Hyperspaces $C(p, X)$ of finite graphs

Florencio Corona-Vázquez, Russell Aarón Quiñones-Estrella, Javier Sánchez-Martínez*, Hugo Villanueva

Universidad Autónoma de Chiapas, Facultad de Ciencias en Física y Matemáticas, Carretera Emiliano Zapata Km. 8, Rancho San Francisco, Terán, C.P. 29050, Tuxtla Gutiérrez, Chiapas, México.

Abstract
Given a continuum $X$ and $p \in X$, we will consider the hyperspace $C(p, X)$ of all subcontinua of $X$ containing $p$ and the family $K(X)$ of all hyperspaces $C(q, X)$, where $q \in X$. In this paper we give some conditions on the points $p, q \in X$ to guarantee that $C(p, X)$ and $C(q, X)$ are homeomorphic, for finite graphs $X$. Also, we study the relationship between the homogeneity degree of a finite graph $X$ and the number of topologically distinct spaces in $K(X)$, called the size of $K(X)$. In addition, we construct for each positive integer $n$, a finite graph $X_n$ such that $K(X_n)$ has size $n$, and we present a theorem that allows to construct finite graphs $X$ with a degree of homogeneity different from the size of the family $K(X)$.

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1. Introduction

A continuum is a nonempty compact connected metric space. Given a continuum $X$, by a hyperspace of $X$ we mean a specified collection of subsets of $X$ endowed with the Hausdorff metric (see Theorem 2.1). In the literature, *Corresponding author

Email addresses: florencio.corona@unach.mx (Florencio Corona-Vázquez), russell.quinones@unach.mx (Russell Aarón Quiñones-Estrella), jsanchezm@unach.mx (Javier Sánchez-Martínez), hugo.villanueva@unach.mx (Hugo Villanueva)
some of the most studied hyperspaces are the following:

\[ 2^X = \{ A \subseteq X : A \text{ is nonempty and closed} \}, \]
\[ C(X) = \{ A \subseteq X : A \text{ is nonempty, connected and closed} \}. \]

\[ 2^X \] is called the \textit{hyperspace of closed subsets} of \( X \) whereas that \( C(X) \) is called the \textit{hyperspace of subcontinua} of \( X \). These have been amply studied to characterize topological properties of \( X \) through them and vice versa. The readers are referred to [2] for more information about this topic.

For the purpose of this paper, given a continuum \( X \), \( A \in C(X) \) and a point \( p \) of \( X \), we focus our attention also to the following hyperspaces:

\[ C(A, X) = \{ B \in C(X) : A \subset B \}, \]
\[ C(p, X) = \{ A \in C(X) : p \in A \}, \]
\[ K(X) = \{ C(x, X) : x \in X \}. \]

The topological structure of the hyperspaces \( C(A, X) \) and \( C(p, X) \) have been recently studied, for example, in [1], [5], [9], [10], and [11]. In [9] P. Pellicer gives characterizations of the class of continua \( X \) for which \( K(X) \) coincides with \( K(I) \) or \( K(S) \), where \( I \) denotes the unit interval and \( S \) is a simple closed curve. As a useful tool to characterize hyperspaces of the form \( K(X) \), is distinguishing the structure of the hyperspaces \( C(p, X) \) where \( p \in X \), see for example [12], in this way, we consider in \( K(X) \) the following natural equivalence relation: \( C(p, X) \sim C(q, X) \) if and only if \( C(p, X) \) is homeomorphic to \( C(q, X) \). Given a positive integer \( n \) and a continuum \( X \), we say that \( K(X) \) \textit{has size} \( n \) if the quotient \( K(X)/\sim \) has cardinality \( n \).

In this paper we study this relation in the class of finite graphs as well as the \textit{size} of \( K(X) \). We show that \( K(X) \) having size \( n \) is not equivalent to \( X \) being \( \frac{1}{n} \)-homogeneous. We present, for each positive integer \( n \), a finite graph \( X_n \) such that \( K(X_n) \) has size \( n \), we also present in Theorem 5.2 a way to construct continua \( X \) with a degree of homogeneity different from the size of \( K(X) \). The homogeneity degree of continua has been studied extensively, see for example [4], [8], [13] and [14], being a useful property to characterize classes of continua. Regarding [12, Corollary 3.12, p. 1006], in Theorem 3.3, we present a similar result in the class of finite graphs, both results aims to find cells in \( C(p, X) \).
2. Preliminaries

