Bulk Dynamics in Confining Gauge Theories

Marcus Berg\textsuperscript{1}, Michael Haack\textsuperscript{1} and Wolfgang M"uck\textsuperscript{2}

\textsuperscript{1} Kavli Institute for Theoretical Physics, University of California
Santa Barbara, California 93106-4030, USA
E-mails: mberg, mhaack@kitp.ucsb.edu

\textsuperscript{2} Dipartimento di Scienze Fisiche, Universit\`a degli Studi di Napoli “Federico II”
via Cintia, 80126 Napoli, Italy
E-mail: mueck@na.infn.it

Abstract

We consider gauge/string duality (in the supergravity approximation) for confining gauge theories. The system under scrutiny is a 5-dimensional consistent truncation of type IIB supergravity obtained using the Papadopoulos-Tseytlin ansatz with boundary momentum added. We develop a gauge-invariant and sigma-model-covariant approach to the dynamics of 5-dimensional bulk fluctuations. For the Maldacena-Nunez subsystem, we study glueball mass spectra. For the Klebanov-Strassler subsystem, we compute the linearized equations of motion for the 7-scalar system, and show that a 3-scalar sector containing the scalar dual to the gluino bilinear decouples in the UV. We solve the fluctuation equations exactly in the "moderate UV" approximation and check this approximation numerically. Our results demonstrate the feasibility of analyzing the generally coupled equations for scalar bulk fluctuations, and constitute a step on the way towards computing correlators in confining gauge theories.
1 Introduction

Gauge/string duality offers an alternative approach to aspects of supersymmetric non-Abelian gauge theories that are hard to describe with conventional techniques. For example, at strong coupling many non-Abelian gauge theories exhibit confinement, the familiar yet still somewhat mysterious phenomenon that the only finite-energy states are singlets under the color gauge group: at colliders, we never see quarks directly, only colorless hadrons. The details of confinement, and of other nonperturbative phenomena such as chiral symmetry breaking, are difficult to capture with conventional gauge theory methods. In the dual picture, the nonperturbative gauge theory regime is typically described by weakly coupled closed strings propagating on a space of higher dimensionality (the bulk), and their dynamics can be approximated by classical supergravity.

One of the most powerful applications of gauge/string duality is the calculation of field theory correlation functions from the dual bulk dynamics. This idea was developed in \[1, 2, 3\] for superconformal gauge theories, whose gravity duals are Anti-de Sitter (AdS) spaces. Since then, in a program known as holographic renormalization (\[4, 5, 6, 7, 8, 9\] and references therein), it has been systematically generalized to gauge theories that are conformal in the ultraviolet (UV), whose gravity duals are asymptotically AdS spaces. There are several reasons to push ahead with this line of research. First, confining gauge theories have duals that are not asymptotically AdS. Despite some progress, the holographic calculation of correlators in confining gauge theories has not yet been carried out in any controlled approximation (we shall specify later what we mean by that). Second, interesting new supergravity solutions have been found recently \[10, 11, 12, 13\]. They are regular and thus qualify as dual configurations of ground states in gauge theories. It is then only natural to investigate the possibility of calculating correlation functions for their dual field theories. In this paper, we report on progress towards this general goal and sharpen some of the remaining challenges.

Some of the asymptotically AdS backgrounds studied in the literature (such as the GPPZ flow \[14\]) were originally envisaged as toy-model duals of confining gauge theories. The obstruction to being full-fledged duals is a naked curvature singularity at finite distance from the boundary into the bulk. One would have liked to interpret this distance as the (dynamically generated) scale of onset of confinement in the dual field theory, but unbounded curvature invalidates the use of the supergravity approximation to string theory. Although string theory appears to cure these curvature singularities by the enhancement mechanism \[15\] or the Myers effect \[16\], correlators are always computed in the supergravity approximation. In practice, this involves imposing regularity conditions on the bulk fluctuations at the curvature singularity. It would be interesting to quantify the precise effect of the string theory resolution on the explicit correlators computed in singular supergravity backgrounds, but it would be simpler if one could compute correlators directly from regular duals.

An even simpler approach to computing correlators is the hard-wall approximation, which has been used in the effort to connect gauge/string duality to real QCD. This
Figure 1: Bulk toy models used in the literature. In the hard-wall approximation, the bulk is exactly AdS, so couplings in the gauge theory do not run (represented by straight sides in the figure). In singular approximations, like the singular conifold, there is logarithmic running, but also a curvature singularity (represented by the black dot).

“AdS/QCD correspondence” studies problems like meson-hadron coupling universality \[17, 18, 19\] and deep inelastic scattering \[20\]. In the hard-wall approximation, one replaces the regular solution by AdS space cut off at a minimal radius \(r_{\text{IR}}\), the idea being that some of the physics should be insensitive to the details of the geometry in the deep infrared (IR) region, while retaining conformal UV behaviour. Then, the issue arises which boundary conditions to impose at the IR boundary. If one were able to compute correlators directly in the regular solution at least for some simple cases, a qualitative picture of which hard-wall boundary conditions best mimic the behaviour in the regular case could be pieced together.\(^1\)

Thus, we are interested in the question to what extent it is feasible to compute correlators directly from regular supergravity duals of confining gauge theories. The first example of such a bulk configuration was the warped deformed conifold solution found by Klebanov and Strassler (KS) \[22\]. The \(\mathcal{N} = 1\) supersymmetric gauge theory dual to this solution undergoes a cascade of Seiberg dualities, as recently explained in greater detail in the lecture notes by Strassler \[23\].\(^2\) Importantly, the curvature remains small everywhere, so the supergravity approximation can be used at all energies. Another example is the wrapped D5-brane, also known as the Maldacena-Nunez (MN) solution \[25\]. In the infrared, it shares many properties with \(\mathcal{N} = 1\) SYM theory \[26, 27, 28, 29\], but it becomes six-dimensional little string theory in the UV. We are mostly interested in the KS solution, since there the supergravity approximation is under full control.

Even before addressing the implementation of gauge/string duality in such bulk config-

\(^1\)Incidentally, in \[21\], cut-off AdS was used to model the dynamics of D-brane inflationary cosmology on the Klebanov-Strassler background. It is not unreasonable to hope that our methods will also prove useful in that context, for the same reasons as for AdS/QCD.

\(^2\)See also \[24\] for a nice review of the KS solution.
urations, it is worth noting that the dual field theory interpretation of non-asymptotically-AdS supergravity configurations poses a conceptual problem (which we do not resolve): in holographic renormalization, the asymptotically AdS bulk region corresponds to the presence of a Wilsonian renormalization group (RG) UV fixed point in the 4-dimensional gauge theory. Thus, in its absence, one might wonder whether the dual gauge theory is well-defined in the Wilsonian sense. Several viewpoints on this are possible. One may defer the UV completion of the field theory to string theory, as in the MN solution. Alternatively, one can attempt to define the field theory by its holographic dual, as advocated in [30]. Another hope may be to embed the KS solution into a more complicated configuration with an asymptotically AdS region, so that the dual field theory is UV-conformal, but there is an intermediate energy range where couplings do run logarithmically as in the KS solution. Here, we adopt a pragmatic approach, somewhat like [30]. We extrapolate from AdS/CFT that the bulk dynamics encodes information about some dual field theory, which might only be an effective theory, and try to see which of its features can be extracted by existing holographic renormalization technology.

Optimistically, then, we would like to investigate whether techniques similar to holographic renormalization can be used to calculate (effective) field theory correlation functions from supergravity duals in non-asymptotically AdS setups. First results in this direction were obtained by Krasnitz [31, 32, 33] for certain 2-point functions in the singular conifold (Klebanov-Tseytlin) background [34], and in the limit of very large energy. No counterterms were obtained, but it was argued that the particular correlators studied would only have received minor corrections, had counterterms been included. Counterterms were recently studied in a tour de force by Aharony, Buchel and Yarom [30], who obtained renormalized one-point functions of the stress-energy tensor.

To address the problem of computing correlators systematically, one must face three interrelated issues:

1. define precisely the duality relations between supergravity fields and field theory operators (the “dictionary” problem);

2. renormalize the bulk prescription for correlation functions, that is, compute the requisite covariant counterterms and show the absence of divergences (the “renormalization” problem);

3. solve for the dynamics of supergravity fluctuations about the background of interest, where the fluctuations must be allowed to vary along the external spacetime coordinates (the “fluctuation” problem).

In this paper, we will mostly address the last issue. We focus on gauge theories dual to the regular supergravity solutions discussed above: the MN solution and especially the KS solution of type IIB supergravity in ten dimensions.

Although it may seem that the first and second issues should be resolved first, this point is moot: ultimately all three questions have to be addressed, and as we shall see, the solution to one may help with the others.
In holographic renormalization, the bulk dynamics is 5-dimensional (for a 4-dimensional gauge theory). Thus, we need to find a sector of type IIB supergravity which can be described by a 5-dimensional system, while allowing for the background solutions we are interested in. Papadopoulos and Tseytlin (PT) [35] found an effective 1-dimensional action (subject to a Hamiltonian constraint) that is general enough to describe both the MN and KS background solutions, where the fields only depend on the radial coordinate. This suggests that one can suitably generalize their ansatz to allow the parameterizing scalar fields to depend also on the coordinates of the 4 dimensions of the gauge theory. In other words, we add boundary momentum to the PT ansatz, which leads to a 5-dimensional effective theory. We will show that, after imposing an integrability constraint that is automatically satisfied for the MN and KS systems, this generalization constitutes a consistent truncation of type IIB supergravity and gives rise to a non-linear sigma model of scalars coupled to 5-dimensional gravity. Moreover, the resulting 5-dimensional system falls into a general class of actions dubbed “fake supergravity” actions in [36], since the scalar potential is determined by a function resembling a superpotential. We will mostly stick to this terminology (i.e., “fake supergravity”), even though the background solutions we consider have been shown to preserve some supersymmetry [37, 38], and one might expect the full system to be embeddable in a supersymmetric system (see Sec. 3 for some further comments on this). These fake supergravity actions are formally similar to those governing holographic RG flow backgrounds in standard AdS/CFT, which suggests that they can be studied using appropriately generalized AdS/CFT techniques.

Thus, we need to study the dynamics of fluctuations about the (MN and KS) background solutions in the effective five-dimensional bulk system. To this end, we generalize the gauge-invariant formalism developed in [39] to generic multi-scalar systems. The gauge-invariant formalism overcomes technical difficulties encountered in early work on correlation functions in holographic RG flows [40, 41, 42]. These difficulties arose from the fact that the fluctuations of “active” scalars (those with a non-trivial radial background profile) couple to the fluctuations of the five-dimensional metric already at the linear level, making it inconsistent to set the metric fluctuations to zero when studying the scalar fluctuations, or vice versa. Consistent treatment of the coupled system typically involved, even in the simplest cases, third-order differential equations containing spurious gauge redundancies that needed to be painstakingly factored out by hand. Happily, fluctuations are manifestly disentangled at the linear level in the gauge-invariant formalism, and their equations of motion are second-order. The formalism was applied to the holographic calculation of three-point functions and scattering amplitudes in [49] (see also [50] for

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4Gauge-invariant variables for linearized scalar-gravity systems have been studied in cosmology since the early 1980s [43, 44]. Those variables are similar to the ones used in holographic renormalization in [45, 46, 4]; typical cosmological backgrounds are themselves very similar to the Poincaré-sliced AdS domain walls used in the simplest RG flow geometries. The connection between the linearized cosmology variables and linearized holographic-renormalization variables was studied in [47]. Also, holography of finite-temperature field theories using linearized gauge-invariant variables was initiated in [48]. Although the applications in this paper are worked out at the linear level, our gauge-invariant formalism is defined non-linearly.
earlier work on three-point functions), and the main ideas for the generalization that we undertake here were presented in \[51\].

These two ingredients—consistent truncation to a five-dimensional fake supergravity system, and a general gauge-invariant formalism to describe its fluctuations—put us in a position to tackle the “fluctuation problem” in the list above, and now we proceed to summarize the new results of our work. We first note that in most cases, one can only expect to solve the fluctuation equations numerically, but there are notable exceptions and simplifying limits where analytical solutions are possible. Still, with the hope that issues 1 and 2 in the above list will be solved in the future, we wish to emphasize that numerical integration of classical gauge-invariant ordinary differential equations is a vastly simpler problem than numerically computing the corresponding correlators by lattice methods directly in the gauge theory, so the formulation of equations well-suited for numerical analysis should be important even in the absence of analytical solutions. Moreover, even without solving issues 1 and 2, there are physical quantities that should not depend on the counterterms and which can, therefore, be addressed directly using our methods. For example, glueball masses in the gauge theory correspond to the existence of normalizable bulk modes and do not depend on renormalization details. As an example, we calculate the mass spectra of states for the \( \mathcal{N} = 1 \) gauge theory dual of the MN solution, up to a caveat discussed in Sec. 5. We note that mass spectra obtained in the literature \[52, 53\] disagree with ours, which will be discussed more thoroughly in Secs. 5 and 7.

For the KS background, one can only hope to obtain numerical results, so we pose the problem in terms of gauge-invariant variables and leave numerical evaluation to future work. However, we can analytically study the scalar fluctuations of the KS system (i.e., the 7 scalars present in the KS ansatz) in the singular Klebanov-Tseytlin (KT) background \[34\], which is a sensible approximation to the ultraviolet region of the KS background. In this case, we observe decoupling between the 4-scalar KT system and the 3 additional scalars that are present in the KS system. We will refer to this group of 3 scalars as the \textit{gluino sector}, because it contains the scalar dual to the gluino bilinear \( \text{tr} \lambda \lambda \). The remaining equations are simple enough to allow for analytical solutions in the “moderate UV” regime considered by Krasnitz \[31, 32, 33\], in terms of combinations of Bessel functions and logarithms. For the ultraviolet physics of the KS gauge theory, we consider the Krasnitz approximation to be controlled, since we will be able to check it numerically by computing the same solutions in KS. We leave a thorough check for future work and content ourselves with comparing our analytical results to numerical solutions of the full equations in the KT background. The result is that the Krasnitz approximation seems to work very well, so we expect our analytical solutions to be useful as guidance in numerical work in the full KS background.

Now, let us outline the rest of the paper. In Sec. 2 we start by reviewing briefly the essentials of holographic renormalization in AdS/CFT, in particular the dictionary and renormalization problems. On the way, we will introduce generic fake supergravity, which is the typical bulk system in holographic RG flows. Then, in Sec. 3 we perform a consistent truncation of type IIB supergravity to a 5-dimensional fake supergravity
Sec. 4 is dedicated to the generalization of the gauge-invariant analysis of bulk fluctuations \cite{39} to a generic “fake supergravity” system, allowing for an arbitrary number of scalars and an arbitrary (but invertible) sigma-model metric. The principle of reparametrization invariance is the beacon that guides us to the main result of this section: a system of (generally coupled) second order differential equations, which describes the dynamics of the scalar fluctuations about Poincaré-sliced domain walls in a manifestly gauge-invariant fashion. The presentation of the gauge-invariant method is intended to be pedagogical: the principal line of argument is explained in the main text, while details are included in the appendices.

In Secs. 5 and 6 we use our techniques to study the MN and KS systems, respectively. For both, we shall first derive the most general background solutions including all integration constants. Although the regular bulk configurations correspond to a unique choice of integration constants, we find it useful to keep the constant governing the resolution of the singularity. It determines the vacuum expectation value of the gluino bilinear, and by tuning it one is able to consider regimes where analytic solutions to the fluctuation equations are possible. In the MN system, we discover a number of normalizable (sub-leading only) modes, which we link to the mass values of glueball states. In the KS system, we perform the calculation in the singular KT background and, in addition, apply the Krasnitz approximation. The resulting solutions are very similar to the ones Krasnitz found in simpler cases, but we refrain from trying to extract correlators given that the “dictionary” and “renormalization” problems have yet to be solved.

Finally, Sec. 7 contains conclusions and a discussion of possible further developments.

2 Review: correlation functions from AdS/CFT

In this section, we briefly review some essentials of holographic renormalization, following the three-pronged list of problems discussed in the introduction. We review how holographic renormalization systematically resolves the "dictionary" and "renormalization" problems for asymptotically AdS bulk geometries. These two steps must ultimately be generalized to non-asymptotically AdS setups. We leave their general resolution to future work (initial progress was made in \cite{39}), but we will comment on some of the specific challenges. A general approach to the third issue in the list, the fluctuation problem, will be described in detail in Sec. 4.

Let us start by introducing a generic “fake supergravity” system, a non-linear sigma model of scalar fields with a particular potential, coupled to gravity in \( d + 1 \) dimensions (typically, \( d = 4 \)). Its action is given by\(^5\)

\[
S = \int d^{d+1}x \sqrt{g} \left[ -\frac{1}{4} R + \frac{1}{2} G_{ab}(\phi) \partial_\mu \phi^a \partial^\mu \phi^b + V(\phi) \right],
\]

\(5\)We follow the curvature conventions of MTW and Wald \cite{54, 55}, i.e., the signature is mostly “+”, and \( R_{jkl}^a = \partial_k \Gamma^a_{jl} + \Gamma^a_{km} \Gamma^m_{jl} - (k \leftrightarrow l) \). This has the opposite sign of the convention used in \cite{39}.
where the potential, $V(\phi)$, follows from a superpotential, $W(\phi)$, by

$$V(\phi) = \frac{1}{2} G^{ab} W_a W_b - \frac{d}{d - 1} W^2 . \quad (2.2)$$

The matrix $G^{ab}(\phi)$ is the inverse of the sigma model metric $G_{ab}(\phi)$. Our notation is as in [39], i.e., derivatives of $W$ with respect to fields are indicated as subscripts, as in $W_a = \partial W/\partial \phi^a$. Moreover, the sigma model metric and its inverse are used to lower and raise field indices.

Actions of the form (2.1) arise in a variety of cases, such as the familiar truncation of $\mathcal{N} = 8$, $d = 5$ gauged supergravity, where several holographic RG flow background solutions have been found. As we shall see in the next subsection, other consistent truncations of type IIB supergravity can also give rise to effective actions of the form (2.1). This richness in applications is our main motivation for considering the generic case in detail.

We are interested in a particular class of solutions of the action (2.1) with $d$-dimensional Poincaré invariance, called Poincaré-sliced domain walls\footnote{As opposed to, for example, the AdS-sliced domain walls studied in [56, 57, 9], where the $d$-dimensional boundary can be AdS instead of flat space.} or holographic RG flow backgrounds:

$$\begin{align*}
\text{ds}^2 &= dr^2 + e^{2A(r)} \eta_{ij} \, dx^i \, dx^j , \\
\phi^a &= \bar{\phi}^a(r) .
\end{align*} \quad (2.3)$$

That is, the radial domain wall in the metric is supported by a radial profile of one or several scalars (the “active” scalars). If the background fields are determined by the following coupled first order equations (which is true in all the cases we consider):

$$\begin{align*}
\partial_r A(r) &= - \frac{2}{d - 1} W(\bar{\phi}) , \\
\partial_r \bar{\phi}^a(r) &= G^{ab} W_b ,
\end{align*} \quad (2.4)$$

the domain wall has been shown to be stable, cf. [58, 36]. These relations do not specify the background uniquely (integration constants!), but they are sufficient for the general analysis carried out in this section. We also note that, although the various backgrounds we study in this paper are “logarithmically warped” and not usually given in the form (2.3), one can always reach this form by a change of radial variable.

For the system (2.4) to admit an asymptotically AdS solution, it is necessary and sufficient that the superpotential $W$ possess a local extremum with a non-zero value, i.e., $W_a(\phi_0) = 0$ for all $a$. Then, $\phi_0$ is called a fixed point. Without loss of generality, we can assume that the fixed-point value of $W$ is negative\footnote{Note that an overall sign change of $W$ can be absorbed by changing the sign of the coordinate $r$.} $W(\phi_0) = -(d - 1)/(2L)$, where $L$ is the characteristic AdS length scale which is often set to $L = 1$.

Let us now briefly review how the issues discussed in Sec. I are addressed in AdS/CFT. We start with issue 1, the dictionary between gauge theory operators and bulk fields. The
action \((2.1)\) is manifestly invariant under field redefinitions—this is indeed the point of the gauge-invariant formalism that we develop in Sec. 3—but this invariance is given up when formulating the one-to-one correspondence between bulk fields and primary conformal operators of the dual gauge theory. As is well known, conformal invariance imposes that two-point functions of primary conformal operators of different weights vanish:

\[
\langle O_{\Delta}(x_1) O_{\Delta'}(x_2) \rangle = 0 \quad \text{for } \Delta \neq \Delta'.
\]

(2.5)

In AdS/CFT, this orthogonality property is achieved for the holographically calculated correlators by the following choice of field variables. Let us consider Riemann normal coordinates (RNCs) \([59, 55]\) in field space around the fixed point \(\phi_0\). This means that we choose field variables such that \(\phi_0 = 0\), and that the sigma model connections (defined later in (4.1)) vanish at \(\phi_0\). This still leaves us the freedom to impose \(G_{ab}(\phi_0) = \delta_{ab}\) and, by means of a rotation, to diagonalize the symmetric matrix of second derivatives of \(W\) at \(\phi_0\). With this choice of parametrization, \(W\) has the following expansion around the fixed point,

\[
W = -\frac{(d-1)}{2L} - \frac{1}{2} \sum_a \lambda_a (\phi^a)^2 + \cdots,
\]

(2.6)

where the ellipsis stands for terms that are at least cubic in \(\phi\). Using the AdS/CFT dictionary, it is now a simple matter to establish that the fields \(\phi^a\) are dual to primary conformal operators of dimensions:

\[
\Delta_a = \frac{d}{2} \pm \left| \frac{d}{2} - \lambda_a \right|.
\]

(2.7)

For pure AdS, \((2.6)\) ensures that the matrix of holographically calculated two-point functions is diagonal, that is, equation \((2.5)\) follows. In general, \((2.6)\) is not enough to unambiguously identify a map between supergravity modes and field theory operators. For operators with the same dimension, one can usually distinguish them by other quantum numbers like transformations under \(R\)-symmetry groups. (When even that fails, one can try to use additional information from the correlators \(7\).) It is fair to say that the dictionary question is well understood in known asymptotically AdS examples.

