BINARY LINEAR CODES VIA 4D DISCRETE IHARA-SELBERG FUNCTION

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ABSTRACT. We express the weight enumerator of each binary linear code, in particular the Ising partition function of an arbitrary finite graph, as a formal infinite product. An analogous result was obtained by Feynman and Sherman in the beginning of 60’s for the special case of the Ising partition function of the planar graphs. A product expression is an important step towards understanding the Ising free energy, i.e., the logarithm of the Ising partition function, for general graphs and in particular for the cubic 3D lattices.

1. INTRODUCTION

A linear code $C$ of length $n$ over the binary field $\mathbb{F}_2$ is a linear subspace of $\mathbb{F}_2^n$. Each vector in $C$ is called a codeword. Let $w : \{1, \ldots, n\} \to \mathbb{R}$ be a real weight function. The weight of a codeword $c$, denoted by $w(c)$, is defined as the sum of the weights of the non-zero entries of $c$. The weight enumerator of $(C, w)$ is defined as

$$W_{C, w}(x) = \sum_{c \in C} x^{w(c)}.$$ 

A k-uniform hypergraph (k-hypergraph for short) is a pair $H = (V, E)$ where $E$ is a set of $k$-element-subsets of $V$ called hyperedges. If a set is given along with its linear order, we say that the set is directed. A directed k-hypergraph is a pair $D = (V, A)$ where $A$ is a set of directed $k$-element-subsets of $V$. We note that one $k$-element-subset may appear several times in $A$, with different linear orders. As an illustration we observe that the directed 2-hypergraphs are exactly the directed graphs.

In this paper we show that the weight enumerator of each binary linear code can be expressed as a formal product in the form of the 4-dimensional discrete Ihara-Selberg function of a directed 4-hypergraph (see Definition 10). The main theorem is as follows (see subsection 4.3 for the proof).

**Theorem 1.** Let $C$ be a binary linear code of length $n$ over the binary field $\mathbb{F}_2$, let $w : \{1, \ldots, n\} \to \mathbb{R}$ be a real weight function and let $W_{C, w}(x)$ be the weight enumerator of $(C, w)$. Then one can construct in polynomial time a directed 4-hypergraph $D = (V, A)$ and weight function $z : A \to \mathbb{R}$, so that $W_{C, w}(x)$ is equal to the 4-dimensional discrete Ihara-Selberg function of $D, z$.

Next we explain motivation of our result which is the Feynman’s formula for the planar Ising partition function.

1.1. **Ising partition function.** Let $G = (V, E)$ be a graph. We say that $E' \subset E$ is even if the graph $(V, E')$ has even degree (possibly zero) at each vertex. Let $\mathcal{E}(G)$ denote the set of all even sets of $G$. It is not difficult to see that the set of the characteristic vectors of the elements of $\mathcal{E}(G)$ form a binary linear code. We assume that a variable $x_e$ is associated with each edge $e$, and define the generating function for even sets, $\mathcal{E}_G$, in $\mathbb{Z}[[x_e]_{e \in \mathcal{E}(G)}]$, as follows:

$$\mathcal{E}_G(x) = \sum_{E' \in \mathcal{E}(G)} \prod_{e \in E'} x_e.$$
Knowing the polynomial $\mathcal{E}_G$ is equivalent to knowing the partition function $Z_G^{\text{Ising}}$ of the Ising model on the graph $G$. The Ising partition function is defined by

$$Z_G(\beta) = Z_G(x)\bigg|_{x_e := e^{\beta J_e}} \forall e \in E(G)$$

where the $J_e$ ($e \in E(G)$) are weights (coupling constants) associated with the edges of the graph $G$, the parameter $\beta$ is the inverse temperature, and

$$Z_G(x) = \sum_{\sigma : V(G) \to \{1, -1\}} \prod_{e = \{u, v\} \in E(G)} x_e^{\sigma(u)\sigma(v)}.$$

The theorem of van der Waerden [13] (see [5] for a proof) states that $Z_G(x)$ is the same as $\mathcal{E}_G(x)$ up to change of variables and multiplication by a constant factor:

$$Z_G(x) = 2^{|V(G)|} \left( \prod_{e \in E(G)} \frac{x_e + x_e^{-1}}{2} \right) \mathcal{E}_G(z)\bigg|_{z_e := \frac{x_e - x_e^{-1}}{x_e + x_e^{-1}}}.$$

1.2. Main contribution. In a suggestion how to prove the formula of Kac, Ward [3] for the planar Ising partition function, Feynman proposed that $\mathcal{E}_G(x)$ is equal to a formal product, which then equals to the square root of the determinant introduced in [3]. The formula of Feynman was proved by Sherman in [11]; Sherman also studies in [11] the logarithm of the Feynman’s product and reproduces results for the free energy of the planar Ising problem.

For a non-planar graph $G$, the Ising partition function can be written as a linear combination of formal products (see [6, 5]), but a single product formulation, crucial for studying the logarithm and the free energy, has been missing. The initial idea of this paper is that for the Ising model where the underlying graph is non-planar, we should consider higher-dimensional underlying structure than the structure of graphs. It turns out that it is advantageous to study the weight enumerators of all binary linear codes, i.e., in particular Ising partition functions of general graphs, as functions on uniform hypergraphs. The relevant functions are hyper-determinants and hyper-permanents of higher-dimensional matrices. Finally, a product formula is obtained by a generalization of Bass’ theorem for 4-dimensional matrices.

The rest of the paper is devoted to the proof of Theorem 1. First in section 2 we explain the background: the Feynman’s formula and the Bass’ formula. In section 3 we show that the weight enumerator of each binary linear code can be written as $\det(I + A)$, where $A$ is the incidence matrix of a directed 4-uniform hypergraph, and $I$ is the 4-dimensional identity matrix. This section is a generalisation and a modification of several previous results (see [9, 10, 7]). Then, in section 4 4-dimensional discrete Ihara-Selberg function is defined and 4-dimensional Bass’ theorem is stated and it is proved in the last section. The results of the last three sections prove the main theorem.

Concluding remarks. (1) An immediate consequence of Theorem 1 is that the Tutte polynomial $T(M, x, y)$ of binary matroid $M$ can be written along the hyperbola $(x - 1)(y - 1) = 2$ as a formal product. (2) In view of [10], the main result of this paper should hold for linear codes over arbitrary finite field $\mathbb{F}_p$, $p$ prime.

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2. Feynman’s formula

Let $G = (V, E)$ be a planar graph properly embedded in the plane and as above we assume that a variable $x_e$ is associated with each edge $e$. Let $D = (V, A)$ be the symmetric orientation of $G$, i.e., each edge of $G$ appears in $D$ with both orientations. If $a$ is orientation of $e$ then we let $x_a = x_e$. Let $\mathcal{R}(D)$ denote the set of the equivalence classes of aperiodic reduced closed walks of $D$: we identify the closed walks that differ only in the starting arc, and we also identify pairs of closed walks that are reverses of each other. A closed walk
is reduced if two orientations of the same edge are never consecutive, and it is aperiodic if it is not a power of a smaller closed walk.

