Abstract

We prove that if two closed disks $X_1$ and $X_2$ of the Riemann sphere are spectral sets for a bounded linear operator $A$ on a Hilbert space, then $X_1 \cap X_2$ is a complete $(2 + 2/\sqrt{3})$-spectral set for $A$. When the intersection $X_1 \cap X_2$ is an annulus, this result gives a positive answer to a question of A.L. Shields (1974).

1 Introduction and the statement of the main results.

Let $X$ be a closed set in the complex plane and let $R(X)$ denote the algebra of bounded rational functions on $X$, viewed as a subalgebra of $C(\partial X)$ with the supremum norm

$$\|f\|_X = \sup\{|f(x)| : x \in X\} = \sup\{|f(x)| : x \in \partial X\}.$$ 

Here $\partial X$ denotes the boundary of the set $X$.

1.1 Spectral and complete spectral sets.

Let $A \in \mathcal{L}(H)$ be a bounded linear operator acting on a complex Hilbert space $H$. For a fixed constant $K > 0$, the set $X$ is said to be a $K$-spectral set for $A$ if the spectrum $\sigma(A)$ of $A$ is included in $X$ and the inequality $\|f(A)\| \leq K\|f\|_X$ holds for every $f \in R(X)$. Notice that, for a rational function $f = p/q \in R(X)$, the poles of $f$ are outside of $X$, and the operator $f(A)$ is naturally defined as $f(A) = p(A)q(A)^{-1}$ or, equivalently, by the Riesz holomorphic functional calculus. The set $X$ is a spectral set for $A$ if it is a $K$-spectral set with $K = 1$. Thus $X$ is spectral for $A$ if and only if $\|\rho\| \leq 1$, where $\rho : R(X) \to \mathcal{L}(H)$ is the homomorphism given by $\rho(f) = f(A)$.

We let $M_n(R(X))$ denote the algebra of $n$ by $n$ matrices with entries from $R(X)$. If we let the $n$ by $n$ matrices have the operator norm that they inherit as linear transformations on the $n$-dimensional Hilbert space $\mathbb{C}^n$, then we can endow $M_n(R(X))$ with the norm

$$\| (f_{ij}) \|_X = \sup\{\| (f_{ij}(x)) \| : x \in X\} = \sup\{\| (f_{ij}(x)) \| : x \in \partial X\}.$$ 

In a similar fashion we endow $M_n(\mathcal{L}(H))$ with the norm it inherits by regarding an element $(A_{ij})$ in $M_n(\mathcal{L}(H))$ as an operator acting on the direct sum of $n$ copies of $H$. For a fixed constant $K > 0$, the set $X$ is said to be a complete $K$-spectral set for $A$ if $\sigma(A) \subset X$ and the inequality $\|(f_{ij}(A))\| \leq K\|(f_{ij})\|_X$ holds for every matrix $(f_{ij}) \in M_n(R(X))$ and every $n$. In terms of the complete bounded norm \cite{14} of the homomorphism $\rho$, this means that $\|\rho\|_{cb} \leq K$. A complete spectral set is a complete $K$-spectral set with $K = 1$.

Spectral sets were introduced and studied by J. von Neumann \cite{12} in 1951. In the same paper von Neumann proved that a closed disk $\{z \in \mathbb{C} : |z - \alpha| \leq r\}$ is a spectral set for $A$ if and only if $\|A - \alpha I\| \leq r$. Also \cite{12}, the closed set $\{z \in \mathbb{C} : |z - \alpha| \geq r\}$ is spectral for $A \in \mathcal{L}(H)$ if and only if $\|(A - \alpha I)^{-1}\| \leq r^{-1}$. We refer to two books \cite{3, 14} for a survey of known properties of spectral and complete spectral sets.
1.2 The annulus as a $K$-spectral set