Let \( X \) be a continuum with metric \( d \). Given \( \varepsilon > 0 \) and \( p \in X \) we denote as customary \( B_\varepsilon(p) = \{ x \in X : d(x,p) < \varepsilon \} \) and if \( A \subseteq X \)
\[
N(\varepsilon,A) = \{ x \in X : \text{there exists } y \in A \text{ such that } x \in B_\varepsilon(y) \}.
\]
If \( A \) and \( B \) are two closed subsets of \( X \), remember that the Hausdorff distance between \( A \) and \( B \) is given by:
\[
H(A,B) = \inf \{ \varepsilon > 0 : A \subseteq N(\varepsilon,B) \text{ and } B \subseteq N(\varepsilon,A) \}.
\]

Theorem 2.1. [2, Theorem 2.2, p. 11] If \((X,d)\) is a metric compact space, then \( H \) is a metric for \( 2^X \).

Since \( 2^X \) equipped with the metric \( H \) is a continuum (cf. [2, Corollary 14.10, p.114]), we also consider \( H_2 \) the Hausdorff metric by \( 2^{2^X} \) induced by \( H \). The hyperspaces \( C(X) \), \( C(p,X) \) and \( C(A,X) \) are considered as subspaces of \( 2^X \) and \( K(X) \) as a subspace of \( 2^{2^X} \). The topological structure of the hyperspace \( K(X) \) was studied first by P. Pellicer in [10], the author shows that the hyperspace \( K(X) \) is not always a continuum and gives conditions to ensure that \( K(X) \) is compact, connected, arcwise connected and locally connected.

By a finite graph we mean a continuum \( X \) which can be written as the union of finitely many arcs, any two of which are either disjoint or intersect only in one or both of their end points. Given a positive integer \( n \), a simple \( n \)-od is a finite graph, denoted by \( T_n \), which is the union of \( n \) arcs emanating from a single point, \( v \), and otherwise disjoint from each another. The point \( v \) is called the vertex of the simple \( n \)-od. A simple \( 3-\text{od}, T_3 \), will be called a simple triod. A tree is a finite graph without simple closed curves. A \( n \)-cell is any space homeomorphic to \([0,1]^n\).

Given a finite graph \( X \), \( p \in X \) and a positive integer \( n \), we say that \( p \) is of order \( n \) in \( X \), denoted by \( \text{ord}(p,X) = n \), if \( p \) has a closed neighborhood which is homeomorphic to a simple \( n \)-od having \( p \) as the vertex. The points of order 1, 2 or \( \geq 3 \) are called end points, ordinary points and ramification points and denoted by \( E(X) \), \( O(X) \) and \( R(X) \), respectively. Also define vertices as the points in \( V(X) := E(X) \cup R(X) \). An edge \( J \) is any arc joining two points \( p,q \in V(X) \) \((p = q \text{ is allowed})\) and containing no other vertices, we will write \( J = pq \) and \( (pq) = J - \{p,q\} \).
Let $A, B \in C(X)$. An order arc from $A$ to $B$ is a mapping (i.e. a continuous function) $\alpha : [0, 1] \to C(X)$ such that $\alpha(0) = A$, $\alpha(1) = B$, and $\alpha(r) \subsetneq \alpha(s)$ whenever $r < s$. Existence and other properties of order arcs if $A \subsetneq B$ can be found in [6, 1.2-1.8].

Given a positive integer $n$, we say that a space $X$ is $\frac{1}{n}$-homogeneous provided that the natural action of the group of homeomorphisms of the space $X$ onto itself has exactly $n$ orbits. In this case we say that $n$ is the homogeneity degree of $X$.

Given continua $X$ and $Y$, we write $X \approx Y$ if there exists an homeomorphism between $X$ and $Y$.

For a mapping $f : X \to Y$ between continua, we consider the induced mapping by $f$, $C(f) : C(X) \to C(Y)$, given by $C(f)(A) = f(A)$. In case that $f$ is a homeomorphism, for each $p \in X$, $C(f) \vert_{C(p, X)} : C(p, X) \to C(f(p), Y)$ is a homeomorphism (see [8, Lemma 3.4, p. 262]). The following result is easy to prove.

**Proposition 2.2.** If $X$ is an $\frac{1}{n}$-homogeneous continuum, then $K(X)$ has size $m \leq n$.

Concerning Proposition 2.2, the equality $m = n$ does not hold in general, for example if $X$ is a 2-cell, $X$ is $\frac{1}{2}$-homogeneous, but for each $p \in X$, $C(p, X)$ is a Hilbert cube ([1, Theorem 4, p. 221]).

