CYLINDER MAPS OF ALGEBRAIC CYCLES ON CUBIC HYPERSURFACES

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Abstract. Let $X \subset \mathbb{P}^{n+1}$ be a smooth cubic hypersurface, and let $F(X)$ be the variety of lines on $X$. We prove the surjectivity of the cylinder maps on the Chow groups of $F(X)$ and $X$ under a mild condition on one-cycles of $X$. Mongardi and Ottem previously proved the integral Hodge conjecture for curve classes on hyperkähler manifolds. Using the cylinder maps, we provide an alternative proof for the $F(X)$ of a smooth complex cubic fourfold $X$, which is a special hyperkähler fourfold. In addition, we confirm the integral Tate conjecture for $F(X)$ of a smooth cubic fourfold $X$ over a finitely generated field.

Contents

Introduction 1
1. The Hilbert square of a cubic hypersurface 3
2. Proof of surjectivity of the cylinder maps 9
3. Integral Hodge conjectures and Tate conjectures 12
References 15

Introduction

Let $X$ be a smooth closed subvariety in the projective space $\mathbb{P}^N_k$ over a field $k$. The variety of lines on $X$, denoted by $F(X)$, is the parameter space of lines in $\mathbb{P}^N_k$ that are contained in $X$. The geometry and topology of $F(X)$ and $X$ are closely related, particularly when $X$ is a Fano variety.

Fix an integer $r \geq 1$. The cylinder map or cylinder homomorphism associated to lines on $X$ is a homomorphism on groups of algebraic cycles:

$$\psi: \mathbb{Z}_{r-1}(F(X)) \to \mathbb{Z}_r(X)$$

such that for a given $(r - 1)$-dimensional cycle $\gamma$ on $F(X)$, the $r$-dimensional cycle $\psi(\gamma)$ on $X$ is the union of lines over $\gamma$. On Chow groups, this map is naturally described via the universal $\mathbb{P}^1$-bundle over $F(X)$, namely, the incidence correspondence

$$P: = \{([\ell], x) \in F(X) \times X \mid x \in \ell \subset X\}.$$

The cylinder map on the Chow groups is then given by the induced homomorphism

(1) $$[P]_* = q_* p^*: \text{CH}_{r-1}(F(X)) \to \text{CH}_r(X).$$

where $p: P \to F(X)$ and $q: P \to X$ are the natural projections.

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Suppose that \( k = \mathbb{C} \). The cohomological cylinder map

\[
q_* p^* : H_{n-2}(F(X), \mathbb{Z}) \to H_n(X, \mathbb{Z})
\]

where \( n = \dim X \) has been studied in various cases, especially for Fano threefolds \([22]\) and related variants \([13]\). When \( X \) is a smooth complex cubic threefold, Clemens and Griffiths \([7]\) proved that \((2)\) is an isomorphism, playing a crucial role in Torelli and irrationality theorem for cubic threefolds. For a smooth complex cubic fourfold \( X \), Beauville and Donagi \([3]\) showed that \( F(X) \) is a hyperkähler manifold and \((2)\) is an isomorphism, which is fundamental in describing the Bogomolov-Beauville-Fujiki form on \( H^2(F(X), \mathbb{Z}) \). More generally, Shimada asserted that \((2)\) is surjective for general Fano complete intersections \([25]\).

In this paper, we concern the cylinder maps for a smooth cubic hypersurface. Our main result is as follows:

**Theorem 0.1** (\(=\)Theorem \(2.3\)). Let \( k \) be a field, and let \( X \subset \mathbb{P}^{n+1}_k \) be a smooth cubic hypersurface of dimension \( n \geq 3 \). Let \( F(X) \) be the variety of lines on \( X \). Assume that \( X \) contains a \( k \)-line. Then the cylinder map

\[
q_* p^* : CH_{n-1}(F(X)) \to CH_r(X)
\]

is surjective except \( r = \dim X - 1 \).

It was a question whether any one-cycle on a smooth cubic hypersurface \( X \) is generated, modulo the rational equivalence, by lines, which is equivalent to the

\[
\Xi
\]

Assume that \( \geq \)

\[
X
\]

folds \([32]\) and related variants \([15]\). When \( X \) is a smooth complex cubic threefold, Beauville and Donagi \([3]\) showed that \( F(X) \) is a hyperkähler manifold and \( (2) \) is an isomorphism, which is fundamental in describing the Bogomolov-Beauville-Fujiki form on \( H^2(F(X), \mathbb{Z}) \). More generally, Shimada asserted that \( (2) \) is surjective for general Fano complete intersections \([25]\).

Very few examples exist where the cylinder maps of higher dimensional cycles are known to be surjective. Mboro \([17]\) established this result for 2-cycles on a cubic hypersurface of dimension \( \geq 5 \). Additional examples can be found in Lewis \([16]\).

Our proof of Theorem \(0.1\) follows Shen’s method in \([22, 23]\) where he established key relations among one-cycles on a cubic hypersurface. These relations allow us to deduce the surjectivity of the cylinder map for \( r = 1 \). Moreover, this method applies to fields that are not necessarily algebraically closed. For higher dimensional cycles, we obtain similar relations in the following form.

**Proposition 0.2** (\(=\)Proposition \(1.1\)). Let \( X \subset \mathbb{P}^{n+1}_k \) be a smooth cubic hypersurface over a field \( k \), and let \( F(X) \) be the variety of lines. Denote by \( h_X \in CH^1(X) \) the hyperplane section class. Assume \( X \) has an one-cycle of degree 1 defined over \( k \).

Then for a given algebraic cycle \( \Gamma \in CH_r(X) \) of dimension \( r > 1 \) with degree \( e = h_X^r \cdot \Gamma \), there exists two \((r-1)\)-cycles \( \gamma_1, \gamma_2 \in CH_{r-1}(F(X)) \) such that

\[
2\Gamma + q_* p^* \gamma_1 \in \mathbb{Z} \cdot h_X^{n-r} - \gamma_2 \in \mathbb{Z} \cdot h_X^{n-r}.
\]

These relations imply that any \( r \)-cycle on \( X \) lies in the image of the cylinder map modulo a multiple of the class \( h_X^{n-r} \). If \( X \) contains a \( k \)-line, we can further show that \( h_X^{n-r} \) is also in the image of the cylinder map, see Lemma \(2.2\). It is noteworthy that Mboro has presented the formula \(4\) when \( \Gamma \) is a smooth closed subvariety in \( X \), see \([17]\) Thm. 0.8.

The proof of Proposition \(0.2\) is based on the birational geometry of the Hilbert scheme of two points on a cubic hypersurface, developed by Galkin-Shinder \([11]\) and Voisin \([31]\). We will review this in Section \(1\).

In Section \(5\) we apply Theorem \(0.1\) to prove the integral Hodge conjecture and integral Tate conjecture for the variety of lines on a smooth cubic fourfold.
The integral Hodge conjecture asks whether the Hodge classes on a given smooth complex projective variety are algebraic with the integral coefficients. Unlike the well-known Hodge conjecture, examples [12, 28, 29] and counterexamples [2, 4, 14, 27] both exist for the integral Hodge conjecture, making it an interesting question to determine whether the conjecture holds for a specific variety.

Let \( X \) be a smooth complex cubic fourfold. Then \( F(X) \) is a hyperkähler manifold of \( K3^{(2)} \)-type. Mongardi and Ottem [19] recently affirmed the integral Hodge conjecture for curve classes on hyperkähler varieties of \( K3 \)-type and the generalized Kummer type. Using Theorem 0.1 and the integral Hodge conjecture for cubic fourfolds by Voisin [30], we provide an alternative proof for the specific hyperkähler fourfold \( F(X) \).