Let $r$ and $R$ be two positive constants with $r < R$. Let $A \in \mathcal{L}(H)$ be an invertible operator such that $\|A\| \leq R$ and $\|A^{-1}\| \leq 1/r$. Then $X_1 = \{ z \in \mathbb{C} : |z| \leq R \}$ and $X_2 = \{ z \in \mathbb{C} : |z| \geq r \}$ are spectral sets for $A$. The annulus

$$X(r, R) = \{ z \in \mathbb{C} : r \leq |z| \leq R \} = X_1 \cap X_2$$

is not necessarily spectral for a given invertible operator $A$. Examples can be found in [21, 11, 13]. Given an invertible operator $A$ with $\|A\| \leq R$ and $\|A^{-1}\| \leq 1/r$, Shields proved in [17] that $X(r, R)$ is a $K$-spectral set for $A$ with $K = 2 + ((R + r) / (R - r))^{1/2}$. The following questions were asked by Shields (see [17, Question 7]):

**Question 1.1.** Find the best constant $K(r, R)$, i.e., the smallest constant $C$ such that $X(r, R)$ is a $C$-spectral set for all invertible $A \in \mathcal{L}(H)$ with $\|A\| \leq R$ and $\|A^{-1}\| \leq r^{-1}$.

**Question 1.2.** Fixing (for instance) $R$, is this best constant bounded (as a function of $r$)?

In analogy with Question 1.1, we will denote by $K_{cb}(r, R)$ the smallest constant $C$ such that $X(r, R)$ is a complete $C$-spectral set. The same proof of Shields (see also [7, 14]) shows that in fact $K_{cb}(r, R) \leq 2 + ((R + r) / (R - r))^{1/2}$.

1.3 Statement of the main results.

The aim of the present note is to study the intersection of two closed disks of the Riemann sphere which are spectral sets for a Hilbert space bounded linear operator. In the case of the annulus we give an estimate for $K(r, R)$ (a partial answer to Question 1.1) which allows to give a positive answer to Question 1.2.

We describe now the main results of this paper. By possibly multiplying the operator by a scalar, we see that $K(r, R) = K(\sqrt{r/R}, \sqrt{R/r})$. This allows to assume, without any loss of generality, that $r = R^{-1}$. We have the following result.

**Theorem 1.3.** Let $R > 1$, $X = X(R^{-1}, R) = \{ z \in \mathbb{C} : R^{-1} \leq |z| \leq R \}$, and denote by $K(R) = K(R^{-1}, R)$ (and $K_{cb}(R) = K_{cb}(R^{-1}, R)$, respectively), the smallest constant $C$ such that $X$ is a $C$-spectral set (and a complete $C$-spectral set, respectively) for any invertible $A \in \mathcal{L}(H)$ verifying $\|A\| \leq R$ and $\|A^{-1}\| \leq R$. Then

$$\frac{2}{1 + R^{-2}} < K(R) \leq K_{cb}(R) \leq 2 + \min\left(\sqrt{\frac{R^2 + 2R + 1}{R^2 + R + 1}}, \sqrt{\frac{R^2 + 1}{R^2 - 1}}\right) \leq 2 + \frac{2}{\sqrt{3}} < 3.2.$$ 

In particular $K(R)$ and $K_{cb}(R)$ are bounded functions of $R$. We obtain the following consequence about normal dilations.

**Corollary 1.4.** Let $R > 1$. Let $A \in \mathcal{L}(H)$ be an invertible operator verifying $\|A\| \leq R$ and $\|A^{-1}\| \leq R$. Let $X = \{ z \in \mathbb{C} : R^{-1} \leq |z| \leq R \}$. Then there exist an invertible operator $L \in \mathcal{L}(H)$ with $\|L\| \cdot \|L^{-1}\| \leq 2 + 2/\sqrt{3}$, a larger Hilbert space $\mathcal{H} \supset H$ and an invertible normal operator $N \in \mathcal{L}(\mathcal{H})$ with $\sigma(N) \subset \partial X$ such that

$$L^{-1}f(A)L = P_Hf(N) \big|_H \quad (f \in R(X)).$$

Here $P_H$ is the orthogonal projection of $\mathcal{H}$ onto $H$.

