Quantization of a Particle on a Two-Dimensional Manifold of Constant Curvature

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Abstract

The formulation of quantum mechanics on spaces of constant curvature is studied. It is shown how a transition from a classical system to the quantum case can be accomplished by the quantization of the Noether momenta. These can be determined by Lie differentiation of the metric which defines the manifold. For the metric examined here, it is found that the resulting Schrödinger equation is separable and the spectrum and eigenfunctions can be investigated in detail.

PACs: 03.65.Ge, 03.65.Ta, 03.65.Aa, 03.65.Ca, 02.40.Hw

Keywords: curvature, vector field, Hamiltonian, quantization, metric, canonical
1 Introduction

The study of a quantum particle on a spherical or hyperbolic space in two-dimensions is amenable to study by treating the scalar curvature as a parameter. This sort of approach has been of great interest recently [1,2]. It is certainly physically relevant since many new phenomenon under investigation occur in two dimensions or on two-dimensional manifolds. A very pertinent example of this is the quantum Hall effect which exhibits the formation of quasiparticles in the course of its operation [3]. The spherical and hyperbolic spaces are characterized by either a positive or negative value for the scalar curvature of the space. In the Euclidean case where the curvature vanishes, the problem is not complicated because the solutions are plane-wave states that are in fact momentum eigenfunctions of the linear momentum operator. Further, here it is the case that plane waves are simultaneous eigenfunctions of both the energy and momentum operators. If the curvature of the space is constant but different from zero, the canonical momenta do not coincide with the Noether momenta, so the Noether momenta do not Poisson commute and as well the quantum versions of these quantities do not commute as operators. These reasons make the situation much more complicated in any space with a nonzero or variable curvature. What is referred to as a plane wave is a Euclidean concept, and it is not clear how to generalize the definition to a curved space. Many of these difficulties can be resolved by adopting a curvature dependent approach.

Many physical situations can be formulated in terms of an underlying, curved manifold. In addition to the quantum Hall effect, the area of quantum dots requires the use of models which are founded on quantum mechanics on constant curvature spaces. There is the very active area which studies polynomial billiards, or systems which are enclosed by geodesic arcs on surfaces with curvature. Some motions that are integrable in the Euclidean case may become ergodic when the curvature of the space becomes negative [4]. The problem under investigation here overlaps with the study of quantum chaos in quantum systems. The entire area of gravitation and cosmology are presently formulated on a geometric basis. Gravity is a manifestation of the curvature of the space-time. A space-time is specified or characterized by defining a metric whose components are
used in the calculation of the curvature of the space-time manifold [5]. There has been a great deal of interest in quantum motion on a curved manifold recently. A very different approach to the one examined here is to view quantum motion as a submanifold problem in a generalized Dirac’s theory of second-class constraints [6].

It is the objective here to first review some $\kappa$-dependent formalisms which are appropriate for the description of the dynamics on the spaces $M^2_\kappa = (S^2_\kappa, E^2_\kappa, H^2_\kappa)$ with constant curvature. It is possible to give a unified approach to both spherical ($\kappa > 0$) and hyperbolic ($\kappa < 0$) spaces so Euclidean dynamics manifests itself when the parameter $\kappa = 0$. These three spaces can be thought of as three different cases arising from a family of Riemannian manifolds $M^2_\kappa$ with the curvature appearing as a parameter. The components of the metric are selected according to this geometric structure. The metric of interest here was not quantized in [7-9] or [10]. Everything can be done in such a way that applications to other types of system whose Lagrangian can be defined explicitly in terms of the components of a metric should be possible.

