A many-body Fredholm index for ground state spaces and Abelian anyons

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We propose a many-body index that extends Fredholm index theory to many-body systems. The index is defined for any charge-conserving system with a topologically ordered p-dimensional ground state sector. The index is fractional with the denominator given by p. In particular, this yields a new short proof of the quantization of the Hall conductance and of Lieb-Schultz-Mattis theorem. In the case that the index is non-integer, the argument provides an explicit construction of Wilson loop operators exhibiting a non-trivial braiding and that can be used to create fractionally charged Abelian anyons.

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Introduction. The use of topology to study condensed matter systems is among the most influential developments of late 20th century theoretical physics [20]. The first major application of topology appeared in the context of the quantum Hall effect [2, 43, 49] in the early 80’s, and topological concepts have since been applied systematically to discover and classify phases of matter [10, 14, 50] and cellular automata [11, 18]. For non-interacting systems, several topological indices can be formulated as Fredholm indices [4, 8, 15] or, equivalently, as transport through a Thouless pump [45]. These formulations have been influential and insightful, in particular, as transport and insight in K-theory [27, 31, 47], but a framework of similar scope is lacking for interacting systems, except possibly in 1 dimension where there is a classification of matrix product states [16, 19, 32, 39, 46, 52, 54]. The full classification for interacting systems is among the most influential developments of late 20th century theoretical physics [20, 33].

The aim of this letter is to provide an interacting counterpart to this formalism. In a natural sense, it also gives rise to fractional indices and to Abelian anyons.

Free fermions. Consider a 2d discrete torus $T_L$ of $L \times L$ sites $i = (i_1, i_2)$ and let $\Gamma$ be the region $0 < i_1 \leq L/2$, see Figure 1. Let $P$ be an orthogonal projection that we think of as a Fermi projection corresponding to a one-particle Hamiltonian on the 2-torus, and let $U$ be a unitary such that $[P, U] = 0$. These are operators on the (spinless) one-fermion space $L^2(T)$. Let $Q$ (charge) be the projector on $\Gamma$: $Q = 1_\Gamma = \sum_{i \in \Gamma} |i \rangle \langle i |$. We consider the charge transported by $U$ out of $\Gamma$ starting from the filled Fermi sea, given by $\text{tr}[P(U^\dagger QU - Q)]$. One immediately checks by using $[P, U] = 0$ and cyclicity of the trace that this vanishes. This is because the transport at $i_1 = 0$ is offset by an opposite flow at $i_1 = L/2$. Separately however, the flows do not need to be trivial. If $U$ is sufficiently local, i.e. the matrix elements $U(i,j)$ decay fast as $|i-j| \to \infty$, then $U^\dagger QU - Q = (U^\dagger QU - Q)_- + (U^\dagger QU - Q)_+$ with $(U^\dagger QU - Q)_\pm$ located around the boundaries $\partial_\pm$ of $\Gamma$. Then the charge transport through $\partial_- \Gamma$ is given by

$$\text{Ind}(P, U) = \text{tr}[P(U^\dagger QU - Q)_-].$$  

If $P$ is also local in the above sense, then $\text{Ind}(P, U)$ is well-defined and it is an integer: $\text{Ind}(P, U) \in \mathbb{Z}$ up to corrections vanishing for large $L$. This formula is insensitive to local changes: if we add to any of $Q, P, U$ an operator $B$ well-localized around $\partial_-$, then the index does not change, reflecting its topological nature. Our presentation, inspired by [29], was stressing the Thouless pump picture, and we refer to SM for the connection to a Fredholm index and the omitted proof. In both cases, the point is that the index is constructed in a general way out of the minimal data provided by $P, U$. In particular, if the index is quantum Hall conductance, its quantization is shown without recourse to any explicit bundle.

Interacting systems. We consider a many-body setting, either of spins or fermions on the discrete torus $T_L$. We say that an observable $O$ has support $X \subset T_L$ if $O = O_X \otimes 1_{X^c}$. A local observable is supported in a fixed, $X$-independent set $X$, up to rapidly vanishing tails [54]. All our equalities hold up to finite size corrections of order $O(L^{-\infty})$, i.e. decaying faster than polynomial in $L$, as was also the case above.

