Finite-approximate controllability of evolution systems via resolvent-like operators

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5 Jan 2018

Abstract

In this work we extend a variational method to study the approximate controllability and finite dimensional exact controllability (finite-approximate controllability) for the semilinear evolution equations in Hilbert spaces. We state a useful characterization of the finite-approximate controllability for linear evolution equation in terms of resolvent-like operators. We also find a control so that, in addition to the approximate controllability requirement, it ensures finite dimensional exact controllability. Assuming the approximate controllability of the corresponding linearized equation we obtain sufficient conditions for the finite-approximate controllability of the semilinear evolution equation under natural conditions. The obtained results are generalization and continuation of the recent results on this issue. Applications to heat equations are treated.

1 Introduction

Controllability is one of the basic qualitative concepts in modern mathematical control theory that play an important role in deterministic and stochastic control theory. From mathematical point of view, exact and approximate controllability problems should be distinguished. Exact controllability enables to steer the system to an arbitrary final state while approximate controllability means that the system can be steered to an arbitrary small neighborhood of final state, and very often approximate controllability is completely adequate in applications. If the semigroup associated with the system is compact, the controllability operator is also compact, and therefore the inverse fails to exist. Hence, the concept of exact controllability is very strong and feasibility is limited; approximately controllability is a weaker concept that is entirely appropriate in practice. We would like to mention some interesting works: Triggiani (22), (23), Bashirov and Mahmudov (3), Yamamoto and Park (25), Naito (19), Zhou (28), (29), Seidman (30), Li and Yong (31), Mahmudov (4), (17). Also, there are many papers on
the approximate controllability of the various types of nonlinear systems under different conditions (see (3)-(31) and references therein).

In this paper we will study a stronger version of controllability concept that is referred to as the finite-approximate controllability problem. It should be stressed out that in the context of abstract linear control systems, finite-approximate controllability problem is a consequence of approximate one, see (7). So these two concepts are equivalent. However in the nonlinear context they are not equivalent, see (6). Recently finite-approximate controllability result for abstract semilinear evolution equations with compact $C_0$-semigroup is presented in (8).

In this paper, we investigate simultaneous approximate and finite-dimensional exact controllability (finite-approximate controllability) of the following semilinear evolution system:

\[
\begin{aligned}
\begin{cases}
y'(t) = A y(t) + B u(t) + f(t, y(t)) + g(t, y(t)), & t \in [0, T], \\
y(0) = y_0,
\end{cases}
\end{aligned}
\]  

where the state variable $y(\cdot)$ takes values in the Hilbert space $X$, $A : D(A) \subset X \to X$ is a family of closed and bounded linear operators generating a strongly continuous semigroup $\U : [0, T] \to L(X)$, where the domain $D(A) \subset X$ which is dense in $X$, the control function $u(\cdot)$ is given in $L^2([0, T], U)$, $U$ is a Hilbert space, $B$ is a bounded linear operator from $U$ into $X$, $f, g : [0, T] \times X \to X$ are given functions satisfying some assumptions specified later and $y_0$ is an element of the Hilbert space $X$.

We present the following definition of mild solutions of system (1).

**Definition 1** $y \in C([0, T], X)$ is called a mild solution of (1) if

\[
y(t) = \U(t) x_0 + \int_0^t \U(t - s) [B u(s) + f(s, y(s)) + g(s, y(s))] ds, \quad t \in [0, T].
\]

Following (1), we define the controllability concepts and controllability operator for the system (1).

**Definition 2** For the system (1), we define the following concepts:

(a) A controllability operator is the bounded linear operator $B_T^0 : L^2([0, T], U) \to X$ defined by

\[
L_T^0 u := \int_0^T \U(T - s) B u(s) ds;
\]

(b) Control system (1) is approximately controllable on $[0, T]$, if for every $y_0, y_f \in X$, and for every $\varepsilon > 0$, there exists a control $u \in L^2([0, T], U)$ such that the mild solution $y$ of the Cauchy problem (1) satisfies $y(0) = y_0$ and $\|y(T) - y_f\| < \varepsilon$.

(c) Let $M$ be a finite dimensional subspace of $X$ and let us denote by $\pi_M$ the orthogonal projection from $X$ into $M$. Control system (1) is finite-approximately
controllable on \([0,T]\), if for every \(y_0, y_f \in \mathcal{X}\), and for every \(\varepsilon > 0\), there exists a control \(u \in L^2([0,T], U)\) such that the mild solution \(y\) of the Cauchy problem satisfies \(y(0) = y_0\) and \(\|y(T) - y_f\| < \varepsilon\) and \(\pi_M y(T) = \pi_M y_f\).

(d) The controllability Gramian is defined by

\[
\Gamma^T_0 := L^T_0 (L^T_0)^* = \int_0^T \mathcal{U}(T-s) BB^* \mathcal{U}^* (T-s) \, ds : \mathcal{X} \to \mathcal{X}.
\]

The rest of this paper is organized as follows. In section 2, we will present some results on properties of positive linear compact operators depending on parameter. We define a resolvent-like operators and give necessary and sufficient conditions for finite-approximate controllability of linear evolution equations. Section 3 is divided into two subsections. In subsection 3.1, using a control defined by resolvent-like operator we define a control operator \(\Theta_\varepsilon\) and show existence of fixed points. In subsection 3.2 we prove our main result on finite-approximate controllability of semilinear evolution system. Finally, we present two examples to demonstrate our main results in section 4.

Several comments are in order:

(i) The variational approach developed in this paper is somewhat different from those applied in the literature and provide another new method to prove simultaneous approximate and exact finite-dimensional controllability for (1).

(ii) The proof of the main result obtained in this paper are based on quasi linearization of semilinear problem and on viewing the finite-approximate controllability problem as a limit of optimal control problems. It combines the methods used in papers (6), (17) and (4).

(iii) Requirement of exactly controlling the finite-dimensional projection introduces new difficulties. To overcome it, we present criteria for finite-approximate controllability of linear systems in terms of resolvent-like operators, and study convergence properties of approximating resolvent-like operators.

(iv) The variational approach developed here is constructive since approximating control can be given explicitly. It is interesting both from the theoretical and the numerical point of view.

(v) One may expect the results of this paper to hold for a class of problems governed by different type of evolution systems such as Caputo fractional differential equations (FDEs), Riemann-Liouville FDEs, stochastic FDEs, Sobolev type FDEs and so on.

2 Finite-approximate controllability of linear systems

In the present section we investigate the finite-approximate controllability of linear evolution system:

\[
\begin{cases}
  y'(t) = Ay(t) + Bu(t), & t \in [0, T], \\
  y(0) = y_0.
\end{cases}
\]

(3)
Finite-approximate controllability concept was introduced in [6]. This property not only says that the distance between \( y(T) \) and the target \( y_f \) is small but also that the projections of \( y(T) \) and \( y_f \) over \( M \) coincide.

It is known that the resolvent operator \( (\varepsilon I + \Gamma T)^{-1} \) is useful in studying the controllability properties of linear and semilinear systems, see [3], [4]. In this respect, we state a useful characterization of the finite-approximate controllability for \( 3 \) in terms of resolvent-like operator. We show that for the linear evolution system \( 3 \) approximate controllability on \([0,T]\) is equivalent to the finite-approximate controllability on \([0,T]\).

Moreover, we present necessary and sufficient conditions for the finite-approximate controllability of linear evolution systems in Hilbert spaces in terms of resolvent-like operators.

Firstly, we present three results on the resolvent operators.

