On Complex Analytic tools, and the Holomorphic Rotation methods

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1 introduction

This paper in honor of Guido Weiss was written posthumously, jointly with him, as we had, all of his initial notes and ideas related to the program described below.

Our task, here, is to recount ideas, explorations, and visions that Guido his collaborators and students, developed over the last 60 years. To point out the connection of ideas between the original views of the interplay between complex and real analysis as envisioned by Zygmund and his students Calderón, Guido Weiss, Eli Stein, and many others, 70 years ago, and the current approaches introducing nonlinear multi layered analysis for the organization and processing of complicated oscillatory functions.

It was Zygmund’s view that harmonic analysis provides the infrastructure linking most areas of analysis, from complex analysis to partial differential equations, to probability, number theory, and geometry.

In particular he pushed forward the idea that the remarkable tools of complex analysis, which include; contour integration, conformal mappings, factorization, tools which were used to provide miraculous proofs in real analysis, should be deciphered and converted to real variable tools. Together with Calderón, they bucked the trend for abstraction, prevalent at the time, and formed a school pushing forward this interplay between real and complex analysis. A principal bridge was provided by real variable methods, multiscale analysis, Littlewood Paley theory, and related Calderon representation...
formulas. Our aim, here, is to elaborate on the "magic" of complex analysis and indicate potential applications in Higher dimensions. An old idea of Calderón and Zygmund, the so called "rotation method", enabled the reductions of the study of \( L^p \) estimates for multi dimensional singular integrals to a superposition, over all directions, of Hilbert transforms. Thereby allowing the use of one complex variable methods. A related idea was the invention of systems of Harmonic functions satisfying generalised Cauchy Riemann equations, such as the Riesz systems, exploiting their special properties. \[6\]

Our goal is to extend these ideas to enable remarkable nonlinear complex analytic tools for the adapted analysis of functions in one variable, to apply in higher dimensions.

Guido has been pushing the idea that factorization theorems like Blaschke products are a key to a variety of nonlinear analytic methods \[3\]. Our goal here is to demonstrate this point, deriving amazing approximation theorems, in one variable, and opening doors to higher dimensional applications. Application in which each harmonic function is the average of special holomorphic functions in planes and constant in orthogonal directions.

We start by describing recent developments in nonlinear complex analysis, exploiting the tools of factorization and composition. In particular we will sketch methods extending conventional Fourier analysis, exploiting both phase and amplitudes of holomorphic functions. The "miracles of nonlinear complex analysis", such as factorization and composition of functions lead to new versions of holomorphic wavelets, and relate them to multiscale dynamical systems.

Our story interlaces the role of the phase of signals with their analytic/geometric properties. The Blaschke factors are a key ingredient, in building analytic tools, starting with the Malmquist-Takenaka orthonormal bases of the Hardy space \( H^2(T) \), continuing with "best" adapted bases obtained through phase unwinding, and describing relations to composition of Blaschke products and their dynamics (on the disc and upper half plane). Specifically we construct multiscale orthonormal holomorphic wavelet bases, generalized scaled holomorphic orthogonal bases, to dynamical systems, obtained by composing Blaschke products.

We also, remark, that the phase of a Blaschke product is a one layer neural net with (arctan as an activation sigmoid) and that the composition is a "Deep Neural Net" whose "depth" is the number of compositions. Our results provide a wealth of related libraries of orthogonal bases.

We sketch these ideas in various "vignette" subsections and refer for more details on analytic methods \[2\], related to the Blaschke based nonlinear phase unwinding decompositions \[4\] \[5\] \[11\]. We also consider orthogonal decompositions of invariant subspaces of Hardy spaces. In particular we constructed a multiscale decomposition, described below, of the Hardy space of the upper half-plane.

Such a decomposition can be carried in the unit disk by conformal mapping. A somewhat different multiscale decomposition of the space \( H^2(T) \)
has been constructed by using Malmquist-Takenaka bases associated with Blaschke products whose zeroes are $(1 - 2^{-n})e^{i\pi j/2^n}$ where $n \geq 1$ and $0 \leq j < 2^n$. Here we provide a variety of multiscale decompositions by considering iterations of Blaschke products.

