Abstract: We present an implementation in the functional programming language Haskell of the PLE decomposition of matrices over division rings. We discover in our benchmarks that in a relevant number of cases it is significantly faster than the C-based implementation provided in FLINT. Describing the guiding principles of our work, we introduce the reader to basic ideas from high performance functional programming.

LINEAR algebra pervades modern algorithms. Today's multitude of applications of linear algebra has spawned an equal multitude of refinements of linear algebra. Dense (as opposed to sparse) linear algebra refers to the computation with matrices or vectors with few expected zero entries. Exact linear algebra (as opposed to approximate linear algebra [KLN96] and numerical linear algebra) refers to computation admitting no approximation error. Need for exact dense linear algebra arises from, among others, cryptography, compression, and “inner” mathematical problems. It is the backbone of symbolic and algebraic-geometric computation facilities.

Major open-source implementations of exact dense linear algebra are available within LinBox [LinBox] and Flint [Flint] (Fast LIbrary for Number Theory). LinBox covers functionality to find solutions to linear equations, to compute invariants of linear operators, and to compute various canonical forms of matrices. The focus is on computation over finite fields and the integers, and extends to rationals via a technique called rational reconstruction. The library is based on black box algorithms [KT90]. Flint offers functionality similar to LinBox, but uses a different way of implementing it. It is mostly based on classical and Strassen approaches [Str69].

Questions about vector spaces and systems of linear equations are mostly addressed via matrix factorizations. One instance of matrix factorization is the PLE decomposition of a matrix M, which in particular provides an echelon form E. In this work, we present an implementation HLinear of PLE decomposition in the functional programming language Haskell. It is competitive with Flint and in some cases outperforms it by a factor of 10. On the other hand, it enjoys typical benefits of programs written in functional languages. For instance, it opens doors to formal verification and painless distributed computation.

The HLinear code is available in the public repository

https://github.com/martinra/hlinear

1 Background

§1.1 Motivation. The need for an efficient PLE decomposition grew out of the second author’s project to compute with (Siegel) modular forms—cf. the last section of [BWR14]. Matrices arising from this application are comparatively large with 10,000 up to several 100,000 rows. To complicate matters, they have entries over number fields. On the plus side, algebraic-geometric methods show that these matrices have PLE decompositions with rather small denominators. The urgent need for parallelization and distributed computing in conjunction with the authors’ desire to formally verify as much of their future computation as possible, rendered impossible the usage of available implementations. The verification requirement, specifically, suggested use of a functional programming language.

§1.2 PLE decomposition. Let $\mathcal{R}$ be a (unital) ring. Given a matrix $M \in \text{Mat}_{m,n}(\mathcal{R})$, we say that $M = PLE$ is a PLE decomposition of $M$, if $P$ is a permutation matrix, $L$ is a lower triangular matrix with diagonal entries 1, and $E$ is a matrix...
in echelon form. Jeannerod, Pernet, and Storjohann [JPS13] explain the PLE and related decompositions in the context of rank-profiles. As an example of PLE decomposition, we record that a $4 \times 6$ matrix could allow for the following factorization:

$$ \begin{pmatrix} 1 & * & * & * & * & * \\ * & 1 & * & * & * & * \\ * & * & 1 & * & * & * \\ * & * & * & 1 & * & * \end{pmatrix} \begin{pmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{pmatrix} $$

where * is an arbitrary entry from $\mathcal{R}$ and $*'$ is a non-zero entry.

If $\mathcal{R} = K$ is a field (or a division ring) then it is possible to pass to normalized echelon forms. In this case, we allow for arbitrary non-zero entries on the diagonal of $L$, and in exchange demand that the pivot entries of $E$ be 1.

$$ \begin{pmatrix} 1 & * & * & * & * & * \\ * & 1 & * & * & * & * \\ * & * & 1 & * & * & * \\ * & * & * & 1 & * & * \end{pmatrix} \begin{pmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{pmatrix} $$

Slightly ambiguously, this decomposition is also called (normalized) PLE decomposition. To obtain a normal form of $M$ with respect to the action of invertible matrices from the left, one may proceed to the reduced echelon normal form by applying to $E$ an upper triangular matrix $U$ with diagonal entries 1. We thus obtain the PLUE decomposition associated with the previous example:

$$ \begin{pmatrix} 1 & * & * & * & * & * \\ * & 1 & * & * & * & * \\ * & * & 1 & * & * & * \\ * & * & * & 1 & * & * \end{pmatrix} \begin{pmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{pmatrix} $$

Gaussian elimination is the most classical algorithm to compute PLE decompositions. It proceeds by iterating through columns: (1) picking a non-zero element in the current column, if possible; (2) permuting the corresponding row to the top unprocessed position; (3) normalizing that row; (4) eliminating entries in the current column below that row. Despite its age, this algorithm continues to be the fundamental building block in the computation of PLE decompositions of modest-sized matrices.