**Corollary 0.3 (=Theorem 3.1).** Let \( X \) be a smooth complex cubic fourfold. The integral Hodge conjecture holds for curve classes on the hyperkähler variety \( F(X) \).

The arithmetic analogue of the Hodge conjecture is the Tate conjecture. Suppose \( V \) is a smooth proper variety over a finitely generated field \( k \). Let \( \bar{k} \) be the separable closure of \( k \) and let \( G := \text{Gal}(\bar{k}/k) \) be the absolute Galois group. Fix a prime \( \ell \) that is invertible in \( k \). The integral Tate conjecture for one-cycles on \( V \) states the following

**Conjecture 0.4.** The cycle class map

\[
\text{cl}_V : \text{CH}_1(V_k) \otimes \mathbb{Z}_\ell \to \lim_{\rightarrow} \text{H}^{2\text{dim}V-2}_\text{et}(V_k, \mathbb{Z}_\ell(i))^G_k.
\]

is surjective, where the direct limit is taken over all immediate finite field extensions \( k \subset k' \subset \bar{k} \).

This conjecture, formulated by Schoen [21], differs from the “naive” integral Tate conjecture, where algebraic cycles are defined over the initial field \( k \). In particular, if \( k \) is a finite field, Schoen confirmed the conjecture under the assumption of the classical Tate conjecture for surfaces over any finite extension of \( k \), see [21, Thm. 0.5].

We verify Schoen’s integral Tate conjecture for variety of lines on a smooth cubic fourfold.

**Corollary 0.5 (=Theorem 3.3).** Let \( k \) be a finitely generated field of characteristic different from 2 and 3. Let \( X \) be a smooth cubic fourfold over \( k \). Then the cycle class map \( (5) \) for one-cycles on \( F(X) \) is surjective.

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1. **The Hilbert square of a cubic hypersurface**

Let \( k \) be a field. Let \( X \subset \mathbb{P}^{n+1}_k \) be a smooth cubic hypersurface, and let \( F(X) \) be the variety of lines on \( X \). Consider the universal \( \mathbb{P}^1 \)-bundle

\[
P : = \{(\ell, x) \in F(X) \times X \mid x \in \ell \subset X\}
\]

with two projections \( p : P \to F(X) \) and \( q : P \to X \), and the induced cylinder homomorphism

\[
P_* = q_*p^* : = \text{CH}_{r-1}(F(X)) \to \text{CH}_r(X), \ r \geq 1.
\]

We aim to prove the following two formulae.
Proposition 1.1. Let $h_X \in CH^1(X)$ be the hyperplane section class. Suppose that $X$ has an one-cycle of degree 1. Then for a given algebraic cycle $\Gamma \in CH_r(X)$ of dimension $r > 1$ and degree $e = h_X^n \cdot \Gamma$, there exists two $(r - 1)$-cycles $\gamma_1, \gamma_2 \in CH_{r-1}(F(X))$ satisfying

\begin{align}
2\Gamma + q_*p^*\gamma_1 &\in Z \cdot h_X^{n-r}; \\
(2e - 3)\Gamma + q_*p^*\gamma_2 &\in Z \cdot h_X^{n-r}.
\end{align}

Let $X^{[2]}$ be the Hilbert scheme of two points on $X$, and let

$$P_X : = \{(\ell, x) \in \mathbb{G}(1, n + 1) \times X | x \in \ell \subset \mathbb{P}^{n+1}\}.$$ 

be the incidence variety of lines in $\mathbb{P}^{n+1}$ meeting with $X$. Galkin and Shinder \cite{11} constructed a birational map

$$\Phi : X^{[2]} \dasharrow P_X$$

as follows: let $\tau \in X^{[2]}$ represent two distinct points in $X$ or a tangent vector supported on a closed point in $X$. If the line $\ell_\tau$ in $\mathbb{P}^{n+1}$ generated by $\tau$ is not contained in $X$, one defines $\Phi(\tau) = (\ell_\tau, z) \in P_X$ where $z$ is the unique residue point of the intersection $\ell_\tau \cap X$.

Apparently $\Phi$ is not defined on the set of points $\tau$ such that $\ell_\tau \subset X$. Let us denote $P_2$ the indeterminacy of $\Phi$. We can view $P_2$ as the relative symmetric product of the $\mathbb{P}^1$-bundle $p : P \to F(X)$. Then $P_2$ is a $\mathbb{P}^2$-bundle over $F(X)$ because the fiber over any $[\ell] \in F(X)$ is the symmetric product $[\ell]^{(2)} \cong \mathbb{P}^2$. In \cite{31} Voisin characterized a resoultion of the birational map $\Phi$.

Proposition 1.2. \cite{31} Proposition 2.9

1. The birational map $\Phi$ can be resolved by the blowing up $\pi_1 : X^{[2]} \to X^{[2]}$ along the smooth center $P_2 \subset X^{[2]}$.

2. The induced map $\tilde{\Phi} : \wt{X}^{[2]} \to P_X$ identifies $\tilde{\Phi}$ with the blow up $\wt{P}_X$ of $P_X$ along the smooth center $P \subset P_X$.

3. The exceptional divisors of the blow ups $\wt{X}^{[2]}$ and $\wt{P}_X$ are identified via the isomorphism $X^{[2]} \cong P_X$.

For the later use, we collect the materials of the resolution in the following diagram.

$$
\begin{array}{ccc}
X^{[2]} & \xrightarrow{\phi} & P_X \\
\downarrow \pi_2 & & \downarrow i_X \\
\wt{X}^{[2]} & \xrightarrow{\tilde{\Phi}} & P_X \\
\downarrow \pi_1 & & \downarrow \phi \\
\mathcal{E} & \xrightarrow{\phi} & P_X \\
\downarrow \pi & & \downarrow \pi_X, \\
P_2 & \xleftarrow{j_*} & X^{[2]},
\end{array}
$$

where $\mathcal{E}$ is the exceptional divisor.

Lemma 1.3. For any algebraic cycle $\Xi \in CH_k(X^{[2]})$, there is a $(k - 2)$-cycle $\gamma \in CH_{k-2}(F(X))$ such that

$$\pi_X^*\tilde{\Phi}_*(\mathcal{E} \cdot \tau^*\Xi) = q_*p^*\gamma$$

in $CH_{k-1}(X)$.

