Newman-Penrose-Debye formalism for fields of various spins in pp-wave backgrounds

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Using Newman-Penrose formalism in tetrad and spinor forms, we perform separation of variables in the wave equations for massless fields of various spins \( s = 1/2, 1, 3/2, 2 \) on the background of exact plane-fronted gravitational wave metrics. Then, applying Wald’s method of conjugate operators, we derive equations for Debye potentials and we find the back-projection operators expressing multicomponent fields in terms of these potentials. For shock wave backgrounds, as a special case of the non-vacuum pp-waves, the exact solutions for Debye potentials are constructed explicitly. The possibilities of generalization to the case of massive fields are discussed, in particular, construction of exact solutions of the Dirac and Proca equations. These results can be used in various supergravity problems on the pp-wave backgrounds, including holographic applications.

Keywords: PP-waves, Newman-Penrose formalism, Debye potentials, shock waves

1. Introduction

Wave equations for fields of different spins in curved space have been extensively studied since the early 1970s. Solutions describing propagation of fields on curved backgrounds are in demand in various contexts including the problem of radiation, supergravity and superstrings theories. Generically, considering fields of higher spins in curved space-time, difficulties arise with splitting the systems of coupled equations into separate ones for certain combinations of the field components. Two efficient methods to do this are known. The first is the application of the Newman-Penrose (NP) formalism\(^1\), in which the intrinsic symmetries of the massless field equations with respect to the Lorentz group are conveniently incorporated. The second efficient tool is the method of Debye potentials\(^2\), which allows to reconstruct the multicomponent fields in terms of solutions of some scalar equations. Debye potentials are complex functions which incorporate two degree of freedom of the massless field of any spin.

Formalism of Debye potentials was suggested in gravity theory in the works by Cohen\(^3\) and Kegeles\(^4\) and then effectively used by Chrzanowski\(^5\) in Kerr spacetime of type D, according to the Petrov classification\(^6\), in combination with the Teukolsky equation\(^7\). In the Kerr metric Debye representations were found for electromagnetic field and gravitational perturbations. Later, formalism of Debye potentials was discussed in detail for Rarita-Schwinger spin 3/2 field by Torres del Castillo\(^8\).
In this work we apply the Newman-Penrose-Debye formalism to the case of metrics of type N. More specifically, we consider the case of pp-wave metrics, which have numerous applications both in astrophysics and in various theoretical aspects, including supergravity and holography.

2. PP-waves

These metrics were introduced by Brinkmann in 1925 and later interpreted as representing the plane-fronted gravitational waves (pp-waves). They are exact solutions of Einstein’s equations of the following form

$$ds^2 = du dv - H(u, \zeta, \bar{\zeta}) du \, du - d\zeta d\bar{\zeta},$$

(1)

where $\zeta = x + iy$ is a complex transverse coordinate, $H$ is arbitrary nonlinear real profile function of $\zeta$, $\bar{\zeta}$, which specifies the nature of the wave. The determinant $g = 1/16$ is coordinate-independent. The scalar curvature turns out to be zero. For $H = 0$ the metric is Minkowski. These metrics describe plane waves with parallel rays belonging to the class of solutions that admits isotropic nonexpanding congruences without shear and twist, as well as the existence of an isotropic Killing vector.

A natural choice of the null tetrad basis for them is

$$l = \sqrt{2} \partial_u, \quad n = \sqrt{2}(\partial_u + H \partial_v), \quad m = \sqrt{2} \partial_\zeta, \quad \bar{m} = \sqrt{2} \partial_{\bar{\zeta}}.$$  

(2)

We will also use the NP covariant derivatives along the null tetrad vectors:

$$D = l^\mu \nabla_\mu, \quad \Delta = n^\mu \nabla_\mu, \quad \delta = m^\mu \nabla_\mu, \quad \bar{\delta} = \bar{m}^\mu \nabla_\mu$$