In the next chapter we will show how with help of an extended Radon transform we can introduce a method of rotations to enable us to lift the one dimensional tools to higher dimensions. In particular the various orthogonal bases of holomorphic functions in one dimension, give rise to orthogonal bases of Harmonic functions in the higher dimensional upper half space.

2 Preliminaries and notation

For $p \geq 1$, $H^p(\mathbb{T})$ stands for the space of analytic functions $f$ on the unit disk $\mathbb{D}$ such that

$$\sup_{0 < r < 1} \int_0^{2\pi} |f(re^{i\theta})| r \frac{d\theta}{2\pi} < +\infty.$$ 

Such functions have boundary values almost everywhere, and the Hardy space $H^p(\mathbb{T})$ can be identified with the set of $L^p$ functions on the torus $\mathbb{T} = \partial \mathbb{D}$ whose Fourier coefficients of negative order vanish. We will alternate between analysis on the disk, and the parallel theory for analytic functions on the upper half plane $\mathbb{H} = \{x + iy : y > 0\}$. The space of analytic functions $f$ on $\mathbb{H}$ such that

$$\sup_{y > 0} \|f(\cdot + iy)\|_{L^p(\mathbb{R})} < +\infty$$

is denoted by $H^p(\mathbb{R})$. These functions have boundary values in $L^p(\mathbb{R})$ when $p \geq 1$. The space $H^p(\mathbb{R})$ is identified to the space of $L^p$ functions whose Fourier transform vanishes on the negative half line $(-\infty, 0)$.

3 Analysis on The upper half plane

We present some known results [2], without proof. In this section one simply writes $H^2$ instead of $H^2(\mathbb{R})$.

Malmquist-Takenaka bases

Let $(a_j)_{1 \leq j}$ be a sequence (finite or not) of complex numbers with positive imaginary parts and such that
The corresponding Blaschke product is

\[
\mathbf{B}(x) = \prod_{j \geq 0} \frac{|1 + a_j^2|}{1 + a_j^2} \frac{x - a_j}{x - \overline{a}_j},
\]

where, 0/0, which appears if \( a_j = i \), should be understood as 1. The factors \( \frac{|1 + a_j^2|}{1 + a_j^2} \) insure the convergence of this product when there are infinitely many zeroes. But, in some situations, it is more convenient to use other convergence factors as we shall see below.

Whether the series (1) is convergent or not, one defines (for \( n \geq 0 \)) the functions

\[
\phi_n(x) = \frac{1}{\sqrt{\pi}} \left( \prod_{0 \leq j < n} \frac{x - a_j}{x - \overline{a}_j} \right) \frac{1}{x - a_n}.
\]

Then these functions form an orthonormal system in \( H^2 \). If the series (1) diverges, it is a Malmquist-Takenaka orthonormal basis of \( H^2 \), otherwise it is a basis of the orthogonal complement of \( \mathcal{B} H^2 \) in \( H^2 \).

We remark that roughly a hundred years ago these bases were constructed [13, 9] through a Gram Schmidt orthogonalization of the list of rational functions with poles in the lower half plane.

Observe that for a rational function with a pole of order \( M \) at \( a \) the corresponding \( M \) basis functions have the form

\[
\phi_n(x) = e^{i n \theta(x)} \frac{1}{x - a_n} \quad (n = 1..M).
\]

These are localized "Fourier like" basis functions around the real part of \( a \) scaled by the imaginary part.

**Example of a multiscale Wavelet decomposition**

The infinite Blaschke products

\[
G_n(x) = \prod_{j \leq n} \frac{j - i}{j + i} \frac{x - j - i}{x - j + i} \quad \text{and} \quad G(x) = \prod_{j \in \mathbb{Z}} \frac{j - i}{j + i} \frac{x - j - i}{x - j + i}
\]

can be expressed in terms of known functions:
\[ G_n(x) = \frac{\Gamma(-i-n)}{\Gamma(i-n)} \frac{\Gamma(x-n+i)}{\Gamma(x-n-i)} \quad \text{and} \quad G(x) = \frac{\sin \pi(i-x)}{\sin \pi(i+x)}. \tag{2} \]

