New trace norm inequalities for $2 \times 2$ blocks of diagonal matrices

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Abstract

Several new trace norm inequalities are established for $2n \times 2n$ block matrices, in the special case where the four $n \times n$ blocks are diagonal. Some of the inequalities are non-commutative analogs of Hanner’s inequality, others describe the behavior of the trace norm under re-ordering of diagonal entries of the blocks.
1 Introduction and statement of results

Hanner’s inequality [4] states that for any complex-valued functions $f$ and $g$, and for $1 \leq p \leq 2$,

$$\|f + g\|^p_p + \|f - g\|^p_p \geq \left(\|f\|^p_p + \|g\|^p_p\right)^p + \left(\|f\|^p_p - \|g\|^p_p\right)^p$$  \hspace{1cm} (1)

For $p \geq 2$ the inequality holds in the reverse direction, with the right side of (1) dominating the left side. It is known that in some cases Hanner’s inequality extends to matrix spaces, with the $L_p$ norms replaced by the trace norm or Schatten norm:

$$\|A\|_p = \text{Tr}|A|^p = \text{Tr}\left(A^*A\right)^{p/2}$$  \hspace{1cm} (2)

Specifically, if $X$ and $Y$ are complex-valued $n \times n$ matrices with both $X + Y$ and $X - Y$ positive semidefinite, then for $1 \leq p \leq 2$

$$\|X + Y\|^p_p + \|X - Y\|^p_p \geq \left(\|X\|^p_p + \|Y\|^p_p\right)^p + \left(\|X\|^p_p - \|Y\|^p_p\right)^p$$  \hspace{1cm} (3)

and again the reverse inequality holds for $p \geq 2$. This was first proved for even integer values of $p$ by Tomczak-Jaegermann [6] and then extended to all $p$ by Ball, Carlen and Lieb [1]. The inequality is also known to hold for any pair of complex-valued matrices $X$ and $Y$ in the intervals $1 \leq p \leq 4/3$ and $p \geq 4$ [1].

The inequality (3) can be re-expressed using $2 \times 2$ block matrices, as follows:

$$\left\|\begin{pmatrix} X & Y \\ Y & X \end{pmatrix}\right\|_p \geq \left\|\begin{pmatrix} \|X\|_p & \|Y\|_p \\ \|Y\|_p & \|X\|_p \end{pmatrix}\right\|_p$$  \hspace{1cm} (4)

This suggests the possibility of trying to extend Hanner’s inequality in a new direction, by replacing the left side of (4) by a general $2 \times 2$ block matrix. It was shown in [5] that for the case of a positive semidefinite matrix $\begin{pmatrix} X & Y \\ Y^* & Z \end{pmatrix}$ the inequality extends in the simplest possible way, that is for $1 \leq p \leq 2$

$$\left\|\begin{pmatrix} X & Y \\ Y^* & Z \end{pmatrix}\right\|_p \geq \left\|\begin{pmatrix} \|X\|_p & \|Y\|_p \\ \|Y\|_p & \|Z\|_p \end{pmatrix}\right\|_p$$  \hspace{1cm} (5)

with the reverse inequality holding for $p \geq 2$. 

[1] Reference not visible in the provided text.

[2] Reference not visible in the provided text.

[3] Reference not visible in the provided text.

[4] Reference not visible in the provided text.

[5] Reference not visible in the provided text.

[6] Reference not visible in the provided text.
It remains an open question whether the analog of (5) holds for a general 2 × 2 block matrix. It is known [5] that a (generally weaker) bound holds; for the special case (3) this weaker bound is
\[ 2^{1/p} \left[ ||X||_p^2 + (p - 1)||Y||_p^2 \right]^{p/2}. \]
Other examples of bounds which relate the p-norms of the matrix and its blocks can be found in [2], [3]. Nevertheless numerical evidence shows that the stronger inequality (5) continues to hold for many non-positive 2 × 2 block matrices.

The purpose of this paper is to establish that Hanner’s inequality does indeed extend in this strong sense for one special class of matrices, namely the 2 × 2 block matrices whose four blocks are all diagonal. This result is the central part of Theorem 1 below.

Given a complex-valued n × n matrix A, we define
\[ |A| = \left( A^* A \right)^{1/2}. \]

**Theorem 1** Let A, B, C, D be diagonal complex-valued n × n matrices. Then for all 1 ≤ p ≤ 2,
\[
\left\| \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right\|_p \geq \left\| \begin{pmatrix} |A| & |B| \\ |C| & |D| \end{pmatrix} \right\|_p \geq \left\| \begin{pmatrix} ||A||_p & ||B||_p \\ ||C||_p & ||D||_p \end{pmatrix} \right\|_p. \tag{6}
\]

For p ≥ 2 all inequalities are reversed.

The inequality (6) was known before in two special cases, namely when A = D and B = C, which is the l_p version of the original Hanner’s inequality [1], and when \( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \) is positive semidefinite, which is a special case of (5).