If \( W \in R(D) \) then we denote by \( A(W) \) the multi-set of the arcs of \( W \). We note that \( W \) may have more than one copy of any arc of \( D \). We note that \( R(D) \) is infinite whenever \( G \) has two cycles which share a vertex. We denote by \( \text{rot}(p) \) the rotation, sometimes called the Whitney index of closed walk \( p \) and note that \( (-1)^{\text{rot}(p)} \) is the same for equivalent closed walks. Feynman suggested and Sherman [11] proved the following (see [5] for a proof).

**Theorem 2.** Let \( G = (V,E) \) be a planar graph properly embedded in the plane. Then

\[
E_G(x) = \prod_{W \in R(D)} [1 - (-1)^{\text{rot}(W)} \prod_{a \in A(W)} x_a].
\]

The product in Theorem 2 is a formal infinite product which we define as follows. If \( I \) is a countable set and \( f_i, i \in I \) are polynomials, then we let

\[
\prod_{i \in I} [1 + f_i] = \sum_{J \subset I \text{ finite}} \prod_{j \in J} f_j.
\]

In Bass’ theorem (see [11] and [2] for several proofs and generalisations), the considerations of Theorem 2 appear in more general setting. This is explained next.

2.1. **Discrete Ihara-Selberg function and Bass’ theorem.** Bass’ theorem expresses the determinant of \( I - A(D) \), where \( I \) is the identity matrix and \( A(D) \) is the adjacency matrix of a digraph \( D \), as formal product over aperiodic closed walks in \( D \).

If \( D = (V,A) \) is a directed graph with no loops then let \( \sharp(D) \) denote the set of closed walks of \( D \) (we identify two closed walks that differ only in the starting arc) that are aperiodic, i.e., they are not a power of a smaller closed walk. If \( W \in \sharp(D) \) then we denote by \( A(W) \) the multi-set of the arcs of \( W \), with multiplicities. We note that \( \sharp(D) \) is infinite whenever \( D \) has two directed cycles which share a vertex. If weights \( w : A \to \mathbb{R} \) on the arcs are given then the adjacency matrix \( A(D,w) \) is the \( V \times V \) matrix defined by \( A(D,w)_{uv} = x_w(uv) \), where \( x \) is a variable.

**Definition 1.** Let \( D = (V,A) \) be a directed graph with no loops and with weight function \( w : A \to \mathbb{R} \). The discrete Ihara-Selberg function \( IS_{(D,w)}(x) \) is the following formal product:

\[
IS_{(D,w)}(x) = \prod_{W \in \sharp(D)} (1 - \prod_{a \in A(W)} x_w(a)).
\]

**Theorem 3** (Bass’ theorem). Let \( D = (V,A) \) be a directed graph with no loops and with weight function \( w : A \to \mathbb{R} \). Then

\[
\det(I - A(D)) = IS_{(D,w)}(x).
\]

Below we include the proof of Theorem 3 along the lines of [12] since later we will use analogous reasoning. The key ingredient of the proof is the following Lemma on coin arrangements of Sherman [11].

**Lemma 1.** (on coin arrangements) Suppose we have a collection of \( N \) objects of which \( m_1 \) are of one kind, \( m_2 \) are of second kind, ..., and \( m_k \) are of \( k \)-th kind. Let \( b_{N,i} = \binom{N}{i,m_1,\ldots,m_k} \) be the number of exhausted unordered arrangements of these objects into \( i \) disjoint, nonempty, circularly ordered sets such that no two circular orders are the same and none is periodic. If \( N > 1 \) then

\[
\sum_{i=1}^{N} (-1)^i b_{N,i} = 0.
\]

We denote by \( F \) the set of all finite sets of aperiodic closed walks, and by \( F' \) the subset of \( F \) consisting of all finite sets of vertex disjoint directed cycles of \( D \). We use a well-known equation:

\[
\det(I - A(D)) = \sum_{c \in \{W_1,\ldots,W_k\} \subseteq F'} (-1)^{|c|} \prod_{i=1}^k \prod_{a \in A(W_i)} x_w(a).
\]
On the other hand, the formal product of the discrete Ihara-Selberg function is defined by

\begin{equation}
\text{IS}_{(D,w)}(x) = \sum_{c=\{W_1,\ldots,W_k\} \in F} (-1)^{|c|} k \prod_{i=1}^{k} \prod_{a \in A(W_i)} x^{w(a)}
\end{equation}

Hence we need to show:

**Proposition 1.**

\[ \sum_{c \in F \setminus F^x} (-1)^{|c|} \prod_{i=1}^{k} \prod_{a \in A(W_i)} x^{w(a)} = 0. \]

**Proof.** Let \( v \) be a vertex of \( D \). Let \( U(v) \) be the subset of \( F \) consisting of all the finite sets \( c \) of aperiodic closed walks of \( D \) such that in \( \cup_{W \in A(W)} \), there is at most one arc entering \( v \) and at most one arc leaving \( v \). We show:

\begin{equation}
\sum_{c=\{W_1,\ldots,W_k\} \in F} (-1)^{|c|} k \prod_{i=1}^{k} \prod_{a \in A(W_i)} x^{w(a)} = \sum_{c=\{W_1,\ldots,W_k\} \in U(v)} (-1)^{|c|} \prod_{i=1}^{k} \prod_{a \in A(W_i)} x^{w(a)}
\end{equation}

Equation 3 can be proved as follows. Let \( \mathcal{S}_v \) be the set of all aperiodic closed walks which have an arc containing \( v \). Each aperiodic closed walk \( W \) of \( \mathcal{S}_v \) can be uniquely decomposed into a cycle of aperiodic closed walks \( (s_1, \ldots, s_l) \) such that each \( s_i \) has exactly one arc entering \( v \) and exactly one arc leaving \( v \). We will call such walks \( s_i \) **stones**. The decomposition goes as follows. We consider walking along \( W \) starting at \( v \) with an arc \( e_1 = (v, w_1) \). If we never get to another arc starting at \( v \) then \( k = 1 \) and the whole \( W \) is a stone. Otherwise we eventually get for the first time to another arc \( e_2 \neq e_1 \) starting at \( v \).

We define two aperiodic closed walks \( W_1 \) and \( W_2 \): \( W_1 \) consists of the arcs which we travelled between \( e_1 \) and \( e_2 \), including \( e_1 \) but not \( e_2 \); we observe that \( W_1 \) is aperiodic closed walk and we declare it to be the stone \( s_1 \). Further, \( W_2 \) consists of the remaining arcs of \( W \). We observe that \( W_2 \) is a closed walk (not necessarily aperiodic) and we continue the decomposition analogously with \( W_2 \). This finishes the definition of the stones.