An orthonormal system

Consider the function \( \phi(x) = \frac{\Gamma(x-1+i)}{\sqrt{\pi} \Gamma(x-i)} \). It is easily checked that

\[ \phi(x-n) = \frac{\Gamma(i-n)}{\Gamma(-i-n)} \frac{G_n(x)}{\sqrt{\pi}(x-(n+1)+i)}. \]

Set \( \phi_n(x) = \phi(x-n) \). For fixed \( m \), the functions \( \phi_n/G_m \), for \( n \geq m \), form a Malmquist-Takenaka basis of \( (G/G_m)H^2 \). In other terms, the functions \( \phi_n \), for \( n \geq m \), form an orthonormal basis of \( G_mH^2 \ominus GH^2 \). This means that the functions \( \phi_n \) (for \( n \in \mathbb{Z} \)) form a Malmquist-Takenaka basis of the orthogonal complement of \( GH^2 \) in \( H^2 \).

Multiscale decomposition

As \(|1-G(2^n x)| \leq C2^n\) all the products \( \mathcal{B}_n(x) = \prod_{j<n} G(2^j x) \) are convergent and \( \lim_{n \to -\infty} \mathcal{B}_n = 1 \) uniformly.

Let \( \mathcal{B} = \mathcal{B}_0 \). Obviously, \( \mathcal{B}_n(x) = \mathcal{B}(2^n x) \). Consider the following subspaces of \( H^2 \): \( E_n = \mathcal{B}_nH^2 \). This is a decreasing sequence. The space \( E_{+\infty} = \bigcap_{n \in \mathbb{Z}} E_n \) is equal to \( \{0\} \) since a function orthogonal to this space would have too many zeros, and the space \( E_{-\infty} = \text{closure of } \bigcup_{n \in \mathbb{Z}} E_n \) is equal to \( H^2 \) since \( \mathcal{B}_n \) converges uniformly to 1 when \( n \) goes to \( -\infty \).

For all \( n \) and \( j \), let

\[ \phi_{n,j}(x) = 2^{n/2} \phi(2^n x - j) \mathcal{B}(2^n x). \]

Then, for all \( n \), \( (\phi_{n,j})_{j \in \mathbb{Z}} \) is an orthonormal basis of \( E_n \ominus E_{n+1} \). We conclude that \( (\phi_{n,j})_{n,j \in \mathbb{Z}} \) is an orthonormal basis of \( H^2 \).

4 Adapted MT bases, ”phase unwinding”

We now find a ”best” adapted Malmquist Takenaka basis to analyze or unwind the oscillations of a given function.
The idea is to peel off the oscillation of a function by dividing by its Blaschke product defined by the zeroes of the function, this procedure is iterated to yield an expansion in an orthogonal collection of functions or Blaschke products which of course are naturally embedded in a MT basis, once the zeroes are ordered.

**The unwinding series.**

There is a natural way to iterate the Blaschke factorization, it is inspired by the power series expansion of a holomorphic function on the disk. If $G$ has no zeroes inside $D$, its Blaschke factorization is the trivial one $G = 1 \cdot G$, however, the function $G(z) - G(0)$ certainly has at least one root inside the unit disk $D$ and will therefore yield some nontrivial Blaschke factorization $G(z) - G(0) = B_1 G_1$. We write

$$F(z) = B(z) \cdot G(z) = B(z) \cdot (G(0) + (G(z) - G(0))$$

$$= B(z) \cdot (G(0) + B_1(z)G_1(z)) = G(0)Bz + B(z)B_1(z)G_1(z).$$

An iterative application gives rise to the *unwinding series*

$$F = a_1 B_1 + a_2 B_1 B_2 + a_3 B_1 B_2 B_3 + a_4 B_1 B_2 B_3 B_4 + \ldots$$

This orthogonal expansion first appeared in the PhD thesis of Michel Nahon [11] and independently by T. Qian in [10] [12]. Detailed approximations in smoothness spaces were derived by S. Steinerberger in [4]. Given a general function $F$ it is not numerically feasible to actually compute the roots of the function; a crucial insight in [11] is that this is not necessary – one can numerically obtain the Blaschke product in a stable way by using a method that was first mentioned in a paper of Guido and Mary Weiss [14] and has been investigated with respect to stability by Nahon [11]. Using the boundedness of the Hilbert transform one can prove easily convergence in $L^p$, $1 < p < \infty$.