The matrix |A| is also diagonal, and its entries are the singular values of A, listed in the order in which they arise in A. Since ||A||_p is independent of this order, and similarly for B, C, D, this raises the interesting question of which ordering of singular values in the four blocks of the middle term in (6) produces the matrix with the smallest (or largest) p-norm. We can answer this question in one case, namely when \( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \) is positive semidefinite. This is the content of Theorem 2 below.
Let $s_1 \geq s_2 \geq \cdots \geq s_n$ be the singular values of $A$ listed in decreasing order. Define the diagonal matrix

$$
\text{Sing}(A) = \begin{pmatrix}
s_1 & 0 & \cdots & 0 \\
0 & s_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & s_n
\end{pmatrix}
$$

(7)

**Theorem 2** Let \( \begin{pmatrix} A & C \\ C^* & B \end{pmatrix} \) be a positive semidefinite matrix in which each block $A, B, C$ is a diagonal matrix. Then for all $1 \leq p \leq 2$,

$$
\left\| \begin{pmatrix} A & C \\ C^* & B \end{pmatrix} \right\|_p \geq \left\| \begin{pmatrix} \text{Sing}(A) & \text{Sing}(C) \\ \text{Sing}(C) & \text{Sing}(B) \end{pmatrix} \right\|_p
$$

(8)

For $p \geq 2$ all inequalities are reversed.

For non-positive matrices the minimal value is generally not attained when the singular values are listed in decreasing order. For example, if

$$
A = \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 7 & 0 \\ 0 & 6 \end{pmatrix}, \ C = \begin{pmatrix} 7 & 0 \\ 0 & 10 \end{pmatrix}
$$

(9)

then (8) does not hold for $1 \leq p < 1.2$. Also, if

$$
A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 5 & 0 \\ 0 & 6 \end{pmatrix}, \ C = \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix}, \ D = \begin{pmatrix} 6 & 0 \\ 0 & 5 \end{pmatrix}
$$

(10)

then for all $1 \leq p < 2$,

$$
\left\| \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right\|_p < \left\| \begin{pmatrix} \text{Sing}(A) & \text{Sing}(B) \\ \text{Sing}(C) & \text{Sing}(D) \end{pmatrix} \right\|_p
$$

(11)

Because the blocks are diagonal, Theorems 1 and 2 can be re-written in terms of $2 \times 2$ matrices, and the proofs reduce to proving certain inequalities for $2 \times 2$ and $4 \times 4$ matrices. The proof of Theorem 1 uses the following two Lemmas. The first one extends the convexity result of Lemma 4 from the paper [5], and the second one is a new ingredient.
Lemma 3 For any $2 \times 2$ matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with nonnegative entries, define

$$g(A) = \text{Tr} \left| \begin{pmatrix} a^{1/p} & b^{1/p} \\ c^{1/p} & d^{1/p} \end{pmatrix} \right|^p$$

where $|X| = (X^*X)^{1/2}$. Then for any $2 \times 2$ matrices $A$ and $B$ with nonnegative entries, and $1 \leq p \leq 2$,

$$g(A + B) \leq g(A) + g(B) \quad (12)$$

For $p \geq 2$, the direction of inequality is reversed.

Lemma 4 Let $a, b, c, d$ be any complex numbers. Then for $1 \leq p \leq 2$,

$$\left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\|_p \geq \left\| \begin{pmatrix} |a| & |b| \\ |c| & |d| \end{pmatrix} \right\|_p \quad (13)$$

For $p \geq 2$, the direction of inequality is reversed.

The proof of Theorem 2 relies on the following re-arrangement lemma for $4 \times 4$ matrices. For real numbers $a$ and $b$ we define

$$a \vee b = \max\{a, b\}$$
$$a \wedge b = \min\{a, b\}$$

Lemma 5 For any positive semidefinite $2 \times 2$ block diagonal matrix

$$M = \begin{pmatrix} a_1 & 0 & c_1 & 0 \\ 0 & a_2 & 0 & c_2 \\ \overline{c_1} & 0 & b_1 & 0 \\ 0 & \overline{c_2} & 0 & b_2 \end{pmatrix} \quad (14)$$

define the rearrangement $M_r$

$$M_r = \begin{pmatrix} a_1 \vee a_2 & 0 & |c_1| \vee |c_2| & 0 \\ 0 & a_1 \wedge a_2 & 0 & |c_1| \wedge |c_2| \\ |c_1| \vee |c_2| & 0 & b_1 \vee b_2 & 0 \\ 0 & |c_1| \wedge |c_2| & 0 & b_1 \wedge b_2 \end{pmatrix} \quad (15)$$
Then for \(1 \leq p \leq 2\):

\[
||M||_p \geq ||M_r||_p
\]  

(16)

For \(p \geq 2\), the direction of inequality is reversed.

The paper is organised as follows. Section 2 uses the results of Lemmas 3, 4 and 5 to prove Theorems 1 and 2. Section 3 contains the bulk of the work in this paper, namely the proof of Lemma 3. Finally Sections 4 and 5 contain the proofs of Lemmas 4 and 5.

