The Law of the Iterated Logarithm for a Class of SPDEs

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Abstract

Using a moderate deviation result, we establish the Strassen’s compact law of the iterated logarithm (LIL) for a class of stochastic partial differential equations (SPDEs). As an application, we obtain this type of LIL for two important population models known as super-Brownian motion and Fleming-Viot process.

KEY WORDS: Law of iterated logarithm, large deviation principle, moderate deviation principle, stochastic partial differential equation, Fleming-Viot process, super-Brownian motion.

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1 Introduction

Large deviations has noticeably become an active area of research with applications in queues, communication theory, exit problems and statistical mechanics. This study began in finance, making many breakthroughs and grew in various other fields. It is the study of very rare events that have probability tending to zero exponentially fast and its goal is to determine the exact form of this rate of convergence. Another closely related area of study is moderate deviations, which is proved for events that have probability going to zero at a rate slower than that of large deviations but faster than the rate for central limit theorem; hence, the name moderate deviations is used. An important application of large and moderate deviations is the law of the iterated logarithm (LIL). Beginning with J. Deuschel, D. Stroock\textsuperscript{13} (Lemma 1.4.3), a notable number of authors have used this connection. For instance, P. Baldi\textsuperscript{2}, G. Divanji, K. Vidyalaxmi\textsuperscript{14}, B. Jing,
Q. Shao, Q. Wang [26], and A. Mogul’skii [28] used their large deviations results to prove LIL, whereas, Y. Chen, L. Liu [6] and R. Wang, L. Xu [35] applied their theorem on moderate deviations. LIL has useful applications in other fields including finance (see for example [24, 41]). In the literature, there are various forms of LIL. They are Classical LIL, Strassen’s Compact LIL, Chover’s type, and Chung’s type which inherited names from the authors who introduced them; namely, A. Khintchine [27], V. Strassen [34], J. Chover [7], and K. Chung [8], respectively. In section two, we provide a description of each type of LIL. For a more detailed introduction and history on each type we recommend [4]. As observed in the papers mentioned here, every type of LIL can be derived from large and moderate deviations by the use of Borel-Cantelli lemmas; however, the most common form for this application is the Stassen’s compact LIL. P. Fatheddin and J. Xiong established the large and moderate deviation principles for the class of SPDEs studied here in [21] and [22], respectively, and used the results to achieve the theories for two important population models: super-Brownian motion (SBM) and Fleming-Viot Process (FVP). In this paper, we extend the result on moderate deviations by achieving the LIL. Since our SPDEs are in a function space, the Strassen’s compact LIL is proved. We note that as in our previous results we only prove LIL in dimension one since the existence and uniqueness of the SPDEs shown by J. Xiong [37], were limited to dimension one.

To achieve the Strassen’s compact LIL, one needs to show that the process multiplied by $\frac{1}{\sqrt{2\log \log t}}$ is relatively compact and then specify the set of limit points. Since our process is real-valued, we obtain the relative compactness property by proving tightness. Moreover, to determine the set of limit points, we apply the result introduced by P. Baldi [1] and implemented in [17, 30, 31]. Other methods have also been used by some authors to establish that their process is relatively compact. A. Dembo, T. Zajic [11] and L. Wu [36] prove this condition by showing that their process is totally bounded. A. Schied [33], attains compact LIL for super-Brownian motion (SBM) in all dimensions, $d \geq 1$, as a corollary to its moderate deviation result also given in [33]. Like all other LIL results derived from LDP or MDP, A. Schied uses the rate function in MDP to form the set of limit points. Since his rate function is a good rate function, he utilizes the compactness property of its level sets and applies Lemma 1.4.3 in [13]. To the best of our knowledge, LIL has not been proven for Fleming-Viot Process (FVP) in the literature.
We begin in section two with notations and spaces used throughout the paper and provide the main theorems. We also give some definitions and background on LIL and the two population models. In section three we focus on deriving the LIL for the class of SPDEs and in section four apply the results to obtain the LIL for SBM and FVP.

## 2 Notations and Main Results

Following the notations given in [21,22], we introduce the space used here as follows. Suppose $(\Omega, \mathcal{F}, P)$ is a probability space and $\{\mathcal{F}_t\}$ is a family of non-decreasing right continuous sub-$\sigma$-fields of $\mathcal{F}$ such that $\mathcal{F}_0$ contains all $P$-null subsets of $\Omega$. We denote $C_b(\mathbb{R})$ to be the space of continuous bounded functions on $\mathbb{R}$ and $C_c(\mathbb{R})$ to be composed of continuous functions in $\mathbb{R}$ with compact support. Throughout the article, we let $K$ be a positive constant that may take different values in different lines.

For $0 < \beta \in \mathbb{R}$, we let $M_{\beta}(\mathbb{R})$ denote the set of $\sigma$-finite measures $\mu$ on $\mathbb{R}$ such that

$$\int e^{-\beta|x|}d\mu(x) < \infty. \quad (1)$$

We endow this space with the topology defined by a modification of the usual weak topology: $\mu^n \to \mu$ in $M_{\beta}(\mathbb{R})$ iff for every $f \in C_b(\mathbb{R})$,

$$\int_{\mathbb{R}} f(x)e^{-\beta|x|}\mu^n(dx) \to \int_{\mathbb{R}} f(x)e^{-\beta|x|}\mu(dx).$$

For $\alpha \in (0,1)$, we consider the space $B_{\alpha,\beta}$ composed of all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $m \in \mathbb{N}$, there exists $K > 0$ with the following conditions:

$$|f(y_1) - f(y_2)| \leq Ke^{\beta m}|y_1 - y_2|^\alpha, \quad \forall |y_1|, |y_2| \leq m \quad (2)$$

$$|f(y)| \leq Ke^{\beta |y|}, \quad \forall y \in \mathbb{R} \quad (3)$$

and with the metric,

$$d_{\alpha,\beta}(u, v) = \sum_{m=1}^{\infty} 2^{-m}(\|u - v\|_{m,\alpha,\beta} \wedge 1), \quad u, v \in B_{\alpha,\beta}$$

where

$$\|u\|_{m,\alpha,\beta} = \sup_{x \in \mathbb{R}} e^{-\beta|x|}|u(x)| + \sup_{y_1 \neq y_2, |y_1|, |y_2| \leq m} \frac{|u(y_1) - u(y_2)|}{|y_1 - y_2|^{\alpha}} e^{-\beta m}. $$
Note that the collection of continuous functions on $\mathbb{R}$ satisfying (3), referred to as $\mathbb{B}_\beta$, is a Banach space with norm,
\[ \|f\|_\beta = \sup_{x \in \mathbb{R}} e^{-|\beta| |x|} |f(x)|. \]

The above space was used to match the setup in [3], where weak convergence approach was introduced to prove large deviation principle and the technical difficulties in time discretization in classical approaches were avoided.

We now give a small background on the two population models under study. For more information see [18]. SBM is the continuous version of branching Brownian motion, the most classical and best known branching process where individuals reproduce according to Galton-Watson process. Since the population is set to evolve as a cloud in $\mathbb{R}^d$, it is a measure-valued process and because of its branching property, we associate a branching rate, denoted as $\epsilon$. For more depth and background on this model we refer the reader to [9, 16, 18, 38]. SBM, denoted as $\{\mu^\epsilon_t\}$, can be characterized by one of the following.

i) $\{\mu^\epsilon_t\}$ having Laplace transform,
\[ \mathbb{E}_{\mu^\epsilon_0} \exp(-\langle \mu^\epsilon_t, f \rangle) = \exp(-\langle \mu^\epsilon_0, v(t, \cdot) \rangle) \]
where $v(\cdot, \cdot)$ is the unique mild solution of the evolution equation:
\[
\begin{cases}
\dot{v}(t, x) = \frac{1}{2} \Delta v(t, x) - v^2(t, x) \\
v(0, x) = f(x)
\end{cases}
\]
for $f \in C^+_p(\mathbb{R}^d)$ where $C_p(\mathbb{R}^d)$ is defined as
\[ C_p(\mathbb{R}^d) := \left\{ f \in C(\mathbb{R}^d) : \sup_{x \in \mathbb{R}^d} \frac{|f(x)|}{\phi_p(x)} < \infty \text{ for } p > d, \phi_p(x) := (1 + |x|^2)^{-\frac{p}{2}} \right\}. \]
See for example [25, 40].

ii) $\{\mu^\epsilon_t\}$ as the unique solution to a martingale problem given as: for all $f \in C_p^2(\mathbb{R})$
\[
M_t(f) := \langle \mu^\epsilon_t, f \rangle - \langle \mu^\epsilon_0, f \rangle - \int_0^t \left\langle \mu^\epsilon_s, \frac{1}{2} \Delta f \right\rangle ds
\]
is a square-integrable martingale with quadratic variation,
\[ \langle M(f) \rangle_t = \epsilon \int_0^t \langle \mu^\epsilon_s, f^2 \rangle ds. \]
For the formulation of this martingale problem see [18] section 1.5.

iii) J. Xiong in [37] studied SBM by its “distribution” function-valued process \( u_t^\epsilon \) defined as

\[
u_t^\epsilon(y) = \int_0^y \mu_t^\epsilon(dx), \quad \forall y \in \mathbb{R}.
\] (4)

Using (4), SBM was characterized by the following stochastic partial differential equation (SPDE),

\[
u_t^\epsilon(y) = F(y) + \int_0^t \int_0^\epsilon W(ds, dy) + \int_0^t \frac{1}{2} \Delta u_s^\epsilon(y)ds
\] (5)

where \( F(y) = \int_0^y \mu_0(dx) \) is the “distribution” function of \( \mu_0 \), \( W \) is an \( \mathcal{F}_t \)-adapted space-time white noise random measure on \( \mathbb{R}^+ \times \mathbb{R} \) with intensity measure \( dsda \).

