Comments on nonunitary conformal field theories

Terry Gannon

Department of Mathematical Sciences
University of Alberta
Edmonton, Canada, T6G 2G1
tgannon@math.ualberta.ca

As is well-known, nonunitary RCFTs are distinguished from unitary ones in a number of ways, two of which are that the vacuum 0 doesn’t have minimal conformal weight, and that the vacuum column of the modular $S$ matrix isn’t positive. However there is another primary field, call it $o$, which has minimal weight and has positive $S$ column. We find that often there is a precise and useful relationship, which we call the Galois shuffle, between primary $o$ and the vacuum; among other things this can explain why (like the vacuum) its multiplicity in the full RCFT should be 1. As examples we consider the minimal WSU(N) models. We conclude with some comments on fractional level admissible representations of affine algebras. As an immediate consequence of our analysis, we get the classification of an infinite family of nonunitary $W_3$ minimal models in the bulk.

1. Introduction

In any RCFT, each chiral sector $\mathcal{H}_a$ (a module of the chiral algebra, i.e. of the vacuum sector $\mathcal{H}_0$) has a Hermitian inner product $\langle , \rangle_a$, obeying $\langle vx, y \rangle_a = \langle x, \omega(v)y \rangle_a$ for $x, y \in \mathcal{H}_a$ and all $v$ in the chiral algebra $V = \mathcal{H}_0$, where $\omega$ is an anti-linear anti-involution. In a unitary RCFT, this inner product is in addition positive-definite (i.e. $\langle x, x \rangle > 0$ when $x \neq 0$).

From the point of view of string theory (being a quantum field theory), the requirement of unitarity seems natural\(^1\), and indeed much of the work on fusion rings and on what we’ll call modular data has assumed it. But from the perspective of e.g. statistical models, at least on a torus, that requirement seems unnecessarily restrictive. Indeed, some of the better known RCFTs are nonunitary, such as the Yang-Lee model $(c = -22/5)$. Presumably (but see \([2]\)!) most RCFTs will be nonunitary — for example the Virasoro minimal model $(p, p')$, with central charge $c = 1 - 6 (p - p')^2/pp'$, is unitary iff $|p - p'| = 1$.

The Wess-Zumino-Witten models \([3]\), describing strings living on Lie group manifolds and corresponding to affine algebras at positive integer levels (see e.g. \([4]\)), will always be unitary, as will their GKO cosets \([5]\). On the other hand, ‘admissible representations’ \([6]\) of affine algebras at fractional levels do not (yet) have a direct interpretation as an RCFT, although their cosets have one as nonunitary RCFTs — in fact this is a powerful way to construct nonunitary theories. For example, the Virasoro minimal $(p, p')$ model corresponds to the coset $(\widehat{su}(2)_m \oplus \widehat{su}(2)_{1})/\widehat{su}(2)_{m+1}$, where $m = \frac{p'}{p-p'} - 2 \in \mathbb{Q}$.

\(^1\) Perhaps though this belief is somewhat naive. As pointed out to the author by Christoph Schweigert, in string theory the matter CFT is coupled to the (super-)ghost CFT, and the latter is of course nonunitary. Physically, what is required is the unitarity of the corresponding BRST cohomology, since the physical states live there. The relationship between the (non)unitarity of the matter CFT and the unitarity of that cohomology space isn’t obvious. For a careful review of the relation of CFT to string theory, see e.g. \([1]\), especially section 6.
In this paper we compare nonunitary and unitary RCFTs. We isolate in §3 what seems to be a key uncertainty in nonunitary theories: the multiplicity in the RCFT (what we will denote below mult$_n(o) = M_{oo}$) of the primary $o$ with minimal conformal weight. We learn that in any RCFT it will be greater than 0, but the question of its precise value turns out to be important. We conjecture in §4 that this multiplicity should equal 1; this would imply, for nonunitary RCFT, irreducibility of nim-reps and finiteness results for modular invariants and nim-reps, among other things. This multiplicity issue did not arise in the Virasoro minimal model classification [7], because as we will see in §4, modular invariance there forces this multiplicity to be 1.

We find that in many theories, the primary $o$ is related to the vacuum 0 in a certain simple way — we’ll say such theories possess the Galois Shuffle (or GS) property. In particular this holds for the nonunitary $W_N = WSU(N)$ and $WSO(2N)$ minimal models, at arbitrary rank $N$ and arbitrary values $p, p'$ (these include of course the Virasoro minimal models). Related comments are made for the affine algebras at fractional level. Because of their role in the quantum Drinfeld-Sokolov reduction [8], this suggests that the GS property may be fairly typical among RCFTs.

Nevertheless, we know that not all all RCFTs possess this property. When the GS property holds for a given RCFT, there are many nontrivial consequences, as we will see in §§4 and 6. One of these will give us for free a nonunitary RCFT classification. Let us briefly describe another consequence.

First, recall that there are several unitary matrices $\tilde{S}$ which diagonalise the fusion rules, i.e. for which the fusion coefficients $N_{ab}^c$ can be formally recovered from Verlinde’s formula when the modular matrix $S$ is replaced with $\tilde{S}$. In particular take the modular matrix $S$ and permute the columns, and multiply each column by an arbitrary phase. But typically the symmetry condition $S = S^t$ will be lost. However, there are two generic ways to find symmetric unitary matrices $\tilde{S}$ which diagonalise the fusions: simple-currents and the Galois symmetry. We will discuss these generic constructions next section.

Now, the GS property implies that the fusion ring of the nonunitary theory can be diagonalised by a symmetric $S$-matrix $\hat{S}$ which obeys all the usual properties of a unitary theory (e.g. the 0th column is strictly positive), and that $\hat{S}$ will be obtained from the ‘sick’ modular $S$ matrix by a generic construction (a ‘Galois shuffle’). (By ‘sick’ here we only mean that the vacuum column of $S$ is not positive.) We call $\hat{S}$ the unitarisation of the nonunitary theory. In other words, the fusion rings of nonunitary and unitary theories are indistinguishable, at least when the nonunitary theory obeys the GS property.

To give a simple explicit example, the fusions of the nonunitary Yang-Lee model are diagonalised of course by its ‘sick’ modular $S$ matrix

$$S = \frac{2}{\sqrt{5}} \begin{pmatrix} -\sin(2\pi/5) & \sin(\pi/5) \\ \sin(\pi/5) & \sin(2\pi/5) \end{pmatrix}.$$

But the Yang-Lee fusions are also diagonalised by the unitarisation

$$\hat{S} = \frac{2}{\sqrt{5}} \begin{pmatrix} \sin(\pi/5) & \sin(2\pi/5) \\ \sin(2\pi/5) & -\sin(\pi/5) \end{pmatrix},$$
which is the modular $S$ matrix of the (unitary) WZW model for the affine algebras $\widehat{G}_2$ and $\widehat{F}_4$ at level 1. However, as we will discuss below, the unitarisation will not in general correspond to a WZW model. Incidentally, the tensor product of the $\widehat{G}_{2,1}$ WZW model with the Yang-Lee model is an example of a theory which doesn’t obey GS.

What does this unitarisation mean at the level of the RCFT? Indirectly, we can see the corresponding unitary theory in e.g. the fusion ring and the list of cylindrical and toroidal partition functions. Typically these partition functions will be in one-to-one correspondence with those of the nonunitary theory, and we will use this correspondence below to obtain the classification of a new infinite family of nonunitary theories. But can we see the unitarisation directly inside the nonunitary theory, perhaps at the level of the chiral algebras? This is still unclear.

2. Background material

The characters $\chi_a(\tau) = \text{Tr}_{H_a} q^{L_0-c/24}$ of an RCFT carry a representation of $\text{SL}_2(\mathbb{Z})$:

\[
\begin{align*}
\chi_a(-1/\tau) &= \sum_b S_{ab} \chi_b(\tau), \\
\chi_a(\tau + 1) &= \sum_b T_{ab} \chi_b(\tau).
\end{align*}
\]

(1a) (1b)

The subscripts $a,b$ here label the (finitely many) chiral primary fields $\phi$. The ‘modular matrices’ $S$ and $T$ are unitary and symmetric, and $T$ is diagonal with entries $\exp[2\pi i (h_a - \frac{c}{24})]$ where $h_a$ are the conformal weights and $c$ is the central charge. The matrix $T$ has finite order, i.e. all numbers $h_a$ and $c$ are rational. The matrix $C = S^2$ is the order-2 (or order-1) permutation matrix called charge-conjugation. Then $(ST)^3 = (TS)^3 = C$. Moreover,

\[ S_{ab}^* = S_{Ca,b} = S_{a,Cb}. \]

(1c)

These properties should hold for any RCFT [9].

The fusion coefficients $N_c^{ac}$ can be expressed using Verlinde’s formula [10]:

\[ N_{ac}^{eb} = \sum_d \frac{S_{ad} S_{bd} S_{de}^*}{S_{0d}}, \]

(2)

where ‘0’ denotes the vacuum. We have $h_0 = 0$ and $C0 = 0$. It is convenient to define the fusion matrices $N_a$ by $(N_a)_{bc} = N_c^{ac}$. Then Verlinde’s formula says that $S$ simultaneously diagonalises all matrices $N_a$, with eigenvalues $S_{ab}/S_{0b}$.

In the case of nonunitary RCFTs (as discussed next section), this so-called ‘vacuum’ 0 does not possess all the usual properties expected of a physical vacuum, and so perhaps a better name for it would be ‘identity’.

