The extremal symmetry of arithmetic simplicial complexes

Benson Farb and Amir Mohammadi

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1 Introduction

Let $K$ be a nonarchimedean local field, for example the $p$-adic numbers $\mathbb{Q}_p$ ($\text{char}(K) = 0$) or the field of Laurent series over a finite field $\mathbb{F}_p((t))$ ($\text{char}(p) > 0$). Let $G = \text{PGL}_n(K)$, or more generally the $K$-points of any absolutely simple, connected, algebraic $K$-group of adjoint form.

There is a natural way to associate to each cocompact lattice $\Gamma$ in $G$ a finite simplicial complex $B_\Gamma$, as follows. Bruhat-Tits theory (see below) provides a contractible, rank $K$-dimensional simplicial complex $X_G$ on which $G$ acts by simplicial automorphisms. The lattice $\Gamma$ acts properly discontinuously on $X_G$ with quotient a simplicial complex $B_\Gamma$.

Margulis proved (see, e.g., [Ma]) that $\text{rank}_K G \geq 2$ implies that every lattice $\Gamma$ in $G$ is arithmetic. We also note that $\text{char}(K) = 0$ implies every lattice in $G(K)$ is cocompact. In this paper we explore one aspect of the theme that, since the complex $B_\Gamma$ is constructed using number theory, it should have remarkable properties. Here we concentrate on the extremal nature of the symmetry of $B_\Gamma$ and all of its covers.

Our first result shows that the simplicial structure of $B_\Gamma$ realizes all simplicial symmetries of any simplicial complex homeomorphic to $B_\Gamma$. For any simplicial complex $C$ we denote by $\text{Aut}(C)$ the group of simplicial automorphisms of $C$. We denote by $|C|$ the simplicial complex $C$ thought of as a topological space, without remembering the simplicial structure.

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1If $\Gamma$ has torsion, one needs to barycentrically subdivide each simplex in $X_G$ in order to make the quotient a true (not orbi) simplicial complex.
Theorem 1.1. Let $K$ be a nonarchimedean local field, and let $G$ be the $K$-points of an absolutely simple, connected algebraic $K$-group of adjoint form with $\text{rank}_KG \geq 2$. Let $\Gamma$ be a cocompact lattice in $G$, and let $B_\Gamma$ be the quotient by $\Gamma$ of the Bruhat-Tits building associated to $G$. Suppose $C$ is any simplicial complex homeomorphic to $|B_\Gamma|$. Then there is an injective homomorphism

$$\text{Aut}(C) \rightarrow \text{Aut}(B_\Gamma).$$

Of course the simplicial structure on the space $|B_\Gamma|$ coming from the Bruhat-Tits building is not the unique simplicial structure satisfying Theorem 1.1. One can, for example, take all the top-dimensional simplices of $B_\Gamma$ and subdivide them in the same way, so that the triangulation restricted to any maximal simplex gives a fixed simplicial isomorphism type. Each of these new triangulations of $|B_\Gamma|$ has automorphism group $\text{Aut}(B_\Gamma)$. We call such a simplicial structure on $B_\Gamma$ an arithmetic simplicial structure.

Our main result is a rigidity theorem characterizing arithmetic simplicial structures among all simplicial structures on $|B_\Gamma|$. It gives a universal constraint on the symmetry of the universal covers of all other simplicial structures on $|B_\Gamma|$.

Theorem 1.2. Let $G$ and $\Gamma$ as in Theorem 1.1 be given. Fix a normalization of Haar measure $\mu$ on $G$. Then there exists a constant $N \geq 1$, depending only on the $\mu(G/\Gamma)$, with the following property: Let $C$ be any simplicial complex homeomorphic to $|B_\Gamma|$, and let $Y$ be the universal cover of $C$ (which therefore inherits a $\Gamma$-equivariant simplicial structure from $C$). Then either:

1. $[\text{Aut}(Y) : \Gamma] < N$, so in particular $\text{Aut}(Y)$ is finitely generated, or

2. $C$ is an arithmetic simplicial structure, and so $\text{Aut}(Y)$ is uncountable and acts transitively on chambers (simplices of maximal dimension.)

Remarks.

1. Theorem 1.2 is not true in the case that $\text{rank}_KG = 1$, i.e. when $X_G$ is a tree. An example is given in Section 4. The obstruction in this case is the fact that $\text{Aut}(X_G)$ is “far” from $G.$
2. One is tempted to weaken the hypotheses of Theorem 1.1 and Theorem 1.2, for example to only require that $C$ is homotopy equivalent to $|B_{\Gamma}|$ rather than homeomorphic to it. However the conclusion of each theorem is not true in this case, even for $C$ of the same dimension as $|B_{\Gamma}|$. One can see this by taking, for any given $n \geq 2$, a triangulation of the closed disk $D^2$ by dividing $D^2$ into $n$ equal sectors based at the origin. This triangulation is invariant by the $2\pi/n$ rotation. Now let $C$ be the complex obtained by attaching the central vertex of $D^2$ to some vertex of $B_{\Gamma}$. It is clear that Aut($C$) contains $\mathbb{Z}/n\mathbb{Z}$. Since $n \geq 2$ was arbitrary, the conclusions both of Theorem 1.1 and of Theorem 1.2 do not hold.

3. Theorem 1.1 (resp. Theorem 1.2) is a simplicial analogue of a theorem of Farb-Weinberger from Riemannian geometry, given in [FW1] (resp. [FW2]). However, the mechanism giving rigidity is different here. Further, the type of generality achieved in the theorems in [FW2] seems not to be possible in the simplicial setting, since counterexamples abound, as the last remark indicates.

