Recovery of the Dirac system from the rectangular Weyl matrix function

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Received 16 June 2011, in final form 26 November 2011
Published 28 December 2011
Online at stacks.iop.org/IP/28/015010

Abstract

Weyl theory for Dirac systems with rectangular matrix potentials is non-classical. The corresponding Weyl functions are rectangular matrix functions. Furthermore, they are non-expansive in the upper semi-plane. Inverse problems are studied for such Weyl functions, and some results are new even for the square Weyl functions. High-energy asymptotics of Weyl functions and Borg–Marchenko-type uniqueness results are derived too.

1. Introduction

The self-adjoint Dirac-type (also called Dirac, ZS or AKNS) system

\[ \frac{d}{dx} y(x, z) = i(zj + JV(x))y(x, z) \quad (x \geq 0), \]  

(1.1)

where

\[ j = \begin{bmatrix} I_{m_1} & 0 \\ 0 & -I_{m_2} \end{bmatrix}, \quad V = \begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix}, \]  

(1.2)

\( I_{m_i} \) is the \( m_i \times m_i \) identity matrix and \( v(x) \) is an \( m_1 \times m_2 \) matrix function, is a classical matrix differential equation. We always assume that \( v \) is measurable and, moreover, locally summable, that is, summable on finite intervals \([0, l] \).

Dirac-type systems are very well known in mathematics and applications, especially in mathematical physics (see, e.g., books [7, 8, 26, 28, 43], recent publications [4–6, 9, 10, 16, 17, 44] and numerous references therein). System (1.1) is used, in particular, in the study of transmission lines and acoustic problems [48]. The most interesting applications are, however, caused by the fact that system (1.1) is an auxiliary linear system for many important nonlinear integrable wave equations and as such it was studied, for instance, in [1–3, 13, 18, 23, 24, 36, 49]. (The name ZS-AKNS system is connected with these applications.) Nonlinear Schrödinger equations, modified Korteweg–de Vries equations and second harmonic generation model, which describe various wave processes (including, e.g., photoconductivity and nonlinear wave processes in water, waveguides, nonlinear optics and on silicon surfaces), are only some of the
well-known examples. The evolution of the Weyl function for these equations is described in terms of Möbius transformations (see, e.g., [32, 36, 37, 41–43] and references therein), which is one of the fruitful approaches to study the corresponding initial-boundary value problems.

The Weyl and spectral theory of the self-adjoint Dirac systems, where \( m_1 = m_2 \), was dealt with, for instance, in [6, 9, 25, 28, 35, 43] (see also various references therein). The ‘non-classical’ Weyl theory for the equally important case \( m_1 \neq m_2 \), which appears in the study of coupled, multicomponent and matrix nonlinear equations, is the subject of this paper.

The \( m_1 \times m_2 \) matrix function \( v(x) \) from (1.2) is called the potential of system (1.1). We put \( m_1 + m_2 =: m \). The fundamental solution of system (1.1) is denoted by \( u(x, z) \), and this solution is normalized by the condition

\[
u(0, z) = I_m.
\]

Section 2 is dedicated to representation of the fundamental solution. In section 3, we follow [14] to introduce the Weyl function and study the high-energy asymptotics of this Weyl function. The solution of the inverse problem to recover the potential (and system) from the Weyl function is given in section 4. Borg–Marchenko-type uniqueness results are contained in that section as well. Finally, section 5 is dedicated to conditions for an analytic matrix function to be the Weyl function of some Dirac-type system. The results from this section were not published before even for the case that \( m_1 = m_2 \), though some of them appeared for that case in the thesis [34].

As usual, \( \mathbb{C} \) stands for the complex plane, \( \mathbb{C}_+ \) for the open upper semi-plane and \( \text{Im} \) stands for image. If \( a \in \mathbb{C} \), then \( \overline{a} \) is its complex conjugate. The notation \( L_{m_1 \times m_2}^2((0, \infty)) \) will be used to denote the space of \( m_2 \times m_1 \) matrix functions with entries belonging to \( L^2((0, \infty)) \). We use \( \overline{H} \) to denote the closure of the space \( H, I \) to denote the identity operator and \( B(H_1, H_2) \) to denote the class of bounded operators acting from \( H_1 \) to \( H_2 \). We write \( B(H_1) \) if \( H_1 = H_2 \). The notation \( B([0, l]) \) will be used to denote the class of functions (or matrix functions), whose derivatives are bounded on \([0, l]\). An \( m_2 \times m_1 \) matrix \( \alpha \) is said to be non-expansive if \( \alpha^* \alpha \leq I_{m_1} \) (or, equivalently, if \( \alpha \alpha^* \leq I_{m_2} \)).

2. Representation of the fundamental solution

The results of this section can be formulated for the Dirac system on a fixed final interval \([0, l]\). We assume that \( v \) is bounded on this interval and put

\[
\beta(x) = \begin{bmatrix} I_{m_2} & 0u(x, 0) \end{bmatrix}, \quad \gamma(x) = \begin{bmatrix} 0 & I_{m_2} \end{bmatrix}u(x, 0).
\]

It follows from \( \sup_{x < l} \|v'(x)\| < \infty \) and from (1.1) that

\[
\sup_{x < l} \|\gamma'(x)\| < \infty, \quad \gamma' := \frac{d}{dx}\gamma.
\]

Moreover, from (1.1)–(1.3), we obtain \( u(x, 0)^*ju(x, 0) = j = u(x, 0)ju(x, 0)^* \). Therefore, (2.1) implies

\[
\beta j\beta^* = I_{m_1}, \quad \gamma j\gamma^* = -I_{m_2}, \quad \beta j\gamma^* = 0.
\]

Next, we need the following similarity result for the Volterra operator:

\[
K = \int_0^x F(t)G(t) \cdot dt, \quad K \in B(L_{m_2}^2([0, l])),
\]

where \( F(x) \) is an \( m_2 \times m \) matrix function, \( G(t) \) is an \( m \times m_2 \) matrix function and

\[
iF(x)G(x) \equiv I_{m_2}.
\]
Proposition 2.1 ([38]). Let $F$ and $G$ be boundedly differentiable and let (2.5) hold. Then, we have

$$K = EAE^{-1}, \quad A := -i \int_0^x \cdot dt, \quad A, E, E^{-1} \in B(L^2_{m_2}(0, l)), \quad (2.6)$$

where $K$ is given by (2.4) and $E$ is a triangular operator of the form

$$(Ef)(x) = \rho(x)f(x) + \int_0^x E(x, t)f(t) \, dt, \quad \frac{d}{dx} \rho = iF'G\rho, \quad \det \rho(0) \neq 0. \quad (2.7)$$

Moreover, the operators $E \pm 1$ map functions with bounded derivatives into functions with bounded derivatives.

The proposition above is a particular case of theorem 1 [38] (see also the later paper [5] for the case that $F$ and $G$ are continuously differentiable).

Set

$$F = \gamma, \quad G = i\gamma^*, \quad \gamma jy^* \equiv -l_{m_2}, \quad \gamma \in B^1[0, l], \quad (2.8)$$

where $B^1$ stands for the class of boundedly differentiable matrix functions. Clearly, $F$ and $G$ in (2.8) satisfy conditions of proposition 2.1. We partition $\gamma(x)$ into two blocks $\gamma = [\gamma_1, \gamma_2]$, where $\gamma_1$ and $\gamma_2$, respectively, are the $m_2 \times m_1$ and $m_2 \times m_2$ matrix functions. Let us show that without loss of generality one can choose such an operator $E$ that

$$E^{-1}\gamma_2 \equiv l_{m_2}, \quad (2.9)$$

where $E^{-1}$ is applied to $\gamma_2$ columnwise.

