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A law of large numbers approximation for Markov population processes with countably many types

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Abstract

When modelling metapopulation dynamics, the influence of a single patch on the metapopulation depends on the number of individuals in the patch. Since the population size has no natural upper limit, this leads to systems in which there are countably infinitely many possible types of individual. Analogous considerations apply in the transmission of parasitic diseases. In this paper, we prove a law of large numbers for quite general systems of this kind, together with a rather sharp bound on the rate of convergence in an appropriately chosen weighted $\ell_1$ norm.

Keywords: Epidemic models, metapopulation processes, countably many types, quantitative law of large numbers, Markov population processes

AMS subject classification: 92D30, 60J27, 60B12

Running head: A law of large numbers approximation

1 Introduction

There are many biological systems that consist of entities that differ in their influence according to the number of active elements associated with them,
and can be divided into types accordingly. In parasitic diseases (Barbour & Kafetzaki 1993, Luchsinger 2001a,b, Kretzschmar 1993), the infectivity of a host depends on the number of parasites that it carries; in metapopulations, the migration pressure exerted by a patch is related to the number of its inhabitants (Arrigoni 2003); the behaviour of a cell may depend on the number of copies of a particular gene that it contains (Kimmel & Axelrod 2002, Chapter 7); and so on. In none of these examples is there a natural upper limit to the number of associated elements, so that the natural setting for a mathematical model is one in which there are countably infinitely many possible types of individual. In addition, transition rates typically increase with the number of associated elements in the system — for instance, each parasite has an individual death rate, so that the overall death rate of parasites grows at least as fast as the number of parasites — and this leads to processes with unbounded transition rates. This paper is concerned with approximations to density dependent Markov models of this kind, when the typical population size $N$ becomes large.

In density dependent Markov population processes with only finitely many types of individual, a law of large numbers approximation, in the form of a system of ordinary differential equations, was established by Kurtz (1970), together with a diffusion approximation (Kurtz, 1971). In the infinite dimensional case, the law of large numbers was proved for some specific models (Barbour & Kafetzaki 1993, Luchsinger 2001b, Arrigoni 2003, see also Léonard 1990), using individually tailored methods. A more general result was then given by Eibeck & Wagner (2003). In Barbour & Luczak (2008), the law of large numbers was strengthened by the addition of an error bound in $\ell_1$ that is close to optimal order in $N$. Their argument makes use of an intermediate approximation involving an independent particles process, for which the law of large numbers is relatively easy to analyse. This process is then shown to be sufficiently close to the interacting process of actual interest, by means of a coupling argument. However, the generality of the results obtained is limited by the simple structure of the intermediate process, and the model of Arrigoni (2003), for instance, lies outside their scope.

In this paper, we develop an entirely different approach, which circumvents the need for an intermediate approximation, enabling a much wider class of models to be addressed. The setting is that of families of Markov population processes $X_N := (X_N(t), t \geq 0), N \geq 1$, taking values in the countable space $\mathcal{X}_+ := \{X \in \mathbb{Z}_{+}^\mathbb{Z}_{+}; \sum_{m \geq 0} X^m < \infty\}$. Each component repre-
sents the number of individuals of a particular type, and there are countably many types possible; however, at any given time, there are only finitely many individuals in the system. The process evolves as a Markov process with state-dependent transitions

\[ X \to X + J \text{ at rate } N{\alpha_J}(N^{-1}X), \quad X \in \mathcal{X}_+, \; J \in \mathcal{J}, \]

(1.1)

where each jump is of bounded influence, in the sense that

\[ \mathcal{J} \subset \{X \in \mathbb{Z}^+; \sum_{m \geq 0} |X^m| \leq J_x < \infty\}, \quad \text{for some fixed } J_x < \infty, \]

(1.2)

so that the number of individuals affected is uniformly bounded. Density dependence is reflected in the fact that the arguments of the functions \( \alpha_J \) are counts normalised by the ‘typical size’ \( N \). Writing \( \mathcal{R} := \mathbb{R}^\mathbb{Z}_+ \), the functions \( \alpha_J: \mathcal{R} \to \mathbb{R}_+ \) are assumed to satisfy

\[ \sum_{J \in \mathcal{J}} \alpha_J(\xi) < \infty, \quad \xi \in \mathcal{R}_0, \]

(1.3)

where \( \mathcal{R}_0 := \{\xi \in \mathcal{R}; \xi_i = 0 \text{ for all but finitely many } i\} \); this assumption implies that the processes \( X_N \) are pure jump processes, at least for some non-zero length of time. To prevent the paths leaving \( \mathcal{X}_+ \), we also assume that \( J_l \geq -1 \) for each \( l \), and that \( \alpha_J(\xi) = 0 \) if \( \xi^l = 0 \) for any \( J \in \mathcal{J} \) such that \( J^l = -1 \). Some remarks on the consequences of allowing transitions \( J \) with \( J^l \leq -2 \) for some \( l \) are made at the end of Section 4.

The law of large numbers is then formally expressed in terms of the system of deterministic equations

\[ \frac{d\xi}{dt} = \sum_{J \in \mathcal{J}} J\alpha_J(\xi) =: F_0(\xi), \]

(1.4)

to be understood componentwise for those \( \xi \in \mathcal{R} \) such that

\[ \sum_{J \in \mathcal{J}} |J^l|\alpha_J(\xi) < \infty, \quad \text{for all } l \geq 0, \]

thus by assumption including \( \mathcal{R}_0 \). Here, the quantity \( F_0 \) represents the infinitesimal average drift of the components of the random process. However, in this generality, it is not even immediately clear that equations (1.4) have a solution.
In order to make progress, it is assumed that the unbounded components in the transition rates can be assimilated into a linear part, in the sense that $F_0$ can be written in the form

$$F_0(\xi) = A\xi + F(\xi), \quad (1.5)$$

again to be understood componentwise, where $A$ is a constant $\mathbb{Z}_+ \times \mathbb{Z}_+$ matrix. These equations are then treated as a perturbed linear system (Pazy 1983, Chapter 6). Under suitable assumptions on $A$, there exists a measure $\mu$ on $\mathbb{Z}_+ \times \mathbb{Z}_+$, defining a weighted $\ell_1$ norm $\| \cdot \|_\mu$ on $\mathcal{R}$, and a strongly $\| \cdot \|_\mu$–continuous semigroup $\{ R(t), t \geq 0 \}$ of transition matrices having pointwise derivative $R'(0) = A$. If $F$ is locally $\| \cdot \|_\mu$–Lipschitz and $\| x(0) \|_\mu < \infty$, this suggests using the solution $x$ of the integral equation

$$x(t) = R(t)x(0) + \int_0^t R(t-s)F(x(s)) \, ds \quad (1.6)$$

as an approximation to $x_N := N^{-1}X_N$, instead of solving the deterministic equations (1.4) directly. We go on to show that the solution $X_N$ of the stochastic system can be expressed using a formula similar to (1.6), which has an additional stochastic component in the perturbation:

$$x_N(t) = R(t)x_N(0) + \int_0^t R(t-s)F(x_N(s)) \, ds + \tilde{m}_N(t), \quad (1.7)$$

where

$$\tilde{m}_N(t) := \int_0^t R(t-s) \, dm_N(s), \quad (1.8)$$

and $m_N$ is the local martingale given by

$$m_N(t) := x_N(t) - x_N(0) - \int_0^t F_0(x_N(s)) \, ds. \quad (1.9)$$

The quantity $m_N$ can be expected to be small, at least componentwise, under reasonable conditions.

To obtain tight control over $\tilde{m}_N$ in all components simultaneously, sufficient to ensure that $\sup_{0 \leq s \leq t} \| \tilde{m}_N(s) \|_\mu$ is small, we derive Chernoff–like bounds on the deviations of the most significant components, with the help of a family of exponential martingales. The remaining components are treated using some general $a$ priori bounds on the behaviour of the stochastic system.
This allows us to take the difference between the stochastic and deterministic equations (1.7) and (1.6), after which a Gronwall argument can be carried through, leading to the desired approximation.

The main result, Theorem 4.7, guarantees an approximation error of order $O(N^{-1/2} \sqrt{\log N})$ in the weighted $\ell_1$ metric $\| \cdot \|_\mu$, except on an event of probability of order $O(N^{-1} \log N)$. More precisely, for each $T > 0$, there exist constants $K_T^{(1)}, K_T^{(2)}, K_T^{(3)}$ such that, for $N$ large enough, if

$$
\| N^{-1} X_N(0) - x(0) \|_\mu \leq K_T^{(1)} \sqrt{\frac{\log N}{N}},
$$

then

$$
P\left( \sup_{0 \leq t \leq T} \| N^{-1} X_N(t) - x(t) \|_\mu > K_T^{(2)} \sqrt{\frac{\log N}{N}} \right) \leq K_T^{(3)} \frac{\log N}{N}.
$$

(1.10)

The error bound is sharper, by a factor of $\log N$, than that given in Barbour & Luczak (2008), and the theorem is applicable to a much wider class of models. However, the method of proof involves moment arguments, which require somewhat stronger assumptions on the initial state of the system, and, in models such as that of Barbour & Kafetzaki (1993), on the choice of infection distributions allowed. The conditions under which the theorem holds can be divided into three categories: growth conditions on the transition rates, so that the $a$ priori bounds, which have the character of moment bounds, can be established; conditions on the matrix $A$, sufficient to limit the growth of the semigroup $R$, and (together with the properties of $F$) to determine the weights defining the metric in which the approximation is to be carried out; and conditions on the initial state of the system. The $a$ priori bounds are derived in Section 2, the semigroup analysis is conducted in Section 3, and the approximation proper is carried out in Section 4. The paper concludes in Section 5 with some examples.

The form (1.8) of the stochastic component $\tilde{m}_N(t)$ in (1.7) is very similar to that of a key element in the analysis of stochastic partial differential equations; see, for example, Chow (2007, Section 6.6). The SPDE arguments used for its control are however typically conducted in a Hilbert space context. Our setting is quite different in nature, and it does not seem clear how to translate the SPDE methods into our context.
2 A priori bounds

We begin by imposing further conditions on the transition rates of the process $X_N$, sufficient to constrain its paths to bounded subsets of $X_+$ during finite time intervals, and in particular to ensure that only finitely many jumps can occur in finite time. The conditions that follow have the flavour of moment conditions on the jump distributions. Since the index $j \in \mathbb{Z}_+$ is symbolic in nature, we start by fixing an $\nu \in \mathbb{R}$, such that $\nu(j)$ reflects in some sense the 'size' of $j$, with most indices being 'large':

$$\nu(j) \geq 1 \text{ for all } j \geq 0 \text{ and } \lim_{j \to \infty} \nu(j) = \infty.$$  

(2.1)

We then define the analogues of higher empirical moments using the quantities $\nu_r \in \mathbb{R}$, defined by $\nu_r(j) := \nu(j)^r$, $r \geq 0$, setting

$$S_r(x) := \sum_{j \geq 0} \nu_r(j)x^j = x^T \nu_r, \quad x \in \mathbb{R}_0.$$  

(2.2)

where, for $x \in \mathbb{R}_0$ and $y \in \mathbb{R}$, $x^T y := \sum_{i \geq 0} x_i y_i$. In particular, for $X \in X_+$, $S_0(X) = \|X\|_1$. Note that, because of (2.1), for any $r \geq 1$,

$$\# \{X \in X_+: S_r(X) \leq K\} < \infty \text{ for all } K > 0.$$

(2.3)

To formulate the conditions that limit the growth of the empirical moments of $X_N(t)$ with $t$, we also define

$$U_r(x) := \sum_{J \in \mathcal{J}} \alpha_J(x)J^T \nu_r; \quad V_r(x) := \sum_{J \in \mathcal{J}} \alpha_J(x)(J^T \nu_r)^2, \quad x \in \mathcal{R}.$$  

(2.4)

The assumptions that we shall need are then as follows.

