\textbf{$q$-analogs of sinc sums and integrals}

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$q$-analogs of sum equals integral relations $\sum_{n \in \mathbb{Z}} f(n) = \int_{-\infty}^{\infty} f(x)dx$ for sinc functions and binomial coefficients are studied. Such analogs are already known in the context of $q$-hypergeometric series. This paper deals with multibasic ‘fractional’ generalizations that are not $q$-hypergeometric functions.

\textbf{Introduction}

Surprising properties of sinc sums and integrals were first discovered by C. Stormer in 1895 \cite{1,2}. The more general properties of band limited functions were known to engineers from signal processing and to physicists. For example, K.S. Krishan viewed them as a rich source for finding identities \cite{3}. R.P. Boas has studied the error term when approximating a sum of a band limited function with corresponding integral \cite{5}. More recently these properties were studied and popularized in a series of papers \cite{6–8}.

The sinc function is a special case of binomial coefficients

$$\left(\frac{2}{1 + x}\right) = \frac{\Gamma(3)}{\Gamma(1 + x)\Gamma(1 - x)} = \frac{2\sin \pi x}{\pi x} = 2\operatorname{sinc}(\pi x).$$

Therefore only sums with binomial coefficients will be studied in the following. It is known that binomial coefficients are band limited (e.g., see \cite{10})

$$\binom{a}{n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} (1 + e^{it})^a e^{-i\alpha t} dt,$$

i.e. their Fourier spectrum is limited to the band $|t| < \pi$. According to general theorems \cite{3,6} whenever Fourier spectrum of a function $f(x)$ is limited to the band $|t| < 2\pi$ one expects that

$$\sum_{n=-\infty}^{\infty} f(n) = \int_{-\infty}^{\infty} f(x)dx. \quad (1)$$

Bandwidth of a product of bandlimited functions is the sum of their bandwidths \cite{8}. In case of binomial coefficients this together with the theorem mentioned above implies that

$$\sum_{n=-\infty}^{\infty} \binom{a - \alpha n}{\alpha n} l = \int_{-\infty}^{\infty} \binom{a}{(\alpha x)} l dx, \quad 0 < \alpha \leq \frac{2}{L}. \quad (2)$$

For a general band limited function the above formula would have been valid only when $\alpha < \frac{2}{L}$. The validity of \cite{2} when $\alpha = \frac{2}{L}$ is explained by the fact that spectral density of binomial coefficient vanishes at boundary values $t = \pm \pi$.

$q$-analog of the Gamma function is defined as

$$\Gamma_q(x) = (q; q)_{\infty} (1 - q)^{1 - x}$$

and the $q$-binomial coefficients

$$\binom{a}{b}_q = \frac{\Gamma_q(a + 1)}{\Gamma_q(b + 1)\Gamma_q(a - b + 1)}.$$
with the standard notations for the $q$-shifted factorials
\[ (a; q)_n = \prod_{k=0}^{n-1} (1 - a q^k), \quad (a_1, \ldots, a_r; q)_n = \prod_{k=1}^{r} (a_k; q)_n, \quad (a; q)_{\infty} = \prod_{k=0}^{\infty} (1 - a q^k). \]

In the limit $q \to 1^-$ one has $\Gamma_q(a) \to \Gamma(a)$, i.e. standard values of the Gamma function and binomial coefficients are recovered.\[11\]

$q$-analog of the property of bandlimitedness has been studied in the literature $[12]$. This paper has a much more narrow scope and only deals with sums of binomial coefficients. We will find that (2) can be brought to the form (2) after a series of simple steps which will be given in Theorem 5.

Main formula and its proof

In the following, we will use a method of functional equations $[13]$ (see also $[11]$, sec. 5.2) combined with an idea due to G. Gasper $[14]$ to find a Laurent series expansion for a certain integral of an infinite product. First we need the following theorem taken from the book $[15]$:

**Theorem 1.** Let
\[ F(z) = \int_{\gamma} f(\zeta, z) d\zeta, \quad (3) \]
where the following conditions are satisfied

1. $\gamma$ is an infinite piecewise continuous curve
2. the function $f(\zeta, z)$ is continuous in $(\zeta, z)$ at $\zeta \in \gamma$, $z \in D$, where $D$ is a domain in the complex $z$ plane,
3. for each fixed $\zeta \in \gamma$ the function $f(\zeta, z)$ viewed as a function of $z$ is regular in $D$.
4. integral $[16]$ converges uniformly in $z \in D'$, where $D'$ is an arbitrary closed subdomain of $D$.

