On the Fundamentals of the Three-dimensional Translation Gauge Theory of Dislocations

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Abstract

We propose a dynamical version of the three-dimensional translation gauge theory of dislocations. In our approach, we use the notions of dislocation density and dislocation current tensors as the translational field strengths and define corresponding response quantities (pseudomoment stress and dislocation momentum flux). For the equations of motion for dislocations, we derive a closed system of field equations in an elegant quasi-Maxwellian form. The dynamical Peach–Koehler force density is derived in this framework as well. Finally, we discuss the similarities and differences between Maxwell field theory and the dislocation gauge theory developed here.

Keywords

dislocation dynamics, gauge theory of dislocations, field theory, Peach–Koehler force

I. Introduction

In recent years there has been growing interest in continuum theories of dislocations. The development of such theories has been driven by the explanation of size effects in small-scale structures and by physically based plasticity theories of dislocations (see, for example, [1]). On the other hand, it has been known for a long time that fast-moving dislocations exhibit properties typical of moving particles and electromagnetic fields [2–4]; the stress field of a moving dislocation is longitudinally contracted, for instance. For this reason, analogies between the theory of dislocations and the Maxwell theory of electromagnetic fields have been discussed frequently in the literature [5–9]. Kröner [5] proposed an analogy between the theory of dislocations and the theory of the magnetic field of distributions of stationary electric currents. Other authors [8, 9] have suggested analogies between the deformation and magnetic fields and between the velocity and electric fields. Moreover, some authors [10, 11] have used the analogy between the magnetic field and the dislocation density as well as the analogy between the electric field and the dislocation current density. They did not, however, make use of the concept of excitations in dislocation theory, which is necessary for obtaining a complete theory with a closed system of equations of motion analogous to those of electromagnetic field theory. Schaefer already pointed out in [12] that constitutive equations are missing from the classical theory of dislocations.

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A dynamical theory of dislocations was formally derived in [13, 14]. We mention that the Lagrangian of dislocations proposed in those papers contains only one material constant for the dislocation density tensor and for the dislocation current tensor. An improved and more realistic dislocation model has been formulated in [15–18]. In [18], the static solutions of screw and edge dislocations were given. The linear version of a dynamical dislocation gauge theory was formulated in [17] and successfully applied to a uniformly moving screw dislocation in both the subsonic and the supersonic regimes [19]. The non-uniform motion of a dislocation is investigated in [20]. Recently, Lazar and Hehl [21] investigated Cartan’s spiral staircase in the gauge theory of dislocations and showed that such a configuration arises naturally as a solution of the three-dimensional theory of dislocations.

The aim of this paper is to develop more systematically a dynamical theory of dislocations which makes use of the concepts of field strengths, excitations and constitutive relations. This theory can be viewed as a kind of axiomatic field theory of dislocations. The nature of such a general dislocation theory is geometrically non-linear. We use the analogies with the axiomatic Maxwell theory given by Hehl and Obukhov [22], and as a mathematical tool we employ the powerful language of differential forms [23]. The framework of metric affine gauge theory developed by Hehl et al. [24] is also utilized in order to obtain a clear description of dislocation systems. The purpose of the paper is to give the differential geometric structure of the theory rather than to discuss solutions of the field equations. For solutions of dislocations in the framework of dislocation gauge theory, we refer the reader to [16, 18–21].

2. Elastodynamics

In finite elasticity theory (see, for example, [25–27]), the material body is identified with a three-dimensional manifold \( M^3 \) embedded in three-dimensional Euclidean space \( \mathbb{R}^3 \). We shall distinguish between the material or final coordinates of \( M^3 \), indexed by \( a, b, c, \ldots = 1, 2, 3 \), and the (holonomic) Cartesian coordinates of the reference system (defect-free or ideal reference system) \( \mathbb{R}^3 \), indexed by \( i, j, k, \ldots = 1, 2, 3 \). A deformation of \( \mathbb{R}^3 \) is a mapping \( \xi : \mathbb{R}^3 \rightarrow M^3 \). A time-dependent family of deformations is called a motion of \( M^3 \). The distortion 1-form is defined by

\[
\vartheta^a = B^a_i \, dx^i = d\xi^a
\]

and is identified with a coframe. In elasticity, \( B^a_i = \partial \xi^a / \partial x^i \) is referred to as the deformation gradient, which is a so-called two-point tensor field because it is defined on two configurations or bases. The dimensions of \( \vartheta^a \) and \( \xi^a \) are \([\vartheta^a] = \text{length}\) and \([\xi^a] = \text{length}\). Here ‘d’ denotes the three-dimensional exterior derivative. If the coframe (or distortion) 1-form has the property that

\[
d\vartheta^a = 0, \tag{2}
\]

then it is said to be holonomic or compatible. Therefore, Equation (2) is a compatibility condition on \( \vartheta^a \). The frame field

\[
e_a = e_a^i \partial_i
\]

is dual to the coframe (1) such that

\[
e_a] \vartheta^b = e_a^i B^b_i = \delta^b_a,
\]

where \( [\cdot] \) denotes the interior product. The frame \( e_a \) has dimension \([e_a] = 1/\text{length}\).

