Basic behavior of a pulse-coupled ring system with three spiking neurons

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Abstract: This paper proposes a pulse-coupled system with three spiking neurons. The neurons operate integrate-and-fire dynamics and output a spike. When this happens, another neuron accepts the spiked output and is compelled to fire: In this way, a ring of three neurons are coupled. As the parameter is varied, the behavior becomes a master-slave conjunction. We clarify the theoretical results for various boundary conditions and present the results of numerical simulations. Using a test circuit, we confirm the typical behavior of a pulse-coupled system.

Key Words: pulse-coupled system, spiking neuron, ring coupling, spike train, bifurcation

1. Introduction

This paper proposes a novel pulse-coupled system that combines three spiking neuron models (SNMs). An SNM is a kind of artificial neuron model, in which the state variable behaves according to integrate-and-fire dynamics that operate between a threshold and a base signal: when the state variable reaches the threshold, it instantaneously fires and is thereby reset to the base level. When it fires, the pulse that is output is referred to as a spike. SNMs exhibit various periodic/chaotic phenomena, including spike trains [1, 2]. Pulse-coupled systems have been studied in order to analyze synchronization phenomena, such as in applications of signal processing and neural networks [3–8]. Our proposed system will further the study of synchronization, and it is an important tool for the fundamental analysis of nonlinear systems.

We begin by introducing the dynamics of a single SNM [1]. Next, we propose a novel pulse-coupled system in which three SNMs are connected as a ring. The dynamics of this system are as follows: If the state of one neuron reaches the threshold, it outputs a spike and resets to the base signal; this is called “self-firing” (SF). At the same time, another neuron accepts the spiked output, and its state is reset instantaneously to the base; this is called “compulsory firing” (CF). In this way, three neurons can each alternate between SF and CF. Based on these firing rules, a spike is output for each firing. For this ring system, we derive the compulsory firing ratio (CFR), which is the ratio of the number of CFs to the total number of spikes. When the parameters are varied, we can see from
the CFR that it is possible that a neuron may not exhibit CF. When this happens, the neuron does not accept the spike as input and behaves like a master. In other words, the interaction changes from ring conjunction behavior (RCB) to master-slave conjunction behavior (MCB). As one of our main results, we calculate the boundary between RCB and MCB under various conditions. Finally, we propose a circuit model that realizes this pulse-coupled system and present laboratory measurements that confirm the existence of RCB and MCB. The significance and purpose of this paper are described as follows.

- Various other systems that output a spike train have been studied. Applications for an analog-to-digital converter, an artificial inner ear, and a self-organizing network have been considered [9–14]. From an application viewpoint, our system is related to these systems and can be developed for application systems with novel functions.

- The SNM used in this paper has a triangular base signal and its state has a fixed slope [1]. This very simple system was chosen to allow theoretical analysis. However, [1, 2] have studied the single SNMs and systems of two coupled SNMs. Our proposed system, which consists of three SNMs, can switch between RCB and MCB; this is not possible in systems with only two SNMs, as is shown in Appendix. The results of this paper are also useful for larger-scale coupling systems.

- The Hodgkin–Huxley model [15] is a well-known artificial neuron model. However, it has a complex nonlinear continuous flow and thus is not suited for implementation with an electronic circuit [14]. On the other hand, an SNM can be implemented by a simple electronic circuit, even if it is a pulse-coupled system. In Section 4, we show an implementation of such a circuit and confirm the expected behavior.

This is the first paper that considers a pulse-coupled system with three SNMs. As the first step, we consider the basic coupling behavior under a given condition and fixed parameters.

