On the Hopf Algebra of Rooted Trees

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Abstract

We find a formula to compute the number of the generators, which generate the $n$-filtered space of Hopf algebra of rooted trees, i.e. the number of equivalent classes of rooted trees with weight $n$. Applying Hopf algebra of rooted trees, we show that the analogue of Andruskiewitsch and Schneider’s Conjecture is not true. The Hopf algebra of rooted trees and the enveloping algebra of the Lie algebra of rooted trees are two important examples of Hopf algebras. We give their representation and show that they have not any nonzero integrals. We structure their graded Drinfeld doubles and show that they are local quasitriangular Hopf algebras.

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0 Introduction

In [CK, BK, Kr1, Kr2], the Hopf algebra of rooted trees $H_R$ was introduced for renormalization theory. Paper [Fo] classified the finite dimensional comodules over $H_R$.

It is well-known that Hopf algebra $H_R$ of rooted trees is the algebra of polynomials over $\mathbb{Q}$, whose indeterminate elements are equivalent classes of rooted trees. That is, $H_R = \mathbb{Q}[RT]$ as algebras, where $RT$ is the set of equivalent classes of all rooted trees (see section 1).

Therefore, it is necessary to find a formula to compute the number of equivalent classes of rooted trees with weight $n$. Fortunately, we complete this in this paper. The Hopf algebra of rooted trees and the enveloping algebra of the Lie algebra of rooted trees are two important examples of Hopf algebras. We give their representation by means of path algebras. In particular, primitive elements of $H_R$ are extraordinary. In this paper we use this special structure to show that the analogue of Andruskiewitsch and Schneider’s Conjecture [AS2 Conjecture 1.4] is not true. That is, there is an infinite
dimensional pointed Hopf algebra, which is not generated by its coradical and its skew-primitive elements. That is, this conjecture is not true when the condition which $H$ is finite dimensional is omitted. Paper [ZGZ] pointed out that we can systematically structure the solutions of Yang-Baxter equations by means of local quasitriangular Hopf algebras. In this paper we structure the graded Drinfeld double of Hopf algebra of rooted trees and show that it is a local quasitriangular Hopf algebra.

Preliminaries

We will use notations of [CK] and [Fo]. We call rooted tree $t$ a connected and simply-connected finite set of oriented edges and vertices such that there is one distinguished vertex with no incoming edge; this vertex is called the root of $t$. The weight of $t$ is the number of its vertices. The fertility of a vertex $v$ of a tree $t$ is the number of edges outgoing from $v$. A ladder is a rooted tree such that every vertex has fertility less than or equal to 1. There is a unique ladder of weight $i$; we denote it by $l_i$.

Let $\mathbb{Z}$, $\mathbb{Z}^+$ and $\mathbb{N}$ denote the sets of all integers, all positive integers and all non-negative integers, respectively. $k = \mathbb{Q}$ denotes rational number field. Let $\mathcal{RT}$ denote the set of equivalent classes of all rooted trees (see section 1). $H_R$ and $\mathcal{L}^1$ denote the Hopf algebra and Lie algebra in [CK] and [Fo, Section 2], called the Hopf algebra of rooted tree and the Lie algebra of rooted tree, respectively. $H_{ladd}$ denotes the Hopf subalgebra of $H_R$ generated by all ladders $l_i$.

We define the algebra $H_R$ as the algebra of polynomials over $\mathbb{Q}$ in $\mathcal{RT}$. The monomials of $H_R$ will be called forests. It is often useful to think of the unit 1 of $H_R$ as an empty forest.

Let $M, N$ be two forests of $\mathcal{H}_R$. We define:

$$M \triangledown N = \begin{cases} \frac{1}{\text{weight}(N)} \sum \text{forests obtained by appending M to every node of N} & \text{if } N \neq 1 \\ 0 & \text{if } N = 1 \end{cases}$$

We can extend $\triangledown$ to a bilinear map from $\mathcal{H}_R \times \mathcal{H}_R$ into $\mathcal{H}_R$.

1 The number of rooted trees

In this section we give a formula to compute the number of equivalent classes of rooted trees with weight $n$.

