Higher dimensional gravity invariant under the Poincarè group

P. Salgado\(^1\)*, M. Cataldo\(^3\)† and S. del Campo\(^4\)‡

\(^1\)Departamento de Física, Universidad de Concepción, Casilla 160-C, Concepción, Chile.
\(^2\)Departamento de Física, Universidad del Bío-Bío, Casilla 5-C, Concepción, Chile.
\(^3\)Instituto de Física, Universidad Católica de Valparaíso, Avda Brasil 2950, Valparaíso, Chile.

It is shown that the Stelle-West Grignani-Nardelli-formalism allows, both when odd dimensions and when even dimensions are considered, constructing actions for higher dimensional gravity invariant under local Lorentz rotations and under local Poincaré translations. It is also proved that such actions have the same coefficients as those obtained by Troncoso and Zanelli in ref. [9].

I. INTRODUCTION

The most general action for the metric satisfying the criteria of general covariance and second-order field equations for \(d > 4\) is a polynomial of degree \([d/2]\) in the curvature known as the Lanczos-Lovelock gravity theory (LL) [1, 2]. The LL lagrangian in a \(d\)-dimensional Riemannian manifold can be defined as a linear combination of the dimensional continuation of all the Euler classes of dimension \(2p < d\) [3, 4].

\[
S = \int \sum_{p=0}^{[d/2]} \alpha_p L^{(p)}
\]  

(1)

where \(\alpha_p\) are arbitrary constants and

\[
L_p = \varepsilon_{a_1a_2\ldots a_d} R^{a_1a_2} \cdots R^{a_{2p-1}a_{2p}} e^{a_{2p+1}} \cdots e^{a_d}
\]

(2)

with \(R^{ab} = d\omega^{ab} + \omega^a \omega^b - \omega^b \omega^a\). The expression (1) can be used both for even and for odd dimensions.

The large number of dimensionful constants in the LL theory \(\alpha_p\), \(p = 0, 1, \ldots, [d/2]\), which are not fixed from first principles, contrast with the two constants of the Einstein-Hilbert action.

In ref. [9] it was found that these parameters can be fixed in terms of the gravitational and the cosmological constants, and that the action in odd dimensions can be formulated as a gauge theory of the AdS group, and in a particular case, as a gauge theory of the Poincaré group. This means that the action is invariant not only under standard local Lorentz rotations \(\delta e^a = \kappa^a \xi^b; \delta \omega^{ab} = -D \kappa^{ab}\), but also under local \(AdS\) boost \(\delta e^a = D \rho^a\); \(\delta \omega^{ab} = \frac{1}{l^{4-d}} (\rho^a \omega^b - \rho^b \omega^a)\), where \(l\) is a length parameter. They also show that this situation is not possible in even dimensions where the action is invariant only under local Lorentz rotations in the same way as is the Einstein-Hilbert action.

On the other hand, in ref. [6] it was found that, in four dimensions, gravity can be formulated as a gauge theory of the Poincaré group. In this reference was considered the Poincaré gauge theory as closely as possible to any ordinary non-Abelian gauge theory.

The basic idea of this formulation is founded on the mathematical definition [10] of the vierbein \(V^a\). This vierbein, also called solder form [11] was considered as a smooth map between the tangent space to the space-time manifold \(M\) at a point \(P\) with coordinates \(x^\mu\), and the tangent space to the internal \(AdS\) space at the point whose \(AdS\) coordinates are \(\xi^a(x)\), as the point \(P\) ranges over the whole manifold \(M\). The fig.1 of ref. [12] illustrates that such a vierbein \(V^a_m(x)\) is the matrix of the map between the tangent space \(T_x(M)\) to the space-time manifold at \(x^\mu\), and the tangent space \(T_{\xi(x)}(\{G/H\}_x)\) to the internal \(AdS\) space \(\{G/H\}_x\) at the point \(\xi^a(x)\), whose explicit form is given by eq. (3.19) of ref. [12].

