On space-like generalized constant ratio hypersurfaces in Minkowski spaces

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Abstract

A hypersurface in a Euclidean space $\mathbb{E}^{n+1}$ is said to be a generalized constant ratio (GCR) hypersurface if the tangential part of its position vector is one of its principle directions. In this work, we move the study of generalized constant ratio hypersurfaces started in [8] into the Minkowski space. First, we get some geometrical properties of non-degenerated GCR hypersurfaces in an arbitrary dimensional Minkowski space. Then, we obtain complete classification of GCR surfaces in the Minkowski 3-space. We also give some explicit examples.

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1 Introduction

The theory of (semi-) Riemannian submanifolds in (semi-) Euclidean spaces is a very active research field which have been recently interesting for mathematicians. In particular, problems related with position vector of submanifolds have cough interest of many geometers so far. In this direction, the notion of constant ratio submanifolds in Euclidean spaces if firstly introduced by B.-Y. Chen in [2], obtained independently by K. Boyadzhiev in [1]. By the definition, a submanifold of Euclidean space is said to be of constant ratio if the ratio of the length of the tangential and normal components of its position vector is constant. It is well known that constant ratio submanifolds are related to the study of the Riemannian convolution manifolds initiated in [3, 4].
Let $M$ be a surface in the Euclidean 3-space, $x$ its position vector and $\theta$ denote the angle function between $x$ and the unit normal vector field of $M$. If the tangential part $x^T$ of $x$ is one of its principal directions, then $M$ is said to be a generalized constant ratio (GCR) surface. One can show that being GCR of $M$ is equivalent to $Y(\theta) = 0$, whenever $Y$ is a tangent vector field orthogonal to $x$ (see [8]). Note that $M$ is a CR surface if and only if it is a GCR surface satisfying $x^T(\theta) = 0$. Furthermore, if the ambient space is Euclidean, being generalized constant ratio of a surface is equivalent to having canonical principal direction (CPD) (see [8, Theorem 1]). Also surfaces with CPD on 3-dimensional Riemannian manifolds were studied in some articles appeared recently, [6, 7, 8, 9, 10, 11]. For example, in [6] and [7] authors obtained the classification of surfaces with canonical principal direction in the spaces $S^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$, respectively.

On the other hand, GCR surfaces in Euclidean 3-space $\mathbb{E}^3$ are also related with constant slope surfaces introduced by M. I. Munteanu in [14], where the author obtain the complete classification of such surfaces in the $\mathbb{E}^3$. Further, similar techniques are used in [12, 13] in order to obtain the complete classification of constant slope surfaces in $\mathbb{E}^3$. We would like to note that an important property of constant slope surfaces in Euclidean 3-space $\mathbb{E}^3$ and Minkowski 3-spaces is the following: Let $U$ and $x$ denote the projection of position vector on the tangent plane of the surface and a generic point in ambient space, respectively. If the projection $U$ makes constant angle with the normal vector of the surface at that point, then $U$ is a canonical principal direction of the surface with the corresponding principal curvature being different from zero.

In the present paper, we would like to move the study of GCR hypersurfaces in Euclidean spaces initiated in [8, 11] into semi-Euclidean spaces by obtaining complete classification of GCR surfaces in Minkowski 3-space. This paper is organized as follows. In Sect. 2, we introduce the notation that we will use and give a brief summary of basic definitions in theory of submanifolds of semi-Euclidean spaces. In Sect. 3, we obtain some of geometrical properties of space-like GCR hypersurfaces in an arbitrary dimensional Minkowski space $\mathbb{E}^n_{1+1}$. In Sect. 4, we obtain the complete classification of space-like GCR surfaces in the Minkowski 3-space.

## Preliminaries

Let $\mathbb{E}^m_s$ denote the pseudo-Euclidean $m$-space with the canonical pseudo-Euclidean metric tensor $g$ of index $s$ given by

$$\bar{g} = \langle \cdot, \cdot \rangle = -\sum_{i=1}^{s} dx_i^2 + \sum_{j=s+1}^{m} dx_j^2,$$

where $(x_1, x_2, \ldots, x_m)$ is a rectangular coordinate system in $\mathbb{E}^3_1$. We put

$$\mathbb{S}^{m-1}_s(r^2) = \{ x \in \mathbb{E}^m_s : \langle x, x \rangle = r^{-2} \},$$
$$\mathbb{H}^{m-1}_s(-r^2) = \{ x \in \mathbb{E}^m_{1+1} : \langle x, x \rangle = -r^{-2} \}.$$
Note that $S^{m-1}(r^2)$ and $H^{m-1}(-r^2)$ are the complete pseudo-Riemannian manifolds of constant curvature $r^2$ and $-r^2$, respectively. Moreover, we will put $H^{m-1}(r^2) = H^{m-1}(-r^2)$.