The second issue, renormalization, is solved in general for bulk systems with asymptotically AdS bulk geometries by holographic renormalization. The reader is referred to the relevant papers \([4, 5, 62, 6, 7, 63, 8, 9]\) and lecture notes \([64]\) for details. Holographic renormalization systematically removes the divergences by first formulating the bulk theory on a bulk space with cut-off boundary located well in the asymptotic UV region. Covariant local counterterms are added to the action so that removing the cutoff yields a finite generating functional, and therefore finite correlation functions. The result of this procedure is most compactly described in terms of the notions of sources and responses, which are the coefficients in front of the leading and sub-leading series in the asymptotic

\(^8\)Usually the upper sign applies. The lower sign can be chosen if \(|d/2 - \lambda_a| < 1\), and is accompanied by imposing irregular boundary conditions on the bulk fields \([60, 61, 7]\).
expansion of the bulk fields, respectively. That is, a bulk scalar that is dual to an operator of dimension $\Delta_a$ (with + sign in (2.7)) displays asymptotic behavior of the schematic form

$$\phi^a(x, r) \approx e^{-(d-\Delta_a)r} \left[ \hat{\phi}^a(x) + \cdots \right] + e^{-\Delta_ar} \left[ \tilde{\phi}^a(x) + \cdots \right],$$

(2.8)

where $\hat{\phi}$ and $\tilde{\phi}$ denote the source and response functions, respectively. Up to the addition of scheme-dependent local terms, which arise from adding finite counterterms to the action, the response function represents the exact one-point function of the dual operator, i.e., the one-point function in the presence of sources. Thus, in order to calculate higher correlation functions, one needs to solve the dynamics of bulk fluctuations up to the required order (e.g., quadratic for 3-point functions), extract the response function from their asymptotic behavior, and differentiate with respect to the sources. It is important to note that although the local terms are scheme-dependent, in general they cannot just be dropped. As was stressed in [4, 5], correlation functions computed in conflicting schemes will in general fail to fulfill the requisite Ward identities. The most efficient renormalization method to date is that presented in [9], that homes in on the minimal calculation needed for each correlator.

Here is an example of a correlation function calculated in this fashion: take the GPPZ flow, which is $\mathcal{N}=4$ SYM deformed by a $\Delta=3$ operator insertion. The result for the two-point function of this operator for arbitrary boundary momentum $p$ is

$$\langle O_\phi(p)O_\phi(-p) \rangle = \frac{N^2}{2\pi^2} p^2 \left[ \psi \left( \frac{3}{2} + \frac{1}{2} \sqrt{1-p^2} \right) + \psi \left( \frac{3}{2} - \frac{1}{2} \sqrt{1-p^2} \right) - 2\psi(1) \right],$$

(2.9)

where $\psi(z) = \Gamma'(z)/\Gamma(z)$. Note that the only scale in this expression is the asymptotic AdS length scale $L$, which has been set to unity and is easily restored replacing $p \to pL$. The ultraviolet ($p^2 \to \infty$) asymptotics is that of the limiting conformal theory, namely $\langle O_\phi(p)O_\phi(-p) \rangle \to p^{2\Delta-4}\log p$. The infrared regime (small $p^2$) encodes the spectrum in a series of poles. It would be very interesting to understand the connection, if any, between AdS/CFT correlators of this type and high-energy correlators computed by integrability in QCD (see e.g. [65]), summing large numbers of certain classes of diagrams. It is intriguing that those correlators also involve the $\psi$ function.

To end this section, let us outline how we imagine approaching the "dictionary" and "renormalization" problems in the non-asymptotically AdS case. The absence of a fixed point of $W$, as in the KS and MN solutions, invalidates some of the strategies discussed above. First, the asymptotic behavior must probably be studied on a case-by-case basis: in general, there may not be a basis in which the scalars decouple asymptotically (we will encounter examples of this later in the MN and KS systems). This means that the bulk field/boundary operator dictionary should be reformulated as finding suitable source functions for the boundary operators. One possibility is to generalize the AdS/CFT

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9The correlators given in these papers differ by a scheme-dependent local term.

10Note that this is so even in generic AdS/CFT, where the bulk fields may not decouple in holographic RG flow backgrounds in some cases, making it ambiguous to speak of the dual bulk field of a specific gauge theory operator.
definition of the source function as follows: A system of \( n \) coupled second-order differential equations for \( n \) scalars has \( 2n \) independent asymptotic solutions, of which \( n \) can be regarded as “leading” and \( n \) as “subleading”. The \( n \) coefficients of the leading solutions can be defined to be sources of dual gauge theory operators. It might be possible to exploit the coupling between the bulk fields to describe operator mixing in the dual gauge theory, but we leave this interesting question for the future. For the KS case, we follow the strategy adopted so far in the literature: to consider fields that are mass eigenstates in the conformal limit. This does not, of course, resolve the dictionary issue in the nonconformal case.

Second, in AdS/CFT, asymptotically AdS behavior implies that the divergent terms of the bulk on-shell action can be ordered into a double expansion in powers of the scalar fields and of the number of boundary derivatives, with higher order terms being less divergent. This means that the number of divergent terms is finite, and that they can be cancelled by adding covariant local counterterms at a cutoff boundary. At present, we have no equivalent prescription for general bulk systems, although the results of [30] are very promising.

A general approach to solving the "fluctuation problem" will be described in detail in Sec. 4. But first, we need to derive the system in which we will study fluctuations.

### 3 Adding boundary momentum to the PT ansatz

The Papadopoulos-Tseytlin (PT) ansatz for type IIB supergravity solutions with fluxes [35] reduces the problem of finding these particular flux solutions to solving the equations of motion deriving from an effective one-dimensional action subject to a zero-energy constraint. This suggests that it should be possible to generalize the PT ansatz in such a way that the scalars that parametrize the 10-dimensional solution depend not only on a “radial” variable, but on all five “external” variables. This corresponds to allowing for non-zero momentum in the boundary theory, as required for computing correlators as functions of momentum. Such a generalization is indeed possible, and we shall present the result in this section, with the technical details given in appendix A. In order not to unnecessarily overload the notation, we deviate slightly from the convention used in the appendix by dropping tildes from the 5-dimensional objects. In the main text, the meaning of the symbols should be clear from the context, whereas a clearer distinction is needed for the detailed calculations in the appendices. The resulting five-dimensional action is of the form (2.1). It will be important for us that in many cases of interest, including the KS and MN systems, a superpotential \( W \) generating the potential \( V \) via (2.2) is known [35].
The equations of motion of type IIB supergravity in the Einstein frame are
\[ R_{MN} = \frac{1}{2} \partial_M \Phi \partial_N \Phi + \frac{1}{2} e^{2\Phi} \partial_M C \partial_N C + \frac{1}{96} g_s^2 \tilde{F}_{MPQRS} \tilde{F}^{PQRS}_N \] (3.1)
\[ + \frac{g_s}{4} (e^{-\Phi} H_{MPQ} H_{N}^{PQ} + e^{\Phi} \tilde{F}_{MPQ} \tilde{F}^{PQ}_N) \]
\[- \frac{g_s}{48} g_{MN} (e^{-\Phi} H_{PQR} H_{PQR} + e^{\Phi} \tilde{F}_{PQR} \tilde{F}^{PQR}_N), \]
\[ d \ast d \Phi = e^{2\Phi} dC \wedge \ast dC - \frac{g_s}{2} e^{-\Phi} H_3 \wedge \ast H_3 + \frac{g_s}{2} e^{\Phi} \tilde{F}_3 \wedge \ast \tilde{F}_3, \] (3.2)
\[ d(e^{2\Phi} \ast dC) = -g_s e^{\Phi} H_3 \wedge \ast \tilde{F}_3, \] (3.3)
\[ d(e^{\Phi} \ast \tilde{F}_3) = g_s F_3 \wedge H_3, \] (3.4)
\[ d \ast (e^{-\Phi} H_3 - C e^{\Phi} \tilde{F}_3) = -g_s F_5 \wedge F_3, \] (3.5)
\[ \ast \tilde{F}_5 = \tilde{F}_5, \] (3.6)

where we have used the notation
\[ F_3 = dC_2, \quad H_3 = dB_2, \quad F_5 = dC_4, \quad \tilde{F}_3 = F_3 - CH_3, \quad \tilde{F}_5 = F_5 + B_2 \wedge F_3. \]

From the last definition follows the Bianchi identity
\[ d \tilde{F}_5 = H_3 \wedge F_3. \] (3.7)

In the following we set \( g_s = 1 \) and \( \alpha' = 1. \)

Our ansatz for a consistent truncation follows PT closely, but allows the scalar fields to depend on all five external coordinates. Thus, we take
\[ ds^2_{10} = e^{2p-x} ds_5^2 + (e^{x+y} + a^2 e^{x-y})(e_1^2 + e_2^2) + e^{x-y}[e_3^2 + e_4^2 - 2a(e_1 e_3 + e_2 e_4)] + e^{-6p-x} e_5^2, \]
\[ ds_5^2 = g_{\mu\nu} dy^\mu dy^\nu, \]
\[ H_3 = h_2 e_5 \wedge (e_1 \wedge e_2 \wedge e_3 \wedge e_1) + dy^\mu \wedge [\partial_\mu h_1 (e_4 \wedge e_3 + e_2 \wedge e_1) + \partial_\mu h_2 (e_4 \wedge e_1 - e_3 \wedge e_2) + \partial_\mu \chi (-e_4 \wedge e_3 + e_2 \wedge e_1)], \]
\[ F_3 = P \{ e_5 \wedge [e_1 \wedge e_3 + e_2 \wedge e_1 - b(e_4 \wedge e_1 - e_3 \wedge e_2)] + dy^\mu \wedge [\partial_\mu b(e_4 \wedge e_2 + e_3 \wedge e_1)] \}, \]
\[ \Phi = \Phi(y), \quad C = 0, \]
\[ \tilde{F}_5 = \mathcal{F}_5 + \ast \mathcal{F}_5, \quad \mathcal{F}_5 = K e_1 \wedge e_2 \wedge e_3 \wedge e_4 \wedge e_5, \] (3.8)

where \( p, x, g, a, b, h_1, h_2, K \) and \( \chi \) are functions of the external coordinates \( y^\mu \), and \( P \) is a constant measuring the units of 3-form flux across the 3-cycle of \( T^{1,1} \) in the UV. For readers familiar with the KS background, it may be useful to note that \( \Phi = \chi = 0 \) in KS, and the other fields have backgrounds as given later in section 6.1.

We are using the KS convention for the forms\(^{11}\), i.e.,
\[ e_1 = -\sin \theta_1 \, d\phi_1, \quad e_2 = d\theta_1, \quad e_3 = \cos \psi \sin \theta_2 \, d\phi_2 - \sin \psi \, d\theta_2, \]
\[ e_4 = \sin \psi \sin \theta_2 \, d\phi_2 + \cos \psi \, d\theta_2, \quad e_5 = d\psi + \cos \theta_1 \, d\phi_1 + \cos \theta_2 \, d\phi_2. \] (3.9)

\(^{11}\)The relation to the PT and MN conventions can be found in footnote 7 of [35].
We note that the first term in the ansatz for $F_3$ is essentially $\omega_3 = g^5 \wedge \omega_2$ in KS notation, and as it will turn out that $b \to 0$ in the UV, we see that the ansatz for $F_3$ indeed describes a flux piercing the 3-cycle of $T^{1,1}$ in the UV. Thus, we have parametrized the 10-d fields of type IIB supergravity by a 5-d metric, $g_{\mu \nu}$, and a set of ten scalars, $\Phi, p, x, g, a, b, h_1, h_2, K$ and $\chi$. As in [35], one finds (again, details are relegated to appendix A) that some of the equations of motion (3.1)–(3.7) impose constraints on this system of fields, namely

$$K = Q + 2P(h_1 + bh_2),$$

(3.10)

for a constant $Q$ that sets the AdS scale when $P = 0$, and

$$\partial_\mu \chi = \frac{(e^{2g} + 2a^2 + e^{-2g}a^4 - e^{-2g})\partial_\mu h_1 + 2a(1 - e^{-2g} + a^2 e^{-2g})\partial_\mu h_2}{e^{2g} + (1 - a^2)^2 e^{-2g} + 2a^2}.$$  

(3.11)

Although this latter constraint is a 5-d generalization of the analogous constraint found by PT, unlike in their case it does not only eliminate $\chi$ from the action, but also imposes restrictions on the possible sets of independent fields. These restrictions arise from the demand of integrability ($\partial_\mu \partial_\nu \chi = \partial_\mu \partial_\nu \chi$) of the five first-order partial differential equations (3.11). Considering the four special cases given in [35], one finds that (3.11) is satisfied for the singular conifold (the KT solution, a special case of the KS system), the deformed conifold, and the resolved conifold (KS), and the wrapped D5-brane (MN), but not in general for fluctuations about the resolved conifold [56]. Further comments on this appear below. Thus, we shall, in the following, consider only the KS and MN systems.

Imposing the constraints (3.10) and (3.11), the remaining equations of motion can be derived from the 5-dimensional action

$$S_5 = \int d^5y \sqrt{g} \left[ -\frac{1}{4} R + \frac{1}{2} G_{ab}(\phi) \partial_\mu \phi^a \partial^\mu \phi^b + V(\phi) \right],$$

(3.12)

with sigma model metric

$$G_{ab}(\phi) \partial_\mu \phi^a \partial^\mu \phi^b = \partial_\mu x \partial^\mu x + \frac{1}{2} \partial_\mu g \partial^\mu g + 6 \partial_\mu p \partial^\mu p + \frac{1}{2} e^{-2g} \partial_\mu a \partial^\mu a + \frac{1}{4} \partial_\mu \Phi \partial^\mu \Phi +$$

$$+ \frac{1}{2} P^2 e^{\Phi - 2x} \partial_\mu b \partial^\mu b + \frac{G}{2} e^{2g} + 2a^2 + e^{-2g}(1 - a^2)^2 \left\{ (1 + 2 e^{-2g} a^2) \partial_\mu h_1 \partial^\mu h_1 +$$

$$+ \frac{1}{2} e^{2g} + 2a^2 + e^{-2g}(1 + a^2)^2) \partial_\mu h_2 \partial^\mu h_2 + 2a \left[ e^{-2g} (a^2 + 1) + 1 \right] \partial_\mu h_1 \partial^\mu h_2 \right\},$$

(3.13)

and potential

$$V(\phi) = -\frac{1}{8} e^{2p - 2x} [e^{\phi} + (1 + a^2) e^{-g}] + \frac{1}{8} e^{-4p - 4x} [e^{2g} + (a^2 - 1)^2 e^{-2g} + 2a^2] +$$

$$+ \frac{1}{2} a^2 e^{-2g + 8p} + \frac{1}{8} P^2 e^{\Phi - 2x + 8p} [e^{2g} + e^{-2g} (a^2 - 2ab + 1)^2 + 2(a - b)^2] +$$

$$+ \frac{1}{4} e^{-\Phi - 2x + 8p} h_2^2 + \frac{1}{8} e^{8p - 4x} [Q + 2P(h_1 + bh_2)]^2.$$  

(3.14)

As emphasized above, we must remember that integrability of (3.11) effectively restricts us to the KS and MN systems. With this restriction, the system (3.12) with kinetic terms
and potential (3.14) represents a consistent truncation of type IIB supergravity. Moreover, the superpotential exists and is known in both cases. We show the consistency of the truncation in appendix A.

It is an interesting and (as far as we know) open question how the truncation of (3.13) and (3.14) to the KS system can be made manifestly supersymmetric. As explained in [35] (and as we review in section 6.1), this truncation introduces one more constraint on the system of ten scalars, cf. (6.1). Together with (3.10) and (3.11) this leaves seven independent scalars. To write down a manifestly supersymmetric effective action for them might require a generalization of the ansatz (3.8). To some readers it may seem discouraging that the number of real scalars in the KS system is odd, as four-dimensional intuition would indicate that the superpotential in a supersymmetric theory ought to be a holomorphic function in complex field variables. However, this intuition does not apply in odd dimensions. In $\mathcal{N} = 2$ theories in five dimensions, the vector multiplet only contains a real scalar, so it is conceivable that a potential of the form (2.2) could be appropriate for a supersymmetric theory, even if the derivatives are with respect to real scalars. A similar situation arises in $\mathcal{N} = 2$ supersymmetric theories in three dimensions (for example, those obtained from Calabi-Yau fourfold compactifications of M-theory). There, the potential is given by an expression similar to (2.2) but involving two functions, one depending on the real scalars of the vector multiplets and the other being a holomorphic function depending on the remaining scalars [67, 68, 69, 70].

It is also interesting to ask whether it is possible (at least in certain cases) to rewrite the general form of the potential in a five-dimensional gauged $\mathcal{N} = 2$ supergravity, given in [71, 72, 73, 74], in a form that resembles (2.2). This question (and its generalization to $\mathcal{N} = 4$) was investigated in [75, 76].

We would also like to connect the discussion above to the work on the Klebanov-Strassler Goldstone mode found in [77, 78] (this mode was predicted already in [79]). Since we argued that the analysis of fluctuations about the resolved conifold does not extend from the one-dimensional to the five-dimensional truncation in any obvious way, one needs to generalize the ansatz to satisfy the integrability constraint if one wants to study the dynamics of the Goldstone mode multiplet. We have no reason to doubt that this is possible, but we will not pursue it further here.

4 Real fluctuations in fake supergravity

4.1 The sigma-model covariant field expansion

It is our aim to study the dynamics of the fake supergravity system (2.1), (2.2) on some known backgrounds of the form (2.3), (2.4). In this section, we shall expand the fields around the background, exploiting the geometric nature of the physical variables to formulate the fluctuation dynamics gauge-invariantly. Our arguments will closely follow the

\[ \text{\ldots} \]
original development of the gauge-invariant method for a single scalar in [39], but im-
portant new ingredients will be needed in order to account for the general sigma model.

As is well known in gravity, reparametrization invariance of spacetime comes at the
price of dragging along redundant metric variables together with the physical degrees of
freedom. One attempts to reduce redundancy by gauge fixing, but as mentioned in the
introduction, such an approach causes problems for fluctuations in holographic RG flows,
due to the coupling between metric and scalar fluctuations. Thus, following [39], we shall
start from a clean slate keeping all metric degrees of freedom and describe in the next
subsection how to isolate the physical ones.

The geometry of the sigma-model target space is characterized by the metric \( G_{ab}(\phi) \),
which we assume to be invertible, the inverse being denoted \( G^{ab}(\phi) \). One can define the
sigma-model connection

\[
G^a_{\ bc} = \frac{1}{2} G^{ad} \left( \partial_c G_{db} + \partial_b G_{dc} - \partial_d G_{bc} \right),
\]

and its curvature tensor

\[
R^a_{\ bcd} = \partial_c G^a_{\ bd} - \partial_d G^a_{\ bc} + G^e_{\ ce} G^a_{\ bd} - G^a_{\ de} G^e_{\ bc}.
\]

We also define the covariant field derivative as usual, e.g.,

\[
D^b A_a \equiv A_{a|b} \equiv \partial_b A_a - G^c_{\ ab} A_c.
\]

All indices after a bar "\( | \)" are intended as covariant field derivatives according to (4.3).
Moreover, field indices are lowered and raised with \( G_{ab} \) and \( G^{ab} \), respectively.

Armed with this notation, it is straightforward to expand the scalar fields in a sigma-
model covariant fashion. The naive ansatz \( \phi^a = \bar{\phi}^a + \varphi^a \), introducing \( \varphi^a \)
simply as the coordinate difference between the points \( \phi \) and \( \bar{\phi} \) in field space, leads to non-covariant
expressions at quadratic and higher orders, because these \( \varphi^a \) do not form a vector in
(tangent) field space. In other words, the coordinate difference is not a geometric object.
However, it is well known that a covariant expansion is provided by the exponential map
[55, 59],

\[
\phi^a = \exp_{\bar{\phi}}(\varphi)^a \equiv \bar{\phi}^a + \varphi^a - \frac{1}{2} G^a_{bc} \varphi^b \varphi^c + \cdots,
\]

where the higher order terms have been omitted, and the connection \( G^a_{bc} \) is evaluated at
\( \bar{\phi} \). Geometrically, \( \varphi \) represents the tangent vector at \( \bar{\phi} \) of the geodesic curve connecting
the points \( \bar{\phi} \) and \( \phi \), and its length is equal to the geodesic distance between \( \bar{\phi} \) and \( \phi \); see Fig. 2.

It is also a standard result that the components \( \varphi^a \) coincide with the Riemann normal
coordinates (RNCs) (with origin at \( \bar{\phi} \)) of the point \( \phi \) (see, e.g., [59]). This fact can be
used to simplify the task of writing equations in a manifestly sigma-model covariant form.
Namely, given a background point \( \bar{\phi} \), we can use RNCs to describe some neighborhood of
it and then employ the following properties at the origin of the RNC system,

\[
G^a_{\ bc} = 0, \quad R^a_{\ bcd} = \partial_c G^a_{\ bd} - \partial_d G^a_{\ bc}.
\]
in order to express everything in terms of tensors. Because the background fields depend on \( r \), we must be careful to use only outside \( r \)-derivatives, but the simplifications are still significant.

Finally, let us also define a “background-covariant” derivative \( D_r \), which acts on sigma-model tensors as, e.g.,

\[
D_r \varphi^a = \partial_r \varphi^a + G^a_{bc} W^b \varphi^c .
\] (4.6)

If a tensor \( A_a \) depends on \( r \) only implicitly through its background dependence, then we find the identity

\[
D_r A_a(\bar{\varphi}) = W^b(\bar{\varphi}) D_b A_a(\bar{\varphi}) .
\] (4.7)

The background-covariant derivative \( D_r \) will be important in our presentation of the field equations in Sec. 4.4.

### 4.2 Gauge transformations and invariants

The form of the background solution (2.3) lends itself well to the ADM (or time-slicing) formalism for parametrizing the metric degrees of freedom [54, 55]. Instead of slicing in time, we shall write a general bulk metric in the radially-sliced form

\[
ds^2 = (n^2 + n_i n^i) \, dr^2 + 2n_i \, dr \, dx^i + g_{ij} \, dx^i \, dx^j
\] (4.8)

where \( g_{ij} \) is the induced metric on the hypersurfaces of constant \( r \), and \( n \) and \( n^i \) are the lapse function and shift vector, respectively. It will be important to us that the objects \( n, n^i \) and \( g_{ij} \) transform properly under coordinate transformations of the radial-slice hypersurfaces. Details concerning the geometry of hypersurfaces are reviewed in appendix B. Again, we will not put tildes on the bulk quantities in the main text, as the meaning of the symbols should be clear from the context. In contrast, tildes are used in the appendices in order to clearly distinguish bulk and hypersurface quantities.

We can now expand the radially-sliced metric around the background configuration:

\[
g_{ij} = e^{2A(r)} \left( \eta_{ij} + h_{ij} \right) ,
\]

\[
n_i = \nu_i ,
\]

\[
n = 1 + \nu ,
\] (4.9)
where $h_{ij}$, $\nu_i$ and $\nu$ denote small fluctuations. Henceforth, we shall adopt the notation that the indices of metric fluctuations, as well as of derivatives $\partial_i$, are raised and lowered using the flat (Minkowski/Euclidean) metric, $\eta_{ij}$.

Now let us turn to the question of isolating the physical degrees of freedom from the set of fluctuations $\{h_{ij}, \nu, \nu^a\}$ introduced so far. In the earlier AdS/CFT literature one usually removed the redundancy following from diffeomorphism invariance by partial gauge fixing, i.e., by placing conditions on certain components of the metric, such as $n \equiv 1$, $n^i \equiv 0$. And indeed, it is always possible to perform a change of coordinates which transforms the metric into a form that satisfies the gauge conditions.

