On the heavenly equation hierarchy and its reductions

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Abstract

Second heavenly equation hierarchy is considered using the framework of hyper-Kähler hierarchy developed by Takasaki [1, 2]. Generating equations for the hierarchy are introduced, they are used to construct generating equations for reduced hierarchies. General N-reductions, logarithmic reduction and rational reduction for one of the Lax-Sato functions are discussed. It is demonstrated that rational reduction is equivalent to the symmetry constraint.

1 Introduction

Plebansky second heavenly equation [3], having its origin in general relativity, has attracted a lot of interest both from the viewpoint of integrability and relativity. It has been intensively studied using different techniques (see e.g. [4, 5, 6, 7, 8, 9, 10]).

In the work [11] we have developed a dressing scheme applicable to second heavenly equation. A very important role was played by a kind of Hirota bilinear identity, which leads to the introduction of the function Θ (analogue of the τ-function for heavenly equation hierarchy) and produces the hierarchy in the form of addition formulae (generating equations) for Θ. This identity also has its natural place in the framework of hyper-Kähler hierarchy developed by Takasaki [1, 2], who demonstrated that it is equivalent to the Lax-Sato equations of the hierarchy. Here we will use this framework to study the reductions of the heavenly equation hierarchy and its symmetry constraints. The ideas and logic of this work are very close to the works [12, 13, 14], where dispersionless hierarchies were considered.

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2 Heavenly equation hierarchy

First we introduce the principal objects and notations. We start from two formal Laurent series in $z$,

$$S^1 = \sum_{n=0}^{\infty} t^1_n z^n + \sum_{n=1}^{\infty} S^1_n(t^1, t^2) z^{-n},$$  \hspace{1cm} (1)

$$S^2 = \sum_{n=0}^{\infty} t^2_n z^n + \sum_{n=1}^{\infty} S^2_n(t^1, t^2) z^{-n},$$  \hspace{1cm} (2)

where the variables $t$ are considered independent and $S^1_n, S^2_n$ are dependent variables. We denote $x = t^1_0, y = t^2_0$, $S = \left( \begin{array}{c} S^1 \\ S^2 \end{array} \right)$, introduce the Poisson bracket $\{ f, g \} := f_x g_y - f_y g_x$ and the projectors $(\sum_{-\infty}^{\infty} u_n z^n)_+ = \sum_{n=0}^{\infty} u_n z^n, (\sum_{-\infty}^{-1} u_n z^n)_- = \sum_{n=-1}^{\infty} u_n z^n$.

Heavenly equation hierarchy is defined by the relation (see [1], [11])

$$(dS^1 \wedge dS^2)_- = 0,$$  \hspace{1cm} (3)

playing a role similar to the role of the famous Hirota bilinear identity for KP hierarchy. This relation is equivalent to the Lax-Sato form of the hierarchy.

Proposition 1 The identity (3) is equivalent to the set of equations

$$\partial^1_n S = - \{ (z^n S^2)_+, S \},$$  \hspace{1cm} (4)

$$\partial^2_n S = \{ (z^n S^1)_+, S \},$$  \hspace{1cm} (5)

$$\{ S^1, S^2 \} = 1.$$  \hspace{1cm} (6)

The proof of this statement is given in [1] for the general hyper-Kähler hierarchy (second heavenly equation hierarchy is its two-component special case). We will not reproduce the complete proof, but will just illustrate some ideas, which will be useful later. It is possible to prove that (3) implies Lax-Sato equations using the following statement.

Lemma 1 Given identity (3), for arbitrary first order operator $\hat{U}$,

$$\hat{U} S = \sum_i (u^1_i(z, t^1, t^2) \partial^1_i + u^2_i(z, t^1, t^2) \partial^2_i) S$$

with ‘plus’ coefficients $(u^1_i)_+ = (u^2_i)_- = 0$, the equality $(\hat{U} S)_+ = 0$ implies that $\hat{U} S = 0$. 

