Unitarity relation and unitarity bounds for scalars with different sound speeds

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Abstract

Motivated by scalar-tensor gravities, we consider a theory which contains massless scalar fields with different sound speeds. We derive unitarity relations for partial wave amplitudes of $2 \rightarrow 2$ scattering, with explicit formulas for contributions of two-particle intermediate states. Making use of these relations, we obtain unitarity bounds both in the most general case and in the case considered in literature for unit sound speed. These bounds can be used for estimating the strong coupling scale of a pertinent EFT. We illustrate our unitarity relations by explicit calculation to the first non-trivial order in couplings in a simple model of two scalar fields with different sound speeds.

1 Introduction

Scalar-tensor theories of gravity with non-trivial scalar kinetic terms and/or non-minimal couplings to metric are commonly used to construct models of inflation \cite{1,2,3,4,5} as well as non-singular cosmological models such as genesis \cite{6,7,8,9,10,11} and bounce \cite{12,13,14,15,16,17,18,19,20}. In these theories, perturbations about non-trivial backgrounds often

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propagate with “sound speeds” different from the speed of light and, moreover, perturbations of different types (e.g., scalar vs tensor in the cosmological context) have different sound speeds. Another feature is that some constructions involve time-dependent couplings which are dangerously large during certain time intervals. An example is Horndeski theory \[ \text{[21]} \] whose subclass admits genesis and bounce with “strong gravity in the past” (effective Planck mass tends to zero as \( t \to -\infty \)) \[ \text{[22]} \]; in this way one evades the no-go theorem of Refs. \[ \text{[23, 24]} \].

An important parameter in an effective QFT is the energy scale of strong coupling, or, in other words, the maximum energy below which the effective QFT description is trustworthy. Scalar-tensor gravities, especially featuring large couplings, are not exceptional in this regard. While the strong coupling energy scale can often be qualitatively estimated by naive dimensional analysis, more accurate estimates are obtained using unitarity bounds that follow from general unitarity relations. This motivates us to derive unitarity relations and unitarity bounds in theories with different sound speeds of different perturbations.

In this paper we consider theories with several scalar fields; theories of particles with spin can be treated in a similar way\[1\]. Also, we study theories in flat space-time and trivial background; this treatment is expected to be relevant also for non-trivial backgrounds, since the classical description of a background is legitimate provided that its classical energy scale is well below the quantum strong coupling scale, in which case the space-time dependence of the background is expected to be negligible when evaluating the quantum scale.

An adequate approach to unitarity relations and unitarity bounds makes use of Partial Wave Amplitudes (PWAs) (see, e.g., Refs. \[ \text{[25, 26, 27, 28]} \]). We follow this approach in our paper. We aim at self-contained presentation and give detailed derivation even though many steps follow closely the analyses existing in literature. In this sense this paper may serve as a pedagogical mini-review of the subject, with the novelty due to the fact that we consider different sound speeds of different excitations.

This paper is organized as follows. We derive in Sec. \[ \text{2} \] the general unitarity relations for PWAs of \( 2 \to 2 \) scattering, paying special attention to two-particle intermediate states. In Sec. \[ \text{2} \] we also derive the unitarity bounds. To this end, in Sec. \[ \text{2.1} \] we describe the class of theories we deal with. We then study separately the cases of a pair of distinguishable particles in the intermediate state (Sec. \[ \text{2.2} \]) and a pair of identical particles (Sec. \[ \text{2.3} \]). Unitarity bounds are derived in Sec. \[ \text{2.4} \]. We give an illustrative example in Sec. \[ \text{3} \] where we explicitly check the validity of the unitarity relation to the leading non-trivial order in a simple model of two real scalar fields. Appendix A is dedicated to the time-reversal invariance and its consequence for PWAs.

\[1\] Bosonic perturbations with spin can often be reduced to effective scalar perturbations at the expense of the violation of Lorentz invariance, which occurs in non-trivial backgrounds anyway.
2 Unitarity relation

2.1 Generalities

In this Section we proceed in the spirit of Ref. [25] and obtain the unitarity relation for $2 \rightarrow 2$ scattering processes in theories with scalar fields $\phi_i$ whose sound speeds $u_i$ are different. Having in mind the issue of strong coupling energy scale, we neglect masses of these particles (if they exist). The quadratic action reads

$$S = \sum_i S_{\phi_i}, \quad S_{\phi_i} = \int d^4x \left( \frac{1}{2} \dot{\phi}_i^2 - \frac{1}{2} u_i^2 (\nabla \phi_i)^2 \right).$$

The linearized equation of motion for $\phi_i$ is

$$\ddot{\phi}_i - u_i^2 \Delta \phi_i = 0,$$

and its solution can be written as follows

$$\phi_i(\vec{x}, t) = \int \frac{d\vec{p}_i}{(2\pi)^3} \frac{1}{\sqrt{2E_{p_i}}} \left( a_{\vec{p}_i} e^{-iE_{p_i}t+i\vec{p}_i\vec{x}} + a_{\vec{p}_i}^\dagger e^{iE_{p_i}t-i\vec{p}_i\vec{x}} \right),$$

where

$$E_{p_i} = u_i p_i,$$

and the operators $a_{\vec{p}_i}$ and $a_{\vec{p}_i}^\dagger$ obey the standard commutational relation

$$[a_{\vec{p}_i}, a_{\vec{p}_j}^\dagger] = (2\pi)^3 \delta^{(3)}(\vec{p}_i - \vec{p}_j) \delta_{ij}. \quad (2)$$

