Parameterizing the Permanent: Hardness for $K_8$-minor-free graphs

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Abstract

In the 1960s, statistical physicists discovered a fascinating algorithm for counting perfect matchings in planar graphs. Valiant later showed that the same problem is $\#P$-hard for general graphs. Since then, the algorithm for planar graphs was extended to bounded-genus graphs, to graphs excluding $K_{3,3}$ or $K_5$, and more generally, to any graph class excluding a fixed minor $H$ that can be drawn in the plane with a single crossing. This stirred up hopes that counting perfect matchings might be polynomial-time solvable for graph classes excluding any fixed minor $H$. Alas, in this paper, we show $\#P$-hardness for $K_8$-minor-free graphs by a simple and self-contained argument.

1 Introduction

A perfect matching in a graph $G$ is an edge-subset $M \subseteq E(G)$ such that every vertex of $G$ has exactly one incident edge in $M$. Counting perfect matchings is a central and very well-studied problem in counting complexity. It already starred in Valiant’s seminal paper [23] that introduced the complexity class $\#P$, where it was shown that counting perfect matchings is $\#P$-complete. The problem has driven progress in approximate counting and underlies the so-called holographic algorithms [24, 6, 4, 5]. It also occurs outside of counting complexity, e.g., in statistical physics, via the partition function of the dimer model [21, 16, 17]. In algebraic complexity theory, the matrix permanent is a very well-studied algebraic variant of the problem of counting perfect matchings [1]. Evaluating permanents is equivalent to counting perfect matchings in bipartite graphs: Given a bipartite input graph $G$ on $n + n$ vertices with its $n \times n$ bi-adjacency matrix $A$, the permanent $\text{per}(A)$ counts exactly the perfect matchings in $G$.

Algorithms for restricted graph classes

A long line of research, dating back to the 1960s, identified structural restrictions on $G$ that facilitate the problem of counting perfect matchings. Statistical physics gave the first examples of useful restrictions: On the graphs $G$ of regular lattices, it turns out that perfect matchings are counted by the determinants of highly structured matrices whose eigenvalues can be derived explicitly [21, 16]. This approach was later generalized to arbitrary planar graphs $G$: More precisely, and very surprisingly, it is possible to flip the signs of some entries in the adjacency matrix of a planar graph $G$ to obtain a matrix $A$ such that $\sqrt{\det(A)}$ counts the perfect matchings in $G$ [17]. The entries to be flipped are determined by a so-called Pfaffian orientation of $G$, which can be computed in linear time for planar graphs. Overall, a polynomial-time algorithm for counting perfect matchings in planar graphs follows, the so-called FKT method (an acronym for the names of their inventors Fisher, Kasteleyn, and Temperley).

Little [18] and Vazirani [26] later generalized the FKT method from planar graphs to the more general class of graphs excluding $K_{3,3}$ as a minor. Such graphs can be obtained inductively by “gluing together” planar

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graphs and $K_5$. Little showed that $K_{3,3}$-free graphs still admit a Pfaffian orientation by combining Pfaffian orientations of the individual parts, and Vazirani later obtained a polynomial-time and poly-logarithmic space algorithm for finding such an orientation.

Still working with Pfaffian orientations, it was shown by Galluccio and Loebl [13] and Tesler [22] that perfect matchings can be counted in time $4^g n^{O(1)}$ for graphs $G$ that are embedded on a surface of genus $g$. In other words, the problem is fixed-parameter tractable in the parameter $g$. These algorithms use Pfaffian orientations to express the number of perfect matchings in $G$ as a linear combination of $4^g$ determinants. A simplified algorithm by the authors [11] bypasses the explicit use of Pfaffian orientations and instead reduces in a black-box manner to $4^g$ instances of counting perfect matchings in planar graphs.

The mold of Pfaffian orientations was broken by Straub, Thierauf and Wagner [20], and Curticapean [7], who independently designed polynomial-time algorithms for counting perfect matchings in graphs excluding a $K_5$-minor. As such graphs do not necessarily admit Pfaffian orientations, a different algorithmic approach was needed: In hindsight, the new algorithms transferred the protrusion replacement technique from parameterized complexity [2] into the counting setting.

Moreover, a standard dynamic programming approach shows that counting perfect matchings is fixed-parameter tractable in graphs of bounded tree-width $t$. This can be improved to $2^t n^{O(1)}$ time [25], where the base 2 is optimal under the strong exponential-time hypothesis [10].