Then $F(z)$ is regular in $D$.

**Lemma 1.** Let $p$ and $q$ be two real numbers that satisfy $0 < p < q < 1$, then
\[ F(z) = \int_{-\infty}^{\infty} \frac{(b q^\zeta a q^{-\zeta}; p)_{\infty}}{(-z q^\zeta, -q^{1-s}/z; q)_{\infty}} d\zeta \]
is regular in the half plane $Re \ z > 0$.

**Proof.** Put in the theorem above $f(\zeta, z) = \frac{(b q^\zeta a q^{-\zeta}; p)_{\infty}}{(-z q^\zeta, -q^{1-s}/z; q)_{\infty}}$, $\gamma = (-\infty, +\infty)$, and $D$ an arbitrary domain in the half plane $Re \ z > 0$. Then (1),(2) and (3) are obviously satisfied. To prove (4) let $p = e^{-\omega}$, $q = p^\alpha$, $\omega > 0$, $0 < \alpha < 1$ and consider the asymptotics of $f(\zeta, z)$ when $\zeta \to +\infty$. In this limit one has $(b q^\zeta a q^{-\zeta}; p)_{\infty} \to 1$, $(-z q^\zeta q^{-\zeta}; q)_{\infty} \to 1$. According to an asymptotic formula ($[11]$, p. 118)
\[ Re[\ln(p^s; p)_{\infty}] = \frac{\omega}{2} (Re s)^2 + \frac{\omega}{2} (Re s) + O(1), \quad Re s \to -\infty, \]
we have
\[ |(a q^{-\zeta}; p)_{\infty}| = |(p^{\alpha \zeta - \omega - 1 \ln a}; p)_{\infty}| = O \left( |a|^{\alpha \zeta q^{-\omega-1}} \right), \]
\[ |(-q^{1-\zeta}/z; q)_{\infty}| = |(q^{1-\zeta}; q)_{\infty}| = O \left( |q/z|^{\zeta q^{-\omega-1}} \right). \]

So
\[ f(\zeta, z) = O \left( |z a^\alpha/q|^{(1-\alpha)/2} \right), \quad \zeta \to +\infty. \]
Similarly
\[ f(\zeta, z) = O \left( \frac{b^{\alpha}}{z|1-\zeta|^{q(1-\alpha)q^2/2}} \right), \quad \zeta \to -\infty. \]

It is now easy to see that the integral (*) converges. Hence according to Weierstrass M-Test integral
\[ F(z) \]
converges uniformly in \( z \) when \( \text{Re} z \geq \delta > 0 \). As a result the function
\[
f(a, b, z) = \frac{(-z, -q/z; q)_{\infty}}{\ln \frac{1}{q}} \int_{0}^{\infty} \frac{(bt/z, pz/at; p)_{\infty}}{(-t, -q/t; q)_{\infty}} \frac{dt}{t}
\]
is regular when \( \text{Re} z > 0 \) \( \square \)

**Lemma 2.** The function
\[
f(a, b, z) = \frac{(-z, -q/z; q)_{\infty}}{\ln \frac{1}{q}} \int_{0}^{\infty} \frac{(bt, a/t; p)_{\infty}}{(-zt, -q/(zt); q)_{\infty}} \frac{dt}{t}
\]
satisfies the functional equations
\[
f(a, b, z) = f(a, bp, z) - bf(a, bp, qz), \quad (4) \\
f(a, b, z) = f(ap, b, z) - af(ap, b, z/q). \quad (5)
\]

**Proof.** After a series of simple manipulations of the infinite products we find
\[
f(a, b, qz) = \frac{(-qz, -1/z; q)_{\infty}}{\ln \frac{1}{q}} \int_{0}^{\infty} \frac{(bt, a/t; p)_{\infty}}{(-qzt, -1/(zt); q)_{\infty}} \frac{dt}{t}
\]
\[
= \frac{(-z, -q/z; q)_{\infty}}{z \ln \frac{1}{q}} \int_{0}^{\infty} \frac{z (bt, a/t; p)_{\infty}}{(-zt, -q/(zt); q)_{\infty}} \frac{dt}{t}
\]
\[
= \frac{p(-z, -q/z; q)_{\infty}}{b \ln \frac{1}{q}} \int_{0}^{\infty} \frac{bt}{p} \frac{(bt, a/t; p)_{\infty}}{(-zt, -q/(zt); q)_{\infty}} \frac{dt}{t}
\]
\[
= \frac{p}{b} (f(a, b, z) - f(a, b/p, z)).
\]

