Risk Sensitive Control of the Lifetime Ruin Problem

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Abstract

We study a risk sensitive control version of the lifetime ruin probability problem. We consider a sequence of investments problems in Black-Scholes market that includes a risky asset and a riskless asset. We present a differential game that governs the limit behavior. We solve it explicitly and use it in order to find an asymptotically optimal policy.

Keywords: Probability of lifetime ruin, optimal investment, risk sensitive control, large deviations, differential games.

1 Introduction

The problem of how an individual should invest her wealth in a risky financial market in order to minimize the probability of outliving her wealth, also known as the probability of lifetime ruin was extensively analyzed, see e.g. [19], [26], [5], [4], [6], [7], and [25]. These works fall naturally within the area of optimally controlling wealth to reach a goal. Research on this topic goes back to the seminal work of [12] and continued with the work of [21], [20], [24], [18], [17], [8], [9], and [10].

In the standard Black-Scholes market that includes a risky asset and a riskless asset, the case of interest is when the investor has more expenses than the potential profit that follows by investing the entire wealth in the riskless asset, that is $c(x) > rx$, when $c(\cdot)$ is the consumption function, $r$ is the constant riskless rate, and $x$ is the current wealth. The other case is trivial, of-course, since by investing the entire wealth in the riskless asset the wealth cannot decrease and ruin is avoided. In case that $c(x) − rx \approx 0^+$ then the investor who wishes to minimize the probability of lifetime ruin should invest almost all of her wealth in the riskless asset. The probability of ruin would be small, yet positive. With the understanding that lifetime ruin is a rare and dramatic event and that one should also avoid living close to the ruin level,

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we study this case, by using a risk sensitive control framework. The risk sensitive control criteria, penalizes such events heavily, and therefore, provides a natural way to address these considerations.

We study the risk sensitive control via large deviations techniques. In [22], Pham provides some applications and methods of large deviations in finance and insurance. Among the studied models, he considers ruin probabilities problems when the initial reserve is large and therefore, the probability of ruin is small. We however study a lifetime ruin problem, which is a different problem, and via risk sensitive control with small noise, as described below, which yields a different analysis.

In order to rigorously treat the mentioned case that \( c(x) - rx \approx 0^+ \) we consider a sequence of models, indexed by \( n \in \mathbb{N} \), that differ from each other only in the consumption function in a way that \( c^n(x) - rx = O(1/n) \), where \( n \) is a large parameter. By using an appropriate time scaling we get a risk sensitive control with small noise as follows. The scaled wealth process under the consumption function \( c^n \) satisfies

\[
d\tilde{W}^n(t) = b(\tilde{W}^n(t), \pi^n(t))dt + \frac{1}{\sqrt{n}}\sigma(\tilde{W}^n(t), \pi^n(t))dB(t), \quad t \geq 0,
\]

\[
\tilde{W}^n(0) = x
\]

for some proper \( b \) and \( \sigma \), where \( \pi^n \) is the investment policy, and \( B \) is a standard Brownian motion. The goal is to choose \( \pi^n \) that minimizes

\[
\frac{1}{n} \ln \mathbb{E} \left[ e^n \left( \int_0^{\tau^n_d \wedge \tau^n_a} l(\tilde{W}^n(s))ds + \rho \mathbb{1}_{\{\tau^n_d \leq \tau^n_a\}} \right) \right],
\]

where \( \tau^n_d \) is the time of death, \( \tau^n_a \) is the time of reaching the ruin level \( a \), \( \rho \) is a penalty for lifetime ruin, and \( l \) is a nonnegative non-increasing function that penalizes low wealth. We present a differential game that governs the limit behavior. We solve it explicitly and use it in order to find an asymptotically optimal policy.

Risk sensitive control for controlled stochastic differential equations with small noise have been studied for example in [15], [16], and [13]. For a survey about the topic the reader is referred to [14]. There are several approaches towards this problem. In [16], Fleming and Soner used differential equations tools and show that the sequence of the appropriate prelimit Hamiltonians converges to the Hamiltonian that is associated with the differential game. Among other requirements, it is assumed that the terminal cost is continuous and that the terminal time is fixed. In our case, the indicator takes the role of the terminal cost, which besides of being not continuous, in this case it also depends on the history of the wealth process. Also, we consider a random terminal time that is independent of the wealth process. Moreover, partial differential equations techniques does not provide asymptotically optimal policies, while we do.

In [13], Dupuis and Kushner approached a risk sensitive control problem of minimizing escape time probabilities by techniques taken from the theory of large deviation. Some of their requirements are that the drift and the diffusion coefficients, \( b \) and \( \sigma \) respectively, are bounded and the latter is also non-degenerate and does not depend on the control. These requirements are essential for the proofs. For example, the non-degeneracy property of \( \sigma \) allows them to use the Girsanov’s change of measure theorem, which is used to prove that the value of the
lower game forms a lower bound for the limit of the value functions of the stochastic models. Also, they use a fixed terminal time. In our model, besides that the terminal time is random, the drift and the diffusion coefficients are assumed to be Lipschitz, but only the diffusion coefficient, $\sigma$, is assumed to be bounded. We allow $\sigma$ to be zero and to depend on the control. In fact, under the asymptotically optimal policy that we suggest the diffusion coefficient can be degenerate.

Recently, in [1] and [2] the authors considered a queuing network problem under the moderate deviation heavy traffic regime. By using a variant of Varadhan’s lemma and some properties of the differential game, an asymptotic optimality in the queueing systems is shown. In these papers, the controlled stochastic processes are not diffusion, but they are relatively close in distribution to a controlled diffusion with small noise. Therefore, the analysis requires some additional tools, and mainly the Skorohod mapping. While the structure of the queuing network in the prelimit raises some difficulties, the approximated diffusion is relatively simple and consists of Brownian motion (reflected Brownian motion, in the second paper) with drift. Although the proof technique also relies on an appropriate modification of Varadhan’s lemma, in contrast to [1] and [2], we need to work with a controlled diffusion process. In the appendix we present a lemma that enables us to use the Freidlin-Wentzell theorem, for a controlled small noise process, under the mentioned less strict conditions on the diffusion coefficient. We use it to show both upper and lower bounds for the limit of the value functions of the stochastic models.

Regarding the random terminal time, the cost function can be referred as a discounted version of the risk sensitive cost. The only model from the above that considered a similar discounted structure is [2]. However, unlike the mentioned paper, we consider a scaled discount factor. The differential game associated with [2] appears in [3] and like in our case, the optimal solution of the game is time-homogeneous. Motivated by this property we analyze discounted risk sensitive control with small noise diffusions further in a future paper.

Let us summarize the contribution of this paper:

- We propose a risk-sensitive cost for a lifetime ruin problem, which can be expressed as a discounted risk sensitive cost. We present a differential game that governs the limiting behavior.
- We solve the differential game explicitly, including finding an optimal policy for the minimizer that leads to an asymptotically optimal policy in the prelimit stochastic model.
- Our assumptions over the diffusion process are weaker than what usually appears in the literature, and yet by supplying a controlled version of the Freidlin-Wentzell theorem we manage to find asymptotically optimal behavior.

The organization of the paper is as follows. In Section 2 we describe the model, introduce the differential game, and state the main results. In Section 3 we analyze the differential game, present an Hamilton–Jacobi–Bellman (HJB) equation, characterize the differential game’s value function as its unique solution, and we provide an explicit expression for the value function. Then we present an explicit optimal control for the minimizer, and a simple control for the maximizer that achieves the value function. In Section 4 we prove the main result by showing that in the limit the differential game describes the stochastic model. We close this section by introducing some frequently used notation.
Notation. We denote $[0, \infty)$ by $\mathbb{R}_+$. For $f : [0, t] \to \mathbb{R}$ let $|f| := \sup_{0 \leq s \leq t} |f(s)|$. For any interval $I$ denote by $\mathcal{AC}(I)$ and $\mathcal{C}(I)$ the spaces of absolutely continuous functions (resp., continuous functions) mapping $I \to \mathbb{R}$. Write $\mathcal{AC}_0(I)$ and $\mathcal{C}_0(I)$ for the subsets of the corresponding function spaces, of functions that start at zero.

2 Model and results

2.1 The stochastic model

We study a sequence of stochastic models, indexed by $n \in \mathbb{N}$ of an investor who trades continuously in a Black-Scholes type financial market with no transaction costs. We allow borrowing and short-selling. We consider a riskless and a risky assets. The price of the riskless asset follows by

$$dV(t) = rV(t)dt,$$

where $r \geq 0$ is the constant riskless rate. We also consider a risky asset whose price follows a geometric Brownian motion process, and given by

$$dS(t) = S(t) [\mu dt + \sigma dB(t)],$$

where $\mu > r$ and $\sigma > 0$ are constants and $(B(t))_{t \geq 0}$ is a standard Brownian motion. For reasons that will be clear onwards we define a sequence of consumption functions, indexed by $n$. For any given $n \in \mathbb{N}$ we assume that there is a consumption process, which is a function of the current wealth and given by $c^n(W^n(t))$, where $W^n(t)$ is the value of the wealth process at time $t$. We assume that there is a function $e : [a, \infty) \to \mathbb{R}$, such that $c^n(\cdot) = r \cdot + \frac{1}{n}e(\cdot)$. We require that $e(\cdot)$ is a Lipschits function and that there is a positive constant $M_0$ such that $e(\cdot) \leq M_0$. Notice that in case that $e$ is linear then also $c^n$ is. For every $n \in \mathbb{N}$ and any given wealth $x$ let $\kappa^n(x)$ be the amount of money that is invested in the risky asset. Then the wealth process satisfies

$$dW^n(t) = (rW^n(t) - c^n(W^n(t)) + (\mu - r)\kappa^n(W^n(t))) dt + \sigma\kappa^n(W^n(t))dB(t), \quad t \geq 0,$$

$W^n(0) = x.$

Now, by using time scaling and by referring to $\tilde{W}^n(\cdot) = W^n(n\cdot)$ we get that

$$d\tilde{W}^n(t) = \left(-e(\tilde{W}^n(t)) + (\mu - r)\pi^n(\tilde{W}^n(t))\right) dt + \frac{1}{\sqrt{n}}\sigma\pi^n(\tilde{W}^n(t))dB(t), \quad t \geq 0,$$

$$\tilde{W}^n(0) = x,$$

where $\pi^n = n\kappa^n$. From now onwards, we refer to $\pi^n$ as the control. We denote by

$$\Pi = \Pi_{M_1, L} := \{\pi : [a, \infty) \to [-M_1, M_1] : \forall x, y \ |\pi(x) - \pi(y)| \leq L|x - y|\},$$

the set of all admissible policies, where $M_1$ and $L$ are positive constants, and take $\pi^n \in \Pi$. By the assumptions on $e(\cdot)$ and $\pi^n$ it follows that for every $x > 0$, the above admits a unique solution. For every $n \in \mathbb{N}$, denote by $\tau^n_a$ the first time that $\tilde{W}^n$ reaches $a \in (0, x)$, which we
will refer to as the ruin level. The investor would like to avoid ruin during her lifetime and also to avoid long living close to the ruin level. Also, let \( \tau^n_d \) be the investor’s random time of death. Due to the time scaling, we assume that \( \tau^n_d \) is exponentially distributed with parameter \( \lambda_n \). The goal of the investor is to minimize the following risk sensitive control cost:

\[
J^n(x, \pi_n) := \frac{1}{n} \ln \mathbb{E} \left[ e^{1_n \left( \int_0^{\tau^n_d \wedge \tau^n} l(W^n(s)) \, ds + \rho 1_{\{\tau^n_d \leq \tau^n\}} \right)} \right] = \frac{1}{n} \ln \left[ \int_0^\infty e^{-\lambda nt} e^{1_n \left( \int_0^{\tau^n_d \wedge t} l(W^n(s)) \, ds + \rho 1_{\{\tau^n_d \leq t\}} \right)} \right] + \frac{1}{n} \ln(\lambda_n),
\]

where \( \rho > 0 \) stands for the punishment cost for being ruin and \( l : [a, \infty) \to [0, \lambda) \) is a non-increasing function. The function \( l \) represents a punishment for the investor when her wealth is close to the ruin level, \( a \). Obviously, we would like to give higher punishment when the wealth is closer to \( a \). Moreover, since we would like that, given \( n \), the function \( J^n \) would be decreasing with respect to the wealth, we require that \( l(\cdot) < \lambda \). Otherwise, \( J^n \) would be increasing around \( a \). This case represents a situation when the punishment of living close to the ruin level dominates the punishment from being ruined. Notice also that the last term on the r.h.s. of the above goes to zero as \( n \) goes to infinity. For this reason we will ignore it in the analysis in Section 4.

