PARAMETRIC KINDS OF GENERALIZED APOSTOL–BERNOULLI POLYNOMIALS AND THEIR PROPERTIES

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ABSTRACT. The purpose of this paper is to define generalized Apostol–Bernoulli polynomials with including a new cosine and sine parametric type of generating function using the quasi-monomiality properties and trigonometric functions. In this study, the Apostol-Bernoulli polynomials with three variable are defined with two new generating functions cosine and sine parameters. Then, we investigate multiplicative and derivative operators, differential equations, some summation formulas and partial differential equations for these polynomials. Moreover, we introduce Gould–Hopper–Apostol–Bernoulli type polynomials, Hermite–Appell–Apostol–Bernoulli type polynomials and truncated exponential Apostol–Bernoulli type polynomials. Finally, the special cases of these new polynomials are investigated, and the corresponding results are expressed.

1. INTRODUCTION

Generating functions play an important role in the investigation of properties of special polynomials and numbers. There are many studies related to polynomials and their generating functions. A few of references to special polynomials and their generating functions are given in the monographs [7, 37, 39, 43]. On the other hand, generating functions have some applications in many fields such as applied mathematics, algebra, statistics, combinatorics, and physics.

The Bernoulli polynomials are one of the important study subject in mathematics. These polynomials are generated by using exponential functions and series expansions. These polynomials have many application areas such as applied mathematics, computer programming, and statistics. The Bernoulli numbers and polynomials are related to many special numbers and polynomials such as the Euler, Genocchi, Hermite, and Stirling polynomials and numbers. The Swiss mathematician Jacob Bernoulli first mentioned the Bernoulli numbers in his work [4]. Then, Tom M. Apostol developed Bernoulli polynomials in [2]. Later, some properties and relations related to these polynomials are obtained [3, 11, 12, 13, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38].

Throughout this paper, we use the notions \( \mathbb{N} = \{1, 2, 3, \ldots\} \), \( \mathbb{N}_0 = \{0\} \cup \mathbb{N} \), R, and \( \mathbb{C} \) to denote the sets of positive integers numbers, non-negative integers numbers, real numbers, and complex numbers, respectively.

The classical Bernoulli polynomials \( B_n(x) \) were defined in [3, p. 1] and [7, p. 3] by the generating function

\[
\sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} = \frac{t e^{xt}}{e^t - 1}, \quad |t| < 2\pi.
\]

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The first five Bernoulli polynomials are

\[ B_0(x) = 1, \quad B_1(x) = x - \frac{1}{2}, \quad B_2(x) = x^2 - x + \frac{1}{6}, \]
\[ B_3(x) = \frac{3}{2}x^3 + \frac{1}{2}x, \quad B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30}. \]

The quantities \( B_n(0) = B_n \) are called the Bernoulli numbers. Due to that the function \( t - 1 - \frac{t}{e^t - 1} \) is even in \( t \in \mathbb{R} \), all of the Bernoulli numbers \( B_{2n+1} \) for \( n \in \mathbb{N} \) equal 0. The first few Bernoulli numbers \( B_{2n} \) are

\[ B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}, \quad B_8 = -\frac{1}{30}, \quad B_{10} = \frac{5}{66}, \]
\[ B_{12} = -\frac{691}{2730}, \quad B_{14} = \frac{7}{6}, \quad B_{16} = -\frac{3617}{510}, \quad B_{18} = \frac{43867}{798}. \]

In [26, 38], some new properties for the ratio \( \frac{B_{2n+2}}{B_{2n}} \) with \( n \in \mathbb{N} \) were discovered.

One of the relations between the Bernoulli numbers and polynomials is

\[ B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}, \quad n \in \mathbb{N}_0 \]

which is a special case of the identity

\[ B_n(x + h) = \sum_{k=0}^{n} \binom{n}{k} B_k(x) h^{n-k}, \quad n \in \mathbb{N}_0 \]

in [1, p. 802, 23.1.7].

The generalized Bernoulli polynomials \( B_n^{(v)}(x) \) of order \( v \) are defined [47, p. 4] by

\[ \sum_{n=0}^{\infty} B_n^{(v)}(x) \frac{t^n}{n!} = \left( \frac{t}{e^t - 1} \right)^v e^{xt}, \quad |t| < 2\pi. \]

It is easy to see that \( B_n^{(1)}(x) = B_n(x) \). In addition, we call \( B_n^{(v)} = B_n^{(v)}(0) \) the Bernoulli numbers of order \( v \). The relation between \( B_n^{(v)} \) and \( B_n^{(v)}(x) \) was given [13, p. 510] by

\[ B_n^{(v)}(x) = \sum_{k=0}^{n} \binom{n}{k} B_k^{(v)} x^{n-k}. \]

The Apostol–Bernoulli polynomials \( B_n(x; \lambda) \) were defined in [3, p. 165] by

\[ \sum_{n=0}^{\infty} B_n(x; \lambda) \frac{t^n}{n!} = \frac{t e^{x t}}{\lambda e^t - 1}, \]
where $|t| < 2\pi$ when $\lambda = 1$ or $|t| < |\ln \lambda|$ when $\lambda \neq 1$ and $\lambda \in \mathbb{C}$. When $x = 0$, we call $B_n(0; \lambda) = B_n(\lambda)$ the Apostol–Bernoulli numbers. These polynomials satisfy

\[ B_n(x; \lambda) = \sum_{k=0}^{n} \binom{n}{k} B_k(\lambda) x^{n-k}, \]

\[ B_n(x+y; \lambda) = \sum_{k=0}^{n} \binom{n}{k} B_k(\lambda) y^{n-k}, \]

\[ \frac{\partial^p}{\partial x^p} B_n(x; \lambda) = \frac{n!}{(n-p)!} B_{n-p}(x; \lambda), \]

\[ \int_a^b B_n(t; \lambda) \, dt = \frac{B_{n+1}(b; \lambda) - B_{n+1}(a; \lambda)}{n+1}, \]

\[ \sum_{k=0}^{m-1} k^n = \frac{\lambda - 1}{n+1} \sum_{k=1}^{m} B_{n+1}(k; \lambda) + \frac{B_{n+1}(m; \lambda) - B_{n+1}(\lambda)}{n+1}, \]

for $m, n \in \mathbb{N}_0$ and $0 \leq p \leq n$. In [2, p. 165], Apostol gave the relation

\[ \lambda B_n(x+1, \lambda) - B_n(x, \lambda) = nx^{n-1}, \quad n \geq 1. \]