One alternative to Gaussian elimination is the slice PLE decomposition (see [BH74] and, for example, [ABP11] for a recent application), which is a hierarchical approach. Splitting up $M$ into column slices $M = (M_0 \cdots M_{r-1})$, one computes $M_0 = P_L 0 E_0$ and sets $M_i = L_0^{-1} P^{−1} M_i$ for $i \geq 1$.

Then we decompose $M''_i = \{E_i, \text{ } M''_{i+1}\}$ row-wise, where the number of rows of $E_i$ equals the rank of $E$. Setting $M'' = (M''_0 \cdots M''_{r-1})$ allows to find $M'' = P''_L E''$ by recursion, and thus build the PLE decomposition of $M$ by rearranging

$$ M = P_0 L_0 E_0 \cdot \begin{pmatrix} E_1 \cdots E_r \end{pmatrix} P''_L E'' $$

This approach and its iterative counterpart have been implemented in M4RI [M4RI] and in a not-yet-released version of LinBox. Any algorithm of this flavor calls for thinking about it in terms of directed acyclic graphs (DAGs). They have not yet been applied to exact dense linear algebra, but they do appear for example in [KE14].

Given that LinBox employs rational reconstruction, we revisit multi-modular linear algebra. The key observation is that the decomposition $M = \text{PLE}$ for $M \in \text{Mat}_{m,n}(\mathbb{Q})$ can be reconstructed from its reductions modulo a large enough integer $N$, coprime to the denominators of $M$, $L$, and $E$. In practice $N$ can be chosen to be a product of distinct word-sized primes, which leads to vastly reduced coefficient size. Of course, this comes at the expense of additional reconstruction steps.

§ 1.3 Implementations of PLE decompositions.

From the three approaches to linear algebra presented above, we see that dense linear algebra requires optimization on at least two scales. First of all, it involves frequent additions and multiplications in the coefficient ring $\mathcal{R}$, and occasional divisions. Such operations are typically optimized at low level. For exact computation, this is addressed in libraries like GMP [GMP], MPIR [MPIR], and FFLAS [DGP08]. Second, the structural dependencies among the operations require optimization at high level. They are traditionally met by studying the algorithms from a theoretical point of view. Modern compilers and libraries facilitate exploitation of fusion, dependency analysis, and even term rewriting.

Other aspects that receive increasing attention are reliability, security, and correctness. They are
of considerable importance to cryptography and “inner” mathematical applications. Correctness is typically addressed by testing, which is supported by various frameworks available for major programming languages. Formal verification provides further reassurance, but it is hard to apply to today’s most popular languages due to, for example, insufficient type systems.

We have mentioned the two implementations LinBox and Flint of exact dense linear algebra. LinBox is more established and aims to be a general-purpose library; it makes heavy use of black box algorithms and rational reconstruction. Flint’s linear algebra implementation is more recent and geared primarily toward the needs of number-theoretic computations, using classical algorithms. For work over the rationals, Flint mostly performs a little bit better than LinBox. The existence and continued development of the two libraries has been very beneficial to users of computational linear algebra. However, both also suffer from certain deficits. Flint has had two “severe bugs” in the last two years affecting primality testing and gcd computation. Parts of LinBox for a certain time were excluded from the computer algebra distribution Sage, because of incorrect results or segmentation faults.

LinBox is written in C++, while Flint is written in C. We notice that Flint cannot rely on the compiler’s ability to simplify structure at a larger scale. Indeed, it focuses on low-level optimizations, and deals with high-level optimizations by hand. LinBox can and does rely on template metaprogramming for achieving a certain level of generality and structural optimization. In connection with correctness, we quote the Google style guide for C++ [Goo16]: “Avoid complicated template programming”, and reasons that:

The techniques used in template metaprogramming are often obscure to anyone but language experts. Code that uses templates in complicated ways is often unreadable, and is hard to debug or maintain.

Both C and C++ have excellent properties when it comes to program overhead; LinBox and Flint make use of them. On the other hand, neither supports the programmer with optimization of large scale structures, testing, or even verification.

§1.4 Haskell. Haskell is a functional programming language, the most popular one besides OCaml. It is successful due to, among other things, the highly-developed and aggressively optimizing Glasgow Haskell Compiler (GHC). While functional programming languages suffer the reputation of being relatively slow, recent progress on fusion [MLPJ13] and plenty of highly developed libraries allowed for implementing the highly-performant webserver Warp [YSV13], beating C code on some numerical applications [MLPJ13], and software being employed in financial industry. Haskell supports the developer by providing compositional code, term rewriting rules, and a strong type system. As a result, code reusability, testing, and verification (in connection with Coq [Coq]) are superior to any other language encountered in an industrial setting.

Haskell has important weak points: (a) it is infamous for its steep learning curve; (b) it does not have a type system as strong as, say, Agda [Nor09]; (c) it has several low hanging fruits to be optimized in its parallelization and distributed computation libraries. Regardless of these imperfections, it currently appears as the best possible choice for functional (in the sense of functional programming) implementations.