The following is an example of a finite graph $X$, such that $K(X)$ has smaller size than the homogeneity degree of $X$.

**Example 2.3.** In the Euclidean plane consider $X = S \cup L \cup J$ where:

- $S = \{(x, y) : (x + 2)^2 + y^2 = 1\}$,
- $L = \{(x, 0) : x \in [-1, 1]\}$,
- $J = \{(1, y) : y \in [-1, 1]\}$. 

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In this case, $K(X)$ has size at most 5 (using the results of the following section and Theorem 5.2 it can be verified that $K(X)$ has size exactly 5) and it is easy to see that $X$ is $\frac{1}{6}$-homogeneous.

3. $C(p, X)$ of finite graphs

In this section we suppose that all finite graphs $X$ have the metric, $d$, given by the arc length, i.e. if $x, y \in X$ the distance in $X$ from $x$ to $y$ will be the length of a shortest path connecting $x$ and $y$ in $X$. We will also suppose that each edge has length equal to 1. In this case, if $x, y \in X$ belongs to the same edge, we say that $z$ is the midpoint of $xy$ if $d(x, z) = d(z, y)$.

Lemma 3.1. Let $X$ be a finite graph, $p \in X$ and $A \in C(p, X)$.

(i) If $p \in E(X)$, then there exists an arc $L \subset C(p, X)$ such that $A \in L$.

(ii) If $p \in O(X)$, then there exists a 2-cell $A \subset C(p, X)$ such that $A \in A$.

Proof. In order to prove (i) take an order arc $L$ in $C(p, X)$ from $\{p\}$ to $X$ through $A$ (see [2, Theorem 14.6, p. 112]).

Let $p \in O(X)$. If $X$ is a simple closed curve it is easy to see that $C(p, X)$ is homeomorphic to an 2-cell. So we can suppose that $X$ is not a simple closed curve. We consider then two cases:

1. If $A - \{p\}$ is not connected, let $C_1, C_2$ be the components of $A - \{p\}$. For $i \in \{1, 2\}$, let $\alpha_i : [0, 1] \to C(p, X)$ be an order arc from $\{p\}$ to $C_i \cup \{p\}$ and define $h : [0, 1] \times [0, 1] \to C(p, X)$ as $h(s, t) = \alpha_1(s) \cup \alpha_2(t)$. Note that $h$ is a embedding. Thus $A = h([0, 1] \times [0, 1])$ is a 2-cell contained in $C(p, X)$ and $h(1, 1) = A$. 

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2. If $A - \{p\}$ is connected, let $J = vw$ be the edge of $X$ containing $p$, where $v$ and $w$ are vertices of $X$.

In case that $p$ is an end point of $A$, we can assume, without loss of generality, that there exists an arc $L \subset pw$ such that $L \cap A = \{p\}$. Let $\alpha, \beta : [0, 1] \rightarrow C(p, X)$ be order arcs from $\{p\}$ to $A$, and from $\{p\}$ to $L$, respectively. Define $h : [0, 1] \times [0, 1] \rightarrow C(p, X)$ as $h(s, t) = \alpha(s) \cup \beta(t)$ and we conclude as above since $h(1, 0) = A$.

If $p \in O(A)$, let $m$ be the midpoint of $pw$ and let $a, b$ be the midpoints of $pm$ and $mw$, respectively. Since $A - \{p\}$ is connected, $A - ab$ is connected and contains $p$. Let $C$ be the closure in $X$ of $A - ab$. Let $\alpha : [0, 1] \rightarrow C(p, X)$ be an order arc from $C$ to $C \cup am$ and $\beta : [0, 1] \rightarrow C(p, X)$ and order arc from $C$ to $C \cup mb$. By defining $h : [0, 1] \times [0, 1] \rightarrow C(p, X)$ as $h(s, t) = \alpha(s) \cup \beta(t)$ we obtain the result since $h(1, 1) = C \cup am \cup mb = A$.

