Scattering of Dirac particles from non-local separable potentials: 
The eigenchannel approach

Remigiusz Augusiak*
Department of Theoretical Physics and Quantum Informatics, 
Faculty of Applied Physics and Mathematics, 
Gdańsk University of Technology, 
Narutowicza 11/12, PL 80-952 Gdańsk, Poland
(Dated: June 26, 2018)

An application of the new formulation of the eigenchannel method [R. Szmytkowski, Ann. Phys. (N.Y.) 311, 503 (2004)] to quantum scattering of Dirac particles from non-local separable potentials is presented. Eigenchannel vectors, related directly to eigenchannels, are defined as eigenvectors of a certain weighted eigenvalue problem. Moreover, negative cotangents of eigenphase-shifts are introduced as eigenvalues of that spectral problem. Eigenchannel spinor as well as bispinor harmonics are expressed throughout the eigenchannel vectors. Finally, the expressions for the bispinor as well as matrix scattering amplitudes and total cross section are derived in terms of eigenchannels and eigenphase-shifts. An illustrative example is also provided.

PACS numbers: 03.65.Nk
Keywords:

I. INTRODUCTION

Recently, Szmytkowski [1] proposed a new formulation of the eigenchannel method for quantum scattering from Hermitian short-range potentials, different from that presented by Danos and Greiner [2]. Some ideas leading to this method were drawn from works on electromagnetism theory by Garbacz [3] and Harrington and Mautz [4]. This method was further extended to the case of zero-range potentials for Schrödinger particles by Szmytkowski and Gruchowski [5] and then for Dirac particles by Szmytkowski [6] (see also [7]).

On the other hand, it is the well-known fact that separable potentials, since they provide analytical solutions to the Lippmann-Schwinger equations [8], have found applications in many branches of physics, both in the non-relativistic and relativistic cases [9]. (It should be noted that much larger effort has been devoted to the separable potentials in the non-relativistic regime.) Especially, their utility was confirmed in nuclear physics by successful use for describing nucleon-nucleon interactions [10]. Moreover, methods allowing one to approximate an arbitrary non-local potential by a separable one are known [11].

In view of what has been said above, it seems interesting to pose the question: how does the new method apply to quantum scattering from non-local separable potentials? Partially, the answer was given by the author by applying the method to quantum scattering of Schrödinger particles from separable potentials [12]. In the present contribution, we extend considerations from [12] to the case of Dirac particles.

This paper is organized as follows. In Section 2 some facts and notions from the theory of potential scattering of Dirac particles (see [13]) are provided. In Section 3 we concentrate on the special class of non-local potentials, namely, separable potentials. In this context, expressions for the bispinor as well as matrix scattering amplitudes are provided. Section 4 contains main ideas and results. Here, we define eigenchannel vectors, directly related to eigenchannels, as solutions to a certain weighted eigenproblem. Moreover, we introduce eigenphase-shifts, relating them to eigenvalues of this spectral problem. Within this approach, we also calculate expressions for the scattering amplitude and the average total cross section. In Section 5, scattering from a rank one delta-like separable potential is discussed as an illustrative example. The paper ends with two appendices.

II. QUANTUM SCATTERING OF DIRAC PARTICLES FROM NON-LOCAL POTENTIALS

Let us assume that a free Dirac particle of energy $E$ (with $|E| > mc^2$) described by the following monochromatic plane wave

$$\phi_i(r) \equiv \langle r | k_i \chi_i \rangle = U_i(k_i) e^{i k_i \cdot r}, \quad (2.1)$$

where

$$U_i(k_i) = \frac{1}{\sqrt{1 + \varepsilon^2}} \left( \varepsilon \sigma \cdot \tilde{k}_i \chi_i \right), \quad (2.2)$$

$$\varepsilon = \sqrt{\frac{E - mc^2}{E + mc^2}} \quad (2.3)$$

is being scattered from a non-local potential given by a kernel $V(r, r')$, which in general may be a $4 \times 4$ matrix. In the above equation, $\chi_i$ stands for a normalized pure
Spinors $\theta_\pm$ constitute an arbitrary orthonormal basis in $\mathbb{C}^2$, i.e., $\Theta_\pm \Theta_\pm = \delta_{ss'}(s, t = -1, +)$ and $\sum_{ss'} \Theta_s \Theta_{s'} = \mathbb{I}_2$.

