THE BINARY QUASIORDER ON SEMIGROUPS

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Abstract. Given two elements \( x, y \) of a semigroup \( X \) we write \( x \lessdot y \) if for every homomorphism \( \chi : X \to \{0, 1\} \) we have \( \chi(x) \leq \chi(y) \). The quasiorder \( \lessdot \) is called the binary quasiorder on \( X \). It induces the equivalence relation \( \equiv \) that coincides with the least semilattice congruence on \( X \). In the paper we discuss some known and new properties of the binary quasiorder on semigroups.

1. Introduction

In this paper we study the binary quasiorder on semigroups. Every semigroup carries many important quasiorders (for example, those generated by the Green relations). One of them is the binary quasiorder \( \lessdot \) defined as follows. Given two elements \( x, y \) of a semigroup \( X \) we write \( x \lessdot y \) if \( \chi(x) \leq \chi(y) \) for any homomorphism \( \chi : X \to \{0, 1\} \). On every semigroup \( X \) the binary quasiorder generates a congruence, which coincides with the least semilattice congruence, and decomposes the semigroup into a semilattice of semilattice-indecomposable semigroups. This fundamental decomposition result was proved by Tamura [34] (see also [25], [26], [37]). Because of its fundamental importance, the least semilattice congruence has been deeply studied by many mathematicians, see the papers [15], [16], [17], [18], [23], [29], [30], [31], [32], [25], [26], [33], [35], [36], surveys [22], [24], and monographs [13], [21], [27]. The aim of this paper is to provide a survey of known and new results on the binary quasiorder and the least semilattice congruence on semigroups. The obtained results will be applied in the theory of categorically closed semigroups developed by the first author in collaboration with Serhii Bardyla, see [3], [4], [5], [6], [7].

2. Preliminaries

In this section we collect some standard notions that will be used in the paper. We refer to [19] for Fundamentals of Semigroup Theory.

We denote by \( \omega \) the set of all finite ordinals and by \( \mathbb{N} \defeq \omega \setminus \{0\} \) the set of all positive integer numbers.

A **semigroup** is a set endowed with an associative binary operation. A semigroup \( X \) is called a **semilattice** if \( X \) is commutative and every element \( x \in X \) is an idempotent which means \( xx = x \). Each semilattice \( X \) carries the natural partial order \( \leq \) defined by \( x \leq y \) iff \( xy = x \). For a semigroup \( X \) we denote by \( E(X) \defeq \{x \in X : xx = x\} \) the set of idempotents of \( X \).

Let \( X \) be a semigroup. For an element \( x \in X \) let

\[
x^n \defeq \{x^n : n \in \mathbb{N}\}
\]

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be the monogenic subsemigroup of \( X \), generated by the element \( x \). For two subsets \( A, B \subseteq X \), let \( AB \defeq \{ ab : a \in A, \, b \in B \} \) be the product of \( A, B \) in \( X \).

For an element \( a \) of a semigroup \( X \), the set
\[
H_a = \{ x \in X : (x X^1 = a X^1) \land (X^1 x = X^1 a) \}
\]
is called the \( H \)-class of \( a \). Here \( X^1 = X \cup \{ 1 \} \) where \( 1 \) is an element such that \( 1 x = x = x 1 \) for all \( x \in X^1 \). By Corollary 2.2.6 [19], for every idempotent \( e \in E(X) \) its \( H \)-class \( H_e \) coincides with the maximal subgroup of \( X \), containing the idempotent \( e \).

### 3. The Binary Quasiorder

In this section we discuss the binary quasiorder on a semigroup and its relation to the least semilattice congruence.

Let \( 2 \) denote the set \( \{ 0, 1 \} \) endowed with the operation of multiplication inherited from the ring \( \mathbb{Z} \). It is clear that \( 2 \) is a two-element semilattice, so it carries the natural partial order, which coincides with the linear order inherited from \( \mathbb{Z} \).

For elements \( x, y \) of a semigroup \( X \) we write \( x \preceq y \) if \( \chi(x) \leq \chi(y) \) for every homomorphism \( \chi : X \to 2 \). It is clear that \( \preceq \) is a quasiorder on \( X \). This quasiorder will be referred to as the binary quasiorder on \( X \). The obvious order properties of the semilattice \( 2 \) imply the following (obvious) properties of the binary quasiorder on \( X \).

**Proposition 3.1.** For any semigroup \( X \) and any elements \( x, y, a \in X \), the following statements hold:

1. if \( x \preceq y \), then \( ax \preceq ay \) and \( xa \preceq ya \);
2. \( xy \preceq yx \preceq xy \);
3. \( x \preceq x^2 \preceq x \);
4. \( xy \preceq x \) and \( xy \preceq y \).