One consequence of Theorem 1.2 is the following. Suppose $B_{\Gamma}$ has more than one top-dimensional simplex; this can always be achieved by passing to a finite index subgroup of $\Gamma$. Now build a new triangulation $C$ of $|B_{\Gamma}|$ by subdividing the top-dimensional simplices of $B_{\Gamma}$, so that the resulting triangulations on some pair of such simplices are not simplicially isomorphic. Then Theorem 1.2 implies that $[\text{Aut}(Y) : \Gamma] < \infty$.

Another way to think of this is that, if we paint the (open) top-dimensional simplices of $B_{\Gamma}$ with colors, and if we use at least 2 distinct colors, the group of color-preserving automorphisms of the universal cover of $B_{\Gamma}$ is discrete, and contains $\Gamma$ as a subgroup of finite index. This result is actually an ingredient in the proof of Theorem 1.2, and so is proven first. Such a result does not hold when rank$_K G = 1$. We give explicit examples of this failure in Section 4.

**Outline of paper.** After giving some preliminary material on Euclidean buildings in §2 we prove the main results in §3. In §4 we give an explicit example of a $B_{\Gamma}$ satisfying the hypotheses of Theorem 1.1 and Theorem 1.2 and a non-example in rank one case.

**Standing assumption.** All simplicial structures considered in this paper are assumed to be locally finite.
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2 Geometry and automorphisms of Euclidean buildings

We now recall some facts from Bruhat-Tits theory which will be needed in this paper. We refer the reader to [AB], [We] and to [Ti2] for these facts and definitions of terms.

2.1 The building $X_G$

Let $K$ be a nonarchimedean local field. Let $G$ be the adjoint form of an absolutely almost simple, connected, simply connected algebraic group defined over $K$. Let $G = G(K)$.

The Bruhat-Tits theory associates a contractible simplicial complex $X_G$ to $G$ on which $G$ acts by simplicial automorphisms. This is easiest to describe if we work with the simply connected cover of $G$. So let $\tilde{G}$ be the simply connected cover of $G$ and let $\tilde{G} = \tilde{G}(K)$.

Let $r := \text{rank}_K(G)$

An Iwahori subgroup $I$ of $\tilde{G}$ is the normalizer of a Sylow pro-$p$-subgroup of $\tilde{G}$. These subgroups are conjugate to each other since the Sylow subgroups are conjugate. The Euclidean (or affine) building $X_G$ associated with $G$ is a simplicial complex defined as follows. The vertices of $X_G$ correspond bijectively with maximal compact subgroups of $\tilde{G}$. A collection of maximal compact subgroups gives a simplex in $X_G$ precisely when their intersection contains an Iwahori subgroup. $X_G$ is a contractible simplicial complex whose dimension equals $\text{rank}_K(G)$. In particular, if $\text{rank}_K(G) = 1$ then $X_G$ is a tree.

We will need the following properties of $X_G$.

1. $X_G$ is thick; that is, any $i$-simplex of $X_G$ with $i < r := \text{dim}(X_G)$ is contained in at least three $(i+1)$-simplices.

2. Given any apartment (maximal flat) $A$ in $X_G$, any $(r-1)$-dimensional simplex lying in $A$ is contained in precisely two $r$-simplices of $A$.

3. Any two simplices of $X_G$ are contained in a common apartment.
2.2 The action of \( G \)

The groups \( \tilde{G} \) and \( G \) act simplicially on \( X_G \) by conjugation. The stabilizer in \( \tilde{G} \) of any vertex of \( X_G \) is a maximal compact subgroup of \( \tilde{G} \). There are \( r + 1 \) orbits of vertices of \( X_G \) under the \( \tilde{G} \)-action. In this way each vertex is given a type. The action of \( \tilde{G} \) on \( X_G \) is type-preserving, and is transitive on the set of chambers (simplexes of maximal dimension) in \( X_G \).

Let \( G^+ \) be the normal subgroup of \( G \) generated by all the unipotent radicals of \( K \)-parabolic subgroups of \( G \). The group \( G^+ \) is the image of \( \tilde{G} \) under the covering map \( \tilde{G} \to G \). For example, if \( G = \text{PGL}_n(K) \) then \( G^+ = \text{PSL}_n(K) \); see e.g. [Ma, Chapter I]. The covering map restricted to the unipotent subgroups is injective since the kernel of the covering map is the center of \( \tilde{G} \). The subgroup \( G^+ \) is cocompact in \( G \), and indeed is finite index when \( \text{char}(K) = 0 \). Further, \( G^+ \) acts by type-preserving automorphisms on \( X_G \).

Denote by \( \text{Aut}_{\text{alg}}(G) \) the group of algebraic automorphisms of \( G \). This group is the semidirect product of \( G \) with the group of automorphisms of the Dynkin diagram for (the Lie algebra corresponding to) \( G \), which is a group of order at most 2 (see [PR, Theorem 2.8]). Let \( \text{Aut}_G(K) \) denote the group of field automorphisms \( \sigma \) of \( K \) such that \( \sigma G \) and \( G \) are \( K \)-isomorphic where \( \sigma G \) is the group obtained from \( G \) by applying \( \sigma \) to the defining equations. The group \( G \) is a locally compact topological group under the topology coming from that of \( K \). We then have (see [BT]) that the group of automorphisms of \( G \) which we denote by \( \text{Aut}(G) \) is an extension of \( \text{Aut}_{\text{alg}}(G) \) by \( \text{Aut}_G(K) \) i.e. the sequence

\[
1 \to \text{Aut}_{\text{alg}}(G) \to \text{Aut}(G) \to \text{Aut}_G(K) \to 1
\]

is an exact sequence. If \( G \) is a \( K \)-split algebraic group, then \( \text{Aut}(G) = \text{Aut}_{\text{alg}}(G) \times \text{Aut}(K) \), see [Tii 5.8, 5.9, 5.10] and references there.

