SUSAN M. COOPER, SABINE E. KHOURY, SARA FARIDI, SARAH MAYES-TANG, SUSAN MOREY, LIANA M. ŞEGA, AND SANDRA SPIROFF

ABSTRACT. The Taylor resolution is almost never minimal for powers of monomial ideals, even in the square-free case. In this paper we introduce a smaller resolution for each power of any square-free monomial ideal, which depends only on the number of generators of the ideal. More precisely, for every pair of fixed integers \( r \) and \( q \), we construct a simplicial complex that supports a free resolution of the \( r^{th} \) power of any square-free monomial ideal with \( q \) generators. The resulting resolution is significantly smaller than the Taylor resolution, and is minimal for special cases. Considering the relations on the generators of a fixed ideal allows us to further shrink these resolutions. We also introduce a class of ideals called "extremal ideals," and show that the Betti numbers of powers of all square-free monomial ideals are bounded by Betti numbers of powers of extremal ideals. Our results lead to upper bounds on Betti numbers of powers of any square-free monomial ideal that greatly improve the binomial bounds offered by the Taylor resolution.

1. INTRODUCTION

Important insight about the underlying structure of an ideal in a polynomial ring is gained from a careful analysis of its minimal free resolution. As such, significant effort has gone into the development of methods to compute resolutions. The approach of leveraging connections between commutative algebra and other fields, such as combinatorics and topology, has proven to be quite fruitful. Diana Taylor’s thesis \([19]\) initiated the exploration of these connections, followed by simplicial resolutions (Bayer, Peeva, and Sturmfels \([1]\)), polytopal complexes (Nagel and Reiner \([15]\)), and cellular complexes (Bayer and Sturmfels \([2]\)), to name just a few. See \([14, 16]\) for an overview of these developments.

The Taylor resolution is powerful: given any ideal \( I \) minimally generated by \( q \) monomials, Taylor constructed a simplicial complex \( \text{Taylor}(I) \) by labeling the vertices of a \((q - 1)\)-simplex with the monomial generators of \( I \). She showed that this complex supports a free resolution of \( I \), in the sense that its simplicial chain complex can be transformed, via a process called homogenization, to a free resolution of \( I \), called the Taylor resolution of \( I \). Unfortunately, even though every monomial ideal has a Taylor resolution, the Taylor resolution is often far from minimal. In particular, for powers of ideals it is almost never minimal due to certain syzygies that are automatically created when taking powers.

The central theme of this paper is to find an analogue for the Taylor complex for powers of square-free monomial ideals. We seek a construction that takes the automatically generated non-minimal syzygies into account and removes them from the Taylor resolution to produce a much smaller free resolution of \( I^r \) that works for any monomial ideal \( I \). Our ultimate goal is to find a uniform combinatorial structure that depends only on the number of generators and the power of the ideal. More precisely, the question at the heart of this paper is the following:

**Question.** Given positive integers \( r \) and \( q \), is it possible to find a simplicial complex (considerably) smaller than the simplex \( \text{Taylor}(I^r) \) that supports a free resolution of \( I^r \), where \( I \) is any ideal generated by \( q \) monomials in a polynomial ring?
When $r = 1$, Taylor($I$) is in fact the optimal answer to the question above, as there are ideals $I$ for which Taylor($I$) supports a minimal resolution. But when $r = 2$, the resolution supported on Taylor($I^2$) is never minimal for any non-principal square-free monomial ideal $I$ (Theorem 4.4). As expected, the resolution supported on Taylor($I^r$) becomes further from minimal as $r$ grows.

Although Taylor’s complex for the $r^{th}$ power of a monomial ideal with $q$ generators can be quite large—a simplex of dimension $(q+r-1)$—we can improve the situation considerably by studying the general relations among the generators of $I^r$ that must always exist for all monomial ideals regardless of the generating set for $I$. Similar to the idea used in Lyubeznik’s resolutions [12], in the case of square-free monomial ideals, this investigation involves detecting and trimming redundant faces of the Taylor complex Taylor($I^r$), bringing us closer to a minimal resolution.

To illustrate our underlying process, let $r = 2$, $q = 3$ and consider any ideal $I = (m_1, m_2, m_3)$ in the polynomial ring $k[x_1, \ldots, x_n]$ where $m_1, m_2$ and $m_3$ are minimal, square-free monomial generators and $k$ is a field. Now $I^2 = (m_1^2, m_2^2, m_3^2, m_1m_2, m_1m_3, m_2m_3)$ and Taylor($I^2$) is a 5-dimensional simplex with 6 vertices, where each vertex is labeled by a generator of $I^2$ and each face is labeled with the least common multiple of its vertices. A non-minimal syzygy occurs when a face and a subface have the same label (see Theorem 3.1). When considering $I^2$, no matter what the monomial generators of $I$ are, when $i, j, k$ are distinct we always have the following:

$$\text{lcm}(m_i^2, m_j^2) = \text{lcm}(m_i^2, m_j m_k, m_i m_k) \text{ and } \text{lcm}(m_i^2, m_j m_k) = \text{lcm}(m_i^2, m_j m_k, m_i m_k).$$

These equalities lead to non-minimal syzygies in the Taylor resolution of $I^2$, and as a result the edges \{ $m_1^2, m_2^2$ \}, \{ $m_1^2, m_3^2$ \}, \{ $m_2^2, m_3^2$ \}, \{ $m_1^2, m_2 m_3$ \}, \{ $m_2^2, m_1 m_3$ \}, \{ $m_3^2, m_1 m_2$ \}, and all faces containing these edges, can be removed from Taylor($I^2$). The resulting 2-dimensional subcomplex $L^2_q$ of Taylor($I^2$) supports a resolution of $I^2$ (Theorem 4.4). This simple observation has a considerable impact on bounding the Betti numbers of $I^2$. For example, in this small case we can conclude that for every ideal $I$ with three square-free monomial generators, the projective dimension of $I^2$ is at most 2 (the dimension of $L^2_q$), versus the bound 5 (the dimension of Taylor($I^2$)).

In this paper we show that a similar argument can be made more generally. If $I$ is minimally generated by square-free monomials $m_1, \ldots, m_q$, then we can identify faces of the simplex Taylor($I^r$) which lead to non-minimal syzygies for $I^r$. Eliminating these faces results in a much smaller subcomplex of Taylor($I^r$), which we call $L^r_q$ (Definition 4.2).

Our main result, Theorem 5.9, shows that $L^r_q$ supports a resolution of $I^r$. This echoes the power of Taylor’s complex in that we have a topological structure depending solely on $r$ and $q$ that supports a resolution of $I^r$ for any square-free monomial ideal $I$ with $q$ minimal generators. By further deleting redundancies specific to the generators of $I$, we obtain a subcomplex $L^r(I)$ of $L^r_q$, so that we have

$$L^r(I) \subseteq L^r_q \subseteq \text{Taylor}(I^r).$$

Theorem 5.9 also shows that $L^r(I)$ supports a free resolution of $I^r$. Our approach to proving this involves showing that $L^r(I)$ and $L^r_q$ are both quasi-trees (see Definition 2.2), meaning (roughly) that they are built from a special ordering on their facets. In Theorem 5.9, we show that for a quasi-tree to support a free resolution one only needs to check that certain induced subcomplexes are connected. This extends the main result of [9].

Our complex $L^r(I)$ gives natural and useful bounds on homological information of $I^r$. Indeed, the Betti numbers of $I^r$ are bounded in terms of the number and size of faces of $L^r(I)$, yielding bounds that are significantly smaller than those given by the Taylor resolution. In the later sections of this paper we examine just how much smaller these bounds on the Betti numbers are and when the resolutions obtained are minimal. We define a class of square-free monomial ideals, which we call extremal ideals, whose Betti numbers of powers bound the Betti numbers of powers of any square-free monomial ideal with the same number of generators. As a result, we reduce the question of minimality of $L^r_q$ to the study of when $L^r_q$ supports a minimal free resolution of the $r^{th}$ power of an extremal ideal.
To put this work in context of the broader literature, studying powers of ideals and bounding their invariants has received much attention in recent years. Powers play an important role in Rees algebras and associated graded rings among other uses, making understanding their behavior desirable but difficult. In another direction, there has been considerable interest in describing minimal topological resolutions for all monomial ideals using a variety of methods, such as using chain maps from multiple simplicial complexes (see [7, 20]). In this paper we combine the two interests and seek, for powers of monomial ideals, resolutions that are supported on a single topological structure which is practical to determine based on the generators of the original ideal $I$.

The paper is organized as follows. Section 2 contains basics of simplicial resolutions. In Section 3 we use simplicial collapsing and the Bayer–Peeva–Sturmfels criterion to prove the above-mentioned criterion for quasi-trees (Theorem 3.6). In Section 4 we introduce the definition of the simplicial complex $L^r_q$, and we prove in Proposition 4.6 that $L^r_q$ is a quasi-tree. In Section 5 we define $L^r_q(I)$, discuss some examples, and then prove the main result, Theorem 5.9. In Section 6 we investigate the bounds on the Betti numbers of $I^r$ that follow from the main result. Finally, Section 7 introduces extremal ideals, which have maximal Betti numbers among powers of square-free monomial ideals. In particular, Proposition 7.11 provides a full characterization of the conditions on $r$ and $q$ that guarantee $L^r_q$ supports a minimal free resolution of $I^r$ for some ideal $I$.

This paper is an extension of the work in [4] where the focus is on the second power of the ideal $I$. The collaboration was initiated at the 2019 workshop “Women in Commutative Algebra” hosted by the Banff International Research Station.

2. SIMPLICIAL RESOLUTIONS

Fix $S = k[x_1, \ldots, x_n]$ to be a polynomial ring over a field $k$. We begin by reviewing necessary background for simplicial complexes and simplicial resolutions and then demonstrate the potential relationship to resolutions of ideals.

A simplicial complex $\Delta$ on a vertex set $V$ is a set of subsets of $V$ such that if $F \in \Delta$ and $G \subseteq F$ then $G \in \Delta$. We use the following terminology for simplicial complexes:

**Definition 2.1.** Let $\Delta$ be a simplicial complex.

1. An element of $\Delta$ is called a face.
2. The facets of $\Delta$ are the maximal faces under inclusion.
3. The dimension of a face $F \in \Delta$ is $\dim(F) = |F| - 1$.
4. The dimension of $\Delta$ is the maximum of the dimensions of its faces.
5. $\Delta$ is called a simplex if it has one facet.
6. The $f$-vector $f(\Delta) = (f_0, \ldots, f_d)$ of a $d$-dimensional simplicial complex $\Delta$ has $f_i = \text{the number of } i$-dimensional faces of $\Delta$.

Note that a simplicial complex can be uniquely determined by its facets. One writes

$$\Delta = \langle F_1, \ldots, F_q \rangle$$

to denote a simplicial complex $\Delta$ with facets $F_1, \ldots, F_q$.

In subsequent sections, we will use the tool of trimming simplicial complexes via certain rules. Essentially, we will delete vertices in a specified fashion. Vertex deletions naturally lead to the consideration of subcomplexes of simplicial complexes, which are defined below, together with additional structures and notions that will be used throughout the paper.

**Definition 2.2.** Let $\Delta$ be a simplicial complex on a vertex set $V$.

1. If $v$ is a vertex of $\Delta$, then the deletion of $v$ from $\Delta$ is the simplicial complex

$$\Delta \setminus \{v\} = \{ \sigma \in \Delta \mid v \notin \sigma \}.$$

2. A subcomplex of $\Delta$ is a subset of $\Delta$ which is also a simplicial complex.
(3) Given $W \subseteq V$, the **induced subcomplex** of $\Delta$ on $W$ is the subcomplex
$$\Delta_W = \{ \sigma \in \Delta \mid \sigma \subseteq W \}.$$  

(4) A **leaf** is a facet $F$ of $\Delta$ such that $F$ is the only facet of $\Delta$, or there is a facet $G$ of $\Delta$ with $F \neq G$ such that
$$F \cap H \subseteq G$$
for all facets $H \neq F$. The facet $G$ is called a **joint** of $F$. (Note that the joint of a leaf need not be unique (see [3]).)

(5) $\Delta$ is called a **quasi-forest** if the facets of $\Delta$ can be ordered as $F_1, \ldots, F_q$ such that for $i = 1, \ldots, q$, the facet $F_i$ is a leaf of the simplicial complex $\langle F_1, \ldots, F_i \rangle$.

(6) $\Delta$ is a **quasi-tree** if it is a connected quasi-forest.

**Example 2.3.** The simplicial complex below is a quasi-tree. The leaf order is $F_1, \ldots, F_5$, meaning that each $F_i$ is a leaf of $\langle F_1, \ldots, F_i \rangle$. In this example the joint of $F_i$ is $F_{i-1}$ for all $i \geq 1$.

Example 2.4. The star-shaped complex drawn below on the left is a quasi-tree, with leaf order $F_0, F_1, F_2, F_3$. In particular, the center facet $F_0$ is the joint of $F_i$ for every $i \geq 1$. This complex is a standard example of a quasi-tree which is not a simplicial tree in the sense of [8]. This particular quasi-tree is shown in [4] to support a free resolution of the second power of any ideal with three square-free monomial generators.