**Theorem 3** Assume that \( \Gamma(\varepsilon), \Gamma: X \to X, \varepsilon > 0 \), are linear positive operators such that

\[
\lim_{\varepsilon \to 0^+} \| \Gamma(\varepsilon) h - \Gamma h \| = 0, \ h \in X.
\]

Then for any sequence \( \{\varepsilon_n > 0\} \) converging to 0 as \( n \to \infty \), we have

\[
\lim_{n \to \infty} \left\| \varepsilon_n (\varepsilon_n I + \Gamma(\varepsilon_n))^{-1} \pi_M \right\| = 0.
\]

**Proof.** It is clear that \( (\varepsilon I + \Gamma(\varepsilon))^{-1} \pi_M \) maps \( X \) into finite dimensional space \( \text{Im} \left((\varepsilon I + \Gamma(\varepsilon))^{-1} \pi_M\right) \) and

\[
0 \leq \left\| \varepsilon (\varepsilon I + \Gamma(\varepsilon))^{-1} \pi_M \right\| \leq 1.
\]

Then for any sequence \( \{\varepsilon_n > 0\} \) converging to 0 as \( n \to \infty \), we have

\[
0 \leq \rho := \lim_{n \to \infty} \left\| \varepsilon_n (\varepsilon_n I + \Gamma(\varepsilon_n))^{-1} \pi_M \right\| \leq 1.
\]

Show that \( \rho = 0 \). Let \( \left\| \varepsilon_n (\varepsilon_n I + \Gamma(\varepsilon_n))^{-1} \pi_M \right\| := \gamma_n \). Then \( 0 \leq \lim_{n \to \infty} \gamma_n = \rho \leq 1 \) and by the definition of \( \gamma_n \) there exists a sequence \( \{h_{n,m} \in X : \|h_{n,m}\| = 1\} \) such that

\[
\varepsilon_n (\varepsilon_n I + \Gamma(\varepsilon_n))^{-1} \pi_M h_{n,m} =: z_{n,m},
\]

\[
0 \leq \|z_{n,m}\| \leq 1, \quad \|z_{n,m}\| \to \gamma_n \quad \text{as} \quad m \to \infty.
\]

It follows that

\[
\varepsilon_n \pi_M h_{n,m} = \varepsilon_n z_{n,m} + \Gamma(\varepsilon_n) z_{n,m}.
\] (4)

Since \( \{\pi_M h_{n,m}\} \) and \( \{z_{n,m}\} \) are bounded sequences of finite dimensional vectors, without loss of generality we may assume that

\[
z_{n,m} \to z_n \quad \text{and} \quad \pi_M h_{n,m} \to h_n \quad \text{strongly as} \quad m \to \infty.
\]
Taking limit as $m \to \infty$ in (4), we get

$$\varepsilon_n h_n = \varepsilon_n z_n + \Gamma (\varepsilon_n) z_n, \quad \|h_n\| \leq 1, \quad \|z_n\| = \gamma_n \leq 1.$$  

(5)

Next, having in mind that $z_n \to z$ along some subsequence, we take limit as $n \to \infty$ (4) to get

$$0 = \lim_{n \to \infty} \Gamma (\varepsilon_n) z_n = \lim_{n \to \infty} (\Gamma (\varepsilon_n) - \Gamma) z + \lim_{n \to \infty} \Gamma (\varepsilon_n) (z_n - z) + \Gamma z = \Gamma z = 0,$$

$$\Gamma z = 0 \implies z = 0.$$

By definition of the positive operator $\Gamma z = 0$ implies that $z = 0$. Thus

$$\rho = \lim_{n \to \infty} \left\| \varepsilon_n (\varepsilon I + \Gamma (\varepsilon_n))^{-1} \pi_M \right\| = \lim_{n \to \infty} \gamma_n = \lim_{n \to \infty} \|z_n\| = \|z\| = 0.$$

The theorem is proved. □

**Theorem 4** Assume that $\Gamma (\varepsilon) : \mathcal{X} \to \mathcal{X}$, $\varepsilon > 0$, are linear positive operators. Then for any $\varepsilon > 0$ we have $\left\| \varepsilon (\varepsilon I + \Gamma (\varepsilon))^{-1} \pi_M \right\| < 1$.

**Proof.** It is clear that $(\varepsilon I + \Gamma (\varepsilon))^{-1} \pi_M$ maps $\mathcal{X}$ into finite dimensional subspace of $\mathcal{X}$ and

$$\left\| \varepsilon (\varepsilon I + \Gamma (\varepsilon))^{-1} \pi_M \right\| \leq 1.$$

Let us show that $\left\| \varepsilon (\varepsilon I + \Gamma (\varepsilon))^{-1} \pi_M \right\| < 1$. Contrary, assume that there exists a sequence $\{h_n \in \mathcal{X} : \|h_n\| = 1\}$ such that

$$\varepsilon (\varepsilon I + \Gamma (\varepsilon))^{-1} \pi_M h_n =: z_n, \quad \|z_n\| \to 1 \text{ as } n \to \infty.$$  

(6)

It follows that $\{z_n\}$ is a sequence of finite dimensional vectors and

$$\varepsilon \pi_M h_n = \varepsilon z_n + \Gamma (\varepsilon) z_n \text{ and } z_n \to z_0 \text{ strongly in } \mathcal{X}.$$  

(7)

$$\langle \pi_M h_n, z_n \rangle = \langle z_n, z_n \rangle + \frac{1}{\varepsilon} \langle \Gamma (\varepsilon) z_n, z_n \rangle,$$

$$\|z_n\|^2 < \langle z_n, z_n \rangle + \frac{1}{\varepsilon} \langle \Gamma (\varepsilon) z_n, z_n \rangle = \langle \pi_M h_n, z_n \rangle \leq \|\pi_M h_n\| \|z_n\| \leq \|z_n\|.$$  

Taking limit as $n \to \infty$ we get

$$1 \leq 1 + \frac{1}{\varepsilon} \langle \Gamma (\varepsilon) z_0, z_0 \rangle \leq 1,$$

$$\langle \Gamma (\varepsilon) z_0, z_0 \rangle = 0 \implies z_0 = 0.$$

Now from (7) it follows that $\|z_n\| \to 0$ as $n \to \infty$. Contradiction. □
Theorem 5 If $\Gamma : \mathcal{X} \to \mathcal{X}$ is a linear nonnegative operator then the operator

$$\varepsilon (I - \pi_M) + \Gamma : \mathcal{X} \to \mathcal{X}$$

is invertible and

$$\|\varepsilon (I - \pi_M) + \Gamma\|^{-1} h \leq \frac{1}{\min (\varepsilon, \delta)} \|h\|, \quad h \in \mathcal{X}, \quad (8)$$

where $\delta = \min \{\|\pi_M \Gamma \pi_M \varphi\| : \|\pi_M \varphi\| = 1\}$. Moreover, if $\Gamma : \mathcal{X} \to \mathcal{X}$ is a linear positive operator then

$$\varepsilon (I - \pi_M) + \Gamma = \varepsilon (I - \pi_M) + (I - \pi_M) \Gamma + \pi_M \Gamma.$$

It is clear that

$$\langle (\varepsilon (I - \pi_M) + \Gamma) \varphi, \varphi \rangle
= \langle (\varepsilon (I - \pi_M) + (I - \pi_M) \Gamma) \varphi, \varphi \rangle + \langle \pi_M \Gamma \varphi, \varphi \rangle
\geq \{\langle \pi_M \Gamma \pi_M \varphi, \varphi \rangle, \quad \varphi \in M,
\langle \varepsilon (I - \pi_M) \varphi + (I - \pi_M) \Gamma (I - \pi_M) \varphi, \varphi \rangle, \quad \varphi \in \mathcal{X} \ominus M
\geq \min (\varepsilon, \delta) \|\varphi\|^2.$$

It follows that $\varepsilon (I - \pi_M) + \Gamma$ is invertible and (8) is satisfied.

