1. Introduction

In this paper, we prove a new kind of modularity lifting theorem for $p$-adic Galois representations. Previous generalizations of the work of Wiles [Wil95] and Taylor–Wiles [TW95] have (essentially) been restricted to circumstances where the automorphic forms in question arise from the middle degree cohomology of Shimura varieties. In particular, such approaches ultimately rely on a “numerical coincidence” (see the introduction to [CHT08]) which does not hold in general, and does not hold in particular for $GL(2)/F$ if $F$ is not totally real. A second requirement of these generalizations is that the Galois representations in question are regular at $\infty$, that is, have distinct Hodge–Tate weights for all $v|p$. Our approach, in contrast, does not a priori require either such assumption.

When considering questions of modularity in more general contexts, there are two issues that need to be overcome. The first is that there do not seem to be “enough” automorphic
forms to account for all the Galois representations. In [CM09, CV, CE], the suggestion is made that one should instead consider integral cohomology, and that the torsion occurring in these cohomology groups may account for the missing automorphic forms. In order to make this approach work, one needs to show that there is “enough” torsion. This is the problem that we solve in some cases. A second problem is the lack of Galois representations attached to these integral cohomology classes. In particular, our methods require Galois representations associated to torsion classes which do not necessarily lift to characteristic zero, where one might hope to apply the recent results of [HLTT]. We do not resolve the problems of constructing Galois representations in this paper, and instead, our results are contingent on a conjecture which predicts that there exists a map from a suitable deformation ring $R^{\text{min}}$ to a Hecke algebra $T$. If one could extend the theorems of [HLTT] to torsion classes then our modularity lifting theorems for imaginary quadratic fields would be unconditional. There are contexts, however, in which the existence of Galois representations is known; in these cases we can produce unconditional results. In principle, our method currently applies in two contexts:

**Betti** To Galois representations conjecturally arising from tempered $\pi$ of cohomological type associated to $G$, where $G$ is reductive with a maximal compact $K$ such that $l_0 = \text{rank}(G) - \text{rank}(K) = 1$.

**Coherent** To Galois representations conjecturally arising from tempered $\pi$ associated to $G$, where $(G, X)$ is a Shimura variety over a totally real field $F$, and such that $\pi_v$ is a holomorphic limit of discrete series at one infinite place and a discrete series at all the other infinite places.

In practice, however, what we really need is that (after localizing at a suitable maximal ideal $m$ of the Hecke algebra $T$) the cohomology is concentrated in only two consecutive degrees. (This is certainly true of the tempered representations which occur in Betti cohomology. According to BW00, the range of cohomological degrees to which they occur has length $l_0$.) We hope that, in the future, the methods of this paper can be adapted to the case of general $l_0$.

The following result is a sample of what can be shown by these methods in case Betti, assuming (Conjecture $A$) the existence of Galois representations. (Our main result in the case Coherent is presented below in Theorem 1.3.) Let $O$ denote the ring of integers in a finite extension of $\mathbb{Q}_p$, let $\varpi$ be a uniformizer of $O$, and let $O/\varpi = k$ be the residue field.

**Theorem 1.1.** Assume Conjecture $A$. Let $F/\mathbb{Q}$ be an imaginary quadratic field. Let $p \geq 3$ be unramified in $F$. Let

$$\rho : G_F \to \text{GL}_2(O)$$

be a continuous semi-stable Galois representation unramified outside finitely many primes. Let $\overline{\rho} : G_F \to \text{GL}_2(k)$ denote the mod-$\varpi$ reduction of $\rho$. Suppose that

1. If $v|p$, the representation $\rho|D_v$ is either Barsotti–Tate or ordinary.
2. The restriction of $\overline{\rho}$ to $F(\zeta_p)$ is absolutely irreducible.
3. $\overline{\rho}$ is modular of level $N(\overline{\rho})$.
4. $\rho$ is minimally ramified.

Then $\rho$ is modular, that is, there exists a regular algebraic cusp form $\pi$ for $\text{GL}(2)/F$ such that $L(\rho, s) = L(\pi, s)$. 

It is important to note that the condition (3) is only a statement about the existence of a mod-p cohomology class of level \( N(\overline{\rho}) \), not the existence of a characteristic zero lift. This condition is the natural generalization of Serre’s conjecture.

It turns out that — even assuming Conjecture [A] — this is not enough to prove that all minimal semi-stable elliptic curves over \( F \) are modular. Even though the Artin conjecture for finite two dimensional solvable representations of \( G_F \) is known, there are no obvious congruences between eigenforms arising from Artin \( L \)-functions and cohomology classes over \( F' \). (Over \( \mathbb{Q} \), this arose from the happy accident that classical weight one forms could be interpreted via coherent cohomology.) One class of mod-p Galois representations known to satisfy (3) are the restrictions of odd Galois representations \( \overline{\rho} : G_Q \to \text{GL}_2(k) \) to \( G_F \). One might imagine that the minimality condition is a result of the lack of Ihara’s lemma; however, Ihara’s lemma and level raising are known for \( \text{GL}(2)/F \) (see [CV]). The issue arises because there is no analogue of Wiles’ numerical criterion for Gorenstein rings of dimension zero.

We deduce Theorem 1.1 from the following more general result.

**Theorem 1.2.** Assume conjecture [A]. Let \( F/\mathbb{Q} \) be an imaginary quadratic field. Let \( p \geq 3 \) be unramified in \( F \). Let

\[
\overline{\rho} : G_F \to \text{GL}_2(k)
\]

be a continuous irreducible representation, and suppose that:

1. If \( \nu | p \), the representation \( \overline{\rho}|D_\nu \) is either finite flat or ordinary.
2. \( \overline{\rho} \) is modular of level \( N = N(\overline{\rho}) \).
3. If \( \overline{\rho} \) is ramified at \( x \), then \( N_{F/Q}(x) \equiv -1 \mod p \), then either \( \overline{\rho}|D_x \) is reducible or \( \overline{\rho}|I_x \) is absolutely irreducible.

Let \( R \) denote the minimal finite flat (respectively, ordinary) deformation ring of \( \overline{\rho} \). Let \( T_m \) be the algebra of Hecke operators acting on \( H_1(Y_0(N), \mathbb{Z}) \) localized at the maximal ideal corresponding to \( \overline{\rho} \). Then there is an isomorphism:

\[
R^{\min} \sim \to T_m,
\]

and there exists an integer \( \mu \geq 1 \) such that \( H_1(Y_0(N), \mathbb{Z})_m \) is free of rank \( \mu \) as a \( T_m \)-module.

If \( H_1(Y_0(N), \mathbb{Z})_m \otimes \mathbb{Q} \cong 0 \), then \( \mu = 1 \). If \( \dim(T_m) = 0 \), then \( T_m \) is a complete intersection.

Note that condition (4) — the non-existence of “vexing primes” \( x \) such that \( N_{F/Q}(x) \equiv -1 \mod p \) — is already a condition that arises in the original paper of Wiles [Wil95]. It could presumably be removed by making the appropriate modifications as in either [Dia96] or [Dia97a] and making the corresponding modifications to Conjecture [A].

Our results are obtained by applying a modification of Taylor–Wiles to the Betti cohomology of arithmetic manifolds. In such a context, it seems difficult to construct Galois representations whenever \( l_0 \neq 0 \). Following [Pil] and [Har12], however, we may also apply our methods to the coherent cohomology of Shimura varieties, where Galois representations are more readily available. In contexts where the underlying automorphic forms \( \pi \) are discrete series at infinity, one expects (and in many cases can prove, see [LS11]) that the integral coherent cohomology localized at at suitably generic maximal ideal \( m \) of \( T \) vanishes outside the middle degree. If \( \pi_\infty \) is a limit of discrete series, however, (so that we are in case **Coherent**) then the cohomology of the associated coherent sheaf can sometimes be shown to be non-zero in exactly two degrees, in which case our methods apply. The most well known example of such a situation is the case of classical modular forms of weight 1. Such modular forms contribute to the cohomology of \( H^0(X_1(N), \omega) \) and \( H^1(X_1(N), \omega) \) in characteristic zero, where
Theorem 1.3. Suppose that \( p \geq 3 \). Let \( \overline{\rho} : G\mathbb{Q} \to GL_2(k) \) be an odd continuous irreducible Galois representation of Serre level \( N \). Assume that \( \overline{\rho} \) is unramified at \( p \). Let \( R^{\text{min}} \) denote the universal minimal unramified-at-\( p \) deformation ring of \( \overline{\rho} \). Then there exists a quotient \( X_U \) of \( X_1(N) \) and a vector bundle \( \mathcal{L}_\sigma \) on \( X_U \) such that if \( T \) denotes the Hecke algebra of \( H^1(X_U, \omega \otimes \mathcal{L}_\sigma^*) \), there is an isomorphism
\[
R^{\text{min}} \xrightarrow{\sim} T_m
\]
where \( m \) is the maximal ideal of \( T \) corresponding to \( \overline{\rho}' \). Moreover, \( H^1(X_U, \omega \otimes \mathcal{L}_\sigma^*)_m \) is free as a \( T_m \)-module.

Note that even the fact that there exists a surjective map from \( R^{\text{min}} \) to \( T_m \) is non-trivial, and requires us to prove a local–global compatibility result for Galois representations associated to Katz modular forms of weight one over any \( \mathbb{Z}_p \)-algebra (see Theorem 3.8). We immediately deduce from Theorem 1.3 the following:

Corollary 1.4. Suppose that \( p \geq 3 \). Suppose also that \( \rho : G\mathbb{Q} \to GL_2(O) \) is a continuous representation satisfying the following conditions.

1. For all primes \( v \), either \( \rho(I_v) \xrightarrow{\sim} \overline{\rho}(I_v) \) or \( \dim(\rho^{I_v}) = \dim(\overline{\rho}^{I_v}) = 1 \).
2. \( \overline{\rho} \) is odd and irreducible.
3. \( \rho \) is unramified at \( p \).

Then \( \rho \) is modular of weight one.

It is instructive to compare this theorem and the corollary to the main theorem of Buzzard–Taylor [BT99] (see also [Buz93]). Note that the hypothesis in that paper that \( \overline{\rho} \) is modular is no longer necessary, following the proof of Serre’s conjecture [KW09]. In both cases, if \( \rho \) is a deformation of \( \overline{\rho} \) to a field of characteristic zero, we deduce that \( \rho \) is modular of weight one, and hence has finite image. The method of [BT99] applies in a non-minimal situation, but it requires the hypothesis that \( \overline{\rho}(\text{Frob}_p) \) has distinct eigenvalues. Moreover, it has the disadvantage that it only gives an identification of reduced points on the generic fibre (equivalently, that \( R^{\text{min}}[1/p]^\text{red} = T_m[1/p] \), although from this by class field theory one may deduce that \( R^{\text{min}}[1/p] = T_m[1/p] \), and says nothing about the torsion structure of \( H^1(X_1(N), \omega) \). Contrastingly, we may deduce the following result:

Corollary 1.5. Suppose that \( p \geq 3 \). Let \( \overline{\rho} : G\mathbb{Q} \to GL_2(k) \) be odd, continuous, irreducible, and unramified at \( p \). Let \( (A, m) \) denote a complete local Noetherian \( O \)-algebra with residue field \( k \) and \( \rho : G\mathbb{Q} \to GL_2(A) \) a minimal deformation of \( \overline{\rho} \). Then \( \rho \) has finite image.

This gives the first results towards Boston’s strengthening of the Fontaine–Mazur conjecture for representations unramified at \( p \) (See [Bos99], Conjecture 2).
It is natural to ask whether our results can be modified using Kisin’s method to yield modularity lifting results in non-minimal level. Although the formalism of this method can be adapted to our context, there is a genuine difficulty in proving that the support of \( \text{Spec}(\mathbb{T}_\infty[1/p]) \) hits each of the components of \( \text{Spec}(R_\infty[1/p]) \) whenever the latter has more than one component. In certain situations, we may apply Taylor’s trick [Tay08], but this cannot be made to work in general. However, suppose one replaces the “minimal” condition away from \( p \) with the following condition:

- If \( \rho \) is special at \( x \nmid p \), and \( \overline{\rho} \) is unramified at \( x \), then \( x \equiv 1 \mod p \).

In this context our methods should yield that the deformation ring \( R \) acts nearly faithfully on \( H^1(X_1(M), \omega)_m \) for an appropriate \( M \). This is sufficient for applications to the conjectures of Fontaine–Mazur and Boston.

In the process of proving our main result, we also completely solved the problem (for \( p \) odd) of determining the multiplicity of an irreducible modular representation \( \overline{\rho} \) in the Jacobian \( J_1(N^\ast)[m] \), where \( N^\ast \) is the minimal level such that \( \overline{\rho} \) arises in weight two. In particular, we prove that when \( \overline{\rho} \) is unramified at \( p \) and \( \overline{\rho}(\text{Frob}_p) \) is a scalar, then the multiplicity of \( \overline{\rho} \) is two (see Theorem 3.25). (In all other cases, the multiplicity was already known to be one — and in the exceptional cases we consider, the multiplicity was also known to be \( \geq 2 \).)

Acknowledgements

We would like to thank Matthew Emerton, Toby Gee, Andrew Snowden, Richard Taylor, Akshay Venkatesh, and Andrew Wiles for conversations. We also thank Toby Gee for comments on a draft of this paper. Some of the ideas of this paper were discovered when both the authors were members of the Institute for Advanced Study in 2010–2011 — the authors would like to thank the IAS for the pleasant and productive environment provided.

Notation

In this paper, we fix a prime \( p \geq 3 \) and let \( \mathcal{O} \) denote the ring of integers in a finite extension \( \mathbb{Q}_p \). We let \( \mathfrak{w} \) denote a uniformizer in \( \mathcal{O} \) and let \( k = \mathcal{O}/\mathfrak{w} \) be the residue field. We denote by \( \mathcal{C}_\mathcal{O} \) the category of complete Noetherian local \( \mathcal{O} \)-algebras with residue field \( k \). The homomorphisms in \( \mathcal{C}_\mathcal{O} \) are the continuous ring homomorphisms. If \( G \) is a group and \( \chi : G \to k^\times \) is a character, we denote by \( \langle \chi \rangle : G \to \mathcal{O}^\times \) the Teichmüller lift of \( \chi \).

If \( F \) is a field we let \( G_F \) denote the Galois group \( \text{Gal}(\overline{F}/F) \) for some choice of algebraic closure \( \overline{F}/F \). We let \( \epsilon : G_F \to \mathbb{Z}_p^\times \) denote the \( p \)-adic cyclotomic character. If \( F \) is a number field and \( v \) is a prime of \( F \), we let \( \mathcal{O}_v \) denote the ring of integers in the completion of \( F \) at \( v \) and we let \( \pi_v \) denote a uniformizer in \( \mathcal{O}_v \). We denote \( G_{F_v} \) by \( G_v \) and let \( I_v \subset G_v \) be the inertia group. We also let \( \text{Frob}_v \in G_v/I_v \) denote the arithmetic Frobenius. If \( R \) is a ring and \( \alpha \in R^\times \), we let \( \lambda(\alpha) : G_v \to R^\times \) denote the unramified character which sends \( \text{Frob}_v \) to \( \alpha \), when such a character exists. We let \( \mathbb{A}_F \) and \( \mathbb{A}_F^\infty \) denote the adeles and finite adeles of \( F \) respectively. If \( F = \mathbb{Q} \), we simply write \( \mathbb{A} \) and \( \mathbb{A}^\infty \).

If \( R \) is a local ring, we will sometimes denote the maximal ideal of \( R \) by \( \mathfrak{m}_R \).

2. Some Commutative Algebra

This section contains one of the main new technical innovations of this paper. The issue, as mentioned in the introduction, is to show that there are enough modular Galois representations. This involves showing that certain modules \( H_N \) (consisting of modular forms) for
the group rings $S_N := \mathcal{O}[\mathbb{Z}/p^N \mathbb{Z}]^q$ compile, in a Taylor–Wiles patching process, to form a module of codimension one over the completed group ring $S_\infty := \mathcal{O}[[\mathbb{Z}_p]^q]]$. The problem then becomes to find a suitable notion of “codimension one” for modules over a local ring that

1. is well behaved for non-reduced quotients of power series rings over $\mathcal{O}$ (like $S_N$),
2. can be established for the spaces $H_N$ in question,
3. compiles well in a Taylor–Wiles system.

It turns out that the correct notion is that of being “balanced”, a notion defined below.

2.1. Balanced Modules. Let $S$ be a Noetherian local ring with residue field $k$ and let $M$ be a finitely generated $S$-module.

**Definition 2.1.** We define the defect $d_S(M)$ of $M$ to be

$$d_S(M) = \dim_k \mathrm{Tor}_0^S(M, k) - \dim_k \mathrm{Tor}_1^S(M, k) = \dim_k M/m_SM - \dim_k \mathrm{Tor}_1^S(M, k).$$

Let

$$\cdots \to P_i \to \cdots \to P_1 \to P_0 \to M \to 0$$

be a (possibly infinite) resolution of $M$ by finite free $S$-modules. Assume that the image of $P_i$ in $P_{i-1}$ is contained in $m_S P_{i-1}$ for each $i \geq 1$. (Such resolutions always exist and are often called ‘minimal.’) Let $r_i$ denote the rank of $P_i$. Tensoring the resolution over $S$ with $k$ we see that $P_i/m_S P_i \cong \mathrm{Tor}^S_1(M, k)$ and hence that $r_i = \dim_k \mathrm{Tor}^S_1(M, k)$.

**Definition 2.2.** We say that $M$ is balanced if $d_S(M) \geq 0$.

If $M$ is balanced, then we see that it admits a presentation

$$S^d \to S^d \to M \to 0$$

with $d = \dim_k M/m_SM$.

2.2. Patching. We establish in this section an abstract Taylor–Wiles style patching result which may be viewed as an analogue of Theorem 2.1 of [Dia97b]. This result will be one of the key ingredients in the proofs of our main theorems.

**Proposition 2.3.** Suppose that

1. $R$ is an object of $\mathcal{C}_\mathcal{O}$ and $H$ is a finite $R$-module which is also finite over $\mathcal{O}$;
2. $q \geq 1$ is an integer, and for each integer $N \geq 1$, $S_N := \mathcal{O}[\Delta_N]$ with $\Delta_N := (\mathbb{Z}/p^N \mathbb{Z})^q$;
3. $R_\infty := \mathcal{O}[[x_1, \ldots, x_{q-1}]]$;
4. for each $N \geq 1$, $\phi_N : R_\infty \to R$ is a surjection in $\mathcal{C}_\mathcal{O}$ and $H_N$ is an $R_\infty \otimes_\mathcal{O} S_N$-module.
5. For each $N \geq 1$ the following conditions are satisfied
   (a) the image of $S_N$ in $\mathrm{End}_\mathcal{O}(H_N)$ is contained in the image of $R_\infty$;
   (b) there is an isomorphism $\psi_N : (H_N)_{\Delta_N} \cong H$ of $R_\infty$-modules (where $R_\infty$ acts on $H$ via $\phi_N$);
   (c) $H_N$ is finite and balanced over $S_N$ (see Definition 2.2).

Then $H$ is a free $R$-module.
Proof. Let $S_\infty = \mathcal{O}[[\mathbb{Z}_p]]$ and let $\mathfrak{a}$ denote the augmentation ideal of $S_\infty$ (that is, the kernel of the homomorphism $S_\infty \rightarrow \mathcal{O}$ which sends each element of $(\mathbb{Z}_p)^d$ to 1). For each $N \geq 1$, let $\mathfrak{a}_N$ denote the kernel of the natural surjection $S_\infty \rightarrow S_N$ and let $\mathfrak{b}_N$ denote the open ideal of $S_\infty$ generated by $\varpi^N$ and $\mathfrak{a}_N$. Let $d = \dim_k(\mathcal{H}/\varpi \mathcal{H})$. Choose a sequence of open ideals $(\mathfrak{d}_N)_{N \geq 1}$ of $R$ such that

- $\mathfrak{d}_N \supset \mathfrak{d}_{N+1}$ for all $N \geq 1$;
- $\mathfrak{d}_N\cap\mathfrak{d}_N = (0)$;  
- $\varpi^N R \subset \mathfrak{d}_N \subset \varpi^N R + \text{Ann}_R(H)$ for all $N$.

Define a patching datum of level $N$ to be a 4-tuple $(\phi, X, \psi, P)$ where

- $\phi : R_\infty \rightarrow R/\mathfrak{d}_N$ is a surjection in $\mathcal{C}_\mathcal{O}$;
- $X$ is an $R_\infty \otimes \mathcal{O} S_\infty$-module such that the action of $S_\infty$ on $X$ factors through $S_\infty/\mathfrak{b}_N$ and $X$ is finite over $S_\infty$;
- $\psi : X/\mathfrak{a}X \xrightarrow{\sim} H/\varpi^N H$ is an isomorphism of $R_\infty$ modules (where $R_\infty$ acts on $H/\varpi^N H$ via $\phi$);
- $P$ is a presentation $$(S_\infty/\mathfrak{b}_N)^d \rightarrow (S_\infty/\mathfrak{b}_N)^d \rightarrow X \rightarrow 0.$$  

We say that two such 4-tuples $(\phi, X, \psi, P)$ and $(\phi', X', \psi', P')$ are isomorphic if

- $\phi = \phi'$;
- there is an isomorphism $X \xrightarrow{\sim} X'$ of $R_\infty \otimes \mathcal{O} S_\infty$ modules compatible with $\psi$ and $\psi'$, and with the presentations $P$ and $P'$.

We note that, up to isomorphism, there are finitely many patching data of level $N$. (This follows from the fact that $R_\infty$ and $S_\infty$ are topologically finitely generated.) If $D$ is a patching datum of level $N$ and $1 \leq N' \leq N$, then $D$ gives rise to patching datum of level $N'$ in an obvious fashion. We denote this datum by $D \mod \mathfrak{d}_N$.

For each pair of integers $(M, N)$ with $M \geq N \geq 1$, we define a patching datum $D_{M,N}$ of level $N$ as follows: the statement of the proposition gives a homomorphism $\phi_M : R_\infty \rightarrow R$ and an $R_\infty \otimes \mathcal{O} S_M$-module $H_M$. We take

- $\phi$ to be the composition $R_\infty \rightarrow R \rightarrow R/\mathfrak{d}_N$;
- $X$ to be $H_M/\mathfrak{b}_N$;
- $\psi : X/\mathfrak{a}X \xrightarrow{\sim} H/\mathfrak{b}_N$ to be the reduction modulo $\varpi^N$ of the given isomorphism $\psi_M : H_M/\mathfrak{a}H_M \xrightarrow{\sim} H$;
- $P$ to be any choice of presentation $$(S_\infty/\mathfrak{b}_N)^d \rightarrow (S_\infty/\mathfrak{b}_N)^d \rightarrow X \rightarrow 0.$$  

(The facts that $H_M/\mathfrak{a}H_M \xrightarrow{\sim} H$ and $d_{S_M}(H_M) \geq 0$ imply that such a presentation exists.)

Since there are finitely many patching data of each level $N \geq 1$, up to isomorphism, we can find a sequence of pairs $(M_i, N_i)_{i \geq 1}$ such that

- $M_i \geq N_i$, $M_{i+1} \geq M_i$, and $N_{i+1} \geq N_i$ for all $i$;
- $D_{M_{i+1}, N_{i+1}} \mod \mathfrak{d}_{N_i}$ is isomorphic to $D_{M_i, N_i}$ for all $i \geq 1$.

For each $i \geq 1$, we write $D_{M_i, N_i} = (\phi_i, X_i, \psi_i, P_i)$ and we fix an isomorphism between the modules $X_{i+1}/\mathfrak{b}_{N_i}X_{i+1}$ and $X_i$ giving rise to an isomorphism between $D_{M_{i+1}, N_{i+1}} \mod \mathfrak{b}_{N_i}$ and $D_{M_i, N_i}$. We define

- $\phi_{i\infty} : R_\infty \rightarrow R$ to be the inverse limit of the $\phi_i$;
3.1. Deformations of Galois Representations. Let

\[ \overline{\rho} : G_Q \to \text{GL}_2(k) \]

be a continuous, odd, absolutely irreducible Galois representation. Let us suppose that \( \overline{\rho}|_{G_p} \) is unramified; this implies that \( \overline{\rho} \) remains absolutely irreducible when restricted to \( G_Q^{(p)} \). Let \( S(\overline{\rho}) \) denote the set of primes of \( Q \) at which \( \overline{\rho} \) is ramified and let \( T(\overline{\rho}) \subset S(\overline{\rho}) \) be the subset consisting of those primes \( x \) such that \( x \equiv -1 \mod p \), \( \overline{\rho}|_{G_x} \) is reducible and \( \overline{\rho}|_{I_x} \) is reducible. Following Diamond, we call the primes in \( T(\overline{\rho}) \) \textit{vexing}. We further assume that if \( x \in S(\overline{\rho}) \) and \( \overline{\rho}|_{G_x} \) is reducible, then \( \overline{\rho}|_{I_x} \neq (0) \). Note that this last condition is always satisfied by a twist of \( \overline{\rho} \) by a character unramified outside of \( S(\overline{\rho}) \).

Let \( Q \) denote a finite set of primes of \( Q \) disjoint from \( S(\overline{\rho}) \cup \{p\} \). (By abuse of notation, we sometimes use \( Q \) to denote the product of primes in \( Q \).) For objects \( R \) in \( C_{\mathcal{O}} \), a deformation of \( \overline{\rho} \) to \( R \) is a \( \ker(\text{GL}_2(R) \to \text{GL}_2(k)) \)-conjugacy class of continuous lifts \( \rho : G_Q \to \text{GL}_2(R) \) of \( \overline{\rho} \). We will often refer to the deformation containing a lift \( \rho \) simply by \( \rho \).

