Pseudo supersymmetric partners for the generalized Swanson model

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Abstract

New non Hermitian Hamiltonians are generated, as isospectral partners of the generalized Swanson model, viz., $H_- = A^\dagger A + \alpha A^2 + \beta A^\dagger 2$, where $\alpha, \beta$ are real constants, with $\alpha \neq \beta$, and $A^\dagger$ and $A$ are generalized creation and annihilation operators. It is shown that the initial Hamiltonian $H_-$, and its partner $H_+$, are related by pseudo supersymmetry, and they share all the eigen energies except for the ground state. This pseudo supersymmetric extension enlarges the class of non Hermitian Hamiltonians $H_{\pm}$, related to their respective Hermitian counterparts $h_{\pm}$, through the same similarity transformation operator $\rho : H_{\pm} = \rho^{-1}h_{\pm}\rho$. The formalism is applied to the entire class of shape-invariant models.

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1. Introduction

Ever since interest in non Hermitian Hamiltonians (with real energies) was revived about a decade ago by Bender and Boettcher [1], quantum systems described by such non Hermitian Hamiltonians have been studied widely [2]. To extend the class of such systems, new exactly solvable (or quasi-exactly solvable) non Hermitian Hamiltonians with real, discrete energies, have been generated using different approaches—e.g., supersymmetry [3], the related intertwining operator method [4], or the Darboux algorithm [5]. In a recent work [6], we had found a class of new non Hermitian models by generalizing the Swanson Hamiltonian $H = a^\dagger a + \alpha a^2 + \beta a^\dagger 2$ where $\alpha, \beta$ are real constants, with $\alpha \neq \beta$. This model was initially proposed by Swanson [7], and later on studied by various authors [8]. For the sake of generalization, we had used generalized creation and annihilation operators, $A^\dagger$ and $A$, in place of Harmonic oscillator creation and annihilation operators $a^\dagger, a$, so that $H_- = A^\dagger A + \alpha A^2 + \beta A^\dagger 2$. The energies of this class of non Hermitian Hamiltonians were found to be real when the parameters satisfy the relations $(\alpha + \beta) < 1$ and $4\alpha\beta < 1$ [6, 7]. In the present work, we shall generate new non Hermitian Hamiltonians $H_{\pm}$, as isospectral
partners of the generalized Swanson Hamiltonian $H_-$. It may be recalled that non Hermiticity may be introduced through a scalar term or by a vector term. In the scalar case non Hermiticity may be introduced by replacing $x$ with $(x \pm i\epsilon)$ or by taking one of the parameters complex in the expression for the potential $V(x)$. Though this looks apparently simple, nevertheless, a similarity transformation of the non Hermitian Hamiltonian ($H_\pm$ with solutions $\psi^{(\pm)}$) maps it to a complicated non-local Hermitian Hamiltonian ($h_\pm$ with solutions $\phi^{(\pm)}$). Consequently, the resulting Hermitian Hamiltonian is not exactly solvable and perturbative techniques may have to be applied for solving the same. Additionally, it may not be possible to determine the exact form of the metric operator $\eta$ explicitly, with respect to which the inner product $\langle \psi_m | \eta | \psi_n \rangle$ is positive definite. On the contrary, if non Hermiticity is introduced through an imaginary vector potential, the similarity transformation yields a Hermitian Schrödinger Hamiltonian (consisting of the standard kinetic term plus a local Hermitian potential term), with possibilities of exact (or quasi-exact) solvability. The metric operator also can be obtained in closed form. In view of the fact that a gauge-like transformation transforms the non Hermitian Hamiltonian $H_-$ to a Hermitian one $h_-$ with Schrödinger form, the imaginary vector potential may be regarded as a trivial way of introducing non Hermiticity. Nevertheless, since the resulting Hamiltonian $H_-$ comes out to be real yet non Hermitian, with real spectrum and possibilities of exact (or quasi-exact) solutions, our interest here is to look for isospectral partners of such a potential.

A couple of recent works deserve special mention here [9, 10]. In the first of these [9], a method was proposed to generate a family of non Hermitian Hamiltonians equivalent to the Swanson Hamiltonian [7], by writing $H$ as a linear combination of $su(1, 1)$ generators $K_0, K_\pm$: i.e. $H = 2K_0 + 2\alpha K_- + 2\beta K_+$. In the second work [10], a quasi-Hermitian supersymmetric extension was proposed for a Harmonic Oscillator Hamiltonian, augmented by a non Hermitian $\mathcal{PT}$ symmetric part. To construct new non Hermitian Hamiltonians related by similarity transformation to Hermitian ones [11, 12], the $su(1, 1)$ Lie algebra needs to be enlarged to a $su(1, 1/1) \sim \text{osp}(2/2, \mathbb{R})$ Lie super algebra. Incidentally, both in [10] and our present formulation, non Hermiticity is introduced not by considering a complex-valued potential, but through a momentum-dependent interaction term. However, our approach is different from either of those employed in [9, 10]. Instead of the Swanson model considered in [9, 10], we deal with its generalized version [6]. So while the non Hermiticity is introduced through a momentum-dependent linear interaction term in [10], viz., $i(\alpha - \beta)(xp + px)$; in our case it is not necessarily linear, to be precise it is of the form $i(\alpha - \beta)[W(x)p + pW(x)]$, thus depending on the particular model considered. Secondly, in our case the Hamiltonian $H_-$ is written in terms of generalized creation and annihilation operators, which are not necessarily Lie algebra generators.

It is worth recalling here that the choice of the metric operator $\eta$ is not unique. In fact many possible such operators exist, obeying the condition $\eta H_- = h_- \eta$. So there are many ways of finding a Hermitian Hamiltonian $h_-$, through a similarity transformation $h_- = \rho H_- \rho^{-1}$ and thus obtain the metric operator $\eta$ [13], from the relation $\eta = \rho^2$ [11]. Each $h_-$ is associated with a different metric, thus invoking a different Hilbert space for each Hermitian map. Our approach gives a simple, straightforward method to determine one such similarity transformation $\rho$ mapping the non Hermitian system $H_-$ to its Hermitian equivalent $h_-$. In this respect our approach is different from [10] where the $su(1, 1)$ generators $K_0, K_\pm$ are used in the construction of $\rho$. Furthermore, the normalization requirement of the wave functions for pseudo Hermitian systems viz., $\langle \psi | \eta | \psi \rangle$ ensures the wave functions to be naturally normalized in our formalism, as we shall see later.

Once a non Hermitian partner Hamiltonian $H_\rho$ of the generalized Swanson Hamiltonian $H_-$ is obtained, it is natural to look for some underlying symmetry between $H_\rho$. Since our
starting Hamiltonian is non Hermitian, the partners cannot be expected to be inter-related through spersymmetry. On the contrary, it is anticipated that they will be related by pseudo supersymmetry [14]. It will also be shown that the pair of non Hermitian Hamiltonians $H_{\pm}$ are related to a pair of Hermitian ones $h_{\pm}$ through the same similarity transformation $\rho$. Finally, we shall apply our formalism to the entire class of shape-invariant potentials, where the parameters of the partner potential are related to those of the initial one through translation [15]. It is worth mentioning here that we have been able to give a general expression for finding the partner Hamiltonian $H_+$ in terms of the parameters of $H_-$, for all the shape-invariant models related through translation of parameters.

