APPROXIMATIONS FOR APERY’S CONSTANT $\zeta(3)$ AND RATIONAL SERIES REPRESENTATIONS INVOLVING $\zeta(2n)$

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Abstract. In this note, using an idea from [4] we derive some new series representations involving $\zeta(2n)$ and Euler numbers. Using a well-known series representation for the Clausen function, we also provide some new representations of Apery’s constant $\zeta(3)$. In particular cases, we recover some well-known series representations of $\pi$.

1. Introduction and Preliminaries

In 1734, Leonard Euler produced a sensation when he proved the following formula:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Later, in 1740, Euler generalized the above formula for even positive integers:

$$\zeta(2n) = (-1)^{n+1} \frac{B_{2n} 2^{2n-1} \pi^{2n}}{(2n)!},$$

where $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ is the celebrated Riemann zeta function. The coefficients $B_n$ are the so-called Bernoulli numbers and they are defined in the following way:

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n, \ |z| < 2\pi.$$

An elementary proof of (1) can be found in [5]. In [4], de Amo, Carrillo and Sanchez produced another proof of Euler’s formula (1) using the Taylor series expansion of the tangent function and Fubini’s theorem.

In this paper, using similar ideas but for other functions, we provide a new proof for the Clausen acceleration formula that will serve as an application in displaying some fast representations for Apery’s constant $\zeta(3)$. Moreover, the Taylor series...
expansion for the secant and cosecant functions combined with some other integration techniques will give us some interesting rational series representations involving $ \zeta(2n) $ and binomial coefficients. Also, we display some particular cases of such series. Some of them are well-known representations of $ \pi $. For the sake of completeness we display the Taylor series for the tangent, cotangent, secant and cosecant functions:

$$
\tan x = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}2^{2n}(2^{2n} - 1)B_{2n}}{(2n)!}x^{2n-1}, \; |x| < \frac{\pi}{2}
$$

$$
\cot x = \sum_{n=0}^{\infty} \frac{(-1)^{n}2^{2n}B_{2n}}{(2n)!}x^{2n-1}, \; |x| < \pi
$$

$$
\sec x = \sum_{n=0}^{\infty} \frac{(-1)^{n}E_{2n}}{(2n)!}x^{2n}, \; |x| < \frac{\pi}{2}
$$

$$
\csc x = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}2(2^{2n-1} - 1)B_{2n}}{(2n)!}x^{2n-1}, \; |x| < \pi
$$

where $ E_n $ are the Euler numbers, and $ B_n $ the Bernoulli numbers.

The Riemann zeta function $ \zeta(s) $ and the Hurwitz (generalized) function $ \zeta(s,a) $ are defined by

$$
\zeta(s) := \begin{cases} 
\sum_{n=1}^{\infty} \frac{1}{n^s} &= \frac{1}{1 - 2^{-s}} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s} \quad \text{Re}(s) > 1, \\
\frac{1}{1 - 2^{-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} &= \frac{1}{1 - 2^{-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} \quad \text{Re}(s) > 0, \; s \neq 1,
\end{cases}
$$

and

$$
\zeta(s,a) := \sum_{n=0}^{\infty} \frac{1}{(n + a)^s}, \; \text{Re}(s) > 1; \; a \neq 0, -1, -2, \ldots
$$

Both of them are analytic over the whole complex plane, except $ s = 1 $, where they have a simple pole. Also, from the two definitions above, one can observe that

$$
\zeta(s) = \zeta(s,1) = \frac{1}{2^s - 1} \zeta\left(s, \frac{1}{2}\right) = 1 + \zeta(s,2).
$$

Clausen’s function (or Clausen’s integral, see [15]) $ \text{Cl}_2(\theta) $ is defined by
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$$\text{Cl}_2(\theta) := \sum_{k=1}^{\infty} \frac{\sin k\theta}{k^2} = -\int_0^\theta \log \left(2 \sin \left(\frac{x}{2}\right)\right) \, dx.$$  

This integral was considered for the first time by Clausen in 1832 ([15]), and it was investigated later by many authors [9, 10, 11, 12, 13, 14, 22, 23, 29, 30]. The Clausen functions are very important in mathematical physics. Some well-known properties of the Clausen function include periodicity in the following sense:

$$\text{Cl}_2(2k\pi \pm \theta) = \text{Cl}_2(\pm \theta) = \pm \text{Cl}_2(\theta).$$

Moreover, it is quite clear from the definition that $\text{Cl}_2(k\pi) = 0$ for $k$ integer. For example, for $k = 1$ we deduce

$$\int_0^\pi \log \left(2 \sin \left(\frac{x}{2}\right)\right) \, dx = 0, \quad \int_0^{\pi/2} \log(\sin x) \, dx = -\frac{\pi}{2} \log 2.$$

By periodicity we have $\text{Cl}_2\left(\frac{\pi}{2}\right) = -\text{Cl}_2\left(\frac{3\pi}{2}\right) = G$, where $G$ is the Catalan constant defined by

$$G := \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \approx 0.9159...$$

More generally, one can express the above integral as the following:

$$\int_0^\theta \log(\sin x) \, dx = -\frac{1}{2} \text{Cl}_2(2\theta) - \theta \log 2,$$

$$\int_0^\theta \log(\cos x) \, dx = -\frac{1}{2} \text{Cl}_2(\pi - 2\theta) - \theta \log 2,$$

$$\int_0^\theta \log(1 + \cos x) \, dx = 2 \text{Cl}_2(\pi - \theta) - \theta \log 2,$$

and

$$\int_0^\theta \log(1 + \sin x) \, dx = 2G - 2 \text{Cl}_2\left(\frac{\pi}{2} + \theta\right) - \theta \log 2.$$

