Efficient Verification of Quantum Gates with Local Operations

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(Dated: November 1, 2019)

Efficient verification of the functioning of quantum devices is a key to the development of quantum technologies, but is a daunting task as the system size increases. Here we propose a simple and general framework for verifying unitary transformations that can be applied to both individual quantum gates and gate sets, including quantum circuits. This framework enables efficient verification of many important unitary transformations, including but not limited to all bipartite unitaries, Clifford unitaries, generalized controlled-Z gates, generalized CNOT gates, and CSWAP gate. The sampling complexity increases at most linearly with the system size and is often independent of the system size. Moreover, little overhead is incurred even if one can only prepare Pauli eigenstates and perform local measurements. Our approach is applicable in many scenarios in which randomized benchmarking (RB) does not apply and is thus instrumental to quantum computation and many other applications in quantum information processing.

Introduction.—Quantum technologies promise to dramatically boost our capability in secure communication, fast computation, and efficient simulation of quantum many-body systems. To harness the power of quantum technologies, it is crucial to characterize and verify quantum devices with a high precision [1]. The problem is particularly pressing in the context of quantum computation, in which efficient characterization of quantum gates and circuits has become a bottleneck. To characterize quantum gates with quantum process tomography, the resource overhead increases exponentially with the system size. As a popular alternative, randomized benchmarking (RB) [2–4] is much more efficient and is robust against state preparation and measurement errors (SPAM). However, RB usually relies on strong assumptions on the set of gates to be benchmarked, such as group structure and the property of being a unitary 2-design. Although tremendous efforts have been directed to relaxing these assumptions [6–11], the applicability of RB is still quite limited.

In this paper we propose a general and efficient framework for verifying unitary processes based on local operations, that is, local state preparation and measurements. Our approach can be applied to individual gates as well as gate sets, including quantum circuits, even in many scenarios in which RB does not apply. It enables efficient verification of many important unitary transformations, including but not limited to all (qubit and qudit) bipartite unitaries, Clifford unitaries, generalized controlled-Z gates, generalized CNOT gates, and CSWAP gate (also known as Fredkin gate) [12]. To achieve infidelity \( \epsilon \) and significance level \( \delta \), the number of tests required (sample complexity) is at most \( 2\text{ne}^{-1}\ln \delta^{-1} \) \( n \) is the number of parties), and it is even independent of system size in many cases of practical interest.

Channel-state duality.—The idea of channel-state duality [13] will play a key role in the study of quantum gate verification (QGV). Let \( \mathcal{H} \) be a Hilbert space of dimension \( d \) and \( \mathcal{B}(\mathcal{H}) \) the space of bounded linear operators on \( \mathcal{H} \). A quantum channel \( \Lambda \) on \( \mathcal{B}(\mathcal{H}) \) is represented by a completely-positive and trace-preserving (CPTP) map \[ \Lambda = \int \Phi \otimes \psi \, d\psi. \]

Quantum channels on \( \mathcal{B}(\mathcal{H}) \) are in one-to-one correspondence with Choi states on \( \mathcal{H} \otimes \mathcal{H} \). More precisely, the Choi state associated with \( \Lambda \) is defined as

\[
\chi_{\Lambda} := (\Lambda \otimes 1)(|\Phi\rangle\langle \Phi|),
\]

where \( \Phi = (\sum_{i,j} |ij\rangle) / \sqrt{d} \) is a maximally entangled state in \( \mathcal{H} \otimes \mathcal{H} \). The two reduced states of \( \chi_{\Lambda} \) are completely mixed, that is, tr(\( \chi_{\Lambda} \)) = \( 1/d \). Conversely, any quantum state on \( \mathcal{H} \otimes \mathcal{H} \) that satisfies this condition determines a quantum channel via the duality relation. Denote by \( \rho^* \) the complex conjugate of \( \rho \) in the computational basis, then \( \Lambda(\rho) = d \text{tr} \chi_{\Lambda}(1 \otimes \rho^*) \).

