Deformations of Semisimple Bihamiltonian Structures of Hydrodynamic Type

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Abstract
We classify in this paper infinitesimal quasitrivial deformations of semisimple bihamiltonian structures of hydrodynamic type.

1 Introduction
A bihamiltonian structure of hydrodynamic type defined on the formal loop space of a manifold \( M \) consists of two compatible Poisson brackets of the form

\[
\{u^i(x), u^j(y)\} = g^{ij}(u(x)) \delta'(x-y) + \Gamma^i_{jk}(u(x))u^k_x \delta(x-y), \quad i, j = 1, \ldots, n. \tag{1.1}
\]

Here \( n = \dim M \), and we assume that \( \det(g^{ij}(u)) \neq 0 \). Such type of Poisson brackets were introduced and classified by Dubrovin and Novikov during the 80’s of the last century [8, 9, 10], they were used to describe the hamiltonian structures of systems of hydrodynamic type. According to the theory of Dubrovin and Novikov, the inverse of \((g^{ij})^{-1}\) must be a flat metric of the manifold \( M \), and the coefficients \( \Gamma^i_{jk} \) be given by the contravariant components of the Levi-Civita connection of this flat metric. Two such Poisson brackets corresponding to two flat metrics \((g^{ij}_1)^{-1}, (g^{ij}_2)^{-1}\) are compatible if these two metrics form a flat pencil [7]. The most well known examples of bihamiltonian structures of hydrodynamic type are possessed by the Whitham equations (in particular, the dispersionless limit) of integrable evolutionary PDEs of KdV type [8, 9, 10, 24].

In the present paper we study the problem of classification of deformations of a given bihamiltonian structure of hydrodynamic type, these deformations depend on a parameter \( \epsilon \) which is called the dispersion parameter. The deformed bihamiltonian structure has the form

\[
\{u^i(x), u^j(y)\}_\epsilon = g^{ij}_\epsilon(u(x)) \delta'(x-y) + \Gamma^i_{jk}(u(x))u^k_x \delta(x-y)
+ \sum_{m \geq 1} \sum_{l=0}^{m+1} \epsilon^m A^i_{m,l,a}(u;u_x,\ldots,u^{(m+1-l)}) \delta^{(l)}(x-y), \quad a = 1, 2. \tag{1.2}
\]

Here \( A^i_{m,l,a} \) are differential polynomials, i.e. they depend polynomially on the \( x \)-derivatives of \( u^i, \ldots, u^n \), and the coefficients of these polynomials are
smooth functions of $u^1, \ldots, u^n$. We also require that $A_{m,l}^{ij}$ are homogeneous polynomials in the sense that if we assign degree $m$ to $u^i = \partial^m u^i$, then $\text{deg} A_{m,l}^{ij} = m + 1 - l$. The class of bihamiltonian structures of the form (1.2) that satisfy some additional conditions is classified in [12]. These additional conditions include the so-called tau-symmetry property and the linearization of the Virasoro symmetries of the corresponding hierarchy of bihamiltonian evolutionary PDEs, they ensure the existence of tau functions for solutions of the hierarchy and the possibility of representing the Virasoro symmetries of the hierarchy by the action of an infinite number of linear differential operators on the tau functions. The moduli space of this class of bihamiltonian structures coincides with the space of semisimple Frobenius manifolds [12]. Here we will study the class of deformed bihamiltonian structures of the form (1.2) without the restriction of these additional properties.

The bihamiltonian structures of hydrodynamic type under our considerations are assumed to be semisimple, i.e., the eigenvalues of the matrix $(g^{ij})^{-1} g^{ij}_2$ are pairwise distinct, here $(g^{ij})^{-1}, (g^{ij}_2)^{-1}$ are the flat metrics corresponding to the given bihamiltonian structure. The simplest example of semisimple bihamiltonian structures of hydrodynamic type has the form

\begin{align*}
\{u(x), u(y)\}_1 &= \delta'(x - y), \\
\{u(x), u(y)\}_2 &= u(x)\delta'(x - y) + \frac{1}{2} u(x)'\delta(x - y),
\end{align*}

(1.3)

it is the dispersionless limit of the bihamiltonian structure of the KdV hierarchy [19, 25, 26]. In [23] Lorenzoni studied its deformations at the approximation up to $\epsilon^4$. He showed that the equivalence classes of all such deformations are parameterized by a smooth function $s(u)$, the bihamiltonian structure of the KdV hierarchy corresponds to the special deformation with a nonzero constant $s(u)$. Here the equivalence relation between deformations of a bihamiltonian structure of hydrodynamic type is defined in [12], two deformations of the form (1.2) are defined to be equivalent if they are related by a Miura-type transformation

$$u^i \mapsto u^i + \sum_{k \geq 1} \epsilon^k F_k^i (u; u_x, \ldots, u^{(k)}), \quad i = 1, \ldots, n$$

(1.4)

where $F_k^i$ are differential polynomials of degree $k$, note that they are not required to depend polynomially on $u^1, \ldots, u^n$. In particular, a deformation (1.4) is called to be trivial if it is equivalent to the undeformed bihamiltonian structure. For the above example, when the function $s(u)$ does not vanish, the corresponding deformation of the bihamiltonian structure (1.3) is nontrivial. Nevertheless, Lorenzoni proved that at the approximation up to $\epsilon^4$ all such deformations are quasitrivial. The notion of quasitriviality was also introduced in [12], a bihamiltonian structure of the form (1.2) is called quasitrivial if it can be obtained from its dispersionless limit by a transformation of the form

$$u^i \mapsto u^i + \sum_{k \geq 1} \epsilon^k G_k^i (u; u_x, \ldots, u^{(m_k)}), \quad i = 1, \ldots, n.$$

(1.5)

Here $G_k^i$ are smooth functions of their arguments, in particular, they are not necessary polynomials of the $x$-derivatives of $u^1, \ldots, u^n$. In [12] it was proved
that all semisimple bihamiltonian structures of the form (1.2) that satisfy the
tau-symmetry property are quasi-trivial. The method given in there can in fact be employed to prove the quasi-triviality of all deformations of (1.3). These results suggest that quasi-triviality could hold true for any deformation (1.2) of a semisimple bihamiltonian structure of hydrodynamic type.

In this paper we will restrict ourselves to study properties of quasi-trivial deformations and leave the discussion on the validity of quasi-triviality for any deformation of a semisimple bihamiltonian structure of hydrodynamic type to a subsequent publication. The main result of the paper is contained in the following two theorems:

**Theorem 1.1** Any two quasi-trivial deformations of a semisimple bihamiltonian structure of hydrodynamic type are equivalent if and only if they are equivalent at the approximation up to $\epsilon^2$.

The semisimplicity of a bihamiltonian structure of hydrodynamic type implies the existence of a coordinate system under which the corresponding two flat metrics are diagonal [15], we call such coordinates the canonical coordinates of the semisimple bihamiltonian structure.

**Theorem 1.2** At the approximation up to $\epsilon^2$, the space of the equivalence classes of all quasi-trivial deformations of a semisimple bihamiltonian structure of hydrodynamic type is parameterized by $n$ smooth functions $c_1(u^1), \ldots, c_n(u^n)$ of its canonical coordinates.

We will prove the above theorems by classifying the infinitesimal quasi-trivial deformations of a given semisimple bihamiltonian structure of hydrodynamic type, it amounts to the calculation of certain modification of the second bihamiltonian cohomology. As a direct consequence of the calculation that will be performed in section 4 we have

**Corollary 1.3** The equivalence classes of infinitesimal quasi-trivial deformations of a semisimple bihamiltonian structure of hydrodynamic type are parameterized by $n$ arbitrary functions of one variable.

The notion of bihamiltonian cohomology was introduced in [12], it provides an efficient tool to study deformations of bihamiltonian structures. We will first recall the notions of Poisson cohomology and bihamiltonian cohomology in section 2 and section 3 respectively, and then give the proof of the main results in section 4, some examples will be given in section 5.

## 2 Local Poisson structures and Poisson cohomologies

We recall in this section the definition of local Poisson structures and Poisson cohomologies that was presented in [12] on the formal loop space $\mathcal{L}(M) = \{S^1 \to M\}$ of a manifold $M$ of dimension $n$, we will closely follow the notations of [12]. Choose a chart $U$ on $M$ with local coordinates $u^1, \ldots, u^n$, we denote by
\( A = A(U) \) the ring of differential polynomials of the form

\[
f(x, u, u_x, \ldots) = \sum_{i_1, s_1; \ldots; i_m, s_m} f_{i_1, s_1; \ldots; i_m, s_m}(x; u) u^{i_1, s_1} \ldots u^{i_m, s_m}.
\]

Here \( u = (u^1, \ldots, u^n) \), \( u^{(s)} = (u^{1, s}, \ldots, u^{n, s}) \) with \( u^{i, s} = \frac{d^s u^i(x)}{dx^s} \), and the coefficients of these differential polynomials are smooth functions on \( S^1 \times M \).

Denote \( A_0 = A/\mathbb{R}, \ A_1 = A_0 dx, \ \Lambda = A_1 / dA_0 \)
where the operator \( d : A_0 \rightarrow A_1 \) is defined by

\[
f \mapsto df = \left( \frac{\partial f}{\partial x} + \sum \frac{\partial f}{\partial u^{i, s}} u^{i, s+1} \right) dx.
\]

Elements of \( \Lambda \) are called local functionals on \( L(M) \), they will be expressed as integrals over \( S^1 \) of a representative differential polynomial

\[
\int f(x; u(x), u_x(x), \ldots, u^{(N)}(x)) dx.
\]

Later in Section 4 we will also use functionals of the above form with densities \( f \) being smooth functions of their arguments instead of being differential polynomials.