Once the metric is known, Killing vector fields on the manifold are determined by means of Lie differentiation of the metric. This procedure results in a coupled system of partial differential equations in the unknown component functions of the vector field which can be easily solved. A Hilbert space can be constructed for the problem by defining a measure which is annihilated upon Lie differentiation with respect to this set of linearly independent vector fields. The Killing vector fields provide the Noether momenta for the system. The Hamiltonian is obtained by means of the usual canonical transformation from the Lagrangian and subsequently written in terms of these momenta. The quantization algorithm can then be applied to the components of the Noether momenta which appear in the Hamiltonian. Thus, the Hamiltonian can be quantized in this way. Once the Hamiltonian has been given, the Schrödinger equation can be written down. It is remarkable to note that for the metric which is defined and used here, the Schrödinger equation is separable, and moreover, the energy and wavefunctions can be calculated from it in closed form.
2 Metric and Associated Hamiltonian

From the geometric point of view, the sphere $S_\kappa$, the Euclidean plane $\mathbb{E}^2$ and the hyperbolic plane $H^2_\kappa$ represent three different objects which make up a family of Riemannian manifolds. These are grouped together and referred to as $M^2_\kappa = (S^2_\kappa, \mathbb{E}^2, H^2_\kappa)$ of curvature $\kappa$. Suppose a general metric is assigned to this class of spaces by making use of the $\kappa$-dependent trigonometric and hyperbolic functions defined as

$$S_\kappa(x) = \begin{cases} \frac{1}{\sqrt{\kappa}} \sin(\sqrt{\kappa}x), & \kappa > 0, \\ x, & \kappa = 0, \\ \frac{1}{\sqrt{\kappa}} \sinh(\sqrt{\kappa}x), & \kappa < 0. \end{cases}$$  \hspace{1cm} (2.1)

From (2.1), the functions $C_\kappa(x)$ and $T_\kappa(x)$ can be defined

$$C_\kappa(x) = \frac{dS_\kappa(x)}{dx}, \quad T_\kappa(x) = \frac{S_\kappa(x)}{C_\kappa(x)}.$$ \hspace{1cm} (2.2)

A general metric in geodesic polar coordinates $(\rho, \varphi)$ on $M^2_\kappa$ is defined in the following way,

$$g = d\rho \otimes d\rho + S^2_\kappa(\rho) d\varphi \otimes d\varphi.$$ \hspace{1cm} (2.3)

Several Lagrangians can be obtained from (2.3) by diffeomorphisms and can be considered to be dynamically equivalent at the classical level. One of these Lagrangians, which has not been examined before, will be used as the starting point for the construction of the Hamiltonian quantum system. Let $M$ be a Riemannian or pseudo-Riemannian manifold whose metric evaluated at a point $p \in M$ is $g(p)$. On the tangent space $TM$, a Lagrangian can be defined by first giving the kinetic energy in terms of the components of the metric [11]

$$T = \frac{1}{2} g_{ij} v^i v^j.$$ \hspace{1cm} (2.4)

The Lagrangian of geodesic motion which corresponds to (2.3) on $M^2_\kappa$ given by (2.4) plus a potential function is

$$L(\kappa) = \frac{1}{2} (v^2_\rho + S^2_\kappa(\rho)v^2_\varphi) + V(\rho).$$ \hspace{1cm} (2.5)

Several diffeomorphic versions of (2.5) can be presented. Consider a $\kappa$-dependent transformation defined by $\rho \to \rho' = T_\kappa(\rho)$. This transformation puts the metric (2.3) into the form

$$g = \frac{1}{(1 + \kappa r^2)^2} dr \otimes dr + \frac{r^2}{1 + \kappa r^2} d\varphi \otimes d\varphi.$$ \hspace{1cm} (2.6)
after writing \( r \) in place of \( r' \), and transforms takes Lagrangian (2.5) into the form,

\[
L_H(\kappa) = \frac{1}{2} \left( \frac{v_r^2}{(1 + \kappa r^2)^2} + \frac{r^2}{1 + \kappa r^2} v_\phi^2 \right) + V(r). \tag{2.7}
\]

It is worth mentioning that (2.7) can be transformed into Cartesian form by means of the following relation,

\[
v_x^2 + v_y^2 + \kappa(x v_y - y v_x)^2 = v_r^2 + r^2(1 + \kappa r^2) v_\phi^2. \tag{2.8}
\]

Consequently, Lagrangian (2.7) is given by

\[
L_H(\kappa) = \frac{1}{2} \left( \frac{1}{1 + \kappa r^2} \right)^2 \left[ v_x^2 + v_y^2 + \kappa(x v_y - y v_x)^2 \right] - \frac{1}{2} \alpha^2 r^2. \tag{2.9}
\]

where a potential which depends on \( r \) has been included and \( r^2 = x^2 + y^2 \).