**Assumption 2.1** There exists a $\nu$ satisfying (2.1) and $r_{\text{max}}^{(1)}, r_{\text{max}}^{(2)} \geq 1$ such that, for all $X \in X_+$,

$$\sum_{J \in \mathcal{J}} \alpha_J(N^{-1}X)|J^T \nu_r| < \infty, \quad 0 \leq r \leq r_{\text{max}}^{(1)},$$

(2.5)

the case $r = 0$ following from (1.2) and (1.3); furthermore, for some non-negative constants $k_{rt}$, the inequalities

$$U_0(x) \leq k_{01}S_0(x) + k_{04},$$

$$U_1(x) \leq k_{11}S_1(x) + k_{14},$$

$$U_r(x) \leq \{k_{r1} + k_{r2}S_0(x)\}S_r(x) + k_{r4}, \quad 2 \leq r \leq r_{\text{max}}^{(1)}.$$

(2.6)
and

\[ V_0(x) \leq k_{03} S_1(x) + k_{05}, \]
\[ V_r(x) \leq k_{r3} S_p(r)(x) + k_{r5}, \quad 1 \leq r \leq r_{\text{max}}^{(2)}, \quad (2.7) \]

are satisfied, where \( 1 \leq p(r) \leq r_{\text{max}}^{(1)} \) for \( 1 \leq r \leq r_{\text{max}}^{(2)} \).

The quantities \( r_{\text{max}}^{(1)} \) and \( r_{\text{max}}^{(2)} \) usually need to be reasonably large, if Assumption 4.2 below is to be satisfied.

Now, for \( X_N \) as in the introduction, we let \( t_X N_n \) denote the time of its \( n \)-th jump, with \( t_X N_0 = 0 \), and set \( t_X N_\infty := \lim_{n \to \infty} t_X N_n \), possibly infinite. For \( 0 \leq t < t_X N_\infty \), we define

\[ S_r^{(N)}(t) := S_r(X_N(t)); \quad U_r^{(N)}(t) := U_r(x_N(t)); \quad V_r^{(N)}(t) := V_r(x_N(t)), \quad (2.8) \]

once again with \( x_N(t) := N^{-1}X_N(t) \), and also

\[ \tau_r^{(N)}(C) := \inf\{t < t_X N_\infty : S_r^{(N)}(t) \geq NC\}, \quad r \geq 0, \quad (2.9) \]

where the infimum of the empty set is taken to be \( \infty \). Our first result shows that \( t_X N_\infty = \infty \) a.s., and limits the expectations of \( S_0^{(N)}(t) \) and \( S_1^{(N)}(t) \) for any fixed \( t \).

In what follows, we shall write \( F_s^{(N)} = \sigma(X_N(u), 0 \leq u \leq s) \), so that \( (F_s^{(N)} : s \geq 0) \) is the natural filtration of the process \( X_N \).

**Lemma 2.2** Under Assumptions 2.1, \( t_X N_\infty = \infty \) a.s. Furthermore, for any \( t \geq 0 \),

\[ \mathbb{E}\{S_0^{(N)}(t)\} \leq (S_0^{(N)}(0) + Nk_{04}t)e^{k_{01}t}; \]
\[ \mathbb{E}\{S_1^{(N)}(t)\} \leq (S_1^{(N)}(0) + Nk_{14}t)e^{k_{11}t}. \]

**Proof.** Introducing the formal generator \( A_N \) associated with \( (1.1) \),

\[ A_N f(X) := N \sum_{J \in \mathcal{J}} \alpha_J (N^{-1}X) \{ f(X + J) - f(X) \}, \quad X \in \mathcal{X}_+ , \quad (2.10) \]

we note that \( NU_t(x) = A_N S_t(Nx) \). Hence, if we define \( M^{(N)}_t \) by

\[ M^{(N)}_t(t) := S^{(N)}_t(t) - S^{(N)}_t(0) - N \int_0^t U^{(N)}_t(u) \, du, \quad t \geq 0, \quad (2.11) \]
for $0 \leq l \leq r_{\max}^{(1)}$, it is immediate from (2.3), (2.5) and (2.6) that the process $(M_l^{(N)}(t \wedge \tau_1^{(N)}(C)))$, $t \geq 0$ is a zero mean $\mathcal{F}^{(N)}$-martingale for each $C > 0$. In particular, considering $M_1^{(N)}(t \wedge \tau_1^{(N)}(C))$, it follows in view of (2.6) that

\[
E\{S_1^{(N)}(t \wedge \tau_1^{(N)}(C))\} \leq S_1^{(N)}(0) + E\left\{\int_0^{t \wedge \tau_1^{(N)}(C)} \{k_{11}S_1^{(N)}(u) + Nk_{14}\} du\right\} \\
\leq S_1^{(N)}(0) + \int_0^t (k_{11}E\{S_1^{(N)}(u \wedge \tau_1^{(N)}(C))\} + Nk_{14}) du.
\]

Using Gronwall’s inequality, we deduce that

\[
E\{S_1^{(N)}(t \wedge \tau_1^{(N)}(C))\} \leq (S_1^{(N)}(0) + Nk_{14}t) e^{k_{14}t}, \quad (2.12)
\]

uniformly in $C > 0$, and hence that

\[
P\left[\sup_{0 \leq s \leq t} S_1(X_N(s)) \geq NC\right] \leq C^{-1}(S_1(x_N(0)) + k_{14}t) e^{k_{14}t}, \quad (2.13)
\]

also. Hence $\sup_{0 \leq s \leq t} S_1(X_N(s)) < \infty$ a.s. for any $t$, $\lim_{C \to \infty} \tau_1^{(N)}(C) = \infty$ a.s., and, from (2.3) and (1.3), it thus follows that $t_N^{(N)} = \infty$ a.s. The bound on $E\{S_1^{(N)}(t)\}$ is now immediate, and that on $E\{S_0^{(N)}(t)\}$ follows by applying the same Gronwall argument to $M_0^{(N)}(t \wedge \tau_1^{(N)}(C))$.

The next lemma shows that, if any $T > 0$ is fixed and $C$ is chosen large enough, then, with high probability, $N^{-1}S_0^{(N)}(t) \leq C$ holds for all $0 \leq t \leq T$.

**Lemma 2.3** Assume that Assumptions 2.7 are satisfied, and that $S_0^{(N)}(0) \leq NC_0$ and $S_1^{(N)}(0) \leq NC_1$. Then, for any $C \geq 2(C_0 + k_{04}T)e^{k_{04}T}$, we have

\[
P\{\{\tau_0^{(N)}(C) \leq T\}\} \leq (C_1 \lor 1)K_{00}/(NC_0^2),
\]

where $K_{00}$ depends on $T$ and the parameters of the model.

**Proof.** It is immediate from (2.11) and (2.6) that

\[
S_0^{(N)}(t) = S_0^{(N)}(0) + N \int_0^t U_0^{(N)}(u) du + M_0^{(N)}(t) \\
\leq S_0^{(N)}(0) + \int_0^t (k_{01}S_0^{(N)}(u) + Nk_{04}) du + \sup_{0 \leq u \leq t} M_0^{(N)}(u). \quad (2.14)
\]
Hence, from Gronwall’s inequality, if \( S_0^{(N)}(0) \leq NC_0 \), then
\[
S_0^{(N)}(t) \leq \left\{ N(C_0 + k_{01}T) + \sup_{0 \leq u \leq t} M_0^{(N)}(u) \right\} e^{k_{01}t}. \tag{2.15}
\]

Now, considering the quadratic variation of \( M_0^{(N)} \), we have
\[
E \left\{ \left\{ M_0^{(N)}(t \wedge \tau_1^{(N)}(C')) \right\}^2 - N \int_0^{t \wedge \tau_1^{(N)}(C')} V_0^{(N)}(u) \, du \right\} = 0 \tag{2.16}
\]
for any \( C' > 0 \), from which it follows, much as above, that
\[
E \left\{ \left\{ M_0^{(N)}(t \wedge \tau_1^{(N)}(C')) \right\}^2 \right\} \leq \int_0^t \left\{ k_{01} E S_1^{(N)}(u \wedge \tau_1^{(N)}(C')) + Nk_{05} \right\} \, du. \tag{2.17}
\]
Using (2.12), we thus find that
\[
E \left\{ \left\{ M_0^{(N)}(t \wedge \tau_1^{(N)}(C')) \right\}^2 \right\} \leq \frac{k_{03}}{k_{11}} N(C_1 + k_{14}T)(e^{k_{11}t} - 1) + Nk_{05}t, \tag{2.17}
\]
uniformly for all \( C' \). Doob’s maximal inequality applied to \( M_0^{(N)}(t \wedge \tau_1^{(N)}(C')) \) now allows us to deduce that, for any \( C', a > 0 \),
\[
P \left[ \sup_{0 \leq u \leq T} M_0^{(N)}(u \wedge \tau_1^{(N)}(C')) > aN \right] \leq \frac{1}{Na^2} \left\{ \frac{k_{03}}{k_{11}} (C_1 + k_{14}T) \{ e^{k_{11}T} - 1 \} + k_{05}T \right\} = \frac{C_1 K_{01} + K_{02}}{Na^2},
\]
say, so that, letting \( C' \to \infty \),
\[
P \left[ \sup_{0 \leq u \leq T} M_0^{(N)}(u) > aN \right] \leq \frac{C_1 K_{01} + K_{02}}{Na^2}
\]
also. Taking \( a = \frac{1}{2} C e^{-k_{01}T} \) and putting the result into (2.15), the lemma follows.

In the next theorem, we control the ‘higher \( \nu \)-moments’ \( S_r^{(N)}(t) \) of \( X_N(t) \).
Assume that Assumptions 2.1 are satisfied, and that $S_1^{(N)}(0) \leq NC_1$ and $S_{p(1)}^{(N)}(0) \leq NC'_1$. Then, for $2 \leq r \leq r_{\text{max}}^{(1)}$ and for any $C > 0$, we have

$$E\{S_r^{(N)}(t \land \tau_0^{(N)}(C))\} \leq (S_r^{(N)}(0) + Nk_{r4}t)e^{(k_{r1}+Ck_{r2})t}, \quad 0 \leq t \leq T. \quad (2.18)$$

Furthermore, if for $1 \leq r \leq r_{\text{max}}^{(2)}$, $S_r^{(N)}(0) \leq NC_r$ and $S_{p(r)}^{(N)}(0) \leq NC'_r$, then, for any $\gamma \geq 1$,

$$P[\sup_{0 \leq t \leq T} S_r^{(N)}(t \land \tau_0^{(N)}(C)) \geq N\gamma C''_{rT}] \leq K_{r0}\gamma^{-2}N^{-1}, \quad (2.19)$$

where

$$C''_{rT} := (C_r + k_{r4}T + \sqrt{(C'_r \lor 1)})e^{(k_{r1}+Ck_{r2})T}$$

and $K_{r0}$ depends on $C, T$ and the parameters of the model.