A local \( k \)-vector on the formal loop space is defined to be a formal infinite sum of the following form

\[
\alpha = \sum_{k!} \partial_{x_1}^{p_1} \ldots \partial_{x_k}^{p_k} A^{i_1 \ldots i_k} \frac{\partial}{\partial u^{i_1, s_1}(x_1)} \wedge \ldots \wedge \frac{\partial}{\partial u^{i_k, s_k}(x_k)}
\]

with the coefficients \( A \)'s having the form

\[
A^{i_1 \ldots i_k} = \sum_{p_2, \ldots, p_k} B_{p_2 \ldots p_k}^{i_1 \ldots i_k} (u(x_1); u_x(x_1), \ldots) \delta^{(p_2)}(x_1 - x_2) \ldots \delta^{(p_k)}(x_1 - x_k).
\]

Here \( B_{p_2 \ldots p_k}^{i_1 \ldots i_k} (u(x_1); u_x(x_1), \ldots) \in A \), and

\[
A^{i_1 \ldots i_k} = A^{i_1 \ldots i_k}(x_1, \ldots, x_k; u(x_1), \ldots, u(x_k), \ldots)
\]

are antisymmetric with respect to the simultaneous permutations \( i_p, x_p \leftrightarrow i_q, x_q \). These coefficients \( A^{i_1 \ldots i_k} \) are called the components of the local \( k \)-vector \( \alpha \). The space of all such local \( k \)-vectors is denoted by \( A^k_{\text{loc}} \). In particular, a local vector field on the formal loop space has the form

\[
\xi = \sum_{i=1}^n \sum_{s \geq 0} \partial_x^s \xi^i(u(x); u_x(x), \ldots) \frac{\partial}{\partial u^{i, s}(x)}
\]

which is also called a translation (along \( x \)) invariant evolutionary vector field.

A local bivector takes the form

\[
\omega = \frac{1}{2} \sum \partial_x^s \partial_y^t \omega^{ij} \frac{\partial}{\partial u^{i, s}(x)} \wedge \frac{\partial}{\partial u^{j, t}(y)}
\]
with
\[ \omega^{ij} = A^{ij}(x - y; u(x), u_x(x), \ldots) = \sum_{k \geq 0} A_{k}^{ij}(u(x); u_x(x), \ldots) \delta^{(k)}(x - y). \quad (2.7) \]

It is assumed that the space \( \Lambda^0_{\text{loc}} \) is the subspace of \( \Lambda \) that consists of local functionals of the form
\[ \bar{f} = \int f(u(x); u_x(x), \ldots)dx, \quad f(u(x); u_x(x), \ldots) \in A_0. \quad (2.8) \]

On the space of local multi-vectors
\[ \Lambda^*_{\text{loc}} = \Lambda^0_{\text{loc}} \oplus \Lambda^1_{\text{loc}} \oplus \Lambda^2_{\text{loc}} \oplus \ldots \quad (2.9) \]
there is defined a bilinear operation of Schouten-Nijenhuis bracket
\[ [\cdot, \cdot] : \Lambda^k_{\text{loc}} \times \Lambda^l_{\text{loc}} \rightarrow \Lambda^{k+l-1}_{\text{loc}}, \quad k, l \geq 0 \quad (2.10) \]
By definition, the Schouten-Nijenhuis bracket of any two elements of \( \Lambda^0_{\text{loc}} \) is equal to zero, and the Schouten-Nijenhuis bracket of a local vector field \( \xi \) of the form (2.5) with a local functional \( \bar{f} \) of the form (2.8) is defined by
\[ [\xi, \bar{f}] = \int \sum \left( \partial_s^i \xi^i \right) \frac{\partial f(u(x); u_x(x), \ldots)}{\partial u^{i,s}} dx = \int \sum_{i=1}^n \xi^i \frac{\delta \bar{f}}{\delta u^i(x)} dx \quad (2.11) \]
where
\[ \frac{\delta \bar{f}}{\delta u^i(x)} = \sum_{s \geq 0} (-1)^s \partial_s^i \left( \frac{\partial f}{\partial u^{i,s}} \right). \quad (2.12) \]
The Schouten-Nijenhuis bracket of two local vector fields is given by their usual commutator
\[ [\xi, \eta] = \sum \left( \xi^{i,t} \frac{\partial \eta^{j,s}}{\partial u^{j,t}} - \eta^{i,t} \frac{\partial \xi^{j,s}}{\partial u^{j,t}} \right) \frac{\partial}{\partial u^{i,s}}, \quad (2.13) \]
and components of the Schouten-Nijenhuis bracket of a bivector \( \omega \) of the form (2.6) with a functional \( I \) and with a local vector field \( \xi \) of the form (2.5) are given respectively by
\[ [\omega, I]^i = \sum_{j,k} A_{k}^{ij} \partial_k A^{ij}(x), \quad (2.14) \]
\[ [\omega, \xi]^{ij} = \sum_{k,t} \left( \partial_x^k \xi^k(u(x); \ldots) \frac{\partial A^{ij}}{\partial u^{k,t}(x)} - \frac{\partial \xi^i(u(x); \ldots)}{\partial u^{k,t}(x)} \partial_x^k A^{ij} \right) \cdot \frac{\partial \xi^j(u(y); \ldots)}{\partial u^{k,t}(y)} \partial_x^k A^{jk} \right) \quad (2.15) \]
The Schouten-Nijenhuis bracket satisfies the following graded Jacobi identity and the antisymmetry property:
\[ (-1)^{km}[[a, b], c] + (-1)^{kl}[[b, c], a] + (-1)^{lm}[[c, a], b] = 0, \quad (2.16) \]
\[ [a, b] = (-1)^{kl}[b, a], \quad a \in \Lambda^k_{\text{loc}}, \quad b \in \Lambda^l_{\text{loc}}, \quad c \in \Lambda^m_{\text{loc}}. \quad (2.17) \]
Definition 2.1 (12) A local bivector $\omega \in \Lambda^2_{\text{loc}}$ of the form (2.6) is called a local Poisson structure on the formal loop space $\mathcal{L}(M)$ if $[\omega, \omega] = 0$.

A local Poisson structure given by a bivector of the form (2.6) can also be represented as an antisymmetric bilinear map from $\Lambda^2$ to $\Lambda$ as follows:

$$\{ f_1, f_2 \} = \int \sum_{k \geq 0} \frac{\delta f_1}{\delta u^j(x)} A^i_j(u(x); u_x(x), \ldots) \frac{\delta f_2}{\delta u^l(x)} dx. \quad (2.18)$$

For a particular choice of the local functionals $\tilde{f}_1 = \int u^i(z) \delta(z - x) dz$, $\tilde{f}_2 = \int u^j(z) \delta(z - y) dz$ we get the usual representation of a Poisson structure

$$\{ u^i(x), u^j(y) \} = \sum_{k \geq 0} A^i_k(u(x); u_x(x), \ldots) \delta^{(k)}(x - y). \quad (2.19)$$

There is a natural gradation on the space of local multi-vectors which is defined by

$$\deg u^{i,s} = s, \quad \deg \frac{\partial}{\partial u^{i,s}} = -s, \quad \deg dx = -1, \quad \deg \delta^{(s)}(x - y) = s + 1. \quad (2.20)$$

To separate monomials of different degree in a local multi-vector, we introduce a formal indeterminate $\epsilon$ and assign to it the degree $-1$. Denote

$$\Omega^k_m = \{ a \in \Lambda^k_{\text{loc}} \mid \deg a = m \},$$

$$\Omega^k = \{ a \in \Lambda^k_{\text{loc}} \otimes \mathbb{C}[\epsilon^{-1}] \mid \deg a = k \}. \quad (2.21)$$

For example, an element of $\Omega^0$ has the form

$$\bar{f} = \int \left( \epsilon^{-1} f_0(u(x)) + \sum_{k=1}^n f_{1,k}(u(x)) u^k_x + \ldots \right) dx. \quad (2.22)$$

The components of a vector field $\xi \in \Omega^1$ has the form

$$\xi^i = \epsilon^{-1} a^i(u) + \sum_{k=1}^n b^i_k(u) u^k_x + \epsilon \left( \sum_{k=1}^n c^i_k(u) u^k_{xx} + \sum_{k,l=1}^n e^i_{kl}(u) u^k_x u^l_x \right) + \ldots \quad (2.23)$$

A Poisson structure $\omega \in \Omega^2$ is of hydrodynamic type and has the representation of the form (12), any Poisson structure of the form $\omega + P(\epsilon) \in \Omega^2$ with $P(\epsilon) = \sum_{k \geq 1} \epsilon^k P_k$, $P_k \in \Omega^{k+2}_k$ is called a deformation of $\omega$.

