The local metric dimension of starbarbell graph, \(K_m \odot P_n\) graph, and Möbius ladder graph

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Abstract. For an ordered set \(W = \{w_1, w_2, ..., w_n\}\) of \(n\) distinct vertices in a nontrivial connected graph \(G\), the representation of a vertex \(v\) of \(G\) with respect to \(W\) is the \(n\)-vector \(r(v|W) = (d(v, w_1), d(v, w_2), ..., d(v, w_n))\). \(W\) is a local metric set of \(G\) if \(r(u|W) \neq r(v|W)\) for every pair of adjacent vertices \(u, v\) in \(G\). Local metric set with minimum cardinality is called local metric basis of \(G\) and its cardinality is the local metric dimension of \(G\) and denoted by \(\text{lmd}(G)\). Starbarbell graph \(SB_{m_1, m_2, ..., m_n}\) is a graph obtained from a star graph \(S_n\) and \(n\) complete graphs \(K_{m_i}\) by merging one vertex from each \(K_{m_i}\) and the \(i\)-th leaf of \(S_n\), where \(m_i \geq 3, 1 \leq i \leq n, n \geq 2\). \(K_m \odot P_n\) graph is a graph obtained from a complete graph \(K_m\) and \(m\) copies of path graph \(P_n\), and then joining by an edge each vertex from the \(i\)-th copy of \(P_n\) with the \(i\)-th vertex of \(K_m\). Möbius ladder graph \(M_n\) is a graph obtained from a cycle graph \(C_n\) by connecting every pair of vertices \(u, v\) in \(C_n\) if \(d(u, v) = \text{diam}(C_n)\) for \(n \geq 5\). In this paper, we determine the local metric dimension of starbarbell graph, \(K_m \odot P_n\) graph, and Möbius ladder graph for even positive integers \(n \geq 6\).

1. Introduction

In 1975, Slater [9] introduced the concept of metric dimension of a graph, where metric generator was called locating set. Then Harary and Melter [3] independently introduced the same concept in 1976, where metric generator was called resolving set. Let \(G\) be a connected graph with vertex set \(V(G)\). A set \(W \subset V(G)\) is called a metric generator of \(G\) if \(d(u, x) \neq d(v, x)\) for any pair of vertices \(u, v, x\) in \(G\) and \(x\) in \(W\), where \(d(u, x)\) is the distance between \(u\) and \(x\). Until now, there are several variations of metric generator, and one of them was introduced by Okamoto et al. [7] in 2010. For an ordered set \(W = \{w_1, w_2, ..., w_n\}\) of \(n\) distinct vertices in a nontrivial connected graph \(G\), the representation of a vertex \(v\) of \(G\) with respect to \(W\) is the \(n\)-vector \(r(v|W) = (d(v, w_1), d(v, w_2), ..., d(v, w_n))\). The set \(W\) is a local metric set of \(G\) if \(r(u|W) \neq r(v|W)\) for every pair of adjacent vertices \(u, v\) in \(G\). The local metric set \(W\) with minimum cardinality is called local metric basis of \(G\) and the cardinality of local metric basis of \(G\) is the local metric dimension of \(G\) and denoted by \(\text{lmd}(G)\).

Some authors have investigated the local metric dimension of some graph classes. In 2014 Kristina et al. [5] found the local metric dimension of comb product of cycle graph and star graph. In the same year, Ningsih et al. [6] found the local metric dimension of comb product of cycle graph and path graph. Then in 2015, Barragán-Ramírez et al. [1] observed the local metric dimension of subgraph amalgamation of graphs. In 2016 Rodríguez-Velázquez et al. [8] obtained the local metric dimension of corona product graphs. In this paper, we determine the
local metric dimension of starbarbell graph $SB_{m_1,m_2,\ldots,m_n}$, $K_m \odot P_n$ graph, and Möbius ladder graph $M_n$ for even positive integers $n$.