We refer the reader who is unfamiliar with Haskell to [Wik16], and illustrate code reusability in Haskell by a design pattern that we will encounter later. It is a common scheme to (i) decompose a data structure into a sequence of relatively simple data structures, and then (ii) recombine these simple structures. In Haskell, this is supported by unfolding and folding. The respective type signatures are

See the News section at flintlib.org

See tickets 6296 and 12629 at trac.sagemath.org

Examples include Barclays Capital [Fra+09], Credit Suisse [Man06], Deutsche Bank [Pol08].

In some of the code listings, we had to violate Haskell’s
That is, unfolding is based on a function that decomposes an instance of a data structure, if possible, into an easier piece of type a and a remainder, which is again of type b.

As a simple example, we formulate the extended Euclidean algorithm for non-negative integers (corresponding to the type Natural) in terms of fold and unfold. Thinking of pairs \((a, b)\) as row vectors, a single reduction step in the Euclidean algorithm can be viewed as writing \((a b) = (b r) T\) for the matrix \(T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\), where \(a = tb + r\) is the result of the division of \(a\) by \(b\). This reduction step is implemented as:

\[
\begin{align*}
\text{reduce} \ (\_, 0) &= \text{Nothing} \\
\text{reduce} \ (0, b) &= \text{Just} \ ((-1), (b, b)) \\
\text{reduce} \ (a, b) &= \text{let} \ (t, r) = \text{quotRem} \ a \ b \\
&\quad \text{in} \ \text{Just} \ (t, (b, r))
\end{align*}
\]

From a pair \((a, b)\) we thus obtain a list \([T_1, \ldots, T_r]\) such that \((a b) = (1 0) T_1 \cdots T_r\). The matrix \( (T_1 \cdots T_r)^{-1} = T_r^{-1} \cdots T_1^{-1}\) is then computed by folding via:

\[
\text{mulinv} \ (a, b, c, d) \ t = \ (b, a-t*b, d, c-t*d)
\]

The extended Euclidean algorithm on a pair \((a, b)\) can thus be cleanly written as

\[
\text{let} \ (x, \_, y, \_) = \text{foldl} \ \text{mulinv} \ (1, 0, 0, 1) \ s \\
\text{unfoldr} \ \text{reduce} \ (a, b)
\]

All intermediate steps are exposed directly to the compiler, which can optimize more aggressively.

§1.5 Implicit configuration via reflection. The configuration problem in functional programming is that data that is given on the outer level of a program needs to be accessed in the inner level. Since functional programming style strives to use pure functions—that is, to exclude side effects—this would a priori require one additional configuration parameter in all functions. For example, the two-argument function

\[
f : : a \rightarrow b \rightarrow c
\]

would be augmented to

\[
f' : : \text{cfg} \rightarrow a \rightarrow b \rightarrow c
\]

and one would need to carry the argument \text{cfg} through all function calls.

One solution to the dynamic configuration problem proposed in [KS04] is to let the type system assist. One introduces a pair of functions
data Proxy s = Proxy
reify :: a \rightarrow (\forall \ (s :: *). Reifies s a \Rightarrow Proxy s \rightarrow r)
\rightarrow r
reflect :: Reifies s a \Rightarrow Proxy s \rightarrow a

Observe that Proxy carries no runtime information, since it has one constructor without any parameter. The first argument of reify is a configuration parameter, and its second argument is a function that requires configuration. Inside that function, one can use reflect to recover the configuration parameter from an instance of Proxy. Prototypical application of this idea would be as follows:

\[
\begin{align*}
\text{import} & \ \text{Data. Reflection} \\
\text{import} & \ \text{Data.Proxy} \\
\text{reify} & \ 4 \ \text{\_} :: \text{Proxy s} \rightarrow \\
& \quad 3 + \text{reflect} \ (\text{Proxy} :: \text{Proxy s})
\end{align*}
\]

Edward Kmett’s library reflection provides a fast implementation of reflection.

2 HLinear

We have split HLinear into three packages: algebraic-structures, HFlint, and the main package HLinear. They rely on various packages authored by others, most prominently the vector package for tuned vector and array manipulation, and Edward Kmett’s reflection package. Testing relies on a combination of QuickCheck and SmallCheck, bundled conveniently in the testing framework Tasty. Benchmarking is based on Criterion.

§2.1 algebraic-structures. The new package algebraic-structures provides classes for algebraic structures ranging from magmas, groups, and actions, to rings, modules, and algebras. For example, magmas are sets together with a binary operator; no further conditions are imposed. The class
The package algebraic-structures makes it easier to implement mathematical ideas in greatest possible generality. For example, the normalized PLE decomposition can be defined for all division rings. And this is exactly the level of generality that HLinear meets.

We conclude with one vague remark. From experiences with HLinear it seems that performance of some programs (specifically HLinear's) can profit from intermediate objections that are defined at mathematical level of rigor. If true, the roots of this observation might be the compiler's ability to rearrange intermediate steps more effectively, i.e. to optimize more aggressively. It can definitely not be related to rewrite rules, which are not included in the current version of algebraic-structures.

