Polynomially-bounded Dehn functions of groups

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Abstract

On the one hand, it is well known that the only subquadratic Dehn function of finitely presented groups is the linear one. On the other hand there is a huge class of Dehn functions $d(n)$ with growth at least $n^4$ (essentially all possible such Dehn functions) constructed in [19] and based on the time functions of Turing machines and S-machines. The class of Dehn functions $n^\alpha$ with $\alpha \in (2;4)$ remained more mysterious even though it has attracted quite a bit of attention (see, for example, [3]). We fill the gap obtaining Dehn functions of the form $n^\alpha$ (and much more) for all real $\alpha \geq 2$ computable in reasonable time, for example, $\alpha = \pi$ or $\alpha = e$, or $\alpha$ is any algebraic number. As in [19], we use S-machines but new tools and new way of proof are needed for the best possible lower bound $d(n) \geq n^2$.

Key words: generators and relations in groups, finitely presented group, Dehn function of group, Turing machine, S-machine, van Kampen diagram.

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1 Introduction

1.1 Formulation of the theorem and corollaries

The minimal non-decreasing function \( d(n) : \mathbb{N} \rightarrow \mathbb{N} \) such that every word \( w \) vanishing in a group \( G = \langle A \mid R \rangle \) and having length \( ||w|| \leq n \) is freely equal to a product of at most \( d(n) \) conjugates of relators from \( R^{\pm 1} \), is called the \textit{Dehn function} of the presentation \( G = \langle A \mid R \rangle \) \cite{5}. In other words, the Dehn function \( d(n) \) of the presentation is the smallest function that bounds from above the areas of loops of length \( \leq n \) in the Cayley complex \( \text{Cay}(G) \), and so by van Kampen’s Lemma, \( d(n) \) is equal to the maximal area of minimal filling diagrams \( \Delta \) with perimeter \( \leq n \). (See Subsection 5.2 for the definitions.)

For a \textit{finitely presented} group (i.e., both sets \( A \) and \( R \) are finite), the Dehn functions exist and it is usually taken up to equivalence to get rid of the dependence on a finite presentation of \( G \) (see \cite{9}). To introduce this \textit{equivalence} \( \sim \), we write \( f \preceq g \) if there is a positive integer \( c \) such that \( f(n) \leq cg(cn) + cn \) for every \( n \in \mathbb{N} \). Two non-decreasing functions \( f \) and \( g \) on \( \mathbb{N} \) are called equivalent if \( f \preceq g \) and \( g \preceq f \).

Note that for many equivalence classes (for example, in \( \sim \)-classes of the functions \( n^\alpha \), \( n^\alpha (\log n)^\beta (\log \log n)^\gamma \), and so on), \( \sim \)-equivalence coincides with \( \Theta \)-\textit{equivalence}, where the symbol \( \Theta \) is borrowed from the theory of computational complexity: one says that \( f(n) = \Theta(g(n)) \) if both properties \( f(n) = \mathcal{O}(g(n)) \) and \( g(n) = \mathcal{O}(f(n)) \) hold.

The Dehn function \( d(n) \) of a finitely presented group \( G \) is also called an isoperimetric function of \( G \) since it is equivalent to the usual isoperimetric function of a simply connected Riemannian manifold \( M \), provided \( G \) acts properly and co-compactly on \( M \) by isometries. So the concept of Dehn function is derived from Geometry, and one can find much more regarding this connection in \cite{6}.

Another connection is to Computational Complexity. The algorithmic word problem in a finitely presented group is decidable if and only if the Dehn function is recursive, and
the Dehn function of a group bounds the computational complexity of the word problem. It was shown in [1] that conversely, every recursively presented group $G$ embeds into a finitely presented group whose Dehn function is only polynomially larger than the computational complexity (the time function) of the word problem in $G$. In particular, groups with word problem in $\mathbf{NP}$ are precisely the subgroups of finitely presented groups with at most polynomial Dehn functions.

Almost all possible Dehn functions $\geq n^4$ of finitely presented groups were described in [19] in terms of time functions of non-deterministic Turing machines. To formulate one of the main corollaries, we recall

**Definition 1.1.** Let $T(m)$ be a function on $\mathbb{N}$. A real number $\alpha$ is computable with time $\leq T(m)$ if there exists a Turing machine which, given a natural number $m$, computes a binary rational approximation of $\alpha$ with an error $O(2^{-m})$, and the time of this computation $\leq T(m)$.

It was proved in [19] that if $\alpha \geq 4$ and the real number $\alpha$ is computable with time $\leq 2^{2^m}$, then there is a finitely presented group with Dehn function equivalent to $n^\alpha$. (One should use the integral part sign for functions on $\mathbb{N}$, but we omit this sign speaking on asymptotic behavior.) The algebraic and many transcendental numbers are computable much faster, and so there are examples of Dehn functions equivalent to $n^{\pi + e}$, and so on. (Note that the computability conditions are inevitable since the set of real numbers is uncountable, while the set of non-isomorphic finitely presented groups is countable.)

Still the class of Dehn functions $< n^4$ was unclear even though it has drawn attention. We note the paper [3] (also the references there), where the Dehn functions equivalent to $n^\alpha$ were constructed for special non-integer exponents of the form $\alpha = \log_2^p q$, where $p$ and $q$ are integers, $p > q > 0$.

Since all finitely presented groups with subquadratic Dehn functions are hyperbolic [5, 11, 2], the only interval $2 < \alpha < 4$ remained misty. To formulate the theorem that fills this gap, we need

**Definition 1.2.** For brevity, we say that a non-decreasing function $f : \mathbb{N} \rightarrow \mathbb{N}$ is suitable if the following properties hold.

- $f(n)^3 = O(n)$
- For every integer $c > 0$, there is $C > 0$ such that $f(cn) \leq Cf(n)$, i.e.,
  \[ f(O(n)) = O(f(n)) \]

- There is a (non-deterministic) Turing machine $M_0$ recognising the values of the function $f(n)$ with computation time $O(n^{1/3})$.

It works as follows. An integer $k \geq 0$ is an input of $M_0$ in the form $a^k$ for a fixed letter $a$. The machine $M_0$ produces a value $f(n)$ for some $n \geq 1$, i.e. it obtains the word $c^{f(n)}$ on a special tape. Then $M_0$ compares $f(n)$ and $k$, accepting $k$ if $k = f(n)$. It can accept an input word $a^k$ if and only if $k = f(n)$ for some natural number $n$.

**Theorem 1.3.** For every suitable function $f(n)$ and every integer $s \geq 2$, the function $F(n) = n^s f(n)^3$ is equivalent to the Dehn function of a finitely presented group.
Given a suitable function \( f(n) \), we denote \( g(n) = f(n)^3 \).

**Corollary 1.4.** If \( \alpha \geq 2 \) and the real number \( \alpha \) is computable with time \( O(2^{2m}) \), then there is a finitely presented group with Dehn function equivalent to \( n^\alpha \).

**Proof.** Assume that \( \alpha \) is computable with time \( O(2^{2m}) \). Clearly, the same property holds for the number \( \beta = \frac{1}{3}(\alpha - s) \), where \( s = [\alpha] \).

Let a (non-deterministic) Turing machine \( M_0 \) produces \( r = [n^{1/3}] \geq 1 \) (in unary) with time \( O(r) \). Then it computes \( \lfloor \log_2 r \rfloor \) and \( m = \lfloor \log_2 \log_2 r \rfloor \) (in binary) with time \( O(r) \) using divisions by 2.

It follows from the assumption of the corollary that one can recursively compute binary rational numbers \( \beta_m \) such that

\[
|\beta - \beta_m| = O(2^{-m}) = O((\log_2 r)^{-1})
\]  
(1.1)

and the time of the computation of \( \beta_m \) is \( O(r) \). Let \( M_0 \) accomplish this computation. In addition, one may assume that the number of digits in the binary expansion of \( \beta_m \) is \( O(m) \). Therefore, the computation of the product \( [\beta_m \log_2 r] \) needs time at most \( O((\log_2 n)^2) \). Since \( r = [n^{1/3}] \), the machine can now obtain the binary presentation \( \gamma_m \) of \( [\beta_m \log_2 n] \) with time \( O((\log_2 n)) \) and error \( O(1) \).

Next, let \( M_0 \) rewrite the binary presentation of \( \gamma_m \) in unary (as a sequence of 1-s). This well-known rewriting (e.g., see p.352 in [19]) has time complexity of the form \( O(\gamma_m) = O(\beta_m \log_2 n) \). One more \( M_0 \)-rewriting of this type applied to the unary presentation of \( \gamma_m \) (considered now as binary one), will have time complexity of the form

\[
\Theta(2^{\gamma_m}) = \Theta(2^{[\beta_m \log_2 n] + O(1)}) = \Theta(2^{[\beta_m \log_2 n]})
\]

One can rewrite as \( \Theta(2^{\beta \log_2 n}) = \Theta(n^{\beta}) = O(n^{1/3}) \) by inequalities (1.1) and \( \beta < 1/3 \), because \( \log_2 n = O(2^m) \).

During the last (deterministic) rewriting, one can count the number \( f(n) \) of commands and obtain a word \( b^f(n) \) on a special tape, where \( f(n) = \Theta(n^{\beta}) \). It is easy to see that the rewriting can be defined so that the function \( f(n) \) to be non-decreasing.

One more tape of the machine \( M_0 \) under construction has the input word \( a^k \). It remains to check whether the lengths of the words \( b^f(n) \) and \( a^k \) are equal or not. This takes the time \( O(f(n)) = O(n^{1/3}) \). Since the time of the entire procedure is \( O(n^{1/3}) \), the function \( f(n) \) is suitable. Now by Theorem 1.3, the function \( n^\alpha f(n)^3 = \Theta(n^\alpha n^{3\beta}) = \Theta(n^\alpha) \) is equivalent to the Dehn function of a finitely presented group. \( \square \)

For example, Corollary 1.4 implies

**Corollary 1.5.** The functions \( n^\alpha \) for every real algebraic \( \alpha \geq 2 \), the functions \( n^{\pi - 1}, n^{\sqrt{\pi} + 1}, \ldots \) are equivalent to Dehn functions of finitely presented groups.

\( \square \)

Corollary 1.4 looks like the best possible because, as was proved in [19], if, conversely, \( n^\alpha \) is equivalent to the Dehn function of a finitely presented group, then \( \alpha \) is computable in time \( 2^{2^{2m}} \), and if, in addition, \( P=NP \), then one can lower that to \( 2^{2^m} \). Thus, this corollary implies all known results about the class of Dehn functions of the form \( n^\alpha \). As we mentioned above, the analog of Corollary 1.4 for \( \alpha \geq 4 \) was proved in [19]. But weakening the restriction to \( \alpha \geq 2 \), although uses \( S \)-machines, as in [19], requires substantial new ideas.
Theorem 1.3 gives a tremendous class of new Dehn functions with growth rate at most \( n^4 \). The following examples can be validated in absolutely similar way as Corollary 1.4.

**Corollary 1.6.** The functions \( n^\alpha (\log n)^\beta, n^\alpha (\log n)^\beta (\log \log n)^\gamma, \ldots \) are equivalent to the Dehn functions of finitely presented groups, provided the real \( \alpha, \beta, \gamma, \ldots \) are computable in time \( O(2^{2^m}) \) and \( \alpha > 2 \) or \( \alpha = 2, \beta > 0 \), or \( \alpha = 2, \beta = 0, \gamma > 0 \), or...

\[ \square \]

Since every finitely presented group is a fundamental group of a connected closed Riemannian manifold \( X \) and therefore acts properly and co-compactly by isometries on its universal cover \( \tilde{X} \), one can use Theorem 1.3 and Corollary 1.4 to formulate one more

**Corollary 1.7.** For every function \( F(n) \) satisfying the assumption of Theorem 1.3, there exists a closed connected Riemannian manifold \( X \) such that the isoperimetric function of the universal cover \( \tilde{X} \) is equivalent to \( F(n) \).

In particular, if a real number \( \alpha \geq 2 \) is computable with time \( O(2^{2^m}) \), then there exists such a universal cover \( \tilde{X} \) with isoperimetric function equivalent to \( n^\alpha \).

\[ \square \]

Since the condition \( \alpha \geq 2 \) is the best possible and the obtained upper bound for the Dehn function must be equivalent to the lower one, all inequalities throughout this paper should be uniformly sharp, up to multiplicative constants.

We collect all the definitions and terms at the end of the paper (see Subject Index). The next subsection presents a short outline of the plan.

### 1.2 Brief description of the proof of Theorem 1.3

The idea of simulation of the commands of a Turing machine by group relations goes back to the works of P. Novikov, W. Boone and many other authors (see [17], [18]). However one has to properly code the work of a Turing machine in terms of group relations, and the interpretation problem for groups remains much harder than for semigroups, because the group theoretic simulation can execute unforeseen computations with non-positive words. Boone and Novikov secured the positiveness of admissible configurations with the help of an additional ‘quadratic letter’ (see [17], Ch.12). However this old trick implies that the constructed group \( G \) contains Baumslag-Solitar subgroups \( B_{1,2} \) and has at least exponential Dehn function. Since we want to obtain at most polynomial Dehn functions, we use the machines introduced in [19]. Those S-machines invented by M. Sapir can work with non-positive words on the tapes and they are polynomially equivalent to classical Turing machines.

According to the original version, S-machines are special rewriting systems. All necessary definitions are given in Subsection 2.1. On the other hand, the state, tape and command letters of an S-machine can be regarded as group generators, and the commands can be interpreted as defining relations (see Subsection 5.1). The obtained group \( M \) is a multiple HNN-extension of a free group. Every computation of the S-machine is simulated by a van Kampen diagram over this group called trapezia (Subsection 5.2).

To construct a finitely presented groups \( G \) with desired Dehn functions, one needs to add a special relation called the **hub**. It consists of state letters. There are very particular van Kampen diagrams, called disks, built of the hub and many trapezia attached around.
It is proved in [19] that for every Turing machine $M_0$ with a time function $T(n)$, there is an equivalent $S$-machine $M_1$ with time function (and the generalized time function) $\sim T(n)^3$. It follows that if an accepting computation starts with an input of length $\sim n$, it has has length $\sim T(n)^3$, and the computational disk has perimeter $\sim n$, and so its area $\sim nT(n)^3$. Since time functions are at least linear, this approach gave the lower bound $\geq n^4$ for the Dehn function of $G$.

To get the linear time of accepting computation, in comparison with the length of the initial configuration for an $S$-machine $M_2$, one can add an additional tape, where the whole history of the forthcoming computation is written. Then every command will erase one letter on this tape. This trick gives disks with quadratic area with respect of their perimeters.

However we want to construct disks with prescribed area $F(n)$, as in Theorem 1.3. In this paper, we first prove this theorem for $s = 2$, i.e. for $F(n) = O(n^3)$, and in the final subsection 10.2, we show that the value of $s$ can be increased, because a non-difficult modification of the main machine $M$ constructed in Subsection 4.1 linearly slows down the work of $M$.

The main machine is composed of the machine $M_2$ repeating the same computation many times and another machine that can stop the computations $M_2$ after $\sim g(n)$ such cycles with subsequent erasing all the tapes and accepting. This gives the lower bound $\sim ng(n)$ for the (general) time function and the lower bound $\sim F(n)$ for the areas of computational disks.

The obtainment of upper bounds is the major job in this paper. First of all, to obtain quadratic upper bound for the areas of trapezia, one needs a linear bound of the space of every computation (i.e. the maximal lengths of all configurations) in terms of the lengths of the first and the last configurations. This task is aggravated by inaccurate simulation of the work of Turing machines by $S$-machines and so by group relations. Standard trapezia correspond to the prescribed work of machines, but there are non-standard ones simulating undesired computations when the same tape is simultaneously changed at both ends. The features of standard (accurate) computations of the main machine and of non-standard ones are considered in Subsections 4.2 and 4.3, respectively. To reduce the effect of non-standard computations, the basic steps of the work alternate with control steps in the definition of the main machine given in Subsection 4.1.

Whereas a non-standard computation has linearly bounded space in terms of the lengths of the first and the last configurations (and so the width of corresponding trapezia is linearly bounded too), there exist much wider trapezia in the standard case. Hence standard trapezia can have too big, cubic areas in the group $M$. The new idea is to make the area of their boundary labels quadratic in the quotient group $G$. We do this in Subsection 10.1 applying the properties of long computations obtained in Subsection 4.5. It turned out that wide trapezia with super-quadratic areas in $M$ can be replaced, preserving the boundary label, by diagrams of quadratic areas over $G$, i.e. by diagrams containing hubs.

However before that, we prescribe artificial $G$-areas to special 'big' subtrapezia (Subsection 6.3), which leads to the definition of $G$-area for arbitrary diagram over $M$ or over $G$. A quadratic upper bound for the $G$-areas of diagrams over $M$ is given in Section 6.3. The induction on the perimeter is based on a non-trivial surgery. (See the proof of Lemma 6.16 in Subsection 6.3, where we cut and paste diagrams.) In fact, we give an upper bound for an area in terms of the perimeter and the mixture of $q$- and $\theta$-letters in the boundary label, because different types of surgeries decrease either perimeter or
the mixture. The mixture is defined for arbitrary necklace with beads of two colors (see Subsection 6.7), and it is bounded by the square of the number of beads, and so we finally obtain a quadratic upper bound in terms of perimeter only.

Note that instead of the combinatorial length of words (length of paths, perimeter) we consider a modified length, where different letters and syllabi have different lengths (Subsection 6.1). With respect to this modified length $|\cdot|$, the length of the top/bottom of every $q$-band $Q$ is just the number of 2-cells in $Q$, and the rim $\theta$-bands with bounded number of $(\theta, q)$-cells can be removed from a diagram with decrease of perimeter. Such properties are exploited in the paper many times. It is easy to reformulate the final results in terms of the combinatorial length $||\cdot||$ since it is $\Theta$-equivalent to $|\cdot|$. Our presentation of the group $G$ (Subsection 5.1) is highly non-aspherical, and so van Kampen diagrams with the same boundary label can differ widely. We choose minimal diagrams in Section 7, i.e. reduced diagrams with minimal number of disks and, for given number of disks, with minimal number of $(\theta, q)$-cells. We do not claim that a minimal diagram has minimal area or minimal $G$-area for fixed boundary label, but to obtain the upper bound for the Dehn function, it suffices to bound from above the $G$-areas of minimal diagrams.

However, even one has quadratic estimates for $G$-areas of disk-free diagrams (i.e. diagrams over $M$) and the required upper bound $\sim F(n)$ for the areas of disks, the 'snow ball' decomposition of diagrams from [19] gives at least cubic upper bound for the $G$-area of the entire minimal diagram. Thus, without new tools one could only hope to weaken the restriction from [19] to $\alpha \geq 3$.

The helpful property is that a minimal diagram cannot contain subdiagram formed by a disk and a very special trapezium connected by a 'shaft' (Section 7). At first sight, it seems that such subdiagrams are extremely rare. But they become ordinary if the work of machine has sufficiently many control steps. Hence the absence of these exotic subdiagrams can help. And it helps indeed provided the sum $\sigma$ of the lengths of all 'shafts' linearly bounded in terms of the perimeter.

To obtain such a linear estimate, we prove a pure combinatorial proposition on the designs formed by two finite sets of segments. There are neither machines nor groups, nor van Kampen diagrams in Section 8, and so the reader can start with that short section. (Is there even shorter proof or a reference to a known property? Although the author did believe that the linear estimate took place, he wasted a time devising a proof.)

Since $\sigma \leq cn$ for some constant $c > 0$, where $n$ is the perimeter, one can estimate the $G$-area of a diagram in terms of the sum $n + \sigma$ instead of $n$ (Section 9), and this is another new tool for obtaining of the required upper bound for the area.

2 General properties of $S$-machines

2.1 $S$-machine as rewriting system

There are several interpretations of $S$-machines in groups. In particular, one can define an $S$-machine as a group that is a multiple HNN extension of a free group. Here we use the original definition [19] and define $S$-machines as rewriting systems working with words in group alphabets. The precise definition of an $S$-machine $S$ is as follows.

A hardware of an $S$-machine $S$ is a pair $(Y, Q)$, where $Q = \sqcup_{i=0}^{N} Q_i$ and $Y = \sqcup_{i=1}^{N} Y_i$ (for convenience we always set $Y_0 = Y_{N+1} = \emptyset$). The elements from $Q$ are called state
letters, the elements from \( Y \) are tape letters. The sets \( Q_i \) (resp. \( Y_i \)) are called parts of \( Q \) (resp. \( Y \)).

The language of admissible words consists of reduced words \( W \) of the form

\[
q_i^{±1}u_1q_2^{±1} \ldots u_kq_k^{±1},
\]

where every subword \( q_i^{±1}u_1q_2^{±1} \) either

- belongs to \( (Q_jF(Y_{j+1})Q_{j+1})^{±1} \) for some \( j \) and \( u_i \in F(Y_{j+1}) \), where \( F(Y_i) \) is the set of reduced group words in the alphabet \( Y_i^{±1} \), or
- has the form \( quq^{-1} \) for some \( q \in Q_j \) and \( u \in F(Y_{j+1}) \), or
- is of the form \( q^{-1}uq \) for \( q \in Q_j \) and \( u \in F(Y) \).

We shall follow the tradition of calling state letters \( q \)-letters and tape letters \( a \)-letters, even though we shall use other letters as state or tape letters. The number of \( a \)-letters in a word \( W \) is the \( a \)-length \( |W|_a \) of \( W \). Usually parts of the set \( Q \) of state letters are denoted by capital letters. For example, a set \( P \) would consist of letters \( p \) with various indices. Then we shall say that letters in \( P \) are \( p \)-letters or \( P \)-letters.

If a group word \( W \) over \( Q \cup Y \) has the form \( q_1u_1q_2u_2 \ldots q_s \), and \( q_i \in Q_j^{±1} \), \( i = 1, \ldots, s \), \( u_i \) are group words in \( Y \), then we shall say that the base of \( W \) is \( \text{base}(W) \equiv Q_j^{±1}Q_j^{±1} \ldots Q_j^{±1} \). Here \( Q_i \) are just letters, denoting the parts of the set of state letters. Note that the base is not necessarily a reduced word, and the sign \( \equiv \) is used for letter-by-letter equality of words. The subword of \( W \) between the \( Q_j^{±1} \)-letter and the \( Q_j^{±1} \)-letter will be called a \( Q_j^{±1}Q_j^{±1} \)-sector of \( W \). A word can have many \( Q_j^{±1}Q_j^{±1} \)-sectors.

Instead of specifying the names of the parts of \( Q \) and their order as in \( Q = Q_0 \cup Q_2 \cup \ldots \cup Q_N \), we say that the standard base of the \( S \)-machine is \( Q_0 \ldots Q_N \).

An \( S \)-machine also has a set of rewriting rules \( \Theta \). Every \( \theta \in \Theta \) is assigned two sequences of reduced words: \([U_0, \ldots, U_m]\), \([V_0, \ldots, V_m]\), and a subset \( Y(\theta) = \cup Y_j(\theta) \) of \( Y \), where \( Y_j(\theta) \subseteq Y_j \). A rule has the form:

\[
[U_1 \rightarrow V_1, \ldots, U_m \rightarrow V_m]
\]

where the following conditions hold:

- Each \( U_i \) and \( V_i \) is a subword of an admissible word with a base \( Q_\ell Q_{\ell+1} \ldots Q_r \) (\( \ell = \ell(i) \leq r = r(i) \))
- \( \ell(i + 1) = r_i + 1 \) for \( i = 1, \ldots, m - 1 \).
- \( U_1 \) and \( V_1 \) must start with a \( Q_1 \)-letter and \( U_m \) and \( V_m \) must end with a \( Q_N \)-letter (so tape letters are not inserted to the left of \( Q_1 \)-letters and to the right of \( Q_N \)-letters).

The pair of words \( U_i, V_i \) is called a part of the rule, and is denoted \([U_i \rightarrow V_i]\).

The notation \( \theta : [U_1 \rightarrow V_1, \ldots, U_m \rightarrow V_m] \) contains all the necessary information about the rule except for the sets \( Y_j(\theta) \). In most cases it will be clear what these sets are, and very often the sets \( Y_j(\theta) \) will be equal to either \( Y_j \) or \( \emptyset \). By default \( Y_j(\theta) = Y_j \). Every \( S \)-rule \( \theta = [U_1 \rightarrow V_1, \ldots, U_m \rightarrow V_m] \) has an inverse \( \theta^{-1} = [V_1 \rightarrow U_1, \ldots, V_m \rightarrow U_m] \) which is also a rule of \( S \); we set \( Y_i(\theta^{-1}) = Y_i(\theta) \). We always divide the set of rules \( \Theta \) of an \( S \)-machine into two disjoint parts, \( \Theta^+ \) and \( \Theta^- \) such that for every \( \theta \in \Theta^+ \),
\( \theta^{-1} \in \Theta^- \) and for every \( \theta \in \Theta^- \), \( \theta^{-1} \in \Theta^+ \) (in particular \( \Theta^{-1} = \Theta \), that is any S-machine is symmetric). The rules from \( \Theta^+ \) (resp. \( \Theta^- \)) are called positive (resp. negative).

To apply an S-rule \( \theta \) to an admissible word \( W \) (2.2) means to check if all tape letters of \( W \) belong to the alphabet \( Y(\theta) \) and then, if \( W \) satisfies this condition, to replace simultaneously every subword \( U_{i}^{\pm 1} = u_{0}q_{i}u_{1}q_{i+1}\ldots q_{r}u_{r-t+1} \) by subword \( V_{i}^{\pm 1} = v_{0}q_{i}v_{1}\ldots q_{r}v_{r-t+1} \) \((i = 1, \ldots, k + 1)\) and to trim a few first and last \( a \)-letters (to obtain an admissible word starting and ending with \( q \)-letters). This replacement is allowed to perform in the form \( q_{i}v_{1}q_{i+1}\ldots q_{r} \to v_{0}q_{i}v_{1}q_{i+1}\ldots q_{r}v_{r-t+1}u_{r-t+1}^{-1} \) followed by the reducing of the resulted word. The following convention is important in the definition of S-machine: After every application of a rewriting rule, the word is automatically reduced. The reducing is not considered a separate step of an S-machine.

If a rule \( \theta \) is applicable to an admissible word \( W \) (i.e., \( W \) belongs to the domain of \( \theta \)) then the word \( W \) is called \( \theta \)-admissible, and \( W \cdot \theta \) denotes the word obtained after the application of \( \theta \).

For recognizing S-machine, we can define input configurations and accept (stop) configurations with standard base. There is one input sector and other sectors are empty for input configurations, and all sectors are empty for the accept one. (However in these paper, some machines have no input or accept configurations.) The state letters of the input (of accept) configuration form a special vector \( s_{0} \) \((v_{0},v_{1},\ldots,v_{l})\) whose letters are zero in one rule only and are completely changed by this rule.

A computation of length or time \( t \geq 0 \) is a sequence of configurations \( W_{0} \to \cdots \to W_{t} \) such that for every \( 0 = 1,\ldots,t-1 \) the machine passes from \( W_{i} \) to \( W_{i+1} \) by applying one of the rules \( \theta_{i} \) from \( \Theta \). The word \( H = \theta_{1}\ldots\theta_{t} \) is called the history of the computation. Since \( W_{t} \) is determined by \( W_{0} \) and the history \( H \), we use notation \( W_{t} = W_{0} \cdot H \).

A computation is called reduced if its history is a reduced word. Clearly, every computation can be made reduced (without changing the start or end configurations of the computation) by removing consecutive mutually inverse rules.

A configuration \( W \) is said to be accepted by a machine \( M \) if there exists at least one computation which starts with \( W \) and ends with the accept configuration.

Assume that \( M \) is an S-machine, and there is only one input sector, say \( Q_{1}Q_{2} \), and there is only one accept configuration. Then the \( a \)-length \( |W|_{a} \) of an input configuration \( W \) is the length of the word \( w \) in its input sector. Let \( \mathcal{L}_{M} \) be the language of acceptable words, i.e. the set of all \( w \) in all accepted input configurations of \( M \). Denote by \( T(w) \) the minimal time of computations accepting \( w \in \mathcal{L}_{M} \). Then the time function \( T(n) = T_{M}(n) \) is defined as \( \max\{T(w) \mid w \in \mathcal{L}_{w}, ||w|| \leq n\} \), where \( ||w|| \) is the length of a word \( w \).

The generalized time function \( T'(n) \) is defined for every machine having a unique accept configuration. The definition is similar to the above definition of time function but one should consider all accepted configurations \( W \), not just input ones. Therefore \( T(n) \leq T'(n) \).

Time functions and generalized time functions are taken up to \( \Theta \)-equivalence.

The above definition of S-machines resembles the definition of multi-tape Turing machine (or algorithm). (The main difference is that a Turing machine has no deal with negative letters.) We do not give an accurate definition here since we will not use Turing machines in the paper (see, for example [4]). It is important that S-machines can simulate the work of Turing machines as follows.

**Lemma 2.1. (19, lemma 3.1 and Proposition 4.1)** Let \( M_{0} \) be a non-deterministic Turing machine accepting the language \( \mathcal{L} \) with a time function \( T(n) \). Then there is an
\( S \)-machine \( M_1 \) accepting the language \( L \) with time function \( \Theta \)-equivalent to \( T(n)^3 \).

Moreover, the \( S \)-machine \( M_1 \) can be constructed so that for every word \( W \in L \) accepted by \( M_0 \) with time \( T(W) \), the machine \( M_1 \) accepts this word with time \( \Theta(T(W)^3) \).

**Remark 2.2.** The part "Moreover..." is not formulated in [19] explicitly, but it follows from Proposition 4.1.3 (b) since every Turing machine can be easily modified so that the length of every accepting computation is \( \Theta \)-equivalent to the space of this computation.

**Remark 2.3.** Later we will assume that the machine \( M_0 \) recognizes the values of some suitable function according to Definition 1.2.

### 2.2 Simplifying the rules of \( S \)-machines

We say that two recognizing \( S \)-machines are equivalent if they have the same language of acceptable words and \( \Theta \)-equivalent time functions.

**Lemma 2.4.** Every \( S \)-machine \( S \) is equivalent to an \( S \)-machine \( S' \), where

1. Every part \( U_i \rightarrow V_i \) of every rule \( \theta \) has 1-letter base: \( U_i \equiv v_1 q_i u_{i+1} \), \( V_i \equiv v'_1 q'_i u'_{i+1} \), where \( q_i, q'_i \) are state letters in \( Q \).
2. In every part \( v_1 q_i u_{i+1} \rightarrow v'_1 q'_i u'_{i+1} \), we have that \(|v_i| + |v'_i| \leq 1\) and \(|u_{i+1}| + |u'_{i+1}| \leq 1\).
3. Moreover, one can construct \( S' \) so that for every rule, we have \( \sum_i (|v_i| + |v'_i| + |u_i| + |u'_i|) \leq 1 \).

**Proof.** (1) Property (1) can be obtained after adding auxiliary state letters and splittings the rules of \( S \). Assume, for example, that the part \( U_1 \rightarrow V_1 \) has 2-letter base: \( q_1 a q_2 \rightarrow q'_1 a' q'_2 \). Then we introduce auxiliary state letters \( q_j(1), q_j(2) \) and replace the rule \( \theta \) by the product of three rules \( \theta_1, \theta_2 \) and \( \theta_3 \), where

(\( \theta_1 \)) For \( \theta_1 \) and \( j > 1 \), we replace base letters \( q \) in \( V_j \) by their \( q(1) \)-copies and obtain the parts \( U_j \rightarrow V_j(1) \), while the part \( U_1 \rightarrow V_1 \) is replaced by the two parts \( q_1 a \rightarrow q_1(1), q_1(2) \); \( q_2 \rightarrow q_2(1) \);

(\( \theta_2 \)) For \( \theta_2 \) and \( j > 1 \), we have now the parts \( V_j(1) \rightarrow V_j(2) \), where \( V_j(2) \) is a copy of \( V_j(1) \) after replacement \( q(1) \rightarrow q(2) \), while the part \( U_1 \rightarrow V_1 \) of \( \theta \) is replaced by two parts \( q_1(1) \rightarrow q_1(2) \) with \( Y_2(\theta_2) = \emptyset \) and \( q_1(1) \rightarrow q_2(2) \);

(\( \theta_3 \)) For \( \theta_3 \) and \( j > 1 \), we have \( V_j(2) \rightarrow V_j \), while the first part splits now as \( q_1(2) \rightarrow q'_1 a' \) and \( q_2(2) \rightarrow q'_2 \).

The key feature of the new \( S \)-machine \( \tilde{S} \) is in the following obvious property.

There is a one-to-one correspondence between computations \( w_0 \rightarrow ... \rightarrow w_t \) of \( \tilde{S} \) (with any base) such that \( w_0, w_t \) do not have auxiliary \( q \)-letters and computations of \( S \) connecting the same words. For every history \( H \) of such computation of \( S \), the corresponding history of computation of \( \tilde{S} \) is obtained from \( H \) by replacing every occurrence of the rule \( \theta \) by the 3-letter word \( (\theta_1 \theta_2 \theta_3)^{\pm 1} \).

Clearly, applying this transformation to an \( S \)-machine \( S \) several times, we obtain an equivalent \( S \)-machine satisfying Property (1).

(2) Suppose Property (2) is not satisfied for a part \( v_1 q_i u_{i+1} \rightarrow v'_1 q'_i u'_{i+1} \). For example, suppose a rule \( \theta \) of an \( S \)-machine \( S \) has the \( i \)-th part of the form \( v_1 q_i u_{i+1} \rightarrow v'_1 q'_i u'_{i+1} \), where \( u_{i+1}, v_i, u'_{i+1}, v'_i \) are words in the appropriate parts of the alphabet of \( a \)-letters, \( v_i \) is not empty, \( a \) is a \( a \)-letter, \( q_i, q'_i \) are \( q \)-letters (a very similar procedure can be done in all other cases). Then we create a new \( S \)-machine \( \tilde{S} \) with the same standard base and the same \( a \)-letters as \( S \). In order to make \( \tilde{S} \), we add one new (auxiliary) \( q \)-letter \( \tilde{q}_i \) to each part.
of the set of \( q \)-letters, and replace the rule \( \theta \) by two rules \( \theta' \) and \( \theta'' \). The rule \( \theta' \) is obtained from \( \theta \) by replacing the part \( v_i a q_i u_{i+1} \rightarrow v'_i q'_i u'_{i+1} \) by \( a q_i u_{i+1} \rightarrow \tilde{q}_j u'_{i+1} \), and all other parts \( U_j \rightarrow V_j \) by \( U_j \rightarrow \tilde{q}_j \) (here \( \tilde{q}_j \) is the auxiliary \( q \)-letter in the corresponding part of the set of \( q \)-letters). The rule \( \theta'' \) is obtained from \( \theta \) by replacing the part \( v_i a q_i u_{i+1} \rightarrow v'_i q'_i u'_{i+1} \) by \( v_i \tilde{q}_j \rightarrow v'_i q'_i \), and all other parts \( U_j \rightarrow V_j \) by \( \tilde{q}_j \rightarrow V_j \).

Note that the sum of lengths of words in all parts of \( \theta' \) (resp. \( \theta'' \)) in \( \tilde{S} \) is smaller than the similar sum for \( \theta \). Therefore, applying this transformation to an \( S \)-machine \( S \) several times, we obtain an equivalent \( S \)-machine satisfying conditions (1) and (2).

(3) Similarly, one can obtain Property (3).

If \( Y_{i+1}(\theta) = \emptyset \) for a machine with Property (1), then the corresponding component \( U_i \rightarrow V_i \) will be denoted \( U_i \xrightarrow{\ell} V_i \) and we shall say that the rule \( Q_i Q_{i+1} \)-sectors. In that case we always assume that \( U_i, V_i \) do not have tape letters to the right of the state letters, i.e., it has the form \( v_i q_i \rightarrow v'_i q'_i \). Similarly, these words have no tape letters to the left of the state letters if the \( Q_{i-1} Q_i \)-sector is locked by the rule.

### 2.3 Some elementary properties of \( S \)-machines

The base of an admissible word is not always a reduced word. However the following is an immediate corollary of the definition of admissible word.

**Lemma 2.5.** If the \( i \)-th component of the rule \( \theta \) has the form \( v_i q_i \rightarrow v'_i q'_i \), i.e. \( Y_{i+1}(\theta) = \emptyset \), then the base of any \( \theta \)-admissible word cannot have subwords \( Q_i Q_{i+1}^{-1} \) or \( Q_{i+1}^{-1} Q_{i+1} \).

\[ \square \]

In this paper we are often using copies of words. If \( A \) is an alphabet and \( W \) is a word involving no letters from \( A^{\pm 1} \), then to obtain a copy of \( W \) in the alphabet \( A \) we substitute letters from \( A \) for letters in \( W \) so that different letters from \( A \) substitute for different letters. Note that if \( U' \) and \( V' \) are copies of \( U \) and \( V \) respectively corresponding to the same substitution, and \( U' \equiv V' \), then \( U \equiv V \).

**Lemma 2.6.** Suppose that the base of an admissible word \( W \) is \( Q_i Q_{i+1} \). Suppose that each rule of a reduced computation starting with \( \equiv q_i u_{i+1} \) and ending with \( \equiv q_i' u_{i+1} \) multiplies the \( Q_i Q_{i+1} \)-sector by a letter on the left (resp. right). And suppose that different rules multiply that sector by different letters. Then

(a) the history of computation is a copy of the reduced form of the word \( u' u^{-1} \) read from right to left (resp. of the word \( u^{-1} u' \) read from left to right). In particular, if \( u \equiv u' \), then the computation is empty;

(b) the length of the history \( H \) of the computation does not exceed \( ||u|| + ||u'|| \);

(c) for every configuration \( q_i u u' q_{i+1} \) of the computation, we have \( ||u'|| \leq \max(||u||, ||u'||) \).

**Proof.** Part (a) is obvious. To prove part (b), we choose a configuration \( W_i \) of the computation with shortest tape word \( u_i \). This factorizes the history as \( H \equiv H_1 H_2 \), where \( H_2 \) is the history of the subcomputation \( W_i \rightarrow W_{i+1} \rightarrow \cdots \rightarrow W' \). It follows that \( ||W_{i+1}|| = ||W_i|| + 1 \). The next rule increases the length of admissible word again since the computation is reduced and different rules multiply the sector by different letters, i.e.

\[ ||W_{i+1}|| = ||W_{i+1}|| + 1. \]

By induction, we have \( ||u'|| \geq ||u_i|| + ||H_2|| \geq ||H_2|| \). To obtain the inequality \( ||u|| \geq ||H_1|| \), we consider the inverse computation \( W' \rightarrow \cdots \rightarrow W \). Hence

\[ ||H|| = ||H_1|| + ||H_2|| \leq ||u|| + ||u'||. \]
The same argument proves Statement (c) since the length of $u''$ is either between $||u_i||$ and $||u'||$ or between $||u_i||$ and $||u||$. \hfill \Box

Lemma 2.7. Suppose the base of an admissible word $W$ is $Q_iQ_{i+1}$. Assume that each rule of a reduced computation starting with $W \equiv q_iuq_{i+1}$ and ending with $W' \equiv q_i'u'q_{i+1}$ multiplies the $Q_iQ_{i+1}$-sector by a letter on the left and by a letter from the right. Suppose different rules multiply that sector by different letters and the left and right letters are taken from disjoint alphabets. Then

(a) for every intermediate configuration $W_j$ of the computation, we have $||W_j|| \leq \max(||W||, ||W'||)$

(b) the length of the history $H$ of the computation does not exceed $\frac{1}{2}(||u|| + ||u'||)$.

Proof. (a) If we choose the configuration $W_i$ of minimal length, then after multiplications of the form $u_i \rightarrow u_{i+1} = au_i b$ we have no cancellation from the left or from the right.

If we have the former option, then we will have no cancellation from the left after the transition $u_{i+1} \rightarrow u_{i+2}$, and therefore $||u_{i+1}|| \leq ||u_{i+2}|| \leq \cdots \leq ||u||$. Hence $||u_j|| \leq ||u'||$ if $j \geq i$. Analogously, $||u_j|| \leq ||u||$ if $j \leq i$.

(b) The word $u'$ results from $u$ after multiplication from the left and from the right by reduced words of length $||H||$: i.e., $u'$ is freely equal to $AuB$, where $||A|| = ||B|| = ||H||$.

There can be cancellations in the products $Au$ and $uB$ but afterwards there are no cancellations since the words $A$ and $B$ are written in disjoint alphabets. Hence the reduced length of $u'$ is at least $||A|| + ||B|| - ||u|| = 2||H|| - ||u||$, whence $2||H|| \leq ||u|| + ||u'||$, as required. \hfill \Box

The following lemma is proved in [13] (Lemma 3.7).

Lemma 2.8. Suppose the base of an admissible word $W$ is $Q_iQ_i^{-1}$ (resp., $Q_i^{-1}Q_i$). Suppose that each rule $\theta$ of a reduced computation starting with $W \equiv q_iuq_i^{-1}$ (resp., $q_i^{-1}uq_i$), where $u \neq 1$, and ending with $W' \equiv q_i'u'(q_i')^{-1}$ (resp., $W' \equiv (q_i')^{-1}u'q_i'$) has a part $q_i \rightarrow a_0q_i'b_0$, where $b_0$ (resp., $a_0$) is a letter, and for different $\theta$-s the $b_0$-s (resp., $a_0$-s) are different. Then the history of the computation has the form $H_1H_2^kH_3$, where $k \geq 0$, $||H_2|| \leq \min(||u||, ||u'||)$, $||H_1|| \leq ||u||/2$, and $||H_3|| \leq ||u'||/2$.

\hfill \Box

3 Auxiliary machines and constructions

3.1 Primitive machines

Here we define a very simple machine $\text{Pr}$, which has neither input nor accept configurations. It will be used to check the order of state letter in bases of computations.

The standard base of $\text{Pr}$ has three letters $Q^1PQ^2$, where $Q^1 = \{q^1\}$, $P = \{p^1, p^2\}$ and $Q^2 = \{q^2\}$. The alphabet $Y$ is $Y^1 \sqcup Y^2$, where $Y^2$ is a copy of $Y^1$. The positive rules of $\text{Pr}$ are defined as follows.

- $\zeta^1(a) = [a^1 \rightarrow q^1, p^1 \rightarrow a^{-1}p^1a', q^2 \rightarrow q^2]$,
  where $a$ is a positive letter from $Y^1$ and $a'$ is its copy from $Y^2$.

Comment. The state letter $p^1$ moves left replacing letters $a$ from $Y^1$ by their copies $a'$ from $Y^2$. 

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\[\zeta^{12} = [q^1 \xrightarrow{\ell} q^1, p^1 \rightarrow p^2, q^2 \rightarrow q^3].\]

Comment. When \(p^1\) meets \(q^1\), it turns into \(p^2\).

\[\zeta^2(a) = [q^1 \rightarrow q^1, p^2 \rightarrow ap^2(a')^{-1}, q^2 \rightarrow q^2]\]

Comment. The state letter \(p^2\) moves right towards \(q^2\) replacing letters \(a'\) from \(Y^2\) by their copies \(a\) from \(Y^1\).

