Replica Approach in Random Matrix Theory

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Abstract. This Chapter outlines the replica approach in Random Matrix Theory. Both fermionic and bosonic versions of the replica limit are introduced and its trickery is discussed. A brief overview of early heuristic treatments of zero-dimensional replica field theories is given to advocate an exact approach to replicas. The latter is presented in two elaborations: by viewing the \( \beta = 2 \) replica partition function as the Toda Lattice and by embedding the replica partition function into a more general theory of \( \tau \) functions.

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1. Introduction

1.1. Resolvent as a field integral

In physics of disorder, all observables depend in highly nonlinear fashion on a stochastic Hamiltonian hereby making a nonperturbative calculation of their ensemble averages very difficult. To determine the average quantities in an interactionless system, one has to know the spectral statistical properties of a single particle Hamiltonian \( H \) contained in the mean product of resolvents

\[
G(z) = \text{tr} \left( z - H \right)^{-1}
\]

defined for a generic, complex valued argument \( z \in \mathbb{C} \setminus \mathbb{R} \). Each of the resolvents can exactly be represented as a ratio of two functional integrals running over an auxiliary vector field \( \psi \) which may consist of either commuting (bosonic, \( \psi = s \)) or anticommuting (fermionic, \( \psi = \chi \)) entries.

In the random matrix theory (RMT) limit, when a system Hamiltonian (or a scattering matrix) is modelled by an \( N \times N \) random matrix \( H \) of prescribed symmetry, the resolvent \( G(z) \) equals

\[
G(z) = \eta \, s_z \int \mathcal{D}[\tilde{\psi}, \psi] \, \tilde{\psi}_j \psi_k e^{i s_z S_H(z; \tilde{\psi}, \psi)} \left( \int \mathcal{D}[\tilde{\psi}, \psi] \, e^{i s_z S_H(z; \tilde{\psi}, \psi)} \right)^{-1}.
\]

Here, \( S_H = \tilde{\psi} (z - H) \psi = \tilde{\psi}_j (z \delta_{jk} - H_{jk}) \psi_k \) (summation over repeated Latin indices is assumed throughout this Section); vector \( \psi \) is defined by \( \psi = (\psi_1, \cdots, \psi_N)^T \), whilst \( \tilde{\psi} = (\tilde{\psi}_1, \cdots, \tilde{\psi}_N) \) is its proper conjugate. The parameter \( \eta \) accounts for the nature of both vectors: it is set to +1 for fermions (\( \psi = \chi \)) and to -1 for bosons (\( \psi = s \)). In the latter case, convergence of field integrals is ensured by the regularizer \( s_z = \text{sgn}(\text{Im} \, z) \). The notation \( \mathcal{D}[\tilde{\psi}, \psi] \) stands for the integration measure \( \int \mathcal{D}[\tilde{\psi}, \psi] = (2\pi)^{-N} \prod_{j=1}^{N} d\bar{\psi}_j d\psi_j \).

Equation (1.1) may conveniently be viewed as a consequence of the identity

\[
G(z) = \eta \frac{\partial}{\partial z} \log \det \eta (z - H),
\]

combined with the field integral representations of the determinant (\( \eta = +1, \, \psi = \chi \)) and/or its inverse (\( \eta = -1, \, \psi = s \)):

\[
\det \eta (z - H) = (i \eta \, s_z)^N \int \mathcal{D}[\tilde{\psi}, \psi] \, e^{i s_z S_H(z; \tilde{\psi}, \psi)}. \tag{1.3}
\]

Although exact, both bosonic and fermionic versions of Eq. (1.1) are a bit too inconvenient for a nonperturbative ensemble averaging due to the awkward random denominator.

To get rid of it, several field theoretic frameworks have been devised by theoretical physicists: the replica trick \([\text{Weg79, Sch80, Efe80}]\), the supersymmetry method \([\text{Efe82a, Efe82b}]\), and the dynamic (Keldysh) approach \([\text{Hor90, Kam99a}]\). Leaving aside

\[\int d\chi \, \chi = \int d\bar{\chi} \, \bar{\chi} = (2\pi)^{1/2}.\]
the supersymmetry and the Keldysh techniques, this Chapter aims to provide an elementary introduction to notoriously known replica approach whose legitimacy has been a point of controversy [Ver85b, Kam99b, Zir99] for over two decades. Recently discovered integrability [Kan02, Kan05, Osi07] of zero-dimensional replica field theories will be a dominant motif of this contribution.

1.2. How replicas arise and why they are tricky

Both fermionic and bosonic replicas are based on the identity

$$\log X = \lim_{n \to 0} \frac{X^n - 1}{n}$$  \hspace{1cm} (1.4)$$

which can be very useful in evaluating the average of a logarithm \(\langle \log X \rangle\) of the random variable \(X\). Indeed, identifying \(X\) with \(\det \eta(z - \mathcal{H})\) in Eq. (1.2), and further combining Eq. (1.4) with the field integral representation Eq. (1.3) of the (inverse) determinant, we formally relate the resolvent

$$G(z) = \eta \lim_{n \to 0} \frac{1}{n} \frac{\partial}{\partial z} \mathcal{Z}_n^{(n)}(z)$$  \hspace{1cm} (1.5)$$
to the ‘partition function’

$$\mathcal{Z}_n^{(n)}(z) = \prod_{\alpha=1}^n \int \mathcal{D}[\bar{\psi}^{(\alpha)}, \psi^{(\alpha)}] \exp \left[ i \bar{\psi}^{(\alpha)} (z - \mathcal{H}) \psi^{(\alpha)} \right]$$  \hspace{1cm} (1.6)$$
of \(n\) copies, or replicas, of the initial random system. The pair of formulae Eqs. (1.5) and (1.6), known as the replica trick, achieve our goal of removing a random denominator from Eq. (1.1) and hint that a nonperturbative calculation of the mean product of resolvents may become feasible.

In order to keep the discussion concrete and set up notation, we further assume that the matrix Hamiltonian \(\mathcal{H}\) is drawn from the paradigmatic Gaussian Unitary Ensemble (GUE) associated with the probability measure [12]

$$P_N[\mathcal{H}] \mathcal{D}[\mathcal{H}] = \pi^{-N^2/2} \exp(-tr \mathcal{H}^2) \prod_{j=1}^N d\mathcal{H}_{jj} \prod_{1 \leq j < k}^N d\mathcal{H}_{jk} d\bar{\mathcal{H}}_{jk}.$$  \hspace{1cm} (1.7)$$

Then, the resolvent \(g(z) = \langle G(z) \rangle\) averaged with respect to the GUE probability measure should be furnished by the limiting procedure

$$g(z) = \eta \lim_{n \to 0} \frac{1}{n} \frac{\partial}{\partial z} \langle \mathcal{Z}_n^{(n)}(z) \rangle,$$  \hspace{1cm} (1.8)$$

‡ An introductory exposition of the supersymmetry method can be found in Chapter 7 of this Handbook as well as in the earlier review papers [Efe83, Ver85a] and the monograph [Efe97]. For a review of the Keldysh approach, the reader is referred to [Kam09]; see also the paper [Alt00] where the Keldysh technique is discussed in the RMT context.

§ Mostly known in condensed matter physics community since the paper by Edwards and Anderson [Edw75] on spin glasses (see also [Eme75]), the recipe of calculating the average of a logarithm based on Eq. (1.4) dates at least as far back as 1934, see the book [Har34] by Hardy, Littlewood and Pólya.

∥ The alert reader might already detect some trickery behind Eqs. (1.5) and (1.6). Their mathematical status will be clarified in Section 1.2.3.

¶ See also Chapter 4 of this Handbook which outlines the method of orthogonal polynomials for unitary invariant random matrix models.
where the (average) replica partition function equals
\[
\langle Z_n^{(\eta)}(z) \rangle = \prod_{\alpha=1}^{n} \int \mathcal{D}[\bar{\psi}^{(\alpha)}, \psi^{(\alpha)}] \exp \left( -i \eta s_z \sigma \right) + \frac{\eta}{4} \text{tr} \sigma^2 \right) .
\] (1.9)

Here, the Hermitean matrix \( \sigma = \sigma^\dagger \) acting in the replica space is defined as
\[
\sigma_{\alpha\beta} = (\psi_j \otimes \bar{\psi}_j)_{\alpha\beta} = \psi_j^{(\alpha)} \bar{\psi}_j^{(\beta)} , \quad \alpha, \beta \in (1, \ldots, n) .
\] (1.10)

Notice that ensemble averaging of \( Z_n^{(\eta)}(z) \) [Eq. (1.6)] has induced an effective \( \psi^4 \) interaction between \( n \) replicated random systems as described by the interaction term \( \text{tr} \sigma^2 = \text{tr} \left[ (\psi_j \otimes \bar{\psi}_j)^2 \right] \) in the action. To make further progress in evaluating \( \langle Z_n^{(\eta)}(z) \rangle \), the interaction term may routinely be decoupled by means of the Hubbard-Stratonovich transformation [Efe97, Ver85a].