If $\Gamma : \mathcal{X} \to \mathcal{X}$ is a linear positive operator then by Theorem 4, $$(I - \varepsilon (\varepsilon I + \Gamma)^{-1} \pi_M)^{-1}$$
exists. On the other hand, since $(\varepsilon I + \Gamma)$ is invertible and

$$\varepsilon (I - \pi_M) + \Gamma = (\varepsilon I + \Gamma) \left(I - \varepsilon (\varepsilon I + \Gamma)^{-1} \pi_M\right),$$

the operator $\varepsilon (I - \pi_M) + \Gamma$ is boundedly invertible and (9) is satisfied.

Next, we present new criteria for the finite-approximately controllability of linear evolution equations.

Theorem 6 The following statements are equivalent:

(i) the system (3) is approximately controllable on $[0, T]$;

(ii) $\Gamma^0_0$ is positive, that is $\langle \Gamma^0_0 x, x \rangle > 0$ for all $0 \neq x \in \mathcal{X}$;

(iii) $\varepsilon (\varepsilon I + \Gamma^0_0)^{-1} \to 0$ as $\varepsilon \to 0^+$ in the strong operator topology;

(iv) $\varepsilon (\varepsilon (I - \pi_M) + \Gamma^0_0)^{-1} \to 0$ as $\varepsilon \to 0^+$ in the strong operator topology;

(v) the system (3) is finite-approximately controllable on $[0, T]$.

Proof. The equivalences (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii) are well known, see (3).

For the equivalence (iii)$\Leftrightarrow$(v), for any $\varepsilon > 0$, $h \in \mathcal{X}$, consider the following functional $J_\varepsilon (\cdot, h) : \mathcal{X} \to \mathbb{R}$:

$$J_\varepsilon (\varphi, h) = \frac{1}{2} \int_0^T \|B^* \Omega^* (T - s) \varphi\|^2 ds + \frac{\varepsilon}{2} \langle (I - \pi_M) \varphi, \varphi \rangle - \langle \varphi, h - \Omega (T) x_0 \rangle.$$

$$= \frac{1}{2} \int_0^T \|B^* \Omega^* (T - s) \varphi\|^2 ds + \frac{\varepsilon}{2} \langle (I - \pi_M) \varphi, \varphi \rangle - \langle \varphi, h - \Omega (T) x_0 \rangle.$$

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Assume that (iii) is satisfied. It is clear that $J_\varepsilon (\tau, h)$ is Gateaux differentiable, $J_\varepsilon' (\varphi, h) = \Gamma_0^T \varphi + \varepsilon (I - \pi_M) \varphi - h + \mathcal{U} (T) x_0$ is strictly monotonic and consequently $J_\varepsilon (\tau, h)$ is strictly convex, since $\Gamma_0^T$ is positive. Thus $J_\varepsilon (\tau, h)$ has a unique minimum and can be found as follows:

$$\Gamma_0^T \varphi + \varepsilon (I - \pi_M) \varphi - h + \mathcal{U} (T) x_0 = 0,$$

$$\varphi_{\min} = - (\varepsilon (I - \pi_M) + \Gamma_0^T)^{-1} (\mathcal{U} (T) x_0 - h).$$

It follows that for the control $u_\varepsilon (s) = B^* \mathcal{U}^* (T - s) \varphi_{\min}$

$$x_\varepsilon (T) - h = \mathcal{U} (T) x_0 + \int_0^T \mathcal{U} (T - s) Bu (s) ds - h$$

$$= \mathcal{U} (T) x_0 - h - \Gamma_0^T (\varepsilon (I - \pi_M) + \Gamma_0^T)^{-1} (\mathcal{U} (T) x_0 - h)$$

$$= \mathcal{U} (T) x_0 - h - (\Gamma_0^T + \varepsilon (I - \pi_M) - \varepsilon (I - \pi_M))$$

$$\times (\varepsilon (I - \pi_M) + \Gamma_0^T)^{-1} (\mathcal{U} (T) x_0 - h)$$

$$= \varepsilon (I - \pi_M) (\varepsilon (I - \pi_M) + \Gamma_0^T)^{-1} (\mathcal{U} (T) x_0 - h).$$

Thus

$$\lim_{\varepsilon \to 0^+} \|x_\varepsilon (T) - h\| = \lim_{\varepsilon \to 0^+} \varepsilon \left\| (I - \pi_M) (\varepsilon (I - \pi_M) + \Gamma_0^T)^{-1} (\mathcal{U} (T) x_0 - h) \right\| = 0,$$

$$\pi_M (x_\varepsilon (T) - h) = 0,$$

that is the system (3) is finite-approximately controllable on $[0, T]$. Thus (iii)$\Rightarrow$(v). The implication (v)$\Rightarrow$(iii) is obvious, since finite-approximate controllability implies the approximate controllability.

For the implication (iii)$\Rightarrow$(iv), suppose that for any $h \in \mathfrak{X}$

$$\lim_{\varepsilon \to 0^+} \left\| (\varepsilon I + \Gamma_0^T)^{-1} h \right\| = 0.$$

From (9) it follows that for any $h \in \mathfrak{X}$

$$\left\| \varepsilon (I - \pi_M) (\varepsilon I + \Gamma_0^T)^{-1} h \right\| \leq \left\| (I - \varepsilon (\varepsilon I + \Gamma_0^T)^{-1} \pi_M)^{-1} \right\| \left\| \varepsilon (\varepsilon I + \Gamma_0^T)^{-1} h \right\|$$

$$\leq \frac{1}{1 - \varepsilon (\varepsilon I + \Gamma_0^T)^{-1} \pi_M} \left\| \varepsilon I + \Gamma_0^T \right\| \left\| \varepsilon (\varepsilon I + \Gamma_0^T)^{-1} h \right\|.$$  

(11)

On the other hand, from

$$\varepsilon_1 (\varepsilon I + \Gamma)^{-1} \pi_M - \varepsilon (\varepsilon I + \Gamma)^{-1} \pi_M$$

$$= \varepsilon_1 (\varepsilon I + \Gamma)^{-1} (I + \varepsilon^{-1} \Gamma - I - \varepsilon^{-1} \Gamma) \varepsilon (\varepsilon I + \Gamma)^{-1} \pi_M$$

$$= \varepsilon_1 (\varepsilon I + \Gamma)^{-1} (\varepsilon^{-1} \Gamma - \varepsilon^{-1} \Gamma) \varepsilon (\varepsilon I + \Gamma)^{-1} \pi_M$$

$$= (\varepsilon I + \Gamma)^{-1} (\varepsilon_1 \Gamma - \varepsilon \Gamma) (\varepsilon I + \Gamma)^{-1} \pi_M$$

$$= (\varepsilon I + \Gamma)^{-1} (\varepsilon_1 - \varepsilon) \Gamma (\varepsilon I + \Gamma)^{-1} \pi_M,$$
it follows that \( \varepsilon (\varepsilon I + \Gamma)^{-1} \pi_M \) is continuous in \( \varepsilon \). Indeed,
\[
\| \varepsilon_1 (\varepsilon_1 I + \Gamma)^{-1} \pi_M - \varepsilon (\varepsilon I + \Gamma)^{-1} \pi_M \| \leq \frac{|\varepsilon_1 - \varepsilon|}{\varepsilon_1} \to 0 \quad \text{as} \quad \varepsilon_1 \to \varepsilon.
\]
By (11), continuity of \( \varepsilon (\varepsilon I + \Gamma)^{-1} \pi_M \) and Theorem 4, we have
\[
\gamma = \max_{0 \leq \varepsilon \leq 1} \| \varepsilon (\varepsilon I + \Gamma_0^T)^{-1} \pi_M \| < 1,
\]
\[
\| \varepsilon (\varepsilon (I - \pi_M) + \Gamma_0^T)^{-1} h \| \leq \frac{1}{1 - \gamma} \| \varepsilon (\varepsilon I + \Gamma_0^T)^{-1} h \|.
\]
Thus \( \varepsilon (\varepsilon (I - \pi_M) + \Gamma_0^T)^{-1} \) converges to zero as \( \varepsilon \to 0^+ \) in the strong operator topology.