**Definition 1.** By *modular data* we mean here unitary matrices $S, T$ such that $T$ is diagonal and of finite order, $S$ is symmetric, $(ST)^3 = S^2 = C$ is an order-2 (or order-1) permutation matrix, $C0 = 0$, and the fusion coefficients $N_{ac}^{ab}$ given by (2) are all nonnegative integers $\mathbb{Z}_+ := \{0, 1, 2, \ldots\}$. 
See e.g. [11,12] for some of the basic properties of modular data, although those papers primarily specialise to what we call below unitary modular data (see Def.2 next section). There are many additional properties which the modular data of a healthy RCFT should obey. For one important example [13], for any primaries \(a, b \in \Phi\), the numbers defined by

\[
Z(a, b) := T_{bb}^{\frac{1}{2}} N_{a,b}^c S_{bc} S_{0d} T_{dd}^{\frac{1}{2}} T_{cc}^{-2}
\]

must always be integers satisfying

\[
\begin{align}
|Z(a, b)| & \leq N_{aa}^b, \\
Z(a, b) & \equiv N_{aa}^b \pmod{2}.
\end{align}
\]

In particular, \(Z(a, 0) \in \{\pm 1, 0\}\) is the Frobenius-Schur indicator. Also, [14] gives formulas for the traces of \(S, ST\) and \((ST)^2\), among other things. These identities and conditions won’t play any special role in this paper, for reasons we’ll give later, and so are ignored in Def.1. We turn next to an additional condition which will be used below.

Form the vector \(\vec{\chi}(i) := (\chi_a(i))\) (each character converges throughout the upper half-plane, so these character values are defined). Since each character

\[
\chi_a(\tau) = q^{h_a - c/24} \sum_{n=0} A_{a,n} q^n
\]

has nonnegative coefficients \(A_{a,n}\), each component \(\chi_a(i)\) of \(\vec{\chi}(i)\) will be strictly positive. Because \(\tau = i\) is a fixed-point of the transformation \(\tau \mapsto -1/\tau\), \(\vec{\chi}(i)\) will be an eigenvector of \(S\) with eigenvalue 1:

\[
\sum_b S_{ab} \chi_b(i) = \chi_a(i).
\]

Fact 1. \(S\) has a strictly positive eigenvector, corresponding to eigenvalue 1.

The one-loop torus partition function [15]

\[
Z(\tau) = \sum_{a,b} M_{ab} \chi_a(\tau) \chi_b(\tau)^*
\]

is modular invariant: \(Z(-1/\tau) = Z(\tau) = Z(\tau + 1)\) etc, so the coefficient matrix \(M\) lies in the commutant of the \(SL_2(\mathbb{Z})\) representation: \(SM = MS, TM = MT\). We also know that the coefficients \(M_{ab}\) are multiplicities, and so lie in \(\mathbb{Z}_+\). Also, the multiplicity of the vacuum must be 1: \(M_{00} = 1\).

A vacuum-to-vacuum amplitude of fundamental importance in boundary CFT, is the cylindrical partition function (see for instance [16,17,18,19] and references therein)

\[
Z_{\alpha\beta}(t) = \sum_a n_{\alpha\beta}^a \chi_a(it).
\]
The coefficients \( n_{aa}^\beta \) will be nonnegative integers. Define matrices \( n_a \) by \( (n_a)_{\alpha\beta} = n_{aa}^\beta \). These matrices are required to form a representation, called a \textit{Nim-rep}, of the fusion ring: \( n_a n_b = \sum_c N_{ab}^c n_c \). We can simultaneously diagonalise the \( n_a \): equation (2) is replaced with

\[
n_{aa}^\beta = \sum_d \frac{\psi_{\alpha d} S_{ad} \psi_{\beta d}^*}{S_{0d}} ,
\]  

(4)

where each \( d \in \Phi \) appears in (4) with a certain multiplicity \( \text{mult}_n(d) \geq 0 \) independent of \( a, \alpha, \beta \).

There is a compatibility condition between the toroidal and cylindrical partition functions: the multiplicity \( \text{mult}_n(d) \in \mathbb{Z}_+ \) of each primary \( d \) in (4) must equal the diagonal entry \( M_{dd} \) of the corresponding modular invariant \( M \). For instance, the diagonal modular invariant \( M = I \) corresponds to the fusion matrix choice \( n_a = N_a \).

Two subtleties which we will avoid this paper are the gluing automorphism of [20], and the fact that the characters \( \chi_a(\tau) \) are usually not linearly independent. These play no special role in our discussion, and the interested reader can consult the relevant literature.

To fix notation, note that we have chosen \( S \) to correspond to the matrix \( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \) rather than its inverse, which also appears in the literature.

Next, let’s review the lesser known Galois action of RCFTs [21]. The entries of \( S \) will lie in some cyclotomic extension of \( \mathbb{Q} \) — that just means each entry \( S_{ab} \) can be written as a polynomial \( p_{ab}(x) \), with rational coefficients, evaluated at some root of unity \( x = \xi_L := \exp[2\pi i/L] \). In fact, \( L \) can be taken to be the order of the matrix \( T \) [22,23]. By the ‘Galois group’ of this cyclotomic extension, we mean the multiplicative group of integers \( \mathbb{Z}/L \mathbb{Z} \) — we write this \( \mathbb{Z}_L \). For example, \( \mathbb{Z}_{12}^\times = \{1,5,7,11\} \). Any such \( \ell \in \mathbb{Z}_L^\times \) acts on the number \( p_{ab}(\xi_L) \) by sending it to \( p_{ab}(\xi_L^\ell) \) — we denote this ‘Galois automorphism’ by \( \sigma_\ell \). For instance: \( \sigma_\ell \) fixes all rationals; \( \sigma_\ell \cos(2\pi \alpha/L) = \cos(2\pi \ell \alpha/L) \); \( \sigma_\ell 1 = \pm i \) for \( \ell \equiv \pm 1 \pmod{4} \), respectively; and \( \sigma_\ell \sqrt{2} = \pm \sqrt{2} \) depending on whether or not \( \ell \equiv \pm 1 \pmod{8} \). What is so special about the \( \sigma_\ell \) is that they are symmetries of the cyclotomic numbers: \( \sigma_\ell (u + v) = \sigma_\ell (u) + \sigma_\ell (v) \) and \( \sigma_\ell (uv) = \sigma_\ell (u) \sigma_\ell (v) \), for any numbers \( u, v \) in the \( L \)th cyclotomic extension \( \mathbb{Q}[\xi_L] \) of \( \mathbb{Q} \).

The point [21] is that to any \( \ell \in \mathbb{Z}_L^\times \), there is a permutation \( \alpha \mapsto \sigma_\ell \alpha \) of the primary fields, and a choice of signs \( \epsilon_\ell(a) = \pm 1 \), such that

\[
\sigma_\ell(S_{ab}) = \epsilon_\ell(a) S_{\sigma_\ell(a,b)} = \epsilon_\ell(b) S_{a,\sigma_\ell(b)} .
\]

(5a)

This Galois action should be regarded as a generalisation of charge-conjugation. In particular, \( \ell = -1 \) corresponds to the familiar Galois automorphism \( \sigma_{-1} = \ast \) (complex conjugation), with signs \( \epsilon_{-1}(a) = +1 \) and permutation given by the charge-conjugation \( \sigma_{-1} a = Ca \). It was proved in [22] that

\[
h_{\sigma_\ell a} - c/24 \equiv \ell^2 (h_a - c/24) \pmod{1} ,
\]

(5b)

where we can take \( \ell \) to be coprime to the order \( L \) of the matrix \( T \) (in fact it suffices here to take \( \ell \) in (5b) coprime to the denominator of the rational number \( h_a - c/24 \)). The partition
functions $M$ and $n$ respect this Galois action [21,24]:

\[ M_{ab} = M_{\sigma_a,\sigma_b} , \quad \text{mult}_n(a) = \text{mult}_n(\sigma a) . \quad (5c) \]

As mentioned in the Introduction, there are infinitely many different unitary matrices $\tilde{S}$ which diagonalise the fusion coefficients. We conclude this section by mentioning a generic way to construct symmetric unitary matrices $\tilde{S}$. Choose any simple-current $J$ and any Galois automorphism $\sigma_\ell$, where the integer $\ell$ is coprime to the order $L$ of $T$. Define the following matrices

\[ \tilde{S}_{ab} := \sigma_\ell(S_{Ja,Jb}) = \exp[2\pi i \ell (Q_J(b) + Q_J(J))] \epsilon_\ell(b) S_{a,\sigma_\ell Jb} , \quad (6a) \]

\[ \tilde{T}_{ab} := \delta_{ab} (T_{Ja,Jb})^\ell . \quad (6b) \]

This implies $\tilde{S} = \sigma_\ell(PSPT^t)$ and $\tilde{T} = \sigma_\ell(PTPT^t)$, where $P$ is the (orthogonal) permutation matrix $P_{ab} = \delta_b,Ja$, and these define a representation of the modular group $\text{SL}_2(\mathbb{Z})$: $(\tilde{S}\tilde{T})^3 = \tilde{S}^2 = C$. This new representation will typically be inequivalent to that of the original $S,T$. Note that $\tilde{S}$ is manifestly symmetric. That $\tilde{S}$ diagonalises the fusions can be verified either by direct calculation (using the fact that the fusion coefficients are rational numbers), or by noting from the right-side of (6a) that the $b$th column of $S$ has been permuted by $\sigma_\ell \circ J$, and multiplied by the phase $\exp[2\pi i \ell (Q_J(b) + Q_J(J))] \epsilon_\ell(b)$. Any such $\tilde{S},\tilde{T}$ will automatically obey the constraints (3) and identities in [14], provided $S,T$ do, and provided we choose the squareroot $\tilde{T}^{1/2} := PT^{1/2}PT^t$ (this is why nothing will be gained in this paper by manifestly imposing those conditions on modular data). Of course an additional (trivial and well-known) construction is to tensor $\tilde{S},\tilde{T}$ by any 1-dimensional representation of $\text{SL}_2(\mathbb{Z})$, i.e. to choose any (not necessarily primitive) 6th root $t$ of unity and to replace $\tilde{S} \mapsto t^{-3} \tilde{S}$, $\tilde{T} \mapsto t\tilde{T}$ (t should be a 6th root of unity, in order to preserve the positivity of $C$).

