A real analogue of the Moore–Tachikawa category

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Abstract

For each complex semisimple group $G_C$, Moore and Tachikawa \cite{MooreTachikawa} conjectured the existence of a certain two-dimensional topological quantum field theory $\eta_{G_C} : \text{Cob}_2 \to \text{MT}$ whose target category has complex Lie groups as objects and holomorphic symplectic varieties with Hamiltonian actions of the groups as morphisms. The conjecture is motivated by string theory, where $\eta_{G_C}$ is obtained by taking the Higgs branch of some supersymmetric quantum field theories (called theories of class $S$) depending on $G_C$ and a Riemann surface. The goal of this paper is to define a real analogue $\text{MT}_R$ of the target category $\text{MT}$ and to rigorously prove that it is a category.

1 Introduction

In 2011, physicists Moore and Tachikawa presented a conjecture at the String-Math Conference \cite{MooreTachikawa}, where they proposed a 2-dimensional topological quantum field theory (TQFT) for each simple and simply-connected complex algebraic group $G_C$, as a functor $\eta_{G_C}$ from the category of 2-bordisms to the category of holomorphic symplectic varieties with Hamiltonian actions of complex semi-simple algebraic groups. In doing so, they defined the Moore–Tachikawa category $\text{MT}$ to be the target category. The goal of this paper is to define a real analogue of the $\text{MT}$ category and prove that it is indeed a category.

In section 2, we will review some important results in symplectic geometry and Lie theory that are necessary to understand the category structure of $\text{MT}_R$. In section 3, we will review the categorical structure of the source category $\text{Cob}_2$ and define a real analogue $\text{MT}_R$ of the Moore–Tachikawa category.

2 Background Material

In this section, we will review some background material in symplectic geometry and Lie theory. In Section 2.1 we derive the canonical symplectic form on the cotangent bundle of any smooth manifold. In Section
In Section 2.3 we review the Marsden–Weinstein–Meyer Theorem and give an example of symplectic reduction. In Section 2.4 we review some properties of submanifolds under Lie group actions.

Remark 2.1. In classical physics, the phase space of a system of \(n\) particles is the space parametrizing the position and momenta of the particles. Mathematical models allow us to view a phase space as a symplectic manifold. A symplectic manifold \(M\) is a smooth manifold with a closed nondegenerate 2-form \(\omega\). In the same way Hamilton’s equations allow us to derive the time evolution of a system from a set of differential equations, the symplectic form allows us to obtain a vector field describing the flow of the system from the differential of a Hamiltonian function \(H\).

2.1 The canonical symplectic form on a cotangent bundle

Given a smooth manifold \(M\), its cotangent bundle \(T^*M\) is a symplectic manifold. That is, there exists a tautological 1-form \(\theta\) on \(T^*M\) such that its negative exterior derivative \(-d\theta\) defines a symplectic 2-form.

Theorem 2.2. Let \(M\) be a smooth manifold. Then, its cotangent bundle \(T^*M\) is a symplectic manifold.

Proof. First, let \((x^i)\) be local coordinates on our base manifold \(M\). Then, let \((x^i, \xi_j)\) be the induced coordinates on the cotangent bundle \(T^*M\). Hence, the cotangent bundle \(T^*M\) is even-dimensional with canonical coordinates \((x^1, \ldots, x^n, \xi_1, \ldots, \xi_n)\) representing the covector \(\sum_{i=1}^n \xi_i dx^i \in T^*M\). Now, we pick an arbitrary point \(m \in T^*M\). By this construction, \(m = (x, \xi)\) for \(x \in M\) and \(\xi \in T^*_x M\). Our 1-form \(\theta\) on \(T^*M\) will “measure” vectors \(v \in T_m(T^*M)\) in the tangent space at each point \(m \in T^*M\). So, it will look something like this:

\[
\theta_m : T_m(T^*M) \longrightarrow \mathbb{R} , \quad \theta_m \in T^*_m(T^*M) \tag{2.1}
\]

Let \(\pi : T^*M \rightarrow M\) be the natural projection of a point on \(T^*M\) to its corresponding base point on \(M\) defined by \((x^i, \xi_j) \mapsto (x^i)\). Then, consider the differential map

\[
d\pi : T(T^*M) \longrightarrow TM. \tag{2.2}
\]

For any tangent vectors \(v \in T_m(T^*M)\), we want to use the original covectors \(\xi_j\) to “measure” the corresponding tangent vectors \(d\pi(v) \in TM\). That is, we want

\[
\theta_m(v) = \xi(d\pi(v)) \tag{2.3}
\]
In local coordinates, the tautological 1-form is given by

\[ \theta_m = \sum_{i=1}^{n} \xi_i \, dx^i \quad (2.4) \]

We want to show that these two formulations (2.3) and (2.4) are equivalent because the coordinate expression will show that \(-d\theta\) is non-degenerate as it is the standard symplectic form on \(\mathbb{R}^{2n}\). Hence, for any vector \(v = a_i \frac{\partial}{\partial x^i} + b_i \frac{\partial}{\partial \xi_i} \in T_m(T^*M)\), we will show

\[ \xi(d\pi(v)) = \xi_i dx^i(v). \quad (2.5) \]

We have

\[ \xi(d\pi(v)) = \langle \xi, d\pi(v) \rangle \]
\[ = \langle \xi, d\pi(a_i \frac{\partial}{\partial x^i} + b_i \frac{\partial}{\partial \xi_i}) \rangle \]
\[ = \langle \xi, a_i(d\pi \frac{\partial}{\partial x^i}) + b_i(d\pi \frac{\partial}{\partial \xi_i}) \rangle \]
\[ = \langle \xi, a_i \frac{\partial}{\partial x^i} \rangle \]
\[ = \langle \sum_j \xi_j dx^j, a_i \frac{\partial}{\partial x^i} \rangle \]
\[ = \xi_i a_i \]
\[ = \xi_i dx^i(v). \]

Then, the canonical symplectic 2-form on \(T^*M\) is

\[ \omega = -d\theta = \sum_{i=1}^{n} dx^i \wedge d\xi_i \quad (2.6) \]

Clearly, \(\omega\) is a closed, non-degenerate 2-form.

**2.2 Lie Group Actions**

In this section, we will review some definitions and theorems related to Lie groups, Hamiltonian actions, and moment maps. The goal of this section is to provide enough background to understand the structure of the \(\text{MT}_\mathbb{R}\) category. In particular, Lie groups appear as the objects and hamiltonian actions (with moment maps) are found in the composition of morphisms. We will also derive an explicit expression for the
induced Hamiltonian action of $G \times G$ on $T^* G$, and determine isomorphisms that identify $T^* G \cong G \times g^*$ and $TG \cong G \times g$.

**Definition 2.3.** A Lie group is a manifold $G$ equipped with a group structure such that the group operations

\[
G \times G \longrightarrow G \\
(a, b) \mapsto a \cdot b
\]

\[
G \longrightarrow G \\
a \mapsto a^{-1}
\]

are smooth maps.

**Definition 2.4.** An action of a Lie group $G$ on a manifold $M$ is a group homomorphism

\[
\psi : G \longrightarrow \text{Diff}(M) \\
g \mapsto \psi_g.
\]

The action $\psi$ is smooth if the evaluation map

\[
ev_{\psi} : M \times G \longrightarrow M \\
(p, g) \mapsto \psi_g(p)
\]

is a smooth map.

**Definition 2.5.** A diffeomorphism between symplectic manifolds $f : (M, \omega) \longrightarrow (N, \omega')$ is called a symplectomorphism if $f^* \omega' = \omega$.