A \( \vartheta \)-basis for 0-, 1-, 2- and 3-forms is

\[
\{ 1, \vartheta^a, \vartheta^{ab} := \vartheta^a \wedge \vartheta^b, \vartheta^{abc} := \vartheta^a \wedge \vartheta^b \wedge \vartheta^c \},
\]
and the Hodge dual $\eta$-basis for 3-, 2-, 1- and 0-forms is specified by

$$
\eta := \star 1 = \frac{1}{3!} \eta_{abc} \vartheta^{abc},
$$

$$
\eta_a := \star \vartheta_a = e_a \eta = \frac{1}{2} \eta_{abc} \vartheta^{bc},
$$

$$
\eta_{ab} := \star (\vartheta_{ab}) = e_b \eta_a = \eta_{abc} \vartheta^c,
$$

$$
\eta_{abc} := \star (\vartheta_{abc}) = e_c \eta_{ab},
$$

where $\star$ denotes the Hodge star, $\wedge$ is the exterior product and $\eta_{abc} = \det (B_{di}) \epsilon_{abc}$ with $\epsilon_{abc}$ being the totally antisymmetric Levi-Civita symbol taking values $\pm 1, 0$.

Simultaneously, the physical velocity 0-form of the motion of the material continuum is given by

$$
\nu^a = \frac{\partial \xi^a}{\partial t} \equiv \dot{\xi}^a,
$$

which is a vector-valued 0-form. It describes the velocity of material points of the continuum and has dimension $[\nu^a] = \text{length/time}$. The time-dependent distortion $\vartheta^a$ and the velocity field $\nu^a$ have to satisfy the following kinematic compatibility condition:

$$
d\nu^a - \dot{\vartheta}^a = 0. \quad (4)
$$

In elasticity, this is just the kinematic compatibility condition between the deformation gradient and the associated velocity field.

The right Cauchy–Green tensor $G$ is defined as the metric of the final state:

$$
G = g_{ab} \vartheta^a \otimes \vartheta^b = g_{ab} B_a^i B_b^j d x^i \otimes d x^j = g_{ij} d x^i \otimes d x^j.
$$

If the coframe is orthonormal, this becomes

$$
G = \delta_{ab} \vartheta^a \otimes \vartheta^b = \delta_{ab} B_a^i B_b^j d x^i \otimes d x^j
$$

with $\delta_{ab} = \text{diag}(++, +)$. The Lagrangian strain tensor is given by

$$
2E = G - 1 = (g_{ij} - \delta_{ij}) d x^i \otimes d x^j.
$$

It measures the change of the metric between the undeformed state and the deformed state.

The question of formulating the response quantities (elastic excitations) is now of interest. This question has a close connection with the elastic field Lagrangian. In the continuum approach, the elastic Lagrangian depends continuously on the elastic distortion and velocity. Thus, the elastic Lagrangian 3-form is given by

$$
\mathcal{L}_{el}(\nu^a, \vartheta^a).
$$

As usual, the elastic Lagrangian density can be given in terms of kinetic and potential energy densities:

$$
\mathcal{L}_{el} = T_k - W, \quad (5)
$$

where $T_k$ is the kinetic energy density 3-form and $W$ is the elastic potential energy density 3-form (or distortion energy). The elastic potential energy density is a measure of the energy stored in the material as a result of elastic deformation. The excitation with respect to the physical velocity 0-form is the elastic momentum 3-form

$$
p_a := \frac{\partial \mathcal{L}_{el}}{\partial \nu^a}.
$$
This is a covector-valued 3-form:

\[ p_a = \frac{1}{3!} p_{ijk} \, dx^i \wedge dx^j \wedge dx^k. \]

It has dimension \([p_a]\) = momentum \(\text{SI} = \text{N s}\), whereas the components have dimension \([p_{ijk}]\) = momentum/(length)\(^3\). The usual momentum vector (covector-valued 0-form) from elasticity theory is given by \(P_a = (1/3!) \eta^{ijk} p_{ijk}\). In elasticity, the linear and isotropic constitutive law for the momentum vector is of the form

\[ P_a = \rho g_{ab} v^b, \]

where \(\rho\) denotes the mass density.

The elastic force stress 2-form is the excitation quantity relative to the distortion 1-form and is defined by

\[ \Sigma_a := \delta \mathcal{L}_{\text{el}} / \delta \dot{\xi}^a = \partial \mathcal{L}_{\text{el}} / \partial \dot{\xi}^a. \]  

Equation (6) is the general constitutive relation for non-linear elasticity. We note that \(\Sigma_a\) is a covector-valued 2-form:

\[ \Sigma_a = \frac{1}{2} \Sigma_{ij} \, dx^i \wedge dx^j. \]

It has dimension \([\Sigma_a]\) = force \(\text{SI} = \text{N}\), and the components have dimension \([\Sigma_{ij}]\) = force/(length)\(^2\) = stress \(\text{SI} = \text{P}\).

