Algebraic topology of spin glasses

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Abstract
We study the topology of frustration in d-dimensional Ising spin glasses with d ≥ 2 with nearest-neighbor interactions. We prove the following. For any given spin configuration, the domain walls on the unfrustration network are all transverse to a frustrated loop on the unfrustration network, where a domain wall is defined to be a connected element of the collection of all the (d−1)-cells which are dual to the bonds having an unfavorable energy, and the unfrustration network is the collection of all the unfrustrated plaquettes. These domain walls are topologically nontrivial because they are all related to the global frustration of a loop on the unfrustration network. Taking account of the thermal stability for the domain walls, we can explain the numerical results that three- or higher-dimensional systems exhibit a spin glass phase, whereas two-dimensional ones do not. Namely, in two dimensions, the thermal fluctuations of the topologically nontrivial domain walls destroy the order of the frozen spins on the unfrustration network, whereas they do not in three or higher dimensions. This may be interpreted as a global topological effect of the frustrations.

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1. Introduction

Theoretical investigations of spin glasses have a long history, starting with the paper by Edwards and Anderson [8]. In particular, Toulouse [20] emphasized the importance of the frustration effect. Along the line, Bovier and Fröhlich [5] studied the distribution of frustration, focusing on the geometrical aspect of frustration. However, such geometrical or topological approaches to spin glasses are still rare, and the nature of ordering at low temperatures still remains controversial, except for the success1 of the mean-field theory.

In this paper, we study the topology of frustration in the Ising spin glasses with nearest-neighbor interactions on the d-dimensional hypercubic lattice \( \mathbb{Z}^d \) with d ≥ 2, in order to

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1 See, for example, [16].
elucidate the nature of ordering in the spin glass phase at low temperatures. We show that algebraic topology is a useful mathematical language to describe the spin glasses. Actually, the frustration function which was introduced by Toulouse is nothing but a one-dimensional cohomology class with $\mathbb{Z}_2$ coefficient as we will see in section 3.

In order to describe our results, we introduce some basic terms. The more precise definitions are given in the following sections. Following Bovier and Fröhlich [5], we classify all the plaquettes into two classes, frustrated and unfrustrated plaquettes. The collection of the frustrated plaquettes is called the frustration network, and the latter collection is called the unfrustration network. For a given spin configuration, consider the collection of all the $(d-1)$-cells which are dual to the bonds having the unfavorable energy. The collection is made of the connected $(d-1)$-dimensional hypersurfaces. We call a connected element of the hypersurfaces the domain wall.

Now let us describe our results. When the system is restricted onto the unfrustration network, there exists a one-to-one correspondence between the homology class of the frustrated loops and the homology class of the domain walls. In more detail, each domain wall is transverse to the corresponding frustrated loop on the unfrustration network. These are topologically nontrivial domain walls because they are all related to the global frustration of a loop. For the whole lattice, the rest of the domain walls are outside of the unfrustration network. Thus, there appear two types of the domain walls for a given spin configuration.

Based on these results, we can heuristically argue the existence of the spin glass phase at low temperatures. First we note that in the same way as in [5], one can show that there appears an infinite, connected cluster of unfrustrated plaquettes for a certain value of the concentration parameter for the positive couplings in dimensions $d \geq 2$. All the bonds of the unfrustration network except for the bonds which intersect a domain wall have a favorable energy. Clearly, in a ground state, the domain walls are determined to minimize their total size. Therefore, the number of the bonds having an unfavorable energy on the unfrustration network becomes a small fraction of the whole bonds of the unfrustration network. From these observations, one notes that the situation on the unfrustration network is very similar to that in the standard ferromagnetic Ising model which is restricted onto the unfrustration network. The difference is the appearance of the two types of the domain walls.

Consider first the effect of the domain walls which are outside of the unfrustration network. They are expected to affect the spin configurations on the unfrustration network as a boundary effect. As we will see in section 4, a ground state on a frustration network is highly degenerate. This suggests that the spins show disorder on the frustration network at finite temperatures. Therefore, the boundary effect may behave as random fields [9, 10] at the boundary of the unfrustration network. But the domain walls minimize their total size in a ground state. This implies that the domain walls show a tendency to confine themselves into a small neighborhood of the frustration network. As a result, the boundary effect is expected to be ignorably small.

Next consider the effect of the domain walls which are transverse to a frustrated loop on the unfrustration network. These topologically nontrivial domain walls may be regarded as an analog of the Dobrushin domain wall [6]. However, the number of the domain walls increases as the size of the system increases. As is well known, a domain wall is unstable against the thermal fluctuation in two dimensions [12], while it is stable in three and higher dimensions. From these observations, we can expect the following. In two dimensions, the thermal fluctuations of the topologically nontrivial domain walls destroy the order of the frozen spins on the unfrustration network. This explains the absence of the spin glass phase in two dimensions. Since these domain walls are related to the global frustration of a loop, destroying the order of the frozen spins may be interpreted as a topological effect.
In three or higher dimensions, the thermal fluctuations of the topologically nontrivial domain walls cannot destroy the order of the frozen spins on the unfrustration network because of the stability of the domain walls. As a result, there appears long-range order of the frozen spins on the unfrustration network at low temperatures. Namely, the system exhibits a spin glass phase in three and higher dimensions at low temperatures.

This paper is organized as follows. In the next section, we describe the models which we consider in this paper, and establish some basic definitions which are related to frustration. In section 3, we study the topology of generic unfrustration networks. As a result, we prove that the homology class of the frustrated loops is isomorphic to the homology class of the domain walls on the unfrustration network. This implies that the topology of the domain walls on the unfrustration network is uniquely determined by the frustrated loops on the unfrustration network, and that the topology is independent of spin configurations. In section 4, we study the topology of certain frustration networks. Their ground states exhibit high degeneracy. In section 5, we study the relation between frustration networks themselves and frustrated loops on unfrustration networks. The resulting relation enables us to elucidate the homology of the domain walls for the whole lattice in section 6. In section 7, we discuss the role of the topologically nontrivial domain walls in the context of the appearance of long-range order of the frozen spins on the unfrustration network at low temperatures.

2. Preliminaries

We consider a short-range Ising spin glass with the Hamiltonian \[ H = - \sum_{i,j} J_{ij} \sigma_i \sigma_j, \] (1)

where \( \sigma_i \) takes the values \( \pm 1 \), and the bond variables \( \{ J_{ij} \} \) form a family of independent and identically distributed random variables; the sum is over the nearest-neighbor pair \( \langle i, j \rangle \) of the sites \( i, j \) on\( \Lambda_1 \) the finite sublattice \( \Lambda \) of the \( d \)-dimensional lattice \( \mathbb{Z}^d \) with \( d \geq 2 \). We choose the distribution of \( J = J_{ij} \) as \[ d\rho(J) = [(1 - x)g(-J) + xg(J)]dJ, \]

where the concentration parameter \( x \) satisfies \( 0 \leq x \leq 1 \), and \( g \) is a nonnegative function with support on the positive reals, and satisfies normalization \[ \int g(J) dJ = 1. \]

A fairly common choice for \( g \) is the singular delta distribution \[ g(J) = \delta(J - J_0) \] with some \( J_0 > 0 \). (2)

The positive coupling \( J \) is distributed according to a Bernoulli bond percolation process with density \( x \).

We denote by \( \hat{J}_{ij} \) the sign of the coupling \( J_{ij} \), i.e. \( \hat{J}_{ij} = J_{ij}/|J_{ij}| \). Following Toulouse [20], we introduce ‘frustration’ which is defined by \[ \phi(\ell) = \prod_{\langle i, j \rangle \in \ell} \hat{J}_{ij} \]

for a loop (or a closed path) \( \ell \) along the bonds of the lattice. (A loop \( \ell \) is made of a collection of bonds.) When \( \phi(\ell) \) takes the value \( -1 \), we say that the loop \( \ell \) is frustrated. We denote by

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2 Although we can treat general lattices which can be identified with a CW-complex, we consider a finite subset \( \Lambda \) of the \( \mathbb{Z}^d \) lattice for simplicity. As general references for cell complexes or CW complexes, see [7, 18].
A plaquette (or elementary 2-cell) of the lattice. Clearly the boundary ∂p of a plaquette p is a loop which consists of the four bonds. When φ(∂p) = −1, we also say that the plaquette p is frustrated. We call a collection of frustrated plaquettes a frustration network, and call a collection of unfrustrated plaquettes p satisfying φ(∂p) = +1 an unfrustration network.

Following Bovier and Fröhlich [5], we associate a frustration network with the collection of the (d − 2)-cells in the dual lattice. Since a plaquette is a 2-cell, its dual is given by a (d − 2)-cell in the dual lattice. For example, in two dimensions, the dual to a plaquette is the center of gravity, and in three dimensions the dual is given by the bond in the dual lattice. A frustration network is dual to a collection of (d − 2)-cells which form (d − 2)-dimensional complexes which are closed or whose boundaries end at the boundaries of the dual lattice Λ∗ to the lattice Λ [5]. To see this, consider a cube (or elementary 3-cell) c. Then, one has

\[ \prod_{p \in c} \phi(\partial p) = \prod_{(i,j) \in c} (\hat{J}_{ij})^2 = 1. \]

This implies that an even number of the plaquettes must be frustrated in c. Thus, no complex made of (d − 2)-cells dual to frustrated plaquettes can end in any cube.

3. Topology of unfrustration networks

In this section, we consider an unfrustration network Ns. By definition, all the plaquettes p of Ns satisfy φ(∂p) = +1 which is called the cocycle condition. In order to study the topology of Ns, we rely on the homology and cohomology theories.

Let us denote by b(ℓ) the set of all the bonds in the loop ℓ. Consider the symmetric difference of the two sets, b(ℓ1) and b(ℓ2), of the bonds

\[ ℓ_1 \oplus ℓ_2 := (b(ℓ_1) \setminus b(ℓ_2)) \cup (b(ℓ_2) \setminus b(ℓ_1)) \]

for two loops, ℓ1 and ℓ2, in Ns. If the symmetric difference ℓ1 ⊕ ℓ2 of the bonds consists of only four bonds of a single unfrustrated plaquette p in Ns, then one has

\[ φ(ℓ_1) = φ(ℓ_2) \]

by using the cocycle condition φ(∂p) = +1. For these two loops, we write ℓ1 ∼ ℓ2. Clearly this relation can be extended to a generic pair of two loops, ℓ and ℓ′, in Ns when there exists a sequence of loops, ℓ = ℓ1, ℓ2, . . . , ℓn = ℓ′, satisfying the above condition ℓi ∼ ℓi+1 for i = 1, 2, . . . , n − 1. For two such loops, we also write ℓ ∼ ℓ′, and we say that the two loops, ℓ and ℓ′, are homologous to each other. Besides we automatically obtain φ(ℓ) = φ(ℓ′) for ℓ ∼ ℓ′ from the definitions. But we stress that the quantity φ(ℓ) is not necessarily equal to +1. For any unfrustrated plaquette p, the boundary ∂p is homologous to the empty set by definition. In this case, we write ∂p ∼ 0, and, more generally, we write ℓ ∼ 0 if the loop ℓ is homologous to the empty set. Clearly we have φ(ℓ) = +1 for ℓ ∼ 0.