Let $k \in \mathbb{N}$. A continuum $Y$ is a $k$-od provided there exists $M \in C(Y)$ such that $Y - M$ has at least $k$ components. In this case, we will say that $M$ is a core of the $k$-od. It is trivial that any $(k+1)$-od is also an $k$-od. The reader should take care that even in the case of finite graphs, for a point $p$ in $X$, there is subtle difference between $ord(p, X) = k$ and $p$ belongs to the core of a $k$-od in $X$, for example, each point in a simple closed curve has order equal 2, but the simple closed curve is not a 2-od. In the Example 2.3, let $M$ be

$$M = \{(x, y) \in X : -2 \leq x < 1\} \cup \{q\}.$$ 

Here $M$ is the core of an 3-od containing $(0, 0)$ and $(0, 0)$ has order equal 2.

Related to these concepts we present the following well known result about $k$-cells in $C(p, X)$ (cf. [12, Corollary 3.12, p. 1006]).

**Theorem 3.2.** Given a continuum $X$ and $p \in X$, $C(p, X)$ contains a $k$-cell if and only if the point $p$ is contained in the core of a $k$-od.

For the class of finite graphs, the following result is similar to the last one and can be considered a generalization in some particular cases.
Theorem 3.3. Let $X$ be a finite graph and $p \in X$. Then for each $A \in C(p, X)$ and $\varepsilon > 0$, there exists a subset $A \subset C(p, X)$ such that $A$ contains an $n$-cell, where $n = \text{ord}(p, X)$ and $H_2(A, \{A\}) < \varepsilon$.

Proof. By Lemma 3.1 it is only necessary to consider the case when $p \in R(X)$.

Suppose $0 < \varepsilon < \min\{d(x, y) : x, y \in V(X), x \neq y\}/4$. Let $L_1, L_2, \cdots, L_n$ be the edges of $X$ emanating from $p$. For each component $C$ of $A - \{p\}$, note that $C \cup \{p\}$ is a subgraph of $X$ such that either

i) $p \in R(C \cup \{p\}) \cup O(C \cup \{p\})$ or

ii) $p \in E(C \cup \{p\})$.

If $p \in O(C \cup \{p\}) \cup R(C \cup \{p\})$, then $m(C) := \text{ord}(p, C \cup \{p\}) \geq 2$. Without loss of generality, suppose that $L_1, L_2, \cdots, L_{m(C)}$ are edges of $X$ contained in $C \cup \{p\}$. Let $m_i$ be the midpoint of $L_i$. For $i \in \{1, 2, \cdots, m(C) - 1\}$, let $a_i, b_i \in L_i$ such that $m_i \in a_i b_i$ and $d(a_i, m_i) = d(m_i, b_i) = \varepsilon/2$. Let $G_C = (C \cup \{p\}) \setminus \bigcup_{i=1}^{m(C)-1} (a_i b_i)$, this is clearly a continuum. Moreover, for each $D \in C(G_C, C \cup \{p\})$, $H(D, C \cup \{p\}) < \varepsilon$.

For each $i \in \{1, 2, \cdots, m(C) - 1\}$, let $a_i', b_i'$ be the midpoints of $a_i, m_i$, and $m_i, b_i$, respectively. Let $J_i^a(t)$ be the arc in $a_i a_i'$ containing $a_i$, such that the length of $J_i^a(t)$ is equal $t$, for $t \in [0, \varepsilon/4]$. We define $J_i^b(t)$ in a similar way, for $b_i b_i'$. Let $\gamma_i : [0, \varepsilon/4] \rightarrow C(G_C, C \cup \{p\})$ be defined as $\gamma_i(s, t) = G_C \cup J_i^a(s) \cup J_i^b(t)$. Define $\gamma_C : [0, \varepsilon/4]^{m(C)-1} \times [0, \varepsilon/4] \rightarrow C(G_C, C \cup \{p\})$ as

$$
\gamma_C(s, t) = \bigcup_{i=1}^{m(C)-1} \gamma_i(s_i, t_i),
$$

where $s = (s_1, s_1, \cdots, s_{m(C)-1})$, $t = (t_1, t_1, \cdots, t_{m(C)-1})$. It is easy to see that $\gamma_C$ is an embedding. Let $C = \{C :$ is component of $A - \{p\}$ satisfying i $\} \cup \{C_1, \cdots, C_l\}$.

Let $\gamma : [0, \varepsilon/4]^{2m-1} \rightarrow \bigcup_{i=1}^{l} C(G_{C_i}, C_i \cup \{p\})$ given by

$$
\gamma(s_1, \cdots, s_l, t_1, \cdots, t_l) = \bigcup_{i=1}^{l} \gamma_{C_i}(s_i, t_i).
$$
It is not difficult to prove that $\gamma$ is an embedding.