Setting \( v_x = \dot{x} \) and \( v_y = \dot{y} \), the canonical momenta are determined to be

\[
px = \frac{\partial L}{\partial v_x} = \frac{1}{(1 + \kappa r^2)^2} [v_x - \kappa y (x v_y - y v_x)], \quad py = \frac{\partial L}{\partial v_y} = \frac{1}{(1 + \kappa r^2)^2} [v_y - \kappa x (x v_y - y v_x)]. \tag{2.10}
\]

These are required in order to compute the classical Hamiltonian. Solving (2.10) for \( v_x \) and \( v_y \), it is found that

\[
v_x = (1 + \kappa r^2)((1 + \kappa x^2)p_x + \kappa x y p_y), \quad v_y = (1 + \kappa r^2)((1 + \kappa y^2)p_y + \kappa x y p_x). \tag{2.11}
\]

The Hamiltonian in the \((x, y)\) coordinates is obtained by means of the usual canonical transformation

\[
H(\kappa) = px v_x + py v_y - L_H(\kappa) = \frac{1}{2} (1 + \kappa r^2) \left( p_x^2 + p_y^2 + \kappa(y p_y + x p_x)^2 \right) + \frac{1}{2} \alpha^2 (x^2 + y^2). \tag{2.12}
\]

The Hamiltonian is simpler and more useful in cylindrical coordinates and it is written in this form now. Introducing \( v_r = \dot{r} \) and \( v_\phi = \dot{\phi} \), the canonical momenta in the cylindrical variables are

\[
p_r = \frac{\partial L}{\partial v_r} = \frac{\dot{r}}{(1 + \kappa r^2)^2}, \quad p_\phi = \frac{\partial L}{\partial v_\phi} = \frac{\dot{r}^2}{1 + \kappa r^2}. \tag{2.13}
\]

Solving (2.13) for \( v_r \) and \( v_\phi \), it is found that

\[
v_r = (1 + \kappa r^2)^2 p_r, \quad v_\phi = \frac{1}{r^2} (1 + \kappa r^2) p_\phi. \tag{2.14}
\]
The Hamiltonian in terms of cylindrical variables is then calculated to be

\[ H(\kappa) = p_r v_r + p_\varphi v_\varphi - L_H(\kappa) = \frac{1}{2}(1 + \kappa r^2)^2 p_r^2 + \frac{1}{2r^2}(1 + \kappa r^2)p_\varphi^2 + \frac{1}{2}\alpha^2 r^2. \]  

(2.15)

All of the calculations given here are easy to verify by means of symbolic manipulation [12].

3 Noether Symmetries

A set of three linearly independent Killing vector fields will be calculated for the metric presented in (2.6). If for a certain vector field the Lie derivative of the metric vanishes, this vector field is called a Killing vector field. This type of vector field can be thought of as an infinitesimal generator of isometries of the \(\kappa\)-dependent metric (2.6).

Let \(X\) be a vector field in terms of the cylindrical \((r, \varphi)\) coordinates defined by

\[ X = f(r, \varphi) \frac{\partial}{\partial r} + h(r, \varphi) \frac{\partial}{\partial \varphi}. \]  

(3.1)

The two functions \(f\) and \(h\) will be determined in such a manner that the Lie derivative of \(g\) vanishes,

\[ \mathcal{L}_X g = 0. \]  

(3.2)

Differentiating the metric with respect to \(X\), it is found that

\[
\mathcal{L}_X g = f \frac{\partial}{\partial r} \left( \frac{1}{(1 + \kappa r^2)^2} \right) dr \otimes dr + \frac{1}{(1 + \kappa r^2)^2} \left( \frac{\partial f}{\partial r} dr \otimes dr + \frac{\partial f}{\partial \varphi} d\varphi \otimes dr + \frac{\partial f}{\partial r} dr \otimes dr + \frac{\partial f}{\partial \varphi} dr \otimes d\varphi \right)
\]

\[
+ f \frac{\partial}{\partial r} \left( \frac{r^2}{1 + \kappa r^2} \right) d\varphi \otimes d\varphi
\]

\[
+ \frac{r^2}{1 + \kappa r^2} \left( \frac{\partial h}{\partial r} dr \otimes d\varphi + \frac{\partial h}{\partial \varphi} d\varphi \otimes dr + \frac{\partial h}{\partial r} d\varphi \otimes d\varphi + \frac{\partial h}{\partial \varphi} d\varphi \otimes d\varphi \right).
\]