Consider a two-dimensional surface s which is made of a collection of unfrustrated plaquettes. Then, the boundaries ∂s of s become loops. From the above argument, if the symmetric difference ℓ1 ⊕ ℓ2 for two loops, ℓ1 and ℓ2, is equal to the boundaries ∂s of a surface s, then one has ℓ1 ∼ ℓ2. We denote by Z1(Ns; Z) the module (additive group) made of all the oriented loops (or cycles) with coefficients Z, and denote by B1(Ns; Z) the submodule made of the boundaries ∂s for all the two-dimensional oriented surfaces s. Then the one-dimensional homology module H1(Ns; Z) is defined to be the quotient module

\[ H_1(Ns; Z) := Z_1(Ns; Z)/B_1(Ns; Z). \]
This module is made of all the classes \([\ell]\) which is represented by a nontrivial loop \(\ell \neq 0\). The frustration \(\phi\) yields the homomorphism\(^5\)

\[ \phi : H_1(N_\Lambda; \mathbb{Z}) \longrightarrow \mathbb{Z}_2, \]

by the relation \(\phi(\ell_1) = \phi(\ell_2)\) for \(\ell_1 \sim \ell_2\).

The following lemma plays a key role in this paper.

**Lemma 1.** Fix the random variable \(\{J_{ij}\}\) in the Hamiltonian \(\mathcal{H}_\Lambda\) of (1) on a finite lattice \(\Lambda\). Suppose that any two sites in \(\Lambda\) are connected by a path of bonds in \(\Lambda\), and suppose that any loop \(\ell\) in \(\Lambda\) satisfies \(\phi(\ell) = +1\), i.e. no loop is frustrated in \(\Lambda\). Then, the Hamiltonian \(\mathcal{H}_\Lambda\) of (1) has exactly two ground states.

**Remark.** There exist some similarities between the present spin glass and an electron gas in a periodic potential. The above lemma states that the vanishing of the frustration leads to the triviality of the \(\mathbb{Z}_2\) bundle for the spin glass. The analog in the electron gas is that the vanishing of the Chern number yields the triviality of the U(1) bundle for the Bloch wavefunctions [17]. This similarity comes from the gauge invariance of the two theories [20].

**Proof.** Let \(i_0\) be a site in \(\Lambda\). Then, any site \(j\) in \(\Lambda\) can be connected with \(i_0\) by a path \(\gamma\) in \(\Lambda\) from the assumption on \(\Lambda\). Write \(\gamma = \{i_0, i_1, \ldots, i_n = j\}\) by using the sequence \(\{i_0, i_1, \ldots, i_n = j\}\) of the sites in \(\Lambda\), where \(\{i_k, i_{k+1}\}\) are bonds of \(\Lambda\) for \(k = 0, 1, \ldots, n - 1\).

Fix the value of the Ising spin \(\sigma_{i_0}\) at the site \(i_0\). Then, we can determine the value of the spin \(\sigma_j\) at the site \(j = i_n\) by using the relations \(J_{i_k i_{k+1}} \sigma_i \sigma_{i+1} = +1\) along the path \(\gamma\). Clearly the value of the spin \(\sigma_j\) is unique for the fixed value of \(\sigma_{i_0}\) and the fixed path \(\gamma\).

In the same way, a different path \(\gamma'\) in \(\Lambda\) gives a value \(\sigma_j'\) of the spin at the site \(j\). We show \(\sigma_j = \sigma_j'\). Namely the value of spin \(\sigma_j\) is independent of paths. Consider the loop \(\ell\) which consists of two paths \(\gamma\) and \(-\gamma'\). Then, we have \(\phi(\ell) = +1\) from the assumption. This implies that, in modulo 2 arithmetic, the number of the negative couplings \(J_{\gamma}\) in \(\gamma\) is equal to that of the negative couplings \(J_{\gamma'}\) in \(\gamma'\). Therefore, \(\sigma_j = \sigma_j'\).

As a result, the whole spin configuration on \(\Lambda\) is uniquely determined by the fixed value of \(\sigma_{i_0}\). Clearly the above condition \(J_{i_k i_{k+1}} \sigma_i \sigma_{i+1} = +1\) implies that the resulting spin configuration is the ground state. Besides, the choice of the value \(\sigma_{i_0}\) is exactly 2. \(\Box\)

In lemma 1, the assumption \(\phi(\ell) = +1\) for any loop \(\ell \subset \Lambda\) is too strong even for \(\Lambda = N_\Lambda\) because a frustrated loop appears in a generic unfustration network. Thus, we must treat the subset \([\ell] \in H_1(N_\Lambda; \mathbb{Z}) | \phi(\ell) = -1\) of the homology module \(H_1(N_\Lambda; \mathbb{Z})\). For this purpose, it is convenient to rely on the cohomology theory on a generic lattice \(\Lambda\) which is a collection of plaquettes.

Consider bond variables \(\tau = \{\tau_{ij}\}_{\langle i,j \rangle}\), where \(\tau_{ij}\) takes the value \(\pm 1\). We denote by \(C_1(\Lambda; \mathbb{Z})\) the module which is made of the oriented bonds (1-chains) with coefficients \(\mathbb{Z}\). An element \(a \in C_1(\Lambda; \mathbb{Z})\) is written as

\[ a = \sum_{\langle i,j \rangle} c_{\langle i,j \rangle} \langle i, j \rangle \quad \text{with} \quad c_{\langle i,j \rangle} \in \mathbb{Z}. \]

Then, \(\tau(a)\) is defined by

\[ \tau(a) = \prod_{\langle i,j \rangle} (\tau_{ij})^{c_{\langle i,j \rangle}}. \]

\(^5\) In this case, a homomorphism is a map \(f\) of an additive group \(A\) into a multiplicative group \(B\) such that \(f(a + b) = f(a)f(b)\) for \(a, b \in A\).
As usual, we define the coboundary operator $\partial^*$ by the adjoint of the boundary operator $\partial$ as

\[(\partial^* \tau)(s) = \tau(\partial s)\]

for a two-dimensional surface $s$. We denote by $Z^1(\Lambda; \mathbb{Z}_2)$ the set of all $\tau$ satisfying $\partial^* \tau = 1$. By the linearity, the condition $\partial^* \tau = 1$ is equivalent to the cocycle condition $\tau(\partial p) = 1$ for any plaquette $p \subseteq \Lambda$. An element of $Z^1(\Lambda; \mathbb{Z}_2)$ is called a cocycle. We also denote by $B^1(\Lambda; \mathbb{Z}_2)$ the set of all $\tau$ such that $\tau$ has $\tau_{ij} = \epsilon_i \epsilon_j$ for all the bonds $(i, j)$ with a site variable $\epsilon_i$ which takes the value $\pm 1$. Since the boundaries $\partial s$ of surfaces $s$ become the loops, one has $\partial^* \tau(s) = \tau(\partial s) = 1$ for any $\tau \in B^1(\Lambda; \mathbb{Z}_2)$. This implies that $B^1(\Lambda; \mathbb{Z}_2) \subset Z^1(\Lambda; \mathbb{Z}_2)$. If two cocycles $\tau, \tau' \in Z^1(\Lambda; \mathbb{Z}_2)$ satisfy $\tau_{ij} = \tau'_{ij} \epsilon_i \epsilon_j$ with site variables $\epsilon_i$, then one has $\tau'(\ell) = \tau(\ell)$ for any loop $\ell$. We say that such two cocycles, $\tau$ and $\tau'$, are cohomologous to each other, and write $\tau \sim \tau'$. (The two sets, $\tau, \tau'$, of the couplings are gauge equivalent to each other.) The one-dimensional cohomology module $H^1(\Lambda; \mathbb{Z}_2)$ is defined to be the quotient module of $Z^1(\Lambda; \mathbb{Z}_2)$ by the submodule $B^1(\Lambda; \mathbb{Z}_2)$.

Now we return to the problem characterizing the collection of the frustrated loops $[[\ell]] \in H_1(\mathcal{N}_\phi; \mathbb{Z})$ which is defined by

\[H_1(\mathcal{N}_\phi; \mathbb{Z}) = \frac{\ker(\partial)}{\text{im}(\partial)}\]

where $\ker(\partial)$ is the submodule of the set of all $\alpha$ for which the boundary operator $\partial$ satisfies $\partial \alpha = 0$ and $\text{im}(\partial)$ is the submodule of the set of all $\partial \beta$ for any $\beta \in Z^0(\mathcal{N}_\phi; \mathbb{Z})$. By the definition of the cohomology module, $H_1(\mathcal{N}_\phi; \mathbb{Z})$ is the set of all homology classes $[\ell]$ in $\mathcal{N}_\phi$ which are defined by loops $\ell$ satisfying $\tau(\ell) = 1$ on the unfrustrated network.

We write $\tau(\ell) = \exp[i \pi \alpha(\ell)]$, where $\alpha = \sum_{(i, j)} \alpha_{ij} \epsilon_i \epsilon_j$ is defined by

\[\alpha = \sum_{(i, j)} \alpha_{ij} \epsilon_i \epsilon_j\]

for a loop $\ell$. Consider the complex $\Sigma$ which consists of $(d - 1)$-cells which are dual to the bonds $(i, j)$ satisfying $\alpha_{ij} = 1$ for a cocycle $\alpha \in Z^1(\mathcal{N}_\phi; \mathbb{Z}_2)$. Here, $Z^1(\mathcal{N}_\phi; \mathbb{Z}_2)$ is the module of all the cocycles $\alpha$ satisfying the cocycle condition $\alpha(\partial p) = 0 \mod 2$. By the cocycle condition, the complex $\Sigma$ cannot end in any plaquette $p \subseteq \mathcal{N}_\phi$. Namely the complex $\Sigma$ forms the $(d - 1)$-dimensional hypersurfaces which are closed or whose boundaries are in the boundaries $\partial \mathcal{N}_\phi^*$ of the dual $\mathcal{N}_\phi^*$ of the unfrustration network $\mathcal{N}_\phi$. We write $\Sigma = \vartheta(\alpha)$ for the homomorphism. The homomorphism $\vartheta$ yields the following isomorphism which is a special case of Poincaré–Lefschetz duality theorem:\footnote{For example, see theorem 3 of section 5.5 of [18].}

A homomorphism is called an isomorphism if it is bijective.\footnote{A homomorphism is called an isomorphism if it is bijective.}

As is well known, $H_0(\mathcal{N}; \mathbb{Z}) \cong \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$ for any network $\mathcal{N}$.\footnote{As is well known, $H_0(\mathcal{N}; \mathbb{Z}) \cong \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$ for any network $\mathcal{N}$.}

\footnote{For example, see section 28 of [13], or theorem 20 of section 6.3 of [18].}

\footnote{For example, see section 28 of [13], or theorem 20 of section 6.3 of [18].}
Proposition 2. The following isomorphism holds:

\[ H^1(\mathcal{N}^*_\tau; \mathbb{Z}_2) \cong H_{d-1}(\partial \mathcal{N}^*_\tau; \mathbb{Z}_2), \]

where the right-hand side is the set of the homology classes for the \((d - 1)\)-dimensional hypersurfaces which are closed or whose boundaries are in the boundaries \(\partial \mathcal{N}^*_\tau\) of the dual \(\mathcal{N}^*_\tau\) of the unfrustration network \(\mathcal{N}_\tau\).