The first six Apostol–Bernoulli numbers are

\[ B_0(\lambda) = 0, \quad B_1(\lambda) = \frac{1}{\lambda-1}, \quad B_2(\lambda) = -\frac{2\lambda}{(\lambda-1)^2}, \quad B_3(\lambda) = \frac{3\lambda(\lambda+1)}{(\lambda-1)^3}, \]

\[ B_4(\lambda) = \frac{4\lambda(\lambda^2 + 4\lambda + 1)}{(\lambda-1)^4}, \quad B_5(\lambda) = \frac{5\lambda(\lambda^3 + 11\lambda^2 + 11\lambda + 1)}{(\lambda-1)^5}. \]

The generalized Apostol–Bernoulli polynomials $B_n^{(v)}(x; \lambda)$ of order $v$ can be generated by

\[ \sum_{n=0}^{\infty} B_n^{(v)}(x; \lambda) \frac{t^n}{n!} = \left( \frac{t}{\lambda e^t - 1} \right)^v e^x, \quad |t + \ln \lambda| < 2\pi \quad (1.1) \]

with

\[ B_n^{(v)}(x; \lambda) = B_n^{(v)}(x; 1) \quad \text{and} \quad B_n^{(v)}(\lambda) = B_n^{(v)}(0; \lambda), \]

where $B_n^{(v)}(\lambda)$ denote the Apostol–Bernoulli numbers of order $v$, see [21, p. 292] and [39, p. 92]. The identity between the Apostol–Bernoulli polynomials $B_n^{(v)}(x; \lambda)$ of order $v$ and the Apostol–Bernoulli numbers $B_n^{(v-1)}(\lambda)$ of order $v - 1$ is given in [21, p. 300] by

\[ B_n^{(v)}(x; \lambda) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k}^{(v-1)}(\lambda) B_k(x; \lambda), \quad n \in \mathbb{N}_0. \]
The Apostol–Bernoulli polynomials $B_n^{(v)}(x; \lambda)$ of order $v$ satisfy

$$B_n^{(0)}(x; \lambda) = x^n,$$

$$B_n^{(v)}(x; \lambda) = \sum_{k=0}^{n} \binom{n}{k} B_k^{(v)}(\lambda) x^{n-k},$$

$$\lambda B_n^{(v)}(x+1; \lambda) - B_n^{(v)}(x; \lambda) = n B_{n-1}^{(v-1)}(x; \lambda),$$

$$B_n^{(v+u)}(x+y; \lambda) = \sum_{k=0}^{n} \binom{n}{k} B_k^{(v)}(x; \lambda) B_{n-k}^{(u)}(y; \lambda),$$

$$\frac{\partial}{\partial x} B_n^{(v)}(x; \lambda) = n B_{n-1}^{(v)}(x; \lambda),$$

$$\int_{a}^{b} B_n^{(v)}(t; \lambda) \, dt = \frac{B_{n+1}^{(v)}(b; \lambda) - B_{n+1}^{(v)}(a; \lambda)}{n+1}.$$

The properties of Apostol–Bernoulli polynomials $B_n^{(v)}(x; \lambda)$ of order $v$ are not limited to these. In the literature, there are many studies on the Apostol–Bernoulli polynomials such as Gaussian hypergeometric function, multiplication formulas, and Fourier expansions. The relations between the Apostol–Bernoulli polynomials, the Euler polynomials, the Genocchi polynomials, and the Stirling numbers have also been investigated in [12, 13, 17, 18, 21, 22, 42].

In [23, p. 3] and [24, p. 4], the Maclaurin expansions of two functions $e^{xt}\cos(yt)$ and $e^{xt}\sin(yt)$ are given by

$$e^{xt}\cos(yt) = \sum_{n=0}^{\infty} C_n(x, y) \frac{t^n}{n!}$$ (1.2)

and

$$e^{xt}\sin(yt) = \sum_{n=0}^{\infty} S_n(x, y) \frac{t^n}{n!}$$ (1.3)

respectively, where

$$C_n(x, y) = \sum_{r=0}^{\lfloor n/2 \rfloor} (-1)^r \binom{n}{2r} x^{n-2r} y^{2r},$$

$$S_n(x, y) = \sum_{r=0}^{\lfloor (n-1)/2 \rfloor} (-1)^r \binom{n}{2r+1} x^{n-2r-1} y^{2r+1},$$

and $\lfloor \lambda \rfloor$ denotes the floor function defined by the largest integer less than or equal to $\lambda \in \mathbb{R}$.

In [44, p. 909], Srivastava et al defined two parametric types Apostol–Bernoulli polynomials $B_n^{(c,v)}(x, y; \lambda)$ and $B_n^{(s,v)}(x, y; \lambda)$ by

$$\left( \frac{t}{\lambda e^t - 1} \right)^v e^{xt}\cos(yt) = \sum_{n=0}^{\infty} B_n^{(c,v)}(x, y; \lambda) \frac{t^n}{n!}$$ (1.4)

and

$$\left( \frac{t}{\lambda e^t - 1} \right)^v e^{xt}\sin(yt) = \sum_{n=0}^{\infty} B_n^{(s,v)}(x, y; \lambda) \frac{t^n}{n!}$$ (1.5)
For \( v \in \mathbb{N}_0 \) and \(|t| < \ln 2\), Srivastava and Kızılateş introduced in [41, p. 3257] two parametric kinds of the Fubini type polynomials \( F_{n}^{(c,v)}(x, y) \) and \( F_{n}^{(s,v)}(x, y) \) by

\[
2^v (2 - e^t)^{2v} e^{xt} \cos(yt) = \sum_{n=0}^{\infty} F_{n}^{(c,v)}(x, y) \frac{t^n}{n!},
\]

and

\[
2^v (2 - e^t)^{2v} e^{xt} \sin(yt) = \sum_{n=0}^{\infty} F_{n}^{(s,v)}(x, y) \frac{t^n}{n!}.
\]