Towards excluding general fixed minors

We observe that any tractable graph class listed above excludes some fixed minor $H$. That is, starting from a graph $G$ in the class, it is not possible to obtain $H$ by deleting edges/vertices and contracting edges. For example, planar graphs exclude $K_{3,3}$ and $K_5$, bounded-genus graphs exclude sufficiently large complete graphs, and bounded-treewidth graphs even exclude large grids. It is therefore natural to ask whether we can count perfect matchings in any graph class excluding fixed minors.

On the positive side, Curticapean [7] and Eppstein and Vazirani [13] lifted the algorithms for $K_{3,3}$-minor-free and $K_5$-minor-free graphs to $H$-minor-free graphs for any graph $H$ that can be drawn in the plane with a single crossing, for example, $H = K_{3,3}$ and $H = K_5$. (Note that excluding single-crossing minors $H$ yields different graph classes than the class of single-crossing graphs themselves, for which an immediate reduction to the FKT method is possible.) In fact, these algorithms run in fixed-parameter tractable time $f(k) n^{O(1)}$ for some function $f$ depending only on the size $k$ of an excluded single-crossing minor. Note that the exponent of $n$ does not grow with $k$.

On the negative side, parameterized complexity rules out such fixed-parameter tractable algorithms for counting perfect matchings in $k$-apex graphs [11]; these are the graphs that are planar up to deleting $k$ vertices. More precisely, it was previously shown by the authors that counting perfect matchings is $\#W[1]$-hard on $k$-apex graphs, suggesting that algorithms for this problem require $n^{\omega(k)}$ time for $g \in \omega(1)$. As $k$-apex graphs exclude $K_{k+5}$-minors, algorithms for counting perfect matchings in $K_t$-free graphs in turn require $n^{\omega(t)}$ time for $g \in \omega(1)$. It could however still be possible that, for any fixed $t \in \mathbb{N}$, some polynomial-time algorithm counts perfect matchings in $K_t$-free graphs, with an exponent depending on $t$.

In fact, such algorithms seemed within close reach: Robertson and Seymour’s graph structure theorem [19] shows that graphs $G$ excluding a fixed $H$-minor can be obtained as clique-sums of particular graphs that are near-embeddable on surfaces of fixed genus. Very roughly speaking, this means that $G$ is glued together from certain pieces, similar to graphs excluding $K_{3,3}$ or $K_5$ but with some “upgrades”. These upgrades involve raising the genus in the decomposition pieces (from 0 to some constant depending on $H$), adding a constant number of apex vertices (vertices that can connect arbitrarily to the remainder of the piece), and adding a constant number of vortices (graphs of bounded path-width that are aligned with the boundary of a face).

Counting perfect matchings supports most of these upgrades: The problem is fixed-parameter tractable in bounded-genus graphs, while apex vertices can be handled by brute-force in $n^{O(1)}$ time, and the gluing operation can be handled as in the simpler case of excluded single-crossing minors. Therefore, it only remains to handle vortices. However, even a minimal example of vortices was unresolved: We say that a ring blowup is a graph obtained from a drawing of a planar graph by cloning each vertex on the outer face into two copies, as shown in the left part of Figure 1. In the terminology of the graph structure theorem, ring blowups are
certain planar graphs with a single vortex. Progress towards polynomial-time algorithms for counting perfect matchings in \( H \)-minor-free graphs was halted because such algorithms were not even known for ring blowups.

Our results

We show that counting perfect matchings remains \#P-hard in ring blowups. Known general results in graph minor theory \[15\] then already imply the existence of a graph \( H \) such that counting perfect matchings is \#P-hard in \( H \)-minor-free graphs. The particularly simple structure of ring blowups however allows us to exclude \( K_8 \)-minors from them in a self-contained way. (Note that ring blowups can contain \( K_7 \)-minors, as Figure 5 shows.) We then obtain our main theorem:

**Theorem 1.** Counting perfect matchings is \#P-hard in graphs excluding \( K_8 \)-minors.

A weaker version of this theorem was previously announced in a survey on parameterized counting \[9\]. To prove Theorem 1 we reduce from the \#P-hard problem of counting perfect matchings in a graph \( G \). Our reduction hinges upon a particular sign-crossing gadget (see Figure 2) that can be used to remove crossings at the cost of disrupting the perfect matching count: After inserting a sign-crossing gadget between crossing edges \( e, f \in E(G) \), perfect matchings that contain both \( e \) and \( f \) are counted with a factor \(-1\), and only perfect matchings containing at most one of \( e \) or \( f \) are counted properly. Such sign-crossing gadgets were previously used in the theory of matchgates \[3\] and in the hardness proof for counting perfect matchings in \( k \)-apex graphs \[11\]. In our proof, sign-crossings are used to transform \( G \) into a ring blowup while preserving the perfect matching count. By a surprisingly simple construction, we can ensure that sign-crossings come in equivalent pairs, so that any \(-1\) factors introduced by sign-crossings cancel via \((-1)^2 = 1\).