1.2.1. Fermionic replicas

In case of fermionic fields \( \psi = \chi, \eta = +1 \), the \( \chi^4 \) interaction can be decoupled via \( n \times n \) Hermitean matrix field \( \Omega_n \) as
\[
\exp \left( \frac{1}{4} \text{tr} \sigma^2 \right) = \pi^{-n^2/2} \int_{\Omega_n = \Omega_n} \mathcal{D}\Omega_n \exp \left[ -\text{tr} \left( \Omega_n^2 + \sigma \Omega_n \right) \right] .
\] (1.11)

Performing the integration over fermionic fields in Eq. (1.9) with the help of Eq. (1.3),
\[
\prod_{\alpha=1}^{n} \int \mathcal{D}[\bar{\chi}^{(\alpha)}, \chi^{(\alpha)}] \exp \left( -i s_z \sigma \right) + \frac{1}{4} \text{tr} \sigma^2 \right) = \pi^{-n^2/2} (i s_z)^{nN} \int_{\Omega_n = \Omega_n} \mathcal{D}\Omega_n e^{-\text{tr} \Omega_n^2} \det^N (z - i s_z \Omega_n) ,
\] (1.12)

we arrive at the fermionic replica limit
\[
g(z) = \lim_{n \to 0} \frac{1}{n} \frac{\partial}{\partial z} \langle Z_n^{(+)}(z) \rangle
\] (1.13)

that relates the average resolvent \( g(z) \) to the fermionic replica partition function
\[
\langle Z_n^{(+)}(z) \rangle = \int_{\Omega_n = \Omega_n} \mathcal{D}\Omega_n e^{-\text{tr} \Omega_n^2} \det^N (z - i s_z \Omega_n) .
\] (1.14)

Before discussing this result, let us turn to the bosonic version of the replica trick.

1.2.2. Bosonic replicas

In case of bosonic fields \( \psi = s, \eta = -1 \), decoupling of the \( s^4 \) interaction can be carried out in a similar fashion. Making use of yet another variant of the Hubbard-Stratonovich transformation
\[
\exp \left( \frac{1}{4} \text{tr} \sigma^2 \right) = \pi^{-n^2/2} \int_{\Omega_n = \Omega_n} \mathcal{D}\Omega_n \exp \left[ -\text{tr} \left( \Omega_n^2 + i \sigma \Omega_n \right) \right] ,
\] (1.15)
Early studies: Heuristic approach to replicas

and integrating out bosonic fields in Eq. (1.9) with the help of Eq. (1.3),
\[
\prod_{\alpha=1}^{n} \int \mathcal{D}[^{\alpha}\bar{s}, s^{(\alpha)}] \exp \left( +i \mathbf{s}_z \mathbf{tr} \sigma - \frac{1}{4} \mathbf{tr} \sigma^2 \right)
\]
\[
= \pi^{-n^2/2} (2i)^n N \int_{\mathbf{Q}_h=\mathbf{Q}_n} \mathcal{D}{\mathbf{Q}_n} e^{-\mathbf{tr} \mathbf{Q}_n^2} \det^{-N} (z - \mathbf{s}_z \mathbf{Q}_n),
\]
we express the average resolvent
\[
g(z) = - \lim_{n \to 0} \frac{1}{n} \frac{\partial}{\partial z} \langle \mathcal{Z}_n^{(-)}(z) \rangle
\]
through the bosonic replica partition function
\[
\langle \mathcal{Z}_n^{(-)}(z) \rangle = \int_{\mathbf{Q}_h=\mathbf{Q}_n} \mathcal{D}{\mathbf{Q}_n} e^{-\mathbf{tr} \mathbf{Q}_n^2} \det^{-N} (z - \mathbf{s}_z \mathbf{Q}_n).
\]

1.2.3. Subtleties of the replica limit

Seemingly innocent at first glance, both fermionic [Eqs. (1.13) and (1.14)] and bosonic [Eqs. (1.17) and (1.18)] replica prescriptions appear to be counterintuitive and rising fundamental mathematical questions [Par03]. Indeed, due to a particular integration measure which makes no sense for \(n\) other than positive integers (\(n \in \mathbb{Z}^+\)), the matrix-integral representation of average replica partition functions
\[
\langle \mathcal{Z}_n^{(\eta)}(z) \rangle = \int_{\mathbf{Q}_h=\mathbf{Q}_n} \mathcal{D}{\mathbf{Q}_n} e^{-\mathbf{tr} \mathbf{Q}_n^2} \det^{-N} (z - \sqrt{1+\eta} \mathbf{s}_z \mathbf{Q}_n)
\]
cannot directly be used to implement the replica limit Eqs. (1.8) as the latter is determined by the behaviour of \(\langle \mathcal{Z}_n^{(\eta)}(z) \rangle\) in a close vicinity of \(n = 0\). This mismatch between the ‘available’ (\(n \in \mathbb{Z}^+\)) and the ‘needed’ (\(n \in \mathbb{R}\)) is at the heart of the trickery the replica field theories have often been charged [Ver85b, Zir99].

The canonical way to bridge this gap is to determine the average replica partition functions \(\langle \mathcal{Z}_n^{(\eta)}(z) \rangle\) for \(n \in \mathbb{Z}^+\), and then attempt to analytically continue them to \(n \in \mathbb{R}\), in general, and to a vicinity of \(n = 0\), in particular. This is a nontrivial task for two major reasons: (i) The analytic continuation of the replica partition function away from \(n\) integers should not necessarily be unique \(\dagger\). (ii) To retain control over the analytic continuation, the latter must rest on an exact calculation of the average replica partition function for \(n \in \mathbb{Z}^+\). Early approaches to replica field theories seem to underestimate these two points bringing a number of pathological results even in the RMT setting (see Section 2).

2. Early studies: Heuristic approach to replicas

Exact evaluation of replica partition functions, whilst welcomed, is quite a challenge. At the same time, their approximate calculation is often feasible in a certain region of

\(\dagger\) For entire functions, the uniqueness is guaranteed by a boundedness property as formulated by Carlson’s theorem [Tit32].
parameter space where a saddle point procedure can be justified. In doing so, one is naturally led to the ‘replica symmetric’ and ‘replica asymmetric’ saddle point manifolds as discussed below.

2.1. Density of eigenvalues in the GUE

In the RMT context, a saddle point evaluation of the replica partition function Eq. (1.19) makes sense if the dimension \( N \) of the random matrix \( \mathbf{H} \) is large enough. For not too large replica parameter \( n \in \mathbb{Z}^+ \) (in particular, \( n \) should not scale with \( N \)), the dominating contribution to \( \langle Z_n^{(n)}(z) \rangle \) is expected to come from the configurations \( Q_n^{(sp)} \) determined by the saddle point equation

\[
\frac{\delta}{\delta Q_n} \text{tr} \left[ Q_n^2 - \eta N \log(z - \sqrt{1+\eta} s_z Q_n) \right] = 0. \tag{2.1}
\]

Its solutions form \( 2^n \) saddle point manifolds

\[
Q_n^{(sp)} = \frac{s_z}{\sqrt{1+\eta}} \sqrt{\frac{N}{2}} \text{diag}(e^{i\kappa_1 \theta}, \ldots, e^{i\kappa_n \theta}) \tag{2.2}
\]

with \( \kappa_\ell \) taking on the values \( \pm 1 \) independently of each other. Here,

\[
e^{i\theta} = z_s + i \sqrt{1 - z_s^2}, \tag{2.3}
\]

where \( z_s \) stands for the scaled energy \( z_s = z/D_{\text{edge}} \) with \( D_{\text{edge}} = \sqrt{2N} \) being the endpoint of the spectrum support. (Hence, the spectrum bulk is situated within the segment \(|\Re z_s| < 1\).)

