Dark Energy from the Fifth Dimension

A. Stern$^1$ and Chuang Xu$^2$

Department of Physics, University of Alabama,
Tuscaloosa, Alabama 35487, USA

Abstract

After extending the Regge-Teitelboim formulation of gravity to include the case where the background embedding space is not flat, we examine the dynamics of the four-dimensional $k = 0$ Robertson-Walker (RW) manifold embedded in various five-dimensional backgrounds. We find that when the background is five-dimensional de Sitter space, the RW manifold undergoes a transition from a de-accelerating phase to an accelerating phase. This occurs before the inclusion of matter, radiation or cosmological constant sources, and thus does not require a balance of different components. We obtain a reasonable two-parameter fit of this model to the Hubble parameter data.

$^1$astern@ua.edu
$^2$cxu24@crimson.ua.edu
Regge-Teitelboim (RT) gravity is an alternative to standard general relativity where the dynamical degrees of freedom are associated with the embeddings of the space-time manifold in a fixed higher dimensional background.\[1–7\] Solutions to Einstein equations satisfy the RT field equations. More generally, the RT formulation of gravity can effectively produce source terms in the standard Einstein equations that are not attributable to the energy-momentum tensor, but rather are a result of the embedding. These RT source terms have the potential of providing an explanation for certain cosmological phenomena, such as for cosmic acceleration. This idea was entertained in \[8\], where a toy model was presented.\[9\] There it was shown that RT gravity can generate cosmic acceleration for a simple class of embeddings of the $k = -1$ Robertson-Walker (RW) metric in a flat five-dimensional background. Moreover, a transition from a de-accelerating to accelerating phase could be observed in a specific, but not physical, example. No cosmological acceleration was found for the case of the $k = 1$ RW metric, and the currently favored $k = 0$ case was not considered.

This article attempts to apply RT gravity in the direction of a more realistic model of the observed cosmological acceleration. The approach taken here relies on embedding the RW manifold in a curved background. While previous discussions of RT gravity have been restricted to flat backgrounds, the formalism can easily be extended to curved background spaces, as is demonstrated here. After developing the formalism, we then apply it to cosmology by embedding the four-dimensional RW manifold in three different five-dimensional background spaces. We specialize to $k = 0$, although the other cases can also be considered as well. The backgrounds we consider are: i) $R^{4,1}$, ii) $AdS_5$ and iii) $dS_5$. As a first approximation, we obtain the evolution of the scale factor on the RW manifold in the absence of matter, radiation or cosmological constant sources. We get that the acceleration of the scale factor is negative for all time for cases i) and ii). On the other hand, for case iii) we find that a transition from the de-accelerating phase to an accelerating phase occurs at a finite time. The evolution in this case is determined by two free parameters, the curvature of the background de Sitter space and the strength of the RT source term. The two parameters allow for a fit to the Hubble parameter data. Unlike in the $\Lambda$CDM model, neither the matter density nor the cosmological constant play a role in the fit, meaning that their contributions should be significantly weaker than the RT source term, and furthermore, that they can have arbitrary strength relative to each other. So here we are able to avoid the coincidence puzzle of the $\Lambda$CDM model, where the matter contribution to Einstein equations, coincidentally, is of the same order of magnitude as the cosmological constant contribution at the current time.

We begin with a very brief discussion of RT gravity, or more precisely, its generalization to the case where the $d$—dimensional background space $\mathbb{M}_d$, $d > 4$, is not necessarily flat. We denote a local set of coordinates on $\mathbb{M}_d$ by $Y^a$, $a, b, \cdots = 0, \ldots, d - 1$, and its
associated metric tensor metric by $g_{ab}(Y)$. Next embed a four-dimensional space-time manifold $M_4$ in $M_d$. This can be done by introducing the set of functions $Y^a = Y^a(x)$, where $x^\mu$, $\mu, \nu, \cdots = 0, \ldots, 3$, span $M_4$. The metric tensor $g_{\mu\nu}(x)$ on $M_4$ is defined to be induced from $g_{ab}(Y)$. So