2 Preliminaries

To give a self-contained proof of Theorem 1 we first state some preliminaries from counting complexity and graph minor theory. Graphs will be undirected and may be edge-weighted; we implicitly consider \( w : E(G) \to \mathbb{Q} \) to be the weight function. Given a vertex \( v \in V(G) \), we write \( I(v) \) for the set of edges incident with \( v \). Furthermore, given a set \( S \subseteq V(G) \), we write \( G[S] \) for the subgraph of \( G \) induced by \( S \).

2.1 Counting perfect matchings and gadgets

We define counting problems as functions \#A : \{0,1\} \* \to \mathbb{Q} \), where inputs (graphs, formulas, numbers) are implicitly encoded as bitstrings. For example, \#SAT asks to count the satisfying assignments to Boolean formulas. Likewise, when given as input a graph with edge-weights from a constant-sized \([\mathbb{Q}] \) set \( W \subseteq \mathbb{Q} \), the problem \#PerfMatch asks to determine the quantity

\[
\#\text{PerfMatch}(G) = \sum_{M \subseteq E(G) \text{ is a perfect matching}} \prod_{e \in M} w(e).
\]

We say that \#A admits a polynomial-time Turing reduction to \#B if \#A can be solved in polynomial time with an oracle for \#B, and we say that \#B is \#P-hard if \#SAT admits such a reduction to \#B. Our hardness proofs are based on the following theorem:

\[1\] The assumption of \( |W| = O(1) \) ensures that \#PerfMatch(G) can be represented with polynomially many bits.
Theorem 2 ([23, 12]). Counting perfect matchings is \#P-hard, even for unweighted 3-regular graphs.

The problem of counting perfect matchings behaves well under insertion of gadgets; in the context of this problem, gadgets are called **matchgates**. For example, given a drawing of a not necessarily planar graph $G$ with a crossing involving edges $e, f \in E(G)$, we can replace the crossing by the planar **sign-crossing gadget** from Cai and Gorenstein [3], as shown in Figure 2a. The resulting graph essentially counts perfect matchings in $G$, but with a significant twist: Any perfect matching $M \subseteq E(G)$ containing both $e$ and $f$ is weighted by an additional factor of $-1$. In other words, every perfect matching $M$ incurs a factor

$$\chi_{e,f}(M) := \begin{cases} -1 & \{e, f\} \subseteq M, \\ 1 & \text{otherwise}. \end{cases} \quad (2)$$

Note that a “perfectly planarizing” crossing gadget that does not introduce negative signs would render counting perfect matchings polynomial-time solvable by reduction to the FKT method. It can even be shown unconditionally that no such gadget exists [3].

The claimed functionality of the sign-crossing gadget follows from standard techniques in the area of so-called Holant problems, see [24, 3, 8]. In the following, we give a self-contained proof.

**Lemma 3.** Let $G$ be a weighted graph that is drawn in the plane with crossing edges $e, f \in E(G)$ of weight 1, and let $G'$ be obtained by inserting a sign-crossing gadget as in Figure 2a. Then, with $\chi_{e,f}$ as in (2), we have

$$\#\text{PerfMatch}(G') = \sum_{M \subseteq E(G) \text{ is a perfect matching}} \chi_{e,f}(M) \prod_{e \in M} w(e).$$

**Proof.** Let $X = \{e_1, e_2, f_1, f_2\}$ be the external edges of the sign-crossing gadget $S$ inserted at $e, f$. Let $G'' = G - V(S)$ be the graph obtained from $G$ by removing all gadget vertices. For a subset $T \subseteq X$ of external edges, write $V(T)$ for the endpoints of edges in $T$. We observe that any perfect matching $M \subseteq E(G'')$ can be obtained by first choosing a subset $T \subseteq X$ and afterwards extending $T$ by *independently* choosing perfect matchings in $S - V(T)$ and $G'' - V(T)$. It follows that

$$\#\text{PerfMatch}(G') = \sum_{T \subseteq X} \left( \prod_{e \in T} w(e) \right) \cdot \#\text{PerfMatch}(S - V(T)) \cdot \#\text{PerfMatch}(G'' - V(T)). \quad (3)$$

We observe that any set $T \subseteq X$ with $f(T) \neq 0$ has even cardinality, as $S - V(T)$ would otherwise have an odd number of vertices, and hence, no perfect matchings. The values of $f(T)$ for sets $T \subseteq X$ of even
cardinality are calculated in Figure 1b. Each column shows such a set \( T \) in the top row and lists the perfect matchings of \( S - V(T) \) below. The value \#\text{PerfMatch}(S - V(T)) \) is then obtained in the bottom row as the \( \pm 1 \)-weighted count of the listed perfect matchings.

