FINITE SOLVABLE GROUPS WHOSE GRUENBERG-KEGEL GRAPH HAS A CUT-SET

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ABSTRACT. Let $\Gamma(G)$ be the Grunberg-Kegel graph of a finite group $G$. We prove that if $G$ is solvable and $\sigma$ is a cut-set for $\Gamma(G)$, then $G$ has a $\sigma$-series of length at most 5, whose factors are controlled. As a consequence, we prove that if $G$ is a solvable group and $\Gamma(G)$ has a cut-vertex $p$, then the Fitting length $\ell_F(G)$ of $G$ is bounded and the bound obtained is the best possible.

Keywords: Grunberg-Kegel graph, prime graph of finite groups, cut-set, solvable groups

1. Introduction

If $G$ is a finite group, we define the Grunberg-Kegel graph $\Gamma(G)$ as follows: the vertices consist in $\pi(G)$, the set of primes that divide $|G|$, and two vertices are joined if and only if there is an element of $G$ with order $pq$. One of the first results that appeared in the context is the celebrated Grunberg-Kegel Theorem, stating that if $G$ is solvable and $\Gamma(G)$ is disconnected, then $G$ is a Frobenius or a 2-Frobenius group and there are exactly two connected components that are complete subgraphs. See Theorem 2.5 below.

If $\Gamma$ is a graph with vertices $V$ and $\sigma$ is a set, then $\Gamma - \sigma$ is graph that has vertices $V \setminus \sigma$ and two vertices of $\Gamma - \sigma$ are adjacent in $\Gamma - \sigma$ if and only if they are adjacent in $\Gamma$. We say that $\sigma$ is a pseudo cut-set of $\Gamma$ if $\Gamma - \sigma$ is disconnected. If $\Gamma$ is connected and $\sigma \subseteq V$ is a pseudo cut-set of $\Gamma$, then we call $\sigma$ a cut-set of $\Gamma$. If $\sigma = \{v\}$ is a cut-set for $\Gamma$, then $v$ is called a cut-vertex of $\Gamma$. The definition of cut-set that we have given is standard in graph theory. The notion of pseudo cut-set is introduced in order to make the proofs smooth, avoiding to distinguish the case where $\Gamma$ is disconnected. In this case, every $\sigma$ such that $\sigma \cap V = \emptyset$ is a pseudo cut-set for $\Gamma$. Note that if $\sigma$ is a cut-set for $\Gamma$, then $\sigma$ is a pseudo cut-set for $\Gamma$.

Theorem A. Let $G$ be a solvable finite group. Suppose that $\sigma \subseteq \pi(G)$ is a pseudo cut-set for $\Gamma(G)$. Then, $\Gamma(G) - \sigma$ consists of two complete connected components with vertex-sets $\pi_1$, $\pi_2$ and, up to exchanging $\pi_1$ and $\pi_2$, a normal series

$$1 \leq G_0 \leq G_1 \leq G_2 \leq G_3 \leq G$$

such that $G_0$ and $G_2/G_1$ are $\sigma$-groups, $G_1/G_0$ is a nilpotent $\pi_1(G)$-group, $G_3/G_2$ is a nilpotent $\pi_2(G)$-group and $G/G_3$ is not nilpotent only if $2 \in \pi_2(G)$ and $G/O(G) \simeq (2.S_4)^-; \text{ in this case, } \ell_F(G/G_3) = 2$.

If $G$ is a finite group and $\sigma$ is a set of primes, then $\ell_\sigma(G)$, the $\sigma$-length of $G$, is the minimal length among all the normal series for $G$ whose factors are $\sigma$-groups or $\sigma'$-groups. As a consequence of Theorem A, we have the following.
Corollary B. Let $G$ be a solvable group. Suppose that $\sigma$ is a cut-set for $\Gamma(G)$, then $\ell_2(p, G) \leq 3$. Moreover, if $\sigma$ consists of a cut-vertex $p$ of $\Gamma(G)$, then $\ell_F(G) \leq 6$ and if $G/O(G) \not\cong (2, S_4)^{-}$, then $\ell_F(G) \leq 5$. The bounds are the best possible.

The motivation of this paper is [11, Lemma 2.3], where it is studied $\ell_G(G)$ when $G$ is solvable and $\Gamma(G)$ is a 3-chain.

2. Preliminary results

Definition 2.1. If $G$ is a group, we say that $G$ is a 2-Frobenius group if there is a normal series $1 < H < K < G$ such that $K$ is a Frobenius group with kernel $H$ and $G/H$ is a Frobenius group with kernel $K/H$. We call $H$ the lower kernel and $K/H$ the upper kernel.