**The fast algorithm of Guido and Mary Weiss [14]**

Our starting point is the theorem that any Hardy function can be decomposed as

$$F = B \cdot G,$$

where $B$ is a Blaschke product, that is a function of the form

$$B(z) = z^n \prod_{i \in I} \frac{\bar{a}_i}{|a_i|} \frac{z - \bar{a}_i}{1 - \bar{a}_iz}.$$
where \( m \in \mathbb{N}_0 \) and \( a_1, a_2, \ldots \in \mathbb{D} \) are zeroes inside the unit disk \( \mathbb{D} \) and \( G \) has no roots in \( \mathbb{D} \). For \( |z| = 1 \) we have \( |B(z)| = 1 \) which motivates the analogy
\[
B \sim \text{frequency and } G \sim \text{amplitude}
\]
for the function restricted to the boundary. However, the function \( G \) need not be constant; it can be any function that never vanishes inside the unit disk. If \( F \) has roots inside the unit disk, then the Blaschke factorization \( F = B \cdot G \) is going to be nontrivial (meaning \( B \not\equiv 1 \) and \( G \not\equiv F \)). \( G \) should be 'simpler' than \( F \) because the winding number around the origin decreases.

In fact since \( |F| = |G| \) and \( \ln(G) \) is analytic in the disk we have formally that \( G = \exp(\ln |F| + i(\ln |F|)^*) = \exp(\mathcal{H}(\ln |F|)) \) where \( \mathcal{H} \) is the projection onto the Hardy space, and \( B = F/G \). \( G \) can be computed easily using the FFT [11].

### A remarkable unwinding

The following is an explicit unwinding of a singular inner function in the upper half plane illustrating this exponentially fast approximation of \( \exp \frac{2i\pi}{x} \):
\[
\exp \frac{2i\pi}{x} = e^{-2\pi} + (1 - e^{-4\pi}) \sum_{n \geq 0} (-1)^n e^{-2n\pi} B(x)^{n+1},
\]
where \( B \) is a Blaschke product whose zeros are \( \{1/(j+i)\}_{j \in \mathbb{Z}} \).

### 5 Geometric function theory: the role of compositions of Blaschke products.

#### Iteration of Blaschke products

We claim that by building Blaschke product through composition we open up rich dynamical structures, and libraries of corresponding Malmquist Takkenaka bases.

We are interested in iteration of finite Blaschke products
\[
B(z) = e^{i\theta} z^\mu \prod_{j=1}^{\nu} \frac{z + a_j}{1 + \overline{a}_j z},
\]
where \( \mu \) and \( \nu \) are nonnegative integers and the \( a_j \) are complex numbers of modulus less than 1.
It is well known that $T$ and $D$ are globally invariant under $B$, as well as the complement of $\mathbb{T}$ in the Riemann sphere.

A careful discussion can be found in [2]. Here is the main result.

**Theorem 1** Let $B$ be a finite Blaschke product with a fixed point $\alpha$ inside the unit disk. Then there exists a sequence $\alpha, a_1, a_2, \ldots, a_j, \ldots$ of complex numbers in the unit disk and an increasing sequence $(\nu_j)_{j \geq 1}$ of positive integers such that $a_1, a_2, \ldots, a_{\nu_n}$ are the zeros, counted according to their multiplicity, of $B_n$ (the $n$th iterate of $B$). Moreover $\sum_{j \geq 1} (1 - |a_j|) = +\infty$. Also, $B_n$ converges towards $\alpha$ uniformly on compact subsets of the open unit disk.

Dynamic Multiscale analysis through composition of Blaschke products

Each Blaschke product $B$ defines invariant subspaces of $H^p$. The projection on this space is given by the kernel $\frac{B(z)\overline{B(w)}}{z - w}$. This projection is continuous for $1 < p < +\infty$.