We denote by \( F_v \) the set of all finite sets of elements of \( \mathcal{S}_v \). We can write

\[ \sum_{R \in F_v} (-1)^{|R|} \prod_{W \in R} \prod_{a \in A(W)} x^{w(a)} = \sum_{T} \alpha(T) \prod_{s \in T} \prod_{a \in A(s)} x^{w(a)}, \]

where \( T \) ranges over all finite multisets of stones and \( \alpha(T) = \sum_{R} (-1)^{|R|} \) where the sum is over all unordered arrangements \( B \) of the stones of \( T \) into collections of distinct aperiodic closed walks. This is the same as arranging the elements of \( T \) into disjoint, nonempty, circularly ordered sets such that no two circular orders are the same and none are periodic. It follows from the Lemma on coin arrangements that \( \alpha(T) = 0 \) whenever \( |T| > 1 \). Hence

\[ \sum_{c=\{W_1,\ldots,W_k\} \in F} (-1)^{|c|} k \prod_{i=1}^{k} \prod_{a \in A(W_i)} x^{w(a)} = \prod_{W \in \mathcal{S}_v} (1 - \prod_{a \in A(W)} x^{w(a)}) \times [1 + \sum_{R \in F_v} (-1)^{|R|} \prod_{W \in R} \prod_{a \in A(W)} x^{w(a)}] = \prod_{W \in \mathcal{S}_v} (1 - \prod_{a \in A(W)} x^{w(a)}) \times [1 - \sum_{W \in \mathcal{S}_v} x^{w(a)}], \]

where \( \mathcal{S}_v \) denotes the set of all aperiodic closed walks which have exactly one arc entering \( v \) and exactly one arc leaving \( v \). This proves 3. Next we proceed analogously for \( F \) replaced by \( U(v) \) and \( v \) replaced by a vertex \( u \neq v \) of \( D \). □
3. Geometric representation of binary linear codes

We say that a k-hypergraph \( H = (V, E) \) is almost disjoint if each pair of hyperedges intersects in at most one vertex. We say that \( H \) is k-partite if the vertex-set \( V \) is partitioned into \( k \) subsets \( V_1, \ldots, V_k \) and each hyperedge of \( E \) intersects each \( V_i \) in exactly one vertex. \( M \subseteq E \) is called perfect matching of \( H \) if the elements of \( M \) are pairwise disjoint and \( \cup M = V \).

3.1. Hypermatrices, their determinants and permanents. We start by introducing basic notions.

- A k-matrix is an array indexed by all k-tuples from its index set \( V_1 \times \ldots \times V_k \).
- Let \( D = (V, A) \) be a directed k-hypergraph and \( w : A \to \mathbb{R} \) a weight function. The adjacency matrix \( A(D, w) \) is the k-matrix with index-set \( V^k \) defined by: \( A(D, w)_a = x^{w(a)} \) for each \( a \in A \), and \( A(D, w)_a = 0 \) otherwise.
- Let \( H = (V_1, \ldots, V_k, E) \) be a k-partite k-hypergraph and \( w : E \to \mathbb{R} \) a weight function. The transition matrix \( T(H, w) \) is the k-matrix with index-set \( U = V_1 \times \ldots \times V_k \) defined by: \( T(H, w)_u = x^{w(u)} \) for each \( u \in U \), and \( A(D, w)_u = 0 \) otherwise.

If \( M \) is a \((k+1)\)-matrix with index-set \( U = V_0 \times \ldots \times V_k \) then its determinant is defined by

\[
\det(M) = \sum_{\alpha_1, \ldots, \alpha_k} \prod_{1 \leq j \leq k} \text{sign}(\alpha_j) \prod_{i \in V_0} M_{(\alpha_1(i), \ldots, \alpha_k(i),i)}. 
\]

Its permanent is defined by

\[
\per(M) = \sum_{\alpha_1, \ldots, \alpha_k} \prod_{i \in V_0} M_{(\alpha_1(i), \ldots, \alpha_k(i),i)}.
\]

As an illustration we observe: if \( H = (V_1, \ldots, V_k, E) \) is a k-partite k-hypergraph and \( w : E \to \mathbb{R} \) is its weight function, then \( \per(T(H, w)) \) is the generating function of the perfect matchings of \( H \), i.e.,

\[
\per(T(H, w)) = \sum_M \prod_{e \in M} x^{w(e)}. 
\]

Finally, we introduce a useful construction, how to get a directed k-hypergraph from an almost disjoint k-partite k-hypergraph.

**Definition 2.** Let \( H = (V_1, \ldots, V_k, E) \) be an almost disjoint k-partite k-hypergraph and let \( P \) be a perfect matching of \( H \). We let \( D(H, P) = (V, A) \) where \( V = V_1 \) and \( A = \{c(e, P) : e \in E \setminus P\} \); it remains to define \( c(e, P) \). For \( 1 \leq i \leq k \) let \( e_i \) denote the vertex of \( e \cap V_i \), let \( M(e)_i \) denote the edge of \( M \) which contains \( e_i \) and let \( a_i \) be the vertex of \( M(e)_i \cap V_1 \). We let \( c(e, P) = (a_1, \ldots, a_k) \).

Next observation follows directly from the definitions.

**Observation 1.** Let \( H = (V_1, \ldots, V_k, E) \) be an almost disjoint k-partite k-hypergraph and let \( P \) be a perfect matching of \( H \). Let \( w : E \to \mathbb{R} \) a weight function where \( w(p) = 0 \) for each \( p \in P \). We let \( w' \) be the weight function of \( D(H, P) \) defined so that for each \( e \in E \setminus P \), \( w'(c(e, P)) = w(e) \). Then

\[
T(H, w) = I + A(D(H, P), w'). 
\]

The main result of this section is the following theorem.

**Theorem 4.** Let \( C \) be a binary linear code of length \( n \) over the binary field \( \mathbb{F}_2 \), let \( w : \{1, \ldots, n\} \to \mathbb{R} \) be a real weight function and let \( W_{C,w}(x) \) be the weight enumerator of \((C, w)\). Then one can construct in polynomial time a directed 4-hypergraph \( D = (V, A) \) and weight function \( z : A \to \mathbb{R} \), so that

\[
W_{C,w}(x) = \det(I + A(D, z)). 
\]

**Proof.** Theorem follows from Theorem 5, Theorem 6 and Observation 1. \( \square \)
3.2. Weight enumerators as 3-permanents. The following theorem follows from Theorem 6, Proposition 6 and Theorem 20 of [9] by setting the weights of all auxiliary triangles to zero, and from Theorem 4 of [7].

**Theorem 5**. Let \( C \) be a binary linear code of length \( n \) over the binary field \( \mathbb{F}_2 \), let \( w : \{1, \ldots, n\} \to \mathbb{R} \) be a real weight function and let \( W_{C,w}(x) \) be the weight enumerator of \((C, w)\). Then one can construct in polynomial time an almost disjoint 3-partite 3-hypergraph \( H = (V, E) \), its perfect matching \( P \) and weight function \( w' : A \to \mathbb{R} \) so that \( w'(p) = 0 \) for each \( p \in P \) and

\[
W_{C,w}(x) = \text{per}(T(H, w')).
\]

3.3. Kasteleyn hyper-matrices, 3-permanents and 4-determinants. The main result of this section is the following theorem.

**Theorem 6**. Let \( H = (V, E) \) be an almost disjoint 3-partite 3-hypergraph, let \( P \) be its perfect matching and let \( w : E \to \mathbb{R} \) be a weight function so that \( w(p) = 0 \) for each \( p \in P \). Then one can construct in polynomial time an almost disjoint 4-partite 4-hypergraph \( H' = (V', E') \), its perfect matching \( P' \) and weight function \( w' : E' \to \mathbb{R} \) so that \( w'(p) = 0 \) for each \( p \in P' \) and

\[
\text{per}(T(H, w)) = \det(T(H', w')).
\]

The reasoning in the proof is analogous to the proof of Theorem 5 of [7], where 2-permanents are turned into 3-determinants. We postpone the proof of Theorem 6 to the end of this subsection.