Alas, as discussed in the introduction, partial gauge fixing can create problems in coupled systems. Instead, we will obtain the equations of motion in gauge-invariant form. Let us start by considering the effect of diffeomorphisms on the fluctuation fields. We consider a diffeomorphism of the form

$$x^\mu = \exp_{x'}[\xi(x')]^\mu = x'^\mu + \xi^\mu(x') - \frac{1}{2} \Gamma^\mu_{\nu\rho}(x')\xi^\nu(x')\xi^\rho(x') + \cdots ,$$

where $\xi$ is infinitesimal. Notice that we found it convenient to apply the diffeomorphism inversely, i.e., we have expressed the old coordinates $x^\mu$ in terms of the new coordinates $x'^\mu$. The use of the exponential map implies that also the transformation laws for the fields can be written covariantly (the functions $\xi^\mu(x')$ are thought of as the components of a vector field). For example, a scalar field transforms as

$$\delta \phi = \xi^\mu \partial_\mu \phi + \frac{1}{2} \xi^\mu \xi^\nu \partial_\nu \phi + \cdots ,$$

whereas a covariant tensor of rank two transforms as

$$\delta E_{\mu\nu} = \xi^\lambda \nabla_\lambda E_{\mu\nu} + (\nabla_\mu \xi^\lambda)(E_{\lambda\nu} + \xi^\rho \nabla_\rho E_{\lambda\nu}) + (\nabla_\nu \xi^\lambda)(E_{\mu\lambda} + \xi^\rho \nabla_\rho E_{\mu\lambda}) +$$

$$+ (\nabla_\mu \xi^\rho)(\nabla_\rho \xi^\lambda) E_{\lambda\nu} + \frac{1}{2} \xi^\rho \xi^\lambda \nabla_\rho \nabla_\lambda E_{\mu\nu} - \nabla^\sigma \xi^\lambda E_{\sigma\nu} - \nabla^\sigma \xi^\lambda E_{\mu\sigma} +$$

$$+ \cdots .$$

Eqs. (4.11) and (4.12) are most easily derived using RNCs around $x'$ and using (4.5). The second order terms in $\xi$ have been included here in order to illustrate the covariance of the transformation laws. For our purposes, the linear terms will suffice.

Splitting the fake supergravity fields into background and fluctuations, as defined in (4.9) and (4.4), the transformations (4.11) and (4.13) become gauge transformations for the fluctuations, to lowest order:

$$\delta \varphi^a = W^a \xi^r + O(f) ,$$

$$\delta \nu = \partial_i \xi^r + O(f) ,$$

$$\delta \nu^i = \partial^i \xi^r + e^{2A} \partial_r \xi^i + O(f) ,$$

$$\delta h_{ij} = \partial_j \xi^r + \partial^r (\eta_{jk} \xi^k) - \frac{4}{d-1} W \delta^r_j \xi^r + O(f) .$$
By $\mathcal{O}(f^n)$ we mean terms of order $n$ in the fluctuations $\{\varphi^a, h_{ij}, \nu_i, \nu\}$. Furthermore, let us decompose $h^i_j$ as follows,

$$h^i_j = h^{TT}^i_j + \partial^i \epsilon_j + \partial_j \epsilon^i + \frac{\partial^i \partial^j H + \frac{1}{d-1} \delta^i_j h}{\Box},$$  \hspace{1cm} (4.15)$$

where $h^{TT}^i_j$ denotes the traceless transverse part, and $\epsilon^i$ is a transverse vector. It is straightforward to obtain from (4.14)

$$\delta h^{TT}^i_j = \mathcal{O}(f),$$  
$$\delta \epsilon^i = \Pi_j^i \xi^j + \mathcal{O}(f),$$  
$$\delta H = 2 \partial^i \xi^i + \mathcal{O}(f),$$  
$$\delta h = -4 W \xi^r + \mathcal{O}(f).$$  \hspace{1cm} (4.16)$$

The symbol $\Pi_j^i$ denotes the transverse projector,

$$\Pi_j^i = \delta_j^i - \frac{1}{\Box} \partial^i \partial_j.$$  \hspace{1cm} (4.17)$$

The main idea of our approach is to construct gauge-invariant combinations from the fields $\{h^{TT}^i_j, \epsilon^i, h, H, \nu, \nu^i, \varphi^a\}$. Using the transformation laws (4.14) and (4.16), this is straightforward, and to lowest order, one finds the gauge-invariant fields.

$$\alpha^a = \varphi^a + W^a \frac{h}{4W} + \mathcal{O}(f^2),$$  \hspace{1cm} (4.18)$$
$$b = \nu + \partial_r \left( \frac{h}{4W} \right) + \mathcal{O}(f^2),$$  \hspace{1cm} (4.19)$$
$$c = e^{-2A} \partial^i \nu^j + e^{-2A} \frac{h}{4W} - \frac{1}{2} \partial_i H + \mathcal{O}(f^2),$$  \hspace{1cm} (4.20)$$
$$d^i = e^{-2A} \Pi_j^i \nu^j - \partial^i \epsilon^j + \mathcal{O}(f^2),$$  \hspace{1cm} (4.21)$$
$$e_j^i = h^{TT}^i_j + \mathcal{O}(f^2).$$  \hspace{1cm} (4.22)$$

The variables $c$ and $d^i$ both arise from $\delta \nu^j$, which has been split into its longitudinal and transverse parts. We chose the \texttt{Fraktur} typeface for the gauge invariant variables in order to avoid confusion with the field indices, and still keep notational similarity with [39]. Notice that $c$ and $d^i$ have been rescaled with respect to [39] for later convenience.

Although we have carried out the construction of gauge-invariant variables only to lowest order, and this is all we will need here, it is necessary for consistency that the preceding analysis can be extended to higher orders, in principle. In this context it becomes clear that the geometric nature of the field expansions, as expressed by the exponential map, is a crucial ingredient of the method.

Finally, let us prepare the ground for the arguments of the next subsection, where we shall analyze the implications of gauge-invariance on the equations of motion. Let

\[\text{The choice of gauge-invariant variables is not unique, of course, as any combination of them will be gauge-invariant as well.}\]
us introduce some more compact notation. Consider the set of gauge-invariant fields, $I = \{a^a, b, c, d^i, e^i_j\}$. From the definitions (4.18)–(4.22) we see that there is a one-to-one correspondence between $I$ and a sub-set of the fluctuation fields, $Y = \{\varphi^a, \nu, \nu^i, h^{TT}{}^j\}$. We also collect the remaining fluctuation variables into a set, $X = \{h, H, \epsilon^i\}$. Henceforth, the symbols $I, X$ and $Y$ shall be used also to denote members of the corresponding sets.

One can better understand the correspondence between $I$ and $Y$ by noting that (4.18)–(4.22) can be re-written as

$$Y = I + y(X) + \mathcal{O}(f^2) \quad ,$$

(4.23)

where $y$ is a linear functional of the fields $X$. Going to quadratic order in the fluctuations, one would find

$$Y = I + y(X) + \alpha(X,X) + \beta(X,I) + \mathcal{O}(f^3) \quad ,$$

(4.24)

where $\alpha$ and $\beta$ are bi-linear in their arguments. Terms of the form $\gamma(I,I)$ do not appear, as they can be absorbed into $I$.

We interpret the gauge-invariant variables $I$ as the physical degrees of freedom, whereas the $(d+1)$ variables $X$ represent the redundant metric variables. This can be seen by observing that one can solve the transformation laws (4.16) for the generators $\xi^\mu$, which yields equations of the form

$$\xi^\mu = z^\mu(\delta X) + \mathcal{O}(f^2) = \delta z^\mu(X) + \mathcal{O}(f^2) \quad ,$$

(4.25)

with $z^\mu$ being a linear functional.

### 4.3 Einstein’s equations and gauge invariance

It is our aim to cast the equations of motion into an explicitly gauge-invariant form. This means that the final equations should contain only the variables $I$ and make no reference to $X$ and $Y$. Reparametrization invariance suggests that this should be possible, and we shall establish the precise details in this subsection.

Let us consider Einstein’s equations, symbolically written as

$$E_{\mu\nu} = 0 \quad ,$$

(4.26)

but it is clear that the arguments given below hold also for the equations of motion for the scalar fields. To start, let us expand the left hand side of (4.26) around the background solution, which yields, symbolically,

$$E_{\mu\nu} = E_{\mu\nu}^{(1)}(X) + E_{\mu\nu}^{(1)2}(Y) + E_{\mu\nu}^{(2)}(X, X) + E_{\mu\nu}^{(2)2}(X, Y) + E_{\mu\nu}^{(2)3}(Y, Y) + \mathcal{O}(f^3) \quad .$$

(4.27)

Here, $E^{(1)}$ and $E^{(2)}$ denote linear and bilinear terms, respectively. The background equations are satisfied identically. Substituting $I$ for $Y$ using (4.24) yields

$$E_{\mu\nu} = \tilde{E}_{\mu\nu}^{(1)}(X) + \tilde{E}_{\mu\nu}^{(1)2}(I) + \tilde{E}_{\mu\nu}^{(2)}(X, X) + \tilde{E}_{\mu\nu}^{(2)2}(X, I) + \tilde{E}_{\mu\nu}^{(2)3}(I, I) + \mathcal{O}(f^3) \quad .$$

(4.28)

Notice that the functionals $E^{(1)2}$ and $E^{(2)3}$ are unchanged ($Y$ is just replaced by $I$), whereas the others are modified by the $X$-dependent terms of (4.24), which we indicate
by adorning them with a tilde. For example, $\tilde{E}^{(2)2}$ receives contributions from $E^{(2)2}$, $E^{(2)3}$ and $E^{(1)2}$.

In order to simplify (4.28), we consider its transformation under the diffeomorphism (4.10). On the one hand, from the general transformation law of tensors (4.12) we find, using also (4.25), that it should transform as

$$\delta E_{\mu\nu} = \left[ \partial_\mu \delta z^\lambda(X) \right] E^{(1)2}_{\lambda\mu} + \left[ \partial_\nu \delta z^\lambda(X) \right] E^{(1)2}_{\mu\lambda} + \delta z^\lambda(X) \partial_\mu E_{\mu\nu} + \mathcal{O}(f^3) \quad (4.29)$$

On the other hand, the variation of (4.28) is

$$\delta E_{\mu\nu} = \tilde{E}^{(1)1}_{\mu\nu}(\delta X) + 2 \tilde{E}^{(2)1}_{\mu\nu}(\delta X, X) + \tilde{E}^{(2)2}_{\mu\nu}(\delta X, I) + \mathcal{O}(f^3) \quad (4.30)$$

Let us compare (4.29) and (4.30) order by order. The absence of first-order terms on the right hand side of (4.29) implies that

$$\tilde{E}^{(1)1}_{\mu\nu}(X) = 0 \quad (4.31)$$

It can easily be checked that this is indeed the case. Then, substituting $E_{\mu\nu} = E^{(1)2}_{\mu\nu}(I) + \mathcal{O}(f^2)$ into the right hand side of (4.29) yields

$$\delta E_{\mu\nu} = \delta \left\{ \left[ \partial_\mu \delta z^\lambda(X) \right] E^{(1)2}_{\lambda\nu}(I) + \left[ \partial_\nu \delta z^\lambda(X) \right] E^{(1)2}_{\mu\lambda}(I) + \delta z^\lambda(X) \partial_\mu E^{(1)2}_{\mu\nu}(I) \right\} + \mathcal{O}(f^3) \quad (4.32)$$

Comparing (4.32) with the second order terms of (4.30), we obtain

$$\tilde{E}^{(2)1}_{\mu\nu}(X, X) = 0 \quad , \quad \tilde{E}^{(2)2}_{\mu\nu}(X, I) = \left[ \partial_\mu \delta z^\lambda(X) \right] E^{(1)2}_{\lambda\nu}(I) + \left[ \partial_\nu \delta z^\lambda(X) \right] E^{(1)2}_{\mu\lambda}(I) + \delta z^\lambda(X) \partial_\mu E^{(1)2}_{\mu\nu}(I) \quad (4.33)$$

Hence, we find that a simple expansion of Einstein’s equations yields gauge-dependent second-order terms, but they contain the (gauge-independent) first order equation, and so can consistently be dropped. Happily, we arrive at the following equation, which involves only $I$:

$$E^{(1)2}_{\mu\nu}(I) + E^{(2)3}_{\mu\nu}(I, I) + \mathcal{O}(f^3) = 0 \quad (4.34)$$

The argument generalizes recursively to higher orders. One will find that the gauge-dependent terms of any given order can be consistently dropped, because they contain the equation of motion of lower orders.

Eq. (4.34) and its higher-order generalizations are obtained using the following recipe:

*Expand the equations of motion to the desired order dropping the fields $X$ and replacing every field $Y$ by its gauge-invariant counterpart $I$.\(^\Box\)*

This rule is summarized by the following substitutions,

$$\varphi^a \to \varphi^a \quad , \quad \nu \to b \quad , \quad e^{-2A} \nu^i \to \delta^i + \partial^i \varphi \quad , \quad h^i_j \to \epsilon^i_j \quad . \quad (4.35)$$

Since $\epsilon^i_j$ is traceless and transverse, the calculational simplifications arising from (4.35) are considerable. For the reader’s reference, the expressions that result from (4.35) for some geometric objects are listed at the end of appendix C.

Let us conclude with the remark that, although the rules (4.35) can be interpreted as the gauge choice $X = 0$, the equations we found are truly gauge invariant.
4.4 Equations of motion

In this section, we shall put the above preliminaries into practice. The equations of motion that follow from the action (2.1) are

\[ \nabla^2 \phi^a + G^a_{\ bc} g^{\mu \nu} (\partial_\mu \phi^b) (\partial_\nu \phi^c) - V^a = 0 \]  

(4.36)

for the scalar fields, and Einstein’s equations

\[ E_{\mu \nu} = -R_{\mu \nu} + 2 G_{ab} (\partial_\mu \phi^a) (\partial_\nu \phi^b) + \frac{4}{d-1} g_{\mu \nu} V^a = 0 . \]  

(4.37)

Notice that we use the opposite sign convention for the curvature with respect to [39, 51].

We are interested in the physical, gauge-invariant content of (4.36) and (4.37) to quadratic order in the fluctuations around an RG flow background of the form (2.3), (2.4). As we saw in the last section, the physical content is obtained by expanding the fields according to (4.9) and (4.4) and then applying the substitution rules (4.35). Since we defined the expansion (4.4) geometrically, we will obtain sigma-model covariant expressions. To carry out this calculation in practice, it is easiest to use RNCs at a given point in field space, so that one can use the relations (1.5) outside \( r \)-derivatives.

In the following, we shall present the linearized equations of motion, and indicate higher order terms as sources, the relevant quadratic terms of which are listed in appendix D. For intermediate steps we refer the reader to appendix C. Let us start with the equation of motion for the scalar fields (4.36), which gives rise to the following fluctuation equation,

\[ \left[ D_r^2 - \frac{2d}{d-1} W D_r + e^{-2A} \square \right] a^a - \left( V^a_c - R^a_{\ bcd} W^b W^d \right) a^c - W^a (c + \partial_a b) - 2V^a b = J^a . \]  

(4.38)

Note the appearance of the field-space curvature tensor in the potential term.

Second, the normal component of Einstein’s equations\(^ {14} \) gives rise to

\[ -4W c + 4W_a (D_r a^a) - 4V_a a^a - 8V b = J . \]  

(4.39)

Third, the mixed components of (4.37) yield

\[ -\frac{1}{2} \square d_i - 2W \partial_i b - 2W_a \partial_i a^a = J_i . \]  

(4.40)

The appearance of the fields \( a^a, b, c \) and \( d^i \) on the left hand sides of (4.38) – (4.40) seems to indicate the coupling between the fluctuations of active scalars (non-zero \( W_a \)) to those of the metric, which is familiar from the AdS/CFT calculation of two-point functions in the literature. However, the gauge-invariant formalism resolves this issue, because (4.39) and (4.40) can be solved algebraically (in momentum space) for the metric fluctuations \( b, \)

\(^{14}\)More precisely, it is the equation obtained by multiplying (4.37) by \( N^\mu N^\nu - g^{ij} X_i^\mu X_j^\nu \).
\(c\) and \(d\), so that the coupling of metric and scalar fluctuations at linear order is completely disentangled. One easily obtains

\[b = -\frac{1}{W}W_a a^a - \frac{1}{2W} \frac{\partial}{\partial i} J^i, \tag{4.41}\]

\[c = \frac{W}{W} \left( \delta^a_b \partial_r - W^a_{|b} + \frac{W^a W_b}{W} \right) d^b - \frac{1}{4W} J + \frac{1}{2} \left( \frac{W_a W^a}{W^2} - \frac{2d}{d-1} \right) \partial^i J_i, \tag{4.42}\]

\[d_i = -\frac{2}{\Box} \Pi^j_i J_j. \tag{4.43}\]

We proceed by substituting (4.41) and (4.42) into (4.38), using also the identities

\[V^a = W^a \left( \delta^a_b \partial_r - W^a_{|b} + \frac{W^a W_b}{W} \right) d^b - \frac{1}{4W} J + \frac{1}{2} \left( \frac{W_a W^a}{W^2} - \frac{2d}{d-1} \right) \partial^i J_i, \tag{4.44}\]

which follow from (2.2) and (2.4), and we end up with the second-order differential equation

\[
\left( \delta^a_b \partial_r + W^a_{|b} - \frac{2d}{d-1} W \delta^a_b \right) \left( \delta^b_c \partial_r - W^b_{|c} + \frac{W^b W_c}{W} \right) \left( \delta^b_c \partial_r - W^b_{|c} + \frac{W^b W_c}{W} \right) + \delta^a_c e^{2A} \Box \right] a^c = \tilde{J}^a, \tag{4.45}\]

where the source term \(\tilde{J}^a\) is related to the sources \(J^a\), \(J\) and \(J_i\) by

\[\tilde{J}^a = J^a - \frac{W^a}{4W} J - \frac{1}{2} \left( \delta^a_b \partial_r + W^a_{|b} - \frac{W^a W_b}{W} - \frac{2d}{d-1} W \delta^a_b \right) \left( \frac{W^b \partial^i}{W} \Box J_i \right). \tag{4.46}\]

Eq. (4.43) implies that we can drop \(d^i\) in the source terms (to quadratic order). Eq. (4.45) is the main result of the gauge-invariant approach and governs the dynamics of scalar fluctuations around generic Poincaré-sliced domain wall backgrounds. Being a system of second order differential equations, one can use the standard Green’s function method to treat the interactions perturbatively.

A feature that is evident from the linearized version of (4.45) is the existence of a background mode in the fluctuations. It is independent of the boundary variables \(x^i\), and is simply given by

\[a^a = \alpha \frac{W^a}{W}, \tag{4.47}\]

where \(\alpha\) is an infinitesimal constant. In standard holographic renormalization, one can use the background mode (4.47) to establish the existence of finite sources (CFT deformations) and vacuum expectation values in the dual field theory. Asymptotically each component of the fluctuation vector is dual to a conformal primary operator (as explained in Sec. 2); a component of \(W^a/W\) that behaves asymptotically as the leading term of the general solution of (4.45) is interpreted as a background source deforming the CFT action by the corresponding dual operator, while a background mode component that behaves
asymptotically as the sub-leading term of the general solution represents a vacuum expectation value of the dual operator. We believe that a statement of this kind can be made also in the general non-asymptotically AdS case, and we shall present an example for the MN system in Sec. 5.

Let us also consider the tangential components of (4.37). Because of the Bianchi identity, their trace and divergence are implied by (4.38), (4.39) and (4.40), which is easily checked at linear order. Thus, we can use the traceless transverse projector,

$$\Pi_{ij}^k = \frac{1}{2} (\Pi^{ik} \Pi_{jl} + \Pi^{jl} \Pi_{ik}) - \frac{1}{d-1} \Pi^i \Pi^j,$$

in order to obtain the independent components. This yields

$$\left( \partial_r^2 - \frac{2d}{d-1} W \partial_r + e^{-2A} \Box \right) e^i_j = J^i_j.$$  

(4.49)

As expected, the physical fluctuations of the metric satisfy the equation of motion of a massless scalar field.

5 The Maldacena-Nuñez system

5.1 Review of the background solution

The MN system is obtained by imposing the following relations on the general effective 5-d action obtained in Sec. 3:

$$Q = 0 , \quad h_1 = h_2 = 0 , \quad b = a ,$$

$$\Phi = -6p - g - 2 \ln P , \quad x = \frac{1}{2} g - 3p.$$  

(5.1)

Together with (5.1), the constraints (3.10) and (3.11) imply also $K = 0$ and $\chi = 0$. (Notice that a constant in $\chi$ is irrelevant.) It is straightforward to check from the equations of motion in appendix A that this truncation is consistent, i.e., the equations of motion for $b$, $h_1$, $h_2$, $\Phi$ and $x$ are satisfied or implied by those for $a$, $p$ and $g$. Notice that, having absorbed the constant $P^2$ into $e^\Phi$, it has disappeared from the equations of motion. Hence, the effective 5-d action reduces to the form (2.1), with three scalar fields $(g,a,p)$, the sigma model metric

$$G_{ab}\partial_\mu \phi^a \partial^\mu \phi^b = \partial_\mu g \partial^\mu g + e^{-2g} \partial_\mu a \partial^\mu a + 24 \partial_\mu p \partial^\mu p ,$$

(5.2)

and the superpotential

$$W = -\frac{1}{2} e^4 \left[ (a^2 - 1)^2 e^{-4g} + 2(a^2 + 1) e^{-2g} + 1 \right]^{1/2}.$$  

(5.3)

Let us briefly summarize the most general Poincaré-sliced domain wall background solution (2.3) for this system. It is obtained by solving (2.4) and coincides with the

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15 We correct formula (5.25) of [35].

16 We have adjusted the overall factor of the superpotential of [35] to our conventions.
family of solutions found in [81]. In the following, \( g, a \) and \( p \) will denote the background fields, while the fluctuations are described by the gauge-invariant variables \( a^a \). Introducing a new radial coordinate, \( \rho \), by
\[
\partial_{\rho} = 2 e^{-4p} \partial_r ,
\]
(5.4)
one can show from the equations for \( g \) and \( a \) that
\[
[(a^2 - 1)^2 + 2(a^2 + 1) e^{2g} + e^{4g}]^{1/2} = 4\rho .
\]
(5.5)
The integration constant arising here has no physical meaning and has been used to fix the origin of \( \rho \). Then, one easily obtains
\[
a = \frac{2\rho}{\sinh(2\rho + c)} , \quad e^{2g} = 4\rho \coth(2\rho + c) - (a^2 + 1) ,
\]
(5.6)
where \( c \) is an integration constant with allowed values \( 0 \leq c \leq \infty \). We shall discuss the interpretation of \( c \) in the next subsection. The MN solution corresponds to \( c = 0 \) and is the only regular solution. All others suffer from a naked curvature singularity.