At least sometimes, the matrices $\tilde{S},\tilde{T}$ in (6) can be realised by ‘characters’. Let $L$ be the order of $T$, and suppose $\ell$ is a perfect square $m^2$ (mod $L$). For instance if $L$ or $L/2$ is a power of any odd prime, then half of the $\ell$’s in $\mathbb{Z}_L^*$ would be perfect squares mod $L$. Let $m'$ be a multiplicative inverse (mod $L$) of $m$, and choose any matrix $A \in \text{SL}_2(\mathbb{Z})$ obeying

\[ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} m & 0 \\ 0 & m' \end{pmatrix} \quad (\text{mod } L) . \]

Such an $A$ will always exist and, with respect to the $\text{SL}_2(\mathbb{Z})$ representation defined by (1), will correspond to the matrix $G_m$ defined by [23,22]

\[ (G_m)_{ab} = (ST^{m'}ST^mST^{m'}C)_{ab} = \epsilon_m(a) \delta_{b,\sigma_\ell(a)} . \]

Define the functions

\[ \tilde{\chi}_a(\tau) := \chi_{Jm'^a}(a\tau + b)_{c\tau + d} = \epsilon_m(a) \chi_{Jm',a}(\tau) . \quad (6c) \]
Then these ‘characters’ realise our matrices $\tilde{S}, \tilde{T}$ in the sense of (1):

$$\tilde{\chi}_a(-1/\tau) = \sum_b \tilde{S}_{ab} \tilde{\chi}_b(\tau),$$

$$\tilde{\chi}_a(\tau + 1) = \sum_b \tilde{T}_{ab} \tilde{\chi}_b(\tau).$$

Note that the coefficients of these ‘characters’ $\tilde{\chi}$ are integers but may not be positive — in fact, $\tilde{S}$ may not satisfy Fact 1 even if this ‘character’ interpretation exists.

More generally however (e.g. for the Yang-Lee model of §1 or the $c = -25/7$ minimal model studied in §6), the modified modular matrices $\tilde{S}, \tilde{T}$ won’t be realised by merely permuting the characters (i.e. reordering the primaries), and their direct meaning in terms of the nonunitary theory is still unclear.

To our knowledge, equations (6) and the related remarks are new (but for some thoughts in this direction see §5 of [25]).

3. Unitary vrs nonunitary RCFTs

In this section we describe the distinctions between the more familiar unitary RCFTs, and the more complicated and common nonunitary RCFTs.

The assumption of unitarity certainly does yield some simplifications. For instance, as is well known, the conformal weight $h = 1$ fields in the vacuum sector will always form a (complex) Lie algebra $g$, and the inner product $\langle , \rangle_0$ will define an invariant symmetric bilinear form for $g$. When the RCFT is unitary, the positive-definiteness of that invariant form implies that $g$ will be reductive [26,27] — i.e. a direct sum of an abelian Lie algebra $\mathbb{C}^m$ with a number of simple Lie algebras. Otherwise, when the theory is nonunitary, that Lie algebra belongs to a much wider class called self-dual. Any such Lie algebra can be constructed from a reductive Lie algebra by a sequence of ‘double extensions’ [28] (see also [29]). Self-dual Lie algebras have appeared in the physics literature: for instance, the 2+1-dimensional Poincaré group ISO(2,1) used by Witten [30] to relate Chern-Simons with 2+1-dimensional gravity has a self-dual Lie algebra; in [31,32,33] and references therein, families of self-dual Lie algebras have been explicitly related to CFT. Self-dual Lie algebras are precisely those for which the Sugawara construction works.

We are more interested here in the differences at the level of modular data, and at the level of the toroidal $M$ and cylindrical $n_a$ partition functions. From this perspective, the key observation is that, in a unitary theory, the vacuum 0 is the unique primary with minimal conformal weight $h$. Hence for $\tau = \epsilon i$ ($\epsilon \approx 0$)

$$0 < \chi_a(\tau) = \sum_b S_{ab} \chi_b(-1/\tau) \approx S_{a0} \chi_0(-1/\tau)$$

and thus $S_{a0} = S_{0a} > 0$.

**Definition 2.** By *unitary modular data*, we mean modular data obeying the additional requirement that $S_{a0} > 0$ for all $a \in \Phi$.

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2 I thank Matthias Gaberdiel for helpful discussions on this point.
A word of warning should be made: the modular data of any unitary RCFT will be unitary modular data. However, the converse may not hold: given some unitary modular data, there may or may not be a unitary RCFT which realises it — e.g. Fact 1 or constraint (3) may be violated.

In any modular data, \( S_{a0} \) is a nonzero real number (this follows because \( S_{a0}^* = S_{a,C0} = S_{a0} \)). What replaces the inequality \( S_{a0} > 0 \) in a nonunitary theory?

First note that there must be one and only one chiral primary, call it \( o \), with

\[
S_{ao} = S_{oa} > 0 \quad \forall a .
\]  

To see this, number all the primaries \( a_1 = 0, a_2, \ldots, a_n \), and take \( m \in \mathbb{R} \) to be large enough that the sum \( N := \sum_{i=1}^{n} m^i N_{ai} \), of fusion matrices has distinct eigenvalues \( \sum_i m^i S_{ai,b}/S_{0b} \) (in fact all these eigenvalues will be distinct, for all but finitely many values of \( m \)). Then for some phase \( \varphi \) and some primary \( o \in \Phi \), the column \( \varphi S_{2o} \) will be the unique (strictly positive) Perron-Frobenius eigenvector [34] of the strictly positive matrix \( N \). But Fact 1 implies that the phase \( \varphi \) equals +1 (if we don’t use Fact 1, i.e. for arbitrary modular data not necessarily corresponding to an RCFT, we only get that \( \varphi = \pm 1 \)). By unitarity of \( S \), there can be only one chiral primary obeying (8).

**Definition 3.** By the *minimal primary* of an RCFT, we mean the unique chiral primary \( o \in \Phi \) obeying (8) for all primaries \( a \in \Phi \).

**Fact 2.** In any RCFT, no primary can have smaller conformal weight than that of the minimal primary \( o \).

To see this, let \( 0', 0'', \ldots \) be the primaries with minimal conformal weight. The argument of (7) shows that for small \( \epsilon > 0 \), the leading terms of \( \chi_a(\epsilon i) \) will be

\[
S_{a0'} \chi_{0'}(\epsilon i) + S_{a0''} \chi_{0''}(\epsilon i) + \cdots ,
\]

that is to say

\[
0 \leq S_{a0'} A_{0',0} + S_{a0''} A_{0'',0} + \cdots
\]

for all \( a \in \Phi \) (recall that \( A_{b,n} \) are the coefficients of \( \chi_b \)). By unitarity of \( S \), and inequality (8), this forces \( o \) to be one of the primaries \( 0', 0'', \ldots \).

Every RCFT has a unique minimal primary. In a unitary RCFT, the minimal primary will be the vacuum 0. In all cases known to this author, there is a unique primary with minimal conformal weight (it does not seem to be known that this should always be the case). Recall that the conformal weight is (essentially) the energy, so it is tempting to identify the minimal primary with the true vacuum. On the other hand, the true vacuum should be invariant under the Poincaré group and in particular translations, yet typically the descendent \( L_{-1} o \) will not vanish (see the explicit example in §6). We will find that in a nonunitary theory, all of the familiar properties usually ascribed to the vacuum are now distributed between 0 and \( o \).
The quantum-dimension is the positive number \( S_{ao}/S_{0o} \). It is the maximal (Perron-Frobenius [34]) eigenvalue of the fusion matrix \( N_a \), and so obeys

\[
S_{ao} \geq S_{0o} , \tag{9a}
\]

\[
|S_{ao} S_{0b}| \geq |S_{0o} S_{ab}| , \tag{9b}
\]

\[
N_{ab}^c \leq \min\{S_{ao}/S_{0o}, S_{bo}/S_{0o}, S_{co}/S_{0o}\} . \tag{9c}
\]

See e.g. [12] for proofs. When there is a unique primary with minimal conformal weight, then that primary of course must be \( o \), and the quantum-dimension \( S_{ao}/S_{0o} \) equals the limit \( \lim_{q \to 1} \chi_a(q)/\chi_0(q) \).

A simple-current \( j \) is by definition any primary with quantum-dimension 1: i.e. \( S_{j,o} = S_{0o} \). Any simple-current corresponds to a permutation \( J \) and a phase ('monodromy charge') \( a \mapsto Q_J(a) \in \mathbb{Q} \) such that \( N_{j,a}^b = \delta_{b,j} \), \( J0 = j \), and [35]

\[
S_{Ja,b} = e^{2\pi i Q_J(b)} S_{ab} . \tag{10a}
\]

We’ll usually identify the simple-current \( j \) with the permutation \( J \). From (10a) we get \( Q_{J,J'}(a) \equiv Q_J(a) + Q_{J'}(a) \pmod{1} \). Hence if \( J' = id. \), then \( Q_J(a) \in \frac{1}{n} \mathbb{Z} \). Also, \( Q_J(\sigma \alpha) \equiv \ell Q_J(a) \pmod{1} \) and \( \sigma J(a) = J' \sigma \alpha \), for all Galois automorphisms \( \sigma \). Although (8) requires \( Q_J(o) \in \mathbb{Z} \) for all simple-currents \( J \), one distinction with the more familiar unitary theories is that the monodromy charge \( Q_J(0) \) of the vacuum can be a half-integer. This introduces the following modifications in the formulas applying to the unitary theories: \( N_{ab}^c \neq 0 \) implies \( Q_J(a) + Q_J(b) \equiv Q_J(c) + Q_J(0) \pmod{1} \), so \( Q_J(J'a) \equiv Q_J(J') + Q_J(a) + Q_J(0) \pmod{1} \). Expanding \( S_{J,J'} \) in two ways gives \( Q_J(J') + Q_J(0) \equiv Q_J(J) + Q_J(0) \pmod{1} \). Finally (see [12] for the arguments)

\[
h_{Ja} - h_a \equiv h_J + Q_J(0) - Q_J(a) \pmod{1} , \tag{10b}
\]

\[
2h_J \equiv Q_J(0) - Q_J(J) \pmod{1} . \tag{10c}
\]

Note that for nonunitary theories, a simple-current is defined by \( S_{jo} = S_{0o} \), rather than the simpler but potentially weaker \( S_{jo} = \pm S_{00} \) (but see Consequence 2(vi) next section).