Taking the limit \(m \to 0\) in such eq. (3.19) of ref. [12] we obtain \(V^a_m(x) = D \mu e^a + \epsilon^a_\mu\) which is the map between the tangent space \(T_x(M)\) to the space-time manifold at \(x^\mu\) and the tangent space \(T_{\xi(x)}(\{ISO(3,1)/SO(3,1)\}_x)\) to the internal Poincaré space \(\{ISO(3,1)/SO(3,1)\}_x\) at the point \(\xi^a(x)\). The same result was obtained in ref. [13] by gauging the action of a free particle defined in the internal Minkowski space. Here \(\xi^a\), called Poincaré coordinates, transform as a vector under Poincaré transformations and can be interpreted as coordinates of an internal Minkowskian space \(M_\xi\) that can locally be made to coincide with the tangent space. Any choice of Poincaré coordinates is equivalent to a gauge that leaves the theory invariant under residual local Lorentz transformations.

It is the purpose of this article to show that it is also possible to find an action for higher dimensional gravity genuinely invariant under the Poincaré group provided that one chooses the vierbein in an appropriate way.
II. THE LANCZOS-LOVELOCK GRAVITY THEORY

In this section we shall review some aspects of higher dimensional gravity. The main point of this section is to display the differences between the invariances of LL action when odd and even dimensions are considered.

A. The local AdS Chern-Simons and Born-Infeld like gravity

The LL action is a polynomial of degree \([d/2]\) in curvature, which can be written in terms of the Riemann curvature and the vielbein \(e^a\) in the form \([1], [2]\). In first order formalism the LL action is regarded as a functional of the vielbein and spin connection, and the corresponding field equations obtained by varying with respect to \(e^a\) and \(\omega^{ab}\) read \([3]\):

\[
\varepsilon_a = \sum_{p=0}^{[(d-1)/2]} \alpha_p (d-2p) \varepsilon^p_a = 0 \tag{3}
\]

\[
\varepsilon_{ab} = \sum_{p=1}^{[(d-1)/2]} \alpha_p p(d-2p) \varepsilon^p_{ab} = 0 \tag{4}
\]

where we have defined

\[
\varepsilon^p_a := \varepsilon_{ab_1 b_2 \ldots b_{d-1}} R^{b_1 b_2} \ldots
\]

\[
\ldots R^{b_2 b_{p-1} b_{p+1} \ldots} e^{b_{d-1}}
\]

\[
\varepsilon^p_{ab} = \varepsilon_{ab a_3 \ldots a_d} R^{a_3 a_4} \ldots
\]

\[
\ldots R^{a_2 a_{p-1} a_p} T^{a_{p+1} a_{p+2} \ldots} + e^{a_{d}}.
\]

Here \(T^a = de^a + \omega^a_b e^b\) is the torsion 2-form. Using the Bianchi identity one finds \([4]\)

\[
D\varepsilon_a = \sum_{p=1}^{[(d-1)/2]} \alpha_{p-1} (d-2p+2)(d-2p+1) e^b \varepsilon^p_{ba}. \tag{7}
\]

Moreover

\[
e^b \varepsilon_{ba} = \sum_{p=1}^{[(d-1)/2]} \alpha_p p(d-2p) e^b \varepsilon^p_{ba}. \tag{8}
\]

From \([5]\) and \([8]\) one finds for \(d = 2n - 1\)

\[
\alpha_p = \alpha_0 \left(\frac{2n-1}{2n-2p-1}\right)^p \left(\frac{n}{p}\right) ; \quad \alpha_0 = \frac{\kappa}{d!d-1};
\]

\[
\gamma = -\text{sign}(\Lambda) \frac{l^2}{2}, \tag{9}
\]

where for any dimensions \(l\) is a length parameter related to the cosmological constant by \(\Lambda = \pm (d-1)(d-2)/2l^2\).

