt{2} - 1)/\sqrt{6}$. This ansatz apparently predicts $\theta_{12} = 45^\circ - \theta_* \simeq 35.3^\circ$, $\theta_{13} = \theta_* \simeq 9.7^\circ$ and $\theta_{23} = 45^\circ$. So it is a nearly tri-bimaximal neutrino mixing pattern with a very appreciable value of $\theta_{13}$. The Jarlskog parameter of leptonic CP violation [17] is given by

$$J = c_{12} s_{12} c_{13} s_{13} c_{23} s_{23} \sin \delta = \frac{\sqrt{2} + 1}{36 \sqrt{3}} \sin \delta \lesssim 0.039 \sin \delta \quad (12)$$

in this scenario. The relatively large $\theta_{13}$ and (perhaps) $|J|$ make scenario A easily testable in a variety of neutrino oscillation experiments in the near future.

Scenario B

In the FX parametrization a generic $3 \times 3$ neutrino mixing matrix $V$ is expressed as

$$V = R_{12}(\theta_i) \otimes R_{23}(\theta) \otimes P_\phi^\dagger \otimes R_{12}^T(\theta_\nu) \otimes P_\nu$$

$$= \left( \begin{array}{ccc}
s_i s_\nu c + c_i c_\nu e^{-i\phi} & s_i c_\nu c - c_i s_\nu e^{-i\phi} & s_i s_\nu \\
-c_i s_\nu c - s_i c_\nu e^{-i\phi} & c_i c_\nu c + s_i s_\nu e^{-i\phi} & c_i s_\nu \\
-s_\nu s & -c_\nu s & c 
\end{array} \right) P_\nu , \quad (13)$$

where $P_\phi = \text{Diag}\{e^{i\phi}, 1, 1\}$ and $P_\nu = \text{Diag}\{e^{i\nu}, e^{i\sigma}, 1\}$ together with $c_i \equiv \cos \theta_i$, $s_i \equiv \sin \theta_i$, $c_\nu \equiv \cos \theta_\nu$, $s_\nu \equiv \sin \theta_\nu$, $c \equiv \cos \theta$ and $s \equiv \sin \theta$. This representation of $V$ has proved to be more convenient and useful than the standard one in deriving the one-loop renormalization-group equations of three neutrino mixing angles and three CP-violating phases [18] and in
linking flavor mixing parameters to the ratios of charge-lepton and neutrino masses [19]. It
coincides with the standard parametrization in the $\theta_l \rightarrow 0$ limit (up to a rearrangement
of the phase convention), in which $\theta_{12} = \theta_\nu$ and $\theta_{23} = \theta$ exactly hold. Hence the democratic
mixing pattern $U_0$ can also be decomposed into a product of $R_{23}(45^\circ + \theta_\nu)$ and $R_{12}(45^\circ)$ in
the FX parametrization with $\theta_\nu = 45^\circ$ and $\theta = 45^\circ + \theta_*$. Here again we assume a universal
angle $\theta_*$ to perturb three mixing angles of $U_0$ in the form of
\[
\Omega_l = R_{12}(\theta_*) \quad \text{and} \quad \Omega_\nu = R^T_{23}(\theta_*) \otimes P^\dagger_\phi \otimes R^T_{12}(\theta_*) .
\]
Then we obtain
\[
V = \Omega_l \otimes R_{23}(45^\circ + \theta_*) \otimes \Omega_\nu \otimes R_{12}(45^\circ) \otimes P_\nu \\
= R_{12}(\theta_*) \otimes R_{23}(45^\circ) \otimes P^\dagger_\phi \otimes R_{12}(45^\circ - \theta_*) \otimes P_\nu \\
= \begin{pmatrix}
\frac{\sqrt{2}}{\sqrt{3}} s_* + \frac{\sqrt{2}}{\sqrt{3}} c_* e^{-i\phi} & \frac{\sqrt{2}}{\sqrt{3}} s_* - \frac{\sqrt{2}}{\sqrt{3}} c_* e^{-i\phi} \\
\frac{\sqrt{2}}{\sqrt{3}} c_* - \frac{\sqrt{2}}{\sqrt{3}} s_* e^{-i\phi} & \frac{\sqrt{2}}{\sqrt{3}} c_* + \frac{\sqrt{2}}{\sqrt{3}} s_* e^{-i\phi}
\end{pmatrix} P_\nu ,
\]
where $c_*$ and $s_*$ have been given below Eq. (11). It is obvious that this ansatz can predict
$\theta_l = \theta_* \simeq 9.7^\circ$, $\theta_\nu = 45^\circ - \theta_* \simeq 35.3^\circ$ and $\theta = 45^\circ$. The Jarlskog invariant of leptonic CP
violation turns out to be
\[
\mathcal{J} = c_l s_l c_\nu s_\nu c s^2 \sin \phi = \frac{1}{36} \sin \phi \lesssim 0.028 \sin \phi
\]
in this scenario, and its maximal magnitude corresponds to $\phi = \pm 90^\circ$.