A non-zero vector $v$ in $E^{m}$ is said to be space-like, time-like and light-like (null) regarding to $\langle v, v \rangle > 0$, $\langle v, v \rangle < 0$ and $\langle v, v \rangle = 0$, respectively. Note that $v$ is said to be causal if it is not space-like.

2.1 Hypersurfaces in the Minkowski space.

Let $M$ be an oriented hypersurface in $E^{n+1}_1$ and $x : M \rightarrow E^{n+1}_1$ an isometric immersion with unit normal vector $N$ associated with the orientation of $M$. The immersion $x$ (or, equivalently hypersurface $M$) is said to be space-like (resp. time-like) if the induced metric $g = \tilde{g}|_M$ of $M$ is Riemannian (resp. Lorentzian). This is equivalent to being time-like (resp. space-like) of $N$ at each point of $M$.

We denote the Levi-Civita connections of $M$ and $E^{n+1}_1$ by $\nabla$ and $\tilde{\nabla}$, respectively. Then, Gauss and Weingarten formulas are given, respectively, by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \quad (2.1)$$
$$\tilde{\nabla}_X N = -S(X) \quad (2.2)$$

for any tangent vector fields $X,Y$, where $h$ and $S$ are the second fundamental form and the shape operator (or Weingarten map) of $M$, respectively. The second fundamental form and the shape operator are related by

$$\langle S(X), Y \rangle = \langle h(X,Y), N \rangle \quad (2.3)$$

We would like to note that if $M$ is space-like, then its shape operator $S$ is diagonalizable, i.e., there exists a local orthonormal frame field $\{e_1, e_2, \ldots, e_n; N\}$ such that $S e_i = k_i e_i, \quad i = 1, 2, \ldots, n$. In this case, the vector field $e_i$ and smooth function $k_i$ are called a principle direction and a principle curvature of $M$.

The Gauss and Codazzi equations are given, respectively, by

$$\langle R(X,Y)Z,W \rangle = \langle h(Y,Z), h(X,W) \rangle - \langle h(X,Z), h(Y,W) \rangle, \quad (2.4)$$
$$\langle \tilde{\nabla}_X h \rangle(Y,Z) = \langle \tilde{\nabla}_Y h \rangle(X,Z), \quad (2.5)$$

where $R$ is the curvature tensor associated with the connection $\nabla$ and $\tilde{\nabla} h$ is defined by

$$\langle \tilde{\nabla}_X h \rangle(Y,Z) = \nabla^\perp_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

The angle function between the position vector and the normal is no more constant. The definition on angle between two vectors in Minkowski 3-space is well-known (see for example [12] [13]).
Definition 1 Let \( v \) and \( w \) be positive (negative) time-like vectors in \( \mathbb{E}^3_1 \). Then, there is a unique non-negative real number \( \theta \) such that
\[
|\langle v, w \rangle| = \|v\| \|w\| \cosh \theta.
\]
The real number \( \theta \) is called the Lorentzian time-like angle between \( v \) and \( w \).

Definition 2 Let \( v \) be a space-like vector and \( w \) a positive time-like vector in \( \mathbb{E}^3_1 \). Then, there is a unique non-negative real number \( \theta \) such that
\[
|\langle v, w \rangle| = \|v\| \|w\| \sinh \theta.
\]
The real number \( \theta \) is called the Lorentzian time-like angle between \( v \) and \( w \).

Before we proceed, we would like to recall the notion of the Lorentzian cross-product \( \wedge : \mathbb{E}^3_1 \times \mathbb{E}^3_1 \to \mathbb{E}^3 \). If \( v_1, v_2, w \in \mathbb{E}^3_1 \), the vector \( v_1 \wedge v_2 \) is defined as the unique one that satisfies
\[
\langle v_1 \wedge v_2, w \rangle = \det (v_1, v_2, w),
\]
where \( \det (v_1, v_2, w) \) is the determinant of the matrix whose columns are the vectors \( v_1, v_2 \) and \( w \) with respect to the usual coordinates.

3 GCR Hypersurfaces in Minkowski spaces

Let \( M \) be a hypersurface in a semi-Euclidean space \( \mathbb{E}^{n+1}_s \) and \( x : M \to \mathbb{E}^{n+1}_s \) an isometric immersion. Since \( x \) can be considered as a vector field defined on \( M \), it can be expressed as
\[
x = x^T + x^\perp,
\]
where \( x^T \) and \( x^\perp \) denote the tangential and normal parts of \( x \). Before we proceed, we would like to recall the following definition. Note that this definition is given in [8] when the ambient space is \( \mathbb{E}^3_1 \).