The plan of the work is as follows. In section 2, new non Hermitian Hamiltonians $H_+$ are generated, which are isospectral to the initial non Hermitian Hamiltonian $H_-$, except for the ground state. It is observed further that both the initial non Hermitian Hamiltonian $H_-$ and its partner $H_+$ so generated, are pseudo Hermitian with respect to the same linear, invertible operator $\eta$. The underlying symmetry between the partners $H_{\pm}$ is studied in section 3. The formalism developed here is actually applied to all the known classes of shape-invariant models mentioned above, in section 4. Finally, Section 5 is kept for conclusions and discussions.

2. Theory

For a better understanding of the topic and to make the paper self-contained, we repeat certain equations from [6] in the initial part of this section. To start with we consider the generalized Swanson model

$$H_- = \mathcal{A}^\dagger \mathcal{A} + \alpha \mathcal{A}^2 + \beta \mathcal{A}^\dagger \mathcal{A}, \quad \alpha \neq \beta$$

(1)

where $\alpha$ and $\beta$ are real, dimensionless constants, with $\alpha \neq \beta$ for $H_-$ to be non Hermitian, and $\mathcal{A}^\dagger$ and $\mathcal{A}$ are generalized creation and annihilation operators, given by

$$\mathcal{A} = \frac{d}{dx} + W(x), \quad \mathcal{A}^\dagger = -\frac{d}{dx} + W(x)$$

(2)

Investigations in this field has revealed that for such non Hermitian Hamiltonians to describe physical systems, they should be necessarily $\eta$-pseudo Hermitian [13],

$$H_-^\dagger = \eta H_- \eta^{-1}, \quad \text{i.e.} \quad H_-^\dagger \eta = \eta H_-$$

(3)

where $\eta$ is a linear, invertible, Hermitian operator. This requirement, along with the criterion for the wave functions to be well behaved in the entire range, the parameters must obey certain conditions [6, 7], viz.,

$$\alpha + \beta < 1, \quad 4\alpha\beta < 1$$

(4)

With the explicit form of (2) and some straightforward algebra, the eigenvalue equation

$$H_- \psi^{(-)}(x) = E \psi^{(-)}(x)$$

(5)

can be cast in the form [6]

$$H_- \psi^{(-)} = \left\{ - (1 - \alpha - \beta) \left( \frac{d}{dx} - \frac{\alpha - \beta}{1 - \alpha - \beta} W \right)^2 + \frac{1 - 4\alpha\beta}{(1 - \alpha - \beta)} W^2 - W' \right\} \psi^{(-)}$$

$$= E \psi^{(-)}$$

(6)

To reduce equation (6) to the well known Schrödinger form

$$h_- \psi^{(-)}(x) = \left( -\frac{d^2}{dx^2} + V_-(x) \right) \phi^{(-)}(x) = \epsilon \phi^{(-)}(x)$$

(7)
one has to apply a transformation of the form [16]

\[ \psi^{(-)}(x) = \rho^{-1} \psi^{(-)}(x) \]  

where

\[ \rho = e^{-\mu \int W(x) dx}, \quad \mu = \frac{\alpha - \beta}{1 - \alpha - \beta}, \quad \alpha + \beta \neq 1 \]  

so that comparison between (6) and (7) gives

\[ V_{-}(x) = \left( \sqrt{1 - 4\alpha\beta} \frac{1}{1 - \alpha - \beta} W(x) \right)^2 - \frac{1}{(1 - \alpha - \beta)} W'(x) \]  

\[ \varepsilon = \frac{E}{1 - \alpha - \beta} \]  

Thus, a quantum system described by a pseudo Hermitian Hamiltonian \( H_{-} \), is mapped to an equivalent system described by its corresponding Hermitian counterpart \( h_{-} \), with the help of a similarity transformation \( \rho \) [6, 11, 12],

\[ h_{-} = \rho H_{-} \rho^{-1} \]  

We, now, take refuge in the formalism of supersymmetric quantum mechanics (SUSYQM) [3], or the equivalent intertwining operator method [4], to find an isospectral partner of \( h_{-} \).

As is well known, \( h_{-} \) can always be written in a factorizable form as a product of a pair of linear differential operators \( \hat{A}, \hat{A}^{\dagger} \), as

\[ h_{-} = \hat{A}^{\dagger} \hat{A} = -\frac{d^2}{dx^2} + w^2 - w' \]  

apart from some factorization energy \( \varepsilon \), where \( \hat{A}, \hat{A}^{\dagger} \) and \( w(x) \) are given by

\[ \hat{A} = \frac{d}{dx} + w(x), \quad \hat{A}^{\dagger} = -\frac{d}{dx} + w(x), \quad w(x) = -\frac{d \ln \phi_{0}^{(-)}(x)}{dx} \]  

\( \phi_{0}^{(-)} \) being the ground state eigenfunction of \( \hat{A}^{\dagger} \hat{A} \) with energy \( \varepsilon_{0} \). Thus \( V_{-}(x) \) in (10) can be identified with \( (w^2 - w') \)

\[ V_{-}(x) = w^2(x) - w'(x) \]  

With the help of (10), the original eigenvalue equation (6) may be written in a more compact form as

\[ H_{-}\psi^{(-)}(x) = (1 - \alpha - \beta) \left\{ -\left( \frac{d}{dx} - \frac{\alpha - \beta}{1 - \alpha - \beta} W(x) \right)^2 + V_{-}(x) \right\} \psi^{(-)}(x) = E\psi^{(-)}(x) \]  

(15)

By the principles of SUSYQM, the hamiltonian \( h_{-} \) is isospectral to its partner Hamiltonian \( h_{+} \), given by

\[ h_{+} = \hat{A}^{\dagger} \hat{A} = -\frac{d^2}{dx^2} + w^2 + w' \]  

i.e.,

\[ h_{+}\phi^{(+)}(x) = \left( -\frac{d^2}{dx^2} + V_{+}(x) \right) \phi^{(+)}(x) = \varepsilon \phi^{(+)}(x) \]  

where

\[ V_{+}(x) = w^2(x) + w'(x) \]  

(18)
Let us now apply the inverse transformation of that given in (8) to (17) above, i.e.,

$$\phi^{(\pm)}(x) = \rho \psi^{(\pm)}(x) = e^{-\mu \int W(x) dx} \psi^{(\pm)}(x)$$

(19)

After some straightforward algebra, equation (17) can be written as

$$H\psi^{(\pm)} = (1 - \alpha - \beta) \left\{ -\left( \frac{d}{dx} + \frac{\alpha - \beta}{1 - \alpha - \beta} W(x) \right)^2 + V_+(x) \right\} \psi^{(\pm)} = E \psi^{(\pm)}$$

(20)

Thus, $H_\pm$ are of the same form, except for the explicit form of $V_\pm(x)$. Evidently, both the initial Hamiltonian $H_-$ as well as its partner $H_+$ are non Hermitian. Since $h_\pm$ share identical energies, except for the ground state, so should $H_\pm$, with the exception of the ground state. Thus, applying the principles of SUSYQM, we obtain a non Hermitian partner Hamiltonian $H_+$ of the initial one $H_-$, sharing identical energies except for the ground state.

