Topological phase diagram of coupled spinless p-wave superconductors

A Maiellaro, F Romeo, R Citro
Dipartimento di Fisica, Università degli Studi di Salerno, via Giovanni Paolo II, Fisciano (Sa) 132-I-84084, IT
Spin-CNR, via Giovanni Paolo II, Fisciano (Sa) 132-I-84084, IT
E-mail: amaiellaro@unisa.it

Abstract. We investigate the topological properties of a ladder given by two one dimensional p-wave superconductors coupled together site to site by transversal hopping $t$. For small $t$, we also derive an effective model whose simpler form is feasible to implement some braiding procedures proposed in the literature.

1. Introduction
Majorana fermions (Mfs), hosted in superconductors [1], may be used as qubit of fault-tolerant topological quantum computers. A crucial challenge towards the topological quantum computer is to implement quantum operation of nearly degenerate quantum states by a dynamical process of Majorana fermions; the principal one is the braiding dynamic in superconducting nanowires [2]. Several proposals have been proposed to perform this operation, for example changing the hopping $t$ and pairing $\Delta$ on chains or between two chains of Kitaev can be used to perform the braiding [3]. Another possibility is crossing them [4] or pairing two wires in the topological phase with an extra wire in the trivial phase [5]. The braiding of Majoranas is another open problem of modern physics and several proposals have been made to this end, as a measured-based scheme for performing braiding operation and detecting the non-Abelian statistics of Mfs without moving or hybridizing them [6]. For example, one proposal is the measurement of conductance that is strictly connected to parity of the systems [5]. Another actual challenge, considered in some reviews, is the analysis of the non-adiabatic corrections in the braiding process of real systems [7].

Along this guide line, a crucial role is played by the study of the topological phases of the various systems that host such excitations and how these phases are modified by tuning the parameters which represent the direct experimental control of a possible future measurement of Majorana zero modes (MzMs). A contribution in this direction is given by the study of a ladder of two Kitaev chains [8].

Our aim in this work is to consider the most simple generalization of the Kitaev chain that exhibits Majorana zero modes. We consider two one dimensional p-wave superconductors that are coupled together site to site by an hopping term $t$. First of all, we study the topological phase of the system and we check the existence of Majorana zero modes and how they are modified by the transversal hopping, with respect to Mfs of the two separated chains.
We also propose an effective model for our system that leads itself to perform some braiding procedures proposed in literature [9].

2. Model

We consider a model in the continuum whose Hamiltonian describes two one dimensional p-wave superconductors coupled together by an hopping term of strength $t$:

$$ H = \frac{1}{2} \int \left[ \sum_{i=1,2} \hat{\Psi}_i^\dagger(x) \left( \frac{p^2}{2m} - \mu \right) \hat{\Psi}_i(x) + \Delta I \left( \hat{\Psi}_i^\dagger(x) \partial_x \hat{\Psi}_i(x) + \text{h.c.} \right) \right] - t \left( \hat{\Psi}_1^\dagger(x) \hat{\Psi}_2(x) + \hat{\Psi}_2^\dagger(x) \hat{\Psi}_1(x) \right) \right] \, dx $$

where $\hat{\Psi}_i$ are the continuum field operators that create and destroy fermions on chain $i$, $\mu$ is the chemical potential, $\Delta$ is the pairing strength of the p-wave superconductor and $l$ is the lattice constant in such a way $\Delta$ has energy dimension.

This Hamiltonian can be rewritten in the following matrix form:

$$ H = \frac{1}{2} \int dx \begin{pmatrix} \hat{\Psi}_1^\dagger(x) & \hat{\Psi}_1(x) & \hat{\Psi}_2^\dagger(x) & \hat{\Psi}_2(x) \end{pmatrix} \begin{pmatrix} \hat{\Psi}_1(x) \\ \hat{\Psi}_2(x) \\ \hat{\Psi}_1^\dagger(x) \\ \hat{\Psi}_2^\dagger(x) \end{pmatrix} H_{\text{BdG}} $$

where $H_{\text{BdG}}$ is the Bogoliubov de Gennes Hamiltonian, whose matrix representation is:

$$ H_{\text{BdG}} = \begin{pmatrix} -\frac{\hbar^2}{2m} \partial_x^2 - \mu & \Delta l \partial_x & -t & 0 \\ -\Delta l \partial_x & \frac{\hbar^2}{2m} \partial_x^2 + \mu & 0 & t \\ -t & 0 & -\frac{\hbar^2}{2m} \partial_x^2 - \mu & \Delta l \partial_x \\ 0 & t & -\Delta l \partial_x & \frac{\hbar^2}{2m} \partial_x^2 + \mu \end{pmatrix} $$

here the Nambu representation has been used [10] and we have assumed that the two wires have the same chemical potential and pairing term (this second choice is made to guarantee an analytical resolution of equations).