This is equivalent to (4). Similarly or using the first functional equation and the formula \( f(a, b, z) = f(b, a, q/z) \) we find
\[
f(a, b, z) = f(b, a, q/z) = f(b, ap, q/z) - af(b, ap, q^2/z)
\]
\[
= f(ap, b, z) - af(ap, b, z/q),
\]
as required. \( \square \)

**Theorem 2.** Let \( p \) and \( q \) be two complex numbers such that \( |p| < |q| < 1 \), then
\[
\sum_{n=-\infty}^{\infty} (bq^n, aq^{-n}; p)_{\infty} z^n q^{n(n-1)/2} = \frac{(-z, -q/z; q)_{\infty}}{\ln \frac{1}{q}} \int_{0}^{\infty} \frac{(bt/z, az/t; p)_{\infty}}{(-t, -q/t; q)_{\infty}} \frac{dt}{t}.
\]

**Proof.** First consider the case \( 0 < p < q < 1 \). The function \( f(a, b, z) \) from Lemma 2 can be written in the form
\[
f(a, b, z) = (-z, -q/z; q)_{\infty} \int_{-\infty}^{\infty} \frac{(bq^{ \zeta} / z, az^{\zeta-1}; p)_{\infty}}{(-q^{\zeta}, -q^{1-\zeta}; q)_{\infty}} d\zeta.
\]

According to Lemma 2, \( f(a, b, z) \) is a regular function of \( z \) in the region \( \text{Re} z > 0 \). As a result \( f(a, b, z) \) has the Laurent series expansion
\[
f(a, b, z) = \sum_{n=-\infty}^{\infty} c_n(a, b) z^n, \quad \text{Re} z > 0.
\]
Functional equation (4) gives the following recursion relation for coefficients $c_n(a, b)$

$$c_n(a, b) = (1 - bq^n)c_n(a, bp).$$

This recursion means that

$$c_n(a, b) = (1 - bq^n;c_n(a,0).$$

The functional equation (5) gives

$$c_n(a, b) = (1 - aq^{-n})c_n(a/p, b),$$

from which one obtains

$$c_n(a, b) = (aq^{-n};p)\infty c_n(0, b).$$

By combining these equations one gets

$$c_n(a, b) = (bq^n;p)\infty c_n(a, 0) = (bq^n, aq^{-n};p)\infty c_n(0, 0).$$

It is known that (11), ex. 6.16

$$\int_0^\infty \frac{1}{(-t, -q/t; q)} \frac{dt}{t} = (q; q)\infty \ln \frac{1}{q}.$$  \hspace{1cm} (6)

According to Jacobi triple product formula

$$(q, -z, -q/z; q)\infty = \sum_{n=-\infty}^\infty z^n q^n(n-1)/2$$

this implies that $c_n(0, 0) = z^n q^n(n-1)/2$, so finally

$$c_n(a, b) = (bq^n, aq^{-n};p)\infty z^n q^n(n-1)/2.$$  \hspace{1cm} (6)

Now one needs to continue the result established for $\text{Re } z > 0, 0 < p < q < 1$ analytically to complex values of parameters $z, p, q$ to complete the proof.

Series containing infinite products $(bq^n, aq^{-n};p)\infty$ have been studied in [12]. It appears that the series in Theorem 3 have been first considered in the paper [17] which also contains a different representation for this sum in terms of an integral over a unit circle.