2.1. Pseudo Hermiticity of $H_+$

If one considers the inverse transformation (19), then it is easy to check that both the Hermitian Hamiltonian $h_\pm$ and their non Hermitian counterparts $H_\pm$ are related by the same similarity transformation as in (11), i.e.,

$$H_\pm = \rho^{-1} h_\pm \rho$$

(21)

Additionally, simple algebra shows that both the non Hermitian Hamiltonian $H_\pm$ are pseudo Hermitian with respect to the same pseudo Hermiticity operator $\eta$

$$H_\pm^\dagger = \eta H_\pm \eta^{-1} \text{ i.e. } H_\pm^\dagger \eta = \eta H_\pm$$

(22)

where $\rho$ and $\eta$ are inter-related through $\rho = \sqrt{\eta}$ [6, 11].

It is interesting to study the behaviour of the wave functions $\psi^{(\pm)}(x)$. Since $H_\pm$ are $\eta$-pseudo Hermitian, the wave functions should be normalized as $\langle \psi^{(\pm)} | \eta | \psi^{(\pm)} \rangle$ [13]. With $\eta = \rho^2$ and $\psi^{(\pm)}(x) = \rho^{-1} \phi^{(\pm)}(x)$, the above normalization condition reduces to the conventional normalization of Hermitian quantum systems, viz., $\langle \phi^{(\pm)} | \phi^{(\pm)} \rangle$, easily available in standard text books of quantum mechanics for the shape-invariant potentials considered here [3].

3. Underlying symmetry between the partners $H_\pm$

To explore the underlying symmetry between the isospectral partners $H_\pm$, we start with their Hermitian counterparts $h_\pm$. Now, $h_\pm$ form a pair of supersymmetric partners, with super Hamiltonian

$$h = \begin{pmatrix} h_+ & 0 \\ 0 & h_- \end{pmatrix}$$

(23)

and generated by supercharges

$$q = \begin{pmatrix} 0 & \tilde{A}^\dagger \\ 0 & 0 \end{pmatrix}, \quad q^\dagger = \begin{pmatrix} 0 & 0 \\ \tilde{A} & 0 \end{pmatrix}$$

(24)

so that

$$h = [q^\dagger, q]$$

(25)

To establish the symmetry relation between $H_\pm$, we return to the similarity transformation between the original non Hermitian Hamiltonian $H_-$ and its Hermitian mapping $h_-$, viz.,

$$H_- = \rho^{-1} h_- \rho$$
If one defines two operators $D_{\pm}$ as
\[
D_+ = (\sqrt{1 - \alpha - \beta}) \rho^{-1} \tilde{A}^{\dagger} \rho \quad D_- = (\sqrt{1 - \alpha - \beta}) \rho^{-1} \tilde{A} \rho
\] (26)
then the isospectral Hamiltonians, $H_{\pm}$, can be written in terms of these operators as
\[
H_- = D_+ D_- \quad H_+ = D_- D_+
\] (27)
so that $D_{\pm}$ play the role of intertwining operators for $H_{\pm}$
\[
D_+ H_+ = H_+ D_+ \quad D_- H_- = H_- D_-
\] (28)
With the help of (2) and (26), $D_{\pm}$ can be written in the explicit form
\[
D_+ = (\sqrt{1 - \alpha - \beta}) \left\{ -\frac{d}{dx} + \mu W(x) + w(x) \right\} \quad D_- = (\sqrt{1 - \alpha - \beta}) \left\{ \frac{d}{dx} - \mu W(x) + w(x) \right\}
\] (29)
It is worth noting here that the functions $W(x)$ and $w(x)$ appearing in the explicit form of $D_{\pm}$ are not independent. Instead, they are related to each other by equations (10) and (14), i.e.
\[
w^2(x) - w'(x) = \left( \frac{1 - 4 \alpha \beta}{1 - \alpha - \beta} W(x) \right)^2 - \frac{1}{(1 - \alpha - \beta)} W'(x)
\] (30)
Since the isospectral partner Hamiltonians $H_{\pm}$ are pseudo Hermitian, we expect them to be embedded in the framework of pseudo supersymmetry [14]. Straightforward algebra shows that the operators $D_{\pm}$ are pseudo-adjoint of one another
\[
(D_+)^2 = \eta^{-1} (D_+)^\dagger \eta = \eta^{-1} (\rho \tilde{A} \rho^{-1}) \eta = \rho^{-1} \tilde{A} \rho = D_-
\] (31)
If we define two operators $Q$ and $Q^\dagger$ as
\[
Q = \begin{pmatrix} 0 & D_+ \\ 0 & 0 \end{pmatrix}, \quad Q^\dagger = \eta^{-1} Q^{\dagger} \eta = \begin{pmatrix} 0 & 0 \\ D_- & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ (D_+)^\dagger & 0 \end{pmatrix}
\] (32)
and construct a new Hamiltonian $\mathcal{H}$ from the partners $H_{\pm}$ as
\[
\mathcal{H} = \begin{pmatrix} H_- & 0 \\ 0 & H_+ \end{pmatrix}
\] (33)
then it is easy to observe that
\[
\mathcal{H} = \{ Q^\dagger, Q \}
\] (34)
Additionally,
\[
\{ Q, Q \} = \{ Q^\dagger, Q^\dagger \} = 0
\] (35)
Thus we obtain the standard pseudo super algebra of non Hermitian supersymmetry [14], with the operators $Q$ and $Q^\dagger$ playing the role of pseudo super charges, the anticommutator of which gives the pseudo super Hamiltonian $\mathcal{H}$. Interestingly, though it may not be possible (in general) to express the new Hamiltonian $H_\pm$ in terms of the generalized annihilation and creation operators $A$ and $A^\dagger$, nevertheless, the isospectral partners $H_{\pm}$ can be shown to be related by pseudo supersymmetry. Furthermore, it is also observed that the super charges $q, q^\dagger$ of conventional supersymmetry are related to the pseudo supercharges $Q, Q^\dagger$ of pseudo supersymmetry through
\[
Q = \rho^{-1} q \rho
\] (36)
1 In case the operators $D_{\pm}$ are defined by taking the negative square root of $(1 - \alpha - \beta)$, then the new operators $D_{\pm}^\text{new}$ so formed are related to $D_{\pm}$ through a constant phase, viz., $D_{\pm}^\text{new} = e^{i\pi} D_{\pm}$. This introduces no change in the expression for $H_{\pm}$.
This follows from the similarity mapping between the non Hermitian Hamiltonians $H_\pm$ and their respective Hermitian counterparts $h_\pm$. We shall devote the next section to construct some non Hermitian Hamiltonians as isospectral partners of the generalized Swanson models based on those shape-invariant potentials where the parameters are related to each other by translation ($a_2 = a_1 + \lambda$) [3, 15].