The implication (iv) \( \Rightarrow \) (v) follows from (10).

\[ \text{Remark 7} \quad \text{Analogue of Theorem 6 is true for different kind of equations such as fractional linear differential equations with Caputo derivative, fractional linear differential equations with Riemann-Liouville derivative, Fredholm type linear integral equations and so on.} \]

\section{3 Finite-approximate controllability of semilinear system}

In this section, we first show that for every \( \varepsilon > 0 \) and every final state \( y_f \in \mathfrak{X} \), the integral equation
\[
z(t) = \mathcal{X}(t,0;F(z))x_0 + \int_0^t \mathcal{X}(t,s;F(z))[B \varepsilon(s,z) + g(s,z(s))]ds,
\]
with the control
\[
u_\varepsilon(t,z) = B^* \mathcal{X}^*(T,t;F(z))(\varepsilon(I - \pi_M) + \Gamma_0^T(F(z)))^{-1}
\times \left( h - \mathcal{X}(T,0;F(z))x_0 - \int_0^T \mathcal{X}(T,s;F(z))g(s,z(s))ds \right)
\]
has at least one solution, say \( y_\varepsilon^* \). Then we can approximate any point \( y_f \in \mathfrak{X} \) by using these solutions \( y_\varepsilon^* \), \( \varepsilon > 0 \).

\subsection{3.1 Existence of fixed point}

We impose the following assumptions:

(S) \( \mathfrak{X} \) and \( U \) are separable Hilbert spaces, \( \mathfrak{U}(t), t > 0 \) is a compact semi-group on \( \mathfrak{X} \) and \( B \in L(U,\mathfrak{X}) \).
(F) \( f : [0, T] \times \mathcal{X} \rightarrow \mathcal{X} \) is continuous and has continuous uniformly bounded Frechet derivative \( f'_z (\cdot, \cdot) \), that is, for some \( L > 0 \),
\[
\| f'_z (t, z) \|_{L(X)} \leq L, \quad \forall (t, z) \in [0, T] \times \mathcal{X}.
\]

(G) \( g : [0, T] \times \mathcal{X} \rightarrow \mathcal{X} \) is continuous and there exists \( m \in C ([0, T], \mathbb{R}^+ ) \) such that
\[
\| g(t, z) \| \leq m(t), \quad \forall (t, z) \in [0, T] \times \mathcal{X}.
\]

(AC) System
\[
y(t) = \Psi(t) y_0 + \int_0^t \Psi(t-s) [B u(s) + G(s) y(s)] \, ds \quad (12)
\]
is approximately controllable for any \( G \in L^2 (0, T; L(\mathcal{X})) \).

It is clear that under the conditions (S), (F) and (G), for any \( y_0 \in \mathcal{X} \) and \( u(\cdot) \in L_2 (0, T; U) \), the system (2) admits a unique solution \( y(\cdot) = y(\cdot, y_0, u) \).

Define
\[
F(t, z) = \int_0^1 f' (t, rz) \, dr, \quad z \in \mathcal{X}. \quad (13)
\]

Thanks to the assumption (F) there exists a constant \( L > 0 \) such that operator \( F \) defined by (13) has the following properties:
\[
F : [0, T] \times \mathcal{X} \rightarrow L(\mathcal{X}),
\]
\[
f(t, z) = F(t, z) z + f(t, 0),
\]
\[
\| F(t, z(t)) \|_{L(X)} \leq L, \quad z(\cdot) \in C ([0, T], \mathcal{X}), \quad t \in [0, T],
\]
\[
F(\cdot, \cdot) \in C ([0, T] \times \mathcal{X}, L(\mathcal{X})).
\]

For simplicity we assume that \( f(t, 0) \equiv 0 \). Then the system (2) can be rewritten as follows
\[
y(t) = \Psi(t) y_0 + \int_0^t \Psi(t-s) [B u(s) + F(s, y(s)) y(s) + g(s, y(s))] \, ds.
\]

For any fixed \( z(\cdot) \in C ([0, T], \mathcal{X}) \), let \( y(\cdot) = y(\cdot, y_0, z, u) \) be the solution of
\[
y(t) = \Psi(t) y_0 + \int_0^t \Psi(t-s) [B u(s) + F(s, z(s)) y(s) + g(s, z(s))] \, ds \quad (14)
\]
or
\[
y(t) = \Psi(t, 0; F(z)) y_0 + \int_0^t \Psi(t, s; F(z)) [B u(s) + g(s, z(s))] \, ds,
\]

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where

\[
\Psi(t, s; F(z)) y = \mathbf{U}(t - s) y + \int_s^t \mathbf{U}(t - r) F(r, z(r)) \Psi(r, s; F(z)) y dr, 0 \leq s \leq t \leq T.
\]

Under the above conditions we are going to show the following:

(i) For any function \( z(\cdot) \in C([0, T], \mathfrak{X}) \), there exists a control \( u_\varepsilon(t, z) \) determined explicitly by \( z(\cdot) \), such that

\[
u_\varepsilon(t, z) = B^* \Psi^*(T, t; F(z)) \left( \varepsilon (I - \pi_M) + \Gamma^T_0 (F(z)) \right)^{-1} \times \left( h - \Psi(T, 0; F(z)) x_0 - \int_0^T \Psi(T, s; F(z)) g(s, z(s)) ds \right).
\]

(ii) For any \( \varepsilon > 0 \) an operator

\[(\Theta^\varepsilon z)(t) = \Psi(t, 0; F(z)) x_0 + \int_0^t \Psi(t, s; F(z)) [B u_\varepsilon(s, z) + g(s, z(s))] ds \]

admits a fixed point, \( y^\varepsilon(\cdot) \in C([0, T], \mathfrak{X}) \).

First we prove several lemmas.

**Lemma 8** For any \( G \in L^2([0, T], L(\mathfrak{X})) \) there exists a unique strongly continuous function \( \Psi : \Delta \rightarrow L(\mathfrak{X}) \), \( \Delta = \{(t, s) : 0 \leq s \leq t \leq T\} \), such that

\[
\Psi(t, t) = I, \quad t \in [0, T],
\]

\[
\Psi(t, r) \Psi(r, s) = \Psi(t, s), \quad 0 \leq s \leq r \leq t \leq T,
\]

\[
\Psi(t, s; G) y = \mathbf{U}(t - s) y + \int_0^t \mathbf{U}(t - r) G(r) \Psi(r, s; G) y dr
\]

\[
= \mathbf{U}(t - s) y + \int_0^t \Psi(t, r; G) G(r) \Psi(r, s; G) y dr.
\]

The operator valued function \( \Psi : \Delta \rightarrow L(\mathfrak{X}) \) is the evolution operator generated by \( A + F(\cdot, z(\cdot)) \).