Besides the annulus, (complete) $K$-spectral sets which are intersections of spectral disks of the complex plane have been considered in [11, 20, 10, 5, 3]; we refer to [3] for a discussion of the best possible constant $K$. In the second part of our paper we consider the more general case of intersection of two closed disks $X_1$ and $X_2$ of the Riemann sphere. We prove the following result.
Theorem 1.5. Let $X_1$ and $X_2$ be two closed disks of the Riemann sphere. If $X_1$ and $X_2$ are spectral sets for a bounded operator $A$ in a Hilbert space, then $X_1 \cap X_2$ is a complete $(2 + 2/\sqrt{3})$-spectral set for $A$.

This theorem extends previously known results concerning the intersection of two disks in $\mathbb{C}$ to not necessarily convex or simply connected $X_1 \cap X_2$. Note that the case of finitely connected compact sets has been studied in [7, 14], however, without a uniform control on the constant $K$.

Note also that, if we consider two distinct bounded, convex and closed subsets $X_1$ and $X_2$ of the complex plane, and if we assume that $X_1$ and $X_2$ are spectral sets for $A$, then $X_1 \cap X_2$ is a complete 11.08-spectral set for $A$. Indeed, the fact that $X_j$ is a spectral set for $A$ implies that the numerical range $W(A) = \{\langle Ax, x \rangle : \|x\| = 1\}$ is included in $X_j$, $j = 1, 2$, and according to [6] the closure of the numerical range $W(A)$ is a complete 11.08-spectral set for $A$. However, the result from [6] does not imply a solution of Shields’ Question 1.2. We refer also to [15, 2, 6] for some normal dilation results for the numerical range, in the spirit of Corollary 1.4.

The remainder of the paper is organized as follows: we first show in §2 that Theorem 1.3 together with some results from [5, 3] implies Theorem 1.5. Our proof of Theorem 1.3 is based on a representation formula for $f(A)$ established in §3. Finally, the proofs of Theorem 1.3 and Corollary 1.4 are provided in §4.

2 Proof of Theorem 1.5 using Theorem 1.3

Let $X_1$ and $X_2$ be two closed disks of the Riemann sphere, which are spectral sets for a bounded linear operator $A$ in a Hilbert space. Here six different situations have to be considered, see Figure 1.

Case 1: $X_1 \cap X_2 = \{\lambda\}$ is a singleton. Then we have $A = \lambda I$ and $X_1 \cap X_2$ clearly is a complete spectral set for $A$.

Case 2: $X_1 \cap X_2$ is a circle or a straight line. Then $A$ is a normal operator with spectrum $\sigma(A)$ contained in $X_1 \cap X_2$. This yields that $X_1 \cap X_2$ is a complete spectral set for $A$.

Case 3: $X_1 \cap X_2$ is a convex sector or a strip of the complex plane. In this case, both $X_1$ and $X_2$ are half-planes, and a closed half-plane $\Pi$ is a spectral set for $A$ if and only if the numerical range $W(A)$ is a subset of $\Pi$. Thus $W(A) \subset X_1 \cap X_2$. It follows from [5] that $X_1 \cap X_2$ is a complete $K$-spectral set, with $K \leq 2 + 2/\sqrt{3}$.

