A NEUMANN EIGENVALUE PROBLEM FOR FULLY NONLINEAR OPERATORS.

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Dedicated to prof. Louis Nirenberg for his 85th birthday.

1. Introduction

In this introduction and in the rest of the paper we quote some works of Louis Nirenberg that are used explicitly in order to give the right definitions and to prove the results; but the influence of his research, here and in all the papers both the authors have written, goes well beyond the citations. His mathematical ideas have been very important for us, specially for the first named author, but his teaching of how to approach mathematical problems has been as important. We are happy to have this opportunity to thank him for his generosity.

In this paper, for $\Omega$ a $C^2$ bounded domain of $\mathbb{R}^n$ and for any $\alpha > 0$, we consider the eigenvalue problem:

$$
\begin{aligned}
\mathcal{M}_{a,A}^+(D^2u) + \lambda u &= 0 \quad \text{in } \Omega, \\
\frac{\partial u}{\partial n} &= \alpha u \quad \text{on } \partial\Omega,
\end{aligned}
$$

where $\mathcal{M}_{a,A}^+$ is the Pucci operator, i.e.

$$
\mathcal{M}_{a,A}^+(M) = \sup_{0 < a I \leq \sigma \leq aI} \text{tr}(\sigma M).
$$

It is useless to emphasize the importance of the concept of eigenvalue for the understanding of the structural properties of the solutions both for linear and non linear equations. The pioneering work of Berestycki, Nirenberg and Varadhan [4] has open the way to enlarge this fundamental concept to non linear operators. Indeed, even if they treat linear equations, their theory is very well adapted to fully nonlinear operators and viscosity solutions being based primarily on the use of the maximum principle. This has been done by many and in many different contests, let us mention the works of Armstrong, Busca, Demengel, Juutinen, Ishii, Quaas, Sirakov, Yoshimura and the authors of this note ([1, 5, 6, 11, 12, 16, 17]). It should be mentioned that P.-L. Lions in [13], with a completely different approach, first introduces what he called demi-eigenvalues. Indeed when the operator is not odd with respect to the Hessian (as is the case of the Pucci operators), eigenvalues corresponding to positive eigenfunctions or to negative eigenfunctions may not coincide and one could interpret these two eigenvalues as a "splitting" of the eigenvalue.

The eigenvalue problem for Robin boundary conditions associated with a fully-nonlinear operator was already treated in [16]. The novelty here is that we consider $\alpha > 0$ which is the "wrong sign" in the sense that the boundary conditions are not "proper", see e.g. [8]. The boundary source and the reaction-diffusion equation are somehow in competition.

In analogy to [4] we define the eigenvalues in the following way:
\( \lambda_+^\alpha := \sup\{ \lambda \in \mathbb{R} \mid \exists v > 0 \text{ on } \overline{\Omega} \text{ bounded viscosity supersolution of} \quad M_{a,A}(D^2v) + \lambda v = 0 \text{ in } \Omega, \frac{\partial v}{\partial n} = \alpha v \text{ on } \partial \Omega \}, \)

\( \lambda_-^\alpha := \sup\{ \lambda \in \mathbb{R} \mid \exists v < 0 \text{ on } \overline{\Omega} \text{ bounded viscosity subsolution of} \quad M_{a,A}(D^2v) + \lambda v = 0 \text{ in } \Omega, \frac{\partial v}{\partial n} = \alpha v \text{ on } \partial \Omega \}. \)

The first step is to prove that there exists \( u_+^\alpha > 0 \) and \( u_-^\alpha < 0 \) solutions of (1) when respectively \( \lambda = \lambda_+^\alpha \) and \( \lambda = \lambda_-^\alpha \) (Proposition 4.1). We shall also prove that below these eigenvalues there are solutions of the equation with a forcing term \( f(x) \) as long as the \( f \) has the right sign, i.e. \( f \leq 0 \) below \( \lambda_+^\alpha \) and \( f \geq 0 \) below \( \lambda_-^\alpha \).

We are mainly interested in the asymptotic behavior with respect to \( \alpha \) of the eigenvalues. When \( \alpha \to 0 \), \( \lambda_+^\alpha \) and \( \lambda_-^\alpha \) tend to 0 which is the principal eigenvalue of the pure Neumann boundary problem

\[
\begin{aligned}
M_{a,A}(D^2u) + \lambda u &= 0 \quad \text{in } \Omega, \\
\frac{\partial u}{\partial n} &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

But our main goal is to study the behavior when \( \alpha \to +\infty \), this is done in our main

**Theorem 1.1.** The following limits hold:

(2) \[ \lim_{\alpha \to +\infty} \frac{\lambda_+^\alpha}{\alpha^2} = A, \]

(3) \[ \lim_{\alpha \to +\infty} \frac{\lambda_-^\alpha}{\alpha^2} = a. \]

Interestingly this asymptotic behavior emphasizes the “splitting” of the eigenvalue. In the linear case, i.e. when \( a = A = 1 \) and the Pucci operator is nothing else but the Laplacian, this problem was treated in [14] by Lou and Zhu with a variational approach. Very recently Daners and Kennedy [9] have proved that this asymptotic behavior is valid for the whole spectrum.

We also prove that for any \( K \subset \subset \Omega \), the normalized eigenfunctions \( u_+^\alpha \) and \( u_-^\alpha \) satisfy

\[ \| u_+^\alpha \|_{L^\infty(K)} \to 0 \quad \text{and} \quad \| u_-^\alpha \|_{L^\infty(K)} \to 0 \quad \text{as } \alpha \to +\infty. \]

So that the eigenfunctions tend to concentrate on the point of the boundary where they reach the sup or the inf.

The idea of the proof of Theorem 1.1 which somehow follows the line adopted in [14], is the following: first we establish that \( u_+^\alpha \) reaches its maximum on the boundary and then we perform a blow up around this point.

Then a key tool will be a Liouville result in the half space (Theorem 5.1). Precisely we prove that for \( \gamma > A \) (respectively \( \gamma > a \)) there are no bounded subsolutions (respectively supersolutions) of

\[
\begin{aligned}
M_{a,A}(D^2u) - \gamma u &= 0 \quad \text{in } \mathbb{R}^n_+, \\
-\frac{\partial u}{\partial x_n} &= u \quad \text{on } \partial \mathbb{R}^n_+.
\end{aligned}
\]

that are positive (respectively negative) somewhere. In [14] the analogous result for the Laplacian is proved using the construction of sub and super solutions in the flavor of what is done in [3]. Let us mention here that it would be interesting to extend the results of
Berestycki, Caffarelli, Nirenberg \[3\] in half spaces, to this class of fully-nonlinear operators and to these boundary conditions.

Lipschitz estimates up to the boundary will be required in the proofs of both the existence results and the asymptotic behavior. These estimates which are interesting in their own right, are established here using an argument inspired by \[10\] (see also Barles and Da Lio \[2\] and Milakis and Silvestre \[15\]).

In the whole paper the fully-nonlinear operator considered is the Pucci operator
\[ M_{a,A} = \inf_{0 < \sigma \leq A I} \tr(\sigma M). \]
but, mutatis mutandis, parallel results can be stated for the Pucci operator
\[ M_{-a,A} = \inf_{0 < \sigma \leq A I} \tr(\sigma M). \]

2. Preliminary results

Let us recall the definition of viscosity sub and supersolution of the Neumann problem associated to a general elliptic operator \( F : \Omega \times \mathbb{R} \times \mathbb{R}^n \times S(n) \to \mathbb{R} \). Here \( S(n) \) is the space of symmetric matrices on \( \mathbb{R}^n \), equipped with the usual ordering. We denote by \( USC(\Omega) \) (resp., \( LSC(\Omega) \)) the set of upper (resp., lower) semicontinuous functions on \( \Omega \).

Let \( f : \Omega \to \mathbb{R} \), \( g : \partial \Omega \times \mathbb{R} \to \mathbb{R} \).

Definition 2.1. A function \( u \in USC(\Omega) \) (resp., \( u \in LSC(\Omega) \)) is called viscosity subsolution (resp., supersolution) of
\[
\begin{cases}
F(x,u,Du,D^2u) = f(x) & \text{in } \Omega, \\
\frac{\partial u}{\partial n} = g(x,u) & \text{on } \partial \Omega,
\end{cases}
\]
if the following conditions hold

(i) For every \( x_0 \in \Omega \), for any \( \varphi \in C^2(\Omega) \), such that \( u - \varphi \) has a local maximum (resp., minimum) at \( x_0 \) then
\[
F(x_0,u(x_0),D\varphi(x_0),D^2\varphi(x_0)) \geq (\text{resp., } \leq) f(x_0).
\]

(ii) For every \( x_0 \in \partial \Omega \), for any \( \varphi \in C^2(\Omega) \), such that \( u - \varphi \) has a local maximum (resp., minimum) at \( x_0 \) then
\[
-(F(x_0,u(x_0),D\varphi(x_0),D^2\varphi(x_0)) - f(x_0)) \land \left( \frac{\partial \varphi}{\partial n}(x_0) - g(x_0,u(x_0)) \right) \leq 0
\]
(resp.,
\[
-(F(x_0,u(x_0),D\varphi(x_0),D^2\varphi(x_0)) - f(x_0)) \lor \left( \frac{\partial \varphi}{\partial n}(x_0) - g(x_0,u(x_0)) \right) \geq 0.
\]

A viscosity solution is a continuous function which is both a subsolution and a supersolution.

One of the motivation for these relaxed boundary conditions is the stability under uniform convergence. Actually, if the domain \( \Omega \) satisfies the exterior sphere condition and \( F \) is uniformly elliptic, viscosity subsolutions (resp., supersolutions) satisfy in the viscosity sense \( \frac{\partial u}{\partial n} \leq (\text{resp., } \geq) g(x,u) \) for any \( x \in \partial \Omega \), see e.g. Proposition 2.1 in \[16\].