Proof. First we show that the map \(\vartheta \circ K\) is well defined, i.e. if \(\tau \sim \tau'\), then \(\vartheta \circ K(\tau) \sim \vartheta \circ K(\tau')\). Let \(\tau, \tau' \in Z^1(\mathcal{N}^*_\tau; \mathbb{Z}_2)\) satisfying \(\tau' \sim \tau\). Then, \(\tau'_i = \tau_i \epsilon_i \epsilon_j\) with the site variables \(\epsilon_i \in \{+1, -1\}\), and \(\vartheta \circ K(\tau') = \vartheta \circ K(\tau) + \vartheta \circ K(\epsilon)\), where \(\epsilon = \{\epsilon_i = \epsilon_i \epsilon_j\}\) by definition. Consider two regions, \(V_\tau = [i | \epsilon_i = +1] \) and \(V_\tau = [i | \epsilon_i = -1]\). Then, the set of the bonds having \(\epsilon_j = -1\) can be identified with the interfaces \(\Sigma\) between the two regions \(V_\tau\) and \(V_-\). Therefore, one has \(\epsilon_j = -1\) for \(i \in V_\tau\) and \(\epsilon_i = +1\) for \(i \notin V_\tau\).

Let \(\Sigma\) be a \((d - 1)\)-dimensional hypersurfaces which are closed or whose boundaries are in the boundaries \(\partial \mathcal{N}^*_\tau\). If a plaquette \(p \subset \mathcal{N}^*_\tau\) intersects \(\Sigma\), an even number of the bonds of \(p\) must intersect \(\Sigma\). For such bonds \((i, j)\), we choose the bond variables \(\tau_{ij} = -1\). Then, \(\tau\) satisfies the cocycle condition, and \(\vartheta \circ K(\tau) = \Sigma\). Thus, the map \(\vartheta\) is surjective.

Finally let us show that the corresponding map is injective. Let \(\tau \in Z^1(\mathcal{N}^*_\tau; \mathbb{Z}_2)\) satisfying \(\vartheta \circ K(\tau) \sim 0\). Then there exists a region \(V\) such that \(\partial V = \vartheta \circ K(\tau) + \sum_{i} \epsilon_i \epsilon_j\). We take the site variables \(\epsilon_i\) that \(\epsilon_i = -1\) for \(i \in V\) and \(\epsilon_i = +1\) for \(i \notin V\).

Let us show that the relation \(\tau_{ij} \epsilon_i \epsilon_j = +1\) holds for all the bonds \((i, j)\). When both of the sites \(i\) and \(j\) are included in the region \(V\), the bond \((i, j)\) does not intersect the surface \(\partial V\). Therefore, one has \(\tau_{ij} = +1\) and \(\tau_{ij} \epsilon_i \epsilon_j = +1\). Similarly, when neither \(i\) nor \(j\) is included in \(V\), the bond \((i, j)\) does not intersect \(\partial V\), too. Therefore, \(\tau_{ij} = +1\) and \(\tau_{ij} \epsilon_i \epsilon_j = +1\). When the bond \((i, j)\) intersects the surface \(\vartheta \circ K(\tau)\), one has \(\tau_{ij} = -1\). In this case, one obtains \(\epsilon_i \epsilon_j = -1\) because one of the two sites, \(i\) and \(j\), is included in \(V\) and the other is not. These two yield \(\tau_{ij} \epsilon_i \epsilon_j = +1\). Thus, the relation \(\tau_{ij} \epsilon_i \epsilon_j = +1\) holds for all the bonds \((i, j)\). This implies \(\tau \sim 1\).

Combining this proposition with (3), one has the following isomorphism:

\[ \text{Hom}(H_1(\mathcal{N}^*_\tau; \mathbb{Z}_2), \mathbb{Z}_2) \cong H_{d-1}(\partial \mathcal{N}^*_\tau; \mathbb{Z}_2). \]  

(5)

This gives the universal relation between the frustrated loops and the domain walls on the unfrustration network \(\mathcal{N}^*_\tau\) as we will see below.

Let \(\beta\) be a cochain satisfying \(\beta \sim k(\phi)\) for the frustration \(\phi\). Namely there exists \(\psi\) such that \(\beta = k(\psi)\), and that \(\psi \sim \phi\). By definition, the cochain \(\beta\) is written in the above form (4) as

\[ \beta = \sum_{(i, j)} \beta_{ij} (i, j)^* \]

with the bond variables \(\beta_{ij} \in \{0, 1\}\). Write \(b(\beta)\) for the set of the bonds \((i, j)\) satisfying \(\beta_{ij} = 1\). Namely the bonds in \(b(\beta)\) are dual to the \((d - 1)\)-cells of the hypersurfaces \(\vartheta(\beta)\). We write \(b(\beta) = \vartheta(\beta)^*\). Clearly we have \(b(\ell) = 0\) for any loop \(\ell \subset \mathcal{N}^*_\tau \setminus \vartheta(\beta)^*\). This implies \(\phi(\ell) = \psi(\ell) = \exp[i \vartheta(\ell)] = 1\) for any loop \(\ell \subset \mathcal{N}^*_\tau \setminus \vartheta(\beta)^*\). Thus, the frustration function \(\phi\) is not frustrated on \(\mathcal{N}^*_\tau \setminus \vartheta(\beta)^*\).

Let \(\sigma \in \{1, -1\}\) be a spin configuration on the sites of the unfrustration network \(\mathcal{N}^*_\tau\). Let \(\phi'\) be a frustration with the random variable \(\tilde{J}_{ij} = \tilde{J}_{ij} \sigma_i \sigma_j\) for the frustration \(\phi\).
with $\hat{J}_{ij}$. By definition, one has $\phi' \sim \phi$ and $\kappa(\phi') \sim \kappa(\phi)$. In addition, the corresponding $(d-1)$-dimensional hypersurfaces $\vartheta \circ \kappa(\phi')$ and $\vartheta \circ \kappa(\phi)$ are homologous to each other, i.e. $\vartheta \circ \kappa(\phi') \sim \vartheta \circ \kappa(\phi)$ from proposition 2. Let $b_{-}(\phi')$ be the set of the bonds $(i, j)$ satisfying $\hat{J}_{ij} = \hat{J}_{ij}\sigma_i\sigma_j = -1$. By the definition of the map $\kappa$, one has $b_{-}(\phi') = b(\kappa(\phi'))$. Therefore, the bonds in $b_{-}(\phi')$ are dual to the $(d-1)$-cells of the hypersurfaces $\vartheta \circ \kappa(\phi')$. If a loop $\ell$ satisfies $\phi(\ell) = -1$, then the number of the bonds of the set $b(\ell) \cap b_{-}(\phi')$ must be odd. This implies that the loop $\ell$ must intersect at least one of the $(d-1)$-cell of the hypersurfaces $\vartheta \circ \kappa(\phi')$. In order to prove this statement, let us assume that the number of the elements of $b(\ell) \cap b_{-}(\phi')$ is even. Then, one has $(\kappa(\phi'))(\ell) = 0$ mod 2. This implies
$$\phi(\ell) = \phi'(\ell) = \exp[i\pi(\kappa(\phi'))(\ell)] = 1.$$ Thus, this is a contradiction.

This fact is rephrased as follows. All the domain walls in the collection $\vartheta \circ \kappa(\phi')$ are transverse to a frustrated loop $\ell$ on the unfrustration network $\tilde{N}_\epsilon$. Here, we define a domain wall as follows. For a given spin configuration, consider the collection of the $(d-1)$-cells which are dual to the bonds $(i, j)$ having $\hat{J}_{ij}\sigma_i\sigma_j = -1$ which yields the unfavorable energy for the bond. We call a connected element of the collection the domain wall. Clearly, the $(d-1)$-dimensional hypersurfaces $\vartheta \circ \kappa(\phi')$ are the set of the domain walls for the spin configuration $\{\sigma_i\}_{i \in \tilde{N}_\epsilon}$.

Take $\{\sigma_i\}_{i \in \tilde{N}_\epsilon}$ to be a spin configuration of the ground state of the Hamiltonian $\mathcal{H}_\Lambda$ on the unfrustration network $\tilde{N}_\epsilon$. Combining the above observations, lemma 1 and the isomorphism (3) or (5), we obtain the following theorem which gives an algorithm to find the ground state.

**Theorem 3.** Fix the random variable $\{J_{ij}\}$ in the Hamiltonian (1). Let $\tilde{N}_\epsilon$ be an unfrustration network which is made of a collection of unfrustrated plaquettes. Suppose that any two sites in $\tilde{N}_\epsilon$ are connected by a path of bonds in $\tilde{N}_\epsilon$. Then there exist $(d-1)$-dimensional connected hypersurfaces $\Sigma_1, \ldots, \Sigma_r \subset \tilde{N}_\epsilon$ satisfying $\partial \Sigma_i \subset \partial \tilde{N}_\epsilon$ for $i = 1, \ldots, r$ such that each hypersurface $\Sigma_i$ is transverse to a homology class of frustrated loops in $\tilde{N}_\epsilon$, and the Hamiltonian $\mathcal{H}_\Lambda$ of (1) restricted to $\Lambda = \tilde{N}_\epsilon \setminus \{\Sigma_1^*, \ldots, \Sigma_r^*\}$ has exactly two ground states. Here $\Sigma_i^*$ is the set of the bonds which are dual to a $(d-1)$-cell of the hypersurfaces $\Sigma_i$. Further, we can choose the hypersurfaces $\Sigma_1, \ldots, \Sigma_r$ so that the spin configurations of the two ground states become a ground-state spin configuration of the Hamiltonian $\mathcal{H}_\Lambda$ of (1) restricted to $\Lambda = \tilde{N}_\epsilon$.

**Remark.** The hypersurfaces, $\Sigma_1, \ldots, \Sigma_r$, for the ground state never contains a connected element which is homologous to zero. In fact, if they contain such an element $\Sigma_i$, then flipping the spins inside the hypersurface $\Sigma_i$ lowers the energy of the ground state. The hypersurfaces, $\Sigma_1, \ldots, \Sigma_r$, are the domain walls for a ground-state spin configuration. Since the choice of the set of the hypersurfaces, $\Sigma_1, \ldots, \Sigma_r$, is not necessarily unique, the degeneracy of the ground state may be larger than 2. In the infinite volume, we can expect that there appear many ground-state spin configurations with different domain walls because an infinitely large domain wall is stable against a local perturbation.