We consider a many-body ground state projector $P$ with some finite rank $p$ (dimension of ground state space). Even though we use the same symbol, this is very different from the Fermi projection above, which is a one-particle concept. In the interesting case $p > 1$, we require the distinct ground states to be locally indistinguishable, a condition that is also called topological order [9, 53].

$$\text{POP} = \text{tr}(PO) \frac{P}{p}$$
for any local operator $O$. The charge operator $Q$ is now the number of fermions in $\Gamma$, i.e. $Q = \sum_{i \in \Gamma} n_i$. This choice is made for the sake of concreteness, the only important feature is that $Q$ is made out of a collection of commuting, local operators with integer spectrum. The operator $U$ is a unitary process that leaves the ground state space invariant $[P, U] = 0$ and that conserves the total number of fermions, but of course not necessarily $Q$. Therefore $U^1 QU - Q$ is again a sum of two contributions $T_\pm \equiv (U^1 QU - Q)_\pm$ located respectively at $\partial_-, \partial_+$. This splitting is in general not uniquely defined and we choose it to satisfy $e^{2\pi i (Q + T_\pm)} = 1$, see below for details and an explanation. Analogously to the free case, we now consider, for any ground state $\psi \in \text{ran} P$,

$$\text{Ind}(P, U) \equiv \langle \psi | (U^1 QU - Q)_- | \psi \rangle.$$  

(2)

The locality that was crucial in the non-interacting setting is now implemented as follows: 1) we require the ground projection $P$ to correspond to a local Hamiltonian (sum of local terms) $H = \sum_{X} H_X$ that is gapped, uniformly in volume, and 2) For any operator $O$, the spatial support of $U^1 OU$ extends beyond the support of $O$ by a distance that is at most $o(L)$, i.e. distance $L \to 0$ as $L \to \infty$.

**Index Theorem.** The index $\text{Ind}(P, U)$ is a multiple of $1/p$, i.e. $\text{Ind}(P, U) \in \mathbb{Z}/p$.

The index $[7]$ is independent of the choice of $\psi$ in the ground state sector, as follows from topological order since $U^1 QU - Q$ is a sum of local terms. The robustness enjoyed by the noninteracting index $[11]$ is also present here. For example, if we add to $Q$ an observable $B$ that is a sum of local terms supported around $\partial_-$, the index changes by $\langle \psi | (U^1 BU - B) | \psi \rangle$. By topological order and the locality of $B$, the expression takes the same value for any ground state and hence it equals $\frac{1}{p} \text{tr} P (U^1 BU - B) P$. By $[P, U] = 0$ and cyclicity of the trace, this vanishes. The index is also additive. If $U_j, j = 1, 2$ are two unitaries satisfying the assumptions with corresponding transported charges $T_j \equiv (U_j QU - Q)_\pm$ then $U_1 U_2 U_1 U_2 U_1 = Q + T_- + T_+$ with $T_- = T_-^{(1)} + U_1 T_-^{(2)} U_1$ and hence we get

$$\text{Ind}(P, U) = \text{Ind}(P, U_1) + \text{Ind}(P, U_2).$$

(3)

Both the non-interacting and the interacting setup can be seen as a Thouless pumps. They construct in a natural way an index out of $P$ and $U$. A significant difference is the possibility of rank $p > 1$, which gives rise to a rational index in $\mathbb{Z}/p$. Related approaches are found in $[6, 37, 38, 14]$.

**Splitting.** As already mentioned, there is a potential ambiguity in the splitting $U^1 QU - Q = T_- + T_+$. Indeed, if $T_\pm$ are valid choices, then so are $T_\pm + j$ for any real number $j$. There is a canonical physical choice in the case that $U = T e^{\int_0^1 ds G(s)}$ (time-ordered exponential) for a family of charge-conserving local Hamiltonians $G(s)$. Indeed, let $G = G - G_m + G_+$ be a splitting of the Hamiltonian $G$ (in charge-conserving terms) according to a partition of $\Gamma$ (see Figure 1), then we can set $T_\pm := T e^{\int_0^1 ds [G(s)]} - Q - Q$. Because of the commutator and charge conservation, this is independent of the chosen splitting of $G$. Since then $Q + T_\pm$ is unitarily conjugated to $Q$, our condition $e^{2\pi i (Q + T_\pm)} = 1$ is indeed satisfied. Together with $U$ being translation on the lattice, this case actually covers all interesting examples known to us. Let us now argue why the condition $e^{2\pi i (Q + T_\pm)} = 1$ can be satisfied in general. We split $Q = Q - Q_m + Q_+$ (see Figure 1) so that the three parts commute and have integer spectrum. We now demand that also $Q - Q_m + Q_+$ has integer spectrum (this is equivalent to $e^{2\pi i (Q + T_\pm)} = 1$) as it represents the total charge that eventually is present in a neighborhood of $\partial_-$. Let’s prove that such choice exists: $U^1 QU = (Q - T_+) + Q_m + (Q_+ + T_\pm)$ where the summands have disjoint supports. Since $U^1 QU$ and $Q_m$ have integer spectrum, the spectrum of $(Q_\pm + T_\pm)$ necessarily lies in $\mathbb{Z} \pm a$ and we can choose $j$ such that $(Q_\pm + T_\pm)$ has integer spectrum. The remaining freedom $j \in \mathbb{Z}$ is harmless to our results.