Consequences of the main formula

**Corollary 1.** The formula in Theorem 2 can be written in symmetric form

$$\sum_{n=-\infty}^\infty \frac{(bq^n, aq^{-n};p)\infty}{(-q^n, -q^{1-n}/z; q)\infty} = \int_{-\infty}^\infty \frac{(bx^a, aq^{-x};p)\infty}{(-zq^x, -q^{1-x}/z; q)\infty} dx,$$

or in terms of $q$-binomial coefficients

$$\sum_{n=-\infty}^\infty \left[ a \right]_{b + \alpha n} \frac{1}{(-zq^n, -q^{1-n}/z; q)\infty} = \int_{-\infty}^\infty \left[ a \right]_{b + \alpha x} \frac{1}{(-zq^x, -q^{1-x}/z; q)\infty} dx,$$

where $q = p^n, 0 < \alpha < 1.$
This gives an example of function for which sum equals integral. The case $|p| = |q| < 1$, $|b/a| < |z| < 1$ was known to Ramanujan. In this case, the series is Ramanujan’s $\psi_1$ sum and the integral is Ramanujan’s $q$-beta integral (\cite{11}, chs. 5, 6).

Now let $z = e^{i\theta}$, $|\theta| < \pi$. Then
\[
\lim_{q \to 1^{-}} \frac{(-z, -q/z; q)_{\infty}}{(-zq^2, -q^{-1}z; q)_{\infty}} = (1 + z)^2(1 + 1/z)^{-x} = z^x.
\]

Let $q \to 1^-$ with $0 < \alpha < 1$ fixed in equation (6). Then formally
\[
\sum_{n=-\infty}^{\infty} \left( \frac{a}{b + \alpha n} \right) e^{i\theta n} = \int_{-\infty}^{\infty} \left( \frac{a}{b + \alpha x} \right) e^{i\theta x} dx, \quad 0 < \alpha < 1.
\]

The range of validity of (7) is $-\pi \alpha < \theta < \pi \alpha$ as in (9), and not $-\pi < \theta < \pi$. Continuing formal manipulations we obtain by using (7) and binomial theorem
\[
\int_{-\infty}^{\infty} \left( \frac{a}{b + \alpha x} \right) e^{i\theta x} dx = \frac{1}{\alpha} e^{-i\theta b/\alpha} \int_{-\infty}^{\infty} \left( \frac{a}{x} \right) e^{i\theta x/\alpha} dx
\]
\[
= \frac{1}{\alpha} e^{-i\theta b/\alpha} \sum_{n=-\infty}^{\infty} \left( \frac{a}{n} \right) e^{i\theta n/\alpha}
\]
\[
= \frac{1}{\alpha} e^{-i\theta b/\alpha} \sum_{n=0}^{\infty} \left( \frac{a}{n} \right) e^{i\theta n/\alpha}
\]
\[
= \frac{1}{\alpha} e^{-i\theta b/\alpha} (1 + e^{i\theta/\alpha})^a, \quad -\pi \alpha < \theta < \pi \alpha.
\]

Finally (7) and (8) imply
\[
\sum_{n=-\infty}^{\infty} \left( \frac{a}{b + \alpha n} \right) v^{b+\alpha n} = \frac{1}{\alpha} (1 + v)^a, \quad |v| = 1, \quad |\arg v| < \pi, \quad 0 < \alpha \leq 1,
\]

which is T. Osler’s generalization of binomial theorem (18). According to Osler (18), the special case $\alpha = 1$ of (9) was first stated by Riemann (24). It also follows from Ramanujan’s $\psi_1$ sum in the limit $q \to 1^{-}$.

It should be noted that while (9) has a closed form, the series in Theorem 2 does not. If $p = q^2, z = 1, b = aq^2$, then one can prove that
\[
\sum_{n=-\infty}^{\infty} (bq^n, p/aq^n; p)_{\infty} z^n q^{n(1-n)/2} = 2 \left( qa, q/a; q^2 \right) _{\infty} \sum_{n=-\infty}^{\infty} \frac{(-1/a)^n q^{n^2+n}}{1 - aq^{2n+1}}.
\]

The sum on the RHS is proportional to Appell-Lerch sum $m(qa^2, q^2, q^2/a)$ in the notation of the paper (19). In general Appell-Lerch sums do not have an infinite product representation. For example, by taking $a = q^{-1/2}$ in $m(qa^2, q^2, q^2/a)$ we get the sum of the type $m(1, q^2, z)$ which is related to mock theta function of order 2 (see formula (4.2) in (19)).

Corollary 2. The series
\[
\sum_{n=-\infty}^{\infty} \frac{(bq^n, p/aq^n; p)_{\infty}}{(-zq^n, -q/zq^n; q)_{\infty}}, \quad |p| < |q|
\]

with $p$ and $q$ fixed depends only on $b/z$ and $az$.

