THE FU-YAU EQUATION IN HIGHER DIMENSIONS

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Dedicated to Professor Gang Tian on the occasion of his 60th birthday

Abstract. In this paper, we prove the existence of solutions to the Fu-Yau equation on compact Kähler manifolds. As an application, we give a class of non-trivial solutions of the modified Strominger system.

1. Introduction

In 1985, Strominger proposed a new system of equations, now referred to as the Strominger system, on 3-dimensional complex manifolds [18]. This system arises from the study on supergravity in theoretical physics. Mathematically, the Strominger system can be regarded as a generalization of the Calabi equation for Ricci-flat Kähler metrics to non-Kähler spaces [22]. It is also related to Reid’s fantasy on the moduli space of Calabi-Yau threefolds [15].

Let us first recall this system. Assume that $X$ is a 3-dimensional Hermitian manifold which admits a nowhere vanishing holomorphic $(3,0)$-form $\Omega$. Let $E \rightarrow X$ be a holomorphic vector bundle with Hermitian metric $H$. The Strominger system is given by

\begin{align*}
&F_H \wedge \omega_X^2 = 0, F_H^{2,0} = F_H^{0,2} = 0; \\
&d^* \omega_X = \sqrt{-1} (\bar{\partial} - \partial) \log \|\Omega\|_{\omega_X}; \\
&\sqrt{-1} \partial \bar{\partial} \omega_X - \frac{\alpha}{4} (\text{tr} R \wedge R - \text{tr} F_H \wedge F_H) = 0,
\end{align*}

where $\omega_X$ is a Hermitian metric on $X$ with the Chern curvature $R$ and $F_H$ is the curvature of Hermitian metrics $(E, H)$. By tr, we denote the trace of Endomorphism bundle of either $E$ or $T X$.

To achieve a supersymmetry theory, both $H$ and $\omega_X$ have to satisfy (1.1) and (1.2). (1.2) is also called the dilation equation. Li and Yau observed that it is equivalent to a conformally balanced condition [12],

\[ d(\|\Omega\|_{\omega_X} \omega_X^2) = 0. \]

The least understood equation of the system is (1.3) known as the Bianchi identity, which is also related to index theory for Dirac operators (4, 21),

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the topological theory of string structures (2, 10, 17) and generalized geometry (1, 5, 9). It is an equation on 4-forms and intertwines $\omega_X$ with the curvatures $R$ and $F_H$, which is very difficult to understand in view of analysis.

We know little about the Strominger in general except for a few special spaces on which one can make use of particular structures. In 12, Li and Yau found the first irreducible smooth solution. They considered a stable holomorphic bundle $E$ of rank $r = 4, 5$ on a Calabi-Yau 3-fold $X$ and constructed a solution of the Strominger system as a perturbation of a Calabi-Yau metric on $X$ and a Hermitian-Einstein metric $H$ on $E$.

In 6, Fu and Yau constructed non-perturbative, non-Kähler solutions of the Strominger system on a toric fibration over a K3 surface constructed by Goldstein-Prokushki. Let us recall this construction. Let $\omega_1$ and $\omega_2$ be two anti-self-dual $(1, 1)$ forms on a K3 surface $(S, \omega_S)$ ($\omega_S$ is a Kähler-Ricci flat metric on $S$) with a nowhere vanishing holomorphic $(2, 0)$-form $\Omega_S$ satisfying: $[\frac{\omega_1}{2}], [\frac{\omega_2}{2}] \in H^{1,1}(S, \mathbb{Z})$. In 7, Goldstein-Prokushki constructed a toric fibration $\pi: X \to S$ which is determined by $\omega_1$, $\omega_2$ and a $(1, 0)$ form $\theta$ on $X$ such that $\Omega = \pi^*(\Omega_S) \wedge \theta$ defines a nowhere vanishing holomorphic $(3, 0)$ form on $X$. Then, for any $\varphi \in C^\infty(S)$, $(X, \omega_{\varphi})$ always satisfies (1.2), where $\omega_{\varphi} = \pi^*(e^\varphi \omega_S) + \sqrt{-1}\theta \wedge \bar{\theta}$.

Thus if $E \to X$ is a degree zero stable holomorphic vector bundle with a Hermitian-Einstein metric $H$ on $E$, $(\pi^*E, \pi^*H, X, \omega_{\varphi})$ satisfies both (1.1) and (1.2). In 5, Fu and Yau showed that (1.3) for $(\pi^*E, \pi^*H, X, \omega_{\varphi})$ is equivalent to the following equation for $\varphi$, also called the Fu-Yau equation,

$$\text{(1.4) } \sqrt{-1}\partial \bar{\partial}(e^\varphi \omega_S - \alpha e^{-\varphi} \rho) + 2\alpha \sqrt{-1}\partial \bar{\partial} \varphi \wedge \sqrt{-1}\partial \bar{\partial} \bar{\varphi} + \mu \frac{\omega_S^2}{2!} = 0,$$

where $\rho$ is a real-valued smooth $(1, 1)$-form, $\mu$ is a smooth function and $\alpha \neq 0$ is a constant called slope parameter. They further proved the existence for (1.4) in the case of $\alpha < 0$ 6 and $\alpha > 0$ 5 on Kähler surfaces, respectively.

In higher dimensions, Fu and Yau proposed a modified Strominger system for $(E, H, X, \omega_X)$,

$$F_H \wedge \omega_X^n = 0, \quad F_H^{2, 0} = F_H^{0, 2} = 0;$$

$$d(\|\Omega\|^{2(\alpha-1)}_{\omega_X^n}) = 0;$$

$$\left(\sqrt{-1}\partial \bar{\partial} \omega_X - \frac{\alpha}{4}(\text{tr}R \wedge R - \text{tr}F_H \wedge F_H)\right) \wedge \omega_X^{n-2} = 0.$$

Here $X$ is an $(n + 1)$-dimensional Hermitian manifold, equipped with a nowhere vanishing holomorphic $(n+1, 0)$ form $\Omega$. Clearly, the modified Strominger system is the same as the original Strominger system when $n = 2$. Given any Calabi-Yau manifold $M$ with a nowhere vanishing holomorphic
(n, 0) form $\Omega_M$, Goldstein-Prokushki’s construction gives rise to a toric fibration $\pi: X \to M$ as in case of K3 surfaces. Fu and Yau showed that the modified Strominger system for $(\pi^*E, \pi^*H, X, \omega_\phi)$ can be reduced to the Fu-Yau equation on $M$,

$$
\sqrt{-1} \partial \bar{\partial} (e^{\varphi} \omega - \alpha e^{-\varphi} \rho) \wedge \omega^{n-2} + n\alpha \sqrt{-1} \partial \bar{\partial} \varphi \wedge \omega^{n-2} + \frac{\mu}{n!} \omega^n = 0.
$$

(1.5)

More recently, Phong, Picard and Zhang proved the existence for (1.5) in higher dimensions when $\alpha < 0$ [14]. However, the solvability of (1.5) in higher dimensions is still open when $\alpha > 0$. The purpose of present paper is to give a complete solution in this case. Actually, we will give a unified way for (1.5) in higher dimensions in both cases $\alpha > 0$ and $\alpha < 0$, more precisely, we prove

**Theorem 1.1.** Let $(M, \omega)$ be an $n$-dimensional compact Kähler manifold. There exists a small constant $A_0 > 0$ depending only on $\alpha$, $\rho$, $\mu$ and $(M, \omega)$ such that for any positive $A \leq A_0$, there exists a smooth solution $\varphi$ of (1.5) satisfying the elliptic condition

$$
\tilde{\omega} = e^{\varphi} \omega + \alpha e^{-\varphi} \rho + 2n\alpha \sqrt{-1} \partial \bar{\partial} \varphi \in \Gamma_2(M),
$$

and the normalization condition

$$
\|e^{-\varphi}\|_{L^1} = A,
$$

(1.7)

where $\Gamma_2(M)$ is the space of 2-th convex $(1,1)$-forms (cf. Section 3).

**Remark 1.2.** We point out that if $\alpha < 0$ and $n = 2$, our normalization condition (1.7) is the same as that in [5]. However, in the case that $\alpha > 0$ and $n = 2$, Fu and Yau [6] solved (1.5) under the normalization condition $\|e^{-\varphi}\|_{L^4} = A$, which is stronger than (1.7). When $\alpha < 0$ and $n > 2$, Phong, Picard and Zhang used a different normalization condition $\|e^{\varphi}\|_{L^1} = \frac{1}{A}$. Hence, our result is also new compared to the results cited above.

As a geometric application of Theorem 1.1, we prove

**Theorem 1.3.** For any $n \geq 2$, there exists a function $\varphi \in C^\infty(M)$ such that the Fu-Yau’s reduction $(\pi^*E, \pi^*H, X, \omega_\phi)$ yields a smooth solution of the modified Strominger system.