What is important for further considerations, the matrix (2.10) possesses the obvious property that $P(k)\Theta_\pm(k)$ and therefore
\begin{equation}
P(k_i)U_i(k_i) = U_i(k_i).
\end{equation}

We shall be exploiting this property in later analysis.

Considering scattering processes we usually tend to find expressions for a scattering amplitude and various cross sections. To this aim we need to find an asymptotic behavior of the relativistic outgoing Green function. From Eq. (2.7), using the projector (2.10), we have
\begin{equation}
G(E, r, r') \sim \frac{E}{2\pi c^2 h^2} P(k_f) e^{i k r} e^{-i k_f r'},
\end{equation}
where $k_f = kr/r$ is a wave vector of the scattered particle. Notice that due to the fact that we deal with elastic processes $|k_i| = |k_f| = k$. After application of Eq. (2.14) to Eq. (2.6), we obtain
\begin{equation}
\psi(r) \sim \frac{E}{2\pi c^2 h^2} \Theta_i \Theta_f \psi(r'').
\end{equation}

Spinors $\Theta_\pm$ constitute an arbitrary orthonormal basis in $\mathbb{C}^2$, i.e., $\Theta_\pm \Theta_\pm = \delta_{ss'}(s, t = -1, +)$ and $\sum_{ss'} \Theta_s \Theta_{s'} = \mathbb{I}_2$.

What is important for further considerations, the matrix (2.10) possesses the obvious property that $P(k)\Theta_\pm(k)$ and therefore
\begin{equation}
P(k_i)U_i(k_i) = U_i(k_i).
\end{equation}

We shall be exploiting this property in later analysis.

Considering scattering processes we usually tend to find expressions for a scattering amplitude and various cross sections. To this aim we need to find an asymptotic behavior of the relativistic outgoing Green function. From Eq. (2.7), using the projector (2.10), we have
\begin{equation}
G(E, r, r') \sim \frac{E}{2\pi c^2 h^2} P(k_f) e^{i k r} e^{-i k_f r'},
\end{equation}
where $k_f = kr/r$ is a wave vector of the scattered particle. Notice that due to the fact that we deal with elastic processes $|k_i| = |k_f| = k$. After application of Eq. (2.14) to Eq. (2.6), we obtain
\begin{equation}
\psi(r) \sim \frac{E}{2\pi c^2 h^2} \Theta_i \Theta_f \psi(r'').
\end{equation}

Spinors $\Theta_\pm$ constitute an arbitrary orthonormal basis in $\mathbb{C}^2$, i.e., $\Theta_\pm \Theta_\pm = \delta_{ss'}(s, t = -1, +)$ and $\sum_{ss'} \Theta_s \Theta_{s'} = \mathbb{I}_2$.

What is important for further considerations, the matrix (2.10) possesses the obvious property that $P(k)\Theta_\pm(k)$ and therefore
\begin{equation}
P(k_i)U_i(k_i) = U_i(k_i).
\end{equation}

We shall be exploiting this property in later analysis.

Considering scattering processes we usually tend to find expressions for a scattering amplitude and various cross sections. To this aim we need to find an asymptotic behavior of the relativistic outgoing Green function. From Eq. (2.7), using the projector (2.10), we have
\begin{equation}
G(E, r, r') \sim \frac{E}{2\pi c^2 h^2} P(k_f) e^{i k r} e^{-i k_f r'},
\end{equation}
where $k_f = kr/r$ is a wave vector of the scattered particle. Notice that due to the fact that we deal with elastic processes $|k_i| = |k_f| = k$. After application of Eq. (2.14) to Eq. (2.6), we obtain
\begin{equation}
\psi(r) \sim \frac{E}{2\pi c^2 h^2} \Theta_i \Theta_f \psi(r'').
\end{equation}

Spinors $\Theta_\pm$ constitute an arbitrary orthonormal basis in $\mathbb{C}^2$, i.e., $\Theta_\pm \Theta_\pm = \delta_{ss'}(s, t = -1, +)$ and $\sum_{ss'} \Theta_s \Theta_{s'} = \mathbb{I}_2$.

What is important for further considerations, the matrix (2.10) possesses the obvious property that $P(k)\Theta_\pm(k)$ and therefore
\begin{equation}
P(k_i)U_i(k_i) = U_i(k_i).
\end{equation}

We shall be exploiting this property in later analysis.