With these coefficients, the LL action is invariant under local Lorentz rotations and under local AdS boosts. For \(d = 2n\) it is necessary to write equation \([9]\) in the form \([10]\)

\[
D\varepsilon_a = T^a \sum_{p=1}^{[n-1]} 2\alpha_{p-1} (n-p+1) T^p_{ab} - \sum_{p=1}^{[n-1]} 4\alpha_{p-1} (n-p+1)(n-p)e^b \varepsilon^p_{ba} \tag{10}
\]

with

\[
T^p_{ab} = \frac{\delta L}{\delta R^p_{ab}} = \sum_{p=1}^{[(d-1)/2]} \alpha_p p T^p_{ab} \tag{11}
\]

where

\[
T^p_{ab} = \varepsilon_{aba_3 \ldots a_d} R^{a_3 a_4} \ldots
\]

\[
\ldots R^{a_2 a_{p-1} a_p} T^{a_{p+1} a_{p+2} \ldots} + e^{a_d}.
\]

The comparison between \([11]\) and \([12]\) leads to \([13]\)

\[
\alpha_p = \alpha_0 (2\gamma)^p \left(\frac{n}{p}\right). \tag{13}
\]

With these coefficients the LL action, in the same way as the Hilbert action, is invariant only under local Lorentz rotations.

B. Theories described by a generalized action

Recently a class of gravitation theories was found \([14]\) described by the action

\[
S = \int \sum_{p=0}^{k} \alpha_p L^{(p)} = \kappa \int \sum_{p=0}^{k} C^k_p L^{(p)} \tag{14}
\]

where

\[
\alpha_p = \kappa C^k_p = \left(\frac{\alpha_0 (2\gamma)^p}{k!}\right) \left(\frac{n-1}{p}\right), \quad p \leq k
\]

with \(1 \leq k \leq [(d-1)/2]\) and where \(L_p\) is given by

\[
L_p = \varepsilon_{a_1 a_2 \ldots a_d} R^{a_1 a_2} \ldots R^{a_{2p-1} a_{2p}} e^{a_{2p+1}} \ldots e^{a_d}
\]

where \(R^{ab} = d\omega^{ab} + \omega^a_c \omega^c_b\) is the curvare 2-form and \(e^a\) is the vielbein 1-form. For a given dimension \(d\), the \(\alpha_p\)
coefficients give rise to a family of inequivalent theories, labeled by the integer \( k \in \{1, \ldots, \left[(d-1)/2\right]\} \) which represents the highest power of curvature in the lagrangian. This set of theories possesses only two fundamental constants, \( \kappa \) and \( l \), related to the gravitational constant \( G \) and the cosmological constant \( \Lambda \). For \( k = 1 \), the Einstein-Hilbert action is recovered, while for the largest value of \( k \), that is \( k = \left[(d-1)/2\right] \), Born-Infeld and Chern-Simons theories are obtained. These three cases exhaust the different possibilities up to six dimensions, and new interesting cases arise for \( d \geq 7 \). For instance, the case with \( k = 2 \), which is described by the action

\[
S_2 = \kappa \int \epsilon_{a_1a_2\cdots a_d} \left( \frac{l^4}{d} e^{a_1} \cdots e^{a_d} + \frac{2l^2}{d-2} R^{a_1a_2} e^{a_3} \cdots e^{a_d} \right)
\]

exists only for \( d > 4 \); in five dimensions this theory is equivalent to Chern-Simons; for \( d = 6 \) it is equivalent to Born-Infeld and Chern-Simons theories are obtained. These three cases exhaust the maximum value of \( k \) is \( n-1 \), and the corresponding lagrangian of the action

\[
S = \int \left[ \sum_{p=0}^{[d/2]} \frac{\kappa}{2l} \left( \frac{n-1}{p} \right)^{2p-d+1} \epsilon_{a_1a_2\cdots a_d} R^{a_1a_2} \right]
\]

\[
\cdots R^{a_2p-1a_2p} e^{a_2p+1} \cdots e^{a_d}
\]

is a Chern-Simons \((2n-1)\)-form. This action is invariant not only under standard local Lorentz rotations

\[
\delta e^a = \kappa b^a e^b, \quad \delta \omega^{ab} = -D \kappa^{ab},
\]

but also under a local \( AdS \) boost

\[
\delta e^a = D \rho^a, \quad \delta \omega^{ab} = \frac{1}{l^2} \left( \rho^a e^b - \rho^b e^a \right)
\]