Although the following lemma is essentially due to Bovier and Fröhlich [5], it is most efficient when applied to a spin configuration of the ground state in theorem 3.

**Lemma 4.** Fix the random variable $\{J_{ij}\}$ in the Hamiltonian (1). Let $\tilde{N}_\epsilon$ be an unfrustration network which is made of a collection of unfrustrated plaquettes. Suppose that any two sites in $\tilde{N}_\epsilon$ are connected by a path of bonds in $\tilde{N}_\epsilon$. Let $S_0$ be the set of all the $(d-1)$-cells of the hypersurfaces, $\Sigma_1, \ldots, \Sigma_r$, for the ground state in theorem 3, and let $S_{-}$ be the set of}
all the \((d - 1)\)-cells which are dual to the bonds with the negative coupling \(J_{ij} < 0\). (The collection of the cells in \(S_\pm\) forms the hypersurfaces \(\partial \circ \kappa(\phi)\) for the frustration \(\phi\).) Then, the symmetric difference, \((S_0 \\setminus S_\pm) \cup (S_\pm \setminus S_0)\), gives the interfaces between up- and down-spins in the ground-state spin configuration of the Hamiltonian on the unfrustration network \(\mathcal{N}_c\). 

Proof. We denote by \(S_0^+\) the set of all the bonds which are dual to a cell in \(S_0\), and \(S_\pm^+\) the set of all the bonds with the negative coupling \(J_{ij} < 0\). 

For \((i, j) \notin S_0^+ \cup S_\pm^+\), one has \(\hat{J}_{ij} \sigma_i \sigma_j = +1\) and \(\hat{J}_{ij} = +1\). These imply \(\sigma_i = \sigma_j\). For \((i, j) \in S_0^+ \cap S_\pm^+\), one obtains \(\hat{J}_{ij} \sigma_i \sigma_j = -1\) and \(\hat{J}_{ij} = -1\). These yield \(\sigma_i = \sigma_j\) again. Therefore, no interface appears outside the symmetric difference \((S_0 \setminus S_\pm) \cup (S_\pm \setminus S_0)\).

For \((i, j) \in S_0^+ \setminus S_\pm^+\), one obtains \(\hat{J}_{ij} \sigma_i \sigma_j = -1\) and \(\hat{J}_{ij} = +1\). Therefore, \(\sigma_i = -\sigma_j\). The corresponding dual cell gives the element of the interfaces. For \((i, j) \in S_\pm^+ \setminus S_0^+\), one has \(\hat{J}_{ij} \sigma_i \sigma_j = +1\) and \(\hat{J}_{ij} = -1\). These yield \(\sigma_i = -\sigma_j\) again. 

For the concentration \(x\) near 1/2, we can expect that there appear many large interfaces between up- and down-spins in a ground-state spin configuration on a large unfrustration network. If all the bonds which are dual to such a connected interface are removed from the network, the network is divided into two large parts. We will discuss this point again in section 7 by relying on a percolation theory. As the concentration \(x\) of the positive coupling increases beyond a critical value \(x_c\), we can expect that there disappear large interfaces which divide a large unfrustration network into two large parts [19]. As a result, the spin glass phase changes to the ferromagnetic one [4].

4. Topology of frustration networks

In order to study frustration networks, we introduce a concept of an unfrustrated pair of frustrated plaquettes as follows. Let \(p_1\) and \(p_2\) be two frustrated plaquettes such that they share only a single bond. In this case, the union \(p_1 \cup p_2\) can be identified with a single unfrustrated plaquette whose boundary is made of the six bonds. Actually, one has

\[
\phi(\partial(p_1 \cup p_2)) = \phi(\partial p_1) \phi(\partial p_2) = 1
\]

for the frustration function \(\phi\). We call such two plaquettes \(p_1, p_2\) an unfrustrated pair of frustrated plaquettes. We write \(e_{ij} = p_1 \cap p_2\) for the common bond of frustrated plaquettes \(p_1, p_2\) which form an unfrustrated pair.

Consider a frustration network \(\mathcal{N}_c\) which is a collection of unfrustrated pairs of frustrated plaquettes. Let \(\mathcal{B}_c(\mathcal{N}_c)\) be the set of all the bonds \((i, j)\) in \(\mathcal{N}_c\) such that the bond \((i, j)\) is not a common bond of two frustrated plaquettes which form an unfrustrated pair. Since the network \(\mathcal{B}_c(\mathcal{N}_c)\) is regarded as an unfrustration network which is made of the unfrustrated pairs, we can apply the argument in the preceding section to it. Let \(\{\sigma_i\}_{i \in \mathcal{N}_c}\) be a spin configuration on the frustration network \(\mathcal{N}_c\). Consider the random variables \(J'_{ij} = \hat{J}_{ij} \sigma_i \sigma_j\) for the bonds \((i, j) \in \mathcal{B}_c(\mathcal{N}_c)\). Then, the random variables \(J'_{ij}\) define the frustration function \(\phi'\) which is cohomologous to the frustration \(\phi\) on \(\mathcal{B}_c(\mathcal{N}_c)\). Let \(b_{\pm}(\phi')\) be the set of the bonds \((i, j) \in \mathcal{B}_c(\mathcal{N}_c)\) satisfying \(\hat{J}_{ij} = \hat{J}_{ij} \sigma_i \sigma_j = -1\). The collection of the \((d - 1)\)-cells which are dual to the bonds in \(b_{\pm}(\phi')\) form the \((d - 1)\)-dimensional hypersurfaces. However, the \((d - 1)\)-cells of these hypersurfaces are not necessarily connected in an unfrustrated pair of frustrated plaquettes. In order to recover the connectivity, we add some \((d - 1)\)-cells to the hypersurfaces as follows. If a frustrated plaquette in \(\mathcal{N}_c\) has an odd number of bonds \((i, j) \in \mathcal{B}_c(\mathcal{N}_c)\) satisfying \(J'_{ij} = -1\), then we take a unique \((d - 1)\)-cell for the corresponding unfrustrated pair so that the cell is dual to the common bond of the two plaquettes. The resulting set of the \((d - 1)\)-dimensional connected hypersurfaces, \(\Sigma_1, \ldots, \Sigma_r\), satisfies \(\partial \Sigma_i \subset \partial \mathcal{N}_c\).
Let us choose a spin configuration $\{\sigma_i\}_{i \in \mathcal{N}}$ to be a ground state of the Hamiltonian on the frustration network $\mathcal{N}_-$. In this case, we cannot expect that, in the ground state, there appears no domain wall (hypersurface) which is homologous to zero. In fact, as we will show below, the existence of the common bonds $e_{ij}$ for the unfrustrated pairs can lower the energy of the bonds which are dual to a domain wall which is homologous to zero.

4.1. Frustration networks in two dimensions

As a concrete example, consider a two-dimensional frustration network $\mathcal{N}_-$ in which the unfrustrated pairs form a subset of the $\mathbb{Z}^2$ lattice as in figure 1. For simplicity, we assume that the network $\mathcal{N}_-$ is simply connected, i.e. any loop in $\mathcal{N}_-$ is homologous to zero, and assume the delta distribution (2) for the random coupling $J_{ij}$, i.e. the bond variables $J_{ij}$ take the value $\pm J_0$. Following the above argument, one can find the spin configuration $\{\sigma_i\}_{i \in \mathcal{N}}$ satisfying $J_{ij}\sigma_i\sigma_j = J_0 > 0$ for all the bonds $\langle i, j \rangle \in \mathcal{B}^+(\mathcal{N}_-)$. This implies that three bonds of each frustrated plaquette have the energy $J_{ij}\sigma_i\sigma_j = J_0 > 0$. Since the plaquette is frustrated, the rest has the energy $-J_0$. For each unfrustrated pair, the common bond for the two plaquettes has the energy $-J_0$. From these observations, we obtain that the spin configuration $\{\sigma_i\}_{i \in \mathcal{N}_-}$ is one of the ground states of the Hamiltonian on $\mathcal{N}_-$. However, the ground states are highly degenerate. In fact, one can obtain another ground state by a local spin-flip. For example, flipping the five spins marked by the close circles in figure 1 does not change the ground state energy. By this spin-flip, there appears the $(d-1)$-dimensional hypersurface which we expected above. In the present two-dimensional case, the hypersurface is the loop encircling the five spins and is homologous to zero. Thus, the hypersurface is not related to any frustrated loop.

4.2. Frustration networks in three dimensions

The situation in three and higher dimensions is slightly different from that in two dimensions. To see this, let us consider the three-dimensional cubic lattice $\mathbb{Z}^3$. We assume that all the plaquettes are frustrated. We want to find a set of unfrustrated pairs of frustrated plaquettes.
so that each unfrustrated pair becomes a 2-cell of the corresponding complex. We denote by $B$ the set of all the bonds of the $\mathbb{Z}^3$ lattice. We define the subset $B_-$ of the bonds as

$$B_- := B_{-,x} \cup B_{-,y} \cup B_{-,z},$$

where the three sets of the bonds on the right-hand side are given by

$$B_{-,x} := \{(i, j) \mid j = i + (1, 0, 0), i \in \mathbb{Z} \times 2\mathbb{Z} \times 2\mathbb{Z}\},$$

$$B_{-,y} := \{(i, j) \mid j = i + (0, 1, 0), i \in 2\mathbb{Z} \times \mathbb{Z} \times 2\mathbb{Z} + (1, 0, 1)\},$$

and

$$B_{-,z} := \{(i, j) \mid j = i + (0, 0, 1), i \in 2\mathbb{Z} \times 2\mathbb{Z} \times \mathbb{Z} + (0, 1, 0)\}.$$  

Then we can take the set of unfrustrated pairs of frustrated plaquettes so that a common bond for frustrated plaquettes which form an unfrustrated pair is always an element of $B_-$. Further, we define a set $B_+$ of bonds as $B_+ := B \setminus B_-$. This set $B_+$ is the collection of the boundary bonds of the unfrustrated pairs. Each unfrustrated pair has six bonds in their boundary.

As shown in figure 2, three bonds (dotted lines) of a cube are an element of $B_-$. Further, we define a set $B_+$ of bonds as $B_+ := B \setminus B_-$. This set $B_+$ is the collection of the boundary bonds of the unfrustrated pairs. Each unfrustrated pair has six bonds in their boundary.

As shown in figure 2, three bonds (dotted lines) of a cube are an element of $B_-$. Further, we define a set $B_+$ of bonds as $B_+ := B \setminus B_-$. This set $B_+$ is the collection of the boundary bonds of the unfrustrated pairs. Each unfrustrated pair has six bonds in their boundary. Figure 3 shows the case of two cubes whose 2-cells are made of five unfrustrated pairs of frustrated plaquettes and one frustrated plaquette. Since the two cubes have 11 frustrated plaquettes, their 2-cells cannot be expressed in terms of unfrustrated pairs only. Figure 4 shows the case of four cubes. The 2-cells are all made of unfrustrated pairs. We stress that the way of choosing unfrustrated pairs as 2-cells is not unique.
Figure 4. Four cubes whose 2-cells are ten unfrustrated pairs of frustrated plaquettes. By reversing the spins marked by the close circles, another ground state can be obtained from the constructed one.