$$g_{\mu\nu}(x) = g_{ab}(Y)\partial_\mu Y^a \partial_\nu Y^b ,$$

$\partial_\mu$ denoting differentiation with respect to $x^\mu$. As is usual $g_{\nu\lambda}$ is required to be invertible, and metric compatible on $M_4$, $\nabla_\mu g_{\nu\lambda} = 0$, $\nabla_\mu$ being the covariant derivative on $M_4$. The latter leads to the identity:

$$g_{ab}\nabla_\lambda \partial_\mu Y^a \partial_\nu Y^b + \frac{1}{2} \frac{\partial g_{ab}}{\partial Y^c} \left( \partial_\mu Y^a \partial_\nu Y^b \partial_\lambda Y^c + \partial_\nu Y^a \partial_\lambda Y^b \partial_\mu Y^c - \partial_\lambda Y^a \partial_\mu Y^b \partial_\nu Y^c \right) = 0$$

(2)

To derive this compute $\nabla_\lambda g_{\mu\nu} + \nabla_\mu g_{\nu\lambda} - \nabla_\nu g_{\lambda\mu}$ using [1], and apply metric compatibility and the Leibniz rule.

RT gravity assumes the usual Einstein-Hilbert action $S_{EH}$ for the gravitational field, however the dynamical degrees of freedom are the embedding functions, not $g_{\mu\nu}$. So upon including source terms $S_{\text{source}}$, one has

$$S = S_{EH} + S_{\text{source}}, \quad S_{EH} = \frac{1}{16\pi G} \int_M d^4x \sqrt{|g|} R ,$$

(3)

with the scalar curvature constructed from [1]. Field dynamics is obtained from variations of $Y^a$. This gives

$$\partial_\mu \left( \sqrt{|g|} E^{\mu\nu} g_{ab} \partial_\nu Y^b \right) - \frac{1}{2} \sqrt{|g|} E^{\mu\nu} \frac{\partial g_{bc}}{\partial Y^c} \partial_\mu Y^b \partial_\nu Y^c = 0 ,$$

$$E^{\mu\nu} = G^{\mu\nu} - 8\pi G T^{\mu\nu} ,$$

(4)

$G^{\mu\nu}$ and $T^{\mu\nu}$ being the Einstein tensor and stress-energy tensor, respectively. As in Einstein gravity, $T^{\mu\nu}$ must be covariantly conserved. To see this one can first re-write the field equations as

$$\nabla_\mu (E^{\mu\nu} g_{ab} \partial_\nu Y^b) - \frac{1}{2} E^{\mu\nu} \frac{\partial g_{bc}}{\partial Y^c} \partial_\mu Y^b \partial_\nu Y^c = 0 ,$$

(5)

and then expand the first term using the Bianchi identity to obtain

$$- 8\pi G \nabla_\mu T^{\mu\nu} g_{ab} \partial_\nu Y^b + E^{\mu\nu} \left( \nabla_\mu (g_{ab} \partial_\nu Y^b) - \frac{1}{2} \frac{\partial g_{bc}}{\partial Y^c} \partial_\mu Y^b \partial_\nu Y^c \right) = 0$$

(6)

Finally contract with $\partial_\lambda Y^a$ and apply [2] to get $\nabla_\mu T^{\mu\lambda}_\lambda = 0$. The field equations (4) are obviously satisfied for solutions to Einstein equations, $E^{\mu\nu} = 0$. More generally, $E^{\mu\nu}$ need not vanish. Alternatively, we can argue that the Einstein equations effectively pick up additional source terms, which we denote by $T^{\mu\nu}_{\text{RT}}$, which are not associated with the
standard stress-energy tensor but rather are due to the embedding in the background space,
\[ G^{\mu\nu} = 8\pi G \left( T^{\mu\nu} + T_{RT}^{\mu\nu} \right), \]
(7)

Obviously, \( T_{RT}^{\mu\nu} \) is covariantly conserved since \( T^{\mu\nu} \) is.

