SUPERCYCLIDIC NETS

ALEXANDER I. BOBENKO, EMANUEL HUHNEN-VENEDEY, AND THILO RÖRIG

Abstract. Supercyclides are surfaces with a characteristic conjugate parametrization consisting of two families of conics. Patches of supercyclides can be adapted to a Q-net (a discrete quadrilateral net with planar faces) such that neighboring surface patches share tangent planes along common boundary curves. We call the resulting patchworks “supercyclidic nets” and show that every Q-net in $\mathbb{RP}^3$ can be extended to a supercyclidic net. The construction is governed by a multidimensionally consistent 3D system. One essential aspect of the theory is the extension of a given Q-net in $\mathbb{RP}^N$ to a system of circumscribed discrete torsal line systems. We present a description of the latter in terms of projective reflections that generalizes the systems of orthogonal reflections which govern the extension of circular nets to cyclidic nets by means of Dupin cyclide patches.

1. Introduction

Discrete differential geometry aims at the development of discrete equivalents of notions and methods of classical differential geometry. One prominent example is the discretization of parametrized surfaces and the related theory, where it is natural to discretize parametrized surfaces by quadrilateral nets (also called quadrilateral meshes). In contrast to other discretizations of surfaces as, e.g., triangulated surfaces, a quadrilateral mesh reflects the combinatorial structure of parameter lines. While unspecified quadrilateral nets discretize arbitrary parametrizations, the discretization of distinguished types of parametrizations yields quadrilateral nets with special properties.

The present work is on the piecewise smooth discretization of classical conjugate nets by supercyclidic nets. They arise as an extension of the well established, integrable discretization of conjugate nets by quadrilateral nets with planar faces, the latter often called Q-nets in discrete differential geometry. Two-dimensional Q-nets as discrete versions of conjugate surface parametrizations were proposed by Sauer in the 1930s [35]. Multidimensional Q-nets are a subject of modern research [14, 8]. The surface patches that we use for the extension are pieces of supercyclides, a class of surfaces in projective 3-space that we discuss in detail in Section 4. Supercyclides possess conjugate parametrizations with all parameter lines being conics and such that there exists a quadratic tangency cone along each such conic. As a consequence, isoparametrically bounded surface patches coming from those characteristic parametrizations (referred to as SC-patches) always have coplanar vertices, which makes them suitable for the extension of Q-nets to piecewise smooth objects. The 2-dimensional case of such extensions has been proposed previously in the context of Computer Aided Geometric Design (CAGD) [31, 34, 12], but to the best of our knowledge has not been worked out so far.

2010 Mathematics Subject Classification. 51A05, 53A20, 37K25, 65D17.

Year words and phrases. Discrete differential geometry, projective geometry, discrete integrability, discrete conjugate nets (Q-nets), fundamental line systems, supercyclides, surface transformations, architectural geometry.

This research was supported by the DFG Collaborative Research Center TRR 109 “Discretization in Geometry and Dynamics” – http://www.discretization.de.
Based on established notions of discrete differential geometry, we describe in this article the multidimensionally consistent, piecewise smooth extension of \( m \)-dimensional Q-nets in \( \mathbb{R}^3 \) by adapted surface patches, such that edge-adjacent patches in one and the same coordinate plane have coinciding tangent planes along a common boundary curve (see Fig. 1 for a 2-dimensional example). Relevant aspects of the existing theory are presented in Sections 2 to 4 and succeeded by new results which are organized as follows:

- We describe the extension of 2D Q-nets to supercyclidic nets in Section 5, emphasizing the underlying 2D system that governs the extension of Q-nets to fundamental line systems.
- The 3D system that governs multidimensional supercyclidic nets is uncovered and analyzed in Section 6. We also present piecewise smooth conjugate coordinate systems that are locally induced by 3D supercyclidic nets and give rise to arbitrary Q-refinements of their support structures.
- We define \( m \)-D supercyclidic nets and develop a transformation theory thereof that appears as a combination of the existing smooth and discrete theories in Section 7.
- Finally, we introduce frames of supercyclidic nets and describe a related integrable system on projective reflections in Section 8.

Figure 1. A Q-net and its extension to a supercyclidic net. Each elementary quadrilateral is replaced by an adapted SC-patch, that is, a supercyclidic patch whose vertices match the supporting quadrilateral.

The present work is a continuation of our previous works on the discretizations of smooth orthogonal and asymptotic nets by cyclidic and hyperbolic nets, respectively [5, 20, 21]. We will now review the corresponding theory for cyclidic nets and illustrate the core concepts that may serve as a structural guideline for this article.

**Cyclidic nets.** Three-dimensional orthogonal nets in \( \mathbb{R}^3 \) are triply orthogonal coordinate systems, while 2-dimensional orthogonal nets are curvature line parametrized surfaces. Based on the well-known discretization of orthogonal nets by circular nets [4, 9, 23, 6], cyclidic nets are defined as circular nets with an additional structure, that is, as circular nets with orthonormal frames at vertices that determine a unique adapted DC-patch for each elementary quadrilateral with the prescribed circular points as vertices. The term “DC-patches” refers to surface patches of Dupin cyclides\(^1\) that are bounded by curvature lines, i.e., circular arcs. For a 2-dimensional cyclidic net its frames are such that the adapted DC-patches constitute a piecewise smooth \( C^1 \)-surface, cf. Fig. 2.

For any DC-patch the tangent planes at the vertices are tangent to a common cone of revolution. Therefore, the tangent planes to a 2D cyclidic net at the vertices constitute a

\(^1\)Dupin cyclides are surfaces in 3-space that are characterized by the property that all curvature lines are circles. They are special instances of supercyclides.
A 2-dimensional cyclidic net may be understood as a piecewise smooth $C^1$-surface composed of DC-patches.

Going beyond the discretization of curvature line parametrized surfaces, higher dimensional cyclidic nets provide a discretization of orthogonal coordinate systems that is motivated by the classical Dupin theorem. The latter states that the coordinate surfaces of a triply orthogonal coordinate system are curvature line parametrized surfaces. Accordingly, a 3-dimensional cyclidic net is a 3D circular net with frames at vertices that describe 2D cyclidic nets in each coordinate plane and such that for each edge of $\mathbb{Z}^3$ one has one unique circular arc that is a shared boundary curve of all adjacent DC-patches, cf. Fig. 3.

Cyclidic nets are piecewise smooth extensions of discrete integrable support structures by surface patches of a particular class that is in some sense as easy as possible but at the same time as flexible as necessary: We use DC-patches for the discretization of curvature line parametrized surfaces and those patches are actually characterized by the geometric property that all curvature lines are circular arcs, that is, the simplest non-straight curves. At the same time, DC-patches are flexible enough in order to be fit together to adapted $C^1$-surfaces. Moreover, the class of DC-patches is preserved by Möbius transformations, which is in line with the transformation group principle for the discretization of integrable geometries [8]. But there is also a deeper reason to choose Dupin cyclides for the extension of circular nets, that is, they actually embody the geometric characterizations of the latter in the following sense: It is not difficult to see that Dupin cyclides are characterized by the fact that arbitrary selections of curvature lines constitute a principal contact element net (a circular net with normals at vertices that are related by reflections in symmetry planes). As a consequence, cyclidic nets induce arbitrary refinements of their support structures within the class of circular nets, by simply adding parameter lines of the adapted patches.
to the support structures. After all, Dupin cyclides provide a perfect link between the theories of smooth and discrete orthogonal nets. This is also reflected by the theory of transformations of cyclidic nets, which arises by combination of the corresponding smooth and discrete theories. It turns out that supercyclidic patches play an analogous role for the piecewise smooth extension of Q-nets.

Analogous to the theory of cyclidic nets is the theory of hyperbolic nets as piecewise smooth discretizations of surfaces in 3-space that are parametrized along asymptotic lines. On a purely discrete level, such nets are properly discretized by quadrilateral nets with planar vertex stars [35, 8]. Based on that discretization, hyperbolic nets arise as an extension of A-nets by means of hyperbolic surface patches, which is analogous to the extension of 2D circular nets to 2D cyclidic nets via DC-patches. Many results concerning cyclidic nets on the one hand and hyperbolic nets on the other hand correspond to each other under Sophus Lie’s line-sphere-correspondence.

**Supercyclides in CAGD and applications of supercyclidic nets.** Supercyclides have been introduced as double Blutel surfaces by Degen in the 1980s [10, 11]. Soon they found attention in Computer Aided Geometric Design for several reasons. For example their pleasant geometric properties, nicely reviewed in [12], allow to use parts of supercyclides for blends between quadrics [1, 18]. Supercyclidic patchworks with differentiable joins have been proposed in the past [31, 12] as a continuation of the discussion of composite $C^1$-surfaces built from DC-patches within the CAGD community (see, e.g., [25, 27, 26, 16, 37] for the latter).

Another field of potential application for supercyclidic nets is architectural geometry (see the comprehensive book [28] for an introduction to that synthesis of architecture and geometry), because they are aesthetically appealing freeform surfaces that can be approximated at arbitrary precision by flat panels. On the other hand, supercyclidic nets extend their supporting Q-net and may be seen as a kind of 3D texture for that support structure. Purely Dupin cyclidic nets in 3-space, which are special instances of supercyclidic nets, have already been discussed in the context of circular arc structures in [3], while [22, 36] provides an analysis of hyperbolic nets that aims at the application thereof in architecture.

## 2. Q-nets and discrete torsal line systems

Q-nets are discrete versions of classical conjugate nets and closely related to discrete torsal line systems. Before starting our discussion of those discrete objects, we recall a characterization of classical conjugate nets and their transformations that can be found in [17] or [8, Chap. 1].

**Definition 1** (Conjugate net). A map $x: \mathbb{R}^m \to \mathbb{R}^N$, $N \geq 3$, is called an $m$-dimensional conjugate net in $\mathbb{R}^N$ if at every point of the domain and for all pairs $1 \leq i \neq j \leq m$ one has $\partial_{ij}x \in \text{span}(\partial_i x, \partial_j x)$ $\iff$ $\partial_{ij}x = c_{ji} \partial_i x + c_{ij} \partial_j x$.

**Definition 2** (F-transformation of conjugate nets). Two $m$-dimensional conjugate nets $x, x^+$ are said to be related by a fundamental transformation (F-transformation) if at every point of the domain and for each $1 \leq i \leq m$ the three vectors $\partial_i x, \partial_i x^+, \delta x = x^+ - x$ are coplanar. The net $x^+$ is called an F-transform of the net $x$.

The above F-transformations of conjugate nets exhibit the following, Bianchi-type permutability properties. The existence of associated transformations with permutability properties of that kind is classically regarded as a key feature of integrable systems.

2Loosely speaking, infinitesimal quadrilaterals formed by parameter lines of a conjugate net are planar.
Theorem 3 (Permutability properties of F-transformations of conjugate nets).

(i) Let \( x \) be an \( m \)-dimensional conjugate net, and let \( x^{(1)} \) and \( x^{(2)} \) be two of its \( F \)-transforms. Then there exists a 2-parameter family of conjugate nets \( x^{(12)} \) that are \( F \)-transforms of both \( x^{(1)} \) and \( x^{(2)} \). The corresponding points of the four conjugate nets \( x, x^{(1)}, x^{(2)} \) and \( x^{(12)} \) are coplanar.

(ii) Let \( x \) be an \( m \)-dimensional conjugate net. Let \( x^{(1)}, x^{(2)} \) and \( x^{(3)} \) be three of its \( F \)-transforms, and let three further conjugate nets \( x^{(12)}, x^{(23)} \) and \( x^{(13)} \) be given such that \( x^{(ij)} \) is a simultaneous \( F \)-transform of \( x^{(i)} \) and \( x^{(j)} \). Then generically there exists a unique conjugate net \( x^{(123)} \) that is an \( F \)-transform of \( x^{(12)}, x^{(23)} \) and \( x^{(13)} \). The net \( x^{(123)} \) is uniquely defined by the condition that for every permutation \( (ijk) \) of \( (123) \) the corresponding points of \( x^{(i)}, x^{(ij)}, x^{(ik)} \) and \( x^{(123)} \) are coplanar.

Although we gave an affine description, conjugate nets and their transformations are objects of projective differential geometry. Accordingly, we consider the theory of discrete conjugate nets in \( \mathbb{R}P^N \) and not in \( \mathbb{R}^N \). Before coming to that, we explain some notation that will be used throughout this article.

**Notation for discrete maps.** Discrete maps are fundamental in discrete differential geometry. We are mostly concerned with discrete maps defined on cells of dimension 0, 1, or 2 of \( \mathbb{Z}^m \), that is, maps defined on vertices, edges, or elementary quadrilaterals.

Let \( e_1, \ldots, e_m \) be the canonical basis of the \( m \)-dimensional lattice \( \mathbb{Z}^m \). For \( k \) pairwise distinct indices \( i_1, \ldots, i_k \in \{1, \ldots, m\} \) we denote by

\[
\mathcal{B}^{i_1\ldots i_k} = \text{span}_\mathbb{Z}(e_{i_1}, \ldots, e_{i_k})
\]

the \( k \)-dimensional coordinate plane of \( \mathbb{Z}^m \) and by

\[
\mathcal{C}^{i_1\ldots i_k}(z) = \{ z + \varepsilon_{i_1} e_{i_1} + \cdots + \varepsilon_{i_k} e_{i_k} \mid \varepsilon_i = 0, 1 \}
\]

the \( k \)-cell at \( z \) spanned by \( e_{i_1}, \ldots, e_{i_k} \), respectively.

We use upper indices \( i_1, \ldots, i_k \) to describe maps on \( k \)-cells as maps on \( \mathbb{Z}^m \) by identifying the \( k \)-cell \( \mathcal{C}^{i_1\ldots i_k}(z) \) with its basepoint \( z \)

\[
f^{i_1\ldots i_k}(z) = f(\mathcal{C}^{i_1\ldots i_k}(z)).
\]

For a map \( f \) defined on \( \mathbb{Z}^m \) we use lower indices to indicate shifts in coordinate directions

\[
f_i(z) = f(z + e_i), \quad f_{ij}(z) = f(z + e_i + e_j), \quad \ldots
\]

Often we omit the argument of \( f \), writing

\[
f = f(z), \quad f_i = f(z + e_i), \quad f_{-i} = f(z - e_i), \quad \ldots
\]

and analogous for maps defined on cells of higher dimension.

