INTRODUCTION TO $Z_N$ SYMMETRY IN $SU_N$ GAUGE THEORIES
AT FINITE TEMPERATURES

NATHAN WEISS
Department of Physics, University of British Columbia
Vancouver, British Columbia, V6T 1Z1, Canada

ABSTRACT

There are several talks at this workshop on the physical interpretation and consequences of the $Z_N$ symmetry which is present in the Euclidean Path Integral formulation of $SU_N$ Gauge Theories at finite temperature. The purpose of this paper is to present an introduction to this subject. After a brief review of Gluodynamics at finite temperature, the nature of the $Z_N$ symmetry in this system is described and its relationship to confinement is discussed. $Z_N$ domain walls and bubbles are then described as is their relationship to the confining–deconfining phase transition. The effect of Fermions is then considered and the presence of metastable extrema of the Effective Potential for the Polyakov–Wilson Line is described. It is then argued that these metastable extrema do not correspond to physically realizable metastable states.

1. Introduction

The purpose of this paper is to present an introduction to the subject of the $Z_N$ symmetry which is present in the study of $SU_N$ Gauge Theories at finite temperature. I begin with a brief review of the Euclidean Path Integral formulation of pure $SU_N$ Gauge Theories (without fermions) and of the Wilson–Polyakov Line which is used as an order parameter for this theory. This is followed by a discussion of the $Z_N$ symmetry, its relationship to Confinement and the breaking of this symmetry at high temperatures. $Z_N$ domain walls and $Z_N$ bubbles are then introduced. I then discuss what happens when fermions are introduced including the presence of metastable configurations in the Euclidean Partition Function. The paper concludes with a discussion of the physical interpretation of these metastable states, and of $Z_N$ domains in general, in Minkowski space.

2. Pure $SU_N$ Gauge Theories

$^*$Talk presented at the 3rd Thermal Fields Workshop held Aug 16 - 27, 1993 in Banff, Canada.
This work was supported in part by the Natural Sciences and Engineering Research Council of Canada.
Pure $SU_N$ Gauge Theories at a finite temperature $T = \beta^{-1}$ are usually studied via the Partition Function

$$Z(\beta) = \text{Tr } e^{-\beta H} \propto \int_{A_i(\tau=0)=A_i(\tau=\beta)} \mathcal{D}A_\mu \exp[-S_E(\beta)]$$

(1)

with the Euclidean Action $S_E(\beta)$ given by

$$S_E(\beta) = \frac{1}{g^2} \int_0^\beta d\tau \int d^3x \frac{1}{4} F_{\mu\nu}^a F^{\mu\nu}_a$$

(2)

where $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + (A_\mu \times A_\nu)^a$ are the Field Strengths. The above expression for $Z(\beta)$ is derived by considering only the states $|\psi\rangle$ which satisfy Gauss’ Law $D_i E_i |\psi\rangle = 0$. It thus represents the Partition Function for the theory in the absence of any external sources. The Free Energy of such a system is given by

$$F(\beta) = -\frac{1}{\beta} \log[Z(\beta)]$$

The Partition Function in the presence of a single external “quark” source (i.e. a static source in the fundamental representation of $SU_N$) at a spatial point $\vec{x}$ is given by

$$Z_q(\beta) \propto \int_{A_i(\tau=0)=A_i(\tau=\beta)} \mathcal{D}A_\mu \exp[-S_E(\beta)] \times \langle \text{Tr } L(\vec{x}) \rangle$$

(3)

where

$$L(\vec{x}) = \left[ \frac{1}{N} \exp \left( i \int_0^\beta A_0(\vec{x}, \tau) d\tau \right) \right]$$

(4)

is called the Wilson–Polyakov Line.