A sub-tree $u$ of rooted tree $t$ is said to be of the height $r$ if the distant from root of $t$ to root of $u$ is $r$, i.e. there exist exactly $r$ arrows from root of $t$ to root of $u$. Assume that $u$ and $u'$ are two sub-trees of rooted tree $t$ with the same height. If we change
the position of \( u \) and \( u' \) in \( t \), we get a new rooted tree \( t' \), said that \( t' \) is obtained by a elementary transformation from \( t \). Two rooted trees \( t \) and \( t' \) are called equivalent if \( t' \) can be obtained by finite elementary transformations from \( t \), written as \( t \sim t' \). Note that \( t \sim t \) for any rooted tree \( t \) since we can choice \( u = u' \). For example,

![Figure 1: the first and second rooted trees are equivalent; the third and fourth rooted trees are equivalent.](image)

Obviously, “ \( \sim \) ” is an equivalent relation in all rooted trees. Let \( \mathcal{RT} \) be the set of equivalent classes of all rooted trees and \( \mathcal{RT}_n \) be the set of equivalent classes of all rooted trees with weight \( n \). Define \( a(n) := \text{card } \mathcal{RT}_n \), i.e. the number of elements in \( \mathcal{RT}_n \).

**Theorem 1.1.**

\[
a(n+1) = \sum_{1\lambda_1 + 2\lambda_2 + \cdots + n\lambda_n = n} C_{\lambda_1 a(1) + \lambda_2}^{\lambda_2} C_{\lambda_3}^{\lambda_3} \cdots C_{\lambda_n}^{\lambda_n}.
\]

**Proof.** Let \( t \) be a rooted tree with weight \( n+1 \). Assume that there exists \( \lambda_i \) subtrees of \( t \) with height 1 and weight \( i \) for \( i = 1, 2, \ldots, n \). Then \( 1\lambda_1 + 2\lambda_2 + \cdots + n\lambda_n = n \) and we have the formula (1) by combination theory. \( \Box \)

**Remark:** The indeterminate elements of algebra \( \mathcal{H}_R \) of polynomials are the equivalent classes of rooted trees, instead of rooted trees. Otherwise, \( B_+ \) in [CK, map (45)] is not a map.

## 2 The representation of the Hopf algebra and the Lie algebra of rooted trees

In this section we give the representation of the Hopf algebra and the Lie algebra of rooted trees.

Let us recall the representation of a quiver. Let \( G = \{e\} \) be a trivial group and \( Q \) the quiver over \( G \) with arrow set \( Q_1 \). Obviously, \( Q \) is a Hopf quiver. Let \( kQ^n \) denote the path algebra of quiver \( Q \). It is clear that the free algebra generated by \( Q_1 \) is isomorphic to the path algebra \( kQ^n \) as algebras. Let \( V \) be a vector space and \( f = \{f_t\}_{t \in Q_1} \) with \( f_t \in \text{End}_k V \) for any \( t \in Q_1 \). We define a map \( \alpha_f \) from \( kQ^n \otimes V \) to \( V \) as follows:

\[
\alpha_f(t_1 t_2 \cdots t_n \otimes v) = t_1 t_2 \cdots t_n \cdot v := f_{t_1} f_{t_2} \cdots f_{t_n}(v)
\]
for any \( v \in V, t_1, t_2, \ldots, t_n \in Q_1 \). It is clear that \((V, \alpha_f)\) is a \(kQ^a\)-module, called a quiver module.

Assume that \( \rho \) is a subset of \(kQ^a\) and \((\rho)\) denotes the ideal generated by \(\rho\). Let \((V, \alpha_f)\) be a quiver module. For any element \( \sigma \in kQ^a \), we write \( f_\sigma : V \to V \) by sending \( x \) to \( \sigma \cdot x \) for any \( x \in V \). Obviously, if \((V, \alpha_f)\) is a quiver module with \( f_\sigma = 0 \) for any \( \sigma \in \rho \), then \((V, \alpha_f)\) is a \(kQ^a/(\rho)\)-module with \( \bar{a} \cdot m = a \cdot m \) for any \( a \in kQ^a, m \in M \). This is called a factor quiver module. Note every quiver module over \(kQ^a\) can be viewed as a factor quiver module over \(kQ^a/(\rho)\) when \(\rho\) is empty.

Assume that \((V, \alpha_f)\) and \((V', \alpha_{f'})\) are two factor quiver modules. If \( h : (V, \alpha_f) \to (V', \alpha_{f'}) \) a homomorphism of \(kQ^a/(\rho)\)-modules, then \( hf_t = f'_t h \) for any \( t \in Q_1 \). Conversely, if \( h \) is a \(k\)-linear map from \( V \) to \( V' \) with \( hf_t = f'_t h \) for any \( t \in Q_1 \), then \( h \) is homomorphism of \(kQ^a/(\rho)\)-modules.