2.1.1. Bosonic replicas

Out of the plethora of saddles Eq. (2.2), only the distinguished replica symmetric manifold

\[
Q_n^{(sp)} \big|_{\text{sym}} = s_z \sqrt{\frac{N}{2}} e^{i\theta} \otimes 1_n \tag{2.4}
\]

contributes the bosonic replica partition function Eq. (1.18). This is so because Eq. (2.4) is the only saddle [Zir99] (i) reachable by continuous deformation of the integration contour in Eq. (1.18) without crossing the hypersurface defined by the singularities of \( \det^{-N}(z - s_z \mathbf{Q}_n) \) and (ii) compatible with analyticity of the average resolvent Eq. (1.17) at infinity. In the leading order in the large parameter \( N \), the bosonic replica partition function is then approximated by

\[
\langle Z_n^{(-)}(z = \epsilon - i0) \rangle \simeq \left( \frac{N}{2} \right)^{nN/2} (2\sin \theta)^{-n^2/2} \exp \left[ \frac{nN}{2} (e^{2\theta} - 2i\theta) - i \frac{n^2}{2} \left( \theta - \frac{\pi}{2} \right) \right]. \tag{2.5}
\]

By derivation, Eq. (2.5) holds for \( n \in \mathbb{Z}^+ \). To retrieve the average density of eigenlevels

\[
\rho(\epsilon) = \frac{1}{\pi} \Im g(\epsilon - i0) \tag{2.6}
\]
through the replica limit Eq. (1.17), one should analytically continue $\langle Z_n^{(-)}(z) \rangle$ away from $n \in \mathbb{Z}^+$. To be on the safe side, such an analytic continuation must rest on an exact integer-$n$ result for $\langle Z_n^{(-)}(z) \rangle$. The latter is sadly unavailable. Not being spoilt for choice, one could try to analytically continue the bosonic replica partition function to the domain $n \in \mathbb{R}^+$ taking the approximate result Eq. (2.5) as a starting point and merely assuming it to hold, as it stands, in the right vicinity of $n = 0$. Then, the replica limit Eq. (1.17) can be taken to yield the Wigner semicircle [Edw80]

$$\varrho(\epsilon_s) = \frac{2}{\pi} \sqrt{1 - \epsilon_s^2}, \quad |\epsilon_s| \leq 1.$$  

Here, $\epsilon_s = \epsilon/D_{\text{edge}}$. The $1/N$ correction to Eq. (2.7) can be obtained in a similar fashion and is known to vanish within the replica symmetric ansatz.

Similarly, replica symmetric saddle point calculations performed for the Gaussian Orthogonal Ensemble (GOE) and the Gaussian Symplectic Ensemble (GSE) of random matrices yield [Edw76, Ver84, Dhe90, Ito97]

$$\varrho(\epsilon_s) = \frac{2}{\pi} \sqrt{1 - \epsilon_s^2} + \frac{1}{2\pi N} \left( 1 - \frac{2}{\beta} \right) \left[ \frac{1}{\sqrt{1 - \epsilon_s^2}} - \pi \delta(\epsilon_s^2 - 1) \right], \quad |\epsilon_s| \leq 1.$$  

Here, $\beta$ is the Dyson symmetry index [Meh04] taking the values $\beta = 1$ for GOE, $\beta = 2$ for GUE, and $\beta = 4$ for GSE.

Importantly, both leading and subleading in $1/N$ terms in Eq. (2.8) for the average densities of eigenlevels contain no terms oscillating on the scale of the mean level spacing. Since the replica asymmetric saddles are inaccessible as explained on general grounds [Zir99] in (i) and (ii) below Eq. (2.4), we are led to conclude that the saddle point approach to bosonic replicas fails to reproduce truly non-perturbative features of the eigenlevel density.

### 2.1.2. Fermionic replicas

Looking for some insight into a possible rôle played by replica asymmetric saddles, we turn to the approximate performance of fermionic replicas Eq. (1.14). In this case, all $2^n$ saddle point manifolds

$$Q_n^{(sp)} = -is_z \sqrt{\frac{N}{2}} \text{diag}(e^{i\kappa_1 \theta}, \ldots, e^{i\kappa_n \theta}), \quad \kappa_\ell = \pm 1,$$

are accessible adding up, for $n = 2m$, to

$$\langle Z_{2m}^{(+)}(z) \rangle \approx \left( \frac{N}{2} \right)^m e^{mN \cos 2\theta} \sum_{q=-m}^{q=m} V_{m,q} \left( \frac{N}{2\pi} \right)^{m^2 - q^2} (2\sin \theta)^{m^2 - 3q^2} \times \exp \left[ iq \left( N(2\theta - \sin 2\theta) + 2m \left( \theta - \frac{\pi}{2} \right) \right) \right].$$

Here, $V_{m,q}$ denotes the volume of Grassmanian [Kam99b, Zir99]

$$V_{m,q} = \text{vol} \left[ \frac{\mathbb{U}(2m)}{\mathbb{U}(m-q) \times \mathbb{U}(m+q)} \right] = \frac{(2\pi)^{m^2 - q^2} \prod_{j=1}^{m+q} \Gamma(j) \prod_{j=1}^{m-q} \Gamma(j)}{\prod_{j=1}^{2m} \Gamma(j)}.$$
The summation index \(q\) in Eq. (2.10) counts \((2m + 1)\) families of saddle point manifolds [Eq. (2.9)], the \(q\)-th family being represented by the configuration

\[
Q_n^{(\text{sp})}[q] = -is_z\sqrt{\frac{N}{2}} \text{diag}(e^{i\theta} \otimes 1_{m-q}, e^{-i\theta} \otimes 1_{m+q})
\]  

(2.12)
taken with the obvious combinatorial weight

\[
\binom{2m}{m+q} = \frac{\Gamma(2m+1)}{\Gamma(m+1-q)\Gamma(m+1+q)}.
\]

Obtained in the large-\(N\) limit, Eq. (2.10) holds for \(m \in \mathbb{Z}^+\) which do not scale with \(N\).

Aimed at deriving the density of eigenlevels through the replica limit, one should first to analytically continue Eq. (2.10) away from \(m \in \mathbb{Z}^+\). Even though making an analytic continuation based on an approximate result is a dangerous ploy and is, with certainty, a mathematically questionable procedure, we embark on the proposal due to [Kam99b] who have spotted that the volume of Grassmanian [Eq. (2.11)] vanishes for all integers \(|q| \geq m + 1\). This observation makes it tempting to extend the summation over \(q\) in Eq. (2.10) to (minus and plus) infinities to end up with the following trial function for ‘analytically continued’ fermionic replica partition function

\[
\langle Z^{(+)}_{2m}(z) \rangle \approx \left(\frac{N}{2}\right)^{mN} e^{mN \cos 2\theta} \sum_{q=-\infty}^{q=+\infty} V_{m,q} \left(\frac{N}{2\pi}\right)^{m^2-q^2} (2\sin \theta)^{m^2-3q^2} \times \exp \left[iq \left(N(2\theta - \sin 2\theta) + 2m \left(\theta - \frac{\pi}{2}\right)\right)\right].
\]  

(2.13)

A close inspection of this result reveals that it is flawed: (i) so-continued replica partition function \(\langle Z^{(+)}_{2m}(z) \rangle\) diverges [Kam99b, Zir99] in the vicinity of \(m = +0\), the region crucially important for retrieving the density of eigenlevels through the replica limit; (ii) due to the \(q \mapsto -q\) symmetry of the summand, \(\langle Z^{(+)}_{2m}(z) \rangle\) must be real thus leaving no room for a non-zero density of states. Indeed, a formally derived small-\(m\) expansion of Eq. (2.13)

\[
\langle Z^{(+)}_{2m}(z) \rangle \approx 1 + m \left[N \left(\cos 2\theta + \log \frac{N}{2}\right) + \frac{1}{4N(\sin \theta)^3} \cos \left[N(2\theta - \sin 2\theta)\right]\right] + \mathcal{O}(m^2)
\]  

(2.14)

considered together with Eq. (2.6) leads us to conclude that the replica limit [Eq. (1.13)] in the above elaboration fails to reproduce even the smooth part of the average density of eigenlevels yielding \(\varrho(\epsilon) = 0\). Notice that the derivation of Eq. (2.14) boldly ignores the divergence of the infinite series Eq. (2.13).

This unphysical result is at odds with the work [Kam99b] where both the Wigner semicircle and the \(1/N\) oscillating correction to it,

\[
\varrho(\epsilon) \approx \frac{2}{\pi} \sqrt{1 - \epsilon^2} \left[1 - \frac{1}{4N} \frac{\cos \left(N(2\theta - \sin 2\theta)\right)}{\sin^2 \theta}\right],
\]  

(2.15)

were reproduced out of fermionic replicas in almost the same elaboration. The only difference between the above treatment and that in [Kam99b] is that its authors used an

+ The group volume \(V_{m,q}\) grows too fast with \(q\) for the series \(\sum_{q=-\infty}^{q=+\infty} \cdots\) to converge.
alternative enumeration of saddle point manifolds Eq. (2.12) contributing the fermionic replication partition function.