For an element \( a \) of a semigroup \( X \) and subset \( A \subseteq X \), consider the following sets:
\[
\uparrow a \defeq \{ x \in X : a \preceq x \}, \quad \downarrow a \defeq \{ x \in X : x \preceq a \}, \quad \ll a \defeq \{ x \in X : a \preceq x \preceq a \},
\]
called the upper \( 2 \)-class, lower \( 2 \)-class and the \( 2 \)-class of \( x \), respectively. Proposition 3.1 implies that those three classes are subsemigroups of \( X \).

For two elements \( x, y \in X \) we write \( x \ll y \) iff \( y \ll x \). Proposition 3.1 implies that \( \ll \) is a congruence on \( X \).

We recall that a congruence on a semigroup \( X \) is an equivalence relation \( \equiv \) on \( X \) such that for any elements \( x \equiv y \) of \( X \) and any \( a \in X \) we have \( ax \equiv ay \) and \( xa \equiv ya \). For any congruence \( \equiv \) on a semigroup \( X \), the quotient set \( X/\equiv \) has a unique semigroup structure such that the quotient map \( X \to X/\equiv \) is a semigroup homomorphism. The semigroup \( X/\equiv \) is called the quotient semigroup of \( X \) by the congruence \( \equiv \).

A congruence \( \equiv \) on a semigroup \( X \) is called a semilattice congruence if the quotient semigroup \( X/\equiv \) is a semilattice. Proposition 3.1 implies that \( \ll \) is a semilattice congruence on \( X \). The intersection of all semilattice congruences on a semigroup \( X \) is a semilattice congruence called the least semilattice congruence, denoted by \( \eta \) in [19], [20] (by \( \xi \) in [35], [22], and by \( \rho_0 \) in [13]). The minimality of \( \eta \) implies that \( \eta \subseteq \ll \). The inverse inclusion \( \ll \subseteq \eta \) will be deduced from the following (probably known) theorem on extensions of \( 2 \)-valued homomorphisms.

**Theorem 3.2.** Let \( \pi : X \to Y \) be a surjective homomorphism from a semigroup \( X \) to a semilattice \( Y \). For every subsemilattice \( S \subseteq Y \) and homomorphism \( f : \pi^{-1}[S] \to 2 \) there exists a homomorphism \( F : X \to 2 \) such that \( F|_{\pi^{-1}[S]} = f \).
Proof. We claim that the function $F : X \to 2$ defined by

$$F(x) = \begin{cases} 1 & \text{if } \exists z \in \pi^{-1}[S] \text{ such that } \pi(xz) \in S \text{ and } f(xz) = 1; \\ 0 & \text{otherwise;} \end{cases}$$

is a required homomorphism extending $f$.

To see that $F$ extends $f$, take any $x \in \pi^{-1}[S]$. If $f(x) = 1$, then for $z = x$ we have $\pi(xz) = \pi(x)\pi(z) = \pi(x) = \pi(x) \in S$ and $f(xz) = f(x)f(z) = f(x)f(x) = 1$ and hence $F(x) = 1 = f(x)$. If $F(x) = 1$, then there exists $z \in \pi^{-1}[S]$ such that $\pi(xz) \in S$ and $f(x)f(z) = f(xz) = 1$, which implies that $f(x) = 1$. Therefore, $F(x) = 1$ if and only if $f(x) = 1$. Since 2 has only two elements, this implies that $f = F|_{\pi^{-1}[S]}$.

To show that $F$ is a homomorphism, fix any elements $x_1, x_2 \in X$. We should prove that $F(x_1x_2) = F(x_1)F(x_2)$.

First assume that $F(x_1x_2) = 0$. If $F(x_1)$ or $F(x_2)$ equals 0, then $F(x_1)F(x_2) = 0$ and we are done. So, assume that $F(x_1) = 1 = F(x_2)$. Then the definition of $F$ yields elements $z_1, z_2 \in \pi^{-1}[S]$ such that $\pi(x_1z_1) \in S$ and $f(x_1z_1) = 1$ for every $i \in \{1, 2\}$. Now consider the element $z = z_1z_2 \in \pi^{-1}[S]$ and observe that

$$\pi(x_1x_2z) = \pi(x_1x_2z_1z_2) = \pi(x_1)\pi(x_2)\pi(z_1)\pi(z_2) = \pi(x_1z_1)\pi(x_2z_2) \in S$$

and $f(z) = f(z_1z_2) = f(z_1)f(z_2) = 1 \cdot 1 = 1$ and hence $F(x_1x_2) = 1$ by the definition of $F$. By this contradicts our assumption.

Next, assume that $F(x_1x_2) = 1$. Then there exists $z \in \pi^{-1}[S]$ such that $\pi(x_1x_2z) \in S$ and $f(x_1x_2z) = 1$. Let $z' = x_1x_2z \in \pi^{-1}[S]$ and observe that for every $i \in \{1, 2\}$ we have $\pi(x_i z) = \pi(x_i)\pi(x_2)\pi(z) = \pi(x_1)\pi(x_2)\pi(z) = \pi(x_1x_2z) \in S$. It follows from $1 = f(x_1x_2z) = f(x_1)f(x_2)f(z) = f(x_1)f(x_1)f(x_2)f(z)$ that $f(x_1) = 1 = f(z')$ and hence $F(x_1) = 1$. Then $F(x_1)F(x_2) = 1 = F(x_1x_2)$, which completes the proof. \qed

Corollary 3.3. Any homomorphism $f : S \to 2$ defined on a subsemilattice $S$ of a semilattice $X$ can be extended to a homomorphism $F : X \to 2$.