We study the problem when \( n \to \infty \). As mentioned in the introduction, both the prelimit stochastic model and the limit suffer from several complexities in the analysis. First, the indicator part of the cost function complicates the analysis because it depends on the history of the process and if we look at it as a terminal cost, then it is not continuous with respect to the terminal wealth. Second, we study a discounted version of the risk sensitive cost. To the best of our knowledge such formulation studied before only in [2] and also in a queueing system framework and with a discount that is free of \( n \). Third, the diffusion coefficient is not necessarily bounded away from zero and it depends on the control. In fact as is shown in Section 2.3, the asymptotically optimal policy may become zero and therefore, so does the volatility coefficient. Therefore, the Hamiltonian method of [16, Chapter XI.7], or change of measure method of [13] do not work here. We find an asymptotically optimal policy for the problem by studying a differential game. We show that as \( n \to \infty \) the optimal risk sensitive cost function converges to the value of the game, and that an asymptotically optimal policy can be deduced from the minimizer’s optimal control in the game.

### 2.2 Differential game setting

In this section, inspired by [16, Chapter XI.7] and [11, Theorem 5.6.7] we describe a differential game associated with the optimal risk sensitive control problem. Given \( \psi \in C_0[0, \infty) \) and \( \pi \in \Pi \), the dynamics associated with the initial condition \( x \) and the data \( \psi \) and \( \pi \) are given by

\[
\begin{align*}
\dot{\varphi}(t) &= -e(\varphi(t)) + (\mu - r)\pi(\varphi(t)) + \sigma\pi(\varphi(t)) \dot{\psi}(t), \quad t \geq 0, \\
\varphi(0) &= x.
\end{align*}
\]
Note the analogy between the above and (2.1). The game payoff is

$$
\sup_{T \in [0, \infty)} \left\{ \int_0^{T \wedge \tau} \left[ -\lambda + l(\varphi(t)) \right] dt - \mathbb{I}(T \wedge \tau, \psi) + \rho 1_{\{\tau \leq T\}} \right\},
$$

where \( \tau \) is the first time that the dynamics hit the ruin level \( a \) and for every \( t > 0 \), \( \mathbb{I}(t, \cdot) \) is a function mapping \( C[0, t] \) to \( \mathbb{R}_+ \cup \{+\infty\} \) defined as

$$
\mathbb{I}(t, \psi) := \begin{cases} 
\frac{1}{2} \int_0^t \dot{\psi}^2(s) ds & \text{if } \psi \in AC_0[0, t], \\
+\infty & \text{otherwise.}
\end{cases}
$$

The function \( \mathbb{I} \) is the rate function with respect to the Brownian motion \( \left( \frac{1}{\sqrt{n}} B(t) \right)_t \), as \( n \to \infty \), see [11, Theorem 5.2.3]. The “sup\( _{T \in [0, \infty)} \)” is the differential game analog to the control problem’s discount factor, \( \lambda n \). The payoff is maximized over \( \psi \) and minimized over \( \pi \). By the definition of the function \( \mathbb{I} \) we may restrict the maximizer only to \( \psi \in AC_0[0, \infty) \). The explicit details are given below.

The control \( \pi \in \Pi \) is taken to be a feedback control and \( \psi \in AC_0[0, \infty) \) and \( T \in \mathbb{R}_+ \) are open-loop controls. We say that \( \pi \) is admissible if the corresponding dynamics (2.3) has a solution on \([0, T] \) for every \( \psi \). From now onwards, in the context of the differential game, we refer to \( \Pi \) as the restriction of (2.2) to admissible controls. We call \( \psi \) the path part of the control and the \( T \) a termination time part of the control. Given \( x \in [a, \infty) \), \( \pi \in \Pi \), \( \psi \in AC_0[0, \infty) \), and \( T \in \mathbb{R}_+ \), we define the cost until time \( T \) by

$$
C(x, \pi, \psi, T) := \int_0^{T \wedge \tau} \left[ -\lambda + l(\varphi(t)) - \frac{1}{2} \dot{\psi}^2(t) \right] dt + \rho 1_{\{\tau \leq T\}}.
$$

The value of the game is defined by

$$
U(x) := \inf_{\pi \in \Pi} \sup_{\psi \in AC_0[0, \infty), T \in \mathbb{R}_+} C(x, \pi, \psi, T).
$$

In the remark below we show that the maximizer can be restricted to a smaller set of controls without any loss. This property serves us in the sequel.

**Remark 2.1**

1. Since \( l(\cdot) < \lambda \) it follows that for every \( \psi \in AC_0[0, \infty) \) and every \( T \in \mathbb{R}_+ \) one has \( \int_0^{T \wedge \tau} [-\lambda + l(\varphi(t)) - \frac{1}{2} \dot{\psi}^2(t)] dt \leq 0 \). Therefore, without any loss for the maximizer she can be restricted to \( \psi \)’s under which \( \tau < \infty \), and \( T \) would be equal to either \( \tau \) or 0.

2. Notice moreover that the maximizer can also be restricted to \( \psi \)’s, for which the dynamics satisfy \( \varphi(t) < \varphi(0) =: x \) for every \( t > 0 \) and by (1) above also \( \tau < \infty \). Indeed, since the integrand on the r.h.s. of (2.4) is negative then the only way that \( U \) is positive is in case that \( \tau < \infty \). Let \( \psi = \psi_\tau \) be such that \( \tau < \infty \). Denote by \( \tau_x \) the last time before time \( \tau \) that
\[ \varphi(t) = x. \] Then,
\[
C(x, \pi, \psi, \tau) = \int_0^\tau \left[ -\lambda + l(\varphi(t)) - \frac{1}{2}\dot{\psi}^2(t) \right] dt + \rho
\]
\[
< \int_{\tau_x}^{\tau - \tau_x} \left[ -\lambda + l(\varphi_x(t)) - \frac{1}{2}(\dot{\psi}_x)^2(t) \right] dt + \rho
\]
\[
= \int_0^{\tau - \tau_x} \left[ -\lambda + l(\varphi(t)) - \frac{1}{2}(\dot{\psi}_x)^2(t) \right] dt + \rho
\]
\[
= C(x, \pi, \psi_x, \tau - \tau_x),
\]
where \( \psi_x(\cdot) := \psi(\tau_x + \cdot) \) and \( \varphi_x(\cdot) := \varphi(\tau_x + \cdot). \) The last equation follows since \( \tau - \tau_x \) is the first time that the dynamics \( \varphi_x \) hits zero. That is, \( \psi_x \) generates a greater payoff for the maximizer and the associated dynamics does not cross \( x \) upwards. Therefore, for every \( x \in [a, b] \)
\[
U(x) = \max \left\{ 0, \inf_{\pi \in \Pi} \sup_{\psi \in A_x, \pi} C(x, \pi, \psi, \tau) \right\}, \tag{2.5}
\]
where from now onwards \( A_{x, \pi} \) is the restriction to absolutely continuous \( \psi \)'s that satisfy the conditions mentioned in (2) above.

### 2.3 Main results

We now state the main theorem, which states that the limit of the value functions of the stochastic model converge to the value function of the game. Moreover, we state an asymptotically optimal policy for the stochastic model.

Set the control
\[
\pi^*(x) = \begin{cases} 
\frac{(\mu - r)e(x)}{\sigma^2(\frac{1}{2}(\frac{\mu - r}{\sigma})^2 + \lambda - l(x))}, & a \leq x < d, \\
0, & d \leq x,
\end{cases} \tag{2.6}
\]
where
\[
d := b \land \inf \left\{ y > a : \rho - \int_a^x \lambda - l(u) + \frac{1}{2}(\frac{\mu - r}{\sigma})^2 e(u) du = 0 \right\},
\]
and\(^1\)
\[
b := \inf \{ x \geq a : e(x) < 0 \}.
\]
In order to simplify the analysis, we assume that for every \( x \in [a, b] \) one has \( e(x) > 0 \). By the definition of \( b \) and by the assumption that \( e(\cdot) \leq M_0 \) it follows that \( \pi^* \in \Pi \). The parameter \( b \) stands for the minimal wealth, \( x \), such that \( e(x) \leq 0 \).

\(^1\)We use the convention that \( \inf \emptyset = \infty \). Also, hereafter, in case that \( b = \infty \) then by the notation \( (x, b] \) and \( [x, b] \) mean \( (x, \infty) \) and \( [x, \infty) \) respectively.
We will show that the control $\pi^*$ is an optimal control for the minimizer in the differential game and the value function, $U$, is given by

$$U(x) = \begin{cases} ho - \int_a^x \frac{\lambda - l(u) + \frac{1}{2}(\mu - r)^2}{e(u)} du, & a \leq x < d, \\ 0, & d \leq x, \end{cases}$$

(2.7)

Since $U(x) = 0$ for every $x \geq d$, the parameter $d$ is referred as the “safe level”. Notice that the punishment cost, $\rho$, affects $\pi^*$ and $U$ through the parameter $d$, which increases as a function of $\rho$. Therefore, if $\rho$ is high then the safe level is greater. Clearly, it also affects $U$ directly linearly on $[a,d)$.

The next theorem connects between the game and the stochastic model.

**Theorem 2.1 (Main Result)** Let $U^n(\cdot) := \inf_{\pi \in \Pi} J^n(\cdot, \pi)$. For every $x \geq a$ one has, $\lim_{n \to \infty} U^n(x) = U(x)$, and moreover, $\lim_{n \to \infty} J^n(x, \pi^*) = U(x)$.

The proof is given in Section 4.

### 3 Solution and analysis of the game

In this section we provide a solution of the game. We start by some basic properties of the value function in Section 3.1. In Section 3.2 we present the HJB equation and a verification lemma. Then we derive the explicit expression for $U$. Finally, in Sections 3.3 and 3.4 we provide an optimal control for the minimizer and a simple control for the maximizer, which assures her the payoff $U(x)$.

#### 3.1 Basic properties

We begin by providing some basic properties that the value function satisfies. These properties are used in the verification lemma below.

**Lemma 3.1** The function $U$ satisfies the following conditions:

i. $0 \leq U(x) \leq \rho$, $x \in [a,\infty)$ and $U(a) = \rho$.

ii. For every $x \geq b$ one has $U(x) = 0$

iii. $U$ is non-increasing.

Due to part ii of the Lemma in the sequel we analyze $U$ on the interval $[a,b)$. It might be the case that $U$ has a discontinuity at $x = b$.