**Proof.** Recalling (2.11), use the argument leading to (2.12) with the martingales $M_t^{(N)}(t \land \tau_1^{(N)}(C') \land \tau_0^{(N)}(C))$, for any $C' > 0$, to deduce that

$$E S_r^{(N)}(t \land \tau_1^{(N)}(C') \land \tau_0^{(N)}(C))$$

$$\leq S_r^{(N)}(0) + \int_0^t \left\{ \{k_{r1} + Ck_{r2}\} E\left\{ S_r^{(N)}(u \land \tau_1^{(N)}(C') \land \tau_0^{(N)}(C)) \right\} + Nk_{r4} \right\} du,$n

for $1 \leq r \leq r_{\text{max}}^{(1)}$, since $N^{-1}S_r^{(N)}(u) \leq C$ when $u \leq \tau_0^{(N)}(C)$: define $k_{12} = 0$.

Gronwall’s inequality now implies that

$$E S_r^{(N)}(t \land \tau_1^{(N)}(C') \land \tau_0^{(N)}(C)) \leq (S_r^{(N)}(0) + Nk_{r4}t)e^{(k_{r1}+Ck_{r2})t}, \quad (2.20)$$

for $1 \leq r \leq r_{\text{max}}^{(1)}$, and (2.18) follows by Fatou’s lemma, on letting $C' \to \infty$.

Now, also from (2.11) and (2.6), we have, for $t \geq 0$ and each $r \leq r_{\text{max}}^{(1)},$

$$S_r^{(N)}(t \land \tau_0^{(N)}(C))$$

$$= S_r^{(N)}(0) + \int_0^{t \land \tau_0^{(N)}(C)} U_r^{(N)}(u) du + M_r^{(N)}(t \land \tau_0^{(N)}(C))$$

$$\leq S_r^{(N)}(0) + \int_0^t \left\{ \{k_{r1} + Ck_{r2}\} S_r^{(N)}(u \land \tau_0^{(N)}(C)) + Nk_{r4} \right\} du$$

$$+ \sup_{0 \leq u \leq t} M_r^{(N)}(u \land \tau_0^{(N)}(C)).$$
Hence, from Gronwall’s inequality, for all \( t \geq 0 \) and \( r \leq r_{\text{max}}^{(1)} \),
\[
S_r^{(N)}(t \wedge \tau_0^{(N)}(C)) \leq \left\{ N(C_r + k_{r4}t) + \sup_{0 \leq u \leq t} M_r^{(N)}(u \wedge \tau_0^{(N)}(C)) \right\} e^{(k_{r1}+Ck_{r2})t}.
\]  
(2.21)

Now, as in (2.16), we have
\[
\mathbb{E}\left\{ \{ M_r^{(N)}(t \wedge \tau_1^{(N)}(C') \wedge \tau_0^{(N)}(C)) \}^2 - N \int_0^{t \wedge \tau_1^{(N)}(C') \wedge \tau_0^{(N)}(C)} V_r^{(N)}(u) \, du \right\} = 0,
\]  
(2.22)
from which it follows, using (2.7), that, for \( 1 \leq r \leq r_{\text{max}}^{(2)} \),
\[
\mathbb{E}\left\{ \{ M_r^{(N)}(t \wedge \tau_1^{(N)}(C') \wedge \tau_0^{(N)}(C)) \}^2 \right\}
\[
\leq \mathbb{E}\left\{ N \int_0^{t \wedge \tau_1^{(N)}(C') \wedge \tau_0^{(N)}(C)} V_r^{(N)}(u) \, du \right\}
\[
= \int_0^t \left\{ k_{r3} \mathbb{E}S_{p(r)}^{(N)}(u \wedge \tau_0^{(N)}(C)) + Nk_{r5} \right\} \, du
\[
\leq \frac{N(C_r' + k_{p(r),4}T)k_{r3}}{k_{p(r),1} + Ck_{p(r),2}} \left( e^{(k_{p(r),1}+Ck_{p(r),2})t} - 1 \right) + Nk_{r5}T,
\]
this last by (2.20), since \( p(r) \leq r_{\text{max}}^{(1)} \) for \( 1 \leq r \leq r_{\text{max}}^{(2)} \). Using Doob’s inequality, it follows that, for any \( a > 0 \),
\[
\mathbb{P}\left[ \sup_{0 \leq u \leq T} M_r^{(N)}(u \wedge \tau_0^{(N)}(C)) > aN \right]
\[
\leq \frac{1}{Na^2} \left\{ \frac{k_{r3}(C_r' + k_{p(r),4}T)}{k_{p(r),1} + Ck_{p(r),2}} \left( e^{(k_{p(r),1}+Ck_{p(r),2})T} - 1 \right) + k_{r5}T \right\}
\[
= \frac{C_r'K_{r1} + K_{r2}}{Na^2}.
\]
Taking \( a = \sqrt[4]{(C_r' \vee 1)} \) and putting the result into (2.21) gives (2.19), with \( K_{r0} = (C_r'K_{r1} + K_{r2})/(C_r' \vee 1) \). 
Note also that \( \sup_{0 \leq t \leq T} S_r^{(N)}(t) < \infty \) a.s. for all \( 0 \leq r \leq r_{\text{max}}^{(2)} \), in view of Lemma 2.3 and Theorem 2.4.

In what follows, we shall particularly need to control quantities of the form \( \sum_{J \in \mathcal{J}} \alpha_J(x_N(s))d(J, \zeta) \), where \( x_N := N^{-1}X_N \) and
\[
d(J, \zeta) := \sum_{j \geq 0} |J_j| \zeta(j),
\]  
(2.23)
for $\zeta \in \mathcal{R}$ chosen such that $\zeta(j) \geq 1$ grows fast enough with $j$: see (4.12). Defining

$$
\tau^{(N)}(a, \zeta) := \inf \left\{ s: \sum_{J \in \mathcal{J}} \alpha_J(x_N(s)) d(J, \zeta) \geq a \right\},
$$

(2.24)
infinite if there is no such $s$, we show in the following corollary that, under suitable assumptions, $\tau^{(N)}(a, \zeta)$ is rarely less than $T$.

**Corollary 2.5** Assume that Assumptions 2.1 hold, and that $\zeta$ is such that

$$
\sum_{J \in \mathcal{J}} \alpha_J(N^{-1}X)d(J, \zeta) \leq \{ k_1N^{-1}S_{r}(X) + k_2 \}^b
$$

(2.25)

for some $1 \leq r := r(\zeta) \leq r^{(2)}_{\text{max}}$ and some $b = b(\zeta) \geq 1$. For this value of $r$, assume that $S^{(N)}_{r}(0) \leq NC_{r}$ and $S^{(N)}_{p(\gamma)}(0) \leq NC'_{r}$ for some constants $C_{r}$ and $C'_{r}$. Assume further that $S^{(N)}_{0}(0) \leq NC_{0}$, $S^{(N)}_{1}(0) \leq NC_{1}$ for some constants $C_{0}$, $C_{1}$, and define $C := 2(C_{0} + k_{04}T)e^{k_{01}T}$. Then

$$
P[\tau^{(N)}(a, \zeta) \leq T] \leq N^{-1}\{ K_{r0}\gamma_{a}^{-2} + K_{00}(C_{1} \vee 1)C^{-2} \},
$$

for any $a \geq \{ k_2 + k_1C'_{rT} \}^b$, where $\gamma_{a} := (a^{1/b} - k_2)/\{ k_1C'_{rT} \}$, $K_{r0}$ and $C'_{rT}$ are as in Theorem 2.4 and $K_{00}$ is as in Lemma 2.3.

**Proof.** In view of (2.28), it is enough to bound the probability

$$
P[\sup_{0 \leq t \leq T} S^{(N)}_{r}(t) \geq N(a^{1/b} - k_2)/k_1].
$$

However, Lemma 2.3 and Theorem 2.4 together bound this probability by

$$
N^{-1} \left\{ K_{r0}\gamma_{a}^{-2} + K_{00}(C_{1} \vee 1)C^{-2} \right\},
$$

where $\gamma_{a}$ is as defined above, as long as $a^{1/b} - k_2 \geq k_1C'_{rT}$. $lacksquare$

If (2.25) is satisfied, $\sum_{J \in \mathcal{J}} \alpha_J(x_N(s)) d(J, \zeta)$ is a.s. bounded on $0 \leq s \leq T$, because $S^{(N)}_{r}(s)$ is. The corollary shows that the sum is then bounded by $\{ k_2 + k_1C'_{rT} \}^b$, except on an event of probability of order $O(N^{-1})$. Usually, one can choose $b = 1$. 12
3 Semigroup properties

We make the following initial assumptions about the matrix $A$: first, that $A_{ij} \geq 0$ for all $i \neq j \geq 0$; $\sum_{j \neq i} A_{ji} < \infty$ for all $i \geq 0$, \hfill (3.1)

and then that, for some $\mu \in \mathbb{R}^Z_+$ such that $\mu(m) \geq 1$ for each $m \geq 0$, and for some $w \geq 0$,

$$A^T \mu \leq w \mu.$$ \hfill (3.2)

We then use $\mu$ to define the $\mu$-norm

$$\|\xi\|_{\mu} := \sum_{m \geq 0} \mu(m)|\xi^m| \text{ on } \mathcal{R}_\mu := \{\xi \in \mathcal{R} : \|\xi\|_{\mu} < \infty\}. \hfill (3.3)$$

Note that there may be many possible choices for $\mu$. In what follows, it is important that $F$ be a Lipschitz operator with respect to the $\mu$-norm, and this has to be borne in mind when choosing $\mu$.