The space

$$\Omega = \Omega^0 \oplus \Omega^1 \oplus \Omega^2 \oplus \ldots \quad (2.24)$$

is closed with respect to the Schouten-Nijenhuis bracket $\epsilon[\cdot, \cdot]$, so a Poisson structure $\omega \in \Omega^2$ defines a differential

$$\epsilon d : \Omega^k \to \Omega^{k+1}, \quad \epsilon da = \epsilon[\omega, a], \quad a \in \Omega^k. \quad (2.25)$$

The cohomology of the complex $(\Omega, \epsilon d)$ is called the Poisson cohomology of the Poisson structure $\omega$, and is denoted by $H^\epsilon(\mathcal{L}(M), \omega)$ [12]. It is a natural generalization of the notion of Poisson cohomology for finite dimensional Poisson structures [22].
3 Bihamiltonian structures and bihamiltonian cohomologies

Assume that we are given two Poisson structures \( \omega_1, \omega_2 \) of hydrodynamic type with components of the form

\[
\omega^{ij}_a = g^{ij}_{k,a} (u) \delta^k(x-y) + \Gamma^{ij}_{k,a} (u) u^k (x-y), \quad \det (g^{ij}_a) \neq 0, \quad a = 1, 2. \quad (3.1)
\]

If the linear combination \( \omega_\lambda = \omega_2 - \lambda \omega_1 \) is also a Poisson structure for an arbitrary parameter \( \lambda \in \mathbb{R} \), then the pair \( (\omega_1, \omega_2) \) is called a bihamiltonian structure of hydrodynamic type. These two Poisson structures define two complexes \((\Omega, \epsilon d_a)\), \(a = 1, 2\). It is proved in [6, 20] that the Poisson cohomologies \( H^*_a (\mathcal{L}(M), \omega_a) \), \(a = 1, 2\) are trivial (also see [12] for a different proof of triviality for the first and the second Poisson cohomologies). Thus any deformation \( \omega_a + P(\epsilon) \in \Omega^2 \) of a single Poisson structure \( \omega_a \) can be obtained from \( \omega_a \) by performing a Miura type transformation of the form (1.4). Instead of the deformations of a single Hamiltonian structure, we are interested in deformations of the bihamiltonian structure \( (\omega_1, \omega_2) \). Due to the triviality of the Poisson cohomology \( H^* (\mathcal{L}(M), \omega_1) \), we can always assume that our deformations keep the first Poisson structure \( \omega_1 \) unchanged.

**Definition 3.1** The pair of bivectors

\[
(\omega_1, \omega_2 + \sum_{m \geq 1} \epsilon^m P_m), \quad P_m \in \Omega^2_{m+2}. \quad (3.2)
\]

is called a deformation of the bihamiltonian structure \( (\omega_1, \omega_2) \) if the equality

\[
[\omega_2 + \sum_{m \geq 1} \epsilon^m P_m - \lambda \omega_1, \omega_2 + \sum_{m \geq 1} \epsilon^m P_m - \lambda \omega_1] = 0 \quad (3.3)
\]

holds true for an arbitrary parameter \( \lambda \). It is called an \( N \)-th order deformation of the bihamiltonian structure \( (\omega_1, \omega_2) \) if the equality (3.3) holds true for an arbitrary parameter \( \lambda \) at the approximation up to \( \epsilon^N \).

**Definition 3.2** We say that two deformations (of order \( N \)) of the bihamiltonian structure \( (\omega_1, \omega_2) \) are equivalent or quasi-equivalent if they are related (resp. at the approximation up to \( \epsilon^N \)) by a Miura type transformation (1.4) or by a quasi-Miura type transformation. A deformation (of order \( N \)) of the bihamiltonian structure \( (\omega_1, \omega_2) \) is called trivial or quasi-trivial if it is equivalent or quasi-equivalent to \( (\omega_1, \omega_2) \) (resp. at the approximation up to \( \epsilon^N \)).

Due to the above definition, for a \( N \)-th order deformation (3.2) the bivectors \( P_m \) must satisfy the conditions

\[
d_1 P_m = 0, \quad 1 \leq m \leq N, \quad (3.4)
\]

\[
d_2 P_1 = 0, \quad 2 d_2 P_m + \sum_{k=1}^{m-1} [P_k, P_{m-k}] = 0, \quad 2 \leq m \leq N. \quad (3.5)
\]

Here the differentials \( d_1, d_2 \) are defined by the Poisson structures \( \omega_1 \) and \( \omega_2 \) respectively as in (2.25), they act on the subspaces \( \Omega^k_m \) as

\[
d_a : \Omega^k_m \to \Omega^{k+1}_{m+2}, \quad k \geq 0, \quad m \geq k-1, \quad a = 1, 2. \quad (3.6)
\]
The notion of bihamiltonian cohomologies \( H^k = \oplus_{m \geq k-1} H^k_m \), \( k \geq 0 \) for \((\omega_1, \omega_2)\) is introduced in [12], they are defined by
\[
H^k_m(\mathcal{L}(M); \omega_1, \omega_2) = \text{Ker}(d_1 d_2|_{\Omega^{k-1}_m})/\text{Im}(d_1|_{\Omega^{k-2}_m}) + \text{Im}(d_2|_{\Omega^{k-2}_m}), \quad k \geq 2,
\]
\[
H^1_m(\mathcal{L}(M); \omega_1, \omega_2) = \text{Ker}(d_1 d_2|_{\Omega^0_m})
\]
\[
H^0_m(\mathcal{L}(M); \omega_1, \omega_2) = \text{Ker}(d_1|_{\Omega^0_m}) \cap \text{Ker}(d_2|_{\Omega^0_m})
\]
(3.7)

It was proved in [12] that the zero-th cohomology coincides with the space of common Casimirs of the Poisson structures \(\omega_1, \omega_2\), the first cohomology corresponds to the space of bihamiltonian vector fields, and the second cohomology corresponds to the space of infinitesimal deformations of the bihamiltonian structure modulo the trivial deformations caused by Miura transformations. Below we list some other simple propositions on the second and third cohomologies.

**Proposition 3.3** 1). The bihamiltonian cohomologies \( H^2_i(\mathcal{L}(M); \omega_1, \omega_2) \) vanish for \( K + 1 \leq i \leq N \) iff any class of deformations of the bihamiltonian structure \((\omega_1, \omega_2)\) of order \( s \leq N \) is uniquely determined by the corresponding class of deformations of order \( K + 2 \). The bihamiltonian cohomologies \( H^2_{2k+1}(\mathcal{L}(M); \omega_1, \omega_2) \) vanish for \( 1 \leq 2k + 1 \leq N \) iff any deformation of the bihamiltonian structure \((\omega_1, \omega_2)\) is equivalent to a deformation of the form (3.2) with \( P_{2l+1} = 0, 2l + 1 \leq N \).

**Proof** Let us first assume that \( H^2_i(\mathcal{L}(M); \omega_1, \omega_2) \) vanishes for \( K + 1 \leq i \leq N \). We need to prove that any two deformations of order \( s \leq N \) of the form
\[
(\omega_1, \omega_2 + \sum_{m=1}^{K} e^m P_m + \sum_{m=K+1}^{s} e^m P^{(1)}_m + \mathcal{O}(\epsilon^{s+1}), \quad l = 1, 2
\]
(3.8)
are equivalent. By using the identities in (3.4), (3.5) we can find \( X, Y \in \Omega^1_{K+1} \) such that
\[
P^{(1)}_{K+1} = d_1 X, \quad P^{(2)}_{K+1} = d_1 Y.
\]
From (3.5) it follows that
\[
d_2 d_1 (X - Y) = 0.
\]
So our assumption implies the existence of \( I, J \in \Omega^0_{K-1} \) such that
\[
X = Y + d_1 I + d_2 J.
\]
Thus after the Miura type transformation
\[
u^i \mapsto u^i - \epsilon^{K+1} d_1 J
\]
the first deformation
\[
(\omega_1, \omega_2 + \sum_{m=1}^{K} e^m P_m + \sum_{m=K+1}^{s} e^m P^{(1)}_m + \mathcal{O}(\epsilon^{s+1})
\]
is transformed to
\[
(\omega_1, \omega_2 + \sum_{m=1}^{K} e^m P_m + \epsilon^{K+1} P^{(2)}_{K+1} + \sum_{m=K+2}^{s} e^m P^{(1)}_m + \mathcal{O}(\epsilon^{s+1})
\]
(3.9)
By repeating the same procedure, we prove the equivalence of the two deformations of (3.8).