**Proof of Lemma 3.1:**

i. By choosing $T = 0$, the maximizer can guarantee $U \geq 0$. On the other hand, since $l(\cdot) - \lambda < 0$ then clearly $U(\cdot) \leq \rho$. By choosing $T = 0$, one easily gets that $U(a) = \rho$.

ii. We show that in case that $x \geq b$ then the control $\pi \equiv 0$, that is, never investing in the risky asset can prevent from the dynamics to hit the ruin level $a$. Indeed, if $\pi \equiv 0$ then $\dot{\varphi} = -e(\varphi)$, $\varphi(0) = x \geq b$. In case that $x = b$ then $\varphi(\cdot) \equiv b$. In case that $x > b$ then since the function $e(\cdot)$
is Lipschitz we get by Picard-Lindelöf theorem that there is a unique \( \varphi \in C^1[0, \infty) \) that solves the ordinary differential equation that is mentioned above. One can easily verify that once \( \varphi \) reaches the level \( b \) it remains at this level from this time onwards. Therefore, \( \varphi \geq b \).

iii. Fix \( x \in (a, \infty) \) and \( y > x \) and set \( \varphi(0) = y \). Let \( \tau_x := \inf\{t \geq 0 : \varphi(t) = x\} \). By a dynamic programming principle and recalling that \( l(\cdot) < \lambda \) we get that

\[
U(y) = \sup_{\psi \in A_{x, \pi}, T \in \mathbb{R}_+} \inf_{\pi \in \Pi} \left[ \int_0^{T \wedge \tau_x} [-\lambda + l(\varphi(t)) - \frac{1}{2} \psi^2(t)] dt + U(x) \right] \leq U(x).
\]

\[\square\]

### 3.2 The HJB equation

In this section we prove that equation (2.5) holds. We start with a verification lemma in which we provide an appropriate HJB equation for the problem. Then we present a solution for the HJB equation. Recall that by Lemma 3.1.ii, \( U(x) = 0 \) for \( x \geq b \). Therefore, we limit ourselves to the interval \( [0, b) \).

**Lemma 3.2 (Verification Lemma)** Let \( V : [a, b) \to [0, \rho] \) be a non-increasing and continuous function, differentiable on \( (a, \beta) \), where \( \beta := b \wedge \inf\{x > a : V(x) = 0\} \), that satisfy also \( V(a) = \rho \). Let \( P : [a, \beta) \to \mathbb{R} \) and \( \Theta : \mathbb{R} \times [a, \beta) \to \mathbb{R} \) be measurable functions. Assume that \( V, P, \) and \( \Theta \) satisfy the following conditions:

(i) for every \( x \in [a, \beta) \) one has

\[
\text{[The HJB equation]} \quad \inf_{p \in \mathbb{R}} \sup_{\theta \in \mathbb{R}} \left\{ V'(x)(-e(x) + (\mu - r)p + \sigma p \theta) - \lambda + l(x) - \frac{1}{2} \theta^2 \right\} = 0; \quad (3.1)
\]

(ii) \( P \in \Pi \) and \( P(x) \) attains the infimum in (i) for every \( x \in [a, \beta) \) and \( \Theta(p, x) \) attains the supremum for every \( p \in \mathbb{R} \) and every \( x \in [a, \beta) \);

(iii) for every \( \pi \in \Pi \), set \( \psi(t) = \Theta(\pi(\varphi(t)), \varphi(t)), t \geq 0 \). Then \( \psi \in A_{x, \pi} \) (see Remark 2.1).

Then \( U = V \) on \( [a, b) \).

Notice that we defined the HJB equation only on the interval \( [a, \beta) \). This structure follows since for every \( x \) for which \( U(x) = 0 \), under optimality of both players, the time part of the maximizer’s control equals zero and the game is terminated immediately. As we show in Proposition 3.2, for every \( x \in [d, b) \), the minimizer’s optimal control would be \( \pi > 0 \). Otherwise, the maximizer can generate a positive payoff in contradiction to the fact that \( U(x) = 0 \) on this interval.

**Proof of Lemma 3.2:** 1. We will prove that for every \( x \in [a, \beta) \) one has \( V(x) \geq U(x) \). As a result \( 0 = V(\beta) \geq U(\beta) \geq 0 \), where the last inequality follows by Lemma 3.1.i. Since \( V \geq 0 \) and \( U \) is non-increasing we get that \( V \geq U \) on \( [a, b) \).
Fix $x \in [a, \beta]$. Set the control $\pi^* = P$. Also, fix a control $\psi \in A_{x, \pi^*}$ and denote by $\varphi^*$ the dynamics associated with $\pi^*$ and $\psi$. Recall that by the definition of $A_{x, \pi^*}$, for every $t > 0$, one has $\varphi^*(t) < \varphi^*(0) = x$ and that $\tau^*$ is the first time that $\varphi^*$ reaches $a$. Recalling moreover that $x < \beta$ we get that for every $t \geq 0, \varphi^*(t) < \beta$. Since $V$ is differentiable on $[a, \beta]$ we can apply the chain rule to $V$ and get

$$V(\varphi^*(t)) - V(\varphi^*(0)) = \int_0^{\tau^*} V'(\varphi^*(t))[-e(\varphi^*(t)) + (\mu - r)\pi^*(\varphi^*(t)) + \sigma\pi^*(\varphi^*(t)) \dot{\psi}(t)]dt.$$  

Using again the inequality $\varphi^*(t) < \beta$ we get by conditions (i) and (ii) that

$$0 = \sup_{\theta \in \mathbb{R}} \left\{ V'(\varphi^*(t))[-e(\varphi^*(t)) + (\mu - r)\pi^*(\varphi^*(t)) + \sigma\pi^*(\varphi^*(t)) \theta] - \lambda + l(\varphi^*(t)) - \frac{1}{2}\theta^2 \right\}$$
$$\geq V'(\varphi^*(t))[-e(\varphi^*(t)) + (\mu - r)\pi^*(\varphi^*(t)) + \sigma\pi^*(\varphi^*(t)) \dot{\psi}(t)] - \lambda + l(\varphi^*(t)) - \frac{1}{2}(\dot{\psi}^*)^2(t).$$

So we have

$$V(x) \geq \int_0^{\tau^*} [-\lambda + l(\varphi^*(t)) - \frac{1}{2}\dot{\psi}^2(t)]dt + \rho.$$  

By taking first $\sup_{\psi \in A_{x, \pi^*}}$ and then $\inf_{\pi \in \Pi}$ on both sides, we get that

$$V(x) \geq \inf_{\pi \in \Pi} \sup_{\psi \in A_{x, \pi^*}} C(x, \pi, \psi, \tau).$$

By the above and recalling that $V \geq 0$ we get by (2.5) that $V(x) \geq U(x)$.

2. By using the assumption that $V$ is non-increasing and that $U \geq 0$ it follows that it is sufficient to prove that $V \leq U$ on $x \in [a, \beta]$. Fix $x \in [a, \beta]$. Set the following control. For every $\pi \in \Pi$ let $\psi^*(t) := \Theta(\pi(\varphi^*(t)), \varphi^*(t))$, where $\varphi^*$ is the dynamics associated with $\pi$ and $\psi^*$. By (iii) $\varphi^*$, reach $a$ is finite, that is, $\tau^* < \infty$. Now, fix an arbitrary $\pi \in \Pi$. By conditions (i) and (ii) we get that

$$0 = \inf_{\rho \in \mathbb{R}} \left\{ V'(\varphi^*(t))[-e(\varphi^*(t)) + (\mu - r)p + \sigma p \dot{\psi}^*(t)] - \lambda + l(\varphi^*(t)) - \frac{1}{2}(\dot{\psi}^*)^2(t) \right\} \quad (3.2)$$
$$\leq V'(\varphi^*(t))[-e(\varphi^*(t)) + (\mu - r)\pi(\varphi^*(t)) + \sigma\pi(\varphi^*(t)) \dot{\psi}^*(t)] - \lambda + l(\varphi^*(t)) - \frac{1}{2}(\dot{\psi}^*)^2(t).$$

Since $l(\cdot) < \lambda$ we get by the above that $V'(\varphi^*(t)) \neq 0$. Therefore,

$$\dot{\varphi}^*(t) = -e(\varphi^*(t)) + (\mu - r)\pi(\varphi^*(t)) + \sigma\pi(\varphi^*(t)) \dot{\psi}^*(t) \leq \frac{\lambda - l(\varphi^*(t)) + \frac{1}{2}(\dot{\psi}^*)^2(t)}{V'(\varphi^*(t))} < 0,$$

where the last inequality follows since $V$ is non-increasing, $V'(\varphi^*(t)) \neq 0$, and since that $l(\cdot) < \lambda$. Therefore, $\varphi^*$ is decreasing and so does not cross $\varphi^*(0)$ upwards. By this argument and since $\tau^* < \infty$ we get that $\psi \in A_{x, \pi^*}$ and $\varphi^*$ never reach $\beta$. By (3.2) we get moreover that

$$V(x) \leq \int_0^{\tau^*} [-\lambda + l(\varphi^*(t)) - \frac{1}{2}(\dot{\psi}^*)^2(t)]dt + \rho.$$
By taking $\sup_{\psi \in A_{x,\pi}}$ first and then $\inf_{\pi \in \Pi}$ on both sides, we get that

$$V(x) \leq \inf_{\pi \in \Pi} \sup_{\psi \in A_{x,\pi}} C(x, \pi, \psi, \tau).$$

By the definition of $\beta$ and since $x < \beta$ it follows that $V(x) > 0$. Therefore,

$$\inf_{\pi \in \Pi} \sup_{\psi \in A_{x,\pi}} C(x, \pi, \psi, \tau) > 0$$

and by (2.5) it follows that $U(x) = \inf_{\pi \in \Pi} \sup_{\psi \in A_{x,\pi}} C(x, \pi, \psi, \tau)$. Therefore, $V(x) \leq U(x)$.

We now use the verification lemma in order to provide an explicit expression for the value function $U$.

**Proposition 3.1**

$$U(x) = \begin{cases} 
\rho - \int_a^x \frac{\lambda - l(u) + \frac{1}{2}(\mu - \frac{\mu}{\sigma})^2}{e(u)} du, & a \leq x < d, \\
0, & d \leq x,
\end{cases} \tag{3.3}$$

where recall that

$$d := b \land \inf \left\{ y > a : \rho - \int_a^x \frac{\lambda - l(u) + \frac{1}{2}(\mu - \frac{\mu}{\sigma})^2}{e(u)} du = 0 \right\}.$$

**Proof**: Recall that by Lemma 3.1.ii, for every $x \geq b$, $U(x) = 0$. We now prove that $U(x)$ satisfies (3.3) for $x \in [a, b]$ using Lemma 3.2. Denote by $V$ the restriction of the r.h.s. of (3.3) to the interval $[a, b]$. Notice that the parameter $\beta$ that appears in the verification lemma is actually $d$. Let $P : [a, d) \rightarrow \mathbb{R}$ be defined by

$$P(x) = \frac{(\mu - \rho)e(x)}{\sigma^2(\frac{1}{2}(\frac{\mu}{\sigma})^2 + \lambda - l(x))}, \tag{3.4}$$

and let $\Theta : \mathbb{R} \times [a, d) \rightarrow \mathbb{R}$ be defined by

$$\Theta(p, x) = \sigma p V'(x). \tag{3.5}$$

We claim that $V$, $P$, and $\Theta$ satisfy the conditions of the verification lemma. Indeed, $V$ is obviously non-increasing, continuous, and differentiable on $(0, d)$. It also satisfies $V(a) = \rho$. One can easily verify that the dynamics under $P$ and any $\psi \in \mathcal{A}_0[0, \infty)$ is well-defined, see [23, Theorem 19.12]. Since moreover $0 \leq l(\cdot) < \lambda$ and since $e$ is Lipschitz it follows that $P \in \Pi$. One can easily verify that the supremum in (3.1) is attained by $\theta^*(p, x) = \sigma p V'(x)$ and the infimum is attained by $p^*(x) = -\frac{\mu - \rho}{\sigma^2 V'(x)}$. Substituting $\theta^*$ and $p^*$ in (3.1) we get the true statement $V'(x) = \frac{\lambda - l(x) + \frac{1}{2}(\frac{\mu}{\sigma})^2}{e(x)}$. Moreover, notice that $\psi(\cdot) : \Theta(\pi(\varphi(\cdot)), \varphi(\cdot)) \in A_{x,\pi}$. Indeed, for every $\varphi(0) = x \in [a, \beta)$ one has $\dot{\varphi}(t) = -\frac{\lambda - l(\varphi(t)) + \frac{1}{2}(\frac{\mu}{\sigma})^2}{e(\varphi(t))}$, $t \geq 0$. $\varphi$ is non-increasing since $\dot{\varphi} < 0$. Moreover, since $\frac{\lambda - l(\varphi(t)) + \frac{1}{2}(\frac{\mu}{\sigma})^2}{e'(\varphi(t))}$ is uniformly continuous on $[a, x]$ we get that there is $C_x < 0$ such that $\dot{\varphi}(t) < C_x$. Thus, $\tau < \infty$. Altogether, $\psi \in A_{x,\pi}$.

