Flocks of Cones: Herds and Herd Spaces

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1 Introduction

This is the first in a series of articles devoted to providing a foundation for a theory of flocks of arbitrary cones in $PG(3, q)$. The desire to have such a theory stems from a need to better understand the very significant and applicable special case of flocks of quadratic cones in $PG(3, q)$. Flocks of quadratic cones have connections with several other geometrical objects, including certain types of generalized quadrangles, spreads, translation planes, hyperovals (in even characteristic), ovoids, inversive planes and quasi-fibrations of hyperbolic quadrics. This rich collection of interconnections is the basis for the strong interest in such flocks. Recent work has shown that some of these connections can be made with other types of cones. The author has attempted incremental generalizations of flocks of quadratic cones ([1],[2]) and the similarity of the results in these investigations indicated the existence of a more general framework. However, this incremental approach leads to more and more difficult algebraic considerations that ultimately make this approach untenable. By jumping to the most general situation and changing our point of view (as we will do in this series of articles) we can transcend those algebraic difficulties and hopefully gain a clearer perspective on the subject. This first paper lays out the fundamentals while later papers will examine special types of flocks.

The reader should be warned that we have taken some liberties with terminology (especially the terms “flock” and “herd”) by redefining objects in a more general context. This was deemed necessary to avoid having to introduce a more cumbersome set of terms. However, the use of modifiers will insure that the new definitions are in agreement with the more commonly used ones in the appropriate context. We should also mention that we have
restricted ourselves to the finite case only out of preference for that setting. Infinite analogs of almost everything that appears here do exist, but we shall leave it to others to develop these.

2 Cones and Flocks

Let $\pi_0$ be a plane and $V$ a point not on $\pi_0$ in $PG(3, q)$. Let $C$ be any set of points in $\pi_0$ (including the empty set). A cone, $\Sigma = \Sigma(V, C)$ is the union of all points of $PG(3, q)$ on the lines $VP$ where $P$ is a point of $C$. $V$ is called the vertex and $C$ is called the carrier of $\Sigma$. $\pi_0$ is the carrier plane and the lines $VP$ are the generators of $\Sigma$. In the event that $C = \emptyset$ we call $\Sigma$ the empty cone and by convention consider it to consist of only the point $V$.

A flock of planes in $PG(3, q)$ is any set of $q$ distinct planes of $PG(3, q)$. As $q$ planes can not cover all the points of $PG(3, q)$, there always are points of the space which do not lie in any of the planes in a flock of planes. If $\Sigma$ is a cone of $PG(3, q)$, then a flock of planes, $\mathcal{F}$, is said to be a flock of $\Sigma$ when the vertex of $\Sigma$ lies in no plane of $\mathcal{F}$ and no two planes of $\mathcal{F}$ intersect at a point of $\Sigma$. Any flock of planes is a flock of a cone, possibly only the empty cone. In general, however, a given flock of planes will be a flock of several cones. In the literature on flocks of quadratic cones, the approach is always to consider a fixed quadratic cone and study the flocks of that cone. We will change the viewpoint and consider, for a fixed flock of planes, the various cones of which it is a flock. In the sequel we shall refer to a flock of planes simply as a flock and it shall be understood that it is always a flock of a cone, even if the cone is not explicitly indicated.

In order to provide an algebraic representation of a flock we will need to introduce coordinates. The standard homogeneous coordinates of a point in $PG(3, q)$ will be given by $\langle x_0, x_1, x_2, x_3 \rangle$ with $\langle \cdots \rangle$ denoting the fact that we are dealing with an equivalence class. When a specific element of this equivalence class is needed, it will be denoted with parentheses. Thus $(a, b, c, d) \in \langle a, b, c, d \rangle$ is a specific representative of the equivalence class.

Let $\mathcal{F}$ be a flock. We can introduce coordinates in $PG(3, q)$ so that the plane $x_3 = 0$ is one of the planes of the flock and the point $V = \langle 0, 0, 0, 1 \rangle$ is not in any plane of the flock. We parameterize the planes of $\mathcal{F}$ with the elements of $GF(q)$ in an arbitrary way except that we will require that 0 is the parameter assigned to the plane $x_3 = 0$. We can now describe the flock as, $\mathcal{F} = \{ \pi_t \mid t \in GF(q) \}$ with $\pi_0: x_3 = 0$. Since $V$ is not in any plane of $\mathcal{F}$, each
of the planes of this flock has an equation of the form $Ax_0 + Bx_1 + Cx_2 - x_3 = 0$. Consider the points $P = \langle 1, 0, 0, 0 \rangle, Q = \langle 0, 1, 0, 0 \rangle$ and $R = \langle 0, 0, 1, 0 \rangle$ of $\pi_0$. The points other than $V$ on the lines $VP, VQ$ and $VR$ are given by $\langle 1, 0, 0, \lambda \rangle, \langle 0, 1, 0, \mu \rangle$ and $\langle 0, 0, 1, \nu \rangle$ respectively, with $\lambda, \mu, \nu$ varying in $GF(q)$. For each $t \in GF(q)$ the plane $\pi_t$ of the flock meets these lines at the points $(1, 0, 0, \lambda_t)$, $(0, 1, 0, \mu_t)$ and $(0, 0, 1, \nu_t)$ respectively (note the use of specific representatives). We define three functions $f, g, h: GF(q) \to GF(q)$ by $f(t) = \lambda_t, g(t) = \mu_t$ and $h(t) = \nu_t$. These functions describe the equations of the planes of the flock, namely $\pi_t: f(t)x_0 + g(t)x_1 + h(t)x_2 - x_3 = 0$. The functions $f, g$ and $h$ are called the coordinate functions of the flock. Note that the requirement on the parameter 0 means that $f(0) = g(0) = h(0) = 0$. If $f, g$ and $h$ are the coordinate functions of the flock $F$ we shall write $F = F(f, g, h)$. We remark that the coordinate functions of a flock depend on both the parameterization of the flock and the procedure used to obtain the functions from the homogeneous coordinates (i.e., the selection of representatives of these coordinates). We will have occasion to change the parameterization of a flock but will never change this standard procedure for obtaining the functions.

All cones under consideration will have vertex $V = \langle 0, 0, 0, 1 \rangle$ and we will consider the plane $\pi_0$ as the carrier plane of the cone. Thus, a cone is determined when its carrier $C$, a point set in $\pi_0$, is specified. Given a flock $F$, there is a largest set $C_0$ of $\pi_0$ such that $F$ is a flock of the cone with carrier $C_0$. This cone is called the critical cone of $F$. If $C$ is any subset of the carrier of the critical cone of a flock $F$, then clearly $F$ is also a flock of the cone with carrier $C$. Thus, determining the critical cone of a flock implicitly determines all cones for which this flock of planes is a flock.

The critical cone of a flock may be fairly “small”. Besides the empty cone, we will consider cones whose carriers consist of collinear points as being “small”. Cones of this type are called flat cones. For the most part, we shall regard flocks whose critical cones are flat as being uninteresting.

## 3 Herds and Herd Spaces

Let $Z$ denote the collection of all functions $f: GF(q) \to GF(q)$ such that $f(0) = 0$. Note that each element of $Z$ can be expressed uniquely as a polynomial in one variable of degree at most $q - 1$ and that $Z$ is a vector space over $GF(q)$. We shall always consider the vectors of $Z$ as polynomials of this
type and use \( \{ t, t^2, \ldots, t^{q-1} \} \) as the standard basis of \( \mathbb{Z} \). Let \( \mathcal{V} = \mathcal{V}(f, g, h) \) denote the subspace of \( \mathbb{Z} \) generated by the vectors \( f, g \) and \( h \) of \( \mathbb{Z} \). Also, let \( \mathcal{U} = \mathbb{GF}(q) \times \mathbb{GF}(q) \times \mathbb{GF}(q) \) considered as a vector space over \( \mathbb{GF}(q) \). Now, the map \( \phi_{f, g, h}: \mathcal{U} \to \mathcal{V} \) given by:

\[
\phi_{f, g, h}(a, b, c) = af + bg + ch
\]

is a non-trivial vector space homomorphism, provided that at least one of the functions is not the constant function (which we shall always assume to be the case). In the sequel we will suppress the subscripts in the name of this homomorphism whenever this will not lead to confusion. In the usual manner we may construct the projective geometries \( P(\mathbb{Z}), P(\mathcal{U}) \) and \( P(\mathcal{V}) \) from the vector spaces \( \mathbb{Z}, \mathcal{U} \) and \( \mathcal{V} \) respectively. The projective space \( P(\mathbb{Z}) \) is isomorphic to \( PG(q-2, q) \) and \( P(\mathcal{U}) \) is just \( PG(2, q) \) in its usual representation. It is clear that \( \phi \) induces a map \( \hat{\phi}: P(\mathcal{U}) \to P(\mathcal{V}) \) given by:

\[
\hat{\phi}(\langle a, b, c \rangle) = \langle af + bg + ch \rangle.
\]

The set of ordered pairs that defines \( \hat{\phi} \) (sometimes called the graph of \( \hat{\phi} \)), which we denote by \( \Gamma(f, g, h) \), i.e.,

\[
\Gamma(f, g, h) = \{(\langle a, b, c \rangle, \langle af + bg + ch \rangle) | (a, b, c) \in P(\mathcal{U})\}
\]

is called the Herd Space of the functions \( f, g \) and \( h \).