§2.2 HFlint. HFlint is a wrapper around some parts of Flint. Specifically, it wraps integers \( \mathbb{Z} \), rationals \( \mathbb{Q} \), polynomial algebras \( \mathbb{Z}[x] \) and \( \mathbb{Q}[x] \), and finite fields \( \mathbb{F}_p \) for primes \( p \). Given the current capabilities of Flint, it is possible to extend this to number fields (via Antic [Har16]), to \( \mathbb{Q}_p \) and \( \mathbb{Z}_p \), and their finite extensions, to all finite fields \( \mathbb{F}_q \) for prime powers \( q \), and to the real and complex numbers \( \mathbb{R} \) and \( \mathbb{C} \) (via Arb [Joh13]). In light of the layout of HFlint, this extension would be feasible with rather little effort.

In Section 1.3, we have discussed low-level and high-level optimizations of linear algebra. While papers like [CSS03; MLPJ13] suggest that the performance barrier between C and Haskell for elementary operations is low or even non-existent, state-of-the-art implementations of, say, integer arithmetic (GMP, MPIR) are very hard to beat. This shall not be the topic of this paper. Consequently, we rely on Flint for fundamental arithmetic. Note that we purposely wrap \( \mathbb{Z} \), which is also represented by \( \text{Integer} \). However, \( \text{Rational} \) (built on top of \( \text{Integer} \)) cannot compete with the corresponding Flint implementation.

Flint objects are either synonyms for an elementary data type (int, long, etc.) or pointers to C structures. Functions for Flint objects may accept a context object storing information about
the type rather than an individual object. For example, rationals are usually accessed via

```c
typedef fmpq fmpq_t [1];
```

and the signature of a typical function is

```c
void fmpq_add(fmpq_t res,
    const fmpq_t op1, const fmpq_t op2);
```

with self-evident meaning of the arguments. There are no context objects attached to rationals.

Finite fields \( \mathbb{F}_p \) do have a context associated to them that keeps track of the prime \( p \). Elements of finite fields of small moduli are encoded by means of an elementary data type `mp_limb_t`, which on most common architectures resolves as `unsigned long`. A typical signature for such finite fields is

```c
mp_limb_t nmod_add(
    mp_limb_t a, mp_limb_t b, nmod_t mod)
```

The first two arguments are the summands and the third one is a pointer to a context.

The disciplined interface style of Flint allows for systematic wrapping. As mentioned above there is a context attached to finite fields. Clearly it is desirable to disallow, say, addition of elements of \( \mathbb{F}_p \) and \( \mathbb{F}_{p'} \) for \( p \neq p' \). A context parameter on the left hand side of the data type declaration takes care of this. Note that the context does not appear on the right.

```c
type FlintLimb = CULong
newtype NMod ctxProxy =
    NMod {unNMod :: FlintLimb}
```

To operate with elements of finite fields, Flint requires the context reference `nmod_t`. The implementation of addition with type signature

```c
(+) :: NMod ctx \rightarrow NMod ctx \rightarrow NMod ctx
```

can therefore be viewed as a dynamic configuration problem. We make `NMod` an instance of the class `FlintPrim`, which contains the function

```c
withFlintPrimCtx :: ReifiesFlintContext ctx ctxProxy \Rightarrow NMod ctxProxy
    \Rightarrow ( CFlintPrim NMod
        \Rightarrow \text{Ptr (CFlintCtx ctx)} \Rightarrow \text{IO b})
    \Rightarrow \text{IO b}
```

The condition `ReifiesFlintContext` is related to the reflection library and is discussed in the next subsection. The second argument corresponds to a wrapped C function. Concretely, addition could be implemented as

```c
(+)
```

```c
a b =
    withFlintPrimCtx a \$ \text{\_} \rightarrow
    withFlintPrimCtx b \$ \text{\_} \rightarrow
    nmod_add ac bc ctxtptr
```

The inner variables `ac` and `bc` arise from the data represented by `a` and `b`, but the context originates from their type via reflection. In particular, it is semantically correct to ignore the context pointer provided by the first call to `withFlintPrimCtx`, since `a` and `b` have the same type, and therefore they have the same context.

To create and employ the context of a finite field one uses `withNModContext`, whose type signature includes the condition `NFData b` for the type `b` of the result. Its first argument is the modulus of the finite field.

```c
withNModContext :: NFData b
    \Rightarrow FlintLimb
    \Rightarrow (\text{forall ctxProxy .}
        \text{ReifiesFlintContext NModCtx ctxProxy \Rightarrow Proxy ctxProxy \rightarrow b})
    \Rightarrow b
```

**Reflection and dynamic types.** The goal of this section is to explain the condition `NFData b`, which might appear unnecessary. Context objects are generally represented by a pointer to a C structure. Most commonly, one would use a `ForeignPtr`, whose finalizer frees the C structure. A typical computation in \( \mathbb{F}_7 \) might look as follows.

```c
unsafePerformIO $ do
    ctx <- newFlintContext $ NModCtxData 7
    withFlintContext ctx $ unNMod $ (NMod 3) + (NMod 4)
```

Invoking in this way an implementation based on foreign pointers and Kmett’s reflection library can and will produce segmentation faults.