Now consider the $\mathbb{Z}^3$ lattice whose plaquettes are all frustrated. As we constructed above, all of the 2-cells can be taken to be an unfrustrated pair of frustrated plaquettes. In order to avoid the difficulty coming from boundary or finite-size effects, we first consider the infinite-volume lattice although the situation is not realistic. Since the $\mathbb{Z}^3$ lattice with the bonds $B_+$ is simply connected, there exists a spin configuration $\{\sigma_i\}_{i \in \mathbb{Z}^3}$ such that $\hat{J}_{ij}\sigma_i\sigma_j = +1$ for all $(i, j) \in B_+$. When the random bond variables $J_{ij}$ take the values $\pm J_0$, this spin configuration is also a ground state of the Hamiltonian on $\mathbb{Z}^3$ with the bonds $B = B_+ \cup B_-$. This is not a unique ground state because one can get another ground state by reversing the spins at the sites $\{i = (k, 1, 1) \mid k \in \mathbb{Z}\}$ on the line as in figure 4. The sites are marked by the closed circles in figure 4. This implies that the ground state is highly degenerate. But, in contrast to two dimensions, the ground states show stability for local perturbations as

**Proposition 5.** Assume that the random bond variables $J_{ij}$ take the values $\pm J_0$, and assume that all of the plaquettes on $\mathbb{Z}^3$ are frustrated. If a spin configuration satisfies that the sum of the bond energies for each cube is minimized, then the spin configuration is a ground state of the Hamiltonian on $\mathbb{Z}^3$. Further the ground state is stable against any local spin-flip, i.e. any local spin-flip costs a finite energy for the ground state.

**Proof.** Write $\{\sigma_i\}_{i \in \mathbb{Z}^3}$ for the spin configuration. For each cube, three bonds must have the energy $J_{ij}\sigma_i\sigma_j = -J_0$ because all the plaquettes are frustrated. Let $V$ be a finite region in which the spins are flipped. The boundary $\partial V$ of $V$ is a two-dimensional surface which encloses the spins on $V$. One can find a cube $c$ such that only a single site of $c$ is in $V$ and that the remaining seven sites of $c$ are outside $V$. Clearly only three bonds of $c$ intersect the boundary $\partial V$ of $V$. By the spin-flip, the three bonds change the energy. As a result, the cube $c$ must have at least four bonds having the energy $-J_0$. This increases the energy of the ground state. □

For a finite subset of the frustration network in proposition 5, the stability of a ground-state spin configuration does not hold because of the existence of the boundaries. As to thermal fluctuations, we can expect the global spin-flip symmetry to remain unbroken at finite temperatures for the spin system having the frustration network on $\mathbb{Z}^3$ because we can flip spins on a large size cluster with a small energy cost as we have seen in the above argument. Thus, we can expect that finite-size effects and thermal fluctuations make spin configurations on frustration networks disorder.
5. Links of frustrations

In this section, we study the relation between frustration networks themselves and frustrated loops in unfrustration networks.

As we showed in theorem 3, all the domain walls in a ground state of the Hamiltonian on an unfrustration network $N_+^*$ are transverse to a frustrated loop in $N_+^*$. Each domain wall is a connected element of the collection of the $(d-1)$-dimensional hypersurfaces whose boundaries are included in the boundaries $\partial N_+^*$ of the dual lattice $N_+^*$ of $N_+$. Let $\Sigma$ be such a domain wall, and let $\ell$ be the frustrated loop which is transverse to the domain wall $\Sigma$.

Consider first the case of $\partial \Sigma \neq \emptyset$. The boundary $\partial \Sigma$ is the $(d-2)$-dimensional closed complex in $\partial N_+^*$. Since $\partial \Sigma$ is outside $N_+$, each $(d-2)$-cell of $\partial \Sigma$ is dual to a plaquette which is inside a frustrated network $N_-$. Outside the lattice $\Lambda$ on which the Hamiltonian is defined. Namely we have $\partial \Sigma \subset N_- \cup \partial \Lambda^*$. Since the frustrated loop $\ell$ is transverse to the domain wall $\Sigma$, one has the linking number

$$\text{Link}(\ell, \partial \Sigma) = 1 \mod 2,$$

when $\partial \Sigma \neq \emptyset$.

If a two-dimensional surface $s$ satisfies $\partial s = \ell$, then the $(d-2)$-complex $\Gamma = \partial \Sigma$ is transverse to the two-dimensional surface $s$. However, it is not clear whether or not there exists a two-dimensional surface $s$ satisfying $\partial s = \ell$. In fact, one can easily construct a ground-state spin configuration with a single domain wall on the two-dimensional torus whose plaquettes are all unfrustrated, and the boundary $\partial \Sigma$ of the domain wall is vanishing. In this example, there exists no surface whose boundary gives the frustrated loop. The existence of the nontrivial frustrated loop is a consequence of the topology of the torus. Instead of the torus, if we consider a rectangular box with a free boundary, we can expect the existence of a surface whose boundary is a frustrated loop.

Since the frustrated loop $\ell$ is defined on an unfrustration network $N_+$, we use the relative homology theory\(^ {10} \) on the union $N_- \cup N_+$ of the two networks. We denote by $Z_2(N_- \cup N_+; \mathbb{Z})$ the module which is made of all the two-dimensional surfaces $s$ satisfying $\partial s \subset N_+$. We also denote by $B_2(N_- \cup N_+; \mathbb{Z})$ the module which is made of all the two-dimensional surfaces $s$ which is written as $s + s_0 = \partial v$, where the two-dimensional surface $s_0$ is included in $N_+$, and $v$ is a three-dimensional complex. Clearly the module $B_2(N_- \cup N_+; \mathbb{Z})$ is a subset of $Z_2(N_- \cup N_+; \mathbb{Z})$. The two-dimensional homology module $H_2(N_- \cup N_+; \mathbb{Z})$ is defined to be the quotient module of $Z_2(N_- \cup N_+; \mathbb{Z})$ by the submodule $B_2(N_- \cup N_+; \mathbb{Z})$.

As is well known, the following lemma holds.

Lemma 6. The following sequence is exact:

$$H_2(N_- \cup N_+; \mathbb{Z}) \xrightarrow{j_*} H_2(N_- \cup N_+; \mathbb{Z}) \xrightarrow{\partial} H_1(N_+; \mathbb{Z}) \xrightarrow{i_*} H_1(N_- \cup N_+; \mathbb{Z}).$$

Namely the image of the inclusion $j_*$ is equal to the kernel of the boundary map $\partial$, and the image of $\partial$ is equal to the kernel of the inclusion $i_*$.

**Proof.** (i) First let us show $\text{Im} \ j_* = \text{Ker} \ \partial$. Clearly one has $\text{Im} \ j_* \subset \text{Ker} \ \partial$. In order to show $\text{Im} \ j_* \supset \text{Ker} \ \partial$, let $s$ be the two-dimensional surfaces $s$ satisfying $\partial s \sim 0$ in $N_+$. One has $\partial s = \partial s_0$ with a two-dimensional surface $s_0 \subset N_+$. Immediately $\partial(s - s_0) = 0$. This implies

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\(^{10}\) See, for example, section 13 of the book [13].
that the homology class \([s - s_0]\) is an element of \(H_2(\mathcal{N}_- \cup \mathcal{N}_+; \mathbb{Z})\). Besides, \(s - s_0 \sim s\) in \(H_2(\mathcal{N}_- \cup \mathcal{N}_+; \mathbb{Z})\).

(ii) Next we show \(\text{Im} \, \partial = \text{Ker} \, i_*\). One can easily obtain \(\text{Im} \, \partial \subseteq \text{Ker} \, i_*\). Let \(\ell\) be a loop in \(\mathcal{N}_+\) such that \(\ell \sim 0\) in \(\mathcal{N}_- \cup \mathcal{N}_+\). Then, there exists a surface \(s\) in \(\mathcal{N}_- \cup \mathcal{N}_+\) such that \(\ell = \partial s\). Thus, \(\ell\) is within the image of \(\partial\). \(\square\)

From this lemma, when \(\mathcal{N}_- \cup \mathcal{N}_+\) is simply connected, there exists a two-dimensional surface \(s\) in \(\mathcal{N}_- \cup \mathcal{N}_+\) such that \(\partial s = \ell\) for any given frustrated loop \(\ell \subseteq \mathcal{N}_+\). The surface \(s\) must contain an odd number of frustrated plaquettes because the loop \(\ell = \partial s\) is frustrated. Correspondingly, there are the odd number of \((d - 2)\)-complexes which pierce the surface \(s\). One of them, the boundary \(\partial \Sigma\), pierces the two-dimensional surface \(s\), too, because the loop \(\ell\) is transverse to the domain wall \(\Sigma\). Since the frustration of the loop \(\ell\) is not affected by an even number of the frustrated plaquettes, we can take \(\Gamma = \partial \Sigma\) to be the representative of the frustration which affects the unfrustration network \(\mathcal{N}_-\).

These observations suggest the existence of a relation between a frustration function of the plaquettes and the \((d - 2)\)-complex \(\Gamma = \partial \Sigma\) for a domain wall \(\Sigma\) on the network \(\mathcal{N}_- \cup \mathcal{N}_+\). In order to explore the relation, we rely on the relative cohomology theory\(^\text{11}\). Since the frustrated loop \(\ell\) is defined on an unfrustration network, we consider the union \(\mathcal{N}_- \cup \mathcal{N}_+\) of the two networks, \(\mathcal{N}_-\) and \(\mathcal{N}_+\), where \(\mathcal{N}_-\) is the collection of all the unfrustration networks which touch \(\mathcal{N}_-\). We do not require the connectivity of \(\mathcal{N}_-\) and of \(\mathcal{N}_+\). Let us introduce plaquette variables \(\eta = \{\eta_p\}_p\) for all the plaquettes, where \(\eta_p\) takes the values \(\pm 1\). Then, the 2-cochain \(\eta\) is defined by

\[
\eta(s) = \prod_{p \subseteq s} \eta_p
\]

for a two-complex \(s\). Since the bond variables \(\hat{J}_{ij}\) defines the plaquette variables \(\eta_p\) by

\[
\eta_p = \prod_{\langle i, j \rangle \subseteq p} \hat{J}_{ij},
\]

one can define the 2-cochain \(\Phi\) for the bond variables \(\hat{J}_{ij}\) by

\[
\Phi(s) = \prod_{p \subseteq s} \prod_{\langle i, j \rangle \subseteq p} \hat{J}_{ij} = \prod_{\langle i, j \rangle \subseteq \partial s} \hat{J}_{ij}
\]

for a two-complex \(s\). Since \(\partial s\) is a collection of loops, we have \(\Phi(s) = \phi(\partial s)\) with the frustration \(\phi\). This yields the homomorphism \(\hat{\alpha} : \phi \longmapsto \Phi\). (See lemma 7 below.) Clearly the 2-cochain \(\Phi\) satisfies the condition \(\Phi(s) = 1\) for \(s \subseteq \mathcal{N}_+\).