4. Models based on shape-invariant potentials

For our formalism to be applicable to specific models, one needs to solve the highly non-trivial Ricatti equation (30). This demands certain restrictions on the forms of $W(x)$ and $w(x)$. If one wants to map a certain type of potential (say Harmonic oscillator) to a different type (say e.g., Pöschl–Teller or Rosen–Morse), while going from the non Hermitian to the Hermitian picture, the corresponding Ricatti equation cannot be solved analytically (or, at least in an obvious way). For the shape invariant class, the function $w(x)$ consists of two parts, denoted by $f(x)$ and $g(x)$, i.e.

$$w(x) = \lambda_1 f(x) + \delta_1 g(x), \quad \text{with } \lambda_1, \delta_1 \text{ constants} \quad (37)$$

For reasons given in the beginning of this section, the function $W(x)$ used in the construction of generalized annihilation and creation operators $A$ and $A^\dagger$ in (2) is assumed to be of the same form as $w(x)$:

$$W(x) = \lambda_2 f(x) + \delta_2 g(x), \quad \text{with } \lambda_2, \delta_2 \text{ constants} \quad (38)$$

Our aim is to write $V_-(x)$ in terms of $w^2(x) - w'(x)$. It is already shown that $W(x)$ and $w(x)$ are inter-related through (30). Substituting (37) and (38) in (30), the expression takes the explicit form

$$\frac{1 - 4a \beta}{(1 - \alpha - \beta)^2} \left\{ \lambda_1^2 f'^2 + \delta_1^2 g'^2 + 2\lambda_1 \delta_1 fg \right\} = \frac{1}{1 - \alpha - \beta} (\lambda_2 f' + \delta_2 g')$$

$$= \lambda_1^2 f'^2 + \delta_1^2 g'^2 + 2\lambda_1 \delta_1 f g - \lambda_1 f' - \delta_1 g' \quad (39)$$

This general expression relates the unknown parameters $\lambda_1, \delta_1$ in terms of the known ones $\lambda_2, \delta_2$, for all shape invariant potentials where the parameters of the original potential and its partner are related to each other by translation. This enables one to write the partner potential $V_+(x)$, and hence the partner Hamiltonian $H_+$, in terms of the parameters of the starting Hamiltonian $H_-$. Now these shape-invariant models can be further classified under different categories, depending on the particular forms of $f(x)$ and $g(x)$. We shall explore these in further detail in the next few subsections.

4.1. Case 1: $g(x) = \text{constant}$, $f^2(x) = c_1 f'(x) + c_2$, with $c_1, c_2 \text{ constants}$

In this subsection, we shall study the models based on the following potentials:

1. Rosen–Morse I (trigonometric) potential

$$V(x) = a(a - 1) \csc^2 x + 2b \cot x - a^2 + \frac{b^2}{a^2}, \quad 0 \leq x \leq \pi \quad (40)$$

with $W(x) = -a_2 \cot x - \frac{b_2}{a_2}, \quad a_2 > 0, \ b_2 > 0 \quad (41)$
(2) Rosen–Morse II (hyperbolic) potential

\[ V(x) = -a(a + 1) \sech^2 x + 2b \tanh x + a^2 + \frac{b^2}{a^2}, \quad b < a^2, \quad -\infty \leq x \leq \infty \tag{42} \]

with \( W(x) = a_2 \tanh x + \frac{b_2}{a_2}, \quad a_2 > 0, \quad b_2 > 0 \tag{43} \)

(3) Eckart potential

\[ V(x) = a(a - 1) \cosech^2 x - 2b \coth x + a^2 + \frac{b^2}{a^2} \quad b > a^2, \quad 0 \leq x \leq \infty \tag{44} \]

with \( W(x) = -a_2 \coth x + \frac{b_2}{a_2}, \quad a_2 > 0, \quad b_2 > 0 \tag{45} \)

For the sake of convenience, we put \( g(x) = 1 \). This simplifies (39) to

\[
(\lambda_1^2 c_1 - \lambda_1) f'(x) + 2\lambda_1 \delta_1 f(x) + \lambda_1^2 c_2 + \delta_1^2 = 
\left( \frac{1}{1 - \alpha - \beta} \right) f'(x) + \frac{2\lambda_2 \delta_2 (1 - 4\alpha \beta)}{(1 - \alpha - \beta)^2} f(x) + \frac{1}{1 - \alpha - \beta} \left( \lambda_2^2 c_2 + \delta_2^2 \right) \tag{46} \]

Equating like terms on both sides, the unknown parameters \( \lambda_1, \delta_1 \) are expressed in terms of the known ones \( \lambda_2, \delta_2 \) through the following:

\[
\lambda_1^2 c_1 - \lambda_1 = \frac{\lambda_2^2 c_1 (1 - 4\alpha \beta)}{(1 - \alpha - \beta)^2} - \frac{\lambda_2}{1 - \alpha - \beta} \tag{47} \]

or, more explicitly,

\[
\lambda_1 = \frac{1 \pm \sqrt{1 + 4\sigma_-}}{2c_1} \tag{48} \]

where

\[
\sigma_- = \frac{\lambda_2^2 c_1 (1 - 4\alpha \beta)}{(1 - \alpha - \beta)^2} - \frac{\lambda_2}{1 - \alpha - \beta} \tag{49} \]

and

\[
\delta_1 = \frac{\lambda_2 \delta_2}{\lambda_1} \frac{1 - 4\alpha \beta}{1 - \alpha - \beta} \tag{50} \]

Since \( \lambda_1 \) and \( \lambda_2 \) should be of the same sign, only the positive sign is allowed in the expression for \( \lambda_1 \) in (48). The pseudo supersymmetric partners \( H_\pm \), expressed as,

\[ H_\pm \psi(\pm)(x) = E^\pm \psi(\pm)(x) \tag{51} \]

or, more explicitly,

\[
H_\pm(x) = (1 - \alpha - \beta) \left\{ -\left( \frac{d}{dx} - \frac{\alpha - \beta}{1 - \alpha - \beta} W(x) \right)^2 + V_\pm(x) \right\} \tag{52} \]

have identical energies except for the ground state, with \( V_\pm(x) \) for this class of potentials reducing to

\[
V_\pm(x) = \left( \lambda_1^2 c_1 \pm \lambda_3 \right) f'(x) + 2\lambda_1 \delta_1 f(x) + \delta_1^2 + c_2 \lambda_1^2 \tag{53} \]
The potentials falling in this category are listed below. In each case the form of $w(x)$ is similar to that of $W(x)$, with $a_2$ and $b_2$ being replaced by $a_1$ and $b_1$. The unknown parameters $a_1$ and $b_1$ are obtained in terms of the known ones $a_2$ and $b_2$ from expressions (48), (49) and (50). It can be shown that the eigen energies of the positive and the negative sector are related through

$$E_n^{(+)} = E_{n+1}^{(-)}, \quad \text{with} \quad E_n^{(\pm)} = (1 - \alpha - \beta)e^{\pm b_2}, \quad n = 0, 1, 2, \ldots$$

