Heat transfer in a free liquid layer under action of additional tangential stresses: numerical modeling

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Abstract. The dynamics and heat transfer processes in a planar layer of viscous incompressible heat-conducting liquid with free boundaries under condition of zero gravity are investigated. The influence of thermocapillary forces and additional tangential stresses on the boundaries caused by the environment is taken into account. A special class of solutions of the Navier-Stokes equations is used to model the flows and layer deformations. The numerical results of investigation of the layer dynamics and temperature field in the three-dimensional case are presented.

1. Introduction
Problems related to the study of the infinite liquid layers with free boundaries have been studied for quite a long time. Modeling of such flows requires consideration of additional shear stresses on the free boundary and the study of the interaction of different mechanisms, causing fluid motion [1, 2]. Formulation of the problem in three dimensions for the study of the deformation of a viscous incompressible liquid layer under the influence of thermocapillary forces is carried out in [3, 4]. The first numerical modeling of deformation of the viscous liquid layer with free boundaries is given in [5]. The problem becomes more complicated, if one need to find the temperature of the liquid in the infinite deformable layer [6, 7].

In this paper the three-dimensional nonstationary liquid flow in an infinite plane-parallel layer is investigated under conditions of zero gravity. Thermocapillary forces and additional shear stresses are considered to have an influence on the dynamics of the liquid flow. Deformation of the layer and the heat transfer process are modeled on the basis of the exact solutions of the Navier-Stokes equations [3]. A numerical algorithm to find the position of layer boundaries, the velocity field and temperature distribution and some results of numerical studies are presented.

2. Statement of the problem
Consider the liquid occupying the infinite plane-parallel layer of incompressible viscous thermally conducting fluid \( \Omega = \{(x, y, z): -\infty < x < +\infty, -\infty < y < +\infty, -Z(t) < z < Z(t)\} \) in zero gravity. Free surfaces \( \pm Z(t) \) remains rigid and parallel at all points in time. The coordinate system is chosen so that the \( Ox \) and \( Oy \) axis are directed along surfaces \( \pm Z(t) \), and \( Oz \) axis is perpendicular to them (see figure 1). The unit vectors of external normals and tangential vectors to the free boundaries are defined as follows: \( n = (0, 0, \pm 1) \) and \( s_1 = (1, 0, 0) \), \( s_2 = (0, 1, 0) \).
Unknown function (the velocity vector $\mathbf{v} = (u, v, w)$, the pressure $p$, the temperature $T$) satisfy the Navier-Stokes and heat transfer equations:

$$
\begin{align*}
\frac{\partial u}{\partial t} + uu_x + vu_y + wu_z &= -p_x + \frac{1}{Re}(u_{xx} + u_{yy} + u_{zz}), \\
\frac{\partial v}{\partial t} + uv_x + vv_y + wv_z &= -p_y + \frac{1}{Re}(v_{xx} + v_{yy} + v_{zz}), \\
\frac{\partial w}{\partial t} + uw_x + vw_y + ww_z &= -p_z + \frac{1}{Re}(w_{xx} + w_{yy} + w_{zz}), \\
u_x + v_y + w_z &= 0, \\
T_t + uT_x + vT_y + wT_z &= \frac{1}{RePr}(T_{xx} + T_{yy} + T_{zz}).
\end{align*}
$$

Here $Re$ is the Reynolds number ($Re = \frac{vl}{v}$), $Pr$ is the Prandtl number ($Pr = \frac{\nu}{\chi}$), $\nu$ and $\chi$ are the kinematic viscosity and thermal diffusivity coefficients, $l$ is the characteristic length of the flow domain, $v_*$ is the characteristic velocity, $t_*$ = $\frac{l}{v_*}$ is the characteristic time.