We recognize that the force stress has nine independent components. Therefore, the first Piola–Kirchhoff stress tensor (a two-point tensor) is represented by \(\Sigma_a^k = (1/2) \eta^{jk} \Sigma_{ijk}\). In elasticity theory, the first Piola–Kirchhoff stress tensor is \(\Sigma_a^k = e_{aj} \sigma^j\), where the Cauchy stress tensor is given by the generalized Hooke law \(\sigma^{jk} = C^{jkmn} E_{mn}\) with elasticity tensor \(C^{jkmn}\).

In elastodynamics, the Euler–Lagrange equations give the equation of motion

\[ \frac{\delta \mathcal{L}_{\text{el}}}{\delta \dot{\xi}^a} \equiv \frac{\partial \mathcal{L}_{\text{el}}}{\partial \dot{\xi}^a} - \frac{d}{d\xi^a} \frac{\partial \mathcal{L}_{\text{el}}}{\partial \xi^a} - \partial_t \frac{\partial \mathcal{L}_{\text{el}}}{\partial \dot{\xi}^a} = 0, \]

which, in terms of the stress 2-form and momentum 3-form, reads

\[ \dot{p}_a + d \Sigma_a = 0. \]

The elastic energy density \(\mathcal{E}_{\text{el}}\) is defined in the framework of field theory to be the Hamiltonian of the elastic system; that is,

\[ \mathcal{E}_{\text{el}} : = p_a \dot{\nu}^a - \mathcal{L}_{\text{el}} = T_k + W. \]

3. **T(3) Gauge Theory of Dislocations**

In this section, we discuss the three-dimensional translation gauge theory of dislocations. In this setting, we consider the translation group \(T(3)\) as a gauge group. We assume that all fields depend on the space and time variables. The \(T(3)\)-transformation acts on \(\xi^a\) as a gauge transformation in the following way:

\[ \xi^a \rightarrow \xi^a - \tau^a(x, t), \]

where \(\tau^a(x, t)\) are local and time-dependent translations. Note that the invariance of the compatible distortion (1) and the invariance of the material velocity (3) are lost under the local \(T(3)\)-transformations (7). In order to compensate for the invariance-violating terms, we need to introduce gauge potentials, i.e. a vector-valued 1-form \(\phi^a = \phi^a \, dx^i\) and a vector-valued 0-form \(\varphi^a\) that transform under the local transformations in a suitable manner:

\[ \phi^a \rightarrow \phi^a + d\tau^a(x, t), \]
\[ \varphi^a \rightarrow \varphi^a + \dot{\tau}^a(x, t). \]
The $\phi^a$ and $\varphi^a$ thus defined are the translational gauge potentials of the dynamical $T(3)$ gauge theory. Since the elastic distortion and the material velocity are state quantities in the field theory of dislocations, they have to be gauge-invariant. Now we redefine the elastic distortion and material velocity in gauge-invariant forms as follows:

$$\vartheta^a := d\xi^a + \phi^a, \quad (8)$$
$$v^a := \dot{\xi}^a + \varphi^a. \quad (9)$$

Hence, $\xi^a$, $\phi^a$ and $\varphi^a$ always appear together in translation-invariant combinations. The gauge fields $\phi^a$ and $\varphi^a$ make the elastic distortion $\vartheta^a$ and the material velocity $v^a$ incompatible, because they are no longer just a simple gradient and a time-derivative of $\xi^a$.

In [15] it was shown that $\phi^a$ in Equation (8) can be interpreted as the translational part of the generalized affine connection in a Weitzenböck space. The underlying geometrical structure of the theory is given by the affine tangent bundle $AM$. At every point, the tangent space is replaced by the affine tangent space. The translation group $T(3)$ acts on the affine space as an internal symmetry. The field $\xi^a$ is also known as Cartan’s ‘radius vector’ and determines the ‘origin’ of the affine space; see [24]. In the context of dynamical $T(3)$ gauge theory, two translational gauge potentials $\phi^a$ and $\varphi^a$ and one translational Goldstone field $\xi^a$ occur, and these are the canonical field quantities. On the other hand, in the ‘classical’ theory of defects (see, for instance, [28]), fields like $\phi^a$ and $\varphi^a$ are called the negative plastic (or initial) distortion and velocity, respectively.