The entanglement fidelity or process fidelity between \( \Lambda \) and \( \mathcal{U} \) is defined as

\[
F_{E}(\Lambda, \mathcal{U}) := \text{tr} \chi_{\Lambda} \chi_{\mathcal{U}} = \langle \Phi | \chi_{\Lambda} (1 \otimes \rho^*) | \Phi \rangle.\]

The entanglement infidelity. By Refs. [15–16], we have

\[
F_{E} = \frac{d F_{E} + 1}{d + 1}, \quad \epsilon_{\Lambda} = \frac{d e_{E}}{d + 1}.\]

When \( \mathcal{U} \) is the identity, \( F_{E}(\Lambda, \mathcal{U}) \) and \( F_{E}(\Lambda, \mathcal{U}) \) are abbreviated as \( F_{E}(\Lambda) \) and \( F_{E}(\Lambda) \), respectively.
Quantum gate verification.—Consider a device that is supposed to perform the unitary transformation $U$, but actually realizes an unknown channel $\Lambda$. To verify whether the channel $\Lambda$ is sufficiently close to the target unitary, we can pick a pure test state $\rho_j = |\psi_j\rangle\langle\psi_j|$ randomly with probability $p_j$ from the ensemble of test states and feed into the channel. Then we can verify whether the output $\Lambda(\rho_j)$ is sufficiently close to the target state $U(\rho_j) = U\rho_j U^\dagger$. To this end, we perform two-outcome tests $\{E_{ij}, 1 - E_{ij}\}$ from a set of accessible tests depending on the input state $\rho_j$ and the test operator $E_{ij}$ corresponds to passing the test and satisfies the condition $E_{ij} U(\rho_j) = U(\rho_j)$, so that the target output state $U(\rho_j)$ can always pass the test. Suppose the test $E_{ij}$ is performed with probability $p_{ij}$ given the test state $\rho_j$, then the passing probability of $\Lambda(\rho_j)$ reads $\text{tr}(\Omega_j \Lambda(\rho_j))$, where $\Omega_j = \sum_{ij} p_{ij} E_{ij}$ is a verification operator for $U(\rho_j)$ \footnotemark[4] \footnotemark[5]. Note that $\Omega_j \geq U(\rho_j)$ since $U(\rho_j)$ is supported in the eigenspace of $\Omega_j$ with the largest eigenvalue $1$. The overall average passing probability reads
\begin{equation}
\sum_{ij} p_j p_{ij} \text{tr}(\Lambda(\rho_j) E_{ij}) = \text{tr}(\tilde{\Theta} \Lambda) = \text{tr}(\Theta \chi),
\end{equation}
where
\begin{equation}
\tilde{\Theta} := d \sum_{j} p_j \Omega_j \otimes \rho_j^*, \quad \Theta := d \sum_{j} p_j U^\dagger \Omega_j U \otimes \rho_j^*
\end{equation}
are called channel verification operators. Here the first variant $\tilde{\Theta}$ is more natural, but the second variant $\Theta$ is more convenient for technical analysis. Note that $\text{tr}(\Theta |\Phi\rangle \langle \Phi|) = \sum_j p_j = 1$, so the passing probability is $1$ if $\Lambda = U$ (which means $\chi = |\Phi\rangle \langle \Phi|$) as expected.

Now we repeat the preparation and test procedure $N$ times and accept the device iff all tests are passed. In general, the channels $\Lambda_1, \Lambda_2, \ldots, \Lambda_N$ realized over the $N$ runs may be different due to inevitable fluctuations, but here we assume that they are independent for simplicity. Then the acceptance probability reads $\prod_{r=1}^N \text{tr}(\Theta \chi_{E_r})$, where $E_r = U^\dagger \circ \Lambda_r$. Our goal is to ensure that the false acceptance probability is smaller than a given threshold, the significance level $\delta$, whenever the average gate fidelity (or entanglement fidelity) is smaller than a given threshold.