Proof. Apply Theorem 3.2 to the identity homomorphism $\pi : X \to X$. \qed

Corollary 3.3 implies the following important fact, first noticed by Petrich [25, 26] and Tamura [35].

Theorem 3.4. The congruence $\updownarrow$ on any semigroup $X$ coincides with the least semilattice congruence on $X$.

Proof. Let $\eta$ be the least semilattice congruence on $X$ and $\eta(\cdot) : X \to X/\eta$ be the quotient homomorphism assigning to each element $x \in X$ its equivalence class $\eta(x) \in X/\eta$. We need to prove that $\eta(x) = \updownarrow x$ for all $x \in X$. Taking into account that $\updownarrow$ is a semilattice congruence and $\eta$ is the least semilattice congruence on $X$, we conclude that $\eta \leq \updownarrow$ and hence $\eta(x) \leq \updownarrow x$ for all $x \in X$. Assuming that $\eta \neq \updownarrow$, we can find elements $x, y \in X$ such that $x \updownarrow y$ but $\eta(x) \neq \eta(y)$. Consider the subsemilattice $S = \{\eta(x), \eta(y), \eta(x)\eta(y)\}$ of the semilattice $X/\eta$. It follows from $\eta(x) \neq \eta(y)$ that $\eta(x)\eta(x) \neq \eta(x)$ or $\eta(x)\eta(y) \neq \eta(y)$. Replacing the pair $x, y$ by the pair $y, x$, we can assume that $\eta(x)\eta(y) \neq \eta(y)$. In this case the unique function $h : S \to 2$ with $h^{-1}(1) = \{\eta(y)\}$ is a homomorphism. By Corollary 3.3, the homomorphism $h : S \to 2$ can be extended to a homomorphism $H : X/\eta \to 2$. Then the composition $\chi \defeq H \circ \eta(\cdot) : X \to 2$ is a homomorphism such that $\chi(x) = 0 \neq 1 = \chi(y)$, which implies that $\updownarrow x \neq \updownarrow y$. But this contradicts the choice of the points $x, y$. This contradicton completes the proof of the equality $\updownarrow = \eta$. \qed
A semigroup \( X \) is called 2-trivial if every homomorphism \( h : X \to 2 \) is constant. Tamura \[35\], \[36\] calls 2-trivial semigroups semilattice-indecomposable (or briefly s-indecomposable) semigroups.

Theorem \[3,2\] implies the following fundamental fact first proved by Tamura \[34\] and then reproved by another method in \[37\], see also \[25\], \[26\].

**Theorem 3.5 (Tamura).** For every element \( x \) of a semigroup \( X \) its 2-class \( \uparrow x \) is a 2-trivial semigroup.

Now we provide an inner description of the binary quasiorder via prime (co)ideals, following the approach of Petrich \[26\] and Tamura \[35\].

A subset \( I \) of a semigroup \( X \) is called

- an ideal in \( X \) if \( (IX) \cup (XI) \subseteq I \);
- a prime ideal if \( I \) is an ideal such that \( X \setminus I \) is a subsemigroup of \( X \);
- a (prime) coideal if the complement \( X \setminus I \) is a (prime) ideal in \( X \).

According to this definition, the sets \( \emptyset \) and \( X \) are prime (co)ideals in \( X \).

Observe that a subset \( A \) of a semigroup \( X \) is a prime coideal in \( X \) if and only if its characteristic function

\[
\chi_A : X \to 2, \quad \chi_A : x \mapsto \chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{otherwise}, \end{cases}
\]

is a homomorphism. This function characterization of prime coideals implies the following inner description of the 2-quasiorder, first noticed by Tamura in \[35\].

**Proposition 3.6.** For any element \( x \) of a semigroup \( X \), its upper 2-class \( \uparrow x \) coincides with the smallest coideal of \( X \) that contains \( x \).

The following inner description of the upper 2-classes is a modified version of Theorem 3.3 in \[26\].

**Proposition 3.7.** For any element \( x \) of a semigroup \( X \) its upper 2-class \( \uparrow x \) is equal to the union \( \bigcup_{n \in \omega} \uparrow_n x \), where \( \uparrow_0 x = \{ x \} \) and

\[
\uparrow_{n+1} x \overset{\text{def}}{=} \{ y \in X : X^1 y X^1 \cap (\uparrow_n x)^2 \neq \emptyset \}
\]

for \( n \in \omega \).