### 3.1. Translational Field Strengths

Because we have two translational gauge potentials $\phi^a$ and $\varphi^a$, we can define two translational field strengths in terms of these gauge potentials. We introduce the well-known quantities of dislocation density and dislocation current as field strengths of the $T(3)$ gauge theory that break the compatibility conditions (2) and (4). The dislocation density 2-form $T^a$ is defined as the object of anholonomity or the torsion 2-form in a teleparallel space (Weitzenböck space). The torsion 2-form (dislocation density 2-form) is given in terms of the translational gauge potential $\phi^a$ by

$$T^a = d\phi^a$$

or, in terms of the anholonomic coframe $\vartheta^a$, by

$$T^a = d\vartheta^a.$$  

Thus, $T^a$ measures how much the gauge potential $\phi^a$ or the coframe $\vartheta^a$ fail to be holonomic or compatible. It is a vector-valued 2-form which can be expressed as

$$T^a = \frac{1}{2} T^a_{\;\;ij} dx^i \wedge dx^j,$$

with dimension $[T^a] = \text{length}$. The torsion tensor $T^a_{\;\;ij}$ has dimension $[T^a_{\;\;ij}] = 1/\text{length}$. The field $T^a_{12}$ measures the number of dislocation lines going in the 3-direction and having Burgers vector $b^3$. The torsion 2-form $T^a$ can describe a continuous distribution of dislocations as well as single dislocations. The usual dislocation density tensor (see [6]) is represented by $\alpha^{ab} = (1/2)\eta^{ijk} T^a_{ij}$. Another translational field strength is the dislocation current. The dislocation current 1-form is defined in terms of the translational gauge potentials $\varphi^a$ and $\phi^a$ as

$$I^a = d\varphi^a - \dot{\phi}^a \quad (10)$$

or, in terms of the incompatible coframe and incompatible velocity, as

$$I^a = dv^a - \dot{\vartheta}^a. \quad (11)$$
From Equation (11), we see that the dislocation current is given in terms of the physical velocity gradient and the rate of elastic distortion. It is a vector-valued 1-form,

$$I^a = I^a_i \, dx^i,$$

with dimension $[I^a] = \text{velocity} \, \text{SI} = \text{ms}^{-1}$. The dislocation current tensor has dimension $[I_a] = 1/\text{time} \, \text{SI} = \text{s}^{-1}$. The translational field strengths $T^a$ and $I^a$ measure the amount of violation of the two compatibility conditions (2) and (4) and, therefore, by how much the elastic distortion $\vartheta^a$ and the physical velocity $v^a$ are incompatible. The dislocation density $T^a$ and the dislocation current $I^a$ are state quantities which can be observed experimentally. The dislocation current is the appropriate quantity for description of the dynamics of dislocations. In the dynamical case, $I^a$ and $T^a$ carry information about the dislocation state of motion.

The two translational field strengths have to satisfy the following Bianchi identities:

$$dT^a = 0, \quad (12)$$
$$dI^a + \dot{T}^a = 0. \quad (13)$$

Equation (12) is the well-known conservation law of dislocations (continuity equation of the dislocation density), and Equation (13) is known as the continuity equation of the dislocation current [29, 30]. Equation (12) says that a dislocation line cannot end inside the body. The evolution of $T^a$ is determined by $I^a$ in closed form. If we integrate (12) over a three-dimensional volume that contains dislocations and apply Stokes’ theorem, we obtain the Burgers vector by means of the Burgers circuit $\partial S$:

$$\int_S T^a = \oint_{\partial S} \vartheta^a = b^a, \quad (14)$$

where $\partial S$ is the boundary of the surface $S$. On the other hand, if we integrate (13) over a two-dimensional surface $S$ and apply Stokes’ theorem, we get the so-called ‘conservation law of the Burgers vector’:

$$\oint_{\partial S} I^a + \dot{b}^a = 0. \quad (15)$$

The integral in (15) determines the flux of the Burgers vector $b^a$ per unit time through the contour $\partial S$. The equation tells us that the time change of the Burgers vector on a surface $S$ is equal to the negative dislocation current over the contour $\partial S$ of the surface $S$.

If the local translation is not time-dependent, i.e. $\tau^a = \tau^a(x)$, then the physical velocity $v^a$ is compatible as in (3) and one may use the gauge condition $\varphi^a = 0$ to eliminate the gauge potential $\varphi^a$. Such a gauge condition is referred to as ‘temporal gauge’ (or ‘Weyl gauge’) in gauge field theories. In this gauge, the dislocation current (10) reads $I^a = -\dot{\varphi}^a$. Such is the case, for example, in [29, 30], if we identify the gauge potential $\varphi^a$ with the negative plastic distortion. In this way, the dislocation current is related to the rate of plastic distortion.