Collecting like terms, it is required that the coefficient of each tensor product in this expression must vanish for (3.2) to hold. This produces the following system of three coupled first order partial differential equations in terms of \(f\) and \(h\),

\[
\frac{\partial f}{\partial r} - \frac{2\kappa r}{1 + \kappa r^2} f = 0, \quad \frac{\partial f}{\partial \varphi} + r^2(1 + \kappa r^2) \frac{\partial h}{\partial r} = 0, \quad r(1 + \kappa r^2) \frac{\partial h}{\partial \varphi} + f = 0. \]  

(3.3)

This system of equations can be readily solved to yield the following general solution for \(f\) and \(h\),

\[ f(r, \varphi) = (1 + \kappa r^2)(C_1 \sin \varphi + C_2 \cos \varphi), \quad h(r, \varphi) = \frac{1}{r}(C_1 \cos \varphi - C_2 \sin \varphi) + C_3. \]  

(3.4)
The three required independent vector fields can be specified by choosing the constants appropriately; for example, \((C_1, C_2, C_3) = (1, 0, 0), (0, 1, 0), (0, 0, 1)\). For this choice, we have the following vector fields,

\[
X_1 = (1 + \kappa r^2) \cos \varphi \frac{\partial}{\partial r} - \frac{1}{r} \sin \varphi \frac{\partial}{\partial \varphi}, \quad X_2 = (1 + \kappa r^2) \sin \varphi \frac{\partial}{\partial r} + \frac{1}{r} \cos \varphi \frac{\partial}{\partial \varphi}, \quad X_J = \frac{\partial}{\partial \varphi}. \quad (3.5)
\]

The Lie commutator brackets for (3.5) can be calculated and are given by

\[
[X_1, X_2] = -\kappa X_J, \quad [X_1, X_J] = X_2, \quad [X_2, X_J] = -X_1. \quad (3.6)
\]

The set of vector fields given by (3.5) are the required Noether symmetries, so the coefficient functions satisfy system (3.3) and the associated constants of the motion are

\[
P_1 = (1 + \kappa r^2) \cos \varphi p_r - \frac{1}{r} \sin \varphi p_\varphi, \quad P_2 = (1 + \kappa r^2) \sin \varphi p_r + \frac{1}{r} \cos \varphi p_\varphi, \quad J = p_\varphi. \quad (3.7)
\]

The classical Poisson bracket of two dynamical quantities \(F\) and \(G\) is defined by

\[
\{F, G\} = \frac{\partial F}{\partial p_r} \frac{\partial G}{\partial p_\varphi} + \frac{\partial F}{\partial p_\varphi} \frac{\partial G}{\partial p_r} - \frac{\partial F}{\partial r} \frac{\partial G}{\partial p_\varphi} - \frac{\partial F}{\partial \varphi} \frac{\partial G}{\partial p_r}. \quad (3.8)
\]

For the case in which the variables \(F, G\) are replaced by \(P_1, P_2\) and \(J\) in (3.8), the following brackets are obtained

\[
\{P_1, P_2\} = \kappa J, \quad \{P_1, J\} = -P_2, \quad \{P_2, J\} = P_1. \quad (3.9)
\]

Using Hamiltonian (2.15), the following Poisson brackets are also found

\[
\{P_1, H\} = 0, \quad \{P_2, H\} = 0, \quad \{J, H\} = 0. \quad (3.10)
\]

At the classical level, it is clear using (3.7) that

\[
P_1^2 + P_2^2 + \kappa J^2 = (1 + \kappa r^2)^2 p_r^2 + \frac{1}{r^2}(1 + \kappa r^2) p_\varphi^2. \quad (3.11)
\]