Lemma 2.2. Let $G$ be a Frobenius group. Then $F(G)$ is the Frobenius kernel.

Proof. Let $L$ be the Frobenius kernel of $G$. Then $L \leq F(G)$ by [4, Theorem 10.3.1, so $Z(F(G)) \leq C_G(L) \leq L$ as $L$ is the Frobenius kernel of $G$. Then $F(G) \leq C_G(Z(F(G))) \leq L$, so in fact $F(G) = L$. □

Proposition 2.3. Let $G$ be a 2-Frobenius group. Then $F(G)$ is the lower kernel and $F_2(G)/F(G)$ is the upper kernel. Moreover, the upper kernel is a cyclic group of odd order, $G/F_2(G)$ is cyclic and the lower kernel is not cyclic.

Proof. By definition, there is a normal series $1 < H < K < G$ of $G$ such that $H$ is a Frobenius kernel of $K$ and $K/H$ is a Frobenius kernel of $G/H$. By Lemma 2.2 we have that $F(G/K) = H/K$, therefore $F(G) \leq H$. By Lemma 2.2 again, we have that $H = F(K)$, therefore $F(G) \leq H \leq F(G)$, so $H = F(G)$ and $K = F_2(G)$. The remaining part of the Proposition can be found in [5, Lemma 2.1]. □

The next proposition, that is known as the "Lucido’s 3 primes Lemma".

Lemma 2.4. [9, Proposition 1] Let $G$ be a solvable group. If $p, q, r$ are distinct primes dividing $|G|$, then $G$ contains an element of order the product of two of these three primes.

The following is a version of Grunberg-Kegel Theorem whose statement is suitable for our scopes.

Theorem 2.5 (Grunberg-Kegel). Let $G$ be a solvable group. Suppose that $\sigma$ is a pseudo cut-set of $\Gamma(G)$ and let $H \in \text{Hall}_\sigma(G)$. Then $\Gamma(H) = \Gamma(G) - \sigma$ and $\Gamma(H)$ consists of two complete connected components. Moreover, one of the following holds.

1. $H$ is a Frobenius group and the vertex-set of one component of $\Gamma(H)$ consists of the primes dividing the size of a Frobenius complement of $H$.
2. $H$ is a 2-Frobenius group and the vertex-set of one component of $\Gamma(H)$ consists of the primes dividing the size of a Frobenius complement of $F_2(H)$.

Proof. Suppose that $\sigma$ is a pseudo cut-set of $G$. Observe that the vertex set of $\Gamma(G) - \sigma$ is equal to the vertex set of $\Gamma(H)$. In addition, two vertices of $\Gamma(H)$ are adjacent in $\Gamma(H)$ if and only if they are adjacent in $\Gamma(G) - \sigma$. This follows from the fact that every $\sigma'$-element is contained in some conjugate of $H$, being $G$ solvable and $H \in \text{Hall}_\sigma(G)$. Therefore, this implies that $\Gamma(G) - \sigma = \Gamma(H)$. By [11, Corollary], $\Gamma(H)$ consists of two components and either part 1 or part 2 of the theorem holds,
where the lower complement of $H$ is the Frobenius complement of $H$ when $H$ is a Frobenius group, and the complement of $F_2(H)$ when $H$ is a 2-Frobenius group. By Lucido’s three Primes Lemma 2.4, these connected components are complete subgraphs of $\Gamma(H)$. 

3. PROOF OF THEOREM 3

According to Theorem 2.5 we give the following definition.

**Definition 3.1.** Let $G$ be a solvable group and $\sigma$ be a set of primes that is a pseudo cut-set of $\Gamma(G)$. Let $H \in \text{Hall}_{\sigma}(G)$. Then, in view of the Theorem 2.5 we adopt the following definitions.

1. If $H$ is a Frobenius group, then $\pi_{2,\sigma}(G)$ consists of primes that divide the order of a Frobenius complement of $H$.
2. If $H$ is a 2-Frobenius group, then $\pi_{2,\sigma}(G)$ consists of primes that divide the order of a Frobenius complement of $F_2(H)$.

Moreover, we define, $\pi_{1,\sigma}(G) = \pi(H) \setminus \pi_{2,\sigma}(G)$. If the pseudo cut-set $\sigma$ is fixed, we write $\pi_1(G) = \pi_{1,\sigma}(G)$ and $\pi_2(G) = \pi_{2,\sigma}(G)$.