In the important case that \( \hat{\phi} \) is bijective, \( P(\mathcal{V}) \) is a projective plane and \( \hat{\phi} \) is just the inverse of the standard coordinate function, when \( \{f, g, h\} \) is considered as an ordered basis of \( \mathcal{V} \). In this case the herd space can be thought of as a plane in \( P(\mathbb{Z}) \) which has been coordinatized in a special way.

For reasons that will become clear later, we say that a herd space \( \Gamma(f, g, h) \) is degenerate if there exist distinct elements \( u, v \in \mathbb{GF}(q) \) such that \( f(u) = f(v), g(u) = g(v) \) and \( h(u) = h(v) \). Thus, if any of \( f, g \) or \( h \) is a permutation (i.e., a permutation polynomial) then \( \Gamma(f, g, h) \) is non-degenerate.

There are \((q - 1)!\) elements of \( \mathbb{Z} \) which are permutation functions. As all non-zero scalar multiples of a permutation are also permutations, there are \((q - 2)!\) points of \( P(\mathbb{Z}) \) which are classes of permutation functions and will be referred to as permutation points. Let \( S \) denote the set of all permutation points in \( P(\mathbb{Z}) \). A corollary to Hermite’s condition for permutation polynomials (see [8]) states that the degree of a permutation polynomial over \( \mathbb{GF}(q) \) is either 1 or it does not divide \( q - 1 \). Thus, \( S \) lies in the hyperplane \( \langle t, t^2, \ldots, t^{q-2} \rangle \) of \( P(\mathbb{Z}) \).
We now define a significant partial function of the herd space \( \Gamma(f, g, h) \), called the *Herd Cover* of \( f, g \) and \( h \), and denoted by \( \mathcal{HC} = \mathcal{HC}(f, g, h) \), where

\[
\mathcal{HC} = \{(\langle a, b, c \rangle, \langle af + bg + ch \rangle) \mid \langle af + bg + ch \rangle \in S\}.
\] (4)

Note that \( \mathcal{HC}(f, g, h) \) may be (and often is, for arbitrary \( f, g \) and \( h \)) the empty set. Also note that, a priori, there is no geometrical significance to a point being a permutation point in \( \mathbf{P}(\mathbf{V}) \), as this is a purely algebraic notion. However, as we shall see below, these points gain an important geometric role in the context of flocks.

For any given herd cover, \( \mathcal{HC} = \mathcal{HC}(f, g, h) \), the image of the standard projection map onto the first coordinates, \( \pi_1: \mathcal{HC} \to \mathbf{P}(\mathbf{U}) \), is called the *point set of* \( \mathcal{HC} \), and denoted by \( \mathbf{P}_{\mathcal{HC}} \). Any map \( \rho: \mathbf{P}(\mathbf{U}) \to \mathbf{U} \) such that \( \rho(\langle a, b, c \rangle) \in \langle a, b, c \rangle \) is called a *herd selection function*. As an aside we note that in the infinite case the existence of these choice functions may require the Axiom of Choice, but since we have restricted ourselves to the finite case, this issue does not arise. Finally, the set of representatives of the herd cover determined by the herd selection function \( \rho \), denoted by \( \rho-\mathcal{H}(f, g, h) \), and given by,

\[
\rho-\mathcal{H}(f, g, h) = \{(\rho(P), \phi(\rho(P))) \mid P \in \mathbf{P}_{\mathcal{HC}}\},
\] (5)

is called the *\( \rho \)-Herd* of the herd cover \( \mathcal{HC}(f, g, h) \). Essentially, a herd is just a representation of a herd cover where the representative of the first coordinate is arbitrary and the representative of the second coordinate depends on the first coordinate choice. It is clear that there are several \( \rho \)-herds associated to a given herd cover and that the herd cover is uniquely determined by any of its \( \rho \)-herds.

Since we have given a formal definition of a \( \rho \)-herd and freely admit that its notation is a bit cumbersome, we will use a simpler alternative notation when it will not lead to confusion, namely,

\[
\rho-\mathcal{H}(f, g, h) = \{f_P \mid P \in \mathbf{P}_{\mathcal{HC}}, f_P = \phi(\rho(P))\}.
\] (6)

In this view, a \( \rho \)-herd is an indexed family of permutation functions where the indexing set is \( \mathbf{P}_{\mathcal{HC}} \).

There are several useful choices for the herd selection function \( \rho \) and we shall discuss a few. The *standardized herd* is the \( \rho-\mathcal{H}(f, g, h) \) such that

\[
f_P = \begin{cases} 
\frac{2}{c}f + \frac{b}{c}g + h, & \text{if } c \neq 0; \\
f + \frac{b}{a}g, & \text{if } c = 0 \text{ and } a \neq 0; \\
g, & \text{if } a = c = 0.
\end{cases}
\] (7)
The herd selection function for the standardized herd is the one which selects the representatives of \( \langle a, b, c \rangle \) of the forms \((x, y, 1), (1, m, 0)\) or \((0, 1, 0)\). We also have the alternate standardized herd which is the \( \rho \)-\( \mathcal{H}(f, g, h) \) such that
\[
 f_P = \begin{cases} 
 f + \frac{b}{a}g + \frac{c}{a}h, & \text{if } a \neq 0; \\
 g + \frac{c}{b}h, & \text{if } a = 0 \text{ and } b \neq 0; \\
 h, & \text{if } a = b = 0.
\end{cases}
\] (8)

The herd selection function here is the one which selects the representatives of \( \langle a, b, c \rangle \) so that the leftmost non-zero coordinate is 1, i.e. the forms \((1, x, y), (0, 1, m)\) or \((0, 0, 1)\). The normalized herd is the \( \rho \)-\( \mathcal{H}(f, g, h) \) such that
\[
 f_P(t) = \frac{af(t) + bg(t) + ch(t)}{af(1) + bg(1) + ch(1)} \quad \text{where } P = \langle a, b, c \rangle, \forall t \in GF(q). \] (9)

The normalized herd has the property that \( f_P(1) = 1, \forall P \in \mathcal{P}_{HC} \). The herd selection function for the normalized herd, unlike the previous examples, involves the functions \( f, g \) and \( h \) and is given by:
\[
 \rho(\langle a, b, c, \rangle) = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \begin{pmatrix} a \\ af(1) + bg(1) + ch(1) \\ af(1) + bg(1) + ch(1) \end{pmatrix}^{-1}.
\]

We will now examine two special classes of herd spaces which play a fundamental role in the theory of flocks. It is the inclusion of these classes which prompts the definition of herd space that we have given.

Consider \( \mathcal{V} = \mathcal{V}(f, g, h) \). If \( \text{rank}(\mathcal{V}) = 1 \) (where \( \text{rank} = \text{vector space dimension} \)), then the functions \( f, g \) and \( h \) are all scalar multiples of the same function. Suppose then that \( f = \alpha F, g = \beta F, \) and \( h = \gamma F \) for some function \( F \in \mathcal{Z} \). In this case, (1) becomes \( \phi(a, b, c) = (\alpha a + \beta b + \gamma c)F \) and the kernel of \( \hat{\phi} \) is the line \( \ell \) with equation \( \alpha x + \beta y + \gamma z = 0 \) in \( \mathbb{P}(U) \). The herd space \( \Gamma(\alpha F, \beta F, \gamma F) \) consists of only two types of ordered pairs, namely, \((P, \langle 0 \rangle)\) when \( P \) is on \( \ell \) (here \( \langle 0 \rangle \) denotes the constant function), and \((P, \langle F \rangle)\) otherwise. The herd cover \( \mathcal{H}(\alpha F, \beta F, \gamma F) \) has a point set \( \mathcal{P}_{HC} \) which is either empty if \( F \) is not a permutation or the complement of \( \ell \) (an affine subplane of \( \mathbb{P}(U) \)) if \( F \) is a permutation. The functions of the \( \rho \)-herds corresponding to the non-empty herd cover are all scalar multiples of \( F \). Herd spaces of this type will be called linear herd spaces.
Proposition 3.1. The point set of the herd cover of a linear herd space is empty if, and only if, the herd space is degenerate.

Proof. Using the notation of the previous paragraph let $P_{HC}$ be the point set of the herd cover of the linear herd space $\Gamma = \Gamma(\alpha F, \beta F, \gamma F)$. Not all of $\alpha, \beta$ and $\gamma$ can be 0, else $\text{rank}(V) = 0$. So, assume w.l.o.g. that $\alpha \neq 0$. Now, suppose that this herd space is degenerate. Then there exist $u \neq v \in GF(q)$ so that $\alpha F(u) = \alpha F(v)$, and $F$ is not a permutation. On the other hand, if $F$ is not a permutation then there exist $r \neq s \in GF(q)$ so that $F(r) = F(s)$. But then, $\alpha F(r) = \alpha F(s), \beta F(r) = \beta F(s)$ and $\gamma F(r) = \gamma F(s)$, and so, $\Gamma$ is degenerate.