Define

\[
\Psi^*(T, t; G) \eta = \mathbf{U}^*(T - t) \eta + \int_t^T \mathbf{U}^*(T - r) G^*(r) \Psi^*(T, r; G) \eta dr.
\]
Lemma 9 Suppose that $g(t), \varphi(t), \psi(t) \geq 0$ and $\omega(t) \geq 0$ are integrable functions. If

$$g(t) \leq \varphi(t) + \psi(t) \int_t^b \omega(r) g(r) \, dr,$$

then

$$g(t) \leq \varphi(t) + \psi(t) \int_t^b \varphi(r) \omega(r) e^{\int_t^r \psi(s) \omega(s) \, ds} \, dr.$$ 

Lemma 10 Let $G_n \in L^2(0,T; L(X))$ and $\eta_n, \eta \in X$ such that

$$\begin{cases}
G_n & \text{is uniformly bounded in } L^2(0,T; L(X)), \\
\eta_n \rightharpoonup \eta & \text{weakly in } X, \text{ as } n \to \infty,
\end{cases} \quad (15)$$

then there exists $G \in L^2(0,T; L(X))$ such that

$$\Xi^*(T, \cdot; G_n) \eta_n \to \Xi^*(T, \cdot; G) \eta \quad \text{in } C(0,T; X),$$

$$\Xi(T, \cdot; G_n) \eta_n \to \Xi(T, \cdot; G) \eta \quad \text{in } C(0,T; X),$$

as $n \to \infty$.

Proof. Let $\{e_m : m \geq 1\}$ be a basis of $X$. By our assumption, there exists $C > 0$ such that for all $n \geq 1$

$$\int_0^T \|G_n(t)\|^2_{L(X)} \, dt \leq C.$$ 

It follows that

$$\int_0^T \|G_n(t) e_m\|^2_X \, dt \leq C.$$

By the "diagonal argument", we know that there exists a subsequence, denoted again by $\{G_n(\cdot) e_m : n \geq 1\}$, which is weakly convergent in $L^2(0,T; X)$ for all $m \geq 1$. Since $\{e_m : m \geq 1\}$ is dense in $X$, we know that the sequence $\{G_n(\cdot) x\}$ is weakly convergent in $L^2(0,T; X)$ for all $x \in X$ to some $G(\cdot) x \in L^2(0,T; X)$.

It is clear that $G(\cdot) \in L^2(0,T; L(X))$.

Denote

$$\xi_n(t) = \Xi^*(T, t; G_n) \eta_n, \quad \xi(t) = \Xi^*(T, t; G) \eta, \quad t \in [0,T].$$

It is easily seen that

$$\xi_n(t) = \Xi^*(T-t) \eta_n + \int_t^T \Xi^*(T-r) G_n(r) \xi_n(r) \, dr, \quad t \in [0,T]. \quad (16)$$

Then by (16) and the Gronwall inequality,

$$\|\xi_n(t)\| \leq M \|\eta_n\| + M \int_t^T \|G_n(r)\|_{L(X)} \|\xi_n(r)\| \, dr, \quad t \geq 0,$$
we have
\[
\|\xi_n(t)\| \leq M\|\eta_n\| + M^2\|\eta_n\|
\times \int_t^T \|G_n(r)\|_{L(X)} \exp \left( M \int_r^t \|G_n(s)\|_{L(X)} \, ds \right) \, dr. \quad (17)
\]

From (15), we have the uniform boundedness of \{\eta_n\}. So from (17) one obtains the uniform boundedness of \{G_n(\cdot)\} in \(C(0, T; X)\) and the uniform boundedness of \{G_n(\cdot)\} in \(L^2(0, T; X)\). Thus, having in mind compactness of \(U(t)\), \(t > 0\), one can show that \{\xi_n(\cdot)\} is relatively compact in \(C(0, T; X)\). On the other hand, for any \(r \in [0, T]\)
\[
U^*(T - r) G_n(r) \xi_n(r) \to U^*(T - r) G(r) \xi(r) \quad \text{in} \quad X.
\]

Passing to the limit in (16) along some proper subsequence, we see that \(\xi(\cdot)\) satisfies
\[
\xi(t) = U^*(T - t) \eta + \int_t^T U^*(T - r) G(r) \xi(r) \, dr, \quad t \in [0, T]. \quad (18)
\]

By uniqueness of the solutions to (18), we obtain that the whole sequence \{\xi_n(\cdot)\} converges to \(\xi(\cdot)\) in \(C(0, T; X)\). Similarly, we may prove that
\[
\Xi(T, \cdot; G_n) \eta_n \to \Xi(T, \cdot; G) \eta \quad \text{in} \quad C(0, T; X) \quad \text{as} \quad n \to \infty.
\]

\begin{lemma}
Let \(G_n \in L^2(0, T; L(X))\) and \(\eta_n, \eta \in X\) such that
\[
\begin{cases}
G_n & \text{is uniformly bounded in} \quad L^2(0, T; L(X)), \\
\eta_n & \to \eta \quad \text{weakly in} \quad X, \quad \text{as} \quad n \to \infty,
\end{cases}
\]

then there exists \(G \in L^2(0, T; L(X))\) such that
\[
\Gamma^T_0 (G_n) \eta_n \to \Gamma^T_0 (G) \eta \quad \text{in} \quad X, \quad \text{as} \quad n \to \infty,
\]
where
\[
\Gamma^T_0 (G_n) \eta_n = \int_0^T \Xi(T, s; G_n) BB^* \Xi^* (T, s; G_n) \eta_n \, ds,
\]
\[
\Gamma^T_0 (G) \eta = \int_0^T \Xi(T, s; G) BB^* \Xi^* (T, s; G) \eta \, ds.
\]
\end{lemma}

\textbf{Proof.} The desired convergence follows from Lemma 10 boundedness of \(\Xi(T, s; G_n)\)
and from the following inequality

$$\| \Gamma_T^n (G_n) \eta_n - \Gamma_T^0 (G) \eta \|$$

$$= \left\| \int_0^T \Xi (T, s; G_n) B B^{*} \Xi^* (T, s; G_n) \eta_n ds - \int_0^T \Xi (T, s; G) B B^{*} \Xi^* (T, s; G) \eta ds \right\|$$

$$\leq \left\| \int_0^T \Xi (T, s; G_n) B B^{*} \Xi^* (T, s; G_n) \eta_n ds - \int_0^T \Xi (T, s; G_n) B B^{*} \Xi^* (T, s; G) \eta ds \right\|$$

$$\leq \int_0^T \| \Xi (T, s; G_n) B B^* \| \| \Xi^* (T, s; G_n) \eta_n - \Xi^* (T, s; G) \eta \| ds$$

$$+ \int_0^T \| \Xi (T, s; G_n) - \Xi (T, s; G) \| \| \Xi^* (T, s; G) \eta \| ds.$$
as $m \to \infty$, and $\lim_{m \to \infty} \gamma_{n,m} = \|z_n\| = \gamma_n$, $\|z_{n,m}\| \leq 2$. By Lemma 11 we have
\[
\gamma_n = \|z_n\| = \left\| \varepsilon (\varepsilon I + \Gamma_{0}^T (G_n))^{-1} (\Gamma_{0}^T (G) - \Gamma_{0}^T (G_n)) (\varepsilon I + \Gamma_{0}^T (G))^{-1} h_0 \right\|
\leq \left\| \varepsilon (\varepsilon I + \Gamma_{0}^T (G_n))^{-1} \right\| \left\| (\Gamma_{0}^T (G) - \Gamma_{0}^T (G_n)) (\varepsilon I + \Gamma_{0}^T (G))^{-1} h_0 \right\|
\leq \left\| (\Gamma_{0}^T (G) - \Gamma_{0}^T (G_n)) (\varepsilon I + \Gamma_{0}^T (G))^{-1} h_0 \right\| \to 0 \text{ as } n \to \infty.
\]
So $\lim_{n \to \infty} \lim_{m \to \infty} \gamma_{n,m} = \lim_{n \to \infty} \gamma_n = 0$.