For $0 \leq \epsilon \leq 1$, define
\begin{equation}
p_{\Lambda}(\Theta, \epsilon) := \max_{F_{\epsilon}(\chi) \leq 1 - \epsilon} \text{tr}(\Theta \chi),
\end{equation}
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p_{\Lambda}(\Theta, \epsilon) := \max_{F_{\epsilon}(\chi) \leq 1 - \epsilon} \text{tr}(\Theta \chi).
\end{equation}

Then the passing probability in Eq. (4) is upper bounded by $p_{\Lambda}(\Theta, \epsilon) \cdot p_{\Lambda}(\Theta, \epsilon)$ whenever the average gate fidelity $F_{\epsilon}(\Lambda, U)$ (entanglement fidelity $F_E(\Lambda, U)$) is upper bounded by $1 - \epsilon$. According to Eq. (3), we have $p_{\Lambda}(\Theta, \epsilon) = p_{\Lambda}(\Theta, (d + 1)\epsilon/d)$, so to compute $p_{\Lambda}(\Theta, \epsilon)$, it suffices to compute $p_{\Lambda}(\Theta, \epsilon)$. In general, it is not easy to derive an analytical formula for $p_{\Lambda}(\Theta, \epsilon)$, but it is easy to compute its value by semidefinite programming (SDP),
\begin{equation}
\max \text{tr}(\Theta \chi) \text{ subject to } \chi \geq 0, \quad \text{tr}(\chi) = 1/d, \quad (\Phi |\chi \rangle \langle \Phi|) \leq 1 - \epsilon.
\end{equation}
The following proposition is a simple implication of the definitions in Eqs. (6) and (7).

**Proposition 1**. $p_{\Lambda}(\Theta, \epsilon)$ and $p_{\Lambda}(\Theta, \epsilon)$ are concave and nonincreasing in $\epsilon$, but are convex in $\Theta$.

Here the concavity of $p_{\Lambda}(\Theta, \epsilon)$ means
\begin{equation}
p_{\Lambda}(\Theta, \epsilon) \geq \mu p_{\Lambda}(\Theta, \epsilon_1) + (1 - \mu) p_{\Lambda}(\Theta, \epsilon_2)
\end{equation}
if $\epsilon = \mu \epsilon_1 + (1 - \mu) \epsilon_2$ and $0 \leq \mu \leq 1$. By contrast, the convexity means
\begin{equation}
p_{\Lambda}(\Theta, \epsilon) \leq \mu p_{\Lambda}(\Theta_1, \epsilon) + (1 - \mu) p_{\Lambda}(\Theta_2, \epsilon)
\end{equation}
if $\Theta = \mu \Theta_1 + (1 - \mu) \Theta_2$. The same holds for $p_{\Lambda}(\Theta, \epsilon)$.

Thanks to Proposition 1, the acceptance probability $\prod_{r=1}^N \text{tr}(\Theta \chi_{E_r})$ can be upper bounded by
\begin{equation}
\prod_{r=1}^N p_{\Lambda}(\Theta, \epsilon_r) \leq \left[ 1 + \frac{1}{N} \sum_{r} p_{\Lambda}(\Theta, \epsilon_r) \right]^N \leq p_{\Lambda}(\Theta, \epsilon)^N,
\end{equation}
where $\epsilon_r = 1 - F_{\epsilon}(E_r)$ and $\bar{\epsilon} = \sum \epsilon_r/N$ are the entanglement infidelities and their average over the $N$ runs. The upper bound in Eq. (11) can be saturated when all $\epsilon_r$ are equal. To verify $U$ within entanglement infidelity $\epsilon$ and significance level $\delta$, which means $p_{\Lambda}(\Theta, \epsilon)^N \leq \delta$ [assuming $p_{\Lambda}(\Theta, \epsilon) < 1$], the number of tests required reads
\begin{equation}
N_{\Lambda}(\epsilon, \delta, \Theta) = \left[ \frac{\ln \delta}{\ln p_{\Lambda}(\Theta, \epsilon)} \right].
\end{equation}
To verify $U$ within average infidelity $\epsilon$, the number is $N_{\Lambda}(\epsilon, \delta, \Theta) = \left[ \ln \delta/\ln p_{\Lambda}(\Theta, \epsilon) \right] = N_{\Lambda}(d + 1)\epsilon/d, \delta, \Theta).

