The Complexity of Concurrent Rational Synthesis

Rodica Condurache and Youssouf Oualhadj

Université Paris Est, LACL(EA 4219), UPEC, 94010 Créteil Cedex, France
rodica.bozianu@lacl.fr
youssouf.oualhadj@lacl.fr

Abstract. In this paper, we investigate the rational synthesis problem for concurrent game structure for a variety of objectives ranging from reachability to Muller condition. We propose a new algorithm that establishes the decidability of the non cooperative rational synthesis problem that relies solely on game theoretic technique as opposed to previous approaches that are logic based. Given an instance of the rational synthesis problem, we construct a zero-sum turn-based game that can be adapted to each one of the afore mentioned objectives thus obtain new complexity results. In particular, we show that reachability, safety, B"uchi, and co-B"uchi conditions are PSPACE-complete, Parity, Muller, Street, and Rabin are PSPACE-hard and in EXPTime.

1 Introduction

The synthesis problem aims at automatically designing a program from a given specification. Several applications for this formal problem can be found in the design of interactive systems i.e. systems interacting with an environment. From a formal point of view, the synthesis problem is traditionally modelled as a zero-sum turn-based game. The system and the environment are modeled by two agents with opposite interest. The goal of the system is the desired specification. Hence, a strategy that allows the system to achieve its goal against any behavior of the environment is a winning strategy and is exactly the program to synthesize.

For a time, the described approach was the standard in the realm of controller synthesis. However, due to the variety of systems to model, such a pessimistic view is not always the most faithful one. For instance, consider a system that consists of a server and $n$ clients. Assuming that all the agents have opposite interests is not a realistic assumption. Indeed, from a design perspective, the purpose of the server is to handle the incoming requests. On the other hand, each client is only concerned with its own request and wants it granted. None of the agents involved in the described interaction have antagonistic purposes. The setting of non-zero-sum games was proposed as model with more realistic assumptions.

In a non zero-sum game, each agent is equipped with a personal objective and the system is just a regular agent in the game. The agents interact together aiming at achieving the best outcome. The best outcome in this setting is often
formalized by the concept of Nash equilibria. Unfortunately, a solution in this setting offers no guarantee that a specification for a given agent is achieved, and in a synthesis context one wants to enforce a specification for one a subset of the agents.

The rational synthesis problem was introduced as a generalization of the synthesis problem to environment with multiple agents [3]. It aims at synthesizing a Nash equilibrium such that the induced behavior satisfies a given specification. This vision enjoys nice algorithmic properties since it matches the complexity bound of the classical synthesis problem. Latter on, yet another version of the problem was proposed where the agents are rational but not cooperative [5]. In the former formalization, the specification is guaranteed as long as the agents agree to behave according to the chosen equilibrium. But anything can happen if not, in particular they can play another equilibrium that does not satisfy the specification. In the Non Cooperative Rational Synthesis (NCRSP), the system has to ensure that the specification holds in any equilibrium (c.f. Section 3 for a formal definition and Figure 1 for an example). A solution for both problems was presented for specifications expressed in Linear Temporal Logic (LTL). The proposed solution relies on the fact that the problem can be expressed in a decidable fragment of a logic called Strategy Logic. The presented algorithm runs in 2-ExpTime. While expressing the problem in a decidable fragment of Strategy Logic gives an immediate solution, it could also hide a great deal of structural properties. Such properties could be exploited in a hope of designing faster algorithms for less expressive objectives. In particular, specifications such as reachability, liveness, fairness, etc.

In [2], the first author took part in a piece of work where they considered this very problem for specific objectives such as reachability, safety, Büchi, etc. in a turn-based interaction model. They established complexity bounds for each objective.

In this paper we consider the problem of non-cooperative rational synthesis with concurrent interaction. We address this problem for a variety of objectives and give exact complexity bounds relying exclusively on techniques inspired by the theory of zero-sum games. The concurrency between agents raises a formal challenge to overcome as the techniques used in [2] do not directly extend. Intuitively, when the interaction is turn-based, one can construct a tree automaton that accepts solutions for the rational synthesis problem. The nodes of an accepted tree are exactly the vertices of the game. This helps a lot in dealing with deviations but cannot be used in concurrent games.

In Section 3.1, we present an alternative algorithm that solves the general problem for LTL specification. This algorithm constructs a zero-sum turn-based game. This fresh game is played between Constructor who tries to construct a solution and Spoiler who tries to falsify the proposed solution. We then show in Section 3 how to use this algorithm to solve the NCRSP for reachability, safety, Büchi, co-Büchi, Parity, Rabin, Street, and Muller conditions. We also observe that we match the complexity results for the NCRSP in turn-based games.
2 Preliminaries

2.1 Concurrent Game Structures

We consider concurrent game structure in which the successor state is determined by the tuple of action of the agents acting on the model. Formally, a game structure is defined as a tuple $G = (St, s_0, Agt, (Act_i)_{i \in Agt}, Tab)$, where $St$ is the set of states in the game, $s_0$ is the initial state, $Agt = \{0, 1, ..., n\}$ is the set of agents, and $Act_i$ is the set of actions of Agent $i$. $Tab : St \times \prod_{i \in Agt} Act_i \rightarrow St$ is the transition table. Note that we consider game structures that are complete and deterministic. That is, from each state $s$ and any tuple of actions $\bar{a} \in \prod_{i \in Agt} Act_i$, there is exactly one successor state $s'$. A play in the game structure is a sequence of states and actions profile $\rho = s_0 \bar{a}_0 s_1 \bar{a}_1 s_2 \bar{a}_2 ...$ in $(St \prod_{i \in Agt} Act_i)^*$ such that $s_0$ is the initial state and for all $j \geq 0, s_{j+1} = Tab(s_j, \bar{a}_j)$. We denote by $\rho[j]$ the $j$-th state along the play $\rho$, $\rho[0..j]$ is the prefix of $\rho$ up to position $j$, by $\rho_{\upharpoonright St}$ we mean the projection of $\rho$ over $St$, and $Plays(G)$ is the set of all the plays in the game structure $G$.

In this paper we allow agents to see the actions played between states. Therefore, they behave depending on the past sequence of states and tuples of actions. A strategy for Agent $i$ is a mapping $\sigma_i : St \left( \prod_{i \in Agt} Act_i, St \right)^* \rightarrow Act_i$. Note that, since the transition’s table is deterministic, we even forget the states along plays and write $\sigma_i : St \left( \prod_{i \in Agt} Act_i \right)^* \rightarrow Act_i$.

A strategy profile is defined as a tuple of strategies $\bar{\sigma} = (\sigma_0, \sigma_1, ..., \sigma_n)$ and by $\bar{\sigma}[i]$ we denote the strategy of $i$-th position (of Agent $i$). Also, $\bar{\sigma}_{-i}$ is a strategy profile $\bar{\sigma}$ from which the strategy of Agent $i$ is ignored and by $\langle \bar{\sigma}_{-i}, \sigma' \rangle$ the strategy profile in which Agent $i$ changes to strategy $\sigma'$. That is, $\bar{\sigma}_{-i} = (\sigma_0, ..., \sigma_{i-1}, \sigma_{i+1}, ..., \sigma_n)$ and $\langle \bar{\sigma}_{-i}, \sigma' \rangle = (\sigma_0, ..., \sigma_{i-1}, \sigma', \sigma_{i+1}, ..., \sigma_n)$.

We say that a play $\rho = s_0 \bar{a}_0 s_1 \bar{a}_1 s_2 \bar{a}_2 ...$ in $(St \prod_{i \in Agt} Act_i)^*$ is compatible with a strategy $\sigma_i$ of Agent $i$ if for every prefix $\rho[0..n]$ of length $n \geq 0$ we have $\sigma(\rho[0..n]) = \bar{a}_{n+1}(i)$ where $\bar{a}_{n+1}(i)$ is the action of Agent $i$ in the vector $\bar{a}_{n+1}$.

We denote by $Plays(\sigma_i)$ the set of all the plays that are compatible with the strategy $\sigma_i$ for Agent $i$. The outcome of an interaction between agents following a certain strategy profile $\bar{\sigma}$ defines a unique play in the game structure $G$ denoted $\text{out}(\bar{\sigma})$. It is the unique play in $G$ compatible with all the strategies in the profile $\bar{\sigma}$ which is an infinite sequence over $(St \prod_{i \in Agt} Act_i)$.

We call history any finite sequence in $St \left( \prod_{i \in Agt} Act_i, St \right)^*$. For a history $h$, we denote by $h \upharpoonright_{St}$ its projection over $St$, and by $\text{Last}_{St}(h)$ the last element of $h \upharpoonright_{St}$. We denote $\sigma_i^h$ the strategy that mimics the strategy $\sigma_i$ when the current history is $h$ i.e. $\sigma_i^h(h') = \sigma_i(h \cdot h')$.

2.2 Payoff and Solution Concepts

Each Agent $i \in Agt$ has an objective expressed as a set $\Theta_i$ of infinite sequences of states in $G$. We say that a play $\rho$ satisfies the objective of Agent $i$ if $\rho \upharpoonright_{St} \in \Theta_i$. 
We slightly abuse notation and write $\rho \in \Theta_i$. We define the payoff function that associates to each play $\rho$ a vector $\text{Payoff}_i(\rho) \in \{0, 1\}^{n+1}$ defined by $\text{Payoff}_i(\rho) = 1$ iff $\rho \mid_{\text{St}}$ is in $\Theta_i$. We borrow game theoretic vocabulary and say that Agent $i$ wins whenever her payoff is 1. We sometimes abuse this notation and write $\text{Payoff}_i(\bar{\sigma})$, this is the payoff of Agent $i$ associated with the play induced by $\bar{\sigma}$.