For every $h \in X$ we have
\[
(\varepsilon (I - \pi_M) + \Gamma_{0}^T (G))^{-1} h - (\varepsilon (I - \pi_M) + \Gamma_{0}^T (G_n))^{-1} h
= (\varepsilon (I - \pi_M) + \Gamma_{0}^T (G_n))^{-1} \left[ I + (\Gamma_{0}^T (G_n) - \Gamma_{0}^T (G)) (\varepsilon (I - \pi_M) + \Gamma_{0}^T (G))^{-1} I \right] h
= (\varepsilon (I - \pi_M) + \Gamma_{0}^T (G_n))^{-1} (\Gamma_{0}^T (G_n) - \Gamma_{0}^T (G)) (\varepsilon (I - \pi_M) + \Gamma_{0}^T (G))^{-1} h.
\]
By (19) and Theorem 4 we have
\[
\lim_{n \to \infty} \left\| (\varepsilon (I - \pi_M) + \Gamma_{0}^T (G_n))^{-1} \right\|
\leq \lim_{n \to \infty} \left\| (I - \varepsilon (\varepsilon I + \Gamma_{0}^T (G_n))^{-1} \pi_M)^{-1} \right\| \left\| (\varepsilon I + \Gamma_{0}^T (G_n))^{-1} \right\|
\leq \frac{1}{\varepsilon} \frac{1}{1 - \lim_{n \to \infty} \varepsilon (\varepsilon I + \Gamma_{0}^T (G_n))^{-1} \pi_M}
= \frac{1}{\varepsilon} \frac{1}{1 - \varepsilon (\varepsilon I + \Gamma_{0}^T (G))^{-1} \pi_M} := \delta (\varepsilon)
\]
(21)

Now desired convergence (21) follows from Lemma 11.
\[
\lim_{n \to \infty} \left\| (\varepsilon (I - \pi_M) + \Gamma_{0}^T (G))^{-1} h - (\varepsilon (I - \pi_M) + \Gamma_{0}^T (G_n))^{-1} h \right\|
\leq \lim_{n \to \infty} \left\| (\varepsilon (I - \pi_M) + \Gamma_{0}^T (G_n))^{-1} \right\|
\times \lim_{n \to \infty} \left\| (\Gamma_{0}^T (G_n) - \Gamma_{0}^T (G)) (\varepsilon (I - \pi_M) + \Gamma_{0}^T (G))^{-1} h \right\|
\leq \delta (\varepsilon) \lim_{n \to \infty} \left\| (\Gamma_{0}^T (G_n) - \Gamma_{0}^T (G)) (\varepsilon (I - \pi_M) + \Gamma_{0}^T (G))^{-1} h \right\| = 0.
\]
Lemma 13 Let \( z(\cdot) \in C(0, T; \mathfrak{X}) \) and \( \mathfrak{H}(t, s; F(z)) \) be the evolution operator generated by \( A + F(z) \), where \( F \) is defined by (13) and let

\[
\begin{align*}
    u_\varepsilon(t, z) &= B^* \mathfrak{H}^* (T, t; F(z)) \left( \varepsilon (I - \pi_M) + \Gamma_0^T (F(z)) \right)^{-1} \\
    &\quad \times \left( h - \mathfrak{H} (T, 0; F(z)) x_0 - \int_0^T \mathfrak{H} (T, s; F(z)) g(s, z(s)) ds \right).
\end{align*}
\]

The control \( z \to u_\varepsilon(t, z) : C(0, T; \mathfrak{X}) \to C(0, T; \mathfrak{X}) \) is continuous and

\[
\|u_\varepsilon(t, z)\| \leq R_\varepsilon := \frac{1}{\varepsilon} M_B M_{\mathfrak{H}} \left( \|h\| + M_{\mathfrak{H}} \|x_0\| + M_{\mathfrak{H}} T \|g\|_C \right),
\]

where

\[
\gamma_\varepsilon := \sup_{z \in C(0, T; \mathfrak{X})} \left\| \varepsilon (I + \Gamma_0^T (F(z)))^{-1} \pi_M \right\| < 1,
\]

and

\[
M_{\mathfrak{H}} := \sup \left\{ \|\mathfrak{H}(t, s; F(z))\| : 0 \leq s \leq t \leq T \right\}.
\]

Proof. To prove continuity of \( u_\varepsilon(\cdot, z) \), let \( \{z_n\} \subset C(0, T; \mathfrak{X}) \) with \( z_n \to z \) in \( C(0, T; \mathfrak{X}) \). By assumption (A2) and (A3) the functions \( F(z) \) and \( g(s, z(s)) \) are continuous. It follows that \( \mathfrak{H}(T, s; F(z)) \) and

\[
h(z) := h - \mathfrak{H}(T, 0; F(z)) x_0 - \int_0^T \mathfrak{H}(T, s; F(z)) g(s, z(s)) ds
\]

are continuous in \( z \). Then from the following equality

\[
\begin{align*}
    u_\varepsilon(t, z_n) - u_\varepsilon(t, z) &= B^* \mathfrak{H}^* (T, t; F(z_n)) \left( \varepsilon (I - \pi_M) + \Gamma_0^T (F(z_n)) \right)^{-1} (h(z_n) - h(z)) \\
    &\quad + B^* \mathfrak{H}^* (T, t; F(z_n)) \\
    &\quad \times \left[ \left( \varepsilon (I - \pi_M) + \Gamma_0^T (F(z_n)) \right)^{-1} - \left( \varepsilon (I - \pi_M) + \Gamma_0^T (F(z)) \right)^{-1} \right] h(z) \\
    &\quad + \left[ B^* \mathfrak{H}^* (T, t; F(z_n)) - B^* \mathfrak{H}^* (T, t; F(z)) \right] \left( \varepsilon (I - \pi_M) + \Gamma_0^T (F(z)) \right)^{-1} h(z)
\end{align*}
\]

it follows that \( u_\varepsilon(t, z_n) \to u_\varepsilon(t, z) \) as \( n \to \infty \) in \( C(0, T; \mathfrak{X}) \). Moreover,

\[
\|u_\varepsilon(t, z)\| \leq M_B M_{\mathfrak{H}} \frac{1}{1 - \varepsilon \left( \varepsilon (I + \Gamma_0^T (F(z)))^{-1} \pi_M \right)} \\
\times \left\| \varepsilon (I + \Gamma_0^T (F(z)))^{-1} \pi_M \right\| \\
\leq \frac{1}{\varepsilon (1 - \gamma_\varepsilon)} M_B M_{\mathfrak{H}} \left( \|h\| + M_{\mathfrak{H}} \|x_0\| + M_{\mathfrak{H}} T \|g\|_C \right) := R_\varepsilon,
\]

where

\[
\gamma_\varepsilon := \sup_{z \in C(0, T; \mathfrak{X})} \varepsilon \left( \varepsilon (I + \Gamma_0^T (F(z)))^{-1} \pi_M \right) < 1.
\]
Show that $\gamma_\varepsilon < 1$. Contrary, assume that there exists a sequence $\{z_n\}$ such that

$$\lim_{n \to \infty} \gamma_\varepsilon(z_n) = 1, \quad \gamma_\varepsilon(z_n) = \left\| \varepsilon \left( \varepsilon I + \Gamma_0^T (F(z_n)) \right)^{-1} \pi_M \right\|. $$

Then since $\|F(t, z_n(t))\| \leq L, F(\cdot, z_n(\cdot)) \in L^2(0, T; L(X))$, then by Lemma 12 there exists $F \in L^2(0, T; L(X))$ such that

$$\lim_{n \to \infty} \gamma_\varepsilon(z_n) = \lim_{n \to \infty} \left\| \varepsilon \left( \varepsilon I + \Gamma_0^T (F(z_n)) \right)^{-1} \pi_M \right\| < 1,$$

which is contradiction. ■

**Theorem 14** For any $\varepsilon > 0$ the operator $\Theta_\varepsilon(z)$ has a fixed point in $C(0, T; \mathfrak{X})$.