The Dirichlet beta function is defined as

$$\beta(s) := \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s}.$$  

Alternatively, one can express the beta function in terms of Hurwitz zeta function by the following formula valid in the whole complex $s$-plane,

$$\beta(s) = \frac{1}{4^s} (\zeta(s, 1/4) - \zeta(s, 3/4)).$$
Clearly, $\beta(2) = G$ (Catalan’s constant), $\beta(3) = \frac{\pi^3}{32}$, and $\beta(2n+1) = \frac{(-1)^n E_{2n} \pi^{2n+1}}{4^{n+1}(2n)!}$, where $E_n$ are the Euler numbers mentioned above which are given by the Taylor series $\frac{2}{e^t + e^{-t}} = \sum_{n=0}^{\infty} \frac{E_n t^n}{n!}$.

1.1. **Organization of the paper.** This note is organized as follows. In the first subsection (2.1) of the next section, we give a new proof for the Clausen acceleration formula. This formula serves as an ingredient for Theorem 2.2 where we provide some ”fast” converging series representations for Apery’s constant $\zeta(3)$. On the other hand, in the second subsection, we produce some rational representations of $\zeta(2n)$ and we also provide some interesting series summations as corollaries.

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2. **Main results**

2.1. **Clausen acceleration formula and some representations of $\zeta(3)$**. We provide a new proof for the classical Clausen acceleration formula [12, 24].

**Proposition 2.1.** We have the following representation for the Clausen function $\text{Cl}_2(\theta)$,

\begin{equation}
\frac{\text{Cl}_2(\theta)}{\theta} = 1 - \log |\theta| + \sum_{n=1}^{\infty} \frac{\zeta(2n)}{(2\pi)^{2n} n(2n + 1)} \theta^{2n}, |\theta| < 2\pi.
\end{equation}

**Proof.** Integrating by parts the function $xy \cot(xy)$ we have

\[
\int_{0}^{\pi} xy \cot(xy) \, dx = \frac{\pi}{2} \log \left( \sin \left( \frac{\pi y}{2} \right) \right) - \int_{0}^{\pi} \log(\sin(xy)) \, dx
\]

\[
= \frac{\pi}{2} \log \left( \sin \left( \frac{\pi y}{2} \right) \right) - \frac{1}{2y} \int_{0}^{\pi y} \log \left( \sin \left( \frac{u}{2} \right) \right) \, du
\]

\[
= \frac{\pi}{2} \log \left( \sin \left( \frac{\pi y}{2} \right) \right) + \frac{\pi}{2} \log 2 + \frac{1}{2y} \text{Cl}_2(\pi y)
\]

\[
= \frac{\pi}{2} \log \left( \frac{\pi y}{2} \right) + \frac{\pi}{2} \log \prod_{k=1}^{\infty} \left( 1 - \frac{y^2}{4k^2} \right) + \frac{\pi}{2} \log 2 + \frac{1}{2y} \text{Cl}_2(\pi y)
\]

\[
= \frac{\pi}{2} \log \left( \frac{\pi y}{2} \right) - \frac{\pi}{2} \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{(\frac{y}{2k\pi})^n}{n} + \frac{\pi}{2} \log 2 + \frac{1}{2y} \text{Cl}_2(\pi y),
\]
where we have used the product formula for sine, \( \sin(\pi x) = \pi x \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{k^2}\right) \).

On the other hand, using the formula \( \theta \cot \theta = 1 - 2 \sum_{n=1}^{\infty} \frac{\zeta(2n)}{\pi^{2n}} \theta^{2n} \) we have

\[
\int_{0}^{\frac{\pi}{2}} xy \cot(xy) \, dx = \frac{\pi}{2} - 2 \sum_{n=1}^{\infty} \frac{\zeta(2n)}{\pi^{2n}(2n+1)} y^{2n}.
\]

Therefore, we obtain

\[
\frac{\pi}{2} - 2 \sum_{n=1}^{\infty} \frac{\zeta(2n)}{\pi^{2n}(2n+1)} y^{2n} = \frac{\pi}{2} \log(\pi y) - \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{\zeta(2n)}{n4^n} y^{2n} + \frac{1}{2y} \text{Cl}_2(\pi y).
\]

This implies that

\[
\text{Cl}_2(\pi y) = 2y \left( \frac{\pi}{2} - \pi \sum_{n=1}^{\infty} \frac{\zeta(2n)}{(2n+1)4^n} y^{2n} - \frac{\pi}{2} \log(\pi y) + \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{\zeta(2n)}{n4^n} y^{2n} \right),
\]

which, after some computations, is equivalent to

\[
\text{Cl}_2(\pi y) = \pi y - \pi y \log(\pi y) + \pi y \sum_{n=1}^{\infty} \frac{\zeta(2n)(\pi y)^{2n}}{4^n n(2n+1)\pi^{2n}}.
\]

Setting \( \alpha = \pi y \), we obtain our result. \( \square \)

**Remarks.** In particular case of \( \theta = \frac{\pi}{2} \), using the fact that \( \text{Cl}_2(\pi) = G \), we obtain

\[
\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n(2n+1)16^n} = \frac{2G}{\pi} - 1 + \log \left(\frac{\pi}{2}\right).
\]