The second case we will consider occurs when $\text{rank}(V) = 2$. In this case the functions $f, g$ and $h$ are linearly dependent over $GF(q)$, but not all are scalar multiples of the same function. Thus, there exist constants $\alpha, \beta$ and $\gamma$, not all zero, so that $\alpha f + \beta g + \gamma h = 0$. The kernel of $\hat{\phi}$ is the point $Q = \langle \alpha, \beta, \gamma \rangle$ of $P(U)$. Now consider a line $\ell$ of $P(U)$ which passes through $Q$. Such a line has an equation of the form $Ax + By + Cz = 0$ where not all the coefficients are zero and $A\alpha + B\beta + C\gamma = 0$. Since not all of $\alpha, \beta$ and $\gamma$ are zero, we can assume w.l.o.g. that $\gamma \neq 0$. If $P = \langle a, b, c \rangle$ is a point of $\ell$ other than $Q$ then we have,

$$\phi(a, b, c) = af + bg + ch$$
$$= \frac{1}{\gamma}(a\gamma f + b\gamma g + c\gamma h)$$
$$= \frac{1}{\gamma}((a\gamma - c\alpha)f + (b\gamma - c\beta)g).$$

From the fact that both $P$ and $Q$ are on the line $\ell$ we can derive that $(a\gamma - c\alpha)A + (b\gamma - c\beta)B = 0$. Now, both of $A$ and $B$ can not be zero, otherwise we would have that $\gamma C = 0$ and our assumption about $\gamma$ would imply that $C = 0$. If $A = 0$ then we must have $b\gamma - c\beta = 0$ and so, we obtain:

$$\phi(a, b, c) = \begin{cases} 
\gamma^{-1}(a\gamma - c\alpha)f, & \text{if } A = 0, \\
\gamma^{-1}(b\gamma - c\beta)(-\frac{B}{A}f + g) & \text{otherwise}. 
\end{cases}$$

(10)

In either case, for all the points on $\ell \setminus \{Q\}$, the associated functions are non-zero multiples of the same non-zero function. The herd space in this case is described by; $(Q, \langle 0 \rangle)$, and, for each line of $P(U)$ through $Q$, the points of the
line other than \( Q \) are associated with the same point of \( P(V) \). If a point \( P \) of \( P(U) \) is in \( P_{HC} \), then all the points of the line \( PQ \) other than \( Q \) are in \( P_{HC} \). The herd spaces of this type are called proper star herd spaces. We define a star herd space to be either a proper star herd space or a linear herd space. Considering that a herd space is a function, we can rephrase this definition: a herd space is a star herd space if, and only if, it is not a bijection.

**Proposition 3.2.** The number of lines in the herd cover of a proper star herd space through the kernel of the herd space is \( | S \cap P(V) | \).

**Proof.** Let \( \Gamma(f, g, h) \) be a proper star herd space. We shall use the notation of the above paragraph. Let \( m \) be any fixed line of \( P(U) \) which does not pass through the kernel \( Q \). The restriction of \( \hat{\phi} \) to \( m \) has a trivial kernel and so, is a bijection between \( m \) and \( P(V) \). If \( R \) is a point of \( m \) such that \( \hat{\phi}\mid_m (R) \in (S) \) then \( R \in P_{HC} \) and all the points other than \( Q \) of the line \( QR \) are in \( P_{HC} \). Any line of \( P(U) \) through \( Q \) which is in the herd cover must intersect \( m \) in a point of \( P_{HC} \), proving the assertion. \( \square \)

### 4 Flocks and Herd Spaces

To each flock \( \mathcal{F}(f, g, h) \) of \( PG(3, q) \) we can naturally associate the (non-degenerate) herd space \( \Gamma(f, g, h) \). By identifying the points of \( P(U) \) of the herd space with the points of the plane \( x_3 = 0 \) (in the flock) of \( PG(3, q) \) we can attach a geometric significance to the concepts introduced in the previous section.

Let \( \mathcal{F} = \mathcal{F}(f, g, h) \) be a flock of \( PG(3, q) \). Naturally identify the points \( \langle a, b, c \rangle \) of \( P(U) \) with the points \( \langle a, b, c, 0 \rangle \) of the plane \( x_3 = 0 \). Let \( \rho \) be a herd selection function on \( P(U) \) extended to \( x_3 = 0 \). For each point \( P \) of \( x_3 = 0 \), \( \rho \) thus fixes a coordinate representative for \( P \). With \( V = (0, 0, 0, 1) \), we fix a coordinate representative of each point other than \( V \) on the line \( VP \) of the form \( (a, b, c, \lambda) \) where \( \rho(P) = (a, b, c, 0) \). Now, each plane \( \pi_t \) of \( \mathcal{F} \) intersects the line \( VP \) in a point other than \( V \). We define a function \( F_P : GF(q) \rightarrow GF(q) \) by \( F_P(t) = \lambda \) if \( \pi_t \cap VP = (a, b, c, \lambda) \). Note that for any \( P \) we have \( F_P(0) = 0 \) since \( x_3 = 0 \) is always the plane \( \pi_0 \) of \( \mathcal{F} \). The defining condition is equivalent to \( af(t) + bg(t) + ch(t) - \lambda = 0 \), i.e., \( F_P(t) = \lambda = af(t) + bg(t) + ch(t) \). Thus, the set of ordered pairs \( \{(P, \langle F_P \rangle) \mid P \in x_3 = 0\} \) defined by the flock \( \mathcal{F} \) is clearly isomorphic to the herd space \( \Gamma(f, g, h) \). Note
that while the definition of \( F_p \) uses a particular herd selection function, the resulting \( \langle F_p \rangle \) of the herd space is independent of that choice.

We now see that \( F_p \) is a permutation function if, and only if, no two planes of \( F \) meet the line \( VP \) at the same point. That is to say, \( VP \) is a generator line of a cone with vertex \( V \) for which \( F \) is a flock. The set of points on all such lines form the critical cone of \( F \). Thus, \( \mathbf{P}_{\mathcal{HC}} \) (under the identification) is the carrier of the critical cone of \( F \). Finally, it should be clear that \( \{ F_p \mid P \in \mathbf{P}_{\mathcal{HC}} \} \) is the \( \rho \)-herd of the herd cover \( \mathcal{HC}(f, g, h) \).

Now, we turn to the question of obtaining a flock from a non-degenerate herd space (degenerate herd spaces do not give rise to \( k \) distinct planes). We first note that a herd space does not determine a unique flock, since for any \( k \in GF(q) \) the flocks \( F(f, g, h) \) and \( F(kf, kg, kh) \) (which are distinct if \( k \neq 1 \), but projectively equivalent) give rise to the same herd space. However, this is the only variation for flocks which give rise to the same herd space.

**Theorem 4.1.** Two flocks, \( F(f, g, h) \) and \( F(f', g', h') \), give rise to the same herd space if, and only if, there exists a non-zero constant \( k \) such that \( f' = kf, g' = kg \) and \( h' = kh \).

**Proof.** Two flocks, \( F(f, g, h) \) and \( F(f', g', h') \), give rise to the same herd space if, and only if, \( \phi_{f', g', h'}(P) = \phi_{f, g, h}(P) \) for each point \( P \in \mathbf{P}(\mathcal{U}) \). That is, for each point \( P = \langle a, b, c \rangle \) we have \( \langle af + bg + ch \rangle = \langle af' + bg' + ch' \rangle \). For the points \( Q = \langle 1, 0, 0 \rangle, R = \langle 0, 1, 0 \rangle \) and \( S = \langle 0, 0, 1 \rangle \) we have \( \langle f \rangle = \langle f' \rangle, \langle g \rangle = \langle g' \rangle \) and \( \langle h \rangle = \langle h' \rangle \) respectively. Thus, there exist constants, \( k_Q, k_R \) and \( k_S \) so that \( f' = k_Q f, g' = k_R g \) and \( h' = k_S h \).