To meet the parameterisation of [Kam99b], we introduce a new summation index, \( p = q + m \), in Eq. (2.10). This amounts to counting saddle point manifolds starting with the ‘replica symmetric’ one,

\[
\tilde{Q}_n^{(sp)}[p] = -is_\zeta \sqrt{\frac{N}{2}} \text{diag}(e^{i\theta} \otimes \mathbb{1}_{2m-p}, e^{-i\theta} \otimes \mathbb{1}_p) \tag{2.16}
\]

as \( p \) varies from 0 to \( 2m \), so that

\[
\left\langle \tilde{Z}_{2m}^{(+)}(z) \right\rangle \simeq \left( \frac{N}{2} \right)^{mN} \left( \frac{2 \sin \theta}{2m^2} \right)^{2m} \sum_{p=0}^{2m} V_{m, p-m} \left( \frac{N}{2\pi} \right)^{p(2m-p)} (2 \sin \theta)^{3p(2m-p)} \times \exp \left[ i(p-m) \left( \frac{N(2\theta - 2 \sin \theta)}{2} + 2m \left( \theta - \frac{\pi}{2} \right) \right) \right]. \tag{2.17}
\]

For \( m \in \mathbb{Z}^+ \), Eqs. (2.10) and (2.17) are trivially identical. However, this is not the case for generic real valued \( m \) after Eq. (2.17) is ‘analytically continued’ by extending the summation over \( p \) to (minus and plus) infinities:

\[
\left\langle \tilde{Z}_{2m}^{(+)}(z) \right\rangle \simeq \left( \frac{N}{2} \right)^{mN} \left( \frac{2 \sin \theta}{2m^2} \right)^{2m} \sum_{p=-\infty}^{+\infty} V_{m, p-m} \left( \frac{N}{2\pi} \right)^{p(2m-p)} (2 \sin \theta)^{3p(2m-p)} \times \exp \left[ i(p-m) \left( \frac{N(2\theta - 2 \sin \theta)}{2} + 2m \left( \theta - \frac{\pi}{2} \right) \right) \right]. \tag{2.18}
\]

Now, contrary to the previous result [Eq. (2.14)], the small-\( m \) expansion of the fermionic replication partition function appears to contain a mysterious imaginary component *,

\[
\left\langle \tilde{Z}_{2m}^{(+)}(z) \right\rangle \simeq 1 + m \left( e^{2i\theta} - 2i\theta + \log \frac{N}{2} \right) + \frac{1}{4N(\sin \theta)^2} e^{iN(2\theta - \sin 2\theta)} + \mathcal{O}(m^2). \tag{2.19}
\]

This expansion coincides with the one in [Kam99b] and does reproduce, through the replica limit, the correct result [Eq. (2.15)] for the average density of eigenvalues in GUE, in both leading and subleading orders in \( 1/N \). As the latter term captures oscillatory behavior of the eigenlevel density, it was assumed in the literature [Kam99b, Kam99c, Yur99] that the replica asymmetric saddles may describe a nonperturbative sector of replica field theories.

Let us stress that neither of the above two treatments of fermionic replicas [resulting in Eqs. (2.14) and (2.19), respectively] can be considered as mathematically satisfactory because they both rely on a nonexisting ‘analytic continuation’ of the replica partition function [Eq. (2.10)] to the vicinity of \( m = +0 \), as explained below Eq. (2.13).

* It was argued in [Zir99] that the procedure of ‘analytic continuation’ that led to Eq. (2.18) and further to Eq. (2.19) favors so-called causal saddle points over their conjugate counterparts called acausal. Such a selectivity is eventually responsible for the correct result for the density of eigenlevels as discussed below Eq. (2.19).
2.2. Brief summary

A brief tour d’horizon on the saddle point approach to replica field theories indicates that the bosonic variation of the replica trick is restricted to the perturbative sector of the theory accounted for by the (only reachable) replica symmetric saddle point manifold. In the fermionic version of the replica trick, both replica symmetric and replica asymmetric saddle point manifolds contribute the replica partition function; however, its analytic continuation to a vicinity of $n = 0$ is ill defined, both in terms of convergence and uniqueness.

These drawbacks of the saddle point approach to replicas are not specific to the GUE description. Similar difficulties arise in replica studies of other random matrix ensembles [Dal01, Nis02] and in analysis of physical systems (notably one-dimensional impenetrable bosons) admitting effective RMT description [Gan01, Nis03, Gan04].

Are the problems surfaced in the above calculation indicative of internal difficulties of the replica method itself or should they be attributed to a particular computational framework? A little thought shows that the approximate evaluation of both bosonic and fermionic replica partition functions is the key point to blame for the inconsistencies encountered in the above elaboration of the replica trick. In such a situation, leaning towards exact calculational schemes in replica field theories is a natural move.

3. Integrable theory of replicas

In this section, we outline an alternative way of treating replica partition functions. A connection [Kan02, Kan05] between zero-dimensional replica field theories and the theory of integrable hierarchies † is central to our formalism.

3.1. Density of eigenvalues in the GUE revisited (easy way)

For illustration purposes ‡, we choose the very same problem of calculating the average density of eigenlevels in the GUE specified by the probability measure Eq. (1.7). For the lack of space, only fermionic replicas will be considered; a bosonic replica treatment can be found in the tutorial paper [Osi09].

3.1.1. Replica partition function as Toda Lattice

Our claim of exact solvability of the replica model Eq. (1.14) and the models of the same ilk rests on two observations. To make the first, we routinely reduce the average

† An introductory exposition of integrability arising in the RMT context can be found in Chapter 10 of this Handbook.
‡ Since the framework to be presented here solely rests on the symmetry underlying the matrix model, it can readily be adopted to other spectral statistics for random matrix ensembles falling into the same $\beta = 2$ Dyson’s symmetry class [Meh04]. The reader interested in a general formulation of the integrable theory of replicas is referred directly to Section 3.2.
fermionic partition function Eq. (1.14) to the $n$-fold integral

$$\langle Z_{n}^{(+)}(z) \rangle = \int_{\mathbb{R}^{n}} \prod_{\ell=1}^{n} d\lambda_{\ell} e^{-\lambda_{\ell}^{2}} (\lambda_{\ell} - iz)^{N} \Delta_{n}^{2}(\lambda)$$

(3.1)

after diagonalising the Hermitean matrix $\mathbf{Q}_{n} = \mathbf{U}_{n} \mathbf{\lambda} \mathbf{U}_{n}^{\dagger}$ by unitary rotation $\mathbf{U}_{n} \in \mathfrak{U}(n)$; here $\mathbf{\lambda}$ is a diagonal matrix $\mathbf{\lambda} = \text{diag}(\lambda_{1}, \ldots, \lambda_{n})$ composed of eigenvalues of $\mathbf{Q}_{n}$, and $\Delta_{n}(\lambda)$ is the Vandermonde determinant

$$\Delta_{n}(\lambda) = \prod_{\ell > k} (\lambda_{\ell} - \lambda_{k})$$

(3.2)

induced by the Jacobian of the transformation $\mathbf{Q}_{n} \mapsto (\mathbf{U}_{n}, \mathbf{\lambda})$. Further, making a proper shift of the integration variables and applying the Andréief-de Bruijn integration formula [And83, deB55]

$$\int_{\mathbb{R}^{n}} \prod_{\ell=1}^{n} d\mu(\lambda_{\ell}) \det[f_{k}(\lambda_{\ell})] \det[g_{k}(\lambda_{\ell})] = n! \det \left[ \int_{\mathbb{R}} d\mu(\lambda) f_{k}(\lambda) g_{\ell}(\lambda) \right]$$

(3.3)

which holds for benign integration measure $d\mu(\lambda)$, one derives:

$$\langle Z_{n}^{(+)}(z) \rangle = \exp \left[ nz^{2} \right] \det_{k,\ell} \left[ \int_{\mathbb{R}} d\lambda \lambda^{N+k+\ell} \exp (-\lambda^{2} - 2iz\lambda) \right] .$$

(3.4)

The latter is equivalent to the remarkable representation

$$\langle Z_{n}^{(+)}(z) \rangle = \exp \left[ nz^{2} \right] \tau_{n}^{(+)}(z)$$

(3.5)

involving the Hankel determinant

$$\tau_{n}^{(+)}(z) = \det \left[ \partial_{z}^{k+\ell} \tau_{1}^{(+)}(z) \right]_{k,\ell=0,\ldots,n-1}$$

(3.6)

with $\tau_{1}^{(+)}(z) = e^{-z^{2}}H_{N}(z)$ being related to the Hermite polynomial $H_{N}(z)$ (see also Chapter 4 of this Handbook). In the above equations, no care was taken of prefactors which tend to unity in the replica limit.

Consequences of the Hankel-determinant-like representation Eq. (3.5) of the fermionic replica partition function $\langle Z_{n}^{(+)}(z) \rangle$ are far reaching. As had first been shown by Darboux [Dar72] a century ago, any set of Hankel determinants meeting the ‘initial condition’ $\tau_{0}^{(+)}(z) = 1$ (which is indeed the case for Eq. (3.6) due to Eq. (3.5) and the normalisation $\langle Z_{0}^{(+)}(z) \rangle = 1$) satisfies the equation

$$\tau_{n}^{(+)}(z) \frac{\partial^{2}}{\partial z^{2}} \tau_{n}^{(+)}(z) - \left( \frac{\partial}{\partial z} \tau_{n}^{(+)}(z) \right)^{2} = \tau_{n-1}^{(+)}(z) \tau_{n+1}^{(+)}(z), \quad n \in \mathbb{Z}^{+}.$$  

(3.7)

Equations (3.5) and (3.7) taken together with the known initial conditions for $\tau_{0}^{(+)}(z)$ and $\tau_{1}^{(+)}(z)$ establish a differential recursive hierarchy between nonperturbative fermionic replica partition functions $\langle Z_{n}^{(+)}(z) \rangle$ with different replica indices $n$. This is an exact alternative to the approximate solution Eq. (2.10) presented in the previous section.