In computations, the following accelerated peeled form [7] is used

\[
\frac{\text{Cl}_2(\theta)}{\theta} = 3 - \log \left(|\theta| \left(1 - \frac{\theta^2}{4\pi^2}\right)\right) - \frac{2\pi}{\theta} \log \left(\frac{2\pi + \theta}{2\pi - \theta}\right) + \sum_{n=1}^{\infty} \frac{\zeta(2n) - 1}{n(2n+1)} \left(\frac{\theta}{2\pi}\right)^{2n}.
\]

It is well-known that \( \zeta(n) - 1 \) converges to zero rapidly for large values of \( n \). In [30], Wu, Zhang and Liu derive the following representation for the Clausen function \( \text{Cl}_2(\theta) \),

\[
\text{Cl}_2(\theta) = \theta - \theta \log \left(2 \sin \left(\frac{\theta}{2}\right)\right) - \sum_{n=1}^{\infty} \frac{2\zeta(2n)}{(2n+1)(2\pi)^{2n}} \theta^{2n+1}.
\]

It is interesting to see that integrating the above formula from 0 to \( \pi/2 \) we have the following representation for \( \zeta(3) \) (see [13]).
\begin{equation}
\zeta(3) = \frac{4\pi^2}{35} \left( \frac{1}{2} + \frac{2G}{\pi} - \sum_{n=1}^{\infty} \frac{\zeta(2n)}{(n+1)(2n+1)16^n} \right).
\end{equation}

Also, in [27], Strivastava, Glasser and Adamchik derive series representations for \(\zeta(2n+1)\) by evaluating the integral \(\int_0^{\pi/\omega} t^{s-1} \cot t dt\), \(s, \omega \geq 2\) integers in two different ways. One of the ways involves the generalized Clausen functions. When they are evaluated in terms of \(\zeta(2n+1)\) one obtains the following formula for \(\zeta(3)\),

\begin{equation}
\zeta(3) = \frac{2\pi^2}{9} \left( \log 2 + 2 \sum_{n=0}^{\infty} \frac{\zeta(2n)}{(2n+3)4^n} \right).
\end{equation}

Another remarkable result which led to Apery’s proof of the irrationality of \(\zeta(3)\) is given by the rapidly convergent series

\begin{equation}
\zeta(3) = \frac{5}{2} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^3 \binom{2n}{n}}.
\end{equation}

Moreover, in [19], Cvijovic and Klinowski derive the following formula

\begin{equation}
\zeta(3) = -\frac{\pi^2}{3} \sum_{n=0}^{\infty} \frac{(2n+5)\zeta(2n)}{(2n+1)(2n+2)(2n+3)2^{2n}}.
\end{equation}

This formula is related to the one found by Ewell [21],

\begin{equation}
\zeta(3) = -\frac{4\pi^2}{7} \sum_{n=0}^{\infty} \frac{\zeta(2n)}{(2n+1)(2n+2)2^{2n}}.
\end{equation}

The following result will provide some new representations for Apery’s constant \(\zeta(3)\). The main ingredients in the proof of this next result are Clausen acceleration formulae and Fubini’s theorem.

**Theorem 2.2.** We have the following series representations

\begin{equation}
\zeta(3) = \frac{4\pi^2}{35} \left( \frac{3}{2} - \log \left( \frac{\pi}{2} \right) + \sum_{n=1}^{\infty} \frac{\zeta(2n)}{n(n+1)(2n+1)16^n} \right),
\end{equation}

\begin{equation}
\zeta(3) = -\frac{64}{3\pi} \beta(4) + \frac{8\pi^2}{9} \left( \frac{4}{3} - \log \left( \frac{\pi}{2} \right) + 3 \sum_{n=1}^{\infty} \frac{\zeta(2n)}{n(2n+1)(2n+3)16^n} \right),
\end{equation}

and
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(19) \[ \zeta(3) = \frac{-64}{3\pi} \beta(4) + \frac{16\pi^2}{27} \left( \frac{1}{2} + \frac{3G}{\pi} - 3 \sum_{n=1}^{\infty} \frac{\zeta(2n)}{(2n+1)(2n+3)16^n} \right), \]

where $G$ is the Catalan constant, and $\beta(s)$ is the Dirichlet beta function.

Proof. For (17), integrating (8) from 0 to $\pi/2$ and using Fubini’s theorem, we have

\[
\int_0^{\pi/2} \text{Cl}_2(y) \, dy = \int_0^{\pi/2} (y - y \log y) \, dy + \int_0^{\pi/2} \sum_{n=1}^{\infty} \frac{\zeta(2n)}{(2\pi)^{2n}n(2n+1)} y^{2n+1} \, dy \\
= \frac{\pi^2}{8} - \left( \frac{\pi^2}{8} \log \left( \frac{\pi}{2} \right) - \frac{1}{2} \int_0^{\pi/2} y \, dy \right) + \sum_{n=1}^{\infty} \frac{\zeta(2n) \left( \frac{\pi}{2} \right)^{2n+2}}{(2\pi)^{2n}n(2n+1)(2n+2)},
\]

which is equivalent to

\[
\int_0^{\pi/2} \text{Cl}_2(y) \, dy = \frac{\pi^2}{8} \left( \frac{3}{2} - \log \left( \frac{\pi}{2} \right) \right) + \sum_{n=1}^{\infty} \frac{\zeta(2n)}{n(n+1)(2n+1)16^n}.
\]