The calculations from Figure 1b show that any set \( T \subseteq X \) with \( f(T) \neq 0 \) is consistent in the sense that it includes none/both of \( \{e_1, e_2\} \) and none/both of \( \{f_1, f_2\} \). Given such a consistent set \( T \), define \( \tilde{T} \subseteq \{e, f\} \) by forgetting the subscripts in \( T \): Include \( e \) into \( \tilde{T} \) iff \( \{e_1, e_2\} \subseteq T \), and likewise for \( f \). The term in (3) corresponding to \( \tilde{T} \) counts precisely those perfect matchings \( M \) in \( G \) with \( M \cap \{e, f\} = \tilde{T} \), except that \( f(T) \) introduces a factor of \(-1\) if \( T = \{e, f\} \). This proves the lemma.

If several crossings are replaced by sign-crossing gadgets, an inductive application of Lemma 3 shows that the factors introduced by sign-crossing gadgets multiply. A particularly interesting situation occurs when edges are drawn as curves rather than straight lines, as two edges \( e \) and \( f \) may then cross more than once. Our corollary supports this case, which will prove very useful in the next section.

**Corollary 4.** Let \( G \) be an unweighted graph that is drawn in the plane. Choose \( t \in \mathbb{N} \) crossings and write \( e_i, f_i \in E(G) \) for the edges involved in the \( i \)-th crossing. Let \( G' \) be obtained by inserting a sign-crossing gadget at each of the \( t \) chosen crossings. Then we have

\[
\#\text{PerfMatch}(G') = \sum_{M \subset E(G) \text{is a perfect matching}} \prod_{i=1}^{t} \chi_{e_i, f_i}(M).
\]

### 2.2 Graph minor theory

A graph \( H \) is a minor of \( G \), written \( H \preceq G \), if \( H \) can be obtained by repeated edge deletions and contractions and vertex deletions. This is equivalent to the existence of a minor model of \( H \) in \( G \):

**Definition 5.** A minor model of \( H \) in \( G \) is a collection of pairwise disjoint branch sets \( S_v \subseteq V(G) \) for \( v \in V(H) \) such that (i) each set \( S_v \) for \( v \in V(H) \) induces a connected subgraph of \( G \), and (ii) for every edge \( uv \in E(H) \), some edge of \( G \) runs between \( S_u \) and \( S_v \).

For example, the colored sets in Figure 1 show minor models of \( K_7 \) in ring blowups. The Hadwiger number \( \eta(G) \) of a graph \( G \) is the maximum \( k \in \mathbb{N} \) with \( K_k \preceq G \).

A plane graph is a planar graph that is given together with a concrete planar embedding. We define a particular graph class from plane graphs by “blowing up” their outer faces, see Figure 2.

**Definition 6.** Given a plane graph \( \hat{Q} \) with outer face \( O \), the blowup of \( \hat{Q} \) is the graph obtained by successively replacing each vertex \( v \in O \) by two clones \( v^1, v^2 \) having the same neighborhood as \( v \), and then adding the edge \( v^1v^2 \). We call \( v^1 \) and \( v^2 \) blowup vertices. A ring blowup is any subgraph of the blowup \( Q \) of a plane graph \( \hat{Q} \). We also call \( Q \) a reduct of \( Q \).

The notion of clique-sums will feature prominently in Section 4.

**Definition 7.** Let \( G \) and \( G' \) be two graphs that share a common vertex-set \( S = V(G) \cap V(G') \), which is a clique in \( G \) and \( G' \). A clique-sum \( G \oplus_S G' \) is any graph that can be obtained from the union \( G \cup G' \) by deleting some edges with both endpoints in \( S \).

A standard separator argument bounds the Hadwiger number of clique-sums by that of their constituents, see [13] Lemma 2.1 for a proof.

**Fact 8.** For any graphs \( G, G' \) and any clique-sum \( G \oplus_S G' \) for \( S = V(G) \cap V(G') \), we have \( \eta(G \oplus_S G') \leq \max\{\eta(G), \eta(G')\} \).

