Soft-wall stabilization

Joan A Cabrer\textsuperscript{1}, Gero von Gersdorff\textsuperscript{2} and Mariano Quirós\textsuperscript{1,3,4}

\textsuperscript{1} IFAE, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Spain
\textsuperscript{2} CERN Theory Division, CH-1211 Geneva 23, Switzerland
\textsuperscript{3} Institució Catalana de Recerca i Estudis Avançats (ICREA), 08010 Barcelona, Spain
E-mail: quiros@ifae.es

New Journal of Physics 12 (2010) 075012 (21pp)
Received 5 August 2009
Published 16 July 2010
Online at http://www.njp.org/
doi:10.1088/1367-2630/12/7/075012

Abstract. We propose a general class of five-dimensional (5D) soft-wall models with AdS metric near the ultraviolet brane and 4D Poincaré invariance, where the infrared scale is determined dynamically. A large UV/IR hierarchy can be generated without any fine-tuning, thus solving the electroweak/Planck scale hierarchy problem. Generically, the spectrum of fluctuations is discrete with a level spacing (mass gap) provided by the inverse length of the wall, similar to RS1 models with standard model fields propagating in the bulk. Moreover, two particularly interesting cases arise. They can describe: (a) a theory with a continuous spectrum above the mass gap, which can model unparticles corresponding to operators of a CFT where the conformal symmetry is broken by a mass gap; and (b) a theory with a discrete spectrum provided by linear Regge trajectories as in AdS/QCD models.

\textsuperscript{4} Author to whom any correspondence should be addressed.
1. Introduction

Warped extra dimensions were introduced by Randall and Sundrum [1] (RS1) as a very elegant way of solving the hierarchy problem by means of the geometry of extra-dimensional theories. The original proposal consisted of a slice of AdS space bounded by two branes: the ultraviolet (UV) brane, located closely to the AdS boundary, and the infrared (IR) one. One of the most exciting aspects of RS1 theories is provided by the AdS/CFT correspondence [2], by which fields living in the UV brane are fundamental fields which can interact with a strongly coupled conformal sector (CFT), while fields living in the IR brane are the dual description of operators of the CFT. In this way, a Higgs boson localized on the IR brane provides a dual description of the composite Higgs as a bound state of an extra strong interaction (technicolor). From the five-dimensional (5D) point of view the Higgs boson mass is simply redshifted from its natural value, of the order of the Planck mass, to the TeV scale by the warp factor. The IR brane spontaneously breaks the conformal symmetry and its location needs to be stabilized by some dynamical mechanism.

A dynamical mechanism to stabilize the IR brane was proposed by Goldberger and Wise [3] (GW) by the introduction of a background scalar field propagating in the 5D bulk and acquiring a coordinate-dependent vacuum expectation value (VEV). This triggered a 4D effective potential for the radion field with a minimum, stabilizing the brane separation at scales of order 1/TeV, determined by the values of the scalar field at both the UV and IR branes, with a modest ~1% fine-tuning. Furthermore, the back-reaction of the scalar field through the 5D Einstein equations generated a deviation from AdS of the metric far away from the UV brane, still preserving the main features of the AdS/CFT correspondence. In fact, this is a general feature since AdS is the only 5D metric consistent with no (or constant) background scalar field, which means that any stabilizing bulk field is expected to back-react on the AdS metric.

Since the presence of bulk scalar fields, required to stabilize the brane distance, is expected to disturb the AdS metric in the far IR, one can introduce phenomenological models with background scalar fields and with the only constraint of describing the AdS geometry near the UV brane. One of them is of course RS1 with GW mechanism, but there are more general models where the IR brane is replaced by a naked curvature singularity [5]–[8]: these models

5 For an earlier analysis of models with exponential (Liouville) dilatonic brane potentials see [4].
are called soft-wall models [9]–[14]. Soft-wall models possess AdS geometry near the UV brane and no IR brane, i.e. they have a non-compact extra dimension as those proposed by Randall and Sundrum with an infinite extra dimensional length [15] (RS2), but with additional background scalar fields. These fields back-react on the metric and generate a singularity at a finite value of the extra dimensions: the (finite) length of the extra dimension provides the location of the soft wall.

Soft-wall models were first introduced to model the Regge behaviour of excited mesons in AdS/QCD models [9, 10], as an alternative to RS1 for electroweak-breaking models solving the hierarchy problem and providing experimental signatures at the Large Hadron Collider (LHC) [12, 13] and also to provide a 5D setup to the theories of unparticles recently proposed by Georgi [16] in the presence of a mass gap [14, 17]. In all these theories, as in GW models, there are background bulk scalar fields and thus a built-in mechanism to stabilize the extra dimension, i.e. the distance between the UV brane and the singularity. However, in the proposed models this distance is naturally of the order of the AdS length and the hierarchy problem does not find a natural solution, i.e. for values of the scalar field at the UV brane of the order of the 5D Planck scale [13].

In this paper, we will propose a set of one-parameter ($\nu$) 5D models with a background scalar field propagating in the bulk of the extra dimension and with the following properties:

- The metric is AdS near the UV brane.
- The hierarchy between the AdS length and the soft-wall length (the electroweak length) is naturally stabilized for values of the scalar field at the UV brane of the order of the 5D Planck scale. The naturalness of this result comes from a double exponential suppression.
- For values of $\nu < 2$ the naked singularity does not contribute to the vacuum energy and thus it does not need to be resolved to satisfy Einstein equations. Moreover, the potential is bounded from above in the solution and the singularity is a physical one according to [7].
- For $1 < \nu < 2$ the spectrum of fluctuations behaves as in RS1 models with standard model (SM) fields propagating in the bulk: for scalar field fluctuations localized near the singularity the hierarchy problem is automatically solved.
- For $\nu = 1$ one can model unparticles with a mass gap provided by the inverse length of the extra dimension. Fluctuations have a continuous spectrum above the mass gap.
- For $0 < \nu < 1$ unparticles without a mass gap emerge: the spectrum is continuous above zero mass.

The paper is organized as follows: In section 2, we review the general construction of backgrounds and the associated question of self versus fine-tuning of the cosmological constant (CC). The general conditions for models with physical singularities are summarized. In section 3, we consider the background solutions for a particularly simple class of models satisfying all physical requirements on the singularity and describing the electroweak/Planck hierarchy without fine-tuned parameters. Fluctuations on the background for the graviton, radion and the scalar field are studied in section 4 where a complete numerical analysis and some analytical approximations are provided. A general class of models having all the good required properties, included the hierarchy determination, is presented in section 5. One of these models has the mass of excitations as $m_n^2 \sim n$ and it is thus a good candidate to model the Regge behaviour in AdS/QCD models. Finally, our conclusions and outlook are drawn in section 6.