If one removes $F_0$ from the center, the remaining complex is shown in the picture on the right. This simplicial complex is not a quasi-tree, since no facet is a leaf.

A **free resolution** of $I$ is an exact sequence of the form
$$0 \to S^{\beta_t} \to S^{\beta_{t-1}} \to \cdots \to S^{\beta_1} \to S^{\beta_0} \to I \to 0,$$
where $S^{\beta_j}$ is a free $S$-module of rank $\beta_j$ and $t \in \mathbb{N}$. When $\beta_j$ is the smallest possible rank of a free module in the $j^{th}$ spot of any free resolution of $I$ for each $j$, the resolution is **minimal**. In this case, the numbers $\beta_j$ are invariants of $I$ and are called the **Betti numbers** of $I$.

In the 1960s, Diana Taylor demonstrated a striking connection between a $(q-1)$-simplex and a resolution of a monomial ideal $I$ in $S$. If $I$ is a monomial ideal in $S$ minimally generated by monomials $m_1, \ldots, m_q$, then Taylor$(I)$ denotes the simplex with $q$ vertices indexed by the set $[q] = \{ 1, \ldots, q \}$, where each vertex $i$ is labeled with one of the monomials $m_i$, and each face $\sigma$ is labeled with the monomial
$$M_{\sigma} = \text{lcm}(m_i : i \in \sigma).$$

In her Ph.D thesis [19], Taylor proved that the simplicial chain complex of Taylor$(I)$ gives rise to a multigraded free resolution of $I$. In particular, the $i^{th}$ Betti number of $I$ is bounded above by the number of $i$-dimensional faces of Taylor$(I)$, which is $\binom{q}{i+1}$. This method has been generalized so that if $\Delta$ is a simplicial or cellular complex whose vertices are labeled with the monomial generators $m_1, \ldots, m_q$ of an ideal $I$ and
whose faces are labeled with the least common multiple of the vertex labels as above, then we say that $\Delta$ supports a free resolution of $I$ if the homogenization of the simplicial (or cellular) chain complex of $\Delta$ is a multi-graded free resolution of $I$, denoted by $F_\Delta$, see [1,2]. The multi-graded complex $F_\Delta$ is described as follows. For each $t \geq 0$, the free module $(F_\Delta)_t$ has basis elements denoted by $e_\sigma$, where $\sigma$ ranges over all faces of $\Delta$ with $|\sigma| = t + 1$, and $e_\sigma$ is considered to have multi-degree $M_\sigma$. The differential is described by

$$\partial(e_\sigma) = \sum_{j=0}^t (-1)^j \frac{M_\sigma}{M_\sigma \setminus \{v_{i_j}\}} e_\sigma \setminus \{v_{i_j}\},$$

(2.4.1)

where $\sigma = \{v_{i_0}, \ldots, v_{i_t}\}$ with $i_0 < i_1 < \cdots < i_t$.

**Example 2.5.** Let $I = (xy, yz, zu)$ in $R = k[x, y, z, u]$. The labeled simplicial chain complex

$\begin{array}{c}
xy \\
xyz \\
yz \\
yzu \\
zu
\end{array}$

supports a free resolution of $I$. The chain complex of $\Delta$ is

$$0 \to k^2 \xrightarrow{\begin{bmatrix} 1 & 0 \\
-1 & 1 \\
0 & -1 \end{bmatrix}} k^3 \to k$$

and homogenization results in the free resolution

$$0 \to R(xyz) \oplus R(yzu) \xrightarrow{\begin{bmatrix} z & 0 & 0 \\
-x & u & y \end{bmatrix}} R(xy) \oplus R(yz) \oplus R(zu) \to I \to 0,$$

where the notation $R(x^a y^b z^c u^d)$ refers to the $R$-free module with one generator in multi-degree $(a, b, c, d)$.

**Remark 2.6.** Simplicial complexes that support a free resolution of a monomial ideal are usually constructed such that the vertices correspond to and are labeled by a minimal set of generators of the ideal. However, one can also work with non-minimal generators, at the expense of producing a larger complex. In particular, one can mimic the construction of $\text{Taylor}(I)$, but use instead any set of monomial generators of $I$. The same considerations show that this complex supports a free resolution of $I$.

There are various combinatorial ways to build subcomplexes $\Delta$ of $\text{Taylor}(I)$ that support a free resolution of $I$. One such well known complex is the Lyubeznik complex, which supports the Lyubeznik resolution of $I$ ([12]). The Lyubeznik resolution is the main inspiration for the complexes $L^r_r(I)$ and $L^r(I)$ which appear later in this paper. We will not define this resolution since it is not used in this paper, but refer the reader to [12,13] for additional information.

### 3. Resolutions Supported on Quasi-Trees

Taylor’s resolution is usually far from minimal. However, for a given monomial ideal $I$, a criterion of Bayer, Peeva and Sturmfels (see Theorem [3,4]) allows one to check if a subcomplex of $\text{Taylor}(I)$ supports a free resolution of $I$. In this section, using the above criterion and by observing that quasi-trees are collapsible, we show that a quasi-tree $\Delta$ supports a free resolution of a given monomial ideal if and only if certain subcomplexes of $\Delta$ are connected.

For a subcomplex $\Delta$ of $\text{Taylor}(I)$ and a monomial $M$ in $S$, let $\Delta_M$ be the subcomplex of $\Delta$ induced on the vertices of $\Delta$ whose labels divide $M$.

The following is the criterion of Bayer, Peeva and Sturmfels [1 Lemma 2.2]; see also [6] Theorem 2.2] for the statement on minimality.
Theorem 3.1 (Criterion for a simplicial complex to support a free resolution). Let \( \Delta \) be a simplicial complex whose vertices are labeled with a monomial generating set of a monomial ideal \( I \) in a polynomial ring \( S \) over a field. Then \( \Delta \) supports a free resolution of \( I \) over \( S \) if and only if for every monomial \( M \), the induced subcomplex \( \Delta_M \) of \( \Delta \) on the vertices whose labels divide \( M \) is empty or acyclic.

Furthermore, a free resolution supported on \( \Delta \) is minimal if and only if \( M_\sigma \neq M_{\sigma'} \) for every proper subface \( \sigma' \) of a face \( \sigma \) of \( \Delta \).

Remark 3.2. The results of Theorem 3.1 are usually stated with the assumption that one uses the minimal monomial generating set of \( I \) for the labels. However, the proof of [1] Lemma 2.2 does not make use of this assumption, hence we formulated the result above to reflect this observation.

In particular, Theorem 3.1 implies that the \( f \)-vector of a complex \( \Delta \) supporting a resolution of a monomial ideal \( I \) is an upper bound for the vector of Betti numbers of \( I \). In other words, for each \( i \leq d = \dim(\Delta) \),

\[
\beta_i(I) \leq f_i \quad \text{where} \quad f(\Delta) = (f_0, \ldots, f_d).
\]

Proposition 3.3 (Induced subcomplexes of quasi-forests are quasi-forests). If a simplicial complex \( \Delta \) is a quasi-forest, then every induced subcomplex of \( \Delta \) is a quasi-forest.

Proof. By [10] Proposition 6, a simplicial complex \( \Delta \) with vertex set \( V \) is a quasi-forest if and only if for every subset \( W \subseteq V \), the induced subcomplex \( \Delta_W \) has a leaf. If \( W \subseteq V \), consider the induced subcomplex \( \Delta_W \) of \( \Delta \). If \( U \subseteq W \), then \( \Delta_U = (\Delta_W)_U \) has a leaf, and hence, \( \Delta_W \) is a quasi-forest.

A face \( \sigma \) of a simplicial complex \( \Delta \) is called a free face if it is properly contained in a unique facet \( F \) of \( \Delta \). A collapse of \( \Delta \) along the free face \( \sigma \) is the simplicial complex obtained by removing the faces \( \tau \) such that \( \sigma \subseteq \tau \subseteq F \) from \( \Delta \). If additionally \( \dim(\sigma) = \dim(F) - 1 \), then the collapse is called an elementary collapse. The simplicial complex \( \Delta \) is called collapsible if it can be reduced to a point via a series of (elementary) collapses.

Example 3.4. Consider the quasi-tree in Example 2.4. To illustrate that this complex is collapsible to a point, we label in Figure 1 the vertices and then demonstrate one collapsing sequence. At each step, the two faces that play the roles of \( \sigma \) and \( F \) in the exposition above are indicated. Note that the first two steps are elementary collapses for demonstration purposes. Alternate collapsing sequences exist.

In general, by starting with a leaf and the face that contains it, any quasi-tree can be collapsed to a point.

Proposition 3.5. Every quasi-tree is collapsible.

Proof. Proceed by induction on \( q \), the number of facets of \( \Delta \). When \( q = 1 \), \( \Delta \) is a simplex, and known to be collapsible (e.g. [9] Proposition 2.7). Suppose the statement holds for all quasi-trees with fewer than \( q \) facets.
We now use Proposition 3.5 and Proposition 3.3 to show that the Bayer-Peeva-Sturmfels criterion in Theorem 3.1 can be extended to the class of quasi-trees.

**Theorem 3.6 (Criterion for a quasi-tree to support a free resolution).** Let $\Delta$ be a quasi-tree whose vertices are labeled with a monomial generating set of a monomial ideal $I$ in a polynomial ring $S$ over a field. Then $\Delta$ supports a free resolution of $I$ over $S$ if and only if for every monomial $M$, the induced subcomplex $\Delta_M$ of $\Delta$ on the vertices whose labels divide $M$ is empty or connected.

**Proof.** By Theorem 3.1, $\Delta$ supports a resolution of $I$ if and only if $\Delta_M$ is empty or acyclic for every monomial $M \in S$. If $M \in S$ is a monomial, then by Proposition 3.3 above, $\Delta_M$ is a quasi-forest. Moreover, by Proposition 3.5 every connected component of $\Delta_M$ is contractible, i.e., homotopy equivalent to a point, and hence acyclic. The only possible homology that $\Delta_M$ could have would be that which comes from it being disconnected. As a result, $\Delta$ supports a resolution of $I$ if and only if $\Delta_M$ is empty or connected for every monomial $M \in S$. □

**4. THE QUASI-TREE $T^r_q$**

Recall that our goal is to find a simplicial complex smaller than Taylor’s complex that supports a free resolution of $I^r$ when $I$ has $q$ square-free monomial generators. In this section we construct a subcomplex...
\(\mathbb{L}_q^r\) of the Taylor simplex, which depends only on the integers \(r\) and \(q\), and contains a subcomplex supporting a free resolution of \(I^r\).

The base case for our construction is the case \(r = 1\). In this case \(\mathbb{L}_q^1\) is the well-known Taylor complex [19]: a simplex with \(q\) vertices that supports a free resolution of any monomial ideal with \(q\) generators. The case \(r = 2\) was investigated in the earlier work of the authors [4]. For example, it was shown in [4] that \(\mathbb{L}_q^2\) is the quasi-tree in Example 2.4.

We now collect and formalize the notation needed for the extension to the case \(r \geq 3\).

**Notation 4.1.** Let \(r\) and \(q\) be two positive integers.

- Let \(e_1, \ldots, e_q\) denote the standard basis vectors for \(\mathbb{R}^q\), i.e., for each \(i \in [q]\), \(e_i\) is the \(q\)-tuple with 1 in the \(i\)th coordinate and 0 elsewhere.

- Define \(N_q^r\) to be the set of points in \(\mathbb{Z}^q_{\geq 0}\) whose coordinates add up to \(r\):
  \[ N_q^r = \{ (a_1, \ldots, a_q) \in \mathbb{Z}^q_{\geq 0} : a_1 + \cdots + a_q = r \} = \{ a_1 e_1 + \cdots + a_q e_q : a_i \in \mathbb{Z}_{\geq 0} \text{ and } a_1 + \cdots + a_q = r \} \]

- Set \(s = \lceil \frac{r}{2} \rceil\); that is, when \(r\) is odd, \(r = 2s - 1\), and when \(r\) is even, \(r = 2s\).

**Definition 4.2 (The simplicial complex \(\mathbb{L}_q^r\) - see also Proposition 4.3).** Let \(r, q \geq 1\) be two integers. Following Notation 4.1, we define \(\mathbb{L}_q^r\) to be the simplicial complex with vertex set \(N_q^r\) whose faces are all subsets of the (not necessarily distinct) sets \(F_1^r, \ldots, F_q^r, G_1^r, \ldots, G_q^r\) defined as

\[ F_i^r = \{ (a_1, \ldots, a_q) \in N_q^r : a_i \leq \max\{r - 1, s\} \text{ and } a_j \leq s \text{ for } j \neq i \} \]

\[ G_i^r = \{ (a_1, \ldots, a_q) \in N_q^r : a_i \geq r - 1 \} = \{ (r - 1) e_i + e_j : j \in [q] \} \]

for each \(i \in [q]\). We refer to the set \(\{ F_1^r, \ldots, F_q^r \}\) as the first layer of \(\mathbb{L}_q^r\) and the set \(\{ G_1^r, \ldots, G_q^r \}\) as the second layer of \(\mathbb{L}_q^r\). We define the base of \(\mathbb{L}_q^r\) to be the face

\[ B^r = \{ (a_1, \ldots, a_q) \in N_q^r : a_i \leq s \text{ for all } i \in [q] \} \]

so that

\[ F_i^r = B^r \cup \{ (a_1, \ldots, a_q) \in N_q^r : s + 1 \leq a_i \leq r - 1 \} \]

In general, \(F_1^r, \ldots, F_q^r, G_1^r, \ldots, G_q^r\) are the facets of \(\mathbb{L}_q^r\) (Proposition 4.3); however, for small values of \(r\) and \(q\), these sets need not be distinct. We summarize these facts in Proposition 4.3 to give a more precise description of the facets of \(\mathbb{L}_q^r\).