If $p \in E(C \cup \{p\})$, take an order arc $\alpha_C$ in $C(C \cup \{p\})$ from $\{p\}$ to $C\cup\{p\}$. By the continuity of $\alpha_C$ in 1 there exists $\delta_C > 0$ such that if $1 - \delta_C \leq t \leq 1$, then $H(\alpha_C(t), C \cup \{p\}) < \varepsilon$.

Let $\mathcal{D} = \{C : C \text{ is component of } A - \{p\} \text{ satisfaying ii) } \}$ and $k = |\mathcal{D}|$.

Let $\alpha : \prod_{C \in \mathcal{D}} [1 - \delta_C, 1] \to \bigcup_{C \in \mathcal{D}} C(C \cup \{p\})$ defined as $\alpha(\vec{x}) = \bigcup_{C \in \mathcal{D}} \alpha(x_C)$ for $\vec{x} = (x_C)_{C \in \mathcal{D}} \in \prod_{C \in \mathcal{D}} [1 - \delta_C, 1]$. It is not difficult to prove that $\alpha$ is an embedding.

Let $L_{n_1}, L_{n_2}, \ldots, L_{n_r}$ be the edges such that there exist an arc $Z_i \subset L_{m_i}$ with $Z_i \cap A = \{p\}$. Observe that $r = n - (m + k)$. For each $i \in \{1, 2, \ldots, r\}$ let $\beta_i$ be an order arc in $C(Z_i)$ from $\{p\}$ to $Z_i$.

By continuity of $\beta_i$ in 0, there exists $\delta_i > 0$ such that if $0 \leq t \leq \delta_i$, then $H(\{p\}, \beta_i(t)) < \varepsilon$ and $\beta_i(t) \cap A = \{p\}$. Let $\beta : \prod_{i=1}^{r} [0, \delta_i] \to \bigcup_{i=1}^{r} L_{m_i}$ de defined as $\beta(\vec{x}) = \bigcup_{i=1}^{r} \beta_i(x_i)$ for $\vec{x} = (x_1, \ldots, x_r) \in \prod_{i=1}^{r} [0, \delta_i]$. It is easy to prove that $\beta$ is an embedding.

Let $h_{2m-l} : [0, 1]^{2m-l} \to [0, \varepsilon/4]^{2m-l}$, $h_k : [0, 1]^k \to \prod_{C \in \mathcal{D}} [1 - \delta_C, 1]$ and $h_k : [0, 1]^r \to \prod_{i=1}^{r} [0, \delta_i]$ be homeomorphisms.

We define $h : [0, 1]^{2m-l} \times [0, 1]^k \times [0, 1]^r \to C(p, X)$ as

$$h(\vec{x}, \vec{y}, \vec{z}) = \gamma \circ h_{2m-l}(\vec{x}) \cup \alpha \circ h_k(\vec{y}) \cup \beta \circ h_r(\vec{z})$$

where $\vec{x} \in [0, 1]^{2m-l}$, $\vec{y} \in [0, 1]^k$ and $\vec{z} \in [0, 1]^r$. Note that $h$ is an embedding, thus, $\mathcal{A} := Im(h)$, the image of $h$, is a $(2m - l + k + r)$-cell contained in $C(p, X)$. Moreover, by the metric in $X$, $H_2(\mathcal{A}, \{A\}) < \varepsilon$. Since $m_i(C) \geq 2$, and $m = \sum_{i=1}^{l} m_i(C) \geq 2l,

2m - l + k + r = 2m - l + k + (n - k - m) = m + n - l \geq n.$

Then $\mathcal{A}$ contains an $n$-cell. \qed

**Corollary 3.4.** Let $X$ be a finite graph and $p, q \in X$ such that $C(p, X) \approx C(q, X)$. Then $ord(p, X) = ord(q, X)$. 

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Proof. Suppose that \( n = \text{ord}(p, X) < \text{ord}(q, X) = m \) and let \( h : C(p, X) \to C(q, X) \) be a homeomorphism. Since \( \{p\} \) has a neighborhood base in \( C(p, X) \) consisting of \( n - 1 \) cells, \( h(\{p\}) \) has a neighborhood base in \( C(q, X) \) consisting of \( n - 1 \) cells which is impossible by Theorem 3.3, thus \( m \leq n \). A similar argument implies that \( n \leq m \). Thus, \( n = m \).