**Definition 2.6** (Hamiltonian actions). Let $M$ be a manifold with symplectic form $\omega$. Suppose that a Lie group $G$ acts on $M$ via symplectomorphisms \{i.e. the action of each $g \in G$ preserves $\omega$\}. Let $x^\#$ be the vector field on $M$ generated by the 1-parameter subgroup $\{\exp(tx) \mid t \in \mathbb{R}\} \subseteq G$. That is, for any $p \in M$,

\[
x^\#_p = \frac{d}{dt}|_{t=0} e^{tx} \cdot p
\]

(2.7)

Then, the action is hamiltonian if there exists a map $\mu : M \rightarrow g^*$ satisfying the following two conditions:

\[
d\mu^x = t_x^\# \omega, \text{ for all } x \in g
\]

(2.8)

\[
\mu \circ \psi_g = Ad^*_g \circ \mu, \text{ for all } g \in G
\]

(2.9)

We call this map $\mu$ a **moment map**.
Now, suppose $G$ is a Lie group acting smoothly on a smooth manifold $M$. Then, there is an induced action of $G$ on $T^*M$ given by the map $\hat{\psi} : G \times T^*M \to T^*M$ such that for any $g \in G$, the map $\hat{\psi}_g : T^*M \to T^*M$ is defined by
\[
(p, \xi) \mapsto (d\psi_{g^{-1}})^*_p(\xi) \in T_{g \cdot p}^*M.
\]
This map $\hat{\psi}$ preserves the symplectic form on $T^*M$ and the action has an associated moment map $\mu : T^*M \to g^*$ defined as follows. For any covector $\xi \in T_p^*M$, we have
\[
\mu(\xi) \in g^* = \{\text{linear maps } g \to \mathbb{R}\}.
\]
For any $x \in g$, we have
\[
\mu(\xi)(x) = \xi(x^\sharp_p) \in \mathbb{R}, \quad \text{where } x^\sharp_p \in T_pM.
\]

**Theorem 2.7.** Let $G$ be a Lie group. Then there is a canonical isomorphism
\[
TG \cong G \times g,
\]
where $g = T_1G$ is the Lie algebra of $G$.

**Proof.** Let $L_g : G \to G$ be the left-multiplication map defined by $a \mapsto ga$. Then, the map $(dL_g)_1 : g = T_1G \to T_gG$ induces an isomorphism
\[
\varphi : G \times g \to TG
\]
\[
(g, x) \mapsto (dL_g)_1(x) \in T_gG.
\]



**Theorem 2.8.** Let $G$ be a Lie group. Then there is a canonical isomorphism
\[
T^*G \cong G \times g^*
\]

**Proof.** Similarly, consider the inverse map $(dL_g^{-1})_1 : T_gG \to T_1G = g$, whose dual map is $(dL_g^{-1})^*_1 : g^* \to T_g^*G$. This map induces an isomorphism $\psi : G \times g^* \to T^*G$ given by $(g, \xi) \mapsto (dL_g^{-1})^*_1(\xi)$. This isomorphism $T^*G \cong G \times g^*$ implies that the tangent spaces are isomorphic as well. Thus,
\[
T_{\psi(g, \xi)}(T^*G) \cong T_{(g, \xi)}(G \times g^*)
\]
\[
\cong T_gG \times T_{\xi}g^*.
\]
We can identify \( T_g G \cong \mathfrak{g} \) via the map \((dL_g^{-1})_1\), and since \( \mathfrak{g}^* \) is a vector space we have \( T_{\xi} \mathfrak{g}^* \cong \mathfrak{g}^* \). Hence, for \( p = \psi(g, \xi) \in T^* G \) we have

\[
T_p(T^* G) \cong \mathfrak{g} \times \mathfrak{g}^*.
\]  

(2.12)

Then, for any \((g, \xi) \in G \times \mathfrak{g}^*\) we can define a 2-form

\[
\omega_{(g, \xi)} : (\mathfrak{g} \times \mathfrak{g}^*) \times (\mathfrak{g} \times \mathfrak{g}^*) \to \mathbb{R}
\]

given explicitly by the following theorem.

**Theorem 2.9** (See e.g. [1]). Let \((u, \eta), (v, \zeta) \in \mathfrak{g} \times \mathfrak{g}^*\). Then, the symplectic 2-form is explicitly given by

\[
\omega_{(g, \xi)}(((u, \eta), (v, \zeta)) = \zeta(u) - \eta(v) + [u, v]).
\]  

(2.13)

**Theorem 2.10** (See e.g. [1] Section 4.4). Let \( G \) be a Lie group, and suppose \( G \times G \) acts on \( G \) by \((a, b) \cdot g = ab^{-1} \). Then, the induced Hamiltonian action of \( G \times G \) on \( T^* G \cong G \times \mathfrak{g}^* \) is given by

\[
(a, b) \cdot (g, \xi) = (agb^{-1}, Ad^*_b \xi),
\]  

(2.14)

with moment map

\[
\mu : T^* G \cong G \times \mathfrak{g}^* \to (\mathfrak{g} \times \mathfrak{g}^*)^* = \mathfrak{g}^* \times \mathfrak{g}^*
\]

\[
(g, \xi) \mapsto (Ad^*_g \xi, -\xi).
\]  

(2.15)

(2.16)

### 2.3 Symplectic Reduction

Symplectic reduction provides us with a way to take quotients of symplectic manifolds under group actions. For example, if we have a free action of \( S^1 \) on a symplectic manifold \( M \), we might hope to find a symplectic structure on the topological quotient \( X/S^1 \). But, clearly this quotient is not symplectic because \( \dim X/S^1 = \dim X - 1 \), which is odd (since \( X \) is even-dimensional). Thus, we can do the following: if the action of \( S^1 \) is Hamiltonian, we can decrease the dimension by 1 by restricting the action to a level set of the moment map. Taking the quotient of this new manifold, we at least get something even-dimensional. To derive a natural symplectic structure on this quotient, we have the following theorem.

**Theorem 2.11** (See e.g. [4] Marsden-Weinstein-Meyer Theorem). Let \( (M, \omega) \) be a symplectic manifold with a Hamiltonian action of a compact Lie group \( G \) and an associated equivariant moment map \( \mu : M \to \mathfrak{g}^* \). If \( 0 \in \mathfrak{g}^* \) is a regular value of \( \mu \) such that \( G \) acts freely on \( \mu^{-1}(0) \), then the orbit space \( M_{red} = M//G := \mu^{-1}(0)/G \) is a symplectic manifold with symplectic structure induced by \( \omega \). That is, there
is a unique symplectic form $\omega_{\text{red}}$ on $M_{\text{red}}$ satisfying $i^*\omega = \pi^*\omega_{\text{red}}$. The pair $(M_{\text{red}}, \omega_{\text{red}})$ is called the \textbf{reduction} of $(M, \omega)$ with respect to $G$ and $\mu$.