We define one-particle state as follows:

$$|\vec{p}_i\rangle \equiv \sqrt{2E_{p_i}} a_{\vec{p}_i}^\dagger |0\rangle,$$

so that one has the standard relation

$$\langle 0 | \phi_i(\vec{x}, t) | \vec{p}_j \rangle = e^{-iE_{p_j}t+i\vec{p}_j\vec{x}} \delta_{ij},$$

while the normalization of this state is given by

$$\langle \vec{p}_j' | \vec{p}_i \rangle = (2\pi)^3 \sqrt{2E_{p_i}} 2E_{p_j} \delta^{(3)}(\vec{p}_i - \vec{p}_j) \delta_{ij}. \quad (3)$$

In the $i$-th one-particle sector one has

$$1 = \int \frac{d^3p_i}{(2\pi)^3 2E_{p_i}} |\vec{p}_i\rangle \langle \vec{p}_i|.$$
The S-matrix and T-matrix are related in the standard way:

\[ S = 1 + iT, \]

and one extracts from T the overall δ-function of 4-momentum conservation:

\[ T = (2\pi)^4 \delta^4(\mathcal{P}^\mu' - \mathcal{P}^\mu) M, \tag{4} \]

where \( \mathcal{P}^\mu = \sum p^\mu_{\text{in}} \) and \( \mathcal{P}^\mu' = \sum p^\mu_{\text{out}} \) are total 4-momenta of the initial and final state, respectively.

Now, we consider an initial state

\[ |\psi, \beta\rangle = \sqrt{2} E_p^1 \sqrt{2} E_p^2 a^\dagger_{p^1} a^\dagger_{p^2} |0\rangle, \tag{5} \]

with two particles of momenta \( \vec{p}_1 \) and \( \vec{p}_2 \), and a final state \( |\psi', \beta'\rangle \) with two particles of momenta \( \vec{p}_1' \) and \( \vec{p}_2' \). Notation \( \beta \) refers to the types of the two particles, \( \beta = \{\phi_i, \phi_j\} \), while notation \( \psi \) is a shorthand for the pair of momenta, \( \psi = \{\vec{p}_1, \vec{p}_2\} \). Thus,

\[ |\psi, \beta\rangle = |\phi_i, \vec{p}_1\rangle \otimes |\phi_j, \vec{p}_2\rangle. \]

In eq. (5) we do not explicitly indicate the type of particle to simplify formulas and write \( a^\dagger_{p^1} \equiv a^\dagger_{i_{p^1}} \), etc.

Our purpose is to derive the unitarity relation for the partial wave amplitudes.

### 2.2 Distinguishable particles

Let us begin with the case of distinguishable particles in a pair \( \beta = \{\phi_i, \phi_j\} \). In the next subsection we consider the case of identical particles.

The scalar product of states \( |\psi', \beta'\rangle \) and \( |\psi, \beta\rangle \) is

\[ \langle \psi', \beta' | \psi, \beta \rangle = (2\pi)^6 2E_{p_1} 2E_{p_2} \delta^{(3)}(\vec{p}_1' - \vec{p}_1)\delta^{(3)}(\vec{p}_2' - \vec{p}_2)\delta_{\beta' \beta}. \tag{6} \]

This follows from the one-particle state normalization (3). In what follows we consider the center-of-mass frame of the two-particle system. In this frame we denote \( \vec{p} \equiv \vec{p}_1 = -\vec{p}_2 \), \( p \equiv |\vec{p}| = |\vec{p}_1| = |\vec{p}_2| \). Let \( \hat{p} = \vec{p}/p \) be the unit vector along \( \vec{p} \) and \( \theta, \phi \) be the corresponding angles. We now replace the variables \( \vec{p}_1, \vec{p}_2 \) in (6) by \( \mathcal{P}^\mu \equiv p^\mu_1 + p^\mu_2 \), \( \theta \) and \( \phi \), where we have in mind that in the vicinity of the center-of-mass frame one has \( \mathcal{P}^\mu \approx (E, 0) \), where \( E = (u_{1,\beta} + u_{2,\beta})p \) and \( u_{1,\beta} \equiv u_i, u_{2,\beta} \equiv u_j \) are sound speeds of the two particles in the pair \( \beta = \{\phi_i, \phi_j\} \). For the volume element we have

\[ d^3 \vec{p}_1 d^3 \vec{p}_2 = d^3 \vec{P} P^2 dp d\hat{p} = \frac{p^2}{(u_{1,\beta} + u_{2,\beta})} d^4 P^\mu d\hat{p}, \]
which gives
\[ \delta^{(3)}(\vec{p}_1' - \vec{p}_1)\delta^{(3)}(\vec{p}_2' - \vec{p}_2)\delta_{\beta \beta'} = \frac{(u_{1\beta} + u_{2\beta})}{p^2} \delta^{(4)}(P^{\mu'} - P^{\mu})\delta^{(2)}(\vec{p}' - \vec{p})\delta_{\beta \beta'}, \]
and hence
\[ \langle \psi', \beta' | \psi, \beta \rangle = (2\pi)^6 \cdot 4u_{1\beta}u_{2\beta}(u_{1\beta} + u_{2\beta}) \cdot \delta^{(4)}(P^{\mu'} - P^{\mu})\delta^{(2)}(\vec{p}' - \vec{p})\delta_{\beta \beta'}. \quad (7) \]