Proposition 2.2. Let $K$ be given by (2.4), where $F$ and $G$ satisfy (2.8), and let $\tilde{E}$ be a similarity operator from proposition 2.1. Introduce $E_0 \in B(L^2_{m_2}(0, l))$ by the equalities

$$(E_0f)(x) = \rho(0)^{-1}\gamma_2(0)f(x) + \int_0^x E_0(x, t)f(t) \, dt, \quad E_0(x) := (\tilde{E}^{-1}\gamma_2)'(x). \quad (2.10)$$

Then, the operator $E := \tilde{E}E_0$ is another similarity operator from proposition 2.1, which satisfies the additional condition (2.9).

Proof. The proof of the proposition is similar to the case $m_1 = m_2$ (see, e.g., [38, pp 103, 104]). Indeed, the next identity can easily be shown directly (and follows also from the fact that $E_0$ is a lower triangular convolution operator):

$$AE_0 = E_0A \quad (A = -i \int_0^t \cdot \, dt). \quad (2.11)$$

Furthermore, because of the third relation in (2.8), we have $\det \gamma_2(0) \neq 0$, and so $E_0$ is invertible. Hence, equalities (2.11) and $K = \tilde{E}AE^{-1}$ imply (2.6):

$$K = EAE^{-1}, \quad E := \tilde{E}E_0. \quad (2.12)$$

Formula (2.10) also leads us to the equality

$$(E_0l_{m_2})(x) = \rho(0)^{-1}\gamma_2(0) + \int_0^x E_0(x-t) \, dt = \rho(0)^{-1}\gamma_2(0) + \int_0^x E_0(t) \, dt$$

$$= \rho(0)^{-1}\gamma_2(0) + (\tilde{E}^{-1}\gamma_2)(x) - (\tilde{E}^{-1}\gamma_2)(0). \quad (2.13)$$

Recalling that $\tilde{E}^{-1}\gamma_2 \in B^1[0, l]$, we obtain the representation

$$(\tilde{E}^{-1}\gamma_2)(x) = \rho(0)^{-1}\gamma_2(0) + i(A(\tilde{E}^{-1}\gamma_2)')(x). \quad (2.14)$$

Using (2.14) we rewrite (2.13) as

$$(E_0l_{m_2})(x) = (\tilde{E}^{-1}\gamma_2)(x), \quad (2.15)$$

and (2.9) follows. It remains to show that $E_0^{\pm 1}$ maps $B^1[0, l]$ into $B^1[0, l]$.
First note that the integral operators $E_0$, $E_0^{-1}$ have bounded kernels and map bounded functions into bounded. In particular, for $E_0^{-1}$ it follows from a series representation of the operator of the form $(I + \int_0^1 k(x, t)\, dt)^{-1}$ (where the kernel $k$ is bounded).

Now, taking into account that (similar to (2.14)) any $f \in B^1[0, l]$ admits representation $f = f(0) + iAf^*$, we see that formulas (2.11) and (2.15) yield the fact that $E_0$ maps $B^1[0, l]$ into $B^1[0, l]$. The identity $AE_0^{-1} = E_0^{-1}A$ is immediate from (2.11). Thus, to prove that $E_0^{-1}$ maps $B^1[0, l]$ into $B^1[0, l]$ we need only to show that $E_0^{-1}I_{m_2} \in B^1[0, l]$, which relation can be derived from (2.14) and (2.15):

$$E_0^{-1}I_{m_2} = (I_{m_2} - iE_0^{-1}A(\tilde{E}^{-1}\gamma(2)^*))\gamma(2)^{-1} \rho(0) \in B^1[0, l].$$

\[\square\]

Remark 2.3. The kernels of the operators $\tilde{E}^{\pm 1}$, which are constructed in [38], as well as the kernels of the operators $E_0^{\pm 1}$ from the proof of proposition 2.2 are bounded. Therefore, without loss of generality we always assume further that the kernels of $E_0^{\pm 1}$ are bounded.

The next lemma easily follows from proposition 2.2 and will be used to construct the fundamental solution.

Lemma 2.4. Let $\gamma$ be an $m_2 \times m$ matrix function, which satisfies the last two relations in (2.8), and set

$$S := E^{-1}(E^{*})^{-1}, \quad \Pi := [\Phi_1 \quad \Phi_2], \quad \Phi_k \in B(C^{m_k}, L^2_{m_k}(0, l));$$

$$\Phi_1 f(x) = \Phi_1(x)f, \quad \Phi_2 f(x) := (E^{-1}(\gamma_1 f(x)), \quad \Phi_2 f(x) = I_{m_2}f \equiv f.$$

(\[2.16\])

where $E$ is constructed (for the given $\gamma$) in proposition 2.2. Then $A$, $S$ and $\Pi$ form an $S$-node, that is (see [39, 40, 43]), the operator identity

$$AS - SA^* = i\Pi j\Pi^*$$

(\[2.18\])

holds. Furthermore, we have

$$\sum_{i=0}^{\infty} \text{Im} \left((A^*)^i S^{-1} \Pi \right) = I^2_{m_2}(0, l).$$

(\[2.19\])

Proof. Because of (2.4), (2.6) and (2.8), we obtain

$$EA E^{-1} - (E^{-1})^* A^* E^* = K - K^* = i\gamma(x) \int_0^1 \gamma(t)^* \, dt.$$ 

(\[2.20\])

Formulas (2.9), (2.16) and (2.17) lead us to the equality

$$\Pi f = (E^{-1}\gamma f(x)).$$

(\[2.21\])

Now, the operator identity (2.18) follows from (2.20), (2.21) and the first equality in (2.16).

To prove (2.19), we will show that

$$\sum_{i=0}^{N} \text{Im} \left((A^*)^i S^{-1} \Pi \right) \supset \sum_{i=0}^{N} \text{Im} \left(S^{-1} A \Pi \right) = S^{-1} \sum_{i=0}^{N} \text{Im} \left(A \Pi \right).$$

(\[2.22\])

For that purpose, we rewrite (2.18) as $S^{-1}A = A^*S^{-1} + iS^{-1} \Pi j\Pi^* S^{-1}$. Hence, for $N_1, N_2 \geq 0$, we obtain

$$\text{Im} \left((A^*)^{N_1} S^{-1} A^{N_2} \Pi \right) + \sum_{i=0}^{N_1+N_2} \text{Im} \left((A^*)^i S^{-1} \Pi \right) \supset \text{Im} \left((A^*)^{N_1} S^{-1} A^{N_2} \Pi \right).$$

(\[2.23\])
Using (2.23), we derive (2.22) by induction. In view of (2.22), it suffices to show that
\[
\sum_{i=0}^{\infty} \text{Im} \left( A^i \Pi \right) = L_{m_2}^2 (0, 1), \tag{2.24}
\]
which, in turn, follows from (2.17). \qed

**Remark 2.5.** Given an \( S \)-node (2.18), we introduce the transfer matrix functions in the Lev Sakhnovich form (see [39, 40, 43]):
\[
w_A (r, z) := I_m + iz \pi S_r^{-1} (I - z A_r)^{-1} P_r \pi, \quad 0 < r \leq l, \tag{2.25}
\]
where \( I \) is the identity operator; \( A_r, S_r \in B(L_{m_2}^2 (0, r)) \),
\[
A_r := P_r A^* P_r, \quad S_r := P_r S^* P_r, \tag{2.26}
\]
\( A \) is given by (2.6), the operators \( S \) and \( \pi \) are given by (2.16) and (2.17) and the operator \( P_r \) is an orthoprojector from \( L_{m_2}^2 (0, l) \) on \( L_{m_2}^2 (0, r) \) such that
\[
(P_r f)(x) = f(x) \quad (0 < x < r), \quad f \in L_{m_2}^2 (0, l). \tag{2.27}
\]
Since \( P_r A = P_r A^* P_r \), it follows from (2.18) that the operators \( A_r, S_r \) and \( P_r \pi \) form an \( S \)-node too, that is, the operator identities
\[
A_r S_r - S_r A_r^* = i P_r \pi \gamma^* P_r^* \tag{2.28}
\]
hold.