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Proof First, (3) implies that
\[ \{S^1, S^2\}_- = 0, \]
and, using (1), (2), we get
\[ \{S^1, S^2\}_+ = \{S^1, S^2\}_+ = 1. \]
Then, using (3) we obtain that
\[ \hat{U} S^1 \cdot S^2_x - \hat{U} S^2 \cdot S^1_x = 0, \]
\[ \hat{U} S^1 \cdot S^2_y - \hat{U} S^2 \cdot S^1_y = 0. \]
If \( \hat{U} S \neq 0 \), then we come to the conclusion that vectors \( S_x, S_y \) are linearly dependent, then \( \{S^1, S^2\}_+ = 0 \), and we come to a contradiction. □

The proof of Proposition 1 (sufficient condition) is then straightforward, one should just check directly that
\[ \left( \partial_1^n S + \{(z^n S^2)_+ + S\}_+ \right) = 0, \]
\[ \left( \partial_2^n S - \{(z^n S^1)_+ + S\}_+ \right) = 0. \]

Function \( \Theta \)
Identity (3) leads also to the introduction of an analogue of the \( \tau \)-function in terms of closed one-form.

Proposition 2 The one-form
\[ \theta = \frac{1}{2\pi i} \text{Res}_\infty (S^2_0 dS^1_+ - S^1_0 dS^2_+) \] (7)
is closed.

Proof Identity (3) implies that
\[ d\theta = \frac{1}{2\pi i} \text{Res}_\infty (dS^2_- \wedge dS^1_+ - dS^1_- \wedge dS^2_+) = 0. \]

Similar to the work [1], we define a \( \tau \)-function \( \Theta(t^1, t^2) \) for heavenly equation hierarchy through closed one-form (7) by the relation \( d\Theta = \theta \). Introducing vertex operators \( D^1(z) = \sum_{n=0}^\infty z^{-n-1} \partial_1^n, \) \( D^2(z) = \sum_{n=0}^\infty z^{-n-1} \partial_2^n, \) it is easy to demonstrate that
\[ S^1_-(z) = -D^2(z) \Theta, \] \[ S^2_-(z) = D^1(z) \Theta. \] (8)
Substituting this representation into (6), we get the equation
\[ D^2(z)\Theta_x - D^1(z)\Theta_y - \{D^1(z)\Theta, D^2(z)\Theta\} = 0 \] (9)
The first nontrivial order of expansion of this equation at \( z \to \infty \) gives exactly the heavenly equation
\[ \Theta_{ty} - \Theta_{tx} - \Theta_{xy}^2 + \Theta_{xx}\Theta_{yy} = 0, \] (10)
where \( t = t_1, \tilde{t} = t_2^* \).
Identity (3) gives also a general set of addition formulae (generating equations in terms of vertex operators) for \( \Theta \) [11],
\[
\frac{1}{z' - z}D^1(z')D^2(z)\Theta + \frac{1}{z'' - z}D^1(z'')D^1(z')\Theta = D^1(z')D^2(z)\Theta + D^1(z')D^1(z)\Theta - D^1(z'')D^1(z)\Theta.
\] (11)
\[
\frac{1}{z'' - z}D^2(z'')D^2(z)\Theta + \frac{1}{z' - z}D^2(z')D^1(z')\Theta = D^2(z')D^2(z)\Theta + D^2(z')D^1(z)\Theta - D^2(z'')D^1(z)\Theta.
\] (11)
\[
\frac{1}{z' - z}D^2(z')D^1(z)\Theta + \frac{1}{z'' - z}D^1(z'')D^2(z)\Theta = D^1(z')D^1(z)\Theta + D^2(z'')D^2(z)\Theta - D^1(z')D^2(z)\Theta.
\] (11)
Expansion of these equations into powers of parameters \( z, z'', z'' \) generates partial differential equations for \( \Theta \) of the heavenly equation hierarchy.

**Generating Lax-Sato equations**

We introduce also generating equations for the Lax-Sato form of the hierarchy,
\[
(z' - z)D^1(z')S(z) = -\{S^2(z'), S(z)\}, \quad (12)
\]
\[
(z' - z)D^2(z')S(z) = \{S^1(z'), S(z)\}, \quad (13)
\]
which are equivalent to the set of equations [11], [5], [10] (that can be checked directly or using Lemma [11]).

It is interesting to note that (12), (13) imply the following symmetric expressions for Poisson brackets:
\[
\{S^1(z'), S^2(z)\} = 1 + (z' - z)D^2(z')D^1(z)\Theta,
\]
\[
\{S^1(z'), S^1(z)\} = (z - z')D^2(z')D^2(z)\Theta,
\]
\[
\{S^2(z'), S^2(z)\} = (z - z')D^1(z')D^1(z)\Theta.
\]
3 Reductions

General $N$-reductions

We will discuss first the properties of general reduction, when one of the functions $S_1^-, S_2^-$ depends on $N$ independent functions of times (i.e., only $N$ coefficients of expansion in $z^{-1}$ are independent). Reductions of this type were studied a lot in dispersionless case (see e.g. [15], [16]).