As the next step we introduce two-particle state of definite angular momentum in the center-of-mass frame. The reason is that the unitarity relations have particularly simple form for the partial-wave amplitudes \([25, 26, 27, 28]\) (PWAs). The relevant state is given by
\[ |l, m, P^{\mu}, \beta \rangle = \frac{1}{\sqrt{4\pi}} \hat{d} \hat{p} Y_l^m(\hat{p}) |\psi, \beta \rangle, \quad (8) \]
where the integration runs over unit sphere and \(Y_l^m\) is the spherical function,
\[ Y_l^m(\hat{p}) = (-1)^{|m|} \frac{1}{2} e^{im\phi} \sqrt{\frac{(2l + 1)(l - |m|)!}{4\pi (l + |m|)!}} P_{l|m|}(\cos \theta), \]
which obeys
\[ \int d\hat{p} Y_{l'}^{m'}(\hat{p}) Y_{l''}^{m''}(\hat{p}) = \delta_{ll'}\delta_{mm'}. \quad (9) \]
One finds the scalar product of these states from (7):
\[ \langle l', m', P^{\mu'}, \beta' | l, m, P^{\mu}, \beta \rangle = 4\pi u_{1\beta}u_{2\beta}(u_{1\beta} + u_{2\beta}) \cdot (2\pi)^4 \delta^{(4)}(P^{\mu'} - P^{\mu})\delta^{(2)}(\vec{p}' - \vec{p})\delta_{\beta \beta'}. \]

Thus, the decomposition of the unit operator reads
\[ 1 = \int d^4P \sum_{l, m, \beta} |l, m, P^{\mu}, \beta \rangle \langle l, m, P^{\mu}, \beta | \frac{1}{N(\beta)} + \ldots, \quad (10) \]
where summation runs over all two-particle states and
\[ N(\beta) = 2(2\pi)^5 u_{1\beta}u_{2\beta}(u_{1\beta} + u_{2\beta}). \quad (11) \]

Dots in (10) stand for terms with multiparticle states. We omit these terms in what follows and comment later on how these terms affect the unitarity relation.

Let us now write the partial wave amplitude,
\[ T_{m'\beta'; m\beta}^{(l)} = \langle l, m', P^{\mu'}, \beta' | T | l, m, P^{\mu}, \beta \rangle. \]
It is given by
\[ T_{m'\beta'; m\beta}^{(l)} = \frac{1}{4\pi} \int d\hat{p} \int d\hat{p}' Y_l^m(\hat{p}) Y_l^{m'}(\hat{p}') \langle \psi', \beta' | T | \psi, \beta \rangle. \]
Due to rotational invariance, the $T$-matrix does not vanish only for $m' = m$ and does not depend on $m$ \[^{29, 30}\]. Thus, we can write

$$T^{(l)}_{m'\beta';m\beta} = \delta_{m'm} \sum_{m=-l}^{l} \frac{T^{(l)}_{m'\beta';m\beta}}{2l+1}.$$  

We recall that

$$\sum_{m=-l}^{l} Y_{l}^{m}(\hat{p}')Y_{l}^{m}(\hat{p}) = \frac{2l+1}{4\pi} P_{l}(\cos \gamma),$$

where $\gamma \equiv \angle(\hat{p}', \hat{p})$ is angle between the two momenta, and arrive at

$$T^{(l)}_{m'\beta';m\beta} = \frac{\delta_{m'm}}{16\pi^{2}} \int d\hat{p}' \int d\hat{p} P_{l}(\cos \gamma) \langle \psi', \beta' | T | \psi, \beta \rangle,$$

where, again due to rotational invariance, $\langle \psi', \beta' | T | \psi, \beta \rangle$ does not depend on angular variables except for $\gamma$. Because of this property, it is straightforward to integrate over all angles but $\gamma$ and obtain

$$T^{(l)}_{m'\beta';m\beta} = \frac{\delta_{m'm}}{2} \int d(\cos \gamma) \cdot P_{l}(\cos \gamma) \langle \psi', \beta' | T | \psi, \beta \rangle.$$

Using (4) one obtains

$$T^{(l)}_{m'\beta';m\beta} = (2\pi)^{4} \delta^{(4)}(P^{\mu'} - P^{\mu}) \frac{\delta_{m'm}}{2} \int d(\cos \gamma) \cdot P_{l}(\cos \gamma) M_{\beta'\beta}.$$

Finally, one defines the partial wave amplitude,

$$a_{l,\beta';\beta} = \frac{1}{32\pi} \int d(\cos \gamma) \cdot P_{l}(\cos \gamma) M_{\beta'\beta}, \quad (12)$$

and finds

$$T^{(l)}_{m'\beta';m\beta} = 16\pi \cdot (2\pi)^{4} \delta^{(4)}(P^{\mu'} - P^{\mu}) \delta_{m'm} a_{l,\beta';\beta}. \quad (13)$$

