Parameter-free $\ell_p$-Box Decoding of LDPC Codes

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Abstract—The Alternating Direction Method of Multipliers (ADMM) decoding of Low Density Parity Check (LDPC) codes has received many attentions due to its excellent performance at the error floor region. In this paper, we develop a parameter-free decoder based on Linear Program (LP) decoding by replacing the binary constraint with the intersection of a box and an $\ell_p$ sphere. An efficient $\ell_2$-box ADMM is designed to handle this model in a distributed fashion. Numerical experiments demonstrate that our decoder attains better adaptability to different Signal-to-Noise Ratio and channels.

Index Terms—parameter-free, $\ell_p$-box, ADMM, LP decoding, LDPC codes

I. INTRODUCTION

Low Density Parity Check (LDPC) codes have been widely applied nowadays [1]. Although various decoding methods have been developed for LDPC, e.g. Belief Propagation (BP) and Min-Sum (MS) [1], [2], nearly all of the existing solutions are approximation-based (not truly Maximum Likelihood solution) and consequently suffer an early error floor, especially in the case of high Signal-to-Noise Ratio (SNR). Therefore, it is important yet challenging to develop an error-floor-free decoding scheme for LDPC with more accurate decoding performance.

For binary LDPC codes over symmetric channels, Feldman introduced a relaxed version [3], [4] of Maximum Likelihood (ML) decoding problem which can be interpreted as a Linear Program (LP) decoding, whose certificate of correctness (ML certificate) and performance are reported in [5]. Compared with other decoding methods, LP decoding does not suffer very much from an error floor [6]. In order to make the decoder behave like BP at low SNRs, Liu et al. [7] developed a penalized-LP decoding model by adding different types of penalty terms in the objective to drive the solution away from a penalized-LP decoding model by adding different types of decoder behave like BP at low SNRs, Liu et al. [7] developed very much from an error floor [6]. In order to make the decoder behave like BP at low SNRs, Liu et al. [7] developed a penalized-LP decoding model by adding different types of penalty terms in the objective to drive the solution away from the error floor, especially in the case of high Signal-to-Noise Ratio (SNR). Therefore, it is important yet challenging to develop an error-floor-free decoding scheme for LP decoding with more accurate decoding performance. For binary LDPC codes over symmetric channels, Feldman introduced a relaxed version [3], [4] of Maximum Likelihood (ML) decoding problem which can be interpreted as a Linear Program (LP) decoding, whose certificate of correctness (ML certificate) and performance are reported in [5]. Compared with other decoding methods, LP decoding does not suffer very much from an error floor [6]. In order to make the decoder behave like BP at low SNRs, Liu et al. [7] developed a penalized-LP decoding model by adding different types of penalty terms in the objective to drive the solution away from the error floor, especially in the case of high Signal-to-Noise Ratio (SNR). Therefore, it is important yet challenging to develop an error-floor-free decoding scheme for LP decoding with more accurate decoding performance.

However, one major hurdle still remains in the penalized-LP decoding is the difficulty in properly choosing the penalty parameter. The effect of the penalty parameter is two-fold. First, the penalty parameter closely affects the performance of the algorithm and an improper value can make the algorithm extremely inefficient. Most importantly, the optimal solution of the penalized-LP decoding problem may vary for different penalty parameter values. As a result, the decoder often find solutions with low accuracy. It is worth noting that the penalized-LP formulation in fact approximates the original problem via penalizing the violation of the binary constraint and therefore provides an inexact solution indeed. Hence its optimal solution is rarely the optimal solution of the original LP decoding problem.

In this paper, we introduce a parameter-free continuous optimization model, which is an exact reformulation of the binary LP decoding problem. The binary constraint in the LP decoding is replaced by the intersection of a box and an $\ell_p$ sphere. We also design an efficient ADMM solver, which can be parallelized to reduce the computational cost. Simulation results demonstrate that the proposed model along with the ADMM solver can further bring down the error floor for large SNRs. Moreover, compared to many other contemporary decoders, the proposed decoding scheme does not need to tune the penalty parameter.