2. Main Results

2.1. The Local Metric Dimension of Starbarbell Graph

The starbarbell graph $SB_{m_1,m_2,\ldots,m_n}$ is a graph obtained from a star graph $S_n$ and $n$ complete graphs $K_{m_i}$ by merging one vertex from each $K_{m_i}$ and the $i^{th}$ leaf of $S_n$, where $m_i \geq 3$, $1 \leq i \leq n$, and $n \geq 2$. The starbarbell graph $SB_{m_1,m_2,\ldots,m_n}$ can be depicted as in Figure 1.

![Figure 1. Starbarbell graph $SB_{m_1,m_2,\ldots,m_n}$](image)

**Theorem 2.1** Let $SB_{m_1,m_2,\ldots,m_n}$ be the starbarbell graph, then $\text{lmd}(SB_{m_1,m_2,\ldots,m_n}) = \sum_{i=1}^{n} (m_i - 2)$.

**Proof.** Let $SB_{m_1,m_2,\ldots,m_n}$ be the starbarbell graph with $m_i \geq 3$, $1 \leq i \leq n$, and $n \geq 2$.

(i) We will show $\text{lmd}(SB_{m_1,m_2,\ldots,m_n}) \geq \sum_{i=1}^{n} (m_i - 2)$.

To prove this case, the vertex set $V(SB_{m_1,m_2,\ldots,m_n})$ is partitioned into $n+1$ sets,

\[
\begin{align*}
V_1 &= \{v_{1,1}, v_{1,2}, \ldots, v_{1,m_1}\}, \\
V_2 &= \{v_{2,1}, v_{2,2}, \ldots, v_{2,m_2}\}, \\
&\vdots \\
V_n &= \{v_{n,1}, v_{n,2}, \ldots, v_{n,m_n}\}, \\
V_{n+1} &= \{u\}.
\end{align*}
\]

The cardinality of $V(SB_{m_1,m_2,\ldots,m_n})$ is $\sum_{i=1}^{n} m_i + 1$. Assume $W \subseteq V(SB_{m_1,m_2,\ldots,m_n})$ is chosen such that $|W| < \sum_{i=1}^{n} (m_i - 2)$, then $|W^c| = |V(SB_{m_1,m_2,\ldots,m_n}) - W| > 2n + 1$.

Based on pigeonhole principle, there will be at least one partition $V_i$ for $1 \leq i \leq n$ which at least three elements of $V_i$ are elements of $W^c$ too. Thus, there will be at least two distinct adjacent vertices $v_{i,j}$ and $v_{i,k}$, with $2 \leq j, k \leq m_i$ where

\[
\begin{align*}
d(v_{i,j}, v_{i,1}) &= d(v_{i,k}, v_{i,1}) = 1, \\
d(v_{i,j}, v_{i,p}) &= d(v_{i,k}, v_{i,p}) = 1, \quad 2 \leq p \leq m_i, j \neq p \neq k, \\
d(v_{i,j}, u) &= d(v_{i,k}, u) = 2, \\
d(v_{i,j}, v_{q,1}) &= d(v_{i,k}, v_{q,1}) = 3, \quad 1 \leq q \leq n, i \neq q, \text{ and} \\
d(v_{i,j}, v_{q,r}) &= d(v_{i,k}, v_{q,r}) = 4, \quad 2 \leq r \leq m_q.
\end{align*}
\]
It means we can not construct $W$ as a local metric set if $|W| < \sum_{i=1}^{n}(m_i - 2)$. So $lmd(SB_{m_1,m_2,...,m_n}) \geq \sum_{i=1}^{n}(m_i - 2)$.

(ii) Assume $W = \{v_{i,j}\}$ where $1 \leq i \leq n$ and $3 \leq j \leq m_i$. The cardinality of $W$ is $\sum_{i=1}^{n}(m_i - 2)$. Then representations of every vertex of $SB_{m_1,m_2,...,m_n}$ with respect to $W$ are