Now we turn to the unitarity relation. Unitarity of $S$-matrix, $SS^\dagger = S^\dagger S = 1$ implies

$$T - T^\dagger = iTT^\dagger = iT^\dagger T.$$ 

Inserting unit operator given by (10) in the right-hand side, we find

$$-i \left( T^{(l)}_{m'\beta';m\beta} - T^{(l)*}_{m\beta;m'\beta'} \right) = \int d^{4}P'' \sum_{m'',\beta''} \frac{1}{N(\beta'')} T^{(l)}_{m''\beta'';m'\beta'} T^{(l)*}_{m\beta;m''\beta''}.$$  

6
One makes use of (13) and recalls the definition of \(N(\beta)\), eq. (11), to obtain the unitarity relation in terms of PWAs:

\[
-\frac{i}{2} (a_{l,\alpha\beta} - a_{l,\beta\alpha}^*) = \sum_{\gamma} \frac{2}{u_{1\gamma}u_{2\gamma}(u_{1\gamma} + u_{2\gamma})} a_{l,\alpha\gamma} a_{l,\beta\gamma},
\]

where \(u_{1\gamma}\) and \(u_{2\gamma}\) are sound speeds of particles in the intermediate state \(\gamma\).

One often assumes time reversal invariance, which gives \(T_{m'\beta':m\beta}^{(l)} = T_{m\beta;m\beta'}^{(l)}\), and hence \(a_{l,\alpha\beta} = a_{l,\beta\alpha}\) (see Appendix A and Refs. [29, 30]). In that case the unitarity relation reads

\[
\text{Im} \, a_{l,\alpha\beta} = \sum_{\gamma} \frac{2}{u_{1\gamma}u_{2\gamma}(u_{1\gamma} + u_{2\gamma})} a_{l,\alpha\gamma} a_{l,\beta\gamma}^*.
\]

For \(u_{1\gamma} = u_{2\gamma} = 1\) this relation coincides with the standard one, see, e.g., Refs. [25, 31].

### 2.3 Identical particles

We now consider the case of identical particles. We again define two-particle states as follows:

\[
|\psi, \beta\rangle = \sqrt{2E_{p_1}} \sqrt{2E_{p_2}} a_{p_1}^+ a_{p_2}^+ |0\rangle,
\]

where \(\beta = \{\phi_i, \phi_i\}\), while the commutational relation is still given by (2). In the case of identical particles the normalization of the two-particle state is different from (6):

\[
\langle \psi', \beta' | \psi, \beta \rangle = (2\pi)^6 2E_{p_1} 2E_{p_2} \left( \delta^{(3)}(\vec{p}_1' - \vec{p}_1) \delta^{(3)}(\vec{p}_2' - \vec{p}_2) + \delta^{(3)}(\vec{p}_2' - \vec{p}_1) \delta^{(3)}(\vec{p}_1' - \vec{p}_2) \right) \delta_{\beta \beta'}.
\]

We proceed along the same lines as in Sec. 2.2. The change of variables in (14) gives

\[
\langle \psi', \beta' | \psi, \beta \rangle = (2\pi)^6 8u^3_{\beta} \cdot \delta^{(4)}(\mathcal{P}_{\mu'} - \mathcal{P}_{\mu}) \left( \delta^{(2)}(\hat{\mathcal{P}}' - \hat{\mathcal{P}}) + \delta^{(2)}(\hat{\mathcal{P}}' + \hat{\mathcal{P}}) \right) \delta_{\beta \beta'},
\]

where \(u_{\beta} \equiv u_i\) is the sound speed of the particle \(\phi_i\). The states of definite angular momentum are still given by (8), but the scalar product of these states is now

\[
\langle l', m', \mathcal{P}_{\mu'}, \beta'| l, m, \mathcal{P}_{\mu}, \beta \rangle = \frac{1}{4\pi} \int d\vec{p} \, (2\pi)^2 8u_{\beta}^3 \cdot (2\pi)^4 \delta^{(4)}(\mathcal{P}_{\mu'} - \mathcal{P}_{\mu}) \delta_{\beta \beta'}
\]

\[
\times \left( Y_l^m(\hat{\mathcal{P}}) Y_{l'}^{m'*}(\hat{\mathcal{P}}) + Y_l^m(\hat{\mathcal{P}}) Y_{l'}^{m'*}(\hat{\mathcal{P}}) \right).
\]

Since identical scalars always have even \(l\), we consider even \(l\) until the end of this subsection without mentioning this explicitly. Making use of the properties of the spherical functions, eqs. (9), and (34a), we get

\[
\langle l', m', \mathcal{P}_{\mu'}, \beta'| l, m, \mathcal{P}_{\mu}, \beta \rangle = 2\pi 8u_{\beta}^3 \cdot (2\pi)^4 \delta^{(4)}(\mathcal{P}_{\mu'} - \mathcal{P}_{\mu}) \delta_{\mu \mu'} \delta_{mm'} \delta_{\beta \beta'},
\]

\[2\text{This can be seen also from eq. (15): the integral in the right hand side vanishes for odd } l.\]
and the contribution of two-particle states with identical particles into the decomposition of the unit operator reads

\[ 1 = \int d^4p \sum_{l,m,\beta} |l,m,\mathcal{P}^\mu,\beta\rangle \langle l,m,\mathcal{P}^\mu,\beta| \frac{1}{N_{identical}(\beta)} + \ldots, \]

where \( N_{identical}(\beta) \) is given by

\[ N_{identical}(\beta) \equiv (2\pi)^5 \cdot 8u_3^3. \]