Definition 3 Let \( M \) be a hypersurface in \( \mathbb{E}^{n+1}_s \). \( M \) is said to be a generalized constant ratio (GCR) hypersurface if the tangential part of its position vector is one of its principal directions.

Remark 4 We want to note that if \( M \) is a surface in \( S^2(1) \times \mathbb{E} \) or \( \mathbb{H}^2(-1) \times \mathbb{E} \), then \( U = x^T \) is called the canonical principle direction of \( M \) by some geometers provided \( U \) to be an eigenvector of the shape operator \( S \) of \( M \), [9, 10].

We also would like to note that if the ambient space is Euclidean, then the position vector \( x \) of \( M \) can be expressed as
\[
x = \mu \sin \theta e_1 + \mu \cos \theta N
\]
for some smooth functions \( \mu \) and \( \theta \). The following proposition obtained in [11] in order to give some equivalent conditions to being GCR (see also [8] for \( n = 2 \)).
**Proposition 5** Let $M$ be an oriented hypersurface in the Euclidean space $\mathbb{E}^{n+1}$, $e_1$ and $\theta$ the unit tangent vector field and the smooth function described in (3.2). Then, the following statements are equivalent:

(i) $M$ is a GCR hypersurface, i.e., $e_1$ is a principal direction of $M$,

(ii) $Y(\theta) = 0$ whenever $Y \in \Gamma(TM)$ is orthogonal to $e_1$,

(iii) All integral curves of $e_1$ are geodesics of $M$.

In the rest of this section, we show that Proposition 5 is still true in some cases if the ambient space is Minkowski. In this direction, we first obtain the following proposition.

**Proposition 6** Let $M$ be an oriented hypersurface in the Minkowski space $\mathbb{E}^{n+1}$ and $x$ its position vector. Assume that $x^T$ is not light-like and $e_1$ is the unit vector field along $x^T$. Then, $M$ is a GCR hypersurface if and only if a curve $\alpha$ is a geodesic of $M$ whenever it is an integral curve of $e_1$.

**Proof.** We will consider being space-like or time-like of $x^T$, separately.

**Case I.** Let $x^T$ be time-like. In this case, $e_1 = x^T / (\langle x^T, x^T \rangle)^{1/2}$ is time-like and $M$ is Lorentzian. Thus, we have

$$x = -(x, e_1)e_1 + (x, N)N.$$  

Since $\nabla_{e_1} x = e_1$, this equation yields

$$e_1 = (1 - \langle x, N \rangle \langle Se_1, e_1 \rangle)e_1 - \langle x, e_1 \rangle \nabla_{e_1} e_1 + \langle x, Se_1 \rangle N - \langle x, N \rangle Se_1.$$  

The tangential part of this equation yields $Se_1 = k_1 e_1$ if and only if $\nabla_{e_1} e_1 = 0$ which is equivalent to being geodesic of all integral curves of $e_1$.

**Case II.** Let $x^T$ be space-like. In this case, $e_1 = x^T / (\langle x^T, x^T \rangle)^{1/2}$ is space-like. Thus, we have

$$x = (x, e_1)e_1 + \varepsilon \langle x, N \rangle N,$$  

where $\varepsilon$ is either 1 or -1 regarding to being time-like or space-like of $M$, respectively.

Similar to Case I, we obtain $Se_1 = k_1 e_1$ if and only if $\nabla_{e_1} e_1 = 0$. 

Now, assume that $M$ is a space-like GCR hypersurface in $\mathbb{E}_1^{n+1}$ and $\{e_1, e_2, \ldots, e_n; N\}$ is a local orthonormal frame field consisting of principle directions of $M$, $k_1, k_2, \ldots, k_n$ are corresponding principle curvatures and $e_1$ is proportional to $x^T$. Note that, since $M$ is non-degenerated, we can, locally assume either $\langle x, x \rangle < 0$ or $\langle x, x \rangle > 0$ on $a$.

**Case I.** $\langle x, x \rangle < 0$. In this case, (3.3) implies

$$x = \mu \sinh \theta e_1 + \mu \cosh \theta N$$  

(3.4)
for some smooth functions $\theta$ and $\mu$. Note that $\mu$ satisfies $\langle x, x \rangle = -\mu^2$ from which we get

\begin{align*}
e_1(\mu) &= -\sinh \theta, \\
e_i(\mu) &= 0 \quad i = 2, 3, \ldots, n.
\end{align*}

(3.5a) (3.5b)

By a direct computation using (3.4), considering (3.5) and Proposition 5, we get

\begin{align*}
\nabla e_1 e_1 &= 0, \\
e_1 &= (-\sinh^2 \theta + \mu \cosh \theta e_1(\theta) - k_1 \mu \cosh \theta) e_1 \\
&\quad - k_1 \mu \sinh \theta - \cosh \theta \cosh \theta + \mu \sinh \theta e_1(\theta)) N, \\
e_i &= \mu \cosh \theta e_i(\theta) e_1 + \mu \sinh \theta \nabla e_1 e_1 \\
&\quad + \mu \sinh \theta e_i(\theta) N - k_1 \mu \cosh \theta e_i, \quad i = 2, 3, \ldots, n.
\end{align*}

