On the maximum size of a \((k, l)\)-sum-free subset of an abelian group

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Abstract

A subset \(A\) of a given finite abelian group \(G\) is called \((k, l)\)-sum-free if the sum of \(k\) (not necessarily distinct) elements of \(A\) does not equal the sum of \(l\) (not necessarily distinct) elements of \(A\). We are interested in finding the maximum size \(\lambda_{k,l}(G)\) of a \((k, l)\)-sum-free subset in \(G\).

A \((2, 1)\)-sum-free set is simply called a sum-free set. The maximum size of a sum-free set in the cyclic group \(\mathbb{Z}_n\) was found almost forty years ago by Diamanda and Yap; the general case for arbitrary finite abelian groups was recently settled by Green and Ruzsa. Here we find the value of \(\lambda_{3,1}(\mathbb{Z}_n)\). More generally, a recent paper of Hamidoune and Plagne examines \((k, l)\)-sum-free sets in \(G\) when \(k - l\) and the order of \(G\) are relatively prime; we extend their results to see what happens without this assumption.

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1 Introduction

Throughout this paper, we let \(G\) be a finite abelian group of order \(n > 1\), written in additive notation; \(v\) will denote the exponent (i.e. largest order of any element) of \(G\).

For subsets \(A\) and \(B\) of \(G\), we use the standard notations \(A + B\) and \(A - B\) to denote the set of all two-term sums and differences, respectively, with one term chosen from \(A\) and one from \(B\). If,
say, $A$ consists of a single element $a$, then we simply write $a + B$ and $a - B$ instead of $A + B$ and $A - B$. For a positive integer $h$ and a subset $A$ of $G$, the set of all $h$-term sums with (not necessarily distinct) elements from $A$ will be denoted by $hA$.

Let $k$ and $l$ be distinct positive integers. A subset $A$ of $G$ is called a \((k, l)\)-sum-free set in $G$ if

$$kA \cap lA = \emptyset,$$

or, equivalently, if

$$0 \notin kA - lA.$$ Clearly, we may assume that $k > l$. We are interested in determining the maximum possible size $\lambda_{k,l}(G)$ of a \((k, l)\)-sum-free set in $G$.

A \((2, 1)\)-sum-free set is simply called a sum-free set. The value of $\lambda_{2,1}(\mathbb{Z}_n)$ was determined by Diamanda and Yap [13] in 1969. It can be proved (see also [31]) that

$$\max_{d \mid n} \left\{ \left\lfloor \frac{d}{3} \right\rfloor \cdot \frac{n}{d} \right\} \leq \lambda_{2,1}(G) \leq \max_{d \mid n} \left\{ \left\lfloor \frac{d}{3} \right\rfloor \cdot \frac{n}{d} \right\},$$

which for cyclic groups immediately implies the following.

**Theorem 1** (Diamanda and Yap [13]) The maximum size $\lambda_{2,1}(\mathbb{Z}_n)$ of a sum-free set in the cyclic group of order $n$ is given by

$$\lambda_{2,1}(\mathbb{Z}_n) = \max_{d \mid n} \left\{ \left\lfloor \frac{d}{3} \right\rfloor \cdot \frac{n}{d} \right\} = \begin{cases} \frac{p+1}{p} \cdot \frac{n}{3} & \text{if } n \text{ is divisible by a prime } p \equiv 2 \mod 3 \\ \left\lfloor \frac{n}{3} \right\rfloor & \text{otherwise.} \end{cases}$$

The problem of finding $\lambda_{2,1}(G)$ for arbitrary $G$ stood open for over 35 years. In a recent breakthrough paper, Green and Ruzsa [15] proved that, as it has been conjectured, the value of $\lambda_{2,1}(G)$ agrees with the lower bound in (1):

**Theorem 2** (Green and Ruzsa [15]) The maximum size $\lambda_{2,1}(G)$ of a sum-free set in $G$ is

$$\lambda_{2,1}(G) = \lambda_{2,1}(\mathbb{Z}_v) \cdot \frac{n}{v} = \max_{d \mid v} \left\{ \left\lfloor \frac{d}{3} \right\rfloor \cdot \frac{n}{d} \right\}.$$ As a consequence, we see that

$$\frac{2}{7}n \leq \lambda_{2,1}(G) \leq \frac{1}{2}n$$

for every $G$, with equality holding in the lower bound when $v = 7$ and in the upper bound when $v$ (iff $n$) is even.

Now let us consider other values of $k$ and $l$. In Section 2 of this paper we generalize [14], and prove the following.
Theorem 3 The maximum size $\lambda_{k,l}(G)$ of a $(k,l)$-sum-free set in $G$ satisfies
\[
\max_{d \mid n} \left\{ \left( \left\lfloor \frac{d - 1 - \delta(d)}{k + l} \right\rfloor + 1 \right) \cdot \frac{n}{d} \right\} \leq \lambda_{k,l}(G) \leq \max_{d \mid n} \left\{ \left( \left\lfloor \frac{d - 2}{k + l} \right\rfloor + 1 \right) \cdot \frac{n}{d} \right\},
\]
where $\delta(d) = \gcd(d, k-l)$.

Note that for $(k, l) = (2, 1)$ Theorem 3 yields (1). Note also that, if $k - l$ is not divisible by $v$, then $\delta(v) = \gcd(v, k - l) \leq v/2$; in particular,
\[
\lambda_{k,l}(G) \geq \frac{n}{2(k+l)} > 0.
\]
If, on the other hand, $k - l$ is divisible by $v$, then clearly $\lambda_{k,l}(G) = 0$, since for any $a \in G$ we have $ka = la$.