The Bogoliubov-de Gennes Hamiltonian has also the particle-hole symmetry that exchanges electrons with holes and is represented by an antiunitary operator $\rho$ that anti-commutates with $H_{\text{BdG}}$:

$$ \rho H_{\text{BdG}} \rho^{-1} = -H_{\text{BdG}} $$

Because of this symmetry, the spectrum of $H_{\text{BdG}}$ must be symmetric around zero energy; indeed, for every eigenvector $\psi$ with energy $E$ there will be a particle-hole symmetric eigenvector $\rho \psi$ with energy $-E$. In our basis this operator is given by:

$$ \rho = (I \otimes \sigma_x)K $$

where $K$ is the complex conjugation and $\sigma_x$ is the Pauli matrix.

3. Topological phases: Analytical resolution

To study the topological phase of the system, we check when this model admits zero energy modes. So, we must solve the BdG equation for zero energy modes: $H_{\text{BdG}} \psi = 0$, where we call $\psi = \sum_{i=1,2} |i \rangle \otimes (u_i \ v_i) \, T$ the topological spinor, $u_1$, $v_1$, $u_2$, $v_2$ are the BdG wavefunctions [11] and $|1 \rangle = (1 \ 0)^T$, $|2 \rangle = (0 \ 1)^T$. The substitution of Eq.(2) into the BdG equation for zero modes gives the following matrix equation:

$$ \partial_x^2 \psi + \frac{2m \mu}{\hbar^2} \psi - \frac{2ml}{\hbar^2} \partial_x [\Delta (I \otimes \sigma_x) \psi] + \frac{2mt}{\hbar^2} (\sigma_x \otimes I) \psi = 0 $$

(4)
we first expand the spinor in the basis given by the eigenstates of \( \sigma_x \otimes \mathcal{I} \) \( (\psi_1, \psi_2, \psi_3, \psi_4) \):
\[
\psi = a\psi_1 + b\psi_2 + c\psi_3 + d\psi_4.
\]
With this choice of basis we decouple the equations for the ladder into equations for two single wires with a shift in the chemical potential. In fact, the substitution of Eq.(5) into Eq.(4) gives:
\[
\begin{align*}
\frac{\partial^2}{\partial x^2} & \begin{pmatrix} a \\ b \end{pmatrix} + k_1^2 \begin{pmatrix} a \\ b \end{pmatrix} - \frac{1}{\xi} \sigma_x \begin{pmatrix} a \\ b \end{pmatrix} = 0 \\
\frac{\partial^2}{\partial x^2} & \begin{pmatrix} c \\ d \end{pmatrix} + k_2^2 \begin{pmatrix} c \\ d \end{pmatrix} - \frac{1}{\xi} \sigma_x \begin{pmatrix} c \\ d \end{pmatrix} = 0
\end{align*}
\]
where \( k_1^2 = \frac{2m(\mu+t)}{\hbar^2} \), \( k_2^2 = \frac{2m(\mu-t)}{\hbar^2} \), \( \frac{1}{\xi} = \frac{\Delta}{\hbar^2} \) and \( \xi \) has the dimensions of a length.
We remark that Eq.(6) are exactly the equations of two single p-wave wires in which we have introduced an effective chemical potential: \( \mu_{\text{eff}} = \mu + t \), for the first of Eq.(6) and \( \mu_{\text{eff}}^2 = \mu - t \), for the second of Eq.(6).
To solve Eq.(6) and compute the topological spinor we expand the vectors: \( (\begin{pmatrix} a \\ b \end{pmatrix} \end{pmatrix}^T, (\begin{pmatrix} c \\ d \end{pmatrix} \end{pmatrix}^T \)
in the basis of the eigenstates of \( \sigma_x \) in such a way we obtain two ordinary linear differential equations of the second order with constant coefficients. We get four independent solutions:
\[
\phi_{1,1} = N_1[e^{i\alpha\sigma_x} - e^{i(\alpha\sigma_x - \alpha\sigma_z)}]L \left[ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right], \quad \phi_{1,2} = N_2[e^{i\theta\sigma_x} - e^{i(\theta\sigma_x - \theta\sigma_z)}]L \left[ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right]
\]
\[
\phi_{2,1} = N_3[e^{i\alpha\sigma_x} - e^{i\alpha\sigma_z}]L \left[ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} i \\ -i \end{pmatrix} \right], \quad \phi_{2,2} = N_4[e^{i\theta\sigma_x} - e^{i\theta\sigma_z}]L \left[ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} i \\ -i \end{pmatrix} \right]
\]
where \( N_1, N_2, N_3, N_4 \) are the normalization coefficients and their analytical expression is present in [12] and we don’t report it here because it is of interest only for the numerical calculation, \( \alpha_j = \frac{j\pm\sqrt{1-4(k_j\xi)^2}}{2k_j}, \beta_j = \frac{j\pm\sqrt{1-4(k_j\xi)^2}}{2k_j}, j = \pm 1 \). These solutions are obtained imposing the boundary conditions, the particle-hole symmetry and the normalization:
\[
\phi_{1,1}(L) = \phi_{1,2}(L) = 0, \quad \phi_{2,1}(0) = \phi_{2,2}(0) = 0, \quad \rho \phi_{i,j} = \phi_{i,j}; \quad \int_0^L |\phi_{i,j}|^2 \, dx = 1
\]
where \( i,j = 1,2 \).
The modes in Eq. (7) are localized at the right-hand side of the ladder and those in Eq. (8) are localized at the left-hand side of the ladder. This distinction between four independent modes is made because, as we have already observed, \( \alpha \) and \( \beta \) are the solutions of two indipendent wires with a shift in the chemical potential and because we have imposed that they are real in the topological phase requirement\(^1\).
A simple remark is needed: these four modes are not the modes of the two single wires of the original Hamiltonian but are a linear combination of them.