II. BACKGROUND

We consider a binary linear LDPC code $C$ of length $N$, which can be defined by an $M \times N$ parity check matrix $H$. Each column of the parity check matrix corresponds to a codeword symbol, indexed by $I := \{1, 2, ..., N\}$. Each row of the parity check matrix corresponds to a check, which specifies a subset of codeword symbols that add to 0 (modulo 2), indexed by $J := \{1, 2, ..., M\}$. The neighborhood of check $j$, denoted as $N(j)$, is the set of variables that check $j$ constrains to add to 0. That is $\{i \in I : H_{j,i} = 1\}$.

For a binary linear code transmitted over a symmetric memoryless channel, let $X = \{0, 1\}$ be the input space, $Y$ be the output space and $P(y|x)$ denotes the probability that the codeword $x \in X$ is sent over the channel and $y \in Y$ is received. Let $C$ be the set of possible codewords. By Bayes’ rule, the decoding problem can be modeled as $\text{arg max}_{x \in C} P(y|x)$. In particular, the Maximum Log-likelihood (ML) decoding problem [3], [4] takes the form

$$\text{arg max}_{x \in C} P(y|x) = \text{arg min}_{x \in C} \sum_{i=1}^{N} \gamma_i x_i = \text{arg min}_{x \in C} \gamma^T x, \quad (1)$$

where $\gamma$ is a length-$n$ vector of Log-likelihood Ratios (LLRs) with $\gamma_i = \log(\frac{P(y_i|0)/P(y_i|1)}{P(y_i|1)/P(y_i|0)})$.

Some work [9], [11], [12] has been done in the past decades focusing on the properties of the convex hull of all possible codewords. Denote the subset of coordinates of $x$ participating in the $j$th check as the matrix $P_j$, so $P_j$ is a binary $d \times N$ matrix consisting of $d$ components participating in the $j$th check. From [9], each local codeword constraint can be relaxed to satisfy $P_j x \in \mathbb{P}_d$, where $\mathbb{P}_d$ is the specific
expression of the convex hull of all possible codewords. The LP decoding problem is then derived in [9], by relaxing the binary constraints

$$\begin{align*}
\min_x & \quad \gamma^T x \\
\text{s.t.} & \quad x \in [0, 1]^N, \quad P_j x \in \mathbb{P}^d, \quad \forall j \in \mathcal{J},
\end{align*}$$

(2)

where $[0, 1]^N$ is the $N$-dimensional box.

The simulation results in [9] suggest that an ideal decoder should perform like BP at low SNRs. To achieve this, a penalized-LP decoding is proposed in [7] by penalizing the fractional solutions, that is, the objective of (2) is replaced with

$$\gamma^T x + \sum_{i=1}^N g(x_i),$$

where the penalty function $g : [0, 1] \to \mathbb{R} \cup \{\pm \infty\}$ has three options, namely, $\ell_1$, $\ell_2$ and log penalty. It is reported that the $\ell_2$ penalty has the best error performance and the fastest convergence among the three options. Other improved penalty functions are proposed and experimented in [13].

### III. $\ell_p$-Box Decoding

In this section, we propose a new formulation for LP decoding. Compared to the existing formulations suitable to decentralized processing, where mixed integer formulation are usually involved, our proposed formulation is to design a continuous and exact reformulation of the LP decoding to circumvent solving an IP problem. The major technique used in our model is the $\ell_p$-box recently proposed in [14] for solving IP problems. The main idea of this technique is to replace the discrete constraint by the intersection of a box and an $\ell_p$-sphere (defined through the $\ell_p$ norm with $p \in (0, \infty)$):

$$x \in \{0, 1\}^N \Leftrightarrow x \in [0, 1]^N \cap \{x : \|x - \frac{1}{2} 1_N\|_p = \frac{N}{2^p}\},$$

(3)

where $1_N$ is the $N$-dimensional vector filled with all 1s. Fig. 1 illustrates this equivalence for $\ell_1$-box and $\ell_2$-box.