Case 4: $\partial X_1 \cap \partial X_2 = \{\lambda_1, \lambda_2\}$ is a set consisting of two distinct points of $\mathbb{C}$. Here $X_1 \cap X_2$ is lens-shaped. If it is in addition convex, then from [3] we know that $X_1 \cap X_2$ is a complete $K$-spectral set, with $K \leq 2 + 2/\sqrt{3}$. The proof for not convex lenses is the same, we repeat here the main idea for the sake of completeness. Let us first assume that $\lambda_1 \notin \sigma(A)$ and set $B = \varphi(A)$ with $\varphi(z) = (\lambda_1 - z)^{-1}$ and $Y_j = \varphi(X_j)$, $j = 1, 2$. Then both $Y_j$ are closed half-planes. The von Neumann inequality for disks shows that $Y_j$ are spectral sets for $B$, see also [16, §154, Lemma 2]. It follows from the previous case that $Y_1 \cap Y_2$ is a complete $K$-spectral set for $B$ and thus $X_1 \cap X_2$ is a complete $K$-spectral set for $A$, with the same constant $K$. Finally, if $\lambda_1 \in \sigma(A)$, we can replace the disk $X_1$ of the Riemann sphere, of radius $R_1$, by a concentric disk $X_1' \supset X_1$, of radius $R_1 \pm \varepsilon$. Then, for $\varepsilon > 0$ small enough, $\partial X_1' \cap \partial X_2 = \{\lambda'_1, \lambda'_2\}$ is still a set with two distinct points of $\mathbb{C}$, the set $X_1'$ is a spectral set for $A$ and $\lambda'_1 \notin \sigma(A)$. We conclude that $X_1 \cap X_2$ is a complete $K$-spectral set for $A$ by letting $\varepsilon \to 0$.

Case 5: $\partial X_1 \cap \partial X_2 = \emptyset$, but $X_1 \cap X_2$ is not a strip. For the special case $X_1 \cap X_2 = \{z \in \mathbb{C}; |z| \leq R\}$, $R > 1$, Theorem 1.3 implies that $X_1 \cap X_2$ is a complete $(2 + 2/\sqrt{3})$-spectral set for $A$. In the general case, we may find $R > 1$ and a linear fractional transformation $\varphi$ such that $\varphi(X_1) = \{z \in \mathbb{C}; |z| \leq R\}$ and $\varphi(X_2) = \{z \in \mathbb{C}; |z| \geq R^{-1}\}$. Then, setting $B = \varphi(A)$ and
Figure 1: The six different cases occurring by considering intersections of closed disks on the Riemann sphere.

\[ Y_j = \varphi(X_j), \quad j = 1, 2, \] we have that \( Y_j \) is a spectral set for \( B \), see also [16, § 154, Lemma 2]. Thus \( \{ z \in \mathbb{C} ; R^{-1} \leq |z| \leq R \} = \varphi(X_1 \cap X_2) \) is a complete \((2 + 2/\sqrt{3})\)-spectral set for \( B \), which is equivalent to \( X_1 \cap X_2 \) is a complete \((2 + 2/\sqrt{3})\)-spectral set for \( A \).

**Case 6:** \( \partial X_1 \cap \partial X_2 = \{ \lambda \} \) is reduced to a single point, but \( X_1 \cap X_2 \) is neither a singleton, nor a sector nor a strip. In this case at least one of the sets \( X_j, \quad j = 1, 2, \) is the interior or the exterior of a disk and the boundaries of the sets \( X_j \) are tangent in one point. We can replace the disk, say \( X_1 \), of radius \( R_1 \), by a concentric disk \( X_1' \supset X_1 \), of radius \( R_1 \pm \varepsilon \). Then, for \( \varepsilon > 0 \) small enough, \( \partial X_1' \cap \partial X_2 = \emptyset \), and we obtain from the previous case that \( X_1 \cap X_2 \) is a complete \( K \)-spectral set for \( A \) by letting \( \varepsilon \to 0 \).

### 3 A decomposition lemma for annuli

In order to give a proof of the upper bound of Theorem [1,3] we need the following representation formula for \( f(A) \).

**Lemma 3.1.** Let \( A \in \mathcal{L}(H) \) be an operator satisfying \( \| A \| < R \) and \( \| A^{-1} \| < R \). We set \( r = 1/R \) and denote by \( X \) the annulus \( X = X(R^{-1}, R) = \{ z \in \mathbb{C} ; r \leq |z| \leq R \} \). For any bounded rational function \( f \) on \( X \), we have the representation formula