In this paper we are interested in winning objectives such as Safety, Reachability, Büchi, coBüchi, Muller and LTL that are defined as follows. Let $\rho$ be a play in a game structure $G$. $\text{occ}(\rho) = \{s \in \text{St} \mid \exists j \geq 0 \text{ s.t. } \rho[j] = s\}$ denotes the set of states that appear along $\rho$ and $\inf(\rho) = \{s \in \text{St} \mid \forall j \geq 0, \exists k \geq j \text{ s.t. } \rho[k] = s\}$ is the set of states appearing infinitely often along $\rho$. Then,

- Reachability: For some $T \subseteq \text{St}$, $\text{Reach}(T) = \{\rho \in \text{St}^{\omega} \mid \text{occ}(\rho) \cap T \neq \emptyset\}$;
- Safety: For some $T \subseteq \text{St}$, $\text{Safe}(T) = \{\rho \in \text{St}^{\omega} \mid \text{occ}(\rho) \subseteq T\}$;
- Büchi: For some $T \subseteq \text{St}$, $\text{Büchi}(T) = \{\rho \in \text{St}^{\omega} \mid \inf(\rho) \cap T \neq \emptyset\}$;
- coBüchi: For some $T \subseteq \text{St}$, $\text{coBüchi}(T) = \{\rho \in \text{St}^{\omega} \mid \inf(\rho) \cap T = \emptyset\}$;
- Parity: For some priority function $\rho : \text{St} \rightarrow \mathbb{N}$, $\text{Parity}(\rho) = \{\rho \in \text{St}^{\omega} \mid \min\{\rho(s) \mid s \in \inf(\rho)\} \text{ is even}\}$;
- Muller: For some boolean formula $\mu$ over $\text{St}$, $\text{Muller}(\mu) = \{\rho \in \text{St}^{\omega} \mid \inf(\rho) \models \mu\}$;
- LTL: For some LTL formula $\varphi$, $\text{LTL}(\varphi) = \{\rho \in \text{St}^{\omega} \mid \rho \models \varphi\}$;

A Nash equilibrium is the formalisation of a situation where every agent is satisfied with its payoff. Formally:

**Definition 1.** A strategy profile $\bar{\sigma}$ is a Nash equilibrium (NE) if for every agent $i$ and any strategy $\sigma'$ of $i$ the following holds true:

$$\text{Payoff}_i(\bar{\sigma}) \geq \text{Payoff}_i(\langle \bar{\sigma}_{-i}, \sigma' \rangle).$$

Throughout this paper, we will assume that Agent 0 is the agent for whom we wish to synthesize the strategy, therefore, we use the concept of 0-fixed Nash equilibria.

**Definition 2.** A profile $\langle \sigma_0, \bar{\sigma}_{-0} \rangle$ is a 0-fixed NE, if for every strategy $\sigma'$ for agent $i$ in $\text{Agt} \setminus \{0\}$ the following holds true:

$$\text{Payoff}_i(\langle \sigma_0, \bar{\sigma}_{-0} \rangle) \geq \text{Payoff}_i(\langle \sigma_0, \bar{\sigma}_{-(0,i)}, \sigma' \rangle).$$

That is under $\sigma_0$, the profile $\bar{\sigma}_{-0}$ is an equilibrium In the sequel we simply write 0-NE.

Using the above definitions, one can define two variants of the Rational Synthesis Problem:

**cooperative:** Is there a 0-NE $\bar{\sigma}$ such that $\text{Payoff}_0(\bar{\sigma}) = 1$?

**non-cooperative:** Is there a strategy $\sigma_0$ for Agent 0 such that for any 0-NE $\bar{\sigma} = \langle \sigma_0, \bar{\sigma}_{-0} \rangle$, we have $\text{Payoff}_0(\bar{\sigma}) = 1$?
In this paper we study computational complexity for the rational synthesis problem in both cooperative and non-cooperative settings. A summary of the obtained complexity results is in Table 1.

For the cooperative rational synthesis, the complexity results are corollaries of existing work. In particular, for Safety, Reachability, Büchi, co-Büchi, Rabin and Muller objectives, we can apply algorithms from [1] to obtain the same complexities for CRSP as for the turn-based models when the number of agents is not fixed. More precisely, in [1] the problem of finding NE in concurrent games is tackled. Brenguier et al. [1] showed that the existence of constrained NE in concurrent games can be solved in \( \text{PTime} \) for Büchi objectives, \( \text{NP} \) for Safety, Reachability and co-Büchi objectives, in \( P^{NP} \) for Rabin objectives and \( \text{PSPACE} \) for Muller objectives. All hardness results are inferred directly from the hardness results in the turn-based setting. This is a consequence of the fact that every turn-based game can be encoded as a concurrent game by allowing at each state at most one agent to have multiple choices. For Streett objectives, by reducing to [1] we only obtain \( \text{PSpace} \)-easiness and the \( \text{NP} \)-hardness comes from the turn-based setting [2].

### 3 Non-Cooperative Case

In the case of non-cooperative rational synthesis, we cannot directly apply the existing results. However, we define an algorithm inspired from the suspect games [1]. The suspect game was introduced to decide the existence of pure NE in concurrent games with \( \omega \)-regular objectives. We inspire ourselves from that approach and design a zero-sum game that combines the behaviors of Agent 0 and an extra entity whose goal is to prove, when needed, that the current play is not the outcome of a 0-NE. We also use the idea in [2] to keep track of deviations. Recall that the non-cooperative rational synthesis problem consists in

| Perfect Information Concurrent Rational Synthesis Problem |
|----------------------------------------------------------|
| Co-operative                                             | Non-Cooperative                        |
| Safety                                                  | \( \text{NP} \) \hspace{1em} \text{NP-h} \hspace{1em} \text{PSPACE} \hspace{1em} \text{PSPACE-h} \hspace{1em} \text{PTime} \hspace{1em} \text{PSpace} \hspace{1em} \text{PSPACE-h} \hspace{1em} \text{ExpTime} \) |
| Reachability                                            | \( \text{NP} \) \hspace{1em} \text{NP-h} \hspace{1em} \text{PSPACE} \hspace{1em} \text{PSPACE-h} \hspace{1em} \text{PTime} \hspace{1em} \text{PSpace} \hspace{1em} \text{PSPACE-h} \hspace{1em} \text{ExpTime} \) |
| Büchi                                                   | \( \text{PTime} \hspace{1em} \text{NP} \hspace{1em} \text{NP-h} \hspace{1em} \text{PSPACE} \hspace{1em} \text{PSPACE-h} \hspace{1em} \text{PTime} \hspace{1em} \text{PSpace} \hspace{1em} \text{PSPACE-h} \hspace{1em} \text{ExpTime} \) |
| co-Büchi                                                | \( \text{NP} \) \hspace{1em} \text{NP-h} \hspace{1em} \text{PSPACE} \hspace{1em} \text{PSPACE-h} \hspace{1em} \text{PTime} \hspace{1em} \text{PSpace} \hspace{1em} \text{PSPACE-h} \hspace{1em} \text{ExpTime} \) |
| Parity                                                  | \( P^{NP} \hspace{1em} \text{NP} \hspace{1em} \text{NP-h} \hspace{1em} \text{ExpTime} \hspace{1em} \text{PSPACE} \hspace{1em} \text{PSPACE-h} \hspace{1em} \text{ExpTime} \hspace{1em} \text{PSpace} \hspace{1em} \text{PSPACE-h} \hspace{1em} \text{ExpTime} \) |
| Streett                                                 | \( \text{PSpace} \hspace{1em} \text{NP} \hspace{1em} \text{NP-h} \hspace{1em} \text{ExpTime} \hspace{1em} \text{PSPACE} \hspace{1em} \text{PSPACE-h} \hspace{1em} \text{ExpTime} \hspace{1em} \text{PSpace} \hspace{1em} \text{PSPACE-h} \hspace{1em} \text{ExpTime} \) |
| Rabin \( P^{NP} \hspace{1em} \text{NP} \hspace{1em} \text{NP-h} \hspace{1em} \text{NP-h} \hspace{1em} \text{ExpTime} \hspace{1em} \text{PSPACE} \hspace{1em} \text{PSPACE-h} \hspace{1em} \text{ExpTime} \hspace{1em} \text{PSpace} \hspace{1em} \text{PSPACE-h} \hspace{1em} \text{ExpTime} \) |
| Muller                                                  | \( \text{PSpace} \hspace{1em} \text{PSPACE-h} \hspace{1em} \text{ExpTime} \hspace{1em} \text{PSPACE} \hspace{1em} \text{PSPACE-h} \hspace{1em} \text{ExpTime} \hspace{1em} \text{PSpace} \hspace{1em} \text{PSPACE-h} \hspace{1em} \text{ExpTime} \) |

Table 1: Our results are written in bold.
designing a strategy $\sigma_0$ for the protagonist (Agent 0 in our case) such that her objective $\Theta_0$ is satisfied by all the plays that are outcomes of 0-NE compatible with $\sigma_0$. This is equivalent to finding a strategy $\sigma_0$ for Agent 0 such that for any play $\rho$ compatible with it, either $\rho$ satisfies $\Theta_0$, or there is no strategy profile $\bar{\sigma} = \langle \sigma_0, \bar{\sigma}_{-0} \rangle$ that is a 0-NE whose outcome is $\rho$. Formally, we have to check the existence of a strategy $\sigma_0$ satisfying of the following formula:

$$\forall \rho \in \text{Plays}(\sigma_0), (\rho \models \Theta_0) \lor (\forall \bar{\sigma} = \langle \sigma_0, \bar{\sigma}_{-0} \rangle \text{ s.t. out}(\bar{\sigma}) = \rho, \bar{\sigma} \text{ is not 0-NE})$$

(1)

**Example 1.** Consider the game of Figure 1 played between three players. Agent 0 wins along any play that reaches a state labeled 1, Agent 1 wins along any play that reaches a state labeled 2, Agent 2 wins along any play that reaches a state labeled 3. The initial state is labeled 0. We want to design a solution for Agent 0, if she plays action $\ell$, then either the play reaches a winning state labeled 1 and her payoff is 1 or it reaches a state labeled 2 and her payoff is 0. Now, notice that in the latter case, the obtained play is the outcome of a 0-NE, hence this strategy for Agent 0 is not a solution for NCRSP (one can check that it satisfies neither of the members in Equation (1)). The other strategy of Agent 0 is to play $r$. By choosing this action, the play ends in the right hand side component of the game regardless of the choices of Agents 1 and 2. In this case the two possible outcomes are not induced by a 0-NE as both agents have profitable deviations.

![Fig. 1: A concurrent game where agent 0 wins by avoiding equilibria](image)

### 3.1 General Algorithm

We will now describe a general algorithm that solves the NCRSP. As a first step in our procedure, we construct a two-player turn-based game. The winning condition for the first player in this game will be derived from the formula of Equation (1). The second player wants to falsify the same formula.