Proof. Note that $\mathcal{E} \cdot \tau^*\Xi = j_*j^*\tau^*\Xi$. The diagram (8) implies that

$$\pi_X^*\tilde{\Phi}_*(\mathcal{E} \cdot \tau^*\Xi) = \pi_X^*\tilde{\Phi}_*j_*j^*\tau^*\Xi = \pi_X^*i_1^*\pi_2^*\tau^*\Xi = q_*\pi_1^*\pi_2^*i_2^*\Xi = q_*p^*\gamma.$$
The projection $\pi_1 : E \rightarrow P$ is a $\mathbb{P}^2$-bundle since $\text{codim}(P, P_X) = 3$. Moreover, the exceptional divisor fits into the cartesian diagram

$$\begin{array}{ccc}
E & \xrightarrow{\pi_1} & P \\
\downarrow{\pi_2} & & \downarrow{p} \\
P_2 & \xrightarrow{\pi_P} & F(X).
\end{array}$$

Hence $\pi_1 \pi_2^* i_2^* \Xi = p^* \pi_F i_2^* \Xi$. Denote by $\gamma$ the cycle $\pi_F i_2^* \Xi \in \text{CH}_{k-2}(F(X))$. It follows that

$$\pi_{X*} \Phi_*(E \cdot \tau^* \Xi) = q^* \pi_{P*} i_2^* \Xi.$$

\[\square\]

It is crucial to describe the divisor class of $E \in \text{Pic}(\widetilde{X^{[2]}})$. For this purpose, we shall consider the natural morphism

$$\varphi : X^{[2]} \rightarrow G(1, n+1),$$

which assigns to $\tau \in X^{[2]}$ the generating line $\ell_\tau \subset \mathbb{P}^{n+1}$. Denote by $\mathcal{U}$ (resp. $\mathbb{P}(\mathcal{U})$) the tautological rank 2 subbundle (resp. $\mathbb{P}^1$-bundle) over $G(1, n+1)$. The pullback $Q := \mathbb{P}(\varphi^* \mathcal{U})$ is a $\mathbb{P}^1$-bundle over $X^{[2]}$. Consider the composition

$$\alpha : Q \rightarrow \mathbb{P}(\mathcal{U}) \xrightarrow{\pi_P} \mathbb{P}^{n+1}$$

where $\pi_P$ is the natural projection. In the proof of Proposition 1.2, Voisin showed that the divisor $\alpha^{-1}(X)$ in $Q$ consists of two components $\widetilde{X}^{[2]}$ and the blow up $\widetilde{X} \times X$ of $X \times X$ along the diagonal $\Delta_X$. Let

$$\rho : \widetilde{X} \times X \rightarrow X \times X$$

denote the blowing up, and $E_{\Delta,X}$ denote the exceptional divisor. The involution $(x, y) \mapsto (y, x)$ on $X \times X$ induces a double covering

$$\sigma : \widetilde{X} \times X \rightarrow X^{[2]}.$$
technical computations. Let us set the following cartesian diagram to track notations in the coming proofs

\[
\begin{array}{ccc}
\widetilde{X} \times \widetilde{X} & \xrightarrow{\Psi \cup \Phi} & P_X \\
\downarrow & & \downarrow \pi_X' \downarrow i_X \\
\mathcal{Q} & \xrightarrow{\pi_F} & \mathbb{P}^{n+1} \\
\downarrow & & \\
X^{[2]} & \xrightarrow{\phi} & \mathbb{G}(1,n+1).
\end{array}
\]

(11)

The map \( \Psi : \widetilde{X} \times \widetilde{X} \to P_X \) given by \( \Psi(x,y) = (x, \ell_{xy}) \) fits into the commutative diagram

\[
\begin{array}{ccc}
X^{[2]} & \xrightarrow{\sigma} & \widetilde{X} \times \widetilde{X} \\
\downarrow \rho & & \downarrow \pi_X \\
X \times X & \xrightarrow{p_1} & X,
\end{array}
\]

(12)

where \( p_1 \) is the projection to the first factor.

**Lemma 1.5.** With the notations in diagrams (8), (11) and (12). Given any algebraic cycle \( Z \in \text{CH}^i(X^{[2]}) \), we have

\[
\pi_X \circ \Phi_* \tau^* Z + \pi_X \circ \Psi_* \sigma^* Z = i_X' \circ \pi_{P_*} \circ \pi_{G^*} \circ \varphi_* Z.
\]

(13)

**Proof.** Regard \( \tilde{X}^{[2]} \) and \( \widetilde{X} \times \widetilde{X} \) as cycle correspondences on \( X^{[2]} \times P_X \). As seen from diagrams (8) and (12), the operator \( \Phi_* \circ \tau^* + \Psi_* \circ \sigma^* \) on the Chow groups is induced by the correspondence \( \tilde{X}^{[2]} + \widetilde{X} \times \widetilde{X} \). It follows from the cartesian diagram (11) that

\[
\Phi_* \circ \tau^* + \Psi_* \circ \sigma^* = i_X' \circ \pi_{G^*} \circ \varphi_*.
\]

The assertion of the lemma follows by composing the pushforward \( \pi_X \) on both sides and \( i_X' \circ \pi_{P_*} = \pi_{X^2} \circ i^* \).

Suppose that \( \eta \) is an one-cycle on \( X \), and \( \Gamma \subset X \) is a closed subvariety of dimension \( r \). Then \( [\Gamma] \otimes \eta + \eta \otimes [\Gamma] \) and \( [\Gamma \times \Gamma] \) are invariant cycles on \( X \times X \) under the involution \( (x,y) \mapsto (y,x) \). By [31, Corollary 2.4] there exist two cycles \( \Sigma_1 \) and \( \Sigma_2 \) of \( X^{[2]} \) satisfying

\[
\rho_* \sigma^* \Sigma_1 = [\Gamma] \otimes \eta + \eta \otimes [\Gamma], \quad \rho_* \sigma^* \Sigma_2 = [\Gamma \times \Gamma].
\]

(14)

More precisely, we set

\[
\Sigma_1 : = \sigma_* \rho^* ([\Gamma] \otimes \eta) \in \text{CH}_{r+1}(X^{[2]}) \\
\Sigma_2 : = [\sigma(\Gamma \times \Gamma)] \in \text{CH}_r(X^{[2]})
\]

where \( \Gamma \times \Gamma \) is the strict transform of \( \Gamma \times \Gamma \) in the blow up \( \widetilde{X} \times \widetilde{X} \). We remark that \( [\sigma(\Gamma \times \Gamma)] \) indicates the cycle of the closed image rather than the pushforward \( \sigma_* [\Gamma \times \Gamma] \). The latter is twice of the former.

Let us check the equations (14). It is direct to see

\[
\rho_* \sigma^* \Sigma_1 = \rho_* \sigma_* \rho^* ([\Gamma] \otimes \eta) = \rho_* \rho^* ([\Gamma] \otimes \eta + \eta \otimes [\Gamma]) = [\Gamma] \otimes \eta + \eta \otimes [\Gamma].
\]

Due to the flatness of \( \sigma \) we have \( \sigma^* | [\sigma(\Gamma \times \Gamma)] = [\sigma^{-1}(\sigma(\Gamma \times \Gamma))] = [\Gamma \times \Gamma] \). Hence

\[
\rho_* \sigma^* \Sigma_2 = \rho_* [\Gamma \times \Gamma] = [\Gamma \times \Gamma].
\]
The next lemma includes necessary computations for the proof of the Proposition 1.1.

Lemma 1.6. Let $X$ be a smooth cubic hypersurface, $h_X$ be the hyperplane section class, and $\delta$ be the half diagonal class on $X^{[2]}$. Let $\Gamma \subset X$ be a closed subvariety of dimension $r > 1$ with degree $e := \Gamma \cdot h_X^{r-1}$. Consider the algebraic cycles $\Sigma_1$ and $\Sigma_2$ associated to $\Gamma$ in (14). Suppose that the degree of the one-cycle $\eta$ is 1.