The following is an easy consequence of the previous corollary, although it is trivial, illustrates necessary conditions for \( C(p, X) \) and \( C(q, X) \) to be homeomorphic.

**Remark 3.5.** Let \( X \) be a finite graph and \( p, q \in X \) such that \( C(p, X) \approx C(q, X) \). Let \( \mathcal{H}(X) \) be one of \( E(X), O(X) \) or \( R(X) \). It holds that if \( p \in \mathcal{H}(X) \) then also \( q \in \mathcal{H}(X) \).

Concerning previous remark, observe that if \( X \) is a finite graph such that \( K(X) \) has size 1, then each point in \( X \) is an ordinary point, thus, by [7, Proposition 9.5] we conclude that \( X \) is a simple closed curve. We rewrite this in the next result, which gives a characterization of simple closed curves.

**Corollary 3.6.** For a finite graph \( X \) the following conditions are equivalent:

1. \( X \) is a simple closed curve,
2. \( K(X) \) has size 1,
3. \( X \) is homogeneous.

Let \( X \) be a finite graph. If \( q \in E(X) \) and \( X \) is not an arc, we denote by \( v(q) \) the unique point in \( R(X) \) such that the component \( C \), of \( X - \{v(q)\} \), containing \( q \), satisfies that \( C \cup \{v(q)\} \) is an arc.

**Proposition 3.7.** Let \( X \) be a finite graph which is not an arc. If \( e_1, e_2 \in E(X) \) and \( C(e_1, X) \approx C(e_2, X) \) then \( \text{ord}(v(e_1), X) = \text{ord}(v(e_2), X) \).

*Proof.* Let \( h : C(e_1, X) \to C(e_2, X) \) be a homeomorphism. Denote by \( l_1 \) and \( l_2 \), the edges of \( X \), \( e_1v(e_1) \) and \( e_2v(e_2) \), respectively. By Theorem 3.3 we have that for each \( A \in C(e_1, l_1) - \{l_1\} \), \( h(A) \in C(e_2, l_2) \). By the continuity of \( h, h(l_1) \in C(e_2, l_2) \). Again by Theorem 3.3, \( h(l_1) = l_2 \) and \( \text{ord}(v(e_1), X) = \text{ord}(v(e_2), X) \). \( \square \)
Definition 3.8. Let $X$ be a finite graph, $p \in O(X)$ and $v, w \in V(X)$ are the end points of the edge containing $p$. Define

$$
\Sigma(p, X) = \begin{cases} 
\text{ord}(v, X) + \text{ord}(w, X), & \text{if } v \neq w, \\
\text{ord}(v, X), & \text{if } v = w.
\end{cases}
$$

The following lemma is a particular case of [12, Corollary 3.5, p. 1006].

Lemma 3.9. Let $X$ be a finite graph. If $A$ is an edge of $X$ and $p \in O(X) \cap A$, then there exist $A \subset C(p, X)$ such that $A$ is contained in a $\Sigma(p, X)$-cell.

Proposition 3.10. Let $X$ be a finite graph. If $p, q \in O(X)$ and $C(p, X) \approx C(q, X)$ then $\Sigma(p, X) = \Sigma(q, X)$.

Proof. Let $h : C(p, X) \rightarrow C(q, X)$ be a homeomorphism and let $l_1$ and $l_2$ be the edges containing $p$ and $q$, respectively. Suppose that $a, b \in l_1$ and $c, d \in l_2$ are vertices in $X$. Using Theorem 3.3 we can see that for each $A \in C(p, l_1)$ such that $A \subset l_1 - \{a, b\}$, $h(A) \in C(q, l_2)$. Then, by the continuity of $h$, $h(l_1) \subset l_2$. By Theorem 3.3 we have that $h(l_1) = l_2$ and by Lemma 3.9 this implies that $\Sigma(p, X) = \Sigma(q, X)$.