Before introducing the game, we state some technical definitions and lemmas.

We say that a history $h$ is a good deviation point for Agent $i$ if she can unilaterally change her last action such that the play reaches a state from which she can win on any continuation. It is formally defined as follows:

**Definition 3.** Let $\rho$ be a play and let $h = s_0 a_0 s_1 a_1 \cdots s_k a_k s_{k+1}$ be a prefix of $\rho$. We say that $h$ is a good deviation point for Agent $i \in \text{Agt} \setminus \{0\}$ if:

- $a_i$ is the last action of $h$.
- $a_i$ is the last action of the play $\rho^i$.
- There exists a 0-NE $\bar{\sigma}$ such that out($\bar{\sigma}$) = $\rho^i$.
- There exists a strategy profile $\bar{\sigma}$ such that out($\bar{\sigma}$) = $\rho^i$.

Thus, a good deviation point for Agent $i$ is a prefix of a play that can be extended to a 0-NE that leads to a winning state for Agent $i$. This definition allows us to solve the NCRSP by finding a strategy for Agent $i$ that leads to a winning state for Agent $i$.
– \( \rho \upharpoonright s \notin \Theta_i \) and,
– there exists a strategy \( \sigma_i' \) of Agent \( i \) from \( [h] \) such that for all \( (\sigma_j)_{j \in \text{Agt}\setminus\{0,i\}} \) we have:
\[
[h] \cdot \text{out} \left( \sigma_0^{[h]}, ..., \sigma_i', ..., \sigma_n^{[h]} \right) \in \Theta_i,
\]
where
\[
[h] = \rho_{[0..k]} \cdot (\bar{a}_{-i}^k, \sigma_i'(\rho_{[0..k]})) \cdot \text{Tab} \left( s_k, (\bar{a}_{-i}^k, \sigma_i'(\rho_{[0..k]})) \right).
\]

We say that \( \rho \) has a good deviation if some prefix \( h \) of \( \rho \) is a good deviation point.

We use the notion of deviation point in the following lemma. This lemma states that a strategy \( \sigma_0 \) is a solution for the NCRSP if any play \( \rho \) compatible with it, either is winning for Agent 0 or some Agent \( i \) would unilaterally deviate and win against any strategy profile of the other agents.

**Lemma 1.** A strategy \( \sigma_0 \) is a solution for NCRSP iff every play \( \rho \) compatible with \( \sigma_0 \) either \( \rho \upharpoonright s \in \Theta_0 \) or, \( \rho \) has a good deviation.

**Intuition of the game:** Using the previous lemma, the procedure to solve the non-cooperative synthesis problem builds a turn-based zero-sum game \( \mathcal{H} \) with two players. In contrast with concurrent games, in a two-player turn-based game the state space is partitioned into two disjoint sets. From each set one and exactly one of the two players can play by choosing the next state of the play. Another property of \( \mathcal{H} \) lies in the objectives of the players. They have opposite goals, the goal of the protagonist (Constructor) is denoted \( \Theta_C \) which is a set of plays in \( \mathcal{H} \), and the goal of the antagonist (Spoiler) is exactly all the plays that avoid \( \Theta_C \).

As already mentioned, we use the following idea presented in [2]; along each play we keep track of some information used to detect deviations. More precisely, we define a set \( D \in 2^{\text{Agt}} \) of agents for which a good deviation point exists in the current play, and the set \( W \in 2^{\text{Agt}} \) of agents that have deviated, those agents are to win. Each agent in the set \( W \) has a winning strategy with respect to the fixed strategy of Agent 0.

The game \( \mathcal{H} \) is played by two players as follows. In each state \((s,W,D)\) Constructor proposes an action for Agent 0 together with the actions corresponding to the winning strategies of the agents in the set \( W \). Then, Spoiler responds with an action profile played by all agents in the environment. In the next step, Constructor knows the entire action profile played by the agents and proposes some new deviations for the agents that do not have a deviation yet (they are neither in \( W \) nor in \( D \)). The last move is performed by Spoiler, it is his role to “check” that the proposed deviations and winning strategies are correct. Therefore, Spoiler can choose any continuation for the game and the sets \( W \) and \( D \) are updated according to the previous choices to some new values \( W' \) and \( D' \). For continuations where some agent \( i \) in \( W \) plays the action proposed by Constructor as being according to the winning strategy, Agent \( i \) remains in the set \( W' \). On all the other continuations, Agent \( i \) is added to the set \( D' \). For any agent \( i \) for which Constructor guesses a deviation point with respect to the current action profile \( \bar{m} \), Agent \( i \) is added to the set \( W' \) on the continuation where Agent \( i \) plays the new action and the other agents do not change their actions. For all
the continuations were all agent but Agent $i$ play $\bar{m}$ and Agent $i$’s action is different from the deviation action, Agent $i$ is added to the set $D'$.

Note that we need the intermediate steps since each deviation is chosen after knowing the actions played by all agents at the current step. Therefore, Constructor has to wait for all actions played at the current state $s$ before proposing a deviation. For instance, in the example of Figure 1, if Agent 0 plays action $r$, the possible continuations are either in state labelled 2 or 3 (in the right hand-side), but both Agent 1 and 2 can deviate. That is, if they play the same action, Agent 1 wins but Agent 2 would deviate to go to state 3 and increase her payoff. Also, if they play different actions, Agent 1 would deviate and change her action to match the action of Agent 2 so that the play reaches state 2.

**Formal definition:** Given a concurrent game $\mathcal{G}$ we construct a turn based game $H = (Q, q_0, \text{Act}_C, \text{Act}_S, \text{Tab}', \Theta_C)$. The game $H$ is obtained as follows:

- The set $Q$ of states is
  \[
  (S \times 2^{\text{Ag}t} \times 2^{\text{Ag}t}) \cup \left( S \times 2^{\text{Ag}t} \times 2^{\text{Ag}t} \times \text{Act}_0 \times \prod_{i=1}^{n}(\text{Act}_i \cup \{\bot\}) \right) \cup \left( S \times 2^{\text{Ag}t} \times 2^{\text{Ag}t} \times \text{Act}_0 \times \prod_{i=1}^{n}(\text{Act}_i \cup \{\bot\}) \times \prod_{i=1}^{n} \text{Act}_i \right) \cup \left( S \times 2^{\text{Ag}t} \times 2^{\text{Ag}t} \times \text{Act}_0 \times \prod_{i=1}^{n}(\text{Act}_i \cup \{\bot\}) \times \prod_{i=1}^{n} \text{Act}_i \times \prod_{i=1}^{n}(\text{Act}_i \cup \{\bot\}) \right).
  \]

- Player Constructor controls states in
  \[
  (S \times 2^{\text{Ag}t} \times 2^{\text{Ag}t}) \quad \text{and} \quad \left( S \times 2^{\text{Ag}t} \times 2^{\text{Ag}t} \times \text{Act}_0 \times \prod_{i=1}^{n}(\text{Act}_i \cup \{\bot\}) \times \prod_{i=1}^{n} \text{Act}_i \right) .
  \]

- Player Spoiler controls states in
  \[
  \left( S \times 2^{\text{Ag}t} \times 2^{\text{Ag}t} \times \text{Act}_0 \times \prod_{i=1}^{n}(\text{Act}_i \cup \{\bot\}) \right) , \quad \text{and} \quad \left( S \times 2^{\text{Ag}t} \times 2^{\text{Ag}t} \times \text{Act}_0 \times \prod_{i=1}^{n}(\text{Act}_i \cup \{\bot\}) \times \prod_{i=1}^{n} \text{Act}_i \times \prod_{i=1}^{n}(\text{Act}_i \cup \{\bot\}) \right).
  \]

The set $\text{Act}'_C$ of actions of player Constructor is as follows:

- From a state $(s, W, D) \in (S \times 2^{\text{Ag}t} \times 2^{\text{Ag}t})$, Constructor plays an action
  \[
  \bar{\beta} \in \text{Act}_0 \times \prod_{i=1}^{n}(\text{Act}_i \cup \{\bot\}) \quad \text{s.t.} \quad \forall 1 \leq i \leq n, \quad \bar{\beta}[i] \in \text{Act}_i \Leftrightarrow i \in W .
  \]
From a state $(s, W, D, \bar{\beta}, \bar{m}) \in \left( S \times 2^{\text{Agt}} \times 2^{\text{Agt}} \times \text{Act}_0 \times \prod_{i=1}^{n} (\text{Act}_i \cup \{\_\}) \times \prod_{i=1}^{n} \text{Act}_i \right)$, he plays an action

$$\bar{\gamma} \in \prod_{i=1}^{n} (\text{Act}_i \cup \{\_\}) \text{ s.t. } \bar{\gamma}[i] \in \text{Act}_i \implies (i \notin W \cup D) .$$

The set $\text{Act}_S$ of actions of player $\text{Spoiler}$ is $\prod_{i=1}^{n} \text{Act}_i$.

The game is played in turns alternating between $\text{Constructor}$ and $\text{Spoiler}$.