The classical Hamiltonian in terms of the variables \(P_1, P_2\) and \(J\) including a mass \(m\) is given by

\[
H = \frac{1}{2m} (P_1^2 + P_2^2 + \kappa J^2) + \frac{1}{2} \alpha^2 r^2. \quad (3.12)
\]
The only measure on the space $\mathbb{R}^2$ that is invariant under the action of the vector fields (3.6) in the sense that the Lie derivative vanishes should be used to construct the Hilbert space. By starting with a function of $r$ times $dr \wedge d\varphi$, the function can be determined by differentiating with respect to these vector fields

$$\mathcal{L}_{X_i} d\mu_\kappa = 0, \quad i = 1, 2, J.$$  

(3.13)

An ordinary differential equation results which can be solved to give the measure as

$$d\mu_\kappa = \frac{r}{(1 + \kappa r^2)^{3/2}} dr \wedge d\varphi.$$  

(3.14)

Thus, the space carries a measure somewhat different from the one in [10]. The quantum Hamiltonian would be self-adjoint, not in the standard space $L^2(\mathbb{R}^2)$, but in the Hilbert space $L^2(d\mu_\kappa)$.

In the spherical case, the space is $L^2(\mathbb{R}^2, d\mu_\kappa)$, and in the hyperbolic case, it is $L^2_0(\mathbb{R}^2_\kappa, d\mu_\kappa)$, where $\mathbb{R}^2_\kappa$ denotes the region $r^2 \leq 1/\kappa$, and functions vanish at the boundary of the region.

4 Quantization of the Hamiltonian and Schrödinger Equation

A procedure which allows the Hamiltonian of the model to be quantized can be formulated based on property (3.13) of the measure. The idea is to consider functions and linear operators which are defined on a related space. This space can be defined by taking the two-dimensional real plane and using the measure (3.14) on it. The quantum operators which will be defined represent the quantum version of the Noether momenta. They must be self-adjoint in the space $L^2(\mathbb{R}, d\mu_\kappa)$.

The transition from classical to quantum mechanics by means of the Noether momenta (3.7) is now represented by the following correspondence

$$P_1 \rightarrow \hat{P}_1 = -i\hbar \{(1 + \kappa r^2) \cos \varphi \frac{\partial}{\partial r} - \frac{1}{r} \sin \varphi \frac{\partial}{\partial \varphi}\},$$

$$P_2 \rightarrow \hat{P}_2 = -i\hbar \{(1 + \kappa r^2) \sin \varphi \frac{\partial}{\partial r} + \frac{1}{r} \cos \varphi \frac{\partial}{\partial \varphi}\},$$

$$J \rightarrow \hat{J} = -i\hbar \frac{\partial}{\partial \varphi}.$$  

(4.1)
Under transformation (4.1), the classical Hamiltonian (3.12) is transformed into the following operator,

\[ \hat{H} = -\frac{\hbar^2}{2m} (1 + \kappa r^2) \left[ (1 + \kappa r^2) \frac{\partial^2}{\partial r^2} + (1 + 2\kappa r^2) \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} \right] + \frac{1}{2} \alpha^2 r^2. \] (4.2)

The Hamiltonian (4.2) immediately yields the Schrödinger equation,

\[ -\frac{\hbar^2}{2m} (1 + \kappa r^2) \left[ (1 + \kappa r^2) \frac{\partial^2}{\partial r^2} + (1 + 2\kappa r^2) \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} \right] \Psi + \frac{1}{2} \alpha^2 r^2 \Psi = E \Psi. \] (4.3)

To solve (4.3), it is advantageous to have (4.3) in a form in which the physical constants have been scaled out of the equation. Introduce the new constant \( \alpha = \sqrt{m\beta} \) into the equation so that

\[ -\frac{\hbar^2}{2m} (1 + \kappa r^2) \left[ (1 + \kappa r^2) \frac{\partial^2}{\partial r^2} + (1 + 2\kappa r^2) \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} \right] \Psi + \frac{1}{2} m\beta^2 r^2 \Psi = E \Psi. \] (4.4)