For \( d = 2n \) and \( k = n-1 \), the action is given by

\[
S = \int \left[ \sum_{p=0}^{[d/2]} \frac{\kappa}{2l} \left( \frac{n}{p} \right)^{2p-d+1} \epsilon_{a_1a_2\cdots a_d} R^{a_1a_2} \right]
\]

\[
\cdots R^{a_2p-1a_2p} e^{a_2p+1} \cdots e^{a_d}
\]

This action is invariant under standard local Lorentz rotations but it is not invariant under local \( AdS \) boosts.

III. LANCZOS-LOVELOCK THEORY AND THE POINCARÈ GROUP

In this section we shall review some aspects of the Stelle-West-Grignani-Nardelli (SWGN)-formalism. This formalism leads to a formulation of general relativity where the Hilbert action is invariant both under local Poincaré translations and under local Lorentz rotations. The main point of this section is to show that the (SWGN) formalism permits, both when odd dimensions and when even dimensions are considered, constructing a higher dimensional gravity action invariant under local Lorentz rotations and under local Poincaré translations. It is also proved that such an action has the same coefficients as those obtained in ref. [3].

A. Einstein gravity invariant under the Poincarè group

1. Invariance of the Hilbert action

The generators of the Poincaré group \( P_a \) and \( J_{ab} \) satisfy the Lie algebra,

\[
[P_a, P_b] = 0;
\]

\[
[J_{ab}, P_c] = \eta_{ac} P_b - \eta_{bc} P_a;
\]

\[
[J_{ab}, J_{cd}] = \eta_{ac} J_{bd} - \eta_{bc} J_{ad} + \eta_{bd} J_{ac} - \eta_{ad} J_{bc}.
\]

Here the operators carry Lorentz indices not related to coordinate transformations. The Yang-Mills connection for this group is given by

\[
A = A^a T_a = e^a P_a + \frac{1}{2} \omega^{ab} J_{ab}.
\]

Using the algebra (2) and the general form for the gauge transformations on \( A \)

\[
\delta A = \nabla \lambda = d \lambda + [A, \lambda]
\]

with

\[
\lambda = \rho^a P_a + \frac{1}{2} \omega^{ab} J_{ab},
\]

we obtain that \( e^a \) and \( \omega^{ab} \), under Poincarè translations, transform as

\[
\delta e^a = D \rho^a; \quad \delta \omega^{ab} = 0,
\]

and under Lorentz rotations, as

\[
\delta e^a = \kappa b^a e^b; \quad \delta \omega^{ab} = -D \kappa^{ab},
\]

where \( D \) is the covariant derivative in the spin connection \( \omega^{ab} \). The corresponding curvature is
The space-time metric is postulated to be
\[ g_{\mu\nu} = \eta_{ab} V^a_{\mu} V^b_{\nu} \]
with \( \eta_{ab} = (-1, 1, 1, 1) \). Thus the corresponding curvature is given by (28), but now (29) does not correspond to the space-time torsion because the vierbein is not given by \( e^a \). The space-time torsion \( \mathcal{T}^a \) is given by
\[ \mathcal{T}^a = D V^a = T^a + R^{ab} \xi_b. \]

3. Hilbert action invariant under the Poincaré group

Within the (SWG) formalism the Hilbert action can be rewritten as
\[ S_{EH} = \int \varepsilon_{abcd} V^a V^c R^{cd} \]
which is invariant under general coordinate transformations, under local Lorentz rotations, as well as under local Poincaré translations. In fact
\[ \delta S_{EH} = \int \varepsilon_{abcd} \delta \left( R^{ab} V^c V^d \right) \]
\[ \delta S_{EH} = 2 \int \varepsilon_{abcd} R^{ab} V^c \delta V^d = 0. \]
Thus the action is genuinely invariant under the Poincaré group without imposing a torsion-free condition.