Note that \( N_{identical}(u_\beta) = 2N(u_{1\beta}, u_{2\beta})|_{u_{1\beta}=u_{2\beta}=u_\beta} \), where \( N \) has been introduced in (11), i.e., if all particles have the same sound speed, then the normalization factor \( N \) is twice larger for identical particles as compared to distinguishable particles. We repeat the calculations in Sec. 2.2 and find that the contribution to PWA unitarity relation from intermediate states \( \gamma \) with two identical particles is given by

\[ -i \frac{1}{2} (a_{l,\alpha\beta} - a_{l,\beta\alpha}^*) = \sum_\gamma \frac{1}{2u_\gamma^3} a_{l,\alpha\gamma} a_{l,\beta\gamma}^* + \ldots. \]

In a \( T \)-invariant theory one has

\[ \text{Im} \ a_{l,\alpha\beta} = \sum_\gamma \frac{1}{2u_\gamma^3} a_{l,\alpha\gamma} a_{l,\beta\gamma}^* + \ldots. \]

This is consistent with Refs. [28, 31]: if all particles have the same sound speed, then the contribution of identical particles in the intermediate state has extra factor 1/2 as compared to distinguishable particles.

### 2.4 Unitary bound

We combine the results of Secs. 2.2 and 2.3 and write the PWA unitarity relation as follows:

\[ -i \frac{1}{2} (a_{l,\alpha\beta} - a_{l,\beta\alpha}^*) = \sum_\gamma g_\gamma a_{l,\alpha\gamma} a_{l,\beta\gamma}^*, \quad (16) \]

where

\[ g_\gamma = \begin{cases} \frac{2}{u_5 u_\gamma (u_5 + u_\gamma)} & \text{distinguishable} \\ \frac{1}{2u_\gamma^3} & \text{identical} \end{cases} \quad (17a, 17b). \]
where eqs. (17a) and (17b) refer to distinguishable and identical particles in the two-particle intermediate state, respectively. We still do not write explicitly contributions due to multi-particle intermediate states. We note in passing that eq. (16) can be written in the matrix form,

\[-\frac{i}{2}(a_l - a_l^\dagger) = a_l \ g \ a_l^\dagger ,\]

where \( g \) is the diagonal matrix with matrix elements \( g_{\gamma\gamma} \), and other notations are self-evident.

To obtain the unitary bound, we introduce rescaled amplitudes \( \tilde{a}_{l,\alpha\beta} \) via

\[ a_{l,\alpha\beta} = \tilde{a}_{l,\alpha\beta} \sqrt{g_{\alpha\alpha} g_{\beta\beta}} . \]  

(18)

In terms of the rescaled amplitudes we write the unitarity relation (16) in a simpler form:

\[-\frac{i}{2} \left( \tilde{a}_{l,\alpha\beta} - \tilde{a}_{l,\beta\alpha}^\ast \right) = \sum_{\gamma} \tilde{a}_{l,\alpha\gamma} \tilde{a}_{l,\beta\gamma}^\ast + \sum_{M} A_{l,\alpha M} A_{l,\beta M}^\ast , \]

(19)

or in matrix form

\[-\frac{i}{2} \left( \tilde{a}_l - \tilde{a}_l^\dagger \right) = \tilde{a}_l \tilde{a}_l^\dagger + A_l A_l^\dagger , \]

(20)

where we restore the contribution of multiparticle intermediate states \( M \) in the right hand side and denote schematically the (rescaled) amplitude \( 2 \to M \) by \( A_{l,\alpha M} \).

Now, let us introduce Hermitean matrices

\[ P_l = -\frac{i}{2} (\tilde{a}_l - \tilde{a}_l^\dagger) , \]

\[ Q_l = \frac{1}{2} (\tilde{a}_l + \tilde{a}_l^\dagger) , \]

so that

\[ \tilde{a}_l = Q_l + iP_l . \]

Then the unitarity relation reads

\[ P_l = P_l^2 + Q_l^2 + A_l A_l^\dagger - i[P, Q] . \]

(22)

We now choose the orthonormal basis in two-particle state space in such a way that the Hermitean matrix \( P_l \) is diagonal,

\[ P_{l,\alpha\beta} = p_{l,\alpha} \delta_{\alpha\beta} . \]

In other words, this basis consists of those linear combinations of states with two particles of definite types which are eigenvectors of \( P_l \). Then the diagonal \( \alpha\alpha \)-component of eq. (22) is (no summation over \( \alpha \))

\[ p_{l,\alpha} = p_{l,\alpha}^2 + (Q_l^2)_{\alpha\alpha} + (A_l A_l^\dagger)_{\alpha\alpha} . \]

9
Diagonal elements of matrices \( Q_l^2 \equiv Q_l Q_l^\dagger \) and \( A_l A_l^\dagger \) are non-negative\(^3\), so that
\[
p_{l,\alpha}^2 - p_{l,\alpha} \leq 0 ,
\]
and therefore
\[
0 \leq p_{l,\alpha} \leq 1 .
\]
To cast this relation into somewhat more familiar form, we come back to the unitarity relation (20), sandwich it between an arbitrary vector \(|\psi\rangle\) of unit norm and write, still using the basis of eigenvectors of \( P_l \),
\[
\langle \psi | \tilde{a}_l \tilde{a}_l^\dagger | \psi \rangle = \sum_\alpha p_{l,\alpha} |\psi_\alpha\rangle^2 - \langle \psi | A_l A_l^\dagger | \psi \rangle .
\]
This gives
\[
\langle \psi | \tilde{a}_l \tilde{a}_l^\dagger | \psi \rangle \leq 1 ,
\]
for all \(|\psi\rangle\), and we arrive at the result that
\[
\text{all eigenvalues of } \tilde{a}_l \tilde{a}_l^\dagger \text{ are not greater than } 1 . \tag{23}
\]