Next we want to apply this dynamical system to the case where the embedded manifold \( \mathcal{M}_4 \) is that of standard cosmology, i.e., it is given by the RW metric tensor. Here we will specialize to the currently favored case of \( k = 0 \)
\[ ds^2 = -dt^2 + a(t)^2 dx^i dx^i, \]
(8)

where \( t = x^0 \) and \( a(t) \) is the scale factor. As a first approximation let us consider source free RT gravity, i.e., \( T^{\mu\nu} = 0 \). From (7) we know that the Einstein tensor need not vanish. \( T_{RT}^{\mu\nu} \) in (7) needs to be computed from the particular choice of embedding, however from consistency with homogeneity and isotropy, we anticipate that its form should be analogous to that of a perfect fluid in the co-moving frame
\[ T_{RT}^{00} = \rho_{RT}, \quad T_{RT}^{11} = T_{RT}^{22} = T_{RT}^{33} = a(t)^2 p_{RT}, \]
(9)

with \( \rho_{RT} \) and \( p_{RT} \) being functions of \( t \). Since it is covariantly conserved we have
\[ \dot{\rho}_{RT} + 3 \frac{\dot{a}}{a} (\rho_{RT} + p_{RT}) = 0, \]
(10)

the dot denoting a \( t \)-derivative.

Substituting (8) into (4) gives
\[ \partial_t \left( F_1(t) g_{ab} \partial_i Y^b \right) - \frac{1}{2} F_1(t) \partial_i Y^b \partial_i Y^c \frac{\partial g_{bc}}{\partial Y^a} = F_2(t) \left( \partial_i \left( g_{ab} \partial_i Y^b \right) - \frac{1}{2} \partial_i Y^b \partial_i Y^c \frac{\partial g_{bc}}{\partial Y^a} \right), \]
(11)

where
\[ F_1(t) = 3a \dot{a}^2 \quad F_2(t) = 2\ddot{a} + \frac{\dot{a}^2}{a} \]
(12)

(11) can produce equations for \( \dot{a} \) and \( \ddot{a} \) which can be written in the form of \( k = 0 \) Friedmann equations
\[ \frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3} \rho_{RT}, \]
(13)
\[ \frac{\ddot{a}}{a} = -\frac{4\pi G}{3} (\rho_{RT} + 3p_{RT}), \]
(14)

allowing us to identify \( \rho_{RT} \) and \( p_{RT} \) in (9). The resulting expressions for \( \rho_{RT} \) and \( p_{RT} \) will in general depend on the background space and the choice of embedding, as we illustrate in the examples that follow. As stated previously, the background spaces we
consider are $R^{4,1}$, $AdS_5$ and $dS_5$. We use the same expression for the embedding in all three cases:

$$
\begin{pmatrix}
Y^0 \\
Y^1 \\
Y^2 \\
Y^3 \\
Y^4
\end{pmatrix} =
\begin{pmatrix}
b(t) \\
x^1 \\
x^2 \\
x^3 \\
h(t)
\end{pmatrix},
$$

(15)

where the functions $b(t)$ and $h(t)$ need to satisfy certain constraints in order to recover the $k = 0$ Robertson-Walker metric on the embedded four-dimensional manifold.

We next deduce $\rho_{RT}$ and $p_{RT}$ for the three different cases.

1. Flat 5-dimensional background $R^{4,1}$

A trivial system results if one chooses Cartesian coordinates for $R^{4,1}$ and maps to $\mathcal{M}_4$ using (15), as this restricts the scale factor in (8) to be one. Alternatively, a nontrivial function $a(t)$ can result from a different coordinatization on $R^{4,1}$, such as is in [10], [11] where

$$(ds^2)_{R^{4,1}} = -(dY^0)^2 + (Y^0 + Y^4)^2 \left( (dY^1)^2 + (dY^2)^2 + (dY^3)^2 \right) + (dY^4)^2$$

(16)

It can be checked that the five-dimensional curvature resulting from this metric is zero. Now using (15) to map to (8) one gets that $b(t)$ and $h(t)$ should satisfy

$$b(t) + h(t) = a(t) \quad \dot{b}^2 - \dot{h}^2 = 1$$

(17)