\[
-r(v_{1,1}|W) = (1, 1, ..., 1, 3, 3, ..., 3, ..., 3, 3, ..., 3),
-r(v_{1,2}|W) = (1, 1, ..., 1, 4, 4, ..., 4, ..., 4, 4, ..., 4),
-\vdots
-r(v_{1,m_1}|W) = (1, 1, ..., 0, 4, 4, ..., 4, ..., 4, 4, ..., 4),
-r(v_{2,1}|W) = (3, 3, ..., 3, 1, 1, ..., 1, ..., 3, 3, ..., 3),
-r(v_{2,2}|W) = (4, 4, ..., 4, 1, 1, ..., 1, ..., 4, 4, ..., 4),
-\vdots
-r(v_{2,m_2}|W) = (4, 4, ..., 4, 1, 1, ..., 0, ..., 4, 4, ..., 4),
-\vdots
-r(v_{n,1}|W) = (3, 3, ..., 3, 3, 3, ..., 3, ..., 1, 1, ..., 1),
-r(v_{n,2}|W) = (4, 4, ..., 4, 4, 4, ..., 4, ..., 1, 1, ..., 1),
-\vdots
-r(v_{n,m_n}|W) = (4, 4, ..., 4, 4, 4, ..., 4, ..., 1, 1, ..., 0),
-r(u|W) = (2, 2, ..., 2, 2, 2, ..., 2, 2, 2, 2, 2).
\]

Every pair of adjacent vertices have distinct representations with respect to $W$, so $W$ is a local metric basis for starbarbell graph $SB_{m_1,m_2,...,m_n}$. \(\Box\)

### 2.2. The Local Metric Dimension of $K_m \odot P_n$ Graph

By using the definition from Frucht and Harary [2], the corona product $K_m \odot P_n$ graph is a graph obtained from a complete graph $K_m$ and $m$ copies of path graph $P_n$, and then joining by an edge each vertex from the $i^{th}$ copy of $P_n$ with the $i^{th}$ vertex of $K_m$. The $K_m \odot P_n$ graph can be depicted as in Figure 2.

![Figure 2. $K_m \odot P_n$ graph](image-url)
Theorem 2.2 For all positive integers $m$ and $n$,

$$lmd(K_m \odot P_n) = \begin{cases} 
1, & m = n = 1; \\
2, & m = 1, 2 \leq n \leq 5; \\
m - 1, & m \geq 2, n = 1; \\
m\left\lceil \frac{n+2}{4} \right\rceil, & \begin{cases} 
m = 1, n \geq 6; \\
m \geq 2, n \geq 2.
\end{cases}
\end{cases}$$

Proof. We consider four cases based on the values of $m$ and $n$.

Case 1. $m = n = 1$.

The corona product $K_1 \odot P_1$ graph is a path with two vertices, so it is easy to say that $lmd(K_m \odot P_n) = 1$ for $m = 1$ and $n = 1$.

Case 2. $m = 1$ and $2 \leq n \leq 5$.

The corona product $K_1 \odot P_n$ graph for $2 \leq n \leq 5$ is a graph which every single vertex in $K_1 \odot P_n$ belongs to some complete graphs $K_3$, so $lmd(K_1 \odot P_n) \neq 1$. Assume $W = \{u_1,v_k\}$ with $k = 2$ for $n = 2$ and $k = 3$ for $3 \leq n \leq 5$, then every pair of adjacent vertices have distinct representations with respect to $W$. Therefore, $W$ is a local metric basis and $lmd(K_m \odot P_n) = 2$ for $m = 1$ and $2 \leq n \leq 5$.

Case 3. $m \geq 2$ and $n = 1$.

$P_1$ is an empty graph. Then by using the results of Rodríguez-Veláquez et al. [8] and Okamoto et al. [7] we have $lmd(K_m \odot P_1) = lmd(K_m) = m - 1$.

Case 4. $m = 1$, $n \geq 6$ and $m \geq 2$, $n \geq 2$.

(i) $m = 1$ and $n \geq 6$.

By the same reason with Case 2, we have $lmd(K_1 \odot P_n) \neq 1$ for $n \geq 6$. Assume $W \subseteq V(K_1 \odot P_n)$ is a non-empty set, then we consider three conditions below.

(a) If $v_{1,1}, v_{1,2}, v_{1,3} \notin W$ then $d(v_{1,1}, u_1) = (v_{1,2}, u_1) = 1$ and $d(v_{1,1}, v_{1,r}) = (v_{1,2}, v_{1,r}) = 2$, for $4 \leq r \leq n$, so $r(v_{1,1}|W) = r(v_{1,2}|W)$. We know that $v_{1,1}$ and $v_{1,2}$ are adjacent vertices, hence $W$ is not a local metric set. In other words, if $W$ is a local metric set then at least one of three vertices $v_{1,1}, v_{1,2}$ or $v_{1,3}$ belongs to $W$.