Some general polynomials have been described with the help of monomiality principle. Khan and Raza defined in [14, p. 4] the 2-variable general polynomials \( T_{n}(x, y) \) by

\[
e^{xt} \mathcal{U}(y, t) = \sum_{n=0}^{\infty} T_{n}(x, y) \frac{t^n}{n!}, \quad T_{0}(x, y) = 1,
\]

where

\[
\mathcal{U}(y, t) = \sum_{n=0}^{\infty} \mathcal{U}_{n}(y) \frac{t^n}{n!}, \quad \mathcal{U}_{0}(y) \neq 0.
\]

The multiplicative and derivative operators for the polynomials \( T_{n}(x, y) \) are defined in [14, p. 4] by

\[
\hat{M}_{T} = x + \frac{\mathcal{U}'(y, D_{x})}{\mathcal{U}(y, D_{x})} \quad \text{and} \quad \hat{P}_{T} = D_{x}.
\]

According to monomiality principle, the polynomials \( T_{n}(x, y) \) satisfy

\[
\hat{M}_{T}\{T_{n}(x, y)\} = T_{n+1}(x, y),
\]

\[
\hat{P}_{T}\{T_{n}(x, y)\} = n \ T_{n-1}(x, y),
\]

\[
\hat{M}_{T}\hat{P}_{T}\{T_{n}(x, y)\} = n \ T_{n}(x, y),
\]

\[
\exp(\hat{M}_{T}t) = \sum_{n=0}^{\infty} T_{n}(x, y) \frac{t^n}{n!}, \quad |t| < \infty.
\]

Among other things, via monomiality principle, Srivastava and five coauthors defined in [45, p. 5] two parametric types of the generalized Fubini type polynomials by

\[
2^v (2 - e^t)^{2v} e^{xt} \mathcal{U}(y, t) \cos(zt) = \sum_{n=0}^{\infty} \mathcal{F}_{n}^{(c,v)}(x, y, z) \frac{t^n}{n!},
\]

and

\[
2^v (2 - e^t)^{2v} e^{xt} \mathcal{U}(y, t) \sin(zt) = \sum_{n=0}^{\infty} \mathcal{F}_{n}^{(s,v)}(x, y, z) \frac{t^n}{n!},
\]

where \( v \in \mathbb{N}_0 \) and \(|t| < \ln 2\).

This paper is organized as follows. In Section 2, by using operational methods and monomiality principle, we define the parametric kinds of generalized Apostol–Bernoulli polynomials and obtain quasi-monomial properties for these kinds of polynomials. In Section 3, we establish various relations, identities, and summation formulas for generalized Apostol–Bernoulli polynomials. In Section 4, we discover partial differential equations and identities for their generating functions. In Section 5, using the generating functions for the parametric kinds of generalized Apostol–Bernoulli polynomials, we present new polynomials and investigate their properties.
2. Parametric kinds of generalized Apostol–Bernoulli polynomials

In this section, by using monomiality principle and operational methods, via generating functions, we give two parametric types of generalized Apostol–Bernoulli type polynomials and introduce multiplicative and derivative operators and differential equations.

**Theorem 2.1.** The generating functions for the parametric kinds of generalized Apostol–Bernoulli polynomials \( \tau B_n^{(c,v)}(x, y, z; \lambda) \) and \( \tau B_n^{(s,v)}(x, y, z; \lambda) \) are defined

\[
\left( \frac{t}{\lambda e^t - 1} \right)^v e^{xt} U(y, t) \cos(zt) = \sum_{n=0}^{\infty} \tau B_n^{(c,v)}(x, y, z; \lambda) \frac{t^n}{n!},
\]

(2.1)

and

\[
\left( \frac{t}{\lambda e^t - 1} \right)^v e^{xt} U(y, t) \sin(zt) = \sum_{n=0}^{\infty} \tau B_n^{(s,v)}(x, y, z; \lambda) \frac{t^n}{n!},
\]

(2.2)

where

\[ |t| < 2\pi, \text{ when } \lambda = 1; |t| < |\log \lambda|, \text{ when } \lambda \neq 1; 1^v := 1. \]

**Proof.** Replacing \( x \) and \( y \) by the multiplication operator \( \hat{M}_T \) and \( z \) in (1.1) gives

\[
\left( \frac{t}{\lambda e^t - 1} \right)^v \exp(\hat{M}_T t) U(y, t) \cos(zt) = \sum_{n=0}^{\infty} B_n^{(c,v)}(x, y, z; \lambda) \frac{t^n}{n!}.
\]

Using equalities in (1.8) and (1.12) results in

\[
\left( \frac{t}{\lambda e^t - 1} \right)^v \left[ \sum_{n=0}^{\infty} T_n(x, y) \frac{t^n}{n!} \right] \cos(zt) = \sum_{n=0}^{\infty} B_n^{(c,v)} \left( x + \frac{U'(y, D_x)}{U(y, D_x)} z; \lambda \right) \frac{t^n}{n!}.
\]

Using the equation (2.1) and denoting the resultant parametric kinds of generalized Apostol–Bernoulli polynomials by \( \tau B_n^{(c,v)}(x, y, z; \lambda) \) yield

\[
\tau B_n^{(c,v)}(x, y, z; \lambda) = B_n^{(c,v)} \left( x + \frac{U'(y, D_x)}{U(y, D_x)} z; \lambda \right).
\]

Therefore, we acquire the assertion in the equation (2.1).