Translating the results of three neutrino mixing angles from the FX parametrization to
the standard parametrization in scenario B, we arrive at
\[
\theta_{13} = \arcsin \left( \sqrt{\frac{2}{3}} s_* \right) = \arcsin \left[ \frac{1}{2 \left( \sqrt{3} + \sqrt{6} \right)} \right] \simeq 6.9^\circ ,
\]
\[
\theta_{23} = \arctan (c_*) = \arctan \left( \frac{1}{2 \sqrt{3} - \sqrt{6}} \right) \simeq 44.6^\circ ,
\]
and
\[
\theta_{12} = \arctan \left[ \frac{\sqrt{2} - t_* e^{i\phi}}{2 + t_* e^{i\phi}} \right] \in \left[ \arctan \left( \frac{4}{6 + \sqrt{2}} \right) , \arctan \left( \frac{4}{3 + \sqrt{2}} \right) \right] \simeq [28.3^\circ , 42.2^\circ] \quad (18)
\]
with $t_* \equiv \tan \theta_* = (\sqrt{2} - 1)/(\sqrt{2} + 1)$. We observe that the value of $\theta_{12}$ in the standard
parametrization depends on the CP-violating phase $\phi$ in the FX parametrization, and its
minimal (or maximal) value corresponds to $\phi = 0^\circ$ (or $\phi = 180^\circ$). Once $\theta_{12}$ is experimentally
determined to a good degree of accuracy, it will be possible to calculate $\phi$ from Eq. (18).
Given $\theta_{12} \simeq 34.5^\circ$ [3] for example,
\[
\phi = \arccos \left[ \frac{2 (1 + t_*^2) - (4 + t_*^2) \tan^2 \theta_{12}}{4 t_* (1 + \tan^2 \theta_{12})} \right] \simeq \pm 61.3^\circ ,
\]
At present both scenarios A and B are compatible with the available neutrino oscillation data. In either scenario the values of three neutrino mixing angles are independent of four independent mass ratios of charged leptons and neutrinos (i.e., $m_e/m_\mu$, $m_\mu/m_\tau$, $m_1/m_2$ and $m_2/m_3$). Of course, this kind of consequence is more or less contrived and it implies some quite special textures of the charged-lepton and neutrino mass matrices which might result from certain flavor symmetries [20]. Instead of going into any details of model building, here we discuss a few possibilities of reconstructing the charged-lepton mass matrix $M_l$ and the neutrino mass matrix $M_\nu$ from a given pattern of the neutrino mixing matrix $V$. For the sake of simplicity, we only concentrate on scenario B to illustrate the salient features of our phenomenological treatment.