(3.6a) (3.6b)

from which we obtain

\begin{align*}
k_1 &= e_1(\theta) - \frac{\cosh \theta}{\mu}, \\
e_i(\theta) &= 0, \\
\nabla e_1 e_1 &= 1 + \frac{k_1 \mu \cosh \theta \mu \sinh \theta}{\mu \cosh \theta} e_i.
\end{align*}

(3.7a) (3.7b) (3.7c)

**Case II.** $\langle x, x \rangle > 0$. In this case, (3.4) implies

\[ x = \mu \cosh \theta e_1 + \mu \sinh \theta N \]

(3.8)

for some smooth functions $\theta$ and $\mu$. By a similar way to case I, we obtain

\begin{align*}
e_1(\mu) &= \cosh \theta, \\
e_i(\mu) &= 0 \quad i = 2, 3, \ldots, n, \\
k_1 &= e_1(\theta) + \frac{\sinh \theta}{\mu}, \\
e_i(\theta) &= 0, \\
\nabla e_1 e_1 &= 1 + \frac{k_1 \mu \sinh \theta}{\mu \cosh \theta} e_i
\end{align*}

(3.9a) (3.9b) (3.9c) (3.9d) (3.9e)

By summing up the results obtained so far, we would like to state following proposition.

**Proposition 7** Let $M$ be a space-like hypersurface in the Minkowski space $\mathbb{R}^{n+1}_{1}$ and $\langle x, x \rangle < 0$ (resp. $\langle x, x \rangle > 0$). Then $M$ is GCR hypersurface if and only if $Y(\theta) = 0$, whenever $\langle Y, xT \rangle = 0$, where $\theta$ is the angle function define in (3.4) (resp. (3.8)).

**Proof.** The necessary condition follows from (3.7) (resp. (3.9)). The converse follows from a direct computation. 

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Remark 8 Proposition 6 and Proposition 7 shows that space-like GCR hypersurfaces in Minkowski spaces satisfies exactly same geometrical conditions with GCR hypersurfaces in Euclidean spaces.

Although, it is out of scope of this paper, we would like to state the following result which is a direct result of (3.7) and (3.9)

Proposition 9 Let $M$ be a space-like GCR hypersurface in the Minkowski space $\mathbb{E}^{n+1}$ and $e_1$ is a unit normal vector field along $x^T$. Then there exists a local coordinate function $s$ such that $e_1 = \partial_s$.

Proof. We consider the case $\langle x, x \rangle < 0$. The other case follows from an analogous computation.

Let $\zeta_1, \zeta_2, \ldots, \zeta_n$ be the dual base of $e_1, e_2, \ldots, e_n$. By a direct computation using (3.6a) and (3.7a), we obtain $d\zeta_1 = 0$, i.e., $\zeta_1$ is closed. Poincaré Lemma (see in [5]) yields that it is exact, i.e., there exists a local coordinate function $s$ such that $\zeta_1 = ds$. \qed

4 Complete classification of space-like GCR surfaces in $\mathbb{E}^3_1$

In this section we classify all space-like GCR surfaces in the Minkowski 3-space $\mathbb{E}^3_1$. Before we proceed, we would like to note that the complete classification of GCR surfaces in the Euclidean 3-space is given in [8].

Let $M$ be an oriented space-like surface in the 3-dimensional Minkowski space. For a generic point $p$ on $M$ immersed $\mathbb{E}^3_1 \setminus \{0\}$, we denote its position vector as $x$. The angle between two vectors in $\mathbb{E}^3_1$ is given in Definition 1 and Definition 2.

Here, we study those space-like surfaces $M$ in $\mathbb{E}^3_1$ which make a constant angle $\theta$ with position vector $x$. For a space-like surface $M$ in $\mathbb{E}^3_1$, the unit normal vector field $N$ is always time-like, i.e., $\langle N, N \rangle = -1$.

We define $\mu = \sqrt{\langle x, x \rangle}$. Since $M$ is space-like, the interior of set of points at which $\mu$ vanishes is empty. Therefore, locally, we assume that $\mu \neq 0$. Hence, we have either $\langle x, x \rangle < 0$ or $\langle x, x \rangle > 0$ over $M$. We will consider these two cases separately.

4.1 Space-like generalized constant ratio surfaces in $\mathbb{E}^3_1$ lying in the time-like cone

In this subsection, we will consider space-like surfaces satisfying $\langle x, x \rangle = -\mu^2 < 0$. In this case, $x$ can be decomposed as given in (3.4) for functions $\theta$ and $\mu$ defined in the previous section.