**Proof.** Claim 1. The operator $\Theta_\varepsilon(z)$ sends $C(0, T; \mathfrak{X})$ into a bounded set.

We need to show that, for any $\varepsilon > 0$ there exists $k(\varepsilon) > 0$ such that $\|(\Theta_\varepsilon z)(t)\| \leq k(\varepsilon)$ for all $z(\cdot) \in C(0, T; \mathfrak{X})$. Indeed,

$$\|(\Theta_\varepsilon z)(t)\| \leq \|\mathfrak{I}(t, 0; F(z))\| \|y_0\|$$

$$+ \int_0^t \|\mathfrak{I}(t, s; F(z))\| \|B\| \|u_\varepsilon(s, z)\| + \|g(s, z(s))\| \|ds\|

\leq L \|y_0\| + LM_B(R_\varepsilon + T \|m\|_C) =: k(\varepsilon).$$

Claim 2. The operator $\Theta_\varepsilon : C(0, T; \mathfrak{X}) \to C(0, T; \mathfrak{X})$ is continuous.

Assume that the sequence $\{z_n\} \subset C(0, T; \mathfrak{X})$ such that $z_n \to z$ in $C(0, T; \mathfrak{X})$. Then the triangle inequality we have

$$\|(\Theta_\varepsilon z_n)(t) - (\Theta_\varepsilon z)(t)\| \leq \|(\mathfrak{I}(t, 0; F(z_n)) y_0 - \mathfrak{I}(t, 0; F(z)) y_0)\|

+ \int_0^t \|\mathfrak{I}(t, s; F(z_n))\| \|B\| \|u_\varepsilon(s, z_n) - u_\varepsilon(s, z)\| \|ds\|

+ \int_0^t \|\mathfrak{I}(t, 0; F(z_n)) - \mathfrak{I}(t, 0; F(z))\| \|B\| \|u_\varepsilon(s, z)\| \|ds\|

+ \int_0^t \|\mathfrak{I}(t, s; F(z))\| \|B\| \|g(s, z_n(s)) - g(s, z(s))\| \|ds\|

+ \int_0^t \|\mathfrak{I}(t, 0; F(z_n)) - \mathfrak{I}(t, 0; F(z))\| \|g(s, z(s))\| \|ds\|.$$

Now from the continuity of $u_\varepsilon(s, \cdot), g(\cdot), \mathfrak{I}(t, s; F(\cdot))$ we get the desired continuity of $\Theta_\varepsilon$.

Claim 3. The family of functions $\{\Theta_\varepsilon z : z \in C(0, T; \mathfrak{X})\}$ is equicontinuous.
We know that compactness of the evolution family $T$ uniform continuity of $p$ pact in $X$ equicontinuous at $t$ we see that $\|X\|$ is relatively compact in $X$. Then from the compactness of $T$ fixed point theorem, the operator $\Theta$ is continuous and compact with uniformly bounded image. By the Schauder operator for any $\varepsilon > 0$ we have that $\|X\|$ is relatively compact in $X$. For $z \in C (0, T; X)$ as $t_2 - t_1 \to 0$. It can be easily shown that the function $\{T_{\varepsilon z} : z \in C (0, T; X)\}$ is equicontinuous at $t = 0$. Hence $\{T_{\varepsilon z} : z \in C (0, T; X)\}$ is equiuniform.

Claim 4. The set $V(t) = \{ (T_{\varepsilon z}) (t) : z (\cdot) \in C (0, T; X) \}$ is relatively compact in $X$.

Obviously, $V(0)$ is relatively compact in $X$. Let $0 < t \leq T$ be fixed and $0 \leq \delta < t$. For $z (\cdot) \in C (0, T; X)$ we define

$$(\Theta_{\varepsilon z}) (t) = T(t, 0; F (z)) x_0 + \int_0^t T(t, s; F (z)) [B_{\varepsilon z} (s, z) + g (s, z (s))] ds$$

$$(\Theta^\delta_{\varepsilon z}) (t) = T(t, 0; F (z)) x_0 + \int_0^{t-\delta} T(t-\delta, s; F (z)) [B_{\varepsilon z} (s, z) + g (s, z (s))] ds.$$  

Then from the compactness of $T(t, t-\delta; F (z)), \delta > 0$ we obtain that $V^\delta (t) = \{ (\Theta^\delta_{\varepsilon z}) (t) : z (\cdot) \in C (0, T; X) \}$ is relatively compact in $X$ for every $\delta, 0 < \delta < t$. Moreover, for every $z (\cdot) \in C (0, T; X)$ we have

$$\| (\Theta_{\varepsilon z}) (t) - (\Theta^\delta_{\varepsilon z}) (t) \| \leq \int_{t-\delta}^t \|T(t, s; F (z)) [B_{\varepsilon z} (s, z) + g (s, z (s))] \| ds.$$  

Therefore, there are relatively compact sets arbitrarily close to the set $V(t)$. Hence $V(t)$ is also relatively compact in $X$.

Thus thanks to the Arzela-Ascoli theorem, the operator $\Theta_{\varepsilon z}$ is a compact operator for any $\varepsilon > 0$. Consequently, the operator $\Theta_{\varepsilon z} : C (0, T; X) \to C (0, T; X)$ is continuous and compact with uniformly bounded image. By the Schauder fixed point theorem, the operator $\Theta_{\varepsilon z}$ has at least one fixed point in $C (0, T; X)$.  

\[ \blacksquare \]
3.2 Finite-approximate controllability

Assume that $y^*_\varepsilon(\cdot)$ is a fixed point of $\Theta_\varepsilon$. We will show that the fixed point $y^*_\varepsilon(\cdot)$ and the corresponding control $u^*_\varepsilon(\cdot)$ satisfies

$$\|y^*_\varepsilon(T; y_0, u^*_\varepsilon) - y_f\| < \varepsilon, \quad \pi_M y^*_\varepsilon(T; y_0, u^*_\varepsilon) = \pi_M y_f \text{ for any } y_f \in \mathcal{X}. $$

Define

$$h(y^*_\varepsilon) = y_f - \mathfrak{T}(T, 0; F(y^*_\varepsilon)) y_0 - \int_0^T \mathfrak{T}(T, s; F(y^*_\varepsilon)) g(s, y^*_\varepsilon(s)) \, ds,$$

$$\varphi_\varepsilon = (\varepsilon (I - \pi_M) + \Gamma^T_0 (F(y^*_\varepsilon)))^{-1} (h(y^*_\varepsilon)).$$

Now we are ready to state and prove the main result on finite-approximate controllability of semilinear evolution systems in this paper.

**Theorem 15** Let $(S), (F), (G),$ and $(AC)$ hold. Then the system $(\mathcal{I})$ is finite-approximately controllable on $[0, T]$.

**Proof.** Let $y^*_\varepsilon \in C(0, T; \mathcal{X})$ be a fixed point of $\Theta_\varepsilon$. Then we have

$$y^*_\varepsilon(T) - y_f = -\varepsilon (I - \pi_M) \varphi_\varepsilon = -\varepsilon (I - \pi_M) (\varepsilon (I - \pi_M) + \Gamma^T_0 (F(y^*_\varepsilon)))^{-1} (h(y^*_\varepsilon)).$$

By the assumption $(A2)$, $\|F(s, y^*_\varepsilon(s))\|_{L(\mathcal{X})} \leq L$. Then by Lemma 10 there exists $\bar{F} \in L^2(0, T; L(\mathcal{X}))$ such that

$$\mathfrak{T}(T, 0; F(y^*_\varepsilon)) y_0 \to \mathfrak{T}(T, 0; \bar{F}) y_0.$$