**Proposition 4.3 (Equivalent definition of \(\mathbb{L}_q^r\)).** The simplicial complex \(\mathbb{L}_q^r\) in Definition 4.2 can be described in terms of its distinct facets:

\[
\mathbb{L}_q^r = \begin{cases}
(F_1^r, F_2^r, \ldots, F_q^r, G_1^r, \ldots, G_q^r) & \text{if } r > 3 \text{ and } q \geq 2 \\
(B^r, G_1^r, \ldots, G_q^r) & \text{if } r = 3 \text{ and } q \geq 2 \text{ or } r = 2 \text{ and } q > 2 \\
(G_1^r, G_2^r) & \text{if } r = 2 \text{ and } q = 2 \\
(N_q^r) & \text{if } r = 1 \text{ or } q = 1.
\end{cases}
\]

**Proof.** When \(r = 1\), then \(s = 1\) and \(N_q^1 = \{ e_1, \ldots, e_q \}\). It follows that

\[ B^1 = F_1^1 = G_1^1 = N_q^1 \quad \text{for all } i \in [q]. \]

In this case \(\mathbb{L}_q^1\) is a simplex with \(q\) vertices (the Taylor simplex).

If \(q = 1\) and \(r > 1\), then

\[ B^r = F_1^r = \emptyset \quad \text{and} \quad G_1^r = N_q^r = \{ (r) \}. \]
If $q = r = 2$, then

$$G_1^2 = \{(2,0),(1,1)\} \quad \text{and} \quad G_2^2 = \{(0,2),(1,1)\}$$

and since $s = 1 = r - 1$,

$$F_1^2 = F_2^2 = B^2 = \{(1,1)\} \subseteq G_1^2 \cap G_2^2.$$ 

When $q \geq 2$ and $r = 3$, or $q > 2$ and $r = 2$, then the $G_i^r$ are distinct as each $G_i^r$ is the unique facet containing the vertex $re_i$. Furthermore, since $s + 1 = r$,

$$F_1^r = \cdots = F_q^r = B^r$$

contains all the vertices $(r - 1)e_i + e_j$ where $i \neq j$, and so $B^r$ is not embedded in any of the $G_i^r$.

Finally, when $q \geq 2$ and $r > 3$, then $s < r - 1$. Therefore, the $G_i^r$ are distinct, since each $G_i^r$ is the unique facet containing the vertex $re_i$.

For $i,j \in [q]$, $i \neq j$, $F^r_j$ is not contained in $F^r_i$ since

$$(s+1)e_i + (r-s-1)e_j \in F^r_j \setminus F^r_i.$$ 

Moreover, $re_j \notin F^r_i$, showing that

$$G_i^r \not\subseteq F^r_j.$$ 

Lastly, no $F_i^r$ can be contained in $G_j^r$ since $B^r \subseteq F_i^r$, but $B^r \cap G_j^r = \emptyset$ when $s < r - 1$. 

**Example 4.4.** The geometric realization of the simplicial complex $L_2^3$ is a path of length 3, as can be seen in the figure below.

![Diagram](image-url)

Since

$$\mathcal{N}_2^3 = \{(3,0),(2,1),(1,2),(0,3)\} = \{3e_1, 2e_1 + e_2, e_1 + 2e_2, 3e_2\},$$

according to Proposition 4.3, the facets of $L_2^3$ are

$$B^3 = \{(2,1),(1,2)\} = \{2e_1 + e_2, e_1 + 2e_2\}$$

$$G_1^3 = \{(3,0),(2,1)\} = \{3e_1, 2e_1 + e_2\}$$

$$G_2^3 = \{(0,3),(1,2)\} = \{3e_2, e_1 + 2e_2\}.$$ 

By contrast, if $q = 2$ and $r \geq 4$ is even, then $B^r$ is a single vertex

$$B^r = \{(r/2)e_1 + (r/2)e_2\} \subseteq F^r_i \quad \text{for all} \quad i \in [q].$$

For instance, if $r = 6$, then $L_2^6$ is pictured below, and $B^6$ is the single point at the middle of the ‘bow-tie’ and only the $F^r_i$'s and $G^r_j$'s are facets.

![Diagram](image-url)

Note that the points in $\mathcal{N}_q^r$ can be viewed as lattice points in $\mathbb{R}^q$. Indeed, they are precisely the integer lattice points in a hyperplane section of the first octant, cut out by the hyperplane whose equation is

$$x_1 + \cdots + x_q = r.$$ 

While for small values of $q$ this gives a natural way to depict the points, it does not illustrate the simplicial structure well. For instance, in Example 4.4, the 6 points would be co-linear, while our depiction emphasizes the existence of the two triangles. Using the combinatorial approach rather than the embedding in $\mathbb{R}^q$ also
allows for a generalized depiction, seen in the following example, that can easily be extended to higher \( q \) and \( r \).

**Example 4.5.** We examine the case \( q = 3 \) in the same manner as above. The simplicial complexes \( L_1^3 \), \( L_2^3 \) and \( L_3^3 \) appear in Figure 2, and \( L_6^3 \) appears in Figure 3.

The following statement generalizes [4, Proposition 3.3], which deals with the special case \( r = 2 \).

**Proposition 4.6 (\( L_q^r \) is a quasi-tree).** The simplicial complex \( L_q^r \) is a quasi-tree.

**Proof.** Following Proposition 4.3 for a description of the facets of \( L_q^r \), we consider each case separately.

When \( r = 1 \) or \( q = 1 \), \( L_1^3 \) is a simplex, and hence by definition a quasi-tree.

When \( r = q = 2 \), \( L_2^3 \) has only two facets, and it is trivially a quasi-tree.
When \( r = 2 \) and \( q > 2 \), then order the facets of \( L_q^2 \) by
\[
B^2, G^2_1, \ldots, G^2_q.
\]
In this case, if \( i \neq j \) with \( i, j \in [q] \), then
\[
G^2_i \cap B^2 \subseteq B^2 \quad \text{and} \quad G^2_i \cap G^2_j = \{e_i + e_j\} \subseteq B^2.
\]
Thus each \( G^2_q \) is a leaf of \( \langle B^2, G^2_1, \ldots, G^2_q \rangle \) with joint \( B^2 \).

When \( r = 3 \) and \( q \geq 2 \), then as in the previous case,
\[
B^3, G^3_1, \ldots, G^3_q
\]
is a leaf order. Since \( G^3_i \cap G^3_j = \emptyset \) when \( i \neq j \), for each \( i \in [q] \) the facet \( G^3_i \) is a leaf of
\[
\langle B^3, G^3_1, \ldots, G^3_q \rangle
\]
with joint \( B^3 \).

When \( r > 3 \) and \( q \geq 2 \), we claim
\[
F^r_1, F^r_2, \ldots, F^r_q, G^r_1, \ldots, G^r_q
\]
is a leaf order for \( L_q^r \), making it a quasi-tree. To see this note that if \( i \neq j, i, j \in [q] \), then
\[
F^r_i \cap G^r_j = \emptyset, \quad F^r_i \cap F^r_j = B^r, \quad G^r_i \cap G^r_j = \emptyset, \quad F^r_i \cap G^r_j = \{(r-1)e_i + e_h : h \neq i\}.
\]
These observations show that for each \( j \in [q] \), \( G^r_j \) is a leaf of
\[
\langle F^r_1, \ldots, F^r_q, G^r_1, \ldots, G^r_j \rangle
\]
with joint \( F^r_j \), and for each \( j \in \{2, \ldots, q\} \) \( F^r_j \) is a leaf of
\[
\langle F^r_1, \ldots, F^r_j \rangle
\]
with joint, say, \( F^r_1 \). Hence (4.6.1) is a leaf order and \( L_q^r \) is a quasi-tree. \( \square \)

5. The quasi-tree \( L^r(I) \) supporting a free resolution of \( I^r \)

Given an ideal \( I \) with \( q \) square-free monomial generators, we now define an induced subcomplex of \( L_q^r \), denoted \( L^r(I) \), which is obtained by deleting vertices representing redundant generators of \( I^r \). We show in Theorem 5.9 that both \( L_q^r \) and \( L^r(I) \) support a free resolution of \( I^r \).

**Definition 5.1 (The simplicial complex \( L^r(I) \)).** Let \( I \) be an ideal minimally generated by monomials \( m_1, \ldots, m_q \) in the polynomial ring \( S \). For \( \mathbf{a} = (a_1, \ldots, a_q) \in N_q^r \) we set
\[
\mathbf{m}^a = m_1^{a_1} \cdots m_q^{a_q}.
\]
Define a partition of \( N_q^r \) into equivalence classes \( V_1, \ldots, V_t \) by
\[
\mathbf{a} \sim \mathbf{b} \iff \mathbf{m}^a = \mathbf{m}^b.
\]
We use these equivalence classes to build an induced subcomplex \( L^r(I) \), of \( L_q^r \) using the following steps:

Step 1. From each equivalence class \( V_i \) pick a unique representative \( \mathbf{c}_i \) in the following way: if \( V_i \cap B^r = \emptyset \), then \( \mathbf{c}_i = B^r \). Otherwise, choose any \( \mathbf{c}_i \in V_i \).

Step 2. From the set \( \{\mathbf{c}_1, \ldots, \mathbf{c}_t\} \), eliminate all \( \mathbf{c}_i \) for which \( \mathbf{m}^{c_j} \mid \mathbf{m}^{c_i} \) for some \( j \neq i \). We call the remaining set, without loss of generality, \( \{\mathbf{c}_1, \ldots, \mathbf{c}_u\} \).

Step 3. Set \( V = \{\mathbf{c}_1, \ldots, \mathbf{c}_u\} \).

Define \( L^r(I) \) to be the induced subcomplex of \( L_q^r \) on the vertex set \( V \).
The complex $L^r(I)$ is a subcomplex of $\text{Taylor}(I^r)$. While $L^r(I)$ is dependent on the choices made when building its vertex set $V$, we will abuse the notation and ignore these choices, as they do not have an impact on our results. Also note that the set of monomials $\{m^e_1, \ldots, m^e_n\}$ forms a minimal monomial generating set for the ideal $I^r$. There are known classes of ideals, for instance, square-free monomial ideals of projective dimension one [3, Proposition 4.1], for which the generating set $\{m^a | a \in N_q\}$ does not contain redundacies, in which case $L^r_q = L^r(I)$. However, in general, $L^r(I)$ will be a proper subcomplex of $L^r_q$. Information about known redundancies in $\{m^a | a \in N_q\}$ can be used to produce $L^r(I)$. As an example, consider edge ideals of graphs. For such ideals, the redundacies in $\{m^a | a \in N_q\}$ are encoded in the ideal of equations of the fiber cone of the ideal, whose generators correspond to primitive even closed walks in the graph (see [22] for relevant definitions and details).

**Example 5.2.** Let $S = k[x, y, z, w]$ and $I = (xy, yz, zw, xw) = (m_1, m_2, m_3, m_4)$. By Proposition 4.3, the facets of $L^2_q$ are a 5-simplex $B^2$ and four tetrahedra $G_i^2$ for $1 \leq i \leq 4$, depicted on the left in the figure below.

By Definition 5.1, since $m_1m_3 = m_2m_4$ is the only equation determining an equivalence class with more than one element, we select the vertex $e_1 + e_3$ to represent this equivalence class and form $L^2(I)$. Then $L^2(I)$ consists of a 4-simplex on the vertices $e_1 + e_2$, $e_1 + e_3$, $e_1 + e_4$, $e_2 + e_3$, $e_3 + e_4$ together with two triangles and two tetrahedra depicted on the right in the figure below. Note that vertex $e_2 + e_4$ has been removed. The edges depicted by dotted lines in the figure of $L^2_q$ do not appear in $L^2(I)$ and higher dimensional faces of $L^4_q$ containing $e_2 + e_4$ have also been deleted.