2 Proof of Theorems

2.1 Proof of Theorem 1

Let \(\{a_i\}, \{b_i\}\) etc denote the diagonal entries of the matrices \(A, B, C, D\). Then (6) is equivalent to

\[
\sum_i \left| \begin{array}{cc}
|a_i|^p & |b_i|^p \\
|c_i|^p & |d_i|^p
\end{array} \right| \geq \sum_i \left| \begin{array}{cc}
|a_i| & |b_i| \\
|c_i| & |d_i|
\end{array} \right|^p
\]  

\[\geq \left| \begin{array}{c}
\left( \sum_i |a_i|^p \right)^{1/p} \\
\left( \sum_i |c_i|^p \right)^{1/p}
\end{array} \right| \left| \begin{array}{c}
\left( \sum_i |b_i|^p \right)^{1/p} \\
\left( \sum_i |d_i|^p \right)^{1/p}
\end{array} \right|^p
\]  

(17)

The first inequality in (17) follows immediately by applying Lemma 4 to each term in the sum. For the second inequality, define for each \(i\) the matrices

\[
X_i = \left| \begin{array}{cc}
|a_i|^p & |b_i|^p \\
|c_i|^p & |d_i|^p
\end{array} \right|
\]  

(18)

Then using the definition of the function \(g\) in Lemma 3 the second inequality can be re-stated as

\[
\sum_i g(X_i) \geq g\left( \sum_i X_i \right)
\]  

(19)

But this follows immediately from the inequality (12).
2.2 Proof of Theorem 2

Using the first inequality in (6) we can assume that $A, B, C$ are all positive. Furthermore by applying permutations if necessary to the blocks on the diagonal, we can assume that the diagonal entries of $A$ and $B$ (which are their singular values) are listed in decreasing order, as described in (7). Let $c_1, \ldots, c_n$ be the diagonal entries of $C$. If $i < j$ and $c_i < c_j$, define the $4 \times 4$ matrix

$$M = \begin{pmatrix}
a_i & 0 & c_i & 0 \\
0 & a_j & 0 & c_j \\
c_i & 0 & b_i & 0 \\
0 & c_j & 0 & b_j
\end{pmatrix} \quad (20)$$

Conjugating by a permutation matrix (which does not depend on the entries of $M$), we can write $(A \ C \ C \ B)$ in the block form

$$\begin{pmatrix} M & 0 \\ 0 & K \end{pmatrix} \quad (21)$$

where the $(2n - 4) \times (2n - 4)$ matrix $K$ depends only on the other entries of $A, B, C$. Using Lemma 5 we replace $M$ by $M_r$, undo the unitary transformation, and deduce that

$$\left\| \begin{pmatrix} A & C \\ C & B \end{pmatrix} \right\|_p \geq \left\| \begin{pmatrix} A & C_r \\ C_r & B \end{pmatrix} \right\|_p \quad (22)$$

where $C_r$ has the entries $c_i$ and $c_j$ swapped. Iterating this procedure eventually lists the singular values of $C$ in decreasing order.

3 Proof of Lemma 3

This Lemma was proved in [5] for the case that the matrices $A$ and $B$ are positive semidefinite. The proof presented below for the general case strengthens and extends the methods introduced in that proof. Notice first that

$$g(A + B) - g(A) = \int_0^1 \frac{d}{dt} g(A + tB) \, dt \quad (23)$$
Replacing $A$ by $A + tB$, it follows that it is sufficient to show that, for any $A, B$ in the domain of $g$,

$$\frac{d}{dt}g(A + tB)|_{t=0} \leq g(B) \quad \text{if } 1 \leq p \leq 2$$  \hspace{1cm} (24)

$$\frac{d}{dt}g(A + tB)|_{t=0} \geq g(B) \quad \text{if } p \geq 2$$  \hspace{1cm} (25)

Let $R$ be the reflection matrix \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) and observe that $g(RA) = g(A)$ for all $A$. This implies that if (24) holds for a matrix $A$ with positive determinant, then it holds for $RA$, since

$$\frac{d}{dt}g(RA + tB)|_{t=0} = \frac{d}{dt}g(R(RA + tB))|_{t=0}$$  \hspace{1cm} (26)

$$= \frac{d}{dt}g(A + tRB)|_{t=0}$$  \hspace{1cm} (27)

$$\leq g(RB) = g(B)$$  \hspace{1cm} (28)

Similarly for (25). So, we can fix matrices $A, B$ and assume det $A \geq 0$:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad B = \begin{pmatrix} x & y \\ w & z \end{pmatrix}$$  \hspace{1cm} (29)

We will first assume that $A$ is in the interior of the domain of $g$, i.e. that all of its entries are nonzero. We will consider the boundary case separately.

Define

$$M = \begin{pmatrix} a^{1/p} & b^{1/p} \\ c^{1/p} & d^{1/p} \end{pmatrix}$$

$$|M| = (M^* M)^{1/2} = (M^T M)^{1/2} = U^T M$$  \hspace{1cm} (30)

where $U$ is some orthogonal $2 \times 2$ matrix. Since det $A \geq 0$, we know det $M \geq 0$ and det $U \geq 0$.

Now, since $g(A) = \text{Tr}(M^T M)^{p/2}$,

$$\frac{d}{dt}g(A + tB)|_{t=0} = \frac{p}{2} \text{Tr}(M^T M)^{(p/2-1)} \frac{d}{dt}(M^T M)$$

$$= \text{Tr}|M|^{p-2}M^T L$$

$$= \text{Tr}|M|^{p-1}U^T L$$  \hspace{1cm} (32)
where $L$ is given by

$$L = p \frac{dM}{dt} \bigg|_{t=0} = \begin{pmatrix} a^{(1-p)/p} & b^{(1-p)/p} \\ c^{(1-p)/p} & d^{(1-p)/p} \end{pmatrix}$$

(33)

In order to prove (24) and (25), we will fix $B$ and consider (32) as a function of $M$. For matrices $M$ in the interior of the domain of $g$, that is matrices whose entries are all nonzero, we will show that (24) hold at all the critical points of $\text{Tr}|M|^{p-1}U^TL$. Assuming the maximum value of (32) occurs in the interior, this will establish the bound for all matrices. Similarly for (25). If the maximum does not occur in the interior, then it must occur on the boundary where some entries of $M$ are zero. We will verify explicitly that the inequalities hold for these cases also, and this will complete the proof.