Considering scattering processes we usually tend to find expressions for a scattering amplitude and various cross sections. To this aim we need to find an asymptotic behavior of the relativistic outgoing Green function. From Eq. (2.7), using the projector (2.10), we have
\begin{equation}
G(E, r, r') \sim \frac{E}{2\pi c^2 h^2} P(k_f) e^{i k r} e^{-i k_f r'},
\end{equation}
where $k_f = kr/r$ is a wave vector of the scattered particle. Notice that due to the fact that we deal with elastic processes $|k_i| = |k_f| = k$. After application of Eq. (2.14) to Eq. (2.6), we obtain
\begin{equation}
\psi(r) \sim \frac{E}{2\pi c^2 h^2} \Theta_i \Theta_f \psi(r'').
\end{equation}
Subsequently, after integration the above over all the directions of \( k_f \), we arrive at the total cross section

\[
\sigma(k_i, \nu_i) = \frac{1}{(4\pi)^2} \int d^2k_f \psi_i^\dagger \chi_f. \tag{2.21}
\]

Finally, averaging over all directions of incidence \( k_i \) and the initial spin orientation \( \nu_i \), one finds the average total cross section

\[
\sigma_i(E) = \frac{1}{(4\pi)^2} \int d^2k_i \int d^2\nu_i \int d^2k_f \psi_i^\dagger \chi_f. \tag{2.22}
\]

Obviously all the mentioned cross sections may be expressed in terms of all the scattering amplitudes \( A_{fi} \), \( A_{if} \), and \( \omega_{fi} \).

### III. SPECIAL CLASS OF NON–LOCAL SEPARABLE POTENTIALS

In this section we employ the above considerations to the special class of non–local separable potentials. As previously mentioned, such a class of potentials allows to find solutions to the Lippmann–Schwinger equations in an analytical way.

Consider the following class of potential kernels:

\[
V(r, r') = \sum_{\mu} \omega_{\mu} u_{\mu}(r) u_{\mu}^\dagger(r') \tag{3.1}
\]

where it is assumed that in general \( N \) may denote the arbitrary finite set of indices, i.e., \( \mu = \{\mu_1, \ldots, \mu_k\} \) and all the coefficients \( \omega_{\mu} \) different from zero. Functions \( u_{\mu}(r) \) are assumed to be four–element columns.

Substitution of Eq. (3.1) to Eq. (2.6) leads us to the Lippmann–Schwinger equation for the separable potentials:

\[
\psi(r) = \phi_i(r) - \sum_{\mu} \omega_{\mu} \int_{\mathbb{R}^3} d^3r' G(E, r, r') u_{\mu}(r') \times \int_{\mathbb{R}^3} d^3r'' u_{\mu}^\dagger(r'') \psi(r''), \tag{3.2}
\]

which may be equivalently rewritten as a set of linear algebraic equations. Indeed, using the Dirac notation one finds

\[
\sum_{\mu} \left[ \delta_{\nu \mu} + \langle u_\nu | \tilde{G}(E) | u_{\mu} \rangle \omega_{\mu} \right] \langle u_{\mu} | \psi \rangle = \langle u_{\nu} | \phi_i \rangle. \tag{3.3}
\]

For further convenience we introduce the following notations

\[
\langle u | \varphi \rangle = \begin{pmatrix} \langle u_1 | \varphi \rangle \\ \langle u_2 | \varphi \rangle \\ \vdots \\ \langle u_N | \varphi \rangle \end{pmatrix}, \quad \langle \varphi | u \rangle = \langle u | \varphi \rangle^\dagger = (\langle \varphi | u_1 \rangle \langle \varphi | u_2 \rangle \ldots ). \tag{3.4}
\]