Both the vacuum 0 and the minimal primary \( o \) obey \( C0 = 0 \) and \( Co = o \), because their \( S \)-columns are real. Also, neither 0 nor \( o \) can be fixed by any simple-current \( J \neq id. \) (because otherwise \( S_{ao} = 0 \) or \( S_{ao} = 0 \) for any \( a \) with \( Q_J(a) \notin \mathbb{Z} \)).

There are arguments valid for unitary RCFTs which break down for nonunitary ones, because the vacuum 0 and the minimal primary \( o \) are distinct. For instance, consider the proof that for a given choice of unitary modular data, there are only a finite number of possible modular invariants \( M \). In particular, we get the bounds [36]

\[
M_{ab} = | \sum_{c,d \in \Phi} S_{ac} M_{cd} S_{bd}^* | \leq \frac{S_{ao} S_{bo}}{S_{0o}^2} \sum_{c,d \in \Phi} S_{0c} M_{cd} S_{0d} = \frac{S_{ao} S_{bo}}{S_{0o}^2} ,
\]

using the triangle inequality, (9b), and the uniqueness of the vacuum \( M_{00} = 1 \). For nonunitary modular data, this argument breaks down, as it yields the bound \( M_{ab} \leq M_{oo} S_{ao} S_{bo}/S_{0o}^2 \).
All other finiteness proofs for modular invariants known to this author, similarly break down for nonunitary modular data, because we don’t have a bound on the multiplicity $M_{\infty}$. Note however that this argument does give us the interesting fact that in any RCFT

$$M_{\infty} \geq 1.$$ 

For example consider the nonunitary modular data

$$S = \frac{1}{6} \left( \begin{array}{cccccccc} 1 & 1 & 2 & 2 & 2 & 2 & -3 & -3 \\ 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 \\ 2 & 2 & 4 & -2 & -2 & -2 & 0 & 0 \\ 2 & 2 & -2 & 4 & -2 & -2 & 0 & 0 \\ 2 & 2 & -2 & -2 & 4 & -2 & 0 & 0 \\ -3 & 3 & 0 & 0 & 0 & 0 & 3 & -3 \\ -3 & 3 & 0 & 0 & 0 & 0 & -3 & 3 \end{array} \right), \quad (11a)$$

$$T = \text{diag}(1, 1, 1, 1, \xi_3, \xi_3^*, -1, 1).$$

Number the primaries from 0 to 7, and let the 0th one be the vacuum. Then the minimal primary $o$ is ‘1’. This data has the same fusions as the quantum-double of the finite group $S_3$ (which is its ‘uniformisation’); although $S$ obeys Fact 1, we know of no RCFT which realises it. It has infinitely many modular invariants: for example, choose any integer $m \geq 0$, then

$$Z = (\sum_{a=0}^{7} |\chi_a|^2) + m |\chi_1 + \chi_3 + \chi_7|^2$$

(11b)

defines a modular invariant. Here, $M_{oo} = m + 1$ can be arbitrarily large.

The same thing was noticed before, in a related context: there are infinitely many modular invariants for the admissible representations of affine algebra $\hat{su}(2)$ at fractional (nonintegral) level $k$ [37]. Why this didn’t happen for the nonunitary Virasoro minimal models, is explained by Fact 3 below. It is somewhat surprising that, although there are infinite numbers of $\hat{su}(2)$ modular invariants at each fractional level, there are only a finite number of modular invariants for their cosets (namely the nonunitary minimal models) — after all we tend to think that a source of coset modular invariants are products of the modular invariants of the component affine algebras. Of course the explanation, ultimately, is field identification.

Another useful fact about modular invariants in unitary modular data, is that for any simple-currents $j, j' \in \Phi$, $M_{j, j'} = 0$ or 1, and $M_{j, j'} = 1$ iff the following selection rule holds for all $a, b \in \Phi$:

$$M_{ab} \neq 0 \quad \implies \quad Q_J(a) \equiv Q_{J'}(b) \pmod{1}. \quad (12)$$

The proof of this also collapses for nonunitary RCFTs, for the same reason; the argument instead tells us that $M_{J_o, J'_o} \leq M_{oo}$ with equality only if the above selection rule holds.

Similar remarks are valid for the cylindrical partition functions (i.e. the NIM-reps). Any NIM-rep is uniquely decomposable into a direct sum of irreducible NIM-reps. For
any \( \text{NIM-rep} \) \( n_a \) of unitary modular data, the multiplicity \( \text{mult}_n(0) \) of the vacuum in (4) precisely equals the number of irreducible \( \text{NIM-reps} \) which compose \( n_a \). Each of these irreducible summands would correspond to a family of boundary conditions which completely decouples from the other families. The uniqueness of the vacuum then tells us that the \( \text{NIM-rep} \) of any unitary RCFT must be indecomposable [24]. The finiteness result for \( \text{NIM-reps} \) of unitary modular data follows from this (see [24] for details). For nonunitary modular data, the number of irreducible \( \text{NIM-reps} \) composing a given \( \text{NIM-rep} \) equals the multiplicity \( \text{mult}_n(o) \) of the minimal primary \( o \). A priori, a \( \text{NIM-rep} \) corresponding to a nonunitary RCFT may be decomposable; hence the finiteness result [24] for \( \text{NIM-reps} \) breaks down unless we know that the multiplicity of \( o \) is bounded.

4. The Galois Shuffle and consequences

For the reasons given last section, it is important to answer the following:

**Question.** What is the multiplicity \( \text{mult}_n(o) = M_{oo} \) of the minimal primary?

The Virasoro minimal models were classified in [7]; checking the answer there, we get multiplicity \( M_{oo} = 1 \) for all of them. We can explain and generalise this result:

**Fact 3.** In any \( W_N \) \( (p,p') \) minimal model, for any positive integers \( N, p, p' \), modular invariance of the 1-loop partition function forces the multiplicity \( \text{mult}_n(o) = M_{oo} \) of the minimal primary to be 1.

We prove this in §6. Fact 3 is new.

**Conjecture.** The multiplicity \( \text{mult}_n(o) = M_{oo} \) of the minimal primary in any RCFT is 1.

This makes some sense, as there are reasons for thinking of the minimal primary as the true vacuum\(^3\). It can also (see Consequence 2(ii)) follow from the Galois Shuffle property given shortly. Collecting results from last section, we get

**Consequence 1.** Suppose the multiplicity \( \text{mult}_n(o) = M_{oo} \) of the minimal primary in a given RCFT is 1. Then:

1(i) There will be only finitely many modular invariants \( M \) with the given (chiral) modular data — in fact we get the inequality \( M_{ab} \leq S_{ao}S_{bo}/S_{oo}^2 \);

1(ii) Any \( \text{NIM-rep} \) corresponding to a modular invariant \( M \) will be indecomposable. There are only finitely many indecomposable \( \text{NIM-reps} \).

In a superficially independent direction, further consequences occur when we assume more.

**Definition 4.** We say an RCFT has the \( GS \) (or Galois shuffle) property, if there is a simple-current \( J_o \) (possibly the identity), and a Galois automorphism \( \sigma_o \) (possibly the

\(^3\) Nevertheless, as pointed out to the author by Philippe Ruelle, curious things can happen in even the nicest unitary RCFTs. For example, the Ising model is unitary, but it has nonunitary observables, according to [38]. That paper also suggests that the conformal weight 0 primary there need not be the identity field. In [39] we see that the 3-state Potts model on the cylinder can have a degenerate vacuum, for some choices of boundary conditions.
identity), such that the minimal primary \( o \) in the theory is related to the vacuum by
\[
o = J_o \sigma_o 0.
\]

Of course this is trivially obeyed by any unitary RCFT. We verify in §6 that:

**Fact 4.** All \( W_N \) minimal models possess the GS property.

The same is true for the minimal \( W_{SO(2N)} \) models, but we won’t prove it here. The modular data for the minimal \( W_{\bar{G}} \) models, for \( \bar{G} \) nonsimply laced, is a little more complicated and we haven’t looked at them yet.

Modular data may not obey this property. For example, the 1-dimensional modular data \( S = (-1), T = (-1) \) does not obey it. On the other hand, that 1-dimensional modular data violates Fact 1 and so can’t be realised by an RCFT. A more serious example is the tensor product of \( \bar{G}_{2,1} \) with the Yang-Lee model. That this doesn’t have the GS property can be seen from Consequence 2(iv) below: this model has central charge \( c = -8/5 \), and four primaries, with conformal weights 0, 1/5, 2/5, so \( h_o = -1/5 \), \( h_J = 0 \) and the congruence in 2(iv) reduces to \( \ell^2 \equiv -2 \) (mod 15), which has no solutions. More generally, the tensor product of a nonunitary model with its unitarisation (defined below) typically won’t satisfy the GS property.

Even though the GS property does not hold for all RCFTs, when it holds for a given RCFT (which is often, as we’ll see) it has several consequences. We begin by listing some of its immediate consequences, without assuming the Conjecture. The most important of these are Consequences 2(ii), (iv), (v) and (viii) — in particular, (iv) reduces to a statement about quadratic residues and becomes a stronger constraint the more distinct primes divide the order of \( T \). Consequences 2(i),(iii),(iv) can be interpreted as constraining the possible simple-currents \( J_o \) and Galois automorphism \( \sigma_o \). Consequences 2(vi),(vii) are of technical interest.

**Consequence 2.** Let 0 be the (chiral) vacuum and \( o \) the minimal primary. Suppose \( o = J_o \sigma_o 0 \) for some Galois automorphism \( \sigma_o \) and some simple-current \( J_o \). Then:

1. **2(i)** The simple-current \( J_o \) has order 1 or 2;
2. **2(ii)** If \( J_0 = id \), then the multiplicity \( \text{mult}_o(o) = M_{oo} \) of the minimal primary must be 1 (i.e. the Conjecture must hold for this RCFT).
3. **2(iii)** For any simple-current \( J \), \( Q_J(J_o) \in \mathbb{Z} \) and \( Q_{J_o}(J) \equiv Q_{J_o}(0) + Q_J(0) \) (mod 1).
4. **2(iv)** The conformal weight \( h_o \) of the minimal primary satisfies

\[
h_o \equiv \frac{c}{24} (1 - \ell^2) + h_J \quad \text{(mod 1)}
\]

for some integer \( \ell \) coprime to the denominator of \( c/24 \), and for some simple-current \( J \) of order 1 or 2 (so \( 4h_J \in \mathbb{Z} \)).