Consider \( \mathcal{V} = \mathcal{V}(f, g, h) \). If the rank of \( \mathcal{V} = 3 \) then at the point \( T = \langle 1, 1, 1 \rangle \) we have \( \langle f + g + h \rangle = \langle f' + g' + h' \rangle \) and so, there is a non-zero constant \( k_T \) so that \( f' + g' + h' = k_T (f + g + h) \). Thus, we obtain \( (k_Q - k_T) f + (k_R - k_T) g + (k_S - k_T) h = 0 \). Since \( f, g \) and \( h \) are linearly independent, we can conclude that \( k_Q = k_R = k_S = k_T \). If the rank of \( \mathcal{V} = 2 \) then there is a unique point \( P = \langle a, b, c \rangle \) at which we have \( af' + bg' + ch' = 0 \). This implies that \( ak_Q f + bk_R g + ck_S h = 0 \) and so, the kernel of \( \phi_{f, g, h} \) is \( (ak_Q, bk_R, ck_S) \). Since the kernel of \( \phi_{f', g', h'} \) must equal the kernel of \( \phi_{f', g', h'} \) we have that \( \langle a, b, c \rangle = \langle ak_Q, bk_R, ck_S \rangle \) and so again can conclude that \( k_Q = k_R = k_S \). Finally, if the rank of \( \mathcal{V} = 1 \) then the functions \( f, g \) and \( h \) are scalar multiples of each other. We can assume w.l.o.g. that there exist scalars \( \alpha \) and \( \beta \) so that \( g = \alpha f \) and \( h = \beta f \). The kernel of \( \phi_{f, g, h} \) is the line with equation \( x + \alpha y + \beta z = 0 \) in \( \mathbf{P}(\mathcal{U}) \). The kernel of \( \phi_{f', g', h'} \) is the line with
equation $k_Q x + \alpha k_R y + \beta k_S z = 0$. Since these lines must be the same, we have $k_Q = k_R = k_S$. □

A non-degenerate herd space together with a herd selection function $\rho$ and the values of $\phi(\rho(P))$ for three non-collinear points $P$ do determine a unique flock. Embed the projective plane $\mathbf{P}(\mathcal{U})$ of the herd space $\Gamma(f,g,h)$ in $\mathbb{P}G(3,q)$ as the plane $x_3 = 0$ by $\langle a,b,c \rangle \mapsto \langle a,b,0 \rangle$. The functions $f, g$ and $h$ can be obtained by using $\rho$ as follows: Let $P_1, P_2$ and $P_3$ be the three given non-collinear points. If $\rho(P_1) = (a_1, a_2, a_3), \rho(P_2) = (b_1, b_2, b_3), \rho(P_3) = (c_1, c_2, c_3)$ and $\phi(\rho(P_i)) = f_i, i = 1,2,3$ we have:

$$\begin{pmatrix} f \\ g \\ h \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}^{-1} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}.$$ 

The matrix inverse exists since the points are non-collinear. Now we can form the flock $\mathcal{F} = \mathcal{F}(f,g,h)$. Note that we obtain $q$ planes since $\Gamma(f,g,h)$ is non-degenerate.

In the construction of a herd space from a flock, a herd selection function was used to define the functions $F_P$. If this herd selection function is used with the constructed herd space to obtain a flock, then it should be clear that the original flock is recaptured if the functions $F_P$ are used as the $f_i$ selections.

The above discussion can be summarized as,

**Theorem 4.2.** Any flock $\mathcal{F}(f,g,h)$ gives rise to a unique herd space $\Gamma(f,g,h)$. On the other hand, any non-degenerate herd space $\Gamma(f,g,h)$ gives rise to several flocks related as in Theorem 4.1. A unique flock can be constructed from a herd space if a herd selection function $\rho$ is given and the value of $\phi(\rho(P))$ is known for three non-collinear points $P$. For each flock that can be constructed from a herd space, the critical cone of the flock has $\mathbf{P}_{HC}$ as its carrier.

**Corollary 4.3** (The General Herd Theorem). If the herd cover of a non-degenerate herd space $\Gamma(f,g,h)$ has a point set which contains three non-collinear points, then any $\rho$-herd of this herd cover gives rise to a unique flock $\mathcal{F}$ whose critical cone has $\mathbf{P}_{HC}$ as its carrier. Conversely, for any flock $\mathcal{F}(f,g,h)$ in $\mathbb{P}G(3,q)$ and any herd selection function $\rho$, there is a $\rho$-herd for which the the corresponding $\mathbf{P}_{HC}$ is the carrier of the critical cone of the flock.
Proof. Since the point set of the herd cover contains three non-collinear points, the \( \rho \)-herd will contain at least three ordered pairs of the form \((\rho(P), \phi(\rho(P)))\) corresponding to these points. This is the data which is needed to construct a unique flock from a non-degenerate herd space. The converse is obvious. \(\square\)

In light of this theorem, we will refer to the (non-degenerate) herd space \(\Gamma(f, g, h)\) as being the \textit{herd space of} \(\mathcal{F} = \mathcal{F}(f, g, h)\), since the choice of \(\rho\) is immaterial. We will refer to the \(\rho\)-herd of \(\mathcal{HC}(f, g, h)\) as being the \textit{herd of} \(\mathcal{F}\), only when the herd selection function \(\rho\) is clearly understood. It will also be convenient to abuse notation and refer to a representative of a point of \(P(V)\) as being a \textit{function of the herd space}.

\section{Equivalence of Herd Spaces}

Two herd spaces \(\Gamma(f, g, h)\) and \(\Gamma(f', g', h')\) are defined to be \textit{equivalent} if there exist a collineation \(\psi: P(U) \rightarrow P(U)\) and a collineation \(\tau: P(Z) \rightarrow P(Z)\) with \(\tau(\langle f, g, h \rangle) = \langle f', g', h' \rangle\) such that the following diagram commutes,

\[
\begin{array}{ccc}
P(U) & \xrightarrow{\hat{\phi}_{f,g,h}} & \langle f, g, h \rangle \\
\downarrow \psi & & \downarrow \tau \\
P(U) & \xrightarrow{\hat{\phi}_{f',g',h'}} & \langle f', g', h' \rangle
\end{array}
\]

and \(\psi(P_{\mathcal{HC}(f,g,h)}) = P_{\mathcal{HC}(f',g',h')}\). If the two herd spaces are equivalent, and \(P_{\mathcal{HC}(f,g,h)} = P_{\mathcal{HC}(f',g',h')}\) then they are said to be \textit{strongly equivalent}. If, in addition to being strongly equivalent, \(\tau\) preserves \(S\), then \(\Gamma(f, g, h)\) and \(\Gamma(f', g', h')\) are said to be \textit{herd equivalent}. Clearly, herd equivalence implies strong equivalence which in turn implies equivalence. Two flocks are said to be \textit{(herd, strongly) equivalent} if their associated herd spaces are (herd, strongly) equivalent.

\textbf{Lemma 5.1.} If in the definition of equivalent herd spaces, diagram (11) commutes, then \(\psi(P_{\mathcal{HC}(f,g,h)}) = P_{\mathcal{HC}(f',g',h')}\) if and only if \(\tau(S \cap \langle f, g, h \rangle) = S \cap \langle f', g', h' \rangle\). In particular, if \(\tau = \text{id}\) then the condition \(\psi(P_{\mathcal{HC}(f,g,h)}) = P_{\mathcal{HC}(f',g',h')}\) is superfluous.
Proof. Let \( T = \mathcal{P}_{\mathcal{HC}}(f,g,h) \) and \( T' = \mathcal{P}_{\mathcal{HC}}(f',g',h') \). By definition, \( \phi_{f,g,h}(T) = S \cap \langle f, g, h \rangle \) and \( \phi'_{f',g',h'}(T') = S \cap \langle f', g', h' \rangle \). Commutativity of diagram (11) gives \( \tau(\phi_{f,g,h}(T)) = \phi'_{f',g',h'}(\psi(T)) \). The RHS = \( S \cap \langle f', g', h' \rangle \) if and only if \( \psi(T) = T' \). \( \square \)

While it is true that star flocks are not generally equivalent, we do have:

**Proposition 5.2.** All linear flocks are equivalent.

Proof. Let \( \mathcal{F}(f,g,h) \) and \( \mathcal{F}(f',g',h') \) be linear flocks. \( \langle f, g, h \rangle \) and \( \langle f', g', h' \rangle \) are points of \( \mathbb{P}(\mathcal{Z}) \) and since the automorphism group of \( \mathbb{P}(\mathcal{Z}) \) acts transitively on its points, we can find a \( \tau \) in this group with \( \tau(\langle f, g, h \rangle) = \langle f', g', h' \rangle \). The kernels of \( \Gamma(f,g,h) \) and \( \Gamma(f',g',h') \) are lines \( \ell \) and \( \ell' \) of \( \mathbb{P}(\mathcal{U}) \) respectively. Since the automorphism group of \( \mathbb{P}(\mathcal{U}) \) acts transitively on its lines, we can find an automorphism \( \psi \) of \( \mathbb{P}(\mathcal{U}) \) with \( \psi(\ell) = \ell' \). Since the point sets of the herd covers of these two linear herd spaces are the complements, in \( \mathbb{P}(\mathcal{U}) \), of these kernels we have \( \psi(\mathcal{P}_{\mathcal{HC}}(f,g,h)) = \mathcal{P}_{\mathcal{HC}}(f',g',h') \). Commutativity of diagram (11) with this choice of \( \psi \) and \( \tau \) is obvious. \( \square \)

**Proposition 5.3.** For any non-zero constants \( A, B \) and \( C \) the flocks \( \mathcal{F}(f,g,h) \) and \( \mathcal{F}(Af,Bg,Ch) \) are equivalent. If \( A = B = C \) then the corresponding herd spaces are herd equivalent (in fact, equal).