The other model studied here is FVP, which observes the evolution of population based on the genetic type of individuals. It is the continuous version of the step-wise mutation model, in which individuals move in \( \mathbb{Z}^d \) according to a continuous time sample random walk. In Biology, mutation is the term given to indicate change in copies of DNA from parents to offsprings, which can cause major problems such as the development of tumors. In FVP, the population is fixed throughout time with each individual having a gene type and every time a mutation occurs the individual changes in gene type and moves to a new location. Therefore, the distribution of gene types is observed, making FVP a probability-measure valued process with mutation rate given by \( \epsilon \). More background on FVP can be found in [18–20]. Similar to SBM, we provide the different ways FVP, denoted also as \( \{\mu_t^\epsilon\} \), is defined in the literature as follows.

i) \( \{\mu_t^\epsilon\} \) a family of probability measure-valued Markov process generated by \( \mathcal{L}^\epsilon \) defined as

\[
\mathcal{L}^\epsilon F(\mu_t^\epsilon) = f'(\langle \mu_t^\epsilon, \phi \rangle) \langle \mu_t^\epsilon, A\phi \rangle + \frac{\epsilon}{2} \int \int f''(\langle \mu_t^\epsilon, \phi \rangle) \phi(x)\phi(y)Q(\mu_t^\epsilon; dx, dy)
\]

for \( \epsilon > 0 \) where

\[
Q(\mu_t^\epsilon; dx, dy) := \mu_t^\epsilon(dx)\delta_x(dy) - \mu_t^\epsilon(dx)\mu_t^\epsilon(dy)
\]

with \( \delta_x \) denoting the Dirac measure at \( x \) and \( A \) being the generator of a Feller process. The operator \( \mathcal{L}^\epsilon \) is given on the set,

\[
\mathcal{D} = \{F(\mu_t^\epsilon) = f(\langle \mu_t^\epsilon, \phi \rangle) : f \in \mathcal{C}_b^2(\mathbb{R}), \phi \in \mathcal{C}(\mathbb{R})\}.
\]
(see [10] and [23] for this formulation).

ii) \( \{\mu^\varepsilon_t\} \) as a unique solution to the following martingale problem: for \( f \in C^2_c(\mathbb{R}) \),

\[
M_t(f) = \langle \mu^\varepsilon_t, f \rangle - \langle \mu^\varepsilon_0, f \rangle - \int_0^t \left( \langle \mu^\varepsilon_s, \frac{1}{2} \Delta f \rangle \right) ds
\]

is a continuous square-integrable martingale with quadratic variation,

\[
\langle M_t(f) \rangle = \varepsilon \int_0^t \left( \langle \mu^\varepsilon_s, f^2 \rangle - \langle \mu^\varepsilon_s, f \rangle^2 \right) ds.
\]

For this martingale problem see [18] Section 1.11.

iii) An alternative SPDE characterization of FVP was also made in [37]. There by using

\[
u^\varepsilon_t(y) = \mu^\varepsilon_t((\infty, y])
\]

FVP was proved to be given by,

\[
u^\varepsilon_t(y) = F(y) + \int_0^t \int_{\mathbb{R}} (1_{a \leq u^\varepsilon_s(y)} - u^\varepsilon_s(y)) W(dsda) + \int_0^t \frac{1}{2} \Delta u^\varepsilon_s(y) ds.
\]

Note that the main difference between (5) and (7) is in the second term. Hence, as in [21, 22], we form the following class of SPDEs and have SBM and FVP as special cases.

\[
u^\varepsilon_t(y) = F(y) + \sqrt{\varepsilon} \int_0^t \int_U G(a, y, u^\varepsilon_s(y)) W(da) + \int_0^t \frac{1}{2} \Delta u^\varepsilon_s(y) ds
\]

where \((U, U, \lambda)\) is a measure space such that \(L^2(U, U, \lambda)\) is separable, \(F\) is a function of \(\mathbb{R}\) and \(u_1, u_2, u, y \in \mathbb{R}\). In addition, \(G : U \times \mathbb{R}^2 \to \mathbb{R}\) satisfies the following conditions,

\[
\int_U |G(a, y, u_1) - G(a, y, u_2)|^2 \lambda(da) \leq K|u_1 - u_2|, \tag{9}
\]

\[
\int_U |G(a, y, u)|^2 \lambda(da) \leq K(1 + |u|^2). \tag{10}
\]

In [22], to prove the moderate deviation principle for \(u^\varepsilon_t(y)\), as \(\varepsilon \to 0\), the centered process,

\[
u^\varepsilon_t(y) = \frac{a(\varepsilon)}{\sqrt{\varepsilon}} (u^\varepsilon_t(y) - u^\varepsilon_0(y))
\]

(11)
was considered where \( a(\epsilon) \) satisfies,

\[
0 \leq a(\epsilon) \to 0, \quad \frac{a(\epsilon)}{\sqrt{\epsilon}} \to \infty \text{ as } \epsilon \to 0. \tag{12}
\]

Moderate deviation principle was proved with speed \( a(\epsilon) \) which can be seen by conditions in (12) to be slower than \( \sqrt{\epsilon} \), the speed for large deviation principle. The controlled PDE version of (11) also referred to as the skeleton version is given by,

\[
S_t(h, y) = \int_0^t \int_U G(a, y, u_s(y))h_s(a)a(s)\lambda(da)ds + \frac{1}{2} \int_0^t \Delta s_s(h, y)ds \tag{13}
\]

where \( h \in L^2([0,1] \times U, ds\lambda(da)) \). It can be shown that for every \( h_s(a) \) there is a unique solution to (13). For MDP, the first term of SPDE (8), \( F(y) \) was assumed to be in space, \( B_{\alpha, \beta} \) where \( \beta_0 \in (0, \beta) \). Applying the method provided by [5], the controlled PDE was used to obtain the following MDP result in [22].

**Theorem 1** (Theorem 1 in [22]). If \( F \in B_{\alpha, \beta_0} \) for \( \alpha \in (0, \frac{1}{2}) \), then family \( \{v^\epsilon\} \) satisfies the LDP in \( \mathcal{C}([0,1]; B_{\beta}) \) with speed \( a(\epsilon) \) and rate function,

\[
I(g) = \frac{1}{2} \inf \left\{ \int_0^1 \int_U |h_s(a)|^2 \lambda(da)ds : g = S_t(h, y) \right\} \tag{14}
\]

which implies family \( \{u^\epsilon_t\} \) obeys the MDP.

To be complete, we provide a definition of large deviation principle (LDP). For more background on large deviations theory we recommend [12],[15].

**Definition 1** (Large Deviation Principle (LDP)). The sequence \( \{X_n\}_{n \in \mathbb{N}} \) satisfies the LDP on \( \mathcal{E} \) with rate function \( I \) if the following two conditions hold.

a. LDP lower bound: for every open set \( U \subset \mathcal{E} \),

\[
- \inf_{x \in U} I(x) \leq \liminf_{n \to \infty} \frac{1}{n} \log P(X_n \in U)
\]

b. LDP upper bound: for every closed set \( C \subset \mathcal{E} \),

\[
\limsup_{n \to \infty} \frac{1}{n} \log P(X_n \in C) \leq - \inf_{x \in C} I(x)
\]
As for population models, the Cameron-Martin space, \( \mathcal{H} \) was used in [22]. Let \( \tilde{\mathcal{H}} \) be the space for which conditions for \( \mathcal{H} \) hold with \( \mathcal{M}_\beta(\mathbb{R}) \) replaced by the space of probability measures \( \mathcal{P}(\mathbb{R}) \), and with the additional assumption,

\[
\left\langle \mu^0_t, \frac{d(\hat{\omega}_t - \frac{1}{2} \Delta^* \omega_t)}{d\mu^0_t} \right\rangle = 0.
\]

In [22], each model being a measure-valued process was denoted as \( \{\mu^\epsilon_t\} \) with \( \epsilon \) being the branching rate or mutation rate based on context and was set to go to zero. Denoting,

\[
\omega^\epsilon_t(dy) := \frac{a(\epsilon)}{\sqrt{\epsilon}} \left( \mu^\epsilon_t(dy) - \mu^0_t(dy) \right)
\]

the following two theorems were given.

**Theorem 2** (Theorem 2 in [22]). If \( \omega^0_0 \in \mathcal{M}_\beta(\mathbb{R}) \) such that \( F \in \mathcal{B}_{\alpha, \beta}^0 \) then super-Brownian motion, \( \{\mu^\epsilon_t\} \), obeys the MDP in \( \mathcal{C}([0,1]; \mathcal{M}_\beta(\mathbb{R})) \) with speed \( a(\epsilon) \) and rate function,

\[
I(\omega) = \begin{cases} 
\frac{1}{2} \int_0^1 \int_{\mathbb{R}} \left| \frac{d(\hat{\omega}_t - \frac{1}{2} \Delta^* \omega_t)}{d\mu^0_t} y \right|^2 \mu^0_t(dy) dt & \text{if } \mu^0_t \in \mathcal{H}_\omega^0 \ \\
\infty & \text{otherwise}.
\end{cases}
\]

**Theorem 3** (Theorem 3 in [22]). Let \( \mathcal{P}_\beta(\mathbb{R}) \) be the probability measure analog of \( \mathcal{M}_\beta(\mathbb{R}) \). If \( \omega^0_0 \in \mathcal{P}_\beta(\mathbb{R}) \) such that \( F \in \mathcal{B}_{\alpha, \beta}^0 \), then, Fleming-Viot process, \( \{\mu^\epsilon\} \), satisfies the MDP on \( \mathcal{C}([0,1]; \mathcal{P}_\beta(\mathbb{R})) \) with speed \( a(\epsilon) \) and rate function,

\[
I(\omega) = \begin{cases} 
\frac{1}{2} \int_0^1 \int_{\mathbb{R}} \left| \frac{d(\hat{\omega}_t - \frac{1}{2} \Delta^* \omega_t)}{d\mu^0_t} y \right|^2 \mu^0_t(dy) dt & \text{if } \mu^0_t \in \tilde{\mathcal{H}}_\omega^0 \ \\
\infty & \text{otherwise}.
\end{cases}
\]

As mentioned in the introduction, there are different types of LIL seen in literature. Below we provide a definition of each type.

**Definition 2** (Law of the Iterated Logarithm (LIL)). Let \( \{X_j\}_{j \geq 1} \) be an i.i.d. sequence of random variables with \( S_n := \sum_{j=1}^n X_j \).

i. Classical LIL: \( \{X_j\}_{j \geq 1} \) is said to satisfy the classical LIL, also referred to as the Khintchine’s LIL, if

\[
\limsup_{n \to \infty} \frac{S_n - n\mu}{\sigma \sqrt{2n \log \log n}} = 1 \text{ a.s.} \quad (18)
\]

\[
\liminf_{n \to \infty} \frac{S_n - n\mu}{\sigma \sqrt{2n \log \log n}} = -1 \text{ a.s.} \quad (19)
\]
for common mean $\mu$ and variance $\sigma^2$. We note that this version is also given by (18) and (19) with $S_n - n\mu$ replaced by $X_n$ with $\mu = 0$ and $\sigma^2 = 1$. For examples of this form see for instance [26, 35].

ii. Strassen’s Compact LIL: A class of functions $F$ satisfies Strassen’s compact LIL with respect to $\{X_j\}_{j \geq 1}$ if there is a compact set $J$ in $\ell_\infty(F)$ such that $\{X_j\}_{j \geq 1}$ is a.s. relatively compact and its limit set is $J$. This is the functional space version of the classical LIL. See for example [1, 11, 36].

iii. Chover-type LIL: $\{X_j\}_{j \geq 1}$ satisfies Chover-type LIL if

$$\limsup_{n \to \infty} \left( \frac{|S_n|}{n^{1/\alpha}} \right) \frac{1}{\log \log n} = e^{1/\alpha} \text{ a.s.}$$

for $0 < \alpha < 2$. For examples of this form see [32, 39].

iv. Chung-type LIL: Let $S_n^* = \max_{k \leq n} |S_k|$. Chung-type LIL for $\{X_j\}_{j \geq 1}$ holds if

$$\liminf_{n \to \infty} \frac{S_n^* \sqrt{\log \log n}}{\sqrt{n}} = \frac{\pi}{\sqrt{8}} \text{ a.s.}$$

For results of this type see for example, [6, 28].