**Lemma 3.2.** Let $G$ be a solvable group and $\sigma$ be a set of primes that is a pseudo cut-set for $\Gamma(G)$. Then $\pi_1(G)$ and $\pi_2(G)$ are the vertex-sets of the connected components of $\Gamma(G) - \sigma$ and $F(G/O_{\sigma}(G))$ is a $\pi_1(G)$-group.

**Proof.** By Theorem 2.5 if $H \in \text{Hall}_{\sigma}(G)$, then $\Gamma(H) = \Gamma(G) - \sigma$ consists of two complete connected components and one of them has $\pi_2(G)$ as vertex-set. So, $\pi_1(G)$ is the vertex set of the other connected component. This is because $\pi(H) = \pi(G) \setminus \sigma = \pi_1(G) \cup \pi_2(G)$. We prove the remaining part of the lemma. Without loss of generality, we can assume that $O_{\sigma}(G) = 1$. Then $F(G)$ is a $\sigma'$-group and hence $F(G) \leq H$. This means that $F(G) \leq F(H)$. It follows from Theorem 2.5 that $F(H)$ is a $\pi_1(G)$-group, so $F(G)$ is a $\pi_1(G)$-group. \qed

The next result stands at the core of our study on Grunberg-Kegel graphs.

**Proposition 3.3.** Let $r$ be a prime, $H$ a solvable group and $V$ a faithful $GF(r)H$-module. Suppose that $\sigma$ is a pseudo cut-set for $\Gamma(HV)$ and $r \not\in \sigma$. Then, $r \in \pi_1(HV)$ and there is $K \leq H$ such that the following holds.

1. $K$ is nilpotent and $\pi(K) \leq \pi_2(HV)$.
2. $F(H/O_{\sigma}(H)) = K\langle O_{\sigma}(H) \rangle / O_{\sigma}(H)$.

**Proof.** Let $\pi_i = \pi_i(HV)$ for $i = 1, 2$. Since $V = F(HV)$ is an $r$-group, we have that $O_{\sigma}(HV) = 1$ and $r \in \pi_1$ by Lemma 3.2. By Theorem 2.5 if $U \in \text{Hall}_{\sigma}(HV)$, we have that $\Gamma(HV) - \sigma = \Gamma(U)$ consists of two complete connected components, having vertex-sets $\pi_1$ and $\pi_2$ by Lemma 3.2. Moreover, $U$ is either a Frobenius group or a 2-Frobenius group. Write $F = F(U)$ and note that $V = F$. Indeed, we have that $V \leq F$. Since $V = F(HV)$, the opposite inclusion also follows: $F = C_U(F) \leq C_U(V) \leq V$.

Consider $\tilde{H} = H/O_{\sigma}(H)$. Then $O_{\sigma}(\tilde{H}) = 1$ and $F(\tilde{H})$ is a $\sigma'$-group. Let $N$ the preimage in $H$ of $F(\tilde{H})$. Then $N/O_{\sigma}(H) = F(\tilde{H})$. Let $U_0 \in \text{Hall}_{\sigma}(HV)$ such that $U_0 \leq U$. Since $V$ is an $r$ group with $r \not\in \sigma$ and $U \in \text{Hall}_{\sigma}(HV)$, we have that $U = VU_0$. Write $K = U_0 \cap N$, so that $KV \leq U$ and $N = O_{\sigma}(H) \times K$. Observe that $K$ is nilpotent because $K \simeq F(\tilde{H})$. Since $V = F(U)$, it follows that $K \leq F_2(U)$. 

If $U$ is a 2-Frobenius group, we have that $V$ is the Frobenius kernel for $F_2(U)$ by Proposition 2.3, so $K$ is contained in a Frobenius complement of $F_2(U)$. If $U$ is a Frobenius group, then $V$ is the Frobenius kernel of $U$ by Lemma 2.2 and therefore $K$ is contained in a Frobenius complement of $U$. In any case, $K$ is a $\pi_2$-group by Definition 3.1. Note that $KO_\sigma(H)/O_\sigma(H) = F(H/O_\sigma(H))$. This concludes the proof. \hfill $\square$

**Definition 3.4.** Let $G$ be a group. We denote with $O(G)$ the largest normal subgroup of $G$ that has odd order.

**Lemma 3.5.** Let $G$ be a solvable group. Suppose that $G/O(G)$ is isomorphic to the extension of $SL_2(3)$ by a cyclic group of order $2q$ with $q$ odd and that a Sylow $2$-subgroup of $G/O(G)$ is a generalized quaternion group. Then $G/O(G)$ is isomorphic to the SmallGroup(48,28).