**Proof of the index theorem**

**Adiabatic Flux Insertion.** Let us define $[7, 22]

$$K := \int dt W(t) e^{itH} i[H, Q] e^{-itH}$$

(4)

with $W$ a real-valued, bounded function satisfying $W(t) = O(|t|^{-\infty})$ and $\tilde{W}(\omega) = \frac{1}{\omega}$ for all $|\omega| \geq \gamma$, with $\gamma$ the spectral gap of the Hamiltonian. The properties of $W$ yield that $[K, P] = [Q, P]$. By the total charge conservation and locality, we see that $[Q, H] = J_- + J_+$, with $J_\pm$ localized around $\partial_\pm$. Altogether, this implies that there are $K_\pm$ localized around $\partial_\pm$ such that

$$\tilde{Q} := Q - K_- - K_+$$

(5)
satisfies $[\bar{Q}, P] = 0$, and hence also $e^{2\pi i \bar{Q}}$ commutes with $P$. Since $Q_m$ has integral spectrum, the unitary decomposes as product of unitaries along $\partial_\phi$, $e^{2\pi i \bar{Q}} = e^{2\pi i Q_+} e^{2\pi i Q_-}$, with $Q_\pm = Q - K_\pm$. By the ‘Locality lemma’ below (we thank Filippo Santi for pointing out an error in the proof in the published version of this article), each of these unitaries alone leaves $P$ invariant. In SM we further explain that $e^{2\pi i \bar{Q}}$ is the ‘quasi-adiabatic’ implementation of $2\pi$ flux threading through $\partial_\phi$, provided that the Hamiltonian remains gapped during this process.

**Locality Lemma.** Let $V = V_- V_+$ with $V_\pm$ unitaries supported around $\partial_\phi$. Then $[P, V] = 0$ implies $[P, V_\pm] = 0$. **Proof:** By exponential clustering $PV\psi = PV_- PV_+ \psi$ for any ground state $\psi \in \text{ran } P$. Then, on one hand $\|PV\psi\| = 1$ by the assumption $[P, V] = 0$, on the other hand $\|PV_-PV_\psi\| \leq \|V_+P\|\|PV_\psi\| \leq 1$. Hence $\|PV_-\psi\| = 1$, and since $\|PV_\psi\| = 1$ by the same argument, we conclude that $[P, V_\pm] = 0$.

**Core argument.** We consider

$$Z_- = U^t e^{2\pi i \bar{Q}} U e^{-2\pi i \bar{Q}},$$

which will reveal the non-commutativity of $U$ and flux insertion $e^{2\pi i \bar{Q}}$. By the locality of $U$, $Z_-$ is supported around $\partial_\phi$. We are going to show that

$$PZ_- P = Pe^{2\pi i \text{tr}(PT_-)}.$$  

(7)

Since the RHS of (7) is a product of 4 unitaries commuting with $P$, we have that $\det(PZ_- P) = 1$, and hence $\text{tr}(PT_-) \in \mathbb{Z}$. The proof is now concluded, since, as noted before, the topological order condition implies that for any ground state $\psi$, $\langle \psi | T_- | \psi \rangle = \frac{1}{2} \text{tr}(PT_-)$.