We are now ready to give the main results. Let

$$Z_\varepsilon^*(y) := \frac{1}{\sqrt{2 \log \log \frac{1}{\varepsilon}}} (u_\varepsilon^*(y) - u_0^*(y))$$

more precisely,

$$Z_\varepsilon^*(y) = \frac{1}{\sqrt{2 \log \log \frac{1}{\varepsilon}}} \int_0^t \int_U G_\varepsilon^*(a, y, Z_\varepsilon^*(y)) W(\text{d}a \text{d}s) + \frac{1}{\varepsilon} \Delta Z_\varepsilon^*(y) \text{d}s$$

where

$$G_\varepsilon^*(a, y, Z_\varepsilon^*(y)) := G \left( a, y, \sqrt{2 \log \log \frac{1}{\varepsilon}} Z_\varepsilon^*(y) + u_0^*(y) \right)$$

Therefore, we have the process $\{v_\varepsilon^*(y)\}$ from moderate deviations used in theorem 1 with $a(\varepsilon) = 1/\sqrt{2 \log \log (1/\varepsilon)}$. One can check that this fulfills the requirements of $a(\varepsilon)$ going to zero as $\varepsilon$ tends to zero, at a rate slower than $\sqrt{\varepsilon}$. Also based on conditions [9] and [10],

$$\int_U \left| G_\varepsilon^*(a, y, Z_\varepsilon^*(y)) - G_\varepsilon^*(a, y, Z_{\varepsilon,2}^*(y)) \right|^2 \lambda(\text{d}a) \leq K \sqrt{2 \log \log \frac{1}{\varepsilon}} |Z_{\varepsilon,1}^*(y) - Z_{\varepsilon,2}^*(y)|$$

9
\[ \int_U |G_s^x(a, y, Z^s_t(y))|^2 \lambda(da) \leq K \left( 1 + \left( 2 \epsilon \log \log \frac{1}{\epsilon} \right) Z^s_t(y)^2 + e^{2\beta_0 |y|} \right) \]  
where we have used the fact that \( F \in \mathbb{B}_{\alpha, \beta_0} \), giving by condition (3), \( |u^\beta_0(y)| \leq K e^{\beta_0 |y|} \).

We point out that the proof of existence and uniqueness of solutions to SPDE \{\( u^t \)\} given in [37] only relies on condition (9), thus we have the existence and uniqueness of solutions to \( Z^t \) and can use its mild solution given as,

\[ Z^t_t(y) := \frac{1}{\sqrt{2 \log \log \frac{1}{\epsilon}}} \int_0^t \int_U P_{t-s} G_s^x(a, y, Z^s_t(y))W(dad) \]  
(27)

where \( P_{t-s} \) is the Brownian semigroup defined as \( P_t f(y) = \int_{\mathbb{R}} p_t(x-y) f(x)dx \) with \( p_t(x-y) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{|x-y|^2}{2t}} \). The following theorems are statements of results of this article.

**Theorem 4.** Process \( \{Z^t\} \) is relatively compact in \( C([0, 1]; \mathbb{B}_\beta) \) and its set of limit points is exactly \( L_1 := \{ g \in C([0, 1]; \mathbb{B}_\beta) : I(g) \leq 1 \} \) where \( I(g) \) is defined by (14).

Similar to MDP result for SBM and FVP, let

\[ \tilde{Z}^\epsilon_t := \frac{1}{\sqrt{2 \log \log \frac{1}{\epsilon}}} (\mu^\epsilon_t(dy) - \mu^0_t(dy)) . \]  
(28)

**Theorem 5.** Process \( \{\tilde{Z}^\epsilon_t\} \) formed by SBM process, \( \{\mu^\epsilon_t\} \) in (28) is relatively compact in \( C([0, 1]; \mathcal{M}_\beta(\mathbb{R})) \) with set of limit points being \( L_2 := \{ \omega \in C([0, 1]; \mathcal{M}_\beta(\mathbb{R})) : I(\omega) \leq 1 \} \), where \( I(\omega) \) is given by (16).

**Theorem 6.** Process \( \{\tilde{Z}^\epsilon_t\} \) formed by FVP process, \( \{\mu^\epsilon_t\} \) in (28) is relatively compact in \( C([0, 1]; \mathcal{P}_\beta(\mathbb{R})) \) with set of limit points being \( L_3 := \{ \omega \in C([0, 1]; \mathcal{P}_\beta(\mathbb{R})) : I(\omega) \leq 1 \} \), where \( I(\omega) \) is given by (17).

### 3 LIL for Class of SPDEs

Our aim in this section is to prove Theorem 4. To prove the relative compactness of \( Z^t_t(y) \), we show its tightness in variables \( t \) and \( \epsilon \) since our space is \( C([0, 1]; \mathbb{B}_\beta) \). We use the following classical theorem.
Theorem 7 (Theorem 12.3 in [3]). The sequence \( \{X_n\} \) is tight in \( C([0,1];\mathbb{R}) \), if it satisfies these two conditions:

(i) The sequence \( \{X_n(0)\} \) is tight

(ii) There exist constants \( \gamma \geq 0 \) and \( \alpha > 1 \) and a nondecreasing, continuous function \( F \) on \([0,1]\) such that

\[
P(|X_n(t_2) - X_n(t_1)| \geq \lambda) \leq \frac{1}{\lambda^\gamma} |F(t_2) - F(t_1)|^\alpha
\]  

holds for all \( t_1, t_2 \) and \( n \) and all positive \( \lambda \).

For our result, we need the subsequent lemma, proof of which is identical to the proof of Lemma 1 in [22].

Lemma 1. Let \( Z_\epsilon^t(y) \) be the unique solution to SPDE (8), then for any \( p \geq 1, \epsilon > 0 \) and \( T > 0 \), there exists a positive constant \( K \) such that,

\[
\sup_{\epsilon > 0} \mathbb{E} \left( \sup_{0 \leq t \leq T} \int_{\mathbb{R}} Z_\epsilon^t(x)^2 e^{-2\beta|x|^2} dx \right)^p \leq K.
\]  

(30)

Theorem 8. The process \( \{Z_\epsilon^t\} \) indexed by \( \epsilon \) takes values in \( C([0,1];\mathbb{B}_\beta) \) and forms a tight family when \( \epsilon \) is in a small neighborhood \((0, \epsilon_0)\).

Proof. It was shown in Lemma 3 of [22] that \( v_\epsilon^t \) defined by (11) takes values in \( C([0,1];\mathbb{B}_\beta) \). Also as noted earlier, we have the existence and uniqueness of solutions to SPDE \( Z_\epsilon^t(y) \) and may use its mild solution given by,

\[
Z_\epsilon^t(y) := \frac{1}{\sqrt{2 \log \log \frac{1}{\epsilon}}} \int_0^t \int_U P_{t-s} G^\epsilon_s(a,y,Z^\epsilon_s(y)) W(dads)
\]  

(31)

where \( P_{t-s} \) is the Brownian semigroup. Let \( \epsilon > 0 \) and \( y \in \mathbb{R} \) be fixed and \( t_1, t_2 \in [0,1] \) be arbitrary with \( t_1 < t_2 \). For \( n > 8 \), we proceed as follows,

\[
\mathbb{E} \left| Z^\epsilon_{t_2}(x) - Z^\epsilon_{t_1}(x) \right|^n
\]

\[
= \mathbb{E} \left| \frac{1}{\sqrt{2 \log \log \frac{1}{\epsilon}}} \int_0^{t_2} \int_U \frac{1}{\sqrt{2\pi (t_2 - s)}} e^{-\frac{|x-y|^2}{2(t_2 - s)}} G^\epsilon_{s}(a,y,Z^\epsilon_{s}(y)) dy W(dads) \right|

- \frac{1}{\sqrt{2 \log \log \frac{1}{\epsilon}}} \int_0^{t_1} \int_U \frac{1}{\sqrt{2\pi (t_1 - s)}} e^{-\frac{|x-y|^2}{2(t_1 - s)}} G^\epsilon_{s}(a,y,Z^\epsilon_{s}(y)) dy W(dads)
\]  

\[
\left. \right|^n
\]
For a better presentation, let $K_0 := \frac{1}{\sqrt{2\log \log \frac{t}{\sqrt{2\pi}}}}$ and $G_s^t(a, y) := G_s^t (a, y, Z_s^t(y))$.

We will call the first convolution integral $I(t_2, t_2)(x)$, where the first $t_2$ appears in the upper limit of the integral and the second is the time parameter in the Gaussian density. Similarly, the second integral is denoted as $I(t_1, t_2)(x)$. Using this notation we have,

$$
\mathbb{E} |Z_{t_2}^t(x) - Z_{t_1}^t(x)|^n
= \mathbb{E} |I(t_2, t_2)(x) - I(t_1, t_1)(x)|^n
\leq 2^{n-1}K_0 [\mathbb{E} |I(t_2, t_2)(x) - I(t_1, t_2)(x)|^n + \mathbb{E} |I(t_1, t_2)(x) - I(t_1, t_1)(x)|^n]
= 2^{n-1}K_0 (J_1 + J_2)
\leq 2^{n-1} (J_1 + J_2).
$$

As for $J_1$, applying the Burkholder-Davis-Gundy inequality yields,

$$
J_1 \leq \mathbb{E} \int_{t_1}^{t_2} \int_U \left( \int_{\mathbb{R}} \frac{1}{\sqrt{t_2 - s}} e^{-\frac{|x-y|^2}{2(t_2-s)}} G_s^t(a, y) dy \right)^2 \lambda(da) ds,
$$

where by Hölder’s inequality and condition (26),

$$
\int_{\mathbb{R}} \left( \int_{\mathbb{R}} \frac{1}{\sqrt{t_2 - s}} e^{-\frac{|x-y|^2}{2(t_2-s)}} G_s^t(a, y) dy \right)^2 \lambda(da) \leq K \int_{\mathbb{R}} \frac{1}{t_2 - s} e^{-\frac{|x-y|^2}{2(t_2-s)}} \int_{\mathbb{R}} G_s^t(a, y)^2 e^{-2\beta_1 |y|} dy \lambda(da)
\leq K \int_{\mathbb{R}} \frac{1}{t_2 - s} e^{-\frac{|x-y|^2}{t_2-s}} e^{2\beta_1 |y|} \int_{\mathbb{R}} \left( 1 + 2 \epsilon \log \log \frac{1}{\epsilon} \right) Z_s^t(y)^2 e^{2\beta_0 |y|} e^{-2\beta_1 |y|} dy
$$

where $\beta_0 < \beta_1 \leq 1/2$. Moreover, note that

$$
\int_{t_1}^{t_2} \int_{\mathbb{R}} \frac{1}{t_2 - s} e^{-\frac{|x-y|^2}{2(t_2-s)}} e^{2\beta_1 |y|} dy ds \leq K \int_{t_1}^{t_2} \int_{\mathbb{R}} \frac{1}{2\sqrt{\pi (t_2-s)}} p_{t_2-s}(x-y) e^{2\beta_1 |y|} dy ds
\leq K e^{2\beta_1 |x|} \int_{t_1}^{t_2} \frac{ds}{\sqrt{t_2-s}}
\leq K e^{2\beta_1 |x|} \sqrt{t_2 - t_1}
$$

therefore, using Lemma 1

$$
J_1 \leq K \mathbb{E} \int_{t_1}^{t_2} \int_{\mathbb{R}} \frac{1}{t_2 - s} e^{-\frac{|x-y|^2}{t_2-s}} e^{2\beta_1 |y|} dy \int_{\mathbb{R}} \left( 2 \epsilon \log \log \frac{1}{\epsilon} \right) Z_s^t(y)^2 e^{-2\beta_1 |y|} dy ds \leq K e^{n\beta_1 |x|} \left| t_2 - t_1 \right|^\frac{2}{n} \int_{\mathbb{R}} \left( 2 \epsilon \log \log \frac{1}{\epsilon} \right) Z_s^t(y)^2 e^{-2\beta_1 |y|} dy ds
\leq K e^{n\beta_1 |x|} \left| t_2 - t_1 \right|^\frac{2}{n} \int_{\mathbb{R}} \left( 2 \epsilon \log \log \frac{1}{\epsilon} \right) Z_s^t(y)^2 e^{-2\beta_1 |y|} dy ds
$$