Now substituting (15) in (11) gives

$$\partial_t(\dot{b}F_1) = 3F_2a$$

(18)

$$\partial_t(\dot{h}F_1) = -3F_2a$$

(19)

The sum of these two equations leads to a constant of motion $\partial_t(\dot{a}F_1) = 0$, from which we get the following expression for $\rho_{RT}$

$$\rho_{RT} = \frac{c_0}{a^3 \dot{a}},$$

(20)

c_0 being a constant. The Friedmann equation (13) then gives $\ddot{a}^3 \propto \frac{1}{a}$, and so there is no acceleration as $a$ increases. One gets a simple solution for the scale factor in this case: $a(t) \propto t^{3/4}$ for $a(0) = 0$. This coincides with the time evolution of the scale factor in the presence of a perfect fluid with equation of state $p = -\frac{2}{3} \rho$. The same result was observed in [9] for a different choice of embedding.

2. $AdS_5$ background
Here we cover a patch of $AdS_5$ using Poincaré coordinates. The background metric is

$$(ds^2)_{AdS_5} = -\frac{(Y^4)^2}{L^2}(dY^0)^2 + \frac{(Y^4)^2}{L^2}\left((dY^1)^2 + (dY^2)^2 + (dY^3)^2\right) + \frac{L^2(dY^4)^2}{(Y^4)^2},$$

the constant $L$ denoting the $AdS_5$ radius of curvature. Utilizing the embedding (15), the $k = 0$ RW metric (8) is recovered provided that

$$h = L a \quad a^2 \dot{b}^2 - L^2 \frac{\dot{a}^2}{a^2} = 1$$

Substituting (15) in (11) gives

$$\partial_t(a^2 \dot{b} F_1) = 0$$

$$L^2 \partial_t \left( \frac{\dot{a}}{a^2} F_1 \right) + \left( a \dot{b}^2 + L^2 \frac{\dot{a}^2}{a^2} \right) F_1 = -3a F_2$$

From (13) and (23) we then get

$$\rho_{RT} = \frac{c_0}{a^3 \sqrt{L^2 \dot{a}^2 - \dot{a}^2}},$$

Note that the form (30) resulting from the flat background is recovered in the limit $L \to \infty$, or more precisely when $|\dot{a}| \gg \frac{1}{L}$.

3. $dS_5$ background

Using the so-called flat slicing the metric for $dS_5$ is

$$(ds^2)_{dS_5} = -(dY^0)^2 + e^{2Y^0/L}\left((dY^1)^2 + (dY^2)^2 + (dY^3)^2\right) + e^{2Y^0/L}(dY^4)^2,$$

$L$ again being the radius of curvature. Now (8) is recovered from the embedding (15) for

$$e^{b/L} = a \quad L^2 \frac{\dot{a}^2}{a^2} - a^2 \dot{h}^2 = 1$$

After substituting (15) in (11)

$$L^2 \partial_t \left( \frac{\dot{a}}{a^2} F_1 \right) + a^2 \dot{h}^2 F_1 - 3a^2 F_2 = 0$$

$$\partial_t(a^2 \dot{h} F_1) = 0$$

From (13) and (29) we then get

$$\rho_{RT} = \frac{c_0}{a^3 \sqrt{L^2 \dot{a}^2 - \dot{a}^2}}$$

c_0 is real which means we need that $|\dot{a}| \gg \frac{1}{L}$. The expression (30) is once again recovered for $|\dot{a}| \gg \frac{1}{L}$. 

