Direct Optimization of Ranking Measures

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Abstract

Web page ranking and collaborative filtering require the optimization of sophisticated performance measures. Current Support Vector approaches are unable to optimize them directly and focus on pairwise comparisons instead. We present a new approach which allows direct optimization of the relevant loss functions. This is achieved via structured estimation in Hilbert spaces. It is most related to Max-Margin-Markov networks optimization of multivariate performance measures. Key to our approach is that during training the ranking problem can be viewed as a linear assignment problem, which can be solved by the Hungarian Marriage algorithm. At test time, a sort operation is sufficient, as our algorithm assigns a relevance score to every (document, query) pair. Experiments show that the our algorithm is fast and that it works very well.

1 Introduction

Ranking, and web-page ranking in particular, has long been a fertile research topic of machine learning. It is now commonly accepted that ranking can be treated as a supervised learning problem, leading to better performance than using one feature alone [Burges et al., 2005, Cao et al., 2006]. Learning to rank can be viewed as an attempt of learning an ordering of the data (e.g. web pages). Although ideally one might like to have a ranker that learns the partial ordering of all the matching web pages, users are most concerned with the topmost (part of the) results returned by the system. See for instance [Cao et al., 2006] for a discussion.

This is manifest in corresponding performance measures developed in information retrieval, such as Normalized Discounted Cumulative Gain (NDCG), Mean Reciprocal Rank (MRR), Precision@n, or Expected Rank Utility (ERU). They are used to address the issue of evaluating rankers, search engines or recommender sytems [Voorhees, 2001, Jarvelin and Kekalainen, 2002, Breese et al., 1998, Basilico and Hofmann, 2004].

Ranking methods have come a long way in the past years. Beginning with vector space models [Salton, 1971, Salton and McGill, 1983], various feature based methods have been proposed [Lee et al., 1997]. Popular set of features include BM25 [Robertson et al., 1994] or its variants [Robertson and Hull, 2000]. Following the intent of Richardson et al. [2006] we show that when combining such methods with machine learning, the performance of the ranker can be increased significantly.

Over the past decade, many machine learning methods have been proposed. Ordinal regression [Herbrich et al., 2000, Chu and Keerthi, 2005] using a SVM-like large margin method and Neural Networks [Burges et al., 2005] were some of the first approaches. This was followed by Perceptrons [Crammer and Singer, 2002], and online methods, such as [Crammer and Singer, 2005, Basilico and Hofmann, 2004]. The state of the art is essentially to describe the partial

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order by a directed acyclic graph and to associate the cost incurred by ranking incorrectly with the edges of this graph. These methods aim at finding the best ordering function over the returned documents. However, it is difficult to express complex (yet commonly used) measures in this framework.

Only recently two theoretical papers [Rudin, 2006, Cossock and Zhang, 2006] discuss the issue of learning ranking with preference to top scoring documents. However, the cost function of [Rudin, 2006] is only vaguely related to the cost function used in evaluating the performance of the ranker. [Cossock and Zhang, 2006] argue that, in the limit of large samples, regression on the labels may be sufficient.

Our work uses the framework of Support Vector Machines for Structured Outputs [Tsochantaridis et al., 2005, Joachims, 2005, Taskar et al., 2004] to deal with the inherent non-convexity of the performance measures in ranking. Due to the capacity control inherent in kernel methods it generalizes well to test observations [Schölkopf and Smola, 2002]. The optimization problem we propose is very general: it covers a broad range of existing criteria in a plug-and-play manner. It extends to position-dependent ranking and diversity-based scores.

Of particular relevance are two recent papers [Joachims, 2005, Burges et al., 2007] to address the complication of the information retrieval loss functions. More specifically, [Joachims, 2005] shows that two ranking-related scores, Precision@n and the Area under the ROC curve, can be optimized by using a variant of Support Vector Machines for Structured Outputs (SVMStruct). We use a similar strategy in our algorithm to obtain a Direct Optimization of Ranking Measures (DORM) using the inequalities proposed in [Tsochantaridis et al., 2005]. [Burges et al., 2007] considered a similar problem without the convex relaxation and instead they optimize the nonconvex cost functions directly by only dealing with their gradients.

Outline: After a summary of structured estimation we discuss performance measures in information retrieval (Section 3) and we express them as inner products. In Section 4 we compute a convex relaxation of the performance criterion and show how it can be solved efficiently using linear assignment. Experiments on web search and collaborative filtering show that DORM is fast and works well.

2 Structured Estimation

In the following we will develop a method to rank objects (e.g. documents $d$) subject to some query $q$ by means of some function $g(d, q)$. Obviously we want to ensure that highly relevant documents will have a high score, i.e. a large value of $g$. At the same time, we want to ensure that the ranking obtained is optimal with respect to the relevant ranking score. For instance for NDCG@10, i.e. a score where only the first 10 retrieved documents matter, it is not very important what particular values of a score $g$ will assign to highly irrelevant pages, provided that they remain below the acceptance threshold.

Obviously, we could use engineering skills to construct a reasonable function (PageRank is an example of such a function). However, we can also use statistics and machine learning to guide us find a function that is optimized for this purpose. This leads to a more optimized way of finding such a function, removing the need for educated guesses. The particular tool we use is max margin structured estimation, as described in Tsochantaridis et al. [2005]. See the original reference for a more detailed discussion.

2.1 Problem Setting

Large margin structured estimation, as proposed by [Taskar et al., 2004, Tsochantaridis et al., 2005], is a general strategy to solve estimation problems of mapping $\mathcal{X} \rightarrow \mathcal{Z}$ by finding related optimization problems. More concretely it solves the estimation problem of finding a matching $z \in \mathcal{Z}$ from a set of (structured) estimates, given patterns $x \in \mathcal{X}$, by finding a function $f(x, z)$
such that
\[
z^*(x) := \arg\max_{z \in Z} f(x, z).
\] (1)

This means that instead of finding a mapping \( X \to Z \) directly, the problem is reduced to finding a real valued function on \( X \times Z \).

In the ranking case, \( x \in X \) corresponds to a set of documents with a corresponding query, whereas \( z \) would correspond to the permutation which orders the documents such that the most relevant ones are ranked first. Consequently \( f \) will be a function of the documents, query, and a permutation. It is then our goal to find such an \( f \) that it is maximized for the “correct” permutation.

To assess how well the estimate \( z^*(x) \) performs we need to introduce a loss function \( \Delta(y, z) \), depending on \( z \) and some reference labels \( y \), which determines the loss at \( z \). For instance, if we want to solve the binary classification problem \( y, z \in \{\pm 1\} \), where \( y \) is be the observed label and \( z \) the estimate we could choose \( \Delta(y, z) = 1 - \delta_{y,z} \). That is, we incur a penalty of 1 if we make a mistake, and no penalty if we estimate \( y = z \). In the regression case, this could be \( \Delta(y, z) = (y - z)^2 \). Finally, in the sequence annotation case, where both \( y \) and \( z \) are binary sequences, [Taskar et al., 2004, Tsochantaridis et al., 2005] use the Hamming loss. In the ranking case, which we will discuss in Section 3, the loss \( \Delta \) will correspond to the relative regret incurred by ranking documents in a suboptimal fashion with respect to WTA, MRR, DCG, NDCG, ERU or a similar criterion. Moreover \( y \) will correspond to the relevance scores assigned to various documents by reference users.