Because $g$ is homogeneous, we can choose $A$ so that $||M|| = 1$, which means that $|M|$ has eigenvalues $1$ and $h$ with $0 \leq h \leq 1$. We can write $|M|$ as a direct sum of orthogonal projections

$$|M| = P_1 + hP_2$$

(34)

If $P_1$ projects onto the vector $(\cos \alpha \sin \alpha)$ and $P_2$ projects onto $(\sin \alpha - \cos \alpha)$, then we can explicitly write

$$|M|^{p-1} = P_1 + h^{p-1}P_2$$

(35)

$$= \begin{pmatrix} \cos^2 \alpha + h^{p-1} \sin^2 \alpha & \frac{1}{2}(1 - h^{p-1}) \sin 2\alpha \\ \frac{1}{2}(1 - h^{p-1}) \sin 2\alpha & \sin^2 \alpha + h^{p-1} \cos^2 \alpha \end{pmatrix}$$

(36)

Note that because $M$ has positive entries, both $M^TM$ and $|M|$ also have positive entries. This means that we can assume

$$0 < \alpha < \frac{\pi}{2}$$

(37)

Since $U$ has positive determinant, it is of the form

$$U = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

(38)

Define

$$\beta = \alpha + \theta$$

$$J = |M|^{p-1}U^T$$

$$= \begin{pmatrix} j_{11} & j_{12} \\ j_{21} & j_{22} \end{pmatrix}$$

(40)
It follows from (36) that

\[ j_{11} = \cos \alpha \cos \beta + h^{p-1} \sin \alpha \sin \beta \]  
\[ j_{12} = \cos \alpha \sin \beta - h^{p-1} \sin \alpha \cos \beta \]  
\[ j_{21} = \sin \alpha \cos \beta - h^{p-1} \cos \alpha \sin \beta \]  
\[ j_{22} = \sin \alpha \sin \beta + h^{p-1} \cos \alpha \cos \beta \]  

This gives an expression for \( \text{Tr}|M|^{p-1}U^T L = \text{Tr}JL \) in terms of \( \alpha, \beta, \) and \( h \):

\[ \text{Tr}JL = F(\alpha, \beta, h) \]

where

\[ F_1(\alpha, \beta, h) = \frac{\cos \alpha \cos \beta + h^{p-1} \sin \alpha \sin \beta}{(\cos \alpha \cos \beta + h \sin \alpha \sin \beta)^{p-1}} \]  
\[ F_2(\alpha, \beta, h) = \frac{\sin \alpha \cos \beta - h^{p-1} \cos \alpha \sin \beta}{(\sin \alpha \cos \beta - h \cos \alpha \sin \beta)^{p-1}} \]  
\[ F_3(\alpha, \beta, h) = \frac{\cos \alpha \sin \beta - h^{p-1} \sin \alpha \cos \beta}{(\cos \alpha \sin \beta - h \sin \alpha \cos \beta)^{p-1}} \]  
\[ F_4(\alpha, \beta, h) = \frac{\sin \alpha \sin \beta + h^{p-1} \cos \alpha \cos \beta}{(\sin \alpha \sin \beta + h \cos \alpha \cos \beta)^{p-1}} \]  

Now we find the critical points of \( F \): looking at each \( F_i \) as a function of \( h \), it has the form

\[ f(h) = \frac{\delta + h^{p-1} \gamma}{(\delta + h \gamma)^{p-1}} \]  

where \( \delta + h \gamma > 0 \) and \( \delta \gamma = (\sin 2\alpha \sin 2\beta)/4 \), so

\[ f'(h) = (p - 1) \frac{\delta \gamma (h^{p-2} - 1)}{(\delta + h \gamma)^p} \]  

and

\[ \frac{\partial F}{\partial h} = \frac{(p - 1)}{4} (h^{p-2} - 1) \sin 2\alpha \sin 2\beta \left( \frac{x}{a} - \frac{y}{b} - \frac{w}{c} + \frac{z}{d} \right) \]
where we use the fact that the denominator of each $F_i$ is the $(p - 1)$ power of an entry of $M$.

To look at partials with respect to $\alpha$ and $\beta$, we return to writing $F(\alpha, \beta, h)$ as $\text{Tr}JL$. For convenience, we also define the matrix $W = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

\[
F(\alpha, \beta, h) = \text{Tr}JL \\
\frac{\partial J}{\partial \alpha} = \begin{pmatrix} -j_{21} & -j_{22} \\ j_{11} & j_{12} \end{pmatrix} = WJ \\
\frac{\partial L}{\partial \alpha} = (p-1) \begin{pmatrix} xa^{-1}b^{1/p} & -yb^{-1}a^{1/p} \\ wc^{-1}d^{1/p} & -zd^{-1}c^{1/p} \end{pmatrix} \\
\frac{\partial F}{\partial \alpha} = \text{Tr}\frac{\partial J}{\partial \alpha}L + \text{Tr}J\frac{\partial L}{\partial \alpha} = \frac{x}{a}(-j_{21}a^{1/p} + (p-1)j_{11}b^{1/p}) + \frac{y}{b}(j_{11}b^{1/p} - (p-1)j_{21}a^{1/p}) \\
+ \frac{w}{c}(-j_{22}c^{1/p} + (p-1)j_{12}d^{1/p}) + \frac{z}{d}(j_{12}d^{1/p} - (p-1)j_{22}c^{1/p}) \\
=: \frac{x}{a}E + \frac{y}{b}F + \frac{w}{c}G + \frac{z}{d}H
\]