The concept of reflection is mathematically sound, but is incompatible with finalizers. A priori, the inner function in the second line does not
contain any reference to the C context instance. This might make the finalizer free it before an actual reference is created by

```haskell```
reflect (Proxy :: Proxy ctx)
```

The point is that the latter call does create runtime data out of type information, which the finalizer of a ForeignPtr cannot keep track of.

Dynamic configuration via reflection is fast, since it allows us to move context information away from the element information. Instances of a hypothetical data type

```
data NMod' = NMod' NModContext FlintLimb
```

will not only double the memory footprint, but also prevent some optimizations for the elementary data type `FlintLimb`.

Dynamic configuration is also convenient, since it transparently prevents accidental combination of elements of different fields. It thus seems worth to introduce an NFDaData b condition to maintain it. Context objects will be implemented by plain `Ptr` instances, and memory allocation should be taken care of manually. The implementation of with `NModContext` in `HFlint` is

```haskell```
withNModContext n f = unsafePerformIO $ do
ctx <- newFlintContext $ NModCxData n
let h = force $ withFlintContext ctx f
seq h $ freeFlintContext ctx
return h
```

§2.3 HLinear. HLinear implements arithmetic with matrices over arbitrary rings, and PLE decomposition of matrices over division rings. It makes heavy use of the vector library, which optimizes function applications and stores elementary data types efficiently.

Matrices keep track (as runtime data) of their dimensions, and their entries as vectors of rows in the same way that Flint does. The corresponding data declaration is

```
data Matrix a =
  Matrix { nmbRows :: !Natural
    , nmbCols :: !Natural
    , rows :: Vector (Vector a)
  }
```

Matrix arithmetic is performed in a straightforward way. In particular, we do not employ Strassen multiplication.

**Gaussian elimination.** Most of the implementation is devoted to the PLE decomposition based on Gaussian elimination. We illustrate it by returning to the example in Section 1.2. If the first column of $M$ is non-zero (which is the case in Section 1.2), we can write $M$ as a product

```
M = PL(E + M')
```

such that $P$ is a permutation matrix and

```
\begin{align*}
L &= \begin{pmatrix}
  \ast & 0 & 0 & 0 \\
  \ast & 1 & 0 & 0 \\
  \ast & \ast & \ast & \ast \\
  \ast & \ast & \ast & \ast
\end{pmatrix}, \\
E &= \begin{pmatrix}
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0
\end{pmatrix}, \\
M' &= \begin{pmatrix}
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0
\end{pmatrix}.
\end{align*}
```

Suppose we have a PLE decomposition of the smaller matrix $M_0' = P_0' L_0' E_0'$. This naturally gives rise to a decomposition $M' = P' L' E'$ by (i) letting $P'$ be the permutation matrix that, when acting from the left, fixes the top row and acts on the remaining rows like $P_0'$, (ii) setting $L'$ and $E'$ to

```
L' = \begin{pmatrix}
  1 & 0 & 0 \\
  0 & 1 & 0 \\
  0 & 0 & 1
\end{pmatrix}, \\
E' = \begin{pmatrix}
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0
\end{pmatrix}.
```

We thus obtain a decomposition

```
M = PL(E + P' L' E').
```

Notice that we have $E = P' L'E$, since both $P'$ and $L'$ fix the top row when acting from the left. In particular, we obtain a PLE decomposition

```
M = PL(E + P' L' E') = PL(P' L'E + P' L' E') = PP'(P'^{-1} L')L'(E + E').
```

It is important to observe that $P'^{-1} L'$ is a lower triangular matrix.

The heart of HLinear is about effectively modeling the computation that we just described. We call a triple of matrices $(P,L,E)$ as in (2.1) a PLE hook.

**Definition 2.1.** A PLE hook of size $n$ is a triple $(P,L,E)$ of a permutation $P$ on $n$ rows, a lower triangular matrix $L$ of size $n \times n$, and a matrix $E$ with...
n rows that is a row sum of a zero matrix and a matrix in echelon form.

We say that the PLE triple \((P, L, E)\) has rank \(r\) and corank \(r'\) if (i) \(P - I_n\) is supported on rows and columns of indices \(n - r - r' < j\); (ii) \(L - I_n\) is supported on columns of indices \(n - r - r' < j \leq n - r'\); and (iii) \(E\) is supported on rows of indices \(n - r - r' < i \leq n - r'\).

**Example 2.2.** The triple of matrices

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 1 & -3 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

is a PLE hook of size 4, rank 2, and corank 1.

The corresponding data type in Haskell is

```haskell
data PLEHook a =
  PLEHook
    { _permutation :: RPermute
    , _left :: LeftTransformation a
    , _echelon :: EchelonForm a
    }
```

Data types for permutations, left transformations, and echelon forms will be explained below.