(33)
where $K$ is independent of $\epsilon$. We continue by estimating $J_2$. Denote,

$$\Delta p(t_2, t_1) := p_{t_2-s}(x-y) - p_{t_1-s}(x-y), \quad (34)$$

then,

$$J_2 = E \left| \int_0^{t_1} \int_\mathbb{R} \Delta p(t_2, t_1) G_s^e(a, y) dy W(da) ds \right|^n$$

$$\leq E \left[ \int_0^{t_1} \int_\mathbb{R} \left( \Delta p(t_2, t_1) \right)^2 e^{2\beta_1|y|} dy \int_\mathbb{R} G_s^e(a, y)^2 e^{-2\beta_1|y|} dy \lambda(da) ds \right]^{1/2}$$

$$\leq K \left| \int_0^{t_1} \int_\mathbb{R} \left( \Delta p(t_2, t_1) \right)^2 e^{2\beta_1|y|} dy ds \right|^{1/2}$$

where steps similar to those taken for estimating $J_1$ were applied. It can be seen that for $0 < \alpha \leq 1/2$,

$$\left( \Delta p(t_2, t_1) \right)^2 = |p_{t_2-s}(x-y) - p_{t_1-s}(x-y)|^\alpha |p_{t_2-s}(x-y) - p_{t_1-s}(x-y)|^{2-\alpha} \quad (35)$$

$$\leq 2^{1-\alpha} |p_{t_2-s}(x-y) - p_{t_1-s}(x-y)|^\alpha (p_{t_2-s}(x-y)^{2-\alpha} + p_{t_1-s}(x-y)^{2-\alpha})$$

Also we can bound $\Delta p(t_2, t_1)$ by

$$\frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{t_2 - s}} e^{-\frac{|x-y|^2}{2(t_2 - s)}} - e^{-\frac{|x-y|^2}{2(t_1 - s)}} + \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{t_2 - s}} - \frac{1}{\sqrt{t_1 - s}}$$

$$=: I_1 + I_2 \quad (36)$$

Using this in (35) we obtain,

$$\left( \Delta p(t_2, t_1) \right)^2 \leq K |I_1 + I_2|^{\alpha} (p_{t_2-s}(x-y)^{2-\alpha} + p_{t_1-s}(x-y)^{2-\alpha})$$

hence,

$$J_2 \leq K E \left| \int_0^{t_1} \int_\mathbb{R} |I_1|^{\alpha} p_{t_2-s}(x-y)^{2-\alpha} e^{2\beta_1|y|} dy ds \right|$$

$$+ \int_0^{t_1} \int_\mathbb{R} |I_2|^{\alpha} p_{t_2-s}(x-y)^{2-\alpha} e^{2\beta_1|y|} dy ds$$

$$+ \int_0^{t_1} \int_\mathbb{R} |I_1|^{\alpha} p_{t_1-s}(x-y)^{2-\alpha} e^{2\beta_1|y|} dy ds$$

$$+ \int_0^{t_1} \int_\mathbb{R} |I_2|^{\alpha} p_{t_1-s}(x-y)^{2-\alpha} e^{2\beta_1|y|} dy ds$$

$$= K E |J_{2,1} + J_{2,2} + J_{2,3} + J_{2,4}|^2$$
We first bound $I_1$ as follows. Using the fact that for $f(u) = e^{-\frac{u^2}{2}}$ its derivative is $f'(u) = -\frac{u}{2}e^{-\frac{u^2}{2}}$ we obtain,

$$|f\left(\frac{1}{t_2-s}\right) - f\left(\frac{1}{t_1-s}\right)| \leq \frac{|x-y|^2}{2} \left|\frac{1}{t_2-s} - \frac{1}{t_1-s}\right|$$

$$= \frac{|x-y|^2}{2} \frac{|t_2-t_1|}{(t_2-s)(t_1-s)}.$$  

Hence,

$$I_1 \leq \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{t_2-s}} \frac{|x-y|^2}{2} \frac{|t_2-t_1|}{(t_2-s)(t_1-s)}. \quad (37)$$

In particular,

$$J_{2,1}$$

\[ \leq K \int_0^{t_1} \int_\mathbb{R} \frac{|x-y|^{2\alpha}}{(2\pi (t_2-s))^{\frac{d}{2}} (t_2-s)^\alpha (t_1-s)^\alpha} |t_2-t_1|^\alpha p_{t_2-s}(x-y)^{2-\alpha} e^{2\beta_1|x|} dy ds \]

\[ \leq K \int_0^{t_1} \int_\mathbb{R} \frac{|t_2-t_1|^\alpha}{(t_2-s)^{\frac{d}{2}} (t_1-s)^\alpha} |x-y|^{2\alpha} p_{t_2-s}(x-y)^{2-\alpha} e^{2\beta_1|x|} dy ds \]

\[ \leq K \int_0^{t_1} \int_\mathbb{R} \frac{|t_2-t_1|^\alpha}{(t_2-s)^{\frac{d}{2}} (t_1-s)^\alpha} |x-y|^{2\alpha} p_{t_2-s}(x-y)^{2-\alpha} e^{\beta_1|x|} dy ds \]

\[ \leq K \int_0^{t_1} \int_\mathbb{R} \frac{|t_2-t_1|^\alpha}{(t_2-s)^{\frac{d}{2}+\alpha} (t_1-s)^\alpha} e^{2\beta_1|x|} ds. \]

So noting the assumption $t_1 < t_2$, we arrive at,

$$\leq Ke^{2\beta_1|x|} |t_2-t_1|^\alpha \int_0^{t_1} (t_1-s)^{-(\frac{d}{2}+2\alpha)} ds$$

$$\leq Ke^{2\beta_1|x|} |t_2-t_1|^\alpha$$

if $2\alpha < 1/2$. Similarly for $J_{2,3}$, we have,

$$J_{2,3} \leq Ke^{2\beta_1|x|} |t_2-t_1|^\alpha$$

if $2\alpha < 1/2$. To determine a bound for $J_{2,2}$ and $J_{2,4}$, we have for $i, j = 1, 2$
with \(i \neq j\),

\[
\int_{\mathbb{R}} \left| \frac{1}{\sqrt{t_1 - s}} - \frac{1}{\sqrt{t_2 - s}} \right|^\alpha p_{t_i - s}(x - y)^{2 - \alpha} e^{2\beta_1 |y|} dy \\
\leq K \int_{\mathbb{R}} \left( \frac{t_2 - t_1}{(t_1 - s) \sqrt{t_2 - s} + (t_2 - s) \sqrt{t_1 - s}} \right)^\alpha \left( \frac{1}{\sqrt{t_1 - s}} \right)^{1 - \alpha} p_{t_i - s}(x - y) e^{2\beta_1 |y|} dy \\
\leq K \frac{|t_2 - t_1|^{\alpha} e^{2\beta_1 |x|}}{(t_j - s)^{\alpha}(t_i - s)^{\frac{\alpha}{2}}}
\]

and

\[
\int_0^{t_1} K \frac{|t_2 - t_1|^{\alpha} e^{2\beta_1 |x|}}{(t_j - s)^{\alpha}(t_i - s)^{\frac{\alpha}{2}}} ds \leq K e^{2\beta_1 |x|} |t_2 - t_1|^{\alpha} \int_0^{t_1} (t_1 - s)^{-(\alpha + \frac{1}{2})} ds \\
\leq K |t_2 - t_1|^{\alpha}
\]

for \(\alpha < 1/2\). Therefore, we need \(\alpha < 1/4\) for \(J_{2,1}\) and \(J_{2,3}\) and \(\alpha < 1/2\) for \(J_{2,2}\) and \(J_{2,4}\). Thus, for \(0 < \alpha < 1/4\),

\[
J_2 \leq K |t_2 - t_1|^{\frac{\alpha n}{2}}
\]

where \(K\) is independent of \(\epsilon\). Furthermore, noting the bound for \(J_1\) in \(33\) we confirm our assumption of \(n > 8\) required to satisfy condition \(29\).

To prove that the limit set of \(\{Z_t^\epsilon\}\) is \(L_1\) given in Theorem 4, we recall the following result which is proved in [29].

**Theorem 9** (Theorem 4 in [29]). Consider the SDE,

\[
X_t^\epsilon = x_0 + \sum_{j=1}^{k} \int_0^t \epsilon \sigma_j(t, s, X_s^\epsilon) \, dW_s^j + \int_0^t b(t, s, X_s^\epsilon) \, ds
\]

where \(x_0 \in \mathbb{R}^d, t \in [0, T]\) and \(\{X^\epsilon, \epsilon > 0\}\) takes values in \(\mathcal{C}([0, T]; \mathbb{R}^d)\) with controlled PDE,

\[
S(h)_t = x_0 + \sum_{j=1}^{k} \int_0^t \sigma_j(t, s, S(h)_s) \, dW_s^j + \int_0^t b(t, s, S(h)_s) \, ds.
\]

Suppose functions \(b(t, s, x), \sigma_j(t, s, x), j = 1, ..., k\) are bounded, measurable, Lipschitz in \(x\), \(\alpha\)-Hölder continuous in \(t\), and there exists a constant \(K\) such
that,
\[ \sum_{j=1}^{k} |\sigma_j(t, s, x) - \sigma_j(r, s, x) - \sigma_j(t, s, y) + \sigma_j(r, s, y)| \leq K |t - r|^\gamma |x - y| \] (38)

where \( t, r \geq s, x, y \in \mathbb{R}^d \) and \( 0 < \gamma \leq 1 \). Then, for any \( h \) in the Cameron Martin space, \( \mathcal{H} \), and \( R, \rho > 0 \), there exists \( \eta > 0 \) and \( \epsilon_0 > 0 \) such that
\[ P \left( \|X^\epsilon - S(h)\|_\infty > \rho, \|\epsilon W - h\|_\infty < \eta \right) \leq \exp \left( -\frac{R}{\epsilon^2} \right) \] (39)

for any \( \epsilon \in (0, \epsilon_0] \).