**Proof.** Observe that for every \( n \in \omega \) and \( y \in \uparrow_n x \) we have \( y y \in X^1 y X^1 \cap (\uparrow_n x)^2 \neq \emptyset \) and hence \( y \in \uparrow_{n+1} x \). Therefore, \( (\uparrow_n x)_{n \in \omega} \) is an increasing sequence of sets. Also, for every \( y, z \in \uparrow_n x \) the we have \( y z \in X^1 y z X^1 \cap (\uparrow_n x)^2 \) and hence \( y z \in \uparrow_{n+1} x \), which implies that the union \( \uparrow \omega x \overset{\text{def}}{=} \bigcup_{n \in \omega} \uparrow_n x \) is a subsemigroup of \( X \).

The definition of the sets \( \uparrow_n x \) implies that the complement \( I = X \setminus \uparrow \omega x \) is an ideal in \( X \). Taking into account that \( \uparrow x \) is the smallest prime coideal containing \( x \), we conclude that \( \uparrow x \subseteq \uparrow \omega x \). To prove that \( \uparrow \omega x \subseteq \uparrow x \), it suffices to check that \( \uparrow_n x \subseteq \uparrow x \) for every \( n \in \omega \). It is trivially true for \( n = 0 \). Assume that for some \( n \in \omega \) we have already proved that \( \uparrow_n x \subseteq \uparrow x \). Since \( \uparrow x \) is a coideal in \( X \), for any \( y \in X \setminus \uparrow x \) we have \( \emptyset = X^1 y X^1 \cap \uparrow x \supseteq X^1 y X^1 \cap \uparrow_n x \), which implies that \( y \notin \uparrow_{n+1} x \) and hence \( \uparrow_{n+1} x \subseteq \uparrow x \). By the Principle of Mathematical Induction, \( \uparrow_n x \subseteq \uparrow x \) for all \( n \in \omega \) and hence \( \uparrow \omega x = \bigcup_{n \in \omega} \uparrow_n x \subseteq \uparrow x \), and finally \( \uparrow_\omega x = \uparrow x \). \( \square \)
For a positive integer \( n \), let
\[
2^{<n} \overset{\text{def}}{=} \bigcup_{k<n} \{0,1\}^k \quad \text{and} \quad 2^{\leq n} \overset{\text{def}}{=} \bigcup_{k\leq n} \{0,1\}^k.
\]

For a sequence \( s = (s_0, \ldots, s_{n-1}) \in 2^n \) and a number \( k \in \{0,1\} \), let
\[
s^k = (s_0, \ldots, s_{n-1}, k) \quad \text{and} \quad k^* = (k, s_0, \ldots, s_{n-1}).
\]

The following proposition provides a constructive description of elements of the sets \( \uparrow_{n} x \) appearing in Proposition 3.7.

**Proposition 3.8.** For every \( n \in \mathbb{N} \) and every element \( x \) of a semigroup \( X \), the set \( \uparrow_{n} x \) coincides with the set \( \uparrow_{n} x \) of all elements \( y \in X \) for which there exist sequences \( \{x_s\}_{s \in 2^{\leq n}}, \{y_s\}_{s \in 2^{\leq n}}, \{a_s\}_{s \in 2^{\leq n}}, \{b_s\}_{s \in 2^{\leq n}} \subseteq X^1 \) satisfying the following conditions:

\begin{enumerate}[(1)]
  \item \( x_s = x \) for all \( s \in 2^n \);
  \item \( y_s = a_s x_s b_s \) for every \( s \in 2^{\leq n} \);
  \item \( y_s = x_s x_{s}^1 \) for every \( s \in 2^{\leq n} \);
  \item \( x_0 = y \) for the unique element \( () \) of \( 2^0 \).
\end{enumerate}

**Proof.** This proposition will be proved by induction on \( n \). For \( n = 1 \), we have
\[
\uparrow_1 = \{ y \in X : x x \in X^1 y X^1 \} = \{ y \in X : \exists a, b \in X^1 \ axb = x x \}
\]
\[
= \{ y \in X : \exists (x_s)_{s \in 2^{\leq 1}}, (y_s)_{s \in 2^{\leq 1}} \subseteq X, \{a_s\}_{s \in 2^{\leq 1}}, \{b_s\}_{s \in 2^{\leq 1}} \subseteq X^1, x_0 = y \}
\]