\textbf{Example 2.12.} Let’s work through an example of symplectic reduction. Consider the symplectic action of the circle $S^1$ on $(\mathbb{C}^2, \omega)$, where each $\lambda \in S^1$ acts on $(x, y) \in \mathbb{C}^2$ by $\lambda \cdot (x, y) = (\lambda x, \lambda y)$. The standard 2-form $\omega$ on $\mathbb{C}^2$ is
\begin{equation}
\sum_{i=1}^{2} r_i dr_i \wedge d\theta_i, \quad (2.17)
\end{equation}
where the polar coordinates $(r_i, \theta_i)$ represent points of $\mathbb{C}^2$. Then, let $X^\# = \frac{\partial}{\partial r_1} + \frac{\partial}{\partial r_2}$ be the vector field generated by the 1-parameter subgroup $\{\exp(tX) : t \in \mathbb{R}\}$ of our Lie group $S^1$. We want to find a moment map $\mu : M = \mathbb{C}^2 \to g^* \cong \mathbb{R}$ such that for each $X \in g$ we have $d\mu X = i_{X^\#}\omega$. This is equivalent to saying that $\mu^X$ is a Hamiltonian function for the vector field $X^\#$. So, we compute
\begin{equation}
i_{X^\#}\omega = \omega(X^\#, \cdot) = -\sum_{i=1}^{2} r_i dr_i \quad (2.18)
\end{equation}
Then, to find the moment map we integrate this expression \(2.18\) to get
\begin{equation}
\mu = -\int \sum_{i=1}^{2} r_i dr_i = -\frac{1}{2}(|r_1|^2 + |r_2|^2) + c \quad (2.19)
\end{equation}
We will set this constant to $c = \frac{1}{2}$ so that the level set of the moment map at the regular value $0 \in g^*$ is $S^3$. The symplectic quotient is then
\begin{equation}
\mu^{-1}(0)/S^1 = S^3/S^1 \cong \mathbb{CP}^1. \quad (2.20)
\end{equation}
Now we want to find the 2-form $\omega_{\text{red}}$ that makes $\mathbb{CP}^1$ a symplectic manifold. By Theorem 2.11, this form satisfies $i^*\omega = \pi^*\omega_{\text{red}}$, for the quotient map $\pi : S^3 \to \mathbb{CP}^1$ that identifies antipodal points by
\begin{equation}
(z_0, z_1) \mapsto [z_0 : z_1]. \quad (2.21)
\end{equation}
This map is the Hopf fibration; the restriction of the projection to $S^3$. Now, consider the diagram
\[
\begin{array}{ccc}
S^3 & \xrightarrow{i} & (\mathbb{C}^2, \omega) \\
\downarrow \pi & & \\
(\mathbb{CP}^1, \omega_{\text{red}})
\end{array}
\]
Using homogeneous coordinates, \( \mathbb{CP}^1 \) is covered by open sets

\[
U_0 = \{ [z_0, z_1] \in \mathbb{CP}^1 : z_0 \neq 0 \}, \quad U_1 = \{ [z_0, z_1] \in \mathbb{CP}^1 : z_1 \neq 0 \}
\]

with corresponding charts

\[
\varphi_0 : U_0 \to \mathbb{C}, \quad [z_0 : z_1] \mapsto \frac{z_1}{z_0}, \quad \varphi_1 : U_1 \to \mathbb{C}, \quad [z_0 : z_1] \mapsto \frac{z_0}{z_1}
\]

We want to find the 2-form \( \tilde{\omega} \) on \( \mathbb{C} \) such that

\[
(\varphi_i \circ \pi)^* \tilde{\omega} = i^* \omega. \tag{2.22}
\]

Well, consider the 2-form

\[
\tilde{\omega} = \frac{i}{2|z|^2} \sum_{i,j=1}^{2} (|z_i|^2 dz_j \wedge d\overline{z}_j - \overline{z}_i z_j dz_i \wedge d\overline{z}_j). \tag{2.23}
\]

Then, the pullback of \( \tilde{\omega} \) via the charts \( \varphi_i \) for \( \mathbb{CP}^1 \) is

\[
\varphi_i^* \tilde{\omega} = \frac{i}{2} \left( \frac{dz \wedge d\overline{z}}{1 + 2|z|^2 + |z|^4} \right)
\]

\[
= \frac{i}{2} \left( \frac{dz \wedge d\overline{z}}{(1 + |z|^2)^2} \right). \tag{2.24}
\]

This is the symplectic 2-form \( \omega_{FS} \) on \( \mathbb{CP}^1 \), known as the Fubini-study form.

### 2.4 Submanifolds and distributions

In this section, we will consider submanifolds of symplectic manifolds and observe how the pullback of the symplectic form (which is the restriction of the form to the tangent bundle of the submanifold) affects \( G \)-orbits of Lie group actions. It turns out that the kernel of the pullback form is a distribution on the submanifold if it has constant rank (i.e. the submanifold is \( \text{pre-symplectic} \)). In particular, we are interested in submanifolds like \( \mu^{-1}(0) \) because the level sets of moment maps appear in symplectic reduction. The following results form the basic ingredients for the proof of the Marsden–Weinstein–Meyer Theorem 2.11.

Let \( (M, \omega) \) be a symplectic manifold with symplectic 2-form \( \omega \in \Omega^2(M) \). Let \( N \subseteq M \) be a submanifold...
(i.e. $\mu^{-1}(0)$) and $i : N \to M$ the inclusion map. Then we have that $i^* \omega \in \Omega^2(N)$ is a closed 2-form since

$$d(i^* \omega) = i^*(d\omega) = 0.$$  

Also, the tangent space of $N$ at each point $p \in N$ is a subspace of the corresponding tangent space of $M$ at $p$, i.e. $T_p N \leq T_p M$. We can restrict the 2-form to the tangent space at each point of our submanifold $N \subseteq M$ by $\omega_{T_p N} = (i^* \omega)_p : T_p N \times T_p N \to \mathbb{R}$. Observe that for any tangent vector $v \in T_p N$,

$$\omega(v, w) = 0, \text{ for all } w \in T_p M \implies v = 0$$

because of the non-degeneracy of $\omega$ on $M$. But, restricted to the submanifold $N$, we see that

$$\omega(v, w) = 0, \text{ for all } w \in T_p N \iff v = 0.$$ 

Hence,

$$\ker(i^* \omega)_p = \{v \in T_p N : \omega(v, w) = 0 \text{ for all } w \in T_p N\}. \quad (2.25)$$

We can view this set as the collection of linear subspaces of the tangent spaces at each point in our submanifold $N \subseteq M$. Letting $D := \ker(i^* \omega)$, we see that $D$ becomes a distribution on $N$ if it has constant rank. The submanifold $N$ is pre-symplectic if $D = \ker(i^* \omega)$ forms a distribution over $N$.

**Theorem 2.13** (Frobenius Theorem). Let $\Gamma(D) := \{Z : N \to D \mid Z(p) \in D_p \text{ for all } p \in N\}$. If for all $X, Y \in \Gamma(D)$ we have $[X, Y] \in \Gamma(D)$, then for all $p \in N$, there exists a submanifold $S \subseteq N$ such that $p \in S$ and $T_q S = D_q$ for all $q \in S$.

Intuitively, this theorem says that if the set of maps “picking” a linear subspace at each point of the tangent space of our submanifold is involutive – closed under the operation of Lie bracket – we can find submanifolds, or leaves, that are tangent to each linear subspace at every point in the submanifold. The following theorem lets us quotient by these leaves to end up with a symplectic manifold.

**Theorem 2.14.** Let $\pi : N \to N/\sim$ be the quotient map into the smooth manifold $N/\sim$, where $p \sim q$ if $p, q \in S$ are in the same leaf. If $\pi$ is a smooth submersion, $D$ satisfies the conditions of the Frobenius Theorem, and the leaf space $N/\sim$ (i.e. submanifolds $S$ such that $T_q S = \ker(i^* \omega)_q, \forall q \in S$) is smooth then $N/\sim$ has a unique symplectic structure $\omega_{\text{red}} \in \Omega^2(N/\sim)$ such that $\pi^* \omega_{\text{red}} = i^* \omega$.

**Proposition 2.15.** Let $G$ be a connected Lie group acting on a symplectic manifold $(M, \omega)$ with moment map $\mu : M \to g^*$, and let $N = \mu^{-1}(0)$. Then,

$$\{\text{leaves of } D = \ker(i^* \omega)\} = \{G\text{-orbits}\}.$$
That is, for each leaf $O_p$ of $N = \mu^{-1}(0)$ at point $p \in N$, we have

$$O_p = G \cdot p = \{g \cdot p : g \in G\}$$

as a result of the conditions \text{[2.19]} and \text{[2.20]} of the moment map. The following theorems will help us show that zero level sets of moment maps are \textit{coisotropic} submanifolds.