For a proof sketch, note that no fixed minor model of \( K_t \) can place some branch sets entirely within \( V(G) \setminus S \) and others entirely within \( V(G') \setminus S \).

It follows that all branch sets intersect \( V(G) \) or all intersect \( V(G') \). In the first case, we can delete all vertices from \( V(G') \setminus S \) without losing edges between branch sets, since \( S \) is a clique. The second case is symmetric.

![Figure 3: A clique-sum.](image)
3 Reducing to ring blowups

In this section, we show how to transform any unweighted graph $G$ into a ring blowup while preserving the value of $\#\text{PerfMatch}$. The main idea, spelled out in Lemma 9 and illustrated in Figure 4, is to arrange the vertices of $G$ on a circle and then bend the edges of $G$ to push crossings across the perimeter of the circle, where blowups will take care of them. As an edge $e$ is bent, it will introduce new crossings with other edges $g$, but our construction ensures that any edge $g$ crossed while bending $e$ is crossed exactly twice. When we then introduce sign-crossing gadgets at these crossings, any $-1$ factors from gadgets are guaranteed to come in pairs, so the overall product of these factors is 1. Hence, going from $G$ to $G'$, the value of $\#\text{PerfMatch}$ is preserved via Corollary 4. In Lemma 10, we then use a standard reduction in counting complexity to remove the $-1$ weights introduced into $G'$ by sign-crossings.

**Lemma 9.** Let $G$ be an unweighted graph with $n$ vertices and $m$ edges. In polynomial time, we can construct a ring blowup $G'$ on $O(n + m^3)$ vertices and edge-weights $\pm 1$ such that $\#\text{PerfMatch}(G) = \#\text{PerfMatch}(G')$ holds and all edges incident with blowup vertices of $G'$ have weight 1.

**Proof.** As shown in Figure 4a, we first place $V(G)$ on a circle $C$ in the plane and draw the edges of $G$ as straight lines inside of $C$. The placement is chosen such that no three edges intersect in the same point and every ray from the center to the perimeter of $C$ contains at most one point that is a crossing or vertex of $G$. Both conditions can be ensured by perturbing an arbitrary placement of $V(G)$ on $C$.

The circle divides the plane into two regions; we call the induced subgraphs of $G$ contained in these regions (both including $C$) the outer and inner part of $G$. Initially, the outer part only contains $C$.

Let $s \in O(m^2)$ be the number of crossings in our drawing of $G$ and let $P_1, \ldots, P_s \in \mathbb{R}^2$ be their locations. For each $i \in [s]$, shoot a ray from the center of $C$ to $P_i$ and write $\ell_i$ for the segment of this ray from $P_i$ to the perimeter of $C$. Note that distinct rays are disjoint and contain no vertices of $G$. For $i \in [s]$ in sequence, write $e_i, f_i \in E(G)$ for the edges involved in crossing $P_i$, write $m_i$ for the number of edges crossed by segment $\ell_i$ and enumerate the crossed edges as $g_{i,1}, \ldots, g_{i,m_i} \in E(G)$. We bend $e_i$ and $f_i$ in a sufficiently narrow neighborhood of $\ell_i$ to cross $C$, as shown in the middle part of Figure 4b. For any $j \in [m_i]$, this process adds two crossings between $e_i, g_{i,j}$ and two crossings between $f_i, g_{i,j}$.

After all original crossings $P_1, \ldots, P_s$ are processed, we insert a sign-crossing gadget at each crossing in the inner part of $G$, as shown in the right part of Figure 4b. Note that no sign-crossing gadgets are inserted at crossings in the outer part. To simplify the subsequent argument, we drag some vertices of the sign-crossing
gadgets onto $C$, as shown in Figure 4b. Overall, we obtain a new graph $G'$ with edge-weights 1 and −1, and we obtain from Corollary 4 that

$$\#\text{PerfMatch}(G') = \sum_{M} \prod_{i=1}^{s} \prod_{j=1}^{m_i} \chi^2_{e_i,g_{i,j}}(M) \cdot \chi^2_{f_j,g_{i,j}}(M) = \#\text{PerfMatch}(G). \quad (4)$$

Taking inventory, while going from $G$ to $G'$, we replaced all crossings from the inner part with planar gadgets and added no new crossings to the inner part, as different segments $\ell_i, \ell_j$ are disjoint. Via sign-crossings, we added $O(\sum_i m_i) = O(sm) = O(m^3)$ vertices into the inner part, where we recall that $m_i$ is the number of crossings between segment $\ell_i$ and the edges in $G$. Each crossing is contained in the outer part and involves edges $v_1v_3$ and $v_2v_4$ for some consecutive block of vertices $v_1, \ldots, v_4$ on the circle $C$. The vertex blocks of different crossings are disjoint, so we can define a reduct of $G'$ by identifying $v_1 = v_2$ and $v_3 = v_4$ in each block, as shown in Figure 4b. Then also no edges of weight −1 are incident with blowup vertices. This shows that $G'$ is a ring blowup satisfying the specifications of the lemma.