Consequently, we may rewrite Eq. (3.3) as a matrix equation \((\mathbb{I} + G\Omega)\langle u | \psi \rangle = \langle u | \phi_i \rangle\) or equivalently as

\[
\langle u | \psi \rangle = (\mathbb{I} + G\Omega)^{-1} \langle u | \phi_i \rangle, \tag{3.5}
\]

with \( G \) being a matrix composed of the elements \( \langle u_\nu | \tilde{G}(E) | u_{\mu} \rangle \) and \( \Omega = \text{diag}[\omega_{\mu}] \). Similarly, substituting Eq. (3.1) to Eq. (2.16) and again using Eq. (2.10), we arrive at the bispinor scattering amplitude for the separable potentials in the form

\[
A_{fi} = \frac{-E}{2\pi c^2 \hbar^2} \mathcal{P}(k_f) \sum_{\mu} \omega_{\mu} \int_{\mathbb{R}^3} d^3r \, e^{-ik_f r} u_{\mu}(r) \times \int_{\mathbb{R}^3} d^3r' \, u_{\mu}^\dagger(r') \psi(r'), \tag{3.6}
\]

which, by virtue of Eqs. (2.11) and (3.5), reduces to

\[
A_{fi} = \frac{-E}{2\pi c^2 \hbar^2} \sum_{s,t=\pm} \Theta_s(k_f) \langle k_f \theta_s | u \rangle (\Omega^{-1} + G)^{-1} \langle u | \phi_i \rangle \tag{3.7}
\]

and, utilizing the fact that for all invertible matrices \( X \) and \( Y \) the relation \( (XY)^{-1} = Y^{-1}X^{-1} \) is satisfied, finally to

\[
A_{fi} = \frac{-E}{2\pi c^2 \hbar^2} \sum_{s,t=\pm} \Theta_s(k_f) \langle k_f \theta_s | u \rangle (\Omega^{-1} + G)^{-1} \langle u | \phi_i \rangle. \tag{3.8}
\]

Subsequently, using the fact that (2.13), we obtain the bispinor scattering amplitude in the following form

\[
A_{fi} = \frac{-E}{2\pi c^2 \hbar^2} \sum_{s,t=\pm} \Theta_s(k_f) \langle k_f \theta_s | u \rangle (\Omega^{-1} + G)^{-1} \langle u | \phi_i \rangle \tag{3.9}
\]

which, after comparison with Eq. (2.18), gives the formulae for the \( 4 \times 4 \) matrix scattering amplitude:

\[
A_{fi} = \frac{-E}{2\pi c^2 \hbar^2} \sum_{s,t=\pm} \Theta_s(k_f) \langle k_f \theta_s | u \rangle (\Omega^{-1} + G)^{-1} \langle u | \phi_i \rangle \tag{3.10}
\]

and finally, after straightforward movements, for the \( 2 \times 2 \) matrix scattering amplitude as

\[
\omega_{fi} = \frac{-E}{2\pi c^2 \hbar^2} \sum_{s,t=\pm} \theta_s \langle k_f \theta_s | u \rangle (\Omega^{-1} + G)^{-1} \langle u | k_f \theta_t \rangle \theta_t^\dagger. \tag{3.11}
\]

### IV. THE EIGENCHANNEL METHOD

Now we are in position to apply the eigenchannel method proposed recently by Szmytkowski [1] to scattering of the Dirac particles from potentials of the form
As we shall see below, such a class of potentials allows us to formulate this method in a simplified algebraic form.

We start from the decomposition of the matrix $\Omega^{-1} + G$ into its Hermitian and non-Hermitian parts, i.e.,

$$\Omega^{-1} + G = A + iB,$$

where matrices $A$ and $B$ are defined through relations

$$A = \Omega^{-1} + \frac{1}{2}(G + G^\dagger), \quad B = \frac{1}{2i}(G - G^\dagger).$$

(4.2)

It is evident from these definitions that both matrices $A$ and $B$ are Hermitian. Moreover, utilizing the fact that

$$\nabla \frac{e^{ik|r-r'|}}{|r-r'|} = \frac{r-r'}{|r-r'|} \left( \frac{e^{ik|r-r'|}}{|r-r'|} - \frac{e^{i{k}|r-r'|}}{|r-r'|^2} \right),$$

(4.3)

where $\varphi = r - r'$, the straightforward calculations lead us to their matrix elements of the form

$$A_{\nu \mu} = \omega_{\mu}^{-1} \delta_{\nu \mu} - \frac{k}{4\pi c^2 \hbar^2} \int_{R^3} d^3r \int_{R^3} d^3r' u_{\nu}(r) \times \left[ ic \hbar k \cdot \frac{\varphi}{|\varphi|} y_1(k|\varphi|) + (\beta mc^2 + E) y_0(k|\varphi|) \right] u_{\mu}(r')$$