**Q-nets and discrete torsal line systems.** Smooth conjugate nets are discretized within discrete differential geometry by quadrilateral meshes with planar faces. The planarity condition is a straightforward discretization of the smooth characteristic property \( \partial_i f \in \text{span}(\partial_i f, \partial_j f) \). In the 2-dimensional case, this discretization has been proposed by Sauer [35] and was later generalized to the multidimensional case [14, 8].

**Definition 4 (Q-net).** A map \( x : \mathbb{Z}^m \to \mathbb{R}P^N, N \geq 3 \), is called an \( m \)-dimensional Q-net or discrete conjugate net in \( \mathbb{R}P^N \) if for all pairs \( 1 \leq i < j \leq m \) the elementary quadrilaterals \( (x_i, x_j, x_{ij}, x_j) \) are planar.
Closely related to Q-nets in $\mathbb{RP}^N$ are configurations of lines in $\mathbb{RP}^N$ with, e.g., $\mathbb{Z}^m$ combinatorics, such that neighbouring lines intersect. We denote the manifold of lines in $\mathbb{RP}^N$ by

$$\mathcal{L}^N := \{\text{Lines in } \mathbb{RP}^N\} \cong \text{Gr}(2, \mathbb{R}^{N+1}),$$

where $\text{Gr}(2, \mathbb{R}^{N+1})$ is the Grassmanian of 2-dimensional linear subspaces of $\mathbb{R}^{N+1}$.

**Definition 5** (Discrete torsal line system). A map $l : \mathbb{Z}^m \to \mathcal{L}^N$, $N \geq 3$, is called an $m$-dimensional discrete torsal line system in $\mathbb{RP}^N$ if at each $z \in \mathbb{Z}^m$ and for all $1 \leq i \leq m$ the neighbouring lines $l$ and $l_i$ intersect. We say that $l$ is generic if

(i) For each elementary quadrilateral of $\mathbb{Z}^m$ the lines associated with opposite vertices are skew (and therefore span a unique 3-space that contains all four lines of the quadrilateral).

(ii) If $m \geq 3$ and $N \geq 4$, the space spanned by any quadruple $(l, l_i, l_j, l_k)$ of lines, $1 \leq i < j < k \leq m$, is 4-dimensional.

A 2-dimensional line system is called a line congruence and a 3-dimensional line system is called a line complex.

Recently, line systems on triangle meshes with non-intersecting neighboring lines were studied [39]. To distinguish the two different types of line systems we call the systems with intersecting neighboring lines torsal. Discrete torsal line systems and Q-nets are closely related [15].

**Definition 6** (Focal net). For a discrete torsal line system $l : \mathbb{Z}^m \to \mathcal{L}^N$ and a direction $i \in \{1, \ldots, m\}$, the $i$-th focal net $f^i : \mathbb{Z}^m \to \mathbb{RP}^N$ is defined by

$$f^i(z) := l(z) \cap l(z + e_i).$$

The planes spanned by adjacent lines of the system are called focal planes. Focal points and focal planes of a discrete torsal line system are naturally associated with edges of $\mathbb{Z}^m$, cf. Fig. 4.

![Figure 4](image)

**Definition 7** (Edge systems of Q-nets and Laplace transforms). Given a Q-net $x : \mathbb{Z}^m \to \mathbb{RP}^N$ and a direction $i \in \{1, \ldots, m\}$, we say that the extended edges of direction $i$ constitute the $i$-th edge system $e^i : \mathbb{Z}^m \to \mathcal{L}^N$. By definition, the $i$-th edge system is a discrete torsal line system, whose different focal nets are called Laplace transforms of $x$. Accordingly, the vertices of the different Laplace transforms are called Laplace points of certain directions with respect to the elementary quadrilaterals of $x$. 

---

A. I. BOBENKO, E. HUHNEN-VENEDEY, AND T. RÖRIG
Supercyclidic nets

Edge systems are called tangent systems in [15]. We prefer to call them edge systems to distinguish them from the tangents systems of supercyclidic nets (formally defined only in Section 8), which consist of tangents to surface patches at vertices of a supporting Q-net.

From simple dimension arguments one obtains the following

**Theorem 8** (Focal nets are Q-nets). For a generic discrete torsal line system \( l : \mathbb{Z}^m \to \mathcal{L}^N, N \geq 4 \), each focal net is a Q-net in \( \mathbb{R}^N \). Moreover, focal quadrilaterals of the type \( (f^i, f^i_j, f^i_k, f^i_l) \) are planar for arbitrary \( N \).

Underlying 3D systems that govern Q-nets and discrete torsal line systems.

Q-nets in \( \mathbb{R}P^N, N \geq 3 \), and also discrete torsal line systems in \( \mathbb{R}P^N, N \geq 4 \), are each governed by an associated discrete 3D system: generic data at seven vertices of a cube determine the remaining 8th value uniquely, cf. Fig. 5. In both cases, this is evident if one considers intersections of subspaces and counts the generic dimensions with respect to the ambient space. For example, consider the seven vertices \( x, x_i, x_{ij}, i, j = 1, 2, 3, i \neq j \), of an elementary cube of a Q-net in \( \mathbb{R}^N \). Those points necessarily lie in a 3-dimensional subspace and determine the value \( x_{123} \) uniquely as intersection point of the three planes \( \pi_{12} = x_3 \lor x_{13} \lor x_{23}, \pi_{23}^1 = x_1 \lor x_{12} \lor x_{13}, \) and \( \pi_{13}^2 = x_2 \lor x_{12} \lor x_{23} \).

**Figure 5.** The seven values \( f, f_i, f_{ij} \) (vertices of a Q-net or lines of a discrete torsal line system) determine the remaining value \( f_{123} \) uniquely.

3D systems of the above type allow to propagate Cauchy data

\[ f|_{B^{ij}}, \quad 1 \leq i < j \leq 3 \]

on the coordinate planes \( B^{12}, B^{23}, B^{13} \) to the whole of \( \mathbb{Z}^3 \) uniquely. In higher dimensions, the fact that the a priori overdetermined propagation of Cauchy data

\[ f|_{B^{ij}}, \quad 1 \leq i < j \leq m \]

to the whole of \( \mathbb{Z}^m \) is well-defined is referred to as multidimensional consistency of the underlying system and understood as its discrete integrability. Simple combinatorial considerations show that for discrete mD systems that allow to determine the value at one vertex of an mD cube from the values at the remaining vertices, the \((m + 1)D\) consistency implies \((m + k)D\) consistency for arbitrary \( k \geq 1 \). Indeed, this is the case for the systems governing Q-nets and discrete torsal line systems, cf. [8]. For future reference, we capture the above in

**Theorem 9.** Discrete torsal line systems in \( \mathbb{R}P^N, N \geq 4 \), as well as Q-nets in \( \mathbb{R}P^M, M \geq 3 \), are each governed by mD consistent 3D systems.

\( ^3 \)For \( m \geq 3 \) there are focal quadrilaterals of the type \( (f^i, f^j_i, f^j_k, f^j_l), i \neq j \neq k \neq i \), which are generically not planar in the case \( N = 3 \).
Multidimensional consistency assures the existence of associated transformations that exhibit Bianchi-type permutability properties in analogy to the corresponding classical integrable systems. In fact, it is a deep result of discrete differential geometry that on the discrete level discrete integrable nets and their transformations become indistinguishable. A prominent example for this scheme are F-transformations of Q-nets. We simply discretize the defining property captured in Definition 2 by replacing the partial derivatives $\partial_i x, \partial_i x^+$ by difference vectors.

**Definition 10 (F-transformation of Q-nets).** Two $m$-dimensional Q-nets $x, x^+: \mathbb{Z}^m \to \mathbb{R}^P_N$ are called F-transforms (fundamental transforms) of one another if at each $z \in \mathbb{Z}^m$ and for all $1 \leq i \leq m$ the quadrilaterals $(x, x_i, x_i^+, x^+)$ are planar.

Obviously, the condition in Definition 10 may be re-phrased as follows: The Q-nets $x$ and $x^+$ are F-transforms of one another if the net $X: \mathbb{Z}^m \times \{0, 1\} \to \mathbb{R}^P_N$ defined by $X(z, 0) = x(z)$ and $X(z, 1) = x^+(z)$ is a two-layer $(m+1)$-dimensional Q-net. Therefore, the existence of F-transforms of Q-nets with permutability properties analogous to the classical situation is a simple consequence of the multidimensional consistency of Q-nets. One obtains

**Theorem 11 (Permutability properties of F-transformations of Q-nets).**

(i) Let $x$ be an $m$-dimensional Q-net, and let $x^{(1)}$ and $x^{(2)}$ be two of its discrete F-transforms. Then there exists a 2-parameter family of Q-nets $x^{(12)}$ that are discrete F-transforms of both $x^{(1)}$ and $x^{(2)}$. The corresponding points of the four Q-nets $x$, $x^{(1)}$, $x^{(2)}$ and $x^{(12)}$ are coplanar. The net $x^{(12)}$ is uniquely determined by one of its points.

(ii) Let $x$ be an $m$-dimensional Q-net. Let $x^{(1)}$, $x^{(2)}$ and $x^{(3)}$ be three of its discrete F-transforms, and let three further Q-nets $x^{(12)}$, $x^{(23)}$ and $x^{(13)}$ be given such that $x^{(ij)}$ is a simultaneous discrete F-transform of $x^{(i)}$ and $x^{(j)}$. Then generically there exists a unique Q-net $x^{(123)}$ that is a discrete F-transform of $x^{(12)}$, $x^{(23)}$ and $x^{(13)}$. The net $x^{(123)}$ is uniquely determined by the condition that for every permutation $(ijk)$ of $(123)$ the corresponding points of $x^{(i)}$, $x^{(ij)}$, $x^{(ik)}$ and $x^{(123)}$ are coplanar.

Theorem 11 yields the classical results captured by Theorem 3 after performing a refinement limit in the “net directions” while keeping the “transformation directions” discrete.

### 3. Fundamental line systems

While generic discrete torsal line systems in $\mathbb{R}^N_P, N \geq 4$, are governed by a 3D system, this is not the case in $\mathbb{R}^3_P$. The reason is that for three generic lines $l_{12}, l_{23}, l_{13}$ in 4-space there is a unique line that intersects all of them, while in $\mathbb{R}^3_P$ there is a whole 1-parameter family of such lines. However, one obtains a 3D system for line systems in $\mathbb{R}^3_P$ by demanding that the focal quadrilaterals be planar, which turns out to be an admissible reduction of the considered configurations (Corollary 19).

**Definition 12 (Fundamental line system).** A discrete torsal line system is called fundamental if its focal nets are Q-nets and if it is generic in the sense that lines associated with opposite vertices of elementary quadrilaterals are skew. In particular, any generic discrete torsal line system in $\mathbb{R}^N_P, N \geq 4$, is fundamental according to Theorem 8.
Remark. The terminology was introduced in [7], where fundamental line systems are discussed in the context of integrable systems. It is shown that elementary cubes of such line systems in \( \mathbb{RP}^3 \) are characterized by the existence of a unique involution \( \tau \) on \( L^3 \), such that lines associated with opposite vertices of the cube are interchanged. The map \( \tau \) is determined by any six lines that remain if one removes a pair of opposite lines. For example, for fixed lines \( l_1, l_2, l_3, l_{12}, l_{23}, l_{13} \) one may express \( l_{123} \) as dependent on \( l \) according to \( l_{123} = \tau(l) \). Moreover, it turns out that \( \tau \) comes from a correlation of \( \mathbb{RP}^3 \) (a projective map from \( \mathbb{RP}^3 \) to its dual, that is, points and planes are interchanged but lines are mapped to lines).

In the following we prove essential properties of fundamental line systems that will play a key role for our later considerations of supercyclidic nets. Many of the presented statements appear in [15, Sect. 2]

Lemma 13. Elementary cubes of a fundamental line system are characterized by any of the following properties:

(i) One focal quadrilateral is planar.
(ii) All focal quadrilaterals are planar.
(iii) The four focal planes of one direction are concurrent, that is, intersect in one point.
(iv) For each direction the four associated focal planes are concurrent.

Proof. If the \( i \)-th focal quadrilateral \( Q^i = (f^i, f^j_i, f^k_i, f^j_k) \) is planar, then the \( j \)-th focal planes all pass through the Laplace point of \( Q^j \) in direction \( j \) and analogous for the \( k \)-th focal planes. Hence also the \( j \)-th and \( k \)-th focal quadrilaterals are planar, i.e., all focal quadrilaterals are planar. This in turn implies that also the \( i \)-th focal planes are concurrent and thus each family of focal planes is concurrent. Finally, this implies \( i \). \( \Box \)

Proposition 14. If one focal net of a discrete torsal line system is a Q-net, the line system is fundamental.

Proof. If the line system is at least 3-dimensional, there are two types of focal quadrilaterals of the \( i \)-th focal net: \( Q^{ij} = (f^i, f^j, f^i_j, f^j_j) \) and \( Q^{jk} = (f^i, f^j, f^i_k, f^j_k) \), \( i \neq j \neq k \neq i \). The vertices of \( Q^{ij} \) are contained in the plane \( l_i \vee l_j \) and hence \( Q^{ij} \) is planar. The planarity of \( Q^{jk} \) follows from Lemma 13 under the assumption of the proposition. \( \Box \)

Laplace transforms of generic Q-nets are Q-nets themselves. This follows from Proposition 14 since the \( i \)-th focal net of the \( i \)-th edge system is the original Q-net. Hence the following

Corollary 15. Edge systems of generic Q-nets are fundamental.