Similarly for $\beta$ we can show

\[
\frac{\partial J}{\partial \beta} = \begin{pmatrix} -j_{12} & j_{11} \\ -j_{22} & j_{21} \end{pmatrix} = -JW \\
\frac{\partial L}{\partial \beta} = (p-1) \begin{pmatrix} xa^{-1}c^{1/p} & yb^{-1}d^{1/p} \\ -wc^{-1}a^{1/p} & zd^{-1}b^{1/p} \end{pmatrix} \\
\frac{\partial F}{\partial \beta} = \frac{x}{a}(-j_{12}a^{1/p} + (p-1)j_{11}c^{1/p}) + \frac{y}{b}(-j_{22}b^{1/p} + (p-1)j_{21}d^{1/p}) \\
+ \frac{w}{c}(j_{11}c^{1/p} - (p-1)j_{12}a^{1/p}) + \frac{z}{d}(j_{21}a^{1/p} - (p-1)j_{22}b^{1/p}) \\
=: \frac{x}{a}P + \frac{y}{b}Q + \frac{w}{c}R + \frac{z}{d}S
\]

Define

\[
v = \begin{pmatrix} x/a \\ y/b \\ w/c \\ z/d \end{pmatrix}
\]
\[
\Phi = \begin{pmatrix}
1 & -1 & -1 & 1 \\
E & F & G & H \\
P & Q & R & S
\end{pmatrix}
\]

Then any critical point of \( F(\alpha, \beta, h) \) will correspond to a solution of

\[
\Phi v = 0
\]

(62)

Note that \( JM = |M|^p \), so

\[
E + F + G + H = p \text{Tr} \frac{\partial J}{\partial \alpha} M = p \text{Tr} W |M|^p = 0
\]

(63)

since \( |M| \) is symmetric and \( W \) skew-symmetric. Likewise,

\[
P + Q + R + S = p \text{Tr} \frac{\partial J}{\partial \beta} M = -p \text{Tr} JW M = -p \text{Tr} W (U|M|^p U^T)
\]

(64)

As a result, we see that \((1, 1, 1, 1)^T\) is a solution to our system of equations (62). For this solution the matrices \( A \) and \( B \) are proportional, say \( A = \lambda B \) with \( \lambda > 0 \). We now want to show that every other solution of the system (62) is a multiple of this one. This will follow if the matrix \( \Phi \) has rank 3. Using column operations, we can see that \( \Phi \) has rank 3 if

\[
\begin{vmatrix}
1 & 0 & 0 \\
E & E + F & E + G \\
P & P + Q & P + R
\end{vmatrix} = (E + F)(P + R) - (P + Q)(E + G) \neq 0
\]

(65)

Explicit calculation yields

\[
E + F = \frac{p}{2} \sin 2\beta(h^{p-1} - h)
\]

\[
P + R = \frac{p}{2} \sin 2\alpha(h^{p-1} - h)
\]

\[
E + G = \frac{p - 2}{2} \sin 2\alpha(1 - h^p)
\]

\[
P + Q = \frac{p - 2}{2} \sin 2\beta(1 - h^p)
\]
and therefore

\[(E + F)(P + R) - (P + Q)(E + G) = \frac{1}{4} \sin 2\alpha \sin 2\beta \left( p^2(h^{p-1} - h)^2 - (p - 2)^2(1 - h^p)^2 \right) \]

Our initial assumption that the entries of \( A \) were strictly positive implies that \( \sin 2\alpha \) and \( \sin 2\beta \) are nonzero and that \( h \neq 1 \), so

\[(E + F)(P + R) - (P + Q)(E + G) = 0 \iff p(h^{p-1} - h) - (2 - p)(1 - h^p) = 0 \]

(this is true for all values of \( p \) since \( h^{p-1} > h \iff 2 > p \)) Viewing the left side as a function of \( h \), it is concave down on \((0, 1)\) and has a solution at \( h = 1 \), so it cannot have any other solutions in the interval. Since \( h < 1 \), the determinant of the matrix must be nonzero.

This allows us to conclude that the rank of \( \Phi \) is 3, so its kernel is simply the span of the vector \((1, 1, 1)^T\). This means that the only interior critical points of \( F(\alpha, \beta, h) \) occur when \( A \) is proportional to \( B \), and for such points

\[\frac{d}{dt}g(\lambda B + tB)|_{t=0} = \frac{d}{dt}(\lambda + t)g(B)|_{t=0} \quad (66)\]

\[= g(B) \quad (67)\]

Assuming that the maximum and minimum of \( F(\alpha, \beta, h) \) are achieved at interior points of the domain of \( g \), this means that \((24)\) and \((25)\) are satisfied for all \( A, B \). Therefore, in order to complete the argument, it only remains to show that \((24)\) and \((25)\) are satisfied on the boundary of the domain. There are three different conditions that define boundary points: \( M \) is not invertible; \( M \) has entries equal to zero but \( |M| \) does not; or \( |M| \) is diagonal. We will examine each of these separately.