**Definition 3.1.** We say that a deformation \( \rho : G_Q \to \text{GL}_2(R) \) of \( \overline{\rho} \) is minimal outside \( Q \) if it satisfies the following properties:

1. The determinant \( \det(\rho) \) is equal to the Teichmüller lift of \( \det(\overline{\rho}) \).
2. If \( x \notin Q \cup S(\overline{\rho}) \) is a prime of \( Q \), then \( \rho|_{G_x} \) is unramified.
3. If \( x \in T(\overline{\rho}) \) then \( \rho(I_x) \to \overline{\rho}(I_x) \).
4. If \( x \in S(\overline{\rho}) \) and \( \overline{\rho}|_{G_x} \) is reducible, then \( \rho|_{I_x} \) is a rank one direct summand of \( \rho \) as an \( R \)-module.
If $Q$ is empty, we will refer to such deformations simply as being minimal.

Note that condition 2 implies that $\rho$ is unramified at $p$. The functor that associates to each object $R$ of $\mathcal{C}_O$ the set of deformations of $\mathfrak{m}$ to $R$ which are minimal outside $Q$ is represented by a complete Noetherian local $O$-algebra $R_Q$. This follows from the proof of Theorem 2.41 of [DDT97]. If $Q = \emptyset$, we will sometimes denote $R_Q$ by $R^\min$. Let $H^1_Q(Q, \text{ad}^0 \mathfrak{m})$ denote the Selmer group defined as the kernel of the map

$$H^1(Q, \text{ad}^0 \mathfrak{m}) \rightarrow \bigoplus_x H^1(Q_x, \text{ad}^0 \mathfrak{m})/L_{Q,x}$$

where $x$ runs over all primes of $Q$ and

- $L_{Q,x} = H^1(G_x/I_x, (\text{ad}^0 \mathfrak{m})^{I_x})$ if $x \notin Q$;
- $L_{Q,x} = H^1(Q_x, \text{ad}^0 \mathfrak{m})$ if $x \in Q$.

Let $H^1_Q(Q, \text{ad}^0 \mathfrak{m}(1))$ denote the corresponding dual Selmer group.

**Proposition 3.2.** The reduced tangent space $\text{Hom}(R_Q/m_Q, k[\epsilon]/\epsilon^2)$ of $R_Q$ has dimension

$$\dim_k H^1_Q(Q, \text{ad}^0 \mathfrak{m}(1)) - 1 + \sum_{x \in Q} \dim_k H^0(Q_x, \text{ad}^0 \mathfrak{m}(1)).$$

**Proof.** The argument is very similar to that of Corollary 2.43 of [DDT97]. The reduced tangent space has dimension $\dim_k H^1_Q(Q, \text{ad}^0 \mathfrak{m})$. By Theorem 2.18 of op. cit. this is equal to

$$\dim_k H^1_Q(Q, \text{ad}^0 \mathfrak{m}(1)) + \dim_k H^0(Q, \text{ad}^0 \mathfrak{m}) - \dim_k H^0(Q, \text{ad}^0 \mathfrak{m}(1))$$

$$+ \sum_x (\dim_k L_{Q,x} - \dim_k H^0(Q_x, \text{ad}^0 \mathfrak{m})) - 1,$$

where $x$ runs over all finite places of $Q$. The final term is the contribution at the infinite place. The second and third terms vanish by the absolute irreducibility of $\mathfrak{m}$ and the fact that $\mathfrak{m}G_p$ is unramified. Finally, as in the proof of Corollary 2.43 of loc. cit. we see that the contribution at the prime $x$ vanishes if $x \notin Q$, and equals $\dim_k H^0(Q_x, \text{ad}^0 \mathfrak{m}(1))$ if $x \in Q$. □

Suppose that $x \equiv 1 \mod p$ and $\mathfrak{m}(\text{Frob}_x)$ has distinct eigenvalues for each $x \in Q$. Then $H^0(Q_x, \text{ad}^0 \mathfrak{m}(1))$ is one dimensional for $x \in Q$ and the preceding proposition shows that the reduced tangent space of $R_Q$ has dimension

$$\dim_k H^1_Q(Q, \text{ad}^0 \mathfrak{m}(1)) - 1 + \#Q.$$

Using this fact and the argument of Theorem 2.49 of [DDT97], we deduce the following result. (We remind the reader that $\mathfrak{m}(G_{Q(p)})$ is absolutely irreducible, by assumption.)

**Proposition 3.3.** Let $q = \dim_k H^0_Q(Q, \text{ad}^0 \mathfrak{m}(1))$. Then $q \geq 1$ and for any integer $N \geq 1$ we can find a set $Q_N$ of primes of $Q$ such that

1. $\#Q_N = q$.
2. $x \equiv 1 \mod p_N$ for each $x \in Q_N$.
3. For each $x \in Q_N$, $\mathfrak{m}$ is unramified at $x$ and $\mathfrak{m}(\text{Frob}_x)$ has distinct eigenvalues.
4. $H^1_Q(Q, \text{ad}^0 \mathfrak{m}(1)) = 0$.

In particular, the reduced tangent space of $R_{Q_N}$ has dimension $q - 1$ and $R_{Q_N}$ is a quotient of a power series ring over $O$ in $q - 1$ variables.
We note that the calculations on the Galois side are virtually identical to those that occur in Wiles’ original paper, with the caveat that the tangent space is of dimension “one less” in our case. On the automorphic side, this –1 will be a reflection of the fact that the Hecke algebras will not (in general) be flat over $\mathcal{O}$ and the modular forms we are interested in will contribute to one extra degree of cohomology.

3.2. Cohomology of Modular Curves.

3.2.1. Modular Curves. We begin by recalling some classical facts regarding modular curves. Fix an integer $N \geq 5$ such that $(N, p) = 1$, and fix a squarefree integer $Q$ with $(Q, Np) = 1$. Let $X_1(N), X_1(N; Q),$ and $X_1(NQ)$ denote the modular curves of level $\Gamma_1(N), \Gamma_1(N) \cap \Gamma_0(Q)$, and $\Gamma_1(N) \cap \Gamma_1(Q)$ respectively as smooth proper schemes over Spec$(\mathcal{O})$. To be precise, we take $X_1(N)$ and $X_1(NQ)$ to be the base change to Spec$(\mathcal{O})$ of the curves denoted by the same symbols in [Gro90] Proposition 2.1]. The proof of this proposition shows that $X_1(NQ)$ represents the functor that assigns to each $\mathcal{O}$-scheme $S$ the set of isomorphism classes of triples $(E, C, \beta)$ where $E/S$ is a generalized elliptic curve, $C \subset E[NQ]$ is a subgroup scheme locally isomorphic to $\mathbb{Z}/NQ\mathbb{Z}$ which meets every irreducible component of every geometric fibre of $E$ and $\beta$ is an isomorphism $\beta : \mathbb{Z}/NQ\mathbb{Z} \cong E[NQ]/C$. Given such a triple, we can naturally decompose $C = C_N \times C_Q$ and $\beta = \beta_N \times \beta_Q$ into their $N$ and $Q$-parts. The group $(\mathbb{Z}/NQ\mathbb{Z})^\times$ acts on $X_1(NQ)$ in the following fashion: $a \in (\mathbb{Z}/NQ\mathbb{Z})^\times$ sends a triple $(E, C, \beta)$ to $(E, C, \beta \circ a)$. We let $X_1(N; Q)$ be the smooth proper curve over Spec$(\mathcal{O})$ classifying triples $(E, C, \beta_N)$ where $E$ and $C$ are as above and $\beta_N$ is an isomorphism $\beta_N : \mathbb{Z}/NZ \cong C_N$. Then $X_1(N, Q)$ is the quotient of $X_1(NQ)$ by the action of $(\mathbb{Z}/N\mathbb{Z})^\times \subset (\mathbb{Z}/NQ\mathbb{Z})^\times$ and the map $X_1(NQ) \rightarrow X_1(N; Q)$ is étale.

For any modular curve $X$ over $\mathcal{O}$, let $Y \subset X$ denote the corresponding open modular curve parametrizing genuine elliptic curves. Let $\pi : \mathcal{E} \rightarrow X$ denote the universal generalized elliptic curve, and let $\omega := \pi_*\Omega^1_{\mathcal{E}/X}$. Then the Kodaira–Spencer map (see [Kat77], A1.3.17) induces an isomorphism $\omega^{\otimes 2} \cong \Omega^1_{\mathcal{E}/\mathcal{O}}$ over $Y$, which extends to an isomorphism $\omega^{\otimes 2} \cong \Omega^1_{X/\mathcal{O}(\infty)}$, where $\infty$ is the reduced divisor supported on the cusps. If $R$ is an $\mathcal{O}$-algebra, we let $X_R = X \times_{\text{Spec} \mathcal{O}} \text{Spec} R$. If $M$ is an $\mathcal{O}$-module and $\mathcal{L}$ is a coherent sheaf on $X$, we let $\mathcal{L}_M$ denote $\mathcal{L} \otimes_{\mathcal{O}} M$.

We now fix a subgroup $H$ of $(\mathbb{Z}/N\mathbb{Z})^\times$. We let $X$ (resp. $X_1(Q)$, resp. $X_0(Q)$) denote the quotient of $X_1(N)$ (resp. $X_1(NQ)$, resp. $X_1(N; Q)$) by the action of $H$. Note that each of these curves carries an action of $(\mathbb{Z}/N\mathbb{Z})^\times / H$. We assume that $H$ is chosen so that $X$ is the moduli space (rather than the course moduli space) of generalized elliptic curves with $\Gamma_H(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) : d \equiv 0 \mod N \in H \right\}$-level structure.

3.2.2. Modular forms with coefficients. The map $j : X_{\mathcal{O}/\omega^m} \rightarrow X$ is a closed immersion. If $\mathcal{L}$ is any $\mathcal{O}$-flat sheaf of $\mathcal{O}_X$-modules on $X$, this allows us to identify $H^0(X_{\mathcal{O}/\omega^m}, j^* \mathcal{L})$ with $H^0(X, \mathcal{L}_{\mathcal{O}/\omega^m})$. For such a sheaf $\mathcal{L}$, we may identify $\mathcal{L}_K/\mathcal{O}$ with the direct limit $\lim_{\rightarrow} \mathcal{L}_{\mathcal{O}/\omega^m}$.

3.2.3. Properties of cohomology groups. Define the Hecke algebra

$$T^m \subset \text{End}_\mathcal{O} H^0(X_1(Q), \omega_K/\mathcal{O})$$

to be the subring of endomorphisms generated over $\mathcal{O}$ by the Hecke operators $T_n$ with $(n, pNQ) = 1$ together with all the diamond operators $\langle a \rangle$ for $(a, NQ) = 1$. (Here “an” denotes anemic.) Let $T$ denote the $\mathcal{O}$-algebra generated by these same operators together
with $U_x$ for $x$ dividing $Q$. If $Q = 1$, we let $T_0 = T_0^\mathrm{an}$ denote $T$. The ring $T^\mathrm{an}$ is a finite $\mathcal{O}$-algebra and hence decomposes as a direct product over its maximal ideals $T^\mathrm{an} = \prod_m T_m^\mathrm{an}$. We have natural homomorphisms

$$T^\mathrm{an} \to T_0^\mathrm{an} = T_0, \quad T^\mathrm{an} \hookrightarrow T$$

where the first is induced by the map $H^0(X, \omega_{K/\mathcal{O}}) \hookrightarrow H^0(X_1(Q), \omega_{K/\mathcal{O}})$ and the second is the obvious inclusion. It will be convenient to introduce the polynomial algebra $T^\mathrm{univ}$ over $\mathcal{O}$ generated by indeterminates $T_n$, $U_x$ and $\langle a \rangle$ for $(n, pNQ) = 1$, $x|Q$ and $(a, NQ) = 1$.

Let $m_\emptyset$ be a non-Eisenstein maximal ideal $T_0$. The ideal $m_\emptyset$ pulls back to ideals of $T^\mathrm{an}$ and $T$ which we also denote by $m_\emptyset$ in a slight abuse of notation. The ideal $m_\emptyset \subset T$ may no longer be maximal – it may give rise to multiple maximal ideals $m$ of $T$ corresponding to different $U_x$-eigenvalues. The following lemma is essentially well known in the construction of Taylor–Wiles primes, we give a detailed proof just to show that the usual arguments apply equally well in weight one.

**Lemma 3.4.** Suppose that for each $x \in Q$ we have that $x \equiv 1 \mod p$ and that the polynomial $X^2 - T_x X + \langle x \rangle \in T_0[X]$ has distinct eigenvalues modulo $m_\emptyset$. Let $m$ denote the maximal ideal of $T$ containing $m_\emptyset$ and $U_x - \alpha_x$ for some choice of root $\alpha_x$ of $X^2 - T_x X + \langle x \rangle \mod m_\emptyset$ for each $x|Q$. Then there is a $T^\mathrm{an}_m$-isomorphism

$$H^0(X, \omega_{K/\mathcal{O}})^{\mathfrak{m}_\emptyset} \cong H^0(X_0(Q), \omega_{K/\mathcal{O}})^m.$$

**Proof.** By induction, we reduce immediately to the case when $Q = x$ is prime. Let $\pi : X_0(x) \to X$ denote the natural projection and let $w : X_0(x) \to X_0(x)$ be the Frobenius involution. Let $E$ denote the universal generalized elliptic curve over $X_0(x)$ and $C \subset E$ the universal subgroup of order $x$. The map $E \to E/C$ and its dual induce maps $w^* : (w^* \circ \pi^*)\omega \to \pi^*\omega$ and $\phi : \pi^*\omega \to (w^*\pi^*)\omega$ over $X_0(x)$. Then, $\psi := (\pi^*, \phi^* \circ (w^* \circ \pi^*))$ and $\psi^\vee := \frac{1}{x}(\tr(\pi) \oplus \tr(\pi) \circ \phi_*)$ give a sequence of morphisms

$$H^0(X, \omega_{K/\mathcal{O}})^2 \to H^0(X_0(x), \omega_{K/\mathcal{O}}) \to H^0(X, \omega_{K/\mathcal{O}})^2,$$

such that the composite map $\psi^\vee \circ \psi$ is given by

$$\left( \begin{array}{cc} x^{-1}(x + 1) & T_x \\ \langle x \rangle^{-1}T_x & x + 1 \end{array} \right).$$

If $\alpha_x$ and $\beta_x$ are the roots of $X^2 - T_x X + \langle x \rangle \mod m_\emptyset$, then $T_x \equiv \alpha_x + \beta_x \mod m_\emptyset$ and $\langle x \rangle \equiv \alpha_x \beta_x \mod m_\emptyset$. It follows that

$$\det(\psi^\vee \circ \psi) = x^{-1}(x + 1)^2 - \langle x \rangle^{-1}T_x^2 \equiv 4 - \langle x \rangle^{-1}T_x^2 \equiv (\alpha_x \beta_x)^{-1}(4\alpha_x \beta_x - (\alpha_x + \beta_x)^2) \equiv - (\alpha_x \beta_x)^{-1}(\alpha_x - \beta_x)^2 \mod m_\emptyset.$$

By assumption, $\alpha_x \neq \beta_x$, and thus, after localizing at $m_\emptyset$, the composite map $\psi^\vee \circ \psi$ is an isomorphism.

We deduce that $H^0(X, \omega_{K/\mathcal{O}})^2$ is a direct factor of $H^0(X_0(x), \omega_{K/\mathcal{O}})^m_\emptyset$ as a $T^\mathrm{an}_m$-module. Consider the action of $U_x$ on the image of $H^0(X, \omega_{K/\mathcal{O}})^2$. The degeneracy map $\psi$ is given explicitly by $q$-expansions by the pair of maps $\{1, V_x\}$ where 1 is the identity map and $V_x(\sum a_n q^n) = \sum a_n q^{nx}$. Moreover, there is an identity of Hecke operators $T_x = U_x + \langle x \rangle V_x$. (More precisely, $\pi^* \circ T_x = U_x \circ \pi^* + \langle x \rangle \phi^* \circ (w^* \circ \pi^*)$.) It follows that the action of $U_x$ on
$H^0(X_0(x), \omega_{K/O})^2_{m_0}$ is given by the matrix

$$A = \begin{pmatrix} T_x & 1 \\ -\langle x \rangle & 0 \end{pmatrix}.$$  

There is an identity $(A - \alpha x)(A - \beta x) \equiv 0 \mod m_0$ in $M_2(\mathbb{T}_{\emptyset, m_0})$. Since $\alpha x \neq \beta x$, by Hensel’s Lemma, there exist $\tilde{\alpha}_x$ and $\tilde{\beta}_x$ in $\mathbb{T}_{\emptyset, m_0}^\times$ such that $(U_x - \tilde{\alpha}_x)(U_x - \tilde{\beta}_x) = 0$ on $(\text{Im } \psi)_{m_0}$. It follows that $U_x - \tilde{\beta}_x$ is a projector (up to a unit) from $(\text{Im } \psi)_{m_0}$ to $(\text{Im } \psi)_m$. We claim that there is an isomorphism of $\mathbb{T}_{m_0}^{\text{univ}}$-modules

$$H^0(X, \omega_{K/O})_{m_0} \cong (\text{Im } \psi)_m = (H^0(X, \omega_{K/O})^2)_m.$$  

It suffices to show that there is a $\mathbb{T}_{m_0}^{\text{univ}}$-equivariant injection from $H^0(X, \omega_{K/O})_{m_0}$ to the module $(\text{Im } \psi)_{m_0}$ such that the image has trivial intersection with the kernel of $U_x - \tilde{\beta}_x$: if there is such an injection then, by symmetry, there is also an injection from $H^0(X, \omega_{K/O})_{m_0}$ to $(\text{Im } \psi)_{m_0}$ whose image intersects the kernel of $U_x - \tilde{\alpha}_x$ trivially; by length considerations both injections are forced to be isomorphisms. We claim that the natural inclusion $\pi^*$ composed with $U_x - \tilde{\beta}_x$ is such a map: from the computation of the matrix above, it follows that

$$(U_x - \tilde{\beta}_x) \begin{pmatrix} f \\ 0 \end{pmatrix} = \begin{pmatrix} T_x f - \tilde{\beta}_x f \\ -x^{-1} f \end{pmatrix},$$

which is non-zero whenever $f$ is by examining the second coordinate.

We deduce that there is a decomposition of $\mathbb{T}_{m_0}^{\text{univ}}$-modules

$$H^0(X_0(x), \omega_{K/O})_m = H^0(X, \omega_{K/O})_{m_0} \oplus V.$$  

Let $v \in V[m]$. The vector $v$ generates a $\text{GL}_2(\mathbb{Q}_x)$-representation which has a $\Gamma_0(x)$-invariant vector but not a $\text{GL}_2(\mathbb{Z}_x)$-invariant vector. It follows that $v$ generates a twist of the Steinberg representation determined (up to an unramified quadratic twist) by the central character, from which we deduce that the operator $U_x$ acts by $U^2_x = \langle x \rangle \mod m_0$. Hence $\alpha^2_x = \langle x \rangle = \alpha_x \beta_x \mod m_0$, and $\alpha_x = \beta_x \mod m_0$, contradicting our assumption.

Recall that there is an étale covering map $X_1(Q) \to X_0(Q)$ with Galois group $(\mathbb{Z}/Q\mathbb{Z})^\times$. Let $\Delta$ be a quotient of $\Delta_Q := (\mathbb{Z}/Q\mathbb{Z})^\times$ and let $X_{\Delta}(Q) \to X_0(Q)$ be the corresponding cover. If $A$ is an $\mathcal{O}$-module, we have an action of the universal polynomial algebra $\mathbb{T}^{\text{univ}}$ on the cohomology groups $H^i(X_{\Delta}(Q), \mathcal{L}_A)$ for $\mathcal{L} = \omega^{\otimes n}$ or $\omega^{\otimes n}(-\infty)$. The ideal $m_0$ gives rise to a maximal ideal $m$ of $\mathbb{T}^{\text{univ}}$ after a choice of eigenvalue mod $m_0$ for $U_x$ for all $x$ dividing $Q$. Extending $\mathcal{O}$ if necessary, we may assume that $\mathbb{T}^{\text{univ}}/m \cong k$.

Let $M \mapsto M^\vee := \text{Hom}_\mathcal{O}(M, K/\mathcal{O})$ denote the Pontryagin duality functor.

**Lemma 3.5.** Let $\Delta$ be a quotient of $\Delta_Q$. Then:

1. $H^1(X_{\Delta}(Q), \mathcal{L}_{K/\mathcal{O}})^\vee$ is $p$-torsion free for $\mathcal{L}$ a vector bundle on $X_{\Delta}(Q)$.
2. For $i = 0, 1$, we have an isomorphism

$$H^i(X_{\Delta}(Q), \omega(-\infty)_{K/\mathcal{O}})_m \cong H^i(X_{\Delta}(Q), \omega_{K/\mathcal{O}})_m.$$  

**Proof.** The first claim is equivalent to the divisibility of $H^1(X_{\Delta}(Q), \mathcal{L}_{K/\mathcal{O}})$. Since $X_{\Delta}(Q)$ is flat over $\mathcal{O}$, there is an exact sequence

$$0 \to \mathcal{L}_k \to \mathcal{L}_{K/\mathcal{O}} \xrightarrow{\alpha} \mathcal{L}_{K/\mathcal{O}} \to 0.$$
Taking cohomology and using the fact that $X_\Delta(Q)$ is a curve and hence $H^2(X_\Delta(Q), \mathcal{L}_k)$ vanishes, we deduce that $H^1(X_\Delta(Q), \mathcal{L}_{K/O})/\bar{\omega} = 0$, from which divisibility follows.

For the second claim, note that there is an exact sequence:

$$0 \to \omega(-\infty) \to \omega \to I \to 0,$$

where $I$ is supported at the cusps. It follows that the cohomology of $I$ is concentrated in degree 0. Yet the action of Hecke on $H^0(X, I)$ is Eisenstein, thus the lemma. □

If $\mathcal{L}$ is a vector bundle on $X_\Delta(Q)$, we define

$$H_i(X_\Delta(Q), \mathcal{L}) := H^i(X_\Delta(Q), (\Omega^1 \otimes \mathcal{L}^*)_{K/O})^\vee$$

for $i = 0, 1$, where $\mathcal{L}^*$ is the dual bundle and $\Omega^1 = \Omega^1_{X_\Delta(Q)/O} = \omega^{\otimes 2}(-\infty)$. If $\Delta \to \Delta'$ are two quotients of $(\mathbb{Z}/\mathbb{Q})^\times$ giving rise to a Galois cover $\pi : X_\Delta(Q) \to X_{\Delta'}(Q)$ and $\mathcal{L}$ is vector bundle on $X_{\Delta'}(Q)$, then there is a natural map $\pi_* : H_i(X_\Delta(Q), \pi^* \mathcal{L}) \to H_i(X_{\Delta'}(Q), \mathcal{L})$ coming from the dual of the pullback $\pi^*$ on cohomology. Verdier duality ([Har66, Corollary 11.2(f)]) gives an isomorphism

$$D : H_i(X_\Delta(Q), \mathcal{L}) \sim \to H^{1-i}(X_\Delta(Q), \mathcal{L})$$

under which $\pi_*$ corresponds to the trace map $\text{tr}(\pi) : H^{1-i}(X_\Delta(Q), \pi^* \mathcal{L}) \to H^{1-i}(X_{\Delta'}(Q), \mathcal{L})$.

When $\mathcal{L} = \omega^{\otimes n}$ or $\omega^{\otimes n}(-\infty)$, we note that the isomorphism $D$ is not Hecke equivariant — we have

$$D(T_x c) = T_x^* D(c) = \langle a \rangle^{-1} T_x D(c)$$
$$D((a) c) = (a)^* = (a)^{-1} D(c)$$
$$D(U_x c) = U_x^* D(c).$$

Here $T_x^*$, $\langle a \rangle$ and $U_x^*$ are defined via the transpose of the correspondences that define $T_x$, $\langle a \rangle$ and $U_x$. For each prime $x \mid N$, let $a_x \in \mathbf{T}_{\text{univ}}/\mathfrak{m} = k$ denote the image of $T_x$ or $U_x$ and let $d_a$ denote the image of $\langle a \rangle$. Let $\mathbf{T}_{\text{univ},^*}$ denote the polynomial algebra over $O$ generated by indeterminates $T_x$, $U_y^*$ and $\langle a \rangle$ for primes $x \mid N\{q, y\}$ and integers $(a, NQ) = 1$. Let $\mathfrak{m}^*$ denote the maximal ideal of $\mathbf{T}_{\text{univ},^*}$ generated by $\mathfrak{c}$, $T_x - d_x^{-1} a_x$, $U_y^* - a_y$, $\langle a \rangle - d_a^{-1}$.

**Proposition 3.6.** Let $\Delta$ be a quotient of $(\mathbb{Z}/\mathbb{Q})^\times$ of $p$-power order. Then the $O[\Delta]$-module $H_0(X_\Delta(Q), \omega)_m$ is balanced (in the sense of Definition [3.4]).