5. **2(v)** The fusion rules are exactly reproduced by modular data \( \hat{S}, \hat{T} \), which is unitary in the sense of Def.2 (this unitarisation is explicitly given in equation (13) below).
6. **2(vi)** A primary \( j \in \Phi \) will be a simple-current, iff \( S_{j0} = \pm S_{00} \).
7. **2(vii)** \( |S_{a0}| \geq S_{o0} \).
The list of modular invariants $M$ of $S, T$ with $M_{oo} = 1$ is identical to that of the modular invariants $\hat{M}$ of the unitarisation $\hat{S}, \hat{T}$ which obey $\hat{M}_{J_0, J_0} = 1$; the (indecomposable) NIM-reps of $S, T$ can be identified with those of the unitarisation $\hat{S}, \hat{T}$.

First note that (1c), $C\sigma = \sigma C$, and $C0 = 0$, tell us that each column $S_{J_0, \sigma 0}$ is real, for any Galois $\sigma$. Hence Consequence 2(i) follows from (10a), because both columns $S_{J_0, \sigma 0} = S_{J_0, -\sigma 0}$ and $S_{T_0}$ are real. Consequence 2(ii) follows from (5c). To see Consequence 2(iii), note that the inequality $S_{J_0} > 0$ requires $Q_J(J_o \sigma_0) \in \mathbb{Z}$. In Consequence 2(iv), take $J = J_o$ and $\sigma_1 = \sigma_o$, and use (5b) and (10b). Since $S_{J_0} = \pm S_{00}$ iff $S_{J_0} = \pm S_{0, J_0}$ iff $\sigma_o(S_{J_0}) = \pm \sigma_0 (S_{0, J_0})$, we get 2(vi). Applying $\sigma_o \circ \sigma_o^{-1} = id.$ to $S_{00}$ and using (9a), gives 2(vii). Since the fusions of $S$ and $\hat{S}$ are identical (as was discussed after eqs.(6)), so are their NIM-reps. The statement in 2(viii) about modular invariants follows from (5c): an integral matrix $M$ commutes with $S$ and $T$ iff the integral matrix $\hat{M} := PMP \ell$ commutes with $\hat{S}$ and $\hat{T}$.

The unitarisation $\hat{S}, \hat{T}$ mentioned in Consequence 2(v) is given explicitly as follows. Recall equations (6a), (6b). Put $J = J_o$ and $\sigma = \sigma_o$. Define the following matrices

\[
\hat{S}_{ab} := \epsilon_\sigma(0) \sigma(S_{Ja, Jb}) = \epsilon_\sigma(0)(-1)^{Q_J(a)+Q_J(b)+Q_J(0)} \sigma(S_{ab}),
\]

\[
\hat{T}_{ab} := \delta_{ab} \epsilon_\sigma(0)(T_{Ja, Jb})^\ell,
\]

where $\ell \in \mathbb{Z}$ corresponds to $\sigma$, i.e. $\sigma$ acts on roots of unity by $\sigma(\xi) = \xi^\ell$. Requiring the vacuum 0 in the unitarised modular data to have conformal weight 0, we find that the unitarisation $\hat{S}, \hat{T}$ has central charge

\[
\hat{c} \equiv \ell c - 24 \ell h j_0 + 12 \ell \quad (\text{mod 24}),
\]

where $(-1)^\ell := \epsilon_\sigma(0)$.

As explained after (6b), the unitarisation obeys all of the conditions stated in Definitions 1 and 2, and an easy Galois argument shows that it automatically obeys the additional conditions (3) and identities given in [14] (provided $S, T$ do). It is conceivable however that $\hat{S}, \hat{T}$ may not always be realisable by an RCFT — for instance $\hat{S}$ could conceivably violate Fact 1. Nevertheless, in the few cases this author has been able to check, the uniformisation always seems to be realised by a completely healthy unitary RCFT. We return to this in the concluding section.

Although the minimal primary $o$ is uniquely determined from the modular data, the simple-current $J_o$ and Galois automorphism $\sigma_o$ may not be unique. There can be more than one unitarisation of nonunitary modular data. In particular, let $G_0$ be the set of all Galois automorphisms $\sigma$ such that $\sigma(S_{00}) = \pm S_{00}$. Then $\sigma_o$ will be unique only up to $G_0$: $\sigma_o$ can be replaced by any other $\sigma$ in its coset $G_0 \sigma_o$; but once the Galois automorphism $\sigma_o$ has been chosen, the simple-current $J_o$ will be uniquely determined.

Further results hold if we assume both the Conjecture and the GS property.

**Consequence 3.** Let 0 be the (chiral) vacuum and $o$ the minimal primary. Suppose $o = J_o \sigma_o 0$ for some Galois automorphism $\sigma_o$ and some simple-current $J_o$. Assume that the multiplicity $\text{mult}_n(o) = M_{oo}$ in the full RCFT is 1. Then:
3(i) The list of possible toroidal and cylindrical partition functions are identical for $S, T$ as for its unitarisation $\hat{S}, \hat{T}$.

3(ii) The multiplicity $\text{mult}_n(J_o) = M_{J_o,J_o} = 1$.

3(iii) We have the symmetry $M_{J_oa,J_ob} = M_{ab}$.

3(iv) We have the selection rule $M_{ab} \neq 0 \Rightarrow Q_{J_o}(a) \equiv Q_{J_o}(b) \pmod{1}$.

3(v) If $J_o \neq \text{id}$, the nim-rep $n$ is 2-colourable.

First let us define 2-colourability of the nim-rep. Let $\mathcal{B}$ be the set labelling the rows and columns of the nim-rep (i.e. the vertices of the fusion graphs, equivalently the labels of the boundary states). We can assign each $x \in \mathcal{B}$ to a number $q_x = 0, \frac{1}{2}$ such that the selection rule

$$n^{y}_{ax} \neq 0 \Rightarrow Q_{J_o}(a) + q_x - q_y \in \mathbb{Z}$$

holds for all $a \in \Phi, x, y \in \mathcal{B}$. For example, the nim-reps of the Virasoro minimal models, at least those compatible with the modular invariants, can be found in [40]. It can be verified explicitly that all of them are 2-colourable. The nim-reps built out of tadpoles aren’t 2-colourable, but they also aren’t compatible with any modular invariants.

Consequence 3 is now easy to prove. 3(i) is just 2(viii). The Galois symmetry (5c) with $\sigma = \sigma_o$ tells us that $1 = M_{00} = M_{J_oJ_o}$. Because $M_{J_oJ_o} = M_{00}$, we get 3(ii), and the rest automatically follows.

Consequence 3(i) reduces the classification of nonunitary models to that of the more familiar unitary ones. In §6 we give the unitarisation of the $W_N$ minimal models, and we find for instance that the affine algebra $\hat{\mathfrak{su}}(3)$ gives the uniformisation of certain $W_3$ models. We thus obtain for free the classification of those $W_3$ models, in the bulk. With a little additional work, the classification of all nonunitary $W_3$ minimal models (in the bulk) can be obtained [41].

**Consequence 4.** The complete list of $W_3$ minimal models at $(p, 4)$ is in exact one-to-one correspondence with the list of $\hat{\mathfrak{su}}(3)$ level $p - 3$ modular invariants. In particular, there are precisely four modular invariants for each value of $p$, except for $p = 3$ (where there is only one) and $p = 5$ (where there are only two). Those four modular invariants are all constructed using simple-currents and/or charge-conjugation in the standard way. Each of those modular invariants has a compatible nim-rep.

Of course $p$ there must be odd (since it must be coprime to 4). It is important to note that Consequence 4 is a theorem independent of the general validity of the conjecture, because Facts 3 and 4 tell us that the minimal $W_3$ models obey all the hypotheses of Consequence 3. The proof of Consequence 4, and the explicit correspondence between the $W_3$ modular invariants and those of $\hat{\mathfrak{su}}(3)$, will be given at the end of §6.

5. Comments on affine algebras at fractional level

Let $\mathfrak{g} = X_{r}^{(1)}$ be any affine algebra. For the relevant facts and notation about integrable representations of $\mathfrak{g}$, and corresponding $\text{SL}_2(\mathbb{Z})$ representations, see e.g. [4].

The so-called *admissible representations* of $\mathfrak{g}$ at fractional level $k = t/u$ [6] share many properties with the better known integrable representations. Their importance for RCFTs, and this paper, lies in the quantum Drinfeld-Sokolov reduction method for constructing
W-algebras (see [8] for a review) and their parallel use in GKO cosets for nonunitary models. But a natural question, which has received much attention (see e.g. [42] and references therein), is: Is there an RCFT which corresponds more directly to the admissible representations, roughly in the way that the integrable data is directly realised by the Wess-Zumino-Witten models [3]? As is well-known, naively placing the admissible $S$ matrix $S^\text{adm}$ into Verlinde’s formula fails to produce nonnegative integer fusions (indeed a general though conjectural expression for these ‘fusions’, manifestly demonstrating that they can be integers of either sign, is given in [43]). This means that the desired correspondence is more subtle, if it exists at all. Indeed, for $\mathfrak{su}(2)$ the generally accepted wisdom of [44,45] has now been challenged by [46] and independently by [47]. For example, [47] suggests that a CFT corresponding to admissible fusions will be both nonunitary and quasi-rational.

In this section we make a couple of preliminary remarks, relating Galois to the admissible $S$ matrix $S^\text{adm}$. This underlies our calculations in §6, and (we believe) lends support to our expectation that the GS property will be quite common among RCFT.