Let us define $M_l$ and $M_\nu$ in the following lepton mass terms by assuming neutrinos to be the Majorana particles:

$$
-\mathcal{L}_{\text{mass}} = (e \mu \tau)_L M_l \begin{pmatrix} e \\ \mu \\ \tau \end{pmatrix}_R + \frac{1}{2} (\nu_e \nu_\mu \nu_\tau)_L M_\nu \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix}_R + \text{h.c.} ,
$$

(20)

where $M_\nu$ is symmetric but $M_l$ is arbitrary. One may diagonalize $M_l$ and $M_\nu$ via the transformations $O_l^T M_l O_l' = \hat{M}_l \equiv \text{Diag}\{m_e, m_\mu, m_\tau\}$ and $O_\nu^T M_\nu O_\nu^* = \hat{M}_\nu \equiv \text{Diag}\{m_1, m_2, m_3\}$, where $O_l$, $O_l'$ and $O_\nu$ are all unitary. Then the flavor mixing matrix $V \equiv O_l^T O_\nu$ will show up in the weak charged-current interactions

$$
-\mathcal{L}_{\text{cc}} = \frac{g}{\sqrt{2}} (e' \mu' \tau')_L \gamma^\mu V \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}_L W^\mu + \text{h.c.} ,
$$

(21)

where $\alpha'$ (for $\alpha = e, \mu, \tau$) and $\nu_i$ (for $i = 1, 2, 3$) stand respectively for the mass eigenstates of charged leptons and neutrinos. Now that both charged-lepton and neutrino sectors may in general contribute to $V$, the reconstruction of $M_l$ and $M_\nu$ crucially depends on how one decomposes a given pattern of $V$ into $O_l$ and $O_\nu$. Taking scenario B for example, we consider four typical possibilities as follows.

**Possibility (1):** $O_l = 1$ and $O_\nu = V$. In this case the mass eigenstates of three charged leptons are identified with their flavor eigenstates, and hence only the neutrino sector is responsible for the effect of flavor mixing. So we have $M_l = \hat{M}_l$ and $M_\nu = V \hat{M}_\nu V^T$. Most authors have taken such a flavor basis for building phenomenological models of $M_\nu$ [20].

**Possibility (2):** $O_\nu = 1$ and $O_l = V^\dagger$. In this case the mass eigenstates of three neutrinos are identified with their flavor eigenstates, and hence only the charged-lepton sector contributes to flavor mixing. So we have $M_\nu = \hat{M}_\nu$ and $M_l = V^\dagger \hat{M}_l O_l^T$. Because $O_l'$ is in general unknown, it is actually difficult to fix the texture of $M_l$ in this flavor basis. One usually assumes $M_l$ to be Hermitian or symmetric so as to reduce the number of degrees of freedom. If $M_l$ is Hermitian, it can be diagonalized via $O_l^T M_l O_l = \hat{M}_l' \equiv \text{Diag}\{\lambda_e, \lambda_\mu, \lambda_\tau\}$ with
\(|\lambda_\alpha| = m_\alpha\) (for \(\alpha = \epsilon, \mu, \tau\)). Then it is possible to reconstruct \(M_t\) through \(M_t = V^\dagger \tilde{M}_t V\). If \(M_t\) is symmetric, we simply set \(O_t' = O_t^* = V^T\) and then arrive at \(M_t = V^\dagger \tilde{M}_t V^*\). A simple example of this kind was originally given in Ref. [4].

Possibility (3): \(O_t = P_\phi \otimes R_{12}^T(\theta_\epsilon)\) and \(O_\nu = R_{23}(45^\circ) \otimes R_{12}(45^\circ - \theta_\epsilon) \otimes P_\nu\). Since \(P_\phi^\dagger\) commutes with \(R_{23}(45^\circ)\), the product \(O_t^\dagger O_\nu\) automatically reproduces the pattern of \(V\) shown in Eq. (15). Assuming \(M_t\) to be symmetric for simplicity, we obtain

\[
M_t = P_\phi R_{12}^T(\theta_\epsilon) \tilde{M}_t R_{12}^T(\theta_\epsilon) P_\phi^T, \\
M_\nu = R_{23}(45^\circ) R_{12}(45^\circ - \theta_\epsilon) P_\nu \tilde{M}_\nu P_\nu^T R_{12}^T(45^\circ - \theta_\epsilon) R_{23}^T(45^\circ).
\]