(3)

and satisfying, in the case of a plane wave metrics, the following commutation relations

$$[\Delta, D] = 0, \quad [\delta, D] = 0, \quad [\delta, \Delta] = -\bar{\nu} D, \quad [\delta, \bar{\delta}] = 0.$$  

(4)

The nonzero tetrad projections of the traceless part of the Ricci tensor, the Weyl scalar, and the only nonzero spin coefficient $\nu$ are

$$\Phi_{22} = \frac{1}{2} \eta^\alpha \eta^\beta R_{\alpha\beta} = -H \zeta \bar{\zeta}, \quad \nu = -\bar{m}^\mu \delta n_\mu = -H \zeta \bar{\zeta},$$

$$\Psi_4 = -\eta^\alpha \bar{m}^\beta \eta^\gamma \bar{m}^\tau C_{\alpha\beta\gamma\tau} = -H \zeta \bar{\zeta}, \quad \Lambda = \frac{1}{24} R = 0,$$  

(5)

where $C_{\alpha\beta\gamma\tau}$ is the Weyl tensor. In the vacuum case, the gravitational field equations $\Phi_{22} = 0$ reduce to the two-dimensional Laplace equation

$$H \zeta \bar{\zeta} = 0.$$  

The d’Alembert operator for the massless scalar field $\Box = \nabla_\mu \nabla^\mu$ reads:

$$\Box \equiv \frac{1}{\sqrt{|g|}} \partial_\mu \left( \sqrt{|g|} g^{\mu\nu} \partial_\nu \right) = 2(D \Delta - \delta \bar{\delta}).$$  

(6)
3. Debye potentials

Here we briefly recall the Wald’s procedure for constructing a solution of the field equations of higher spins, which we then apply to the metrics of plane waves. Consider some multicomponent field $f$ (tensor or spinor) satisfying the field equation $\mathcal{E}(f) = 0$, where $\mathcal{E}$ is an appropriate matrix linear differential operator. This is generically a non-quadratic matrix $n \times m$ taking the $m$-component field into the column of $n$ differential equation. To solve this system of equations one has to disentangle it, which in general is not possible in the closed form. However, it can be possible to decouple a separate equation (or several equations) for some combination $\phi$ of the components of the initial field $f$ by the action of another linear partial differential operator $T$, defining a scalar $\phi = T(f)$. Then one can define a pair of linear operators $S$ and $O$ such, that

$$SE(f) = OT(f) = O(\phi).$$

The problem of finding such operators is facilitated if one knows the source terms in the inhomogeneous equations both for an initial multicomponent field $\mathcal{E}(f) = J$, and in the decoupled scalar equation $O(\phi) = S(J)$. The number of decoupled equations depends on the number of principal null direction of the metric.

The final step is the construction of the adjoint operator $S^\dagger$ with respect to a suitably defined functional scalar product such that, in the matrix form,

$$\int \phi^A_n M_{A_n B_m} \psi^B_m = \int \psi^B_n M^{\dagger}_{B_n A_m} \phi^A_m,$$

where the indices $A_n, B_m$ take $n$ and $m$ values respectively and the integration measure is supressed. Note that complex conjugation is not used.

Now it can be verified that the solution of the homogeneous field equation $\mathcal{E}(f) = 0$ will be guaranteed if

$$f = S^\dagger \psi, \quad O^\dagger \psi = 0.$$  \hfill (7)

The last equation is the Debye potential equation, the relevant operator is therefore the adjoint to one used in the equation for a decoupled scalar. It there are several such decoupled equations (e.g. two in type D metrics), one will have two different representations for the initial multicomponent field which are usually related by some tetrad transformation. In type N case the Debye potential is unique.

The complex Debye representation for for real-valued massless fields reflects the existence of two independent polarizations, which can be obtained as the real and imaginary parts of the complex $f$ in the Debye form.