Setting

$$Q_{ij} := A^T_{ij} \mu(j)/\mu(i) - w \delta_{ij},$$ \hfill (3.4)

where $\delta$ is the Kronecker delta, we note that $Q_{ij} \geq 0$ for $i \neq j$, and that

$$0 \leq \sum_{j \neq i} Q_{ij} = \sum_{j \neq i} A^T_{ij} \mu(j)/\mu(i) \leq w - A_{ii} = -Q_{ii},$$

using (3.2) for the inequality, so that $Q_{ii} \leq 0$. Hence $Q$ can be augmented to a conservative $Q$-matrix, in the sense of Markov jump processes, by adding a coffin state $\partial$, and setting $Q_{i\partial} := -\sum_{j \geq 0} Q_{ij} \geq 0$. Let $P(\cdot)$ denote the semigroup of Markov transition matrices corresponding to the minimal process associated with $Q$; then, in particular,

$$Q = P'(0) \text{ and } P'(t) = QP(t) \text{ for all } t \geq 0 \hfill (3.5)$$

(Reuter 1957, Theorem 3). Set

$$R^T_{ij}(t) := e^{wt} \mu(i)P_{ij}(t)/\mu(j).$$ \hfill (3.6)
Theorem 3.1 Let $A$ satisfy Assumptions (3.1) and (3.2). Then, with the above definitions, $R$ is a strongly continuous semigroup on $R_\mu$, and

$$\sum_{i \geq 0} \mu(i) R_{ij}(t) \leq \mu(j) e^{ut} \quad \text{for all } j \text{ and } t. \quad (3.7)$$

Furthermore, the sums $\sum_{j \geq 0} R_{ij}(t)A_{jk} = (R(t)A)_{ik}$ are well defined for all $i, k$, and

$$A = R'(0) \quad \text{and} \quad R'(t) = R(t)A \quad \text{for all } t \geq 0. \quad (3.8)$$

Proof. We note first that, for $x \in R_\mu$,

$$\|R(t)x\|_\mu \leq \sum_{i \geq 0} \mu(i) \sum_{j \geq 0} R_{ij}(t)|x_j| = e^{ut} \sum_{i \geq 0} \sum_{j \geq 0} \mu(j) P_{ji}(t)|x_j|$$

$$\leq e^{ut} \sum_{j \geq 0} \mu(j)|x_j| = e^{ut}\|x\|_\mu, \quad (3.9)$$

since $P(t)$ is substochastic on $\mathbb{Z}_+$; hence $R: R_\mu \to R_\mu$. To show strong continuity, we take $x \in R_\mu$, and consider

$$\|R(t)x - x\|_\mu = \sum_{i \geq 0} \mu(i) \left| \sum_{j \geq 0} R_{ij}(t)x_j - x_i \right| = \sum_{i \geq 0} e^{ut} \sum_{j \geq 0} \mu(j) P_{ji}(t)x_j - \mu(i)x_i$$

$$\leq (e^{ut} - 1) \sum_{i \geq 0} \sum_{j \geq 0} \mu(j) P_{ji}(t)x_j + \sum_{i \geq 0} \sum_{j \neq i} \mu(j) P_{ji}(t)x_j + \sum_{i \geq 0} \mu(i)x_i(1 - P_{ii}(t))$$

$$\leq (e^{ut} - 1) \sum_{j \geq 0} \mu(j)x_j + 2 \sum_{i \geq 0} \mu(i)x_i(1 - P_{ii}(t)),$$

from which it follows that $\lim_{t \to 0} \|R(t)x - x\|_\mu = 0$, by dominated convergence, since $\lim_{t \to 0} P_{ii}(t) = 1$ for each $i \geq 0$.

The inequality (3.7) follows from the definition of $R$ and the fact that $P$ is substochastic on $\mathbb{Z}_+$. Then

$$(A^T R^T(t))_{ij} = \sum_{k \neq i} Q_{ik} \frac{\mu(i)}{\mu(k)} e^{ut} \frac{\mu(k)}{\mu(j)} P_{kj}(t) + (Q_{ii} + w)e^{ut} \frac{\mu(i)}{\mu(j)} P_{ij}(t)$$

$$= \frac{\mu(i)}{\mu(j)} [(QP(t))_{ij} + wP_{ij}(t)] e^{ut},$$
with \( (QP(t))_{ij} = \sum_{k \geq 0} Q_{ik} P_{kj}(t) \) well defined because \( P(t) \) is sub-stochastic and \( Q \) is conservative. Using (3.5), this gives

\[
(A^T R^T(t))_{ij} = \frac{\mu(i)}{\mu(j)} \frac{d}{dt} [P_{ij}(t) e^{wt}] = \frac{d}{dt} R^T_{ij}(t),
\]

and this establishes (3.8).

\[
\begin{align*}
4 \text{ Main approximation} \\
\text{Let } X_N, N \geq 1, \text{ be a sequence of pure jump Markov processes as in Section 1,} \\
\text{with } A \text{ and } F \text{ defined as in (1.4) and (1.5), and suppose that } F: R_\mu \to R_\mu, \\
\text{with } R_\mu \text{ as defined in (3.3), for some } \mu \text{ such that Assumption (3.2) holds.} \\
\text{Suppose also that } F \text{ is locally Lipschitz in the } \mu\text{-norm: for any } z > 0,
\end{align*}
\]

\[
\sup_{x \neq y: ||x||_\mu, ||y||_\mu \leq z} ||F(x) - F(y)||_\mu/||x - y||_\mu \leq K(\mu, F; z) < \infty. \tag{4.1}
\]

Then, for \( x(0) \in R_\mu \) and \( R \) as in (3.6), the integral equation

\[
x(t) = R(t)x(0) + \int_0^t R(t - s)F(x(s)) \, ds. \tag{4.2}
\]

has a unique continuous solution \( x \) in \( R_\mu \) on some non-empty time interval \([0, t_{\max})\), such that, if \( t_{\max} < \infty \), then \( ||x(t)||_\mu \to \infty \) as \( t \to t_{\max} \) (Pazy 1983, Theorem 1.4, Chapter 6). Thus, if \( A \) were the generator of \( R \), the function \( x \) would be a mild solution of the deterministic equations (1.4). We now wish to show that the process \( X_N := N^{-1}X_N \) is close to \( x \). To do so, we need a corresponding representation for \( X_N \).

To find such a representation, let \( W(t), t \geq 0, \) be a pure jump path on \( X_+ \) that has only finitely many jumps up to time \( T \). Then we can write

\[
W(t) = W(0) + \sum_{j: \sigma_j \leq t} \Delta W(\sigma_j), \quad 0 \leq t \leq T, \tag{4.3}
\]

where \( \Delta W(s) := W(s) - W(s-) \) and \( \sigma_j, j \geq 1, \) denote the times when \( W \) has its jumps. Now let \( A \) satisfy (3.1) and (3.2), and let \( R(\cdot) \) be the associated semigroup, as defined in (3.6). Define the path \( W^*(t), 0 \leq t \leq T, \) from the equation

\[
W^*(t) := R(t)W(0) + \sum_{j: \sigma_j \leq t} R(t - \sigma_j) \Delta_j - \int_0^t R(t - s)A W(s) \, ds, \tag{4.4}
\]

15
where $\Delta_j := \Delta W(\sigma_j)$. Note that the latter integral makes sense, because each of the sums $\sum_{j \geq 0} R_{ij}(t)A_{jk}$ is well defined, from Theorem 3.1 and because only finitely many of the coordinates of $W$ are non-zero.

**Lemma 4.1** $W^*(t) = W(t)$ for all $0 \leq t \leq T$.

**Proof.** Fix any $t$, and suppose that $W^*(s) = W(s)$ for all $s \leq t$. This is clearly the case for $t = 0$. Let $\sigma(t) > t$ denote the time of the first jump of $W$ after $t$. Then, for any $0 < h < \sigma(t) - t$, using the semigroup property for $R$ and (4.4),

$$W^*(t + h) - W^*(t) = (R(h) - I)W(0) + \sum_{j : \sigma_j \leq t} (R(h) - I)R(t - \sigma_j)\Delta_j$$

$$- \int_0^t (R(h) - I)R(t - s)AW(s) ds - \int_t^{t+h} R(t + h - s)AW(t) ds,$$

where, in the last integral, we use the fact that there are no jumps of $W$ between $t$ and $t + h$. Thus we have

$$W^*(t + h) - W^*(t)$$

$$= (R(h) - I) \left\{ R(t)W(0) + \sum_{j : \sigma_j \leq t} R(t - \sigma_j)\Delta_j - \int_0^t R(t - s)AW(s) ds \right\}$$

$$- \int_t^{t+h} R(t + h - s)AW(t) ds$$

$$= (R(h) - I)W(t) - \int_t^{t+h} R(t + h - s)AW(t) ds.$$  \hspace{1cm} (4.6)

But now, for $x \in X_+$,

$$\int_t^{t+h} R(t + h - s)Ax ds = (R(h) - I)x,$$

from (3.8), so that $W^*(t + h) = W^*(t)$ for all $t + h < \sigma(t)$, implying that $W^*(s) = W(s)$ for all $s < \sigma(t)$. On the other hand, from (4.4), we have

$W^*(\sigma(t)) - W^*(\sigma(t) -) = \Delta W(\sigma(t))$, so that $W^*(s) = W(s)$ for all $s \leq \sigma(t)$. Thus we can prove equality over the interval $[0, \sigma_1]$, and then successively over the intervals $[\sigma_j, \sigma_{j+1}]$, until $[0, T]$ is covered. $\blacksquare$
Now suppose that \( W \) arises as a realization of \( X_N \). Then \( X_N \) has transition rates such that

\[
M_N(t) := \sum_{j: \sigma_j \leq t} \Delta X_N(\sigma_j) - \int_0^t AX_N(s) \, ds - \int_0^t NF(x_N(s)) \, ds
\]

(4.7)
is a zero mean local martingale. In view of Lemma 4.1, we can use (4.4) to write

\[
X_N(t) = R(t)X_N(0) + \tilde{M}_N(t) + N \int_0^t R(t-s)F(x_N(s)) \, ds,
\]

(4.8)
where

\[
\tilde{M}_N(t) := \sum_{j: \sigma_j \leq t} R(t - \sigma_j) \Delta X_N(\sigma_j)
\]

\[
- \int_0^t R(t-s)AX_N(s) \, ds - \int_0^t R(t-s)NF(x_N(s)) \, ds.
\]

(4.9)

Thus, comparing (4.8) and (4.2), we expect \( x_N \) and \( x \) to be close, for \( 0 \leq t \leq T < t_{\text{max}} \), provided that we can show that \( \sup_{t \leq T} \|\tilde{M}_N(t)\|_\mu \) is small, where \( \tilde{m}_N(t) := N^{-1} \tilde{M}_N(t) \). Indeed, if \( x_N(0) \) and \( x(0) \) are close, then

\[
\|x_N(t) - x(t)\|_\mu
\]

\[
\leq \|R(t)(x_N(0) - x(0))\|_\mu
\]

\[
+ \int_0^t \|R(t-s)[F(x_N(s)) - F(x(s))]\|_\mu \, ds + \|\tilde{m}_N(t)\|_\mu
\]

\[
\leq e^{wt}\|x_N(0) - x(0)\|_\mu
\]

\[
+ \int_0^t e^{w(t-s)} K(\mu, F; 2\Xi_T)\|x_N(s) - x(s)\|_\mu \, ds + \|\tilde{m}_N(t)\|_\mu,
\]

(4.10)
by (3.3), with the stage apparently set for Gronwall’s inequality, assuming that \( \|x_N(0) - x(0)\|_\mu \) and \( \sup_{0 \leq t \leq T} \|\tilde{m}_N(t)\|_\mu \) are small enough that then \( \|x_N(t)\|_\mu \leq 2\Xi_T \) for \( 0 \leq t \leq T \), where \( \Xi_T := \sup_{0 \leq t \leq T} \|x(t)\|_\mu \).