Next, consider $I^3$. The equation $m_1m_3 = m_2m_4$ produces four non-trivial equivalence classes from Definition 5.1, namely the sets

$\{2e_1 + e_3, e_1 + e_2 + e_4\} \quad \{e_1 + e_2 + e_3, 2e_2 + e_4\} \quad \{e_1 + 2e_3, e_2 + e_3 + e_4\} \quad \{e_1 + e_3 + e_4, e_2 + 2e_4\}$

One can check that the above relations are the only ones. Since $r = 3$, we have $s = 2$ and so all the above duplicated vertices are in $B^3$. In particular, $L^3(I)$ will be the induced subcomplex of $L^4_q$ on a vertex set $V$ with 16 vertices. The sets below are two different possible sets of vertices $V$ for $L^3(I)$:

$V = \{3e_1, 2e_1 + e_2, 2e_1 + e_3, 2e_2 + e_1, 2e_2 + e_3, 3e_3, 3e_4, 2e_4 + e_1, 2e_4 + e_3, e_1 + e_2 + e_3, e_1 + e_2 + e_4, e_1 + e_3 + e_4, e_2 + e_3 + e_4\}$

or

$V = \{3e_1, 2e_1 + e_2, 2e_1 + e_3, 2e_1 + e_4, 3e_2, 2e_2 + e_1, 2e_2 + e_3, 2e_2 + e_4, 3e_3, 2e_3 + e_1, 2e_3 + e_2, 2e_3 + e_4, 3e_4, 2e_4 + e_1, 2e_4 + e_2, 2e_4 + e_3\}$
Example 5.3. Let $S = \mathbb{k}[a, b, c, d, e, f, x, y, z]$ and

$$I = (xyz, abc, def, xza, xzb, xyc, xyd, yze, yzf) = (m_1, m_2, \ldots, m_9).$$

We have the following relation:

$$m_1^4 m_2 m_3 = m_4 m_5 m_6 m_7 m_8 m_9.$$ 

For $r = 6$ we have $s = 3$, and so

$$4e_1 + e_2 + e_3 \in F_1^6 \setminus B^6 \quad \text{but} \quad e_4 + e_5 + e_6 + e_7 + e_8 + e_9 \in B^6.$$ 

Therefore, the induced subcomplex $L'/(I)$ would not contain the vertex $4e_1 + e_2 + e_3$ corresponding to $m_1^4 m_2 m_3$.

While the examples above show that, in general, $L'(I)$ is smaller than $L'_q$, there are also cases when the two complexes are the same. In Section 7 we identify an ideal $E_q$ with $L'_q = L'_q$ for each $q$. The two complexes are also equal for all $I_q$ with $q \leq 3$, as shown below.

Proposition 5.4. Let $I$ be an ideal minimally generated by square-free monomials $m_1, \ldots, m_q$ in the polynomial ring $S$. If $q \leq 3$ then $L_{q+1}(I) = L_q(I)$ for all $r \geq 1$.

Proof. If $q = 1$, then $I = (m_1^r)$ for all $r$, and both $L'_r(I)$ and $L'_q$ consist of a single point, so the equality holds.

Assume $1 < q \leq 3$. By Definition 5.1 it suffices to show that $L'_q(I)$ and $L'_r$ have the same vertex set, or in other words

$$m^a | m^b \implies a = b \quad \text{for all} \quad a, b \in \mathcal{N}_q^{-}.$$ 

Suppose $q = 2$ and $m^a | m^b$ with

$$a = (a_1, a_2), \quad b = (b_1, b_2), \quad \text{and} \quad a_1 + a_2 = b_1 + b_2 = r.$$ 

If $a \neq b$, then we may assume $a_1 > b_1$ and $a_2 < b_2$. Since the monomials $m_i$ are square-free, we have

$$m_1^{a_1 + b_1} m_2^{a_2 + b_2} \implies m_1^{a_1 - b_1} m_2^{b_2 - a_2} \implies m_1^1 m_2^2,$$

which contradicts the minimality of the generators of $I$.

Suppose $q = 3$ and $m^a | m^b$ with

$$a = (a_1, a_2, a_3), \quad b = (b_1, b_2, b_3) \quad \text{and} \quad a_1 + a_2 + a_3 = b_1 + b_2 + b_3 = r.$$ 

Assume $a \neq b$. If $a_i = b_i$ for some $i$ then we can reduce to the case $q = 2$, so we may assume $a_i \neq b_i$ for all $i$. Without loss of generality, assume $a_1 > b_1$.

We have three cases:

1. Suppose $a_2 > b_2$. In this case, we must also have $a_3 < b_3$. Then

$$m_1^{a_1 + m_2^{a_2} m_3^{a_3}} | m_1^{b_1 + m_2^{b_2} m_3^{b_3}} \implies m_1^{a_1 - b_1} m_2^{a_2 - b_2} m_3^{b_3 - a_3} \implies m_1^{a_1 - b_1} m_3^{b_3 - a_3}.$$ 

Since $m_1$ and $m_3$ are square-free, this implies that $m_1 | m_3$ which gives a contradiction.

2. Suppose $a_2 < b_2$ and $a_3 < b_3$. We have

$$m_1^{a_1 + m_2^{a_2} m_3^{a_3}} | m_1^{b_1 + m_2^{b_2} m_3^{b_3}} \implies m_1^{a_1 - b_1} m_2^{b_2 - a_2} m_3^{b_3 - a_3}.$$ 

Let $x$ be a variable such that $x | m_1$. It follows that $x | m_2$ or $x | m_3$. Assume $x \not| m_3$. We then have $x^{a_1 - b_1} | m_2^{b_2 - a_2}$. Since $m_2$ is square-free, it follows that $b_2 - a_2 \geq a_1 - b_1$. This is a contradiction, since

$$a_1 - b_1 = b_2 - a_2 + b_3 - a_3 > b_2 - a_2.$$ 

A similar contradiction is obtained when $x \not| m_2$. We conclude that for every $x | m_1$, we must have $x | m_2$ and $x | m_3$. This implies $m_1 | m_2$ and $m_1 | m_3$, which is a contradiction.

3. Suppose $a_2 < b_2$ and $a_3 > b_3$. After relabeling, this case reduces to Case (1).
Remark 5.5. Note that the square-free assumption is necessary in Proposition [5.4] for if \( I = (x^2y, y^2z, xyz) \), then \( m_1m_2 | m_3^2 \) in \( I^2 \), but \((1, 1, 0) \neq (0, 0, 2) \) in \( \mathbb{N}_2^3 \).

The next proposition shows that the vertices labeled by \( re_i \) belong to the induced subcomplex \( L^r(I) \) for all \( i \in [q] \), regardless of the choices made in Definition [5.1]. Moreover, if \( q \geq 2 \), then for each \( i \in [q] \) there exists some \( j \in [q] \setminus \{i\} \) such that the vertex labeled by \( (r - 1)e_i + e_j \) belongs to \( L^r(I) \).

In what follows we use the standard notation \( \text{LCM}(I^r) \) to denote the \( \text{lcm} \) lattice of \( I^r \), which is the set of all least common multiples of subsets of the minimal monomial generating set of \( I^r \), partially ordered by division.

Proposition 5.6. Let \( r \geq 1 \), \( I \) an ideal in \( S \) minimally generated by square-free monomials \( m_1, \ldots, m_q \), and \( i \in [q] \).

(i) If \( m^a | m^r_i \), then \( a = re_i \), for any \( a \in \mathbb{N}_q^r \).

(ii) If \( q \geq 2 \) and \( m^r_i - 1 \mid M \) for some monomial \( M \neq m^r_i \) with \( M \in \text{LCM}(I^r) \), then there exists \( j \in [q] \setminus \{i\} \) such that \( m_j \mid M \) and, for every \( a \in \mathbb{N}_q^r \),

\[
\text{LCM}(a) \mid m^r_i - m_j \iff \text{LCM}(a) = m^r_i - m_j \iff a = (r - 1)e_i + e_j.
\]

In particular, for all \( i \), and all \( j \) as in (ii),

\[
re_i \quad \text{and} \quad (r - 1)e_i + e_j
\]

are vertices of \( L^r(I) \), and

\[
m_i \quad \text{and} \quad m_i - m_j
\]

are minimal monomial generators of \( I^r \).

Proof. We first observe that, for \( g, h > 0 \) and \( u, v \in [q] \), we have:

\[
m_u^h \mid m_v^g \implies u = v \tag{5.6.1}
\]

Indeed, since \( m_u \) is square-free, if \( m_v^h \) divides \( m_u^g \), then \( m_u \mid m_v \), and we conclude \( u = v \) because \( m_u, m_v \) are minimal generators.

Let \( a = (a_1, \ldots, a_q) \in \mathbb{N}_q^r \), and suppose \( m^a \mid m^r_i \). If, for some \( j \in [q] \), \( a_j \neq 0 \) then \( m_i^{a_j} \mid m^r_i \), which by (5.6.1) implies that \( j = i \), which results in \( a = re_i \). Thus (i) holds.

We now prove (ii). If \( r = 1 \) the statement is trivial, so assume \( r, q \geq 2 \) and \( M \in \text{LCM}(I^r) \) satisfies \( m_i - 1 \mid M \) but \( M \neq m_i'^r \). Then there exists \( k \in [q] \setminus \{i\} \) with \( k \neq i \) and \( m_k \mid M \).

Let

\[
A = \{ k \in [q] : m_k \mid M \}.
\]

Choose \( j \in A \setminus \{i\} \) such that \( m_j \) has the fewest number of variables not dividing \( m_i \).

Now suppose for some \( a = (a_1, \ldots, a_q) \in \mathbb{N}_q^r \),

\[
m^a = m_1^{a_1} \cdots m_q^{a_q} \mid m_i - m_j \tag{5.6.2}
\]

If \( a_j \geq 2 \), by canceling a copy of \( m_j \) in (5.6.2) we will have \( m_j^{a_j - 1} \mid m_i - m_j \) which by (5.6.1) implies that \( i = j \), a contradiction. Similarly if \( a_i = r \) then by canceling a copy of \( m_i - 1 \) in (5.6.2) we obtain \( m_i \mid m_j \), again a contradiction since \( m_i, m_j \) are minimal generators. So we must have \( a_j \leq 1 \) and \( a_i \leq r - 1 \).

Now we claim that we must have \( a_j = 1 \) or \( a_i = r - 1 \). Otherwise, if \( a_j = 0 \) and \( a_i \leq r - 2 \), since \( a_1 + \cdots + a_q = r \), there must exist \( c, d \in [q] \setminus \{i, j\} \) with \( a_c, a_d > 0 \) (\( c \) and \( d \) could be equal). In particular by (5.6.2)

\[
m_c m_d \mid m_i - m_j \tag{5.6.3}
\]

Let

\[
m_c = \gcd(m_c, m_i) \cdot m_c' \quad m_d = \gcd(m_d, m_i) \cdot m_d' \quad \text{and} \quad m_j = \gcd(m_j, m_i) \cdot m_j'.
\]

Let

\[
m_c m_d \mid m_i - m_j \tag{5.6.3}
\]
From (5.6.4) one can see that $m'_c$ and $m'_d$ do not share any variables with $m_i$, and so by (5.6.3)

$$m'_c m'_d \mid m'_j$$

and also

$$m'_c \mid m'_j \mid m_j \mid M \quad \text{and} \quad m'_d \mid m'_j \mid m_j \mid M.$$  \hfill (5.6.6)

On the other hand $m_i \mid M$, so (5.6.4) and (5.6.6), together with the fact that $m_c$ and $m_d$ are square-free, imply that

$$m_c \mid M \quad \text{and} \quad m_d \mid M,$$

which in turn implies that $c, d \in A$. Assume without loss of generality that $\deg m'_c \leq \deg m'_d$. The fact that $j \in A$ was picked so that $m_j$ has the fewest number of variables outside $m_i$ and (5.6.3) together imply

$$\deg m'_c + \deg m'_d = \deg m'_j \leq \deg m'_c \leq \deg m'_d;$$

a contradiction since $m'_c, m'_d$ and $m'_j$ all have positive degrees.

So the only possibilities are $a_j = 1$ or $a_j = r - 1$. If $a_j = 1$ then by (i) we have $m^n = m_i r^{-1} m_j$, and we are done. If $a_i = r - 1$ we have $m^n = m_i r^{-1} m_k$ for some $k \in [q]$, hence by cancellation $k = j$ and $m^n = m_i r^{-1} m_j$.

The remaining statements now follow immediately from (i), (ii) and Definition 5.1. □

One additional lemma is necessary before the statement of our main result.

**Lemma 5.7.** Using Notation 4.7, let $i, j \in [q]$ with $i \neq j$, and let $a = (a_1, \ldots, a_q)$ and $b = (b_1, \ldots, b_q)$ be in $\mathcal{N}_q^r$. If $\alpha$ is a non-negative integer such that

$$a_i \geq \alpha \quad \text{and} \quad b_j > r - \alpha$$

Then

(i) $m^b \nmid m^a$;

(ii) $m^{a_i} m_j^{a_i - \alpha} \frac{m^n}{m_i^{a_i}} \mid \text{lcm}(m^n, m^b)$;

(iii) $\deg(m^{a_i} m_j^{a_i - \alpha} \frac{m^n}{m_i^{a_i}}) \leq \deg(m^n) \iff \deg(m_j) \geq \deg(m_j)$ or $a_i = \alpha$.