**Theorem 2.16** (Quotient Manifold Theorem). [See e.g. \text{[3, Theorem 21.10]}] Let $G$ be a Lie group, and suppose $G$ acts on a smooth manifold $M$ smoothly, freely, and properly. Then, the orbit space $M/G$ is a topological manifold of dimension $\dim M - \dim G$, and it has a unique smooth structure such that $\pi : M \to M/G$ is a smooth submersion.

**Proposition 2.17.** Let $G$ be a Lie group acting freely on a submanifold $N \subseteq M$. Let $\pi : N \to N/G$ be a smooth submersion. Then

$$T_pO_p = \ker(d\pi_p)$$ \hspace{1cm} (2.26)

$$T_pN/T_pO_p \cong T_{\pi(p)}(N/G)$$ \hspace{1cm} (2.27)

**Proof.** Consider the exact sequence

$$0 \xrightarrow{\psi} T_pO_p \xrightarrow{i} T_pN \xrightarrow{d\pi_p} T_{\pi(p)}(N/G) \xrightarrow{\phi} 0 \hspace{1cm} (2.28)$$

By definition, we have $\ker(d\pi_p) = \text{im}(i) = T_pO_p$. This proves \text{[2.26]}. Since $\pi$ is a smooth submersion, the differential $d\pi_p : T_pN \to T_{\pi(p)}(N/G)$ is surjective. Hence, by the First Isomorphism Theorem we have that $T_pN/\ker(d\pi_p) \cong \text{im}(d\pi_p)$. Then by \text{[2.26]} and the surjectivity of $d\pi_p$, it follows that

$$T_pN/T_pO_p \cong T_{\pi(p)}(N/G).$$

\[\square\]

Now, consider the setup in Theorem 2.11. For any subspace $V \subseteq T_pM$, we define

$$V^\omega = \{u \in T_pM : \omega(u, v) = 0, \text{ for all } v \in V\}. \hspace{1cm} (2.29)$$

Thus, by definition \text{[2.29]}, we know

$$(T_pO_p)^\omega = T_p\mu^{-1}(0). \hspace{1cm} (2.30)$$

By Proposition 2.17,

$$T_pO_p = \ker(d\pi_p) = \ker(i^*\omega_p), \hspace{1cm} (2.31)$$
where
\[
\ker(i^*\omega_p) = T_p\mu^{-1}(0) \cap (T_p\mu^{-1}(0))^\omega.
\] (2.32)

Then \([2.30]\) implies that
\[
(T_p\mu^{-1}(0))^\omega = T_pO_p \subseteq (T_pO_p)^\omega = T_p\mu^{-1}(0).
\]

Hence, we have shown
\[
(T_p\mu^{-1}(0))^\omega \subseteq T_p\mu^{-1}(0),
\] (2.33)

which makes \(\mu^{-1}(0)\) a coisotropic submanifold of \(M\). That is, at each \(p \in \mu^{-1}(0)\) the tangent space to the orbit \(T_pO_p\) is a coisotropic subspace of the symplectic vector space \((T_pM, \omega_p)\). As for zero level sets of moment maps, in symplectic reduction we assume that \(G\) acts smoothly, freely, and properly so we can apply Theorem \([2.16]\) to get a smooth submersion \(\pi: \mu^{-1}(0) \to \mu^{-1}(0)/G\). Then, we can use Proposition \([2.17]\) to prove \(\mu^{-1}(0)\) is coisotropic.

### 3 2d TQFT Functor

In 2011, physicists Moore and Tachikawa conjectured the existence of a functor \(\eta_G: \text{Cob}_2 \to \text{MT}\) such that the gluing of the boundaries of the 2-bordisms corresponds to the holomorphic symplectic quotient with respect to the diagonal action of a simple, simply-connected complex algebraic group \(G\). We will construct a real analogue \(\text{MT}_R\) of the target category \(\text{MT}\) and prove that it is a category. First we will review the data of the source and target categories, then we will prove some properties of \(\text{MT}_R\).

#### 3.1 Bordism category \(\text{Cob}_2\)

This is the source category for the functor \(\eta_{Gc}\). Let’s briefly discuss its category structure.

- **Objects:** Closed oriented 1-dimensional manifolds (i.e. disjoint unions of multiple \(S^1\)’s)

- **Morphisms:** A morphism from \(B_1\) to \(B_2\) is a 2-dimensional oriented manifold \(C\) whose boundary is \(B_1 \cup (-B_2)\). We denote these manifolds as bordisms.

- **Composition:** When we compose morphisms \(B_1 \xrightarrow{C} B_2 \xrightarrow{D} B_3\) in this category, we glue their boundaries together in such a way that preserves the orientation of the bordisms.

\(\text{Cob}_2\) is a symmetric monoidal category with duality under the standard operations.
3.2 Real analogue of Moore–Tackiwawa category MTₐ

Moore and Tachikawa define the data of the MT category as follows. The objects are complex semi-simple algebraic groups and the morphisms are holomorphic symplectic varieties with Hamiltonian action of the groups. The composition of morphisms is defined as the symplectic reduction of their product by the common algebraic group. We will consider a real analogue of the Moore–Tachikawa category, MTᵣ, to be the category of symplectic manifolds together with a Hamiltonian action. We will first define the data of the category, then go on to verify its properties and prove that MTᵣ is a category.

3.3 Data

Let us begin with the category data:

- The **objects** are Lie groups (including the trivial group 1).

- A morphism \( G \xrightarrow{\mathcal{M}} H \) is a triple \([\mathcal{M}, G, H]\), where \( M \) is a symplectic manifold with a Hamiltonian action of \( G \times H \) on \( M \) and a corresponding moment map \( \mu : M \rightarrow (\mathfrak{g} \times \mathfrak{h})^* \). Informally, we write \( M \in \text{Hom}(G, H) \) as a morphism with an ordered pair of Lie groups \( G \) and \( H \). The equivalence class \([M]\) identifies \((M, \omega) \sim (M', \omega')\) if there exists a symplectomorphism \( f : M \rightarrow M' \) that is compatible with the \( G \times H\)-actions and moment maps. That is, for any \( p \in M \), we have

  \[
  f((g, h) \cdot p) = (g, h) \cdot f(p),
  \]

  and the following diagram commutes

\[
\begin{array}{ccc}
M & \xrightarrow{f} & M' \\
\downarrow{\mu} & & \downarrow{\mu'} \\
(\mathfrak{g} \times \mathfrak{h})^* & & \\
\end{array}
\]

- Suppose we are given \( G \xrightarrow{M} H \xrightarrow{N} K \). The **composition** of morphisms \( N \circ M \) in this category is defined as the symplectic reduction

  \[
  N \circ M := (M \times N) \sslash H.
  \]  

This is the symplectic reduction of \( M \times N \) by the common Lie group \( H \), which guarantees that the
resulting morphism is indeed a symplectic manifold.

- The identity $\text{id}_G \in \text{Hom}(G, G)$ is $T^*G$ with the Hamiltonian $G \times G$-action introduced in Theorem 2.10

**Theorem 3.1.** $\text{MT}_\mathbb{R}$ is a category.

Our goal is to prove that $\text{MT}_\mathbb{R}$ is a category. We will discuss some of its categorical properties and prove that they are indeed true based on our definitions above.