**Proof.** Let $M = H_0(X_\Delta(Q), \omega)_m$ and $S = O[\Delta]$. Let $\mathcal{L} = \mathcal{O}_X$ (in the proof of Theorem 3.3, $\mathcal{L}$ will be non-trivial). Consider the exact sequence of $S$-modules (with trivial $\Delta$-action):

$$0 \to O \xrightarrow{\mathfrak{c}} O \to k \to 0.$$

Tensoring this exact sequence over $S$ with $M$, we obtain an exact sequence:

$$0 \to \text{Tor}_1^S(M, O)/\mathfrak{c} \to \text{Tor}_1^S(M, k) \to M_\Delta \to M \to M \otimes_S k \to 0.$$

Let $r$ denote the $O$-rank of $M_\Delta$. Then this exact sequence tells us that

$$d_S(M) := \dim_k M \otimes_S k - \dim_k \text{Tor}_1^S(M, k) = r - \dim_k \text{Tor}_1^S(M, O)/\mathfrak{c}.$$

We have a second quadrant Hochschild–Serre spectral sequence ([Mil80 Theorem III.2.20, Remark III.3.8])

$$H^i(\Delta, H^j(X_\Delta(Q), (\Omega^1 \otimes \omega^{-1} \otimes \mathcal{L}^*)_{K/O})) \Rightarrow H^{i+j}(X_0(Q), (\Omega^1 \otimes \omega^{-1} \otimes \mathcal{L}^*)_{K/O}).$$
Applying Pontryagin duality, we obtain a third quadrant spectral sequence
\[ H_i(\Delta, H_j(X_\Delta(Q), \omega \otimes \mathcal{L})) = \text{Tor}^S_i(H_j(X_\Delta(Q), \omega \otimes \mathcal{L}), \mathcal{O}) \implies H_{i+j}(X_0(Q), \omega \otimes \mathcal{L}). \]
The action of \( T^{\text{univ}} \) on cohomology can be lifted to a homotopy action on any chain complex computing cohomology; it follows that there is an induced action of Hecke on the spectral sequence. Localizing at \( \mathfrak{m} \), we obtain an isomorphism \( M_\Delta \cong H_0(X_0(Q), \omega \otimes \mathcal{L})_\mathfrak{m} \) and an exact sequence
\[ (H_1(X_\Delta(Q), \omega \otimes \mathcal{L}))_\mathfrak{m} \to H_1(X_0(Q), \omega \otimes \mathcal{L})_\mathfrak{m} \to \text{Tor}^S_1(M, \mathcal{O}) \to 0. \]
To show that \( d_S(M) \geq 0 \), we see that it suffices to show that \( H_1(X_0(Q), \omega \otimes \mathcal{L})_\mathfrak{m} \) is free of rank \( r \) as an \( \mathcal{O} \)-module. The module \( H_1(X_0(Q), \omega \otimes \mathcal{L}) \) is \( p \)-torsion free by Lemma \[\ref{lemma:p-torsion-freeness} \]. It therefore suffices to show that \( \dim_K H_1(X_0(Q), \omega \otimes \mathcal{L})_\mathfrak{m} \otimes K = r \). However, \( H_1(X_0(Q), \omega \otimes \mathcal{L})_\mathfrak{m} \otimes K = \text{Hom}_K(H^1(X_0(Q)_K, \Omega^1 \otimes \omega^{-1} \otimes \mathcal{L}^*)_\mathfrak{m}, K) \) which in turn is isomorphic, by Serre duality, to \( H^0(X_0(Q)_K, \omega \otimes \mathcal{L})^{\mathfrak{m}}_\mathfrak{m} \), where \( \mathfrak{m}^* \) is defined as above. By definition, we have
\[ r = \dim_K H_0(X_0(Q), \omega \otimes \mathcal{L})_\mathfrak{m} \otimes K = \dim_K H^0(X_0(Q)_K, \Omega^1 \otimes \omega^{-1} \otimes \mathcal{L}^*)_\mathfrak{m}. \]
We are therefore reduced to showing that
\[ \dim_K H^0(X_0(Q)_K, \omega \otimes \mathcal{L})^{\mathfrak{m}}_\mathfrak{m} = \dim_K H^0(X_0(Q)_K, \Omega^1 \otimes \omega^{-1} \otimes \mathcal{L}^*)_\mathfrak{m}. \]
By Kodaira–Spencer, this reduces to showing that
\[ \dim_K H^0(X_0(Q)_K, \omega \otimes \mathcal{L})^{\mathfrak{m}}_\mathfrak{m} = \dim_K H^0(X_0(Q)_K, \omega(-\infty) \otimes \mathcal{L}^*)_\mathfrak{m}. \]
Yet, since \( \mathcal{L} = \mathcal{O}_X \), by Lemma \[\ref{lemma:Kodaira-Spencer-formula} \], it suffices to show that
\[ \dim_K H^0(X_0(Q)_K, \omega \otimes \mathcal{L})^{\mathfrak{m}}_\mathfrak{m} = \dim_K H^0(X_0(Q)_K, \omega \otimes \mathcal{L}^*)_\mathfrak{m}. \]
However, these spaces are obtained from each other by twisting, and hence they have the same dimension. (This dimension does not depend on the choice of \( U_x \) by Lemma \[\ref{lemma:dimension-independence} \].) \( \square \)

3.3. **Galois Representations.** Let \( N = N(\mathfrak{p}) \) where \( N(\mathfrak{p}) \) is the Serre level of \( \mathfrak{p} \). We let \( H \) denote the \( p \)-part of \( (\mathbb{Z}/N(\mathfrak{p})\mathbb{Z})^\times \). Having fixed \( N \) and \( H \), we let \( X \) denote the modular curve defined at the beginning of §3.2. (We note that \( N \geq 5 \) by Serre’s conjecture.) \( \square \) We add, for now, the following assumption:

**Assumption 3.7.** Assume that:

1. The set \( T(\mathfrak{p}) \) is empty.

We address how to remove this assumption in §3.10. The only point at which this assumption is employed is in §3.4.

By Serre’s conjecture \[\cite{KW09} \] and by the companion form result of Gross \[\cite{Gro90} \] and Coleman–Voloch \[\cite{CV92} \], there exists a maximal ideal \( \mathfrak{m}_0 \) of \( T_0 \) corresponding to \( \mathfrak{p} \). The ideal \( \mathfrak{m}_0 \) is generated by \( \pi, T_x - \text{Trace}(\mathfrak{p}(\text{Frob}_x)) \) for all primes \( x \) with \( \langle x, Np \rangle = 1 \) and \( \langle x \rangle = \text{det}(\mathfrak{p}(\text{Frob}_x)) \) for all \( x \) with \( \langle x, N \rangle = 1 \). Extending \( \mathcal{O} \) if necessary, we may assume \( T_0/\mathfrak{m}_0 = k. \) Let \( Q \) be as in Section 3.2. For each \( x \in Q \), assume that the polynomial

\[ \text{If } p = 3 \text{ it may happen that with this choice of } H, \text{ the moduli of elliptic curves with } \Gamma_H(N) \text{-level structure is not represented by a scheme. Presumably, with care, our arguments can also be adapted to deal with this technicality. Alternatively, choose a prime } q \neq 1 \mod p \text{ with } q \geq 5 \text{ such that } \mathfrak{p} \text{ is unramified at } q \text{ and such that the ratio of the eigenvalues of } \mathfrak{p}(\text{Frob}_q) \text{ is neither } 1 \text{ nor } q \text{ nor } q^{-1} \text{ (the existence of such a prime is guaranteed by the Chebotarev density theorem). If we add } \Gamma_1(q) \text{ level structure then the arguments proceed almost entirely unchanged (the assumption on } q \text{ implies that any deformation of } \mathfrak{p} \text{ is unramified at } q). \]
\[ X^2 - T_x X + \langle x \rangle \] has distinct roots in \( T_\emptyset / \mathfrak{m}_\emptyset = k \) and choose a root \( \alpha_x \in k \) of this polynomial. Let \( m \) denote the maximal ideal of \( T \) generated by \( \mathfrak{m}_\emptyset \) and \( U_x - \alpha_x \) for \( x \in \mathbb{Q} \).

**Theorem 3.8** (Local–Global Compatibility). There exists a deformation

\[ \rho_Q : G_\mathbb{Q} \to \text{GL}_2(T_m) \]

of \( \mathfrak{p} \) unramified outside \( N\mathbb{Q} \) and determined by the property that for all primes \( x \) satisfying \((x, pN\mathbb{Q}) = 1\), \( \text{Trace}(\rho_Q(Frob_x)) = T_x \). Let \( \rho'_Q = \rho_Q \otimes \eta \), where \( \eta^2 = \langle \det(\mathfrak{p}) \rangle \det(\rho_Q)^{-1} \).

Then \( \rho'_Q \) is a deformation of \( \mathfrak{p} \) minimal outside \( \mathbb{Q} \).

**Remark 3.9.** Under the hypothesis that \( \mathfrak{p}(\text{Frob}_p) \) has distinct eigenvalues, this result can be deduced using an argument similar to that of [Gro90]. Under the hypothesis that \( \mathfrak{p}(\text{Frob}_p) \) has repeated eigenvalues but is not scalar, we shall deduce this using an argument of Wiese [Wie11] and Buzzard. When \( \mathfrak{p}(\text{Frob}_p) \) is trivial, however, we shall be forced to find a new argument using properties of local deformation rings. In the argument below, we avoid using the fact that the Hecke eigenvalues for all primes \( l \) determine a modular eigenform completely. One reason for doing this is that we would like to generalize our arguments to situations in which this fact is no longer true; we apologize in advance that this increases the difficulty of the argument slightly (specifically, we avoid using the fact that \( T_p \) in weight one can be shown to live inside the Hecke algebra \( T \), although this will be a consequence of our results).

**Proof.** For each \( m > 0 \), we have \( H^0(X_1(\mathbb{Q}), \omega_{\mathcal{O}/\mathfrak{m}^m})_m \cong H^0(X_1(\mathbb{Q}), \omega_{K/\mathcal{O}})_m[\mathfrak{m}^m] \), and we let \( I_m \) denote the annihilator of this space in \( T_m \). Since \( T_m = \lim_{\to} T_m/I_m \), it suffices to construct, for each \( m > 0 \), a representation \( \rho_{Q,m} : G_{\mathbb{Q}} \to \text{GL}_2(T_m/I_m) \) satisfying the conditions of the theorem.

Fix \( m > 0 \) and let \( A \) be a lift of (some power of) the Hasse invariant such that \( A \equiv 1 \) mod \( \mathfrak{m}^m \); let \( n - 1 \) denote the weight of \( A \). Multiplication by \( A \) induces a map:

\[
\begin{align*}
H^0(X_1(\mathbb{Q}), \omega_{K/\mathcal{O}}) &\xrightarrow{\phi} H^0(X_1(\mathbb{Q}), \omega_{\mathfrak{m}}^n) \\
K/\mathcal{O}[[q]] &\xrightarrow{\psi} K/\mathcal{O}[[q]]
\end{align*}
\]

This map is Hecke equivariant away from \( p \) on \( \mathfrak{m}^m \) torsion.

Consider the map

\[ \psi : H^0(X_1(\mathbb{Q}), \omega_{\mathcal{O}/\mathfrak{m}^m})^2[\mathfrak{m}^m] \to H^0(X_1(\mathbb{Q}), \omega_{\mathcal{O}/\mathfrak{m}^m}^n)[\mathfrak{m}^m] \]

defined by \( \psi = (\phi, \phi \circ T_p - T_p \circ \phi) \). For ease of notation, we let \( T_p \) (or \( \phi \circ T_p \)) exclusively refer to the Hecke operator in weight one, and let \( U_p \) denote the corresponding Hecke operator in weight \( n \). (The operator \( U_p \) has the expected effect on \( q \)-expansions, since the weight \( n \) is sufficiently large with respect to \( m \).) On \( q \)-expansions modulo \( \mathfrak{m}^m \), we may compute that \( \psi = (\phi, (p)V_p) \). We claim that \( \psi \) is injective. It suffices to check this on the \( \mathcal{O} \)-socle, namely, on \( \mathfrak{m} \)-torsion. On \( q \)-expansions, \( \phi \) is the identity and \( V_p(\sum a_n q^n) = \sum a_n q^{np} \). Suppose we have
an identity \( \langle p \rangle V_p f = g \). It follows that \( \theta g = 0 \) in \( H^0(X_1(Q), \omega_{O/\mathcal{O}^m})[\varpi] = H^0(X_1(Q), \omega_k) \).

By a result of Katz [Kat77], the \( 0 \) map has no kernel in weight \( \leq p - 2 \), and so in particular no kernel in weight 1. Hence \( \psi \) is injective.

The action of \( U_p \) in weight \( n \) on \( H^0(X_1(Q), \omega_{O/\mathcal{O}^m})^2 \) via \( \psi^{-1} \) is given by

\[
\begin{pmatrix}
  T_p & 1 \\
  -\langle p \rangle & 0 
\end{pmatrix},
\]

where here \( T_p \) is acting in weight one (cf. Prop 4.1 of [Gro90]), and hence satisfies the quadratic relation \( X^2 - T_p X + \langle p \rangle = 0 \). Note that the action of \( U_p + \langle p \rangle U_p^{-1} \) on the image of \( \psi \) is given by

\[
\begin{pmatrix}
  T_p & 0 \\
  0 & T_p 
\end{pmatrix}.
\]

By Proposition 12.1 and the remark before equation (4.7) of [Gro90], we see that \( \langle p \rangle = \alpha \beta \mod m \) and \( (U_p - \alpha)(U_p - \beta) \) acts nilpotently on \( \psi(H^0(X_1(Q), \omega_{O/\mathcal{O}^m})^2) \), where \( \alpha \) and \( \beta \) denote the (possibly non-distinct) eigenvalues of \( \overline{\varphi}(\text{Frob}_p) \) and \( \overline{\alpha} \), \( \overline{\beta} \) are any lifts of \( \alpha \) and \( \beta \) to \( \mathcal{O} \).

Let \( T_n^{\text{an}} \) denote the subalgebra of \( \text{End}_\mathcal{O}(H^0(X_1(Q), \omega_\mathcal{O}^n)) \) generated over \( \mathcal{O} \) by the operators \( T_x \) for primes \( (x, NP) = 1 \) and diamond operators \( \langle a \rangle \) for \( (a, NQ) = 1 \). Let \( T_n \) denote the subalgebra of \( \text{End}_\mathcal{O}(H^0(X_1(Q), \omega_\mathcal{O}^n)) \) generated by \( T_n \) and \( U_x \) for \( x \) dividing \( Q \), and let \( T_n \) denote the subalgebra generated by \( T_n \) and \( U_p \) (recall that we are denoting \( T_p \) in weight \( n \) by \( U_p \)). By a slight abuse of notation, let \( m_0 \) denote the maximal ideal of \( T_n^{\text{an}} \) corresponding to \( \overline{\varphi} \). Similarly, let \( m \) denote the maximal ideal of \( T_n \) generated by \( m_0 \) and \( U_x - \alpha_x \) for \( x \in Q \).

Let \( \overline{m}_\alpha \) and \( \overline{m}_\beta \) denote the ideals of \( T_n \) containing \( m \) and \( U_p - \alpha \) or \( U_p - \beta \) respectively. If \( \alpha = \beta \), we simply write \( \overline{m} = \overline{m}_\alpha = \overline{m}_\beta \). Note that since \( n > 1 \), we have

\[
H^0(X_1(Q), \omega_\mathcal{O}^n) \otimes \mathcal{O}/\mathcal{O}^m \cong H^0(X_1(Q), \omega_\mathcal{O}^n/\mathcal{O}^m)
\]

and hence we may regard the latter as a module for \( T_n \) (and its sub-algebras \( T_n \) and \( T_n^{\text{an}} \)). The proof of Theorem 3.8 will be completed in Sections 3.4–3.8.

### 3.4 Interlude: Galois Representations in higher weight.

In this section, we summarize some results about Galois representations associated to ordinary Hecke algebras in weight \( n \geq 2 \). As above, let \( \alpha \) and \( \beta \) be the eigenvalues of \( \overline{\varphi}(\text{Frob}_p) \). There is a natural map \( T_{n,m} \to \overline{T}_{n,\overline{m}} \). If \( \alpha = \beta \) this map is injective, otherwise, write \( T_{n,m_\alpha} \) for the image. There are continuous Galois representations

\[
\rho_{n,\alpha} : G_Q \to \text{GL}_2(T_{n,m_\alpha})
\]

\[
\overline{\rho}_{n,\alpha} : G_Q \to \text{GL}_2(\overline{T}_{n,\overline{m}_\alpha})
\]

with the following properties:

(a) The representation \( \overline{\rho}_{n,\alpha} \) is obtained from \( \rho_{n,\alpha} \) by composing \( \rho_{n,\alpha} \) with the natural inclusion map \( T_{n,m_\alpha} \to \overline{T}_{n,\overline{m}_\alpha} \).

(b) \( \rho_{n,\alpha} \) and \( \overline{\rho}_{n,\alpha} \) are unramified at all primes \( (x, pNQ) = 1 \) and the characteristic polynomial of \( \rho_{n,\alpha}(\text{Frob}_x) \) for such \( x \) is

\[
X^2 - T_x X + x^{n-1}(x).
\]
(c) If $E$ is a field of characteristic zero, and $\phi : \widetilde{T}_{n,\widetilde{m}} \to E$ is a homomorphism, then $\phi \circ \widetilde{\rho}_{n,\alpha}|_{G_p}$ is equivalent to a representation of the form
\[
\begin{pmatrix}
\epsilon^n-1\lambda(p)/U_p & * \\
0 & \lambda(U_p)
\end{pmatrix}
\]
where $\lambda(z)$ denotes the unramified character sending Frob$_p$ to $z$.

These results follow from standard facts about Galois representations attached to classical ordinary modular forms, together with the fact that there is an inclusion
\[
\mathbb{T}_{n,m} \hookrightarrow \mathbb{T}_{n,\widetilde{m}} \hookrightarrow \prod E_i
\]
with $E_i$ running over a finite collection of finite extensions of $K$ corresponding to the ordinary eigenforms of weight $n$ and level $\Gamma_1(NQ)$.

**Theorem 3.10.** Under Assumption 3.7, $\rho_{n,\alpha}$ is a deformation of $\overline{\phi}$ that satisfies conditions (2) and (4) of Definition 3.1, with the exception of the condition of being unramified at $p$.

This follows from the choice of $N$ and $H$ together with the local Langlands correspondence and results of Diamond–Taylor and Carayol (see [CDT99] Lemma 5.1.1). (The choice of $H$ ensures that for each prime $x \neq p$, $\det(\rho_{n,\alpha})|_{I_x}$ has order prime to $p$.) Note that without any assumption, the representation $\rho_{n,\alpha}$ still satisfies condition (2) of Definition 3.1; the issue is that $\rho_{n,\alpha}$ may have extra ramification at those primes not satisfying Assumption 3.7.

We now fix one of the eigenvalues of $\overline{\phi}(\text{Frob}_p)$, $\alpha$, say, and write $\widetilde{m} = \widetilde{m}_\alpha$. The existence of $\widetilde{\rho}_{n,\alpha}$ gives $B := \mathbb{T}_{n,\widetilde{m}}^2$ the structure of a $\mathbb{T}_{n,\widetilde{m}}[G_Q]$-module. Recall that $G_p$ is the decomposition group of $G_Q$ at $p$.

**Lemma 3.11.** Suppose that $\overline{\phi}(\text{Frob}_p)$ is not a scalar. Then there exists an exact sequence of $\mathbb{T}_{n,\widetilde{m}}[G_p]$-modules
\[
0 \to A \to B \to C \to 0
\]
such that:

1. $A$ and $C$ are free $\mathbb{T}_{n,\widetilde{m}}$-modules of rank one.
2. The sequence splits $B \simeq A \oplus C$ as a sequence of $\mathbb{T}_{n,\widetilde{m}}$-modules.
3. The action of $G_p$ on $C$ factors through $G_F = G_p/I_p$, and Frobenius acts via the operator $U_p \in \mathbb{T}_{n,\widetilde{m}}$.
4. The action of $G_p$ on $A$ is via $\epsilon^n-1(p)U_p^{-1}$.

**Proof.** Let $C$ denote the maximal $\mathbb{T}_{n,\widetilde{m}}$-quotient on which Frob$_p$ acts by $U_p$. The construction of $C$ is given by taking a quotient, and thus its formulation is preserved under taking quotients of $B$. Since $B$ is free of rank two, $B/\widetilde{m}$ has dimension 2. The action of $U_p$ on $C/\widetilde{m}$ is, by definition, given by the scalar $\alpha$. Yet $B/\widetilde{m}$ as a $G_p$-representation is given by $\overline{\phi}$, and $\overline{\phi}(\text{Frob}_p)$ either has distinct eigenvalues $\alpha$ and $\beta$ or is non-scalar by definition. Hence dim $C/\widetilde{m} = 1$, and by Nakayama’s lemma, $C$ is cyclic. On the other hand, by considering the Galois action on classical modular forms, we see that $C \otimes \mathbb{Q}$ has rank one, and thus $C$ is free as a $\mathbb{T}_{n,\widetilde{m}}$-module. It follows that $B \to C$ splits, and that $A$ is also free of rank one. Considering once more the local Galois structure of representations arising from classical modular forms, it follows that $G_p$ acts on $A \otimes \mathbb{Q}$ via $\epsilon^n-1(p)U_p^{-1}$. Since $\mathbb{T}_{n,\widetilde{m}}$ is $\mathcal{O}$-flat, the action of $G_p$ on $A$ itself is given by the same formula, proving the lemma. \qed
When \( \overline{\rho}(\text{Frob}_p) \) is scalar, there does not exist such a decomposition. Instead, in \( \S\ 3.5 \) we shall study the local properties of \( \rho_{n,\alpha} \) and \( \overline{\rho}_{n,\alpha} \) using finer properties of local deformation rings.

3.5. **Proof of Theorem 3.8:** Case 1: \( \alpha \neq \beta, \overline{\rho}(\text{Frob}_p) \) has distinct eigenvalues. We claim there is an isomorphism

\[
H^0(X_1(Q), \omega_{Q/{\mathcal{O}}})_m \simeq (\psi H^0(X_1(Q), \omega_{Q/{\mathcal{O}}})^2)_{\bar{m}_n},
\]

obtained by composing \( \phi \) with the \( \overline{T}_m \)-equivariant projection onto \( (\psi H^0(X_1(Q), \omega_{Q/{\mathcal{O}}})^2)_{\bar{m}_n} \). The argument is similar to the proof of Lemma 3.4. We know that \( U_p \) satisfies the equation \( X^2 - T_1X + \langle p \rangle = 0 \) on the image of \( \psi \) but we may not use Hensel’s Lemma to deduce that there exist \( \widehat{\alpha} \) and \( \widehat{\beta} \) in \( T_m/I_m \) such that \( (U_p - \widehat{\alpha})(U_p - \widehat{\beta}) = 0 \) on the \( m \)-part of the image of \( \psi \). Since we do not know \emph{a priori} that \( T_p \) lies in \( T_m \), Instead, we note the following. Since \( U_p \) acts invertibly on the image of \( \psi \), we deduce from the equality \( T_p = U_p^{-1}(p) + U_p \) that \( T_p - \alpha - \beta \) lies in \( \overline{m}_n \) and \( \overline{m}_3 \), and thus acts nilpotently on \( H^0(X_1(Q), \omega_{Q/{\mathcal{O}}})_m \). It follows from the result just established that \( T_m/I_m \subset \text{End}(H^0(X_1(Q), \omega_{Q/{\mathcal{O}}})_m) \) is a local ring with maximal ideal \( \overline{m} \) which acts on \( H^0(X_1(Q), \omega_{Q/{\mathcal{O}}})_m \). The operator \( U_p \) does satisfy the quadratic relation \( X^2 - T_pX + \langle p \rangle = 0 \) over \( T_m/I_m [T_p] \), and hence by Hensel’s Lemma there exists \( \widehat{\alpha} \) and \( \widehat{\beta} \) in \( T_m/I_m [T_p] \) such that \( (U_p - \widehat{\alpha})(U_p - \widehat{\beta}) = 0 \) on the \( m \)-part of the image of \( \psi \). The argument then proceeds as in the proof of Lemma 3.4, noting (tautologically) that \( H^0(X_1(Q), \omega_{Q/{\mathcal{O}}})_m = H^0(X_1(Q), \omega_{Q/{\mathcal{O}}})_m \).

It follows from the result just established that \( T_m/I_m [T_p] \subset \text{End}(\text{Im}(\psi)_m) \) is a quotient of \( \overline{T}_n, \overline{\rho}_{n,\alpha} \), and \( T_m/I_m \) is the corresponding quotient of \( T_n, \rho_{n,\alpha} \). Note that the trace of any lift of Frobenius on this quotient is equal to \( U_p + \langle p \rangle U_p^{-1} \mod \mathcal{O} = T_p \), and so this in particular implies that \( T_p \in T_m/I_m \). We now define \( \rho_{Q,m} \) to be the composition of \( \rho_{n,\alpha} \) with the surjection \( T_{n,\alpha} \rightarrow T_m/I_m \), and \( \overline{\rho}_{Q,m} \) to be \( \overline{\rho}_{n,\alpha} \) on the corresponding quotient \( T_m/I_m [U_p] \) of \( T_n, \overline{\rho}_{n,\alpha} \).