**Proof.** To prove (i), assume $m^b \nmid m^a$. Set

$$a' = a - a_i e_i \in \mathcal{N}_q^{r - a_i}$$

so that

$$m^n = m^a m_i^{a_i}.$$  

Let $x$ be a variable such that $x \mid m_j$. Then $x^{b_j} \mid m^b$ and hence $x^{b_j} \mid m^n$. Suppose $x$ does not divide $m_i$. Then $x^{b_j} \mid m^n$. Since $m^n$ is a product of $r - a_i$ square-free monomials, we have then

$$b_j \leq r - a_i.$$

The hypothesis on $a_i$ and $b_j$ gives

$$r - a_i \leq r - \alpha < b_j.$$

The contradiction that arises shows $x \mid m_i$. We conclude $m_j \mid m_i$, hence $i = j$, a contradiction.

Now we prove (ii). Set

$$a' = a - a_i e_i \in \mathcal{N}_q^{r - a_i} \quad \text{and} \quad b' = b - b_j e_j \in \mathcal{N}_q^{r - b_j}$$

so that

$$m^n = m^a m_i^{a_i} \quad \text{and} \quad m^b = m^b m_j^{b_j}.$$  

It will be shown that

$$\deg_x(m^{a_i} m_j^{a_i - \alpha} m^n) \leq \max\{\deg_x(m^n), \deg_x(m^b)\}$$

for all variables $x$, where for a monomial $M$, $\deg_x(M) = \max\{t : x^t \mid M\}$. 

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Let \( u = \deg_x(m^a) \) and \( v = \deg_x(m^b) \). Since \( m^a \) is a product of \( r - a_i \) square-free monomials, and \( m^b \) is a product of \( r - b_j \) square-free monomials, we have
\[
0 \leq u \leq r - a_i \quad \text{and} \quad 0 \leq v \leq r - b_j.
\]

Also note that
\[
\deg_x(m_i^a m_j^a m_j^{a_i-a} m^a) = \alpha \cdot \deg_x(m_i) + (a_i - \alpha) \cdot \deg_x(m_j) + u.
\]

- If \( x \mid m_i \), then
\[
\deg_x(m_i^a m_j^a m_j^{a_i-a} m^a) = \alpha + (a_i - \alpha) \deg_x(m_j) + u
\leq \alpha + (a_i - \alpha) + u = a_i + u
= \deg_x(m^a)
\leq \max\{\deg_x(m^a), \deg_x(m^b)\}.
\]

- If \( x \not\mid m_i \), but \( x \mid m_j \), then
\[
\deg_x(m_i^a m_j^a m_j^{a_i-a} m^a) = (a_i - \alpha) + u
\leq (a_i - \alpha) + (r - a_i) \quad \text{by (5.7.1)}
= r - \alpha < b_j \leq b_j + v
= \deg_x(m^b)
\leq \max\{\deg_x(m^a), \deg_x(m^b)\}.
\]

- If \( x \not\mid m_i \) and \( x \not\mid m_j \), then
\[
\deg_x(m_i^a m_j^a m_j^{a_i-a} m^a) = \deg_x(m^a) = \deg_x(m^a) \leq \max\{\deg_x(m^a), \deg_x(m^b)\}.
\]

This finishes the proof of (ii).

Finally, (iii) follows directly from the fact that
\[
\deg(m_i^a m_j^a m_j^{a_i-a} m^a) = \alpha \cdot \deg m_i + (a_i - \alpha) \cdot \deg m_j + \deg(m^a)
= \alpha \cdot \deg m_i + (a_i - \alpha) \cdot \deg m_j + \deg(m^a) - a_i \cdot \deg m_i
= \deg(m^a) - (a_i - \alpha)(\deg(m_i) - \deg(m_j)).
\]

Remark 5.8. Recall that \( s = \left\lceil \frac{r}{2} \right\rceil \). When \( \alpha = s \), the condition on \( b_j \) in Lemma 5.7 can be translated as follows:
\[
b_j > r - s \iff \begin{cases} b_j \geq s & \text{when } r \text{ is odd} \\ b_j > s & \text{when } r \text{ is even.} \end{cases}
\]

Recall that, by Theorem 5.6 if \( \Delta \) is a quasi-tree with vertices labeled with the monomial generating set of a monomial ideal \( I \subseteq k[x_1, \ldots, x_n] \), then \( \Delta \) supports a resolution of \( I \) if and only if \( \Delta_M \) is connected for every monomial \( M \). It is easy to check that the connectivity need only be verified for \( M \) in LCM(\( I \)). We now use this criterion to prove the main result of the paper.

Theorem 5.9 (Main Result). Let \( q \geq 1 \) and let \( I \) be a monomial ideal minimally generated by square-free monomials \( m_1, \ldots, m_q \). By labeling each vertex \( a \) of the simplicial complexes below with the monomial \( m^a \), the following hold for any \( r \geq 1 \):
\[
\begin{align*}
(1) \; & \Downarrow^r_q \; \text{supports a free resolution of } I^r; \\
(2) \; & \Downarrow^{r(1)} I \; \text{supports a free resolution of } I^r.
\end{align*}
\]
Claim 1: (The second layer facets of \( L \) intersect the first layer facets.) For any \( i \in [q] \) we have:

\[
G_i^r \cap V_M \neq \emptyset \implies G_i^r \cap F_i^r \cap V_M \neq \emptyset.
\]

Proof of Claim 1. Assume \( G_i^r \cap V_M \neq \emptyset \) for some \( i \in [q] \). By Definition 4.2, there exists \( a \in [q] \) such that

\[
d = (r-1)e_i + e_a \in G_i^r \cap V_M.
\]

If \( i \neq a \), then \( d \in F_i^r \) as well, and hence \( G_i^r \cap F_i^r \cap V_M \neq \emptyset \) as desired.

Assume \( a = i \). Then \( d \in V_M \) implies \( m_i \mid M \). Since \( M \neq m_i \), Proposition 5.6 guarantees that there exists \( j \in [q] \) with \( j \neq i \) and \( m_j \mid M \) so that \( (r-1)e_i + e_j \in V \).

Since \( m_i \mid M \) and \( m_j \mid M \), we see that \( m_i^{r-1}m_j \mid M \). Indeed, we set \( M = m_i^{r-1}m_j \). Since \( m_j \) is square-free, we have \( m_j \mid m_i \). Thus we have

\[
(r-1)e_i + e_j \in G_i^r \cap F_i^r \cap V_M.
\]

Claim 2: (The first layer facets connect through the base of \( L_M \).) For any \( i, j \in [q] \) with \( i \neq j \) we have:

\[
(F_i^r \cup G_i^r) \cap V_M \neq \emptyset \quad \text{and} \quad (F_j^r \cup G_j^r) \cap V_M \neq \emptyset \implies B^r \cap V_M \neq \emptyset.
\]

Proof of Claim 2. Assume for some \( i, j \in [q] \) with \( i \neq j \) that

\[
(F_i^r \cup G_i^r) \cap V_M \neq \emptyset \quad \text{and} \quad (F_j^r \cup G_j^r) \cap V_M \neq \emptyset.
\]

Without loss of generality, assume \( \deg(m_i) \geq \deg(m_j) \). We choose

\[
a = (a_1, \ldots, a_q) \in (F_i^r \cup G_i^r) \cap V_M \quad \text{and} \quad b = (b_1, \ldots, b_q) \in (F_j^r \cup G_j^r) \cap V_M
\]

such that

\[
\deg(m^a) = \min \{ \deg(m^a) \mid d \in (F_i^r \cup G_i^r) \cap V_M \}.
\]
Assume, by way of contradiction, that $B^r \cap V_M = \emptyset$. Since $a, b \in V_M$, we have then $a, b \notin B^r$. Therefore, 
\[ s < a_i, b_j \leq r \quad \text{and} \quad m^a, m^b \mid M. \]

Set 
\[ a' = a - a_i e_i \in N_q^{-a_i} \quad \text{and} \quad c := (c_1, \ldots, c_q) = s e_i + (a_i - s) e_j + \sum_{1 \leq k \leq q, k \neq i} a_k e_k \in N_q^r. \]

Since $b_j > s \geq r - s$, by Lemma 5.7(ii) and Remark 5.8, 
\[ m^c = m_i^s m_j^{a_i-s} m^{a'} | \text{lcm}(m^a, m^b) | M. \] (5.9.3) 
Moreover $c \in B^r$, because 
\[ c_i = s \]
\[ c_j = a_i - s + a_j \leq a_i - s + (r - a_i) = r - s \leq s \quad \text{since} \ a_i + a_j \leq r \]
\[ c_k = a_k \leq r - a_i < r - s \leq s \quad \text{for all} \ k \neq i, j. \]

If $L = L^r_q$, then $V = N_q^r$. Since (5.9.3) shows $c \in V_M$, we conclude $c \in B^r \cap V_M$, a contradiction. This finishes the proof of part (1).

Assume now $L = L^r(I)$. Since we assumed $B^r \cap V_M = \emptyset$, we must have $c \notin V$. Definition 5.1 implies then 
\[ m^{c'} \mid m^c \quad \text{for some} \quad c' := (c'_1, \ldots, c'_q) \in V. \]

Note that $m^{c'} \mid M$ by (5.9.3) and hence $c' \in V_M$. Since $B^r \cap V_M = \emptyset$, we must have $c' \notin B^r$. Using the notation in Definition 5.1 let $V_a$ be the equivalence class of which $c'$ is a representative. If $m^{c'} = m^c$, then $c \in V_a$ as well, and hence $V_a \cap B^r \neq \emptyset$. Then Step 1 in Definition 5.1 implies $c' \in B^r$, a contradiction. Hence $m^{c'} \neq m^c$.

Since $c' \notin B^r$, there exists $k \in [q]$ such that $c'_k > s \geq r - s$. Since $c_i = s$ and $m^{c'} \mid m^c$, Lemma 5.7(i) implies $k = i$. In particular, $c'_i > s$, and thus 
\[ c' \in (F^r_i \cup G^r_i) \cap V_M. \]

Since $m^{c'}$ strictly divides $m^c = m_i^s m_j^{a_i-s} m^{a'}$, using Lemma 5.7(iii), we have 
\[ \deg(m^{c'}) < \deg(m^c) \leq \deg(m^a), \]
contradicting the assumption made in (5.9.2). This finishes the proof of part (2). \hfill \Box

Remark 5.10. One may feel that part (1) of the statement above is weaker than (2), since $L^r(I) \subseteq L^r_q$. However, the remarkable aspect of (1) is that, before labeling, $L^r_q$ does not depend on the ideal $I$. Thus the same topological structure, depending only on $r$ and $q$, supports a free resolution of the $r^{th}$ power of any ideal generated by $q$ square-free monomials. The idea is that $L^r_q$ provides an alternative notion to the Taylor complex for powers of $I$: when $r = 1$, $L^r_q$ is the Taylor simplex, and when $r > 1$, $L^r_q$ is significantly smaller than the Taylor simplex but still supports a resolution of $I^r$.

6. Bounds on Betti Numbers

One of the key applications of Theorem 5.9 is that we are able to improve the bounds on Betti numbers for powers of ideals from that given by the standard Taylor resolution of $I^r$ since 
\[ L^r(I) \subseteq L^r_q \subseteq \text{Taylor}(I^r). \]

This section contains computations that illustrate the extent to which our results improve the Taylor bounds.

When $r = 2$, we were able to provide a concrete formula for the number of faces of $L^r(I)$ in [4], and as a result we provided bounds for Betti numbers of $I^2$, which are shown in Proposition 5.11 to be sharp. When $r > 2$, however, such formulas are not as easy to write and the numbers are very large even for small
examples. In this case we shift our attention to \( \mathbb{L}^r_q \). While the bound on the Betti numbers stemming from \( \mathbb{L}^r(I) \) is dependent on the relations among the generators of \( I \), one can use the structure of \( \mathbb{L}^r_q \) to get a general, albeit weaker, bound on \( \beta_i(I^r) \).

**Theorem 6.1 (Bounds on Betti numbers of \( I^r \)).** If \( I \) is a square-free monomial ideal with \( q \) minimal generators, then the Betti numbers of \( I^r \) for \( r \geq 2 \) are bounded above by

\[
\beta_i(I^r) \leq q \left( \binom{q-1}{t} + \binom{f}{t+1} - \binom{b}{t+1} \right) + \binom{b}{t+1}
\]

where \( t \geq 0 \), \( b \) is the coefficient of \( x^r \) in the expansion of

\[
(1 + x + x^2 + \cdots + x^{\left\lceil \frac{s}{2} \right\rceil} )^q,
\]

and

\[
f = \frac{(q+r-1) - b - q + b}{q}.
\]

In particular, \( \text{pd}(I^r) \leq \max\{q - 1, f - 1\} \).

**Proof.** Let \( s = \left\lceil \frac{r}{2} \right\rceil \). Note that \( \beta_i(I^r) \) is bounded above by the number of faces of dimension \( t \) of \( \mathbb{L}^r_q \). To count the faces of dimension \( t \), we use the sets \( B^r, F^r, G^r \) from Definition 4.2.