Similarly, we can prove the assertion in the equation (2.2). The proof of Theorem 2.1 is complete. \( \square \)

In order to find the quasi-monomial properties of parametric kinds of generalized Apostol–Bernoulli polynomials \( \tau B_n^{(c,v)}(x, y, z; \lambda) \) and \( \tau B_n^{(s,v)}(x, y, z; \lambda) \), we prove the following properties.

**Theorem 2.2.** The parametric kinds of generalized Apostol–Bernoulli polynomials \( \tau B_n^{(c,v)}(x, y, z; \lambda) \) and \( \tau B_n^{(s,v)}(x, y, z; \lambda) \) are quasi-monomial with respect to the following multiplicative and derivative operators:

\[
\hat{M}_{TB,c} = x + \frac{U'(y, D_x)}{U(y, D_x)} + \frac{v \lambda e^{D_x}(1 - D_x) - 1}{\lambda e^{D_x} - 1} - z \tan(zD_x),
\]

(2.3)

\[
\hat{P}_{TB,c} = D_x,
\]

(2.4)

\[
\hat{M}_{TB,s} = x + \frac{U'(y, D_x)}{U(y, D_x)} + \frac{v \lambda e^{D_x}(1 - D_x) - 1}{\lambda e^{D_x} - 1} + z \cot(zD_x),
\]

(2.5)

\[
\hat{P}_{TB,s} = D_x.
\]

(2.6)
The parametric kinds of generalized Apostol–Bernoulli polynomials

and

Theorem 2.3. The parametric kinds of generalized Apostol–Bernoulli polynomials $\tau B_{n}^{(c,v)}(x, y; z; \lambda)$ and $\tau B_{n}^{(s,v)}(x, y; z; \lambda)$ satisfy

$$\left[ x D_{x} + \frac{U'(y, D_{x})}{U(y, D_{x})} D_{x} + \frac{\lambda e^{D_{x}} (1 - D_{x}) - 1}{\lambda e^{D_{x}} - 1} - z \tan(z D_{x}) D_{x} - n \right] \tau B_{n}^{(c,v)}(x, y; z; \lambda) = 0$$

and

$$\left[ x D_{x} + \frac{U'(y, D_{x})}{U(y, D_{x})} D_{x} + \frac{\lambda e^{D_{x}} (1 - D_{x}) - 1}{\lambda e^{D_{x}} - 1} + z \cot(z D_{x}) D_{x} - n \right] \tau B_{n}^{(s,v)}(x, y; z; \lambda) = 0.$$
and
\[ T B_n^{(s,v)}(x,y,z;\lambda) = \sum_{r=0}^{n} \binom{n}{r} B_{n-r}^{(s,v)}(0,z;\lambda) T_r(x,y). \]  

**Proof.** From (1.4), (1.7), and (2.1), we have
\[ T B_n^{(s,v)}(x,y,z;\lambda) = \sum_{r=0}^{\infty} \binom{n}{r} B_{n-r}^{(s,v)}(0,z;\lambda) T_r(x,y) \sum_{\infty}^{n} \frac{n!}{n!}. \]

Equating coefficients of \( \frac{n!}{n!} \) on both sides of this equation yields the assertion in the equation (3.1). Similarly, we can verify the assertion in the equation (3.2). The proof of Theorem 3.1 is complete. \( \square \)

**Theorem 3.2.** The parametric kinds of generalized Apostol–Bernoulli polynomials \( T B_n^{(c,v)}(x,y,z;\lambda) \) and \( T B_n^{(s,v)}(x,y,z;\lambda) \) satisfy
\[ T B_n^{(c,v)}(x,y,z;\lambda) = \sum_{r=0}^{n} \binom{n}{r} U_r(y) B_{n-r}^{(c,v)}(x,z;\lambda) \]  
and
\[ T B_n^{(s,v)}(x,y,z;\lambda) = \sum_{r=0}^{n} \binom{n}{r} U_r(y) B_{n-r}^{(s,v)}(x,z;\lambda). \]

**Proof.** By virtue of (2.1), (3.7), and (2.1), we have
\[ T B_n^{(c,v)}(x,y,z;\lambda) \sum_{\infty}^{n} \frac{n!}{n!} = \sum_{n=0}^{\infty} \sum_{r=0}^{n} \binom{n}{r} U_r(y) B_{n-r}^{(c,v)}(x,z;\lambda). \]

Comparing the coefficients of \( \frac{t^n}{n!} \) on both sides of the above equation, we confirm the assertion in the equation (3.3). Similarly, we can verify the assertion in the equation (3.4). The proof of Theorem 3.2 is complete. \( \square \)

**Theorem 3.3.** The parametric kinds of generalized Apostol–Bernoulli polynomials \( T B_n^{(c,v)}(x,y,z;\lambda) \) satisfy the implicit summation formula
\[ T B_n^{(c,v)}(x+k,y,z;\lambda) = \sum_{r=0}^{n} \binom{n}{r} T B_{n-r}^{(c,v)}(x,y,z;\lambda) k^r. \]  

**Proof.** In the equation (2.1), replacing \( x \) by \( x+k \) and using the equation (2.1) and the series expansion of \( e^{kt} \) in the resultant equation, we obtain
\[ \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} T B_n^{(c,v)}(x+k,y,z;\lambda) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} T B_n^{(c,v)}(x,y,z;\lambda) k^r \frac{t^{n+r}}{n! r!}, \]
which, upon replacing \( n \) by \( n-r \) and then comparing the coefficients of \( \frac{t^n}{n!} \) on both sides of the resultant equation yields the assertion in (3.5). \( \square \)

**Remark 1.** Setting \( k=x \) in the equation (3.5), we derive
\[ T B_n^{(c,v)}(2x,y,z;\lambda) = \sum_{r=0}^{n} \binom{n}{r} T B_{n-r}^{(c,v)}(x,y,z;\lambda) x^r. \]