The partner potentials $V_\pm(x)$ are given in table 1 while the (unnormalized) solutions of the original Hamiltonian $H_-$ are given in table 2. The solutions of its partner $H_+$ can be obtained by applying the transformation

$$\psi_n^{(+)}(x) = \rho^{-1}\phi_n^{(+)}(x)$$

### Table 1. The partner potentials $V_\pm(x)$ for the Rosen--Morse I and II, and the Eckart potentials. $\epsilon_n^{(b)}$ stands for $\epsilon_n^{(b)} = \frac{1}{2}\epsilon_n^{(s)}$. $E_n^{(+)} = E_{n+1}^{(-)}$.

| Model          | $f(x)$ | $W(x)$ | $V_\pm(x)$ | $\epsilon_n^{(-)}$ |
|----------------|--------|--------|------------|-------------------|
| Rosen--Morse I | $-\left(\frac{a_1^2 - b_2^2}{a_2^2}\right)^{1-4\alpha\beta/\left(1-\alpha\beta\right)^2}$ | $+a_1(a_1 \pm 1)\cos^2x$ | $+(a_1 + n)^2$ | $c_1 = c_2 = -1$ |
| $\lambda_2 = -a_2$ | cot $x$ | $-a_2 \cot x$ | $+\alpha(\alpha_1 \pm 1)\csc^2 x$ | $-(a_1 + n)^2$ |
| $\delta_2 = -\frac{b_1}{a_2}$ | $-\frac{b_2}{a_2}$ | $+2b_2\frac{1-4\alpha\beta}{\left(1-\alpha\beta\right)^2}\cot x$ | $-\frac{b_1}{\alpha_1 - a_1}$ | $n = 0, 1, 2, \ldots$ |
| | $\frac{1}{a_2}$ | | | |
| Rosen--Morse II | $\left(\frac{a_1^2 + b_2^2}{a_2^2}\right)^{1-4\alpha\beta/\left(1-\alpha\beta\right)^2}$ | $-a_1(a_1 \pm 1)\text{sech}^2 x$ | $-(a_1 - n)^2$ | $c_1 = -1, c_2 = 1$ |
| $\lambda_2 = a_2$ | tanh $x$ | $a_2 \tanh x$ | $+\alpha(\alpha_1 \pm 1)\tanh^2 x$ | $-(a_1 - n)^2$ |
| $\delta_2 = \frac{b_1}{a_2}$ | $\frac{b_2}{a_2}$ | $+2b_2\frac{1-4\alpha\beta}{\left(1-\alpha\beta\right)^2}\tanh x$ | $-\frac{b_1}{\alpha_1 - a_1}$ | $n = 0, 1, 2, \ldots < a_1$ |
| | $\frac{1}{a_2}$ | | | |
| Eckart | $\left(\frac{a_1^2 + b_2^2}{a_2^2}\right)^{1-4\alpha\beta/\left(1-\alpha\beta\right)^2}$ | $-a_1(a_1 \pm 1)\text{coth}^2 x$ | $-(a_1 + n)^2$ | $c_1 = -1, c_2 = 1$ |
| $\lambda_2 = -a_2$ | coth $x$ | $-a_2 \coth x$ | $+\alpha(\alpha_1 \pm 1)\coth^2 x$ | $-(a_1 + n)^2$ |
| $\delta_2 = \frac{b_1}{a_2}$ | $\frac{b_2}{a_2}$ | $-2b_2\frac{1-4\alpha\beta}{\left(1-\alpha\beta\right)^2}\coth x$ | $-\frac{b_1}{\alpha_1 - a_1}$ | $n = 0, 1, 2, \ldots$ |

### Table 2. The (unnormalized) solutions of the original Hamiltonian $H_-$, for the Rosen--Morse I and II, and the Eckart potentials. The solutions of their respective partners $H_+$ can be obtained by applying the transformation $\phi_n^{(+)}(x) = \rho^{-1}\phi_n^{(+)}(x)$, where $\phi_n^{(+)}$ are the solutions of the supersymmetric partner Hamiltonian $h_+$. In the expression for $\phi_n^{(+)}$, the different parameters stand for $\mu_1 = \alpha_1 \mu = \alpha_1 + \frac{a_2 - b_2}{a_2 - b_1}$, $\mu_2 = \frac{b_2}{a_2} - \frac{b_2(a_1 - a_2)}{a_2(a_1 - b_1)}$.

| Model          | $y$ | $s_\pm$ | $\phi_n^{(-)}$ |
|----------------|-----|--------|----------------|
| Rosen--Morse I | $i \cot x$ | $-a_1 - n \pm \frac{b_1}{a_1 - a_2}$ | $\frac{a_1}{a_1 - a_2} e^{\frac{b_1}{a_1 - a_2}y} \sin^{b_2} x P_n^{(\epsilon_n, \pm)} (y)$ |
| Rosen--Morse II | tanh $x$ | $a_1 - n \pm \frac{b_1}{a_1 - a_2}$ | $\left(1 - y \frac{a_1}{a_1 - a_2}\right) e^{\frac{b_2(a_1 - a_2)}{a_1 - a_2}y} P_n^{(\epsilon_n, \pm)} (y)$ |
| Eckart | coth $x$ | $\pm \frac{a_1}{a_1 - a_2} - n - a_1$ | $\left(1 - y \frac{a_1}{a_1 - a_2}\right) e^{\frac{b_2(a_1 - a_2)}{a_1 - a_2}y} P_n^{(\epsilon_n, \pm)} (y)$ |
where $\phi^{(+)}_n$ are the solutions of the supersymmetric partner Hamiltonian $h_+$. In the expression for $\psi^{(-)}_n$, the different parameters stand for

$$\mu_1 = a_1\mu = a_1\frac{\alpha - \beta}{1 - \alpha - \beta}, \quad \mu_2 = \frac{b_1}{a_1}\mu = \frac{b_1(\alpha - \beta)}{a_1(1 - \alpha - \beta)} \quad (56)$$

It is evident from the explicit expressions for $\psi^{(-)}_n(x)$ that for its well-defined behaviour, $\alpha$ and $\beta$ must obey additional constraints; e.g., for the Rosen–Morse II and Eckart models,

$$\alpha < \beta \quad (57)$$

while the Rosen–Morse I model requires

$$a_1 + n + \mu_2 > 0, \quad \frac{b_1}{a_1 + n} < \mu_1 \quad (58)$$

which, in turn, implies

$$\alpha > \beta \quad (59)$$

4.2. Case 2: $f^2(x) = c_1 + c_2 g^2(x)$, $f'(x) = c_3 g^2(x)$, $g'(x) = c_4 f(x) g(x)$ with $c_1, c_2, c_3, c_4$ constants

The models falling in this category are based on the:

(1) Scarf I (trigonometric potential)

$$V(x) = k_1 \tan^2 x - k_2 \sec x \tan x, \quad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \quad (60)$$

with $W(x) = \lambda_2 \tan x - \delta_2 \sec x \tan x \quad (61)$

(2) Scarf II (hyperbolic potential)

$$V(x) = k_1 \sech^2 x + k_2 \sech x \tanh x, \quad -\infty \leq x \leq \infty \quad (62)$$

with $W(x) = \lambda_2 \tanh x + \delta_2 \sech x \tanh x \quad (63)$

(3) Pöschl–Teller potential

$$V(x) = k_1 \coth^2 x - k_2 \coth x \coth \tanh x, \quad 0 \leq x \leq \infty \quad (64)$$

with $W(x) = \lambda_2 \tanh x - \delta_2 \coth x \coth \tanh x \quad (65)$

Thus (39) gets simplified to

$$\lambda_2^2 c_1 + (\lambda_2^2 c_2 + \delta_2^2 - \lambda_1 c_1) g^2(x) + (2\lambda_1 \delta_1 - \delta_1 c_4) f(x) g(x) + \lambda_1^2 c_1$$

\[\begin{aligned}
&= \left[ \frac{(1 - 4\alpha\beta)}{(1 - \alpha - \beta)^2} (\lambda_2^2 c_2 + \delta_2^2) - \frac{\lambda_2 c_3}{1 - \alpha - \beta} \right] g^2(x) + \frac{\lambda_2^2 c_1 (1 - 4\alpha\beta)}{(1 - \alpha - \beta)^2} \\
&\quad + \left[ 2\lambda_2 \delta_2 \frac{1 - 4\alpha\beta}{(1 - \alpha - \beta)^2} - \frac{\delta_2 c_4}{1 - \alpha - \beta} \right] f(x) g(x) \\
&\quad \text{Equating like terms on both sides, the unknown parameters $\lambda_1, \delta_1$ are obtained by solving the following two coupled equations simultaneously:}
\end{aligned}\]

$$\delta_1^2 + \lambda_1^2 c_2 - \lambda_1 c_1 = \left( \lambda_2^2 c_2 + \delta_2^2 \right) \frac{(1 - 4\alpha\beta)}{(1 - \alpha - \beta)^2} - \frac{\lambda_2 c_3}{1 - \alpha - \beta} \quad (67)$$

$$2\lambda_1 \delta_1 - \delta_1 c_4 = 2\lambda_2 \delta_2 \frac{1 - 4\alpha\beta}{(1 - \alpha - \beta)^2} - \frac{\delta_2 c_4}{1 - \alpha - \beta} \quad (68)$$

Once again, the pseudo supersymmetric partner Hamiltonians $H_\pm$, given by (52), have identical energies except for the ground state, with $V_\pm(x)$ for this class of potentials assuming
Table 3. The partner potentials \( V_\pm(x) \) for the Scarf I and II, and the Pöschl–Teller potentials. \( \epsilon_n^{(\pm)} \) is the same as that defined in table 1.

| Model        | \( f(x) \)          | \( g(x) \)          | \( V_\pm(x) \)          | \( \epsilon_n^{(\pm)} \) |
|--------------|----------------------|----------------------|--------------------------|--------------------------|
| Scarf I      | \(-\lambda_2^2 \frac{1 - 4\alpha\beta}{(1 - \alpha - \beta)^2} - \lambda_1^2 \) | \(-\lambda_2^2 \frac{1 - 4\alpha\beta}{(1 - \alpha - \beta)^2} \) | \(-\lambda_2^2 \frac{1 - 4\alpha\beta}{(1 - \alpha - \beta)^2} \) | \(-\lambda_2^2 \frac{1 - 4\alpha\beta}{(1 - \alpha - \beta)^2} \) |
| \( c_1 = -1 \), \( c_2 = 1 \) | \( \tan x \)         | \(-\sec x \)         | \(+ (\delta_1^2 + x^2) \) | \(-\lambda_1^2 + (\lambda_1^2 + n^2) \) |
| \( c_3 = 1 \), \( c_4 = 1 \) | \( -\delta_1 (2\lambda_1 \pm 1) \sec x \tan x \) | \( n = 0, 1, 2, \ldots \) |
| Scarf II     | \( \frac{\lambda_2^2}{2} \frac{1 - 4\alpha\beta}{(1 - \alpha - \beta)^2} + \lambda_1^2 \) | \( \frac{\lambda_2^2}{2} \frac{1 - 4\alpha\beta}{(1 - \alpha - \beta)^2} \) | \( \frac{\lambda_2^2}{2} \frac{1 - 4\alpha\beta}{(1 - \alpha - \beta)^2} \) | \( \frac{\lambda_2^2}{2} \frac{1 - 4\alpha\beta}{(1 - \alpha - \beta)^2} \) |
| \( c_1 = 1 \), \( c_2 = -1 \) | \( \tanh x \)         | \( \sech x \)         | \(+ (\delta_1^2 + x^2) \) | \(+\lambda_1^2 - (\lambda_1 - n)^2 \) |
| \( c_3 = 1 \), \( c_4 = -1 \) | \( +\delta_1 (2\lambda_1 \mp 1) \sech x \tan x \) | \( n = 0, 1, 2, \ldots < \lambda_1 \) |
| Pöschl–Teller | \( \frac{\lambda_2^2}{2} \frac{1 - 4\alpha\beta}{(1 - \alpha - \beta)^2} + \lambda_1^2 \) | \( \frac{\lambda_2^2}{2} \frac{1 - 4\alpha\beta}{(1 - \alpha - \beta)^2} \) | \( \frac{\lambda_2^2}{2} \frac{1 - 4\alpha\beta}{(1 - \alpha - \beta)^2} \) | \( \frac{\lambda_2^2}{2} \frac{1 - 4\alpha\beta}{(1 - \alpha - \beta)^2} \) |
| \( c_1 = 1 \), \( c_2 = 1 \) | \( \coth x \)          | \( -\csch x \)        | \(+ (\delta_1^2 + x^2) \) | \(+\lambda_1^2 - (\lambda_1 - n)^2 \) |
| \( c_3 = -1 \), \( c_4 = -1 \) | \( -\delta_1 (2\lambda_1 \mp 1) \csch x \coth x \) | \( n = 0, 1, 2, \ldots < \lambda_1 \) |

Table 4. The (unnormalized) solutions of the original Hamiltonian \( H_\pm \), for the Scarf I and II, and the Pöschl–Teller potentials, with \( \mu_1 = \lambda_2 \mu \), \( \mu_2 = \delta_2 \mu \). The solutions of their respective partners \( H_\pm \) can be obtained in the same way as given in table 2.

| Model        | \( y \)          | \( s_\pm \)          | \( \psi_n^{(\pm)} \)          |
|--------------|------------------|----------------------|--------------------------|
| Scarf I      | \( \sin x \)     | \( \lambda_1 \pm \delta_1 - \frac{1}{2} \) | \( \sec x + \tan x)^{-\mu_2} \) | \(+\lambda_1 \delta_1 - \delta_1 \mu \)(y) |
| Scarf II     | \( \sinh x \)    | \( \pm i\delta_1 - \lambda_1 - \frac{1}{2} \) | \( 1 + y \) \( e^{2\beta} \) | \(+\lambda_1 \delta_1 - \delta_1 \mu \)(y) |
| Pöschl–Teller | \( \cosh x \)    | \( \pm i\delta_1 - \lambda_1 - \frac{1}{2} \) | \( y - 1 \) \( e^{2\beta} \) | \(+\lambda_1 \delta_1 - \delta_1 \mu \)(y) |

the form

\[
V_\pm(x) = \lambda_1^2 c_1 + (\lambda_2^2 c_2 + \delta_1^2 \pm \lambda_1 c_3) g^2(x) + (2\lambda_1 \delta_1 \pm \delta_1 c_4) f(x) g(x) + \lambda_2^2 c_1 \frac{1 - 4\alpha\beta}{(1 - \alpha - \beta)^2} 
\]

(69)

In each of the cases of the three different models in this category, the form of \( w(x) \) is similar to that of \( W(x) \), with \( \lambda_2 \) and \( \delta_2 \) being replaced by \( \lambda_1 \) and \( \delta_1 \). The unknown parameters \( \lambda_1 \) and \( \delta_1 \) are obtained in terms of the known ones \( \lambda_2 \) and \( \delta_2 \) from expressions (48), (49) and (50).