Definition 16. Let \( x: \mathbb{Z}^m \rightarrow \mathbb{RP}^N \) be a quadrilateral net and \( l: \mathbb{Z}^m \rightarrow L^N \) be a discrete line system. If \( x(z) \in l(z) \) for all \( z \in \mathbb{Z}^m \) we say that \( l \) is an extension of \( x \) and, conversely, that \( x \) is inscribed in \( l \).

Throughout this article, the extension of Q-nets to discrete torsal line systems as well as the construction of Q-nets inscribed in discrete torsal lines systems plays a prominent role (line systems and Q-nets related as in Definition 16 are called conjugate in [15]). In fact, it turns out that a discrete torsal line system is fundamental if and only if it is the extension of a Q-net. For a concise treatment of this relation and for later reference, we formulate the following evident
Lemma 17 (Construction schemes).
(Q) Construction of a planar quadrilateral inscribed in a torsal line congruence quadrilateral \((l, l_1, l_1, l_2, l_2)\): Three given points \(x \in l, x_1 \in l_1, \) and \(x_2 \in l_2\) determine the fourth point \(x_{12} = l_{12} \cap (x \vee x_1 \vee x_2)\) uniquely.
(L) Extension of a planar quadrilateral \((x, x_1, x_1, x_2, x_2)\) to a torsal line congruence quadrilateral: Three given lines \(l \ni x, l_1 \ni x_1, \) and \(l_2 \ni x_2\) determine the fourth line \(l_{12} = (x_{12} \vee l_1) \cap (x_{12} \vee l_2)\) uniquely.

Each of the above constructions \((Q)\) and \((L)\) describes a 2D system in the sense that 1-dimensional Cauchy data along two intersecting coordinate axes of \(\mathbb{Z}^2\) propagates uniquely onto the entire lattice.

Let \(l\) be a generic discrete torsal line system in \(\mathbb{R}P^N\) for \(N \geq 4\), which is automatically fundamental due to Theorem 8. Obviously the central projection of \(l\) from a point to a 3-dimensional subspace preserves the planarity of focal quadrilaterals and hence yields a fundamental line system in \(\mathbb{R}P^3\). It turns out that the converse is also true.

Theorem 18 (Characterization of fundamental line systems in \(\mathbb{R}P^3\)). For a discrete torsal line system \(l : \mathbb{Z}^n \rightarrow \mathcal{L}^3\), the following properties are equivalent:

(i) \(l\) is fundamental.
(ii) \(l\) is the projection of a generic discrete torsal line system in \(\mathbb{R}P^4\).
(iii) \(l\) is the extension of a generic Q-net.
(iv) The construction of Q-nets inscribed into \(l\) is consistent.

Proof. \([\text{ll}] \implies [\text{lll}]\). We start with the observation that for each elementary 3-cube of a fundamental line system \(l\) in \(\mathbb{R}P^3\), any seven lines determine the eighth line uniquely via the planarity condition for focal quadrilaterals. For the following, we embed \(l\) into \(\mathbb{R}P^4\) by identification of \(\mathbb{R}P^3\) with any hyperplane \(\pi \subset \mathbb{R}P^4\) and in a first step show that each individual cube may be derived via projection. More precisely, for its eigth lines \(l, l_1, l_2, l_{13}, l_{123}\) in \(\pi\) we show how to construct lines \(l_1, l_1, l_{123}\) in \(\mathbb{R}P^4\) that constitute a fundamental line complex cube, such that \(l = \tau(l_1, l_2, l_{123})\) with \(\tau\) being the projection of \(\mathbb{R}P^4\) from a point \(p \notin \pi\) to the hyperplane \(\pi\). To begin with, we may choose \(p\) to be any point not in \(\pi\). Further, we define \(l = l, l_1 = l_1, l_2 = l_2, l_{12} = l_{12} \in \pi\). The lift \(l_3\) may then be chosen to be any line in the plane \(l_3 \vee p\) that passes through \(l_1 \cap l_2\) and lies outside \(\pi\). This determines the lifts \(l_{13}\) and \(l_{23}\) as each of them is spanned by one focal point in \(\pi\) and one lifted focal point on \(l_3\), e.g., \(l_{13} = (l_1 \cap l_{13}) \vee (l_3 \cap ((l_{13} \cap l_3) \vee p))\). The seven lifted lines constitute generic Cauchy data for a fundamental line complex cube in \(\mathbb{R}P^4\) and hence there exists a unique eighth line \(l_{123}\) that may be constructed as

\[ l_{123} = (l_{12} \vee l_{13}) \cap (l_{12} \vee l_{23}) \cap (l_{13} \vee l_{23}) \]

The central projection \(\tau : \mathbb{R}P^4 \setminus \{p\} \rightarrow \pi\) then yields a fundamental line complex cube in \(\pi\). As seven lines coincide with the original cube by construction, the eighth line has to coincide by uniqueness. It remains to observe that the construction works also globally, since the lift is generic and thus the suggested construction may be extended to Cauchy data for the whole fundamental line system \(l\).

\([\text{lll}] \implies [\text{ll}]\). We have to show the 3D consistency of the 2D system that is described by Lemma 17 (Q) under the given assumptions. So consider an elementary cube of \(l\) and let \(x, x_1, x_2, x_3\) be four generic points on the lines \(l, l_1, l_2, l_3\), respectively. As before, consider a lift \(i\) of \(l\) to \(\mathbb{R}P^4\) and accordingly lifted points \(\hat{x}, \hat{x}_1, \hat{x}_2, \hat{x}_3\), the latter spanning a 3-dimensional subspace \(\hat{x} \vee \hat{x}_1 \vee \hat{x}_2 \vee \hat{x}_3\). This subspace defines a unique Q-cube inscribed
into \( \hat{l} \) via intersection with the lines of \( \hat{l} \). The projection of the \( Q \)-cube in \( \mathbb{RP}^4 \) yields a \( Q \)-cube in \( \mathbb{RP}^3 \), that is, the 2D system is 3D, and hence \( mD \) consistent.

If the construction of inscribed \( Q \)-nets is consistent, then we can construct an inscribed \( Q \)-net.

It is sufficient to show that each elementary 3-cube of \( l \) may be obtained as the projection of a fundamental line complex cube in \( \mathbb{RP}^4 \), since this implies that all focal nets are \( Q \)-nets. So assume you are given a torsal line complex cube \((l_1, \ldots, l_{123})\) with an inscribed \( Q \)-cube \( X = (x, \ldots, x_{123}) \) in \( \mathbb{RP}^3 \). The first thing to note is that the vertices \( X \) together with the lines \( l, l_1, l_2, \) and \( l_3 \) determines the remaining lines uniquely according to Lemma 17 (L). Now identify \( \mathbb{RP}^3 \) with any hyperplane \( \pi \) in \( \mathbb{RP}^4 \) and choose a point \( p \) outside \( \pi \), which defines the central projection \( \tau : \mathbb{RP}^4 \setminus \{p\} \rightarrow \pi \). The next step is to fix the vertices \( X \) in \( \pi \) and construct generic lifts \( \hat{l}, \hat{l}_1, \hat{l}_2, \hat{l}_3 \not\subset \pi \) of the lines \( l, l_1, l_2, \) and \( l_3 \) through the corresponding vertices that satisfy

\[
\hat{l} \cap \hat{l}_i \neq \emptyset, \quad \tau(\hat{l}) = l, \quad \tau(\hat{l}_i) = l_i, \quad i = 1, 2, 3.
\]

We will now show that Lemma 17 (L) consistently determines lines \( \hat{l}_{12}, \hat{l}_{23}, \hat{l}_{13}, \) and \( \hat{l}_{123} \) through the corresponding vertices of \( X \) for the given Cauchy data \( l, l_1, l_2, \) and \( l_3 \). This implies the assertion, since the solution \((\hat{l}, \ldots, \hat{l}_{123})\) is mapped under \( \tau \) to the solution \((l, \ldots, l_{123})\) of the original Cauchy problem because of [1] and \( \tau(X) = X \).

To see the consistency in \( \mathbb{RP}^4 \), first construct the lines \( \hat{l}_{ij} \) according to Lemma 17 (L). As we considered a generic lift, the intersection

\[
\hat{l}_{123} = (\hat{l}_{12} \vee \hat{l}_{13}) \cap (\hat{l}_{12} \vee \hat{l}_{23}) \cap (\hat{l}_{13} \vee \hat{l}_{23})
\]

of three generic hyperplanes in \( \mathbb{RP}^4 \) is 1-dimensional and completes a fundamental line complex cube in \( \mathbb{RP}^4 \). By construction, we also find \( x_{123} \in \hat{l}_{123} \).

The above proof of Theorem 18 reveals the following

**Corollary 19 (mD consistent systems related to fundamental line systems).**

(i) Fundamental line systems are governed by an mD consistent 3D system.

(ii) The construction of \( Q \)-nets inscribed in a discrete torsal line system according to Lemma 17 (Q) is mD consistent if and only if the line system is fundamental.

(iii) The extension of \( Q \)-nets to discrete torsal line systems according to Lemma 17 (L) is mD consistent and always yields fundamental line systems.

### 4. Supercyclides

Let \( \mathcal{M} \) be a surface in \( \mathbb{RP}^3 \) that is generated by a 1-parameter family of conics. If along each conic the tangent planes to \( \mathcal{M} \) envelop a quadratic cone, \( \mathcal{M} \) is called a **surface of Blutel**, cf. Fig. 6 left. If the conjugate curves to the conics of a surface of Blutel are also conics, the Blutel property is automatically satisfied for the conjugate family [10]. Accordingly, such surfaces had been introduced by Degen as **double Blutel surfaces** originally, but eventually became known to a wider audience under the name of **supercyclides**.

\(^4\)Most double Blutel surfaces are (complex) projective images of Dupin cyclides [11], so Degen later referred to double Blutel surfaces as “generalized cyclides” [12] – although this term was already used by Casey and Darboux for quartic surfaces that have the imaginary circle at infinity as a singular curve and hence generalize Dupin cyclides in a different way. Therefore, Pratt proposed the name “supercyclides” for a major subclass of double Blutel surfaces, which is characterized by a certain quartic equation and contains the projective images of quartic Dupin cyclides [31, 33]. Eventually, the term supercyclides was used for the whole class of double Blutel surfaces by Pratt and Degen [32, 13, 34].
12 A. I. BOBENKO, E. HUHNEN-VENEDEY, AND T. RÖRIG

Figure 6. Left: The tangent planes along a characteristic conic on a supercyclide envelop a quadratic cone. The generators of the cone are the tangents of the conjugate curves. Right: Restriction of a supercyclide to an SC-patch, that is, a surface patch bounded by parameter lines of a characteristic parametrization.

Definition 20 (Supercyclides and SC-patches). A surface in $\mathbb{R}P^3$ that is generated by two conjugate families of conics, such that the tangent planes along each conic envelop a quadratic cone, is called a supercyclide. We refer to those conics as characteristic conics and accordingly call a conjugate parametrization $f : U \to \mathbb{R}P^3$ of a supercyclide characteristic if its parameter lines are characteristic conics.

A (parametrized) supercyclidic patch (SC-patch for short) is the restriction of a characteristic parametrization to a closed rectangle $I_1 \times I_2 \subset U$, cf. Fig. 6, right.

In [10] it is shown that for each of the two families of characteristic conics on a supercyclide the supporting planes form a pencil. The axis of such a pencil consists of the vertices of the cones that are tangent to the respective conjugate family of characteristic conics.

Definition 21 (Characteristic lines). The axes of the pencils supporting the families of characteristic conics of a supercyclide are the characteristic lines.

Examples of supercyclides. Any non-degenerate quadric $Q$ in $\mathbb{R}P^3$ is a supercyclide in a manifold way: Simply take two lines $a^1$ and $a^2$ that are polar with respect to $Q$ and consider the plane pencils through those lines. The intersections of those planes with $Q$ constitute a conjugate network of conics that satisfy the tangent cone property. Another prominent subclass of supercyclides is given by Dupin cyclides, the latter being characterized by the fact that all curvature lines are circles. They are supercyclides, since a net of curvature lines is a special conjugate net and, moreover, the tangent cone property is satisfied. As special instances of Dupin cyclides it is worth to mention the three types of rotationally symmetric tori (ring, horn, and spindle – having 0, 1, and 2 singular points, respectively) and to recall that all other Dupin cyclides may be obtained from such tori by inversion in a sphere. Finally, since the defining properties of supercyclides are projectively invariant, it is clear that any projective image of a Dupin cyclide is a supercyclide.

A complete classification of supercyclides is tedious, see the early contributions [11, 2] and also the later [33, 32]. One fundamental result is that supercyclides are algebraic surfaces of, at most, degree four. While [11, 2] follow a classical differential geometric approach that relies essentially on convenient parametrizations derived from the double Blutel property, the treatment of supercyclides in [33, 32] is more of an algebraic nature. On the other hand, there is also a unified approach to the construction and classification of supercyclides that starts with a certain ruled 3-manifold in projective 5-space from which all supercyclides may be obtained by projection and intersection with a hyperplane [13]. The basis for this unified treatment can be found already in [11].

Genericity assumption. In this paper we only consider generic supercyclides, that is, supercyclides of degree four with skew characteristic lines.
Proposition 22 (Properties of SC-patches).

(i) The vertices of an SC-patch are coplanar.
(ii) Isoparametric points on opposite boundary curves are perspective from the non-corresponding Laplace point of the vertex quadrilateral, cf. Fig. 7, left.
(iii) For each coordinate direction, the tangents to the corresponding boundary curves at the four vertices of an SC-patch intersect cyclically. In particular, the eight tangents constitute an elementary hexahedron of a fundamental line complex, cf. Fig. 7, middle and right.
(iv) The tangent planes at the four vertices intersect in one point.

Figure 7. Opposite boundary curves of an SC-patch are perspective from the non-corresponding Laplace point of the vertex quadrilateral (left). The tangents to the boundary of an SC-patch at vertices constitute an elementary hexahedron of a fundamental line complex (middle and right: geometric and combinatorial, respectively).