**Proof of (7).** By integrality of $Q_m + Q_+$, we can replace $\bar{Q}_-$ by $Q = K_-$ in the first exponential of (6). Bringing $U^t(\cdot)U$ inside the exponential, we write

$$U^t(Q - K_-)U = (Q_+ + T_- - K_-^\prime) + Q_m + (Q_+ + T_+)$$

where we use a notation $O^t = U^t O U$ and the three bracketed terms commute, see Figure 4. The exponential of the second/third term is 1 by integrality/our constraint $e^{2\pi i (Q + T \pm)} = 1$. The exp of the first term leads to the identity $Z_- = e^{2\pi i (Q_+ + T_- - K_-^\prime) e^{-2\pi i \bar{Q}}}$. We now interpolate between 1 and $Z_-$ by the operator $Z_- (\phi) = e^{i\phi(Q + T_- - K_-^\prime) e^{-i\phi \bar{Q}}}$, and we prove that $Z_- (\phi, P) = 0$ for all $\phi$. Indeed, let us introduce the corresponding anti-twist $Z_+ (\phi) = e^{i\phi(Q_+ - T_+ - K_+^\prime) e^{-i\phi \bar{Q}} + \phi}$ Then we see that $Z_- (\phi, \bar{Q}_+ (\phi) = U^t e^{i\phi \bar{Q} U e^{-i\phi \bar{Q}} = Z(\phi)$ because far from $\partial_\phi$, the charge is unaffected by $U$. By $[Q, P] = 0, Z(\phi)$ commutes with $P$ and hence, by the Locality Lemma above, so do both $Z_\pm (\phi)$ as claimed.

We now differentiate $Z_- (\phi)$ w.r.t. $\phi$,

$$\partial_\phi (PZ_- (\phi) P) = PZ_- (\phi) e^{i\phi \bar{Q}} - i (T_- - K_-^\prime) e^{-i\phi \bar{Q}} - P.$$  

(8)

The quantity in (7), which we name $D_-$ is localized around $\partial_\phi$ so we can replace $e^{i\phi \bar{Q}} D_- e^{-i\phi \bar{Q}}$ by $e^{i\phi \bar{Q}} D_- e^{-i\phi \bar{Q}}$ and subsequently commute $e^{-i\phi \bar{Q}}$ with $P$. Using also $[Z_- (\phi), P] = 0$, we then rewrite (8) as

$$\partial_\phi (PZ_- (\phi) P) = iPZ_- (\phi) e^{i\phi \bar{Q}} P D_- e^{-i\phi \bar{Q}}.$$  

We now note that $PD_- P = PT_- P$ since $[U, P] = 0$. Furthermore, $T_-$ is a sum of local terms and hence topological order yields $PT_- P = P^{1/2} \text{tr}(P T_-)$. This means that we can also drop the factors $e^{i\phi \bar{Q}}$, because of $[P, Q] = 0$. We hence end up with a simple differential equation whose solution, evaluated at $\phi = 2\pi$, is (7).

Examples

We focus on applications of the index theorem to systems with degenerate ground state manifold.

**Fractional Lieb-Schultz-Mattis theorem.** Let $U$ be spatial translation by a one site to the left. Then $T_- = Q \{x_1 = -1\}$ is the charge operator in the hyperplane $\{x_1 = -1\}$. If the Hamiltonian is translation-invariant then $\rho = \langle \psi | T_- | \psi \rangle$ is the charge in any plane $\{x_1 = k\}$ and the theorem implies that $\rho \in \mathbb{Z}/p$. Of course, $p$ should scale $\propto L$ but the result is still meaningful as the equality holds up to $O(L^{-\infty})$. This theorem was already basically contained in the original treatments.

**Quanitzation of Hall conductance.** Here we rename $U_1 \equiv U = e^{2\pi i \bar{Q}}$ and we let $U_2$ be the analogous operator with $\partial_\phi$ being replaced by the orthogonal loop $\{x_2 = -1/2\}$, i.e. $U_2$ is constructed as $U_1$ upon replacing $Q$ by $\sum_{i \in \Gamma_2} n_i$, with $\Gamma_2 = \{0 < i_2 \leq L/2\}$. Now $T_-$ is simply the charge transported by threading a unit of flux in the 2-direction. This equals the Hall conductance $\sigma$ by the well-known Laughlin argument. Putting back physical units, our result is that

$$\sigma = \frac{q e^2}{p h}.$$  

This gives a mathematically rigorous proof of fractional quantization of $\sigma$ in an interacting setting that is shorter than previous arguments in $\{15, 17, 25\}$.