If \( M \) is not invertible, then \( h = 0 \). Looking at \((52)\), we see that for \( p < 2 \), \( f'(h) > 0 \), which means that for all \( h > 0 \),

\[\lim_{h \to 0} f(h) < f(h) \quad (68)\]

\[\lim_{h \to 0} F(\alpha, \beta, h) < F(\alpha, \beta, h) \quad (69)\]

so \( F \) cannot be maximized at \( h = 0 \). Likewise for \( p > 2 \), \( f'(h) < 0 \) and \( F \) cannot be minimized at \( h = 0 \).
Note that this analysis works even if $M$ has some entries equal to zero; the only difference is that $\alpha$ and $\beta$ must be written as functions of $h$ so that $\alpha$ and $\beta$ remain strictly in the first quadrant while $h > 0$. The sign of $f'(h)$ is unaffected by this adjustment.

To address the invertible cases, we write $M$ explicitly in terms of $\alpha$, $\beta$, and $h$, with $h > 0$:

$$M = \begin{pmatrix} \cos \alpha \cos \beta + h \sin \alpha \sin \beta & \cos \alpha \sin \beta - h \sin \alpha \cos \beta \\ \sin \alpha \cos \beta - h \cos \alpha \sin \beta & \sin \alpha \sin \beta + h \cos \alpha \cos \beta \end{pmatrix} \quad (70)$$

Since the off-diagonals of $M$ are nonnegative,

$$\cos \alpha \sin \beta \geq h \sin \alpha \cos \beta \quad (71)$$
$$\sin \alpha \cos \beta \geq h \cos \alpha \sin \beta \quad (72)$$

As $\alpha$ is in the first quadrant and $h$ is positive, (71) and (72) imply that $\beta$ is also in the first quadrant. In fact, $\beta$ is strictly in the first quadrant if and only if $\alpha$ is.

If $|M|$ is invertible and not diagonal, then $M$ has at most one zero entry. Since $\det M > 0$, this must be off the diagonal, say the upper right entry. Fix $\alpha, \beta$ in the first quadrant and define

$$\rho = \frac{\tan \beta}{\tan \alpha} \quad (73)$$

$M$ becomes diagonal if $h = 1$ or $\sin 2\alpha = 0$, so we can assume

$$0 < \rho < 1 \quad (74)$$

We wish to look at $F(\alpha, \beta, h)$ as $h \to \rho$ from above. Examining the definition of $F_3$ in (48), we see that the denominator goes to zero and the numerator is positive or negative depending on $p$. So:

$$\lim_{h \to \rho} F_3(\alpha, \beta, h) = \begin{cases} -\infty & p < 2 \\ +\infty & p > 2 \end{cases} \quad (75)$$

Since the limits of the other $F_i$ are finite, this determines the behavior of $F$ at $h = \rho$, and in every case (24) and (25) are satisfied.
If $|M|$ is diagonal, then $\sin 2\alpha = \sin 2\beta = 0$; in fact, the inequalities (71) and (72) imply that in this case $\alpha = \beta$. For fixed $h < 1$

$$\lim_{\sin 2\alpha \to 0} F_1(\alpha, \alpha, h) = \lim_{\sin 2\alpha \to 0} F_4(\alpha, \alpha, h) = 1 \quad (76)$$

$$F_2(\alpha, \alpha, h) = F_3(\alpha, \alpha, h) = \frac{1 - h^{p-1}}{(1-h)^{p-1}} \left(\frac{1}{2} \sin 2\alpha\right)^{2-p} \quad (77)$$

$$\lim_{\sin 2\alpha \to 0} F_2(\alpha, \alpha, h) = \lim_{\sin 2\alpha \to 0} F_3(\alpha, \alpha, h) = \begin{cases} 0 & p < 2 \\ \infty & p > 2 \end{cases} \quad (78)$$

$$\lim_{\sin 2\alpha \to 0} F(\alpha, \alpha, h) = \begin{cases} x + z & p < 2 \\ \infty & p > 2 \end{cases} \quad (79)$$

This proves the desired result since

$$(x + z)^{1/p} = \left\| \begin{pmatrix} x^{1/p} & 0 \\ 0 & z^{1/p} \end{pmatrix} \right\|_p \quad (80)$$

$$= \frac{1}{2} \left\| \begin{pmatrix} x^{1/p} & y^{1/p} \\ w^{1/p} & z^{1/p} \end{pmatrix} + \begin{pmatrix} x^{1/p} & -y^{1/p} \\ -w^{1/p} & z^{1/p} \end{pmatrix} \right\|_p \quad (81)$$

$$\leq \frac{1}{2} \left( \left\| \begin{pmatrix} x^{1/p} & y^{1/p} \\ w^{1/p} & z^{1/p} \end{pmatrix} \right\|_p + \left\| \begin{pmatrix} x^{1/p} & -y^{1/p} \\ -w^{1/p} & z^{1/p} \end{pmatrix} \right\|_p \right) \quad (82)$$

$$= g(B)^{1/p} \quad (83)$$

Note that the analysis in the diagonal case also works if $h = 1$; instead of fixing $h$, let it approach 1 as $\sin 2\alpha \to 0$.

