Reconstruction of the potential from $I$-function. *†

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Abstract

If $f(x, k)$ is the Jost solution and $f(k) = f(0, k)$, then the $I$-function is

$$I(k) := \frac{f'(0, k)}{f(0, k)}.$$

It is proved that $I(k)$ is in one-to-one correspondence with the scattering triple $S := \{S(k), k_j, s_j, \ 1 \leq j \leq J\}$ and with the spectral function $\rho(\lambda)$ of the Sturm-Liouville operator $l = -\frac{d^2}{dx^2} + q(x)$ on $(0, \infty)$ with the Dirichlet condition at $x = 0$ and $q(x) \in L_{1,1} := \{q : q = \mathbb{1}, \int_0^\infty (1 + x)|q(x)|dx < \infty\}$.

Analytical methods are given for finding $S$ from $I(k)$ and $I(k)$ from $S$, and $\rho(\lambda)$ from $I(k)$ and $I(k)$ from $\rho(\lambda)$. Since the methods for finding $q(x)$ from $S$ or from $\rho(\lambda)$ are known, this yields the methods for finding $q(x)$ from $I(k)$.

1 Introduction

Let $q(x) \in L_{1,1} := \{q : q = \mathbb{1}, \int_0^\infty (1 + x)|q(x)|dx < \infty\}$ and $\ell u := -u'' + q(x)u$ be the selfadjoint operator defined in $L^2(0, \infty)$ by the boundary condition $u(0) = 0$. Let $\rho(\lambda)$ be its (uniquely defined) spectral function and

$$S := \{S(k), k_j, s_j, 1 \leq j \leq J\}$$

be its scattering data.

Here $k_j > 0, -k_j^2$ are the negative eigenvalues of $\ell$, $J$ is the number of these eigenvalues, $s_j > 0$ are the norming constants,

$$s_j := \|f_j(x)\|^{-2}, \ \| \cdot \| = \| \cdot \|_{L^2(0,\infty)}, \ f_j(x) := f(x, ik_j),$$

$f(x, k)$ is the unique solution of the problem

$$(\ell - k^2)f(x, k) = 0, \ x > 0; \ f(x, k) = e^{ikx} + o(1) \ as \ x \to +\infty.$$
Let $f(k) := f(0, k)$. Then $S(k) := \frac{f(-k)}{f(k)}$.

It is known \[^2\] that:

$$d\rho(\lambda) = \begin{cases} \sqrt{\lambda} d\lambda & \lambda > 0 \\ \sum_{j=1}^{J} c_j \delta(\lambda + k_j^2) d\lambda & \lambda < 0, \end{cases} \tag{1.2}$$

where $c_j := \|\varphi(x, ik_j)\|^{-2}$, the function $\varphi(x, k)$ is the unique solution to the problem:

$$(l - k^2)\varphi = 0, \quad x > 0; \quad \varphi(0, k) = 0, \quad \varphi'(0, k) = 1. \tag{1.3}$$

The function $f(x, k)$ is an analytic function of $k$ in $\mathbb{C}_+ := \{k : k \in \mathbb{C}, \text{Im} k > 0\}$, and $\varphi(x, k)$ is an entire function of $k$. The numbers $ik_j, 1 \leq j \leq J$, are simple zeros of $f(k)$ and $f(k)$ has no other zeros in $\mathbb{C}_+$, but it may have a simple zero at $k = 0$.

Define the I-function by the formula

$$I(k) := \frac{f'(0, k)}{f(k)}. \tag{1.4}$$

This function is of interest in applications and in mathematics: in applications this function is an impedance function (a ratio of a component of the electric field and a component of the magnetic field), and in mathematics it is the Weyl function.

Recall that the Weyl function $m(k)$ is defined as such a function of $k \in \mathbb{C}_+$ that

$$W(x, k) := \theta(x, k) + m(k)\varphi(x, k) \in L^2(0, \infty), \quad \text{Im} k > 0. \tag{1.5}$$

Here $\varphi(x, k)$ was defined above and $\theta(x, k)$ is the solution to the problem

$$(l - k^2)\theta = 0, \quad \theta(0, k) = 1, \quad \theta'(0, k) = 0. \tag{1.6}$$

If $q \in L_{1,1}$, then $W(x, k) = c(k)f(x, k), c(k) \neq 0$. Therefore $I(k) = \frac{W'(0, k)}{W(k)} = m(k)$ as claimed.