**Remark 2.** Letting \( k=1 \) in the equation (3.5), we acquire
\[ T B_n^{(c,v)}(x+1,y,z;\lambda) = \sum_{r=0}^{n} \binom{n}{r} T B_{n-r}^{(c,v)}(x,y,z;\lambda). \]
Theorem 3.4. The parametric kinds of generalized Apostol–Bernoulli polynomials $\mathcal{T}B_n^{(s,v)}(x, y, z; \lambda)$ satisfy

$$\mathcal{T}B_n^{(s,v)}(x + k, y, z; \lambda) = \sum_{r=0}^{n} \binom{n}{r} B_{n-r}^{(s,v)}(x, z; \lambda) T_r(k, y).$$

Proof. In the equation (2.1), first replacing $x$ by $x + k$, then using the equations (1.7) and (2.7) in the resultant equation, we discover

$$\sum_{n=0}^{\infty} \mathcal{T}B_n^{(s,v)}(x + k, y, z; \lambda) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{r=0}^{n} \mathcal{T}B_n^{(s,v)}(x, z; \lambda) T_r(k, y) \frac{t^{n+r}}{n!r!},$$

which, upon replacing $n$ by $n - r$ and then comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the resultant equation, implies the desired result. □

Theorem 3.5. The parametric kinds of generalized Apostol–Bernoulli polynomials $\mathcal{T}B_n^{(c,v)}(x, y, z; \lambda)$ satisfy the relation

$$\mathcal{T}B_n^{(c,v)}(\omega, y, z; \lambda) = \sum_{\ell=0}^{n} \sum_{m=0}^{r} \binom{n}{\ell} \binom{r}{m} (\omega - x)^{\ell+m} \mathcal{T}B_n^{(c,v)}(x, y, z; \lambda) \frac{(t+s)^n}{n!}.$$  (3.6)

Proof. Making use of the identity

$$\sum_{m=0}^{\infty} f(m) \frac{(x+y)^m}{m!} = \sum_{r=0}^{\infty} \frac{f(s+r)}{s!r!} t^r,$$  (3.7)

in [16, p. 1375], replacing $t$ by $t+s$ in the generating function in (2.1), and utilizing the identity (3.7), we find

$$\left( \frac{t+s}{\lambda e^{(t+s)} - 1} \right)^v e^{(t+s)} U(y, t+s) \cos[z(t+s)] = \sum_{n=0}^{\infty} \mathcal{T}B_n^{(c,v)}(x, y, z; \lambda) \frac{(t+s)^n}{n!}.  \quad (3.8)$$

In (3.8), replacing $x$ by $\omega$ and then expanding the exponential function, we acquire

$$\sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \mathcal{T}B_n^{(c,v)}(\omega, y, z; \lambda) \frac{t^n s^r}{n!r!} = \sum_{k=0}^{\infty} \frac{(\omega - x)^k (t+s)^k}{k!} \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \mathcal{T}B_n^{(c,v)}(x, y, z; \lambda) \frac{t^n s^r}{n!r!}.$$  (3.9)

Using the identity (3.7) on the right hand side of the above equation, replacing $n$ by $n - \ell$, $r$ by $r - m$ on the right hand side of the resultant equation give

$$\sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \mathcal{T}B_n^{(c,v)}(x, y, z; \lambda) \frac{t^n s^r}{n!r!} = \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \sum_{\ell=0}^{\ell} \sum_{m=0}^{m} (\omega - x)^{\ell+m} \mathcal{T}B_n^{(c,v)}(x, y, z; \lambda) \frac{t^n s^r}{\ell! m! (n-\ell)! (k-m)!}}.$$  (3.10)

Comparing the coefficients of $t^n$ and $s^r$ on both sides of (3.9) yields the assertion in (3.6). □

Remark 3. Taking $z = 0$ and replacing $\omega$ by $\omega + x$ in (3.6) reduces to

$$\mathcal{T}B_n^{(c,v)}(\omega + x, y, 0; \lambda) = \sum_{\ell=0}^{n} \sum_{m=0}^{r} \binom{n}{\ell} \binom{r}{m} \omega^{\ell+m} \mathcal{T}B_n^{(c,v)}(x, y; \lambda).$$  (3.11)

Theorem 3.6. Let $v, \alpha \in \mathbb{N}_0$. Then we have

$$\mathcal{T}B_n^{(c,v+\alpha)}(x, y, z; \lambda) = \sum_{r=0}^{n} \binom{n}{r} B_r^{(c)}(\lambda) \mathcal{T}B_n^{(c,v)}(x, y, z; \lambda).$$  (3.12)

and

$$\mathcal{T}B_n^{(c,v+\alpha)}(x, y, z; \lambda) = \sum_{r=0}^{n} \binom{n}{r} B_r^{(c)}(\lambda) \mathcal{T}B_n^{(c,v)}(x, y, z; \lambda).$$  (3.13)
Proof. Replacing $v$ by $v + \alpha$ in (2.1), we find
\[
\sum_{n=0}^\infty T B_n^{(c,\alpha+\alpha)}(x, y, z; \lambda) \frac{t^n}{n!} = \left( \frac{t}{\lambda e^t - 1} \right)^{v+\alpha} e^{zt} U(y, t) \cos(zt)
\]
which, upon using Apostol–Bernoulli numbers of order $v$ and (2.1) and comparing the coefficients of $\frac{t^n}{n!}$ on both sides, implies (3.10). Similarly, we can prove the assertion in (3.11). $\square$

Theorem 3.7. Let $n \in \mathbb{N}_0$ and $i = \sqrt{-1}$. Then we have
\[
T B_n^{(\nu)}(x + iz, y) = T B_n^{(\nu)}(x, y, z; \lambda) + i T B_n^{(s, \nu)}(x, y, z; \lambda).
\]

Proof. Taking $z = 0$ and replacing $x$ by $x + iz$ in (2.3), we conclude
\[
\sum_{n=0}^\infty T B_n^{(\nu)}(x + iz, y) \frac{t^n}{n!} = \left( \frac{t}{\lambda e^t - 1} \right)^{v} e^{(x+iz)t} U(y, t)
\]
which, upon using (2.2) on the right hand side and comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the resultant equation, yields the assertion in (3.12). $\square$