Proof. We start with a fact concerning Blutel surfaces that may be found in [11], that is, for any surface of Blutel the conjugate curves induce projective transformations between the generating conics. From that it may be deduced easily that for a double Blutel surface those projective maps are perspectivities. Applied to opposite boundary curves of an SC-patch, we conclude that the patch vertices are coplanar and that the centers of perspectivity have to be the Laplace points of the vertex quadrilateral.

To verify the perspectivity statement, consider two characteristic conics $c_1$ and $c_2$ of one family on a supercyclide for which we assume that they are contained in distinct planes $\pi_1$ and $\pi_2$. According to the above, the conjugate family of conics induces a projective map $\tau: \pi_1 \rightarrow \pi_2$ that identifies isoparametric points on $c_1$ and $c_2$. In order to see that $\tau$ is a perspectivity, take three points $x_1, y_1, z_1$ on $c_1$ and denote the tangents in those points by $g_1, h_1, l_1$, respectively. Further let $(x_2, y_2, z_2, g_2, h_2, l_2) = \tau(x_1, y_1, z_1, g_1, h_1, l_1)$ be the corresponding points and tangents with respect to $c_2$. As, e.g., $x_1$ and $x_2$ are isoparametric, the tangents $g_1$ and $g_2$ intersect at the tip $x$ of the cone that is tangent to the supercyclide along the conjugate curve through $x_1$ and $x_2$. One obtains three tangent cone vertices $x, y, z$ of that kind and also the further intersection points $q_i = g_i \cap h_i$, $r_i = g_i \cap l_i$, $s_i = h_i \cap l_i$, $i = 1, 2$, see Fig. 8.

From the observation $\tau(x, y, z, q_1, r_1, s_1) = (x, y, z, q_2, r_2, s_2)$ it follows that the restrictions of $\tau$ to the lines $g_1, h_1$, and $l_1$, all being projective maps, are determined by $(x, q_1, r_1) \mapsto (x, q_2, r_2)$, $(x, q_1, s_1) \mapsto (x, q_2, s_2)$, and $(x, s_1, r_1) \mapsto (x, s_2, r_2)$, respectively. On the other hand, the three planes $g_1 \vee h_2, h_1 \vee h_2,$ and $l_1 \vee l_2$ intersect in one point $O$, therefore the pairs $(q_1, q_2), (r_1, r_2),$ and $(s_1, s_2)$ of corresponding tangent intersections are each perspective from $O$. From this we may conclude that $\tau: \pi_1 \rightarrow \pi_2$ is the perspectivity from $O$. 
Figure 8. Triangles formed by the tangents to $c_1$ and $c_2$ in corresponding points $x_1, y_1, z_1$ and $x_2, y_2, z_2$, are perspective.

It remains to verify the assertions (iii) and (iv). Now the intersection statement of (iii) follows immediately from the tangent cone property and planarity of the characteristic conics and simply means that the eight tangents to the boundary curves at vertices constitute a line complex cube. The vertex quadrilateral of the patch is a focal quadrilateral for that complex cube and planarity of vertices implies that the complex cube is fundamental due to Lemma 13 (i). Therefore, the equivalent statement (iv) of Lemma 13 also holds and yields (iv) of this proposition. □

Remark 23. In the complexified setting, a generic quartic supercyclide possesses four isolated singularities – two at each characteristic line – and one singular conic [33]. In fact, it can be easily seen that real intersections of a supercyclide with its characteristic lines must be singularities: Suppose that the characteristic line of the first family (the axis of the pencil of planes supporting the first family of conics) intersects the cyclide in a point $x$. Generically, $x$ lies on a non-degenerate conic of the first family and Proposition 22 (ii) implies that all conics of the first family pass through $x$, that is, $x$ is a singular point. On the other hand, if a generic pair of conics of one family intersects (automatically at the corresponding characteristic line), the intersection points are singular points of the cyclide by the same argument.

Notation and terminology for SC-patches. We denote the vertices of an SC-patch $f : [a_0, a_1] \times [b_0, b_1] \to \mathbb{RP}^3$ by

$$x = f(a_0, b_0), \quad x_1 = f(a_1, b_0), \quad x_{12} = f(a_1, b_1), \quad x_2 = f(a_0, b_1),$$

which gives rise to the (oriented) vertex quadrilateral $(x, x_1, x_{12}, x_2)$. Conversely, we say that an SC-patch is adapted to a quadrilateral if it is the vertex quadrilateral of the patch. We define

$$\mathcal{A}^3 = \{ \text{conic arcs in } \mathbb{RP}^3 \}$$

and write the boundary curve of an SC-patch that connects the vertices $x$ and $x_i$ as $b^i \in \mathcal{A}^3$. The supporting plane of $b^i$ in $\mathbb{RP}^3$ is usually denoted by $\pi^i$ and we refer to those planes as boundary planes of the patch. Further, we denote by $t^i$ and $t^i_i$ the tangents to $b^i$ at $x$ and at $x_i$, respectively, as depicted in Fig. 9

Usually we refer to the characteristic lines of an SC-patch as $a^1$ and $a^2$, where $a^i$ is the line through the Laplace point

$$y^i = (x \lor x_i) \cap (x_j \lor x_{12}), \quad i \neq j.$$
Figure 9. The boundary of an SC-patch, labeled according to our usual terminology.

Obviously, the characteristic lines may be expressed in terms of tangents as

\[ a^i = (t^i \lor t^i_j) \cap (t^i_j \lor t^i_{12}), \quad i \neq j. \]

**Extension of planar quadrilaterals to SC-patches.** A supercyclic patch is uniquely determined by its boundary, since points \( \tilde{x}_i \in b^i, i = 1, 2 \), on the boundary of an SC-patch parametrize points \( \tilde{x}_{12} \) in the interior as follows: Given the characteristic lines \( a^i \) and \( a^2 \), the corresponding point \( \tilde{x}_{12} \) may be obtained as intersection of the three planes \((x \lor \tilde{x}_1 \lor \tilde{x}_2)\), \((\tilde{x}_1 \lor a^2)\), and \((\tilde{x}_2 \lor a^1)\).

According to Proposition 22, valid boundaries of SC-patches that are adapted to a given vertex quadrilateral may be constructed as follows: Two adjacent conical boundary curves may be chosen arbitrarily as well as the two opposite boundary planes. The remaining boundary curves are then determined as perspective images of the initial boundary curves from the corresponding Laplace points of the vertex quadrilateral. It turns out that it is very convenient to emphasize the tangents to boundary curves at vertices in the suggested construction, which gives rise to the following

**Construction of adapted SC-patches.** Denote the given (planar) vertex quadrilateral by \((x, x_1, x_{12}, x_2)\). For each \(i = 1, 2\), first choose \(t^i \ni x\) freely and then choose \(t^i_1 \ni x_1\) and \(t^i_2 \ni x_2\) such that \(t^i \cap t^i_j \neq \emptyset\). According to Proposition 22 (iii), the tangent \(t^i_{12}\) is then determined subject to \(t^i_{12} = (x_{12} \lor t^i_1) \cap (x_{12} \lor t^i_2)\). Further, for each coordinate direction choose a conic segment \(b^i\) in the plane \(\pi^i = t^i \lor t^i_1\) that is tangent to \(t^i\) at \(x\) and to \(t^i_1\) at \(x_1\), respectively. The opposite segment \(b^i_j\) is then obtained as the perspective image \(\tau^j(b^i)\), where

\[ \tau^j : \pi^i \to \pi^i_j, \quad p \mapsto (p \lor y^j) \cap \pi^i_j \]

is the perspectivity from the Laplace point \(y^j = e^j \lor e^j_i\).

**Remark 24.** In the above construction, we have exploited 10 independent degrees of freedom: For each direction \(i\), 4 DOF are used in the construction of the corresponding tangents and 1 DOF is used for the adapted boundary curve \(b^i\). The latter corresponds to the free choice of one additional point in the prescribed boundary plane, modulo points on the same segment.\(^5\)

\(^5\)Recall that a conic is uniquely determined by 5 points, where two infinitesimally close points correspond to a point and a tangent in that point (a 1-dimensional contact element).
5. 2D supercyclidic nets

We will now analyze the extension of a given $Q$-net $x: \mathbb{Z}^2 \to \mathbb{R}P^3$ by one adapted supercyclidic patch for each elementary quadrilateral. At the common edge of two neighboring quadrilaterals we require that the tangent cones coincide.

**Definition 25** (Tangent cone continuity). Let $Q$ and $\tilde{Q}$ be two planar quadrilaterals that share an edge $e$. Two adapted SC-patches $f$ and $\tilde{f}$ with common boundary curve $b$ that joins the vertices of $e$ are said to form a tangent cone continuous join, or TCC-join for short, if the tangency cones to $f$ and $\tilde{f}$ along $b$ coincide. We also say that $f$ and $\tilde{f}$ have the TCC-property and, accordingly, satisfy the TCC-condition.

![Figure 10. Pairs of SC-patches that form TCC joins, that is, tangency cones coincide along common boundary curves. As depicted, this does not exclude cusps.](image)

The TCC-condition yields a consistent theory for the extension of discrete conjugate nets by surface patches parametrized along conjugate directions in $\mathbb{R}P^3$. As a quadratic cone is determined by one non-degenerate conic section and two generators, one immediately obtains the following characterization of TCC-joins between SC-patches.

**Lemma 26.** Two SC-patches $f, \tilde{f}$ with common boundary curve $b$ form a TCC-join if and only if their tangents in the common vertices coincide.

Consider coordinates so that $\tilde{Q} = Q_i$ is the shift of $Q$ in direction $i$ and accordingly rename $(\tilde{f}, b) \sim (f_i, b'_i)$. If the relevant tangents coincide, they may be expressed as intersections of boundary planes $t^i_i = \pi^i_i \cap \pi^i_j$, $t^i_{ij} = \pi^i_i \cap \pi^i_{ij}$, and we see that the TCC-property implies that the four involved boundary planes of direction $i$ intersect in one point. Obviously, this means that the two characteristic lines $a^i = \pi^i_i \cap \pi^i_j$, $a^i_i = \pi^i_i \cap \pi^i_{ij}$ associated with the direction $i$ intersect – the point of intersection being the tip of the coinciding tangency cone.

**Definition 27** (2D supercyclidic net). A 2-dimensional $Q$-net together with SC-patches that are adapted to its elementary quadrilaterals is called a 2D supercyclidic net (2D SC-net) if each pair of patches adapted to edge-adjacent quadrilaterals has the TCC-property.

**Affine supercyclidic nets and $C^1$-joins.** An interesting subclass of supercyclidic nets are nets that form piecewise smooth $C^1$-surfaces, i.e., the joins of adjacent patches are not allowed to form cusps and the individual patches must not contain singularities (see Fig. 10). Concise statements concerning the $C^1$-subclass of tangent cone continuous supercyclidic patchworks are tedious to formulate and to prove. As a starting point for future investigation of the $C^1$-subclass we only capture some observations in the following.
A Q-net \( x : \mathbb{Z}^2 \rightarrow \mathbb{R}P^3 \) does not contain information about edges connecting adjacent vertices. In the affine setting in turn one has unique finite edges, which allow to distinguish between the different types of non-degenerate vertex quadrilaterals: convex, non-convex, and non-embedded. We note the following implications for adapted SC-patches:

(i) Finite SC-patches adapted to non-embedded vertex quadrilaterals always contain singular points: The intersection point of one pair of edges is the Laplace point and it lies on the characteristic line of the pencil containing the conic arcs. But if these arcs are finite they have to intersect the axis and hence contain a singular point of the surface, see Remark 23 and Fig. 11 left.

(ii) If the vertex quadrilateral is non-convex an adapted SC-patch cannot be finite: For any SC-patch, opposite boundary curves are perspective and hence lie on a unique cone whose vertex is one of the Laplace points of the vertex quadrilateral. For non-convex vertex quadrilaterals, the Laplace points are always contained in edges of the quadrilateral. Thus, the previously mentioned cones always give rise to a situation as depicted in Fig. 11 right. We observe that one of the edges \((x,x_2)\) or \((x_1,x_12)\) has to lie outside of the cone and the other one inside. Clearly, the conic boundary arc that connects the vertices of the outside segment has to be unbounded.

(iii) If the vertex quadrilateral is convex (and embedded) then there exist finite patches without singularities, but adapted patches have singularities if opposite boundary arcs intersect.

![Figure 11. SC-patches adapted to different types of affine quadrilaterals. Left: For finite patches adapted to non-convex quadrilaterals, two opposite arcs have to intersect. Right: Different shades show the possible pairs of perspective conic arcs for a non-convex vertex quadrilateral related by the perspectivity with respect to the Laplace point \(y^1\).](image)

Accordingly, only affine Q-nets composed of convex quadrilaterals may be extended to bounded \(C^1\)-surfaces (projective images thereof also being \(C^1\), but possibly not bounded). In that context, note that the cyclidic nets of [5] are defined as extensions of circular nets with embedded quadrilaterals (automatically convex) in order to avoid singularities.

**Discrete data for a 2D supercyclidic net.** We have seen that supercyclidic patches are completely encoded in their boundary curves and that the TCC-property may be reduced to coinciding tangents in common vertices. Accordingly, a 2-dimensional supercyclidic net is completely encoded in the following data:

(i) The supporting Q-net \( x : \mathbb{Z}^2 \rightarrow \mathbb{R}P^3 \).

(ii) The congruences of tangents \( t^1, t^2 : \mathbb{Z}^2 \rightarrow \mathcal{C}^3 \).

(iii) The boundary curves \( b^1, b^2 : \mathbb{Z}^2 \rightarrow \mathcal{A}^3 \) with \( b^i \) tangent to \( t^i \) at \( x \) and to \( t^i_x \) at \( x_i \) and such that for each quadrilateral opposite boundary curves are perspective from the corresponding Laplace point.
Remark 28. In order to obtain discrete data that consists of points and lines only, for given Q-net $x$ and tangents $t^i$ the conic segment $b^i$ may be represented by one additional point in its supporting plane $\pi^i$ (see also Remark 24). Accordingly, the boundary curves are then described by maps $b^i : \mathbb{Z}^2 \to \mathbb{R}P^3$ with $b^i \in t^i \vee t^i$. This representation does not only fix the conic arc but also a notion of parametrized supercyclidic nets.