Let $F$ be a Blaschke product of degree at least 2 with a fixed point inside the unit disk. Its iterates define a hierarchy of nested invariant subspaces $E_n = F_n H^2$.

Due to Theorem 1, $\bigcap_{n \geq 1} E_n = \{0\}$.

The Takenaka construction provides orthonormal bases of $E_n \ominus E_{n+1}$. But this is not canonical as it depends on an ordering of the zeros of $F_{n+1}/F_n$.

Figure 1 shows 1st, 3rd, and 5th iterates of $F(z) = z(z - 2^{-1})/(1 - 2^{-1}z)$. Figure 2 displays the phase for the fourth iterate of $F(z) = z^4(z - 2^{-2})/(1 - 2^{-2}z)$. The upper pictures display the phases modulo $2\pi$ (values in the interval $(-\pi, \pi]$) of these Blaschke products while the lower pictures display minus the logarithms of their absolute value. The coordinates $(x, y)$ correspond to the point $e^{-y+i\pi}$. On these figures it is easy to locate the zeros, specially by looking at the phase which then has an abrupt jump.

Remarks on Iteration of Blaschke products as a ”Deep Neural Net”

In the upper half plane let $(a_j)_{1 \leq j}$ be a finite sequence of complex numbers with positive imaginary parts. The corresponding Blaschke product on the line is

$$B(x) = \prod_{j \geq 0} \frac{x - a_j}{x - \overline{a_j}}.$$
We can write $B(x) = \exp(i\theta(x))$, where

$$\theta(x) = \sum_{j \geq 0} \sigma((x - \alpha_j)/\beta_j)$$

with $a_j = \alpha_j + i\beta_j$ and $\sigma = \arctan x + \pi/2$ is a sigmoid.

This is precisely the form of a single layer in a Neural Net, each unit has a weight and bias determined by $a_j$. We obtain the various layers of a deep net through the composition of each layer with a preceding layer. In our preceding examples we took a single short layer given by a simple Blaschke term with two zeroes in the first layer that we iterated to obtain an orthonormal Malmquist Takenaka basis (we could have composed different elementary products at each layer), demonstrating the versatility of the method to generate highly complex functional representations.

As an example let $F(z)$ be mapped from $G$, (2) in the section on wavelet construction.

$$F(z) = G(w) = \frac{\sin(\pi(i - w))}{\sin(\pi(i + w))} \text{ with } w = \frac{i(1 - z)}{1 + z}.$$  

We can view the phase of $F$ as a neural layer which when composed with itself results in a phase which is a two layer neural net represented graphically in fig 3.

Where each end of a color droplet corresponds to one zero or unit of the two layer net.

We refer to Daubechies et al. [7] for a description of a similar iteration for piecewise affine functions in which simple affine functions play the role of a Blaschke product.

6 Higher dimensions, $\theta$-holomorphy

Our goal is to explore methodologies to use the remarkable analytic approximation theorems described above to enable deeper understanding of real analysis, in higher dimensions. We know that Blaschke factorization do not exist, nevertheless there are remarkable bases that can be lifted.

We start by observing that $Z_\theta = (x \cdot \theta + iy) = t + iy$ is harmonic in $(x, y)$ (in 3 dimensions) and so is $Z_\theta^k$. This is a harmonic homogeneous polynomial of degree $k$ in $(x_1, x_2, y)$ that is constant in the direction perpendicular to $\theta$. Here we identified $\theta$ with $(\cos \theta, \sin \theta)$. It is well known [6] that

$$\frac{1}{2\pi} \int_0^{2\pi} e^{-il\theta} Z_\theta^k d\theta = Y^k_l(x_1, x_2, y) \quad (|l| \leq k)$$

is the standard orthogonal basis of spherical Harmonics in 3 dimensions.
As a consequence we see that any Harmonic function $U(x, y)$ is a superposition over $\theta$ of holomorphic functions in $Z_\theta$, more explicitly a Power series in $Z_\theta$ with coefficients depending on $\theta$.

$$U(x, y) = \frac{1}{2\pi} \int_0^{2\pi} F_\theta(Z_\theta) \, d\theta = \frac{1}{2\pi} \int_0^{2\pi} \sum_{k \geq 0} a_k(\theta) Z_\theta^k \, d\theta$$

where $a_k(\theta)$ is a trigonometric polynomial of degree $k$.