The basic results of this paper are the methods and formulas for finding $S(k)$ and $\rho(\lambda)$ from $I(k)$ and $I(k)$ from either $S(k)$ or $\rho(\lambda)$.

Let us describe the results in more detail. Suppose $S(k)$ is known. Then, as we prove, $f(k)$ is uniquely and analytically determined by solving the Riemann problem:

$$f(k) = S(-k)f(-k). \tag{1.7}$$

This problem is solved in section 3, see formula (3.2).

If $f(k)$ is found, one finds $I(k)$, and therefore $f'(0, k)$, by solving the following problem:

$$\frac{f'(0, k)}{f(k)} - \frac{f'(0, -k)}{f(-k)} = \frac{2ik}{|f(k)|^2}, \tag{1.8}$$

which is an immediate consequence of the Wronskian formula:

$$f'(0, k)f(-k) - f'(0, -k)f(k) = 2ik. \tag{1.9}$$
Problem (1.8) is solved in section 3, see formulas (3.9) and (3.12).

If \( \rho(\lambda) \) is given, then \( I(k) \) is uniquely determined as follows: one determines \( |f(k)| \) and then analytically, \( f(k) \), since \( \rho(k) \) determines explicitly the numbers \( k_j \) and \( J \), where \( 1 \leq j \leq J \). The function \( f(k) \) is determined analytically if its modulus on the real axis and its zeros \( ik_j, 1 \leq j \leq J \), are known. If \( f(k) \) is found then \( f'(0, k) \) can be found from the Riemann problem (1.9). Alternatively, \( I(k) \) can be found as the solution of (1.8), which is a Riemann-type problem, the problem of finding a section-meromorphic function with finitely many known simple poles, located at the points \( ik_j, 1 \leq j \leq J \), and known residues at these poles, from its jump across the contour, the real axis in our case. In fact, one can solve (1.8) for \( I(k) \) if the modulus of \( f(k) \), the numbers \( k_j, 1 \leq j \leq J \), and the residues (1.10) of the function \( I(k) \) at its simple poles \( ik_j \) are known.

Conversely, if \( I(k) \) is known for all \( k > 0 \), then \( k_j \) are uniquely determined since \( k_j \) are the (simple) poles of \( I(k) \) in \( \mathbb{C}_+ \). The number \( J \) of these poles is also uniquely defined. One has

\[
I_j := \text{Res}_{k=ik_j} I(k) = \frac{f''(0, ik_j)}{f(ik_j)}, \quad k_j > 0.
\]  

(1.10)

It is known (see \[4\], \[3\]) that

\[
s_j = -\frac{2ik_j}{f(ik_j)f'(0, ik_j)}, \quad c_j = \frac{-2ik_jf'(0, ik_j)}{f(ik_j)},
\]  

(1.11)

so

\[
s_j = -\frac{2ik_j}{f^2(ik_j)I_j}, \quad c_j = -2ik_jI_j.
\]  

(1.12)

To determine \( S \), or \( \rho(\lambda) \), it remains to determine \( f(k) \) from \( I(k) \). This is done by solving a Riemann problem:

\[
f(k) = \frac{k}{\text{Im} I(k)} \frac{1}{f(-k)}.
\]  

(1.13)

In sections 2 and 3 some analytic formulas are derived for the solutions of (1.8) and (1.13).

Recovery of the potential \( q(x) \) from \( I(k) \) can be considered done, if one recovers either \( S \) or \( \rho(\lambda) \) from \( I(k) \).

Indeed, if \( S \) is recovered, then the known procedure (see e.g. \[2\]):

\[
S \Rightarrow F \Rightarrow A \Rightarrow q
\]  

(1.14)

recovers \( q(x) \).

This procedure (the Marchenko method) is analyzed in \[4\] and \[3\], where it is proved that each step of this procedure is invertible

\[
S \Leftrightarrow F \Leftrightarrow A \Leftrightarrow q.
\]
The function $F(x)$ is defined as

$$F(x) = \sum_{j=1}^{J} s_j e^{-k_j x} + \frac{1}{2\pi} \int_{-\infty}^{\infty} [1 - S(k)] e^{ikx} dk.$$  \hfill (1.15)

The kernel $A = A(x, y)$ is the transformation kernel:

$$f(x, k) = e^{ikx} + \int_x^{\infty} A(x, y) e^{iky} dy.$$  \hfill (1.16)

It is related to $F(x)$ by the Marchenko equation:

$$A(x, y) + F(x + y) + \int_x^{\infty} A(x, s) F(s + y) ds = 0, \quad 0 \leq y \leq x < \infty,$$  \hfill (1.17)

which is uniquely solvable for $A(x, y)$ if $S$ (and therefore $F(x)$) comes from $q \in L_{1,1}$.