\[
 f(A) = \int_0^{2\pi} f(Re^{i\theta}) \mu(\theta, A) d\theta + \int_0^{2\pi} f(re^{i\theta}) \mu(-\theta, A^{-1}) d\theta + \int_0^{2\pi} f(e^{i\theta}) M(\theta, A^*)^{-1} d\theta,
\]
where
\[
\mu(\theta, A) = \frac{1}{4\pi}((1+e^{-i\theta}rA)(1-e^{-i\theta}rA)^{-1} + (1+e^{i\theta}rA^*)(1-e^{i\theta}rA^*)^{-1}), \quad \text{and}
\]
\[
M(\theta, A^*) = \frac{2\pi}{R^2 - r^2}(R^2 + r^2 - (e^{i\theta}A^*)^{-1} - e^{i\theta}A^*).
\]

**Proof.** We get from the Cauchy formula
\[
f(A) = \frac{1}{2\pi i} \int_{\partial X} f(\sigma) (\sigma - A)^{-1} d\sigma - (\sigma - A^*)^{-1} d\sigma) + \frac{1}{2\pi i} \int_{\partial X} f(\sigma) (\sigma - A^*)^{-1} d\sigma = F_1 + F_2.
\]
Let us set \(\Gamma_\rho = \{\rho e^{i\theta}; \theta \in [0, 2\pi]\}\). The part \(\Gamma_R\) of \(\partial X\) is counterclockwise oriented and, with \(\sigma = Re^{i\theta}\), we have
\[
\frac{1}{2\pi i}((\sigma - A)^{-1} d\sigma - (\sigma - A^*)^{-1} d\sigma) = \frac{1}{2\pi i}((Re^{i\theta} - A)^{-1} Re^{i\theta} + (Re^{-i\theta} - A^*)^{-1} Re^{-i\theta}) d\theta
\]
\[
= \frac{1}{2\pi i}((1 - e^{-i\theta}rA)^{-1} + (1 - e^{i\theta}rA^*)^{-1}) d\theta
\]
\[
= \frac{1}{2\pi} d\theta + \mu(\theta, A) d\theta.
\]
The other component \(\Gamma_r\) is clockwise oriented and, with \(\sigma = re^{i\theta}\), we have
\[
\frac{1}{2\pi i}((\sigma - A)^{-1} d\sigma - (\sigma - A^*)^{-1} d\sigma) = \frac{1}{2\pi i}((re^{i\theta} - A)^{-1} re^{i\theta} + (re^{-i\theta} - A^*)^{-1} re^{-i\theta}) d\theta
\]
\[
= \frac{1}{2\pi} d\theta - \mu(-\theta, A^{-1}) d\theta.
\]
Noticing that \(\int_0^{2\pi} f(Re^{i\theta}) d\theta = \int_0^{2\pi} f(re^{i\theta}) d\theta\), we obtain that
\[
F_1 = \int_0^{2\pi} f(Re^{i\theta}) \mu(\theta, A) d\theta + \int_0^{2\pi} f(re^{i\theta}) \mu(-\theta, A^{-1}) d\theta.
\]
We consider now the second term \(F_2\). On the component \(\Gamma_R\) we have \(\bar{\sigma} = R^2/\sigma\), and thus
\[
\frac{1}{2\pi i} \int_{\Gamma_R} f(\sigma) (\bar{\sigma} - A^*)^{-1} d\bar{\sigma} = -\frac{1}{2\pi i} \int_{\Gamma_R} f(\sigma) (R^2 - \sigma A^*)^{-1} \frac{R^2}{\sigma} d\sigma
\]
\[
= -\frac{1}{2\pi i} \int_{\Gamma_1} f(\sigma) (R^2 - \sigma A^*)^{-1} \frac{R^2}{\sigma} d\sigma.
\]
Indeed, the last integrand is holomorphic in \(\sigma\). Hence we can replace the integration path \(\Gamma_R\) by \(\Gamma_1\) (counterclockwise oriented). We similarly have for the second component
\[
\frac{1}{2\pi i} \int_{\Gamma_r} f(\sigma) (\bar{\sigma} - A^*)^{-1} d\bar{\sigma} = \frac{1}{2\pi i} \int_{\Gamma_1} f(\sigma) (r^2 - \sigma A^*)^{-1} \frac{r^2}{\sigma} d\sigma
\]
by taking into account the opposite orientation of \(\Gamma_r\). Therefore
\[
F_2 = \frac{1}{2\pi i} \int_{\Gamma_1} f(\sigma) ((r^2 - \sigma A^*)^{-1} \frac{r^2}{\sigma} - (R^2 - \sigma A^*)^{-1} \frac{R^2}{\sigma}) d\sigma
\]
\[
= \int_0^{2\pi} f(e^{i\theta}) M(\theta, A^*)^{-1} d\theta,
\]
which completes the proof of the lemma. \(\square\)
4 The complete bound in an annulus