The the kinematic and dynamic conditions on the free boundaries are determined as follows:

$$
\begin{align*}
\mathbf{v} \cdot \mathbf{n} &= \pm w|_z=\pm Z(t) = \frac{dZ}{dt}, \\
-p + \frac{2}{Re}\mathbf{n} \cdot \mathbf{D}(\mathbf{v})|_z=\pm Z(t) &= -P_y, \\
2s_1 \cdot \mathbf{n} \cdot \mathbf{D}(\mathbf{v})|_z=\pm Z(t) &= \tau_1(x,t) - \frac{Ma}{RePr}T_x, \\
2s_2 \cdot \mathbf{n} \cdot \mathbf{D}(\mathbf{v})|_z=\pm Z(t) &= \tau_2(y,t) - \frac{Ma}{RePr}T_y.
\end{align*}
$$

Temperature and shear stresses on free boundaries are the given functions of time and longitudinal coordinate:

$$
T(x, y, \pm Z(t), t) = \frac{1}{2}A(t)x^2 + \frac{1}{2}B(t)y^2 + \Theta(t), \quad \tau_1(x,t) = x\tilde{\tau}(t), \quad \tau_2(y,t) = y\tilde{\tau}(t).
$$

Here $\tau_1(x,t)$ and $\tau_2(y,t)$ are the shear stresses induced by the external environment, $A(t)$, $\Theta(t)$, $\tilde{\tau}(t)$ are the arbitrary functions that depend on time, $\mathbf{D}(\mathbf{v})$ is the tensor of deformation.
rate, \( P_g \approx \frac{2}{Re} \rho v_0 \cdot (v_g) \cdot n \big|_{z=\pm Z(t)} \); \( P_g \) is the external pressure, \( \rho \), \( \nu \) is the relation of the densities and kinematic viscosity coefficients of gas and liquid, \( Ma \) is the Marangoni number \((Ma = \frac{\sigma_T T^d}{\rho \nu \chi})\), \( T* \) is the characteristic temperature, \( p* = \rho v_*^2 \) is the characteristic pressure, \( \rho \) is the liquid density, \( v_g \) is the gas velocity, \( \sigma_T \) is the temperature coefficient of surface tension \((\sigma = \sigma_0 - \sigma_T (T - T_0))\), \( \sigma_T > 0 \). It is assumed that the action of normal stresses from the external environment may be neglected.

Components of the velocity vector will be determined in the form \([3]\)

\[
    u(x,z,t) = (f + g)x, \quad v(y,z,t) = (f - g)y, \quad w(z,t) = -2 \int_0^z f(\alpha,t) d\alpha,
\]

where the functions \( f(z,t) \) and \( g(z,t) \) are the solutions of the system of equations

\[
    f_t + f^2 + g^2 - 2fz \int_0^z f(\alpha,t) d\alpha - \frac{1}{Re} f_{zz} = 0, \quad (6)
\]

\[
    g_t + 2fg - 2gz \int_0^z f(\alpha,t) d\alpha - \frac{1}{Re} g_{zz} = 0. \quad (7)
\]

The position of the free boundary \( Z(t) \) satisfies the following integro-differential equation:

\[
    \frac{dZ}{dt} = -\int_0^{Z(t)} f(z,t) dz. \quad (8)
\]

Dynamic conditions \((3), (4)\) for the functions \( f(z,t), g(z,t) \) and \( Z(t) \) are written in the form

\[
    f_z(Z(t),t) = \hat{\tau}(t) - \frac{Ma}{Re Pr}(A(t) + B(t)), \quad g_z(Z(t),t) = \hat{\tau}(t) - \frac{Ma}{Re Pr}(A(t) - B(t)).
\]

If we are considering a symmetric flow then the line of symmetry must satisfy the condition \( f_z(0,t) = 0 \). The initial position of the layer is determined from the following equations:

\[
    f(z,0) = f_0(z) = 0, \quad g(z,0) = g_0(z) = 0 (0 \leq z \leq Z_0); \quad Z(0) = Z_0 (Z_0 > 0).
\]

Also the problem has been supplemented by conditions at infinity.

The unknown functions of the problem are determined numerically. Functions \( Z(t), f(z,t), g(z,t) \) can be found using the ”predictor-corrector” algorithm. The position of the free boundary \( Z(t) \) is determined using the following algorithm:

\[
    Z^{k+1} = Z^{k-1} - 2\Delta t \int_0^{Z^k} f(z) dz, \quad (9)
\]

\[
    \dot{Z}^{k+1} = Z^k - \Delta t \left[ \int_0^{Z^{k+1}} f^{k+1}(z) dz + \int_0^{Z^k} f^k(z) dz \right], \quad (10)
\]

where (9) is the predictor and (10) is the corrector scheme \([5, 7]\). The new spatial grid is determined on the each temporal layer.