$\square$
3.3 Optimal control for the minimizer

The following proposition shows that the control $\pi^*$ that is given by (2.6) is an optimal control for the minimizer in the differential game.

**Proposition 3.2** For every $x \in [a, \infty)$ one has $U(x) = \sup_{\psi \in A_{x, \pi^*}, T \in \mathbb{R}^+} C(x, \pi^*, \psi, T)$.

Recall that the HJB equation in the verification lemma refers only to the interval $[a, d)$. In fact we could have shown in the proof of the verification lemma that $\pi^*$ is optimal for $x \in [a, d)$.

We now show that $\pi^*$ is optimal for any $x$ in $[a, \infty)$. 

**Proof of Proposition 3.2:** Recall that $U(x) = \inf_{\pi \in \Pi} \sup_{\psi \in A_{x, \pi}, T \in \mathbb{R}^+} C(x, \pi^*, \psi, T)$. Therefore, $\sup_{\psi \in A_{x, \pi}, T \in \mathbb{R}^+} C(x, \pi^*, \psi, T) \geq U(x)$. Hence, in what follows we shall prove that $U(x) \geq \sup_{\psi \in A_{x, \pi}, T \in \mathbb{R}^+} C(x, \pi^*, \psi, T)$.

In case that $x = a$ then the maximizer can force the maximal payoff $\rho$ by choosing $T = 0$, and there is nothing to prove. In case that $x \geq b$ we already showed in the proof of Lemma 3.1 that the control $\pi = 0$ guarantees a payoff greater or equal to $U(x)$. Indeed, the dynamics would not cross $b$ downwards, so the payoff would be negative unless the maximizer stops immediately. Therefore, in this case $U(x) \geq \sup_{\psi \in A_{x, \pi^*}, T \in \mathbb{R}^+} C(x, \pi^*, \psi, T)$. We split the analysis on the interval $(a, b)$ into two cases $x \in (a, d]$ and $(d, b)$.

The proof in case $x \in (a, d]$ follows by similar arguments to the ones that appear in the proof of Theorem 4.1 and therefore omitted.

Now, fix $x \in (d, b)$. Denote by $\tau_d := \inf\{t \geq 0 : \varphi(t) = d\}$. If the maximizer does not stop by that time, the dynamics satisfy $\dot{\varphi} = -e(\varphi) < 0$ independently of the chosen $\psi$ on the time interval $[0, \tau_d]$. By a dynamic programming principle

$$C(x, \pi^*, \psi, T) = \int_0^{\tau_d} \left[-\lambda + l(\varphi(t)) + \frac{1}{2} \dot{\psi}^2(t)\right] dt + C(d, \pi^*, \psi, T).$$

The r.h.s. of the above is bounded from above by 0. Indeed, the first integrand is negative and the second term is bounded from above by $U(d)$, which equals 0 by the previous paragraph. Therefore, the maximizer would prefer to stop immediately and we get that in this case

$$\sup_{\psi \in A_{x, \pi^*}, T \in \mathbb{R}^+} C(x, \pi^*, \psi, T) = 0 = U(x).$$



3.4 Saddle point property

One can characterize an optimal control for the minimizer. Instead we provide a control (for the maximizer) that is independent of $\pi$ and that assures her the payoff $U(x)$. This control serves us in the sequel, proving the lower bound in Section 4.1. Due to its simplicity the analysis in the proof of Theorem 4.1 will be simpler. The path part of the control of the maximizer, $\psi^*$, is a linear function, which neutralize the impact of $\pi$ in the dynamics. The time control $T^*$ equals to $\tau$ or zero. Moreover, we give a simple upper bound for $T^*$.
Set
\[ \psi^*(t) = -\frac{\mu - r}{\sigma} t, \quad t \geq 0 \quad (3.6) \]
and let \( T^* = \tau \) in case \( x < d \) and \( T^* = 0 \) otherwise. Notice that \( \psi^* \) is independent of the control \( \pi \). Moreover, notice that under \( \psi^* \) the dynamics satisfy \( \dot{\varphi} = -e(\varphi) \) and therefore \( \varphi \) and \( T^* \) are also independent of the choice of \( \pi \).

**Proposition 3.3** For every \( x \in [a, \infty) \) one has \( U(x) = \inf_{\pi \in \Pi} C(x, \pi, \psi^*, T^*) \). Moreover,
\[ T^* \leq \frac{\rho - U(x)}{\lambda + l(a) - \frac{1}{2}(\frac{\mu - r}{\sigma})^2}. \quad (3.7) \]

**Proof:** For \( x \geq d \) one has \( U(x) = 0 \) and by definition \( T^* = 0 \), so that \( \inf_{\pi} C(x, \pi, \psi^*, T^*) = 0 \). Set \( x < d \). We first prove (3.7). By Proposition 3.1 \( U(x) > 0 \). Since moreover \( l \) is non-increasing we get by (2.5) that
\[
U(x) = \int_0^{T^*} [-\lambda + l(\varphi(t)) - \frac{1}{2} (\frac{\mu - r}{\sigma})^2] dt + \rho \\
\leq \int_0^{T^*} [-\lambda + l(a) - \frac{1}{2} (\frac{\mu - r}{\sigma})^2] dt + \rho \\
= -T^* [\lambda - l(a) + \frac{1}{2} (\frac{\mu - r}{\sigma})^2] + \rho
\]
and (3.7) follows. Under \( \psi^* \), the dynamics is independent of \( \pi \) and hits \( a \) in a finite time as we just showed. Hence, for every \( \pi \in \Pi \)
\[
C(x, \pi, \psi^*, T^*) = \int_0^{T^*} [-\lambda + l(\varphi(t)) - \frac{1}{2} (\dot{\psi}^*)^2(t)] dt + \rho \\
= -\int_a^x \lambda - l(u) + \frac{1}{2} (\frac{\mu - r}{e(u)})^2 du + \rho \\
= U(x),
\]
where the second equality follows by the change of variables \( u = \varphi(t) \), and the last equality follows by Proposition 3.1. Hence, the first part of the theorem is proved.

\[ \square \]

4 Proof of Theorem 2.1

The proof of Theorem 2.1 follows the idea of the proof of Varadhan’s lemma. We show in two separate theorems that \( U(x) \) is a lower (resp., upper) bound to \( \liminf_{n \to \infty} U^n(x) \) (resp., \( \limsup_{n \to \infty} U^n(x) \)), where \( U^n(\cdot) := \inf_{\pi \in \Pi} J^n(\cdot, \pi) \). Moreover, we show that the policy \( \pi^* \) is asymptotically optimal.
4.1 Lower bound

**Theorem 4.1** For every \( x \geq a \) one has \( \liminf_{n \to \infty} U^n(x) \geq U(x) \).

**Proof:** Recall that \( U(x) = 0 \) for every \( x \geq d \). Since \( l \geq 0 \) and also \( r > 0 \) it follows that for any sequence of policies \( \{\pi^n\}_n \) one has

\[
\frac{1}{n} \ln E \left[ n \left( \int_0^1 \sigma_{\pi^n}^a W(t) dt + \rho_1(\tau^{\pi^n}_{d+1}) \right) \right] \geq 0.
\]

Fix \( x \in (a, d) \) and fix an arbitrary sequence of policies \( \{\pi^n\}_n \subseteq \Pi \). We show that for every \( \varepsilon > 0 \) there is \( N > 0 \) such that for every \( n > N \) one has \( J^n(x, \pi^n) \geq U(x) - w_0(\varepsilon) \), where \( w_0(\varepsilon) \to 0 \) as \( \varepsilon \to 0 \). We start with introducing some notation. Fix \( m \in \mathbb{N}, T > 0 \), and a policy \( \pi \in \Pi \) and denote by \( \hat{b}(x) = \hat{b}_\pi(x) = -e(x) + (\mu - r)\pi(x) \) and \( \hat{\sigma}(x) = \hat{\sigma}_\pi(x) = \sigma_\pi(x) \). Let the functions \( F_\pi : \mathcal{AC}_0[0, T] \to \mathcal{C}_0[0, T] \) and \( F^m_\pi : \mathcal{C}_0[0, T] \to \mathcal{C}_0[0, T] \) to be defined via \( \varphi = F_\pi(\psi) \) and \( \varphi^m = F^m_\pi(\psi) \), where

\[
\dot{\varphi}(t) = \hat{b}(\varphi(t)) + \hat{\sigma}(t)(\dot{\varphi}(t)), \quad t \in [0, T] \quad (4.1)
\]

and

\[
\varphi^m(t) = \varphi \left( \frac{k}{m} \right) + \hat{b} \left( \varphi \left( \frac{k}{m} \right) \right) \left( t - \frac{k}{m} \right) + \hat{\sigma} \left( \varphi \left( \frac{k}{m} \right) \right) (\varphi(t) - \psi \left( \frac{k}{m} \right)) \quad (4.2)
\]

\[
t \in \left[ \frac{k}{m}, \frac{k + 1}{m} \right], \quad k = 0, \ldots, m, \quad \varphi(0) = x.
\]

Also define the function \( G_\pi : \mathcal{C}_0[0, T] \to \mathcal{C}_0[0, T] \), via \( W = G_\pi(\omega) \), where

\[
dW(t) = \hat{b}(W(t))dt + \hat{\sigma}(W(t))d\omega(t), \quad t \in [0, T] \quad (4.3)
\]

\[
W(0) = x.
\]

Notice that \( G_\pi \left( \frac{\omega}{\sqrt{n}} \right) \) is the SDE associated with the wealth process, where \( \omega(\cdot) \) stands for a path of a standard Brownian motion. Hereafter, we use \( \omega \) to represent a realization of a standard Brownian motion which with probability one belongs to \( \mathcal{C}_0[0, \infty) \).

Let \( \psi^* \) be the function from (3.6) and let

\[
\dot{\psi}^*(t) = \begin{cases} -e(\psi^*(t)), & 0 \leq t \leq T^*, \\ -e(a), & T^* \leq t, \end{cases} \quad (4.4)
\]

with \( \psi^*(0) = x \), where \( T^* \) is the first time that \( \varphi^* \) hits zero. Note that up to time \( T^* \), \( \varphi^* \) are the dynamics of the differential game associated with \( \psi^* \) and any control \( \pi \in \Pi \).

Set \( \varepsilon_1 > 0 \). Since \( l \) is uniformly continuous then there exists \( \gamma_1 > 0 \) such that for every \( y, z \in [a, \infty) \)

\[
|y - z| < 3\gamma_1 \quad \text{implies} \quad |l(y) - l(z)| < \varepsilon_1. \quad (4.5)
\]
Since moreover $\varphi^*$ is continuous and for $t > T^*$, $\varphi^*(t) < 0$ it follows that one may choose $\gamma_1$ such that
\[
|\varphi - \varphi^*|_{T^* + 2\varepsilon_1} < 3\gamma_1 \quad \text{implies} \quad |\tau_a[\varphi] - T^*| < \varepsilon_1, \tag{4.6}
\]
where $\tau_a[\varphi] := \inf\{t \geq 0 : \varphi(t) = a\}$. Indeed, recall that $\varphi(0) = x \in (a, d)$. Now, since $e(\cdot)$ is positive on $[a, d)$ we get by (4.4) that the dynamics $\varphi^*$ is strictly decreasing on $[0, T^* + 2\varepsilon_1]$, touching $a$ only at $T^*$ and continuing to decrease on $[T^*, T^* + 2\varepsilon_1]$.