### 3.4 Properties

In order to find the symplectic reduction $N \circ M$, we must know how $H$ acts on $M \times N$ and define the moment map $\lambda : M \times N \to \mathfrak{h}^*$. Well, we know $H$ acts on both $M$ and $N$ because it is the common Lie group in $G \xrightarrow{M} H \xrightarrow{N} K$. Hence, $H$ has separate Hamiltonian actions on $M$ and $N$ with respective moment maps $\mu^H_M : M \to \mathfrak{h}^*$ and $\mu^H_N : N \to \mathfrak{h}^*$. Putting these two actions together, we let $H$ act diagonally on $M \times N$ by

$$h \cdot (m, n) = (h \cdot m, h \cdot n),$$

so that the moment map $\lambda : M \times N \to \mathfrak{h}^*$ becomes

$$\lambda(m, n) = \mu^H_M(m) + \mu^H_N(n). \tag{3.2}$$

Now, using this moment map of the Hamiltonian action of $H$ on $M \times N$, we can take the symplectic reduction $(M \times N) /\!/ H$ at the level set $\lambda^{-1}(0)$. The symplectic quotient is then

$$N \circ M := (M \times N) /\!/ H = \{ (m, n) \in M \times N \mid \mu_M(m) + \mu_N(n) = 0 \} / H, \tag{3.3}$$

where $\mu^H_M : M \to \mathfrak{h}^*$ and $\mu^H_N : N \to \mathfrak{h}^*$ are moment maps of the Hamiltonian $H$-action on $M$ and $N$, respectively. Our reduced space will consist of equivalence classes $[(m, n)] \in N \circ M$ of points in the same $H$-orbits.

**Notation 3.2.** If $\mu : M \to \mathfrak{g}^*$ and $\nu : N \to \mathfrak{g}^*$ are two Hamiltonian manifolds, we define the fibre product

$$M \times \mathfrak{g}^* N := \{ (m, n) \in M \times N : \mu(m) + \nu(n) = 0 \}.$$

Now, we will show that $N \circ M$ is indeed a morphism in the category $\text{MT}_\mathbb{R}$.

**Proposition 3.3 (Composition).** The symplectic reduction $N \circ M$ is a morphism in the $\text{MT}_\mathbb{R}$ category, i.e. it has canonical Hamiltonian action of $G \times K$. 

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Proof. The composition of morphisms $G \xrightarrow{M} H \xrightarrow{N} K$ results in a morphism $G \xrightarrow{N \circ M} K$, where the canonical Hamiltonian action of $G \times K$ on $N \circ M$ is given by

$$\psi_{(g,k)} : G \times K \times (N \circ M) \to N \circ M$$

$$(g,k) \cdot [(m,n)] = [(g \cdot m, k \cdot n)].$$

The corresponding moment map is

$$\mu : N \circ M \to \mathfrak{g}^* \times \mathfrak{k}^*$$

$$[(m,n)] \mapsto (\mu^G_M(m), \mu^K_N(n)),$$

where the maps $\mu^G_M : M \to \mathfrak{g}^*$ and $\mu^K_N : N \to \mathfrak{k}^*$ are the components of the moment maps for the actions of $G \times H$ on $M$ and $H \times K$ on $N$, respectively. Now, we will check that this moment map is well-defined and smooth. Well-defined means that points in the same equivalence classes ($H$-orbits) are mapped to the same point. Let $[(m_1, n_1)] = [(m_2, n_2)] \in N \circ M$ so that $(m_1, n_1) = h \cdot (m_2, n_2)$, for some $h \in H$. We will show that

$$\mu([(m_1, n_1)]) = \mu([(m_2, n_2)]). \quad (3.4)$$

Well, we have that

$$\mu([(m_1, n_1)]) = (\mu^G_M(m_1), \mu^K_N(n_1))$$

$$= (\mu^G_M(h \cdot m_2), \mu^K_N(h \cdot n_2))$$

$$= (\text{Ad}_h^* \circ \mu^G_M(m_2), \text{Ad}_h^* \circ \mu^K_N(n_2))$$

$$= \text{Ad}_h^* \circ (\mu^G_M(m_2), \mu^K_N(n_2))$$

$$= \mu([(m_2, n_2)]).$$

Then, we will show $\mu$ is smooth. Consider the following theorem.

**Theorem 3.4** (Passing Smoothly to the Quotient). [See e.g. [3, Theorem 4.30]] Suppose $M$ and $N$ are smooth manifolds and $\pi : M \to N$ is a surjective smooth submersion. If $P$ is a smooth manifold and $F : M \to P$ is a smooth map that is constant on the fibers of $\pi$, then there exists a unique smooth map $\tilde{F} : N \to P$ such that $\tilde{F} \circ \pi = F$.
Then, we have the following diagram (using Notation 3.2):

\[
\begin{array}{ccc}
M & \xrightarrow{F} & P \\
\downarrow{\pi} & & \\
N & \xrightarrow{\tilde{\mu}} & (g \times \mathfrak{k})^* \\
\end{array}
\]

where \( \pi : M \times h^* N \to N \circ M \) is the quotient map of the symplectic reduction and \( \tilde{\mu} : M \times h^* N \to (g \times \mathfrak{k})^* \) given by \((m,n) \mapsto (\mu_M^G(m), \mu_N^K(n)) \) is a smooth map (since each component is smooth). By Theorem 3.4, since \( \tilde{\mu} \) is constant on the fibers of \( \pi \), we have that \( \mu \) is smooth and well-defined.

Now, we will check that the map \( \mu \) satisfies conditions (2.8) and (2.9) of moment maps. First we show that for all \((x,y) \in g \times \mathfrak{k}\),

\[
d\mu^{(x,y)} = i_{(x,y)^\#} \omega, \tag{3.5}
\]

where \( \omega \in \Omega^2_{N \circ M} \) is the symplectic 2-form on \( N \circ M \). Note that \( d\mu^{(x,y)} : T_{[(m,n)]}(N \circ M) \to \mathbb{R} \) is a 1-form on \( N \circ M \). Well, we have

\[
d\mu^{(x,y)} = ((d\mu_M^G)^x, (d\mu_N^K)^y) \\
= (i_x^\# \omega_M, i_y^\# \omega_N) \\
= (\omega_M(x^\#, \cdot), \omega_N(y^\#, \cdot)) \\
= \omega((x,y)^\#, \cdot) \\
= i_{(x,y)^\#} \omega,
\]

where \( \omega_M(x^\#, \cdot) : T_{[(m,n)]}(N \circ M) \to T_m M \to \mathbb{R} \in \Omega^1_M \) is a 1-form on \( N \circ M \) that descends from the projection \( q_M : M \times N \to M \). Similarly, \( \omega_N(y^\#, \cdot) : T_{[(m,n)]}(N \circ M) \to T_n N \to \mathbb{R} \in \Omega^1_N \) is a 1-form that descends from the projection \( q_N : M \times N \to N \).

\[\square\]
Then, we will show that for all \((g,k) \in G \times K\),

\[
\mu \circ \psi_{(g,k)}([(m,n)]) = Ad^*_{(g,k)} \circ \mu([(m,n)])
\] (3.6)

Well, we have

\[
\mu \circ \psi_{(g,k)}([(m,n)]) = \mu([(g \cdot m, k \cdot n)])
\]

\[
= (\mu_M(g \cdot m), \mu_N(k \cdot n))
\]

\[
= Ad^*_{(g,k)} \circ (\mu_M(m), \mu_N(n))
\]

\[
= Ad^*_{(g,k)} \circ \mu([(m,n)]).
\]

**Proposition 3.5 (Identity).** The symplectic manifold \(T^*G\) with the Hamiltonian \(G \times G\)-action stated in Theorem 2.10 is an identity in \(MT_R\).