From the view point of PDE, (1.5) can be written as a 2-nd Hessian equation of the form

$$
\sigma_2(\tilde{\omega}) = F(z, \varphi, \partial \varphi),
$$

(1.8)

where

$$
F(z, \varphi, \partial \varphi) = \frac{n(n-1)}{2} (e^{2\varphi} - 4\alpha e^{\varphi} |\partial \varphi|^2) + \frac{n(n-1)}{2} f(z, \varphi, \partial \varphi)
$$

and $f(z, \varphi, \partial \varphi)$ satisfies (cf. (3.1)),

$$
|f(z, \varphi, \partial \varphi)| \leq C(e^{-2\varphi} + e^{-\varphi} |\nabla \varphi|^2 + 1).
$$
There are many interesting works for the $k$-th complex Hessian equation of the form:

$$
\sigma_k(\omega + \sqrt{-1}\partial\bar{\partial}\varphi) = F(z).
$$

(1.9)

For examples, Hou, Ma and Wu proved the second order estimate for (1.9) \cite{11}; Combining Hou-Ma-Wu’s estimate with a blow-up argument, Dinew and Kołodziej solved (1.9) \cite{3}; Székelyhidi, and also Zhang, obtained analogous result in the Hermitian case \cite{19}, \cite{23}.

However, for the Fu-Yau equation (1.5), new difficulties arise because the right hand side $F$ of (1.8) depends on $\partial\varphi$. Moreover, (1.5) may become degenerate when $\alpha > 0$. This makes a big difference between the case $\alpha > 0$ and the case $\alpha < 0$. When $\alpha < 0$, there is no issue on non-degeneracy. However, when $\alpha > 0$, one needs to establish a non-degeneracy estimate. In dimension 2, Fu and Yau obtained such an estimate \cite{6}. Unfortunately, their arguments do not work in higher dimensions. It has been a main obstacle to solving (1.5) in higher dimensions when $\alpha > 0$.

In this paper, we find a new method for establishing the non-degeneracy estimate. This estimate is different from either Fu-Yau’s one in \cite{5, 6} or Phong-Picard-Zhang’s one in \cite{13, 14}. We regard the first and second order estimates as a whole and derive the required non-degeneracy estimate. To be more specific, assuming that

$$
|\partial\bar{\partial}\varphi|_g \leq D_0,
$$

where $D_0$ is a constant (depending only on $n$, $\alpha$, $\rho$, $\mu$ and $(M, \omega)$) to be determined later, we derive a stronger gradient estimate (independent of $D_0$) by choosing a small number $A$ in (1.7) (cf. Proposition 3.1). Then using this stronger gradient estimate, we obtain an improved estimate (cf. Proposition 4.1),

$$
|\partial\bar{\partial}\varphi|_g \leq \frac{D_0}{2}.
$$

This can be used to obtain an a prior $C^2$-estimate and consequently the non-degeneracy estimate via the continuity method (cf. (5.8) in Section 5).

From the proof of Theorem 1.1, we also prove the following uniqueness result of (1.5).

**Theorem 1.4.** The solution $\varphi$ of (1.5) is unique if it satisfies (1.6), (1.7) and

$$
e^{-\varphi} \leq \delta_0, \ |\partial\bar{\partial}\varphi|_g \leq D, \ D_0 \leq D \text{ and } A \leq \frac{1}{C_0M_0D},
$$

where $C_0$ is a uniform constant, and $\delta_0$, $M_0$ and $D_0$ are constants determined in Proposition 2.1, Proposition 4.1, respectively.

Since the normalization condition (1.7) is different from ones in the previous works as in \cite{5, 6, 13, 14}, etc., we shall also derive $C^0, C^1, C^2$-estimates for solutions of (1.8) step by step.
The paper is organized for each estimate in one section. Theorems 1.1 and 1.4 are both proved in last section, Section 5.

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2. Zero order estimate

In this section, we use the iteration method to derive the following zero order estimate of \( \varphi \) to (1.5).

**Proposition 2.1.** Let \( \varphi \) be a smooth solution of (1.5). There exist constants \( A_0 \) and \( M_0 \) depending only on \( \alpha, \rho, \mu \) and \( (M, \omega) \) such that if
\[
e^{-\varphi} \leq \delta_0 := \sqrt{\frac{1}{2\alpha\|\rho\|_{C^0} + 1}} \text{ and } \|e^{-\varphi}\|_{L^1} = A \leq A_0,
\]
then
\[
\frac{1}{M_0 A} \leq e^{\inf_M \varphi} \text{ and } e^{\sup_M \varphi} \leq \frac{M_0}{A}.
\]

**Proof.** We first do the infimum estimate. The supremum estimate depends on the established infimum estimate. By the choice of \( \delta_0 \) and the condition \( e^{-\varphi} \leq \delta_0 \), it is clear that
\[
\omega + \alpha e^{-2\varphi} \rho \geq \frac{1}{2} \omega.
\]

By the elliptic condition (1.6), we have for \( k \geq 2 \),
\[
k \int_M e^{-k\varphi} \sqrt{-1} \partial \varphi \wedge \overline{\partial} \varphi \wedge \omega^{n-2} \geq 0.
\]

By the Stokes’ formula, it follows that
\[
-k \int_M e^{-k\varphi}(e^{\varphi} \omega + \alpha e^{-\varphi} \rho) \wedge \sqrt{-1} \partial \varphi \wedge \overline{\partial} \varphi \wedge \omega^{n-2} \leq 2n\alpha k \int_M e^{-k\varphi} \sqrt{-1} \partial \varphi \wedge \overline{\partial} \varphi \wedge \sqrt{-1} \overline{\partial} \varphi \wedge \omega^{n-2}
\]
\[
= -2n\alpha \int_M \sqrt{-1} \partial \varphi \sqrt{-1} \partial \varphi \wedge \omega^{n-2}
\]
\[
= 2n\alpha \int_M e^{-k\varphi} \sqrt{-1} \partial \varphi \wedge \sqrt{-1} \overline{\partial} \varphi \wedge \omega^{n-2}
\]
\[
= -2 \int_M e^{-k\varphi} \sqrt{-1} \overline{\partial} \varphi (e^{\varphi} \omega - \alpha e^{-\varphi} \rho) \wedge \omega^{n-2} - 2 \int_M e^{-k\varphi} \mu \frac{\omega^n}{n!}.
\]

In the last equality, we used the equation (1.5).
For the first term of right hand side in (2.3), we compute
\[
-2 \int_M e^{-k\varphi} \sqrt{-1} \overline{\partial \overline{\partial}}(e^\varphi \omega - \alpha e^{-\varphi} \rho) \wedge \omega^{n-2} = -2k \int_M e^{-k\varphi} \sqrt{-1} \partial \overline{\partial}(e^\varphi \omega - \alpha e^{-\varphi} \rho) \wedge \omega^{n-2}
\]
(2.4)
\[
= -2k \int_M e^{-k\varphi} (e^\varphi \omega + \alpha e^{-\varphi} \rho) \wedge \sqrt{-1} \partial \overline{\partial} \wedge \omega^{n-2} + 2\alpha k \int_M e^{-(k+1)\varphi} \sqrt{-1} \partial \varphi \wedge \overline{\partial} \wedge \omega^{n-2}.
\]
Substituting (2.4) into (2.3), we see that
\[
k \int_M e^{-k\varphi} (e^\varphi \omega + \alpha e^{-\varphi} \rho) \wedge \sqrt{-1} \partial \overline{\partial} \wedge \omega^{n-2} \leq 2\alpha k \int_M e^{-(k+1)\varphi} \sqrt{-1} \partial \varphi \wedge \overline{\partial} \wedge \omega^{n-2} - 2 \int_M e^{-k\varphi} \mu \omega^n.
\]
Combining (2.5) with (2.2) and the Cauchy-Schwarz inequality, it follows that
\[
k \int_M e^{-(k-1)\varphi} |\partial \varphi|_g^2 \omega^n \leq Ck \int_M (e^{-(k+1)\varphi} |\partial \varphi|_g + e^{-k\varphi}) \omega^n \leq \frac{k}{2} \int_M e^{-(k+1)\varphi} |\partial \varphi|_g^2 \omega^n + Ck \int_M (e^{-(k+3)\varphi} + e^{-k\varphi}) \omega^n.
\]
Recalling \( e^{-\varphi} \leq \delta_0 \), we get
\[
\frac{k}{2} \int_M e^{-(k-1)\varphi} |\partial \varphi|_g^2 \omega^n \leq Ck(\delta_0^4 + \delta_0) \int_M e^{-(k-1)\varphi} \omega^n,
\]
which implies
\[
\int_M |\partial e^{-\frac{(k-1)\varphi}{2}}|_g^2 \omega^n \leq Ck^2 \int_M e^{-(k-1)\varphi} \omega^n.
\]
Replacing \( k - 1 \) by \( k \), for \( k \geq 1 \), we deduce
\[
\int_M |\partial e^{-\frac{k\varphi}{2}}|_g^2 \omega^n \leq Ck^2 \int_M e^{-k\varphi} \omega^n.
\]
Hence, by the Moser iteration together with (1.7), we obtain
\[
\|e^{-\varphi}\|_{L^{\infty}} \leq C\|e^{-\varphi}\|_{L^1} = CA.
\]
As a consequence, we prove
\[
\frac{1}{M_0 A} \leq e^{\inf_M \varphi}.
\]
(2.6)
Next we do the supremum estimate. By the similar calculation of (2.3)-(2.5), for $k \geq 1$, we have

$$k \int_M e^{k\varphi}(e^\varphi \omega + \alpha e^{-\varphi} \rho) \wedge \sqrt{-1} \partial \varphi \wedge \overline{\partial} \varphi \wedge \omega^{n-2} \leq 2\alpha k \int_M e^{(k-1)\varphi}\sqrt{-1} \partial \varphi \wedge \overline{\partial} \rho \wedge \omega^{n-2} + 2 \int_M e^{k\varphi} \mu \omega^n / n!.$$ 