1. With the notations in diagram (12), we show
   \[ \pi_X_* \Psi_* \sigma^*(\Sigma_1) = 0, \]
   \[ \pi_X_* \Psi_* \sigma^*(h_X \otimes 1 \cdot \Sigma_1) = \Gamma, \]
   \[ \pi_X_* \Psi_* \sigma^*(\delta \cdot \Sigma_1) = 0. \]

2. With the notations in diagram (12), we show
   \[ \pi_X_* \Psi_* \sigma^*(\Sigma_2 \cdot (h_X \otimes 1)^k \cdot \delta^l) = \begin{cases} 0, & \forall 0 \leq k, l \leq r - 1, \\ e \cdot \Gamma, & k = r, l = 0, \\ (-1)^{r+1} \Gamma, & k = 0, l = r. \end{cases} \]

Proof. By the diagram (12) we have $\pi_X_* \Psi_* = p_{1*} \rho_*$. It follows from (14) that
   \[ \pi_X_* \Psi_* \sigma^*(\Sigma_1) = p_{1*} \rho_* \sigma^*(\Sigma_1) = p_{1*}(\Gamma \otimes \eta \otimes \Gamma) = 0. \]

Recall that $\sigma^*(\alpha \otimes \beta) = \sigma^* \sigma_* \rho^*(\alpha \otimes \beta) = \rho^*(\alpha \otimes \beta \otimes \alpha)$. Therefore
   \[ \pi_X_* \Psi_* \sigma^*(h_X \otimes 1 \cdot \Sigma_1) = p_{1*} \rho_* (\sigma^*(h_X \otimes 1) \cdot \sigma^*(\Sigma_1)) = p_{1*} \rho_* (h_X \otimes 1 + 1 \otimes h_X) \cdot \rho_\sigma^*(\Sigma_1).
   
   (projection formula) \]
   \[ = p_{1*} ((h_X \otimes 1 + 1 \otimes h_X) \cdot \rho_\sigma^*(\Sigma_1)). \]

Note that $\dim(h_X \cdot \Gamma) > 0$ since $\dim \Gamma = r > 1$. Hence the last pushforward equals to $\deg(h_X \cdot \eta) \cdot \Gamma = \Gamma$.

To prove $\pi_X_* \Psi_* \sigma^*(\delta \cdot \Sigma_1) = 0$ we shall use the diagram

\[ E_{\Delta, X} \xrightarrow{j_E} X \times X \xrightarrow{\sigma} X^{[2]} \]
\[ \xrightarrow{\pi} X \xrightarrow{\iota_{\Delta}} X \times X \]

where $\iota_{\Delta}$ is the diagonal embedding. Then we have

\[ \pi_X_* \Psi_* \sigma^*(\delta \cdot \Sigma_1) = \pi_X_* \Psi_* (E_{\Delta, X} \cdot \sigma^* \Sigma_1) = p_{1*} \rho_* (j_E j_E^* \sigma^* \Sigma_1) \]

It follows from $\rho \circ j_E = \iota_{\Delta} \circ \pi_{\Delta}$ and $p_{1} \circ \iota_{\Delta} = \text{id}_X$ that

\[ p_{1*} \rho_* (j_E j_E^* \sigma^* \Sigma_1) = \pi_{\Delta, X} j_E j_E^* \sigma^* \Sigma_1 \]
\[ = \pi_{\Delta, X} j_E^* \rho^*(\Gamma \otimes \eta \otimes \Gamma) \]
\[ = \pi_{\Delta, X} \pi_{\Delta}^* \iota_{\Delta}^* \sigma^* (\Gamma \otimes \eta \otimes \Gamma) \]
\[ = 2 \pi_{\Delta, X} \pi_{\Delta}^* (\Gamma \cdot \eta). \]

Note that $\pi_{\Delta}: E_{\Delta, X} \to X$ is a projective bundle. Hence $\pi_{\Delta, X} \pi_{\Delta}^* = 0$. Therefore assertion (1) is complete.

For the assertion (2), we first note
\[
\sigma^*(h_X \otimes 1)^k \equiv \rho^*(h_X \otimes 1 + 1 \otimes h_X)^k = \sum_{s=0}^{k} \binom{k}{s} \rho^*(h_X^{s} \otimes h_X^{k-s}).
\]
Recall that the cycle class $\sigma^*\Sigma_2$ is represented by $[\Gamma \times \Gamma]$. Similar argument as above yields to

$$\pi_X \Psi_*\sigma^*(\Sigma_2 \cdot (h_X \otimes 1)^k \cdot \delta^l) = p_{1*}(\rho_*([\Gamma \times \Gamma] \cdot E_{\Delta,X}^l) \cdot \sum_{s=0}^k \binom{k}{s} h_X^s \otimes h_X^{k-s}).$$

Let us carry out computations case by case.

- Suppose that $l = 0, k \leq r$. It follows from $\rho_*\sigma^*\Sigma_2 = [\Gamma \times \Gamma]$ that
  $$\pi_X \Psi_*\sigma^*(\Sigma_2 \cdot (h_X \otimes 1)^k) = p_{1*}(\rho_*([\Gamma \times \Gamma] \cdot E_{\Delta,X}^l) \cdot \sum_{s=0}^k \binom{k}{s} h_X^s \otimes h_X^{k-s})$$
  $$= \sum_{s=0}^k \binom{k}{s} p_{1*}(h_X^s \cdot \Gamma \otimes h_X^{k-s} \cdot \Gamma)$$
  $$= \begin{cases} 0, & k < r; \\ \deg \Gamma \cdot \Gamma, & k = r. \end{cases}$$

We see $p_{1*}(h_X^s \cdot \Gamma \otimes h_X^{k-s} \cdot \Gamma)$ is nontrivial if and only if $k = r$ and $s = 0$ since the dimension of $\Gamma$ is $r$.

- Suppose that $1 \leq l \leq r - 1$. We claim that $\rho_*([\Gamma \times \Gamma] \cdot E_{\Delta,X}^l) = 0$.

  Consider the excess normal bundle $Q$ of the exceptional divisor $E_{\Delta,X}$. It is the quotient bundle given by the exact sequence

$$0 \to \mathcal{O}_E(-1) \to \pi^*_X N_{X/X} \to Q \to 0 \tag{16}$$

where $\mathcal{O}_E(1)$ is tautological line bundle. Fulton’s blow-up formula [10] Thm. 6.7 associated with the blow-up diagram [15] presents the cycle class of the strict transform $\Gamma \times \Gamma$ as follows

$$[\Gamma \times \Gamma] = \rho^*([\Gamma \times \Gamma]) - j_{E*}\{c(Q) \cap \pi\Delta s(\Gamma, \Gamma \times \Gamma)\}_{2r}$$

where $s(\Gamma, \Gamma \times \Gamma)$ is the Segre class of the closed subvariety $\Gamma \subset \Gamma \times \Gamma$, and the notation $\{a\}_{2r}$ indicates the $2r$-th dimensional part of a given total class $a \in \bigoplus_{k \geq 0} \text{CH}_k$. Using the blow-up formula, the projection formulae for the morphism $\rho$ and $j_E$, and $\pi_\Delta \circ \iota_\Delta = \rho \circ j_E$, the class $\rho_*([\Gamma \times \Gamma] \cdot E_{\Delta,X}^l)$ equals to

$$[\Gamma \times \Gamma] \cdot \rho_* E_{\Delta,X}^l - \iota_\Delta \pi^*_\Delta,\{c(Q) \cap \pi\Delta s(\Gamma, \Gamma \times \Gamma)\}_{2r} \cdot j_E^l E_{\Delta,X}^l.$$

For the intersection class $E_{\Delta,X}^l$ we have

$$\rho_* E_{\Delta,X}^l = \rho_* j_E j_E^l E_{\Delta,X}^{l-1} = \iota_\Delta \pi^*_\Delta, c_1(\mathcal{O}_E(-1))^{l-1}. $$

The class $\pi_\Delta c_1(\mathcal{O}_E(-1))^{l-1}$ is zero, because $l$ is less than the rank of the normal bundle $N_{X/X \times X}$.