Observe that the coefficient \( b \) of \( x^r \) in the expansion of

\[
(1 + x + x^2 + \cdots + x^s)^q
\]

is exactly the number of \( q \)-tuples \( (a_1, \ldots, a_q) \) with \( a_i \leq s \) and \( a_1 + \cdots + a_q = r \), in other words \( b = |B^r| \).

Note that the vertices of \( \mathbb{L}^r_q \) are formed by selecting \( r \) of the original \( q \) generators, so using the standard formula for counting with repetition, we have

\[
|V(\mathbb{L}^r_q)| = \binom{q + r - 1}{r}.
\]

Now \( |F^r_i| = |F^r_j| \) for all \( i, j \), so let \( f = |F^r_i| \). Since \( |G^r_i \setminus F^r_i| = 1 \), there are \( |V(\mathbb{L}^r_q)| - q \) vertices in \( \cup_{i=1}^q F^r_i \). Also \( F^r_i \cap F^r_j = B^r \) for all \( i, j \) such that \( i \neq j \), so we have

\[
f = |F^r_i| = \frac{1}{q} \left( |V(\mathbb{L}^r_q)| - q \right) - |B^r| = \frac{\left( q + r - 1 \right) - q + b}{q} + b.
\]

(6.1.1)

To count the number of faces of dimension \( t \), that is, faces with \( t + 1 \) vertices, in \( \mathbb{L}^r_q \), we separate the faces into three distinct types.

1. **Faces containing** \( m_i \) **for some** \( i \): By the definition of \( \mathbb{L}^r_q \), faces of this type must be contained in \( G^r_i \). Since the vertex corresponding to \( m_i \) has been fixed, \( t \) additional vertices of \( G^r_i \) are needed. There are \( q - 1 \) such vertices since all vertices of \( G^r_i \) have the form \( (r - 1)e_i + e_j \) where \( 1 \leq j \leq q \). Since there are \( q \) choices for \( i \), there are

\[
q \binom{q - 1}{t}
\]

such faces.

2. **Faces contained in** \( B^r \): Setting \( b = |B^r| \) as above, there are

\[
\binom{b}{t+1}
\]

such faces.
(3) Faces contained in $F^r_i$ but not in $B^r$: Recalling that $f = |F^r_i|$ and $B^r \subseteq F^r_i$, there are

$$ q \left( \binom{f}{t+1} - \binom{b}{t+1} \right) $$

such faces.

Combining the three cases, we see that there are

$$ q \left( \binom{q-1}{t} + \binom{f}{t+1} - \binom{b}{t+1} \right) + \binom{b}{t+1} $$

faces of $L^r_q$ of dimension $t$. Thus for $I^r$,

$$ \beta_t(I^r) \leq q \left( \binom{q-1}{t} + \binom{f}{t+1} - \binom{b}{t+1} \right) + \binom{b}{t+1}.$$

In particular, if $t > q - 1$ and $t + 1 > f$, we must have $\beta_t(I^r) = 0$. Thus

$$ \text{pd}(I^r) \leq \max\{q - 1, f - 1\}. \quad (6.1.2) $$

\[ \Box \]

Corollary 6.2. If an ideal $I$ is minimally generated by $q$ square-free monomials, then the Betti numbers of $I^r$ are bounded by

$$ \beta_t(I^r) \leq q \left( \binom{q-1}{t} + \binom{b}{t+1} \right) $$

for $r = 2, 3$, where $b$ is as defined in Theorem 6.1. In the case where $r = 2$, $b = \binom{q}{2}$ and the bound reduces to the bound given in [4].

Proof. When $r = 2$ or $r = 3$, then $F^r_i = B^r$ for all $i$, so $b = f$ and the result follows immediately from simplifying the formula in Theorem 6.1. Moreover, when $r = 2$, the coefficient of $x^2$ in the binomial expansion of $(1 + x)^q$ is $\binom{q}{2}$ as stated. The reduction is then evident. \[ \Box \]

Notice that when $r = 2$, the bound is known to be sharp; it agrees with the result in [4]. In Proposition 7.11 we characterize the values of $r$ and $q$ for which these bounds are sharp, i.e. can be realized by some ideal.

Example 6.3. As a first example, we examine cases where $r$ or $q$ is small. We first consider $b$ and $f$ for small values of $q$.

- For $q = 2$, computing $(1 + x + x^2 + \cdots + x^s)^2$ shows that $b = 1$ when $r = 2s$ is even and $b = 2$ when $r = 2s - 1$ is odd. If $r$ is even, we then have

$$ f = \frac{(2^r - 2 - 1)}{2} + 1 = \frac{r + 1 - 2 - 1}{2} + 1 = \frac{r}{2} = s. \quad (6.3.1) $$

A similar computation shows that when $r$ is odd, $f = s$ as well.

- If $q = 3$ and $r = 3$ or $r = 4$ (so that $s = 2$), then computing $(1 + x + x^2)^3$ yields $b = 7$ and $f = 7$ when $r = 3$ and $b = 6$ and $f = 8$ when $r = 4$.

Applying the equations from Theorem 6.1 yields:

- For $r = 2$, any $q$, and any $t$,

$$ \beta_t(I^r) \leq q \left( \binom{q-1}{t} + \frac{2q(q-1)}{t+1} \right). $$

- For any $r$, $q = 2$, and any $t \geq 2$, if $s = \lceil r/2 \rceil$,

$$ \beta_t(I^r) \leq 2 \binom{s}{t+1}. $$
• For \( r = q = 3 \) and any \( t \),
\[
\beta_t(I^r) \leq 3 \binom{2}{t} + \binom{7}{t+1}.
\]

• For \( r = 4, q = 3 \) and any \( t \geq 3 \),
\[
\beta_t(I^r) \leq 3 \binom{8}{t+1} - 2 \binom{6}{t+1}.
\]

**Remark 6.4.** In (6.1.2) it is useful to understand which of the integers \( q - 1, f - 1 \) achieves the maximum. For small values of \( q \) and \( r \), it is possible to have \( q \geq f \). However, the opposite holds in most cases. More precisely, we show below that if \( r, q \) satisfy any of the following assumptions:

1. \( q = 2 \) and \( r \geq 5 \);
2. \( q = 3 \) and \( r \geq 3 \);
3. \( q \geq 4 \) and \( r \geq 2 \),
then \( f > q \), and hence
\[
\text{pd}(I^r) \leq f - 1 = \dim L_q^r.
\]

In the case \( q = 2 \), (6.3.1) shows \( f = s = \lceil \frac{r}{2} \rceil \). If \( r \geq 5 \), we must have \( s > 2 \), thus \( f = s > 2 = q \). This settles (1).

Now suppose \( q > 2 \). If \( r = 2 \), then Corollary 6.2 shows that \( f = b = \binom{q}{2} \). When \( q \geq 4 \), we have \( f = \binom{q}{2} > q \). Thus, to show (2) and (3) it remains to consider the case when \( q \geq 3 \) and \( r \geq 3 \). Observe that when \( r \geq 3, s \geq 2 \). In this scenario, since \( b \) is the coefficient of \( x^r \) in \((1 + x + \cdots + x^q)^9\) and \( q > 2 \), then \( b \geq 2 \) and \( bq - b > q \).

To see that \( f > q \) holds, we will show \( f - q > 0 \), which by (6.1.1) amounts to
\[
\frac{(q+r-1) - q - b}{q} + b - q = \frac{(q+r-1) - q - bq - q^2}{q} > 0.
\]

It is sufficient to show the numerator is positive, so we observe the following, where the first inequality results from \( bq - b > q \) and the second follows since \( r \geq 3 \):
\[
\frac{(r + q - 1)}{q - 1} - q - b + bq - q^2 > \frac{(r + q - 1)}{q - 1} - q^2
\]
\[
\geq \frac{q + 2}{q - 1} - q^2
\]
\[
= \frac{(q + 2)(q + 1)q}{6} - q^2 = \frac{q(q-1)(q-2)}{6} > 0.
\]

This ends our argument.

**Example 6.5.** In general, the bounds from Theorem 6.1 will be considerably smaller than those provided by the Taylor complex, which is a simplex, and the bound on the projective dimension will also decrease significantly. We display this phenomenon in the table below.

| Bound Comparisons |
|-------------------|
|                  |
| \( q = 2, r = 3 \) | \( q = 3, r = 3 \) | \( q = 3, r = 4 \) |
|-------------------|
| \( \beta_0(I^r) \leq \) | 4 | 10 | 15 |
| \( \beta_1(I^r) \leq \) | 3 | 6 | 27 |
| \( \beta_2(I^r) \leq \) | 0 | 4 | 38 |
| \( \text{pd}(I^r) \leq \) | 1 | 3 | 6 |
The bounds on Betti numbers and projective dimension given by the complex $L_q^r$ in Theorem 6.1 are most effective when the generating set $\{m^a \mid a \in N_q^r\}$ does not contain redundancies. When there are redundancies, using $L^r(I)$ will yield improved bounds. We illustrate how to use this improvement by continuing Example 5.2.

**Example 6.6.** Let $I = (xy, yz, zw, xw) = (m_1, m_2, m_3, m_4)$ as in Example 5.2. Counting faces of size $i$ in the complex $L^2(I)$ provides a bound on the $i^{th}$ Betti number of $I^2$. Note that in general, these improvements in the Betti numbers follow from knowledge of the redundancies in $\{m^a \mid a \in N_q^r\}$ and can often be computed from that information using the equivalence classes without explicitly constructing $L^r(I)$. For instance, a comparison of the bounds is summarized in the table below for $I^2$ and $I^3$ using the first of the two vertex sets for $L^i(I)$ given in Example 5.2.

| Bound Comparisons | $r = 2$ | $r = 3$ |
|-------------------|--------|--------|
| $L^2(I)$          | 8-simplex | 9-simplex | $L^3(I)$          | 15-simplex | 19-simplex |
| $\beta_0(I^r)$ $\leq$ | 9 | 10 | 16 | 20 |
| $\beta_1(I^r)$ $\leq$ | 20 | 27 | 45 | 132 | 190 |
| $\beta_2(I^r)$ $\leq$ | 18 | 32 | 120 | 224 | 572 | 1,140 |
| $pd(I^r)$ $\leq$ | 4 | 5 | 9 | 11 | 15 | 19 |

7. **Extremal ideals: When does $L^r_q$ support a minimal resolution?**

Based upon the above work, a natural question that arises is the following: for given $r$ and $q$, can one find ideals $I$ for which $I^r$ has a minimal free resolution supported on $L^r_q$ itself? When $r = 1$, $L^r_q$ is the Taylor complex, which one can easily see always supports a minimal free resolution of the ideal generated by $q$ variables $(x_1, x_2, \ldots, x_q)$. In the case where $r > 1$, Proposition 7.11 describes exactly when the bounds for Betti numbers in Theorem 6.1 and Corollary 6.2 are sharp, in the sense that there exist ideals for which equality is attained. We call the ideals which realize these bounds $q$-extremal ideals and denote them by $E_q$.

In fact we can show a much stronger statement in Theorem 7.9: the powers $E_q^r$ have maximal Betti numbers among the ideals $I^r$ where $I$ is generated by $q$ square-free monomials.

**Definition 7.1 (Extremal ideals).** Let $q$ be a positive integer. For every set $A$ with $\emptyset \neq A \subseteq [q]$, introduce a variable $x_A$, and consider the polynomial ring

$$S_E = k[x_A : \emptyset \neq A \subseteq [q]]$$

over a field $k$. For each $i \in [q]$ define a square-free monomial $\epsilon_i$ in $S_E$ as

$$\epsilon_i = \prod_{A \subseteq [q] \atop i \in A} x_A.$$

The square-free monomial ideal $E_q = (\epsilon_1, \ldots, \epsilon_q)$ is called a $q$-extremal ideal.

When it is unlikely to lead to confusion, we will simplify the notation by writing $x_1$ for $x_{\{1\}}$, $x_{12}$ for $x_{\{1,2\}}$, etc., and refer to a $q$-extremal ideal simply as an extremal ideal. The ring $S_E$ has $2^q - 1$ variables, corresponding to the power set of $[q]$ (excluding $\emptyset$), and each $\epsilon_i$ is the product of $2^{q-1}$ variables; that is, those corresponding to the subsets of $[q]$ that contain $i$. The following example illustrates how this works for $E_4$. 


Example 7.2. When \( q = 4 \), the ideal \( E_q \) is generated by the monomials

\[
\begin{align*}
\epsilon_1 &= x_1 x_2 x_3 x_4 x_{12} x_{13} x_{14} x_{23} x_{24} x_{34} x_{123} x_{124} x_{134} x_{1234}; \\
\epsilon_2 &= x_1 x_2 x_3 x_4 x_{12} x_{13} x_{14} x_{23} x_{24} x_{34} x_{123} x_{124} x_{134} x_{1234}; \\
\epsilon_3 &= x_1 x_2 x_3 x_4 x_{12} x_{13} x_{14} x_{23} x_{24} x_{34} x_{123} x_{124} x_{134} x_{1234}; \\
\epsilon_4 &= x_1 x_2 x_3 x_4 x_{12} x_{13} x_{14} x_{23} x_{24} x_{34} x_{123} x_{124} x_{134} x_{1234}.
\end{align*}
\]

in the polynomial ring \( k[x_1, x_2, x_3, x_4, x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}, x_{123}, x_{124}, x_{134}, x_{1234}] \).