Splitting off the first non-zero column and regarding the remaining bottom right matrix as a matrix of smaller size yields a list of PLE hooks when iterating. In Haskell, we can obtain this list as

```haskell
unfoldr splitOffHook
```

based on the function

```haskell
splitOffHook :: ( DecidableZero a , DivisionRing a ) 
  => Matrix a -> Maybe (PLEHook a, Matrix a)
```

Equation (2.2) in the above discussion of PLE hooks yields a partially-defined associative product on PLE hooks. Given PLE hooks \((P_1, L_1, E_1)\) and \((P_2, L_2, E_2)\) of ranks \(r_1\), \(r_2\) and coranks \(r'_1\) and \(r'_2\), we set

\[
(P_1, L_1, E_1) \cdot (P_2, L_2, E_2) = \left( P_1 P_2, P_2^{-1} L_1 P_2 L_2, E_1 + E_2 \right)
\]

if \(r'_1 \geq r_2 + r'_2\). One verifies that this condition is satisfied for all sequences of PLE hooks that arise from unfolding a matrix with splitOffHook. As a result, we can formulate PLE decomposition of a matrix \(m\) as

```haskell
foldl (\(*\)) (firstHook nrs ncs) $
  unfold$ splitOffHook m
```

**PLE hooks.** PLE hooks consist of three elements, \(P, L,\) and \(E\). The implementation is designed for general division rings, with no particular optimizations for rational numbers.

For permutations, we rely on the library permutation, which internally makes use of IntArray.

Their action on Vector is implemented directly via invoking functionality of the vector library. Since permutations make up for only a small part of the computation, we discuss them in no more detail.

Efficient data types for \(L\) and \(E\), on the other hand, are crucial to good overall performance.

Matrices \(L\) are encoded as left transformations, whose declaration is as follows.

```haskell
data LeftTransformation a =
  LeftTransformation
    { nmbRows :: Natural
    , columns :: Vector (LeftTransformationColumn a)
    }
```

The first parameter of LeftTransformation refers to the number of rows of \(L\), which is the same as its number of columns. The second parameter is a list of columns, not necessarily exhausting all.

For example, a left transformation with 4 rows and only two columns listed will be of the form

```haskell
[ (*' 0 0 0)
  (*' 0 0 0)
  (*' 1 0 0)
  (*' 0 1 0)
]
```

Columns of left transformations keep track of their column index \(j\), which is referred to as offset (from the top). The headNonZero of a column is its \(j\)-th element. The newtype wrapper NonZero ensures that it is not zero, which over division rings...
is equivalent to being invertible. All remaining entries of a left transformation column are stored in a vector tail.

Implementing left transformations with offsets ensures that they are stored as compactly as possible. The separate saving of the column index, which would a priori be deducible from the container columns, makes some operations more localizable.

Echelon forms are stored in a way that is similar to left transformations.

```haskell
data EchelonForm a =
    EchelonForm
        { nmbRows :: Natural
        , nmbCols :: Natural
        , rows :: Vector (EchelonFormRow a) }

data EchelonFormRow a =
    EchelonFormRow
        { offset :: Natural
        , row :: Vector a }
```

We need to keep track of both the number of rows and columns of \( E \). A row of vectors has an offset as left transformation columns do.

### 3 Usage

We illustrate usage of HLinear via the computation of one example.

```haskell
import HFlin . F M P Q
import HLinear . Matrix as M
import HLinear . PLE . Decomposition as D

let m = M . fromListsUnsafe
    [[ 84 , 168 , 588 , - 252 , 336 , 49 ]
    , [ 672 , 1344 , 4704 , - 1992 , 4722 , 2552 ]
    , [ - 504 , - 1008 , - 3528 , 2100 , - 1575 , - 4998 ]
    , [ 168 , 336 , 1176 , - 168 , 1428 , - 27720 ]
    , [ 1 2 7
        - 3
        0 0 0 1
        0 0 0 0 1
        0 0 0 0 0
    ] :: Matrix F M P Q

D . toMatrices $ D . pleDecomposition m
```

Reformatting the output slightly, this yields

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
84 & 0 & 0 & 0 & 0 & 0 \\
672 & 24 & 0 & 0 & 0 & 0 \\
- 504 & 588 & - 49392 & 0 & 0 & 0 \\
168 & 336 & - 27720 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}
\]

Behind the scenes, \( D . \) pleDecomposition invokes the fold-unfold implementation of PLE decomposition. It returns a PLE decomposition object, which can be converted to a PLE hook via \( D . \) unPLEDecomposition, or directly to the triple of matrices \( (P, L, E) \) via \( D . \) toMatrices.

### 4 Performance

We compared the performance of HLinear to that of Flint via a suite of benchmarks (all run on one core of a 2.70GHz Intel Xeon E5-4650 processor).

**Increasing fractions.** The first test case considers the family of special matrices of size \( n \) whose \((i, j)\) entry is given by

\[
\frac{i^2 + 2}{(n - j)^3 + 1}.
\]

Comparison of FLINT and HLinear is given in Table 1 on page 11.

