A Full Determination of the Neutrino Mass Spectrum from Two-zero Textures of the Neutrino Mass Matrix

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Abstract

We show that it is possible to fully determine the neutrino mass spectrum from two-zero textures of the neutrino mass matrix. As a consequence, definite predictions can be obtained for the neutrinoless double beta decay.
In the flavor basis where the charged lepton mass matrix is diagonal, a phenomenological
analysis of two-zero textures and Majorana CP-violating phases of the neutrino mass matrix
has recently been carried out by the author [1]. Seven of such textures were found to
be in accord with current experimental data on atmospheric [2], solar [3] and reactor [4]
neutrino oscillations. A classification of two-zero textures of the neutrino mass matrix was
originally done by Frampton, Glashow and Marfatia [5]. In two follow-up works, Frampton,
Oh and Yoshioka [6] have proposed a model with three Higgs triplets to gain texture zeros
of the neutrino mass matrix; and Kageyama, Kaneko, Shimoyama and Tanimoto [7] have
incorporated those phenomenologically acceptable textures with the seesaw mechanism.

As an addendum to Ref. [1], the present letter aims to show that it is actually possible
to fully determine the neutrino mass spectrum from two-zero textures of the neutrino mass
matrix. This important point was not observed in Ref. [1] and Refs. [5, 6, 7], where only
the ratios of three neutrino masses were calculated. Given the neutrino mass spectrum, a
definite prediction for the neutrinoless double beta decay can be made. Taking account of
current experimental data, which remain rather rough, we illustrate the typical magnitudes
of neutrino masses and that of the neutrinoless double beta decay for each of the seven
two-zero textures of the neutrino mass matrix.

The Majorana neutrino mass matrix can be written as

\[
M = V \begin{pmatrix}
m_1 & 0 & 0 \\
0 & m_2 & 0 \\
0 & 0 & m_3
\end{pmatrix} V^T
\]

in the flavor basis where the charged lepton mass matrix is diagonal, where \( m_i \) (for \( i = 1, 2, 3 \))
are physical neutrino masses and \( V \) stands for the lepton flavor mixing matrix linking the
neutrino mass eigenstates \( (\nu_1, \nu_2, \nu_3) \) to the neutrino flavor eigenstates \( (\nu_e, \nu_\mu, \nu_\tau) \). Following
Ref. [1], we parametrize \( V \) as

\[
V = \begin{pmatrix}
c_x c_z & s_x c_z & s_z \\
-c_x s_y s_z - s_x c_y e^{-i\delta} & -s_x s_y s_z + c_x c_y e^{-i\delta} & s_y c_z \\
-c_x c_y s_z + s_x s_y e^{-i\delta} & -s_x c_y s_z - c_x s_y e^{-i\delta} & c_y c_z
\end{pmatrix}
\begin{pmatrix}
e^{i\rho} & 0 & 0 \\
0 & e^{i\sigma} & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

where \( s_x \equiv \sin \theta_x \), \( c_y \equiv \cos \theta_y \), and so on. In this “standard” parametrization of \( V \), the
Dirac phase \( \delta \) controls the strength of CP violation in normal neutrino oscillations, while
the Majorana phases \( \rho \) and \( \sigma \) are relevant to the neutrinoless double beta decay [8]. Without
loss of generality, three mixing angles \( (\theta_x, \theta_y, \theta_z) \) can all be arranged to lie in the first
quadrant. Arbitrary values are possible for three CP-violating phases \( (\delta, \rho, \sigma) \). Note that \( M \)
is symmetric, thus it consists totally of six independent complex entries. Assuming two of
them to vanish, one may figure out fifteen distinct textures of \( M \). Only seven textures, as
listed in Table 1, were found to be compatible with current experimental data and empirical
hypotheses [1, 5]. Their predictions for the ratios of three neutrino masses,

\[
\xi \equiv \frac{m_1}{m_3}, \quad \zeta \equiv \frac{m_2}{m_3},
\]