If $A(x, y)$ is found from (1.17), then

$$q(x) = -2 \frac{dA(x, x)}{dx}.$$  \hfill (1.18)

If $\rho(\lambda)$ is found from $I(k)$, then the Gelfand-Levitan procedure recovers $q(x)$:

$$\rho \Rightarrow L(x, y) \Rightarrow K(x, y) \Rightarrow q(x).$$

Here

$$L(x, y) = \int_{-\infty}^{\infty} \varphi_0(x, \sqrt{\lambda}) \varphi_0(y, \sqrt{\lambda}) d(\rho(\lambda) - \rho_0(\lambda)),$$  \hfill (1.19)

$$\varphi_0(x, \sqrt{\lambda}) = \frac{\sin(\sqrt{\lambda} x)}{\sqrt{\lambda}}, \rho_0(\lambda)$$ is the spectral function of the operator $\ell_0$, that is, of operator $\ell$ with $q(x) = 0, d\rho_0(\lambda) = 0$ for $\lambda < 0, d\rho_0(\lambda) = \frac{\sqrt{\lambda} d\lambda}{\pi}, \lambda > 0$.

The kernel $K(x, y)$ is uniquely determined as the solution to the Gelfand-Levitan equation:

$$K(x, y) + L(x, y) + \int_0^x K(x, s) L(s, y) ds = 0, \quad 0 \leq y \leq x.$$  \hfill (1.20)

If $K(x, y)$ is found from (1.20), then

$$q(x) = 2 \frac{dK(x, x)}{dx}.$$  \hfill (1.21)

The kernel $K(x, y)$ is the transformation kernel:

$$\varphi(x, k) = \varphi_0(x, k) + \int_0^x K(x, y) \varphi_0(y, k) dy, \quad k > 0,$$  \hfill (1.21)

where $\varphi(x, k)$ solves (1.3), and $\varphi_0(x, k) = \frac{\sin(kx)}{k}$ solves (1.3) with $q(x) = 0$.  

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It is proved in [2], [3] that
\[ \rho(\lambda) \Leftrightarrow L(x, y) \Leftrightarrow K(x, y) \Leftrightarrow q, \] (1.23)
for a very wide class of \( q(x) \), namely for the real-valued \( q(x) \) for which the equation \((l - z)u = 0, \text{Im } z > 0\), has exactly one solution \( u \in L^2(0, \infty) \) (the limit point at infinity case).

In [4] a necessary and sufficient condition is obtained for a function \( I(k) \) to be the \( I \)-function corresponding to an operator \( l \) with the potential \( q(x) \) which is locally \( C^{(m)} \)-smooth and a method is outlined for the recovery of \( q \) from \( I(k) \).

In [3], [5] various properties of \( I(k) \) are studied.

The main results of this paper are:
1) it is proved that \( I(k) \) is in one-to-one correspondence with the scattering data \( S \) and with the spectral function \( \rho(\lambda) \).
2) formulas are given for finding \( I(k) \) from \( S \) or from \( \rho(\lambda) \) and vice versa ((2.11), (2.13), (2.15), (2.16), (3.2), (3.4), (3.5), (3.9), (3.12)).

## 2 Recovery of \( S \) and \( \rho \) from \( I(k) \).

The function \( I(k) \) is meromorphic in \( \mathbb{C}_+ \). Its values on the real axis determine uniquely \( I(k) \) in \( \mathbb{C}_+ \).

Given \( I(k) \forall k > 0 \), one finds its poles in \( \mathbb{C}_+ \). These poles are exactly the points \( ik_j, k_j > 0 \). \( I(k) \) may have a pole at \( k = 0 \). If \( k = 0 \) is a pole of \( I(k) \), then \( |I(0)| = \infty \) since \( f'(0,0) \neq 0 \). The number \( J \) of the points \( ik_j \) is uniquely determined by \( I(k) \).