It is also easy to show from (2.4) that
\[
e^{-2A} e^{-8p} = C^2 ,
\]
(5.7)
where the integration constant \( C \) determines the 4-d reference scale. We shall set \( C^2 = 1/4 \) for later convenience. The explicit solution for \( p \) can be found by plugging \( \Phi \) from the literature into (5.1), but it will not be needed here.

5.2 The role of \( c \)
The family of background solutions of the MN system suffers from naked singularities for all \( c \) except for the case \( c = 0 \), which is regular. Hence, on the supergravity side the integration constant \( c \) governs the resolution of the singularity. However, the scalar \( a(\rho) \) is the dual of the gluino bilinear \( \lambda^2 \) [26], so \( c \), which enters \( a(\rho) \) in (5.6), also determines the “measured” value of the gluino condensate, \( \langle \lambda^2 \rangle \), which is of non-perturbative field theory origin. In other words, \( c \) identifies the “amount” of non-perturbative physics that is captured by the supergravity solution.

In this subsection, we will attempt to flesh out this picture qualitatively, applying Mathur’s coarse graining argument [82, 83], before we analyze the fluctuations in the next subsection. Although only regular solutions qualify as gravity duals of (pure) field theory quantum states, the coarse graining argument indicates that certain singular solutions have a meaning as an approximation to the duals of mixed states. In this point of view, singularities appear because the “space-time foam” that is dual to the mixture of pure states cannot be resolved by supergravity. (We are using the terminology of [82, 83] here. See also [84] for some earlier discussion of the admissibility of singular solutions.) In the case at hand, the possible pure states are naturally identified as the \( N \) equivalent vacua of \( \text{SU}(N) \mathcal{N} = 1 \) SYM theory, which are distinguished by a phase angle in the gluino
Let us denote these $N$ vacua by $|n\rangle$, where $n = 0, 1, 2, \ldots, N - 1$. The gluino condensate in these vacua takes the values

$$\langle n|\lambda^2|n\rangle = \Lambda^3 e^{2\pi in/N},$$  \hspace{1cm} (5.8)

where we have absorbed the $\theta$ angle of the gauge theory in the phase of $\Lambda^3$.

Now, let us form mixed states by defining the density matrix

$$\varrho = \sum_{n=0}^{N-1} p_n |n\rangle\langle n|,$$

with \(\sum_{n=0}^{N-1} p_n = 1\).  \hspace{1cm} (5.9)

Clearly, for equal weights, $p_n = 1/N$, we would measure $\langle \lambda^2 \rangle = \text{tr}(\lambda^2 \varrho) = 0$. For a generic mixed state, the measured value $\langle \lambda^2 \rangle$ lies somewhere within the $N$-polygon spanned by the $N$ pure-state values (5.8). Using standard thermodynamics arguments, it is straightforward to determine the unique distribution $\{p_n\}$ maximizing the entropy for a given fixed $\langle \lambda^2 \rangle$. Notice, however, that the $N$ vacua are equivalent, and that, for large $N$, which is the regime described by the supergravity approximation, the vacuum values of $\langle \lambda^2 \rangle$ effectively span a circle of radius $|\Lambda^3|$. Thus, up to $1/N$ corrections, the phase of some given $\langle \lambda^2 \rangle$ is irrelevant, making the relevant parameter space for the probability distribution $\{p_n\}$ effectively one-dimensional.

Thus, from the point of view advocated in [82, 83], the integration constant $c$ can be interpreted as a parameter that interpolates between the uniform distribution $(c = \infty)$ and a pure state $(c = 0)$, with fixed phase of $\langle \lambda^2 \rangle$. It would be interesting to make this interpretation precise by attempting to match the statistical entropy of a mixed state with the area of the apparent horizon surrounding the dual “space-time foam”. We leave such investigations for the future.

Instead, let us confirm the role of $c$ in determining the measured value of the gluino condensate from the perspective of holographic renormalization. Being a one-point function, the gluino condensate should appear as a background response function in a supergravity field (cf. the discussion in Sec. 2). Thus, consider the background mode (4.47) of the fluctuation equation for an arbitrary value of $c$. As noted in Sec. 4.4, the background mode, $W^a/W$, is always a solution of (4.45) independent of $x^i$. Let us determine its asymptotic behaviour (large $\rho$) and see whether it is leading or sub-leading. For an arbitrary value of $c$, we obtain

$$\frac{W^a}{W} \sim \left(-\frac{1}{2\rho}, 8 e^{-c} \rho e^{-2\rho}, \frac{1}{6}\right).$$  \hspace{1cm} (5.10)

The first and third components are independent of $c$, i.e., universal for all background solutions, and they are leading compared to the general solutions that we shall find in the next subsection. We have, at present, no specific interpretation of their role, although the

\[17\] In the 10-d MN solution, the location of the Dirac string for the magnetic 2-form $C_2$ is specified by an angular variable $\psi$ that can take $2N$ different values for the same field theory $\theta$-angle, but the solution is symmetric under a shift by $\pi$ of $\psi$, leaving $N$ different configurations. Equivalently, one has $N$ different ways of placing probe $D5$-branes in the background, in order to obtain the same field theory action [25, 27, 29].
arguments outlined in Sec. 2 indicate that they should correspond to finite field theory sources (couplings). In contrast, the second component is sub-leading and depends on $c$. Hence, we argue in analogy with AdS/CFT (again, we refer to Sec. 2) that its coefficient represents a response function, so it determines the vacuum expectation value of the dual operator. In this case, the dual operator is the gluino bilinear. Restoring dimensions, this yields

$$\langle \lambda^2 \rangle = \Lambda^3 e^{-c} , \quad (5.11)$$

which fits nicely with the preceding discussion involving mixed states.

### 5.3 Fluctuations and mass spectra

In the following, we shall consider the equation of motion for scalar fluctuations about the singular background with $c = \infty$. Although we argued in the introduction that singular solutions as supergravity duals should be taken with a grain of salt, doing so is quite instructive and serves mainly two purposes: First, this solution elegantly describes the asymptotic region (large $\rho$) of all background solutions, including the regular MN solution, so that we can learn something about the asymptotic behaviour of the field fluctuations, which will be important for the “dictionary” and “renormalization” problems. Second, the matrix equation for fluctuations becomes diagonal and analytically solvable. Thus, we can hope to get a qualitative glimpse of the particle spectrum of the dual field theory.

Consider the equation of motion for scalar fluctuations (4.45). In terms of $\rho$ and going to 4-d momentum space, as well as neglecting the source terms on the right hand side, (4.45) becomes

$$\left[ (\partial^a_{\rho} \partial_\rho + 2M^a_b)(\partial^b_\rho - 2N^b_c) - k^2 \right] a^a = 0 , \quad (5.12)$$

where we have fixed the 4-d scale by the choice $C^2 = 1/4$, which will turn out convenient later. The matrices $M^a_b$ and $N^a_b$ are given by

$$N^a_b = e^{-4p} \left( \partial_b W^a - \frac{W^a W^b}{W} \right) ,$$

$$M^a_b = N^a_b + 2 e^{-4p} \left( G^a_{bc} W^c - W \delta^a_b \right) . \quad (5.13)$$

Notice that the $p$-dependence in $M^a_b$ and $N^a_b$ cancels out. For the case $c = \infty$, the matrices $M^a_b$ and $N^a_b$ are diagonal,

$$N^a_b = \text{diag} \left( -\frac{1}{2\rho}, \frac{1}{2\rho}, \frac{1}{2\rho}, -1, 0 \right) ,$$

$$M^a_b = \frac{1}{4\rho - 1} \text{diag} \left( 4\rho - 2 + \frac{1}{2\rho}, 1 - \frac{1}{2\rho}, 4\rho \right) . \quad (5.14)$$

We are mostly interested in the field $a^2$ (the middle component), since its dual operator is the gluino bilinear (+ its hermitian conjugate). From (5.12) and (5.14), its equation of motion reads

$$\left( \frac{\partial^2}{\partial^2_{\rho}} + 4 \frac{2\rho - 1}{4\rho - 1} \partial_\rho + \frac{4}{4\rho - 1} - k^2 \right) a^2 = 0 . \quad (5.15)$$
Performing a change of variable by defining
\[
\rho - \frac{1}{4} = \alpha z ,
\] (5.16)
with a constant \(\alpha\) to be determined later, and using the following ansatz for the solution,
\[
a^2 = e^{\alpha z} z^b f(z) ,
\] (5.17)
with constant \(a\) and \(b\), we find that the choices
\[
a = -\alpha , \quad b = \frac{1}{4} , \quad \alpha^2 (1 + k^2) = \frac{1}{4} \quad \text{(5.18)}
\]
lead to the equation
\[
\left( \partial_z^2 - \frac{1}{4} + \frac{3\alpha}{2z} + \frac{5}{16z} \right) f = 0 .
\] (5.19)
This can be recognized as Whittaker’s equation, the solutions of which are linear combinations of the two Whittaker functions
\[
f = \left\{ M_{\frac{3}{2}\alpha, \frac{1}{4}}(z) , \quad M_{\frac{3}{2}\alpha, -\frac{1}{4}}(z) \right\} .
\] (5.20)
Hence, using (5.17) and the relation of Whittaker’s functions to confluent hypergeometric functions \(\Phi\) and \(\Psi\) [85, 86], we find
\[
a^2 \sim e^{-(\alpha+1/2)z} \begin{cases} (\alpha z)^{3/2} \Phi \left( \frac{5}{4} - \frac{3}{2}\alpha, \frac{5}{2}; z \right) , \\ \Phi \left( -\frac{1}{4} - \frac{3}{2}\alpha, -\frac{1}{2}; z \right) . \end{cases} \quad \text{(5.21)}
\]
In standard AdS/CFT, one would impose a regularity condition in the bulk interior in order to obtain a linear combination of the two solutions, which uniquely fixes the relation between the response and the source functions. Here, however, we were not able to find such a condition, probably due to the curvature singularity of the background. However, there is a useful feature that can guide us in the choice of suitable solutions. From (5.16), we should demand that the solution be invariant under a simultaneous change of sign of \(z\) and \(\alpha\). Due to the identity [85]
\[
\Phi(a, b; z) = e^z \Phi(b - a, b; -z) , \quad \text{(5.22)}
\]
the particular solutions (5.21) are invariant under this symmetry. This implies two things. First, we are free to choose the solution for \(\alpha\) in the last equation of (5.18) such that \(\text{Re} \alpha > 0\), which implies also \(\text{Re} \ z > 0\). Notice that the square root in the definition of \(\alpha\) demands a branch cut in \(k^2\)-space, which we place at \(k^2 + 1 < 0\). This branch cut is an indication for a continuum in the particle spectrum, for \(m^2 = -k^2 > 1\). (Notice that this is relative to a reference scale, since we are working in dimensionless variables. With the earlier choice \(C^2 = 1/4\) we place the onset of the continuum conveniently at the branch point \(k^2 = -1\).) Second, linear combinations of the solutions should also reflect
this symmetry implying that proportionality factors can depend only on $\alpha^2$. In particular, the choice of the functions $\Psi(a, b; z)$ instead of $\Phi(a, b; z)$ is not allowed, cf. [85].

It is instructive to consider the asymptotic behavior of the solutions. Let $\alpha$ be generic and fixed, so that we can consider large $z$. One finds that both solutions in (5.21), and any generic linear combination of them, behave as

$$a^2 \sim e^{(1/2-\alpha)z} z^{1/4-3\alpha/2} , \quad (5.23)$$

but there are notable exceptions. Indeed, the confluent hypergeometric functions $\Phi(a, b; z)$ reduce to polynomials (Laguerre polynomials, to be precise), if the first index, $a$, is zero or a negative integer. In these cases, the generic leading terms (5.23) are absent. Generalizing the AdS/CFT argument [87], we interpret the corresponding values of $-k^2$ as discrete particle masses in the spectrum of the dual field theory.

Hence, the two solutions (5.21) give rise to two different discrete spectra

$$m_n^2 = 1 - \frac{9}{(4n+3)^2} , \quad n = 0, 1, 2, \ldots , \quad (5.24)$$

and

$$m_n^2 = 1 - \frac{9}{(4n+5)^2} , \quad n = 0, 1, 2, \ldots . \quad (5.25)$$

Notice that there is a massless state, for $n = 0$ in (5.24). Moreover, both spectra approach the branch point, $-k^2 = 1$, for $n \to \infty$.

Similarly, we consider the other components. The equation of motion for $a^3$ is

$$\left( \partial^2 + \frac{8\rho}{4\rho - 1} \partial_\rho - k^2 \right) a^3 = 0 , \quad (5.26)$$

for which we obtain the solutions

$$a^3 \sim e^{-(\alpha+1/2)z} \begin{cases} \left( \alpha z \right)^{1/2} \Phi \left( \frac{3}{4} + \frac{1}{2} \alpha, \frac{3}{2}; z \right) , \\ \Phi \left( \frac{1}{4} + \frac{1}{2} \alpha, \frac{1}{2}; z \right) . \end{cases} \quad (5.27)$$

As before, $z$ and $\alpha$ are defined by (5.16) and (5.18), respectively. Hence, we find again a continuum of states for $-k^2 > 1$. However, although the solutions (5.27) are similar to (5.21), the sign in front of the $\alpha$-terms in the first index of the confluent hypergeometric functions does not allow them to reduce to polynomials. (Remember that $\text{Re} \alpha > 0$.) Hence, there is no discrete spectrum of states.

We would like to note that the solution (5.27) is very similar to (3.17) of [52]. They considered fluctuations of the dilaton about the MN background and introduced a hard-wall cut-off, and found an unbounded discrete spectrum of glueball masses. This procedure was subsequently criticized in [53]. Due to the discussion in the previous paragraph, we do not infer glueball masses from the component $a^3$.

The treatment of component $a^4$ is slightly more complicated. Its equation of motion is

$$\left( \partial^2 + \frac{8\rho}{4\rho - 1} \partial_\rho - \frac{2}{\rho^2} + \frac{8}{4\rho - 1} - k^2 \right) a^4 = 0 . \quad (5.28)$$
The awkward double pole in \( \rho \) can be removed by setting \( a^1 = \rho^{-1} f(\rho) \), which yields the equation
\[
\left[ \partial_{\rho}^2 + \left( 2 + \frac{2}{4\rho - 1} - \frac{2}{\rho} \right) \partial_{\rho} - k^2 \right] f = 0 .
\] (5.29)

After changing variables to \( z \) by using (5.16) and making the ansatz
\[
f(z) = e^{cz} \tilde{f}(z) ,
\] (5.30)
we find that the choice
\[
c = \frac{1}{2} - \alpha ,
\] (5.31)
where \( \alpha \) is defined as before, leads to the equation
\[
\left\{ 4\alpha z \left[ z\partial_z^2 + \left( -\frac{3}{2} - z \right) \partial_z + \frac{3}{4} + \frac{3}{2} \alpha \right] + \left[ z\partial_z^2 + \left( \frac{1}{2} - z \right) \partial_z - \frac{1}{4} - \frac{1}{2} \alpha \right] \right\} \tilde{f} = 0 .
\] (5.32)

The two terms in square brackets represent differential equations for confluent hypergeometric functions, which gives us a nice hint for solving the equation. Indeed, we can explicitly find the solutions, which, combined with (5.30) and \( a^1 = \rho^{-1} f \), result in
\[
a^1 \sim \frac{e^{-(\alpha+1/2)z}}{\alpha z + 1/4} \left\{ \Phi \left( -\frac{3}{4} - \frac{3}{2} \alpha, -\frac{3}{2}; z \right) - \frac{4\alpha^2 - 1}{3} z^2 \Phi \left( \frac{5}{4} - \frac{3}{2} \alpha, \frac{5}{2}; z \right) , \right. \\
\left. \left( \alpha z \right)^{1/2} \left[ \Phi \left( -\frac{1}{2} - \frac{3}{2} \alpha, -\frac{1}{2}; z \right) + \frac{36\alpha^2 - 1}{5} z^2 \Phi \left( \frac{3}{4} - \frac{3}{2} \alpha, \frac{3}{2}; z \right) \right] \right\} .
\] (5.33)

Notice that both solutions respect the symmetry of simultaneously changing the signs of \( \alpha \) and \( z \). The sign of the \( \alpha \)-terms in the first index of the confluent hypergeometric functions indicates that, in addition to the continuum from the branch cut, we have again a discrete spectrum of states for those values of \( \alpha \), where these functions reduce to polynomials. The corresponding spectra are given again by (5.24) and (5.25), but in (5.24) only values \( n = 1, 2, 3, \ldots \) are allowed, which implies that the massless state is absent.

To conclude this section, let us discuss whether we can trust the mass spectrum we have found. This question arises since the calculation was performed in the singular background with \( c = \infty \), but the true supergravity dual of a field theory vacuum is the MN solution, with \( c = 0 \). Moreover, one typically expects the boundary conditions in the interior to influence the dual IR physics, but we have not directly imposed any conditions except symmetry of simultaneously changing the signs of \( \alpha \) and \( z \). However, there are only three things that can happen to each particular mass value when the regular background with \( c = 0 \) is considered. First, there could exist a corresponding regular and sub-leading solution for which the mass value changes as we go from \( c = \infty \) to \( c = 0 \). Second, there could exist a corresponding regular and sub-leading solution with the same mass. Third, the corresponding sub-leading solution may not be regular at \( \rho = 0 \), in which case that particular mass value would not be in the spectrum. In the following, we will argue that the first of these scenarios is excluded. Remember that the background with \( c = \infty \), which we have considered here, correctly describes the asymptotic region of the regular
background. Hence, the asymptotic behaviour of the fluctuations we found is valid also for \( c = 0 \), implying that the mass spectra (5.24) and (5.25) are unchanged. One can verify this by a series expansion in \( e^{-c} \) of the equations of motion. Also, it is a straightforward but important check that the component \( a^3 \) decouples from the other two for any value of \( c \) and, therefore, cannot spoil the sub-leading behaviour. (Remember that the solutions for \( a^3 \) did not give rise to mass spectra.)

However, it might happen that imposing a regularity condition on the fluctuations, which is required to calculate 2-point functions, does not allow for the solution that corresponds to a given mass value. This mechanism can be summarized as follows: For given \( k^2 \), \( a^1 \) and \( a^2 \) give four independent solutions, two of which give rise to the mass spectrum (5.24), the other two leading to (5.25). These solutions evolve as we go from \( c = \infty \) to \( c = 0 \), but their asymptotic behaviour does not change. For \( c = 0 \), imposing regularity conditions will select two linear combinations of these four solutions. If such a linear combination involves only the two solutions corresponding to the same mass spectrum, then this spectrum will survive. If, in contrast, the linear combination involves solutions corresponding to different mass spectra, no mass values will result from it. A particularly interesting case is the massless state, which belongs to the spectrum (5.24), but arises only from the component \( a^2 \), not from \( a^1 \), in the analysis above. One does not expect a massless glueball state to exist, and in fact, it is likely to be excluded by this mechanism. It is less likely that only single masses, as opposed to an entire spectrum, will survive this mechanism. This is in contrast to the result of [53], where only a single glueball state was found. We will not answer these interesting questions in this paper, but we intend to come back to them.

6 The Klebanov-Strassler system

In this section we review the warped deformed conifold, or the Klebanov-Strassler solution [22]. We will be particularly interested in the “gluino sector”, the 3-scalar system of fluctuations that contains the field dual to the gluino bilinear \( \text{tr} \lambda \lambda \).

6.1 Review of the background

The KS system is obtained from the general PT system by relating the fields \( a \) and \( g \) by the relation

\[
a = \tanh y , \quad e^{-g} = \cosh y \quad (\text{KS}) ,
\]

whereby a new field \( y \) (not to be confused with the 5-d coordinates \( y^\mu \) used in Sec. 3) is introduced. This relation renders the constraint (3.11) integrable and implies \( \chi = 0 \). Moreover, one can check that the equations of motion for \( a \) and \( g \), (A.30) and (A.31), become equivalent.

There exists an even more restricted truncation, which gives rise to the singular 10-d conifold background of KT and certain fluctuations thereof. It contains four scalars and
is obtained by imposing
\[ a = b = g = h_2 = 0 \quad (\text{KT}), \quad (6.2) \]
which also implies \( \chi = 0 \). We shall not consider the KT system separately, but discuss the KS system in a way similar to the treatment of the MN system in Sec. 5. That is, we will consider a class of background solutions characterized by a parameter \( c \), that formally interpolates between the KT and KS backgrounds. As in the MN case, the background solutions are typically singular, except for the KS endpoint of the family.

Table 1: Comparison of symbol and field conventions used by Apreda \[88\] (see also \[89\]), Papadopoulos and Tseytlin \[35\], and Klebanov and Strassler (KS) \[22\]. The entries N/A mean that these fields do not appear explicitly in the KS paper. Apreda’s fields diagonalize the mass matrix in the AdS background for \( P = 0, Q = 2/\sqrt{27} \). The last two columns contain, respectively, the mass squared of the bulk fields and the conformal dimensions of the dual operators in the AdS background for \( P = 0 \).

| Field | Apreda | PT | KS | \( m^2 \) | \( \Delta \) |
|-------|--------|----|----|----------|----------|
| \( q \) | \( \frac{1}{6}(x - 2p) + \frac{3}{29} \ln(3) + \frac{1}{10} \ln(2) \) | N/A | 32 | 8 |
| \( f_{\text{Apreda}} \) | \( \frac{1}{6}(x + 3p) + \frac{1}{10} \ln(2/3) \) | N/A | 12 | 6 |
| \( y \) | \( \sinh^{-1}(ae^{-g}) \) | N/A | -3 | 3 |
| \( \Phi \) | \( \Phi \) | \( \Phi \) | 0 | 4 |
| \( s \) | \(-2h_1\) | \( M(k + f_{\text{KS}}) \) | 0 | 4 |
| \( N_1 \) | \(-h_2 - P_{\text{PT}}(b + 1)\) | \( \frac{M}{2}(k - f_{\text{KS}}) - MF \) | 21 | 7 |
| \( N_2 \) | \(-h_2 + P_{\text{PT}}(b + 1)\) | \( \frac{M}{2}(k - f_{\text{KS}}) + MF \) | -3 | 3 |
| \( P_{\text{Apreda}} \) | \(-P_{\text{PT}} \equiv -P\) | \( M/2 \) | - | - |

For the remaining fields of the KS system, there exist a variety of conventions in the literature, some of which we list for reference in Tab. 1.\(^{18}\) For the purpose of rederiving the background solutions, we shall start with the PT variables \((x, p, y, \Phi, b, h_1, h_2)\), where \( y \) was introduced in (6.1). The sigma-model metric (3.13) for the KS system reduces to

\[
G_{ab} \partial_\mu \phi^a \partial^\mu \phi^b = \partial_\mu x \partial^\mu x + 6 \partial_\mu p \partial^\mu p + \frac{1}{2} \partial_\mu y \partial^\mu y + \frac{1}{4} \partial_\mu \Phi \partial^\mu \Phi + \frac{P^2}{2} e^{\Phi - 2x} \partial_\mu b \partial^\mu b +
\frac{1}{4} e^{-\Phi - 2x} \left[ e^{-2y} \partial_\mu (h_1 - h_2) \partial^\mu (h_1 - h_2) + e^{2y} \partial_\mu (h_1 + h_2) \partial^\mu (h_1 + h_2) \right], \quad (6.3)
\]

\(^{18}\)Note that there are typos in the first three equations of (5.24) in \[35\], which relate the variables used in that paper to those used in \[22\]. The correct relations can be read off from Tab. 1.
and the superpotential reads \[35\]

\[
W = -\frac{1}{2} (e^{-2p-2x} + e^{4p} \cosh y) + \frac{1}{4} e^{4p-2x} (Q + 2Ph_2 + 2Ph_1) .
\]

(6.4)

Setting \( P = 0 \), there exists an AdS fixed point. The choice \( Q = 2/\sqrt{27} \) leads to the corresponding AdS background with unit length scale, and Aprea’s fields vanish in this background.