**Random matrices.** The other benchmarks use random square matrices. To accommodate our focus on matrices with bounded denominators, we generate random matrices whose denominators are products \( d_1 \cdots d_n \) for random numbers \( d_i \).

We use the following parameters:

- matrix size;
- snum: upper bound on the size of the numerators of the matrix entries (in bytes);
- nden: upper bound on the number of factors used to generate denominators of the matrix entries;
- sden: upper bound on the size of the factors used to generate denominators of the matrix entries (in bytes).

Both Flint and HLinear are run on the same random matrices. Tables 2 through 5 show results for one varying parameter at a time.

### 5 Conclusion

We have demonstrated that the implementation of Gaussian elimination in a functional programming language can compete with C implementations and even outperform them. The design of our implementation was guided by the algebraic structure of intermediate steps. In particular, we
exposed the iteration scheme of Gaussian elimination by unfolding explicitly a matrix to a vector of PLE hooks. We believe that this feature made it possible for the compiler to rearrange them more easily while optimizing the code. Potential for such rearrangement is generally advertised as a fundamental advantage of functional programming, and our example shows how it comes into effect in a practical case.

Our fold-unfold implementation of Gaussian elimination is general enough to cover all division rings. With slight modification it can be optimized for rationals (fraction free PLE) or extended to discrete valuation rings (e.g. the local ring \( \mathbb{Z}_p \)). Despite being very general it performs well in practice.

The splitting up into algebraically modeled intermediate steps also opens doors to formal verification. The main obstacle for this is the partially defined multiplication of PLE hooks. The introduction of some type parameters in conjunction with type literals would allow to remedy this, but it would lead to a heterogeneously typed list. While implementation of this list is possible without problems in Haskell, it would not profit from the extensive optimizations in the vector library. It would thus defeat our central aim to provide an implementation of linear algebra that can compete with major contemporary ones. Annotations, e.g. Liquid Haskell, might provide a usable compromise.

\[ \text{[ABP11]} \quad \text{M. Albrecht, G. Bard, and C. Pernet.} \\
\quad \text{Efficient dense Gaussian elimination over the finite field with two elements.} 2011. \]

\[ \text{[BHL74]} \quad \text{James R. Bunch and John E. Hopcroft.} \\
\quad \text{“Triangular factorization and inversion by fast matrix multiplication”.} \text{ Math. Comp. 28} (1974). \text{issn: 0025-5718.} \]

\[ \text{[BWR14]} \quad \text{J. Bruinier and M. Westerholt-Raum.} \\
\quad \text{Kudla’s Modularity Conjecture and Formal Fourier-Jacobi Series.} \\
\quad \text{arXiv:1408.4996. 2014.} \]

\[ \text{[Coq]} \quad \text{The Coq development team.} \\
\quad \text{The Coq proof assistant reference manual, version 8.4. 2014.} \]

\[ \text{[CSS03]} \quad \text{K. Claessen, M. Sheeran, and S. Singh.} \\
\quad \text{“Using Lava to design and verify recursive and periodic sorters”.} \\
\quad \text{International Journal on Software Tools for Technology Transfer 4.3 (2003).} \]

\[ \text{[DGP08]} \quad \text{J-G. Dumas, P. Giorgi, and C. Pernet.} \\
\quad \text{“Dense linear algebra over word-size prime fields: the FFLAS and FFPACK packages”.} \text{ ACM Transactions on Mathematical Software (TOMS) 35.3 (2008).} \]

\[ \text{[Flint]} \quad \text{W. Hart, F. Johansson, and S. Pancratz.} \\
\quad \text{FLINT: Fast Library for Number Theory Version 2.5.2.} \text{ http://flintlib.org. 2015.} \]

\[ \text{[Fra+09]} \quad \text{S. Frankau, D. Spinellis, N. Nassuphis, and C. Burgard.} \\
\quad \text{“Commercial Uses: Going Functional on Exotic Trades”.} \text{ J. Funct. Program. 19.1 (Jan. 2009).} \]

\[ \text{[GMP]} \quad \text{T. Granlund and the GMP development team.} \\
\quad \text{GNU MP: The GNU Multiple Precision Arithmetic Library.} \\
\quad \text{http://gmplib.org.} \]

\[ \text{[Goo16]} \quad \text{Google.} \\
\quad \text{ Google C++ style guide.} \text{ http://google.github.io/styleguide/cppguide.html. 2016.} \]

\[ \text{[Har16]} \quad \text{W. Hart.} \\
\quad \text{“Antic – Algebraic Number Theory In C”.} \\
\quad \text{https://github.com/wbhart/antic. 2016.} \]

\[ \text{[Joh13]} \quad \text{F. Johansson.} \\
\quad \text{“Arb: a C library for ball arithmetic”.} \text{ ACM Communications in Computer Algebra 47.4 (2013).} \]

\[ \text{[JPS13]} \quad \text{C.-P. Jeannerod, C. Pernet, and A. Storjohann.} \\
\quad \text{“Rank-profile revealing Gaussian elimination and the CUP matrix decomposition”.} \text{ J. Symbolic Comput. 56 (2013).} \]