**Fractional Avron-Dana-Zak relations.** A fractional quantum Hall sample pierced by a rational flux $\phi$ has a Hamiltonian that is invariant under magnetic translations, which is a composition of a translation and threading the torus by $-\phi$ flux. Combining the discussions of FOHE and Lieb-Schultz-Mattis theorem, and relying on the additivity property of our index, we get the constraint $\rho - \phi \sigma \in \mathbb{Z}/p$. This relation was derived in $\{12\}$ for non-interacting systems (hence $p = 1$) and in $\{37, 51\}$ for interacting systems.
Braiding relations and Abelian anyons

Let $U_1, U_2$ be as above in the example of the FQHE. That is, they correspond to threading a unit of flux in the 1, 2-direction. Then, the four unitaries in (9) satisfy, by (7),

$$U_1^\dagger U_2 U_1^\dagger P = e^{2\pi i\frac{q}{p}} P,$$

and we recall that each of them remains unitary when restricted to $\text{ran} P$. Note that these restricted unitaries are naturally associated to oriented loops winding around the torus. If $\frac{q}{p}$ is noninteger, then (9) gives a nontrivial commutation relation between those loops, see [1, 34, 42]. In the case when $p > 1$ and $q, p$ are coprime, and the topological quantum field theory (TQFT) describing the ground state sector $\text{ran} P$ is a $U(1)$-Chern Simons theory, these loops can be identified with Wilson loops. In particular, the action of $PU_1, PU_2$ on any ground state $\psi$ generates the full ground state sector. This follows because there is no representation of $\mathbb{Z}$ on a space of dimension smaller than $p$. As far as we know, our approach is the first explicit construction of such loop operators in generic microscopic models, cf. 29, 33.

**Anyonic quasiparticles.** To any region $\Omega$, we associate $Q^\Omega = Q^\Omega - K^\Omega$, where the notation is a reminder of the fact that $K^\Omega$, defined as in (11) with $Q \rightarrow Q^\Omega$, is an operator supported on the boundary of the domain. By the integrality of the spectrum of $Q^\Omega$, $U^\Omega = e^{2\pi i Q^\Omega}$ is a loop operator supported around $\partial \Omega$. We can write it explicitly as $U^\Omega = Te^{-i \int_{\partial \Omega} d\gamma K^\Omega(\gamma)}$ where $K^\Omega(\phi) = e^{-i d\gamma K^\Omega} K^\Omega e^{i d\gamma K^\Omega}$. Since $K^\Omega$ is a sum of local terms, we can choose, albeit not in any canonical way, to retain only the terms associated to an open string $\gamma \subset \partial \Omega$ and this defines $U_\gamma$. Since all local terms commute with the total charge, so does $U_\gamma$. For a ground state $\psi$, $\varphi = U_\gamma \psi$ is a state with two localized excitations at the endpoints of $\gamma$, see Figure 2. Indeed, $\varphi$ and $\psi$ are locally indistinguishable away from the endpoints of $\gamma$. The charge of an excitation is the excess charge in a region $R$ around the excitation that does not extend to the other endpoint. It is given by

$$\epsilon = \langle \varphi | Q_R | \varphi \rangle - \langle \psi | Q_R | \psi \rangle = \langle \psi, (U_\gamma^\dagger Q_R U_\gamma - Q_R) \psi \rangle.$$

By charge conservation, $U_\gamma^\dagger Q_R U_\gamma - Q_R$ is supported at the intersection $\partial R \cap \gamma$, so that the excitation has a fractional charge

$$\epsilon = \frac{q}{p}$$

by applying the index theorem. The excitation at the other end point has opposite charge.

The factor $q/p$ also appears when braiding the excitations. For a closed contractible path $\alpha$, $U\alpha \psi$ is proportional to $\psi$, and we set the phase to be 0. When an excitation is present inside $\alpha$, the loop is not contractible anymore, and we obtain by (9)

$$U_\alpha \varphi = U_\gamma (U_\alpha^\dagger U_\gamma U_\alpha^\dagger) \psi = e^{2\pi i \frac{\varphi}{\psi}} \varphi.$$

Hence, the created excitations are Abelian anyons.

**Conclusions**

We described an index for systems with $U(1)$ symmetry (charge conservation), reminiscent of the Fredholm index. The index is associated to a charge transported across a hypersurface and it is rational, with denominator $p$ being the dimension of a topologically ordered ground state sector. We relate the index to a commutation relation on the ground state space, and show that the relation reveals the existence of anyonic excitations whenever the index is non-integer.