Let $g$ be the dual Coxeter number of $\overline{g} = X_r$. Write the level $k = t/u$, where $\gcd(t, u) = 1$, $t \in \mathbb{Z}$, $u \in \{1, 2, 3, \ldots\}$. Put $k^I = u(k + g) - g$ and $k^F = u - 1$, for reasons to be clear soon. The integrable case is recovered when the denominator $u = 1$. Admissibility requires $k \geq (\frac{1}{u} - 1)g$, and we will assume for most of this section an additional coprimeness condition: for $\overline{g} = \mathfrak{su}(N)$, the denominator $u$ must be coprime to $N$; for $\overline{g} = \mathfrak{so}(N)$, $\mathfrak{sp}(N)$, $F_4$, or $E_7$, $u$ must be odd; and for $\overline{g} = E_6$ or $G_2$, $u$ must not be a multiple of 3. This coprimeness condition is necessary for the direct Galois interpretation to be given shortly; when it fails the Galois interpretation is slightly more subtle (see the discussion on [48] when $u$ is even, given at the end of this section).

The admissible highest weights $P^k_\text{adm}$ consist of pairs $\lambda = (\lambda^I, \lambda^F)$, where $\lambda^I \in P^k_+$ is an integrable highest weight of $g$ of level $k^I \in \mathbb{Z}_+$, and where $\lambda^F$ belongs to a certain finite set of level $k^F \in \mathbb{Z}_+$ weights of $g$. For instance, for $\mathfrak{su}(2)$, $\lambda^F \in P^k_+$, so $P^k_\text{adm} = P^k_+ \times P^k_+$. For $\mathfrak{su}(3)$, $\lambda^F$ belongs to this disjoint union of $P^k_+$ with the subset of $P^k_+$ with first Dynkin label $\lambda_1^F \geq 1$. In general, the cardinality of $P^k_\text{adm}$ will be $||P^k_+|| u^r$ where $r$ is the rank of the Lie algebra $\overline{g}$. The modular $S$ matrix $S^\text{adm}$ is given by [6]

$$S^\text{adm}_{\lambda \mu} = \pm F_m \exp[2\pi i \{ (\lambda^I + \rho|\mu^F) + (\lambda^F|\mu^I + \rho) - (k + g)(\lambda^F|\mu^F) \}]$$

$$\times \sum_{w \in W} \det(w) \exp[\frac{-2\pi i}{k + g} (w(\lambda^I + \rho)|\mu^I + \rho)] = \varphi_{\lambda \mu} S^{(k)}_{\lambda \mu},$$

(14)

where $F_m$ is some constant (independent of $\lambda, \mu$), and the sign $\pm 1$ depends on $\lambda, \mu$ (it is given explicitly in e.g. eq.(2.46) of [49]). $W$ here is the (finite) Weyl group of $\overline{g}$. The modulus $|\varphi_{\lambda \mu}| = u^{-r/2}$ is constant, and the matrix $S^{(k)}$ is the sum over the Weyl group $W$, normalised by $u^{r/2} F_m$.

$S^\text{adm}$ is unitary and symmetric, but $S$ does not have a column of constant phase, and therefore Verlinde’s formula (2) will yield negative ‘fusions’ $N^\nu_{\lambda \mu}$. So the admissible modular matrices $S^\text{adm}, T^\text{adm}$ don’t constitute modular data (although they define a representation of $\text{SL}_2(\mathbb{Z})$ as before). In modular data, the charge-conjugation matrix $C = S^2$ is a permutation matrix; for the admissible data, $C = S^2$ is a signed permutation matrix.
Typically, there won’t be a unique admissible weight \( \lambda \in P^k_{\text{adm}} \) with minimal conformal weight.

Returning to formula (14), our observation in this section is simply that, up to a sign independent of \( \lambda, \mu, S^{(k)}_{\lambda\mu} = \pm \sigma S^{(k)}_{\lambda\mu} \), where \( \sigma \) is a Galois automorphism depending on \( u \) but not \( \lambda, \mu \), and where \( S^{(k)} \) is the integrable ‘WZW’ \( S \) matrix for \( \mathfrak{g} \) at integer level \( k \).

What this means is that the \( \lambda \) part of admissible representations is understandable. If for instance we adjusted the phases \( \phi \to \phi' \) in (14) appropriately, the admissible \( S \) matrix \( S^\text{adm} \) would become a factorised matrix \( S^\text{fac} = \phi' \otimes S^{(k)} \) which, when placed in Verlinde’s formula (2), would yield the factorised nonnegative integer fusions \( N^\text{adm}_{\lambda\mu} \phi' \otimes \nu' \). Where \( N^{(k)} \) are the usual integrable WZW fusions for \( \mathfrak{g} \) at level \( k \), and where the fractional part \( N' \) has simple-current fusions.

For example, for \( \tilde{\mathfrak{su}}(2) \) at level \( k = t/u \), write \( \lambda = (\lambda^I, \lambda^F) =: (l, l'), \mu =: (m, m') \), where \( \lambda^I \in P^{+2u-2}_t(\mathfrak{su}(2)) = \{0, 1, \ldots, t + 2u - 2\} \) and \( \lambda^F \in \{0, 1, \ldots, 2u - 1\} \). Then

\[
S^\text{adm}_{\lambda\mu} = \sqrt{\frac{2}{u^2(2k+2)}}(-1)^t(m+1)+m'(l+1)e^{-\pi lt'm'/u} \sin(\frac{\pi (l+1)(m+1)}{k+2}).
\]

If we were to drop the \((-1)^t(m+1)+m'(l+1)\) factor, then the resulting factorised \( S \) matrix \( S^\text{fac} \) would yield the factorised fusions \( N'_{\mu',\nu'} n' N^I_{lm} \) \( k \) where \( N'_{\mu',\nu'} n' = \delta_{n',l'+m'} (\mod 2u) \) and \( N^{(t+2u-2)} \) are \( \tilde{\mathfrak{su}}(2)_{t+2u-2} \) fusions. The Galois automorphism \( \sigma \) involved here corresponds to any \( \ell \) coprime to \( u \) satisfying \( \ell \equiv u (\mod 8(t+2u)). \)

A more subtle idea along these lines was suggested in [48], who in effect changed the sign \((-1)^t(m+1)+m'(l+1)\) to \((-1)^u(m'+1)+m'(l+1)\) (for \( u \) odd) and to \((-1)^t(l'-1)(m+1)+m'(l+1)\) (for \( u \) even). The resulting fusions are again in the factorised form: \( \tilde{\mathfrak{su}}(2)_{t+2u-2} \) fusions times \( \mathbb{Z}_{2u} \) fusions. Although these fusions differ from those in e.g. [44,45], they are consistent with the in-depth analysis of [46] for \( \tilde{\mathfrak{su}}(2) \) at level \( k = -4/3 \). Incidentally, this modular data of [48] is nonunitary; its unitarisation corresponds to the Galois automorphism with \( \ell \) odd and congruent to \( u (\mod t+2u) \).

What value if any the observation in this section has, is unclear at this point. However it is implicit in the following section.

6. Examples

The classic examples of nonunitary RCFTs come from the Virasoro minimal models. Let \( p > p' \geq 2 \) be coprime integers. The primary fields \( \varphi_{rs} \) here are parametrised by all pairs \((r, s)\) where \( 1 \leq r \leq p' - 1, 1 \leq s \leq p - 1 \), and \( p's < pr \). So the number of primaries is \((p-1)(p'-1)/2\). The conformal weight \( h_{rs} \) of \( \varphi_{rs} \) is

\[
h_{rs} = \frac{(pr - p's)^2 - (p - p')^2}{4pp'}. \quad (15a)
\]

The \( S \) matrix has entries

\[
S_{rs;rs'} = 2 \sqrt{\frac{2}{pp'}}(-1)^{1+sr'+rs'} \sin(\frac{p}{p'}rr') \sin(\frac{p'}{p}ss'). \quad (15b)
\]
The central charge of the \((p,p')\) minimal model is \(c = 1 - 6(p-p')^2/pp'\). It is unitary iff \(p-p' = 1\). The vacuum 0 corresponds to primary \((r,s) = (1,1)\). The simple-current is \(j = (p'-1,1)\), corresponding to permutation \(J(r,s) = (p'-r,s)\) (if \(pr + p's < pp')\) or \((r,p-s)\) (otherwise); the monodromy charge in (10a) is \(Q(r,s) = (1+pr+p's)/2\). There is a unique primary \(o = (r_o,s_o)\) with minimal conformal weight, corresponding to the unique \((r,s) \in \Phi\) obeying \(pr - p's = 1\).

The Galois group here can be taken to be \(\mathbb{Z}_{8pp'}^\times\). When \(r_o\) and \(s_o\) are both odd, choose any \(\ell\) obeying \(\ell \equiv r_o \pmod{2p'}\) and \(\ell \equiv s_o \pmod{2p}\), and take \(J_o = (1,1)\). When instead \(r_o\) is even, so both \(s_o\) and \(p'\) will be odd, choose \(\ell \equiv p'-r_o \pmod{2p'}\) and \(\ell \equiv s_o \pmod{2p}\), and \(J_o = (p'-1,1)\). Finally, if \(s_o\) is even, then both \(r_o\) and \(p\) will be odd, and choose \(\ell \equiv r_o \pmod{2p'}\) and \(\ell \equiv p-s_o \pmod{2p}\), and again \(J_o = (p'-1,1)\). In all cases, by the Chinese remainder theorem, there's a unique solution in the range \(1 \leq \ell \leq 2pp'\); it will lie in \(\mathbb{Z}_{8pp'}^\times\) (i.e. be coprime to 2, \(p\) and \(p'\)), so let \(\sigma_o\) be the corresponding Galois automorphism. Then \(o = J_o\sigma_o(1,1)\), showing that the GS property holds here.