Let us now consider cyclic groups. When $G \cong \mathbb{Z}_n$ and $n$ and $k - l$ are relatively prime, then Theorem 3 gives
\[
\lambda_{k,l}(\mathbb{Z}_n) = \max_{d \mid n} \left\{ \left( \left\lfloor \frac{d - 2}{k + l} \right\rfloor + 1 \right) \cdot \frac{n}{d} \right\}.
\]
This result was already established by Hamidoune and Plagne in [17]. Their method was based on a generalization of Vosper’s Theorem [30] on critical pairs where arithmetic progressions, that is, sets of the form
\[
A = \{a, a + d, \ldots, a + c \cdot d\}
\]
play a crucial role. In particular, Hamidoune and Plagne proved that, if $G \cong \mathbb{Z}_n$ and $n$ and $k - l$ are relatively prime, then
\[
\lambda_{k,l}(\mathbb{Z}_n) = \max_{d \mid n} \left\{ \alpha_{k,l}(\mathbb{Z}_d) \cdot \frac{n}{d} \right\},
\]
where $\alpha_{k,l}(\mathbb{Z}_n)$ is the maximum size of a $(k,l)$-sum-free arithmetic progression in $\mathbb{Z}_n$. Hamidoune and Plagne deal only with the case when $n$ and $k - l$ are relatively prime; as they point out, “in the absence of this assumption, degenerate behaviors may appear”, and we concur with this assessment. Nevertheless, we attempt to treat the general case; in Section 3 of this paper we prove that (3) remains valid even without the assumption that $n$ and $k - l$ are relatively prime:

Theorem 4 For arbitrary positive integers $k$, $l$, and $n$ we have
\[
\lambda_{k,l}(\mathbb{Z}_n) = \max_{d \mid n} \left\{ \alpha_{k,l}(\mathbb{Z}_d) \cdot \frac{n}{d} \right\}.
\]

Let us now move on to general abelian groups. Hamidoune and Plagne conjecture in [17] that
\[
\lambda_{k,l}(G) = \lambda_{k,l}(\mathbb{Z}_v) \cdot \frac{n}{v}
\]
holds when $n$ and $k-l$ are relatively prime. They prove this assertion with the additional assumption that at least one prime divisor of $v$ is not congruent to 1 (mod $k+l$). We generalize this result for the case when $n$ and $k - l$ are not necessarily relatively prime:
Theorem 5 As before, for a positive integer \( d \), we set \( \delta(d) = \gcd(d, k - l) \). If \( v \) possesses at least one divisor \( d \) which is not congruent to any integer between 1 and \( \delta(d) \) (inclusive) \((\mod k + l)\), then
\[
\lambda_{k,l}(G) = \lambda_{k,l}(\mathbb{Z}_v) \cdot \frac{n}{v}.
\]

We closely follow some of the fundamental work of Hamidoune and Plagne in [17]; in fact, Section 3 of this paper can be considered an extension of [17] for the case when \( n \) and \( k - l \) are not assumed to be relatively prime.

In Section 4 we employ Theorem 4 to establish the value of \( \lambda_{3,1}(\mathbb{Z}_n) \) explicitly. As an analogue to Theorem 1 we prove the following.

Theorem 6 The maximum size \( \lambda_{3,1}(\mathbb{Z}_n) \) of a \((3,1)\)-sum-free set in the cyclic group of order \( n \) is given by
\[
\lambda_{3,1}(\mathbb{Z}_n) = \max_{d|n} \left\{ \left\lfloor \frac{d}{4} \right\rfloor \cdot \frac{n}{d} \right\} = \begin{cases} 
\frac{p+1}{p} \cdot \frac{n}{4} & \text{if } n \text{ is divisible by a prime } p \equiv 3 \pmod{4} \\
\frac{n}{4} & \text{otherwise.}
\end{cases}
\]

As a consequence, we see that
\[
\frac{1}{9}n \leq \lambda_{3,1}(\mathbb{Z}_n) \leq \frac{1}{3}n,
\]
with equality holding in the lower bound when \( n \in \{5,10\} \) and in the upper bound when \( n \) is divisible by 3.

In our final section, Section 5, we provide some further comments and discuss several open questions about \((k,l)\)-sum-free sets.

2 Bounds for the size of maximum \((k,l)\)-sum-free sets

In this section we prove Theorem 3.

We will use the following easy lemma.

Lemma 7 Suppose that \( A \) is a maximal \((k,l)\)-sum-free set in \( G \). Let \( K \) denote the stabilizer subgroup of \( kA \). Then
\[
(i) \ k(A + K) = kA;
(ii) \ A + K \ is \ a \ (k,l)\)-sum-free set in \( G \);
(iii) \ A + K = A;
(iv) \ A \ is \ the \ union \ of \ cosets \ of \ K.
\]
Proof. (i) The inclusion $kA \subseteq k(A+K)$ is obvious. Suppose that $a_1, \ldots, a_k \in A$ and $h_1, \ldots, h_k \in K$. Then
\[(a_1 + \cdots + a_k) + (h_1 + \cdots + h_k) \in kA,
\]so $k(A+K) \subseteq kA$.

(ii) Suppose, indirectly, that $k(A+K) \cap l(A+K) \neq \emptyset$; by (i) this implies $kA \cap l(A+K) \neq \emptyset$.
Then we can find elements $a_1, \ldots, a_k \in A$, $a'_1, \ldots, a'_l \in A$, and $h_1, \ldots, h_l \in K$ for which
\[a_1 + \cdots + a_k = a'_1 + \cdots + a'_l + h_1 + \cdots + h_l.
\]But
\[a'_1 + \cdots + a'_l = a_1 + \cdots + a_k - h_1 - \cdots - h_l \in kA,
\]and this contradicts the fact that $A$ is $(k, l)$-sum-free.

(iii) Since $A \subseteq A + K$ and $A$ is a maximal $(k, l)$-sum-free set in $G$, by (ii) we have $A + K = A$.

(iv) We need to show that for any $a \in A$, we have $a + K \subseteq A$. But $a + K \subseteq A + K$, so the claim follows from (iii). \qed

For the upper bound in Theorem 3 we need the following result which is essentially due to Kneser.