Now introduce the following set of new variables \((\bar{r}, \bar{\kappa}, \mathcal{E})\) which are defined to be

\[ r = \sqrt{\frac{\hbar}{m\beta}} \bar{r}, \quad \kappa = \frac{m\beta}{\hbar} \bar{\kappa}, \quad E = \hbar \beta \mathcal{E}. \] (4.5)

Consequently, \( \kappa r^2 = \bar{\kappa} \bar{r}^2 \) and upon substituting (4.5) into Schrödinger equation (4.4), it transforms into

\[ -\frac{\hbar^2}{2m} (1 + \kappa \bar{r}^2) \frac{m \beta}{\hbar} \frac{\partial^2}{\partial \bar{r}^2} + (1 + 2\kappa \bar{r}^2) \frac{m \beta}{\hbar} \frac{1}{\bar{r}} \frac{\partial}{\partial \bar{r}} + \frac{m \beta}{\hbar} \frac{1}{\bar{r}^2} \frac{\partial^2}{\partial \varphi^2} \Psi + \frac{1}{2} m\beta^2 h \frac{m \beta}{\hbar} \bar{r}^2 \Psi = E \Psi. \]

Removing the physical constants from the equation and dropping the bars from the variables, we obtain

\[ (1 + \kappa \bar{r}^2) \left[ (1 + \kappa \bar{r}^2) \frac{\partial^2}{\partial \bar{r}^2} + (1 + 2\kappa \bar{r}^2) \frac{1}{\bar{r}} \frac{\partial}{\partial \bar{r}} + \frac{1}{\bar{r}^2} \frac{\partial^2}{\partial \varphi^2} \right] \Psi - r^2 \Psi + 2\mathcal{E} \Psi = 0. \] (4.6)

5 Energies and Wavefunctions for the Model.

It is particularly interesting to observe that the spectrum and the structure of the wave functions for this Hamiltonian can be investigated for the system defined by metric (2.6). This is mainly due to the fact that the Schrödinger equation (4.6) is separable. It will be shown that there exist solutions to it of the form,

\[ \Psi(r, \varphi) = R(r)\Phi(\varphi), \] (5.1)
where $R$ and $\Phi$ are functions of the variables $r$ and $\varphi$, respectively. Substitute (5.1) into Schrödinger equation (4.6) so it takes the form

$$\Phi(\varphi)(1 + \kappa r^2)\{(1 + \kappa r^2)R'' + (1 + 2\kappa r^2)\frac{R'}{r}\} + \frac{1}{r^2} (1 + \kappa r^2) R\ddot{\Phi} - r^2 R\Phi + 2\mathcal{E} R\Phi = 0. \quad (5.2)$$

It is possible to separate this equation by first introducing a separation constant $\beta$. Equation (5.2) takes the form,

$$\frac{r^2}{R}\[(1 + \kappa r^2)R'' + (1 + 2\kappa r^2)\frac{R'}{r}\] - \frac{r^4}{1 + \kappa r^2} + 2\frac{r^2}{1 + \kappa r^2}\mathcal{E} = -\frac{\ddot{\Phi}}{\Phi} = \beta^2. \quad (5.3)$$

This result is equivalent to the following pair of equations for $\Phi$ and $R$,

$$\ddot{\Phi} + \beta^2 \Phi = 0,$$

$$r^2 (1 + \kappa r^2)R'' + r(1 + 2\kappa r^2)R' - \frac{r^4}{1 + \kappa r^2} R + \frac{2r^2}{1 + \kappa r^2}\mathcal{E} R - \beta^2 R = 0. \quad (5.4)$$

The equation for $\Phi$ has the exponential solutions of the form

$$\Phi(\varphi) = e^{\pm i\beta \varphi}.$$

The parameter $\kappa$ is relegated to the radial equation. The radial solution factorizes to the form,

$$R(r, \kappa) = F(r, \kappa)(1 + \kappa r^2)^s. \quad (5.5)$$

The radial equation then becomes an equation satisfied by the function $F(r) = F(r, \kappa)$,

$$r^2 (1 + \kappa r^2)^2 F''(r) + r(1 + \kappa r^2)(2\kappa(1 + 2s)r^2 + 1)F'(r) + (2s(1+s)\kappa^2 - 1)r^4 + (4\kappa s - \kappa \beta^2 + 2\mathcal{E}) r^2 - \beta^2 F = 0. \quad (5.6)$$