**Definition 3**. We say that a \( k \)-matrix \( A \) is Kasteleyn if there is a \( k \)-matrix \( A' \) obtained from \( A \) by changing signs of some entries so that \( \text{per}(A) = \det(A') \).

If \( A \) is the transition matrix of a 2-partite 2-hypergraph, i.e., of a bipartite graph, then already Kasteleyn [4] noticed that \( A \) is Kasteleyn provided the bipartite graph is planar. Kasteleyn 2-matrices were characterised in a seminal paper [5]: the set of Kasteleyn 2-matrices is severely restricted and does not go far beyond the transition matrices of planar bipartite graphs. An important observation of [7] is that Kasteleyn 3-matrices form a rich class. Next we generalize the reasoning of [7] to Kasteleyn k-matrices, \( k \geq 3 \).

We first introduce a necessary condition for a hyper-matrix to be Kasteleyn. Let \( A \) be a \( V_0 \times \cdots \times V_k \) hyper-matrix, \( k \geq 2 \), and \(|V_i| = |V_j|\) for all \( i \neq j \). We first define \( k \) bipartite graphs \( G_1, \ldots, G_k \) as follows. We let, for \( 1 \leq i \leq k \), \( G_i = (V_i, V_i, E_i) \) where

\[
E_i = \{(v_0, v_i) | v_0 \in V_0, v_i \in V_i \text{ and there is } v = (v_0, \ldots, v_{i-1}, v_i, \ldots, v_k) \text{ such that } A_0 \neq 0\}.
\]

**Theorem 7**. If \( A \) is such that all \( G_1, G_2, \ldots, G_k \) are planar bipartite graphs then \( A \) is Kasteleyn.

**Proof**. Let \( M_i \) be the transition matrix of \( G_i \) and let \( \text{sign}_i : E(G_i) \to \{-1, 1\} \) be the signing of the entries of \( M_i \), which defines matrix \( M'_i \) such that \( \text{per}(M_i) = \det(M'_i) \). We recall that such signing exists by the above mentioned result of Kasteleyn. We define hyper-matrix \( A' \) by

\[
A'_{{v_0, \ldots, v_k}} = \left[ \prod_{1 \leq i \leq k} \text{sign}_i({v_0, v_i}) A_{(v_0, \ldots, v_k)}. \right]
\]

We have

\[
\det(A') = \sum_{\sigma_1, \ldots, \sigma_{k-1}} \left( \prod_{1 \leq i < k} \text{sign}(\sigma_i) \right) \times \sum_{\sigma_k} \text{sign}(\sigma_k) \prod_{j \in V_0} \prod_{1 \leq i < k} \text{sign}_i({j, \sigma_i(j)}) |\text{sign}_k({j, \sigma_1(j), \ldots, \sigma_{k-1}(j)})| A_{j \sigma_1(j) \ldots \sigma_{k-1}(j)}.
\]

By the construction of \( \text{sign}_k \) we have that for each \( \sigma_k \) and each \( \sigma_1, \ldots, \sigma_{k-1} \), if \( \prod_{j \in V_0} A_{(j \sigma_1(j) \ldots \sigma_{k-1}(j))} \neq 0 \) then

\[
\text{sign}(\sigma_k) \prod_{j \in V_0} \text{sign}_k({j, \sigma_k(j)}) = 1.
\]

Hence

\[
\det(A') = \sum_{\sigma_k} \sum_{\sigma_1, \ldots, \sigma_{k-1}} \left( \prod_{1 \leq i < k} \text{sign}(\sigma_i) \right) \prod_{j \in V_0} \prod_{1 \leq i < k} \text{sign}_i({j, \sigma_i(j)}) |\text{sign}_k({j, \sigma_1(j), \ldots, \sigma_{k-1}(j)})| A_{(j \sigma_1(j) \ldots \sigma_{k-1}(j))}.
\]

Continuing this way for \( k - 1, \ldots, 1 \) we get \( \det(A') = \text{per}(A) \).

\[\Box\]
Proof of Theorem 6. Let $V = (V_1, V_2, V_3)$ be the 3-partition of $V$. We note that existence of a perfect matching implies that all three parts have the same size which we denote by $n$.

We next define four sets of vertices $R_0, R_1, R_2, R_0$ and 4-partite 4-hypergraph $H' = (V', E')$ as follows:

We first introduce four vertices $u^0, u^1, u^2, u^3$ of each vertex $u$ of $V$. Then, for each edge $e \in E$ we introduce twelve vertices $e^i_j, j = 0, 1, 2, 3$ and $i = 1, 2, 3$.

For $i = 1, 2, 3$ let $R_j = \{u^i; i = 1, 2, 3, u \in V_j\} \cup \{e^i_j; i = 1, 2, 3, e \in E\}$, and $R_0 = \{u^0; u \in V\} \cup \{e^0_i; i = 1, 2, 3, e \in E\}$. We let $V' = R_0 \cup R_1 \cup R_2 \cup R_3$. Next, the set of edges $E'$ consists of

$$E' = \{(e^0_1, u^1, v^i, w^i); e = (u, v, w) \in E\} \cup \{(e^0_1, e^1_2, e^3_1); e \in E\} \cup \{(u^0, e^1_2, e^3_1), (u^1, e^2_3, e^3_1), (u^2, e^3_1, e^3_1); e = (u, v, w) \in Q\} \cup \{(e^0_1, e^1_2, e^3_1); e \notin Q\}.$$  

Clearly, $H' = (V', E')$ is almost disjoint. For $e = (u, v, w)$ and $i = 1, 2, 3$ we let $w'(e, v^i, w^i) = 1/3w(e)$ and $w'(x) = 0$ otherwise.

For each perfect matching $Q$ of $H$ we define

$$Q' = \{(e^0_1, u^1, v^i, w^i); e = (u, v, w) \in Q\} \cup \{(u^0, e^1_2, e^3_1), (u^1, e^2_3, e^3_1), (u^2, e^3_1, e^3_1); e = (u, v, w) \in Q\} \cup \{(e^0_1, e^1_2, e^3_1); e \notin Q\}.$$  

We observe that $Q'$ is a perfect matching of $H'$ and each perfect matching of $H'$ is equal to $S'$ for some perfect matching $S$ of $H$. Summarising,

$$\text{per}(T(H, w)) = \text{per}(T(H', w')).$$

Next we show that $T(H', w')$ is Kasteleyn: by Theorem 7 it suffices to observe that the corresponding three bipartite graphs $G_1, G_2, G_3$ are planar; this follows since each connectivity component of these bipartite graphs consists of paths of three edges which share exactly the end-vertices.