For the KS system, the background equations (2.3), (2.4) become

\[
\partial_r (x + 3p) = \frac{3}{2} e^{-2p-2x} - e^{4p} \cosh y ,
\]

\[
\partial_r (x - 6p) = 2 e^{4p} \cosh y - \frac{3}{2} e^{4p-2x} [Q + 2Ph_2 + 2Ph_1] ,
\]

\[
\partial_r y = - e^{4p} \sinh y ,
\]

\[
\partial_r \Phi = 0 ,
\]

\[ (6.5) \]

We shall, in the following, redervive the background solutions of this system by following the calculations of KS \[22\], but adding the relevant integration constants. From (6.5) we can immediately read off \( \Phi = \Phi_0 = \text{const.} \), and after introducing the KS radial coordinate \( \tau \) by

\[
\partial_\tau = e^{-4p} \partial_r ,
\]

(6.6)

we easily find

\[
e^\nu = \tanh \frac{\tau + c}{2} .
\]

(6.7)

For generality, we shall keep the integration constant \( c \). In particular, \( c \) takes the values \( \infty \) and 0 for the KT and KS solutions, respectively. Similar to the parameter \( c \) in the MN solution discussed in Sec. 5, it determines whether the supergravity solution is regular \((c = 0)\) or not \((c \neq 0)\). We note that (6.7) restricts the range of \( \tau \) to \( \tau > -c \).

From the equations for \( b, h_1 \) and \( h_2 \) one can derive the differential equation

\[
\partial_\tau b = b \cosh(2y) - \sinh(2y) ,
\]

(6.8)

whose general solution is

\[
b = b_1 \cosh(\tau + c) - \frac{(b_1 + 1)\tau + b_2}{\sinh(\tau + c)} .
\]

(6.9)

We must set \( b_1 = 0 \) in order to avoid the exponential blow-up for large \( \tau \), and \( b_2 \) can be absorbed into a redefinition of \( \tau \) and \( c \). Hence, we have

\[
b = -\frac{\tau}{\sinh(\tau + c)} ,
\]

(6.10)

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from which follows immediately
\[ h_2 = P e^{\Phi_0} \frac{\tau \coth(\tau + c) - 1}{\sinh(\tau + c)}. \]  
(6.11)

Then, we obtain also
\[ h_1 = P e^{\Phi_0} \coth(\tau + c) [\tau \coth(\tau + c) - 1] + \tilde{h}, \]  
(6.12)

where \( \tilde{h} \) is an integration constant.

The functions \( b, h_1 \) and \( h_2 \) determine the function \( K \), which measures the 5-form flux in the 10-d configuration (3.8). From (3.10), (6.10), (6.11) and (6.12) we find
\[ K = K_0 + P^2 e^{\Phi_0} \frac{\tau \coth(\tau + c) - 1}{\sinh^2(\tau + c)} \left[ \sinh(2\tau + 2c) - 2\tau \right], \]  
(6.13)

where we have abbreviated
\[ K_0 = Q + 2\tilde{P}\tilde{h}. \]  
(6.14)

Now, let us calculate the background fields \( x \) and \( p \). It is convenient to use Apreda’s fields \( f \) and \( q \), the definitions of which are given in Tab. 1. Then, from the equation for \((x + 3p)\) we find
\[ 5\partial_\tau f = e^{-10f} - \cosh y, \]  
(6.15)

with the general solution
\[ e^{10f} = \coth(\tau + c) - \frac{\tau + f_0}{\sinh^2(\tau + c)}, \]  
(6.16)

where \( f_0 \) is again an integration constant. The remaining background equation gives rise to
\[ \left( \partial_\tau - \frac{4}{3} \coth y \right) e^{6q - 8f/3} = -2 \cdot 3^{1/2} K e^{-20f/3}, \]  
(6.17)

where \( K \) is given by (6.13). Isolating the homogeneous solution by the ansatz
\[ e^{6q - 8f/3} = 2^{4/3} e^{-4c/3} \sinh^{4/3}(\tau + c) h(\tau), \]  
(6.18)

we obtain from (6.17) that \( h(\tau) \) satisfies
\[ \partial_\tau h = -2^{1/3} 3^{1/2} e^{4c/3} \left[ \sinh(2\tau + 2c) - 2\tau - 2f_0 \right]^{-2/3} \times \]
\[ \times \left\{ K_0 + P^2 e^{\Phi_0} \frac{\tau \coth(\tau + c) - 1}{\sinh^2(\tau + c)} \left[ \sinh(2\tau + 2c) - 2\tau \right] \right\}. \]  
(6.19)

It is instructive to consider the limit \( c \to \infty \), which describes the large-\( \tau \) behaviour of all background solutions. In this case, we obtain explicitly
\[ h = \frac{1}{2} 3^{3/2} e^{-4\tau/3} \left[ K_0 + 2P^2 e^{\Phi_0} \left( \tau - \frac{1}{4} \right) \right] + h_0, \]  
(6.20)

\(^{19}\)Note that this is not the \( K \) of KS.

\(^{20}\)The constant factor \( 2^{4/3} e^{-4c/3} \) has been inserted to normalize the forefactor to unity in the \( c \to \infty \) limit.
The choice $h_0 = 0$, needed in order to avoid the exponential growth in (6.18), removes the asymptotically flat region from the 10-d solution.

Finally, one can show that the equation for the warp factor $A$ in (2.3) yields

$$e^{-2A} = C^2 e^{-2x/3}(2 e^{-\tau})^{-2/3} \sinh^{-2/3}(\tau + c) ,$$

where the integration constant $C$ sets the 4-d scale and will be fixed later. \footnote{Note that in this formula most of the complicated $\tau$-dependence of $e^{-2A}$ is hidden in the factor $e^{-2x/3}$.}

The regular KS solution is given by fixing the integration constants as follows:

$$c = f_0 = K_0 = 0 ,$$

and imposing vanishing $h$ for large $\tau$, which yields

$$h = 2^{1/3}3^{1/2} P^2 e^{\Phi_0} \int_\tau^\infty d\vartheta \frac{\vartheta \coth \vartheta - 1}{\sinh^2 \vartheta} [\sinh(2\vartheta) - 2\vartheta]^{1/3} .$$

Note that our definition of $h$ differs from the one in [22] by a constant involving a factor $e^{-8/3}$. (Although [22] fix $\epsilon$ to a numerical value early on, it is clear from (65) in [24] that their $h \sim e^{-8/3}$. ) Our constant $C^2$ of (6.21), which appears in front of the external 4-dimensional metric in (2.3), corresponds to $\epsilon^{-4/3}$ of [22] up to numerical factors.

### 6.2 Fluctuation equations

We are now in a position to write down the equations of motion for fluctuations (4.45) about the background solutions found in the previous subsection. Let us begin by expressing the equation of motion in terms of the KS radial coordinate $\tau$. After multiplying (4.45) by $e^{-8p}$ and using (6.6) we obtain

$$\left[(\partial_\tau + M)(\partial_\tau - N) + e^{-8p-2A} \square\right] a = 0 ,$$

where the matrices $M$ and $N$ are given by

$$N^a_b = e^{-4p} \left( \partial_b W^a - \frac{W^a W_b}{W} \right) ,$$

$$M^a_b = N^a_b + 2 e^{-4p} \left( G^a_{bc} W^c + e^{-2p-2x} \delta^a_b \right) .$$

When we substitute the KS background in (6.25), the matrices become quite complicated and are relegated to Appendix E. We view it as an important step to have obtained them explicitly, and we intend to come back to a more detailed study of them at a later date.

In the following, we shall consider fluctuations about the KT background, which is given by the choice of integration constants

$$c = \infty , \quad K_0 = f_0 = h_0 = \Phi_0 = 0 .$$

\footnote{Note that in this formula most of the complicated $\tau$-dependence of $e^{-2A}$ is hidden in the factor $e^{-2x/3}$.}
The motivation for this choice is essentially the same as for the MN system: this background describes correctly the asymptotic region of the KS solution, and the equations of motion have a simpler form, which can be treated analytically (with a further approximation described in Sec. 6.3).

For the background specified by the integration constants (6.26), the matrices $M$ and $N$ have quite a simple form. Using Apreda’s variables for the fluctuation fields, $a = \delta(q, f, \Phi, s, y, N_1, N_2)$, and $P \equiv P_{PT} = -P_{Apreda}$, we find

$$M = \begin{pmatrix} \frac{4(8\tau-3)}{3(4\tau+1)} & 0 & 0 & \frac{32(\tau-1)}{45P(16\tau^2-1)} & 0 & 0 & 0 \\ 0 & -2 & 0 & -\frac{4}{15P(4\tau-1)} & 0 & 0 & 0 \\ 0 & 0 & \frac{4}{3} & -\frac{8}{3P(4\tau-1)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{3P(4\tau-1)}{8(2\tau+1)(4\tau-3)} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{8}{3(4\tau-1)} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{32(\tau-1)}{4(4\tau+1)} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{16(\tau-1)}{3(4\tau+1)} \end{pmatrix}, \quad (6.27)$$

$$N = \begin{pmatrix} \frac{16(\tau-1)}{3(4\tau+1)} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (6.28)$$

The block-diagonal form of these matrices is a nice feature of the KT background. Remember that (6.2) defines a consistent truncation of the KS to the KT system, so that the lower left $3 \times 4$ off-diagonal blocks of $M$ and $N$ are expected to be zero, but vanishing of the upper right block is a bonus feature. It is a particularly welcome bonus, since the gluino sector $\delta(y, N_1, N_2)$ is where we would expect much of the interesting physics to be encoded.

We also see from (6.27) and (6.28) that the UV limit $\tau \to \infty$ and the conformal limit $P \to 0$ do not commute. One might have considered performing an expansion in $P$ to study a “near-conformal” regime, but the order of limits would pose a problem. This is not surprising, because among other things we have imposed $K_0 = 0$ on the solution, which is not possible for $P = 0$, as can be seen from (6.13). It is of course possible to study the conformal (Klebanov-Witten [30]) system directly, but this would require changing field variables.

For the KT background, it is useful to change the radial variable by introducing $\tau = 3 \ln \sigma + \frac{1}{4}$. 

Using (6.21), (6.18) and (6.20), we find that the term in (6.24) with the 4-dimensional

\[22\]Our $\sigma$ corresponds to $r$ of KT up to a multiplicative factor, whereas our $r$ corresponds to their $u$. 

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box operator is proportional to
\[ e^{-2A-8p} \sim \frac{C^2P^2}{\sigma^2} \ln \sigma , \]  
(6.30)
where we have suppressed a numerical factor. Hence, from (6.24) and with a suitable choice of the constant C follows, in momentum space,
\[ \left( \sigma \partial_{\sigma} + 3M \right) \left( \sigma \partial_{\sigma} - 3N \right) - \frac{k^2P^2}{\sigma^2} \ln \sigma \right] a = 0 . \]  
(6.31)
We see that fixing C indeed sets the 4-dimensional energy scale, as claimed in the previous section.

We further introduced
\[ v = \frac{kP}{\sigma} , \]  
(6.32)
in terms of which (6.31) becomes
\[ \left[ v^3 \partial_v v^{-3} \partial_v - Y v^{-1} \partial_v - Z v^{-2} - \ln \frac{kP}{v} \right] a = 0 , \]  
(6.33)
where the matrices Y and Z are given by
\[ Y = 3(M - N) - 4 , \quad Z = 9MN + 3\sigma \partial_{\sigma} N . \]  
(6.34)
In [32], fluctuations of the 4-scalar KT system were studied in a particular gauge, leading to equations more complicated than, but presumably equivalent to (6.33).

6.3 “Moderate UV” approximation

Despite its apparent simplicity, equation (6.33) has no analytic solution. A method to extract the response functions at leading order in the high-energy limit was developed by Krasnitz [31, 32]. We proceed to briefly review this method, but first we pause for a short comment on our motivation to use the method in the first place.

We are, of course, ultimately interested in all energy ranges and the confining phase, not just the high-energy limit. Nevertheless, we have seen that the matrices in appendix E are prohibitively complicated for analytical work, so we view the approximation in this subsection as a simple way to get a handle on the full problem in one particular regime (high energy), which should provide good cross-checks for a numerical treatment. In addition, since renormalization is a UV problem, KT counterterms should be sufficient to renormalize KS correlators, so the UV regime seems a good place to start.

Here is the brief review. In [31 32], the KT solution was divided into two overlapping regions, which we will call “moderate UV” (or “mUV”) region and “extreme UV” (or “xUV”) region. For the purposes of this discussion, let us set \( P = 1 \); it can be restored by \( k \rightarrow kP \). In the mUV region, \(|\log v| \ll |\log k|\), so we can approximate the troublesome

\[ ^{23}\text{Our } v \text{ corresponds to Krasnitz’s } y . \]
log(k/v) in (6.33) by a constant log k. This clearly does not work for v too small, hence “moderate” UV, but when it does work, exact solutions of the approximated equation can be found [31, 32]. In the xUV region, [31, 32] treats log(k/v) as a perturbation, and expands iteratively in it. Then, there is an intermediate overlap region (see Fig. 3) where both solutions should be valid simultaneously. For large k, the solutions naively appear to differ appreciably in the intermediate regime, unless there is some relation between large-k terms in the two solutions: this allows us to match the leading-order terms in k. Analytic correlators can in principle be extracted from this matching at leading order in k, but we reiterate that the dictionary and renormalization problems should be completely solved before any gauge theory correlators can be quoted with certainty. (Thus we will not perform the xUV analysis here, but we mentioned it for completeness).

Here, we will show that the 7-scalar system is analytically solvable in the mUV approximation, generalizing the analysis of [31, 32] to the present case. We will then check our solutions numerically.

The mUV regime is obtained in two steps. First, we consider the UV region, i.e., large τ, which implies large σ. To leading order, the matrices Y and Z become

\[
Y = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
36P & 24P & 6P & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -6P & 0 & 0 \\
0 & 0 & 0 & 0 & -6P & 0 & 0
\end{pmatrix},
\]

(6.35)
and

\[
Z = \begin{pmatrix}
32 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 12 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
336P & -96P & -24P & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -3 & 0 & 0 \\
0 & 0 & 0 & 0 & 42P & 21 & 0 \\
0 & 0 & 0 & 0 & 6P & 0 & -3 \\
\end{pmatrix}.
\]  

(6.36)

As a check, we see that \( Z \) reproduces the masses in table 1 in the conformal limit \( P \to 0 \), although we noted earlier that one would need rescaled field variables to study this limit. (The mass does not depend on the field normalization.)

Second, as discussed earlier, the mUV region is isolated by considering large external momenta \( | \log k | \gg | \log v | \). (As in the discussion above, note that this limits \( v \) from below as well as from above.). This means that we can neglect \( \ln v \) from (6.33). When this is done, \( k \) is easily removed from (6.33) by defining

\[
z = \sqrt{\ln(kP)} v ,
\]

(6.37)

so that we obtain the equation

\[
\left[ z^3 \partial_z z^{-3} \partial_z - Y z^{-1} \partial_z - Z z^{-2} - 1 \right] a = 0 .
\]

(6.38)

The variable \( z \) blows up in the conformal limit \( P \to 0 \) (cf. the order-of-limits discussion in the previous subsection). If needed, one can always go back to (6.31) and set \( P = 0 \) there.

With \( Y \) and \( Z \) given by the constant matrices (6.35) and (6.36), the equation (6.38) admits analytical solutions. We are, as usual in AdS/CFT, interested in the solutions that are regular for large \( z \).

---

24We recall that we use dimensionless variables.
For the four scalars of the KT system, we obtain

\[ a_1 = z^2 K_6(z) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 6P \\ 0 & 0 & 0 \end{pmatrix} - 6P \left[ 4z^2 K_4(z) + 6z K_1(z) + 6 K_0(z) - 3z^2 K_2(z) \ln z \right] \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \]

\[ a_2 = z^2 K_4(z) \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - 12P \left[ z^2 K_4(z) + 2z K_1(z) + 2 K_0(z) - z^2 K_2(z) \ln z \right] \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \]

\[ a_3 = z^2 K_2(z) \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - 3P \left[ 2z K_1(z) + 2 K_0(z) - z^2 K_2(z) \ln z \right] \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \]

\[ a_4 = z^2 K_2(z) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \]

(6.39)

where the \( K_n \) are Bessel functions of order \( n \). For the gluino sector, we find

\[ a_5 = z^2 K_1(z) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - 3P \left[ z^2 K_5(z) \ln z + 21z K_4(z) + \frac{7}{6} z^2 K_1(z) + \frac{80}{z} K_3(z) + \frac{240}{z^2} K_1(z) + \frac{384}{z^3} K_0(z) \right] \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} -

- 3P \left[ z^2 K_1(z) \ln z + z K_0(z) \right] \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \]

\[ a_6 = z^2 K_5(z) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \]

\[ a_7 = z^2 K_1(z) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \]

(6.40)

A few comments on these solutions are in order. We note that from the approximate matrices \( Y \) and \( Z \), given in (6.35) and (6.36), it could have been gleaned already that the component \( y \) is a source for \( N_1 \) and \( N_2 \), but not the other way around. Hence, it is not surprising that the solutions \( a_6 \) and \( a_7 \), where only \( N_1 \) or \( N_2 \) are non-zero, are significantly
simpler than \( a_5 \), where also the \( y \)-component is turned on. Our next observation is that \( y \) sources the other two gluino-sector fields by terms linear in \( P \). Indeed, the matrices \( Y \) and \( Z \) in (6.35), (6.36) make manifest the fact that the Apreda basis diagonalizes the mass matrix in the conformal \((P = 0)\) limit. It makes it equally manifest that the gauge/gravity dictionary problem is significantly more pressing in the \( P \neq 0 \) case than in the conformal limit.

These analytical solutions are remarkably simple. In the face of dark *Shelob* horror like the full KS matrices shown in appendix [E] these solutions may prove to be our saving *Eärendil* light\(^\text{25}\) provided we can convince ourselves that they actually do solve the exact KT equation (6.33) in a suitably approximate sense. The Krasnitz approximation is valid for very large \( k \), so we give a representative check for moderately large \( k \), when the approximation should just begin to work.

![Figure 4: Moderate-UV analysis: comparison of the analytical solutions (6.40) of equation (6.38) with the corresponding numerical solution of (6.33) found by shooting (marked by crosses) for \( k = 10^3 \), \( P = 1 \). The “response functions” agree to an accuracy of 8%.

The numerical solutions were found by shooting for approximately regular solutions of (6.33), that is, minimizing field values at the grid endpoint by tuning the derivative at a UV cutoff\(^\text{26}\). Superimposing a linear combination of the solutions \( a_5, a_6, a_7 \) of the approximate equation (6.38), and normalizing them to unity at the cutoff, we find good qualitative agreement in Fig. 4. The derivatives at the cutoff essentially give the response functions for the given value of \( k \) (since we normalized the fields at the cutoff to unity). The numerical responses agree with the analytical solutions of the approximate

\(^{25}\)see wikipedia.org

\(^{26}\)Further details will be given in future work.
equation within fairly good accuracy, and the accuracy will improve with energy. Above and beyond any matching procedure à la Krasnitz, we take the good agreement in Fig. 4 as evidence that the solutions (6.39), (6.40) may give us the crutch we need as we embark on a numerical study of the full KS system.

7 Outlook

In this paper we have investigated aspects of the bulk dynamics of supergravity fluctuations about the duals of confining gauge theories, in particular the KS and MN backgrounds. In our to-do list in the introduction, we called this the “fluctuation problem”. This sets the stage for addressing the problem of calculating correlators in confining gauge theories from non-asymptotically AdS supergravity backgrounds. To be able to perform our analysis we derived a consistent truncation of type IIB supergravity to a set of scalars coupled to 5-d gravity, which is general enough to deal with fluctuations about the KS and MN backgrounds. Importantly, we also developed a gauge-invariant and sigma-model-covariant formulation of the dynamics of the field fluctuations in generic, “fake-supergravity”-type systems, which should find many applications amongst the various configurations studied in the literature. Moreover, we point out that the gauge-invariant formalism naturally includes higher order interactions. Hence, once the “dictionary” and “renormalization” problems for holographic renormalization of confining gauge theories (as introduced in the introduction) are understood, the calculation of three-point functions and scattering amplitudes (along the lines of [49]) should become straightforward.

Concerning our particular results, there are many open issues that could and will be addressed in the near future. For the MN system, the most interesting question is to check the validity of the mass spectra (5.24) and (5.25) by numerically solving the fluctuation equation in the regular background. As discussed in detail at the end of Sec. 5.3, it is only the existence of the discrete masses, not their particular values, which is in doubt. This question is of particular interest also in view of the contrasting results of [52, 53]. It is an interesting point, though, that the existence of an upper bound on the masses, as is the branch point in our case, was also found in [53]. In any case, all MN results should be considered in light of the fact that the supergravity approximation is not under control in the UV region of the MN solution.

For the KS system, it should be straightforward to generalize the numerical analysis of Sec. 6.3 to the full KS background. This will not only lead to a better understanding of the range of validity of our approximate analytical solutions of Sec. 6.3, but also pave the way for the extraction of some dual physics, once progress has been made on the dictionary and renormalization problems. Moreover, a more detailed analysis of the fluctuation equations in the “extreme UV” region, for instance the asymptotic expansion used by Krasnitz [31, 32] which we have not performed here, might shed further light on these problems.