To conclude this section, we remark that negative edge-weights can be removed from the graphs constructed before while staying in the graph class of ring blowups.

**Lemma 10.** The problem #PerfMatch restricted to the graphs from Lemma 2 (that is, ring blowups with edge-weights ±1 such that edges of weight −1 are not incident with blowup vertices) admits a polynomial-time Turing reduction to #PerfMatch in unweighted ring blowups.

A standard proof of this lemma, see e.g. [8], replaces occurrences of the weight −1 with an indeterminate $x$; this turns the number of perfect matchings in an $n$-vertex graph into a polynomial $p \in \mathbb{Z}[x]$ of degree at most $d = n/2$. This polynomial can be evaluated at non-negative integer inputs $0, \ldots, d$ via planar gadgets, and the value $p(-1)$ can then be recovered from $p(0), \ldots, p(d)$ via polynomial interpolation. As Lemma 9 guarantees that edges of weight −1 are not incident with blowup vertices, the planar gadgets introduced in Lemma 10 can be contained within the inner part of the resulting graphs.

Combining Theorem 2 (the #P-hardness of counting perfect matchings in 3-regular graphs) with Lemmas 9 and 10 we immediately obtain:

**Theorem 11.** The problem #PerfMatch is #P-hard in unweighted ring blowups.

In fact, our proof even shows that #PerfMatch is #P-hard in graphs $G$ that are obtained from plane graphs by adding edges $v_1v_3$ and $v_2v_4$ between disjoint blocks of consecutive vertices $v_1, \ldots, v_4$ on the outer face. For the arguments in Section 1, we however need fully general ring blowups.

## 4 Bounding the Hadwiger number

In this section, we bound the Hadwiger number of ring blowups by 7. First, we show in Lemma 13 that it suffices to consider simple ring blowups, that is, blowups of plane graphs that have all but ≤ 3 vertices on the outer face. In Lemma 15 we iteratively remove certain complications from simple ring blowups. When this process terminates, we obtain graphs that can be handled by trivial arguments in Lemma 16.

**Definition 12.** A simple ring is a plane graph $Q$ with outer face $O$ such that $Q - O$ is a complete graph with ≤ 3 vertices. We call $W = V(Q) \setminus O$ the inner face of $Q$. A simple ring blowup is any subgraph of the blowup of a simple ring. (Equivalently, a graph is a simple ring blowup if it has a simple ring as reduct.)

In the following lemma, we adapt an argument by Joret and Wood [15, Lemma 3.4] to make ring blowups simple without decreasing their Hadwiger number. See Figure 5 for an illustration of the process; the last drawing shows an example of a simple ring blowup with a single vertex on the inner face. The lemma repeatedly uses the fact that, given a minor model of a graph $H$ in another graph $G$, contracting an edge contained a branch set canonically induces a minor model of $H$ in the resulting graph.

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2 Technically, this need not be a face.
Lemma 13. For any ring blowup \( G \), there is a simple ring blowup \( Q \) with \( \eta(G) \leq \eta(Q) \).

Proof. We abbreviate \( t = \eta(G) \). The lemma holds for \( t \leq 4 \), since there clearly are simple ring blowups containing \( K_4 \)-minors. We may therefore assume \( t \geq 5 \) in the following.

Let \( \hat{G} \) be a reduct of \( G \), with outer face \( O \), and let \( G \) be drawn as a blowup of \( \hat{G} \). Fix a minor model \( S_1, \ldots, S_t \subseteq V(G) \) of \( K_t \) in \( G \), define \( G' := G \) and proceed as follows.

1. Delete from \( G' \) all vertices not contained in \( S_1 \cup \ldots \cup S_t \). Then the sets \( S_1, \ldots, S_t \) still yield a minor model of \( K_t \) in the resulting graph, still called \( G' \), as no edges between branch sets were deleted.