We simply denote by $s_i$ the $i$-th Segre class of $s(\Gamma, \Gamma \times \Gamma)$. Then the $2r$-th dimensional part of the total class $c(Q) \cdot \pi^*_\Delta s(\Gamma, \Gamma \times \Gamma)$ is

$$\sum_{i + t = n - r - 1} c_t(Q) \cdot \pi^*_\Delta s_i \in \text{CH}_{2r}(E_{\Delta,X}).$$

By the projection formula $\pi^*_\Delta(c(Q) \cdot \pi^*_\Delta s_i \cdot j_E^l E_{\Delta,X})$ is equal to

$$s_i \cdot \pi^*_\Delta(c(Q) \cdot c_1(\mathcal{O}_E(-1))^l).$$

Denote by $\xi$ the first Chern class $c_1(\mathcal{O}_E(1))$, and by $N$ the normal bundle $N_{X/X \times X}$. Using the exact sequence [10], the total Chern class of the excess
normal bundle $Q$ equals to

$$c(Q) = \frac{\pi^*c(N)}{c(O_E(-1))} = \pi^*c(N) \cdot \sum_{i \geq 0} \xi^i,$$

which implies that the $t$-th term $c_t(Q) = \sum_{j=0}^t \pi^*_\Delta c_j(N) \cdot \xi^{t-j}$. As a consequence, we have

$$\pi^*_{\Delta*}(c_t(Q) c_1(O_E(-1)^t)) = (-1)^t \sum_{j=0}^t c_j(N) \cdot \pi^*_\Delta \xi^{t+j}.$$

Note that $t \leq n - r - 1$ and $l \leq r - 1$. Hence we have that $l + j < n - 1$ for all $0 \leq j \leq l$, thus $\pi^*_\Delta \xi^{t+j} = 0$.

- Suppose that $k = 0, l = r$ and $r < n$. By the same reason as above, we compute the following cycle

$$[\Gamma \times \Gamma] \cdot \rho^* \xi^* \delta - \iota_{\Delta*} \pi^*_\Delta \{(c(Q) \cap \pi^* N(\Gamma, \Gamma \times \Gamma))_{2r} \cdot j^*_E \xi^* \delta \}.$$ 

Since $r$ is less than the rank of $N_{X/\overline{X}}$ we obtain $\rho^* \xi^* \delta \cdot \pi^*_\Delta = 0$. For the second term

$$\sum_{i + t = n - r - 1} \iota_{\Delta*} (s_i \cdot \pi^*_\Delta (c_t(Q) c_1(O_E(-1)^t)))$$

we have shown via (17) that $\pi^*_\Delta \xi^{r+t-j}$ is nonzero only if $j = 0$, $t = n - r - 1$, and $i = 0$. Recall that the 0-th Segre class $s_0$ is $[\Gamma]$. Hence

$$\pi^*_\Delta \pi^*_\Sigma \sigma^*(\Sigma_2 \cdot \delta^*) = p_{1*} \xi^* \delta \cdot c_0(N) \cdot \pi^*_\Delta \pi^*_1 c_1(O_E(-1)^n-1))$$

$$= (-1)^{r+t} s_0 \cap [X]$$

$$= (-1)^{r+1} [\Gamma].$$

When $k = 0, l = r = n$, the variety $\Gamma$ is the total space $X$. Then the cycle $\Sigma_2$ is total class $[X^{[2]}]$, and $\sigma^* \Sigma_2 = [\Gamma \times X]$. It is straightforward to see that

$$\pi^*_X \pi^*_\Sigma \sigma^*(\Sigma_2 \cdot \delta^*) = p_{1*} \rho_*(\Gamma \times X) \cdot \xi^* \delta \cdot E_{\Delta*}^n$$

$$= \pi^*_\Delta \pi^*_\Sigma \sigma^*(\Sigma_2 \cdot \delta^*)$$

$$= \pi^*_\Delta \pi^*_\Sigma \sigma^*(\Sigma_2 \cdot \delta^*)$$

$$= (-1)^{n+1} [X].$$

\[\square\]

Remark 1.7. Over a non-closed field, a cubic hypersurface not necessarily contains one-cycles of degree 1, see [9].

2. Proof of surjectivity of the cylinder maps

Proof of Proposition [14] Suppose that $\Gamma \subset X$ is an irreducible closed subvariety. Let $\Sigma_1$ be the algebraic cycle given in [14]. By Lemma [13]

$$\pi^*_X \pi^*_\Sigma (E \cdot \tau^* \Sigma_1) = q^* \rho^* \gamma_1,$$

where $\gamma_1 = \pi^*_E \pi^*_1 \Sigma_1 \in \text{CH}_{n-1}(F(X))$. Recall from Lemma [14] that the exceptional divisor $E = h \xi^* \delta \Sigma_1 \in \text{CH}_{n-1}(F(X))$. Hence we have

$$\pi^*_X \pi^*_\Sigma (E \cdot \tau^* \Sigma_1) = -\pi^*_X \pi^*_\Sigma (h \xi^* \delta \Sigma_1) + \pi^*_X \pi^*_\Sigma (\tau^* (2h \Sigma_1 \otimes 1 - 3\delta) \cdot \tau^* \Sigma_1).$$
Note that $h_Q = \alpha^* h$. We see from the commutative diagram (11) that

$$\pi X_* \Phi_*(h_Q|_{X^{(2)}} \cdot \tau^* \Sigma_1) = h_X \cdot \pi X_* \Phi_* \tau^* \Sigma_1.$$ 

It follows from Lemma 1.5 and Lemma 1.6 that

$$\pi X_* \Phi_* \tau^* \Sigma_1 = i_X^* \pi_p \pi G^* \varphi_* \Sigma_1 - \pi_* \Psi_* \sigma^* \Sigma_1$$

$$= i_X^* \pi_p \pi G^* \varphi_* \Sigma_1.$$ 

Note that $\pi_p \pi G^* \varphi_* \Sigma_1 \in \text{CH}^{n-r-1}(\mathbb{P}^{n+1})$. Hence $\pi X_* \Phi_*(h_Q|_{X^{(2)}} \cdot \tau^* \Sigma_1)$ must be a multiple of the class $h_X^{n-r}$. With the same process we have

$$\pi X_* \Phi_* (\tau^*(2h_X \cdot 1 - 3\delta) \cdot \tau^* \Sigma_1)$$

$$= i_X^* \pi_p \pi G^* \varphi_* ((2h_X \cdot 1 - 3\delta) \cdot \Sigma_1) - \pi_* \Psi_* \sigma^* ((2h_X \cdot 1 - 3\delta) \cdot \Sigma_1)$$

$$\equiv - \pi_* \Psi_* \sigma^* ((2h_X \cdot 1 - 3\delta) \cdot \Sigma_1) \pmod Z \cdot h_X^{n-r}.$$ 

Again Lemma 1.6 yields $\pi_* \Psi_* \sigma^* ((2h_X \cdot 1 - 3\delta) \cdot \Sigma_1) = 2\Gamma$. Therefore the class $2\Gamma + q_s p^* \gamma$ is a multiple of $h_X^{n-r}$.