4. Numerical results
We study the topological phase diagram of the model by tuning the transversal hopping \( t \).
As is depicted in Fig.(1) and Fig.(2), when \( t \) increases we have a phase transition between two topological phases, one with four modes and one with two modes. This is because the stronger the transversal hopping, the greater the coupling between the two wires of the ladder and so it appears as a single wire with an internal structure and therefore exhibits only two topological modes.
\(^1\) This last requirement implies that \( k_1\xi < \frac{1}{2}, k_2\xi < \frac{1}{2} \).
Topological phase with two modes

Figure 1. Topological phase diagram in the $\mu-t$ plane in unit of $\Delta$ with the fixed parameter $L = 75$. In the region of interest we can distinguish between two regions: a topological phase with four modes and a topological region with two modes.

Figure 2. The square modulus of the four modes at increasing $t$. This plots are realized with: $\xi = 1$, $\frac{L}{\xi} = 75$ and $k_F\xi = \frac{\sqrt{2m\mu}}{\hbar} = 0.45$ and for $\frac{t}{\mu} = 0$, 0.3, 0.55, 0.9 respectively for the panels (a), (b), (c), (d). In the panel (a) the four modes are overlapped, in the other panels two of the four modes are delocalizing. In the panel (d) two modes are recombining themselves and the model exhibits only two topological modes.

5. Topological effective Hamiltonian

For small $t$, we can consider the projection of Hamiltonian of Eq.(2) onto the Hilbert subspace identified by the four localized eigenstates of two separated wires. This new Hamiltonian will represent an effective Hamiltonian for the system; in fact, in the limit in which the system remains in the zero-degenerate variety the mapping $H \rightarrow H_{eff}$ is a change of basis. The utility of this Hamiltonian is, as we will see, that in its regime of validity it has a simpler expression than Eq.(2) and exhibits, for a long ladder, four "Majorana zero energy modes" (MzMs) and then it can be used to perform braiding procedures [9].