Using the terminology of \([13]\), it naturally follows that \( E_q \) is precisely the nearly Scarf ideal of a \( (q - 1) \)-simplex: we see this by matching the variable \( x_{[q] \setminus B} \) with the face \( B \) of the simplex.

Following Notation \([4,1]\) let \( r \) and \( q \) be positive integers and \( I \) an ideal generated by square-free monomials \( m_1, \ldots, m_q \). If \( a = (a_1, \ldots, a_q) \in \mathbb{N}^q \), set

\[
m^a = m_1^{a_1} \cdots m_q^{a_q} \quad \text{and} \quad \epsilon^a = e_1^{a_1} \cdots e_q^{a_q}.
\]

Observe

\[
\epsilon^a = \prod_{j \in [q]} \prod_{A \subseteq [q], j \in A} x_A^{a_j} = \prod_{\emptyset \neq A \subseteq [q]} (x_A)^{\sum_{j \in A} a_j}.
\]

The \( r^{th} \) powers \( I^r \) and \( E_q^r \) are generated by monomials of the form \( m^a \) and \( \epsilon^a \), respectively, with \( a \in \mathbb{N}^r \).

Proposition \([7.3]\) demonstrates that for \( E_q^r \), the elements \( \epsilon^a \) with \( a \in \mathbb{N}^r \) form a minimal generating set. In fact, all faces of \( \mathbb{L}^r_q \) are necessary in \( \mathbb{L}^r_q(E_q) \); i.e., none may be removed when constructing \( \mathbb{L}^r_q(E_q) \).

Proposition 7.3. Let \( r \) and \( q \) be positive integers, and \( a = (a_1, \ldots, a_q) \), \( b = (b_1, \ldots, b_q) \in \mathbb{N}^q \). Then

\[
\begin{align*}
(i) & \quad e^b | e^a \iff b_i \leq a_i \quad \text{for each} \quad i \in [q], \\
(ii) & \quad \text{If } a, b \in \mathbb{N}^r_q, \text{ then } e^b | e^a \iff b = a.
\end{align*}
\]

In particular, if \( a, b \in \mathbb{N}^r_q \), then \( \mathbb{L}^r_q = \mathbb{L}^r(E_q) \).

Proof. To prove (i), use \((7.1.1)\) to justify the first equivalence below.

\[
e^b | e^a \iff \sum_{j \in A} b_j \leq \sum_{j \in A} a_j \quad \text{for each} \quad \emptyset \neq A \subseteq [q] \iff b_j \leq a_j \quad \text{for each} \quad j \in [q].
\]

Now (ii) follows from (i) and the added assumption

\[
b_1 + \cdots + b_q = a_1 + \cdots + a_q = r.
\]

The final claim follows directly from Definition \([5,1]\) \( \square \)

In general, given the discussions above (see also \([13]\)), for every \( r \) and \( q \) we have

\[
\mathbb{L}^r(E_q) = \mathbb{L}^r_q \subseteq \text{Taylor}(E_q^r),
\]

where we assume that every vertex \( a \) of \( \mathbb{L}^r_q \) is labeled with a generator \( e^a \) of \( E_q^r \), and each face \( \sigma \) is labeled with the lcm of the labels of its vertices, denoted \( M_\sigma \). The following observation will be useful when working with the monomial label \( M_\sigma \) of \( \sigma \in \text{Taylor}(E_q^r) \).

Remark 7.4. Let \( \sigma = \{e^{a_1}, \ldots, e^{a_t}\} \in \text{Taylor}(E_q^r) \) and set \( a_i = (a_{i1}, a_{i2}, \ldots, a_{iq}) \) for \( i \in [t] \). Using \((7.1.1)\), we then have

\[
M_\sigma = \text{lcm}\left( \prod_{\emptyset \neq A \subseteq [q]} (x_A)^{\sum_{j \in A} a_{ij}} : i \in [t] \right) = \prod_{\emptyset \neq A \subseteq [q]} (x_A)^{\max_{1 \leq i \leq t} \sum_{j \in A} a_{ij}}.
\]

Furthermore, if \( c = (c_1, \ldots, c_q) \in \mathbb{N}^r_q \), \((7.1.1)\) and \((7.4.1)\) give

\[
e^c | M_\sigma \iff \sum_{j \in A} c_j \leq \max_{1 \leq i \leq t} \sum_{j \in A} a_{ij} \quad \text{for all} \quad \emptyset \neq A \subseteq [q].
\]
We show in Theorem 7.9 that powers of extremal ideals attain maximal Betti numbers among powers of all square-free monomial ideals with the same number of generators. To prove this, we first define a ring homomorphism $S_q \rightarrow S$ and discuss its properties.

**Definition 7.5 (The ring homomorphism $\psi$).** Let $I$ be an ideal of the polynomial ring $S = k[x_1, \ldots, x_n]$ minimally generated by square-free monomials $m_1, \ldots, m_q$. For each $k \in [n]$, set

$$A_k = \{ j \in [q] : x_k \mid m_j \}.$$ 

We have thus

$$j \in A_k \iff x_k \mid m_j.$$ 

Define $\psi$ to be the ring homomorphism

$$\psi : S_q \rightarrow S \quad \text{where} \quad \psi(x_A) = \begin{cases} \prod_{k \in [n]} x_k & \text{if } A = A_k \text{ for some } k \in [n], \\ 1 & \text{otherwise.} \end{cases}$$

Before proceeding directly with our work on extremal ideals, we illustrate how the map $\psi$ works in a sample case where there are four generators and seven variables.

**Example 7.6.** Let $I$ be the ideal of $k[x_1, \ldots, x_7]$ generated by the square-free monomials

$$m_1 = x_1x_2x_5x_7, \quad m_2 = x_2x_3x_7, \quad m_3 = x_3x_4x_6, \quad m_4 = x_1x_4.$$ 

Since $n = 7$ and $q = 4$, it follows that

$$A_1 = \{1, 4\}, \quad A_2 = \{1, 2\}, \quad A_3 = \{2, 3\}, \quad A_4 = \{3, 4\}, \quad A_5 = \{1\}, \quad A_6 = \{3\}, \quad A_7 = \{1, 2\}.$$ 

The function

$$\psi : k[x_A : \emptyset \neq A \subseteq [q]] \rightarrow k[x_1, \ldots, x_7]$$ 

maps

$$\psi(x_{\{1, 4\}}) = x_1, \quad \psi(x_{\{1, 2\}}) = x_2x_7, \quad \psi(x_{\{2, 3\}}) = x_3, \quad \psi(x_{\{3, 4\}}) = x_4,$$

$$\psi(x_{\{1\}}) = x_5, \quad \psi(x_{\{3\}}) = x_6,$$

and

$$\psi(x_A) = 1 \text{ otherwise}.$$ 

Observe that

$$\psi(\epsilon_1) = \psi\left(\prod_{A \subseteq [q], \, 1 \in A} x_A\right) = \prod_{A \subseteq [q], \, 1 \in A} \psi(x_A) = \psi(x_{\{1, 4\}})\psi(x_{\{1, 2\}})\psi(x_{\{1\}}) = x_1x_2x_7x_5 = m_1.$$ 

Generalizing the properties of $\psi$ from the example, we arrive at the following lemma.

**Lemma 7.7.** Let $I$ be an ideal of the polynomial ring $S = k[x_1, \ldots, x_n]$, minimally generated by square-free monomials $m_1, \ldots, m_q$. Then

(i) $\psi(e^a) = m^a$ for each $a \in N_q^r$;

(ii) $\psi(E_q(S)) = \Gamma_q^r$ for every $r > 0$;

(iii) $\psi$ preserves least common multiples, that is:

$$\psi(\operatorname{lcm}(e^{a_1}, \ldots, e^{a_t})) = \operatorname{lcm}(m^{a_1}, \ldots, m^{a_t}) \quad \text{for all } a_1, \ldots, a_t \in N_q^r, \quad t \geq 1.$$ 

**Proof.** By Definitions 7.1 and 7.5 for every $j \in [q]$

$$\psi(e_j) = \psi\left(\prod_{A \subseteq [q], \, j \in A} x_A\right) = \prod_{A \subseteq [q], \, j \in A} \psi(x_A) = \prod_{k \in [n]} x_k = \prod_{k \in [n]} x_k = \prod_{k \in [n]} x_k = m_j.$$ 

It follows that $\psi(e^a) = m^a$ for all $a \in N_q^r$, which establishes (i) and also (ii). It remains to show (iii).
Set \( a_i = (a_{i1}, a_{i2}, \ldots, a_{iq}) \), for \( i \in [t] \). Using (7.4.1) in the first equality below, we have:

\[
\psi_I(\text{lcm}(e^{a_1}, \ldots, e^{a_t})) = \psi_I \left( \prod_{\emptyset \neq A \subseteq [q]} \left( \prod_{k \in [n]} x_k \right)_{j \in A_k} \right)
\]

\[
= \prod_{\emptyset \neq A \subseteq [q]} \left( \prod_{k \in [n]} x_k \right)_{j \in A_k}^{\max_j} \sum_{j \in A} a_{ij}
\]

\[
= \prod_{k \in [n], A_k \neq \emptyset} \left( x_k \right)_{j \in A_k}^{\max_j} \sum_{j \in A_k} a_{ij}
\]

\[
= \text{lcm} \left( \prod_{k \in [n], A_k \neq \emptyset} \left( x_k \right)_{j \in A_k}^{\max_j} \sum_{j \in A_k} a_{ij} \right)
\]

\[
= \text{lcm} \left( \prod_{j \in [q]} \prod_{k \in [n]} x_k^{a_{1j}}, \ldots, \prod_{j \in [q]} \prod_{k \in [n]} x_k^{a_{ij}} \right)
\]

\[
= \text{lcm} \left( \prod_{j \in [q]} m_j^{a_{1j}}, \ldots, \prod_{j \in [q]} m_j^{a_{ij}} \right)
\]

\[
= \text{lcm}(m^{a_1}, \ldots, m^{a_t}).
\]

\[ \square \]

Theorem 7.9 below demonstrates why the ideals from Definition 7.5 are called extremal: they have the greatest Betti numbers among all ideals minimally generated by \( q \) square-free monomials. The following lemma provides the technical preliminaries necessary for the proof of Theorem 7.9.

**Lemma 7.8.** Let \( I \) be an ideal minimally generated by \( q \) square-free monomials in a polynomial ring \( k[x_1, \ldots, x_n] \). With notation as in Definition 7.5 if \( S \) is viewed as an \( S_E \)-module via the ring homomorphism \( \psi_I : S_E \rightarrow S \), then

\[
S_E / \mathcal{E}_q^r \otimes_{S_E} S \cong S / I^r \quad \text{and} \quad \text{Tor}_i^{S_E}(S_E / \mathcal{E}_q^r, S) = 0 \quad \text{for all} \ i > 0.
\]

**Proof.** First, note that

\[
\frac{S_E}{\mathcal{E}_q^r} \otimes_{S_E} S \cong \frac{S}{(\mathcal{E}_q^r)S} = \frac{S}{\psi_I(\mathcal{E}_q^r)S} = \frac{S}{I^r}.
\]

To compute \( \text{Tor}_i^{S_E}(S_E / \mathcal{E}_q^r, S) \), use the Taylor complex \( \text{Taylor}(\mathcal{E}_q^r) \), which supports a free resolution \( F \) of \( S_E / \mathcal{E}_q^r \); see (2.4.1) for a description of the differentials of \( F \). Since the homomorphism \( \psi_I \) changes the labels \( e^a \) to the labels \( m^a \) and preserves least common multiples, the chain complex \( F \otimes_{S_E} S \) is isomorphic to a homogenization of the chain complex associated to the simplex with vertices corresponding to \( a \in N_q^r \) and labeled with the monomials \( m^a \). This is a version of the Taylor resolution of \( S/I^r \) defined starting with a possibly non-minimal generating set of \( I^r \). Such a non-minimal version is a free resolution of \( S/I^r \) as well, hence \( \text{Tor}_i^{S_E}(S_E / \mathcal{E}_q^r, S) = 0 \) when \( i > 0 \). (See Remark 2.6) \[ \square \]

**Theorem 7.9 (Powers of extremal ideals have maximal Betti numbers).** Given positive integers \( r \) and \( q \),

\[
\beta_i^S(I^r) \leq \beta_i^{S_E}(\mathcal{E}_q^r)
\]

for any \( i \geq 0 \) and any ideal \( I \) minimally generated by \( q \) square-free monomials in a polynomial ring \( S \).
Proposition 7.10

The following statements are equivalent:

1. The simplicial complex $L_q^r$ supports a minimal free resolution of the ideal $E_q^r$.
2. $\text{pd}_{S_E}(E_q^r) = \dim \mathbb{L}_{q}^r$ and the values of $r$ and $q$ do not satisfy the conditions in (3).

Proof. (1) $\implies$ (2): This implication is clear.