We assume throughout the paper that \( \Omega \) is a bounded domain of \( \mathbb{R}^n \) of class \( C^2 \).
Theorem 2.1 (Strong Comparison Principle, [16] Theorem 3.1). Assume that $c$ and $f$ are continuous on $\overline{\Omega}$. Let $u \in USC(\Omega)$ and $v \in LSC(\Omega)$ be respectively a sub and a supersolution of

$$
\begin{cases}
\mathcal{M}_{a,A}^+(D^2 u) + c(x)u = f(x) & \text{in } \Omega, \\
\frac{\partial u}{\partial n} = \alpha u & \text{on } \partial \Omega.
\end{cases}
$$

If $u \leq v$ on $\overline{\Omega}$ then either $u < v$ on $\overline{\Omega}$ or $u \equiv v$ on $\overline{\Omega}$.

Proposition 2.2 (Maximum Principle for $\lambda < \lambda_a^+$, [16] Theorem 4.5). Assume $\lambda < \lambda_a^+$. Let $v \in USC(\Omega)$ be a viscosity subsolution of

$$
\begin{cases}
\mathcal{M}_{a,A}^+(D^2 v) + \lambda v = 0 & \text{in } \Omega, \\
\frac{\partial v}{\partial n} = \alpha v & \text{on } \partial \Omega,
\end{cases}
$$

then $v \leq 0$ on $\overline{\Omega}$.

Proposition 2.3 (Minimum Principle for $\lambda < \lambda_a^-$, [16] Remark 4.6). Assume $\lambda < \lambda_a^-$. Let $v \in LSC(\Omega)$ be a viscosity supersolution of

$$
\begin{cases}
\mathcal{M}_{a,A}^+(D^2 v) + \lambda v = 0 & \text{in } \Omega, \\
\frac{\partial v}{\partial n} = \alpha v & \text{on } \partial \Omega,
\end{cases}
$$

then $v \geq 0$ on $\overline{\Omega}$.

3. Lipschitz estimates

In this section we shall prove a local Lipschitz regularity result for solutions of the Neumann problem associated to general uniformly elliptic operators, that we will use in the next sections. Let us consider the Neumann problem

$$
\begin{cases}
F(x,u,Du,D^2 u) = f(x) & \text{in } \Omega, \\
\frac{\partial u}{\partial n} = g(x) & \text{on } \partial \Omega,
\end{cases}
$$

where the operator $F$ is supposed to be continuous on $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times S(n)$ and satisfying the following assumptions:

(F1) There exist $b,c > 0$ such that for $x \in \overline{\Omega}$, $r,s \in \mathbb{R}$, $p,q \in \mathbb{R}^n$, $X,Y \in S(n)$

$$
\mathcal{M}_{a,A}^-(Y - X) - b|p - q| - c|r - s| \leq F(x,r,p,Y) - F(x,s,q,X) \\
\leq \mathcal{M}_{a,A}^+(Y - X) + b|p - q| + c|r - s|.
$$

(F2) There exists $C_1 > 0$ such that for all $x,y \in \overline{\Omega}$ and $X \in S(n)$

$$
|F(x,0,0,0) - F(y,0,0,0)| \leq C_1|x - y|^\frac{2}{p}||X||.
$$

Proposition 3.1. Assume that (F1) and (F2) hold. Let $f: \overline{\Omega} \to \mathbb{R}$ be bounded, $g: \partial \Omega \to \mathbb{R}$ be Lipschitz continuous. Let $u \in C(\overline{\Omega})$ be a viscosity solution of (5), then, for any $x_0 \in \overline{\Omega}$ and for any $\rho > 0$, there exists $K > 0$ such that

$$
|u(x) - u(y)| \leq (MK + |g|_{L^\infty(\partial \Omega)})|x - y| \quad \forall x, y \in B_\rho(x_0) \cap \overline{\Omega}
$$

and

$$
K^2 - bK \leq C\left[c|u|_{L^\infty(B_\rho(x_0) \cap \overline{\Omega})} + |f|_{L^\infty(\overline{\Omega})} + (1 + b)|g|_{C^0(\partial \Omega)} \frac{b}{\rho^2} + 1\right],
$$

where $C$ is independent of $x_0$, $\rho$, and $K$. 

where $M \leq C(|u|_{L^\infty(\Omega^0(x_0) \cap \Omega)} + |g|_{L^\infty(\partial \Omega)} + 1)$ and $C$ depends on $a$, $A$, $C_1$, $n$ and $\Omega$.

**Corollary 3.2.** Assume $\lambda \in \mathbb{R}$ and $\alpha \geq 0$. Let $u \in C(\overline{\Omega})$ be a viscosity solution of

$$
\begin{cases}
M_{x,A}(D^2u) + \lambda u = f(x) & \text{in } \Omega,
\frac{\partial u}{\partial \nu} = \alpha u & \text{on } \partial \Omega.
\end{cases}
$$

Then, for any $\rho > 0$, there exists $K > 0$ such that for any $x, y \in \Omega_{\rho} := \{x \in \overline{\Omega} \mid d(x) \leq \rho\}$

$$
|u(x) - u(y)| \leq (\alpha |e^{\alpha d}(x)u|_{L^\infty(\Omega_{\rho})} + MK)|x - y|
$$

and

$$
K^2 - C\alpha K \leq C \left[ (\alpha + \alpha^2 + |\lambda|) |e^{\alpha d}(x)u|_{L^\infty(\Omega_{\rho})} + |e^{\alpha d}(x)f|_{L^\infty(\Omega)} + \frac{\alpha}{\rho} + \frac{1}{\rho^2} + 1 \right],
$$

where $M \leq C(|e^{\alpha d}(x)u|_{L^\infty(\Omega_{\rho})} + 1)$ and $C$ depends on $a$, $A$, $n$ and $\Omega$.

**Proof of Proposition 3.1** We follow the proof of Proposition III.1 of [10], that we modify taking test functions which depend on the distance function and that are suitable for the Neumann boundary conditions. Moreover, as in [2], we are going to use a regularization of $g$. In order to do so, it is convenient to introduce the following classical lemma.

**Lemma 3.3.** Assume $\rho \in C^\infty(\mathbb{R}^n)$, $\rho > 0$, $\text{supp}(\rho) \subset B_1(0)$ and $\int_{\mathbb{R}^n} \rho(y)dy = 1$. If $g \in C^{0,1}(\mathbb{R}^n)$ and $g$ is bounded, then the function $\tilde{g} : \mathbb{R}^n \times [0, +\infty) \to \mathbb{R}$ defined by

$$
\tilde{g}(x, \epsilon) := \int_{\mathbb{R}^n} g(z)\rho \left( \frac{x - z}{\epsilon} \right) \frac{1}{\epsilon^n}dz, \quad \epsilon > 0,
$$

$\tilde{g}(x, 0) = g(x)$ for $x \in \mathbb{R}^n$, is in $C^{0,1}(\mathbb{R}^n \times [0, +\infty))$. Moreover, the function $\tilde{g}$ is in $C^2(\mathbb{R}^n \times (0, +\infty))$ with

$$
|D_x\tilde{g}(x, \epsilon)|, |D_{xx}\tilde{g}(x, \epsilon)| \leq C_0,
$$

$$
|D_{xx}^2\tilde{g}(x, \epsilon)|, |D_{x\epsilon}^2\tilde{g}(x, \epsilon)|, |D_{\epsilon\epsilon}^2\tilde{g}(x, \epsilon)| \leq \frac{C_0}{\epsilon} \quad \text{in } \mathbb{R}^n \times (0, +\infty)
$$

for some positive constant $C_0 \leq C|g|_{C^{0,1}(\mathbb{R}^n)}$, with $C$ depending only on $\rho$ and $n$.

We first extend $g$ to a $C^{0,1}$ function of $\mathbb{R}^n$ and we still denote by $g$ this extension. Then, we consider the function $\tilde{g}$ associated to $g$ as in Lemma 3.3.

Since $\Omega$ is a domain of class $C^2$, it satisfies the uniform exterior sphere condition, i.e., there exists $r > 0$ such that $B(x + r\overline{\nu}(x), r) \cap \Omega = \emptyset$ for any $x \in \partial \Omega$. From this property it follows that

$$
\langle \overline{\nu}(x), y - x \rangle \leq \frac{1}{2r}|y - x|^2 \quad \text{for } x \in \partial \Omega \text{ and } y \in \overline{\Omega}.
$$

Moreover, the $C^2$-regularity of $\Omega$ implies the existence of a neighborhood of $\partial \Omega$ in $\overline{\Omega}$ on which the distance from the boundary

$$
d(x) := \inf \{|x - y|, y \in \partial \Omega\}, \quad x \in \overline{\Omega}
$$

is of class $C^2$. We still denote by $d$ a $C^2$ extension of the distance function to the whole $\overline{\Omega}$. Without loss of generality we can assume that $|Dd(x)| \leq 1$ on $\overline{\Omega}$. 