Assume that $M$ is a GCR surface, i.e., the function $\theta$ is satisfying $e_2(\theta) = 0$. Then, we have (3.5)-(3.7) for $n = 2$. 

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Lemma 10 The Levi-civita connection $\nabla$ and the shape operator $S$ of $M$ is given by

$$
\nabla e_1 e_1 = \nabla e_1 e_2 = 0, \\
\nabla e_2 e_1 = \frac{1 + \mu k_2 \cosh \theta}{\mu \sinh \theta} e_2, \\
\nabla e_2 e_2 = -\frac{1 + \mu k_2 \cosh \theta}{\mu \sinh \theta} e_1,
$$

(4.1a)

for a function $k_2$ satisfying

$$
S = \begin{pmatrix}
e_1(\theta) - \frac{\cosh \theta}{\mu} & 0 \\
0 & k_2
\end{pmatrix}
$$

(4.1c)

for a function $k_2$ satisfying

$$
e_1(k_2) = \left( e_1(\theta) - \frac{\cosh \theta}{\mu} - k_2 \right) \left( 1 + \frac{\mu k_2 \cosh \theta}{\mu \sinh \theta} \right)
$$

(4.2)

Proof. Equations given in (4.1) directly follow from (3.6a) and (3.7c) for $n = 2$. Moreover, by combining these equations with the Codazzi equation (2.5), we obtain (4.2).

Now, we are ready to obtain one of our main theorems.

Theorem 11 Let $x : M \rightarrow \mathbb{E}^3_1$ be a space-like surface immersed in the 3-dimensional Minkowski space $\mathbb{E}^3_1$. Also, assume that $M$ is lying in the time-like cone of $\mathbb{E}^3_1$. Then, $M$ is a GCR space-like surface if and only if it can be parametrized by

$$
x(s, t) = s (\cosh u \varphi(t) + \sinh u \varphi(t) \wedge \varphi'(t)),
$$

(4.3)

for a smooth function $u = u(s)$ and an arclength parametrized curve $\varphi = \varphi(t)$ lying on $\mathbb{H}^2(-1)$. In this case, $x$ can be decomposed as

$$
x = -s (\sinh \theta e_1 + \cosh \theta N)
$$

(4.4)

for the function $\theta$ given by

$$
\coth \theta = su',
$$

(4.5)

Proof. Let $M$ be a space-like GCR surface with the position vector $x$ satisfying $\langle x, x \rangle = -\mu^2$. Since $M$ is a GCR surface, $x$ can be decomposed as given in (3.4). Moreover, we have (3.6a), (3.7a) for $n = 2$.

It follows from (4.1a), (4.1b) that

$$
[e_1, e_2] = -\frac{1 + \mu k_2 \cosh \theta}{\mu \sinh \theta} e_2.
$$

(4.6)

Thus, if $m$ is a non-vanishing smooth function on $M$ satisfying

$$
\mu \sinh \theta e_1(m) = m(1 + \mu k_2 \cosh \theta),
$$

(4.7)

then we have $\left[ \frac{1}{\sinh \theta} e_1, me_2 \right] = 0$. Therefore, there exists a local coordinate system $(s, t)$ on $M$ such that $\frac{\partial}{\partial s} = \frac{1}{\sinh \theta} e_1$ and $\frac{\partial}{\partial t} = me_2$ which yields that the induced metric tensor $M$ is

$$
g = \frac{1}{\sinh^2 \theta} ds^2 + m^2 dt^2.
$$

(4.8)

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On the other hand, the angle $\theta$ is a function depending only on the variable $s$. Consequently, (3.5) for $n = 2$ and (4.7) become

$$\mu_s = -1, \mu_t = 0, \quad (4.9)$$

$$m_s = m \frac{1 + \mu k_2 \cosh \theta}{\mu \sinh^2 \theta} \quad (4.10)$$

Solving (4.9) one gets $\mu = -s + c_0$. Up to an appropriated translation on $s$, we may choose $c_0 = 0$.