Equation (3.7), known as the positive Toda Lattice equation [Tod67] in the theory of integrable hierarchies * [Mor94], is the first indication of exact solvability of replica

* See also Chapter 10 of this Handbook.
field theories. Importantly, the emergence of the Toda Lattice hierarchy is eventually due to the $\beta = 2$ symmetry of the fermionic replica field theory encoded into the squared Vandermonde determinant in Eq. (3.1).

3.1.2. From Toda Lattice to Painlevé transcendent

While important from conceptual point of view, the positive Toda Lattice equation for the fermionic replica partition function $\langle Z_n^{(+)}(z) \rangle$, if taken alone, is not much helpful in performing the replica limit.

Fortunately, here the second observation, borrowed from [For01], comes in. Miraculously, the same Toda Lattice equation governs the behaviour of so-called $\tau$-functions arising in the Hamiltonian formulation [Oka86] of the six Painlevé equations †† [Cla03, Nou04], which are yet another fundamental object in the theory of nonlinear integrable systems. The Painlevé equations contain the hierarchy (or replica) index $n$ as a parameter. For this reason, they serve as a proper starting point [Kan02] for building a consistent analytic continuation of replica partition functions away from $n$ integers.

The aforementioned Painlevé reduction [Oka86, For01] of the Toda Lattice equation Eq. (3.7) materialises in the exact representation [For01]

$$
\langle Z_n^{(+)}(z) \rangle = \langle Z_n^{(+)}(0) \rangle \exp \left( \int_0^t \, dt \, \sigma_{IV}(t) \right) \quad (3.8)
$$

which holds as soon as $n \in \mathbb{Z}^+$. It involves the fourth Painlevé transcendent $\sigma_{IV}(t)$ satisfying the Painlevé IV equation § in the Jimbo-Miwa-Okamoto form [Jim81, Oka86]

$$(\sigma_{IV}'' - 4t\sigma_{IV}' - \sigma_{IV})^2 + 4\sigma_{IV}'(\sigma_{IV}' + 2n)(\sigma_{IV}' - 2N) = 0. \quad (3.9)$$

The boundary condition is $\sigma_{IV}(t) \sim (nN/t)(1 + O(t^{-1}))$ as $t \to +\infty$. Note that Eq. (3.9), and therefore Eq. (3.8), contain the replica index $n$ as a parameter.

By derivation, Eq. (3.8) holds for $n$ positive integers only and, strictly speaking, there is no a priori reason to expect it to stay valid away from $n \in \mathbb{Z}^+$. It can be shown, however, that it is legitimate to extend Eq. (3.8), as it stands, beyond $n \in \mathbb{Z}^+$ and consider this extension as a proper analytic continuation to $n \in \mathbb{R}^+$ we are looking for (the reader is referred to [Kan02] for a detailed discussion).

As the result, the fermionic replica limit Eq. (1.13) can now safely be implemented to bring, via Eq. (2.6), the average density of eigenlevels [Meh60]

$$
\varrho(\epsilon) = \frac{1}{2^N \Gamma(N) \sqrt{\pi}} \, e^{-\epsilon^2} \left[ H_N'(\epsilon)H_{N-1}(\epsilon) - H_N(\epsilon)H_{N-1}'(\epsilon) \right] \quad (3.10)
$$

expressed in terms of Hermite polynomials. This is the famous GUE result firmly established by other methods [Meh04, Guh91]. Technically, the derivation of Eq. (3.10) is based on the small-$n$ expansion of the Hamiltonian representation [Nou04] of the fourth Painlevé transcendent. The details of this somewhat cumbersome calculation

†† See also Chapter 9 of this Handbook.

§ In the original paper [Kan02], Eq. (3.9) appears to have incorrect signs in front of $n$ and $N$. I thank Nicholas Witte for bringing this fact to my attention.
can be found in [Osi09] where the nonperturbative result Eq. (3.10) is also re-derived within the bosonic variation of the replica trick.

3.1.3. Brief summary

The above treatment was largely based on a wealth of ‘ready-for-use’ results (Andréief-de Bruijn formula, Darboux theorem, and a connection between the Toda Lattice and Painlevé transcendents) which surprisingly well fitted our goal of a nonperturbative evaluation of the particular replica partition function Eq. (3.1). Since existence of such an ‘easy way’ is clearly the exception rather than the rule, a regular yet flexible formalism is needed for a nonperturbative description of a general class of replica partition functions.

3.2. The \( \tau \) function theory of replicas (\( \beta = 2 \))

In this section, we outline such a regular formalism [Osi07, Osi09] tailor-made for an exact analysis of both fermionic and bosonic zero-dimensional replica field theories belonging to the broadly interpreted \( \beta = 2 \) Dyson symmetry class.

3.2.1. From replica partition function to \( \tau \) function

Let us concentrate on the fermionic and/or bosonic zero-dimensional replica field theories whose partition functions admit the eigenvalue representation

\[
\langle Z_n^{(\pm)}(\varsigma) \rangle = \int_{D^n} \prod_{\ell=1}^{n} d\lambda_{\ell} \Gamma(\varsigma; \lambda_{\ell}) e^{-V_n(\lambda_{\ell})} \Delta_n^2(\lambda), \quad n \in \mathbb{Z}^+.
\] (3.11)

Here \( V_n(\lambda) \) is a ‘confinement potential’ which may depend on the replica index \( \pm n \); \( \Gamma(\varsigma; \lambda) \) is a function accommodating relevant physical parameters \( \varsigma = (\varsigma_1, \varsigma_2, \ldots) \) of the theory (e.g., energies in the multi-point spectral correlation functions). In order to treat the fermionic and bosonic replicas on the same footing, the integration domain \( D \) was chosen to be \( \dagger \)

\[
D = \bigcup_{j=1}^{r} [c_{2j-1}, c_{2j}].
\] (3.12)

To determine the replica partition function \( \langle Z_n^{(\pm)}(\varsigma) \rangle \) nonperturbatively, we adopt the ‘deform-and-study’ approach, a standard string theory method of revealing hidden structures. Its main idea consists of ‘embedding’ \( \langle Z_n^{(\pm)}(\varsigma) \rangle \) into a more general theory of \( \tau \) functions

\[
\tau_n^{(s)}(\varsigma; t) = \frac{1}{n!} \int_{D^n} \prod_{\ell=1}^{n} d\lambda_{\ell} \Gamma(\varsigma; \lambda_{\ell}) e^{-V_{n-s}(\lambda_{\ell})} e^{t(\lambda_{\ell})} \Delta_n^2(\lambda)
\] (3.13)

\( \dagger \) Notice that \( D = [-1, +1] \) for (compact) fermionic replicas, and \( D = [0, +\infty) \) for (noncompact) bosonic replicas. A more general setting Eq. (3.12) does not complicate the theory.
which posses the infinite-dimensional parameter space \( t = (t_1, t_2, \cdots) \) arising as the result of the \( t \)-deformation
\[
v(t; \lambda) = \sum_{j=1}^{\infty} t_j \lambda^j
\]
of the confinement potential. The auxiliary parameter \( s \) is assumed to be an integer, \( s \in \mathbb{Z} \). Studying the evolution of \( \tau \) functions in the extended \((n, s, t, \xi)\) space allows us to identify the highly nontrivial, nonlinear differential hierarchical relations between them. Projection of these relations, taken at \( s = 0 \), onto the hyperplane \( t = 0 \),
\[
\langle Z_n^{(\pm)}(\xi) \rangle = n! \tau_n^{(s)}(\xi; t) \bigg|_{s=0, t=0},
\]
will generate, among others, a closed nonlinear differential equation for the replica partition function \( \langle Z_n^{(\pm)}(\xi) \rangle \). Since this \textit{nonperturbative} equation appears to contain the replica (or hierarchy) index \( n \) as a parameter, it is expected [Kan02] to serve as a proper starting point for building a consistent analytic continuation of \( \langle Z_n^{(\pm)}(\xi) \rangle \) away from \( n \) integers.

Having formulated the crux of the method, let us turn to its exposition. The two key ingredients of the exact theory of \( \tau \) functions are (i) the bilinear identity [Dat83, Adl95] and (ii) the (linear) Virasoro constraints [Mir90].