**Lemma 3.1.** Let \(C: W_0 \rightarrow \cdots \rightarrow W_t\) be a reduced computation of the machine \(\Pr\) with the standard base and with \(t \geq 1\). Then

1. if \(|W_i|_a > |W_{i-1}|_a\) for some \(i = 1, \ldots, t - 1\), then \(|W_{i+1}|_a > |W_i|_a\);
2. \(|W_i|_a \leq \max(|W_0|_a, |W_t|_a)\) for every \(i = 0, 1, \ldots, t\);
3. if \(W_0 \equiv q^1u^1p^1q^2\) and \(W_t \equiv q^1v^p^2q^2\) for some words \(u, v\), then \(u \equiv v\), \(|W_i|_a = |W_0|_a\) for every \(i = 0, \ldots, t\), \(t = 2k + 1\), where \(k = |W_0|_a\), and \(p^1, p^2\) meets \(q^1\) in \(W_k\) (in \(W_{k+1}\)) and the sector \(Q^1P\) is empty in \(W_k\) and in \(W_{k+1}\); moreover, the history \(H\) of \(\mathcal{C}\) is uniquely determined by \(W_0\) (by \(W_t\)); provided \(W_0\) and \(W_t\) have the form \(q^1u^1p^1q^2\) and \(q^1v^p^2q^2\); vice versa, the history \(H\) uniquely determines words \(u\) and \(v\) under this assumption.
4. it is not possible that \(W_0 \equiv q^1u^1p^1q^2\) and \(W_t \equiv q^1v^p^2q^2\) for some \(u, v\), and it is not possible that \(W_0 \equiv q^1u^1p^2q^2\) and \(W_t \equiv q^1v^p^2q^2\);
5. if \(W_0 \equiv q^1u^1p^1q^2\) or \(W_0 \equiv q^1p^1u^1q^2\), or \(W_0 \equiv q^1u^2p^2q^2\), or \(W_0 \equiv q^1p^2u^2q^2\) for some word \(u\), then \(|W_i|_a \geq |W_0|_a\) for every \(i = 0, \ldots, t\).

**Proof.** Note that each of the rules \((\zeta^j)_{\pm 1}(a), (j = 1, 2)\) either moves the state letter left or moves it right, or deletes one letter from left and one letter from right, or insert letters from both sides. In the later case, the next rule of a computation must be again \((\zeta(j)_{\pm 1}(b))\) for some \(b\), and if the computation is reduced, it again must increase the length of the configuration by two. Therefore Statement (1) is true and (2) is also true since one can choose a shortest \(W_i\) and consider the subcomputation \(W_j \rightarrow \cdots \rightarrow W_t\) and the inverse subcomputation \(W_j \rightarrow \cdots \rightarrow W_0\). Since \(\zeta^{12}\) locks \(Q^1P\)-sector, the \(p\)-letter must reach \(q_1\) moving always left to change \(p^1\) by \(p^2\), and so \(W_k \equiv q_1^1p_1\ldots\). The next rule of the form \((\zeta^1(a))_{\pm 1}\) could increase the length of the configuration, which would imply that all consecutive rules have to have the same type and \(p^1\) would never been replaced by \(p^2\), a contradiction. Hence the next rule is \(\zeta^{12}\), and arguing in this way, one uniquely reconstructs the whole computation in case (3) for given \(W_0\) or \(W_t\), and vice versa, the history \(H\) determines both \(u\) and \(v\). Property (4) holds for same reasons.

Note that no rule of \(\Pr\) changes the projection of a configuration onto the free group with basis \(Y_1\) if the state letter are mapped to 1 and the letters from \(Y_2\) are mapped to their copies from \(Y_1\). Since the word \(u\) is mapped to itself, we have \(|q^1u^1p^2|_a = ||u|| \leq |W_i|_a\). The other cases of (5) are similar. \(\square\)

**Remark 3.2.** Similar tricks will be used later as the projection argument.

**Lemma 3.3.** If \(W_0 \rightarrow \cdots \rightarrow W_t\) is a reduced computation of \(\Pr\) with base \(Q^1P \rightarrow Q^1P^{-1}(Q^1)^{-1}\) or \((Q^2)^{-1}P^{-1}PQ^2\) and \(W_0 \equiv q^1p^1u(p^1)^{-1}(q^1)^{-1} or \ W_0 \equiv (q^2)^{-1}(p^2)^{-1}v(p^2)q^2\) for some words \(u, v\), then \(|W_i|_a \geq |W_0|_a\) for every \(i = 0, \ldots, t\).
Proof. The statement follows from the projection argument 3.2. \qed

**Remark 3.4.** Also we will use the right analog $\text{Pr}^*$ of $\text{Pr}$: now $p^1$ should move right, meet $q^2$ locking $PQ^2$-sector and turning into $p^2$, and move back towards $Q^1$.

**Remark 3.5.** Assume that a standard base has two (or more) subwords of the form $Q^1PQ^2$, for example $Q^1PQ^2P^iQ^3$. Then one can define parallel or sequential work of two primitive machines.

For the parallel composition, the same rule changes both subwords $Q^1PQ^2$ and $Q^2PQ^3$. One assumes that the tape alphabet of the sector $Q^2P^i$ (of $P^iQ^3$) is a copy of the tape alphabet of the sector $Q^1P$ (of $PQ^2$, resp.) and the rules of $\text{Pr}$ change simultaneously the subwords with bases $Q^1PQ^2$ and $Q^2P^iQ^3$ (e.g. simultaneously moves left both $p^1$ and $(p')^1$, and so on). Every rule (e.g $\xi^1(a)$) is applicable to a configuration iff it is applicable to both these subwords.

In case of sequential work, we have a primitive machine working with one of these two bases, say with $Q^1PQ^2$, while the sector $P^iQ^3$ is locked with state letter $(p')^1$.

The second primitive machine can compute with base $Q^2P^iQ^3$ when the sector $PQ^2$ is locked (and so the first machine stays idle) with state $p^2$. For this goal, one needs a connecting rule $\xi^{21}$; it locks the sector $PQ^2$ and changes the state $P$-letters for new ones to switch off the first primitive machine and to switch on the second one.

It is clear that in the same way one can define a more complex compositions $\mathbf{P}$ of primitive machines, with several stages of parallel and sequential work. We will consider compositions $\mathbf{P}$, where every sector can be changed at one stage only.

**Lemma 3.6.** Let we have a composition $\mathbf{P}$ of primitive machines with parallel or sequential work, and $\mathcal{C}: W_0 \to \cdots \to W_t$ be a reduced computation of $\mathbf{P}$ with standard base. Then

(a) $||W_j||_a \leq \max(||W_0||_a, ||W_t||_a)$ for every configuration $W_j$ of $\mathcal{C}$; moreover, $||W_0||_a \leq \cdots \leq ||W_t||_a$ if every $P$-letter neighbors some $Q$-letter in the word $W_0$;

(b) $t \leq ||W_0|| + ||W_t||_a - 4$, moreover, $t \leq 2||W_t||_a - 4$ if every $P$-letter neighbors some $Q$-letter in the word $W_0$.

Proof. (a) Let $W_r$ be a shortest word of the computation. Then either $|W_r|_a = |W_{r+1}|_a = \cdots = |W_t|_a$, or $|W_r|_a = |W_{r+1}|_a = \cdots = |W_s| < |W_{s+1}|_a$ for some $s$. It follows that the number of sectors increasing their lengths by two at the transition $W_s \rightarrow W_{s+1}$ is greater than the number of the sectors decreasing the lengths by 2. Now it follows from Lemma 3.1 (1) that the same primitive machine will continue increasing the lengths of the whole configurations, i.e., $|W_{s+1}|_a < |W_{s+2}|_a < \cdots$. So for every $j \geq r$, we have $|W_j|_a \leq |W_t|_a$. Similarly, we have $|W_r|_a \leq |W_0|_a$ for $j \leq r$. Under the additional assumption about $P$-letters, $W_0$ is the shortest configuration by the projection argument.

(b) If the rules of $\mathbf{P}$ do not change the lengths of configurations, then every control letter runs back and forward only one time, and the inequality follows. (One takes into account that the base has length at least 3.) If $|W_r|_a < ||W_{r+1}||_a$ for some $r$, then every next transition keeps increasing the length by Lemma 3.1 (1), and so the inequality holds as well. \qed
3.2 S-machine with historical sectors

To control the space of computations, we endow a given S-machine with historical sectors. Let us assume that an S-machine $M_1$ satisfies the conditions of Lemma 2.4 and has hardware $(Q, Y)$, where $Q = \sqcup_{i=0}^{m} Q_i$, and the set of rules $\Theta$. The new machine $M_2$ has hardware

$$Q_{0, r} \sqcup Q_{1, \ell} \sqcup Q_{1, r} \sqcup Q_{2, \ell} \sqcup Q_{2, r} \sqcup \cdots \sqcup Q_{m, \ell}, \ Y_h = Y_1 \sqcup X_1 \sqcup Y_2 \sqcup \cdots \sqcup X_{m-1} \sqcup Y_m$$

where $Q_{i, \ell}$ and $Q_{i, r}$ (left and right) copies of $Q_i$ are a disjoint union of two copies of $\Theta^*$, namely $X_{i, \ell}$ and $X_{i, r}$. (There is neither $Q_{0, \ell}$, nor $Q_{m, r}$, nor $X_0$, nor $X_m$.) The positive rules of $M_2$ are in one-to-one correspondence with the positive rules of $M_1$.

If $\theta = [U_0 \rightarrow V_0, \ldots, U_m \rightarrow V_m]$ is a positive rule of $M_1$ with parts $U_i \rightarrow V_i$ of the form $v_i q_i u_{i+1} \rightarrow v'_i q'_i u'_{i+1}$, then the corresponding two parts of the rule $\theta_h$ are $U_{i, \ell} \rightarrow V_{i, \ell}$ and $U_{i, r} \rightarrow V_{i, r}$, with

$$U_{i, \ell} \equiv v_i q_i a_{\theta, i}, V_{i, \ell} \equiv v'_i q'_i \text{ and } U_{i, r} \equiv q_i r u_{i+1}, V_{i, r} \equiv b_{\theta, i} q'_i u'_{i+1},$$

where $a_{\theta, i}$ (resp., $b_{\theta, i}$) is the copy of $\theta$ in $X_{i, \ell}$ (in $X_{i, r}$). We also claim that a sector $Q_{i, \ell} Q_{i+1, \ell}$ is locked by $\theta_h$ if and only if the sector $Q_i Q_{i+1}$ is locked by $\theta (i = 1, \ldots, m-1)$.

Comment. Every computation of machine $M_2$ with history $H$ coincide with the a computation of $M_1$ if one observes it only in working sectors $Q_{i, r} Q_{i+1, r}$. In the standard base, the working sectors of $M_2$ alternate with historical sectors $Q_{i, \ell} Q_{i, r}$. Every positive rule $\theta_h$ multiples the content of the historical sector $Q_{i, \ell} Q_{i, r}$ by the corresponding letter $b_{\theta, i}$ from the right and by letter $a_{\theta, i}$ from the left.

Remark 3.7. The state letters of the machine $M_1$ split when passing to $M_2$. There is a rule $\theta_h$ corresponding to the start (to the accept) rule $\theta$ of $M_1$. By definition, the set of letters $Y_h(\theta_h)$ has no letters from right alphabets $X_{i, r}$ (from left alphabets $X_{i, \ell}$) if $\theta$ is the start (resp., the stop) rule of $M_1$.

However we do not define input/stop configurations of $M_2$ since the historical sectors are never locked. By definition, every $Q_{i-1, r} Q_{i, l}$ is the working sector of $M_2$. The input sector of $M_2$ is the working sector corresponding to the input sector of $M_1$.

Remark 3.8. It follows from the definition of $M_2$ that only Properties (1) and (2) of Lemma 2.4 hold for $M_2$, but not Property (3).

The sectors of the form $Q_{i, \ell} Q_{i, \ell}^{-1}$ and $Q_{i, r}^{-1} Q_{i, r}$ (in a non-standard base) are also called historical. Historical sectors help to give a linear estimate of the space of every computation $W_0 \rightarrow \cdots \rightarrow W_t$ in terms of $|W_0|$ and $|W_t|$.

Lemma 3.9. Let $W_0 \rightarrow \cdots \rightarrow W_t$ be a reduced computation of $M_2$ with base $Q_{i, \ell} Q_{i, r}$ and history $H$. Assume that the a-special letters of $W_0$ belong to one of the alphabets $X_{i, \ell}$, $X_{i, r}$. Then $||H|| \leq |W_t|_a$ and $|W_0|_a \leq |W_t|_a$.

Proof. Let $W_0 = q_0 v_0 q'_0$ and assume that $v_0$ has no letters from $X_{i, r}$. Then $v_t = uv_0 u'$, where $u$ is a copy of $H^{-1}$ in the alphabet $X_{i, \ell}$ and $u'$ is a copy of $H$ in $X_{i, r}$. So no letter of $u'$ is cancelled in the product $uv_0 u'$. Therefore $|W_t|_a \geq ||u|| = ||H||$ and $|W_t|_a \geq |W_0|_a$.

Lemma 3.10. If the base of an admissible word of the machine $M_2$ has length at least 3, then it contains a historical sector.
Proof. The base contains a subword of the form $Q'Q''Q'''$ with three letters from $Q^{\pm 1}$. It follows from the definition of admissible word that either $Q'Q''$-sector or $Q''Q'''$-sector is historical since every non-historical sector of the machine $M_2$ has to have neighbor historical sectors.

Lemma 3.11. Let a reduced computation $W_0 \to W_1 \to \cdots \to W_t$ of the machine $M_2$ has 2-letter base and the history of the form $H \equiv H_1 H_2^k H_3$ ($k \geq 0$). Then for each tape word $w_i$ between two state letters of $W_i$ ($i = 0, 1, \ldots, t$), we have inequality $||w_i|| \leq ||w|| + ||w_{i+1}|| + 2h_1 + 3h_2 + 2h_3$, where $h_j = ||H_j||$ ($j = 1, 2, 3$).

Proof. By Lemma 2.4 (2) and the definition of $M_2$, we have $||w_i|| - ||w_{i-1}|| \leq 2$ for every $i = 1, \ldots, t$. Therefore for $i \leq h_1$, we have $||w_i|| \leq ||w|| + 2h_1$. Similarly, $||w_i|| \leq ||w|| + 2h_3$ for $i \geq t - h_3$. It remains to assume that $h_1 < i < t - h_3$.

Denote by $W_j$ the words $w_i$ with $i = h_1 + jh_2$, $j = 0, 1, \ldots, k$. If $W_1 = W_0 H_2 = uW_0 v$ for some words $u$ and $v$ depending on $H_2$ and on the sector, then $W_2 = uW_1 v = u^2 W_0 v^2$ in free group, since the histories of the computations $W_0 \to \cdots \to W_1$ and $W_1 \to \cdots \to W_2$ are both equal to $H_2$. Hence $W_j = u^j W_0 v^j$, where both $u$ and $v$ have length at most $h_2$ by Lemma 2.4 (2) and the definition of $M_2$.

By Lemma 8.1 from [15], the length of an arbitrary word $W_j$ is not greater than $||u|| + ||v|| + ||W_0|| + ||W_k||$ provided $0 \leq j \leq k$.

If $|i - jh_2| \leq h_2/2$ for some $j$, then $||w_i|| - ||W_j|| \leq h_2$, and therefore $||w_i|| \leq ||u|| + ||v|| + ||W_0|| + ||W_k|| + h_2$. Since $||W_0|| \leq ||w|| + 2h_1$ and $||W_k|| \leq ||w|| + 2h_3$, we obtain

$||w_i|| \leq ||u|| + ||v|| + ||w|| + ||w|| + 2h_1 + 2h_3 + h_2 \leq ||w|| + ||w|| + 2h_1 + 2h_3 + 3h_2$ for every $i$, as required.

\[\square\]

Lemma 3.12. For any reduced computation $W_0 \to \cdots \to W_t$ of machine $M_2$ with base of length at least 3, we have $||W_i|| \leq 9(||W_0|| + ||W_t||)$ ($0 \leq i \leq t$).

Proof. Let $Q_{i_1}^{\pm 1} \cdots Q_{i_m}^{\pm 1}$ be the base of the computation. There are computations with the same history $H$ and bases $Q_{i_1}^{\pm 1} \cdots Q_{i_m}^{\pm 1}$, $Q_{i_1}^{\pm 1} \cdots Q_{i_m}^{\pm 1}$, $Q_{i_1}^{\pm 1} \cdots Q_{i_m-1}^{\pm 1}$, $Q_{i_1}^{\pm 1} \cdots Q_{i_m}^{\pm 1}$, where each base has length 3 or 4. Hence it suffices to prove the lemma for any computation with base of the form $Q'Q''Q'''$ or $Q''Q'''Q''''$. By Lemma 3.10, every such computation contains a historical sector, say $Q''Q'''$. Consider two cases.

1. The historical sector has the form $Q_{i,\ell} Q_{i,\ell}$. By Lemma 2.7, we have $||H|| \leq \frac{1}{2}(||W_0|| + ||W_t||)$. It follows from Lemma 2.4 (2) that $||W_{i+1}|| - ||W_i|| \leq 6$ for every neighbor configurations. Therefore

$||W_i|| \leq \max(||W_0||,||W_t||) + 3||H|| \leq \max(||W_0||,||W_t||) + \frac{3}{2}(||W_0|| + ||W_t||)$.

2. The historical sector has form $Q_{i,\ell} Q_{i,\ell}^{-1}$ or $Q_{i,\ell} Q_{i,\ell}^{-1}$. Then one can apply Lemma 2.8 to the sector $Q''Q'''$ and obtain the factorization $H \equiv H_1 H_2^k H_3$, with $k \geq 0$, $h_2 \leq \min(||u||,||u'||)$, $h_1 \leq ||u||/2$, and $h_3 \leq ||u'||/2$, where $u$ and $u'$ are the $a$-words of $W_0$ and $W_t$, respectively in the historical sector and $h_j = ||H_j||$ for $j = 1, 2, 3$. Since every $W_i$ has at most three sectors, applying Lemma 3.11 to each of them, we obtain:

$||W_i|| \leq ||W_0|| + ||W_t|| + 3(2h_1 + 3h_2 + 2h_3)$.
Here we start with a machine $D_1$. This S-machine has two input sectors with words $a^k$ and $b^l$ and checks whether $2k$ divides $\ell$ or not.

The standard base of $D_1$ is $S(1)S(2)T(1)T(2)$. The first input sector $S(1)S(2)$ has one-letter alphabet $\{a\}$, the second input sector $T(1)T(2)$ has alphabet $\{b\}$. Also we have one-letter alphabet $\{a'\}$ for the sector $S(2)T(1)$. We omit some parts of rules in the list below if these parts do not change configurations (e.g., $s \to s$ for $s \in S(1)$ is a part of $\tau_1$).

- $\tau_1: [s_1 \to a^{-1}s_1a'], [t_1 \to t_1b^{-1}], s_1 \in S(2), t_1 \in T(1)$

  *Comment.* The state letter $s_1$ moves left changing letter $a$ by its copy $a'$, while $t_1$ erases one letter $b$.

- $\tau_{12}: [s \to s, s_1 \to s_2] \ s \in S(1), s_2 \in S(2)$

  *Comment.* The rule $\tau_{12}$ locks the sector $S(1)S(2)$ and replaces the state letter $s_1$ by $s_2$.

- $\tau_2: [s_2 \to as_2(a')^{-1}] \ [t_1 \to t_1b^{-1}]$

  *Comment.* $s_2$ moves right toward $T(1)$ replacing $a'$ by $a$, while $t_1$ erases the letter $b$.

- $\tau_{21}: [s_2 \xrightarrow{\ell} s_1]$

  *Comment.* This rule locks the sector $S(2)T(1)$ and replaces $s_2$ by $s_1$.

- $\tau_3: [s_1 \xrightarrow{\ell} t_1], [t_1 \xrightarrow{\ell} t_2]$

  *Comment.* The state $t_2$ can appear if both sectors $S(2)T(1)$ and $T(1)T(2)$ are empty.

We call a transition $W \to W'$ given by a rule $\tau_1^{\pm 1}$ or $\tau_2^{\pm 1}$ wrong if it increases the lengths of both sectors $S(1)S(2)$ and $S(2)T(1)$.

**Lemma 3.13.** Let $W_0 \to \cdots \to W_t$ be a reduced computation of the machine $D_1$ with standard base and the first transition $W_0 \to W_1$ is wrong. Then all subsequent transitions are wrong too.

*Proof.* Let the first rule $\theta$ be $\tau_1^{\pm 1}$. Then the first transition (restricted to the sectors $S(1)S(2)$ and $S(2)T(1)$) has the form $su_1v \to sua^{\mp 1}s_1(a')^{\mp 1}v$, where the words $u, v, ua^{\mp 1}, (a')^{\mp 1}v$ are reduced. It follows that the only possible rule for the next transition is $\theta$ again, $W_2$ contains reduced form $sua^{\mp 2}s_1(a')^{\mp 2}v$, and so on. The case $\theta = \tau_2^{\pm 1}$ is similar.

**Lemma 3.14.** (1) Let we have a reduced computation $W_0 \to \cdots \to W_r$ of $D_1$, where $W_0 \equiv sa^ks_1t_1b^lt'$, where $t' \in T(2)$, and $W_r \equiv \cdots t_it' \ (i = 1, 2, \ i.e. \ the \ sector \ T(1)T(2) \ is \ empty)$. Then the exponent $\ell$ is divisible by $2k$.

(2) Conversely, if $2k$ divides $\ell$, then there is a computation $sa^ks_1t_1b^lt' \to \cdots \to sa^ks_2t_2l'$ of length $|\ell| + |\ell/k| + 1$ for $k \neq 0$ and of length 1 for $k = 0$. 

\[ |W_0|_a + |W_t|_a + 3|W_0|_a + 9 \min(|W_0|_a, |W_t|_a) + 3|W_t| \leq 9(|W_0|_a + |W_t|_a) \]
Remark 3.15. Thus, one can repeat the argument from the proof Lemma 3.14 (1,2)
three times to conclude that starting with an input configuration with standard base,
absolutely similar to the rules of state letters and all rules since we do not need them.)
The rules of the third part interchange their roles: a state letter from part. The difference
between these two parts is that the sectors D base as which, in turn is necessitated by Lemma 2.1.
The machine can empty all the sectors, except for sector D k when the analog of D k
becomes empty. Thus, Statement (1) is proved by contradiction.
(2) If ℓ = 2km or ℓ = −2km for some m ≥ 0, the required computations can be
immediately constructed according to the samples from part (1) of the proof.

Now we want to modify the machine D 3 as follows. To define machine D 2 we add one
more part T(3) to the standard base. The sector T(2)T(3) serves to count the number of
cycles of the machine D 1. So it is empty for the start configuration of D 2, and the
rule τ 12 of D 1 extended to the rule of D 2 has one more part: |t′ → t′c|, where t′ ∈ T(2).
Clearly, Properties (1) and (2) of Lemma 3.14 holds for D 2 as well. Moreover, repeating
the proof of property (2), we see that one obtains c m in sector T(2)T(3) with m = ℓ
when the sector T(1)T(2) becomes empty.
The further modification is needed since we should check the divisibility by (2k) 3,
which, in turn is necessitated by Lemma 2.1. The machine D 3 has the same standard base
as D 2 but it check divisibility by 2k three times, so its rules subdivided in three parts.
The rules of the first part are exactly the rules of the machine D 1. The rule τ 3 of D 2
serves as a connecting rule between the rules of the first part and the rules of the second
part. The difference between these two parts is that the sectors T(1)T(2) and T(2)T(3)
interchange their roles: a state letter from T(2) erases letters in the sector T(2)T(3)
when the analogs of τ 1 and τ 2 work, and a state letter from T(1) add one letter to the
sector T(1)T(2) when the analog of τ 12 is applied. (We do not introduce notation for all
state letters and all rules since we do not need them.) The rules of the third part are
absolutely similar to the rules of D 1, i.e. the sectors T(2)T(3) is locked.

Remark 3.15. Thus, one can repeat the argument from the proof Lemma 3.14 (1,2)
three times to conclude that starting with an input configuration with standard base,
the machine D 3 can empty all the sectors, except for sector S(1)S(2) if and only if the
exponent ℓ is divisible by (2k) 3.

Finally, we add a rule τ erasing a-letters of the sector S(1)S(2) with non-trivial part
[s → sa −1] for s ∈ S(1) locking other sectors, and if all the sectors become empty, one
more rule $\tau_0$ locks all the sectors and changes all the state letters for the letters of the stop configuration. Let us denote the obtained machine by $D_4$.

**Lemma 3.16.** (1) Let we have a reduced accepting computation $W_0 \to \cdots \to W_r$ of $D_4$, where $W_0 \equiv sa^kst_1bt't''$ is an input configuration. Then the exponent $\ell$ is divisible by $(2k)^3$.

(2) Conversely, if $(2k)^3$ divides $\ell$, then there is an accepting computation starting with $W_0 \equiv sa^kst_1bt't''$ of length $\Theta(|\ell| + |k|)$.

**Proof.** Properties (1), (2) follows from similar properties of $D_3$ mentioned in Remark 3.15.

The next modification is obtained by adding historical sectors to the standard base of $D_4$. The approach is similar to that described in subsection 3.2. The standard base of $D_5$ is

$$S(1)_rS(2)_\ell S(2)_r T(1)_r T(1)_r T(2)_r T(2)_\ell T(3)_r$$

and the rules of $D_5$ are the extensions of the rules of $D_4$ to historical sectors as this was defined in subsection 3.2. The following lemma is an analog of Lemma 3.12.

**Lemma 3.17.** For any reduced computation $W_0 \to \cdots \to W_r$ of the machine $D_5$ with base of length at least 3, we have $|W_i|_a \leq 9(|W_0|_a + |W_r|_a) \ (0 \leq i \leq r)$.

### 3.4 Control state letters

The work of the main machine will be checked by control state letters running back and forward along the sectors from time to time. The control letters behaves as $p$-letters of the primitive machines $Pr$ or $Pr^\ast$.

Suppose $M$ is an $S$-machine with a standard base $Q_0Q_1 \ldots Q_s$. We denote by $M_c$ the machine with standard base

$$\{P_0\} \sqcup \{Q_0\} \sqcup \{R_0\} \sqcup \{P_1\} \sqcup \{Q_1\} \sqcup \{R_1\} \sqcup \cdots \sqcup \{P_s\} \sqcup \{Q_s\} \sqcup \{R_s\}$$

For every rule $\theta$ of $M$, its $i$-th part $[v_iq_iu_{i+1} \to v'_iq'_iu'_{i+1}]$ is replaced in $M_c$ with three parts

$$[v_ip^i \xrightarrow{k} v'_ip'^i], [q_i \xrightarrow{k} q'_i], [r^iu_{i+1} \to r'u'_{i+1}]$$

(3.3)

($i = 0, \ldots, s, p^i \in P_i, r^i \in R_i$). Here we should use one more $\xrightarrow{k}$ if there is $\xrightarrow{k}$ in the definition of the component of $M$.

**Comment.** Thus, the sectors $P_iQ_i$ and $Q_iR_i$ are always locked, and three state letters $p^i, q_i, r^i$ work together in $M_c$ as the single $q_i$ in $M$. Of course, such a modification is useless for solo work of $M$. But it will be helpful when one constructs a composition of $M_c$ with other machines, because the control letters from the parts $P_i$ and $R_i$ will work when $M_c$ stands idle.
4 Main machine.

4.1 Definitions of machines $M_3 - M_6$

We use the machine $M_2$ from Section 3.2 and auxiliary machines to compose the main machine needed for this paper.

At first we add control state letters to $M_2$ and obtain machine $M_3$ as it was defined in Section 3.4. Let the standard base of $M_2$ be $Q_0Q_1\ldots Q_s$, where sectors $Q_0Q_1, Q_2Q_3,\ldots Q_s-1Q_s$ are working sectors, $Q_{s-1}Q_s$ is the input sector, and $Q_1Q_2, Q_3Q_4,\ldots$ are historical sectors. Then the standard base of $M_3$ is

$$P_0Q_0R_0P_1Q_1R_1\ldots P_sQ_sR_s,$$

where $P_i$ (resp., $R_i$) contains control $p$-letters ($r$-letters), $i = 0, \ldots, s$.

Since the rules of $M_3$ treat every syllable $P_iQ_iR_i$ as a single base letter, the working and the historical sectors for $M_3$ are of the form $R_{s-1}P_i$. In particular, every historical sector has the form $R_{i-1}P_i$ with even $i$.

The rules of the next machine $M_4$ will be partitioned in subsets corresponding to ten steps with auxiliary rules $\theta(ij)$ connecting $i$-th and $j$-th steps. The state letters are also disjoint for different steps. Therefore we need $Q_0$, which is the disjoint union of ten subsets, $P_0$, which is the disjoint union of ten subsets, and so on. Thus, the rules of different steps of a computation on $M_4$ must be separated by connecting rules.

We want to combine the machines $M_5$, the machine $D_5$ from subsection 3.3 (which is now endowed with control state letters too), and compositions of primitive machines introduced in subsection 3.1. We interbreed the input sector of $M_3$ and the first sectors of $D_5$. Namely the state letters from $Q_{s-1}$ and from $S(1)_r$ will be included in the part $Q_{s-1}$ of the new machine $M_4$, $Q_s$ and $S(2)_r$ will be included in $Q_s$. The reader will see that at some steps of computations, the part of base $Q_{s-1}P_sQ_s$ works as $M_3$ while at other steps it works as $D_5$.

The new S-machine $M_4$ repeats the computation of $M_3$ many times and $D_5$ bounds the number of such cycles. The standard base of $M_4$ is

$$P_0Q_0R_0P_1Q_1R_1\ldots P_sQ_{s-1}P_sQ_sR_s \times P\times S(2)_r \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \time
Thus, to define the control machine of Step \(2^-\) below, one should define the order of the work of primitive components for \(P_{12^-}\) and \(P_{22^-}\) and choose either \(p\)-letters or \(r\)-letter to be control letters for these primitive components.

**Remark 4.1.** The control steps are used for double purpose. If the base of a computation is standard, then the history of a control step restores all the configurations (Lemma 4.5 (1)). If the base is not reduced, then the control steps and the right order of the work of their primitive components redound to a linear bound of the space of the computation in terms of the lengths of the first and the last configurations (Lemma 4.13).

By default, every *connecting* rule \(\theta(ij)\) locks a sector if this sector is locked by all rules from \(\Theta_i\) or if it is locked by all rules from \(\Theta_j\). It also changes all state letters used at step \(i\) by there copies from the disjoint set of state letters used at step \(j\).

**Step \(1^-\).** This is a control step between Steps 2 and 1. So we define below the canonical work of the machines \(P_{2^-1}\) and \(P_{1^-1}\), keeping in mind that the last rule (as \(\zeta^{21}\) in Remark 3.5) of the canonical computation of \(P_{2^-1}\) switches on the inverse computation for the canonical one of \(P_{1^-1}\). This conjunction of the machines \(P_{2^-1}\) and \(P_{1^-1}\) is the machine of Step \(1^-\) denoted by \(P_{1^-}\).

Let us define \(P_{2^-1}\). At first, we have the parallel work of primitive machines in all big historical sectors, and control \(r\)-letters run forward and back according the rules from subsection 3.1. The tape alphabet for every such primitive machine is the left alphabet \(X_{i,\ell}\) of the big historical sector.

The next primitive machine (see Remark 3.5 for the definition of composition) starts working similarly in the input subsector of \(M_3\) after the above mentioned primitive machines stop working.

Then we have parallel work of primitive machines in the small historical sectors. Again, the running control letters are \(r\)-letters.

Finally, the primitive machine is switched on that checks the input word (of \(D_5\)) between \(R^{1,r}\) and \(P^{2,\ell}\) (with running state letters from \(R_s\)).

The running control letters of \(P_{11^-}\) are \(r\)-letters again (not \(p\)-letters), and this machine is a copy of \(P_{21^-}\) with another set of state letters.

The transition rule \(\theta(1^-1)\) changes all state letters of Step \(1^-\) by their copies in Step 1, which contain the letters of the start vector \(\bar{s}_1\) of \(M_1\). It locks all sectors except for historical sectors, the sector \(R_{s-1}P_s\) (we can call it the input since it comes from the input sectors of \(M_3\)) and the sector \(R^{1,r}P^{2,\ell}\).

The \(\theta(1^-1)\)-admissible words may involve the copies \(p_{i,s}^{1,*}\) and \(r_{i,s}^{1,*}\) of the letter \(p^1\) of a primitive machine \(Pr\), but no copies of \(p^2\). They may contains letters from alphabets \(X_{i,\ell}\) but not from \(X_{i,r}\).
Step 1. The rules $\theta(M_4)$ from $\Theta_1^+$ restricted to the base $P_0Q_0R_0 \ldots P_sQ_sR_s$ are just the (copies of the) positive rules $\theta$ of $M_3$. They do not change other sectors and lock the sector $R^{2,r}\mathcal{P}^{3,\ell}$. 

Comment. At Step 1, $M_4$ works as machine $M_3$.

The connecting rule $\theta(12^-)$ changes the state letters by their copies in disjoint alphabet, in particular, the letters from the accept vector $\vec{s}_0$ of $M_1$ are replaced by their copies. The $\theta(12^-)$-admissible words have no letters from 'left' alphabets $X_{i,\ell}$.

Besides, the rule $\theta(12^-)$ 'removes' one letter in the sector $R^{1,r}\mathcal{P}^{2,\ell}$: $[bp_{2,\ell}^1 \rightarrow (p_{2,\ell}^1)]$.

Step 2$^-$. 

This Step is similar to Step 1$, the difference in the definition of machine $P_{2^-}$ working at Step 2$^-$ is that alphabets $X_{i,\ell}$ should be replaced by alphabets $X_{i,r}$ ($i = 1, \ldots, s$) and the control letters are $p$-letters (not $r$-letters).

Comment. The copies of primitive machine check several sectors again.

The transition rule $\theta(2^-2)$ replaces all state letters of Step 2$^-$ with their copies in Step 2, and the letters of the accept vector $\vec{s}_1$ of $M_1$ are among them. It locks non-historical sectors except for the sector $R^{1,r}\mathcal{P}^{2,\ell}$.

Step 2. The positive rules from $\Theta_2$ are just copies of the negative rules from $\Theta_1$.

Comment. $M_4$ works as at Step 1, but reverses the computation procedure.

The connecting rule $\theta(21^-)$ completes the cycle.

The connecting rule $\theta(23^-)$ makes possible final Steps $3^- - 6$.

Step 3$^-$. As at Step 1$, the machine $P_{3^-}$ is the conjunction of two machines: $P_{21^-}$ and $P_{33^-}$, where the first one is just a copy of $P_{21^-}$ with different set of state letters. Historical sectors are checked, and

For the machine $P_{33^-}$, the running state letters are $r$-letters too. Its canonical work is as follows. At first, the primitive machines simultaneously check the small historical sectors. Then the next primitive machine checks the sector $R^{1,r}\mathcal{P}^{2,\ell}$, then the input sector $R_{s-1}P_s$ is checked, and finally the big historical sectors are simultaneously checked.

The connecting rule $\theta(3^-3)$ cannot be applied to a configuration having $a$-letters from right alphabets of historical sectors.

Step 3. The rules from $\Theta_3$ extend the rules of machine $D_5$ as follows. The rules of $D_5$ on the configurations with base

$$S(1)_{r}S(2)_{\ell}T(1)_{r}T(2)_{r}T(2)_{\ell}T(3)_{\ell}$$

are now the rules of Step 3 on the base

$$P_{s-1}Q_{s-1}R_{s-1}P_sQ_sR_sP\mathcal{S}(2)_{r}R_{x}$$

$$P^{1,\ell}_{r}\mathcal{T}(1)_{r}P^{1,\ell}_{r}\mathcal{R}(1)_{r}P^{2,\ell}_{r}\mathcal{P}^{2,\ell}_{r}\mathcal{T}(2)_{r}P^{2,\ell}_{r}\mathcal{R}^{2,\ell}_{r}\mathcal{P}^{3,\ell}_{r}\mathcal{T}(3)_{r}R^{3,\ell}_{r}$$

with control letters, according to the definition given in subsection 3.4, but the control letters do not work at this step, and so $P^{1,\ell}_{r}\mathcal{T}(1)_{r}P^{1,\ell}_{r}\mathcal{R}^{1,\ell}_{r}$ in this base behaves as $T(1)_{r}$ in (4.4), and so on.

The rules of $\Theta_3$ do not change big historical sectors and lock non-historical sectors of $M_3$, except for the input sector $R_{s-1}P_s$.

Comment. After standard work with consecutive Steps $1^- - 2$ and control Step $3^-$, the (copy of) machine $D_5$ checks if the length of $a$-word in the sector $R^{1,\ell}\mathcal{P}^{2,\ell}$ divisible by the eight cubes of the $a$-length of the input sector.
Remark 4.3. The rule $\theta(34^-)$ locks all sectors except for historical ones. It cannot be applied to a configuration having $a$-letters from left alphabets of small historical sectors.

**Step 4.** The tape alphabets of the control machine $P_{4^-}$ are right alphabets $X_{i,r}$ for small historical sectors and left alphabets for big historical sectors. All working sectors are locked.

The first half of $P_{4^-}$ is the machine $P_{34^-}$. Its running control letters are $p$-letters, and the canonical work starts with parallel work of primitive machines in all small historical sectors. followed by the parallel work of primitive machines in the big historical sectors.

The second machine $P_{44^-}$ starts with parallel work of primitive machines in all small historical sectors with control $r$-letters followed by parallel work of primitive machines in all big historical sectors with control $p$-letters.

**Step 4.** The rules of $\Theta_4$ simultaneously erase the letters from small historical sectors. The corresponding parts of the positive rules are $r_{j,\ell}1x \rightarrow r_{j,\ell}1 (j = 1, 2, 3)$ for every positive letter $x$ from the right alphabet of a small historical sector.

The connecting rule $\theta(45^-)$ locks all sectors excepts for big historical sectors.

**Step 5.** We define $P_{5^-} = P_{45^-}$. The control $p$-letters simultaneously check the big historical sectors, while all other sectors are locked.

The connecting rule $\theta(5^-5)$ locks all sectors excepts for big historical sectors.

**Step 5.** The rules of $\Theta_5$ simultaneously erase the letters from big historical sectors. The corresponding parts of the positive rules are $xp_{j}^1 \rightarrow p_{j}^1 (j = 1, \ldots, s)$ for every positive letter $x$ from the left alphabet of a big historical sector.

The accept command $\theta_0$ from $\Theta_5$ can be applied when all the sector are empty. So it locks all the sectors, changes the state letters and terminates the work of $M_4$.

So $M_4$ has a unique accept configuration.

**Lemma 4.2.** Let a computation $C$ of a machine $P_{1^-}$ $(i = 1, \ldots, 5)$ start with a connecting rule $\theta$ and end with a connecting rule $\theta' \neq \theta^{-1}$. Then for every sector of the standard base, there is a rule in the history of $C$ locking this sector.

*Proof.* We consider $P_{1^-}$ only. Since the computation starts with $(21^-)$ and ends with $(1^-1)$, all the primitive machines listed in the definition of the machine $P_{1^-}$ have to start and finish their work. So every sector of the standard base is locked either by $\theta(21^-)$ or by $\theta(1^-1)$, or by a rule of a primitive machine of the form $\zeta^{12}$ - see subsection 3.1 and Lemma 3.1 there, because every sector unlocked by these connecting rules are checked by one of the primitive machines forming the machine $P_{1^-}$. □

**Remark 4.3.** Every cycle of the Steps $1^{-1}, 1, 2^-, 2$ just changes the length of the sector $R^{1, r}P^{2, \ell}$ by 1. (See the definition of the connecting rule $\theta(12^-)$.) If this length $\ell$ becomes divisible by $8k^3$, where $k$ is the length of the input sector of $M_4$, the copy of machine $D_5$ can accept at Step 3, and one can obtain the stop configuration of $M_4$ after Steps $4^-, 4, 5^-, 5$. Hence the shortest accepting computation has at most $\Theta(k^3) = \Theta(g(n))$ cycles of the Steps $1^{-1}, 1, 2^-, 2$ if the length of the sector $R^{1, r}P^{2, \ell}$ is $\Theta(n)$.

This is an informal answer to the question why the division machine is needed. Indeed, if an auxiliary machine just checks if $\ell$ is equal to $k^3$, then the number of cycles as above could be $\Theta(n)$, which would lead to at least cubic Dehn function.

Another question: Since we want to repeat the cycle of the Steps $1^{-1}, 1, 2^-, 2 \Theta(f(n)^3)$ times, why does $M_4$ recognizes the values $f(n)$ instead of $g(n) = f(n)^3$? - Because the Turing machine has to take time at least $\Theta(g(n))$ to recognize $g(n)$. By Lemma 2.1,
the S-machine $M_1$ should work as long as $\Theta(g^3(n))$ or longer for the same goal. If $g(n)$ is \('\text{almost}'\) linear function, then this approach makes the time function of $M_1$ almost cubic, and the growth of the Dehn function we are going to construct, becomes almost biquadratic.

Let $B$ be the standard base of $M_4$ and $B'$ be its disjoint copy. By $M_5$ we denote the S-machine with standard base $B(B')^{-1}$ and rules $\theta(M_5) = [\theta, \theta]$, where $\theta \in \Theta$ and $\Theta$ is the set of rules of $M_4$. So the rules of $\Theta(M_5)$ are the same for $M_4$-part of $M_5$ and for the mirror copy of $M_4$. Therefore we will denote $\Theta(M_5)$ by $\Theta$ as well, assuming that the sector between $B$ and $(B')^{-1}$ is locked by any rule from $\Theta$.

Finally, the main machine $M = M_6$ is a cyclic analog of $M_5$. We add one more base letter $\{t\}$. So the standard base of $M_6$ it $\{t\}B(B')^{-1}\{t\}$, where the part $\{t\}$ has only one letter $t$ and the first part $\{t\}$ is identified with the last part. (For example, $\{t\}B(B')^{-1}\{t\}B(B')^{-1}$ can be a base of an admissible word for $M_6$. Furthermore, sectors involving $t^{\pm 1}$ are locked by every rule from $\Theta$. The stop word $W_0(M)$ is defined accordingly: every letter in the standard base $B(B')^{-1}$ of $M_5$ is replaced by the corresponding letters from the stop word of $M_5$.

The 'mirror' symmetry of the base of $M$ will be used in Lemma 9.22. For a different purpose mirror symmetry of machines was used in the papers of W.W.Boon and P.S. Novikov (see [17]).

### 4.2 Standard computations of $M$

The history $H$ of a reduced computation of $M$ can be factorized so that every factor corresponds to one of the Steps $1^-\to 5$. If, for example, $H = H'H''H'''$, where $H'$ is a product of rules from Step 2, $H''$ has only one rule $\theta(21^-)$ and $H'''$ is a product of rules from Step $1^-$, then we say that the \textit{step history} of the computation (or its type) is $(2)(21^-)(1^-)$ or just $(2)(1^-)$ since the only rule connecting the computations of Steps 2 and 1 is $\theta(21^-)$ and for the most asymptotic estimates of the length of steps (e.g., $||H'''||$) or of the lengths of their configurations, it does not matter to which of the two possible steps the connecting rule is attributed.