We would also like to comment on the question of the glueball spectrum in the KS theory. This has already been studied in [91, 92], where it was argued that the glueball
spectrum is an IR quantity. As the 3-cycle is an $S^3$ in the IR, the fields were expanded in harmonics of $S^3$ in these papers, which was argued to lead to a decoupling of, for example, the dilaton. From (A.10), (A.7) and (A.8) it is obvious that this decoupling can never be exact. The complicated dependence on the internal coordinates present in the PT ansatz simply drops out of the 10-d equation of motion for the dilaton, leaving the 5-d equation given in (A.10). It might still be that the expansions performed in [91, 92] are approximately correct, but it would be important to check to what extent this is really a controllable approximation of the IR physics of the KS gauge theory. We are optimistic that our formalism presents a useful starting point to attempt such a check by solving the linearized gauge-invariant equations for the scalars numerically.

Another interesting open issue was brought up in Sec. 3. In the PT ansatz, there is an additional scalar field, which does not appear in the KS system: This is the superpartner of the Goldstone mode predicted in [79] and studied in [77, 78]. Even though it seems to be an ideal candidate for addressing the problem of calculating 2-point functions in the KS background (at linear level it decouples from the other scalars as long as it depends only on the radial variable [77]), it turns out that the dynamics of this mode requires a generalization of the PT ansatz, once we allow the field to depend on all five external coordinates, since then it does not satisfy the integrability constraint (3.11) in general. It would be very interesting to find a generalization of the PT ansatz that would lead to a 5-dimensional consistent truncation of the type IIB equations of motion and include this additional mode. Attempts along those lines might also lead to a form of the 5-dimensional effective theory which is manifestly supersymmetric.

In all, we have found an efficient approach to (at least an important subset of) the dynamics of fluctuations about the supergravity duals of confining gauge theories, and demonstrated its applicability in a number of examples. This is an important step towards a full understanding of the “fluctuation” problem for holographic renormalization. We are hopeful that our results make also the “dictionary” and “renormalization” problems more accessible. We look forward to the day when exciting new physics of confining gauge theories can be reliably extracted from gauge/string duality.

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A Consistent truncation of type IIB supergravity

In this appendix, we present details of the consistent truncation of type IIB supergravity to the effective 5-d system (3.12). In particular, we shall show how the constraints (3.10) and (3.11) and the 5-d equations of motion derivable from (3.12) arise from the 10-d supergravity equations. When comparing the results of this appendix with (3.12)-(3.14), one has to bear in mind that we omitted the tildes in those formulas for esthetic reasons.

To start, let us identify our conventions and present some useful formulas. We use the metric and curvature conventions of MTW, Polchinski and Wald, i.e., the signature is mostly plus and

\[ R_{MNP}^Q = \partial_N \Gamma_{MP}^Q - \partial_M \Gamma_{NP}^Q + \Gamma_{MP}^S \Gamma_{SN}^Q - \Gamma_{NP}^S \Gamma_{SM}^Q , \]

\[ R_{MN} = R_{MPN}^P , \] (A.1)

where the Christoffel symbols are defined as

\[ \Gamma_{MN}^P = \frac{1}{2} g^{PQ} (\partial_M g_{NQ} + \partial_N g_{MQ} - \partial_Q g_{MN}) . \] (A.2)

With these conventions one has the following transformation rules of the Ricci tensor and Ricci scalar of a \( D \)-dimensional manifold under a conformal transformation \( \hat{g}_{MN} = \Omega^2 g_{MN} \), cf. appendix D of [55]

\[ \hat{R}_{MN} = R_{MN} - (D-2) \nabla_M \partial_N \ln \Omega - g_{MNPQ} \nabla_P \partial_Q \ln \Omega + (D-2)(\partial_M \ln \Omega)(\partial_N \ln \Omega) - (D-2) g_{MN} g^{PQ} (\partial_P \ln \Omega)(\partial_Q \ln \Omega) , \]

\[ \hat{R} = \Omega^{-2} \left[ R - 2(D-1) g^{PQ} \nabla_P \partial_Q \ln \Omega - (D-2)(D-1) g^{PQ} (\partial_P \ln \Omega)(\partial_Q \ln \Omega) \right] . \] (A.3)

Moreover, our conventions for the Hodge star of a \( p \)-form \( \omega_p \) are

\[ \star \omega_p = \frac{1}{p! (D-p)!} \omega_{M_1 \ldots M_p} \epsilon^{M_1 \ldots M_p N_1 \ldots N_{D-p}} \ dx^{N_1} \wedge \ldots \wedge dx^{N_{D-p}} . \] (A.4)

Finally, we adopt the convention to adorn with a tilde objects derived from the metric \( ds_5^2 = \tilde{g}_{\mu\nu} \ dy^\mu \ dy^\nu \) (again, note the difference in notation compared to (3.8); in the main text we suppressed the tildes for readability). For example, \( \tilde{\nabla} \) denotes the covariant derivative with respect to \( \tilde{g}_{\mu\nu} \), and \( \tilde{F}^{ij} = \tilde{g}_{\mu\nu} g^{ik} g^{jl} F_{\nu\lambda} \). Note the relation of \( \tilde{g}_{\mu\nu} \) to the external components of the metric, \( g^{(\text{ext})}_{\mu\nu} = e^{2p-x} \tilde{g}_{\mu\nu} \), as follows from (3.8). \( g^{(\text{int})} \) denotes
the remaining internal part, although we usually omit the superscript \(^{\text{int}}\) if it is clear from the indices \(i, j, \ldots\) that we mean an internal component. Also note that the usage of the index \(i\) to label the internal coordinates in the 10-dimensional metric (3.8), i.e. \(i \in \{\psi, \theta_1, \theta_2, \phi_1, \phi_2\}\), differs from the usage in Sec. I and Appendices [B, C and D].

Let us turn to the analysis of the 10-d equations of motion. The equation of motion for the RR scalar \(C\), (3.3), is satisfied, because
\[
H_{M_1\ldots M_3} F^{M_1\ldots M_3} = 0.
\]

The equation of motion for \(\tilde{F}_5\), (3.7), leads to
\[
\partial \mu K = 2 P \partial \mu (h_1 + bh_2), \tag{A.5}
\]
from which the constraint (3.10) follows.

The second constraint, (3.11), arises from (3.5), in particular, from the mixed components
\[
\partial_M \left( e^{-\Phi} H^{M\mu_3} \sqrt{-g} \right) = \partial_k \left( e^{-\Phi} H^{k\mu_3} \sqrt{-g} \right) = 0. \tag{A.6}
\]
Eq. (A.6) follows from (3.5), because
\[
\varepsilon^{M_1\ldots M_{10}} F_{M_1\ldots M_5} F_{M_6\ldots M_8}, \tag{A.7}
\]
and
\[
C \equiv 0. \tag{A.8}
\]
Furthermore, in the first equality of (A.6) we have used that the components of \(H\) have at most one external index. One can show that all components of (A.6) are fulfilled, once (3.11) is imposed.\(^{27}\)

The equation of motion for the dilaton can be checked as follows. If we denote
\[
I_1 := 2 e^{8p} h_2^2 + 2 \partial_\mu h_2 \tilde{\partial}^\mu h_2
+ 4(1 + 2 e^{-2g} a^2) \partial_\mu h_1 \tilde{\partial}^\mu h_1 + 8 e^{-2g} a^2 \partial_\mu h_2 \tilde{\partial}^\mu h_2 + 8a[e^{-2g}(a^2 + 1) + 1] \partial_\mu h_1 \tilde{\partial}^\mu h_2
\]
\[
= \frac{1}{6} e^{2p+x} H_{MNP} H^{MNP}, \tag{A.9}
\]
\[
I_2 := P^2 \left\{ 2 \partial_\mu b \tilde{\partial}^\mu b + e^{8p} [e^{2g} + e^{-2g}(a^2 - 2ab + 1)^2 + 2(a - b)^2] \right\}
\]
\[
= \frac{1}{6} e^{2p+x} F_{MNP} F^{MNP}, \tag{A.10}
\]
then the dilaton-dependent terms in the 5-d action (3.13) are given by
\[
S^{\text{dil}}_5 = \int d^5y \sqrt{g} \left( \frac{1}{8} \partial_\mu \Phi \tilde{\partial}^\mu \Phi + \frac{1}{8} e^{-\Phi-2x} I_1 + \frac{1}{8} e^{\Phi-2x} I_2 \right). \tag{A.11}
\]

The equation of motion that follows from (3.2) and the constraint (3.11) is
\[
e^{x-2p} \tilde{\nabla}^2 \Phi = -\frac{1}{2} e^{-\Phi-2p-x} I_1 + \frac{1}{2} e^{\Phi-2p-x} I_2. \tag{A.12}
\]
Obviously, (A.12) is precisely the equation of motion that one would derive from the 5-d action (A.9).

Let us next consider the equation of motion for \(F_3\), (3.4), which reads
\[
\nabla_M (e^\Phi F^{MNP}) = -\frac{1}{3! \sqrt{g^{(\text{ext})}}} F_{y_1\ldots y_3} H_{ijke^{ijkNP}}. \tag{A.13}
\]
The right hand side is only non-vanishing if $N$ and $P$ are internal indices. One can verify that the same holds also for the left hand side. To see this, note that from the ansatz of $F_3$ and the block structure of the metric, one only has to check
\[ \partial_k (F^{klm} \sin \theta_1 \sin \theta_2) = 0, \quad (A.12) \]
because $F_3$ can have at most one external index. The validity of (A.12) can easily be checked with the help of a computer. Thus, the non-trivial part of the equation of motion for $F_3$ boils down to
\[ \nabla_M (e^\Phi F^{Mlm}) = -\frac{1}{3!} K e^{3p-\frac{4}{2}x} H_{ijk} \epsilon^{ijklm}, \quad (A.13) \]
where we have made use of $F_{y_1...y_5} = K e^{3p-x} \sqrt{g^{\text{ext}}}$. It turns out that the angle dependences of the left and right hand sides in (A.13) coincide for each value of $l$ and $m$. Moreover, the components only differ in their angle dependence. More precisely, on the one hand we have
\[ \nabla_M (e^\Phi F^{Mlm}) = e^\Phi \sin \theta_1 \sin \theta_2 \partial_k (\sin \theta_1 \sin \theta_2 H_{klm}) + e^{x-2p} \nabla_\mu (e^\Phi \hat{F}^{\mu lm}) \]
\[ = P \left\{ e^\Phi [e^{6p-2x} b - a] - a(a^2 - 2ab + 1) \right\} f^{lm}(\psi, \theta_1, \theta_2, \phi_1, \phi_2), \quad (A.14) \]
where $f^{lm}$ is some simple rational expression involving trigonometric functions of the angles, whose precise form depends on $l$ and $m$. On the other hand,
\[ -\frac{1}{3!} K e^{3p-\frac{4}{2}x} H_{ijk} \epsilon^{ijklm} = -K e^{6p-3x} h_2 f^{lm}(\psi, \theta_1, \theta_2, \phi_1, \phi_2). \]
Taking (A.14) and (A.15) together leads to the equation of motion for $b$
\[ P^2 \nabla_\mu (e^\Phi \hat{\partial}^\mu b) = PK e^{8p-4x} h_2 + P^2 e^{8p-2x} [b - a - a e^{-2g(a^2 + 1 - 2ba)]}, \quad (A.16) \]
which is exactly what one would derive from (3.12).

Now, we come to the equation of motion for $H_3$, (3.3), which is equivalent to
\[ \nabla_M (e^\Phi H^{MNP}) = \frac{1}{3! \sqrt{g^{\text{ext}}}} F_{\eta_1...\eta_5} F_{ijk} \epsilon^{ijklN}. \quad (A.17) \]
Again, the right hand side is only non-vanishing for internal components $N$ and $P$. As we already said above, the same is true for the left hand side after imposing the constraint (3.11), cf. (A.6). Thus, the non-trivial part of (A.17) becomes
\[ \nabla_M (e^\Phi H^{Mlm}) = e^\Phi \sin \theta_1 \sin \theta_2 \partial_k (\sin \theta_1 \sin \theta_2 H^{klm}) + e^{x-2p} \nabla_\mu (e^\Phi \hat{H}^{\mu lm}) \]
\[ = \frac{1}{3!} K e^{3p-\frac{4}{2}x} F_{ijk} \epsilon^{ijklm}. \]
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That is, inserted into (A.18), add up to give the equation of motion for \( f \) where \( f \) whereas the coefficients of \( \tilde{\theta} \) have satisfied, because both sides of (3.1) are identically zero. For the relevant internal com-
ponents we notice that \( f_1 \) always (i.e., for all values of \( l \) and \( m \)) contains a factor \( \cos(\psi) \) or \( \sin(\psi) \), whereas \( f_2 \) is independent of \( \psi \). Furthermore,
\[
\frac{e^{-\Phi}}{\sin \theta_1 \sin \theta_2} \partial_k (\sin \theta_1 \sin \theta_2 H^{klm}) = -e^{-\Phi+6p-x} h_2 f^{lm}_1(\psi, \theta_1, \theta_2, \phi_1, \phi_2),
\]
where \( f_1 \) and \( f_2 \) differ in such a way that \( f_1 \) always (i.e., for all values of \( l \) and \( m \)) contains a factor \( \cos(\psi) \) or \( \sin(\psi) \), whereas \( f_2 \) is independent of \( \psi \). Furthermore,
\[
\frac{e^{-\Phi}}{\sin \theta_1 \sin \theta_2} \partial_k (\sin \theta_1 \sin \theta_2 H^{klm}) = -e^{-\Phi+6p-x} h_2 f^{lm}_1(\psi, \theta_1, \theta_2, \phi_1, \phi_2),
\]
and
\[
e^{-2p} \tilde{\nabla}_\mu (e^{-\Phi} \tilde{H}^{ilm}) = e^{-2p} \tilde{\nabla}_\mu \left\{ e^{-\Phi-2x} \times \left[ \frac{2a(1 + e^{-2g}(1 + a^2))}{e^{2g} + 2a^2 + e^{-2g}(1 - a^2)^2} \tilde{\mu} h_1 + \frac{e^{2g} + 2a^2 + e^{-2g}(1 + a^2)^2}{e^{2g} + 2a^2 + e^{-2g}(1 - a^2)^2} \tilde{\mu} h_2 \right] + 2 e^{-\Phi-2x} \left[ \frac{1 + 2a^2 e^{-2g}}{e^{2g} + 2a^2 + e^{-2g}(1 - a^2)^2} \tilde{\mu} h_1 + \frac{a(1 + e^{-2g}(1 + a^2))}{e^{2g} + 2a^2 + e^{-2g}(1 - a^2)^2} \tilde{\mu} h_2 \right] \right\} f^{lm}_1 + \tilde{f}^{lm}_2.\]
It is not difficult to verify that the coefficients of \( f^{lm}_1 \) in (A.19), (A.20) and (A.21), when inserted into (A.18), add up to give the equation of motion for \( h_2 \), as derived from (3.12). That is,
\[
\tilde{\nabla}_\mu \left\{ e^{-\Phi-2x} \left[ \frac{2a(1 + e^{-2g}(1 + a^2))}{e^{2g} + 2a^2 + e^{-2g}(1 - a^2)^2} \tilde{\mu} h_1 + \frac{e^{2g} + 2a^2 + e^{-2g}(1 + a^2)^2}{e^{2g} + 2a^2 + e^{-2g}(1 - a^2)^2} \tilde{\mu} h_2 \right] \right\} = e^{-\Phi+8p-2x} h_2 + PK e^{8p-4x} b,\]
whereas the coefficients of \( f^{lm}_2 \) give the equation of motion for \( h_1 \), as derived from (3.12), i.e.,
\[
\tilde{\nabla}_\mu \left\{ 2 e^{-\Phi-2x} \left[ \frac{1 + 2a^2 e^{-2g}}{e^{2g} + 2a^2 + e^{-2g}(1 - a^2)^2} \tilde{\mu} h_1 + \frac{a(1 + e^{-2g}(1 + a^2))}{e^{2g} + 2a^2 + e^{-2g}(1 - a^2)^2} \tilde{\mu} h_2 \right] \right\} = PK e^{8p-4x}.\]
Finally, we consider Einstein’s equation, (3.1). The mixed components are trivially satisfied, because both sides of (3.1) are identically zero. For the relevant internal components we notice that
\[
R_{ij} = R_{ikj}^k + R_{ij}^\mu \mu.\]
Using the fact that the only non-vanishing Christoffel symbols besides the pure components \( \Gamma^\mu_{jk} \) and \( \Gamma^\mu_{\nu\rho} \) are
\[
\Gamma^\mu_{ij} = -\frac{1}{2} g^{\mu\nu} \partial_\nu g_{ij}, \quad \Gamma^i_{j\mu} = \frac{1}{2} g^{il} \partial_l g_{j\mu} = \Gamma^i_{\mu j},\]
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we derive
\[ R_{ij} = R_{ij}^{(\text{int})} - \frac{1}{4}(\partial^\mu g_{ij})(\partial_\mu \ln g^{(\text{int})}) + \frac{1}{2}(\partial_\mu g_{lk})(\partial^\mu g_{lj})g^{kl} - \frac{1}{2}\nabla_\mu \partial^\mu g_{ij}, \quad (A.26) \]
where \( g^{(\text{int})} \) denotes the internal block of the metric, and \( R_{ij}^{(\text{int})} \) is the Ricci tensor that follows from it. Notice the absence of tildes in (A.26).

Hence, the internal components of Einstein’s equation are
\[ 0 = R_{ij}^{(\text{int})} - \frac{1}{4}(\partial^\mu g_{ij})(\partial_\mu \ln g^{(\text{int})}) + \frac{1}{2}(\partial_\mu g_{lk})(\partial^\mu g_{lj})g^{kl} - \frac{1}{2}\nabla_\mu \partial^\mu g_{ij} \quad (A.27) \]
\[ - \frac{1}{96} F_{im1...m4}F_{j}^{m1...m4} - \frac{1}{4}(e^{-\varphi}H_{iPQ}H_{j}^{PQ} + e^{\varphi}F_{iPQ}F_{j}^{PQ}) + \frac{1}{8}g_{ij}(e^{-\varphi-2p-x}I_{1} + e^{\varphi-2p-x}I_{2}) \equiv S_{ij}, \]
where \( I_{1} \) and \( I_{2} \) were defined in (A.7) and (A.8). The expressions are quite complicated, but using \textit{Maple} we checked that all components of (A.27) are satisfied once \( S_{\psi\psi}, S_{\theta_{i}\theta_{i}}, S_{\phi_{i}\phi_{i}} \), and \( S_{\psi\phi_{j}} \), for instance, are zero. Moreover, taking these four components and solving for the second derivatives \( \tilde{\nabla}^{2}p, \tilde{\nabla}^{2}x, \tilde{\nabla}^{2}a \) and \( \tilde{\nabla}^{2}g \), leads to the same expressions as derived from the action (3.12), i.e.,
\[ \tilde{\nabla}_{\mu}\partial_{\mu}p = -\frac{1}{6}\left\{ e^{p-2x-g}[e^{2g}+(1+a^{2})] + \frac{1}{2}e^{-4p-4x}[e^{2g}+(a^{2}-1)^{2}e^{-2g}+2a^{2}] - \right. \]
\[ - 2a^{2}e^{-2g+8p} - 2e^{-\varphi-2x+8p}h_{2}^{2} - e^{8p-4x}K^{2} - \]
\[ - P^{2}e^{\varphi-2x+8p}[e^{2g} + e^{-2g}(a^{2} - 2ab + 1)^{2} + 2(a - b)^{2}] \right\}, \quad (A.28) \]
\[ \tilde{\nabla}_{\mu}\partial_{\mu}x = e^{2p-2x-g}[e^{2g}+(1+a^{2})] - \frac{1}{2}e^{-4p-4x}[e^{2g}+(a^{2}-1)^{2}e^{-2g}+2a^{2}] - \]
\[ - \frac{1}{4}P^{2}e^{\varphi-2x}\left\{ e^{8p}[e^{2g} + e^{-2g}(a^{2} - 2ab + 1)^{2} + 2(a - b)^{2}] + 2\partial_{\mu}b\partial^{\mu}b \right\} - \]
\[ - \frac{1}{2}e^{8p-4x}K^{2} - \frac{1}{4}e^{-\varphi-2x}\left\{ 2e^{8p}h_{2}^{2} + 2\partial_{\mu}h_{2}\partial^{\mu}h_{2} + \right. \]
\[ + \frac{4(1 + 2e^{-2g}a^{2})\partial_{\mu}h_{1}\partial^{\mu}h_{1} + 8e^{-2g}a^{2}\partial_{\mu}h_{2}\partial^{\mu}h_{2} + 8a[e^{2g}(a^{2} + 1) + 1]\partial_{\mu}h_{1}\partial^{\mu}h_{2}}{e^{2g} + 2a^{2} + e^{-2g}(1 - a^{2})^{2}} \right\}, \quad (A.29) \]
\[ \tilde{\nabla}_{\mu}(e^{-2g}\partial_{\mu}a) = -e^{-\varphi-2x}[e^{2g} + 2a^{2} + e^{-2g}(1 - a^{2})^{2}]^{-2} \times \]
\[ \times \left\{ 4ae^{-2g}(a^{2} - 1)[1 + (a^{2} + 1)e^{-2g}]\partial_{\mu}h_{1}\partial^{\mu}h_{1} + 4a[e^{-2g}(a^{4} - 1) - 1]\partial_{\mu}h_{2}\partial^{\mu}h_{2} + \right. \]
\[ + 2[e^{4g}(a^{6} + 5a^{4} - 5a^{2} - 1) - e^{2g} + e^{-2g}(a^{4} - 1) - a^{2} - 1]\partial_{\mu}h_{1}\partial^{\mu}h_{2} \right\} - \]
\[ - 2ae^{2p-2x-g} + ae^{-4p-4x}[e^{2g}(a^{2} - 1) - e^{-2g} + 1] + ae^{-2g+8p} + \]
\[ + P^{2}(a - b)e^{\varphi-2x+8p}[e^{-2g}(a^{2} - 2ab + 1) + 1], \quad (A.30) \]
\[ \tilde{\nabla}_\mu \tilde{\nabla}^\mu g = -e^{-2\phi} \partial_\mu a \tilde{\nabla}^\mu a - 2 e^{-\phi} e^{-2x} [e^{2g} + 2a^2 + e^{-2g}(1 - a^2)]^{-2} \times \]
\[ \times \left\{ [e^{2g} + 4a^2 + e^{-2g}(3a^2 + 2a^2 - 1)] \partial_\mu h_1 \tilde{\nabla}^\mu h_1 + 4a^2 (1 + a^2 e^{-2g}) \partial_\mu h_2 \tilde{\nabla}^\mu h_2 + \right. \]
\[ + 2a[e^{2g} + 2(a^2 + 1) + e^{-2g}(a^2 + 4a^2 - 1)] \partial_\mu h_1 \tilde{\nabla}^\mu h_2 \right\} - \]
\[ - e^{2p-2x-g} [e^{2g} - (1 + a^2)] + \frac{1}{2} e^{-4p-4x} [e^{2g} - (a^2 - 1)^2 e^{-2g}] - \]
\[ - a^2 e^{-2g+8p} + \frac{1}{2} p^2 e^{1-2x+8p} [e^{2g} - e^{-2g}(a^2 - 2ab + 1)^2] . \] (A.31)

Thus, the equations of motion for \((p, x, a, g)\) arising from (3.12) guarantee that all internal components of Einstein’s equation are satisfied.