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[54] X.-G. Wen, F. Wilczek, and A. Zee. Chiral spin states and superconductivity. *Phys. Rev. B*, 39(16):11413, 1989.

[55] This is literally true for spin systems. For fermionic systems, an observable $O$ has support $X$ if it is a function of the field operators $c_i, c_i^\dagger$, $i \in X$.

[56] The natural notion here is that $O = \sum_{n \geq 0} O_n$, where $O_n$ is supported in the $n$-fattening of $X$ and $\|O_n\|^k \to 0$ as $n \to \infty$, for any $k$ and uniformly in $L$. 
SUPPLEMENTARY MATERIAL

Firstly, we present the proof, inspired by [30], of the index theorem for free fermions. Secondly, we give an explicit expression for the unitary associated with the process of quasi-adiabatic flux insertion. Although the expression is new, all its properties are well-known, see [24, 25].

INDEX THEOREM FOR FREE FERMIONS

We briefly review the setup. Let $\mathbb{T} = \mathbb{T}_L$ be the $L \times L$ discrete torus. We say that an operator $O$ on $L^2(\mathbb{T})$ has uniform rapid decay if

$$\sup_{i,j, |i-j| \geq \ell} |O_{ij}| = O(e^{-\ell}) ,$$

where $| \cdot |$ under the sup is the graph distance on $\mathbb{T}$. A restriction of the operator to a region $\Omega$ is given by $\Pi_{\Omega} A \Pi_{\Omega}$ with $\Pi_{\Omega} = \sum_{i \in \Omega} |i \rangle \langle i |$, and $A \mapsto A_{\pm}$ is the restriction to a region of width $\ell$ about $\partial_{\pm}$, where $\ell \to \infty$ but $\ell/L \to 0$.

Let $P = P(\ell) = P^2$ be a projection and $U$ a unitary such that both have rapid decay, in the sense above, and such that $[P, U] = O(L^{-\infty})$. We define

$$\text{Ind}(P, U) = \text{tr}[P(U^\dagger QU - Q)_{-}] .$$

The precise statement of the result in the main text is that

$$\text{dist}(\text{Ind}(P, U), Z) = O(L^{-\infty}) .$$

(10)

We remark that with the conditions given so far we cannot conclude that $\text{Ind}(P, U)$ converges to a fixed integer, as $L \to \infty$, because we did not demand that $P, U$ converge in any way. This could of course easily be done, but it would distract from the main point. We now prove [10] in an approach pioneered by [30].

We revert to the convention used in the main text that equalities hold up to $O(L^{-\infty})$ corrections and that $Q = \Pi_{\Gamma}$ is the charge of the region $\Gamma$ with boundaries $\partial_{\pm}$. By rapid decay of $P$,

$$K = PQ(1 - P) + (1 - P)QP$$

is of the form $K = K_- + K_+$, i.e. supported only at $\partial_- \cup \partial_+$. The operator

$$\bar{Q} = Q - K_- - K_+,$$

commutes with $P$. By rapid decay of $U$ and $[P, U] = 0$, we also have that

$$U^\dagger \bar{Q} U = Q + (U^\dagger QU - Q)_- + (U^\dagger QU - Q)_+ - K_- - K_+$$

commutes with $P$. Here we have again used the shorthand $OU = U^\dagger OU$. The two operators $\bar{Q}$ and $U^\dagger \bar{Q} U$ hence commute with $P$. On the other hand, their commutator with $P$ can naturally be decomposed into two terms supported at $\partial_{\pm}$. These two terms hence have to vanish independently. We conclude that the operator

$$N = Q + (U^\dagger QU - Q)_- - K_- - K_+$$

also commutes with $P$. Next, we consider the expression

$$Z_- = U^\dagger e^{2\pi i \bar{Q} U} e^{-2\pi i \bar{Q} U} -$$

with $\bar{Q} = \bar{Q} - K_-$. We note that by rapid decay $[e^{2\pi i \bar{Q} U}, P] = 0$. Let $\text{det}_P(A) = \text{det}(PA + (1 - P))$. Then, since $Z_-$ is a product of four unitaries commuting with $P$, we have

$$\text{det}_P(U^\dagger e^{2\pi i \bar{Q} U} e^{-2\pi i \bar{Q} U}) = 1$$

by the product rule for determinants. On the other hand we have

$$U^\dagger e^{2\pi i \bar{Q} U} e^{-2\pi i \bar{Q} U} = e^{2\pi i (Q - K_+ + (U^\dagger QU - Q)_-) - 2\pi i \bar{Q} U} = e^{2\pi i N} e^{-2\pi i \bar{Q} U} .$$