Theorem 8 (Kneser [20]; see Theorem 4.4 in [25]) Suppose that $A$ is a non-empty subset of $G$ and, for a given positive integer $h$, let $H$ be the stabilizer of $hA$. Then we have
\[|hA| \geq h \cdot |A| - (h-1) \cdot |H|.
\]

Proof of the upper bound in Theorem 3 Let $A$ be a $(k, l)$-sum-free set in $G$ with $|A| = \lambda$; then we have
\[kA \cap lA = \emptyset
\]and therefore
\[n \geq |kA| + |lA|.
\]As before, let $K$ and $L$ be the stabilizer subgroups of $kA$ and $lA$, respectively. Then, by Theorem 8 we have
\[|kA| \geq k \cdot |A| - (k-1) \cdot |K|
\]and
\[|lA| \geq l \cdot |A| - (l-1) \cdot |L|;
\]thus, from (4) we get
\[n \geq (k + l) \cdot |A| - (k - 1) \cdot |K| - (l - 1) \cdot |L|.
\]Without loss of generality we can assume that $|K| \geq |L|$, so
\[n \geq (k + l) \cdot |A| - (k + l - 2) \cdot |K|
\]
or
\[ \frac{|A|}{|K|} \leq \frac{1}{k+l} \left( \frac{n}{|K|} + (k+l-2) \right). \]

Now \(|A| = \lambda\); in particular, \(A\) is maximal, so by Lemma \(\Box\) (iv), \(\frac{|A|}{|K|}\) must be an integer. Therefore, with \(d\) denoting the index of \(K\) in \(G\), we get
\[ \lambda n/d \leq \frac{1}{k+l} \left( d + k + l - 2 \right), \]
from which our claim follows. \(\square\)

**Proposition 9** Let \(d\) be a positive integer, and set \(\delta(d) = \gcd(d, k-l)\). Suppose that \(c\) is a positive integer for which
\[(k+l) \cdot c \leq d - 1 - \delta(d).\]
Then there exists an element \(a \in \mathbb{Z}_d\) for which the set
\[A = \{a, a+1, a+2, \ldots, a+c\}\]
is a \((k, l)\)-sum-free in \(\mathbb{Z}_d\) of size \(c+1\).

**Proof.** By the Euclidean Algorithm, we have unique integers \(q\) and \(r\) for which
\[l \cdot c = \delta(d) \cdot q - r\]
and \(1 \leq r \leq \delta(d)\). We also know the existence of integers \(u\) and \(v\) for which
\[\delta(d) = (k-l) \cdot u + d \cdot v.\]
Now set \(a = u \cdot q\). We will show that
\[A = \{a, a+1, a+2, \ldots, a+c\}\]
is a \((k, l)\)-sum-free in \(\mathbb{Z}_d\). (Here, and elsewhere, we consider integers as elements of \(\mathbb{Z}_d\) via the canonical homomorphism \(\mathbb{Z} \rightarrow \mathbb{Z}_d\).)

First note that, for any integer \(i\) with \(-l \cdot c \leq i \leq k \cdot c\), our assumption about \(c\) implies
\[1 \leq r \leq l \cdot c + i + r \leq (k+l) \cdot c + r \leq (k+l) \cdot c + \delta(d) \leq d - 1,\]
and therefore, considering
\[B = \{l \cdot c + i + r \mid -l \cdot c \leq i \leq k \cdot c\}\]
as a subset of \(\mathbb{Z}_d\), we have \(0 \notin B\).

Furthermore, in \(\mathbb{Z}_d\) we have
\[(k-l) \cdot a = (k-l) \cdot u \cdot q = \delta(d) \cdot q - d \cdot v \cdot q = \delta(d) \cdot q = l \cdot c + r,\]
and therefore
\[kA - lA = \{(k-l) \cdot a + i \mid -l \cdot c \leq i \leq k \cdot c\} = B.\]
Since \(0 \notin B\), \(A\) is indeed \((k, l)\)-sum-free in \(\mathbb{Z}_d\).

Furthermore, since \(c < d\), we see that \(|A| = c+1\), as claimed. \(\square\)

The lower bound in Theorem \(\Box\) now follows from Proposition \(\Box\) and the following lemma.
Lemma 10  Suppose that $d$ is a divisor of $v$. Then
\[
\lambda_{k,l}(G) \geq \lambda_{k,l}(\mathbb{Z}_d) \cdot \frac{n}{d}.
\]

Proof. Since $d$ is a divisor of $v$, there is a subgroup $H$ of $G$ of index $d$ for which
\[ G/H \cong \mathbb{Z}_d. \]
Let $\Phi : G \to G/H$ be the canonical homomorphism from $G$ to $G/H$, and let $\Psi : G/H \to \mathbb{Z}_d$ be the isomorphism from $G/H$ to $\mathbb{Z}_d$. Then, for any $(k, l)$-sum-free set $A \subseteq \mathbb{Z}_d$, the set $\Phi^{-1}(\Psi^{-1}(A))$ is a $(k, l)$-sum-free set in $G$ and has size $\frac{n}{d} \cdot |A|$. \[\Box\]

3  $(k, l)$-sum-free sets in cyclic groups

In this section we analyze $(k, l)$-sum-free arithmetic progressions in $\mathbb{Z}_n$ and prove Theorems 4 and 5. This was carried out by Hamidoune and Plagne in [17] with the assumption that $n$ and $k - l$ are relatively prime; here we drop that assumption but follow their approach.

A subset $A$ of $\mathbb{Z}_n$ is an arithmetic progression of difference $d \in \mathbb{Z}_n$, if
\[ A = \{a, a + d, \ldots, a + c \cdot d\} \]
for some $a \in \mathbb{Z}_n$ and non-negative integer $c$. We let $A_{k,l}(n)$ be the set of $(k, l)$-sum-free arithmetic progression in $\mathbb{Z}_n$. We also let $B_{k,l}(n)$ and $C_{k,l}(n)$ be the sets of those sequences in $A_{k,l}(n)$ whose difference is not relatively prime to $n$, and relatively prime to $n$, respectively. Note that a sequence can belong to both $B_{k,l}(n)$ and $C_{k,l}(n)$ only if it contains exactly 1 term, and that sequences in $B_{k,l}(n)$ are each contained in a proper coset in $\mathbb{Z}_n$, while no sequence in $C_{k,l}(n)$ with more than one term is contained in a proper coset.