Note that the noise in SPDE (23) is not Lipschitz continuous and does not satisfy (38). Here we modify the proof of the above theorem to match our setting. Based on our process and setup, we prove that for \( h \in L^2 ([0, 1] \times U, ds\lambda(da)) \),
\[ P \left( \|Z^\epsilon_t - S_t(h, y)\|_\infty > \rho, \left\| \frac{1}{\sqrt{2 \log \log \frac{1}{\epsilon}}} \right\|_\infty < \eta \right) \leq \exp \left( -2R \log \log \frac{1}{\epsilon} \right) \] (40)
in place of (39). We define,
\[ Y^\epsilon_t(y) = \frac{1}{\sqrt{2 \log \log \frac{1}{\epsilon}}} \int_0^t \int_U G^\epsilon_s(a, y, Y^\epsilon_s(y)) W(da) ds + \int_0^t \frac{1}{2} \Delta Y^\epsilon_s(y) ds \\
+ \int_0^t \int_U G^\epsilon_s(a, y, Y^\epsilon_s(y)) h_s(a) \lambda(da) ds. \] (41)

As shown in [29], the proof of Theorem 9 is reduced to attaining the result of the next proposition (similar to Proposition 5 of [29]) and applying the Girsanov’s theorem with no condition on the coefficients needed. Therefore, our aim is to prove this proposition based on our setting and hence obtain the result of Theorem 9 for our SPDE.

**Proposition 1.** For all \( h \in L^2 ([0, 1] \times U, ds\lambda(da)) \), \( R, \rho > 0 \), there exist \( \eta > 0, \epsilon_0 \in (0, 1] \), such that for all \( \epsilon \in (0, \epsilon_0] \),
\[ P \left( \|Y^\epsilon_t - S_t(h, y)\|_\infty > \rho, \left\| \frac{1}{\sqrt{2 \log \log \frac{1}{\epsilon}}} \right\|_\infty < \eta \right) \leq \exp \left( -2R \log \log \frac{1}{\epsilon} \right) \] (42)
First we need to verify the result of Lemma 1 for $Y_t^\epsilon(y)$ to use in estimates later.

**Lemma 2.** Suppose $Y_t^\epsilon(y)$ is the unique solution to SPDE (41), then for every $p \geq 1$, $\epsilon > 0$ and $T > 0$, there exists a positive constant $K$ such that,

$$
\sup_{\epsilon > 0} \mathbb{E} \left( \sup_{0 \leq t \leq T} Y_t^\epsilon(x)^2 e^{-2\beta|x|} \right)^p \leq K. \tag{43}
$$

**Proof.** Denote the Hilbert space, $L^2 \left( \mathbb{R}, e^{-2\beta|x|} dx \right)$ by $\chi_0$, defined by the inner product,

$$
\langle f, g \rangle_{\chi_0} = \int_{\mathbb{R}} f(x)g(x)e^{-2\beta|x|} \, dx \tag{44}
$$

for $f, g \in L^2(\mathbb{R})$. We apply an induction argument as follows. Define the process $Y_t^{\epsilon,0}(x) = 0$ for all $t > 0$, $x \in \mathbb{R}$ and $\epsilon > 0$. For a complete orthonormal system, $\{f_j\}$ with $f_j \in C_\infty(\mathbb{R}) \cap \chi_0$ for all $j$, define,

$$
\left< Y_t^{\epsilon,n+1}, f_j \right>_{\chi_0} = -\frac{1}{\sqrt{2 \log \log \frac{1}{\epsilon}}} \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} G_s^\epsilon(a,y,Y_s^{\epsilon,n}(y)) f_j(y)e^{-2\beta|y|} \, dy \, W(da\, ds) \\
+ \int_0^t \left< \frac{1}{2} \Delta Y_t^{\epsilon,n+1}, f_j \right>_{\chi_0} \, ds \\
+ \int_0^t \int_{\mathbb{R}} G_s^\epsilon(a,y,Y_s^{\epsilon,n}(y)) h_s(a)f_j(y)e^{-2\beta|y|} \, dy \, \lambda(da) \, ds
$$

Suppose (43) holds for $Y_t^{\epsilon,n}(x)$ and suppose,

$$
\sum_{k=0}^{2} \int_{\mathbb{R}} \left( Y_t^{\epsilon,n+1} \right)^{(k)}(x)^2 e^{-2\beta|x|} \, dx < \infty \tag{45}
$$
where \( f^{(k)}(x) \) denotes the \( k\text{th} \) derivative of \( f \). By Ito’s formula,

\[
\left\langle Y^\epsilon_{t+1}, f_j \right\rangle^2_{X_0} = \frac{2}{\sqrt{2 \log \log \frac{1}{\epsilon}}} \int_0^t \left\langle Y^\epsilon_{s+1}, f_j \right\rangle_{X_0} \int_U \int_{\mathbb{R}} G^\epsilon_s(a, y, Y^\epsilon_{s+1}(y)) f_j(y) e^{-2\beta|y|} dy W(dads) \\
+ \int_0^t \left\langle Y^\epsilon_{s+1}, f_j \right\rangle_{X_0} \left\langle \Delta Y^\epsilon_{s+1}, f_j \right\rangle_{X_0} ds \\
+ 2 \int_0^t \left\langle Y^\epsilon_{s+1}, f_j \right\rangle_{X_0} \int_U \int_{\mathbb{R}} G^\epsilon_s(a, y, Y^\epsilon_{s+1}(y)) h_s(a) f_j(y) e^{-2\beta|y|} dy \lambda(da) ds \\
+ \frac{1}{2 \sqrt{2 \log \log \frac{1}{\epsilon}}} \int_0^t \int_U \left( \int_{\mathbb{R}} G^\epsilon_s(a, y, Y^\epsilon_{s+1}(y)) f_j(y) e^{-2\beta|y|} dy \right)^2 \lambda(da) ds
\]

Summing over \( j \) we obtain,

\[
\left\| Y^\epsilon_{t+1} \right\|^2_{X_0} = \frac{2}{\sqrt{2 \log \log \frac{1}{\epsilon}}} \int_0^t \int_U \left\langle Y^\epsilon_{s+1}, G^\epsilon_s(a, y, Y^\epsilon_{s+1}(y)) \right\rangle_{X_0} W(dads) \\
+ \int_0^t \left\langle Y^\epsilon_{s+1}, \Delta Y^\epsilon_{s+1} \right\rangle_{X_0} ds \\
+ 2 \int_0^t \int_U \left\langle Y^\epsilon_{s+1}, G^\epsilon_s(a, y, Y^\epsilon_{s+1}(y)) h_s(a) \right\rangle_{X_0} \lambda(da) ds \\
+ \frac{1}{2 \sqrt{2 \log \log \frac{1}{\epsilon}}} \int_0^t \int_U \int_{\mathbb{R}} G^\epsilon_s(a, y, Y^\epsilon_{s+1}(y))^2 e^{-2\beta|y|} dy \lambda(da) ds \quad (46)
\]
Applying Ito's formula again this time to \((46)\) gives,

\[
\|Y_{t}^{\epsilon,n+1}\|_{\chi_{0}}^{2p} = \frac{2p}{\sqrt{2\log\log\frac{1}{\epsilon}}} \int_{0}^{t} \|Y_{s}^{\epsilon,n+1}\|_{\chi_{0}}^{2(p-1)} \int_{U} \langle Y_{s}^{\epsilon,n+1}, G_{s}^{\epsilon} (a, y, Y_{s}^{\epsilon,n}(y)) \rangle_{\chi_{0}} W(dads) + p \int_{0}^{t} \|Y_{s}^{\epsilon,n+1}\|_{\chi_{0}}^{2(p-1)} \int_{U} \langle Y_{s}^{\epsilon,n+1}, \Delta Y_{s}^{\epsilon,n+1} \rangle_{\chi_{0}} ds
\]

\[+ 2p \int_{0}^{t} \int_{U} \|Y_{s}^{\epsilon,n+1}\|_{\chi_{0}}^{2(p-1)} \langle Y_{s}^{\epsilon,n+1}, G_{s}^{\epsilon} (a, y, Y_{s}^{\epsilon,n}(y)) h_{s}(a) \rangle_{\chi_{0}} \lambda(da)ds + \frac{p}{2\log\log\frac{1}{\epsilon}} \int_{0}^{t} \int_{\mathbb{R}} \|Y_{s}^{\epsilon,n+1}\|_{\chi_{0}}^{2(p-1)} \int_{\mathbb{R}} G_{s}^{\epsilon} (a, y, Y_{s}^{\epsilon,n}(y))^{2} e^{-2\beta|y|} dy \lambda(da)ds
\]

\[+ \frac{p(p-1)}{\log\log\frac{1}{\epsilon}} \int_{0}^{t} \|Y_{s}^{\epsilon,n+1}\|_{\chi_{0}}^{2(p-2)} \int_{U} \langle Y_{s}^{\epsilon,n+1}, G_{s}^{\epsilon} (a, y, Y_{s}^{\epsilon,n}(y)) \rangle_{\chi_{0}}^{2} \lambda(da)ds
\]

For simplicity of notation, let \(g(y) := e^{-2\beta|y|}\) and observe that,

\[
\langle Y_{s}^{\epsilon,n+1}, \Delta Y_{s}^{\epsilon,n+1} \rangle_{\chi_{0}} = \int_{\mathbb{R}} Y_{s}^{\epsilon,n+1}(y) \Delta Y_{s}^{\epsilon,n+1}(y) g(y) dy
\]

\[= \int_{\mathbb{R}} (Y_{s}^{\epsilon,n+1})''(y) Y_{s}^{\epsilon,n+1}(y) g(y) dy
\]

\[= -\int_{\mathbb{R}} (Y_{s}^{\epsilon,n+1})'(y) Y_{s}^{\epsilon,n+1}(y) g'(y) dy
\]

\[-\int_{\mathbb{R}} \left((Y_{s}^{\epsilon,n+1})'(y)\right)^{2} g(y) dy
\]

where,

\[
\int_{\mathbb{R}} (Y_{s}^{\epsilon,n+1})'(y) Y_{s}^{\epsilon,n+1}(y) g'(y) dy = -\int_{\mathbb{R}} (Y_{s}^{\epsilon,n+1}(y))^{2} g''(y) dy
\]

\[-\int_{\mathbb{R}} Y_{s}^{\epsilon,n+1}(y) (Y_{s}^{\epsilon,n+1})'(y) g'(y) dy
\]

which leads to,

\[-\int_{\mathbb{R}} (Y_{s}^{\epsilon,n+1})'(y) Y_{s}^{\epsilon,n+1}(y) g'(y) dy = \frac{1}{2} \int_{\mathbb{R}} (Y_{s}^{\epsilon,n+1}(y))^{2} g''(y) dy
\]