There are no computations of some types, say $(1)(3)$, as it immediately follows from the definition of connecting rules (and from fig. 1). In this subsection, we eliminate some other subwords in step histories.

\textbf{Lemma 4.4. 1} There are no reduced computations of $M$ with step histories $(1^-)(1)(1^-1)$, $(21^-)(2)(21^-)$ and $(23^-)(2)(23^-)$ (with $(12^-)(1)(12^-)$ and $(2^-2)(2)(2^-2)$) if the base of the computation contains at least one big historical sector \ldots $P_*$ (resp., big historical sector $\mathcal{R}_*$ \ldots ).

\textbf{2} There are no reduces computations of $M$ with step histories $(3^-3)(3)(3^-3)$ (with $(3^-4)(3)(3^-4)$) if the base of the computation contains a small historical sectors \ldots $P^*$ (resp., a small historical sectors $\mathcal{R}^*$ \ldots ).

\textbf{Proof.} (1) We consider only the type $(1^-1)(1)(1^-1)$ since other variants are similar. Indeed, if the history $H$ of the part $(1)$ is non-empty, then a state $p$-letter inserts a copy of $H^{-1}$ in historical letters of the alphabet $X_{i,r}$ (see subsection 3.2). Recall that the words with non-empty subwords over $X_{i,r}$ are not $\theta_{1^-1}$-admissible, but we should have $\theta_{1^-1}$-admissible words both in the beginning and at the end of the subcomputation with history $H$, a contradiction.

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(2) The proof is similar.\hfill\Box

**Lemma 4.5.** (1) There are no reduced computations of $M$ with step histories of the form $(i^-j)(i^-j)(i^-j)$ for $i = 1, \ldots, 5$ if the base is standard.

(2) Let the base of a computation $C : W_0 \to \cdots \to W_t$ be standard and $C$ has one of the step histories $(21^-)(1^-)(1^-1), (12^-)(2^-)(2^-2), (23^-)(3^-)(3^-3), (34^-)(4^-)(4^-4), (45^-)(5^-)(5^-5)$. Then all configurations of $C$ are uniquely defined by the history $H$ of $C$, $|W_0|_a = |W_1|_a = \cdots = |W_t|_a$, and $||H|| \leq 4||W_0||$.

**Proof.** (1) Consider only the step history $(21^-)(1^-)(21^-)$ and the work of the primitive machines switched on by the rule $(21^-)$. If one of the rules $ζ^1(a)^±1$ of this primitive machine (see subsection 3.1) does not move a control state letter right/left, but instead just insert $a^±1$ from the right and $a^±1$ from the left, then the rule $ζ^{12}$ is not applicable since the sector is not locked. So the next rule should be $ζ^1(b)^±1$ which is not the inverse one for $ζ^1(a)^±1$. Hence the control state letter has to insert letters from both sides without cancellations, and neither $ζ^{12}$ nor $(ζ^{12})^{-1}$ can be ever applied, a contradiction.

Therefore the primitive machine must work canonically, as it was described in subsection 3.1 (also see Lemma 3.1 (2)). Hence the history of its work uniquely restores the words in the sectors controlled by this primitive machine.

Hence the first control primitive machine has to complete its canonical work and switch on the next primitive machine according to the definition of $P_{1^-}$ for Step $(1^-)$, and so on. Thus, one never obtains a $θ(21^-)^{-1}$-admissible word, a contradiction.

(2) By Lemma 3.1 (3), the histories of the primitive machines subsequently restore the tape words in all unlocked sectors. Lemma 3.6 applied to $C$ and to the inverse computation, implies equalities $|W_0|_a = |W_1|_a = \cdots = |W_t|_a$, and gives $||H|| \leq 4||W_0||$ if one takes into account that both the machine $P_{21^-}$ and $P_{21^-}$ control each of the sectors. \hfill\Box

**Lemma 4.6.** There are no reduced computations with standard base and step histories $(5^-)(4)(5^-)$.

**Proof.** Assume such a computation exists. Note that the small historical sectors are empty at the both transitions $(45^-)$. However every rule of Step 4 multiplies the word in the small historical sector by a letter $x$, and one obtains no cancellations of these letters since the computation is reduced and different rules multiply by different letters. Hence the part (4) is empty, and the lemma is proved by contradiction. \hfill\Box

Below we need a rougher subdivision of the history of a reduced computation with standard base. We say that the Steps $1 - 4 -$ are *fundamental* steps and the Steps $4, 5-, 5$ are *erasing* steps. So the *block history* of every computation of $M$ is a subword of $(F)(E)(F)(E)\ldots$, where $(F)$ (where $(E)$) are maximal parts of the history with fundamental (resp., erasing) steps only. The separating rules for neighbor blocks are $θ(4^-4)^±1$.

**Lemma 4.7.** If the block history of a computation is $(E)$, then its step history is a subword of the word $(4)(5^-)(5)(5^-)(4)$.

**Proof.** Proving by contradiction and taking into account Lemma 4.5 (1), we should get a subword $(5^-)(4)(5^-)$ in the step history, contrary to Lemma 4.6. \hfill\Box
Lemma 4.8. Let $W_0 \rightarrow \cdots \rightarrow W_t$ be a computation with block history $(E)$. Then

(1) $|W_j|_a \leq \max(|W_0|_a,|W_t|_a)$ for $j = 0, 1, \ldots, t$;

(2) $t \leq 10(||W_0|| + ||W_t||)$.

Proof. (1) If the history has only one Step 4 or 5, then Statement (1) follows from Lemma 2.6 (c). For single Step 5− it follows from Lemma 3.6 (a).

If there is Step 5 in the computation, then by Lemma 4.7 we have only one maximal subcomputation $W_k \rightarrow \cdots \rightarrow W_\ell$ of Step 5. Here $|W_k|_a \leq |W_0|_a$ since $W_k$ has no non-empty sectors except for big historical sectors, which are unchanged at Steps 4, while Steps 5− cannot make these sectors shorter by projection argument (see Remark 3.2). Hence it suffices to prove Statement (1) for subcomputations with step histories $H'$ and $H''$, where $H'(5)H''$ is the step history. Therefore we may prove Statement (1) under assumption that there are no Steps 5 in the step history.

For the step history $(4)(5−)$, the a-length of the configuration separating two steps cannot be longer than the final configuration by projection argument, which reduce the proof to one-step histories.

(2) By Lemma 4.7, we have at most 5 steps, and by Property (1) it suffices to prove (2) for one-step histories but with coefficient 2. Indeed such estimates for the lengths of histories are obtained for Steps 4, 5−, 5 in Lemmas 2.6 (b) and 3.6 (b).

Lemma 4.9. Let $W_0 \rightarrow \cdots \rightarrow W_t$ be a computation with block history $(F)$.

(1) Then the step history of this computation is a subword of the word $W(m) \equiv (4−)(3)(3−)((1−)(1)(2−)(2))^m(3−)(3)(4−)$ for some non-zero integer $m$. (Here we define $(1−)(1)(2−)(2)^{-1} = (2)(2−)(1−).$

(2) If the step history is equal to $W(m)$ for some $m \neq 0$, then the exponent $m$ is divisible by $(2k)^3$, where $k$ is the a-length of the input sector $R_{s−1}P_s$ after the application of the connecting rule $\theta(3−3)$.

(3) If the step history is equal to $((1−)(1)(2−)(2))^m(3−)(3)(4−)$ and the sistory starts with a connecting rule, then the exponent $m$ is congruent to $l$ modulo $(2k)^3$, where $k$ is as in Statement (2) and $b^l$ is the tape word of the sector $R^{1,r}P^{2,\ell}$ in the beginning of the computation.

Proof. (1) Since the block history is $(F)$, Lemmas 4.5 (1) and 4.4 forbid ‘reverse moves’ $\theta^{±1}H\theta^{±1}$ in the history, where $H$ is a one-step history and $\theta$ is a connecting rule, Statement (1) follows from the definition of connecting rules between steps (see figure 1).

(2) Let us restrict the subcomputation with step history $(3−3)(3)(3−3)$ to the input sector $R_{s−1}P_s$ and the sector $R^{1,r}P^{2,\ell}$. Then we can apply Lemma 3.14 to conclude that the exponent $l$ of the tape word $b^l$ in the sector $R^{1,r}P^{2,\ell}$ at the beginning of this computation is divisible by $(2k)^3$. Similarly, for $b^{l'}$ at the end of the computation with step history $(3−3)(3)(3−3)$, we obtain that $l'$ is divisible by $(2k)^3$, where $k'$ is the a-length of the sector $R_{s−1}P_s$.

Note that a computation with step history $(1−1)(1)(2−)(2)(21−)$ or $(1−1)(1)(2−)(2)(23−)$ does not change the a-length of the sector $R_{s−1}P_s$ since it preserved by the machine $P_{2−}$ by Lemma 3.1 and the history of Step 2 is inverse (of the copy) of the history of Step 1 here. The same is true for the subcomputations with step history $(21−)(1−)(1−1)$ and $(23−)(3−3)(3−3)$ by Lemma 3.1. So $k' = k$ and $l' \equiv l$ is divisible by $(2k)^3$ by Lemma 3.14.

On the other hand, every rule $\theta(12−)$ multiplies the tape word in the sector $R^{1,r}P^{2,\ell}$, by $b^{−1}$. Therefore $l'-l = m$, whence $m$ is divisible by $(2k)^3$, as required.
(3) The proof is similar to the proof of Statement (2).

4.3 Computations with faulty bases

Lemma 4.10. If the step history of a reduced computation $C$ of $M$ is $(21^-)(1^-)(1^-1)$ or $(12^-)(2^-)(2^-2)$, or $(23^-)(3^-)(3^-3)$, or $(34^-)(4^-)(4^-4)$, or $(45^-)(5^-)(5^-5)$, then
(1) the base of $C$ is a reduced word;
(2) the first letter of the base, the length of the base and the history $H$ of $C$ completely determine the computation $C$.

Proof. (1) This follows from Lemmas 4.2 and 2.5.
(2) Consider for example the step history $(21^-)(1^-)(1^-1)$. By Property (1) and the definition of admissible word, the base of $C$ is determined by its length and the first letter. Since every sector unlocked by $\theta(21^-)$ has to be checked by a primitive machine, the copy of the content of this sector is contained in the history of the computation as a product of the letters of the form $\zeta_1(a)\pm^1$ defined for the particular primitive machine.

Lemma 4.11. Suppose that a reduced computation $W_0 \rightarrow \cdots \rightarrow W_t$ of Step 1 or 2 (of Step 3) starts with a connecting rule. Assume that the length of its base $B$ is bounded form above by a constant $N_0$, and $B$ has a big historical sector (a small historical sector, resp.) of the form $\mathcal{RP}$ (with indices). There is a constant $c = c(N_0)$ such that $|W_0|_a \leq c|W_t|_a$.

Proof. Let $V_0 \rightarrow \cdots \rightarrow V_t$ be the restriction of the computation to the sector with base $\mathcal{RP}$. By Lemma 3.9, we have $t \leq |V_t|_a$ and $|V_0|_a \leq |V_t|_a$.

It follows from lemma 2.4 (2) that
\[
|W_0|_a \leq |W_t|_a + 2N_0t \leq |W_t|_a + 2N_0|V_t|_a \leq (2N_0 + 1)|W_t|_a
\]

It suffices to choose $c = 2N_0 + 1$.

Definition 4.12. We call a base of $M$ faulty if

- it starts and ends with the same base letter,
- it has no proper subwords with this property, and
- it is a not a reduced word.

The main lemma of this subsection is

Lemma 4.13. There is a constant $C = C(M)$, such that for every reduced computation $C: W_0 \rightarrow \cdots \rightarrow W_t$ of $M$ with a faulty base and every $j = 0, 1, \ldots, t$, we have $|W_j|_a \leq C(|W_0|_a + |W_t|_a)$.

Proof. 1. By Lemma 4.10, the step history (and the inverse step history) has no subwords $(21^-)(1^-)(1^-1)$, $(12^-)(2^-)(2^-2)$, $(23^-)(3^-)(3^-3)$, $(34^-)(4^-)(4^-4)$ and $(45^-)(5^-)(5^-5)$.

2. Assume that the history has only one Step $(2^-)$ and the base is $pp^{-1}p$ for some control state letter $p$ (or $r$). (Such a base is faulty and cannot be a subbase of a longer faulty base.) Obviously, the running letter $p$ cannot lock any sector and so every rule has the same type (either $\zeta_1(\star)$ or $\zeta_2(\star)$ as in subsection 3.1).

If $|W_{j+1}|_a > |W_j|_a$ for some $j$, then one of two sectors (e.g., the first one) increases its length by 2, while another one does not decrease the length under the transition.
$W_j \rightarrow W_{j+1}$. But then the first sector of $W_{j+2}$ has to increase again by 2 since the computation is reduced. It follows that $|W_{j+1}| \leq |W_{j+2}| \leq \cdots \leq |W_i|$. Hence the length of every configuration does not exceed $\max(|W_0|_a, |W_f|_a)$.

If the base has no subwords of the form $(p p^{-1} p)^{\pm 1}$, then the same inequality for a computation with history of Step 2 $\Rightarrow$ follows from lemmas 3.1 and 3.3. Thus, one may assume further that the step history is not $(2^-)$. Similarly, it is neither $(1^-)$ nor $(3^-)$, nor $(4^-)$, nor $(5^-)$.

3. Assume there is (1) in the step history. Then by item 1, the set of steps is either (a) $\{(1)\}$ or (b) $\{(1), (1^-)\}$, or (c) $\{(1), (2^-)\}$, or (d) we have the subword $(1^-)(1)(2^-)$ in the step history (or in the inverse step history).

3a. In this case, the required inequality follows from Lemma 3.12. (Recall that by definition the connecting rule $\theta(12^-)$ inserts/deletes one letter, but this small change of length is not an obstacle.)

3c. Assume that the step history is $(2^-)(1)$ (or $(1)(2^-)$) and $W_0 \rightarrow \cdots \rightarrow W_j$ is a maximal subcomputation with step history $(2^-)$. Then $|W_0|_a \geq |W_j|_a$ by the projection argument. Therefore it suffices to prove the statement of the lemma for a subcomputations with step histories $(2^-)$ and $(1)$. For the case $(2^-)$, we refer to item 2. The case $(1)$ is considered in item 3a. Thus, we assume further that the step history has length at least 3.

Suppose the step history has a subword $(2^-)(1)(2^-)$. Then the base (or the inverse base) has no big historical sectors $\mathcal{R} \ldots$ (with indices) by Lemma 4.4. So the only possible bases for big historical sectors have form $\mathcal{P}^{-1}\mathcal{P}$. The state control letters of such sector should start running at Step $(2^-)$ after the application of the connecting rule $\theta(12^-)$ but they cannot ever meet state letter from $\mathcal{R}$ and will run forever, and $|W_f|_a \geq |W_j|_a$ by Lemma 3.3, if $W_j$ is obtained at the application of $\theta(12^-)$. Hence one can cut the computation at $W_j$ reducing the problem to items 2 and 3a. If there are no big historical sectors, then there are no working sectors of the machine $M_3$. The other sectors of $M$ (which could come from the base of $D_5$) do not work at Steps $(1)$ and cannot decrease their length starting from a connecting rule at Steps $(2^-)$ by Lemma 3.1. This makes the statement of the lemma obvious for them.

Hence one may assume that there are no subwords $(2^-)(1)(2^-)$, and so the step history is $(1)(2^-)(1)$.

Suppose the base has a big historical sector $\mathcal{R}\mathcal{P}$ (with indices). Then for the maximal subcomputation $W_r \rightarrow \cdots \rightarrow W_t$ of Step 1, we have $|W_r|_a \leq c|W_t|_a$ by Lemma 4.11 because the length of a faulty base is bounded by a constant $N_0$ depending on the machine $M$ only. Hence one can reduce our task to the subcomputations with the step histories $(1)$ and $(2^-)$. (Of course, the constant in the desired inequality changes when we pass to step histories involving more types of steps.) Hence we assume further that the base has no big historical sectors $\mathcal{R}\mathcal{P}$.

Also there are no big historical sectors $\mathcal{P}^{-1}\mathcal{P}$ because state control letters of such sector should start running after the application of the connecting rule $\theta(12^-)$ but they cannot ever meet state letter from $\mathcal{R}$ and will run forever by Lemma 3.3; the last Step 1 will not be reached.

So all big historical sectors (if any) are of the form $\mathcal{R}^{-1}\mathcal{R}$. Recall that the alphabets for the $\theta(12^-)^{\pm 1}$-admissible words are $X_{i,r}$, and so the word in this alphabet will be conjugated at Step 1 by the letters from the disjoint alphabets $X_{i,\ell}$ in sectors with bases $\mathcal{R}^{-1}_{i-1}\mathcal{R}_{i-1}$. Hence after application of $\theta(12^-)^{-1}$, each rule of Step 1 will increase the
length of such sector by 2. By Lemmas 3.10 and 2.4, we have $|W_t|_a \leq \cdots \leq |W_t|_a$ if the last Step 1 starts with $W_r$. This reduces the problem to the subcomputations with the step histories (1) and $(2^-)$ again.

If there are no such sectors, then we have no working sectors except for the sectors of the machine $D_5$ because one may assume that the left-most sector of the standard base of $M_2$ is always locked and because the base is faulty. The other sectors (which could come from the base of $D_5$) do not work at Steps (1) and cannot decrease their length towards $W_0$ or $W_t$ at Steps $(2^-)$ by Lemma 3.1. This makes the statement of the lemma obvious for them and completes case 3c.

3b. This case is similar to 3c up to exchange of $R$ with $P$ and $X_{i,t}$ with $X_{i,r}$.

3d. Assume that the step history has a subword $(1^-)(1)(2^-)$. Then there are no big historical sectors $P^{-1}P$ and $RR^{-1}$ since the conjugation in free group given by Step 1 cannot transform a non-trivial word in the alphabet $X_{i,t}$ in a word in the alphabet $X_{i,r}$. So all big historical sectors have base of the form $RP$ (or inverse one).

Consider the word $W_j$ obtained after the application of the last connecting rule $\theta(12^-)^{\pm1}$. Only big historical sectors of $W_j$ are not locked (except for the sectors of the machine $D_5$, which are not touched by Step 1). If the next step is 1, then its rules cannot make the historical sector shorter by Lemma 3.9. If the next step is $2^-$, then no sector becomes shorter by Lemma 3.1. The same is true for Step 1 if it follows Step 1. Repeating this procedure, we have $|W_j|_a \leq |W_t|_a$. Therefore one can reduce Case 3d to previous cases by subdivision of the computation along the transitions between Steps 1 and $2^-$. (The occurrences of $\theta(12^-)$ and $\theta(12^-)^{-1}$ in the history of the computation have to alternate in Case 3d inserting/deleting the same letter, and so this does not affect the desired inequality.)

Thus, we may assume further that there are no Steps 1 in the computation.

4. We may also assume that there are no Steps 2 in the computation. The proof copies the proof at item 3 with subcases (a) \{(2)\} (b) \{(2), (2^-)\}, (c) \{(2), (3^-)\} and (d), where one considers the subword $(2^-)(2)(3^-)$ and the configuration $W_j$ provided by the last connecting rule $\theta(2^-2)^{\pm1}$.

5. If there is Step 5 in the step history, then we cannot have steps except for 5 and $5^-$ by item 1. It follows from the definition of Step 5^- and Lemma 3.1 (4) that the subword $(5^-)(5^-)(5)$ in the step history is not possible if there is a big historical sector with base of type $RP$. By Lemma 3.3, historical sectors with base $P^{-1}P$ are not possible too. However the historical sectors with base $RR^{-1}$ do not change configuration at Step 5. It follows that the word $W_j$ obtained after the transition from $(5^-)$ to $(5)$ is not longer than $W_t$ (see similar argument at item 3d). So one reduces the task to shorter step histories.

Thus, one may assume that there are no subwords $(5^-)(5^-)(5)$ in the step history. Therefore assuming that there is $(5)$ in the step history, we should consider only the history $(5^-)(5^-)(5^-)$ or its subwords. Again the rules $\theta(5^-5)$ defines a configuration $W_j$, which is not longer than $W_t$ by Lemma 3.1. This reduce the task to step histories $(5^-)$ and $(5)$. For $(5^-)$, the problem is solved in item 2, and it is solved by Lemma 2.6 (c) for $(5)$.

Thus one may assume from now that there are no Steps $(5)$ in the step history.

6. If there is $(5^-)$ in the step history, then by items 1 and 5, there are no other steps except for (4) and $(4^-)$. However the transition $(45^-)$ provides us with the shortest configurations in the computations since neither computations of Step $5^-$ nor those
from (4) can make big historical sectors shorter by lemmas 3.1 and 3.3. So cutting the computation along such transitions, we can decrease the number of steps. Since a single Step $5^-$ is eliminated in item 2, we may further assume that there are no Steps $(5^-)$ in the computation.

7. If there is (4) in the step history, then by items 1 and 6, there are no other steps except for $(4^-)$. For such histories we will repeat some arguments from item 3c using now small histories instead of big ones.

The cases with a single Step $(4^-)$ or (4) are eliminated by item 4 and Lemma 3.12, respectively. Assume that the step history is $(4^-)(4)$ and $W_0 \rightarrow \cdots \rightarrow W_j$ is a maximal subcomputation with step history $(4^-)$. Then $|W_0|_a \geq |W_j|_a$ by the projection argument. Therefore the problem is reduced to single step histories. Thus, we assume further that the step history has length at least 3.

Assume that there is a subword $(4)(4^-)(4)$ in the step history and there is the letter $R_s$ in the base. Then it cannot follow by $R_s^{-1}$, because the letter from $R_s$ must start running right after the connecting rule $\theta(4^-)-1$ and it cannot ever reach the part $P$, so the next connecting rule $\theta(4^-)$ cannot appear, a contradiction. Hence there is a sector $R_s P$. It follows that there is $R_1^{1,\ell}$ in the base since the sectors between $P$ and $R_1^{1,\ell}$ are locked by $\theta(4^-)$. Then we obtain $P^{1,\ell}$ and all other parts of small historical sectors.

Also there is the part $P_{s-1}$ in the base, because all the sectors between $P_{s-1}$ and $R$ are locked by $\theta(4^-)$, and the base is faulty. Hence right after the sector $R_s P$ is checked by the first primitive machine of $P_{4-}$ a letter from $P_{s-1}$ starts running checking the big historical sectors, and again, there should be the part $R_s P_{s-2}$ in the base, since otherwise the next occurrence of $\theta(4^-)$ does not happen.

This implies it turn, that there is $P_{s-2}$ in the base, and so on, that is we have all the sectors of the standard base of $M$ and the base of our computation has no cancellation, contrary to the definition of faulty base.

Hence there are no parts $R_s$ in the base, provided $(4)(4^-)(4)$ is a subword of the step history. Similar argument shows that there are no $R_1^{1,\ell}$ and $R_2^{3,\ell}$. (For example, if there is $R_1^{1,\ell}$, then there is a rule locking the sector $R_1^{1,\ell} P^{1,\ell}$ by the definition of primitive machines; this rule has to lock sector $R_s P$ too, and so the part $R_s$ occurs in the base too.) Hence nothing changes at steps 4, since only $R$-letters can erase the small historical sectors. The transition $(44^-)$ provides us with the shortest configurations in the computations since neither computations of Step $4^-$ nor those from (4) can make small historical sectors shorter by lemmas 3.1 and 3.3. So cutting the computation along such transitions, we can decrease the number of steps. Since a single Step $4^-$ is eliminated in item 2, we may further assume that there are no subwords $(4)(4^-)(4)$ in the step history.

Thus, the step history is $(4^-)(4)(4^-)$, and as above it can be subdivided in one-step histories. Therefore we may assume from now that there are no Steps 4 in the step history.

8. Now we assume that there is Step $4^-$ in the Step history.

Suppose the step history has a subword $(4^-)(3)(4^-)$. Then the base has no small historical sectors $R^s \cdots$ by Lemma 4.4 (2). So the only possible bases for small historical sectors are $(P^{i,r})^{-1} P^{i,r}$ or $P^{-1} P$. The state control letters of such sectors should start running after the application of the connecting rule $\theta(34^-)$ but they cannot ever meet state $R$-letters and will run forever, whence $|W_{i,a}| \geq |W_j|_a$ by Lemma 3.3, if $W_j$ is obtained at the application of $\theta(34^-)$. So the whole step history is $(4^-)(3)(4^-)$. Hence one can cut the computation at $W_j$ reducing the problem to item 2 and Lemma 3.17. If
there are no small historical sectors, then there are no sectors of the machine $D_5$ changed at step 3. The other sectors (which could come from the base of $M_2$) do not work at steps 3 and cannot decrease their length when starting from a connecting rule at steps $4^-$ by Lemma 3.1. As usual, this allows induct on the number of steps.

Hence one may assume that there are no subwords $(4^-)(3)(4^-)$. The subwords $(3^-)(3)(3^-)$ are eliminated in the same way.

Assume that the step history has a subword $(3^-)(3)(4^-)$. Then there are no small historical sectors with non-reduced bases $UU^{-1}$ since the conjugation in free group given by step (3) cannot transform a non-trivial word in the alphabet $X_i \ell$ in a word in the alphabet $X_i r$. So all small historical sectors have base of the form $RP$ (or inverse ones).

Consider the word $W_j$ obtained after the application of the last connecting rule $\theta(34^-)^{+1}$. Only small historical sectors of $W_j$ are not locked (except for the sectors of machine $M_3$, which are not touched by step 3). If the next step is (3), then its rules cannot make the historical sector shorter by Lemma 3.9. If the next step (4$^-$), then no sector becomes shorter by Lemma 3.1. The same is true for step $3^-$ if it follows Step 3. Repeating this procedure, we have $|W_j|_a \leq |W_t|_a$. Therefore one can reduce the problem to shorter step histories.

It remains to assume that the step history is $(3)(4^-)(3)$. It resembles item 3c, but below we consider small historical sectors instead of the big ones.

Suppose the base has a small historical sector $i,\ell P^{i,r}$ or $R_s P$. Let $W_r \rightarrow \cdots \rightarrow W_t$ be the maximal subcomputation with step history (3). Then we obtain inequality $|W_r|_a \leq c|W_t|_a$ by Lemma 4.11. Hence one can reduce our task to the subcomputations with the step histories (3) and $(4^-)$. Therefore we assume further that the base has no small historical sectors of the form $RP$.

Also there are no small historical sectors $(P^{i,r})^{-1}P^{i,r}$ (or $P^{-1}P$), because state control letters of such sector should start running after the application of the connecting rule $\theta(34^-)$ but they cannot ever meet state letter from $R$ and will run forever by Lemma 3.3; the last Step 3 will not be reached.

So all small historical sectors are of the form $R^{i,\ell}(R^{i,\ell})^{-1}$ (or $R_s R_s^{-1}$) But then the word from such a sector in ‘right’ alphabet will be conjugated at Step 3 by the letters from a left alphabet. Hence after application of $\theta(3^-4)^{-1}$, each rule of Step 3 will increase the length of such sector by 2. By Lemmas 3.10 and 2.4, we have $|W_r|_a \leq \cdots \leq |W_t|_a$ if the last Step 3 starts with $W_r$. This reduces the problem to the subcomputations with the step histories (4) and $(4^-)$ again.

If there are no such sectors, then there are no sectors of the machine $D_5$ changed at step 3 since the base is faulty. The other sectors (which could come from the base of $M_2$) do not work at steps 3 and cannot decrease the length towards $W_0$ or $W_t$ at Steps 3. This makes the statement of the lemma obvious.

Hence one may assume from now that there are no Steps $4^-$ in the computation.

9. If there is Step 3 in the step history, then there are no steps except for 3 and $(3^-)$. As in item 8, one may assume that the length of step history is at least 3. Then the subwords $(3^-)(3)(3^-)$ can be eliminated by the same argument we used in item 8 to eliminate subwords $(4^-)(3)(4^-)$. It remains to consider computations with step history $(3)(3^-)(3)$. Again, one refer to item 8 since one can eliminate such history in the way the histories $(3)(4^-)(3)$ were eliminated in item 8.

The lemma is proved.
4.4 Space and time of M-computations with standard base

Lemma 4.14. Let $C : W_0 \rightarrow \cdots \rightarrow W_t$ be a computation with standard base and step history $(21^{-})(1^{-})(1)(2^{-})(2)(21^{-})$. Then the configuration $W_t$ is a copy of $W_0$ except for the sector $R^{1,r}P^{2,l}$ and the mirror copy of it, whose lengths in $W_0$ and $W_t$ differ by one.

If $H(1^{-})$ and $H(2^{-})$ are the histories of the subcomputations $C(1^{-})$ and $C(2^{-})$ of $C$ with step histories $(21^{-})(1^{-})(1)(1^{-})$ and $(12^{-})(2^{-})(21^{-})$, respectively, then $||H(1^{-})|| - ||H(2^{-})|| = 2$.

Proof. The parts subcomputations $C(1^{-})$ and $C(2^{-})$ do not change the $a$-words in the historical sectors by Lemma 3.6(a), and so the histories of Steps 1 and 2 are inverse copies of each other. Taking into account that the transition $(12^{-})$ changes the length of the sector $R^{1,r}P^{2,l}$ (and the mirror it) by 1, we obtain the first statement of the lemma.

After the sector $R^{1,r}P^{2,l}$ changes length by one, the primitive machine checking this sector changes the computation time by 2, as it follows from Lemma 3.1 (3). This proves the second statement.

Lemma 4.15. (1) If a configuration $W_0$ is accepted and $\theta$-admissible for a rule $\theta$ from block $(E)$, then there is a reduced accepting computation $W_0 \rightarrow \cdots \rightarrow W_t$ with block history $(E)$ and $t \leq 3||W_0||$;

(2) There is a constant $c_1$ depending on $M$ only, such that for any computation $C : W_0 \rightarrow \cdots \rightarrow W_t$ of $M$, which is the beginning of a reduced accepting computation with block history $(E)$ or $(F)(E)$, we have $||W_j|| \leq c_1(||W_0|| + s)$ for every $j = 0, \ldots, t$, where $s$ is the length of the step history.

Proof. (1) Note that all the sectors of $W_t$ are empty. It follows that all the words in big historical sectors of $W_0$ are the copies of the same word since every rule of $M$ multiplies the tape words of these sectors by the copies of the same letter or does not change all these words. Similarly, the tape words in the small historical sectors (and in there mirror sectors) are copies of each other.

If the small historical sectors of $W_0$ are non-empty, then $\theta$ is a rule of Step 4. So there is an accepting computation erasing all the letters of this sector (and of its mirror copy). The length of the next control Step $5^{-}$ will be at most $2||W_0||$ by the definition of the machine $P_5$, and Lemma 3.1 (3). Then the rules of Step 5 can erase all tape letters in the big historical sectors (and their mirror copies). This gives the total upper estimate $t \leq 2||W_0|| + ||W_0||$, as required. If $\theta$ is a rule of Step $5^{-}$ or 5, then the estimate is even better.

(2) If the computation is accepting and has type $(E)$, then the step history is a suffix of the word $(4)(5^{-})(5)$ by Lemma 4.7. The rules of Step $5^{-}$ cannot increase the lengths of configurations by Lemma 3.1. Clearly, the rules of Steps 4 and 5 cannot insert letters too. Hence $||W_j|| \leq ||W_0||$, and so it suffices to prove the same inequality under the assumption that $j \leq r$, where $W_0 \rightarrow \cdots \rightarrow W_r$ is the maximal subcomputation with block history $(F)$.

By Lemma 4.9, the step history of this subcomputation is a suffix of the word $W(m) \equiv (4^{-})(3)(3^{-})(1^{-})(1)(2^{-})(2)(3)(3^{-})(3)(4^{-})$ for some non-zero integer $m$.

At first we consider the subcomputations $C_i$ with step histories $((21^{-})(1^{-})(1)(2^{-})(2)(21^{-}))^{\pm 1}$. By Lemma 4.14, we conclude that in the beginning and at the end of $C_i$, the difference of lengths of configurations is equal to $\pm 2$. The number of such subcomputations $C_i$ does not exceed $s/2$. 

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The number of one-step subcomputations, which are not subcomputations of any \( C_i \), is at most 7. The transitions of some of them \((1^-, 2^-, 3^-, 4^-)\) do not increase the lengths of configurations by Lemma 3.6 (a). The all transitions of each other step \((1, 2, 3)\) can increase the \( a \)-length but Lemma 4.11 bounds from above possible enlargement. This proves Statement (2).

**Lemma 4.16.** Let a history \( H \) of a computation \( C : W_0 \rightarrow \cdots \rightarrow W_t \) with standard base has type \((F)\) and ends with a connecting rule. Suppose that \( C \) is a beginning of a reduced accepting computation and there are at most 10 steps in \( C \). Then for a constant \( c_2 = c_2(M) \), we have \( ||H|| \leq c_2||W_0|| \).

**Proof.** Recall that the length of the history is linearly bounded for each of the steps in terms of the lengths of their configurations (Lemma 2.6 for Steps 4 and 5, Lemma 3.9 for Steps 1, 2 and 3, Lemmas 4.5(2) and 3.1 (1) for control steps.) The lengths of these configurations are linearly bounded in Lemma 4.15 (2). Taking into account that the number of steps is at most 10 we come to the desired inequality. 

**Lemma 4.17.** There is a constant \( c_3 \) such that for any accepted configuration \( W_0 \), which is \( \theta(2,1^-)^{\pm 1}-\)admissible, there is an accepting computation with block history \((F)(E)\) of length at most \( c_3(k^3+1)(||W_0|| + k^3) \), where \( k \) is the \( a \)-length of the input sector \( R_s P \) of the word \( W_0 \). The number of steps in this computation is less than \( 32k^3 + 4 \).

**Proof.** By Lemma 4.15 (1), an accepted word \( \theta \)-admissible for a rule \( \theta \) from block \((E)\) can be accepted by a computation having only one block. Hence it suffices to consider computations with block histories \((F)(E)\).

By Lemma 4.9, the step history of block \((F)\) is a word \( ((1^-)(1)(2^-)(2))^m(3^-)(3)(4^-) \) for some integer \( m \) since \( W_0 \) is \( \theta(21^-)^{\pm 1} \) admissible.

Assume that \( |m| > (2k)^3 \). Then the rule \( \theta(12^-) \) occurs at least \((2k)^3 + 1\) times in the history. The rule \( \theta(12^-) \) permanently changes the length of the sector \( R_s P^{2,\ell} \) by one multiplying it by the same letter. On the other hand, by Lemma 4.14, the subcomputations with step histories \( ((21^-)(1^-)(1)(2^-)(2)(21^-))^\pm 1 \) do not change the \( a \)-word of length \( k \) in the input sector \( R_s P \). Therefore there is a transition (not the last one), where the \( a \)-length of the sector \( R_s P^{2,\ell} \) is divisible by \((2k)^3\), and so the transition to Step 3 was possible earlier by Lemma 3.16 (2). Hence it suffices to prove the lemma under the assumption \( |m| \leq (2k)^3 \), and so the number of steps does not exceed \( 4|m| + 3 \leq 32k^3 + 3 \) by Lemma 4.7.

The length of the history is linearly bounded for each of the steps in terms of the lengths of the configurations (Lemma 2.6 for Steps 4 and 5, Lemma 3.9 for Steps 1, 2 and 3, Lemmas 4.5(2) and 3.1 (1) for control steps.) The lengths of these configurations are linearly bounded in Lemma 4.15 (2). Taking into account that the number of steps is linearly bounded in terms of \( m \) and \(|m| \leq (2k)^3\), we come to the desired inequality.

We will consider a suitable function function \( f(n) \) and the functions \( g(n) \) and \( F(n) \) from Definition 1.2 under assumption that \( s = 2 \) in that definition. (The inequality \( s \geq 3 \) will appear in the last Subsection 10.2.) So the recognizing Turing machine \( M_0 \) is taken from that definition and \( M_1 \) is given by Lemma 2.1.

Without loss of generality, one may assume that the values of \( f(n) \) are greater than some constant, for instance, for every integer \( n \geq 1 \), we have
\[ f(n) \geq 5 \] (4.5)

Besides, it is convenient to enlarge the domain of those functions assuming that they are defined for every real \( x \geq 0 \). One may assume that \( f(x) \) is still positive for \( x > 0 \) and non-decreasing.

Lemma 4.18. For every \( x > 0 \) and \( d \in [0, x) \), we have \( F(x) - F(x - d) \geq \frac{d}{x} F(x) = dxg(x) \).

Proof. Note that \( F(x)/x^2 \geq F(x - d)/(x - d)^2 \) since the function \( g(x) \) is non-decreasing. Hence

\[
F(x) - F(x - d) \geq F(x) \left(1 - \frac{(x - d)^2}{x^2}\right) \geq F(x) \frac{2dx - d^2}{x^2} = \frac{d}{x} F(x)\]

\[\square\]

Lemma 4.19. Let \( C : W_0 \rightarrow \cdots \rightarrow W_t \) be a computation of \( M \) with block history \( (E) \) and the step history of this computation contain a subword \( ((21^-)(1^-)(1)(2^-)(2)(21^-))^\pm \). Let \( k \) be the \( a \)-length of the input sector \( R_sP \) of a \( \theta (2-1)^{\pm} \)-admissible configuration \( W_s \) in \( C \). Then (a) \( k = f(n) \) for some \( n \geq 1 \), (b) \( k = O(f(||W_0||)) \), and (c) \( k^3 = O(||W_0||) \).

Proof. Claim (c) follows from (b) since \( f^3(n) = O(n) \).

To prove (b) statement, it suffices to prove that \( k = O(f(r)) \), where \( r \) is the length of a big historical sector of \( W_0 \). We have \( r' \leq r \), where \( r' \) is the length of a big historical sector of \( W_s \) by Lemma 4.9; indeed, the computation \( W_s \rightarrow \cdots \rightarrow W_0 \) cannot decrease the length of it by Lemma 3.6 (a) applied to the control steps and Lemma 3.9 applied to Steps 1 and 2. (Step 3 does not change this sector.) For the same reason \( r'' \leq r' \), and \( k = k'' \) where \( r'' \) is the length of a big historical sector of the first configuration \( D_0 \) of a subcomputation \( D \) of \( C^{\pm} \) with the step history \( (1^-1)(1)(12^-) \) and \( k'' \) is the \( a \)-length of the input sector \( R_sP \) of \( D_0 \). Hence it suffices to prove that \( k'' = O(f(r'')) \).

The subcomputation of Step 1 (restricted to the base of \( M_0 \)) is actually the computation of \( M_2 \). If we ignore the historical sectors, we have the accepting computation of \( M_1 \) with input sector \( a^{k''} \). By the definition of \( M_0 \) and \( M_1 \), \( k'' = f(n) \) for some \( n \geq 1 \), the accepting computation of \( M_0 \) has length \( \Theta(n^{1/3}) \), and so the number of rules at Step 1 is \( \Theta(n) \) by Lemma 2.1. Hence \( r'' \geq \Theta(n) \), because \( D_0 \) contains the history of Step 1 written in big historical sectors. Hence \( k'' = f(n) = O(f(r'')) \). Since \( k = k'' \), Property (a) is obtained as well. \[\square\]

Lemma 4.20. Let \( W_0 \) be an accepted word and \( C : W_0 \rightarrow \cdots \rightarrow W_t \) be a reduced computation of \( M \) with block history \( (E) \), where the computation either length-non-increasing or length-non-decreasing, or \( (F)(E) \), where the block \( E \) is a length-non-increasing sub-computation, or \( (F) \). Then

(a) \( ||W_j|| \leq c_4 \max(||W_0||, ||W_t||) \) \( (j = 0, \ldots, t) \), where \( c_4 = c_4(M) \), or
(b) there are accepting computations for \( W_0 \) and \( W_t \) with histories \( H_0 \) and \( H_t \) such that \( ||H_0|| + ||H_t|| < t/100 \), the history \( H \) of \( C \) has a factorization \( H = H(1)H(2)H(3) \), where \( ||H(1)||, ||H(2)|| < t/100 \), \( H(2) \) is of type \( ((21^-)(1^-)(1)(2^-)(2)(21^-))^m \) with \( |m| \geq c_3 \) and the lengths of the subhistories of \( H(2) \) of type \( ((21^-)(1^-)(1)(2^-)(2)(21^-))^\pm \) are less than \( t/10 \).
Proof. If \( C \) is of type \((E)\), then Property (a) follows since \( C \) either non-increases or non-decreases the lengths of configurations.

Let \( C \) have type \((F)(E)\). Again we obtain Property (a) by Lemma 4.15 (2) if the number of steps in block \((F)\) is less than 10.

Otherwise, by Lemmas 4.7 and 4.9, block \((F)\) contains a subword \(((21^-)(1^-)(2^-)(2)(21^-))^\pm 1\) in the step history, and by Lemma 4.14, the corresponding computations \( C_i \) just multiply the words in the sector \( \mathcal{R}^{1,r}_{P_{2},\ell} \) (and in its mirror copy) by a letter \( b \) or by \( b^{-1} \) depending on the sign of the exponent, while the length \( k \) of the word in the sector \( \mathcal{R}s_{P} \) is not changed.

The history corresponding to the block \((F)\) is \( H_1 H_2 H_3 \), where the length of the step history at most 4 for \( H_1 \) and \( H_3 \), \( H_2 \) has the form \(((21^-)(1^-)(2^-)(2)(21^-))^m \), and \( H_2 \) is the history of the subcomputation \( W_{s_0} \rightarrow \cdots \rightarrow W_{s_1} \rightarrow \cdots \rightarrow W_{s_m} \) starting with \( \theta(21^-)^{\pm 1}, \) and every \( W_{s_i} \) is a \( \theta(21^-)^{\pm 1}\)-admissible word. If \( w = \max(||W_{s_0}||, ||W_{s_m}||) \), then by Lemma 4.14,

\[
\max(\sum_{i=0}^{m-1} ||W_{s_i}||, \sum_{i=1}^{m} ||W_{s_i}||) \geq w + |w - 2| + \cdots + |w - 2| m - 1| \geq |m| w/2 \tag{4.6}
\]

Hence the sum of the lengths of the subcomputations \( C_0, \ldots, C_{m-1} \) is at least \( |m| w \) by Lemma 3.1 for the Steps 1\(^-\) and 2\(^-\) of these subcomputations. Therefore \( ||H_2|| > |m| w. \)

Now if \( \max(||W_{0}||, ||W_{1}||) = w' \geq w \), then the length of every \( W_{j} \) with \( j \leq s_0 \) or \( j \geq s_m \) is bounded by \( c_1 (w' + 10) \) by Lemma 4.15 (2), and the same estimate works if \( W_{j} \) is a configuration of some subcomputation \( C_i. \) Hence we have the inequality of item (a) of the lemma.

So we assume now that \( w' < w \). Let \( k \) be the \( a \)-length of the input sector \( \mathcal{R}s_{P} \) of the word \( W_{s_0}. \) If \( m \geq c_3 k^3 \), then \( t \geq ||H_2|| > c_3^3 k^3 w. \) We also have that each of the subcomputations \( C_i \) with step histories \(((21^-)(1^-)(2^-)(2)(21^-))^\pm 1\) have length less than \( t/10 \). This follows from the property, that the difference of the lengths of \( C_i \) and \( C_{i+1} \) is at most 4 (the computational time for control step can change by 2 by Lemma 4.14, but the number of such subcomputations is at least \( c_3^3 > 1000 \).