4. Finite graphs with $K(X)$ having size $n$

The purpose of this section is to use the previous results to construct for each positive integer $n$, a finite graph $X_n$ such that $K(X_n)$ has size $n$. For this, we will start introducing notation for some sets in the Euclidean Plane, $\mathbb{R}^2$. For each $n \in \mathbb{N}$:

- $I_n = [n - 1, n] \times \{0\}$;
- $C_n$ will denote the circle with radius $\frac{1}{4}$ and center $(n, \frac{1}{4})$;
- $C$ will denote the circle with radius $\frac{1}{2}$ and center $(2, \frac{1}{8})$;
- for each $i \in \mathbb{N}$, we consider $l_i^{(n)}$ as the linear segment joining $(n, 0)$ and $(n - \frac{1}{4}, -1)$;
• if \( n \geq 3 \), \( S_n = I_n \cup C_n \cup \bigcup_{i=1}^{2n} l_i^{(n)} \);

• \( P_1 = I_2 \cup C_2 \cup C \);

• \( P_2 = I_2 \cup C_1 \cup C_2 \cup C \);

• \( P_3 = I_2 \cup C_2 \cup C \cup \bigcup_{i=1}^{3} l_i^{(1)} \);

• \( P_4 = I_2 \cup C_1 \cup C_2 \cup C \cup l_1^{(1)} \);

• \( P_5 = I_2 \cup C_2 \cup C \cup l_1^{(1)} \cup l_2^{(1)} \cup l_1^{(2)} \cup l_2^{(2)} \).

Next we give a picture of these sets.

We define \( X_n \) as follows:

1. Let \( X_1 \) be a simple closed curve, \( X_2 \) an arc and \( X_3 \) a simple triod.

2. \( X_{i+3} = P_i \) for \( i \in \{1, 2, 3, 4, 5\} \),

3. for each \( n \geq 9 \) we define \( X_n \) recursively as follows, write \( n \) as \( n = 5(k+1)+r \), where \( r \in \{-1, 0, 1, 2, 3\} \), then

\[
X_n = X_{5k+r} \cup S_{k+2}.
\]

To clarify the last formula in 3 we draw some pictures.

By using Remark 3.5 and Propositions 3.7 and 3.10 we obtain the following result.
**Theorem 4.1.** For each $n$, $K(X_n)$ has size $n$.

Concerning Theorem 4.1, note that for each $n$, $X_n$ is a $\frac{1}{n}$-homogeneous continuum. On the other hand Example 2.3 shows a $\frac{1}{n}$-homogeneous finite graph $X$, such that $K(X)$ has size 5. We will prove this fact in the next section by proving a more general result.

By using the same ideas of the previous section, it can be proved that the continuum in the next picture has countably many topologically distinct types of $C(p, X)$.

![Continuum Diagram]

**Question 4.2.** Does there exist a continuum $X$ such that the cardinality of $K(X)/\sim$ is equal to $\mathfrak{c}$, the cardinality of the continuum?

By the arguments given above the Proposition 2.2, such a continuum must have uncountably many orbits.

It is known by [3, Theorem 9] that the pseudocircle $X$ has uncountably many orbits, however since it is hereditarily indecomposable each element of $K(X)$ is an arc ([9, Lemma 3.19]).

5. $\frac{1}{n}$-homogeneous continua with size smaller than $n$

We have seen in the last section that for every positive integer $n$ there exist a finite graph $X_n$ with $K(X_n)$ having size $n$. In our construction, all the examples $X_n$ are $1/n$-homogeneous. Since the homogeneity degree is an upper bound for the size of $K(X)$, it is natural to ask if it is possible to give families of $1/n$-homogeneous finites graphs $Y_n$ such that $K(Y_n)$ has size $< n$.

**Definition 5.1.** Given a continuum $X$ and $p, q \in X$ we say that $X$ is **pseudo–symmetric with respect to $p$ and $q$, provided that there exists a homeomorphism $\varphi : C(p, X) \to C(q, X)$ such that $\varphi(\{p\}) = \{q\}$ and $\varphi(X) = X$.

As examples of pseudo–symmetric continua we have the following:
• The continuum $X$ in Example 2.3 is pseudo–symmetric with respect to the end points $(1, 1)$ and $(1, -1)$.

• Each homogeneous continuum is pseudo–symmetric with respect to each pair of its points.

• By [9, Lemma 3.19] every hereditarily indecomposable continuum is pseudo–symmetric with respect to each pair of its points.

**Theorem 5.2.** Let $Y$ be a pseudo–symmetric continuum with respect to $p$ and $q$. Let $X = L \cup Y \cup K$, where $L$ and $K$ are continua such that $L \cap Y = \{p\}$, $K \cap Y = \{q\}$ and $K \cap L = \emptyset$. Suppose that there exists a homeomorphism $f : C(p, L) \to C(p, K)$ such that $f(\{p\}) = \{q\}$ and $f(L) = K$. Then $C(p, X)$ is homeomorphic to $C(q, X)$.