Combining this with (2.2), we have

$$\int_M e^{(k+1)\varphi} |\partial \varphi|^2 g \omega^n \leq C \int_M (e^{(k-1)\varphi} |\partial \varphi|^2 g + e^{k\varphi}) \omega^n.$$ 

Using $e^{-\varphi} \leq \delta_0$ and the Cauchy-Schwarz inequality, it then follows that

$$(2.7) \quad \int_M e^{(k+1)\varphi} |\partial \varphi|^2 g \omega^n \leq C \int_M e^{k\varphi} \omega^n.$$ 

Moreover, by (2.6), we get

$$(2.8) \quad \int_M e^{k\varphi} |\partial \varphi|^2 g \omega^n \leq C \int_M e^{k\varphi} \omega^n.$$ 

We will use (2.8) to do the iteration. We need

**Claim 1.**

$$(2.9) \quad \|e^\varphi\|_{L^1} \leq \frac{C}{A}.$$ 

Without loss of generality, we assume that $\text{Vol}(M, \omega) = 1$. We define a set by

$$U = \{ x \in M \mid e^{-\varphi(x)} \geq \frac{A}{2} \}.$$ 

Then by (2.6) and (1.7), we have

$$A = \int_M e^{-\varphi} \omega^n = \int_U e^{-\varphi} \omega^n + \int_{M \setminus U} e^{-\varphi} \omega^n \leq e^{-\inf_M \varphi} \text{Vol}(U) + \frac{A}{2} (1 - \text{Vol}(U)) \leq \left( M_0 - \frac{1}{2} \right) A \text{Vol}(U) + \frac{A}{2}.$$ 

It implies

$$(2.10) \quad \text{Vol}(U) \geq \frac{1}{C_0}.$$ 

On the other hand, by the Poincaré inequality and (2.7) (taking $k = 1$), we have

$$\int_M e^{2\varphi} \omega^n - \left( \int_M e^\varphi \omega^n \right)^2 \leq C \int_M |\partial e^\varphi|^2 g \omega^n \leq C \int_M e^\varphi \omega^n.$$
By (2.10) and the Cauchy-Schwarz inequality, we obtain
\[
\left( \int_M e^\varphi \omega^n \right)^2 \leq (1 + C_0) \left( \int_U e^\varphi \omega^n \right)^2 + \left( 1 + \frac{1}{C_0} \right) \left( \int_{M \setminus U} e^\varphi \omega^n \right)^2 \\
\leq \frac{4(1 + C_0)}{A^2} (\text{Vol}(U))^2 + \left( 1 + \frac{1}{C_0} \right) (1 - \text{Vol}(U))^2 \int_M e^{2\varphi} \omega^n \\
\leq \frac{4(1 + C_0)}{A^2} + \left( 1 - \frac{1}{C_0} \right) \left( \left( \int_M e^\varphi \omega^n \right)^2 + C \int_M e^\varphi \omega^n \right).
\]

Clearly, the above implies (2.9).

By Claim 1, (2.8) and the Moser iteration, we see that
\[
\| e^\varphi \|_{L^\infty} \leq C \| e^\varphi \|_{L^1} \leq \frac{C}{A}.
\]

Thus
\[
\| e^\varphi \|_{L^\infty} \leq \frac{C}{A}.
\]

\[\square\]

### 3. First order estimate

In this section, we give the first order estimate of \( \varphi \). For convenience, in this and next section, we say a constant is uniform if it depends only on \( \alpha, \rho, \mu \) and \( (M, \omega) \).

**Proposition 3.1.** Let \( \varphi \) be a solution of (1.5) satisfying (1.6). Assume that
\[
\frac{A}{M_0} \leq e^{-\varphi} \leq M_0 A \quad \text{and} \quad |\partial^2 \varphi|_g \leq D,
\]
where \( M_0 \) is a uniform constant. Then there exists a uniform constant \( C_0 \) such that if
\[
A \leq A_D := \frac{1}{C_0 M_0 D},
\]
then
\[
|\partial \varphi|_g^2 \leq M_1,
\]
where \( M_1 \) is a uniform constant.

**Remark 3.2.** The key point in Proposition 3.1 is that \( M_1 \) is independent of \( D \). The constant \( D \) can be chosen arbitrary large and the constant \( A_D \) depends on \( D \). This will play an important role in the second order estimate next section. In fact, we will determine \( D \) so that \( A \) can be determined (cf. Proposition 4.1).

As usually, for any \( \eta = (\eta_1, \eta_2, \ldots, \eta_n) \in \mathbb{R}^n \), we define
\[
\sigma_k(\eta) = \sum_{1 < i_1 < \cdots < i_k < n} \eta_{i_1} \eta_{i_2} \cdots \eta_{i_k},
\]
\[
\Gamma_2 = \{ \eta \in \mathbb{R}^n \mid \sigma_j(\eta) > 0 \text{ for } j = 1, 2 \}.
\]
Clearly $\sigma_2$ is a 2-multiple functional. Then one can extend it to $A^{1,1}(M)$ by

$$\sigma_k(\alpha) = \binom{n}{k} \frac{\alpha^k \wedge \omega^{n-k}}{\omega^n}, \quad \forall \alpha \in A^{1,1}(M),$$

where $A^{1,1}(M)$ is the space of smooth real (1,1) forms on $(M, \omega)$. Define a cone $\Gamma_2(M)$ on $A^{1,1}(M)$ by

$$\Gamma_2(M) = \{ \alpha \in A^{1,1}(M) \mid \sigma_j(\alpha) > 0 \text{ for } j = 1, 2 \}.$$

Then, (1.5) is equivalent to (1.8) while the function $f(z, \varphi, \partial \varphi)$ satisfies

$$f \omega^n = 2\alpha \rho \wedge \omega^{n-1} + \alpha^2 e^{-2\varphi} \rho^2 \wedge \omega^{n-2} - 4n\alpha \mu \frac{\omega^n}{n!} + 4n\alpha^2 e^{-\varphi} \sqrt{-1} \left( \partial \varphi \wedge \overline{\partial} \varphi \wedge \rho - \partial \varphi \wedge \overline{\partial} \rho - \overline{\partial} \varphi \wedge \partial \rho \wedge \omega^{n-2} \right).$$

We will use (1.8) to apply the maximum principle to the quantity

$$Q = \log |\partial \varphi|^2_g + \frac{\varphi}{B},$$

where $B > 1$ is a large uniform constant to be determined later.

Assume that $Q$ achieves a maximum at $x_0$. Let $\{e_i\}_{i=1}^n$ be a local unitary frame in a neighbourhood of $x_0$ such that, at $x_0$,

$$\tilde{g}_{ij} = \delta_{ij} \tilde{g}_{ii} = \delta_{ij} (e^\varphi + \alpha e^{-\varphi} \rho_i \rho_i + 2n\alpha \varphi \rho_i).$$

For convenience, we use the following notation:

$$\hat{\omega} = e^{-\varphi} \omega, \tilde{g}_{ij} = e^{-\varphi} \tilde{g}_{ij} \text{ and } F^{ij} = \frac{\partial \sigma_2(\hat{\omega})}{\partial \tilde{g}_{ij}}.$$ 

Since $\tilde{g}_{ij}(x_0)$ is diagonal at $x_0$, it is easy to see that

$$F^{ij} = \delta_{ij} F^{ii} = \delta_{ij} e^{-\varphi} \sum_{k \neq i} \tilde{g}_{ik}.$$ 

By the assumption of Proposition 3.1 at the expense of increasing $C_0$, we have

$$e^{-\varphi} |\partial \varphi|_g \leq M_0 DA D \leq \frac{1}{1000Bn^3|\alpha|}.$$ 

Combining this with (3.2) and (3.3), we get

$$|F^{ii} - (n-1)| \leq \frac{1}{100}.$$

We need to estimate the lower bound of $F^{ij} \varepsilon^j_\varepsilon (|\partial \varphi|^2_g)$, where we are summing over repeated indices. Note

$$|\partial \varphi|^2_g = \sum_k \varphi_k \varphi_k,$$
Lemma 3.3. At \( x_0 \), we have

\[
\sum_k F^{ij} e_i e_j (|\partial \varphi|^2_g) \geq \sum_k F^{ij} (e_i e_j (\varphi) + e_i \bar{e}_j (\varphi)) \varphi_k.
\]