For every $n, m \in \mathbb{N}$ define
\[
E_{n,m} := \left\{ \omega \in C_0[0, T^* + 2\varepsilon_1] : \left| F_{\pi^n}(\frac{\omega}{\sqrt{n}}) - G_{\pi^n}\left(\frac{\omega}{\sqrt{n}}\right) \right|_{T^* + 2\varepsilon_1} \leq \gamma_1 \right\}.
\]

For every $\delta > 0$ define
\[
B^n_\delta := \left\{ \omega \in C_0[0, T^* + 2\varepsilon_1] : \left| \frac{\omega}{\sqrt{n}} - \psi^* \right|_{T^* + 2\varepsilon_1} < \delta \right\}.
\]

Also set
\[
k = k(\varepsilon_1) := \lfloor T^* + 2\varepsilon_1, \psi^*) + 1 + \varepsilon = \frac{1}{2} \left( \frac{\mu - r}{\sigma} \right)^2 (T^* + 2\varepsilon_1) + 1 + \varepsilon_1.
\]

The following lemma consists some technical arguments, which are required for the proof of the theorem.

**Lemma 4.3** There exist $M = M(\gamma_1, k) \in \mathbb{N}$, $\delta_1 = \delta_1(M)$, and $N = N(M, \delta_1) \in \mathbb{N}$ such that for every $n \geq N$ the following hold.

\[
P((E^{n,M})^c) \leq e^{-nk}, \tag{4.7}
\]

$\omega \in B^n_\delta \cap E^{n,M}$ implies
\[
|F_{\pi^n}(\psi^*) - G_{\pi^n}\left(\frac{\omega}{\sqrt{n}}\right)|_{T^* + 2\varepsilon_1} \leq 3\gamma_1, \tag{4.8}
\]

and
\[
P(B^n_\delta) \geq e^{-n(k-1)}. \tag{4.9}
\]

The r.h.s. of (4.8) can be simply expressed as
\[
|\varphi^* - \tilde{W}^n|_{T^* + 2\varepsilon_1} \leq 3\gamma_1, \tag{4.10}
\]

where $\tilde{W}^n$ is the realization of (2.1) given the observation $\omega(\cdot)$.

**Proof of Lemma 4.3:** The proof relies on Lemma A.1. To use it, notice that given $x > 0$, for every $\pi \in \Pi(M_1, L)$ one has $(\hat{b}_\pi, \hat{\sigma}_\pi) \in BL(x, L, \sigma M_1, M_2)$ (see the definition in the appendix), where $\hat{b}_\pi(x) := -e(x) + (\mu - r)\pi(x)$, $\hat{\sigma}_\pi(x) := \sigma\pi(x)$, and $M_2 = (\mu - r)M_1 + \sup_{x \geq a} e(x)$, which
is finite by the assumptions in the model. By (A.2) and (A.3) in the appendix it follows that there is $M(\gamma_1, k)$ sufficiently large such that

$$\sup_{n \in \mathbb{N}} F_{\pi^n}(\psi^*) - F_{\pi^n}(\psi^*)|_{T^* + 2\varepsilon_1} \leq \gamma_1.$$  \hspace{1cm} (4.11)

and

$$\limsup_{n \to \infty} \frac{1}{n} \ln P\left( \left| F_{\pi^n}(\frac{\omega}{\sqrt{n}}) - G_{\pi^n}(\frac{\omega}{\sqrt{n}}) \right|_{T^* + 2\varepsilon_1} > \gamma_1 \right) \leq -k$$

Hence, there exists $N(M)$ sufficiently large such that for every $n > N(M)$ inequality (4.7) holds. By (A.1) in the appendix it follows that there exists $\delta_1(M) > 0$ sufficiently small such that

$$\omega \in B^n_{\delta_1} \implies \sup_{n > N(M)} \left| F_{\pi^n}(\psi^*) - F_{\pi^n}(\frac{\omega}{\sqrt{n}}) \right|_{T^* + 2\varepsilon_1} \leq \gamma_1.$$ \hspace{1cm} (4.12)

Fix $n > N(M)$ and $\omega \in B^n_{\delta_1} \cap E^{n, M}$. By the triangle inequality we get

$$\left| F_{\pi^n}(\psi^*) - G_{\pi^n}(\frac{\omega}{\sqrt{n}}) \right|_{T^* + 2\varepsilon_1} \leq \left| F_{\pi^n}(\psi^*) - F_{\pi^n}(\psi^*)|_{T^* + 2\varepsilon_1} \right| + \left| F_{\pi^n}(\psi^*) - F_{\pi^n}(\frac{\omega}{\sqrt{n}}) \right|_{T^* + 2\varepsilon_1}$$

$$+ \left| F_{\pi^n}(\psi^*) - F_{\pi^n}(\frac{\omega}{\sqrt{n}}) \right|_{T^* + 2\varepsilon_1}.$$ \hspace{1cm} (4.13)

From (4.11) and (4.12) it follows that each of the first two terms on the r.h.s. of the above is bounded from above by $\gamma_1$. The third term admits the same upper bound by the definition of $E^{n, M}$. Hence, (4.8) follows. Finally, the lower bound of (4.9) follows by Schilder’s theorem, from which it follows that there exists $N(\delta_1)$ sufficiently large such that (4.9) holds for every $n > N(\delta_1)$. At last, take $N(M, \delta_1) = \max\{N(M), N(\delta_1)\}$.

We are now ready to bound from below $J^n(x, \pi^n)$. Fix $n > N$ then

$$J^n(x, \pi^n) = \mathbb{E} \left[ \int_0^\infty e^{-\lambda n t} e^{n \int_0^{r_n \vee t} l(\tilde{W}^n(s)) ds + \rho_1(\tilde{W}^n(s)) \vee \varepsilon_1) dt \right]$$

$$\geq \mathbb{E} \left[ \int_{T^* + \varepsilon_1}^{T^* + 2\varepsilon_1} e^{-\lambda n t} e^{n \int_0^{r_n \vee (T^* + \varepsilon_1)} l(\tilde{W}^n(s)) ds + \rho_1(\tilde{W}^n(s)) \vee \varepsilon_1) dt \right]$$

$$\geq \mathbb{E} \left[ e^{n \left( -\lambda(T^* + 2\varepsilon_1) + \int_0^{r_n \vee (T^* + \varepsilon_1)} l(\tilde{W}^n(s)) ds + \rho_1(\tilde{W}^n(s)) \vee \varepsilon_1) \right) 1_{B^n_{\delta_1} \cap E^{n, M}} \right]$$

$$\geq \mathbb{E} \left[ e^{n \left( -\lambda(T^* + 2\varepsilon_1) + \int_0^{T^* - \varepsilon_1} l(\tilde{W}^n(s)) ds + \rho \right) 1_{B^n_{\delta_1} \cap E^{n, M}} \right]$$

$$\geq \mathbb{E} \left[ e^{n \left( -\lambda(T^* + 2\varepsilon_1) + \int_0^{T^* - \varepsilon_1} \left| l(\tilde{W}^n(s)) - \omega \right| ds + \rho \right) 1_{B^n_{\delta_1} \cap E^{n, M}} \right].$$
Lemma 4.4

The last three inequalities are easy to check. The forth inequality follows by (4.6) and (4.10). The last inequality follows by (4.5) and (4.10). Notice that the terms on the last exponent are non-random. By taking \( \liminf \frac{1}{n} \ln \) on both sides of the last sequence of inequalities and reorganizing the terms, we get that

\[
\liminf \frac{1}{n} \ln J^n(x, \pi^n) = \liminf \frac{1}{n} \ln \mathbb{E} \left[ \int_0^\infty e^{-\lambda nt} e^n (f_0^{n \wedge t} l(\tilde{W}^n(s)) ds + \rho l(\epsilon \leq t)) \right] \\
\geq -\lambda T^* + \int_0^{T^*} l(\psi^*(s)) ds + \rho - 2\epsilon_1 \lambda - \epsilon_1 (T^* - \epsilon_1) - \int_{T^* - \epsilon_1}^{T^*} l(\psi^*(s)) ds \\
+ \liminf \frac{1}{n} \ln \mathbb{P}(B^n_{\delta_1} \cap \mathcal{E}^{n,M}) \\
= U(x) + \frac{1}{2} \int_0^{T^*} (\psi^*)^2(s) ds + w_1(\epsilon_1) + \liminf \frac{1}{n} \ln \mathbb{P}(B^n_{\delta_1} \cap \mathcal{E}^{n,M}),
\]

where \( w_1(\epsilon_1) := -2\epsilon_1 \lambda - \epsilon_1 (T^* - \epsilon_1) - \int_{T^* - \epsilon_1}^{T^*} l(\psi^*(s)) ds \). The equality follows by Proposition 3.3. We now show that the sum of the second and the last terms of the above is bounded from below by \( w_2(\epsilon_1) := -\epsilon_1 (\frac{\mu - r}{\sigma})^2 - \epsilon_1 \). By setting \( w_0 = w_1 + w_2 \) the proof ends. To this end, by (4.7) and since \( k = \mathbb{I}(T^* + 2\epsilon_1, \psi^*) + 1 + \epsilon_1 \) we get that

\[
\liminf \frac{1}{n} \ln \left( \mathbb{P}(B^n_{\delta_1} \cap \mathcal{E}^{n,M}) \right) \\
\geq \liminf \frac{1}{n} \ln \left( \mathbb{P}(B^n_{\delta_1}) - \mathbb{P}(\mathcal{E}^{n,M}) \right) \\
\geq \liminf \frac{1}{n} \ln \left( e^{-n(\mathbb{I}(T^* + 2\epsilon_1, \psi^*) + \epsilon_1)} - e^{-nk} \right) \\
= -\mathbb{I}(T^* + 2\epsilon_1, \psi^*) - \epsilon_1 \\
= -\frac{1}{2} \int_0^{T^*} (\psi^*)^2(s) ds - \epsilon_1 \left( \frac{\mu - r}{\sigma} \right)^2 - \epsilon_1.
\]

\[\square\]

### 4.2 Asymptotically optimal policy

In this section we show that the optimal policy in the game, which was defined in (2.6) is an asymptotically optimal policy in the stochastic model.

**Theorem 4.2** For every \( x \geq a \) one has \( \limsup_{n \to \infty} J^n(x, \pi^*) \leq U(x) \).

**Proof:** Fix \( x \geq a \). The first step in the proof is to bound the discounted cost from above by an alternative cost. Set \( T := \rho/(\lambda - l(a)) \).