Assume that for some \( n \in \mathbb{N} \) the equality \( \uparrow_n x = \uparrow'_n x \) has been proved. To check that \( \uparrow_{n+1} x \subseteq \uparrow'_{n+1} x \), take any \( x_0 \in \uparrow_{n+1} x \). The definition of \( \uparrow_{n+1} x \) ensures that \( X^1 x_0 x_1 \cap (\uparrow_n x)^2 \neq \emptyset \) and hence \( a_0 x_0 b_0 = x_0 x_1 \) for some \( a_0, b_0 \in X^1 \) and \( x_0 x_1 \in \uparrow_n x = \uparrow'_n x \). By the definition of the set \( \uparrow'_n x \), for every \( k \in \{0,1\} \), there exist sequences \( \{x_k^s\}_{s \in 2^{\leq n}}, \{y_k^s\}_{s \in 2^{\leq n}} \subseteq X \) and \( \{a_k^s\}_{s \in 2^{\leq n}}, \{b_k^s\}_{s \in 2^{\leq n}} \subseteq X^1 \) such that

\begin{itemize}
  \item \( x_{k^*} = x \) for all \( s \in 2^n \);
  \item \( y_{k^*} = a_k^s x_{k^*} b_k^s \) for every \( s \in 2^{\leq n} \);
  \item \( y_{k^*} = x_{k^*} x_{k^*}^1 \) for every \( s \in 2^{\leq n} \).
\end{itemize}

Then the sequences \( \{x_s\}_{s \in 2^{\leq n+1}}, \{x_s\}_{s \in 2^{\leq n+1}} \subseteq X \) and \( \{a_s\}_{s \in 2^{\leq n+1}}, \{b_s\}_{s \in 2^{\leq n+1}} \subseteq X^1 \) witness that \( x_0 \in \uparrow_{n+1} x \), which completes the proof of the inclusion \( \uparrow_{n+1} x \subseteq \uparrow'_{n+1} x \).

To prove that \( \uparrow'_{n+1} x \subseteq \uparrow_{n+1} x \), take any \( x_0 \in \uparrow'_{n+1} x \) and by the definition of \( \uparrow'_{n+1} x \), find sequences \( \{x_s\}_{s \in 2^{\leq n+1}}, \{x_s\}_{s \in 2^{\leq n+1}} \subseteq X \) and \( \{a_s\}_{s \in 2^{\leq n+1}}, \{b_s\}_{s \in 2^{\leq n+1}} \subseteq X^1 \) satisfying the conditions (1.1)–(3.1). The for every \( k \in \{0,1\} \), the sequences \( \{x_{k^*}^s\}_{s \in 2^{\leq n}}, \{x_{k^*}^s\}_{s \in 2^{\leq n}} \subseteq X \) and \( \{a_{k^*}^s\}_{s \in 2^{\leq n}}, \{b_{k^*}^s\}_{s \in 2^{\leq n}} \subseteq X^1 \) witness that \( x_0 x_1 x_0 x_1 \subseteq \uparrow_{n+1} x \) and then the equalities \( a_0 x_0 b_0 = x_0 x_1 \) imply that \( X^1 x_0 X^1 \cap (\uparrow_n x)^2 \neq \emptyset \) and hence \( x_0 \in \uparrow_{n+1} x \), which completes the proof of the equality \( \uparrow_{n+1} x = \uparrow'_{n+1} x \). \( \Box \)

A semigroup \( X \) is called duo if \( aX = Xa \) for every \( a \in X \). Observe that each commutative semigroup is duo.

The upper 2-classes in duo semigroups have the following simpler description.

**Theorem 3.9.** For any element \( a \in X \) of a duo semigroup \( X \) we have
\[
\uparrow a = \{ x \in X : a^\mathbb{N} \cap X x X \neq \emptyset \}.
\]
Proof. First we prove that the set \( \frac{a_N}{X} \) is a subsemigroup. Given any elements \( x, y \) we have \( X^1 x = x X^1 = X^1 x^1 \). If \( x, y \in \frac{a_N}{X} \), then
\[
X^1 x \cap a_N = X^1 x^1 \cap a_N \neq \emptyset \neq X^1 y = y X^1 \cap a_N = y X^1 \cap a_N
\]
and hence \( X^1 xyX^1 \in a_N \neq \emptyset \), which means that \( xy \in \frac{a_N}{X} \). Therefore, \( \frac{a_N}{X} \) is a subsemigroup of \( X \). The definition of \( \frac{a_N}{X} \) ensures that \( X \setminus \frac{a_N}{X} \) is an ideal in \( X \). Then \( \frac{a_N}{X} \subseteq \uparrow a \) is a prime coideal in \( X \) and \( \frac{a_N}{X} = \uparrow a \), by the minimality of \( \uparrow a \).

For viable semigroups Putcha and Weissglass [32] proved the following simplification of Proposition 3.10. Following Putcha and Weissglass [32], we define a semigroup \( X \) to be viable if for any elements \( x, y \in X \), the set \( \{ x, y \} \subseteq E(X) \), we have \( xy = yx \). For various equivalent conditions to the viability, see [2].

**Proposition 3.10 (Putcha–Weissglass).** If \( X \) is a viable semigroup, then for every idempotent \( e \in E(X) \) we have \( \uparrow e = \{ x \in X : e \in X^1 x \} \).