On the other hand, by the relation (see e.g. [10])

\[
\varphi_{ij} = \partial \bar{\partial} \varphi (e_i, e_j) = e_i \bar{e}_j (\varphi) - [e_i, e_j]^{(0,1)} (\varphi),
\]

we have

\[
e_k (\varphi_{ij}) = e_k e_i \bar{e}_j (\varphi) - e_k [e_i, e_j]^{(0,1)} (\varphi) = e_i \bar{e}_j e_k (\varphi) + [e_k, e_i] e_j (\varphi) - e_i [e_k, e_j]^{(0,1)} (\varphi).
\]

Thus combining this with (3.5), we get

\[
\sum_k F^{ij} e_i e_k (\varphi) \varphi_k + e_i \bar{e}_j e_k (\varphi)
\]

\[
\geq \sum_k F^{ij} (e_k (\varphi_{ij}) + e_i \bar{e}_j (\varphi) - e_k [e_i, e_j]^{(0,1)} (\varphi)) - C |\partial \varphi|^2_g
\]

\[
\geq \sum_k F^{ij} (e_k (\varphi_{ij}) + e_i \bar{e}_j (\varphi)) - \frac{1}{10} \sum_{i,j} (e_i e_j (\varphi))^2 + C |\partial \varphi|^2_g.
\]

Hence, we obtain

\[
F^{ij} e_i e_j (|\partial \varphi|^2_g) \geq \frac{4}{5} \sum_{i,j} (e_i e_j (\varphi))^2 - C |\partial \varphi|^2_g
\]

\[
+ \sum_k \left( F^{ij} e_k (\varphi_{ij}) + e_i \bar{e}_j (\varphi) \varphi_k \right).
\]

Next, we use equation (1.8) to deal with the third order terms in (3.7).

**Lemma 3.3.** At \( x_0 \), we have

\[
\sum_k (e_k (\varphi_{ij}) + e_i \bar{e}_j (\varphi)) \varphi_k
\]

\[
\geq - \frac{1}{5} \sum_{i,j} (e_i e_j (\varphi))^2 - 2 (n - 1) \text{Re} \left( \sum_k (|\partial \varphi|^2_g) \varphi_k \right)
\]

\[
- \left( C e^{-\varphi} + \frac{1}{D} \right) |\partial \varphi|^4_g - C |\partial \varphi|^2_g - C.
\]

**Proof.** By (1.8), we have

\[
\sigma_2 (e^{-\varphi} \hat{\omega}) = e^{-2\varphi} F (z, \varphi, \partial \varphi).
\]
Differentiating (3.8) along $\epsilon_k$ at $x_0$, we get

$$F_{\mathcal{J}}e_k(g_{\mathcal{J}} + \alpha e^{-\varphi} \rho_{\mathcal{J}} + 2\alpha e^{-\varphi} \varphi_{\mathcal{J}})$$

$$= -2\alpha n(n - 1) \left( -e^{-\varphi} |\partial \varphi|^2 f_k + e^{-\varphi} (|\partial \varphi|^2 g) + \frac{n(n - 1)}{2} (e^{-2\varphi} f_k) \right).$$

Then

$$2n\alpha F_{\mathcal{J}}e_k(\varphi_{\mathcal{J}}) = 2\alpha e^{-\varphi} \varphi_k F_{\mathcal{J}} \rho_{\mathcal{J}} - \alpha e^{-\varphi} F_{\mathcal{J}} e_k(\rho_{\mathcal{J}}) + 2n\alpha \varphi_k F_{\mathcal{J}} \varphi_{\mathcal{J}}$$

$$- 2\alpha n(n - 1) (|\partial \varphi|^2 g) + 2\alpha n(n - 1) |\partial \varphi|^2 g$$

$$- n(n - 1) e^{-\varphi} f_k + \frac{n(n - 1)}{2} e^{-\varphi} f_k.$$  

It follows that

$$\sum_k F_{\mathcal{J}} (e_k(\varphi_{\mathcal{J}}) \varphi_{\mathcal{J}} + \bar{e}_k(\varphi_{\mathcal{J}}) \varphi_{\mathcal{J}})$$

$$= \frac{2}{n} e^{-\varphi} |\partial \varphi|^2 F_{\mathcal{J}} \rho_{\mathcal{J}} - \frac{1}{n} e^{-\varphi} \text{Re} \left( \sum_k F_{\mathcal{J}} e_k(\rho_{\mathcal{J}}) \varphi_{\mathcal{J}} \right) + 2 |\partial \varphi|^2 F_{\mathcal{J}} \varphi_{\mathcal{J}}$$

$$- 2(n - 1) \text{Re} \left( \sum_k (|\partial \varphi|^2 g) \varphi_{\mathcal{J}} \right) + 2(n - 1) |\partial \varphi|^4 - \frac{n - 1}{\alpha} e^{-\varphi} |\partial \varphi|^2 f_k$$

$$+ \frac{n - 1}{2\alpha} e^{-\varphi} \text{Re} \left( \sum_k f_k \varphi_{\mathcal{J}} \right)$$

$$\geq - Ce^{-\varphi} |\partial \varphi|^2 - Ce^{-\varphi} |\partial \varphi|^2 + 2 |\partial \varphi|^2 F_{\mathcal{J}} \varphi_{\mathcal{J}} - 2(n - 1) \text{Re} \left( \sum_k (|\partial \varphi|^2 g) \varphi_{\mathcal{J}} \right)$$

$$+ 2(n - 1) |\partial \varphi|^4 - \frac{n - 1}{\alpha} e^{-\varphi} |\partial \varphi|^2 f_k + \frac{n - 1}{2\alpha} e^{-\varphi} \text{Re} \left( \sum_k f_k \varphi_{\mathcal{J}} \right),$$

where we used (3.5) in the last inequality. On the other hand, by (3.1), a direct calculation shows that

$$\sum_k F_{\mathcal{J}} (e_k(\varphi_{\mathcal{J}}) \varphi_{\mathcal{J}} + \bar{e}_k(\varphi_{\mathcal{J}}) \varphi_{\mathcal{J}})$$

$$\geq - \frac{n - 1}{\alpha} e^{-\varphi} |\partial \varphi|^2 f_k + \frac{n - 1}{2\alpha} e^{-\varphi} \text{Re} \left( \sum_k f_k \varphi_{\mathcal{J}} \right)$$

$$\geq - C \left( e^{-2\varphi} |\partial \varphi|^2 g + e^{-2\varphi} |\partial \varphi|^2 \right) \sum_{i,j} (|e_i \overline{e_j}(\varphi)| + |e_i e_j(\varphi)|) - Ce^{-\varphi} |\partial \varphi|^4$$

$$- Ce^{-\varphi} |\partial \varphi|^3 - Ce^{-\varphi} |\partial \varphi|^2 - Ce^{-\varphi} |\partial \varphi|$$

$$\geq - \frac{1}{10} \sum_{i,j} (|e_i \overline{e_j}(\varphi)|^2 + |e_i e_j(\varphi)|^2) - Ce^{-\varphi} |\partial \varphi|^4 - Ce^{-\varphi}. $$
where we used the Cauchy-Schwarz inequality in the last inequality. Thus substituting (3.10) into (3.9), we derive

$$\sum_k F_k^\ast (e_k(\varphi_\ast)\varphi_\ast + \bar{e}_k(\varphi_\ast)\varphi_k)$$

(3.11)\[\geq 2|\partial\varphi|^2_g F_k^\ast \varphi_\ast^2 + 2(n-1)|\partial\varphi|^4_g - 2(n-1)Re \left( \sum_k (|\partial\varphi|^2_g)_{k\varphi_k} \right)\]

$$- \frac{1}{10} \sum_{i,j} (|e_i e_j(\varphi)|^2 + |e_i e_j(\varphi)|^2) - Ce^{-\varphi}|\partial\varphi|^4_g - Ce^{-\varphi}.$$

By (3.3), we have

(3.12)\[2|\partial\varphi|^2_g F_k^\ast \varphi_\ast \]

$$= 2|\partial\varphi|^2_g \sum_i \sum_{k \neq i} (g_{k\bar{k}} + \alpha e^{-2\varphi} p_{k\bar{k}} + 2\alpha e^{-\varphi} \varphi_{k\bar{k}}) \varphi_\ast$$

$$= 2(n-1)|\partial\varphi|^2_g \Delta \varphi + 4\alpha |\partial\varphi|^2_g e^{-\varphi} \sum_{k \neq i} \varphi_\ast^2 \varphi_{k\bar{k}} + 2\alpha e^{-\varphi} \partial^2 \varphi_g \sum_{k \neq i} \varphi_\ast^2 p_{k\bar{k}}$$

$$\geq 2(n-1)|\partial\varphi|^2_g \Delta \varphi - 4n^2 (n-1)\alpha |\varphi|^2_g \partial \varphi_g^2 \partial \varphi_g^2 - Ce^{-\varphi} |\partial\varphi|^4_g \partial \varphi_g^2.$$ 