Proof. Since \( \langle f, g, h \rangle = \langle Af, Bg, Ch \rangle \) we may take \( \tau = id \). Let \( \psi \) be the homography defined by \( \psi((x_0, x_1, x_2)) = (\frac{x_0}{A}, \frac{x_1}{B}, \frac{x_2}{C}) \). Consider the point \( P = \langle a, b, c \rangle \) in \( \mathbb{P}(\mathcal{U}) \), \( \tau(\phi_{f,g,h}(P)) = (af + bg + ch) \) and \( \phi_{Af,Bg,Ch}(\psi(P)) = (\frac{a}{A}(Af) + \frac{b}{B}(Bg) + \frac{c}{C}(Ch)) = (af + bg + ch) \), so diagram (11) commutes. The result now follows from Lemma 5.1. If \( A = B = C \) then \( \psi \) is just the identity map of \( \mathbb{P}(\mathcal{U}) \). \( \square \)

The last result is a special case of the following:

**Proposition 5.4.** If \( \mathcal{F} = \mathcal{F}(f,g,h) \) is a non-star flock and \( \{f',g',h'\} \) is any basis of \( \langle f, g, h \rangle \) then \( \mathcal{F}(f',g',h') \) is equivalent to \( \mathcal{F} \).

Proof. Since \( \langle f, g, h \rangle = \langle f', g', h' \rangle \) we may take \( \tau = id \). Let \( \psi \) be the homography of \( \mathbb{P}(\mathcal{U}) \) given by the change of basis matrix for the ordered basis \( \{f',g',h'\} \) to the ordered basis \( \{f,g,h\} \) acting on points. A straight-forward calculation shows that diagram (11) commutes, and the result follows from Lemma 5.1. \( \square \)

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An analogous result for star flocks will be given in the next section.

**Proposition 5.5.** Let $\mathcal{F}$ and $\mathcal{F}'$ be two flocks with the same critical cone $\mathcal{K}$ in $\mathrm{PG}(3,q)$ whose carrier contains at least 3 non-collinear points. If there is a collineation of $\mathrm{PG}(3,q)$ stabilizing $\mathcal{K}$ and mapping $\mathcal{F}$ to $\mathcal{F}'$ then $\mathcal{F}$ and $\mathcal{F}'$ are strongly equivalent.

**Proof.** We will provide a proof only for the case that $\mathcal{F}$ and $\mathcal{F}'$ are non-star flocks and leave the star flock case to the reader.

Let $\mathcal{F} = \mathcal{F}(f,g,h)$ and $\mathcal{F}' = \mathcal{F}(f',g',h')$ and suppose that $\sigma$ is a collineation of $\mathrm{PG}(3,q)$ stabilizing $\mathcal{K}$ and mapping $\mathcal{F}$ to $\mathcal{F}'$. $\sigma$ induces a map that maps $f \mapsto f'$, $g \mapsto g'$ and $h \mapsto h'$. Since $\mathcal{F}$ is a non-star flock, $f,g$ and $h$ are linearly independent vectors of $\mathcal{Z}$ and this set can be extended to an ordered basis $\mathcal{B} = \{\mu_0 = f, \mu_1 = g, \mu_2 = h, \mu_3, \ldots, \mu_{q-2}\}$ of $\mathcal{Z}$. Similarly, we can construct an ordered basis $\mathcal{B}' = \{\mu'_0 = f', \mu'_1 = g', \mu'_2 = h', \mu'_3, \ldots, \mu'_{q-2}\}$. The linear extension of the map given by $\mu_i \mapsto \mu'_i$, $0 \leq i \leq q - 2$, is a vector space automorphism. If we let $\tau$ be the collineation of $\mathcal{P}(\mathcal{Z})$ induced by this automorphism, it is clear that $\tau(\langle f,g,h \rangle) = \langle f',g',h' \rangle$. Since $\tau(\langle af + bg + ch \rangle) = \langle a'f' + b'g' + c'h' \rangle$ for all $a,b,c \in GF(q)$, if we take $\psi = id$ then diagram (11) commutes. As $\mathcal{F}$ and $\mathcal{F}'$ have the same critical cone, the point sets of the herd covers of their herd spaces are identical and so, preserved by $\psi$. Thus, $\mathcal{F}$ and $\mathcal{F}'$ are strongly equivalent. \hfill $\square$

Let $\mathcal{F} = \{\pi_t \mid t \in GF(q)\}$ be a flock of planes. A reparameterization of $\mathcal{F}$ is a reassignment of the elements of $GF(q)$ to the planes of $\mathcal{F}$, with 0 still assigned to the plane $x_3 = 0$. If after a reparameterization we have $\mathcal{F} = \{\pi_s \mid s \in GF(q)\}$ then there exists a permutation $r$ of $GF(q)$ so that $\pi_{rt} = \pi_{r(s) - r(0)}$. In terms of coordinate functions, if $\mathcal{F} = \mathcal{F}(f,g,h)$ then after this reparameterization we would have $\mathcal{F} = \mathcal{F}(f',g',h')$ where $f'(s) = f(r(s) - r(0))$, $g'(s) = g(r(s) - r(0))$ and $h'(s) = h(r(s) - r(0))$. By changing the parameterization we obtain a different set of coordinate functions, but we haven’t changed the planes, so the corresponding herd spaces should be equivalent in a very strong sense. This is the content of the next proposition.

**Proposition 5.6.** The herd space of a flock and the herd space of any reparameterization of that flock are herd equivalent.

**Proof.** Let $\mathcal{F}(f',g',h')$ be a reparameterization of $\mathcal{F}(f,g,h)$ by the permutation $r$ of $GF(q)$. That is, $f'(s) = f(r(s) - r(0))$, $g'(s) = g(r(s) - r(0))$
and \( h'(s) = h(r(s) - r(0)) \). Now, \( p(s) = r(s) - r(0) \) for \( s \in GF(q) \) is also a permutation of \( GF(q) \). Define \( \tau: \mathbf{P}(\mathcal{Z}) \to \mathbf{P}(\mathcal{Z}) \) by \( \tau((F)) = (F \circ p) \). This is easily seen to be well defined and obviously, \( F \circ p(0) = 0 \). \( \tau \) is a collineation of \( \mathbf{P}(\mathcal{Z}) \) and since the composition of permutations is a permutation, \( \tau(\mathcal{S}) = \mathcal{S} \). Clearly, \( \tau((f, g, h)) = (f', g', h') \). Let \( \psi = id \). Since \( \tau((af(t)+bg(t)+ch(t))) = (af(p(s))+bg(p(s))+ch(p(s))) = \hat{\phi}_{f', g', h'}((a, b, c)) \), diagram (I) commutes. 

In order to compare our results with the extant literature concerning flocks of quadratic (and other) cones, the following definition will be useful. Let \( \mathcal{C} \) be a subset of \( \mathcal{P}_{\mathcal{HC}(f, g, h)} \) of the herd space \( \Gamma(f, g, h) \). The herd spaces \( \Gamma(f, g, h) \) and \( \Gamma(f', g', h') \) are defined to be \( \mathcal{C} - \)equivalent if in the definition of equivalent herd spaces the condition \( \psi(\mathcal{P}_{\mathcal{HC}(f, g, h)}) = \mathcal{P}_{\mathcal{HC}(f', g', h')} \) is replaced by \( \psi(\mathcal{C}) \subseteq \mathcal{P}_{\mathcal{HC}(f', g', h')} \). These herd spaces are said to be strongly \( \mathcal{C} - \)equivalent if they are \( \mathcal{C} - \)equivalent and \( \psi(\mathcal{C}) = \mathcal{C} \). We refer to two flocks as being (strongly) \( \mathcal{C} - \)equivalent if their herd spaces are (strongly) \( \mathcal{C} - \)equivalent. Note that equivalent herd spaces are \( \mathcal{C} - \)equivalent for any appropriate set \( \mathcal{C} \), but the converse need not be true. Consider the following example of \( \mathcal{C} - \)equivalence.

**Example 5.7.** The flocks \( \mathcal{F}(t, 5t^3, 5t^5) \) and \( \mathcal{F}(t + 7t^3 + 3t^5, 5t^3 + 14t^5, 5t^5) \) of \( PG(3, 17) \) are equivalent, but not strongly equivalent flocks. The equivalence is given by the collineations \( \psi(x_0, x_1, x_2) = (x_0, 2x_0 + x_1, 4x_0 + 4x_1 + x_2) \) and \( \tau = id \). \( \mathcal{P}_{\mathcal{HC}(t, 5t^3, 5t^5)} \) consists of the conic \( \mathcal{C}: x_0x_2 = x_1^2 \) together with the point \( (0, 1, 0) \), while \( \mathcal{P}_{\mathcal{HC}(t + 7t^3 + 3t^5, 5t^3 + 14t^5, 5t^5)} \) consists of the conic \( \mathcal{C}: x_0x_2 = x_1^2 \) together with the point \( (0, 13, 1) \). Since these point sets are not equal, the flocks are not strongly equivalent. However, since \( \psi \) stabilizes the conic \( \mathcal{C} \), these two flocks are strongly \( \mathcal{C} - \)equivalent.