The tangents at vertices of a supercyclidic net are obviously determined by the conic splines composed of the patch boundaries. For the formulation of a related Cauchy problem it is more convenient to capture the tangents and boundary curves separately.

Cauchy data for a 2D supercyclidic net. According to Lemma 17 (L), the tangents of a 2D SC-net are uniquely determined by the tangents along coordinate axes if the supporting Q-net is given. On the other hand, if all the tangents are known, we know all boundary planes and may propagate suitable initial boundary splines according to the perspectivity property of SC-patches that is captured by Proposition 22 (ii). Hence Cauchy data of a 2D supercyclidic net consists of, e.g.,

(i) The supporting Q-net $x : \mathbb{Z}^2 \to \mathbb{R}P^3$.

(ii) The tangents along two intersecting coordinate lines for each coordinate direction $t^i|_{B^j}$, $i, j = 1, 2$.

(iii) Two conic splines $b^i|_{B^i}$, $i = 1, 2$, such that $b^i$ is tangent to $t^i$ at $x$ and to $t^i$ at $x_i$, cf. Fig. 12.

![Figure 12. Cauchy data for a 2D cyclidic net.](image)

Rigidity of supercyclidic nets. An SC-patch is determined by its boundary curves uniquely. Further, one observes that for fixed Q-net and fixed tangents the variation of a boundary curve propagates along a whole quadrilateral strip due to the perspectivity property. Moreover, a local variation of tangents is not possible for a prescribed Q-net, as each tangent is determined uniquely by the contained vertex of the Q-net and two adjacent tangents. This shows that it is not possible to vary a supercyclidic net locally. However, we know that local deformation of 2D Q-nets is possible\cite{19}, which gives rise to the question whether effects of the (global) variation of a given 2D SC-net might be minimized by incorporation of suitable local deformations of the supporting Q-net.

\footnote{Local deformation of higher dimensional Q-nets is not possible as they are governed by a 3D system.}
Q-refinement induced by 2D supercyclidic nets. Obviously, any selection of characteristic conics on an SC-patch induces a Q-refinement of its vertex quadrilateral since also the vertices of the induced “sub-patches” are coplanar. Accordingly, any continuous supercyclidic patchwork induces an arbitrary Q-refinement of the supporting 2D Q-net. This is interesting in the context of the convergence of discrete conjugate nets to smooth conjugate nets, but also a noteworthy fact from the perspective of architectural geometry: 2D supercyclidic nets may be realized at arbitrary precision by flat panels.

Implementation on a computer. We have implemented two ways to experiment with supercyclidic nets. The first is in java and solves the Cauchy problem described on page 18 using the framework of jReality and VaryLab. We start with a Q-net with $\mathbb{Z}_2$ combinatorics. To create the two tangent congruences we prescribe initial tangents at the origin and propagate them along each coordinate axis via reflections in the symmetry planes of the edges (which does not exploit all possible degrees of freedom). The prescribed tangent congruences on the coordinate axes can then be completed to tangent congruences on the entire Q-net according to Lemma 17 (L). The conic splines on the coordinate axes are described as rational quadratic Bezier curves with prescribed weight. The initial weights are then propagated to all edges by evaluating suitable determinant expressions, which take into account the Q-net, the focal nets, the weights, and the intersection points of the four tangent planes at each quadrilateral. This yields a complete description of the resulting supercyclidic net in terms of rational quadratic Bezier surface patches. See Fig. 13 for two different SC-nets adapted to one and the same supporting Q-net that are obtained that way. The second approach to the computation of supercyclidic nets is based on a global variational approach and uses the framework of Tang et al. It also makes use of quadratic Bezier surface patches, where quadratic constraints involving auxiliary variables guarantee that those patches constitute an SC-net.

Figure 13. Two different supercyclidic nets with same supporting Q-net. The Q-net with tangents at vertices is drawn on top of the corresponding supercyclidic net.

7http://www.jreality.de and http://www.varylab.com
6. 3D supercyclidic nets

Motivated by a common structure behind integrable geometries that is reflected by the “multidimensional consistency principle” of discrete differential geometry, see [8], we start out with

**Definition 29 (3D supercyclidic net).** A $Q$-net $x: \mathbb{Z}^3 \to \mathbb{R}P^3$ together with SC-patches that are adapted to its elementary quadrilaterals – such that patches adapted to edge-adjacent quadrilaterals meet in a common boundary curve associated with the corresponding edge – is called a **3D supercyclidic net** if the restriction to any coordinate surface is a 2D supercyclidic net.

While Definition 29 appears natural, it is not obvious that one can consistently (that is, with coinciding boundary curves) adapt SC-patches even to a single elementary hexahedron of a 3D $Q$-net. Before proving that this is indeed possible (Theorem 31), we introduce some convenient terminology.

**Definition 30 (Common boundary condition and supercyclidic cubes).** Consider several quadrilaterals that share an edge. We say that SC-patches adapted to those quadrilaterals satisfy the **common boundary condition** if their boundary curves associated with the common edge all coincide. Moreover, we call a 3-dimensional $Q$-cube with SC-patches adapted to its faces, such that the common boundary condition is satisfied for each edge, a **supercyclidic cube** or **SC-cube** for short, cf. Fig. 14.

![Figure 14. A supercyclidic cube: Six SC-patches with shared boundary curves whose vertices form a cube with planar faces.](image)

**Theorem 31 (Supercyclidic 3D system).** Three faces of an SC-cube that share one vertex, that is, three generic SC-patches with one common vertex and cyclically sharing a common boundary curve each, determine the three opposite faces (“SC 3D system”). Accordingly, the extension of a $Q$-cube to an SC-cube is uniquely determined by the free choice of three SC-patches that are adapted to three faces that meet at one vertex and satisfy the common boundary condition.

**Proof.** First note that the three given SC-patches determine a unique supporting $Q$-cube $C$, since $Q$-nets are governed by a 3D system and seven vertices of the cube are already given. Further, denote the vertices of $C$ in the standard way by $x, \ldots, x_{123}$ so that the vertex quadrilaterals of the three patches are $Q^{ij} = (x, x_i, x_{ij}, x_j)$, $1 \leq i < j \leq 3$, and the common vertex is $x$.

Assume that the three given patches may be completed to an SC-cube. Then through each edge of $C$ in direction $i$ there is a unique boundary plane that contains the associated boundary curve. We denote those planes by

$$\pi^i = t^i \lor t^i_i \supset e^i = x \lor x_i$$
and the respective shifts. Now consider the perspectivities between those planes that identify the opposite boundary curves of the adapted patches. They are central projections through the Laplace points (cf. Fig. 15 left)

\[ y^{ij} = (x \lor x_j) \cap (x_i \lor x_{ij}), \]

which we denote by

\[ \tau^{ij} : \pi^i \to \pi^{ij}, \quad p \mapsto (p \lor y^{ij}) \cap \pi^{ij}, \]

so that

\[ b^{ij}_j = \tau^{ij}(b^i). \]

Here we make the genericity assumption that the tangents are such, that the boundary planes do not contain the Laplace points.

Now fix one direction \( i \) and consider the corresponding "maps around the cube" as depicted in Fig. 15, right.

Figure 15. Laplace projections around a cube.

On use of the fact that the four involved Laplace points \( y^{ij}_i, y^{ij}_k, y^{ik}_i, y^{ik}_j \) are contained in the line

\[ l^i = Q^{jk} \cap Q^{ij}_i, \]

one verifies that the relations (3) imply

\[ \tau^{ik}_k \circ \tau^{ij}_i = \tau^{ij}_i \circ \tau^{ik}_k. \]

So, given the three patches adapted to \( Q^{12}, Q^{23}, \) and \( Q^{13} \), for any direction \( i \) we know already the three planes \( \pi^i, \pi^{ij}, \pi^{ik}, \{i, j, k\} = \{1, 2, 3\} \), and the claim of Theorem 31 follows if we show that given those planes there exists a unique plane \( \pi^{jk}_i \) through \( e^{ij}_{jk} \) such that (4) holds. The fact that \( \pi^{jk}_i \) is unique, if it exists at all, follows again from the collinearity of \( y^{ij}_i, y^{ij}_k, y^{ik}_i, y^{ik}_j \) and \( y^{jk}_i \) as this guarantees that the construction

\[ p_{jk} = (\tau^{ij}_i(p) \lor y^{ik}_i) \cap (\tau^{ik}_k(p) \lor y^{ij}_j) \subset l^i \lor p \]

yields a well-defined map

\[ \tau^i : \mathbb{R}P^3 \setminus \{l^i\} \to \mathbb{R}P^3, \quad p \mapsto p_{jk}. \]

To see the existence of \( \pi^{jk}_i \), note that the above construction may be reversed and that \( \tau^i \) maps lines to lines according to

\[ l \mapsto l_{jk} = (\tau^{ik}_k(l) \lor y^{ik}_j) \cap (\tau^{jk}_i(l) \lor y^{ij}_j). \]
Thus \( \tau^i \) is a bijection between open subsets of \( \mathbb{R}P^3 \) that maps lines to lines and therefore the restriction of a projective transformation \( \hat{\tau}^i \) due to the fundamental theorem of real projective geometry. Accordingly, \( \pi^i_{jk} = \hat{\tau}^i(\pi^i) \) is the well-defined plane we are looking for.

**Remark 32.** Observe that in the above proof the point \( q^i = \pi^i \cap \pi^i_j \cap \pi^i_k \) is a fixed point of \( \tau^i \), and hence \( q^i \in \pi^i_{jk} \). Accordingly, the plane \( \pi^i_{jk} \) may be constructed as \( \pi^i_{jk} = q^i \lor e^i_{jk} \) and \( q^i \) becomes the common intersection of the four characteristic lines of the direction \( i \), that is, \( q^i = a^i \cap a^i_j \cap a^i_{jk} \cap a^i_k \). This shows that for an SC-cube in fact all planes supporting the characteristic conics of its supercyclic faces associated with one and the same direction \( i \) contain the point \( q^i \).

**Corollary 33.** Let \( \pi^i, \pi^i_j, \pi^i_k, \text{ and } \pi^i_{jk} \) be four planes through the respective \( i \)-edges of a Q-cube \((x, x_i, x_{ij}, x_{123})\) in \( \mathbb{R}P^3 \), such that each plane contains exactly one edge. Then the central projections \( \pi^i, \pi^i_j, \pi^i_k, \text{ and } \pi^i_{jk} \) commute in the sense of (4) if and only if the four planes are concurrent.

**Theorem 34** (Properties of SC-cubes). Characteristic conics around an SC-cube close up and bound (unique) SC-patches as indicated in Fig. 16. Patches obtained this way constitute three families \( \mathcal{F}^i, i = 1, 2, 3 \), where the notation is such that patches of the family \( \mathcal{F}^i \) interpolate smoothly between opposite cyclidic faces of the cube with respect to the direction \( i \). The induced SC-patches have the following properties:

(i) For each direction \( i \) there exists a unique point \( q^i \), which is the common intersection of the characteristic lines \( a^i \) of all patches \( \mathcal{F}^j \cup \mathcal{F}^k, \{i, j, k\} = \{1, 2, 3\} \).

(ii) Patches \( f^i \in \mathcal{F}^i \) and \( f^j \in \mathcal{F}^j, i \neq j \), intersect along a common characteristic conic that is associated with their shared net direction.

(iii) Patches \( f^i, \tilde{f}^i \in \mathcal{F}^i \) are classical fundamental transforms of each other. In particular, this holds for opposite cyclidic faces of the initial SC-cube.

**Proof.** The proof of Theorem 31 shows that going along characteristic conics of the supercyclic faces of an SC-cube closes up as indicated in Fig. 16. Generically, one obtains four conic arcs that intersect in four coplanar vertices\(^8\) as the four centers of perspectivity that relate isoparametric points on the patch boundaries are collinear (cf. Fig. 15). In order to

---

\(^8\)If opposite boundary curves of a supercyclic face intersect, those intersections are singular points of the patch (Remark 23). Accordingly, one obtains only three or less vertices when going around the cube if one passes through such points. However, as the singularities are isolated, the presented argumentation may be kept valid by incorporation of a continuity argument.
verify that those arcs constitute the boundary of an SC-patch, it remains to show that opposite arcs are related by a “Laplace perspectivity” with respect to the constructed vertex quadrilateral. So consider Fig. 16 and note that the vertices $x, x_1, x_{13}, x_3, \tilde{x}_2, \tilde{x}_{123}, \tilde{x}_{23}$ constitute a Q-cube. Moreover, there are unique planes $\pi^1, \tilde{\pi}^2, \tilde{\pi}^1_{23}, \pi^3$ through its edges of direction 1 that support the characteristic conics of the original cyclidic faces and those planes intersect in a point $q^1$ according to Remark 32. We know already that some of the characteristic conics in those planes are related by Laplace perspectivities, namely

$$\tau^{1,3}_1: \pi^1 \rightarrow \tilde{\pi}^1_3, \quad \tau^{1,2}_{3}: \pi^1 \rightarrow \tilde{\pi}^1_2, \quad \tau^{1,2}_{3}: \pi^1_3 \rightarrow \tilde{\pi}^1_{23}.$$

Therefore, also the curves in the planes $\tilde{\pi}^1_2$ and $\tilde{\pi}^1_{23}$ are related by the Laplace perspectivity $\tilde{\tau}^{1,3}_2: \tilde{\pi}^1_2 \rightarrow \tilde{\pi}^1_{23}$ according to Corollary 33.

We conclude that characteristic conics of the supercyclidic faces of an SC-cube induce three families of SC-patches, each family interpolating between opposite faces. Denote by $F^i$ the family that interpolates in the direction $i$. The construction implies that the patches of the family $F^i$ may be parametrized smoothly by points $\tilde{x}^i$ of the boundary arc $b^i$ that connects $x$ and $x_i$. Note that, as a consequence, any intermediate patch $f^i \in F^i$ splits the original SC-cube into two smaller SC-cubes, for which $f^i$ is a common face.

Proof of (i): This is an immediate consequence of Remark 32.