We introduce the following notations.
\[
\alpha_{k,l}(\mathbb{Z}_n) = \max \{|A| \mid A \in A_{k,l}(n)\}
\]
\[
\beta_{k,l}(\mathbb{Z}_n) = \max \{|A| \mid A \in B_{k,l}(n)\}
\]
\[
\gamma_{k,l}(\mathbb{Z}_n) = \max \{|A| \mid A \in C_{k,l}(n)\}
\]
Clearly, $\alpha_{k,l}(\mathbb{Z}_n) = \max\{\beta_{k,l}(\mathbb{Z}_n), \gamma_{k,l}(\mathbb{Z}_n)\}$.

We also let $D(n)$ be the set of all divisors of $n$ which are greater than 1. Furthermore, we separate the elements of $D(n)$ into subsets $D_1(n)$ and $D_2(n)$ according to whether they do not or do divide $k - l$, respectively. Then the following are clear:

- $D_1(n) = \emptyset$ if, and only if, $k - l$ is divisible by $n$;
- $D_2(n) = \emptyset$ if, and only if, $k - l$ and $n$ are relatively prime; and
- $D_1(n) \neq \emptyset$ and $D_2(n) \neq \emptyset$ if, and only if, $1 < \gcd(n, k - l) < n$. 

The next three propositions summarize our results on $\alpha_{k,l}(\mathbb{Z}_n)$, $\beta_{k,l}(\mathbb{Z}_n)$, and $\gamma_{k,l}(\mathbb{Z}_n)$. We start with $\beta_{k,l}(\mathbb{Z}_n)$.

**Proposition 11** The maximum size $\beta_{k,l}(\mathbb{Z}_n)$ of a $(k,l)$-sum-free arithmetic progression in $\mathbb{Z}_n$ whose difference is not relatively prime to $n$ satisfies the following.

(i) If $k - l$ is divisible by $n$, then $\beta_{k,l}(\mathbb{Z}_n) = 0$.

(ii) If $k - l$ and $n$ are relatively prime, then $\beta_{k,l}(\mathbb{Z}_n) = \frac{n}{p}$ where $p$ is the smallest prime divisor of $n$.

(iii) If $1 < \gcd(n, k - l) < n$, then we have

$$\frac{n}{\rho_1} \leq \beta_{k,l}(\mathbb{Z}_n) \leq \max \left\{ \frac{n}{\rho_1}, \frac{n}{2\rho_2} \right\},$$

where $\rho_1$ and $\rho_2$ are the smallest elements of $D_1(n)$ and $D_2(n)$, respectively.

**Proof.** If $n$ divides $k - l$, then for any $a \in \mathbb{Z}_n$ we have $ka = la$. This implies (i). Statements (ii) and (iii) will follow from the following three claims.

**Claim 1.** Suppose that $d \in D_1(n)$. Then the set

$$A = \{1 + i \cdot d \mid 0 \leq i \leq \frac{n}{d} - 1\}$$

is an arithmetic progression in $B_{k,l}(n)$, has size $|A| = \frac{n}{d}$, and is $(k,l)$-sum-free.

**Proof of Claim 1.** Clearly, $A$ belongs to $B_{k,l}(n)$ and has size $|A| = \frac{n}{d}$. Furthermore,

$$kA - lA = \left\{(k - l) + d \cdot j \mid -l \cdot \left(\frac{n}{d} - 1\right) \leq j \leq \frac{n}{d} - 1\right\}. $$

Since $d|n$ but $d \not| (k - l)$, we have $0 \notin kA - lA$ which means that $A$ is $(k,l)$-sum-free.

**Claim 2.** Suppose that $H$ is a subgroup of $\mathbb{Z}_n$ of index $d$, and that $A$ is a $(k,l)$-sum-free subset of $\mathbb{Z}_n$ (not necessarily an arithmetic progression) which lies in a single coset of $H$. Then $|A| \leq \frac{n}{2d}$.

**Proof of Claim 2.** Clearly, $A \subseteq a + H$ implies $|A| \leq |H| = \frac{n}{d}$.

**Claim 3.** Suppose again that $H$ is a subgroup of $\mathbb{Z}_n$ of index $d$, and that $A$ is a $(k,l)$-sum-free subset of $\mathbb{Z}_n$ which lies in a single coset of $H$. If $d \in D_2(n)$, then $|A| \leq \frac{n}{2d}$.

**Proof of Claim 3.** Note that $H$ is a cyclic group of order $n/d$ and

$$H = \{0, d, 2d, \ldots, \frac{n}{d} - 1\}. $$

Since $A$ lies in a single coset of $H$, so do $kA$ and $lA$. But $k - l$ is divisible by $d$, so $ka - la \in H$, and therefore the sets $kA$ and $lA$ lie in the same coset of $H$. Thus we have

$$|kA \cup lA| \leq |H| = \frac{n}{d}. $$

But $A$ is $(k,l)$-sum-free, so $kA$ and $lA$ must be disjoint, hence

$$|kA| + |lA| \leq \frac{n}{d}. $$
Now clearly $(k - 1)a + A \subseteq kA$, so $|A| \leq |kA|$; similarly, $|A| \leq |lA|$. This implies that

$$|A| + |A| \leq \frac{n}{d}.$$  

\[\square\]

Next, we turn to $\gamma_{k,l}(\mathbb{Z}_n)$.

**Proposition 12** The maximum size $\gamma_{k,l}(\mathbb{Z}_n)$ of a $(k, l)$-sum-free arithmetic progression in $\mathbb{Z}_n$ whose difference is relatively prime to $n$ satisfies

$$\left\lfloor \frac{n - 1 - \delta}{k + l} \right\rfloor + 1 \leq \gamma_{k,l}(\mathbb{Z}_n) \leq \left\lfloor \frac{n - 2}{k + l} \right\rfloor + 1,$$

where $\delta = \gcd(n, k - l)$.