**Proof.** Call $H = G/O(G)$ and observe that $O(H) = 1$. Suppose that $H$ is the extension of $SL_2(3)$ by a cyclic group of order $2q$, with $q$ odd. Let $N \trianglelefteq G$ such that $H/N$ is cyclic of order $q$. Then $N$ contains a subgroup of index $2$ that is normal in $G$ and that is isomorphic to $SL_2(3)$. Moreover, a Sylow $2$-subgroup of $N$ is a generalized quaternion group. Therefore, by direct check with the software GAP, up to isomorphism there is only one such a group, namely the SmallGroup(48,28).

Note that $\text{Aut}(N) = C_2 \times S_4$. Let $R \in \text{Hall}_2(H)$, then $R$ acts on $N$ and $R/R_0 \trianglelefteq \text{Aut}(N)$, where $R_0 = C_N(R)$. Therefore, $[R : R_0] \leq 3$. Take $x \in N$ of order $3$; therefore $x$ acts non trivially on $N$ and hence $(x) \cap R_0 = 1$. It follows that $R = (x) \times R_0$. Since $R_0$ is cyclic, we have that $NR \leq C_H(R_0)$ and $R_0 \leq Z(H)$. Thus, $R_0 \leq O(H) = 1$.

**Definition 3.6.** We denote by $(2.S_4)^-$ the group SmallGroup(48,28). Following \cite{1}, we call $(2.S_4)^-$ the binary octaedral group.

We now prove Theorem 3.7 that we restate for convenience.

**Theorem 3.7.** Let $G$ be a solvable group. Suppose that $\sigma$ is a set of primes that is a pseudo cut-set for $\Gamma(G)$. Then there is a normal series

$$1 \leq G_0 \leq G_1 \leq G_2 \leq G_3 \leq G$$

such that $G_0$ and $G_2/G_1$ are $\sigma$-groups, $G_1/G_0$ is a nilpotent $\pi_1(G)$-group, $G_3/G_2$ is a nilpotent $\pi_2(G)$-group and $G/G_3$ is not nilpotent only if $2 \in \pi_2(G)$ and $G/O(G) \simeq (2.S_4)^-$; in this case, $\ell_F(G/G_3) = 2$.