From the description of \( X_G \) given above, one sees that the group \( \text{Aut}(G) \) acts on the \( X_G \) by simplicial automorphisms, giving a representation

\[
\rho : \text{Aut}(G) \to \text{Aut}(X_G).
\]

The central theorem about automorphisms of buildings is the following.

**Theorem 2.1** (Tits [Tii]). Assume that \( \text{rank}_K G > 1 \). Then the representation

\[
\rho : \text{Aut}(G) \to \text{Aut}(X_G)
\]

is an exact sequence.
is an isomorphism.

Note that $G$, which is subgroup of index at most 2 in $\text{Aut}_{\text{alg}}(G)$, is a normal subgroup of $\text{Aut}(X_G)$. The group $\text{Aut}(X_G)$ is a locally compact group with respect to the compact-open topology. This topology coincides with the topology on $\text{Aut}(X_G)$ determined by the property that the sequences of neighborhoods about the identity map correspond to sets of automorphisms that are the identity on larger and larger balls in $X_G$. On the other hand, the groups $G$ and $\text{Aut}_G(K)$ inherit a topology from the topology on $K$. The isomorphism given in Theorem 2.1 is an isomorphism of topological groups.

### 2.3 Apartments and root subgroups

The apartments (maximal flats) in $X_G$ correspond to maximal diagonalizable subgroups in $\tilde{G}$. Suppose $S$ is a maximal diagonalizable subgroup of $\tilde{G}$, and let $A$ be the corresponding apartment in $X_G$. Then $S$ acts on $A$ by translation. The root subgroups corresponding to $S$ acts on $X_G$ as follows. Any root subgroup determines a family of parallel hyperplanes in $A$. If $u$ lies in the root subgroup it will fix a half-apartment of $A$, i.e. one component of the complement of some hyperplane $P$ in $A$. Moreover, $P$ is an intersection of apartments, and the action of the root group is transitive on the link of $P$ (see §1.4 and §2.1 of [Ti2] or, alternatively, Proposition 18.17 of [We]). In particular we have the following.

**Fact 2.2.** Let $G$ be as above. Then for any $(r - 1)$-simplex $\sigma$ of $X_G$, and for any three $r$-simplices $\alpha_1, \alpha_2, \alpha_3$ having $\sigma$ as their common intersection, there exists an element $\phi \in G^+$ fixing $\alpha_1$ and switching $\alpha_2$ and $\alpha_3$.

As an example consider $G = \text{PGL}_2(\mathbb{Q}_p)$. Then $X_G$ is a $(p + 1)$-regular tree. Let $\ell$ be the apartment in $X_G$ corresponding to the diagonal group of $G$. In this case $\ell$ is a bi-infinite geodesic in $X_G$. Let $\ell(0)$ be the vertex corresponding to $\text{PGL}_2(\mathbb{Z}_p)$, i.e. the vertex corresponding to the standard lattice $\mathbb{Z}_p^2$. The geodesic ray $\ell((0, \infty))$ is a half-apartment based at $\ell(0)$. The above fact gives that there are elements $u_1, \ldots, u_{p-1}$ in one of the corresponding root groups (more precisely $u_i$’s are strictly upper triangular matrices) which map the ray $\ell((-\infty, 0])$ to the other $(p - 1)$-rays based at $\ell(0)$ intersecting $\ell$ only at that point. These may be taken to be the representatives of nontrivial cosets in $\mathbb{Z}_p/p\mathbb{Z}_p$ if we identify the root group with the additive group $\mathbb{Q}_p$.

In this paper we will assume $\text{rank}_K G > 1$. Suppose $\Gamma$ is a lattice in $G$. Then the Margulis Superrigidity Theorem, proved in positive characteristic by Venkataramana [Ve]
(see also [Ma]), implies (by an argument of Margulis) that $\Gamma$ is superrigid and hence arithmetic.

3 Proving extremal symmetry

We begin by proving some lemmas and propositions that are used in the proof of both Theorem 1.1 and Theorem 1.2.

3.1 Topological (non)rigidity of $X_G$

The following is a kind of topological rigidity result for $X_G$: it gives that the topological structure of $X_G$ remembers the simplicial structure. It is worth mentioning that in section 3.1 we only need $Y$ to be locally compact simplicial complex homeomorphic to $X_G$.

**Proposition 3.1.** Let $f : X_G \rightarrow X_G$ be a homeomorphism. Then $f$ maps $k$-dimensional simplices of $X_G$ onto $k$-dimensional simplices for each $0 \leq k \leq \text{dim}(X_G)$. Hence there is a natural homomorphism

$$\psi : \text{Homeo}(X_G) \rightarrow \text{Aut}(X_G).$$

**Proof.** We call a point $x \in X_G$ a $k$-manifold point of $X_G$ if $x$ has some neighborhood homeomorphic to $\mathbb{R}^k$, and $k$ is the maximal such number so that this is true. As mentioned above, $X_G$ is a thick building, that is for each $k < \text{dim}(X_G)$, every $k$-dimensional simplex of $X_G$ is the face of at least three $(k+1)$-dimensional simplices of $X_G$. From this we clearly have the following:

Let $x \in X_G$ be any point. Then $x$ is a $k$-manifold point if and only if $x$ lies in the interior of a $k$-simplex of $X_G$.