Theorem 3.
\[
\int_{-\infty}^{\infty} \frac{(bq^x, aq^{-x}; p)_{\infty}}{(-q^x, -q^{-1-x}; q)_{\infty}} e^{ixy} dx
\]
\[
= 2\pi i / \log q \left( (-q, -q, e^{iy}, qe^{-iy}; q)_{\infty} \sum_{n=-\infty}^{\infty} \frac{(bq^n, aq^{-n}; p)_{\infty}}{(-q^n, -q^{-1-n}; q)_{\infty}} e^{iny} \right).
\]
Theorem 4. \[ \lim_{n \to -\infty} (bq^n, aq^{-n}; p)_{\infty} e^{izy} dz \]

where \( C \) is rectangle with vertices at \((\pm R, 0), (\pm R, -2\pi i / \log q)\). In view of asymptotics found in the proof of Lemma 1 integrals over the vertical segments vanish in the limit \( R \to +\infty \). Integrals over the horizontal segments are convergent and related by a factor of \(-e^{2\pi y / \log q}\). The integrand has simple poles at \( z = n - \pi i / \log q \) with residues

\[-\frac{e^{\pi y / \log q}}{(q; q)_{\infty}^2 \log q} (-bq^n, -aq^{-n}; p)_{\infty} (-1)^n q^{n(n-1)/2} e^{izy} \]

Application of the residue theorem yields

\[ \int_{-\infty}^{\infty} \frac{(bq^n, aq^{-n}; p)_{\infty}}{(-q^2, -q^{1-x}; q)_{\infty}} e^{izy} dx = \frac{\pi i / \log q}{(q; q)_{\infty}^2 \sinh \frac{\pi y}{\log q}} \sum_{n=\infty}^{\infty} (-bq^n, -aq^{-n}; p)_{\infty} (-1)^n q^{n(n-1)/2} e^{izy} \]

According to Corollary 2

\[ \sum_{n=\infty}^{\infty} (-bq^n, -aq^{-n}; p)_{\infty} (-1)^n q^{n(n-1)/2} e^{izy} = \frac{(e^{iy}, qe^{-iy}; q)_{\infty}}{(-e^{iy}, -qe^{-iy}; q)_{\infty}} \sum_{n=\infty}^{\infty} (bq^n, aq^{-n}; p)_{\infty} q^{n(n-1)/2} e^{izy} \]

To complete the proof observe that

\[ \sum_{n=\infty}^{\infty} (bq^n, aq^{-n}; p)_{\infty} q^{n(n-1)/2} e^{izy} = (-1, -q; q)_{\infty} \sum_{n=\infty}^{\infty} (bq^n, aq^{-n}; p)_{\infty} (-q^{n}, -q^{1-n}; q)_{\infty} e^{izy} \]

and \((-1, -q; q)_{\infty} = 2(q; q)_{\infty}^2\).

One can see from Theorem 3 that the function \( g(x) = \frac{(bq^x, aq^{-x}; p)_{\infty}}{(-q^x, -q^{1-x}; q)_{\infty}} \)

is not band limited. However Fourier transform of \( g(x) \) vanishes at frequencies \( y = 2\pi m \), where \( m \neq 0 \)

is an integer. Hence according to Poisson summation formula [20]

\[ \sum_{n=\infty}^{\infty} g(x) = \sum_{n=\infty}^{\infty} \int_{-\infty}^{\infty} g(x) e^{-2\pi i n x} dx = \int_{-\infty}^{\infty} g(x) dx \]

in agreement with Corollary 4.

The fact that bilateral summation formulas in the theory of \( q \)-hypergeometric functions give examples of functions of the type \( \mathbb{I} \)

has been recognized in the literature.

Corollary 3. Let \(|p| < |q|\) and \( m \in \mathbb{Z} \), then

\[ \int_{-\infty}^{\infty} \frac{(bq^x, aq^{-x}; p)_{\infty}}{(-q^x, -q^{1-x}; q)_{\infty}} q^{mn} dx = \sum_{n=\infty}^{\infty} \frac{(bq^n, aq^{-n}; p)_{\infty}}{(-q^n, -q^{1-n}; q)_{\infty}} q^{mn} \]

Proof. Resolve the \( \frac{0}{0} \) ambiguity at the rhs of the formula of Theorem 2 using L’Hospital’s Rule.

Next we apply the method due to Bailey [22] to the identity in Theorem 2.