To be more explicit,

\[
M_t = \begin{pmatrix}
[m_\epsilon c_\epsilon^2 + m_\mu s_\mu^2] e^{2i\phi} & [m_\epsilon - m_\mu] c_\epsilon s_\epsilon e^{i\phi} & 0 \\
[m_\epsilon - m_\mu] c_\epsilon s_\epsilon e^{i\phi} & m_\epsilon s_\epsilon^2 + m_\mu c_\mu^2 & 0 \\
0 & 0 & m_\tau
\end{pmatrix}, \\
M_\nu = \begin{pmatrix}
\frac{2}{3} \tilde{m}_1 + \frac{1}{3} \tilde{m}_2 & \frac{1}{3} [\tilde{m}_2 - \tilde{m}_1] & \frac{1}{3} [\tilde{m}_1 - \tilde{m}_2] \\
\frac{1}{3} [\tilde{m}_2 - \tilde{m}_1] & \frac{1}{6} \tilde{m}_1 + \frac{1}{3} \tilde{m}_2 + \frac{1}{2} m_3 & \frac{1}{2} m_3 - \frac{1}{2} \tilde{m}_1 - \frac{1}{3} \tilde{m}_2 \\
\frac{1}{3} [\tilde{m}_1 - \tilde{m}_2] & \frac{1}{6} \tilde{m}_1 - \frac{1}{3} \tilde{m}_2 & \frac{1}{6} \tilde{m}_1 + \frac{1}{3} \tilde{m}_2 + \frac{1}{2} m_3
\end{pmatrix},
\]

where \(\tilde{m}_1 \equiv m_\epsilon e^{2i\phi}\) and \(\tilde{m}_2 \equiv m_\mu e^{2i\sigma}\). It is interesting to notice that the Dirac CP-violating phase \(\phi\) is attributed to \(M_t\) while the Majorana CP-violating phases \(\rho\) and \(\sigma\) come from \(M_\nu\) in this decomposition. One may derive both the textures of \(M_t\) and \(M_\nu\) from certain flavor symmetries [20]. For instance, the non-Abelian \(A_4\) flavor symmetry has been used to derive the form of \(M_\nu\) in Eq. (23) [21].

Possibility (4): \(O_t = P_\phi \otimes R_{23}^T(\theta_\epsilon) \otimes R_{12}^T(\theta_\epsilon)\) and \(O_\nu = R_{23}(45^\circ - \theta_\epsilon) \otimes R_{12}(45^\circ - \theta_\epsilon) \otimes P_\nu\). Here again the product \(O_t^\dagger O_\nu\) can reproduce the pattern of \(V\) in Eq. (15). Assuming \(M_t\) to be symmetric, we analogously arrive at

\[
M_t = P_\phi R_{23}^T(\theta_\epsilon) R_{12}^T(\theta_\epsilon) \tilde{M}_t R_{12}^T(\theta_\epsilon) R_{23}^T(\theta_\epsilon) P_\phi^T, \\
M_\nu = R_{23}(45^\circ - \theta_\epsilon) R_{12}(45^\circ - \theta_\epsilon) P_\nu \tilde{M}_\nu P_\nu^T R_{12}^T(45^\circ - \theta_\epsilon) R_{23}^T(45^\circ - \theta_\epsilon).
\]