**Lemma 2.1.** (see [22, Theorem 2.9] or [23, Theorem 2.9]) (i) \((V, \alpha)\) is a left \(kQ^a\)-module if and only if there exist \( f_t \in \text{End}_k(V) \) for any \( t \in Q_1 \) such that \( \alpha = \alpha_f \). That is to say every \(kQ^a\)-module is a quiver module.

(ii) \((V, \alpha)\) is a left \(kQ^a/(\rho)\)-module if and only if there exist \( f_t \in \text{End}_k(V) \) for any \( t \in Q_1 \) such that \( \alpha = \alpha_f \) and \( f_\sigma = 0 \) for any \( \sigma \in \rho \). That is to say every \(kQ^a/(\rho)\)-module is a factor quiver module.

Obviously, \( \mathcal{H}_R \cong kQ^a/(\rho) \) as algebras with \( Q_1 = RT, \rho := \{tt' - t't | t, t' \in RT\} \). Using Lemma 2.1 we have the following:

**Theorem 2.2.** \((V, \alpha)\) is a left \(\mathcal{H}_R\)-module if and only if there exists \( f_t \in \text{End}_k(V) \) for any \( t \in RT \) such that \( \alpha = \alpha_f \) with \( f = \{f_t\}_{t \in RT} \) and \( f_t f_{t'} = f_{t'} f_t \) for any \( t, t' \in RT \).

Explicitly, in the cases above,

\[
\alpha_f(t \otimes v) = t \cdot v := f_t(v)
\]  

(3)

for any \( v \in V, t \in RT \).

Let \( \{p_i | i \in \mathbb{Z}^+\} \) be a homogeneous basis of \( Prim(\mathcal{H}_R) \), the set of all primitive elements. By the proof of [20, Proposition 8.1], \( \{1\} \cup \bigcup_{i=1}^{\infty} \{p_{i_1} \cdots p_{i_n} | (i_1, i_2, \ldots, i_n) \in (\mathbb{Z}^+)^n\} \) is a basis of \( \mathcal{H}_R \). Let \( \{1\} \cup \bigcup_{i=0}^{\infty} \{\xi_i | i = (i_1, i_2, \ldots, i_n) \in (\mathbb{Z}^+)^n\} \) be its dual basis in \( \mathcal{H}_R^d \) with \( < \xi_i, p_{i_1} \cdots p_{i_n} > = \delta_{ij} \) for any \( i, j \in \bigcup_{i=0}^{\infty} (\mathbb{Z}^+)^n \cup \{0\} \), where \( p_{i_1} \cdots p_{i_n} \) and \( \xi_i \) denote 1 when \( j = i = 0 \).

Let \( l_{(i_1, i_2, \ldots, i_n)} \in U(\mathcal{L}^1) \) with \( \Psi(l_{(i_1, i_2, \ldots, i_n)}) = \xi_{(i_1, i_2, \ldots, i_n)} \) for any \( i \in \bigcup_{n=1}^{\infty} (\mathbb{Z}^+)^n \cup \{0\} \), where \( \Psi \) is the same as in [20, Corollary 3.3]. Therefore, \( \{l_i | i \in \bigcup_{n=1}^{\infty} (\mathbb{Z}^+)^n \cup \{0\} \} \) is a basis of \( U(\mathcal{L}^1) \) and \( U(\mathcal{L}^1) \) is the free algebra generated by \( \{l_i | i \in \bigcup_{n=1}^{\infty} (\mathbb{Z}^+)^n \cup \{0\} \} \).

**Theorem 2.3.** \((V, \alpha)\) is a left \(U(\mathcal{L}^1)\)-module (or \(\mathcal{L}^1\)-module) if and only if there exists \( f_i \in \text{End}_k(V) \) for any \( i \in \bigcup_{n=1}^{\infty} (\mathbb{Z}^+)^n \cup \{0\} \) such that \( \alpha = \alpha_f \) with \( f = \{f_i\}_{i \in \bigcup_{n=1}^{\infty} (\mathbb{Z}^+)^n \cup \{0\}} \).
Explicitly, in the cases above,
\[\alpha_f(l_i \otimes v) = l_i \cdot v := f_i(v)\]  \hspace{1cm} (4)

for any \(v \in V, i \in (\bigcup_{i=1}^{\infty} (\mathbb{Z}^+)^i) \cup \{0\}\).