The pseudo supersymmetric partner Hamiltonians of the form given in (52), have energies \( E^{(\pm)}(n) \), related to \( e^{(\pm)} \) through (54). The partner potentials \( V_{\pm}(x) \) are given in table 3 while the solutions are given in table 4, with

\[
\mu_1 = \lambda_2 \mu, \quad \mu_2 = \delta_2 \mu
\]

(70)

From the explicit expressions for the solutions, it is evident that well defined behaviour is assured only when the parameters satisfy additional constraints. For example, for the Pöschl–Teller model, this condition reduces to

\[
\mu_2 < 0 \quad \text{i.e.} \quad \alpha < \beta
\]

4.3. Case 3: \( g(x) = 1 \), and \( f'(x) = k f(x) \), with \( k = -1 \)

The Morse potential, given by

\[
V(x) = a_1^2 + b_1^2 \exp(-2x) - b_1(2a_1 + 1) \exp(-x), \quad -\infty \leq x \leq \infty
\]

(71)
belongs to this class of potentials, with
\[ W(x) = a_2 - b_2 \exp(-x) \]  
Thus, for this particular model, \( \lambda_2 = -b_2, \delta_2 = a_2 \), with \( f(x) = \exp(-x) \), so that equation (39) reduces to
\[ b_1 = b_2 \frac{\sqrt{1-4\alpha\beta}}{1-\alpha-\beta} \]  
\[ a_1 = \frac{1}{2b_1} \left\{ \frac{b_2}{(1-\alpha-\beta)^2} [2a_2 (1-4\alpha\beta) + (1+\alpha+\beta)] - b_1 \right\} \]  
Thus
\[ V_\pm(x) = a_1^2 + b_1^2 \exp(-2x) - b_1 (2a_1 \mp 1) \exp(-x) + a_2^2 \frac{1-4\alpha\beta}{(1-\alpha-\beta)^2} \]  
admit energies
\[ \varepsilon_n^{(-)} = a_1^2 - (a_1 - n)^2 + a_2^2 \frac{1-4\alpha\beta}{(1-\alpha-\beta)^2}, \quad n = 0, 1, 2, \ldots < a_1 \]  
\[ \varepsilon_n^{(+)} = \varepsilon_n^{(-)} \]  
The solutions of the original non Hermitian Hamiltonian \( H_- \) are given by
\[ \psi_n^{(-)}(x) \approx y^{\lambda_1-n} e^{(\frac{\mu_1}{2}-1)\frac{1}{2}} L_n^{2\lambda_1-2n}(y) \]  
where \( \mu_1 \) and \( \mu_2 \) are defined in equation (70) and
\[ y = 2\delta_1 e^{-x} \]  

4.4. Case 4: \( g(x) = 1 \), and \( f(x) = x \)

These values represent the \textit{Shifted Oscillator}, denoted by the potential
\[ V(x) = \frac{a^2}{4} \left( x - \frac{2b}{a} \right)^2 - \frac{a}{2}, \quad \infty \leq x \leq \infty \]  
with
\[ W(x) = \frac{1}{2}a_2x - b_2 \]  
Proceeding in a similar fashion, and assuming \( w(x) \) to be of the same form as \( W(x) \), with \( a_2, b_2 \) replaced by \( a_1, b_1 \), we obtain the following results:
\[ a_1 = \frac{a_2 \sqrt{1-4\alpha\beta}}{1-\alpha-\beta}, \quad b_1 = \frac{b_2 \sqrt{1-4\alpha\beta}}{1-\alpha-\beta} \]  
so that
\[ V_\pm(x) = \frac{1}{4}a_1^2 \left( x - \frac{2b_1}{a_1} \right)^2 \pm \frac{a_1}{2} - \frac{a_2}{2(1-\alpha-\beta)} \]  
with energy
\[ \varepsilon_n^{(-)} = a_1 n - \frac{a_2}{2(1-\alpha-\beta)}, \quad n = 0, 1, 2, \ldots \]  
\[ \varepsilon_n^{(+)} = \varepsilon_{n+1}^{(-)} \]  
Writing the solutions of \( H_- \) directly
\[ \psi_n^{(-)}(x) \approx e^{(\mu_1-n)\frac{1}{2}x} L_n^{2\lambda_1-2n}(y) \]
where $H_n(y)$ are the Hermite polynomials, $y = \sqrt{\frac{a_1}{a_2}} (x - \frac{2b_1}{a_1})$ and $\mu_1 = \mu a_2$, $\mu_2 = \mu b_2$. It can be checked that for the solutions to behave properly in the entire interval, the parameters should obey the condition $|\alpha + \beta| < 1$.

5. Conclusions

To conclude, we have developed a formalism to find an isospectral partner Hamiltonian $H_+$ of the generalized Swanson model, viz., $H_- = A^\dagger A + \alpha A^2 + \beta A^4$. Though both the initial Hamiltonian $H_-$ as well as its partner $H_+$ are non Hermitian, nevertheless they have real energies for certain range of parameter values. It is observed that $H_-$ form a pair of pseudo super symmetric partners of a pseudo super Hamiltonian $H$, and share identical energies except for the ground state. Furthermore, the same similarity transformation operator $\rho$ maps the pair of non Hermitian Hamiltonians $H_\pm$ to their respective Hermitian counterparts $h_\pm$, through $H_\pm = \rho^{-1} h_\pm \rho$, and these Hermitian maps form a pair of supersymmetric partners, generated by supercharges $q, q^\dagger$. The pseudo super charges $Q, Q^\dagger$ generating the pseudo super algebra of $H$ are also related to $q, q^\dagger$ through the similarity transformation: $Q = \rho^{-1} q \rho$.