**Proposition 5.8.** Let \( \mathcal{C} \) be a set of points in the plane \( x_3 = 0 \) of \( PG(3, q) \) whose automorphism group acts transitively on the lines of \( x_3 = 0 \) which do not meet \( \mathcal{C} \). Then, all linear flocks of the cone of \( PG(3, q) \) with carrier \( \mathcal{C} \) are strongly \( \mathcal{C} - \)equivalent.

**Proof.** Let \( \mathcal{F}(f, g, h) \) and \( \mathcal{F}(f', g', h') \) be linear flocks of the cone with carrier \( \mathcal{C} \). \( (f, g, h) \) and \( (f', g', h') \) are points of \( \mathbf{P}(\mathcal{Z}) \) and since the automorphism group of \( \mathbf{P}(\mathcal{Z}) \) acts transitively on its points, we can find a \( \tau \) in this group with \( \tau((f, g, h)) = (f', g', h') \). We identify the plane \( x_3 = 0 \) with \( \mathbf{P}(\mathcal{U}) \) in the usual manner. The kernels of \( \Gamma(f, g, h) \) and \( \Gamma(f', g', h') \) are lines \( \ell \) and \( \ell' \) respectively, of \( \mathbf{P}(\mathcal{U}) \) which do not intersect \( \mathcal{C} \). By the hypothesis we can find
a \psi in the automorphism group of \mathcal{C} so that \psi(\ell) = \ell'. Commutativity of diagram (11) is obvious. Since \psi(\mathcal{C}) = \mathcal{C} \subseteq P_{HC(f',g',h')}, we see that \mathcal{F}(f, g, h) and \mathcal{F}(f', g', h') are strongly \mathcal{C}-equivalent. \hfill \Box

In most of the applications, \mathcal{C} will be a conic or other oval of \(x_3 = 0\). When \mathcal{C} is a conic, strong \mathcal{C}-equivalence coincides with the concept of flock equivalence found in the literature on the quadratic cone case. The proposition above is thus a generalization of the well known result that all linear flocks of quadratic cones are “equivalent”, while Proposition 5.2 is not. From a geometrical point of view, all linear flocks should be equivalent since they are structurally the same, but this will not be the case for non-quadratic cones, in general, if the notion of equivalence requires that the cone be stabilized. This consideration has led us to the more general notion of equivalence that we have adopted.

6 Star Flocks

We can easily characterize star flocks in terms of their herd spaces. A more detailed examination of star flocks can be found in [3].

**Proposition 6.1.** A flock \(\mathcal{F}\) is a star flock if, and only if, its herd space contains the constant function. A herd space is a star herd space if and only if its associated flock is a star flock.

**Proof.** Let \(\mathcal{F}\) be a star flock with the point \(Q\) common to all the planes of \(\mathcal{F}\). Since \(x_3 = 0\) is in the flock, \(Q\) is in this plane. Clearly, \(f_Q(t) = 0, \forall t \in GF(q)\), so the herd space of \(\mathcal{F}\) contains a constant function (i.e., \((Q, \langle 0 \rangle)\)). On the other hand, if the herd space of \(\mathcal{F}\) contains a constant function, it must be the zero function. The point associated with the zero function lies in all planes of the flock, and so, the flock is a star flock. Since the kernel of \(\hat{\phi}\) is not trivial, the herd space in this case is a star herd space. \hfill \Box

**Proposition 6.2.** A flock \(\mathcal{F}\) is a linear flock if and only if its herd space contains at least two constant functions. In this case, the herd space is a linear herd space.

**Proof.** Suppose that the herd space of flock \(\mathcal{F}\) contains \((P, \langle 0 \rangle)\) and \((Q, \langle 0 \rangle)\) with \(P \neq Q\). Then by Proposition 6.1 both \(P\) and \(Q\), and hence the entire line \(PQ\), lie in all planes of \(\mathcal{F}\). The converse is clear. The kernel of \(\hat{\phi}\) has dimension 1, and the herd space is a linear herd space. \hfill \Box
Proposition 6.3. Any star flock is equivalent to one of the form $F(f, \langle 0 \rangle, h)$. Any linear flock is equivalent to one of the form $F(f, \langle 0 \rangle, \langle 0 \rangle)$.

Proof. Consider the flock $F = F(f, \langle 0 \rangle, \langle 0 \rangle)$ where $f$ is a non-constant function. By Proposition 6.2, $F$ is a linear flock. If $F'$ is any linear flock, then $F'$ is equivalent to $F$ by Proposition 5.2.

Now, suppose that $F = F(f, g, h)$ is a proper star flock. $\langle f, g, h \rangle$ is a line of $P(Z)$ and there exist constants $a, b$ and $c$, not all zero, so that $af + bg + ch = 0$. The kernel of $\Gamma(f, g, h)$ is the point $Q = \langle a, b, c \rangle$ in $P(U)$. Consider the points $P = \langle 1, 0, 0 \rangle$ and $R = \langle 0, 0, 1 \rangle$ of $P(U)$. If $Q$ is not on the line $PR$, i.e., $b \neq 0$, then the homography $\psi_1 \in PGL(3, q)$ given by the matrix,

$$
\begin{pmatrix}
1 & 0 & 0 \\
-a & 1 & -c \\
0 & 0 & 1
\end{pmatrix},
$$

acting on points, fixes $P$ and $R$ and $\psi_1(Q) = \langle 0, 1, 0 \rangle$. Taking $\tau = id$ and $\psi = \psi_1$ shows that the flock $F$ is equivalent to $F(f, \langle 0 \rangle, h)$. If $Q$ is on the line $PR$ but not equal to $P$ (thus, $b = 0, c \neq 0$), then the homography $\psi_2 \in PGL(3, q)$ given by the matrix,

$$
\begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
-a & 1 & 0
\end{pmatrix},
$$

acting on points, fixes $P$, $\psi_2(Q) = \langle 0, 1, 0 \rangle$ and $\psi_2(\langle 0, 1, 0 \rangle) = R$. Taking $\tau = id$ and $\psi = \psi_2$ shows that the flock $F$ is equivalent to $F(f, \langle 0 \rangle, g)$. Finally, if $Q = P$, then the homography $\psi_3 \in PGL(3, q)$ given by the matrix,

$$
\begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix},
$$

acting on points, fixes $R$, $\psi_3(Q) = \langle 0, 1, 0 \rangle$ and $\psi_3(\langle 0, 1, 0 \rangle) = P$. Taking $\tau = id$ and $\psi = \psi_3$ shows that the flock $F$ is equivalent to $F(g, \langle 0 \rangle, h)$. □

We can also characterize star flocks in terms of other functions in their herd spaces.

Proposition 6.4. A flock $F$ is a star flock if and only if the function classes associated to two distinct points in its herd space are equal, i.e., the associated functions are scalar multiples.
Proof. The herd space of the flock \( F \) contains \((P, \langle f \rangle)\) and \((Q, \langle f \rangle)\) with \( P \neq Q \) if and only if the herd space is not a bijection. The herd space is a star herd space and so, its kernel is not empty. The statement now follows from Proposition 6.1.

Proposition 6.5. A flock \( F \) is a linear flock if and only if there exist three non-collinear points in its herd space whose associated function classes are non-constant and equal.

Proof. If \( F \) is a linear flock then all points of the herd cover of its herd space are associated to the same permutation. Choose any three non-collinear points of the affine plane which is the point set of the herd cover to satisfy the condition. On the other hand, suppose that \((P, \langle f \rangle), (S, \langle f \rangle)\) and \((R, \langle f \rangle)\) are in the herd space of \( F \) with \( P, S \) and \( R \) distinct non-collinear points of \( P(U) \) and \( f \) a non-constant function. By Proposition 6.4, \( F \) is a star flock. The restriction of \( \hat{\phi} \) to the line \( PS \) is not a bijection, so this restriction has a nontrivial kernel. Thus, there is a point \( Q_1 \) on \( PS \) which is in the kernel of \( \hat{\phi} \). Similarly, there is a point \( Q_2 \) on the line \( PR \) in the kernel of \( \hat{\phi} \). If \( Q_1 = Q_2 \) then they would be the point \( P \) contradicting the fact that \( P \) is not in the kernel since \( f \) is non-constant. Thus, there are at least two constant functions in the herd space and the result follows from Proposition 6.2.

7 Some Non-Star Flocks

In this section we will present some examples of the herd spaces of various non-star flocks. Only a selected few are examined to illustrate the techniques and ideas concerning herds and herd spaces. To determine the herd covers of these herd spaces, we appeal to Dickson [8] for the required information on permutation polynomials. He has determined all permutation polynomials of degree \( \leq 5 \) and we rely heavily on this classification.

