POSITIVITY PROPERTIES OF THE MATRIX \([(i + j)^{i+j}]\)

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Abstract. Let \(p_1 < p_2 < \cdots < p_n\) be positive real numbers. It is shown that the matrix whose \(i, j\) entry is \((p_i + p_j)^{p_i + p_j}\) is infinitely divisible, nonsingular and totally positive.

1. Introduction

Matrices whose entries are obtained by assembling natural numbers in special ways often possess interesting properties. The most famous example of such a matrix is the Hilbert matrix \(H = \left[ \frac{1}{i+j-1} \right]\) which has inspired a lot of work in diverse areas. Some others are the min matrix \(M = [\min(i, j)]\), and the Pascal matrix \(P = [\binom{i+j}{i}]\). There is a considerable body of literature around each of these matrices, a sample of which can be found in [3], [5] and [7].

In this note we initiate the study of one more matrix of this type. Let \(A\) be the \(n \times n\) matrix with its \((i, j)\) entry equal to \((i + j - 1)^{i+j-1}\). Thus

\[
A = \begin{bmatrix}
1 & 2^2 & 3^3 & \cdots & n^n \\
2^2 & 3^3 & 4^4 & \cdots & (n+1)^{n+1} \\
3^3 & 4^4 & 5^5 & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
n^n & \cdots & \cdots & \cdots & (2n-1)^{2n-1}
\end{bmatrix}.
\]  

More generally, let \(p_1 < p_2 < \cdots < p_n\) be positive real numbers, and consider the \(n \times n\) matrix

\[
B = \left[ (p_i + p_j)^{p_i + p_j} \right].
\]  

The special choice \(p_i = i - 1/2\) in (2) gives us the matrix (1). We investigate the behaviour of these matrices with respect to different
kinds of positivity.

A real symmetric matrix $S$ is said to be \textit{positive semidefinite} (psd) if for every vector $x$, we have $\langle x, Sx \rangle \geq 0$. Further if $\langle x, Sx \rangle = 0$ only when $x = 0$, then we say $S$ is \textit{positive definite}. This is equivalent to saying that $S$ is psd and nonsingular. If $S$ is a psd matrix, then for every positive integer $m$, the $m$th \textit{Hadamard power} (entrywise power) $S^m = [s_{ij}^m]$ is also psd. Now suppose $s_{ij} \geq 0$. We say that $S$ is \textit{infinitely divisible} if for every real number $r > 0$, the matrix $S^r = [s_{ij}^r]$ is psd. (See [3], Chapter 5 of [4], and Chapter 7 of [9] for expositions of this topic.) The principal minors of a psd matrix are nonnegative. This may not be so for other minors. A matrix with nonnegative entries is called \textit{totally positive} if all its minors are nonnegative. It is called \textit{strictly totally positive} if all its minors are positive. We recommend the books [8, 10, 11] and the survey article [1] for an account of totally positive matrices.

Our main result is the following:

\textbf{Theorem.} Let $p_1 < p_2 < \cdots < p_n$ be positive real numbers. Then the matrix $B$ defined in (2) is infinitely divisible, nonsingular and totally positive.

There is another way of stating this. Let $X$ be a subset of $\mathbb{R}$. A continuous function $K : X \times X \to \mathbb{R}$ is said to be a \textit{positive definite kernel} if for every $n$ and for every choice $x_1 < \cdots < x_n$ in $X$, the matrix $[K(x_i, x_j)]$ is positive definite. In the same way we can define infinitely divisible and totally positive kernels. Our theorem says that the kernel $K(x, y) = (x + y)^{x+y}$ on $(0, \infty) \times (0, \infty)$ is infinitely divisible and totally positive. This is an addition to the examples given in [3, 8, 10, 11]. The three matrices in the first paragraph also have the properties mentioned in the theorem.

2. Proof

Let $H_1$ be the space of all vectors $x = (x_1, \ldots, x_n)$ with $\sum x_i = 0$. A real symmetric matrix $S$ is said to be \textit{conditionally positive definite} (cpd) if $\langle x, Sx \rangle \geq 0$ for all $x \in H_1$. If $-S$ is cpd, then $S$ is said to be \textit{conditionally negative definite} (cnd). According to a theorem of C. Loewner, a matrix $S = [s_{ij}]$ is infinitely divisible if and only if the matrix $[\log s_{ij}]$ is cpd. See Exercise 5.6.15 in [4].
By Loewner’s theorem cited above, in order to prove that the matrix $B$ defined in (2) is infinitely divisible it is enough to show that the matrix
\[ C = [(p_i + p_j) \log(p_i + p_j)] \] (3)
is cpd. It is convenient to use the formula
\[ \log x = \int_0^\infty \left( \frac{1}{1 + \lambda} - \frac{1}{x + \lambda} \right) d\lambda, \quad x > 0, \]
which can be easily verified. Using this we can write our matrix $C$ as
\[ C = \left[ \int_0^\infty \left( \frac{p_i + p_j}{1 + \lambda} - \frac{p_i + p_j}{p_i + p_j + \lambda} \right) d\lambda \right]. \]
We will show that the matrix $[p_i + p_j]$ is cpd, and the matrix $\left[ \frac{p_i + p_j}{p_i + p_j + \lambda} \right]$ is cnd for each $\lambda > 0$. From this it follows that $C$ is a cpd matrix.