Alternatively, we can integrate the Clausen function using its definition given in the introduction and changing the order of integration as follows:

\[
\int_0^{\pi/2} \text{Cl}_2(y) \, dy = - \int_0^{\pi/2} \int_0^{y} \log \left( 2 \sin \left( \frac{x}{2} \right) \right) \, dx \, dy = - \int_0^{\pi/2} \int_x^{\pi/2} \log \left( 2 \sin \left( \frac{x}{2} \right) \right) \, dy \, dx \\
= - \int_0^{\pi/2} \frac{\pi}{2} \log \left( 2 \sin \left( \frac{x}{2} \right) \right) \, dx + \int_0^{\pi/2} x \log 2 \, dx + \int_0^{\pi/2} x \log \left( \sin \left( \frac{x}{2} \right) \right) \, dx.
\]

Using the definition of the Clausen function again and, after the substitution $x = 2u$, using the identity $\int_0^{\pi/4} u \log(\sin(u)) \, du = \frac{35}{128} \zeta(3) - \frac{\pi G}{8} - \frac{\pi^2}{32} \log 2$ (see [13]), we have

\[
\int_0^{\pi/2} \text{Cl}_2(y) \, dy = \frac{\pi}{2} \text{Cl}_2 \left( \frac{\pi}{2} \right) + \frac{\pi^2}{2} \log 2 + 4 \left( \frac{35}{128} \zeta(3) - \frac{\pi G}{8} - \frac{\pi^2}{32} \log 2 \right) = \frac{35}{32} \zeta(3).
\]

Setting the two results equal to each other and solving for $\zeta(3)$ gives us our result. For (18), we proceed similarly to the previous method. We will begin with

\[
\int_0^{\pi^2/4} \text{Cl}_2(y) \, dy = \int_0^{\pi^2/4} (\sqrt{y} - \sqrt{y} \log(\sqrt{y})) \, dy + \int_0^{\pi^2/4} \sum_{n=1}^{\infty} \frac{\zeta(2n)y^{n+1/2}}{n(2n+1)(2\pi)^{2n}} \, dy \\
= \frac{\pi^3}{12} - \left( \frac{\pi^3}{12} \log \left( \frac{\pi}{2} \right) \right) - \frac{1}{3} \int_0^{\pi^2/4} \sqrt{y} \, dy + \sum_{n=1}^{\infty} \frac{\zeta(2n) \left( \frac{\pi}{2} \right)^{n+3/2}}{n(2n+1)(n+3/2)(2\pi)^{2n}},
\]

where $\beta(s)$ is the Dirichlet beta function.
which is equivalent to
\[
\int_0^{\pi^2/4} \text{Cl}_2(\sqrt{y}) \, dy = \frac{\pi^3}{12} \left( \frac{4}{3} - \log \left( \frac{\pi}{2} \right) \right) + 3 \sum_{n=1}^{\infty} \frac{\zeta(2n)}{n(2n+1)(2n+3)16^n}.
\]

On the other hand, using the definition of the Clausen function and changing the order of integration, we see
\[
\int_0^{\pi^2/4} \text{Cl}_2(\sqrt{y}) \, dy = -\int_0^{\pi^2/4} \int_0^{\sqrt{y}} \log \left( 2 \sin \left( \frac{x}{2} \right) \right) \, dx \, dy
= -\int_0^{\pi^2/4} \int_{x^2}^{\pi^2/4} \log \left( 2 \sin \left( \frac{x}{2} \right) \right) \, dy \, dx
= \frac{\pi^2G}{4} + \frac{1}{168} \left( 72\pi \zeta(3) - 192\pi^2G + \psi_3 \left( \frac{1}{4} \right) - \psi_3 \left( \frac{3}{4} \right) \right),
\]
where \( \psi_3 \) is the trigamma function. Using the identity \( \psi_n(z) = (-1)^{n+1}n! \zeta(n+1, z) \) where \( \zeta(k, z) \) is the Hurwitz zeta function, and the relationship between the Hurwitz zeta function and the beta function, we have that
\[
\int_0^{\pi^2/4} \text{Cl}_2(\sqrt{y}) \, dy = \frac{3\pi}{32} \zeta(3) + 2\beta(4).
\]

Setting this result equal to the previous result of the integral and solving for \( \zeta(3) \), we see (18) is indeed true. For (19), instead of integrating the Clausen function using (8), we will integrate it using (11). This gives us,
\[
\int_0^{\pi^2/4} \text{Cl}_2(\sqrt{y}) \, dy = \int_0^{\pi^2/4} \sqrt{y} \, dy - \int_0^{\pi^2/4} \sqrt{y} \log \left( 2 \sin \left( \frac{\sqrt{y}}{2} \right) \right) \, dy
+ 2 \int_0^{\pi^2/4} \sum_{n=1}^{\infty} \frac{\zeta(2n)y^{n+1/2}}{(2n+1)(2\pi)^{2n}} \, dy
= \frac{\pi^3}{12} - 2 \int_0^{\pi^2/4} u^2 \log \left( 2 \sin \left( \frac{u}{2} \right) \right) \, du + 2 \sum_{n=1}^{\infty} \frac{\zeta(2n)\left( \frac{\pi^2}{4} \right)^{n+3/2}}{(2n+1)(n+3/2)(2\pi)^{2n}}
= \frac{\pi^3}{12} - \frac{1}{384} \left( 72\pi \zeta(3) - 192\pi^2G + 3!4^4\beta(4) \right) - \frac{\pi^3}{2} \sum_{n=1}^{\infty} \frac{\zeta(2n)}{(2n+1)(2n+3)16^n}.
\]
Setting this result equal to the previous Clausen formula yields
\[
\left( \frac{3\pi}{32} + \frac{3\pi}{16} \right) \zeta(3) = \frac{9\pi}{32} \zeta(3) = \frac{\pi^3}{12} + \frac{\pi^2G}{2} - 6\beta(4) - \frac{\pi^3}{2} \sum_{n=1}^{\infty} \frac{\zeta(2n)}{(2n+1)(2n+3)16^n},
\]
and from here, (19) follows. □