![Fig. 1. Geometric illustration of the equivalence between $\ell_p$-box technique and the set of binary points in $\mathbb{R}^3$ with $p \in \{1, 2\}$](image)

Now consider this technique in the LP decoding problem (2). Besides the existing box constraint $[0, 1]^N$, we only need an additional $\ell_p$-sphere constraint to enforce the solution to be binary, yielding the $\ell_p$-box decoding problem:

$$\begin{align*}
\min_x & \quad \gamma^T x \\
\text{s.t.} & \quad P_j x \in \mathbb{P}^d, \quad \forall j \in \mathcal{J} \\
& \quad x \in [0, 1]^N, \quad \|x - \frac{1}{2} 1_N\|_p = \frac{N}{2^p}.
\end{align*}$$

(4)

For the penalized-LP decoding, the choice of parameters is critical to the solution performance and, unfortunately, an appropriate choice can be very intricate to obtain. If the penalty parameter is set too large, the algorithm may quickly converge to a binary point far from the global solution. On the other hand, for a too small penalty parameter, the algorithm may converge to a fractional point. Compared to the parameterized penalized-LP formulation, there is no parameter involved in (4), rescuing users from adjusting the penalty parameters according to SNR. It should be noticed that (4) is equivalent to the binary decoding problem. Therefore, the optimal solution of the binary decoding problem is the global optimal solution of (4). Furthermore, (4) is a continuous problem, and thus can be attacked by various continuous optimization algorithms.

There are many options for the selection of $p$, and (4) becomes nonsmooth with $p \in (0, 1]$. In this paper, we use $p = 2$ in order to obtain a smooth function and avoid using higher order polynomial constraints. Moreover, for $p = 2$, the projection of a point onto an $\ell_2$ sphere can be carried out very efficiently, which is simply rescaling the magnitude of this point.

### IV. $\ell_2$-Box ADMM

In this section, we design an efficient algorithm for solving the proposed $\ell_2$-box decoding problem by incorporating the $\ell_2$-box technique in the ADMM framework.

As discussed in previous section, we set $p = 2$. In order to apply the ADMM, we introduce two sets of auxiliary variables $y$ and $z$ to decouple the box and the $\ell_2$ sphere, resulting in an $\ell_2$-box decoding problem:

$$\begin{align*}
\min_{x, y, z} & \quad \gamma^T x \\
\text{s.t.} & \quad x \in [0, 1]^N \\
& \quad y = x, \quad \|y - \frac{1}{2} 1_N\|_2^2 = \frac{N^2}{4} \\
& \quad z_j = P_j x, \quad z_j \in \mathbb{P}^d, \quad \forall j \in \mathcal{J},
\end{align*}$$

(5)

where $y$ is a $N$-dimensional vector and $z_j \in \mathbb{P}^d$ for all $j \in \mathcal{J}$. The ADMM works with the augmented Lagrangian

$$L_{\mu_1, \mu_2}(x, y, z, \lambda_1, \lambda_2) = \gamma^T x + \sum_{j \in \mathcal{J}} \lambda_{1, j}^T (P_j x - z_j) + \frac{\mu_1}{2} \sum_{j \in \mathcal{J}} \|P_j x - z_j\|_2^2 + \lambda_2^T (x - y) + \frac{\mu_2}{2} \|x - y\|_2^2$$

with constants $\mu_1$ and $\mu_2$ all positive. Here $\lambda_1$ is the dual variable associated with the constraint $z_j = P_j x$ and its dimension is the cardinality of $|\mathcal{J}|$, and the $N$-dimensional $\lambda_2$ is the dual variable associated with the coupling constraint $y = x$.