Now we assume that any class of deformations of the bihamiltonian structure \((\omega_1, \omega_2)\) of order \(s \leq N\) is uniquely determined by the corresponding class of deformations of order \(K\). For any

\[
X \in \text{Ker}(d_1 d_2|\Omega^s_1), \ K + 1 \leq s \leq N
\]

we have a \(s\)-th order deformation of the form

\[
(\omega_1, \omega_2 + \epsilon^s d_1 X).
\] (3.9)

It follows from our assumption that there exists a Miura type transformation

\[
u^i \mapsto u^i + \sum_{j=1}^s \epsilon^j A^i_j, \ A_j \in \Omega^1_j
\]

that transforms the bihamiltonian structure \((\omega_1, \omega_2)\) to \((\tilde{\omega}_1, \omega_2)\), i.e.,

\[
\begin{align*}
\omega_1 &= e^{-\epsilon^s d_1 \tilde{A}_s} \cdots e^{-\epsilon d_1 \tilde{A}_1} \omega_1 + O(\epsilon^{s+1}), \\
\omega_2 + \epsilon^s d_1 X &= e^{-\epsilon^s d_1 \tilde{A}_s} \cdots e^{-\epsilon d_1 \tilde{A}_1} \omega_2 + O(\epsilon^{s+1}).
\end{align*}
\] (3.10)

Here we represent, modulo \(\epsilon^{s+1}\), the Miura transformation as the composition of the one parameter transformation groups

\[
u^i \mapsto e^{\epsilon^k \tilde{A}^i_k} u, \ k = 1, \ldots, s
\]

and correspond to the vector fields

\[
\begin{align*}
\tilde{A}_1^i &= A_1^i, \quad \tilde{A}_2^i = A_2^i - \frac{1}{2} \sum_{j=1}^n \sum_{t \geq 0} \partial x^j A^j_1 & \cdots \frac{\partial x^t A^t_1}{\partial u^i} A^i_1.
\end{align*}
\]

From the identities in (3.10) we obtain

\[
d_1 \tilde{A}_s = 0, \quad d_2 \tilde{A}_s = d_1 X.
\]

The first equality yields the existence of \(I \in \Omega^0_{s-2}\) such that \(\tilde{A}_s = d_1 I\), and from the second equality it follows that \(X \in \text{Im}(d_1|\Omega^0_{s-2}) \oplus \text{Im}(d_2|\Omega^0_{s-2})\). Thus we proved the first part of the proposition. The second part can be proved in a similar way. The proposition is proved. \(\square\)

**Proposition 3.4** If the bihamiltonian cohomology \(H^3_{N+3}(\mathcal{L}(M); \omega_1, \omega_2)\) vanishes then any \(N\)-th order deformation of the bihamiltonian structure \((\omega_1, \omega_2)\) can be extended to a \(N + 1\)-th order deformation.

**Proof** Any \(N\)-th order deformation can be represented as

\[
(\omega_1, \omega_2 + \sum_{i=1}^N \epsilon^i d_1 X_i) + O(\epsilon^{N+1}), \quad X_i \in \Omega^1_i.
\]

In order to extend it to a deformation of order \(N + 1\) we need to find a local vector field \(X_{N+1} \in \Omega^1_{N+1}\) such that

\[
d_1 d_2 X_{N+1} = \frac{1}{2} \sum_{i=1}^N [d_1 X_i, d_1 X_{N+1-i}],
\] (3.11)
Denote by $Q$ the r.h.s. of the above equation. Then by using the graded Jacobi identity (2.16) of the Schouten-Nijenhuis bracket and the equalities

$$d_1d_2X_m = \frac{1}{2} \sum_{i=1}^{m-1} [d_1X_i, d_1X_{m-i}], \quad m = 1, \ldots, N$$

we obtain

$$d_1Q = d_2Q = 0.$$  

So there exists $R \in \Omega^{2N+3}_{N+3}$ such that

$$Q = d_1R.$$  

Now it follows from the equality $d_1d_2R = 0$ and our assumption of the proposition that

$$R = d_1A + d_2B, \quad A, B \in \Omega^{1}_{N+1}.$$  

So the equation now takes the form

$$d_1d_2X_{N+1} = d_1(d_1A + d_2B)$$

and it has a solution $X_{N+1} = B$. The proposition is proved. \(\square\)

Due to the above propositions, the problem of classification of deformations of the hydrodynamic bihamiltonian structures is reduced to the computation of bihamiltonian cohomology. We can also consider certain modification of the bihamiltonian cohomology in order to deal with quasitrivial deformations of the hydrodynamic bihamiltonian structures, we will do this in the next section.

### 4 Computation of a modified bihamiltonian cohomology and the proof of the main theorems

We consider in this section the problem of classification of infinitesimal quasitrivial deformations of a semisimple bihamiltonian structure $(\omega_1, \omega_2)$ with components of the form (3.1). Let us choose the coordinates $u^1, \ldots, u^n$, called the canonical coordinates of the semisimple bihamiltonian structure, such that both metrics $g_{ij}^1$ and $g_{ij}^2$ are diagonal under these coordinates, and the identities $g_{ii}^2 = u^i g_{ii}^1$. In terms of these coordinates the bihamiltonian structure can be expressed as

$$\omega_{ij}^1 = f^i \delta^{ij} \delta(x-y) + \frac{1}{2} f^i_x \delta^{ij} \delta(x-y) + A^{ij} \delta(x-y),$$

$$\omega_{ij}^2 = g^i \delta^{ij} \delta(x-y) + \frac{1}{2} g^i_x \delta^{ij} \delta(x-y) + B^{ij} \delta(x-y).$$

Here $f^i = f^i(u^1, \ldots, u^n)$, $g^i = u^i f^i$, $f^i_x = \partial_x f^i$, $g^i_x = \partial_x g^i$, and

$$A^{ij} = \frac{1}{2} \left( \frac{f^i}{f^j} f^j_x u^i_x - \frac{f^j}{f^i} f^i_x u^j_x \right), \quad B^{ij} = \frac{1}{2} \left( \frac{u^i f^j}{f^i} f^i_x u^j_x - \frac{u^j f^i}{f^j} f^j_x u^i_x \right)$$

where $f^a = \frac{\partial f}{\partial u}.$

Denote by $\tilde{\Omega}^0$ the space of local functionals of the form

$$\tilde{f} = \int f(u, u_x, \ldots, u^{(N)})dx$$
Then \( \hat{G}, H \) are the densities of the bihamiltonian structure \((\omega_1, \omega_2)\). We will use the symbol
\[
\omega \equiv \{ A \}
\]
to indicate that the difference of the functions \( \{ A \} \) is indivisible by \( u \).

Theorem 4.1 We have \( \hat{H}_m^2 = 0 \) for \( m = 1, 3, 4, \ldots \) and
\[
\hat{H}_2^2 = \sum_{i=1}^{n} \left( d_2 \int (c_i(u^i)u_x^i \log u_x^i) dx - d_1 \int (u^i c_i(u^i)u_x^i \log u_x^i) dx \right). \tag{4.5}
\]
Here \( d_1, d_2 \) are the differentials defined by the Poisson structures \( \omega_1 \) and \( \omega_2 \) respectively, \( c_i(u^i) \) are arbitrary smooth functions of \( u^i \). Moreover, two sets of functions \( \{ c_i \} \) and \( \{ \tilde{c}_i \} \) define the same element in \( \hat{H}_2^2 \) iff \( c_i = \tilde{c}_i \).

We will use the symbol
\[
A(u, u_x, \ldots, u^{(N)}) \sim B(u, u_x, \ldots, u^{(N)})
\]
to indicate that the difference of the functions \( A \) and \( B \) is a differential polynomial. In order to prove the above theorem we first need to prove some lemmas.

Lemma 4.2 Let \( X = d_2 I - d_1 J \in \hat{H}_2^2 \) with
\[
I = \int G(u, u_x, \ldots, u^{(N)}) dx, \quad J = \int H(u, u_x, \ldots, u^{(N)}) dx, \quad N \geq 2.
\]
Then the densities \( G, H \) can be chosen to have the form
\[
G \sim \sum_{i=1}^{n} \frac{(u_i^{(N)})^2}{u_x^i} P^i(u; u_x, \ldots, u^{(N)}; u_i^{i.N-1}) + Q(u, \ldots, u^{(N-1)}), \tag{4.6}
\]
\[
H \sim \sum_{i=1}^{n} \frac{(u_i^{(N)})^2}{u_x^i} u^i P^i(u; u_x, \ldots, u^{(N-2)}; u_i^{i.N-1}) + R(u, \ldots, u^{(N-1)}). \tag{4.7}
\]
Here \( P^i(u; u_x, \ldots, u^{(N-2)}; u_i^{i.N-1}) \) are differential polynomials, \( Q, R \) are smooth functions, and any nonzero differential polynomial \( P^i(u; u_x, \ldots, u^{(N-2)}; u_i^{i.N-1}) \) is indivisible by \( u_x^i \).

Proof Denote by \( X^i, i = 1, \ldots, n \) the components of the local vector field \( X \), from our assumption we know that they are differential polynomials. We are to use this property repeatedly to prove the lemma. Let us start with the polynomials of \( \frac{\partial X^i}{\partial u^{i,j}} \). Denote
\[
X_{j,m}^i = \frac{\partial X^i}{\partial u^{j,m}}, \quad G_{i,p;j,q} = \frac{\partial^2 G}{\partial u^{i,p} \partial u^{j,q}}, \quad H_{i,p;j,q} = \frac{\partial^2 H}{\partial u^{i,p} \partial u^{j,q}}.
\]
By using the simple identity
\[
\frac{\partial}{\partial u^{i,k}} \partial_x^m = \sum_{l=0}^{m} \binom{m}{l} \partial_x^l \frac{\partial}{\partial u^{i,k-m+l}}
\]

where \( f \) is a smooth function of all of its arguments. Define
\[
\hat{H}_2^2(\mathcal{L}(M); \omega_1, \omega_2) = \oplus_{m \geq 1} \hat{H}_m^2,
\]
\[
\hat{H}_m^2 = H_m^2(\mathcal{L}(M); \omega_1, \omega_2) \cap (d_1 \hat{\Omega}^0 \oplus d_2 \hat{\Omega}^0).
\tag{4.4}
\]
and the form \( [12] \), \( [23] \) of the bihamiltonian structure \((\omega_1, \omega_2)\) we obtain the following formulae:

\[
(-1)^N X^i_{j+2N+1} = g^i G_{i,N;j,N} - f^i H_{i,N;j,N}. \tag{4.8}
\]

It follows that the functions \( G \) and \( H \) satisfy the relations

\[
u^i G_{i,N;j,N} = H_{i,N;j,N} ~ 0, ~ (u^i - u^j) G_{i,N;j,N} ~ 0. \tag{4.9}
\]