**Proof.** Since $K$ and $L$ are disjoint continua, $L \cap Y = \{p\}$ and $K \cap Y = \{q\}$, then $p$ and $q$ are cut points of $X$. Moreover $p$ is a cut point of $L \cup Y$ and $q$ is a cut point of $K \cup Y$. Thus it is easy to see that the function $g : C(Y, X) \to C(p, L) \times C(q, K)$ defined as $g(A) = (A \cap L, A \cap K)$, for each $A \in C(Y, X)$ is a homeomorphism.

Since $Y$ is pseudo–symmetric with respect to $p$ and $q$, there exists a homeomorphism $\varphi : C(p, Y) \to C(q, Y)$ such that $\varphi(\{p\}) = \{q\}$ and $C(Y) = Y$.

Let $h : C(p, X) \to C(q, X)$ be defined as

$$h(A) = \begin{cases} \varphi(A \cap Y) \cup f(A \cap L), & \text{if } A \in C(p, L \cup Y); \\ g^{-1}(f^{-1}(A \cap K), f(A \cap L)), & \text{if } A \in C(Y, X). \end{cases}$$

Note that if $A \in C(p, L \cup Y) \cap C(Y, X)$ then $A \cap K = \{q\}$, and $A \cap Y = Y$. Thus

$$g^{-1}(f^{-1}(A \cap K), f(A \cap L)) = g^{-1}(f^{-1}(\{q\}), f(A \cap L))$$

$$= Y \cup f(A \cap L) = \varphi(A \cap Y) \cup f(A \cap L).$$

Hence, $h$ is well-defined and since $\varphi$, $f$ and $g$ are homeomorphisms, we conclude that $h$ is a homeomorphism.

\[ \square \]

Using the notation of the previous theorem, if we suppose that in addition $X$ is $\frac{1}{n}$-homogeneous and, $p$ and $q$ are in different orbits under the
accion of the homeomorphism group, then \( K(X) \) has size smaller than \( n \). As a consequence, it is easy to see that the continuum \( X \) in Example 2.3 is pseudo–symmetric with respect to \( p \) and \( q \).

Next we will construct a family of finite graphs with hyperspace \( K(X) \) having size \( n \) and homogeneity degree greater than \( n \). The main idea is to use the Theorem 5.2 pasting two disjoint continua \( L \) and \( K \) in a pseudo–symmetric continuum \( X \) in points \( p \) and \( q \), respectively, such that \( C(p, L) \) and \( C(q, K) \) are homeomorphic, but in such a way that there is no homeomorphism between \( L \) and \( K \) sending \( p \) to \( q \). In order to do this, observe that each one of the graphs \( P_i \) given in the last section are pseudo–symmetric with respect to the points \((2, \frac{1}{2})\) and \((2, \frac{1}{4})\), \( i \in \{1, 2, 3, 4, 5\} \). We will make a modification to these graphs, attaching an arc and a circumference in the points \((2, \frac{1}{4})\) and \((2, \frac{1}{2})\), respectively, getting the new graphs \( Q_i \), which are illustrated below:

Similar to the construction of the graphs \( X_n \), by using the same symbols as in the last section, we define for each \( n \geq 4 \), a finite graph \( Y_n \) as follows:

1. \( Y_{i+3} = Q_i \) for \( i \in \{1, 2, 3, 4, 5\} \);

2. for each \( n \geq 9 \), we write \( n \) as \( n = 5(k+1)+r \), where \( r \in \{-1, 0, 1, 2, 3\} \), and define

\[
Y_n = Y_{5k+r} \cup S_{k+2}.
\]
We consider furthermore the following finite graphs:

\[ Y_1 \quad Y_2 \quad Y_3 \]

By using Theorem 5.2, we have that for each \( n \), \( Y_n \) is a \( \frac{1}{n+5} \)-homogeneous finite graph such that \( K(Y_n) \) has size \( n + 4 \). By this and Corollary 3.6, it remains to solve the following question.

**Question 5.3.** If \( n \in \{2, 3, 4\} \), does there exists a \( \frac{1}{n+1} \)-homogeneous finite graph, \( X \), such that \( K(X) \) has size \( n \)?

We conclude this paper with the following natural problem, which is still open.

**Problem 5.4.** Characterize the class of finite graphs \( X \), such that \( X \) is \( \frac{1}{n} \)-homogeneous and \( K(X) \) has size \( n \), where \( n \) is a positive integer.

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