Let us derive a formula for finding \( f(k) \) from \( I(k) \). Start with formula (1.18), which we write as:
\[ f(k) = \frac{k}{\text{Im } I(k)} \frac{1}{f(-k)}. \] (2.1)

Define
\[ w(k) := \prod_{j=1}^{J} \frac{k - ik_j}{k + ik_j}, \text{ if } f(0) \neq 0, \quad w_1(k) := \frac{k}{k+i} w(k) \text{ if } f(0) = 0, \quad f_0(k) := \frac{f(k)}{w(k)}. \] (2.2)

Note that \( f_0(k) \) is analytic in \( \mathbb{C}_+ \), has no zeros in \( \mathbb{C}_+ \), and \( f_0(\infty) = 1 \). Also
\[ w(k) = w(-k) = w^{-1}(k), \text{ and } |w(k)| = 1 \text{ if } k \in \mathbb{R}. \] (2.3)

Equation (2.1) can be written as
\[ f_0(k) = \frac{k}{\text{Im } I(k)} \frac{1}{f_0(-k)w(k)w(-k)} = g(k)f_0^{-1}(-k), \] (2.4)

where
\[ g(k) := \frac{k}{\text{Im } I(k)} > 0, \quad k \in \mathbb{R}, \] (2.5)
and the relation \( w(k)w(-k) = 1 \) for \( k \in \mathbb{R} \) was used.

The function \( f_0^{-1}(-k) \) is analytic in \( \mathbb{C}_- := \{ k : \in \mathbb{C}, \text{Im} \, k < 0 \} \), does not have zeros in \( \mathbb{C}_- \), and \( \lim_{|k| \to \infty, k \in \mathbb{C}_-} f_0^{-1}(-k) = 1 \).

Therefore \( \ln f_0(k) \) and \( \ln f_0^{-1}(-k) \) are functions analytic in \( \mathbb{C}_+ \) and \( \mathbb{C}_- \), respectively, vanishing at infinity and

\[
\ln f_0(k) = \ln g(k) + \ln f_0^{-1}(-k). \tag{2.6}
\]

Therefore

\[
\ln f_0(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\ln g(s)ds}{s - z}, \quad \text{Im} \, z > 0, \tag{2.7}
\]

since (2.7) implies that the section-analytic function (2.7) satisfies (2.6), and there is exactly one such function: if there were another one, then their difference \( \Phi(z) \) would be analytic in \( \mathbb{C}_+ \) and in \( \mathbb{C}_- \), \( \Phi(\infty) = 0 \) in \( \mathbb{C}_+ \) and in \( \mathbb{C}_- \) and \( \Phi_+(k) = \Phi_-(k) \), where \( \Phi_+(k) = \Phi(k + i0) \), \( \Phi_-(k) = \Phi(k - i0) \), \( k \in \mathbb{R} \). Thus \( \Phi(z) \) is analytic in \( \mathbb{C} \) and \( \Phi(\infty) = 0 \).

By the Liouville theorem, \( \Phi(z) \equiv 0 \). This argument is well-known (see, e.g., [1]), and is given to make the presentation self-contained.

Let us summarize the result:

**Lemma 2.1.** Given \( I(k) \), one finds \( k_j \) and \( J \), and then \( f(z) \) in \( \mathbb{C}_+ \):

\[
f(z) = \exp \left( \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\ln \text{Im} \, I(s) ds}{s - z} \right) w(z), \quad \text{Im} \, z > 0, \tag{2.8}
\]

where \( w(z) \) is defined in (2.2).

Note that \( g(s) = g(-s) \), since \( \text{Im} \, I(s) = -\text{Im} \, I(-s) \), \( s \in \mathbb{R}_- \), so one has:

\[
\int_{-\infty}^{\infty} \frac{\ln g(s)ds}{s - z} = 2 \int_{0}^{\infty} \frac{z \ln g(s)ds}{s^2 - z^2}, \quad \text{Im} \, z > 0, \tag{2.9}
\]

and (2.8) can be written as:

\[
f(z) = \exp \left( \frac{1}{i\pi} \int_{0}^{\infty} \frac{z \ln \text{Im} \, I(s) ds}{s^2 - z^2} \right) w(z), \quad \text{Im} \, z > 0. \tag{2.10}
\]