Now, in a way similar to [31, 32], the fundamental solution \( w \) of the system
\[
\frac{d}{dx} w(x, z) = iz \gamma^* \gamma (x) w(x, z), \quad w(0, z) = I_m, \tag{2.29}
\]
is constructed.

**Theorem 2.6.** Let \( \gamma \) be an \( m_2 \times m \) matrix function, which satisfies the last two relations in (2.8). Then, the fundamental solution \( w \) given by (2.29) admits the representation
\[
w(r, z) = w_A (r, z), \tag{2.30}
\]
where \( w_A (r, z) \) is defined in remark 2.5 (see (2.25)).

**Proof.** The statement of the theorem follows from the continual factorization theorem (see [43, p 40]). More precisely, our statement follows from a corollary of the continual factorization theorem, namely from theorem 1.2 [43, p 42]. Using lemma 2.4, we easily check that the conditions of theorem 1.2 [43, p 42] are fulfilled. Therefore, if \( \pi S_r^{-1} P_r \pi \) is boundedly differentiable, we have
\[
\frac{d}{dr} w_A (r, z) = iz \gamma^* \gamma (r) w_A (r, z), \quad \lim_{r \to 0^+} w_A (r, z) = I_m, \tag{2.31}
\]
\[
H(r) := \frac{d}{dr} (\pi S_r^{-1} P_r \pi), \tag{2.32}
\]
where \( w_A \) is given by (2.25). Since \( E^{\pm 1} \) are lower triangular operators, we see that
\[
P_r E^* P_r = P_r E, \quad (E^{-1})^* P_r = P_r (E^{-1})^* P_r. \tag{2.33}
\]
Hence, formulas (2.16) and (2.26) lead us to
\[
S_r^{-1} = E^*_r E_r, \quad E_r := P_r E^* P_r. \tag{2.34}
\]
Therefore, taking into account (2.21), we rewrite (2.32) as
\[ H(r) = \gamma'(r)^* \gamma(r). \] (2.35)
Formulas (2.29), (2.31) and (2.35) imply (2.30).

Now, consider again the case of the Dirac system. Because of (2.1) and (2.3), we obtain
\[ u(x, 0) j \gamma(x)^* \gamma(x) u(x, 0)^{-1} = - \begin{bmatrix} 0 & 0 \\ 0 & I_{m_2} \end{bmatrix}. \] (2.36)
Hence, direct calculation shows that the following corollary of theorem 2.6 is true.

**Corollary 2.7.** Let \( u(x, z) \) be the fundamental solution of a Dirac system with the bounded potential \( v \) and let \( \gamma \) be given by (2.1). Then, \( u(x, z) \) admits the representation
\[ u(x, z) = \alpha^{(x)} u(x, 0) w(x, 2z), \] (2.37)
where \( w \) has the form (2.30) and the S-node generating the transfer matrix function \( u_A \) is recovered from \( \gamma \) in lemma 2.4.

**Remark 2.8.** For the case that \( \gamma \) is given by (2.1), it follows from (1.1) and (2.3) that
\[ \gamma'(x) \gamma(x)^* = -i[v(x)^* 0] u(x, 0) j \gamma(x)^* = -i[v(x)^* \beta(x) j \gamma(x)^* \equiv 0. \] (2.38)
Thus, from (2.8) and (2.38) we see that \( \rho \) in (2.7) is a constant matrix. Therefore, since \( \gamma_2(0) = I_{m_2} \), equality (2.21) implies that \( \rho(x) = I_{m_2} \), and formula (2.7) can be rewritten in the form
\[ (Ef)(x) = f(x) + \int_0^x E(x, t) f(t) \, dt, \quad E \in B(L^2_{m_2} (0, 1)). \] (2.39)

Recalling that \( \det \gamma_2(x) \neq 0 \), we can rewrite (2.38) as
\[ \gamma_2' = \gamma_2'( \gamma_2^{-1} \gamma_1)^*. \] (2.40)
Using (2.40), we recover \( \gamma_2 \) and \( \gamma_1 = \gamma_2(\gamma_2^{-1} \gamma_1) \) from \( \gamma_2^{-1} \gamma_1 \). Indeed, we have
\[ (\gamma_2^{-1} \gamma_1)' = -\gamma_2^{-1} \gamma_2^{-1} \gamma_1 + \gamma_2^{-1} \gamma_1', \quad \text{i.e.,} \quad \gamma_1' = \gamma_2(\gamma_2^{-1} \gamma_1) + \gamma_2(\gamma_2^{-1} \gamma_1)'. \]
Because of (2.1), (2.40) and the formula above, we obtain
\[ \gamma_2'(I_{m_2} - (\gamma_2^{-1} \gamma_1)(\gamma_2^{-1} \gamma_1)^*) = \gamma_2(\gamma_2^{-1} \gamma_1)'(\gamma_2^{-1} \gamma_1)^*, \quad \gamma_2(0) = I_{m_2}. \]
Therefore, taking into account that \( I_{m_2} - (\gamma_2^{-1} \gamma_1)(\gamma_2^{-1} \gamma_1)^* \) is invertible, we obtain
\[ \gamma_2 = \gamma_2(\gamma_2^{-1} \gamma_1)'(\gamma_2^{-1} \gamma_1)^* (I_{m_2} - (\gamma_2^{-1} \gamma_1)(\gamma_2^{-1} \gamma_1)^*)^{-1}, \quad \gamma_2(0) = I_{m_2}. \] (2.41)

**Remark 2.9.** If the conditions of remark 2.8 hold, then \( \gamma_2, \gamma \) and Hamiltonian \( H = \gamma^* \gamma \) are consecutively recovered from \( \gamma_2^{-1} \gamma_1 \) via the differential equation (2.41). Furthermore, from the third relation in (2.8) we see that \( (\gamma_2^{-1} \gamma_1)(\gamma_2^{-1} \gamma_1)^* \leq I_{m_2} \). Thus, the matrix function \( \gamma_2^{-1} \gamma_1 \) is a continuous analogue of Schur coefficients for the canonical system (2.29). (Compare this with remark 3.1 from the paper [17] on the canonical systems.)
3. Weyl function: high-energy asymptotics

To define Weyl functions, we introduce the class of nonsingular \( m \times m \) matrix functions \( P(z) \) with property- \( j \), which are an immediate analogue of the classical pairs of parameter matrix functions. Namely, the matrix functions \( P(z) \) are meromorphic in \( \mathbb{C}_+ \) and satisfy (excluding, possibly, a discrete set of points) the following relations:

\[
\begin{align*}
P(z)^*P(z) & > 0, \quad P(z)^*jP(z) \geq 0 \quad (z \in \mathbb{C}_+). \\
\end{align*}
\]

(3.1)

It is immediate from (1.1) that

\[
\frac{d}{dx} (u(x,z)^*ju(x,z)) = i(z - \sigma)u(x,z)^*u(x,z) < 0, \quad z \in \mathbb{C}_+. 
\]

(3.2)

Relations (3.1) and (3.2) imply

\[
\det([I_{m_1} 0]u(x,z)^{-1}P(z)) \neq 0. 
\]

(3.3)

**Definition 3.1.** The set \( \mathcal{N}(x,z) \) of Möbius transformations is the set of values at \( x \) and \( z \) of matrix functions

\[
\varphi(x,z,P) = [0 \quad I_{m_1}u(x,z)^{-1}P(z) ([I_{m_1} 0]u(x,z)^{-1}P(z))^{-1},
\]

where \( P(z) \) are nonsingular matrix functions with property- \( j \).