(2) $\implies$ (3): Assume that $\text{pd}_{S_E}(E_q^r) = \dim \mathbb{L}_{q}^r$ and the values of $r$ and $q$ do not satisfy the conditions in (3). In particular, we must have $q \geq 2$ and $r \geq 5$, or $q \geq 3$ and $r = 3, 4$. By Remark [6.4] and Equation (6.11), for all $i, j \in [q]$, $|F_{i}^r| = f > q = |G_{j}^r|$ and so $F_1^r, \ldots, F_q^r$ are the facets of $L_q^r$ of highest dimension, with the caveat that $F_q^r = \cdots = F_{q+1}^r$ when $r = 3$, $q$, and thus $\dim \mathbb{L}_{q}^r = f - 1$. Since we assumed $\text{pd}_{S_E}(E_q^r) = \dim \mathbb{L}_{q}^r$, we have $\text{pd}_{S_E}(E_q^r) = f - 1$ as well. Let $F$ denote the free resolution supported on $\mathbb{L}_{q}^r$, and let $\partial$ denote its differential, which is described in (2.4.1). Since $\text{pd}_{S_E}(E_q^r) = f - 1$ and a minimal free resolution of $E_q^r$ is isomorphic to a direct summand of $F$, there must exist a basis element $e \in F_{f-1} \setminus (S_E)_{\geq 1} F_{f-1}$ such that $\partial(e) \in \sigma$.

As in (2.4.1), let $e_{\sigma}$ denote the basis element in $F$ corresponding to $\sigma \in \mathbb{L}_{q}^r$. We write $e = \sum_{F \in \mathbb{L}_{q}^r, |F| = f} \alpha_F e_F$ with $\alpha_F \in S_E$. The assumption $e \notin \sigma$ implies that $\alpha_F$ is a unit for some $F \in \mathbb{L}_{q}^r$ with $|F| = f$. By Equation (7.10.1), we see that $F = F_i^r$ for some $i \in [q]$. Without loss of generality, assume $i = 1$.

Recall that $M_{\sigma}$ denotes the lcm of the monomial labels of the vertices in $\sigma \in \mathbb{L}_{q}^r$.

Claim. There exists $c \in F_1^r \setminus G_1^r$ such that $M_{F_1^r} = M_{F_1^r \setminus \{c\}}$.

Proof of Claim. Assume first $r > 4$, and $q \geq 2$, so that $s = \left\lceil \frac{r}{2} \right\rceil \geq 3$. Let $a, b, c \in \mathbb{N}_{q}^r$ be such that $e^a = \epsilon_1^{r-s} \epsilon_2^s$, $e^b = \epsilon_1^{r-s+2} \epsilon_2^{s-2}$, and $e^c = \epsilon_1^{r-s+4} \epsilon_2^{s-1}$, so that $a, b, c$ are distinct vertices of $L_q^r$. Note that $e^c | \text{lcm}(e^a, e^b)$. Indeed, after removing the common factors, this divisibility is equivalent to $\epsilon_1 \epsilon_2 | \text{lcm}(\epsilon_1^2, \epsilon_2^2)$, and can be verified using (7.4.2).
Now let \( r = 3 \) or \( r = 4 \), and \( q \geq 3 \). Then \( s = 2 \), and if one sets
\[
\epsilon^a = \epsilon_1 r^{-2} \epsilon_2^2, \quad \epsilon^b = \epsilon_1 r^{-1} \epsilon_3 \quad \text{and} \quad \epsilon^c = \epsilon_1 r^{-2} \epsilon_2 \epsilon_3,
\]
then we see that \( \epsilon^c \mid \text{lcm}(\epsilon^a, \epsilon^b) \). Indeed, after removing the common factors, this divisibility is equivalent to
\[
\epsilon_2 \epsilon_3 \mid \text{lcm}(\epsilon_2^2, \epsilon_1 \epsilon_3),
\]
and can be verified using (7.4.2).

In both cases, we have \( a, b, c \in F'_r \) and \( c \notin G'_r \). The divisibility \( \epsilon^c \mid \text{lcm}(\epsilon^a, \epsilon^b) \) establishes the conclusion of the Claim.

We finish the proof of (2) \( \implies \) (3) as follows. We have
\[
\partial(e) = \sum_{F \in \mathbb{L}_q^r, |F| = f} \alpha_F \partial(\epsilon_F) = \sum_{F \in \mathbb{L}_q^r, |F| = f} \alpha_F \left( \sum_{e' \notin F} \frac{M_F}{M_F \setminus \{e'\}} \epsilon_F \setminus \{e'\} \right).
\] (7.10.3)
Let \( c \) be as in the Claim and let \( F \in \mathbb{L}_q^r \) with \( |F| = f \). By Equation (7.10.1), we have \( F = F'_j \) for some \( j \). If \( c' \in F \), observe
\[
F'_j \setminus \{c\} = F \setminus \{c'\} \iff F'_j = F \text{ and } c = c'.
\] (7.10.4)
To prove this statement, we refer to Section 4 for basic properties of the sets \( F'_j, G'_j \), and \( B' \). Indeed, (7.10.4) is clear when \( r = 3 \), since \( F'_j = B' = F'_r \) for all \( j \) in \( [q] \) in this case. Assume now \( r > 3 \), and recall that \( q \geq 2 \). Assuming \( F'_j \setminus \{c\} = F'_j \setminus \{c'\} \) and \( j \neq 1 \), we use \( F'_j \cap F'_j = B' \), and conclude \( F'_j = B' \cup \{c\} \). Since \( B' \cap G'_j = \emptyset \) and \( c \notin G'_j \), this contradicts the fact that \( F'_j \cap G'_j \neq \emptyset \) when \( r > 3 \) and \( q \geq 2 \). Thus, (7.10.4) must hold.

In view of the Claim, (7.10.4) and (7.10.3), we see that the coefficient of \( e_{F'_j \setminus \{c\}} \) in \( \partial(e) \) is equal to \( \pm \alpha_{F'_j} \) hence it is a unit. This contradicts (7.10.2).

(3) \( \implies \) (1): Theorem 5.9 shows that \( \mathbb{L}_q^r \) supports a free resolution of \( \mathbb{E}_q^r \) for all \( r, q \geq 1 \). To show minimality, according to Theorem 3.1 it suffices to show that \( M_\sigma \neq M_{\sigma'} \) for any faces \( \sigma, \sigma' \in \mathbb{L}_q^r \) with \( \sigma \neq \sigma' \), or, in other words, that each monomial label appears only once.

If \( q = 1 \), then \( F^r = (m_1^r) \) for all \( r \), and all complexes in (7.3.1) are one point, so each supports a minimal resolution by default.

If \( q = 2 \), then \( \mathbb{E}_2^2 = (x_1 x_{12}, x_2 x_{12}), \mathbb{L}_2^2(\mathbb{E}_2), \mathbb{L}_2^3(\mathbb{E}_2), \) and \( \mathbb{L}_2^4(\mathbb{E}_2) \) are shown below. Observe that each monomial label appears once in each complex, hence \( \mathbb{L}_2^2(\mathbb{E}_2), \mathbb{L}_2^3(\mathbb{E}_2), \) and \( \mathbb{L}_2^4(\mathbb{E}_2) \) support a minimal free resolution of \( \mathbb{E}_2^2, \mathbb{E}_2^3 \), and \( \mathbb{E}_2^4 \), respectively.

Now assume \( q \geq 3 \) and \( 1 \leq r \leq 2 \). We will show that, for every \( c \in \mathbb{N}_q^r \) and \( \sigma \in \mathbb{L}_q^r \),
\[
\epsilon^c \mid M_\sigma \iff c \in \sigma.
\] (7.10.5)
When \( r = 1 \) observe that for every \( i \in [q] \),
\[
\epsilon_i \mid \text{lcm}(\epsilon_{k_1}, \ldots, \epsilon_{k_p}) \quad \text{for all} \quad i \in [q], \quad k_1, \ldots, k_p \in [q] \setminus \{i\}.
\] (7.10.6)
This can be seen by noting that the right-hand side of (7.4.2) becomes \( 1 \leq 0 \) for \( A = \{i\} \), \( \epsilon^e = \epsilon_i \) and \( \sigma = \{\epsilon_{k_1}, \ldots, \epsilon_{k_p}\} \). Therefore \( \mathbb{L}_q^1 = \mathbb{L}_q^1(\mathbb{E}_q) = \text{Taylor}(\mathbb{E}_q) \), and (7.10.5) follows immediately.
Now let \( r = 2, q \geq 3, \sigma \subseteq \mathbb{Z}_q^2 \) and \( c \subseteq \mathbb{Z}^2_q \), with \( e^c \mid M_\sigma \). Since \( r = 2 \), \( e^c = \epsilon_i \epsilon_j \) for some \( i, j \in [q] \). Pick \( A = \{i, j\} \) in the right-hand side of (7.4.2).

- If \( i = j \), then \( e^c = \epsilon_i^2 \) and \( \epsilon_A = \{i\} \). By (7.4.2) we must have \( c \subseteq \sigma \).
- If \( i \neq j \), then by (7.4.2) there exists \( b \in \sigma \) with \( e^b = \epsilon_i \epsilon_j, \epsilon_i^2 \) or \( \epsilon_j^2 \).

If \( e^b = \epsilon_i \epsilon_j \) then \( c = b \subseteq \sigma \), as desired. Suppose \( e^c \notin \sigma \), so without loss of generality \( e^b = \epsilon_i^2 \). As \( \sigma \subseteq \mathbb{Z}_q^2 \), we must have \( \sigma \subseteq \mathbb{Z}_q^2 \). So there exist \( k_1, \ldots, k_p \in [q] \setminus \{i, j\} \) such that
\[
M_\sigma = \text{lcm}(\epsilon_i^2, \epsilon_i \epsilon_{k_1}, \ldots, \epsilon_i \epsilon_{k_p}) = \epsilon_i \text{lcm}(\epsilon_i, \epsilon_{k_1}, \ldots, \epsilon_{k_p}).
\]

Since \( \epsilon_i \epsilon_j \mid M_\sigma \), it follows that \( \epsilon_j \mid \text{lcm}(\epsilon_i, \epsilon_{k_1}, \ldots, \epsilon_{k_p}) \), which contradicts (7.10.6). Thus \( c \subseteq \sigma \).

We have now proved the statement in (7.10.5), and by Theorem 6.1 we conclude that \( \mathbb{Z}_q^2 \) supports a minimal free resolution of \( E_q^r \) for every \( q \geq 3 \).

\[ \square \]

A direct consequence of our work in Proposition 7.10 is the following statement.

**Proposition 7.11 (When \( \mathbb{Z}_q^r \) supports a minimal resolution of some \( I^r \)).** If \( r \) and \( q \) are positive integers, then \( \mathbb{Z}_q^r \) supports a minimal free resolution of \( I^r \) for some ideal \( I \) minimally generated by \( q \) square-free monomials if and only if one of the following holds:

- \( q = 1 \) and \( r \geq 1 \);
- \( q = 2 \) and \( 1 \leq r \leq 4 \);
- \( q \geq 3 \) and \( 1 \leq r \leq 2 \).

**Proof.** For any square-free monomial ideal \( I \) with \( q \) minimal generators the Betti numbers of \( I^r \) are bounded above by the Betti numbers of \( E_q^r \) by Theorem 7.9. So the question is reduced to when \( \mathbb{Z}_q^r \subseteq \mathbb{Z}_q^2 \) supports a minimal free resolution of \( E_q^r \). The rest follows from Proposition 7.10. \[ \square \]

When \( r = 2 \), the fact that these bounds are sharp had been previously announced in [4]. The search for sharp(er) bounds when \( r > 2 \) is continued in [5].

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MANITOBA, 520 MACRAY HALL, 186 DYSART ROAD, WINNIPEG, MB, CANADA R3T 2N2

Email address: susan.cooper@umanitoba.ca

DEPARTMENT OF MATHEMATICS, AMERICAN UNIVERSITY OF BEIRUT, BLISS HALL 315, P.O. BOX 11-0236, BEIRUT 1107-2020, LEBANON

Email address: se24@aub.edu.lb

DEPARTMENT OF MATHEMATICS & STATISTICS, DALHOUSIE UNIVERSITY, 6316 COBURG RD., PO BOX 15000, HALIFAX, NS, CANADA B3H 4R2

Email address: faridi@dal.ca

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TORONTO, 40 ST. GEORGE STREET, ROOM 6290, TORONTO, ON, CANADA M5S 2E4

Email address: smt@math.toronto.edu

DEPARTMENT OF MATHEMATICS, TEXAS STATE UNIVERSITY, 601 UNIVERSITY DR., SAN MARCOS, TX 78666, E.S.A.

Email address: morey@txstate.edu

DIVISION OF COMPUTING, ANALYTICS AND MATHEMATICS, UNIVERSITY OF MISSOURI-KANSAS CITY, KANSAS CITY, MO 64110 U.S.A.

Email address: segal@umkc.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MISSISSIPPI, HUME HALL 335, P.O. BOX 1848, UNIVERSITY, MS 38677 USA

Email address: spiroff@olemiss.edu