The variations of the action with respect to \( \xi^a, e^a, \omega^{ab} \) lead to the following equations:
\[ \varepsilon_{abcd} T^b R^{cd} = 0 \]
\[ \varepsilon_{abcd} V^b R^{cd} = 0 \]
\[ \varepsilon_{[acd]b} V^c R^{cd} + \varepsilon_{abcd} V^c T^d = 0 \]
which reproduce the correct Einstein equations:
\[ T^a = D V^a = 0 \]
\[ \varepsilon_{abcd} V^b R^{cd} = 0. \]

The commutator of two local Poincaré translations is given by
\[ [\delta(\rho_2), \delta(\rho_1)] = 0 \]
i.e. the local Poincaré translations now commute. The rest of the algebra is unchanged. Thus the Poincaré algebra closes off-shell. This fact has deep consequences in supergravity.

2. The Stelle-West Grignani-Nardelli formalism

The central ingredient of the (SWGN) formalism are the so called Poincaré coordinates \( \xi^a(x) \) which behave as vectors under ISO(3,1) and are involved in the definition of the 1-form vierbein \( V^a \), which is not identified with the component \( e^a \) of the gauge potential, but is given by
\[ V^a = D \xi^a + e^a = d \xi^a + \omega_0^a e^b + e^a. \]

Since \( \xi^a, e^a, \omega^{ab} \) under local Poincaré translations change as
\[ \delta \xi^a = -\rho^a, \quad \delta e^a = D \rho^a, \quad \delta \omega^{ab} = 0; \]
and under local Lorentz rotations change as
\[ \delta \xi^a = \kappa^a_b \xi^b, \quad \delta e^a = \kappa^a_b e^b, \quad \delta \omega^{ab} = -D \kappa^{ab}; \]
we have that the vierbein \( V^a \) is invariant under local Poincaré translations
\[ \delta V^a = 0 \]
and, under local Lorentz rotations, transforms as
\[ \delta V^a = \kappa^a_b V^b. \]
B. Lanczos-Lovelock Theory invariant under the Poincaré group

1. Invariance of $LL$ action

Within the (SWGN)-formalism the action for $LL$ gravity can be rewritten as

$$S = \int \sum_{p=0}^{[d/2]} \alpha_p \varepsilon_{a_1 a_2 \ldots a_d} R^{a_1 a_2} \ldots$$

$$\ldots R^{a_2 p-1 a_2 p} V^{a_2 p+1} \ldots V^{a_d}$$

with $R^{ab} = d\omega^{ab} + \omega_c^{[ab} \omega_c^{ab]}$ and $V^a$ is given in (34). This action is, both when odd dimensions and when even dimensions are considered, invariant under local Lorentz rotations and under local Poincaré translations as well as under diffeomorphism. In fact

$$\delta S = \int \sum_{p=0}^{[d/2]} \alpha_p \varepsilon_{a_1 a_2 \ldots a_d} \delta R^{a_1 a_2} \ldots$$

$$\ldots R^{a_2 p-1 a_2 p} \delta V^{a_2 p+1} \ldots \delta V^{a_d} = 0$$

where we have used the invariance of $V^a$ under local Poincaré translation (27). Thus the $LL$ action is genuinely invariant under the Poincaré group.