are quoted from Ref. [1] and listed in the same table. Our present purpose is to determine
\( m_1, m_2 \) and \( m_3 \) separately, so as to fully fix the neutrino mass spectrum.
Given $\xi$ and $\zeta$, the absolute values of three neutrino masses can be extracted from the measured mass-squared differences of solar and atmospheric neutrino oscillations (i.e., $\Delta m^2_{\text{sun}}$ and $\Delta m^2_{\text{atm}}$). Because current Super-Kamiokande and SNO data strongly support the hypothesis that solar and atmospheric neutrino oscillations are dominated respectively by $\nu_e \rightarrow \nu_\mu$ and $\nu_\mu \rightarrow \nu_\tau$ transitions [2, 3], one may simply define $\Delta m^2_{\text{sun}}$ and $\Delta m^2_{\text{atm}}$ as

$$\begin{align*}
\Delta m^2_{\text{sun}} &\equiv |m_2^2 - m_1^2|, \\
\Delta m^2_{\text{atm}} &\equiv |m_3^2 - m_2^2|.
\end{align*} \quad (4)$$

In Refs. [1, 5], the large-angle MSW solution to the solar neutrino problem has been taken into account. Thus $R_\nu \equiv \Delta m^2_{\text{sun}}/\Delta m^2_{\text{atm}} \sim O(10^{-2})$ has been used as a crucial criterion to single out the textures of $M$ which are phenomenologically favored. From Eqs. (3) and (4), it is straightforward to obtain

$$m_3 = \sqrt{\Delta m^2_{\text{atm}}/|\xi^2 - 1|}, \quad (5a)$$

or equivalently

$$m_3 = \sqrt{\Delta m^2_{\text{sun}}/|\xi^2 - \zeta^2|}. \quad (5b)$$

Once $m_3$ is determined from Eq. (5a) or (5b), the magnitudes of $m_1$ and $m_2$ can be fixed by use of Eq. (3) or with the help of the formulas presented in Ref. [9].

To be specific, let us calculate $m_3$ for each of the seven textures of $M$ listed in Table 1. We use $\Delta m^2_{\text{atm}} \approx 3 \times 10^{-3}$ eV$^2$ [2] and $\Delta m^2_{\text{sun}} \approx 5 \times 10^{-5}$ eV$^2$ [10] as typical inputs. Note that the number of $\Delta m^2_{\text{sun}}$ results from the best global fit of present solar neutrino oscillation data (including the latest SNO neutral current measurement [3]) in the large-angle MSW mechanism, and the corresponding value of $\theta_x$ is $\theta_x \sim 30^\circ$. Furthermore, $\theta_y \sim 40^\circ$ and $\theta_z \sim 5^\circ$ are typically taken, although the latter may be much smaller. We assume the unknown CP-violating phase $\delta$ to be around $90^\circ$ in most cases. As the uncertainty associated with $\Delta m^2_{\text{atm}}$ is expected to be smaller than that associated with $\Delta m^2_{\text{sun}}$, we choose to use Eq. (5a) instead of Eq. (5b) in the calculation of $m_3$.

**Pattern A$_1$:** Because of $s_z \ll 1$ and $t_x \sim t_y$, $\zeta^2 \ll 1$ is naturally anticipated. Therefore we obtain

$$m_3 \approx \sqrt{\Delta m^2_{\text{atm}}} \approx 5.5 \times 10^{-2} \text{ eV}. \quad (6)$$

Using the typical inputs $\theta_x = 30^\circ$, $\theta_y = 40^\circ$ and $\theta_z = 5^\circ$, we arrive at

$$m_1 \approx 2.3 \times 10^{-3} \text{ eV}, \quad m_2 \approx 7.0 \times 10^{-3} \text{ eV}. \quad (7)$$

This quasi-hierarchical spectrum of neutrino masses implies that it is in practice impossible to detect the neutrinoless double beta decay.

**Pattern A$_2$:** Once again $\zeta^2 \ll 1$ holds. Then we obtain the same value of $m_3$ as that given in Eq. (6). Using the same inputs as above, we find

$$m_1 \approx 3.3 \times 10^{-3} \text{ eV}, \quad m_2 \approx 1.0 \times 10^{-2} \text{ eV}. \quad (8)$$

One can see that the neutrino mass spectra of Patterns A$_1$ and A$_2$ are quite similar.
Pattern B\(_1\):  With the help of Eq. (5a), we obtain

\[
m_3 \approx \sqrt{\Delta m_{\text{atm}}^2 / |t_y^4 - 1|} \approx 7.7 \times 10^{-2} \text{ eV ,}
\]

where \(\theta_y = 40^\circ\) has been used. In the lowest-order approximation, we have

\[
m_1 \approx m_2 \approx 5.4 \times 10^{-2} \text{ eV .}
\]

This quasi-degenerate spectrum of neutrino masses might have useful hints at possible flavor symmetries and their breaking schemes, which are expected to be responsible for the generation of lepton masses [11] and associated with the origin of leptogenesis [12]. It is also worth mentioning that the values of \(m_i\) in Eqs. (9) and (10) are compatible with the present direct-mass-search experiments [13], in particular for the electron neutrino.