We can rewrite (3.4) in an equivalent form, which will be used later on,

\[
\left[
\begin{array}{c}
I_{m_1} \\
\varphi(x,z,P)
\end{array}
\right] = u(x,z)^{-1}P(z) ([I_{m_1} 0]u(x,z)^{-1}P(z))^{-1}. 
\]

(3.5)

**Proposition 3.2** ([14]). Let the Dirac system \((1.1)\) on \([0, \infty)\) be given and assume that \( v \) is locally summable. Then the sets \( \mathcal{N}(x,z) \) are well defined. There is a unique matrix function \( \varphi(z) \) in \( \mathbb{C}_+ \) such that

\[
\varphi(z) = \bigcap_{x < \infty} \mathcal{N}(x,z). 
\]

(3.6)

This function is analytic and non-expansive.

In view of proposition 3.2, we define the Weyl function of the Dirac system similar to the canonical system case [43].

**Definition 3.3** ([14]). The Weyl–Titchmarsh (or simply Weyl) function of the Dirac system \((1.1)\) on \([0, \infty)\), where potential \( v \) is locally summable, is the function \( \varphi \) given by (3.6).

By proposition 3.2 the Weyl–Titchmarsh function always exists. Clearly, it is unique.

**Corollary 3.4** ([14]). Let the conditions of proposition 3.2 hold. Then the Weyl function is the unique function, which satisfies the inequality

\[
\int_0^\infty [I_{m_1} \varphi(z)^*u(x,z)^*u(x,z) \left[ I_{m_1} \varphi(z) \right] ] dx < \infty. 
\]

(3.7)

**Remark 3.5.** In view of corollary 3.4, inequality (3.7) can be used as an equivalent definition of the Weyl function. The definition of the form (3.7) is a more classical one and deals with the solutions of \((1.1)\) which belong to \( L^2(0, \infty) \). Compare this with the definitions of Weyl–Titchmarsh or \( M \)-functions for discrete and continuous systems in [10, 28, 29, 32, 33, 43, 46, 47] (see also references therein).
Important works by Gesztesy and Simon [19, 20, 45] gave rise to a whole series of interesting papers and results on the high-energy asymptotics of Weyl functions and local Borg–Marchenko-type uniqueness theorems (see, e.g., [9, 11, 12, 17, 27, 35] and references therein). Here, we generalize the high-energy asymptotics result from [35] for the case that the Dirac system (1.1) has a rectangular $m_1 \times m_2$ potential $v$, where $m_1$ is not necessarily equal to $m_2$. For that, we recall first that $S > 0$ and $\Phi_1$ is boundedly differentiable in lemma 2.4. Furthermore, recalling that $\gamma_1 (0) = 0$ and taking into account (2.17) and remark 2.3, we have
\[ \Phi_1 (+0) = 0. \] (3.8)
Therefore, using theorem 2.5 from [15] we get the statement below.

**Theorem 3.6.** Let $\Pi = [\Phi_1 \quad \Phi_2]$ be constructed in lemma 2.4. Then there is a unique solution
\[ S \in B (L^2_{m_1} (0, l)) \] of the operator identity (2.18); this $S$ is strictly positive (i.e. $S > 0$) and is defined by the equalities

\[ (Sf)(x) = f(x) - \int_0^l s(x, t) f(t) \, dt, \quad s(x, t) := \int_0^{\min (x, l)} \Phi_1' (x - \xi) \Phi_1' (t - \xi)^* d\xi. \] (3.9)

Continuous operator kernels of the form above (with a possible jump at $x = t$) were considered in section 2.4 from [21], where they were called ‘close to displacement kernels’ (see also references therein).

Now, we will apply the $S$-node scheme, which was used in [32] for the skew-self-adjoint case, to derive the high-energy asymptotics of the Weyl function of the Dirac system with a rectangular potential.

**Theorem 3.7.** Assume that $\varphi \in \mathcal{N}(l, z)$, where $\mathcal{N}(l, z)$ is defined in definition 3.1 and the potential $v$ of the corresponding Dirac system (1.1) is bounded on $[0, l]$. Then (uniformly with respect to Re(z)) we have
\[ \varphi (z) = 2iz \int_0^l e^{2izc} \Phi_1 (x) \, dx + O (2z e^{2iz} / \sqrt{\Im(z)}), \quad \Im(z) \to \infty. \] (3.10)

**Proof.** To prove the theorem, we consider the matrix function
\[ U(z) = [I_{m_1} \quad \varphi (z)^* (j - w_A (l, 2z)^* j w_A (l, 2z)) ] [I_{m_1} \varphi (z) ] \] (3.11)
Because of (2.30), (2.37) and (3.11), we have

\[ U(z) = I_{m_1} - \varphi (z)^* \varphi (z) - e^{i(lz - \xi)} [I_{m_1} \varphi (z)^* ] [I_{m_1} \varphi (z) ] \] (3.12)
Taking into account (3.5), we rewrite the formula above as
\[ U(z) = I_{m_1} - \varphi (z)^* \varphi (z) - e^{i(lz - \xi)} \times ([I_{m_1} 0] u(l, z)^{-1} P(z)^{-1})^{-1} \varphi (z)^* j P(z) ([I_{m_1} 0] u(l, z)^{-1} P(z)^{-1})^{-1}. \] (3.12)
From (3.1) and (3.12), we see that
\[ U(z) \leq I_{m_1}. \] (3.13)
It easily follows from (2.18) and (2.25) (see, e.g., [17, 40]) that
\[ w_A (l, z)^* j w_A (l, z) = j + i (z - \bar{z}) \Pi^* (I - \bar{z} A^*)^{-1} \Pi (I - z A)^{-1} \Pi. \] (3.14)
Now, formulas (3.11), (3.13) and (3.14) imply
\[ 2i(z - \eta)[I_{m_1}, \varphi(z)^*]\Pi^*(I - 2\zeta A)^{-1} S^{-1}_1(I - 2\zeta A)^{-1} \Pi \left[ I_{m_1} \varphi(z) \right] \leq I_{m_1}. \tag{3.15} \]

Since \( S \) is positive and bounded, inequality (3.15) yields
\[ \left\| (I - 2\zeta A)^{-1} \Pi \left[ I_{m_1} \varphi(z) \right] \right\| \leq \frac{C}{\sqrt{\text{Im} z}} \quad \text{for some} \quad C > 0. \tag{3.16} \]

After applying \(-i\Phi_2^*\) to the operator on the left-hand side of (3.16), we derive
\[ -i\Phi_2^*(I - 2\zeta A)^{-1}\Phi_2\varphi(z) = i\Phi_2^*(I - 2\zeta A)^{-1}\Phi_1 + O\left(\frac{1}{\sqrt{\text{Im}(z)}}\right). \tag{3.17} \]

We easily check directly (see also these formulas in the works on the case \( m_1 = m_2 \)) that
\[ \Phi_2^*(I - 2\zeta A)^{-1} f = \int_0^1 e^{2i(x - l)z} f(x) \, dx, \quad \Phi_2^*(I - 2\zeta A)^{-1} \Phi_2 = \frac{i}{2\zeta}(e^{-2iz} - 1)I_{m_2}. \tag{3.18} \]

Because of (3.17) and (3.18), we obtain
\[ \frac{1}{2\zeta}(e^{-2iz} - 1)\varphi(z) = ie^{-2iz}\int_0^1 e^{2iz}\Phi_1(x) \, dx + O\left(\frac{1}{\sqrt{\text{Im}(z)}}\right). \tag{3.19} \]

Since \( \varphi \) is non-expansive, we see from (3.19) that (3.10) holds.

Now, consider a potential \( v \), which is locally bounded, that is, bounded on all finite intervals \([0, l]\). The following integral representation is essential in interpolation and inverse problems.