The choice of \( n \) ensures that \( \epsilon^{n-1} \) is trivial modulo \( \mathcal{O} \). The character \( \nu \) defined by the formula \( \nu := (\det \overline{\rho}) \det(\rho_Q)^{-1} \) is thus unramified outside \( Q \) and of \( p \)-power order. Since \( p > 2, \nu \) admits a square root \( \lambda \) (also unramified outside \( Q \)). We have established that \( \rho_{Q,m} := \rho_{Q,m} \otimes \eta \) satisfies all the conditions of Definition 3.1 except the condition that it be unramified at \( p \). Equivalently, it suffices to show that \( \overline{\rho}_{Q,m} \) is unramified at \( p \). By Lemma 3.11, we may write

\[
\overline{\rho}_{Q,m} | G_p \cong \begin{pmatrix} \lambda(\beta) & * \\ 0 & \lambda(\alpha) \end{pmatrix}.
\]

By symmetry, we could equally well have defined \( \rho_{Q,m} \) by regarding \( T_m/I_m \) as a quotient of \( T_{n,\alpha} \). (Note that the Chebotarev density theorem and [Car94, Théorème 1] imply that \( \rho_{Q,m} \) is uniquely determined by the condition that \( \text{Trace} \rho_Q(\text{Frob}_x) = T_x \) for all \( (x, pNQ) = 1 \).) It follows that we also have

\[
\overline{\rho}_{Q,m} | G_p \cong \begin{pmatrix} \lambda(\alpha) & * \\ 0 & \lambda(\beta) \end{pmatrix}.
\]

Since \( \alpha \neq \beta \), this forces \( \rho_{Q,m} | G_p \) to split as a direct sum of the unramified character \( \lambda(\alpha) \) and \( \lambda(\beta) \). (Moreover, we see that \( T_p = U_p^{-1}(p) + U_p = \widehat{\alpha} + \widehat{\beta} = \text{Trace}(\rho_{Q,m}(\text{Frob}_p)) \in T_m \).)
3.6. **Proof of Theorem 3.8** Case 2: \( \alpha = \beta, \mathfrak{p}(\mathrm{Frob}_p) \) **non-scalar**. Let us now assume that \( \alpha = \beta \) is a generalized eigenvalue of \( \mathfrak{p}(\mathrm{Frob}_p) \), and furthermore that \( \mathfrak{p}(\mathrm{Frob}_p) \) is non-scalar. We follow the argument of [Wie11].

**Lemma 3.12 (Doubling).** The action of \( \mathbf{T}_{n, \tilde{m}} \) on \( \psi(\mathcal{H}^0(\mathcal{X}_1(Q), \omega_{\mathcal{O}/\mathfrak{m}})^2) \) factors through a quotient isomorphic to

\[
\mathbf{T}_m/I_m[T_p][X]/(X^2 - T_pX + \langle p \rangle),
\]

where \( U_p \) acts by \( X \).

**Proof.** The action of \( \mathbf{T}_{n, \tilde{m}} \) certainly contains \( T_p := U_p + \langle p \rangle U_p^{-1} \), and moreover \( U_p \) also satisfies the indicated relation. Thus it suffices to show that \( U_p \) does not satisfy any further relation. Such a relation would be of the form \( AU_p + B = 0 \) for operators \( A, B \) in \( \mathbf{T}_m/I_m[T_p] \). By considering the action of \( U_p \) as a matrix on the image of \( \psi \), however, this would imply an identity:

\[
A \begin{pmatrix} T_p & 1 \\ -\langle p \rangle & 0 \end{pmatrix} + B \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},
\]

from which one deduces that \( A = B = 0 \) (the fact that one can deduce the vanishing of the entries is equivalent to the injectivity of \( \psi \)). \( \square \)

Following Wiese, we call this phenomenon “doubling”, because the corresponding quotient of \( \mathbf{T}_{n, \tilde{m}} \) contains two copies of the image of \( \mathbf{T}_{n, m} \). We show that this implies that the corresponding Galois representation is unramified.

Note that the trace of any lift of Frobenius on \( \mathbf{T}_m/I_m \) is equal to \( U_p + \langle p \rangle U_p^{-1} \mod \mathfrak{m} = T_p \), and so this in particular implies that \( T_p \in \mathbf{T}_m/I_m \). The image of \( \mathbf{T}_{n, m} \) under the map

\[
\mathbf{T}_{n, m} \to \mathbf{T}_{n, \tilde{m}} \to \mathbf{T}_m/I_m[T_p][U_p]
\]

is given by \( \mathbf{T}_m/I_m = \mathbf{T}_m/I_m[T_p] \). We thus obtain a Galois representation

\[
\rho_{Q, m} : G_Q \to \text{GL}_2(\mathbf{T}_m/I_m)
\]

As in §3.5 it suffices to prove that \( \rho_{Q, m} \) is unramified at \( p \). Consider the Galois representation \( \tilde{\rho}_{Q, m} : G_Q \to \text{GL}_2(\mathbf{T}_m/I_m[U_p]) \) obtained by tensoring over \( \mathbf{T}_{n, m} \) with \( \mathbf{T}_{n, \tilde{m}} \). By Lemma 3.12 there is an isomorphism

\[
\mathbf{T}_m/I_m[U_p] \simeq \mathbf{T}_m/I_m \oplus \mathbf{T}_m/I_m
\]

as a \( \mathbf{T}_m/I_m \)-module. Since \( \tilde{\rho}_{Q, m} \) is obtained by \( \rho_{Q, m} \) by tensoring with a doubled module, it follows that there is an isomorphism \( \tilde{\rho}_{Q, m} \simeq \rho_{Q, m} \oplus \rho_{Q, m} \).

**Lemma 3.13.** Let \( (\mathcal{R}, m) \) be a local ring, and let \( N, M, \) and \( Q \) be \( \mathcal{R}[G_p] \)-modules which are free \( \mathcal{R} \)-modules of rank two. Suppose there is an exact sequence of \( \mathcal{R}[G_p] \)-modules

\[
0 \to N \to M \oplus M \to Q \to 0
\]

which is split as a sequence of \( \mathcal{R} \)-modules. Suppose that \( Q/m \) is indecomposable as a \( \mathcal{R}[G_p] \)-module. Then \( N \simeq M \simeq Q \) as \( \mathcal{R}[G_p] \)-modules.

**Proof.** This is Proposition 4.4 of Wiese [Wie11]. (The lemma is stated for \( \mathbb{F}_p \)-algebras \( \mathcal{R} \) and the stated condition is on the sub-module \( N \) rather than the quotient \( Q \), but the proof is exactly the same.) \( \square \)
We apply this as follows. Consider the sequence of $\tilde{T}_{n,m}$-modules considered in Lemma\ref{lem:3.11}. If we tensor this sequence with the quotient of $\tilde{T}_{n,m}$ corresponding to the doubling isomorphism

$$T_m/I_m[U_p] = T_m/I_m \oplus T_m/I_m,$$

then the corresponding quotient of $B$ is $\tilde{\rho}_{Q,m}$, which, by doubling, is free of rank 4 and as a $G_p$-module and is given by $\rho_{Q,m} \oplus \rho_{Q,m}$. The corresponding quotients $A$ and $C$ are similarly free over $T_m/I_m$ of rank 2. The action of $\text{Frob}_p$ on $Q/m$ is given by $U_p$. Since $U_p$ does not lie in $m$ — as this would contradict doubling — it follows that $(U_p - \alpha)$ acts nilpotently but non-trivially on $Q/m$, and hence $Q/m$ is indecomposable. Hence, by the lemma above, there are isomorphisms $Q \simeq \rho_{Q,m}$ as a $G_p$-module. Yet $Q$ is a quotient of $C$, which is by construction unramified, and thus $\rho_{Q,m}$ is also unramified. Finally, we note that the trace of Frobenius at $p$ is given by $U_p + \langle p \rangle U_p^{-1} = T_p$, so $T_p = \text{Trace}(\rho_{Q,m}(\text{Frob}_p))$.

3.7. **Proof of Theorem 3.8.** Case 3: $\alpha = \beta$, $\bar{\rho}(\text{Frob}_p)$ scalar. The construction of the previous section gives a representation $\rho_{Q,m}$ which satisfies all the required deformation properties with the possible exception of knowing that $\rho_{Q,m}$ is unramified at $p$. In order to deal with the case when $\bar{\rho}(\text{Frob}_p)$ is scalar, we shall have to undergo a closer study of local deformation rings. Suppose that

$$\bar{\rho}: G_p \to \text{GL}_2(k)$$

is trivial. (If $\bar{\rho}$ is scalar, it is trivial after twisting.) We introduce some local framed universal deformation rings associated to $\bar{\rho}$. Fix a lift $\phi_p \in G_p$ of $\text{Frob}_p$.

**Definition 3.14.** For $A$ in $\mathcal{C}_\mathcal{O}$, let $D(A)$ denote the set framed deformations of $\bar{\rho}$ to $A$, and let $\tilde{D}(A)$ denote the framed deformations together with an eigenvalue $\alpha$ of $\phi_p$. Let these functors be represented by rings $R^\text{univ}$ and $\tilde{R}^\text{univ}$ respectively.

There is a natural inclusion $R^\text{univ} \to \tilde{R}^\text{univ}$ and $\tilde{R}^\text{univ}$ is isomorphic to a quadratic extension of $R^\text{univ}$ (given by the characteristic polynomial of $\phi_p$). Kisin constructs certain quotients of $R^\text{univ}$ which capture characteristic zero quotients with good $p$-adic Hodge theoretic properties. Let $\epsilon$ denote the cyclotomic character, let $\omega$ denote the Teichmüller lift of the mod-$p$ reduction of $\epsilon$, and let $\chi = \epsilon \omega^{-1}$ so $\chi \equiv 1 \mod p$. We modify the choice of $\phi_p$ if necessary so that $\chi(\phi_p) = 1$. Let $R^\text{univ,} \chi^n$ and $\tilde{R}^\text{univ,} \chi^n - 1$ denote the quotients of $R^\text{univ}$ and $\tilde{R}^\text{univ}$ corresponding to deformations with determinant $\chi^n - 1$.

**Theorem 3.15.** Fix an integer $n \geq 2$.

1. There exists a unique reduced $\mathcal{O}$-flat quotient $\tilde{R}^\dagger$ of $R^\text{univ,} \chi^n - 1$ such that points on the generic fiber of $\tilde{R}^\dagger$ correspond to representations $\rho: G_p \to \text{GL}_2(E)$ such that:

$$\rho \sim \begin{pmatrix} \chi^{n-1} \lambda(\alpha^{-1}) & * \\ 0 & \lambda(\alpha) \end{pmatrix}$$

2. The ring $\tilde{R}^\dagger$ is an integral domain which is normal, Cohen–Macaulay, and of relative dimension 4 over $\mathcal{O}$.

**Proof.** Part 1 of this theorem is due to Kisin. For part 2, the fact that $\tilde{R}^\dagger$ is an integral domain follows from the proof of Lemma 3.4.3 of \cite{Ger}. The rest of part 2 follows from the method and results of Snowden (\cite{Snd}). More precisely, Snowden works over an arbitrary finite extension of $\mathbb{Q}_p$ containing $\mathbb{Q}_p(\zeta_p)$, and assumes that $n = 2$, so $\chi = \epsilon \omega^{-1} = \epsilon$. However, this is exactly the hardest case — since for us $p \neq 2$, $\chi^{n-1} \neq \epsilon$, our deformation problem consists of a single
potentially crystalline component. In particular, the arguments of [Sno] show that $\widetilde{R}^t \otimes k$ is an integral normal Cohen–Macaulay ring of dimension four, which is not Gorenstein, and is identified (in the notation of *ibid.*) with the completion of $B_1$ at $b = (1, 1; 0)$.

Let $R^t$ denote the image of $R^\text{univ}$ in $\widetilde{R}^t$. We also define the following rings:

**Definition 3.16.** Let $R^\text{unr}$ denote the largest quotient of $R^t$ corresponding to unramified deformations of $\overline{\rho}$. Let $R^\text{unr}$ denote the corresponding quotient of $R^t$.

We are now in a position to define two ideals of $R^\text{univ}$.

**Definition 3.17.** The unramified ideal $\mathcal{I}$ is the kernel of the map $R^\text{univ} \to R^\text{unr}$. The doubling ideal $\mathcal{J}$ is the annihilator of $R^t/R^t$ as an $R^\text{univ}$-module.

**Lemma 3.18.** There is an inclusion $\mathcal{I} \subset \mathcal{J}$.

**Proof.** By definition, $R^\text{univ}/\mathcal{I}$ acts faithfully on $R^t/R^t$, and it is the largest such quotient. Hence it suffices to show that $R^\text{univ}/\mathcal{I}$ acts faithfully on

$$\left( \frac{R^t}{R^t} \right) \otimes R^\text{univ}/\mathcal{I} \simeq \left( \frac{R^t}{\mathcal{I}} \right) / \left( \frac{R^t}{\mathcal{J}} \right).$$

Since $\widetilde{R}^t/\mathcal{J} \simeq R^\text{unr}$ and $R^t/\mathcal{I} \simeq R^\text{unr}$, it suffices to show that $R^\text{unr}/R^\text{unr}$ is a faithful $R^\text{univ}/\mathcal{I}$-module. Let $\varpi^m$ denote the exact power of $\varpi$ dividing $(\chi(g) - 1)O$ for all $g$ in the decomposition group at $p$. By considering determinants, $R^\text{unr}$ (and $R^\text{unr}$) is annihilated by $\varpi^m$. Moreover, $R^\text{unr}$ is explicitly given by the moduli space of matrices $\phi$ in $O/\varpi^m$ which are trivial modulo $\varpi$ and have determinant one, that is, with

$$\phi = \begin{pmatrix} 1 + \phi_1 & \phi_2 \\ \phi_3 & 1 + \phi_4 \end{pmatrix},$$

one has:

$$R^\text{unr} \simeq O/\varpi^m[(\phi_1, \phi_2, \phi_3, \phi_4)]/(\phi_1 + \phi_4 + \phi_1\phi_4 - \phi_2\phi_3),$$

whereas $\widetilde{R}^\text{unr}$ consists of such matrices together with an eigenvalue $\alpha = 1 + \beta$, so

$$\widetilde{R}^\text{unr} \simeq R^\text{unr}[\beta]/(\beta^2 - (\phi_1 + \phi_4)\beta - (\phi_1 + \phi_4)) \simeq R^\text{unr} \oplus R^\text{unr},$$

where the last isomorphism is as a $R^\text{unr}$-module. Clearly $R^\text{unr}$ acts faithfully on $(R^\text{unr} \oplus R^\text{unr})/R^\text{unr} \simeq R^\text{unr}$, the lemma is proven.

What is less obvious is that there is actually an inclusion in the other direction. That is, there is an equality $\mathcal{J} = \mathcal{I}$. As in Snowden, the ring $\widetilde{R}^t$ is defined by the following equations:

1. $\phi \in M_2(A)$ has determinant 1.
2. $\alpha$ is a root of the characteristic polynomial of $\phi$.
3. $\text{Trace}(g) = \chi^{n-1}(g) + 1$ for $g \in I_p$.
4. $(g - 1)(g' - 1) = (\chi^{n-1}(g) - 1)(g' - 1)$ for $g, g' \in I_p$.
5. $(g - 1)(\phi - \alpha) = (\chi^{n-1}(g) - 1)(\phi - \alpha)$ for $g \in I_p$.
6. $(\phi - \alpha)(g - 1) = (\alpha^{n-1} - \alpha)(g - 1)$ for $g \in I_p$.

Morally, we have:

$$\rho(\phi_g) = \phi = \begin{pmatrix} \alpha^{-1} & * \\ 0 & \alpha \end{pmatrix}, \quad \rho(g) = \begin{pmatrix} \chi^{n-1}(g) & * \\ 0 & 1 \end{pmatrix}, \quad g \in I_p,$$

Instead of writing down a presentation of this ring, it will suffice to note the following: The ring $\widetilde{R}^t$ is generated over $O$ by the following elements which all lie in the maximal ideal:
(1) Parameters $\phi_i$ (for $i = 1$ to 4) corresponding to the image of $\rho(\phi_p) - 1$,
(2) Parameters $x_{ij}$ for $i = 1$ to 4 and a finite number of $j$ corresponding to the image of
inertial elements $m_j = \rho(g_j) - 1$.
(3) An element $\beta$, where $\alpha = 1 + \beta$ is an eigenvalue of $\rho(\phi_p)$.

Moreover, $R^\dagger$ is generated as a sub-algebra by $\phi_i$ and the $x_{ij}$, and $\beta$ satisfies
\[ \beta^2 - (\phi_1 + \phi_4)\beta - (\phi_1 + \phi_4) = 0. \]

Since the determinant of $\phi$ is one, it follows that $\alpha + \alpha^{-1} = 2 + \phi_1 + \phi_4$. By definition, there
is a decomposition of $R^\dagger$-modules $\tilde{R^\dagger} / \mathcal{J} = \tilde{R^\dagger} / \mathcal{J} \oplus \beta \tilde{R^\dagger} / \mathcal{J}$ with each summand being free
over $R^\dagger / \mathcal{J}$. From the equality (6), we deduce that the relation
\[ (\phi - 1)m_j - (\phi_1 + \phi_4)m_j = (\alpha^{-1} - 1 - \phi_1 - \phi_4)m_j = -(\alpha - 1)m_j = -\beta m_j \]
holds in $\tilde{R^\dagger}$, and hence in $\tilde{R^\dagger} / \mathcal{J}$. Yet by assumption, over $R^\dagger / \mathcal{J}$, the modules $\tilde{R^\dagger} / \mathcal{J}$ and $\beta \tilde{R^\dagger} / \mathcal{J}$ have trivial intersection, from which it follows that $\beta m_j = 0$ in $\beta \tilde{R^\dagger} / \mathcal{J}$. In particular, since the latter module is generated by $\beta$, we must have $x_{ij} \in \mathcal{J}$ for all $i$ and $j$. Since $\mathcal{J}$ is generated by $x_{ij}$, we deduce that $\mathcal{J} \subset \mathcal{J}$, and hence from Lemma 3.18 that $\mathcal{J} = \mathcal{J}$.

**Remark 3.19.** Why might one expect an equality $\mathcal{J} = \mathcal{J}$? One reason is as follows. The

doubling ideal $\mathcal{J}$ represents the largest quotient of $R^\dagger$ on which the eigenvalue of Frobenius $\alpha$ cannot be distinguished from its inverse $\alpha^{-1}$. It is clear that this is true for unramified representations. Similarly, for a ramified ordinary quotient, one might expect that the $\alpha$ can be distinguished from $\alpha^{-1}$ by looking at the “unramified quotient line” of the representation. Indeed, for characteristic zero representations this is clear — one even has $R^\dagger[1/\omega] \simeq \tilde{R^\dagger}[1/\omega]$.

**Lemma 3.20.** There is a surjection $T_{n,m} \otimes_{R^\dagger} \tilde{R^\dagger} \to \tilde{T}_{n,\tilde{m}}$.

**Proof.** Recall that $\tilde{T}_{n,\tilde{m}} = T_{n,m}[U_p]$. Since $U_p$ is given as an eigenvalue of Frobenius, $\tilde{T}_{n,\tilde{m}}$

naturally has the structure of a $R^\text{univ}$-algebra. We claim that the action of $R^\text{univ}$ factors through $\tilde{R^\dagger}$. Since $T_{n,\tilde{m}}$ acts faithfully on a space of modular forms, there is an injection:
\[ \tilde{T}_{n,\tilde{m}} \hookrightarrow \prod_i E_i \]
into a product of fields corresponding to the Galois representations associated to the ordinary
modular forms of weight $n$ and level $\Gamma_1(NQ)$. By the construction of $\tilde{R^\dagger}$, it follows that the action of $\tilde{R^\text{univ}}$ on this latter module acts via $\tilde{R^\dagger}$. This also implies that the action of $R^\text{univ}$ on $\tilde{T}_{n,\tilde{m}}$ (and hence on $T_{n,m}$) factors through $\tilde{R^\dagger}$, and hence there exists a map
\[ T_{n,m} \otimes_{R^\dagger} \tilde{R^\dagger} \to \tilde{T}_{n,\tilde{m}}, \]
sending $\alpha \in \tilde{R^\dagger}$ to $U_p$. Yet the image of this map contains both $T_{n,m}$ and $U_p$, and is thus
surjective. \qed

**Definition 3.21.** Let the global doubling ideal $\mathcal{J}^\text{glob}$ be the annhilator of $\tilde{T}_{n,\tilde{m}}/T_{n,m}$ as an

$R^\dagger$-module.

Since there is a surjection $T_{n,m} \otimes_{R^\dagger} \tilde{R^\dagger} \to \tilde{T}_{n,\tilde{m}}$, it follows that $\tilde{T}_{n,\tilde{m}}/T_{n,m}$ is a quotient of

\[ (T_{n,m} \otimes_{R^\dagger} \tilde{R^\dagger})/T_{n,m} = (T_{n,m} \otimes_{R^\dagger} \tilde{R^\dagger})/T_{n,m} \otimes_{R^\dagger} R^\dagger \simeq T_{n,m} \otimes_{R^\dagger} \tilde{R^\dagger}/R^\dagger \]
as an $R^\ell$-module. In particular, by considering the action on the last factor, we deduce that $\mathcal{F} \subset \mathcal{F}^\text{glob}$. In particular, $\mathcal{F} \subset \mathcal{F}^\text{glob}$, or equivalently, on any quotient of $T_{n, m}$ on which the corresponding quotient of $\tilde{T}_{n, m}$ is doubled, the action of the Galois group at $p$ is unramified. In particular, by Lemma 3.12 this implies to the quotient of $\tilde{T}_{n, m}$ given by $T_m/I_m[T_p][U_p]$. Specifically, as in the the previous sections, we obtain corresponding Galois representations:

$$\rho_{Q, m} : G_Q \to \text{GL}_2(T_m/I_m) \quad \tilde{\rho}_{Q, m} : G_Q \to \text{GL}_2(T_m/I_m[U_p]),$$

(The trace of any lift of Frobenius on this quotient is equal to $U_p + (p)U_p^{-1} \mod \varpi^m = T_p$, and so $T_p \in T_m/I_m$.) From the discussion above, we deduce that $\tilde{\rho}_{Q, m}$ and thus $\rho_{Q, m}$ is unramified at $p$, and that $\text{Trace}(\rho_{Q, m}(\text{Frob}_p)) = T_p$. The rest of the argument follows as in Case 3.5 and this completes the proof of Theorem 3.8.

3.8. Multiplicity Two. Although this is not needed for our main results, we deduce in this section some facts about global multiplicity of Galois representations in modular Jacobians. Recall that $k$ denotes a finite field of odd characteristic, $\mathcal{O}$ denotes the ring of integers of some finite extension $K$ of $\mathbb{Q}_p$ with uniformizer $\varpi$ and $\mathcal{O}/\varpi = k$.

We recall some standard facts about Cohen–Macaulay rings from [Eis95], § 21.3 (see also [HK71]). Let $(A, m, k)$ be a complete local Cohen–Macaulay ring of dimension $n$. Then $A$ admits a canonical module $\omega_A$. Moreover, if $(x_1, \ldots, x_m)$ is a regular sequence for $A$, and $B = A/(x_1, \ldots, x_m)$, then

$$\omega_B := \omega_A \otimes_A B$$

is a canonical module for $B$. It follows that

$$\omega_A \otimes_A A/m = \omega_A \otimes_A (B \otimes_B B/m) = \omega_B \otimes_B B/m.$$

If $m = n$, so $B$ is of dimension zero, then $\text{Hom}(\ast, \omega_B)$ is a dualizing functor, and so

$$\dim_k B[m] = \dim_k \omega_B \otimes B/m = \dim_k \omega_A \otimes A/m.$$

Moreover, we have the following:

**Lemma 3.22.** Let $A$ be a finite flat local $\mathbb{Z}_p$-algebra. Suppose that $A$ is Cohen–Macaulay. Then $\text{Hom}_{\mathbb{Z}_p}(A, \mathbb{Z}_p)$ is a canonical module for $A$.

**Proof.** More generally, if $A$ is a module-finite extension of a regular (or Gorenstein) local ring $R$, then (by Theorem 21.15 of [Eis95]) $\text{Hom}_R(A, R)$ is a canonical module for $A$. \qed

Finally, we note the following:

**Lemma 3.23.** If $B$ is a complete local Cohen–Macaulay $\mathcal{O}$-algebra and admits a dualizing module $\omega_B$ with $\mu$ generators, then the same is true for the power series ring $A = B[[T_1, \ldots, T_n]]$. Moreover, the same is also true for $B \otimes_\mathcal{O} C$, for any complete local $\mathcal{O}$-algebra $C$ which is a complete intersection.

**Proof.** For power series rings this is a special case of the discussion above. Consider now the case of $B \otimes_\mathcal{O} C$. By assumption, $C$ is a quotient of $\mathcal{O}[[T_1, \ldots, T_n]]$ by a regular sequence. Hence $B \otimes_\mathcal{O} C$ is a quotient of $B[[T_1, \ldots, T_n]]$ by a regular sequence, and the result follows from the discussion above applied to the maps $B[[T_1, \ldots, T_n]] \to B$ and $B[[T_1, \ldots, T_n]] \to B \otimes_\mathcal{O} C$ respectively. \qed
As an example, this applies to $B[\Delta]$ for any finite abelian group $\Delta$ of $p$-power order, since $\mathcal{O}[\Delta]$ is a complete intersection.

Let $\overline{\rho}: G_p \to \text{GL}_2(k)$ be unramified with $\overline{\rho}(\text{Frob}_p)$ scalar. Let $\tilde{R}^!$ denote the framed deformation ring of ordinary representations of weight $n$ over $\mathcal{O}$-algebras with fixed determinant (together with a Frobenius eigenvalue $\alpha$ acting on an “unramified quotient”) as in Theorem 3.15.

**Theorem 3.24.** $\tilde{R}^!$ is a complete normal local Cohen–Macaulay ring of relative dimension 4 over $\mathcal{O}$. Let $\omega_{\tilde{R}^!}$ denote the canonical module of $\tilde{R}^!$. Then $\dim_k \omega_{\tilde{R}^!}/m = 3$.