3 Some properties about Hopf algebras

In this section we give some properties of the Hopf algebra of rooted trees.

3.1 Integrals

Proposition 3.1. \(\mathcal{H}_R, H_{ladd}\) and \(U(\mathcal{L}^1)\) have not any nonzero integrals.

Proof. Let \(l_1\) denote the rooted tree with weight \((l_1) = 1\). It is clear that subalgebra \(k[l_1]\) of \(\mathcal{H}_R\) generated by \(l_1\) is a Hopf subalgebra of \(\mathcal{H}_R\). By [DNR, Proposition 5.6.11], \(k[l_1]\) has not any nonzero integral. Therefore, it follows from [DNR, Corollary 5.3.3] that \(\mathcal{H}_R\) has not any nonzero integral. Similarly, we can show the other. \(\square\)

3.2 Andruskiewitsch and Schneider’s Conjecture

Proposition 3.2. \(H_{ladd}\) can not be generated by primitive elements of \(H_{ladd}\) as algebras.

Proof. By [Fo, Proposition 9.3 and Theorem 9.5], \(l_2\) can not generated by set \(\{P_i \mid i = 1, 2, \ldots\}\), where \(P_i\) is defined in [Fo, Proposition 9.3]. \(\square\)

N. Andruskiewitsch and H. J. Schneider in [AS2, Conjecture 1.4] gave a conjecture: every finite dimensional pointed Hopf algebra can be generated by its coradical and its skew-primitive elements. By Proposition 3.2, analogue of this conjecture for infinite dimensional pointed Hopf algebra does not hold.

Furthermore, \(H_{ladd}\) and \(\mathcal{H}_R\) are not Nichols algebras over trivial group since \((H_{ladd})_{(1)} \neq \text{Prim}(H_{ladd})\) and \((\mathcal{H}_R)_{(1)} \neq \text{Prim}(\mathcal{H}_R)\). They also are not strictly graded coalgebras (see [Sw, P 232]).

3.3 Hopf subalgebra

A Hopf algebra \(H\) is called trivial, if \(\dim H = 1\).

Proposition 3.3. (i) If \(H\) is a nontrivial pointed irreducible Hopf algebra, then \(H\) has not any finite dimensional nontrivial Hopf subalgebra;

(ii) \(\mathcal{H}_R\) and \(U(\mathcal{L}^1)\) have not any finite dimensional nontrivial Hopf subalgebra.
Lemma 3.5] or [AS1, Lemma 3.3], set \( \{ h \} \) for any \( A \)

By [Fo, Theorem 3.1], there is a bilinear form \( \Phi \) with \( \dim H < \sigma \) where \( \sigma \) is a graded Hopf algebra. Considering [Ni, Pro.1.5.1], we have to prove only that it is a graded Hopf algebra.

In this section we structure the graded Drinfeld double of Hopf algebra of rooted trees and show that it is a local quasitriangular Hopf algebra.

3.4 Graded duality

By [Fo, Theorem 3.1], there is a bilinear form \( \langle , \rangle \) on \( U(\mathcal{L}^1) \times \mathcal{H}_R \) such that

\[
\langle 1, h \rangle = \varepsilon(h), \quad \langle Z_t, h \rangle = (\frac{\partial}{\partial t} h) |_{t=0},
\]

and \( \langle Z_t Z_{t'}, h \rangle = \langle Z_t \otimes Z_{t'}, \Delta(h) \rangle \)

for any \( h \in \mathcal{H}_R, t, t' \in RT \).

Lemma 3.4. (see [Su, Section 11.2]) Assume that \( H = \sum_{i=0}^{\infty} H_{(i)} \) is a local finite graded Hopf algebra (i.e. \( \dim H_{(i)} < \infty \) for any \( i \)). Define \( H^g := \sum_{i=0}^{\infty} (H_{(i)})^* \), called graded dual of \( H \). Then \( H^g \) is a graded Hopf algebra and \( H \cong (H^g)^g \) as graded Hopf algebra.

Proof. It is clear that \( \sum_{i=0}^{\infty} (H_{(i)})^* \subseteq H^g \). Now we show that \( H^g = \sum_{i=0}^{\infty} (H_{(i)})^* \) is a graded Hopf algebra. Considering [Ni, Pro.1.5.1], we have to prove only that it is a graded bialgebra. However, it is easy. Therefore, \( (H_{(i)})^* = (H^g)_{(i)} \) for \( i = 0, 1, 2, \ldots \).