Let $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ denote the feasible regions for variables $x$, $y$ and $z$ respectively, i.e., $\mathcal{X} = \mathcal{Y} = [0, 1]^N$ and

$$\mathcal{Z} = \frac{\mathbb{P}P_{|\mathcal{N}(j)|} \times \ldots \times \mathbb{P}P_{|\mathcal{N}(j)|}}{\text{number of checks } \mathcal{J}}.$$

(6)

The ADMM iteration then can be elaborated as follows

$$\begin{align*}
\text{x-update :} & \quad x^{k+1} = \arg \min_{x \in \mathcal{X}} L_{\mu_1, \mu_2}(x, y^k, z^k, \lambda_1^k, \lambda_2^k), \\
\text{subject to} & \quad \|y - \frac{1}{2} 1_N\|_2^2 = \frac{N^2}{4} \\
\text{y-update :} & \quad y^{k+1} = \arg \min_{y \in \mathcal{Y}} L_{\mu_1, \mu_2}(x^{k+1}, y, z^k, \lambda_1^k, \lambda_2^k), \\
\text{subject to} & \quad \|y - \frac{1}{2} 1_N\|_2^2 = \frac{N^2}{4} \\
\text{z-update :} & \quad z^{k+1} = \arg \min_{z \in \mathcal{Z}} L_{\mu_1, \mu_2}(x^{k+1}, y^k, z, \lambda_1^k, \lambda_2^k), \\
\text{subject to} & \quad z_j \in \mathbb{P}^d, \quad \forall j \in \mathcal{J},
\end{align*}$$

(7)
(λ₁, λ₂)-update : \[
\begin{cases}
\lambda_{1,j}^{k+1} = \lambda_{1,j}^k + \mu_1 (P_j x_j^{k+1} - z_j^{k+1}), \quad \forall j \in J, \\
\lambda_2^{k+1} = \lambda_2^k + \mu_2 (x_j^{k+1} - y_j^{k+1}).
\end{cases}
\]

Next, we discuss the solution of the ADMM subproblems.
(a) The \(x\)-update subproblem
The \(x\)-update is to minimize \(L_{\mu_1, \mu_2}(x, y, z, \lambda_1, \lambda_2)\) subject to \(x \in [0, 1]^N\) with fixed \((y, z, \lambda_1, \lambda_2)\). The \(i\)th component of the solution of the \(x\)-update subproblem can be given explicitly by
\[
x_i^{k+1} = \Pi_{[0,1]^N} \left( \frac{d^k - \gamma_i - \lambda_{2,j}^k + \mu_2 y_i^k}{\mu_1 |N(i)| + \mu_2} \right).
\]

Here \(d^k = \sum_{j \in N(i)} \mu_1 (P_j^T z_j^k)^{(i)} - (P_j^T \lambda_1^k)^{(i)}\) where \((P_j^T \lambda_1^k)^{(i)}\) denotes the \(i\)th component of \(P_j^T \lambda_1^k\), and so forth for \((P_j^T \lambda_1^k)^{(i)}, x_i, \lambda_2,i, i, y, j\).

Define \(P = \sum_{j \in J} P_j^T P_j\) and \(\Pi_{[0,1]^N}\) as the projection operator onto the \(N\)-dimensional box \([0, 1]^N\). Note that for any \(j, P = \sum_{j \in J} P_j^T P_j\) is an \(N \times N\) diagonal matrix with non-zero entries at \((i, i)\) if and only if \(i\) participates in the \(j\)th parity check, i.e., \(i \in N(j)\). This implies that \(P = \sum_{j \in J} P_j^T P_j\) is a diagonal matrix with the \((i, i)\)th entry equal to \(|N(i)|\). Hence \(P^{-1}\) is diagonal with \(1/|N(i)|\) as the \(i\)th diagonal entry. See details of this calculation in [2].
(b) The \((y, z)\)-update subproblem
Notice that the two variables \(y\) and \(z\) are independent to each other, thus they can be updated separately. By fixing \((x, z, \lambda_1, \lambda_2)\), the \(y\)-update subproblem is given by
\[
y^{k+1} = \arg \min_{y \in \mathbb{Y}} L_{\mu_1, \mu_2}(x^{k+1}, y, z^k, \lambda_1^k, \lambda_2^k),
\]
s.t. \(|y - \frac{1}{2} 1_N|^2 = \frac{N}{4} \).