**Remark.** To prove (17), one could solve for \( G \) in (9) and plug it into (12) and rearrange. Also, to prove (19), one could solve for \( \log \left( \frac{\pi}{2} \right) \) in (9) and plug that into...
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(18) and rearrange. Further, if we integrate (10), we arrive at a rapidly converging series representation for $\zeta(3)$, that is

$$\zeta(3) = \frac{2\pi^2}{35} \left( 9 + 138 \log 2 - 18 \log 3 - 50 \log 5 - 2 \log \pi + 2 \sum_{n=1}^{\infty} \frac{\zeta(2n) - 1}{n(2n+1)(n+1)16^n} \right).$$

2.2. Rational series representations involving $\zeta(2n)$ and binomial coefficients. As it has been already been highlighted in [7] one can relate the rational $\zeta$-series with various Dirichlet $L$-series. A rational $\zeta$-series can be accelerated for computational purposes provided that one solves the exact sum

$$\sum_{n=2}^{\infty} \frac{q_n}{a^n},$$

where $a = 2, 3, 4, \ldots$. In fact, it has been already highlighted in [7] we shall call rational $\zeta$-series of a real number $x$, the following representation:

$$x = \sum_{n=2}^{\infty} q_n \zeta(n, m),$$

where $q_n$ is a rational number and $\zeta(n, m)$ is the Hurwitz zeta function. For $m > 1$ integer, one has

$$x = \sum_{n=2}^{\infty} q_n \left( \zeta(n) - \sum_{j=1}^{m-1} j^{-n} \right).$$

In the particular case $m = 2$, one has the following series representations:

$$1 = \sum_{n=2}^{\infty} (\zeta(n) - 1)$$

$$1 - \gamma = \sum_{n=2}^{\infty} \frac{1}{n} (\zeta(n) - 1)$$

$$\log 2 = \sum_{n=2}^{\infty} \frac{1}{n} (\zeta(2n) - 1),$$

where $\gamma$ is the Euler-Mascheroni constant. For other rational zeta series representations we recommend [3, 13, 27].
Theorem 2.3. The following representation is true

\[ \sum_{n=1}^{\infty} \frac{\zeta(2n)}{n4^n} \left( \frac{2n}{m} \right) = \begin{cases} \frac{1}{m} & \text{m odd,} \\ \frac{1}{m} \left( 2\zeta(m) \left( 1 - \frac{1}{2^m} \right) - 1 \right) & \text{m even.} \end{cases} \]  

Proof. We start by integrating \( xy \csc(xy) \) two different ways. Applying integration by parts, L’Hospital’s rule, and properties of logarithm, we find

\[
\int_0^{\pi/2} xy \csc(xy) \, dx = -\frac{\pi}{2} \left( \log \left( 1 + \cos \left( \frac{\pi y}{2} \right) \right) - \log \left( \sin \left( \frac{\pi y}{2} \right) \right) \right) \\
+ \int_0^{\pi/2} \log(1 + \cos(xy)) \, dx - \int_0^{\pi/2} \log(\sin(xy)) \, dx \\
= -\frac{\pi d}{2} \frac{d}{d\alpha} \left( 2\text{Cl}_2(\pi - \alpha) + \frac{1}{2} \text{Cl}_2(2\alpha) \right) + \frac{1}{y} \left( 2\text{Cl}_2(\pi - \alpha) + \frac{1}{2} \text{Cl}_2(2\alpha) \right),
\]

where \( \alpha = \frac{\pi y}{2} \). Applying (8) to the result and simplifying, we get

\[
\int_0^{\pi/2} xy \csc(xy) \, dx = \pi \sum_{n=1}^{\infty} \frac{\zeta(2n)(\pi^2)^{2n}(2 - y)^{2n}}{n(2\pi)^{2n}} - \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{\zeta(2n)(\pi y)^{2n}}{n(2\pi)^{2n}} - \frac{\pi}{2} \\
+ \frac{2\pi}{y} \left( 1 - \log \left( \frac{\pi}{2} \right) - \log(2 - y) \right) + \frac{2}{y} \sum_{n=1}^{\infty} \frac{\zeta(2n)(\pi y)^{2n+1}(2 - y)^{2n+1}}{n(2n + 1)(2\pi)^{2n}} + \frac{1}{2y} \sum_{n=1}^{\infty} \frac{\zeta(2n)(\pi y)^{2n+1}}{(2\pi)^{2n}}.
\]

On the other hand, we can apply Fubini’s theorem and integrate its power series term by term. Thus, we will have

\[
\int_0^{\pi/2} xy \csc(xy) \, dx = \int_0^{\pi/2} \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(4^n - 2)B_{2n}(\pi y)^{2n}}{(2n)!} dx \\
= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(4^n - 2)B_{2n}(\pi y)^{2n}}{(2n + 1)(2n)!} y^{2n}.
\]