**Lemma 4.4**

\[
\limsup \frac{1}{n} \ln \mathbb{E} \left[ \int_0^\infty e^{-\lambda nt} e^n (f_0^{n \wedge t} l(\tilde{W}^n(s)) ds + \rho l(\epsilon \leq t)) dt \right] \\
\leq \limsup \frac{1}{n} \ln \left( \sup_{0 \leq t \leq T} \mathbb{E} \left[ e^n (f_0^{n \wedge t} l(\tilde{W}^n(s)) - \lambda ds + \rho l(\epsilon \leq t)) \right] \right).
\]

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Proof: The lemma follows by taking “lim sup $\frac{1}{n}\ln$” of the following sequence of inequalities.

$$
\mathbb{E}
\left[
\int_0^\infty e^{-\lambda nt} e^n \left( f^{n,\lambda,\ell}_t (\tilde{W}^{n}(s)ds + \rho_1 (\tau^n_\ell \leq t) \right) dt
\right]
\leq \mathbb{E}
\left[
\int_0^T e^{-\lambda nt} e^n \left( f^{n,\lambda,\ell}_t (\tilde{W}^{n}(s)ds + \rho_1 (\tau^n_\ell \leq t) \right) dt + \int_T^\infty e^{nt(l(a) - \lambda)} dt
\right]
= \mathbb{E}
\left[
\int_0^T e^{-\lambda nt} e^n \left( f^{n,\lambda,\ell}_t (\tilde{W}^{n}(s)ds + \rho_1 (\tau^n_\ell \leq t) \right) dt + \frac{1}{n(\lambda - l(a))} e^{nT(l(a) - \lambda)}
\right]
\leq \mathbb{E}
\left[
\int_0^T e^n \left( f^{n,\lambda,\ell}_t [l(\tilde{W}^{n}(s)) - \lambda]ds + \rho_1 (\tau^n_\ell \leq t) \right) dt + \frac{1}{n(\lambda - l(a))}
\right]
\leq \limsup_{0 \leq t \leq T} \mathbb{E}
\left[
\frac{1}{n} \left( f^{n,\lambda,\ell}_t [l(\tilde{W}^{n}(s)) - \lambda]ds + \rho_1 (\tau^n_\ell \leq t) \right) dt + \frac{1}{n(\lambda - l(a))}
\right].
$$

The first inequality follows since $l$ is non-increasing and since that by eliminating the indicator on the second exponent we only increase the cost. The second inequality follows by the choice of $T$ and since $-\lambda nt \geq -\lambda n(\tau^n_\ell \wedge t)$. The other relations are trivial. \hfill \Box

Fix $\varepsilon_1 > 0$. As mentioned earlier, in (4.5), since $l$ is uniformly continuous then there exists $\gamma_1 > 0$ such that for every $y, z \in \{a, \infty\}$

$$
|y - z| < 3 \gamma_1 \quad \text{implies} \quad |l(y) - l(z)| < \varepsilon_1.
$$

We use below the functions $F_{\pi^*}, F_m^{\pi^*}$, and $G_{\pi^*}$ that were defined through (4.1)–(4.3) with respect to the policy $\pi^*$. Since we refer in this section to the single policy $\pi^*$, we omit the subindex $\pi^*$ from the mentioned functions.

Compact level set: As mentioned at the beginning of Section 4 we follow the idea of the proof of Varadhan’s lemma. For every $J > 0$ define

$$
D_J := \{\psi \in \mathcal{C}_0[0, 2T] : I(2T, \psi) \leq J\}.
$$

By [11, Theorem 5.2.3] $\mathbb{I}$ is a good rate function, and therefore $D_J$ is compact.

We now provide for each $\tilde{\psi} \in D_J$ a neighborhood and some related parameters. For every $\tilde{\psi} \in \mathcal{C}_0[0, 2T]$ and particularly for $\tilde{\psi} \in D_J$, the function $F(\tilde{\psi})(t)$ is continuous as a function of $t$ and hence, we get that there exists $\gamma_1(\tilde{\psi})$ such that $\gamma_1(\tilde{\psi})$ that satisfies

$$
|h - F(\tilde{\psi})|_{2T} \leq 3 \gamma_1 \quad \text{one has} \quad \tau_a[F(\tilde{\psi})] - \varepsilon_1 \leq \tau_a[h],
$$

where $\tau_a[h] := \inf \{0 \leq t \leq T : h(t) \leq a\}$ and $\tau_a[h] := \infty$ in case that $h > a$ on the time interval $[0, T]$. By Lemma A.1 in the appendix there exists also $M = M(J, \gamma_1(\tilde{\psi}))$ such that

$$
\limsup_{n \to \infty} \frac{1}{n} \ln \mathbb{P} \left( \left| F^M \left( \frac{\omega}{\sqrt{n}} \right) - G \left( \frac{\omega}{\sqrt{n}} \right) \right|_{2T} > \gamma_1 \right) \leq -\rho - 2
$$

and such that

$$
\sup_{\psi \in D_J} |F^M(\psi) - F(\psi)|_{2T} \leq \gamma_1.
$$
Since $F^M(\psi)$ is continuous as a function of $\psi$ (see (A.1) in the appendix) and since $I$ is upper semi-continuous it follows that there exists a fixed constant $\nu > 0$, such that for every $t \in [0, 2T]$, 

$$\inf_{\psi \in \hat{B}_{\psi,t}} I(t, \psi) \geq I(t, \hat{\psi}) - \varepsilon_1,$$  

and also

$$\sup_{\psi \in \hat{B}_{\psi,2T}} |F^M(\psi) - F^M(\hat{\psi})|_{2T} \leq \gamma_1,$$  

where $\hat{B}_{\psi,t} := \{ \psi \in \mathcal{AC}[0, \infty) : |\psi - \hat{\psi}|_t < \nu \}$.

For every $n \in \mathbb{N}$, $\hat{\psi} \in D_J$, and $t \in [0, 2T]$ define

$$\mathcal{E}_{\gamma_1(\hat{\psi})}^{n,M(J,\gamma_1(\hat{\psi}))} := \{ \omega \in \mathcal{C}_0[0, 2T] : \left| F^M \left( \frac{\omega}{\sqrt{n}} \right) - G \left( \frac{\omega}{\sqrt{n}} \right) \right|_{2T} \leq \gamma_1 \}$$

and

$$\mathcal{B}_{\hat{\psi},t}^n := \{ \omega \in \mathcal{C}_0[0, 2T] : \frac{\omega}{\sqrt{n}} \in \hat{B}_{\psi,t} \}.$$ 

Notice that as in (4.13), by the triangle inequality, and also by the definitions of $\mathcal{E}_{\gamma_1(\hat{\psi})}^{n,M(J,\gamma_1(\hat{\psi}))}$ and $\mathcal{B}_{\hat{\psi},t}^n$ and by (4.20) and (4.22) we may choose the neighborhood to be sufficiently small such that

$$\omega \in \mathcal{E}_{\gamma_1(\hat{\psi})}^{n,M(J,\gamma_1(\hat{\psi}))} \cap \mathcal{B}_{\hat{\psi},2T}^n \quad \text{implies} \quad \left| G \left( \frac{\omega}{\sqrt{n}} \right) - F(\hat{\psi}) \right|_{2T} \leq 3\gamma_1.$$  

Also by (4.18)

$$\tau_a \left[ F \left( \frac{\hat{\psi}}{\sqrt{n}} \right) \right] - \varepsilon_1 \leq \tau_a \left[ \frac{\omega}{\sqrt{n}} \right]$$  

Now, since $D_J$ is compact, one can find a finite number of members $\psi^1, \ldots, \psi^K \in D_J$ such that $D_J \subseteq \cup_{k=1}^K \mathcal{B}_{\psi^k,2T}$. Since the only restrictions about the neighborhood come from (4.21) and (4.22) it follows that $K$ is independent of $n$. Hereafter, we use to following notation

$$\mathcal{E}_k^n = \mathcal{E}_{\gamma_1(\hat{\psi})}^{n,M(J,\gamma_1(\psi^k))} \quad \text{and} \quad \mathcal{B}_{k,2T}^n = \mathcal{B}_{\psi^k,2T}^n,$$

where $t \in [0, 2T]$. We are now ready to bound from above the r.h.s. of (4.16). Fix $0 \leq t \leq T$. Then,

$$\mathbb{E} \left[ e^{n \left( \int_0^T \lambda_1 \mu_l (\hat{W} - \lambda) ds + \rho_1 (\tau^0 \leq t) \right) } \right] \leq \sum_{k=1}^K \mathbb{E} \left[ e^{n \left( \int_0^T \lambda_1 \mu_l (\hat{W} - \lambda) ds + \rho_1 (\tau^0 \leq t) \right) } 1_{\mathcal{B}_{k,2T}^{\mathcal{E}_k^n}} \right] + \sum_{k=1}^K \mathbb{E} \left[ e^{n \rho_1 (\mathcal{E}_k^n) \varepsilon} \right] + \mathbb{E} \left[ e^{n \rho_1 1_{\mathcal{B}_n}} \right].$$
where $\mathcal{B}^n := \{ \omega \in C_0[0, 2T] : \frac{\omega}{\sqrt{n}} \in \bigcup_{k=1}^{K} B_{\psi^k, 2T} \}$. We now show that for sufficiently large $n$ the following three inequalities hold

$$\sup_{0 \leq t \leq T} \sum_{k=1}^{K} \mathbb{E} \left[ e^{n\left(\int_0^{t} [I(W^n(s)) - \lambda] ds + \rho_1(\tau_a \leq t)\right) 1_{B_{\psi^k, 2T} \cap \mathcal{E}^n_k}} \right] \leq K e^{n(U(x) + w(\epsilon_1)) + \epsilon_1}, \quad (4.25)$$

$$\sum_{k=1}^{K} \mathbb{E} \left[ e^{n\rho(\mathcal{E}^n_k)} \right] \leq K e^{-n}, \quad (4.26)$$

$$\mathbb{E} \left[ e^{n\rho(\mathcal{B}^n)} \right] \leq e^{n(r - J + \epsilon_1)}, \quad (4.27)$$

where $w_3(\epsilon) \to 0$ as $\epsilon \to 0$. Lemma 4.4 and the last three inequalities imply that

$$\limsup \frac{1}{n} \ln \mathbb{E} \left[ \int_0^\infty e^{-\lambda nt} e^{n\left(\int_0^{t} [I(W^n(s)) - \lambda] ds + \rho_1(\tau_a \leq t)\right) dt} \right] \leq \max \{ U(x) + w(\epsilon_1), -1, \rho - J + \epsilon_1 \}. \quad \text{By taking } J = \rho \text{ and } \epsilon_1 \to 0 \text{ the proof of the theorem is done.} \quad \text{We now prove the three inequalities mentioned above.}$$

**Inequality (4.25):** It is sufficient to show that each of the terms in the sum on the l.h.s. of the first inequality is bounded from above by $e^{n(U(x) + w(\epsilon_1))} + \epsilon_1$. Fix $k \in \{1, \ldots, K\}$. Recall that $G\left(\frac{\omega}{\sqrt{n}}\right) = \tilde{W}^n$, so $\tau_a \left[ \frac{\omega}{\sqrt{n}} \right] = \tau_a \left[ \tilde{W}^n \right]$. Also denote $\varphi^k := F(\psi^k)$. Then

$$\sup_{0 \leq t \leq T} \mathbb{E} \left[ e^{n\left(\int_0^{t} [I(W^n(s)) - \lambda] ds + \rho_1(\tau_a \leq t)\right) 1_{B_{\psi^k, 2T} \cap \mathcal{E}^n_k}} \right] \leq \sup_{0 \leq t \leq T} \mathbb{E} \left[ e^{n\left(\int_0^{t} [I(W^n(s)) - \lambda] ds + \rho_1(\tau_a \leq t) + T \epsilon_1\right) 1_{B_{\psi^k, 2T} \cap \mathcal{E}^n_k}} B_{\psi^k, 2T} \right]$$

$$\leq \sup_{0 \leq t \leq T} e^{n\left(\int_0^{t} [I(W^n(s)) - \lambda] ds + \rho_1(\tau_a \leq t) + T \epsilon_1\right) 1_{B_{\psi^k, 2T} \cap \mathcal{E}^n_k}} B_{\psi^k, 2T} \right].$$

The first inequality follows by (4.17), (4.23), since the integrand is negative, and also by (4.24). Indeed, assume w.l.o.g. that $\epsilon_1 < T$, which yields $T + \epsilon_1 < 2T$. Then, if $\omega \in B_{\psi^k, 2T} \cap \mathcal{E}^n_k$ and for some $t \in [0, T]$, $\tau_a \left[ \frac{\omega}{\sqrt{n}} \right] < t$ we get that $\tau_a \left[ \psi^k \right] < \infty$ and by (4.24) and the definition of $\tau_a \left[ \psi^k \right]$ below (4.18), also $\tau_a \left[ \psi^k \right] < \infty$ and $\tau_a \left[ \psi^k \right] - \epsilon_1 < t$. Take $t^k \in [0, T]$ to be such that the rightmost-hand side is bounded from above by

$$e^{n\left(\int_0^{t^k} [I(W^n(s)) - \lambda] ds + \rho_1(\tau_a \leq t^k) + T \epsilon_1\right) 1_{B_{\psi^k, 2T} \cap \mathcal{E}^n_k}} B_{\psi^k, 2T} \right] + \epsilon_1. \quad (4.28)$$