In summary, it is our goal to find some function \( f \) to minimize the error incurred by \( f \) on a set of observations \( X = \{x_1, \ldots, x_m\} \) and reference labels \( Y = \{y_1, \ldots, y_m\} \)
\[
R_{\text{emp}}[f, X, Y] := \sum_{i=1}^{m} \Delta(y_i, \arg\max_{z \in Z} f(x_i, z)).
\] (2)

We will refer to \( R_{\text{emp}}[f, X, Y] \) as the empirical risk. Direct minimization of the latter with respect to \( f \) is difficult:

- It is a highly nonconvex optimization problem. This makes practical optimization extremely difficult, as the problem has many local minima.
- Good performance with respect to the empirical risk \( R_{\text{emp}}[f, X, Y] \) does not result in good performance on an unseen test set. In practice, strict minimization of the empirical risk virtually ensures bad performance on a test set due to overfitting. This issue has been discussed extensively in the machine learning literature (see e.g. [Vapnik, 1982]).

To deal with the second problem we will add a regularization term (here a quadratic penalty) to the empirical risk. To address the first problem, we will compute a convex upper bound on the loss \( \Delta(y_i, \arg\max_{z \in Z} f(x_i, z)) \).

2.2 Convex Upper Bound

The key inequality we exploit in obtaining a convex upper bound on the problem of minimizing the loss \( \Delta(y, z) \) is the following lemma, which is essentially due to [Tsochantaridis et al., 2005].

Lemma 1 Let \( f : X \times Z \to \mathbb{R} \) and \( \Delta : Y \times Z \to \mathbb{R} \) and let \( z_0 \in Z \). Moreover let \( \xi \in \mathbb{R} \). In this case, if \( \xi \) satisfies
\[
f(x, z_0) - f(x, z) \geq \Delta(y, z) - \Delta(y, z_0) - \xi \text{ for all } z \in X,
\]
then \( \xi \geq \Delta(y, \arg\max_{z \in Z} f(x, z)) - \Delta(y, z_0) \). Moreover, the constraints on \( \xi \) and \( f \) are linear.
Proof  Linearity is obvious, as $\xi$ and $f$ only appear as individual terms. To see the first claim, denote by $z^*(x) := \arg\max_{z \in Z} f(x, z)$. Since the inequality needs to hold for all $z$, it holds in particular for $z^*(x)$. This implies

$$0 \geq f(x, z_0) - f(x, z^*(x)) \geq \Delta(y, z^*(x)) - \Delta(y, z_0) - \xi.$$  

The first inequality holds by construction of $z^*(x)$. Rearrangement proves the claim.  

Typically one chooses $z_0$ to be the minimizer of $\Delta$ and one assumes that the loss for $z_0$ vanishes. In this case $\xi \geq \Delta(z^*)$. Note that this convex upper bound is tight for $z_0 = z^*$ and if the minimal $\xi$ satisfying this inequality is chosen.

2.3 Kernels

The last ingredient to obtain a convex optimization problem is a suitable function class for $f$. In principle, any class, such as Decision Trees, Neural Networks, convex combinations of weak learners as they occur in Boosting, etc. would be acceptable. For convenience we choose $f$ via

$$f(x, z) = \langle \Phi(x, z), w \rangle.$$  

(3)

Here $\Phi(x, z)$ is a feature map and $w$ is a corresponding weight vector. The advantage of this formulation is that by choosing different maps $\Phi$ it becomes possible to incorporate prior knowledge efficiently. Moreover, it is possible to express the arising optimization and estimation problem in terms of the kernel functions

$$k((x, z), (x', z')) := \langle \Phi(x, z), \Phi(x', z') \rangle.$$  

(4)

without the need to evaluate $\Phi(x, z)$ explicitly. This allows us to work with infinite-dimensional feature spaces while keeping the number parameters of the optimization problem finite. Moreover, we can control the complexity of $f$ by keeping $\|w\|$ sufficiently small (for binary classification this amounts to large margin classification).

Denote by $z_i$ the value of $z$ for which $\Delta(y_i, z)$ is minimized. Moreover, let $C > 0$ be a regularization constant specifying the trade-off between empirical risk minimization and the quest for a “simple” function $f$. Combining (2), Lemma 1, and (3) one arrives at the following optimization problem:

$$\min_{w, \xi} \frac{1}{2} \|w\|^2 + C \sum_{i=1}^{m} \xi_i$$  

subject to $\langle w, \Phi(x_i, z_i) - \Phi(x_i, z) \rangle \geq \Delta(y_i, z) - \xi_i$  

for all $\xi_i \geq 0$ and $z \in Z$ and $i \in \{1, \ldots, m\}$.

In the ranking case we will assume (without loss of generality) that the documents are already ordered in decreasing order of relevance. In this case $z_i$ will correspond to the unit permutation which leaves the order of documents unchanged.

2.4 Optimization

One may show [Taskar et al., 2004] that the solution of (5) is given by

$$f(x', z') = \sum_{i, z} \alpha_{iz} k((x_i, z), (x', z')).$$  

(6)
Algorithm 1 Column Generation

Input: data $x_i$, labels $y_i$, sample size $m$, tolerance $\epsilon$

Initialize $S_i = \emptyset$ for all $i$, and $w = 0$.

repeat
  for $i = 1$ to $m$ do
    $w = \sum_1^m \sum_{z \in S_i} \alpha_{iz} \Phi(x_i, z)$
    $z^* = \arg\max_{z \in \mathbb{Z}} \langle w, \Phi(x_i, z) \rangle + \Delta(y_i, z)$
    $\xi = \max(0, \max_{z \in S_i} \langle w, \Phi(x_i, z) \rangle + \Delta(y_i, z))$
    if $\langle w, \Phi(x_i, z^*) \rangle + \Delta(y_i, z) > \xi + \epsilon$ then
      Increase constraint set $S_i \leftarrow S_i \cup z^*$
      Optimize (7) using only $\alpha_{iz}$ where $z \in S_i$.
    end if
  end for
until $S$ has not changed in this iteration

This fact is also commonly referred to as the Representer Theorem [Schölkopf and Smola, 2002]. The coefficients $\alpha_{iz}$ are obtained by solving the dual optimization problem of (5):

$$\min_\alpha \frac{1}{2} \sum_{i,j,z,z'} \alpha_{iz} \alpha_{jz'} k((x_i, z), (x_j, z')) - \sum_{i,z} \Delta(y_i, z) \alpha_{iz}$$

subject to $\sum_z \alpha_{iz} \leq C$ and $\alpha_{iz} \geq 0$ for all $i$ and $z$. (7b)

Solving the optimization problem (7) presents a formidable challenge. In particular, for large $\mathbb{Z}$ (e.g. the space of all permutations over a set of documents) the number of variables is prohibitively large and it is essentially impossible to find an optimal solution within a practical amount of time. Instead, one may use column generation [Tsochantaridis et al., 2005] to find an approximate solution in polynomial time. The key idea in this is to check the constraints (5b) to find out which of them are violated for the current set of parameters and to use this information to improve the value of the optimization problem. That is, one needs to find

$$\arg\max_{z \in \mathbb{Z}} \Delta(y_i, z) + \langle w, \Phi(x_i, z) \rangle,$$

as this is the term for which the constraint (5b) becomes tightest. If $\mathbb{Z}$ is a small finite set of values, this is best achieved by brute force evaluation. For binary sequences, one often uses dynamic programming. In the case of ranking, where $\mathbb{Z}$ is the space of permutations, we shall see that (8) can be cast as a linear assignment problem.

The Algorithm 1 has good convergence properties. It follows from [Tsochantaridis et al., 2005] that it terminates after adding at most

$$\max\left[\frac{2n\Delta}{\epsilon}, \frac{8CR^2}{\epsilon^2}\right]$$

steps, where $\Delta$ is an upper bound on the loss $\Delta(y_i, z)$ and $R$ is an upper bound on $\|\Phi(x_i, z)\|$.