To minimize the number of tests, we need to minimize $p_{\Lambda}(\Theta, \epsilon)$ or $p_{\Lambda}(\Theta, \epsilon)$ over the channel verification operator $\Theta$. Let $V$ be an arbitrary unitary operator on $\mathcal{H}$ and $\Theta_V$ be the verification operator constructed from $\Theta$ by replacing $\rho_j$ with $V \rho_j V^\dagger$ and $E_{ij}$ with $UV U^\dagger E_{ij} U V^\dagger U^\dagger$. Then we have $p_{\Lambda}(\Theta_V, \epsilon) = p_{\Lambda}(\Theta, \epsilon)$ and $p_{\Lambda}(\Theta_V, \epsilon) = p_{\Lambda}(\Theta, \epsilon)$ so averaging over unitarily equivalent strategies cannot decrease the efficiency. In addition, the efficiency cannot decrease if we replace $\Omega_j$ with $U \rho_j U^\dagger$ given that $\Omega_j \geq U \rho_j U^\dagger$ by assumption. Therefore, $p_{\Lambda}(\Theta, \epsilon)$ and $p_{\Lambda}(\Theta, \epsilon)$ are minimized when $\Omega_j = U \rho_j U^\dagger$ and the ensemble $\{\rho_j, \rho_j \}_j$ of test states is distributed according to the Haar measure. Then we have
\begin{equation}
\Theta = \Theta_P := d \sum_{j} p_j \rho_j \otimes \rho_j^* = \frac{d |\Phi\rangle \langle \Phi| + 1}{d + 1},
\end{equation}
where $\Theta_P$ is the preparation operator. The same conclusion holds if the ensemble $\{\rho_j, p_j\}_j$ forms a 2-design \[21, 22\]. The minimums of $p_E(\Theta, \epsilon)$ and $p_A(\Theta, \epsilon)$ read

$$p_E(\Theta, \epsilon) = 1 - \frac{d}{d+1}\epsilon, \quad p_A(\Theta, \epsilon) = 1 - \epsilon,$$

so that $N_E(\epsilon, \delta, \Theta) = N_A(d\epsilon/(d+1), \delta, \Theta)$ and

$$N_A(\epsilon, \delta, \Theta) = \left[\frac{\ln \delta}{\ln(1 - \epsilon)}\right] \leq \left[\frac{1}{\epsilon} \ln \frac{1}{\epsilon}\right].$$

Note that these results are independent of the unitary transformation to be verified.

According to the above analysis, in general we have $p_A(\Theta, \epsilon) \geq 1 - \epsilon$, and the number of tests required is bounded from below by Eq. (14). Denote by $\beta_j$ the second largest eigenvalue of $\Omega_j$, that is, $\beta_j := \|\Omega_j - U\rho_j U^\dagger\|$, and let $\beta_M := \max_j \beta_j$, $\nu_M := 1 - \beta_M$. Then $\Omega_j \leq \nu_j U\rho_j U^\dagger + \beta_j$, so that

$$\Theta \leq d \sum_j p_j (\nu_j \rho_j + \beta_j) \otimes \rho_j^* \leq \nu_M \Theta_P + \beta_M \otimes \sum_j dp_j \rho_j^*.$$  

(16)

This inequality is very instructive to understanding the efficiency of a general verification strategy.

**Balanced state preparation.**—The ensemble of test states $\{\rho_j, p_j\}_j$ is balanced if $d \sum_j p_j \rho_j = 1$, that is, if $\{dp_j \rho_j\}_j$ is formally a positive operator-valued measure. Then the verification strategy and the operators $\Theta, \Theta_P$ are also called balanced. In this case, we have $\Theta_P \leq \Theta \leq 1$, and the maximally entangled state $|\Phi\rangle$ is an eigenstate of $\Theta_P$ and $\Theta$ with the largest eigenvalue 1. Formally $\Theta_P, \Theta$ are verification operators of $|\Phi\rangle$ \[22\], so many results presented in Ref. \[22\] can be applied to the current study. A simple way for constructing balanced ensembles is to use orthonormal bases: If $\{\langle \psi_j|\psi_j\rangle\}_j$ forms an orthonormal basis in $\mathcal{H}$, then the ensemble $\{|\psi_j\rangle\}_j$ is balanced. In this case, $\{U|\psi_j\rangle\}_j$ also forms an orthonormal basis, which is often useful to simplify the verification of the output states.