Thus, by eliminating the constants, (7) can be written as
\[
\min_y \left( \mu_2 (x^{k+1} - \frac{1}{2} 1_N) + \lambda_2^k \right)^T y \quad \text{s.t.} \quad \|y - \frac{1}{2} 1_N\|_2 = \frac{N}{4}.
\]

This simplified \(y\)-update subproblem is merely a rescaling problem, which has an explicit solution:
\[
y^{k+1} = \frac{\mu_2 (x^{k+1} - \frac{1}{2} 1_N) + \lambda_2^k}{\|\mu_2 (x^{k+1} - \frac{1}{2} 1_N) + \lambda_2^k\|_2} \times \frac{\sqrt{N}}{2} + \frac{1}{2}.
\]

For updating \(z\), the subproblem is to solve
\[
\min_{z_j} \frac{\mu_1}{2} \|P_j x_j^{k+1} - z_j\|_2^2 - \lambda_{1,j}^k z_j, \quad \text{s.t.} \quad z_j \in \mathbb{D}_d.
\]

for each \(j \in J\). The solution of this subproblem can be given explicitly [7]:
\[
z_j^{k+1} = \Pi_{\mathbb{D}_d} \left( P_j x_j^{k+1} + \frac{1}{\mu_1} \lambda_{1,j}^k \right), \quad \forall j \in J,
\]
where \(\Pi_{\mathbb{D}_d}\) is the projection operator onto the codeword polytope.
(c) \((\lambda_1, \lambda_2)\)-update
Note that \(\lambda_1\) and \(\lambda_2\) are also independent to each other, and thus can be updated separately:
\[
\begin{align*}
\lambda_{1,j}^{k+1} &= \lambda_{1,j}^k + \mu_1 (P_j x_j^{k+1} - z_j^{k+1}), \quad \forall j \in J, \\
\lambda_2^{k+1} &= \lambda_2^k + \mu_2 (x_j^{k+1} - y_j^{k+1}).
\end{align*}
\]

The entire \(\ell_2\)-box ADMM decoding algorithm is stated in Algorithm 1.

Algorithm 1 \(\ell_2\)-box ADMM decoding
1: Given a \(M \times N\) parity check matrix \(H\), and parameters \(\mu_1 > 0, \mu_2 > 0\), tolerance \(\epsilon > 0\).
2: Construct the log-likelihood vector \(\gamma\) and the \(d \times N\) matrix \(P_j\) for all \(j \in J\).
3: Initialize \(y, \lambda_2, z_j, \lambda_{1,j}\) for all \(j \in J\).
4: Set \(k = 0\).
5: repeat
6: Update \(x\): \(x_i^{k+1} = \Pi_{[0,1]^N} \left( \frac{d^k - \gamma_i - \lambda_{2,j}^k + \mu_2 y_i^k}{\mu_1 |N(i)| + \mu_2} \right)\).
7: Update \(y\):
\[
y_j^{k+1} = \frac{\mu_2 (x_j^{k+1} - \frac{1}{2} 1_N) + \lambda_2^k}{\|\mu_2 (x_j^{k+1} - \frac{1}{2} 1_N) + \lambda_2^k\|_2} \times \frac{\sqrt{N}}{2} + \frac{1}{2}.
\]
8: Update \(z\):
\[
\text{for all } j \in J \text{ do } z_j^{k+1} = \Pi_{\mathbb{D}_d} \left( P_j x_j^{k+1} + \frac{1}{\mu_1} \lambda_{1,j}^k \right).
\]
9: Update \(\lambda_1\):
\[
\text{for all } j \in J \text{ do } \lambda_{1,j}^{k+1} = \lambda_{1,j}^k + \mu_1 (P_j x_j^{k+1} - z_j^{k+1})
\]
10: Update \(\lambda_2\):
\[
\lambda_2^{k+1} = \lambda_2^k + \mu_2 (x_j^{k+1} - y_j^{k+1})
\]
11: Set \(k = k + 1\)
12: until \(\min \|P_j x_j^{k+1} - z_j^{k+1}\|_{\infty} < \epsilon \) and \(\|x_j^{k+1} - y_j^{k+1}\| < \epsilon\)
13: return \(x\).