**Corollary 3.8.** Let \( \varphi \) be the Weyl function of the Dirac system (1.1) on \([0, \infty)\), where the potential \( v \) is locally bounded. Then, we have
\[ \varphi(z) = 2iz \int_0^\infty e^{2iz}\Phi_1(x) \, dx, \quad \text{Im}(z) > 0. \tag{3.20} \]

**Proof.** Since \( \varphi \) is analytic and non-expansive in \( \mathbb{C}_+ \), for any \( \varepsilon > 0 \) it admits (see, e.g., [30, theorem VI]) a representation
\[ \varphi(z) = 2iz \int_0^\infty e^{2iz}\Phi(x) \, dx, \quad \text{Im}(z) > \varepsilon > 0, \tag{3.21} \]
where \( e^{-2iz}\Phi(x) \in L^2_{m_2 \times m_1}(0, \infty) \). Because of (3.10) and (3.21), we obtain
\[ \tilde{Q}(z) := \int_0^l e^{2i(x - l)z}\left(\Phi_1(x) - \Phi(x)\right) \, dx \]
\[ = \int_l^\infty e^{2i(x - l)z}\Phi(x) \, dx + O(1/\sqrt{\text{Im}(z)}). \tag{3.22} \]

From (3.22), we see that \( \tilde{Q}(z) \) is bounded in some semi-plane \( \text{Im}(z) \geq \eta_0 > 0 \). Clearly, \( \tilde{Q}(z) \) is also bounded in the semi-plane \( \text{Im}(z) < \eta_0 \). Since \( \tilde{Q} \) is analytic and bounded in \( \mathbb{C} \) and tends to zero on some rays, we have
\[ \tilde{Q}(z) = \int_0^l e^{2i(x - l)z}\left(\Phi_1(x) - \Phi(x)\right) \, dx \equiv 0. \tag{3.23} \]
It follows from (3.23) that \( \Phi_1(x) \equiv \Phi(x) \) on all finite intervals \([0, l]\). Hence, (3.21) implies (3.20). \( \square \)
Remark 3.9. Since \( \Phi_1 \equiv 0 \), we obtain that \( \Phi_1(x) \) does not depend on \( l \) for \( l > x \). Compare this with the proof of proposition 4.1 in [17], where the fact that \( E(x,t) \) (and so \( \Phi_1 \)) does not depend on \( l \) follows from the uniqueness of the factorizations of operators \( S^e_j \). See also section 3 in [5] on the uniqueness of the accelerant. Furthermore, since \( \Phi_1 \equiv 0 \) we recover the proof of corollary 3.8 also implies that \( e^{-i\xi}\Phi_1(x) \in L^2_{m,x,m_1}(0,\infty) \) for any \( \varepsilon > 0 \).

Remark 3.10. From (3.20) we see that \( \Phi'_1 \) is an analogue (for the case of the Dirac system) of \( A \)-amplitude, which was studied in [20, 45]. On the other hand, \( \Phi'_1 \) is closely related to the so-called accelerant, which appears for the case that \( m_1 = m_2 \) in papers by Krein (see, e.g., [5, 25, 26]). See also remark 2.3 in [17], where \( \Phi'_1 \) is discussed again for the case that \( m_1 = m_2 \).

4. Inverse problem and Borg–Marchenko-type uniqueness theorem

Taking into account Plancherel theorem and remark 3.9, we apply the inverse Fourier transform to formula (3.20) and obtain

\[
\Phi_1(x) = \frac{1}{\pi} e^{x\xi} \lim_{m \to \infty} \int_{-\infty}^{\infty} e^{-i\xi} \frac{\varphi(\xi + i\eta)}{2(\xi + i\eta)} \, d\xi, \quad \eta > 0. \tag{4.1}
\]

Here, \( \lim \) stands for the entrywise limit in the norm of \( L^2(0,\infty) \). (Note that if we put additionally \( \Phi_1(x) = 0 \) for \( x < 0 \), equality (4.1) holds for \( \lim \) as the entrywise limit in \( L^2(-r,r) \).) Thus, operators \( S \) and \( \Pi \) are recovered from \( \varphi \).

Since Hamiltonian \( H = \gamma^* \gamma \) is recovered from \( S \) and \( \Pi \) via formula (3.22), we also recover \( \gamma \). Indeed, first we recover the Schur coefficient (see remark 2.9 for the motivation of the term ‘Schur coefficient’):

\[
\begin{pmatrix} 0 & I_{m_2} \end{pmatrix} H \begin{pmatrix} I_{m_2} & 0 \end{pmatrix}^{-1} = (\gamma_2^* \gamma_2)^{-1} \gamma_2^* \gamma_1 = \gamma_2^{-1} \gamma_1. \tag{4.2}
\]

Next, we recover \( \gamma_2 \) from \( \gamma_2^{-1} \gamma_1 \) using (2.41), and finally, we easily recover \( \gamma \) from \( \gamma_2 \) and \( \gamma_2^{-1} \gamma_1 \).

To recover \( \beta \) from \( \gamma \), we partition \( \beta \) into two blocks \( \beta = [\beta_1 \quad \beta_2] \), where \( \beta_k \) (\( k = 1, 2 \)) is an \( m_1 \times m_k \) matrix function. We put

\[
\tilde{\beta} = [I_{m_1} \quad \gamma_2^{-1} \gamma_1]. \tag{4.3}
\]

Because of (2.3) and (4.3), we have \( \beta^* j \gamma^* = \tilde{\beta}^* j \gamma^* = 0 \), and so

\[
\beta(x) = \beta_1(x) \tilde{\beta}(x). \tag{4.4}
\]

It follows from (1.1) and (2.1) that

\[
\beta'(x) = iv(x) \gamma(x), \tag{4.5}
\]

which implies

\[
\beta^* j \beta^* = 0, \quad \beta^* j \gamma^* = -iv. \tag{4.6}
\]

Formula (4.4) and the first relations in (2.3) and (4.6) lead us to

\[
\tilde{\beta}^* j \tilde{\beta}^* = \beta_1^{-1} (\beta_1^*)^{-1}, \quad \beta^* j \beta^* = \beta_1^{-1} + \beta_1 (\tilde{\beta}^* j \tilde{\beta}^* \beta_1^*) = 0. \tag{4.7}
\]

According to (1.3) and (4.7), \( \beta_1 \) satisfies the first-order differential equation

\[
\beta'_1 = -\beta_1 (\tilde{\beta}^* j \tilde{\beta}^*) (\tilde{\beta}^* j \tilde{\beta}^* \beta_1^*)^{-1}, \quad \beta_1(0) = I_{m_1}. \tag{4.8}
\]

Relations (4.1)–(4.6) and (4.8) give us a procedure to construct the solution of the inverse problem.
**Theorem 4.1.** Let \( \varphi \) be the Weyl function of the Dirac system (1.1) on \([0, \infty)\), where the potential \( v \) is locally bounded. Then, \( v \) can be uniquely recovered from \( \varphi \) via the formula
\[
v(x) = i\beta'(x)j\gamma'(x)^*.
\] (4.9)

Here, \( \beta \) is recovered from \( \gamma \) using (4.3), (4.4) and (4.8); \( \gamma \) is recovered from the Hamiltonian \( H \) using (4.2) and equation (2.41); the Hamiltonian is given by (2.32), and \( \Pi \) and \( S \) in (2.32) are expressed via \( \Phi_1(x) \) in formulas (2.17) and (3.9). Finally, \( \Phi_1(x) \) is recovered from \( \varphi \) using (4.1).

There is another way to recover \( \beta \) and \( \gamma \).

**Remark 4.2.** We can recover \( \beta \) directly from \( \Pi \) and \( S \) as described in the proposition below, and then recover \( \gamma \) from \( \beta \) in the same way that \( \beta \) is recovered from \( \gamma \).