Since both operators in the exponentials commute with $P$, we have

$$\text{det}_P(U^\dagger e^{2\pi i \bar{Q} U} e^{-2\pi i \bar{Q} U}) = e^{2\pi i (\text{tr}(P(N - \bar{Q} U)))}$$

by the relation between determinant and trace. Plugging the definition of $N$ and using $\text{tr}(PK_+) = \text{tr}(PK_-)$ by $[P, U] = 0$, this exponential equals $e^{2\pi i (\text{tr}(P(U^\dagger QU - Q)_-))}$. It follows that $\text{tr}(P(U^\dagger QU - Q)_-) = 1$ is an integer, as was to be proven.

ADIABIATIC FLUX THREADING

This section refers to the interacting many-body setup. Therefore, the symbols $P, Q, U$ have now a different meaning than the ones in the previous section. We use a unitary modelling adiabatic flux threading through the loop $\partial_-$. Let $H(\phi) = e^{i\phi Q} He^{-i\phi Q}$ be a gauge equivalent ‘twist-antitwist’ Hamiltonian corresponding to threading flux $\phi$ through $\partial_-$ and removing it at $\partial_+$. The ground state projection is then $P(\phi) = e^{i\phi Q} Pe^{-i\phi Q}$ and the adiabatic evolution is generated by $Q$. Following [29], an alternative ‘quasi-adiabatic’ generator $K(\phi)$ was constructed in [7]

$$K(\phi) = \int dt W(t) e^{itH(\phi)} \partial_\phi H(\phi) e^{-itH(\phi)} ,$$

(11)

with $W$ a real-valued, bounded, integrable function satisfying $W(t) = O(|t|^{-\infty})$ and $\hat{W} (\omega) = \frac{1}{\sqrt{\gamma}}$ for all $|\omega| \geq \gamma$, with $\gamma$ the spectral gap of the Hamiltonian. It satisfies

$$\partial_\phi P(\phi) = i[K(\phi), P(\phi)] .$$
The advantage of the quasi-adiabatic generator is that it is manifestly supported only in those regions of space where the Hamiltonian actually changes. For the present, charge conserving Hamiltonian, this means that $K(\phi) = K_-(\phi) + K_+(\phi)$, with $K_\pm(\phi)$ localized around the loops $\partial_\pm$. Furthermore, it satisfies $K(\phi) = e^{i\phi Q} K e^{-i\phi Q}$ (we write $K = K(0)$) and from this it follows that the unitary

$$V(\phi) = e^{i\phi(Q-K)} e^{-i\phi Q}$$

(12) implements the ground state evolution: $P(\phi) = V(\phi)^\dagger P V(\phi)$.

Of course, the physically more interesting deformed Hamiltonian is one where the flux through $\partial_-$ is not removed at $\partial_+$. It is denoted by $H_-(\phi)$ and defined to be equal to $H(\phi)$ around $\partial_-$ and to $H$ otherwise. Unlike $H(\phi)$, it is not unitarily equivalent to $H$. If the gap remains open for $H_-(\phi)$ then $[11]$ with $H$ replaced by $H_-$ is the quasi-adiabatic generator associated to $H_-(\phi)$ and by locality it is equal to $K_-(\phi)$. It follows that the ground state projection $P_-(\phi)$ of $H_-(\phi)$ is obtained by replacing $K \rightarrow K_-$ in (12), i.e.

$$P_-(\phi) = V_-(\phi) P V_-(\phi)^\dagger, \quad V_-(\phi) = e^{i\phi(Q-K_-)} e^{-i\phi Q}.$$

In this case and by integrality of the spectrum of $Q$, $e^{2\pi i(Q-K_-)}$ corresponds to a $2\pi$ flux insertion across $\partial_-$, leaving the GS invariant:

$$[e^{2\pi i\bar{Q}_-}, P] = O(L^{-\infty}), \quad \bar{Q}_- = Q - K_-.$$

(13)

A remarkable fact [25] is that (13) holds even if the gap closes at some $\phi \neq 0$. 