Note that by (1.5) it holds

$$\sqrt{-1} \partial \bar{\partial} (e^{\varphi} \omega - \alpha e^{-\varphi} \rho) \wedge \omega^{n-2} \geq -n|\alpha||\partial \varphi_g^2 - C,$$

which implies

(3.13)\[\Delta \varphi + |\partial\varphi|^2_g \geq -n|\alpha| e^{-\varphi} |\partial \varphi_g^2 - C e^{-2\varphi} |\partial \varphi_g^2 - C e^{-2\varphi} |\partial\varphi|^2_g - C.$$ 

Then substituting (3.13) into (3.12), we get

$$2|\partial\varphi|^2_g F_k^\ast \varphi_\ast \geq -2(n-1)|\partial\varphi|^4_g - 5n^3 |\alpha| e^{-\varphi} |\partial\varphi|^2_g \partial \varphi_g^2 \partial \varphi_g^2$$

$$- Ce^{-2\varphi} |\partial\varphi|^2_g \partial \varphi_g^2 - Ce^{-2\varphi} |\partial\varphi|^4_g - C |\partial\varphi|^2_g.$$

Thus by (3.3) and the Cauchy-Schwarz inequality, we derive

(3.14)\[2|\partial\varphi|^2_g F_k^\ast \varphi_\ast + 2(n-1)|\partial\varphi|^4_g \]

$$\geq -5n^3 |\alpha| (e^{-\varphi} |\partial \varphi_g^2| + |\partial \varphi|^2_g)$$

$$- C (e^{-\varphi} |\partial \varphi_g^2| (e^{-\varphi} |\partial\varphi|^2_g) - Ce^{-2\varphi} |\partial\varphi|^4_g - C |\partial\varphi|^2_g$$

$$\geq - \frac{1}{B} |\partial \varphi_g^2| - \left( Ce^{-2\varphi} + \frac{1}{B} \right) |\partial\varphi|^4_g - C |\partial\varphi|^2_g$$

$$\geq - \frac{1}{10} \sum_{i,j} |e_i e_j(\varphi)|^2 - \left( Ce^{-2\varphi} + \frac{1}{B} \right) |\partial\varphi|^4_g - C |\partial\varphi|^2_g.$$
Combining (3.11) and (3.14), we prove Lemma 3.3 immediately. □

By (3.7) and Lemma 3.3, we get a lower bound for $F_{i\overline{j}} e_i \overline{e}_j \partial\varphi$ at $x_0$ as follows,

$$F_{i\overline{j}} e_i \overline{e}_j \partial\varphi \geq \frac{3}{5} \sum_{i,j} (|e_i \overline{e}_j(\varphi)|^2 + |e_i e_j(\varphi)|^2) - 2(n-1)Re \left( \sum_k (|\partial\varphi|_g^2)_{k\overline{k}} \right)$$

$$- \left( Ce^{-\varphi} + \frac{1}{B} \right) |\partial\varphi|_g^4 - C|\partial\varphi|_g^2 - C. \quad (3.15)$$

Now we are in a position to prove Proposition 3.1.

Proof of Proposition 3.1. Without loss of generality, we assume that $|\partial\varphi|_g^2 \geq 1$. By (3.15) and the maximum principle, at $x_0$, we see that

$$0 \geq F^\overline{\tau} e_i \overline{e}_j (Q)$$

$$= \frac{F^\overline{\tau} e_i \overline{e}_j (|\partial\varphi|_g^2)}{|\partial\varphi|_g^2} - \frac{F^\overline{\tau} e_i (|\partial\varphi|_g^2) \overline{e}_j (|\partial\varphi|_g^2)}{|\partial\varphi|_g^2} + \frac{1}{B} F^\overline{\tau} e_i \overline{e}_j (\varphi)$$

$$\geq \frac{1}{2|\partial\varphi|_g^2} \sum_{i,j} (|e_i \overline{e}_j(\varphi)|^2 + |e_i e_j(\varphi)|^2) - \frac{2(n-1)Re \left( \sum_k (|\partial\varphi|_g^2)_{k\overline{k}} \right)}{|\partial\varphi|_g^2}$$

$$- \frac{F^\overline{\tau} e_i (|\partial\varphi|_g^2)}{|\partial\varphi|_g^4} - \left( Ce^{-\varphi} + \frac{1}{B} \right) |\partial\varphi|_g^4 - C + \frac{1}{B} F^\overline{\tau} e_i \overline{e}_i (\varphi). \quad (3.16)$$

The second and third terms in (3.16) can be controlled by the relation $dQ(x_0) = 0$. Namely, we have

$$- \frac{2(n-1)Re \left( \sum_k (|\partial\varphi|_g^2)_{k\overline{k}} \right)}{|\partial\varphi|_g^2} = \frac{2(n-1)}{B} |\partial\varphi|_g^2 \quad (3.17)$$

and

$$- \frac{F^\overline{\tau} e_i (|\partial\varphi|_g^2)}{|\partial\varphi|_g^4} = - \frac{1}{B^2} F^\overline{\tau} \varphi \overline{\varphi} \geq - \frac{C}{B^2} |\partial\varphi|_g^2, \quad (3.18)$$

where we used (3.5) in the last inequality. On the other hand, by (3.5) and the Cauchy-Schwarz inequality, we have

$$\frac{1}{B} F^\overline{\tau} e_i \overline{e}_i (\varphi) \geq - \frac{1}{4|\partial\varphi|_g^2} \sum_{i,j} |e_i \overline{e}_j(\varphi)|^2 - \frac{C}{B^2} |\partial\varphi|_g^2. \quad (3.19)$$
Thus substituting (3.17), (3.18), (3.19) into (3.16), we get
\[
0 \geq \frac{1}{4|\partial \varphi|^2_g} \sum_{i,j} \left( |e_i e_j(\varphi)|^2 + |e_i e_j(\varphi)|^2 \right) - C_0 \\
+ \left( \frac{2n - 3}{B} - \frac{C_0}{B^2} - C_0 e^{-\varphi} \right) |\partial \varphi|^2_g,
\]
where $C_0$ is a uniform constant.

We choose the number $B = 2C_0$ in (3.20). Moreover, by the assumption in the proposition we may also assume
\[
C_0 e^{-\varphi} \leq \frac{1}{8C_0}.
\]
Then, we get
\[
|\partial \varphi|^2_g(x_0) \leq 8C_0^2.
\]
Hence, by Proposition 2.1, we obtain
\[
\max_M |\partial \varphi|^2_g \leq e^{\frac{1}{8} (\sup_M \varphi - \inf_M \varphi)} |\partial \varphi|^2_g(x_0) \leq C,
\]
as desired. \hfill \Box

The following lemma will be used in the next section.

**Lemma 3.4.** For a uniform constant $C_1$, we have
\[
F^i_j e_i e_j(|\partial \varphi|^2_g) \geq \frac{1}{2} \sum_{i,j} \left( |e_i e_j(\varphi)|^2 + |e_i e_j(\varphi)|^2 \right) - C_1.
\]

**Proof.** This lemma is an immediate consequence of (3.15), Proposition 3.1 and the Cauchy-Schwarz inequality. \hfill \Box

## 4. Second order estimate

This section is denoted to $C^2$-estimate. We prove

**Proposition 4.1.** Let $\varphi$ be a solution of (1.5) satisfying (1.6) and $A_{M_0} \leq e^{-\varphi} \leq M_0 A$ for some uniform constant $M_0$. There exist uniform constants $D_0$ and $C_0$ such that if
\[
|\partial \varphi|^2_g \leq D, \ D_0 \leq D \text{ and } A \leq A_D := \frac{1}{C_0M_0D},
\]
then
\[
|\partial \varphi|^2_g \leq \frac{D}{2}.
\]
We consider the following quantity
\[
Q = |\partial \varphi|^2_g + B|\partial \varphi|^2_g,
\]
where $B > 1$ is a uniform constant to be determined later. As in Section 3, we assume that $Q(x_0) = \max_M Q$ and a local $g$-unitary frame $\{e_i\}_{i=1}^n$ for $T^{(1,0)}_C M$ around $x_0$ such that $\tilde{g}_{ij}(x_0)$ is diagonal. By the following notations,

$$
\tilde{\omega} = e^{-\varphi} \hat{\omega}, \tilde{g}_{ij} = e^{-\varphi} \hat{g}_{ij}, F^{\tilde{\omega}}_{ij} = \frac{\partial \sigma_2(\tilde{\omega})}{\partial \tilde{g}_{ij}}, \quad \text{and} \quad F^{\tilde{\omega}}_{ijkl} = \frac{\partial^2 \sigma_2(\tilde{\omega})}{\partial \tilde{g}_{ij} \partial \tilde{g}_{kl}},
$$

we have

$$
F^{\tilde{\omega}} = \delta_{ij} F^{\tilde{\omega}} = \delta_{ij} e^{-\varphi} \sum_{k \neq i} \tilde{g}_{ik},
$$

and

$$
F^{\tilde{\omega}}_{ijkl} = \begin{cases} 
1, & \text{if } i = j, k = l, i \neq k; \\
-1, & \text{if } i = l, k = j, i \neq k; \\
0, & \text{otherwise.}
\end{cases}
$$

By the assumption of Proposition 4.1 at the expense of increasing $C_0$, we may also assume that

$$
e^{-\varphi} |\partial \varphi|_g \leq \frac{1}{1000 n^3 |\alpha| B}.
$$

Hence, we get

$$
|F^{\tilde{\omega}} - (n - 1)| \leq \frac{1}{100} \quad \text{and} \quad |F^{\tilde{\omega}}_{ijkl}| \leq 1.
$$

We need the following lemma.