**Proof.** Following the previous discussion, to determine $\dim_k \omega_{\tilde{R}^!}/m$, it suffices to find a regular sequence of length 5 ($= \dim \tilde{R}^!$), take the quotient $B$, and compute $\dim_k B[m]$. Since $\tilde{R}^!$ is $\mathcal{O}$-flat, $\omega$ is regular, and thus we may choose $\omega$ as the first term of our regular sequence. Yet the method of Snowden shows that $\tilde{R}^! \otimes k$ is given by the following relations (the localization of $B_1$ at $b = (1,1)$ in the notation of *ibid.*):

$$m = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, n = \phi - \text{id} = \begin{pmatrix} \phi_1 & \phi_2 \\ \phi_3 & \phi_4 \end{pmatrix}, \beta = \alpha - 1, mn = \beta n, P_\phi(\alpha) = 0, m^2 = 0, \det(\phi) = 1$$

Explicitly, in terms of equations, this is given by the quotient of $k[[a,b,c,\phi_1,\phi_2,\phi_3,\phi_4,\beta]]$ by the following equations:

$$\phi_1 + \phi_4 + \phi_1 \phi_4 - \phi_2 \phi_3 = 0, \quad \beta^2 - (\phi_1 + \phi_4)\beta - (\phi_1 + \phi_4) = 0,$$

$$a\phi_1 + b\phi_3 = a\beta, a\phi_2 + b\phi_4 = b\beta, -a\phi_3 + c\phi_1 = c\beta, a\phi_4 - c\phi_2 = a\beta, a^2 + bc = 0.$$ 

We find that a regular sequence in this ring is given by the following elements:

$$\{a, b + \beta, c + \phi_1, \phi_2 + \phi_3\}.$$ 

Taking the quotient $B$ of $A$ by these elements, we arrive at the corresponding zero dimensional algebra $B$ given by the quotient of ring $k[[\phi_1, \phi_2, \phi_4, \beta]]$ by the following relations:

$$\phi_1 + \phi_4 + \phi_1 \phi_4 - \phi_2 \phi_3 = 0, \quad \beta^2 - (\phi_1 + \phi_4)\beta - (\phi_1 + \phi_4) = 0,$$

$$\beta \phi_2 = 0, \beta \phi_4 = \beta^2, \phi_1^2 = \phi_1 \beta, \phi_1 \phi_2 = 0, \beta \phi_1 = 0.$$ 

We compute that $B[m] = (\beta, \phi_4, \phi_3)$, and hence dim $B[m] = 3$. In fact,

$$B \simeq k[x, y, z]/(x^2, y^2, z^2, xy, xz, yz),$$

the simplest example of a non-Gorenstein ring.

From now until the end of Section 3.8 we let $\overline{\pi}: G_\mathcal{Q} \to \text{GL}_2(k)$ be an absolutely irreducible modular (= odd) representation of Serre conductor $N = N(\overline{\pi})$ and Serre weight $k(\overline{\pi})$ with $p + 1 \geq k(\overline{\pi}) \geq 2$. This is an abuse of notation as we have already fixed a representation $\overline{\rho}$ in Section 3.1 but we hope it will not lead to confusion. Assume that $\overline{\pi}$ has minimal conductor amongst all its twists at all other primes (one can always twist $\overline{\pi}$ to satisfy these condition.) One knows that $\overline{\pi}$ occurs as a the mod-$p$ reduction of a modular form of weight 2 and level $N^*$, where $N^* = N$ if $k = 2$ and $Np$ otherwise. Let $\mathbf{T}$ denote the ring of endomorphisms of $J_1(N^*)/\mathcal{Q}$ generated by the Hecke operators $T_l$ for all primes $l$ (including $p$), and let $\mathfrak{m}$ denote the maximal ideal of $\mathbf{T}$ corresponding to $\overline{\pi}$. Assume that $p \geq 3$.

**Theorem 3.25** (Multiplicity one or two). If $\overline{\pi}$ is either ramified at $p$ or unramified at $p$ and $\overline{\pi}(\text{Frob}_p)$ is non-scalar, then $J_1(N^*)[\mathfrak{m}] \simeq \overline{\pi}$, that is, $\mathfrak{m}$ has multiplicity one. If $\overline{\pi}$ is unramified at $p$ and $\overline{\pi}(\text{Frob}_p)$ is scalar, then $J_1(N^*)[\mathfrak{m}] \simeq \overline{\pi} \oplus \overline{\pi}$, that is, $\mathfrak{m}$ has multiplicity two.
Remark 3.26. By results of Mazur [Maz77] (Prop. 14.2), Mazur–Ribet [MR91] (Main Theorem), Gross [Gro90] (Prop. 12.10), Edixhoven [Edi92] (Thm. 9.2), and Buzzard [RS01], the theorem is known except in the case when $\rho$ is unramified at $p$ and $\overline{\rho}(\text{Frob}_p)$ is scalar. In this case, Wiese [Wie07] has shown that the multiplicity is always at least two. Thus our contribution to this result is to show that the multiplicity is exactly two in the scalar case.

Remark 3.27. It was historically the case that multiplicity one was an ingredient in modularity lifting theorems, e.g., Theorem 2.1 of [Wil95]. It followed that the methods used to prove such theorems required a careful study of the geometry of $J_1^1(N^*)$. However, a refinement of the Taylor–Wiles method due to Diamond showed that one could deduce multiplicity one in certain circumstances while simultaneously proving a modularity theorem (see [Dia97b]). Our argument is in the spirit of Diamond, where it is the geometry of a local deformation ring rather than $J_1^1(N^*)$ that is the crux of the matter.

Proof. Let $G$ denote the part of the $p$-divisible group of $J$ which is associated to $m$. By [Gro90], Prop 12.9, as well as the proof of Prop 12.10, recall there is an exact sequence of groups

$$0 \to T_pG^0 \to T_pG \to T_pG^e \to 0$$

which is stable under $T_m$. Moreover, $T_pG^e$ is free of rank one over $T_p$, and $T_pG^0 = \text{Hom}(T_pG^e, \mathbb{Z}_p)$.

We may assume that $\overline{\rho}$ is unramified at $p$ and $\overline{\rho}(\text{Frob}_p)$ is scalar. Thus $N^* = pN$. Let $M$ denote the largest factor of $N$ which is only divisible by the so called “harmless” primes, that is, the primes $v$ such that $\rho|I_v$ is absolutely irreducible or such that $\rho|G_v$ is absolutely irreducible and $v \not\equiv -1 \mod p$. Define the group $\Phi$ as follows:

$$\Phi := \mathbb{Z}_p \otimes \prod_{x|M}(\mathbb{Z}/x\mathbb{Z})^\times$$

$\Phi$ measures the group of Dirichlet characters (equivalently, characters of $G_{\mathbb{Q}}$) congruent to 1 mod $\varpi$ which preserve the space $\Gamma_1^1(N)$ under twisting (by assumption, $\overline{\rho}$ has minimal conductor amongst its twists, so an easy exercise shows that these are the only twists with this property). Extending $\mathcal{O}$ if necessary, we may assume that each character $\phi \in \hat{\Phi} = \text{Hom}(\Phi, \overline{\mathbb{Q}}_p^\times)$ is valued in $\mathcal{O}^\times$. Let $\phi \in \hat{\Phi}$, and let $\chi_\phi$ denote the character $\epsilon \cdot (\overline{\rho}^{-1})_\phi$ of $G_{\mathbb{Q}}$. For $v|N^*$ we define a quotient $R_v = R_{v,\phi}$ of the universal framed deformation ring with determinant $\chi_\phi$ of $\overline{\rho}|G_v$ as follows:

1. When $v = p$, $R_v = R_{v,\phi}$ is the ordinary framed deformation ring $R^\dagger$ of Section 3.7 (with $n = 2$).
2. When $v \neq p$, $R_v = R_{v,\phi}$ is the unrestricted framed deformation ring with determinant $\chi_\phi$.

The isomorphism types of these deformation rings do not depend on $\phi$. Let $R = R_{\phi}$ denote the (global) universal deformation ring of $\overline{\rho}$ corresponding to deformations with determinant $\chi_\phi$ which are unramified outside $N^*$ and which are classified (after a choice of framing) by $R_v$ for each $v|N^*$. Let $R^\square$ denote the framed version of $R$, with framings at each place $v|N^*$. Let $T^\text{an}_\phi$ be the anemic weight 2, level $\Gamma_1(N^*)$ ordinary Hecke algebra (so it does not contain $U_p$) which acts on

$$S_\phi := \bigoplus_{\chi} S_2^{\text{ord}}(\Gamma_1(N^*), \chi, \mathcal{O}),$$
where $\chi$ runs over all the characters of $(\mathbb{Z}/Np\mathbb{Z})^\times$ with $\chi|\Phi = \phi$. Note that
\[
S_2^{\text{corg}}(\Gamma_1(N^\times), \mathcal{O}) \otimes \mathbb{Q} = \bigoplus_{\phi} S_\phi \otimes \mathbb{Q}.
\]

The reason for dealing with the harmless primes in this manner is as follows. For all non-harmless $v \neq p$, the conductor-minimal deformation ring of $\mathfrak{p} G_v$ (which classifies deformations which appear at level $\Gamma_1(N)$) is isomorphic to the ring $R_{v, \phi}$ (up to unramified twists). However, the naïve conductor-minimal deformation deformation ring at a harmless prime is equal to the unrestricted deformation ring and does not have fixed inertial determinant if $v \equiv 1 \mod p$. However, one needs the determinant to be fixed for the Taylor–Wiles method to work correctly.

Consider the Galois representation $\rho : G_{\mathbb{Q}} \to \text{GL}_2(\mathbb{T}_{\phi,m}^{\text{an}})$ associated to eigenforms in $S_\phi$. The character $\epsilon^{-1} \det \rho$ can be regarded as a character $\chi : (\mathbb{Z}/Np\mathbb{Z}) \to (\mathbb{T}_{\phi,m}^{\text{an}})^\times$ with $\chi|\Phi = \phi$. Let $\psi$ denote the restriction of $\chi$ to $\mathcal{O}_{\mathfrak{p}} \otimes \prod_{x \mid M}(\mathbb{Z}/x\mathbb{Z})^\times$, which we may regard as a character of $G_{\mathbb{Q}}$. After twisting $\rho$ by $\psi^{-1/2}$, we obtain a Galois representation
\[
G_{\mathbb{Q}} \to \text{GL}_2(\mathbb{T}_{\phi,m}^{\text{an}})
\]
with determinant $\chi_\phi = \epsilon \cdot (\mathfrak{p}c^{-1})\phi$ which is classified by $R = R_{\phi}$. Kisin’s improvement of the Taylor-Wiles method yields an isomorphism $R_{\phi}[1/p] \simeq T_{\phi,m}^{\text{an}}[1/p]$. (Here we apply the Taylor-Wiles type patching results, Prop. 3.3.1 and Lemma 3.3.4 of [Kis09], except that the rings denoted $B$ and $D$ in the statements of these results may no longer be integral domains in our situation (though their generic fibres will be formally smooth over $\tilde{K}$ by Lemma 3.28 below). This is due to the fact that the rings $R_{v, \phi}$ defined above may have multiple irreducible components for certain $v \neq p$. On the other hand, the only place in [Kis09] where the assumption that $B$ and $D$ be integral domains is used is in the first paragraph of the proof of Lemma 3.3.4. In our case, it will suffice to show that each irreducible component of $R_{v, \phi}$ is in the support of $S_\phi$. This follows from now standard results on the existence of modular deformations which prescribed local inertial types.)

Now, $R^\square$ is (non-canonically) a power series ring over $R$, and is realized as a quotient of
\[
\left( R^\square \otimes_{\mathbb{Z}[1/N]} R_v \right) [[x_1, \ldots, x_n]] \to R^\square
\]
by a sequence of elements that can be extended to a system of parameters (this last fact follows from the proof of Prop 5.1.1 of [Sno] and the fact that $R$ is finite over $\mathcal{O}$). As a variant of this, we may consider deformations of $\mathcal{P}$ together with an eigenvalue $\alpha$ of Frobenius at $p$. Globally, this now corresponds to a modified global deformation ring $\tilde{R} = \tilde{R}_\phi$ and the corresponding framed version $\tilde{R}^\square$, where we now map to the full Hecke algebra $\mathbb{T}_m$. There are surjections:
\[
R_{\text{loc}}[[x_1, \ldots, x_n]] := \left( \tilde{R}^\square \otimes_{\mathbb{Z}[1/N]} R_v \right) [[x_1, \ldots, x_n]] \to \tilde{R}^\square \to \tilde{R} \to \tilde{R}/\pi.
\]
Since $\tilde{R}$ is finite over $\mathcal{O}$, it follows that $\tilde{R}/\pi$ is Artinian. Again, as in the proof of Prop 5.1.1 of [Sno] (see also Prop. 4.1.5 of [KW09b]), the kernel of the composition of these maps is given by a system of parameters, one of which is $\pi$. On the other hand, we have:

**Lemma 3.28.** The rings $R_v = R_{v, \phi}$ for $v \neq p$ are complete intersections. Moreover, their generic fibres $R_v[1/p]$ are formally smooth over $K$.
Proof. There are three cases in which \( R_v \) is not smooth. In two of these cases, we shall prove that \( R_v \) is a power series ring over \( \mathcal{O}[\Delta] \) for some finite cyclic abelian \( p \)-group \( \Delta \). Since \( \mathcal{O}[\Delta] \) is manifestly a complete intersection with formally smooth generic fibre, this suffices to prove the lemma in these cases. In the other case, we will show that \( R_v \) is a quotient of a power series ring by a single relation. This shows that it is a complete intersection. The three situations in which \( R_v \) is not smooth correspond to primes \( v \) such that:

1. \( v \equiv 1 \mod p \), \( \overline{p}G_v \) is reducible, and \( \overline{p}I_v \simeq \chi \oplus 1 \) for some ramified \( \chi \).
2. \( v \equiv -1 \mod p \), \( \overline{p}G_v \) is absolutely irreducible, and \( \overline{p}I_v \simeq \zeta \oplus \xi^p \).
3. \( v \equiv 1 \mod p \), \( \overline{p}^{\times} \) is 1-dimensional and \( \overline{p}^{\times}|G_v \) is unramified.

Suppose that \( v \) is a vexing prime (the second case). Any conductor-minimal deformation of \( \overline{p} \) is induced from a character of the form \( \langle \xi \rangle \psi \) over the quadratic unramified extension of \( \mathcal{O}_v \), where \( \psi \mod \varpi \) is trivial. It follows that \( \psi \) is tamely ramified, and in particular, up to twisting by Frobenius, it may be identified with a character of \( F_{v^2}^{\times} \) of \( p \)-power order. We may therefore write down the universal framed deformation explicitly, which identifies \( R_v \) with a power series ring over \( \mathcal{O}[\Delta] \), where \( \Delta \) is the maximal \( p \)-quotient of \( F_{v^2}^{\times} \).

Suppose that we are in the first case, and so, after an unramified twist, \( \overline{p}G_v \cong \chi \oplus 1 \). All \( R_v \)-deformations of \( \overline{p} \) are of the form \( \langle \chi \rangle \psi \otimes \psi^{-1} \otimes \langle \phi \rangle \langle \chi \rangle^{-1} \rangle^{1/2} \), where \( \psi \equiv 1 \mod \varpi \). It follows that \( \psi \) is tamely ramified, and in particular, decomposes as an unramified character and a character of \( F_{v^2}^{\times} \) of \( p \)-power order. We may therefore write down the universal framed deformation explicitly, which identifies \( R_v \) with a power series ring over \( \mathcal{O}[\Delta] \), where \( \Delta \) is the maximal \( p \)-quotient of \( F_{v^2}^{\times} \).

In the third case, we note that \( R_{v,\phi} \) is a quotient of a power series ring over \( \mathcal{O} \) in \( \dim Z^1(G_v, \text{ad}^0 \overline{p}) = 4 \) variables by at most \( \dim H^2(G_v, \text{ad}^0 \overline{p}) = 1 \) relation. Points on the generic fiber of \( R_{v,\phi} \) correspond to lifts \( \rho \) of \( \overline{p}G_v \) which are either unramified twists of the Steinberg representation or lifts which decompose (after inverting \( p \)) into a sum \( \chi \phi \psi \otimes \psi^{-1} \) with \( \psi|I_v \) of \( p \)-power order. The completion of \( R_v[1/p] \) at such a point is the corresponding characteristic 0 deformation ring of the lift \( \rho \). In each case, we have \( \dim H^2(G_v, \text{ad}^0 \rho) = 0 \) and hence this ring is a power series ring (over the residue field at the point) in \( \dim Z^1(G_v, \text{ad}^0 \rho) = 3 \) variables. It follows that \( R_v \cong \mathcal{O}[[x_1, x_2, x_3, x_4]]/(r) \) for some \( r \neq 0 \) and \( R_v[1/p] \) is formally smooth over \( K_{\varpi}^{c} \). This concludes the proof of the lemma.

By Lemma 3.23, it follows that \( R_{\text{loc}}[[x_1, \ldots, x_n]] \) is Cohen–Macaulay, and hence the sequence of parameters giving rise to the quotient \( \tilde{R}/\varpi \) is a regular sequence. In particular, \( \tilde{R} \) is Cohen–Macaulay and \( \varpi \)-torsion free. Moreover, again by Lemma 3.23, the number of generators of the canonical module of \( R_{\text{loc}}[[x_1, \ldots, x_n]] \) (and hence of \( \tilde{R} \)) is equal to the number of generators of the canonical module of \( \tilde{R}^{\dagger} \), which is 3, by Theorem 3.24. Since patching arguments may also be applied to the adorned Hecke algebras \( T_m \) (the extra data of \( U_p \) can easily be patched), the method of Kisin yields an isomorphism \( \tilde{R}[1/p] = T_{\phi,m}[1/p] \) (note that \( \tilde{R}^{\dagger} \) is a domain). Since (as proven above) \( \tilde{R} \) is \( \mathcal{O} \)-flat, it follows that \( \tilde{R} \simeq T_{\phi,m} \). In particular, we deduce that \( T_{\phi,m} \) is Cohen–Macaulay, and that \( \dim \omega T_{\phi,m}/m = 3 \). If \( T_{{\phi,m}} \) is the

\footnote{One may take \( r \) to be \( C(T) - T \), where \( C \) is the Chebyshev-type polynomial determined by the relation \( C(t + t^{-1}) = t^r + t^{-r} \), and \( T \) is the trace of a generator of tame inertia (note that \( T - 2 \in \mathfrak{m}_{R_v} \)). The generic fibre of \( R_v \) has \( (q + 1)/2 \) components, where \( q \) is the largest power of \( p \) dividing \( v - 1 \). One component corresponds to lifts of \( \overline{p} \) on which inertia is nilpotent, and in particular has trace \( T = 2 \). The remaining \( (q - 1)/2 \) components correspond to representations which are finitely ramified of order dividing \( q \), on which \( T = \zeta^i + \zeta^{-i} \) for some \( q \)-th root of unity \( \zeta \neq 1 \).}
Hecke ring at full level $\Gamma_1(N^*)$, then from the isomorphism

$$S_2(\Gamma_1(N^*), \mathcal{O})_m \otimes \mathbb{Q} = \bigoplus_{\phi} S_{\phi, m} \otimes \mathbb{Q} \simeq (S_{\phi, m} \otimes \mathbb{Q}) \otimes_{\mathcal{O}} \mathcal{O}[\Phi]$$

we deduce that $T_m \simeq T_{\phi, m} \otimes_{\mathcal{O}} \mathcal{O}[\Phi]$. Hence, applying Lemma 3.23 once more, we deduce that $T_m$ is Cohen–Macaulay and $\dim \omega_{T_m}/m = 3$

Since $T_m$ is finite over $\mathbb{Z}_p$, we deduce by Lemma 3.22 that $\text{Hom}(T_m, \mathbb{Z}_p)$ is the canonical module of $T_m$, and thus $\text{Hom}(T_m, \mathbb{Z}_p)/m$ also has dimension three. Yet we have identified this module with $T_p G^0$, and it follows that $\dim G^0[m] = 3$, and hence

$$\dim J_1(N^*)[m] = \frac{1}{2} \dim \mathcal{O}[\Phi] = \frac{1}{2} \dim G^0[m] = \frac{3 + 1}{2} = 2.$$

□

Remark 3.29. If $X_H(N^*) = X_1(N^*)/H$ is the smallest quotient of $X_1(N^*)$ where one might expect $\bar{\rho}$ to occur, a similar argument shows that $J_H(N^*)[m]$ has multiplicity two if $\bar{\rho}$ is unramified and scalar at $p$, and has multiplicity one otherwise, providing that $p \neq 3$ and $\bar{\rho}$ is not induced from a character of $\mathbb{Q}(\sqrt{-3})$. The only extra ingredient required is the result of Carayol (see [Car89], Prop. 3 and also [Edi97], Prop. 1.10).

These methods also yield the following alternate proof of the companion form result of Coleman–Voloch [CV92]:

Lemma 3.30 (Coleman–Voloch). Suppose that $f$ is an ordinary cuspidal eigenform on $X_1(N)$ of weight $k$ where $2 < k \leq p$. Then the representation $\bar{\rho}_f$ is tamely ramified above $p$ if and only if $f$ has a companion form.

Proof. The “if” part of this result is proved by Wiese in [Wie11]. It suffices to show that if $\bar{\rho} : G_{\mathbb{Q}} \to \text{GL}_2(k)$ is trivial at $p$, and $\bar{T}_{p, \tilde{m}}$ denotes the corresponding ordinary deformation ring in weight $p$ and level $\Gamma_1(N)$ (containing $U_p$), then $\bar{T}_{p, \tilde{m}} \neq T_{p, m}$. That this suffices follows, for example, by Prop. 6.2 of [Edi06], since if $\tilde{m} = m \bar{T}_{p, \tilde{m}}$, then Nakayama’s Lemma implies that $T_{p, m} \to \bar{T}_{p, \tilde{m}}$ is an isomorphism. If there was equality, then the global doubling ideal $\mathfrak{g}_{\text{glob}}$ would consist of the entire ring $\bar{R}^\dagger$. Yet, as in the proof of the previous theorem, we deduce that $\bar{T}_{p, \tilde{m}} \simeq \bar{R} \simeq R \otimes_{R^\dagger} \bar{R}^\dagger$. In particular, $\bar{R}/R = R \otimes_{R^\dagger} \bar{R}^\dagger / \bar{R}^\dagger \neq 0$. Yet $T_{p, m}$ is the image of $R$ in $\bar{R}$, it follows that $\bar{T}_{p, \tilde{m}}/T_{p, m}$ is also non-zero, and thus the global doubling ideal is a proper, a contradiction. □

Remark 3.31. We expect that these arguments should also apply in principle when $p = 2$; the key point is that one should instead use the quotient $\bar{R}^\dagger_3$ or $\bar{R}^\dagger$ (in the notation of [Sno], § 4), corresponding to crystalline ordinary deformations. The special fibre of $\bar{R}^\dagger_3$ is (in this case) also given by $B_1$, and thus one would deduce that the multiplicity of $\bar{\rho}$ is two when $\bar{\rho}$(Frob) is scalar, assuming that $\bar{\rho}$ is not induced from a quadratic extension. The key point to check is that the arguments above are compatible with the modifications to the $R = T$ method for $p = 2$ developed by Khare–Wintenberger and Kisin (in particular, this will require that $\bar{\rho}$ is not dihedral).
3.9. Modularity Lifting. We now return to the situation of Section 3.3. Taking $Q = 1$ in Theorem 3.8 we obtain a minimal deformation $\rho : G_{\overline{Q}} \to \text{GL}_2(T_{\emptyset, m_0})$ of $\overline{\rho}$ and hence a homomorphism $\varphi : R_{\text{min}} \to T_{\emptyset, m_0}$ which is easily seen to be surjective. Recall that Assumption 3.7 is still in force.

Theorem 3.32. The map $\varphi : R_{\text{min}} \to T_{\emptyset, m_0}$ is an isomorphism and $T_{\emptyset, m_0}$ acts freely on $H_0(X, \omega)_{m_0}$.

Proof. We view $H_0(X, \omega)_{m_0}$ as an $R_{\text{min}}$-module via $\varphi$. Since $\varphi$ is surjective, to prove the theorem, it suffices to show that $H_0(X, \omega)_{m_0}$ is free over $R_{\text{min}}$. To show this, we will apply Proposition 2.3.

We set $R = R_{\text{min}}$ and $H = H_0(X, \omega)_{m_0}$ and we define

$$q := \dim_k H_{\emptyset}^1(G_{\overline{Q}}, \text{ad} \overline{\rho}).$$

Note that $q \geq 1$ by Proposition 3.3. As in Proposition 2.3 we set $S_N = \mathcal{O}((\mathbb{Z}/p^N\mathbb{Z})^g)$ for each integer $N \geq 1$ and we let $R_{\infty}$ denote the power series ring $\mathcal{O}[[x_1, \ldots, x_{q-1}]]$. For each integer $N \geq 1$, fix a set of primes $Q_N$ of $\mathbb{Q}$ satisfying the properties of Proposition 3.3. We can and do fix a surjection $\phi_N : R_{\infty} \to R_{Q_N}$ for each $N \geq 1$. We let $\tilde{\phi}_N$ denote the composition of $\phi_N$ with the natural surjection $R_{Q_N} \to R_{\text{min}}$. Let

$$\Delta_{Q_N} = \prod_{x \in Q_N} (\mathbb{Z}/x)^\times$$

and choose a surjection $\Delta_{Q_N} \to \Delta := (\mathbb{Z}/p^N\mathbb{Z})^g$. Let $X_{\Delta_N}(Q_N) \to X_0(Q_N)$ denote the corresponding Galois cover. For each $x \in Q_N$, choose an eigenvalue $\alpha_x$ of $\overline{\rho}$(Frob$_x$). We let $T_{Q_N}$ denote the Hecke algebra denoted in Section 3.2.3 with the $Q$ of that section taken to be the current $Q_N$. We let $\mathfrak{m}$ denote the maximal ideal of $T_{Q_N}$ generated by $\mathfrak{m}_0$ and $U_x - \alpha_x$ for each $x \in Q_N$. We set $H_N := H_0(X_{\Delta_N}(Q_N), \omega)_{\mathfrak{m}}$. Then $H_N$ is naturally an $\mathcal{O}[\Delta_N] = S_N$-module. By Theorem 3.8, we deduce the existence of a surjective homomorphism $R_{Q_N} \to T_{Q_N, \mathfrak{m}}$. Since $T_{Q_N, \mathfrak{m}}$ acts on $H_N$, we get an induced action of $R_{\infty}$ on $H_N$ (via $\phi_N$ and the map $R_{Q_N} \to T_{Q_N, \mathfrak{m}}$). We can therefore view $H_N$ as a module over $R_{\infty} \otimes_{\mathcal{O}} S_N$.