Setting the two results equal to each other and simplifying more, we see

\[
\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n4^n} \sum_{k=0}^{2n} \binom{2n}{k} \frac{(-1)^k}{2^k} y^k - \sum_{k=1}^{\infty} \frac{\zeta(2k)}{(2k + 1)4^k} y^{2k} + \frac{2}{y} (1 - \log \pi) + 2 \sum_{k=1}^{\infty} \frac{1}{k2^k} y^{k-1} \\
- \frac{1}{2} \sum_{n=1}^{\infty} \frac{\zeta(2n)}{n(2n + 1)4^n} \sum_{k=0}^{2n+1} \binom{2n+1}{k} \frac{(-1)^k}{2^k} y^{k-1} = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}(2^{2k-1} - 1)B_{2k} \pi^{2k}}{4^k(2k + 1)!} y^{2k}.
\]
Now we group the coefficients on both sides. The odd powers of $y$ (i.e., $2j - 1$ for $j = 1, 2, 3, ...$), we find

$$
\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n4^n} \left( \frac{2n}{2j-1} \right) \frac{(-1)^{2j-1}}{2^{2j-1}} + \frac{2}{(2j)2^{2j}} + 2 \sum_{n=1}^{\infty} \frac{\zeta(2n)}{n(2n+1)4^n} \frac{(-1)^{2j}}{2^{2j}} \left( \frac{2n+1}{2j} \right) = 0.
$$

Multiplying by $2^{2j-1}$, we see

$$\frac{1}{2j} = \sum_{n=1}^{\infty} \frac{\zeta(2n)}{n4^n} \left( \frac{2n}{2j-1} \right) \frac{1}{2n+1} \left( \frac{2n+1}{2j} \right) = \frac{2j}{2j} - 1 \sum_{n=1}^{\infty} \frac{\zeta(2n)}{n4^n} \left( \frac{2n}{2j-1} \right).
$$

Setting $m = 2j - 1$, we arrive at the first part of the theorem. For the even powers of $y$ (i.e., $2j$ for $j = 1, 2, 3, ...$), we arrive at the following:

$$
\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n4^n} \left( \frac{2n}{2j} \right) \frac{(-1)^{2j}}{2^{2j}} \left( \frac{2n+1}{2j} \right) = \frac{\zeta(2j)}{(2j+1)4^j} + \frac{2}{(2j+1)2^{2j+1}}
$$

$$+ 2 \sum_{n=1}^{\infty} \frac{\zeta(2n)}{n(2n+1)4^n} \frac{(-1)^{2j+1}}{2^{2j+1}} \left( \frac{2n+1}{2j+1} \right) = \frac{(-1)^{j+1}(2^{2j-1} - 1)B_{2j}\pi^{2j}}{4^j(2j+1)!}.
$$

Multiplying by $4^j$ and using (1) to replace the Bernoulli numbers by $\zeta(2j)$, we have

$$
\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n4^n} \left( \frac{2n}{2j} \right) \frac{1}{2n+1} \left( \frac{2n+1}{2j+1} \right) = \frac{\zeta(2j)}{2j+1} - \frac{1}{2j+1} \zeta(2j) + \frac{\zeta(2j)}{(2j+1)} \left( 1 - \frac{2}{2^{2j}} \right)
$$

Using the binomial identity \( \binom{2n}{2j} - \frac{1}{2n+1} \binom{2n+1}{2j+1} = \frac{2j}{2j+1} \binom{2n}{2j} \), this simplifies to

$$
\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n4^n} \left( \frac{2n}{2j} \right) = \frac{1}{2j} \left( \frac{2\zeta(2j)}{1 - \frac{1}{2^{2j}}} \right).
$$

Letting $m = 2j$ gives the final result of the theorem and thus, the proof is complete. \(\square\)

**Corollary 2.4.** (28, 3) We have

$$
(22) \quad \sum_{n=1}^{\infty} \frac{\zeta(2n)}{n(2n+1)4^n} = \log \pi - 1.
$$
Proof. This follows immediately by setting the coefficients of \( y^{-1} \) equal to each other on both sides. □

**Corollary 2.5.** We have the following series representations

\[(23) \quad \sum_{n=1}^{\infty} \frac{\zeta(2n)}{4^n} = \frac{1}{2},\]

\[(24) \quad \sum_{n=1}^{\infty} \frac{\zeta(2n)(2n-1)(2n-2)}{4^n} = 1,\]

\[(25) \quad \sum_{n=1}^{\infty} \frac{\zeta(2n)(2n-1)}{4^n} = \frac{\pi^2}{8} - \frac{1}{2},\]

\[(26) \quad \sum_{n=1}^{\infty} \frac{\zeta(2n)n}{4^n} = \frac{\pi^2}{16},\]

and

\[(27) \quad \sum_{n=1}^{\infty} \frac{\zeta(2n)n^2}{4^n} = \frac{3\pi^2}{32}.\]

Proof. Define \( F(k) := \sum_{n=1}^{\infty} \frac{\zeta(2n)n^k}{4^n} \). Letting \( m = 1, 3, 2 \), we obtain (23), (24), and (25), respectively. Using (23), note that (25) can be rewritten as \( 2F(1) - F(0) = \frac{\pi^2}{8} - F(0) \), and so, (26) follows immediately. Finally, using (23) and (26), note that (24) can be rewritten as \( 4F(2) - 6F(1) + 2F(0) = 2F(0) \). From this, (27) follows immediately. □