Let $\Sigma_2$ be the second algebraic cycle in (14). To simplify the notations, let $g$ denote the divisor class $c_1(\mathcal{U}^*) = -h_X \cdot 1 + \delta$ in Lemma 1.4. The second formula (4) will be derived by computing the following cycle

$$\pi X_* \Phi_* (E \cdot \tau^* (\Sigma_2 \cdot g^{r-1})).$$

We see $\dim(\Sigma_2 \cdot g^{r-1}) = r + 1$. Again by Lemma 1.6

(18) $$\pi X_* \Phi_* (E \cdot \tau^* (\Sigma_2 \cdot g^{r-1})) = q_s p^* \gamma_2,$$

where $\gamma_2 = \pi F_* i_* (\Sigma_2 \cdot g^{r-1}) \in \text{CH}_{r-1}(F(X))$. The same argument as above yields

$$\pi X_* \Phi_* (E \cdot \tau^* (\Sigma_2 \cdot g^{r-1})) = -h_X \cdot \pi X_* \Phi_* \tau^* (\Sigma_2 \cdot g^{r-1}) + \pi X_* \Phi_* \tau^* ((2h_X \cdot 1 - 3\delta) \cdot \Sigma_2 \cdot g^{r-1}).$$

It follows from Lemma 1.5 that

$$\pi X_* \Phi_* \tau^* (\Sigma_2 \cdot g^{r-1}) = i_X^* \pi_p \pi G^* \varphi_* (\Sigma_2 \cdot g^{r-1}) - \pi_* \Psi_* \sigma^* (\Sigma_2 \cdot g^{r-1}).$$

The first term on the right side is a multiple of the class $h_X^{n-r-1}$. The second term is equal to

$$\pi X_* \Psi_* \sigma^* (\Sigma_2 \cdot \sum_{k=0}^{r-1} \binom{r-1}{k} (-h_X \cdot 1)^k \cdot \delta^{r-k-1}).$$

This class vanishes as a result of the assertion (2) of Lemma 1.6. Therefore the class $h_X \cdot \pi X_* \Phi_* \tau^* (\Sigma_2 \cdot g^{r-1})$ is a multiple of $h_X^{n-r}$. By the same argument we can see $\pi X_* \Phi_* \tau^* ((2h_X \cdot 1 - 3\delta) \cdot \Sigma_2 \cdot g^{r-1})$ equals to

$$\pi X_* \Psi_* \sigma^* ((2h_X \cdot 1 - 3\delta) \cdot \Sigma_2 \cdot \sum_{k=0}^{r-1} \binom{r-1}{k} (-h_X \cdot 1)^k \cdot \delta^{r-k-1}),$$

modulo a multiple of $h_X^{n-r}$. The assertion (2) of Lemma 1.6 shows that the above class is

$$(-1)^{r-1} 2 \deg \Gamma \cdot \Gamma + (-1)^{r-1} \gamma_2.$$ 

To conclude the formula (7) we may replace $\gamma_2$ by $(-1)^{r-1} \gamma_2$. Then the above computation implies that

$$(2 \deg \Gamma - 3) \Gamma \equiv q_s p^* \gamma_2 \pmod Z \cdot h_X^{n-r}.$$ 

Through the linear combination, one easily extend relations (6) and (7) to cycles.

The following corollary is immediate.
Corollary 2.1. Let \( X \subset \mathbb{P}^{n+1} \) be a smooth cubic hypersurface of dimension \( n \geq 3 \) over a field \( k \), and let \( F(X) \) be the variety of lines on \( X \). Assume that \( X \) has an one-cycle of degree one. Fix any \( r \geq 1 \), the cylinder map
\[
q_*p^* : \text{CH}_r-1(F(X)) \to \text{CH}_r(X)
\]
is surjective modulo the subgroup generated by the class \( h_X^{n-r} \).

Proof. Suppose that \( \Gamma = \sum n_i \Gamma_i \) is an algebraic cycle on \( X \) with irreducible components \( \Gamma_i \). Let \( e_i \) be the degree of each \( \Gamma_i \). Let \( \gamma_{i1} \) and \( \gamma_{i2} \) be the \((r-1)\)-cycles associated with \( \Gamma_i \) satisfying the two formulae in Proposition 1.1. It follows that
\[
\Gamma_i = q_*p^*((e_i - 1)\gamma_{i1} - \gamma_{i2}) + a_i h_X^{n-r}.
\]
for some \( a_i \in \mathbb{Z} \). Hence \( \Gamma \) is contained in the subgroup generated by \( q_*p^*\text{CH}_r-1(F(X)) \) and \( \mathbb{Z} \cdot h_X^{n-r} \).

The case for 1-cycles had been proved by Shen [23 Proposition 4.2]. \( \square \)

It remains to show that the class of hyperplane intersections \( h_X^{n-r} \) is also contained in the image of the cylinder map. Following the idea in Corollary 2.1 we shall seek for a pair of coprime integers \((a, b)\) and corresponding \((r-1)\)-cycles \( \alpha_{r-1} \) and \( \beta_{r-1} \) satisfying
\[
q_*p^*\alpha_{r-1} = ah_X^{n-r}, \quad q_*p^*\beta_{r-1} = bh_X^{n-r},
\]
which is the aim of the next Lemma.

Lemma 2.2. Let \( X \) be a smooth hypersurface of dimension \( n \geq 3 \) over a field \( k \). Let \( h_X \) be the hyperplane section class on \( X \). Assume that \( X \) contains a line defined over \( k \). Then for any \( 1 \leq r < n - 1 \), there exists cycles \( \alpha_{r-1} \) and \( \beta_{r-1} \) in \( \text{CH}_{r-1}(F(X)) \) such that
\[
q_*p^*\alpha_{r-1} = 2h_X^{n-r}, \quad q_*p^*\beta_{r-1} = 5h_X^{n-r}.
\]

Proof. We first prove the assertion when \( k \) is an algebraically closed field. Let \( x \in X \) be a closed point. Consider the closed subvariety
\[
C_x = \{ [\ell] \in F(X) \mid x \in \ell \}
\]
of lines meeting with the point \( x \). If \( x \) is a generic point in \( X \), the dimension of \( C_x \) is \( n - 3 \). Fix a coordinate \((x_0, \ldots, x_{n+1})\) of \( \mathbb{P}^{n+1} \). By projective transformation let \( x = (1: 0: \ldots: 0) \). Then the defining equation of \( X \) is of the form \( x_0^3L + x_0Q + C \) where \( L \) (resp. \( Q \) and \( C \)) is a homogeneous polynomial in \( k[x_1, \ldots, x_{n+1}] \) of degree \( 1 \) (resp. \( 2 \) and \( 3 \)). Each line in \( X \) meeting with the point \( x \) corresponds to a point in \( \mathbb{P}(x_1: \ldots: x_{n+1}) \) cut out by the equations \( L, Q, \) and \( C \). The closed subvariety \( q_*p^*C_x \) in \( X \) is simply defined by \( L = Q = 0 \). Hence we have
\[
q_*p^*[C_x] = 2h_X^2 \in \text{CH}^2(X).
\]
Consider a generic linear section \( X_{r+2} := \mathbb{P}^{r+3} \cap X \) of codimension \( n - r - 2 \) in \( X \). Define
\[
C_xX_{r+2} := \{ [\ell] \in F(X) \mid x \in \ell \subset X_{r+2} \}
\]
the closed subvariety of the lines in \( X_{r+2} \) meeting with a generic point \( x \in X_{r+2} \). The subvariety \( C_xX_{r+2} \) is thus contained in \( F(X_{r+2}) \). It follows that
\[
q_*p^*[C_xX_{r+2}] = 2i_*h_X^{r+2} = 2h_X^{n-r}
\]
with the inclusion \( i : X_{r+2} \hookrightarrow X \). Taking \( \alpha_{r-1} = [C_xX_{r+2}] \) yields to the first assertion.