As being a $k$-manifold point is clearly a topological property for any fixed $k$, it follows that any homeomorphism $f : X_G \rightarrow X_G$ maps $k$-manifold points to themselves, and therefore $f$ maps open $k$-simplices into open $k$-simplices, for each $0 \leq k \leq \text{dim}(X_G)$. Applying the same argument to $f^{-1}$, we see that $f$ maps each open $k$-simplex of $X_G$ homeomorphically onto an open $k$-simplex of $X_G$. 


As $f$ is a homeomorphism it preserves adjacencies between simplices, and so $f$ induces a simplicial automorphism of $X_G$. This association of $f$ to the simplicial automorphism it induces is clearly a homomorphism. ♦

Recall that $Y$ and $X_G$ are homeomorphic. Thus we have that

$$\text{Aut}(Y) \subseteq \text{Homeo}(Y) \approx \text{Homeo}(X_G).$$

We will denote by $\iota$ the restriction to $\text{Aut}(Y)$ of the homomorphism $\psi$ defined in Proposition 3.1.

**Lemma 3.2.** Let $\text{Aut}(Y)$ and $\text{Aut}(X_G)$ be endowed with the compact-open topology. Then the homomorphism $\iota : \text{Aut}(Y) \to \text{Aut}(X_G)$ is proper and continuous.

**Proof.** Continuity follows from the definitions. To see that $\iota$ is proper, first note that $\text{Aut}(Y)$ is locally compact since $Y$ is assumed to be locally finite, and that for any compact set $K$ in $\text{Aut}(X_G)$ we have that $\iota^{-1}(K)$ is bounded. Now suppose that we are given any sequence $\varphi_n \in \text{Aut}(Y)$ such that $\{\iota(\varphi_n)\}$ converges to $\tau \in \text{Aut}(X_G)$. Then $\{\varphi_n\}$ is pre-compact in $\text{Aut}(Y)$, and for any limit point $\varphi_\infty$ we have that $\iota(\varphi_\infty) = \tau$. Hence $\iota$ is a proper map. ♦

In contrast to rigidity, it is easy to see that the kernel of $\psi$ is huge. Indeed it clearly contains the infinite product, over all maximal simplices $\sigma$, of the group of homeomorphisms of the closed $\text{dim}(X_G)$-disk which are the identity on $\partial \sigma$. On the other hand, when restricted to the subgroup $\text{Aut}(Y)$, the map $\psi$ is injective.

**Proposition 3.3.** The homomorphism $\iota : \text{Aut}(Y) \to \text{Aut}(X_G)$ is injective.

**Proof.** Let $\varphi \in \ker(\iota)$. We will argue inductively on the dimension $k \geq 0$ that $\varphi$ is the identity on the $k$-skeleton of $Y$. Since $\iota(\varphi) = \text{id}$, we get $\varphi(v) = v$ for any vertex $v \in X_G$. Now assume that $\varphi$ is identity on each $j$-simplex of $X_G$ for each $j < k$. Let $D$ be any $k$-simplex of $X_G$. Since $\iota(\varphi) = \text{id}$, we have from the definition of $\iota$ that $\phi(D) \subseteq D$. By induction we have that $\phi(x) = x$ for each $x \in \partial D$.

Since $Y$ is a locally finite complex and $\varphi$ is simplicial automorphism of $Y$, we have that orbits of points under $\varphi$ are discrete. Since $D$ is compact, there are only $m$ simplices
of $Y$ intersecting $D$ for some $m < \infty$. It follows that there exists $n$, depending only on $m$, so that $\phi^n(x) = x$ for any $x \in D$. Let $\tau := \phi|_D$.

Suppose that $\tau \not= \text{id}$. Then after raising $\tau$ to a power we can (and will) assume that $\tau$ has order $p$ for some prime $p$.

Since we have a $p$-group $\langle \tau \rangle$ acting on a closed disk $D$, we can apply Smith Theory to this action. The pair $(D, \partial D)$ is of course a homology $k$-ball. By Smith’s Theorem (see, e.g. [Br], Theorem III.5.2), the pair $(\text{Fix}(\tau), \text{Fix}(\tau|_{\partial D}))$ is a mod-$p$ homology $r$-ball for some $0 \leq r \leq k$. Since $\tau|_{\partial D} = \text{id}$ by the induction hypothesis, we have that $\text{Fix}(\tau|_{\partial D}) = \partial D$, it follows that $r = k$.

Now suppose that $\text{Fix}(\tau) \not= D$. Pick $x \in D$ in the complement of $\text{Fix}(\tau)$. Then radial projection away from $x$ to $\partial D$ gives a homotopy equivalence of pairs

$$(\text{Fix}(\tau), \text{Fix}(\tau|_{\partial D})) \simeq (\partial D, D).$$

But this contradicts the fact that $(\text{Fix}(\tau), \text{Fix}(\tau|_{\partial D}))$ is a mod-$p$ homology $k$-disk with $k > 0$, since as such, we have

$$H_k(\text{Fix}(\tau), \text{Fix}(\tau|_{\partial D}); \mathbb{Z}/p\mathbb{Z}) \neq 0 = H_k(D, \partial D; \mathbb{Z}/p\mathbb{Z}).$$

Thus it must be that $\text{Fix}(\tau) = D$; that is, $\tau = \text{id}$. We have just proven that $\phi|_D = \text{id}$ for each $k$-simplex $D$ of $X_G$, so by the induction on $k$ we have $\phi = \text{id}$, as desired. ☐

3.2 The extremal symmetry of $B_G$

With the results from subsection 3.1 in hand, we can now prove Theorem 1.1.