To adapt the above framework to the ranking setting, we need to address three issues: a) we need to derive a loss function $\Delta$ for ranking, b) we need to develop a suitable feature map $\Phi$ which takes document collections, queries, and permutations into account, and c) we need to find an algorithm to solve (8) efficiently.

3 Ranking and Loss Functions

3.1 A Formal Description

For efficiency, commercial rankers, search engines, or recommender systems, usually employ a document-at-a-time approach to answer a query $q$: a list of candidate documents is evaluated (while retaining a heap of the $n$ top scoring documents) by evaluating the relevance for
| Variable | Meaning |
|----------|---------|
| $x = (q, D)$ | document-query pair |
| $q_i$ | $i$-th query |
| $l_i$ | number of documents for $q_i$ |
| $D_i = \{d_{i1}, \ldots, d_{il_i}\}$ | documents for $q_i$ |
| $r_{ij} \in [0, \ldots, r_{\max}]$ | relevance of document $d_{ij}$ |
| $y_i = \{r_{i1}, \ldots, r_{il_i}\}$ | reference label |
| $f(x, \pi)$ | global scoring function |
| $g(q_i, d_{ij})$ | individual scoring function |
| $m$ | number of queries for training |

For computational efficiency (not for theoretical reasons) it is not desirable that $f$ depends on $\{d_1, \ldots, d_l\}$ jointly.

a (document, query)-pair one at a time. For this purpose a score function $g(d, q)$ is needed, which assigns a score to every document given the query.\footnote{For computational efficiency (not for theoretical reasons) it is not desirable that $f$ depends on $\{d_1, \ldots, d_l\}$ jointly.} Performance of the ranker is typically measured by means of a set of labels $y := \{r_1, \ldots, r_l\}$ with $r_i \in [0, \ldots, r_{\max}]$, where 0 corresponds to 'irrelevant' and $r_{\max}$ corresponds to 'highly relevant'. Training instances contain document query pairs that are labelled by experts. Such data for commercial search engines or recommender systems often have less than ten levels of relevance.

At training time we are given $m$ instances of queries $q_i$, document collections $D_i$ of cardinality $l_i$ and labels $y_i$ with $|y_i| = |D_i|$. In the context of the previous section a set of documents $D_i$ in combination with a matching query $q_i$ will play the role of a pattern, i.e. $x_i := (q_i, d_{i1}, \ldots, d_{il_i})$. Likewise the reference labels $r_{ij} \in y_i$ consist of the corresponding expert ratings for the documents $d_{ij}$.

We want to find some mapping $f$, such that the ordering of a new collection of documents $d_1, \ldots, d_l$ obtained by sorting $g(d_i, q)$ agrees well with $y$ in expectation. We would like to obtain a single ranking which will perform well on the query for a given performance measure, unlike [Matveeva et al., 2006] who use a cascade of rankers.

Note that there is also a processing step associated with ranking documents: for each document query pair $(d, q)$, we have to construct a feature vector $x$. In this paper, we assume that the feature vector is given, and also use $(d, q)$ to mean $x$. For instance BM25 [Robertson et al., 1994, Robertson and Hull, 2000], date, click-through logs [Joachims, 2002] have proved to be an effective set of features.

Many widely-used performance measures in the information retrieval community are irreducibly multivariate and permutation based. By permutation-based, we mean that the performance measure can be computed by comparing the two sets of ordering. For example, 'Winner Takes All' (WTA), Mean Reciprocal Rank (MRR) [Voorhees, 2001], Mean Average Precision (MAP), and Normalized Discounted Cumulative Gain (NDCG) [Jarvelin and Kekalainen, 2002] all fulfill this property.

It is our goal to find a suitable permutation $\pi(D, q, g)$ obtained for the collection $D$ of documents $d_i$ given the query $q$ and the scoring function $g$. We will drop the arguments $D, q, g$, wherever it is clear from the context. Moreover, given a vector $v \in \mathbb{R}^m$ we denote by $v(\pi)$ the permutation of $v$ according to $\pi$, i.e. $v(\pi)_i = v_{\pi(i)}$. Finally, without loss of generality (and for notational convenience) we assume that $y$ is sorted in descending order, i.e. most relevant documents first. That is, the identical permutation $\pi = 1$ will correspond to the sorting which returns the most relevant documents first with respect to the reference labeling.

Note that $\pi$ will play the role of $z$ of Section 2. Likewise we will denote by $\Pi$ the space of all permutations (i.e. $Z = \Pi$).
3.2 Scores and Loss

**Winner Takes All (WTA):** If the first document is relevant, i.e. if \( y(\pi)_1 = r_1 \) the score is 1, otherwise 0. We may rewrite this as

\[
\text{WTA}(\pi, y) = \langle a(\pi), b(y) \rangle
\]

where \( a_i = \delta(i, 1) \) and \( b_i = \delta(r_i, r_1) \).

Note that there may be more than one document which is considered relevant. In this case \( \text{WTA}(\pi, y) \) will be maximal for several classes of permutations.

**Mean Reciprocal Rank (MRR):** We assume that there exists only one top ranking document. We have

\[
\text{MRR}(\pi, y) = \langle a(\pi), b(y) \rangle
\]

where \( a_i = 1/i \) and \( b_i = \delta(i, 1) \).

In other words, the reciprocal rank is the inverse of the rank assigned to document \( d_1 \), the most relevant document. MRR derives its name from the fact that this quantity is typically averaged over several queries.

**Discounted Cumulative Gain (DCG and DCG@n):** WTA and MRR use only a single entry of \( \pi \), namely \( \pi(1) \), to assess the quality of the ranking. Discounted Cumulative Gains are a more balanced score:

\[
\text{DCG}(\pi, y) = \langle a(\pi), b(y) \rangle
\]

where \( a_i = 1/\log(i + 1) \) and \( b_i = 2^{r_i} - 1 \).

Here it pays if a relevant document is retrieved with a high rank, as the coefficients \( a_i \) are monotonically decreasing. Variants of DCG, which do not take all ranks into account, are DCG@\( n \). Here \( a_i = 1/\log(i + 1) \) if \( i \leq n \) and \( a_i = 0 \) otherwise. That is, we only care about the \( n \) top ranking entries. In search engines the truncation level \( n \) is typically 10, as this constitutes the number of hits returned on the first page of a search.

**Normalized Discounted Cumulative Gain (NDCG):** A downside of DCG is that its numerical range depends on \( y \) (e.g. a collection containing many relevant documents will yield a considerably larger value at optimality than one containing only irrelevant ones). Since \( y \) is already sorted it follows that DCG is maximized for the identity permutation \( \pi = 1 \):

\[
\text{NDCG}(\pi, y) := \frac{\text{DCG}(\pi, y)}{\text{DCG}(1, y)}
\]

and \( \text{NDCG}@n(\pi, y) := \frac{\text{DCG}@n(\pi, y)}{\text{NDCG}@n(1, y)} \).

This allows us to define

\[
\text{NDCG}(\pi, y) = \langle a(\pi), b(y) \rangle
\]

where \( a_i = \frac{1}{\log(i + 1)} \) and \( b_i = \frac{2^{r_i} - 1}{\text{DCG}(1, y)} \).