Let us go back to the general setting for 2-cochains. We denote by \(Z^2(\mathcal{N}_- \cup \mathcal{N}_+; \mathbb{Z}_2)\) the two-dimensional cohomology module which is made of all the 2-cochains \(\eta\) which satisfy the cocycle condition \(\eta(\partial c) = 1\) for any cube \(c\). We also denote by \(B^2(\mathcal{N}_- \cup \mathcal{N}_+; \mathbb{Z}_2)\) the module which is made of all the 2-cochains \(\eta\) which are given by

\[
\eta_p = \prod_{\langle i, j \rangle \subseteq p} \epsilon_{ij}
\]

with bond variables \(\epsilon_{ij}\) which take the values \(\pm 1\). The two-dimensional cohomology module \(H^2(\mathcal{N}_- \cup \mathcal{N}_+; \mathbb{Z}_2)\) is defined to be the quotient module \(Z^2(\mathcal{N}_- \cup \mathcal{N}_+; \mathbb{Z}_2) / B^2(\mathcal{N}_- \cup \mathcal{N}_+; \mathbb{Z}_2)\). Further, we denote by \(Z^2(\mathcal{N}_- \cup \mathcal{N}_+; \mathbb{Z}_2)\) the module which is made of all the 2-cocycles \(\eta \in Z^2(\mathcal{N}_- \cup \mathcal{N}_+; \mathbb{Z}_2)\) satisfying the condition \(\eta(p) = 1\) for \(p \subseteq \mathcal{N}_+\), and \(B^2(\mathcal{N}_- \cup \mathcal{N}_+; \mathbb{Z}_2)\) the module which is made of all the 2-cocycles \(\eta \in B^2(\mathcal{N}_- \cup \mathcal{N}_+; \mathbb{Z}_2)\).

\(^\text{11}\) See, for example, the book [13].
with the bond variables $\{\epsilon_{ij}\}$ of (6) satisfying the condition $\epsilon_{ij} = 1$ for $(i, j) \subset \mathcal{N}_+$. From the definitions, one has

$$B^2(\mathcal{N}_- \cup \mathcal{N}_+, \mathcal{N}_+; \mathbb{Z}_2) \subset Z^2(\mathcal{N}_- \cup \mathcal{N}_+, \mathcal{N}_+; \mathbb{Z}_2).$$

Similarly, the two-dimensional cohomology module $H^2(\mathcal{N}_- \cup \mathcal{N}_+, \mathcal{N}_+; \mathbb{Z}_2)$ is defined to be the quotient module as

$$H^2(\mathcal{N}_- \cup \mathcal{N}_+, \mathcal{N}_+; \mathbb{Z}_2) := Z^2(\mathcal{N}_- \cup \mathcal{N}_+, \mathcal{N}_+; \mathbb{Z}_2)/B^2(\mathcal{N}_- \cup \mathcal{N}_+, \mathcal{N}_+; \mathbb{Z}_2).$$

If $\eta, \eta' \in Z^2(\mathcal{N}_- \cup \mathcal{N}_+, \mathcal{N}_+; \mathbb{Z}_2)$ satisfy the relation $\eta'_p = \eta_p \epsilon_p$ with $\epsilon = \{\epsilon_p\}_p \in B^2(\mathcal{N}_- \cup \mathcal{N}_+, \mathcal{N}_+; \mathbb{Z}_2)$ for any plaquette $p$, then one has $\eta'(s) = \eta(s)$ for any two-dimensional surface $s$ satisfying $\partial s \subset \mathcal{N}_+$. Let $s, s'$ be two-dimensional surfaces which satisfy $s - s' \in B_2(\mathcal{N}_- \cup \mathcal{N}_+, \mathcal{N}_+; \mathbb{Z}_0)$. Namely, there exists a two-dimensional surface $s_0 \subset \mathcal{N}_+$ and a three-dimensional complex $v$ such that $s - s' + s_0 = \partial v$. Then, one has $\eta(s) = \eta(s')$ for any $\eta \in Z^2(\mathcal{N}_- \cup \mathcal{N}_+, \mathcal{N}_+; \mathbb{Z}_2)$. Thus, a cohomology class $[\eta] \in H^2(\mathcal{N}_- \cup \mathcal{N}_+, \mathcal{N}_+; \mathbb{Z}_2)$ defines the homomorphism:

$$[\eta] : H_2(\mathcal{N}_- \cup \mathcal{N}_+, \mathcal{N}_+; \mathbb{Z}) \rightarrow \mathbb{Z}_2.$$

The cohomology version of lemma 6 is

**Lemma 7.** The following sequence is exact:

$$H^1(\mathcal{N}_- \cup \mathcal{N}_+, \mathcal{N}_+; \mathbb{Z}_2) \xrightarrow{i^*} H^1(\mathcal{N}_+; \mathbb{Z}_2) \xrightarrow{\partial^*} H^2(\mathcal{N}_- \cup \mathcal{N}_+, \mathcal{N}_+; \mathbb{Z}_2) \xrightarrow{j^*} H^2(\mathcal{N}_- \cup \mathcal{N}_+, \mathcal{N}_+; \mathbb{Z}_2).$$

Here, $i^*$ and $j^*$ are the inclusion maps.

**Proof.** (i) From the definitions, one has $\text{Im } i^* \subset \text{Ker } \partial^*$. In order to show $\text{Im } i^* \supset \text{Ker } \partial^*$, let $\tau \in Z^1(\mathcal{N}_+; \mathbb{Z}_2)$ satisfy $\partial^* \tau \sim 1$. Then, one has

$$(\partial^* \tau)(p) = \prod_{(i, j) \subset p} \tau_{ij} = \prod_{(i, j) \subset p} \epsilon_{ij}$$

for any plaquette $p$, where the bond variables $\epsilon_{ij}$ satisfy the condition $\epsilon_{ij} = 1$ for $(i, j) \subset \mathcal{N}_+$. Set $\tau'_{ij} = \tau_{ij} \epsilon_{ij}$. Then, the corresponding cocycle $\tau'$ is an element of $Z^1(\mathcal{N}_- \cup \mathcal{N}_+, \mathbb{Z}_2)$, and $\tau'_{ij} = \tau_{ij}$ for $(i, j) \subset \mathcal{N}_+$. Further, one has $(\partial^* \tau')(p) = 1$ for any plaquette $p$.

(ii) Clearly one has $\text{Im } \partial^* \subset \text{Ker } j^*$. Let us show $\text{Im } \partial^* \supset \text{Ker } j^*$. Let $\eta \in Z^2(\mathcal{N}_- \cup \mathcal{N}_+, \mathcal{N}_+; \mathbb{Z}_2)$ satisfy $(j^* \eta) \sim 1$. Then there exist bond variables $\epsilon_{ij}$ which satisfy

$$(j^* \eta)_p = \prod_{(i, j) \subset p} \epsilon_{ij}.$$

From $\eta \in Z^2(\mathcal{N}_- \cup \mathcal{N}_+, \mathcal{N}_+; \mathbb{Z}_2)$, one has $(j^* \eta)_p = \eta_p = 1$ for $p \subset \mathcal{N}_+$. Combining these, one obtains

$$\prod_{(i, j) \subset p} \epsilon_{ij} = 1 \quad \text{for } p \subset \mathcal{N}_+.$$

This implies that $\epsilon = \{\epsilon_{ij}\} \in Z^1(\mathcal{N}_+; \mathbb{Z}_2)$ and that $\eta = \partial^* \epsilon$. \qed

The following proposition is a special case of Poincaré–Lefschetz duality, too. This is nothing but the relation which we have explored, i.e. the relation between the frustration function of plaquettes and $(d - 2)$-complexes.
**Proposition 8.** The following isomorphism is valid:

$$H^2(N_+ \cup N_-, N_*; \mathbb{Z}_2) \cong H_{d-2}(N_+^*, \partial N_* \cap \partial \Lambda^*; \mathbb{Z}_2),$$

where the right-hand side is the set of all the homology classes of the $(d - 2)$-complexes $\Gamma$ such that $\Gamma \cap N_* = \emptyset$ and that $\partial \Gamma \subset \partial N_* \cap \partial \Lambda^*$.

**Proof.** Let $\eta$ be a $2$-cochain. Then, we can define the $(d - 2)$-dimensional complex $\Gamma$ which consists of the $(d - 2)$-cells which is dual to the plaquettes $p$ having the plaquette variable $\eta_p = -1$. This is an extension of the dual map from frustration networks to $(d - 2)$-dimensional complexes. We write $\Gamma = \xi(\eta)$ for this map $\eta \to \Gamma$. This map $\xi$ gives the desired isomorphism of the proposition. Let $\eta, \eta' \in Z^2(N_+ \cup N_-, N_*; \mathbb{Z}_2)$ satisfy $\eta' \sim \eta$. First, we want to show $\xi(\eta') \sim \xi(\eta)$. Namely the map of the proposition is well defined. From the definitions of the $2$-cocycles, there exist bond variables $\epsilon = \{\epsilon_{ij}\}$ satisfying

$$\eta_p' = \eta_p \prod_{(i,j) \in P} \epsilon_{ij}$$

with the condition $\epsilon_{ij} = 1$ for $(i,j) \subset N_+$. Therefore, one has $\xi(\eta') = \xi(\eta) + \xi(\epsilon)$. The collection of the $(d - 1)$-cells which are dual to the bonds $(i,j)$ with $\epsilon_{ij} = -1$ forms $(d - 1)$-dimensional hypersurfaces $\Sigma$. Since $\Sigma$ ends at a plaquette $p$ satisfying $\epsilon(\partial p) = -1$ or at $\partial N_* \cap \partial \Lambda^*$, the boundary $\partial \Sigma$ is made of the collection of the $(d - 2)$-cells which are dual to the plaquettes $p$ satisfying $\epsilon(\partial p) = -1$ or end at $\partial N_* \cap \partial \Lambda^*$. Namely, $\partial \Sigma = \xi(\epsilon) + \Gamma_0$ with $\Gamma_0 \subset \partial N_* \cap \partial \Lambda^*$. This implies $\xi(\epsilon) \sim 0$. Immediately, $\xi(\eta') \sim \xi(\eta)$.

Let $\Gamma \subset N_*^*$ be a $(d - 2)$-dimensional complex satisfying $\partial \Gamma \subset \partial N_*^* \cap \partial \Lambda^*$. If a cube $c \subset N_+ \cup N_-$ intersects $\Gamma$, an even number of the plaquettes of $c$ must intersect $\Gamma$. For such a plaquette $p$, we choose the plaquette variables $\eta_p = 1$. Then, the corresponding $2$-cochain $\eta$ satisfies the cocycle condition $\eta(\partial c) = 1$ for any cube $c$, and $\xi(\eta) = \Gamma$. Thus, the map of the proposition is surjective.