6 A preliminary version of this work can be found in [18].
2. The five-dimensional scalar gravity system

In this section, we would like to review the construction of backgrounds with 4D Poincaré invariance proposed in [19, 20], and the associated question of self- versus fine-tuning of the CC [5]–[8], [21]. Let us consider 5D gravity with a scalar field, and look for the most general solutions to this system that preserve 4D Poincaré invariance, i.e. a background of the form

\[ ds^2 = e^{-2A(y)} dx^\mu dx^\nu \eta_{\mu\nu} + dy^2, \]  

(2.1)

with a ‘mostly plus’ flat Mikowskian metric \( \eta_{\mu\nu} = \text{diag} (-, +, +, +, +) \) and an arbitrary warp factor \( A(y) \). We will introduce a single brane sitting at \( y = 0 \) and impose the orbifold \( \mathbb{Z}_2 \) symmetry \( y \to -y \) under which \( A \) and \( \phi \) are even. We will thus consider the bulk plus brane action

\[ S = \int d^5x \sqrt{-g} \left[ M^3 R - 3(\partial \phi)^2 - V(\phi) \right] - \int d^4x \sqrt{-g_{\text{ind}}} \lambda(\phi). \]  

(2.2)

We have introduced arbitrary bulk and brane potentials \( V \) and \( \lambda \), and \( M \) denotes the 5D Planck mass, which we will set to unity for the remainder of the paper. Note the non-canonical form of the kinetic term for the scalar that will simplify future formulae. The bulk equations of motion (EOMs) that follow from the action in equation (2.2) are

\[ 6\phi'' - 24A'\phi' - \partial_\phi V(\phi) = 0, \]  

(2.3)

\[ A'' - \phi^2 = 0, \]  

(2.4)

\[ 12A'^2 - 3\phi^2 + V = 0. \]  

(2.5)

This system has three integration constants\(^7\). One of them is \( A(0) \) that remains totally free. The other two can be fixed from the boundary conditions (BCs) that follow from the boundary pieces of the EOM,

\[ A'(0_+) = 16 \lambda(\phi_0), \]  

(2.6)

\[ \phi'(0_+) = 16 \partial_\phi \lambda(\phi_0), \]  

(2.7)

where \( \phi_0 = \phi(0) \). Using equations (2.6) and (2.7) in equation (2.5) determines \( \phi_0 \),

\[ \frac{1}{12} [\partial_\phi \lambda(\phi_0)]^2 - \frac{1}{3} \lambda(\phi_0)^2 = V(\phi_0), \]  

(2.8)

which could be used to replace (2.7).

The authors of [20] introduced the following ‘trick’ to obtain solutions to this system by defining the so-called ‘superpotential’ via the differential equation

\[ 3(\partial_\phi W)^2 - 12W^2 = V, \]  

(2.9)

and writing

\[ A' = W(\phi), \]  

(2.10)

\[ \phi' = \partial_\phi W(\phi), \]  

(2.11)

\(^7\) Differentiating the third equation one obtains a linear combination of the first two, such that (say) the first can be discarded. The system is then first order in \( \phi \) and second order in \( A \).
while the BCs are satisfied if
\[
W(φ_0) = \frac{1}{2} \lambda(φ_0), \quad \partial_φ W(φ_0) = \frac{1}{6} \partial_φ \lambda(φ_0).
\] (2.12)

Again, the system of equations (2.9)–(2.11) has three integration constants and in principle every solution to equations (2.3)–(2.5) can be constructed in this way [20]. One integration constant is the trivial additive constant \(A(0)\) that does not enter in equation (2.12). We are left with the integration constant in equation (2.9) and the value \(φ_0\) to fix the two constraints equation (2.12).

The equation for \(W\) is a complicated nonlinear differential equation, and in practice it is often easier to start with a particular superpotential satisfying the BCs and deduce the potential needed to reproduce it.

A peculiarity of the scalar-gravity system with one brane is the appearance of naked curvature singularities at finite proper distance. In particular, it can easily be checked that if the superpotential \(W\) grows faster than \(e^φ\) at large \(φ\), the profile \(φ(y)\) diverges at finite value of \(y = y_s\). Moreover, the 5D curvature scalar along the fifth dimension can be written as
\[
R(y) = 8(\partial_φ W(φ(y)))^2 - 20W(φ(y))^2,
\] (2.13)

so that the curvature in general diverges at \(y = y_s\). The interpretation is that spacetime ends at \(y_s\).

Having three integration constants but only two constraints, it seems that one can obtain flat 4D solutions with fairly generic brane and bulk potentials without fine-tuning. This miraculous self-tuning property of the scalar-gravity system was first pointed out in [5, 6] and it was further scrutinized in several papers [7, 8, 21]. In particular, Forste et al [21] pointed out that the on-shell Lagrangian, integrated over the fifth dimension can be written as
\[
\mathcal{L}^{on-shell} = \frac{1}{2} \lambda(φ_0) + \frac{2}{3} \int dy \ e^{-4A(y)} V(φ[y])
\] (2.14)

(we have set \(A(0) = 0\)). They then make particular choices for \(V\) and \(λ\) and show that the result is non-vanishing. The interpretation of this apparent contradiction to the existence of a flat background is simple: having dynamically generated a new boundary at the singularity, we must ensure that the boundary pieces of the equations of motion vanish at \(y = y_s\). If this is not the case, the resulting ‘solution’ does in fact not extremize the action, resulting in a nonzero 4D CC. This can actually be seen in rather general terms. Making use of the equations for the superpotential, we write
\[
e^{-4A(y)} V(φ[y]) = 3 \frac{d}{dy} \left[ e^{-4A(y)} W(φ[y]) \right],
\] (2.15)

leading to
\[
\mathcal{L}^{on-shell} = \frac{1}{3} \lambda(φ_0) - 2W(φ_0) + 2 \lim_{y \to y_s} e^{-4A(y)} W(φ[y]).
\] (2.16)

The first two terms cancel if equation (2.12) is satisfied, while the last one depends on the particular form of the superpotential. In order for the last term to vanish, \(W\) needs to grow more slowly than \(e^{2φ}\) at large \(φ\); this can be seen by using the field \(φ\) itself as a coordinate. The position of the singularity moves to \(φ(y_s) = ∞\) and the equation for \(A(φ)\) becomes \(A' = W/W'\). It follows that the last term in equation (2.16) goes to a constant for \(W \sim e^{2φ}\), while it goes to infinity (zero) when \(W\) grows faster (slower) than \(e^{2φ}\). We thus arrive at a simple criterion for the existence of singular solutions:

\[
A \text{ singularity with } φ(y_s) → ∞ \text{ is allowed if and only if } W(φ) \text{ grows asymptotically more slowly than } e^{2φ}.
\] (2.17)
Note that a potential $V$ growing more slowly than $e^{4\phi}$ is only necessary but not sufficient for the validity of (2.17), the trivial counterexample being $V \equiv 0$, which has the general solution $W = ce^{2\phi}$. It is instructive to compare our criterion with the one found in [7] where AdS-CFT duality was used to classify physical singularities. According to Gubser [7] admissible singularities are those whose potential is bounded above in the solution. Inspection of equation (2.9) shows that singularities fulfilling (2.17) have a potential that goes to $-\infty$, whereas those that fail (2.17) go to $+\infty$. Although we here employ a much more basic condition (a consistent solution to the Einstein equations), it is good to know that our allowed solutions have potentially consistent interpretations as 4D gauge theories at finite temperature.

However, achieving a superpotential that grows more slowly than $e^{2\phi}$ requires a hidden fine-tuning of the CC. To see this it suffices to consider a potential that behaves asymptotically as

$$V \sim be^{2\nu \phi}.$$  \hfill (2.18)

Writing $W$ as

$$W(\phi) = w(\phi)e^{\nu \phi},$$  \hfill (2.19)

we can express the solutions for $w$ as the roots of

$$e^{(4-\nu^2)(\phi-c)} = \left(2w + \sqrt{b+4w^2}\right)^{\pm 2} \left(vw \mp \sqrt{b+4w^2}\right)^{\nu},$$  \hfill (2.20)

where $c$ is an integration constant. For $\nu > 2$ this implies that $w$ asymptotes to a constant

$$w \sim \pm \frac{\sqrt{b}}{\nu^2 - 4}$$  \hfill (2.21)

at large $\phi$ and for $b > 0$. However, for $0 < \nu < 2$, $w$ generically behaves as

$$w \sim e^{(2-\nu)\phi}.$$  \hfill (2.22)

Only if we adjust $c \rightarrow \infty$, we can achieve that $w$ behaves as in equation (2.21). In this case, $b$ has to be negative in order to have a real solution for $W$.