Finally, we show that \( \sigma_H \) is a graded Hopf algebra isomorphism from \( H \) to \( H^g \), where \( \langle \sigma_H(h), f \rangle = \langle f, h \rangle \) for any \( f \in H^g, h \in H \). Indeed, we have to show only that \( \sigma_H(H) = H^g \). For any \( h \in H_{(i)}, f \in H_{(j)}^g = (H^g)_{(j)} \) with \( i \neq j \), we have that \( \langle \sigma_H(h), f \rangle = \langle f, h \rangle = 0 \), so \( \sigma_H(h) \in (H^g)_{(i)} \). That implies \( \sigma_H(H) = H^g \).

Corollary 3.5. \( H_{(g)}^g \cong \mathcal{H}_R \) and \( U(\mathcal{L}^1)^g \cong U(\mathcal{L}^1) \) as graded Hopf algebras.

Recall [Fo, Corollary 3.3], \( \Phi : \mathcal{H}_R \rightarrow U(\mathcal{L}^1)^g \) and \( \Psi : U(\mathcal{L}^1) \rightarrow (\mathcal{H}_R)^g \) are a coalgebra isomorphism and an algebra isomorphism, respectively. However, we have

Theorem 3.6. \( \Phi \) is not algebraic and \( \Psi \) is not coalgebraic.

Proof. Let \( t \) and \( t' \) are two rooted trees, which are in different equivalent classes. It is clear that \( \Phi(tt')(Z_t) = t' \) and \( < (\Phi(t) \ast \Phi(t')), Z_t > = 0 \), so \( \Phi(tt') \neq \Phi(t) \ast \Phi(t') \). That is, \( \Phi \) is not algebraic. Similarly, \( \Psi \) is not coalgebraic.

4 The graded Drinfeld double

In this section we structure the graded Drinfeld double of Hopf algebra of rooted trees and show that it is a local quasitriangular Hopf algebra.
Let \( V = \bigoplus_{i=0}^{\infty} V_i \) be a local finite graded vector space and \( V_n := \sum_{i=0}^{n} V_{(i)} \) for any \( n \in \mathbb{N} \). Let \( \text{ev}_{V(n)} := d_{V(n)} \) and \( \text{coev}_{V(n)} := b_{V(n)} \) denote the evaluation and coevaluation of \( V_{(n)} \), respectively. If we denote by \( \{e_1^{(n)}, e_2^{(n)}, \ldots, e_r^{(n)}\} \) a basis of \( V_{(n)} \) and \( \{f_1^{(n)}, f_2^{(n)}, \ldots, f_r^{(n)}\} \) a its dual basis in \((V_{(n)})^*\), then

\[
b_{V_{(n)}} := \sum_{i=1}^{r_n} (e_i^{(n)} \otimes f_i^{(n)}) \quad \text{and} \quad b_{V_{(n)}} := \sum_{i=0}^{n} b_{V_{(i)}}
\]

are coevaluations of \( V_{(n)} \) and \( V_n \), respectively, for any \( n \in \mathbb{N} \). Let \( C_{U,V} \) denote the flip from \( U \otimes V \) to \( V \otimes U \) by sending \( u \otimes v \) to \( v \otimes u \) for any \( u \in U \), \( v \in V \).

**Lemma 4.1.** Let \( H = \bigoplus_{i=0}^{\infty} H_{(i)} \) be a local finite graded Hopf algebra. Under notation above, set \( A = (H^{(n)})^{op} \), \( \tau = \text{ev}_HC_{H,H} \), \( P_n = b_{H_n} \) and \( R_n := [P_n] = 1 \otimes P_n \otimes 1 = \sum_{r=0}^{\infty} \sum_{i=1}^{r_n} e_i^{(n)} \otimes f_i^{(n)} \otimes 1 \). Then \( (D(H), \{R_n\}) \) is a local quasitriangular Hopf algebra, where \( D(H) = A \bowtie \tau \), called graded Drinfeld double of \( H \).