The predictor-corrector algorithm of second-order accuracy is used to calculate functions \( f(t,z), g(t,z) \):

\[
    q^{k+1/4} = q^k + 0.5\Delta t[-\Lambda_1 q^{k+1/4} + \Phi^k],
\]

\[
    q^{k+1/2} = q^{k+1/4} + 0.5\Delta t[-\Lambda_2 q^{k+1/2}],
\]
\[ q^{k+1} = q^k + \Delta t[-\Lambda q^{k+1/2} + \Phi^k]. \]

Here \( q^k \) are the values of the unknown function \( f \) or \( g \) at the grid points, \( \Lambda = \Lambda_1 + \Lambda_2 \), \( \Lambda_i \) are the finite-difference operators corresponding to differential operators \((-1/Re)\partial^2/\partial z^2\) and \(-F\partial/\partial z\), functions \( F \) and \( \Phi^k \) are determined from (8). The integrals in the equations (6)-(8) are determined by quadrature formulas. The interpolation formulas are used for the transition to a new temporal layer.

3. Numerical algorithm for heat transfer simulation

The values of temperature distribution in the layer can be determined in the following parallelepiped (figure 1):

\[ \Omega = \{-N < x < N, -L < y < L, -Z < z < Z\}. \]  

(11)

The conditions (1)-(6) will be imposed at \( z = \pm Z \). "Soft" conditions resulting conditions at infinity will be imposed on the "lateral walls" of a parallelepiped. As such conditions the conditions of the following type can be used:

\[ T_{xx} = 0, \quad T_{yy} = 0. \]

These expressions are the result of conditions \( T \to T_\infty (T_\infty = const) \) at infinity on the assumption \( T_x \to 0, T_y \to 0 \).

Numerical investigation of heat transfer process is based on the stabilizing corrections method. Finite-difference scheme of second order approximation for the solution of the heat transfer equation can be represented as follows [8]:

\[ \frac{T^{k+1/3} - T^k}{\Delta t} = K_1 T^{k+1/3} + K_2 T^k + K_3 T^k + S^k, \]

\[ \frac{T^{k+2/3} - T^{k+1/3}}{\Delta t} = K_2 (T^{k+2/3} - T^k), \]

\[ \frac{T^{k+1} - T^{k+2/3}}{\Delta t} = K_3 (T^{k+1} - T^k). \]

(12)

Here \( T^k(x,y,z) = T(x,y,z,t^k) \), \( K_i \) are the finite-difference analogues of the corresponding differential operators \((K_1 \sim \frac{1}{RePr}\partial^2/\partial z^2, K_2 \sim \frac{1}{RePr}\partial^2/\partial x^2, K_3 \sim \frac{1}{RePr}\partial^2/\partial y^2)\), \( S^k \) is the finite-difference analog of convective term \((S \sim -u \frac{\partial T}{\partial x} - v \frac{\partial T}{\partial y} - w \frac{\partial T}{\partial z})\), \( \Delta t \) is the time step. The grid \((x_n, y_l, z_m)\) is introduced to implement the presented finite-difference scheme: \( x_n = (n-1)h_x(n = 1, ..., N + 1), h_x = N/N, y_l = (l - 1)h_y(l = 1, ..., L + 1), h_y = L/L \). This grid is moving along the vertical coordinate: \( z_m = (m - 1)h_z(m = 1, ..., M + 1), h_z = Z/M, \) \( Z = Z^{k+1} \) is the new position of the free boundary on the time layer \( k + 1 \). We assume that the flow is symmetric with respect to \( OZ \) (see [1]-[5]). Then the mathematical modeling can be devoted to construction of the solution in the half-layer \( 0 < z < Z(t) \). The conditions \( T_x = 0, T_y = 0 \) on the boundary \( z = 0 \) will be imposed in this case.