2. Equations of motions

The variations of the action with respect to $e^a, \omega^{ab}, \xi^a$ lead to the following equations of motion:

$$\sum_{p=0}^{[(d-1)/2]} \alpha_p (d - 2p) \varepsilon_{a_1 a_2 \ldots a_d} R^{a_1 a_2} \ldots$$

$$\ldots R^{a_2 p-1 a_2 p} V^{a_2 p+1} \ldots V^{a_d} \cdot 1 = 0$$

$$\sum_{p=1}^{[(d-1)/2]} p(d - 2p) \varepsilon_{a_1 a_2 \ldots a_d} R^{a_1 a_2} \ldots$$

$$\ldots R^{a_2 p-1 a_2 p} V^{a_2 p+1} \ldots V^{a_d}$$

$$\varepsilon_{a_1 \ldots a_d} R^{a_1 a_2} \ldots R^{a_2 p-1 a_2 p} V^{a_2 p+1} \ldots V^{a_d-1} \xi_b = 0$$

$$\sum_{p=0}^{[(d-1)/2]} \alpha_p (d - 2p)(d - 2p - 1) \varepsilon_{a_1 a_2 \ldots a_d} R^{a_1 a_2} \ldots$$

$$\ldots R^{a_2 p-1 a_2 p} V^{a_2 p+1} \ldots V^{a_d-2} T^{a_d-1} = 0.$$ (54)

The field equation corresponding to the variation of the action (50) with respect to $\xi^a$ is not an independent equation. In fact, taking the covariant derivative operator $D$ of equation (53) we obtain the same equation that one obtains by varying the action (50) with respect to $\xi^a$. Furthermore, equations (52),(53) are also not completely independent. In fact taking the product of (52) with $-\xi_b$ and then taking the addition with (53) we obtain

$$\sum_{p=1}^{[(d-1)/2]} p(d - 2p) \alpha_p \varepsilon_{a_1 a_2 \ldots a_d} R^{a_1 a_2} \ldots$$

$$\ldots R^{a_2 p-1 a_2 p} T^{a_2 p+1} V^{a_2 p+2} \ldots V^{a_d} = 0.$$ (55)

This means that the equations (52),(53) are independent. The equations (52),(53) can be rewritten in the form (3), (4), where now

$$\varepsilon_a^p := \varepsilon_{a b_1 b_2 \ldots b_{d-1}} R^{b_1 b_2} \ldots$$

$$\ldots R^{b_2 p-1 b_2 p} V^{b_2 p+1} \ldots V^{b_{d-1}}$$ (56)

$$\varepsilon_{a b} := \varepsilon_{a b a_3 \ldots a_d} R^{a_3 a_4} \ldots$$

$$\ldots R^{a_2 p-1 a_2 p} T^{a_2 p+1} V^{a_2 p+2} \ldots V^{a_d}$$ (57)

with $T^a = D V^a$.

It is easy to check that, taking the covariant derivative of (52) and using the Bianchi identities, one obtains

$$D \varepsilon_a^p = (d - 2p - 1) V^b \varepsilon_{a b}^{p+1},$$ (58)

for $0 \leq p \leq \lfloor (d - 1)/2 \rfloor$, which leads to the following off-shell identity:

$$D \varepsilon_a = \sum_{p=1}^{\lfloor (d+1)/2 \rfloor} \alpha_{p-1} (d - 2p + 2)(d - 2p + 1) V^b \varepsilon_{a b}^p$$ (59)

which, by consistency with (3), must also vanish. On the other hand, taking the exterior product of (4) with $V^b$ one obtains

$$V^b \varepsilon_{b a} = \sum_{p=1}^{\lfloor (d-1)/2 \rfloor} \alpha_{p+1} (d - 2p) V^b \varepsilon_{a b}^{p-1}$$ (60)

which, by consistency with (4), must also vanish.