Under a gauge transformation $U(\vec{x}, \tau)$, $L(\vec{x})$ transforms as

$$L(\vec{x}) \rightarrow U(\vec{x}, 0)L(\vec{x})U(\vec{x}, \beta)$$

(5)

It thus follows that $\text{Tr } L(\vec{x})$ is invariant under gauge transformations which are periodic in time i.e. $U(\vec{x}, 0) = U(\vec{x}, \beta)$. The increase in Free Energy $\delta F$ when adding such an external source is thus given by

$$e^{-\beta(\delta F)} = \langle \text{Tr } L(\vec{x}) \rangle = \frac{Z_q(\beta)}{Z(\beta)}$$

(6)

Similarly the excess Free Energy $V(|\vec{x} - \vec{y}|)$ of a “quark” and an “antiquark” source at locations $\vec{x}$ and $\vec{y}$ respectively is given by

$$\langle \text{Tr } L^+(\vec{y}) \text{ Tr } L(\vec{x}) \rangle \propto e^{-\beta V(|\vec{x} - \vec{y}|)}$$

(7)

It follows from the above discussion that $\langle \text{Tr } L(\vec{x}) \rangle$ is a useful order parameter for distinguishing a confining from a deconfining phase in Gauge Theories without fermions. If $\langle \text{Tr } L \rangle = 0$ then $\delta F$ is infinite and it costs an infinite amount of energy to introduce a single source. Furthermore $\langle \text{Tr } L^+(\vec{y}) \text{ Tr } L(\vec{x}) \rangle \rightarrow 0$ as $|\vec{x} - \vec{y}| \rightarrow \infty$ so that $V(|\vec{x} - \vec{y}|) \rightarrow \infty$ at large distances. This signals a confining phase. If, on the other hand, $\langle \text{Tr } L \rangle \neq 0$ then the potential energy $V$ is finite at large distance. This signals a deconfining phase.
3. \( Z_N \) Symmetry

It so happens that the Euclidean Path Integral possesses a discrete symmetry which implies that \( \langle \text{Tr} \, L \rangle = 0 \). For \( SU_N \) this symmetry transforms

\[
L \rightarrow e^{2\pi ik/N} \times L \tag{8}
\]

for \( k = 1, 2...N \) leaving the action \( S_E \) invariant. To see this note that in our formalism the Partition Function is a sum over only periodic Gauge Potentials \( A_i \). If, however, we consider a nonperiodic Gauge Transformation \( U(\vec{x}, \tau) \) such that

\[
U(\vec{x}, \tau = 0) = I; \quad U(\vec{x}, \tau = \beta) = M(\vec{x}) \neq I \tag{9}
\]

then the periodic boundary conditions on \( A_i \) will be maintained if and only if \( M(\vec{x}) \) commutes with any \( A_i \). This can only happen if \( M \) is in the Center of the group \( SU_N \)

\[
M = e^{2\pi ik/N} \times I \tag{10}
\]

for \( k = 1, 2...N \). This is the group \( Z_N \). Under this transformation both the action and the boundary conditions are invariant but the order parameter \( \langle \text{Tr} \, L \rangle \) is not invariant but transforms as in Eq. (5). This then implies that \( \langle \text{Tr} \, L \rangle = 0 \) which seems to imply confinement. The loophole is that this symmetry is a discrete, global symmetry and thus it is entirely possible that this symmetry is spontaneously broken at some temperatures. It thus follows that the issue of Confinement is intimately tied to the question of whether the \( Z_N \) symmetry is spontaneously broken. Spontaneous breaking of the symmetry leads to \( \langle \text{Tr} \, L \rangle \neq 0 \) which signals a deconfined phase.

It has been shown\(^4\) that the \( Z_N \) symmetry is indeed broken in weak coupling perturbation theory which is expected to be valid at high temperatures. The basic reason for this is that perturbation theory is an expansion around \( A_\mu = 0 \) which implies that \( \langle \text{Tr} \, L \rangle = 1 \) - (perturbative corrections). Perturbative calculations of the Effective Potential for \( A_0 \) or, equivalently for \( L \) show a minimum at \( \langle \text{Tr} \, L \rangle = 1 \) and at the \( Z_N \) symmetric points \( \langle \text{Tr} \, L \rangle = \exp(2\pi ik/N) \). A typical curve (for \( SU_2 \)) is shown in Figure 1. The presence of this minimum in the deconfined (high temperature) phase and its absence in the confined (low temperature) phase is confirmed by numerical Lattice simulations.

It is interesting to note that, unlike most other symmetries, this \( Z_N \) symmetry is broken in the low temperature phase and unbroken in the high temperature phase.