![Fig. 1. Single SNM. Left figure: Dynamics. Right figure: Sketch of the SNM. The circle denotes the SNM, and the arrow is in the direction of the spiked output.](image)

![Fig. 2. Typical time waveforms and outputs with k = 2.5. (a) Periodic waveform with period 1 for s = 1.33. (b) Periodic waveform with period 3 for s = 2.66.](image)
2. Single spiking neuron model (SNM)

First, we introduce the SNM [1]. Figure 1 shows the dynamics of the SNM, where \( x(\tau) \), \( y(\tau) \), and \( \tau \) denote the dimensionless state variable, the output, and the time, respectively. The dynamics of the SNM are as follows. If an initial state \( x(0) \) is given, the state \( x(\tau) \) increases in time with slope \( s \). When \( x \) reaches the threshold level \( x = 0 \), it is reset instantaneously to the base signal \( b(\tau) \), and, simultaneously, the spike \( y(\tau) = 1 \) is output. We call this operation “firing”. By repeating this behavior, the neuron outputs a spike train. These dynamics and the firing rules are described by

\[
\begin{cases}
  \frac{dx}{d\tau} = s, & y(\tau) = 0 \quad \text{for} \quad x(\tau) < 0, \\
  x(\tau^+) = b(\tau^+), & y(\tau^+) = 1 \quad \text{for} \quad x(\tau) \geq 0.
\end{cases}
\]

(1)

We assume that the base signal is a triangular waveform with period one:

\[
b(\tau) = \begin{cases} 
  -k(\theta - \frac{1}{4}) - 1 & \text{for } 0 \leq \theta < \frac{1}{2}, \\
  k(\theta - \frac{3}{4}) - 1 & \text{for } \frac{1}{2} \leq \theta < 1;
\end{cases}
\]

\[
\theta \equiv \tau \mod 1, \quad b(\tau) = b(\tau + 1),
\]

(2)

where \( \theta \) is the phase of \( \tau \). This system has two parameters \( s \) and \( k \) that correspond to the slopes of the state and base signals. For simplicity, we assume \( 0 < k < 4, 0 < s, \) and \( -2 < x(0) < 0 \). The right side of Fig. 1 shows a sketch of the SNM system in which the SNM appears as a circle. The arrow shows the direction of the spike that is output. In this case, the spiked output of the SNM returns to the same SNM. Typical waveforms are shown in Fig. 2. Figures 2(a) and (b) are periodic waveforms with periods 1 and 3, respectively.

In order to analyze this phenomenon we need to define the time at which the spike occurs. Let \( \tau(n) \) be the moment at which the \( n \)-th spike occurs (firing moment), where \( n \) is a positive integer; see Fig. 1. The position in time of the next spike \( \tau(n+1) \) is determined uniquely by \( \tau(n) \), and we can define the spike position map \( f \):

\[
\tau_{n+1} \equiv f(\tau_n) = \tau_n + \frac{|b(\tau_n)|}{s}, \quad \tau_n \equiv \tau(n).
\]

(3)

\[ Fig. 3. \] Examples of the spike position map and spike phase map with \( k = 2.5 \): (a) \( s = 1.33 \); (b) 1-SPO for \( s = 1.33 \). The orbit converges to the black circle (stable fixed point); (c) \( s = 2.66 \); (d) 3-SPO for \( s = 2.66 \).
Typical spike position maps are shown in Figs. 3(a) and (c). These maps are piecewise linear. Considering \( f(\tau + 1) = f(\tau) + 1 \), we can derive the spike phase map \( F \) as follows:

\[
\theta_{n+1} \equiv F(\theta_n) = F_1(\theta_n) \mod 1, \quad F: I \rightarrow I \equiv [0, 1),
\]

\[
F_1(\theta_n) = \begin{cases} 
(1 + \frac{5}{2})\theta_n + \frac{1}{2}(1 - \frac{4}{k}) & \text{for } 0 \leq \theta_n < \frac{1}{2}, \\
(1 - \frac{k}{4})\theta_n + \frac{1}{2}(1 + \frac{3k}{4}) & \text{for } \frac{1}{2} \leq \theta_n < 1,
\end{cases}
\]

where \( \theta_n \) denotes the phase of the n-th spike \( \theta_n \equiv \tau_n \mod 1 \). According to nonlinear dynamical theory, we have the following definitions.