We finally observe that Kasteleyn signing is trivial. Let $D_1$ be the orientation of $G_1$ in which each edge is directed from $R_0$ to $R_1$. In each planar drawing of $G_1$, each inner face is a cycle of length 6, and thus it has an odd number (three) of edges directed in $D_1$ clockwise. This means that $D_1$ is a Pfaffian orientation of $G_1$, and $\text{per}(A) = \det(A)$. See e.g. [5] for basic facts on Kasteleyn orientations (called Pfaffian orientations in [5]) and Kasteleyn signings. This finishes the proof of Theorem 6.

4. 4D-Bass’ theorem

We first reformulate the discrete Ihara-Selberg function in a way that is easier to generalise.

4.1. Back to the discrete Ihara-Selberg function.

Definition 4. Let $D = (V, A)$ be a directed graph. Let $A$ be the multiset of arcs, where each arc of $D$ appears infinitely many times. Vertex connector in $D$ is a triple $(e_1, e_2, v)$ where $v$ is a vertex and $e_1, e_2$ are elements of $A$ such that $e_1$ ends in $v$ and $e_2$ starts in $v$.

A 2-circuit in $D$ is a pair $(C, S)$ where $S \subset A$ is a multi-set of arcs and $C$ is a set of vertex connectors so that

(1) Each arc of $S$ is in exactly one vertex connector of $C$ as the entering arc and in exactly one connector of $C$ as the leaving arc,

(2) Each arc of each vertex connector of $C$ belongs to $S$,

(3) The digraph induced by the arcs of $S$ is weakly connected,

(4) Each vertex of $V$ is in at most one vertex connector.

A 2-circulation in $D$ is a pair $(C, S)$ satisfying (1), (2), (3) and (5) It is not possible to write $S$ as disjoint union of $S_1, S_2$ and $C$ as disjoint union of $C_1, C_2$ such that both $(S_i, C_i), i = 1, 2$, are 2-circulations.

The following observation is straightforward.


Definition 5. Let \( D = (V, A) \) be a directed 4-hypergraph. 4.2. The 2-circuits of \( D \) are exactly directed cycles of \( D \). The 2-circulations of \( D \) are exactly closed walks of \( D \) (we identify two closed walks which differ only in the starting arc).

The following view will also be instructive: given digraph \( D \) we define a new digraph so that its vertices are all the vertex connectors, and there is arc \( a' = (c, c') \) for each arc \( a \in A \) such that \( a \) belongs to \( c \) as leaving arc, and \( a \) belongs to \( c' \) as entering arc. We call the directed cycles of the new digraph connector cycles of \( D \).

Observation 2. The 2-circuits of \( D \) are exactly directed cycles of \( D \). The 2-circulations of \( D \) are exactly closed walks of \( D \) (we identify two closed walks which differ only in the starting arc).

The following view will also be instructive: given digraph \( D \) we define a new digraph so that its vertices are all the vertex connectors, and there is arc \( a' = (c, c') \) for each arc \( a \in A \) such that \( a \) belongs to \( c \) as leaving arc, and \( a \) belongs to \( c' \) as entering arc. We call the directed cycles of the new digraph connector cycles of \( D \).

Observation 3. The 2-circulations of \( D \) are exactly the connector cycles of \( D \).

Next we need to define periodic 2-circulations.

(6) We say that 2-circulation \((S, C)\) is periodic if there is \( k > 1 \) and partitions \( S = S_1 \cup \ldots \cup S_k \) and \( C = C_1 \cup \ldots \cup C_k \) so that, after identifying the different copies of each arc of \( D \), the pairs \((S_i, C_i), i = 1, \ldots, k\) are all equal to the same 2-circulation. A 2-circulation is aperiodic if it is not periodic.

Using these notions, the discrete Ihara-Selberg function can be expressed as follows:

Observation 4. Let \( D = (V, A) \) be a directed graph with no loops and with weight function \( w : A \to \mathbb{R} \). Then

\[
IS_D(y) = \prod_{(S, C) \in \mathcal{S}(D)} (1 + (-1)^{m(S, C)}) \prod_{a \in S} y_a,
\]

where \( \mathcal{S}(D) \) is the set of all aperiodic 2-circulations of \( D \) and \( m(S, C) \) denotes the number of the connector cycles of \((S, C)\).

Next we generalise the introduced concepts to 4 dimensions.

4.2. Directed 4-hypergraphs. Let \( D = (V, A) \) be a directed 4-hypergraph. We consider \( D \) given along with weights \( w : A \to \mathbb{R} \).

Definition 5. Let \( a = (a_1, a_2, a_3, a_4) \in A \). We say that each \( a_i \) is a vertex of \( a \). We introduce new vertex \( v(a) \) called edge-vertex, and four colored arcs: white \((v(a), a_1)\), red \((v(a), a_2)\) and green \((v(a), a_3)\) start in \( v(a) \) and blue \((a_4, v(a))\) enters \( v(a) \). If \( v \) is a vertex of directed arc \( a \) then the colored arc between \( v(a) \) and \( v \) will be denoted by \( a(a, v) \).

Definition 6. Let \( D = (V, A) \) be a directed 4-hypergraph and let \( A \) be the multiset of directed hyperedges of \( D \) where each element of \( D \) appears infinitely many times, with the same vertices. We usually denote the elements of \( A \) by \( a \) and the elements of \( A \) by \( h \) or by \( r \).

(1) Let \( S \subset A \). We let \( O(S) \) be the directed graph with the vertex-set consisting of the subset \( V(S) \subset V \) of the vertices contained in at least one element of \( S \), and all \( v(h), h \in S \). The arcs of \( O(S) \) are exactly the colored arcs of the elements of \( S \).

(2) Connector in \( D \) is a 5-tuple \((h_1, h_2, h_3, h_4, v)\) where \( v \in V \) and \( h_1, h_2, h_3, h_4 \) are directed hyperedges of \( A \) incident with \( v \) and such that \( a(h_1, v) \) is white, \( a(h_2, v) \) is red, \( a(h_3, v) \) is green and \( a(h_4, v) \) is blue. Hence \( a(h_4, v) \) leaves \( v \) and the other three arcs enter \( v \).

Definition 7. A 4-circuit in \( D \) is a pair \((S, C)\) where \( S \subset A \) and \( C \) is a set of connectors so that

(1) For each \( h = (v_1, v_2, v_3, v_4) \in S \) and \( i \in \{1, 2, 3, 4\} \) there is exactly one connector \((h_1, h_2, h_3, h_4, v)\) of \( C \) such that \( h = h_i \); moreover, \( v = v_i \).

(2) Each hyperedge of each connector of \( C \) belongs to \( S \).

(3) The digraph \( O(S) \) induced by the arcs of the directed hyperedges of \( S \) is weakly connected.

(4) Each vertex of \( V \) is in at most one connector.

A 4-circulation in \( D \) is a pair \((S, C)\) satisfying (1), (2), (3) and

(5) It is not possible to write \( S \) as disjoint union of \( S_1, S_2 \) and \( C \) as disjoint union of \( C_1, C_2 \) such that both \((S_i, C_i)\) are 4-circulation.