If $\Theta$ is balanced, then $\Theta \leq \nu_M \Theta_P + \beta_M$ according to Eq. (16). Let $\beta = \beta(\Theta)$ be the second largest eigenvalue of $\Theta$ and $\nu = \nu(\Theta) := 1 - \beta(\Theta)$; then $\nu \geq \nu_P \nu_M$, where $\nu_P := 1 - \beta_P$ and $\beta_P = \beta(\Theta_P)$ is the second largest eigenvalue of the preparation operator $\Theta_P$. Therefore,

$$p_E(\Theta, \epsilon) \leq 1 - \nu \epsilon \leq 1 - \nu_P \nu_M \epsilon.$$  

(17)

Here the first inequality follows from the definition in Eq. (7) and can be proved using a similar idea used to prove Eq. (1) in Ref. \[19\] (cf. Ref. \[17\]). As a consequence, the minimum number of tests required satisfies

$$N_E(\epsilon, \delta, \Theta) \leq \left[\frac{\ln \delta}{\ln(1 - \epsilon)}\right] \leq \left[\frac{1}{\nu_P \nu_M \epsilon}\right] \leq \left[\frac{\ln(\delta^{-1})}{\nu_P \nu_M \epsilon}\right].$$  

(18)

The efficiency is mainly determined by $\nu_P$ and $\nu_M$, which characterize the performances of state preparation and measurements, respectively. This simple observation is crucial to constructing efficient verification protocols.

The upper bound 1 for $\nu_M$ is saturated iff $\Omega_j = U\rho_j U^\dagger$ for all $j$. Given that $\tr(\Theta_P) = d$, we can deduce that $\beta_P \geq 1/(d+1)$ and $\nu_P \leq d/(d+1)$. The bound for $\nu_P$ is saturated if $\Theta_P = (d/\tr(\Phi)\{|\Phi\rangle\langle \Phi|\})/(d+1)$, which means the ensemble $\{\rho_j, p_j\}_j$ is a 2-design. When $d \geq 3$, a 2-design can be constructed from $(1/(d-1)^2) + 1$ bases \[21, 22\]. When $d$ is a prime power, the set of stabilizer states forms a 2-design. In addition, a 2-design can be constructed from a complete set of $d + 1$ mutually unbiased bases (MUB) \[21, 24, 25\]. Two bases $\{|\psi_j\rangle\}_j$ and $\{|\varphi_k\rangle\}_k$ are mutually unbiased if $\langle \psi_j|\varphi_k\rangle^2 = 1/d$ for all $j, k$. If instead the ensemble $\{\rho_j, p_j\}_j$ is constructed from $r$ MUB with uniform weights, then we have $\nu_P = (r-1)/r$ (cf. Proposition 2 in Ref. \[22\]). The case of two MUB was studied in Ref. \[29\] (cf. Ref. \[30\]).

Define the phase operator $Z$ and shift operator $X$ on $\mathcal{H}$ as follows,

$$Z|j\rangle = e^{2\pi i/d}|j\rangle, \quad X|j\rangle = |j+1\rangle, \quad j \in \mathbb{Z}_d,$$

(19)

where $\omega = e^{2\pi i/d}$ and $\mathbb{Z}_d$ is the ring of integers modulo $d$. Then the respective eigenvalues of the three operators $Z, X, Z^2$ are mutually unbiased \[28\]. To be specific, the eigenbasis of $Z$ is just the standard computational basis; the eigenbasis of $X$ is the Fourier basis and is composed of $\{\langle \psi_j|\psi_j\rangle\}_j$. Then the eigenbasis of $XZ$ is composed of $\{\langle \psi_j|\psi_j\rangle\}_j$, where $\tau = -e^{2\pi i/d}$. If the dimension $d$ is a prime, then the respective eigenvalues of $Z, X, Z^2, ..., XZ^{d-1}$ constitute a complete set of MUB \[28\].

**Local state preparation and measurements.**—In practice, it is often not easy to prepare entangled test states or to perform entangling measurements. It is thus of fundamental interest to clarify the limitation of local operations on QGV. Suppose the Hilbert space $\mathcal{H}$ is a tensor product of $n$ Hilbert spaces $\mathcal{H} = \bigotimes_{k=1}^n \mathcal{H}_k$ with $\dim(\mathcal{H}_k) = d_k$ and $d = \prod_{k=1}^n d_k$. Let $U(\mathcal{H}_k)$ be the group of unitary operators on $\mathcal{H}_k$. Suppose we can only prepare product test states. Then $\nu(\Theta_P)$ is maximized when the product test states are distributed according to the Haar measure induced by $\bigotimes_{k=1}^n U(\mathcal{H}_k)$ given that $\nu(\Theta_P)$ is concave in $\Theta_P$. In this case, we have

$$\Theta_P = \bigotimes_{k=1}^n \frac{d_k|\Phi_k\rangle\langle \Phi_k| + 1}{d_k + 1}, \quad \nu(\Theta_P) = \frac{d_{\min}}{d_{\min} + 1}.$$  