6
To summarize, the source term $\rho_{RT}$ for the three different backgrounds has the form

$$\rho_{RT} = \frac{c_0}{a^3 \sqrt{L^2 \dot{a}^2 - k_5 a^2}}$$

(31)

where $k_5$ defines the curvature of the five-dimensional background space: $k_5 = 0, -1, 1$ for $R^{1,1}$, $AdS_5$, and $dS_5$, respectively. Moreover, from (13) one has that

$$\dot{a}^2 a \sqrt{L^2 \dot{a}^2 - k_5 a^2} = \text{constant}$$

(32)

$p_{RT}$ can be determined from the conservation law (10) leading to a time-dependent equation of state

$$p_{RT} = -\frac{a}{3a} \rho_{RT} - \rho_{RT} = \frac{a (L^2 \dot{a} - k_5 a)}{3 (L^2 \dot{a}^2 - k_5 a^2)} \rho_{RT}$$

(33)

The evolution of the scale factor for the three cases $k_5 = 1, 0, -1$ is obtained from (32). As stated previously, for $k_5 = 0$ one gets $a(t) \propto t^{3/4}$. We resort to numerical integration to obtain solutions for the other two cases, $k_5 = \pm 1$. The results for all three cases are plotted in Fig. 1, using the initial condition $a(0) = 0$. All three cases agree for small $t$, i.e. $a(t) \propto t^{3/4}$ as $L \dot{a} \rightarrow \infty$, and so $\ddot{a} < 0$. For cases $k_5 = 0$ and $-1$, we find that $\ddot{a} < 0$, for all $t$. The situation is more interesting for $k_5 = 1$, corresponding to the de Sitter background. In this case, $\ddot{a}$ vanishes at finite $t$, when $L \dot{a} = \sqrt{2}$, thus signaling a transition from the de-accelerating phase to an accelerating phase. We get that $L \dot{a}$ goes asymptotically to one in the $t \rightarrow \infty$ limit, where the scale factor undergoes an exponential expansion at leading order,

$$a(t) \rightarrow a_1 e^{t/L} \left(1 - a_2 e^{-8t/L} + \cdots\right), \quad \text{as } t \rightarrow \infty,$$

(34)

$a_1$ and $a_2$ being positive constants. From (33) we can obtain the equation of state for the RT source as a function of time. The ratio $p_{RT}/\rho_{RT}$, standardly denoted by $w$, goes from $-\frac{1}{3}$, near $t = 0$, to $-\frac{1}{3}$ at the transition, to $-1$, in the limit $t \rightarrow \infty$. Note that unlike in the $\Lambda$CDM model, here we get a transition from the de-accelerating phase to an accelerating phase even without the inclusion of a matter component or cosmological constant component to the Friedmann equations.

Finally, we proceed with a fit of the $k_5 = 1$ case to observational data. (32) gives an algebraic relation between the Hubble parameter $H = \dot{a}/a$ and the redshift parameter $z = a_0/a - 1$, where $a_0$ is the scale parameter at the current time. It is

$$L^2 H^2 \sqrt{L^2 H^2 - 1} = \bar{c}_0 (1 + z)^4,$$

(35)

where $\bar{c}_0 = \frac{8\pi G}{3} L^2 a_0^{-4} c_0$. In Fig. 2(a) we fit the real solution to eq. (35) to observed results for $H$ versus $z$ using the data in Table 1. The best fit occurs for $\bar{c}_0 \approx .26$ and

--

4Here we have done a rescaling of the constant $c_0$ for the case $k_5 = 0$.

5The case $k_5 = 0$ is an exception. After using (32) one gets the simple relation $p_{RT} = -\frac{1}{3} \rho_{RT}$.
Figure 1: Plot of $t$ vs $a$ for three different five-dimensional background spaces: $R^{4,1}$, $AdS_5$ and $dS_5$. (Here we set $L = 1$.)