Thus, equation (4.2) turns into

$$s^2 \sinh^2 \theta (k_2)_s = (s \sinh \theta \theta' + \cosh \theta - s k_2) (s k_2 \cosh \theta - 1) \quad (4.11)$$

If we put

$$k_2(s, t) = \frac{\sigma \sinh \theta + \cosh \theta}{s},$$

then the equation (4.11) reduces to

$$s \sinh \theta \sigma_s = \cosh \theta \left(1 - \sigma^2\right). \quad (4.12)$$

Note that the solution of equation (4.12) is

$$\sigma(s, t) = \tanh(u(s) + \phi(t))$$

where a smooth function $u$ satisfying (4.5) and an arbitrary function $\phi$ smooth on $M$. Hence, the function $k_2$ becomes

$$k_2(s, t) = \frac{1}{s} \left(\cosh \theta + \sinh \theta \tanh (u(s) + \phi(t))\right). \quad (4.13)$$

Substituting (4.13) into (4.10) and since $\mu = -s$

$$m(s, t) = s \cosh (u(s) + \phi(t)) \varphi(t) \quad (4.14)$$

for another non-zero function $\varphi$ depending only $t$. Considering the equation (4.8), one can obtain the Levi-Civita connection on $M$ as regards local coordinates $(s, t)$ on $M$

$$\nabla_{\partial_s} \partial_s = -\coth \theta \theta' \partial_s, \nabla_{\partial_s} \partial t = \frac{m_s}{m} \partial t, \nabla_{\partial t} \partial t = \left(-mm_s \sinh^2 \theta\right) \partial s + \frac{m_t}{m} \partial t.$$  

The equation (3.6a) and the first equation above applying into the Gauss formula, we have

$$\nabla_{\partial_s} \partial s = -\coth \theta \theta' \partial s + \left(\operatorname{csch} \theta \theta' + \operatorname{csch} \theta \frac{\coth \theta}{s}\right) N. \quad (4.15)$$

By taking the decomposition (3.4) into account and reordering (4.15), we get

$$s^2 \sinh^2 \theta \cosh \theta x_s + (s^2 \sinh \theta \theta' - s \sinh^2 \theta \cosh \theta) x_s - (\cosh \theta + s \sinh \theta \theta') x = 0. \quad (4.16)$$
Putting \( x = \Psi(s,t) \cdot s \), the previous equation turns into
\[
s^2 \sinh^2 \theta \cosh \theta \Psi_{ss} + (s \sinh^2 \theta \cosh \theta + s^2 \sinh \theta \theta') \Psi_s - \cosh^3 \theta \Psi = 0. \tag{4.17}
\]
Considering \( \Psi \), the PDE \( (4.17) \) can be rewritten as
\[
\Psi_{uu} - \Psi = 0.
\]
Solving this equation, we find that the position vector \( x \) can be expressed as
\[
x(s,t) = s (\cosh u \varphi(t) + \sinh u \psi(t)) , \tag{4.18}
\]
where both \( \varphi \) and \( \psi \) are vector-valued functions depending only on \( t \) in \( \mathbb{R}^3 \).

Consequently, by using the metric tensor \( \text{HES} \) we get first
\[
(cosh u + coth \theta \sinh u)^2 \langle \varphi(t), \varphi(t) \rangle + (sinh u + coth \theta \cosh u)^2 \langle \psi(t), \psi(t) \rangle + 2 \coth \theta (cosh u + \sinh u)^2 \langle \varphi(t), \psi(t) \rangle = \frac{1}{\sinh^2 \theta},
\]
which implies that
\[
- \langle \varphi(t), \varphi(t) \rangle = \langle \psi(t), \psi(t) \rangle = 1, \langle \varphi(t), \psi(t) \rangle = 0.
\]
Moreover, we have
\[
\langle \varphi'(t), \varphi'(t) \rangle = \phi^2(t) \cosh^2 \phi(t),
\]
\[
\langle \psi'(t), \psi'(t) \rangle = \phi^2(t) \sinh^2 \phi(t),
\]
\[
\langle \varphi'(t), \psi'(t) \rangle = \phi^2(t) \sinh \phi(t) \cosh \phi(t),
\]
\[
\langle \varphi'(t), \psi(t) \rangle = \langle \varphi(t), \psi'(t) \rangle = 0.
\]
By redefining \( t \) suitable, we may assume \( \langle \varphi'(t), \varphi'(t) \rangle = 1 \). Then, \( \{ \psi, \varphi, \varphi' \} \) form an orthonormal base. Therefore, we have \( \psi = \varphi \wedge \varphi' \). Hence, \( (4.18) \) becomes \( \text{HES} \) which yields that \( M \) can be parametrized as given in the theorem.

Conversely, assume that \( M \) is the surface parametrized by \( (4.3) \) for a non-vanishing smooth function \( \theta \) and a smooth function \( u \) satisfying \( (4.5) \). Then, there exists a function \( C(t) \) such that \( \varphi \wedge \varphi' = C(t) \varphi' \). Therefore, vector fields \( e_1 = \sinh \theta \partial_s \) and \( e_2 = \frac{1}{\text{coth} u + C(t) \sinh u} \partial_t \) spans the tangent bundle of \( M \) and the unit normal vector field \( N \) of \( M \) is given by
\[
N = \cosh(\theta + u) \varphi + \sinh(\theta + u) \varphi \wedge \varphi'. \tag{4.19}
\]
A further computation yields that \( e_1 \) is a principle direction of \( M \) and \( x \) can be decomposed as given in \( (4.4) \).