To apply Proposition 2.3, it remains to check points (a)–(c). We check these conditions one by one:

(a) The image of $S_N$ in $\text{End}_{\mathcal{O}}(H_N)$ is contained in the image of $R_{\infty}$ by construction (see Theorem 3.8).

(b) As in the proof of Proposition 3.3, we have a Hochschild-Serre spectral sequence

$$\text{Tor}^S_i(H_j(X_{\Delta_N}(Q_N), \omega)_{\mathfrak{m}}, \mathcal{O}) \Rightarrow H_{i+j}(X_0(Q_N), \omega)_{\mathfrak{m}}.$$

We see that $(H_N)_{\Delta_N} \cong H_0(X_0(Q_N), \omega)_{\mathfrak{m}}$. Then, by Lemma 3.4 we obtain an isomorphism $(H_N)_{\Delta_N} \cong H_0(X_0, \omega)_{\mathfrak{m}} = H$, as required.

(c) The module $H_N$ is finite over $\mathcal{O}$ and hence over $S_N$. Proposition 3.6 implies that $d_{S_N}(H_N) \geq 0$.

We may therefore apply Proposition 2.3 to deduce that $H$ is free over $R$, completing the proof.

We now deduce Theorem 1.3 under Assumption 3.7 from the previous result. In the statement of Theorem 1.3, we take $X_{T'} = X = X_1(N)/H$ and $L_\varphi = \mathcal{O}_X$ and, as in the statement, we let $T$ be the Hecke algebra of $H^1(X, \omega)$ (generated by prime-to-$p$ Hecke operators) and $\mathfrak{m}$ the maximal ideal of $T$ corresponding to $\overline{\rho}'$. The Verdier duality isomorphism
$D : H_0(X, \omega) \sim H^1(X, \omega)$ which transforms the action of $T_x$ to the action of $T^*_x$ (see the discussion preceding Prop. 3.3) gives rise to an isomorphism $T_{\emptyset, m_0} \sim T_n$.

We also show that $H_0(X, \omega)_{m_0}$ has rank one as a $T_{\emptyset, m_0}$-module: this follows by multiplicity one for $GL(2)/\mathbb{Q}$ if $H_0(X, \omega_K)_{m_0}$ is non-zero. In the finite case, we argue as follows. By Nakayama’s lemma it suffices to show that $H^0(X, \omega_k(-\infty))[m_0]$ has dimension one. Since we have shown that $T_p \in T_{\emptyset, m_0}$, we may deduce this from the fact that $q$-expansion is completely determined by the Hecke eigenvalues $T_x$ for all $x$.

3.10. Vexing Primes. In this section, we detail the modifications to the previous arguments which are required to deal with vexing primes. To recall the difficulty, recall that a prime $x$ different from $p$ is vexing if:

1. $\overline{\rho}|D_x$ is absolutely irreducible.
2. $\overline{\rho}|I_x \simeq \xi \otimes \xi^c$ is reducible.
3. $x \equiv -1 \mod p$.

The vexing nature of these primes can be described as follows: in order to realize $\overline{\rho}$ automorphically, one must work with $\Gamma_1(x^n)$ structure where $x^n$ is the Artin conductor of $\overline{\rho}|D_x$. However, according to local Langlands, at such a level we also expect to see non-minimal deformations of $\overline{\rho}$, namely, deformations with $\rho|I_x \simeq \psi(\xi) \oplus \psi^{-1}(\xi^c)$, where $\psi$ is a character of $(F_x)^\times$ of $p$-power order. Diamond [Dia96] was the first to address this problem by observing that one can cut out a smaller space of modular forms by using the local Langlands correspondence. The version of this argument in [CDT99] can be explained as follows. By Shapiro’s Lemma, working with trivial coefficients at level $\Gamma(x^n)$ is the same as working at level prime to $x$ where one now replaces trivial coefficients $\mathcal{Z}$ by a local system $\mathcal{F}$ corresponding to the group ring of the corresponding geometric cover. In order to avoid non-minimal lifts of $\overline{\rho}$, one works with a smaller local system $\mathcal{F}_\sigma$ cut out of $\mathcal{F}$ by a representation $\sigma$ of the Galois group of the cover to capture exactly the minimal automorphic lifts of $\overline{\rho}$. The representation $\sigma$ corresponds to a fixed inertial type at $x$. In our setting (coherent cohomology) we may carry out a completely analogous construction. Thus, instead, we shall construct a vector bundle $L_\sigma$ on $X$. We then replace $H^*(X_1(N), \omega)$ by the groups $H^*(X_1(N), \omega \otimes L_\sigma)$. The main points to check are as follows:

1. The spaces $H^0(X_1(N), \omega^{\otimes n} \otimes L_\sigma)$ for $n \geq 1$ do indeed cut out the requisite spaces of automorphic forms.
2. This construction is sufficiently functorial so that all the associated cohomology groups admit actions by Hecke operators.
3. These cohomology groups inject into natural spaces of $q$-expansions.
4. This construction is compatible with arguments involving the Hochschild–Serre spectral sequence and Verdier duality.

We start by discussing some more refined properties of modular curves, in the spirit of §3.2. Let $S(\overline{\rho})$, $T(\overline{\rho})$ and $Q$ be as in Section 3.1. Let $P(\overline{\rho})$ denote the set of $x \in S(\overline{\rho}) - T(\overline{\rho})$ where $\overline{\rho}$ is ramified and reducible.

We will now introduce compact open subgroups $V \triangleleft U \subset GL_2(\mathbb{A}^\infty)$ and later we will fix a representation $\sigma$ of $U/V$ on a finite free $\mathcal{O}$ module $W_\sigma$. (In applications, $U$, $V$ and $\sigma$ will be chosen to capture all minimal modular lifts of $\overline{\rho}$. If the set of vexing primes $T(\overline{\rho})$ is empty, then $U = V$ and all minimal lifts of $\overline{\rho}$ will appear in $H^0(X_U, \omega)$. As indicated above, there is a complication if $T(\overline{\rho})$ is non-empty. In this case, minimal modular lifts of $\overline{\rho}$ will appear in the $\sigma^\ast := \text{Hom}(\sigma, \mathcal{O})$-isotypical part of $H^0(X_V, \omega)$.)
For each prime \( x \in S(\mathfrak{p}) \), let \( c_x \) denote the Artin conductor of \( \mathfrak{p}|G_x \). Note that \( c_x \) is even when \( x \in T(\mathfrak{p}) \). For \( x \in S(\mathfrak{p}) \), we define subgroups \( V_x \subset U_x \subset \text{GL}_2(\mathbb{Z}_x) \) as follows:

- If \( x \in P(\mathfrak{p}) \), we let
  \[
  U_x = V_x = \left\{ g \in \text{GL}_2(\mathbb{Z}_x) : g \equiv \begin{pmatrix} * & * \\ 0 & d \end{pmatrix} \mod x^{c_x}, d \in (\mathbb{Z}/x^{c_x})^\times \text{ has } p\text{-power order} \right\}.
  \]

- If \( x \in T(\mathfrak{p}) \), then let \( U_x = \text{GL}_2(\mathbb{Z}_x) \) and
  \[
  V_x = \ker \left( \text{GL}_2(\mathbb{Z}_x) \rightarrow \text{GL}_2(\mathbb{Z}/x^{c_x/2}) \right).
  \]

- If \( x \in S(\mathfrak{p}) - T(\mathfrak{p}) \cup P(\mathfrak{p}) \),
  \[
  U_x = V_x = \left\{ g \in \text{GL}_2(\mathbb{Z}_x) : g \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \mod x^{c_x} \right\}.
  \]

For \( x \) a prime not in \( S(\mathfrak{p}) \), we let
\[
U_x = V_x = \text{GL}_2(\mathbb{Z}_x)
\]
Finally, if \( x \) is any rational prime, we define subgroups \( U_{1,x} \subset U_{0,x} \subset \text{GL}_2(\mathbb{Z}_x) \) by:
\[
U_{0,x} = \left\{ g \in \text{GL}_2(\mathbb{Z}_x) : g \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod x \right\},
\]
\[
U_{1,x} = \left\{ g \in \text{GL}_2(\mathbb{Z}_x) : g \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \mod x \right\}.
\]

We now set
\[
U = \prod_x U_x, \quad U_i(Q) = \prod_{x \not\in Q} U_x \times \prod_{x \in Q} U_{i,x},
\]
\[
V = \prod_x V_x, \quad V_i(Q) = \prod_{x \not\in Q} V_x \times \prod_{x \in Q} U_{i,x},
\]
for \( i = 0, 1 \). For \( W \) equal to one of \( U, V, U_i(Q) \) or \( V_i(Q) \), we have a smooth projective modular curve \( X_W \) over \( \text{Spec}(\mathcal{O}) \) which is a moduli space of generalized elliptic curves with \( W \)-level structure.\(^3\) Let \( Y_W \subset X_W \) be the open curve parametrizing genuine elliptic curves and let \( j : Y_W \hookrightarrow X_W \) denote the inclusion. As in Section 3.2 we let \( \pi : \mathcal{E} \rightarrow X_W \) denote the universal generalized elliptic curve, we let \( \omega := \pi_*\Omega^1_{\mathcal{E}/X_W} \) and we let \( \infty \) denote the reduced divisor supported on the cusps. If \( M \) is an \( \mathcal{O} \)-module and \( \mathcal{L} \) is a sheaf of \( \mathcal{O} \)-modules on \( X_M \), then we denote by \( \mathcal{L}_M \) the sheaf \( \mathcal{L} \otimes_\mathcal{O} M \) on \( X_W \). If \( R \) is an \( \mathcal{O} \)-algebra, we will sometimes denote \( X_W \times_{\text{Spec}(\mathcal{O})} \text{Spec}(R) \) by \( X_{W,R} \).

There is a natural right action of \( U/V \) on \( X_V \) coming from the description of \( X_V \) as a moduli space of generalized elliptic curves with level structure ([DR73, §IV]). It follows from [DR73, IV 3.10] that we have \( X_V/(U/V) \tilde{\rightarrow} X_U \). Away from the cusps, the map \( Y_V \rightarrow Y_U \) is \( \text{étale and Galois with Galois group } U/V \) and the map \( X_V \rightarrow X_U \) is tamely ramified. Similar remarks apply to the maps \( X_{V_i}(Q) \rightarrow X_{U_i}(Q) \) for \( i = 0, 1 \).

The natural map \( X_{U_1}(Q) \rightarrow X_{U_0}(Q) \) is \( \text{étale and Galois with Galois group} \)
\[
\Delta_Q := \prod_{x \in Q} U_{0,x}/U_{1,x} \cong \prod_{x \in Q} (\mathbb{Z}/x)^\times.
\]

\(^3\)Again, in order to obtain a representable moduli problem, we may need to introduce auxiliary level structure at a prime \( q \) as in Section 3.3. This would be necessary if every prime in \( S(\mathfrak{p}) \) were vexing, for example.
3.10.1. Cutting out spaces of modular forms. Let $G = U/V = \prod_{T \in T} \text{GL}_2(\mathbb{Z}/x^e)^2$ and let $\sigma$ denote a representation of $G$ on a finite free $\mathcal{O}$-module $W_\sigma$. We will now proceed to define a vector bundle $L_\sigma$ on $X$ such that

$$H^0(X_U, \omega^{\otimes n} \otimes_{\mathcal{O}_X} L_\sigma) \simto (H^0(X_V, \omega^{\otimes n}) \otimes_{\mathcal{O}} W_\sigma)^G = \text{Hom}_{\mathcal{O}[G]}(W_\sigma, H^0(X_V, \omega^{\otimes n})),$$

where $W_\sigma^*$ is the $\mathcal{O}$-dual of $W_\sigma$. The sheaf $L_\sigma$ will thus allow us to extract the $W_\sigma^*$-part of the space of modular forms at level $V$.

To begin, we define a locally constant sheaf of $\mathcal{O}$-modules $F_\sigma$ on the small étale site of $Y_U$ as follows: if $Z \to Y_U$ is a finite étale cover of a Zariski open subset of $Y_U$, we set

$$F_\sigma(Z) = \{ f : \pi_0(Y_V \times Y_U) \to W_\sigma : f(Cg) = \sigma(g)^{-1}f(C) \text{ for each } g \in G \}.$$

The pullback of $F_\sigma$ to $X_{U_i(Q)}$ or $X_{V_j(Q)}$ for $i = 0, 1$ will also be denoted $F_\sigma$.

The sheaf $\mathcal{O}_{Y_U} \otimes_{\mathcal{O}} F_\sigma$ is locally free of finite rank on $Y_U$. We will now extend it to a locally free sheaf on $X_U$ as follows. First, we let $f$ denote the natural map $X_V \to X_U$. The sheaf $f^*F_\sigma$ is a $G$-equivariant constant sheaf with stalks isomorphic to $W_\sigma$ on $Y_V$ and hence $j_*(f^*F_\sigma)$ is $G$-equivariant constant sheaf on $X_V$ with stalks isomorphic to $W_\sigma$ (since $X_V$ is normal, for example). We now define a coherent sheaf $L_\sigma$ on $X_U$ by setting

$$L_\sigma := f_*(\mathcal{O}_{X_V} \otimes_{\mathcal{O}} j_*(f^*F_\sigma))^G.$$

For $i = 0, 1$ we denote the pull back of $L_\sigma$ to $X_{U_i(Q)}$ also by $L_\sigma$.

Lemma 3.33. Let $X$ denote $X_U$ (resp. $X_{U_i(Q)}$ for $i = 0$ or 1), let $X(V)$ denote $X_V$ (resp. $X_{V_j(Q)}$) and let $f$ denote the map $X(V) \to X$. Let $Y$ denote the non-cuspidal open subscheme of $X$. Then

1. The sheaf $L_\sigma$ is locally free of finite rank on $X$ and we have

$$L_\sigma|_Y \simto \mathcal{O}_Y \otimes_{\mathcal{O}} F_\sigma.$$

2. If $A$ is an $\mathcal{O}$-algebra and $V$ is a locally free sheaf of $\mathcal{O}_{X_A}$-modules, then

$$H^0(X_A, V \otimes L_\sigma) \simto (H^0(X(V)_A, f^*V) \otimes_{\mathcal{O}} W_\sigma)^G.$$

3. If $n > 1$, then

$$H^1(X, \omega^{\otimes n} \otimes L_\sigma) = (0)$$

and hence

$$H^0(X, \omega^{\otimes n} \otimes L_\sigma \otimes_{\mathcal{O}} \mathcal{O}/\omega^m) = (H^0(X(V), \omega^{\otimes n}) \otimes_{\mathcal{O}} W_\sigma)^G \otimes_{\mathcal{O}} \mathcal{O}/\omega^m.$$

Moreover, the analogous result holds for $n > 2$ if we replace $\omega^{\otimes n}$ by $\omega^{\otimes n}(-\infty)$.

Proof. First of all, consider the case where $X = X_U$. We have $L_\sigma|_Y = f_*(\mathcal{O}_{Y_V} \otimes_{\mathcal{O}} f^*F_\sigma)^G$ since $Y \to X$ is flat. It then follows from [Mum70, §III.12 Theorem 1 (B)] (and its proof) that $L_\sigma|_Y \simto \mathcal{O}_Y \otimes_{\mathcal{O}} F_\sigma$. To show that $L_\sigma$ is locally free of finite rank on $X$, it remains to check that its stalks at points of $X - Y$ are free. Let $x$ be a point of $X - Y$. We can and do assume that for each point $x'$ of $X_V$ lying above $x$, the natural map on residue fields is an isomorphism. We have

$$L_{\sigma,x} = \left( \bigoplus_{x' \to x} \mathcal{O}_{X_V, x'} \otimes W_\sigma \right)^G.$$
Choose some point \( x' \mapsto x \) and let \( I(x'/x) \subset G \) be the inertia group of \( x' \). Then projection onto the \( x' \)-component defines an isomorphism
\[
\left( \bigoplus_{x'' \mapsto x} \mathcal{O}_{X,v''} \otimes W_\sigma \right)^G \xrightarrow{\sim} (\mathcal{O}_{X,v} \otimes W_\sigma)^{I(x'/x)}.
\]

Now \( I(x'/x) \) is abelian of order prime to \( p \). Extending \( \mathcal{O} \), we may assume that each character \( \chi \) of \( I(x'/x) \) is defined over \( \mathcal{O} \). Let \( W_{\sigma,\chi} \) and \( \mathcal{O}_{X,v,x',\chi} \) denote the \( \chi \)-parts of \( W_\sigma \) and \( \mathcal{O}_{X,v,x'} \). Then \( W_\sigma \cong \bigoplus_\chi W_{\sigma,\chi} \) and similarly \( \mathcal{O}_{X,v,x'} \cong \bigoplus_\chi \mathcal{O}_{X,v,x',\chi} \). Each \( W_{\sigma,\chi} \) (resp. \( \mathcal{O}_{X,v,x',\chi} \)) is free over \( \mathcal{O} \) (resp. \( \mathcal{O}_{X,v} \)), being a summand of a free module. (Note that \( f \) is finite flat.) We now have
\[
\mathcal{L}_{\sigma,x} \xrightarrow{\sim} (\mathcal{O}_{X,v,x} \otimes W_\sigma)^{I(x'/x)} \xrightarrow{\sim} \bigoplus_\chi W_{\sigma,\chi} \otimes \mathcal{O}_{X,V,x',\chi^{-1}},
\]
which is free over \( \mathcal{O}_{X,x} \). This establishes part (1).

We now turn to part (2). We first of all note that \( X_{V,A}/G \xrightarrow{\sim} X_A \). (When \( A = \mathcal{O} \), this follows from \cite{DR73} IV Prop. 3.10). The same argument works when \( A = k \). These two cases, and the flatness of \( \mathcal{O}_{X,v} \) over \( \mathcal{O} \), imply the result when \( A = \mathcal{O}/\mathfrak{m}^n \). Then general result follows from this by \cite{KM85} Prop. A7.1.4.) From this we see that
\[
H^0(X_A, V \otimes \mathcal{L}_\sigma) \xrightarrow{\sim} H^0(X_{V,A}, f^*V \otimes f^*\mathcal{L}_\sigma)^G.
\]
It thus suffices to show that \( f^*\mathcal{L}_\sigma \cong \mathcal{O}_{X,v} \otimes \mathcal{O}_{W_\sigma} \) as a \( G \)-equivariant sheaf on \( X_V \).

Let \( \mathcal{M} = \mathcal{O}_{X,v} \otimes j_*(f^*\mathcal{F}_\sigma) \). Since \( j_*(f^*\mathcal{F}_\sigma) \) is the constant sheaf \( W_\sigma \) with \( G \)-equivariant structure coming from \( \sigma \), we have \( \mathcal{M} = \mathcal{O}_{X,v} \otimes W_\sigma \). We have a natural map
\[
f^*\mathcal{L}_\sigma = f^*(j_*(\mathcal{M}))^G \rightarrow \mathcal{M}
\]
and it suffices to prove that this natural map is an isomorphism. Again, by \cite{Mum70} III.12 Theorem 1 (B)], and its proof, it is certainly an isomorphism over \( Y_V \). Thus, it suffices to show that we have an isomorphism on stalks at each point of \( X_V - Y_V \). Let \( x' \) be a point of \( X_V - Y_V \) with image \( x \in X \). As above, we can and do assume that for each \( x'' \in X_V \) mapping to \( x \), the map on residue fields is an isomorphism. Then, \( (f^*\mathcal{L}_\sigma)_{x'} \cong \mathcal{O}_{X,V,x'} \otimes \mathcal{O}_{x,x} \mathcal{L}_{\sigma,x} \) and we saw above that \( \mathcal{L}_{\sigma,x} \cong (\mathcal{O}_{X,V,x} \otimes \mathcal{O}_{W_\sigma})^{I(x'/x)} \). The map \( (f^*\mathcal{L}_\sigma)_{x'} \rightarrow \mathcal{M}_{x'} \) is thus given by the natural map
\[
\mathcal{O}_{X_V,x'} \otimes \mathcal{O}_{x,x} \cong (\mathcal{O}_{X_V,x'} \otimes \mathcal{O}_{W_\sigma})^{I(x'/x)} \rightarrow \mathcal{O}_{X,V,x'} \otimes \mathcal{O}_{W_\sigma}.
\]

However, in the notation used above, we have
\[
(\mathcal{O}_{X_V,x'} \otimes \mathcal{O}_{W_\sigma})^{I(x'/x)} = \bigoplus_\chi \mathcal{O}_{X,V,x',\chi^{-1}} \otimes \mathcal{O}_{W_{\sigma,\chi}}.
\]
Moreover, each \( \mathcal{O}_{X,V,x',\chi} \) is free of rank 1 over \( \mathcal{O}_{x,x} \). Thus the map \( (f^*\mathcal{L}_\sigma)_{x'} \rightarrow \mathcal{M}_{x'} \) is given by the natural map
\[
\bigoplus_\chi (\mathcal{O}_{X,V,x'} \otimes \mathcal{O}_{W_{\sigma,\chi}}) \rightarrow \mathcal{O}_{X,V,x'} \otimes \mathcal{O}_{W_\sigma}
\]
which is certainly an isomorphism.

The second statement of part (3) follows immediately from the first and from part (2) by considering the long exact sequence in cohomology associated to the short exact sequence
\[
0 \rightarrow \omega^{\otimes n} \otimes \mathcal{O}_X \mathcal{L}_\sigma \xrightarrow{\omega^{\otimes m}} \omega^{\otimes n} \otimes \mathcal{O}_X \mathcal{L}_\sigma \rightarrow (\omega^{\otimes n} \otimes \mathcal{O}_X \mathcal{L}_\sigma)/\omega^{\otimes m} \rightarrow 0.
\]
To prove the first statement, it suffices to show that $H^1(X, \omega \otimes L_\sigma \otimes \mathcal{O}_k) = (0)$. By Serre duality, this is equivalent to the vanishing of $H^0(X_k, \omega \otimes (2-n)(-\infty) \otimes \mathcal{O}_X L_\sigma^*)$. By part 3.10.2, we have

$$H^0(X_k, \omega \otimes (2-n)(-\infty) \otimes \mathcal{O}_X L_\sigma^*) \sim (H^0(X_{V,k}, \omega \otimes (2-n)(-\infty) \otimes \mathcal{O} W_\sigma)^G$$

which vanishes since $n > 1$. The case where $\omega \otimes n$ is replaced by $\omega \otimes n(-\infty)$ is proved in exactly the same way.

It remains to consider the case where $X = X_{U_i(Q)}$ for $i = 0, 1$. However, this case can be treated in the same way as the case $X = X_U$. To see this, consider the Cartesian square:

$$\begin{array}{ccc}
X_{V_i(Q)} & \xrightarrow{g'} & X_V \\
\downarrow f' & & \downarrow f \\
X_{U_i(Q)} & \xrightarrow{g} & X
\end{array}$$

It suffices to show that

$$g^* L_\sigma \sim f'_*(O_{X_{V_i(Q)}} \otimes O j_*((f')^*(g^* F_\sigma)))^G.$$

Since $g$ is flat, we have

$$g^* L_\sigma = f'_*((g')^*(O_{X_V} \otimes O j_* (f^* F_\sigma)))^G = f'_*(O_{X_{V_i(Q)}} \otimes O (g')^* j_* (f^* F_\sigma))^G.$$

It is not hard to see that the natural map

$$(g')^* (j_* (f^* F_\sigma)) \longrightarrow j_* (g')^* (f^* F_\sigma) \cong j_* ((f')^* (g^* F_\sigma))$$

is an isomorphism. The desired isomorphism follows. \qed

3.10.2. Hecke operators. Having defined the sheaf $L_\sigma$, we now proceed to define Hecke operators on the spaces $H^i(U, \omega \otimes L_\sigma)$ and $H^i(U, \omega \otimes L_\sigma)$.