More explicitly, we have

\[
M_t = \begin{pmatrix}
[m_\epsilon c_\epsilon^2 + m_\mu s_\mu^2] e^{2i\phi} & [m_\epsilon - m_\mu] c_\epsilon s_\epsilon e^{i\phi} & [m_\epsilon - m_\mu] c_\epsilon s_\epsilon e^{i\phi} \\
[m_\epsilon - m_\mu] c_\epsilon s_\epsilon e^{i\phi} & m_\epsilon s_\epsilon^2 + m_\mu c_\mu^2 & [m_\epsilon s_\epsilon^2 + m_\mu c_\mu^2] c_\epsilon s_\epsilon e^{i\phi} + [m_\epsilon s_\epsilon^2 + m_\mu c_\mu^2] c_\epsilon s_\epsilon e^{i\phi} + [m_\epsilon s_\epsilon^2 + m_\mu c_\mu^2] c_\epsilon s_\epsilon e^{i\phi} \\
[m_\epsilon - m_\mu] c_\epsilon s_\epsilon e^{i\phi} & [m_\epsilon s_\epsilon^2 + m_\mu c_\mu^2] c_\epsilon s_\epsilon e^{i\phi} + [m_\epsilon s_\epsilon^2 + m_\mu c_\mu^2] c_\epsilon s_\epsilon e^{i\phi} + [m_\epsilon s_\epsilon^2 + m_\mu c_\mu^2] c_\epsilon s_\epsilon e^{i\phi} & [m_\epsilon s_\epsilon^2 + m_\mu c_\mu^2] c_\epsilon s_\epsilon e^{i\phi} + [m_\epsilon s_\epsilon^2 + m_\mu c_\mu^2] c_\epsilon s_\epsilon e^{i\phi}
\end{pmatrix},
\]

\[
M_\nu = \begin{pmatrix}
\frac{2}{3} \tilde{m}_1 + \frac{1}{3} \tilde{m}_2 & \frac{2}{3} \sqrt{\frac{1}{3} [\tilde{m}_2 - \tilde{m}_1]} & \frac{1}{3} \sqrt{\frac{2}{3} [\tilde{m}_1 - \tilde{m}_2]} \\
\frac{2}{3} \sqrt{\frac{1}{3} [\tilde{m}_2 - \tilde{m}_1]} & \frac{2}{3} \tilde{m}_1 + \frac{1}{3} \tilde{m}_2 + \frac{1}{2} m_3 & \frac{1}{2} m_3 - \frac{1}{2} \tilde{m}_1 - \frac{1}{3} \tilde{m}_2 \\
\frac{1}{3} \sqrt{\frac{2}{3} [\tilde{m}_1 - \tilde{m}_2]} & \frac{1}{2} m_3 - \frac{1}{2} \tilde{m}_1 - \frac{1}{3} \tilde{m}_2 & \frac{1}{6} \tilde{m}_1 + \frac{1}{3} \tilde{m}_2 + \frac{1}{2} m_3
\end{pmatrix},
\]

where \(\tilde{m}_1\) and \(\tilde{m}_2\) have been defined above. In this case the textures of \(M_t\) and \(M_\nu\) are more or less parallel to each other, implying that they could arise from a common flavor symmetry or dynamic mechanism.
Indeed, there are infinite possibilities of reconstructing $M_l$ and $M_\nu$ for a given pattern of $V$ which is consistent with current experimental data. From a phenomenological point of view, we hope to make the textures of $M_l$ and $M_\nu$ as simple as possible, or much easier to link with an underlying flavor symmetry. In this sense the above examples just serve for illustration. Theoretically, a viable neutrino mass model should predict or constrain the proper forms of $M_l$ and $M_\nu$ from which the flavor mixing matrix $V$ can be derived. But the inverse approach discussed above (i.e., starting from $V$ to reconstruct the textures of $M_l$ and $M_\nu$, based on a few assumptions) remains very useful because it is at least possible to help give a ballpark estimate of the flavor structure that a viable model ought to possess.