The transition table $\text{Tab}'$ is obtained as follows:

- The initial state is $(s_0, \emptyset, \emptyset) \in (S \times 2^{\text{Agt}} \times 2^{\text{Agt}})$, where $s_0$ is the initial state in $G$.
- From a state $(s, W, D)$, $\text{Constructor}$ plays by proposing an action $\bar{\beta}$ and the next state is $(s, W, D, \bar{\beta})$. Intuitively, $\text{Constructor}$ proposes an action for Agent 0 and one action for each agent that is in the set $W$. The action proposed for each Agent $i \in W$ is assumed to be compatible with the winning strategy from the current history.
- From the new state $\text{Spoiler}$ proposes an action profile $\bar{m}$ for all agents but agent 0 and the new state is $(s, W, D, \bar{\beta}, \bar{m})$. The strategy profile $\bar{m}$ corresponds to the actual strategy played by the agents.
- It is again player $\text{Constructor}$ turn to play an action $\bar{\gamma} \in \prod_{i=1}^{n} (\text{Act}_i \cup \{\_\})$ and the new state is $(s, W, D, \bar{\beta}, \bar{m}, \bar{\gamma})$. By playing the action $\bar{\gamma}$ s.t. $\bar{\gamma}[i] \in \text{Act}_i$ only for some $i \notin W \cup D$, $\text{Constructor}$ proposes some deviations for agents for which he did not guess one in the past.
- Finally, from the state $(s, W, D, \bar{\beta}, \bar{m}, \bar{\gamma})$, $\text{Spoiler}$ proposes again an action profile $\bar{\alpha}$ for the coalition $\text{Agt}_{\neq 0}$ and the new state is $(s', W', D')$ such that:

$$W' = W \cup \{i \notin W \cup D \mid \bar{\gamma}[i] \in \text{Act}_i \text{ and } \bar{\alpha}[i] = \bar{\gamma}[i] \text{ and } \forall i \in \text{Agt} \setminus \{0, i\}, \bar{\alpha}[i] = \bar{m}[i] \} \setminus \{i \in W \mid \bar{\alpha}[i] \neq \bar{\beta}[i] \} ,$$

and

$$D' = D \cup \{i \in W \mid \bar{\alpha}[i] \neq \bar{\beta}[i] \} \cup \{i \notin W \cup D \mid \bar{\gamma}[i] \in \text{Act}_i \text{ and } \bar{\alpha}[i] \neq \bar{\gamma}[i] \text{ and } \forall i \in \text{Agt} \setminus \{0, i\}, \bar{\alpha}[i] = \bar{m}[i] \} .$$

Intuitively, at this step, $\text{Spoiler}$ tries to prove that the winning strategies proposed using $\bar{\beta}$ and the deviations (with respect to the action profile $\bar{m}$) proposed by $\text{Constructor}$ using $\bar{\gamma}$ are not correct.

For a play $\eta \in \text{Plays}(H)$, we define by $\eta \mid_C$ the restriction of $\eta$ to states in $(S \times 2^{\text{Agt}} \times 2^{\text{Agt}})$.

We also define
\[
\begin{align*}
\text{Proj}_{St}: \ St \times 2^{Agt} \times 2^{Agt} & \rightarrow \ St \\
(s, W, D) & \mapsto s . \\
\text{Proj}_{W}: \ St \times 2^{Agt} \times 2^{Agt} & \rightarrow 2^{Agt} \\
(s, W, D) & \mapsto W . \\
\text{Proj}_{D}: \ St \times 2^{Agt} \times 2^{Agt} & \rightarrow 2^{Agt} \\
(s, W, D) & \mapsto D .
\end{align*}
\]

The above functions are naturally extended over sequences in \((St \times 2^{Agt} \times 2^{Agt})^\omega\).

Remark 1. Let \(\eta\) in \(\text{Plays}(\mathcal{H})\) be a play. The sequences \(\text{Proj}_{D}(\eta \mid C)\), \(\text{Proj}_{W}(\eta \mid C)\) eventually stabilize. We denote by \(\lim \text{Proj}_{D}(\eta \mid C)\), \(\lim \text{Proj}_{W}(\eta \mid C)\) resp. their limit values.

The above remark is a consequence of the following observation. Once an agent is added in the set \(D\) along the play \(\eta\), it is never removed. Also, Agent \(i\) may be added in the set \(W\), and then removed by choosing an action such that the obtained history is not a prefix of \(\eta\). At this stage, Agent \(i\) is added to the set \(D\) along the play \(\eta\) and never added again in the set \(W\) (since Constructor only guesses new deviations for agents \(i \notin W \cup D\)).

The objective of Player Constructor is given by the following set of plays in \(\mathcal{H}\):

\[
\Theta_C = \{ \eta \in Q^\omega \mid \left(\left(\text{Proj}_{St}(\eta \mid C) \in \Theta_0\right) \text{ or } \left(\exists i \in \lim \text{Proj}_{D}(\eta \mid C) \text{ s.t. } \text{Proj}_{St}(\eta \mid C) \notin \Theta_i\right)\right) \text{ and } \left(\forall i \in \lim \text{Proj}_{W}(\eta \mid C) \implies \text{Proj}_{St}(\eta \mid C) \in \Theta_i\right)\} .
\]

### 3.2 Correctness

Let \(\mathcal{G}\) be an instance of the NCRSP, and let \(\mathcal{H}\) be the game obtained from \(\mathcal{G}\) using the construction above.

**Proposition 1.** Let \(\sigma_C\) be a winning strategy for player Constructor in \(\mathcal{H}\), then there exists a solution for the NCRSP in \(\mathcal{G}\).

**Proof.** Let \(\sigma_C\) be a winning strategy and \(\text{Plays}(\sigma_C)\) the set of plays in \(\mathcal{H}\) compatible with \(\sigma_C\). We will construct a solution \(\sigma_0\) for the NCRSP in \(\mathcal{G}\). We define the strategy \(\sigma_0\) inductively as follows: \(\sigma_0(s_0) = \sigma_C(s_0, \emptyset, \emptyset)\), and for every history of length less than \(n\), \(s_0a_0s_1a_1...s_na_n\) we define \(\sigma_0\) by:

\[
\sigma_0(s_0a_0s_1a_1...s_n) = \sigma_C(\text{Tr}_{\mathcal{G} \rightarrow \mathcal{H}}(s_0a_0s_1a_1...s_n))[0] ,
\]
where $\text{Tr}_{G \rightarrow H} : \text{Plays}(G) \rightarrow \text{Plays}(H)$ transforms each finite history $h = s_0 \bar{a}_0 s_1 \bar{a}_1 \ldots s_n$ in $\text{St} \left( \prod_{t \in \text{Agent}} \text{Act}_t \right)^* \text{St}$ of $G$ into a finite history of $H$ such that:

$$\text{Tr}_{G \rightarrow H}(h) = (s_0, \emptyset, \emptyset) \beta_0 q_0^1 m_0 q_0^2 \gamma_0 q_0^3 \bar{a}_0 (s_1, W_1, D_1) \ldots (s_n, W_n, D_n),$$

where

$$q_0^1 = (s_0, \emptyset, \emptyset, \beta_0), \quad q_0^2 = (s_0, \emptyset, \emptyset, m_0), \quad q_0^3 = (s_0, \emptyset, \beta_0, m_0, \gamma_0).$$

defined inductively as follows:

1. $\text{Tr}_{G \rightarrow H}(s_0) = (s_0, \emptyset, \emptyset)$,
2. For any $0 \leq k \leq n$, let $h[0..k] = s_0 \bar{a}_0 s_1 \bar{a}_1 \ldots s_k$ be the prefix of $h$ up to position $k$ and $(s_k, W_k, D_k)$ be the last state of $\text{Tr}_{G \rightarrow H}(h[0..k])$. Then

$$\text{Tr}_{G \rightarrow H}(h[0..k+1]) = \text{Tr}_{G \rightarrow H}(h[0..k]) \beta_k q_k^1 m_k q_k^2 \gamma_k q_k^3 \bar{a}_k (s_{k+1}, W_{k+1}, D_{k+1}),$$

where

$$q_k^1 = (s_k, W_k, D_k, \beta_k), \quad q_k^2 = (s_k, W_k, D_k, \bar{a}_k, m_k), \quad q_k^3 = (s_k, W_k, D_k, \bar{a}_k, m_k, \gamma_k),$$

and:

- $\beta_k[0] = \bar{a}_k[0]$,
- $\beta_k[i] = \sigma_G(\text{Tr}_{G \rightarrow H}(h[0..k]))[i]$ for $i \neq 0$,
- $m_k[i] = \bar{a}_k[i]$ for all $i \in \text{Agent} \setminus \{0\}$,
- $\gamma_k = \sigma_G(\text{Tr}_{G \rightarrow H}(h[0..k])) \beta_k(s_k, W_k, D_k, \bar{a}_k, m_k)$,
- $\bar{a}_k = m_k$ and
- $(s_{k+1}, W_{k+1}, D_{k+1}) = \text{Tab}'((s_k, W_k, D_k, \bar{a}_k, m_k, \gamma_k), \bar{a}_k)$ (Tab' is the transition relation in $H$).

Note, from the definition of $\sigma_0$, that if along the history $h$ there is a position $k$ such that $a_k \neq \sigma'(\text{Tr}_{G \rightarrow H}(s_0 \bar{a}_0 s_1 \bar{a}_1 \ldots s_k))[0]$, we are not along an execution compatible with the winning strategy and we ignore it (play as if it would have been along an execution compatible with a winning strategy). Therefore, along any history $h$ where Agent 0 plays according to $\sigma_0$, it holds that $a_k[0] = \sigma_0(h[0..k]) = \sigma_G(\text{Tr}_{G \rightarrow H}(h[0..k]))[0] = \beta_k[0]$.

Also, note that we construct the play $\text{Tr}_{G \rightarrow H}(\pi)$ by considering that $\text{Spoiler}$ does not change the action profile $a$ he first proposed at each index $(a_k = m_k$ for any $k)$. This is because the other action profiles are used only to verify that the deviations proposed by Constructor are profitable deviations.

Let us show that $\sigma_0$ defined above is a solution for the NCRSP in $G$. We know that for each $\eta \in \text{Plays}(\sigma_G)$ is in $\Theta_C$, and hence either

$$\text{Proj}_{S_k}(\eta | C) \in \Theta_0, \quad \text{or}$$

$$\exists i \in \lim \text{Proj}_{D}(\eta | C) \text{ s.t. } \text{Proj}_{S_k}(\eta | C) \not\in \Theta_i$$

(2)

(3)

Also, we have that

$$\forall i \in \lim \text{Proj}_{W}(\eta | C) \implies \text{Proj}_{S_k}(\eta | C) \in \Theta_i$$

(4)
If Equation (2) holds, the execution in $G$ that follows $\sigma_0$ and the environment agents play the actions $\tilde{\alpha}_k$ at each position, ensures the same outcome, the correction follows.

If Equation (3) holds, then let $i$ be the agent for which this equation holds. In the outcome of $\sigma_0$, Agent $i$ has a payoff of 0 thus has an incentive to deviate. Now, since it is in the set $\lim \text{Proj}_D(\eta \mid C)$, from the definition of transitions in $H$, there is a position $k$ where the Constructor guessed a deviation for Agent $i$ by first playing some action $\gamma_k[i]$. Therefore, the outcome of $\sigma_0$ in $G$ is not compatible with a Nash equilibrium since Agent $i$ can unilaterally deviate.