Bounding \( \sup_{0 \leq t \leq T}\|\tilde{M}_N(t)\|_\mu \) is, however, not so easy. Since \( \tilde{M}_N \) is not itself a martingale, we cannot directly apply martingale inequalities to control its fluctuations. However, since

\[
\tilde{M}_N(t) = \int_0^t R(t-s) \, dM_N(s),
\]

(4.11)
we can hope to use control over the local martingale $M_N$ instead. For this
and the subsequent argument, we introduce some further assumptions.

**Assumption 4.2**

1. There exists $r = r_\mu \leq r^{(2)}_{\text{max}}$ such that $\sup_{j \geq 0} \{\mu(j)/\nu_r(j)\} < \infty$.

2. There exists $\zeta \in \mathcal{R}$ with $\zeta(j) \geq 1$ for all $j$ such that (2.25) is satisfied for some $b = b(\zeta) \geq 1$ and $r = r(\zeta)$ such that $1 \leq r(\zeta) \leq r^{(2)}_{\text{max}}$, and that

$$Z := \sum_{k \geq 0} \frac{\mu(k)(|A_{kk}| + 1)}{\sqrt{\zeta(k)}} < \infty. \quad (4.12)$$

The requirement that $\zeta$ satisfies (4.12) as well as satisfying (2.25) for some $r \leq r^{(2)}_{\text{max}}$ implies in practice that it must be possible to take $r^{(1)}_{\text{max}}$ and $r^{(2)}_{\text{max}}$ to be quite large in Assumption 2.1; see the examples in Section 5.

Note that part 1 of Assumption 4.2 implies that $\lim_{j \to \infty} \{\mu(j)/\nu_r(j)\} = 0$ for some $r = \tilde{r}_\mu \leq r_\mu + 1$. We define

$$\rho(\zeta, \mu) := \max\{r(\zeta), p(r(\zeta)), \tilde{r}_\mu\}, \quad (4.13)$$

where $p(\cdot)$ is as in Assumptions 2.1. We can now prove the following lemma, which enables us to control the paths of $\tilde{M}_N$ by using fluctuation bounds for the martingale $M_N$.

**Lemma 4.3** Under Assumption 4.2

$$\tilde{M}_N(t) = M_N(t) + \int_0^t R(t-s)AM_N(s)\,ds.$$ 

**Proof.** From (3.8), we have

$$R(t-s) = I + \int_0^{t-s} R(v)A\,dv.$$ 

Substituting this into (4.11), we obtain

$$\tilde{M}_N(t) = \int_0^t R(t-s)\,dM_N(s)$$
\begin{align*}
&= M_N(t) + \int_0^t \left\{ \int_0^t R(v) A_1_{[0,t-s]}(v) \, dv \right\} dM_N(s) \\
&= M_N(t) + \int_0^t \left\{ \int_0^t R(v) A_1_{[0,t-s]}(v) \, dv \right\} dX_N(s) \\
&\quad - \int_0^t \left\{ \int_0^t R(v) A_1_{[0,t-s]}(v) \, dv \right\} F_0(x_N(s)) \, ds.
\end{align*}

It remains to change the order of integration in the double integrals, for which we use Fubini’s theorem.

In the first, the outer integral is almost surely a finite sum, and at each jump time \( t_{X_N}^i \) we have \( dX_N(t_{X_N}^i) \in J \). Hence it is enough that, for each \( i, m \) and \( t, \sum_{j \geq 0} R_{ij}(t) A_{jm} \) is absolutely summable, which follows from Theorem 3.1. Thus we have

\[
\int_0^t \left\{ \int_0^t R(v) A_1_{[0,t-s]}(v) \, dv \right\} dX_N(s) = \int_0^t R(v) A \{ X_N(t-v) - X_N(0) \} \, dv.
\]

For the second, the \( k \)-th component of \( R(v) A F_0(x_N(s)) \) is just

\[
\sum_{j \geq 0} R_{kj}(v) \sum_{l \geq 0} A_{jl} \sum_{J \in J} J^l \alpha_J(x_N(s)).
\]

Now, from (3.7), we have \( 0 \leq R_{kj}(v) \leq \mu(j)e^{\mu v}/\mu(k) \), and

\[
\sum_{j \geq 0} \mu(j) |A_{jl}| \leq \mu(l)(2|A_{ll}| + w),
\]

because \( A^T \mu \leq w \mu \). Hence, putting absolute values in the summands in (4.15) yields at most

\[
\frac{e^{\mu v}}{\mu(k)} \sum_{J \in J} \alpha_J(x_N(s)) \sum_{l \geq 0} |J^l| \mu(l)(2|A_{ll}| + w).
\]

Now, in view of (4.12) and since \( \zeta(j) \geq 1 \) for all \( j \), there is a constant \( K < \infty \) such that \( \mu(l)(2|A_{ll}| + w) \leq K \zeta(l) \). Furthermore, \( \zeta \) satisfies (2.25), so that, by Corollary 2.5, \( \sum_{J \in J} \alpha_J(x_N(s)) \sum_{l \geq 0} |J^l| \zeta(l) \) is a.s. uniformly bounded in \( 0 \leq s \leq T \). Hence we can apply Fubini’s theorem, obtaining

\[
\int_0^t \left\{ \int_0^t R(v) A_1_{[0,t-s]}(v) \, dv \right\} F_0(x_N(s)) \, ds = \int_0^t R(v) A \left\{ \int_0^{t-v} F_0(x_N(s)) \, ds \right\} \, dv,
\]
and combining this with (4.14) proves the lemma.

We now introduce the exponential martingales that we use to bound the fluctuations of $M_N$. For $\theta \in \mathbb{R}^{\mathbb{Z}_+}$ bounded and $x \in \mathcal{R}_\mu$, 

$$Z_N,\theta(t) := e^{\theta^T x_N(t)} \exp \left\{ - \int_0^t g_{N\theta}(x_N(s-)) \, ds \right\}, \quad t \geq 0,$$

is a non-negative finite variation local martingale, where 

$$g_{N\theta}(\xi) := \sum_{J \in J} N\alpha_J(\xi) \left( e^{N^{-1}\theta^T J} - 1 \right).$$

For $t \geq 0$, we have

$$\log Z_N,\theta(t) = \theta^T x_N(t) - \int_0^t g_{N\theta}(x_N(s-)) \, ds = \theta^T m_N(t) - \int_0^t \varphi_{N,\theta}(x_N(s-), s) \, ds, \quad (4.17)$$

where

$$\varphi_{N,\theta}(\xi) := \sum_{J \in J} N\alpha_J(\xi) \left( e^{N^{-1}\theta^T J} - 1 - N^{-1}\theta^T J \right), \quad (4.18)$$

and $m_N(t) := N^{-1}M_N(t)$. Note also that we can write

$$\varphi_{N,\theta}(\xi) = N \int_0^1 (1 - r) D^2v_N(\xi, r\theta)[\theta, \theta] \, dr, \quad (4.19)$$

where

$$v_N(\xi, \theta') := \sum_{J \in J} \alpha_J(\xi)e^{N^{-1}(\theta')^T J},$$

and $D^2v_N$ denotes the matrix of second derivatives with respect to the second argument:

$$D^2v_N(\xi, \theta')[\zeta_1, \zeta_2] := N^{-2} \sum_{J \in J} \alpha_J(\xi)e^{N^{-1}(\theta')^T J} \zeta_1^T JJ^T \zeta_2, \quad (4.20)$$

for any $\zeta_1, \zeta_2 \in \mathcal{R}_\mu$.

Now choose any $B := (B_k, k \geq 0) \in \mathcal{R}$, and define $\tilde{\tau}_k^{(N)}(B)$ by

$$\tilde{\tau}_k^{(N)}(B) := \inf \left\{ t \geq 0 : \sum_{J : J_k \neq 0} \alpha_J(x_N(t-)) > B_k \right\}.$$

Our exponential bound is as follows.
Lemma 4.4 For any \( k \geq 0 \),
\[
P \left[ \sup_{0 \leq t \leq T \wedge \tau_k^{(N)}(B)} |m_N^k(t)| \geq \delta \right] \leq 2 \exp(-\delta^2 N/2B_kK_*T).
\]
for all \( 0 < \delta \leq B_kK_*T \), where \( K_* := J_*^2 e^{J_*} \), and \( J_* \) is as in (1.2).

Proof. Take \( \theta = e^{(k)\beta} \), for \( \beta \) to be chosen later. We shall argue by stopping the local martingale \( Z_N,\theta \) at time \( \sigma^{(N)}(k, \delta) \), where
\[
\sigma^{(N)}(k, \delta) := T \land \tau_k^{(N)}(B) \land \inf \{ t : m_N^k(t) \geq \delta \}.
\]
Note that \( e^{N^{-1}\theta T J} \leq e^{J_*} \), so long as \( |\beta| \leq N \), so that
\[
D^2 v_N(\xi, r\theta)[\theta, \theta] \leq N^{-2} \left( \sum_{J: J \neq 0} \alpha_J(\xi) \right) \beta^2 K_*.
\]
Thus, from (4.19), we have
\[
\varphi_{N,\theta}(x_N(u-)) \leq \frac{1}{2} N^{-1} B_k \beta^2 K_* , \quad u \leq \tau_k^{(N)}(B),
\]
and hence, on the event that \( \sigma^{(N)}(k, \delta) = \inf \{ t : m_N^k(t) \geq \delta \} \leq (T \land \tau_k^{(N)}(B)) \), we have
\[
Z_{N,\theta}(\sigma(k, \delta)) \geq \exp \{ \beta \delta - \frac{1}{2} N^{-1} B_k \beta^2 K_* T \}.
\]
But since \( Z_{N,\theta}(0) = 1 \), it now follows from the optional stopping theorem and Fatou’s lemma that
\[
1 \geq \mathbb{E} \left[ Z_{N,\theta}(\sigma^{(N)}(k, \delta)) \right] \geq \mathbb{P} \left[ \sup_{0 \leq t \leq T \land \tau_k^{(N)}(B)} m_N^k(t) \geq \delta \right] \exp \{ \beta \delta - \frac{1}{2} N^{-1} B_k \beta^2 K_* T \}.
\]
We can choose \( \beta = \delta N/B_kK_*T \), as long as \( \delta/B_kK_*T \leq 1 \), obtaining
\[
\mathbb{P} \left( \sup_{0 \leq t \leq T \land \tau_k^{(N)}(B)} m_N^k(t) \geq \delta \right) \leq \exp(-\delta^2 N/2B_kK_*T).
\]
Repeating with
\[
\tilde{\sigma}^{(N)}(k, \delta) := T \land \tau_k^{(N)}(B) \land \inf \{ t : -m_N^k(t) \geq \delta \},
\]
we can choose
and choosing $\beta = \delta N/B_k K_* T$, gives the lemma.

The preceding lemma gives a bound for each individual component of $M_N$. We need first to translate this into a statement for all components simultaneously. For $\zeta$ as in Assumption 4.2 we start by writing

$$
Z^{(1)}_* := \max_{k \geq 1} k^{-1} \# \{ m : \zeta(m) \leq k \}; \quad Z^{(2)}_* := \sup_{k \geq 0} \frac{\mu(k)(|A_{kk}| + 1)}{\sqrt{\zeta(k)}}.
$$

(4.21)

$Z^{(2)}_*$ is clearly finite, because of Assumption 4.2, and the same is true for $Z^{(1)}_*$ also, since $Z$ of Assumption 4.2 is at least $\# \{ m : \zeta(m) \leq k \}/\sqrt{k}$, for each $k$.