**Lemma 4.2.** At $x_0$, we have

$$
|F^{\tilde{\omega}}_{ij} e_i \bar{e}_j(\varphi)_{kl}| \leq 8n |\alpha| e^{-\varphi} \sum_{i,j,p} |e_p e_i \bar{e}_j(\varphi)|^2 + C \sum_{i,j,p} |e_p e_i \bar{e}_j(\varphi)| + C \sum_{i,j} (|e_i \bar{e}_j(\varphi)|^2 + |e_i e_j(\varphi)|^2) + C.
$$

**Proof.** Differentiating (3.8) twice along $e_k$ and $\bar{e}_l$ at $x_0$, we have

$$
F^{\tilde{\omega}}_{ij} e_k (e^{-\varphi} \tilde{g}_{ij}) \bar{e}_l (e^{-\varphi} \tilde{g}_{kl}) + F^{\tilde{\omega}}_{ijkl} e_k (e^{-\varphi} \tilde{g}_{ij})
$$

$$
= -2n(n - 1) \alpha e_k \bar{e}_l (e^{-\varphi} |\partial \varphi|^2_g) + \frac{n(n - 1)}{2} e_k \bar{e}_l (e^{-2\varphi} f).
$$

Let

$$
I_1 = -F^{\tilde{\omega}}_{ij} e_k (e^{-\varphi} \tilde{g}_{ij}) \bar{e}_l (e^{-\varphi} \tilde{g}_{kl}),
$$

$$
I_2 = -2n(n - 1) \alpha e_k \bar{e}_l (e^{-\varphi} |\partial \varphi|^2_g),
$$

$$
I_3 = \frac{n(n - 1)}{2} e_k \bar{e}_l (e^{-2\varphi} f).
$$

Then (4.3) becomes

$$
F^{\tilde{\omega}}_{ijkl} e_k \bar{e}_l (e^{-\varphi} \tilde{g}_{ij}) = I_1 + I_2 + I_3.
$$
We estimate each term in (4.4) below. For $I_1$, by (4.2), Proposition 3.1 and the Cauchy-Schwarz inequality, we have

$$|I_1| \leq \sum_{i,j,k} \left| e_k (\alpha e^{-2\varphi} \rho_{ij} + 2n\alpha e^{-\varphi} \varphi_{ij}) \right|^2$$

$$\leq 2 \sum_{i,j,k} \left| e_k (2n\alpha e^{-\varphi} \varphi_{ij}) \right|^2 + 2 \sum_{i,j,k} \left| e_k (\alpha e^{-2\varphi} \rho_{ij}) \right|^2$$

$$\leq 8n^2 \alpha^2 e^{-2\varphi} \sum_{i,j,k} |e_k e_j \overline{\varphi}_j (\varphi) - e_k [e_i, \overline{\varphi}_j]^{(0,1)} (\varphi) - \varphi_{ij} \varphi_{ij}|^2 + C e^{-4\varphi}$$

$$\leq 16n^2 \alpha^2 e^{-2\varphi} \sum_{i,j,k} |e_k e_j \overline{\varphi}_j (\varphi)|^2 + C e^{-2\varphi} \sum_{i,j} \left( |e_i \overline{\varphi}_j (\varphi)|^2 + |e_i e_j (\varphi)|^2 \right) + C e^{-\varphi},$$

where we used (3.6) in the last inequality. Similarly, for $I_2$ and $I_3$, we get

$$|I_2| \leq C e^{-\varphi} \sum_{i,j,p} |e_p e_i \overline{\varphi}_j (\varphi)| + C e^{-\varphi} \sum_{i,j} \left( |e_i \overline{\varphi}_j (\varphi)|^2 + |e_i e_j (\varphi)|^2 \right) + C e^{-\varphi}$$

and

$$|I_3| = \frac{n(n-1)}{2} e^{-2\varphi} \left| 4 \varphi_{ij} \varphi_{kl} f - 2 e_k e_l \overline{\varphi}_l (\varphi) f - 2 \varphi_{ij} f_k - 2 \varphi_k f_l + e_k e_l f \right|$$

$$\leq C e^{-2\varphi} \sum_{i,j,p} |e_p e_i \overline{\varphi}_j (\varphi)| + C e^{-2\varphi} \sum_{i,j} \left( |e_i \overline{\varphi}_j (\varphi)|^2 + |e_i e_j (\varphi)|^2 \right) + C e^{-2\varphi},$$

where we used Proposition 3.1 and (3.1). Thus substituting these estimates into (4.4), we obtain

$$|F^\varphi_{ij} e_k \overline{\varphi}_l (e^{-\varphi} \overline{g}_{ij})|$$

$$\leq 16n^2 \alpha^2 e^{-2\varphi} \sum_{i,j,p} |e_p e_i \overline{\varphi}_j (\varphi)|^2 + C e^{-\varphi} \sum_{i,j,p} |e_p e_i \overline{\varphi}_j (\varphi)|$$

$$+ C e^{-\varphi} \sum_{i,j} \left( |e_i \overline{\varphi}_j (\varphi)|^2 + |e_i e_j (\varphi)|^2 \right) + C e^{-\varphi}. \tag{4.5}$$

On the other hand, by the definition of $\overline{g}_{ij}$ and (3.6), we have

$$F^\varphi_{ij} e_k \overline{\varphi}_l (e^{-\varphi} \overline{g}_{ij}) = \alpha F^\varphi_{ij} e_k \overline{\varphi}_l (e^{-2\varphi} \rho_{ij} - 2n\alpha F^\varphi_{ij} e_k \overline{\varphi}_l (e^{-\varphi} \varphi_{ij})$$

$$= \alpha F^\varphi_{ij} e_k \overline{\varphi}_l (e^{-2\varphi} \rho_{ij} - 2n\alpha F^\varphi_{ij} e_k \overline{\varphi}_l (e^{-\varphi} e_i \overline{\varphi}_i (\varphi))$$

$$- 2n\alpha F^\varphi_{ij} e_k \overline{\varphi}_l (e^{-\varphi} \overline{g}_{ij})^{(0,1)} (\varphi).)$$

Then by (4.2) and Proposition 3.1 it follows that

$$|2n\alpha e^{-\varphi} F^\varphi_{ij} e_k \overline{\varphi}_l e_i \overline{\varphi}_i (\varphi) | \leq |F^\varphi_{ij} e_k \overline{\varphi}_l (e^{-\varphi} \overline{g}_{ij})| + C e^{-\varphi} \sum_{i,j,p} |e_p e_i \overline{\varphi}_j (\varphi)|$$

$$+ C e^{-\varphi} \sum_{i,j} \left( |e_i \overline{\varphi}_j (\varphi)|^2 + |e_i e_j (\varphi)|^2 \right) + C e^{-\varphi}.$$

Thus substituting (4.5) into the above inequality, we derive

\[
|F^{i}e_{k}e_{i}e_{k}(\varphi)| \leq 8n|\alpha|e^{-\varphi} \sum_{i,j,p} |e_{p}e_{i}e_{j}(\varphi)|^{2} + C \sum_{i,j,p} |e_{p}e_{i}e_{j}(\varphi)| \\
+ C \sum_{i,j} (|e_{i}e_{j}(\varphi)|^{2} + |e_{i}e_{j}(\varphi)|^{2}) + C.
\]

(4.6)

Note that

\[
e_{i}e_{k}e_{k}(\varphi) = e_{k}e_{i}e_{i}(\varphi) + e_{k}[e_{i}, e_{i}e_{i}e_{i}(\varphi)] + [e_{i}, e_{k}][e_{i}e_{i}e_{i}(\varphi)] \\
+ e_{i}e_{k}[e_{i}, e_{i}e_{i}e_{i}(\varphi)] + e_{i}[e_{i}, e_{k}e_{i}e_{i}e_{i}(\varphi)].
\]

(4.7)

Since \((M, \omega)\) is Hermitian, near \(x_{0}\), \([e_{i}, e_{k}]\) is a (1,0) vector field and \([e_{i}, e_{i}]\) is a (0,1) vector field. By (4.2) and (4.6), we see that

\[
|F^{i}e_{i}e_{k}e_{k}(\varphi)| \leq |F^{i}e_{k}e_{i}e_{i}(\varphi)| + C \sum_{i,j,p} |e_{p}e_{i}e_{j}(\varphi)| \\
+ C \sum_{i,j} (|e_{i}e_{j}(\varphi)|^{2} + |e_{i}e_{j}(\varphi)|^{2}) + C.
\]

As a consequence, we obtain

\[
|F^{i}e_{i}e_{k}(\varphi)| \leq |F^{i}e_{i}e_{k}e_{k}(\varphi)| + |F^{i}e_{i}e_{i}e_{k}(\varphi)| + |F^{i}e_{i}e_{i}e_{i}e_{k}(\varphi)| \\
\leq 8n|\alpha|e^{-\varphi} \sum_{i,j,p} |e_{p}e_{i}e_{j}(\varphi)|^{2} + C \sum_{i,j,p} |e_{p}e_{i}e_{j}(\varphi)| \\
+ C \sum_{i,j} (|e_{i}e_{j}(\varphi)|^{2} + |e_{i}e_{j}(\varphi)|^{2}) + C.
\]

The lemma is proved. 