Let $x_0 \in \overline{\Omega}$ and $\rho > 0$. Let us denote $\overline{B}_\|_x(x_0, \rho) := \overline{B}_\|_x(x_0) \cap \overline{\Omega}$ and $B_\|_x(x_0, \rho) := B_\|_x(x_0) \cap \overline{\Omega}$. We are going to show that $u$ is Lipschitz continuous on $B_\|_x(x_0, \rho)$. For this aim, let us introduce the functions

$$
\Phi(x) = MK|x| - M(K|x|)^2,
$$

$$
\Psi_1(x, y) = e^{-L(d(x)+d(y))}\Phi(x - y),
$$

$$
\Psi_2(x, y) = \tilde{g}\left(\frac{x + y}{2}, (\delta^2 + |x - y|^2)\frac{1}{2}\right)(d(x) - d(y)),
$$

and

$$
\varphi(x, y) = \Psi_1(x, y) - \Psi_2(x, y),
$$

where $L$ is a fixed number greater than $\frac{1}{r}$ with $r$ the radius in (10), $K$ and $M$ are positive constants to be chosen later and $\delta$ is a small parameter. We also use the notation

$$
\tilde{g}(Z, T) = \tilde{g}\left(\frac{x + y}{2}, (\delta^2 + |x - y|^2)\frac{1}{2}\right).
$$

If $K|x| \leq \frac{1}{T}$, then

$$
\Phi(x) \geq \frac{3}{4}MK|x|.
$$

We define

$$
\Delta_K := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : |x - y| \leq \frac{1}{4K}\}.
$$

We fix $M > 1$ and $j > 0$ such that

$$
\max_{\overline{B}_\|_x(x_0, \rho)^2} |u(x) - u(y)| + 2d_0(|g|_\infty + C_0\delta) \leq e^{-2Ld_0}\frac{M}{8},
$$

$$
j = \frac{M}{\rho^2},
$$

where $d_0 = \max_{x \in \overline{\Omega}} d(x)$, and we claim that taking $K$ large enough, for any small $\delta$ one has

$$
u(x) - u(y) - \varphi(x, y) - je^{-Ld}|x - x_0|^2 \leq 0 \quad \text{for } (x, y) \in \Delta_K \cap \overline{B}_\|_x(x_0, \rho)^2.
$$

To show (13) we suppose by contradiction that the maximum of $u(x) - u(y) - \varphi(x, y) - je^{-Ld}|x - x_0|^2$ on $\Delta_K \cap \overline{B}_\|_x(x_0, \rho)^2$ is positive. Then, for $\delta$ small enough, there is $(\overline{x}, \overline{y}) \in \Delta_K \cap \overline{B}_\|_x(x_0, \rho)^2$ such that $\overline{\tau} \neq \overline{y}$ and

$$
u(\overline{x}) - u(\overline{y}) - \tilde{\varphi}(\overline{x}, \overline{y}) = \max_{\Delta_K \cap \overline{B}_\|_x(x_0, \rho)^2} (u(x) - u(y) - \tilde{\varphi}(x, y)) > 0,
$$

where

$$
\tilde{\varphi}(x, y) = \varphi(x, y) + je^{-Ld}|x - x_0|^2 - C_0\delta(d(x) + d(y)),
$$

with $C_0$ the constant defined as in Lemma 3.3.

The point $(\overline{x}, \overline{y})$ belongs to $\text{int}(\Delta_K) \cap \overline{B}_\|_x(x_0, \rho)^2$. Indeed, if $|x - y| = \frac{1}{4K}$, by (12) and (11), we have

$$
u(x) - u(y) \leq e^{-2Ld_0}\frac{M}{8} - 2d_0(|g|_\infty + C_0\delta) \leq e^{-L(d(x)+d(y))}\frac{1}{2}MK|x - y| - \Psi_2(x, y) - C_0\delta(d(x) + d(y)) \leq \tilde{\varphi}(x, y).
$$
On the other hand, if $|x - x_0| = \rho$, then

$$u(x) - u(y) \leq e^{-Ld_0} M - 2d_0(|g|_\infty + C_0\delta) \leq e^{-Ld(x)} \frac{M}{\rho^2}|x - x_0|^2 + \Psi_1(x, y) - \Psi_2(x, y) - C_0\delta(d(x) + d(y)) = \tilde{\varphi}(x, y).$$

Similarly, if $|y - x_0| = \rho$ and $K > K_0/\rho$, for some constant $K_0 > 0$, then $u(x) - u(y) \leq \tilde{\varphi}(x, y)$. Hence, $(\tilde{x}, \tilde{y}) \in \text{int}(\Delta_K) \cap B_{1}\tilde{r}(x_0, \rho)^2$.

Since $\tilde{x} \neq \tilde{y}$ we can compute the derivatives of $\tilde{\varphi}$ at $(\tilde{x}, \tilde{y})$ obtaining

$$D_x\tilde{\varphi}(\tilde{x}, \tilde{y}) = -Le^{-L(d(\tilde{x})+d(\tilde{y}))} MK |\tilde{x} - \tilde{y}| (1 - K|\tilde{x} - \tilde{y}|) Dd(\tilde{x}) + e^{-L(d(\tilde{x})+d(\tilde{y}))} MK (1 - 2K|\tilde{x} - \tilde{y}|) \frac{(|\tilde{x} - \tilde{y}|)}{|\tilde{x} - \tilde{y}|} - C_0\delta Dd(\tilde{x})$$

$$- jLe^{-Ld(\tilde{x})} |\tilde{x} - x_0|^2 Dd(\tilde{x}) + D_x\psi_2(\tilde{x}, \tilde{y}),
$$

$$D_y\tilde{\varphi}(\tilde{x}, \tilde{y}) = -Le^{-L(d(\tilde{x})+d(\tilde{y}))} MK |\tilde{x} - \tilde{y}| (1 - K|\tilde{x} - \tilde{y}|) Dd(\tilde{y}) - e^{-L(d(\tilde{x})+d(\tilde{y}))} MK (1 - 2K|\tilde{x} - \tilde{y}|) \frac{(|\tilde{x} - \tilde{y}|)}{|\tilde{x} - \tilde{y}|} - C_0\delta Dd(\tilde{y}) - D_y\psi_2(\tilde{x}, \tilde{y}),$$

where

$$D_x\psi_2(\tilde{x}, \tilde{y}) = \frac{d(\tilde{x}) - d(\tilde{y})}{2} D_z\tilde{g}\left(\frac{\tilde{x} + \tilde{y}}{2}, (\delta^2 + |\tilde{x} - \tilde{y}|^2)^{\frac{1}{2}}\right)$$

$$+ \left(\frac{d(\tilde{x}) - d(\tilde{y})}{2}\right) D_T\tilde{g}\left(\frac{\tilde{x} + \tilde{y}}{2}, (\delta^2 + |\tilde{x} - \tilde{y}|^2)^{\frac{1}{2}}\right)$$

and

$$D_y\psi_2(\tilde{x}, \tilde{y}) = \frac{d(\tilde{x}) - d(\tilde{y})}{2} D_z\tilde{g}\left(\frac{\tilde{x} + \tilde{y}}{2}, (\delta^2 + |\tilde{x} - \tilde{y}|^2)^{\frac{1}{2}}\right)$$

$$- \left(\frac{d(\tilde{x}) - d(\tilde{y})}{2}\right) D_T\tilde{g}\left(\frac{\tilde{x} + \tilde{y}}{2}, (\delta^2 + |\tilde{x} - \tilde{y}|^2)^{\frac{1}{2}}\right)$$

Observe that

$$|D_x\tilde{\varphi}(\tilde{x}, \tilde{y})|, |D_y\tilde{\varphi}(\tilde{x}, \tilde{y})| \leq C(MK + C_0 + j\rho).$$

Here and henceforth $C$ denotes various positive constants independent of $K, b, c, f, g$ and $u$.

By Lemma 3.3

$$\left|\tilde{g}\left(\frac{\tilde{x} + \tilde{y}}{2}, (\delta^2 + |\tilde{x} - \tilde{y}|^2)^{\frac{1}{2}}\right) - g(\tilde{x})\right| \leq C_0(2|\tilde{x} - \tilde{y}| + \delta),$$