Until now we worked in full generality. To the best of our knowledge, previous analyses not only were restricted to unit sound speed, but also studied somewhat less general situation, see, e.g., Refs. [31, 32, 33, 34]. Namely, (i) the matrix \( \tilde{a}_{l,\alpha\beta} \) was assumed to be symmetric due to \( T \)-invariance, \( \tilde{a}_{l,\alpha\beta} = \tilde{a}_{l,\beta\alpha} \). Then \( Q_l \) and \( P_l \) are its real and imaginary parts, respectively. (ii) One assumed further that \( P_l \) and \( Q_l \) are simultaneously diagonalizable. The latter property holds, in particular, when there is just one type of particles, and also when the contribution of multiparticle states is negligible in (20): in the latter case the imaginary part of eq. (22) gives \([P, Q] = 0\). In this situation eq. (23) tells that any eigenvalue \( \tilde{a}_{\alpha\alpha} \) of matrix \( \tilde{a} \) obeys \( |\tilde{a}_{\alpha\alpha}| \leq 1 \). In fact, in this case one obtains slightly stronger bound \([35]\). In the basis of eigenvectors of \( \tilde{a} \) (i.e., common eigenvectors of \( Q \) and \( P \)), one writes the diagonal part of the unitarity relation (19) for each \( \alpha \) (no summation over \( \alpha \)):
\[
\text{Im} \tilde{a}_{l,\alpha\alpha} = \tilde{a}_{l,\alpha\alpha} \tilde{a}_{l,\alpha\alpha}^* + \sum_M A_{l,\alpha M} A_{l,M\alpha}^* ,
\]
Again, the contribution of multi-particle intermediate states is non-negative, so we arrive at the inequality
\[
\text{Im} \tilde{a}_{l,\alpha\alpha} \geq |\tilde{a}_{l,\alpha\alpha}|^2 .
\]
This gives
\[
\left( \text{Im} \tilde{a}_{l,\alpha\alpha} - \frac{1}{2} \right)^2 + (\text{Re} \tilde{a}_{l,\alpha\alpha})^2 \leq \frac{1}{4} ,
\]
\(^3\)Because, e.g., \( 0 \leq \langle \psi^{(\alpha)} | A_l A_l^\dagger | \psi^{(\alpha)} \rangle = (A_l A_l^\dagger)_{\alpha\alpha} \) for \( \psi^{(\alpha)}_\beta = \delta_{\alpha\beta} \).
and, therefore,
\[ |\text{Re } \tilde{a}_{\ell,\alpha}| \leq \frac{1}{2}, \tag{24} \]
for any eigenvalue of \( \tilde{a} \).

The latter special situation is particularly relevant when it comes to perturbative unitarity and the estimate of the strong coupling scale \( [32, 33, 34, 35] \). In that case the multiparticle intermediate states (almost) always give contributions to (20) which are indeed suppressed by extra powers of the couplings, while the matrix \( \tilde{a} \) is real at the tree level. Perturbative unitarity then requires that the inequality (24) holds for the tree level amplitudes. Note, however, that the bounds (23) and (24) are qualitatively the same even in this situation.

3 Example: theory of two real scalar fields

In this Section we show explicitly that the unitarity relation (16) holds at the lowest non-trivial order in a model of two real scalar fields with the Lagrangian
\[ \mathcal{L} = \frac{1}{2} \left( \dot{\phi}_1^2 - u_1^2 (\mathbf{\nabla} \phi_1)^2 \right) + \frac{1}{2} \left( \dot{\phi}_2^2 - u_2^2 (\mathbf{\nabla} \phi_2)^2 \right) + \frac{\lambda_1}{4!} \phi_1^4 + \frac{\lambda_2}{4!} \phi_2^4 + \frac{\lambda_3}{4} \phi_1^2 \phi_2^2, \tag{25} \]
where \( u_1 \) and \( u_2 \) are the two sound speeds. The scalar potential in eq. (25) is a general fourth-order homogeneous polynomial symmetric under the transformation \( \phi_{1,2} \rightarrow -\phi_{1,2} \). In this theory, the PWA matrix \( a_{\alpha\beta} \) is symmetric due to \( T \)-invariance, so the unitarity relation is
\[ \text{Im } a_{t,\alpha\beta} = \sum_{\gamma} g_{\alpha\gamma} a_{t,\alpha\gamma} a_{t,\gamma\beta}^*, \]
or in matrix form
\[ \text{Im } a_{t} = \sum_{\gamma} a_{t} g_{\gamma}, \tag{26} \]
where elements of the diagonal matrix \( g \) are still given by eq. (17).