**Proposition 3** Following three statements are equivalent:

1) $S_1^-(z, t_1, t_2) = S_1^-(z, f_1(t_1, t_2), \ldots, f_N(t_1, t_2))$,

2) $\frac{\partial^2 N}{\partial N S_1^-(z, t_1, t_2)} - \sum_{i=0}^{N-1} \phi_i(t_1, t_2) \frac{\partial^2 S_1^-}{\partial^2_1}(z, t_1, t_2) = 0$,

3) $\frac{S_1^y}{S_1^x}$ is a rational function with $N$ poles,

$$\frac{S_1^y}{S_1^x} = \sum_{i=1}^{N} \frac{u_i}{z - v_i},$$

where $f_i, \phi_i, u_i, v_i$ are some functions of times.

**Proof** 1 $\Leftrightarrow$ 2 is evident and it is not connected with equations of the hierarchy; 1 $\Rightarrow$ 2 requires some linear algebra, and 2 $\Rightarrow$ 1 is proved by the method of characteristics. The absence of minus projector in (15) (in contrast with (14)) is connected with the fact that $S_1^-$ is of the form (1) and $\frac{\partial^2 S_1^-}{\partial^2_1} = \frac{\partial^2 S_1^-}{\partial^2_1}$.

2 $\Rightarrow$ 3 Using equations of the hierarchy (15), one obtains

$$\frac{\partial^2 S_1^-}{\partial^2_1} = H_2^2 \frac{S_1^y}{S_1^x} - H_2^2_{xy},$$

where $H_2^2 = (z^n S_1^+)$. Substituting these expressions to relation (15) divided by $S_1^x$, one gets

$$\frac{S_1^y}{S_1^x} = \frac{H_2^2_{N_y} - \sum_{i=0}^{N-1} \phi_i H_2^2_{i_y}}{H_2^2_{N_x} - \sum_{i=0}^{N-1} \phi_i H_2^2_{i_x}},$$

that is evidently a rational function with $N$ poles.

3 $\Rightarrow$ 2 Using equations of the hierarchy and formula (16), we come to the conclusion that all ratios $\frac{\partial^2 S_1^-}{\partial^2_1}$ are rational functions in $z$ with $N$ poles.
in the same points, that implies relation (15).

A short comment on the Proposition. Formula (14) gives a standard definition of $\mathcal{N}$-reduction similar to the dispersionless case (see e.g. [15], [16]). Equivalent formulation (15) suggests invariance of the hierarchy under the action of some vector field and it is probably useful for geometric interpretation of $\mathcal{N}$-reduction. And finally, statement 3 gives analytic characterization of the reduction in terms of Lax-Sato functions. This statement implies also that all ratios $\frac{\partial^{2} S^1}{\partial z \partial S^1}$ are rational functions of $z$.

Similar statements are also known in the dispersionless case [17].

Generating equations for reduced hierarchy

To obtain linear equations of the reduced hierarchy, we use (16) to express $S^1_y$ through $S^1_x$ in generating Lax-Sato equations,

\[
(z - z')D^1(z')S^1(z) = \left(S^2_x(z') \sum_i \frac{u_i}{z - v_i} - S^2_y(z')\right) \partial_x S^1(z) = U^1 \partial_x S^1(z),
\]

(18)

\[
(z' - z)D^2(z')S^1(z) = \left(S^1_x(z') \sum_i \frac{z - v_i}{u_i} - S^1_y(z')\right) \partial_x S^1(z) = U^2 \partial_x S^1(z),
\]

(19)

\[
\partial_y S^1 = \sum_{i=1}^{N} \frac{u_i}{z - v_i} \partial_x S^1 = V \partial_x S^1.
\]

(20)

Compatibility conditions for these equations are

\[
(z - z')D^1(z')V - \partial_y U^1 + VU^1_x - V_x U^1 = 0,
\]

(21)

\[
(z' - z)D^2(z')V - \partial_y U^2 + VU^2_x - V_x U^2 = 0.
\]

(22)

First, both equations (zero order term at $z = \infty$) give an important relation

\[
\Theta_{yy} = -\sum_i u_i
\]