First of all, consider the case of $X := X_U$. For each rational prime $p \not\in S \cup \{p\}$, let $X_0(x)$ denote the modular curve over $O$ associated to $\prod_{y \neq x} U_x \times U_{0,x}$ and let $\pi_1 : X_0(x) \rightarrow X$ denote the map which forgets the level structure at $x$. Let $E$ denote the universal generalized elliptic curve over $X_0(x)$ and let $C \subset E[x]$ denote the universal subgroup of rank $x$. The Atkin-Lehner involution $w_x : X_0(x) \rightarrow X_0(x)$ is described away from the cusps by sending $(E, C)$ to $(E/C, E[x]/C)$. Let $\phi$ denote the isogeny $E \rightarrow E/C$ and let $\pi_2 = \pi_1 \circ w_x : X_0(x) \rightarrow X$. For each integer $n$, $\phi$ induces a morphism $\phi^n : \pi_1^* \omega \rightarrow \pi_1^* \omega$ over $Y_0(x)$ which extends over $X_0(x)$. Similarly, there is a morphism $\phi : \pi_1^* \omega \rightarrow \pi_1^* \omega$ induced by $tr(\phi)$. We recall that the classical Hecke operator $T_x$ on $H^i(U, \omega \otimes L_\sigma)$ is defined to be $\frac{1}{x} tr(\pi_1) \circ \phi^n \circ \pi_2^n$. Similarly, the operator $T_x^*$ is defined to be $\frac{1}{x} tr(\pi_2) \circ \phi \circ \pi_1^*$. The map $\pi_2^* \mathcal{F}_\sigma \rightarrow \pi_1^* \mathcal{F}_\sigma$ over $Y_0(x)$ which is defined as follows: let $Z \rightarrow Y_0(x)$ be étale. Then, we have

$$\pi_1^*(\mathcal{F}_\sigma)(Z) = \{ f : \pi_0(Y_{V,0}(x) \times Y_0(x) \rightarrow W_{\sigma} : f(Cg) = \sigma(g)^{-1} f(C) \text{ for each } g \in G \}$$

$$\pi_2^*(\mathcal{F}_\sigma)(Z) = \{ f : \pi_0(Y_{V,0}(x) \times w_x Y_0(x) \rightarrow W_{\sigma} : f(Cg) = \sigma(g)^{-1} f(C) \text{ for each } g \in G \}$$
(where in the second case, \( Y_{V,0}(x) \times_{w,Y_0(x)} Z \) is the fibre product of \( Z \to Y_0(x) \) and \( Y_{V,0}(x) \to Y_0(x) \)). The isomorphism \( \iota \) is induced by the \( G \)-equivariant isomorphism

\[
Y_{V,0}(x) \times_{Y_0(x)} Z \xrightarrow{w_x \times \text{id}} Y_{V,0}(x) \times_{w,Y_0(x)} Z.
\]

For \( i = 1, 2 \), consider the Cartesian square:

\[
\begin{array}{ccc}
X_{V,0}(x) & \xrightarrow{\pi'_i} & X_V \\
\downarrow{f'} & & \downarrow{f} \\
X_0(x) & \xrightarrow{\pi_i} & X
\end{array}
\]

We have an isomorphism

\[
\pi_i^* L_\sigma \xrightarrow{\sim} f'_i(O_{X_{V,0}(x)} \otimes \sigma(j_*(\phi^*(\pi_i^* F_\sigma))))^G
\]

(see the last paragraph of the proof of Lemma 3.33). The isomorphism \( \iota : \pi_2^* F_\sigma \xrightarrow{\sim} \pi_1^* F_\sigma \) induces a \( G \)-equivariant isomorphism \((\pi_2^*)^*(\sigma(j_* (\phi^* F_\sigma))) \xrightarrow{\sim} (\pi_1^*)^*(\sigma(j_* (\phi^* F_\sigma)))\) and hence an isomorphism

\[
\pi_2^* L_\sigma \xrightarrow{\sim} \pi_1^* L_\sigma
\]

which we also denote by \( \iota \).

If \( M \) is an \( \mathcal{O} \)-module, we now define Hecke operators \( T_x \) and \( T_x^* \) on \( H^i(X,(\omega^{\otimes n} \otimes_{\mathcal{O}_X} L_\sigma)_M) \) in the usual fashion:

\[
T_x = \frac{1}{x} \text{tr}(\pi_1) \circ (\phi^* \otimes \iota) \circ \pi_1 : H^i(X,(\omega^{\otimes n} \otimes_{\mathcal{O}_X} L_\sigma)_M) \to H^i(X,(\omega^{\otimes n} \otimes_{\mathcal{O}_X} L_\sigma)_M)
\]

\[
T_x^* = \frac{1}{x} \text{tr}(\pi_2) \circ (\phi^* \otimes \iota^{-1}) \circ \pi_1^* : H^i(X,(\omega^{\otimes n} \otimes_{\mathcal{O}_X} L_\sigma)_M) \to H^i(X,(\omega^{\otimes n} \otimes_{\mathcal{O}_X} L_\sigma)_M).
\]

For each integer \( a \) not divisible by any element of \( S \), there is an automorphism \( \alpha : X \to X \) which corresponds to the map which sends an elliptic curve \( E \) with level \( U \)-structure represented by \( \alpha : (\mathbb{Z}/N\mathbb{Z})^2 \to E[N] \) to \( \alpha \circ \text{diag}(a,a) \). We denote by \( \langle a \rangle \) the operator \( a^* \) on \( H^j(X, (\omega^{\otimes n} \otimes_{\mathcal{O}_X} L_\sigma)_M) \). We note that \( T_x^* = \langle x \rangle^{-1} T_x \) and we let \( \langle a \rangle^{-1} \) denote \( \langle a \rangle^{-1} \).

In the same way, we have operators \( T_x \) and \( \langle a \rangle \) on \( H^i(X_U(j),\omega^{\otimes n} \otimes L_\sigma)_M \) for each \( j = 0, 1 \), each prime \( x \notin S \cup Q \cup \{p\} \) and each integer \( a \) not divisible by any prime in \( S \cup Q \). In addition, we have the operators \( U_x \) and \( U_x^* \) on \( H^i(X_J(j),\omega^{\otimes n} \otimes L_\sigma)_M \) for \( x \in Q \) which are defined, as in the case of \( T_x \) above, by incorporating \( \iota : \pi_2^* \mathcal{L}_\sigma \xrightarrow{\sim} \pi_1^* \mathcal{L}_\sigma \) (where the \( \pi_i \) are the appropriate degeneracy maps) into the usual definition.

3.10.3. The proof of Theorem 1.3 in the presence of vexing primes. To complete the proof of Theorem 1.3, it suffices to note the various modifications which must be made to the argument. For vexing primes \( x \), let \( c_x \) denote the conductor of \( \mathfrak{p} \) (which is necessarily even). We define a \( \mathcal{O} \)-representation \( W_{\sigma_s} \) of \( \text{GL}_2(\mathbb{Z}/x^{c_s/2}\mathbb{Z}) \) to be the representation \( \sigma_x \) as in \( \S \) 5 of [CDT99]. The collection \( \sigma = (\sigma_x)_{x \in T(\mathfrak{p})} \) gives rise to a sheaf \( L_\sigma \) on \( X_U \) as above. Let \( T_{\emptyset} \) denote the ring of Hecke operators acting on \( H_0(X_U, \omega \otimes L_\sigma) \) generated by Hecke operators away from \( S(\overline{\mathfrak{p}}) \cup \{p\} \). The analogue of Theorem 3.32 is as follows:

**Theorem 3.34.** The map \( \varphi : R \to T_{\emptyset,m_0} \) is an isomorphism and \( T_{\emptyset,m_0} \) acts freely on \( H_0(X_U, \omega \otimes L_\sigma)_{m_0} \).
Proof. The proof is the same as the proof of Theorem 3.32, we indicate below the modifications that need to be made.

(1) (Lemma 3.4): Exactly the same argument shows that there is an isomorphism of Hecke modules:
\[ H^0(X_U, \omega_K \otimes L_\sigma)_m \cong H^0(X_{U_0}(Q), \omega_K \otimes L_\sigma)_m. \]

(2) (Proposition 3.6): We need to show that the \( O[\Delta] \)-module \( M = H^0(X_{\Delta}(Q), \omega \otimes L_\sigma) \) is balanced. With \( L = L_\sigma \), the proof proceeds in exactly the same manner, up to the point where it suffices to show that
\[ \dim_K H^0(X_{U_0}(Q), K, \omega \otimes L_\sigma)_m = \dim_K H^0(X_{U_0}(Q), K, \omega(-\infty) \otimes L_\sigma)_m. \]

By Lemma 3.33(2) and Serre duality, these spaces are both identified with spaces of classical modular forms of weight 1. As in the proof of Proposition 3.6, Serre duality introduces a twisting into the Hecke action, but this is resolved exactly as in the proof of \textit{ibid} by global twisting. (Note that the module \( W_\sigma \) is itself self-dual up to twist.)

(3) (q-expansions): Let \( A \) be an \( O \)-module and let \( x \) be a cusp of \( X_U \). Let \( x' \) be a cusp of \( X_V \) lying over \( x \). Then the proof of Lemma 3.33 shows that (after possibly extending \( O \)) the formal completion of \( L_\sigma \) at \( x \) is given by
\[ L_{\sigma,x}^\wedge \cong \bigoplus_{\chi} O_{X_{V'}, x'_{\chi}}^\wedge \otimes W_{\sigma, \chi^{-1}} \]
where \( \chi \) runs over the characters of the inertia group \( I(x'/x) \subset U/V \), which is abelian of order prime to \( p \). Choosing an isomorphism \( O_{X_U, x}^\wedge \cong O_{X_{V'}, x'_{\chi}}^\wedge \) for each \( \chi \) gives an isomorphism
\[ O_{X_U, x}^\wedge \otimes W_\sigma \cong L_{\sigma,x}^\wedge. \]
Combining this with the proof of the usual q-expansion principal for modular forms gives an injection
\[ H^0(X_U, \omega_A \otimes L_\sigma) \hookrightarrow A[[q]] \otimes W_\sigma. \]

(4) (Theorem 3.10): We need to prove that the representations \( \rho_{n, \alpha} \) constructed from \( H^0(X_{U_1}(Q), \omega^\otimes n \otimes L_\sigma \otimes O/\varpi^m)_m \) are minimal deformations of \( \overline{\pi} \) at primes dividing \( S \). This is exactly the point of the construction of the sheaves \( L_\sigma \), namely, that part \( 3 \) of Lemma 3.33 guarantees that this space is a quotient of the space cut out from \( H^0(X_{V_1}(Q), \omega^\otimes n)_m \) by the automorphic representations which, by local Langlands, give rise to minimal deformations (see [CDT99, Lemma 5.1.1]).

(5) (Existence of the operator \( T_p \)): It suffices to note that for each \( O \)-algebra \( A \) the operator \( T_p \) is independent of (and thus commutes with) the action of \( U/V \), and thus the action on \( H^0(X_V, \omega_A)_m \otimes W_\sigma \) preserves the \( U/V \)-invariants. Yet the \( U/V \) invariants are exactly \( H^0(X_U, \omega_A \otimes L_\sigma)_m \).

\[ \square \]

Theorem 1.3 follows from the previous result and Verdier duality as in Section 3.9. We remark that \( H_0(X, \omega \otimes L_\sigma)_{m_0} \) is of rank one over \( T_{0,m_0} \) when \( H_0(X, \omega_K \otimes L_\sigma)_{m_0} \) is non-zero. This follows from multiplicity one for \( \text{GL}(2) \) and [CDT99 Lemma 4.2.4(3)].
4. Imaginary Quadratic Fields

In this section, we apply our methods to Galois representations of regular weight over imaginary quadratic fields. The argument, formally, is very similar to what happens to weight one Galois representations over $G_\mathbb{Q}$. The most important difference is that we are not able to prove the existence of Galois representations associated to torsion classes in cohomology, and so our results are predicated on a conjecture that suitable Galois representations exist (Conjecture [A]).

4.1. Deformations of Galois Representations. Let $F$ be an imaginary quadratic field, and let $p \geq 3$ be a prime that is unramified in $F$. Suppose that $v|p$ is a place of $F$ and $A$ is an Artinian local $\mathcal{O}$-algebra. We say that a continuous representation $\rho : G_v \to \text{GL}_2(A)$ is finite flat if there is a finite flat group scheme $F/\mathcal{O}_F$ such that $\rho \cong F(\mathcal{T}_v)$ as $\mathbb{Z}_p[G_v]$-modules. We say that $\rho$ is ordinary if $\rho$ is conjugate in $\text{GL}_2(A)$ to a representation of the form

$$
\begin{pmatrix}
\epsilon \chi_1 & * \\
0 & \chi_2
\end{pmatrix}
$$

where $\chi_1$ and $\chi_2$ are unramified.

Let

$$
\overline{\rho} : G_F \to \text{GL}_2(k)
$$

be a continuous absolutely irreducible Galois representation. Let $S(\overline{\rho})$ denote the set of primes not dividing $p$ where $\overline{\rho}$ is ramified. Assume that $\overline{\rho}$ does not admit any vexing primes, i.e., if $\overline{\rho}$ is ramified at $x$ where $N_F/Q(x) \equiv -1 \mod p$, then either $\overline{\rho}|D_x$ is reducible or $\overline{\rho}|I_x$ is absolutely irreducible. After twisting by a global character, then, after possibly enlarging the set $S(\overline{\rho})$, we may assume that for all $x \in S(\overline{\rho})$ such that $\overline{\rho}|D_x$ is reducible, either $I_x$ acts unipotently, is of the form $1 \oplus \psi$, or acts through scalars via a tamely ramified character (over a field of class number $> 1$, one might not be able to globally twist away from such a situation). Let us suppose that $\overline{\rho}$ is either finite flat or ordinary for any prime $v|p$ in $\mathcal{O}_F$.

Let $Q$ denote a finite set of primes in $\mathcal{O}_F$ not containing any primes above $p$ and not containing any primes at which $\overline{\rho}$ is ramified. For objects $R$ in $\mathcal{C}_\mathcal{O}$, we may consider lifts $\rho : G_F \to \text{GL}_2(R)$ of $\overline{\rho}$ with the following properties:

1. $\text{det}(\rho) = \epsilon(\overline{\rho})^{-1}(\text{det}(\overline{\rho}))$.
2. If $v|p$, then $\rho \otimes R (R/m^n_R)|G_v$ is finite flat or ordinary for all $n \geq 1$.
3. If $v|p$ and $\overline{\rho}|G_v$ is finite flat, then $\rho \otimes R (R/m^n_R)|G_v$ is finite flat for all $n \geq 1$.
4. If $x \not\in Q \cup S(\overline{\rho}) \cup \{v|p\}$, then $\rho|G_x$ is unramified.
5. $S$ : If $x \in S(\overline{\rho})$, and $\overline{\rho}|I_x$ is unipotent, then $\rho|I_x$ is unipotent.
6. $P$ : If $x \in S(\overline{\rho})$, and $\overline{\rho}|I_x \cong 1 \oplus \psi$, then $\rho|I_x \cong 1 \oplus \langle \psi \rangle$.
7. $M$ : If $x \in S(\overline{\rho})$, $\overline{\rho}|D_x$ is irreducible, and $\overline{\rho}|I_x = \psi_1 \oplus \psi_2$ is reducible, then $\rho|I_x = \langle \psi_1 \rangle \oplus \langle \psi_2 \rangle$.
8. $\text{Sc}$ : If $x \in S(\overline{\rho})$, and $\overline{\rho}|I_x$ acts by scalars via a tamely ramified character $\psi$, then $\rho|I_x$ acts by a tamely ramified character. Equivalently, $\rho|I_x$ acts by $\langle \psi \rangle$. (The equivalence arises from the fact that we are fixing the determinant and $p \neq 2$.)
9. $H$ : If $\overline{\rho}|I_x$ is irreducible, then $\rho(I_x) \sim \overline{\rho}(I_x)$. (This also follows automatically from the determinant condition.)

(For $x \in S(\overline{\rho})$ we say that $\overline{\rho}|D_x$ is of type Special, Principal, Mixed, Scalar, or Harmless respectively if is of the type indicated above. Note that primes of type $M$ are called vexing by [Dia97a], but we have eliminated the most troublesome of the vexing primes, namely
In particular, the reduced tangent space of $R_Q$ (this follows from the proof of Theorem 2.41 of [DDT97]). If $Q = \emptyset$, we will sometimes denote $R_Q$ by $R^{\min}$. Let $H^1_Q(F, \text{ad}^0 \overline{\rho})$ denote the Selmer group defined as the kernel of the map

$$H^1(F, \text{ad}^0 \overline{\rho}) \longrightarrow \bigoplus_x H^1(F_x, \text{ad}^0 \overline{\rho})/L_{Q,x}$$

where $x$ runs over all primes of $F$ and

- $L_{Q,x} = H^1(G_x/I_x, (\text{ad}^0 \overline{\rho})^I_x)$ if $x \not\in Q \cup \{v|p\}$;
- $L_{Q,x} = H^1(F_x, \text{ad}^0 \overline{\rho})$ if $x \in Q$ and $x \nmid p$;
- $L_{Q,v} = H^1(F_v, \text{ad}^0 \overline{\rho})$ if $v|p$ and $v \not\in Q$;

(The group $H^1_1(F_v, \text{ad}^0 \overline{\rho})$ is defined as in §2.4 of [DDT97].) Let $H^1_Q(F, \text{ad}^0 \overline{\rho}(1))$ denote the corresponding dual Selmer group.

**Proposition 4.1.** The reduced tangent space $\text{Hom}(R_Q/\mathfrak{m}_Q, k[\epsilon]/\epsilon^2)$ of $R_Q$ has dimension at most

$$\dim_k H^1_Q(F, \text{ad}^0 \overline{\rho}(1)) - 1 + \sum_{x \in Q} \dim_k H^0(F_x, \text{ad}^0 \overline{\rho}(1)).$$

**Proof.** The argument follows along the exact lines of Corollary 2.43 of [DDT97]. The only difference in the calculation occurs at $v|p$ and at $v = \infty$. Specifically, when $v|p$ and $p$, the contribution to the Euler characteristic formula (Theorem 2.19 of [DDT97]) is

$$\sum_{v|p} (\dim_k H^1_1(F_v, \text{ad}^0 \overline{\rho}) - \dim_k H^0(F_v, \text{ad}^0 \overline{\rho})), $$

which, by Proposition 2.27 of [DDT97], is at most 2. However, the contribution at the prime at $\infty$ is $- \dim_k H^0(C, \text{ad}^0 \overline{\rho}) = -3$. When $p$ is inert, the contribution at $p$ is

$$\dim_k H^1_1(F_p, \text{ad}^0 \overline{\rho}) - \dim_k H^0(F_p, \text{ad}^0 \overline{\rho})$$

which is also at most 2 (see, for instance, Corollary 2.4.3 of [CHT08] and note that there is an inclusion $H^1(G_{F_p}/I_{F_p}, k) \subset H^1_1(F_p, \text{ad} \overline{\rho}) \cap H^1(F_p, k)$ where we view $k$ as the scalar matrices in $\text{ad} \overline{\rho}$).

Suppose that $N_{F/Q}(x) \equiv 1 \mod p$ and $\overline{\rho}(\text{Frob}_x)$ has distinct eigenvalues for each $x \in Q$. Then $H^0(F_x, \text{ad}^0 \overline{\rho})$ is one dimensional for $x \in Q$ and the preceding proposition shows that the reduced tangent space of $R_Q$ has dimension at most

$$\dim_k H^1_Q(F, \text{ad}^0 \overline{\rho}(1)) - 1 + \#Q.$$

By adjoining Taylor–Wiles primes to kill the dual Selmer group, one deduces the following.

**Proposition 4.2.** Let $q = \dim_k H^1_0(F, \text{ad}^0 \overline{\rho}(1))$ and suppose that $\overline{\rho}|G_{F(\zeta_q)}$ is absolutely irreducible. Then $q \geq 1$ and for any integer $N \geq 1$ we can find a set $Q_N$ of primes of $F$ such that

1. $\#Q_N = q$.
2. $N_{F/Q}(x) \equiv 1 \mod p^N$ for each $x \in Q_N$.
3. For each $x \in Q_N$, $\overline{\rho}$ is unramified at $x$ and $\overline{\rho}(\text{Frob}_x)$ has distinct eigenvalues.
4. $H^1_{Q_N}(F, \text{ad}^0 \overline{\rho}(1)) = (0)$.

In particular, the reduced tangent space of $R_{Q_N}$ has dimension at most $q - 1$ and $R_{Q_N}$ is a quotient of a power series ring over $\mathcal{O}$ in $q - 1$ variables.
Proof. That \( q \geq 1 \) follows immediately from Proposition 4.1. Now suppose that \( Q \) is a finite set of primes of \( F \) containing no primes dividing \( p \) and no primes where \( \overline{p} \) is ramified. Suppose that \( \overline{p}(\text{Frob}_x) \) has distinct eigenvalues and \( N_{F/Q}(x) \equiv 1 \mod p \) for each \( x \in Q \). Then we have an exact sequence

\[
0 \rightarrow H^1_{Q}(F, \text{ad}^0\overline{p}(1)) \rightarrow H^1_{\emptyset}(F, \text{ad}^0\overline{p}(1)) \rightarrow \bigoplus_{x \in Q} H^1(G_x/I_x, \text{ad}^0\overline{p}(1)).
\]

Moreover, for each \( x \in Q \), the space \( H^1(G_x/I_x, \text{ad}^0\overline{p}(1)) \) is one-dimensional over \( k \) and is isomorphic to \( \text{ad}^0\overline{p}/(\overline{p}(\text{Frob}_x) - 1)(\text{ad}^0\overline{p}) \) via the map which sends a class \( [\gamma] \) to \( \gamma(\text{Frob}_x) \). It follows that we may ignore condition (1): if we can find a set \( \hat{Q}_N \) satisfying conditions (2), (3) and (4), then \( \#\hat{Q}_N \geq q \) and by removing elements of \( \hat{Q}_N \) if necessary, we can obtain a set \( Q_N \) satisfying (1)–(4).

By the Chebotarev density theorem, it therefore suffices to show that for each non-zero class \( [\gamma] \in H^1_{\emptyset}(F, \text{ad}^0\overline{p}(1)) \), we can find an element \( \sigma \in G_F \) such that

- \( \sigma|_{G_F(\mathbb{Q}_p)} = 1 \);
- \( \overline{p}(\sigma) \) has distinct eigenvalues;
- \( \gamma(\sigma) \notin (\overline{p}(\sigma) - 1)(\text{ad}^0\overline{p}) \).

The existence of such a \( \sigma \) can be established exactly as in the proof of Lemma 2.5 of [Tay06]. \( \square \)

### 4.2. Homology of Arithmetic Quotients

Let \( \mathbb{A} \) denote the adeles of \( Q \), and \( \mathbb{A}_F \) the finite adeles. Similarly, let \( \mathbb{A}_F \) and \( \mathbb{A}_F^\infty \) denote the adeles and finite adeles of \( F \). Let \( G = \text{Res}_{F/Q}\text{GL}(2) \), and write \( G_\infty = \text{G}(\mathbb{R}) = \text{GL}_2(\mathbb{C}) \). Let \( A \) be a maximal \( Q \)-split torus in the centre of \( G \), and let \( A^0_\mathbb{Q} \) denote the connected component of the real points of \( A \). Furthermore, let \( K_\infty \) denote a maximal compact of \( G_\infty \) with connected component \( K_\infty^0 \). For any compact open subgroup \( K \) of \( G(\mathbb{A}_\infty) \), we may define an arithmetic manifold \( Y(K) \) as follows:

\[
Y(K) := G(\mathbb{Q})\backslash G(\mathbb{A})/A^0_\mathbb{Q}K_\infty K.
\]

We will specifically be interested in the following \( K \). Let \( S(\overline{p}) \) and \( Q \) be as above.

#### 4.2.1. Arithmetic Quotients

If \( v \) is a place of \( F \) and \( c \geq 1 \) is an integer, we define

\[
\Gamma_0(v^c) = \left\{ g \in \text{GL}_2(\mathbb{O}_v) \mid g \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod \pi_v^c \right\}
\]

\[
\Gamma_1(v^c) = \left\{ g \in \text{GL}_2(\mathbb{O}_v) \mid g \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \mod \pi_v^c \right\}
\]

\[
\Gamma_p(v^c) = \left\{ g \in \text{GL}_2(\mathbb{O}_v) \mid g \equiv \begin{pmatrix} * & * \\ 0 & d \end{pmatrix} \mod \pi_v^c, \ d \text{ has } p\text{-power order} \right\}
\]

\[
\Gamma_\delta(v) = \left\{ g \in \text{GL}_2(\mathbb{O}_v) \mid g \equiv \begin{pmatrix} \alpha & * \\ 0 & \alpha^{-1} \end{pmatrix} \mod \pi_v \right\}
\]

Let \( K_Q = \prod_v K_{Q,v} \) and \( L_Q = \prod_v L_{Q,v} \) denote the open compact subgroups of \( G(\mathbb{A}) \) such that:

1. If \( v \in Q \), \( K_{Q,v} = \Gamma_1(v) \).
2. If \( v \notin Q \), \( L_{Q,v} = \Gamma(v) \).
3. If \( v \) is not in \( S(\overline{p}) \cup \{v|p\} \cup Q \), then \( K_{Q,v} = L_{Q,v} = \text{GL}_2(\mathbb{O}_v) \).
(4) If \( v \mid p \), then \( K_{Q,v} = L_{Q,v} = \text{GL}_2(\mathcal{O}_v) \) if \( \mathfrak{p} \mid D_v \) is finite flat. Otherwise, \( K_{Q,v} = L_{Q,v} = \Gamma_0(v) \).

(5) If \( v \in S(\mathfrak{p}) \), \( K_{Q,v} = L_{Q,v} \) is defined as follows:
   a) If \( \mathfrak{p} \) is of type \( S \) at \( v \), then \( K_{Q,v} = \Gamma_0(v) \).
   b) If \( \mathfrak{p} \) is of type \( P, M \) or \( H \) at \( v \), then \( K_{Q,v} = \Gamma_p(v^c) \), where \( c \) is the conductor of \( \mathfrak{p} \).
   c) If \( \mathfrak{p} \) is of type \( SC \) at \( v \), then \( K_{Q,v} = \Gamma_\delta(v) \).

We define the arithmetic quotients \( Y_0(Q) \) and \( Y_1(Q) \) to be \( Y(L_Q) \) and \( Y(K_Q) \) respectively.