Now we show that, following the same method used in ref. [5], the action (50) has the same coefficients as those obtained in ref. [6] for the action (1).
3. Coefficients $\alpha_p$ for $d = 2n - 1$

Following the same procedure of ref. 3 one can see that in odd dimensions, (59) and (60) have the same number of terms because the last term in (59) vanishes. Now, if (59) and (60) are to impose no further algebraic constraint on $R_{ab}$ and $T^a$, then $D\varepsilon_a$ and $V^b\varepsilon_{ba}$ must be proportional term by term, which implies the following recursion relation for the coefficients:

$$\frac{\alpha_p}{\alpha_{p+1}} = \frac{(p+1)(d-2p+2)}{(d-2p)(d-2p+1)}$$

whose solution is given by (61). Thus, the action contain only two fundamental constants $\alpha_0$ and $\gamma$, related to the gravitational and the cosmological constant.

Therefore, the use of the (SWG) formalism does not change the coefficients $\alpha_p$ of the action already obtained in ref. 3, but now the $LL$ action for $d = 2n - 1$ is genuinely invariant under the Poincaré group.

From the action (54) and the equations (52), (53), (54) we can see that, once the gauge $\xi = 0$ is chosen, from equations (54), (57) it follows that $V^a = \varepsilon^a$, $T^a = T^a = D\varepsilon^a$, and that the action (54) takes the form (1) and the equations (52), (53), (54) take the forms of the equations for $LL$ gravity theory as developed in refs. 4, 5.

We must also note that the action (1) for $d = 2n - 1$ is invariant under the Poincaré group only when $\alpha_p = \alpha_n = 1$ and $\alpha_p = 0 \forall p \neq n$. In this case both formalisms coincide. In fact, for $\alpha_n = 1$ and $\alpha_p = 0$, (54) takes the form

$$S = \int \varepsilon_{a_1a_2\ldots a_d}R^{a_1a_2} \ldots R^{a_{d-2}a_{d-1}}V^{a_d}$$

with $V^d = \varepsilon^d + D\xi^d$. Using the Bianchi identities it is direct to see that, up to a surface term,

$$S = \int \varepsilon_{a_1a_2\ldots a_d}R^{a_1a_2} \ldots R^{a_{d-2}a_{d-1}}\varepsilon^{a_d}.$$

4. Coefficients $\alpha_p$ for $d = 2n$

In even dimensions, equation (59) has one more term than (60). This means that (59) and (60) cannot be compared term by term. Following the same procedure of ref. 3 we find that for $d = 2n$

$$\varepsilon_{ab} = D\chi_{ab}$$

where

$$\chi_{ab} = \sum_{p=1}^{[d-1]/2} p\alpha_p\chi_{ab}^p$$

and

$$\chi_{ab}^p = \varepsilon_{ab_1a_2\ldots a_d}R^{a_1a_2} \ldots \varepsilon_{a_{d-p}a_{d-p+1}}V^{a_{d-p+2}} \ldots V^{a_d}$$

$$R^{a_2p-1a_2p}V^{a_2p+1} \ldots V^{a_d}$$

$$V^b\chi_{ab}^p = \varepsilon^b_{a_1a_2} \ldots \chi_{ab_1a_2} \ldots \varepsilon^{a_d}$$

$$D\chi_{ab}^p = (d - 2p)\varepsilon_{ab}.$$

From (59) and (60) we find that (54) can also be written

$$D\varepsilon_a = \sum_{p=1}^{[n-1]} 2\alpha_{p-1}(n-p+1)\chi_{ab}^p T^b$$

$$- \sum_{p=1}^{[n-1]} 4\alpha_{p-1}(n-p+1)(n-p)V^b\varepsilon_{ba}.$$

This equation can be compared with (see 61)

$$V^b\varepsilon_{ba} = \sum_{p=1}^{n-1} 2p\alpha_p(n-p)V^b\varepsilon_{ba}.$$

Both (64) and (65) can be zero if either $T^a = 0$, or $\chi_{ab} = 0$. These conditions are excessive for the vanishing of (60). In the same way as in ref. 3, it is sufficient to impose the weaker conditions $\varepsilon_{ab} = 0$, and, at the same time, that the second term in (69) be proportional to (70). Now, (64) and (65) possess the same number of terms, which implies the following recursion relation for the coefficients:

$$2\gamma(n-p+1)\alpha_{p-1} = p\alpha_p,$$

whose solution is given by (61).