**Proof of Theorem 1.1.** First note that any simplicial automorphism of $B_G$ induces an automorphism of $\pi_1(B_G) = \Gamma$, well-defined up to conjugacy. We thus have a homomorphism

$$\nu : \text{Aut}(B_G) \to \text{Out}(\Gamma)$$

where $\text{Out}(\Gamma)$ is the group of *outer automorphisms* of $\Gamma$, i.e. the quotient of $\text{Aut}(\Gamma)$ by inner automorphisms.

We claim that $\nu$ is injective. Suppose $f \in \ker(\nu)$. Since $B_G$ is aspherical and $f_*$ acts trivially (up to conjugation) on $\pi_1(B_G)$, it follows that $f$ is freely homotopic to the identity map. Metrize $B_G$ so that it has the path metric induced by giving each simplex
the standard Euclidean metric; $X_\Gamma$ then inherits a unique path metric making the covering $X_\Gamma \to B_\Gamma$ a local isometry.

Since $f$ is homotopic to the identity and since $B_\Gamma$ is compact, each track in this homotopy moves points of $B_\Gamma$ some uniformly bounded distance $D$. Thus $f$ has some lift $\tilde{f} \in \text{Aut}(X_G)$ such that $\tilde{f}$ moves each point of $X_G$ at most a distance $D$. We claim that the only element of $\text{Aut}(X_G)$ that moves all points of $X_G$ at most a uniformly bounded distance is the identity automorphism. Given this claim, it follows that $\tilde{f}$, and hence $f$, is the identity, so that $\nu$ is injective.

The claim is well known, but for completeness we indicate a proof. The building $X_G$ admits a nonpositively curved (in the CAT(0) sense) metric with the property that $\text{Aut}(X_G) = \text{Isom}(X_G)$. Now, the boundary $\partial X_G$ of $X_G$ as a nonpositively curved space, namely the set of Hausdorff equivalence classes of infinite geodesic rays, can be identified with the spherical Tits building associated to $G$ (see [We, Theorem 8.24 and Chapter 28]). By the nonpositive curvature condition, infinite geodesic rays in $X_G$ either stay a uniformly bounded distance from each other, hence represent the same equivalence class in $\partial X_G$, or diverge with distance between point being unbounded. If an element $\phi \in \text{Aut}(X_G)$ moves all points of $X_G$ a uniformly bounded distance, it follows that $\phi$ induces the identity map on $\partial X_G$. But the natural homomorphism $\text{Aut}(X_G) \to \text{Aut}(\partial X_G)$ is injective (see [Ti1] or [We, Theorem 12.30 and Section 28.29]), from which it follows that $\phi$ is the identity, proving the claim.

We now claim that $\nu$ is surjective, and thus is an isomorphism. To see this, note that by the assumptions on $G$, we can apply the Margulis Superrigidity Theorem, proved in positive characteristic by Venkataramana [Ve], see [Ma], to the lattice $\Gamma$ in $G$. This gives in particular that $\Gamma$ satisfies strong (Mostow-Prasad) rigidity, which means that any automorphism of $\Gamma$ can be extended to a continuous homomorphism of $G$. Note that the group of continuous automorphisms of $G$ is precisely $\text{Aut}(X_G)$. Thus given any $h \in \text{Out}(\Gamma)$, there is some $h' \in \text{Aut}(X_G)$ extending (a representative of) $h$, and so preserving $\Gamma$ in $G$. Thus $h'$ descends to the desired automorphism of $B_\Gamma$, proving that $\nu$ is surjective. We have thus shown that $\nu$ is an isomorphism.

Now each $\varphi \in \text{Aut}(C)$ induces an automorphism of $\pi_1(C) = \Gamma$, which is well-defined up to conjugacy. Hence we obtain a homomorphism

$$\alpha : \text{Aut}(C) \to \text{Out}(\Gamma).$$
Since we just proved that \( \nu^{-1} : \text{Out}(\Gamma) \rightarrow \text{Aut}(B_{\Gamma}) \) is an isomorphism, to prove the theorem it is enough to prove that \( \alpha \) is injective. To this end, consider \( \varphi \in \ker(\alpha) \). Let \( Y \) denote the universal cover of \( C \). Then, just as in the argument above, \( \varphi \) lifts to some \( \tilde{\varphi} \) moving points a bounded distance from the identity. Here we are using the metric induced from the simplicial structure on \( X_{\Gamma} \), not on \( Y \) (although these metrics are uniformly comparable, so it doesn’t actually matter). By Proposition 3.3 the homomorphism \( \iota : \text{Aut}(Y) \rightarrow \text{Aut}(X_{\Gamma}) \) is one to one. As was mentioned above the only element of \( \text{Aut}(X_{\Gamma}) \) moving points a uniformly bounded distance is the identity, we have \( \tilde{\varphi} = \text{id} \), so that \( \varphi = \text{id} \) and \( \alpha \) is injective, as desired. \( \diamond \)

### 3.3 Characterizing \( X_{\Gamma} \) among all simplicial structures

The following result, crucial to our proof of Theorem 1.2, gives the consequence discussed at the end of the introduction.

**Proposition 3.4** (Coloring rigidity). Let the notation and assumptions be as above. Then precisely one of the following holds:

(i) \( \text{Aut}(Y) \) is discrete.

(ii) \( G^+ \subseteq \iota(\text{Aut}(Y)) \), where \( \iota \) is the monomorphism in Proposition 3.3.