**Def:** A point \( \theta_p \in I \) is said to be periodic with period \( k \) if \( \theta_p = F^k(\theta_p) \), \( \theta_p \neq F^j(\theta_p) \) for \( 1 \leq j < k \), where \( F^k \) denotes the \( k \)-fold composition of \( F \) (\( j \) does not exist for \( k = 1 \)). We say that these points with period \( k \) form a \( k \)-periodic orbit. A \( k \)-periodic orbit is said to be unstable or stable for an initial state if \( |DF^k(\theta_p)| > 1 \) or \( |DF^k(\theta_p)| < 1 \), respectively, where \( DF \equiv \frac{dF}{d\theta} \). A stable \( k \)-periodic orbit is also referred to as a \( k \)-SPO.

Figures 3(b) and (d) illustrate spike phase maps. These maps correspond to Figs. 2(a) and (b), respectively, and the orbit of \( F \) corresponds to the waveform \( x(\tau) \). The maps show the 1-SPO and 3-SPO systems for \( s = 1.33 \) and \( s = 2.66 \). Note that this single SNM exhibits various stable periodic and chaotic orbits as the parameter varies [1].

### 3. Pulse-coupled system with three neurons

In this section, we introduce a novel system of multiple coupled SNMs. This paper focuses on a ring of three spike-coupled neurons. Note that a pulse-coupled system of two SNMs is discussed in Appendix.

#### 3.1 Basic dynamics and coupling method

Figure 4 shows the dynamics and firing rule for the proposed pulse-coupled system, where \( x_m \) and \( y_m \) are the state variable and the output, and each neuron is indicated by \( m \) where \( m = (1, 2, 3) \). For simplicity, we assume that each neuron has a common base signal \( b(\tau) \) and that the threshold levels
are the same: \( x_1 = x_2 = x_3 = 0 \). As shown in Fig. 4(a), the state variables \( x_m(\tau) \) increase from the initial state with slope \( s_m \). If state \( x_2(\tau) \) reaches the threshold level \( x_2 = 0 \) at time \( \tau = \tau_{2(1)} \), states \( x_2 \) and \( x_3 \) simultaneously and instantaneously fire and reset to their bases: the state \( x_3 \) accepts the spike \( y_2 = 1 \) that is output by neuron 2 and is compelled to reset. In the same manner, if neuron 1 or 3 reaches the threshold level, state \( x_2 \) or \( x_1 \) is compelled to reset to the base signal. The spiked outputs thus create a pulse-coupled system, which is a kind of ring coupling.

A sketch of this coupling method is shown in Fig. 4(b), where the neurons 1, 2, and 3 are denoted by \( N_1, N_2, \) and \( N_3 \) (this notation will be used hereafter). The arrow shows the firing rule and the coupling direction. The dynamics and firing rules are shown as Eqs. (5) and (6), respectively, and the base signal is shown as Eq. (7):

\[
\dot{x}_m = s_m, \quad y_m(\tau) = 0 \text{ for } x_1 < 0, x_2 < 0 \text{ and } x_3 < 0, \tag{5}
\]

\[
\begin{align*}
  x_1(\tau^+) &= b(\tau^+), \quad y_1(\tau^+) = 1 \\
  x_2(\tau^+) &= b(\tau^+), \quad y_2(\tau^+) = 1 \\
  x_3(\tau^+) &= b(\tau^+), \quad y_3(\tau^+) = 1
\end{align*}
\quad \text{if } \begin{cases} 
  x_1(\tau) \text{ and/or } x_3(\tau) \geq 0, \\
  x_2(\tau) \text{ and/or } x_1(\tau) \geq 0, \\
  x_3(\tau) \text{ and/or } x_2(\tau) \geq 0;
\end{cases} \tag{6}
\]

\[
b(\tau) = \begin{cases} 
  -k(\theta - \frac{1}{4}) - 1 & \text{for } 0 \leq \theta < \frac{1}{2}, \\
  k(\theta - \frac{3}{4}) - 1 & \text{for } \frac{1}{2} \leq \theta < 1;
\end{cases} \tag{7}
\]

where \( x'_m = \frac{dx_m}{d\tau} \). This system has four parameters \( s_m \) (\( m = 1, 2, 3 \)) and \( k \). For simplicity, we assume the following conditions:

\[
0 < s_1, \quad 0 < s_2, \quad 0 < s_3, \quad s_1 < s_2, \quad 0 < k < 4. \tag{8}
\]

In a pulse-coupled system, we can consider two kinds of firing. One situation is that in which the state is reset by its own spike, which is known as “self-firing” (SF). The other case is that in which the state is reset by a spike that is output by another neuron; this is called “compulsory firing” (CF). As shown in Fig. 4, only the SF spikes control (reset) the neighboring neurons. A waveform example is shown in Fig. 5. In this case, all the neurons exhibit both SF and CF.