4. Domain Walls and Bubbles

The form of the Effective Potential shown in Figure 1 in the high temperature phase implies the existence of domain walls in the Euclidean Path Integral. If we imagine forcing boundary conditions on our system such that for \( x_3 \rightarrow -\infty \) the system sits near the minimum at \( \langle \text{Tr} \, L \rangle = 1 \) whereas for \( x_3 \rightarrow +\infty \) the system is near another of the minima of the Effective Potential, say at \( \langle \text{Tr} \, L \rangle = \exp(2\pi ik/N) \)
then there will be a Domain Wall separating these two “vacua”. Calculations of these Domain Wall energies have been carried out both in perturbation theory and numerically and they will be discussed in another paper at this workshop by C. Korthals–Altes.

Figure 1

Free energy versus $q \propto \log \langle \text{Tr} \ L \rangle$ for $SU(2)$ with no Fermions

A consequence of the existence of these domain walls is that there also exist $Z_N$ bubbles. These bubbles are regions in space for which $\langle \text{Tr} \ L \rangle \simeq 1$ outside the bubble but at the center of the bubble $\langle \text{Tr} \ L \rangle \simeq \exp(2\pi i k/N)$. As the temperature $T$ decreases it becomes more likely to form these bubbles. This happens because the probability to form a bubble is proportional to $\exp(-S/g^2)$. The action is proportional to $T^4$ and $g^2$ increases as $T$ is decreased. As the temperature is lowered more of these bubbles will be present and they will eventually “randomize” $L$ until at some critical temperature the symmetry is restored and $\langle \text{Tr} \ L \rangle = 0$. This is the confining–deconfining transition.

5. Gauge Theories with Fermions

The situation described above for pure Gauge Theories change significantly when Fermions (in the fundamental representation of $SU_N$) are introduced. We consider a theory with $N_f$ flavours of “quarks” described by Fermionic fields $\psi$. The Partition
Function is given by
\[
Z_f(\beta) = e^{-\beta F_f(\beta)} \propto \int_{A_i(0) = A_i(\beta)} DA_\mu D\psi D\psi^\dagger \exp \left[-S_E^f(\beta)\right]
\]
(11)

where \(S_E^f(\beta)\) is the usual Euclidean Action for QCD. Note the antiperiodic boundary conditions which \(\psi\) satisfies.

We expect, both on physical and on mathematical grounds, that \(\langle \text{Tr} L \rangle \neq 0\) in this case. Physically this is a result of the fact that the potential energy \(V(r)\) of two static sources separated by a distance \(r\) does not grow as \(r \to \infty\) due to screening by the dynamical fermions. Furthermore the ground state in the presence of a single quark source has a finite energy since it includes a dynamical antiquark to which it is bound.

This physical expectation is realized mathematically by the fact that there is no \(Z_N\) symmetry for this system and thus there is no mathematical reason to suppose that \(\langle \text{Tr} L \rangle = 0\). (There is, in fact, no known order parameter which distinguishes a confining from a deconfining phase in the presence of quarks.) The reason for the loss of \(Z_N\) symmetry is that even though both the Action and the periodic boundary conditions on \(A_i\) are maintained by the transformation (9) the antiperiodic boundary conditions on \(\psi\) are not preserved since
\[
\psi(\beta) \to U(\beta)\psi(\beta) = -e^{2\pi i k/N}\psi(0)
\]
(12)

which is a simple reflection of the fact that fields transforming under the fundamental representation of \(SU_N\) are not invariant under its center.

This lack of \(Z_N\) invariance can be vividly demonstrated by computing the Effective Potential for \(\langle \text{Tr} L \rangle\). The specific case of \(N = 3\) is plotted in Figure 2 for various values of \(N_f\). In this figure \(F_f/T^4\) is plotted as a function of a variable \(q \propto \log(\langle \text{Tr} L \rangle)\). Note the presence of “metastable minima” at \(q = 1\) and \(2\). These minima actually become maxima for sufficiently large \(N_f\) although this is not shown in the figure.

6. Interpretation of the \(Z_N\) Domains in Minkowski Space

It is tempting to treat the metastable extrema of an Effective Potential such as that of Figure 2 as a physical metastable state in Minkowski space. There have been several attempts to do this and interesting physical and cosmological consequences of this have been suggested. Despite these attempts I believe that these metastable Euclidean extrema have no direct physical interpretation.