(b) By the same reason with (a), if $W$ is a local metric set then at least one of three vertices $v_{1,(n-2)}, v_{1,(n-1)}$ or $v_{1,n}$ belongs to $W$.

(c) Assume a vertex $v_1,t$, for $1 \leq t \leq n - 4$, belongs to $W$. If all of the vertices $v_{1,(t+1)}, v_{1,(t+2)}, v_{1,(t+3)}$, and $v_{1,(t+4)}$ do not belong to $W$, then $d(v_{1,(t+1)}, u_1) = (v_{1,(t+2)}, u_1) = 1$ and $d(v_{1,(t+2)}, v_{1,r}) = (v_{1,(t+3)}, v_{1,r}) = 2$, with $1 \leq r \leq t$ or $t + 5 \leq r \leq n$, so $r(v_{1,(t+2)}|W) = r(v_{1,(t+3)}|W)$. We know that $v_{1,(t+2)}$ and $v_{1,(t+3)}$ are two adjacent vertices, then $W$ is not a local metric set. In other words, if $W$ is a local metric set and $v_{1,t} \in W$, then at least one of four vertices $v_{1,(t+1)}, v_{1,(t+2)}, v_{1,(t+3)}$, or $v_{1,(t+4)}$ belongs to $W$.

Based on three conditions above, the construction of $W$ such that $W$ is a local metric basis is by choosing every vertices $v_{1,r}$, for $r \equiv 3$ (mod 4) and $1 \leq r \leq n$, as the elements of $W$. Then, if $n - r_{\text{max}} = 3$, we have to choose one of the vertices $v_{1,(n-2)}, v_{1,(n-1)}$, or $v_{1,n}$ as the element of $W$. Hence the cardinality of $W$ is $\left\lceil \frac{n+2}{4} \right\rceil$.

(ii) $m \geq 2$ and $n \geq 2$.

By using the results of Rodríguez-Veláquez et al. [8], for $2 \leq n \leq 5$ we have $lmd(K_m \odot P_n) = m \cdot lmd(K_1 + P_n) - 1 = m \cdot (lmd(K_1) + P_n - 1) = m \cdot (2 - 1) = m$. For $2 \leq n \leq 5$ we have $m = m \left\lceil \frac{n+2}{4} \right\rceil$. Then for $n \geq 6$, again, by using the results of Rodríguez-Veláquez et al. [8], we have $lmd(K_m \odot P_n) = m \cdot lmd(K_1 + P_n) = m \cdot lmd(K_1 \odot P_n) = m \left\lceil \frac{n+2}{4} \right\rceil$. $\square$
2.3. The Local Metric Dimension of Möbius Ladder Graph

Harary and Guy [4] defined the Möbius ladder graph $M_n$ is a graph obtained from a cycle graph $C_n$ by connecting every pair of vertices $u, v$ in $C_n$ if $d(u, v) = diam(C_n)$ for $n \geq 5$. The Möbius ladder graph $M_n$ can be depicted as in Figure 3 and Figure 4.

![Figure 3. Möbius ladder graph $M_n$ for even positive integers $n$](image1)

![Figure 4. Möbius ladder graph $M_n$ for odd positive integers $n$](image2)

**Theorem 2.3** Let $M_n$ be the Möbius ladder graph. Then for even positive integers $n \geq 6$

$$lmd(M_n) = \begin{cases} 
1, & n \equiv 2 \pmod{4}; \\
2, & n \equiv 0 \pmod{4}.
\end{cases}$$

**Proof.** We consider two cases based on the values of $n$.

**Case 1.** $n \equiv 2 \pmod{4}$.

The Möbius ladder graph $M_n$ for $n \equiv 2 \pmod{4}$ is a bipartite graph. Then by using the result of Okamoto et al. [7] we have $lmd(M_n) = 1$.