**Proof.** Let $\pi_i = \pi_i(G)$ for $i = 1, 2$. Call $G_0 = O_\sigma(G)$ and $\bar{G} = (G/G_0)/\Phi(G/G_0)$. Suppose that there is a normal series $1 \leq \bar{G}_1 \leq \bar{G}_2 \leq \bar{G}_3 \leq \bar{G}$ such that $\bar{G}_1 = F(G)$ is a $\pi_1$-group, $\bar{G}_1/\bar{G}_2$ is a $\sigma$-group, $\bar{G}_3/\bar{G}_2$ is a nilpotent $\pi_2$-group and $\bar{G}/\bar{G}_3$ is not nilpotent if and only if $2 \in \pi_2$ and $\bar{G}/O(\bar{G})$ is isomorphic to $(2.S_4)^-$. Consider $G_i$ the preimage of $\bar{G}_i$ in $G$. Then, $(G_1/G_0)/\Phi(G(G_0)) = F(G/G_0)/\Phi(G/G_0)$ by \cite{3} III Satz 3.5] and it follows that $G_1/G_0 = F(G/G_0)$, that is a nilpotent $\pi_1$-group. For $i \geq 2$, it is easy to see that $G_i/G_{i-1} \simeq G_i/\bar{G}_i \simeq G_{i-1}$ and $1 \leq G_0 \leq G_1 \leq G_2 \leq G_3 \leq G$ satisfies the thesis of the theorem; in particular, $G/G_3$ is not nilpotent if and only if $\bar{G}/\bar{G}_3$ is not nilpotent. This happens if and only if $2 \in \pi_2$ and $\bar{G}/O(\bar{G}) \simeq (2.S_4)^-$. Since $\bar{G} = (G/O_\sigma(G))/(\Phi(G/O_\sigma(G)))$ and $\pi_2 = \pi(G) \setminus (\sigma \cup \pi_1)$, we have that $\bar{G}$ is the quotient of $G$ by a normal $\sigma \cup \pi_1$-group. Observe that $2 \notin \sigma \cup \pi_1$; so, we deduce that $O(\bar{G})$ is a quotient of $O(G)$ and $G/O(G) \simeq (2.S_4)^-$. We proved
that if \( \bar{G} \) possesses a series that satisfies the thesis of the Theorem, then the same is true for \( G \). Hence, it is no loss to assume \( O_\sigma(G) = 1 \), \( \Phi(G) = 1 \) and \( F(G) \) is a \( \pi_1 \)-group. By Gaschütz’s Theorem \([10, 1.12]\), \( F(G) \) has a complement \( H \) in \( G \) and \( F(G) \) is a faithful completely reducible \( H \)-module, possibly of mixed characteristic. Write \( F(G) = M_1 \times \cdots \times M_n \) as the product of irreducible \( H \)-modules, so that \( M_i \) is an elementary abelian \( r_i \)-group for \( r_i \in \pi_1 \). Call \( H_i = H/C_H(M_i) \) and \( \bar{H} = \prod H_i \). Note that \( H_i \leq \bar{H} \), since \( \bigcap H_i = C_H(M_i) = C_H(F(G)) = 1 \). The group \( M_i \) is a faithful irreducible \( H_i \)-module. Note that \( M_i H_i = G/C_H(M_i) \bigcap_{j \neq i} M_j \), so \( \Gamma(M_i H_i) \) is a subgraph of \( \Gamma(G) \). Let \( L \in \text{Hall}_{\pi_2}(H) \); since no vertex in \( \pi_2 \) is adjacent in \( \Gamma(G) \) to any vertex in \( \pi_1 \), we have that \( L \cap C_L(M_i) = 1 \). Therefore, for every \( i \), \( H_i \) contains a subgroup that is isomorphic to \( L \). In particular, \( \pi_2 \leq \pi(H_i) \). Since \( r_i \in \pi_1 \), \( H_i \) is isomorphic to an extension of the group \( C_2 \) by \( G \). By Proposition \([5, 3]\), for every \( i \), there are \( H_{i,2}, H_{i,3} \leq H_i \) such that \( H_{i,2} = O_{\pi}(H_i) \) and \( F(H_i/H_{i,2}) = H_{i,3}/H_{i,2} \). Moreover, we have that \( H_{i,3} = K_i H_{i,2} \), where \( K_i \in \text{Hall}_{\pi_2}(H_{i,3}) \) and \( K_i \) is nilpotent. Suppose that \( \sigma \) is a set of primes, \( a \)-length is well-behaved
with subgroups and quotient. Moreover, if $\ell_F(G) = n$, then $\ell_\sigma(G) \leq n - 1$. Now the first part of Corollary \[3.3\] easily follows from Theorem \[3.7\].

**Corollary 3.8.** Let $G$ be a solvable group. Suppose that $\sigma$ is a cut-set for $\Gamma(G)$, then $\ell_\sigma(G) \leq 3$. Moreover, if $\sigma$ consists of a cut-vertex $p$ of $\Gamma(G)$, then $\ell_F(G) \leq 6$ and if $G/O(G) \not\cong (2.S_4)^-$, then $\ell_F(G) \leq 5$. The bounds are the best possible.

**Proof.** By Theorem \[3.7\] there is a series $1 \leq G_0 \leq G_1 \leq G_2 \leq G_3 \leq G$ such that $G_0$ and $G_2/G_1$ are $\sigma$-groups, $G_1/G_0$ is a $\pi_1(G)$-group and $G_3/G_2$ is a $\pi_2(G)$-group. Now, $G/G_3$ is not nilpotent if and only if $G/O(G)$ is isomorphic to $(2.S_4)^-$. It is easy to see that $\ell_\sigma(G/G_3) \leq 1$. Thus, $\ell_\sigma(G) \leq 3$.

Suppose now that $\sigma = \{p\}$. Clearly $G_0$ and $G_2/G_1$ are nilpotent because they are $p$-groups. The factor $G/G_3$ is not nilpotent if and only if $G/O(G) \simeq (2.S_4)^-$. In this case, $\ell_F(G/G_3) = 2$. So $\ell_F(G) \leq 5$ except when $\ell_F(G/G_3) = 2$. In this case, $G/O(G) \simeq (2.S_4)^- \text{ and } \ell_F(G) \leq 6$.

For proving that the bounds obtained are the best possible, it suffices to assume that $\sigma = \{p\}$, where $p$ is a cut-vertex. If $G$ is the group constructed in \[4.1\] Remark 2, then $G$ is solvable, 3 is a cut-vertex of $\Gamma(G)$ and $\ell_\sigma(G) = 3$, where $\sigma = \{3\}$. Moreover, $\ell_F(G) = 6$. Observe that $G/O(G) \simeq (2.S_4)^-$. Suppose that $G/O(G) \not\simeq (2.S_4)^-$, then Theorem \[4.5\] below provides an example of a group $G$ of odd order such that $\ell_F(G) = 5$ and $\Gamma(G)$ has a cut-vertex.