First note that if we do interpret these metastable states physically we run into very serious trouble. The reason is that the Free Energy \(F_f \propto T^4\). For example for \(SU_3\) with \(N_f = 4\), the metastable extremum at \(q = 1\) has a free energy \(F = \gamma T^4\) with \(\gamma > 0\). This happens for a very large class of values for \(N_f\) and \(N\). This is a disaster if these points represent true metastable states of the system. The reason
is that if $F = |\gamma| T^4$ then the pressure $p = -|\gamma| T^4$, the internal energy $E = -3|\gamma| T^4$, the specific heat $c = -12|\gamma| T^3$ but worst of all the entropy

$$S = \frac{E - F}{T} = -4|\gamma| T^3$$

(13)

This negative entropy implies, among other things, that there is less than one state available to the system. The above values for the thermodynamic quantities are clearly unphysical.

This problem (which is caused by the “wrong” sign of $F$) can artificially fixed by adding extra particles to the system whose entropy $S > 0$. This is what happens, for example, in the Standard Electroweak Model. But this does not solve the fundamental problem that a physical interpretation of these states is untennable.

Figure 2

Free energy versus $q \propto \log \langle \text{Tr} \ L \rangle$ for $SU(3)$ with various numbers of fermion flavours

This problem (which is caused by the “wrong” sign of $F$) can artificially fixed by adding extra particles to the system whose entropy $S > 0$. This is what happens, for example, in the Standard Electroweak Model. But this does not solve the fundamental problem that a physical interpretation of these states is untennable.

The basic problem is that $\langle \text{Tr} \ L \rangle$ is fundamentally a Euclidean object. It is, in fact, nonlocal in Euclidean time. In fact a Euclidean $A_0 \neq 0$ (which leads to a nontrivial $\langle \text{Tr} \ L \rangle$) corresponds to an imaginary $A_0$ in Minkowski space. Thus in Minkowski space a constant $A_0$ looks like a purely imaginary chemical potential for “color” charge! This is clearly unphysical for a Minkowski object. In fact if we
use as an example the case when $N$ is even and $q = N/2$ (i.e. $\langle \text{Tr } L \rangle = -1$) then the Fermi distribution in such a constant background $A_0$ field turns into a Bose distribution but with

$$n(p) = -\frac{1}{e^{\beta E(p)} - 1}$$

(14)

Note the minus sign in front of the expression! This is related to the negative entropy of the system.

The lesson of the above discussion is that $Z_N$ bubbles should not be interpreted as physical bubbles any more than Instantons should be interpreted as real Minkowski objects. But these $Z_N$ bubbles do contribute to the Partition Function and to expectation values of observables if they are calculated using the Euclidean Path Integral. They should thus be included in a non–perturbative analysis of the thermodynamics of QCD. The physical interpretation of these bubbles is further discussed in a paper by A.V. Smilga presented at this Workshop.

7. Summary

- The $Z_N$ symmetry for $SU_N$ gauge theories plays an important role in the Euclidean analysis of its thermodynamics.
- $Z_N$ bubbles are likely to play a role in the confining–deconfining phase transition.
- Fermions in the Fundamental representation break the $Z_N$ symmetry.
- The resulting metastable extrema of the Effective Potential should not be interpreted as physically attainable states nor should the $Z_N$ bubbles which attain these extrema in their cores.
- These metastable extrema do however contribute to the thermodynamics of the system.

REFERENCES

1. G. ’t Hooft, Nucl. Phys. B138 1 (1987)
2. See for example D.J. Gross, R.D. Pisarski, and L.G. Yaffe, Rev. Mod. Phys. 53 43 (1981) and references therein.
3. A.M. Polyakov, Phys. Lett. 72B 477 (1978); B.Svetitsky and L.G. Yaffe, Nucl. Phys. B210 423 (1982); L. McLaren, B. Svetitsky, Phys. Rev. D24 450 (1981); J. Kuti, J. Polonyi, K. Szlachanyi, Phys. Lett. 98B 199 (1981)
4. Nathan Weiss, Phys. Rev. D24 475 (1981); Phys. Rev. D25 2667 (1982)
5. T. Bhattacharya, A. Gocksch, C. Korthals Altes and R.D. Pisarski, Phys. Rev. Lett. 66 998 (1991); Nucl. Phys. B383 497 (1992)
6. V. Dixit, M.C. Ogilvie, Phys. Lett. B269 353 (1991); J. Ignatius, K. Kajantie, K. Rummakainen, Phys. Rev. Lett. 68 737 (1992)
7. V.M. Belyaev, I. Kogan, G.W. Semenoff, Nathan Weiss, Phys. Lett. B277 331 (1992)