Finally, \( \text{NDCG}@n \), the measure which this paper focuses on, is given by \( \langle a(\pi), b(y) \rangle \) where

\[
a_i = \begin{cases} 
\frac{1}{\log(i+1)} & \text{if } i \leq n \\
0 & \text{else} 
\end{cases} \quad \text{and} \quad b_i = \frac{2^{r_i}-1}{\text{DCG}@n(1, y)}. \]
**Precision@n:** Note that this measure, too, can be expressed by $\langle a(\pi), b(y) \rangle$. Here we define

$$
a_i = \begin{cases} 
1/n & \text{if } i \leq n \\
0 & \text{else}
\end{cases} \quad \text{and} \quad b_i = \begin{cases} 
1 & \text{if } r_i \text{ correct} \\
0 & \text{else}
\end{cases}
$$

The main difference to NDCG is that Precision@n has no decay factor, weighing the top $n$ answers equally.

**Expected rank utility:** It has an exponential decay in the top ranked items and it can be represented as

$$
\text{ERU}(\pi, y) = \langle a(\pi), b(y) \rangle
$$

where $a_i = 2^{1-i\alpha-1}$ and $b_i = \max(r_i - d, 0)$

Here $d$ is a neutral vote and $\alpha$ is the viewing half-life. The normalized ERU can also be defined in a similar manner to NDCG. The (normalized) ERU is often used in collaborative filtering for recommender systems where the lists of items are often very short.

It is commonly accepted that NDCG@n is a good model of a person’s judgment of a search engine: the results on the first page matter, between them there should be a decay factor. NDCG has another advantage that it is more structured and more general than WTA and MRR. For collaborative and content filtering, ERU is more popular [Breese et al., 1998, Basilico and Hofmann, 2004].

Since we set out to design a loss function as described in Section 2.1, we now define the relative loss incurred by any score of the form $\langle a(\pi), b(y) \rangle$. For convenience we assume (again) that $\pi = 1$ is the optimal permutation:

$$
\Delta(y, \pi) := \langle a(1), b(y) \rangle - \langle a(\pi), b(y) \rangle
$$

### 3.3 Scoring Function

The final step in our problem setting is to define a suitable function $f(x, \pi)$ (where $x = (q,D)$) which is maximized for the “optimal” permutation. As stated in Section 3.1 we require a function $g(d,q)$ which will assign a relevance score to every (document, query) pair independently at test time. The Polya-Littlewood-Hardy inequality tells us how we can obtain a suitable class of functions $f$, given $g$:

**Theorem 2** Let $a, b \in \mathbb{R}^n$ and let $\pi \in \Pi$. Moreover, assume that $a$ is sorted in decreasing order. Then $\langle a, b(\pi) \rangle$ is maximal for the permutation sorting $b$ in decreasing order.

Consequently, if we define

$$
f(x, \pi) = \sum_i g(d_i, q)c(\pi)_i
$$

for some decreasing sequence $c$, the maximizer of $f$ will be the one which sorts the documents in decreasing order of relevance, as assigned by $g(d_i, q)$. The expansion (18) also acts as a guidance when it comes to designing a feature map $\Phi(x, \pi)$, which will map all documents $d_i$, the query $q$, and the permutation $\pi$ jointly into a feature space. More to the point, it will need to reflect the decomposition into terms related to individual pairs $(d_i, q)$ only.
3.4 General Position Dependent Loss

Before we proceed to solving the ranking problem by defining a suitable feature map, let us briefly consider the most general case we are able to treat efficiently in our framework.\footnote{More general cases, such as a quadratic dependency on positions, while possible, will typically lead to optimization problems which cannot be solved in polynomial time.} Assuming that we are given a ranking $\pi$ of documents $\{d_1, \ldots, d_l\}$, we define the loss as

$$\Delta(\pi, y) := \sum_{i,j} \pi_{ij} C_{ij}(y) + \text{const.} \quad (19)$$

That is, for every position $j$ we have a cost $C_{ij}(y)$ which is incurred by placing document $i$ at position $j$. Clearly (17) falls into this category: simply choose $C_{ij} = a_i b(y)_j$. For instance, we might have a web page ranking problem where the first position should contain a result from a government-related site, the second page should contain a relevant page from a user-created site, etc. In other words, this setting would apply to cases where specific positions in the ranked list are endowed with specific meanings.

The problem with this procedure is, however, that estimation and optimization are somewhat more costly than merely sorting a list of relevance scores: we would want to have a different scoring function for each position. In other words, the computational cost is dramatically increased, both in terms of the number of functions needed to compute the scores of an element and in terms of the optimization required to find a permutation which minimizes the loss $\Delta(\pi, y)$.

4 Learning Ranking

4.1 The Featuremap

We now expand on the ansatz of (18). The linearity of the inner product $f(x, \pi) = \langle \Phi(x, \pi), w \rangle$, as given by (3) requires that we should be able to write $\Phi$ as

$$\Phi(x, \pi) = \sum_{i=1}^l c(\pi)_i \phi(d_i, q) \quad \text{where } c \in \mathbb{R}^m. \quad (20)$$

In this case $\langle w, \Phi(D, q, \pi) \rangle = \langle c(\pi), p_i \rangle$ where $p_i = \langle w, \phi(d_i, q) \rangle$. The problem of choosing $\phi(d_i, q)$ is ultimately data dependent, as we do not know in general what data type the documents and queries are composed of. We will discuss several choices in the context of the experiments in Section 6.

Eq. (20) implies that $f$ is given by

$$f(x, \pi) = \langle \Phi(x, \pi), w \rangle = \sum_{i=1}^l c(\pi)_i \langle \phi(d_i, q), w \rangle. \quad (21)$$

Hence we can apply the Polya-Littlewood-Hardy inequality and observe that it is maximized by sorting the terms $\langle \phi(d_i, q), w \rangle$ in decreasing order just as $c$. Note that this permutation is easily obtained by applying QuickSort in $O(l \log l)$ time. To obtain only the $n$ top-ranking terms we can do even better and only need to expend $O(l + n \log n)$ time, using a QuickSort-style divide and conquer procedure.

This leaves the issue of choosing $c$. In general we want to choose it such that a) the margin of (5) is maximized and that b) the average length $\|\Phi(x, \pi)\|^2$ is small for good generalization and fast convergence of the implementation.

Since $\Phi$ is linear in $c$, we could employ a kernel optimization technique to obtain an optimal value of $c$, such as those proposed in [Bousquet and Herrmann, 2002, Ong et al., 2003].
principle not impossible, this leads rather inevitably to a highly constrained (at worst semidefinite) optimization problem in terms of $c$ and $w$. Obtaining an efficient algorithm for this more general problem is topic of current research. We report experimental results for different choices of $c$ in Section 6.

The above reasoning is sufficient to apply the optimization problem described in Section 2 to ranking problems. All that changes with respect to the general case is the choice of loss function $\Delta$ and the feature map $\Phi(x, \pi)$. In order to obtain an efficient optimization algorithm we need to overcome one last hurdle: we need to find an efficient algorithm for finding constraint violators in (8).

4.2 Finding Violated Constraints

Recall (8). In the context of ranking this means that we need to find the permutation $\pi$ which maximizes

$$\langle \Phi(x, \pi), w \rangle + \Delta(y, \pi)$$

$$= \sum_{i=1}^{l} \langle \phi(d_i, q), w \rangle c(\pi)_i + \langle a(1), b(y) \rangle - \langle a(\pi), b(y) \rangle$$

$$= \langle c(\pi), g \rangle - \langle a(\pi), b(y) \rangle + \text{const.} \quad (23)$$

where $g_i = \langle \phi(d_i, q), w \rangle$. Note that (23) is a so-called linear assignment problem which can be solved by the Hungarian Marriage method: the Kuhn-Munkres algorithm in cubic time. Maximizing (23) amounts to solving

$$\text{argmax}_{\pi \in \Pi} \sum_{i=1}^{m} C_{i,\pi_i} \text{ where } C_{ij} = c_j g_i - a_j b_i. \quad (24)$$

Note that there is a special case in which the problem (8) can be solved by a simple sorting operation: whenever $a = c$ the problem reduces to maximizing $\langle a(\pi), g - b(y) \rangle$. This choice, however, is not always desirable as $a$ may be rather degenerate (i.e. it may contain many terms with value zero).