\[ \text{[KE14]} \quad \text{K. Kim and V. Eijkhout.} \\
\quad \text{“A Parallel Sparse Direct Solver via Hierarchical DAG Scheduling”.} \text{ ACM Trans. Math. Softw. 41.1 (Oct. 2014).} \]

\[ \text{[KLN96]} \quad \text{V. Kreinovich, A. Lakeyev, and S. Noskov.} \\
\quad \text{“Approximate linear algebra is intractable”.} \text{ Linear Algebra Appl. 232 (1996).} \]

\[ \text{[KS04]} \quad \text{O. Kiselyov and C. Shan.} \\
\quad \text{“Functional pearl: implicit configurations--or, type classes reflect the values of types”.} \\
\quad \text{Proceedings of the 2004 ACM SIGPLAN workshop on Haskell. ACM. 2004.} \]

\[ \text{[KT90]} \quad \text{E. Kaltofen and B. M. Trager.} \\
\quad \text{“Computing with polynomials given by black boxes for their evaluations: greatest common divisors, factorization, separation of numerators and denominators”.} \text{ J. Symbolic Comput. 9.3 (1990).} \]

\[ \text{[LinBox]} \quad \text{The LinBox Group.} \\
\quad \text{ LinBox – Exact Linear computational linear algebra Version 1.3.2.} \text{http://linalg.org. 2015.} \]

\[ \text{[M4RI]} \quad \text{M. Albrecht and G. Bard.} \\
\quad \text{ The M4RI Library. http://m4ri.sagemath.org.} \\
\quad \text{The M4RI Team.} \]
Table 1: Matrices of increasing fractions

| Matrix size n | CPU time in milliseconds |
|---------------|--------------------------|
| 100           | 143                      |
| 200           | 2 111                    |
| 300           | 6 599                    |
| 400           | 26 804                   |
| 500           | 38 704                   |
| 600           | 77 866                   |
| 700           | 132 205                  |
| 800           | 229 307                  |
| 900           | 371 995                  |
| 1 000         | 541 234                  |
| 1 200         | 103 509                  |
| 1 400         | 174 117                  |
| 1 600         | 255 548                  |
| 1 800         | 370 306                  |
| 2 000         | 514 827                  |
| 2 500         | 1 078 127                |
| 3 000         | 2 329 201                |

Table 2: Varying matrix size: snum = 50, nden = 10, sden = 20

| Matrix size n | CPU time in milliseconds |
|---------------|--------------------------|
| 10            | 5                        |
| 20            | 200                      |
| 30            | 30 898                   |
| 40            | 810                      |
| 50            | 2 817                    |
| 60            | 290 978                  |
| 70            | 24                       |
| 80            | 1 265 749                |
| 90            | 2 011 192                |
| 100           | 1 897 532                |
| 110           | 5 127                    |
| 120           | 3 063 023                |
| 130           | 8 180 156                |
| 140           | 8 004 673                |
| 150           | 10 055                   |

Alexandru Ghitza
School of Mathematics and Statistics, University of Melbourne, Parkville, VIC 3010, Australia
E-mail: aghitza@alum.mit.edu
Homepage: http://aghitza.org

Martin Westerholt-Raum
Chalmers tekniska högskola och Göteborgs Universitet, Institutionen för Matematiska vetenskaper, SE-412 96 Göteborg, Sweden
E-mail: martin@raum-brothers.eu
Homepage: http://raum-brothers.eu/martin
### Table 3: Varying numerator size (in bytes):

| Size of numerator | CPU time in milliseconds |
|-------------------|--------------------------|
|                   | Flint | HLinear |
| 10                | 19    | 51      |
| 20                | 543 843 | 39 064 |
| 30                | 509 743 | 55 712 |
| 40                | 1 277 | 144 710 |
| 50                | 3 178 | 2 921   |
| 60                | 729 967 | 97 205 |
| 70                | 1 342 | 776     |
| 80                | 1 091 764 | 133 734 |
| 90                | 184 072 | 107 426 |
| 100               | 1 396 408 | 80 657 |

### Table 4: Varying number of factors in denominator:

| Factors in denominator | CPU time in milliseconds |
|------------------------|--------------------------|
|                        | Flint | HLinear |
| 5                      | 4 208 | 6 176   |
| 10                     | 3 216 | 2 961   |
| 15                     | 1 471 252 | 51 824 |
| 20                     | 2 148 879 | 200 091 |

### Table 5: Varying denominator size (in bytes):

| Size of denominator | CPU time in milliseconds |
|---------------------|--------------------------|
|                     | Flint | HLinear |
| 10                  | 189 859 | 22 582 |
| 20                  | 3 179  | 2 914 |
| 30                  | 1 112 938 | 131 965 |
| 40                  | 4 136 898 | 228 384 |
| 50                  | 5 578 141 | 140 004 |
| 60                  | 5 297 293 | 110 420 |
| 70                  | 11 836 301 | 201 168 |
| 80                  | 13 269 407 | 441 943 |