In what follows we apply this procedure to the pp-wave background with the only one nonzero spin coefficient $\nu$. Clearly, the adjoint to the product of two operators will be the product of the adjoints to each of them in reverse order. So to construct an adjoint of some polynomial product it will suffice to know the adjoint to basic operators. The adjoint NP differential operators will read

$$D^\dagger = -D, \quad \Delta^\dagger = -\Delta, \quad \delta^\dagger = -\delta, \quad \bar{\delta}^\dagger = -\bar{\delta}. \hfill (8)$$
4. Maxwell field

Maxwell’s equations in the Newman-Penrose formalism are written in terms of the projections of the electromagnetic field tensor onto the bivector constructed of null tetrad as follows:

\[ \Phi_0 = F_{lm}, \quad 2\Phi_1 = (F_{ln} + F_{lm}), \quad \Phi_2 = F_{mn}. \]  (9)

A sourceless decoupled equation for the scalar \( \Phi_0 \) in the case of vacuum pp-waves was obtained in the Refs. 11 and 12. According to above procedure, we have to construct the source term to this equation. For this, one should act with an appropriate NP differential operators on the first pair of the system of equations and exclude the variable \( \Phi_1 \) from them using the commutation relations. This gives

\[ \Box \Phi_0 = 2\pi J_0, \quad J_0 = 2[\delta J_l - DJ_m]. \]  (10)

Since metrics of type N have only one principal null direction (contrary to the type D, where there are two), the NP component \( \Phi_0 \) will be the only one for which the decoupling can be made. Further, having written down both adjoint operators, one can easily construct a solution for vector \( A^\mu \), satisfying the Lorentz gauge condition, in terms of the Debye potential:

\[ A^\mu = [\bar{m}^\mu D - l^\mu \delta] \psi, \quad \Box \psi = 0. \]  (11)

It is easy to see that the contraction of the Ricci tensor with the expression for the vector potential gives zero; therefore, the solution is valid also for the non-vacuum pp-waves.

5. Weyl field

For the Weyl equations of spin 1/2, one uses the two-component spinor version of the NP formalism. In this case, we will no longer deal with a tetrad, but with a spinor dyad. The massless spin 1/2 equation has the following form:

\[ \nabla_A B^A \chi_A = 0. \]  (12)

Writing this in components and carrying out transformations similar those for the vector field, we obtain the decoupled equation with the source term:

\[ \Box \chi_0 = N_0, \quad N_0 = 2[Dj_1 - \delta j_0], \]  (13)

where the spinor source is denoted as \( j_A \). Then performing the conjugation, we construct a solution in terms of the Debye potential, which, in turn, satisfies the d’Alembert equation:

\[ \chi_0 = -D\psi, \quad \chi_1 = -\bar{\delta}\psi, \quad \Box \psi = 0. \]  (14)
6. Rarita-Schwinger field

The spin 3/2 field is described by the Rarita-Schwinger equation for the spinor-vector $\psi_\mu$, which satisfies the equations

$$\gamma^\mu (\nabla_\mu \psi_\nu - \nabla_\nu \psi_\mu) = 0, \quad \gamma^\rho \psi_\rho = 0. \quad (15)$$

This system is consistent only in the case of the vacuum metrics, the non-vacuum case refers to the supergravity context.

The field of an arbitrary spin $s$ in the two-component spinor formalism is described by a totally symmetric spinor of valence $2s$ satisfying the equation of motion

$$\nabla^A AB' \phi_{ABC} = 0.$$  

If $s \geq 3/2$ the Buchdahl consistency constraint

$$\Psi^{ABC}_{(D\phi_{EF...})ABC} = 0$$

must be satisfied, where $\Psi_{SABC}$ denotes the Weyl spinor. But there exists also an alternative approach, developed in the work, where the consistency constraint is satisfied automatically. This method can be easily adapted to the case of the pp-wave backgrounds.