**4. Examples**

**Example 4.1.** Let $G$ be a solvable group and $\sigma$ a cut-set for $G$. If $|\sigma| = 1$, then there is a bound for $\ell_F(G)$ by Corollary \[3.3\]. If $|\sigma| \geq 2$, then there is no such bound for $\ell_F(G)$. In fact, let $n \geq 2$ be a large integer. It is not difficult to find a group $G_1$ of order $p^aq^b$ with Fitting length $n$, where $p, q \geq 5$ are two primes. Consider $G_2 = S_3$, so $G_2$ is the union of two connected components that consist of the prime 2 and the prime 3. Let $G = G_1 \times G_2$, then $\{p, q\}$ is a cut-set for $\Gamma(G)$ and $\ell_F(G) = n$.

In the hypotheses of Corollary \[3.8\], if $G/O(G) \not\simeq (2.S_4)^-$, then the bound obtained is the best possible. In fact, in Theorem \[4.5\] we construct a group $G$, of odd order and arbitrarily large derived length, such that $\Gamma(G)$ has a cut-vertex and $\ell_F(G) = 5$.

Some concepts of representation theory are involved; we adopt the notation in \[7\] Ch. 9. Let $r$ be a prime, $\mathbb{R}$ be the ring of local integers for the prime $r$ (see \[7\] pag. 265) and $*: \mathbb{R} \to \mathbb{R}/M$ be the projection map of $\mathbb{R}$ on the quotient $\mathbb{R}/M$, where $M$ is a maximal ideal of $\mathbb{R}$ containing $r$. In this section, $F$ denotes the field $\mathbb{R}/M$.

**Lemma 4.2.** Let $G$ be a group and $x \in G$. Let $\chi$ be the ordinary character of $G$ afforded by the representation $\mathcal{X}: G \to GL(V)$. Then $[\chi_{\langle x \rangle}, 1_{\langle x \rangle}] = 0$ if and only if $\mathcal{X}(x)$ acts fixed-point-freely on $V$.

**Proof.** The principal character $1_{\langle x \rangle}$ appears among the irreducible constituents of $\chi_{\langle x \rangle}$ if and only if $\mathcal{X}(x)$ has one eigenvector with eigenvalue 1, namely there is a fixed point. 

Let $\mathcal{X}$ a complex representation of a group $G$ and suppose that, for every $g \in G$, $\mathcal{X}(g)$ has entries in $\mathbb{R}$. Then, following \[7\] pag. 266, if $F = \mathbb{R}/M$, we can construct
an $\mathbb{F}$-representation $\mathfrak{X}^*$ of $G$ by setting $\mathfrak{X}^*(g) = \mathfrak{X}(g)^*$, where $\mathfrak{X}(g)^*$ is the matrix obtained by applying $\ast$ to every entry of $\mathfrak{X}(g)$.

If $E \subseteq \mathbb{F}$ is a subfield and $\mathfrak{Z}$ is an $E$-representation of $G$, then $\mathfrak{Z}$ maps $G$ into a group of non-singular matrices over $E$. We may, therefore, view $\mathfrak{Z}$ as an $F$-representation of $G$. As such we denote it by $\mathfrak{Z}^F$ (see [7, pag. 144]). If $\mathfrak{X}$, $\mathfrak{Y}$ are two $G$-representations that are similar over some field, we write $\mathfrak{X} \simeq \mathfrak{Y}$.

**Lemma 4.3.** Let $\mathfrak{X}$ be an irreducible $\mathbb{C}$-representation of a group $G$. Suppose that $r$ is a prime and $r \nmid |H|$. Then, there is a finite field $E \subseteq F$, a $\mathbb{C}$-representation $\mathfrak{Y}$ similar to $\mathfrak{X}$ that takes values in $\bar{F}$ and an absolutely irreducible $E$-representation $\mathfrak{Z}$ such that $\mathfrak{Y}^* \simeq \mathfrak{Z}^F$.