4.3 Solving the Linear Assignment Problem

It is well known that there exists a convex relaxation of the problem of maximizing $\sum_i C_{i,\pi_i}$ into a linear problem which leads to an optimal solution.

$$\text{maximize } \pi \quad \text{tr} \pi^\top C$$

subject to $\sum_i \pi_{ij} = 1$ and $\sum_j \pi_{ij} = 1$ and $\pi_{ij} \in \{0, 1\}$

More specifically, the integer linear program can be relaxed by replacing the integrality constraint $\pi_{ij} \in \{0, 1\}$ by $\pi_{ij} \geq 0$ without changing the solution, since the constraint matrix is totally unimodular. Consequently the vertices of the feasible polytope are integral, hence also the solution. The dual is

$$\text{minimize } \sum_i u_i + v_i \quad \text{subject to } u_i + v_i \geq C_{ij}. \quad (26)$$

The solution of linear assignment problems is a well studied subject. The original papers by Kuhn [1955] and Munkres [1957] implied an algorithm with $O(l^3)$ cost in the number of terms. Later, Karp [1980] suggested an algorithm with expected quadratic time in the size of the
Algorithm 2 Direct Optimization of Ranking Measures

**Input:** Document collections $D_i$, queries $q_i$, ranks $y_i$, sample size $m$, tolerance $\epsilon$

Initialize $S_i = \emptyset$ for all $i$, and $w = 0$.

repeat
  for $i = 1$ to $m$ do
    $w = \sum_{\pi \in S_i} \alpha_{i\pi} \Phi(x_i, \pi)$
    $\pi^* = \arg\max_{\pi \in \Pi} \langle w, \Phi(x_i, \pi) \rangle + \Delta(\pi, y_i)$
    $\xi = \max(0, \max_{\pi \in S_i} \langle w, \Phi(x_i, \pi) \rangle + \Delta(\pi, y_i))$
    if $\langle w, \Phi(x_i, \pi^*) \rangle + \Delta(\pi^*, y_i) > \xi + \epsilon$ then
      Increase constraint set $S_i \leftarrow S_i \cup \pi^*$
      Optimize (7) using only $\alpha_{i\pi}$ where $\pi \in S_i$.
  end if
end for

until $S$ has not changed in this iteration

assignment problem (ignoring log-factors). Finally, Orlin and Lee [1993] propose a linear time algorithm for large problems. Since in our case the number of pages is fairly small (in the order of 50 to 200), we used an existing implementation due to Jonker and Volgenant [1987]. See Section 6.3 for runtime details. The latter uses modern techniques for computing the shortest path problem arising in (26).

This means that we can check whether a particular set of documents and an associated query $(D_i, q_i)$ satisfies the inequality constraints of the structured estimation problem (5). Hence we have the subroutine necessary to make the algorithm of Section 2 work. In particular, this is the only subroutine we need to replace in SVMStruct [Tsochantaridis et al., 2005].

4.4 In a Nutshell

Before describing the experiments, let us briefly summarize the overall structure of the algorithm. In completely analogy to (5) the primal optimization problem can be stated as

$$\begin{align*}
\text{minimize} & \quad \frac{1}{2} \|w\|^2 + C \sum_{i=1}^{m} \xi_i \\
\text{subject to} & \quad \langle w, \Phi(x_i, 1) - \Phi(x_i, \pi) \rangle \geq \Delta(y_i, \pi) - \xi_i \\
& \quad \text{for all } \xi_i \geq 0 \text{ and } \pi \in \Pi \text{ and } i \in \{1, \ldots, m\}
\end{align*}$$

(27b)

The dual problem of (5) is given by

$$\begin{align*}
\text{minimize} & \quad \frac{1}{2} \sum_{i,j,\pi,\pi' \in \Pi} \alpha_{i\pi} \alpha_{j\pi'} k((x_i, \pi), (x_j, \pi')) - \sum_{i,\pi \in \Pi} \Delta(y_i, \pi) \alpha_{i\pi} \\
\text{subject to} & \quad \sum_{\pi \in \Pi} \alpha_{i\pi} \leq C \text{ and } \alpha_{i\pi} \geq 0 \text{ for all } i \text{ and } \pi.
\end{align*}$$

(28b)

This problem is solved by Algorithm 2. Finally, documents on a test set are ranked by $g(d, q) = \langle w, \phi(d, q) \rangle$, where $w = \sum_{i,\pi} \alpha_{i\pi} \Phi(x_i, \pi)$.  

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5 Extensions

5.1 Diversity Constraints

Imagine the following scenario: when searching for 'Jordan', we will find many relevant webpages containing information on this subject. They will cover a large range of topics, such as a basketball player (Mike Jordan), a country (the kingdom of Jordan), a river (in the Middle East), a TV show (Crossing Jordan), a scientist (Mike Jordan), a city (both in Minnesota and in Utah), and many more. Clearly, it is desirable to provide the user with a diverse mix of references, rather than exclusively many pages from the same site or domain or topic range.

One way to achieve this goal is to include an interaction term between the items to be ranked. This leads to optimization problems of the form

$$\min_{\pi \in \Pi} \sum_{ijkl} \pi_{ij} \pi_{kl} c_{ij,kl}$$

(29)

where $c_{ij,kl}$ would encode the interaction between items. This is clearly not desirable, since problems of the above type cannot be solved in polynomial time. This would render the algorithm impractical for swift ranking and retrieval purposes.

However, we may take a more pedestrian approach, which will yield equally good performance in practice, without incurring exponential cost. This approach is heavily tailored towards ranking scores which only take the top $n$ documents into account. We will require that among the top $n$ retrieved documents no more than one of them may come from the same source (e.g. topic, domain, subdomain, personal homepage). Nonetheless, we would like to minimize the ranking scores subject to this condition. Formally, we would like to find a matrix $\pi \in \{0,1\}^{l \times n}$ such that

$$\sum_{i,j} \pi_{ij} a_i b(y)_j$$

is maximized, subject to the constraints

$$\sum_i \pi_{ij} = 1 \text{ for all } j \in \{1, n\}$$

(31)

$$\sum_j \sum_{i \in B_s} \pi_{ij} \leq 1 \text{ for all } B_s.$$  

(32)

Here the disjoint sets $B_s$ which form a partition of $\{1, \ldots, l\}$, correspond to subsets of documents (or webpages) which must not be retrieved simultaneously. In the above example, for instance all webpages retrieved from the domain `http://www.jordan.govoffice.com` would be lumped together into one set $B_s$. Another set $B_{s'}$ would cover, e.g. all webpages from `http://www.cs.berkeley.edu/~jordan/`.

It is not difficult to see that during training, we need to solve an optimization problem of the form

$$\max_{\pi} \text{ tr } \pi^\top C$$

(33a)

subject to $\sum_i \pi_{ij} = 1$ and $\sum_j \sum_{i \in B_s} \pi_{ij} \leq 1$ where $\pi \in \{0,1\}^{l \times n}$.  