\[\leq K \int_{\mathbb{R}} (Y_{s}^{\epsilon,n+1}(y))^{2} g(y) dy = K \|Y_{s}^{\epsilon,n+1}\|_{\chi_{0}}^{2}.
\]
Therefore,
\[
\langle Y^{\varepsilon,n+1}_s, \Delta Y^{\varepsilon,n+1}_s \rangle_{\chi_0} \leq K \left\| Y^{\varepsilon,n+1}_s \right\|_{\chi_0}^2.
\]
Taking expectations of (47) and using the Burkholder-Davis-Gundy inequality on its first term, we arrive at,
\[
\mathbb{E} \sup_{0 \leq s \leq t} \left\| Y^{\varepsilon,n+1}_s \right\|_{\chi_0}^{2p} \leq K_1 \mathbb{E} \int_0^t \left\| Y^{\varepsilon,n+1}_s \right\|_{\chi_0}^{4(p-1)} \left( \int_U \left( \int Y^{\varepsilon,n+1}_s(y)G^{\varepsilon}_s(a,y,Y^{\varepsilon,n}_s(y)) L^{-2\beta\|y\|}dy \right)^2 \right) \lambda(da)ds + K_2 \mathbb{E} \int_0^t \left\| Y^{\varepsilon,n+1}_s \right\|_{\chi_0}^{2(p-1)} \left\| Y^{\varepsilon,n+1}_s \right\|_{\chi_0}^2 ds + K_3 \mathbb{E} \int_0^t \left\| Y^{\varepsilon,n+1}_s \right\|_{\chi_0}^{2(p-1)} \left( \int_U \left( \int Y^{\varepsilon,n+1}_s(y)G^{\varepsilon}_s(a,y,Y^{\varepsilon,n}_s(y)) \right)^2 e^{-2\beta\|y\|}dy \lambda(da)ds + K_4 \mathbb{E} \int_0^t \left\| Y^{\varepsilon,n+1}_s \right\|_{\chi_0}^{2(p-2)} \left( \int_U \left( \int Y^{\varepsilon,n+1}_s(y)G^{\varepsilon}_s(a,y,Y^{\varepsilon,n}_s(y)) e^{-2\beta\|y\|}dy \right)^2 \lambda(da)ds
\]
Notice that,
\[
\int_U \left( \int Y^{\varepsilon,n+1}_s(y)G^{\varepsilon}_s(a,y,Y^{\varepsilon,n}_s(y)) e^{-2\beta\|y\|}dy \right)^2 \lambda(da) = \int_U \left( \int Y^{\varepsilon,n+1}_s(y)e^{-\beta\|y\|}G^{\varepsilon}_s(a,y,Y^{\varepsilon,n}_s(y)) e^{-\beta\|y\|}dy \right)^2 \lambda(da) \leq \int_U \left\| Y^{\varepsilon,n+1}_s \right\|_{\chi_0}^{2} \left( \int U \left( \int Y^{\varepsilon,n+1}_s(y)G^{\varepsilon}_s(a,y,Y^{\varepsilon,n}_s(y)) e^{-2\beta\|y\|}dy \right)^2 \lambda(da) \leq K \left\| Y^{\varepsilon,n+1}_s \right\|_{\chi_0}^{2} \int U \left( 1 + 2e \log \log \frac{1}{\varepsilon} \right) Y^{\varepsilon,n}_s(y)^2 e^{-2\beta\|y\|}dy \leq K \left\| Y^{\varepsilon,n+1}_s \right\|_{\chi_0}^{2} (\left\| Y^{\varepsilon,n}_s \right\|_{\chi_0}^{2} + 1)
\]
by remembering that the symbol \( K \) denotes a suitable constant that can vary from line to line. In addition, for the third term recall that \( h \in \)
Thus, inequality (48) is simplified to,

\[ L_2(([0, 1] \times U, ds\lambda(da)) \text{ so,} \]

\[
K_E \int_0^t \|Y_{s,n}^{\epsilon,n+1}\|_{X_0}^{2(p-1)} \left( \int_U \left( \int_{Y_{s,n}^{\epsilon,n+1}(y)G_\epsilon(a, y, Y_{s,n}^{\epsilon,n}(y)) \lambda_c(a) \right) e^{-2|y| \|d_y \lambda(ds)ds} \right)^{\frac{1}{2}}
\]

\[
\leq K_E \int_0^t \|Y_{s,n}^{\epsilon,n+1}\|_{X_0}^{2(p-1)} \left( \int_U \left( \int_{Y_{s,n}^{\epsilon,n+1}(y)G_\epsilon(a, y, Y_{s,n}^{\epsilon,n}(y)) e^{-2|y| \|d_y \lambda(ds)ds} \right)^2 \lambda(ds) \right)^{\frac{1}{2}}
\]

\[
\leq K_E \int_0^t \|Y_{s,n}^{\epsilon,n+1}\|_{X_0}^{2(p-1)} \left( \int_U \left( \int_{Y_{s,n}^{\epsilon,n+1}(y)G_\epsilon(a, y, Y_{s,n}^{\epsilon,n}(y)) e^{-2|y| \|d_y \lambda(ds)ds} \right)^2 \lambda(ds) \right)^{\frac{1}{2}}
\]

Thus, inequality (48) is simplified to,

\[
\mathbb{E} \sup_{0 \leq s \leq t} \|Y_{s,n}^{\epsilon,n+1}\|_{X_0}^{2p} \leq K_E \int_0^t \|Y_{s,n}^{\epsilon,n+1}\|_{X_0}^{2p} ds
\]

\[
+ K_E \int_0^t \|Y_{s,n}^{\epsilon,n+1}\|_{X_0}^{4p-2} (\|Y_{s,n}^{\epsilon,n}\|_{X_0}^{2} + 1) ds
\]

\[
+ K_E \int_0^t \|Y_{s,n}^{\epsilon,n+1}\|_{X_0}^{2p-1} (\|Y_{s,n}^{\epsilon,n}\|_{X_0} + 1) ds
\]

\[
+ K_E \int_0^t \|Y_{s,n}^{\epsilon,n+1}\|_{X_0}^{2p-1} (\|Y_{s,n}^{\epsilon,n}\|_{X_0}^{2} + 1) ds
\]

Now the result of this lemma can be concluded by noting the induction hypothesis and using Gronwall’s inequality. \( \Box \)

In order to prove the result of proposition[11] we apply a time discretization of \( Y_t^\epsilon \). For \( n \in \mathbb{N}, i = 0, 1, ..., n \), let \( \Delta_t^n = \left[ t^n_i, t^n_{i+1} \right] \) then by the following two estimates we can obtain inequality (42).

\begin{align}
P \left( \|Y_t^\epsilon - Y_{t_i}^\epsilon\|_\infty > \mu \right) & \leq \exp \left( -2R \log \log \frac{1}{\epsilon} \right) \quad (49) \\
P \left( \|Y_t^\epsilon - S_t(h, y)\|_\infty > \rho, \|W\|_\infty < \eta, \|Y_t^\epsilon - Y_{t_i}^\epsilon\|_\infty \leq \mu \right) & \leq \exp \left( -2R \log \log \frac{1}{\epsilon} \right) \quad (50)
\end{align}
Thus, we aim to prove (49) and (50) by the following lemmas. To this end, we recall the theorem below, which is also shown in [29].

**Theorem 10** (Theorem 2 in [29]). Let \( \sigma : [0, T] \times [0, T] \times \Omega \to \mathbb{R}^d \times \mathbb{R}^k \) be a \( \mathcal{B}([0, T]) \otimes \mathcal{B}([0, T]) \otimes \mathcal{F} \)-measurable process satisfying,

i. \( \sigma(t, s) = 0 \) if \( t < s \)

ii. \( \sigma(t, s) \) is \( \mathcal{F}_s \)-measurable

iii. there exists a positive random variable \( \xi \) and \( \alpha \in (0, 2] \), such that for all \( t, r \in [0, T] \),

\[
\int_0^{\min\{t, r\}} |\sigma(t, s) - \sigma(r, s)|^2 \, ds \leq \xi |t - r|^\alpha \tag{51}
\]

then for any \( 0 < \beta \leq \min\{1, \alpha\} \), there exist positive constants \( K_\sigma, C_\sigma, K \) such that,

\[
P \left( \sup_{0 \leq t \leq T} \left| \int_0^t \sigma(t, s)dW_s^j \right| > L, \|\sigma\|_\infty \leq K_\sigma, \xi \leq C_\sigma \right) \leq \exp \left( -\frac{L^2}{(TK_\sigma^2 + T^\alpha C_\sigma)K} \right) \tag{52}
\]

for all \( L \geq 0, C_\sigma \geq 0 \) such that,

\[
\frac{L}{(T^{\alpha-\beta}C_\sigma + T^{1-\beta}K_\sigma^2)^{1/2}} \geq \max \left\{ 2^{1+\frac{11}{T}} \sqrt{\pi} \beta^{-\frac{1}{2}}, 2^{\frac{25}{11}} (1 + T) T^{\frac{\beta}{2}} \right\} \tag{53}
\]

We now continue by proving estimates (49) and (50).

**Lemma 3.** For all \( R > 0, \mu > 0 \) there exists \( n_0 \in \mathbb{N} \) such that for all \( n \geq n_0 \), and \( \epsilon \in (0, 1] \),

\[
P \left( \left\| Y^\epsilon_t - Y^\epsilon_{t^n} \right\|_\infty > \mu \right) \leq \exp \left( -2R \log \log \frac{1}{\epsilon} \right) . \tag{54}
\]
Proof. Let $n$ be fixed and $t \in \Delta^n_i$ then using notation (34), we estimate,
\[
\left| Y_t^i - Y_{t_i}^n \right| \leq K \left| \frac{1}{\sqrt{2 \log \log t}} \int_{t_i}^{t} \int_{U} P_{t-s} G_s^e(a, y, Y_s^e(y)) W(da) ds \right|
\]
\[
+ K \left| \frac{1}{\sqrt{2 \log \log t}} \int_{t_i}^{t} \int_{U} \int_{\mathbb{R}} \Delta p(t, t_i^n) G_s^e(a, x, Y_s^e(x)) dx W(da) ds \right|
\]
\[
+ K \left| \int_{t_i}^{t} \int_{U} \int_{R} \Delta p(t, t_i^n) G_s^e(a, x, Y_s^e(x)) h_s(a) \lambda(da) ds \right|
\]
\[
+ K \left| \int_{t_i}^{t} \int_{U} \int_{R} \Delta p(t, t_i^n) G_s^e(a, x, Y_s^e(x)) h_s(a) dx \lambda(da) ds \right|
\]
\[
= I_1 + I_2 + I_3 + I_4
\]
leading to,
\[
P \left( \left\| Y_t^i - Y_{t_i}^n \right\|_{\infty} > \mu \right) = P \left( \sup_{t \in \Delta^n_i} \left| Y_t^i - Y_{t_i}^n \right| > \mu \right) = \sum_{i=1}^{4} P \left( I_i > \frac{\mu}{4} \right)
\]
By techniques used in proving estimates in theorem 8 and noting the domain $L^2([0,1] \times U, ds \lambda(da))$ of $h_s(a)$, we have,
\[
P \left( I_3 > \frac{\mu}{4} \right) \leq K \mathbb{E} \left| I_3 \right|^2 \leq K \left| t - t_i^n \right|^2
\]
and
\[
P \left( I_4 > \frac{\mu}{4} \right) \leq K \mathbb{E} \left| I_4 \right|^2 \leq K \left| t - t_i^n \right|^2
\]
giving,
\[
P \left( I_3 > \frac{\mu}{4} \right) + P \left( I_4 > \frac{\mu}{4} \right) \leq K \left| t - t_i^n \right|^2
\]
which for sufficiently large $n$ tends to zero and can be disregarded. Contin-
uing, we obtain,

\[
P \left( \left\| Y_t^\varepsilon - Y_{t_i}^\varepsilon \right\|_\infty > \mu \right) 
\leq P \left( I_1 > \frac{\mu}{4} \right) + P \left( I_2 > \frac{\mu}{4} \right) 
\leq K \sum_{i=1}^{n} P \left( \sup_{t \in \Delta t_i} K \left| \frac{1}{\sqrt{2 \log \log \frac{1}{\varepsilon}}} \int_{t_i}^{t} \int_U \left| P_{t-s} G_s^\varepsilon(a, y, Y_s^\varepsilon(y)) W(dads) \right| > \frac{\mu}{4} \right) 
+ P \left( \sup_{t \in \Delta t_i} K \left| \frac{1}{\sqrt{2 \log \log \frac{1}{\varepsilon}}} \int_{t_i}^{t} \int_U \int_{\mathbb{R}} \Delta p(t, t_i^n) G_s^\varepsilon(a, x, Y_s^\varepsilon(x)) dx W(dads) \right| > \frac{\mu}{4} \right) 
= \sum_{i=1}^{n} P(A_i^1) + P(A_i^2) 
\]