The generic solution to equation (2.9) thus grows as $W \sim e^{\nu \phi}$ for $\nu \geq 2$ and $W \sim e^{2\phi}$ for $\nu \leq 2$. However, it is possible to arrange for $W \sim e^{\nu \phi}$ in the latter case by picking a particular value for the integration constant in equation (2.9)\textsuperscript{8}. There are thus two possible scenarios.

- The superpotential $W$ grows as $e^{2\phi}$ or faster and the EOMs are not satisfied at the singularity. The only consistent way out is to resolve the singularity, for instance by introducing a second brane located at $y_s$. In that case fine-tuning of the CC is restored as we introduce two more conditions analogous to equations (2.6) and (2.7) or equation (2.12), respectively, but do not increase the number of free parameters [21].

- The superpotential grows as $e^{\nu \phi}$ with $\nu < 2$, or slower. The EOMs are satisfied at the singularity, and there is no need to resolve it. The price one pays is the adjustment of the integration constant in equation (2.9). In this case, we lose one of our parameters needed to satisfy the BC equation (2.12), resulting in a fine-tuning of the brane tension.

\textsuperscript{8} Similar reasonings apply to potentials that grow even more slowly, e.g. as a power. The generic solution behaves as $e^{2\phi}$, but particular solutions may exist that behave as $\sqrt{V}$ and hence allow for consistent, yet fine-tuned, flat backgrounds.
It is important to realize that either fine-tuning precisely corresponds to the fine tuning of the CC. In the second possibility above this is particularly obvious: the superpotential is completely specified by the bulk potential and the BC at $\phi \to \infty$. Equation (2.12) is then simply the minimization of the 4D potential

$$V_4(\phi) \equiv \lambda(\phi) - 6W(\phi)$$

under the condition that $V_4(\phi)$ vanishes at the minimum $\phi = \phi_0$. In fact, the brane potential $\lambda(\phi)$ should be determined by physics localized at the UV brane interacting with the (dilaton) field $\phi$. For example, if the SM Higgs field $H$ is localized at the UV brane it will generate a brane potential as $\lambda(\phi, H)$, which will in turn provide the effective brane potential $\lambda(\phi) \equiv \lambda(\phi, \langle H \rangle)$ after electroweak symmetry breaking. So after the electroweak phase transition there will be a $\phi$-dependent vacuum energy, which will require retuning the CC to zero and possibly a shift in the minimum of equation (2.23).

What matters to us here is that there exist consistent solutions to the equations of motion in the full closed interval $[0, y_s]$ that, although requiring a fine-tuning of the CC, do not demand the introduction of a second brane or any other means of resolving the singularity.

### 3. A model of soft-wall stabilization

Our goal in this section is to find solutions to the 5D scalar-gravity system that:

1. require only one brane at $y = 0$,
2. behave as AdS$_5$ near that brane,
3. have a soft wall at a value $y = y_s$, i.e. give rise to finite volume,
4. possess a mass gap or level spacing hierarchically smaller than the Planck scale without fine-tuning of parameters.

The last requirement might be called the problem of soft-wall stabilization. We will now show that all these requirements can be met by starting from the simple superpotential

$$W = k(1 + e^{\nu \phi}),$$  \hspace{1cm} (3.1)

where $k$ is some arbitrary dimensionful constant of the order of the 5D Planck scale, and $\nu < 2$. The solution can be immediately written down

$$A(y) = ky - \frac{1}{\nu^2} \log \left( \frac{1 - y}{y_s} \right),$$

$$\phi(y) = -\frac{1}{\nu} \log [\nu^2 k(y_s - y)].$$

At the point $y = y_s$, we encounter a naked curvature singularity as explained in section 2. For $y \ll y_s$, i.e. near the boundary at $y = 0$, the geometry is AdS$_5$.

The bulk potential that corresponds to the superpotential (3.1) is given by

$$V(\phi) = (3k^2 \nu^2 - 12k^2) e^{2\nu \phi} - 24k^2 e^{\nu \phi} - 12k^2.$$  \hspace{1cm} (3.3)

---

9 Other phase transitions, such as the QCD phase transition, should have a similar effect.
For $\nu \leq 2$ the potential is bounded from above. More precisely, it satisfies the condition

$$V(\phi(y)) \leq V(\phi_0),$$

(3.4)

necessary [7] for the corresponding bulk geometry to support finite temperature in the form of black hole horizons\(^{10}\). Moreover for $\nu < 2$, as we have seen in the previous section, the equations of motion are satisfied at the singularity and there is no need to resolve it.

- For $\nu > 2$ the equations of motion are not satisfied at the singularity and the latter would need to be resolved to fine-tune to zero the 4D CS. Finally, the potential is not bounded from above and finite temperature is not supported in the dual theory.

The location of the singularity depends exponentially on the brane value of $\phi$,

$$k y_s = \frac{1}{\nu^2} e^{-\nu \phi_0}.$$  

(3.5)

As we will see in the next section the relevant mass scale for the 4D spectrum is not the inverse volume but rather the ‘warped down’ quantity

$$\rho \equiv k (k y_s)^{-1/\nu^2} e^{-k y_s}.$$  

(3.6)

All we need in order to create the electroweak hierarchy is thus $\phi_0 < 0$ but otherwise of order unity. This can be achieved with a fairly generic brane potential, for instance by choosing a suitable $\lambda(\phi)$ such that the second of equation (2.12) is satisfied for our superpotential\(^{11}\). For negative $\phi_0$, the ratio of scales $k/\rho$ exhibits a double exponential behaviour

$$\log \frac{k}{\rho} \sim \frac{e^{v(-\phi_0)}}{\nu^2} + \cdots,$$

(3.7)

and we can create a huge hierarchy with very little fine-tuning. In figure 1, we plot $\rho/k$ as a function of $|\phi_0|$ for different values of $\nu$ and also as a function of $\nu$ for a fixed value $k y_s = 30$, which generates a hierarchy of about 14 orders of magnitude.

A comment about the choice of our superpotential is in order here. Its particular form, equation (3.1), guarantees full analytic control over our solution. A more detailed analysis of other possiblilities will be postponed to section 5.

It will be useful in the following to define the metric also in conformally flat coordinates defined by the line element

$$ds^2 = e^{-2A(z)}(dx^\mu dx^\nu \eta_{\mu\nu} + dz^2),$$

(3.8)

where $A(z) \equiv A[y(z)]$, the relationship between $z$- and $y$-coordinates being given by $\exp[A(y)]dy = dz$. One easily finds, for $\nu > 0$, that

$$\rho(z - z_0) = \Gamma(1 - 1/\nu^2, k y_s - k y) - \Gamma(1 - 1/\nu^2, k y_s),$$

(3.9)

where $z_0$ corresponds to the location of the UV brane that we assume to be at $z_0 = 1/k$ and $\Gamma(a, x)$ is the incomplete gamma function. Since we are taking $e^{k y_s} \gg 1$ and hence $k/\rho \gg 1$, we can approximate $\Gamma(1 - 1/\nu^2, k y_s) \simeq \rho/k$ and (3.9) simplifies to

$$\rho z \simeq \Gamma(1 - 1/\nu^2, k y_s - k y).$$

(3.10)

\(^{10}\) A characteristic feature of physical singularities [7].