3.2. Gauge Field Momenta: Dislocation Field Excitations

To complete the field theory of dislocations, we have to define the excitations with respect to the dislocation density and the dislocation current. The dislocation Lagrangian density is of the form

$$L_{\text{disl}} = L_{\text{disl}}(v^a, \vartheta^a, I^a, T^a).$$

For the time being, let us leave the explicit form of $L_{\text{disl}}$ open. It is then necessary to introduce the field momenta that are canonically conjugated to the translational field strengths; these field momenta are called dislocation excitations. The excitation with respect to the torsion 2-form is defined by

$$H_a := -\frac{\partial L_{\text{disl}}}{\partial T^a} \quad (16)$$
and is the specific response to $T^a$. Another excitation is the 2-form $D_a$ given by

$$D_a := \frac{\partial \mathcal{L}_{\text{disl}}}{\partial T^a},$$  \hspace{1cm} (17)$$

which is the response of the dislocation Lagrangian to $T^a$. We can interpret $H_a$ and $D_a$ as, respectively, the pseudomoment stress and the dislocation momentum flux caused by the motion of dislocations. Here, $H_a$ is a covector-valued 1-form:

$$H_a = H_{ai} \, dx^i.$$

The dimension of $H_a$ is $[H_a] = \text{force}$, and its components have dimension $[H_{ai}] = \text{force/length}$. The dislocation momentum flux $D_a$ is a covector-valued 2-form:

$$D_a = \frac{1}{2} D_{aj} \, dx^j \wedge dx^i.$$

It has dimension $[D_a] = \text{momentum}$, and the components have dimension $[D_{aj}] = \text{momentum/(length)}^2$. Equations (16) and (17) are constitutive relations of the non-linear dislocation field theory.

In addition, we define the dislocation stress as a covector-valued 2-form by

$$E_a := \frac{\partial \mathcal{L}_{\text{disl}}}{\partial \theta^a},$$  \hspace{1cm} (18)$$

with dimension $[E_a] = \text{force}$. Further, the dislocation momentum 3-form is

$$\pi_a := \frac{\partial \mathcal{L}_{\text{disl}}}{\partial \varphi^a},$$  \hspace{1cm} (19)$$

which is a covector-valued 3-form with dimension $[\pi_a] = \text{momentum}$. Equations (18) and (19) are valid for the dislocation stress and dislocation momentum in non-linear dislocation field theory. The dislocation stress 2-form is explicitly given by

$$E_a = e_a \, L_{\text{disl}} + (e_a) \, T^b \wedge H_b - (e_a) \, P^b D_b,$$  \hspace{1cm} (20)$$

and the dislocation momentum 3-form reads

$$\pi_a = (e_a) \, T^b \wedge D_b = -(e_a) \, D_b \wedge T^b.$$  \hspace{1cm} (21)$$

In continuum mechanics, (20) and (21) are called the Eshelby stress and the pseudomomentum, respectively; see, for example, [31, 27].

### 3.3. Dislocation Field Equations

We are now ready to derive the Yang–Mills-type field equations that govern the dynamics of dislocations. The total Lagrangian density is of the form

$$\mathcal{L} = \mathcal{L}_{\text{disl}} + \mathcal{L}_{\text{el}}.$$  \hspace{1cm} (22)$$

The variations of the total Lagrangian with respect to the Goldstone field $\xi^a$ and the translational gauge potentials $\varphi^a$ and $\phi^a$ give the Euler–Lagrange equations. According to the extremal action principle, the field equations are found to be

$$\frac{\delta \mathcal{L}}{\delta \xi^a} = \frac{\partial \mathcal{L}}{\partial \xi^a} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\xi}^a} = 0,$$  \hspace{1cm} (23)$$

$$\frac{\delta \mathcal{L}}{\delta \varphi^a} = \frac{\partial \mathcal{L}}{\partial \varphi^a} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\varphi}^a} = 0,$$  \hspace{1cm} (24)$$