Theorem 3.8. For $n \in \mathbb{N}_0$, we have the following summation formula
\[
\sum_{r=0}^n \binom{n}{r} B_{n-r}^{(\nu)}(x; \lambda) T B_r^{(s, \beta)}(x, y, 2z; \lambda) = 2 \sum_{r=0}^n \binom{n}{r} B_r^{(s, \nu)}(x, z; \lambda) T B_r^{(c, \beta)}(x, y, z; \lambda).
\]

Proof. In view of the equations (1.1) and (2.2), we have
\[
\sum_{n=0}^\infty B_n^{(c)}(x; \lambda) \frac{t^n}{n!} \sum_{r=0}^\infty T B_r^{(s, \beta)}(x, y, 2z; \lambda) \frac{t^r}{r!} = \left( \frac{t}{\lambda e^t - 1} \right)^{v} e^{zt} \left( \frac{t}{\lambda e^t - 1} \right)^{\beta} e^{zt} U(y, t) \sin(2zt)
\]
\[
= 2 \left( \frac{t}{\lambda e^t - 1} \right)^{v} e^{zt} \left( \frac{t}{\lambda e^t - 1} \right)^{\beta} e^{zt} U(y, t) \cos(zt)
\]
which, upon using the equations (1.3) and (2.1) and comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the resultant equation, yields the assertion in (3.13). $\square$

4. Partial derivative equations

In this section, we introduce the partial derivative equations and identities including the parametric kinds of generalized Apostol–Bernoulli polynomials $T B_n^{(c, \nu)}(x, y, z; \lambda)$ and $T B_n^{(s, \nu)}(x, y, z; \lambda)$. We will apply derivative operators and the generating functions in (2.1) and (2.2).

For $n \in \mathbb{N}_0$ and in view of the equations (2.1) and (2.2), we derive the following results:
\[
\frac{\partial}{\partial x} [T B_n^{(c, \nu)}(x, y, z; \lambda)] = n T B_{n-1}^{(c, \nu)}(x, y, z; \lambda),
\]
\[
\frac{\partial}{\partial x} [T B_n^{(s, \nu)}(x, y, z; \lambda)] = n T B_{n-1}^{(s, \nu)}(x, y, z; \lambda),
\]
\[
\frac{\partial}{\partial z} [T B_n^{(c, \nu)}(x, y, z; \lambda)] = -n T B_{n-1}^{(s, \nu)}(x, y, z; \lambda),
\]
\[
\frac{\partial}{\partial z} [T B_n^{(s, \nu)}(x, y, z; \lambda)] = n T B_{n-1}^{(c, \nu)}(x, y, z; \lambda).
\]
The above equations show that we have
\[ \frac{\partial}{\partial x} \left[ T B_n^{(c,v)}(x, y, z; \lambda) \right] = \frac{\partial}{\partial z} \left[ T B_n^{(s,v)}(x, y, z; \lambda) \right] \]
and
\[ \frac{\partial}{\partial x} \left[ T B_n^{(s,v)}(x, y, z; \lambda) \right] = -\frac{\partial}{\partial z} \left[ T B_n^{(c,v)}(x, y, z; \lambda) \right]. \]

**Theorem 4.1.** Let \( m, n \in \mathbb{N} \) and \( n \geq m \). Then we have
\[
\frac{\partial^m}{\partial x^m} [T B_n^{(s,v)}(x + \alpha, y, z + \beta; \lambda)] = \sum_{r=0}^{n} m! \binom{n}{r} \left( \begin{array}{c} r \cr m \end{array} \right) T B_{r-m}^{(s,v)}(x, y, z; \lambda) C_{n-r}(\alpha, \beta) + T B_{r-m}^{(c,v)}(x, y, z; \lambda) S_{n-r}(\alpha, \beta). \tag{4.1}
\]

**Proof.** Replacing \( x \) by \( x + \alpha \) and \( z \) by \( z + \beta \) in (2.2) and then applying the derivative operator \( \frac{\partial^m}{\partial x^m} \) to the resultant equation, we find
\[
\sum_{n=0}^{\infty} \frac{\partial^m}{\partial x^m} [T B_n^{(s,v)}(x + \alpha, y, z + \beta; \lambda)] \frac{t^n}{n!} = t^m \left( \frac{t}{\lambda e^t - 1} \right)^v e^{(x+\alpha)t} U(y, t) \sin[(z + \beta)t] + t^m \left( \frac{t}{\lambda e^t - 1} \right)^v e^{xt} U(y, t) \cos(\beta t) e^{\alpha t} \sin(\beta t).
\]

Next, in view of the equations (1.2), (1.3), (2.1), and (2.2), the above equation becomes
\[
\sum_{n=0}^{\infty} \frac{\partial^m}{\partial x^m} [T B_n^{(s,v)}(x + \alpha, y, z + \beta; \lambda)] \frac{t^n}{n!} = \sum_{n=0}^{\infty} m! \binom{n}{r} \left( \begin{array}{c} r \cr m \end{array} \right) T B_{r-m}^{(s,v)}(x, y, z; \lambda) C_{n-r}(\alpha, \beta) \frac{t^n}{n!} + \sum_{n=0}^{\infty} m! \binom{n}{r} \left( \begin{array}{c} r \cr m \end{array} \right) T B_{r-m}^{(c,v)}(x, y, z; \lambda) S_{n-r}(\alpha, \beta) \frac{t^n}{n!}.
\]

Comparing the coefficients of \( \frac{t^n}{n!} \) on both sides of the resultant equation yields the assertion in (4.1). \( \square \)