Since we have introduced non Hermiticity through an imaginary vector potential, the Hermitian maps $h_\pm$ obtained by similarity transformation are Schrödinger operators comprising of the standard kinetic term plus a local real Hermitian potential. It may be mentioned here that though two Hamiltonians may be related by similarity transformations, yet they can reveal different physical aspects of the dynamical system. In fact, for a particular class of potentials, certain physical properties are expected to emerge more distinctly in the non Hermitian framework. For example, exceptional points, or branch-point singularities of the spectrum and eigenfunctions, are associated with non Hermitian operators [17]. However, when one goes from the non Hermitian to the corresponding Hermitian picture, the exceptional points are lost, and consequently the entire information related to such phenomena. Additionally, though the super symmetric partners $h_\pm$ of a Hermitian Hamiltonian can always be mapped to non Hermitian ones (say $H_\pm$) by a similarity transformation, there is absolutely no way to determine whether $H_\pm$ are isospectral or not. This is due to the fact that to write the pseudo Hermitian partner Hamiltonian $H_\pm$ in terms of the generalized annihilation and creation operators $A$ and $A^\dagger$ is still an open problem. As a result, while $h_\pm$ look similar in appearance (being expressed in terms of the creation and annihilation operators $A^\dagger$ and $A$), $H_\pm$ are not look-aliases. Nevertheless, we have been able to express $H_\pm$ in terms of the operators $D_\pm$, thus proving them to be related by pseudo super symmetry, sharing identical energies, barring the ground state.

We have applied our formalism successfully to all the known classes of shape-invariant models where the parameters of the original potential and its shape-invariant partner are related through translation. A general formula has been obtained for generating the respective pseudo supersymmetric partner Hamiltonians for such cases. Interestingly, the wave functions are automatically normalized following the normalization criterion for pseudo Hermitian systems [13]. We have intentionally left out the 3-dimensional shape-invariant models falling in this category, viz. 3-dimensional oscillator and Coulomb models, as we have restricted this work to deal with one-dimensional systems only. However, the radial part of these models can be studied in this framework, with $0 \leq r \leq \infty$.

This work deals with real Hamiltonians that are nevertheless non Hermitian. We can make a straightforward extension of our formalism to map a complex non Hermitian Hamiltonian $H$ to a Schrödinger Hamiltonian which is also complex but $\mathcal{P}\mathcal{T}$ symmetric. However, in such a case $H$ will be weakly pseudo Hermitian [18]. Finally we would like to note that in this
work we have studied shape-invariant models with unbroken supersymmetry. It would be interesting to study models with broken supersymmetry, too, in this framework.

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References

[1] Bender C M and Boettcher S 1998 Phys. Rev. Lett. 80 5243
Bender C M and Boettcher S 1998 J. Phys. A: Math. Gen. 31 L273

[2] Znojil M 2000 J. Phys. A: Math. Gen. 33 4561
Léviá G and Znojil M 2000 J. Phys. A: Math. Gen. 33 7165
Dorey P, Dunning C and Tateo R 2001 J. Phys. A: Math. Gen. 34 5679
Bender C M, Boettcher S, Jones H F, Meisinger P N and Simsek M 2001 Phys. Lett. A 291 197
Ahmed Z 2001 Phys. Lett. A 282 343
Ahmed Z 2001 Phys. Lett. A 287 295
Bagchi B and Quesne C 2000 Phys. Lett. A 273 285
Bagchi B and Quesne C 2002 Phys. Lett. A 300 18
Bender C M, Brody D C and Jones H F 2002 Phys. Rev. Lett. 89 270401
Bender C M, Brody D C and Jones H F 2004 Phys. Rev. Lett. 92 119902(E)
Bender C M and Tan B 2006 J. Phys. A: Math. Gen. 39 1945
Caliceti E, Cannata F and Graffi S 2006 J. Phys. A: Math. Gen. 39 10019 (and references therein)
Quesne C 2007 J. Phys. A: Math. Theor. 40 F745–51
Cannata F, Ioffe M V and Nishnianidze D N 2007 Phys. Lett. A 369 9
Bender C M 2007 Rep. Prog. Phys. 70 947
Assis P E G and Fring A 2008 J. Phys. A: Math. Theor. 41 244001
For various works on $\text{PT}$ symmetric/non Hermitian quantum systems see link http://gemma.ujf.cas.cz/znojil/conf/index.html. This contains the special issues on Conf. Proc. on pseudo-Hermitian Hamiltonians in Quantum Physics:
2004 Czech. J. Phys. 54
2005 Czech J. Phys. 55
2006 J. Phys. A: Math. Gen. 39
2006 Czech J. Phys. 56
2008 J. Phys. A: Math. Theor. 41
[3] Cooper F, Khare A and Sukhatme U 2001 Supersymmetry in Quantum Mechanics (Singapore: World Scientific)
Bagchi B 2000 Supersymmetry in Quantum and Classical Mechanics (London: Chapman and Hall)
Kalka H and Soff G 1997 Supersymmetry (Teuber)
[4] Infeld L and Hull T E 1951 Rev. Mod. Phys. 23 21
[5] Fatteev V V and Salle M A 1994 Darboux Transformations and Solitons (New York: Springer)
[6] Sinha A and Roy P 2007 J. Phys. A: Math. Theor. 40 10599
[7] Swanson M S 2004 J. Math. Phys. 45 585
[8] Musumbu D P, Geyer H B and Heiss W D 2007 J. Phys. A: Math. Theor. 40 F75
Geyer H B, Snyman I and Scholtz F G 2004 Czech. J. Phys. 54 1069
Jones H F 2005 J. Phys. A: Math. Gen. 38 1741
Bagchi B, Quesne C andRoychoudhury R 2005 J. Phys. A: Math. Gen. 38 L647
Scholtz F G and Geyer H B 2006 Phys. Lett. B 634 84
Scholtz F G and Geyer H B 2006 J. Phys. A: Math. Gen. 39 10189
Geyer H B, Heiss W D and Znojil M (eds) 2006 J. Phys. A: Math. Gen. 39 (11)
[9] Quesne C 2007 J. Phys. A: Math. Theor. 40 F745
[10] Quesne C 2008 J. Phys. A: Math. Theor. 41 244022
[11] Jones H F 2005 J. Phys. A: Math. Gen. 38 1741
[12] Mostafazadeh A and Batal A 2004 J. Phys. A: Math. Gen. 37 11645
[13] Mostafazadeh A 2002 J. Math. Phys. 43 205
Mostafazadeh A 2002 J. Math. Phys. 43 2814
Mostafazadeh A 2002 J. Math. Phys. 43 3944

[14] Mostafazadeh A 2002 Nucl. Phys. B 640 419

Znojil M 2002 Annihilation and creation operators in non-Hermitian supersymmetric quantum mechanics Preprint hep-th/0012002
Znojil M 2002 PT symmetry and supersymmetry Preprint hep-th/0209062
Znojil M 2002 J. Phys. A: Math. Gen. 35 2341
Znojil M, Cannata F, Bagchi B and Roychoudhury R 2000 Phys. Lett. B 483 284
Dorey P, Dunning C and Tateo R 2001 J. Phys. A: Math. Gen. 34 L391
Kleefeld F 2004 Non-Hermitian quantum theory and its Holomorphic representation: Introduction and some applications Preprint hep-th/0408028

[15] Gendenshtein L 1983 JETP Lett. 38 356
[16] Faria C F M and Fring A 2006 J. Phys. A: Math. Gen. 39 9269
[17] Heiss W D 2003 Exceptional points of non Hermitian operators Preprint quant-ph/0304152v1
[18] Solombrino L 2002 Weak pseudo-Hermiticity and antilinear commutant Preprint quant-ph/0203101