Consider the herd space \( \Gamma(t, t^2, t^3) \). The functions of this herd space are of the forms: \( t^3 + at^2 + bt \), \( at^2 + t \) and \( t^2 \) where \( a, b \in GF(q) \). \( t^2 \) is a permutation polynomial if, and only if \( q = 2^e \). That is to say, \((0, 1, 0) \in P_{HC} \iff q = 2^e \). \( at^2 + t \) is a permutation polynomial if \( a = 0 \), in which case it is a permutation polynomial for all \( q \). Finally, there are two cases for which \( t^3 + at^2 + bt \) is a permutation polynomial. The first case occurs when \( q \equiv -1 \mod 3 \) and \( a = 3c, b = 3c^2 (\forall c \in GF(q)) \), and the second when \( q = 3^e \) and \( a = 0, b = -n \) for \( n \) a non-square in \( GF(q) \). We summarize the possibilities for \( P_{HC} \) of
\[ \begin{array}{|c|c|} \hline q & \mathbf{P}_{HC} \\ \hline q \equiv 1 \mod 3, \; q \text{ odd} & (1, 0, 0) \\ q \equiv 1 \mod 3, \; q = 2^{2k} & (1, 0, 0), (0, 1, 0) \\ q \equiv -1 \mod 3, \; q \text{ odd} & \text{conic, } 3x_0x_2 = x_1^2 \\ q \equiv -1 \mod 3, \; q = 2^{2k+1} & \text{hyperconic, } x_0x_2 = x_1^3 \cup (0, 1, 0) \\ q = 3^e & (1, 0, 0), (0, 0, 1), \{(-n, 0, 1) \mid n \text{ a non-square} \} \\ \hline \end{array} \]

Table 1: \( \mathbf{P}_{HC} \) of \( \mathcal{H}(t, t^2, t^3) \), \( q \geq 5 \)

\( \mathcal{H}(t, t^2, t^3) \) for all \( q \geq 5 \) in Table 1 For completeness, note that all flocks with \( q < 5 \) having more than two points in the carriers of their critical cones are star flocks, as will be shown in the next section.

We illustrate a few herds of the flock \( \mathcal{F}(t, t^2, t^3) \), in the case that \( q \equiv -1 \mod 3, \; q \) odd . The standard herd is \( \{ f_{(1,0,0,0)}(t) = t \} \cup \{ f_{(3c^2,3c,1,0)}(t) = 3c^2t + 3ct^2 + t^3 \mid c \in GF(q) \} \), while the alternate standard herd is given by \( \{ f_{(1,0,0,0)}(t) = t \} \cup \{ f_{(3c^2,3c,1,0)}(t) = t + \frac{1}{c} t^2 + \frac{1}{3c^2} t^3 \mid c \in GF(q)^* \} \cup \{ f_{(0,0,1,0)}(t) = t^3 \} \). The normalized herd in this case is \( \{ f_{(1,0,0,0)}(t) = t \} \cup \{ f_{(3c^2,3c,1,0)}(t) = \frac{3c^2t + 3ct^2 + t^3}{3c^2 + 3c + 1} \mid c \in GF(q) \} \).

Continuing with this example, we note that the homography of \( PG(3, q) \), \( q \neq 3^e \) represented by the matrix,

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 3 & 0 & 0 \\
0 & 0 & 3 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]

acting on planes (on the left), will map the flock \( \mathcal{F}(t, t^2, t^3) \) to the flock \( \mathcal{F}(t, 3t^2, 3t^3) \) and the conic \( 3x_0x_2 = x_1^2 \) to the conic \( x_0x_2 = x_1^2 \) in the plane \( x_3 = 0 \) (which is stabilized by this collineation). Thus, for \( q \equiv -1 \mod 3 \), \( \mathcal{F}(t, t^2, t^3) \) is equivalent (Proposition 5.3) to the Fisher-Thas-Walker (FTW) flock of a quadratic cone as represented in [11]. If \( q = 2^{2k+1} \), for \( k \geq 1 \), then \( t \mapsto t^k \) is a permutation of \( GF(q) \) which fixes 0. If we reparameterize the planes in the the flock \( \mathcal{F}(t, t^2, t^3) \) using this permutation, we obtain the herd equivalent flock \( \mathcal{F}(t^2, t^3, t^4) \) (Proposition 5.6). The herd space of this flock is herd equivalent to the original herd space, and so, has the same \( \mathbf{P}_{HC} \). The normalized herd of this flock is therefore, \( \{ f_{(1,0,0,0)}(t) = t^k, f_{(0,1,0,0)}(t) = t^k \} \cup \{ f_{(3c^2,3c,1,0)}(t) = \frac{c^2t^k + c^2t^2 + c^2t}{c^2 + c + 1} \mid c \in GF(q) \} \). Observe that all the functions...
in the normalized herd are o-polynomials. This herd (without $f_{(0,1,0,0)}$, and with a different indexing) is called a **herd of ovals** in [4].

We now consider the more complex herd space of the flock $\mathcal{F}(t, t^3, t^5)$. The functions of this herd space are of the forms: $t^5 + at^3 + bt$, $at^3 + t$ and $t^3$ where $a, b \in GF(q)$. Using Dickson [8], we may again list all possible herd covers of this herd space. For $q \leq 5$, this flock is a star flock. The possibilities for $\mathbf{P}_{\mathcal{HC}}$ of $\mathcal{HC}(t, t^3, t^5)$ for all $q \geq 7$ are summarized in Table 2.

| $q$ | $\mathbf{P}_{\mathcal{HC}}$ |
|-----|--------------------------------|
| $q \equiv 1 \pmod{15}$ | $(1,0,0)$ |
| $q \equiv 2, 8 \pmod{15}$ | $(0,1,0)$, **conic** |
| $q \equiv 3, 12 \pmod{15}, q = 3^{2k+1}$ | $(0,1,0)$, **conic**, **collinear set**$_1$ |
| $q \equiv 4 \pmod{15}$ | $(1,0,0), (0,0,1)$ |
| $q \equiv 5 \pmod{15}, q = 5^{2k+1}$ | $(0,1,0)$, **collinear set**$_2$, **partial conic** |
| $q \equiv 6 \pmod{15}, q = 3^{2k}$ | $(0,1,0)$, **collinear set**$_1$ |
| $q \equiv 7, 13 \pmod{15}$ | **conic** |
| $q \equiv 9 \pmod{15}, q = 3^{2k+2}$ | $(1,0,0), (0,1,0), (0,0,1)$, **collinear set**$_1$ |
| $q \equiv 10 \pmod{15}, q = 5^{2k}$ | **collinear set**$_2$, **partial conic** |
| $q \equiv 11 \pmod{15}$ | $(1,0,0), (0,1,0)$ |
| $q \equiv 14 \pmod{15}$ | $(1,0,0), (0,1,0), (0,0,1)$ |

**Table 2**: $\mathbf{P}_{\mathcal{HC}}$ of $\mathcal{HC}(t, t^3, t^5)$, $q \geq 7$

As we see from the table, the only cases in which $\mathbf{P}_{\mathcal{HC}}$ contains a conic occur when $q \equiv 2, 3, 7, 8, 12, 13 \pmod{15}$ which simplifies to $q \equiv \pm 2 \pmod{5}$. These cases give rise to flocks of quadratic cones. For $q$ odd, these are equivalent to the Kantor K2 flocks [13] and for $q$ even (more precisely, $q = 2^{2k+1}$, occurring when $q$ is even and $q \equiv 2, 8 \pmod{15}$) they are known as the Payne [12] flocks. As the description of this flock is characteristic-free, we would prefer to call this flock the Kantor-Payne flock (as has been done elsewhere in the literature).

Another example, illustrating a flock whose coordinate functions are not all monomial, can be obtained by considering the function $f(x) = x^5 + 2nx^3 +$
where $n$ is a non-square in $GF(q)$. Dickson [8] has shown that this is a permutation polynomial if and only if $q = 5^e$. This implies that over $GF(5^e)$,
\[ f(x + a) - (2na^3 + n^2a) = x^5 + 2nx^3 + 6nax^2 + (6na^2 + n^2)x \]
is a permutation polynomial $\forall a \in GF(5^e)$. Consider the flock $\mathcal{F}(t, t^2, t^3 + 2nt^4)$. For $q = 5^e$, $\mathcal{P}_{HC}$ of this flock consists of \{(0, 0, 0)\} $\cup$ \{(na^2 + n^2, na, 1, 0) $|$ $a \in GF(5^e)$\}, in other words, the points of the conic $nx_0x_2 = x_1^2 + n^3x_2^2$. These flocks are equivalent to the K3 or “Kantor likeable” flocks due to Gevaert and Johnson [10].