Then, using the definition (2.24) of $\tau(N(a, \zeta))$, note that, for every $t < \tau(N(a, \zeta))$ and any $h \in \mathcal{R}$, and that, for any $K \subseteq \mathbb{Z}_+$,

$$
\sum_{k \in K} \sum_{J : J \neq 0} \alpha_J(x_N(t)) h(k) \leq \sum_{k \in K} \sum_{J : J \neq 0} \frac{\alpha_J(x_N(t)) h(k) d(J, \zeta)}{|J|^k \zeta(k)} \leq \frac{ah(k)}{\zeta(k)},
$$

(4.22)

for any $t < \tau(N(a, \zeta))$ and any $h \in \mathcal{R}$, and that, for any $K \subseteq \mathbb{Z}_+$,

$$
\sum_{k \in K} \sum_{J : J \neq 0} \alpha_J(x_N(t)) h(k) \leq \sum_{k \in K} \sum_{J : J \neq 0} \frac{\alpha_J(x_N(t)) h(k) d(J, \zeta)}{|J|^k \zeta(k)} \leq \frac{ah(k)}{\min_{k \in K}(\zeta(k)/h(k))}.
$$

(4.23)

From (4.22) with $h(k) = 1$ for all $k$, if we choose $B := (a/\zeta(k), k \geq 0)$, then $\tau(N(a, \zeta)) \leq \tilde{\tau}_k(N(B)$ for all $k$. For this choice of $B$, we can take

$$\delta^2_k := \delta^2_k(a) := \frac{4aK_* T \log N}{N \zeta(k)} = \frac{4B_k K_* T \log N}{N},
$$

(4.24)

in Lemma 4.4 for $k \in \kappa_N(a)$, where

$$
\kappa_N(a) : = \{ k : \zeta(k) \leq \frac{1}{4} aK_* T N / \log N \} = \{ k : B_k \geq 4 \log N/K_* T N \},
$$

(4.25)

since then $\delta_k(a) \leq B_k K_* T$. Note that then, from (4.12),

$$
\sum_{k \in \kappa_N(a)} \mu(k) \delta_k(a) \leq 2Z \sqrt{aK_* T N^{-1} \log N},
$$

(4.26)

with $Z$ as defined in Assumption 4.2 and that

$$|\kappa_N(a)| \leq \frac{1}{4} aZ_*^{(1)} K_* T N / \log N.
$$

(4.27)
Lemma 4.5  If Assumptions 4.2 are satisfied, taking \( \delta_k(a) \) and \( \kappa_N(a) \) as defined in (4.24) and (4.25), and for any \( \eta \in \mathcal{R} \), we have

1. \( P \left[ \bigcup_{k \in \kappa_N(a)} \left\{ \sup_{0 \leq t \leq T \wedge \tau(N)(a, \zeta)} |m_N(t)| \geq \delta_k(a) \right\} \right] \leq \frac{a Z_1^* K_* T}{2N \log N} ; \)

2. \( P \left[ \sum_{k \notin \kappa_N(a)} X_N^k(t) = 0 \text{ for all } 0 \leq t \leq T \wedge \tau(N)(a, \zeta) \right] \geq 1 - \frac{4 \log N}{K_* N} ; \)

3. \( \sup_{0 \leq t \leq T \wedge \tau(N)(a, \zeta)} \left\{ \sum_{k \notin \kappa_N(a)} \eta(k) |F^k(x_N(t))| \right\} \leq \frac{a J_*}{\min_{k \notin \kappa_N(a)} (\zeta(k)/\eta(k))}. \)

Proof. For part 1, use Lemma 4.4 together with (4.24) and (4.27) to give the bound. For part 2, the total rate of jumps into coordinates with indices \( k \notin \kappa_N(a) \) is

\[
\sum_{k \notin \kappa_N(a)} \sum_{J : J^k \neq 0} \alpha_J(x_N(t)) \leq \frac{a}{\min_{k \notin \kappa_N(a)} \zeta(k)},
\]

if \( t \leq \tau(N)(a, \zeta) \), using (4.23) with \( \mathcal{K} = (\kappa_N(a))^c \), which, combined with (4.25), proves the claim. For the final part, if \( t \leq \tau(N)(a, \zeta) \),

\[
\sum_{k \notin \kappa_N(a)} \eta(k) |F^k(x_N(t))| \leq \sum_{k \notin \kappa_N(a)} \eta(k) \sum_{J : J^k \neq 0} \alpha_J(x_N(t)) J_*,
\]

and the inequality follows once more from (4.23). \( \square \)

Let \( B_N^{(1)}(a) \) and \( B_N^{(2)}(a) \) denote the events

\[
B_N^{(1)}(a) := \left\{ \sum_{k \notin \kappa_N(a)} X_N^k(t) = 0 \text{ for all } 0 \leq t \leq T \wedge \tau(N)(a, \zeta) \right\};
\]

\[
B_N^{(2)}(a) := \left( \bigcap_{k \in \kappa_N(a)} \left\{ \sup_{0 \leq t \leq T \wedge \tau(N)(a, \zeta)} |m_N(t)| \leq \delta_k(a) \right\} \right), \tag{4.28}
\]

and set \( B_N(a) := B_N^{(1)}(a) \cap B_N^{(2)}(a) \). Then, by Lemma 4.5, we deduce that

\[
P[B_N(a)^c] \leq \frac{a Z_1^* K_* T}{2N \log N} + \frac{4 \log N}{K_* N}, \tag{4.29}
\]

23
of order $O(N^{-1} \log N)$ for each fixed $a$. Thus we have all the components of $M_N$ simultaneously controlled, except on a set of small probability. We now translate this into the desired assertion about the fluctuations of $\tilde{m}_N$.

**Lemma 4.6** If Assumptions 4.2 are satisfied, then, on the event $B_N(a)$,

$$\sup_{0 \leq t \leq T \wedge \tau(N)} \|\tilde{m}_N(t)\|_\mu \leq \sqrt{a K_{4.6}} \sqrt{\frac{\log N}{N}},$$

where the constant $K_{4.6}$ depends on $T$ and the parameters of the process.

**Proof.** From Lemma 4.3 it follows that

$$\sup_{0 \leq t \leq T \wedge \tau(N)} \|\tilde{m}_N(t)\|_\mu \leq \sup_{0 \leq t \leq T \wedge \tau(N)} \|m_N(t)\|_\mu + \sup_{0 \leq t \leq T \wedge \tau(N)} \int_0^t \|R(t-s)Am_N(s)\|_\mu ds.$$  

For the first term, on $B_N(a)$ and for $0 \leq t \leq T \wedge \tau(N)(a,\zeta)$, we have

$$\|m_N(t)\|_\mu \leq \sum_{k \in \kappa_N(a)} \mu(k)\delta_k(a) + \int_0^t \sum_{k \notin \kappa_N(a)} \mu(k)|F^k(x_N(u))| du.$$  

The first sum is bounded using (4.26) by $2Z_a T \sqrt{aK_*N^{-1/2} \log N}$, the second, from Lemma 4.5 and (4.25), by

$$\frac{TaJ_*}{\min_{k \notin \kappa_N(a)}(\zeta(k)/\mu(k))} \leq Z^{(2)}_{aJ_*} \sqrt{\frac{Ta}{K_4}} \sqrt{\frac{\log N}{N}}.$$  

For the second term in (4.30), from (3.7) and (4.16), we note that

$$\|R(t-s)Am_N(s)\|_\mu \leq \sum_{k \geq 0} \mu(k) \sum_{l \geq 0} R_{kl}(t-s) \sum_{r \geq 0} |A_{lr}|m^r_N(s)|$$  

$$\leq e^{w(t-s)} \sum_{l \geq 0} \mu(l) \sum_{r \geq 0} |A_{lr}|m^r_N(s)|$$  

$$\leq e^{w(t-s)} \sum_{r \geq 0} \mu(r) \{2|A_{rr}| + w\}m^r_N(s)|.$$
On $B_N(a)$ and for $0 \leq s \leq T \wedge \tau^{(N)}(a, \zeta)$, from (4.12), the sum for $r \in \kappa_N(a)$ is bounded using

\[
\sum_{r \in \kappa_N(a)} \mu(r) \{2|A_{rr}| + w\} \{m_N^r(s)\} \leq \sum_{r \in \kappa_N(a)} \mu(r) \{2|A_{rr}| + w\} \delta_r(a) \\
\leq \sum_{r \in \kappa_N(a)} \mu(r) \{2|A_{rr}| + w\} \sqrt{\frac{4aK_*T \log N}{N \zeta(r)}} \\
\leq (2 \lor w) Z \sqrt{4aK_*T} \sqrt{\frac{\log N}{N}}.
\]

The remaining sum is then bounded by Lemma 4.5 on the set $B_N(a)$ and for $0 \leq s \leq T \wedge \tau^{(N)}(a, \zeta)$, giving at most

\[
\sum_{r \notin \kappa_N(a)} \mu(r) \{2|A_{rr}| + w\} \{m_N^r(s)\} \leq \sum_{r \notin \kappa_N(a)} \mu(r) \{2|A_{rr}| + w\} \int_0^s \|F^r(x_N(t))\| dt \\
\leq \frac{(2 \lor w) saJ_*}{\min_{k \notin \kappa_N(a)} \{\zeta(k)/\mu(k)\} (|A_{kk}| + 1)} \\
\leq (2 \lor w) Z_*^{(2)} Z J_* \sqrt{\frac{T a}{K_*}} \sqrt{\frac{\log N}{N}}.
\]

Integrating, it follows that

\[
\sup_{0 \leq t \leq T \wedge \tau^{(N)}(a, \zeta)} \int_0^t \|R(t - s)Am_N(s)\|_\mu ds \\
\leq (2T \lor 1) e^{wT} \left\{ \sqrt{4aK_*T Z} + Z_*^{(2)} J_2 J_* \sqrt{\frac{T a}{K_*}} \right\} \sqrt{\frac{\log N}{N}},
\]

and the lemma follows.

This has now established the control on $\sup_{0 \leq t \leq T} \|\tilde{m}_N(t)\|_\mu$ that we need, in order to translate (4.10) into a proof of the main theorem.
Then, from (4.29), for some constant $c$ follows that for some constant $c$ the constants

\[ \|B\| \leq C \]

for (4.32) to hold. Then, taking $r = C$ in Theorem 2.4, since we can take $r = C$ for some $C_* < \infty$. Recalling the definition (4.13) of $\rho(\zeta, \mu)$, for $\zeta$ as given in Assumption 4.2, suppose that $\rho(\zeta, \mu)(0) \leq NC_* for some $C_* < \infty$.

Let $x$ denote the solution to (4.2) with initial condition $x(0)$ satisfying $S_{\rho(\zeta, \mu)}(x(0)) < \infty$. Then $t_{\max} = \infty$.