(20)

where $|\Phi_k\rangle$ is a maximally entangled state in $\mathcal{H}_k \otimes \mathcal{H}_k$ and $d_{\min} = \min_{k} d_k$ is the minimum of the local dimensions. Note that $\Theta_P$ acts on $\bigotimes_{k=1}^n \mathcal{H}_k^{\otimes 2}$, which is isomorphic to $\bigotimes_{k=1}^n \mathcal{H}_k^{\otimes 2}$. The above equation also applies if the ensemble of test states is constructed from a product 2-design, that is, the tensor product of 2-designs for
individual local Hilbert spaces. Any other preparation operator $\Theta_p$ based on product test states satisfies the bound $\nu(\Theta_p) \leq d_{\text{min}}/(d_{\text{min}} + 1)$.

Suppose we can construct $r$ MUB $B_1^{(k)}, B_2^{(k)}, \ldots, B_r^{(k)}$ for each $\mathcal{H}_k$. Then the $r$ bases $\bigotimes_{k=1}^n B_s^{(k)}$ in $\mathcal{H}$ for $s = 1, 2, \ldots, r$ are mutually unbiased. These bases can be used to construct an ensemble of test states, which can achieve preparation spectral gap $\nu_p = (r - 1)/r$. If each $d_k$ for $k = 1, 2, \ldots, n$ is a prime power, then we can construct $d_{\text{min}}$ mutually unbiased product bases in this way, so the preparation spectral gap can attain the maximum $d_{\text{min}}/(d_{\text{min}} + 1)$ (over local preparation strategies). In general, we can construct at least three mutually unbiased product bases given the discussion after Eq. 19, so the preparation spectral gap achievable in this way is at least 2/3. The restriction to local state preparation thus has little influence on the verification efficiency.

By contrast, the limitations of local measurements depend on the unitary transformation to be verified and the test states used. Nevertheless, many important gates and circuits can be verified efficiently using local measurements and local state preparation, as shown below.

Applications.—To illustrate the power of the general framework of QGV proposed above, here we construct efficient verification protocols for a number of important quantum gates and circuits using local operations.

Let us start with bipartite unitary transformations. The test states can be constructed from a product two-design, so that the preparation spectral gap can attain the maximum $d_{\text{min}}/(d_{\text{min}} + 1)$ (over local state preparation). Alternatively, we can use mutually unbiased product bases to achieve a similar performance. To verify general bipartite unitary transformations efficiently with local measurements, we need to verify general bipartite pure states efficiently with local measurements. Fortunately, the later problem has been solved recently [23, 31, 32]. For any bipartite pure state, we can construct a verification strategy using local measurements which can achieve a spectral gap of at least 2/3. The optimal strategy can be constructed by performing the standard test (based on projective measurements on the Schmidt basis) and a number of adaptive local projective tests with two-way communication. In addition, the measurement spectral gap can reach the value of 1/2 even if we can perform only two distinct tests based on one-way communication [28]. The number of tests required by such a simplified strategy is comparable to the optimal strategy. Therefore, all bipartite unitary transformations can be verified efficiently with local state preparation and measurements as summarized in Table I. Note that RB is not applicable except for some special cases.