**Proof.** The lower bound follows directly from Proposition 9.

For the upper bound, suppose that $d \in \mathbb{Z}_n$ and $\gcd(d, n) = 1$, and let $a \in \mathbb{Z}_n$. We need to show that, if the set

$$A = \{a, a + d, \ldots, a + c \cdot d\}$$

is $(k, l)$-sum-free in $\mathbb{Z}_n$, then

$$(k + l) \cdot c \leq n - 2.$$  

Suppose, indirectly, that $$(k + l) \cdot c \geq n - 1;$$

then we have

$$\{(k - l) \cdot a + i \cdot d \mid -l \cdot c \leq i \leq k \cdot c\} \supseteq \{(k - l) \cdot a + j \cdot d \mid 0 \leq j \leq n - 1\}.$$  

Now the left-hand side equals $kA - lA$. Since $\gcd(d, n) = 1$, the right-hand side equals the entire group $\mathbb{Z}_n$. But then $kA - lA$ must contain 0, which is a contradiction. \[\square\]

We can now combine Propositions 11 and 12 to get results for the maximum size of $(k, l)$-sum-free arithmetic progressions in $\mathbb{Z}_n$.

**Proposition 13** The maximum size $\alpha_{k,l}(\mathbb{Z}_n)$ of a $(k, l)$-sum-free arithmetic progression in $\mathbb{Z}_n$ satisfies the following.

(i) If $k - l$ is divisible by $n$, then $\alpha_{k,l}(\mathbb{Z}_n) = 0$.

(ii) If $k - l$ and $n$ are relatively prime, then

$$\alpha_{k,l}(\mathbb{Z}_n) = \max \left\{ \frac{n}{p}, \left\lfloor \frac{n - 2}{k + l} \right\rfloor + 1 \right\},$$

where $p$ is the smallest prime divisor of $n$.

(iii) If $1 < \gcd(n, k - l) < n$, then we have

$$\max \left\{ \frac{n}{\rho_1}, \left\lfloor \frac{n - 1 - \delta}{k + l} \right\rfloor + 1 \right\} \leq \alpha_{k,l}(\mathbb{Z}_n) \leq \max \left\{ \frac{n}{\rho_1}, \frac{n}{2 \rho_2}, \left\lfloor \frac{n - 2}{k + l} \right\rfloor + 1 \right\},$$

where $\delta = \gcd(n, k - l)$, and $\rho_1$ and $\rho_2$ are the smallest elements of $D_1(n)$ and $D_2(n)$, respectively.
It is easy to see that the bounds in Proposition 13 are tight.

Now we are ready to prove Theorem 4. Due to the following result in [17], our task is not difficult.

**Theorem 14 (Hamidoune and Plagne, [17])** Let $\epsilon$ be 0 if $n$ is even and 1 if $n$ is odd. Then we have the following bounds.

$$\max_{d|v} \left\{ \alpha_{k,l}(\mathbb{Z}_d) \cdot \frac{n}{d} \right\} \leq \lambda_{k,l}(G) \leq \max_{d|v} \left\{ \frac{n-\epsilon}{k+l} \cdot \max_{d|v} \left\{ \alpha_{k,l}(\mathbb{Z}_d) \cdot \frac{n}{d} \right\} \right\}$$

**Proof of Theorem 4.** If $k - l$ is divisible by $n$, Theorem 4 obviously holds as both sides equal zero, so let’s assume otherwise. By Theorem 14, it suffices to prove that

$$\left\lfloor \frac{n-\epsilon}{k+l} \right\rfloor \leq \max_{d|n} \left\{ \alpha_{k,l}(\mathbb{Z}_d) \cdot \frac{n}{d} \right\}.$$ 

By Proposition 13, this statement follows once we prove

$$\left\lfloor \frac{n-\epsilon}{k+l} \right\rfloor \leq \max_{d|n} \left\{ \frac{d}{\rho_1(d)} \cdot \left\lfloor \frac{d-1-\delta(d)}{k+l} \right\rfloor + 1 \right\} \cdot \frac{n}{d}, \quad (5)$$

where $\rho_1(d)$ is the smallest divisor of $d$ which does not divide $k - l$. (Note that in the case when $\delta = 1$, $\rho_1(d)$ is simply the smallest prime dividing $d$, thus we do not need to consider cases (ii) and (iii) of Proposition 13 separately.)

Now $\rho_1 = \rho_1(n)$ does not divide $k - l$, so we must have $\delta(\rho_1) = \gcd(\rho_1, k - l) < \rho_1$. Therefore, since $\rho_1$ divides $n$, we have

$$\max_{d|n} \left\{ \left\lfloor \frac{d-1-\delta(d)}{k+l} \right\rfloor + 1 \right\} \cdot \frac{n}{d} \geq \left\lfloor \frac{\rho_1-1-\delta(\rho_1)}{k+l} \right\rfloor + 1 \right\} \cdot \frac{n}{\rho_1} \geq \frac{n}{\rho_1}.$$

We then have

$$\max_{d|n} \left\{ \frac{d}{\rho_1(d)} \cdot \left\lfloor \frac{d-1-\delta(d)}{k+l} \right\rfloor + 1 \right\} \cdot \frac{n}{d} =$$

$$= \max_{d|n} \left\{ \frac{n}{\rho_1(d)} \right\} \cdot \max_{d|n} \left\{ \left\lfloor \frac{d-1-\delta(d)}{k+l} \right\rfloor + 1 \right\} \cdot \frac{n}{d} \right\} =$$

$$= \max_{d|n} \left\{ \left\lfloor \frac{d-1-\delta(d)}{k+l} \right\rfloor + 1 \right\} \cdot \frac{n}{d} \right\} \cdot \frac{n}{d} \right\} =$$

$$\max_{d|n} \left\{ \left\lfloor \frac{d-1-\delta(d)}{k+l} \right\rfloor + 1 \right\} \cdot \frac{n}{d} \right\}.$$