On the other hand by the assumption $(A3)$, $\|g(s, y^*_\varepsilon(s))\| \leq m(s)$, and Dunford-Pettis Theorem, we have that the sequence $\{g(s, y^*_\varepsilon(s))\}$ is weakly compact in $L^2(0, T; \mathcal{X})$, so there is a subsequence, still denoted by $\{g(s, y^*_\varepsilon(s))\}$ that weakly converges to, say, $g$ in $L^2(0, T; \mathcal{X})$. Denote

$$\bar{h} = y_f - \mathfrak{T}(T, 0; \bar{F}) y_0 - \int_0^T \mathfrak{T}(T, s; \bar{F}) g(s) \, ds,$$

$$\varphi_\varepsilon(t, s) = \mathfrak{T}(t, s; F(y^*_\varepsilon)) g(s, y^*_\varepsilon(s)) - \mathfrak{T}(t, s; \bar{F}) g(s),$$

$$\psi_\varepsilon(t, s) = \int_s^t \mathfrak{M}(t - r) \left[ F(r, y^*_\varepsilon(r)) - \bar{F}(r) \right] \mathfrak{T}(r, s; \bar{F}(r)) g(s) \, dr.$$

Then by the definition of $\mathfrak{T}$ and the Gronwall inequality we have

$$\|\varphi_\varepsilon(t, s)\| \leq \|\psi_\varepsilon(t, s)\| + M \int_s^t \|\varphi_\varepsilon(r, s)\| \, dr \Rightarrow$$

$$\|\varphi_\varepsilon(t, s)\| \leq \|\psi_\varepsilon(t, s)\| + M \int_s^t \|\psi_\varepsilon(r, s)\| \, dr.$$
It is clear that
\[ \psi_\varepsilon (r, s) \text{ is uniformly bounded and } \| \psi_\varepsilon (r, s) \| \to 0. \]

Then
\[ \varphi_\varepsilon (T, s) = \Xi (T, s; F (y_\varepsilon^*)) g (s, y_\varepsilon^* (s)) - \Xi (T, s; \bar{F}) g (s) \to 0 \text{ as } \varepsilon \to 0^+. \]

So
\[
\| h (y_\varepsilon^*) - \bar{h} \| \leq \| \Xi (T, 0; F (y_\varepsilon^*)) y_0 - \Xi (T, 0; \bar{F}) y_0 \| \\
+ \int_0^T \| \Xi (T, s; F (y_\varepsilon^*)) g (s, y_\varepsilon^* (s)) - \Xi (T, s; \bar{F}) g (s) \| ds \to 0 \text{ as } \varepsilon \to 0^+. \tag{22}
\]

To prove the strong convergence recall that
\[
\| y_\varepsilon^* (T) - y_f \| \leq \varepsilon \left\| (I - \pi_M) \left( \varepsilon (I - \pi_M) + \Gamma_T^0 (F (y_\varepsilon^*)) \right)^{-1} \left( h (y_\varepsilon^*) - \tilde{h} \right) \right\| \\
+ \varepsilon \left\| (I - \pi_M) \left[ \varepsilon (I - \pi_M) + \Gamma_T^0 (F (y_\varepsilon^*)) \right]^{-1} \left( \varepsilon (I - \pi_M) + \Gamma_T^0 (\bar{F}) \right)^{-1} \left( \tilde{h} \right) \right\| \tag{23}
\]

By (22) and the uniform boundedness of \( \| \varepsilon (I - \pi_M) + \Gamma_T^0 (F (y_\varepsilon^*)) \| \), the first term goes to zero. The second term approaches zero thanks to Lemma 12. The last converges to zero according to Theorem 6. Thus taking limit in (23) we complete the proof. \( \blacksquare \)

Remark 16 If in (1) \( f = 0 \) we get the finite-approximate controllability of semilinear system with bounded nonlinear term (cf (8)). If \( g = 0 \) we get the finite-approximate controllability of semilinear system with the Lipschitz nonlinear term. So the result is new even for the case \( g = 0 \).

4 Applications

Example 1. Consider the partial differential system of the form
\[
\frac{\partial}{\partial t} y (t, \theta) = \frac{\partial^2}{\partial \theta^2} y (t, \theta) + m (\theta) u (t, \theta) + f (y (t, \theta)) + g (y (t, \theta)), (t, \theta) \in (0, T) \times (0, \pi) \\
y (t, \theta) = 0, (0, T) \times \{0, \pi\}, \tag{24} \\
y (0, \theta) = y_0 (\theta), \theta \in [0, \pi], \quad y_0 \in L^2 [0, \pi],
\]

\( m \) is the characteristic function of an open subset \( \omega \subset [0, \pi] \). We assume that \( f \in C^1 (R) \) and \( |f' (r)| \leq L \) for all \( r \in R \). So \( f \) is globally Lipschitz. Moreover assume that \( g \in C (R) \) and \( |g (r)| \leq M \) for all \( r \in R \).
To write the system (24) in a semigroup form define $X = U = L^2[0, \pi]$ and $A : D(A) \subset X \to X$ to be $Ay = y''$, where $D(A) = H^1_0[0, \pi] \cap H^2[0, \pi]$. We also define the operators $F, G : X \to X$ by $(Fy)(\theta) = f(y(\theta))$, $(Gy)(\theta) = g(y(\theta))$ for almost every $\theta \in [0, \pi]$ and the bounded linear control operator $B : X \to X$ by $(Bu)(\theta) = m(\theta)u(t, \theta)$ for almost every $\theta \in [0, \pi]$.

Taking into account all these notations, the state system (24) becomes

$$
\begin{align*}
\frac{\partial y}{\partial t} + Ay &= Bu + F(y) + G(y) \quad \text{on} \quad (0, T) \\
y(0) &= y_0.
\end{align*}
$$

We know that $A$ generates a compact $C_0$-semigroup, $F$ is globally Lipschitz on $X$ and $B$ is bounded. So, for each $u \in L^2(0, T; X)$ and $y_0 \in X$, (24) has a unique mild solution $y \in C(0, T; X)$.

It is known that the linearized system associated with (24) is finite-approximately controllable on $[0, \pi]$ and $[0, \pi]$ is globally Lipschitz on $X$. Thus by Theorem 15, the system (24) is finite-approximately controllable on $[0, T]$.

**Example 1.** Take $X = L^2[0, \pi]$ the bounded linear operator $B \in L^2[0, \pi]$ is defined as in Example 1. Define the operator $B$ as follows

$$
Bu(t) = \sum_{n=1}^{\infty} \pi_n(t)e_n,
$$

where $\pi_n(t)$ are the eigenvalues of $A$.

Consider the following initial-boundary value problem of parabolic control system

$$
\begin{align*}
\frac{\partial y}{\partial t} + Ay(t, \theta) &= \frac{\partial^2}{\partial \theta^2}y(t, \theta) + Bu(t, \theta) + g(t, y(t, \theta)) , &t \in [0, 1], \theta \in [0, \pi], \\
y(t, 0) &= y(t, \pi) = 0, &t \in [0, 1], \\
y(0, \theta) &= y_0(\theta), &t \in [0, 1], \theta \in [0, \pi].
\end{align*}
$$

Take $X = U = L^2[0, \pi]$ and the operator $A : D(A) \subset X \to X$ is defined as in Example 1. Define the operator $B$ as follows

$$
Bu(t) = \sum_{n=1}^{\infty} \pi_n(t)e_n.
$$
where
\[ u(t) = \sum_{n=1}^{\infty} \langle u(t), e_n \rangle e_n, \]
\[ \pi_n(t) = \begin{cases} 
0, & 0 \leq t < 1 - \frac{1}{n^2}, \\
\langle u(t), e_n \rangle, & 1 - \frac{1}{n^2} \leq t \leq 1,
\end{cases} \]
then, one can easily obtain that \( \|Bu\| \leq \|u\| \), which implies that \( B \) is bounded. It is known that the linear system corresponding to (26) is approximately controllable. By Theorem 15, the system (26) is finite-approximately controllable on \([0, T]\), provided that condition (G) is satisfied.

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