then, if $\tilde{x} \in \partial\Omega$ we have

$$-\langle \tilde{\nu}(\tilde{x}), D_x\psi_2(\tilde{x}, \tilde{y})\rangle - g(\tilde{x}) \geq -C_0(4|\tilde{x} - \tilde{y}| + \delta).$$
Hence, using (10), if $\mathbf{x} \in \partial \Omega$ we get
\[
\langle \nabla_{\mathbf{x}}(\mathbf{x}), D_x \tilde{\varphi}(\mathbf{x}, \mathbf{y}) \rangle - g(\mathbf{x}) = L e^{-L_d(\mathbf{y})} M K |\mathbf{x} - \mathbf{y}|(1 - K |\mathbf{x} - \mathbf{y}|)
\]
\[
+ e^{-L_d(\mathbf{y})} M K (1 - 2 K |\mathbf{x} - \mathbf{y}|) \langle \nabla_{\mathbf{x}}(\mathbf{x}), \frac{(\mathbf{x} - \mathbf{y})}{|\mathbf{x} - \mathbf{y}|} \rangle
\]
\[
+ j L |\mathbf{x} - x_0|^2 + 2 j \langle \nabla_{\mathbf{x}}(\mathbf{x}), \mathbf{x} - x_0 \rangle
\]
\[
- \langle \nabla_{\mathbf{x}}(\mathbf{x}), D_x \Psi_2(\mathbf{x}, \mathbf{y}) \rangle - g(\mathbf{x}) + C_0 \delta
\]
\[
\geq \frac{1}{2} e^{-L_d(\mathbf{y})} M K |\mathbf{x} - \mathbf{y}| \left( \frac{3}{2} L - \frac{1}{r} \right) + j |\mathbf{x} - x_0|^2 \left( L - \frac{1}{r} \right)
\]
\[
- 4 C_0 |\mathbf{x} - \mathbf{y}| > 0,
\]
for $MK > \frac{16C_{\text{loc}}^{L_d}}{\delta r L - 2}$, since $\mathbf{x} \neq \mathbf{y}$ and $L > \frac{1}{r}$. Similarly, if $\mathbf{y} \in \partial \Omega$ then
\[
\langle \nabla_{\mathbf{x}}(\mathbf{y}), -D_y \tilde{\varphi}(\mathbf{x}, \mathbf{y}) \rangle - g(\mathbf{y}) \leq \frac{1}{2} e^{-L_d(\mathbf{y})} M K |\mathbf{x} - \mathbf{y}| \left( -\frac{3}{2} L + \frac{1}{r} \right) + 4 C_0 |\mathbf{x} - \mathbf{y}| < 0.
\]
Then, by definition of sub and supersolution
\[
F(\mathbf{x}, u(\mathbf{x}), D_x \tilde{\varphi}(\mathbf{x}, \mathbf{y}), X) \geq f(\mathbf{x}), \quad \text{if} \quad (D_x \tilde{\varphi}(\mathbf{x}, \mathbf{y}), X) \in \mathcal{J}^{2,+} u(\mathbf{x}),
\]
\[
F(\mathbf{y}, u(\mathbf{y}), -D_y \tilde{\varphi}(\mathbf{x}, \mathbf{y}), Y) \leq f(\mathbf{y}), \quad \text{if} \quad (-D_y \tilde{\varphi}(\mathbf{x}, \mathbf{y}), Y) \in \mathcal{J}^{2,+} u(\mathbf{y}).
\]
Since $(\mathbf{x}, \mathbf{y}) \in \text{int} \Delta_K \cap B_{\overline{\Omega}}(x_0, \rho)^2$, it is a local maximum point of $u(x) - u(y) - \tilde{\varphi}(x, y)$ in $\overline{\Omega}^2$. Then applying Theorem 3.2 in [3], for every $\epsilon > 0$ there exist $X, Y \in S(n)$ such that
\[
(D_x \tilde{\varphi}(\mathbf{x}, \mathbf{y}), X - C_0 \delta D^2 d(\mathbf{x}) + D^2(\epsilon e^{-L_d(x)} |x - x_0|^2)) \in \mathcal{J}^{2,+} u(\mathbf{x}),
\]
\[
(-D_y \tilde{\varphi}(\mathbf{x}, \mathbf{y}), Y + C_0 \delta D^2 d(\mathbf{y})) \in \mathcal{J}^{2,-} u(\mathbf{y})
\]
and
\[
\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq D^2 \varphi(\mathbf{x}, \mathbf{y}) + \epsilon(D^2 \varphi(\mathbf{x}, \mathbf{y}))^2
\]
\[
\leq D^2 \Psi_1(\mathbf{x}, \mathbf{y}) - D^2 \Psi_2(\mathbf{x}, \mathbf{y}) + 2 \epsilon(D^2 \Psi_1(\mathbf{x}, \mathbf{y}))^2 + 2 \epsilon(D^2 \Psi_2(\mathbf{x}, \mathbf{y}))^2.
\]
Now we want to estimate the matrix on the right-hand side of the last inequality.

Using Lemma 3.3, it is easy to check that
\[
-C C_0 \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq D^2 \Psi_2(\mathbf{x}, \mathbf{y}) \leq C C_0 \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.
\]
Next, let us estimate $D^2 \Psi_1(\mathbf{x}, \mathbf{y})$.
\[
D^2 \Psi_1(\mathbf{x}, \mathbf{y}) = \Phi(\mathbf{x} - \mathbf{y}) D^2(e^{-L(d(\mathbf{x})+d(\mathbf{y}))}) + D(e^{-L(d(\mathbf{x})+d(\mathbf{y}))} \otimes D(\Phi(\mathbf{x} - \mathbf{y}))
\]
\[
+ D(\Phi(\mathbf{x} - \mathbf{y})) \otimes D(e^{-L(d(\mathbf{x})+d(\mathbf{y}))}) + e^{-L(d(\mathbf{x})+d(\mathbf{y}))} D^2(\Phi(\mathbf{x} - \mathbf{y})).
\]
We set
\[
A_1 := \Phi(\mathbf{x} - \mathbf{y}) D^2(e^{-L(d(\mathbf{x})+d(\mathbf{y}))}),
\]
\[
A_2 := D(e^{-L(d(\mathbf{x})+d(\mathbf{y}))}) \otimes D(\Phi(\mathbf{x} - \mathbf{y})) + D(\Phi(\mathbf{x} - \mathbf{y})) \otimes D(e^{-L(d(\mathbf{x})+d(\mathbf{y}))}),
\]
\[
A_3 := e^{-L(d(\mathbf{x})+d(\mathbf{y}))} D^2(\Phi(\mathbf{x} - \mathbf{y})).
\]
Observe that
\[
A_1 \leq C M K |\mathbf{x} - \mathbf{y}| \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.
\]
For $A_2$ we have the following estimate
\begin{equation}
A_2 \leq CMK \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + CMK \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} \leq CMK \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.
\end{equation}
Indeed for $\xi, \eta \in \mathbb{R}^n$ we compute
\begin{align*}
\langle A_2(\xi, \eta), (\xi, \eta) \rangle &= 2Le^{-L(d(\bar{x}+,d(\bar{y})))} \{ (Dd(\bar{x}) \otimes D\Phi(\bar{x} - \bar{y})(\eta - \xi), \xi) \\
&\quad + (Dd(\bar{y}) \otimes D\Phi(\bar{x} - \bar{y})(\eta - \xi), \eta) \} \\
&\leq CMK (|\xi| + |\eta|)|\eta - \xi| \\
&\leq CMK (|\xi|^2 + |\eta|^2) + CMK|\eta - \xi|^2.
\end{align*}
Now we consider $A_3$. The matrix $D^2(\Phi(\bar{x} - \bar{y}))$ has the form
\begin{equation*}
D^2(\Phi(\bar{x} - \bar{y})) = \begin{pmatrix} 2D^2\Phi(\bar{x} - \bar{y}) & -2D^2\Phi(\bar{x} - \bar{y}) \\ -2D^2\Phi(\bar{x} - \bar{y}) & 2D^2\Phi(\bar{x} - \bar{y}) \end{pmatrix},
\end{equation*}
and the Hessian matrix of $\Phi(x)$ is
\begin{equation}
D^2\Phi(x) = \frac{MK}{|x|} \left( I - \frac{x \otimes x}{|x|^2} \right) - 2MK^2 I.
\end{equation}
If we choose
\begin{equation}
\epsilon = \frac{|\bar{x} - \bar{y}|}{4MK e^{-L(d(\bar{x}+,d(\bar{y})))}},
\end{equation}
then we have the following estimates
\begin{equation}
\epsilon A_1^2 \leq CMK|\bar{x} - \bar{y}|^2 I_{2n}, \quad \epsilon A_2^2 \leq CMK|\bar{x} - \bar{y}|^2 I_{2n},
\end{equation}
\begin{equation}
\epsilon(A_1 A_2 + A_2 A_1) \leq CMK|\bar{x} - \bar{y}|^2 I_{2n},
\end{equation}
\begin{equation}
\epsilon(A_1 A_3 + A_3 A_1) \leq CMK|\bar{x} - \bar{y}|^2 I_{2n}, \quad \epsilon(A_2 A_3 + A_3 A_2) \leq CMK I_{2n},
\end{equation}
where $I_{2n} := \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$. Then using (18), (19), (20), (23) and observing that
\begin{equation*}
(D^2(\Phi(\bar{x} - \bar{y})))^2 = \begin{pmatrix} 2(D^2\Phi(\bar{x} - \bar{y}))^2 & -2(D^2\Phi(\bar{x} - \bar{y}))^2 \\ -2(D^2\Phi(\bar{x} - \bar{y}))^2 & 2(D^2\Phi(\bar{x} - \bar{y}))^2 \end{pmatrix},
\end{equation*}
from (17) we can conclude that
\begin{equation*}
\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq (MO(K) + CC_0) \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + \begin{pmatrix} B & -B \\ -B & B \end{pmatrix},
\end{equation*}
where
\begin{equation}
B = e^{-L(d(\bar{x}+,d(\bar{y})))} \left[ D^2\Phi(\bar{x} - \bar{y}) + \frac{|\bar{x} - \bar{y}|}{MK} (D^2\Phi(\bar{x} - \bar{y})) \right].
\end{equation}
The last inequality can be rewritten as follows
\begin{equation*}
\begin{pmatrix} \tilde{X} & 0 \\ 0 & \tilde{Y} \end{pmatrix} \leq \begin{pmatrix} B & -B \\ -B & B \end{pmatrix},
\end{equation*}
with $\tilde{X} = X - (MO(K) + CC_0)I$ and $\tilde{Y} = Y + (MO(K) + CC_0)I$.