$$\frac{\delta \mathcal{L}}{\delta \phi^a} = \frac{\partial \mathcal{L}}{\partial \phi^a} + \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\phi}^a} = 0.$$  \hspace{1cm} (25)$$
Alternatively, we can take the variation of the total Lagrangian with respect to the gauge-invariant quantities, namely the incompatible velocity $v^a$ and the coframe $\vartheta^a$, and obtain
\[
\frac{\delta L}{\delta v^a} \equiv \frac{\partial L}{\partial v^a} - d\frac{\partial L}{\partial d\vartheta^a} - \partial_t \frac{\partial L}{\partial \dot{v}^a} = 0,
\]
\[
\frac{\delta L}{\delta \vartheta^a} \equiv \frac{\partial L}{\partial \vartheta^a} + d\frac{\partial L}{\partial d\vartheta^a} + \partial_t \frac{\partial L}{\partial \dot{\vartheta}^a} = 0,
\]
which can further be expressed as
\[
\frac{\delta L}{\delta v^a} \equiv \frac{\partial L}{\partial v^a} - d\frac{\partial L}{\partial I^a} = 0, \tag{26}
\]
\[
\frac{\delta L}{\delta \vartheta^a} \equiv \frac{\partial L}{\partial \vartheta^a} + d\frac{\partial L}{\partial T^a} + \partial_t \frac{\partial L}{\partial \dot{I}^a} = 0. \tag{27}
\]
These field equations (26) and (27) of dislocation theory may be expressed in terms of the response quantities (16), (17), (18) and (19) to take the form
\[
d D_a - \pi_a = p_a, \tag{28}
\]
\[
d H_a - \dot{D}_a - E_a = \Sigma_a. \tag{29}
\]
Equations (28) and (29) are two field equations of Yang–Mills type, with the momentum $(p_a + \pi_a)$ and the force stress $(\Sigma_a + E_a)$ as sources of the dislocation excitations $D_a$ and $H_a$, respectively. The translational gauge fields themselves produce ‘gauge’ sources $\pi_a$ and $E_a$, thereby contributing to their own elastic sources $p_a$ and $\Sigma_a$. Owing to the complexity of the dislocation gauge field interaction, self-couplings involving the dislocation momentum $\pi_a$ and the dislocation stress $E_a$ arise. Thus, the field equations (28) and (29) are non-linear. In addition, Equations (28) and (29) constitute a closed system of twelve independent field equations for the state quantities $v^a$ and $\vartheta^a$. Equation (28) can be interpreted as the equation of equilibrium between the dislocation momentum flux and the momenta (i.e. the balance equation of momenta); in the static case, (28) is vanishing. Equation (29) is the equation of equilibrium between the pseudomoment stress, dislocation momentum flux and force stresses (i.e. the balance equation of stresses). From (28) and (29) we obtain the following conservation law:
\[
\dot{p}_a + \dot{\pi}_a + d(\Sigma_a + E_a) = 0. \tag{30}
\]
Equation (30) is none other than the Euler–Lagrange equation (23), which is the force equilibrium condition when dislocations are present (the continuity equation of force stresses). It determines the exchange of momentum and stress between the elastic and the dislocation subsystems. Observe that, in contrast to standard elasticity theory, the conserved quantities are the total momentum and the total force stress of the system, not the elastic quantities themselves. Linear solutions of the field equations (28) and (29) for moving dislocations were given by Lazar in [19, 20].

In order to complete the framework of dislocation field theory, we introduce the Peach–Koehler force 3-form,
\[
f_a := -\pi_a - dE_a = \dot{p}_a + d\Sigma_a = (e_a)_{[l}p_{b]} + (e_a)_{[T}T_{b]} \wedge \Sigma_b, \tag{31}
\]
which represents the force density acting on dislocations. Equation (31) is the dynamical form of the Peach–Kohler force, which is analogous to the Lorentz force (see [22]) in Maxwell’s theory of electromagnetic fields. The force density 3-form is a covector-valued 3-form,
\[
f_a = \frac{1}{3!} f_{aijk} \, dx^i \wedge dx^j \wedge dx^k,
\]
and the dimension of the force density tensor is $[f_{aijk}] = \text{force}/(\text{length})^3$. 

The moment stress 2-form $\tau_{ab}$ is related to the excitation $H_a$ via

$$\tau_{ab} := \partial_{[a} H_{b]},$$  \hspace{1cm} (32)

and its dimension is $[\tau_{ab}] = \text{force} \times \text{length}$. In components, the moment stress 2-form reads

$$\tau_{ab} = \frac{1}{2} \tau_{abij} d x^i \wedge d x^j,$$

where the dimension of the moment stress tensor is $[\tau_{abij}] = \text{force}/\text{length}$. The formula (32) can be inverted as follows:

$$H_a = -2 e_b \rfloor \tau_a^b + \frac{1}{2} \partial_a \wedge (e_b \rfloor e_c \rfloor \tau^{bc}).$$  \hspace{1cm} (33)

Deriving the moment equilibrium requires some algebra. We start with the field equation (29) and compute the antisymmetric piece of the total stress; then we use (32) and finally find that

$$d \tau_{ab} - T_{[a} \wedge H_{b]} + \partial_{[a} \wedge \dot{D}_{b]} + \partial_{[a} \wedge (E_{b]} + \Sigma_{b}) = 0.$$  \hspace{1cm} (34)

Apart from the terms $-T_{[a} \wedge H_{b]}$ and $\partial_{[a} \wedge \dot{D}_{b]}$, this is exactly the expected law from continuum mechanics.