Fix any $T$, and define $\Xi_T := \sup_{0 \leq t \leq T} \|x(t)\|$. If $\|x_N(0) - x(0)\|_{\mu} \leq \frac{1}{2} \Xi_T e^{-\varpi T}$, where $k_* := e^{2T K(\mu, F; 2\Xi_T)}$, then there exist constants $c_1, c_2$ depending on $C_*, T$ and the parameters of the process, such that for all $N$ large enough

\[
P \left( \sup_{0 \leq t \leq T} \|x_N(t) - x(t)\|_{\mu} > \left( e^{wT} \|x_N(0) - x(0)\|_{\mu} + c_1 \sqrt{\log N} \right) e^{k_* T} \right) \leq \frac{c_2 \log N}{N}. \tag{4.31} \]

**Proof.** As $S_{\rho(\zeta, \mu)}(0) \leq NC_*$, it follows also that $S_{\rho(\zeta, \mu)}(0) \leq NC_*$ for all $0 \leq r \leq \rho(\zeta, \mu)$. Fix any $T < t_{\max}$, take $C := 2(C_* + k_0 T) e^{c_{k_1} T}$, and observe that, for $r \leq \rho(\zeta, \mu) \wedge r_{\max}^{(2)}$, and such that $p(r) \leq \rho(\zeta, \mu)$, we can take

\[
C_{P(T)} := 2(C_* \vee 1) + k_4 T \right) e^{(k_4 + C_{k_4}) T}, \tag{4.32} \]

in Theorem 2.4, since we can take $C_*$ to bound $C_r$ and $C'_{\rho}$. In particular, $r = r(\zeta)$ as defined in Assumption 4.2 satisfies both the conditions on $r$ for (4.32) to hold. Then, taking $a := \{k_2 + k_1 C_{\rho(\zeta)} T\} b(\zeta)$ in Corollary 2.5 it follows that for some constant $c_3 > 0$, on the event $B_N(a)$,

\[
P [\tau^{(N)}(a, \zeta) \leq T] \leq c_3 N^{-1}. \]

Then, from (4.25), for some constant $c_4$, $P [B_N(a)] \leq c_4 N^{-1} \log N$. Here, the constants $c_3, c_4$ depend on $C_*, T$ and the parameters of the process.

We now use Lemma 4.6 to bound the martingale term in (4.10). It follows that, on the event $B_N(a) \cap \{ \tau^{(N)}(a, \zeta) > T \}$ and on the event that $\|x_N(s) - x(s)\|_{\mu} \leq \Xi_T$ for all $0 \leq s \leq t$,

\[
\|x_N(t) - x(t)\|_{\mu} \leq \left( e^{wT} \|x_N(0) - x(0)\|_{\mu} + \sqrt{a} K_{4.6} \sqrt{\log N} \right) + k_* \int_0^t \|x_N(s) - x(s)\|_{\mu} ds,
\]

26
where \( k_* := e^{wT}K(\mu, F; 2\Xi_T) \). Then from Gronwall's inequality, on the event \( B_N(a) \cap \{ \tau(N)(a, \zeta) > T \} \),

\[
\|x_N(t) - x(t)\|_\mu \leq \left( e^{wT} \|x_N(0) - x(0)\|_\mu + \sqrt{a} K^{1.6} \sqrt{\frac{\log N}{N}} \right) e^{k_*t},
\]

for all \( 0 \leq t \leq T \), provided that

\[
\left( e^{wT} \|x_N(0) - x(0)\|_\mu + \sqrt{a} K^{1.6} \sqrt{\frac{\log N}{N}} \right) \leq \Xi_T e^{-k_*T}.
\]

This is true for all \( N \) sufficiently large, if \( \|x_N(0) - x(0)\|_\mu \leq \frac{1}{2} \Xi_T e^{-(w+k_*)T} \), which we have assumed. We have thus proved (4.31), since, as shown above, \( P(B_N(a)^c \cup \{ \tau(N)(a, \zeta) > T \}^c) = O(N^{-1} \log N) \).

We now use this to show that in fact \( t_{max} = \infty \). For \( x(0) \) as above, we can take \( x^*_j(0) := N^{-1} [Nx^j(0)] \leq x^j(0) \), so that \( S_{\rho(\zeta, \mu)}(0) \leq NC_* \) for \( C_* := S_{\rho(\zeta, \mu)}(x(0)) < \infty \). Then, by (4.13), \( \lim_{j \to \infty} \{ \mu(j)/\nu_{\rho(\zeta, \mu)}(j) \} = 0 \), so it follows easily using bounded convergence that \( \|x_N(0) - x(0)\|_\mu \to 0 \) as \( N \to \infty \). Hence, for any \( T < t_{max} \), it follows from (4.31) that \( \|x_N(t) - x(t)\|_\mu \to_D 0 \) as \( N \to \infty \), for \( t \leq T \), with uniform bounds over the interval, where ‘\( \to_D \)’ denotes convergence in distribution. Also, by Assumption 4.2 there is a constant \( c_5 \) such that \( \|x_N(t)\|_\mu \leq c_5 N^{-1} S_{\rho(\zeta, \mu)}(t) \) for each \( t \), where \( r_\mu \leq r^{(2)}_\mu \) and \( r_\mu \leq \rho(\zeta, \mu) \). Hence, using Lemma 2.3 and Theorem 2.4, \( \sup_{0 \leq t \leq 2T} \|x_N(t)\|_\mu \) remains bounded in probability as \( N \to \infty \). Hence it is impossible that \( \|x(t)\|_\mu \to \infty \) as \( T \to t_{max} < \infty \), implying that in fact \( t_{max} = \infty \) for such \( x(0) \).

**Remark.** The dependence on the initial conditions is considerably complicated by the way the constant \( C \) appears in the exponent, for instance in the expression for \( C_{r,T} \) in the proof of Theorem 4.7. However, if \( k_r \) in Assumptions 2.1 can be chosen to be zero, as for instance in the examples below, the dependence simplifies correspondingly.

There are biologically plausible models in which the restriction to \( J' \geq -1 \) is irksome. In populations in which members of a given type \( l \) can fight one another, a natural possibility is to have a transition \( J = -2e^{(l)} \) at a rate proportional to \( X'(X^l - 1) \), which translates to \( \alpha_J = \alpha_J^{(N)} = \gamma X'(x^l - N^{-1}) \), a function depending on \( N \). Replacing this with \( \alpha_J = \gamma (x^l)^2 \) removes the
$N$-dependence, but yields a process that can jump to negative values of $X^l$. For this reason, it is useful to be able to allow the transition rates $\alpha_j$ to depend on $N$.

Since the arguments in this paper are not limiting arguments for $N \to \infty$, it does not require many changes to derive the corresponding results. Quantities such as $A$, $F$, $U_r(x)$ and $V_r(x)$ now depend on $N$; however, Theorem 4.7 continues to hold with constants $c_1$ and $c_2$ that do not depend on $N$, provided that $\mu$, $w$, $\nu$, the $k_{lm}$ from Assumption 2.1 and $\zeta$ from Assumption 4.2 can be chosen to be independent of $N$, and that the quantities $Z_s^{(l)}$ from (4.21) can be bounded uniformly in $N$. On the other hand, the solution $x = x^{(N)}$ of (4.2) that acts as approximation to $x_N$ in Theorem 4.7 now itself depends on $N$, through $R = R^{(N)}$ and $F = F^{(N)}$. If $A$ (and hence $R$) can be taken to be independent of $N$, and $\lim_{N \to \infty} \|F^{(N)} - F\|_\mu = 0$ for some fixed $\mu$–Lipschitz function $F$, a Gronwall argument can be used to derive a bound for the difference between $x^{(N)}$ and the (fixed) solution $x$ to equation (4.2) with $N$-independent $R$ and $F$. If $A$ has to depend on $N$, the situation is more delicate.

5 Examples

We begin with some general remarks, to show that the assumptions are satisfied in many practical contexts. We then discuss two particular examples, those of Kretzschmar (1993) and of Arrigoni (2003), that fitted poorly or not at all into the general setting of Barbour & Luczak (2008), though the other systems referred to in the introduction could also be treated similarly. In both of our chosen examples, the index $j$ represents a number of individuals — parasites in a host in the first, animals in a patch in the second — and we shall for now use the former terminology for the preliminary, general discussion.

Transitions that can typically be envisaged are: births of a few parasites, which may occur either in the same host, or in another, if infection is being represented; births and immigration of hosts, with or without parasites; migration of parasites between hosts; deaths of parasites; deaths of hosts; and treatment of hosts, leading to the deaths of many of the host’s parasites. For births of parasites, there is a transition $X \to X + J$, where $J$ takes the form

$$J_l = 1; \quad J_m = -1; \quad J_j = 0, \quad j \neq l, m,$$

(5.1)
indicating that one \( m \)-host has become an \( l \)-host. For births of parasites within a host, a transition rate of the form \( b_{l\rightarrow m}X_m \) could be envisaged, with \( l > m \), the interpretation being that there are \( X_m \) hosts with parasite burden \( m \), each of which gives birth to \( s \) offspring at rate \( b_s \), for some small values of \( s \). For infection of an \( m \)-host, a possible transition rate would be of the form

\[
X_m \sum_{j \geq 0} N^{-1}X_j \lambda p_{j,l-m},
\]

since an \( m \)-host comes into contact with \( j \)-hosts at a rate proportional to their density in the host population, and \( p_{jr} \) represents the probability of a \( j \)-host transferring \( r \) parasites to the infected host during the contact. The probability distributions \( p_j \) can be expected to be stochastically increasing in \( j \). Deaths of parasites also give rise to transitions of the form \( \text{(5.1)} \), but now with \( l < m \), the simplest form of rate being just \( dmX_m \) for \( l = m-1 \), though \( d = d_m \) could also be chosen to increase with parasite burden. Treatment of a host would lead to values of \( l \) much smaller than \( m \), and a rate of the form \( \kappa X_m \) for the transition with \( l = 0 \) would represent fully successful treatment of randomly chosen individuals. Births and deaths of hosts and immigration all lead to transitions of the form

\[
J_l = \pm 1; \quad J_j = 0, \quad j \neq l.
\]

For deaths, \( J_l = -1 \), and a typical rate would be \( d'X_l \). For births, \( J_l = 1 \), and a possible rate would be \( \sum_{j \geq 0} X_j b_{jl} \) (with \( l = 0 \) only, if new-born individuals are free of parasites). For immigration, constant rates \( \lambda_l \) could be supposed. Finally, for migration of individual parasites between hosts, transitions are of the form

\[
J_l = J_m = -1; \quad J_{l+1} = 1; \quad J_{m-1} = 1; \quad J_j = 0, \quad j \neq l, m, l+1, m-1,
\]

a possible rate being \( \gamma mX_mN^{-1}X_l \).