4.2.2. Hecke Operators. We recall the construction of the Hecke operators. Let \( g \in \mathcal{G}(\mathbb{A}^\infty) \) be an invertible matrix. For \( K \subset \mathcal{G}(\mathbb{A}^\infty) \) a compact open subgroup, the Hecke operator \( T(g) \) is defined on the homology modules \( H_\bullet(Y(K), \mathcal{Z}) \) by considering the composition:

\[
H_\bullet(Y(K), \mathcal{Z}) \to H_\bullet(gKg^{-1} \cap K, \mathcal{Z}) \to H_\bullet(Y(K \cap g^{-1}Kg), \mathcal{Z}) \to H_\bullet(Y(K), \mathcal{Z}),
\]

the first map coming from the corestriction map, the second coming from the map \( Y(gKg^{-1} \cap K, \mathcal{Z}) \to Y(K \cap g^{-1}Kg, \mathcal{Z}) \) induced by right multiplication by \( g \) on \( \mathcal{G}(\mathbb{A}) \) and the third coming from the natural map on homology. The Hecke operators act on \( H_\bullet(Y(K), \mathcal{Z}) \) but do not preserve the homology of the connected components. Indeed, the action on the component group \( A_K := F^\times \backslash \mathbb{A}^\infty_F^\times / \text{det}(K) \) of \( Y(K) \) is via the determinant map on \( \mathcal{G}(\mathbb{A}^\infty) \) and the natural action of \( \mathbb{A}^\infty_F^\times \) on \( A_K \). For \( \alpha \in \mathbb{A}^\infty_F^\times \), we define the Hecke operator \( T_\alpha \) by taking

\[
g = \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}.
\]

We may define the diamond operators \( \langle \alpha \rangle \) via the action of the centre: \( \mathbb{A}^\infty_F^\times = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \) inside \( \mathcal{G}(\mathbb{A}^\infty) \). (The operator \( \langle \alpha \rangle \) acts non-trivially on the component group via the element \( \alpha^2 \in \mathbb{A}^\infty_F^\times \).) If \( K = K_Q \) or \( L_Q \) and \( x \in Q \), then we set \( U_x := T_{\pi_x} \) (which does not depend on the choice of \( \pi_x \)). We now define the Hecke algebra.

Definition 4.3. Let \( T_{\mathcal{Q}}^{\alpha} \) denote the subring of \( \text{End} H_1(Y_1(Q), \mathcal{O}) \) generated by Hecke endomorphisms \( T_\alpha \) for all \( \alpha \) away from primes in \( Q \cup S(\mathfrak{p}) \cup \{v|p\} \), and by \( \langle \alpha \rangle \) for all \( \alpha \) coprime to the level. Let \( T_{\mathcal{Q}} \) denote the \( \mathcal{O} \)-algebra generated by the same operators together with \( U_x \) for \( x \in Q \). If \( Q = \emptyset \), we write \( T = T_\emptyset \) for \( T_{\mathcal{Q}} \).

If \( \epsilon \in \mathcal{O}_F^\times \) is a global unit, then \( T_\epsilon \) acts by the identity. If \( \mathfrak{a} \subset \mathcal{O}_F \) is an ideal, we may, therefore, define the Hecke operator \( T_\alpha \) as \( T_\alpha \) where \( \alpha \in \mathbb{A}^\infty_F^\times \) represents the ideal \( \mathfrak{a} \).

4.3. Conjectures on Existence of Galois Representations. Let \( \mathfrak{m} \) denote a maximal ideal of \( T_{\mathcal{Q}} \), and let \( T_{\mathcal{Q},\mathfrak{m}} \) denote the completion. It is a local ring which is finite (but not necessarily flat) over \( \mathcal{O} \).

Definition 4.4. We say that \( \mathfrak{m} \) is Eisenstein if \( T_\alpha - 2 \in \mathfrak{m} \) for all but finitely primes which split completely in some fixed abelian extension of \( F \). We say that \( \mathfrak{m} \) is non-Eisenstein if it is not Eisenstein.
We say that $m$ is associated to $\rho$ if for each $\lambda \not\in S(\overline{\rho}) \cup \{v|p\}$, we have $T_{\lambda} - \text{Trace}(\overline{\rho}(\text{Frob}_{\lambda})) \in m$ and $\langle \lambda \rangle_{N_F/Q}(\lambda) - \det(\overline{\rho}(\text{Frob}_{\lambda})) \in m$.

**Conjecture A.** If $m$ is non-Eisenstein and is associated to $\overline{\rho}$, then there exists a continuous Galois representation $\rho = \rho_{m} : G_F \to \text{GL}_2(T_{Q,m})$ with the following properties:

1. If $\lambda \not\in S(\overline{\rho}) \cup \{v|p\}$ is a prime of $F$, then $\rho$ is unramified at $\lambda$, and the characteristic polynomial of $\rho(\text{Frob}_{\lambda})$ is
   \[ X^2 - T_{\lambda}X + \langle \lambda \rangle_{N_F/Q}(\lambda) \in T_{Q,m}[X]. \]

2. If $v \in S(\overline{\rho})$, then:
   (a) If $\overline{\rho}D_v$ is of type $S$, then $\overline{\rho}|I_v$ is unipotent.
   (b) If $\overline{\rho}D_v$ is of type $P$, so that $\overline{\rho}|I_v \cong 1 \oplus \psi$, then $\rho|I_v \cong 1 \oplus \langle \psi \rangle$.
   (c) If $\overline{\rho}D_v$ is of type $SC$, then $\rho|I_v$ acts by a tamely ramified character.
3. If $v \in Q$, then $\rho|D_v \cong \chi_1 \oplus \chi_2$ where $\chi_1$ is unramified and $\chi_1(\text{Frob}_{v}) = U_v$.
4. If $v|p$, then $\overline{\rho}D_v$ is ordinary or finite flat, and is finite flat if $\overline{\rho}D_v$ is finite flat.

Some form of this conjecture has been suspected to be true at least as far back as the investigations of F. Grunewald in the early 70's (see [Gru72, GHM78]). Related conjectures about the existence of $\overline{\rho}_m$ were made for GL$(n)/Q$ by Ash [Ash92], and for GL$(2)/F$ by Figueiredo [Fig99].

**Lemma 4.5.** Assume Conjecture A. Assume that there exists a maximal ideal $m$ of $T_Q$ associated to $\overline{\rho}$. Then there exists a representation: $\rho_Q : G_F \to \text{GL}_2(T_{Q,m})$ whose traces generate $T_{Q,m}$ such that $\rho_Q$ is a minimal deformation of $\overline{\rho}$ outside $Q$.

**Proof.** By Conjecture A the representation $\rho_Q := \rho_m$ is such a representation up to the determinant. We let $\rho_Q := \rho_Q \otimes \eta$, where $\eta^2 = \epsilon(\overline{\rho})^{-1}(\det(\overline{\rho}))\det(\rho_Q)^{-1}$. (Note that $\det(\rho_Q)$ is equal to $\epsilon(\epsilon)^{-1}(\det(\overline{\rho}))$ up to a finite $p$-power order character which is unramified at all $v \not\in Q$. This follows from Conjecture A (11) and the definition of $K_Q$.)

4.3.1. **Properties of homology groups.** Let $m_0$ denote a non-Eisenstein maximal ideal of $T_0$. We have natural homomorphisms

\[ T_{Q}^\text{an} \to T_0^\text{an} = T_0, \quad T_{Q}^\text{an} \hookrightarrow T_Q \]

induced by the map $H_1(Y_1(Q), O) \to H_1(Y, O)$ and by the natural inclusion. The ideal $m_0$ of $T_0$ pulls back to an ideal of $T_Q^\text{an}$ which we also denote by $m_0$ in a slight abuse of notation. The ideal $m_0$ may give rise to multiple maximal ideals $m$ of $T_Q$.

If $x \not\in S(\overline{\rho}) \cup \{v|p\}$ is a prime of $F$ such that $N_{F/Q}(x) \equiv 1 \mod p$ and $\overline{\rho}(\text{Frob}_{x})$ has distinct eigenvalues, then the representation $\overline{\rho}G_x$ does not admit ramified semistable deformations. The following lemma is the homological manifestation of this fact.

**Lemma 4.6.** Suppose that for each $x \in Q$ we have that $N_{F/Q}(x) \equiv 1 \mod p$ and that the polynomial $X^2 - T_xX + x \in T_0[x]$ has distinct eigenvalues modulo $m_0$. Let $m$ denote the maximal ideal of $T_Q$ containing $m_0$ and $U_x - \alpha_x$ for some choice of root $\alpha_x$ of $X^2 - T_xX + x$ mod $m$ for each $x \in Q$. Then there is an isomorphism

\[ H_1(Y_0(Q), O)_m \cong H_1(Y, O)_{m_0}. \]

**Proof.** The argument proceeds exactly as in the proof of Lemma 3.1 (note that $x \equiv 1 \mod m$). We deduce that there is a decomposition

\[ H_1(Y_0(x), O)_m = H_1(Y, O)_{m_0} \oplus V. \]
By the universal coefficient theorem, we deduce that there is also an isomorphism

\[ H^1(Y_0(x), K/O)_m = H^1(Y, K/O)_{\mathfrak{m}_\mathfrak{g}} \otimes V^\vee. \]

Any \( v \in V^\vee \) will generated a twist of a Steinberg representation (as in Lemma \[3.4\]) from which we may deduce a contradiction as in that case. \( \square \)

There is a natural covering map \( Y_1(Q) \to Y_0(Q) \) with Galois group

\[ \Delta_Q := \prod_{x \in Q} (O_F/x)^\times. \]

If \( \mu \) is a finitely generated \( O[\Delta_Q] \)-module, it gives rise to a local system on \( Y_0(Q) \). We have an action of \( T^\text{univ} \) (the polynomial algebra defined as in Section \[3.2.3\]) on the homology groups \( H_i(Y_0(Q), \mu) \) and the Borel-Moore homology groups \( H^B_i(Y_0(Q), \mu) \). The ideal \( \mathfrak{m}_\mathfrak{g} \) gives rise to a maximal ideal \( \mathfrak{m} \) of \( T^\text{univ} \) after a choice of eigenvalue mod \( \mathfrak{m} \) for \( U_x \) for all \( x \) dividing \( Q \).

**Lemma 4.7.** Let \( \mu \) be a finitely generated \( O[\Delta_Q] \)-module. Then:

1. \( H_i(Y_0(Q), \mu)_\mathfrak{m} = (0) \) for \( i = 0, 3 \).
2. If \( \mu \) is \( p \)-torsion free, then \( H_2(Y_0(Q), \mu)_\mathfrak{m} \) is \( p \)-torsion free.
3. For all \( i \), we have an isomorphism

\[ H_i(Y_0(Q), \mu)_\mathfrak{m} \cong H^B_i(Y_0(Q), \mu)_\mathfrak{m}. \]

**Proof.** Consider part (1). By dévissage, we reduce to the case when \( \mu = k \). Yet \( H_3(Y_0(Q), k) = 0 \) and the action of Hecke operators on \( H_0(Y_0(Q), k) \) which preserve the connected components is via the degree map, which is Eisenstein. For part (2), since \( \mu \) is \( O \)-flat (by assumption), there is an exact sequence

\[ 0 \to \mu \to \mu \to \mu/\varpi \to 0. \]

Taking cohomology, localizing at \( \mathfrak{m} \), and using the vanishing of \( H_3(Y_0(Q), \mu)_\mathfrak{m} \) from part (1), we deduce that \( H^2(Y_0(Q), \mu)_\mathfrak{m}[\varpi] = 0 \), hence the result. For part (3), there is a long exact sequence

\[ \ldots \to H_i(\partial Y_0(Q), \mu) \to H_i(Y_0(Q), \mu) \to H^B_i(Y_0(Q), \mu) \to H_{i-1}(\partial Y_0(Q), \mu) \to \ldots \]

from which we observe that it suffices to show that \( H_i(\partial Y_0(Q), \mu)_\mathfrak{m} \) vanishes for all \( i \). By dévissage, we once more reduce to the case when \( \mu = k \). The cusps are given by tori (specifically, elliptic curves with CM by some order in \( O_F \)), and since the cohomology with constant coefficients of tori is torsion free, the case when \( \mu = k \) reduces to the case when \( \mu = O \) and then \( \mu = K \). Yet the action of \( T^\text{univ} \) on the cusps in characteristic zero given by Grossencharacters for the field \( F \) (see [Har77], Theorem 1, p.56), and this action is Eisenstein. \( \square \)

We let \( \Delta \) denote a quotient of \( \Delta_Q \) and \( Y_\Delta(Q) \to Y_0(Q) \) the corresponding Galois cover. Further suppose that \( \Delta \) is a \( p \)-power order quotient of \( \Delta_Q \). Then \( O[\Delta] \) is a local ring. Note that by Shapiro’s Lemma there is an isomorphism \( H_1(Y_0(Q), O[\Delta]) \cong H_1(Y_\Delta(Q), O) \).

**Proposition 4.8.** The \( O[\Delta] \)-module \( H_1(Y_0(Q), O[\Delta])_\mathfrak{m} \cong H_1(Y_\Delta(Q), O)_\mathfrak{m} \) is balanced (in the sense of Definition \[2.3\]).

**Proof.** The argument is almost identical to the proof of Proposition \[3.6\]. Let \( M \) denote the module \( H_1(Y_0(Q), O[\Delta])_\mathfrak{m} \) and \( S = O[\Delta] \). Consider the exact sequence of \( S \)-modules (with trivial \( \Delta \)-action):

\[ 0 \to O \xrightarrow{\varpi} O \to k \to 0 \]
where \( \varpi \) denotes a uniformizer in \( \mathcal{O} \). Tensoring this exact sequence over \( S \) with \( M \), we obtain an exact sequence:
\[
0 \to \text{Tor}^S_1(M, \mathcal{O})/\varpi \to \text{Tor}^S_1(M, k) \to M_\Delta \to M \otimes_S k \to 0.
\]
Let \( r \) denote the \( \mathcal{O} \)-rank of \( M_\Delta \). Then this exact sequence tells us that
\[
d(S)(M) = \dim_k M \otimes_S k - \dim_k \text{Tor}^S_1(M, k) = r - \dim_k \text{Tor}^S_1(M, \mathcal{O})/\varpi.
\]
We have a Hochschild–Serre spectral sequence
\[
H_i(\Delta, H_j(Y_0(Q), S)) = \text{Tor}^S_i(H_j(Y_0(Q), S), \mathcal{O}) \implies H_{i+j}(Y_0(Q), \mathcal{O}).
\]
Localizing at \( \mathfrak{m} \), and using the fact that \( H_i(Y_0(Q), S)_{\mathfrak{m}} = (0) \) for \( i = 0, 3 \) by Lemma 4.7(1), we obtain an exact sequence
\[
(H_2(Y_0(Q), S)_{\mathfrak{m}})_\Delta \to H_2(Y_0(Q), \mathcal{O})_{\mathfrak{m}} \to \text{Tor}^S_1(M, \mathcal{O}) \to 0.
\]
To show that \( d(S)(M) \geq 0 \), we see that it suffices to show that \( H_2(Y_0(Q), \mathcal{O})_{\mathfrak{m}} \) is free of rank \( r \) as an \( \mathcal{O} \)-module. By Lemma 4.7(2), it then suffices to show that \( \dim_k H_2(Y_0(Q), K)_{\mathfrak{m}} = r \). Inverting \( \varpi \) and applying Hochschild–Serre again, we obtain isomorphisms
\[
(H_i(Y_0(Q), S \otimes_\mathcal{O} K)_{\mathfrak{m}})_\Delta \sim H_i(Y_0(Q), K)_{\mathfrak{m}}
\]
for \( i = 1, 2 \). It follows that \( r = \dim_k H_1(Y_0(Q), K)_{\mathfrak{m}} \). By Poincaré duality, we have
\[
\dim_k H_2(Y_0(Q), K)_{\mathfrak{m}} = \dim_k H_1^{BM}(Y_0(Q), K)_{\mathfrak{m}}.
\]
Finally, by Lemma 4.7(3) we have
\[
\dim_k H_1(Y_0(Q), K)_{\mathfrak{m}} = \dim_k H_1^{BM}(Y_0(Q), K)_{\mathfrak{m}},
\]
as required. \( \square \)

4.4. Modularity Lifting. We now associate to \( \mathfrak{p} \) the ideal \( \mathfrak{m}_\emptyset \) of \( T_\emptyset \) which is generated by \( (\varpi, T_\lambda - \text{Trace}(\overline{\varphi}(\text{Frob}_\lambda)), (\lambda)N_{F/\mathbb{Q}}(\lambda) - \det(\overline{\varphi}(\text{Frob}_\lambda))) \) where \( \lambda \) ranges over all primes \( \lambda \not\in S(\mathfrak{p}) \cup \{v|p\} \) of \( F \). We make the hypothesis that \( \mathfrak{m}_\emptyset \) is a proper ideal of \( T_\emptyset \). In other words, we are assuming that \( \overline{\varphi} \) is ‘modular’ of minimal level and trivial weight. Since \( T_\emptyset/\mathfrak{m}_\emptyset \to k \) it follows that \( \mathfrak{m}_\emptyset \) is maximal. Since \( \overline{\varphi} \) is absolutely irreducible, it follows by Chebotarev density that \( \mathfrak{m}_\emptyset \) is non-Eisenstein.

We now assume that Conjecture \( \mathcal{A} \) holds for \( \mathfrak{m}_\emptyset \). In other words, there is a continuous Galois representation
\[
\rho_{\mathfrak{m}} : G_F \to \text{GL}_2(T_{\mathfrak{m}_\emptyset})
\]
satisfying the properties of Conjecture \( \mathcal{A} \). The definition of \( \mathfrak{m}_\emptyset \) and the Chebotarev density theorem imply that \( \rho_{\mathfrak{m}_\emptyset} \mod \mathfrak{m}_\emptyset \) is isomorphic to \( \overline{\varphi} \). Properties (1)–(4) of Conjecture \( \mathcal{A} \) then imply that \( \rho_{\mathfrak{m}_\emptyset} \) gives rise to a homomorphism
\[
\varphi : R_{\mathfrak{m}_\emptyset}^{\text{min}} \to T_{\mathfrak{m}_\emptyset}
\]
such that the universal deformation pushes forward to \( \rho_{\mathfrak{m}_\emptyset} \). The following is the main result of this section.

**Theorem 4.9.** If we make the following assumptions:
1. the ideal \( \mathfrak{m}_\emptyset \) is a proper ideal of \( T_\emptyset \), and
2. Conjecture \( \mathcal{A} \) holds for all \( Q \),
then the map \( \varphi : R_{\mathfrak{m}_\emptyset}^{\text{min}} \to T_{\mathfrak{m}_\emptyset} \) is an isomorphism and \( T_{\emptyset, \mathfrak{m}_\emptyset} \) acts freely on \( H_1(Y, \mathcal{O})_{\mathfrak{m}_\emptyset} \).
Proof. By property (1) of Conjecture A the map \( \varphi : R_{\min} \to T_{0,m_0} \) is surjective. To prove the theorem, it therefore suffices to show that \( H_1(Y, \mathcal{O})_{m_0} \) is free over \( R_{\min} \) (where we view \( H_1(Y, \mathcal{O})_{m_0} \) as an \( R_{\min} \)-module via \( \varphi \)). To show this, we will apply Proposition 2.3.

We set \( R = R_{\min} \) and \( H = H_1(Y, \mathcal{O})_{m_0} \) and define

\[
q := \dim_k H_0^1(G_F, \text{ad}^0\mathfrak{p}).
\]

Note that \( q \geq 1 \) by Proposition 4.2. As in Proposition 2.3, we set \( \Delta_\infty = \mathbb{Z}_p^d \) and let \( \Delta_N = (\mathbb{Z}/p^N\mathbb{Z})^d \) for each integer \( N \geq 1 \). We also let \( R_\infty \) denote the power series ring \( \mathcal{O}[[x_1, \ldots, x_{q-1}]] \). It remains to show that condition 5 of Proposition 2.3 is satisfied. For this we will use the existence of Taylor–Wiles primes together with the results established in Section 4.2.

For each integer \( N \geq 1 \), fix a set of primes \( Q_N \) of \( F \) satisfying the properties of Proposition 4.2. We can and do fix a surjection \( \tilde{\phi}_N : R_\infty \to R_{Q_N} \) for each \( N \geq 1 \). We let \( \phi_N \) denote the composition of \( \tilde{\phi}_N \) with the natural surjection \( R_{Q_N} \to R_{\min} \). Let

\[
\Delta_{Q_N} = \prod_{x \in Q_N} (\mathcal{O}_F/x)^{\times}
\]

and choose a surjection \( \Delta_{Q_N} \to \Delta_N \). Let \( Y_{\Delta_N}(Q_N) \to Y_0(Q_N) \) denote the corresponding Galois cover. We set \( H_N := H_1(Y_{\Delta_N}(Q_N), \mathcal{O})_{m_0} \) where \( m_0 \) is the ideal of \( T_{Q_N} \) which contains \( m_0 \) and \( U_x - \alpha_x \) for each \( x \in Q \), for some choice of \( \alpha_x \). Then \( H_N \) is naturally an \( \mathcal{O}[\Delta_N] = S_N \)-module. Applying Conjecture A to \( T_{Q_N} \), we deduce the existence of a surjective homomorphism \( R_{Q_N} \to T_{Q_N,m} \). Since \( T_{Q_N,m} \) acts on \( H_N \), we get an induced action of \( R_\infty \) on \( H_N \) (via \( \tilde{\phi}_N \)) and the map \( R_{Q_N} \to T_{Q_N,m} \). We can therefore view \( H_N \) as a module over \( R_\infty \otimes_{\mathcal{O}} S_N \). To apply Proposition 2.3, it remains to check points (a)–(c) in part 5. We check these conditions one by one:

(a) The image of \( S_N \) in \( \text{End}_\mathcal{O}(H_N) \) is contained in the image of \( R_\infty \) by Conjecture A because it is given by the image of the diamond operators.

(b) We have a Hochschild-Serre spectral sequence

\[
\text{Tor}^{S_N}_i(H_j(Y_{\Delta_N}(Q_N), \mathcal{O})_{m_0}) \Rightarrow H_{i+j}(Y_0(Q_N), \mathcal{O})_{m_0}.
\]

Applying part (1) of Lemma 4.7, we see that \( (H_N)_{\Delta_N} \cong H_1(Y_0(Q_N), \mathcal{O})_{m_0} \). Then, by Lemma 4.6, we see that \( (H_N)_{\Delta_N} \cong H_1(Y, \mathcal{O})_{m_0} = H \), as required.

(c) \( H_N \) is finite over \( \mathcal{O} \) and hence over \( S_N \). Proposition 4.8 implies that \( d_{S_N}(H_N) \geq 0 \).

We may therefore apply Proposition 2.3 to deduce that \( H \) is free over \( R \) and the theorem follows. \( \square \)

If \( H_1(Y, \mathcal{O})_{m_0} \otimes \mathbb{Q} \neq 0 \), then we may deduce that the multiplicity \( \mu \) for \( H \) as a \( T_{0,m_0} \)-module is one by multiplicity one for \( \text{GL}(2)/F \). The proof also exhibits \( T_{0,m_0} \) as a quotient of a power series ring in \( q-1 \) variables by \( q \) elements. In particular, if \( \dim(T_{0,m_0}) = 0 \), then \( T_{0,m_0} \) is a complete intersection. From these remarks we see that Theorem 4.2 follows from Theorem 4.9.

Remark 4.10. Our methods may easily be modified to prove an \( R_{\min} = T_m \) theorem for ordinary representations in weights other than weight zero (given the appropriate modification of Conjecture A). In weights which are not invariant under the Cartan involution (complex conjugation), one knows a priori for non-Eisenstein ideals \( m \) that \( T_m \) is finite. Note that in
this case it is sometimes possible to prove unconditionally that $R_{\text{min}}[1/p] = 0$, see Theorem 1.4 of [Cal11].

5. Other Potential Applications, and Various Remarks

One technical tool that is conspicuously absent in this paper is the technique of solvable base change. When proving modularity results for $GL(2)$ over totally real fields, for example, one may pass to a finite solvable extension to avoid various technical issues, such as level lowering (see [SW01]). However, if $F$ is an imaginary quadratic field, then every non-trivial extension $H/F$ has at least two pairs of complex places, and the corresponding invariant $l_0 = \text{rank}(G) - \text{rank}(K)$ for $GL(2)/H$ is at least 2 (more precisely, it is equal to the number of complex places of $H$). This means that our techniques are mostly confined to the approach used originally by Wiles, Taylor–Wiles, and Diamond ([Wil95, TW95, Dia97b]).

Our techniques also apply to some other situations in which $l_0 = 1$ (the Betti case). One may, for example, consider 2-dimensional representations over a field $F$ with one complex place. If $[F : \mathbb{Q}]$ is even, there exists an inner form for $GL(2)/F$ which is compact at all real places of $F$, and the corresponding arithmetic quotient is a finite volume arithmetic hyperbolic manifold which is compact if $[F : \mathbb{Q}] > 2$. (If $[F : \mathbb{Q}]$ is odd, one would have to require that $\overline{p}$ be ramified with semi-stable reduction at at least one prime $\lambda \not\mid p$.) None the less, we obtain minimal lifting theorems in these cases, modulo Conjecture A. Similarly, our methods immediately produce minimal lifting theorems for $GL(3)/\mathbb{Q}$, modulo an appropriate version of Conjecture A. Similarly, our methods should also apply to other situations in which $\pi_{\infty}$ is a holomorphic limit of discrete series (the Coherent case). One case to consider would be odd ordinary irreducible Galois representations $\rho : G_F \to GL_2(\mathbb{Q}_p)$ of a totally real field $F$ which conjecturally arise from Hilbert modular forms exactly one of whose weights is one. Other examples of particular interest include the case in which $\rho : G_\mathbb{Q} \to GSp_4(\mathbb{Q}_p)$ is the Galois representation associated to an abelian surface $A/\mathbb{Q}$, or $\rho : G_E \to GL_3(\mathbb{Q}_p)$ is the Galois representation associated to a Picard Curve (see the appendix to [Til06]). We hope to return to these examples in future work.

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