For a concrete example, consider the \((7,3)\) minimal model \((c = -25/7)\), with primaries

\[
\{(1,1), (1,2), (2,1), (2,2), (2,3), (2,4)\}
\]

taken in that order. Then

\[
S_{r,s;r',s'} = \sqrt{\frac{2}{7}} \begin{pmatrix}
-d & a & d & -a & -b & b \\
 a & b & a & b & d & d \\
d & a & d & -b & -b \\
-a & b & a & -b & d & -d \\
-b & d & -b & d & -a & -a \\
d & b & -d & -a & a & -a
\end{pmatrix},
\]

\[
T_{r,s;r',s'} = \text{diag}\{\exp[\pi i 25/84], \exp[\pi i -5/84], \exp[\pi i 235/84], \exp[\pi i 121/84], \exp[\pi i 43/84], \exp[\pi i 1/84]\},
\]

where \(a = \sin(\pi/7) \approx 0.434\), \(b = \sin(2\pi/7) \approx 0.782\), and \(d = \sin(3\pi/7) \approx 0.975\). The positive column (namely the second) corresponds to the minimal primary \(o = (1,2)\) (note that \(7 \cdot 1 - 3 \cdot 2 = 1\)). The Galois automorphism \(\sigma_o\) corresponds to \(\ell = 19\), and the simple-current \(J_o = (2,1)\) is also nontrivial. The unitarisation is

\[
\hat{S} = \sqrt{\frac{2}{7}} \begin{pmatrix}
a & b & a & b & d & d \\
d & b & -d & -a & a \\
a & -b & a & b & -d & d \\
b & -d & b & a & a \ a \\
d & -a & -d & a & b & -b \\
d & a & d & a & -b & -b
\end{pmatrix},
\]

\[
\hat{T} = \text{diag}\{\exp[\pi i 97/84], \exp[\pi i -53/84], \exp[\pi i -29/84], \exp[\pi i 73/84], \exp[\pi i 19/84], \exp[\pi i -23/84]\}.
\]

Incidentally, this unitarisation is the modular data for affine algebra \(\text{su}(2) \oplus E_8\) at level \((5,1)\); more generally, the \((p,3)\) Virasoro minimal model has unitarisation corresponding
to $\mathfrak{su}(2)$ at level $p - 2$, whenever $p$ is odd (ignoring a certain number of $\widehat{E}_{8,1}$’s, whose only purpose is to adjust $c$ by an appropriate multiple of 8).

The characters $\chi_{rs}$ of the minimal models have been explicitly calculated [50]: for example, for the $(7,3)$ model the characters of the vacuum and minimal primaries are

$$\chi_0 = \chi_{11} = q^{25/168} (1 + q^2 + q^3 + 2q^4 + \cdots),$$

$$\chi_o = \chi_{12} = q^{-5/168} (1 + q + q^2 + 2q^3 + \cdots).$$

In particular we see that, although the translation operator $L_{-1}$ kills the vacuum 0, it doesn’t kill the minimal primary $o$. This illustrates one of the ways the minimal primary doesn’t behave like a vacuum.

More generally, we can consider the minimal $W_N$ models (cf. [51,52,49,43]). These theories are also parametrised by a pair $p > p' \geq N$ of coprime integers. As before, they are unitary iff $p = p' + 1$. The primaries consist of all $(J,J)$-orbits $(J^i\lambda, J^i\mu)$, where $\lambda \in P_{++}^p, \mu \in P_{++}^{p'}$ — by ‘$P_m^{++}$’ we mean all $N$-tuples $\nu_i$ of positive integers, with $\sum_{i=0}^{N-1} \nu_i = m$. Algebraically, $P_{++}^m$ are the level $m - N$ highest weights of $\widehat{\mathfrak{su}}(N)$, shifted by the Weyl vector $\rho = (1,1,\ldots,1)$. The simple-current $J$ of $\widehat{\mathfrak{su}}(N)$ takes weight $\nu = (\nu_0, \nu_1, \ldots, \nu_{N-1}) \in P_{++}^m$ to $(\nu_{N-1}, \nu_0, \nu_1, \ldots, \nu_{N-2}) \in P_{++}^{m'}$. So there are a total of $(N-1)!/N!$ $N$-primary. The matrix $T$ is given by the conformal weights

$$h_{\lambda \mu} \equiv \frac{|p\lambda - p'\mu|^2 - (p - p')^2(N - 1)N(N + 1)/12}{2pp'} \pmod{1}$$

and central charge $c = (N - 1)(1 - N(N + 1)(pp'p')/pp'^2)$. The $S$ matrix is

$$S_{(\lambda,\mu)(\lambda',\mu')} = \alpha \exp(-2\pi i [t(\lambda) t(\mu') + t(\mu) t(\lambda')]/N) S_{\lambda,\lambda'}^{(p/p')} S_{\mu,\mu'}^{(p'/p)}$$

where $\alpha$ is some irrelevant constant, where $t(\lambda) = \sum_i i\lambda_i$, and where $S^{(m)}$ here is the usual (‘integrable’) affine $S$ matrix, expressed as usual as an alternating sum over the (finite) Weyl group (see e.g. eq.(14.57) of [4]), formally evaluated at (fractional) level $m - N$. The vacuum 0 corresponds to the $(J,J)$-orbit containing $(\rho, \rho)$, or more precisely $((p' - N + 1, 1, \ldots, 1), (p - N + 1, 1, \ldots, 1))$ (recall that $(k+1,1,\ldots,1) \in P_{++}^{k+N}$ corresponds to the vacuum in the $\widehat{\mathfrak{su}}(N)_k$ WZW theory, and projects to the Weyl vector $\rho$ of $\mathfrak{su}(N)$). An order-$N$ simple-current of this minimal model is $(J, id)$ (or rather its $(J, J)$-orbit); it has monodromy charge $Q(\lambda, \mu) \equiv \frac{p t(\lambda) - p' t(\mu) + N - 1}{2} \pmod{1}$. As we will see below, it generates all of the simple-currents of the model.

The Galois group here can be taken to be the multiplicative group $\mathbb{Z}_{p} \times \mathbb{Z}_{p'}$. Choose any odd integer $\ell$, coprime to $N$, obeying the congruences $\ell p \equiv 1 \pmod{p'}$ and $\ell p' \equiv 1 \pmod{p}$ (this is always possible, by the Chinese remainder theorem and the fact that $\gcd(p, p') = 1$). Let $\sigma_\ell$ be the Galois automorphism corresponding to $\ell$, and write $(\rho', o'') \in P_{++}^p \times P_{++}^{p'}$ for the image $\sigma_\ell 0$ of the vacuum under $\sigma_\ell$. We claim that $(J^i\rho', o'')$ is the minimal primary $o$, for some $i$. 

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To see this, first note that for any primaries $(\lambda, \mu), (\lambda', \mu')$, we have

\[
\left| \frac{S_{\lambda, \mu}(\lambda', \mu')}{S_{0}(\lambda', \mu')} \right| = \left| \frac{S_{\lambda'}^{(p/p')}}{S_{\mu'}^{(p/p')}} \right| = \left| \sigma_{\ell}^{-1} \frac{S_{\lambda'}^{(p)}}{S_{\mu'}^{(p)}} \right| = \left| \sigma_{\ell}^{-1} \frac{S_{\lambda'}^{(p')}}{S_{\mu'}^{(p')}} \right| = \left| \frac{S_{\lambda'}^{(p)}}{S_{\mu'}^{(p')}} \right|,
\]

where \(\sigma_{\ell}^{-1}(\lambda', \mu') = (\lambda'', \mu'')\). Implicit in this calculation is that we can write \(S_{\lambda'}^{(p)} = \psi_{\lambda, \lambda'} S_{\lambda, \lambda'}\) (similarly for \(S_{\mu'}^{(p')}\)), where \(\psi_{\lambda, \lambda'} = \exp[2\pi i t(\lambda + \rho)/pN]\) is a root of unity, and \(S_{\lambda, \lambda'}\) is in the number field \(\mathbb{Q}[\exp(2\pi i/p)]\). This means that \(|\sigma_{m} S_{\lambda, \lambda'}^{(p)}| = |\sigma_{m} S_{\lambda, \lambda'}^{(p')}|\) depends only on the value of \(m\) (mod \(p\)). The maximum value of the right-side is the product of WZW quantum dimensions, achieved by the vacuum \((\lambda'', \mu'') = 0\) (recall (9b)). In particular, the choice \((\lambda', \mu') = (\alpha', \alpha'')\) achieves this maximum, for all primaries \((\lambda, \mu)\), as must the minimal primary \((\lambda', \mu') = 0\) (again by (9b)). Thus, for every primary \((\lambda, \mu)\) there is a phase \(\varphi_{\lambda, \mu}\) such that \(S_{\lambda, \mu}(\alpha', \alpha'')/S_{0}(\alpha', \alpha'') = \varphi_{\lambda, \mu} S_{\lambda, \mu, 0}/S_{0, 0}\). Taking the absolute value of

\[
\frac{S_{\lambda, \mu}(\alpha', \alpha'')}{S_{0}(\alpha', \alpha'')} = \sum_{(\lambda'', \mu'')} N_{(\lambda', \mu')(\lambda'', \mu'')} S_{\lambda'', \mu''}(\alpha', \alpha'')/S_{0}(\alpha', \alpha'')
\]

and using the triangle inequality, we find that the assignment \((\lambda, \mu) \mapsto \varphi_{\lambda, \mu}\) defines a grading on the \(W_{N}\) fusion ring at \((p, p')\). Hence \((\lambda, \mu) \mapsto \varphi_{\lambda, \mu} S_{\lambda, \mu, 0}/S_{0, 0}\) defines a 1-dimensional representation of the fusion ring, and so

\[
\varphi_{\lambda, \mu} \frac{S_{\lambda, \mu, 0}}{S_{0, 0}} = \frac{S_{\lambda, \mu, A}}{S_{0, A}}
\]

for some primary \(A\). We want to show \(A\) is a simple-current, i.e. that \(S_{A, 0} = S_{0, 0}\). By unitarity of the matrix \(S\), the norm of the 0-column must equal that of the \(A\)-th column, and so (17) implies \(|S_{0, 0}| = |S_{0, A}|\). Substituting \(\theta\) for \((\lambda, \mu)\) in (17) and using positivity (8) now concludes the proof of Fact 4: \(A\) is a simple-current and \(A \theta = (\alpha', \alpha'')\).