Figure 2: The solid purple curve in figure (a) represents a fit of eq. (35) with the Hubble parameter data, while the dashed red curve is $\Lambda$CDM. $H$ is given in units of km s$^{-1}$Mpc$^{-1}$. The best fit occurs for $\tilde{c}_0 \approx .26$ and $1/L \approx 72$ km s$^{-1}$Mpc$^{-1}$. From figure (b) the minimum of $H/(1 + z)$ for the best fit occurs at $z \approx .675$, corresponding to the transition from a de-accelerating phase to an acceleration phase.
$1/L \approx 72 \text{ km s}^{-1}\text{Mpc}^{-1}$. For $H$ evaluated at $z = 0$ one gets $H(0) \approx 74 \text{ km s}^{-1}\text{Mpc}^{-1}$. Our fit in Fig. 2(a) is compared to that of $\Lambda$CDM, where the expression for the Hubble parameter is given by $H = H_0 \sqrt{(1+z)^3 \Omega_m + \Omega_\Lambda}$, with $\Omega_m = .3$, $\Omega_\Lambda = .7$ and $H_0 \approx 68.92 \text{ km s}^{-1}\text{Mpc}^{-1}$. $H/(1+z)$ (which is proportional to $\dot{a}$) versus $z$ is plotted in Fig. 2(b), using our fit for $H$ in Fig 2(a). It shows that the transition from a de-acceleration phase to acceleration phase occurs at $z \approx .675$, which is similar to the value predicted by $\Lambda$CDM.

We now summarize some of the features of this model. After generalizing RT gravity to curved backgrounds, we found universal formulas for the effective density and pressure, (31) and (33), respectively, resulting from embedding the $k = 0$ RW manifold in three different five-dimensional background spaces. We suspect that the results found here are dependent on the choice of embedding (in addition to the choice of background space), although we have not found specific examples of this.

A reasonable fit to the Hubble parameter data was obtained in the case where the background was de Sitter space. This is true even without considering the usual stress-energy contributions to the Einstein equations, which on the other hand, play an essential role for $\Lambda$CDM. Such components can easily be included in our model by adding appropriate terms to (12) and consequent equations. For the case of nonrelativistic matter, one ends up with the following modification to (35):

$$\frac{L^2 H^2}{(1+z)^3} - \frac{\tilde{c}_0 (1+z)}{\sqrt{L^2 H^2 - 1}} = \tilde{c}_1,$$

where $\tilde{c}_1$ is an additional constant which quantifies the nonrelativistic matter component. The inclusion of the additional parameter $\tilde{c}_1$ does not appear to improve the previous fit in any significant manner.

The presence of the square root in (35) [and also in (36)] gives a lower bound on the Hubble parameter, $H(z) > 1/L$, which is in agreement with observation.

The fit we obtained to the Hubble parameter data holds for values of $z$ up to approximately 2.36. Concerning $z > 2.36$, the deviation of our fit in Fig. 2 with that of $\Lambda$CDM grows when extrapolating to higher $z$. However, our fit did not include contributions from the stress-energy tensor, which can play a more significant role at large $z$. For example, if one considers the matter density $\rho_m$ which is proportional to $a^{-3}$, then its relative contribution is $\rho_m/\rho_{RT} \propto \sqrt{L^2 H^2 - 1}/(z + 1)$, which grows like $LH/z$ for large $z$. Also, there is no reason to assume that the 5d de Sitter background is valid for all $z$. This among other issues is open for further investigation/speculation.

References

[1] T. Regge and C. Teitelboim, “General Relativity \`a la string: a progress report,” in Proceedings of the First Marcel Grossmann Meeting (Trieste, Italy, 1975), 77-87,
Table 1: Data used for fit in Fig. 2. Columns 1-4 are $z$, $H$, error in $H$ and citation respectively. Columns 2&3 are in units of km s$^{-1}$Mpc$^{-1}$. Data was selected with $\sigma_H < .15H$.

| $z$ | $H$  | $\sigma_H$ | Citation |
|-----|------|------------|----------|
| 0   | 74.03| 1.42       | [12]     |
| .17 | 83   | 8          | [13]     |
| .1791 | 75 | 4          | [14]     |
| .1993 | 75 | 5          | [14]     |
| .38 | 81.5 | 1.9       | [15]     |
| .4783 | 80.9 | 9          | [16]     |
| .51 | 90.4 | 1.9       | [15]     |
| .5929 | 104 | 13         | [14]     |
| .61 | 97.3 | 2.1       | [15]     |
| .6797 | 92 | 8          | [14]     |
| .7812 | 105 | 12         | [14]     |
| .8754 | 125 | 17         | [14]     |
| 1.037 | 154 | 20         | [14]     |
| 1.3 | 168 | 17         | [13]     |
| 1.43 | 177 | 18         | [13]     |
| 1.53 | 140 | 14         | [13]     |
| 2.34 | 222 | 7          | [17]     |
| 2.36 | 226 | 8          | [18]     |