We now try to apply Theorem 10 to obtain the needed upperbounds for \( P(A_i^1) \) and \( P(A_i^2) \). For \( KI_1^2 \),

\[
\sigma(t, s) = \frac{1}{\sqrt{2 \log \log \frac{1}{\varepsilon}}} \int_{\mathbb{R}} p_{t-s}(x-y) G_s^\varepsilon(a, x, Y_s^\varepsilon(x)) dx \tag{55}
\]

where \( p_{t-s}(x-y) = \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{|x-y|^2}{2(t-s)}} \) is not real-valued for \( t < s \) and so these valued for \( t \) are excluded from our domain. Also \( \sigma(t, s) \) is \( \mathcal{F}_s \)-adapted since \( Y_s^\varepsilon(x) \) is \( \mathcal{F}_s \) adapted. In addition, \( \sigma(t, s) \) is bounded a.s. since,

\[
\mathbb{E} \int_U \sigma(t, s) \lambda(da) = \frac{1}{\sqrt{2 \log \log \frac{1}{\varepsilon}}} \mathbb{E} \int_{\mathbb{R}} \int_U p_{t-s}(x-y) G_s^\varepsilon(a, x, Y_s^\varepsilon(x)) dx \lambda(da) \leq K \sigma \tag{56}
\]

by applying Hölder’s inequality and Lemma 2 as in previous estimates. Moreover, recall from the proof of Theorem 8,

\[
J_2 = \mathbb{E} \left| \int_0^{t_1} \int_U \int_{\mathbb{R}} \Delta p(t_2, t_1) G_s^\varepsilon(a, y) dy W(dads) \right|^n 
\leq K |t_2 - t_1|^\frac{n}{2}. 
\]
for $0 < \alpha \leq 1/4$. Therefore,

$$
\mathbb{E} \int_0^{\min\{t,r\}} \int_U |\sigma(t,s) - \sigma(r,s)|^2 \lambda(da) ds 
\leq \mathbb{E} \int_0^{\min\{t,r\}} \int_U \frac{1}{\sqrt{2 \log \log \frac{1}{\epsilon}}} \int_\mathbb{R} \Delta p(t,r) G_s^x(a,x,Y_s^x(x)) dx \left| \lambda(da) ds \right|^2
\leq C_\sigma |t - r|
$$

where here we have, $n = 2$. Thus, according to theorem 10, for $\beta < 1$,

$$P(A_1^i) \leq \exp \left( -\frac{\mu^2 (2 \log \log \frac{1}{\epsilon})}{16 \left( \frac{1}{n} K_\sigma^2 + \frac{1}{n} C_\sigma \right)} \right)
$$

whenever,

$$\frac{\mu \sqrt{2 \log \log \frac{1}{\epsilon}}}{4 \left( \left( \frac{1}{n} \right)^{1-\beta} C_\sigma + \left( \frac{1}{n} \right)^{1-\beta} K_\sigma^2 \right)^{1/2}} \geq \max \left\{ 2^{\frac{11}{2}} \sqrt{\pi} \beta^{-\frac{1}{2}}, 2^{\frac{25}{4}} \left( 1 + \frac{1}{n} \right) \left( \frac{1}{n} \right)^{\frac{1}{2}} \right\}
$$

where we have used the length of $\Delta^n_{\tilde{t}}$, $1/n$ with $T = 1$ and bound $K_\sigma$ in (56). Along the same lines of reasoning, we have for $\beta < 1$,

$$P(A_2^i) \leq \exp \left( -\frac{\mu^2 (2 \log \log \frac{1}{\epsilon})}{16 \left( \frac{1}{n} K_\sigma^2 + \frac{1}{n} C_\sigma \right)} \right)
$$

whenever, (58) holds with analogous bounds $K_\sigma$ and $C_\sigma$ and so we obtain the result for large enough $n$.

**Lemma 4.** For all $R > 0$, $\rho > 0$, $n \in \mathbb{N}$, there exist $\mu_0, \eta_0 > 0$ such that for all $\mu \leq \mu_0, \eta \leq \eta_0$, and $\epsilon \in (0,1]$,

$$P \left( \|Y_t^\epsilon - S_t(h,y)\|_\infty > 2\rho, \frac{1}{\sqrt{2 \log \log \frac{1}{\epsilon}}} W < \eta, \|Y_t^\epsilon - Y_t^{\epsilon_0}\|_\infty \leq \mu \right) \leq 2 \exp \left( -2 R \log \log \frac{1}{\epsilon} \right)$$

**Proof.** For the simplicity of notation, we let,

$$\Delta G_s^x(v(x),w(x)) := G_s^x(a,x,v(x)) - G_s^x(a,x,w(x)).$$
Using the mild solution of \( S_t(h, y) \) based on its uniqueness of solutions we have,

\[
\|Y^*_t - S_t(h, y)\|_\infty^2 \leq K \left( \sup_{0 \leq t \leq 1} \frac{1}{\sqrt{2 \log \log \frac{1}{\epsilon}}} \int_0^t \int_U P_{t-s} G_s(a, y, Y^*_s(y)) W(dads) \right)^2 \\
+ K \left( \sup_{0 \leq t \leq 1} \int_0^t P_{t-s} \Delta G_s(Y^*_s(y), 0) h_s(a) \lambda(da) ds \right)^2.
\]

Denoting the integral in the second term on the right side as \( I_t \), we can bound \( I_t^2 \) for all \( t \) as follows:

\[
I_t^2 \leq \left[ \int_0^t \left( \int_U h_s^2(a) \lambda(da) \right)^{1/2} \left\{ \int_U \left( \int_\mathbb{R} p_{t-s}(y - x)e^{2\beta|x|} dx \right) \{ \int_{\mathbb{R}} (G_s^*(a, x, Y^*_s(x)) - G_s^*(a, x, 0))^2 e^{-2\beta|x|} dx \} \lambda(da) \right\}^{1/2} ds \right] \leq \left[ \int_0^t Ke^{\beta|y|} \frac{1}{(\sqrt{t-s})^{1/2}} \left( \int_U h_s^2(a) \lambda(da) \right)^{1/2} (2\epsilon \log \log(1/\epsilon))^{1/2} \left( \int_{\mathbb{R}} |Y_s^*(x)| e^{-2\beta|x|} dx \right)^{1/2} \right]^2 \\
\leq K^2 e^{\beta|y|} (2\epsilon \log \log(1/\epsilon)) \sup_{0 \leq s \leq 1} \left( \int_{\mathbb{R}} |Y_s^*(x)| e^{-2\beta|x|} dx \right) \left[ \int_0^t \frac{1}{(\sqrt{t-s})^{1/2}} \left( \int h_s(a)^2 \lambda(da) \right)^{1/2} ds \right]^2 \\
\leq K(2\epsilon \log \log(1/\epsilon)) \sup_{0 \leq s \leq 1} \left( \int_{\mathbb{R}} |Y_s^*(x)| e^{-2\beta|x|} dx \right).
\]

If we refer to the above bound as

\[ K(2\epsilon \log \log(1/\epsilon)) A, \]

then

\[
P(|K(2\epsilon \log \log(1/\epsilon)) A| > \rho) \leq \frac{K}{\rho}(2\epsilon \log \log(1/\epsilon) E(|A|)) \\
\leq C \epsilon \log \log(1/\epsilon)
\]
for a suitable constant $C$. For $\epsilon$ small enough,

$$C \epsilon \log \log(1/\epsilon) \leq e^{-2R \log \log(1/\epsilon)}.$$ 

by L'Hôpital's rule. Thus, the inequality (59) is equivalent to showing that

$$P\left( \sup_{0 \leq t \leq 1} \frac{1}{\sqrt{2 \log \log \frac{1}{\epsilon}}} \left| \int_0^t \int_U P_{t-s} G_s^\epsilon(a, y, Y^\epsilon_s(y)) W(dads) \right| > \rho, \right.$$ 

$$\left\| \frac{1}{\sqrt{2 \log \log \frac{1}{\epsilon}}} W \right\|_\infty < \eta, \left\| Y_t^\epsilon - Y_{t_1}^\epsilon \right\|_\infty \leq \mu \right) \leq \exp \left( -2R \log \log \frac{1}{\epsilon} \right)$$

Notice that,

$$P\left( \sup_{0 \leq t \leq 1} \frac{1}{\sqrt{2 \log \log \frac{1}{\epsilon}}} \left| \int_0^t \int_U P_{t-s} G_s^\epsilon(a, y, Y^\epsilon_s(y)) W(dads) \right| > \rho, \right.$$ 

$$\left\| \frac{1}{\sqrt{2 \log \log \frac{1}{\epsilon}}} W \right\|_\infty < \eta, \left\| Y_t^\epsilon - Y_{t_1}^\epsilon \right\|_\infty \leq \mu \right) \leq P\left( \sup_{0 \leq t \leq 1} \frac{1}{\sqrt{2 \log \log \frac{1}{\epsilon}}} \left| \int_0^t \int_U P_{t-s} \Delta G_s^\epsilon(Y_s^\epsilon(y), Y_{s_1}^\epsilon(y)) W(dads) \right| > \frac{\rho}{2}, \right.$$ 

$$\left\| Y_t^\epsilon - Y_{t_1}^\epsilon \right\|_\infty \leq \mu \right)$$

$$+ P\left( \sup_{0 \leq t \leq 1} \frac{1}{\sqrt{2 \log \log \frac{1}{\epsilon}}} \left| \int_0^t \int_U P_{t-s} G_s^\epsilon(a, y, Y^\epsilon_s(y)) W(dads) \right| > \frac{\rho}{2}, \right.$$ 

$$\left\| \frac{1}{\sqrt{2 \log \log \frac{1}{\epsilon}}} W \right\|_\infty < \eta \right) \right) = P_1 + P_2$$