To solve this, a specific value for the parameter $s$ is taken. Consider the case in which $s$ is given by

$$s = \frac{1}{4} - \frac{q(\kappa)}{4\kappa}, \quad q = q(\kappa) = \sqrt{\kappa^2 + 8\kappa\mathcal{E} + 4}. \quad (5.7)$$

In this instance, it follows that in the small $\kappa$ limit,

$$\lim_{\kappa \to 0} R(r, \kappa) = F(r)e^{-r^2/2}. $$
This choice for $s$ gives an equation in which the $r^4$ dependence has disappeared from the last term of (5.6) and it becomes,

$$r^2(1 + \kappa r^2)F''(r) + ((3\kappa - q)r^2 + 1) r F'(r) + ((2\mathcal{E} + \kappa - q)r^2 - \beta^2) F(r) = 0. \quad (5.8)$$

The indicial equation for (5.8) implies that there is a regular solution at $r = 0$ of the form,

$$F(r) = r^\beta f(r).$$

Substituting this form into (5.8), it is found that $f(r)$ must satisfy,

$$r(1 + \kappa r^2) f''(r) + (1 + 2\beta + (3\kappa + 2\beta\kappa - q)r^2) f'(r) + r(\kappa(\beta + 1)^2 - q(\beta + 1) + 2\mathcal{E}) f(r) = 0. \quad (5.9)$$

In the Euclidean case, the curvature scalar $\kappa = 0$, and this equation reduces to

$$r f''(r) + (2\beta + 1 - 2r^2) f'(r) - 2(1 + \beta - \mathcal{E}) r f(r) = 0. \quad (5.10)$$

The solution which is regular at $r = 0$ is the Kummer-M function

$$f(r) = c_0 K_M(a; c; r^2), \quad (5.11)$$

where $c_0$ is a constant and the parameters $a$ and $c$ are defined by

$$a = \frac{1}{2}(1 + \beta - \mathcal{E}), \quad c = \beta^2 + 1. \quad (5.12)$$

The physically acceptable solutions are the polynomial solutions that appear when $a = -n_r$, $n_r = 0, 1, 2, \cdots$. This choice gives rise to a quantization condition on the energy spectrum.

To obtain a recursion relation for the coefficients $a_n(\kappa)$, write $f(r)$ in the form of a power series,

$$f(r) = \sum_{n=0}^{\infty} a_n(\kappa)r^n. \quad (5.13)$$

This function is substituted into (5.9), and the required recursion relation is determined to be, $a(\kappa) = 0$ and,

$$a_{n+1}(\kappa) = \frac{(n + \beta)q - (n + \beta)^2\kappa - 2\mathcal{E}}{(n + 1)(n + 2\beta + 1)} a_{n-1}(\kappa), \quad n = 1, 2, \cdots. \quad (5.14)$$
Since $a_0(\kappa)$ does not depend on $r$, (5.14) implies that (5.13) is made up of even powers of $r$. A form for $f(r)$ can also be obtained in terms of the hypergeometric function by putting the equation in hypergeometric form. Introduce then the variable $t = r^2$ so that (5.9) becomes

$$t(1 + \kappa t)f_{tt} + (\beta + 1 + (2\kappa + \beta\kappa - \frac{q}{2}) t)f_t + \frac{1}{4}((\beta + 1)^2\kappa - q(\beta + 1) + 2\mathcal{E})f = 0. \quad (5.15)$$

For the last step, introduce the new variable $s = -\kappa t$ so the equation becomes,

$$s(1 - s)f_{ss} + (\beta + 1 - \frac{1}{2\kappa}(2\kappa(2 + \beta) - q)s)f_s - \frac{1}{4\kappa}(\kappa(\beta + 1)^2 - (\beta + 1)q + 2\mathcal{E})f = 0. \quad (5.16)$$