(33b)

We will show below that the constraint matrix of (33) is totally unimodular. This means that a linear programming relaxation of the constraint set, i.e. the change from $\pi_{ij} \in \{0,1\}$ to $\pi_{ij} \in [0,1]$ will leave the solution of the problem unchanged. This can be seen as follows:

**Theorem 3 (Heller and Tompkins [1956])** An integer matrix $A$ with $A_{ij} \in \{0, \pm 1\}$ is totally unimodular if no more than two nonzero entries appear in any column, and if its rows can be partitioned into two sets such that:
1. If a column has two entries of the same sign, their rows are in different sets;
2. If a column has two entries of different signs, their rows are in the same set.

**Corollary 4** The linear programming relaxation of (33) has an integral solution.

**Proof** All we need to show is that in (33b) each term $\pi_{ij}$ only shows up exactly twice with coefficient 1. This is clearly the case since $B_s$ is a partition of $\{1, \ldots, l\}$, which accounts for one occurrence, and the assignment constraints which account for the other occurrence. Hence Theorem 3 applies.

Note that we could extend this further by requiring that weighted combinations over topics $\sum_{ij} \pi_{ij} w_{is} \leq 1$, where now the weights $w_{is}$ may be non-integral and the domains where $w_{is}$ is nonzero might overlap. In this case, obviously the optimization problem cannot be relaxed easily any more. However, it will still provide useful results when used in combination with integer programming codes, such as Bonmin [Bonami et al., 2005].

Finally, note that at test stage, it is very easy to take the constraints (33b) into account: Simply pick the highest ranking document from each set $B_s$ and use the latter to obtain an overall ranking.

### 5.2 Ranking Matrix Factorization

An obvious application of our framework is matrix factorization for collaborative filtering. The work of Srebro and Shraibman [2005], Rennie and Srebro [2005], Srebro et al. [2005b] suggests that regularized matrix factorizations are a good tool for modeling collaborative filtering applications. More to the point, Srebro and coworkers assume that they are given a sparse matrix $X$ arising from collaborative filtering, which they would like to factorize.

More specifically, the entries $X_{ij}$ denote ratings by user $i$ on document/movie/object $j$. The matrix $X \in \mathbb{R}^{m \times n}$ is assumed to be sparse, where zero entries correspond to (user,object) pairs which have not been ranked yet. The goal is to find a pair of matrices $U, V$ such that $UV^\top$ is close to $X$ for all nonzero entries. Or more specifically, such that the entries $[UV^\top]_{ij}$ can be used to recommend additional objects.

However, this may not be a desirable approach, since it is, for instance, completely irrelevant how accurate our ratings are for undesirable objects (small $X_{ij}$), as long as we are able to capture the users preferences for desirable objects (large $X_{ij}$) accurately. In other words, we want to model the user’s *likes* well, rather than his *dislikes*. In this sense, any indiscriminate minimization, e.g. of a mean squared error, or a large margin error for $X_{ij}$ is inappropriate.

Instead, we propose to use a ranking score such as those proposed in Section 3.2 to evaluate an entire row of $X$ at a time. That is, we want to ensure that $X_{ij}$ is well reflected as a whole in the estimates for all objects $j$ for a fixed user $i$. This means that we should be minimizing

$$R_{\text{emp}}[U, V, X] := \frac{1}{m} \sum_i \Delta([UV^\top]_{i\cdot}, X_{i\cdot})$$

where $\Delta$ is defined as in (17) and it is understood that it is evaluated over the nonzero terms of $X_{ij}$ only. This is a highly nonconvex optimization problem. However, we can, again, find a convex upper bound by the methods described in (5), yielding a function $\tilde{R}_{\text{emp}}[U, V, X]$. The technical details are straightforward and therefore omitted.

Note that by linearity this upper bound is convex in $U$ and $V$ respectively, whenever the other argument remains fixed. Moreover, note that $\tilde{R}[U, V, X]$ decomposes into $m$ independent problems in terms of the users $U_i$, whenever $V$ is fixed, whereas no such decomposition holds in terms of $V$.

In order to deal with overfitting, regularization of the matrices $U$ and $V$ is recommended. The trace norm $\|U\|_F^2 + \|V\|_F^2$ can be shown to have desirable properties in terms of generalization [Srebro et al., 2005a]. This suggests an iterative procedure for collaborative filtering:
For fixed $V$ solve $m$ independent optimization problems in terms of $U_i$, using the Frobenius norm regularization on $U$.

For fixed $U$ solve one large-scale convex optimization problem in terms of $V$.

Since the number of users is typically considerably higher than the number of objects, it is possible to deal with the optimization problem in $V$ efficiently. Details are subject to future research.

6 Experiments

We address a number of questions: a) is learning needed for good ranking performance, b) how does DORM (our algorithm) perform with respect to other algorithms on a number of datasets of different size and truncation level of the performance criteria, c) how fast is DORM when compared to similar large margin ranking algorithms, and d) how important is the choice of $c$ for good performance?

6.1 Datasets and Experimental Protocol

UCI We choose PageBlock, PenDigits, OptDigits, and Covertype from the UCI repository mainly to increase the number of different datasets on which we may compare DORM to other existing approaches. Since they are not primary ranking data, we will discuss the outcomes only briefly for illustrative purposes. For PageBlock, PenDigits and OptDigits we sample 50 queries and 100 documents for each query. For Covertypes we sample 500 queries and 100 documents.

Web Search Our web search dataset (courtesy of Chris Burges at Microsoft Research) consists of 1000 queries each for training, validation and testing. They are provided and selected from a larger pool of training data used for a search engine. Figure 1 shows a histogram of the number of documents per query (the median is approximately 50). Documents are ranked according to five levels of relevance (1:Bad, 2:Fair, 3:Good, 4:Excellent, 5:Perfect). Unlabelled documents are treated as Bad. The ratio between the five categories (1 to 5) is approximately 75:17:15:2:1. The length of the feature vectors is 367 (i.e. we are using BM25). We evaluate our algorithm with respect to three goals: performance in terms of NDCG@$n$, MRR and WTA performance.

EachMovie This collaborative filtering dataset consists of 2811983 ratings by 72916 users on 1628 movies. To prove the point that our improvement is not due to an improved choice of a kernel but rather in the improved choice of a loss function, we follow the experimental setup, choice of kernels, and pre-processing of [Basilico and Hofmann, 2004] and compare performance using ERU. We also use the experimental setup of [Yu et al., 2006] and compare performance using NDCG and NDCG@10. In both cases we are able to improve the results considerably. The datasets for both experiments are as provided by Thomas Hofmann at Google Research, and Shipeng Yu at Siemens Research respectively.

Protocol Since WebSearch provided a validation set, we used the latter for model selection. Otherwise, 10-fold cross validation was used to adjust the regularization constant $C$. We used linear kernels throughout, except for the EachMovie datasets, where we followed the protocols of [Basilico and Hofmann, 2004] and [Yu et al., 2006]. This was done to show that the performance improvement we observe is due to our choice of a better loss function rather than the function class. Note that NDCG, MRR were rescaled from $[0, 1]$ to $[0, 100]$ for better visualization.
6.2 UCI Datasets

Since the UCI data does not come in the form of multiple queries, we permute the datasets and randomly subsample documents for each query.

We compare DORM against SVM classification, Precision@n and ROCArea as reported by [Joachims, 2005] and use NDCG@10 as a performance measure. Table 1 shows that our algorithm significantly outperforms other methods. \( c \) was chosen to be \( c_i = 1/(i+1) \) in DORM.

The experiment results show how hard it is to optimize NDCG, traditional methods perform poorly when we use NDCG as the performance metric while direct optimization of the correct criterion works very well.