(6) In addition we require a feasibility property. Let \( v \) be a vertex of \( V(S) \) and let \( a = a(h_4, v) \) be the blue edge of a connector \((h_1, h_2, h_3, h_4, v)\) of \( C \). Let \((V_a, O_a)\) be the subdigraph of \( O(S) \) which is minimum with the properties: (1) it contains \( a \), (2) each edge-vertex \( v(h) \in V_a \) is incident with exactly one arc of \( O_a \) of each of the four colors, and (3) let \( v' \neq v, v' \in V_a \) and \( a(h, v') \in O_a \). Let
Proof. Conditions (1), (2), (3) of (6) of Definition 7 which contradicts the minimality of $D$ by property (1) of Definition 7, the connector cycles of $(S, C)$ contains both $v, h'$ and $w, r'$ in case $c$ contains hyperedge $h$ and vertex $v$. The red directed cycles and the green directed cycles are defined analogously. We call these cycles connector cycles in analogy with the 2-dimensional case discussed in subsection 4.1. If $z$ is such a connector cycle then we let

$$\text{sign}(z) = (-1)^{|z|/2-1}.$$ 

Next we are ready to introduce the 4-dimensional discrete Ihara-Selberg function.

**Definition 8.** We say that 4-circulation $(S, C)$ is periodic if there is $k > 1$ and partitions $S = S_1 \cup \ldots \cup S_k$, $C = C_1 \cup \ldots \cup C_k$ so that, after identifying the different copies of each element of $A$, the pairs $(S_i, C_i), i = 1, \ldots, k$ are all equal to the same 4-circulation. A 4-circulation is aperiodic if it is not periodic.

We denote by $\Delta(D)$ the set of all aperiodic 4-circulations of $D$.

**Definition 9.** Let $D = (V, A)$ be a directed 4-hypergraph and let $A$ be as above the multiset of directed hyperedges of $D$. We define the white, red and green directed cycles on the vertex-set given by the connectors of $D$ and the edge-vertices $v(h), h \in A$: the white directed cycles contain alternately blue and white arcs. By abuse of notation we identify here the arc between connector $c$ and edge-vertex $v(h)$ with colored arc $a(v, h)$ in case $c$ contains hyperedge $h$ and vertex $v$. The red directed cycles and the green directed cycles are defined analogously. We call these cycles connector cycles in analogy with the 2-dimensional case discussed in subsection 4.1. If $z$ is such a connector cycle then we let

$$\text{sign}(z) = (-1)^{|z|/2-1}.$$

Now we are ready to introduce the 4-dimensional discrete Ihara-Selberg function.

**Definition 10.** Let $D = (V, A)$ be a directed 4-hypergraph. We associate an independent variable $y_a$ with each directed hyperedge of $A$ and we let $y = (y_a)_{a \in A}$ be the vector of these variables. The 4-dimensional discrete Ihara-Selberg function $4IS_{(D, w)}(y)$ is the following formal product:

$$4IS_{(D, w)}(x) = \prod_{(S, C) \in \Delta(D)} (1 + (-1)^{m((S, C))}) \prod_{h \in S} y_h,$$

where $m((S, C))$ is the number of the connector cycles of $(S, C)$.

Next we study properties of the digraphs $(V_a, O_a)$ of (6) of Definition 7. Let $(S, C)$ be a 4-circulation of $D$ and let $c$ be a connector of $C$. Let $h$ be a hyperedge of $c$ and let $v$ be the vertex of $c$. By property (1) of Definition 7 no other connector of $(S, C)$ contains both $v$ and $h$. Hence colored arc $a(h, v)$ uniquely determines connector $c$ of $(S, C)$ and we say that $a(h, v)$ belongs to connector $c$. We also observe that, again by property (1) of Definition 7, the connector cycles of $(S, C)$ of the same color are disjoint.

**Observation 5.** Let $(S, C)$ be a 4-circulation of $D$. Let $a = a(r_1, v)$ and $a' = (r_4', v)$ be the blue edges of connectors $c = (r_1, r_2, r_3, r_4, v)$ and $c' = (r_1', r_2', r_3', r_4', v)$ of $C$. Then

1. $O_a \cap O_{a'} = \emptyset$.
2. $O_a$ contains a white arc $a_w = (v(r), v)$, a red arc $a_r = (v(r'), v)$ and a green arc $a_g = (v(r''), v)$. By the feasibility property of the 4-circulations, see Definition 4 we have that all $r, r', r''$ belong to the same connector $c' = (r, r', r'', r''', v)$ of the 4-circulation $(S, C)$.
3. Let $S_a$ be the set of the directed hyperedges $h \in S$ such that $v(h)$ is incident with at least one arc of $O_a$ and let $C_a$ be the set of the connectors which contain at least one arc of $O_a$. Let $C'_a = C_a \setminus \{c\} \cup \{(r, r', r'', r_4, v)\}$. Then $(S_a, C'_a)$ is an aperiodic 4-circulation.

**Proof.** Let $O = O_a \cap O_{a'} \neq \emptyset$. We observe that $a \notin O$ by the feasibility property for $a'$. Then $O_a \setminus O$ satisfies conditions (1), (2), (3) of (6) of Definition 4 which contradicts the minimality of $O_a$. This proves (1).

Since $a \in O_a$ and by the defining properties of $O_a$, we have that $O_a$ has all the arcs of the segment of the white (red, green respectively) connector cycle starting by $a$ and ending at $v$. This proves (2).

In order to show (3), we first observe that $(S_a, C'_a)$ satisfies (1), (2), (3) of Definition 4. If $(S_a, C'_a)$ can be written as disjoint union of 4-circulations then the one containing triangle connector $(r, r', r'', r_4, v)$
contradicts minimality of \( O_a \). Hence \((S_a, C'_a)\) satisfies (5) of Definition 7. We also observe that \((S_a, C'_a)\) is aperiodic since it has exactly one connector containing \( v \).

Hence it remains to observe that (6) of Definition 7 holds. This is clearly true for arc \( a \), hence let us assume for a contradiction that (6) does not hold for blue arc \( a' \neq a \) of \((S_a, C'_a)\). Let \( a' \) start at vertex \( v' \).

We first note a general observation about the feasibility property (6) of Definition 7: if its first point is violated then the second one is violated as well. Namely, in the notation of Definition 7, for each blue arc \( z \in O_a \) starting at \( v \) there are, by the definition of \( O_a \), segments of the white, red and green connector cycles which start with \( z \) and end at \( v \). We recall that the connector cycles of the same color are arc-disjoint. Hence, if \( z \neq a \) then necessarily \( O_a \) has at least two white, red and green arcs entering \( v \) which violates the second point of the feasibility property.

Hence we can assume that the second point of the feasibility property is violated for \( a' \neq a \) in \((S_a, C'_a)\). We let \( O_1 \) denote the set \( O_{a'} \) in \((S_a, C'_a)\). Since original \( O_{a'} \) satisfies the feasibility property \(((S, C)\) is a 4-circulation) but \( O_1 \) does not, necessarily there is an arc, say \( x \), of \( O_1 \setminus O_{a'} \) entering the initial vertex \( v' \) of \( a' \).