Remark. Letting \( m = 2k \) and \( m = 2k - 1 \) \((k = 1, 2, 3,\ldots)\) in the even and odd parts of (21) respectively, we can add the two results to find

\[\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n4^n} \left( \frac{2n}{2k-1} + \frac{2n}{2k} \right) = \frac{\zeta(2k)}{k} \left( 1 - \frac{1}{4^k} \right) + \frac{1}{2k-1} - \frac{1}{2k},\]

which is equivalent to

\[(28) \quad \sum_{n=1}^{\infty} \frac{\zeta(2n)}{n4^n} \left( \frac{2n+1}{2k} \right) = \frac{\zeta(2k)}{k} \left( 1 - \frac{1}{4^k} \right) - \frac{1}{2k(2k-1)}.\]
Theorem 2.6. We have the following series representation

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n16^n} \binom{2n}{m} = \begin{cases} 
\frac{1}{m} \left( 1 - \zeta_E(m - 1) \left( 1 - \frac{1}{2^{m-1}} \right) \right) & \text{m odd,} \\
\frac{1}{m} \left( \zeta(m) \left( 1 - \frac{1}{2^{m}} \right) - 1 \right) & \text{m even,}
\end{cases}$$

where $\zeta_E(2k) = \frac{(-1)^{k+1} E_{2k} \pi^{2k+1}}{4(1 - 4^k)(2k)!}$ and $E_{2k}$ are the Euler numbers.

Proof. Similar to the previous theorem’s proof, we integrate $\sec(xy)$ two different ways. First, integrating regularly and using properties of logarithm, we have

$$\int_0^{\pi/2} \sec(xy) \, dx = \frac{1}{y} \log \left( 1 + \sin \left( \frac{\pi y}{2} \right) \right) - \frac{1}{y} \log \left( \cos \left( \frac{\pi y}{2} \right) \right)$$

$$= \frac{1}{y} \frac{d}{d\alpha} \left( -2 Cl_2 \left( \frac{\pi}{2} + \alpha \right) - \alpha \log 2 \right) - \frac{1}{y} \log \left( \prod_{n=1}^{\infty} \left( 1 - \left( \frac{y}{2n-1} \right)^2 \right) \right),$$

where $\alpha = \frac{\pi y}{2}$ and we have used the product formula $\cos(\beta) = \prod_{k=1}^{\infty} \left( 1 - \left( \frac{2\beta}{\pi(2k-1)} \right)^2 \right)$.

Using equation (8) in the first term and applying properties of logarithm and its power series to the second term, the integral becomes

$$\int_0^{\pi/2} \sec(xy) \, dx = \frac{1}{y} \left( 2 \log \left( \frac{\pi}{2} \right) + 2 \log(1 + y) - \log 2 \right)$$

$$- \frac{2}{y} \sum_{n=1}^{\infty} \frac{\zeta(2n)(\pi^2)^{2n}(1+y)^{2n}}{n(2\pi)^{2n}} + \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{y^{2k}}{k(2n-1)^{2k}},$$

and

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^{2k}} = \sum_{n=1}^{\infty} \frac{1}{n^{2k}} - \sum_{n=1}^{\infty} \frac{1}{(2n)^{2k}} = \left( 1 - \frac{1}{4^k} \right) \zeta(2k).$$

Alternatively, we can apply Fubini’s theorem once again and integrate the power series of the secant function term by term. Doing so will give us the following:

$$\int_0^{\pi/2} x y \sec(xy) \, dx = \int_0^{\pi/2} \sum_{n=0}^{\infty} \frac{(-1)^n E_{2n}(xy)^{2n}}{(2n)!} \, dx = \sum_{n=0}^{\infty} \frac{(-1)^n E_{2n}(\pi y)^{2n+1}}{(2n+1)(2n)!} y^{2n}.$$

Setting the two results equal to each other and simplifying more, we see...
\[
\frac{2}{y} \log \left( \frac{\pi}{2\sqrt{2}} \right) + 2 \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} y^{k-1} - 2 \sum_{n=1}^{\infty} \frac{\zeta(2n)}{n^{16^n}} \sum_{k=0}^{2n} \left( \begin{array}{c} 2n \\ k \end{array} \right) y^{k-1} \\
+ \sum_{k=1}^{\infty} \frac{\zeta(2k)}{k} \left( 1 - \frac{1}{4^k} \right) y^{2k-1} = \sum_{k=0}^{\infty} \frac{(-1)^k E_{2k} \left( \frac{\pi}{2k+1} \right)^{2k+1}}{(2k+1)!} y^{2k}.
\]

Now we can group coefficients. For the even powers of \( y \) (i.e., \( 2j \) for \( j = 1, 2, 3, ... \)), we have

\[
2 \sum_{n=1}^{\infty} \frac{\zeta(2n)}{n^{16^n}} \left( \begin{array}{c} 2n \\ 2j+1 \end{array} \right) = \frac{(-1)^j E_{2j} \pi^{2j+1}}{(2j+1)! 2^{2j+1}}.
\]

Define \( \zeta_E(2j) := \frac{(-1)^j E_{2j} \pi^{2j+1}}{4(1 - 4^j)(2j)!} \). The motivation comes from Euler’s formula for \( \zeta(2n) \) and the asymptotic formula \( B_{2n} \sim \frac{\pi E_{2n}}{2^{2n+1}(1 - 4^n)}; n \to \infty \). For future computations, note that \( \zeta_E(2) = \frac{\pi^3}{24} \). Now, simplifying the above expression,

\[
\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n^{16^n}} \left( \begin{array}{c} 2n \\ 2j+1 \end{array} \right) = \frac{1}{2j+1} + \frac{4\zeta_E(2j)(1 - 4^j)(2j)!}{4(2j+1)! 4^j} = \frac{1}{2j+1} \left( 1 - \zeta_E(2j) \left( 1 - \frac{1}{4^j} \right) \right).
\]