**Theorem 4.2.** Let \( m, n \in \mathbb{N} \) and \( n \geq m \). Then parametric kinds of generalized Apostol–Bernoulli polynomials satisfy
\[
\frac{\partial^m}{\partial x^m} [T B_n^{(c,v)}(x, y, z; \lambda)] = \sum_{r=0}^{n-m} m! \binom{n}{r} \left( \begin{array}{c} n-m \cr r \end{array} \right) B_r^{(\delta)}(\lambda) T B_{n-r-m}^{(c,v-\delta)}(x, y, z; \lambda) \tag{4.2}
\]
and
\[
\frac{\partial^m}{\partial x^m} [T B_n^{(s,v)}(x, y, z; \lambda)] = \sum_{r=0}^{n-m} m! \binom{n}{r} \left( \begin{array}{c} n-m \cr r \end{array} \right) B_r^{(\delta)}(\lambda) T B_{n-r-m}^{(s,v-\delta)}(x, y, z; \lambda). \tag{4.3}
\]
Proof. Using the derivative operator \( \frac{\partial^m}{\partial x^m} \) in (2.1), we arrive at
\[
\sum_{n=0}^{\infty} \frac{\partial^m}{\partial x^m} \left[ \tau B_n^{(c, v)}(x, y, z; \lambda) \right] = \frac{\partial^m}{\partial x^m} \left[ \left( \frac{t}{\lambda e^t - 1} \right)^v e^{xt} U(y, t) \cos(zt) \right]
\]
\[
= t^m \left[ \left( \frac{t}{\lambda e^t - 1} \right)^v e^{xt} U(y, t) \cos(zt) \right]
\]
\[
= t^m \left[ \left( \frac{t}{\lambda e^t - 1} \right)^{\delta} \left( \frac{t}{\lambda e^t - 1} \right)^{v-\delta} e^{xt} U(y, t) \cos(zt) \right]
\]
\[
= \sum_{n=0}^{\infty} \sum_{r=0}^{n-m} m! \binom{n}{m} \binom{n-m}{r} B_r^{(\delta)}(\lambda) \tau B_{n-r-m}^{(c, v-\delta)}(x, y, z; \lambda) \frac{t^n}{n!}
\]
Comparing the coefficients of \( \frac{t^n}{n!} \) on both sides, we derive the equation (4.2). Similarly, we can conclude the equation (4.3). \( \square \)

5. Examples and special cases

In this section, we demonstrate several families of polynomials, including the Apostol–Bernoulli type polynomial families, by substituting different functions for \( U(y, t) \), and investigate their properties.

To give examples of the parametric kinds of generalized Apostol–Bernoulli polynomials \( \tau B_n^{(c, v)}(x, y, z; \lambda) \) and \( \tau B_n^{(s, v)}(x, y, z; \lambda) \), we recall the generating functions of some special polynomials.

(1) The Gould–Hopper polynomials \( H_n^{(m)}(x, y) \) are defined \([8, pp. 51–63]\) by
\[
e^{xt+ytm} = \sum_{n=0}^{\infty} H_n^{(m)}(x, y) \frac{t^n}{n!}
\]
where \( m \in \mathbb{N} \). If taking \( m = 2 \) in the above generating function, we derive the generating function of the Hermite–Kampé de Feriét polynomials \( H_n(x, y) \), see \([3]\).

(2) The Hermite–Appell polynomials \( H_n^{(m)}(x, y) \) are generated \([17]\) by
\[
A(t) e^{xt+ytm^2} = \sum_{n=0}^{\infty} H_n^{(m)}(x, y) \frac{t^n}{n!}
\]
What is the symbol \( A(t) \)?

(3) The truncated exponential polynomials \( e_n^{(m)}(x, y) \) of order \( m \) are generated in \([1]\) by
\[
e^{xt} \frac{1}{1-ytm} = \sum_{n=0}^{\infty} e_n^{(m)}(x, y) \frac{t^n}{n!}
\]

5.1. Gould–Hopper–Apostol–Bernoulli type polynomials. When taking \( U(y, t) = e^{yt} \) in (2.1) and (2.2), we deduce the Gould–Hopper–Apostol–Bernoulli type polynomials \( H_n^{(m)}B_n^{(c, v)}(x, y, z; \lambda) \) and \( H_n^{(m)}B_n^{(s, v)}(x, y, z; \lambda) \) which are generated by
\[
\left( \frac{t}{\lambda e^t - 1} \right)^v e^{xt+yt} \cos(zt) = \sum_{n=0}^{\infty} H_n^{(m)}B_n^{(c, v)}(x, y, z; \lambda) \frac{t^n}{n!}
\] (5.1)
and
\[
\left( \frac{t}{\lambda e^t - 1} \right)^v e^{xt+yt} \sin(zt) = \sum_{n=0}^{\infty} H_n^{(m)}B_n^{(s, v)}(x, y, z; \lambda) \frac{t^n}{n!}
\] (5.2)
According to (2.3), (2.4), (2.5), and (2.6), we see that the polynomials $\mathcal{H}_n^{m}(x, y, z; \lambda)$ and $\mathcal{H}_n^{s,v}(x, y, z; \lambda)$ are quasi-monomial with respect to the following multiplication and derivative operators:

$$
\begin{align*}
\hat{M}_{n,B_x} &= x + ymD_x^{m-1} + \frac{\lambda e^{D_x}(1 - D_x)}{\lambda e^{D_x} - 1} - z\tan(zD_x), \\
\hat{P}_{n,B_x} &= D_x,
\end{align*}
$$

respectively. From (2.8) and (2.9), using the derivative and multiplicative operators given above, we conclude

$$
\begin{align*}
\hat{M}_{n,B_x} &= x + ymD_x^{m-1} + \frac{\lambda e^{D_x}(1 - D_x)}{\lambda e^{D_x} - 1} + z\cot(zD_x), \\
\hat{P}_{n,B_x} &= D_x,
\end{align*}
$$

If letting $m = 2$ in (5.1) and (5.2), we can derive the Hermite–Kampé de Fériet–Apostol–Bernoulli type polynomials $\mathcal{H}_n^{m}(x, y, z; \lambda)$ and $\mathcal{H}_n^{s,v}(x, y, z; \lambda)$. These polynomials satisfy the properties stated between (5.3) and (5.4).