□
Proof of Proposition 4.1. By Lemma 4.2 and the Cauchy-Schwarz inequality, at \( x_0 \), we have

\[
F^\pi e_i \overline{\varepsilon}_i (|\partial \varphi|_g^2) = 2 \sum_{k,l} F^\pi e_i \overline{\varepsilon}_i (\varphi_{k\ell}) \varphi_{\ell i} + 2 \sum_{k,l} F^\pi e_i (\varphi_{k\ell}) \overline{\varepsilon}_i (\varphi_{\ell i})
\]

\[
\geq -2 |\partial \varphi|_g \sum_{k,l} |F^\pi e_i \overline{\varepsilon}_i (\varphi_{k\ell})| + \frac{1}{2} \sum_{i,j,p} |e_p e_i \overline{\varepsilon}_j (\varphi)|^2
\]

\[
- C \sum_{i,j} (|e_i \overline{\varepsilon}_j (\varphi)|^2 + |e_i e_j (\varphi)|^2) + C
\]

\[
\geq \left( \frac{1}{4} - 8n^3 |\alpha| e^{-\varphi} |\partial \varphi|_g \right) \sum_{i,j,p} |e_p e_i \overline{\varepsilon}_j (\varphi)|^2 - C
\]

\[
- C (|\partial \varphi|_g + 1) \sum_{i,j} (|e_i \overline{\varepsilon}_j (\varphi)|^2 + |e_i e_j (\varphi)|^2).
\]

Recalling (4.1) and \( |\partial \varphi|_g \leq D \). Thus

\[
F^\pi e_i \overline{\varepsilon}_i (|\partial \varphi|_g^2) \geq -C_0 (D + 1) \sum_{i,j} (|e_i \overline{\varepsilon}_j (\varphi)|^2 + |e_i e_j (\varphi)|^2) - C_0,
\]

where \( C_0 \) is a uniform constant. On the other hand, by Lemma 3.3, we have

\[
F^\pi e_i \overline{\varepsilon}_i (|\partial \varphi|_g^2) \geq \frac{1}{2} \sum_{i,j} (|e_i \overline{\varepsilon}_j (\varphi)|^2 + |e_i e_j (\varphi)|^2) - C_1.
\]

Hence, by the maximum principle, at \( x_0 \), we get

\[
0 \geq F^\pi e_i \overline{\varepsilon}_i (Q)
\]

\[
= F^\pi e_i \overline{\varepsilon}_i (|\partial \varphi|_g^2) + BF^\pi e_i \overline{\varepsilon}_i (|\partial \varphi|_g^2)
\]

\[
\geq \left( \frac{B}{2} - C_0 D - C_0 \right) \sum_{i,j} (|e_i \overline{\varepsilon}_j (\varphi)|^2 + |e_i e_j (\varphi)|^2) - C_0 - C_1 B.
\]

Choose \( B = 8C_0 D + 8C_0 \). It follows that

\[
|\partial \varphi|_g^2 (x_0) \leq C.
\]

Therefore, by Proposition 3.1, at the expense of increasing \( D_0 \), we obtain

\[
\max_M |\partial \varphi|_g^2 \leq |\partial \varphi|_g^2 (x_0) + BC \leq CD \leq \frac{D^2}{4}.
\]

\( \square \)
5. Proofs of Theorem 1.1 and Theorem 1.4

In this section, we prove Theorem 1.1 and Theorem 1.4. We use the continuity method and consider the family of equations \( t \in [0, 1] \),

\[
\sqrt{-1} \partial \bar{\partial} (e^\varphi \omega - t\alpha e^{-\varphi} \rho) \wedge \omega^{n-2} + n\alpha \sqrt{-1} \partial \bar{\partial} \omega \wedge \omega^{n-2} + t\mu \frac{\omega^n}{n!} = 0,
\]

where \( \varphi \) satisfies the elliptic condition,

\[
e^{\varphi} \omega + t\alpha e^{-\varphi} \rho + 2n\alpha \sqrt{-1} \partial \bar{\partial} \varphi \in \Gamma_2(M)
\]

and the normalization condition

\[
\|e^{-\varphi}\|_{L^1} = A.
\]

We shall prove that (5.1) is solvable for any \( t \in [0, 1] \). As the Fu-Yau equation (1.5), (5.1) is equivalent to a 2-nd Hessian type equation as (1.8).

For a fixed \( \beta \in (0, 1) \), we define the following sets of functions on \( M \),

\[
B = \{ \varphi \in C^{2,\beta}(M) \mid \|e^{-\varphi}\|_{L^1} = A \},
\]

\[
B_1 = \{ (\varphi, t) \in B \times [0, 1] \mid \varphi \text{ satisfies (5.2)} \},
\]

\[
B_2 = \{ u \in C^\beta(M) \mid \int_M u \omega^n = 0 \}.
\]

Then \( B_1 \) is an open subset of \( B \times [0, 1] \). Since \( \int_M \mu \omega^n = 0 \), we introduce a map \( \Phi : B_1 \to B_2 \),

\[
\Phi(\varphi, t)\omega^n = \sqrt{-1} \partial \bar{\partial} (e^\varphi \omega - t\alpha e^{-\varphi} \rho) \wedge \omega^{n-2} + n\alpha \sqrt{-1} \partial \bar{\partial} \varphi \wedge \omega^{n-2} + t\mu \frac{\omega^n}{n!}.
\]

Let \( I \) be the set

\[
\{ t \in [0, 1] \mid \text{there exists } (\varphi, t) \in B_1 \text{ such that } \Phi(\varphi, t) = 0 \}.
\]

Thus, to prove Theorem 1.1 it suffices to prove that \( I = [0, 1] \). Note that \( \varphi_0 = -\ln A \) is a solution of (5.1) at \( t = 0 \). Hence, we have \( 0 \in I \). In the following, we prove that the set \( I \) is both open and closed.

5.1. Openness. Suppose that \( \hat{t} \in I \). By the definition of the set \( I \), there exists \( (\hat{\varphi}, \hat{t}) \in B_1 \) such that \( \Phi(\hat{\varphi}, \hat{t}) = 0 \). Let \( (D_\varphi \Phi)(\hat{\varphi}, \hat{t}) \) be the linearized operator of \( \Phi \) at \( \hat{\varphi} \). Then we have

\[
(D_\varphi \Phi)(\hat{\varphi}, \hat{t}) : \{ u \in C^{2,\beta}(M) \mid \int_M u e^{-\hat{\varphi}} \omega^n = 0 \} \to \{ v \in C^\beta(M) \mid \int_M v \omega^n = 0 \}
\]

and

\[
(D_\varphi \Phi)(\hat{\varphi}, \hat{t}) u \omega^n = \sqrt{-1} \partial \bar{\partial} (ue^{\hat{\varphi}} \omega + \hat{t} \alpha e^{-\hat{\varphi}} \rho) \wedge \omega^{n-2} + 2n\alpha \sqrt{-1} \partial \bar{\partial} e^{\hat{\varphi}} \wedge \sqrt{-1} \partial \bar{\partial} u \wedge \omega^{n-2}.
\]
We use the implicit function theorem to prove the openness of $I$. It suffices to prove that $(D_\phi \Phi)(\hat{\phi}, \hat{t})$ is injective and surjective. For convenience, we let $L : C^{2,\beta}(M) \to C^\beta(M)$ be an extension operator of $(D_\phi \Phi)(\hat{\phi}, \hat{t})$. First we compute the formal $L^2$-adjoint of $L$ in the following.