(23)

connecting $S^1, S^2$ with $u_i$,

\[
S^1(z') = S^1_+(z') + D^2(z')\Theta, \quad S^2(z') = S^2_+(z') - D^1(z')\Theta
\]

(24)
Considering equation (21) at \( z = v_j \), one obtains a system

\[
(z' - v_j) D^1(z') u_j - u_j D^1(z') v_j = -(S^2_x u_j)_y + 2S^2_{xx}(z') \sum_{i(i \neq j)} \frac{u_i}{v_j - v_i} - (u_j S^2_{xy} - u_{jx} S^2_y)
\]

\[
(z' - v_j) D^1(z') v_j = S^2_{xx} u_j + (S^2_y v_{jx} - S^2_x v_{jy}).
\]

Taking into account expressions (24), this is a closed (2+1)-dimensional system of equations generating \( t_n^1 \) flows of reduced hierarchy.

Equation (21) gives a system generating flows connected with \( t_n^2 \),

\[
(z' - v_j) D^2(z') u_j - u_j D^2(z') v_j = (S^1_x u_j)_y - 2S^1_{xx}(z') \sum_{i(i \neq j)} \frac{u_i}{v_j - v_i} + (u_j S^1_{xy} - u_{jx} S^1_y)
\]

\[
(z' - v_j) D^2(z') v_j = -S^1_{xx} u_j - (S^1_y v_{jx} - S^1_x v_{jy}).
\]

Let us consider the first systems of the reduced hierarchy. The first order of expansion of (18), (19) in \( z' - 1 \) provides linear equations for these systems (plus (20)),

\[
(\partial_t - z \partial_x) S^1 = \left( -\Theta_{xx} \sum_i \frac{u_i}{z - v_i} + \Theta_{xy} \right) \partial_x S^1(z),
\]

\[
(\partial_t - z \partial_y) S^1 = \left( -\Theta_{xy} \sum_i \frac{u_i}{z - v_i} + \Theta_{yy} \right) \partial_x S^1(z).
\]

The first order of expansion of (25), (26) in \( z' - 1 \) gives the first systems of reduced hierarchy,

\[
(\partial_t - v_j \partial_x) u_j - u_j \partial_x v_j = -(\Theta_{xx} u_j)_y + 2\Theta_{xxx} \sum_{i(i \neq j)} \frac{u_i}{v_j - v_i} - (u_j \Theta_{xx} u_j - u_{jx} \Theta_{xy})
\]

\[
(\partial_t - v_j \partial_x) v_j = \Theta_{xx} u_j + (\Theta_{xy} v_{jx} - \Theta_{xx} v_{jy})
\]

(27)

and

\[
(\partial_t - v_j \partial_y) u_j - u_j \partial_y v_j = -(\Theta_{xy} u_j)_y + 2\Theta_{xyy} \sum_{i(i \neq j)} \frac{u_i}{v_j - v_i} - (u_j \Theta_{xy} u_j - u_{jx} \Theta_{yy})
\]

\[
(\partial_t - v_j \partial_y) v_j = \Theta_{xy} u_j + (\Theta_{yy} v_{jx} - \Theta_{xy} v_{jy}),
\]

(28)
where $\Theta$ is defined by relation (23).

Now we will consider some simple special cases of the general reduction, when the function $S^1$ has simple analytic properties in $z$.

**Logarithmic reduction**

In this case $S^1$ is of the form

$$S^1 = S^1_+ - \sum_{i=1}^{N} c_i \ln(1 - \frac{u_i}{z}).$$

Generating equations for the reduced hierarchy read

$$(z' - u_j)D^1(z')u_j = -\{D^1(z')\Theta, u_j\},$$

$$(z' - u_j)D^2(z')u_j = -\{D^2(z')\Theta, u_j\},$$

$$D^2(z')\Theta = \sum_{i=1}^{N} c_i \ln(1 - \frac{u_i}{z'}).$$

The first two (2+1)-dimensional systems of reduced hierarchy are

$$\partial_{t} u_{k} = u_{k} \partial_{y} u_{k} + \sum_{i} c_{i} \{u_{i}, u_{k}\}$$

and

$$\partial_{t} u_{k} = u_{k} \partial_{z} u_{k} - (u_{k})_{x} \partial_{x} \sum_{i} c_{i} u_{i} - \Theta_{xx} \partial_{y} u_{k},$$

$$\Theta_{y} = -\sum_{i} c_{i} u_{i}.$$

Common solution to these systems gives a solution $\Theta$ to heavenly equation (2).