For the external components of Einstein’s equation we note that
\[ R_{\mu\nu} = R_{\mu\nu}^{ext} + R^{ext}_{\mu\nu} , \] (A.32)
where \(R^{ext}_{\mu\nu}\) stands for the purely external part of the Ricci-tensor, i.e., the one that is calculated solely with \(\Gamma_\nu^{\mu\rho}\). Using (A.3) and
\[ \frac{1}{4} g^{mn} g^{ik} (\partial_\mu g_{il}) (\partial_\nu g_{mk}) = e^{-2g} \partial_\mu a \partial_\nu a + \frac{3}{2} \partial_\mu p \partial_\nu x + \frac{3}{2} \partial_\mu x \partial_\nu p + \]
\[ + \partial_\mu g \partial_\nu g + 9 \partial_\mu p \partial_\nu p + \frac{5}{4} \partial_\mu x \partial_\nu x , \] (A.33)
one arrives at
\[ R_{\mu\nu} = \tilde{R}_{\mu\nu} - 2 \partial_\mu x \partial_\nu x - 12 \partial_\mu p \partial_\nu p - e^{-2g} \partial_\mu a \partial_\nu a - \partial_\mu g \partial_\nu g - \tilde{g}_{\mu\nu} \tilde{\nabla}_\rho \tilde{\nabla}^\rho x + \frac{1}{2} \tilde{g}_{\mu\nu} \tilde{\nabla}_\rho \tilde{\nabla}^\rho x \]
\[ = \frac{1}{2} \partial_\mu \Phi \partial_\nu \Phi + \frac{1}{96} F_{\mu \rho \sigma \alpha \beta} F_{\nu}^{\rho \sigma \alpha \beta} + 4 e^{-\phi} H_{\mu \nu}^{MNP} H^{MNP} + 4 e^{\phi} F_{\mu \nu}^{MNP} F^{MNP} - \]
\[ - \frac{1}{48} g_{\mu\nu} (e^{-\phi} H_{MNP} H^{MNP} + e^{\phi} F_{MNP} F^{MNP}) . \] (A.34)

Finally, using
\[ \frac{1}{96} F_{\mu \rho \sigma \alpha \beta} F_{\nu}^{\rho \sigma \alpha \beta} = -\frac{1}{4} K^2 e^{8p-4x} \tilde{g}_{\mu\nu} , \] (A.35)
\[ H_{\mu \nu}^{MNP} H^{MNP} = \frac{8 e^{-2x}}{e^{2g} + 2a^2 + e^{-2g}(1 - a^2)^2} \left\{ (1 + 2 e^{-2g} a^2) \partial_\mu h_1 \partial_\nu h_1 + \right. \]
\[ + \frac{1}{2} [e^{2g} + 2a^2 + e^{-2g}(1 + a^2)^2] \partial_\mu h_2 \partial_\nu h_2 + \]
\[ + a[e^{-2g}(a^2 + 1) + 1] (\partial_\mu h_1 \partial_\nu h_2 + \partial_\mu h_2 \partial_\nu h_1) \right\} , \] (A.36)
\[ F_{\mu \nu}^{MNP} F^{MNP} = 4P^2 e^{-2x} \partial_\mu b \partial_\nu b , \] (A.37)
the relations (A.7) and (A.8), as well as (A.28) and (A.29) in order to dispose of the second derivatives of \(x\) and \(p\) in (A.34), one verifies that
\[ \tilde{R}_{\mu\nu} = 2G_{ab} \partial_\mu \phi^a \partial_\nu \phi^b + \frac{4}{3} \tilde{g}_{\mu\nu} V , \] (A.38)
with \(G_{ab}\) and \(V\) given in (3.13) and (3.14).
Geometric relations for hypersurfaces

The time-slicing (or ADM) formalism [55, 54], which we employ in our analysis of Einstein’s equations, makes essential use of the geometry of hypersurfaces [94]. Therefore, we shall begin with a review of the basic relations governing their geometry.

A hypersurface in a space-time with coordinates \(X^\mu (\mu = 0, \ldots, d)\) and metric \(\tilde{g}^{\mu\nu}\) is defined by a set of \(d + 1\) functions, \(X^\mu(x^i)\) (\(i = 1, \ldots, d\)), where the \(x^i\) are a set of coordinates on the hypersurface (note the difference to appendix A where \(i\) labeled the internal coordinates of the 10-dimensional space-time). The tangent vectors, \(X_i^\mu \equiv \partial_i X^\mu\), and the normal vector, \(N^\mu\), of the hypersurface can be chosen such that they satisfy the following orthogonality relations,

\[
\tilde{g}^{\mu\nu} X_i^\mu X_j^\nu = g_{ij},
\]

\[
X_i^\nu N_\mu = 0,
\]

\[
N^\mu N_\mu = 1,
\]

(B.1)

where \(g_{ij}\) represents the (induced) metric on the hypersurface. Henceforth, a tilde will be used to label quantities characterizing the \((d + 1)\)-dimensional space-time manifold, whereas those of the hypersurface remain unadorned.

The equations of Gauss and Weingarten define the second fundamental form, \(K_{ij}\), of the hypersurface,

\[
\partial_i X_j^\mu + \tilde{\Gamma}^\mu_{\lambda\nu} X_\lambda X_j^\nu - \Gamma_{ij}^k X_k^\mu = K_{ij} N^\mu ,
\]

\[
\partial_i N^\mu + \tilde{\Gamma}^\mu_{\lambda\nu} X_\lambda N^\nu = -K_{ij} X_j^\mu.
\]

(B.2)

(B.3)

The second fundamental form describes the extrinsic curvature of the hypersurface, and is related to the intrinsic curvature by another equation of Gauss,

\[
\tilde{R}_{\mu\nu\lambda\rho} X_i^\mu X_j^\nu X_k^\lambda X_l^\rho = R_{ijkl} + K_{il} K_{jk} - K_{ik} K_{jl}.
\]

(B.4)

Furthermore, it satisfies the equation of Codazzi,

\[
\tilde{R}_{\mu\nu\lambda\rho} X_i^\mu X_j^\nu N^\lambda X_k^\rho = \nabla_i K_{jk} - \nabla_j K_{ik}.
\]

(B.5)

The symbol \(\nabla\) denotes covariant derivatives with respect to the induced metric \(g_{ij}\).

The above formulas simplify if (as in the familiar time-slicing formalism), we choose space-time coordinates such that

\[
X^0 = \text{const.}, \quad X^i = x^i.
\]

(B.6)

Then, the tangent vectors are given by \(X_i^0 = 0\) and \(X_i^i = \delta_i^i\). One conveniently splits up the space-time metric as (shown here for Euclidean signature)

\[
\tilde{g}_{\mu\nu} = \begin{pmatrix} n_i n^i + n^2 & n_j \\ n_i & g_{ij} \end{pmatrix},
\]

(B.7)
whose inverse is given by
\[
\tilde{g}^{\mu\nu} = \frac{1}{n^2} \begin{pmatrix}
1 & -n^i \\
-n^i & n^2 g^{ij} + n^i n^j
\end{pmatrix}.
\] (B.8)

The matrix \(g^{ij}\) is the inverse of \(g_{ij}\), and is used to raise hypersurface indices. The quantities \(n\) and \(n^i\) are the lapse function and shift vector, respectively.

The normal vector \(N^\mu\) satisfying the orthogonality relations (B.1) is given by
\[
N^\mu = (n, 0), \quad N^\mu = \frac{1}{n} (1, -n^i) .
\] (B.9)

Then, one can obtain the second fundamental form from the equation of Gauss (B.2) as
\[
\mathcal{K}_{ij} = n \tilde{\Gamma}_{ij}^0 = -\frac{1}{2n} \left( \partial_0 g_{ij} - \nabla_i n_j - \nabla_j n_i \right) .
\] (B.10)

We are interested in expressing all bulk quantities in terms of hypersurface quantities. Using the equations of Gauss and Weingarten, some Christoffel symbols can be expressed as follows,
\[
\tilde{\Gamma}_{ij}^k = \Gamma_{ij}^k - \frac{n^k}{n} \mathcal{K}_{ij} ,
\] (B.11)
\[
\tilde{\Gamma}_{i0}^0 = \frac{1}{n} \partial_i n + \frac{n^k}{n} \mathcal{K}_{ij} ,
\] (B.12)
\[
\tilde{\Gamma}_{i0}^k = \nabla_i n^k - \frac{n^k}{n} \partial_i n - n \mathcal{K}_{ij} \left( g^{jk} + \frac{n^j n^k}{n^2} \right) .
\] (B.13)

The remaining components, \(\tilde{\Gamma}_{00}^0\) and \(\tilde{\Gamma}_{00}^k\), are easily found from their definitions using (B.7) and (B.8),
\[
\tilde{\Gamma}_{00}^0 = \frac{1}{n} \left( \partial_0 n + n^i \partial_i n + n^i n^j \mathcal{K}_{ij} \right) ,
\] (B.14)
\[
\tilde{\Gamma}_{00}^k = \partial_0 n^k + n^i \nabla_i n^k - n \nabla^k n - 2n \mathcal{K}_{ij} n^i - n^k \tilde{\Gamma}_{00}^0 .
\] (B.15)

## C Intermediate steps

In this appendix, we provide the equations of motion in terms of the geometric variables characterizing the time-slice hypersurfaces introduced in appendix B. The equations of motion that follow from the action (2.1) are\(^{28}\)
\[
\tilde{\nabla}^2 \phi^a + G_{b(c}^a \tilde{g}^{d(a} \partial_d \phi^{b)} - V^a = 0
\] (C.1)
for the scalar fields, and Einstein’s equations
\[
E_{\mu\nu} = -\tilde{R}_{\mu\nu} + 2G_{ab}(\partial_a \phi^b)(\partial_c \phi^c) + \frac{4}{d-1} \tilde{g}_{\mu\nu} V = 0 .
\] (C.2)

\(^{28}\)Note that, as opposed to the main text, we use a tilde here to denote 5d quantities in order to distinguish them from the hypersurface quantities.
In terms of hypersurface quantities, (C.1) takes the form

\[ \{ \partial^2_r - 2n^i \partial_i \partial_r + n^2 \nabla^2 + n^i n^j \nabla_i \partial_j - (n K^j_i + \partial_j \ln n - n^i \partial_i \ln n) \partial_r + \\
+ \left[ n \nabla^2 n - \partial_i n^i + n \nabla_j n^i + n^i (n K^j_i + \partial_j \ln n - n^j \partial_i \ln n) \right] \phi^a + \\
+ G^a_{bc} [(\partial_r \phi^b)(\partial_r \phi^c) - 2n^i (\partial_i \phi^b)(\partial_r \phi^c) + (n^2 g^{ij} + n^i n^j)(\partial_i \phi^b)(\partial_j \phi^c)] - n^2 \frac{\partial V}{\partial \phi^a} = 0. \]  

(C.3)

Eq. (C.2) splits into components that are normal, mixed, and tangential to the hypersurface, obtained by projecting with \( N^\mu N^\nu - g^{ij} X^i_\mu X^j_\nu, N^\mu X^i_\nu \) and \( X^i_\mu X^j_\nu \), respectively. The normal components become

\[ (n K^j_i)(n K^i_j) - (n K^j_i)^2 + n^2 R - 4n^2 V + \\
+ 2G_{ab} \left[ \left( \partial_r \phi^a \right)(\partial_r \phi^b) - 2n^i (\partial_i \phi^a)(\partial_r \phi^b) + (n^i n^j - n^2 g^{ij})(\partial_i \phi^a)(\partial_j \phi^b) \right] = 0. \]  

(C.4)

The mixed components are

\[ \partial_i (n K^j_i) - \nabla_j (n K^i_j) - n K^j_i \partial_i \ln n + n K^i_j \partial_j \ln n - 2G_{ab} \left( \partial_i \phi^a - n^i \partial_j \phi^a \right) \left( \partial_i \phi^b \right) = 0. \]  

(C.5)

For the tangential components one obtains

\[ - \partial_r (n K^j_i) + n^i \nabla_k (n K^j_i) + n K^j_i (n K^j_k + \partial_k \ln n - n^k \partial_k \ln n) + n \nabla^i \partial_j n + \\
+ n K^i_k \nabla_j n^k - n K^j_k \nabla_k n^i - n^2 R^j_i + 2n^2 G_{ab}(\nabla^i \phi^a)(\partial_j \phi^b) + \frac{4n^2 V}{d - 1} \delta^j_i = 0. \]  

(C.6)

The equations of motion given in Sec. 4.4 are obtained from (C.3)–(C.6) upon expanding the fields and using the substitution rules (C.35). For this, the following expressions for geometric hypersurface quantities, up to quadratic order in the gauge-invariant fluctuations, are useful. The extrinsic curvature tensor is

\[ n K^j_i \rightarrow \frac{2}{d - 1} W \delta^j_i - \frac{1}{2} \partial_i \epsilon^k_j + \frac{1}{2} \left( \partial^i \partial_j + \partial_j \partial^i + 2 \frac{\partial \partial_j}{\Box} \epsilon \right) + \frac{1}{2} \epsilon^j_k \partial_i \epsilon^i_k \\
- \frac{1}{2} \left[ \epsilon^j_k \left( \partial^k \partial_j + \partial_j \partial^k + 2 \frac{\partial \partial_k}{\Box} \epsilon \right) + \left( \partial^k + \frac{\partial \partial_k}{\Box} \epsilon \right) \left( \partial_i \epsilon_{jk} + \partial_j \epsilon_{ik} - \partial_k \epsilon_{ij} \right) \right], \]  

(C.7)

and its trace is

\[ n K^j_j \rightarrow \frac{2d}{d - 1} W + \epsilon + \frac{1}{2} \epsilon^i_k \partial_i \epsilon^k_j - \epsilon^i_k \left( \partial^i \partial_j + \frac{\partial \partial_j}{\Box} \epsilon \right). \]  

(C.8)

The intrinsic Ricci tensor is replaced by

\[ R^j_i \rightarrow -\frac{1}{2} e^{-2A} \left[ \Box \epsilon^j_i + \epsilon^k_i \left( \partial^i \partial_k \epsilon^j + \partial_j \partial^i \epsilon^k - \partial_k \partial^j \epsilon^i \right) - \epsilon^k_j \Box \epsilon^k \right. \\
- \left. \frac{1}{2} (\partial^i \epsilon^k_j)(\partial_j \epsilon^i_k) + (\partial_i \epsilon^k_j)(\partial_j \epsilon^i_k) - (\partial_i \epsilon^k_j)(\partial_j \epsilon^i_k) \right], \]  

(C.9)

and the Ricci scalar becomes

\[ R \rightarrow e^{-2A} \left[ \epsilon^j_i \Box \epsilon^j_i + \frac{3}{4} \partial_i \epsilon^j \partial^j \epsilon^k - \frac{1}{2} \partial_i \epsilon^j \partial^j \epsilon^k \right]. \]  

(C.10)
In this appendix, we provide the explicit expressions for the source terms $J_a$, $J$, $J^i$ and $J^j$ to quadratic order, which appear in the equations of motion (4.38), (4.39), (4.40) and (4.41), respectively. The field $\mathbf{b}'$ has been dropped everywhere, since its solution (4.43) is of second order. Moreover, we have used the linear equations of motion in order to eliminate terms, in particular the relation

\[ \partial_r \mathbf{c} - \frac{2d}{d-1} W \mathbf{c} - e^{-2A} \nabla \mathbf{b} = 0, \]  

which follows from (4.41), (4.42) and (4.45).

\[ J^a = \frac{1}{2} \left[ V^a_{bc} - R^a_{bcd} V^d - (R^a_{bcd} - R^a_{debc}) W^d W^c \right] a^b a^c - 2 R^a_{bcd} W^d (D_r a^b) a^c + 2 V^a_{ab} a^b + (D_r a^a)(c + \partial_r b) + 2 V^a b^2 + 2 \left( \frac{\partial^j c}{c} \right) \partial_r D_a a^a - e^{-2A} \left( 2 b \nabla a^a - c^j \partial_r \partial^i a^a \right) - \frac{V^a b^2}{W^a} - W^a \left[ -b \partial_r b + \frac{1}{2} c^j \partial_r c^i - c^j \frac{\partial^i c}{c} - (\partial_r b) \frac{\partial^j c}{c} \right]. \]  

\[ J = 2 V_{a b} a^a a^b - 2 (D_r a^a)(D_r a_a) + 2 R^a_{bcd} W_a W^c a^b a^d + 2 e^{-2A} (\partial^a a^a)(\partial_r a_a) + 8 V_a a^b + 4 W_a (\partial_a a^a) \frac{\partial^j c}{c} + 8 V b^2 - 4 V b^2 - (\Pi_1 c) \left( \frac{\partial^j c}{c} \right) + \frac{1}{4} \left( \partial_r c^j \right) (\partial_r c^i) + 2 W c^j \partial_r c^i - e^{-2A} \left[ c^j \nabla c^i + \frac{3}{4} (\partial_r c^j) (\partial^i c^k) - \frac{1}{2} (\partial_r c^j) (\partial^k c^j) \right]. \]  

\[ J_i = 2 (\partial_r a^a) D_r a_a - 2 W b \partial_r b + (\partial_r b) \Pi_1 c + \frac{1}{2} (\partial_r b) \partial_r c^i - \frac{1}{2} \left( \frac{\partial_r c}{c} \right) \nabla c^i - \frac{1}{4} (\partial_r c) (\partial^j c^k) + \frac{1}{2} c^j \partial_r \partial_r c^i + \frac{1}{4} (\partial_r c^j) (\partial_r c^j). \]  

The terms underlined in (D.2), (D.3) and (D.4) can be eliminated by the field redefinitions

\[ \mathbf{b} \rightarrow \mathbf{b} + \frac{1}{2} b^2, \]  

\[ \mathbf{c} \rightarrow \mathbf{c} - \frac{1}{2} c^j \partial_r c^i + (\partial_r b) \frac{\partial^j c}{c} + c^j \frac{\partial^i c}{c}. \]  

The terms underlined in (D.2), (D.3) and (D.4) can be eliminated by the field redefinitions

\[ \mathbf{b} \rightarrow \mathbf{b} + \frac{1}{2} b^2, \]  

\[ \mathbf{c} \rightarrow \mathbf{c} - \frac{1}{2} c^j \partial_r c^i + (\partial_r b) \frac{\partial^j c}{c} + c^j \frac{\partial^i c}{c}. \]  

The terms underlined in (D.2), (D.3) and (D.4) can be eliminated by the field redefinitions
\[ J^j_i = \Pi^{ik}_{jl} \left\{ 2 \left( \frac{\partial^l \partial_m \xi}{\Box} \right) \left( \frac{\partial^m \partial_k \xi}{\Box} \right) - 2 \left( \frac{\partial^l \partial_k \xi}{\Box} \right) (c + \partial_r b) + 
\]
\[ + 2(\partial_m \partial_r e'_k) \left( \frac{\partial^m}{\Box} \right) + (\partial_r e'_k)(c + \partial_r b) + \frac{(\partial_r c'_m)(\partial_r e^m)}{\Box} + 
\]
\[ + e^{-2A} \left[ 2(\partial^l b)(\partial_k b) - 4(\partial^l a^n)(\partial_k a_n) - 2b \Box e'_k - 2e^m \partial^l \partial_m e'_k + 
\]
\[ + e^m \partial_m \partial^m e'_k - \frac{1}{2}(\partial^m c'_n)(\partial_k e^m_n) - (\partial_m c'_n)(\partial^m e'_k) + \frac{(\partial_m c'_n)(\partial^m e'_k)}{\Box} \right\} . \]

The underlined terms in (D.7) can be eliminated by the field redefinition

\[ e'_j \rightarrow e'_j + \frac{1}{2} \Pi^{ik}_{jl}(e'_m e'^m_k) . \]  