Now we want to get a good estimate for $\text{tr}(\tilde{X} - \tilde{Y})$, as in [10]. For that aim let
\begin{equation*}
0 \leq P := \frac{(\bar{x} - \bar{y}) \otimes (\bar{x} - \bar{y})}{|\bar{x} - \bar{y}|^2} \leq I.
\end{equation*}
Since $\tilde{X} - \tilde{Y} \leq 0$ and $\tilde{X} - \tilde{Y} \leq 4B$, we have
\[ \text{tr}(\tilde{X} - \tilde{Y}) \leq \text{tr}(P(\tilde{X} - \tilde{Y})) \leq 4\text{tr}(PB). \]

We have to compute $\text{tr}(PB)$. From (21), observing that the matrix $(1/|x|^2)x \otimes x$ is idempotent, i.e., $[(1/|x|^2)x \otimes x] = (1/|x|^2)x \otimes x$, we compute
\[ (D^2\Phi(x))^2 = \frac{M^2K^2}{|x|^2} (1 - 4K|x|) \left( I - \frac{x \otimes x}{|x|^2} \right) + 4M^2K^4I. \]

Then, since $\text{tr}P = 1$ and $4K|\bar{x} - \bar{y}| \leq 1$, we have
\[ \text{tr}(PB) = e^{-L(d(\bar{x}) + d(\bar{y}))}MK^2(-2 + 4K|\bar{x} - \bar{y}|) \leq -e^{-L(d(\bar{x}) + d(\bar{y}))}MK^2 < 0. \]

This gives
\[ |\text{tr}(\tilde{X} - \tilde{Y})| = -\text{tr}(\tilde{X} - \tilde{Y}) \geq 4e^{-L(d(\bar{x}) + d(\bar{y}))}MK^2 \geq CMK^2. \]

Since $\|B\| \leq \frac{CMK}{|\bar{x} - \bar{y}|}$, we have
\[ \|B\|^{\frac{1}{2}}|\text{tr}(\tilde{X} - \tilde{Y})|^{\frac{1}{2}} \leq \left( \frac{CMK}{|\bar{x} - \bar{y}|} \right)^{\frac{1}{2}} |\text{tr}(\tilde{X} - \tilde{Y})|^{\frac{1}{2}} \leq \frac{C}{K^{\frac{1}{2}}|\bar{x} - \bar{y}|^{\frac{1}{2}}} |\text{tr}(\tilde{X} - \tilde{Y})|. \]

The Lemma III.I in [10] ensures the existence of a universal constant $C$ depending only on $n$ such that
\[ \|\tilde{X}\|, \|\tilde{Y}\| \leq C\{|\text{tr}(\tilde{X} - \tilde{Y})| + \|B\|^{\frac{1}{2}}|\text{tr}(\tilde{X} - \tilde{Y})|^{\frac{1}{2}}\}. \]

Thanks to the above estimates we can conclude that
\[ \|\tilde{X}\|, \|\tilde{Y}\| \leq C|\text{tr}(\tilde{X} - \tilde{Y})| \left( 1 + \frac{1}{K^{\frac{1}{2}}|\bar{x} - \bar{y}|^{\frac{1}{2}}} \right). \]

Now, using assumptions (F1) and (F2) concerning $F$, the definition of $\tilde{X}$ and $\tilde{Y}$ and the fact that $u$ is sub and supersolution we compute
\[
\begin{align*}
    f(\bar{x}) & \leq F(\bar{x}, u(\bar{x}), D_x\varphi(\bar{x}, \bar{y}), X - C_0\delta D^2d(\bar{x}) + D^2(je^{-Ld(x)}|x - x_0|^2)) \\
    & \leq F(\bar{x}, u(\bar{x}), D_x\varphi(\bar{x}, \bar{y}), \tilde{X}) + MO(K) + CC_0 + Cj \\
    & \leq F(\bar{x}, u(\bar{y}), -D_y\varphi(\bar{x}, \bar{y}), \tilde{Y}) + c|u(\bar{x}) - u(\bar{y})| + b|D_x\varphi(\bar{x}, \bar{y}) + D_y\varphi(\bar{x}, \bar{y})| \\
    & \quad + atr(\tilde{X} - \tilde{Y}) + MO(K) + CC_0 + Cj \\
    & \leq F(\bar{y}, u(\bar{y}), -D_y\varphi(\bar{x}, \bar{y}), \tilde{Y}) + C_1|\bar{x} - \bar{y}|^{\frac{1}{2}}|\tilde{Y}| + 2c|u(\bar{y})| + 2b|D_y\varphi(\bar{x}, \bar{y})| \\
    & \quad + c|u(\bar{x}) - u(\bar{y})| + b|D_x\varphi(\bar{x}, \bar{y}) + D_y\varphi(\bar{x}, \bar{y})| + atr(\tilde{X} - \tilde{Y}) + MO(K) + CC_0 + Cj \\
    & \leq f(\bar{y}) + C_1|\bar{x} - \bar{y}|^{\frac{1}{2}}|\tilde{Y}| + 2c|u(\bar{y})| + 2b|D_y\varphi(\bar{x}, \bar{y})| + c|u(\bar{x}) - u(\bar{y})| \\
    & \quad + b|D_x\varphi(\bar{x}, \bar{y}) + D_y\varphi(\bar{x}, \bar{y})| + atr(\tilde{X} - \tilde{Y}) + MO(K) + CC_0 + Cj.
\end{align*}
\]
From these inequalities, using (15), (20) and (25), for \( K > \overline{K} \), where \( \overline{K} \) is a constant depending only on \( a, A, C_1, n \) and \( \Omega \), we get

\[
-2 |f|_{L^\infty(\Omega)} - 4c|u|_{L^\infty(\overline{\Omega}(x_0,\rho))} - C|g|_{C^{0,1}(\partial \Omega)}
\]

\[
\leq Cb |D\varphi|_\infty + MO(K) + Cj + C|\text{tr}(\tilde{x} - \tilde{y})||x - y|^\frac{1}{2} + K^{-\frac{1}{2}} + a\text{tr}(\tilde{x} - \tilde{y})
\]

\[
\leq CM \left(-K^2 + bK + \frac{b}{\rho} + \frac{1}{\rho^2}\right) + Cb|g|_{C^{0,1}(\partial \Omega)}.
\]

Then, since we have chosen \( M > 1 \), for \( K > \overline{K} \) we obtain

\[
K^2 - bK \leq C \left(|f|_{L^\infty(\Omega)} + c|u|_{L^\infty(\overline{\Omega}(x_0,\rho))} + (1 + b)|g|_{C^{0,1}(\partial \Omega)} + \frac{b}{\rho} + \frac{1}{\rho^2}\right),
\]

and this is a contradiction for \( K \) large enough. This implies that there exists \( K \) satisfying (28), such that (13) holds true. Next, choosing \( x = x_0 \), (13) gives

\[ u(x_0) - u(y) \leq \varphi(x_0, y) \quad \forall y \in \overline{B_\rho(x_0, \rho) \cap \Omega}. \]

Repeating the proof in \( \overline{B_\rho(x, 2\rho)} \) for any \( x \in \overline{B_\rho(x_0, \rho)} \), we finally find the \( u \) satisfies (6) and (7).

**Proof of Corollary 3.2** Let us define

\[ v(x) := e^{\alpha d(x)}u(x). \]

Then, \( v \) is a solution of

\[
\begin{cases}
F(x, v, Dv, D^2v) = e^{\alpha d(x)} f(x) & \text{in } \Omega, \\
\frac{\partial v}{\partial n} = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where

\[ F(x, r, p, X) = M_{a,A}^+ \{X - \alpha (Dd \otimes p + p \otimes Dd) + \alpha^2 r (Dd \otimes Dd) - \alpha r D^2d\} + \lambda r. \]

It is easy to check that \( F \) satisfies assumptions (F1) and (F2) with \( C_1 = 0 \), and

\[ c = C(\alpha^2 + \alpha + |\lambda|), \quad b = C\alpha, \]

where \( C \) depends on \( a, A, n \) and \( \Omega \). Then, by Proposition 3.1, the Lipschitz constant of \( v \) on \( \Omega_\rho \) is bounded from above by \( M_v K_v \), where \( M_v \leq C(|e^{\alpha d(x)}u|_{L^\infty(\Omega_\rho)} + 1) \) and \( K_v \) satisfies (2). Hence, for any \( x, y \in \Omega_\rho \), we have

\[ |u(x) - u(y)| \leq |e^{-\alpha d(x)} - e^{-\alpha d(y)}||v(x)| + e^{-\alpha d(y)}|v(x) - v(y)| \leq (\alpha |e^{\alpha d(x)} u|_{L^\infty(\Omega_\rho)} + M_v K_v)|x - y|, \]

and this concludes the proof.

4. **Properties of the principal eigenvalues**

**Proposition 4.1** (Existence of principal eigenfunctions). There exists \( u^+_\alpha > 0 \) and \( u^-_\alpha < 0 \) on \( \overline{\Omega} \) respectively viscosity solution of

\[
\begin{cases}
M_{a,A}^+(D^2u^+_\alpha) + \lambda^+_\alpha u^+_\alpha = 0 & \text{in } \Omega, \\
\frac{\partial u^+_\alpha}{\partial n} = \alpha u^+_\alpha & \text{on } \partial \Omega, 
\end{cases}
\]

\[
\begin{cases}
M_{a,A}^+(D^2u^-_\alpha) + \lambda^-_\alpha u^-_\alpha = 0 & \text{in } \Omega, \\
\frac{\partial u^-_\alpha}{\partial n} = \alpha u^-_\alpha & \text{on } \partial \Omega.
\end{cases}
\]
Proof. We follow the arguments of [5]. To show the existence of positive eigenfunctions, the first step is to prove that if \( f \) is a continuous function such that \( f \leq 0, f \neq 0 \), then for any \( \lambda < \lambda_0^+ \) there exists a positive solution of

\[
\begin{aligned}
\mathcal{M}_{a,A}^+(D^2u) + \lambda u &= f(x) \quad \text{in } \Omega, \\
\frac{\partial u}{\partial n} &= \alpha u \quad \text{on } \partial \Omega.
\end{aligned}
\]