To summarise: we have now examined all the possibilities, and we see that $rac{d}{dt} g(A + tB)|_{t=0}$ achieves its maximum on the interior of the set of nonnegative matrices if $p < 2$, and it achieves its minimum on the interior if $p > 2$. Furthermore these extremes occur when $A$ is proportional to $B$, in which case

$$\frac{d}{dt} g(\lambda B + tB)|_{t=0} = \frac{d}{dt}(\lambda + t)g(B)|_{t=0} \quad (84)$$

$$= g(B) \quad (85)$$

Therefore the result is proved.

## 4 Proof of Lemma 4

We can first multiply $M$ by a matrix of the form $\begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{i\beta} \end{pmatrix}$; this changes neither the $p$-norm of the matrix nor the absolute values of the entries. Choosing
\( \alpha \) and \( \beta \) appropriately, we can reduce the proof to the case where the diagonal entries of \( M \) are nonnegative real numbers.

The norm of \( M \) is a function of the eigenvalues of \( M^*M \), so we can write

\[
||M||_p = \Tr(M^*M)^{\frac{p}{2}}
\]

(86)

\[
= \left( \frac{1}{2} \right)^{\frac{p}{2}} \left( (T + \sqrt{T^2 - 4D})^{\frac{p}{2}} + (T - \sqrt{T^2 - 4D})^{\frac{p}{2}} \right)
\]

(87)

where \( T = \Tr M^*M = a^2 + |b|^2 + |c|^2 + d^2 \), and \( D = \det M^*M = |ad - bc|^2 \). By the same reasoning,

\[
||N||_p = \left( \frac{1}{2} \right)^{\frac{p}{2}} \left( (T + \sqrt{T^2 - 4D'})^{\frac{p}{2}} + (T - \sqrt{T^2 - 4D'})^{\frac{p}{2}} \right)
\]

(88)

where \( D' = \det N^*N = (ad - |bc|)^2 \). By the triangle inequality,

\[
D \geq D'
\]

(89)

and so for \( 1 \leq p \leq 2 \) the concavity of \( x^{\frac{p}{2}} \) implies

\[
||M||_p \geq \left( \frac{1}{2} \right)^{\frac{p}{2}} \left( (T + \sqrt{T^2 - 4D'})^{\frac{p}{2}} + (T - \sqrt{T^2 - 4D'})^{\frac{p}{2}} \right)
\]

(90)

\[
\geq \left( \frac{1}{2} \right)^{\frac{p}{2}} \left( (T + \sqrt{T^2 - 4D})^{\frac{p}{2}} + (T - \sqrt{T^2 - 4D})^{\frac{p}{2}} \right)
\]

(91)

\[
= ||N||_p
\]

(92)

If \( p \geq 2 \), \( x^{\frac{p}{2}} \) is convex, so \( ||M||_p \leq ||N||_p \).

5 Proof of Lemma 5

The lemma is clearly true if \( p = 2 \), so we will first address the case \( 1 \leq p < 2 \). As usual, we will assume \( M \) is invertible and allow the general result to follow by continuity. We can conjugate \( M \) with an appropriate diagonal unitary matrix to replace \( c_i \) with \( |c_i| \) for \( i = 1, 2 \), so we will hence assume that \( c_1, c_2 \geq 0 \). Also, a permutation of the basis elements allows us to rewrite \( M \) as a block diagonal matrix and to assume that \( a_1 \geq a_2 \) and \( b_1 \geq b_2 \).

\[
M = \begin{pmatrix}
a_1 & 0 & c_1 & 0 \\
0 & a_2 & 0 & c_2 \\
c_1 & 0 & b_1 & 0 \\
0 & c_2 & 0 & b_2
\end{pmatrix} \sim \begin{pmatrix}
a_1 & c_1 & 0 & 0 \\
c_1 & b_1 & 0 & 0 \\
0 & 0 & a_2 & c_2 \\
0 & 0 & c_2 & b_2
\end{pmatrix} = \begin{pmatrix}A & 0 \\
0 & B
\end{pmatrix}
\]

(93)
where \( \sim \) indicates unitary equivalence. Noting that \( M_r = M \) if \( c_1 \geq c_2 \), we will now assume that \( c_2 > c_1 \). We apply the same basis permutation to \( M_r \) that we did to \( M \) to get

\[
M_r \sim \begin{pmatrix}
  a_1 & c_2 & 0 & 0 \\
  c_2 & b_1 & 0 & 0 \\
  0 & 0 & a_2 & c_1 \\
  0 & 0 & c_1 & b_2 \\
\end{pmatrix} = \begin{pmatrix} A_r & 0 \\ 0 & B_r \end{pmatrix}
\]

Since \( a_1 b_1 \geq a_2 b_2 > c_2^2 \geq c_1^2 \), we see that \( M_r \) is positive definite. Also, note that

\[
||M||_p = (||A||_p^p + ||B||_p^p)^{1/p}
\]

Define:

\[
y = \frac{c_1 + c_2}{2}
\]

\[
A(h) = \begin{pmatrix} a_1 & y + h \\ y + h & b_1 \end{pmatrix}, \quad B(h) = \begin{pmatrix} a_2 & y + h \\ y + h & b_2 \end{pmatrix}
\]

\[
f(h) = ||A(h)||_p^p + ||B(-h)||_p^p - ||A(-h)||_p^p - ||B(h)||_p^p
\]

\[
f(c_2 - y) = ||M_r||_p^p - ||M||_p^p
\]

The final step uses the fact that \( A(h) \) and \( B(h) \) are positive definite for \( |h| \leq c_2 - y \). Noting that

\[
f(c_2 - y) = 0
\]

the lemma follows if \( f(c_2 - y) \leq 0 \). Since \( f(0) = 0 \), it suffices to show that \( f'(h) \leq 0 \) for all \( h \in [0, c_2 - y] \).