\(^{11}\) In order to satisfy the first of equation (2.12), we still need a fine-tuning, for instance by adding a $\phi$-independent term to $\lambda(\phi)$. This is precisely the tuning of the 4D CS discussed above that of course has nothing to do with the electroweak hierarchy we want to explain here.

New Journal of Physics 12 (2010) 075012 (http://www.njp.org/)
For \( \nu > 1 \) the singularity at \( y_s \) translates into a singularity at \( z_s \) given by
\[
\rho z_s \simeq \Gamma(1 - 1/\nu^2).
\] (3.11)

For \( 0 < \nu \leq 1 \) the singularity at \( y_s \) translates into a singularity at \( z_s \rightarrow \infty \).

As we will see in the next section the case \( 0 \leq \nu < 1 \) provides continuous spectra without any mass gap\(^{12}\), i.e. typically it leads from a 4D perspective to unparticles \([16]\). The case \( \nu = 1 \) corresponds to continuous spectra with a mass gap provided by \( \rho \) in equation (3.7) leading in 4D to unparticles with mass gaps \([14, 17]\). Finally, the case \( 1 < \nu < 2 \) corresponds to discrete spectra with typical level spacing controlled by \( \rho \), as in AdS models with two branes \([1]\).

4. Fluctuations and the four-dimensional spectrum

In this section, we study the fluctuations of the metric and scalar around the classical background solutions. A general ansatz to describe all gravitational excitations of the model is, with the appropriate gauge choice \([22]\)
\[
\phi(x, y) = \phi(y) + \varphi(x, y),
\]
\[
dx^2 = \, e^{-2A(y) - 2F(x, y)} (\eta_{\mu\nu} + h_{\mu\nu}^{TT}) \, dx^\mu \, dx^\nu + (1 + G(x, y))^2 \, dy^2,
\] (4.2)
where \( \phi(y) \) is the background solution given in equation (3.2). The Einstein equations that arise from this ansatz have the spin-two fluctuations decoupled from the spin-zero fluctuations, so we can proceed to study them independently.

4.1. The graviton

Let us first consider the graviton as the transverse traceless fluctuations of the metric
\[
dx^2 = \, e^{-2A(y)} (\eta_{\mu\nu} + h_{\mu\nu}(x, y)) \, dx^2 + dy^2,
\] (4.3)
\(^{12}\) The case \( \nu = 0 \) is just the RS2 model \([15]\) with a constant dilaton \( \phi \).
where \( h_\mu^\nu = \partial_\mu h_\nu^{\mu\nu} = 0 \). In order to respect the orbifold symmetry and to keep the possibility of a constant profile zero mode, we will consider \( h(y) = h(-y) \), which leads to the BC at the brane \( h'(0) = 0 \). The part of the action quadratic in the graviton fluctuations becomes

\[
S = \int d^4x \, dy \sqrt{\gamma} R
\]

\[
- \frac{1}{4} \int d^4x \, dy \, e^{-2A(y)} \left( \partial_\mu h_{\nu\rho} \partial^\rho h_\nu^{\mu\nu} + e^{-2A(y)} \partial_\rho h_{\mu\nu} \partial^\rho h_\nu^{\mu\nu} \right).
\]  

(4.4)

Using the ansatz

\[
h_{\mu\nu}(x, y) = h_{\mu\nu}(x) h(y),
\]

(4.5)

one can obtain the EOM for the wavefunctions \( h(y) \), which is given by

\[
h''(y) - 4A'(y)h'(y) + e^{2A(y)} m^2 h(y) = 0.
\]

(4.6)

After an integration by parts in (4.4), one finds an additional equation due to boundary terms at \( y = y_s \),

\[
e^{-4A(y_s)} h'(y_s) = 0.
\]

(4.7)

In addition, one has to impose that the solutions are normalizable, i.e.

\[
\int_0^{y_s} dy \, e^{-2A(y)} h^2(y) < \infty.
\]

(4.8)

It is now convenient to change to conformally flat coordinates, as defined in (3.9). In this frame, rescaling the field by \( \bar{h}(z) = e^{-3A(z)/2} h(z) \), equation (4.6) can be written as a Schroedinger-like equation,

\[
\ddot{\bar{h}}(z) + V_h(z) \bar{h}(z) = m^2 \bar{h}(z),
\]

(4.9)

where a dot denotes derivation with respect to \( z \), and the potential is given by

\[
V_h(z) = \frac{9}{4} \dot{\bar{A}}^2(z) - \frac{3}{2} \ddot{\bar{A}}(z).
\]

(4.10)

The boundary equations are written in the \( z \)-frame as

\[
e^{-3A(z)} h(z) \bigg|_{z_0, z_0} = e^{-3A(z)/2} \left( \dot{\bar{h}}(z) + \frac{3}{2} \ddot{\bar{A}}(z) \bar{h}(z) \right) \bigg|_{z_0, z_0} = 0,
\]

(4.11)

and the normalizability condition is

\[
\int_{z_0}^{z_s} dz \, e^{-3A(z)} h^2(z) = \int_{z_0}^{z_s} dz \, \bar{h}^2(z) < \infty.
\]

(4.12)

In the case of study, it is only possible to obtain an analytic expression for the potential in the \( y \)-frame, where it reads

\[
V_h(z[y]) = \frac{3e^{-2ky} \left( 1 - \frac{z}{y} \right)^{2\nu^2} \left[ 5\nu^4 k^2 (y - y_s)^2 - 10\nu^2 k (y - y_s) - 2\nu^2 + 5 \right]}{4\nu^4 (y - y_s)^2}.
\]

(4.13)
Figure 2. Behaviour of $V_h(z)$ for different values of $\nu$. Here, $V_0 \equiv V_h(1/k)$. For the radion, $V_F(z)$ has the same behaviour with the exception that figure 2(c) applies for all $\nu > 1$.

It is however possible to invert numerically the coordinate change (3.9), and so to plot (4.13). Its behaviour for different values of $\nu$ is shown in figure 2. One can distinguish three possible situations:

- $\nu < 1$ (figure 2(a)) In this case, $z$ extends to infinity where $V_h \to 0$. The mass spectrum is continuous from $m = 0$, leading to unparticles without a mass gap. However, conformal symmetry is broken due to the occurrence of the scale $y_s$.
- $\nu = 1$ (figure 2(b)) $z$ also extends to infinity but $V_h \to (9/4)\rho^2$. This leads to unparticles with a mass gap $m_g = (3/2)\rho$.
- $\nu > 1$ (figures 2(c) and (d)) $z_s$ is finite and thus the mass spectrum is discrete. The potential diverges at $z_s$ changing sign at $\nu^2 = 5/2$, but this does not have observable consequences in the mass spectrum as we will see.

Equations (4.6) and (4.9) do not have analytic solutions. However, for $\nu > 1$ one can find approximations for the wavefunction in the regions near the brane and near the singularity. Let us first consider the region near the brane ($ky \simeq 0$). Assuming $ky_s \gg 1$, the potential (4.13) is approximated as

$$V_h|_{y=0} \simeq \frac{15k^2}{4} e^{-2ky} \simeq \frac{15}{4} \frac{1}{z^2},$$

(4.14)

13 Similar potentials were considered in [23].
where the coordinate change is given by (3.10), which is approximated for \( \nu > 1 \) as
\[
k \nu \simeq e^{ky}.
\] (4.15)

One can see that (4.14) corresponds to an AdS metric. With this approximated potential, equation (4.9) is solved by
\[
\tilde{h}(z)|_{z=z_0} = c_1 \sqrt{kz} J_2(mz) + c_2 \sqrt{kz} Y_2(mz).
\] (4.16)

The two coefficients can be determined by the normalization and the BC (4.11) at \( z_0 \), which yields
\[
\frac{c_2}{c_1} = -\frac{J_1(m/k)}{Y_1(m/k)} \sim \left( \frac{m}{k} \right)^2 \simeq 0,
\] (4.17)

since we expect the first mass modes to be of order \( m \simeq (z_s - z_0)^{-1} \), and in our approximation \( k(z_s - z_0) \gg 1 \).