**Proof.** Suppose \(G\) is a Lie group. We want a symplectic manifold \(id_G\) with a Hamiltonian action of \(G \times G\) such that for all \(G \xrightarrow{M} H\),

\[
M \circ id_G = (id_G \times M) \parallel G \cong M.
\]

We claim that \(id_G := T^*G\) satisfies these conditions. By Theorem 2.2, we know the cotangent bundle \(T^*G\) has a canonical symplectic structure arising from its tautological 1-form. Hence, \(T^*G\) is a symplectic manifold. Now, we must determine the Hamiltonian action of \(G \times G\) on \(T^*G\) such that the composition of \(T^*G\) with any other morphism \(M \in \text{Hom}(G, H)\) is isomorphic to \(M\). Our setup is given by the composition of morphisms

\[
G \xrightarrow{T^*G} G \xrightarrow{M} H.
\]

Recall that \(T^*G \cong G \times \mathfrak{g}^*\), and consider the action of \(G \times G\) on \(G \times \mathfrak{g}^*\) given in (2.14) by

\[
(a, b) \cdot (g, \xi) = (agb^{-1}, Ad^*_b \xi),
\]

with an associated moment map

\[
\mu : T^*G \longrightarrow (\mathfrak{g} \times \mathfrak{g})^*
\]

\[
(g, \xi) \mapsto (Ad^*_g \xi, -\xi).
\]

We will restrict this action to the Lie subgroup \(\{1\} \times G \leq G \times G\), where we identify \(G \cong \{1\} \times G\). Thus,
the action of $G$ on $T^*G$ is defined by

$$b \cdot (g, \xi) = (gb^{-1}, Ad_b^\ast \xi).$$

(3.7)

For the morphism $M : G \to H$, we have the action of $G \times H$ on $M$ given by component-wise left multiplication

$$(a, b) \cdot m = amb^{-1},$$

with moment map

$$\nu : M \longrightarrow (\mathfrak{g} \times \mathfrak{h})^* \cong \mathfrak{g}^* \times \mathfrak{h}^*$$

$$m \longmapsto (\nu_G(m), \nu_H(m)).$$

Now, we construct the action of $G$ on $T^*G \times M$ by combining the actions of $G$ on $T^*G$ and on $M$ from above, and restricting each moment map to the subgroup $G$ (identified with $\{1\} \times G$).

Hence, the action of $G$ on $T^*G \times M$ is defined by

$$a \cdot (g, \xi, m) = (ga^{-1}, Ad_a^\ast \xi, a \cdot m)$$

(3.8)

with moment map

$$\lambda : T^*G \times M \longrightarrow \mathfrak{g}^*$$

$$(g, \xi, m) \longmapsto \mu_G(g, \xi) + \nu_G(m)$$

$$= -\xi + \nu_G(m).$$

Thus, the zero level set of the moment map $\lambda$ is explicitly given by

$$\lambda^{-1}(0) = \{(g, \xi, m) \in T^*G \times M : \xi = \nu_G(m)\}.$$  

(3.9)

We have used symplectic reduction to get the reduced space

$$(T^*G \times M)\# G = \lambda^{-1}(0)/G.$$  

Now, we want to find a symplectomorphism $M \cong \lambda^{-1}(0)/G$. Well, consider the map

$$\varphi : M \longrightarrow \lambda^{-1}(0)/G$$

$$m \longmapsto [(1, \nu_G(m), m)].$$
First we will show $\varphi$ is a diffeomorphism. Recall that the moment map $\nu_G : M \to \mathfrak{g}^*$ corresponds to the action of $G$ on $M$, so it satisfies the condition

$$d\nu^x_G = i_{x \#} \omega,$$

where $\omega$ is the symplectic form on the $M$. We know $\varphi$ is a smooth map as each of its component maps, the identity, the moment map $\nu_G$, and the quotient map $\lambda^{-1}(0) \to \lambda^{-1}(0)/G$ are smooth. Then, we define the map

$$\tilde{\psi} : \lambda^{-1}(0)/G \to M$$

by

$$[(g, \xi, m)] \mapsto g \cdot m,$$

which is clearly well-defined because any two elements in the same equivalence class lie in the same $G$-orbit. It is a smooth map by Theorem 3.4 because we have the following diagram:

$$\begin{array}{ccc}
\lambda^{-1}(0) & \xrightarrow{\pi} & \lambda^{-1}(0)/G \\
\downarrow & & \downarrow \tilde{\psi} \\
\lambda^{-1}(0)/G & \xrightarrow{\pi} & M
\end{array}$$

where the quotient map $\pi : \lambda^{-1}(0) \to \lambda^{-1}(0)/G$ and the composition of maps (inclusion and projection) $q_2 \circ i : \lambda^{-1}(0) \to M$ are smooth.

To show $\tilde{\psi} : \lambda^{-1}(0)/G \to M$ is a symplectomorphism, it suffices to show $\tilde{\psi}^\ast \omega = \eta$ (by Definition 2.5), where $\eta$ is the symplectic form on $\lambda^{-1}(0)/G$. We know that the product of symplectic manifolds is symplectic, so we have that $(T^*G \times M, q_1^\ast \Omega + q_2^\ast \omega)$ is symplectic, where $q_1, q_2$ are the natural projections $q_1 : T^*G \times M \to T^*G, q_2 : T^*G \times M \to M$. From the symplectic reduction, Theorem 2.11 tells us that $\eta$ is the unique form that satisfies

$$\pi^\ast \eta = i^\ast (q_1^\ast \Omega + q_2^\ast \omega).$$

Hence, if we substitute $\eta = \tilde{\psi}^\ast \omega$ and show condition (3.11) holds, we are done. Well, we have

$$\pi^\ast (\tilde{\psi}^\ast \omega) = i^\ast (q_1^\ast \Omega + q_2^\ast \omega)$$

and

$$(\tilde{\psi} \circ \pi)^\ast \omega = (q_1 \circ i)^\ast \Omega + (q_2 \circ i)^\ast \omega$$
\[ \psi^* \omega = (q_1 \circ i)^* \Omega + (q_2 \circ i)^* \omega, \]  
(3.12)

where \( \psi = \tilde{\psi} \circ \pi \) Now, let’s make a few observations. By Theorem 2.8, we have that

\[
T_{(g, \xi, p)} \lambda^{-1}(0) \subseteq T_{(g, \xi, p)}(T^*G \times M)
\]
\[
= T_{(g, \xi)}(T^*G) \times T_p M
\]
\[
\cong T_{(g, \xi)}(G \times g^*) \times T_p M
\]
\[
= g \times g^* \times T_p M.
\]

These identifications made in Section 2.2 make the following computations much cleaner. Let \( u = (x, \eta, U) \) and \( v = (y, \zeta, V) \) be elements of the tangent space \( T_{(g, \xi, p)} \lambda^{-1}(0) \subseteq g \times g^* \times T_p M \). Then, evaluating (3.12) at \( (u, v) \), we want to show

\[
\psi^* \omega(u, v) = (q_1 \circ i)^* \Omega(u, v) + (q_2 \circ i)^* \omega(u, v).
\]  
(3.13)

By definition of the pullback, this becomes

\[
\omega(d\psi_{(g, \xi, p)}(x, \eta, U), d\psi_{(g, \xi, p)}(y, \zeta, V)) = \Omega(d(q_1 \circ i)(u), d(q_1 \circ i)(v)) + \omega(d(q_2 \circ i)(u), d(q_2 \circ i)(v))
\]  
(3.14)

Let’s evaluate the RHS. First, observe the composition maps

\[
q_1 \circ i : \lambda^{-1}(0) \rightarrow T^*G, \quad (g, \xi, p) \mapsto (g, \xi)
\]
\[
q_2 \circ i : \lambda^{-1}(0) \rightarrow T^*G, \quad (g, \xi, p) \mapsto p
\]

The exterior derivative of each map at the point \( u = (x, \eta, U) \) is

\[
d(q_1 \circ i)_{(g, \xi, p)}(x, \eta, U) = (x, \eta)
\]
\[
d(q_2 \circ i)_{(g, \xi, p)}(x, \eta, U) = U
\]

Hence, by Theorem 2.9 in Section 2.2, our expression (3.14) simplifies to

\[
\omega(d\psi_{(g, \xi, p)}(x, \eta, U), d\psi_{(g, \xi, p)}(y, \zeta, V)) = \zeta(x) - \eta(y) + \xi([x, y]) + \omega(U, V).
\]  
(3.15)

Now, recall the following lemma for regular values of smooth maps between manifolds.