Note that

\[
A'(h) = B'(h) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]

We can then differentiate \( f \):

\[
\frac{1}{p} f'(h) = \text{Tr} \left( (A(h)^{p-1} - B(-h)^{p-1} + A(-h)^{p-1} - B(h)^{p-1}) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right)
\]

\[
= 2\left( A(h)_{12}^{p-1} - B(-h)_{12}^{p-1} + A(-h)_{12}^{p-1} - B(h)_{12}^{p-1} \right)
\]
So, to show that \( f'(h) \leq 0 \), it suffices to compare the off-diagonal entries and show that \( A(h)^{p-1}_{12} \leq B(h)^{p-1}_{12} \) and \( A(-h)^{p-1}_{12} \leq B(-h)^{p-1}_{12} \).

We now derive an expression for the off-diagonal entries of the \((p-1)\) power of a general \(2 \times 2\) positive definite matrix \(K\) using the integral representation

\[
K^{p-1} = \gamma_p \int_0^\infty t^{p-2} \frac{K}{t+K} dt
\]

where \( \gamma_p = \left( \sin((p-1)\pi) \right) / \pi > 0 \) for \(1 < p < 2\). Since \(K\) is a \(2 \times 2\) matrix,

\[
(t + K)^{-1} = \frac{1}{\det(t + K)} \begin{pmatrix} t + K_{22} & -K_{12} \\ -K_{21} & t + K_{11} \end{pmatrix} \tag{103}
\]

\[
= \frac{t + K^{-1} \det K}{\det(t + K)} \tag{104}
\]

\[
\frac{K}{t + K} = \frac{tK + \det K}{\det(t + K)} \tag{105}
\]

This gives an expression for \(K^{p-1}\) in terms of the original matrix \(K\).

\[
K^{p-1} = \gamma_p \int_0^\infty t^{p-2} \frac{tK + \det K}{\det(t + K)} dt \tag{106}
\]

\[
K^{p-1}_{12} = \gamma_p \int_0^\infty t^{p-2} \frac{tK_{12}}{\det(t + K)} dt \tag{107}
\]

Using the expression for the derivative, we see

\[
A(h)^{p-1}_{12} - B(h)^{p-1}_{12} = \gamma_p \int_0^\infty t^{p-2} \left( \frac{tA(h)_{12}}{\det(t + A(h))} - \frac{tB(h)_{12}}{\det(t + B(h))} \right) dt \tag{108}
\]

\[
= \gamma_p \int_0^\infty t^{p-2} \left( \frac{t(y + h)}{\det(t + A(h))} - \frac{t(y + h)}{\det(t + B(h))} \right) dt
\]

\[
= \gamma_p \int_0^\infty t^{p-1}(y + h) \left( \frac{1}{\det(t + A(h))} - \frac{1}{\det(t + B(h))} \right) dt
\]

\[
\leq 0
\]

since \(\det(t + A(h)) \geq \det(t + B(h))\) for all \(t \geq 0\). Likewise,

\[
A(-h)^{p-1}_{12} - B(-h)^{p-1}_{12} = \gamma_p \int_0^\infty t^{p-1}(y - h) \left( \frac{1}{\det(t + A(-h))} - \frac{1}{\det(t + B(-h))} \right) dt
\]

\[
\leq 0
\]

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But this implies that $f'(h) \leq 0$ for all $h \in [0, c_2 - y]$, so
\[
\|M_r\|_p^p - \|M\|_p^p = f(c_2 - y) \quad (110)
\]
\[
= f(0) + \int_0^{c_2 - y} f'(h)dh 
\]
\[
\leq f(0) 
\]
\[
= 0 
\]
which completes the proof for $1 \leq p \leq 2$.

Now set $p > 2$ and let $q$ be its conjugate index. Let $M = \begin{pmatrix} A & B \\ 0 & B \end{pmatrix} \geq 0,$
where $A, B$ are $2 \times 2$ matrices. There exists a positive matrix $N = \begin{pmatrix} K & 0 \\ 0 & L \end{pmatrix}$
s.t
\[
\|M\|_p \|N\|_q = \text{Tr} MN 
\]
\[
= \text{Tr}(AK + BL) 
\]
\[
= \sum_{i,j=1}^2 A_{ij}K_{ji} + B_{ij}L_{ji} 
\]

For any numbers $a, b, x, y,$
\[
a x + b y \leq (|a| \lor |b|)(|x| \lor |y|) + (|a| \land |b|)(|x| \land |y|) 
\]
(114)
so we can write
\[
\sum_{i,j} A_{ij}K_{ji} + B_{ij}L_{ji} \leq \sum_{i,j}(|A_{ij}| \lor |B_{ij}|)(|K_{ji}| \lor |L_{ji}|) 
\]
(115)
\[
+ \sum_{i,j}(|A_{ij}| \land |B_{ij}|)(|K_{ji}| \land |L_{ji}|)
\]
\[
= \text{Tr} M_r N_r 
\]
\[
\leq \|M_r\|_p \|N_r\|_q 
\]
Since $q < 2$, $\|N_r\|_q \leq \|N\|_q$, which implies that $\|M_r\|_p \geq \|M\|_p$.

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