Since $\Pi$ is a good rate function and by (4.21) we get that for sufficiently large $n$,

$$\frac{1}{n} \ln \mathbb{P}(B_{\psi^k, t^k + \epsilon_1}^n) \leq - \inf_{\psi \in B_{\psi^k, t^k + \epsilon_1}^n} \Pi(t^k + \epsilon_1, \psi) + \epsilon_1 \leq -\Pi(t^k + \epsilon_1, \psi^k) + 2 \epsilon_1.$$
Therefore, (4.28) is bounded from above by
\[ e^{n \left( \int_0^{\tau_0[\varphi^k \wedge t^k]} [l([\varphi^k(s)] - \lambda)ds + r_1 \{ \tau_0[\varphi^k \wedge t^k] \leq t^k \} - \mathbb{I}(t^k + \varepsilon_1, \psi^k) + (2 + T)\varepsilon_1 \} \right) + \varepsilon_1}. \]

Since \( l(\cdot) - \lambda \) is bounded, one can easily verify that the last expression is bounded from above by
\[ e^{n \left( \int_0^{\tau_0[\varphi^k \wedge t^k]} [l([\varphi^k(s)] - \lambda)ds + r_1 \{ \tau_0[\varphi^k \wedge t^k] \leq t^k \} - \mathbb{I}(t^k + \varepsilon_1, \psi^k) + (2 + T)\varepsilon_1 \} \right) + \varepsilon_1, \]

where \( w_3(\varepsilon_1) := (2 + T)\varepsilon_1 + \int_0^{\tau_0[\varphi^k \wedge (t^k + \varepsilon_1)]} [l([\varphi^k(s)] - \lambda)ds \text{ and } w_3(\varepsilon_1) \to 0 \text{ as } \varepsilon_1 \to 0. \) The term on the last exponent equals to \( C(x, \pi^*, \psi^k, t^k + \varepsilon_1) + w_3(\varepsilon_1), \) which by the definition of the value function \( U \) is bounded from above by \( U(x) + w_3(\varepsilon_1). \) Thus, the first inequality follows.

**Inequality (4.26):** Fix \( k \in \{1, \ldots, k\}. \) By (4.19) we get that for sufficiently large \( n, \)
\[ \frac{1}{n} \ln P(E^n_k) \leq -\rho - 1. \]

Hence,
\[ E[e^{n\rho}1_{E^n_k}] \leq e^{-n}. \]

Summing up over all the \( k \)'s and the second inequality follows.

**Inequality (4.27):** Since \( \mathbb{I} \) is a good rate function we get that for sufficiently large \( n, \)
\[ \frac{1}{n} \ln P(B^n) \leq -\inf_{\psi \in \left( \cup_{k=1}^{K} B_{\psi^k, 2T} \right)^c} \mathbb{I}(T, \psi) + \varepsilon_1. \]

Together with the fact that \( \cup_{k=1}^{K} B_{\psi^k} \subseteq D^c_j \) and the definition of \( D_J \) we get that for large \( n, \)
\[ E[e^{n\rho}1_{B^n}] \leq \exp \left( n \left( \rho - \inf_{\psi \in \left( \cup_{k=1}^{K} B_{\psi^k, 2T} \right)^c} \mathbb{I}(T, \psi) + \varepsilon_1 \right) \right) \leq e^{n(\rho-J+\varepsilon_1)}. \]

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**Appendix**

Fix \( x, L, M_1, M_2 > 0. \) Denote by \( BL_x = BL(x, L, M_1, M_2) \) the set of all functions \( \hat{b} : \mathbb{R} \to \mathbb{R}, \)
\( \hat{\sigma} : \mathbb{R} \to [-M_1, M_1] \) such that
\[ \forall y, z \in \mathbb{R} \ |\hat{b}(y) - \hat{b}(z)| \leq L|y - z|, \ |\hat{\sigma}(y) - \hat{\sigma}(z)| \leq L|y - z| \text{ and } |\hat{b}(x)| \leq M_2. \]
Notice that we do not force \( \hat{b} \) to be bounded, we only require that for the specific \( x \), which will be considered in the sequel as the initial state of some stochastic processes and some dynamics. The \( \hat{b}'s \) that we consider are such that \( |\hat{b}(x)| \) is bounded by \( M_2 \).

Let \( F^m, F, \) and \( G \) be the functions from (4.1)–(4.3) with general \((\hat{b}, \hat{\sigma}) \in BL_x \). Fix \( S > 0 \) and for every \( J > 0 \) define

\[ D_J = D_{J,S} := \{ \psi \in C_0[0,S] : \mathbb{I}(S, \psi) \leq J \}. \]

**Lemma A.1** Fix \( x > 0 \).

1. For every \( m \in \mathbb{N} \), every \( \gamma > 0 \), and every \( g_1 \in C_0[0,S] \) there exists \( \delta > 0 \) such that

\[ |g_1 - g_2|_S \leq \delta \implies \sup_{(\hat{b}, \hat{\sigma}) \in BL_x} |F^m_{(\hat{b}, \hat{\sigma})}(g_1) - F^m_{(\hat{b}, \hat{\sigma})}(g_2)|_S \leq \gamma. \quad (A.1) \]

2. For every \( J > 0 \) one has

\[ \lim_{m \to \infty} \sup_{\psi \in D_J} \sup_{(\hat{b}, \hat{\sigma}) \in BL_x} |F^m_{(\hat{b}, \hat{\sigma})}(\psi) - F_{(\hat{b}, \hat{\sigma})}(\psi)|_S = 0. \quad (A.2) \]

3. For every \( \gamma > 0 \) one has

\[ \lim_{m \to \infty} \limsup_{n \to \infty} \sup_{(\hat{b}, \hat{\sigma}) \in BL_x} \frac{1}{n} \ln \mathbb{P} \left( \left| F^m_{(\hat{b}, \hat{\sigma})} \left( \left( \frac{\omega}{\sqrt{n}} \right) \right) - G_{(\hat{b}, \hat{\sigma})} \left( \left( \frac{\omega}{\sqrt{n}} \right) \right) \right|_S > \gamma \right) = -\infty. \quad (A.3) \]

**Proof:** For the entire proof fix \( x, J, L, M_1, M_2 > 0 \).

**Proof of (1):** Fix \( m \in \mathbb{N}, \gamma > 0, (\hat{b}, \hat{\sigma}) \in BL_x, \) and \( g_1, g_2 \in C_0[0,S] \). We show that there exist a \( \delta > 0 \) that depends on \( g_1 \) and on \((\hat{b}, \hat{\sigma})\) only through the parameters \( L \) and \( M_1 \) such that (A.1) holds. Denote by \( h_i := F^m(g_i), \ i = 1, 2 \) and set \( y(t) := |h_1(t) - h_2(t)|, \ t \in [0,S] \). For every \( \hat{b} \in \left[ \begin{array}{c} k \left\{ m \right\} \\ k \end{array} \right] \) one has:

\[
y(t) \leq y \left( \frac{k}{m} \right) + \hat{b} \left( h_1 \left( \frac{k}{m} \right) \right) - \hat{b} \left( h_2 \left( \frac{k}{m} \right) \right) - \frac{1}{m} + 2|g_1|_S \cdot \hat{\sigma} \left( h_1 \left( \frac{k}{m} \right) \right) - \hat{\sigma} \left( h_2 \left( \frac{k}{m} \right) \right) + 2M_1 |g_1 - g_2|_S
\]

Set \( C := \max\{1 + L \left( \frac{1}{m} + 2|g_1|_S \right), 2M_1 \}, \) and define \( \bar{y}_k := \sup\{y(t) : t \in \left[ \frac{k}{m}, \frac{k+1}{m} \right] \}, \) for \( k = 0, \ldots, m - 1 \). Then, by the above \( \bar{y}_{k+1} \leq C(\bar{y}_k + |g_1 - g_2|_S) \) and \( \bar{y}_0 \leq C(y(0) + |g_1 - g_2|_S) = C|g_1 - g_2|_S \), the last equality follows since \( y(0) = 0 \). By induction one gets that for every \( k = 0, \ldots, m, \bar{y}_k \leq C \cdot \frac{c^{k+1}}{c-1} |g_1 - g_2|_S \leq \frac{c^{m+1}}{c-1} |g_1 - g_2|_S \). Therefore, \( \delta = \gamma / \left( C \cdot \frac{c^{m+1}}{c-1} \right) \), which depends solely on \( m, L, M_1 \), and \( g_1 \) satisfies (A.1).
Proof of (2): Fix \( \psi \in D_J \) and \((\hat{b}, \hat{\sigma}) \in \mathcal{BL}_x\). Denote by \( p_k = p_{k,m} := \sup \{ h(t) - h \left( \frac{k}{m} \right) : t \in \left[ \frac{k}{m}, \frac{k+1}{m} \right) \} \), where \( h := F^m(\psi) \), and \( k = 0, \ldots, m - 1 \). Clearly,

\[
 h(t) = \int_0^t \hat{b} \left( h \left( \frac{m s}{m} \right) \right) ds + \int_0^t \hat{\sigma} \left( h \left( \frac{m s}{m} \right) \right) \psi(s) ds, \quad t \in [0, S].
\]

By the Cauchy-Schwartz inequality and since \(|\hat{\sigma}| \leq M_1\) we get that \( p_k \leq (b_k + 2JM_1)/m \), where \( b_k = b_{k,m} := \left| \hat{b} \left( h \left( \frac{k}{m} \right) \right) \right| \). By the definition of \( \mathcal{BL}_x \) we have that \( b_0 \leq M_2 \) and that \(|b_k - b_{k-1}| \leq L \left| h \left( \frac{k}{m} \right) - h \left( \frac{k-1}{m} \right) \right| \leq Lp_k \). Therefore, \( b_k \leq b_{k-1} + Lp_k \). By induction one gets that \( p_k \leq (M_2 + 2JM_1 + L(p_0 + \ldots + p_{k-1}))/m \). Set \( \delta_m := \max \{ M_2 + 2JM_1, L \} / m \) then \( p_k \leq \delta_m (1 + \delta_m)^k \). Since \( (1 + \delta_m)^m \leq e^{\max \{ M_2 + 2JM_1, L \}} = C < \infty \) as \( m \to \infty \) it follows that for every \( m \in \mathbb{N} \) and every \( k = 0, \ldots, m \), one has \( 1 + \delta_m)^k \leq (1 + \delta_m)^m < C \). Therefore, for every \( m \in \mathbb{N} \) one has \( \bar{p}_m := \sup_{0 \leq t \leq 1} |h(t) - h \left( \frac{m t}{m} \right) | = \sup \{ p_k : k = 0, \ldots, m \} \leq \delta_m C \). Set \( f := F(\psi) \). By the Cauchy-Schwartz inequality and recalling that \( \psi \in D_J \) we get that

\[
|f(t) - h(t)| \leq \int_0^t \left| \hat{b} (f(s)) - \hat{b} \left( h \left( \frac{m s}{m} \right) \right) \right| ds + \int_0^t \left| \hat{\sigma} (f(s)) - \hat{\sigma} \left( h \left( \frac{m s}{m} \right) \right) \right| \cdot |\psi(s)| ds
\]

\[
\leq \int_0^t L \left| f(s) - h \left( \frac{m s}{m} \right) \right| ds + L \sqrt{2J} \left( \int_0^t \left| f(s) - h \left( \frac{m s}{m} \right) \right|^2 ds \right)^{1/2}
\]

\[
\leq L \sqrt{t} \left( \int_0^t \left| f(s) - h \left( \frac{m s}{m} \right) \right|^2 ds \right)^{1/2} + L \sqrt{2J} \left( \int_0^t \left| f(s) - h \left( \frac{m s}{m} \right) \right|^2 ds \right)^{1/2}
\]

\[
= L \sqrt{S + \sqrt{2J}} \left( \int_0^t \left| f(s) - h \left( \frac{m s}{m} \right) \right|^2 ds \right)^{1/2}.
\]