Observe that \( v \equiv 1 \) is a positive subsolution of (1) for \( \lambda \geq 0 \). This implies, by Proposition 2.2, that if \( \lambda < \lambda_0^+ \) then \( \lambda < 0 \). Let \((v_n)_n\) be the sequence defined by \( v_1 = 0 \) and \( v_{n+1} \) be the solution of

\[
\begin{aligned}
F(x, v_{n+1}, Dv_{n+1}, D^2v_{n+1}) - (c - \lambda)v_{n+1} &= e^{\alpha d(x)}f(x) - cv_n \quad \text{in } \Omega, \\
\frac{\partial v_{n+1}}{\partial n} &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

where

\[
F(x, r, p, X) = \mathcal{M}_{a,A}^+(X - \alpha(Dd \otimes p + p \otimes Dd) + \alpha^2 r(Dd \otimes Dd) - \alpha rD^2d)
\]

and \( c = C(\alpha^2 + \alpha) \). By comparison, the sequence is positive and increasing. Let \((u_n)_n\) be the sequence defined by \( u_n(x) := e^{-\alpha d(x)}v_n(x) \), then \( u_{n+1} \) is solution of

\[
\begin{aligned}
\mathcal{M}_{a,A}^+(D^2u_{n+1}) - (c - \lambda)u_{n+1} &= f(x) - cu_n \quad \text{in } \Omega, \\
\frac{\partial u_{n+1}}{\partial n} &= \alpha u_{n+1} \quad \text{on } \partial \Omega.
\end{aligned}
\]

We claim that \((u_n)_n\) is bounded. Suppose that it is not, then defining \( w_n := \frac{u_n}{|u_n|_\infty} \), one gets that \( w_{n+1} \) is a solution of

\[
\begin{aligned}
\mathcal{M}_{a,A}^+(D^2w_{n+1}) - (c - \lambda)w_{n+1} &= \frac{f(x)}{|u_{n+1}|_\infty} - c \frac{|u_n|_\infty}{|u_{n+1}|_\infty}w_n \quad \text{in } \Omega, \\
\frac{\partial w_{n+1}}{\partial n} &= \alpha w_{n+1} \quad \text{on } \partial \Omega.
\end{aligned}
\]

By Corollary 3.2 \((w_n)_n\) converges along a subsequence to a positive function \( w \) which satisfies

\[
\begin{aligned}
\mathcal{M}_{a,A}^+(D^2w) + \lambda w &= c(1 - k)w \geq 0 \quad \text{in } \Omega, \\
\frac{\partial w}{\partial n} &= \alpha w \quad \text{on } \partial \Omega,
\end{aligned}
\]

where \( k := \lim_{n \to +\infty} \frac{|u_n|_\infty}{|u_{n+1}|_\infty} \leq 1 \). This contradicts the Maximum Principle, Proposition 2.2. Then \((u_n)_n\) is bounded and letting \( n \) go to infinity, by the compactness result, the sequence converges uniformly to a function \( u \) which is a solution of (31). Moreover, \( u \) is positive by the Strong Comparison Principle, Theorem 2.1.

We are now in position to construct a sequence \((u_n)_n\) of positive solutions of

\[
\begin{aligned}
\mathcal{M}_{a,A}^+(D^2u_n) + \lambda_n u_n &= -1 \quad \text{in } \Omega, \\
\frac{\partial u_n}{\partial n} &= \alpha u_n \quad \text{on } \partial \Omega,
\end{aligned}
\]

where \((\lambda_n)_n\) is an increasing sequence which converges to \( \lambda_0^+ \). The sequence \((u_n)_n\) is unbounded, otherwise one would contradict the definition of \( \lambda_0^+ \) (see Theorem 8 of [5]). Then, up to subsequence, \( |u_n|_\infty \to +\infty \) as \( n \to +\infty \) and defining \( \phi_n := \frac{u_n}{|u_n|_\infty} \), one gets that \( \phi_n \) satisfies

\[
\begin{aligned}
\mathcal{M}_{a,A}^+(D^2\phi_n) + \lambda_n \phi_n &= -\frac{1}{|u_n|_\infty} \quad \text{in } \Omega, \\
\frac{\partial \phi_n}{\partial n} &= \alpha \phi_n \quad \text{on } \partial \Omega.
\end{aligned}
\]
By Corollary 3.2 an extracted subsequence converges to a function $u^+_{\alpha}$ with $|u^+_{\alpha}|_{\infty} = 1$, which is a solution of \eqref{eq:29}. Moreover, by Theorem 2.1 $u^+_{\alpha} > 0$ on $\Omega$.

Similar arguments show the existence of negative solutions of \eqref{eq:30}.

\begin{proof}
\end{proof}

**Proposition 4.2** (Simplicity of the first eigenvalues, [16] Proposition 7.1). Let $v \in C(\overline{\Omega})$ be a viscosity subsolution (resp. supersolution) of \eqref{eq:29} (resp. \eqref{eq:30}), then there exists $t \in \mathbb{R}$ such that $v \equiv tu^+_{\alpha}$ (resp. $v \equiv tu^-_{\alpha}$).

**Remark 4.3.** Remark that

\begin{equation}
\lambda^+_\alpha < -A\alpha^2,
\end{equation}

\begin{equation}
\lambda^-_{\alpha} < -a\alpha^2.
\end{equation}

Indeed, the function $v(x) := e^{\alpha x_1}$, where $x_1$ is the first coordinate of $x \in \mathbb{R}^n$, is a positive subsolution of

\begin{equation}
\begin{cases}
\mathcal{M}^+_{a,A}(D^2v) - A\alpha^2v = 0 & \text{in } \Omega, \\
\frac{\partial v}{n} = \alpha v & \text{on } \partial\Omega.
\end{cases}
\end{equation}

Then the Maximum Principle, Proposition 2.2 implies that $\lambda^+_\alpha \leq -A\alpha^2$. If $\lambda^+_\alpha = -A\alpha^2$, then by Proposition 4.2 $v(x)$ is a solution of \eqref{eq:29} and this implies that $\Omega = \mathbb{R}^n$. Hence \eqref{eq:32} holds true. Similarly, inequality \eqref{eq:33} is a consequence of the Minimum Principle, Proposition 2.3, of Proposition 4.2 and the fact that $-v(x)$ is a negative supersolution of \eqref{eq:34} with $A$ replaced by $a$.

**Remark 4.4.** Since $\lambda^+_\alpha, \lambda^-_{\alpha} < 0$ the operator $\mathcal{M}^+_{a,A}(D^2u) + \lambda u$, with $\lambda = \lambda^+_\alpha$ or $\lambda = \lambda^-_{\alpha}$ satisfies the Dirichlet Comparison Principle.

**Proposition 4.5.** The sequences $(\lambda^+_\alpha)_\alpha$ and $(\lambda^-_{\alpha})_\alpha$ are decreasing.

**Proof.** Let us prove that $(\lambda^+_\alpha)_\alpha$ is decreasing. Consider $0 < \alpha_1 < \alpha_2$ and let $u^+_{\alpha_1}$ be a solution of \eqref{eq:29} with $\alpha = \alpha_1$. Then $u^+_{\alpha_1}$ is a positive subsolution of

\begin{equation}
\begin{cases}
\mathcal{M}^+_{a,A}(D^2u) + \lambda^+_\alpha u = 0 & \text{in } \Omega, \\
\frac{\partial u}{n} = \alpha_2 u & \text{on } \partial\Omega,
\end{cases}
\end{equation}

and the Maximum Principle, Proposition 2.2 implies $\lambda^+_\alpha \geq \lambda^+_\alpha$. The strict inequality $\lambda^+_\alpha > \lambda^+_\alpha$ follows from Proposition 4.2 \hfill \square

**Lemma 4.6.** Let $u^+_{\alpha}$ and $u^-_{\alpha}$ be respectively a positive solution of \eqref{eq:29} and a negative solution of \eqref{eq:30}, then

\begin{align*}
u^+_{\alpha}(x) &< \max_{\partial\Omega} u^+_{\alpha} & \forall x \in \Omega, \\
u^-_{\alpha}(x) &> \min_{\partial\Omega} u^-_{\alpha} & \forall x \in \Omega.
\end{align*}

**Proof.** Let us show the result for $u^+_{\alpha}$. Suppose by contradiction that the maximum of $u^+_{\alpha}$ is attained at some point $x_0 \in \Omega$ and let $v(x) := u^+_{\alpha}(x) - u^+_{\alpha}(x_0)$. Since $u^+_{\alpha}(x_0) > 0$ and $\lambda^+_{\alpha} < 0$, $v$ satisfies

\begin{equation}
\mathcal{M}^+_{a,A}(D^2v) + \lambda^+_{\alpha} v \geq 0 & \text{in } \Omega
\end{equation}

and $v \leq 0$ in $\Omega$, $v(x_0) = 0$. Then the Strong Maximum Principle implies $u^+_{\alpha} \equiv u^+_{\alpha}(x_0)$ in $\Omega$ and this contradicts the fact that $u^+_{\alpha}$ solves \eqref{eq:29}. \hfill \square
5. Liouville type results

For \( \gamma > 0 \) let us introduce the system

\[
\begin{aligned}
\mathcal{M}_{\nu}^+(D^2 u) - \gamma u &= 0 \quad \text{in } \mathbb{R}^n_+,
\quad \text{on } \partial \mathbb{R}^n.
\end{aligned}
\]

**Theorem 5.1.** If \( \gamma > A \), any bounded subsolution of (35) is non-positive in \( \mathbb{R}^n_+ \).

If \( \gamma > a \), any bounded supersolution of (35) is non-negative in \( \mathbb{R}^n_+ \).

Hence, if \( \gamma > A \) there are no, non trivial bounded solutions of (35).

**Remark 5.2.** It turns out that Theorem 5.1 is sharp: \( u(x) = e^{-\gamma x} \) (resp., \( u(x) = -e^{-\gamma x} \)) is a positive bounded subsolution (resp., negative bounded supersolution) of (35) for every \( \gamma \leq A \) (resp., \( \gamma \leq a \)).

Theorem 5.1 also fails without the boundedness condition. Indeed, \( u(x) = e^{\nu x} \) (resp., \( u(x) = -e^{\nu x} \)), with \( \nu = (\nu_1, ..., \nu_{n-1}, -1) \), \( |\nu| > 1 \), is an unbounded subsolution (resp., supersolution) of (35) for \( A < \gamma \leq A|\nu|^2 \) (resp., \( a < \gamma \leq |\nu|^2 a \)).