### 3.2 Compulsory firing ratio (CFR) and conjunction behavior

In order to consider the characteristics of the dynamics, we define the CFR \( R_m \) of neuron \( m \):

\[
R_m = \frac{\text{CF}_m + \text{AF}_m}{\text{SF}_m + \text{CF}_m + \text{AF}_m}, \quad m = 1, 2, 3, \tag{9}
\]
Fig. 6. Compulsory firing ratios for $s_3$. $s_1 = 1.33, s_2 = 2.66, k = 2.5, x_1(0) = x_2(0) = x_3(0) = -0.25$. (a) $N_1$, (b) $N_2$, and (c) $N_3$.

Fig. 7. Sketch of the firing rules in each region. The arrows represent the firing rules and the coupling directions. (a) Conjunction behavior in Region A (MCB), (b) Conjunction behavior in Region C (RCB), (c) Conjunction behavior in Region B (MCB).

where $SF_m$ and $CF_m$ are the numbers of SFs and CFs, respectively, in neuron $m$. $AF$ is the number of simultaneous SFs and CFs. Note that numbers of individual $SF_m$ and $CF_m$ do not include their combinations $AF$. Figure 6 shows characteristics of the CFR for $s_3$. On the whole, since $N_3$ tends to exhibit SF as $s_3$ increases, $R_3$ decreases and $R_1$ increases. Roughly speaking, we can see that the characteristics of the CFR can be divided into three regions, as shown in Fig. 6. The properties of the three regions are as follows:

Region A: $R_1 = 0$ and $R_3 = 1$. Since the slope $s_3$ of $N_3$ is small, neither $N_3$ nor $N_1$ exhibits SF or CF. $N_1$ does not accept the spike of $N_3$ at any time; that is, $N_3$ does not control $N_1$. In other words,
Fig. 8. Bifurcation diagrams for $s_3$. $s_1 = 1.33, s_2 = 2.66, k = 2.5, x_1(0) = x_2(0) = x_3(0) = -0.25$. (a) $N_1$, (b) $N_2$, and (c) $N_3$.

the firing rule (arrow) from $N_3$ to $N_1$ disappears, as shown in Fig. 7(a). In this case, the system has only one-way coupling and $N_1$ behaves like a master, and so we refer to this as MCB.

Region B: $R_1 = 1$ and $R_2 = 0$. The slope $s_3$ is larger than $s_2$, and $s_1$ is the smallest slope. Neither $N_1$ nor $N_2$ exhibits SF or CF. $N_2$ does not accept the spike of $N_1$ at any time; that is, $N_1$ does not control $N_2$. In this case, the firing rule (arrow) from $N_1$ to $N_2$ disappears, as shown in Fig. 7(c). $N_2$ behaves like a master, and the system operates as a MCB. Note that $s_3$ is the steepest, but $R_3 \neq 0$.

Region C: the area that is not in Region A or B. All the neurons exhibit SF and CF, and they control their neighbor neurons, as shown in Fig. 7(c). We believe this is true ring-coupling behavior, and so we refer to it as RCB. The CFR $R_m$ varies in a complex way.

We note that this pulse-coupled system can switch from RCB to MCB when the parameters vary.