(4.4)

and

$$B_{\nu \mu} = \frac{k}{4\pi c^2 \hbar^2} \int_{R^3} d^3r \int_{R^3} d^3r' u_{\nu}(r) \times \left[ ic \hbar k \cdot \frac{\varphi}{|\varphi|} j_1(k|\varphi|) + (\beta mc^2 + E) j_0(k|\varphi|) \right] u_{\mu}(r'),$$

(4.5)

where $j_0(z)$, $j_1(z)$, $y_0(z)$ and $y_1(z)$ are, respectively, the Bessel and Neumann spherical functions [14]. Recall that in general $j_0(z) = (\sin z)/z$, $y_0(z) = (\cos z)/z$ and

$$j_1(z) = -\frac{\cos z}{z} + \frac{\sin z}{z^2}, \quad y_1(z) = -\frac{\sin z}{z} - \frac{\cos z}{z^2}.$$  (4.6)

The main idea of the present paper, adopted from [1], is to construct the following weighted spectral problem:

$$AX_{\gamma}(E) = \lambda_{\gamma}(E)BX_{\gamma}(E),$$

(4.7)

where $X_{\gamma}(E)$ and $\lambda_{\gamma}(E)$ is, respectively, an eigenvector and an eigenvalue. Thereafter the eigenvectors $\{X_{\gamma}(E)\}$ will be called eigenchannel vectors. They are directly related to eigenchannels defined in [1] as state vectors. In fact, they constitute a projection of eigenchannels onto subspace spanned by $\{u_{\mu}(r)\}$.

Using the fact that matrices $A$ and $B$ are Hermitian and, as it is proven in Appendix A, the matrix $B$ is positive semi-definite, one finds that the eigenvalues $\{\lambda_{\gamma}(E)\}$ are real, i.e., $\lambda_{\gamma}^*(E) = \lambda_{\gamma}(E)$. Moreover, the eigenchannels associated with different eigenvalues obey the orthogonality relation

$$X_{\gamma_{\gamma}}(E)B X_{\gamma}(E) = 0 \quad (\lambda_{\gamma}(E) \ne \lambda_{\gamma}(E)).$$

(4.8)

In case of degeneration of some eigenvalues one may always choose the corresponding eigenvectors to be orthogonal according to the above relation. Then, imposing the normalization $X_{\gamma_i}(E)B X_{\gamma_j}(E) = 1$, one obtains the following orthonormality relation

$$X_{\gamma_i}(E)B X_{\gamma_j}(E) = \delta_{\gamma_i \gamma_j}.$$  (4.9)

From Eqs. (4.7) and (4.9) one infers that the eigenvalues $\{\lambda_{\gamma}(E)\}$ may be related to the matrix $A$ as follows

$$\lambda_{\gamma}(E) = X_{\gamma}(E)A X_{\gamma}(E).$$

(4.10)

Similar reasoning may be carried out employing the matrices $A$ and $\Omega^{-1} + G$. Indeed, after algebraic manipulations we arrive at

$$X_{\gamma}(E)A X_{\gamma}(E) = \lambda_{\gamma}(E) \delta_{\gamma \gamma}, \quad \lambda_{\gamma}(E) = X_{\gamma}(E)X_{\gamma}(E)(\Omega^{-1} + G) X_{\gamma}(E) = i + \lambda_{\gamma}(E)) \delta_{\gamma \gamma},$$

(4.11)

and

$$\lambda_{\gamma}(E) = X_{\gamma}(E)(\Omega^{-1} + G) X_{\gamma}(E) - i. \quad \text{Since the eigenchannels } \{X_{\gamma}(E)\} \text{ are the solutions of the Hermitian eigenvalue problem, they may satisfy the following closure relations}

$$\sum_{\gamma} X_{\gamma}(E)X_{\gamma}(E)B = I, \quad \sum_{\gamma} \lambda_{\gamma}^{-1}(E) X_{\gamma}(E)X_{\gamma}(E)A = I,$$

(4.12)

and

$$\sum_{\gamma} \frac{1}{i + \lambda_{\gamma}(E)} X_{\gamma}(E)X_{\gamma}(E)(\Omega^{-1} + G) = I,$$

(4.13)

where $I$ is an identity matrix, which dimension depends on the dimension of the matrix $G$. For purposes of further analyzes the above closure relations are assumed to hold.