For any $u, v \in C^\infty(M)$, we have
\[
\int_M v L(u) \omega^n = \int_M v \left( \sqrt{-1} \partial \bar{\partial}(u e^{\hat{\phi}} \omega + i \alpha u e^{-\hat{\phi}} \rho) + 2 n \alpha \sqrt{-1} \partial \bar{\partial} \hat{\phi} \land \sqrt{-1} \partial \bar{\partial} u \right) \land \omega^{n-2} = \int_M u \left( (e^{\hat{\phi}} \omega + i \alpha e^{-\hat{\phi}} \rho) \land \sqrt{-1} \partial \bar{\partial} v + 2 n \alpha \sqrt{-1} \partial \bar{\partial} \hat{\phi} \land \sqrt{-1} \partial \bar{\partial} v \right) \land \omega^{n-2}.
\]

This implies that
\[
L^*(v) \omega^n = \sqrt{-1} \partial \bar{\partial} v \land \left( (e^{\hat{\phi}} \omega + i \alpha e^{-\hat{\phi}} \rho) + 2 n \alpha \sqrt{-1} \partial \bar{\partial} \hat{\phi} \right) \land \omega^{n-2}.
\]

By the strong maximum principle, it follows
\[\text{Ker} L^* = \{\text{Constant functions on } M\}.\]

Since the index of $L$ is zero, we see that $\dim \text{Ker} L = 1$. Combining this with the theory of linear elliptic equations, there exists a positive function $u_0 \in C^{2,\beta}(M)$ such that
\[\text{Ker} L = \{c u_0 \mid c \in \mathbb{R}\}.
\]

Hence,
\[
\int_M u_0 e^{-\hat{\phi}} \omega^n > 0 \text{ and } u_0 \notin \{u \in C^{2,\beta}(M) \mid \int_M u e^{-\hat{\phi}} \omega^n = 0\},
\]

which implies $(D_\phi \Phi)(\hat{\phi}, \hat{t})$ is injective.

Next, for any $v \in C^{\beta}(M)$ such that $\int_M v \omega^n = 0$, by the Fredholm alternative, there exists a weak solution $u$ of the equation $Lu = v$. Moreover, by the theory of linear elliptic equations, we see that $u \in C^{2,\beta}(M)$. Taking
\[
c_0 = \frac{-\int_M u e^{-\hat{\phi}} \omega^n}{\int_M u_0 e^{-\hat{\phi}} \omega^n}.
\]

Then
\[
(D_\phi \Phi)(\hat{\phi}, \hat{t})(u + c_0 u_0) = L(u + c_0 u_0) = v \text{ and } \int_M (u + c_0 u_0) e^{-\hat{\phi}} \omega^n = 0,
\]

which implies $(D_\phi \Phi)(\hat{\phi}, \hat{t})$ is surjective.

5.2. Closeness. Since $0 \in I$ and $I$ is open, there exists $t_0 \in (0, 1]$ such that $[0, t_0) \subset I$. We need to prove $t_0 \in I$. It suffices to prove the following proposition.
Proposition 5.1. Let $\varphi_t$ be the solution of (5.1). If $\varphi_t$ satisfies (5.2) and (5.3), there exists a constant $C_A$ depending only on $A$, $t_0$, $\rho$, $\mu$, $\alpha$, $\beta$ and $(M, \omega)$ such that

$$\|\varphi_t\|_{C^{\alpha, \beta}} \leq C_A.$$  

Proof. First, we prove the zero order estimate. In fact, we have

Claim 2.  

(5.4) \(\sup_M e^{-\varphi_t} \leq 2M_0A, \quad t \in [0, t_0),\) 

where $M_0$ is the constant in Proposition 2.1.  

Note that $\varphi_0 = -\ln A$. Then $\sup_M e^{-\varphi_0} \leq M_0A$, which satisfies (5.4). Thus, if (5.4) is false, there will exist $\tilde{t} \in (0, t_0)$ such that

(5.5) \(\sup_M e^{-\varphi_{\tilde{t}}} = 2M_0A.\)  

We may assume that $2M_0A \leq \delta_0$, where $\delta_0 = \sqrt{\frac{1}{2|\alpha|\|\rho\|_{C^0} + 1}}$ is chosen as in Proposition 2.1. Namely, $e^{-\varphi_{\tilde{t}}} \leq \delta_0$. Hence, we can apply Proposition 2.1 to $\varphi_{\tilde{t}}$ while $\rho$ and $\mu$ are replaced by $t\rho$ and $t\mu$, respectively, and we obtain

$$e^{-\varphi_{\tilde{t}}} \leq M_0A,$$

which contradicts to (5.5). This proves (5.4). Combining (5.4) and Proposition 2.1 we obtain the zero order estimate

Next, we use the similar argument to prove the second order estimate

(5.6) \(\sup_M |\partial\varphi_{\tilde{t}}|_g \leq D_0,\)  

for any $t \in (0, t_0)$, where $D_0$ is the constant as in Proposition 4.1. If (5.6) is false, there exists $\tilde{t} \in (0, t_0)$ such that

$$\sup_M |\partial\varphi_{\tilde{t}}|_g = D_0.$$  

Recalling Proposition 4.1 we get

$$\sup_M |\partial\varphi_{\tilde{t}}|_g \leq \frac{D_0}{2},$$

which is a contradiction. Thus (5.6) is true.

By (5.6) and Proposition 3.1 we have the first order estimate

(5.7) \(\sup_M |\partial\varphi_{\tilde{t}}|_g^2 \leq C.\)  

Combining (5.6) and (5.7) with equation (5.1) (Note that (5.1) is equivalent to a 2-nd Hessian type equation as (1.8)), we get

(5.8) \(\left|\sigma_2(\tilde{\omega}) - \frac{n(n-1)}{2} e^{2\varphi}\right| \leq Ce^{\varphi}.\)  

Then, by the zero order estimate, we deduce

$$\frac{1}{CA^2} \leq \sigma_2(\tilde{\omega}) \leq \frac{C}{A^2}. $$
Hence, (5.1) is uniformly elliptic and non-degenerate. By the $C^{2,\alpha}$-estimate (cf. [20, Theorem 1.1]), we obtain
\begin{equation}
\|\varphi_t\|_{C^{2,\alpha}} \leq C_A.
\end{equation}
\qed

5.3. Uniqueness. In this subsection, we give the proof of Theorem 1.4. First, we show the uniqueness of solutions to (5.1) when $t = 0$.

Lemma 5.2. When $t = 0$, (5.1) has a unique solution
\[ \varphi_0 = -\ln A. \]

Proof. By the similar calculation of (2.5) (taking $k = 1$), we obtain
\begin{align*}
\int_M e^{-\varphi} (e^{\varphi} \omega + \alpha e^{-\varphi} t \rho) \wedge (-1) \partial \varphi \wedge \bar{\partial} \varphi \wedge \omega^{n-2} & \leq 2\alpha \int_M e^{-2\varphi} \sqrt{-1} \partial \varphi \wedge \bar{\partial} (t \rho) \wedge \omega^{n-2} - 2 \int_M e^{-\varphi} t \mu \frac{\omega^n}{n!}.
\end{align*}

When $t = 0$, it is clear that
\[ \int_M \sqrt{-1} \partial \varphi \wedge \bar{\partial} \varphi \wedge \omega^{n-1} = 0. \]
Combining this with the normalization condition $\|e^{-\varphi}\|_{L^1} = A$, we obtain
\[ \varphi_0 = -\ln A. \]
\qed

Proof of Theorem 1.4. Assume that we have two solutions $\varphi$ and $\varphi'$ of (1.5). We use the continuity method to solve (5.1) from $t = 1$ to 0. Note that $\varphi$ and $\varphi'$ are both solutions when $t = 1$. Then by the implicit function theorem as in Subsection 5.1, there is a smooth solution $\varphi^1_t$ (or $\varphi^2_t$) of (5.1) for any $t \in (t_0, 1]$ ($t_0 < 1$) with the property $\varphi^1_1 = \varphi$ (or $\varphi^2_1 = \varphi'$). Set
\[ J_\varphi = \{ t \in [0, 1] \mid \text{there exists a family of smooth solutions } \varphi^1_{t'} \text{ of (5.1)} \text{ for any } t \in [t_0, 1] \text{ such that } \varphi^1_1 = \varphi \}. \]

From the argument in Section 2-4, we see that Proposition 2.1, Proposition 3.1, and Proposition 4.1 are still true for $\varphi^1_t$. As a consequence, Proposition 5.1 holds for $\varphi^1_t$. Thus $J_\varphi = [0, 1]$. Similarly, $J_{\varphi'} = [0, 1]$. On the other hand, thanks to Lemma 5.2, we have
\[ \varphi^1_0 = \varphi^2_0 = -\ln A. \]
Hence $\varphi^1_t = \varphi^2_t$ for any $t \in [0, 1]$. Theorem 1.4 is proved.
\qed

It seems that the condition (1.10) in Theorem 1.4 can be removed. In precise, we have the following conjecture.

Conjecture 5.3. The solution $\varphi$ of (1.5) in Theorem 1.4 is unique.
Remark 5.4. We remark that conjecture 5.3 is true if \( \alpha < 0 \) and \( \rho \geq 0 \) in equation (1.5). In fact, by modifying the argument in the proof of Proposition 2.1, we can get the \( C^0 \)-estimate for the solution \( \varphi_t \) of equation (5.1) by the assumption of (1.6) and (1.7) in this case. Then by the \( C^2 \)-estimate in [14, Proposition 5, Proposition 6], we can also obtain (5.9). We will discuss it for details somewhere.

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