**Proposition 4.3.** Let the Dirac system (1.1) on \([0, \infty)\) be given. Assume that \( v \) is locally bounded, and \( \Pi \) and \( S \) are operators constructed in lemma 2.4. Then the matrix function \( \beta \), which is defined in (2.1), satisfies the equality
\[
\beta(x) = [I_{m_1} 0] + \int_0^x (S^{-1}_x \Phi_1'(t)) (t)^* [\Phi_1(t) I_{m_2}] \, dt.
\] (4.10)

**Proof.** First, we fix an arbitrary \( l \) and rewrite (2.21) (for \( x < l \)) in the form
\[
\gamma(x) = (E[\Phi_1 I_{m_2}]) (x).
\] (4.11)

It follows that
\[
i(EAE^{-1}E\Phi_1')(x) = \gamma_1(x) - \gamma_2(x)\Phi_1(+0).
\] (4.12)

Since (3.8) holds, we rewrite (4.12) as
\[
i(EAE^{-1}E\Phi_1')(x) = \gamma_1(x).
\] (4.13)

Next, we substitute \( K = EAE^{-1} \) from (2.6) into (4.13) and (using (2.4) and (2.8)) we obtain
\[
\gamma_1(x) = -\gamma(x)j \int_0^x \gamma(t)^*(E\Phi_1')(t) \, dt.
\] (4.14)

Formulas (4.11) and (4.14) imply
\[
\gamma_1(x) = -\gamma(x)j \int_0^x (E[\Phi_1 I_{m_2}]) (t)^* (E\Phi_1')(t) \, dt.
\] (4.15)

Because of (2.33), (2.34) and (4.15), we see that
\[
\gamma(x)j\theta(x)^* \equiv 0, \quad \theta(x) : = [I_{m_1} 0] + \int_0^x (E\Phi_1')(t)^* (E[\Phi_1 I_{m_2}]) (t) \, dt
\]
\[
= [I_{m_1} 0] + \int_0^x (S^{-1}_x \Phi_1')(t)^* [\Phi_1(t) I_{m_2}] \, dt;
\] (4.16)

where \( S_x := P_S \pi_x \)). We shall show that \( \theta = \beta \).

In view of (4.11) and the second equality in (4.16), we have
\[
\theta'(x) = (E\Phi_1')(x)^* \gamma(x).
\] (4.17)

Therefore, (2.3) leads us to
\[
\beta(x)j\theta(x)^* \equiv 0.
\] (4.18)

Furthermore, compare (2.3) with the first equality in (4.16) to see that
\[
\theta(x) = \kappa(x)\beta(x).
\] (4.19)
where $x(x)$ is an $m_1 \times m_1$ matrix function, which is boundedly differentiable on $[0, l]$. Now, equalities (4.18) and (4.19) and the first relations in (2.3) and (4.6) yield that $x' \equiv 0$ (i.e. $x$ is a constant). It follows from (2.1), (4.16) and (4.19) that $x(0) = I_{m_1}$, and so $x = I_{m_1}$; that is, $\theta \equiv \beta$. Thus, (4.10) is immediate from (4.16).

Remark 4.4. Because of (4.5), (4.17) and equality $\beta = \theta$, we see that the potential $v$ can be recovered via the formula

$$v(x) = (iE \Phi')^*(x). \quad (4.20)$$

Formally applied, formulas (2.34), (2.39) and (4.20) yield

$$v(x) = (iS^{-1}_x \Phi')^*(x), \quad (4.21)$$

though one needs a proper ‘pointwise’ definition of matrix functions $S^{-1}_x \Phi'$ for (4.21) to hold.

The last statement in this section is a Borg–Marchenko-type uniqueness theorem, which follows from theorems 3.7 and 4.1.

Theorem 4.5. Let $\varphi$ and $\hat{\varphi}$ be the Weyl functions of two Dirac systems on $[0, \infty)$ with locally bounded potentials, which are denoted by $v$ and $\hat{v}$, respectively. Suppose that on some ray $\text{Re} z = c \text{Im} z$ ($c \in \mathbb{R}$, $\text{Im} z > 0$), the equality

$$\|\varphi(z) - \hat{\varphi}(z)\| = O(e^{2|rz|}) \quad (\text{Im} z \to \infty) \quad \text{for all} \quad 0 < r < l \quad (4.22)$$

holds. Then we have

$$v(x) = \hat{v}(x), \quad 0 < x < l. \quad (4.23)$$

Proof. Since Weyl functions are non-expansive, we obtain

$$\|e^{-2|rz|}(\varphi(z) - \hat{\varphi}(z))\| \leq c_1 e^{2|rz|}, \quad \text{Im} z \geq c_2 > 0, \quad (4.24)$$

for some $c_1$ and $c_2$, and the matrix function $e^{-2|rz|}(\varphi(z) - \hat{\varphi}(z))$ is bounded on the line $\text{Im} z = c_2$. Furthermore, formula (4.22) implies that $e^{-2|rz|}(\varphi(z) - \hat{\varphi}(z))$ is bounded on the ray $\text{Re} z = c \text{Im} z$. Therefore, applying the Phragmen–Lindelöf theorem in the angles generated by the line $\text{Im} z = c_2$ and the ray $\text{Re} z = c \text{Im} z$ ($\text{Im} z \geq c_2$), we see that

$$\|e^{-2|rz|}(\varphi(z) - \hat{\varphi}(z))\| \leq c_3, \quad \text{Im} z \geq c_2 > 0. \quad (4.25)$$

Functions associated with $\hat{\varphi}$ will be written with a hat (e.g. $\hat{\Phi}_1$). Because of formula (3.10), its analogue for $\hat{\varphi}$, $\hat{\Phi}_1$ and the inequality (4.25), we have

$$\left\| \int_0^r e^{2i\langle x - r \rangle z}(\Phi_1(x) - \hat{\Phi}_1(x)) \, dx \right\| \leq c_4, \quad \text{Im} z \geq c_2 > 0. \quad (4.26)$$

Clearly, the left-hand side of (4.26) is bounded in the semi-plane $\text{Im} z < c_2$ and tends to zero on some rays. Thus, we derive

$$\int_0^r e^{2i\langle x - r \rangle z}(\Phi_1(x) - \hat{\Phi}_1(x)) \, dx \equiv 0, \quad \text{i.e.} \quad \Phi_1(x) \equiv \hat{\Phi}_1(x) \quad (0 < x < r). \quad (4.27)$$

Since (4.27) holds for all $r < l$, we obtain $\Phi_1(x) \equiv \hat{\Phi}_1(x)$ for $0 < x < l$. In view of theorem 4.1, the last identity implies (4.23).
5. Weyl function and positivity of $S$

In this section, we discuss some sufficient conditions for a non-expansive matrix function $\varphi$, which is analytic in $\mathbb{C}_+$, to be a Weyl function of the Dirac system on the semi-axis. For the case that $\varphi$ is a scalar and Weyl functions $\varphi$ are the so-called Nevanlinna functions (i.e. $\text{Im} \varphi \geq 0$), sufficient condition for $\varphi$ to be a Weyl function can be given in terms of the spectral function [28, 29], which is connected with $\varphi$ via the Herglotz representation. Furthermore, a positive operator $S$ is also recovered from the spectral function (see [43, chapters 4, 10] and [35]). The invertibility of the convolution operators, which is required in [4, 5, 25], provides their positivity too, and the spectral problem is treated in this way. Here, we formulate conditions on the $m_2 \times m_1$ non-expansive matrix functions, again in terms of $S$. To derive our sufficient conditions, we apply the procedure to recover the Dirac system from its Weyl function (see section 4). Recall that $\Phi_1$ in section 4 is the Fourier transform of $\varphi$, that is, it is given by (4.1).

First, consider a useful procedure to recover $\varphi$ from $\mu$ as mentioned in remark 4.2.