Setting \( m = 2j+1 \), we achieve the first result of the theorem. For the odd powers of \( y \) (i.e., \( 2j-1 \) for \( j = 1, 2, 3, ... \)), we find

\[
2 \sum_{n=1}^{\infty} \frac{\zeta(2n)}{n^{16^n}} \left( \begin{array}{c} 2n \\ 2j \end{array} \right) + \frac{\zeta(2j)}{j} \left( 1 - \frac{1}{4^j} \right) = 0,
\]

and rearranging this gives the second part of the theorem. \( \square \)

\textbf{Corollary 2.7.} We have the following representations

(30) \[ \sum_{n=1}^{\infty} \frac{\zeta(2n)}{n^{16^n}} = \log \left( \frac{\pi}{2\sqrt{2}} \right), \]

and

(31) \[ \sum_{n=1}^{\infty} \frac{\zeta(2n)}{16^n} = \frac{4 - \pi}{8}. \]
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Proof. Formula (30) follows immediately by setting the coefficients of $y^{-1}$ equal to each other on both sides. For (31), we set the constant terms on both sides equal to find

$$2 - 4 \sum_{n=1}^{\infty} \frac{\zeta(2n)}{16^n} = \frac{\pi}{2}. $$

From here, (31) follows immediately. One could also use the theorem for $m = 2k + 1$ for $k = 0$ and note that $\lim_{k \to 0} \zeta_E(2k)(1 - 4^k) = \pi/4$. □

Remark. In the previous theorem, if we used the Clausen identity for $\log(\cos(x))$ given in the introduction rather than the cosine product formula, we obtain the following relation:

$$\log \left( \frac{\pi}{4} \right) - \sum_{n=1}^{\infty} \frac{\zeta(2n)}{n4^n} \sum_{k=0}^{2n} \binom{2n}{k} y^k \left( \frac{2}{4^n} - (-1)^k \right) + \sum_{k=1}^{\infty} \frac{y^k}{k} (1 - 2(-1)^k)$$

$$= \sum_{k=0}^{\infty} \frac{2\zeta_E(2k)}{2k+1} \left( 1 - \frac{1}{4^k} \right) y^{2k+1}. $$

Gathering the coefficients of $y^0$, we find

$$\log \left( \frac{\pi}{4} \right) - \sum_{n=1}^{\infty} \frac{\zeta(2n)}{n4^n} \left( \frac{2}{4^n} - 1 \right) = 0. $$

Rearranging and using (30), we can see that

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n4^n} = \log \left( \frac{\pi}{2} \right),$$

which is a very nice result. This series appears by setting the coefficients of $y^0$ equal to each other in the proof of Theorem 2.3; however, the series will cancel and you arrive at a truth statement. Here, we are able to recover that series.

Corollary 2.8. We have

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)(2n - 1)}{16^n} = \frac{\pi^2}{16} - \frac{1}{2},$$

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)(2n - 1)(2n - 2)}{16^n} = 1 - \frac{\pi^3}{96},$$

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)n}{16^n} = \frac{\pi}{16} \left( \frac{\pi}{2} - 1 \right).$$
and

\[ \sum_{n=1}^{\infty} \frac{\zeta(2n) n^2}{16^n} = \frac{\pi}{32} \left( \frac{3\pi}{2} - \frac{\pi^2}{4} - 1 \right). \]

**Proof.** Define \( G(k) := \sum_{n=1}^{\infty} \frac{\zeta(2n) n^k}{16^n} \). Letting \( m = 2, 3 \) in (29) gives us formulas (33) and (34), respectively. Using (31), we can rewrite (33) as

\[ 2G(1) - G(0) = \frac{\pi^2}{16} - G(0) - \frac{\pi}{8} \]

and from this, (35) follows immediately. Lastly, for (36), expanding (34) using (31), we have

\[ 4G(2) - 6G(1) + 2G(0) = \frac{\pi}{4} + 2G(0) - \frac{\pi^3}{96}. \]

Using (35), we achieve the desired result. \( \square \)

**Corollary 2.9.** We have the following series

\[ \sum_{n=1}^{\infty} \frac{\zeta(2n)}{n^4} \left( 1 - \frac{1}{4^n} \right) \binom{2n}{2k} = \frac{\zeta(2k)}{2k} \left( 1 - \frac{1}{4^k} \right), \]

and

\[ \sum_{n=1}^{\infty} \frac{\zeta(2n)}{n^4} \left( 1 - \frac{1}{4^n} \right) \binom{2n}{2k+1} = \zeta_E(2k) \frac{2k+1}{2k+1} \left( 1 - \frac{1}{4^k} \right). \]

**Proof.** Denote (29.1) and (29.2) as well as (21.1) and (21.2) as the formula for \( m \) odd and \( m \) even, respectively. Letting \( m = 2k \) for \( k = 1, 2, 3... \) and subtracting (29.2) from (21.2), formula (37) follows immediately. Similarly, letting \( m = 2k + 1 \) for \( k = 0, 1, 2, ... \) and subtracting (29.1) from (21.1), the obtain (38). \( \square \)

**Remark.** You can add (37) and (38) together to get \( \frac{2n+1}{2k+1} \) inside the series. You can also change \( k \) to \( k - 1 \) in (38) and add (37) and (38) again to give you \( \frac{2n+1}{2k} \) in the series.

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