We have paid our attention to how to confront a constant neutrino mixing pattern, which may be motivated by a certain flavor symmetry and can predict $\theta_{13} = 0^\circ$ in the symmetry limit, with a relatively large value of $\theta_{13}$ (e.g., $5^\circ \lesssim \theta_{13} \lesssim 12^\circ$). The latter seems quite possible, at least not to be impossible, according to some preliminary experimental hints extracted from current neutrino oscillation data. We have shown that reasonable perturbations to the democratic mixing pattern $U_0$, which is geometrically related to the tri-bimaximal mixing pattern $V_0$ through an equal shift $\theta_\ast \simeq 9.7^\circ$ of two large mixing angles, can naturally produce a nearly tri-bimaximal neutrino mixing matrix $V$ with relatively large $\theta_{13}$. We have proposed two simple but viable scenarios of $V$ for illustration: one of them is based on the standard parametrization of $U_0$ and predicts $\theta_{13} \simeq 9.7^\circ$, and the other relies on the FX parametrization of $U_0$ and predicts $\theta_{13} \simeq 6.9^\circ$. Both scenarios are in good agreement with current neutrino oscillation data, and they can soon be tested in a variety of more accurate neutrino oscillation experiments.

In this work we have tried not to go into any details of model building. But we have discussed a few phenomenological possibilities of reconstructing the charged-lepton mass matrix $M_l$ and the neutrino mass matrix $M_\nu$ for a given neutrino mixing matrix $V$ with a relatively large value of $\theta_{13}$. Both the charged-lepton and neutrino sectors may in general have significant contributions to $V$, and hence a specific lepton flavor model should be able to determine the textures of $M_l$ and $M_\nu$ so as to give testable predictions for both neutrino mixing angles and CP-violating phases.

Finally, we stress that it is not impossible to obtain a sufficiently large value of $\theta_{13}$ at the electroweak scale from finite quantum corrections to a given constant neutrino mixing pattern with $\theta_{13} = 0^\circ$ [16]. It is also possible to generate $\theta_{13}$ via the renormalization-group running effects from the conventional seesaw scales of $O(10^{14})$ GeV down to the electroweak scale [22], in particular when the seesaw threshold effects are taken into account [23]. But it seems more likely to achieve relatively large $\theta_{13}$ from relatively significant symmetry breaking terms at a given scale where the constant neutrino mixing pattern can be derived on the basis of a certain flavor symmetry. In this sense our speculations and discussions are expected to be phenomenologically useful and suggestive.

Acknowledgments: I am indebted to K.K. Phua for his warm hospitality and H. Fritzsch for sharing many joys of life at the IAS of NTU in Singapore, where this paper was written. I am also grateful to T. Araki, Y. Koide, and Y.F. Li for discussions. This work was supported in part by the National Natural Science Foundation of China under grant No. 10875131.
REFERENCES