6.3 NDCG for Web Search

We compare DORM to a range of kernel methods for ranking: multiclass SVM classifiers, SVM for ordinal regression (RSVM) [Joachims, 2002, Herbrich et al., 2000], SVM for information retrieval (SVM-IR-QP) [Cao et al., 2006], Precision@n, ROCArea [Joachims, 2005] and DORM.

We use NDCG to assess the performance of the ranking algorithms. BM25 [Robertson et al., 1994, Robertson and Hull, 2000] is used as a baseline (it also constitutes the feature vector for the other algorithms).

In the first experiment, we use the full training set (parameter selection on the validation

| Dataset          | ROCArea | SVM   | Prec@10 | DORM    |
|------------------|---------|-------|---------|---------|
| PageBlocks       | 35.9 ± 7| 46.5 ± 7| 44.0 ± 7| 63.7 ± 6|
| PenDigits        | 26.2 ± 8| 41.5 ± 4| 15.6 ± 3| 85.2 ± 3|
| OptDigits        | 26.0 ± 9| 26.1 ± 3| 26.2 ± 3| 76.1 ± 6|
| Covertypes       | 47.0 ± 2| 48.5 ± 2| 42.1 ± 1| 58.8 ± 2|

Table 1: Performance on UCI data. Bold indicates a high significance of \( p < 0.0001 \) by paired \( t \)-test.

![Figure 1: Number of documents per query.](image)
Figure 2: NDCG@n scores on the web search dataset at different truncation levels n.

Figure 3: NDCG@10 scores on the web search dataset for different sample sizes. Note that the performance of NDCG on 100 observations is larger than of any other competing method for 1000 observations.
Figure 4: Maximum difference in NDCG@10 for different choices of $c$ with respect to $1/\sqrt{m}$.

set) to train our model and report the performance on the designated test set. We report the average prediction results for NDCG with various truncation levels ranging from 1 to 10 in Figure 2. Note that DORM consistently outperforms other methods for ranking by $2−3\%$. We chose $c_i = (i + 1)^{-1/2}$.

In a second experiment (same choice of $c_i$), we investigate the effects of increasing training set size using the first $m = 100, 300, 500, 700$ or all 1000 queries for training. Using the same experimental protocol as above we report the average prediction results for NDCG@10 in Figure 3. Our results confirm that the same gain for NDCG@10 can be achieved if we increase the training set size. Note that for many methods, doubling the sample size increases NDCG by less than 1% (Figure 3). DORM achieves the same gain without the need to double $m$. In fact, DORM using 100 queries for training beats all other methods using 1000 queries!

Since expert-judged datasets can be very expensive, DORM is more cost-effective. This confirms that learning to rank is beneficial, as using a combination of many features for ranking algorithms could result in better performance than using one or some several features individually, such as BM25.

The choice of $c$ critically determines the feature map. At the moment, we have little theoretical guidance with regard to this matter, hence we investigated the effect of choosing schemes of $c$ experimentally. Clearly $c$ needs to be a monotonically decreasing function. We chose $c_i = (i + 1)^{-d}$ for $d \in \{\frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1, 2, 3\}$ and $c_i = 1/\log(i + 2)$ and $c_i = 1/\log \log(i + 2)$.

We found experimentally that the differences between the various schemes are not as dramatic as the improvement obtained by using DORM instead of other algorithms. To summarize the results we show the difference in performance in Figure 4 for NDCG@10. Note that the difference in terms of NDCG accuracy resulted by taking different $c$ will decrease when the sample size increases. The rate of convergence is suspected to be $1/\sqrt{m}$. An possible interpretation is that the choice of $c$ can be considered prior knowledge. Thus with increasing sample size, we will need to rely less on this prior knowledge and a reasonable choice of $c$ will suffice.
6.4 MRR and WTA for Web Search

MRR  DORM is effective not only for NDCG but also for other performance measures. We compare it using the Mean Reciprocal Rank (MRR) and Winner Takes All (WTA) scores on the same dataset. For comparison we use Precision@n (with \(n = 3\), since this yielded the best results experimentally), DORM minimizing NDCG (which in this case is the incorrect criterion), and the previous methods.

As before, we use the validation set to adjust the regularization parameter \(C\). We picked \(c_i = 1/(i+1)\) (the influence of the particular choice of \(c\) was rather minor, as demonstrated in Figure 4). The average results for varying sample size (from 100 to 1000) are reported in Figure 5.

It can be seen from Figure 5 that DORM for MRR outperforms all other models including DORM for NDCG. This is not surprising, since DORM for MRR optimizes MRR directly while other methods do not. DORM for MRR beats other methods by 1% to 2% which is quite significant given that if we double the dataset size, the gain is only around 1%. The fact that the gains in optimizing MRR are less than when optimizing MRR is probably due to the fact that MRR is less structured than NDCG.

WTA  This is the least structured of all scores, as it only takes the top ranking document retrieved into account. This means that for a ranking dataset where 5 different degrees of relevance are available, only the top scoring ones are chosen. This transformation discards a great deal of information in the labels (i.e. the gradations among the lower-scoring documents), which leads to the suspicion that minimizing a related cost function taking all levels into account should perform better. It turns out, experimentally, that indeed a direct optimization of the WTA loss function leads to bad performance. In order to amend this issue, we decided to minimize a modified NDCG score instead of the straight WTA score. This improved the performance significantly.

The truncation level in the NDCG@n score should be closer to 1 rather than 10. We devise a heuristic to find the truncation level: for queries that have more than 2 level of preference at the top 3 items, the truncation is 3; for queries that have only one level of preference at the top 3 items, the truncation level is after position where the next level of preference appears in the ranked list, as for documents with a large number of top ranking documents we want to include at least one lower-ranking document in the list. We call the method mWTA (modified DORM for WTA).

Experiments with different decay terms for \(c\) indicated that \(c_i = 1/\sqrt{i+1}\) yields best results. We compare the new method with various methods using model selection on the validation set and report the total number of correct predictions in Figure 6. RSVM and RSVM-IR-QP perform poorly on this task and we omit their results due to space constraints. While direct optimization of WTA is unsatisfactory, mWTA outperforms other methods significantly.

6.5 Runtime Performance

One might suspect that our formulation is potentially slow, as the Hungarian marriage algorithm takes cubic time \(O(l^3)\). However, each such optimization problem is relatively small (on average in the order of 50 documents per query), which means that the overall computational time is well controlled.

For practical results we carried out experiments to measure the time for training plus cross validation. We used a modified version of SVMStruct [Tsochantaridis et al., 2005]. The algorithms are all written in C and the code was run on a Pentium 4 3.2GHz workstation with 1GB RAM, running Linux and using GCC 3.3.5. As can be seen in Figure 7, DORM outruns most other methods, except for multiclass SVM, and BM25 (which does not require training). Note that ordinal regression algorithms are significantly slower than DORM, as they need to deal
Figure 5: MRR scores on the web search dataset for different sample sizes. We compare eight methods (including DORM), using $c_i = 1/(i + 1)$.

Figure 6: WTA scores (“I’m feeling lucky”) on the web search dataset for different sample sizes. DORM (mWTA) minimizing an adaptive version of NDCG outperforms straight WTA minimization, as it makes better use of the label information.
with a huge number of simple inequalities rather than a smaller number of more meaningful ones.

Table 2 has a comparison of the number of support vectors (the more the slower) number of column generation iterations (the more the slower), percent of time spent in the QP solver. DORM is faster than other algorithms since it has a sparser solution. In terms of number of iterations a percent of time in the QP solver, DORM is a well balanced solution between Precision@n and ROCArea. Results are similar when optimizing MRR (this is reported in the bottom of Table 2). Note that since all models use linear functions, prediction times is less than 0.5s for 1000 queries.