Another example, taking

$$F_\theta(Z_\theta) = e^{-2i\pi r Z_\theta} e^{-2i\pi k \theta}.$$

we get the harmonic function

$$J_k \left( 2\pi r \sqrt{x_1^2 + x_2^2} \right) e^{-2i\pi k \phi} e^{-2\pi yr}$$

**Radon and Fourier in the upper Half space**

This relationship between holomorphic functions in planes as generating all harmonic functions can most easily be explored through Fourier analysis. We define the Radon transform, and relate it to the Fourier transform to lead to the $\theta$-holomorphic version.

Let

$$R_\theta f(t) = \int_{\mathbb{R}^n} f(t\theta + x) \, dx.$$  \hspace{1cm} (3)

Obviously $R_{-\theta} f(t) = R_\theta f(-t)$.

For $f \in L^1(\mathbb{R}^n)$, consider its harmonic extension $u$ to $\mathbb{R}^{n+1}_+$. For $x \in \mathbb{R}^n$ and $y > 0$ we have

$$u(x, y) = f * P_y(x) = \int e^{2i\pi x \cdot \xi} e^{-2i\pi \|\xi\|y} \hat{f}(\xi) \, d\xi$$

$$= \int_{S_{n-1}} \left( \int_0^\infty e^{2i\pi r(x \cdot \theta + iy)} \hat{f}(r\theta) r^{n-1} \, dr \right) \, d\theta$$

$$= \int_{S_{n-1}} F_\theta(x \cdot \theta + iy) \, d\theta,$$

where

$$F_\theta(z) = \int_0^\infty e^{2i\pi r z} \hat{f}(r\theta) r^{n-1} \, dr = \int_0^\infty e^{2i\pi r z} \hat{R}_\theta f(r) \, r^{n-1} \, dr. \hspace{1cm} (4)$$

When $n = 2$, we have
\[ \hat{F}_\theta(t) = \mathcal{R}_\theta f(t) \mathbf{1}_{(0, +\infty)}(t) = \frac{1}{2\pi i} \overline{\mathcal{D}_\theta f(t)} \mathbf{1}_{(0, +\infty)}(t). \]

So, for \( \Im z > 0 \),
\[ F_\theta(z) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \frac{d(R_\theta f(t))}{t - z} = \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \frac{R_\theta f(t)}{(t - z)^2} dt. \]

For general \( n \) we get
\[ F_\theta(z) = \frac{(n - 1)!}{(2\pi)^n} \int_{-\infty}^{+\infty} \frac{R_\theta f(t)}{(t - z)^n} dt. \]

**Some isometries**

We describe some computations in the case when \( n = 2 \) and mention the case of higher dimension at the end of this section.

In view of (4)
\[ \hat{F}_\theta(\cdot + iy)(r) = \hat{f}(r \theta) e^{-2\pi r y} \mathbf{1}_{(0, +\infty)}(r). \] (5)

Hence, the Plancherel identity yields
\[ \int_{0}^{\infty} dy \int_{-\infty}^{+\infty} |F_\theta(t + iy)|^2 dt = \int_{0}^{\infty} \int_{0}^{+\infty} |\hat{f}(r \theta)|^2 r^2 e^{-4\pi ry} dr dy = \frac{1}{4\pi} \int_{0}^{\infty} |\hat{f}(r \theta)|^2 dr. \]

Let \( \|F\|_{B}^2 = \int_{0}^{\infty} \int_{-\infty}^{+\infty} |F_\theta(t + iy)|^2 dy dt \) (this is the norm of the Bergman space on the upper half plane). Then
\[ 4\pi \int_{0}^{2\pi} \|F_\theta\|_{B}^2 d\theta = \int_{(0, +\infty) \times (0, 2\pi)} |\hat{f}(r \theta)|^2 r dr d\theta = \|f\|_{L^2_{\mathbb{R}^n}}^2. \] (6)