Next, consider the verification of a Clifford unitary. Here we only require Pauli operations: preparation of eigenstates of local Pauli operators and measurements of local Pauli operators. The ensemble of test states can be constructed from three mutually unbiased product bases, namely, the standard basis, eigenbasis of all $X$ operators for individual qubits, and eigenbasis of all $Y = iXZ$ operators. In this way, the preparation spectral gap can attain the maximum of 2/3. Under the action of the Clifford unitary, each test state is turned into a stabilizer state, which can be verified efficiently with Pauli measurements. More precisely, we can measure all nontrivial stabilizer operators with an equal probability, and the resulting measurement spectral gap is $2^{-n}/(2^n - 1) \geq 1/2 [17, 19]$, where $n$ is the number of qubits. These results apply to both individual gates and gate sets, including whole Clifford circuits. In addition, by virtue of Ref. [19], they can be generalized to systems with odd-prime local dimensions. If the local dimension is not a prime, the output stabilizer state can be verified by measuring $n$ stabilizer generators with an equal probability; the resulting measurement spectral gap is 1/n. The performance may be improved by virtue of the cover or coloring protocol proposed in Ref. [33]. Therefore, Clifford gates and circuits can be verified efficiently even if the local dimension is not a prime (see Table I), in which case the Clifford group is not a unitary 2-design and RB does not apply. In addition, the number of tests required by our protocol (for a given infidelity) is quadratically fewer than direct fidelity estimation [34].

Next, consider the verification of the generalized controlled-$Z$ gate $C^{(n-1)}Z$ with $n-1$ control qubits. Now test states can be constructed from two mutually unbiased product bases, namely, the computational basis and the eigenbasis of all $X$ operators for individual qubits. The resulting preparation spectral gap is 1/2. Under

| unitary                  | $\nu_p$ | $\nu_M$ | $N_E(\epsilon, \delta)$ |
|-------------------------|---------|---------|--------------------------|
| bipartite               | $d_{\text{min}} + 1$ | $\frac{3}{2}d_{\text{min}} + 1$ | $\frac{3}{2}d_{\text{min}} + 1$ |
| Clifford(2)             | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{3}{2}d_{\text{min}} + 1$ |
| Clifford($d_1$ odd prime) | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{3}{2}d_{\text{min}} + 1$ |
| Clifford($d_1$)         | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{3}{2}d_{\text{min}} + 1$ |
| $C^{(n-1)}Z$            | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{3}{2}d_{\text{min}} + 1$ |
| $C^{(n-1)}X$            | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{3}{2}d_{\text{min}} + 1$ |
| CSWAP                   | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{3}{2}d_{\text{min}} + 1$ |
the action of $C^{(n-1)}Z$, each state in the computational basis is invariant and so can be verified by performing the projective measurement on the computational basis. By contrast, each state in the other basis is turned into basis is invariant and so can be verified by performing the action of $\text{Ref. } [33]$; the resulting measurement spectral gap is $1/n$. The same method can also be applied to verifying circuits composed of generalized controlled-$Z$ gates, which is useful to studying IQP circuits and to demonstrating quantum supremacy $^{57, 38}$. With minor modification, this approach can be applied to verifying the generalized controlled-$X$ (CNOT) gate, including the Toffoli gate $^{12}$ given that $C^{(n-1)}X = H_n C^{(n-1)} Z H_n$, where $H_n$ is the Hadamard gate acting on the $n$th qubit. Generalization to the qudit setting will be presented elsewhere.

Finally, for the CSWAP gate, the test states can be constructed from three mutually unbiased product bases as in the verification of Clifford unitaries. Under the action of the CSWAP gate, each state in the computational basis still belongs to the computational basis and so can easily be verified. Each state in the other two bases is turned into a product state or a Greenberger-Horne-Zeilinger (GHZ) state; the former can be verified by a Pauli measurement, while the latter can be verified efficiently using the coloring protocol proposed in $^{57, 38}$; the resulting measurement spectral gap is $2/3$. So the CSWAP gate can be verified efficiently as well.

Summary. — We proposed a simple and general framework for verifying quantum circuits and that can be applied to both individual gates as well as gate sets and quantum circuits. Our approach enables efficient and scalable verification of all bipartite unitaries, Clifford unitaries, generalized controlled-$Z$ gates, generalized CNOT gates, and CSWAP gate using local state preparation and measurements. It is applicable in many scenarios in which RB does not apply and is thus instrumental to quantum computation and many other applications in quantum information processing. On the other hand, our approach is still not robust against SPAM and is not a replacement of RB. Instead combining the merits of our approach and RB may lead to a more powerful tool. We hope that our work will stimulate further progresses in this direction.

This work is supported by the National Natural Science Foundation of China (Grant No. 11875110).

Note added.—Upon completion of this work, we became aware of a related work by Liu et al. $^{41}$.

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