Let \( W_{s_0} \equiv V_0 \rightarrow \cdots \rightarrow V_d \) be a shortest accepting computation for \( W_{s_0} \) with a history \( H_0. \) To estimate \( d \) from above, we may assume by Lemma 4.15 (1) that its block history is either \((E)\) or \((F)(E)\). The step history of block \((E)\) has length at most 3 by Lemma 4.7.

If the number of steps in \( H_0 \) is at most 10, then \( ||H_0|| = O(||W_0||) \) by Lemma 4.16. Otherwise by Lemma 4.9, \( H_0 = H'H'' \), where \( H' \) has step history \(((21^-)(1^-)(2^-)(2)(21^-))^\pm 1 \) and \( H'H'' \) starts with a \( \theta(21^-)^{\pm 1}\)-admissible configuration \( W_{s_0}. \), By Lemma 4.17,

\[
||H'H''|| \leq c_3 (k^3 + 1)(||W_{s_0}|| + k^3) \leq c_3 (k^3 + 1)(w + k^3) = c_3 (k^3 + 1)O(w) \tag{4.7}
\]

since \( k^3 = O(w) \) by Lemma 4.19.

Thus by (4.7), \( ||H'H''|| \leq c_3^2 k^3 w/400 \leq ||H_2||/400 \leq (s - r)/400 \) if \( c_3 \) is large enough. To estimate the length of the shortest accepting computation for \( W_0 \), it remains to estimate \( s_0 = ||H_1|| \), but this value does not exceed \( c_2 w \leq c_2^2 k^3 w/800 \leq t/400 \) by Lemma 4.16. Therefore \( ||H_0|| \leq t/400 + t/400 = t/200 \leq ||H_2||/200 \). Similarly, we obtain \( ||H_t|| \leq t/200 \leq ||H_2||/200. \) Now define \( H(1) = H_1, H(2) = H_2 \) and \( H(3) = H_3 \tilde{H} \), where \( \tilde{H} \) is of type \((E)\) and so has at most 3 steps, and \( H(3) \) has less than 10 steps. To obtain Property (2) of the lemma if \( m \geq c_3^3 k^3 \), it remains to estimate \( H(3). \) Indeed, by lemmas 4.15 (2) and 4.16, we have \( ||H(3)|| = ||H_3\| + ||\tilde{H}|| \leq c_2 w + 3w \leq t/100. \)
Assume now that $|m| < c_3^2 k^3$. Then the number of steps $s$ in the subcomputation $C$ does not exceed $10c_3^2 k^3 + 20$. Therefore for every configuration $W_j$ of $C$, we obtain from Lemmas 4.15 (2) and 4.19 that $||W_j|| \leq c_1 (w' + 10c_3^2 k^3 + 20) = c_1 c_2 O(w') \leq c_4 w'$, if $c_4$ is big enough, and we have Property (a).

The same argument works if the block history is just $(F)$.

\[\square\]

**Lemma 4.21.** Let $W_0$ be an accepted word and $C : W_0 \rightarrow \cdots \rightarrow W_t$ be a reduced computation of $M$ with block history of the form $(E)(F)(E)(F)(E)$, where the first (the last) block $(E)$ is a length-non-decreasing (resp., length-non-increasing) subcomputation. Then there are accepting computations for $W_0$ and $W_t$ with histories $H_0$ and $H_t$ such that $||H_0|| + ||H_t|| < t/100$.

**Proof.** (1) Since the word $W_0$ is $\theta$-admissible for a rule $\theta$ of block $E$, we have $||H_0|| \leq 3||W_0||$ by Lemma 4.15 (1), and $||W_0|| \leq ||W_s||$, where $C' : W_s \rightarrow \cdots \rightarrow W_r$ is the subcomputation corresponding to the first occurrence of $(F)$ in the block history of $C$, because the rules of the first block $(E)$ does not decrease the lengths.

Let $k$ and $m$ be the parameters of $C'$ defined as in the proof of Lemma 4.20. Note that $m \neq 0$ by Lemmas 4.9 (1) and 4.5 (1) since $C'$ starts with $\theta (4 \cdot 4)^{-1}$. Due to the maximality of $C'$, one can apply Lemmas 4.9 (2) and 4.19 (a) to $C'$ and obtain $|m| \geq (2k)^3 \geq 1000$ since the length $k$ of any word accepted by $M_1$ and $M_2$ is at least 5 by (4.5). Then, as in Lemma 4.20, we obtain that the length of $C'$ is at least $2||W_s|| + 2||W_s|| - 2 + \cdots + 2||W_s|| - 999$, which is at least $1000||W_s||$ and at least $1000||W_0|| > 300||H_0||$.

Similarly we have that the length of $C''$ is greater than $300||H_t||$, where the computation $C''$ corresponds to the last occurrence of $F$ in the block history of $C$. It follows that $||H_0|| + ||H_t|| < t/100$.

\[\square\]

**Lemma 4.22.** Let $W_0$ be an accepted word, $C : W_0 \rightarrow \cdots \rightarrow W_t$ be a reduced computation of $M$ and $H_0$, $H_t$ be the histories of the shortest computations accepting $W_0$ and $W_t$, respectively. Then

1. (a) $C$ is a product of at most three subcomputations $C_1 : W_0 \rightarrow \cdots \rightarrow W_{n_1}$, $C_2 : W_{n_1} \rightarrow \cdots \rightarrow W_{n_1+n_2}$ and $C_3 : W_{n_1+n_2} \rightarrow \cdots \rightarrow W_{n_1+n_2+n_3}$, where $n_1 + n_2 + n_3 = t$, and for every $C_i$ ($i = 1, 2, 3$) either

   (a) $||W_j|| \leq c_4 \max(||W_0||, ||W_t||)$, for every configuration $W_j$ of $C_i$, where $c_4 = c_4(M)$

   or

   (b) there are accepting computations for the first and the last configuration of $C_i$ with histories $H'_i$ and $H''_i$ such that $||H'_i|| + ||H''_i|| < n_i/100$.

2. (2) The sum of lengths of all maximal subcomputations of $C$ with block history $E$ does not exceed $3(||W_0|| + ||W_t||) + t/100$.

**Proof.** (1) If the block history of $C$ is $(F)$, then the entire computation $C$ satisfies either (a) or (b) by Lemma 4.20. If the block history of $C$ is $(E)(F)$, we consider the subcomputation $C' : W_0 \rightarrow \cdots \rightarrow W_s$, corresponding to the first block $(E)$ of the step history. If $W_{n_1}$ has minimal length in $C'$, then the subcomputation $C'' : W_{n_1} \rightarrow \cdots \rightarrow W_s$ is length-non-decreasing; this follows from Lemma 2.6 for Steps 4 and 5 and from Lemma 3.6 for Step 4. Similarly, the subcomputation $W_0 \rightarrow \cdots \rightarrow W_{n_1}$ is length-non-increasing. Since $||W_{n_1}|| \leq ||W_0||$ we can again apply Lemma 4.20 to the subcomputations $W_0 \rightarrow \cdots \rightarrow W_{n_1}$ and $W_{n_1} \rightarrow \cdots \rightarrow W_t$.
Therefore we assume that there are at least three blocks.

Consider the subcomputation $C' : W_r \rightarrow \cdots \rightarrow W_s$ corresponding to the first block $(E)$ occurred in the step history. It has a maximal length-non-increasing part $W_r \rightarrow \cdots \rightarrow W_{n_1}$ as in the previous paragraph.

Observe that $\|W_r\| \leq \|W_0\|$. Indeed, only historical sectors can be unlocked in $W_r$, but neither control machine can increase the lengths of these sectors in the computation $W_r \rightarrow \cdots \rightarrow W_0$ with block history $(F)$ by Lemma 3.1, nor the computations of Steps 1,2,3 can do this by Lemma 3.9.

Since $\|W_{n_1}\| \leq \|W_r\| \leq \|W_0\|$, we can apply Lemma 4.20 to the subcomputation $C_1 : W_0 \rightarrow \cdots \rightarrow W_{n_1}$ and obtain one of the properties (a) or (b) for it. Similarly, we consider the last block $(E)$ in the block history of $C$ and define the subcomputation $C_3$ starting with $W_{n_1+n_2}$ whose length does not exceed the length of $W_t$, and so either (a) or (b) holds for $C_3$. We have $n_2 = 0$ if the block history is $(F)(E)(F)$.

If there are at least two blocks $(E)$, then the middle computation $C_2$ satisfies the assumptions of Lemma 4.21, and so Property (b) holds for it.

(2) The subdivision of each subcomputation corresponding to a $(E)$ according the sample of part (1) gives the required estimate. Namely, if a subcomputation $D$ is a product $D_1D_2D_3$ with block history $(E)(F)(E)$ satisfying Lemma 4.21, then we obtain that the length of $D_1$ plus the length of $D_3$ is less than 0.01 of the length of $D_2$. If $D$ has brief history $(E)$ or $(F)(E)$, or $(E)(F)$, the we refer to Lemma 4.20.

Lemma 4.23. For every accepted word $W_0$ of length at most $n$ there is an accepting computation of length $O(nf(n)^3)$ with number of steps $O(f(n)^3)$. The generalized time function $T'(n)$ of $M$ is equivalent to $O(nf(n)^3)$.

Proof. By Lemma 4.15 (1), given an accepted word $W_0$ of length $n$, there is a shortest accepting computation $W_0 \rightarrow \cdots \rightarrow W_t$ with block history either $(E)$ or $(F)(E)$. We denote by $H$ its history. The step history of block $(E)$ has length at most $3\|W_0\|$ by Lemma 4.7 and contains at most 3 steps.

If the number of steps in $H$ is at most 10, then $\|H\| = O(\|W_0\|) = O(n)$ by Lemma 4.16. Otherwise by Lemma 4.9, $H = H_1H_2H_3$, where $H_1$ has less than 10 steps and $H_2$ has step history $((1^-)(1)(2^-)(2))^\pm1$ and $H_2H_3$ starts with a $\theta(21^-)^{\pm1}$-admissible configuration $W_j$. By Lemma 4.17, $\|H_2H_3\| \leq c_3(k^3 + 1)(\|W_j\| + k^3)$, where $k$ is the length of the sector $\mathcal{R}_s\mathcal{P}$ in $W_j$, and the number of steps in $H_2H_3$ is $O(k^3)$ (and so the number of steps in the entire $H$ is $O(k^3)$). Here $\|W_j\| \leq c_1(\|W_0\|a + 10) = O(n)$ by Lemma 4.15 (2).

Since $k = O(f(\|W_j\|)) = O(n)$ and $k^3 = O(\|W_j\|)$ by Lemma 4.19, we have $O(k^3) = O(f(n)^3)$ for the number of steps and

$$\|H_2H_3\| \leq c_3(k^3 + 1)(\|W_j\| + k^3) = c_3(f(n)^3)O(n) = O(n)g(n)$$

by the definition of the functions $f(n)$ and $g(n)$.

The length of each of the steps in $H_1$ is bounded by $4c_1(\|W_0\|a + 10)$ (use Lemma 3.6 (b) for Steps $1^-,2^-,3^-,4^-,5^-$, Lemma 2.7 (b) for Steps 1, 2 and 3, and Lemma 2.6 for Steps 4 and 5. Hence the length of the whole history $H$ is also $O(n)g(n)$, as required.

To bound $T'(n)$ from below, we will construct a series of accepted words $V(n)$ of length $\Theta(n)$. The base of every $V(n)$ is standard, and $V(n)$ is $\theta(2^-1)$-acceptable. The input sector $\mathcal{R}_s\mathcal{P}$ contains $a^k$, where $k = f(n) > 0$, the word in the sector $\mathcal{R}_s^{2,\ell}\mathcal{P}^{2,\ell}$ is $b^l$, where $l = \Theta(n) > 8k^3$ and $l$ congruent to $4k^3$ modulo $8k^3$. (There is such $l$ since

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\( k^3 = f(n)^3 = O(n). \) Each of the big historical sectors of \( V(n) \) contains the history of an accepting computation for \( M_2 \), written in the alphabets \( X_{i,t} \). The length of this history is \( O(n) \) by the definition of the suitable function \( f(n) \), the definition of the machines \( M_0 - M_2 \) and Lemma 2.1. Each of the small historical sectors contains the history of the computation of \( D_3 \) (also in left alphabets) that checks that \( l - 4k^3 \) is divisible by \( 8k^3 \). Since \( l = \Theta(n) \), this history has length \( O(n) \) by Lemma 3.16. Thus, we have \( ||V_n|| = \Theta(n) \).

Every word \( V(n) \) is accepted. Indeed, the rules of Step 1 can check all the sectors since the base is standard. Then the rules of Step 1 can accept \( f(n) \), \( \theta(12^{-}\) replaces \( b^l \) with \( b^{l-1} \). The rules of Step 2 check the sectors again, the history of Step 2 copies the inverse history of Step 1, it restores the alphabets \( X_{i,t} \) in big historical sectors. Then we repeat the cycle decreasing the exponent at \( b \) by one again. After \( 4k^3 \) such cycles we obtain \( b^{l-4k^3} \) in the sector \( R^{2,\ell}P^{3,\ell} \), where \( l - 4k^3 \) is divisible by \( 8k^3 \), and therefore after Step 3, Step 3 can completes its work by Lemma 3.16 (b). It remains to erase all tape letters using the rules of block \( (E) \) and stop computing after the rule \( \theta_0 \) is applied.

Now let us estimate from below the length of arbitrary (reduced) computation \( V(n) \equiv W_0 \rightarrow \cdots \rightarrow W_t \). By Lemma 4.9, we have the block \( F \) of the form

\[
((1^-)(1)(2^-)(2)^m(3^-)(3)(4^-) \text{ in the history, where } m \equiv 4k^3 \text{ (mod } 8k^3) \) \text{ by Lemma 4.9 (2). Hence } |m| \geq 4k^3 = 4f(n)^3.
\]

The history of every subcomputation with step history \( ((1^-)(1)(2^-)(2))^{\pm 1} \) has length at least \( \Theta(n) \) for the following reason. Every configuration of it has a word in the sector \( R^{2,\ell}P^{3,\ell} \) of length \( \Theta(n) \) since this length belongs in the segment \( [l-4k^3, l] \). So by Lemma 3.1, one needs \( \Theta(n) \) rules to check this sector at the control steps 1\(^- \) and 2\(^- \).

Therefore the length of the computation \( W_0 \rightarrow \cdots \rightarrow W_t \) is at least \( 4k^3\Theta(n) = \Theta(nf(n)^3) \), as desired. Since we obtain the required lower bound for every \( n \) and \( ||V_n|| = \Theta(n) \), the lemma is proved. \( \square \)

**Remark 4.24.** A subcomputation with block history \( ((21^-)(1^-)(1)(2^-)(2)(21^-))^{\pm 1} \) does not change the length of the sector \( R^{2,\ell}P^{3,\ell} \) by Lemma 3.6 applied to steps \( 1^- \) and \( 2^- \). Hence we have the same property for computations \( ((1^-)(1)(2^-)(2))^{m} \) starting and ending with connecting rules. Thus, above we obtained \( \Theta(nf(n)^3) \) configurations of length at least \( \Theta(n) \) for any computation accepting the word \( V(n) \).

We call a base \( B \) of a reduced computation (and the computation itself) *revolving* if \( B \equiv xvx \) for some letter \( x \) and a word \( v \), and \( B \) has no proper subword of this form.

If \( v \equiv v_1zv_2 \) for some letter \( z \), then the word \( zv_2xv_1z \) is also revolving. One can cyclically permute the sectors of revolving computation with base \( xvx \) and obtain a uniquely defined computation with the base \( zv_2xv_1z \), which is called a cyclic permutation of the original computation. The history and lengths of configurations do not change when one cyclically permutes a computation.

**Lemma 4.25.** There is a constant \( c_4 = c_4(M) \) such that following holds. For any computation \( C : W_0 \rightarrow \cdots \rightarrow W_t \) of \( M \) with a revolving base \( xvx \) either

1. we have inequality \( ||W_j|| \leq c_4 \max(||W_0||, ||W_t||) \), for every configuration \( W_j \) of \( C \), where \( c_4 = c_4(M) \), or
2. we have the following properties:
   1. \( \text{the word xy is a cyclic permutation of the standard base } B = B(M) \text{ or of } B^{-1} \) and
   2. the corresponding cyclic permutations \( W'_0 \) and \( W'_t \) of the words \( W_0 \) and \( W_t \) are accepted words, and
(c) the step history of $C$ (or of the inverse computation) contains subwords $(21^-)(1^-)(1^-)$ and $(12^-)(2^-)(2^-)$, and

(d) $C$ is a product of at most three subcomputations $C_1 : \quad W_0' \to \cdots \to W_n'$, $C_2 : \quad W_{n_1}' \to \cdots \to W_{n_1+n_2}$ and $C_3 : \quad W_{n_1+n_2}' \to \cdots \to W_{n_1+n_2+n_3}'$ ($n_1 + n_2 + n_3 = t$), where

\[
\max(||W_{n_1}'||, ||W_{n_1+n_2}'||) \leq \max(||W_0'||, ||W_l'||) \quad \text{and for each } C_i, \quad \text{either}
\]

(d1) \quad $||W_i'|| \leq c_4 \max(||W_0'||, ||W_l'||)$, for every configuration $W_j$ of $C_i$ or

(d2) there are accepting computations for the first and the last configuration of $C_i$ with histories $H_i'$ and $H_i''$ such that $||H_i'|| + ||H_i''|| < n_i$.

\[\]

Proof. If the computation is faulty, then Property (1) is given by Lemma 4.13 since $c_4 > C$. If it is non-faulty, then we have all sectors of the base in the same order as in the standard base (or its inverse), and we obtain Property (2a). Therefore we may assume now that the base $xv$ is standard and Property (1) does not hold.

If the block history of $C$ is $(E)$, we obtain a contradiction with Lemma 4.8 since $c_4 > 1$.

If the computation has only one step of type $(F)$, then Property (1) follows from Lemmas 3.6, 3.12 and 3.17, a contradiction again. So there is a connecting rule $\theta$ from block $(F)$ in the history.

Assume there is a block $(F)$ in the block history of $C$, and this block has at least 8 steps. Then by Lemma 4.9, the step history of $C$ has a subword $((21^-)(1^-)(1^-)(2^-)(2^-)(21^-))^{\pm 1}$, and Property (c) follows. Moreover the words at the big (small) history sectors are copies of the same word since Step $(1^-)$ (or $2^-$) has simultaneously controlled these sectors. Therefore after a number of such cycles one can obtain the length of the sector $\mathcal{R}_{s'}^P \mathcal{R}_{s'}^P$ divisible by $8k^3$ (where $k$ is the length of the sector $\mathcal{R}_{s'}^P$), which by Lemma 4.9 (2), makes possible to accept after the Steps $3^-, 3^-, 4^-, 4, 5$. So one obtains Properties (a), (b) and (c). Then Property (d) follows from Lemma 4.22.

If there are no such blocks $(F)$, then there are no subwords $(E)(F)(E)$ in the block history by Lemmas 4.9 and 4.5. Hence the block history is $(F)(E)(F)$ or a subword of this word. Let configuration $W_r'$ and $W_s'$ subdivide $C$ in single block computations. Then $||W_r'|| < c_3||W_0'||$, because there are at most 7 steps in the subcomputation $W_r' \to \cdots \to W_0'$, and each step transition from $W_r'$ towards $W_0'$ can multiply the length by at most $c$. (See Lemma 3.6 for control steps and Lemma 4.11 for Steps 1, 2 and 3.) Analogously, we have $||W_s'|| < c_3||W_0'||$. Since for every step the lengths of all configurations are linearly bounded in terms the first and the last configurations (see Lemmas 3.12, 3.17 3.6 (a)) we have $||W_r'|| \leq c_4 \max(||W_0'||, ||W_s'||)$ if $j \leq r$ or $j \geq s$. So to obtain Property (1) (and a contradiction), it suffices to linearly bound the configurations in the subcomputation $W_r' \to \cdots \to W_s'$ in terms of $\max(||W_r'||, ||W_s'||)$. This is done in Lemma 4.8 (1). Thus, the proof is complete.

4.5 Two more properties of standard computations

Here we prove two lemmas needed for the estimates in Subsection 9. The first one says (due to Lemma 4.5 (2)) that if a standard computation $C$ is very long in comparison with the lengths of the first and the last configuration, then it can be completely restored if one knows the history of $C$, and the same is true for the long subcomputations of $C$. This makes the auxiliary parameter $\sigma_\lambda(\Delta)$ useful for some estimates of areas of diagrams $\Delta$. The second lemma is also helpful for the proof of Lemma 9.23 in Subsection 9.
Lemma 4.26. Let $\mathcal{C} : W_0 \rightarrow \cdots \rightarrow W_t$ be a reduced computation with standard base, where $t \geq c_5 \max(||W_0||, ||W_t||)$ for sufficiently large constant $c_5 = c_5(M)$. Suppose the word $W_0$ is accepted. Then any subcomputation $\mathcal{D} : W_r \rightarrow \cdots \rightarrow W_s$ of $\mathcal{C}$ (or the inverse for $\mathcal{D}$) of length at least $0.4t$ contains one of the words $(21^{-})(1^{-})(1^{-}1), (1^{-})(2^{-})(2^{-}2), (23^{-})(3^{-})(3^{-}3), (34^{-})(4^{-})(4^{-}4)$ in its step history.

Proof. If the block history of $\mathcal{C}$ is $(F)$, we refer to Lemma 4.20 as follows.

Assume that Property (a) of that lemma holds. Then every step of the computation $\mathcal{D}$ has lengths at most $4c_4 \max(||W_0||, ||W_t||)$ by Lemma 3.9 for Steps 1, 2 and 3 and by Lemma 3.6 (b) for other steps. Hence the number of steps in $\mathcal{D}$ has to be at least 10 since the length of its history is at least $0.4c_5 \max(||W_0||, ||W_t||)$ and $c_5$ can be chosen large enough. It follows from Lemma 4.9 (1) that the step history of $\mathcal{D}$ contains subwords $(21^{-})(1^{-})(1^{-}1)$ and $(1^{-})(2^{-})(2^{-}2)$, as required.

If Property (a) of Lemma 4.20 fails, then by Property (b), we have a subcomputation of length $> 0.98t$ with step history $((1^{-})(1^{-})(2^{-})(2^{-}))^{\pm m}$, where every cycle with block history $((1^{-})(1^{-})(2^{-})(2^{-}))^{\pm 1}$ has length $< t/10$. Then the subcomputation $\mathcal{D}$ of length $\geq 0.4t$ has to contain such a cycle, and so the step history of $\mathcal{D}$ contains $(12^{-})(2^{-})(2^{-}2)$, as required. Thus, we may assume that the block history of $\mathcal{C}$ is not $(F)$.

If the block history of $\mathcal{D}$ contains a subword $(E)(F)(E)$, then the statement follows from Lemma 4.9 (where Lemma 4.5 eliminates the case $m = 0$). So the block history of $\mathcal{D}$ is a subword of $(E)(F)(E)$. By Lemma 4.22 (2), the length of the $(E)$-subcomputation of $\mathcal{D}$ is less that $t/100 + 3(||W_0|| + ||W_t||) \leq t/50$. So one of the $(F)$-subcomputations of $\mathcal{D}$ has length $> (0.4 - 0.02)t/2 = 0.19t$.

Case 1. Assume that there is a block $(E)$ in the block history of $\mathcal{D}$, and without loss of generality, we may assume that the computation $\mathcal{D}$ has a subcomputation $\mathcal{D}' : W_r \rightarrow \cdots \rightarrow W_j$ of type $(F)$ with $j - r > 0.19t$ and the subcomputation of type $(E)$ occurs after $\mathcal{D}'$ in $\mathcal{D}$. Proving by contradiction, we conclude that the step history of $\mathcal{D}'$ is $(4^{-})$ since a longer step history would provide us with the subword $(34^{-})(4^{-})(4^{-}4)$ in the step history of $\mathcal{D}$.

Suppose the subcomputation $W_0 \rightarrow \cdots \rightarrow W_r$ also has a block $(E)$. Then $\mathcal{C}$ has a subcomputation with block history $(E)(F)(E)$. Let the subcomputation $\mathcal{C}$ corresponds to the middle block $(F)$. Then the first and the last configurations of $\mathcal{C}$ are admissible for some rules of type $(E)$. Therefore by Lemma 4.15 (1), one can construct an auxiliary computation $\hat{\mathcal{C}} = \hat{\mathcal{C}}' \hat{\mathcal{C}}''$, where the first factor (the third one) starts (resp. ends) with Step (5). Then by Lemmas 4.9 and 4.19 for $\hat{\mathcal{C}}$, the subcomputation $\hat{\mathcal{C}}$ of $\mathcal{C}$ has a subcomputation of type $(F)$ with step history $((1^{-})(1^{-})(2^{-})(2^{-}))^{m_1}(3^{-})(3^{-})(4^{-})$. $|m_1| \geq 8t^3 \geq 1000$, and the subcomputation corresponding to the last Step $(4^{-})$ has length $> 0.19t$. It follows that at least 1000 control of the form $(1^{-})$ or $(2^{-})$ should have length $> 0.19t$ since their control machines have to check the big and small historical sectors too (and the length of the historical sectors are unchanged by the rules of $((1^{-})(1^{-})(2^{-})(2^{-}))^{m_1}(3^{-})(3^{-})(4^{-}))$. We obtain a contradiction since $1000 \times 0.19t > t$.

Thus, the computation $\hat{\mathcal{E}} : W_0 \rightarrow \cdots \rightarrow W_j$ is of type $F$. Hence $||W_j|| \leq ||W_0||$, because only historical sectors can be unlocked in $W_j$, but neither control machine can decrease the lengths of these sectors in the computation $W_j \rightarrow \cdots \rightarrow W_0$ with block history $(F)$ by Lemma 3.1, nor the computations of Steps 1,2,3 can do this by Lemma 3.9.

If the step history of $\mathcal{E}$ ends with $(34^{-})(4^{-})(4^{-}4)$, then $||W_0|| \geq ||W_j|| \geq 0.19t/4 > 0.04t$ by Lemma 3.6, which contradicts to the assumption of the lemma. Hence the step
history of $\mathcal{E}$ is $4^-$, and so $0.19t \leq 4||W_0||$ by Lemma 3.6 (b), a contradiction again.

**Case 2.** The step history of $\mathcal{D}$ is $(F)$. Since the block history of $\mathcal{C}$ is not $(F)$ but a subword of $(F)(E)(F)$, we conclude without loss of generality, that $\mathcal{C}$ begins with a maximal subcomputation $\mathcal{E} : W_0 \to \cdots \to W_u$ of type $(F)$, where $r \leq u < 1$. Then as in Case 1, we have $||W_u|| \leq ||W_0||$.

Now consider the options (a) and (b) provided by Lemma 4.20 for $\mathcal{E}$. The option (b) is eliminated exactly as in the second paragraph of the proof of the current lemma, where $t$ can be replaced by $0.4t$ since $u \geq r - s \geq 0.4t$. Hence we have by (a) that every configuration $W_j$ of $\mathcal{E}$ satisfies the inequality $||W_j|| \leq c_4||W_0||$ since $||W_u|| \leq ||W_0||$. Then the length of every single step of $\mathcal{E}$ cannot exceed $4c_4||W_0||$ (see Lemma 3.6 (b) for control steps and 2.7 for Steps 1, 2 and 3). Here we have $4c_4 < c_5/10$ by the choice of $c_5$.

Since $s - r \geq 0.4t \geq 0.4c_5||W_0||$, we see that the length of the step history of $\mathcal{D}$ is at least 4. It follows from Lemma 4.9 that the step history of $\mathcal{D}$ contains one of the words mentioned in the formulation of Lemma 4.26.

**Lemma 4.27.** Let a reduced computation $W_0 \to \cdots \to W_i$ starts with an accepted word $W_0$, have standard base, and have step history of length 1. Assume that for some index $j$, we have $|W_j|_a > 3|W_0|$. Then there is a sector $QQ'$ such that a state letter from $Q$ or from $Q'$ inserts a letter increasing the length of this sector after any transition of the subcomputation $W_j \to \cdots \to W_i$.

**Proof.** If we have a control step, then the statement follows from Lemma 3.1 (1).

Let the step history be (1). Note that all big historical sectors of any configuration $W_i$ have the same content (up to taking a copy) since the word $W_0$ is accepted. Assume that no rule of the subcomputation $\mathcal{D} : W_0 \to \cdots \to W_j$ increases the length of big historical sectors. Then by Lemma 2.7 (b) the lengths of the history of $\mathcal{D}$ does not exceed $h$, where $h$ is the $a$-length of such sectors.

Every rule of the subcomputation $\mathcal{D}$ can change the length of any working sector at most by 1. (See Lemma 2.4 (3)). Hence if its length in $W_0$ is $\ell$, its length in $W_j$ is at most $\ell + h$. It follows that $|W_j|_a \leq 3|W_0|_a$, because the working sectors of $M_2$ and its historical sectors alternate in the standard base. This contradicts to the assumption of the lemma.

Thus, there is a rule in the history of $\mathcal{D}$ increasing the length of a big historical sector $QQ'$. It has to insert a letter from $X_{i,\ell}$ from the left and a letter from $X_{i,r}$ from the right. Since the obtained word is not a word over one of these alphabets, Step 1 is not over, and the next rule has to increase the lengths of the sector again in the same manner since the computation is reduced. This procedure will repeat until one gets $W_i$. This proves the statement. The same proof works for Steps 2 and 3. (In the later case, one will consider small historical sectors.)

It follows from the definition of Step 4 (of Step 5) that every rule either increase or decrease the length of small (resp., of big) history sectors. If any rule increases it, then all the next rules will increase the lengths of these sectors too. Hence the argument of the previous paragraph works for Steps 4 and 5 as well.

For the control Steps $1^- - 5^-$, the statement of the lemma follows from Lemma 3.1 (1): if we have a transition of a primitive machine, where the control state letter increases the length of a sector, then it will keep increasing it in any reduced computation.
5 Groups and diagrams

5.1 The groups

Every S-machine can be simulated by finitely presented group (see [19] and also [16], [14]). Here we apply such a construction to the machine $M$. To simplify formulas, it is convenient to change the notation. From now on we shall denote by $N$ the length of the standard base of $M$.

Thus the set of state letters is $Q = \sqcup_{i=0}^{N} Q_i$ (where $Q_N = Q_0 = \{ \ell \}$) $Y = \sqcup_{i=1}^{N} Y_i$, and $\Theta$ is the set of rules of the S-machine $M$.

The finite set of generators of the group $M$ (the same letter as for the machine) consists of $q$-letters corresponding to the states $Q$, $a$-letters corresponding to the tape letters from $Y$, and $\theta$-letters corresponding to the rules from the positive part $\Theta^+$ of $\Theta$.

The relations of the group $M$ correspond to the rules of the machine $M$; for every rule $\theta = [U_0 \to V_0, \ldots U_N \to V_N] \in \Theta^+$, we have

$$U_i \theta_{i+1} = \theta_i V_i, \quad \theta_j a = a \theta_j, \quad i, j = 0, \ldots, N$$  \hspace{1cm} (5.8)

for all $a \in Y_j(\theta)$. (Here $\theta_N \equiv \theta_0$.) The first type of relations will be called $(\theta, q)$-relations, the second type - $(\theta, a)$-relations.

Finally, the required group $G$ is given by the generators and relations of the group $M$ and by one more additional relation, namely the $hub$-relation

$$(W_M)^L = 1,$$  \hspace{1cm} (5.9)

where $W_M$ is the accept word (of length $N$) of the machine $M$ and the exponent $L$ is a large enough integer. (It depends on $M$ and will be made more precise later.)

5.2 Van Kampen diagrams

Recall that a van Kampen diagram $\Delta$ over a presentation $P = \langle A \mid R \rangle$ (or just over the group $P$) is a finite oriented connected and simply–connected planar 2–complex endowed with a labeling function $\text{Lab} : E(\Delta) \to A^{\pm 1}$, where $E(\Delta)$ denotes the set of oriented edges of $\Delta$, such that $\text{Lab}(e^{-1}) = \text{Lab}(e)^{-1}$. Given a cell (that is a 2-cell) $\Pi$ of $\Delta$, we denote by $\partial\Pi$ the boundary of $\Pi$; similarly, $\partial\Delta$ denotes the boundary of $\Delta$. The labels of $\partial\Pi$ and $\partial\Delta$ are defined up to cyclic permutations. An additional requirement is that the label of any cell $\Pi$ of $\Delta$ is equal to (a cyclic permutation of) a word $R^{\pm 1}$, where $R \in R$. The label and the combinatorial length $||p||$ of a path $p$ are defined as for Cayley graphs.

The van Kampen Lemma states that a word $W$ over the alphabet $A^{\pm 1}$ represents the identity in the group $P$ if and only if there exists a diagram $\Delta$ over $P$ such that $\text{Lab}(\partial\Delta) \equiv W$; in particular, the combinatorial perimeter $||\partial\Delta||$ of $\Delta$ equals $||W||$. ([8], Ch. 5, Theorem 1.1; our formulation is closer to Lemma 11.1 of [10]). The word $W$ representing 1 in $P$ is freely equal to a product of conjugates to the words from $R^{\pm 1}$. The minimal number of factors in such products is called the area of the word $W$. The area of a diagram $\Delta$ is the number of cells in it. By van Kampen Lemma, Area($W$) is equal to the area of a diagram having the smallest number of cells among all diagrams with boundary label $\text{Lab}(\partial\Delta) \equiv W$.

We will study diagrams over the groups $M$ and $G$. The edges labeled by state letters ($= q$-letters) will be called $q$-edges, the edges labeled by tape letters ($= a$-letters) will be called $a$-edges, and the edges labeled by $\theta$-letters are $\theta$-edges.
We denote by $|p|_a$ (by $|p|_\theta$, by $|p|_q$) the $a$-length (resp., the $\theta$-length, the $q$-length) of a path/word $p$, i.e., the number of $a$-edges/letters (the number of $\theta$-edges/letters, the number of $q$-edges/letters) in $p$.

The cells corresponding to relation (5.9) are called hubs, the cells corresponding to $(\theta, q)$-relations are called $(\theta, q)$-cells, and the cells are called $(\theta, a)$-cells if they correspond to $(\theta, a)$-relations.

A van Kampen diagram is reduced, if it does not contain two cells (= closed 2-cells) that have a common edge $e$ such that the boundary labels of these two cells are equal if one reads them starting with $e$ (if such pairs of cells exist, they can be removed to obtain a diagram of smaller area and with the same boundary label). To study (van Kampen) diagrams over the group $G$ we shall use their simpler subdiagrams such as bands and trapezia, as in [12], [19], [1], etc. Here we repeat one more necessary definition.

**Definition 5.1.** Let $Z$ be a subset of the set of letters in the set of generators of the group $M$. A $Z$-band $B$ is a sequence of cells $\pi_1, \ldots, \pi_n$ in a reduced van Kampen diagram $\Delta$ such that

- Every two consecutive cells $\pi_i$ and $\pi_{i+1}$ in this sequence have a common boundary edge $e_i$ labeled by a letter from $Z^{\pm 1}$.
- Each cell $\pi_i$, $i = 1, \ldots, n$ has exactly two $Z$-edges in the boundary $\partial \pi_i$, $e_i^{-1}$ and $e_i$ (i.e. edges labeled by a letter from $Z^{\pm 1}$) with the requirement that either both $\text{Lab}(e_{i-1})$ and $\text{Lab}(e_i)$ are positive letters or both are negative ones.
- If $n = 0$, then $B$ is just a $Z$-edge.

The counter-clockwise boundary of the subdiagram formed by the cells $\pi_1, \ldots, \pi_n$ of $B$ has the factorization $e^{-1}q_1f^{-1}q_2$ where $e = e_0$ is a $Z$-edge of $\pi_1$ and $f = e_n$ is an $Z$-edge of $\pi_n$. We call $q_1$ the bottom of $B$ and $q_2$ the top of $B$, denoted $\text{bot}(B)$ and $\text{top}(B)$. Top/bottom paths and their inverses are also called the sides of the band. The $Z$-edges $e$ and $f$ are called the start and end edges of the band. If $n \geq 1$ but $e = f$, then the $Z$-band is called a $Z$-annulus.

We will consider $q$-bands, where $Z$ is one of the sets $Q_i$ of state letters for the machine $M$, $\theta$-bands for every $\theta \in \Theta$, and $a$-bands, where $Z = \{a\} \subseteq Y$. The convention is that $a$-bands do not contain $(\theta, q)$-cells, and so they consist of $(\theta, a)$-cells only.

**Remark 5.2.** To construct the top (or bottom) path of a band $B$, at the beginning one can just form a product $x_1 \ldots x_n$ of the top paths $x_i$'s of the cells $\pi_1, \ldots, \pi_n$ (where each $\pi_i$ is a $Z$-bands of length 1). No $\theta$-letter is being canceled in the word $W \equiv \text{Lab}(x_1) \ldots \text{Lab}(x_n)$ if $B$ is a $q$- or $a$-band since otherwise two neighbor cells of the band would make the diagram non-reduced. For similar reason, there are no cancellations of $q$-letters $W$ if $B$ is a $\theta$-band.

If $B$ is a $\theta$-band then a few cancellations of $a$-letters (but not $\theta$-letters) are possible in $W$. (This can happen if one of $\pi_i, \pi_{i+1}$ is a $(\theta, q)$-cell and another one is a $(\theta, a)$-cell.) We will always assume that the top/bottom label of a $\theta$-band is a reduced form of the word $W$. This property is easy to achieve: by folding edges with the same labels having the same initial vertex, one can make the boundary label of a subdiagram in a van Kampen diagram reduced (e.g., see [10] or [19]).

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If the path \((e^{-1}q_1f)^{\pm 1}\) or the path \((fq_2^{-1}e^{-1})^{\pm 1}\) is the subpath of the boundary path of \(\Delta\) then the band is called a *rim* band of \(\Delta\). We shall call a \(Z\)-band *maximal* if it is not contained in any other \(Z\)-band. Counting the number of maximal \(Z\)-bands in a diagram we will not distinguish the bands with boundaries \(e^{-1}q_1fq_2^{-1}\) and \(fq_2^{-1}e^{-1}q_1\), and so every \(Z\)-edge belongs to a unique maximal \(Z\)-band.

We say that a \(Z_1\)-band and a \(Z_2\)-band *cross* if they have a common cell and \(Z_1 \cap Z_2 = \emptyset\).

Sometimes we specify the types of bands as follows. A \(q\)-band corresponding to one of the letters \(t\) of the base is called a \(t\)-band.

The papers [12], [1], [15] contain the proof of the following lemma in a more general setting. (In contrast to Lemmas 6.1 [12] and 3.11 [15], we have no \(x\)-cells here.)

**Lemma 5.3.** A reduced van Kampen diagram \(\Delta\) over \(M\) has no \(q\)-annuli, no \(\theta\)-annuli, and no \(a\)-annuli. Every \(\theta\)-band of \(\Delta\) shares at most one cell with any \(q\)-band and with any \(a\)-band.

\(\square\)

If \(W = x_1...x_n\) is a word in an alphabet \(X\), \(X'\) is another alphabet, and \(\phi: X \to X' \cup \{1\}\) (where 1 is the empty word) is a map, then \(\phi(W) = \phi(x_1)...\phi(x_n)\) is called the *projection* of \(W\) onto \(X'\). We shall consider the projections of words in the generators of \(M\) onto \(\Theta\) (all \(\theta\)-letters map to the corresponding element of \(\Theta\), all other letters map to 1), and the projection onto the alphabet \(\{Q_0 \cup \cdots \cup Q_{N-1}\}\) (every \(q\)-letter maps to the corresponding \(Q_i\), all other letters map to 1).

**Definition 5.4.** The projection of the label of a side of a \(q\)-band onto the alphabet \(\Theta\) is called the *history* of the band. The step history of this projection is the *step history* of the \(q\)-band. The projection of the label of a side of a \(\theta\)-band onto the alphabet \(\{Q_0, ..., Q_{N-1}\}\) is called the *base* of the band, i.e., the base of a \(\theta\)-band is equal to the base of the label of its top or bottom.

As for words, we may use representatives of \(Q_j\)-s in base words.

**Definition 5.5.** Let \(\Delta\) be a reduced diagram over \(M\), which has boundary path of the form \(p_1^{-1}q_1p_2q_2^{-1}\), where \(p_1\) and \(p_2\) are sides of \(q\)-bands, and \(q_1\), \(q_2\) are maximal parts of the sides of \(\theta\)-bands such that \(\text{Lab}(q_1), \text{Lab}(q_2)\) start and end with \(q\)-letters.

Then \(\Delta\) is called a *trapezium*. The path \(q_1\) is called the *bottom*, the path \(q_2\) is called the *top* of the trapezium, the paths \(p_1\) and \(p_2\) are called the *left and right sides* of the trapezium. The history (step history) of the \(q\)-band whose side is \(p_2\) is called the *history*
(resp., step history) of the trapezium; the length of the history is called the *height* of the trapezium. The base of $\text{Lab}(q_1)$ is called the *base* of the trapezium.

**Remark 5.6.** Notice that the top (bottom) side of a $\theta$-band $T$ does not necessarily coincide with the top (bottom) side $q_2$ (side $q_1$) of the corresponding trapezium of height 1, and $q_2$ ($q_1$) is obtained from $\text{top}(T)$ (resp. $\text{bot}(T)$) by trimming the first and the last $a$-edges if these paths start and/or end with $a$-edges. We shall denote the *trimmed top* and bottom sides of $T$ by $t\text{top}(T)$ and $t\text{bot}(T)$. By definition, for arbitrary $\theta$-band $T$, $t\text{top}(T)$ is obtained by such a trimming only if $T$ starts and/or ends with a $(\theta, q)$-cell; otherwise $t\text{top}(T) = \text{top}(T)$. The definition of $t\text{bot}(T)$ is similar.

By Lemma 5.3, any trapezium $\Delta$ of height $h \geq 1$ can be decomposed into $\theta$-bands $T_1, \ldots, T_h$ connecting the left and the right sides of the trapezium. The word written on the trimmed top side of one of the bands $T_i$ is the same as the word written on the trimmed bottom side of $T_{i+1}$, $i = 1, \ldots, h$. Moreover, the following lemma claims that every trapezium simulates the work of $M$. It summarizes the assertions of Lemmas 6.1, 6.3, 6.9, and 6.16 from [15]. For the formulation (1) below, it is important that $M$ is an $S$-machine. The analog of this statement is false for Turing machines. (See [14] for a discussion.)