**Proof.** Let $\mathfrak{X}$ be a $\mathbb{C}$-representation of a group $H$ and $\chi$ be its complex character. By [7, Theorem 15.8], there exists a $\mathbb{C}$-representation $\mathfrak{Y}$, similar to $\mathfrak{X}$, that takes values in $\bar{F}$, namely $\mathfrak{Y}(g) \in M_{\chi(1)}(\bar{F})$ for all $g \in G$. Moreover $\mathfrak{Y}^*$ is an $\mathbb{F}$-representation of $G$ and $\tilde{\chi}$ is its Brauer character. Since $r \nmid |G|$, by [7, Theorem 15.13] we have $\tilde{\chi} = \chi$ and $\tilde{\chi}$ is irreducible. Hence $\mathfrak{Y}^*$ is irreducible. The field $\mathbb{F}$ is algebraically closed over its prime field $\mathbb{F}_p$ by [7, Lemma 15.1c]), so $\mathfrak{Y}^*$ is absolutely irreducible by [7, Corollary 9.4]. Let $E \subseteq \mathbb{F}$ a splitting field for the polynomial $x^{[G]} - 1 \in \mathbb{F}_p[x]$. Note that $E$ is a finite-degree extension of the prime field of $\mathbb{F}$, therefore $E$ is finite.

For every $g \in G$, $\chi^*(g)$ is a sum of $|G|$-roots of unity, so $\chi^*(g) \in \mathbb{E}$ for every $g \in G$. By [7, Theorem 9.14] there exists an absolutely irreducible $E$-representation $\mathfrak{Z}$ of $G$ such that $\mathfrak{Z}^E \simeq \mathfrak{Y}^*$. □

**Proposition 4.4.** Let $\mathfrak{X}$ be a $\mathbb{C}$-representation for a group $G$ and $W$ the associated $\mathbb{C}[G]$-module. Then, there is a finite splitting field $E$ of $G$ and a $E[G]$-module $V$, such that $(|G|, |V|) = 1$ and

$$\dim_{\mathbb{E}} C_W(x) = \dim_{\mathbb{E}} C_V(x)$$

for every $x \in G$.

**Proof.** Let $r$ be a prime that does not divide $|G|$ and denote $\bar{R}$ the ring of local integers at the prime $r$. By Lemma 4.3, there is a $\mathbb{C}$-representation $\mathfrak{Y}$, similar to $\mathfrak{X}$, that takes values in $\bar{R}$, a finite field $E \subseteq \bar{F} = \bar{R}/M$ (where $M$ is the unique maximal ideal of $\bar{R}$) and absolutely irreducible $E$-representation $\mathfrak{Z}$ such that $\mathfrak{Y}^* \simeq \mathfrak{Z}^F$. Call $W$ the $C[G]$-module associated to $\mathfrak{X}$ and $V$ the $E[G]$-module associated to $\mathfrak{Z}$. Note that $V$ is finite since $E$ is finite, moreover $(|G|, |V|) = 1$. Let $x \in G$, the number $m = \dim_{\mathbb{E}} C_W(x)$ is the geometric multiplicity of the eigenvalue 1 of the matrix $\mathfrak{Y}(x)$, that is equal to the algebraic multiplicity of 1 of the matrix $\mathfrak{Y}(x)$. This is because the characteristic of $W$ is 0 and thus the action of $\langle x \rangle$ on $W$ is completely reducible by Maschke’s Theorem. Hence, following the proof of [7, Lemma 2.15], $\mathfrak{Y}(x)$ is diagonalizable and, therefore, algebraic and geometric multiplicities of $\mathfrak{Y}(x)$ coincide. Note that $m$, as the algebraic multiplicity of 1 in $\mathfrak{Y}(x)$, is equal to the algebraic multiplicity of the 1 in $\mathfrak{Y}^*(x)$. Since $\mathfrak{Y}^*(x)$ and $\mathfrak{Z}^F(x)$ are similar, $m$ is the algebraic multiplicity of 1 for the matrix $\mathfrak{Z}^F(x)$, that is the algebraic multiplicity of 1 in $\mathfrak{Z}(x)$. Using the same argument as above, since the characteristic of $V$ does not divide $|G|$, $m$ is the geometric multiplicity of 1 for $\mathfrak{Z}(x)$, that is equal to $\dim_{\mathbb{E}} C_V(x)$. □
Theorem 4.5. Let \( n \geq 5 \) be an integer. Then, there is a solvable group \( G \) of odd order, with derived length greater than \( n \), such that \( \ell_F(G) = 5 \) and \( \Gamma(G) \) is the graph in Figure 7.