Therefore, (5) is equivalent to

$$\left\lfloor \frac{n-\epsilon}{k+l} \right\rfloor \leq \max_{d|n} \left\{ \left\lfloor \frac{d-1-\delta(d)}{k+l} \right\rfloor + 1 \right\} \cdot \frac{n}{d} \right\}.$$
But this inequality clearly holds, since
\[
\max_{d|n} \left\{ \left( \left\lfloor \frac{d - 1 - \delta(d)}{k + l} \right\rfloor + 1 \right) \cdot \frac{n}{d} \right\} \geq \left\lfloor \frac{n - 1 - \delta}{k + l} \right\rfloor + 1
\]
\[
= \left\lfloor \frac{n - 1 - (k - l)}{k + l} \right\rfloor + 1
\]
\[
= \left\lfloor \frac{n + (2l - 1)}{k + l} \right\rfloor
\]
\[
\geq \left\lfloor \frac{n - \epsilon}{k + l} \right\rfloor.
\]

\[
(3, 1)\text{-sum-free sets in cyclic groups}
\]

In this section we prove Theorem 6 and find \( \lambda_{3,1}(\mathbb{Z}_n) \) explicitly. First, we evaluate \( \alpha_{3,1}(\mathbb{Z}_n) \). We note that, while Proposition 13 (ii) readily yields
\[
\alpha_{2,1}(\mathbb{Z}_n) = \begin{cases} 
\frac{n}{2} & \text{if } 2|n; \\
\left\lfloor \frac{n+1}{4} \right\rfloor & \text{if } 2 \not| n;
\end{cases}
\]
evaluating \( \alpha_{3,1}(\mathbb{Z}_n) \) requires a bit more work.
Proposition 15 The maximum size $\alpha_{3,1}(\mathbb{Z}_n)$ of a $(3,1)$-sum-free arithmetic progression in $\mathbb{Z}_n$ is given as follows:

$$\alpha_{3,1}(\mathbb{Z}_n) = \begin{cases} 
\frac{n}{3} & \text{if } 3|n; \\
\left\lfloor \frac{n+2}{4} \right\rfloor & \text{if } 3 \nmid n \text{ and } n \not\equiv 2 \pmod{8}; \\
\frac{n-2}{4} & \text{if } 3 \nmid n \text{ and } n \equiv 2 \pmod{8}.
\end{cases}$$

Proof. Let $\alpha_{3,1}(n) = \alpha$. If $n = 2$, the claim holds, so we assume that $n \geq 3$. We distinguish several cases.

Case 1: $2 \nmid n$ and $3 \nmid n$. In this case Proposition 13 (ii) applies, and

$$\alpha = \left\lfloor \frac{n+2}{4} \right\rfloor.$$

Case 2: $2 \nmid n$ and $3|n$. Proposition 13 (ii) applies again; we get

$$\alpha = \max \{ n, \left\lfloor \frac{n+2}{4} \right\rfloor \} = \frac{n}{3}.$$

Case 3: $2|n$ and $3|n$. In this case Proposition 13 (iii) applies with $\delta = 2$, $\rho_1 = 3$, and $\rho_2 = 2$; we get

$$\max \left\{ \frac{n}{3}, \left\lfloor \frac{n+1}{4} \right\rfloor \right\} \leq \alpha \leq \max \left\{ \frac{n}{3}, \left\lfloor \frac{n+2}{4} \right\rfloor \right\},$$

which again implies

$$\alpha = \frac{n}{3}.$$

Case 4: $4|n$ and $3 \nmid n$. Again Proposition 13 (iii) applies — this time with $\delta = 2$, $\rho_1 = 4$, and $\rho_2 = 2$. Therefore we get

$$\max \left\{ \frac{n}{4}, \left\lfloor \frac{n+1}{4} \right\rfloor \right\} \leq \alpha \leq \max \left\{ \frac{n}{4}, \left\lfloor \frac{n+2}{4} \right\rfloor \right\},$$

which gives

$$\alpha = \frac{n}{4}.$$

Case 5: $n \equiv 2 \pmod{4}$ and $3 \nmid n$. Again Proposition 13 (iii) applies — this time with $\delta = 2$, $\rho_1 \geq 5$, and $\rho_2 = 2$. Therefore we get

$$\max \left\{ \frac{n}{\rho_1}, \left\lfloor \frac{n+1}{4} \right\rfloor \right\} \leq \alpha \leq \max \left\{ \frac{n}{\rho_1}, \left\lfloor \frac{n+2}{4} \right\rfloor \right\},$$

which yields only

$$\alpha \in \left\{ \frac{n-2}{4}, \frac{n+2}{4} \right\}.$$
Case 5.1. Let us first consider the case when \( n \equiv 6 \pmod{8} \). With \( a = \frac{n+2}{3} \) and \( c = \frac{n-2}{3} \), we let
\[
A = \{a, a + 1, \ldots, a + c\}.
\]
Then
\[
3A - A = \{2a - c + i \mid 0 \leq i \leq 4c\} = \{1 + i \mid 0 \leq i \leq n - 2\} = \mathbb{Z}_n \setminus \{0\},
\]
so \( A \) is \((3,1)\)-sum-free in \( \mathbb{Z}_n \) of size \( c + 1 = \frac{n+2}{3} \).