**Lemma 5.7.** (1) Let $\Delta$ be a trapezium with history $\theta_1 \ldots \theta_d$ ($d \geq 1$). Assume that $\Delta$ has consecutive maximal $\theta$-bands $\mathcal{T}_1, \ldots, \mathcal{T}_d$, and the words $U_j$ and $V_j$ are the trimmed bottom and the trimmed top labels of $\mathcal{T}_j$ ($j = 1, \ldots, d$). Then the history of $\Delta$ is a reduced word, $U_j$, $V_j$ are admissible words for $M$, and

$$V_1 \equiv U_1 \cdot \theta_1, U_2 \equiv V_1, \ldots, U_d \equiv V_{d-1}, V_d \equiv U_d \cdot \theta_d$$

(2) For every reduced computation $U \rightarrow \cdots \rightarrow U \cdot H \equiv V$ of $M$ with $||H|| \geq 1$ there exists a trapezium $\Delta$ with bottom label $U$, top label $V$, and with history $H$.

Using Lemma 5.7, one can immediately derive properties of trapezia from the properties of computations obtained earlier.

If $H' \equiv \theta_1 \ldots \theta_d$ is a subword of the history $\theta_1 \ldots \theta_d$ from Lemma 5.7 (1), then the bands $\mathcal{T}_1, \ldots, \mathcal{T}_j$ form a subtrapezium $\Delta'$ of the trapezium $\Delta$. This subtrapezium is uniquely defined by the subword $H'$ (more precisely, by the occurrence of $H'$ in the word $\theta_1 \ldots \theta_d$), and $\Delta'$ is called the $H'$-*part* of $\Delta$.

We say that a trapezium $\Delta$ is *standard* if the base of $\Delta$ is the standard base $B$ of $M$ or $B^{-1}$, and the step history of $\Delta$ (or the inverse one) contains one of the words $(21^-)(1^-)(1^-1), (12^-)(2^-)(2^-2), (23^-)(3^-)(3^-3), (34^-)(4^-)(4^-4), (45^-)(5^-)(5^-5)$.

**Remark 5.8.** By Lemmas 5.7 and 4.10 (2), given the history $H$, one can reconstruct the entire standard trapezium $\Delta$.

**Definition 5.9.** We say that a trapezium $\Gamma$ is *big* if

1. the base of $\Delta$ or the inverse word has the form $xvx$, where $xv$ a cyclic shift of the $L$-s power of the standard base;
2. the diagram $\Gamma$ contains a standard trapezium $\Delta$.

**Lemma 5.10.** Let $\Delta$ be a trapezium whose base is $xvx$, where $x$ occurs in $v$ exactly $L - 1$ times and other letters occur $< L$ times each. Then either $\Delta$ is big or the length of a side of every $\theta$-band of $\Delta$ does not exceed $c_5(||W|| + ||W'||)$, where $W, W'$ are the labels of its top and bottom, respectively.
Proof. The diagram $\Delta$ is covered by $L$ subtrapezia $\Gamma_i$ ($i = 1, \ldots, L$) with bases $xv_ix$.

Assume that the step history of $\Delta$ (or inverse step history) contains one of the subwords $(21^-)(1^-)(1^-)$, $(12^-)(2^-)(2^-)$, $(23^-)(3^-)(3^-)$, $(34^-)(4^-)(4^-)$. Then by Lemma 4.5 (2) (and 5.7), the base of $\Delta$ has the form $(xu)^Lx$, where $xu$ is a cyclic shift of the standard base (or the inverse one) and the diagrams $\Gamma_i$-s ($i = 1, \ldots, L$) are just copies of each other. Each of them contains a standard subtrapezia, and so $\Delta$ is big.

Now, under the assumption that the step history has no subwords mentioned in the previous paragraph, it suffices to bound the length of a side of every $\theta$-band of arbitrary $\Gamma_i$ by $\leq c_4(||V|| + ||V'||)$, where $V$ and $V'$ are the labels of the top and the bottom of $\Gamma_i$.

Assume that the word $xv_ix$ has a proper subword $yuy$, where $u$ has no letters $y$, and any other letter occurs in $u$ at most once. Then the word $yuy$ is faulty since $v_i$ has no letters $x$. By Lemma 4.13, we have $|U_j| \leq C \max(|U_0|, |U_L|)$ for every configuration $U_j$ of the computation given by Lemma 5.7 restricted to the base $yuy$. Since $c_4 > C$, it suffices to obtain the desired estimate for the computation whose base is obtained by deleting the subword $yu$ from $xv_ix$. Hence inducting on the length of the base of $\Gamma_i$, one may assume that it has no proper subwords $yuy$, and so the base of $\Gamma_i$ is revolving. Now the required upper estimate for $\Gamma_i$ follows from Lemma 4.25 (see (1) and (2c) there).

5.3 Parameters

The following constants will be used for the proofs in this paper.

$$\lambda^{-1}, N \ll c_0 \ll \cdots \ll c_5 \ll L_0 \ll L \ll K \ll J \ll \delta^{-1} \ll c_6 \ll c_7 \ll$$

$$N_1 \ll N_2 \ll N_3 \ll N_4 \quad (5.10)$$

For each of the inequalities of this paper, one can find the highest constant (with respect to the order $\ll$) involved in the inequality and see that for fixed lower constants, the inequality is correct as soon as the value of the highest one is sufficiently large. This principle makes the system of all inequalities used in this paper consistent.

6 Diagrams without hubs

6.1 A modified length function

Let us modify the length function on the words and paths. The standard length of a word (a path) will be called its combinatorial length. From now on we use the word 'length' for the modified length. We set the length of every $q$-letter equal 1, and the length of every $a$-letter equal a small enough number $\delta$ so that

$$J\delta < 1 \quad (6.11)$$

We also set to 1 the length of every word of length $\leq 2$ which contains exactly one $\theta$-letter and no $q$-letters (such words are called $(\theta,a)$-syllables). The length of a decomposition of an arbitrary word in a product of letters and $(\theta,a)$-syllables is the sum of the lengths of the factors. The length $|w|$ of a word $w$ is the smallest length of such decompositions. The length $|p|$ of a path in a diagram is the length of its label. The
perimeter $|\partial \Delta|$ of a van Kampen diagram is similarly defined by cyclic decompositions of the boundary $\partial \Delta$.

The next statement follows from the above definitions and from the property of $(\theta, q)$-relations and their cyclic shifts: The subword between two $q$-letters (between $\theta$-letters) in arbitrary $(\theta, q)$-relation is a syllable (has at most one $q$-letter and at most two $a$-letters).

**Lemma 6.1.** Let $s$ be a path in a diagram $\Delta$ having $c$ $\theta$-edges and $d$ $a$-edges. Then

(a) $|s| \geq \max(c, c + (d - c) \delta)$;

(b) $|s| = c$ if $s$ is a top or a bottom of a $q$-band.

(c) For any product $s = s_1 s_2$ of two paths in a diagram, we have

\[ |s_1| + |s_2| \geq |s| \geq |s_1| + |s_2| - \delta \]  \hspace{1cm} (6.12)

(d) Let $T$ be a $\theta$-band with base of length $l_b$. Let $l_a$ be the number of $a$-edges in the top path $\text{toplevel}(T)$. Then the length of $T$ (i.e., the number of cells in $T$) is between $l_a - l_b$ and $l_a + 3l_b$.

\[ \square \]

**Lemma 6.2.** Let $\Delta$ be a van Kampen diagram whose rim band $\mathcal{T}$ has base with at most $K$ letters. Denote by $\Delta'$ the subdiagram $\Delta \setminus \mathcal{T}$. Then $|\partial \Delta| - |\partial \Delta'| > 1$.

**Proof.** Let $s$ be the top side of $\mathcal{T}$ and $s \subset \partial \Delta$. Note that the difference between the number of $a$-edges in the bottom $s'$ of $\mathcal{T}$ and the number of $a$-edges in $s$ cannot be greater than $2K$, because every $(\theta, q)$-relator has at most two $a$-letters. Hence $|s'| - |s| \leq 4LN\delta$. However, $\Delta'$ is obtained by cutting off $\mathcal{T}$ along $s'$, and its boundary contains two $\theta$-edges fewer than $\Delta$. Hence we have $|s_0| - |s_0'| \geq 2 - 2\delta$ for the complements $s_0$ and $s_0'$ of $s$ and $s'$, respectively, in the boundaries $\partial \Delta$ and $\partial \Delta'$. Finally,

\[ |\partial \Delta| - |\partial \Delta'| \geq 2 - 2\delta - 2K\delta - 4\delta > 1 \]

by (6.11) and (6.12). \[ \square \]

We call a base word $w$ tight if

(1) for some letter $x$ the word $w$ has the form $uxvx$, where the letter $x$ does not occur in $u$ and $x$ occurs in $v$ exactly $L - 1$ times,

(2) every proper prefix $w'$ of $w$ does not satisfy property (1).

**Lemma 6.3.** If a base $w$ of a $\theta$-band has no tight prefixes, then $||w|| \leq K_0$, where $K_0 = 2LN$.

**Proof.** The hub base includes every base letter $L$ times. Hence every word in this group alphabet of length $\geq K_0 + 1$ includes one of the letters $L + 1$ times. \[ \square \]

From now on we shall fix a constant $K$

\[ K > 2K_0 = 4LN \]  \hspace{1cm} (6.13)

**Definition 6.4.** We say that a reduced diagram $\Gamma$ is a comb if it has a maximal $q$-band $Q$ (the handle of the comb), such that

(C1) $\text{bott}(Q)$ is a part of $\partial \Gamma$, and every maximal $\theta$-band of $\Gamma$ ends at a cell in $Q$. 47
If in addition the following properties hold:

(C$_2$) one of the maximal $\theta$-bands $\mathcal{T}$ in $\Gamma$ has a tight base (if one reads the base towards the handle) and

(C$_3$) other maximal $\theta$-bands in $\Gamma$ have tight bases or bases without tight prefixes

then the comb is called **tight**.

The number of cells in the handle $Q$ is the **height** of the comb, and the maximal length of the bases of the $\theta$-bands of a comb is called the **basic width** of the comb.

Notice that every trapezium is a comb.

**Lemma 6.5.** ([16], Lemma 4.10) Let $l$ and $b$ be the length and the basic width of a comb $\Gamma$ and let $T_1, \ldots, T_l$ be consecutive $\theta$-bands of $\Gamma$ (as in Figure 3). We can assume that $\text{bot}(T_i)$ and $\text{top}(T_i)$ are contained in $\partial \Gamma$. Denote by $\alpha = |\partial \Gamma|$ the number of a-edges in the boundary of $\Gamma$, and by $\alpha_1$ the number of a-edges on $\text{bot}(T_1)$. Then $\alpha + 2lb \geq 2\alpha_1$, and the area of $\Gamma$ does not exceed $c_0bl^2 + 2\alpha l$ for some constant $c_0 = c_0(M)$.

We say that a subdiagram $\Gamma$ of a diagram $\Delta$ is a **subcomb** of $\Delta$ if $\Gamma$ is a comb, the handle of $\Gamma$ divides $\Delta$ in two parts, and $\Gamma$ is one of these parts.

**Lemma 6.6.** Let $\Delta$ be a reduced diagram over $G$ with non-zero area, where every rim band has base of length at least $K$. Assume that

1. $\Delta$ is a diagram over the group $M$ or
2. $\Delta$ has a subcomb of basic width at least $K_0$.

Then there exists a maximal $q$-band $Q$ dividing $\Delta$ in two parts, where one of the parts is a tight subcomb with handle $Q$.

**Proof.** Let $T_0$ be a rim band of $\Delta$ (fig.1). Its base $w$ is of length at least $K$, and therefore $w$ has disjoint prefix and suffix of lengths $K_0$ since $K > 2K_0$ by (6.13). The prefix of this base word must have its own tight subprefix $w_1$, by Lemma 6.3 and the definition of tight words. A $q$-edge of $T_0$ corresponding to the last $q$-letter of $w_1$ is the start edge of a maximal $q$-band $Q'$ which bounds a subdiagram $\Gamma'$ containing a band $\mathcal{T}$ (a subband of $T_0$) satisfying property (C$_2$). It is useful to note that a minimal suffix $w_2$ of $w$, such
that $w_2^{-1}$ is tight, allows us to construct another band $Q''$ and a subdiagram $\Gamma''$ which satisfies $(C_2)$ and has no cells in common with $\Gamma'$.

Thus, there are $Q$ and $\Gamma$ satisfying $(C_2)$. Let us choose such a pair with minimal $\text{Area}(\Gamma)$. Assume that there is a $\theta$-band in $\Gamma$ which does not cross $Q$. Then there must exist a rim band $T_1$ which does not cross $Q$ in $\Gamma$. Hence one can apply the construction from the previous paragraph to $T_1$ and construct two bands $Q_1$ and $Q_2$ and two disjoint subdiagrams $\Gamma_1$ and $\Gamma_2$ satisfying the requirement $(C_2)$ for $\Gamma$. Since $\Gamma_1$ and $\Gamma_2$ are disjoint, one of them, say $\Gamma_1$, is inside $\Gamma$. But the area of $\Gamma_1$ is smaller than the area of $\Gamma$, and we come to a contradiction. Hence $\Gamma$ is a comb and condition $(C_1)$ is satisfied.

Assume that the base of a maximal $\theta$-band $T$ of $\Gamma$ has a tight proper prefix (we may assume that $T$ terminates on $Q$), and again one obtain a $q$-band $Q'$ in $\Gamma$, which provides us with a smaller subdiagram $\Gamma'$ of $\Delta$, satisfying $(C_2)$, a contradiction. Hence $\Gamma$ satisfies property $(C_3)$ as well.

(2) The proof is shorter since a comb is given in the very beginning.

6.2 Mixture on the boundary of a diagram

We will need a parameter of diagrams introduced in [13]. I was called mixture.

Let $O$ be a circle with two-colored finite set of points (or vertices) on it, more precisely, let any vertex of this finite set be either black or white. We call $O$ a necklace with black and white beads on it. We want to introduce the mixture of this finite set of beads.

Assume that there are $n$ white beads and $n'$ black ones on $O$. We define sets $P_j$ of ordered pairs of distinct white beads as follows. A pair $(o_1, o_2)$ ($o_1 \neq o_2$) belongs to the set $P_j$ if the simple arc of $O$ drawn from $o_1$ to $o_2$ in clockwise direction has at least $j$ black beads. We denote by $\mu_J(O)$ the sum $\sum_{j=1}^{J} \text{card}P_j$ (the $J$-mixture on $O$). Below similar sets for another necklace $O'$ are denoted by $P'_j$. In this subsection, $J \geq 1$, but later on it will be a fixed large enough number $J$ from the list (5.10).

Lemma 6.7. ([13], Lemma 6.1) (a) $\mu_J(O) \leq J(n^2 - n)$.

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(b) Suppose a necklace $O'$ is obtained from $O$ after removal of a white bead $v$. Then $\text{card} \mathbf{P}'_j - n < \text{card} \mathbf{P}_j \leq \text{card} \mathbf{P}'_j$ for every $j$, and $\mu_\mathbf{J}(O) - Jn < \mu_\mathbf{J}(O') \leq \mu_\mathbf{J}(O)$.

(c) Suppose a necklace $O'$ is obtained from $O$ after removal of a black bead $v$. Then $\text{card} \mathbf{P}'_j \leq \text{card} \mathbf{P}_j$ for every $j$, and $\mu_\mathbf{J}(O') \leq \mu_\mathbf{J}(O)$.

(d) Assume that there are three black beads $v_1, v_2, v_3$ of a necklace $O$, such that the clockwise arc $v_1 - v_3$ contains $v_2$ and has at most $J$ black beads (excluding $v_1$ and $v_3$), and the arcs $v_1 - v_2$ and $v_2 - v_3$ have $m_1$ and $m_2$ white beads, respectively. If $O'$ is obtained from $O$ by removal of $v_2$, then $\mu_\mathbf{J}(O') \leq \mu_\mathbf{J}(O) - m_1m_2$.

\[\square\]

For any diagram $\Delta$ over $G$, we introduce the following invariant $\mu(\Delta) = \mu_\mathbf{J}(\partial \Delta)$ depending on the boundary only. To define it, we consider the boundary $\partial \Delta$, as a necklace, i.e., we consider a circle $O$ with $||\partial \Delta||$ edges labeled as the boundary path of $\Delta$. By definition, the white beads are the mid-points of the $\theta$-edges of $O$ and black beads are the mid-points of the $q$-edges $O$. Then, by definition, the mixture on $\partial \Delta$ is $\mu(\Delta) = \mu_\mathbf{J}(O)$.

### 6.3 Quadratic upper bound for quasi-areas of the diagrams over $M$.

The Dehn function of the group $M$ is greater than the required function. For example, it is cubic if $f(n) = \text{const}$. However we are going to find the Dehn function of $G$, and first we want to bound the areas of the words vanishing in $M$ with respect to the presentation of $G$. For this goal we artificially introduce the concept of $G$-area. The $G$-area of a big trapezium can be much less than the real area of it in $M$. This concept will be justified at the end of this paper, where some big trapezia are replaced by diagrams with hubs, but having lesser areas.

**Definition 6.8.** The $G$-area $\text{Area}_G(\Gamma)$ of a big trapezium $\Gamma$ is, by definition, the minimum of the half of its area (i.e., the number of cells) and the product

$$ c_5h(||\text{top}(\Gamma)|| + ||\text{bot}(\Gamma)||), $$

where $h$ is the height of $\Gamma$ and $c_5$ is the constant from (5.10).

To define the $G$-area of a diagram $\Delta$ over $G$, we consider a family $\mathbf{S}$ of big subtrapezia (i.e. subdiagrams, which are trapezia) and single cells of $\Delta$ such that every cell of $\Delta$ belongs to a member $\Sigma$ of this family, and if a cell $\Pi$ belongs to different $\Sigma_1$ and $\Sigma_2$ from $\Sigma$, then both $\Sigma_1$ and $\Sigma_2$ are big subtrapezia of $\Delta$ with bases $xv_1x$, $xv_2x$, and $\Pi$ is a $(\theta, x)$-cell. (In the later case, the intersection $\Sigma_1 \cap \Sigma_2$ must be an $x$-band.) There is such a family 'covering' $\Delta$, e.g. just the family of all cells of $\Delta$.

The $G$-area of $\mathbf{S}$ is the sum of $G$-areas of all big trapezia from $\mathbf{S}$ plus the number of single cells from $\mathbf{S}$ (i.e. the $G$-area of a cell $\Pi$ is $\text{area}(\Pi) = 1$). Finally, the $G$-area $\text{Area}_G(\Delta)$ is the minimum of the $G$-areas of all 'covering' $\mathbf{S}$ as above.

It follows from the definition that $\text{Area}_G(\Delta) \leq \text{Area}(\Delta)$ since the $G$-area of a big trapezium does not exceed a half of its area.

**Lemma 6.9.** Let $\Delta$ be a reduced diagram, and every cell $\pi$ of $\Delta$ belongs in one of subdiagrams $\Delta_1, \ldots, \Delta_m$, where any intersection $\Delta_i \cap \Delta_j$ either has no cells or it is a $q$-band, Then $\text{Area}_G(\Delta) \leq \sum_{i=1}^{m} \text{Area}_G(\Delta_i)$.
Proof. Consider the families $S_1, \ldots, S_m$ given by the definition of $G$-areas for the diagrams $\Delta_1, \ldots, \Delta_m$. Then the family $S = S_1 \cup \cdots \cup S_m$ 'covers' the entire $\Delta$ according to the above definition. This implies the required inequality for $G$-areas.

We will show that for some constant $N_2$ and $N_1$ the $G$-area of any reduced diagram $\Delta$ over $M$ with perimeter $n$ does not exceed $N_2n^2 + N_1\mu(\Delta)$. (Using the quadratic upper bound for $\mu(\Delta)$ from Lemma 6.7 (a), one deduces that the $G$-area is bounded by $M'n^2$ for some constant $M'$.) Roughly speaking, we are doing the following. We use induction on the perimeter of the diagram. First we remove rim $\theta$-bands (those with one side and both ends on the boundary of the diagram) with short bases. This operation decreases the perimeter and preserves the sign of $N_2n^2 + N_1\mu(\Delta) - Area_G(\Delta)$, so we can assume that the diagram does not have such bands. Then we use Lemma 6.6 and find a tight comb inside the diagram with a handle $C$. We also find a long enough $q$-band $C'$ that is close to $C$. We use a surgery which amounts to removing a part of the diagram between $C'$ and $C$ and then gluing the two remaining parts of $\Delta$ together. The main difficulty is to show that, as a result of this surgery, the perimeter decreases and the measure and the mixture change in such a way that the expression $N_2n^2 + N_1\mu(\Delta) - Area_G(\Delta)$ does not change its sign. In the proof, we need to consider several cases depending on the shape of the subdiagram between $C'$ and $C$. Note that neither $N_2n^2$ nor $N_1\mu(\Delta)$ nor $Area_G(\Delta)$ alone behave in the appropriate way as a result of the surgery, but the expression $N_2n^2 + N_1\mu(\Delta) - Area_G(\Delta)$ behaves as needed.

Remark 6.10. We introduced the surgery and used induction mentioned above in [16] (Lemma 6.2) to obtain a worse upper bound $n^2 \log n$ for the area. But there were neither mixture, nor $G$ (just $M$), nor $G$-area in [16], and a different definition for length $|*|$ was used there. Besides, we will use an auxiliary function $\Phi(x)$ in the proof to be able to repeat in part our argument later, for diagrams over $G$. So we shall prove Lemma 6.16 anew to obtain the better estimate.

So, $N_1$ and $N_2$ are big enough constants from the list (5.10). Here “big enough” means that they satisfy the inequalities used in the proof of Lemma 6.16 (such that (6.23), (6.24), (6.25), (6.36), (6.40), (6.42), (6.44), (6.49)). Each of them has the form $N_i > * (i = 1, 2)$, where the right-hand side $*$ does not depend on $N_i$ (but depends on the constants introduced earlier). Since the number of inequalities is finite, the right choice of $N_1, N_2$ is possible.

Let $\Phi(x)$ be arbitrary function defined for real $x \geq 0$ such that

$$\Phi(x) = x^2 \phi(x)$$

for a non-decreasing function $\phi(x) > 0$ with $\phi(1) \geq 1$ and

$$\Phi(x) - \Phi(x - y) \geq xy \phi(x) \quad \text{for} \quad 0 \leq y \leq x. \quad (6.14)$$

Remark 6.11. For this section, it suffices to take quadratic $\Phi(x)$ and $\phi(x) = const$, but to estimate the $G$-area of diagrams with hubs, we will take the functions $\Phi(x) = F(x)$ and $\phi(x) = g(x)$, satisfying inequality (6.14) by Lemma 4.18.

We are going to prove that the $G$-area of a reduced diagram $\Delta$ over $M$ does not exceed $N_2\Phi(n) + N_1\phi(n)\mu(\Delta)$, where $n = |\partial\Delta|$. Arguing by contradiction in the remaining part of this section, we consider a counter-example $\Delta$ with minimal perimeter $n$. Of course, its $G$-area is positive, and, by Lemma (NoAnnul), we have at least 2 $\theta$-edges on the boundary $\partial\Delta$, and so $n \geq 2$. 

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If \( \Gamma \) is a comb with handle \( C \) and \( \mathcal{B} \) is another maximal \( q \)-band in \( \Gamma \), then \( \mathcal{B} \) cut up \( \Gamma \) in two parts, where the part that does not contain \( C \) is a comb \( \Gamma' \) with handle \( \mathcal{B} \). It follows from the definition of comb, that every maximal \( \theta \)-band of \( \Gamma \) crossing \( \mathcal{B} \) connects \( \mathcal{B} \) with \( C \). If \( \mathcal{B} \) and \( C \) can be connected by a \( \theta \)-band containing no \((\theta,q)\)-cells, then \( \Gamma' \) is called the derivative subcomb of \( \Gamma \). Note that no maximal \( \theta \)-band of \( \Gamma \) can cross the handles of two derivative subcombs.

**Lemma 6.12.** (1) The diagram \( \Delta \) has no two disjoint subcombs \( \Gamma_1 \) and \( \Gamma_2 \) of basic widths at most \( K \) with handles \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \) such that some ends of these handles are connected by a subpath \( x \) of the boundary path of \( \Delta \), where \( x \) has no \( q \)-edges.

(2) The boundary of every subcomb \( \Gamma \) with basic width \( s \leq K \) has \( 2s q \)-edges.

**Proof.** We will prove the statements (1) and (2) using simultaneous induction on \( A = \text{Area}(\Gamma_1) + \text{Area}(\Gamma_2) \) (resp., on \( A = \text{Area}(\Gamma) \)). Arguing by contradiction, we consider a counter-example with minimal \( A \).

(1) Since the area of \( \Gamma_i \) (\( i = 1, 2 \)) is less than \( A \), we may use Statement (2), and so we have at most \( 2K q \) edges in \( \partial \Gamma_i \).

Let \( h_1 \) and \( h_2 \) be the lengths of the handles \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \) of \( \Gamma_1 \) and \( \Gamma_2 \), resp. Without loss of generality, we assume that \( h_1 \leq h_2 \). Denote by \( y_i z_i \) the boundaries of \( \Gamma_i \) (\( i = 1, 2 \)), where \( z_i \) is the part of the boundary path of \( \Delta \), see fig. 5 (1)). Then each of the \( \theta \)-edges \( e \) of \( y_1 \) is separated in \( \partial \Delta \) from every \( \theta \)-edge \( f \) of \( y_2 \) by less than \( 4K < J q \)-edges. Hence every such pair \((e,f)\) (or the pair of white beads on these edges) makes a contribution to \( \mu(\Delta) \).

Let \( \Delta' \) be the diagram obtained by deleting the subdiagram \( \Gamma_1 \) from \( \Delta \). When passing from \( \partial \Delta \) to \( \partial \Delta' \), one replaces the \( \theta \)-edges from \( z_1 \) by the \( \theta \)-edge of \( y_1 \) belonging to the same maximal \( \theta \)-band. The same is true for white beads.

But each of the \( h_1 h_2 \) pairs in the corresponding set \( P' \) of white beads is separated in \( \partial \Delta' \) by less number of black beads than the pair defined by \( \Delta \). Indeed, since the handle of \( \Gamma_1 \) is removed when one replaces \( \partial \Delta \) by \( \partial \Delta' \), two black bead at the ends of this handle are removed, and therefore

\[
\mu(\Delta) - \mu(\Delta') \geq h_1 h_2
\]  

by Lemma 6.7 (d).

Let \( \alpha \) be the number of \( a \)-edges in \( \partial \Gamma_1 \). It follows from Lemma 6.5 that the area, and so the \( G \)-area of \( \Gamma_1 \), does not exceed \( C_1(h_1)^2 + 2\alpha h_1 \), where \( C_1 = c_0 K \).
Remark 6.13. The constants $C_1, C_2, C_{12}, C_3$ are not included in the list (5.10) since their values chosen here make sense only in the present subsection.

Since the boundary of $\Delta'$ has at least two $q$-edges fewer than $\Delta$ and $|z_1| = h_1 \leq |y_1|$, we have $|\partial \Delta'| \leq |\partial \Delta| - 2$. Moreover, we have from Lemma 6.1 (a) and Lemma 5.3 that

$$|\partial \Delta| - |\partial \Delta'| \geq \gamma = \max(2, \delta(\alpha - 2h_1)) \quad (6.16)$$

because the top/the bottom of $B_1$ has at most $h_1$ $a$-edges.

This inequality, inequality (6.15), and the inductive assumption related to $\Delta'$, imply that the $G$-area of $\Delta'$ is not greater than

$$N_1 \phi(n) - N_2 \phi(n) \mu(\Delta) - N_1 \phi(n) h_1 h_2$$

Adding the $G$-area of $\Gamma_1$ and using inequality 6.14, we see that the $G$-area of $\Delta$ does not exceed

$$N_2 \phi(n) - N_2 \gamma n + N_1 \phi(n) \mu(\Delta) - N_1 \phi(n) h_1 h_2 + C_1 h_1^2 + 2\alpha h_1.$$

Since $h_1 \leq h_2$ and $\phi(n) \geq 1$, this will contradict the choice of the counter-example $\Delta$ when we prove that

$$- N_2 \gamma n - N_1 h_1^2 + C_1 h_1^2 + 2\alpha h_1 < 0 \quad (6.17)$$

If $\alpha \leq 4h_1$, then inequality (6.17) follows from the inequalities $\gamma \geq 2$ and

$$N_1 \geq C_1 + 8 \quad (6.18)$$

Assume that $\alpha > 4h_1$. Then by (6.16), we have $\gamma \geq \frac{1}{2} \delta \alpha$ and $N_2 \gamma n > 2\alpha h_1$ since $n \geq 2h_1$ by Lemma 5.3, and

$$N_2 > 2\delta^{-1}. \quad (6.19)$$

Since $N_1 h_1^2 > C_1 h_1^2$ by (6.18), the inequality (6.17) follows.

(2) If there are at least two derivative subcombs of $\Gamma$, then one can find two of them satisfying the assumptions of Statement (1), and $\text{Area}(\Gamma_1) + \text{Area}(\Gamma_2) < \text{Area}(\Gamma) = A$, a contradiction. Therefore there is a most one derivative subcomb $\Gamma''$ in $\Gamma$ (fig 5 (2)). In turn, $\Gamma'$ has at most one derivative subcomb $\Gamma'''$, and so one. It follows that there are no maximal $q$-bands in $\Gamma$ except for the handles of $\Gamma', \Gamma'', \ldots$. Since the basic width of $\Gamma$ is $s$, we have $s$ maximal $q$-bands in $\Gamma$, and the lemma is proved.

**Lemma 6.14.** There are no pair of subcombs $\Gamma$ and $\Gamma'$ in $\Delta$ with handles $\mathcal{X}$ and $\mathcal{X}'$ of length $\ell$ and $\ell'$ such that $\Gamma'$ is a subcomb of $\Gamma$, the basic width of $\Gamma$ does not exceed $K_0$ and $\ell' \leq \ell/2$.

**Proof.** Proving by contradiction, one can choose $\Gamma'$ so that $\ell'$ is minimal for all subcombs in $\Gamma$ and $\Gamma'$ has no proper subcombs, i.e. its basic width is 1 (fig. 6). It follows from Lemma 6.5 that for $\alpha = |\Gamma'|_a$, we have

$$\text{Area}_{G} (\Gamma') \leq \text{Area}(\Gamma') \leq c_0 (l')^2 + 2\alpha l' \quad (6.20)$$

Let $\Delta'$ be the diagram obtained after removing the subdiagram $\Gamma'$ from $\Delta$. The following inequality is the analog of (6.16) (where $h_1$ is replaced by $\ell'$)

$$|\partial \Delta| - |\partial \Delta'| \geq \gamma = \max(2, \delta(\alpha - 2\ell')) \quad (6.21)$$
The q-band $X$ contains a subband $C$ of length $l'$. Moreover one can choose $C$ so that all maximal $\theta$-bands of $\Gamma$ crossing the handle $X'$ of $\Gamma'$, start from $C$. These $\theta$-bands form a comb $\Gamma''$ contained in $\Gamma$, and in turn, $\Gamma''$ contains $\Gamma'$. The two parts of the compliment $X \setminus C$ are the handles of two subcombs $E_1$ and $E_2$ formed by maximal $\theta$-bands of $\Gamma$, which do not cross $X'$. Let the length of these two handles be $\ell_1$ and $\ell_2$, respectively, and so we have $\ell_1 + \ell_2 = l - l' > l'$. ($E_1$ or $E_2$ can be empty; then $\ell_1$ or $\ell_2$ equals 0.)

It will be convenient to assume that $\Gamma$ is drawn from the left of the vertical handle $X$. Denote by $yz$ the boundary path of of $\Gamma$, where $y$ is the right side of the band $X$.

Thus, there are $\ell_1$ (resp., $\ell_2$) $\theta$-edges on the common subpath $x_1$ (subpath $x_2$) of $z$ and $\partial E_1$ (and $\partial E_2$).

By Lemma 6.12 (2), the path $z$ contains at most $2K_0 q$-edges, because the basic width of $\Gamma$ is at most $K_0$.

Consider the factorization $z = x_2x x_1$, where $x$ is a subpath of $\partial \Gamma'$. It follows that between every white bead on $x_1$ (i.e. the middle point of the $\theta$-edges on $x_1$) and a white bead on $x$ we have at most $2K_0$ black beads (i.e. the middle points of the $q$-edges of the path $x$). Since $J$ is greater than $2K_0$, every pair of white beads, where one bead belongs in $x_1$ and another one belongs in $x_2$ (or, similarly, in $x_2$) contributes 1 to $\mu(\Delta)$. Let $P$ denote the set of such pairs. By the definition of $E_1$ and $E_2$, we have $\text{card} P = l'(l_1 + l_2) = l'(l - l') > (l')^2$.

When passing from $\partial \Delta$ to $\partial \Delta'$, one replaces the left-most $\theta$-edges of every maximal $\theta$-band from $\Gamma'$ with the right-most $\theta$-edges lying on the right side of $X'$. The same is true for white beads. But each of the $l'(l - l')$ pairs in the corresponding set $P'$ of white beads is separated in $\partial \Delta'$ by less number of black beads since the q-band $X'$ is removed. Therefore every pair from $P'$ gives less by 1 contribution to the mixture, as it follows from the definition of mixture. Hence $\mu(\Delta) - \mu(\Delta') \geq l'(l - l') \geq (l')^2$. This inequality, inequality (6.21), and the inductive assumption related to $\Delta'$, imply that the $G$-area of $\Delta'$ is not greater than

$$N_2 \Phi(n - \gamma) + N_1 \phi(n) \mu(\Delta) - N_1 \phi(n)(l')^2$$

Adding the $G$-area of $\Gamma'$ (6.20) and applying inequality (6.14), we see that the $G$-area of

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Figure 6: Lemma 6.14

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Figure 7: Rim \( \theta \)-band

\( \Delta \) does not exceed

\[
N_2 \Phi(n) + N_1 \phi(n) \mu(\Delta) - N_2 \gamma n - N_1 \phi(n)(l')^2 + c_0(l')^2 + 2\alpha l'.
\]

This will contradict the choice of the counter-example \( \Delta \) when we prove that

\[
-N_2 \gamma n - N_1 (l')^2 + c_0(l')^2 + 2\alpha l' < 0,
\]

because \( \phi(n) \geq 1 \). Consider two cases.

(a) Let \( \alpha \leq 4l' \). Then inequality (6.22) follows from the inequalities \( \gamma \geq 2 \) and

\[
N_1 \geq c_0 + 8.
\]

(b) Assume that \( \alpha > 4l' \). Then by (6.21) we have \( \gamma \geq \frac{1}{2} \delta \alpha \) and \( N_2 \gamma n > 2\alpha l' \) since \( n \geq 2l \geq 4l' \) by Lemma 5.3, and

\[
N_2 > \delta^{-1}.
\]

Since \( N_1 (l')^2 > c_0(l')^2 \) by (6.23), the inequality (6.22) follows.

Thus, the lemma is proved by contradiction.

Lemma 6.15. \( \Delta \) has no rim \( \theta \)-band whose base has \( s \leq K \) letters.

Proof. Assume by contradiction that such a rim \( \theta \)-band \( \mathcal{T} \) exists, and \( \text{top}(\mathcal{T}) \) belongs in \( \partial(\Delta) \) (fig.7). When deleting \( \mathcal{T} \), we obtain, by Lemma 6.2, a diagram \( \Delta' \) with \( |\partial\Delta'| \leq n-1 \). Since \( \text{top}(\mathcal{T}) \) lies on \( \partial \Delta \), we have from the definition of the length , that the number of \( a \)-edges in \( \text{top}(\mathcal{T}) \) is less than \( \delta^{-1}(n - s) \). By Lemma 5.2, the length of \( \mathcal{T} \) is at most \( 3s + \delta^{-1}(n - s) < \delta^{-1}n \). Thus, by applying the inductive hypothesis to \( \Delta' \), we have that \( G \)-area of \( \Delta \) is not greater than \( N_2 \Phi(n-1) + N_1 \phi(n) \mu(\Delta) + \delta^{-1}n \) because \( \mu(\Delta') \leq \mu(\Delta) \) by Lemma 6.7 (b). But the first term of this sum does not exceed \( N_2 \Phi(n) - N_2 n \) by (6.14), and so the entire sum is bounded by \( N_2 \Phi(n) + N_1 \phi(n) \mu(\Delta) \) provided

\[
N_2 \geq \delta^{-1}.
\]

This contradicts to the choice of \( \Delta \), and the lemma is proved.

Lemma 6.16. The \( G \)-area of a reduced diagram \( \Delta \) over \( M \) does not exceed \( N_2 \Phi(n) + N_1 \phi(n) \mu(\Delta) \), where \( n = |\partial \Delta| \).
Proof. We continue studying the hypothetical minimal counter-example $\Delta$. By Lemma 6.15, now we can apply Lemma 6.6 (1). By that lemma, there exists a tight subcomb $\Gamma' \subset \Delta$. Let $T$ be a $\theta$-band of $\Gamma$ with a tight base.

The basic width of $\Gamma$ is less than $K_0$ by Lemma 6.3. Since the base of $\Gamma$ is tight, it is equal to $uxvx$ for some $x$, where the last occurrence of $x$ corresponds to the handle $Q$ of $\Gamma$, the word $u$ does not contain $x$, and $v$ has exactly $L - 1$ occurrences of $x$. Let $Q'$ be the maximal $x$-band of $\Gamma$ crossing $T$ at the cell corresponding to the first occurrence of $x$ in $uxvx$ (fig. 8 (a)).

We consider the smallest subdiagram $\Gamma'$ of $\Delta$ containing all the $\theta$-bands of $\Gamma$ crossing the $x$-band $Q'$. It is a comb with handle $Q_2 \subset Q$. The comb $\Gamma'$ is covered by a trapezium $\Gamma_2$ placed between $Q'$ and $Q$, and a comb $\Gamma_1$ with handle $Q'$. The band $Q'$ belongs to both $\Gamma_1$ and $\Gamma_2$. The remaining part of $\Gamma$ is a disjoint union of two combs $\Gamma_3$ and $\Gamma_4$ whose handles $Q_3$ and $Q_4$ contain the cells of $Q$ that do not belong to the trapezium $\Gamma_2$. The handle of $\Gamma'$ is the composition of handles $Q_3, Q_2, Q_4$ of $\Gamma_3, \Gamma_2, \Gamma_4$ in that order.

Let the lengths of $Q_3$ and $Q_4$ be $l_3$ and $l_4$, respectively. Let $l'$ be the length of the handle of $\Gamma'$. Then by Lemma 6.14, we have

$$l' > 1/2 \quad \text{and} \quad l = l' + l_3 + l_4 \quad (6.26)$$

For $i \in \{3, 4\}$ and $\alpha_i = |\partial \Gamma_i|_a$, Lemma 6.5 gives inequalities

$$A_i \leq C_1 l_i^2 + 2\alpha_i l_i, \quad (6.27)$$

where $A_i$ is the $G$-area of $\Gamma_i$. (We take into account that $G$-area cannot exceed area.)

Let $p_3, p_4$ be the top and the bottom of the trapezium $\Gamma_2$. Here $p_3^{-1}$ (resp. $p_4^{-1}$) shares some initial edges with $\partial \Gamma_3$ (with $\partial \Gamma_4$), the rest of these paths belong to the boundary of $\Delta$. We denote by $d_3$ the number of $a$-edges of $p_3$ and by $d_4$ the number of the $a$ edges of $p_4$ which do not belong to $\Gamma_3$. Similarly, we introduce $d_3'$ and $d_4'$.

Let $A_2$ be the $G$-area of $\Gamma_2$. Then by Lemma 5.10 and the definition of the $G$-area for big trapezia, we have

$$A_2 \leq C_2 l'(d_3 + d_4 + 1) \quad (6.28)$$

for some constant $C_2 < 1$, because the basic width of $\Gamma_2$ is less than $K$. 

Figure 8: Lemma 6.16.
Now we observe that the handle $Q_2$ of $\Gamma'$ is a copy of $Q'$ because both maximal $q$-bands of the trapezium $\Gamma_2$ correspond to the same basic letter $x$.

This makes the following surgery possible. The diagram $\Delta$ is covered by two subdiagrams: $\Gamma$ and another subdiagram $\Delta_1$, having only the band in common. We construct a new auxiliary diagram by attaching of $\Gamma_1$ to $\Delta_1$ with identification of the of the band $Q'$ of $\Gamma_1$ and the band $Q_2$. We denote the constructed diagram by $\Delta_0$.

$\Delta_0$ is a reduced diagram because every pair of its cells having a common edge, has a copy either in $\Gamma_1$ or in $\Delta_1 \cup Q$. Now we need the auxiliary

**Lemma 6.17.** The $G$-area $A_0$ of $\Delta_0$ is at least the sum of the $G$-areas of $\Gamma_1$ and $\Delta_1$ minus $l'$.

**Proof.** Consider a minimal 'covering' $S$ of $\Delta_0$ from the definition of $G$-area, and assume that there is a big trapezium $E \in S$, such that neither $\Gamma_1$ nor $\Delta_1$ contains it. Then $E$ has a base $ywy$, where $(yw)^{\pm 1}$ is a cyclic permutation of the $L$-th power of the standard base, and the first $y$-band of $E$ is in $\Gamma_1$, but it is not a subband of $Q'$.

Since the history $H$ of the big trapezium $E$ is a subhistory of the history of $\Gamma_2$, and $H$ uniquely determines the base starting with given letter by Remark 5.8, we conclude that $\Gamma_2$ is a big trapezium itself, and therefore $(xv)^{\pm 1}$ is an $L$-th power of the standard base. Since the first $y$ occurs in $uxvx$ before the first $x$ it follows that we have the $(L+1) - th$ occurrence of $y$ before the last occurrence of $x$ in the word $uxvx$. But this contradicts the definition of tight comb $\Gamma$.

Hence every big trapezium from $S$ entirely belongs either in $\Gamma_1$ or in $\Delta_1$. Therefore one can obtain 'coverings' $S'$ and $S''$ of these two diagrams if (1) every $\Sigma$ from $S$ is assigned either to $S'$ or to $S''$ and then (2) one add at most $l'$ single cells since the common band $Q'$ in $\Delta_0$ should be covered twice in disjoint diagrams $\Gamma_1$ and $\Delta_1$. These construction complete the proof of the lemma.

By Lemma 6.9, the $G$-area of $\Delta$ does not exceed the sum of $G$-areas of the five subdiagrams $\Gamma_1$, $\Gamma_2$, $\Gamma_3$, $\Gamma_4$ and $\Delta_1$. But the direct estimate of each of these values is not efficient. Therefore we will use Lemma 6.31 to bound the $G$-area of the auxiliary diagram $\Delta_0$ built of two pieces $\Gamma_1$ and $\Delta_1$.

It follows from our constructions and lemmas 6.9, 6.31, that

$$\text{Area}_G(\Delta) \leq A_2 + A_3 + A_4 + A_0 + l'$$

(6.29)

Now we continue proving Lemma 6.16.

Let $p^3$ be the segment of the boundary $\partial \Gamma_3$ that joins $Q$ and $\Gamma_2$ along the boundary of $\Delta$ (fig. 8 (b)). It follows from the definition of $d_3$, $d'_3$, $l_3$ and $\alpha_3$, that the number of $a$-edges lying on $p^3$ is at least $\alpha_3 - (d_3 - d'_3) - l_3$.