Pattern B\(_2\):  In analogy to Eq. (9), \(m_3\) reads as

\[
m_3 \approx t_y^2 \sqrt{\Delta m_{\text{atm}}^2 / |t_y^4 - 1|} \approx 5.4 \times 10^{-2} \text{ eV ,}
\]

where \(\theta_y = 40^\circ\) has been used. To lowest order, we obtain

\[
m_1 \approx m_2 \approx 7.7 \times 10^{-2} \text{ eV .}
\]

We see that the neutrino mass spectra of Patterns B\(_1\) and B\(_2\) have much similarity too.

Pattern B\(_3\):  To lowest order, the neutrino mass spectrum in this pattern is identical to that in Pattern B\(_1\).

Pattern B\(_4\):  To lowest order, the neutrino mass spectrum in this pattern is identical to that in Pattern B\(_2\). We have seen that the phenomenological consequences of Patterns B\(_1\), B\(_2\), B\(_3\) and B\(_4\) are nearly the same [1] [4]. It is particularly difficult to distinguish between Patterns B\(_1\) and B\(_3\), or between Patterns B\(_2\) and B\(_4\).

Pattern C:  In the lowest-order approximation, we obtain

\[
m_3 \approx |t_{2y}|s_z \sqrt{\Delta m_{\text{atm}}^2 / t_x |t_x + 2t_{2y}s_z c_\delta|} .
\]

One can see that the magnitude of \(m_3\) depends not only upon three mixing angles (\(\theta_x, \theta_y, \theta_z\)) but also upon the Dirac phase \(\delta\). Therefore a careful analysis of the parameter space is required for Pattern C [14], so as to fit current experimental data and to fix the allowed range of \(m_3\). The values of \(\delta\) and \(\theta_z\) are unfortunately unknown at present. Just for the purpose of illustration, we typically take \(\theta_x = \theta_y = 44.8^\circ, \theta_z = 5^\circ\) and \(\delta = 90^\circ\). In this special case, one may get \(R_\nu \approx 0.03\) – a correct order of \(\Delta m_{\text{sun}}^2 / \Delta m_{\text{atm}}^2\) [1]. Then we arrive at

\[
m_3 \approx \frac{t_{2y}s_z}{t_x} \sqrt{\Delta m_{\text{atm}}^2} \approx 0.7 \text{ eV .}
\]

Using the same inputs, we get \(m_1 \approx m_2 \approx m_3 \approx 0.7 \text{ eV}\) to a high degree of accuracy. This result indicates that three neutrino masses are essentially degenerate, and their magnitude can be of \(O(1)\) eV. Therefore it is rather sensitive to the neutrinoless double beta decay.
Indeed two-zero textures of the neutrino mass matrix allow us to obtain definite predictions for the neutrinoless double beta decay, whose effective mass term is a simple function of neutrino masses and flavor mixing parameters:

\[
\langle m \rangle_{ee} = m_3 |V_{e1}^2 \xi + V_{e2}^2 \zeta + V_{e3}^2|.
\]

Using the parametrization of \( V \) in Eq. (2) and the expressions of \( \xi \) and \( \zeta \) in Table 1, we can calculate \( \langle m \rangle_{ee} \) for each of the seven patterns of \( M \). The approximate analytical results are listed in Table 2. Some comments are in order.

1. \( \langle m \rangle_{ee} \approx 0 \) holds for Patterns A\(_1\) and A\(_2\). This is obviously true, as \( M_{ee} = 0 \) has been taken in both patterns.

2. The sizes of \( \langle m \rangle_{ee} \) in Patterns B\(_1\) and B\(_3\) are essentially identical: \( \langle m \rangle_{ee} \approx m_1 \approx m_2 \approx 5.4 \times 10^{-2} \) eV for \( \theta_y = 40^\circ \). So are the sizes of \( \langle m \rangle_{ee} \) in Patterns B\(_2\) and B\(_4\): \( \langle m \rangle_{ee} \approx m_1 \approx m_2 \approx 7.7 \times 10^{-2} \) eV for \( \theta_y = 40^\circ \).