### 3.4. Quadratic Gauge Field Lagrangian

We are now ready to specify constitutive laws. In order to give concrete expressions for the excitations, we need to specify the constitutive relations between the field strengths ($T^a$, $I^a$) and the excitations ($H_a$, $D_a$). For a local, linear, isotropic continuum we have

$$H_a = g_{ab} \sum_{l=1}^{3} a_l \, ^{(l)}T^b,$$  \hspace{1cm} (35)

where the $\, ^{(l)}T^a$ are the irreducible pieces (36), (37) and (38) of the torsion and $a_1$, $a_2$ and $a_3$ are constitutive moduli that have dimension $[a_l] = \text{force}$. The quantity $^*T^a$ is dimensionless. We can decompose the torsion into three SO(3)-irreducible pieces as follows: $T^a = \, ^{(1)}T^a + \, ^{(2)}T^a + \, ^{(3)}T^a$, with the number of independent components being $9 = 5 \oplus 3 \oplus 1$. These three pieces (the tensor, trace-vector and axial-vector pieces of the torsion) are defined by

$$\begin{align*}
\, ^{(1)}T^a &:= T^a - \, ^{(2)}T^a - \, ^{(3)}T^a \quad \text{(tentor)}, \hspace{1cm} (36) \\
\, ^{(2)}T^a &:= \frac{1}{2} \partial^a \wedge (e_b \rfloor T^b) \quad \text{(trator)}, \hspace{1cm} (37) \\
\, ^{(3)}T_a &:= \frac{1}{3} e_a \rfloor (\partial^b \wedge T^b) \quad \text{(axitor)}. \hspace{1cm} (38)
\end{align*}$$

For the local, linear, isotropic continuum we additionally have the following constitutive relation between $D_a$ and $I^a$: 

$$D_a = g_{ab} \sum_{l=1}^{3} f_l \, ^{(l)}I^b,$$  \hspace{1cm} (39)

where the $\, ^{(l)}I^a$ are the irreducible pieces (40), (41) and (42) of the dislocation current and $[^*I^a] = (\text{length})^2/\text{time}$. The $f_1$, $f_2$ and $f_3$ are constitutive moduli that have dimension $[f_l] = \text{mass}/\text{length} = \text{SI} = \text{kgm}^{-1}$. 
Table 1. The correspondence between Maxwell’s theory and dislocation field theory.

| Maxwell field theory                                                                 | Dislocation field theory |
|--------------------------------------------------------------------------------------|--------------------------|
| $B$  magnetic field strength                                                          | $T^a$  dislocation density |
| $E$  electric field strength                                                          | $I^a$  dislocation current |
| $H$  magnetic excitation                                                             | $H_a$  pseudomoment stress |
| $D$  electric excitation                                                             | $D_a$  dislocation momentum flux |
| $A$  magnetic potential 1-form                                                        | $\phi^a$  dislocation potential 1-form |
| $\varphi$  potential 0-form                                                         | $\varphi^a$  dislocation potential 0-form |
| $f$  gauge function                                                                  | $\xi^a$  deformation mapping |
| $B = dA, A' = A + df$                                                                 | $T^a = d\phi^a, \phi^{a'} = \phi^a + d\xi^a$ |
| $E = \varphi - A, \varphi' = \varphi + f$                                           | $I^a = d\phi^a - \phi^a, \varphi^{a'} = \varphi^a + \xi^a$ |
| $\Phi$  magnetic flux of magnetic vortices                                           | $b^a$  Burgers vector of dislocations |
| $\int_S B = \Phi$                                                                    | $P_a^I = p_a + \pi_a$ total momentum density |
| $\rho$  electric charge density                                                      | $\Sigma^I_a = \Sigma_a + E_a$ total force stress |
| $j$  electric current                                                                | continuity equation of dislocation density: |
| magnetic field closed:                                                               | $dT^a = 0$ |
| $dB = 0$                                                                             | continuity equation of dislocation current: |
| Faraday law:                                                                         | $dT^a + T^a = 0$ |
| $dE + B = 0$                                                                         | continuity equation of dislocation moment flux: |
| Gauss law:                                                                           | $dD_a = p_a^I$ |
| $dD = \rho$                                                                          | continuity equation of moment stress: |
| Oersted–Ampère law:                                                                  | $dH_a - D_a = \Sigma^I_a$ |
| $dH - D = j$                                                                         | continuity equation of force stress: |
| continuity equation of current:                                                      | $p_a^I + d\Sigma^I_a = 0$ |
| $\rho + dj = 0$                                                                      | constitutive laws: |
| $H = H(B), D = D(E)$                                                                 | $H_a = H_a(T^b), D_a = D_a(T^b)$ |
| Lorentz force density:                                                                | Peach–Koehler force density: |
| $f_a = (e_a)E\rho + (e_a)B \wedge j$                                                | $f_a = (e_a)I^b p_b + (e_a)T^b \wedge \Sigma_b$ |

The three $SO(3)$-irreducible pieces $I^a = (1) I^a + (2) I^a + (3) I^a$, with the number of independent components being $9 = 5 \oplus 3 \oplus 1$, are given by

\begin{align}
(1) I^a & := I^a - (2) I^a - (3) I^a \quad \text{(symmetric and traceless)}, \\
(2) I^a & := \frac{1}{2} e_a^b (\varphi^b \wedge I^b) \quad \text{(antisymmetric)}, \\
(3) I^a & := \frac{1}{3} \varphi^a (e_b) I^b \quad \text{(trace)}. 
\end{align}