5.2. Hermite–Appell–Apostol–Bernoulli type polynomials. If taking $U(y, t) = A(t)e^{yt^2}$ in (2.3) and (2.2), then we derive the Hermite–Appell–Apostol–Bernoulli type polynomials $\mathcal{H}_n^{(c,v)}(x, y, z; \lambda)$ and $\mathcal{H}_n^{(s,v)}(x, y, z; \lambda)$ which are generated by

$$
\left( \frac{t}{\lambda e^{t^2} - 1} \right)^v e^{xt+yt^2} A(t) = \sum_{n=0}^{\infty} \mathcal{H}_n^{(c,v)}(x, y, z; \lambda) \frac{t^n}{n!},
$$

and

$$
\left( \frac{t}{\lambda e^{t^2} - 1} \right)^v e^{xt+yt^2} A(t) = \sum_{n=0}^{\infty} \mathcal{H}_n^{(s,v)}(x, y, z; \lambda) \frac{t^n}{n!}.
$$

From (2.3), (2.4), (2.5), and (2.6), we see that the polynomials $\mathcal{H}_n^{(c,v)}(x, y, z; \lambda)$ and $\mathcal{H}_n^{(s,v)}(x, y, z; \lambda)$ are quasi-monomial with respect to the following multiplication and derivative operators:

$$
\begin{align*}
\hat{M}_{A,B_x} &= x + 2yD_x + \frac{A'(D_x)}{A(D_x)} + \frac{\lambda e^{D_x}(1 - D_x)}{\lambda e^{D_x} - 1} - z\tan(zD_x), \\
\hat{P}_{A,B_x} &= D_x,
\end{align*}
$$

respectively.

According to (2.8) and (2.9), using the derivative and multiplicative operators given above, we obtain the differential equations

$$
\begin{align*}
\hat{M}_{A,B_x} &= x + 2yD_x + \frac{A'(D_x)}{A(D_x)} + \frac{\lambda e^{D_x}(1 - D_x)}{\lambda e^{D_x} - 1} + z\cot(zD_x), \\
\hat{P}_{A,B_x} &= D_x,
\end{align*}
$$

respectively.
and

\[ xD_x + yD_x^2 + \frac{A'(D_x)}{A(D_x)} + v \frac{\lambda e^{D_x} (1 - D_x) - 1}{\lambda e^{D_x} - 1} + z \cot(zD_x)D_x - n \] \[ n \lambda B_n^{(c,v)}(x, y, z; \lambda) = 0. \]

5.3. Truncated exponential Apostol–Bernoulli type polynomials. If \( U(y, t) = \frac{1}{1 - yt} \) in (2.1) and (2.2), we derive the truncated exponential Apostol–Bernoulli type polynomials \( e^{(m)} B_n^{(c,v)}(x, y, z; \lambda) \) and \( e^{(m)} B_n^{(s,v)}(x, y, z; \lambda) \) which are generated by

\[ \left( \frac{t}{\lambda e^t - 1} \right)^v \frac{1}{1 - yt^m} \cos(zt) = \sum_{n=0}^{\infty} e^{(m)} B_n^{(c,v)}(x, y, z; \lambda) \frac{t^n}{n!} \]

and

\[ \left( \frac{t}{\lambda e^t - 1} \right)^v \frac{1}{1 - yt^m} \sin(zt) = \sum_{n=0}^{\infty} e^{(m)} B_n^{(s,v)}(x, y, z; \lambda) \frac{t^n}{n!}. \]

Considering (2.3), (2.4), (2.5), and (2.6), we see easily that the polynomials \( e^{(m)} B_n^{(c,v)}(x, y, z; \lambda) \) and \( e^{(m)} B_n^{(s,v)}(x, y, z; \lambda) \) are quasi-monomial with respect to the following multiplication and derivative operators:

\[
\begin{align*}
\hat{M}_{e^{(m)} B_x} &= x + v \frac{\lambda e^{D_x} (1 - D_x) - 1}{\lambda e^{D_x} - 1} + \frac{myD_x^{m-1}}{1 - yD_x^m} - z \tan(zD_x), \\
\hat{P}_{e^{(m)} B_x} &= D_x,
\end{align*}
\]

\[
\begin{align*}
\hat{M}_{e^{(m)} B_x} &= x + \frac{v \lambda e^{D_x} (1 - D_x) - 1}{\lambda e^{D_x} - 1} + \frac{myD_x^{m-1}}{1 - yD_x^m} + z \cot(zD_x), \\
\hat{P}_{e^{(m)} B_x} &= D_x,
\end{align*}
\]

respectively.

From (2.8) and (2.9), considering the derivative and multiplicative operators given above, we acquire

\[
\begin{align*}
\left[ xD_x + v \frac{\lambda e^{D_x} (1 - D_x) - 1}{\lambda e^{D_x} - 1} + \frac{myD_x^m}{1 - yD_x^m} - z \tan(zD_x)D_x - n \right] e^{(m)} B_n^{(c,v)}(x, y, z; \lambda) = 0
\end{align*}
\]

and

\[
\begin{align*}
\left[ xD_x + v \frac{\lambda e^{D_x} (1 - D_x) - 1}{\lambda e^{D_x} - 1} + \frac{myD_x^m}{1 - yD_x^m} + z \cot(zD_x)D_x - n \right] e^{(m)} B_n^{(s,v)}(x, y, z; \lambda) = 0.
\end{align*}
\]

6. Conclusion

In this paper, we have introduced the new generating functions using Euler’s formula for parametric kinds of generalized Apostol–Bernoulli polynomials which are of quasi-monomial properties. Then we have investigated quasi-monomial properties and identities, some addition and summation formulas and partial derivative formulas. Then, in the special cases of \( U(y, t) \), we construct new subpolynomial families such as the Gould–Hopper–Apostol–Bernoulli type polynomials, the Hermite–Appell–Apostol–Bernoulli type polynomials and truncated exponential Apostol–Bernoulli type polynomials. Lastly, we investigated the multiplicative and derivative operators and differential equations of these new families.

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