To calculate \( f(k) \), \( k > 0 \), one takes \( z = k + i0 \) in (2.8) or (2.10) and uses the well-known formula:

\[
\frac{1}{s - k - i0} = P \frac{1}{s - k} + i\pi \delta(s - k), \tag{2.11}
\]

where \( P \frac{1}{s} \) is the principle value of \( \frac{1}{s} \) and \( \delta(s) \) is the delta-function, to get from (2.8):

\[
f(k) = \exp \left( \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\ln \text{Im} \, I(s) ds}{s - k} + \frac{1}{2} \ln \frac{k}{\text{Im} \, I(k)} \right) w(k), k > 0. \tag{2.12}
\]
Thus, given \( I(k) \forall k > 0 \), we recovered uniquely \( f(k) \forall k > 0, k_j, \) and \( J \). The \( s_j \) are found by the first formula (1.12) and (1.10). So \( S \) is recovered and \( q(x) \) is recovered by the procedure (1.14).

Formula (2.8) holds if \( f(0) \neq 0 \) and if \( f(0) = 0 \). If \( f(0) = 0 \), then one can suggest an equivalent to (2.8) formula which looks differently. Namely, if \( f(0) = 0 \), then (2.4) holds with \( g_1 := g(k + 1) \) in place of \( g \) and \( f_1(k) := f(k) w_1(k) \) in place of \( f_0(k) \). Note that \( 0 < |g_1(0)| < \infty \) if \( f(0) = 0 \). Indeed, \( \dot{f}(0) \neq 0 \) and, by (2.1), \( g(k) = |f(k)|^2 \), so \( 0 < |g_1(0)| < \infty \), as claimed. Formula (2.8) holds if \( f(0) = 0 \) and if \( \frac{k}{\text{Im} I(s)} \) is replaced by \( \frac{s^2 + 1}{\text{Im} I(s)} \) and \( w(k) \) is replaced by \( w_1(k) \).

Let us show now how to recover \( \rho(\lambda) \) from \( I(k) \). As above, one recovers \( k_j \) and \( J \).

One has

\[
|f(k)| = \left( \frac{k}{\text{Im} I(k)} \right)^{\frac{1}{2}}, \quad k > 0, \tag{2.13}
\]

where the square root is positive and the function \( \frac{k}{\text{Im} I(k)} > 0, k > 0 \).

To find \( c_j \) (see formula (1.12)) one uses second formula (1.12) and formula (1.10). If \( \rho(\lambda) \) is found, then \( q(x) \) is found by the Gelfand-Levitan method (1.23).

### 3 Recovery of \( I(k) \) from \( S \) or \( \rho \).

Let us first explain how to calculate \( I(k) \) given \( S \).

Assume \( S \) is given. Then \( f(k) \) can be recovered analytically as follows.

Write (1.7), using (2.3), as

\[
f_0(k) = S(-k) w^{-2}(k) f_0(-k), \quad f(0) \neq 0. \tag{3.1}
\]

The function \( f_0(k) \) is analytic and has no zeros in \( \mathbb{C}_+ \), and \( f_0(\infty) = 1 \), and \( f_0(-k) \) has similar properties in \( \mathbb{C}_- \). Therefore one solves analytically the Riemann problem (3.1) and gets:

\[
f(z) = \exp \left( \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\ln S(-t)}{w^2(t)} dt \right) w(z), \quad \text{Im} z > 0. \tag{3.2}
\]

Note that the index \( \text{ind}_R \frac{S(-k)}{w^2(k)} = 0 \) if \( f(0) \neq 0 \), so \( f(z) \) is uniquely defined by formula (3.2). As in section 2, one calculates \( f(k) \) for \( k > 0 \) by taking \( z = k + 0 \) in (3.2) and using (2.11).

Formula (3.2) holds also when \( f(0) = 0 \). In this case \( \text{ind}_R \frac{S(-k)}{w^2(k)} = 1 \), and \( \text{ind}_R S_1(-k) = 0 \), where \( S_1(k) := S(k) \frac{k-i}{k+i} \). To see that formula (3.2) holds in the case \( f(0) = 0 \), one can use the above argument and the following formula:

\[
\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\log \frac{s+i t}{s-z}}{s-z} ds = \log \frac{z}{z+i t}, \tag{3.3}
\]

which holds if \( t > 0 \) and \( \text{Im} z > 0 \).
Alternatively, if \( f(0) = 0 \), then one replaces in formula (3.1) \( f_0(k) \) by \( f_1 := \frac{f(k)}{w_1(k)} \), where \( w_1 \) is defined in (2.2), and \( S(k) \) by \( S_1(k) := S(k)\frac{k-i}{k+i} \). Formula (3.2) in this case takes the form:

\[
f(z) = \exp \left( \frac{1}{2\pi i} \int_{-\infty}^{\infty} \ln \frac{S_1(-t)}{w_1^2(t)} \, dt \right) w_1(z), \quad \text{Im } z > 0, \tag{3.2'}
\]
equivalent to (3.2).