### 3.2.2. Bilinear identity and integrable hierarchies

The bilinear identity encodes an infinite set of hierarchically structured nonlinear differential equations in the variables \( \{t_j\} \). For the model introduced in Eq. (3.13), the bilinear identity reads [Adl95, Tu96, Osi07, Osi09]:
\[
\oint_{\mathcal{C}_\infty} dz e^{a v(t-t';z)} \left( \tau^{(s)}_n(t - [z^{-1}]) \frac{\tau^{(m+1+s-n)}_{m+1}(t' + [z^{-1}])}{z^{m+1-n}} \right) e^{v(t-t';z)}
\]
\[
-\tau^{(m+s-n)}_{m}(t' - [z^{-1}]) \frac{\tau^{(s+1)}_{n+1}(t + [z^{-1}])}{z^{n+1-m}} = 0.
\]
Here, \( a \in \mathbb{R} \) is a free parameter; the integration contour \( \mathcal{C}_\infty \) encompasses the point \( z = \infty \); the notation \( t \pm [z^{-1}] \) stands for the infinite set of parameters \( \{t_j \pm \frac{z^{-j}}{j}\} \); for brevity, the physical parameters \( \xi \) were dropped from the arguments of \( \tau \) functions.

Being expanded in terms of \( t' - t \) and \( a \), Eq. (3.16) generates various integrable hierarchies. One of them, the Kadomtsev-Petviashvili (KP) hierarchy in the Hirota form \( ^\dagger \)
\[
\frac{1}{2} D_k D_k \tau^{(s)}_n(t) \circ \tau^{(s)}_n(t) = s_{k+1}(\{D\}) \tau^{(s)}_n(t) \circ \tau^{(s)}_n(t)
\]
\( ^\dagger \) In Eq. (3.17), the \( j \)-th component of the infinite-dimensional vector \( \{D\} \) equals \( j^{-1} D_j \); the functions \( s_k(t) \) are the Schur polynomials [Mac98] defined by the expansion
\[
\exp \left( \sum_{j=1}^{\infty} t_j x^j \right) = \sum_{k=0}^{\infty} x^k s_k(t).
\]
The operator symbol \( D_j f(t) \circ g(t) \) stands for the Hirota derivative \( \partial_{x_j} f(t + x) g(t - x) \big|_{x=0} \).
$(k \geq 3)$ is of primary importance for the exact theory of replicas. The first nontrivial member of the KP hierarchy reads
\[
\left( \frac{\partial^4}{\partial t_1^4} + 3 \frac{\partial^2}{\partial t_2^2} - 4 \frac{\partial^2}{\partial t_1 \partial t_3} \right) \log \tau_n^{(s)}(\varsigma; t) + 6 \left( \frac{\partial^2}{\partial t_1^2} \log \tau_n^{(s)}(\varsigma; t) \right)^2 = 0. \tag{3.18}
\]
In what follows, it will be shown that its projection onto $s = 0$ and $t = 0$ [Eq. (3.15)] gives rise to a nonlinear differential equation for the replica partition function $\langle Z_n^{(\pm)}(\varsigma) \rangle$.

### 3.2.3. Virasoro constraints

Since we are interested in deriving a differential equation for $\langle Z_n^{(\pm)}(\varsigma) \rangle$ in terms of the derivatives over physical parameters $\{\varsigma_j\}$, we have to seek an additional block of the theory that would make a link between the $\{t_j\}$ derivatives in Eq. (3.18) taken at $t = 0$ and the derivatives over physical parameters $\{\varsigma_j\}$. The study [Adl95] by Adler, Shiota, and van Moerbeke suggests that the missing block is the Virasoro constraints $^*$ which reflect the invariance of the $\tau$ function [Eq. (3.13)] under a change of the integration variables.

In the present context, it is useful to demand the invariance under an infinite set of transformations
\[
\lambda_j \rightarrow \mu_j + \epsilon \mu_j^{q+1} f(\mu_j) \prod_{k=1}^m (\mu_j - c'_k), \quad q \geq -1, \tag{3.19}
\]
labeled by integers $q$. Here, $\epsilon > 0$, the vector $c'$ is $c' = \{c_1, \ldots, c_{2r}\} \setminus \{\pm \infty, N\}$ with $N$ denoting a set of zeros of $f(\lambda)$, and $m = \dim (c')$. The function $f(\lambda)$ is, in turn, related to the confinement potential $V_{n-s}(\lambda)$ through the parameterisation
\[
\frac{dV_{n-s}}{d\lambda} = \frac{g(\lambda)}{f(\lambda)} \cdot \quad g(\lambda) = \sum_{k=0}^\infty b_k \lambda^k, \quad f(\lambda) = \sum_{k=0}^\infty a_k \lambda^k \tag{3.20}
\]
in which both $g(\lambda)$ and $f(\lambda)$ depend on $n - s$ as do the coefficients $b_k$ and $a_k$ in the above expansions. The transformation Eq. (3.19) induces the Virasoro constraints $[Osi07]
\[
\left[ \hat{L}_q^V(t) + \hat{L}_q^\Gamma(\varsigma; t) \right] \tau_n^{(s)}(\varsigma; t) = 0, \tag{3.21}
\]
where the differential operator
\[
\hat{L}_q^V(t) = \sum_{\ell=0}^\infty \sum_{k=0}^{m} s_{m-k} (e'_m) \left( a_{\ell} \hat{L}_{q+k+\ell}(t) - b_{\ell} \frac{\partial}{\partial t_{q+k+\ell+1}} \right), \tag{3.22}
\]
acting in the $t$-space, is expressed in terms of the Virasoro operators $^\S$
\[
\hat{L}_q(t) = \sum_{j=1}^\infty j t_j \frac{\partial}{\partial t_{q+j}} + \sum_{j=0}^q \frac{\partial^2}{\partial t_j \partial t_{q-j}}, \tag{3.23}
\]

$^*$ See also Chapter 10 of this Handbook.

$^\S$ Equation (3.23) assumes that $\partial/\partial t_0$ is identified with the multiplicity of the matrix integral in Eq. (3.13), $\partial/\partial t_0 \equiv n$. 

obeying the Virasoro algebra
\[ [\hat{L}_p, \hat{L}_q] = (p - q)\hat{L}_{p+q}, \quad p, q \geq -1. \] (3.24)
The notation \( s_k(-p_m(c')) \) stands for the Schur polynomial and \( p_m(c') \) is an infinite dimensional vector
\[ p_m(c') = \left( \text{tr}_m(c')^1, \frac{1}{2}\text{tr}_m(c')^2, \ldots, \frac{1}{k}\text{tr}_m(c')^k, \ldots \right), \] (3.25)
where \( \text{tr}_m(c')^k = \sum_{j=1}^{m} (c'_j)^k \).

While very similar in spirit, the calculation of \( \hat{L}_q^\Gamma(t) \), the second ingredient in Eq. (3.21), is more of an art since the function \( \Gamma(\varsigma; \lambda) \) in Eq. (3.13) may significantly vary from one replica model to the other.

### 3.2.4. From \( \tau \) function to replica partition function

Remarkably, for \( t = 0 \), the two equations [Eqs. (3.18) and (3.21)] can be solved jointly to bring a closed nonlinear differential equation for \( \langle Z_n^\pm(\varsigma) \rangle \). It is this equation which, being supplemented by appropriate boundary conditions, provides a truly nonperturbative description of the replica partition functions and facilitates performing the replica limit.

### 3.3. Exact (bosonic) replicas at work: Density of eigenlevels in the chiral GUE

#### 3.3.1. Definitions

To see the above formalism at work, let us consider the chiral Gaussian Unitary Ensemble (chGUE) of \( N \times N \) random matrices
\[ \mathcal{H}_D = \begin{pmatrix} 0 & W \\ W^\dagger & 0 \end{pmatrix} \] (3.26)
known to describe the low-energy sector of \( \mathfrak{su}(N_c \geq 3) \) QCD in the fundamental representation \( \mathfrak{su}(N_c \geq 3) \) QCD in the fundamental representation [Ver93]. Composed of rectangular \( n_L \times n_R \) random matrices \( W \) with the Gaussian distributed complex valued entries
\[ P_{n_L,n_R}(W) = \left( \frac{2\pi}{N\Sigma^2} \right)^{n_Ln_R} \exp \left[ -\frac{N\Sigma^2}{2} \text{tr} W^\dagger W \right], \] (3.27)
where \( N = n_L + n_R \), the matrix \( \mathcal{H}_D \) has exactly \( \nu = |n_R - n_L| \) zero eigenvalues identified with the topological charge \( \nu \); the remaining eigenvalues occur in pairs \( \{ \pm \lambda_j \} \); the parameter \( \Sigma \) denotes the chiral condensate.