Finally we show that the map is also injective. Let $\eta \in Z^2(N_+ \cup N_-, N_*; \mathbb{Z}_2)$ satisfy $\xi(\eta) \sim 0$. Then there exists a $(d - 1)$-dimensional hypersurface $\Sigma \subset N_*^*$ such that $\partial \Sigma = \xi(\eta) + \Gamma_0$ with $\Gamma_0 \subset \partial N_*^* \cap \partial \Lambda^*$. We take bond variables $\epsilon = \{\epsilon_{ij}\}$ so that $\epsilon_{ij} = -1$ if the bond $(i,j)$ pierces $\Sigma$, and $\epsilon_{ij} = +1$ otherwise. It is sufficient to show that

$$\eta_p \prod_{(i,j) \in P} \epsilon_{ij} = 1.$$ 

When $\eta_p = -1$, an odd number of the bond of $p$ pierces $\Sigma$ because $\partial \Sigma$ ends at $p$. When $\eta_p = +1$, an even number of the bond of $p$ pierces $\Sigma$. Thus, in both of the two cases, the above relation holds.

The main result of this section is summarized as follows.

**Theorem 9.** We have the commutative diagram

$$
\begin{array}{ccc}
H^1(N_*; \mathbb{Z}_2) & \xrightarrow{\delta^*} & H^2(N_+ \cup N_-, N_*; \mathbb{Z}_2) \\
\downarrow \cong & & \downarrow \cong \\
H_{d-1}(N_*^* \cup N_*; \mathbb{Z}_2) & \xrightarrow{\delta} & H_{d-2}(N_*^* \cap \partial \Lambda^*; \mathbb{Z}_2).
\end{array}
$$

**Proof.** Let $\phi \in Z^1(N_*; \mathbb{Z}_2)$ with the bond variables $\hat{J}_{ij}$, and let $\{\sigma_i\}_{i \in N_*}$ be a spin configuration of the ground state of the Hamiltonian on $N_+ \cup N_-$. Then the cocycle $\phi' = \{\hat{J}_{ij}' = \hat{J}_{ij} \sigma_i \sigma_j\}$
determines the domain walls as we showed in proposition 2, and the boundaries of the domain walls are the \( (d - 2) \)-dimensional complexes.

On the other hand, since \( \partial \sigma' \) is written in terms of only the bond variables \( J'_{ij} \), we can find a 2-cocycle \( \eta \) which is cohomologous to \( \partial \sigma' \) and written in terms of only the bond variables \( \tau_{ij} \) which satisfy \( \tau_{ij} = J'_{ij} \) for \( \langle i, j \rangle \in \mathcal{N}_+ \), and \( \tau_{ij} = 1 \) otherwise. Therefore, the bond variables \( \tau_{ij} = J'_{ij} = -1 \) on \( \partial \mathcal{N}_+ \) yield the frustrated plaquettes \( p \) with \( \eta_p = -1 \) in \( \mathcal{N}^- \). These frustrated plaquettes determine the same \( (d - 2) \)-dimensional complexes as the above complexes by proposition 8.

\[ \blacksquare \]

6. Homology of domain walls

Relying on theorem 9, we discuss the topology of the domain walls for a ground state.

Consider a generic spin configuration \( \{ \sigma_i \} \) on \( \Lambda = \mathcal{N}_+ \cup \mathcal{N}_- \). We write \( \tilde{E}_{ij} = J_{ij} \sigma_i \sigma_j \). Then, the sign of the bond energy is given by \( -\tilde{E}_{ij} \). Consider the collection of all the \( (d - 1) \)-cells which are dual to the bonds \( \langle i, j \rangle \) having \( \tilde{E}_{ij} = -1 \). Then, the domain walls for the spin configuration are given by the connected elements of the collection. If a plaquette \( p \) is frustrated, then the number of the bonds having \( \tilde{E}_{ij} = -1 \) in \( p \) is odd. Therefore, the corresponding domain wall ends at the plaquette \( p \). The \( (d - 2) \)-cell which is dual to the plaquette \( p \) becomes the boundary of the domain wall. If \( p \) is unfrustrated, the number of the bonds having \( \tilde{E}_{ij} = -1 \) in \( p \) becomes even. In this case, the boundary of the domain wall does not appear at the plaquette \( p \). Thus, a domain wall is a connected, \( (d - 1) \)-dimensional hypersurface whose boundary is a collection of \( (d - 2) \)-dimensional complexes dual to the frustrated plaquettes or ends at the boundary of the lattice \( \Lambda^\times \).

From the proof of theorem 9, one notes the following fact. All of the domain walls are outside the unfrustration network except for the domain walls which are transverse to a frustrated loop in the unfrustration network. In order to show this, we introduce the bond variables \( \tau'_{ij} \) which satisfy \( \tau'_{ij} = 1 \) for \( \langle i, j \rangle \in \mathcal{N}_+ \), and \( \tau'_{ij} = J'_{ij} \) otherwise. Clearly one has the decomposition, \( J'_{ij} = J_{ij} \sigma_i \sigma_j = \tau_{ij} \tau'_{ij} \), where \( \tau_{ij} \) is the cochain in the proof of theorem 9. This relation implies that \( \tau_{ij} \) yields the set of the domain walls \( \theta \circ \kappa(\tau) \) which are transverse to a frustrated loop in the unfrustration network \( \mathcal{N}_+ \) as we have seen in the proof of theorem 9, and that \( \tau'_{ij} \) yields the rest of the domain walls \( \theta \circ \kappa(\tau') \) which are outside the unfrustration network \( \mathcal{N}_+ \) from the condition \( \tau'_{ij} = 1 \) for \( \langle i, j \rangle \in \mathcal{N}_- \).

To summarize, we obtain the following description of the domain walls for the ground state. The homology class of the hypersurfaces \( \theta \circ \kappa(\tau) \) is determined by the frustrated loops on \( \mathcal{N}_+ \) as in relation (5). Besides, the size of the hypersurfaces \( \theta \circ \kappa(\tau) \) cannot become large for the ground state because the position and profile of the domain walls are determined to minimize the total domain wall energy. On the other hand, the rest of the domain walls \( \theta \circ \kappa(\tau') \) are not expected to show such a similar, simple structure as we showed in section 4. But the outstanding feature is that they are all outside the unfrustration network.

The spins on the unfrustration network prefer to maintain their relative orientation although the neighboring pairs of the spins for the bonds on the domain walls cannot take their favorable orientation. Therefore, we can expect that the ground state exhibits order of the frozen spins on the unfrustration network even if the size of the frustration network is large. On the frustration network, there appear many domain walls whose boundaries end at a frustrated plaquette. If the ground state on the frustration network is highly degenerate, then the spin configurations yield many patterns of the domain walls on the frustration network. In such a situation, the order of the frozen spins on the frustration network cannot be expected.

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7. Topological effect and the spin glass phase

As summary and concluding remarks, we discuss the role of topologically nontrivial domain walls which are transverse to a frustrated loop in the unfrustration network in the context of the appearance of the spin glass phase at finite temperatures.

7.1. Absence of the spin glass phase in two dimensions

Let us consider the system on the square lattice $\mathbb{Z}^2$. A naive application of a percolation argument as in [5] to the unfrustration network yields the existence of the long-range order of the frozen spins on the unfrustration network when we ignored the effect of the domain walls which are transverse to a frustrated loop on the unfrustration network. However, it is widely believed that there is no spin glass phase in two dimensions at finite temperatures\(^{13}\). Thus, the percolation argument alone cannot explain the absence of the spin glass phase in two dimensions. As a result, we can expect that the thermal fluctuations of the domain walls which are transverse to a frustrated loop on the unfrustration network play an essential role in the growth process of the long-range order. Actually, since a large domain wall is known to be unstable in the Ising ferromagnet on the $\mathbb{Z}^2$ lattice at finite temperatures [12], the thermal fluctuations of the topologically nontrivial domain walls can be expected to destroy the long-range order of the frozen spins on the unfrustration network in two dimensions. This is essentially the global, topological effect for the unfrustration network because the sizes of the domain walls are expected to be ignorably small in the unfrustration network.

7.2. The spin glass phase in three and higher dimensions

We specialize to the simple cubic lattice $\mathbb{Z}^3$. Our argument in higher dimensions or on other lattices is the same. In the same way as in Bovier and Fröhlich [5], one can show that the unfrustration network percolates for $x$ near 1/2. When we restrict the present system to the unfrustration network with infinite volume, the situation is very similar to that of the Ising ferromagnet except that there appear many topologically nontrivial domain walls on the unfrustration network. But, unlike the square lattice $\mathbb{Z}^2$, the domain walls in the Ising ferromagnet is known to be stable against the thermal fluctuation on $\mathbb{Z}^3$ at low temperatures [6]. Thus, one can expect that such topologically nontrivial domain walls are stable against the thermal fluctuation on the unfrustration network at low temperatures too. These observations suggest that the thermal fluctuation of the topologically nontrivial domain walls cannot destroy the long-range order of the frozen spins on the unfrustration network. Namely, we can expect the existence of the spin glass phase on the $\mathbb{Z}^3$ lattice for the density $x$ near 1/2.

Appendix A. Proof of relation (3)

In order to make the paper self-contained, we give a proof of (3).

For a map $f$, we denote by $\text{Im } f$ the image of the map $f$, and denote by $\text{Ker } f$ the kernel of $f$. Let $\Lambda \subset \mathbb{Z}^d$ be a collection of plaquettes. Consider the exact sequence on $\Lambda$:

$$0 \longrightarrow B_1(\Lambda; \mathbb{Z}) \overset{i}{\longrightarrow} Z_1(\Lambda; \mathbb{Z}) \overset{p}{\longrightarrow} H_1(\Lambda; \mathbb{Z}) \longrightarrow 0. \quad (A.1)$$

\(^{12}\) As is well known, dilute Ising ferromagnets are proved to show ferromagnetic long-range order above the percolation threshold [1, 2].

\(^{13}\) See [3, 14, 15] for a mathematical analysis of the ground states in two dimensions.
Namely, Ker $i = 0$, Im $i = \text{Ker } p$ and the map $p$ is surjective. Here the map $i$ is the inclusion, $B_1(\Lambda; \mathbb{Z}) \subset Z_1(\Lambda; \mathbb{Z})$, and $p$ is the projection. We write
$$H^a_i = \text{Hom}(H_1(\Lambda; \mathbb{Z}), \mathbb{Z}_2),$$
$$Z^a_i = \text{Hom}(Z_1(\Lambda; \mathbb{Z}), \mathbb{Z}_2),$$
and
$$B^a_i = \text{Hom}(B_1(\Lambda; \mathbb{Z}), \mathbb{Z}_2).$$
We choose the additive group $\mathbb{Z}_2 = \{0, 1\}$ for the representation of $\mathbb{Z}_2$. We define the map $p^a : H^a_i \rightarrow Z^a_i$ by the adjoint $p^a$ of the projection $p$ as
$$(p^a \alpha)(\ell) = \alpha(p(\ell))$$
for $\alpha \in H^a_i$ and $\ell \in Z_1(\Lambda; \mathbb{Z})$. Similarly we define $i^a : Z^a_i \rightarrow B^a_i$ by
$$(i^a z)(\partial s) = z(i(\partial s))$$
for $z \in Z^a_i$ and $\partial s \in B_1(\Lambda; \mathbb{Z})$.