Since \( f(k) \) is found, to recover \( I(k) \) one can use formula (1.8):

\[
I(k) = I(-k) + \frac{2ik}{|f(k)|^2}. \tag{3.3}
\]
The function \( I(k) \) is meromorphic in \( \mathbb{C}_+ \). Let us subtract from \( I(k) \) its principal parts at the poles and get a holomorphic in \( \mathbb{C}_+ \) function \( J(k) \) defined as:

\[
J(k) := I(k) - ik - \sum_{j=0}^{J} \frac{I_j}{k - ik_j}, \tag{3.4}
\]
where \( k_0 := 0 \) and the term with \( j = 0 \) is included only if \( f(0) = 0 \). The numbers \( I_j, j > 0 \) are calculated by the first formula (1.12) if \( S \) and \( f(k) \) are known. The number \( I_0 = \frac{f''(0,0)}{f(0)} \). This number can be calculated if one knows how to calculate \( f''(0,0) \).

One has, differentiating (1.16):

\[
f'(0, k) = ik - A(0, 0) + \int_{0}^{\infty} A_x(0, y)e^{iky}dy.
\]
The number \( A(0, 0) \) can be calculated:

\[
f(k) = 1 + \int_{0}^{\infty} A(0, y)e^{iky}dy = 1 + A(0, 0)\frac{e^{iky} |_{0}}{iky}
\]

\[
-\frac{1}{ik} \int_{0}^{\infty} A_y(0, y)e^{iky}dy = 1 - \frac{A(0, 0)}{ik} - \frac{1}{ik} \int_{0}^{\infty} A_y(0, y)e^{iky}dy.
\]
Thus

\[
A(0, 0) = -\lim_{k \to \infty} [ik(f(k) - 1)] \tag{3.5}
\]
To calculate \( f'(0,0) \), we divide the Wronskian formula (1.9) by \( k \) and let \( k \to 0 \). This yields:

\[
f'(0,0) = \frac{-i}{f(0)}, \tag{3.6}
\]
so that the residue of \( I(k) \) at \( k = 0 \), when \( f(0) = 0 \), is:

\[
I_0 = \frac{f''(0,0)}{f(0)} = -\frac{i}{[f(0)]^2}. \tag{3.7}
\]
Existence of $\dot{f}(0)$ for $q \in L_{1,1}$ is proved in [2].

The function $\mathcal{J}(k)$ solves the following Riemann problem:

$$\mathcal{J}(k) = \mathcal{J}(-k) + \frac{2ik}{|f(k)|^2} - 2ik - \sum_{j=0}^{J} I_j \left( \frac{1}{k - ik_j} + \frac{1}{k + ik_j} \right), k \in \mathbb{R},$$  \hspace{1cm} (3.8)

and $\mathcal{J}(k)$ is analytic in $\mathbb{C}_+$, $\mathcal{J}(\infty) = 0$, while $\mathcal{J}(-k)$ has similar properties in $\mathbb{C}_-$. Thus (3.8) implies

$$\mathcal{J}(z) = \frac{1}{\pi} \text{v.p.} \int_{-\infty}^{\infty} \frac{dt}{t - z} \left[ t \left( |f(t)|^{-2} - 1 \right) + i \sum_{j=0}^{J} I_j t (t^2 + k_j^2)^{-1} \right], \quad \text{Im } z > 0,$$  \hspace{1cm} (3.9)

and taking $z = k + i0, k > 0$, one can calculate $\mathcal{J}(k)$ for $k > 0$ using formula (2.11).

If $\mathcal{J}(k)$ is found for all $k > 0$, then $I(k)$ is calculated by formula (3.4).

Therefore $\mathcal{S}$ determines $I(k)$ uniquely and analytically, since $\mathcal{S}$ determines $k_j, J, I_j$, and $\mathcal{J}(k)$ uniquely.

Let us explain how to calculate $I(k)$ given $\rho(\lambda)$.