Introducing the projection operator onto our new basis $\Pi = \sum_{i=1}^{4} |i\rangle \langle i|$, we get:

$$H_{eff} = \Pi H_{BdG} \Pi = \sum_{i,j} H_{i,j} |i\rangle \langle j|$$

for $i, j = 1, \ldots, 4$ and where $H_{i,j} = \langle i|H_{BdG}|i\rangle$ are the Hamiltonian matrix elements in the new basis. The states $|1\rangle$, $|2\rangle$, $|3\rangle$, $|4\rangle$ are the MzMs of two separated wires and are well-defined in literature [12]. Using the expression of the four modes of the two separated wires we numerically calculate the matrix elements of Eq.(9) and we get (from now on we express the parameters of
Hamiltonian in unit of $\Delta$ and $l$):

$$H_{\text{eff}} = \begin{pmatrix} 0 & -ib & 0 & -ict \\ ib & 0 & ict & 0 \\ 0 & -ict & 0 & -ib \\ ict & 0 & ib & 0 \end{pmatrix}$$

where we have explicited the dependence on the transversal hopping $t$ and included the other dependencies inside the coefficients $b = b(L, \xi, \mu)$ and $c = c(L, \xi, \mu)$. The analytical expressions of $c$ and $d$ are present in [12]. Numerical diagonalization of Eq.(10) gives us the eigenvalues and eigenvectors.

As is depicted in Fig.(3), we have a pair of values of $\mu$ and $t$ for which two modes have exactly zero energy, in particular for $\mu = 0$, the effect of transverse hopping is that of removing two modes from zero while when $\mu$ increases there is an exact value of $t$ for which two modes go to zero.

Indeed, in Fig.(3) we can see that all the four modes have a very small energy, in such a way we can approximately interpret all modes are Majorana-like modes. The regime of validity of $H_{\text{eff}}$ is along the vertical cuts for small $t$ in the phase diagram of Fig.(1). In this limit, we can consider the linear combination of the eigenstates of $H_{\text{eff}}$ in Eq.(11) that are quasi-MzMs. Fig.(4) exhibits the plots of the square modulus of this quasi-MzMs. The particle-hole operator in this new basis: $\rho = IK$ confirms the Majorana character of this modes.

$$|\psi_1 \rangle + |\psi_2 \rangle = \frac{1}{\sqrt{2}} \left[ \left( \begin{array}{c} 1 \\ 1 \end{array} \right) \otimes \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \right], \quad |\psi_1 \rangle - |\psi_2 \rangle = \frac{1}{\sqrt{2}} \left[ \left( \begin{array}{c} 1 \\ 1 \end{array} \right) \otimes \left( \begin{array}{c} -i \\ 0 \end{array} \right) \right],$$

$$|\psi_3 \rangle + |\psi_4 \rangle = \frac{1}{\sqrt{2}} \left[ \left( \begin{array}{c} 1 \\ -1 \end{array} \right) \otimes \left( \begin{array}{c} 0 \\ -1 \end{array} \right) \right], \quad |\psi_3 \rangle - |\psi_4 \rangle = \frac{1}{\sqrt{2}} \left[ \left( \begin{array}{c} 1 \\ -1 \end{array} \right) \otimes \left( \begin{array}{c} -i \\ 0 \end{array} \right) \right]$$

where $|\psi_1 \rangle$, $|\psi_2 \rangle$, $|\psi_3 \rangle$, $|\psi_4 \rangle$ are the eigenvectors of Eq.(10).
Figure 4. Square modulus of the linear combination of eigenstates of $H_{\text{eff}}$. We have that: $|\psi_1 + \psi_2|^2$ is the blue mode, $|\psi_1 - \psi_2|^2$ is the red mode, $|\psi_3 + \psi_4|^2$ is the green mode and $|\psi_3 - \psi_4|^2$ is the black mode. The plots are realized in units of $\Delta$, for fixed $\xi = 10$, $\frac{t}{\Gamma} = 750$.

6. Conclusions
In this work, we have presented a generalization of the well known topological model of a one dimensional p-wave superconductor to the case of two p-wave superconductors coupled together site to site via transversal hopping. We have found that when $t$ increases the system exhibits two topological phases one with four zero energy modes and one with two zero energy modes as the effect of a stronger coupling. This is a check of the robustness of topological phase against possible perturbations lead by the variation of parameters. We have also derived an effective model that exhibits four Majorana zero energy modes. In the limit of small $t$, this effective model represents a good candidate to perform some braiding procedures proposed in literature [9].

7. Acknowledgements
R.C. acknowledges the project Quantox (Grant Agreement N. 731473) of Horizon 2020 Programme.

8. References
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