We have
\[ \frac{\partial u(x, y)}{\partial y} = -2\pi \int_{\mathbb{R}^2} e^{2i\pi \xi \cdot x} |\xi| e^{-2\pi |\xi| y} \hat{f}(\xi) d\xi. \]
\[
\int \int_{\mathbb{R}^3_+} \left| \frac{\partial u(x,y)}{\partial y} \right|^2 \, dx \, dy = (2\pi)^2 \int_{\mathbb{R}^2} \left( \int_0^{\infty} e^{-4\pi|\xi|y} \, dy \right) |\hat{f}(\xi)|^2 |\xi|^2 \, d\xi
\]
\[
= \pi \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 |\xi| \, d\xi
\]  
(7)

Equation (5) yields \[
\int_{-\infty}^{\infty} |F_\theta(t)|^2 \, dt = \int_0^{\infty} |\hat{f}(r\theta)|^2 r^2 \, dr, \quad \text{and}
\]
\[
\int_0^{2\pi} \|F_\theta\|^2_{H^2(\mathbb{R})} d\theta = \int_{\mathbb{R}^2} |\hat{f}(\xi)|^2 |\xi| \, d\xi.
\]  
(8)

Formulas (7) and (8) together give \[
\int_0^{2\pi} \|F_\theta\|^2_{H^2(\mathbb{R})} d\theta = \frac{1}{\pi} \int \int_{\mathbb{R}^3_+} \left| \frac{\partial u(x,y)}{\partial y} \right|^2 \, dx \, dy,
\]  
(9)

In higher dimension formulas (5) and (9) become
\[
\int_{S_{n-1}} \, d\theta \int \int_{\mathbb{R}^{n+1}} |F_\theta(t + iy)|^2 y^{n-2} \, dtdy = \frac{1}{(4\pi)^{n-1}} \|f\|^2_{L^2(\mathbb{R}^n)}
\]
and
\[
\int \int_{\mathbb{R}^{n+1}} \left| \frac{\partial^k u(x,y)}{\partial y^k} \right| y^{2k-n} \, dx \, dy
\]
\[
= \frac{(4\pi)^{n-1} \Gamma(2k - n + 1)}{2^{2k}} \int_{S_{n-1}} \|F_\theta\|^2_{H^2(\mathbb{R})} \, d\theta.
\]  
(10)

Remarks; ”lifted Analysis” of Harmonic functions

One of our goals is to enable the application of some of the one dimensional analytic approximation tools to higher dimensions. We refer to Michel Nahon’s thesis [11] in which he decomposes a function in the plane as a sum of functions whose Fourier transform live in thin wedges, as a tool to extract features (gradients of Phase) from an image of a fingerprint. This illustrates potential variants of our current approach.

We envision a function in two variables represented as a superposition of \(F_\theta(t + iy)\), each of which is approximated to error \(\epsilon\) leading to a harmonic function approximation of error \(\epsilon\) in the Dirichlet space. Similar estimations with different mix of Hilbert spaces can be easily derived as in [4], leading to faster rates of convergence (when more regularity is present).
Another obvious application is the representation of a Calderón Zygmond operator given by a Fourier multiplier homogeneous of degree 0, \( \Omega(\theta) \), simply by averaging \( \Omega(\theta)F_\theta(t + iy) \).

The representation of these operators is a version of the rotation method (no parity required on \( \Omega \)). Also it provides a local representation method for generalized conjugate functions or C-Z operators, just by using the local spherical Harmonics version of the \( \theta \)-holomorphic representation. In particular Riesz transforms correspond to \( \Omega(\theta) = (\cos \theta, \sin \theta) \).

**A natural ortho-basis in the Dirichlet space**

We now use identity (9) to transfer an orthonormal basis of the Hardy space \( H^2 \) to an orthonormal system in the Dirichlet space in \( \mathbb{R}^3_+ \).