**Proof.** Recall that \( \iota \) is a proper map. Hence \( \iota(\text{Aut}(Y)) \) is a closed subgroup of \( \text{Aut}(X_{\Gamma}) \) with respect to the compact-open topology. The continuity of \( \iota \) together with Proposition 3.3 imply that if \( \iota(\text{Aut}(Y)) \) is discrete then \( \text{Aut}(Y) \) is discrete and so (i) holds and we are done.

We thus assume now that (i) does not hold. So there is a sequence of elements \( \varphi_n \in \text{Aut}(Y) \) such that \( \{g_n = \iota(\varphi_n)\} \) converges to the identity in \( \text{Aut}(X_{\Gamma}) \).

Note that \( \Gamma \subseteq \text{Aut}(Y) \) and, with this abuse of notation, \( \iota(\Gamma) = \Gamma \). Note that \( H := G \cap \iota(\text{Aut}(Y)) \) is closed normal subgroup of \( \iota(\text{Aut}(Y)) \) containing \( \Gamma \). We claim that \( H \) is indiscrete. Assume the contrary and let \( g_n \) be as above. Since \( g_n \) converge to identity it follows that \( g_n \gamma g_n^{-1} \rightarrow \gamma \) for any \( \gamma \in \Gamma \). Since \( H \) is normal and discrete, and since \( \Gamma \subseteq H \), it follows that \( g_n \gamma g_n^{-1} = \gamma \) for \( n \) large enough.

By the assumption \( \text{rank}_K G \geq 2 \), the group \( G \) has Kazhdan’s property T, and so the lattice \( \Gamma \) in \( G \) is finitely generated. Hence there exists some \( n_0 \) such that if \( n > n_0 \) then \( g_n \gamma g_n^{-1} = \gamma \) for all \( \gamma \in \Gamma \).
Thus for each $n \geq n_0$ we have that $g_n$ centralizes $\Gamma$. Such $g_n$ however is the trivial isometry. This follows, for example, from the proof of Theorem 1.1. To be more explicit $g_n$ centralizes $\Gamma$ thus it induces the trivial isometry of $B_\Gamma$, since $\text{Aut}(B_\Gamma)$ and $\text{Out}(\Gamma)$ are isomorphic, as we showed in loc. cit. Note now that $g_n$ centralizes $\Gamma$ so the action of $g_n$ on $X_\Gamma$ is trivial. Thus $g_n$ is identity if $n \geq n_0$, which is a contradiction. Hence $H$ is indiscrete.

Recall that $G/\Gamma$ has a finite $G$-invariant measure and $\Gamma \subset H$, hence $G/H$ has a finite $G$-invariant measure, namely the direct image of the measure on $G/\Gamma$ under the natural map $G/\Gamma \to G/H$. Also we showed above that $H$ is an indiscrete subgroup of $G$. Now [Ma, Chapter II, Theorem 5.1] states that such a subgroup must contain $G^+$, as we wanted to show.

We are now ready to prove the main theorem of this paper.

**Proof of Theorem 1.2.** There is nothing to prove if $\text{Aut}(Y)$ is discrete, so suppose this is not the case. By Proposition 3.3 and Proposition 3.4 there exists a subgroup $H \subseteq \text{Aut}(Y)$ such that $\iota : H \to G^+$ is an isomorphism. It follows from the definition of $\iota$ that if $\varphi \in \text{Aut}(Y)$ then $\iota(\varphi)(v) = \varphi(v)$ for any vertex $v \in X_\Gamma$. Actually, the proof of Proposition 3.1 immediately gives: if $D$ is any simplex in $X_\Gamma$ then $\iota(\varphi)(D) = \varphi(D)$ is a simplex of $X_\Gamma$ whose vertices are the $\varphi$-images of the vertices of $D$.

We claim that for any simplex $\sigma$ of $Y$, there is some chamber (simplex of maximal dimension) $C$ of $X_\Gamma$ such that $\sigma \subseteq C$. We prove this by induction on the dimension $k \geq 0$ of the cell $\sigma$. When $k = 0$ this is trivial. Now assume the claim is true up to dimension $k - 1$.

Let $C(k)$ be the standard Euclidean $k$-dimensional simplex, and $\beta : C(k) \to \sigma$ be a simplicial parameterization. The induction hypothesis guarantees that any simplex of $\beta(\partial(C(k)))$ is contained some chamber of $X_\Gamma$. Of course this chamber may not be unique. If $\beta(C(k))$ is not contained in a single chamber, then there exists $x \in C(k)^0$, a neighborhood $B_\delta(x) \subseteq C(k)^0$, and two adjacent chambers $C_0$ and $C_1$ of $X_\Gamma$ such that $B_\delta(x) \cap C_i^0 \neq \emptyset$ for each $i = 0, 1$. Without loss of generality we can (and will) assume that $\beta(x) \in C_0 \cap C_1$.

Since the building $X_\Gamma$ is thick, there exists a chamber $C_2$ distinct from $C_0$ and $C_1$ such that the facet (i.e. codimension one face) $C_0 \cap C_1$ is a facet of $C_2$ also. By Fact 2.2 above, there is an element $u \in G^+$ such that $u|_{C_0} = \text{id}$ and $u(C_1) = C_2$. Let $\varphi \in H$ be such that
\( \iota(\varphi) = u \). We have \( \varphi(C_0) = C_0 \) and \( \varphi(C_1) = C_2 \). We also have \( \varphi(v) = v \) for each vertex \( v \) of \( C_0 \).