Proof of (ii): As indicated in Fig. 17, the patches $f^i$ and $f^j$ determine a pair of corresponding points $p$ and $p_k$ on the opposite faces of the SC-cube with respect to the common net direction $k$ of $f^i$ and $f^j$. Together with the vertices of $f^i$ and $f^j$, the points $p$ and $p_k$ induce a refinement of the original supporting Q-cube into four smaller Q-cubes. This follows from the aforementioned property that any patch $f^i \in F^i$ different from the opposite faces of the SC-cube splits the cube into two smaller SC-cubes, together with the fact that the characteristic conics around an SC-cube close up. We conclude that for two of the induced SC-cubes the vertices $p$ and $p_k$ are connected by a characteristic conic of $f^i$, while for the other two cubes they are connected by a characteristic conic of $f^j$. The fact that those conics coincide is again an implication of Remark 32 together with Corollary 33.

Proof of (iii): It is sufficient to prove the claim for opposite faces of the SC-cube, since any pair $f^i, f^j \in F^i$ may be understood as a pair of opposite faces after a possible 2-step reduction of the initial SC-cube to a smaller SC-cube that is induced by $f^i$ and $f^j$. Now, corresponding points $p$ and $p_k$ on opposite faces, top and bottom, say, may be identified by using ”vertical” patches as indicated in Fig. 17. Thus property (ii) implies that corresponding tangents in those points intersect due to the tangent cone property for the vertical patches.

\[\square\]

**Figure 17.** Opposite faces of an SC-cube form a fundamental pair.
Supercyclidic coordinate systems. We say that an SC-cube is regular if its six supercyclidic faces do not contain singularities and opposite faces are disjoint. Note that this implies that all patches of all three families are regular, which may be derived from the fact that isolated singularities of SC-patches appear as intersections of opposite boundary curves, cf. Remark 23. This in turn shows that for a regular SC-cube, any two patches $f^i, \tilde{f}^i \in F^i$ are disjoint. Otherwise there would exist a patch $f$ of another family through $p \in f^i \cap \tilde{f}^i$ for which the same reasoning as above together with a “cutting down argument” as in the proof of Theorem 34 would imply that $p$ is a singular point of $f$. Having observed this, we see that for a regular SC-cube the property (ii) of Theorem 34 gives rise to classical conjugate coordinates on the region that is covered by the SC-patches of the families $F_1, F_2, F_3$, cf. Fig. 18. Associated with those coordinates, planarity of vertex quadrilaterals of sub-patches of the coordinate surfaces induces arbitrary Q-refinements of the original Q-cube.

Figure 18. Regular SC-cubes induce smooth conjugate coordinates and give rise to arbitrary Q-refinements of the supporting Q-cube.

TCC-reduction of the supercyclidic 3D system. Theorem 31 shows that it is possible to equip elementary quadrilaterals of a 3D Q-net consistently with adapted SC-patches that satisfy the common boundary condition, such an extension being determined by its restriction to three coordinate planes (one of each family). We will now demonstrate that imposition of the TCC-condition in coordinate planes, i.e., to require that each 2D layer is a 2D supercyclidic net is an admissible reduction of the underlying 3D system, which in turn shows the existence of 3D supercyclidic nets. In fact, it turns out that the TCC-reduction is multidimensionally consistent, which will later give rise to multidimensional SC-nets as well as fundamental transformations thereof that exhibit the same Bianchi-permutability properties as fundamental transformations of classical conjugate nets and their discrete counterparts.

Theorem 35. Imposition of the TCC-property on 2D coordinate planes is an admissible reduction of the SC 3D system. This means, propagation of admissible TCC-Cauchy data for coordinate planes $B^{ij}$ according to Theorem 31 yields a Q-net with SC-patches adapted to elementary quadrilaterals, such that the common boundary condition is satisfied and the TCC-property holds in all 2D coordinate planes. In particular, the reduction is multidimensionally consistent.

Proof. Denote the supporting Q-net by $x$ and the adapted patches by $f$ and assume TCC-Cauchy data $(x, f)|_{\partial B^{ij}}, 1 \leq j < k \leq m$, to be given (admissible in the sense that the common boundary condition is satisfied along coordinate axes). To begin with, note that $x$ is uniquely determined by the Cauchy data $x|_{\partial B^{ij}}$ as Q-nets are governed by an
mD consistent 3D system. Accordingly, we may assume that \( x \) is given and focus on the Cauchy data \( f|_{B_{ij}} \) for adapted SC-patches.

The propagation of adapted patches may be understood as propagation of their defining boundary curves and it has been shown (Remark 32 and Corollary 33) that, on a 3D cube, the propagation of boundary curves by Laplace perspectivities is consistent if and only if the boundary planes satisfy

\[
\pi^i_j = (\pi^i \cap \pi^j \cap \pi^k) \cup x_{jk} \cup x_{ijk} = q^i \cup e^j_{jk}, \quad i \neq j \neq k \neq i. 
\]

Accordingly, it has to be shown that the propagation of boundary planes subject to (6) is multidimensionally consistent. Moreover, we have to assure that the induced lines \( t^i = \pi^i_{j-i} \cap \pi^i \) at vertices \( x \) constitute a discrete torsal line system. This is the case if and only if they are compatible with adapted SC-patches, while automatically implying the TCC-property in coordinate planes (see Proposition 22 (iii) and Lemma 26). Both suggested properties may be verified by a reverse argument: Instead of considering the Cauchy problem for boundary planes, consider the evolution of tangents subject to Lemma 17 (L) and for the admissible Cauchy data \( t^i|_{B_{ij}}, \ j = 1, \ldots, m \). Due to Corollary 19 the propagation is multidimensionally consistent and one obtains a unique fundamental line system \( t^i \). It remains to note that the solution \( t^i \) encodes the unique solution \( \pi^i = t^i \cup t^i_i \) of the original Cauchy problem, for the reason that (6) is satisfied by virtue of Lemma 13 (iv).

\[\square\]

7. mD supercyclidic nets and their fundamental transformations

The previous considerations motivate the following definition of \( m \)-dimensional supercyclidic nets, whose existence is verified by Theorem 35.

**Definition 36 (mD supercyclidic net).** Let \( x : \mathbb{Z}^m \rightarrow \mathbb{R}P^3 \), \( m \geq 2 \), be a Q-net and assume that \( f : \{2\text{-cells of } \mathbb{Z}^m\} \rightarrow \{\text{supercyclidic patches in } \mathbb{R}P^3\} \) describes SC-patches, which are adapted to the elementary quadrilaterals of \( x \) and satisfy the common boundary condition. The pair \((x, f)\) is called an \( m \)-D supercyclidic net if the restriction to any coordinate surface is a 2D supercyclidic net, that is, if the TCC-property is satisfied in each 2D coordinate plane.

**Discrete data for an mD supercyclidic net.** Analogous to the 2-dimensional case, an \( m \)-dimensional supercyclidic net is completely encoded in the following data:

(i) The supporting Q-net \( x : \mathbb{Z}^m \rightarrow \mathbb{R}P^3 \),
(ii) The discrete torsal tangent systems \( t^i : \mathbb{Z}^m \rightarrow \mathcal{L}^3 \), \( i = 1, \ldots, m \),
(iii) The boundary curves \( b^i : \mathbb{Z}^m \rightarrow \mathcal{A}^3 \), \( i = 1, \ldots, m \), with \( b^i \) tangent to \( t^i \) at \( x_i \) and such that for each quadrilateral opposite boundary curves are perspective from the corresponding Laplace point.

See also Remark 28 on the fully discrete description of boundary curves in terms of points and lines.

**Cauchy data for an mD supercyclidic net.** According to Theorem 35, Cauchy data for an \( m \)-D supercyclidic net is given by a collection \((x^{ij}, f^{ij})\) of 2D supercyclidic nets for the 2D coordinate planes \( B^{ij}, 1 \leq i < j \leq m \), which has to be admissible in the sense that the common boundary condition is satisfied along coordinate lines. Independent Cauchy
data in turn is given by Cauchy data for such compatible 2D supercyclidic nets and we conclude that the following constitutes Cauchy data for an $mD$ supercyclidic net.

(i) Cauchy data for the supporting Q-net $x: \mathbb{Z}^m \to \mathbb{R}^3$, e.g.,

$x|_{\mathbb{Z}^m}, \quad 1 \leq i < j \leq m$.

(ii) Cauchy data for the discrete torsal tangent systems $t^i$, $i = 1, \ldots, m$, e.g.,

$t^i|_{\mathbb{Z}^m}, \quad i, j = 1, \ldots, m$.

(iii) One adapted spline of conic segments for each direction, e.g.,

$b^i|_{\mathbb{Z}^m}, \quad i = 1, \ldots, m$,

such that $b^i$ is tangent to $t^i$ at $x$ and to $t^i_j$ at $x_j$.

**Fundamental transformations of supercyclidic nets.** Motivated by the theories of cyclidic and hyperbolic nets, we define fundamental transformations of supercyclidic nets by combination of the according smooth and discrete fundamental transformations of the involved SC-patches and supporting Q-nets, respectively.

**Definition 37** (F-transformation of supercyclidic nets). Two $mD$ supercyclidic nets $(x, f)$ and $(x \pm, f \pm)$ are called fundamental transforms of each other if the supporting Q-nets $x$ and $x \pm$ form a discrete fundamental pair and each pair of corresponding SC patches $f, f \pm$ forms a classical fundamental pair.

It turns out that two $mD$ supercyclidic nets are fundamental transforms if and only if they may be embedded as two consecutive layers of an $(m + 1)$-dimensional supercyclidic net. While the possibility of such an embedding implies the fundamental relation according to Theorem 34 (iii), the converse has to be shown. It is not difficult to see (think of Cauchy data for $mD$ supercyclidic nets) that this is an implication of the following Proposition 38.

**Proposition 38.** Consider a 3D cube $Q$ with planar faces and SC-patches $f, f \pm$ adapted to one pair of opposite faces of $Q$. Then the following statements are equivalent:

(i) The patches $f$ and $f \pm$ are classical fundamental transforms of each other.

(ii) Corresponding boundary curves $b$ and $b \pm$ of $f$ and $f \pm$ are perspective from the associated Laplace point $(x \cup x^\pm) \cap (y \cup y^\pm)$ of the face of $Q$ that consists of the vertices $x, y$ of $b$ and $x^\pm, y^\pm$ of $b^\pm$.

(iii) The configuration can be extended to an SC-cube. The extension is uniquely determined by the choice of the four missing boundary planes that have to be concurrent, and a conic segment that connects corresponding vertices in one of those planes.

For the proof of Proposition 38 we will use the following

**Lemma 39.** Let $f, f^\pm : U \subset \mathbb{R}^2 \to \mathbb{R}P^3$ be smooth conjugate nets that are fundamental transforms of each other and let $b$ and $b^\pm$ be corresponding parameter lines, that is, tangents to $b$ and $b^\pm$ in corresponding points intersect. Further denote by $c$ the curve that consists of intersection points of the conjugate tangents along $b$ and $b^\pm$. If $b$ and $b^\pm$ both are planar and $c$ is regular, then $c$ is also a planar curve. Moreover, the supporting planes of $b, b^\pm$, and $c$ intersect in a line.

**Proof of Lemma 39** Denote by $t$ and $t^\pm$ the tangents to $b$ and $b^\pm$ and by $g$ and $g^\pm$ the conjugate tangents to $f$ and $f^\pm$. As $b$ and $b^\pm$ are planar, corresponding tangents $t$ and $t^\pm$ intersect along the line $l$ that is the intersection of the supporting planes of $b$ and $b^\pm$. In order to see that the curve $c$ composed of the intersection points $g \cap g^\pm$ is planar, note the following: The tangent planes $\pi$ to $f$ along $b$ are of the form $t \cup g$ and analogous for the
tangent planes $\pi^+$ of $f^+$ one has $\pi^+ = t^+ \lor g^+$. So $\pi \cap \pi^+ = h = (t \cap t^+) \lor (g \cap g^+)$ and we conclude $h \cap l = t \cap t^+$. Moreover, the lines $h$ are the tangents of $c$, since the planes $\pi$ along $b$ envelop a torsal ruled surface $\sigma$ whose rulings are precisely the conjugate tangents $g^+$. Accordingly, $c$ is a curve on $\sigma$ and thus in each point of $c$ its tangent is contained in the corresponding tangent plane $\pi$ of $\sigma$. Of course the same argumentation holds if we consider the planes $\pi^+$ and hence the tangents to $c$ must be of the form $\pi \cap \pi^+$. Finally, for regular $c$ the planarity follows from the fact that continuity of a regular curve $\gamma$ implies that $\gamma$ is planar if and only if there exists a line that intersects all of its tangents. Clearly, the supporting plane of $c$ also passes through $l$. □

Proof of Proposition 38. A Q-cube with one conic segment per edge that connects the incident vertices encodes an SC-cube if and only if for each quadrilateral the conic segments that connect the vertices of opposite edges are “Laplace perspective”, cf. Fig. 15. Thus, (ii) is necessary for (iii). It is also sufficient since the suggested construction in (iii) always yields an SC-cube according to Corollary 33 and actually covers all possible extension as a consequence of the proof of Theorem 31. On the other hand, (iii) \(\implies\) (i) is covered by Theorem 34 and it remains to show (i) \(\implies\) (ii).

So consider two corresponding boundary curves $b$ and $b^+$ of the SC-patches $f$ and $f^+$ that connect four coplanar vertices $x, y, x^+, y^+$. As the patches $f$ and $f^+$ are fundamental transforms, the tangents to $b$ and $b^+$ in isoparametric points $p$ and $p^+$ intersect along the line $l$ that is the intersection of the two supporting planes $\pi$ and $\pi^+$ of $b$ and $b^+$, respectively. On the other hand, also the conjugate tangents $g$ and $g^+$ in $p$ and $p^+$ intersect in $q = g \cap g^+$ because of the fundamental relation in the other direction and those points $q$ trace a curve $c$. Moreover, all conjugate tangents along $b$ pass through the apex $a$ of the tangency cone $C$ to $f$ along $b$ and analogously all conjugate tangents along $b^+$ pass through the apex $a^+$ of the tangency cone $C^+$ to $f^+$ along $b^+$, cf. Fig. 19. The curve $c$ is contained in the intersection $C \cap C^+$ and thus is regular. As $b$ and $b^+$ are planar, the conditions of Lemma 39 are met and we may conclude that $c$ is planar (actually, a conic arc) and contained in a plane $\pi_c$ that passes through the line $l$.