Equation (4) checks the validity of the deviation point. This is performed by checking for every Agent $i$ with $i \in \lim \text{Proj}_W(\eta \mid C)$ that $\text{Proj}_G(\eta \mid C) \in \Theta_i$ holds.

Note that when Constructor decides the existence of a deviation point at position $k$ along a play for Agent $i$, he chooses an action $\gamma_k[i]$ that is the first action to deviate from the current play. Then, Agent $i$ is added in the set $W$ on all successors where Agent $i$ unilaterally deviates by playing $\gamma_k[i]$. Then, Constructor keeps on guessing a winning strategy for Agent $i$ by playing $\beta[i]$ at any state where $i \in W$ and $i$ remains in $W$ on all possible successors with this action for Agent $i$. Now, since there is at most one addition and one removal for each agent in/from the set $W$, when $i \in \lim \text{Proj}_W(\eta \mid C)$, the current play is compatible with the deviating strategy that has to be profitable (winning). The fact that the deviation is profitable is verified since the strategy $\sigma_C$ is winning and Spoiler can verify deviations by playing any at position any action profile $\bar{\alpha}$ that may be different from the initial action profile $\bar{m}$ proposed.

Given this, the strategy $\sigma_0$ defined above is a winning strategy in the game $G$.

**Proposition 2.** If there is a solution $\sigma_0$ for the NCRSP in $G$, Constructor has a winning strategy in $H$.

**Proof.** Let $\sigma_0$ be a solution for the NCRSP problem in $G$. Therefore, any play $\rho$ compatible with $\sigma_0$, either satisfies $\rho \in \Theta_0$ or it is not the outcome of a 0-NE.

Let $\rho \in \text{St}([\prod_{i \in \text{Agt} \cup \text{Act}} \text{St}])^\omega$ be a play compatible with $\sigma_0$. If $\rho \not\in \Theta_0$, there is a position $k$ such that some agent $i \in \text{Agt} \setminus \{0\}$ has a profitable deviation.

We define the strategy $\sigma_C$ of Constructor such that he plays $\sigma_0$ for the actions of Agent 0 and proposes the actions for the other agents according to the strategies played by agents in $\text{Agt} \setminus \{0\}$ from deviation points. Also, Constructor guesses at most one deviation point for each agent along an execution.

Formally, Constructor’s strategy is defined as follows.

$$\sigma_C(s_0, 0, 0) = \tilde{\beta} \text{ s.t. } \tilde{\beta}[0] = \sigma_0(s_0) \text{ and } \tilde{\beta}[i] = \bar{\alpha}_i \text{ for all } i \neq 0$$

Let us first define a mapping $\text{Tr}_{H \rightarrow G} : \text{Plays}(H) \rightarrow \text{Plays}(G)$ that associates to each finite history

$$g = (s_0, 0, 0)\tilde{\delta}_0(s_0, 0, 0, \tilde{\beta}_0, \bar{\alpha}_0)\tilde{m}_0(s_0, 0, 0, \tilde{\beta}_0, \bar{\alpha}_0)\gamma_0(s_0, 0, 0, \tilde{\beta}_0, \bar{\alpha}_0)\alpha_0(s_1, W_1, D_1)\ldots(s_n, W_n, D_n)$$

that correctly follows transitions in $H$, the corresponding history in $G$:

$$\text{Tr}_{H \rightarrow G}(g) = s_0(\tilde{\beta}_0[0], \bar{\alpha}_0)s_1(\tilde{\beta}_1[0], \bar{\alpha}_1)\ldots s_n$$
that is the sequence of states and actions taken by agents.

Then, for every finite history $g$ in $H$, for every $i \in W_n$, there is an index $k \leq n$ such that $i \notin W_k \cup D_k$ and $\gamma_h[i] \in \text{Act}_i$. We assume (and will see in the rest of the definition of $\sigma_C$) that this is "the" good deviation point guessed by Constructor when Agent $i$ had a winning strategy. Let $\sigma_i^{\text{Tr}_{H\rightarrow G}(g[0..k]), \bar{m}_k}$ be the winning strategy played by Agent $i$ from the deviation point $h = \text{Tr}_{H \rightarrow G}(g[0..k]) \cdot \bar{m}_k \cdot \text{Tab}(s_k, (\beta_k[0], \bar{m}_k))$. Note that the strategy $\sigma_i^{\text{Tr}_{H \rightarrow G}(g[0..k]), \bar{m}_k}$ exists since $\sigma_0$ is winning and $h$ is a good deviation point for Agent $i$. Then, the strategy $\sigma_C$ from $g$ is defined as:

$$\sigma_C(g) = \bar{\beta} \quad \text{where} \quad \bar{\beta}[0] = \sigma_0(\text{Proj}_{g[1..]}(g | c)) \quad \text{and} \quad \bar{\beta}[i] = \sigma_i^{\text{Tr}_{H \rightarrow G}(g[0..k]), \bar{m}_k}(\text{Proj}_{g[1..]}(g | c))$$

Now, from a history $g' = g \cdot \bar{\beta}_n(s_n, W_n, D_n, \bar{\beta}_n)\bar{m}_n(s_n, W_n, D_n, \bar{\beta}_n, \bar{m}_n)$, Constructor plays $\bar{\gamma}_n$ that correspond to deviating strategies of agents for which $h' = \text{Tr}_{H \rightarrow G}(g) \cdot \bar{m}_n \cdot \text{Tab}(s_n, (\beta_n[0], \bar{m}_n))$ is a first good deviation point of Agent $i$ from some continuation of $g'$. Therefore,

$$\sigma_C(g') = \bar{\gamma}_n$$

such that

$$\bar{\gamma}_n[i] = \begin{cases} \sigma_i^{\text{Tr}_{H \rightarrow G}(g[0..n]), \bar{m}_n} & \text{if } h = \text{Tr}_{H \rightarrow G}(g) \cdot \bar{m}_n \cdot \text{Tab}(s_n, (\beta_n[0], \bar{m}_n)) \text{ is a good deviation point for Agent } i, \\ \cdot \text{otherwise}. \end{cases}$$

Since the strategy $\sigma_0$ is winning, for any prefix of a play in $H$ that follows $\sigma_C$, Constructor correctly guesses the deviation points such that every continuation either satisfies $\Theta_0$ or there is at least one Agent $i$ that is losing but it has a good deviation point. Then, from the construction of $H$, since $i \in D$ when a deviation is guessed, we have that for every infinite play $\eta$ in $H$, either $\text{Proj}_{\eta[1..]}(\eta | c) \in \Theta_0$ or $\exists i \in \lim \text{Proj}_{\eta[1..]}(\eta | c)$ s.t. $\text{Proj}_{\eta[1..]}(\eta[0..k]) \notin \Theta_i$. Also, since Constructor plays the winning strategies $\sigma_i^{\text{Tr}_{H \rightarrow G}(\eta[0..k]), \bar{m}_k}$ for each Agent $i$ that deviates in some deviation point $h = \text{Tr}_{H \rightarrow G}(\eta[0..k]) \cdot \bar{m}_k \cdot \text{Tab}(s_k, (\beta_k[0], \bar{m}_k))$ and since $i \in \lim \text{Proj}_{\eta[1..]}(\eta | c)$ only on plays $\eta$ compatible with the the winning strategies, we have that $\text{Proj}_{\eta[1..]}(\eta | c) \in \Theta_i$. Therefore, the strategy $\sigma_C$ is a winning strategy for Constructor in the game $H$.

As a consequence of the two previous propositions, we have the following theorem that establishes the correctness of our solution.

**Theorem 1.** There is a solution for the NCRSP in $G$ if and only if Constructor has a winning strategy in the corresponding game $H$.

### 4 Computational Complexity

In this section, we take advantage of the construction presented in the previous section to give complexity bounds for numerous winning conditions. In fact,
we can adapt the technique used in [2] in order to establish the upper bound complexity for NCRSP. For some of the objectives we consider, we reduce the game $\mathcal{H}$ to a finite duration game. This is done by first exploiting the winning condition in such a way that it can be transformed into a finite horizon condition in finite duration game.

In order to obtain the finite duration game, we transform the game $\mathcal{H}$ (or simply rewrite the winning objective) into a game $\mathcal{H}'$ with Büchi/Parity objective and then define the finite duration game $\mathcal{H}^f$ by stopping the game $\mathcal{H}'$ after the first loop.

The following lemma, proves that the game $\mathcal{H}'$ with Parity condition can be transformed into a finite duration game that stops after the first loop. Let $pr : St' \rightarrow \mathbb{N}$ be the priority function in $\mathcal{H}'$. Then, the finite duration game $\mathcal{H}^f$ is defined over the same game arena as $\mathcal{H}'$, but each play stops after the first loop. A play $\rho = xqqy$ is winning for Constructor if $\min\{pr(yq[j]) \mid 0 \leq j < |yq|\}$ is even.

**Lemma 2.** Constructor has a winning strategy in the game $\mathcal{H}'$ with the parity condition $pr$ if and only if he has a winning strategy in the game $\mathcal{H}^f$.

In the case of Reachability, Safety, Büchi and coBüchi conditions, we exploit the structure of the accepting condition and construct the game $\mathcal{H}'$ by introducing some counters in the game $\mathcal{H}$. Then, the finite duration game $\mathcal{H}^f$ whose plays have polynomial length can be solved using an alternating Turing machine and therefore we obtain PSPACE complexity.

In the case of Muller conditions, we have to use Least Appearance Record (LAR) construction to get the parity game $\mathcal{H}'$ and then the finite duration game would have plays with exponential length in the size of the initial game. This approach would give ExpSpace complexity. Fortunately, the parity condition in the game $\mathcal{H}'$ that we obtain after applying the LAR construction has exponential number of states but only a polynomial number of priorities. Then, by using the result from [4,6], we obtain ExpTime complexity.

**Theorem 2.** Deciding if there is a solution for the non-cooperative rational synthesis problem in concurrent games is in PSPACE for Safety, Reachability, Büchi and coBüchi objectives and ExpTime for Muller objectives.