Let $u_3$ be the part of $\partial \Delta$ that contains $p^3$ and connects $Q$ with $Q'$. It has $l_3$ $\theta$-edges. Hence we have, by Lemma 6.1, that

$$|u_3| \geq \max(l_3, l_3 + \delta(|p^3_a| - l_3)) \geq \max(l_3, l_3 + \delta(\alpha_3 - (d_3 - d'_3) - 2l_3)).$$

Since $u_3$ includes a subpath of length $d'_3$ having no $\theta$-edges, we also have by Lemma 6.1 (c) that $|u_3| \geq l_3 + \delta(d'_3 - 1)$.

One can similarly define $p^4$ and $u_4$ for $\Gamma_4$. When passing from $\partial \Delta$ to $\partial \Delta_0$ we replace the end edges of $Q'$, $u_3$ and $u_4$ by two subpaths at $\partial Q$ having lengths $l_3$ and $l_4$. Let $n_0 = |\partial \Delta_0|$. Then it follows from the previous paragraph that
\[ n - n_0 \geq 2 + \delta (\max(0, d_3' - 1, \alpha_3 - (d_3 - d_3') - 2l_3) + \max(0, d_4' - 1, \alpha_4 - (d_4 - d_4') - 2l_4)) \]  
(6.30)

In particular, \( n_0 \leq n - 2 \). By the inductive hypothesis,

\[ A_0 \leq N_2 \Phi(n_0) + N_1 \phi(n_0) \mu(\Delta_0) \]  
(6.31)

We note that the mixture \( \mu(\Delta_0) \) of \( \Delta_0 \) is not greater than \( \mu(\Delta) - l'(l - l') \). Indeed, by Lemma 6.14 (2), one can use the same trick as in Lemma 6.14 as follows. For every pair of white beads, where one bead corresponds to a \( \theta \)-band of \( \Gamma_2 \) and another one to a \( \theta \)-band of \( \Gamma_3 \) or \( \Gamma_4 \), the contribution of this pair to \( \mu(\Delta_0) \) is less than the contribution to \( \Delta \). It remains to count the number of such pairs: \( l'(l_3 + l_4) = l(l - l') \).

Therefore, by inequalities (6.31) and (6.14), the \( G \)-area of \( \Delta \) is not greater than

\[ N_2 \Phi(n) + N_1 \phi(n) \mu(\Delta) - N_2 n(n - n_0) - N_1 \phi(n) l'(l - l') + A_2 + A_3 + A_4 + l' \]  
(6.32)

In view of inequalities (6.28), (6.27) and (6.29), to obtain the desired contradiction, it suffices to prove that

\[ N_2 n(n - n_0) + N_1 l'(l - l') \geq C_{12} l'(d_3 + d_4 + 1) + C_{12} (l_3^2 + l_4^2) + 2\alpha_3 l_3 + 2\alpha_4 l_4 + l' \]  
(6.33)

where \( C_{12} = \max(C_1, C_2) \).

Since \( l = l' + l_3 + l_4 \), it suffices to prove that

\[ N_2 n(n - n_0) + N_1 l'(l - l') \geq C_3 l'(d_3 + d_4 + 1) + C_3 (l_3^2 + l_4^2) + 2\alpha_3 l_3 + 2\alpha_4 l_4 \]  
(6.34)

where \( C_3 = C_{12} + 1 \).

Note that we can assume that

\[ C_3 \gg 1. \]  
(6.35)

First we can choose \( N_1 \) big enough so that \( N_1 l'(l - l') / 3 \geq C_3 (l_3 + l_4)^2 \geq C_3 (l_3^2 + l_4^2) \). Indeed, by (6.26), we obtain \( \frac{N_1}{3} l'(l - l') \geq \frac{N_1}{3} (l_3 + l_4)(l_3 + l_4) \), so it is enough to assume that

\[ N_1 > 3C_3. \]  
(6.36)

We also have that

\[ \frac{N_2}{2} n(n - n_0) \geq C_3 l' \]  
(6.37)

because \( n - n_0 \geq 2, n \geq 2l' \) and \( N_2 \geq C_3 \) by (6.36).

It remains to prove that

\[ \frac{N_2}{2} n(n - n_0) + \frac{2N_1}{3} l'(l - l') > C_3 l'(d_3 + d_4) + 2\alpha_3 l_3 + 2\alpha_4 l_4. \]  
(6.38)

We assume without loss of generality that \( \alpha_3 \geq \alpha_4 \), and consider two cases.

\[ \text{(a) Suppose } \alpha_3 \leq 2C_3 (l - l'). \]

Since \( d_i \leq \alpha_i + d_i' \) for \( i = 3, 4 \), by inequality (6.30), we have

\[ d_3 + d_4 \leq \alpha_3 + \alpha_4 + d_3' + d_4' < 4C_3 (l - l') + \delta^{-1} (n - n_0) + 2 - 2\delta^{-1} < 4C_3 (l - l') + \delta^{-1} (n - n_0). \]

Therefore
\[ \frac{N_1}{3} l'(l - l') + \frac{N_2}{2} n(n - n_0) \geq 4C_3^2 l'(l - l') + C_3 \delta^{-1} (n - n_0) l' > C_3 l''(d_3 + d_4) \]  
(6.39)

since we can assume that

\[ N_1 > 12C_3^2, \]  
\[ N_2/2 > C_3 \delta^{-1}. \]  
(6.40)

We also have by (6.26): 

\[ \frac{N_1}{3} l'(l - l') \geq \frac{N_1}{3} (l_3 + l_4)(l_3 + l_4) \geq \frac{N_1}{3} \frac{\alpha_3 + \alpha_4}{4C_3} (l_3 + l_4) > 2\alpha_3 l_3 + 2\alpha_4 l_4 \]  
(6.41)

since we can assume that

\[ N_1 > 24C_3. \]  
(6.42)

The sum of inequalities (6.39) and (6.41) gives us the desired inequality (6.38).

(b) Assume now that \( \alpha_3 > 2C_3(l - l') \). Then, applying Lemma 6.5 to the comb \( \Gamma_3 \), we obtain

\[ d_3 - d'_3 < \frac{1}{2} \alpha_3 + K_0 l_3 \leq \frac{5}{6} \alpha_3 \]  
(6.43)

since \( l_3 \leq l - l' < \frac{\alpha_3}{2C_3} \) and

\[ C_3 > 3K_0. \]  
(6.44)

We also have \( d_4 - d'_4 < \frac{1}{2} \alpha_4 + K_0 l_4 \leq \frac{5}{6} \alpha_4 \). These two inequalities and inequality (6.30) lead to

\[ d_3 + d_4 \leq \frac{5}{3} \alpha_3 + \delta^{-1} (n - n_0) \]  
(6.45)

It follows from (6.43) that

\[ \alpha_3 - (d_3 - d'_3) - 2l_3 \geq \frac{1}{6} \alpha_3 - \frac{2}{2C_3} \alpha_3 \geq \frac{1}{7} \alpha_3, \] \[ \text{since } l_3 \leq l - l' < \frac{\alpha_3}{2C_3} \] \[ \text{and } C_3 > 42 \text{ by (6.35). Therefore, by (6.30),} \] \[ n - n_0 \geq \frac{1}{7} \delta \alpha_3. \]  
(6.46)

Thus, by (6.45),

\[ d_3 + d_4 < 13 \delta^{-1} (n - n_0). \]  
(6.47)

Since \( 2l' < n \) and \( n - n_0 \geq 2 \), inequality (6.47) implies

\[ \frac{N_2}{3} n(n - n_0) > C_3 l'(d_3 + d_4) \]  
(6.48)

because we can assume that

\[ N_2 >> C_3 \delta^{-1} \]  
(6.49)

\( (N_2 > 21C_3 \delta^{-1} \text{ is enough}). \)

Inequalities (6.46), (6.49), \( \alpha_3 \geq \alpha_4 \), and \( 4(l_3 + l_4) \leq n \) give us

\[ \frac{N_2}{6} n(n - n_0) \geq \frac{7}{2} C_3 \delta^{-1} (n - n_0)n \geq 2\alpha_3 (l_3 + l_4) \geq 2\alpha_3 l_3 + 2\alpha_4 l_4 \]  
(6.50)

The inequality (6.38) follows now from inequalities (6.48), and (6.50).
7 Minimal diagrams over $G$

Given a reduced diagram $\Delta$ over the group $G$, one can construct a planar graph whose vertices are the hubs of this diagram plus one improper vertex outside $\Delta$, and the edges are maximal $t$-bands of $\Delta$.

Let us consider two hubs $\Pi_1$ and $\Pi_2$ in a reduced diagram, connected by two neighbor $t$-bands $C_i$ and $C_{i+1}$, where there are no other hubs between these $t$-bands. These bands, together with parts of $\partial \Pi_1$ and $\partial \Pi_2$, bound either a subdiagram having no cells, or a trapezium $\Psi$ of height $\geq 1$ (fig. 9). The former case is impossible since in this case the hubs have a common $t$-edge and, the diagram is not reduced since all cyclic permutations of the word $(W_M)^L$ starting with $t$ are equal. We want to show that the latter case is not possible either if the diagram $\Delta$ is chosen with minimal number of hubs among the diagrams with the same boundary label.

Indeed, both the top and the bottom of $\Psi$ are the subwords of the word $(W_Mt)^{\pm 1}$, and removing $C'$ from $\Psi$, we get a subdiagram $\Psi'$ with top/bottom label $(W_M)^{\pm 1}$ and the same label $V$ of its sides. It follows that $W_M$ and $V$ commute in the group $M$. Hence the word $U \equiv (W_M)^{1-L}V(W_M)^L-1V^{-1}$ is equal to 1 in $M$. But $U$ is the boundary label of a subdiagram $\Gamma$ containing $\Psi'$ and both $\Pi_1$ and $\Pi_2$. Hence one can replace $\Gamma$ with a diagram over $M$, decreasing the number of hubs in $\Delta$, a contradiction.

If $W$ is an $M$-accepted word, then the word $(W)^L$ is equal to 1 in $G$. To see this, one can glue up $L$ copies $\Delta_1, \ldots, \Delta_L$ of the trapezium corresponding to the accepting computation of $W$, identifying the right side of each $\Delta_i$ and the left side of $\Delta_{i+1}$ (indices are taken modulo $L$). The obtained annulus has inner boundary labeled by the hub word $(W_M)^L$, and so the hole can be glued up by a hub cell.

We will increase the set of relations of $G$ by adding the (infinite) set of disk relators $(W)^L$ for every accepted word $W$. So we will consider diagrams with disks, where every disk cell (or just disk) is labeled by such a word $(W)^{\pm L}$. (In particular, the hub is a disk.)

Again, if two disks are connected by two $t$-bands and there are no other hubs between these $t$-bands, then one can reduce the number of discs in the diagram. For this aid, it suffices to replace the disks with hubs and the cells corresponding to the defining relations of $M$, and apply the trick exploited above.

**Definition 7.1.** We will call a reduced diagram $\Delta$ **minimal** if

1. the number of disks is minimal for the diagrams with the same boundary label and
2. $\Delta$ has minimal number of $(\theta, t)$-cells among the diagrams with the same boundary label and with minimal number of disks.
Clearly, a subdiagram of a minimal diagram is minimal itself.

Thus, no two disks of a minimal diagram are connected by two $t$-bands, such that the subdiagram bounded by them contains no other disks. This property makes the disk graph of a reduced diagram hyperbolic, in a sense, if the degree $L$ of every proper vertex (= disk) is high ($L >> 1$). Below we give a more precise formulation (proved for diagrams with such a disk graph, in particular, in [19], Lemma 11.4 and in [12], Lemma 3.2).

**Lemma 7.2.** If a a minimal diagram contains at least one disk, then there is a disk $\Pi$ in $\Delta$ such that $L - 3$ consecutive maximal $t$-bands $B_1, \ldots, B_{L-3}$ start on $\partial \Pi$, end on the boundary $\partial \Delta$, and for any $i \in [1, L - 4]$, there are no disks in the subdiagram $\Gamma_i$ bounded by $B_i, B_{i+1}, \partial \Pi$, and $\partial \Delta$ (fig. 10).

□

A maximal $q$-band starting on a disk of a diagram is called a *spoke*. Lemma 7.2 implies by induction on the number of hubs:

**Lemma 7.3.** (see [13], Lemma 5.19) If a reduced diagram $\Delta$ has $m \geq 1$ hubs, then the number of spokes in $\Delta$, and therefore the number of $q$-edges in the boundary path of $\Delta$, is greater than $mLN/2$.

□

Recall the following transformation for diagrams with disks, exploited earlier in [12], [19]. Assume there is a disk $\Pi$ and a $\theta$-band $T$ subsequently crossing some spokes $B_1, \ldots, B_k$ which start (say, clockwise) from $\Pi$. Assume that $k \geq 2$ and there are no other cells between $\Pi$ and the bottom of $T$, and so there is a subdiagram $\Gamma$ formed by $\Pi$ and $T$.

We describe the *transposition* of the disk and the band as follows. We have a word $V \equiv tW_1t \ldots tW_{k-1}t$ written on the top of the subband $T'$ of $T$, that starts on $B_1$ and ends on $B_k$. The bottom $q_2$ of $T'$ is the subpath of the boundary path $q_2q_3$ of $\Pi$ (fig. 11).

Note that $W_1 \equiv W_2 \equiv \cdots \equiv W_{k-1}$ and $tW_1$ is an accepted word by Lemma 5.7. Therefore one can construct a new disk $\bar{\Pi}$ with boundary label $(tW_1)^L$ and boundary $s_1s_2$, where $\text{Lab}(s_1) \equiv V$. Also one construct an auxiliary band $T''$ with top label $W^{-1}t^{-1} \cdots t^{-1}W_1^{-1}$, where the number of occurrences of $t^{-1}$ is $L - k$, and attach it to $s_2^{-1}$, which has the same label. Finally we replace the subband $T'$ by $T''$ (and make
cancellations in the new $\theta$-band $\bar{T}$ if any appear). The new diagram $\bar{\Gamma}$ formed by $\bar{\Pi}$ and $\bar{T}$ has the same boundary label as $\Gamma$.

**Remark 7.4.** After the transposition, the first $(\theta,t)$-cells of $t$-spokes $B_1, \ldots, B_k$ are removed and the total number of common $(\theta,t)$-cells of the new spokes $\bar{B}_1, \ldots, \bar{B}_k$ of $\bar{\Pi}$ and $\bar{T}$ is less than the number of common $(\theta,t)$-cells of $B_1, \ldots, B_k$ and $T$ at least by $k$.

In particular, if $k > L - k$, then the number of $(\theta,t)$-cells in $\bar{\Gamma}$ is less than the number of $(\theta,t)$-cells in $\Gamma$. This observation implies

**Lemma 7.5.** Let $\Delta$ be a minimal diagram.

1. Assume that a $\theta$-band $T_0$ crosses $k$ $t$-spokes $B_1, \ldots, B_k$ starting on a disk $\Pi$, and there are no disks in the subdiagram $\Delta_0$, bounded by these spokes, by $T_0$ and by $\Pi$. Then $k \leq L/2$.

2. Assume that there are two disjoint $\theta$-bands $T$ and $S$ whose bottoms are parts of the boundary of a disk $\Pi$ and these bands correspond to the same rule $\theta$ (if their histories are read towards the disk). Suppose $T$ crosses $k \geq 2$ $t$-spokes starting on $\partial \Pi$ and $S$ crosses $\ell \geq 2$ $t$-spokes starting on $\partial \Pi$. Then $k + \ell \leq L/2$.

3. $\Delta$ contains no $\theta$-annuli.

4. A $\theta$-band cannot cross a maximal $q$-band (in particular, a spoke) twice.

**Proof.** (1) Since every cell, except for disks, belongs to a maximal $\theta$-band, it follows from Lemma 5.3 that there is a $\theta$-band $T$ such that $T$ crosses all $B_1, \ldots, B_k$ and $\Delta_0$ has no cells between $T$ and $\Pi$. If $k > L/2$, then by Remark 7.4, the transposition of $\Pi$ and $T$ would decrease the number of $(\theta,t)$-cells in $\Delta$, a contradiction, since $\Delta$ is a minimal diagram.

(2) As above, let us transpose $T$ and $\Pi$. This operation removes $k$ $(\theta,t)$-cells but add $L - k$ new $(\theta,t)$-cells in $T$. However $\ell$ $(\theta,t)$-cells of $S$ and $\ell$ $(\theta,t)$-cells of $T$ will form mirror pairs, and so after cancellations one will have at most $L - k - 2\ell$ new $(\theta,t)$-cells. This number is less than $k$ if $k + \ell > L/2$ contrary to the minimality of the original diagram. Therefore $k + \ell \leq L/2$.

(3) Proving by contradiction, consider the subdiagram $\Delta'$ bounded by a $\theta$-annulus. It has to contain disks by Lemma 5.3. Hence it must contain spokes $B_1, \ldots, B_{L-3}$ introduced in Lemma 7.2. But this contradicts to item (1) of the lemma since $L - 3 > L/2$.

(4) The argument of item (3) works if there is a subdiagram $\Delta'$ of $\Delta$ bounded by an $q$-band and a $\theta$-band. □
The transposition transformation will be used for extrusion of disks from quasi-trapezia. The definition of a quasi-trapezium sounds as the definition of trapezium, but quasi-trapezia may contain disks. (So a quasi-trapezium without disks is a trapezium.)

**Lemma 7.6.** Let a minimal diagram $\Gamma$ be a quasi-trapezium with standard factorization of the boundary as $p_1^{-1}q_1p_2q_2^{-1}$. Then there is a diagram $\Gamma'$ such that

1. the boundary of $\Gamma'$ is $(p'_1)^{-1}q'_1p'_2(q'_2)^{-1}$, where $\text{Lab}(p'_j) \equiv \text{Lab}(p_j)$ and $\text{Lab}(q'_j) \equiv \text{Lab}(q_j)$ for $j = 1, 2$;
2. the numbers of hubs and $(\theta, q)$-cells in $\Gamma'$ are the same as in $\Gamma$;
3. the vertices $(p'_1)_-$ and $(p'_2)_-$ (the vertices $(p'_1)_+$ and $(p'_2)_+$) are connected by a simple path $s_1$ (by $s_2$, resp.) such that we have three subdiagrams $\Gamma_1, \Gamma_2, \Gamma_3$ of $\Gamma'$, where $\Gamma_2$ is a trapezium with standard factorization of the boundary $(p'_1)^{-1}s_1p'_2s_2^{-1}$ and all cells of the subdiagrams $\Gamma_1$ and $\Gamma_3$ with boundaries $q'_1s_1^{-1}$ and $s_2(q'_1)^{-1}$ are disks;
4. All maximal $\theta$-bands of $\Gamma$ and all maximal $\theta$-bands of $\Gamma_2$ have the same number of $(\theta, t)$-cells (equal for $\Gamma$ and $\Gamma_2$).

**Proof.** Every maximal $\theta$-band of $\Gamma$ must connect an edge of $p_1$ with an edge of $p_2$; this follows from Lemma 7.5 (3). Hence we can enumerate these bands from bottom to top: $T_1, \ldots, T_h$, where $h = |p_1| = |p_2|$.

If $\Gamma$ has a disk, then by Lemma 7.2, there is a disk $\Pi$ such that at least $L - 3$ $t$-spokes of it end on $q_1$ and $q_2$, and there are no disks between the spokes ending on $q_1$ (and on $q_2$). By Lemma 7.5 (2), at least $L - 3 - L/2 \geq 2$ of these spokes must end on $q_1$ (resp., on $q_2$).

If $\Pi$ lies between $T_j$ and $T_{j+1}$, then the number of its $t$-spokes crossing $T_j$ (crossing $T_{j+1}$) is at least 2. So one can make a transposition of $\Pi$ with each of these two $\theta$-bands. So we can move the disk toward $q_1$ (or toward $q_2$) until the disk is extruded from the quasi-trapezium. (We use that if $k$ $t$-spokes $B_1, \ldots, B_k$ of $\Pi$ end on $q_1$, then after transposition, we again have $k$ $t$-spokes $\bar{B}_1, \ldots, \bar{B}_k$ of $\Pi$ ending on $q_1$. - See the notation of Remark 7.4.)

No pair $T_j$ and $T_{j+1}$ corresponds to two mutual inverse letters of the history. This is clear if there are no discs between these $\theta$-bands. If there is a disk, then this is impossible too by Lemma 7.6 (2) since one could choose a disk $\Pi$ as in the previous paragraph. So the projection of the label of $p_1$ on the history is reduced.

Let us choose $i$ such that the number $m$ of $(\theta, t)$-cells in $T_i$ is minimal. It follows that $\Gamma$ has at least $hm$ $(\theta, t)$-cells.

If the disk $\Pi$ lies above $T_i$, we will move it upward using transpositions. So after a number of transpositions all such (modified) disks will be placed above the $\theta$-band number $h$ and form the subdiagram $\Gamma_1$. Similarly we can form $\Gamma_3$ moving other disks downward.

In the obtaining diagram $\Gamma_2$ lying between $\Gamma_1$ and $\Gamma_3$, every $\theta$-band is reduced by the definition of transpositions. The neighbor maximal $\theta$-band of $\Gamma_2$ cannot correspond to mutual inverse letters of the history since the labels of $p_1$ and $p'_1$ are equal. It follows that the diagram $\Gamma_2$ (without disks) is a reduced diagram, and so it is a trapezium of height $h$.

The $\theta$-band $T_i$ did not participate in the transpositions. Therefore it is a maximal $\theta$-band of $\Gamma_2$. Hence the trapezium $\Gamma_2$ contains exactly $mh$ $(\theta, t)$-cells, which does not exceed the number of $(\theta, t)$-cells in $\Gamma$. In fact these two numbers are equal since $\Gamma$ is a
minimal diagram. So every maximal \( \theta \)-band of \( \Gamma \) and every maximal \( \theta \)-band of \( \Gamma_2 \) has \( m(\theta, t) \)-cells.

We say that a history \( H \) is *standard* if there is a standard trapezium with history \( H \).

**Definition 7.7.** Let we have a disk \( \Pi \) with boundary label \((tW)^L\) and \( B \) be a \( t \)-spoke starting on \( \Pi \). Suppose there is a subband \( C \) of \( B \), which also starts on \( \Pi \) and has a standard history \( H \), for which the word \( tW \) is \( H \)-admissible. Then we call the \( t \)-band \( C \) a shaft.

For a constant \( \lambda \in [0; 1/2) \) we also define a stronger concept of \( \lambda \)-shaft at \( \Pi \) as follows. A shaft \( C \) with history \( H \) is a \( \lambda \)-shaft if for every factorization of the history \( H = H_1H_2H_3 \), where \(|H_1| + |H_3| < \lambda|H|\), the middle part \( H_2 \) is still a standard history. (So a shaft is a \( 0 \)-shaft).

**Lemma 7.8.** Let \( \Pi \) be a disk in a minimal diagram \( \Delta \) and \( C \) be a \( \lambda \)-shaft at \( \Pi \) with history \( H \). Then \( C \) has no factorizations \( C = C_1C_2C_3 \) such that

(a) the sum of lengths of \( C_1 \) and \( C_3 \) do not exceed \( \lambda|H| \) and

(b) \( \Delta \) has a quasi-trapezium \( \Gamma \) such that top (or bottom) label of \( \Gamma \) has \( L+1 \) occurrences of \( t \) and \( C_2 \) starts on the bottom and ends on the top of \( \Gamma \).

**Proof.** Proving by contradiction, we first replace \( \Gamma \) by a trapezium \( \Gamma' \) according to Lemma 7.6. The transpositions used for this goal do not affect neither \( \Pi \) nor \( C \) since \( C \) crosses all the maximal \( \theta \)-bands of \( \Gamma \). Also one can replace \( \Gamma' \) by a trapezium with shorter base and so we assume that the base of it starts and ends with letter \( t \).

For the beginning, we assume that \( C \) is a shaft (i.e. \( \lambda = 0 \)). Then it follows from the definition of shaft and Remark 5.8 that \( \text{bot}(\Gamma') \) is labeled by \((tW)^L t \), where \((tW)^L \) is the boundary label of \( \Pi \). One can remove the first or the last maximal \( t \)-band from \( \Gamma' \) and obtain a subtrapezium \( \Gamma'' \) whose boundary label coincides with the label of \( \partial \Pi \) (up to cyclic permutation), and \( \partial\Gamma'' \) shares a \( t \)-edge with \( \partial\Pi \) (fig.12 with \( \lambda = 0 \)). However \( \text{Lab}(\text{top}(\Gamma'')) = (tW')^L \), where \( tW' = (tW) \cdot H \) by Lemma 5.7, and so there is a disk \( \Pi' \) with boundary label \((tW')^L \). Therefore the subdiagram \( \Delta' \) can be replaced by a single disk. So we decrease the number of \((\theta, t)\)-cells contrary to the minimality of \( \Delta \).

Now we consider the general case, where \( C = C_1C_2C_3 \). As above, we replace \( \Gamma \) by a trapezium \( \Gamma' \) and obtain a trapezium \( \Gamma'' \) after removing of one \( t \)-band in \( \Gamma' \). To obtain
a contradiction, it suffices to consider the diagram \( \Delta' = \Pi \cup C_1C_2 \cup \Gamma'' \) (forgetting of the compliment of \( \Delta' \) in \( \Delta \)) and find another diagram \( \Delta'' \) with one disk and fewer \((\theta,t)\)-cells such that \( \text{Lab}(\partial\Delta'') = \text{Lab}(\partial\Delta') \) in the free group.

Since both histories \( H \) and \( H_2 \) (and so \( H_1H_2 \)) are standard, one can enlarge \( \Gamma'' \) and construct a trapezium \( \Gamma''' \) with history \( H_1H_2 \). (The added parts \( E_1 \) and \( E_2 \) are dashed in figure 12 (with \( \lambda > 0 \)). Note that we add \( < \lambda\)\(||H||L \) new \((\theta,t)\)-cells since every maximal \( \theta \)-band of \( \Gamma''' \) has \( L \) such cells. As in case \( \lambda = 0 \), this trapezium \( \Gamma''' \) and the disk \( \Pi \) can be replaced by one disk \( \Pi' \). However to obtain the boundary label equal to \( \text{Lab}(\partial\Delta') \), we should attach the mirror copies \( \exists_1 \) and \( \exists_2 \) of \( E_1 \) and \( E_2 \) to \( \Pi' \). The obtained diagram \( \Delta'' \) has at most \( \lambda\)\(||H_1||L \) \((\theta,t)\)-cells, while \( \Delta' \) has at least \( ||H_2||L \geq (1 - \lambda)||H|| \) \((\theta,t)\)-cells. Since \( \lambda < 1 - \lambda \), we have the desired contradiction.

Lemma 7.8 will be used to obtain a linear bound, in terms of perimeter \( ||\partial\Delta|| \), for the sum of lengths \( \sigma = \sigma_\lambda(\Delta) \) of all \( \lambda \)-shafts in a minimal diagram \( \Delta \), which make possible to exploit \( \sigma_\lambda \) as an inductive parameter along with \( ||\partial\Delta|| \). One more tool needed to linearly bound \( \sigma_\lambda \), is a combinatorial proposition of two finite systems of disjoint segments on Euclidean plane proved in the next section.

8 Designs in topological disk

By \( \mathcal{D} \), we denote Euclidean closed disk of radius 1. Let \( \mathbf{T} \) be a finite set of disjoint chords (plain lines in fig. 13) and \( \mathbf{Q} \) a finite set of disjoint simple curves in \( \mathcal{D} \) (dotted lines in fig. 13). One may think that a curve is a non-oriented broken line, i.e. it is built from finitely many finite segments. To distinguish the elements from \( \mathbf{T} \) and \( \mathbf{Q} \), we will say that the elements of \( \mathbf{Q} \) are \textit{arcs}.

We shall assume that the arcs belong in the open disk \( D^o \), an arc may cross a chord transversally at most once, and the intersection point cannot coincide with one of the two ends of an arc.

Under these assumptions, we shall say that the pair \((\mathbf{T}, \mathbf{Q})\) is a \textit{design}. 

Figure 13: Design
By definition, the length $|C|$ of an arc $C$ is the number of the chords crossing $C$. The term subarc will be used in natural way; obviously one has $|D| \leq |C|$ if $D$ is a subarc of an arc $C$.

We say that an arc $C_1$ is parallel to and arc $C_2$ and write $C_1 \parallel C_2$ if every chord (from $T$) crossing $C_1$ also crosses $C_2$. So the relation $\parallel$ is transitive. (The arc of length 2 is parallel to the arc of length 5 in fig. 13.)

**Definition 8.1.** Given $\lambda \in (0; 1)$ and an integer $n \geq 1$, the property $P(\lambda, n)$ of a design says that for any $n$ different arcs $C_1, \ldots, C_n$, there exist no subarcs $D_1, \ldots, D_n$, respectively, such that $|D_i| > (1 - \lambda)|C_i|$ for every $i = 1, \ldots, n$ and $D_1 \parallel D_2 \parallel \cdots \parallel D_n$.

By definition, the length $\ell(Q)$ of the set of arcs $Q$ is defined by the equality

$$\ell(Q) = \sum_{C \in Q} |C| \quad (8.51)$$

The number of chords will be denoted by $\#T$. The goal of this subsection is to prove the following

**Theorem 8.2.** There is a constant $c = c(\lambda, n)$ such that for any design $(T, Q)$ with property $P(\lambda, n)$, we have

$$\ell(Q) \leq c(\#T) \quad (8.52)$$

To prove Theorem 8.2, we may assume that $Q$ has no arcs of length 0 and that every chord is crossed by an arc. Also we may assume that $\#T > 1$ since otherwise all the arc are parallel, and Property $P(\lambda, n)$ implies that the number of arcs is at most $n - 1$; therefore one can take $c = \frac{1}{n-1}$.

Every chord $T$ divides the disk $D$ in two half-disks. If one of these half-disks contains no other chords, we call the chord $T$ peripheral and denote the peripheral half-disk (without chord) by $O_T$.

An arc $D$ is called an extension of an arc $C$ if $C$ is a subarc of $D$. (An extension need not be an element of $Q$.) We will consider only extensions of $C \in Q$ such that replacing $C$ by $D$ we again obtain a design $(T, Q)$ (but the property $P(\lambda, n)$ can be violated for the new design).

An arc $C$ of a design is called maximal if there exists no extension $D$ of $C$ with $|D| > |C|$.

**Lemma 8.3.** Let $(T, Q)$ be a design with $\#T \geq 1$. Then all the arcs $C_1, C_2, \ldots$ from $Q$ have maximal extensions $D_1, D_2, \ldots$ forming a set of arcs $Q'$ such that the design $(T, Q')$ has the following property: for every arc $D_i$, its ends belong to two different peripheral half-disks.

**Proof.** Since no arc can cross a chord twice and the set of chords is finite, there is a system of maximal arcs $D_1, D_2, \ldots$ such that every $D_i$ is an extension of $C_i$. It suffices to prove that the ends of every $D_i$ belong in peripheral half-disks.

Assume that we have an end $o$ of an arc $D = D_i$ and $o$ belongs to no peripheral half-disk. Let us choose the direction for $D$ toward $o$, and assume that $T$ is the last chord crossed by $D$. Let $H$ be the half-disk defined by $T$, where the point $o$ belongs in. The half-disk $H$ is not peripheral, and so it contains a chord $T' \neq T$. None of such $T'$ could be crosses by $D$ because otherwise $D$ had to cross the chord $T$ at least twice. We
may assume that $T'$ is the closest to $o$ in the sense that one can connect $o$ and $T'$ by a path $p$ inside $H$, $p$ consequently intersect $\ell$ arcs $D_{i_1}, \ldots, D_{i_\ell}$ from the set $\{D_1, D_2, \ldots\}$, and the number $\ell$ is minimal.

If $\ell = 0$, then using $p$ one could extend $D$ so that the extension crosses $T'$. If $\ell > 0$ we come to a contradiction too. Indeed, let us consider a small neighborhood $U$ of $D_{i_1}$, which contains neither points of other arcs, nor the boundary points of $D$. If one can bypath $D_{i_1}$ in $U$ moving around it, thereby replacing $p$ by a path $p'$ having $\ell - 1$ intersections with arcs (see fig. 14 (a)), we get a contradiction. Otherwise going around $D_{i_1}$ in $U$ clockwise or counter-clockwise one will cross an interval of some chord $T'' \neq T$ from $H$ (fig 14 (b)). Then one can use $U$ to connect $o$ with the chord $T''$ and an extention of $D$ crosses $T''$, a contradiction again.

To continue the proof of Theorem 8.2, we modify the number $\# T$, taking every chord $T$ with a weight $\nu(T)$. By definition,

- $\nu(T) = 1$, if $T$ crosses exactly one arc from $Q$,
- $\nu(T) = 2$, if $T$ crosses exactly two arcs,
- $\nu(T) = 2n - 2$, if $T$ crosses exactly $2n - 2$ arcs,
- $\nu(T) = 2n - 1$, if $T$ crosses at least $2n - 1$ arcs.

Clearly, we have

$$\# T \leq \nu(T) = (\text{by definition}) = \sum_{T \in T} \nu(T) \leq (2n - 1) \# T$$

Therefore, instead of (8.52), it suffices to prove the following inequality

$$\ell(Q) \leq d \nu(T) \quad (8.53)$$

for some $d = d(\lambda, n) > 0$. We will prove (8.53) for any $d \geq \frac{1}{\lambda}$ by induction on the number of arcs in $Q$. If there is only one arc, then it is nothing to prove since $\ell(Q) \leq \# T$ in this case. So we will assume that there are at least two arcs.
Assume that there is an arc $C$, which, after one choose a direction, can be factorized as $C_1C_2C_3$, where

(a) $|C_1| + |C_3| \geq \lambda |C|$ and
(b) every chord crossing $C_1$ or $C_3$ has weight at most $2n - 2$.

Let us remove $C$ from $Q$. We obtain a new design $(T', Q')$. Here $T'$ have the same chords as $T$, but their weights change. Obviously, Property $P(\lambda, n)$ holds for the design $(T', Q')$. Hence by inductive hypothesis, the inequality $\ell(Q') \leq d \nu(T')$ is true.

Also we have $\ell(Q) = \ell(Q') + |C|$ and

$$\nu(T) \geq \nu(T') + |C_1| + |C_3| \geq \nu(T') + \lambda |C|$$

since all chords crossing $C_1$ and $C_3$ loss their weight by 1. It follows that

$$\ell(Q) = \ell(Q') + |C| \leq d \nu(T') + |C| \leq d(\nu(T') + \lambda |C|) \leq d \nu(T),$$

as desired, since $d \geq \lambda^{-1}$.

It remains to obtain a contradiction assuming that no arc $C \in Q$ has a factorization with Properties (a) and (b). In other words, every arc $C \in Q$ has a subarc $D$ maximal with respect to the following properties:

(A) $|D| > (1 - \lambda)|C|$ and
(B) the first and the last chords crossing $D$ have weight $2n - 1$.

We denote by $(\bar{T}, \bar{Q})$ the design obtained after the transition $C \to D$ for every arc $C$. Here we assume that $\bar{T}$ contains the chords from $T$, which cross some arcs from $\bar{Q}$. Observe that all chords of weight $2n - 1$ from $T$ have the same weight in $\bar{T}$, as it follows from the definition of $\bar{Q}$. (We do not claim Property $P(\lambda, n)$ for $(\bar{T}, \bar{Q})$.)

Let $\bar{T}$ be a peripheral chord from $\bar{T}$. It is crossed by (at least) $2n - 1$ arcs $D_1, \ldots, D_{2n-1}$ since $\bar{T}$ is the first/last chord crossing the arcs.

By Lemma 8.3, there are maximal extensions $\bar{D}_1, \ldots, \bar{D}_{2n-1}$. Moreover, such extensions can be constructed for every peripheral chord, and the two ends of every extension must belong to different peripheral half-disks of the design $(\bar{T}, \bar{Q})$.

Suppose one can chose $n$ extension, say $\bar{D}_1, \ldots, \bar{D}_n$ starting in $O_{\bar{T}}$ and ending in the same half-disk $O_{T'}$ ($T' \in \bar{T}$). Then every chord of $\bar{T}$ crossing $\bar{D}_i$ has to cross $\bar{D}_j$ for $1 \leq i, j \leq n$. The same is true for the chords of $T$ disposed between $T$ and $T'$. Since $\bar{D}_i$ starts with $D_i$ and $\bar{D}_j$ starts with $D_j$, the inequality $|D_i| \leq |D_j|$ implies that every chord of $T$ crossing $D_i$ has to cross $D_j$ too. Therefore assuming that $|D_1| \leq \cdots \leq |D_n|$, we have $D_1 || \cdots || D_n$. However this violates Property $P(\lambda, n)$ for the arcs $C_1, \ldots, C_n$ since $|D_i| > (1 - \lambda)|C_i|$ for every $i = 1, \ldots, n$.

Thus, there are no $n$ arcs among $\bar{D}_1, \ldots, \bar{D}_{2n-1}$ connecting the half-disk $O_{\bar{T}}$ with the peripheral half-disk. We see that for every peripheral half-disk $O_{\bar{T}}$, one can find three arcs, say $\bar{D}_i, \bar{D}_j, \bar{D}_k$ connecting $O_{\bar{T}}$ with three different peripheral half-disks.

Now let us choose one vertex in every peripheral half-disk (e.g., on the boundary of the disk $D$) and connect it with three other vertices using the triples of arcs obtained in the previous paragraph. We obtain an outerplanar graph with at least four vertices, where every vertex has degree at least 3. However there exist no such graphs ([7], Corollary 11.9). The obtained contradiction completes the proof of Theorem 8.2. □

Remark 8.4. One may allow the ends of arcs to belong in the boundary of $D$, and then the same inequality (8.52) holds since one can cut off the ends of every arc $C$ preserving the length of $C$.  

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Let we have a parameter $\lambda \in [0, 1/2)$. For every $t$-spoke $B$ of a minimal diagram $\Delta$, we choose the $\lambda$-shaft of maximal length in it (if a $\lambda$-shaft exists). If $B$ connects two disks $\Pi_1$ and $\Pi_2$, then there can be two $\lambda$-shafts: at $\Pi_1$ and at $\Pi_2$. We denote by $\sigma_\lambda(\Delta)$ the sum of lengths of all $\lambda$-shafts in this family.

**Lemma 8.5.** There is a constant $c = c(\lambda)$ such that $\sigma_\lambda(\Delta) \leq c|\partial \Delta|$ for every minimal diagram $\Delta$ over the group $G$.

**Proof.** Let us associate the following design with $\Delta$. We say that the middle lines of the maximal $\theta$-bands (they cross $\theta$-edges of the bands in the middle points) are the chords and the middle lines of the maximal $\lambda$-shafts are the arcs. Here we use two disjoint middle lines if two maximal $\lambda$-shafts share a $(\theta, t)$-cell. By Lemma 7.5 (3), (4), we obtain a design, indeed.

Observe that the length $|C|$ of an arc is the number of cells in the $\lambda$-shaft and $\# T \leq |\partial \Delta|/2$ since every maximal $\theta$-band has two $\theta$-edges on $\partial \Delta$.

Thus, by Theorem 8.2, it suffices to show that the constructed design satisfies to the condition $P(\lambda, n)$, where $n$ does not depend on $\Delta$.

Let $n = 2L + 1$. If the property $P(\lambda, n)$ is violated, then we have $n$ maximal $\lambda$-shafts $C_1, \ldots, C_n$ and a subband $D$ of $C_1$, such that $|D| > (1 - \lambda)|C_1|$, and every maximal $\theta$-band crossing $D$ must cross each of $C_2, \ldots, C_n$. (Here $|B|$ is the length of a $t$-band $B$.) It follows that each of these $\theta$-bands crosses at least $L + 1$ maximal $t$-bands. (See Lemma 7.5 (3,4). Here we take into account that the same $t$-spoke can generate two arcs in the design.) Hence using the $\lambda$-shaft $C_1$ one can construct a quasi-trapezium of height $|D|$, which contradicts to the statement of Lemma 7.8. \hfill $\square$

### 9 Upper bound for $G$-areas of diagrams over the group $G$.  

By definition, the $G$-area of a disk $\Pi$ is just the minimum of areas of diagrams over the presentation $(5.8, 5.9)$ of $G$ having the same boundary label as $\Pi$.

**Lemma 9.1.** There is a constant $c_6$ such that the $G$-area of any disk does not exceed $c_6F(|\partial \Pi|)$.

**Proof.** The disk $\Pi$ can be built of a hub and $L$ standard accepting trapezia over $M$. By Lemma 4.23, and the definition of the functions $f(n), g(n)$, there are such trapezia of height $O(|\partial \Pi|/L)g(|\partial \Pi|/L) = O(|\partial \Pi|)g(|\partial \Pi|)$. The step history of these trapezia has length $O(f(|\partial \Pi|)^3) = O(|\partial \Pi|)$ by Lemma 4.23. Therefore the length of every $\theta$-band in it is $O(|\partial \Pi|)$ by Lemmas 4.17 and 6.1 (a, d). The statement of the lemma follows. \hfill $\square$

By definition, the $G$-area of a minimal diagram $\Delta$ over $G$ is the sum of $G$-areas of its disks plus the $G$-area of the compliment. For the compliment, as in subsection 6.3, we consider a family $S$ of big subtrapezia and single cells of $\Delta$ such that every cell of $\Delta$ belongs to a member $\Sigma$ of this family, and if a cell $\Pi$ belongs to different $\Sigma_1$ and $\Sigma_2$ from $\Sigma$, then both $\Sigma_1$ and $\Sigma_2$ are big subtrapezia of $\Delta$ with bases $xv_1x, xv_2x$, and $\Pi$ is an $(\theta, x)$-cell.) Hence the statement of Lemma 6.9 holds for minimal diagrams over $G$ as well.

We want to prove that for big enough constants $N_3$ and $N_4$, $\text{Area}(\Delta) \leq N_4F(n + \sigma_\lambda(\Delta)) + N_3\mu(\Delta)g(n)$ for every minimal diagram $\Delta$ with perimeter $n$. For this goal, we will argue by contradiction in this section and study a counter-example with minimal $n + \sigma_\lambda(\Delta)$.
Lemma 9.2. The diagram $\Delta$ has no rim $\theta$-bands with base of length at most $K$.

Proof. The functions $F(x)$ and $g(x)$ satisfy the definition given for $\Phi(x)$ and $\phi(x)$, and the inequality (6.14) by Lemma 4.18. Hence the proof of Lemma 6.15 works for the minimal counter-example over $G$. It suffices to replace $N_2$ and $N_1$ by $N_4$ and $N_3$, replace $n$ by $n + \sigma_\lambda(\Delta)$, and notice that the value of $\sigma_\lambda$ does not increase when passing from $\Delta$ to $\Delta'$.

By Lemma 6.16, $\Delta$ has at least one disk. Using Lemma 7.2, we fix a disk $\Pi$ in $\Delta$ such that $L - 3$ consecutive maximal $t$-bands $B_1, \ldots, B_{L-3}$ start on $\partial \Delta$, end on the boundary $\partial \Pi$, and for any $i \in [1, L - 4]$, there are no disks in the subdiagram bounded by $B_i, B_{i+1}, \partial \Pi$, and $\partial \Delta$. (See fig. 10.)