Thus, the action for $d = 2n$ contains only two fundamental constants $\alpha_0$ and $\gamma$, related to the gravitational and the cosmological constants.

Therefore, in the same way as for $d = 2n - 1$, the use of the (SWG) formalism does not change the coefficients $\alpha_p$ of the action already obtained in ref. 3. However, now the $LL$ action for $d = 2n$ is genuinely invariant under the Poincaré group.

From the action (54) and the equations (52), (53), (54) we can see that, once the gauge $\xi = 0$ is chosen, from equations (54), (57) it follows that $V^a = \varepsilon^a$, $T^a = T^a = D\varepsilon^a$, and that the action (54) takes the form (1) and the equations (52), (53), (54) take the forms of the equations for $LL$ gravity theory as developed in refs. 4, 5.

We must to note that, within the SWG formalism, the action (14) can be rewritten as

$$S = \sum_{p=1}^{K} \frac{\gamma}{d-2p} \left(\begin{array}{c} k \\frac{p}{2} \end{array}\right)^2 \varepsilon_{a_1a_2\ldots a_d}R^{a_1a_2} \ldots \varepsilon_{a_{d-p}a_{d-p+1}}V^{a_{d-p+2}} \ldots V^{a_d}$$

which is invariant under local Lorentz rotations as well as under Poincaré translations.
IV. COMMENTS

We have shown in this work that the successful formalism used by Stelle-West and Grignani-Nardelli to construct an action for (3 + 1)-dimensional gravity invariant under the Poincaré group can be generalized to arbitrary dimensions. The main result of this paper is that we have shown that the (SWGN) formalism permits, both when odd dimensions and when even dimensions are considered, constructing a higher dimensional gravity action invariant under local Lorentz rotations and under local Poincare translations. This means using the vierbein $V^a$ which involves in its definition the so called "Poincare coordinates" $\xi^a(x)$. It is also proved that such actions have the same coefficients as those obtained in ref.

It is perhaps interesting to note that if one considers, following ref., $g_{\mu\nu} = V^a_{\mu}V^b_{\nu}\eta_{ab}$, one can write the lagrangian of the action in the form that was written in ref. This means that, if one considers the theory constructed in terms of the space-time metric $g_{\mu\nu}$, ignoring the underlying formulation, the theory described in our manuscript is completely equivalent to the theory developed in refs. No trace of the new structure of the vierbein existing in the underlying formulation of the theory can be found at the metric level.

Acknowledgments

This work was supported in part by FONDECYT through Grants # 1010485 (M.C and P.S) and # 1000305 (S del C) and in part by UCV through Grant UCV-DGIP # 123.752/00.

[1] C. Lanczos, Ann. Math. 39 (1938) 842.
[2] D. Lovelock, J. Math. Phys. 12 (1971) 498.
[3] B. Zumino, Phys. Rep. 137 (1986) 109.
[4] C. Teitelboim and J. Zanelli, Class. and Quantum Grav. 4 (1987) L125.
[5] R. Troncoso, J. Zanelli, Class. Quantum Grav. 17 (2000) 4451.
[6] G. Grignani and G. Nardelli, Phys. Rev. D45 (1992) 2719.
[7] S. Kobayashi and K. Nomizu, Foundations of Differential Geometry, Volume 1, J. Wiley 1963, Chapter III.
[8] E. Witten, Nucl. Phys. B 311 (1988) 46.
[9] K.S. Stelle, P.C. West, Phys. Rev. D21 (1980) 1466.
[10] P. Salgado, M. Cataldo and S. del Campo, Phys. Rev D65 (2002) 0840XX; Preprint gr-qc/0110097.
[11] J. Crisostomo, R. Troncoso, J. Zanelli, Phys. Rev. D62 (2000) 084013
[12] M. Bañados, R. Troncoso, J. Zanelli, Phys. Rev. D54 (1996) 2605.