Thus, for a linear continuum, the dislocation Lagrangian has the bilinear form

$$
\mathcal{L}_{\text{disl}} = \frac{1}{2} I^a \wedge D_a - \frac{1}{2} T^a \wedge H_a. 
$$

The pure dislocation energy is defined to be the Hamiltonian of the dislocation system; that is,

$$
\mathcal{E}_{\text{disl}} := I^a \wedge D_a - \mathcal{L}_{\text{disl}} = \frac{1}{2} I^a \wedge D_a + \frac{1}{2} T^a \wedge H_a. 
$$

A more physical interpretation of $\mathcal{E}_{\text{disl}}$ is as the dislocation core energy. Moreover, it can be seen from Equations (43) and (44) that the first term plays the role of the kinetic dislocation energy while the second term can be identified with the potential energy of dislocations. Thus, the excitation $D_a$ can be viewed as a kind of ‘momentum’, and $I^a$ plays the role of a generalized ‘velocity’ of the dislocation motion.
4. Discussion and Conclusions

In this paper, we have proposed a dynamical field theory of dislocations based on $T(3)$ gauge theory. In order to obtain a closed field theory, we used concepts of field strengths, excitations and constitutive laws analogous to those in electromagnetic field theory. All dislocation field quantities can be described by $\mathbb{R}^3$-valued exterior differential forms. The translation field strengths are even (or polar) differential forms, while the excitations (stresses and momenta) are odd (or axial) forms. We have shown that the excitations corresponding to the dislocation density and dislocation current are necessary for establishing a realistic physical dislocation field theory. Moreover, we have demonstrated how the excitations have to fit into the Maxwell-type field equations; this is in contrast to [10], where it was claimed that in dislocation theory there exist no analogues to the second pair of Maxwell equations. A summary of the electromagnetic quantities and their corresponding dislocation quantities is given in Table 1. The gauge theory of dislocations is a closed field theory; however, there are important differences from the Maxwell theory. In Maxwell theory the field quantities are scalar-valued forms, while in our dislocation field theory all field quantities are vector-valued (or covector-valued) forms. Owing to self-couplings, the field equations for dislocations are non-linear. The electromagnetic current $j$ depends on an exterior material field, whereas the stress $(E_a + \Sigma_a)$ is interior and depends on the coframe (distortion) itself.

It is known that dislocations in crystals move in two different modes, known as ‘glide’ (conservative motion) and ‘climb’ (non-conservative motion). The volume and mass density of the crystal, for instance, are not changed by the gliding motion of dislocations. On the other hand, climbing dislocations interchange with point defects such as vacancies and interstitials. Additionally, if dislocations cut across one another, they form networks of dislocations. In our dynamical dislocation theory, we have neglected dissipation effects (friction and radiation damping) and the interaction with point defects. To take into account the energy dissipated and converted into heat, one can use a Lagrangian extended by a dissipation function (see, for example, [32]).

In summary, the state quantities in dislocation dynamics are the physical velocity $v^a$, the elastic distortion $\vartheta^a$, the dislocation density $T^a$ and the dislocation current $I^a$. A dislocation field theory based on these state quantities yields the Euler–Lagrange equations and response quantities presented in this paper. In addition, one can combine such a dislocation theory with the so-called ‘multiplicative decomposition’ [33–35] widely used and accepted in engineering science. Nevertheless, the geometric or field-theoretic arena of dislocation field theory is the gauge theory of the three-dimensional translation group. This is a consequence of the fact that dislocations locally break translation symmetry in a crystal and that the dislocation density tensor is none other than a realization of Cartan’s torsion tensor [36, 37] in three dimensions, which was originally found by Kondo [38] (see also [34]).

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Conflict of Interest

None declared

References

[1] Abu Al-Rub, RK, and Voyiadis, GZ. A physically based gradient plasticity theory. International Journal of Plasticity, 22, 654–684 (2006).
[2] Frank, FC. On the equations of motion of crystal dislocations. Proceedings of the Physical Society, Section A, 62, 131–134 (1949).
[3] Eshelby, JD. Uniformly moving dislocations. Proceedings of the Physical Society, Section A, 62, 307–314 (1949).
[4] Leibfried, G, and Dietze, H.-D. Zur Theorie der Schraubenversetzung. Zeitschrift für Physik, 126, 790–808 (1949).
[5] Kröner, E. Die inneren Spannungen und der Inkompatibilitätsensor in der Elastiziätstheorie. Zeitschrift für angewandte Physik, 7, 249–257 (1955).
[6] Kröner, E. Continuum theory of defects, in ed. R. Balian, M. Kléman and J.-P. Poirier, Physics of Defects (Proceedings of the Les Houches Summer School, Session 35), pp. 215–315, North-Holland, Amsterdam, 1981.