By minimality of \( O_1 \) and existence of \( x \) we also have that there is a white, red or green arc of \( O_{a'} \setminus O_1 \). By the defining properties of \( O_1 \) this implies that there is a white, red or green arc, say \( u \), of \( O_{a'} \setminus O_1 \) which enters \( v' \). Tracing back the colored cycle of \( u \) in \( O_{a'} \), we get that necessarily it contains \( v \) and its segment \( P \) between \( v \) and \( u \) starts by blue edge \( a'' \neq a \) incident with \( v \). Let \( a' \) be the connector containing \( u \) and \( v' \). If an arc of \( d' \) belongs to \( O_a \) then \( P \) belongs to \( O_a \) but by the feasibility property \( O_a \) cannot have \( a'' \): a contradiction. Hence no arc of \( d' \) belongs to \( O_a \). This means that in \( O_{a'} \), all three colored segments starting by \( a' \) and ending by an arc of \( d' \) enter \( v \) and leave \( v \) by a blue arc different from \( a \). Let \( W_1, R_1, G_1 \) be initial parts of these segments between \( a' \) and \( v \). By the defining property of \( O_1 \) we have that all \( W_1, R_1, G_1 \) are subsets of \( O_1 \).

Let \( d \) be any connector at \( v' \) which contains an arc of a white, red or green path of \( O_1 \) starting by \( a' \) and ending in \( v' \), and let \( d \) do not contain \( x \). Since no arc of \( d' \) belongs to \( O_a \), we have that \( d \neq d' \). It follows that each such segment contains \( v \). Let \( W_2, R_2, G_2 \) be initial parts of these segments between \( a' \) and \( v \), and let \( W'_2, R'_2, G'_2 \) be the terminal segments starting by \( a \). Then necessarily \( W_1 = W_2, R_1 = R_2, G_1 = G_2 \) and \( x \) is a consequence of \( W'_2, R'_2, G'_2 \) since \( x \notin O_{a'} \).

We observe, by traversing back in \((S, C)\) the connector cycles which contain the white, red and green arcs of \( d \), that there is a blue arc \( z \) leaving \( v' \) such that set \( O_z \) of the original 4-circuits \((S, C)\) contains the white, red and green arc of \( d \). Hence \( O_z \) contains \( W'_2, R'_2, G'_2 \). But then by the above reasoning it also contains \( x \), since \( x \) is a consequence of \( W'_2, R'_2, G'_2 \). This is a contradiction with the feasibility property for \( O_z \) in \((S, C)\).

\[ \square \]

Next, we turn our attention to the 4D-matrices. First we recall the case of the classical matrices. If \( D = (V, A) \) is a directed graph, \( w \) its weight function and \( A(D, w) \) its adjacency matrix, then we get, from the definition of the determinant, that \( \det(A(D, w)) \) is equal to the sum, over all sets \( S = \{C_1, \ldots, C_k\} \) of vertex disjoint directed cycles of \( D \) covering all the vertices, of \( \text{sign}(S) \prod_{a \in \cup_i C_i} x^{w(a)} \); \( \text{sign}(S) \) is the sign of the permutation with the cycles given by the cycles of \( S \). Clearly, \( \text{sign}(S) = (-1)^m \) where \( m \) is the number of even-length cycles of \( S \). These consideration immediately imply that

\[ \det(I - A(D)) = \sum_{R=\{C_1, \ldots, C_i\}} (-1)^l \prod_{a \in \cup_i C_i} x^{w(a)}, \]

where the sum is over all sets of vertex disjoint directed cycles of \( D \), not necessarily covering all the vertices of \( D \). This observation is used in the proof of the Bass’ theorem described in Introduction. We first develop these statements for the determinant of the 4-matrices.

**Proposition 2.** Let \( A(D, w) \) be the adjacency 4-matrix of a directed 4-hypergraph \( D = (V, A) \) with weight function \( w \).

\[ \det(A(D, w)) = \sum_P \prod_{c_i} \text{sign}(c_i) \prod_{\text{hyperedge of } c_i} x^{w(c)}, \]

where \( P \) ranges over all sets of vertex-disjoint 4-circuits containing directed hyperedges of \( A \) only, which cover all vertices of \( D \), and \( \text{sign}(c_i) = \prod_i \text{sign}(x) \), where \( x \) ranges over all connector cycles of \( c_i \).
Proof. We study the term of the defining expansion of $\det(A(D, w))$ corresponding to a triplet of permutations $\alpha_1, \alpha_2, \alpha_3$. We find out that the set of the cycles of $\alpha_1$ corresponds to a collection of white connector cycles where each directed hyperedge belongs to $A$, each edge-vertex is in at most one such cycle and each vertex of $V$ is in exactly one connector of these connector cycles. Moreover the sign of $\alpha$ is the product of the signs (defined in Definition 3) of the connector cycles of the collection.

Let $Z$ be the set of the collections $z$ of connector cycles satisfying: if edge-vertex $v(a)$ belongs to a connector cycle of $z$ then (1) $a \in A$ and (2) for each color $\in \{\text{white}, \text{red}, \text{blue}\}$, $v(a)$ belongs to unique connector cycle of the color. Next, each connector of a connector cycle of $z$ belongs to exactly one connector cycle of each color. Finally each vertex of $V$ is in exactly one connector of a connector cycle of $z$. If $z \in Z$ then let $R(z)$ be the set of the directed hyperedges of the elements of $z$ and let $\text{sign}(z)$ be the product of the signs of the connector cycles of the elements of $z$. It follows that

$$\det(M(D)) = \sum_{z \in Z} \text{sign}(z) \prod_{a \in R(z)} x^{w(a)}.$$ 

Clearly, the collection of the connector cycles of a set of vertex-disjoint 4-circuits which cover all the vertices of $V$ belongs to $Z$. In order to finish the proof, we argue that each $z \in Z$ can be partitioned into the sets of the connector cycles of vertex-disjoint 4-circuits: indeed, the set of the directed hyperedges and the set of the connectors of $z$ satisfy all the properties of a 4-circuits (see Definition 4) but property (3).

**Corollary 1.** Let $A(D, w)$ be the adjacency 4-matrix of a directed 4-hypergraph $D = (V, A)$ with weight function $w$.

$$\det(I - A(D, w)) = \sum_{Q = \{c_1, \ldots, c_k\}} (-1)^{m(Q)} \prod_i \prod_{a \in c_i} x^{w(a)},$$

where $Q$ ranges over all sets of vertex-disjoint 4-circuits and $m(Q)$ is the number of the connector cycles of $Q$.

Proof. We use Proposition 1. It remains to prove that the signs are correct. Each $Q$ contributes $(-1)^{n(Q)}$, where $n(Q) = \sum_c \text{connector cycle of } Q \left( |c|/2 - 1 \right) + r(Q)$, where $r(Q)$ is the number of the directed hyperedges of $Q$. We further notice that, since each hyperedge of $Q$ contributes 6 to the total length of the connector cycles, we have $\sum_c \text{connector cycle of } Q \left|c\right|/2 = 3r(Q)$.

Next we state our 4D-Bass’ theorem.

**Theorem 8.** Let $D = (V, A)$ be a directed 4-hypergraph and let $w : A \rightarrow \mathbb{R}$ be its weight function. Then $\det(I - M(D, w)) = 4ISD(y)|_{y_w = x^{w(a)}}$.