Let us now move on to consider the region next to the singularity (\( y \simeq y_s \)). In this case, the potential is approximated by
\[
V_h|_{y=y_s} \simeq \frac{3(5 - 2\nu^2)}{4v^4} \frac{\rho^2}{[k(y_s - y)]^{2-2/v^2}} \simeq \frac{3(5 - 2\nu^2)}{4(1-v^2)^2 (z_s - z)^2},
\] (4.18)

where we have used that, for \( \nu > 1 \), the coordinate change (3.9) is approximated by
\[
\rho(z_s - z) = \frac{\nu^2}{v^2 - 1} [k(y_s - y)]^{1-1/v^2}.
\] (4.19)

With this approximation, equation (4.9) yields the solution
\[
\tilde{h}(z) = c_J \sqrt{m \Delta z} J_{\alpha}(m \Delta z) + c_Y \sqrt{m \Delta z} Y_{\alpha}(m \Delta z),
\] (4.20)

where
\[
\alpha = \frac{4 - \nu^2}{2(v^2 - 1)}
\] (4.21)
and \( \Delta z \equiv z_s - z \). The two integration constants can be obtained by imposing the BC at the singularity and normalizability and by matching this solution to the solution for the intermediate region between the brane and the singularity. Near the singularity (4.20) behaves like
\[
\tilde{h}(z) \sim c_J^{(1)} (\Delta z)^{3/(2v^2-2)} + c_J^{(2)} (\Delta z)^{(4v^2-1)/(2v^2-2)} + c_Y^{(1)} (\Delta z)^{(2v^2-5)/(2v^2-2)},
\] (4.22)

where numerical factors are being absorbed in the constants \( c_J \). We have included the next to leading order in the expansion of \( J_{\alpha} \) as we need it for computing the BC, which reads
\[
e^{-3A/2} \left( \frac{\rho(z_s - z)}{z \Delta z} \tilde{h}(z) + \frac{3}{z} \Delta z \tilde{h}'(z) \right) \sim c_J^{(2)} (\Delta z)^{(v^2+2)/(v^2-1)} + c_Y^{(1)} (\Delta z)^0.
\] (4.23)

Again numerical factors have been absorbed in \( c_J' \). Note that the BC is only satisfied when \( c_Y = 0 \), and that this condition also ensures that the solution (4.22) is normalizable when \( v^2 < 2 \).

The BCs provide the quantization of the mass eigenstates for \( \nu > 1 \). In order to compute the mass spectrum for the graviton one should match the solutions at the ends of the space with a solution for the intermediate region. Unfortunately, for the parameter range we are interested in we do not have good analytic control for this region. However, we can extract a generic property
of the spectrum by looking at the potential equation (4.13) and using the form of the coordinate transformation equation (3.10) to deduce that, assuming $e^{k y_s} \gg 1$, the potential has the form\(^{14}\)

$$V_h(z) = \rho^2 v_h(\rho z),$$

(4.24)

where $v_h$ is some dimensionless function of the dimensionless variable $\rho z$. In other words, we have eliminated the two scales $k$, $y_s$ in favour of the single scale $\rho$ given in equation (3.6). The spectrum is therefore of the form

$$m_n(v, k, y_s) = \mu_n(v) \rho(v, k, y_s),$$

(4.25)

where the pure numbers $\mu_n$ only depend on the parameter $v$ but not on the parameters $k$ or $y_s$.

Moreover, one can find an expression for the spacing of the mass eigenstates by approximating the potential as an infinite well, which is valid for $m^2 \gg V_h$. The result of this approximation is

$$\Delta m \simeq \frac{\rho \pi}{\Gamma(1 - 1/v^2)} = \frac{\pi}{z_s}.$$  

(4.26)

Note that the mass spectrum is linear ($m_n \sim n$), and that as one approaches $v = 1$

$$\lim_{v \to 1} \Delta m = 0,$$

(4.27)

recovering the expected continuous spectrum at this value (for $v < 1$ the spectrum is continuous too, since (4.26) is only valid for $v > 1$). The numerical result for the mass eigenvalues is shown in figure 3, where these behaviours can be observed. Some profiles for the graviton computed numerically using the EOM (4.6) and the BCs (4.7) are shown in figure 4.

\(^{14}\) This behaviour can actually be seen in the limiting cases equations (4.14) and (4.18).

\(^{15}\) Numerically one finds that the scaling property in equation (4.25) ceases to be valid for $k y_s \lesssim 3$, as discrepancies from this behaviour become greater than 1%.
4.2. The radion-scalar system

Now we consider the spin-zero fluctuations of the system. This is

$$\phi(x, y) = \phi(y) + \varphi(x, y),$$

(4.28)

$$ds^2 = e^{-2A(y)} e^{-2F(x, y)} \eta_{\mu\nu} dx^\mu dx^\nu + (1 + G(x, y))^2 dy^2.$$ (4.29)

With an appropriate gauge choice, the equations of motion for the y-dependent part of the KK modes form a coupled system with only one degree of freedom. The derivation of the equations is given with detail in [22], and the result is

$$F'' - 2 A' F' - 4 A'' F = -m^2 e^{2A} F,$$ (4.30)

$$\phi' \varphi = F' - 2 A' F,$$ (4.31)

$$G = 2 F.$$ (4.32)

The boundary equations on the brane depend on the brane tension $\lambda(\phi)$. The precise form of the dependence can be found in [22]. At the singularity, similarly to the graviton case, one obtains the boundary equation

$$e^{-4A(y)} \varphi'(y)|_{y_s} = 0,$$ (4.33)

and the normalizability condition

$$\int_0^{y_s} dy e^{-2A(y)} \varphi^2(y) = \int_{z_0}^{z_s} dz \tilde{\varphi}^2(z) < \infty,$$ (4.34)

where the field has been rescaled by $\tilde{\varphi}(z) \equiv e^{-3A/2} \varphi(z)$. 

**Figure 4.** KK graviton profiles in the $z$-frame for $k y_s = 30$ and $v = 3/2$, using the normalization $\int dz \tilde{h}^2 = 1$. The massless mode ($n = 0$) is peaked near the brane. The two first massive modes ($n = 1, 2$) are also shown. The zero mode becomes more peaked near the brane in comparison with the massive modes as $k y_s$ increases.
It is convenient, as for the graviton, to use conformally flat coordinates. Rescaling the field by $\tilde{F}(z) = e^{-3A(z)/2} F(z)/\phi(z)$, (4.32) can be written as the Schroedinger equation
\[ -\ddot{\tilde{F}}(z) + V_F(z) \tilde{F}(z) = m^2 \tilde{F}(z), \tag{4.35} \]
where
\[ V_F(z) = \frac{9}{4} \dot{A}^2 + \frac{5}{2} \ddot{A} - \dot{A} \frac{\ddot{\phi}}{\phi} - \dddot{\phi} + 2 \left( \frac{\ddot{\phi}}{\phi} \right)^2. \tag{4.36} \]

The relation between the rescaled field $\tilde{F}$ and the scalar field $\phi$ is
\[ \phi(z) = e^{3A/2} \left[ \dot{\tilde{F}} + \left( \frac{\dot{\phi}}{\phi} - \frac{1}{2} \dot{A} \right) \tilde{F} \right]. \tag{4.37} \]

In the $y$-frame, equation (4.36) is given by
\[ V_F(z[y]) = e^{-2ky} \left( 1 - \frac{y}{y_s} \right)^{2/\nu^2} \left[ 3 \nu^4 k^2 (y - y_s)^2 + (-6 \nu^2 + 8 \nu^4) k (y - y_s) + 6 \nu^2 + 3 \right] \frac{e^{2ky}}{4 \nu^4 (y - y_s)^2}. \tag{4.38} \]

This potential has similar form to the graviton potential, and the three situations presented above also apply for the radion (with the same mass gap for $\nu = 1$). A difference is that this potential does not change the sign of divergence for $\nu > 1$ but, as said before, this does not have any observable consequences.