The beginning of the calculation follows textbooks. There are three two-particle states \( \alpha = (\phi_1, \phi_1) \), \( \beta = (\phi_1, \phi_2) \), and \( \gamma = (\phi_2, \phi_2) \) in this theory. The tree-level matrix elements make a matrix
\[ M_{\text{tree}} = \begin{pmatrix} M_{\alpha\alpha} & M_{\alpha\beta} & M_{\alpha\gamma} \\ M_{\beta\alpha} & M_{\beta\beta} & M_{\beta\gamma} \\ M_{\gamma\alpha} & M_{\gamma\beta} & M_{\gamma\gamma} \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & \lambda_3 \\ 0 & \lambda_3 & 0 \\ \lambda_3 & 0 & \lambda_2 \end{pmatrix}. \]

Since these matrix elements do not depend on scattering angle \( \gamma \), the only non-zero PWA, as given by eq. (12), is \( a_0 \), i.e., scattering occurs in \( s \)-wave. The matrix of these PWAs is
given by
\[
a_{0,\text{tree}} = \frac{1}{32\pi} \int_{-1}^{1} d(\cos\theta) P_0(\cos\theta) M_{\text{tree}} = \frac{M_{\text{tree}}}{16\pi} = \frac{1}{16\pi} \begin{pmatrix} \lambda_1 & 0 & \lambda_3 \\ 0 & \lambda_3 & 0 \\ \lambda_3 & 0 & \lambda_2 \end{pmatrix}.
\]

As usual in QFT, the right-hand side of (26) is of order \(\lambda_i \lambda_j\), so \(a_i\) obtains its lowest-order contribution at one loop. This contribution comes from \(s\)-channel diagrams shown in Fig. 1, while \(t\)- and \(u\)-channel diagrams give no contribution to imaginary part at one loop.

![Figure 1: One-loop \(s\)-channel diagrams in the theory with the Lagrangian (25).](image)

We begin with the first diagram in Fig. 1. It gives the one-loop contribution to matrix element
\[
iM_{1\text{-loop}}^{(1)} = \frac{\lambda_3^2}{2} \int \frac{d^4 q}{(2\pi)^4} \frac{1}{[(E - q^0)^2 - u_2^2 q^2 - i\epsilon] [((E + q^0)^2 - u_2^2 q^2 + i\epsilon]},
\]
where \(E\) is still the total energy in the center-of-mass frame. Upon rescaling \(u_2 \bar{q} \to \bar{q}\), a textbook calculation gives
\[
\text{Im } M_{1\text{-loop}}^{(1)} = \frac{\lambda_3^2}{32\pi u_2^3}.
\]
Likewise, the diagrams 2–6 in Fig. 1 give
\[
\text{Im } M_{1\text{-loop}}^{(2)} = \frac{\lambda_3^2}{32\pi u_3^3}, \quad \text{Im } M_{1\text{-loop}}^{(3)} = \frac{\lambda_2 \lambda_3}{32\pi u_3^3}, \quad \text{Im } M_{1\text{-loop}}^{(4)} = \frac{\lambda_1 \lambda_3}{32\pi u_1^3}, \quad \text{Im } M_{1\text{-loop}}^{(5)} = \frac{\lambda_2^2}{32\pi u_2^3}, \quad \text{Im } M_{1\text{-loop}}^{(6)} = \frac{\lambda_3^2}{32\pi u_1^3}.
\]
We now turn to the diagram 7 in Fig. 1. Unlike others, it has two different particles in the loop. One writes

\[ iM_{1\text{-loop}}^{(7)} = \lambda_3^2 \int \frac{d^4q}{(2\pi)^4} \frac{1}{[(\frac{E}{2} - q^0)^2 - u_1^2q^2 + i\epsilon] \left[ (\frac{E}{2} + q^0)^2 - u_2^2q^2 + i\epsilon \right]} . \]  

(29)

There are four poles of the integrand at

\[ q_{1,2}^0 = \frac{E}{2} \pm u_1|\vec{q}| \mp i\epsilon, \]

\[ q_{3,4}^0 = -\frac{E}{2} \pm u_2|\vec{q}| \mp i\epsilon . \]

Without loss of generality we assume

\[ u_1 \geq u_2 . \]

Then it is convenient to close the integration contour as shown in Fig. 2; the poles inside it are at \( q_1^0 \) and \( q_3^0 \). We integrate over \( q^0 \) and get

\[ iM_{1\text{-loop}}^{(7)} = \lambda_3^2 \int \frac{d^3q}{(2\pi)^4} (-2\pi i) \left[ \frac{1}{2qu_1(E + q(u_1 - u_2))(E + q(u_1 + u_2))} \right. \]

\[ + \left. \frac{1}{(-2qu_2)(-E + q(u_1 + u_2) - i\epsilon)(E + q(u_1 - u_2))} \right] , \]
The first term in the integrand does not contribute to $\text{Im} M_{1\text{-loop}}^{(7)}$. Imaginary part due to the second term is calculated using Sokhotski-Plemelj formula

$$\lim_{\epsilon \to 0^+} \left( \frac{1}{x \pm i \epsilon} \right) = \mp i \pi \delta(x) + P \left( \frac{1}{x} \right),$$

where $P$ stands for principal value. We find

$$\text{Im} M_{1\text{-loop}}^{(7)} = \lambda_3^2 \int \frac{d^3q}{(2\pi)^3} \frac{1}{2qu_2(E + q(u_1 - u_2))} \cdot \pi \delta (-E + q(u_1 + u_2)),$$

and, finally,

$$\text{Im} M_{1\text{-loop}}^{(7)} = \frac{\lambda_3^2}{8\pi u_1 u_2(u_1 + u_2)}.$$