Figure 19. Two boundary curves of a fundamental pair of SC-patches with coplanar vertices $x, y, x^+, y^+$.  

---

This is a classical, purely geometric description of conjugate tangents to a smooth surface (cf. [29]).
It is obvious that isoparametric points on $b$ and $c$ are perspective from $a$ (along the generators of $C$) and analogously $b^+$ and $c$ are perspective from $a^+$. Accordingly, those perspective relations between curves are restrictions of the perspective projective transformations

\[
\begin{align*}
\tau_a : \pi &\rightarrow \pi_c, \quad p \mapsto q \quad (\text{central projection through } a) \\
\tau_{a^+} : \pi_c &\rightarrow \pi^+, \quad q \mapsto p^+ \quad (\text{central projection through } a^+).
\end{align*}
\]

Further denote by $\tau = \tau_{a^+} \circ \tau_a$ the composition of those perspectivities so that

\[
\tau : \pi \rightarrow \pi^+, \quad p \mapsto p^+
\]

identifies corresponding points on $b$ and $b^+$. The fact that $\tau$ is also a perspectivity may be seen as follows. First note, that $\tau$ is determined by four points in general position in $\pi$ and their images in $\pi^+$. We know already $\tau(x) = x^+$ as well as $\tau(y) = y^+$ and, moreover, that those corresponding points are perspective from $o = (x \vee x^+) \cap (y \vee y^+)$. It remains to observe that $l = \pi \cap \pi^+$ consists of fixed points of $\tau$ because of $l \subset \pi_c$ and to consider two further points $r, s \in l$ such that $x, y, r, s$ are in general position.

\[\square\]

**Remark.** The center of perspectivity $o$ for $\tau$ lies on the line $a \lor a^+$. This follows from the fact that each triple $p, p^+, q$ of corresponding points is contained in a plane that is spanned by intersecting generators $g$ and $g^+$ of $C$ and $C^+$, respectively. All planes of that kind are contained in the pencil of planes through the line that is spanned by the apices of $C$ and $C^+$, that is, $a \lor a^+$.

**Construction of fundamental transforms.** Let $s = (x, f)$ be an $m$D supercyclidic net. According to our previous considerations, the construction of a fundamental transform $s^+ = (x^+, f^+)$ of $s$ corresponds to the extension of $s$ to a 2-layer $(m + 1)$-dimensional supercyclidic net. Any such extension may be obtained as follows. First, construct an F-transform $x^+$ of $x$, that is, extend $x$ to a 2-layer $(m + 1)$-dimensional Q-net $X : \mathbb{Z}^m \times \{0, 1\} \rightarrow \mathbb{R}P^3$ with $X(\cdot, 0) = x$ so that $x^+ = X(\cdot, 1)$. Cauchy data for such an extension is given by the values of $x^+$ along the coordinate axes $B_i$, $i = 1, \ldots, m$, which has to satisfy the condition that the quadrilaterals $(x, x_i, x^+_i, x^+)$ are planar. Further, the section on Cauchy data for an $m$D supercyclidic net shows that the remaining relevant data are the tangents of $s^+$ at one point $x_0^+$, which have to intersect the corresponding tangents of $s$ at the corresponding point $x_0$. The boundary curves of $s^+$ are then obtained as perspective images of the boundary curves of $s$.

**Remark.** In the above construction there are some degrees of freedom left for the construction of compatible “vertical” patches in the resulting 2-layer $(m + 1)$D SC-net, but this does not affect the net $s^+$.

**Permutability properties.** Our previous considerations of $m$D supercyclidic nets immediately yield the following permutability properties for F-transforms of supercyclidic nets. Corollary 30 is a literal translation of the analogous Theorem 11 for Q-nets and follows from the fact that the missing tangents of the involved supercyclidic nets are uniquely determined by their supporting Q-nets (subject to the $m$D consistent 2D system that describes the extension of Q-nets to torsal line systems, cf. Corollary 19).

**Corollary 40** (Permutability properties of F-transformations of supercyclidic nets).

(i) Let $s = (x, f)$ be an $m$-dimensional supercyclidic net, and let $s^{(1)}$ and $s^{(2)}$ be two of its F-transforms. Then there exists a 2-parameter family of SC-nets $s^{(12)}$ that are F-transforms of both $s^{(1)}$ and $s^{(2)}$. The corresponding vertices of the four SC-nets $s$, $s^{(1)}$, $s^{(2)}$ and $s^{(12)}$ are coplanar. The net $s^{(12)}$ is uniquely determined by one of its vertices.
(ii) Let $s = (x, f)$ be an $m$-dimensional supercyclidic net. Let $s^{(1)}$, $s^{(2)}$ and $s^{(3)}$ be three of its $F$-transforms, and let three further SC-nets $s^{(12)}$, $s^{(23)}$ and $s^{(13)}$ be given such that $s^{(ij)}$ is a simultaneous $F$-transform of $s^{(i)}$ and $s^{(j)}$. Then generically there exists a unique SC-net $s^{(123)}$ that is an $F$-transform of $s^{(12)}$, $s^{(23)}$ and $s^{(13)}$. The net $s^{(123)}$ is uniquely determined by the condition that for every permutation $(ijk)$ of $(123)$ the corresponding vertices of $s^{(i)}$, $s^{(ij)}$, $s^{(ik)}$ and $s^{(123)}$ are coplanar.

8. Frames of supercyclidic nets via projective reflections

Definition 41 (Tangent systems and frames of supercyclidic nets). Let $x : \mathbb{Z}^m \to \mathbb{R}P^3$ be the supporting $Q$-net of a supercyclidic net and $t^i : \mathbb{Z}^m \to \mathcal{L}^3$, $i = 1, \ldots, m$, be the tangents to the supercyclidic patches at vertices of $x$. We call

$$(t^1, \ldots, t^m) : \mathbb{Z}^m \to \mathcal{L}^3 \times \ldots \times \mathcal{L}^3$$

the tangent system of the supercyclidic net and $(t^1, \ldots, t^m)(z)$ the tangents at $x(z)$. For $m \in \{2, 3\}$ we refer to the tangents as frames of the supercyclidic net.

It turns out that the integrable structure behind the tangent systems of supercyclidic nets (in the sense of Corollary [1]) has a nice expression in terms of projective reflections that are associated with the edges of the supporting $Q$-net. In fact, the uncovered integrable system on projective reflections governs the simultaneous extension of a single $Q$-net to multiple fundamental line systems also in the multidimensional case and in that sense is more general than the theory of supercyclidic nets which is rooted in 3-space.

Before going into details, let us recall that any pair of disjoint subspaces $U, V \subset \mathbb{R}P^N$ with $U \cup V = \mathbb{R}P^N$ induces a unique projective reflection $f : \mathbb{R}P^N \to \mathbb{R}P^N$ as follows. Points of the union $U \cup V$ are defined to be fixed points of $f$. For all other points $x$ there exists a unique line $l$ through $x$ that intersects $U$ and $V$, which yields points $p = U \cap l$ and $q = V \cap l$. The image $f(x)$ is then defined by the cross-ratio condition

$$\text{cr}(x, p, f(x), q) = -1.$$  

Note that any projective reflection is an involution, $f^2 = \text{id}$, since $\text{cr}(x, p, f(x), q) = -1$ if and only if $\text{cr}(f(x), p, x, q) = -1$.

Relevant for our purpose is the particular subclass of projective reflections that we denote by

$$\mathcal{F}_{o, \pi}(N) = \{ \text{Projective reflections in } \mathbb{R}P^N \text{ induced by a point } o \text{ and a hyperplane } \pi \neq o \}.$$  

Now the starting point for the following considerations is the observation that for a generic 3D supercyclidic net any two frames $T = (t^1, t^2, t^3)$ and $T_i = (t^1_i, t^2_i, t^3_i)$ at adjacent vertices $x$ and $x_i$ are related by a unique projective reflection $f^i \in \mathcal{F}_{o, \pi}(3)$, that is

$$f^i(T) = T_i \iff f^i(T_i) = T.$$  

To verify this claim, first note that (7) implies $f^i(x) = x_i$. Further let $f^i \sim (o^i, \pi^i) \in \mathcal{F}_{o, \pi}(3)$ be any projective reflection with $f^i(x) = x_i$ and denote $p^i = t^i \cap \pi^i$. As $x \neq x_i$, we have $x \notin \pi$ and thus

$$f^i(t^i) = f^i(x \vee p^i) = x_i \vee p^i.$$  

Therefore, (7) implies that the $p^i$ have to be the three focal points $p^j = t^j \cap t^j_i$, $j = 1, 2, 3$, and $\pi^i = p^i \vee p^2 \vee p^3$ is uniquely determined. On the other hand, the condition $f^i(x) = x_i$ determines $o^i \in x \vee x_i$ uniquely according to $\text{cr}(x, q^i, x_i, o^i) = -1$, where $q^i = \pi^i \cap (x \vee x_i)$ is distinct from $x$ and $x_i$.  

SUPERCYCLIDIC NETS 29
The next step is to consider frames $T, T_i, T_j, T_{ij}$ at the four vertices $x, x_i, x_j, x_{ij}$ of a quadrilateral. One obtains four induced projective reflections $f^i, f^j, f^i_j, f^j_i \in \mathcal{F}_{o,\pi}(3)$ for which

$$(f^i_j \circ f^i)(T) = T_{ij} = (f^j_i \circ f^j)(T)$$

and in particular

$$(f^i(x) = x_i, \quad f^j(x) = x_j, \quad f^j_i(x_j) = f^i_j(x_i) = x_{ij}).$$

Clearly, $f^i, f^j,$ and $T$ determine the frames $T_i, T_j,$ and $T_{ij}$ subject to $T_i = f^i(T), T_j = f^j(T), \text{ and Lemma } 17 \text{ (L). Accordingly, the projective reflections } f^i_j \text{ and } f^j_i \text{ may be constructed from } f^i \text{ and } f^j \text{ but the construction depends on } T.$ However, it turns out that this constructive propagation

$$(f^i, f^j) \mapsto (f^i_j, f^j_i)$$

is independent of $T$ modulo (8). Before we show this, note that the previous considerations and described constructions also apply for the simultaneous extension of a Q-net $x : \mathbb{Z}^n \to \mathbb{R}P^N$ to fundamental line systems $t^1, \ldots, t^N$ and yield unique projective reflections that relate the lines at adjacent vertices. Accordingly, we will now go beyond the setting of supercyclic nets in $\mathbb{R}P^3$ and consider the multidimensional case in $\mathbb{R}P^N.$ Moreover, it is evident that the frame dependent construction of reflections is mD consistent because of the multidimensional consistency of the extension of Q-nets to fundamental line systems, cf. Corollary 19.

In order to show that the propagation (9) depends only on the supporting Q-net, still inducing an mD consistent 2D system with variables in $\mathcal{F}_{o,\pi}(N)$ on edges, we will need the following

**Lemma 42.** Let $Q = (x, x_1, x_{12}, x_2)$ be a planar quadrilateral in $\mathbb{R}P^N$ and $(t^1, t^j_1, t^i_1, t^1_1), i = 1, \ldots, N,$ be $N$ simultaneous extensions of $Q$ to torsal line congruence quadrilaterals subject to Lemma 17 (L). Further denote by $\pi^j = \bigvee_i (t^i \cap t^1_i)$ the hyperplane spanned by focal points that are associated with the edge $x \vee x_j \text{ (cf. Fig. 20)}$ and let $X = \pi^1 \cap \pi^2 \cap \pi^1_1 \cap \pi^2_1.$ Then $\text{codim}(X) = 2.$

![Figure 20](image)

*Figure 20.* Hyperplanes spanned by focal points of line congruence quadrilaterals that extend the same planar quadrilateral.

**Proof.** For $1 \leq i < j \leq N$ the eight corresponding lines $t^i, t^j, \ldots, t^i_1, t^j_1$ form a line complex cube for which $Q$ is a focal quadrilateral. Now consider the four lines $t^{1,ij} = (t^i \cap t^j_1) \vee (t^j \cap t^1_i) \subset \pi^1, t^{2,ij} \subset \pi^2, t^{1,ij}_1 \subset \pi^1_1, \text{ and } t^{2,ij}_1 \subset \pi^2_1 \text{ that are spanned by the remaining focal points of the complex cube. Lemma 13 implies that those four lines}
intersect in one point \( p^{ij} = l^{1,ij} \cap l^{2,ij} \cap l^1_{i,j} \cap l^2_{i,j} \in X \) due to the planarity of \( Q \). As always, we assume that the data is generic so that \( \tilde{X} = \bigcup_{i \neq j} p^{ij} \) is a subspace of codimension 1 in \( \pi^1 = \bigvee_{i \neq j} l^{1,ij} \) and thus has codimension 2 in \( \mathbb{R}P^N \). Now \( \tilde{X} \subset X \subset \pi^1 \cap \pi^2 \) together with \( \text{codim}(\pi^1 \cap \pi^2) = 2 \) shows \( X = \tilde{X} \) and hence proves the lemma. \( \square \)

**Theorem 43** (Extension of \( Q \)-nets to fundamental line systems via projective reflections). Let \((x, x_1, x_{12}, x_2)\) be a planar quadrilateral in \( \mathbb{R}P^N \), \( N \geq 3 \), and let \( f^1, f^2 \in F_{o,\pi}(N) \) be projective reflections with \( f^i(x) = x_i \). Then there exist unique projective reflections \( f^1_2, f^2_1 \in F_{o,\pi}(N) \), which are determined by the conditions

\[
(10) \quad f^i_j(x_j) = x_{12} \quad \text{and} \quad (f^1_2 \circ f^i)(l) = (f^i_1 \circ f^2)(l) \quad \text{for all} \ l \in L^N \text{ with } x \in l.
\]

Further, let \( x : \mathbb{Z}^m \to \mathbb{R}P^N \), \( N \geq 3 \), be a generic \( Q \)-net. Then the induced propagation

\[
\tau : \mathbb{Z}^m \times F_{o,\pi}(N) \times F_{o,\pi}(N) \to F_{o,\pi}(N) \times F_{o,\pi}(N), \quad (f^i, f^j) \mapsto (f^i_j, f^j_i)
\]

of projective reflections is \( mD \)-consistent, that is, admissible Cauchy data \( f^i|_{\mathbb{Z}^1} \), \( i = 1, \ldots, m \), (mapping adjacent vertices onto each other) determines all remaining projective reflections uniquely.