### E  Matrices for the KS background

In this appendix we give the explicit form of the matrices appearing in (6.25) in the KS background. In order to keep the formulas under (typographical) control, we introduce the following notation

\[ H_1 = e^{6q+4f} , \]
\[ H_2 = e^{10q+6f} , \]
\[ H_3 = e^{-4q+4f} , \]
\[ \Upsilon_1 = e^{-\Phi}[2 \cosh(2y)P + \sinh(2y)(2P - N_2 + N_1)] , \]
\[ \Upsilon_2 = e^{-\Phi}[2 \sinh(2y)P + \cosh(2y)(2P - N_2 + N_1)] , \]
\[ \Upsilon_3 = Q - P(s - N_1 - N_2) + \frac{1}{2}(N_1^2 - N_2^2) , \]
\[ \Upsilon_4 = -4e^{6q} - 6e^{10f+6q} \cosh y + 5\sqrt{27} e^{6f} \Upsilon_3 , \]
\[ \Upsilon_5 = \cosh y - e^{-10f} , \]
\[ \Upsilon_6 = e^{-\Phi}(N_1 + N_2) , \]
\[ \Upsilon_7 = -\frac{2}{15} \left[ 8e^{-10f} + 12 \cosh y - \frac{25\sqrt{27}\Upsilon_3}{H_1} \right] , \]

where the fields denote the background values given in Sec. 6.1. With these abbreviations, the matrices are given by
\[ G_{\text{b}} W^e = H_3 \times \]
\[
\begin{pmatrix}
0 & 0 & 0 & -\sqrt{27} P_{H_1} & 0 & \frac{\sqrt{27} N_1 + P}{H_1} & -\frac{\sqrt{27} N_2 - P}{H_1} \\
0 & 0 & 0 & -\sqrt{27} P_{H_1} & 0 & \frac{\sqrt{27} N_1 + P}{H_1} & -\frac{\sqrt{27} N_2 - P}{H_1} \\
0 & 0 & 0 & -\sqrt{27} P_{H_1} & 0 & \frac{\sqrt{27} N_1 + P}{H_1} & -\frac{\sqrt{27} N_2 - P}{H_1} \\
9\Upsilon_1 & 6\Upsilon_1 & \frac{3}{2} \Upsilon_1 & \sqrt{27}(\Upsilon - 2e^{(4)}) & 3\Upsilon_2 & -3 \sinh y & -3 \sinh y \\
0 & 0 & 0 & -\frac{\sqrt{27}}{2} N_1 - N_2 + 2P_{H_1} & 0 & \frac{\sqrt{27} P}{H_1} & \sqrt{27} P_{H_1} \\
-\frac{9}{2}(\Upsilon_2 + \Upsilon_6) & -3(\Upsilon_2 + \Upsilon_6) & -\frac{3}{4}(\Upsilon_2 - \Upsilon_6) & -\frac{3}{2} \sinh y & -\frac{3}{2} \Upsilon_1 & \frac{\sqrt{27} \Upsilon_2 - 2 e^{(4)}}{H_1} & 0 \\
-\frac{9}{2}(\Upsilon_2 - \Upsilon_6) & -3(\Upsilon_2 - \Upsilon_6) & -\frac{3}{4}(\Upsilon_2 + \Upsilon_6) & -\frac{3}{2} \sinh y & -\frac{3}{2} \Upsilon_1 & 0 & \frac{\sqrt{27} \Upsilon_2 - 2 e^{(4)}}{H_1} \\
\end{pmatrix},
\]

\[ \partial_0 W^a = H_3 \times \]
\[
\begin{pmatrix}
\Upsilon_7 & \frac{3}{5} \Upsilon_5 & 0 & \frac{\sqrt{27}}{2} \sinh y & -\frac{\sqrt{27} N_1 + P}{H_1} & \frac{\sqrt{27} N_2 - P}{H_1} \\
12 \Upsilon_5 & -\frac{6}{5}(2 \cosh y + 3 e^{-10 f}) & 0 & 0 & \frac{3}{2} \sinh y & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
12 \sinh y & -12 \sinh y & -3 \Upsilon_1 & 0 & -6 \Upsilon_2 & -3 \sinh(2y) e^\Phi & 3 \sinh(2y) e^\Phi \\
12 \sinh y & -12 \sinh y & -3 \Upsilon_1 & 0 & -6 \Upsilon_2 & -3 \sinh(2y) e^\Phi & 3 \sinh(2y) e^\Phi \\
-6(\Upsilon_2 + \Upsilon_6) & 6(\Upsilon_2 + \Upsilon_6) & \frac{3}{2}(\Upsilon_2 - \Upsilon_6) & 0 & 3 \Upsilon_1 & \frac{3}{2}[e^{-\Phi} + e^\Phi \cosh(2y)] & \frac{3}{2}[e^{-\Phi} - e^\Phi \cosh(2y)] \\
-6(\Upsilon_2 - \Upsilon_6) & 6(\Upsilon_2 - \Upsilon_6) & \frac{3}{2}(\Upsilon_2 + \Upsilon_6) & 0 & 3 \Upsilon_1 & \frac{3}{2}[e^{-\Phi} - e^\Phi \cosh(2y)] & \frac{3}{2}[e^{-\Phi} + e^\Phi \cosh(2y)] \\
\end{pmatrix},
\]

\[ W^a W_6 = H_3^2 \times \]
\[
\begin{pmatrix}
\frac{\Upsilon_7^2}{15 H_2^2 H_3} & \frac{2 \Upsilon_7 \Upsilon_5}{5 H_2 H_3} & 0 & \frac{\sqrt{27} P \Upsilon_5}{10 H_1 H_2 H_3} & \frac{\Upsilon_4 \sinh y}{10 H_2 H_3} & \frac{\sqrt{3} (N_1 + P) \Upsilon_5}{10 H_1 H_2 H_3} & \frac{\sqrt{3} (N_2 - P) \Upsilon_5}{10 H_1 H_2 H_3} \\
3 \Upsilon_5 \Upsilon_5 & \frac{18 \Upsilon_5^2}{5 H_2 H_3} & 0 & \frac{3 \sqrt{27} P \Upsilon_5}{10 H_1} & \frac{9}{10} \sinh y \Upsilon_5 & -3 \frac{\sqrt{27} (N_1 + P) \Upsilon_5}{10 H_1} & \frac{3 \sqrt{27} (N_2 - P) \Upsilon_5}{10 H_1} \\
0 & 0 & 0 & 0 & \frac{3}{4} \sinh y \Upsilon_5 & 0 & 0 \\
3 \Upsilon_4 \Upsilon_5 & 18 \Upsilon_5 \sinh y & 0 & \frac{3 \sqrt{27} P \sinh y}{2 H_1} & \frac{9}{2} \sinh^2 y & -3 \frac{\sqrt{27} (N_1 + P) \sinh y}{2 H_1} & \frac{3 \sqrt{27} (N_2 - P) \sinh y}{2 H_1} \\
3 \Upsilon_4 \Upsilon_5 \sinh y & 18 \Upsilon_5 \sinh y & 0 & \frac{3 \sqrt{27} P \sinh y}{2 H_1} & \frac{9}{2} \sinh^2 y & -3 \frac{\sqrt{27} (N_1 + P) \sinh y}{2 H_1} & \frac{3 \sqrt{27} (N_2 - P) \sinh y}{2 H_1} \\
\frac{3 \Upsilon_4 (\Upsilon_2 + \Upsilon_6)}{2 H_2 H_3} & -9 \Upsilon_5 (\Upsilon_2 + \Upsilon_6) & 0 & -\frac{3 \sqrt{27} P (\Upsilon_5 + \Upsilon_6)}{4 H_1} & -\frac{9}{4} (\Upsilon_2 + \Upsilon_6) \sinh y & \frac{3 \sqrt{27} (N_1 + P) (\Upsilon_2 + \Upsilon_6)}{4 H_1} & \frac{3 \sqrt{27} (N_2 - P) (\Upsilon_2 + \Upsilon_6)}{4 H_1} \\
\frac{3 \Upsilon_4 (\Upsilon_2 - \Upsilon_6)}{2 H_2 H_3} & -9 \Upsilon_5 (\Upsilon_2 - \Upsilon_6) & 0 & -\frac{3 \sqrt{27} P (\Upsilon_5 + \Upsilon_6)}{4 H_1} & -\frac{9}{4} (\Upsilon_2 - \Upsilon_6) \sinh y & \frac{3 \sqrt{27} (N_1 + P) (\Upsilon_2 - \Upsilon_6)}{4 H_1} & \frac{3 \sqrt{27} (N_2 - P) (\Upsilon_2 - \Upsilon_6)}{4 H_1} \\
\end{pmatrix},
\]

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References

[1] J. M. Maldacena, *The large N limit of superconformal field theories and supergravity*, Adv. Theor. Math. Phys. 2 (1998) 231–252, [hep-th/9711200](https://arxiv.org/abs/hep-th/9711200).

[2] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, *Gauge theory correlators from non-critical string theory*, Phys. Lett. B428 (1998) 105–114, [hep-th/9802109](https://arxiv.org/abs/hep-th/9802109).

[3] E. Witten, *Anti-de Sitter space and holography*, Adv. Theor. Math. Phys. 2 (1998) 253–291, [hep-th/9802150](https://arxiv.org/abs/hep-th/9802150).

[4] M. Bianchi, D. Z. Freedman, and K. Skenderis, *How to go with an RG flow*, JHEP 08 (2001) 041, [hep-th/0105276](https://arxiv.org/abs/hep-th/0105276).

[5] M. Bianchi, D. Z. Freedman, and K. Skenderis, *Holographic renormalization*, Nucl. Phys. B631 (2002) 159–194, [hep-th/0112119](https://arxiv.org/abs/hep-th/0112119).

[6] D. Martelli and W. Mück, *Holographic renormalization and Ward identities with the Hamilton-Jacobi method*, Nucl. Phys. B654 (2003) 248–276, [hep-th/0205061](https://arxiv.org/abs/hep-th/0205061).

[7] M. Berg and H. Samtleben, *Holographic correlators in a flow to a fixed point*, JHEP 12 (2002) 070, [hep-th/0209191](https://arxiv.org/abs/hep-th/0209191).

[8] I. Papadimitriou and K. Skenderis, *AdS/CFT correspondence and geometry*, [hep-th/0404176](https://arxiv.org/abs/hep-th/0404176).

[9] I. Papadimitriou and K. Skenderis, *Correlation functions in holographic RG flows*, JHEP 10 (2004) 075, [hep-th/0407071](https://arxiv.org/abs/hep-th/0407071).

[10] H. Lin, O. Lunin, and J. Maldacena, *Bubbling AdS space and 1/2 BPS geometries*, JHEP 10 (2004) 025, [hep-th/0409174](https://arxiv.org/abs/hep-th/0409174).

[11] O. Lunin and J. Maldacena, *Deforming field theories with U(1) X U(1) global symmetry and their gravity duals*, JHEP 05 (2005) 033, [hep-th/0502086](https://arxiv.org/abs/hep-th/0502086).

[12] U. Gursoy and C. Nuñez, *Dipole deformations of N = 1 SYM and supergravity backgrounds with U(1) X U(1) global symmetry*, [hep-th/0505100](https://arxiv.org/abs/hep-th/0505100).

[13] N. Halmagyi, K. Pilch, C. Romelsberger, and N. P. Warner, *Holographic duals of a family of N = 1 fixed points*, [hep-th/0506206](https://arxiv.org/abs/hep-th/0506206).

[14] L. Girardello, M. Petrini, M. Porrati, and A. Zaffaroni, *The supergravity dual of N = 1 super Yang-Mills theory*, Nucl. Phys. B569 (2000) 451–469, [hep-th/9909047](https://arxiv.org/abs/hep-th/9909047).

[15] C. V. Johnson, A. W. Peet, and J. Polchinski, *Gauge theory and the excision of repulson singularities*, Phys. Rev. D61 (2000) 086001, [hep-th/9911161](https://arxiv.org/abs/hep-th/9911161).

[16] R. C. Myers, *Dielectric-branes*, JHEP 12 (1999) 022, [hep-th/9910053](https://arxiv.org/abs/hep-th/9910053).
[17] S. Hong, S. Yoon, and M. J. Strassler, On the couplings of vector mesons in AdS/QCD, hep-th/0409118.

[18] S. Hong, S. Yoon, and M. J. Strassler, On the couplings of the rho meson in AdS/QCD, hep-ph/0501197.

[19] T. Sakai and S. Sugimoto, More on a holographic dual of QCD, hep-th/0507073.

[20] J. Polchinski and M. J. Strassler, Deep inelastic scattering and gauge/string duality, JHEP 05 (2003) 012, hep-th/0209211.

[21] S. Kachru, R. Kallosh, A. Linde, J. Maldacena, L. McAllister, and S. Trivedi, Towards inflation in string theory, JCAP 0310 (2003) 013, hep-th/0308055.

[22] I. R. Klebanov and M. J. Strassler, Supergravity and a confining gauge theory: Duality cascades and χSB-resolution of naked singularities, JHEP 08 (2000) 052, hep-th/0007191.

[23] M. Strassler, The duality cascade, hep-th/0505153.

[24] C. P. Herzog, I. R. Klebanov, and P. Ouyang, Remarks on the warped deformed conifold, hep-th/0108101.

[25] J. M. Maldacena and C. Nuñez, Towards the large N limit of pure N = 1 superyang mills, Phys. Rev. Lett. 86 (2001) 588–591, hep-th/0008001.

[26] R. Apreda, F. Bigazzi, A. L. Cotrone, M. Petrini, and A. Zaffaroni, Some comments on N = 1 gauge theories from wrapped branes, Phys. Lett. B536 (2002) 161–168, hep-th/0112236.

[27] P. Di Vecchia, A. Lerda, and P. Merlatti, N = 1 and N = 2 super yang-mills theories from wrapped branes, Nucl. Phys. B646 (2002) 43–68, hep-th/0205204.

[28] M. Bertolini and P. Merlatti, A note on the dual of N = 1 super Yang-Mills theory, Phys. Lett. B556 (2003) 80–86, hep-th/0211142.

[29] W. Mück, Perturbative and non-perturbative aspects of pure N = 1 super yang-mills theory from wrapped branes, JHEP 02 (2003) 013, hep-th/0301171.

[30] O. Aharony, A. Buchel, and A. Yarom, Holographic renormalization of cascading gauge theories, hep-th/0506002.

[31] M. Krasnitz, A two point function in a cascading N = 1 gauge theory from supergravity, hep-th/0011179.

[32] M. Krasnitz, Correlation functions in a cascading N = 1 gauge theory from supergravity, JHEP 12 (2002) 048, hep-th/0209163.
[33] M. Krasnitz, *Correlation functions in supersymmetric gauge theories from supergravity fluctuations*. PhD thesis, Princeton, 2003. UMI-30-68792.

[34] I. R. Klebanov and A. A. Tseytlin, *Gravity duals of supersymmetric SU(N) X SU(N+M) gauge theories*, Nucl. Phys. B578 (2000) 123–138, [hep-th/0002159](https://arxiv.org/abs/hep-th/0002159).

[35] G. Papadopoulos and A. A. Tseytlin, *Complex geometry of conifolds and 5-brane wrapped on 2-sphere*, Class. Quant. Grav. 18 (2001) 1333–1354, [hep-th/0012034](https://arxiv.org/abs/hep-th/0012034).

[36] D. Z. Freedman, C. Nuñez, M. Schnabl, and K. Skenderis, *Fake supergravity and domain wall stability*, Phys. Rev. D69 (2004) 104027, [hep-th/0312055](https://arxiv.org/abs/hep-th/0312055).

[37] S. S. Gubser, *Supersymmetry and F-theory realization of the deformed conifold with three-form flux*, [hep-th/0010010](https://arxiv.org/abs/hep-th/0010010).

[38] M. Graña and J. Polchinski, *Supersymmetric three-form flux perturbations on AdS(5)*, Phys. Rev. D63 (2001) 026001, [hep-th/0009211](https://arxiv.org/abs/hep-th/0009211).

[39] M. Bianchi, M. Prisco, and W. Mück, *New results on holographic three-point functions*, JHEP 11 (2003) 052, [hep-th/0310129](https://arxiv.org/abs/hep-th/0310129).

[40] O. DeWolfe and D. Z. Freedman, *Notes on fluctuations and correlation functions in holographic renormalization group flows*, [hep-th/0002226](https://arxiv.org/abs/hep-th/0002226).

[41] G. Arutyunov, S. Frolov, and S. Theisen, *A note on gravity-scalar fluctuations in holographic RG flow geometries*, Phys. Lett. B484 (2000) 295–305, [hep-th/0003116](https://arxiv.org/abs/hep-th/0003116).

[42] W. Mück, *Correlation functions in holographic renormalization group flows*, Nucl. Phys. B620 (2002) 477–500, [hep-th/0105270](https://arxiv.org/abs/hep-th/0105270).

[43] J. M. Bardeen, *Gauge invariant cosmological perturbations*, Phys. Rev. D22 (1980) 1882–1905.

[44] J. M. Bardeen, P. J. Steinhardt, and M. S. Turner, *Spontaneous creation of almost scale - free density perturbations in an inflationary universe*, Phys. Rev. D28 (1983) 679.

[45] O. DeWolfe, D. Z. Freedman, S. S. Gubser, and A. Karch, *Modeling the fifth dimension with scalars and gravity*, Phys. Rev. D62 (2000) 046008, [hep-th/9909134](https://arxiv.org/abs/hep-th/9909134).

[46] M. Bianchi, O. DeWolfe, D. Z. Freedman, and K. Pilch, *Anatomy of two holographic renormalization group flows*, JHEP 01 (2001) 021, [hep-th/0009156](https://arxiv.org/abs/hep-th/0009156).

[47] F. Larsen and R. McNees, *Holography, diffeomorphisms, and scaling violations in the CMB*, JHEP 07 (2004) 062, [hep-th/0402050](https://arxiv.org/abs/hep-th/0402050).
[48] P. K. Kovtun and A. O. Starinets, *Quasinormal modes and holography*, hep-th/0506184.

[49] W. Mück and M. Prisco, *Glueball scattering amplitudes from holography*, JHEP 04 (2004) 037, hep-th/0402068.

[50] M. Bianchi and A. Marchetti, *Holographic three-point functions: One step beyond the tradition*, Nucl. Phys. B686 (2003) 261–284, hep-th/0302019.

[51] W. Mück, *Progress on holographic three-point functions*, Fortschr. Phys. 53 (2005) 948–954, hep-th/0412251.

[52] L. Ametller, J. M. Pons, and P. Talavera, *On the consistency of the N = 1 SYM spectra from wrapped five-branes*, Nucl. Phys. B674 (2003) 231–258, hep-th/0305075.

[53] E. Caceres and C. Nuñez, *Glueballs of super Yang-Mills from wrapped branes*, hep-th/0506051.

[54] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation*. Freeman, San Francisco, 1973.

[55] R. M. Wald, *General Relativity*. University of Chicago Press, Chicago, 1984.

[56] G. Lopes Cardoso, G. Dall’Agata and D. Lüst, *Curved BPS domain wall solutions in five-dimensional gauged supergravity*, JHEP 0107, 026 (2001) hep-th/0104156.

[57] A. B. Clark, D. Z. Freedman, A. Karch, and M. Schnabl, *The dual of Janus: an interface cft*, Phys. Rev. D71 (2005) 066003, hep-th/0407073.

[58] K. Skenderis and P. K. Townsend, *Gravitational stability and renormalization-group flow*, Phys. Lett. B468, 46 (1999) hep-th/9909070.

[59] A. Z. Petrov, *Einstein Spaces*. Pergamon Press, Oxford, 1969.

[60] P. Breitenlohner and D. Z. Freedman, *Stability in gauged extended supergravity*, Ann. Phys. 144 (1982) 249.

[61] I. R. Klebanov and E. Witten, *AdS/CFT correspondence and symmetry breaking*, Nucl. Phys. B556 (1999) 89–114, hep-th/9905104.

[62] M. Berg and H. Samtleben, *An exact holographic RG flow between 2d conformal fixed points*, JHEP 05 (2002) 006, hep-th/0112154.

[63] M. Berg, *Holography and infrared conformality in two dimensions*, in *Cargese 2002, Progress in string, field and particle theory* (L. Baulieu et. al., eds.), Kluwer Academic, 2003. hep-th/0212197.
[64] K. Skenderis, *Lecture notes on holographic renormalization*, *Class. Quant. Grav.* **19** (2002) 5849–5876, [hep-th/0209067].

[65] A. V. Belitsky, V. M. Braun, A. S. Gorsky, and G. P. Korchemsky, *Integrability in QCD and beyond*, *Int. J. Mod. Phys.* **A19** (2004) 4715–4788, [hep-th/0407232].

[66] L. A. Pando Zayas and A. A. Tseytlin, *3-branes on resolved conifold*, *JHEP* **11** (2000) 028, [hep-th/0010088].

[67] M. Haack and J. Louis, *M-theory compactified on Calabi-Yau fourfolds with background flux*, *Phys. Lett.* **B507** (2001) 296–304, [hep-th/0103068].

[68] M. Berg, M. Haack, and H. Samtleben, *Calabi-Yau fourfolds with flux and supersymmetry breaking*, *JHEP* **04** (2003) 046, [hep-th/0212255].

[69] B. de Wit, I. Herger, and H. Samtleben, *Gauged locally supersymmetric D = 3 nonlinear sigma models*, *Nucl. Phys.* **B671** (2003) 175–216, [hep-th/0307066].

[70] O. Hohm and J. Louis, *Spontaneous N = 2 to N = 1 supergravity breaking in three dimensions*, *Class. Quant. Grav.* **21** (2004) 4607–4624, [hep-th/0403128].

[71] M. Gunaydin, G. Sierra and P. K. Townsend, *Gauging The D = 5 Maxwell-Einstein Supergravity Theories: More On Jordan Algebras*, *Nucl. Phys.* **B253** (1985) 573.

[72] M. Gunaydin and M. Zagermann, *The gauging of five-dimensional, N = 2 Maxwell-Einstein supergravity theories coupled to tensor multiplets*, *Nucl. Phys.* **B572**, (2000) 131 [hep-th/9912027].

[73] A. Ceresole and G. Dall’Agata, *General matter coupled N = 2, D = 5 gauged supergravity*, *Nucl. Phys.* **B585** (2000) 143–170, [hep-th/0004111].

[74] E. Bergshoeff, S. Cucu, T. de Wit, J. Gheerardyn, S. Vandoren and A. Van Proeyen, *N = 2 supergravity in five dimensions revisited*, *Class. Quant. Grav.* **21**, 3015 (2004) [Class. Quant. Grav. **23**, 7149 (2006)] [hep-th/0403045].

[75] A. Celi, A. Ceresole, G. Dall’Agata, A. Van Proeyen and M. Zagermann, *On the fakeness of fake supergravity*, *Phys. Rev.* **D71** (2005) 045009 [hep-th/0410126].

[76] M. Zagermann, *N = 4 fake supergravity*, *Phys. Rev.* **D71**, (2005) 125007 [hep-th/0412081].

[77] S. S. Gubser, C. P. Herzog, and I. R. Klebanov, *Symmetry breaking and axionic strings in the warped deformed conifold*, *JHEP* **09** (2004) 036, [hep-th/0405282].

[78] S. S. Gubser, C. P. Herzog, and I. R. Klebanov, *Variations on the warped deformed conifold*, *Comptes Rendus Physique* **5** (2004) 1031–1038, [hep-th/0409186].
[79] O. Aharony, A note on the holographic interpretation of string theory backgrounds with varying flux, JHEP 03 (2001) 012, [hep-th/0101013].

[80] A. Butti, M. Graña, R. Minasian, M. Petrini, and A. Zaffaroni, The baryonic branch of Klebanov-Strassler solution: a supersymmetric family of su(3) structure backgrounds, [hep-th/0412187].

[81] S. S. Gubser, A. A. Tseytlin, and M. S. Volkov, Non-Abelian 4-d black holes, wrapped 5-branes, and their dual descriptions, JHEP 09 (2001) 017, [hep-th/0108205].

[82] S. D. Mathur, Where are the states of a black hole?, [hep-th/0401115].

[83] S. D. Mathur, The fuzzball proposal for black holes: An elementary review, [hep-th/0502050].

[84] S. S. Gubser, Curvature singularities: The good, the bad, and the naked, Adv. Theor. Math. Phys. 4 (2002) 679 [hep-th/0002160].

[85] I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series and Products. Academic Press, New York, 5 ed., 1994.

[86] M. Abramowitz and I. A. Stegun, eds., Handbook of Mathematical Functions. Dover Publ., New York, 1965.

[87] E. Witten, Anti-de Sitter space, thermal phase transition, and confinement in gauge theories, Adv. Theor. Math. Phys. 2 (1998) 505–532, [hep-th/9803131].

[88] R. Apreda, Non supersymmetric regular solutions from wrapped and fractional branes, [hep-th/0301118].

[89] F. Bigazzi, L. Girardello, and A. Zaffaroni, A note on regular type 0 solutions and confining gauge theories, Nucl. Phys. B598 (2001) 530–542, [hep-th/0011041].

[90] I. R. Klebanov and E. Witten, Superconformal field theory on threebranes at a calabi-yau singularity, Nucl. Phys. B536 (1998) 199–218, [hep-th/9807080].

[91] E. Caceres and R. Hernandez, Glueball masses for the deformed conifold theory, Phys. Lett. B504 (2001) 64–70, [hep-th/0011204].

[92] X. Amador and E. Caceres, Spin two glueball mass and glueball regge trajectory from supergravity, JHEP 11 (2004) 022, [hep-th/0402061].

[93] GRTensor, Maple package, [http://grtensor.phy.queensu.ca/].

[94] L. P. Eisenhart, Riemannian Geometry. Princeton University Press, Princeton, 1964.