We keep the notation from the previous section. The following lemma shows that \( \text{Re} M(\theta, A^*) \) is a positive operator.

**Lemma 4.1.** Assume that \( \|A\| < R \) and \( \|A^{-1}\| < R \). Let \( r = R^{-1} \). Then we have the lower bound

\[
\text{Re} M(\theta, A^*) \geq N(\theta) := \frac{2\pi}{R^2 - r^2} \left( (R^2 + r^2 - R - r) + \frac{R + r + 2}{4} \left( 2 - e^{i\theta}U^* - e^{-i\theta}U \right) \right),
\]

where \( U \) denotes the unitary operator such that \( A = UG \), with \( G \) self-adjoint positive definite. Also, \( N(\theta) \) is a positive invertible operator.

**Proof.** We have

\[
\frac{R^2 - r^2}{2\pi} \text{Re} M(\theta, A^*) = \frac{R^2 + r^2 - \text{Re}((e^{-i\theta}A)^{-1} + e^{i\theta}A^*)}{2\pi} = \frac{R^2 + r^2 - \text{Re} \left( e^{i\theta}(G^{-1} + G)U^* \right)}{2\pi} = \frac{R^2 + r^2 - \frac{R+r+2}{2} \text{Re} \left( e^{i\theta}U^* \right) - \text{Re} \left( e^{i\theta}(G^{-1} + G - \frac{R+r+2}{2})U^* \right)}{2\pi}
\]

We note that the assumptions \( \|A\| \leq R \) and \( \|A^{-1}\| \leq R \) are equivalent to \( \|G\| \leq R \) and \( \|G^{-1}\| \leq R \). Since \( G \) is self-adjoint, this means that \( r \leq G \leq R \), and hence

\[
\|G^{-1} + G - \frac{R+r+2}{2}\| \leq \sup_{r \leq x \leq R} |x^{-1} + x - \frac{R+r+2}{2}| = \frac{R+r-2}{2}.
\]

It follows that

\[
\frac{R^2 - r^2}{2\pi} \text{Re} M(\theta, A^*) \geq \frac{R^2 + r^2 - \frac{R+r+2}{2} \text{Re} \left( e^{i\theta}U^* \right) - \frac{R+r-2}{2}}{2\pi} = R^2 + r^2 - R - r + \frac{R+r+2}{2} \text{Re} \left( 1 - e^{i\theta}U^* \right),
\]

which completes the proof of the lemma.

**Proof of the upper bound of Theorem 1.3.** We can suppose that \( \|A\| < R \) and \( \|A^{-1}\| < R \). Using the notation of Lemma 3.1, it follows from the condition \( \|A\| < R \) that \( \mu(\theta, A) \geq 0 \) for all \( \theta \in \mathbb{R} \). Therefore we have

\[
\left\| \int_0^{2\pi} f(Re^{i\theta}) \mu(\theta, A) \, d\theta \right\| \leq \left\| \int_0^{2\pi} \mu(\theta, A) \, d\theta \right\| \|f\|_x = \|f\|_x.
\]

Here we have used that \( \int_0^{2\pi} \mu(\theta, A) \, d\theta = 1 \), which follows from the residue formula. Similarly we have \( \mu(-\theta, A^{-1}) \geq 0 \) and we get the estimate

\[
\left\| \int_0^{2\pi} f(re^{i\theta}) \mu(-\theta, A^{-1}) \, d\theta \right\| \leq \|f\|_x.
\]