Start from the basis \( \frac{1}{\sqrt{2\pi}} \left( \frac{z-i}{z+i} \right)^n \frac{1}{z+i} \) of \( H^2 \) (this corresponds to the Fourier basis in the disc, mapped to the upper half plane Hardy space). We consider the generating function

\[
F(z) = \frac{i}{\sqrt{2\pi}} \sum_{n \geq 0} (z - i)^n t^n / ((z + i)^n + 1)
\]

and compute \( G(x, y) = \int_0^{2\pi} F(z\theta) \, d\theta \). We get

\[
G(x, y) = \sqrt{2/\pi} / \sqrt{(\rho^2 + 2y + 1)t^2 - 2(1 - \rho^2)t + \rho^2 - 2y + 1},
\]

where \( \rho = \sqrt{x_1^2 + x_2^2 + (y + 1)^2} \).

This also can be written as

\[
G(x, y) = \frac{\sqrt{2/\pi}}{\sqrt{\rho^2 + 2y + 1} \sqrt{a^2t^2 - 2bat + 1}}.
\]

If one sets \( a = \sqrt{\rho^2 - 2y + 1} \) and \( b = \frac{\rho^2 - 1}{\sqrt{(\rho^2 + 1)^2 - 4y^2}} \).

It results that the functions \( \sqrt{2/\pi} a^n P_n(b) \), where the \( P_n \) are the Legendre polynomials, form an orthonormal system in the Dirichlet space.

To get an orthonormal basis of the Dirichlet space in 3 dimensions, it suffices to take \( \sqrt{2/\pi} 2a^n P_n(b) e^{ik\theta} / \sqrt{\rho^2 + 2y + 1} \), with \( k \in \mathbb{Z} \) and \( n \geq 0 \).
Of course such computation can be done in higher dimension: isometry \([10]\) allows to transfer orthonormal bases of \(H^2\) to orthonormal systems in a suitable Dirichlet space.

**Concluding remarks and potential applications**

As we all know complex methods, such as interpolation of operators, or the remarkable proofs by Calderón of the boundedness in \(L^2\) of commutators with the Hilbert transform, or the Cauchy integral on Lipschitz curves are powerful tools. Over the years the goal has been to convert them into real variable methods. In parallel the quest for higher dimensional complex tools is continuing, see the examples \([6]\) in which various systems generalizing holomorphic functions to higher dimension are studied. The point here, is that the infinite dimensional \(\theta\)-holomorphic functions generate all of these systems through the choice of appropriate multipliers (as described for the Riesz system).

Our goal here was to describe recent nonlinear analytic tools in the classical setting, and transfer them to the higher dimensional real setting. Together with Guido Weiss we had observed \([6]\) that all harmonic functions in higher dimensions are combinations of holomorphic functions on subplanes which are constant in normal directions. The recent developments in one dimension as well as the isometries described here, and the corresponding efficient approximation methods, open the door for applications in higher dimensions, such as image denoising. See \([5]\) for the impact of unwinding on precision Doppler analysis in 1 dimension, which we expect to carry over to 2 or 3 dimensions.

Observe also that, for simplicity, we restricted our attention to 2 dimensions in cylindrical coordinates. We could have defined more generally power series in the variable \(Z_\epsilon = (x \cdot \epsilon)\) where \(\epsilon\) satisfying \((\epsilon \cdot \epsilon) = 0\), represents a point on the complex quadric with \(|\Re \epsilon| = 1, |\Im \epsilon| = 1\), or a two dimensional plane spanned by \(\Re \epsilon, \Im \epsilon\).

Clearly we can extend the preceding discussion to this setting. Where; \(Z_\epsilon = (x \cdot \epsilon)\) is the point \(t+i\) in the complex plane with coordinate \(t\Re \epsilon + i\Im \epsilon\).

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Fig. 1 The argument and the absolute value of $F(z)$, $F^{(3)}(z)$, and $F^{(5)}(z)$, with $F(z) = \frac{z(z-2^{-1})}{1-2^{-1}z}$ and $z = \exp(-y + ix)$. 
Fig. 2 The multiscale view of the argument of $F^{(4)}(z)$, with $F(z) = \frac{z^2(2-z^2)}{1-2z^2}$ and $z = \exp(-y+iz)$.

Fig. 3 Two iterations of $F(z) = G(w) = \frac{\sin(\pi(1-w))}{\sin(\pi(1+w))}$ with $w = \frac{i(1-z)}{(1+z)}$. 