[2] S. Deser, F. A. E. Pirani and D. C. Robinson, “Imbedding the G-String,” Phys. Rev. D 14, 3301-3303 (1976).
[3] M. Pavsic, “On the Quantization of Gravity by Embedding Space-time in a Higher Dimensional Space,” Class. Quant. Grav. 2, 869 (1985).
[4] V. Tapia, “GRAVITATION A LA STRING,” Class. Quant. Grav. 6, L49 (1989).
[5] M. D. Maia, “On the Integrability Conditions for Extended Objects,” Class. Quant. Grav. 6, 173-183 (1989).
[6] I. A. Bandos, “String - like description of gravity and possible applications for F theory,” Mod. Phys. Lett. A 12, 799-810 (1997).
[7] R. Capovilla, A. Escalante, J. Guven and E. Rojas, “Hamiltonian dynamics of extended objects: Regge-Teitelboim model,” [arXiv:gr-qc/0603126 [gr-qc]].
[8] S. Fabi, A. Stern and C. Xu, “Cosmic acceleration in Regge-Teitelboim gravity,” [arXiv:2202.09453 [gr-qc]].
[9] A. Davidson, “Lambda = 0 cosmology of a brane - like universe,” Class. Quant. Grav. 16, 653-659 (1999).
[10] I. Robinson, “On plane waves and nullicles,” in *From SU(2) to Gravity: Festschrift in Honor of Yuval Ne’eman*, Cambridge Press, (1985).

[11] M. M. Akbar, “Embedding FLRW geometries in pseudo-Euclidean and anti-de Sitter spaces,” Phys. Rev. D 95, no.6, 064058 (2017).

[12] A. G. Riess, S. Casertano, W. Yuan, L. M. Macri and D. Scolnic, “Large Magellanic Cloud Cepheid Standards Provide a 1% Foundation for the Determination of the Hubble Constant and Stronger Evidence for Physics beyond $\Lambda$CDM,” Astrophys. J. 876, no.1, 85 (2019).

[13] D. Stern, R. Jimenez, L. Verde, M. Kamionkowski and S.A. Stanford, “Cosmic chronometers: constraining the equation of state of dark energy. I: $H(z)$ measurements” JCAP 1002, 008 (2010).

[14] M. Moresco, A. Cimatti, R. Jimenez, L. Pozzetti, G. Zamorani, M. Bolzonella, J. Dunlop, F. Lamareille, M. Mignoli and H. Pearce, et al. “Improved constraints on the expansion rate of the Universe up to $z$=1.1 from the spectroscopic evolution of cosmic chronometers,” JCAP 08, 006 (2012).

[15] S. Alam et al. [BOSS], “The clustering of galaxies in the completed SDSS-III Baryon Oscillation Spectroscopic Survey: cosmological analysis of the DR12 galaxy sample,” Mon. Not. Roy. Astron. Soc. 470, no.3, 2617-2652 (2017).

[16] M. Moresco, L. Pozzetti, A. Cimatti, R. Jimenez, C. Maraston, L. Verde, D. Thomas, A. Citro, R. Tojeiro and D. Wilkinson, “A 6% measurement of the Hubble parameter at $z \sim 0.45$: direct evidence of the epoch of cosmic re-acceleration,” JCAP 05, 014 (2016).

[17] T. Delubac et al. [BOSS], “Baryon acoustic oscillations in the Ly$\alpha$ forest of BOSS DR11 quasars,” Astron. Astrophys. 574, A59 (2015).

[18] A. Font-Ribera et al. [BOSS], “Quasar-Lyman $\alpha$ Forest Cross-Correlation from BOSS DR11: Baryon Acoustic Oscillations,” JCAP 05, 027 (2014).