The second-order finite-difference approximations for second derivatives and the central difference scheme for approximation of convective terms are applied The numerical scheme (12) can be represented as a system of linear algebraic equations. At each time sublayer the systems of equations are implemented by the sweep method. The transition to the new spatial grid at the new time step is performed using the Newton interpolation formula [7].

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4. Numerical results

Numerical results characterize the spreading of an incompressible viscous layer and the liquid temperature distribution in the considered domain. Figure 3 shows the change of the free boundary position over time in the case of different time dependent functions $A(t) = A_0/(t^2 + 1)$, $B(t) = B_0/(t^2 + 1)$, $\tau(t) = A_\tau \tau_0/(t^2 + 1)$, 2 — the case when $A(t) = A_0 t \exp(t)$, $B(t) = B_0 t \exp(t)$, $\tau(t) = A_\tau \tau_0 t \exp(t)$.

Figure 4. Spreading of the layer, $A(t) = A_0/(t^2 + 1)$, $B(t) = B_0/(t^2 + 1)$, $\tau(t) = A_\tau \tau_0/(t^2 + 1)$, $t = 0.05$.

Figure 5. The temperature distribution in the XZ plane ($y=2.5$), $t=0.02$.

Figure 6. The temperature distribution in the XZ plane ($y=2.5$), $t=0.05$.

Figure 7. The temperature distribution in the YZ plane ($x=5$), $t=0.02$.

Figure 8. The temperature distribution in the YZ plane ($x=5$), $t=0.05$. 
Position of the free liquid boundary at the initial time is defined as the \( Z_0 = 0.5 \), \( \Theta \) is assumed to be 10. In the case when the dependencies \( A(t) \), \( B(t) \) and \( \tilde{\tau}(t) \) have the form \( A(t) = A_0 t \exp(t), B(t) = B_0 t \exp(t), \tilde{\tau}(t) = A_0 t \exp(t) \) the layer is spreading much slower than in the case when \( A(t) = A_0 / (t^2 + 1), B(t) = B_0 / (t^2 + 1), \tilde{\tau}(t) = A_0 t \exp(t) / (t^2 + 1) \) (here \( A_0 = -0.1, B_0 = -0.01, A_\tau = 400, \tau_0 = -0.1 \)). Figure 4 demonstrates an example of three-dimensional spreading of a liquid layer at time \( t=0.05 \).

The temperature distribution is defined in the parallelepiped where \( N = L = 5 \) (see (11)). Figures 5 and 6 demonstrates temperature distribution in the spreading liquid layer in the XZ plane when the dependencies \( A(t) \), \( B(t) \) and \( \tilde{\tau}(t) \) have the form \( A(t) = A_0 / (t^2 + 1), B(t) = B_0 / (t^2 + 1), \tilde{\tau}(t) = A_0 t \exp(t) / (t^2 + 1) \) at time \( t=0.02 \) and \( t=0.05 \), respectively. In figures 7 and 8 shown the ”lateral wall” \( x = 5 \) (YZ plane) in the case of the same dependency functions \( A(t), B(t) \) and \( \tilde{\tau}(t) \) of time as the figures 5 and 6. Note that the temperature redistribution in the layer is observed over time. At the parallelepiped boundaries the temperature lower than in the central part of the investigated area. The qualitative distribution of the temperature is retained within the whole layer despite the decrease in temperature at the boundaries of the parallelepiped.

Numerical investigations of the temperature field in a liquid layer were also carried out for the case when boundary conditions at the interfaces are coordinated with the boundary conditions on the artificial boundaries. The conditions \( T_{xx} = A(t) \) and \( T_{yy} = B(t) \) were used as the conditions on the lateral walls. The temperature distribution in the layer remains qualitatively be the same; there are only the small quantitative differences (approximately 3%) in the values of the temperature functions.

5. Conclusion

The processes of deformation of a viscous incompressible liquid layer with free boundaries and heat transfer in the layer are modeled using exact solutions of the Navier-Stokes equations. A numerical algorithm for solving the problem of the heat distribution in the parallelepiped domain with moving boundaries in three dimensions. The change of the heat distribution in a spreads liquid layer is investigated.

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