\( \mathfrak{su}(N_c \geq 3) \) QCD in the fundamental representation [Ver93]. Composed of rectangular \( n_L \times n_R \) random matrices \( W \) with the Gaussian distributed complex valued entries

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3.3.2. Nonperturbative calculation of bosonic replica partition function

To determine the (microscopic) spectral density from the bosonic replicas, we define the replica partition function

$$
\langle Z_n^{(-)}(\varsigma) \rangle = \langle \det^{-n}(\varsigma + i\mathcal{H}_D) \rangle_W
$$

(3.28)

(the angular brackets denote averaging with respect to the probability density Eq. (3.27)) and map it onto a bosonic field theory. In the half-plane \( \mathfrak{Re} \varsigma > 0 \), the partition function \( \langle Z_n^{(-)}(\varsigma) \rangle \) reduces to \cite{Dal01, Fyo02}

$$
Z_n^{(-)}(\omega) = \int_{S_n} d\Omega_n \det^{\nu-n}\Omega_n \exp \left[ -\frac{\omega}{2} \text{Tr}(\Omega_n + \Omega_n^{-1}) \right],
$$

(3.29)

where the integration domain \( S_n \) spans all \( n \times n \) positive definite Hermitean matrices \( \Omega_n \). Equation (3.29) was derived in the thermodynamic limit \( N \to \infty \) with the spectral parameter \( \omega = \varsigma N \Sigma \) being kept fixed (\( \mathfrak{Re} \omega > 0 \)).

Spotting the invariance of the integrand in Eq. (3.29) under the unitary rotation of the matrix \( \Omega_n \), one readily realises that \( Z_n^{(-)}(\omega) \) belongs to the class of \( \tau \) functions specified by Eq. (3.13) where \( D \) is set to \( \mathbb{R}^+ \), the potential \( V_{n-s}^{\nu}(\lambda) = (n-s-\nu) \log \lambda \),

(3.30)

and \( \Gamma(\varsigma; \lambda) \) is replaced with

$$
\Gamma(\omega; \lambda) = \exp \left[ -\frac{\omega}{2} \left( \lambda + \frac{1}{\lambda} \right) \right].
$$

(3.31)

This observation implies that the associated \( \tau \) function \( \tau_n^{(s)}(\omega; t) \) satisfies both the first KP equation (3.18) and the Virasoro constraints Eq. (3.21) with \cite{Osi07}

$$
\hat{L}_q^V(t) = \hat{L}_{q+1}(t) + (\nu - n + s) \frac{\partial}{\partial t_{q+1}},
$$

(3.32)

$$
\hat{L}_q^V(\omega; t) = -\frac{\omega}{2} \frac{\partial}{\partial t_{q+2}} - \delta_{q,-1} \left( \omega \frac{\partial}{\partial \omega} + \frac{\omega}{2} \frac{\partial}{\partial t_1} \right) + [1 - \delta_{q,-1}] \frac{\omega}{2} \frac{\partial}{\partial t_q}.
$$

(3.33)

Projecting Eq. (3.18) taken at \( s = 0 \) onto \( t = 0 \), and expressing the partial derivatives therein via the derivatives over \( \omega \) with the help of Eqs. (3.21), (3.32) and (3.33), we conclude that

$$
h_n(\omega) = \frac{\partial}{\partial \omega} \log Z_n^{(-)}(\omega)
$$

(3.34)

obeys the differential equation \cite{Osi07}

$$
h_n'' + \frac{2}{\omega} h_n' - \left( 4 + \frac{1 + 4(n^2 + \nu^2)}{\omega^2} \right) h_n' + 6(h_n')^2 + \frac{1 - 4(n^2 + \nu^2)}{\omega^3} h_n - \frac{2}{\omega^2} (h_n)^2 + \frac{4}{\omega} h_nh_n' + \frac{4n^2}{\omega^2} = 0
$$

(3.35)

that can be reduced to the Painlevé III.

Considered together with the boundary conditions \( h_n(\omega \to 0) \simeq -n\nu/\omega \) and \( h_n(\omega \to \infty) \simeq -n - n^2/(2\omega) \), following from Eq. (3.29), the nonlinear differential
equation Eq. (3.35) provides a nonperturbative characterisation of the bosonic replica partition function \( \mathcal{Z}_n^{(-)}(\omega) \) for all \( n \in \mathbb{Z}^+ \).

### 3.3.3. Implementing the replica limit

To pave the way for the replica calculation of the resolvent \( g(\omega) \) determined by the replica limit

\[
g(\omega) = -\lim_{n \to 0} \frac{1}{n} \frac{\partial}{\partial \omega} \log \mathcal{Z}_n^{(-)}(\omega) = -\lim_{n \to 0} \frac{1}{n} h_n(\omega),
\]

one has to analytically continue \( h_n(\omega) \) away from \( n \) integers. Previous studies [Kan02, Kan05] suggest that the sought analytic continuation is given by the very same Eq. (3.35) where the replica parameter \( n \) is let to explore the entire real axis. This leap makes the rest of the calculation straightforward. Representing \( h_n(\omega) \) in the vicinity of \( n = 0 \) as

\[
h_n(\omega) = \sum_{p=1}^{\infty} n^p a_p(\omega),
\]

we conclude that \( g(\omega) = -a_1(\omega) \) satisfies the equation

\[
\omega^3 g'' + 2\omega^2 g' - \left(1 + 4\nu^2 + 4\omega^2\right) \omega g + \left(1 - 4\nu^2\right) g = 0.
\]

(3.37)

Its solution, subject to the boundary conditions consistent with those specified below Eq. (3.35),

\[
g(\omega) = \frac{\nu}{\omega} + \omega \left[ I_\nu(\omega) K_\nu(\omega) + I_{\nu+1}(\omega) K_{\nu-1}(\omega) \right],
\]

(3.38)

brings the microscopic spectral density \( \varrho(\omega) = \pi^{-1} \text{Re} g(i\omega + 0) \) in the form

\[
\varrho(\omega) = \nu \delta(\omega) + \frac{\omega}{2} \left[ J^2_\nu(\omega) - J_{\nu-1}(\omega) J_{\nu+1}(\omega) \right].
\]

(3.39)

In the above formula (which is one of the celebrated RMT results originally obtained in [Ver93] within the orthogonal polynomial technique), the function \( J_\nu \) denotes the Bessel function of the first kind, whilst \( I_\nu \) and \( K_\nu \) are the modified Bessel function of the first and second kind, respectively. Let us stress that the approximate, saddle point approach to bosonic replicas [Dal01] fails to produce Eq. (3.39).

### 4. Concluding remarks

Concluding this brief excursion into integrable theory of replica field theories, we wish to mention another important development due to Splittorff and Verbaarschot [Spl03, Spl04], not reviewed here for a lack of space. These authors showed that nonperturbative results for various RMT correlation functions at \( \beta = 2 \) can be derived by taking the replica limit of the graded Toda Lattice equation whose positive \((n \in \mathbb{Z}^+)\)
and negative \((n \in \mathbb{Z}^-)\) branches describe fermionic and bosonic replica partition functions, respectively. Being a supersymmetric in nature, this approach greatly simplifies calculations of spectral correlation functions through a remarkable fermionic-bosonic factorisation. For further details, the reader is referred to the original papers [Spl03, Spl04], the lecture notes [Ver05], Chapter 32 of this Handbook, and the tutorial paper [Osi09].

Certainly, more effort is needed to accomplish integrable theory of zero-dimensional replica field theories. In particular, its extension to the \(\beta = 1\) and \(\beta = 4\) Dyson symmetry classes is very much called for.

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References

[Adl95] M. Adler, T. Shiota, and P. van Moerbeke, Random matrices, vertex operators and the Virasoro algebra, Phys. Lett. A 208, 67 (1995).

[Alt00] A. Altland and A. Kamenev, Wigner-Dyson statistics from the Keldysh σ model, Phys. Rev. Lett. 85, 5615 (2000).

[And83] C. Andréief, Note sur une relation les intégrales définies des produits des fonctions, Mém. de la Soc. Sci., Bordeaux 2, 1 (1883).

[Cla03] P. A. Clarkson, Painlevé equations – nonlinear special functions, J. Comp. Appl. Math. 153, 127 (2003).

[Dal01] D. Dalmazi and J. J. M. Verbaarschot, The replica limit of unitary matrix integrals, Nucl. Phys. B 592[FS], 419 (2001).

[Dar72] G. Darboux, Lecons sur la Theorie Generale des Surfaces et les Applications Geometriques du Calcul Infinitesimal, Vol. II: XIX (Chelsea, New York, 1972).

[Dat83] E. Date, M. Kashiwara, M. Jimbo, and T. Miwa, Transformation groups for soliton equations, in: Nonlinear Integrable Systems – Classical Theory and Quantum Theory, edited by M. Jimbo and T. Miwa (World Scientific, Singapore, 1983).

[deB55] N. G. de Bruijn, On some multiple integrals involving determinants, J. Indian Math. Soc. 19, 133 (1955).

[Dhe90] G. S. Dhesi and R. C. Jones, Asymptotic corrections to the Wigner semicircular eigenvalue spectrum of a large real symmetric random matrix using the replica method, J. Phys. A: Math. Gen. 23, 5577 (1990).