In order to analyze these phenomena, let $\tau_{m(n)}$ be the position of the $n$-th spike of the neuron $m$, as shown in Fig. 4(a). Let $\theta_{m(n)}$ be the phase of the $n$-th spike of the neuron $m$: $\theta_{m(n)} \equiv \tau_{m(n)} \mod 1$. The bifurcation diagrams of $\theta_{m(n)}$ are shown in Fig. 8. When $s_3$ is small (in Region A), $N_1$ becomes the master and exhibits the same phenomena as a single neuron, because $N_1$ does not exhibit CF. Recalling Figs. 2(a) and 3(b), $N_1$ exhibits 1-SPO with $s_1 = 1.33$ and $k = 2.5$. Since the master $N_1$ is periodic, the other neurons exhibit periodic phenomena. If there is a periodic neuron $m$ with $R_m = 0$, the overall behavior of this pulse-coupled system is periodic. In Region B in Fig. 8 ($s_3 > s_2$), $N_2$ becomes the master and exhibits a 3-SPO that is the same as in Figs. 2(b) and 3(d). In Region C in
Fig. 9. A key object for consideration is the mechanism by which $N_3$ generates SF. We assume $s_2 > s_1$ and $s_2 > s_3$.

Fig. 10. Behavior of the coupling system in $s_2$–$s_3$ space, where $k = 2.5$ and $s_1 = 1.33$. A solid line indicates the boundary $B_{mr}$ from MCB to RCB. (a) The left region of $B_{mr}$ is MCB: $R_1 = 0$, $R_3 = 1$; the broken line indicates $s_2 = s_3$. (b) Behavior of $N_2$ in Region A; the $n$-SPO denotes the phenomena of $N_2$. (c) Enlargement figure of a square written broken line in Fig. (b); black circles indicate the time waveforms shown in Fig. 11.

Fig. 8, each neuron exhibits SF and CF, and we can see that this system exhibits very complicated periodic/nonperiodic phenomena when the parameter varies. It should be noted that, in RCB, the system exhibits nonperiodic behavior even if the parameters are set so that each of the three single neurons is periodic. Analysis of the RCB is difficult, and it is left as an area of future investigation.

3.3 Boundary from MCB to RCB
In this subsection, we consider the boundary at which Region A (MCB) changes to Region C (RCB), as $s_3$ increases from a small value. Since theoretical analysis of this is difficult, we will simplify it by assuming various conditions, as follows:

$$s_1 = 1.33, \quad s_2 > s_1, \quad s_2 > s_3.$$  \hspace{1cm} (10)

We will also assume that $N_1$ exhibits only SF and that the 1-SPO is in a steady state; see Fig. 2(a). In this pulse-coupled system, the spike phase $\theta_{m(n+1)}$ depends on $\theta_{m(n)}$ and the state of another neuron, which then does not reset to the base. Therefore, the spike phase map is generally a two-dimensional map. This map is different from the map in Section 2, and it is very complex, which makes it difficult to use the map to analyze this situation. Although this analysis is left to future study, we will consider the boundary by examining the trajectories in time space, as follows.

As shown in Fig. 9, let us assume that $N_1$ exhibits SF and that $N_2$ exhibits CF at $\tau = \tau_{2(0)}$. Let
\( \tau_{2(1)} \) be the position of the spike of \( N_2 \) prior to \( \tau_{2(0)} \). If there is SF of \( N_3 \), then this must occur between the CF and SF of \( N_2 \), namely, it must occur at \( \tau_{3(0)} \) before the time \( \tau_{2(1)} \), as shown in Fig. 9, the condition \( \tau_{3(0)} < \tau_{2(1)} \) must be satisfied. Considering \( \tau_{3(-1)} + \frac{[b(\tau_{3(-1)})]}{s_3} = \tau_{3(0)} \) from Eq. (3), we note that SF of \( N_3 \) can occur when the following condition is satisfied:

\[
\tau_{3(-1)} + \frac{[b(\tau_{3(-1)})]}{s_3} < \tau_{2(1)}, \quad \text{where} \quad \tau_{2(1)} = \tau_{2(-1)} + d + \frac{[b(\tau_{2(0)})]}{s_2}, \quad d \equiv \tau_{2(0)} - \tau_{2(-1)}.
\]

(11)

Therefore, one boundary \( B_{mr} \) from MCB to RCB is given as \( \frac{[b(\tau_{3(-1)})]}{s_3} = d + \frac{[b(\tau_{2(0)})]}{s_2} \). Solving for \( s_3 \), we obtain \( s_3 = \frac{[b(\tau_{3(-1)})]}{[b(\tau_{2(0)})/s_2 + d]} \). The boundary \( B_{mr} \) can be described as in Eq. (12):

\[
B_{mr} = \{ (s_2, s_3) \mid s_3 = \frac{s_2[b(\tau_{3(-1)})]}{[b(\tau_{2(0)})/s_2 + d]} \}.
\]