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For $P_1$ we have,

$$
E \int_0^{\min\{t,r\}} \int_{\mathbb{R}} \frac{1}{\sqrt{2 \log \log \frac{1}{\epsilon}}} \int_{\mathbb{R}} p_{t-s}(x-y) \Delta G_{s}^{\varepsilon} \left(Y_s^{\varepsilon}(x), Y_{s_n}^{\varepsilon}(x)\right) dx
$$

$$
- \frac{1}{\sqrt{2 \log \log \frac{1}{\epsilon}}} \int_{\mathbb{R}} p_{r-s}(x-y) \Delta G_{s}^{\varepsilon} \left(Y_s^{\varepsilon}(x), Y_{s_n}^{\varepsilon}(x)\right) dx \lambda(da)ds
$$

$$
= E \int_0^{\min\{t,r\}} \int_{\mathbb{R}} \frac{1}{\sqrt{2 \log \log \frac{1}{\epsilon}}} \Delta p(t,r) \Delta G_{s}^{\varepsilon} \left(Y_s^{\varepsilon}(x), Y_{s_n}^{\varepsilon}(x)\right) dx \lambda(da)ds
$$

$$
\leq K E \int_0^{\min\{t,r\}} \int_{\mathbb{R}} \frac{1}{\sqrt{2 \log \log \frac{1}{\epsilon}}} \Delta p(t,r) G_{s}^{\varepsilon} \left(a, x, Y_s^{\varepsilon}(x)\right) dx \lambda(da)ds
$$

$$
+ K E \int_0^{\min\{t,r\}} \int_{\mathbb{R}} \frac{1}{\sqrt{2 \log \log \frac{1}{\epsilon}}} \Delta p(t,r) G_{s}^{\varepsilon} \left(a, x, Y_{s_n}^{\varepsilon}(x)\right) dx \lambda(da)ds
$$

$$
\leq C_\sigma |t-r|
$$

by applying Hölder’s inequality similar to (57). The other two conditions of theorem 9 can be verified as well and it can be shown that

$$
E \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{\sqrt{2 \log \log \frac{1}{\epsilon}}} p_{t-s}(x-y) \Delta G_{s}^{\varepsilon} \left(Y_s^{\varepsilon}(x), Y_{s_n}^{\varepsilon}(x)\right) dx \lambda(da) \leq K_\sigma
$$

so there exist positive constants, $C_\sigma, K_\sigma, K$ such that

$$
P_1 \leq \exp \left( - \frac{\rho^2 \left( 2 \log \log \frac{1}{\epsilon} \right)}{4 (K_\sigma + C_\sigma)} K \right)
$$

for $\beta \leq 1$ whenever,

$$
\frac{\rho \sqrt{2 \log \log \frac{1}{\epsilon}}}{2 (C_\sigma + K_\sigma^2)^{1/2}} \geq \max \left\{ 2^{\frac{11}{12}} \sqrt{\pi} \beta^{-\frac{1}{2}}, 2^{\frac{29}{30}} \right\}
$$  \hspace{1cm} (60)
Moreover, for $P_2$ we see that,

$$
P_2 = P \left( \sup_{0 \leq t \leq 1} \left| \frac{1}{\sqrt{2 \log \log \frac{1}{\varepsilon}}} \sum_{i=1}^{n} \int_{[0,t] \cap \Delta_i^n} \int_{U} P_{t-s} G_s^e \left( a, y, Y_{s_i}^\varepsilon (y) \right) W(dads) \right| > \frac{\rho}{2}, \right.

\left. \right| \frac{1}{\sqrt{2 \log \log \frac{1}{\varepsilon}}} W \left. \right|_\infty < \eta \right) \right)

\leq \sum_{i=1}^{n} P \left( \sup_{0 \leq t \leq 1} \left| \frac{1}{\sqrt{2 \log \log \frac{1}{\varepsilon}}} \int_{[0,t] \cap \Delta_i^n} \int_{U} P_{t-s} G_s^e \left( a, y, Y_{s_i}^\varepsilon (y) \right) W(dads) \right| > \frac{\rho}{2n}, \right.

\left. \right| \frac{1}{\sqrt{2 \log \log \frac{1}{\varepsilon}}} W \left. \right|_\infty < \eta \right) \right) = \sum_{i=1}^{n} P(B_i)

we have,

$$
E \int_{0}^{\min \left\{ [0,t] \cap \Delta_i^n, [0,r] \cap \Delta_i^n \right\}} \int_{U} \left| \frac{1}{\sqrt{2 \log \log \frac{1}{\varepsilon}}} \int_{\mathbb{R}} p_{t-s}(x-y) G_s^e \left( a, x, Y_{s_i}^\varepsilon (x) \right) dx \right|^2 \lambda(da) ds

- \frac{1}{\sqrt{2 \log \log \frac{1}{\varepsilon}}} \int_{\mathbb{R}} p_{t-s}(x-y) G_s^e \left( a, x, Y_{s_i}^\varepsilon (x) \right) dx \left. \right|_\lambda \left( da \right) ds

= E \int_{0}^{\min \left\{ [0,t] \cap \Delta_i^n, [0,r] \cap \Delta_i^n \right\}} \int_{U} \left| \frac{1}{\sqrt{2 \log \log \frac{1}{\varepsilon}}} \int_{\mathbb{R}} \Delta p(t, r) G_s^e \left( a, x, Y_{s_i}^\varepsilon (x) \right) dx \right|^2 \lambda(da) ds

\leq C_\sigma |t-r|

with,

$$
E \int_{U} \int_{\mathbb{R}} \left| \frac{1}{\sqrt{2 \log \log \frac{1}{\varepsilon}}} p_{t-s}(x-y) G_s^e \left( a, x, Y_{s_i}^\varepsilon (x) \right) dx \lambda(da) \right| \leq K_\sigma
$$
then according to theorem 10, we have for \( \beta \leq 1 \),

\[
P(B_i) \leq \exp \left( -\frac{\rho^2 (2\log \log \frac{1}{\epsilon})}{4n^2 (K_\sigma + C_\sigma)} \right)
\]

whenever,

\[
\rho \sqrt{2 \log \log \frac{1}{\epsilon}} \geq \max \left\{ 2^{\frac{1}{12}} \sqrt{\pi \beta^{-\frac{1}{2}}} , 2^{\frac{29}{4}} \right\}
\]

Hence we achieve the result for large \( n \) and sufficiently small \( \epsilon > 0 \).

**Lemma 5.** For any \( g \in L_1 \), \( \epsilon > 0 \), and \( c > 1 \), there exists \( j_0 \in \mathbb{N} \) such that for every \( j > j_0 \), \( P(\|Z_{1j}c - g\| \leq \epsilon \text{ i.o.}) = 1 \).

**Proof.** Let \( g \in L_1 \) and \( h \in L^2([0,1] \times U, ds\lambda(da)) \) such that \( g = S_t(h,y) \) and \( \frac{1}{2} \int_0^t \int_U |h_s(a)|^2 \lambda(da) ds \leq 1 \). Denote,

\[
F_j := \left\{ \|Z_{1j}^{1/2} - g\| \leq \epsilon \right\} \quad \text{and} \quad G_j := \left\{ \left\| \frac{1}{\sqrt{\log \log c^j}} W_{c^j} - h \right\| \leq \eta \right\}
\]

for some constant \( \eta > 0 \). We need to prove that \( P \left( \limsup_{j \to \infty} F_j \right) = 1 \).

Strassen [34] proved the compact LIL for Brownian paths and so \( P \left( \limsup_{j \to \infty} G_j \right) = 1 \). Let \( R > 2 \), then by Theorem 9 we have,

\[
P \left( F_j^c \cap G_j \right) \leq \exp \left( -R \log \log c^j \right) = \frac{K_R}{j^R} \leq \frac{K_R}{j^2} \quad (61)
\]

where we have \( 1/\sqrt{\log \log c^j} \) for \( \epsilon \) in (39) and used the fact that for \( k \in \mathbb{R} \),

\[
\exp \left( -k \log \log c^j \right) = \frac{K_k}{j^k}
\]

Now by Borel-Cantelli lemma applied to (61), we arrive at

\[
P \left( \limsup_{j \to \infty} F_j^c \cap G_j \right) = 0
\]

Thus we have,

\[
1 = P \left( \limsup_{j \to \infty} G_j \right) \leq P \left( \limsup_{j \to \infty} G_j \cap F_j \right) + P \left( \limsup_{j \to \infty} G_j \cap F_j^c \right)
\]
\[
\leq P \left( \limsup_{j \to \infty} F_j \right)
\]

obtaining the result. \( \square \)
4 LIL for SBM and FVP

In this section we apply the results from section three to derive the Strassen’s compact LIL for SBM and FVP. Based on the SPDE characterization of the two population models, SBM and FVP are given by class of SPDE (8) with

\[ G(a, y, u_s(y)) := \begin{cases} 
1_{0<a<u_s^c(y)} + 1_{u_s^c(y)<a<0} & \text{if}\ G(a, y, u_s(y)) := 1_{a<u_s^c(y)} - u_s^c(y), \\
\end{cases} \]

respectively. Since in both cases, \( G(a, y, u_s(y)) \) satisfies conditions (9) and (10), then by results in previous section we have tightness for process (28) with \( \{\mu^t_s\} \) representing each of the models in the corresponding space specified in theorems 5 and 6.

It is left to show that \( L_2 \) and \( L_3 \) in Theorems 5 and 6, respectively, are the limit sets of \( \{\tilde{Z}_t^j\} \) for the respective population models. In [22], in order to obtain the rate function for SBM (FVP), relation \( u_t^c(y) = \int_0^t \mu_t^s(dx) \) \( (u_t^c(y) = \int_0^\infty \mu_t^s(dx)) \) was used in controlled PDE (13). For \( h \in L^2([0,1] \times U, ds\lambda(da)) \) an expression was derived for \( \frac{1}{2} \int_0^1 \int_U |h_s(a)|^2 \lambda(da)ds \) by letting \( a = u_t^0(y) \). Thus, we let \( \omega \in \mathcal{C}([0,1]; M_{\beta}(\mathbb{R})) \) and \( h \in L^2([0,1] \times U; ds\lambda(da)) \) such that

\[
\frac{1}{2} \int_0^1 \int_U |h_s(a)|^2 \lambda(da)ds = \frac{1}{2} \int_0^1 \int_{\mathbb{R}} \left| \frac{d(\dot{\omega} - \frac{1}{2} \Delta^s \omega)}{d\mu_t^0} \right| \mu_t^0(dy)dt
\]

where \( \mu_t^0 \in \mathcal{H}_{\omega_0} \left( \mu_t^0 \in \tilde{\mathcal{H}}_{\omega_0} \right) \). Then the same argument as in the proof of Lemma 5 can be followed with \( F_j \) replaced by

\[
\tilde{F}_j := \left\{ \left| \tilde{Z}_t^j - \omega \right| \leq \epsilon \right\}
\]

to give, \( P \left( \left| \tilde{Z}_t^j - \omega \right| \leq \epsilon \text{ i.o.} \right) = 1 \) for every \( \omega \in \mathcal{C}([0,1]; M_{\beta}(\mathbb{R})) \) in the case of SBM and \( \omega \in \mathcal{C}([0,1]; \mathcal{P}_{\beta}(\mathbb{R})) \) for FVP, thus establishing the Strassen’s compact LIL for SBM and FVP.

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