**Lemma 3.6** (See e.g. [3, Proposition 5.38]). Suppose \( F : M \rightarrow N \) is a smooth map and let \( q_0 \in N \). If \( dF_p \)
is surjective for all \( p \in F^{-1}(q_0) \), then \( F^{-1}(q_0) \) is a smooth manifold with

\[
T_p F^{-1}(q_0) = \ker(dF_p) \leq T_p M.
\]

By Lemma 3.6 we see that \( T_{(g,\mu(p),p)} \lambda^{-1}(0) = \ker d\lambda_{(g,\mu(p),p)} \), where the differential of the moment map \( \lambda \) is

\[
d\lambda_{(g,\xi,p)}(x,\eta,U) = d\mu_p(U) - \eta.
\]

For notational convenience, we will let \( \mu = \nu_G \). Then, we have that

\[
\ker d\lambda_{(g,\mu(p),p)} = \{(x,\eta,U) \in g \times g^* \times T_p M : \eta = d\mu_p(U)\}. \tag{3.16}
\]

Hence, by Lemma 3.6 and (3.16) we see that

\[
\zeta(x) = d\mu_p(V)(x)
\]

\[
\eta(y) = d\mu_p(U)(y),
\]

so the RHS of expression (3.15) is

\[
= d\mu_p(V)(x) - d\mu_p(U)(y) + \xi([x,y]) + \omega(U,V). \tag{3.17}
\]

**Lemma 3.7.** Let \( \mu : M \to V^* \), where \( V \) is a vector space over \( \mathbb{R} \). Let \( x \in V \), so \( \mu^x := \langle \mu, x \rangle : M \to \mathbb{R} \) is defined by \( p \mapsto \langle \mu(p), x \rangle = \mu(p)(x) \). Then, for any \( v \in T_{\mu(p)} V \cong V \),

\[
(d\mu^x)_p(v) = d\mu_p(v)(x) \in \mathbb{R}. \tag{3.18}
\]

**Proof.** By the definition of \( \mu^x(p) := \langle \mu(p), x \rangle \) as the component of \( \mu \) along \( x \), we have

\[
(d\mu^x)_p(v) = (d\langle \mu, x \rangle)_p(v)
\]

\[
= \langle d\mu_p(v), x \rangle
\]

\[
= d\mu_p(v)(x)
\]

\[ \square \]
Using Lemma 3.7, we can simplify (3.17) to

\[(d\mu^x)_p(V) - (d\mu^y)_p(U) + \xi([x, y]) + \omega(U, V),\]

then use condition (2.8) of moment maps to get

\[(i_x\#\omega)_p(V) - (i_y\#\omega)_p(U) + \xi([x, y]) + \omega(U, V)\]

\[= \omega(x_p^#, V) - \omega(y_p^#, U) + \xi([x, y]) + \omega(U, V).\] (3.19)

Now, we can rewrite (3.15) as

\[\omega(d\psi(g, \xi, p)(x, \eta, U), d\psi(g, \xi, p)(y, \zeta, V)) = \omega(x_p^#, V) - \omega(y_p^#, U) + \mu_p([x, y]) + \omega(U, V),\] (3.20)

where \(\xi = \mu_p\) by (3.9). Now, we will compute the differential \(d\psi\) using the following definition of computing differentials with curves.

For a smooth map \(F : M \to N\) and any \(v \in T_pM\), if \(\gamma : \mathbb{R} \to M\) is a curve such that \(\gamma'(0) = v\) and \(\gamma(0) = p\), then

\[dF_p(v) = \frac{d}{dt} \bigg|_{t=0} F(\gamma(t))\] (3.21)

Also, recall that \((g, \mu(p), p) \in \lambda^{-1}(0)\) is the point on the submanifold \(\lambda^{-1}(0) \subseteq T^*G \times M \cong G \times g^* \times M\) and the corresponding tangent vector is \((x, \eta, U) \in T(g, \mu(p), p)\lambda^{-1}(0)\). Using (3.21), we must find a curve

\[\gamma : \mathbb{R} \to \lambda^{-1}(0)\] such that \(\gamma(0) = (g, \mu(p), p)\) and \(\gamma'(0) = (x, \eta, U)\)

Well, consider the curve

\[\gamma(t) = (ge^{tx}, \gamma_2(t), \gamma_3(t)),\]

which satisfies the initial conditions. Then, we compute

\[d\psi(g, \mu(p), p)(x, \eta, U) = \frac{d}{dt} \bigg|_{t=0} \psi(\gamma(t))\]

\[= \frac{d}{dt} \bigg|_{t=0} \psi(ge^{tx}, \gamma_2(t), \gamma_3(t))\]

\[= \frac{d}{dt} \bigg|_{t=0} ge^{tx} \cdot \gamma_3(t)\]

\[= \frac{d}{dt} \bigg|_{t=0} ge^{tx} \cdot \gamma_3(0) + \frac{d}{dt} \bigg|_{t=0} g \cdot \gamma_3(t)\]
Now we know the initial conditions $\gamma_3(0) = p$ and $\gamma'_3(0) = U$, and by definition (2.7) this simplifies to

$$\frac{d}{dt} \bigg|_{t=0} ge^{tx} \cdot p + g \cdot \gamma'_3(0) = g \cdot x^p + g \cdot U = g \cdot (x^p + U)$$

Similarly for the second argument on the LHS of (3.20), we have

$$d\psi_{(g,\mu(p),\pi)}(y, \zeta, V) = g \cdot (y^p + V) \in T_{g\cdot p}M$$

Then, the expression (3.20) simplifies to

$$\omega(g \cdot (x^p + U), g \cdot (y^p + V)) = \omega(x^p, V) - \omega(y^p, U) + \mu_p([x, y]) + \omega(U, V)$$

where we know $\omega(g \cdot a, g \cdot b) = \omega(a, b)$ as a result of the symplectic action (the 2-form $\omega$ is invariant under elements within the same $G$-orbit). Thus, we have

$$\omega(x^p + U, y^p + V) = \omega(x^p, V) - \omega(y^p, U) + \mu_p([x, y]) + \omega(U, V)$$

where the skew-symmetry of $\omega$ means $\omega(U, y^p) = -\omega(y^p, U)$. Hence, we have

$$\omega(x^p, y^p) + \omega(x^p, V) - \omega(y^p, U) + \omega(U, V) = \omega(x^p, V) - \omega(y^p, U) + \mu_p([x, y]) + \omega(U, V).$$