Proof. Let \( t, q \) be two odd primes and \( T, Q \) two cyclic groups of order respectively \( t^2 \) and \( q^2 \). Let \( T_0 \leq T \) the subgroup of order \( t \) and \( Q_0 \leq Q \) the subgroup of order \( q \). Choosing \( t \) to be a prime divisor of \( q^2 - 1 \), it is no loss to assume that there is an action of \( T \) on \( Q \) that has kernel \( T_0 \). Consider \( L = Q \times T \), \( L_0 = Q_0T_0 \) and \( \bar{L} = L/L_0 \). Let \( p \) be an odd prime. By [3, Theorem 22.25], there is a \( p \)-group \( P \) of derived length \( n \) that has a faithful irreducible character \( \theta \). If \( P_0 \) is the base group of \( P \) on \( \bar{L} \), there is an action of \( L \) on \( P_0 \) and the kernel of such an action is \( L_0 \). Call \( H = P_0 \times L \), note that \( F(H) = P_0L_0 \), \( F_2(H) = P_0F(L) \) and \( F_3(H) = H \). Moreover, the derived length of \( H \) is greater than \( n \). Now, \( P_0 = \prod_{j \in \mathbb{N}} P_j \) with \( P_j \simeq P \) for every \( j \) and there is a character \( \theta \) of \( P_j \) that is isomorphic to \( \theta \). Consider \( \psi = \prod_j \theta^j \). By construction, \( \psi \) is a faithful irreducible character of \( P_0 \). Consider now two non-trivial characters \( \lambda \in \text{Irr}(Q_0) \) and \( \mu \in \text{Irr}(T_0) \) such that \( \lambda \mu \) is a faithful irreducible character of \( L_0 \). Note that \( L_0P_0 = P_0 \times Q_0 \times T_0 \), hence \( \psi \lambda \mu \in \text{Irr}(P_0L_0) \). Let \( \chi \in \text{Irr}(H \mid \psi \lambda \mu) \), it is easy to see that \( \chi \) is faithful. Indeed \( \ker \chi \cap P_0 = 1 \) since \( [\chi_{P_0}, \psi] \neq 0 \) and \( \psi \) is faithful. Moreover, if \( q \) divides \( |\ker \chi| \), we have that \( Q_0 \leq \ker \chi \), but \( [\chi_{Q_0}, \lambda] \neq 0 \) and this is impossible. Replacing \( q \) by \( t \), we have that \( t \) does not divide \( |\ker \chi| \). So, we have that \( \ker(\chi) = 1 \) and \( \chi \) is faithful. The same argument also implies that \( [\chi_{Q_0}, 1_{Q_0}] = [\chi_{T_0}, 1_{T_0}] = 0 \). Therefore, by Lemma 4.2, if \( \chi \) is a representation for \( H \) that affords \( \chi \) and \( x \) is an element of order \( t \) or \( q \), then \( x \) is contained in of either \( T_0 \) or \( Q_0 \) and \( \chi(x) \) acts fixed point freely on \( W \), the \( C[H] \)-module associated to \( \chi \). Let \( r \) be an odd prime such that \( r \mid |H| \). By Proposition 4.4, there is a finite field \( \mathbb{E} \) of characteristic \( r \) and a finite \( \mathbb{E}[H] \)-module \( V \) such that, for every \( x \in H \)

\[
\dim_{\mathbb{C}} C_W(x) = \dim_{\mathbb{E}} C_V(x).
\]

Note that, since \( W \) is faithful, \( V \) is faithful. Moreover, an element \( x \in H \) acts fixed-point-freely on \( V \) whenever it does on \( W \). So, every element in \( H \) of order \( t \) or \( q \) acts fixed-point-freely on \( V \) and \( HV \) does not contain any element of order \( tr \) or \( qr \).

On the other hand, the subgroup \( P_0 \) has an elementary abelian subgroup of order \( p^2 \). Hence the action of \( P_0 \) on \( V \) is not regular by [3, Theorem 10.3.1]. So, there is one element of order \( rp \). In addition, in \( HV \) there are elements of order \( tp, qp \) and \( tq \) since \( F(H) = P_0L_0 \). This means that \( p \) is a cut-vertex for the connected graph \( \Gamma(HV) \). Note that \( F(HV) = V \) and \( HV \) has Fitting length 4. Now consider \( C_p \) a group of order \( p \) and let \( G = C_p \ltimes (HV) \). Note that \( G \) has odd order, \( \ell_F(G) = 5 \) and that \( G \) has derived length greater than \( n \). \( \square \)
The above theorem shows that bound for $\ell_F(G)$ obtained in Corollary 3.8 is the best possible and that is independent from the derived length of the group.

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