(12)

Using numerical simulation, the boundary \( B_{mr} \) was determined, and it is indicated by a solid line (serration line) in the \( s_2-s_3 \) space of Fig. 10(a). The region to the left of \( B_{mr} \) has MCB (\( N_1 \) exhibits only SF). It should be noted that, for Fig. 10, we assumed as an initial state that the 1-SPO of \( N_1 \) was in a steady state.

An example of the trajectories in front of and behind the boundary \( B_{mr} \) are shown in Fig. 10(c) and Fig. 11, respectively. The values in Figs. 11(a) and (b) correspond to the black circles in Fig. 10(c). Figure 11(a) is to the left of \( B_{mr} \) and exhibits MCB. If \( s_3 \) increases and crosses the boundary \( B_{mr} \), its behavior is changed into that shown in Fig. 11(b), and the system exhibits RCB. In Fig. 11(a), \( N_3 \) cannot exhibit SF after it fires at \( \tau = \tau_{3(-1)} \), because \( \tau_{3(-1)} + \frac{[b(\tau_{3(-1)})]}{s_3} > \tau_{2(1)} \) is satisfied. On the other hand, in Fig. 11(b), \( s_3 \) becomes large and \( \frac{[b(\tau_{3(-1)})]}{s_3} \) becomes small. In this case, \( \tau_{3(-1)} + \frac{[b(\tau_{3(-1)})]}{s_3} < \tau_{2(1)} \) is satisfied and \( N_3 \) can exhibit SF before \( \tau = \tau_{2(1)} \). Using Eqs. (7), (11), and (12), each value can be calculated for the parameters \( s_1 = 1.33, s_2 = 2.66, k = 2.5 \) in Fig. 11(a), as follows:

\[
\tau_{2(0)} \mod 1 \approx 0.618, \quad [b(\tau_{2(0)})] \approx 1.33, \quad \tau_{2(-1)} \mod 1 = \tau_{3(-1)} \mod 1 \approx 0.37, \quad d \approx 0.248, \quad [b(\tau_{3(-1)})] \approx 1.3, \quad B_{mr} : s_3 \approx 1.74.
\]

(13)

The results conform with those shown in Figs. 10(c) and 11.
Fig. 12. Circuit model of the pulse-coupled system. The subcircuits of dotted boxes represent the neurons. COMP, MM, and SW denote a comparator, a monostable multivibrator, and an analog switch, respectively. OR gates realize each SF and CF, and current sources are realized by OTAs.

Fig. 13. Observed waveforms, observed spike-train outputs, and the corresponding waveforms of the numerical simulations. In the waveform, $v_1$, $v_2$, $v_3$, and the base signal are indicated by violet, light blue, green, and yellow, respectively. In the spike-train outputs, the SF output of $v_1$, $v_2$, and $v_3$ are indicated by violet, light blue, and green, respectively. It should be noted that we only show the spike-train outputs of the SFs for the hardware experiments and the numerical simulations. In this implementation, we set $C \approx 47 \text{ nF}$, $V_T \approx 1 \text{ V}$, $T \approx 1 \text{ ms}$, $K \approx 2500 (k \approx 2.5)$, $I_1 \approx 0.063 \text{ mA} \ (s_1 \approx 1.33)$, $I_2 \approx 0.125 \text{ mA} \ (s_2 \approx 2.66)$, $I_3 \approx 0.047 \text{ mA} \ (s_3 \approx 1.00)$. Here, we have MCB, and the master is $N_1$.

Roughly speaking, at the boundary shown in Fig. 10, as $s_2$ increases, $|b(\tau_{3(0)})|$ becomes small and $B_{mr}$ becomes large. However, there is a case where $B_{mr}$ becomes small as $s_2$ increases. In this case, the base $b(\tau_{3(-1)})$ has slope $-k$. As $s_2$ increases, the timing $\tau_{3(-1)}$ is earlier and $|b(\tau_{3(-1)})|$ becomes smaller so $B_{mr}$ also becomes smaller; see Eq. (12).