We denote by $\Psi = cl(\Pi, B_1, B_{L-3})$ the subdiagram without disks bounded by the spokes $B_1, B_{L-3}$ (and including them) and by subpaths of the boundaries of $\Delta$ and $\Pi$, and call this subdiagram a clove. Similarly one can defined the cloves $\Psi_{ij} = cl(\Pi, B_i, B_j)$ if $1 \leq i < j < L - 3$.

Lemma 9.3. The clove $\Psi = cl(\Pi, B_1, B_{L-3})$ has no subcombs of basic width at least $K_0$.

Proof. Proving by contradiction, we may assume that there is a tight subcomb $\Gamma$ by Lemma 6.6 (2). Then contradiction appears exactly as in cases 1 and 3 of the proof of Lemma 6.16, since Lemma 4.18 allows us to define $\Phi(x) = F(x)$. (Case 2 is eliminated by Lemma 9.2.) It suffices to replace $N_2$ and $N_1$ by $N_4$ and $N_3$, replace $n$ by $n + \sigma_\lambda(\Delta)$, and notice that the value of $\sigma_\lambda$ does not increase when passing from $\Delta$ to $\Delta_0$ since no $t$-band of $\Gamma$ is a part of a spoke.

The statements of auxiliary lemmas 6.12, 6.14 and 6.31 holds as well for the minimal counter-example over $G$. Below we use the following analog of Lemma 6.12:

Lemma 9.4. (1) The counter-example $\Delta$ has no two disjoint subcombs $\Gamma_1$ and $\Gamma_2$ of basic widths at most $K$ with handles $C_1$ and $C_2$ such that some ends of these handles are connected by a subpath $x$ of the boundary path of $\Delta$, where $x$ has no $q$-edges.

(2) The boundary of every subcomb $\Gamma$ with basic width $s \leq K$ has $2s$ $q$-edges.

Lemma 9.5. (1) Every maximal $\theta$-band of $\Psi$ crosses either $B_1$ or $B_{L-1}$.

(2) There exists $r$, $L/2 - 3 \leq r \leq L/2$, such that the $\theta$-bands of $\Psi$ crossing $B_{L-3}$ do not cross $B_r$, and the $\theta$-bands of $\Psi$ crossing $B_1$ do not cross $B_{r+1}$.

Proof. (1) If the claim were wrong, then one could find a rim $\theta$-band $T$ in $\Psi$, which crosses neither $B_1$ nor $B_{L-3}$. By Lemma 9.2, the basic width of $T$ is greater than $K$. Since (1) a disk has $LN$ spokes, (2) no $q$-band of $\Psi$ intersects $T$ twice by Lemma 5.3 (3), $T$ has at least $K$ $q$-cells, and (4) $K > 2K_0 + LN$, there exists a maximal $q$-band $C'$ such that a subdiagram $\Gamma'$ separated from $\Psi$ by $C'$ contains no edges of the spokes of $\Pi$ and the part of $T$ belonging to $\Gamma'$ has at least $K_0$ $q$-cells (fig. 15).

If $\Gamma'$ is not a comb, and so a maximal $\theta$-band of it does not cross $C'$, then $\Gamma'$ must contain another rim band $T'$ having at least $K$ $q$-cells. This makes possible to find a subdiagram $\Gamma''$ of $\Gamma'$ such that a part of $T'$ is a rim band of $\Gamma''$ containing at least $K_0$ $q$-cells, and $\Gamma''$ does not contain $C'$. Since $\text{Area}(\Gamma') > \text{Area}(\Gamma'') > \ldots$, such a procedure must stop. Hence, for some $i$, we obtain a subcomb $\Gamma^{(i)}$ of basic width $\geq K_0$, contrary to Lemma 9.3.
(2) Assume there is a maximal \( \theta \)-band \( T \) of \( \Psi \) crossing the spoke \( B_1 \). Then assume that \( T \) is the closest to the disk \( \Pi \), i.e. the intersection of \( T \) and \( B_1 \) is the first cell of the spoke \( B_1 \). If \( B_1, \ldots, B_r \) are all the spokes crossed by \( T \), then \( r \leq L/2 \) by Lemma 6.3. Since the band \( T \) does not cross the spoke \( B_{r+1} \), no other \( \theta \)-band of \( \Psi \) crossing \( B_1 \) can cross \( B_{r+1} \). The same argument shows that \( r + 1 \geq L/2 - 2 \) if there is a \( \theta \)-band of \( \Psi \) crossing the spoke \( B_{L-3} \). 

For the clove \( \Psi = cl(\pi, B_1, B_{L-3}) \) in \( \Delta \), we denote by \( p(\Psi) \) the common subpath of \( \partial \Psi \) and \( \partial \Delta \) starting with the \( t \)-edge of \( B_1 \) and ending with the \( t \)-edge of \( B_{L-3} \). Similarly we define the (outer) path \( p_{ij} = p(\Psi_{ij}) \) for every smaller clove \( \Psi_{ij} \).

**Lemma 9.6.** Every path \( p_{i,i+1} \) \((i = 1, \ldots, L - 4)\) has less than \( 3K_0 q \)-edges.

**Proof.** Let a maximal \( q \)-band \( C \) of \( \Psi \) starts on \( p_{i,i+1} \) and does not end on \( \Pi \). Then is has to end on \( p_{i,i+1} \) too. If \( \Gamma \) is the subdiagram without disks separated by \( C \), then every maximal \( \theta \)-band of \( \Gamma \) has to cross the \( q \)-band \( C \) since its extension in \( \Psi \) must cross either \( B_1 \) or \( B_{L-3} \) by Lemma 9.5. Therefore \( \Gamma \) is a comb with handle \( C \).

Consider the \( q \)-bands of this kind defining maximal subcombs \( \Gamma_1, \Gamma_2, \ldots, \Gamma_k \) in \( \Psi_{i,i+1} \). The basic width of each of them is less than \( K_0 \) by Lemma 9.3. Therefore \( k \leq 1 \) since otherwise one can get two subcombs contradicting to Lemma 9.4 (1), because there are at most \( N + 1 \) maximal \( q \)-bands starting on \( \partial \Pi \) in \( \Psi_{i,i+1} \). By Lemma 9.4 (2), such a subcomb has at most \( 2K_0 q \)-edges in the boundary. Hence there are at most \( 2K_0 + N < 3K_0 \) \( q \)-edges in the path \( p_{i,i+1} \). 

We denote by \( \bar{\Delta} \) the subdiagram formed by \( \Pi \) and \( \Psi \), and denote by \( \bar{\mathbf{p}} \) the path \( \text{top}(B_1)u^{-1}\text{bot}(B)_{L-3}^{-1} \), where \( u \) is a subpath of \( \partial \Pi \), such that \( \bar{\mathbf{p}} \) separates \( \bar{\Delta} \) from the remaining subdiagram \( \Psi' \) of \( \Delta \) (fig. 16).

Similarly we define subdiagrams \( \bar{\Delta}_{ij} \), paths \( \bar{\mathbf{p}}_{ij} = \text{top}(B_1)u_{ij}^{-1}\text{bot}(B)_j^{-1} \), where \( u_{ij} \) is a subpath of \( \partial \Pi \), and the subdiagrams \( \Psi'_{ij} \).

We denote by \( H_1, \ldots, H_{L-3} \) the histories of the spokes \( B_1, \ldots, B_{L-3} \) (read starting from the disk \( \Pi \)) and by \( h_1, \ldots, h_{L-3} \) their lengths, i.e. the numbers of \((\theta, t)\)-cells. By Lemma 9.5, these lengths non-increase and then non-decrease as follows:
Figure 16: Boundaries of Ψ and Ψ′

\[ h_1 \geq h_2 \geq \cdots \geq h_r; \quad h_{r+1} \leq \cdots \leq h_{L-3} \quad (L/2 - 3 \leq r \leq L/2), \quad (9.54) \]
and therefore \( H_{i+1} \) is a prefix of \( H_i \) \((H_j \text{ is a prefix } H_{j+1})\) for \( i = 1, \ldots, r - 1 \) (resp., for \( j = r + 1, \ldots, L - 4 \)).

Recall that the boundary label of \( \partial \Pi \) has the form \( W^L \), i.e., it is the \( L \)-th power of an accepted word \( W \).

Lemma 9.7. We have the following inequalities

\[ |\mathbf{p}_{ij}| \leq h_i + h_j + (L - j + i)|W| - 1 \]
and, if \( i \leq r \) and \( j \geq r + 1 \), then

\[ |\mathbf{p}_{ij}| \geq |\mathbf{p}_{ij}|_q \geq h_i + h_j + (j - i)N + 1 \]

Proof. The first inequality follows from Lemma 6.1 (b) since the path \( \mathbf{u}_{ij} \) has \( L - j + i - 1 \) \( t \)-edges. To prove the second inequality, we observe that the path \( |\mathbf{p}_{ij}| \) has \( (j - i)N + 1 \) \( q \)-edges and it has \( h_i + h_j \) \( \theta \)-edges by Lemma 9.5.

The large constants \( L \) and \( L_0 \) are chosen so that

\[ L_0^3 \leq L \quad (9.55) \]

Lemma 9.8. If \( j - i > L/2 \), then we have \( \mu(\Delta) - \mu(\Psi'_{ij}) > -2Jn(h_i + h_j) \geq -2Jn|\mathbf{p}_{ij}|. \)

Proof. The number of \( q \)-edges in the path \( \mathbf{p}_{ij} \) (or in the path \( \mathbf{u}_{ij} \)) does not exceed the similar number for \( \mathbf{p}_{ij} \), provided \( j - i \geq L/2 \). Therefore any two white beads \( o, o' \) of the necklace on \( \partial \Delta \), provided they both do not belong to \( \mathbf{p}_{ij} \), are separated by at least the same number of black beads in the necklace for \( \Delta \) as in the necklace for \( \Psi'_{ij} \) (either the clockwise arc \( o - o' \) includes \( \mathbf{p}_{ij} \) or not). So such a pair contributes to \( \mu(\Delta) \) at least the amount it contributes to \( \mu(\Psi'_{ij}) \). Thus, to estimate \( \mu(\Delta) - \mu(\Psi'_{ij}) \) from below, it
suffices to consider the contribution to $\mu(\Psi')$ for the pairs $o, o'$, where one of the two beads lies on $p_{ij}$. The number of such (unordered) pairs is bounded by $n(h_i + h_j)$ Taking into account the definition of $\mu$ for diagrams and inequalities (9.54), we get the required inequality.

\[ \text{Lemma 9.9. If } j - i > L/2, \text{ then the following inequality holds: } |p_{ij}| < (1 + \varepsilon)|\bar{p}_{ij}|, \text{ where } \varepsilon = N_4^{-1/2}. \text{ Moreover, we have } |p_{ij}| + \sigma_\lambda(\bar{\Delta}_{ij}) < (1 + \varepsilon)|\bar{p}_{ij}|. \]

\[ \text{Proof. It suffices to prove the second statement. Let } d \text{ be the difference } |p_{ij}| + \sigma_\lambda(\bar{\Delta}_{ij}) - |\bar{p}_{ij}| \text{ and assume to contradiction that } d \geq \varepsilon|\bar{p}_{ij}|. \text{ Then } d \geq (1 + \varepsilon)^{-1}(d)\geq (1 + \varepsilon)^{-1}(d)|p_{ij}| + \sigma_\lambda(\bar{\Delta}_{ij}) \geq \frac{\varepsilon}{2}|p_{ij}| + \sigma_\lambda(\bar{\Delta}_{ij}) \geq \frac{\varepsilon y}{2}, \] where by definition, $y = p_{ij} + \sigma_\lambda(\bar{\Delta}_{ij})$.

We have $(d|p_{ij}| + \sigma_\lambda(\bar{\Delta}_{ij})) \geq d > 0$, because $|d| - |\partial \Psi'_{ij}| \geq |p_{ij}| - |\bar{p}_{ij}|$ and $\sigma_\lambda(\bar{\Delta}_{ij}) + \sigma_\lambda(\bar{\Delta}_{ij}) \leq \sigma_\lambda(\Delta)$. Therefore for $x = n + \sigma_\lambda(\Delta)$, we obtain from the minimality of the counter-example $\Delta$ that

\[ \text{Area}_G(\Psi'_i) \leq N_4 F(x - d) + N_3 \mu(\Psi'_{ij})g(n) \leq N_4 F(x) - N_4 F(x) x^{-1} d + N_3 \mu(\Delta) g(n) \]

\[ + 2N_3 J_n|p_{ij}|g(n) \leq N_4 F(x) + N_3 \mu(\Delta) g(n) - N_4 F(x) x^{-1} d + 2N_3 J_n y g(n) \quad (9.57) \]

by Lemma 9.8, inequality $\sigma_\lambda(\Psi'_{ij}) \leq \sigma_\lambda(\Delta)$, and Lemma 4.18. By Lemma 9.7, $|\bar{p}_{ij}| < |p_{ij}| + |\partial \Pi|$, and so the perimeter $|\partial \Psi'_{ij}|$ is less than $2|p_{ij}| + |\partial \Pi|$. Since $|\partial \Pi| \leq L|\bar{p}_{ij}| < L(|p_{ij}| + \sigma_\lambda(\bar{\Delta}_{ij}))$, we have

\[ |\partial \Psi_{ij}| < (2 + L)|p_{ij}| + L \sigma_\lambda(\bar{\Delta}_{ij}) \leq (L + 2)y \quad (9.58) \]

By the inequalities $N_2 > N_1$, (9.58), lemmas 6.16 and 6.7 (a), the $G$-area of $\Psi'_{ij}$ does not exceed

\[ N_2(2 + L)^2 y^2 + N_1 \mu(\Psi'_{ij}) \leq N_2(J + 1)(2 + L)^2 y^2 \quad (9.59) \]

By Lemma 9.1, the $G$-area of $\Pi$ does not exceed $c_6 F(|\partial \Pi|) \leq c_6 F((L + 2)y)$, and by definition of the functions $f$ and $F$, there is a constant $c_7 = c_7(L)$ such that $\text{Area}_G(\Pi) \leq c_7 F(y)$.

This estimate and (9.59) give the inequality $\text{Area}_G(\bar{\Delta}_{ij}) \leq N_2(J + 1)(2 + L)^2 y^2 + c_7 F(y)$, and we obtain with (9.57) that

\[ \text{Area}_G(\Delta) \leq N_4 F(x) + N_3 \mu(\Delta) g(n) \]

\[ - N_4 F(x) x^{-1} d + 2N_3 J_n g(n) + N_2(J + 1)(2 + L)^2 y^2 + c_7 F(y) \]

To obtain the desired contradiction, it suffices to show that here, the number $T = N_4 F(x) x^{-1} d/3$ is greater than each of the last three summands. Recall that $F(x) x^{-1} = x g(x) \geq n g(n)$, $d > \varepsilon y/2$ by (9.56), $\varepsilon = N_4^{-1/2}$, and so $T > 2N_3 J_n g(n)$ if $N_4$ is large enough in comparison with $N_3$ and other constant chosen earlier. Also we have $T > N_2(J + 1)(2 + L)^2 y^2$, because $x = n + \sigma_\lambda(\Delta) > p_{ij} + \sigma_\lambda(\bar{\Delta}_{ij}) = y$, and so $F(x) x^{-1} d > x g(x) \varepsilon y/2 \geq \varepsilon y^2/2$. Finally, $T > c_7 F(y)$ since

\[ F(x) x^{-1} d > x g(x) \varepsilon y/2 \geq y^2 g(y) \varepsilon/2 = \varepsilon F(y)/2 \]
For every path $p_{i,i+1}$ we will fix a shortest path $q_{i,i+1}$ homotopic to $p_{i,i+1}$ in the subdiagram $\Psi_{ij}$, such that the first and the last $t$-edges of $q_{i,i+1}$ coincide with the first and the last $t$-edges of $p_{i,i+1}$. For $j > i + 1$ the path $q_{i,j}$ is formed by $q_{i,i+1}, \ldots q_{j-1,j}$. The following lemma is similar to the second part of Lemma 9.7.

**Lemma 9.10.** If $i \leq r$ and $j \geq r + 1$, then

$$|q_{ij}| \geq |q_{ij}|_{\theta} + |q_{ij}|_{q} \geq h_{i} + h_{j} + (j - i)N + 1$$

□

Let $\Psi_{ij}^{0}$ (let $\Psi^{0}$, $\Delta^{0}$) be the subdiagram of $\Psi_{ij}$ (of $\Psi$, of $\Delta$) obtained after replacement of the subpath $p_{ij}$ (of $p$) by $q_{ij}$ (by $q = q_{1,L-3}$, resp.) in the boundary.

**Lemma 9.11.** The subdiagram $\Psi_{ij}^{0}$ has no maximal $q$-bands except for the $q$-spokes starting from $\partial \Pi$. Every $\theta$-band of $\Psi_{ij}^{0}$ is crossed by the path $q_{ij}$ at most once.

**Proof.** Assume there is a $q$-band $Q$ of $\Psi_{ij}^{0}$ starting and ending on $q_{ij}$. Then $j = i + 1$ and $q_{i,i+1} = u e f w v$, where $Q$ starts with the $q$-edge $e$ and ends with the $q$-edge $f$. Let $Q$ has length $\ell$. Then $|v| \geq \ell$ since every maximal $\theta$-band of $\Psi_{i,i+1}^{0}$ crossing $Q$ has to end on the subpath $v$. So one has $|e f w v| = \ell + 2$, and replacing the subpath $e f w v$ by a side of $Q$ of length $\ell$ one replaces the path $q_{i,i+1}$ with a shorter homotopic path by Lemma 6.1. This contradicts to the choice of $q_{i,i+1}$, and so the first statement is proved. The prove of the second statement is similar. □

It follows from Lemma 9.5 that between the spokes $B_{j}$ and $B_{j+1}$ (1 $\leq j \leq r - 1$), there is a trapezium $\Gamma_{j}$ of height $h_{j+1}$ with the side $t$-bands $B_{j+1}$ and $B_{j}'$, where $B_{j}'$ is the beginning of length $h_{j+1}$ of the $t$-spoke $B_{j}$. Similarly, we have trapezia $\Gamma_{j}$ for $r + 1$ $\leq j \leq L - 4$. By Lemma 9.11 (2), every trapezium $\Gamma_{j}$ is contained in both $\Psi_{j,j+1}$ and $\Psi_{j+1,j+1}$. The bottoms $y_{j}$ of all trapezia $\Gamma_{j}$ belong to $\partial \Pi$ and have the same label $W t$.

We will use $z_{j}$ for the tops of these trapezia. Since $\Gamma_{j}$ and $\Gamma_{j-1}$ (2 $\leq j \leq r - 1$) have the same bottom labels and the history $H_{j}$ is a prefix of $H_{j-1}$, by Lemma 5.7, $h_{j}$ different $\theta$-bands of $\Gamma_{j-1}$ form the copy $\Gamma_{j}'$ of the trapezium $\Gamma_{j}$ with top and bottom paths $z_{j}'$ and $y_{j}' = y_{j-1}$.

We denote by $E_{j}$ (by $E_{j}^{0}$) the comb formed by the maximal $\theta$-bands of $\Psi_{j,j+1}$ (of $\Psi_{j,j+1}^{0}$, resp.) crossing the $t$-spoke $B_{j}$ but not crossing $B_{j+1}$ (1 $\leq j \leq r - 1$, see fig. 17). Its handle $C_{j}$ of height $h_{j} - h_{j+1}$ is contained in $B_{j}$. The boundary $\partial E_{j}$ consists of the side of this handle, the path $z_{j}$ and the path $p_{i,j+1}$ (the path $q_{i,j+1}$).

Assume that a maximal $a$-band $A$ of $E_{j}^{0}$ (2 $\leq j \leq r - 1$) starts on the path $z_{j}$ and ends on a $a$-edge of a maximal $q$-band $C$ of $E_{j}^{0}$. Then $A$, a part of $C$ and a part $z$ of $z_{j}$ bound a comb $\nabla$.

**Lemma 9.12.** There is a copy of the comb $\nabla$ in the trapezium $\Gamma = \Gamma_{j-1}\setminus \Gamma_{j}'$.

**Proof.** The subpath $z$ of $z_{j}$ starts with and $a$-edge $e$ and ends with a $q$-edge $f$. There is a copy $z'$ of $z$ in $z_{j}'$ starting with $e'$ and ending with $f'$. Note that the $\theta$-cells $\pi$ and $\pi'$ attached to $f$ and to $f'$ in $\nabla$ and in $\Gamma$ are copies of each other since they correspond to the same letter of the history. Now moving from $f$ to $e$, we see that the whole maximal $\theta$-band $T_{j}$ of $\nabla$ containing $\pi$ has a copy in $\Gamma$. Similarly we obtain a copy of the next maximal $\theta$-band $T_{2}$ of $\nabla$, and so on. □
Lemma 9.13. At most $N$ $a$-bands starting on the path $y_j$ can end on the $(\theta, q)$-cells of the same $\theta$-band. This property holds for the $a$-bands starting on $z_j$ too.

Proof. We will prove the second claim only since the proof of the first one is similar. Assume that the $a$-bands $A_1, \ldots, A_s$ start from $z_j$ and end on some $(\theta, q)$-cells of a $\theta$-band $T$. Let $T_0$ be the minimal subband of $T$, where the $a$-bands $A_2, \ldots, A_{s-1}$ end and $\bar{z}_j$ be the minimal subpath of $z_j$, where they start. Then by Lemma 5.3, every maximal $q$-band starting on $\bar{z}_j$ has to cross the band $T_0$ and vice versa. Hence the base of $T_0$ is a subbase of the standard base (or of its inverse). Since every rule of $M$ can change at most $N-2$ $a$-letters in a word with standard base, all $(\theta, q)$-cells of $T_0$ have at most $N-2$ $a$-edges, and the statement of the lemma follows.

Without loss of generality, we assume that

$$h = h_{L_0+1} \geq h_{L-L_0-3}$$

(9.60)

Lemma 9.14. If $h \leq L_0^2|W|_a$, then the number of trapezia $\Gamma_j$ with the properties $|z_j|_a \geq |W|_a/c_5N$ for $j \in [L_0 + 1, r - 1]$ or $j \in [r + 1, L - L_0 - 5]$, is less than $L/5$.

Proof. Consider $\Gamma_j$ as in the assumption of the lemma with $j \in [L_0 + 1, r - 1]$. The subcomb $E_j^0$ has at most $N$ maximal $q$-bands by Lemma 9.11. So there are at most $N$ maximal $a$-bands starting on $z_j$ and ending on each of the $\theta$-bands of $E_j^0$. Proving by contradiction, we have at least $L|W|_a/5c_5N$ such $a$-bands for all $j \in S$, where $S$ the set of integers in $[L_0 + 1, r - 1] \cup [r + 1, L - L_0 - 5]$; denote this set of $a$-bands by $A$. But the number of maximal $\theta$-bands in all such subcombs $E_j^0$ does not exceed $2h$. Therefore at least $L|W|_a/5c_5N - 2hN$ bands from $A$ end on the subpaths $q_{j,j+1}$ for $j \in S$. Therefore by Lemmas 9.10 and 6.1, we have

$$|p_{L_0+1,L-L_0-5}| \geq |q_{L_0+1,L-L_0-5}| \geq h_{L_0+1} + h_{L-L_0-5} + LN/2 + \delta(L|W|_a/5c_5N - 2hN)$$

$$\geq h_{L_0+1} + h_{L-L_0-5} + LN/2 + \delta L|W|_a/10c_5N$$

(9.61)
since \( 2hN \leq 2L_0^2N|W|_a < L_0^3|W|_a/10c_5N \leq L|W|_a/10c_5N \) by the choice of \( L_0 \) and \( L \) (9.55).

Also by Lemma 9.7, we have

\[
\left| p_{L_0+1,L-L_0-5} \right| \leq h_{L_0+1} + h_{L-L_0-5} + 3L_0N + 3L_0\delta|W|_a
\]

\[
\leq h_{L_0+1} + h_{L-L_0-5} + 3L_0N + \delta L|W|_a/20c_5N,
\]

(9.62)

because by the choice of \( L, 3L_0 < L/20c_5N \). Since \( h_{L_0+1} + h_{L-L_0-5} \leq 2h \leq 2L_0^2|W|_a < L|W|_a \), \( L \) is chosen after \( c_5N \), and \( \varepsilon = N_4^{-1/2} \) is chosen after \( L \), the inequality

\[
\left| p_{L_0+1,L-L_0-5} \right| \geq 1 + \frac{\delta}{20c_5N} > 1 + \varepsilon
\]

follows from (9.61, 9.62), contrary to Lemma 9.9. The lemma is proved by contradiction. \( \square \)

**Lemma 9.15.** If \( h \leq L_0^3|W|_a \), then the histories \( H_1 \) and \( H_{L-3} \) have different first letters.

**Proof.** Let \( T \) and \( S \) be the maximal \( \theta \)-bands of \( \Psi \) crossing \( B_1 \) and \( B_{L-3} \), respectively, and the closest to the disk \( \Pi \). Let they cross \( k \) and \( \ell \) spokes of \( \Pi \), respectively. By Lemma 9.14, \( k + \ell > L - L/5 - 3L_0 > 2L/3 \), and also \( k, \ell \geq 2 \) since \( L/2 - 3 \leq r \leq L/2 \). It follows from Lemma 7.5 that the first letters of \( H_1 \) and \( H_{L-3} \) are different. \( \square \)

**Lemma 9.16.** We have \( h > L_0^3|W|_a \).

**Proof.** If this inequality is wrong, then by Lemma 9.14, there are at least \( L - L/5 - 3L_0 > 0.7L \) trapezia \( \Gamma_j \) with \( |z_j|_a < |W|_a/c_5N \), and one can choose two such trapezia \( \Gamma_k \) and \( \Gamma_\ell \) such that \( k < r, \ell \geq r + 1 \) and \( \ell - k > 0.6L \). Since \( H_{k+1} \) (resp. \( H_\ell \)) is a prefix of \( H_1 \) (of \( H_{L-3} \)), it follows from Lemma 9.15 that the first letters of \( H_{k+1} \) and \( H_\ell \) are different. Since the bottoms of \( \Gamma_k \) and \( \Gamma_\ell \) (which belong in \( \partial \Delta \)) have the same label, one can construct an auxiliary trapezium \( E \) identifying the bottom of a copy of \( \Gamma_k \) and the bottom of a mirror copy of \( \Gamma_\ell \). The history of \( E \) is \( H^{-1}_\ell H_{k+1} \), which is a reduced word since the first letters of \( H_k \) and \( H_\ell \) are different, i.e. \( E \) is a trapezium indeed by Lemma 5.7.

The top and the bottom of \( E \) have \( a \)-lengths less than \( |W|_a/c_5N \). Without loss of generality, one may assume that \( h_{k+1} \geq h_\ell \), and so \( h_{k+1} \geq t/2 \), where \( t \) is the height of \( E \).

Note that the difference of \( a \)-lengths \( |W|_a - |W|_a/c_5N > |W|_a/2 \), and so

\[
h_{k+1}, h_\ell > |W|_a/2N
\]

(9.63)

since the difference of \( a \)-lengths for the top and the bottom of every maximal \( \theta \)-band of \( E \) does not exceed \( N \). Therefore \( t > |W|_a/N \), and the computation corresponding \( E \) satisfies the assumption of Lemma 4.26.

So for every factorization \( H'H''H''' \) of the history of \( \Gamma_k \), where \( ||H'|| + ||H'''|| < \lambda ||H'H''H'''|| \), we have \( ||H'''|| > 0.4t \), since \( \lambda < 1/5 \). Therefore by Lemma 4.26, the spoke \( B_{k+1} \) is a \( \lambda \)-shaft.

Using Lemma 9.7, we obtain:

\[
|p_{k+1,|\sigma_\lambda(\Delta_{k+1,\ell})|} \geq h_{k+1} + h_\ell + 0.6LN + h_{k+1}
\]

(9.64)
By inequality (9.63), we have $\delta L|W|_a \leq 2LN \delta h_{k+1} < h_{k+1}$ by the choice of $\delta$ and by Lemma 9.7,

$$|\bar{p}_{k+1,l}| \leq h_{k+1} + h_{\ell} + 0.4LN + 0.4L \delta |W|_a \leq h_{k+1} + h_{\ell} + h_{k+1}/2$$ (9.65)

The right-hand side of the inequality (9.64) divided by the right-hand side of (9.65) is greater than 1.1 (because $h_{k+1} \geq h_{\ell}$), which contradicts to Lemma 9.9. Thus, the lemma is proved.

**Lemma 9.17.** We have $h_i > \delta^{-1}$ for every $i = 1, \ldots, L_0$.

*Proof.* By inequalities (9.60) and (9.54), we have $h_i \geq h_{L-L_0-3}$. Proving by contradiction, we obtain $|W|_a < h_i \leq \delta^{-1}$ by Lemma 9.16. Then

$$|\bar{p}_{i,L-L_0-3}| < h_i + h_{L-L_0-3} + 3L_0(N + \delta^{-1}\delta) \leq h_i + h_{L-L_0-3} + 4L_0N$$

by Lemma 9.7, and $|\bar{p}_{i,L-L_0-3}| \geq h_i + h_{L-L_0-3} + LN/2$. Since $h_i + h_{L-L_0-3} \leq 2\delta^{-1}$ and $4L_0N < LN/4$, we see that $|\bar{p}_{i,L-L_0-3}| > 1 + \delta > 1 + \varepsilon$ contrary to Lemma 9.9. The lemma is proved by contradiction.

**Lemma 9.18.** None of the spokes $B_1, \ldots, B_{L_0}$ contains a $\lambda$-shaft at $\Pi$ of length at least $\delta h$. 

*Proof.* On the one hand, by Lemmas 9.7 and 9.16,

$$|\bar{p}_{L_0+1,L-L_0-3}| < h_{L_0+1} + h_{L-L_0-3} + 3L_0(N + \delta |W|_a) < h_{L_0+1} + h_{L-L_0-3} + 3L_0(N + \delta L_0^{-2}h)$$ (9.66)

On the other hand, by Lemma 9.7,

$$|\bar{p}_{L_0+1,L-L_0-3}| > h_{L_0+1} + h_{L-L_0-3} + (L - 3L_0)N$$ (9.67)

If the statement of the lemma were wrong, then we would have $\sigma_\lambda(\bar{\Delta}) \geq \delta h$, and inequalities (9.66) and (9.67) would imply that

$$|\bar{p}_{L_0+1,L-L_0-3}| - |\bar{p}_{L_0+1,L-L_0-3}| + \sigma_\lambda(\bar{\Delta}) \geq (L - 6L_0)N - 3L_0^{-1}\delta h + \delta h \geq LN/2 + \delta h/2$$

The right-hand side of the last inequality divided by the right-hand side of (9.66) is greater than $\varepsilon = N^{-1/2}$, because $h \geq h_{L_0+1}, h_{L-L_0-3}$, which would contradict to Lemma 9.9. Thus, the lemma is proved.

**Lemma 9.19.** For every $j \in \{1, L_0 - 1\}$, we have $|z_j|_a > h_{j+1}/c_5$.

*Proof.* If $|z_j|_a \leq h_{j+1}/c_5$, then the computation $C : W_0 \rightarrow \cdots \rightarrow W_t$ corresponding to the trapezium $\Gamma_j$ satisfies the assumption of Lemma 4.26, since $t = h_{j+1} > c_5 |W_t|_a = c_5 |z_j|_a$ and by Lemma 9.16, $t = h_{j+1} \geq L_0^2|W_0|_a \geq c_5 |W|_a$ since $L_0 > c_5$. Hence $B_{j+1}$ is a $\lambda$-shaft by Lemma 4.26 since $\lambda < 1/2$. We obtain a contradiction with Lemma 9.18 since $\delta h \leq h \leq h_{j+1}$. Thus, the lemma is proved.

**Lemma 9.20.** For every $j \in \{1, L_0 - 1\}$, we have $h_{j+1} < (1 - \frac{1}{10ckN})h_j$. 

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Proof. By Lemma 9.19, we have \(|z_j|_a \geq h_{j+1}/c_5\). Let us assume that \(h_{j+1} \geq (1 - 1/10c_5)h_j\), that is the handle \(C_j\) of \(E_j\) has height at most \(h_j/10c_5 N\). By Lemma 9.13, at most \(h_j/10c_5\) maximal \(a\)-bands of \(E_j\) starting on \(z_j\) can end on the \((\theta, q)\)-cells of \(E_j\). Hence at least

\[
|z_j|_a - h_j/10c_5 \geq |z_j|_a - 2h_{j+1}/10c_5 \geq 0.8h_{j+1}/c_5 > 0.7h_j/c_5
\]
of them have to end on the path \(p_{j,j+1}\).

The path \(p_{j,j+1}\) has at most \(h_j/10c_5\) \(\theta\)-edges. Hence by Lemma 6.1,

\[
|p_{j,j+1}| \geq h_j - h_{j+1} + \delta(0.7h_j/c_5 - h_j/10c_5 N) \geq h_j - h_{j+1} + 0.6\delta h_j/c_5,
\]
and therefore by Lemma 9.7, \(|p_{j,j+1}| \geq LN/2 + h_j + h_{L-L_0-3} + 0.6\delta h_j/c_5\). On the other hand by Lemma 9.7, we have

\[
|\bar{p}_{j,L-L_0-3}| \leq h_j + h_{L-L_0-3} + 3NL_0 + 3L_0\delta|W|_a \leq h_j + h_{L-L_0-3} + 3NL_0 + 3L_0\delta h_j + 1
\]
by Lemma 9.16 and inequality \(h \leq h_{j+1}\). Hence \(|\bar{p}_{j,L-L_0-3}| \geq (1 + \delta/10c_5)\) since \(h_{L-L_0-3} \leq h_{L_0+1} \leq h_{j+1} \leq h_j\) and \(L_0 >> c_5\). We have a contradiction with Lemma 9.9 since \(\delta/10c_5 > \varepsilon\). The lemma is proved by contradiction.

The proof of the next lemma is similar.

Lemma 9.21. For every \(j \in [2, L_0 - 1]\), we have \(|z_j|_a \leq 2Nh_j\),

Proof. Assume that \(|z_j|_a \geq 2Nh_j\). By Lemma 9.13, at most \(Nh_j\) maximal \(a\)-bands of \(E_j\) starting on \(z_j\) can end on the \((\theta, q)\)-cells of \(E_j\). Hence at least \(|z_j|_a - Nh_j \geq Nh_j\) of them has to end on the path \(p_{j,j+1}\). The path \(p_{j,j+1}\) has at most \(Nh_j\) \(\theta\)-edges. Hence by Lemma 6.1, \(|p_{j,j+1}| \geq h_j - h_{j+1} + \delta(Nh_j - h_j) = h_j - h_{j+1} + \delta(N - 1)h_j\) and therefore by Lemma 9.7, \(|p_{j,j+1}| \geq LN/2 + h_j + h_{L-L_0-3} + \delta(N - 1)h_j\). On the other hand by Lemmas 9.7 and 9.16, we have

\[
|\bar{p}_{j,L-L_0-3}| \leq h_j + h_{L-L_0-3} + 3NL_0 + 3L_0\delta|W|_a \leq h_j + h_{L-L_0-3} + 3NL_0 + \frac{3\delta h_j}{L_0}
\]
because \(h \leq h_j\). Since \(h_j \geq h \geq h_{L-L_0-3}\), we have \(|\bar{p}_{j,L-L_0-3}| \geq (1 + \varepsilon)\), a contradiction by Lemma 9.9.

Lemma 9.22. There is no \(i \in [2, L_0 - 3]\) such that the histories \(H_{i-1} = H_i H' = H_{i+1} H'' H' = H_{i+2} H''' H'' H'\) and the computation \(C\) with history \(H_i\) corresponding to the trapezium \(\Gamma_{i-1}\) satisfy the following condition:

(*) The history \(H''' H'' H'\) has only one step, and for the subcomputation \(D\) with this history, there is a sectors \(Q' Q\) such that a state letter from \(Q\) or from \(Q'\) inserts a letter increasing the length of this sector after every transition of \(D\).

Proof. Recall that the standard base of \(M\) is built of the standard base \(B\) of \(M_1\) and its inverse copy \((B')^{-1}\) (plus letter \(t\)). Due to this mirror symmetry of the standard base, we have mirror symmetry for any accepting computation, in particular, for \(C\) and \(D\). Therefore proving by contradiction, we may assume that the \(a\)-letters are inserted from the left of \(Q\).
Let $Q$ be the maximal $q$-spoke of the subdiagram $E_i^0 \subset \Gamma_i$ corresponding to the base letter $Q$. If $Q'$ is the neighbor from the left $q$-spoke for $Q$ (the spokes are directed from the disk $\Pi$), then the subpath $x$ of $z_i$ between these two $q$-spokes has at least $h_{i+1} - h_{i+2} = ||H''||$ $a$-letters. Indeed, $\Gamma_i$ contains a copy $\Gamma'_{i+1}$ of $\Gamma_{i+1}$, the bottom of the trapezium $\Gamma_i \setminus \Gamma'_{i+1}$ is the copy $z_{i+1}$ of $z_{i+1}$ and the top of it is the $z_i$, and so the subcomputation with history $H''$ has already increased the length of the $Q'Q$-sector. Thus, by lemmas 9.20, 9.16 and the choice of $L_0 > 100c_5N$, we have

$$|x|_a \geq h_{i+1} - h_{i+2} \geq \frac{1}{10c_5N}h_{i+1} \geq 10L_0 |W|_a$$ (9.68)

Note that an $a$-band $A$ starting on $x$ cannot end on a $(\theta,q)$-cell from $Q$. Indeed, by Lemma 9.12, there is a copy of this configuration in the diagram $\Gamma_{i-1}$, i.e., the copy of $A$ ends on the copy of $Q$ contrary the assumption that the rules of computation with history $H'\Gamma H'$ do not delete $a$-letters.

Let us consider the comb bounded by $Q$, $Q'$, $x$ and the boundary path of $\Delta^0$ (without the cells from $Q'$). If the lengths of $Q$ and $Q'$ are $s$ and $s'$, respectively, then there are $|x| + s$ maximal $a$-bands starting on $x$ and $Q$ and ending either on $Q'$ or on $\partial \Delta^0$ since the comb has no maximal $q$-bands by Lemma 9.11. At most $s' < s$ of these $a$-bands can end on $Q'$. Therefore at least $|x| + s - s'$ of them end on the segment of the boundary path of $\Delta^0$ lying between the ends of $Q'$ and $Q$.

Since this segment has $s - s'$ $\theta$-edges, its length is at least $s - s' + \delta|x|_a$ by Lemma 6.1. This inequality, Lemma 9.10 and inequality (9.68) imply

$$|p_{i,L-L_0-3}| \geq |q_{i,L-L_0-3}| \geq LN/2 + h_i + h_{L-L_0-3} + \frac{\delta}{10c_5N}h_{i+1}$$

$$\geq LN/2 + h_i + h_{L-L_0-3} + 10\delta L_0 |W|_a$$

Therefore

$$|p_{i,L-L_0-3}| - \frac{7\delta}{100c_5N}h_{i+1} > 3L_0N + h_i + h_{L-L_0-3} + 3\delta L_0 |W|_a \geq |\tilde{p}_{i,L-L_0-3}|, \quad (9.69)$$

by Lemma 9.7, and since $\Delta$ is a minimal counter-example, we obtain by the definition of $F(x)$, $g(x)$ and inequality (6.14) that

$$\text{Area}_{\tilde{G}}(\Psi'_{i+1,L-L_0-3}) \leq N_4 F(n + \sigma_{\lambda}(\Delta)) - \frac{7\delta}{100c_5N}h_{i+1} + N_3 g(n)\mu(\Psi'_{i+1,L-L_0-3})$$

$$\leq N_4 F(n + \sigma_{\lambda}(\Delta)) - N_4 \frac{7\delta n}{100c_5N}h_{i+1} + N_3 g(n)\mu(\Psi'_{i+1,L-L_0-3})$$ (9.70)

By Lemma 9.16, $|W|_a \leq L_0^{-2}h_i$, and by Lemma 9.17, $h_i > \delta^{-1} > 100L_0N$, whence $|\tilde{p}_{i,L-L_0-3}| \leq 2h_i + 3L_0N + 3\delta L_0 |W|_a \leq (2 + 0.03 + \frac{3\delta}{L_0})h_i \leq 2.1h_i$ and by Lemma 9.9, we have

$$|p_{i,L-L_0-3}| \leq (1 + \varepsilon)|\tilde{p}_{i,L-L_0-3}| < 2.2h_i$$ (9.71)

By lemmas 6.16 and 6.7 (a) and inequalities (9.71) and (9.69), the $G$-area of $\Psi_{i,L-L_0-3}$ does not exceed

$$N_2(2|p_{i,L-L_0-3}|)^2 + N_1\mu(\Psi_{i,L-L_0-3}) \leq N_2(4J + 4)|p_{i,L-L_0-3}|^2 \leq 5N_2(4J + 4)h_i^2$$ (9.72)
By Lemma 9.1, the G-area of $\Pi$ is bounded by $c_6 F(|\partial \Pi|)$. The inequalities (9.69) and (9.71) implies the inequality $|\partial \Pi| < L|\mathbf{p}_{i,L-L_0-3}| < L|\mathbf{p}_{i,L-L_0-3}| < 3Lh_i$. Therefore one may assume that the constant $c_7$ is chosen so that

$$\text{Area}_G(\Pi) < c_6 F(|\partial \Pi|) < c_7 F(h_i) = c_7 h_i^2 g(h_i) \leq c_7 h_i^2 g(n)$$  \hspace{1cm} (9.73)$$

(Recall that $h_i \leq n/2$ here since $h_i$ is the number $\theta$-bands crossing $\mathcal{B}_i$; they start and end on $\partial \Delta$.) It follows from (9.72) and (9.73) that

$$\text{Area}_G(\Delta_{i,L-L_0-3}) \leq 5N_2 (4J + 4)h_i^2 + c_7 h_i^2 g(n)$$  \hspace{1cm} (9.74)$$

We need an estimate for $\mu(\Psi'_{i+1,L-L_0-3}) - \mu(\Psi'_{i,L-L_0-3})$ now. To obtain it, we observe that by Lemma 9.5, the common $q$-edge $f$ of the spoke $\mathcal{B}_i$ and $\partial \Delta$ separates at least $h_{i-1} - h_i = m_1$ $\theta$-edges of the path $\mathbf{p}_{i-1,i}$ and $m_2$ ones lying on $\mathbf{p}_{i,L-L_0-3}$, where $m_2 = h_i + h_{i,L-L_0-3}$ (see fig. 18). Since the number of $q$-edges of $\mathbf{p}$ is less than $3K_0L < J$ by Lemma 9.6, one decreases $\mu(\Psi'_{i+1,L-L_0-3})$ at least by $m_1 m_2$ when erasing the black bead on $f$ in the necklace on $\partial \Psi'_{i+1,L-L_0-3}$ by Lemma 6.7 (d,b,c). Hence

$$\mu(\Psi'_{i+1,L-L_0-3}) - \mu(\Psi'_{i,L-L_0-3}) \geq m_1 m_2$$

by Lemma 9.20. This inequality and Lemma 9.8 applied to $\Psi_{i+1,L-L_0-3}$, imply

$$\mu(\Delta) - \mu(\Psi'_{i,L-L_0-3}) \geq -2Jn(h_{i+1} + h_{L-L_0-3}) + \frac{1}{10c_5 N} h_{i-1}(h_i + h_{L-L_0-3})$$

Note that $(h_{i+1} + h_{L-L_0-3}) \leq 2h_{i+1}$ by (9.54) and (9.60). Hence

$$N_3 \mu(\Delta) - N_3 \mu(\Psi'_{i,L-L_0-3}) \geq -4N_3 Jn(h_{i+1} + \frac{N_3}{10c_5 N} h_{i-1}(h_i + h_{L-L_0-3}) \hspace{1cm} (9.75)$$

Figure 18: $\mu(\Psi'_{i+1,L-L_0-3}) - \mu(\Psi'_{i,L-L_0-3})$
It follows from (9.74, 9.75, 9.70) that
\[
\text{Area}_G(\Delta) \leq \text{Area}_G(\Psi_{i,L-L_0-3}) + \text{Area}_G(\Delta_{i,L-L_0-3}) \leq \\
N_4 F(n+\sigma_\lambda(\Delta)) - N_4 \frac{7\delta n}{100c_5N} h_{i+1} g(n) + N_3 g(n)\mu(\Psi_{i,L-L_0-3}) + 5N_2(4J+4)h_i^2 + c7h_i^2 g(n) \\
\leq N_4 F(n+\sigma_\lambda(\Delta)) + N_3\mu(\Delta)g(n) - N_4 \frac{7\delta n}{100c_5N} h_{i+1} g(n) - \frac{N_3}{10c_5N} h_{i-1}(h_i + h_{L-L_0-3}) g(n) \\
+ 4N_3 Jnh_{i+1}g(n) + 5N_2(4J+4)h_i^2 + c7h_i^2 g(n)
\]

We come to a contradiction since we obtain inequality \(\text{Area}_G(\Delta) \leq N_4 F(n+\sigma_\lambda(\Delta)) + N_3\mu(\Delta)g(n)\), because \(N_4 \frac{7\delta}{100c_5N} > 4N_3 J\) and \(\frac{N_3}{10c_5N} > 5N_2(4J+4) + c7\).