So there exist smooth functions \( a_i, b_i, c \), such that

\[
G \sim \sum_{i=1}^{n} a_i (u, \ldots, u^{i-1}, u^i),
\]

\[
H \sim \sum_{i=1}^{n} (u^i a_i (u, \ldots, u^{i-1}, u^i) + b_i (u, \ldots, u^{i-1}) u^i) + c(u, \ldots, u^{i-1}).
\]

By substituting these expressions into the relations \((-1)^N \frac{\partial X^i}{\partial u^{i+2N}} \sim 0\) we obtain

\[
-(N + 1) \frac{\partial^2 a_i}{\partial u^i \partial u^{i+2N}} \sim 0.
\]

Thus we can find differential polynomials \( p_i(u, \ldots, u^{i-1}, u^i) \) and smooth functions \( q_i(u, \ldots, u^{i-1}), r_i(u, \ldots, u^{i-1}) \) such that

\[
a_i = \frac{p_i(u, \ldots, u^{i-1}, u^i)}{u^i} + q_i(u, \ldots, u^{i-1}) u^i + r_i(u, \ldots, u^{i-1}). \tag{4.10}
\]

Now the functions \( G, H \) can be written in the form

\[
G \sim \sum_{i=1}^{n} \left( \frac{p_i(u, \ldots, u^{i-1})}{u^i} + q_i(u, \ldots, u^{i-1}) u^i \right)
\]

\[
H \sim \sum_{i=1}^{n} \left( \frac{p_i(u, \ldots, u^{i-1})}{u^i} + s_i(u, \ldots, u^{i-1}) u^i \right) \tag{4.11}
\]

Here \( s_i, c \) are some smooth functions. \textit{In the above expression of} \( G, H \), \textit{we assume that the differential polynomials} \( p_i \) \textit{do not contain terms that are linear and constant with respect to} \( u^i \), \textit{such terms can be absorbed into the functions} \( q_i, s_i, c \) \textit{and} \( e \).

Assuming the form \( [12] \) \) \textit{and} \( [11] \) \textit{of the functions} \( G, H \) \textit{we continue to use the polynomiality of} \((-1)^N \frac{\partial X^i}{\partial u^{i+2N}}\) \textit{with} \( i \neq j \) \textit{to obtain}

\[
u^i \left( G_{i,N;j,N-1} - G_{j,N;i,N-1} \right) - \left( H_{i,N;j,N-1} - H_{j,N;i,N-1} \right) \sim 0.
\]

From these relations \textit{it follows that for indices} \( i \neq j \) \textit{we have}

\[
H_{i,N;j,N-1} - H_{j,N;i,N-1} \sim 0, ~ G_{i,N;j,N-1} - G_{j,N;i,N-1} \sim 0 \tag{4.12}
\]

\[
G_{i,N;i,N;j,N-1} - G_{i,N;i,N-1;j,N} \sim G_{i,N;i,N;j,N-1} \sim 0 \tag{4.13}
\]

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12
The relation (4.13) shows that we can adjust the differential polynomials \( p_i \) so that they have the form

\[
p_i = p_i(u, \ldots, u^{(N-2)}, u^{i,N-1}, u^{i,N}), \quad i = 1, \ldots, n.
\]

Now by substituting the expression (4.11) for the function \( H \) into the first relation of (4.12) we arrive at

\[
\frac{\partial s_i}{\partial u^{i,N-1}} - \frac{\partial s_j}{\partial u^{j,N-1}} \sim 0,
\]

by using the Poincaré lemma we can find differential polynomials \( \hat{s}_1, \ldots, \hat{s}_n \) such that the identity

\[
\frac{\partial}{\partial u^{i,N-1}} (s_i - \hat{s}_i) = 0
\]

hold true. This identity implies the existence of a function \( W(u, \ldots, u^{(N-1)}) \) satisfying

\[
s_i \sim \frac{\partial W}{\partial u^{i,N-1}}, \quad i = 1, \ldots, n.
\]

So by adjusting the density \( H \) of the functional \( J \) to \( H - \partial_x W \) we can assume that in the expression (4.11) for the function \( H \) the second term \( \sum_{i=1}^{n} s_i u^{i,N} \) does not appear. In a similar way, we can also assume that the term \( \sum_{i=1}^{n} q_i u^{i,N} \) in the expression (4.10) of the density of the functional \( I \) vanishes.

Finally, the relation \((-1)^N \frac{\partial^2 X}{\partial u^{i,N} \partial u^{j,N}} \sim 0\) implies that

\[
N^2 \frac{f^i u^{i,xx}}{2} \frac{\partial^3 p_i}{\partial u^{i,N} \partial u^{i,N} \partial u^{i,N}} \sim 0, \quad i = 1, \ldots, n. \tag{4.14}
\]

So we can adjust the densities \( G, H \) of the functionals \( I, J \) so that they have the forms (4.6), (4.7). The lemma is proved.

\( \square \)

Let us introduce the operators

\[
Z_{ij}^m = \sum_{p \geq m} (-1)^p \binom{p}{m} \frac{\partial^2}{\partial u^{i,N} \partial u^{j,N+m-p}}, \quad 1 \leq i, j \leq n, \quad m \geq 0.
\]

It is easy to verify that these operators satisfy the identities \([\partial_x, Z_{ij}^m] = Z_{ij}^{m-1}\) and, moreover, we have the following lemma:

**Lemma 4.3** For a functional \( I = \int G(u, u^{(1)}, \ldots) \) \( dx \), denote

\[
I_k = \frac{\delta I}{\delta u^k}, \quad k = 1, \ldots, n. \tag{4.15}
\]

Then for any indices \( i, j, m \), the following formulae hold true

\[
Z_{ij}^m I_k = \sum_{s \geq 0} \binom{s + m}{s} (-\partial_x)^s \frac{\partial}{\partial u^{k,s+m}} \left( \frac{\partial I_i}{\partial u^{i,2N}} \right)
\]

**Proof** It is well known from the theory of variational calculus that for any functional \( I \) we have the following identities:

\[
\frac{\partial}{\partial u^{i,p}} \left( \frac{\delta I}{\delta u^k} \right) = \sum_{t \geq p} (-1)^t \binom{t}{p} \frac{\partial^{t-p}}{\partial u^{k,t}} \left( \frac{\delta I}{\delta u^l} \right)
\]
From which it follows that
\[
\frac{\partial^2 I_k}{\partial u^{i,p} \partial u^{j,2N+m-p}} = \sum_{s \geq 0} \sum_{t \geq p} (-1)^{s+t} \binom{s+t}{p} \binom{s+t-p}{s} \frac{\partial^s}{\partial u^{k,s+t}} \frac{\partial^t}{\partial u^{j,2N+m-t}}.
\]

By using this identity we obtain
\[
Z_{ij}^m I_k = \sum_{p \geq 0} (-1)^p \binom{p}{m} \frac{\partial^2 I_k}{\partial u^{i,p} \partial u^{j,2N+m-p}}
= \sum_{p \geq 0} (-1)^p \binom{p}{m} \sum_{s \geq 0} \sum_{t \geq p} (-1)^{s+t} \binom{s+t}{p} \binom{s+t-p}{s} \binom{t}{p} \frac{\partial^s}{\partial u^{k,s+t}} \frac{\partial^t}{\partial u^{j,2N+m-t}}
= \sum_{s \geq 0} (-\partial_x)^s \sum_{t \geq 0} \binom{s+t}{s} \left[ \sum_{p=0}^t (-1)^p \binom{p}{m} \binom{t}{p} \right] \frac{\partial^s}{\partial u^{k,s}} \frac{\partial^t}{\partial u^{j,2N}}.
\]

Here we assumed \( \binom{p}{m} = 0 \) when \( p \leq m - 1 \) and we used the identity
\[
\sum_{p=0}^t (-1)^p \binom{p}{m} \binom{t}{p} = (-1)^t \delta_{tm}.
\]

The lemma is proved. \( \square \)

**Lemma 4.4** The polynomials \( P^i \) defined in Lemma 4.2 must vanish.

**Proof** Let \( m \) be the highest order of the \( x \)-derivatives of \( u^1, \ldots, u^n \) that appear in the polynomials \( P^i \). We first prove, by using the polynomiality of \( Z_{ij}^{m-1} X^k \), that \( m \) must less than 3. To this end, let’s assume at the moment that \( m \geq 3 \). From the form \( \text{[11]}. \text{[12]} \) of the bihamiltonian structure \( \omega_1, \omega_2 \) we know that the components of the vector field \( X = d_2 I - d_1 J \) can be expressed as
\[
X^k = g^k \partial_x \delta I + \frac{\partial_x g^k}{2} \delta I + \sum_{\alpha=1}^n B^{k\alpha} \delta I - f^k \partial_x \delta J - \frac{\partial_x f^k}{2} \delta J - \sum_{\alpha=1}^n A^{k\alpha} \delta J.
\]