Proof. We present a short proof of this theorem, for convenience of the reader. Let \( \uparrow e = \{ x \in X : e \in X^1 x \} \). By Proposition 3.7, \( \uparrow e \subseteq \uparrow e \). The reverse inclusion will follow from the minimality of the prime coideal \( \uparrow e \) as soon as we prove that \( \uparrow e \) is a prime coideal in \( X \). It is clear from the definition that \( \frac{a_N}{X} \) is a prime coideal. So, it remains to check that \( \uparrow e \) is a subsemigroup. Given any elements \( x, y \in \uparrow e \), find elements \( a, b, c, d \in X^1 \) such that \( axb = e = cyd \). Then \( axb = e = cyd \) and \( (beax)(beax)ax = b(axb)ax = beax = beax \), which means that \( beax \) is an idempotent. By the viability of \( X \), \( axbe = e = beax \). By analogy we can prove that \( ecyd = e = ydec \). Then \( aeaxydex = ee = e \) and hence \( xy \in \uparrow e \).

Proposition 3.10 has an important corollary, proved in [32].

**Corollary 3.11 (Putcha–Weissglass).** If \( X \) is a viable semigroup, then for every \( x \in X \) its 2-class \( \uparrow x \) contains at most one idempotent.

Proof. To derive a contradiction, assume that the semigroup \( \uparrow x \) contains two distinct idempotents \( e, f \). By Proposition 3.10, there are elements \( a, b, c, d \in X^1 \) such that \( e = afb \) and \( f = ced \). Observe that \( afbe = ee = e \) and \( (beaf)(beaf) = be(afbe)af = beaf = beaf \) and hence \( afbe \) and \( beaf \) are idempotents. The viability of \( X \) ensures that \( afbe = beaf \). By analogy we can prove that \( eafb = ee = eafbe \), \( cedf = f = dfce \) and \( fced = f = edfc \). These equalities imply that \( H_e = H_f \) and hence \( e = f \) because the group \( H_e = H_f \) contains a unique idempotent. But the equality \( e = f \) contradicts the choice of the idempotents \( e, f \).

4. THE STRUCTURE OF 2-TRIVIAL SEMIGROUPS

Tamura’s Theorem 3.5 motivates the problem of a deeper study of the structure of 2-trivial semigroups. This problem has been considered in the literature, see, e.g. [26], §3. Proposition 3.6 implies the following simple characterization of 2-trivial semigroups.

**Theorem 4.1.** A semigroup \( X \) is 2-trivial if and only if every nonempty prime ideal in \( X \) coincides with \( X \).
Observe that a semigroup \( X \) is 2-trivial if and only if \( X = \uparrow x \) for every \( x \in X \). This observation and Propositions 3.7 and 3.8 imply the following characterization.

**Proposition 4.2.** A semigroup \( X \) is 2-trivial if and only if for every \( x, y \in X \) there exists \( n \in \mathbb{N} \) and sequences \( \{a_s\}_{s \leq n}, \{b_s\}_{s \leq n} \subseteq X^1 \) and \( \{x_s\}_{s \leq n}, \{y_s\}_{s \leq n} \subseteq X \) satisfying the following conditions:

1. \( x_s = x \) for all \( s \in 2^n \);
2. \( y_s = a_s x_s b_s \) for every \( s \in 2^\leq n \);
3. \( y_s = x_s 0 x_s 1 \) for every \( s \in 2^{<n} \);
4. \( x() = y \) for the unique element ( ) of \( 2^0 \).

A semigroup \( X \) is called Archimedean if for any elements \( x, y \in X \) there exists \( n \in \mathbb{N} \) such that \( x^n \in X y X \) for some \( a, b \in X \). A standard example of an Archimedean semigroup is the additive semigroup \( \mathbb{N} \) of positive integers. For commutative semigroups the following characterization was obtained by Tamura and Kimura in [38].

**Theorem 4.3.** A duo semigroup \( X \) is 2-trivial if and only if \( X \) is Archimedean.

**Proof.** If \( X \) is 2-trivial, then by Theorem 3.9 for every \( x, y \in X \) there exists \( n \in \omega \) such that \( x^n \in X y X \), which means that \( X \) is Archimedean.

If \( X \) is Archimedean, then for every \( e \in X \), we have

\[
\uparrow x = \{y \in X : x^n \cap (X x X) \neq \emptyset\} = X,
\]

see Theorem 3.9 which means that the semigroup \( X \) is 2-trivial. \( \square \)

Following Tamura [36], we define a semigroup \( X \) to be unipotent if \( X \) contains a unique idempotent.

**Theorem 4.4** (Tamura, 1982). For the unique idempotent \( e \) of an unipotent 2-trivial semigroup \( X \), the maximal group \( H_e \) of \( e \) in \( X \) is an ideal in \( X \).

**Proof.** This theorem was proved by Tamura in [36]. We present here an alternative (and direct) proof. To derive a contradiction, assume that \( H_e \) is not an ideal in \( X \). Then the set

\[
I \triangleq \{x \in X : \{ex, xe\} \not\subseteq H_e\}
\]

is not empty. We claim that \( I \) is an ideal in \( X \). Assuming the opposite, we could find \( x \in I \) and \( y \in X \) such that \( xy \not\in I \) or \( yx \not\in I \).