In Fig. 10(b), $n$-SPO represents the behavior of $N_2$. If the period $n$ increases to $n+1$, $d$ decreases and $B_{mr}$ suddenly becomes large. We can see that the value $s_3$ on the boundary suddenly becomes large when the period of $N_2$ increases. We have confirmed that $N_1$ exhibits CF in Region A when the initial state is changed, even if the parameters are the same. That is, phenomena coexist. Consideration of these effects is left to future study.
4. Experiments

In order to realize the pulse-coupled system described in Section 3, we propose a circuit model, as shown in Fig. 12. It should be noted that an implementation of the single SNM is shown in [1]. The subcircuit indicated by the dotted boxes denotes each neuron, and the capacitor voltages, $v_1(t)$, $v_2(t)$, and $v_3(t)$, denote the state of the respective neurons. Voltages $V_T$ and $B(t)$ correspond to the threshold.
and base signal, respectively. The comparators COMPs (LM339), monostable multivibrators MMs (TC4538), switches SWs (NJU4066), and OR gates (TC4071) realize the firing rules. Current sources are realized by the operational transconductance amplifiers OTAs (13600). Operation of this circuit model is described as follows. The capacitor voltage \( v_m \) increases due to the current source \( I_m \), where \( m = (1,2,3) \) is the index of the neurons. If the voltage \( v_1 \) reaches the threshold voltage \( V_T \), the comparator outputs a high-level voltage. This triggers the MM, which then outputs a pulse. This causes the OR gate to output a pulse to close; this happens nearly instantaneously. Next, \( v_1 \) and \( v_2 \) are reset to their respective base signals \( B(t) \). The switches immediately reopen, and \( v_1 \) and \( v_2 \) again begin to increase. The circuit equation, firing rules, and base signal are shown in Eqs. (14) and (15):

\[
\begin{align*}
C^\frac{dv_1}{dt} &= I_1, Y_1(t) = -E \\
C^\frac{dv_2}{dt} &= I_2, Y_2(t) = -E \quad \text{for} \quad \begin{cases} v_1(t) < V_T \text{ and } v_3(t) < V_T, \\ v_2(t) < V_T \text{ and } v_1(t) < V_T, \\ v_3(t) < V_T \text{ and } v_2(t) < V_T; \end{cases} \\
C^\frac{dv_3}{dt} &= I_3, Y_3(t) = -E \\
\end{align*}
\]

\[
\begin{align*}
\begin{cases} v_1(t^+) = B(t^+), Y_1(t^+) = E \\ v_2(t^+) = B(t^+), Y_2(t^+) = E \quad \text{if} \quad \begin{cases} v_1(t) \text{ or } v_3(t) \geq V_T, \\ v_2(t) \text{ or } v_1(t) \geq V_T, \\ v_3(t) \text{ or } v_2(t) \geq V_T; \end{cases} \\ v_3(t^+) = B(t^+), Y_3(t^+) = E \end{cases} \\
B(t) = B(t + T), \quad B(t) = \begin{cases} -K(t - \frac{1}{4}T) \quad \text{for } 0 \leq t < \frac{1}{2}T, \\ K(t - \frac{3}{4}T) \quad \text{for } \frac{1}{2}T \leq t < T, \end{cases}
\end{align*}
\]

where we assume that the parasitic resistors in the circuit can be ignored. For simplicity, we also ignore the delays in the switch and the pulse output. Equations (14) and (15) can be transformed into Eqs. (5), (6), and (7), using the following equations:

\[
\begin{align*}
\tau &= \frac{t}{T}, \quad x_m = \frac{v_m - V_T}{V_T}, \quad y_m = \frac{Y_m + E}{2E}, \quad k = \frac{KT}{V_T}, \quad s_m = \frac{I_m T}{C V_T}.
\end{align*}
\]

Figures 13, 14, and 15 show waveforms and spike-trains that are typical of those observed, and the corresponding results of the numerical simulation. Figures in the left and right columns show the hardware and numerical simulations, respectively. In the observed waveforms, violet, light blue, green, and yellow represent \( v_1, v_2, v_3 \), and \( B(t) \), respectively. In the observed spike-train outputs, violet, light blue, and green represent the SF output of \( v_1, v_2, \text{ and } v_3 \), respectively. It should be noted that we only show the SF spike-train outputs for the hardware experiments and the numerical simulations. We observed MM output \( Y_m' \) in the hardware experiments.