We assume that \( u(x) \) is a bounded subsolution of (35) with \( \gamma > 0 \), which is positive someplace. We normalize \( u \) so that

\[
\sup_{\mathbb{R}^n_+} u = 1.
\]

Then \( u \) is a viscosity subsolution of

\[
\begin{aligned}
\mathcal{M}_{\nu}^+(D^2 u) - \gamma u &= 0 \quad \text{in } \mathbb{R}^n_+,
\quad -\partial_{x_n} u = 1 \quad \text{on } \partial \mathbb{R}^n.
\end{aligned}
\]

**Proposition 5.3.** Assume \( \gamma > 0 \) and \( k \in \mathbb{R} \). Let \( u \in \text{USC}(\mathbb{R}^n_+) \) and \( v \in \text{LSC}(\mathbb{R}^n_+) \) be respectively bounded viscosity sub and supersolution of

\[
\begin{aligned}
\mathcal{M}_{\nu}^+(D^2 u) - \gamma u &= 0 \quad \text{in } \mathbb{R}^n_+,
\quad -\partial_{x_n} u = k \quad \text{on } \partial \mathbb{R}^n.
\end{aligned}
\]

Then \( u \leq v \) in \( \mathbb{R}^n_+ \).

**Proof.** Suppose by contradiction that \( \sup_{\mathbb{R}^n_+} (u - v) = M > 0 \). Let \( \psi \) be a smooth positive function with bounded derivatives and such that \( \psi(x) \to +\infty \) as \( |x| \to +\infty \). Let \( \chi(x) = \chi(x_n) \) be a smooth function such that \( \chi(x_n) = x_n \) for \( |x_n| \leq 1 \) and \( \chi(x_n) \equiv 0 \) for \( |x_n| > 2 \). Let

\[
\varphi(x, y) = \frac{j}{2} |x - y|^2 - k(x_n - y_n) + \beta \psi(x) - \epsilon (\chi(x) + \chi(y)).
\]

Then, for \( \beta \) and \( \epsilon \) small enough and \( j > 0 \), the supremum of the function \( u(x) - v(y) - \varphi(x, y) \) is greater than \( \frac{M}{2} \) and it is reached at some point \((\overline{x}, \overline{y}) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ \).

If \( \overline{x} \in \partial \mathbb{R}^n_+ \) then, for \( \overline{y} \in \mathbb{R}^n_+ \),

\[
-\partial_{x_n} \varphi(\overline{x}, \overline{y}) - k = -j(\overline{x}_n - \overline{y}_n) + k - \beta \partial_{x_n} \psi(\overline{x}) + \epsilon - k = j\overline{y}_n - \beta \partial_{x_n} \psi(\overline{x}) + \epsilon > 0
\]

for \( \epsilon > \beta |D \psi|_\infty \).

If \( \overline{y} \in \partial \mathbb{R}^n_+ \) then, for \( \overline{x} \in \mathbb{R}^n_+ \),

\[
\partial_{y_n} \varphi(\overline{x}, \overline{y}) - k = -j(\overline{x}_n - \overline{y}_n) - \epsilon = -j\overline{x}_n - \epsilon < 0.
\]

Both inequalities contradict the definition of sub and supersolution, therefore \( \overline{x}, \overline{y} \in \mathbb{R}^n_+ \).
Applying Theorem 3.2 of [8], there exist \( X, Y \in S(n) \) such that \((D_x \varphi(x, y), X + \beta D^2 \psi(x) - \epsilon D^2 \chi(x)) \in J^{2^+}_u(x, y), (D_y \varphi(x, y), Y + \epsilon D^2 \chi(y)) \in J^{2^-}_v(y)\) and
\[
-3j \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq 3j \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.
\]
Since \( u \) and \( v \) are respectively sub and supersolution, we have
\[
M^+_{a, A}(X + \beta D^2 \psi(x) - \epsilon D^2 \chi(x)) \geq \gamma u(x),
\]
\[
M^+_{a, A}(Y + \epsilon D^2 \chi(y)) \leq \gamma v(y).
\]
Subtracting the two previous inequalities, using the properties of Pucci's operators and that
\[
u(x) - v(y) > \frac{M}{2} + \frac{j}{2} |x_n - y_n|^2 - k(x_n - y_n) - \epsilon (\chi(x) + \chi(y)) \geq \frac{M}{2} - \frac{k^2}{2j} - C\epsilon,
\]
we finally get
\[
\frac{\gamma}{2} \left( M - \frac{k^2}{j} - C\epsilon \right) < \gamma (u(x) - v(y)) \leq M^+_{a, A}(X + \beta D^2 \psi(x) - \epsilon D^2 \chi(x))
\]
\[
- M^+_{a, A}(Y + \epsilon D^2 \chi(y)) \leq M^+_{a, A}(X - Y) + \beta M^+_{a, A}(D^2 \psi(x)) - \epsilon M^-_{a, A}(D^2 \chi(x) + D^2 \chi(y))
\]
\[
\leq \beta M^+_{a, A}(D^2 \psi(x)) - \epsilon M^-_{a, A}(D^2 \chi(x) + D^2 \chi(y)).
\]
This is a contradiction for \( \beta \) and \( \epsilon \) small enough and \( j \) large. Then \( u \leq v \) in \( \mathbb{R}^n_+ \).

\[\square\]

**Proof of Theorem 5.1** The function
\[
v(x) = \sqrt{A} e^{-\sqrt{\pi} x_n}
\]
is the bounded viscosity solution of (37). Then by Proposition 5.3
\[
u(x) \leq \sqrt{A} e^{-\sqrt{\pi} x_n}, \text{ for any } x \in \mathbb{R}^n_+.
\]
It follows from (36) that
\[
1 = \sup_{\mathbb{R}^n_+} u \leq \sqrt{\frac{A}{\gamma}},
\]
i.e. \( \gamma \leq A \).

Similarly, if \( u \) is a negative supersolution of (35), normalized so that \( \min_{\mathbb{R}^n_+} u = -1 \), then \( u \) is a supersolution of
\[
\begin{cases} 
M^+_{a, A}(D^2 u) - \gamma u = 0 & \text{in } \mathbb{R}^n_+, \\
-\frac{\partial u}{\partial x_n} = -1 & \text{on } \partial \mathbb{R}^n_+,
\end{cases}
\]
and by comparison
\[
u(x) \geq \sqrt{\frac{a}{\gamma}} e^{-\sqrt{\pi} x_n}.
\]
This implies \( \gamma \leq a \) and Theorem 5.1 is proved.
\[\square\]
Proposition 6.1. \( \lim_{\alpha \to 0} \lambda^\pm_\alpha = 0 \).

**Proof.** By Proposition 4.5, \( \lambda^+_\alpha \) increases to some value \( \lambda_0 \leq 0 \). On the other hand, the sequence of normalized solutions \( (u^+_\alpha)_\alpha \), by the Lipschitz estimates Corollary 3.2, converges to \( u_0 \) a positive solution of
\[
\begin{cases}
M^+_{a,A}(D^2u) + \lambda_0 u = 0 & \text{in } \Omega, \\
\frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega,
\end{cases}
\]
which satisfies \(|u_0| = 1\). Recall that 0 is the principal eigenvalue for the Neumann problem. If \( \lambda_0 < 0 \), the Maximum Principle below the first eigenvalue, i.e. Proposition 2.2, implies that \( u_0 \leq 0 \) a contradiction. \( \square \)

We consider now the asymptotic behavior at infinity. By Remark 4.3, it is enough to show that
\[
\limsup_{\alpha \to +\infty} \frac{\lambda^+_\alpha}{-\alpha^2} \leq A,
\]
and
\[
\limsup_{\alpha \to +\infty} \frac{\lambda^-_\alpha}{-\alpha^2} \leq a.
\]

We are going to show (39). For \( \alpha > 0 \), let \( u^+_\alpha \) be a positive solution of (29). By Lemma 4.6, we know that \( u^+_\alpha \) attains its maximum at \( x_\alpha \in \partial\Omega \). After normalization, we can assume that \( \max_{\Omega} u^+_\alpha = 1 \) and \( x_\alpha \to 0 \) as \( \alpha \to +\infty \). Furthermore, we can assume that there is a \( C^2 \) function \( \phi \) and \( r > 0 \) such that
\[
x_n = \phi(x'), \quad \forall (x', x_n) \in \partial\Omega \cap B_r(0) \\
x_n > \phi(x'), \quad \forall (x', x_n) \in \Omega \cap B_r(0) \\
\phi(0) = 0, \\
\partial x_i \phi(0) = 0, \quad \text{for } i = 1, \ldots, n - 1.
\]

We flatten \( \partial\Omega \) near the origin. Let \( \Phi(x) : \Omega \cap B_r(0) \to \Omega_\Phi := \Phi(\Omega \cap B_r(0)), \) be such that
\[
\Phi_i(x) = x_i, \quad i = 1, \ldots, n - 1, \\
\Phi_n(x) = x_n - \phi(x').
\]

Denote by \( x = \Psi(y) \) the inverse of \( y = \Phi(x) \). The function
\[
v_\alpha(y) = u^+_\alpha(\Psi(y))
\]
is solution of
\[
\begin{cases}
M^+_{a,A} \left[ \sum_{i,k=1}^n \partial^2_{y_ky_i} v_\alpha \partial x_j \Phi_k(\Psi(y)) \partial x_i \Phi_i(\Psi(y)) \right]_{ij} \\
+ \left( \sum_{k=1}^n \partial_{y_k} v_\alpha \partial^2_{x_ix_j} \Phi_k(\Psi(y)) \right)_{ij} + \lambda^+_\alpha v_\alpha = 0 & y \in \Omega_\Phi, \\
\sum_{k,j=1}^n \partial_{y_k} v_\alpha \partial x_j \Phi_k(\Psi(y)) \overrightarrow{n}_j(\Psi(y)) = \alpha v_\alpha & y \in \partial\Omega_\Phi.
\end{cases}
\]
Since the exterior normal \( \vec{n}(x) \) at \( x \in \partial \Omega \cap B_r(0) \) is 
\[
\vec{n}(x) = \frac{(D \phi(x'), -1)}{\sqrt{|D \phi(x')|^2 + 1}},
\]
by (41), the boundary condition in (42) can be rewritten as follows 
\[
\frac{1}{\sqrt{|D \phi(y')|^2 + 1}} \sum_{k=1}^{n-1} \partial y_k v_\alpha \partial x_k \phi(y') - \left( \sqrt{|D \phi(y')|^2 + 1} \right) \partial y_n v_\alpha = \alpha v_\alpha, \quad y \in \partial \Omega_\Phi.
\]
Notice that, since \( D \phi(x') \to 0 \) as \( x' \to 0 \), \( D \Phi(\Psi(y)) \to I \) as \( y \to 0 \), where \( I \) is the identity matrix of \( S(n) \).