### 4.1 Fixed Number of Players

In the case of fixed number of players, the game $\mathcal{H}$ that we build has polynomial size in the size of the initial game $\mathcal{G}$ (when considering that the transitions are given explicitly in the table $\text{Tab}$ - since we build nodes in $\mathcal{H}$ for each possible action profile). This lowers the complexities that we obtain for the rational synthesis problem. The below theorem holds since the game $\mathcal{H}$ has polynomial size and the Constructor’s objective is fixed.

**Theorem 3.** Deciding if there is a solution for the non-cooperative rational synthesis in concurrent games with a fixed number of players and Safety, Reachability, Büchi or co-Büchi objectives is in PTIME.
The above argument does not work in the case of Muller conditions. However, as it is also done in [2], it is easily rewritten as a two-player Muller game of polynomial size. Moreover, two-player Muller games can be solved using polynomial space. This leads to the following complexity result.

**Theorem 4.** Deciding if there is a solution for the non-cooperative rational synthesis in concurrent games with a fixed number of players and Muller objectives is in PSpace.

5 Conclusion

We studied the complexity of the non-cooperative rational synthesis for concurrent game structures. We established tight complexity results for Reachability, Safety, Büchi, and Muller. For Parity, Rabin, Street, and Muller we still suffer a gap between a PSPACE lower bound and an ExpTime upper bound. Tightening this gap requires efficient algorithm for solving two-player games where the winning condition is given by a boolean combination of Parity objectives which is itself an interesting research direction.

References

1. Patricia Bouyer, Romain Brenguier, Nicolas Markey, and Michael Ummels. Pure nash equilibria in concurrent deterministic games. *Logical Methods in Computer Science*, 11(2), 2015.
2. Rodica Condurache, Emmanuel Filiot, Raffaella Gentilini, and Jean-François Raskin. The complexity of rational synthesis. In *43rd International Colloquium on Automata, Languages, and Programming, ICALP 2016, July 11-15, 2016, Rome, Italy*, volume 55 of LIPIcs, pages 121:1–121:15. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2016.
3. Dana Fisman, Orna Kupferman, and Yoad Lustig. Rational synthesis. In *Tools and Algorithms for the Construction and Analysis of Systems, 16th International Conference, TACAS 2010, Held as Part of the Joint European Conferences on Theory and Practice of Software, ETAPS 2010, Paphos, Cyprus, March 20-28, 2010. Proceedings*, volume 6015 of Lecture Notes in Computer Science, pages 190–204. Springer, 2010.
4. Marcin Jurdzinski, Mike Paterson, and Uri Zwick. A deterministic subexponential algorithm for solving parity games. In *Proceedings of the Seventeenth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2006, Miami, Florida, USA, January 22-24, 2006*, pages 117–123, 2006.
5. Orna Kupferman, Giuseppe Perelli, and Moshe Y. Vardi. Synthesis with rational environments. In *Multi-Agent Systems - 12th European Conference, EUMAS 2014, Prague, Czech Republic, December 18-19, 2014, Revised Selected Papers*, pages 219–235, 2014.
6. Sven Schewe. Solving parity games in big steps. In *FSTTCS 2007: Foundations of Software Technology and Theoretical Computer Science, 27th International Conference, New Delhi, India, December 12-14, 2007. Proceedings*, volume 4855 of Lecture Notes in Computer Science, pages 449–460. Springer, 2007.
7. Michael Ummels. The complexity of nash equilibria in infinite multiplayer games. In Foundations of Software Science and Computational Structures, 11th International Conference, FOSSACS 2008, Held as Part of the Joint European Conferences on Theory and Practice of Software, ETAPS 2008, Budapest, Hungary, March 29 - April 6, 2008. Proceedings, volume 4962 of Lecture Notes in Computer Science, pages 20–34. Springer, 2008.
A Proof of Lemma 1

Proof. We start by establishing the if direction, let $\sigma_0$ be a solution for the NCRSP. If any outcome $\rho \in \text{Plays}(\sigma_0)$ is such that $\rho |_{S_t} \in \Theta_0$ then there is nothing to prove. Let $\rho$ be a play in $\text{Plays}(\sigma_0)$ such that $\rho$ is not in $\Theta_0$. Assume toward a contradiction that $\rho$ does not contain a good deviation point. Then by Definition 3 we know that for any prefix $h$ of $\rho$, any agent $i \neq 0$ such that $\text{Payoff}_i(\rho) = 0$, and any strategy $\tau_i$ of $i$ there exists $\sigma_1, \ldots, \sigma_n$ strategies for agents 1 to $n$ such that the following holds:

$$[h] \cdot \text{out}(\sigma_0^h, \sigma_1^h, \ldots, \tau_i^h, \ldots, \sigma_n^h) \notin \Theta_i$$.

The above equation implies that Agent $i$ does not have a profitable deviation under the strategy $\sigma_0$, hence the profile $<\sigma_0, \ldots, \sigma_n>$ is a 0-fixed NE contradicting the fact that $\sigma_0$ is a solution for the NCRSP.

For the only if direction, let $\sigma_0$ be a strategy for agent 0, assume that every $\rho$ in $\text{Plays}(\sigma_0)$ satisfies

1. $\rho |_{S_t} \in \Theta_0$ or,
2. $\rho$ has a good deviation.

If every play $\rho$ in $\text{Plays}(\sigma_0)$ is in $\Theta_0$ then $\sigma_0$ is a solution for NCRSP. Let $\rho$ be a play in $\text{Plays}(\sigma_0)$ such that it is not in $\Theta_0$. By assumption, $\rho$ has a good deviation point i.e. there exists an Agent $i \neq 0$ and a strategy $\tau_i$ for the same agent such that: i) $\rho |_{S_t} \notin \Theta_i$ and ii) after a finite prefix $h$ of $\rho$ for any tuple of strategies $(\sigma_j)_{j \in \text{Agt}\setminus\{0,i\}}$ the following holds:

$$[h] \cdot \text{out}(\sigma_0^h, \sigma_1^h, \ldots, \tau_i^h, \ldots, \sigma_n^h) \in \Theta_i$$.

Hence, $\rho$ is not the outcome of a 0-fixed NE and therefore $\sigma_0$ is a solution for the NCRSP.

B Proofs of Section 4

The following lemma establishes the fact that inside a cycle in the game $\mathcal{H}$, the values of the sets $W$ and $D$ do not change.

**Lemma 3.** Let $\eta$ be a play in $\mathcal{H}$. Then, each loop along $\eta$ contains only one value on states for the sets $W$ and $D$.

Proof. Let $\eta = xqyqz$ be an infinite play in $\mathcal{H}$. But from the definition of the transition relation in $\mathcal{H}$, $\eta' = x(qy)^\omega$ is also a valid play in $\mathcal{H}$. Then, suppose there are two states $i qy$ having different values on states $W$ or $D$. It means that there is at least one player $i$ that is added or removed to/from $W$ or $D$. Therefore, along $\eta'$ we would have an infinite number of additions or removals to/from $W$ or $D$. But, according to the transition relation, this is not possible since once a player is removed from $W$, it is added to the set $D$ and never added to $W$ again along $\eta'$. Also, once a player added in the set $D$, he is never removed. Therefore, along each path, along each loop, the values of $W$ and $D$ do not change.
B.1 Proof of Lemma

The proof of the above lemma is an adaptation from [2] we reproduce it here for the sake of completeness.

Proof. From right to left, if Constructor has a winning strategy $\sigma^f_C$ in $\mathcal{H}^f$, then for all strategies $\sigma^f_S$ of Spoiler, $\text{out}(\sigma^f_C, \sigma^f_S) = xqqy$ is such that $\min\{\text{pr}(yq[j]) \mid 0 \leq j < |yq|\}$ is even. We define Constructor’s strategy $\sigma_C$ in $\mathcal{H}'$ as $\sigma_C(hq) = \sigma^f_C(h'q)$ where $h'$ is obtained from $h$ by removing all the loops. We prove that $\sigma_C$ is winning for Constructor in $\mathcal{H}'$.

Let $\eta$ be a play compatible with $\sigma_C$. By the definition of $\sigma_C$, we can decompose $\eta$ in $\eta = \eta_1 \eta_2 \eta_3$... s.t. $\eta_j$ is a suffix of a play $\eta'_j$ in $\mathcal{H}'$ compatible with $\sigma^f_C$. Moreover, there is a decomposition of the suffixes $\eta_j$ such that by reordering the resulting fragments of all suffixes appearing infinitely often, we obtain an infinite sequence of loops being suffixes of plays in $\mathcal{H}^f$ compatible with $\sigma^f_C$ preceded by a finite prefix. Then, since $\sigma^f_C$ is winning in $\mathcal{H}^f$, on all loops the minimum priority is even and therefore the minimum priority appearing infinitely often in $\eta$ is even and $\eta$ is winning for Constructor.

On the other direction, if there is no winning strategy for Constructor in $\mathcal{H}^f$, by determinacy, there is a winning strategy $\sigma^f_S$ for Spoiler such that $\forall \sigma^f_C, \text{out}(\sigma^f_C, \sigma^f_S) = xqqy$ is such that $\min\{\text{pr}(yq[j]) \mid 0 \leq j < |yq|\}$ is odd. Let $\sigma_S$ be the strategy of Spoiler in $\mathcal{H}'$ defined as $\sigma_S(hq) = \sigma^f_S(h'q)$ where $h'$ is obtained from $h$ by removing all cycles. We prove that $\sigma_S$ is winning for Spoiler in $\mathcal{H}'$.

Let $\eta$ be a play compatible with $\sigma_S$. Doing the same reasoning as before, we can decompose $\eta$ and rearrange the components such that we obtain an infinite sequence of loops being suffixes of plays in $\mathcal{H}^f$ compatible with $\sigma^f_S$ preceded by a finite prefix. Then, since $\sigma^f_S$ is winning in $\mathcal{H}^f$ for Spoiler, all the loops have the minimum priority odd and then the priority that appears infinitely often in $\eta$ is odd. Therefore, $\eta$ is winning for Spoiler in $\mathcal{H}'$.

B.2 Reachability and Safety Objectives

In order to verify Reachability (or Safety) objectives, we keep along plays in the game $\mathcal{H}$ a set $P$ of players in the environment that won (lost in the case of Safety objectives). Then, the resulting game $\mathcal{H}'$ has states in $\mathcal{S}' \times 2^{\mathcal{A}_t}$ where $\mathcal{S}'$ is the set of states in the game $\mathcal{H}$.