Now if we argue as in the proof of Proposition \[3.3\] we get \( \varphi|_{C_0} = \text{id}. \) This implies that \( \varphi|_{\beta(B_\delta(x) \cap C_0)} = \text{id} \) and \( \varphi(\beta(B_\delta(x) \cap C_1)) \subseteq C_2 \). Recall however that \( \varphi \) is a simplicial automorphism of \( Y \), so that \( \varphi(\sigma) \) is another \( k \)-cell of \( Y \). Further, \( \varphi(\beta(x)) = \beta(x) \) is an interior point for two different \( k \)-cells of \( Y \), namely \( \sigma \) and \( \varphi(\sigma) \). This is a contradiction. Thus the claim that \( \sigma \) lies in some chamber of \( X_G \) follows.

Let \( C \) be a chamber which is a fundamental domain for the standard action of \( G^+ \) on \( X_G \). As a consequence of the claim above, we have that the restriction of the simplicial structure of \( Y \) to each chamber of \( X_G \) gives a simplicial subdivision of the chamber. In particular \( C \) is simplicially subdivided by \( Y \). Recall that since \( \text{Aut}(Y) \) is indiscrete, there is a subgroup \( H \) which is isomorphically mapped to \( G^+ \) by \( \iota \). Now as \( G^+ \) acts transitively on chambers, we have that if \( C' \) is any chamber of \( X_G \) then there is some \( \varphi \in H \) such that \( \iota(\varphi)(C) = C' \). By the remark we made in the beginning of the proof, we have \( \varphi(C) = C' \). The proof of the theorem is now complete. \( \diamond \)

4 Explicit examples

In this section we give explicit examples of the arithmetic complexes to which Theorem [1.1] and Theorem [1.2] apply. We then give examples in the rank one case where the loc. cit. do not apply.

**An explicit example where Theorems [1.1] and [1.2] apply.** The explicit construction of these examples is given [LSV], using lattices constructed in [CS]. These examples where constructed as explicit examples of “Ramanujan complexes”. Similar (explicit) constructions of complexes for which the above theorems holds are possible in characteristic zero using lattices constructed in [CMSZ1], [CMSZ2] and [MS].

Let \( G = \text{PGL}_3(F_2((y))) \). We want to describe a quotient of \( X_G \) by a lattice \( \Gamma \) which is a congruence subgroup of a lattice \( \Gamma' \), where \( \Gamma' \) acts simply transitively on the vertices of \( X_G \). Note that the building \( X_G \) is in fact a clique complex, that is, any set of \( k + 1 \) vertices is a cell if and only if every two vertices form a 1-cell. This property holds for quotient complexes as well. Thus, in order to describe the simplicial complex \( B_\Gamma \), it suffices to describe the Caley graph of \( \Gamma'/\Gamma \) with an explicit set of generators.
Let $t$ be a generator for the field of 16 elements whose minimal polynomial is $t^4 + t + 1$. In other words, $F_{16} = F_2[t]/(t^4 + t + 1)$. The following set $S$ of seven matrices generates $\text{PGL}_2(F_{16})$. The clique of the Caley graph of $\text{PGL}_2(F_{16})$ with respect to this set of generators is the complex obtained by taking the quotient of $X_G$ by a lattice $\Gamma$, as above. This lattice is a congruence lattice of a lattice $\Gamma'$ which is constructed using a division algebra which splits at all places except at $1/y$ and $1/(y+1)$, at which it remains a division algebra.

The set $S$ consists of the following seven matrices:

$$
\begin{pmatrix}
    t^2 + t^3 & t^2 & t + t^2 \\
    t & t^3 & 1 + t + t^2 \\
    t + t^2 & 1 + t^2 & 1 + t^3
\end{pmatrix},
\begin{pmatrix}
    1 + t^2 + t^3 & t + t^2 & 1 + t^2 \\
    1 + t & t^2 + t^3 & 1 \\
    1 + t^2 & t & t^3
\end{pmatrix},
\begin{pmatrix}
    1 + t^2 + t^3 & 1 + t^2 & t \\
    1 + t + t^2 & t + t^2 + t^3 & t + t^2 \\
    1 + t & 1 + t + t^2 & t + t^3
\end{pmatrix},
\begin{pmatrix}
    1 + t^3 & 1 + t & 1 + t + t^2 \\
    t^2 & 1 + t^2 + t^3 & 1 + t^2 \\
    1 + t + t^2 & 1 & 1 + t^2 + t^3
\end{pmatrix},
\begin{pmatrix}
    t^3 & 1 + t^2 & 1 \\
    t + t^2 & t + t^2 + t^3 & t \\
    1 & t^2 & 1 + t^2 + t^3
\end{pmatrix},
\begin{pmatrix}
    t^2 + t^3 & 1 & t^2 \\
    1 + t^2 & 1 + t^3 & 1 + t \\
    t^2 & x + x^2 & t + t^2 + t^3
\end{pmatrix}
$$

**An example in rank one case.** We begin with an example of an (arithmetic) lattice $\Lambda$ in $G = \text{PGL}_2(Q_5)$, given with a symmetric generating set of $\Lambda$ with 6 elements, which acts simply transitively on $X_G$. In other words $X_G$, which is a 6-regular tree, is the Caley graph of $\Lambda$. This lattice $\Lambda$ is also used in [LPS] to construct explicit examples of “Ramanujan graphs”. Let

$$
H(Z) = \{ \alpha = a_0 + a_1 i + a_2 j + a_3 k : a_i \in Z \}
$$

where $i^2 = j^2 = k^2 = -1$ and $ij = -ji = k$. For any $\alpha \in H(Z)$ we let $\overline{\alpha} = a_0 - a_1 i - a_2 j - a_3 k$ and let $N(\alpha) = \alpha \overline{\alpha}$. Let

$$
\Lambda' = \{ \alpha \in H(Z) : N(\alpha) = 5^k, k \in Z \text{ and } \alpha \equiv 1 \}
$$