For the second assertion, let \( \ell \) be a generic line contained in \( X \). We define \( S_\ell \subset F(X) \) to be the closed subvariety as the closure of the subset
\[
\{ [\ell'] \in F(X) \mid \ell' \cap \ell \neq \emptyset, \ell' \neq \ell \}
\]
of lines meeting with \( \ell \). The dimension of \( S_\ell \) is \( n - 2 \). So that \( q_\ast p^\ast [S_\ell] = a h_X \) for some integer \( a \). To determine \( a \), take \( \ell \) another generic line on \( X \), the adjointness implies that
\[
a = q_\ast p^\ast [S_\ell] \cdot \ell = [S_\ell] \cdot p_\ast q^\ast \ell = [S_\ell] \cdot [S_\ell].
\]
Thus the integer \( a \) indicates the number of lines on \( X \) meeting with two generic lines \( \ell \) and \( \ell' \). It has been proved in [22, Lemma 3.10] or [13, § 5, Lem. 1.14] that \( a = 5 \).

Again let \( X_{r+1} = \mathbb{P}^{r+2} \cap X \) be a generic linear section of codimension \( n - r - 1 \) in \( X \). Let \( \ell \) be a generic line contained in \( X_{r+1} \). Define
\[
S_{\ell,r+1} := \{ [\ell'] \in F(X) \mid \ell' \subset X_{r+1}, \ell' \cap \ell \neq \emptyset \}.
\]
The subvariety \( S_{\ell,r+1} \) is contained in \( F(X_{r+1}) \). It follows that
\[
q_\ast p^\ast [S_{\ell,r+1}] = 5 i_* h_{X_{r+1}} = 5 h_X^{n-r}, i: X_{r+1} \hookrightarrow X.
\]
Hence the second assertion holds by taking \( \beta_{r+1} = [S_{\ell,r+1}] \).

The above argument can apply to a general field \( k \) if the existences of \( k \)-points and \( k \)-lines in generic position are satisfied. For this reason, we need a result [8, p. 599] suggested by Colliot-Thélène. It says that for any non-empty Zariski open subset \( U \subset V \) of a smooth \( k \)-variety \( V \), every zero-cycle on \( V \) is rationally equivalent to a zero-cycle supported in \( U \).

Now let \( X \) contain a \( k \)-line \( L \), and \( p \in L \) be any \( k \)-point. Then \( p \) is rationally equivalent to a zero-cycle \( z \) of degree \( r \) supported in generic position on \( X \). Hence the cycle \( C_z := p, q^\ast z \) on \( F(X) \) is of dimension \( n - 3 \). Note that \( \deg z = 1 \). Then we have \( q_\ast p^\ast C_z = 2 h_X^{n-r} \). The line \( L \) gives a zero-cycle \( [L] \) on \( F(X) \). Since \( F(X) \) is smooth, \( [L] \) is rationally equivalent to a zero-cycle \( w := \sum n_i \ell_i \) where all \( \ell_i \) are supported in generic position on \( F(X) \) and the degree \( \sum n_i = 1 \). Then the cycle \( S_w := \sum n_i S_{\ell_i} \) on \( F(X) \) has dimension \( n - 2 \). Because \( \deg w = 1 \) we have \( q_\ast p^\ast S_w = 5 h_X \). To close the proof we repeat the argument in the above.

\[\square\]

**Theorem 2.3.** Let \( X \subset \mathbb{P}^{n+1}_k \) be a smooth cubic hypersurface of dimension \( n \geq 3 \) over a field \( k \), and let \( F(X) \) be the variety of lines on \( X \). Assume that \( X \) contains a \( k \)-line. For any \( 1 \leq r < n - 1 \). The cylinder map
\[
q_\ast p^\ast : \text{CH}_{r-1}(F(X)) \rightarrow \text{CH}_r(X)
\]
is surjective.

**Proof.** It immediately follows from Corollary [2.4] and Lemma [2.2].

\[\square\]

**Remark 2.4.** The result does not hold for divisor classes on cubic threefolds. It is mentioned in [13, §5, Rmk.1.22.] that for a generic cubic threefold \( X \), the Néron-Severi group \( \text{NS}(F(X)) \) is generated by the primitive class \([S_\ell] \). Since \( q_\ast p^\ast [S_\ell] = 5 h_X \) no divisor on \( F(X) \) sent to \( h_X \) via the cylinder map.

However, when \( n \geq 4 \), there exists certain \((n - 2)\)-cycle \( \theta_{n-2} \) on \( F(X) \) such that \( q_\ast p^\ast \theta_{n-2} = a h_X \) and \( a \) is coprime to 5 e.g., see [24, Lem. A.1].

3. **Integral Hodge conjectures and Tate conjectures**

**Theorem 3.1.** The integral Hodge conjecture holds for one-cycles on the variety \( F(X) \) of lines on a smooth complex cubic fourfold \( X \).

**Proof.** Suppose that \( \alpha \in H^6(F(X), \mathbb{Z}) \) is an integral Hodge class of type \((3,3)\). Then \( q_\ast p^\ast \alpha \in H^4(X, \mathbb{Z}) \) is an integral Hodge class of type \((2,2)\). The integral Hodge conjecture for a smooth cubic 4-fold is proved by Voisin, see [30]. Hence there exists a 2-cycle \( \gamma \in \text{CH}_2(X) \) such that the cohomology class \([\gamma] = q_\ast p^\ast \alpha \). As
a result of Theorem 2.3, there exists 1-cycle $\Gamma \in \text{CH}_1(F(X))$ such that $q_*p^*\Gamma = \gamma$. By the commutative diagram of the cylinder homomorphisms and cycle class maps
\[
\begin{array}{c}
\text{CH}_1(F(X)) \\
\downarrow \\
H^6(F(X), \mathbb{Z}) \\
\downarrow \\
H^4(X, \mathbb{Z}),
\end{array}
\]
we see $q_*p^*(\lfloor \Gamma \rfloor - \alpha) = 0$. The cylinder map $q_*p^*$ on the cohomology groups is an isomorphism since it is dual to the Abel-Jacobi isomorphism
\[p_*q^*: H^4(X, \mathbb{Z}) \to H^2(F(X), \mathbb{Z})\]
by Beauville and Donagi [3]. Therefore $[\Gamma] = \alpha$ is an algebraic class. \hfill \Box

Let $V$ be a smooth proper variety over a field $k$ which is finitely generated over its prime field. Let $\overline{k}$ be the separable closure of $k$, and $G$ be the absolute Galois group $\text{Gal}(\overline{k}/k)$ of $k$. Fix a prime number $\ell$ which is invertible in $k$. The Tate conjecture states that any $G$-invariant class in the $\ell$-adic cohomology space $H^2(V_{\overline{k}}, \mathbb{Q}_\ell(i))$ is spanned by the algebraic classes of codimension $i$ on $V_{\overline{k}}$.