**Case 2.** $n \equiv 0 \pmod{4}$.

(i) We will show that $lmd(M_n) \neq 1$.

Without loss of generality, assume $W = \{v_1\}$, then we obtain

$$r(v_{n+4}|W) = r(v_{n+8}|W) = r(v_{3n}|W) = r(v_{3n+4}|W) = \left(\frac{n}{3}\right).$$

Every pair of vertices $\{v_{n+4}, v_{n+8}\}$, $\{v_{3n}, v_{3n+4}\}$, and $\{v_{n+4}, v_{3n+4}\}$ are adjacent, hence $W$ is not local metric set. So $lmd(M_n) \neq 1$, for $n \equiv 0 \pmod{4}$.
(ii) Assume \( W = \{v_1, v_2\} \), then representations of every vertex of \( M_n \) with respect to \( W \) are

\[
\begin{align*}
r(v_1|W) &= (0, 1),
r(v_2|W) &= (1, 0),
r(v_3|W) &= r(v_{\frac{n}{2}+2}|W) = (2, 1),
r(v_4|W) &= r(v_{\frac{n}{2}+3}|W) = (3, 2),
& \quad \vdots
r(v_{\frac{n}{4}+4}|W) &= r(v_{\frac{n}{4}+6}|W) = (\frac{n}{4}, \frac{n}{4} - 1),
r(v_{\frac{n}{4}+8}|W) &= r(v_{\frac{n}{4}+12}|W) = (\frac{n}{4}, \frac{n}{4}),
& \quad \vdots
r(v_{\frac{n}{2}}|W) &= r(v_{n-1}|W) = (2, 3),
r(v_{\frac{n}{2}+1}|W) &= r(v_{n}|W) = (1, 2).
\end{align*}
\]

Every pair of adjacent vertices have distinct representation, so \( W \) is a local metric basis and \( lmd(M_n) = 2 \), for \( n \equiv 0(\text{mod}4) \). \( \square \)

3. Conclusion

It can be concluded that the strong metric dimension of a starbarbell graph \( SB_{m_1,m_2;\ldots;m_n} \), \( K_m \odot P_n \) graph, and a Möbius ladder graph \( M_n \) for even positive integers \( n \geq 6 \) are as stated in Theorem 2.1, Theorem 2.2, and Theorem 2.3, respectively.

**Open Problem**: Determine the local metric dimension of Möbius ladder graph \( M_n \) for odd positive integers \( n \geq 5 \).

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References

[1] Barragán-Ramírez, G. A., R. Simanjuntak, S. W. Saputro, and S. Uttunggadewa, *The Local Metric Dimension of Subgraph-Amalgamation of Graphs*, Universitat Rovira i Virgili and Institut Teknologi Bandung (2015).

[2] Frucht, R. and F. Harary, *On The Corona of Two Graphs*, Aequationes Mathematicae 4 (1970), 322-325.

[3] Harary, F. and R. A. Melter, *On The Metric Dimension of A Graph*, Ars Combinatoria 2 (1976), 191-195.

[4] Harary, F. and R. K. Guy, *On The Möbius Ladders*, Canad. Math. Bull. 10 (1967), 493-496.

[5] Kristina, M., N. Estuningsih, and L. Susilowati, *Dimensi Metrik Lokal pada Graf Hasil Kali Comb dari Graf Siklus dan Graf Bintang*, Jurnal Matematika 1 (2014), 1-9.

[6] Ningsih, E. U. S., N. Estuningsih, and L. Susilowati, *Dimensi Metrik Lokal pada Graf Hasil Kali Comb dari Graf Siklus dan Graf Lintasan*, Jurnal Matematika 1 (2014), 24-33.

[7] Okamoto, F., B. Phinezy, and P. Zhang, *The Local Metric Dimension of A Graph*, Matematica Bohemica 135 (2010), 239-255.

[8] Rodríguez-Velázquez, J. A., G. A. Barragán-Ramírez, and C. G. Gómez, *On The Local Metric Dimension of Corona Product Graphs*, Bull. Malays. Math. Sci. Soc. 39 (2016), 157-173.

[9] Slater, P. J., *Leaves of Trees*, Congressus Numerantium 14 (1975), 549-569.