**Proof.** First observe that the maps \( f^i|_{\mathbb{Z}^1} \), \( i = 1, \ldots, m \), together with one line \( t(0) \ni x(0) \) are Cauchy data for the \((mD \text{ consistent})\) extension of \( x \) to a fundamental line system. Accordingly, a frame \( T(0) = (t^1, \ldots, t^N)(0) \) at \( x(0) \) yields well defined frames at every vertex of \( x \). Those frames in turn yield one well-defined projective reflection for each edge as described before: The hyperplane \( \pi^i \) of the reflection \( f^i \sim (o^i, \pi^i) \) associated with the edge \((x, x_i)\) is spanned by the corresponding focal points, \( \pi^i = \bigvee_{j=1}^N (t^j \cap t^i_j) \), which in turn allows to construct the center \( o^i \) from the condition \( f^i(x) = x_i \). It remains to show that the construction is independent of \( T(0) \), that is, the propagation \( (f^i, f^j) \mapsto (f^i_j, f^j_i) \) according to \( (10) \) is well-defined.

So consider a planar quadrilateral \( Q = (x, x_1, x_{12}, x_2) \) in \( \mathbb{R}P^N \) with one generic frame \( T = (t^1, \ldots, t^N) \) at \( x \) and let \( f^1, f^2 \in F_{o,\pi}(N) \) with \( f^i(x) = x_i \) be given. One obtains frames \( T_1 = f^1(T) \) at \( x_1 \) and \( T_2 = f^2(T) \) at \( x_2 \) and Lemma \([17](L)\) yields a unique frame \( T_{12} \) at \( x_{12} \). As described above this induces projective reflections \( f^1_2, f^2_1 \) such that

\[
(11) \quad (f^1_2 \circ f^1)(T) = T_{12} = (f^2_1 \circ f^2)(T)
\]

and we have to show that the so obtained maps satisfy \( (10) \). By construction we have \( f^1_2(x_j) = x_{12} \). On the other hand, the second condition \( (f^2_1 \circ f^1)(l) = (f^2 \circ f^1)(l) \) for all \( l \in L^N(x) = \{ l \in L^N \mid x \in l \} \) may be rewritten as

\[
(12) \quad (f^2 \circ f^1)(l) = l,
\]

since projective reflections are involutions\(^{10}\). In order to verify \( (12) \) we identify lines through vertices of \( Q \) with their intersections with the hyperplanes of the projective reflections according to \( x \in l \leftrightarrow p = l \cap \pi^1 \), \( x_1 \in l_1 \leftrightarrow p_1 = l_1 \cap \pi^1_1 \), \( x_{12} \in l_{12} \leftrightarrow p_{12} = l_{12} \cap \pi^1_2 \), \( x_2 \in l_2 \leftrightarrow p_2 = l_2 \cap \pi^2_2 \).

\(^{10}\)Note that the identity \( (12) \) on sets does not imply \( f^2 \circ f^1 \circ f^2 \circ f^1 = \text{id} \). The composition could also be a homothety from an affine point of view.
After this identification, the images of lines through the vertices of $Q$ under the considered projective reflections correspond to the images of their representing points under certain central projections:

\[
\begin{align*}
  f^1 : \mathcal{L}^N(x) &\rightarrow \mathcal{L}^N(x_1) &\leftrightarrow& \tau^1 : \pi^1 \rightarrow \pi^1_1 \quad \text{(central projection through $x_1$)} \\
  f_1^1 : \mathcal{L}^N(x_1) &\rightarrow \mathcal{L}^N(x_{12}) &\leftrightarrow& \tau_1^2 : \pi^2_1 \rightarrow \pi^1_2 \quad \text{(central projection through $x_{12}$)} \\
  f_2^1 : \mathcal{L}^N(x_{12}) &\rightarrow \mathcal{L}^N(x_2) &\leftrightarrow& \tau_1^2 : \pi^2_2 \rightarrow \pi^1_1 \quad \text{(central projection through $x_2$)} \\
  f^2 : \mathcal{L}^N(x_2) &\rightarrow \mathcal{L}^N(x) &\leftrightarrow& \tau^2 : \pi^2 \rightarrow \pi^1 \quad \text{(central projection through $x$)}
\end{align*}
\]

The identity (12) may then be rewritten as

\[
\tau = \tau^2 \circ \tau^1_2 \circ \tau^1_1 \circ \tau^1 = \text{id}.
\]

As $\tau : \pi^1 \rightarrow \pi^1$ is a projective transformation and $\dim(\pi^1) = N - 1$, it is sufficient to show that there are $N + 1$ fixed points of $\tau$ in general position. In fact, we already have $N$ fixed points because of (11), that is

\[
\tau(p^i) = p^i, \quad i = 1, \ldots, N,
\]

for $p^i = t^i \cap \pi^1$. The existence of one further fixed point is guaranteed by Lemma 42. □

The analytic description of Q-nets with frames at vertices via reflections might be useful as it turned out to be in the special case of circular nets. For circular nets, orthonormal frames at vertices were introduced in [6] and played an essential role in the proof of the $C^\infty$-convergence of circular nets to smooth orthogonal nets. The Dupin cyclidic structure behind was discovered later and gave rise to the introduction of cyclidic nets in [5]. In that case, the orthonormal frames at adjacent vertices of the supporting circular net are related by a reflection in the Euclidean symmetry plane, which is a special instance of a projective reflection (clearly, cyclidic nets in $\mathbb{R}^3$ are special instances of supercyclidic nets). Note that the Euclidean reflections in the cyclidic case are uniquely determined by the supporting circular net, thus a Dupin cyclidic net is uniquely determined by its supporting circular net and the frame in one vertex.

References

[1] S. Allen and D. Dutta. Supercyclides and blending. Comput. Aided Geom. Des., 14(7):637–651, 1997.
[2] M. Barner. Eine differentialgeometrische Kennzeichnung der allgemeinen Dupinschen Zykliden. Archiv der Mathematik, 34:277–286, 1987.
[3] P. Bo, H. Pottmann, M. Kilian, W. Wang, and J. Wallner. Circular arc structures. ACM Trans. Graphics, 30-#101,1–11, 2011. Proc. SIGGRAPH.
[4] A. I. Bobenko. Discrete conformal maps and surfaces. In P. A. Clarkson and F. W. Nijhoff, editors, Symmetries and integrability of difference equations (Canterbury 1996), volume 255 of London Math. Soc. Lecture Notes, pages 97–108. Cambridge University Press, 1999.
[5] A. I. Bobenko and E. Huhnen-Venedey. Curvature line parametrized surfaces and orthogonal coordinate systems: discretization with dupin cyclides. Geometriae Dedicata, 159(1):207–237, 2012.
[6] A. I. Bobenko, D. Matthes, and Y. B. Suris. Discrete and smooth orthogonal systems: $C^\infty$-approximation. Int. Math. Res. Not., 45:2415–2459, 2003.
[7] A. I. Bobenko and W. K. Schief. Discrete line complexes and integrable evolution of minors. Preprint, http://arxiv.org/abs/1410.5794, 2014.
[8] A. I. Bobenko and Y. B. Suris. Discrete Differential Geometry. Integrable structure, volume 98 of Graduate Studies in Mathematics. AMS, 2008.
[9] J. Cieslinski, A. Doliwa, and P. M. Santini. The integrable discrete analogues of orthogonal coordinate systems are multi-dimensional circular lattices. Phys. Lett. A, 235:480–488, 1997.
[10] W. Degen. Surfaces with a conjugate net of conics in projective space. Tensor, New Ser., 39:167–172, 1982.
[11] W. Degen. Die zweifachen Blutelschen Kegelschnittflächen. Manuscr. Math., 55:9–38, 1986.
[12] W. Degen. Generalized cyclides for use in CAGD. In The mathematics of surfaces IV. Proceedings of the 4th IMA conference held in Bath, GB, September 1990, pages 349–360. Oxford Clarendon Press, 1994.

[13] W. Degen. On the origin of supercyclides. In The mathematics of surfaces VIII. Proceedings of the 8th IMA conference held in Birmingham, GB, August and September 1998, pages 297–312. Winchester: Information Geometers, 1998.

[14] A. Doliwa and P. M. Santini. Multidimensional quadrilateral lattices are integrable. Phys. Lett. A, 233:265–372, 1997.

[15] A. Doliwa, P. M. Santini, and M. Mañas. Transformations of quadrilateral lattices. J. Math. Phys., 41(2):944–990, 2000.

[16] D. Dutta, R. Martin, and M. Pratt. Cyclides in surface and solid modeling. IEEE Computer Graphics and Applications, 13:53–59, 1993.

[17] L. Eisenhart. Transformations of surfaces. Princeton, N. J. Princeton University Press, 1923.

[18] S. Foufou and L. Garnier. Obtainment of Implicit Equations of Supercyclides and Definition of Elliptic Supercyclides. Machine Graphics and Vision, 14(2):123–144, 2005.

[19] T. Hoffmann. On local deformations of planar quad-meshes. In K. Fukuda, J. Hoeven, M. Joswig, and N. Takayama, editors, Mathematical Software – ICMS 2010, volume 6327 of Lecture Notes in Computer Science, pages 167–169. Springer Berlin Heidelberg, 2010.

[20] E. Huhnen-Venedey and T. Rörg. Discretization of asymptotic line parametrizations using hyperboloid surface patches. Geometriae Dedicata, 168(1):265–289, 2013.

[21] E. Huhnen-Venedey and W. K. Schief. On Weingarten transformations of hyperbolic nets. Int. Math. Res. Not., pages 1–61, 2014.

[22] F. Käferböck and H. Pottmann. Smooth surfaces from bilinear patches: discrete affine minimal surfaces. Comput. Aided Geom. Des., 30(5):476–489, 2013.

[23] B. G. Konopelchenko and W. K. Schief. Three-dimensional integrable lattices in Euclidean spaces: conjugacy and orthogonality. R. Soc. Lond. Proc. Ser. A, 454:3075–3104, 1998.

[24] Y. Liu, H. Pottmann, J. Wallner, Y.-L. Yang, and W. Wang. Geometric modeling with conical meshes and developable surfaces. ACM Trans. Graphics, 25(3):681–689, 2006. Proc. SIGGRAPH.

[25] R. Martin. Principal patches – a new class of surface patch based on differential geometry. Eurographics Proceedings, pages 47–55, 1983.

[26] R. Martin, J. de Pont, and T. Sharrock. Cyclide surfaces in computer aided design. In J. Gregory, editor, The mathematics of surfaces, pages 253–267, Clarendon Press, 1986.

[27] D. McLean. A method of Generating Surfaces as a Composite of Cyclide Patches. The Computer Journal, 28(4):433–438, 1985.

[28] H. Pottmann, A. Asperl, M. Hofer, and A. Kilian. Architectural Geometry. Bentley Institute Press, 2007.

[29] H. Pottmann and J. Wallner. Computational line geometry. Mathematics and Visualization. Springer-Verlag, Berlin, 2001.

[30] H. Pottmann and J. Wallner. The focal geometry of circular and conical meshes. Adv. Comput. Math., 29(3):249–268, 2008.

[31] M. J. Pratt. Dupin cyclides and supercyclides. In The mathematics of surfaces VI. Based of the 6th IMA mathematics of surfaces international conference, Brunel Univ., Uxbridge, Middlesex, GB, September 1994, pages 43–66. Oxford: Oxford Univ. Press, 1996.

[32] M. J. Pratt. Classification and characterization of supercyclides. In The mathematics of surfaces VII. Proceedings of the 7th conference, Dundee, Great Britain, September 1996, pages 25–41. Winchester: Information Geometers, Limited, 1997.

[33] M. J. Pratt. Quartic supercyclides I: Basic theory. Comput. Aided Geom. Des., 14(7):671–692, 1997.

[34] M. J. Pratt. Quartic supercyclides for geometric design. In From geometric modeling to shape modeling. IFIP TC5 WG5. 2 7th workshop on geometric modeling: Fundamentals and applications, Parma, Italy, October 2–4, 2000, pages 191–208. Boston: Kluwer Academic Publishers, 2002.

[35] R. Sauer. Projektive Liniengeometrie. Berlin, W. de Gruyter & Co. (Göschen’s Lehrbücherei I. Gruppe, Bd. 23), 1937.

[36] L. Shi, J. Wang, and H. Pottmann. Smooth surfaces from rational bilinear patches. Comput. Aided Geom. Des., 31(1):1–12, 2014.

[37] Y. Srinivas, V. Kumar, and D. Dutta. Surface design using cyclide patches. Computer-Aided Design, 28(4):263–276, 1996.

[38] C. Tang, X. Sun, A. Gomes, J. Wallner, and H. Pottmann. Form-finding with polyhedral meshes made simple. ACM Trans. Graphics, 33(4), 2014. Proc. SIGGRAPH.
[39] J. Wang, C. Jiang, P. Bompas, J. Wallner, and H. Pottmann. Discrete line congruences for shading and lighting. *Computer Graphics Forum*, 32/5, 2013.

Alexander I. Bobenko, Emanuel Huhnen-Venedey, Thilo Röric
Institute of Mathematics, Secr. MA 8-4, TU Berlin, 10623 Berlin, Germany
Email: {bobenko, huhnen, roerig}@math.tu-berlin.de