3. If \( \theta_x = \theta_y = 44.8^\circ, \theta_z = 5^\circ \) and \( \delta = 90^\circ \) are typically taken, one will arrive at \( \langle m \rangle_{ee} \approx m_1 \approx m_2 \approx m_3 \approx 0.7 \) eV for Pattern C\(_1\). This result is apparently consistent with the alleged evidence for the neutrinoless double beta decay\( ^{15} \), 0.05 eV \( \leq \langle m \rangle_{ee} \leq 0.84 \) eV, at the 95% confidence level. Nevertheless, it might be premature to take this measurement too seriously\( ^{19} \). To be more conservative, we take account of the relatively reliable experimental upper bound \( \langle m \rangle_{ee} < 0.35 \) eV (at the 90% confidence level\( ^{17} \)) and find that the typical result \( \langle m \rangle_{ee} = 0.7 \) eV is actually disfavored. It should be noted, however, that there does exist the appropriate parameter space for Pattern C\(_3\), in which both \( R_\nu \sim \mathcal{O}(10^{-2}) \) and \( \langle m \rangle_{ee} < 0.35 \) eV can be satisfied.

As pointed out in Ref.\( ^{1} \), the seven patterns of \( M \) can be classified into three distinct categories: A (with A\(_1\) and A\(_2\)), B (with B\(_1\), B\(_2\), B\(_3\) and B\(_4\)), and C. A definite measurement of the neutrinoless double beta decay at the level \( \langle m \rangle_{ee} \sim \mathcal{O}(0.01) \) eV to \( \mathcal{O}(0.1) \) eV will rule out category A. On the other hand, more precise data of neutrino oscillations will help reduce the uncertainties associated with \( \langle m \rangle_{ee} \) predicted in categories B and C. It might even be possible to distinguish between categories B and C (and therefore discard one of them), if the magnitude of \( \langle m \rangle_{ee} \) is experimentally determined in the future.

In summary, we have shown that a full determination of the neutrino mass spectrum is indeed possible from two-zero textures of the neutrino mass matrix \( M \). This important observation indicates that two-zero textures of \( M \) have much more predictability than previously expected. In particular, one can get definite predictions for the neutrinoless double beta decay. We hope that a variety of proposed precision measurements of neutrino oscillations and lepton-number-violating processes may finally allow us to pin down the unique texture of lepton mass matrices.

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\(^{1}\)At this point we notice that there is a typing error associated with \( |M_{ee}| \) in Eq. (29) of Ref.\( ^{1} \). The correct result should be \( |M_{ee}| \approx m_3 \sqrt{1 - 4c_3/(t_{2x}t_{2y}s_z) + 4/(t_{2x}^2 t_{2y}^2 s_z^2)} \).
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Table 1: Seven patterns of the neutrino mass matrix $M$ with two independent vanishing entries, which were found to be in accord with current experimental data and empirical hypotheses [1, 2]. Analytical results for the neutrino mass ratios $\xi \equiv m_1/m_3$ and $\zeta \equiv m_2/m_3$ are given in terms of the flavor mixing parameters $\theta_x, \theta_y, \theta_z$ and $\delta$ [1], where $t_x \equiv \tan \theta_x$, $t_{2y} \equiv \tan 2\theta_y$, $s_z \equiv \sin \theta_z$, $s_{2x} \equiv \sin 2\theta_x$, $c_\delta \equiv \cos \delta$ and so on.