4.3. **Proof of Theorem 8**. It follows from Theorem 3 and Theorem 8 that

$$W_{c, w}(x) = 4ISD(y)|_{y_w = x^{z(a)}},$$

where $D$ and $z$ are as in Theorem 4.

5. **Proof of 4D-Bass’ theorem.**

We denote by $G$ the set of all finite sets of aperiodic 4-circulations, and by $G'$ the subset of $G$ consisting of the finite sets $c = \{c_1, \ldots, c_k\}$ of 4-circulations so that each vertex of $D$ is at most once in a connector of some $c_i$; equivalently, $G'$ is the set of all finite sets of vertex-disjoint 4-circuits of $D$. By Corollary 1

$$\det(I - A(D, w)) = \sum_{c = \{(S_1, C_1), \ldots, (S_k, C_k)\} \in G'} (-1)^{m(c)} \prod_{i=1}^{k} \prod_{a \in S_i} x^{w(a)}.$$ 

Hence, analogously as in the 2-dimensional case, we need to show:

**Proposition 3.**

$$\sum_{c = \{(S_1, C_1), \ldots, (S_k, C_k)\} \in G' \setminus G'} (-1)^{m(c)} \prod_{i=1}^{k} \prod_{a \in S_i} x^{w(a)} = 0.$$
Proof. Let $t$ be a vertex of $D$. Let $U(t)$ be the subset of $G$ consisting of all the finite sets $c$ of aperiodic 4-circulations of $D$ such that $t$ appears at most once at a connector of an element of $c$. We show:

$$
\sum_{c=(S_1,C_1),\ldots,(S_k,C_k)\in G} (-1)^{m(c)} \prod_{i=1}^{k} \prod_{a\in S_i} x^{w(a)} = \sum_{c=(S_1,C_1),\ldots,(S_k,C_k)\in U(t)} (-1)^{m(c)} \prod_{i=1}^{k} \prod_{a\in S_i} w(a)
$$

Equation (5) can be proved as follows. Let $\Delta$ be the set of all aperiodic 4-circulations which have a connector containing $t$. Each aperiodic 4-circulation $(S,C)$ of $\Delta$ can be uniquely decomposed into a cycle of 4-circulations $(s_1,\ldots,s_k)$ (where each $s_i$ contains a unique connector at $t$). We will call such $s_i$ stones. The decomposition goes as follows.

We consider a process which starts with a connector $c = (r_1,r_2,r_3,r_4,t)$ of $(S,C)$ and its blue arc $a$; we recall that $r_1,r_2,r_3,r_4$ are directed hyperedges of $D$ incident with $t$ and such that $a(r_1,t)$ is white, $a(r_2,t)$ is red, $a(r_3,t)$ is green and $a(r_4,t)$ is blue. Arc $a = a(r_4,t)$ leaves $t$ and the other three colored arcs enter $t$.

We recall definition of $O_t$ in the feasibility property (see Definition 7). Let $S_a$ be the set of directed hyperedges of $S$ that are incident with at least one arc of $O_a$ and let $C_a$ be the set of the connectors which contain at least one arc of $O_a$.

By Observation 6 we have that $O_t$ contains a white arc $a_w$, a red arc $a_r$, and a green arc $a_g$ which enter vertex $t$. By the feasibility property of the 4-circulations, see Definition 7, we have that all three hyperedges of these colored arcs belong to the same triangle connector of the 4-circulation $(S,C)$. Let us denote this connector as $c' = (r,r',r'',r'''$, $t)$. If $c' = c$ then $k = 1$ and the whole $(S,C)$ is a stone.

Otherwise let $a = a_1$ and let $a_2$ be the blue arc of $c'$. Let us denote $c = c = (r_1',r_2',r_3',r_4',t)$ and $c' = c'' = (r_1'',r_2'',r_3'',r_4'',t)$. If $c'' = c''$ then $k = 2$, otherwise let $a_2$ be the blue arc of $c''$. We continue like this until, for $k \geq 2$, $c_k = c_k$. We have that $O_a \cap O_a = 0$ for $i \neq j$, and $\cup_{k=1}^{k} S_a = S$ by property (5) of Definition 7. We define pairs $(S_i,C_i)$ by: for each $i \leq k$, $S_i = S_{a_i}$, for $i < k$, $C_i = C_{a_i} \setminus \{c_i,c_{i+1}\} \cup \{(r_1^{i+1},r_2^{i+1},r_3^{i+1},r_4^{i},t)\}$, and finally $C_k = C_{a_k} \setminus \{c_k,c_{k+1}\} \cup \{(r_1^{k+1},r_2^{k+1},r_3^{k+1},r_4^{k},t)\}$. By Observation 6 we get that each $(S_i,C_i), i = 1,\ldots,k$, is an aperiodic 4-circulation. This finishes the definition of the stones $s_i = (S_i,C_i)$, $i = 1,\ldots,k$.

**Observation 6.** $m((S,C)) = [\sum_{i=1}^{k} m((S_i,C_i))] - 3(k-1)$.

**Proof.** In the construction of $(S_i,C_i), i = 1,\ldots,k$, we replace three connector cycles of $(S,C)$ by $3k$ connector cycles of $(S_1,C_1),\ldots,(S_k,C_k)$.

We denote by $G_t$ the set of all finite sets of elements of $\Delta$. We can write

$$\sum_{R \in G_t} (-1)^{m(R)} \prod_{(S,C) \in R \in S} x^{w(a)} = \sum_{T \in T} \beta(T) \prod_{a \in S} (-1)^{m(s)} \prod_{a \in S} x^{w(a)},$$

where $T$ ranges over all finite multisets of stones and $\beta(T) = \sum_{W} (-1)^{|W|}$ where the sum is over all unordered arrangements $W$ of the stones of $T$ into collections of distinct aperiodic 2-circulations. This is the same as arranging the elements of $T$ into disjoint, nonempty, circularly ordered sets such that no two circular orders are the same and none are periodic. It follows from the Lemma on coin arrangements that $\beta(T) = 0$ whenever $|T| > 1$. Hence

$$\sum_{c=(S_1,C_1),\ldots,(S_k,C_k)\in G} (-1)^{m(c)} \prod_{i=1}^{k} \prod_{a\in S_i} x^{w(a)} =$$

$$\prod_{(S,C) \in \Delta \setminus \Delta_t} (1 + (-1)^{m((S,C))} \prod_{a \in S} x^{w(a)}) \times \left[ 1 + \sum_{R \in G_t} (-1)^{m(R)} \prod_{(S,C) \in R \in S} x^{w(a)} \right] =$$

$$\prod_{(S,C) \in \Delta \setminus \Delta_t} (1 + (-1)^{m((S,C))} \prod_{a \in S} x^{w(a)}) \times \left[ 1 + \sum_{(S,C) \in \Delta_t'} (-1)^{m((S,C))} \prod_{a \in S} x^{w(a)} \right],$$

where $\Delta_t'$ denotes the set of all aperiodic 2-circulations which have exactly one connector containing $t$. This proves (5). Next we proceed analogously for $G$ replaced by $U(t)$ and $t$ replaced by a vertex $t' \neq t$ of $D$. This finishes the proof of Proposition 3 and the 4D-Bass’ theorem (Theorem 3).

□
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