Reachability Objectives In the case of Reachability objectives, the set $P$ is initially equal to the set of players for which the initial state is in their target set. Let $R_i \subseteq \mathcal{S}$ be the target set of Player $i$. Then, $P_0 = \{i \mid s_0 \in R_i\}$ and the initial state in the resulting game $\mathcal{H}'$ is $(s_0, 0, 0, P_0)$.

The set $P$ is updated as follows:

- if $(q, q') \in \text{Tab}'$ in $\mathcal{H}$ and $q' = (s, W, D) \in \mathcal{S} \times 2^{\mathcal{A}_t} \times 2^{\mathcal{A}_t}$, then $((q, P), (q', P \cup \{i \mid s \in R_i\}))$ is the corresponding transition in $\mathcal{H}'$. 
if \((q, q') \in \text{Tab}' \) in \(\mathcal{H}\) and \(q' \not\in \text{St} \times 2^{\text{Agt}} \times 2^{\text{Agt}}\), then \(((q, P), (q', P))\) is the corresponding transition in \(\mathcal{H}'\).

Note that the set \(P\) also eventually stabilizes since it only increases and there is a finite number of players in \(\mathcal{G}\). Let \(\lim_{\eta} \text{Proj}_P(\eta |_{C})\) be the limit along the play \(\eta\).

Then, the objective of Constructor in the game \(\mathcal{H}\) is equivalently written in \(\mathcal{H}'\) as the Büchi condition

\[
F^R = \{(s, W, D, P) \mid (0 \in P \text{ or } D \setminus P \neq \emptyset) \text{ and } (W \subseteq P)\}
\]

Now, the Büchi objective \(\text{B"uchi}(F^R)\) can be expressed as the parity objective \(\text{Parity}(pr)\) with

\[
pr(v) = 0 \text{ if } v = (s, W, D, P) \in F^R \text{ and } pr(v) = 1 \text{ otherwise.}
\]

We now define the finite duration game \(\mathcal{H}^f\) over the same game arena as \(\mathcal{H}\), but each play stops when the first state in \(\text{St} \times 2^{\text{Agt}} \times 2^{\text{Agt}} \times 2^{\text{Agt}}\) is repeated. Then, each play is of the form \(\eta = xqyq\), with \(q \in \text{St} \times 2^{\text{Agt}} \times 2^{\text{Agt}} \times 2^{\text{Agt}}\). Constructor wins in the game \(\mathcal{H}^f\) if \(q = (s, W, D, P)\) is such that \((0 \in P \text{ or } D \setminus P \neq \emptyset)\) and \((W \subseteq P)\). Equivalently, thanks to Lemma 3 and because the value of \(P\) does not change from the same reason as \(W\) and \(D\), Constructor wins if \(\min\{pr(yq[j]) \mid 0 \leq j < |yq|\}\) is even.

**Lemma 4.** All plays in the game \(\mathcal{H}^f\) has polynomial length in the size of the initial game.

**Proof.** Since \(D\) and \(P\) are monotone, there are at most \(|\text{Agt}| + 1\) different values that they can take on a path of \(\mathcal{H}\). Also, in the set \(W\) we can have at most one addition and one removal for each player \(i \in \text{Agt}\) and hence \(2|\text{Agt}| + 1\) different values for \(W\). Therefore, along a play \(\pi\) there are at most \(r = 1 + (2|\text{Agt}| + 1) \cdot (|\text{Agt}| + 1)^2 \cdot |\text{St}|\) different states in \(\text{St} \times 2^{\text{Agt}} \times 2^{\text{Agt}} \times 2^{\text{Agt}}\). Then, between two states in \(\text{St} \times 2^{\text{Agt}} \times 2^{\text{Agt}} \times 2^{\text{Agt}}\), there are three intermediate states. Therefore, since all the plays in \(\mathcal{H}^f\) stop after the first cycle, the length of each play is of at most \(4r + 1\) states since there is only one state that appears twice. Therefore, all plays in \(\mathcal{H}^f\) have polynomial length in \(\text{Agt}\) and \(\text{St}\) of the initial play \(\mathcal{G}\).

**Theorem 5.** Deciding if there is a solution for the non-cooperative rational synthesis in concurrent games with Reachability objectives is in \(\text{PSPACE}\).

**Proof.** Using Lemmas 2 and 4 solving the non-cooperative rational synthesis problem, reduces to solving the finite duration game \(\mathcal{H}^f\) which has polynomial length plays. This can be done in \(\text{PSPACE}\) using an alternating Turing machine running in \(\text{PTIME}\).

**Safety Objectives** In the case of Safety objectives, we use the set \(P\) with the following semantics: all players that are in the set \(s\) already lost by passing through an unsafe state.

Initially, the set \(P\) equals to the set of players for which the initial state is unsafe. Let \(S_i\) be the set of safe states of Player \(i\). Then, \(P_0 = \{i \mid s_0 \not\in S_i\}\).

The set \(P\) is updated as follows:
with Safety objectives is in PSpace

written as a parity objective. Note that this constructions are similar to the one

where

\( q, q' = (s, W, D) \in \text{St} \times 2^{\text{Agt}} \times 2^{\text{Agt}} \), then \((q, P), (q', P \cup \{i \mid s \notin S_i\})\) is the corresponding transition in \( \mathcal{H}' \)

- if \((q, q') \in \text{Tab}'\) in \( \mathcal{H} \), and \(q' = (s, W, D) \in \text{St} \times 2^{\text{Agt}} \times 2^{\text{Agt}} \), then \((q, P), (q', P)\) is the corresponding transition in \( \mathcal{H} \)

Note that in this case the set \( P \) is also increasing and eventually stabilizes to a limit \( \lim \text{Proj}_P(\eta \upharpoonright C) \) along the play \( \eta \).

Then, using the fact that the sets \( W, D \) and \( P \) eventually stabilize, the objective of Constructor can be rewritten as the Büchi objective \( \text{BUCHI}(F^S) \) where

\[
F^S = \{(s, W, D, P) \mid (0 \notin P \text{ or } D \cap P \neq \emptyset) \text{ and } (W \cap P = \emptyset)\}
\]

Using a similar proof as in the case of Reachability objectives, we can prove that we can reduce to solving a finite duration game having plays of polynomial length. Therefore, the following theorem holds.

**Theorem 6.** Answering the rational synthesis problem in concurrent games with Safety objectives is in PSpace.

### B.3 Büchi and co-Büchi Objectives

In the case of Büchi and co-Büchi objectives, the idea is to transform the game \( \mathcal{H} \) by possibly adding some counters such that Constructor’s objective can be written as a parity objective. Note that this constructions are similar to the one in [2].

**Büchi Objectives** In the case of Büchi objectives, Constructor’s objective is

\[
\Theta_C = \{\eta \in Q^\omega \mid ((\text{Proj}_{St}(\eta \upharpoonright C) \models \Diamond \Box F_i) \text{ or } (\exists i \in \lim \text{Proj}_D(\eta \upharpoonright C) \text{ s.t. } \text{Proj}_{St}(\eta \upharpoonright C) \models \Diamond \Box F_i) \}
\]

and \((\forall i \in \lim \text{Proj}_W(\eta \upharpoonright C) \implies \text{Proj}_{St}(\eta \upharpoonright C) \models \Box \Diamond F_i)\}).

In order to reduce to the Parity objectives, we first make some small changes on Constructor’s objective as follows. We exploit the monotonicity of the sets \( D \) and \( W \) to rewrite the formulas

\[
\phi_D \equiv \exists i \in \lim \text{Proj}_D(\eta \upharpoonright C) \text{ s.t. } \text{Proj}_{St}(\eta \upharpoonright C) \not\models \Box \Diamond F_i \quad \text{and}
\]

\[
\phi_W \equiv \forall i \in \lim \text{Proj}_W(\eta \upharpoonright C) \implies \text{Proj}_{St}(\eta \upharpoonright C) \models \Box \Diamond F_i
\]

First, the negation of \( \phi_D \) says that for all players in \( \lim \text{Proj}_D(\eta \upharpoonright C) \), holds \( \text{Proj}_{St}(\eta \upharpoonright C) \models \Box \Diamond F_i \). Since the set \( D \) stabilize, and the formulas to be verified inside \( \phi_D \) is a tale objectives, instead of \( \lim \text{Proj}_D(\eta \upharpoonright C) \), we can consider the current value of the set \( D \) and use a counter \( c_D \) to wait for each player \( i \in D \) (on turns) a state \( q' = (s', D', W') \in \text{St} \times 2^{\text{Agt}} \times 2^{\text{Agt}} \) s.t. \( s' \in F_i \). Then, the formula \( \neg \phi_D \) is satisfied if either \( D = \emptyset \) or we visit infinitely often a state \( (q, c_d) \) with \( q = (s, D, W) \) and the counter \( c_D \) takes the smallest value in \( D \) and \( s \in F_{c_D} \).
For the formula $\phi_W$ we proceed in the same way. We consider the value of the set $W$ along executions and use a counter $c_W$ to "check" the appearance of a state $q = (s, D, W)$ such that $s \in F_i$ for each player $i \in W$.

Formally, the obtained game is $\tilde{\mathcal{H}}$ is as follows: the set of states $\tilde{Q}$ consists of tuples $(q, c_D, c_W)$ where $q$ is a state in $\mathcal{H}$; $((s_0, \emptyset, \emptyset), -1, -1)$ is the initial state; and transition between states is as follows:

- $(q, c_D, c_W) \rightarrow (q', c_D, c_W)$ iff $(q, q')$ is a transition in $\mathcal{H}$ and $q \in \text{St} \times 2^{2\text{Ag}} \times 2^{\text{Ag}}$
- $(q, c_D, c_W) \rightarrow (q', c_D', c_W')$ $(q', c')$ is a transition in $\mathcal{H}$ and $q' = (s', D', W') \in \text{St} \times 2^{2\text{Ag}} \times 2^{\text{Ag}}$ and

\[
\begin{align*}
    c_D' &= \begin{cases} \min \{(c_D + \ell) \mod k \mid \ell > 0\} & \text{if } D' = 0 \\ c_D & \text{otherwise} \end{cases} \\
    c_W' &= \begin{cases} \min \{(c_W + \ell) \mod k \mid \ell > 0\} & \text{if } W' = 0 \\ c_W & \text{otherwise} \end{cases}
\end{align*}
\]

Also, for a play $\eta \in \tilde{Q}^\omega$ we have that $\text{Proj}_{\text{St}}(\eta \upharpoonright c) \models \Box \Diamond F_0$ if the corresponding play $\tilde{\eta}$ for $\eta$ in $\mathcal{H}$ satisfies $\tilde{\eta} \models \Box \Diamond T_0$ where $T_0 = \{(q = (s, W, D, c_D, c_W) \in \text{St} \times 2^{2\text{Ag}} \times 2^{\text{Ag}} \mid (\text{Agt} \cup \{-1\}) \times (\text{Agt} \cup \{-1\}) \mid s \in F_0\}$.