We now consider two different cases. 

**Case 1.**
\[
\limsup_{\alpha \to +\infty} \frac{\lambda^+_{\alpha}}{-\alpha^2} = \gamma < +\infty.
\]
Without loss of generality, we may assume that \( \frac{\lambda^+_{\alpha}}{-\alpha^2} \to \gamma \) as \( \alpha \to +\infty \), and \( u^+_{\alpha}(x_\alpha) = \max_{\Omega} u^+_{\alpha} = 1, x_\alpha \to 0 \) as \( \alpha \to +\infty \). We let 
\[
z = \alpha(y - y_\alpha),
\]
where \( y_\alpha = \Phi(x_\alpha) = (x_\alpha', 0) \). We set 
\[
w_\alpha(z) = v_\alpha(y) = u^+_{\alpha}(x),
\]
then for any \( R > 0 \), as \( \alpha \) becomes sufficiently large, \( w_\alpha \) is solution of 
\[
\begin{align}
\mathcal{M}^+_\alpha \left[ \left( \sum_{i,k=1}^{n} \partial^2_{z_i z_k} w_\alpha \partial x_i \Phi_k(\Psi(y)) \partial x_i \Phi_i(\Psi(y)) \right)_{ij} 
+ \frac{1}{\alpha} \left( \sum_{k=1}^{n} \partial z_k w_\alpha \partial^2_{x_i x_j} \Phi_k(\Psi(y)) \right)_{ij} 
+ \frac{\lambda^+_{\alpha}}{\alpha^2} w_\alpha = 0 \right] 
\quad z \in B^+_R,
\end{align}
\]
where 
\[
y = y(z) = \frac{z}{\alpha} + y_\alpha
\]
and 
\[
B^+_R := B_R(0) \cap \mathbb{R}^n_+, \quad \Gamma_R := B_R(0) \cap \partial \mathbb{R}^n_+.
\]
Since for \( z \in B^+_R \), \( z/\alpha + y_\alpha \to 0 \) as \( \alpha \to +\infty \) and \( \partial z_i \phi(0) = 0 \) for \( i = 1, \ldots, n-1 \), for \( \alpha \) sufficiently large, \( I/2 \leq D\Psi(z/\alpha + y_\alpha) \leq 2I \). Hence, if \( L_\alpha \) is the Lipschitz constant of \( u^+_{\alpha} \) in the set \( \{ x = \Psi(z/\alpha + y_\alpha), \ |z| \leq R \} \), we have 
\[
|w_\alpha(z_1) - w_\alpha(z_2)| = \left| u^+_{\alpha} \left( \Psi \left( \frac{z_1}{\alpha} + y_\alpha \right) \right) - u^+_{\alpha} \left( \Psi \left( \frac{z_2}{\alpha} + y_\alpha \right) \right) \right| \leq \frac{2L_\alpha}{\alpha} |z_1 - z_2|.
\]
Remark that if \( |z| \leq R \), then \( d(x) \leq CR/\alpha \) for \( x = \Psi(z/\alpha + y_\alpha) \), where \( C \) depends on \( \phi \). Hence, since for \( \rho = CR/\alpha \), \( |e^{ad(x)} u^+_{\alpha}|_{L^\infty(\Omega_\alpha)} \leq e^{3CR} \), Corollary 3.2 gives 
\[
L_\alpha \leq Ce^{3CR}(\alpha + K_\alpha),
\]
where \( K_\alpha \) satisfies 
\[
K_\alpha^2 - CaK_\alpha \leq C \left[ (\alpha + \alpha^2 + |\lambda^+_{\alpha}|)e^{3CR} + \frac{\alpha^2}{CR^2} + 1 \right].
\]
This implies that the sequence \((w_\alpha)_{\alpha}\) is bounded in the space of Lipschitz continuous functions of \(\overline{B_R^+}\) for any fixed \(R > 0\), and then, up to subsequence, \(w_\alpha \to w_0\) uniformly on \(\overline{B_R^+}\), with \(\sup_{\mathbb{R}^n} w = 1\), viscosity solution of (35). Moreover by the Strong Comparison Principle, \(w > 0\) on \(\mathbb{R}^n\). Then, by Theorem 5.1 \(\gamma \leq A\) and this proves (39).

**Case 2.**

\[
\limsup_{\alpha \to +\infty} \frac{\lambda_0^+}{-\alpha^2} = +\infty.
\]

Let \(u_\alpha^+\) be the sequence of positive solutions of (29) such that

\[
\frac{\lambda_0^+}{-\alpha^2} = l_\alpha \to +\infty \quad \text{as} \quad \alpha \to +\infty,
\]

and \(u_\alpha^+(x_\alpha) = \max_{\Omega} u_\alpha^+ = 1\). Define

\[
z = \sqrt{l_\alpha} \alpha (y - y_\alpha) \quad \text{and} \quad w_\alpha(z) = u_\alpha(x),
\]

where \(y = \Phi(x)\) and \(y_\alpha = \Phi(x_\alpha)\). Then, for any \(R > 0\), as \(\alpha\) becomes sufficiently large, \(w_\alpha\) satisfies

\[
\begin{align*}
\mathcal{M}_{a,A}^+ \left[ \left( \sum_{l,k=1}^n \partial_{x_l x_k} w_\alpha \partial_x \Phi_k(\Psi(y)) \partial_x \Phi_l(\Psi(y)) \right)_{ij} 
+ \frac{1}{\sqrt{-\lambda_0^+}} \left( \sum_{k=1}^n \partial_x w_\alpha \partial_{x,x_j} \Phi_k(\Psi(y)) \right)_{ij} \right] - w_\alpha = 0 & \quad \text{in} \quad B_R^+, \\
\frac{1}{\sqrt{|D\Phi(y')|^2 + 1}} \sum_{k=1}^n \partial_{x_k} w_\alpha \partial_x \Phi(y') - \sqrt{|D\Phi(y')|^2 + 1} \partial_{x^2} w_\alpha = \frac{1}{\sqrt{l_\alpha}} w_\alpha & \quad \text{for} \quad z \in \Gamma_R,
\end{align*}
\]

where \(y = y(z) = \Phi + y_\alpha\). As in Case 1, we can show that \(w_\alpha \to w_0\) which is a bounded positive viscosity solution of

\[
(44) \quad \begin{cases} \mathcal{M}_{a,A}^+(D^2 w_0) - w_0 = 0 & \quad \text{in} \quad \mathbb{R}_+^n, \\
-\partial_{x_0} w_0 = 0 & \quad \text{on} \quad \partial \mathbb{R}^n.
\end{cases}
\]

On the other hand, by Proposition 5.3, the only bounded viscosity solution of (44) is \(u \equiv 0\) and we reach a contradiction. \(\square\)

**Proposition 6.2.** Let \(u_\alpha^+\) and \(u_\alpha^-\) be respectively the normalized solution of (29) and (30), i.e. \(\|u_\alpha^+\|_\infty = 1\), then for any compact set \(K \subset \Omega\)

\[
\|u_\alpha^+\|_{L^\infty(K)} \to 0 \quad \text{as} \quad \alpha \to +\infty.
\]

**Proof.** Let \(u_\alpha^+\) be the normalized solution of (29) and let \(K\) be a compact set contained in \(\Omega\). Let \(x_\alpha \in K\) be such that \(\max_K u_\alpha^+ = u_\alpha^+(x_\alpha)\). Define \(z = \alpha(x - x_\alpha)\) and \(w_\alpha(z) = u_\alpha^+(x)\) for \(|z| < \alpha r\) where \(r = \text{dist}(K, \partial \Omega)\). Then for any \(R > 0\), as \(\alpha\) becomes large, \(w_\alpha(z)\) satisfies

\[
\mathcal{M}_{a,A}^+(D^2 w_\alpha) + \frac{\lambda_0^+}{\alpha^2} w_\alpha = 0 \quad \text{in} \quad B_{2R},
\]

and \(\|w_\alpha\|_\infty \leq 1\). By standard elliptic estimates, see e.g. [7] and Theorem 1.1 \(w_\alpha \to w_0\) non-negative solution of

\[
\mathcal{M}_{a,A}^+(D^2 w_0) - Aw_0 = 0 \quad \text{in} \quad \mathbb{R}^n.
\]

It is well-known that there are no nontrivial bounded solutions of the above equation, see e.g. [8], hence \(w_\alpha(0) = \max_K u_\alpha^+ \to 0\) as \(\alpha \to +\infty\) and Proposition 6.2 is proved. \(\square\)
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