In the spinor equivalent of the Eq. (15)

$$\nabla^A AB' \psi^{ACD'} - \nabla^C CD' \psi^A_{AB'} = 0,$$  

one has to pass from field $\psi^{ACD'}$ to the symmetric rank three spinor arriving at the modified equations of motion

$$\nabla^A AB' \phi_{ABC} = \psi_s^{(B)} \psi^{A}_{C} R', \quad \phi_{ABC} = \nabla^{(B)[R'\psi^A_{C}]} R'.$$

We write down the complete system of equations with the sources for the symmetric spinor field $\phi_{ABC}$ in the component form, applicable to the type N metrics:

$$\delta \phi_{000} - D \phi_{100} = \delta J_{000} - D J_{010}; \quad \Delta \phi_{000} - \delta \phi_{100} = \delta J_{010} - D J_{110};$$

$$\delta \phi_{110} - D \phi_{010} = \frac{1}{2} [\Delta J_{000} + \delta J_{001} - \delta J_{010} - D J_{011}];$$

$$\Delta \phi_{110} - \delta \phi_{110} - \nu \phi_{000} = \frac{1}{2} [\Delta J_{000} + \delta J_{001} - \delta J_{010} - D J_{011} - \nu J_{000};$$

$$\delta \phi_{110} - D \phi_{100} = \Psi_{100} + \Delta J_{000} - \frac{1}{2} \delta (J_{001} + J_{011} - \nu J_{000};$$

$$\Delta \phi_{111} - \delta \phi_{111} - 2 \nu \phi_{100} = \Psi_{100} - \delta J_{000} + \frac{1}{2} \Delta (J_{011} + J_{001}) - \nu J_{000} - \nu J_{010}.$$

One can exclude the component $\phi_{100}$ from the first pair of equations, obtaining the decoupled equation for $\phi_{000}$ with the source term:

$$\Box \phi_{000} = K_0, \quad K_0 = 2 [D \delta (J_{010} + J_{000}) - D^2 J_{010} - \delta^2 J_{000}]. \quad (18)$$
Using this, one can further represent a spin-vector in terms of the Debye potential as follows:

\[
\psi_\mu = \left( \bar{m}_\mu D \delta - l_\mu \bar{\delta}^2 \right) \psi, \quad \square \psi = 0.
\]  

The resulting expression satisfies the system (19). Note that the equation for the Debye potential is the same for the vector field, the Weyl field and, as we will see in the next section, also for the tensor field.

7. Gravitational perturbations

Starting with the Einstein’s equations and expanding the metric on the background

\[ g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu}, \]

one derives the Lichnerowitz equation for the spin 2 field in curved space-time:\[ \nabla^\alpha \nabla_\alpha \psi_{\mu\nu} + 2R^\alpha_\mu\tau \nu (0) \psi_{\sigma\tau} - 2R^\sigma_{(\nu} \psi_{\mu)\sigma} = 0, \quad \nabla_\mu \psi^{\mu\nu} = 0, \]  

where

\[ \psi_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} g_{\mu\nu} h, \quad h = g^{\mu\nu} h_{\mu\nu}, \quad \psi = g^{\mu\nu} \psi_{\mu\nu}. \]

In the case of non-vacuum background, the additional gauge fixing condition \( \psi = 0 \) should be imposed.

To apply the NP formalism, similar splitting has to be performed in the null tetrad vectors, spin coefficients, Weyl scalars and Ricci tensor deviators, equipping the first order perturbations with an index one. The complete set of gravitational perturbation equations in the Newman-Penrose formalism is obtained by linearizing the Bianchi identities. We present here two equations from the resulting system, which are relevant for pp-waves:

\[ \delta \Psi^{(1)}_0 - D \Psi^{(1)}_1 = 4\pi \left[ \delta T^{(1)}_{\mu\nu} - D T^{(1)}_{\mu\nu} \right], \quad \Delta \Psi^{(1)}_0 - \delta \Psi^{(1)}_1 = 4\pi \left[ \delta T^{(1)}_{\mu\nu} - D T^{(1)}_{\mu\nu} \right]. \]