If \( xy \not\in I \), then \( \{exy, xyx\} \subseteq H_e \). Taking into account that \( exy \) and \( xyx \) are elements of the group \( H_e \), we conclude that \( exy = xyx = yxe \). Let \( g \) be the inverse element to \( xyx \) in the group \( H_e \). Then \( eyx = xyx = yxe \). Replacing \( y \) by \( yg \), we can assume that \( ye = y \) and \( xy = e \). Observe that \( yxyx = y(xy)x = yex = (ye)x = yx \), which means that \( yx \) is an idempotent in \( S \). Since \( e \) is a unique idempotent of the semigroup \( X \), \( ye = e = xy \). It follows that \( xe = x(yx) = (xy)x = ex \) and \( ey = (yx)y = y(xy) = ye = y \). Using this information it is easy to show that \( xe = ex \in H_e \). By analogy we can show that the assumption \( yx \not\in I \) implies \( ex = xe \not\in H_e \). So, in both cases we obtain \( ex = xe \in H_e \), which contradicts the choice of \( x \in I \).

This contradiction shows that \( I \) is an ideal in \( S \). Observe that for any \( x, y \in X \setminus I \) we have \( \{ex, xe, ey, ye\} \subseteq H_e \). Then also \( xy = x(ey)(xe)(ye) \in H_e \) and \( ey = (exe)(y) = (ex)(ey) \in H_e \), which means that \( xy \in X \setminus I \) and hence \( I \) is a nontrivial prime ideal in \( X \). But the existence of such an ideal contradicts the 2-triviality of \( X \). \( \square \)

An element \( z \) of a semigroup \( X \) is called central if \( zx = xz \) for all \( x \in X \).
Corollary 4.5. The unique idempotent \( e \) of a unipotent 2-trivial semigroup \( X \) is central in \( X \).

Proof. Let \( e \) be a unique idempotent of the unipotent semigroup \( X \). By Tamura’s Theorem \[14\], the maximal subgroup \( H_e \) of \( e \) is an ideal in \( X \). Then for every \( x \in X \) we have \( xe, ex \in H_e \). Taking into account that \( xe \) and \( ex \) are elements of the group \( H_e \), we conclude that \( ex = exe = xe \). This means that the idempotent \( e \) is central in \( X \). \( \square \)

As we already know a semigroup \( X \) is 2-trivial if and only if each nonempty prime ideal in \( X \) is equal to \( X \).

A semigroup \( X \) is called

- simple if every nonempty ideal in \( X \) is equal to \( X \);
- 0-simple if contains zero element 0, \( XX \neq \{0\} \) and every nonempty ideal in \( X \) is equal to \( X \) or \( \{0\} \);
- congruence-free if every congruence on \( X \) is equal to \( X \times X \) or \( \Delta \) where \( \Delta \) is the diagonal.

It is clear that a semigroup \( X \) is 2-trivial if \( X \) is either simple or congruence-free. On the other hand the additive semigroup of integers \( \mathbb{N} \) is 2-trivial but not simple.

Remark 4.6. By \[1\]–\[14\], there exists an infinite 0-simple congruence-free monoid \( X \). Being congruence-free, the semigroup \( X \) is 2-trivial. On the other hand, \( X \) contains at least two central idempotents: 0 and 1. The 2-trivial monoid \( X \) is not unipotent and its center \( Z(X) = \{ z \in X : \forall x \in X \ (xz = zx) \} \) is not 2-trivial. The polycyclic monoids (see \[10\], \[11\], \[8\], \[9\]) have the similar properties. By Theorem 2.4 in \[10\], for \( \lambda \geq 2 \) the polycyclic monoid \( P_\lambda \) is congruence-free and hence 2-trivial, but its center \( Z(P_\lambda) = \{0, 1\} \) is not 2-trivial.