For all the above transitions, we can take \( J_* = 2 \) in \( \text{(1.2)} \), and \( \text{(1.3)} \) is satisfied in biologically sensible models. \( \text{(3.1)} \) and \( \text{(3.2)} \) depend on the way in which the matrix \( A \) can be defined, which is more model specific; in practice, \( \text{(3.1)} \) is very simple to check. The choice of \( \mu \) in \( \text{(3.2)} \) is influenced by the need to have \( \text{(4.1)} \) satisfied. For Assumptions \( \text{(2.1)} \), a possible choice of \( \nu \) is to take \( \nu(j) = (j+1) \) for each \( j \geq 0 \), with \( S_1(X) \) then representing the number of hosts plus the number of parasites. Satisfying \( \text{(2.5)} \) is then easy for
transitions only involving the movement of a single parasite, but in general requires assumptions as to the existence of the \( r \)-th moments of the distributions of the numbers of parasites introduced at birth, immigration and infection events. For (2.6), in which transitions involving a net reduction in the total number of parasites and hosts can be disregarded, the parasite birth events are those in which the rates typically have a factor \( mX_m \) for transitions with \( J_m = -1 \), with \( m \) in principle unbounded. However, at such events, an \( m \)-individual changes to an \( m + s \) individual, with the number \( s \) of offspring of the parasite being typically small, so that the value of \( J^T \nu_r \) associated with this rate has magnitude \( m^r - 1 \); the product \( mX_m m^r - 1 \), when summed over \( m \), then yields a contribution of magnitude \( S_r(X) \), which is allowable in (2.6). Similar considerations show that the terms 

\[ N^{i-1} S_0(X) S_r(X) \]

accommodate the migration rates suggested above. Finally, in order to have Assumptions 4.2 satisfied, it is in practice necessary that Assumptions 2.1 are satisfied for large values of \( r \), thereby imposing restrictions on the distributions of the numbers of parasites introduced at birth, immigration and infection events, as above.

### 5.1 Kretzschmar’s model

Kretzschmar (1993) introduced a model of a parasitic infection, in which the transitions from state \( X \) are as follows:

\[
\begin{align*}
J &= e^{(i-1)} - e^{(i)} \quad \text{at rate} \quad Ni\mu x^i, \quad i \geq 1; \\
J &= -e^{(i)} \quad \text{at rate} \quad N(\kappa + i\alpha)x^i, \quad i \geq 0; \\
J &= e^{(0)} \quad \text{at rate} \quad N\beta \sum_{i \geq 0} x^i \theta^i; \\
J &= e^{(i+1)} - e^{(i)} \quad \text{at rate} \quad N\lambda x^i \varphi(x), \quad i \geq 0,
\end{align*}
\]

where \( x := N^{-1}X, \varphi(x) := \|x\|_{11}\{c + \|x\|_1\}^{-1} \) with \( c > 0 \), and \( \|x\|_{11} := \sum_{j \geq 1} j|x|^j \); here, \( 0 \leq \theta \leq 1 \), and \( \theta^i \) denotes its \( i \)-th power (our \( \theta \) corresponds to the constant \( \xi \) in [7]). Both (1.2) and (1.3) are obviously satisfied. For Assumptions (3.1), (3.2) and (4.1), we note that equation corresponding to (1.5) has

\[
\begin{align*}
A_{ii} &= -\{\kappa + i(\alpha + \mu)\}; \quad A_{i,i-1}^T = i\mu \quad \text{and} \quad A_{10}^T = \beta \theta^i, \quad i \geq 2; \\
A_{11} &= -\{\kappa + \alpha + \mu\}; \quad A_{10}^T = \mu + \beta \theta; \\
A_{00} &= -\kappa + \beta, \quad i \geq 1,
\end{align*}
\]
with all other elements of the matrix equal to zero, and
\[ F^i(x) = \lambda(x^{i-1} - x^i)\varphi(x), \quad i \geq 1; \quad F^0(x) = -\lambda x^0\varphi(x). \]

Hence Assumption (3.1) is immediate, and Assumption (3.2) holds for \( \mu = (j + 1)^s \) for any \( s \geq 0 \), with \( w = (\beta - \kappa)_+ \). For the choice \( \mu(j) = j + 1 \), \( F \) maps elements of \( R_\mu \) to \( R_\mu \), and is also locally Lipschitz in the \( \mu \)-norm, with \( K(\mu,F;\Xi) = c^{-2}\lambda\Xi(2c + \Xi) \).

For Assumptions (2.3) choose \( \nu = \mu \); then (2.5) is a finite sum for each \( r \geq 0 \). Turning to (2.6), it is immediate that \( U_0(x) \leq \beta S_0(x) \). Then, for \( r \geq 1 \),
\[
\sum_{i=0}^{\infty} \lambda\varphi(N^{-1}X)X^i((i+2)^r - (i+1)^r) \leq \frac{S_1(X)}{S_0(X)} \sum_{i=0}^{\infty} rX^i(i+2)^{r-1} \leq r2^{r-1}\lambda S_r(X),
\]
since, by Jensen’s inequality, \( S_1(X)S_{r-1}(X) \leq S_0(X)S_r(X) \). Hence we can take \( k_{r2} = k_{r4} = 0 \) and \( k_{r1} = \beta + r2^{r-1}\lambda \) in (2.6), for any \( r \geq 1 \), so that \( r^{(1)}_{\max} = \infty \). Finally, for (2.7),
\[
V_0(x) \leq (\kappa + \beta)S_0(x) + \alpha S_1(x),
\]
so that \( k_{03} = \kappa + \beta + \alpha \) and \( k_{05} = 0 \), and
\[
V_r(x) \leq r^2(\kappa S_{2r}(x) + \alpha S_{2r+1}(x) + \mu S_{2r-1}(x) + 2^{2(r-1)}\lambda S_{2r-1}(x)) + \beta S_0(x),
\]
so that we can take \( p(r) = 2r + 1 \); \( k_{r3} = \beta + r^2(\kappa + \alpha + \mu + 2^{2(r-1)}\lambda) \), and \( k_{r5} = 0 \) for any \( r \geq 1 \), and so \( r^{(2)}_{\max} = \infty \). In Assumptions 4.2, we can clearly take \( r_\mu = 1 \) and \( \zeta(k) = (k + 1)^2 \), giving \( r(\zeta) = 8 \), \( b(\zeta) = 1 \) and \( \rho(\zeta,\mu) = 17 \).

### 5.2 Arrigoni’s model

In the metapopulation model of Arrigoni (2003), the transitions from state \( X \) are as follows:
\[
\begin{align*}
J &= e^{(i-1)} - e^{(i)} & \text{at rate } Nix^i(d_i + \gamma(1 - \rho)), \quad i \geq 2; \\
J &= e^{(0)} - e^{(1)} & \text{at rate } Nx^1(d_1 + \gamma(1 - \rho) + \kappa); \\
J &= e^{(i+1)} - e^{(i)} & \text{at rate } Nib_i x^i, \quad i \geq 1; \\
J &= e^{(0)} - e^{(i)} & \text{at rate } N\alpha x^i \kappa, \quad i \geq 2; \\
J &= e^{(k+1)} - e^{(k)} + e^{(i-1)} - e^{(i)} & \text{at rate } Nix^k \rho \gamma, \quad k \geq 0, \quad i \geq 1;
\end{align*}
\]
as before, \( x := N^{-1}X \). Here, the total number \( N = \sum_{j \geq 0} X_j = S_0(X) \) of patches remains constant throughout, and the number of animals in any one patch changes by at most one at each transition; in the final (migration) transition, however, the numbers in two patches change simultaneously. In the above transitions, \( \gamma, \rho, \kappa \) are non-negative, and \((d_i), (b_i)\) are sequences of non-negative numbers.

Once again, both \((1.2)\) and \((1.3)\) are obviously satisfied. The equation corresponding to \((1.4)\) can now be expressed by taking

\[
A_{ii} = -\{\kappa + i(b_i + d_i + \gamma)\}; \quad A_{i,i-1}^T = i(d_i + \gamma); \quad A_{i,i+1}^T = ib_i, \quad i \geq 1;
\]

\[A_{00} = -\kappa,\]

with all other elements of \( A \) equal to zero, and

\[
F_i(x) = \rho\gamma\|x\|_{11}(x^{i-1} - x^i), \quad i \geq 1; \quad F_0(x) = -\rho\gamma x^0\|x\|_{11} + \kappa,
\]

where we have used the fact that \( N^{-1} \sum_{j \geq 0} X_j = 1 \). Hence Assumption \((3.1)\) is again immediate, and Assumption \((3.2)\) holds for \( \mu(j) = 1 \) with \( w = 0 \), for \( \mu(j) = j + 1 \) with \( w = \max_i (b_i - d_i - \gamma - \kappa)_+ \) (assuming \((b_i)\) and \((d_i)\) to be such that this is finite), or indeed for \( \mu(j) = (j + 1)^s \) with any \( s \geq 2 \), with appropriate choice of \( w \). With the choice \( \mu(j) = j + 1 \), \( F \) again maps elements of \( \mathcal{R}_\mu \) to \( \mathcal{R}_\mu \), and is also locally Lipschitz in the \( \mu \)-norm, with \( K(\mu,F;\Xi) = 3\rho\gamma\Xi \).

To check Assumptions \((2.1)\) take \( \nu = \mu \); once again, \((2.5)\) is a finite sum for each \( r \). Then, for \((2.6)\), it is immediate that \( U_0(x) = 0 \). For any \( r \geq 1 \), using arguments from the previous example,

\[
U_r(x) \leq r2^{r-1}\left\{ \sum_{i \geq 1} ib_i x^i(i + 1)^{r-1} + \sum_{i \geq 1} \sum_{k \geq 0} i\rho\gamma x^i x^k (k + 1)^{r-1} \right\}
\]

\[
\leq r2^{r-1}\{\max_i b_i S_r(x) + \rho\gamma S_1(x) S_{r-1}(x)\}
\]

\[
\leq r2^{r-1}\{\max_i b_i S_r(x) + \rho\gamma S_0(x) S_r(x)\},
\]

so that, since \( S_0(x) = 1 \), we can take \( k_{r1} = r2^{r-1}(\max_i b_i + \rho\gamma) \) and \( k_{r2} = k_{r4} = 0 \) in \((2.6)\), and \( r_{(1)}^{(1)}_{\text{max}} = \infty \). Finally, for \((2.7)\), \( V_0(x) = 0 \) and, for \( r \geq 1 \),

\[
V_r(x)
\leq r2^{r}\left\{ 2^{2r-1}\max_i b_i S_{2r-1}(x) + \max_i (i^{-1} d_i) S_{2r}(x) + \gamma(1 - \rho) S_{2r-1}(x) + \rho\gamma(2^{2(r-1)} S_1(x) S_{2r-2}(x) + S_0(x) S_{2r-1}(x)) \right\} + \kappa S_{2r}(x),
\]

32
so that we can take \( p(r) = 2r \), and (assuming \( i^{-1}d_i \) to be finite)

\[
k_{r;3} = \kappa + r^2 \{ 2^{2(r-1)} \left( \max_i b_i + \rho \gamma \right) + \max_i (i^{-1}d_i) + \gamma \},
\]

and \( k_{r;5} = 0 \) for any \( r \geq 1 \), and \( r_{\text{max}}^{(2)} = \infty \). In Assumptions 4.2 we can again take \( r_{\mu} = 1 \) and \( \zeta(k) = (k + 1)^7 \), giving \( r(\zeta) = 8 \), \( b(\zeta) = 1 \) and \( \rho(\zeta, \mu) = 16 \).

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