2. Contract every branch set \( S_i \) that does not contain blowup vertices. This yields a minor model of \( K_t \) in the resulting graph, still called \( G' \). The \( \ell \leq t \) vertices \( W = \{w_1, \ldots, w_\ell\} \) resulting from the contraction induce a planar drawing of \( K_\ell \), so we have \( \ell \leq 4 \). We may even assume \( \ell = 3 \): Otherwise, if \( \ell = 4 \), then one of the vertices, say \( w_4 \), would be enclosed by the cycle on \( W \setminus \{w_4\} \) in the drawing of \( G' \). But then \( w_4 \) cannot have edges to the \( t - \ell \) other branch sets, so the \( t \) overall branch sets cannot form a minor model of \( K_t \) for \( t \geq 5 \).

3. For every edge \( uv \) fully contained in a branch set, where \( u \) is a blowup vertex and \( v \) is not, contract \( uv \) into \( u \). This still yields a minor model of \( K_t \) in the resulting graph \( G'' \).

Summing up, we see that \( G' \) contains a \( K_t \)-minor, so it suffices to prove that \( G' \) is a simple ring blowup. Note that the \( \leq 3 \) vertices in \( W \) form a clique in \( G'' \), and all other vertices are blowup vertices. By applying the operations used to transform \( G \) into \( G' \) on the reduct of \( G \), we obtain a reduct of \( G'' \) that is a simple ring, thus proving the lemma. 


diagram

By removing certain structures that we call “complications”, simple ring blowups can be simplified even further. In the following, it will be useful to work with the reducts.

Definition 14. Given a simple ring \( \hat{Q} \) with outer face \( O \) and inner face \( W \), a complication in \( \hat{Q} \) is

(a) any edge between vertices \( u, v \in O \) that are not consecutive in the cyclic order of \( O \), or

(b) given a vertex \( w \in W \), any neighbor \( o \in O \) of \( w \) after its first two neighbors in the cyclic order of \( O \).

Figure 6 illustrates complications of the different types. We show in the following lemma that it suffices to bound the Hadwiger number of the blowups of triangulated complication-free simple rings. Here, we say that a graph is triangulated if every face but its outer face is a triangle.
Lemma 15. If $\eta(Z) \leq 7$ holds for the blowup $Z$ of any triangulated and complication-free simple ring $\hat{Z}$, then $\eta(Q) \leq 7$ holds for any simple ring blowup $Q$.

Proof. Let $\hat{Q}$ be a simple ring with complications, outer face $O$, and inner face $W$. Let $Q$ be the blowup of $\hat{Q}$. We may assume $Q$ to be triangulated; this can be ensured by adding edges, which does not decrease $\eta(Q)$.

The goal is to decompose $\hat{Q}$ along clique-sums into simple rings with strictly less complications; the triangulation will be kept intact along the way. Using Fact 8 inductively, it will therefore suffice to bound $\eta(Z)$ when $Z$ is the blowup of a triangulated and complication-free ring $\hat{Z}$. The following notation will be useful. For vertex sets $S \subseteq V(Q)$, let $B(S)$ be obtained by replacing each vertex $v \in S \cap O$ with $v^1, v^2$.

We start by removing all type-a complications. To this end, consider a type-a complication involving $u, v \in O$. Then there exist sets $L, R \subseteq V(\hat{Q})$ such that $\hat{Q}$ is a clique-sum $\hat{Q}[L] \oplus_{\{u,v\}} \hat{Q}[R]$, as shown in the left part of Figure 6. This in turn means that the blowup $Q$ is a clique-sum $Q[B(L)] \oplus_{B(S)} Q[B(R)]$. Via Fact 8, it suffices to show that $Q[B(L)]$ and $Q[B(R)]$ are $K_8$-minor-free. But $Q[B(L)]$ and $Q[B(R)]$ are the blowups of $\hat{Q}[L]$ and $\hat{Q}[R]$, which are triangulated simple rings with at least one type-a complication less. By induction, we may therefore assume in the following that all type-a complications are processed.

Now consider a type-b complication, with $w \in W$ and consecutive neighbors $u_1, u_2, u_3 \in O$ of $w$ in the cyclic order of $O$. Note that we may indeed assume the neighbors to be consecutive, since all type-a complications were processed. Let us define $S = \{w, u_1, u_3\}$ and $T = S \cup \{u_2\}$. Then $Q$ is a clique-sum $\hat{Q} \oplus_S \hat{Q}[T]$ for the simple ring $P$ obtained from $\hat{Q}$ by removing $u_2$ and adding the edge $u_1u_3$, as shown in the right part of Figure 6. This in turn implies that $Q$ is a clique-sum $P \oplus_{B(S)} Q[B(T)]$, where $P$ is the blowup of $\hat{P}$. We observe that $Q[B(T)]$ has only 7 vertices, so it suffices to exclude $K_8$ from $P$, where $\hat{P}$ is a triangulated simple ring that still has no type-a complications, and one type-b complication less. By induction, we may therefore assume that $\hat{Q}$ has no complications at all. This proves the lemma.