[Edw75] S. F. Edwards and P. W. Anderson, Theory of spin glasses, J. Phys. F: Met. Phys. 5, 965 (1975).

[Edw76] S. F. Edwards and R. C. Jones, The eigenvalue spectrum of a large symmetric random matrix, J. Phys. A: Math. Gen. 9, 1595 (1976).

[Edw80] S. F. Edwards and M. Warner, The effect of disorder on the spectrum of a Hermitian matrix, J. Phys. A: Math. Gen. 13, 381 (1980).

[Efe80] K. B. Efetov, A. I. Larkin, and D. E. Khmelnitskii, Interaction between diffusion modes in localization theory, Zh. Êksp. Teor. Fiz. 79, 1120 (1980) [Sov. Phys. JETP 52, 568 (1980)].

[Efe82a] K. B. Efetov, Supersymmetry method in localisation theory, Zh. Êksp. Teor. Fiz. 82, 872 (1982) [Sov. Phys. JETP 55, 514 (1982)].

[Efe82b] K. B. Efetov, Statistics of the levels in small metallic particles, Zh. Êksp. Teor. Fiz. 83, 833 (1982) [Sov. Phys. JETP 56, 467 (1982)].

[Efe83] K. B. Efetov, Supersymmetry and theory of disordered metals, Adv. Phys. 32, 53 (1983).]

[Efe97] K. B. Efetov, Supersymmetry in disorder and chaos (Cambridge University Press, Cambridge, 1997).

[Eme75] V. J. Emery, Critical properties of many-component systems, Phys. Rev. B 11, 239 (1975).

[For01] P. J. Forrester and N. S. Witte, Application of the τ-function theory of Painlevé equations to random matrices: PIV, PII and the GUE, Commun. Math. Phys. 219, 357 (2001).

[Fyo02] Y. V. Fyodorov, Negative moments of characteristic polynomials of random matrices: Ingham-Siegel integral as an alternative to Hubbard-Stratonovich transformation, Nucl. Phys. B 621 [PM], 643 (2002).

[Gan01] D. M. Gangardt and A. Kamenev, Replica treatment of the Calogero-Sutherland model, Nucl. Phys. B 610[PM], 578 (2001).

[Gan04] D. M. Gangardt, Universal correlations of trapped one-dimensional impenetrable bosons, J. Phys. A: Math. and Gen. 37, 9335 (2004).

[Guh91] T. Guhr, Dyson’s correlation functions and graded symmetry, J. Math. Phys. 32, 336 (1991).

[Har34] G. H. Hardy, J. E. Littlewood, and G. Pólya, Inequalities (Cambridge University Press, Cambridge, 1934).

[Hor90] M. Horbach and G. Schön, Dynamic nonlinear sigma model of localization theory, Physica A
[Ito97] C. Itoi, H. Mukaida, and Y. Sakamoto, Replica method for wide correlators in Gaussian orthogonal, unitary and symplectic random matrix ensembles, J. Phys. A.: Math. Gen. 30, 5709 (1997).

[Jim81] M. Jimbo and T. Miwa, Monodromy preserving deformation of linear differential equations with rational coefficients, Physica D 2, 407 (1981).

[Kam99a] A. Kamenev and A. Andreev, Electron-electron interactions in disordered metals: Keldysh formalism, Phys. Rev. B 60, 3944 (1999).

[Kam99b] A. Kamenev and M. Mézard, Wigner-Dyson statistics from the replica method, J. Phys. A: Math. and Gen. 32, 4373 (1999).

[Kam99c] A. Kamenev and M. Mézard, Level correlations in disordered metals: the replica σ-model, Phys. Rev. B 60, 3944 (1999).

[Kam09] A. Kamenev and A. Levchenko, Keldysh technique and non-linear sigma model: Basic principles and applications, Adv. Phys. 58, 197 (2009).

[Kan02] E. Kanzieper, Replica field theories, Painlevé transcendents, and exact correlation functions, Phys. Rev. Lett. 89, 250201 (2002).

[Kan05] E. Kanzieper, Exact replica treatment of non-Hermitean complex random matrices, in: O. Kovras (ed.) Frontiers in Field Theory, p. 23 (Nova Science Publishers, New York, 2005).

[Leu92] H. Leutwyler and A. Smilga, Spectrum of Dirac operator and role of winding number in QCD, Phys. Rev. D 46, 5607 (1992).

[Mac98] I. G. Macdonald, Symmetric Functions and Hall Polynomials (Oxford University Press, Oxford, 1998).

[Meh60] M. L. Mehta and M. Gaudin, On the density of eigenvalues of a random matrix, Nucl. Phys. B 18, 420 (1960).

[Meh04] M. L. Mehta, Random Matrices (Elsevier, Amsterdam, 2004).

[Mir90] A. Mironov and A. Morozov, On the origin of Virasoro constraints in matrix model: Lagrangian approach, Phys. Lett. B 252, 47 (1990).

[Mor94] A. Morozov, Integrability and matrix models, Uspekhi Fiz. Nauk 164, 3 (1994) [Physics-Uspekhi (UK) 37, 1 (1994)].

[Nis02] S. M. Nishigaki and A. Kamenev, Replica treatment of non-Hermitean disordered Hamiltonians, J. Phys. A: Math. and Gen. 35, 4571 (2002).

[Nis03] S. M. Nishigaki, D. M. Gangardt, and A. Kamenev, Correlation functions of the BC Calogero-Sutherland model, J. Phys. A: Math. and Gen. 36, 3137 (2003).

[Nou04] M. Noumi, Painlevé Equations through Symmetry (AMS, Providence, 2004).

[Oka86] K. Okamoto, Studies on the Painlevé equations, III. Second and fourth Painlevé equations, PII and PIV, Math. Ann. 275, 221 (1986).

[Osi07] V. Al. Osipov and E. Kanzieper, Are bosonic replicas faulty? Phys. Rev. Lett. 99, 050602 (2007).

[Osi09] V. Al. Osipov and E. Kanzieper, Correlations of RMT characteristic polynomials and integrability: I. Hermitean matrices, in preparation (2009).

[Par03] G. Parisi, Two spaces looking for a geometer, Bull. Symbolic Logic 9, 181 (2003).

[Sch80] L. Schäfer and F. Wegner, Disordered system with n orbitals per site: Lagrange formulation, hyperbolic symmetry, and Goldstone modes, Z. Phys. B 38, 113 (1980).

[Smi95] A. Smilga and J. J. M. Verbaarschot, Spectral sum rules and finite volume partition function in gauge theories with real and pseudoreal fermions, Phys. Rev. D 51, 829 (1995).

[Spl03] K. Splittorff and J. J. M. Verbaarschot, Replica limit of the Toda lattice equation, Phys. Rev. Lett. 90, 041601 (2003).

[Spl04] K. Splittorff and J. J. M. Verbaarschot, Factorization of correlation functions and the replica limit of the Toda lattice equation, Nucl. Phys. B 683 [FS], 467 (2004).

[Tit32] E. C. Titchmarsh, The theory of functions (Oxford University Press, Oxford, 1932).

[Tod67] M. Toda, Vibration of a chain with nonlinear interaction, J. Phys. Soc. Japan 22, 431 (1967).
[Tu96] M. H. Tu, J. C. Shaw, and H. C. Yen, A note on integrability in matrix models, Chinese J. Phys. 34, 1211 (1996).

[Ver84] J. J. M. Verbaarschot and M. R. Zirnbauer, Replica variables, loop expansion, and spectral rigidity of random matrix ensembles, Ann. Phys. (N. Y.) 158, 78 (1984).

[Ver85a] J. J. M. Verbaarschot, H. A. Weidenmüller, and M. R. Zirnbauer, Grassmann integration in stochastic quantum mechanics: The case of compound-nucleus scattering, Phys. Rep. 129, 367 (1985).

[Ver85b] J. J. M. Verbaarschot and M. R. Zirnbauer, Critique of the replica trick, J. Phys. A: Math. and Gen. 17, 1093 (1985).

[Ver93] J. J. M. Verbaarschot and I. Zahed, Spectral density of the QCD Dirac operator near zero virtuality, Phys. Rev. Lett. 70, 3852 (1993).

[Ver05] J. J. M. Verbaarschot, The supersymmetric method in random matrix theory and applications to QCD, AIP Conf. Proc. 744, 277 (2005).

[Weg79] F. Wegner, The mobility edge problem: Continuous symmetry and a conjecture, Z. Phys. B 35, 207 (1979).

[Yur99] I. V. Yurkevich and I. V. Lerner, Nonperturbative results for level correlations from the replica nonlinear $\sigma$ model, Phys. Rev. B 60, 3955 (1999).

[Zir99] M. R. Zirnbauer, Another critique of the replica trick, arXiv: cond-mat/9903338 (1999).