6.6 EachMovie and Collaborative Filtering

ERU Past published results on collaborative filtering use Expected Rank Utility (ERU), NDCG and NDCG@10 as reference scores. In order to show that the improvement in performance is truly due to a better loss function rather than a different kernel we use the same kernels and experimental protocol as proposed by [Basilico and Hofmann, 2004] using the same parameter combinations in the context of ERU. Table 3 shows the merit of DORM: it outperforms JRank [Basilico and Hofmann, 2004] and PRank [Crammer and Singer, 2002]. In experiment 1 we used user features in combination with item correlations. In experiment 2 we used item features in combination with user ratings. In both cases, results are averaged over 100 trials with 100 training users, 2000 input users and 800 training items.

Having used a kernel which is optimal for JRank we expect that optimizing the kernel further would lead to better results, as there is no reason to assume that the model class optimal for JRank would be the best choice for DORM, too.

NDCG In a second experiment, we mimicked the experimental protocol of [Yu et al., 2006] on EachMovie. Here, we treat each movie as a document and each user as a query. After filtering out all the unpopular documents and queries (as in [Yu et al., 2006]) we have 1075 documents and 100 users.

For each user, we randomly select 10, 20 and 50 labeled items for training and perform prediction on the rest. The process is repeated 10 times independently. The methods for
Table 2: Number of support vectors, iterations in column generation, and time spent in the Quadratic Programming loop for various SVM style optimization algorithms. Top: optimization for NDCG@10, using $c_i = \frac{1}{\sqrt{i} + 1}$. Bottom: optimization for MRR and NDCG, using $c_i = \frac{1}{(i + 1)}$. Corresponding methods have different numbers due to the different choices of truncation level (Precision@n) and $c$ (DORM NDCG).

| Method          | # SVs | # Iter | % in QP |
|-----------------|-------|--------|---------|
| Precision@10    | 1037  | 44     | 89.02   |
| ROCArea         | 997   | 12     | 19.64   |
| DORM (NDCG)     | 561   | 22     | 28.76   |

| Method          | # SVs | # Iter | % in QP |
|-----------------|-------|--------|---------|
| Precision@3     | 1000  | 11     | 46.80   |
| ROCArea         | 997   | 12     | 19.64   |
| DORM (NDCG)     | 520   | 23     | 61.66   |
| DORM (MRR)      | 550   | 17     | 1.76    |

Table 3: Expected rank utility scores for three methods. Results averaged over 100 trials with 100 training users, 2000 input users and 800 training items.

| Experiment | PRank | JRank | DORM       |
|------------|-------|-------|------------|
| 1          | 70.8  | 75.3  | 76.5 ± 0.43|
| 2          | 73.4  | 76.2  | 76.7 ± 0.32|

Table 4: NDCG optimization on EachMovie dataset. Comparison between 6 methods using unpaired $t$-test with values of $p$ shown (best score vs. second best score).

| N    | Method | NDCG   | NDCG@10     |
|------|--------|--------|-------------|
| 10   | GPR    | 83.41 ± 0.22 | 45.58 ± 1.51|
|     | CGPR   | 86.39 ± 0.24 | 57.34 ± 1.44|
|     | GPOR   | 80.59 ± 0.03 | 36.92 ± 0.25|
|     | CGPOR  | 80.83 ± 0.11 | 37.89 ± 1.05|
|     | MMMF   | 84.34 ± 0.48 | 47.46 ± 3.42|
|     | DORM   | **87.17 ± 0.24** | **61.75 ± 1.83**|
|     |        | $p < 0.0001$ | $p < 0.0001$ |
| 20   | GPR    | 84.12 ± 0.15 | 48.49 ± 0.66 |
|     | CGPR   | 86.98 ± 0.16 | 59.89 ± 1.18 |
|     | GPOR   | 80.48 ± 0.05 | 36.78 ± 0.30 |
|     | CGPOR  | 80.78 ± 0.13 | 37.81 ± 0.56 |
|     | MMMF   | 84.85 ± 0.28 | 47.86 ± 1.39 |
|     | DORM   | **87.63 ± 0.37** | **62.82 ± 1.9**|
|     |        | $p < 0.0001$ | $p = 0.0006$  |
| 50   | GPR    | 85.15 ± 0.23 | 53.75 ± 0.89 |
|     | CGPR   | 87.82 ± 0.21 | 63.41 ± 1.14 |
|     | GPOR   | 80.10 ± 0.04 | 36.63 ± 0.24 |
|     | CGPOR  | 80.45 ± 0.06 | 37.74 ± 0.41 |
|     | MMMF   | 86.13 ± 0.38 | 54.78 ± 2.11 |
|     | DORM   | 87.84 ± 0.32 | **65.05 ± 1.27**|
|     |        | $p = 0.8706$ | $p = 0.006$   |
comparison are the standard Gaussian process regression (GPR) [Rasmussen and Williams, 2006], Gaussian Process ordinal regression (GPOR) [Chu and Ghahramani, 2005], and their collaborative extensions (CPR, CGPOR) [Yu et al., 2006], MMMF [Rennie and Srebro, 2005] and DORM (for NDCG). The figures for the first 5 methods are extracted from [Yu et al., 2006] and scaled by 100 to fit our convention of showing NDCG results. We perform an unpaired t-test for significance (see Table 4).

Note that there is no cross validation or model selection involved in [Yu et al., 2006]. Thus to be fair, we fix the following parameters $c_i = (i + 1)^{-0.25}$ (performing slightly better in this dataset), $C = 0.01$ and $n = 10$ (the truncation level of NDCG). The results show that DORM performs very well for predicting the ranking for new items, especially when the number of labeled items is small.

7 Summary and Discussion

In this paper we proposed a general scheme to deal with a large range of criteria commonly used in the context of web page ranking and collaborative filtering. Unlike previous work, which mainly focuses on pairwise comparisons we aim to minimize the multivariate performance measures (or rather a convex upper bound on them) directly. This has both computational savings, leading to a faster algorithm and practical ones, leading to better performance. In a way, our work follows the mantra of [Vapnik, 1982] of estimating directly the desired quantities rather than optimizing a surrogate function. There are clear extensions of the current work:

- The key point of our paper was to construct a well-designed loss function for optimization. In this form it is completely generic and can be used as a drop-in replacement in many settings. We completely ignored language models [Ponte and Croft, 1998] to parse the queries in any sophisticated fashion.
- Although the content of the paper is directed towards ranking, the method can be generalized for optimizing many other complicated multivariate loss functions.
- We could use our method directly for information retrieval tasks or authorship identification queries. In the latter case, the query $q_i$ would consist of a collection of documents written by one author.
- We may add personalization to queries. This is no major problem, as we can simply add some personal data $u_i$ to $\phi(q_i, d_i, u_i)$ and obtained personalized ranking.
- Online algorithms along the lines of [Shalev-Shwartz and Singer, 2006] can easily be accommodated to deal with massive datasets.
- The present algorithm can be extended to learn matching problems on graphs. This is achieved by extending the linear assignment problem to a quadratic one. The price one needs to pay in this case is that the Hungarian Marriage algorithm is no longer feasible, as the optimization problem itself is NP hard.

Note that the choice of a Hilbert space for the scoring functions is done for reasons of convenience. If the applications demand Neural Networks or similar (harder to deal with) function classes instead of kernels, we can still apply the large margin formulation. That said, we find that the kernel approach is well suited to the problem.

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