Let us now proceed to find the approximation for the wavefunction near the UV brane. Taking $ky \simeq 0$ and $ky_s \gg 1$ and using (4.15), (4.38) is given by
\[ V_F|_{y=0} \simeq \frac{3k^2}{4} e^{-2ky} \simeq \frac{3}{4} \frac{1}{z^2}, \tag{4.39} \]
and hence the solution to (4.35) is
\[ \tilde{F}(z)|_{z \gg z_0} = c_1 \sqrt{kz} J_1(mz) + c_2 \sqrt{kz} Y_1(mz). \tag{4.40} \]

The coefficients $c_1$ are to be determined using the BCs at the brane [22]. As an example, using the condition\(^ {16} \) $\phi(y = 0) = 0$ (which we will use for the numerical computation) yields
\[ \frac{c_2}{c_1} = - \frac{J_2(m/k)}{Y_2(m/k)} \simeq 0. \tag{4.41} \]

Next to the singularity, using (4.19) the potential is approximated by
\[ V_F|_{y=y_s} \simeq \frac{6 \nu^2 + 3}{4 \nu^4} \frac{\rho^2}{[k(y_s - y)]^{2 - 2/v^2}} \simeq \frac{6 \nu^2 + 3}{4(1 - \nu^2)^2 (z_s - z)^2}, \tag{4.42} \]
that gives the solution
\[ \tilde{F}(z) = c_J \sqrt{m \Delta z} J_\alpha(m \Delta z) + c_Y \sqrt{m \Delta z} Y_\alpha(m \Delta z), \tag{4.43} \]
with
\[ \alpha = \frac{2 + \nu^2}{2 \nu^2 - 2}. \tag{4.44} \]

\(^ {16} \) This condition holds for brane potentials satisfying $\partial^2 \lambda/\partial \phi^2 \gg 1$ [22].
The behaviour of this solution near the singularity is

\[ \tilde{F}(z) \sim c_j^{(1)}(\Delta z)^{(2\nu^2+1)/(2\nu^2-2)} + c_j^{(2)}(\Delta z)^{(6\nu^2-3)/(2\nu^2-2)} + c_y^{(1)}(\Delta z)^{-3/(2\nu^2-2)}. \]  

(4.45)

Using (4.37) we can compute the behaviour of the field and apply the normalizability condition (4.34),

\[ \tilde{\phi}(z) \sim c_j^{(1)}(\Delta z)^{3/(2\nu^2-2)} + c_y^{(1)}(\Delta z)^{-(2\nu^2+1)/(2\nu^2-2)}, \]  

(4.46)

and the BC (4.33),

\[ e^{-3A(z)} \phi(z) \sim c_j^{(2)}(\Delta z)^{(\nu^2+2)/(\nu^2-1)} + c_y^{(1)}(\Delta z)^{(-2\nu^2+2)/(\nu^2-1)}. \]  

(4.47)

Again, the condition \( c_y = 0 \) is sufficient to ensure both the fulfillment of the BCs and the normalizability. The scaling of the mass eigenvalues equation (4.25) and the approximation (4.26) for the spacing of the mass modes also holds for the radion. The numerically obtained values for the masses are shown in figure 5. In comparison to the graviton mass modes of figure 3, note that the first mode for the radion is lighter than the first massive mode of the graviton. This can be understood recalling that the radion does not have a zero mode\(^{17}\). Some profiles of the scalar fluctuations of the field \( \tilde{\phi} \) are shown in figure 6.

5. Other soft walls with a hierarchy

The particular form of \( W \), equation (3.1), guarantees full analytic control over our solution but may seem a little ad hoc. It is natural to ask what are the essential ingredients of our stabilization

\(^{17}\) One can in fact show that for \( \nu \to \infty \), which we can only take if we resolve the singularity, the lightest mode tends to be massless, corresponding to the radion profile in [24].
mechanism and whether it is possible to generalize it to other potentials or superpotentials. The location of the singularity, and hence the size of the extra dimension, is given by

$$y_s = \int_{\phi_0}^{\infty} \frac{d\phi}{W'(\phi)}. \tag{5.1}$$

Here and in the following, we will assume that $W$ is a monotonically increasing function of $\phi$, i.e. $W'(\phi) > 0$. The integral is finite whenever $W$ diverges faster than $W \sim \phi^2$.

However, the inverse volume $y_s^{-1}$ is, in general, not the 4D KK scale nor the mass gap as there might be a strong AdS warping near the UV brane. The KK scale is given by the inverse conformal volume $z_s^{-1}$ (when it is finite), calculated as

$$z_s = \int_0^{y_s} e^{A(y)} \, dy. \tag{5.2}$$

It is easy to warp the geometry near the brane without affecting $y_s$ by adding a positive constant of $O(k)$ to the superpotential, leading to

$$A(y) \rightarrow A(y) + ky, \quad \text{for} \quad W \rightarrow W + k. \tag{5.3}$$

Notice that $A(y)$ is a monotonically increasing function of $y$, such that

$$kz_s > e^{ky_s}. \tag{5.4}$$

One sees that the KK scale is warped down with respect to the compactification scale, a phenomenon well known in RS models with two branes \[1\]. In order to obtain, e.g. the TeV from the Planck scale we need

$$ky_s = \int_{\phi_0}^{\infty} \frac{k}{W'(\phi)} \, d\phi \simeq 37. \tag{5.5}$$

This is not hard to achieve in a natural manner. In our model, equation (3.1), it works so well because the exponential behaviour that was introduced for large values of $\phi$ is also valid at $O(1)$ negative values and dominates the integral, leading to equation (3.5).