**Rational reduction**

We consider $S^1$ of the form

$$S^1 = S^1_+ + \sum_{i=1}^{N} \frac{u_i}{z - z_i}.$$
Generating equations for the reduced hierarchy read
\[(z' - z_j)D^1(z')u_j = -\{D^1(z')\Theta, u_j\},\]
\[(z' - z_j)D^2(z')u_j = -\{D^2(z')\Theta, u_j\},\]
\[D^2(z')\Theta = -\sum_{i=1}^{N} \frac{u_i}{z' - z_i}.\]

The first two (2+1)-dimensional systems of reduced hierarchy are
\[\partial_t u_k = z_k \partial_y u_k + \sum_i \{u_i, u_k\}\]
and
\[\partial_t u_k = z_k \partial_x u_k - (u_k) \partial_x \sum_i u_i - \Theta xx \partial_y u_k,\]
\[\partial_y \Theta = -\sum_i u_i.\]

(1+1)-dimensional reductions

If we use rational or logarithmic reduction for both \(S^1, S^2\), we obtain (1+1) dimensional systems of equations for coefficients directly from (6). The reduction with both \(S^1, S^2\) rational was considered in [18].

Let us use logarithmic reduction for both \(S^1, S^2\),
\[S^1 = S^1_+ - \sum_{i=1}^{N} c_i \ln(1 - \frac{u_i}{z}),\]
\[S^2 = S^2_+ - \sum_{i=1}^{M} c_i \ln(1 - \frac{v_i}{z}).\]

Then from (6) we get a (1+1)-dimensional system of equations
\[\partial_x u_k + \sum_i c_i \frac{\{u_k, v_i\}}{u_k - v_i} = 0,\]
\[\partial_y v_j - \sum_i c_i \frac{\{v_j, u_i\}}{v_j - u_i} = 0.\]

Using the expressions
\[S^1_n = \sum_{i=1}^{N} c_i \frac{u_i}{i}, \quad S^2_n = \sum_{i=1}^{M} c_i \frac{v_i}{i},\]

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we obtain the systems defining the dependence of \( u_k, v_j \) on higher times, the first two of them are

\[
\begin{align*}
\partial_t u_k &= u_k \partial_x u_k - \sum_i c_i \{ v_i, u_k \}, \\
\partial_t v_j &= v_j \partial_x v_j - \sum_i c_i \{ v_i, v_j \},
\end{align*}
\]

\[
\begin{align*}
\partial_{\tilde{t}} u_k &= u_k \partial_y u_k + \sum_i c_i \{ u_i, u_k \}, \\
\partial_{\tilde{t}} v_j &= v_j \partial_y v_j + \sum_i c_i \{ u_i, v_j \}.
\end{align*}
\]

## 4 Symmetry constraints

In this section we will consider symmetries of the heavenly equation hierarchy defined through the wave functions of the hierarchy (solutions to linear equations of the hierarchy) and symmetry constraints connected with these symmetries. Symmetries of this type were discussed in the work [11] starting from explicit formula for the function \( \Theta \). Similar symmetry constraints are well known in KP hierarchy case (see e.g. [19]) as well as in dispersionless case [14]. We will demonstrate that symmetry constraint is equivalent to rational reduction (for one of the functions \( S^1, S^2 \)) of the heavenly equation hierarchy.

We introduce a set of wave functions \( \sigma_i(t_1, t_2) \) depending only on the times of the hierarchy (no dependence on \( z \)),

\[
\begin{align*}
(z - z_i)D^1(z)\sigma_i &= -\{ S^2(z), \sigma_i \}, \quad (29) \\
(z - z_i)D^2(z)\sigma_i &= \{ S^1(z), \sigma_i \}, \quad (30)
\end{align*}
\]

where \( z_i, 1 \leq i \leq N \), is some fixed set of points.

**Proposition 4** \( \delta \Theta = \sigma_i \) is an infinitesimal symmetry for \( \Theta \) (i.e., it satisfies linearized equations of the hierarchy).