Using Lemma 3.1 and the positivity of \( \text{Re} M(\theta, A^*) \) for all \( \theta \in \mathbb{R} \) (Lemma 4.1) we obtain the estimate

\[
\|f(A)\| \leq K \|f\|_x, \quad \text{with} \quad K = 2 + \left\| \int_0^{2\pi} (\text{Re} M(\theta, A^*))^{-1} \, d\theta \right\|.
\]

Let \( \rho : \mathcal{R}(X) \mapsto \mathcal{L}(H) \) be the homomorphism given by \( \rho(f) = f(A) \). Therefore the norm of \( \rho \) is bounded by \( K \). Furthermore, since we only have used arguments based on positivity of operators, it is easily seen that the complete bounded norm \( \|\rho\|_{cb} \) is also bounded by \( K \).
Taking into account the bound of Shields [17], for establishing the upper bound of Theorem 1.3 it suffices now to show that
\[
\left\| \int_0^{2\pi} (\text{Re} \, M(\theta, A^*))^{-1} \, d\theta \right\| \leq \sqrt{\frac{R^2 + 2R + 1}{R^2 + R + 1}} \leq \frac{2}{\sqrt{3}} \tag{1}
\]
Consider the function
\[
J(z) := \frac{R^2 - r^2}{2\pi} \int_0^{2\pi} \left( (R^2 + r^2 - R - r) + \frac{R + r + 2}{4} \left( 2 - e^{i\theta}z^{-1} - e^{-i\theta}z \right) \right)^{-1} \, d\theta.
\]
Since \(U\) is a unitary operator, it follows from Lemma 4.1 that
\[
\left\| \int_0^{2\pi} (\text{Re} \, M(\theta, A^*))^{-1} \, d\theta \right\| \leq \left\| \int_0^{2\pi} (N(\theta))^{-1} \, d\theta \right\| = \|J(U)\| = \sup \left\{ |J(e^{i\phi})| : e^{i\phi} \in \sigma(U) \right\}.
\]
On the other hand, we have
\[
J(e^{i\varphi}) = \frac{R^2 - r^2}{2\pi} \int_0^{2\pi} \left( (R^2 + r^2 - R - r) + \frac{R + r + 2}{4} \left( 2 - 2\cos(\theta - \varphi) \right) \right) \, d\theta
\]
\[
= \frac{R^2 - r^2}{2\pi} \int_{-\infty}^{\infty} \left( (R^2 + r^2 - R - r)(1 + s^2) + (R + r + 2)s^2 \right) \, ds
\]
\[
= \frac{R^2 - r^2}{2\pi} \int_{-\infty}^{\infty} \left( (R^2 + r^2 - R - r) + (R^2 + r^2 + 2)s^2 \right) \, ds
\]
\[
= \sqrt{\frac{R^2 + 2R + 1}{R^2 + R + 1}} \leq \frac{2}{\sqrt{3}}
\]
which implies (1). This gives a proof of the upper bound of Theorem 1.3 for \(K_{cb}(R)\).