Denote by \( y(t) := |f(t) - h(t)|^2 \). Then by the above we get that

\[
y^2(t) \leq L \sqrt{S + \sqrt{2J}} \left( \int_0^t \left( |f(s) - h(s)| + \left| h(s) - h \left( \frac{m s}{m} \right) \right| \right)^2 ds \right)^{1/2}
\]

\[
\leq L^2 \left( \sqrt{S + \sqrt{2J}} \right)^2 \left( \int_0^t \left( |f(s) - h(s)| + C \delta_m \right)^2 ds \right)
\]

\[
\leq 2L^2 \left( \sqrt{S + \sqrt{2J}} \right)^2 \int_0^t \left( y^2(s) + C^2 \delta_m^2 \right) ds
\]

\[
\leq B \left( \int_0^t y^2(s) ds + \delta_m^2 \right),
\]

where \( B := 2L^2 \left( \sqrt{S + \sqrt{2J}} \right)^2 \max \{ 1, C^2 S \} \). Hence, by Gronwall’s lemma, \( y(t) \leq \delta_m^2 B e^{Bt} \). Consequently, we get that

\[
\sup_{\psi \in D_J} \left| F_{(\hat{b}, \hat{\sigma})}^m (\psi) - F_{(\hat{b}, \hat{\sigma})} (\psi) \right| S \leq \delta_m \sqrt{B e^{BS}}.
\]

Since the parameters \( B \) and \( \delta_m \) depend on \((\hat{b}, \hat{\sigma})\) only through \( x, L, M_1 \), and \( M_2 \), we get that

\[
\sup_{(\hat{b}, \hat{\sigma}) \in \mathcal{BL}_x} \sup_{\psi \in D_J} \left| F_{(\hat{b}, \hat{\sigma})}^m (\psi) - F_{(\hat{b}, \hat{\sigma})} (\psi) \right| S \leq \delta_m \sqrt{B e^{BS}}.
\]
By taking \( \lim_{m \to \infty} \) we get that (A.2) holds.

**Proof of (3):** To simplify the notation we define the following processes. Let \( \chi_{t}^{n,m}, m = 1, 2, \ldots \) be the solution of the stochastic differential equation

\[
dx_{t}^{n,m} = \hat{b}(\chi_{[nt]}^{n,m}) dt + \frac{1}{\sqrt{n}} \hat{\sigma}(\chi_{[nt]}^{n,m}) d\omega_t, \quad \chi_{0}^{n,m} = x.
\]

Also set \( \chi_{t}^{n}(\omega) = G_{(\hat{b},\hat{\sigma})}(\omega)(t), t \geq 0. \)

Fix \( \gamma > 0 \) and \( (\hat{b},\hat{\sigma}) \in BL_x. \) Let \( z_t := \chi_{t}^{n,m} - \chi_{t}^{n} = \hat{b}_t dt + \hat{\sigma}_t dB_t, \) where \( \hat{b}_t := \hat{b}\left(\chi_{[nt]}^{n,m}\right) - \hat{b}\left(\chi_{[nt]}^{n}\right) \) and \( \hat{\sigma}_t := \hat{\sigma}\left(\chi_{[nt]}^{n,m}\right) - \hat{\sigma}\left(\chi_{[nt]}^{n}\right). \) Define:

\[
\tau_1 := \inf\{t : |\chi_{[nt]}^{n,m} - \chi_{[nt]}^{n}| \geq \varrho\} \wedge S.
\]

Notice that for every \( t \in [0, \tau] \) one has

\[
\max\{\varrho_t^2, \hat{b}_t^2\} \leq L^2 |\chi_{[nt]}^{n,m} - \chi_{[nt]}^{n}|^2 \leq L^2 \left(|\chi_{[nt]}^{n,m} - \chi_{[nt]}^{n}|^2 + |\chi_{[nt]}^{n,m} - \chi_{[nt]}^{n}|^2\right) \leq L^2 (\varrho^2 + |z_t|^2).
\]

Hence, by [11, Lemma 5.6.18] we get that for every \( n > 1 \) one has

\[
\frac{1}{n} \ln \mathbb{P}\left( \sup_{t \in [0, \tau_1]} |\chi_{[nt]}^{n,m} - \chi_{[nt]}^{n}| \geq \gamma \right) \leq 2L + 3L^2 + \ln \left( \frac{\varrho^2}{\varrho^2 + \gamma^2} \right).
\]

Since we only used the Lipschitz constant in the calculation and no other properties of \( \hat{b} \) and \( \hat{\sigma} \) we get that the following is also true

\[
\sup_{(\hat{b},\hat{\sigma}) \in BL_x} \frac{1}{n} \ln \mathbb{P}\left( \sup_{t \in [0, \tau_1]} |\chi_{[nt]}^{n,m} - \chi_{[nt]}^{n}| \geq \gamma \right) \leq 2L + 3L^2 + \ln \left( \frac{\varrho^2}{\varrho^2 + \gamma^2} \right).
\]

Hence by considering first \( n \to \infty \) and then \( \varrho \to 0, \)

\[
\lim_{\varrho \to 0} \sup_{m \geq 1} \sup_{(\hat{b},\hat{\sigma}) \in BL_x} \frac{1}{n} \ln \mathbb{P}\left( \sup_{t \in [0, \tau_1]} |\chi_{[nt]}^{n,m} - \chi_{[nt]}^{n}| \geq \gamma \right) = -\infty.
\]

Now, since

\[
\{|\chi_{t}^{n,m} - \chi_{t}^{n}|s \geq \gamma\} \subseteq \{\tau_1 < S, |\chi_{t}^{n,m} - \chi_{t}^{n}|_{\tau_1} < \gamma, |\chi_{t}^{n,m} - \chi_{t}^{n}|s > \gamma\} \cup \{|\chi_{t}^{n,m} - \chi_{t}^{n}|_{\tau_1} \geq \gamma\} \subseteq \{\tau_1 < S\} \cup \{|\chi_{t}^{n,m} - \chi_{t}^{n}|_{\tau_1} \geq \gamma\}.
\]

We finish the argument as soon as we show that for each \( \varrho > 0, \)

\[
\lim_{m \to \infty} \sup_{n \to \infty} \sup_{(\hat{b},\hat{\sigma}) \in BL_x} \frac{1}{n} \ln \mathbb{P}\left( \sup_{t \in [0, S]} |\chi_{[nt]}^{n,m} - \chi_{[nt]}^{n,m}| \geq \varrho \right) = -\infty.
\]
To this end, fix \( n, m \in \mathbb{N} \). For every \( k = 0, \ldots, m - 1 \), denote by 
\[
q_k = q_k(n, m) := \sup_{0 \leq s \leq \frac{1}{m}} \left| \chi_{k+s, m} - \chi_{k, m} \right| \quad \text{and} \quad H_k := \sup_{0 \leq s \leq \frac{1}{m}} \left| \omega_{k+s, m} - \omega_{k, m} \right| .
\]
Then,
\[
q_k \leq \frac{1}{m} (b_0 + L(q_0 + \ldots + q_{k-1})) + \frac{M_1}{\sqrt{n}} H_k, \quad k = 0, \ldots, m - 1.
\]

By the definition of \( BL \), \( b_0 \leq M_2 \). Therefore, by setting \( \delta_m := \frac{\max\{M_2, L\}}{m} \) one gets by induction that

\[
q_k \leq \delta_m (1 + \delta_m)^k + \frac{1}{\sqrt{n}} H_k + \delta_m \frac{1}{\sqrt{n}} \sum_{i=0}^{k-1} H_i (1 + \delta_m)^{k-1-i}
\]

and since that \( (1 + \delta_m)^m \not\geq e^{\max\{M_2, L\}} := C \) as \( m \to \infty \) we get that

\[
q_k \leq C \left( \delta_m + \frac{1}{\sqrt{n}} H_k + \delta_m \frac{1}{\sqrt{n}} \sum_{i=0}^{k-1} H_i \right)
\]

Now,

\[
P \left( \sup_{0 \leq t \leq 1} \left| \chi_{t, m} - \chi_{t \cdot \frac{m}{m}, m} \right| \geq \varrho \right) \leq \sum_{k=0}^{m-1} P \left( \sup_{0 \leq s \leq \frac{1}{m}} \left| \chi_{s+k, m} - \chi_{k, m} \right| \geq \varrho \right)
\]

\[
= \sum_{k=0}^{m-1} P(q_k \geq \varrho)
\]

\[
\leq \sum_{k=0}^{m-1} P \left( \frac{1}{\sqrt{n}} H_k + \delta_m \frac{1}{\sqrt{n}} \sum_{i=0}^{k-1} H_i \geq \frac{\varrho}{C} - \delta_m \right).
\]

Without loss of generality we may assume that \( m > \frac{\max\{M_2, L\} \varrho}{2C} \), so that \( \delta_m < \frac{\varrho}{2C} \). Therefore, the l.h.s. of the above satisfies

\[
\sum_{k=0}^{m-1} P \left( \frac{1}{\sqrt{n}} H_k + \delta_m \frac{1}{\sqrt{n}} \sum_{i=0}^{k-1} H_i \geq \frac{\varrho}{C} - \delta_m \right)
\]

\[
\leq \sum_{k=0}^{m-1} P \left( H_k + \delta_m \sum_{i=0}^{k-1} H_i \geq \frac{\varrho \sqrt{n}}{2C} \right)
\]

\[
= \sum_{k=0}^{m-1} P \left( H_{m-1} + \delta_m \sum_{i=0}^{k-1} H_i \geq \frac{\varrho \sqrt{n}}{2C} \right)
\]

\[
\leq mP \left( \frac{1}{\max\{M_2, L\}} H_{m-1} + \frac{1}{m} \sum_{i=0}^{m-2} H_i \geq \frac{\varrho \sqrt{n}}{2C \max\{M_2, L\}} \right),
\]

where the equality follows since the \( H_i \)'s are i.i.d. The second inequality follows by the definition of \( \delta_m \) and since \( H_0, \ldots, H_{m-1} \) are i.i.d. and therefore, the last element in the second sum above
is the greatest. To simplify the argument, assume without loss of generality that $M_2 > 1$. Now, notice that if
\[ \frac{1}{\max\{M_2, L\}} H_{m-1} + \frac{1}{m} \sum_{i=0}^{m-2} H_i \geq \frac{\sqrt{\theta n}}{2C \max\{M_2, L\}} \] holds then at least one of the $H_i$’s must be greater or equal to $\frac{\sqrt{n}}{4C \max\{M_2, L\}}$. Therefore,
\[
m^2 \mathbb{P} \left( H_{m-1} + \frac{1}{m} \sum_{i=0}^{m-2} H_i \geq \frac{\sqrt{n}}{2C \max\{M_2, L\}} \right)
\leq m \sum_{i=0}^{m-1} \mathbb{P} \left( H_i \geq \frac{\sqrt{n}}{4C \max\{M_2, L\}} \right)
= m^2 \mathbb{P} \left( H_0 \geq \frac{\sqrt{n}}{2C \max\{M_2, L\}} \right)
\leq 4m^2 e^{-\left(\frac{\sqrt{n}}{4C \max\{M_2, L\}}\right)^2},
\]
where we used again that the $H_i$’s are identically distributed. The last inequality follows by [11, Lemma 5.2.1]. Therefore,
\[
\lim_{m \to \infty} \limsup_{n \to \infty} \frac{1}{n} \sup_{(b, \delta) \in \mathcal{B}_L} \mathbb{P} \left( \sup_{t \in [0,S]} \left| \chi_t^{n,m} - \chi_{\lfloor nt \rfloor}^{n,m} \right| \geq \theta \right)
\leq \lim_{m \to \infty} \limsup_{n \to \infty} \sup_{(b, \delta) \in \mathcal{B}_L} \left[ \frac{1}{n} \ln \left( 4m^2 \right) - \left( \frac{\theta \sqrt{n}}{4C \max\{M_2, L\}} \right)^2 \frac{m}{2} \right] = -\infty.
\]

\[\square\]

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