**Proof** Taking vertex cross-derivatives of (29), (30), we get

\[
\{ S^1(z), D^1(z)\sigma_i \} + \{ S^2(z), D^2(z)\sigma_i \} = 0.
\]

Then, using the representation of \( S^1, S^2 \) in terms of \( \Theta \) [5], we obtain

\[
D^2(z)\partial_x \sigma_i - D^1(z)\partial_y \sigma_i - \{ D^1(z)\sigma_i, D^2(z)\Theta \} - \{ D^1(z)\Theta, D^2(z)\sigma_i \} = 0,
\]

that is exactly the linearization of equation (3). In a similar manner, it possible to prove that \( \sigma_i \) satisfies the linearization of a general set of addition formulae. \( \square \)
Then it is possible to introduce the symmetry constraint

$$\Theta_x = \sum_{i=1}^{N} \sigma_i.$$  \hspace{1cm} (31)

**Proposition 5** The constraint (31) is equivalent to

$$S^2(z) = S^2_+(z) + \sum_{i=1}^{N} \frac{\sigma_i}{z - z_i}.$$  \hspace{1cm} (32)

**Proof** First, it is straightforward to demonstrate (using (8)) that constraint (31) is a necessary condition for $S^2$ to be of the form (32). To prove that it is sufficient, we will prove first the uniqueness of $S^2$ satisfying the set of linear equations associated with the heavenly equation hierarchy.

**Lemma 2** If the function $s^2(z, t^1, t^2)$ satisfies linear equations

$$(z' - z)D^1(z')s^2(z) = -\{S^2(z'), s^2(z)\}$$  \hspace{1cm} (33)

$$(z' - z)D^2(z')s^2(z) = \{S^1(z'), s^2(z)\}$$  \hspace{1cm} (34)

(or, equivalently, the set of linear equations associated with (4), (5)) and

$$(S^2)_+ = (s^2)_+,$$ then $s^2 = S^2$ (up to a function of $z$ only).

**Proof** (Lemma 2) Taking (33), (34) at $z = z'$, we get

$$\{S^2(z), s^2(z)\} = 0, \quad \{S^1(z), s^2(z)\} = 1.$$

Taking into account that $\{S^1, S^2\} = 1$, we come to the conclusion that

$$s^2(z) = S^2_+(z) + \phi,$$

where $\phi_x = \phi_y = 0$. Substituting $s^2$ to (33), (34) and taking into account that $\phi$ annihilates the Poisson bracket, we obtain that

$$D^1(z')\phi(z) = D^2(z')\phi(z) = 0,$$

thus $\phi$ is independent of all times of the hierarchy, so it doesn’t influence the dynamics and reflects a freedom in the definition of $S^2$. \qed
To finish the proof of Proposition 5, it is enough to demonstrate that under the constraint (31) function in the r.h.s. of (32) satisfies equations (33), (34). Substituting this function to (34), we get

\[ 1 + (z' - z) \sum_i \frac{1}{z - z_i} \left\{ S^1(z'), \sigma_i \right\} = S^1_x(z') + \sum_i \frac{1}{z - z_i} \left\{ S^1(z'), \sigma_i \right\}. \]

Both l.h.s. and r.h.s. are rational in \( z \), the coefficients of the poles at \( z_i \) are evidently equal, and the condition at \( z = \infty \) is (we use \( z \) instead of \( z' \))

\[ (S^1_x)_- + \sum_i \frac{1}{z - z_i} \left\{ S^1, \sigma_i \right\} = 0. \]

Using the equations for \( \sigma_i \) (29), (30) to get

\[ (S^1_x(z))_- + D^2(z) \sum_i \sigma_i = 0, \]

and applying the constraint (31) we discover that the condition at \( z = \infty \) is indeed satisfied, so the function \( S^2(z) + \sum_i \sigma_i \) satisfies (34). In a similar manner, it is possible to prove that this function satisfies (33). Then, using Lemma 2 we come to the conclusion that

\[ S^2(z) = S^2_+(z) + \sum_{i=1}^N \frac{\sigma_i}{z - z_i}. \]

Finally, we will formulate a more general statement; the proof is completely analogous.

**Proposition 6** The constraint

\[ \partial^1_n \Theta = \sum_{i=1}^N \sigma_i. \]  

is equivalent to

\[ S^2(z) = S^2_+(z) + \sum_{j=1}^n \nu_j \frac{1}{z} + \sum_{i=1}^N \frac{\sigma_i}{z - z_i}, \]

where the functions \( \nu_j \) are defined by the relations

\[ \partial^1_{j-1} \Theta = \nu_j. \]
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