V. Numerical Experiments
In this section, we design numerical experiments to test the proposed decoding model and algorithm. We use the experimental framework of Liu [7] in our test, and design a simulator that the \([2640, 1320]\) “Margulis” code [13] with randomly added noise transmitting over an Additive White Gaussian Noise (AWGN) channel.

Many acceleration techniques exist for ADMM [16], [17], and the \(\ell_2\)-penalized ADMM LP decoder in our comparison is in fact an accelerated version [17]. Our algorithm, however, is simply the basic ADMM without using acceleration, since our primary focus is on the effectiveness of our proposed decoding model. The ADMM parameters in our implementation are roughly tuned to ensure the algorithm achieve an acceptable performance.

A. Effectiveness of the parameter-free model
To emphasize the advantage of our new formulation, we first show for the \(\ell_2\)-penalized ADMM LP decoder how the penalty parameter value could affect the decoding accuracy and running time.

We consider the \(\ell_2\)-penalized ADMM LP decoder [7] in our comparison, which is reported to be the most effective and efficient among three penalty terms in [7]. In this experiment, we set \(\text{SNR} = 1.6dB\), and the maximum number of iterations to be 1000. The decoder is terminated whenever 200 errors have been encountered. We record the Word error rate (WER) and the running time of the \(\ell_2\)-penalized ADMM for penalty parameter value varying from 0.24 to 5.0, and depict the results in Fig. 2(a) and 2(b).

From Fig. 2(a) and Fig. 2(b), we can see that the penalty parameter for the \(\ell_2\)-penalized ADMM LP decoder plays an important role in the decoding accuracy and efficiency. The algorithm becomes extremely slowly for too small or too large penalty parameter, and the WER could also be extremely large.
We propose a parameter-free decoding model, no parameter tuning is needed for the users. It should be noticed that the WER and running time of our decoder are stable in different environments. In real situations, generally SNR cannot be accurately estimated, and it is not realistic to adjust penalty parameter in real time for broadcast transmission. These disadvantages may limit the practicability of the $\ell_2$-penalized ADMM LP decoder, while our proposed decoder does not suffer from such issue and thus is more practical in real applications.

B. WER performance

In the second experiment, we test our proposed decoder against existing standard algorithms including the penalized ADMM decoder in [7] and the BP decoder in [9]. The decoding model parameters for each decoder are tuned to achieve the best performance, and the algorithms are terminated once 200 errors are encountered. The WER of each decoder is depicted in Fig. 3.

From Fig. 3, we can have the following observations. First of all, the overall performance of our proposed $\ell_2$-box ADMM decoder achieves lower WER than all other contemporary decoders. In particular, for small SNR between 1 and 1.4, the performance of each decoder does not differ from each other very much. As the SNR turns large, the difference between decoders becomes increasingly significant, and this is especially the case when the SNR is between 1.8 and 2.2.

VI. CONCLUSION

In this paper, we have proposed a parameter-free $\ell_p$-box formulation for LDPC decoding and implemented the new formulation with an ADMM solver. The binary constraint in the binary LP decoding is replaced with the intersection of a box and an $\ell_p$ sphere. The new decoding formulation is equivalent to the original binary LP decoding, in other words, an exact reformulation. We have also developed an efficient ADMM-based algorithm to solve this newly formulated problem in a distributive manner. It should be emphasized that the proposed decoder can be easily applied to various situations with different SNR and channels since the decoding model is parameter-free. This advantage as well as the efficiency of our ADMM is demonstrated by numerical experiments.

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