If $\rho(\lambda)$ is given, then (see formula (1.2)), the function $|f(k)|$ is known for all $k > 0$, and the numbers $k > 0$, and the numbers $k_j, J$ and $c_j$ are known.

Therefore the function $f(z)$ can be calculated analytically. Indeed, $|f_0(k)| = |f(k)|$ if $k \in \mathbb{R}$, where $f_0(k) = \frac{f(k)}{w(k)}$. The function $f_0(z)$ is analytic and has no zeros in $\mathbb{C}_+$, and $f_0(\infty) = 1$. Therefore $\ln f_0(z)$ is analytic in $\mathbb{C}_+$ and vanishes at infinity. So it can be recovered by the Schwarz formula:

$$\ln f_0(z) = \frac{1}{i\pi} \int_{-\infty}^{\infty} \frac{\ln |f_0(t)|}{t - z} dt, \quad \text{Im } z > 0,$$  \hspace{1cm} (3.10)

which constructs an analytic function in $\mathbb{C}_+$ given its real part on the real axis. Since $|f_0(t)| = |f(t)|$, one gets

$$f(z) = w(z) \exp \left\{ \frac{1}{i\pi} \int_{-\infty}^{\infty} \frac{\ln |f(t)| dt}{t - z} \right\}, \quad \text{Im } z > 0.$$  \hspace{1cm} (3.11)

Taking $z = k + i0, k > 0$, in (3.11), and using (2.14) yields $f(k)$ for $k > 0$. One calculates the numbers $I_j$ by the second formula (1.12).

If $k_j, J, I_j$ and $f(k)$ are known, then one calculates $\mathcal{J}(z)$ by formula (3.9), and $\mathcal{J}(k)$ for all $k > 0$ by taking $z = k + i0$ in (3.9) and using (2.11). If $\mathcal{J}(k)$ is known then $I(k)$ is calculated by formula (3.4).

This completes the description of the formulas for finding $I(k)$ given $\mathcal{S}$ or $\rho(\lambda)$.

In conclusion let us make a remark concerning numerical aspects of finding $k_j$ given $I(k)$.
Since $I(k) = \overline{I(-k)}$ for $k \in \mathbb{R}$, the knowledge of $I(k)$ for $k > 0$ yields the values of $I(k)$ for all $k \in \mathbb{R}$. The function $I(k)$ is meromorphic in $\mathbb{C}_+$ and is of the form (see formula (3.4)):

$$I(k) = ik + \sum_{j=0}^{J} \frac{I_j}{k - ik_j} + \mathcal{J}(k), \quad (3.12)$$

where $\mathcal{J}(k)$ is analytic in $\mathbb{C}_+$ and is $o\left(\frac{1}{|k|}\right)$ as $|k| \to \infty$, $k \in \mathbb{C}_+$. Therefore the numbers $J$, $I_j$ and $ik_j$ can be calculated if $I(k)$ is known for all $k > 0$. A method for calculating $J$, $I_j$ and $k_j$ can be based on the formula:

$$\frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{ikt} [I(k) - ik] \, dk = \frac{I_0}{2} + \sum_{j=1}^{J} I_j e^{-k_j t}, \quad t > 0. \quad (3.13)$$

The behavior of $I(k)$ as $k \to +\infty$ one can obtain from the formulas:

$$f'(0, k) = ik - A(0, 0) + \tilde{A}_1(k), \quad \tilde{A}_1(k) := \int_{0}^{\infty} A_x(0, y) e^{iky} dy$$

$$f(k) = 1 + \tilde{A}(k), \quad \tilde{A}(k) = \int_{0}^{\infty} A(0, y) e^{iky} dy.$$  

The functions $A(y) := A(0, y)$ and $A_1(y) := A_x(0, y)$ belong to $L^1(0, \infty)$. Thus

$$I(k) = \frac{f'(0, k)}{f(k)} = [ik - A(0, 0) + \tilde{A}_1][1 + \tilde{A}(k)]^{-1} = ik - A(0, 0) - ik\tilde{A} + \cdots = ik + o(1) \quad (3.14)$$

where the dots stand for the terms of higher order of smallness as $k \to +\infty$. The important point is: the constant term $-A(0, 0)$ is cancelled in the asymptotics since

$$-ik\tilde{A} = A(0, 0) + o(1), \quad \text{as } k \to +\infty, \quad (3.15)$$

as follows from the calculation preceding formula (3.5).
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