This argument also showed that the quantum-dimension \(S_{\lambda, \mu, 0}/S_{0, 0}\) equals the product of the WZW \(\mathfrak{su}(N)\) quantum-dimensions of \(\lambda\) and \(\mu\) at levels \(p - N\) and \(p' - N\), respectively. Since the only simple-currents of \(\mathfrak{su}(N)\) are the ones \(J^{i}\) corresponding to the order-\(N\) cyclic symmetry of the \(\mathfrak{su}(N)\) Dynkin diagram [53], the only simple-currents of the \(W_{N}\) minimal models are the \((J, J)-orbit of those primaries \((J^{i}, id)\). Hence \(A = (J^{i}, id)\) for some \(i\).

In fact, we know the order of \(A = J_{o}\) must be 1 or 2 (see e.g. Consequence 2(i), although this can be easily seen directly). Thus when \(N\) is odd, \(J_{o} = id\), and when \(N\) is even, the only possibilities are \(J_{o} = id\) or \(J_{o} = (J^{N/2}, id)\).

Next, let’s give a proof of Fact 3. By Consequence 2(ii) it is now automatic for \(N\) odd, since then \(J_{o} = id\). So let \(N\) be even and \(J_{o} = (J^{N/2}, id)\). Let \(M\) be any modular invariant of \(W_{N}\). From \(MT = TM\) we see that if \(M_{\lambda, \mu}(\lambda', \mu') \neq 0\), then (16a) requires \(|p \lambda - p' \mu|^{2} \equiv |p \lambda' - p' \mu'|^{2} \text{ (mod } 2pp')\). But an easy calculation shows (for \(\mathfrak{su}(N)\)) that \(N |\lambda|^{2} \equiv -t(\lambda)^{2} \text{ (mod } N)\). Thus if \(M_{\lambda, \mu}(\lambda', \mu') \neq 0\), then the monodromies \(Q_{J_{o}} = NQ/2\)
of \((\lambda, \mu)\) and \((\lambda', \mu')\) are equal mod 1. Note then from \(M = SMS^*\) that

\[
M_{J_o, J_o} = \sum_{(\lambda, \mu), (\lambda', \mu')} S_{J_o, (\lambda, \mu)} M_{(\lambda, \mu)(\lambda', \mu')} S^*_{(\lambda', \lambda'), J_o} = \sum_{(\lambda, \mu), (\lambda', \mu')} S_{0, (\lambda, \mu)} M_{(\lambda, \mu)(\lambda', \mu')} S^*_{(\lambda', \lambda'), 0} = M_{00} = 1 .
\]

Thus by (6c) and the fact that \(o = \sigma \ell J_o\), we get that \(M_{oo} = 1\), which is Fact 3.

We conclude this section, and this paper, with the proof of Consequence 4. It is helpful to remember that the simple-current \(J_o\) for \(W_3\) must be trivial, so all we have to account for here is the Galois automorphism. The minimal \((p, 4)\) \(W_3\) models have unitarisation \(\hat{S}\) equal to the modular \(S\) matrix of the affine algebra \(\hat{su}(3)\) at level \(p - 3\). Explicitly, the primary \(\lambda = (\lambda_0, \lambda_1, \lambda_2) \in P_{++}^{p-3}(\hat{su}(3))\) corresponds to the \(W_3\) primary given by the \((J, J)\)-orbit \([\lambda + \rho] := (J^i(2, 1, 1), J^i(\lambda_0 + 1, \lambda_1 + 1, \lambda_2 + 1))\). The \(T\) matrix enters into the modular invariant classification only via the selection rule

\[
M_{ab} \neq 0 \text{ iff } h_a \equiv h_b \text{ (mod 1) , } \forall a, b \in \Phi ,
\]

and in both our cases \(T\)-invariance reduces to (in the \(W_3\) notation) the selection rule

\[
M_{[\lambda][\mu]} \neq 0 \text{ iff } \lambda_1^2 + \lambda_1 \lambda_2 + \lambda_2^2 \equiv \mu_1^2 + \mu_1 \mu_2 + \mu_2^2 \text{ (mod 3p) , } \forall \lambda, \mu \in P_{++}^p .
\]

Thus, using that primary field correspondence, the modular invariants and \(\text{NIM-reps}\) of these two different pairs of modular data can be identified. The modular invariants are classified in [54]. Of course the exceptional \(\hat{su}(3)\) modular invariants all occur for odd level and so don’t arise here. Many \(\text{NIM-reps}\) for \(\hat{su}(3)\) were first given in [55], and other \(\text{NIM-reps}\) were obtained in other papers, culminating with the announcement in [56] that there is a unique \(\text{NIM-rep}\) for each \(\hat{su}(3)\) modular invariant. Ocneanu’s argument requires the full structure of his von Neumann subfactor theory (see e.g. [57] for a review) and it is not yet known if all this has a necessary counterpart in RCFT. In any case a \(\text{NIM-rep}\) for each \(\hat{su}(3)\) modular invariant (and hence each \(W_3\) \((p, 4)\) model) can be found in e.g. [40].

More generally, the \(W_N\) minimal models at \((p, p') = (p, N + 1)\) will for most \(p\) correspond to the WZW \(\hat{su}(N)\) model at level \(p - N\).

7. Concluding remarks

This paper compares nonunitary and unitary RCFT. In a nonunitary theory, some of the properties we would like to ascribe to the vacuum 0 instead belong to what we call the minimal primary \(o\), which corresponds to the positive column of the modular \(S\) matrix.

**Question 1.** Must there be exactly one primary with minimal conformal weight?

In all examples of (healthy) nonunitary RCFTs known to this author, the answer to Question 1 is ‘yes’ and it is tempting to conjecture that it always must be. The affine algebras at fractional level can have several different admissible highest weight representations with minimal conformal weight (see e.g. [49]), but they don’t have a direct interpretation
as an RCFT and so don’t constitute a counterexample. When there is only one primary
with minimal conformal weight, that primary must be the minimal primary \( o \) (hence the
name).

A fundamental question here is the multiplicity of the minimal primary \( o \) in the full
RCFT. We showed that this multiplicity must be 1 for any minimal \( W_N \) model, and we
conjectured in §4 that it always equals 1.

The relation between 0 and \( o \) lies at the heart of this paper. We identify a property
which many (but not all) RCFTs obey, namely that the minimal primary \( o \) and the vacuum
0 in the chiral theory are related to each other by what we call the Galois shuffle. Once
again, all minimal \( W_N \) models obey this GS property.

**Question 2.** Is there a characterisation of the nonunitary RCFTs which obey the GS
property? How typical is it?

Our reasons for suspecting that it’s quite typical are that we know many RCFTs
which obey it, and also that many \( W \)-algebras can be constructed from the affine algebras
at fractional level, and the latter respects it (see §5). On the other hand Consequence 2(iv)
in §4 indicates that it is quite nontrivial.

We learned in §4 that the GS property has many nice consequences. Roughly speak-
ing, it means that the nonunitary theory behaves almost like a unitary theory. That
 corresponding theory (or rather its modular data) is called the ‘unitarisation’ of the uni-
tary theory. The unitarisation obeys the same fusion rules as the nonunitary theory, and
their toroidal and cylindrical partition functions will be in one-to-one correspondence.

**Question 3.** Can we see the unitarisation directly inside the nonunitary RCFT? In partic-
ular, it makes sense at the chiral level, so can we see it at the level of the chiral algebras?

**Question 4.** Will the unitarisation always equal the modular data of a completely healthy
unitary RCFT?

As mentioned in §4, the unitarisation will always satisfy conditions (3) and the iden-
tities in [14], and all the properties of unitary modular data, but nevertheless it still might
not be realised as the modular data of an actual theory. A simple critical case is given
by the Virasoro minimal models at \((p,2)\), where \( p \geq 1 \) is odd: we find in §6 that its
unitarisation can be expressed as

\[
\tilde{S}_{ab} = \frac{2}{\sqrt{p}} \sin(\pi \frac{ab}{p}) ,
\]

\[
\tilde{T}_{ab} = \delta_{ab} \exp\left[-\pi \left(\pm \frac{a^2}{2p} + \frac{3 \pm 1}{12}\right)\right] ,
\]

where the primaries \( a, b \) consist of all odd numbers \( 1 \leq a, b \leq p - 2 \), and where we take
the upper signs (i.e. ‘+’) in the formula for \( T \) if \( p \equiv +1 \pmod{4} \), and otherwise take the
lower signs (i.e. ‘−’). The question is, can this modular data be realised by the characters
of an RCFT. It obeys all properties (e.g. Fact 1) this author has been able to check.
For \( p = 3, 5, 9, 11 \), this data coincides with that of the WZW models with affine algebras
\( E_8 \oplus E_8 \oplus E_8 \) at level \((1,1,1)\), \( \hat{F}_4 \) at level 1, \( \hat{G}_2 \) at level 2, and \( \hat{F}_4 \) at level 2. For the other
values of $p$, this author has been unable to find a healthy RCFT which realises it (but that certainly doesn’t mean that none exists). Curiously, for arbitrary (odd) $p$, that matrix $S$ coincides with the matrix $\psi$ diagonalising (in the sense of eq.(4) above) the spurious $\hat{su}(2)$ level $p - 2$ NIM-rep called the tadpole in [55]. In any case, if an RCFT realisation can be found for this modular data for all odd $p$, this would lend support to the thought that the uniformisation of a nonunitary RCFT is the modular $S$ and $T$ matrices for a healthy unitary RCFT.

Another reason the existence of a unitarisation might be interesting is the following. A deep relationship between von Neumann subfactors and RCFT has been developed by Ocneanu, Evans, and others (see e.g. [57,36] for reviews). Although subfactors cannot recover the entire RCFT (e.g. they don’t see the chiral algebra or even its character), they can realise for example the fusions, the modular $S$- and $T$-matrices, the 1-loop modular invariant partition functions, and the NIM-reps of RCFTs. The subfactor picture (at present) can only realise unitary RCFTs. It is tempting to speculate that perhaps subfactors realise instead the unitarisation $\hat{S}, \hat{T}$ and corresponding modular invariants and NIM-reps, even if the unitarisation were not to correspond to a completely healthy RCFT.

Question 5. Not all RCFTs have a unitarisation. However, given any nonunitary RCFT, can we always find a unitary RCFT with identical fusion rules?

Again, this is true of all examples known to this author. Of course it holds for any RCFT obeying the GS property.

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