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References

\[1\] F. Al-Kharousi, A.J. Cain, V. Maltcev, A. Umar, A countable family of finitely presented infinite congruence-free monoids. Acta Sci. Math. (Szeged) 81 (2015), no. 3-4, 437–445.
\[2\] T. Banakh, E-Separated semigroups, preprint.
\[3\] T. Banakh, S. Bardyla, Complete topologized posets and semilattices, Topology Proc. 57 (2021) 177–196.
\[4\] T. Banakh, S. Bardyla, Characterizing categorically closed commutative semigroups, Journal of Algebra. 591 (2022) 84–110.
\[5\] T. Banakh, S. Bardyla, Categorically closed countable semigroups, preprint (arxiv.org/abs/1806.02869).
\[6\] T. Banakh, S. Bardyla, Categorically closed commutative semigroups, in preparation.
\[7\] T. Banakh, S. Bardyla, Categorically closed Clifford semigroups, in preparation.
\[8\] S. Bardyla, Classifying locally compact semitopological polycyclic monoids, Math. Bull. Shev. Sci. Soc. 13 (2016), 13–28.
\[9\] S. Bardyla, On universal objects in the class of graph inverse semigroups, Eur. J. Math. 6 (2020), 4–13.
\[10\] S. Bardyla, O. Gutik, On a semitopological polycyclic monoid, Alg. Discr. Math. 21:2 (2016), 163–183.
\[11\] S. Bardyla, O. Gutik, On a complete topological inverse polycyclic monoid, Carp. Math. Publ. 8:2 (2016), 183–194.
[12] S. Bogdanović, M. Ćirić, Primitive π-regular semigroups, Proc. Japan Acad. Ser. A Math. Sci. 68:10 (1992), 334–337.
[13] S. Bogdanović, M. Ćirić, Z. Popović, Semilattice decompositions of semigroups, University of Niš, Niš, 2011. viii+321 pp.
[14] A. Cain, V. Maltcev, A simple non-bisimple congruence-free finitely presented monoid, Semigroup Forum 90:1 (2015), 184–188.
[15] M. Ćirić, S. Bogdanović, Decompositions of semigroups induced by identities, Semigroup Forum 46:3 (1993), 329–346.
[16] M. Ćirić, S. Bogdanović, Semilattice decompositions of semigroups, Semigroup Forum 52:2 (1996), 119–132.
[17] J. Galbiati, Some semilattices of semigroups each having one idempotent, Semigroup Forum 55:2 (1997), 206–214.
[18] R. Gigoň, η-simple semigroups without zero and η'-simple semigroups with a least non-zero idempotent, Semigroup Forum 86:1 (2013), 108–113.
[19] J. Howie, Fundamentals of Semigroup Theory, Clarendon Press, Oxford, 1995.
[20] J. Howie, G. Lallement, Certain fundamental congruences on a regular semigroup, Proc. Glasgow Math. Assoc. 7 (1966), 145–159.
[21] M. Mitrović, Semilattices of Archimedean semigroups, With a foreword by Donald B. McAlister. University of Niš. Faculty of Mechanical Engineering, Niš, 2003. xiv+160 pp.
[22] M. Mitrović, On semilattices of Archimedean semigroup – a survey, Semigroups and languages, 163–195, World Sci. Publ., River Edge, NJ, 2004.
[23] M. Mitrović, D.A Romano, M. Vincić, A theorem on semilattice-ordered semigroup, Int. Math. Forum 4:5–8 (2009), 227–232.
[24] M. Mitrović, S. Silvestrov, Semilattice decompositions of semigroups. Hereditariness and periodicity—an overview, Algebraic structures and applications, 687–721, Springer Proc. Math. Stat., 317, Springer, Cham, 2020.
[25] M. Petrich, The maximal semilattice decomposition of a semigroup, Bull. Amer. Math. Soc. 69 (1963), 342–344.
[26] M. Petrich, The maximal semilattice decomposition of a semigroup, Math. Z. 85 (1964), 68–82.
[27] M. Petrich, Introduction to semigroups, Merrill Research and Lecture Series. Charles E. Merrill Publishing Co., Columbus, Ohio, 1973. viii+198 pp.
[28] M. Petrich, N.R. Reilly, Completely regular semigroups, A Wiley-Intersci. Publ. John Wiley & Sons, Inc., New York, 1999.
[29] Z. Popović, S. Bogdanović, M. Ćirić, A note on semilattice decompositions of completely π-regular semigroups, Novi Sad J. Math. 34:2 (2004), 167–174.
[30] M. Putcha, Semilattice decompositions of semigroups, Semigroup Forum 6:1 (1973), 12–34.
[31] M. Putcha, Minimal sequences in semigroups, Trans. Amer. Math. Soc. 189 (1974), 93–106.
[32] M. Putcha, J. Weissglass, A semilattice decomposition into semigroups having at most one idempotent, Pacific J. Math. 39 (1971), 225–228.
[33] R. Šulka, The maximal semilattice decomposition of a semigroup, radicals and nilpotency, Mat. Časopis Sloven. Akad. Vied 20 (1970), 172–180.
[34] T. Tamura, The theory of construction of finite semigroups, I. Osaka Math. J. 8 (1956), 243–261.
[35] T. Tamura, Semilattice congruences viewed from quasi-orders, Proc. Amer. Math. Soc. 41 (1973), 75–79.
[36] T. Tamura, Semilattice indecomposable semigroups with a unique idempotent, Semigroup Forum 24:1 (1982), 77–82.
[37] T. Tamura, J. Shafer, Another proof of two decomposition theorems of semigroups, Proc. Japan Acad. 42 (1966), 685–687.
[38] T. Tamura, N. Kimura, On decompositions of a commutative semigroup, Ködai Math. Sem. Rep. 6 (1954), 109–112.