Figure 6. KK profiles of the rescaled scalar fluctuations $\tilde{\phi}(z)$ for $ky_s = 30$ and $\nu = 3/2$, using the normalization $\int dz \tilde{\phi}^2 = 1$. The first three massive modes $(n = 0, 1, 2)$ are shown.
Moreover, there are many cases where \( z_s \) is infinite, even though \( y_s \) is finite. There can still be mass gaps or even a discrete spectrum, but \( z_s \) is clearly inadequate to characterize the energy levels. One such example is the case \( W = k e^\phi \) that leads to a mass gap. Let us be slightly more general and consider the class of superpotentials given by

\[
W(\phi) = k e^\phi (\phi - \phi_1)^\beta ,
\]

with \( \phi_1 < \phi_0 \). This superpotential is monotonically increasing for \( \beta \geq 0 \) and has infinite \( z_s \) for \( \beta \leq \frac{1}{2} \), so we will assume \( 0 \leq \beta \leq \frac{1}{2} \). The volume \( y_s \) is approximately

\[
y_s \approx e^{\phi_0},
\]

so, again, \( k y_s \) is (mildly) exponentially enhanced when \( |\phi_0| = O(1 - 10) \), \( \phi_0 < 0 \). In order to estimate the spectrum, we need the asymptotic behaviour of the warp factor in conformally flat coordinates. For large \( z \), it is given by

\[
A(z) \approx (\rho z)^{1/(1 - 2\beta)},
\]

where \( \rho = O(y_s^{-1}) \). The coordinate change is given by

\[
z(y) = \int e^A(y) \, dy.
\]

Using our trick of adding warping while keeping \( y_s \) unchanged, equation (5.3), we see that near the singularity

\[
z(y) \to z_w(y) \approx z(y) e^{k y_s}.
\]

On the other hand, adding the warping leaves \( A(y) \) nearly unchanged near the singularity (adding a constant \( k y_s \) to infinity makes no difference). Demanding thus \( A_w(y) \approx A(y) \) near \( y = y_s \) leads to

\[
[\rho z(y)]^{1/(1 - 2\beta)} = [\rho_w z_w(y)]^{1/(1 - 2\beta)} = [\rho_w z(y) e^{k y_s}]^{1/(1 - 2\beta)},
\]

and hence

\[
\rho_w = \rho e^{-k y_s} \approx e^{-k y_s} y_s.
\]

Combining this with equation (5.7), we find a strong suppression of \( \rho_w / k \) resulting just from \( O(1) \) numbers. The quantity \( \rho_w \) sets the scale for the KK spectrum in this case. In fact, metrics of the form equation (5.8) have been studied in [13]. A Wentzel–Kramers–Brillouin (WKB) approximation shows that the spectrum can be approximated by

\[
m_n \approx \rho_w n^{2\beta}.
\]

We see that \( \rho_w \) indeed sets the scale of the 4D masses, which are hence parametrically suppressed with respect to \( k \). The complete superpotential that accomplishes a hierarchy and leads to the spectrum equation (5.13) is

\[
W(\phi) = k (1 + e^\phi [\phi - \phi_1]^{\beta}).
\]

In particular, the case \( \beta = 1/4 \) generates the linear Regge trajectory spectrum \( m_n^2 \approx \rho_w^2 n \) appropriate for AdS/QCD models as in [9]. In this case, one would obtain the linear confinement behaviour of, e.g. \( \rho \)-mesons by considering an additional piece in our action

\[
\int d^4 x \, dz \, \sqrt{-g} \, e^{-(1/2)\phi} \mathcal{L}_{\text{mesons}}.
\]

The fact that asymptotically \( A(z) \approx \phi(z) \approx z^2 \) guarantees that the resonances of the vector mesons follow the same linear law as the ones for the scalars and tensors.

*New Journal of Physics* **12** (2010) 075012 (http://www.njp.org/)
Let us conclude this section by noting that there are certainly other ways to obtain the mild hierarchy $k_y$, including moderate fine-tunings of parameters. What is completely generic, though, is the fact that adding warping as in equation (5.3) leaves $k_y$ manifestly unchanged but suppresses the masses by an additional warp factor $e^{k_y}$.

6. Conclusions and outlook

In this paper, we have studied the stabilization of soft walls, i.e. 4 + 1 dimensional geometries with 4D Poincaré invariance, that are only bounded by a single three-brane but that nevertheless exhibit a finite volume for the extra dimensional coordinate. The second brane is typically replaced by a naked curvature singularity at a finite proper distance. In particular, we have studied how these soft walls arise in models with a single scalar field, and classified the type of models that can be realized as full solutions to the Einstein equations without destabilizing contributions at the singularities. We have proven that all admissible solutions result in a fine-tuning of the CC.

Our main objective has been to show how to stabilize the position of the singularity at parametrically large values compared with the 5D Planck length. We have employed the superpotential method of [19, 20] and proposed a family of models that accomplishes this goal. Our stabilizing superpotential allows three types of 4D spectra: continuous, continuous with a mass gap and discrete. The 4D mass scale $\rho$, controlling the mass gap in the continuous case and the spacing in the discrete one, depends in a double exponential manner on the value of the scalar field at the brane, equation (3.7), and can thus be naturally suppressed with respect to the 5D mass scale $k$ without fine-tuning.

Next, we have studied in detail the spectra resulting from fluctuations around our family of solutions. As with the case of the background, we have paid close attention to the BCs at the singularity, projecting out solutions that would give contributions at the dynamically generated boundary. We have given analytical forms of the wavefunctions near the brane and the singularity, as well as numerical values for the lowest lying excitations and their profiles.

We have also given a constructive recipe of how to obtain superpotentials that accomplish the stabilization of the hierarchy, a desired spectrum, and an ‘end-of-the-world singularity’ that is consistent with the equations of motion. In a first step, one chooses the asymptotic (i.e. large $\phi$) behaviour of $W$. This will determine the asymptotic form of the spectrum. To ensure consistency of the equations of motion, the divergence should be milder than $W \sim e^{2\phi}$. The different asymptotic forms of $W$ and corresponding spectra are summarized in table 1. In a second step, one completes $W$ for smaller values of $\phi$ in such a way as to accomplish a mild hierarchy of the proper distance $y_s$ with respect to the fundamental 5D scale $k$, given by the simple relation equation (5.5). Notice that many of the interesting spectra require some kind of exponential behaviour at large $\phi$, such that this region does not contribute at all to $k_y$. At the same time one minimizes the effective 4D potential, equation (2.23), at the brane at $y = 0$ to find the vacuum value of $\phi_0 \equiv \phi(0)$. In a third and final step, one adds a constant of $O(k)$ to the superpotential. This adds strong warping near the UV brane, but has no effect whatsoever on the determination of $k_y$ and $\phi_0$. We have shown that in this way one can warp down the parameter setting the overall scale for the spectrum by a factor $e^{k_y}$, leading to the desired hierarchy.

A classification of superpotentials giving rise to confining backgrounds was performed in [10].
Table 1. Spectra resulting from different asymptotic forms of the superpotential. In the first row, we give the asymptotic behaviour of $W(\phi)$, with the strength of the divergence increasing from left to right ( > means ‘diverges faster than’, etc). Second and third rows show the finiteness of $y_s$ and $z_s$, with the behaviour changing at $W \sim \phi^2$ and $W \sim e^{e \phi^{1/2}}$, respectively. The third row shows the spectrum, while in the last one, we indicate the consistency of the solution.

| $W(\phi)$ | $\phi$ | $e^\phi$ | $e^{\phi \phi}$ | $e^{e \phi^{1/2}}$ |
|-----------|--------|---------|----------------|------------------|
| $y_s$     | $\infty$ | Finite |
| $z_s$     | $\infty$ | Finite |

Mass spectrum | Continuous | Continuous with mass gap | Discrete |
|--------------|-----------|-------------------------|---------|
| $m_n \sim n^{2\beta}$ | $m_n \sim n$ |

Consistent solution | Yes | No |

There are a number of phenomenological applications that are outside the scope of the present paper but which are worthy of future investigations. For the range of the parameter $1 < \nu < 2$ these applications are common with two brane models, as RS1, but with some peculiarities. In particular, graviton (and radion) KK modes are at the TeV scale and they can be produced and decay at LHC by their interaction with matter $\sim h_{\mu \nu} T^{\mu \nu}$, so they are expected to be produced through gluon annihilation [25]. Since there is no IR brane, for soft-wall models to solve the gauge hierarchy problem the Higgs boson (either a scalar doublet or the fifth component of a gauge field in a gauge-Higgs unified model) has to propagate in the bulk and it has to be localized near the singularity for its mass to feel the warping. On the other hand, fermions with sizable Yukawa couplings (third generation fermions) should be localized near the singularity as well while first and second generation fermions can propagate at (or near the) UV brane. As we have seen that the first graviton KK mode is localized near the singularity, once produced it is expected to decay into either Higgs or $t\bar{t}$ pairs. For $\nu = 1$ the mass spectrum of fields propagating in the bulk is a continuum above an $O(\text{TeV})$ mass gap. This continuum (endowed with a given conformal dimension) can interact with SM fields propagating in the UV brane as operators of a CFT, where the conformal invariance is explicitly broken at a scale given by the mass gap, and can model and describe the unparticle phenomenology. In particular, the Higgs embedded into such 5D background can describe the unHiggs theory of Stancato and Terning [26] in the presence of a mass gap. In all those cases, the strength of electroweak constraints should be an issue. Finally, the case where the spectrum is described by linear Regge trajectories ($\nu = 1, \beta = 1/4$) can give rise to a phenomenological description of AdS/QCD, similar to that of [9], where the QCD scale can be naturally stabilized by the scalar field.