[1] Particle Data Group, K. Nakamura et al., J. Phys. G 37, 075021 (2010).
[2] H. Fritzsch and Z.Z. Xing, Phys. Lett. B 517, 363 (2001); Z.Z. Xing, Int. J. Mod. Phys. A 19, 1 (2004).
[3] M.C. Gonzalez-Garcia, M. Maltoni, and J. Salvado, JHEP 1004, 056 (2010).
[4] H. Fritzsch and Z.Z. Xing, Phys. Lett. B 372, 265 (1996); Phys. Lett. B 440, 313 (1998); Phys. Rev. D 61, 073016 (2000).
[5] P.F. Harrison, D.H. Perkins, and W.G. Scott, Phys. Lett. B 530, 167 (2002); Z.Z. Xing, Phys. Lett. B 533, 85 (2002); P.F. Harrison and W.G. Scott, Phys. Lett. B 535 (2002); X.G. He and A. Zee, Phys. Lett. B 560, 87 (2003).
[6] For a recent review, see: G. Altarelli and F. Feruglio, arXiv:1002.0211; Z.Z. Xing, D. Yang, and S. Zhou, Phys. Lett. B 690, 304 (2010).
[7] G.L. Fogli, E. Lisi, A. Marrone, A. Palazzo, and A.M. Rotunno, Phys. Rev. Lett. 101, 141801 (2008).
[8] The KamLAND Collaboration, A. Gando et al., arXiv:1009.4771.
[9] S.F. King, Phys. Lett. B 675, 347 (2009).
[10] Z.Z. Xing, plenary talk given at ICHEP 2008, Philadelphia, August 2008; Int. J. Mod. Phys. A 23, 4255 (2008).
[11] H. Fritzsch and Z.Z. Xing, Phys. Lett. B 413, 396 (1997); Phys. Rev. D 57, 594 (1998).
[12] S. Pakvasa, W. Rodejohann, and T.J. Weiler, Phys. Rev. Lett. 100, 111801 (2008).
[13] S.F. King, Phys. Lett. B 659, 244 (2008).
[14] M. Abbas and A.Yu. Smirnov, Phys. Rev. D 82, 013008 (2010).
[15] See, e.g., S. Goswami, S.T. Petcov, S. Ray, and W. Rodejohann, Phys. Rev. D 80, 053013 (2009); and references therein.
[16] T. Araki, C.Q. Geng, and Z.Z. Xing, arXiv:1012.2970.
[17] C. Jarlskog, Phys. Rev. Lett. 55, 1039 (1985); H. Fritzsch and Z.Z. Xing, Nucl. Phys. B 556, 49 (1999).
[18] Z.Z. Xing, Phys. Lett. B 633, 550 (2006).
[19] See, e.g., H. Fritzsch and Z.Z. Xing, Phys. Lett. B 634, 514 (2006); Phys. Lett. B 682, 220 (2009).
[20] For recent reviews with extensive references, see: H. Fritzsch and Z.Z. Xing, Prog. Part. Nucl. Phys. 45, 1 (2000); S.F. King, Rept. Prog. Phys. 67, 107 (2004); Altarelli and F. Feruglio, New J. Phys. 6, 106 (2004); R.N. Mohapatra and A.Yu. Smirnov, Ann. Rev. Nucl. Part. Sci. 56, 569 (2006); A. Strumia and F. Vissani, hep-ph/0606054.
[21] For the latest review in this connection with many references, see: G. Altarelli and F. Feruglio, arXiv:1002.0211. An intrinsic deviation from the tri-bimaximal neutrino mixing in a class of $A_4$ models has been discussed by T. Araki, J. Mei, and Z.Z. Xing, Phys. Lett. B 695, 165 (2011).
[22] See, e.g., S. Antusch, J. Kersten, M. Lindner, and M. Ratz, Phys. Lett. B 544, 1 (2002); S. Antusch and M. Ratz, JHEP 0211, 010 (2002); S. Luo and Z.Z. Xing, Phys. Lett. B 632, 341 (2006).
[23] J.W. Mei and Z.Z. Xing, Phys. Rev. D 70, 053002 (2004); Phys. Lett. B 623, 227 (2005); J.W. Mei, Phys. Rev. D 71, 073012 (2005); J. Bergstrom, T. Ohlsson, and H. Zhang, arXiv:1009.2762.
FIG. 1. A geometric relationship between the democratic mixing pattern $U_0$ (with $\theta_{12}^{(0)} = 45^\circ$ and $\theta_{23}^{(0)} = 45^\circ + \theta_*$) and the tri-bimaximal mixing pattern $V_0$ (with $\vartheta_{12}^{(0)} = 45^\circ - \theta_*$ and $\vartheta_{23}^{(0)} = 45^\circ$), where $\theta_* = \arctan(\sqrt{2}) - 45^\circ = 45^\circ - \arctan(1/\sqrt{2}) \simeq 9.7^\circ$. Four nonzero mixing angles of $U_0$ and $V_0$ correspond to four inner angles of two right triangles $\triangle ABC$ and $\triangle A'BC$ in the real plane.