After some manipulations using the perturbed Ricci identities in the NP formalism, we obtain the decoupled equation for the perturbation of \( \Psi_0 \) with the corresponding source term:

\[ \square \Psi^{(1)}_0 = 4\pi T^{(1)}_0, \quad T^{(1)}_0 = 2 \left[ 2D \delta T^{(1)}_{\mu\nu} - D^2 T^{(1)}_{\mu\nu} - \delta^2 T^{(1)}_{\mu\nu} \right]. \]

Then using the conjugate operator we can write down the solution in terms of the Debye potential:

\[ \psi^{\mu\nu} = \left[ 2l^{(\mu \nu)} D - \bar{m}^{\mu} \bar{m}^{\nu} D^2 - \bar{m}^{\mu} \bar{m}^{\nu} \bar{\delta}^2 \right] \psi, \quad \square \psi = 0. \]

This expression satisfies the Lichnerowitz equation with the gauge condition (22). As in the case of the vector field, this construction will be also valid for the non-vacuum pp-waves. This can be easily verified by the direct substitution.
8. Shock wave backgrounds

An important subclass of pp-waves is generated via boosting the black hole metrics to an infinite-momentum frame. The line element of the resulting metrics (parametrized by the real transverse coordinates $x_i, i = 1, 2$) reads

$$ds^2 = \delta(u)f(x)du^2 + dudv - dx_i^2.$$  \hfill (24)

This is an exact solution of the Einstein equations representing the gravitational shock wave. When $u \neq 0$, the space-time is flat, but for $u = 0$ it has a delta-like singularity. But the field equations on such a background are still meaningful due to lack of singularities in the metric determinant. The function $f(x)$ describes the gravitational wave profile. In particular, for the case of the boosted Schwarzschild solution (the Aichelburg-Sexl metric)\(^{15}\)

$$f(x) = -8p_m\ln \rho, \quad \rho = \sqrt{x_1^2 + x_2^2},$$

where $p_m$ denotes the energy of shock wave. For the case of a boosted Einstein-Maxwell-dilaton solution\(^{16}\)

$$f(x) = -8p_m\ln \rho + \frac{(3 - 4a^2)\pi p_e}{(1 - a^2)\rho},$$

where $a$ - the dilaton coupling constant, $p_e$ - the electric charge. For boosted Taub-NUT\(^{17}\)

$$f(x) = -8p_n\tan^{-1} \frac{x_1}{x_2},$$

where $p_n$ is the NUT charge. Also known in the literature are the boosted Kerr-Newman solution\(^{18}\), the boosted Schwarzschild-anti-de Sitter\(^{19}\) space and some other metrics.

Here we present the solution of the massive Klein-Gordon equation on the background of the singular shock-wave metrics:

$$\left(\Box + m^2\right)\phi = 2(D\Delta - \delta\bar{\delta})\psi + m\psi = \left(4\partial_u\partial_v + 4 \kappa \delta(u)f(x)\partial_v^2 - \partial_v^2 + m^2\right)\phi = 0. \hfill (25)$$

In spite of presence of the singularity, there exists an exact solution of this equation containing the Heaviside function discontinuity only:

$$\phi = \int \exp[i \mathcal{W}] \, d\mathcal{G}, \quad d\mathcal{G} = \frac{dq}{(2\pi)^2} \, dx,$$

$$\mathcal{W} = -\frac{k_v}{2} \left[v - \kappa \theta(u)f(x')\right] - \frac{(k - q)^2 + m^2}{2k_v}u + kx + q(x' - x), \hfill (26)$$

where we have introduced the notation $k = (k_{x_1}, k_{x_2}), \quad q = (q_{x_1}, q_{x_2}), \quad x = (x_1, x_2)$ for the two-dimensional transverse vectors. If we put the mass equal to zero, then we obtain the solution of the equation for the Debye potential.
9. Massive field with spin $\frac{1}{2}$