| Pattern | Texture of $M$ | Results of $\xi$ and $\zeta$ |
|---------|----------------|--------------------------------|
| A₁      | $\begin{pmatrix} 0 & 0 & x \\ 0 & x & x \\ x & x & x \end{pmatrix}$ | $\xi \approx t_xt_ys_z$, $\zeta \approx \frac{t_y}{t_x} s_z$ |
| A₂      | $\begin{pmatrix} 0 & x & 0 \\ x & x & x \\ 0 & x & x \end{pmatrix}$ | $\xi \approx \frac{t_x}{t_y} s_z$, $\zeta \approx \frac{1}{t_xt_y} s_z$ |
| B₁      | $\begin{pmatrix} x & 0 & x \\ 0 & x & 0 \\ 0 & x & x \end{pmatrix}$ | $\xi \approx \zeta \approx t_y^2$, $\zeta - \xi \approx \frac{4 s_z c_\delta}{s_{2x}s_{2y}}$ |
| B₂      | $\begin{pmatrix} x & 0 & x \\ x & x & 0 \\ 0 & x & x \end{pmatrix}$ | $\xi \approx \zeta \approx \frac{1}{t_y^2}$, $\zeta - \xi \approx \frac{4 s_z c_\delta}{s_{2x}s_{2y}}$ |
| B₃      | $\begin{pmatrix} x & 0 & x \\ 0 & x & 0 \\ x & x & 0 \end{pmatrix}$ | $\xi \approx \zeta \approx t_y^2$, $\zeta - \xi \approx \frac{4t_y^2 s_z c_\delta}{s_{2x}s_{2y}}$ |
| B₄      | $\begin{pmatrix} x & x & 0 \\ x & x & 0 \\ 0 & 0 & 0 \end{pmatrix}$ | $\xi \approx \zeta \approx \frac{1}{t_y^2}$, $\zeta - \xi \approx \frac{4 s_z c_\delta}{s_{2x}s_{2y}t_y^2}$ |
| C       | $\begin{pmatrix} x & x & x \\ x & 0 & x \\ x & x & 0 \end{pmatrix}$ | $\xi \approx \sqrt{1 - \frac{2c_\delta}{t_xt_y s_z} + \frac{1}{t_x^2 t_y^2 s_z^2}}$, $\zeta \approx \sqrt{1 + \frac{2t_x c_\delta}{t_y s_z} + \frac{t_x^2}{t_y^2 s_z^2}}$ |
Table 2: Seven patterns of the neutrino mass matrix $M$ with two independent vanishing entries, and their predictions for $\langle m \rangle_{ee}$ of the neutrinoless double beta decay, in which $t_x \equiv \tan \theta_x$, $t_{2y} \equiv \tan 2\theta_y$, $s_z \equiv \sin \theta_z$, $c_{\delta} \equiv \cos \delta$ and so on.

| Pattern | Texture of $M$ | Result of $\langle m \rangle_{ee}$ |
|---------|----------------|-------------------------------------|
| A1      | $\begin{pmatrix} 0 & 0 & \times \\ 0 & \times & \times \\ \times & \times & \times \end{pmatrix}$ | 0 |
| A2      | $\begin{pmatrix} 0 & \times & 0 \\ \times & \times & \times \\ 0 & \times & \times \end{pmatrix}$ | 0 |
| B1      | $\begin{pmatrix} \times & \times & 0 \\ \times & 0 & \times \\ 0 & \times & \times \end{pmatrix}$ | $t_y^2 \sqrt{\frac{\Delta m^2_{\text{atm}}}{|t_y^4 - 1|}}$ |
| B2      | $\begin{pmatrix} \times & 0 & \times \\ 0 & \times & \times \\ \times & \times & 0 \end{pmatrix}$ | $\sqrt{\frac{\Delta m^2_{\text{atm}}}{|t_y^4 - 1|}}$ |
| B3      | $\begin{pmatrix} \times & 0 & \times \\ 0 & 0 & \times \\ \times & \times & \times \end{pmatrix}$ | $t_y^2 \sqrt{\frac{\Delta m^2_{\text{atm}}}{|t_y^4 - 1|}}$ |
| B4      | $\begin{pmatrix} \times & \times & 0 \\ \times & \times & \times \\ 0 & \times & \times \end{pmatrix}$ | $\sqrt{\frac{\Delta m^2_{\text{atm}}}{|t_y^4 - 1|}}$ |
| C       | $\begin{pmatrix} \times & \times & \times \\ \times & 0 & \times \\ \times & \times & 0 \end{pmatrix}$ | $\frac{|t_{2y}| s_z}{t_x} \sqrt{\frac{\Delta m^2_{\text{atm}}}{|t_x + 2t_{2y}s_zc_{\delta}|}} \left[ 1 - \frac{4c_{\delta}}{t_{2x}t_{2y}s_z} + \frac{4}{t_{2x}^2t_{2y}^2s_z^2} \right]$ |