In order to present an explicit example, we would like to give the following classification of flat GCR surfaces.
Corollary 12 Let \( x : M \rightarrow \mathbb{E}^3_1 \) be a space-like surface immersed in the 3-dimensional Minkowski space \( \mathbb{E}^3_1 \). Also, assume that \( M \) is lying in the time-like cone of \( \mathbb{E}^3_1 \). Then, \( M \) is a flat GCR space-like surface if and only if it can be parametrized as given in (4.3) for the function \( u \) given by

\[
  u(s) = c_1 - \cosh^{-1} \frac{c_2}{s},
\]

where \( c_1 \) and \( c_2 \) are some constants.

**Proof.** Let \( M \) be the GCR surface given by (4.3) for a function \( u \) and \( \theta \) the angle function. Then, \( \theta \) satisfies (4.5).

Consider the unit tangent normal vector fields \( e_1 = \sinh \theta \partial_s \) and \( e_2 = \frac{1}{\cosh u + C(t) \sinh u} \partial_t \) given in the proof of Theorem 11 and unit normal vector field \( N \) given in (4.19). Then, by a direct computation we see that \( e_1 \) and \( e_2 \) are principle directions of \( M \) with corresponding principle curvatures

\[
  k_1 = \theta' + u'
\]

and \( k_2 \), respectively. Note that, by considering (4.2), one can see that \( K = k_1 k_2 = 0 \) implies \( k_1 \equiv 0 \) on \( M \).

Now, assume that \( M \) is flat, i.e., \( K = 0 \). Then, (4.21) implies \( \theta' + u' = 0 \). From this equation and (4.5) we obtain

\[
  u' = \frac{\coth(-u + c_1)}{s}
\]

for a constant \( c_1 \). By solving this equation, we get (4.20). \( \blacksquare \)

4.2 Space-like generalized constant ratio surfaces in \( \mathbb{E}^3_1 \) lying in the space-like cone

In this subsection, we will consider study space-like surfaces with the position vector \( x \) satisfying \( \langle x, x \rangle < 0 \). In this case \( x \) can be decomposed as given in (3.8) for some smooth functions \( \theta \) and \( \mu \).

Now suppose that \( M \) is a GCR surface, i.e., the function \( \theta \) is satisfying \( e_2(\theta) = 0 \). Then, we have (3.9) for \( n = 2 \).

First we would like to state the following lemma which is obtained by an exactly similar way with Lemma 10.

**Lemma 13** The Levi-civita connection \( \nabla \) and the shape operator \( S \) of \( M \) is given by

\[
  \nabla_{e_1}e_1 = \nabla_{e_2}e_2 = 0,
\]

\[
  \nabla_{e_2}e_1 = \frac{1 + \mu k_2 \sinh \theta}{\mu \cosh \theta} e_2,
\]

\[
  \nabla_{e_2}e_2 = -\frac{1 + \mu k_2 \sinh \theta}{\mu \cosh \theta} e_1,
\]

\[
  S = \begin{pmatrix}
    e_1(\theta) + \frac{\sinh \theta}{\mu} & 0 \\
    0 & k_2
  \end{pmatrix}
\]
for a function \( k_2 \) satisfying
\[
e_1(k_2) = \left( e_1(\theta) + \frac{\sinh \theta}{\mu} - k_2 \right) \left( \frac{1 + \mu k_2 \sinh \theta}{\mu \cosh \theta} \right) \quad (4.23)
\]

Before we proceed, we would like to obtain the following corollary directly obtained from Lemma 13.

**Corollary 14** Let \( M \) be a space-like GCR surface in \( \mathbb{E}^3_1 \) lying in the space-like cone. Then, \( M \) is flat if and only if \( \theta \) satisfies
\[
e_1(\theta) = \frac{\sinh \theta}{\mu}.
\]

**Proof.** If \( M \) is a flat GCR surface, then (4.22c) implies
\[
\left( e_1(\theta) + \frac{\sinh \theta}{\mu} \right) k_2 = 0. \quad (4.24)
\]
Consider the open subset \( \mathcal{O} \) of \( M \) given by
\[
\mathcal{O} = \left\{ p \in M | e_1(\theta) + \frac{\sinh \theta}{\mu} \neq 0 \text{ at } p \right\}.
\]
Because of (4.24), we have \( k_2 = 0 \) on \( \mathcal{O} \). However, because of (4.23), this is a contradiction unless \( \mathcal{O} \) is empty. ■

Now, we are ready to obtain the following theorem.