Case 5.2. Now suppose that \( n \equiv 2 \pmod{8} \). We prove that \( \alpha = \frac{n-2}{3} \). Suppose, indirectly, that \( \alpha = \frac{n+2}{3} \) and there is a \((3,1)\)-sum-free arithmetic progression
\[
A = \{a, a + d, \ldots, a + c \cdot d\}
\]
in \( \mathbb{Z}_n \) of size \( c + 1 = \frac{n+2}{4} \). Similarly to above,
\[
3A - A = \{2a - c \cdot d + i \cdot d \mid 0 \leq i \leq 4c\} = \{2a - c \cdot d + i \cdot d \mid 0 \leq i \leq n - 2\}.
\]
By Proposition\( \Box \) (iii), we have
\[
\beta_{3,1}(n) \leq \max \left\{ \frac{n}{\rho d}, \frac{n}{d} \right\} = \frac{n}{4};
\]
so we have \( \beta_{3,1}(n) < \alpha \). Therefore, we must have \( \gcd(d,n) = 1 \), which implies that
\[
|3A - A| = n - 1.
\]
Since \( A \) is \((3,1)\)-sum-free, \( 0 \not\in 3A - A \), and this can only occur if
\[
2a - c \cdot d + (n - 1) \cdot d \equiv 0 \pmod{n}.
\]
A simple parity argument provides a contradiction: \( 2a - c \cdot d + (n - 1) \cdot d \) is odd, so it cannot be divisible by \( n \). \( \Box \)

Proof of Theorem\( \Box \) As previously, we let \( D(n) \) be the set of divisors of \( n \) which are greater than 1. We introduce the following six (potentially empty) subsets of \( D(n) \), as well as some notations.

\[
\begin{align*}
E_1(n) &= \{ d \in D(n) \mid 3 \nmid d \} & e_1 &= \max_{d \in E_1(n)} \left\{ \frac{d}{4} \cdot \frac{n}{d} \right\} \\
E_2(n) &= \{ d \in D(n) \mid d \equiv 3(4), 3 \nmid d \} & e_2 &= \max_{d \in E_2(n)} \left\{ \frac{d+1}{4} \cdot \frac{n}{d} \right\} \\
E_3(n) &= \{ d \in D(n) \mid 4 \nmid d, 3 \nmid d \} & e_3 &= \max_{d \in E_3(n)} \left\{ \frac{d}{4} \cdot \frac{n}{d} \right\} \\
E_4(n) &= \{ d \in D(n) \mid d \equiv 1(4), 3 \nmid d \} & e_4 &= \max_{d \in E_4(n)} \left\{ \frac{d-1}{4} \cdot \frac{n}{d} \right\} \\
E_5(n) &= \{ d \in D(n) \mid d \equiv 6(8), 3 \nmid d \} & e_5 &= \max_{d \in E_5(n)} \left\{ \frac{d+2}{4} \cdot \frac{n}{d} \right\} \\
E_6(n) &= \{ d \in D(n) \mid d \equiv 2(8), 3 \nmid d \} & e_6 &= \max_{d \in E_6(n)} \left\{ \frac{d-2}{4} \cdot \frac{n}{d} \right\}
\end{align*}
\]
(We have the understanding that \( \max\emptyset = 0 \).)

Then we have
\[
D(n) = \bigcup_{i=1}^{6} E_i(n);
\]
Furthermore, by Theorem 4 and Proposition 15 we have
\[ \lambda_{3,1}(\mathbb{Z}_n) = \max\{e_i | 1 \leq i \leq 6\}. \]

For any \( i \in \{1, 2, \ldots, 6\} \) for which \( E_i(n) \neq \emptyset \), we let
\[ p_i = \min\{E_i(n)\} \]
and
\[ n_i = \max\{E_i(n)\}. \]

Now suppose that \( E_5(n) \neq \emptyset \). Then \( E_2(n) \neq \emptyset \), and \( p_5 = 2 \cdot p_2 \). Therefore
\[ e_5 = \frac{p_5 + 2}{4} \cdot \frac{n}{p_5} = \frac{p_2 + 1}{4} \cdot \frac{n}{p_2} = e_2. \]
We can similarly show that, if \( E_6(n) \neq \emptyset \), then \( E_4(n) \neq \emptyset \) and \( e_6 = e_4 \). Therefore, we see that
\[ \lambda_{3,1}(\mathbb{Z}_n) = \max\{e_i | 1 \leq i \leq 4\}. \]

Next, observe that, if \( E_i(n) \neq \emptyset \) for some \( i \in \{1, 2, 3\} \), then \( e_i \geq e_j \) for all \( i < j \leq 4 \).

Now we consider the following cases.

**Case 1.** Suppose that \( n \) has divisors which are congruent to 3 mod 4, and let \( p \) be the smallest such divisor. If \( p = 3 \), then \( E_1(n) \neq \emptyset \), thus
\[ \lambda_{3,1}(\mathbb{Z}_n) = e_1 = \frac{n}{3}. \]
If, on the other hand, \( p > 3 \), then \( E_1(n) = \emptyset \) but \( E_2(n) \neq \emptyset \), thus
\[ \lambda_{3,1}(\mathbb{Z}_n) = e_2 = \frac{p + 1}{p} \cdot \frac{n}{4}. \]

**Case 2.** Suppose that \( n \) has no divisors which are congruent to 3 mod 4, but that \( n \) is divisible by 4. In this case, \( E_1(n) = E_2(n) = \emptyset \) but \( E_3(n) \neq \emptyset \), thus
\[ \lambda_{3,1}(\mathbb{Z}_n) = e_3 = \frac{n}{4}. \]

**Case 3.** Suppose that \( n \) has no divisors which are congruent to 3 mod 4, and that \( n \) is not divisible by 4. In this case, \( E_1(n) = E_2(n) = E_3(n) = \emptyset \) but \( E_4(n) \neq \emptyset \), thus
\[ \lambda_{3,1}(\mathbb{Z}_n) = e_4 = \frac{n_4 - 1}{4} \cdot \frac{n}{n_4}. \]
If \( n \) is odd, then \( n_4 = n \); if \( n \) is even, then (since \( n \) is not divisible by 4), \( n_4 = \frac{n}{2} \). In either case, we get
\[ \lambda_{3,1}(\mathbb{Z}_n) = e_4 = \frac{n_4 - 1}{4} \cdot \frac{n}{n_4} = \left\lfloor \frac{n}{4} \right\rfloor. \]

The claims of Theorem 6 now readily follow. □
5 Further comments and open questions

In this final section, we discuss some interesting open questions.