In Fig. 13, \( N_3 \) only has CF, and the system exhibits MCB. This corresponds to Region A in Fig. 8. In Fig. 14, all the neurons have both SF and CF, and the system exhibits RCB. This corresponds to Region C in Fig. 8. In Fig. 15, \( N_1 \) has only CF, and the system exhibits MCB. This corresponds to Region B in Fig. 8. We can see that the observed waveforms and spike-train outputs conform with those obtained in the numerical simulations.

5. Conclusions
We have proposed a pulse-coupled system with three SNMs. The coupling method was designed so that the SNM accepted each spiked output as an input. Such a coupling method generates SF and CF for each neuron. As the parameter varies, the pulse-coupled system exhibits both periodic and nonperiodic behavior. Using the CFR, we can see that this system changes from RCB to MCB. As one of our main results, we have determined the boundary between RCB and MCB for a particular set of conditions. Finally, we have proposed a circuit model and confirmed the occurrence of MCB and RCB. In the future, we intend to consider the details of RCB, the effects of the initial state, the mechanism of the bifurcation phenomenon, and an expansion of this model into a large-scale system.
Fig. 16. Dynamics and firing rules of a pulse-coupled system with two SNMs. (a) Waveforms. (b) Sketch of coupling method. \( N_1 \) and \( N_2 \) indicate neurons 1 and 2, respectively. The arrow shows the direction of the spiked output.

Fig. 17. Typical waveforms for \( s_2 = 1.8, k = 2.5, x_1(0) = x_2(0) = -0.25 \). (a) MCB (\( N_2 \) is the master) for \( s_1 = 1.3 \); (b) MCB (\( N_1 \) is the master) for \( s_1 = 2.2 \).

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Appendix

A. Pulse-coupled system with two SNMs

Here, we consider a pulse-coupled system with two SNMs. Figure 16 shows the dynamics for the proposed pulse-coupled system. The definition of each variable is the same as in Section 3. For simplicity, the base and threshold signals are the same as in Section 3. As shown in Fig. 16(a), the state variables \( x_m(\tau) \) increase from the initial state with slope \( s_m \) where \( m = (1, 2) \). If state \( x_1(\tau) \) reaches the threshold level \( x_1 = 0 \), state \( x_2 \) simultaneously and instantaneously fires and resets to its base: the state \( x_2 \) accepts the spike \( y_1 = 1 \) that is output by \( N_1 \) and is compelled to reset. In the same manner, if \( x_2 \) reaches the threshold level, state \( x_1 \) is compelled to reset to the base signal. A sketch of this coupling method is shown in Fig. 16(b), where the arrow shows the firing rule and the coupling direction. This is a kind of ring (or mutual) coupling. The dynamics and firing rules are shown as Eqs. (A-1) and (A-2), respectively:

\[
x'_m = s_m, \quad y_m(\tau) = 0 \quad \text{for} \quad x_1 < 0, \quad \text{and} \quad x_2 < 0,
\]

\[
x_1(\tau^+) = x_2(\tau^+) = b(\tau^+), \quad y_1(\tau^+) = y_2(\tau^+) = 1, \quad \text{if} \quad x_1(\tau) \text{ and/or } x_2(\tau) \geq 0,
\]
where $b(\tau)$ is the same as in Eq. (7). This system has three parameters: $s_1$, $s_2$, and $k$. Waveform examples are shown in Fig. 17. In this case, the neuron having large (or small) slope only exhibits SF (or CF). Therefore, the neuron with the large slope is the master, and this system exhibits MCB if $s_1 \neq s_2$; this system does not exhibit RCB if $s_1$ is not exactly equal to $s_2$. It should be noted that it is difficult to achieve $s_1 = s_2$ exactly in an actual circuit.

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