To sum up, we collect all results in one matrix

$$\text{Im} a_{0,1\text{-loop}} = \frac{1}{16\pi} \text{Im} M_{1\text{-loop}} = \frac{1}{16\pi} \begin{pmatrix} \frac{\lambda_1^2}{32\pi u_1} + \frac{\lambda_3^2}{32\pi u_2} & 0 & \frac{\lambda_1 \lambda_3}{32\pi u_1^2} + \frac{\lambda_2 \lambda_3}{32\pi u_2^2} \\ 0 & \frac{\lambda_2^2}{32\pi u_2} & \frac{\lambda_2 \lambda_3}{32\pi u_2^2} + \frac{\lambda_3^2}{32\pi u_1^2} \\
\frac{\lambda_1 \lambda_3}{32\pi u_1^2} + \frac{\lambda_2 \lambda_3}{32\pi u_2^2} & \frac{\lambda_2 \lambda_3}{32\pi u_1^2} + \frac{\lambda_3^2}{32\pi u_1^2} & 0 \\
\frac{\lambda_1 \lambda_3}{32\pi u_1^2} + \frac{\lambda_2 \lambda_3}{32\pi u_2^2} & \frac{\lambda_2 \lambda_3}{32\pi u_1^2} + \frac{\lambda_3^2}{32\pi u_1^2} & 0 \end{pmatrix}.$$  \hspace{1cm} (31)

Now, eq. (17) gives for the matrix $g$ in (26)

$$g = \text{diag} \left( \frac{1}{2u_1^3}, \frac{2}{u_1 u_2(u_1 + u_2)}, \frac{1}{2u_2^3} \right).$$  \hspace{1cm} (32)

Making use of eqs. (27), (31) and (32), one finds that

$$\text{Im} a_{0,1\text{-loop}} = a_{0,\text{tree}} g a_{0,\text{tree}},$$

i.e., the unitarity relation (26) is indeed valid to the lowest non-trivial order in the couplings.

4 Summary

In this paper we found PWA unitarity relations (16) in a theory containing massless scalar fields with different sound speeds. We illustrated these relations in a model with the Lagrangian (25), to the lowest non-trivial order in the couplings. When written in terms of rescaled amplitudes (18), the unitarity relations have particularly simple form (19), which is formally the same as in a theory with unit sound speeds.

Using the unitarity relations, we derived the unitarity bounds, which in the most general case have the form (23), and in (still quite general) case considered in literature reduce to
the familiar form (24) (but written in terms of rescaled amplitudes). The latter form is particularly useful for evaluating the quantum strong coupling scale in pertinent EFT.

Our study has been motivated by models with “strong gravity in the past” [22]. One obvious future direction is to make use of our results to further study models from this class. We anticipate, however, that the results of this paper may have applications in other theories where different perturbations about non-trivial backgrounds propagate with different sound speeds.

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Appendix A: Time-reversal invariance and symmetry of S' matrix

In this Appendix we show that $T$-invariance of $S$-matrix implies the symmetry of the partial-wave amplitudes,

$$T^{(l)}_{\beta'\beta} = T^{(l)}_{\beta\beta'}. \quad (33)$$

$T$-invariance of $S$-matrix is invariance under exchange of initial and final states and sign reversal of all spatial momenta:

$$\langle \vec{p}'', \beta'' | S | \vec{p}', \beta' \rangle = \langle -\vec{p}, \beta | S | -\vec{p}', \beta' \rangle.$$

We make use of this property to write (we work in the center-of-mass frame)

$$\langle l, m; \beta' | S | l, m; \beta \rangle = \frac{1}{4\pi} \int d^3\hat{p}' \ d^3\hat{p} \ Y_l^m(\hat{p}) Y_l^m(\hat{p}') \langle \vec{p}', \beta' | S | \vec{p}, \beta \rangle$$

$$= \frac{1}{4\pi} \int d^3\hat{p}' \ d^3\hat{p} \ Y_l^m(\hat{p}) Y_l^m(\hat{p}') \langle -\vec{p}, \beta | S | -\vec{p}', \beta' \rangle$$

$$= \frac{1}{4\pi} \int d^3(-\hat{p}'') \ d^3(-\hat{p}) \ Y_l^m(-\hat{p}) Y_l^m(-\hat{p}') \langle \vec{p}, \beta | S | \vec{p}', \beta' \rangle.$$

Now, the spherical functions obey

$$Y_l^m(-\hat{p}) = (-1)^l Y_l^m(\hat{p}), \quad (34a)$$

$$Y_l^m(\hat{p}) = (-1)^m Y_l^{-m}(\hat{p}), \quad (34b)$$
so that
\[ Y_{l}^{m,*(−\tilde{p})} = (-1)^{l+m} Y_{l}^{−m}(\tilde{p}). \]

This gives
\[
\langle lm; β′|S|lm; β \rangle = \frac{1}{4π} \int \int d^3\tilde{p}′ d^3\tilde{p} Y_{l}^{-m}(\tilde{p}′)Y_{l}^{-m,*}(\tilde{p})\langle \tilde{p}, β|S|\tilde{p}′, β' \rangle
\]
\[= \langle l, −m; β|S|l, −m; β′ \rangle. \]

Since these matrix elements are actually independent of \( m \), this proves the relation (33).

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