Finally, an elementary case distinction allows us to handle complication-free ring blowups.

Lemma 16. For any triangulated and complication-free simple ring $\hat{Z}$ with blowup $Z$, we have $\eta(Z) \leq 7$.

Proof. Let $O$ and $W$ with $|W| \leq 3$ be the outer and inner faces of $\hat{Z}$. Every edge $e \in E(\hat{Z})$ either has both endpoints in $W$, one endpoint in $O$ and $W$ each, or both endpoints of $e$ lie consecutively on $O$ (since $Z$ has no type-a complications.) If $W$ is empty, then we have $|V(\hat{Z})| \leq 3$, since $Z$ is triangulated. This already implies $\eta(Z) \leq |V(Z)| \leq 6$.

If $W$ is non-empty, then any vertex $w \in W$ has $\leq 2$ neighbors in $O$, since $\hat{Z}$ has no type-b complications. Furthermore, since $\hat{Z}$ is triangulated, any pair of vertices in $W$ shares a neighbor in $O$, because $\hat{Z}$ would otherwise contain a chordless cycle of length 4. This implies the following:
• If $|W| \leq 2$, then $|O| \leq 2$ and thus $\eta(Z) \leq |V(Z)| \leq 6$.

• If $|W| = 3$, then $\hat{Z}$ and its blowup $Z$ are the following graphs, where $|V(Z)| = 9$ and $|E(Z)| = 30$.

To obtain a $K_8$-minor from $Z$, one vertex must be deleted or one edge must be contracted. Deleting a vertex removes at least 6 edges. Contracting an edge reduces the number of edges by least 3, since every edge in $Z$ is contained in some $K_4$-subgraph (shown above as colored blobs) and contracting an edge of $K_4$ yields a $K_3$, thus losing 3 edges. It follows that any 8-vertex minor of $Z$ has at most $27 < |E(K_8)|$ edges and therefore cannot be $K_8$.

This covers all cases for $|W|$, thus proving the lemma.

The proof of Theorem 1 is now immediate.

Proof of Theorem 1. By Theorem 11, the problem $\#\text{PerfMatch}$ is $\#P$-hard in unweighted ring blowups $G$. It remains to show that $G$ excludes $K_8$-minors. By Lemma 13, we have $\eta(G) \leq \eta(Q)$ for some simple ring blowup $Q$. By Lemma 15, we have $\eta(Q) \leq 7$ if $\eta(Z) \leq 7$ holds for all blowups of triangulated and complication-free ring graphs $\hat{Z}$, which in turn is true by Lemma 16.

5 Conclusion and outlook

We showed that the FKT method for planar graphs cannot be extended to graphs excluding arbitrary fixed minors. However, our work leaves open an exhaustive classification of the minors whose exclusion renders $\#\text{PerfMatch}$ polynomial-time solvable. This is not an artifact of our analysis: As Figure 5 shows, the graphs constructed by our reduction can contain $K_7$-minors, so our reduction inherently fails to address the open case of $K_7$-minor-free $\#\text{PerfMatch}$. This prompts the obvious question:

**Question 17.** What is the complexity of $\#\text{PerfMatch}$ in graphs excluding $K_6$ or $K_7$? More generally, given any fixed graph $H$, what is the complexity of $H$-minor-free $\#\text{PerfMatch}$?

Turning towards a bigger picture, it is also interesting to investigate which other counting problems benefit from excluded minors. This can be studied systematically in the framework of Holant problems, of which counting perfect matchings constitutes a representative example.

In a future version of this paper, we rule out $\exp(o(\sqrt{n}))$ time algorithms for $\#\text{PerfMatch}$ with edge-weights $\pm 1$ under the exponential-time hypothesis. This lower bound introduces various complications that we eschewed here in favor of a self-contained and simple presentation. Note that graphs excluding fixed minors have tree-width $O(\sqrt{n})$, and therefore standard algorithms for counting perfect matchings in graphs of bounded tree-width yield matching $\exp(O(\sqrt{n}))$ time upper bounds on $H$-minor-free graphs.

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