**Proof of the lower bound of Theorem 1.3** For \(t \in \mathbb{C}\), let \(A(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\) with inverse \(A(t)^{-1} = \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix}\) acting on the Hilbert space \(\mathbb{C}^2\). For \(t_0 = R - R^{-1}\) we have \(\|A(t_0)\| = \|A(t_0)^{-1}\| = R\) (compare with [14] p. 152). We will make use of the following result from geometric function theory about the infinitesimal Carathéodory metric: it is shown by Simha in [18] Example (5.3)] that
\[
\sup \left\{ \frac{|f'(1)|}{\|f\|_X} : f \text{ analytic in } X \text{ and } f(1) = 0 \right\} = \frac{2}{R} \prod_{n=1}^{\infty} \left( \frac{1 - R^{-8n}}{1 - R^{-4n}} \right)^2,
\]
with the supremum being attained for some function \(f_0\) analytic in \(X\), with \(\|f_0\|_X = 1\) and \(f_0(1) = 0\). Therefore
\[
K(R) \geq \frac{1}{\|f_0\|_X} \|f_0(A(t_0))\| = \left\| \begin{pmatrix} f_0(1) & t_0 f'_0(1) \\ 0 & f'_0(1) \end{pmatrix} \right\| = t_0 |f'_0(1)| = \gamma(R)
\]
with
\[
\gamma(R) := 2(1 - R^{-2}) \prod_{n=1}^{\infty} \left( \frac{1 - R^{-8n}}{1 - R^{-4n}} \right)^2 = \frac{2}{1 + R^{-2}} \prod_{n=1}^{\infty} \frac{(R^{4n} - R^{-4n})^2}{(R^{4n} - R^{-4n})^2} = \frac{2}{1 + R^{-2}} \prod_{n=1}^{\infty} \left( 1 - \frac{(R^2 - R^{-2})^2}{(R^{4n} - R^{-4n})^2} \right)^{-1}.
\]
Figure 2: The two upper bounds and the lower bound for $K(R)$ from Theorem 1.3 and the lower bound $\gamma(R)$ from the proof of Theorem 1.3.

This yields the estimate

$$K(R) > \frac{2}{1 + R^{-2}},$$

as claimed in Theorem 1.3. It remains to justify why we are allowed to take for a lower bound of $K(R)$ the function $f_0$ which is not a rational function. Indeed, by using instead of $f_0$ partial sums of the Laurent expansion of an extremal function for the infinitesimal Carathéodory metric on the annulus $1/R' < |z| < R'$ for some $R' > R$ we obtain the same conclusion after taking the limit $R' \to R$. \[\square\]

Remark 4.2. The final estimate (2) of the preceding proof is not very sharp for $R$ close to one (see Figure 2), and $\gamma(R)$ is a sharper but less readable lower bound for $K(R)$. For instance, for $R \to 1$ the lower bound $2/(1 + R^{-2})$ of Theorem 1.3 tends to 1 but

$$\lim_{R \to 1} \gamma(R) = \lim_{R \to 1} \prod_{n=1}^{\infty} \left(1 - \frac{(R^2 - R^{-2})^2}{(R^{4n} - R^{-4n})^2}\right)^{-1} = \prod_{n=1}^{\infty} \left(1 - \frac{1}{4n^2}\right)^{-1} = \frac{\pi}{2}.$$

In contrast, for our fixed matrix $A(t_0)$, it follows from [9, Theorem 1] and [18] that the function $f_0$ is extremal within the class of functions analytic in $X$.

Proof of Corollary 1.4. We use the terminology of Paulsen’s book [14]. Let $\rho : R(X) \to \mathcal{L}(H)$ be the homomorphism given by $\rho(f) = f(A)$. Theorem 1.3 implies that the complete bounded norm $\|\rho\|_{cb}$ of $\rho$ is bounded by $2 + 2/\sqrt{3}$. Using a theorem of Paulsen [14, Theorem 9.1], there exists an invertible operator $L$ with $\|L\| \cdot \|L^{-1}\| = \|\rho\|_{cb} \leq 2 + 2/\sqrt{3}$ such that $L^{-1} \rho(\cdot)L$ is a unital completely contractive homomorphism. Thus $X$ is a complete spectral set for $L^{-1}AL$. Therefore, as a consequence of Arveson’s extension theorem (see [14, Corollary 7.8]), $L^{-1}AL$ has a normal dilation with spectrum included in $\partial X$, as claimed in Corollary 1.4. \[\square\]

Remark 4.3. According to a deep result due to Agler [1], if $X$ is a spectral set for $A$, then $X$ is a complete spectral set for $A$, and thus $A$ has a normal dilation with spectrum included in $\partial X$. The analogue of Agler’s theorem is not true for triply connected domains (see [8]).

8
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