Since the highest order of the \( x \)-derivatives of \( u^p \) that appear in \( \binom{d^t}{m} \) is 2\( N \), we have
\[
Z_{ij}^{m-1} X^k
= g^k (\partial_x Z_{ij}^{m-1} - Z_{ij}^{m-2}) I_k + \frac{\partial_x g^k}{2} Z_{ij}^{m-1} I_k + \sum_{\alpha=1}^n B^{k\alpha} Z_{ij}^{m-1} I_\alpha
- f^k (\partial_x Z_{ij}^{m-1} - Z_{ij}^{m-2}) J_k - \frac{\partial_x f^k}{2} Z_{ij}^{m-1} J_k - \sum_{\alpha=1}^n A^{k\alpha} Z_{ij}^{m-1} J_\alpha.
\]
Here $I_k, J_k$ are defined as in [4.7]. By using Lemma [4.2] and [4.3] we know that
\[
\frac{\partial I_i}{\partial u^j 2^N} \sim (-1)^N \frac{2 P^i}{u_x} \delta_{ij},
\]
and
\[
Z_{ij}^{m-1} I_k \sim \left( \frac{\partial}{\partial u^{k,m-1}} - m \frac{\partial x}{\partial u^{k,m}} \right) \frac{\partial I_i}{\partial u^j 2^N},
\]
\[
Z_{ij}^{m-2} I_k \sim \left( \frac{\partial}{\partial u^{k,m-2}} - (m-1) \frac{\partial x}{\partial u^{k,m-1}} + \frac{m(m-1)}{2} \frac{\partial^2 x}{\partial u^{k,m}} \right) \frac{\partial I_i}{\partial u^j 2^N}.
\]
We can get similar expression for $Z_{ij}^{m-1} J_k$ and $Z_{ij}^{m-2} J_k$. By using these formulae, we see that for the case $i = j \neq k$ the term with the highest power of $\frac{1}{u_x}$ in the expression of $Z_{ij}^{m-1} X^k$ is given by
\[
(-1)^N 2m(m+1) f^k (u^i - u^k) \frac{(u^{i,x})^2}{(u_x^i)^3} \frac{\partial P^i}{\partial u^{k,m}}.
\]
(4.16)
From the fact that $P^i$ is indivisible by $u_x^i$ and $Z_{ii}^{m-1} X^k$ is a differential polynomial it follows that $P^i$ does not depend on $u^{k,m}$ for $k \neq i$. In the case when $i = j = k$ we have
\[
Z_{ij}^{m-1} X^k \sim (-1)^{N+1} m^2 f^i u_x^{i,x} \frac{\partial P^i}{\partial u_x^{i,x}}.
\]
(4.17)
So $P^i$ does not depend on $u^{i,m}$ either. Thus we proved that the highest order $m$ of the $x$-derivatives of $u^1, \ldots, u^n$ that appear in the polynomial $P^i$ must less than 3. To complete the proof of the lemma we use the polynomiality of $Z_{ii}^{1} X^k$.

In the same way as we did above, we can prove that the terms (4.16) for the case of $m = 2$ is a differential polynomial, so $P^i$ does not depend on $u_x^{i,x}$ for $i \neq k$. Then the counterpart of (4.17) for the case of $m = 2$ has the form
\[
Z_{ii}^{1} X^i \sim \frac{(-1)^{N+1} f^i}{u_x^i} \left( 4 u_x^{i,x} \frac{\partial P^i}{\partial u_x^{i,x}} + (2N-2) P^i \right)
\]
(4.18)
which implies $P^i = 0$. The lemma is proved. \hfill \Box

Now we can prove the main result of this section.

**Proof of Theorem [4.7]** By using the above lemma, we know that for any element of $H^2$ we can choose its representative $X \in \text{Ker}(d_1 d_2)$ of the form
\[
X = d_2 I - d_1 J, \quad I = \int G(u, u_x) dx, \quad J = \int H(u, u_x) dx.
\]
(4.19)
Then the polynomiality of
\[
\frac{\partial X^i}{\partial u^{j,3}} = f^i \frac{\partial^2 H}{\partial u_x^i \partial u_x^j} - g^i \frac{\partial^2 G}{\partial u_x^i \partial u_x^j}
\]
(4.20)
allows us to adjust the vector field $X$ such that the functions $G$ and $H$ have the expression
\[
G = \sum_{i=1}^n h_i(u^1, \ldots, u^n, u_x^i), \quad H = \sum_{i=1}^n u^i h_i(u^1, \ldots, u^n, u_x^i).
\]
(4.21)
By using the identity
\[ \frac{\partial X^i}{\partial u^i_x} = \frac{3}{2} f^i u^i_x \frac{\partial^2 h_i}{\partial u^i_x \partial u^i_x} \] (4.22)
we see that the functions \( h_i \) must take the form
\[ h_i = c_i(u) u^i_x \log u^i_x + \text{differential polynomial.} \] (4.23)
Now from the explicit form of \( \frac{\partial X^i}{\partial u^i_x} \) we know that
\[ (u^i - u^j) \frac{\partial c_j}{\partial u^i} \log u^j_x \] (4.24)
are differential polynomials, thus we have \( \frac{\partial c_j}{\partial u^i} = 0 \) for \( i \neq j \), and \( c_i \) depend only on \( u^i \). So we proved that any element of \( \hat{H}^2 \) has a representative of the form given in the right hand side of (4.5).

On the other hand, given any vector field \( X \) with the form given in the right hand side of (4.5), we can easily verify that its components have the expressions
\[ X^i = \sum_{j=1}^{n} \left[ \left( \frac{1}{2} \delta_{ij} f^i + A^{ij} \right) c_j u^j_x + (2 \delta_{ij} f^i - L^{ij}) \frac{\partial}{\partial u^i} \left( c_j u^j_x \right) \right]. \] (4.25)
Here
\[ L^{ij} = \frac{1}{2} \delta_{ij} f^i + \frac{(u^i - u^j) f^i}{2 f^j} \frac{\partial f^j}{\partial u^i}. \] (4.26)
It shows that \( X^i \) are differential polynomials and thus \( X \) is a representative of an element of \( \hat{H}^2 \).

Finally, we are left to show that a vector field \( X \) of the form given in the right hand side of (4.5) is trivial if and only if \( c_1 = \cdots = c_n = 0 \). From the expression (4.25) it follows that the triviality of the vector field \( X \) is equivalent to the existence of functions \( \alpha_i(u), \beta_i(u), i = 1, \ldots, n \) such that the vector fields \( X \) can be expressed as \( \bar{X} = d_2 \tilde{I} - d_1 \tilde{J} \), where the functionals \( \tilde{I} \) and \( \tilde{J} \) have the form
\[ \tilde{I} = \int \sum_{i=1}^{n} \alpha_i(u) u^i_x \, dx, \quad \tilde{J} = \int \sum_{i=1}^{n} \beta_i(u) u^i_x \, dx. \] (4.27)
The coefficient of \( u^i_{xx} \) of the \( i \)-th component of \( X \) is given by \( 2 f^i c_i \), while that of \( \bar{X} \) equals zero. Thus we must have \( c_i = 0, i = 1, \ldots, n \). The theorem is proved.

\[ \square \]

**Proof of Theorem 1.1 and Theorem 1.2** Let us assume that the hydrodynamic bihamiltonian structure \( (\omega_1, \omega_2) \) has two \( N \)-th order quasitrivial deformations of the form
\[ (\omega_1, \omega_2 + \sum_{m=1}^{N} e^m P_m) + \mathcal{O}(e^{N+1}), \] (4.28)
\[ (\omega_1, \omega_2 + \sum_{m=1}^{N} e^m P_m + e^N Q) + \mathcal{O}(e^{N+1}). \] (4.29)
Here \( P_m \in \Omega_{m+2}^2 \), \( Q \in \Omega_{N+2}^2 \). Due to our assumption, we can find a quasi-Miura transformation of the form (4.25) that transforms the bihamiltonian structure (4.28) to \((\omega_1, \omega_2) + \mathcal{O}(\epsilon^{N+1})\). Then this same quasi-Miura transformation transforms the bihamiltonian structure (4.29) to \((\omega_1, \omega_2 + \epsilon N Q) + \mathcal{O}(\epsilon^{N+1})\).  

It is also a quasi-trivial deformation of the bihamiltonian structure \((\omega_1, \omega_2)\), so we are able to find a quasi-Miura transformation that transforms \((\omega_1, \omega_2)\) to (4.30). Such a quasi-Miura transformation can be represented by some vector fields \(Y_1, \ldots, Y_N\) in the form

\[
\omega_1 = e^{-\epsilon N \text{ad}_Y} \omega_1 + \mathcal{O}(\epsilon^{N+1}), \\
\omega_2 + \epsilon N Q = e^{-\epsilon N \text{ad}_Y} \omega_2 + \mathcal{O}(\epsilon^{N+1}).
\]

From the above identities it follows that \(d_1 Y_N = 0, Q + d_2 Y_N = 0\), so there exists a functional \(I\) such that \(Y_N = d_1 I, Q = d_1 d_2 I\). On the other hand, the compatibility of \((\omega_1, \omega_2 + \epsilon N Q) + \mathcal{O}(\epsilon^{N+1})\) implies the existence of a vector field \(X \in \Omega_N^1\) satisfying \(Q = d_1 X\). From the above two expressions of \(Q\) we see that we can express the vector field \(X\) as

\[X = d_2 I - d_1 J\]

with certain functional \(J \in \tilde{H}_0^0\).

Now the results of Theorem 4.1 lead to the following conclusions:

1. If \(N \neq 2\), then \(I \) and \(J \) must be differential polynomials, so the two deformations (4.28) and (4.29) are related by a Miura transformation

\[u \mapsto u + \epsilon N d_1 d_2 I.\]  

Theorem 1.1 is proved.