14
Now let
\[
\Lambda = \Lambda'/\sim
\]
where
\[
\alpha \sim \beta \ 	ext{if} \ 5^{k_1} \alpha = \pm 5^{k_2} \beta \ 	ext{for some} \ k_1, k_2 \in \mathbb{Z}.
\]
Note that \(\Lambda\) is an (arithmetic) subgroup of \(\text{PGL}_2(\mathbb{Q}_5)\) and \([\alpha][\overline{\alpha}] = 1\). It is easy to see, and is shown in [LPS, Section 3], that \(\Lambda\) is actually a free group in \(\{\alpha_1, \alpha_2, \alpha_3\}\), where \(N(\alpha_i) = 5\) and \(a_0 > 0\) for each \(i = 1, 2, 3\). We identify \(X_G\) with the Cayley graph of \(\Lambda\) with respect to the generating set \(S = \{\alpha_1, \overline{\alpha_1}, \alpha_2, \overline{\alpha_2}, \alpha_3, \overline{\alpha_3}\}\).

Now let \(\Gamma\) be the kernel of the map \(\Lambda \to \mathbb{Z}/4\mathbb{Z}\) given by \(\alpha_i \mapsto i\) for \(i = 1, 2, 3\). Then \(B_\Gamma = X_G/\Gamma\) is the Cayley graph of \(\mathbb{Z}/4\mathbb{Z}\) with respect to this generating set; that is, it is the complete graph with 4 vertices. We now color the edges of \(B_\Gamma\) with 3 different colors so that the edges emanating from a vertex have 3 different colors, and we lift this to a coloring of \(X_G\) using the \(\Gamma\) action.

Fix an arbitrarily large ball in \(X_G\). Consider the automorphism \(\phi\) of the tree \(X_G\) which fixes this ball pointwise and flips two rays corresponding to \(\alpha_1\) and \(\overline{\alpha_3}\) emanating from a vertex on the sphere and is identity everywhere else. Then \(\phi\) lies in the group of color-preserving automorphisms of this tree. As the large ball was chosen arbitrarily, this argument proves that the group of color-preserving automorphisms of \(X_G\) is not discrete. Of course we can replace different “colors” by different simplicial isomorphism types of triangulations of the corresponding simplices. We thus have a contrast with the conclusion of Theorem 1.2.

References

[BT] A. Borel, J. Tits. Homomorphismes “abstraits” de groupes algebriques simples Ann. of Math., Second Series, 97, no. 3 (1973) 499-571

[AB] P. Abramenko and K. Brown, Buildings. Theory and applications, Graduate Texts in Mathematics, 248. Springer, New York, 2008.

[Br] G. Bredon, Introduction to Compact Transformation Groups, Academic Press, 1972.
[CMSZ1] D. I. Cartwright, A. M. Mantero, T. Steger, A. Zappa, *Groups acting simply transitively on the vertices of a building of type $\tilde{A}_2$, I*, Geometriae Dedicata **47** (1993) 143-166.

[CMSZ2] D. I. Cartwright, A. M. Mantero, T. Steger, A. Zappa, *Groups acting simply transitively on the vertices of a building of type $\tilde{A}_2$, II*, Geometriae Dedicata **47** (1993) 143-166.

[CS] D. I. Cartwright, T. Steger, *A family of $\tilde{A}_n$-groups*, Israel Journal of Math. **103** (1998) 125-140.

[FW1] B. Farb, S. Weinberger, *Hidden symmetries and arithmetic manifolds*, Geometry, spectral theory, groups, and dynamics, 111–119, Contemp. Math., 387, Amer. Math. Soc., Providence, RI, 2005.

[FW2] B. Farb, S. Weinberger, *Isometries, rigidity and universal covers*. Ann. of Math. (2) 168 (2008), no. 3, 915–940.

[LPS] A. Lubotzky, R. Phillips, P. Sarnak, *Ramanujan Graphs*, Combinatorica, **8** (3) (1988) 261-277.

[LSV] A. Lubotzky, B. Samuels, U. Vishne, *Explicit construction of Ramanujan complexes of type $A_d$*, Europ. J. of Combinatorics. **26** (2005) 965-993.

[Ma] G. A. Margulis, Discrete subgroups of semisimple Lie groups, Ergeb. Math. Grenzgeb. **17**, Springer, Berlin, 1991.

[MS] A. Mohammadi, A. Salehi Golsefidy, Discrete vertex transitive actions on Bruhat-Tits building, Preprint.

[PR] V. Platanov, A. Rapinchuk, Algebraic groups and number theory, Academic Press, 1993.

[Ti1] J. Tits, Building of Spherical type and finite B-N pairs, Lecture Notes in Mathematics **386**, Springer-Verlag, Berlin-New York 1974.

[Ti2] J. Tits, Reductive groups over local fields, in *Automorphic Forms, Representations and $L$-Functions* (Corvallis, Ore., 1977), I, Proc. Sympos. Pure Math. 33, Amer. Math. Soc., Providence, 1979, 29–69.
[Ve] T. N. Venkataramana, *On superrigidity and arithmeticity of lattices in semisimple groups over local fields of arbitrary characteristic*, Invent. Math. 92 (1988), 255-306.

[We] R. Wiess, *The structure of affine buildings*, Ann. of Math. Studies, 168, 2009.

Dept. of Mathematics  
University of Chicago  
5734 University Ave.  
Chicago, IL 60637  
E-mail: farb@math.uchicago.edu, amirmo@math.uchicago.edu