This equation is exactly the Gauss hypergeometric equation,

$$s(1 - s)f_{ss} + (c - (1 + a_\kappa + b_\kappa)s)f_s - a_\kappa b_\kappa f = 0, \quad (5.17)$$

where the constants are defined as

$$c = \beta + 1, \quad a_\kappa + b_\kappa = \frac{2(\beta + 1)\kappa - q}{2\kappa}, \quad a_\kappa b_\kappa = \frac{1}{4\kappa}((\beta + 1)^2\kappa - (\beta + 1)q + 2\mathcal{E}). \quad (5.18)$$

The solution of (5.16) for $f(r)$ which is regular at $r = 0$ can be expressed in terms of the generalized hypergeometric function,

$$f(r) = \, _2F_1(a_\kappa; b_\kappa; c; r^2), \quad (5.19)$$

where $a_\kappa$ and $b_\kappa$ are given by

$$a_\kappa = \frac{2(1 - \beta)\kappa - \sqrt{\kappa^2 + 4} - q}{4\kappa}, \quad b_\kappa = \frac{2(1 - \beta)\kappa + \sqrt{\kappa^2 + 4} - q}{4\kappa}, \quad c = \beta + 1. \quad (5.20)$$

Physically acceptable solutions which are determined as eigenfunctions of the singular $\kappa$-dependent Sturm-Liouville problem appear when one of the two $\kappa$-dependent coefficients $a_\kappa$ or $b_\kappa$ coincides with zero or a negative integer,

$$a_\kappa = -N_r, \quad b_\kappa = -N_r, \quad N_r = 0, 1, 2, \cdots. \quad (5.21)$$

This restricts the energy to one of the following values

$$\mathcal{E}(\kappa) = \frac{1}{2}(2N_r + \beta + 1)((2N_r + \beta + 1)\kappa - \sqrt{\kappa^2 + 4}). \quad (5.22)$$
The hypergeometric series should reduce to a polynomial of degree $N_r$. Introducing the quantum number $n = 2N_r + \beta$, the energy levels are given by

$$E(\kappa) = (n + 1)\frac{1}{2}(n + 1)\kappa - \sqrt{\kappa^2 + 4}).$$

The wavefunctions for the Schrödinger equation on a space with constant curvature can be summarized as

$$\Psi_{N_r,\beta} = C_{\kappa} r^{\beta} (1 + \kappa r^2)^{\frac{1}{4}} e^{\frac{-\kappa}{2}} {}_2F_1(-N_r; b_r; \beta + 1; -\kappa r^2) e^{\pm i\beta \phi}. \quad (5.23)$$

In (5.23), $C_{\kappa}$ is a normalization constant. The energies $E$ are recovered by using (4.5) as

$$E_n(\kappa) = \frac{\hbar \beta}{\sqrt{m}} (n + 1)\frac{1}{2}(n + 1)\kappa - \sqrt{\kappa^2 + 4}). \quad (5.24)$$

The total energy (5.24) is a linear function of the curvature $\kappa$ and depends, as in the Euclidean case, on the combination $n = 2N_r + \beta$ of the quantum numbers $N_r$ and $\beta$.

6 Summary

The motion of the quantum free particle has been studied on spherical and hyperbolic spaces using a curvature dependent approach. The geometric approach was outlined at the start, and an important step was the determination of three Killing vector fields by means of Lie differentiation of the metric and then the associated Noether symmetries. A Hilbert space was defined by calculating a measure which is invariant under the same process of Lie differentiation with respect to these Killing vector fields. It is worth noting again that this measure is different from the measure that appeared in [10]. It is a bit reminiscent of the change in the path integral measure in formulations of non-abelian gauge theories which give rise to anomalies. Quantization of the three Noether momenta as self-adjoint operators with respect to the $\kappa$-dependent measure was carried out here, and the construction of the quantum Hamiltonian based on them in terms of the related operators $\hat{P}_1$, $\hat{P}_2$ and $\hat{J}$ obtained from the Noether symmetries. Finally, it was found that the Schrödinger equation can be separated. This has led to a determination of both the spectrum and the eigenfunctions of the quantum Hamiltonian for the choice of metric.
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