Our first question is about a possible generalization of Theorems 1 and 6. Note that, according to Theorem 3, we have

\[ \lambda_{k,l}(\mathbb{Z}_n) \leq \max_{d \mid n} \left\{ \left\lfloor \frac{d - 2}{k + l} \right\rfloor + 1 \right\} \cdot \frac{n}{d}. \]

**Question 1** Let \( D(n) \) be the set of divisors of \( n \) (which are greater than 1). Given distinct positive integers \( k \) and \( l \), is there a subset \( D_{k,l}(n) \) of \( D(n) \) so that

\[ \lambda_{k,l}(\mathbb{Z}_n) = \max_{d \in D_{k,l}(n)} \left\{ \left\lfloor \frac{d - 2}{k + l} \right\rfloor + 1 \right\} \cdot \frac{n}{d}. \]

As we see from (2), Question 1 holds with \( D_{k,l}(n) = D(n) \) when \( n \) and \( k - l \) are relatively prime, in particular, for sum-free sets. According to Theorem 6, the set

\[ D_{3,1}(n) = \{ d \in D(n) \mid d \not\equiv 2 \pmod{4} \} \]

works for \((k, l) = (3, 1)\). (Note that, if it exists, \( D_{k,l}(n) \) is not necessarily unique.)

Moving on to general abelian groups, we observe that, by Lemma 10, we have

\[ \lambda_{k,l}(G) \geq \lambda_{k,l}(\mathbb{Z}_v) \cdot \frac{n}{v}. \]

Then one of course wonders the following.

**Question 2** Given distinct positive integers \( k \) and \( l \), is

\[ \lambda_{k,l}(G) = \lambda_{k,l}(\mathbb{Z}_v) \cdot \frac{n}{v}. \]

According to Theorem 4, Question 2 is equivalent to asking: is

\[ \lambda_{k,l}(G) = \max_{d \mid v} \left\{ \alpha_{k,l}(\mathbb{Z}_d) \cdot \frac{n}{d} \right\}. \]

Note that Theorem 2 of Green and Ruzsa affirms Question 2 for sum-free sets. Theorem 5 exhibits some other cases when the equality also holds. In particular, as a consequence of Theorem 5, we see that

\[ \lambda_{3,1}(G) = \lambda_{3,1}(\mathbb{Z}_v) \cdot \frac{n}{v} \]

holds when \( v \) (iff \( n \)) has at least one prime divisor which is congruent to 3 mod 4, or when \( v \) is divisible by 4. So the only cases left open are when \( v = P \) or \( v = 2P \) where \( P \) is the product of primes all of whom are congruent to 1 mod 4.

Next, we are interested in characterizing all \((k, l)\)-sum-free subsets of maximum size.
Question 3  What are the \((k, l)\)-sum-free subsets \(A\) of \(G\) with size \(|A| = \lambda_{k,l}(G)|\)?

A pleasing answer is given by Bier and Chin [5] for the case when \(k \geq 3\) and \(G \cong \mathbb{Z}_p\) where \(p\) is an odd prime: in this case \(A\) is an arithmetic progression. The same answer was given by Di ananda and Yap [13] earlier for the case when \((k, l) = (2, 1)\) (that is, when \(A\) is sum-free) and \(G \cong \mathbb{Z}_p\) with \(p\) not congruent to 1 mod 3; however, for \(p = 3m + 1\) the set
\[
A = \{m, m + 2, m + 3, \ldots, 2m - 1, 2m + 1\}
\]
is also sum-free with maximum size. More generally, the answer to Question 3 is known for \((k, l) = (2, 1)\) and when \(n\) has at least one divisor not congruent to 1 mod 3: in this case \(A\) is the union of arithmetic progressions of the same length. More precisely, there is a subgroup \(H\) in \(G\) so that \(G/H\) is cyclic and
\[
A = \{(a + H) \cup (a + d + H) \cup \cdots \cup (a + c \cdot d + H)\}
\]
for some \(a, d \in G\) and integer \(c\). These and other results can be found in [31].

More ambitiously, one may ask for a characterization of all “large” (but not necessarily maximal) \((k, l)\)-sum-free sets in \(G\). Can one, for example, describe explicitly all \((k, l)\)-sum-free sets of size greater than \(n/(k + l)\)? Hamidoune and Plagne [17] carry this out for sum-free sets of size at least \(n/3\) in arbitrary groups. Other results can be found in the papers of Davydov and Tombak [12] and Lev [21], [22].

Our final question is about the number of \((k, l)\)-sum-free subsets in \(G\), which we here denote by \(N_{k,l}(G)\).

Question 4  What is the cardinality \(N_{k,l}(G)\) of the set of \((k, l)\)-sum-free subsets in \(G\)?

Clearly, any subset of a \((k, l)\)-sum-free set is also \((k, l)\)-sum-free, so the answer to Question 4 is at least
\[
N_{k,l}(G) \geq 2^{\lambda_{k,l}(G)}.
\]
But there are indications that the number is not much larger. In fact, for sum-free sets we have the following result of Green and Ruzsa [15]:
\[
N_{2,1}(G) = 2^{\lambda_{2,1}(G) + o(1)n},
\]
where \(o(1)\) approaches zero as \(n\) goes to infinity. They have a more accurate approximation for the case when \(n\) has a prime divisor which is congruent to 2 mod 3. (This result had been established for even \(n\) earlier by Lev, Luczak, and Schoen [23] and independently by Sapozhenko [28].)

In closing, we mention that the analogues of our questions about the maximum size, the structure, and the number of \((k, l)\)-sum-free sets (especially sum-free sets) have been investigated in non-abelian groups (see Kedlaya’s papers [18] and [19]) and, more extensively, among the positive integers (see the works of Alon [1], Bilu [6], Calkin [7], Calkin and Taylor [8], Cameron [9], Cameron and Erdős [10] and [11], and Luczak and Schoen [24]). General background references on related questions include Nathanson’s book [25], Guy’s book [19], and Ruzsa’s papers [26] and [27]; see also [3] and [4].
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