2. Any second order deformation \((\omega_1, \omega_2 + \epsilon P_1 + \epsilon^2 P_2) + \mathcal{O}(\epsilon^3)\) is equivalent to a second order deformation of the form \((\omega_1, \omega_2 + \epsilon^2 \tilde{P}_2) + \mathcal{O}(\epsilon^3)\). By applying the results of Theorem 4.1 to the case with \(N = 2\), we see that modulo a Miura transformation the deformed bihamiltonian structure can be represented in the form

\[(\omega_1, \omega_2 + \epsilon^2 d_1 (d_2 I - d_1 J)) + \mathcal{O}(\epsilon^3)\]

for some functionals \(I, J\) defined by

\[I = \int \sum_{i=1}^n c_i(u^i) u_x^i \log u_x^i \, dx, \quad J = \int u^i c_i(u^i) u_x^i \log u_x^i \, dx.\]

On the other hand, it is easy to see that any functionals \(I, J\) of the above form define a second order quasi-trivial deformation of the bihamiltonian \((\omega_1, \omega_2)\).  

Theorem 1.2 is proved.

From the proof of the main theorems it follows that the any equivalence class of quasi-trivial deformations of the bihamiltonian structure \((\omega_1, \omega_2)\) has a unique representative of the form (4.33), (4.34) which corresponds to an element of the modified cohomology \(\tilde{H}^2\).
5 Some examples

In this section, we consider as examples the deformations of the bihamiltonian structures of hydrodynamic type that are related to the KdV and the nonlinear Schrödinger equations, these deformations yield the bihamiltonian structures for the Camassa-Holm hierarchy [2, 3, 10, 17, 18] and its generalization.

Let us first consider deformations of the bihamiltonian structure (1.3). The class of deformations that corresponds to the element of $\hat{H}^2$ (see Theorem 4.1) with $c(u) = -\frac{1}{24}$ has a representative

$$\{u(x), u(y)\}_1 = \delta'(x - y),$$
$$\{u(x), u(y)\}_2 = u(x)\delta'(x - y) + \frac{1}{2}u(x)'\delta(x - y) + \frac{\epsilon^2}{8}\delta''(x - y). \quad (5.1)$$

Here we reddenote $u^1 = u, c_1(u) = c(u)$. It is just the well known bihamiltonian structure for the KdV hierarchy [19, 25, 26]. Now if we take $c(u) = -\frac{1}{24}u$, then the corresponding class of deformations has the following representative

$$\{u(x), u(y)\}_1 = \delta'(x - y) - \frac{\epsilon^2}{8}\delta''(x - y),$$
$$\{u(x), u(y)\}_2 = u(x)\delta'(x - y) + \frac{1}{2}u(x)'\delta(x - y). \quad (5.2)$$

In fact, it is equivalent to the bihamiltonian structure

$$\omega_1, \omega_2 + \epsilon^2 d_1(d_2 I - d_1 J) + O(\epsilon^3) \quad (5.3)$$

under the Miura transformation

$$u \mapsto u + \frac{\epsilon^2}{16}u''.$$  

Here $(\omega_1, \omega_2)$ denotes the bihamiltonian structure [13] and the functionals $I$ and $J$ are defined by

$$I = -\frac{1}{24}\int u(x)u'(x) \log u'(x)dx, \quad J = -\frac{1}{24}\int u(x)^2u'(x) \log u'(x)dx.$$

The related bihamiltonian hierarchy of integrable systems is the Camassa-Holm hierarchy that is well known in soliton theory. It can be expressed by the following bihamiltonian recursion relations:

$$\frac{\partial u}{\partial \tau^q} = \{u(x), H_q\}_1 = \frac{2}{2q + 1}\{u(x), H_{q-1}\}_2, \quad q \geq 0. \quad (5.4)$$

Here we start from the Casimir $H_{-1} = \int u(x)dx$ of the first Poisson bracket, and then determine the Hamiltonians $H_q, q \geq 0$ recursively from the above relation. The recursive procedure of finding the Hamiltonians $H_q$ is guaranteed by the triviality of the first Poisson cohomology of the Poisson structure $\omega_1$ [6, 12, 20]. The first nontrivial flow $\frac{\partial}{\partial \tau} = \frac{\partial}{\partial \tau^1}$ of the hierarchy can be put into the form

$$v = \frac{\epsilon^2}{8}v_{xx}t = vv_x - \frac{\epsilon^2}{12}v_xv_{xx} - \frac{\epsilon^2}{24}vv_{xxx}. \quad (5.5)$$
Here the dependent variable $v$ is defined by

$$u = v - \frac{\epsilon^2}{8}v_{xx}. \quad (5.6)$$

If we change the time variable as $t \mapsto t = -\frac{1}{4}t^1$ and put $\epsilon^2 = 8$, then the resulting equation is just the Camassa-Holm shallow water wave equation \[16 17 18\], which possesses most of the important properties of an integrable system. In particular, it has the following Lax pair representation

$$\epsilon^2 \phi_{xx} = \left(2 - \frac{8v - \epsilon^2 v_{xx}}{2\lambda}\right) \phi, \quad (5.7)$$

$$\phi_t = \frac{1}{3}(\lambda + v)\phi_x - \frac{v_x}{6} \phi \quad (5.8)$$

and its initial value problems can be solved by using the inverse scattering method. The Camassa-Holm equation also possesses some features that are distinguished from the usual KdV-type integrable systems, such as the existence of peaked solitons, the nonlinear dependence of the arguments of its algebraic-geometric solutions on the spatial variable $x$ \[1\] and the non-existence of tau function \[12\]. We will call the equation (5.5) and the hierarchy (5.4) the Camassa-Holm equation and the Camassa-Holm hierarchy respectively.

The quasitriviality of the bihamiltonian structure \[5.1\], \[5.2\] can be deduced from a result of \[12\] on the quasitriviality of a general class of bihamiltonian structures. Details on this aspect will be given in a subsequent publication.

For the choice of a general smooth function $c(u)$, we do not have at this moment an explicit expression of the correspondent class of deformations of the bihamiltonian structure \[123\]. At the approximation up to $\epsilon^4$ Lorenzoni obtained the expression of a representative of the corresponding class of deformations, and we can in fact go further to show that his result can be modified to reach the approximation up to higher orders of $\epsilon$. This fact strongly indicates the existence of a full deformation of the bihamiltonian structure \[123\] for any smooth function $c(u)$, or equivalently, to the vanishing of the third bihamiltonian cohomologies $H^3_m(C; \omega_1, \omega_2)$, $m \geq 5$ of the bihamiltonian structure \[123\].

We now consider the deformations of the following bihamiltonian structure

$$\{w_1(x), w_1(y)\}_1 = \{w_2(x), w_2(y)\}_1 = 0,$$

$$\{w_1(x), w_2(y)\}_1 = \delta(x - y),$$

$$\{w_1(x), w_1(y)\}_2 = 2\delta(x - y),$$

$$\{w_1(x), w_2(y)\}_2 = w_1(x)\delta'(x - y) + w_1'(x)\delta(x - y),$$

$$\{w_2(x), w_2(y)\}_2 = [w_2(x)\partial_x + \partial_x w_2(x)] \delta(x - y). \quad (5.9)$$

It is related to the Frobenius manifold with potential \[5\]

$$F = \frac{1}{2}w_1^2 w_2 + \frac{1}{2}w_2^2 \left(\log \frac{w_2 - \frac{3}{2}}{2}\right).$$

The canonical coordinates of this bihamiltonian structure are given by

$$u^{1,2} = w_1 \pm 2\sqrt{w_2}. \quad (5.11)$$
Let us consider the following two classes of deformations:

**Case 1.** We take the element of $\hat{H}^2$ with $c_1(u) = c_2(u) = -\frac{1}{24}$, then the corresponding class of deformations has a representative

$$
\{w_1(x), w_1(y)\}_1 = \{w_2(x), w_2(y)\}_1 = 0,
$$

$$
\{w_1(x), w_2(y)\}_1 = \delta'(x - y).
$$

(5.12)

$$
\{w_1(x), w_1(y)\}_2 = 2\delta'(x - y),
$$

$$
\{w_1(x), w_2(y)\}_2 = w_1(x)\delta'(x - y) + w_1'(x)\delta(x - y) - \epsilon\delta''(X - Y),
$$

$$
\{w_2(x), w_2(y)\}_2 = [w_2(x)\partial_x + \partial_x w_2(x)]\delta(x - y).
$$

(5.13)

To see this, let us denote by $\omega_1, \omega_2$ the two bivectors of the bihamiltonian structure (5.12), (5.13), and by $I, J$ the functionals

$$
I = -\int \frac{1}{24} \left( \epsilon w_1^1 \log u_1^1 + u_2^2 \log u_2^2 \right) dx,
$$

$$
J = -\int \frac{1}{24} \left( \epsilon u_1^1 \log u_1^1 + u_2^2 \log u_2^2 \right) dx,
$$

(5.14)

then by a direct computation it can be verified that the bihamiltonian structure (5.12), (5.13) is equivalent to the bihamiltonian structure

$$
(\omega_1, \omega_2 + \epsilon^2 d_1(d_2 I - d_1 J)) + O(\epsilon^3)
$$

(5.15)

under the Miura transformation

$$
w_1 \mapsto w_1 + \epsilon \frac{w_2}{2\sqrt{3}} \frac{w_{2,x}}{w_2} + \epsilon^2 \left( \frac{1}{12} - \frac{1}{4\sqrt{3}} \right) \left( \frac{w_{1,xx}}{w_2} - \frac{w_{1,x}w_{2,x}}{w_2^2} \right),
$$

$$
w_2 \mapsto w_2 + \epsilon \left( \frac{1}{2} + \frac{1}{2\sqrt{3}} \right) w_{1,x}.
$$

(5.16)

The bihamiltonian hierarchy of integrable systems that is related to this bihamiltonian structure is called the extended NLS hierarchy, the algebraic properties of this hierarchy together with its relation to the $CP^1$ topological sigma model is studied in detail in [3, 13]. It is also shown in [5] that this hierarchy is equivalent to the extended Toda hierarchy [21, 27] which contains the standard Toda lattice hierarchy.