Let $D$ be the diagonal matrix $D = \text{diag}(p_1, ..., p_n)$ and $E$ the matrix with all its entries equal to 1. Then $[p_i + p_j] = DE + ED$. Every vector $x$ in $H_1$ is annihilated by $E$. Hence $\langle x, (DE + ED)x \rangle = \langle x, DEx \rangle + \langle Ex, Dx \rangle = 0$. So, the matrix $[p_i + p_j]$ is cpd. Using the identity
\[ \frac{p_i + p_j}{p_i + p_j + \lambda} = 1 - \frac{\lambda}{p_i + p_j + \lambda}, \]
we can write
\[ \left[ \frac{p_i + p_j}{p_i + p_j + \lambda} \right] = E - \lambda C_\lambda, \]
where $C_\lambda = \left[ \frac{1}{p_i + p_j + \lambda} \right]$. This is a Cauchy matrix (see [4]) and is positive definite. Hence it is also cpd. The matrix $E$ annihilates $H_1$, and therefore is cnd. Hence $E - \lambda C_\lambda$ is cnd for every $\lambda > 0$. This completes the proof of the assertion that $C$ is cpd, and $B$ infinitely divisible.

Since $C_\lambda$ is positive definite, $\langle x, C_\lambda x \rangle > 0$ for every non zero vector $x$. If $x \in H_1$, then $Ex = 0$, and $\langle x, (DE + ED)x \rangle = 0$. So, the arguments given above also show that $\langle x, Cx \rangle > 0$ for every non zero vector $x$ in $H_1$. By Lemma 4.3.5 in [2], this condition is necessary and sufficient for $C$ to be nonsingular. Using the next proposition, we can conclude that $B$ is nonsingular.

**Proposition.** If $C$ is a nonsingular conditionally positive definite matrix, then the matrix $[e^{c_{ij}}]$ is positive definite.
Proof. By Proposition 5.6.13 of [4] we can express $C$ as

$$C = P + YE + YE,$$

where $P$ is a psd matrix and $Y$ is a diagonal matrix. By Problem 7.5.P.25 in [9] the matrix $[e_{ij}]$ is positive definite unless $P$ has two equal columns. Suppose the $i$th column of $P$ is equal to its $j$th column.

Let $x$ be any vector with coordinates $x_i = -x_j 
eq 0$, and all other coordinates zero. Then $x \in H_1$ and $Px = 0$. Hence $\langle x,Cx \rangle = 0$. This is not possible since $C$ is a nonsingular cpd matrix. \hfill \Box

We have proved that the matrix $B$ is infinitely divisible and nonsingular. These properties are inherited by the matrix $A$ defined in (1). This is, moreover, a Hankel matrix; i.e, each of its antidiagonals has the same entry. Theorem 4.4 of [11] gives a simple criterion for strict total positivity of such a matrix. According to this a Hankel matrix $A$ is strictly totally positive if and only if $A$ is positive definite and so is the matrix $\tilde{A}$ obtained from $A$ by deleting its first column and last row. For the matrix $A$ in (1), $\tilde{A}$ is the $(n-1) \times (n-1)$ matrix whose $(i,j)$ entry is $(i+j)^{i+j}$. Both $A$ and $\tilde{A}$ are positive definite. Hence $A$ is strictly totally positive. In fact we have shown that for every $r > 0$, the matrix $A^{or}$ is strictly totally positive.

Now let $k_1 < k_2 < \cdots < k_n$ be positive integers. The matrix $K$ with entries $k_{ij} = (k_i+k_j)^{k_i+k_j}$ is principal submatrix of $A$. Hence it is infinitely divisible and strictly totally positive. The same holds for $K^{or}$ for every $r > 0$. Next let $0 < q_1 < q_2 < \cdots < q_n$ be rational numbers. Let $q_j = \frac{l_j}{m_j}$, where $l_j$ and $m_j$ are positive integers. Let $m$ be the LCM of $m_1, \ldots, m_n$ and $k_j = m q_j$. Then $k_1 < k_2 < \cdots < k_n$, and as seen above, the matrix $K = [(k_i+k_j)^{k_i+k_j}]$ is infinitely divisible and strictly totally positive. Now consider the matrix $Q = [(q_i+q_j)^{q_i+q_j}]$. Then for each $r > 0$

$$Q^{or} = [(q_i+q_j)^{(q_i+q_j)r}],$$

$$= \left[\frac{(k_i+k_j)^{(k_i+k_j)r/m}}{m^{l_i/m} m^{l_j/m}}\right],$$

$$= XK^{or}/m X^*,$$

where $X$ is the positive diagonal matrix with entries $\frac{1}{m^{l_i/m}}, \ldots, \frac{1}{m^{l_n/m}}$ on its diagonal. We have seen that the matrix $K^{or}/m$ is positive definite.
and strictly totally positive. Hence, so is the matrix \( Q^r \). A continuity argument completes the proof of the theorem.

We believe that the matrix \( B \) in (2) is strictly totally positive. However, the continuity argument that we have invoked at the last step only shows that it is a limit of such matrices.

Like for the other matrices mentioned in the opening paragraph, it would be interesting to have formulas for the determinant of \( A \).

In a recent work [6] of ours, we have studied spectral properties of the matrices \([ (p_i + p_j)^r \] \), where \( r \) is any positive real number.

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