Let $C_1 = \text{St} \times 2^{2\text{Ag}} \times 2^{\text{Ag}} \times (\text{Agt} \cup \{-1\}) \times (\text{Agt} \cup \{-1\})$ be the set of states $q = (s, W, D, c_D, c_W)$ of Constructor and $\eta \upharpoonright C_1$ be the restriction of $\tilde{\eta} \in \text{Plays}(\tilde{\mathcal{H}})$ on $C_1$. Then, the objective $\Theta_C$ can be equivalently written in the game $\mathcal{H}$ as

\begin{align*}
\Theta_C = \{\tilde{\eta} \in \tilde{Q}^\omega \mid \tilde{\eta} \upharpoonright C_1 \models (\Box \Diamond T_0 \lor \Box \Box \neg T_d) \land \Box \Diamond T_w\}
\end{align*}

where $T_d = \{(s, W, D, c_D, c_W) \mid D = 0 \lor (s \in F_{c_D} \land c_D = \min \{i \in D\})\}$ and $T_w = \{(s, W, D, c_D, c_W) \mid W = 0 \lor (s \in F_{c_W} \land c_W = \min \{i \in W\})\}$.

To continue, the formula $(\Box \Diamond T_0 \lor \Box \Box \neg T_d) \land \Box \Diamond T_w$ is equivalent to $(\Box \Diamond T_0 \land \Box \Diamond T_w) \lor (\Box \Box \neg T_d \land \Box \Diamond T_w)$ and we also use a counter (bit) $b \in \{0, 1\}$ to verify $\Box \Diamond T_0 \land \Box \Diamond T_w$ and therefore the set of states in the new game $\mathcal{H}'$ denoted $Q'$ consists of tuples of the form $(q, c_D, c_W, b)$. Initially, $b = 0$ and the transition relation is as follows: $(q, c_D, c_W, b) \rightarrow (q', c_D, c_W, b')$ iff $(q, c_D, c_W) \rightarrow (q', c_D, c_W)$ is a transition in $\mathcal{H}$ and

\[
    b' = \begin{cases} 1 & \text{if } b = 0 \text{ and } (q, c_D, c_W) \in T_0 \\ 0 & \text{if } b = 1 \text{ and } (q, c_D, c_W) \in T_w \\ b & \text{otherwise} \end{cases}
\]

Then, considering $C_1' = C_1 \times \{0, 1\}$, the winning objective is

\begin{align*}
\Theta_C' = \{\eta' \in Q'^\omega \mid \eta' \upharpoonright C_1' \models (\Box \Diamond T_0' \lor (\Box \Box \neg T_d' \land \Box \Diamond T_w')
\end{align*}

where $T_0' = \{(q, c_D, c_W, 0) \mid (q, c_D, c_W) \in T_0\}$, $T_d' = T_d \times \{0, 1\}$ and $T_w' = T_w \times \{0, 1\}$. 
And finally, we have that a a play \( q' \) satisfies \( q' \models c_1 \mid \models \Box \Diamond T_0 \lor (\Diamond \Box \neg T_d') \land \Box \Diamond T_w' \) iff the Parity condition \( \text{Parity}(pr) \) is satisfied by \( q' \) where the priority function \( pr \) is defined as follows: For \( q' = (s, D, W, c_d, c_w, b) \in St \times 2^{Agst} \times 2^{Agst} \times (Agt \cup \{-1\}) \times (Agt \cup \{-1\}) \times \{0,1\} \),

\[
pr(q') = \begin{cases} 
0 & \text{if } q' \in T_0' \\
1 & \text{if } q' \not\in T_0' \land q' \in T_d' \\
2 & \text{if } q' \not\in T_0' \land q' \not\in T_d' \land q' \in T_w' \\
3 & \text{if } q' \not\in T_0' \land q' \not\in T_d' \land q' \not\in T_w'
\end{cases}
\]

For \( q' \not\in St \times 2^{Agst} \times 2^{Agst} \times (Agt \cup \{-1\}) \times (Agt \cup \{-1\}) \times \{0,1\} \), \( pr(q') = 4 \).

Since each play in the parity game \( \mathcal{H}' \) has polynomial number of different states, we can use Lemmas and obtain a finite duration game whose plays have polynomial length. This gives the following result:

**Theorem 7.** Deciding if there is a solution for the non-cooperative rational synthesis in concurrent games with Büchi objectives is in PSPACE.

**co-Büchi Objectives** For co-Büchi objectives, the winning condition for Constructor in the game \( \mathcal{H} \) is

\[
\Theta_C = \{ \eta \in Q^\omega \mid (\forall i \in \text{lim} \text{Proj}_W(\eta \mid c) \Rightarrow \text{Proj}_W(\eta \mid c) \models \Box \Diamond \neg F_i) \}
\]

We use again the fact that the sets \( D \) and \( W \) stabilize along a play \( \eta \) and the fact that co-Büchi objectives are tail objectives. Let \( C_1 = St \times 2^{Agst} \times 2^{Agst} \) be the set of states \( (s, D, W) \) of Constructor. Then, \( \forall i \in \text{lim} \text{Proj}_D(\eta \mid c) \) s.t. \( \text{Proj}_D(\eta \mid c) \models \Box \Diamond F_i \) is equivalent to \( \eta \models \Box C_1 \models \Box \Diamond T_d \) where \( T_d = \{ q = (s, D, W) \mid s \in \bigcup_{i \in D} F_i \} \).

Further, \( \forall i \in \text{lim} \text{Proj}_W(\eta \mid c) \Rightarrow \text{Proj}_W(\eta \mid c) \models \Box \Diamond \neg F_i \) is equivalent to \( \eta \models \Box \Diamond T_w \) where \( T_w = \{ q = (s, D, W) \mid s \in \bigcup_{i \in W} F_i \} \).

Therefore, the winning condition for Constructor in the game \( \mathcal{H} \) is equivalently written as

\[
\theta_C = \{ \eta \in Q^\omega \mid \eta \models (\Diamond \Box \neg T_0 \lor \Diamond \Box T_d) \land \Diamond \Box \neg T_w \}
\]

This can be written as the Parity condition \( \text{Parity}(pr) \) where the priority function \( pr \) is defined as follows: For \( q \in St \times 2^{Agst} \times 2^{Agst} \),

\[
pr(q) = \begin{cases} 
1 & \text{if } q \in T_w \\
2 & \text{if } q \not\in T_w \land q \in T_d \\
3 & \text{if } q \not\in T_w \land q \not\in T_d \land q \in T_0 \\
4 & \text{if } q \not\in T_w \land T_0
\end{cases}
\]

For \( q \not\in St \times 2^{Agst} \times 2^{Agst} \), \( pr(q) = 6 \).
Now, applying Lemma 2 on the game $H$ with parity objective $Parity(pr)$, and since each play in $H$ has a polynomial number of distinct states, we get the following complexity result.

**Theorem 8.** Deciding if there is a solution for the non-cooperative rational synthesis in concurrent games with co-Büchi objectives is in $PSpace$.

### B.4 Muller Objectives

Let $\mu_i$ be the Muller objective of Player $i$. Then, the Constructor’s objective in the game $H$ is $\Theta_C = \{ \eta \in Q^\omega \mid (ProjSt(\eta |_C) \in Muller(\mu_0) \text{ or } \exists i \in \lim ProjD(\eta |_C) \text{ s.t. } ProjSt(\eta |_C) \notin Muller(\mu_i)) \}

Contrary to the previous cases, in the case of Muller objectives, we cannot directly reduce to a finite duration game with plays having polynomial length. Instead, as also proceeded in [2], we use Least Appearance Record (LAR) construction to reduce the objective $\Theta_C$ to a parity objective with a polynomial number of priorities. That is, each state in the obtained game $H'$ is of form $(q, (m, h))$ where $q$ is a state in $H, m \in P(St)$ is a permutation of states in $St$ and $h \in \{0, 1, ..., |St| - 1\}$ is the position in $m$ of the last state $s$ that appeared in $q$.

The transition between states is defined by:

- $(q, (m, h)) \rightarrow (q', (m, h))$ if $q \rightarrow q'$ in $H$ and $q' \notin St \times 2^{Agt} \times 2^{Agt}$
- $(q, (m, h)) \rightarrow (q', (m', h'))$ if $q \rightarrow q'$ in $H$ and $q' \in St \times 2^{Agt} \times 2^{Agt}$ where, assuming $q' = (s, D, W)$ and $m = x_1sx_2$ for some $x_1, x_2 \in St^*$, $(m', h') = (x_1x_2s, |x_1|)$

Finally, the priority function $pr$ over states in $H'$ is defined as:

- for $q = (s, D, W) \in St \times 2^{Agt} \times 2^{Agt}$,

  $$pr((s, D, W), (m, h)) = \begin{cases} 2h & \text{if } \forall i \in W\{m[l] \mid l \geq h\} \models \mu_i \text{ and } \\
 2h + 1 & \text{otherwise} \end{cases}$$

- for $q \notin St \times 2^{Agt} \times 2^{Agt}$, $pr(q, (m, h)) = 2|St| + 2$.

Note that in this case, if we use the reduction to finite duration game, we obtain exponential size plays. Instead, we use the fact that the game $H'$ has exponential number of states in the size of the original game $G$, but it has a Parity objective with polynomial number of priorities. Then, the results in [46] prove the following theorem:

**Theorem 9.** Deciding if there is a solution for the non-cooperative rational synthesis in concurrent games with Muller objectives is in $ExpTime$. 