Now consider other massive fields, starting from the spin 1/2. In the two-component notation, the Dirac bispinor equation $\imath \gamma^\mu \nabla_\mu \psi - \mu \psi = 0$ with $\psi = (\xi^A, \eta^A)$, splits into two equations

$$\nabla_{AB'} \xi^A - \imath \mu \eta_{B'} = 0, \quad \nabla^{AB'} \eta_{B'} - \imath \mu \xi^A = 0. \quad (27)$$

Their solution in the case of shock-wave backgrounds is a generalization of the previously obtained expression (14) for the massless field and can be written in the following form

$$\eta^1 = 2 \int \left( [k_{x_1} - q_{x_1}] + \imath [k_{x_2} - q_{x_2}] - \mu \right) \exp[\imath \mathcal{W}] \, d\mathcal{G}, \quad \eta^0 = - \int k_v \exp[\imath \mathcal{W}] \, d\mathcal{G},$$

$$\xi^0 = 2 \int \left( [k_{x_1} - q_{x_1}] - \imath [k_{x_2} - q_{x_2}] + \mu \right) \exp[\imath \mathcal{W}] \, d\mathcal{G}, \quad \xi^1 = \int k_v \exp[\imath \mathcal{W}] \, d\mathcal{G}, \quad (28)$$

where $\mathcal{W}$ defined in (26).

10. Proca equation

For the massive spin-1 field the gauge invariance of Maxwell’s field is broken by the mass term. Instead of the gauge fixing condition, we deal with the Lorentz-like dynamical constraint, so we have two equations:

$$\nabla_\mu F^{\mu \nu} + m^2 A^\nu = 0, \quad \nabla_\mu A^\mu = 0. \quad (29)$$

The massive vector field is no longer transverse, possessing three physical degrees of freedom. The first two polarizations in the case of shock wave background are realized by modified expressions with Debye potentials (the real and imaginary parts of the solution):

$$A^\mu_{(1,2)} = \int K^\mu_{(1,2)} \exp[\imath \mathcal{W}] \, d\mathcal{G}, \quad K^\mu_{(1,2)} = \left\{ 0, 2[k_{x_1} - q_{x_1}] - 2\imath [k_{x_2} - q_{x_2}], k_v, -ik_v \right\}. \quad (30)$$

For the third polarization we solve the dynamical constraint acting by the covariant derivative on the massive scalar field, obtaining

$$A^\mu_{(3)} = \int K^\mu_{(3)} \exp[\imath \mathcal{W}] \, d\mathcal{G} + c.c,$$

$$K^\mu_{(3)} = \left\{ k_v, \frac{(-\mathbf{q})^2 - m^2}{k_v} + k_v \delta(u) f(x), k_{x_1} - q_{x_1}, k_{x_1} - q_{x_1} \right\}. \quad (31)$$

It can be seen that this expression satisfies the constraint indeed. But this expression gives us only one additional polarization, because of real multipliers in front of the exponent. Therefore the complete solution of the Proca equation in AS metric is a sum $A^\mu = A^\mu_{(1,2)} + A^\mu_{(3)}$. To write down the solution for massless electromagnetic field, it is sufficient to set the mass to zero.
11. Conclusions

This work is devoted to some novel applications of the Newman-Penrose formalism and the method of Debye potentials. Previously this technique was successfully used to construct solutions of equations for massless fields of different spins on the background of vacuum black hole solutions of Petrov type D. Here it was applied to solutions of type N. Unlike the case D, where the metric has two principal null directions and, accordingly, two decoupled equations for NP projections can be derived, in the metrics of type N only one decoupled equation exists. Namely, one can decouple the $\chi_0$-equation for the Weyl field, the $\phi_{000}$-equation for the Rarita-Schwinger field, and the equations for perturbations of $\Phi_0$ and $\Psi_0$ of the vector and tensor fields respectively. It is still enough to construct the Debye representation for the solutions obtaining the universal equation for the Debye potential for all spins.

We also managed to generalize our construction to the case of massive fields on the background of shock-wave metrics. Our formulas can be used for quantization in shock wave backgrounds and in some holographic applications.

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10

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