**Proof.** It follows from [ZGZ, Lemma 3.4] and [ZGZ, Lemma 3.6]. □

Let \( \{p_i^{(n)} \mid 1 \leq i \leq r_n\} \) be a basis of \((\text{Prim}(\mathcal{H}_R))_{(n)}\) with \( n \in \mathbb{Z}^+ \). By the proof of [8], Proposition 8.1, \( \{p_i^{(j_1)} \otimes p_i^{(j_2)} \otimes \cdots \otimes p_i^{(j_s)} \mid j_1 + j_2 + \cdots + j_s = n, j_1, j_2, \ldots, j_s \in \mathbb{Z}^+; 1 \leq i \leq r_{j_1}, \ldots, 1 \leq i_s \leq r_{j_s}; s \in \mathbb{Z}^+\} \) is a basis of \((\mathcal{H}_R)_{(n)}\). Let \( e_{j_1, j_2, \ldots, j_s}^{(n)} := p_{i_1}^{(j_1)} \otimes p_{i_2}^{(j_2)} \otimes \cdots \otimes p_{i_s}^{(j_s)} \). Therefore, \( \{e_{j_1, j_2, \ldots, j_s}^{(n)} \mid j_1 + j_2 + \cdots + j_s = n, j_1, j_2, \ldots, j_s \in \mathbb{Z}^+; 1 \leq i_1 \leq r_{j_1}, \ldots, 1 \leq i_s \leq r_{j_s}; s \in \mathbb{Z}^+\} \) is a dual basis of \((\mathcal{H}_R)_{(n)}\). Let \( \{f_{j_1, j_2, \ldots, j_s}^{(n)} \mid j_1 + j_2 + \cdots + j_s = n, j_1, j_2, \ldots, j_s \in \mathbb{N}; 1 \leq i_1 \leq r_{j_1}, \ldots, 1 \leq i_s \leq r_{j_s}; s \in \mathbb{Z}^+\} \) is another dual basis in \((\mathcal{H}_R)_{(n)}\). Applying these basis and Lemma above we have

**Theorem 4.2.** Let \( H = \mathcal{H}_R = \bigoplus_{i=0}^{\infty} H_{(i)} \). Then \( (D(H), \{R_n\}) \) is a local quasitriangular Hopf algebra with \( R_n = [P_n] = 1 \otimes P_n \otimes 1 = 1 \otimes 1 \otimes 1 \otimes 1 + \sum_{m=1}^{n} (1 \otimes e_{i_1, \ldots, i_s}^{(m)} \otimes f_{j_1, \ldots, j_s}^{(m)} \otimes 1 \mid j_1 + j_2 + \cdots + j_s = m, j_1, j_2, \ldots, j_s \in \mathbb{N}; 1 \leq i_1 \leq r_{j_1}, \ldots, 1 \leq i_s \leq r_{j_s}; s \in \mathbb{Z}^+) \). Furthermore \( D(H) = A \bowtie \tau \cdot H = A \bowtie \mathcal{H} \cdot \mathcal{H} \) (i.e. (right) smash product), i.e. \( (a \bowtie h)(b \bowtie g) = \sum_{(b)} a_{(b)}^1 \otimes \beta(h, b_2)g \) and \( \beta(h, a) = \sum_{(a), (h)} < a_1, h_1 > < a_2, S(h_3) > h_2 \) for any \( a, b \in A, h, g \in \mathcal{H} \).

**Theorem 4.3.** Let \( H = U(L_1) = \bigoplus_{i=0}^{\infty} H_{(i)} \). Then \( (D(H), \{R_n\}) \) is a local quasitriangular Hopf algebra. Furthermore, \( D(H) = A \bowtie \tau \cdot H = A \bowtie \mathcal{H} \) (i.e. smash product), i.e. \( (a \bowtie h)(b \bowtie g) = \sum_{(b)} a_{(b)}^1 a_{(1)}h_1 \otimes h_2 g \) and \( \alpha(h, a) = \sum_{(a), (h)} < a_1, h_1 > < a_3, S(h_2) > a_2 \) for any \( a, b \in A, h, g \in \mathcal{H} \).

5 Appendix

A rooted tree \( t \) is called an \( r \)-branch tree if the fertility of every vertex of \( t \) is less than or equal to \( r \). Let \( a_r(n) \) denote the number of equivalent classes of \( r \)-branch rooted trees
with weight $n$. We can show the following by the method similar to the proof of Theorem 1.1.

Theorem 5.1.

$$a_r(n+1) = \sum \{ C_{a_r(1)+\lambda_1-1}^{\lambda_1} C_{a_r(2)+\lambda_2-2}^{\lambda_2} \cdots C_{a_r(n)+\lambda_n-1}^{\lambda_n} \mid 1\lambda_1 + 2\lambda_2 + \cdots + n\lambda_n = n; \text{ card } \{ i \mid \lambda_i \neq 0 \} \leq r \}. \quad (6)$$

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