**Case 2.** Let us take the element of $\hat{H}^2$ with $c_1(u) = -\frac{(u_1^1)^2}{24}, c_2(u) = -\frac{(u_2^2)^2}{24}$, then the correspondent class of deformations has a representative of the form

$$
\{w_1(x), w_1(y)\}_1 = \{w_2(x), w_2(y)\}_1 = 0,
$$

$$
\{w_1(x), w_2(y)\}_1 = \delta'(x - y) - \epsilon\delta''(x - y).
$$

(5.17)

$$
\{w_1(x), w_1(y)\}_2 = 2\delta'(x - y),
$$

$$
\{w_1(x), w_2(y)\}_2 = w_1(x)\delta'(x - y) + w_1'(x)\delta(x - y),
$$

$$
\{w_2(x), w_2(y)\}_2 = [w_2(x)\partial_x + \partial_x w_2(x)]\delta(x - y).
$$

(5.18)

Denote by $I, J$ the functionals

$$
I = -\int \frac{1}{24} \left( (u_1^1)^2 u_1^1 \log u_1^1 + (u_2^2)^2 u_2^2 \log u_2^2 \right) dx,
$$

$$
J = -\int \frac{1}{24} \left( (u_1^1)^3 u_1^1 \log u_1^1 + (u_2^2)^3 u_2^2 \log u_2^2 \right) dx,
$$

(5.19)
then it can be verified that the bihamiltonian structure \((5.17), (5.18)\) is equivalent to the bihamiltonian structure
\[
(\omega_1, \omega_2 + \epsilon^2 d_1(d_2 I - d_1 J)) + O(\epsilon^3)
\]
modulo a Miura transformation of the form
\[
w_1 \mapsto w_1 + \epsilon^2 \left( \frac{w_1^2 + 4w_2}{24w_2} w_{1,x} \right) + O(\epsilon^3)
\]
\[
w_2 \mapsto w_2 + \epsilon \left( \frac{w_1^2}{4} - w_2 \right) - \epsilon^2 \left( \frac{w_1^2 + 4w_2}{24w_2} - 1 \right) w_{2,x} + O(\epsilon^3)
\]

A hierarchy of integrable systems can be obtained by using the bihamiltonian recursion relation
\[
\{w_i(x), H_{q-1}\}_2 = (q + 1)\{w_i(x), H_q\}_1, \quad q \geq 0. \tag{5.20}
\]
Here we start from the Casimir
\[
H_{-1} = \int w_2(x)dx \quad \text{of the first Poisson bracket},
\]
and then determine the Hamiltonians \(H_q, q \geq 0\) recursively by using the above relation. The flows of the bihamiltonian hierarchy is then given by
\[
\frac{\partial w_i}{\partial H^q} = \{w_i(x), H_q\}_1, \quad q \geq 0. \tag{5.21}
\]
The first flow \(\frac{\partial}{\partial H_0}\) corresponds to the translation along the spatial variable \(x\), and the second flow \(\frac{\partial}{\partial H_1} = \frac{\partial}{\partial \epsilon}\) has the form
\[
(\varphi_1 - \epsilon \varphi_{1,x})_t = (\varphi_2 + \frac{1}{2} \varphi_1^2 - \frac{\epsilon}{2} \varphi_1 \varphi_{1,x})_x, \tag{5.22}
\]
\[
(\varphi_2 + \epsilon \varphi_{2,x})_t = (\varphi_1 \varphi_2 + \frac{\epsilon}{2} \varphi_1 \varphi_{2,x})_x. \tag{5.23}
\]
Here \(\varphi_1, \varphi_2\) are defined by
\[
w_1 = \varphi_1 - \epsilon \varphi_{1,x}, \quad w_2 = \varphi_2 + \epsilon \varphi_{2,x}.
\]
By introducing the new variables
\[
v_1 = \varphi_1, \quad v_2 = \varphi_2 + \epsilon \varphi_{2,x} - \frac{1}{4} (\varphi_1 - \epsilon \varphi_{1,x})^2
\]
we can rewrite the above system of equations in the following form
\[
(v_1 - \epsilon^2 v_{1,xx})_t = \left( v_2 + \frac{3}{4} v_1^2 - \epsilon^2 \left( \frac{1}{2} v_1 v_{1,xx} + \frac{1}{4} v_1^3 \right) \right)_x, \tag{5.24}
\]
\[
v_{2,t} = \frac{1}{2} v_1 v_{2,x} + v_2 v_{1,x}. \tag{5.25}
\]
It easily follows from the above expression that the system of equations \((5.24), (5.25)\) is reduced to the Camassa-Holm equation \((5.5)\) under the constraint
\[
v_2 = 0 \tag{5.26}
\]
together with the rescaling \(t \mapsto \frac{3}{2} t, \quad \epsilon^2 \mapsto \frac{1}{3} \epsilon^2\). So we can view the hierarchy \((5.21)\) as a natural 2-component generalization of the Camassa-Holm hierarchy.
The following Lax pair formalism of the system (5.24), (5.25) manifests the above observation:

\[ \varepsilon^2 \phi_{xx} = \left( \frac{1}{4} - \frac{v_1 - \varepsilon^2 v_1,xx}{2\lambda} \right) \phi, \tag{5.27} \]

\[ \phi_t = \frac{1}{2} (\lambda + v_1) \phi_x - \frac{v_1,x}{4} \phi. \tag{5.28} \]

When we put \( v_2 = 0 \) this Lax pair is reduced to the one that is given in (5.7), (5.8).

The quasitriviality of the bihamiltonian structure (5.12), (5.13) can be verified by using the method given in [12]. However, at this moment we do not have a proof for the quasitriviality of the bihamiltonian structure (5.17), (5.18). In order to use the approach of [12] to prove its quasitriviality we need to construct a bihamiltonian hierarchy of the form (5.21) that corresponds to the Casimir \( \int w_1(x) dx \) of the first Poisson bracket, since this functional is also a Casimir of the second Poisson bracket, the usual bihamiltonian recursion procedure fails to yield the needed Hamiltonians in a direct way. We will consider in detail the properties of the above 2-component Camassa-Holm hierarchy and its further generalizations in a separate publication.

6 Concluding remarks

For any semisimple bihamiltonian structure of hydrodynamic type, we classify its infinitesimal quasitrivial deformations. We show that the equivalence classes of its second order quasitrivial deformations are parameterized by \( n \) arbitrary functions of one variable, and we prove that any class of its quasitrivial deformations is uniquely determined by its corresponding class of second order deformations. We end this paper with the following two remarks:

Remark 1. At a first glance the condition of quasitriviality seems to be highly non-trivial, however, a careful study shows that any deformation of the semisimple bihamiltonian structure of the form (4.1), (4.2) is quasitrivial at least for the case of \( n = 1 \), this fact together with the quasitriviality of any tau-symmetric bihamiltonian structure [12] indicates the validity of quasitriviality for any deformation of the semisimple bihamiltonian structure of the form (4.1), (4.2). An even more optimistic conjecture is the existence of a full deformation of a semisimple bihamiltonian structure of hydrodynamic type with a given second order deformation. In the language of bihamiltonian cohomology we can formulate the above conjectures as follows:

Conjecture 6.1 For any semisimple bihamiltonian structure of hydrodynamic type \( (\mathcal{L}(M); \omega_1, \omega_2) \) we have \( H^2(\mathcal{L}(M); \omega_1, \omega_2) = \tilde{H}^2(\mathcal{L}(M); \omega_1, \omega_2) \), and the third bihamiltonian cohomologies \( H^3_m(\mathcal{L}(M); \omega_1, \omega_2) \) for \( m \geq 5 \) are trivial.

Remark 2. On the formal loop space of any semisimple Frobenius manifold there is defined a semisimple bihamiltonian structure of hydrodynamic type [7], a class of deformations of such bihamiltonian structure was constructed in [12], these deformations correspond to the element of the second cohomology \( H^2 \) with \( c_1 = \cdots = c_n = -\frac{1}{24} \), they are compatible with the universal identities satisfied
by the Gromov-Witten invariants of smooth projective varieties, for this reason we call them the topological deformations. The corresponding bihamiltonian hierarchy of integrable systems satisfies, in the sense of [12], the properties of tau-symmetry and linearization of the Virasoro symmetries. If we drop the requirement of linearization of the Virasoro symmetries, then the resulting tau symmetric bihamiltonian structure must correspond to an element of the second cohomology $H^2$ with constant $c_1(u) = c_1, \ldots, c_n(u) = c_n$. An example of such bihamiltonian structures is given by the one that is obtained by using the Drinfeld-Sokolov construction for the affine Lie algebra of type $B_2$ [4, 11, 14]. in this case the corresponding element of the second cohomology $H^2$ is determined by the constant functions $c_1 = -\frac{1}{6}, c_2 = -\frac{1}{12}$.

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