**Lemma 9.23.** There exists no counter-example \(\Delta\), and therefore \(\text{Area}_G(\Delta) \leq N_4 F(n + \sigma_\lambda(\Delta)) + N_3\mu(\Delta)g(n)\) for any minimal diagram \(\Delta\) with \(|\partial\Delta| = n\).

**Proof.** Recall that for \(j = 1, \ldots, L_0 - 1\), we have \(h_{j+1} < (1 - \frac{1}{10c_5N})h_j\) by Lemma 9.20, and by lemmas 9.19 and 9.21, we have inequalities \(|z_j|_a \geq h_{j+1}/c_5\) and \(|z_k|_a \leq 2Nh_k\).

One can choose an integer \(\rho = \rho(M)\) (it depends on the machine \(M\) only as \(c_5\) and \(N\)) so that \((1 - \frac{1}{10c_5N})^\rho < \frac{1}{6Nc_5}\), and so \(h_{j+1} > 6Nc_5h_k\) if \(k - j - 1 \geq \rho\). Hence
\[
|z_j|_a \geq h_{j+1}/c_5 \geq 6Nh_k > 3|z_k|
\]

If \(L_0\) is large enough, say \(L_0 > 2000\rho\), one can obtain 1000 indices \(j_1 < j_2 < \cdots < j_{1000} < L_0\) such that for \(i = 2, \ldots, 1000\), one obtains inequalities
\[
|z_{j_i-1}| > 3|z_{j_i}| \text{ and } h_{j_i-1} \geq h_{j_i-1+1} > 6c_5Nh_{j_i}
\]  
(9.76)

Let \(C : W \equiv W_0 \rightarrow \cdots \rightarrow W_i\) be the computation corresponding to the trapezium \(\Gamma_{j_2}\). Since it contains the copy \(\Gamma'_{j_2+1}\) of \(\Gamma_{j_2+1}\), which in turn contains a copy of \(\Gamma_{j_2+2}\) and so on, we have some configurations \(W(k)\) in \(C\) \((k = 1, \ldots, 999)\), that are the labels of some \(z_k\) and \(|W(k+1)|_a \geq 3|W(k)|_a\) for \(k = 1, \ldots, 998\). If for some \(k\) we were obtain one-step subcomputation \(W(k) \rightarrow \cdots \rightarrow W(k+4)\), then the statement of Lemma 4.27 would give a subcomputation \(W(k+1) \rightarrow \cdots \rightarrow W(k+4)\) contradicting to the statement of Lemma 9.22. Hence no five consecutive words \(W(k)-s\) are configuration of a one-step subcomputation, and so the number of steps in \(W(1) \rightarrow \cdots \rightarrow W(999)\) in at least 100.

It follows now from Lemmas 4.9 and 4.7 that the step history of \(\Gamma_{j_2}\), where \(\Gamma\) is the copy of \(\Gamma_{L_0}\) in \(\Gamma_{j_2}\), has a subword \(((21^-)(1^-)1^-)(2^-)(21^-))^{\pm 1}\). Without loss of generality we assume that the exponent is +1. Therefore the history \(H_{j_2+1}\) of \(\Gamma_{j_2}\) can be decomposed as \(H'H''H''\), where \(H''\) has step history \((12^-)(2^-)(2^-), ||H''|| \geq ||H''||\) by Lemma 4.14 and \(||H''|| \geq h\) since \(\Gamma_{L_0}\) is a prefix of \(H''\).

Since \(h_{j_2+1} > 2h_{j_2}\) by (9.76), the history \(H_{j_2+1}\) of \(\Gamma_{j_2}\) has a prefix \(H'H''H^*\), where \(||H^*|| = ||H'|| \geq ||H''||\), and so the t-spoke \(B_{j_2+1}\) has a t-subband \(C\) starting with \(\partial\Pi\) and having the history \(H'H''H^*\).

For any factorization \(C = C_1C_2C_3\) with \(||C_1|| + ||C_2|| \leq ||C||/3\), the history of \(C_2\) contains the substhistory \(H''\), since \(||H^*|| = ||H'|| \geq ||H''||\). It follows that \(C\) is a \(\lambda\)-shaft, because \(\lambda < 1/3\). The shaft has length at least \(||H'|| \geq h\) contrary to Lemma 9.18. We come to the final contradiction in this section.

□
10 Proof of Theorem 1.3

10.1 Dehn function of the group \( G \)

Lemma 10.1. For every big trapezia \( \Delta \), there is a diagram \( \tilde{\Delta} \) over \( G \) with the same boundary label, such that the area of \( \tilde{\Delta} \) does not exceed \( 2 \text{Area}_G(\Delta) \).

Proof. Consider the computation \( C : V_0 \to \cdots \to V_t \) corresponding to \( \Delta \). According to Definition 6.8, one may assume that \( \text{Area}_G(\Delta) = c_5 h(||V_0|| + ||V_t||) \) since otherwise \( \tilde{\Delta} = \Delta \).

\( \Delta \) is the covered by \( L \) trapezia \( \Delta_1, \ldots, \Delta_L \) with base \( xvy \), where \( xy \) (or the inverse word) is a cyclic shift of the standard base of \( M \). By remark 5.8 all \( \Delta_1, \ldots, \Delta_L \) are copies of each other. Let us apply Lemma 4.4 to any of them, say to \( \Delta_1 \), whose top and bottom have labels \( W_0 \) and \( W_2 \). If we have Property (1) of that lemma, then the area of \( \Delta_1 \) does not exceed \( c_4 h(||V_0|| + ||V_t||) \) since every maximal \( \theta \)-band of \( \Delta_1 \) has at most \( c_4(||V_0|| + ||V_t||) \) cells in this case. Hence area of \( \Delta \) does not exceed \( Lc_4 h(||V_0|| + ||V_t||) \leq 2c_4 h(||W_0|| + ||W_t||) < c_5 h(||W_0|| + ||W_t||) = \text{Area}_G(\Delta) \), i.e. \( \tilde{\Delta} = \Delta \) in this case too.

Hence one may assume that Property (2) of Lemma 4.25 holds for \( \Delta_1 \). By that Lemma, items (b,d), the corresponding cyclic shifts \( W'_0 \) and \( W'_t \) are accepted, and there is a factorization of \( C = C_1 C_2 C_3 \) (we use the same letter for the computations corresponding to \( \Delta \), to \( \Delta_1 \), and for the revolving computation with standard base), where \( C_1 : W'_0 \to \cdots \to W'_{n_1} \), \( C_2 : W'_{n_1} \to \cdots \to W'_{n_1+n_2} \) and \( C_3 : W'_{n_1+n_2} \to \cdots \to W'_{n_1+n_2+n_3} \) (\( n_1 + n_2 + n_3 = t \)), where \( \max(||W'_{n_1}||, ||W'_{n_1+n_2}||) \leq \max(||W_0||, ||W_t||) \) and for each \( C_i \), either

(d1) \( ||W_j'|| \leq c_4 \max(||W_0'||, ||W_t'||) \), for every configuration \( W_j \) of \( C_i \) or

(d2) there are accepting computations for the first and the last configuration of \( C_i \)

with histories \( H'_i \) and \( H''_i \) such that \( ||H'_i|| + ||H''_i|| < n_i \)

So \( \tilde{\Delta} \) is built of at most three trapezia, where \( \Delta(i) (i = 1, 2, 3) \) corresponds to the computations \( C_i \). Since their tops and bottoms have lengths at most \( \max(||V_0||, ||V_t||) \), it suffices to estimate the area of \( \tilde{\Delta}(i) \) for \( i = 1, 2, 3 \). Again, we have \( \tilde{\Delta}(i) = \Delta(i) \) in the case (d1).

Assume that we have Property (d2) for \( C_i \). Denote by \( U_1 \) and \( U_2 \) the first and the last configurations of \( C_i \) with standard base. By (d2), \( ||U_1||, ||U_2|| \leq \max(||W_0||, ||W_t||) \).

By Property (d2), there is an accepting computations \( D \) of length \( \leq n_i \) for \( U_1 \), and we may assume that \( D \) is the shortest such computation. Then case (b) of Lemma 4.20 gives a contradiction of the form \( \ell < \ell/100 \) for the length \( \ell \) of the computation \( D \). Hence we should have case (a), and so every configuration of \( D \) has length at most \( c_4 \||U_1|| \). If \( \Gamma \) is the trapezium corresponding to \( D \) with bottom (top) label \( U_1 \) (resp., \( W_M \)), then the lengths of its \( \theta \)-bands are less than \( 2c_4 ||U_1|| \) by Lemma 6.1 and therefore the area of \( \Gamma \) is less than \( 2n_i c_4 ||U_1|| - 1 \). Therefore \( L \) copies of \( \Gamma \) can be attached to an auxiliary hub so that one gets an auxiliary disk \( \Pi_1 \) of area \( \leq 2Ln_ic_4 ||U_1|| \leq 3c_4 n_i ||V(1)|| \), where \( V(1) \) is the label of the bottom of \( \Delta(i) \) up to cyclic permutations. Thus, the word \( V(1) \) is equal to the boundary label of \( \Pi_1 \). Similarly, one can construct a disk of area \( \leq 3c_4 n_i ||V(2)|| \) for the top of \( \Delta(2) \).

Denote by \( \Delta_- \) the diagram \( \Delta \) without maximal rim \( x \)-band. So \( \Delta_- \) has the boundary \( p_1 q_1 q_2^{-1} q_2^{-1}, \) where \( \text{Lab}(p_1) \) and \( \text{Lab}(p_2) \) are the boundary labels of the disks \( \Pi_1 \) and \( \Pi_2 \) (up to cyclic permutations) and \( \text{Lab}(q_1) = \text{Lab}(q_2) \) since two \( x \)-bands with the same history have the same boundary labels.
If we attach disks $\Pi_1$ and $\Pi_2$ along their boundaries to the top and the bottom of $\Delta_-$, we obtain a diagram, whose boundary label is trivial in the free group. Hence there is a diagram $E$ with two disks whose boundary label is equal to the boundary label of $\Delta_-$ and the area is less than $\leq 3c_4n_i(||V(1)||+||V(2)||)$. If we attach one $\epsilon$-band of length $n_i$ to $E$, we construct the required diagram $\Delta(i)$ of area at most $\leq 3c_4n_i(||V(1)||+||V(2)||)+n_i < c_5n_i(||V(1)||+||V(2)||)$.

**Lemma 10.2.** The Dehn function $d(n)$ of the group $G$ is equivalent to $F(n)$.

**Proof.** To obtain the upper bound for $d(n)$ (with respect to the finite presentation of $G$ given in Section 5), it suffices, for every word $W$ vanishing in $G$ with $||W|| \leq n$, to find a diagram over $G$ of area $O(F(n))$ with boundary label $W$. Since $|W| \leq ||W||$, van Kampen’s lemma and Lemma 9.16 provide us with a minimal diagram $\Gamma$ every in this way, we should add on the presentation of $G$ given in Section 5), it suffices, for every word $W$ vanishing in $G$ with $||W|| \leq n$, to find a diagram over $G$ of area $O(F(n))$ with boundary label $W$. Since $|W| \leq ||W||$, van Kampen’s lemma and Lemma 9.16 provide us with a minimal diagram $\Delta$ such that $\text{Area}_{\Gamma}(\Delta) \leq N_4F(n+\sigma_3(\Delta))+N_3\mu(\Delta)g(n)$ for some constants $N_3$ and $N_4$ depending on the presentation of $G$. By Lemmas 8.5, 6.7 (a) and the definition of $\mu(\Delta)$, the right-hand side does not exceed $N_4F((1+\epsilon)n)+N_3Jn^2g(n)$. Since $F(O(n)) = O(F(n))$ and $n^2g(n) = F(n)$, we conclude that $\text{Area}_{\Gamma}(\Delta) \leq C_0F(n)$ for some constant $C_0$.

Recall that in the definition of $G$-area, the subdiagrams, which are big trapezia $\Gamma, \Gamma', \ldots$, can have common cells in their rim $q$-bands only. By Lemma 10.1, any big trapezia $\Gamma$ from this list with top $p_1$ and bottom $p_2$ can be replaced by a diagram $\Gamma$ with (combinatorial) area at most $2\text{Area}_{\Gamma}(\Gamma)$. When replacing all big trapezia $\Gamma, \Gamma', \ldots$, in this way, we should add $q$-bands for the possible intersection of big trapezia, but for every $\Gamma$ of height $h$, we add at most $2h$ new cells. So the area of the modified diagram $E$ is at most $3\text{Area}_{\Gamma}(\Delta) \leq 3C_0F(n)$. Hence the required diagram is found for given word $W$.

To obtain the lower bound for $d(n)$, we will use the series of $M$-accepted words $V(n)$ of (combinatorial) length $\Theta(n)$ constructed in the proof of Lemma 4.23. Since $V(n)^L = 1$ in $G$, there is a reduced diagram $\Delta$ over the presentation of $G$ with boundary label $V(n)^L$. A $t$-band starting on the boundary of $\Delta$ cannot end on it since all occurrences of $t$ in $V(n)^L$ are positive. Therefore there are hubs in $\Delta$. Furthermore, since the number of $q$-edges on the boundary of $\Delta$ is $LN$, there is only one hub $\pi$ in $\Delta$ by Lemma 7.3, and all the spokes of $\pi$ end on $LN q$-edges of $\partial\Pi$. Hence $\Delta$ has $L$ trapezia corresponding to an accepting reduced computation $C$ for $V(n)$, and it suffices to get a lower bound for the area of one trapezium $\Gamma$.

By Remark 4.24, $\Gamma$ has at least $\Theta(ng(n))$ maximal $\theta$-bands of length at least $\Theta(n)$, Therefore the area of $\Gamma$ is at least $\Theta(F(n))$. Since $||\partial(\Delta)|| = \Theta(n)$, the Dehn function $d(n)$ is bounded from below by a function equivalent to $F(n)$, as required.

### 10.2 Supercubic Dehn functions

Here we show that for the Dehn functions $F(n)$ obtained earlier, one can construct a finitely presented group with Dehn function $nF(n)$. This will complete the proof of Theorem 1.3. For this goal we modify the control machines used in the definition of the machine $M$. The unnecessary extra-control will just slow down the work of $M$. The construction resembles the one from subsection 4.2 of [13]. We will modify only the machine $P$ defined in Subsection 3.1. The copies of the auxiliary primitive machine $Z(A)$ will work between the applications of the (copies of) the rules of $P$. 

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For every set of letters $A$, let $A', A''$ and $A'''$ be disjoint copies of $A$, the maps $a \mapsto a'$, $a \mapsto a''$ and $a \mapsto a'''$ identify $A$ with $A'$, $A''$ and $A'''$, resp. Let $\overrightarrow{Z}$ be the $S$-machines with tape alphabet $A' \sqcup A'' \sqcup A'''$, state alphabet $\{L \sqcup K \sqcup P \sqcup \{R\}\}$, with the following positive rules.

$$\chi_1(a) = [L \xrightarrow{\ell} L, k(1) \rightarrow k(1), p(1) \rightarrow (a'')^{-1}p(1)a''', R \rightarrow R], a \in A$$

$$\chi_2 := [L \rightarrow L, k(1) \rightarrow k(2), p(1) \rightarrow p(2), R \rightarrow R]$$

$$\chi_3(a) = [L \rightarrow L, k(2) \rightarrow k(2), p(2) \rightarrow a''p(2)(a''')^{-1}, R \rightarrow R]$$

$$\chi_4 = [L \rightarrow L, k(2) \rightarrow k(3), p(2) \rightarrow p(3), R \rightarrow R]$$

The rules of $\overrightarrow{Z}$ are similar, but the moving base letter is $K$, while the sector $PR$ is locked.

To define the composition $P \circ \{Z, \overrightarrow{Z}\}$ we insert the base of $\overrightarrow{Z}$ (and $\overleftarrow{Z}$) between every two consecutive state letters of $P$. In this subsection, we assume that $P$ has the standard base $Q_0Q_1...Q_N$ (and forget more detailed earlier notations).

For every $i = 1,...,N$, we make copies $Y_i', Y_i''$ and $Y_i'''$ of the alphabet $Y_i$ of $P$ ($i = 1,...,N$). Let $\Theta$ be the set of positive commands of $P$. The set of state letters of $P \circ \{Z, \overrightarrow{Z}\}$ is

$$S_0 \sqcup K_1 \sqcup P_1 \sqcup S_1 \sqcup K_2 \sqcup P_2 \sqcup ... \sqcup P_N \sqcup S_N$$

where $P_i = \{p^{(i,1)}, p^{(i,2)}, p^{(i,3)} | \theta \in \Theta\}$, $K_i$ is defined similarly for $i = 1,...,N$, $S_i = Q_i \cup (Q_i \times \theta)$. Thus the state letters $L$ and $R$ of the copies of the machines $\overrightarrow{Z}$ and $\overleftarrow{Z}$ are identified with the corresponding $S$-letters. We shall call the state letters from $P_i$-s and $K_i$-s the $p$-letters, and the other state letters (i.e. the copies of the state letters of $P$), the basic state letters.

The set of tape letters of $P \circ \{Z, \overrightarrow{Z}\}$ is $Y = Y_1 \sqcup ... \sqcup Y_{3N} = Y_1' \sqcup Y_1'' \sqcup Y_1''' \sqcup Y_2' \sqcup Y_2'' \sqcup Y_2''' \sqcup ... \sqcup Y_N' \sqcup Y_N'' \sqcup Y_N'''$.

Assume $\theta$ is a positive $P$-rule of the form

$$[s_0u_1 \rightarrow s_0'v_1, s_1s_2 \rightarrow s_1's_2',... , v_Ns_N \rightarrow v_N's_N]$$

where $s_i, s_i' \in Q_i$, and $v_i$-s, $u_i$-s are words in $Y$. Then this rule is replaced in $P \circ \{Z, \overrightarrow{Z}\}$ by positive

$$\overrightarrow{\theta} = \left[ s^{(\theta,0)} \xrightarrow{\ell} s^{(\theta,0)}', k^{(\theta,1)}(3)u_1 \rightarrow k^{(\theta,1)}(1)u_1', v_1p^{(\theta,1)}(3) \rightarrow v_1'p^{(\theta,1)}(1), \right]\left[ s^{(\theta,1)} \xrightarrow{\ell} s^{(\theta,1)}', k^{(\theta,2)}(3)u_2 \rightarrow k^{(\theta,2)}(1)u_2', v_2p^{(\theta,2)}(3) \rightarrow v_2'p^{(\theta,2)}(1), \right]$$

with $Y_{3i-2}(\overrightarrow{\theta}) = \emptyset$, $Y_{3i-1}(\overrightarrow{\theta}) = Y_i'''(\theta)$ and $Y_{3i}(\overrightarrow{\theta}) = Y_i''(\theta)$.

Now we want to describe the alternating work of the auxiliary machines $\overrightarrow{Z}^{(\theta,i)}$ and $\overleftarrow{Z}^{(\theta,i)}$. Normally each of them is switched on exactly once in the frame of the rule $\theta$, but the sequence of their turning on depends on $\theta$. First, we need the following transition rule $\chi_{-}(\theta)$. This rule adds $\theta$ to all state letters and turns all $k^{(j)}$ and $p^{(j)}$ into $k^{(\theta,j)}(1)$ and $p^{(\theta,j)}(1)$:
\[ s_i \xrightarrow{\ell} s^{(\theta,i)}, k^{(j)} \rightarrow k^{(\theta,j)}(1), p^{(j)} \xrightarrow{\ell} p^{(\theta,j)}(1), i = 0, \ldots, N, j = 1, \ldots, N \]

Then the machines \( \overrightarrow{Z}^{(\theta,1)}, \ldots, \overrightarrow{Z}^{(\theta,N)} \) and \( \overleftarrow{Z}^{(\theta,1)}, \ldots, \overleftarrow{Z}^{(\theta,N)} \) are switched on in a specific order (defined below) after the rule \( \chi_-(\theta) \) is applied. So the state letters \( k^{(\theta,j)}(1), p^{(\theta,j)}(1) (j = 1, \ldots, N) \) successively turn into \( k^{(\theta,j)}(3), p^{(\theta,j)}(3) \), find themselves just after \( s_{i-1} \) and before \( s_i \)-letters, respectively, and the rule \( \theta \) can be applicable.

After an application of \( \bar{\theta} \), the machines \( \overrightarrow{Z}^{(\theta,1)}, \ldots, \overrightarrow{Z}^{(\theta,N)} \) and \( \overleftarrow{Z}^{(\theta,1)}, \ldots, \overleftarrow{Z}^{(\theta,N)} \) are switched on again in the following order.

Assume that the rule \( \theta \) is a rule of a primitive machine \( P \). (Recall that \( P \) is composed from primitive machines.) The machine \( P \) can work in several sectors. (For example, \( \theta \) can be the control rule checking all the big historical sectors simultaneously.) Let \( i \) be the minimal index such that \( S_i \) has a control running state letter of \( P \). Then the rule \( \bar{\theta} \) first switches on the machines \( \overrightarrow{Z}^{(\theta,j)} \) for \( j = i \) and simultaneously for all other \( j \)-s, where \( S_j \) also has a control running state letter of \( P \). The last rule \( \chi_4 \) of this machine switches on the machine \( \overrightarrow{Z}^{(\theta,i-1)} \) and similar machines in similar sectors (e.g. in all small historical sectors if the sectors \( S_{i-2} \text{ and } S_{i-1} \) is a small historical sector). The next machine is \( \overrightarrow{Z}^{(\theta,i-2)} \), if it did not work earlier, and so on. Then the machines \( \overrightarrow{Z}^{(\theta,i+1)}, \ldots \) subsequently work, except for the sectors, where the auxiliary \( Z \)-machines worked earlier.

The same machines work after the application of the rule \( \chi_-(\theta) \) but they are switched on in the inverse order.

Finally, the transition rule \( \chi_+(\theta) \) removes the index \( \theta \) from all state letters, and turns all \( p^{(\theta,j)}(3) \) into \( p^{(j)} \):

\[ s^{(\theta,i)} \xrightarrow{i} s_i, k^{(j)} \rightarrow k^{(j)}, p^{(j)} \rightarrow p^{(j)}, i = 0, \ldots, N, j = 1, \ldots, N \]

For every admissible word \( W \) of \( P \) with standard base, let \( \iota(W) \) be the admissible word of \( P \circ \{ \overrightarrow{Z}, \overleftarrow{Z} \} \) obtained by inserting state letters \( k^{(i)} \) and \( p^{(i)} \) next to the right of each \( s_{i-1} \) and next to the left of each \( s_i \), \( i \leq N \).

Assume that \( W \rightarrow W \cdot \theta \) is a computation of the machine \( P \) with standard base and a positive rule \( \theta \). Then, by the definition of \( P \circ \{ \overrightarrow{Z}, \overleftarrow{Z} \} \), we have the canonically defined reduced computation \( \cdots \rightarrow \iota(W) \rightarrow \iota(W) \cdot \theta \rightarrow \cdots \) starting and ending with words whose state letters have no \( \theta \)-indices and all other words do have \( \theta \)-indices. The computation of \( P \circ \{ \overrightarrow{Z}, \overleftarrow{Z} \} \) with these properties is unique by Lemma 3.1 (3) since the machines \( \overrightarrow{Z} \) and \( \overleftarrow{Z} \) are primitive. Thus the following claim is true.

**Lemma 10.3.** (similar to Lemma 4.24 from [13]). For every computation \( W \rightarrow W \cdot \theta \) of the machine \( P \circ \{ \overrightarrow{Z}, \overleftarrow{Z} \} \) with standard base and a positive rule \( \theta \), there is a unique reduced \( \overrightarrow{Z}, \overleftarrow{Z} \) -computation \( \cdots \rightarrow \iota(W) \rightarrow \iota(W) \cdot \theta \rightarrow \cdots \) starting and ending with words whose state letters have no \( \theta \)-indices and all other words have \( \theta \)-indices. The history of this computation starts with \( \chi_-(\theta) \) and ends with \( \chi_+(\theta) \).

For every admissible word \( W \) of \( P \circ \{ \overrightarrow{Z}, \overleftarrow{Z} \} \) with the standard base, let \( \pi(W) \) be the word obtained by removing state \( k \)- and \( p \)-letters, \( \theta \)-indices of state letters, and the indices that distinguish \( a \)-letters from the left and from the right of \( k \)- and \( p \)-letters. After possible cancellations of \( a \)-letters, we obtain an admissible word of \( P \). Note that we have \( \pi(\iota(W)) \equiv w \).

Given a computation \( C \) of the machine \( P \circ \{ \overrightarrow{Z}, \overleftarrow{Z} \} \) with standard base and history \( H \) involving a \( \bar{\theta} \)-rule, we define the projection \( \pi(C) \) of it, which is a computation of \( P \). To
obtain it, one removes all transitions given by \( \chi \)-rules and replaces the configurations by their projections. Note that this operation makes sense since \( \chi \)-rules do not change the projection of the word onto a word in \( Y_i \). The projection \( \pi(H) \) of the history is defined in obvious way: one forgets the \( \chi \)-rules and removes bars over \( \zeta \)-rules.

**Lemma 10.4.** (also see Lemma 4.28 in [13]). If \( C \) is a reduced computation of \( P \circ \{ \vec{Z}, \vec{Z} \} \) with standard base, then \( \pi(C) \) is a reduced computation of \( P \).

Proof. Assume that we have a subword \( \vec{\theta}H\vec{\theta}^{-1} \) in the history \( H \) of \( C \), where \( H' \) has no rules of the form \( \vec{\theta} \). If \( H' \) is non-empty, then one obtains a contradiction by Lemma 3.1 (4) applied to the work of the primitive machines \( \vec{Z}^{(\theta, i)} \) and \( \vec{Z}^{(\theta, i)} \). Hence \( \pi(H) \) is a reduced history.

Below we change some formulations of Subsection 3.1 as applied to \( P \circ \{ \vec{Z}, \vec{Z} \} \).

**Lemma 10.5.** (duplicate of 3.1). Let \( C : C_0 \to \cdots \to C_t \) be a reduced computation of the machine \( P \circ \{ \vec{Z}, \vec{Z} \} \) with the standard base and with \( t \geq 1 \). Then the following properties hold.

1. If \( |C_i|_a > |C_{i-1}|_a \) for some \( i = 1, \ldots, t-1 \), then \( |C_i|_a \leq |C_{i+1}|_a \leq |C_{i+2}|_a, \leq \ldots \)
2. \( |C_i|_a \leq \max(|C_0|_a, |C_2|_a) \) for every \( i = 0, 1, \ldots, t \).
3. Assume that the words \( C_0 \) and \( C_t \) have \( a \)-letters only from the subalphabets \( Y_i'' \) \( (i = 1, \ldots, N) \) and that for the primitive machines \( P \) forming \( P \), all their subconfigurations in \( \pi(C_0) \) and \( \pi(C_t) \) look like in Lemma 3.1 (3), i.e. as \( q^1v^pq^q \) and \( v^p q^2 \) for some words \( u, v \) in the notation of Lemma 3.1. Then \( a \)-words in the corresponding sectors of \( C_0 \) and \( C_t \) are equal, \( |C_i|_a = |C_0|_a \) for every \( i = 0, \ldots, t \) and \( t = \Theta(s^2) \), where \( s = |C_0|_a \).
4. Assume that the words \( C_0 \) and \( C_t \) have \( a \)-letters only from the subalphabets \( Y_i'' \) \( (i = 1, \ldots, N) \) and that for the primitive machines \( P \) forming \( P \), all their subconfigurations in \( \pi(C_0) \) and \( \pi(C_t) \) look like in Lemma 3.1 (4). Then it is not possible that the configurations \( C_0 \) and \( C_t \) have the same set of state letters.
5. If \( C_0 \) (or \( C_t \)) satisfies the assumptions of item (3), then \( |C_i|_a \geq |C_0|_a \) (respectively, \( |C_i|_a \geq |C_i|_a \) for every \( i = 0, \ldots, t \).

Proof. If \( C_i = C_{i-1} \chi_1^{(\theta, i)}(a)^{\pm 1} \) (or \( C_i = C_{i-1} \chi_2^{(\theta, i)}(a)^{\pm 1} \)), then \( p(1)^{(\theta, i)} \) inserts letters from both sides and the next rule of the computation must be again \( \chi_1^{(\theta, i)}(a)^{\pm 1} \) (or \( \chi_2^{(\theta, i)}(a)^{\pm 1} \)) for some \( c \).

It again must increase the length of the configuration by two, and so on.

If \( C_i = C_{i-1} \vec{\theta} \) for some \( \vec{\theta} \)-rule, then the transition \( \pi(C_{i-1}) \to \pi(C_i) \) increases the length by Lemmas 10.4 and 3.1 (1). The work of \( \vec{Z}^{(\theta, i)} \vec{Z}^{(\theta, i)} \) cannot decrease configuration length by Lemma 3.1 (5) for these primitive machines. Therefore Statement (1) is true and Statement (2) is also true since one can choose a shortest \( C_j \) and consider the subcomputation \( C_j \to \cdots \to C_t \) and inverse subcomputation \( C_j \to \cdots \to C_0 \).

To prove equalities \( |C_i|_a = |C_0|_a \) in Statement (3), one just apply Lemma 3.1 (3) first, to \( \pi(C) \) and second, to the maximal subcomputations of \( Z \)-machines. Since the length of \( \pi(C) \) is \( 2k+1 \) by that Lemma, and if \( k > 0 \) the maximal subcomputations of \( Z \)-machines work with configurations of lengths, \( k-1, k-2, \ldots, 1 \), we obtain the later claim of (3) by summation.

Property (4) follows from Lemma 3.1 (4) applied to the computation \( \pi(C) \) and the maximal subcomputations of \( Z \)-machines. Property (5) follows from the projection argument as in Lemma 3.1 (5).
Lemma 10.6. (duplicate of 3.3) If \( C_0 \to \cdots \to C_t \) is a reduced computation of \( P \) with base \( S_{i-1}K_iK_i^{-1}S_{i-1}^{-1} \) or \( S_i^{-1}P_i^{-1}P_iS_i \) and \( C_0 \) has \( a \)-letters from the alphabet \( Y_i'' \) only. Then \( |C_i|_a \geq |C_0|_a \) for every \( i = 0, \ldots, t \).

**Proof.** The statement follows from the projection argument 3.2 as in Lemma 3.3. \( \Box \)

Let us call the constructed machine \( P \circ \{ \overline{Z}, \overline{Z} \} \) biprimitive.

To define the modified machine \( M_3 \) we insert two more base letters in each pair \( R_{i-1} \) and \( P_i \) of the standard base, i.e. this base has the subwords \( Q_{i-1}R_{i-1}R_i'P_iQ_i \); now the first big historical sector is \( R_0^rP_1 \) (instead of \( R_0P_1 \)). In the definition of Step 1', we now replace the rules of the primitive control machines with the rules of the corresponding biprimitive machines. By definition, at all other steps the control machines are just primitive, i.e. the sectors \( R_{i-1}R_i' \) and \( P_i'P_i \) are locked, there are no \( \chi \)-rules, and two base letters of the locked sectors work as one letter of a primitive machine.

Thus, at Step 1', the machine \( P' \) works, that is a composition of biprimitive machines.

Lemma 10.7. (duplicate of 3.6) Let \( \mathcal{C} : W_0 \to \cdots \to W_t \) be a reduced computation of \( P' \) with standard base. Then

(a) \( |W_j|_a \leq \max(|W_0|_a, |W_t|_a) \) for every configuration of \( \mathcal{C} \); moreover, \( |W_0|_a \leq \cdots \leq |W_t|_a \) if every control \( P \) - and \( R \) -letter neighbors some \( Q \)-letter and \( P' \)-letter (\( R' \)-letter) neighbors a \( P \)-letter (resp., an \( R \)-letter) in the word \( W_0 \);

(b) we have \( t = O(|W_0|^2 + |W_t|^2) \), moreover, \( t = O(|W_t|^2) \) if every \( P \) -letters and control letters has a neighbor in the word \( W_0 \) as in item (a).

**Proof.** (a) The first property is given by Lemma Under the additional assumption for control letters and \( p \)-letters, \( W_0 \) is the shortest configuration by the projection argument.

(b) Let us say that a subcomputation without \( \theta \)-rules in the history is a \( Z \)-subcomputation. If the computation is a \( Z \)-subcomputation, then the statement (b) follows from Lemma 3.6 (b) for the (composition of) primitive machines. Otherwise we chose a \( \theta \)-transition \( W_r \to W_{r+1} \) with minimal \( \min(|W_r|, |W_{r+1}|) \). Without loss of generality, one may assume that this minimum is \( |W_r| \). Lemma 3.6 (a) for \( \pi(\mathcal{C}) \) and for \( Z \)-subcomputations implies that \( |W||r \leq |W_i| \) for every \( i \), and therefore it suffices to bound the histories of length-non-decreasing computations \( \mathcal{C}' : W_r \to \cdots \to W_t \) and \( \mathcal{C}'' : W_r \to \cdots \to W_0 \).

The length of \( \pi(\mathcal{C}') \) is at most \( 2||W_r|| \) by Lemma 3.6 (b) for the machine \( P \). By the same Lemma for maximal \( Z \)-subcomputations, the length of every maximal \( Z \)-subcomputation is at most \( 2||W_t|| \) too. Since the number of such maximal subcomputations is at most \( 2||W_t|| \), we have that \( t - r = O(||W_t||^2) \). Similarly, we obtain \( r = O(||W_0||^2) \), and the the first estimate of (b) is obtained. For the second estimate, one can choose \( r = 0 \), since the whole computation \( \mathcal{C} \) is length-non decreasing by Lemma 10.5 (5,1). \( \Box \)

We see that Lemmas 10.5 (3) and 10.7 (b) provide us with quadratic estimate of the computation time for the biprimitive machine against the linear time for primitive machines. This have a few consequences mentioned below.

For the same function \( f(x) \) recognized by the original Turing machine and \( F(x) = x^2g(x) \), we define the function \( g(x) \) to be equivalent to \( xf(x)^3 \) now.

The extra sectors of the biprimitive machine do not affect the work of all other steps except for \( 1^- \) since they are locked therein.

The formulation of Lemma 4.5(1) is unchanged but the proof is now based on Lemma 10.5 (4). For the step history \( (21^-)(1^-)(1^-1) \) in the formulation of Lemma 4.5(2), we
have now $|H| = O(|W_0|^2)$, which follows from Lemma 10.7 (b). The estimates of $|H|$ for other step histories mentioned in Lemma 4.5(1) remain unchanged. Also we add an item to the formulation of Lemma 4.5:

(3) Let the history of a reduced computation $C : W_0 \rightarrow \cdots \rightarrow W_t$ with standard base has a subword $\theta_1 H \theta_2$, where $\theta_1$ and $\theta_2$ are $\theta$-rules of the machine $P'$ and $H$ has only $\chi$-rules of $P'$. Then all configurations of $C$ are uniquely determined by $H$, $|W_1|_a = \cdots = |W_{t-1}|_a$ and $|W_j| = \Theta(||H||)$ for $j \leq t$.

The proof of this statement follows the proof of Lemma 4.5 (2), but now one refers to Lemma 10.3 instead of Lemma 3.1 (3).

To the assumption of Lemma 4.10, we add: "or the history of the computation $C$ has a subword $\theta_1 H \theta_2$, where $\theta_1$ and $\theta_2$ are $\theta$-rules of the machine $P'$ and $H$ has only $\chi$-rules of $P'$". In the proof of Lemma 4.10 (2), one should use the following property. The computation of the biprimitive machine does not change the length of configurations by Lemma 10.7 (a). Hence the reduced computation of it is canonical by Lemma 10.3, and the history restores the tape words.

Consider the single Step 1− in item 2 of the proof of Lemma 4.13. If we have the work of primitive machines only, then the proof is unchanged. If there is a $\theta$-rule of the biprimitive machine in the computation $C$, we consider the projection $\pi(C)$, where the $a$-lengths of all configurations are at most $C(|W_0|_a + |W_1|_a)$ by Lemma 4.13. It remains to consider maximal subcomputation $W_r \rightarrow \cdots \rightarrow W_s$ of $Z$-machines, where $|W_r|_a, |W_s|_a \leq C(|W_0| + |W_t|)_a$. By Lemma 3.6 (a) we have $|W_i|_a \leq \max(|W_r|_a, |W_s|_a) \leq C(|W_r| + |W_s|)_a$ for $r \leq i \leq s$, as required.

In item 3 of the proof of Lemma 14.13, the base letters of the first big historical sector should be replaced by their dashed duplicates.

The estimate $|H| \leq c_2|W_0|$ of Lemma 4.16 changes now by the quadratic estimate $|H| \leq c_2|W_0|^2$ due to the application of Lemma 10.7 (b). Respectively, the upper bound $c_3(k^3 + 1)(|W_0| + k^3)$ of Lemma 14.17 is now replaced by $c_3(k^3 + 1)(|W_0|^2 + k^3)$.

The formulation of Lemma 4.20 does not change since comparing the lengths of histories, we now increase all of them. In particular, we have now in equation (10.2) that $2w^2 + 2(w-1)^2 + \cdots + 2(w-(m-1))^2 \geq |m|w^2/2$, we have that the difference of lengths of subcomputations $C_{t-1}$ and $C_t$ does not exceed $10w$, we have $|H_0| = O(w^2)$ and obtain $|H''''| \leq c_2(k^3 + 1)O(w^2)$, which leads to the same estimates for $|H_0|$ and $|H_t|$ since the constant $c_4$ can be chosen large enough.

The change of linear estimates by quadratic ones in the proof of Lemma 4.21 just sharpens the required inequalities.

Lemma 4.23 claims now that the generalized time function $T'(n)$ of the machine $M'$ is equivalent to $n^2 f^3(n)$. Thus, we multiply the generalized time function of $M$ by $n$. This is sufficient for the upper bound. Indeed the replacements of the form $|W_j| \rightarrow |W_j|^2$ in the proof of the modified Lemma 4.23 can multiply the generalized time function at most by $n$ since it is shown there that $|W_j| = O(n)$ for every configuration $W_j$. The lower bound obtained in the original Lemma 4.23 must also be multiplied by $n$ now. Indeed, the time of the constructed subcomputations with step history $(21^−)(1^−)(1)(2^−)(2)(21^−)^{−1}$ will be at least $\Theta(n^2)$ now (instead of $\Theta(n)$) since by Lemma 10.5 (3), we have such lower bound for the Step 1−.

Similar replacement should be made in Remark 4.24: There are $\Theta(n^2 f(n)^3)$ configurations of length at least $\Theta(n)$ for any computation accepting the word $V(n)$.

The statement of Lemma 4.26 is modified now by adding the words "or the history of $C$ has a subword $\theta_1 H \theta_2$, where $\theta_1$ and $\theta_2$ are $\theta$-rules of the machine $P'$ and $H$ has only
χ-rules of P". If the step history of C is (F) and Property (a) of Lemma 4.20 holds we have the same proof as in the original Lemma 4.26 since the length of a subcomputation of Step 1− is at most 5 max(||W0||a, ||Wt||a) provided the history of D has no subwords θ1Hθ2. Indeed, in this case, the history of any subcomputation of Step 1− is a subword of H′θH′′, where H′ and H′′ have only χ-rules of P′, and one can bound each of ||H′|| and ||H′′|| by 2c4(||W0||a, ||Wt||a) applying Lemma 3.6 (b).

The formulation of Lemma 4.27 remains unchanged for all steps except for Step 1−. If the step is 1−, we add the assumption that the history has no θ-rules of the machine P′. Since the biprimitive machine P′ works as a primitive one in the later case, the proof does not change.

The set of standard trapezia is enlarged now, namely, a trapezium with standard base having a subword θ1Hθ2 in the history, where θ1 and θ2 are θ-rules of the machine P′ and H has only χ-rules of P′, is also standard by definition.

The above modification of Lemma 4.27 changes the proof of Lemma 9.23 as follows. One obtains a contradiction if for some k, there is one-step subcomputation W(k) → ··· → W(k + 4), where the step differs from 1− or the step is 1− and the history of this subcomputation has no θ-rules of the machine P′. Hence either every subcomputation W(k) → ··· → W(k + 8) has at least two steps, and so the computation W(1) → ··· → W(999) has at least 100 steps, which leads to a contradiction as in the original proof of Lemma 9.23, or for some k, one obtains at least two θ-rules in the history of W(k) → ··· → W(k + 8), i.e., the history of this subcomputation has a subword θ1Hθ2, where θ1 and θ2 are θ-rules of the machine P′ and H has only χ-rules of P′. In the later case one should use the modified Lemma 4.10 (instead of the original Lemma 4.10) to complete the proof.

Lemma 9.23 proves Theorem 1.3 for s = 2. In the present subsection, we has modified the main machine and the corresponding groups M and G so that the modified Lemma 9.23 provides us with the statement of Theorem 1.3 for s = 3. One can make further modifications, which similarly slow down the work of the previously modified machines. This will give the proofs of Theorem 1.3 for s = 4, 5, .... However we can leave the details for the reader taking into account that for s ≥ 4, Corollary 1.4 is obtained in [19]. Thus, the proof of Theorem 1.3 is complete.

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