For an intermediate field $k \subset k' \subset \overline{k}$ of finite degree over $k$, the Galois group $G_{k'} := \text{Gal}(k'/k)$ is an open subgroup of the profinite group $G$. The group $G_{k'}$ acts on the $\mathbb{Z}_\ell$-module $H^2_{et}(V_{k'}, \mathbb{Z}_\ell(i))$. For codimension $i$ algebraic cycles on $V_{k'}$, there is the cycle class map
\[c_{V'}^i: \text{CH}^i(V_{k'}) \otimes \mathbb{Z}_\ell \to \lim_{\text{et}} \lim_{G_{k'}} H^2_{et}(V_{k'}, \mathbb{Z}_\ell(i))^{G_{k'}}\]
where the direct limit is over all intermediate fields $k \subset k' \subset \overline{k}$ of finite degrees over $k$. Schoen proposed the following integral analog of the Tate conjecture for one-cycles.

**Conjecture 3.2.** The cycle class map $c_{V'}^i$ is surjective for $i = \dim V - 1$.

The Tate conjecture is true for divisors on the variety $F(X)$ of lines on a smooth cubic fourfold $X$ over a number field or a finite field of characteristic $p \geq 5$, see [1] and [5]. Thus the Tate conjecture holds for one-cycles on $F(X)$ by the hard Lefschetz’s theorem. Let us prove the integral Tate conjecture for the one-cycles on $F(X)$.

**Theorem 3.3.** Let $k$ be a finitely generated field of characteristic different from 2 and 3. Denote by $F$ the variety $F(X)$ of lines on a smooth cubic fourfold $X$ defined over $k$. Fix a prime $\ell$ invertible in $k$, the cycle class map
\[c_{F}^3: \text{CH}_1(F_{\overline{k}}) \otimes \mathbb{Z}_\ell \to \lim_{\text{et}} H^6_{et}(F_{k'}, \mathbb{Z}_\ell(3))^{G_{k'}}\]
is surjective, where $U$ runs over all open subgroups of $\text{Gal}(\overline{k}/k)$.

To imitate the proof of Theorem 3.1 we need the following lemma.

**Lemma 3.4.** Let $k$ be a number field or a finite field. Let $U$ be any open subgroup of the Galois group $\text{Gal}(k/k)$. Fix a prime $\ell$ invertible in $k$. Then the cylinder map on the $\ell$-adic cohomology
\[H^6_{et}(F_{\overline{k}}, \mathbb{Z}_\ell(3)) \to H^4_{et}(X_{\overline{k}}, \mathbb{Z}_\ell(2))\]
is a $U$-equivariant isomorphism.

**Proof.** Let $W(k)$ be the ring of Witt vectors of a finite field $k$. It is known any smooth cubic fourfold $X$ over $k$ can lift to a family $\mathcal{X}$ of smooth cubic fourfolds over $W(k)$. Denote by $F$ the relative variety of lines of $\mathcal{X}$ over $W(k).$ The fraction
field of $W(k)$ is of characteristic zero, which can be embedded into the complex numbers $C$. Hence the base change $\mathcal{X}_C$ is a complex smooth cubic fourfold, and $\mathcal{F}_C$ is the variety of lines on $\mathcal{X}_C$.

By the smooth and proper base change and comparison theorem, we have canonical isomorphisms

$$H^2_{\text{et}}(X_{k}, \mathbb{Z}_\ell(i)) \simeq H^2_{\text{et}}(X_{C}, \mathbb{Z}_\ell(i)) \simeq H^2_{\text{et}}(X_{\mathbb{C}}^n, \mathbb{Z}(i)) \otimes \mathbb{Z}_\ell.$$

The cylinder homomorphisms over different bases are defined by the associated universal $\mathbb{P}^1$-bundles. Hence they commute with the base change and the cohomology comparison, and are $U$-equivariant. Therefore our assertion follows from the isomorphism

$$H^6(X_{\mathbb{C}}^n, \mathbb{Z}(3)) \to H^4(\mathcal{F}_C^n, \mathbb{Z}(2)).$$

Suppose that $k$ is a number field. Then $\bar{k} \subset \mathbb{C}$ both are separably closed fields. It follows from [15 §VI, Cor. 4.3.] that

$$H^2_{\text{et}}(X_{k}, \mathbb{Z}_\ell(i)) \simeq H^2_{\text{et}}(X_{C}, \mathbb{Z}_\ell(i)).$$

Then the assertion for number fields also follows from the cohomology comparison. 

The integral Tate conjecture for 2-cycles of cubic fourfolds has been proved by Charles and Pirutka.

**Theorem 3.5.** [6 Theorem 1.1] Let $k$ be finitely generated field of characteristic different from 2 and 3, and let $\bar{k}$ be the separable closure of $k$. Let $X$ be a smooth cubic fourfold over $k$. Fix a prime $\ell$ invertible in $k$, the cycle class map

$$c^2_X : \text{CH}^2(X_{\bar{k}}) \otimes \mathbb{Z}_\ell \to \lim_{U} H^4_{\text{et}}(X_{\bar{k}}, \mathbb{Z}(2))^U$$

is surjective, where the direct limit is over all open subgroups $U$ of $\text{Gal}(\bar{k}/k)$.

**Proof of Theorem** Let $k_0$ be the prime subfield of $k$. Let $R$ be the domain of finite type over $k_0$ such that its fraction field is $k$. After shrinking $\text{Spec}(R)$ to an open affine subset, we may extend the $X$ defined over $k$ to a family of smooth cubic fourfolds $\pi : X \to R$. Then the étale sheaf $R^4\pi_*\mathbb{Z}_\ell(2)$ is locally constant on $\text{Spec}(R)$. Regarding $X_{\bar{k}}$ as the geometric generic fiber of $\pi$, and $X_{\bar{k}_0}$ as the geometric fiber over a closed point in $\text{Spec}(R)$. Since $\pi$ is smooth and proper, the specialization map

$$H^4(X_{\bar{k}}, \mathbb{Z}_\ell(2)) \to H^4(X_{\bar{k}_0}, \mathbb{Z}_\ell(2))$$

is an isomorphism, see Corollary [15 §VI, Cor. 4.2.]. The specialization maps for $X$ and $F$ commute with the cylinder maps because the later are defined by the universal $\mathbb{P}^1$-bundle. Since $k_0$ is either a finite field or the field of rational numbers, it implies by Lemma [3.5] that

$$H^4_\text{et}(F_{\bar{k}}, \mathbb{Z}_\ell(3)) \to H^4(X_{\bar{k}}, \mathbb{Z}_\ell(2))$$

is an isomorphism, which is meanwhile $U$-equivariant.

Let $\alpha \in H^4_\text{et}(F_{\bar{k}}, \mathbb{Z}_\ell(3))^U$ denote a representative of any given cohomology class in the direct limit $\lim_{U} H^4_\text{et}(F_{\bar{k}}, \mathbb{Z}_\ell(3))^U$. By Theorem 3.5, there exists a 2-cycle $\Gamma \in \text{CH}^2(X_{\bar{k}}) \otimes \mathbb{Z}_\ell$ such that $c^2_X(\Gamma) = q_*p^*\alpha$ in $H^4_\text{et}(X_{\bar{k}}, \mathbb{Z}_\ell(2))^U$. By the surjectivity of the cylinder map, there exists an one-cycle $\gamma \in \text{CH}_1(X_{\bar{k}}) \otimes \mathbb{Z}_\ell$ such that $q_*p^*\gamma = \Gamma$. Since $c^2_X(\gamma) = \alpha$. 

□
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