**Proposition 5.1.** Let a given $m_1 \times m$ matrix function $\beta(x)$ ($0 \leq x \leq l$) be boundedly differentiable and satisfy relations

$$\beta(0) = [I_{m_1} \ 0], \quad \beta' j \beta^* = 0. \tag{5.1}$$

Then there is a unique $m_2 \times m$ matrix function $\gamma$, which is boundedly differentiable and satisfies relations

$$\gamma(0) = [0 \ I_{m_2}], \quad \gamma' j \gamma^* \equiv 0, \quad \gamma j \beta^* \equiv 0. \tag{5.2}$$

Moreover, this $\gamma$ is given by the formula

$$\gamma = \gamma_2 \tilde{\gamma}, \quad \tilde{\gamma} = [\tilde{\gamma}_1 \ I_{m_2}], \quad \tilde{\gamma}_1 = \beta_1^* (\beta_1^*)^{-1}, \tag{5.3}$$

where $\det \beta_1(x) \neq 0$, $\det (I_{m_2} - \tilde{\gamma}_1(x) \tilde{\gamma}_1(x)^*) \neq 0$ and $\gamma_2$ can be recovered via the linear differential system and initial condition below:

$$\gamma'_2 = \gamma_2 \tilde{\gamma}_1^* (I_{m_2} - \tilde{\gamma}_1 \tilde{\gamma}_1^*)^{-1}, \quad \gamma_2(0) = I_{m_2}. \tag{5.4}$$

**Proof.** Because of (5.1) we have $\beta j \beta^* \equiv I_{m_1}$ (and so $\det \beta_1 \neq 0$). On the other hand, formula (5.3) implies $\tilde{\gamma} j \beta^* = 0$. Therefore, we obtain

$$\gamma j \beta^* = 0, \quad \tilde{\gamma} j \tilde{\gamma}_1^* < 0. \tag{5.5}$$

In particular, we see that $\det (I_{m_2} - \tilde{\gamma}_1 \tilde{\gamma}_1^*) \neq 0$. According to (5.4) we have $\det \gamma_2 \neq 0$. Furthermore, any $\gamma$ satisfying (5.2) admits representation $\gamma = \gamma_2 \tilde{\gamma}$ with some boundedly differentiable $\gamma_2$, such that $\gamma_2(0) = I_{m_2}$, and $\gamma_2(x) \neq 0$. Thus, it remains to rewrite $\gamma' j \gamma^*$ in the equivalent form

$$\left(\gamma'_2 \tilde{\gamma} + [\gamma_2 \tilde{\gamma}_1^* \ 0]\right) j \tilde{\gamma}_1^* = 0, \tag{5.6}$$

which, in turn, is equivalent to the first equality in (5.4). Clearly, (5.4) uniquely defines $\gamma_2$. □

Now, we formulate the main statement in this section.

**Theorem 5.2.** Let an $m_2 \times m_1$ matrix function $\varphi(z)$ be analytic and non-expansive in $\mathbb{C}_+$. Furthermore, let matrix function $\Phi_1(x)$ and operators $S_l$, which are given by (4.1) and (3.9), respectively, be such that $\Phi_1$ is boundedly differentiable on each finite interval $[0, l]$ and satisfies equality $\Phi_1(x) = 0$ for $x \leq 0$, whereas operators $S_l$ are boundedly invertible for all $0 < l < \infty$.

Then, $\varphi$ is the Weyl function of some Dirac system on $[0, \infty)$. The operators $S_l^{-1}$ admit unique factorizations

$$S_l^{-1} = E_{\Phi_1}^* E_{\Phi_1}, \quad E_{\Phi_1} = I + \int_0^x E_\Phi(x, t) \cdot dt \in B(L_{m_2}^2(0, l)). \tag{5.7}$$
where $E_\Phi(x,t)$ is continuous (with respect to $x$ and $t$) and does not depend on $l$, and the potential of the Dirac system is constructed via the formula

$$v(x) = (iE_\Phi, \Phi')(x)^*, \quad 0 < x < l.$$  

(5.8)

Note that formula (5.8) in theorem 5.2 is similar to (4.20). To prove this theorem we need an auxiliary proposition 2.1 from [15] on operator $S$.

**Proposition 5.3** ([15]). Let $\Phi_1(x)$ be an $m_2 \times m_1$ matrix function, which is boundedly differentiable on the interval $[0, l]$ and satisfies equality $\Phi_1(0) = 0$. Then the operator $S$, which is given by (3.9), satisfies the operator identity (2.18), where $\Pi := \begin{bmatrix} \Phi_1 & \Phi_2 \end{bmatrix}$ is expressed via formulas

$$\Phi_k \in B(\mathbb{C}^{m_k}, L_{m_2}^2(0, l)), \quad (\Phi_1 f)(x) = \Phi_1(x)f, \quad \Phi_2 f = I_{m_2} f.$$  

(5.9)

Factorization results from [22, pp 185–6] yield the following lemma (see [15, theorem 4.2]).

**Lemma 5.4.** Let matrix function $\Phi_1(x)$ and operators $S_l$, which are expressed via $\Phi_1$ in (3.9), be such that $\Phi_1$ is boundedly differentiable on each finite interval $[0, l]$ and satisfies equality $\Phi_1(0) = 0$, whereas operators $S_l$ are boundedly invertible for all $0 < l < \infty$. Then the operators $S_l$ admit factorizations (5.7), where $E_\Phi(x,t)$ is continuous with respect to $x$, $t$ and does not depend on $l$. Furthermore, all the factorizations (5.7) with continuous $E_\Phi(x,t)$ are unique.

Now, we consider our procedure to solve the inverse problem.

**Lemma 5.5.** Let the conditions of lemma 5.4 hold. Then the matrix functions

$$\beta_\Phi(x) := [I_{m_1}, 0] + \int_0^x \left( S_l^{-1} \Phi_1'(t) \right) (t)^* \Phi_1(t) \, I_{m_2} \, dt,$$  

(5.10)

$$\gamma_\Phi(x) := [\Phi_1(x) \quad I_{m_2}] + \int_0^x E_\Phi(x,t) \Phi_1(t) \, I_{m_2} \, dt$$  

(5.11)

are boundedly differentiable and satisfy the conditions

$$\beta_\Phi(0) := \beta_\Phi(+0) = [I_{m_1}, 0], \quad \beta_\Phi^0 j \beta_\Phi^0 = 0;$$  

(5.12)

$$\gamma_\Phi(0) = [0 \quad I_{m_2}], \quad \gamma_\Phi^0 j \gamma_\Phi^0 = 0; \quad \gamma_\Phi^0 j \beta_\Phi^0 = 0.$$  

(5.13)

**Proof.** Step 1. The first equalities in (5.12) and (5.13) are immediate from (5.10) and (5.11), respectively. Furthermore, (5.11) is equivalent to the equalities

$$\gamma_\Phi(x) = (E_\Phi, I_1 \Phi_1 I_{m_2})(x), \quad 0 \leq x \leq l \quad (\text{for all } l < \infty).$$  

(5.14)

Next, fix any $0 < l < \infty$ and recall that according to proposition 5.3 the operator identity

$$A S_l - S_l A^* = i\Pi j \Pi^*$$  

(5.15)

holds, where $\Pi$ is expressed via (5.9). Hence, taking into account (5.7) and (5.14), and turning around the proof of (5.15) in lemma 2.4, we obtain

$$E_\Phi A E_\Phi^{-1} = (E_\Phi^{-1})^* A^* E_\Phi^0 = i\gamma_\Phi(x) j \int_0^l \gamma_\Phi(t)^* \cdot dt, \quad \text{i.e.}$$  

$$E_\Phi A E_\Phi^{-1} = i\gamma_\Phi(x) j \int_0^x \gamma_\Phi(t)^* \cdot dt \quad \text{(for } E_\Phi = E_{\Phi,l}).$$  