Theorem 4.

\[ \sum_{n=\infty}^{\infty} (b_1 q^n, b_2 q^n, a_1 q^{-n}, a_2 q^{-n}; p)_{\infty} z^n q^{n(n-1)} = z \sum_{n=\infty}^{\infty} (b_1 q^n / z, b_2 q^n / z, a_1 z q^{-n}, a_2 z q^{-n}; p)_{\infty} z^{-n} q^{n(n-1)}. \]
Proof. Multiplying the equations

\[
\sum_{n=-\infty}^{\infty} (b_1 q^n, a_1 q^{-n}; p) e^{i\theta n} q^{n(n-1)/2} = \frac{(-e^{i\theta}, -q e^{-i\theta}; q)_{\infty}}{\ln \frac{1}{q}} \int_{\frac{1}{2}}^{\infty} \frac{(b_1 t e^{-i\theta}, a_1 e^{i\theta}/t; p)_{\infty} dt}{(-t, -q/t; q)_{\infty}},
\]

and integrating with respect to \(\theta\) one obtains

\[
\sum_{n=-\infty}^{\infty} (b_1 q^n, b_2 q^n, a_1 q^{-n}, a_2 q^{-n}; p)_{\infty} z^n q^{n(n-1)}
\]

and

\[
= \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \frac{(-e^{i\theta}, -q e^{-i\theta}; q)_{\infty}}{\ln \frac{1}{q}} \int_{0}^{\infty} \frac{(b_1 t e^{-i\theta}, a_1 e^{i\theta}/t_1; p)_{\infty} dt_1}{(-t_1, -q/t_1; q)_{\infty}}
\]

\[
\times \frac{(-ze^{-i\theta}, -q e^{-i\theta}/z; q)_{\infty}}{\ln \frac{1}{q}} \int_{0}^{\infty} \frac{(b_2 t_2 e^{i\theta}/z, a_2 e^{-i\theta}/t_2; p)_{\infty} dt_2}{(-t_2, -q/t_2; q)_{\infty}}
\]

\[
= z \sum_{n=-\infty}^{\infty} (b_1 q^n/z, b_2 q^n/z, a_1 q^{-n}, a_2 q^{-n}; p)_{\infty} z^{-n} q^{n(n-1)}.
\]

It is straightforward to rewrite the formula of Theorem 4 in terms of \(q\)-binomial coefficients:

**Corollary 4.** Let \(0 < q < 1\) and \(0 < \alpha < 1\), then

\[
\sum_{n=-\infty}^{\infty} \left[ \begin{array}{c} a_1 \\ b_1 + \alpha n \end{array} \right]_p \left[ \begin{array}{c} a_2 \\ b_2 + \alpha n \end{array} \right]_p q^{\alpha n(n-1)+\alpha n} = p^\theta \sum_{n=-\infty}^{\infty} \left[ \begin{array}{c} a_1 \\ b_1 - \theta + \alpha n \end{array} \right]_p \left[ \begin{array}{c} a_2 \\ b_2 - \theta + \alpha n \end{array} \right]_p q^{\alpha n(n-1)-\theta n}.
\]

Theorem 2 can be generalized.

**Theorem 5.** Let \(q = p_1^{\alpha_1} = p_2^{\alpha_2}\) where \(\alpha_1 > 0\), \(\alpha_2 > 0\), and \(\alpha_1 + \alpha_2 < 1\). Then

\[
\sum_{n=-\infty}^{\infty} \left[ \begin{array}{c} a_1 \\ b_1 + \alpha n \end{array} \right]_{p_1} \left[ \begin{array}{c} a_2 \\ b_2 + \alpha n \end{array} \right]_{p_2} \frac{1}{-z q^{n}, q^{1-n}/z; q)_{\infty}}
\]

\[
= \int_{-\infty}^{\infty} \left[ \begin{array}{c} a_1 \\ b_1 + \alpha x \end{array} \right]_{p_1} \left[ \begin{array}{c} a_2 \\ b_2 + \alpha x \end{array} \right]_{p_2} \frac{dx}{-z q^x, q^{1-x}/z; q)_{\infty}}.
\]

In fact, the number of \(q\)-binomial coefficients in this formula can be arbitrary as long as the parameters \(\alpha_j > 0\) are subject to the constraint \(\sum_j \alpha_j < 1\).

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