**Theorem 15** Let \( x : M \to \mathbb{E}^3_1 \) be a space-like surface immersed in the 3-dimension Minkowski space \( \mathbb{E}^3_1 \). Also, assume that \( M \) is lying in the space-like cone of \( \mathbb{E}^3_1 \). Then, \( M \) is a GCR space-like surface if and only if it can be parametrized by
\[
x(s, t) = s (\cosh u \varphi(t) + \sinh u \varphi(t) \wedge \varphi'(t)), \quad (4.25)
\]
for a smooth function \( u = u(s) \) and an arc-length parametrized curve \( \varphi = \varphi(t) \) lying on \( S^2_1(1) \). In this case, \( x \) can be decomposed as
\[
x = s (\cosh \theta e_1 + \sinh \theta N). \quad (4.26)
\]
for the function \( \theta \) given by
\[
tanh \theta = su' \quad (4.27)
\]

**Proof.** Let \( M \) be a GCR space-like surface with the position vector \( x \) satisfying \( \langle x, x \rangle = \mu^2 \). Consequently, \( x \) can be decomposed as given in (3.8) and we also have (4.9) for \( n = 2 \) and (4.22).

In a similar way with the proof of Theorem 11 by using (4.22a) and (4.22b) we see that there is a local coordinate system \((s, t)\) on \( M \) such that
\[
\frac{1}{\cosh \theta} e_1 \quad \text{and} \quad \frac{\partial}{\partial t} = me_2 \quad \text{for a smooth non-vanishing function } m \text{ satisfying}
\]
\[
\mu \cosh \theta e_1(m) - m (1 + \mu k_2 \sinh \theta) = 0. \quad (4.28)
\]
Therefore, the induced metric tensor $g$ of $M$ becomes

$$g = \frac{1}{\cosh^2 \theta} ds^2 + m^2 dt^2. \quad (4.29)$$

Moreover, (3.9a), (3.9b) for $n = 2$, (4.23) and (4.28) give

$$\mu_s = 1, \mu_t = 0, \quad (4.30)$$

$$\mu \cosh^2 \theta m_s - m (1 + \mu k_2 \sinh \theta) = 0, \quad (4.31)$$

$$s^2 \cosh^2 \theta (k_2)_s - (s \cosh \theta \theta' + \sinh \theta - sk_2) (1 + sk_2 \sinh \theta) = 0. \quad (4.32)$$

Because of (4.30), up to an appropriated translation, we may assume $\mu = s$.
Moreover, by defining $\sigma$ as

$$\sigma(s, t) = \cosh (u(s) + \phi(t)), \quad (4.33)$$

we see that (4.32) become

$$s \cosh \theta \sigma_s + \sinh \theta (\sigma^2 - 1) = 0. \quad (4.33)$$

By solving (4.33), we obtain

$$\sigma(s, t) = \tanh(u(s) + \phi(t)), \quad (4.33)$$

where $\phi$ and $u$ are some smooth functions satisfying (4.27).

On the other hand, since $\mu = s$ and

$$k_2(s, t) = \frac{1}{s} (\sinh \theta + \cosh \theta \tanh(u(s) + \phi(t))), \quad (4.31)$$

gives

$$m = s \cosh(u(s) + \phi(t)) \varphi(t) \quad (4.34)$$

for a non-vanishing function $\varphi = \varphi(t)$.

Next, in a similar way with the proof of Theorem 11, we get $\nabla_{\partial_s} \partial_s, \nabla_{\partial_t} \partial_t$ and $\nabla_{\partial_t} \partial_t$. Then, by considering (1.22), we see that the position vector $x$ of $M$ satisfies

$$s^2 \cosh^2 \theta \sinh \theta x_s - (s^2 \cosh \theta \theta' - s \cosh^2 \theta \sinh \theta) x_s + (\sinh \theta + s \cosh \theta \theta') x = 0. \quad (4.35)$$

Next, we define $\Psi$ by $x = \Psi(s, t) \cdot s$ and consider (4.27) to obtain

$$\Psi_{uu} - \Psi = 0$$

from (4.33). By solving this equation we get

$$x(s, t) = s (\cosh u \varphi + \sinh u \psi), \quad (4.36)$$

where $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are smooth $\mathbb{E}^3_1$ valued functions.

By a similar way to proof of Theorem 11 we get $\langle \varphi(t), \varphi(t) \rangle = - \langle \psi(t), \psi(t) \rangle = 1$. Further, by redefining $t$ suitable, we may assume $\langle \varphi(t), \varphi(t) \rangle = 1$ which yields that $\varphi$ is an arc-length parametrized curve lying on $S^1_2(1)$ and $\psi = \varphi \wedge \varphi'$.

Hence, (4.36) turns into (4.26) which completes the proof of necessary condition.

The proof of sufficient condition can be established by a direct computation similar to the proof of Theorem 11.

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