Cancelling out equal terms on both sides, we are left with

$$\omega(x^p, y^p) = \mu_p([x, y]). \quad (3.22)$$

In order to prove (3.22) holds, we will use the properties of the moment map $\mu$ and the facts above. Starting with the right hand side, we have

$$\mu_p([x, y]) = \langle \mu(p), [x, y] \rangle$$

$$= \langle \mu(p), \frac{d}{dt} \bigg|_{t=0} Ad_{\exp(tx)}y \rangle$$

$$= \langle \frac{d}{dt} \bigg|_{t=0} Ad^*_\exp(-tx) \circ \mu(p), y \rangle$$

$$= \frac{d}{dt} \bigg|_{t=0} Ad^*_\exp(-tx) \circ \mu(p), y \rangle$$

$$= \omega(x^p, y^p)$$
and by the equivariance condition of moment maps \((\text{Ad}_g^* \circ \mu = \mu \circ \psi_g)\), this becomes
\[
\left. \frac{d}{dt} \right|_{t=0} \mu \circ \psi_{\exp(-tx)}(p), y \right) = \langle d\mu_p \left( \left. \frac{d}{dt} \right|_{t=0} \psi_{\exp(-tx)}(p) \right), y \rangle \tag{3.23}
\]
where we know \(\left. \frac{d}{dt} \right|_{t=0} \mu \circ \psi(t) = d\mu_p(\psi'(0))\), so we have
\[
\left. \frac{d}{dt} \right|_{t=0} \psi_{\exp(-tx)}(p) = \left. \frac{d}{dt} \right|_{t=0} e^{-tx} \cdot p = -x^\#_p
\]
so (3.23) simplifies to
\[
\langle d\mu_p(-x^\#_p), y \rangle = -d\mu_Y(x^\#_p) = -(i_{y^\#}(x^\#_p)) = -\omega(y^\#_p, x^\#_p) = \omega(x^\#_p, y^\#_p)
\]
by the skew-symmetry of \(\omega\). This is equivalent to the LHS of (3.22), so we are done.

Finally, we have shown that the map \(\tilde{\psi} : \lambda^{-1}(0)/G \rightarrow M\) is a symplectomorphism. This proves that \(T^*G\) is the identity morphism in the category \(\mathcal{M}_\mathbb{R}\).

\[\square\]

**Proposition 3.8** (Associativity). The composition of morphisms in \(\mathcal{M}_\mathbb{R}\) is **associative**. That is, given a composition of morphisms \(G \xrightarrow{X} H \xrightarrow{Y} K \xrightarrow{Z} L\), we want to show
\[
Z \circ (Y \circ X) = (Z \circ Y) \circ X. \tag{3.24}
\]

**Proof.** By definition (3.1), we have that \(Y \circ X = (X \times Y)//H\) and \(Z \circ Y = (Y \times Z)//K\), so each side of (3.24) is
\[
Z \circ (Y \circ X) = ((X \times Y)//H \times Z)//K \tag{3.25}
\]
\[
(Z \circ Y) \circ X = (X \times (Y \times Z)//K)//H. \tag{3.26}
\]
Then, by definition (3.3), we can explicitly write expressions (3.25) and (3.26) as the sets
\[
Z \circ (Y \circ X) = \{(x, [y, z]) \in (X \times Y)//H \times Z : \tilde{\mu}_Y^K([(x, y)]) + \tilde{\mu}_Z^K(z) = 0\}/K \tag{3.27}
\]
\[
(Z \circ Y) \circ X = \{(x, [(y, z)]) \in X \times (Y \times Z)//K : \tilde{\mu}_X^K(x) + \tilde{\mu}_Y^K([(y, z)]) = 0\}/H \tag{3.28}
\]
where \(\tilde{\mu}_Y^K : (X \times Y)//H \rightarrow \mathfrak{k}^*\) and \(\tilde{\mu}_Z^K : Z \rightarrow \mathfrak{k}^*\) are the moment maps of the Hamiltonian \(K\)-action on

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symplectic manifolds \((X \times Y)_{//H}\) and \(Z\), respectively. And, where \(\mu_X^H : X \to h^*\) and \(\tilde{\mu}_Z^H : (Y \times Z)_{//K} \to h^*\) are the moment maps of the Hamiltonian \(K\)-action on \((X \times Y)_{//H}\) and \(Z\), respectively.

Now, we must show these sets (3.27) and (3.28) are equivalent. We will define a symplectomorphism
\[
\varphi : Z \circ (Y \circ X) \longrightarrow (Z \circ Y) \circ X
\]
\[
[([(x, y), z)] \mapsto [(x, [y, z])].
\]

This is well-defined and smooth. It is well-defined because for \([(x_1, y_1), z_1)] = \([(x_2, y_2), z_2)] \in Z \circ (Y \circ X)\) such that \((x_1, y_1) = h \cdot (x_2, y_2)\) and \(z_1 = k \cdot z_2\), we have
\[
\varphi([(x_1, y_1), z_1)]) = [(x_1, [y_1, z_1])]
\]
\[
= [(h \cdot x_2, [(h \cdot y_1, k \cdot z_1)])]
\]
\[
= [(x_2, ([y_2, z_2])]
\]
\[
= \varphi([(x_2, y_2), z_2)]).
\]

To show the map \(\varphi\) is smooth, we will use Theorem 3.4 twice and diagram chase using the definition of symplectic reduction. Consider the diagram:

\[
\begin{array}{ccc}
X \times h^* Y \times Z & \xrightarrow{\pi_1} & (Y \circ X) \times h^* Z \\
\downarrow & & \downarrow \pi_2 \\
(Y \circ X) \times h^* Z & \xrightarrow{\pi_3} & X \times h^* (Z \circ Y) \\
\downarrow \pi_4 & & \downarrow \pi_4 \circ \tilde{\pi}_2 \\
Z \circ (Y \circ X) & \xrightarrow{\varphi} & (Z \circ Y) \circ X
\end{array}
\]

where each quotient map \(\pi_i\) corresponds to a smooth map coming from symplectic reduction. By using Theorem 3.4 on the upper half of the diagram, we have that \(\tilde{\pi}_2\) is smooth. Then, applying Theorem 3.4 again, we have that \(\varphi\) is smooth because the composition of smooth maps \(\pi_4 \circ \tilde{\pi}_2\) is smooth.

Let \(\omega_1\) be the symplectic 2-form on \((Y \circ X) \times Z\), let \(\omega_2\) be the 2-form on \(X \times (Z \circ Y)\), and let \(\omega_3\) be the 2-form on \(Z \circ (Y \circ X)\). Let \(i_1 : (Y \circ X) \times h^* Z \to (Y \circ X) \times Z\) and \(i_2 : X \times h^* (Z \circ Y) \to X \times (Z \circ Y)\) be the natural inclusion maps. Thus, we have that \(\pi_4^* \omega_4 = i_2^* \omega_2\) and \(\pi_3^* \omega_3 = i_1^* \omega_1\). Now, we must show that \(\varphi^* \omega_4 = \omega_3\) where \(\omega_4\) is the symplectic 2-form on \((Z \circ Y) \circ X\). This will follow from the uniqueness of
symplectic reduction given by Theorem 2.11. Using the diagram, we see that

\[(\pi_4 \circ \tilde{\pi}_2)^* \omega_4 = (\varphi \circ \pi_3)^* \omega_4.\]

This implies that

\[\tilde{\pi}_2^* (i_2^* \omega_2) = \pi_3^* (\varphi^* \omega_4),\]

where

\[\tilde{\pi}_2^* (i_2^* \omega_2) = \pi_3^* (\omega_3).\]

Hence, we have that

\[\varphi^* \omega_4 = \omega_3.\]

This concludes the proof, and we have shown that $MT_\mathbb{R}$ is indeed a category.

\[\square\]

**Remark 3.9.** The Moore–Tachikawa conjecture is motivated by String Theory, where $\eta_{G_C}$ is obtained by taking the Higgs branch of some supersymmetric quantum field theories (called theories of class $S$) depending on $G_C$ and a Riemann surface. The Higgs branch of class $S$ supersymmetric theories naturally comes equipped with a hyperkähler structure. Thus, it seems more natural to define the target category $HK$ of hyperkähler manifolds instead of one with a weaker holomorphic symplectic structure. However, it turns out that $\eta_G$ in the hyperkähler sense is not quite a topological quantum field theory because the hyperkähler quotient changes the overall factor of the metric.
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