**Lemma 10.** The following sequence is exact:
$$0 \rightarrow H^a_i \xrightarrow{p^a} Z^a_i \xrightarrow{i^a} B^a_i.$$  

**Proof.** First we show Ker $p^a = 0$. Assume $p^a \alpha = 0$ for $\alpha \in H^a_i$. From the exact sequence (A.1), we have that for any $a \in H_1(\Lambda; \mathbb{Z})$, there exists $\ell \in Z_1(\Lambda; \mathbb{Z})$ such that $a = p(\ell)$. Then, one obtains
$$0 = (p^a \alpha)(\ell) = \alpha(p(\ell)) = \alpha(a)$$
for any $a \in H_1(\Lambda; \mathbb{Z})$. This implies $\alpha = 0$.

Next we show Im $p^a \subset$ Ker $i^a$. Let $\alpha \in H^a_i$ and $\partial s \in B_1(\Lambda; \mathbb{Z})$. Then, one has
$$(i^a \circ p^a \alpha)(\partial s) = (p^a \alpha)(i(\partial s)) = \alpha(p \circ i(\partial s)) = 0$$
from the exact sequence (A.1).

Finally we show Ker $i^a \subset$ Im $p^a$. Assume $i^a z = 0$ for $z \in Z^a_i$. Let $\ell, \ell' \in Z_1(\Lambda; \mathbb{Z})$ such that $p(\ell) = p(\ell')$. Then $\ell - \ell' \in \text{Ker } p = \text{Im } i$ from (A.1). Combining this with the assumption $i^a z = 0$, one has $z(\ell - \ell') = 0$. Thus, if $p(\ell) = p(\ell')$, then $z(\ell) = z(\ell')$. This defines $\alpha \in H^a_i$ by
$$\alpha(p(\ell)) = z(\ell).$$
This left-hand side is equal to $(p^a \alpha)(\ell)$. \hfill $\Box$

Clearly this lemma yields the following isomorphism:
$$H^a_i = \text{Hom}(H_1(\Lambda; \mathbb{Z}), \mathbb{Z}_2) \cong \text{Ker } i^a.$$  

We denote by $B_0(\Lambda; \mathbb{Z})$ the module made of the boundaries $\partial \gamma$ for all the oriented path $\gamma$ in $\Lambda$ with the coefficients $\mathbb{Z}$. Consider the exact sequence
$$0 \rightarrow Z_1(\Lambda; \mathbb{Z}) \xrightarrow{j} C_1(\Lambda; \mathbb{Z}) \xrightarrow{\partial} B_0(\Lambda; \mathbb{Z}) \rightarrow 0$$  

(3.3) 

where the map $j$ is the inclusion $Z_1(\Lambda; \mathbb{Z}) \subset C_1(\Lambda; \mathbb{Z})$. We write
$$B^a_0 = \text{Hom}(B_0(\Lambda; \mathbb{Z}), \mathbb{Z}_2)$$
and
$$C^a_1 = \text{Hom}(C_1(\Lambda; \mathbb{Z}), \mathbb{Z}_2).$$
We define the map \( \partial^\#: B_0^\# \longrightarrow C_1^\# \) by the adjoint \( \partial^\# \) of the boundary operator \( \partial \) as
\[
(\partial^\# \beta)(c) = \beta(\partial c)
\]
for \( \beta \in B_0^\# \) and \( c \in C_1(\Lambda; \mathbb{Z}) \). Further, we define \( j^\#: C_1^\# \longrightarrow Z_1^\# \) by
\[
(j^\# f)(\ell) = f(j(\ell))
\]
for \( f \in C_1^\# \) and \( \ell \in Z_1(\Lambda; \mathbb{Z}) \).

**Lemma 11.** The following exact sequence is valid:
\[
0 \longrightarrow B_0^\# \xrightarrow{\partial^\#} C_1^\# \xrightarrow{j^\#} Z_1^\# \longrightarrow 0.
\]

**Proof.** First we show \( \text{Ker} \partial^\# = 0 \). Assume \( \partial^\# \beta = 0 \) for \( \beta \in B_0^\# \). From the exact sequence (A.3), one has that for any \( b \in B_0(\Lambda; \mathbb{Z}) \), there exists \( c \in C_1(\Lambda; \mathbb{Z}) \) such that \( b = \partial c \). Therefore, one obtains
\[
0 = (\partial^\# \beta)(c) = \beta(\partial c) = \beta(b)
\]
for any \( b \in B_0(\Lambda; \mathbb{Z}) \). This implies \( \beta = 0 \).

Next we show \( \text{Im} \partial^\# = \text{Ker} j^\# \). Let \( \beta \in B_0^\# \) and \( \ell \in Z_1(\Lambda; \mathbb{Z}) \). Then,
\[
(j^\# \circ \partial^\# \beta)(\ell) = \beta(\partial \circ j(\ell)) = 0.
\]
This implies \( \text{Im} \partial^\# \subset \text{Ker} j^\# \).

In order to prove \( \text{Ker} j^\# \subset \text{Im} \partial^\# \), we assume \( j^\# f = 0 \) for \( f \in C_1^\# \). Let \( c, c' \in C_1(\Lambda; \mathbb{Z}) \) such that \( \partial c = \partial c' \). Then one has \( c - c' \in \text{Ker} \partial = \text{Im} j \) from the exact sequence (A.3). Combining this with the assumption \( j^\# f = 0 \), one has \( f(c - c') = 0 \). Thus, if \( \partial c = \partial c' \), then \( f(c) = f(c') \). This defines \( \beta \in B_0^\# \) by
\[
\beta(\partial c) = f(c).
\]
This left-hand side is equal to \( (\partial^\# \beta)(c) \). Therefore, \( \text{Ker} j^\# \subset \text{Im} \partial^\# \).

Combining this with the above \( \text{Im} \partial^\# \subset \text{Ker} j^\# \), the desired result \( \text{Im} \partial^\# = \text{Ker} j^\# \) is obtained.

Finally, we show that the map \( j^\# \) is surjective. Let \( \{b_\lambda\}_\lambda \) be a basis of the module \( B_0(\Lambda; \mathbb{Z}) \). Then, for each \( b_\lambda \), there exists a chain \( c_\lambda \in C_1(\Lambda; \mathbb{Z}) \) such that \( b_\lambda = \partial c_\lambda \). This defines the map
\[
\overline{\partial} : B_0(\Lambda; \mathbb{Z}) \longrightarrow C_1(\Lambda; \mathbb{Z}).
\]
One can note that any chain \( c \in C_1(\Lambda; \mathbb{Z}) \) can be decomposed into two parts as \( c = \overline{\partial}(b) + j(\ell) \) with \( b \in B_0(\Lambda; \mathbb{Z}) \) and \( \ell \in Z_1(\Lambda; \mathbb{Z}) \). Using this decomposition, we define the map \( \overline{j} : C_1(\Lambda; \mathbb{Z}) \longrightarrow Z_1(\Lambda; \mathbb{Z}) \) by \( \overline{j}(c) = \ell \). Clearly one has \( \overline{j} \circ j = 1 \). Further, we can define the map \( \overline{j}^\# : Z_1^\# \longrightarrow C_1^\# \) by
\[
(j^\# z)(\ell) = z(\overline{j}(c))
\]
for \( z \in Z_1^\# \) and \( c \in C_1(\Lambda; \mathbb{Z}) \). Then, one has \( j^\# \circ \overline{j}^\# = 1 \). Actually one can easily show
\[
(j^\# \circ \overline{j}^\#)(\ell) = (j^\# z)(j(\ell)) = z(\overline{j} \circ j(\ell)) = z(\ell)
\]
for any \( z \in Z_1^\# \) and for any loop \( \ell \in Z_1(\Lambda; \mathbb{Z}) \). The result \( j^\# \circ \overline{j}^\# = 1 \) implies that the map \( j^\# \) is surjective.

**Lemma 12.** The following sequence is exact:
\[
0 \longrightarrow H^1(\Lambda; \mathbb{Z}_2) \xrightarrow{j^\#} Z_1^\# \xrightarrow{j^\#} B_1^\#.
\]
Proof. We choose the multiplicative group $\mathbb{Z}_2 = \{1, -1\}$ for the representation of $\mathbb{Z}_2$. Let $\alpha \in B^1(\Lambda; \mathbb{Z}_2)$. Then, $(j^*\alpha)(\ell) = \alpha(j(\ell)) = 1$ for any $\ell \in Z_1(\Lambda; \mathbb{Z})$. Thus, the adjoint $j^*$ of the inclusion $j : Z_1(\Lambda; \mathbb{Z}) \subset C_1(\Lambda; \mathbb{Z})$ is well defined.

Let us show $\text{Ker } j^* = 0$. Assume $j^*\alpha = 1$ for $\alpha \in H^1(\Lambda; \mathbb{Z}_2)$. Then, one has

$$1 = (j^*\alpha)(\ell) = \alpha(j(\ell)) = \alpha(\ell)$$

for any loop $\ell \in Z_1(\Lambda; \mathbb{Z})$. Therefore, we have $\alpha \in B^1(\Lambda; \mathbb{Z}_2)$ from the proof of lemma 1.

Next we show $\text{Im } j^* \subset \text{Ker } i^*$. Let $\alpha \in H^1(\Lambda; \mathbb{Z}_2)$ and $\delta s \in B_1(\Lambda; \mathbb{Z})$. Then

$$(i^* \circ j^*\alpha)(\delta s) = (j^*\alpha)(i(\delta s)) = \alpha(j \circ i(\delta s)) = \alpha(\delta s) = (\delta^*\alpha)(s) = 1.$$

Finally we show $\text{Ker } i^* \subset \text{Im } j^*$. Assume $i^*\alpha = 1$ for $\alpha \in Z_1^*$. Then, one has

$$1 = (i^*\alpha)(\delta s) = z(i(\delta s)) = z(\delta s).$$

From lemma 11, there exists $f \in C_1^i$ such that $z = j^* f$. Substituting this into the above equality, one obtains

$$1 = (j^* f)(\delta s) = f(j(\delta s)) = f(\delta s) = (\delta^* f)(s)$$

for any $s$. This implies that $f$ can be identified with $\alpha \in H^1(\Lambda; \mathbb{Z}_2)$. Thus, $z = j^*\alpha$. □

Clearly this lemma yields the isomorphism

$$H^1(\Lambda; \mathbb{Z}_2) \cong \text{Im } j^* = \text{Ker } i^*.$$

Combining this with the isomorphism (A.2), one obtains the desired isomorphism

$$H^1(\Lambda; \mathbb{Z}_2) \cong \text{Hom}(H_1(\Lambda; \mathbb{Z}); \mathbb{Z}_2).$$

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