(5.16)
Introducing the resolvent kernel $\Gamma_\Phi$ by $E_\Phi^{-1} = I + \int_0^t \Gamma_\Phi(x,t) \cdot dt$, we rewrite (5.16) in the form of an equality for kernels:

$$I_m + \int_t^x (E_\Phi(x,r) + \Gamma_\Phi(r,t)) \cdot dr + \int_t^x \int_\xi^x E_\Phi(x,r) \cdot dr \cdot \Gamma_\Phi(\xi,t) \cdot d\xi$$

$$= -\gamma_\Phi(x)j\gamma_\Phi(t)^*.$$  \hspace{3cm} (5.17)

In particular, formula (5.17) for the case that $x = t$ implies

$$\gamma_\Phi(x)j\gamma_\Phi(x)^* \equiv -I_m.$$  \hspace{3cm} (5.18)

In a way quite similar to the first part of the proof of proposition 4.3, we use equalities (5.14), (5.16) and $\Phi_1(0) = 0$ to derive

$$\gamma_\Phi(x)j\beta_\Phi(x)^* \equiv 0$$  \hspace{3cm} (5.19)

(compare (5.19) with (4.16)). Because of (5.7), (5.10) and (5.14) we have

$$\beta_\Phi'(x) = (E_\Phi \Phi_1')(x)^*\gamma_\Phi(x).$$  \hspace{3cm} (5.20)

In view of (5.19) and (5.20), $\beta_\Phi$ is boundedly differentiable and the last relation in (5.12) holds.

Step 2. It remains to show that $\gamma_\Phi$ is boundedly differentiable and the identity $\gamma_\Phi j\gamma_\Phi^* \equiv 0$ holds. For that purpose note that $\beta_\Phi$ satisfies conditions of proposition 5.1, and so there is a boundedly differentiable matrix function $\tilde{\gamma}$ such that

$$\tilde{\gamma}(0) = [0 \quad I_m], \quad \tilde{\gamma}' j\tilde{\gamma}^* \equiv 0; \quad \tilde{\gamma} j\beta_\Phi^* \equiv 0.$$  \hspace{3cm} (5.21)

Formulas (5.12) and (5.21) yield

$$\beta_\Phi j\beta_\Phi^* \equiv I_m, \quad \tilde{\gamma} j\tilde{\gamma}^* \equiv -I_m,$$  \hspace{3cm} (5.22)

respectively. That is, the rows of $\beta_\Phi$ (the rows of $\tilde{\gamma}$) are linearly independent. Hence, the last relations in (5.21) and (5.22) and formulas (5.18) and (5.19) imply that there is a unitary matrix function $\omega$ such that

$$\gamma_\Phi(x) = \omega(x)\tilde{\gamma}(x), \quad \omega(x)^* = \omega(x)^{-1}.$$  \hspace{3cm} (5.23)

Moreover, formulas (5.12), (5.21) and (5.22) lead us to the relations

$$\tilde{\alpha}' j\tilde{\alpha}^* = ij\begin{bmatrix} 0 & \tilde{\nu} \\ \tilde{\nu}^* & 0 \end{bmatrix}, \quad \tilde{\alpha}(x) := \begin{bmatrix} \beta_\Phi(x) \\ \tilde{\gamma}(x) \end{bmatrix}, \quad \tilde{\nu} := j\beta_\Phi^*.$$  \hspace{3cm} (5.24)

According to (5.24) and (5.25) the matrix function $\tilde{\alpha}(x)$ the fundamental solution of the Dirac system, where $\tilde{\nu}$ is the bounded potential and the spectral parameter $z$ equals zero. In other words, $\beta_\Phi$ and $\tilde{\gamma}$ correspond to $\tilde{\nu}$ via equalities (2.1), and we can apply the results of section 2. Therefore, according to proposition 2.2 and remark 2.8, there is an operator $E$ of the form (2.39), such that

$$EA = i\tilde{\gamma}(x)j \int_0^x \tilde{\gamma}(t)^* \cdot dt \cdot E, \quad \tilde{\gamma}_2 = EI_m.$$  \hspace{3cm} (5.26)

It follows from (5.23) and (5.26) that

$$\omega EA = i\gamma_\Phi(x)j \int_0^x \gamma_\Phi(t)^* \cdot dt \cdot \omega E, \quad \gamma_{2,-\Phi} = \omega EI_m,$$  \hspace{3cm} (5.27)

where $\omega$ also denotes the operator of multiplication by the matrix function $\omega$. On the other hand, formulas (5.14) and (5.16) lead us to

$$E_\Phi A = i\gamma_\Phi(x)j \int_0^x \gamma_\Phi(t)^* \cdot dt \cdot E_{\Phi}, \quad \gamma_{2,-\Phi} = E_{\Phi}I_m.$$  \hspace{3cm} (5.28)
It is easy to see that \( \sum_{i=0}^{\infty} \text{Im} \left( A^i I_m \right) = L^2_{m_2}(0, 1) \). Hence, equalities (5.27) and (5.28) imply \( E_{\Phi} = \omega E \). Taking into account that according to (2.39) and (5.7) the expressions for both operators \( E \) and \( E_{\Phi} \) include the term \( I \), we see that

\[
\omega(x) \equiv I_{m_2} , \quad E_{\Phi} = E .
\] (5.29)

Furthermore, because of (5.23) and (5.29) we have \( \gamma_{\Phi} = \hat{\gamma} \), and so \( \gamma_{\Phi} \) is boundedly differentiable and satisfies (5.13). \( \square \)

**Proof of theorem 5.2.** By the assumptions of theorem 5.2 the conditions of lemmas 5.4 and 5.5 are fulfilled. Therefore, the theorem’s statements about operators \( S_l \) and \( E_{\Phi, l} \) are true. Moreover, because of (5.18), (5.20), (5.24) and equality \( \gamma_{\Phi} = \hat{\gamma} \), we have \( v = \hat{v} \) for \( v \) given by (5.8) and \( \hat{v} \) from lemma 5.5. It follows that \( v \) is bounded on \([0, l]\), \( \gamma_{\Phi} = \hat{\gamma} = \gamma \) and the operator \( E \), which is recovered from \( v \) in section 2, satisfies (5.29). Thus, we derived

\[
v = \hat{v} , \quad \gamma_{\Phi} = \gamma , \quad E_{\Phi} = E.
\] (5.30)

Using (2.17), (5.14) and (5.30), we see that \( \Phi_1 \), which is recovered from \( v \) and \( E \) in section 2, coincides with \( \Phi_1 \) in the statement of the theorem. In view of proposition 3.2 and corollary 3.8, there is a unique Weyl function \( \psi_W \) of the Dirac system with the potential \( v \) and this Weyl function is given by (3.20).

It remains to show that \( \psi_W \) equals the function \( \varphi \), which generates \( \Phi_1 \) via (4.1). Recalling that (4.1) also holds for \( \psi_W \), we see that

\[
\text{l.i.m.}_{a \to \infty} \int_{-a}^{a} e^{-i \xi \eta} \left( \varphi(\xi + i \eta) - \psi_W(\xi + i \eta) \right) d\xi \equiv 0, \quad \eta > 0 , \quad (5.31)
\]

where l.i.m. stands for the entrywise limit in the norm of \( L^2(0, r) \) \((0 < r \leq \infty)\). Therefore, we obtain \( \psi_W = \varphi \), that is, \( \varphi \) is the Weyl function of the Dirac system, where the potential is given by (5.8). \( \square \)

**Acknowledgments**

The work of IYaR was supported by the German Research Foundation (DFG) under grant no KI 760/3-1 and the work of ALS was supported by the Austrian Science Fund (FWF) under grant no Y330.

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