Acknowledgments

This work was supported in part by the European Commission under the European Union through the Marie Curie Research and Training Network ‘UniverseNet’ (MRTN-CT-2006-035863); by the Spanish Consolider-Ingenio 2010 Programme CPAN (CSD2007-00042); and
by CICYT, Spain, under contract FPA 2008-01430. GG thanks IFAE for hospitality during part of this project. The work of JAC is supported by the Spanish Ministry of Education through a FPU grant.

References

[1] Randall L and Sundrum R 1999 Phys. Rev. Lett. 83 3370 (arXiv:hep-ph/9905221)
[2] Maldacena J M 1998 Adv. Theor. Math. Phys. 2 231
   Maldacena J M 1999 Int. J. Theor. Phys. 38 1113 (arXiv:hep-th/9711200)
   Gubser S S, Klebanov I R and Polyakov A M 1998 Phys. Lett. B 428 105 (arXiv:hep-th/9802109)
   Witten E 1998 Adv. Theor. Math. Phys. 2 253 (arXiv:hep-th/9802150)
[3] Goldberger W D and Wise M B 1999 Phys. Rev. Lett. 83 4922 (arXiv:hep-ph/9907447)
   Goldberger W D and Wise M B 2000 Phys. Lett. B 475 275 (arXiv:hep-ph/9911457)
[4] Chamblin H A and Reall H S 1999 Nucl. Phys. B 562 133 (arXiv:hep-th/9903225)
[5] Arkani-Hamed N, Dimopoulos S, Kaloper N and Sundrum R 2000 Phys. Lett. B 480 193 (arXiv:hep-th/0001197)
[6] Kachru S, Schulz M B and Silverstein E 1999 Adv. Theor. Math. Phys. 2 253 (arXiv:hep-th/9902264)
   Gubser S S, Klebanov I R and Polyakov A M 1998 Phys. Lett. B 428 105 (arXiv:hep-th/9802109)
   Witten E 1998 Adv. Theor. Math. Phys. 2 253 (arXiv:hep-th/9802150)
[7] Gursoy U and Kiritsis E 2008 J. High Energy Phys. JHEP02(2008)032 (arXiv:0707.1324 [hep-th])
[8] Brandhuber A and Sfetsos K 1999 J. High Energy Phys. JHEP10(1999)013 (arXiv:hep-th/9908116)
[9] DeWolfe O, Freedman D Z, Gubser S S and Karch A 2000 Phys. Rev. D 62 046008 (arXiv:hep-th/9909134)
[10] Forste S, Lalak Z, Lavignac S and Nilles H P 2000 Phys. Lett. B 481 360 (arXiv:hep-th/9911200)
[11] Brandhuber A and Sfetsos K 1999 J. High Energy Phys. JHEP10(1999)013 (arXiv:hep-th/9908116)
[12] DeWolfe O, Freedman D Z, Gubser S S and Karch A 2000 Phys. Rev. D 62 046008 (arXiv:hep-th/9909134)
[13] Forste S, Lalak Z, Lavignac S and Nilles H P 2000 Phys. Lett. B 481 360 (arXiv:hep-th/9911200)
[14] Brandhuber A and Sfetsos K 1999 J. High Energy Phys. JHEP10(1999)013 (arXiv:hep-th/9908116)
[15] DeWolfe O, Freedman D Z, Gubser S S and Karch A 2000 Phys. Rev. D 62 046008 (arXiv:hep-th/9909134)
[16] Forste S, Lalak Z, Lavignac S and Nilles H P 2000 Phys. Lett. B 481 360 (arXiv:hep-th/9911200)
[17] Brandhuber A and Sfetsos K 1999 J. High Energy Phys. JHEP10(1999)013 (arXiv:hep-th/9908116)
[18] DeWolfe O, Freedman D Z, Gubser S S and Karch A 2000 Phys. Rev. D 62 046008 (arXiv:hep-th/9909134)
[19] Forste S, Lalak Z, Lavignac S and Nilles H P 2000 Phys. Lett. B 481 360 (arXiv:hep-th/9911200)
[20] Brandhuber A and Sfetsos K 1999 J. High Energy Phys. JHEP10(1999)013 (arXiv:hep-th/9908116)
[21] DeWolfe O, Freedman D Z, Gubser S S and Karch A 2000 Phys. Rev. D 62 046008 (arXiv:hep-th/9909134)
[22] Forste S, Lalak Z, Lavignac S and Nilles H P 2000 Phys. Lett. B 481 360 (arXiv:hep-th/9911200)
[23] Brandhuber A and Sfetsos K 1999 J. High Energy Phys. JHEP10(1999)013 (arXiv:hep-th/9908116)
[24] DeWolfe O, Freedman D Z, Gubser S S and Karch A 2000 Phys. Rev. D 62 046008 (arXiv:hep-th/9909134)
[25] Forste S, Lalak Z, Lavignac S and Nilles H P 2000 Phys. Lett. B 481 360 (arXiv:hep-th/9911200)
[26] Forste S, Lalak Z, Lavignac S and Nilles H P 2000 Phys. Lett. B 481 360 (arXiv:hep-th/9911200)
[27] Brandhuber A and Sfetsos K 1999 J. High Energy Phys. JHEP10(1999)013 (arXiv:hep-th/9908116)
[28] DeWolfe O, Freedman D Z, Gubser S S and Karch A 2000 Phys. Rev. D 62 046008 (arXiv:hep-th/9909134)
[29] Forste S, Lalak Z, Lavignac S and Nilles H P 2000 Phys. Lett. B 481 360 (arXiv:hep-th/9911200)
[30] Brandhuber A and Sfetsos K 1999 J. High Energy Phys. JHEP10(1999)013 (arXiv:hep-th/9908116)
[31] DeWolfe O, Freedman D Z, Gubser S S and Karch A 2000 Phys. Rev. D 62 046008 (arXiv:hep-th/9909134)
[32] Forste S, Lalak Z, Lavignac S and Nilles H P 2000 Phys. Lett. B 481 360 (arXiv:hep-th/9911200)
[33] Brandhuber A and Sfetsos K 1999 J. High Energy Phys. JHEP10(1999)013 (arXiv:hep-th/9908116)
[34] DeWolfe O, Freedman D Z, Gubser S S and Karch A 2000 Phys. Rev. D 62 046008 (arXiv:hep-th/9909134)
[35] Forste S, Lalak Z, Lavignac S and Nilles H P 2000 Phys. Lett. B 481 360 (arXiv:hep-th/9911200)