Consider the special case of $q = 5$. Since $x^5 = x$ over $GF(5)$, the permutation polynomial reduces to $x^3 + 3ax^2 + (3a^2 + 3n + 1)x$. This in turn implies that the Kantor likeable flocks in $PG(3, 5)$ are equivalent to the FTW flock, a point that is implied, but never stated in [6].

Our last example is restricted to even characteristic. Let $q = 2^e$ and chose $i < e$ so that $(2^i + 1, q - 1) = 1$. In this case, $f(x) = x^{2^i + 1}$ is a permutation polynomial over $GF(2^e)$. Therefore, $f(x + a) - a^{2^i + 1} = x^{2^i + 1} + ax^{2^i} + a^2x$ is a permutation polynomial over $GF(q)$, $\forall a \in GF(q)$. The $\mathcal{P}_{HC}$ of the flock $\mathcal{F}(t, t^{2^i}, t^{2^i + 1})$ contains the points of $nx_0x_2 = x_1^{2^i}$. If $(i, e) = 1$ this curve is a translation oval and $\mathcal{F}(t, t^{2^i}, t^{2^i + 1})$ is an $\alpha$-flock [1]. This situation arises only when $e$ is odd. This is an alternate (and simpler) proof of Theorem 5 in [1], whose proof simplified that of a result of Fisher and Thas [9]. When $(i, e) > 1$ we obtain flocks of non-oval cones of a type that we have called $\beta$-flocks [2].

8 The Classification of Flocks for $q \leq 7$

All flocks of arbitrary cones can be easily determined for small $q$, by examining herd spaces. Since there exist cones which do not admit any flock [3], all references to cones in this section are to non-empty cones which admit at least one flock. The material in this section should be compared to [13] and [7] where the flocks of quadratic cones for $q$ in this range are determined by other methods.

Remark: We are not attempting a complete classification of these flocks, although the methods used here could be used to do so. Rather, we are trying to classify the flocks of “interesting” cones, that is, cones whose carriers are not too small and/or contained in just a few lines. While it is possible to define “interesting”, such a definition is bound to be arbitrary in nature and so we will not do so here. Also, we will not try to refine the classification of
the star flocks other than to indicate when they are linear. Star flocks are examined in more detail in \cite{3}.

8.1 \( q = 2 \)

Since any two distinct planes of \( PG(3, q) \) meet in a line, a flock of any cone in \( PG(3, 2) \) is a linear flock.

8.2 \( q = 3 \)

\( P(\mathbb{Z}) \) is isomorphic to \( PG(1, 3) \), the projective line, and \( S \) is a point of this line. Obviously, all flocks in \( PG(3, 3) \) are star flocks. The point set of the herd cover of any proper star herd space consists of just one line by Proposition \cite{3.2}. Therefore, if the carrier of a cone contains at least three non-collinear points, a flock of the cone must be linear. Thus the flocks of all quadratic cones in \( PG(3, 3) \) are linear.

8.3 \( q = 4 \)

\( P(\mathbb{Z}) \) is isomorphic to \( PG(2, 4) \), the projective plane of order 4, and \( S \) is a pair of points in this plane. If a cone has more than two points in its carrier, then it can only admit star flocks. A herd cover of a proper star herd space can contain at most two lines, so if the carrier of a cone contains a 5-arc, the cone can admit only linear flocks. The only non-star flock is the FTW flock, but its critical cone is flat. We again have that the flocks of all quadratic cones in \( PG(3, 4) \) are linear.

8.4 \( q = 5 \)

\( P(\mathbb{Z}) \) is isomorphic to \( PG(3, 5) \) and \( S \) consists of 6 points which lie in a hyperplane, i.e., they are coplanar in this space. The degree of a permutation polynomial over \( GF(5) \) can only be 1 or 3. The six permutation points are \( \langle t \rangle \) and the five points \( \langle 3at + 3a^2t^2 + t^3 \rangle \) for \( a \in GF(5) \). The point set of the herd cover is the conic \( 2x_0x_2 = x_1^2 \) in the herd space \( \Gamma(t, t^2, t^3) \). Any other plane of \( P(\mathbb{Z}) \) can contain at most two permutation points. Thus, the only non-star flock of a cone of \( PG(3, 5) \) whose carrier contains at least three points is the FTW flock (up to equivalence). There are at most two lines in the point set of the herd cover of any proper star herd space. A star flock
of a cone in $PG(3,5)$ whose carrier contains a 5-arc must be a linear flock. The flocks of quadratic cones are either linear or FTW.

8.5 $q = 7$

In this case $P(Z)$ is isomorphic to $PG(5,7)$ and contains $5! = 120$ permutation points. The permutation polynomials over $GF(7)$ can only have degrees of 5, 4 or 1. Using Dickson’s list, we can explicitly exhibit all the permutation polynomials over $GF(7)$. This is done in Table 3.

As points of $P(Z)$ these 120 permutation points lie in the hyperplane $P(Z_1) = \langle t, t^2, t^3, t^4, t^5 \rangle$. Any plane of $P(Z)$ which does not lie in this hyperplane must meet the hyperplane in a line. The permutation points of the non-star herd spaces that correspond to these planes must all lie on a line, and so, such flocks can only have flat cones as critical cones. A non-star flock of a non-flat cone would therefore correspond to a plane which lies in $P(Z_1)$. A simple calculation shows that all the permutation points lie on the quadric $Q(4,7)$ of $P(Z_1)$ given by $3x_2^2 = x_0x_4 + x_1x_3$, $x_5 = 0$. At each point of the hyperbolic quadric, there are 8 generator lines which form a quadratic cone that lies in a unique 3-space whose only intersection with the hyperbolic quadric is this cone. For example, at the point $\langle t \rangle$ on the quadric, the type $V$ points lie on seven generators of such a cone with vertex $\langle t \rangle$, and the 3-space containing this cone is $\langle t, t^2, t^3, t^4 \rangle$. The type II points lie in seven planes, $\pi_a : \langle t, (t+a)^3 - a^3, (t+a)^5 - a^5 \rangle, a \in GF(q)$, the points in each plane together with $\langle t \rangle$ forming a conic. The remaining points come in plus/minus pairs, and each such pair is collinear with a unique point of type II (those with $b = 0$ or $b$ a non-square in $GF(q)$). These lines are generator lines of

| Type | Num. | Permutation Polynomial |
|------|------|------------------------|
| I    | 1    | $t$                    |
| II   | 49   | $(5a^4 + 3ab + 3b^2)t + (3a^3 + 3ab)t^2 + (3a^2 + b)t^3 + 5at^4 + t^5$, $\forall a, b \in GF(7)$ |
| III  | 52   | $(5a^4 + 3a^2n \pm 2a + 3n^2)t + (3a^4 + 3an \pm 1)t^2 + (3a^2 + n)t^3 + 5at^4 + t^5$, $\forall a \in GF(7), n$ a nonsquare |
| IV   | 14   | $(5a^4 \pm 4a)t + (3a^3 \pm 2)t^2 + 3a^2t^3 + 5at^4 + t^5$, $\forall a \in GF(7)$ |
| V    | 14   | $(4a^3 \pm 3)t + 6a^2t^2 + 4at^3 + t^4$, $\forall a \in GF(7)$ |

Table 3: Representative Permutation Polynomials over $GF(7)$
the hyperbolic quadric (lying completely in the quadric). To determine all possible non-star flocks of non-empty cones we need only examine the planes of $P(Z_1)$ which contain $\langle t \rangle$. There are only four types of planes, with respect to the hyperbolic quadric, through a point of the quadric. Namely, planes whose intersection with the hyperbolic quadric consists of precisely 1 generator, 2 generators, a conic or a single point. Thus, the herd cover of any non-star flock that is not contained in two lines must be an arc which lies on a conic. A simple computer-aided calculation shows that the only planes through $\langle t \rangle$ which contain at least seven points are the $\pi_a$. The flocks that correspond to these planes are all projectively equivalent, as the following calculation shows:

$$
\begin{bmatrix}
1 & 0 & 0 & -a \\
0 & 1 & 0 & -a^3 \\
0 & 0 & 1 & -a^5 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
t \\
t^3 \\
t^5 \\
1
\end{bmatrix}
= 
\begin{bmatrix}
t - a \\
t^3 - a^3 \\
t^5 - a^5 \\
1
\end{bmatrix}
= 
\begin{bmatrix}
s \\
(s + a)^3 - a^3 \\
(s + a)^5 - a^5 \\
1
\end{bmatrix},
$$

where the last step is just a reparameterization of the flock by $s = t - a$. We can therefore conclude that the only non-star flocks of quadratic cones in $PG(3, 7)$ are equivalent to $F(t, t^3, t^5)$, the Kantor-Payne Flock.

A line can intersect $S$ in at most 3 points (a generator of the hyperbolic quadric). So, there are at most three lines in the point set of the herd cover of any proper star herd space. A star flock of a cone in $PG(3, 7)$ whose carrier contains a 7-arc must be a linear flock. The flocks of quadratic cones are either linear or Kantor-Payne.

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