Below, we employ the above reasoning to the derivation of the scattering amplitudes. From Eq. (4.13) one deduces that

$$(\Omega^{-1} + G)^{-1} = \sum_{\gamma} \frac{1}{i + \lambda_{\gamma}(E)} X_{\gamma}(E)X_{\gamma}^\dagger(E).$$

(4.14)

After substitution of Eq. (4.14) to Eq. (3.10) and rearranging terms, we have

$$\mathcal{A}_{\gamma} = \frac{-E}{2\pi^2 \hbar^2} \sum_{\gamma} \sum_{i + \lambda_{\gamma}(E)} \Theta_{\gamma}(k_f)(k_f\theta_s|u)X_{\gamma}(E)$$

$$\times \sum_{i} X_{\gamma_i}(E)(u|k_i\theta_i)\Theta_{\gamma_i}(k_i).$$

(4.15)

Let us define the following angular functions

$$\mathcal{Y}_{\gamma}(k) = \sqrt{\frac{E}{8\pi^2 c^2 \hbar^2}} \sum_{s=\pm} \Theta_{\gamma}(k\theta_s|u) X_{\gamma}(E),$$

(4.16)
hereafter termed the *eigenchannel bispinor harmonics*. The functions \( \{ Y_\gamma(k) \} \) are orthonormal on the unit sphere (for proof, see Appendix B), i.e.,

\[
\int d^2k \, \bar{Y}_\gamma^\dagger(k) Y_\gamma(k) = \delta_{\gamma/\gamma}. \tag{4.17}
\]

Application of Eq. (4.16) to Eq. (4.15) yields

\[
\mathcal{A}_{fi} = \frac{4\pi}{k} \sum_\gamma e^{i\delta_\gamma(E)} \sin \delta_\gamma(E) Y_\gamma(k_f) Y_\gamma^\dagger(k_i), \tag{4.18}
\]

where \( \{ \delta_\gamma(E) \} \) are called *eigenphase-shifts* and are related to \( \{ \lambda_\gamma(E) \} \) according to

\[
\lambda_\gamma(E) = -\cot \delta_\gamma(E). \tag{4.19}
\]

Similar considerations may be carried out for the 2 \( \times \) 2 matrix scattering amplitude \( \mathcal{A}_{fi} \). Indeed, in virtue of Eq. (2.19) we may rewrite Eq. (3.11) in the form

\[
\mathcal{A}_{fi} = \frac{4\pi}{k} \sum e^{i\delta_\gamma(E)} \sin \delta_\gamma(Y_\gamma(k_f) Y_\gamma^\dagger(k_i)), \tag{4.20}
\]

where the angular functions \( \{ Y_\gamma(k) \} \), hereafter called *eigenchannel spinor harmonics*, are defined as follows

\[
Y_\gamma(k) = \sqrt{\frac{E_k}{8\pi^2 c^2 \hbar^2}} \sum_{\pm} \theta_\gamma(k \theta_\gamma | u) X_\gamma(E). \tag{4.21}
\]

Moreover, they are orthogonal on the unit sphere (for proof, see Appendix B)

\[
\int d^2k \, \bar{Y}_\gamma^\dagger(k) Y_\gamma(k) = \delta_{\gamma/\gamma}, \tag{4.22}
\]

and, as one can verify, are related to the eigenchannel bispinor harmonics \( \{ Y_\gamma(k) \} \) via the relation

\[
Y_\gamma(k) = \frac{1}{\sqrt{1+\varepsilon^2}} \left( \begin{array}{c} Y_\gamma(k) \\ \varepsilon \sigma \cdot \hat{k} Y_\gamma(k) \end{array} \right). \tag{4.23}
\]

Now we are in position to compute scattering cross-sections. Substitution of Eq. (4.20) to Eq. (2.20) and integration over all directions of scattering \( \hat{k}_f \), by virtue of relation (4.22), yields

\[
\sigma(k_i, \nu_i) = \frac{16\pi^2}{k^2} \sum_\gamma \sin^2 \delta_\gamma(E) \left| \chi_\gamma^\dagger Y_\gamma(k_i) \right|^2. \tag{4.24}
\]

To compute the total cross-section averaged over all arrangements of spin of the incident particle, we have to notice that the projector onto the pure state \( \chi_\gamma \) may be written as \( \chi_\gamma^\dagger \chi_\gamma = (1/2) [\mathbb{1}_2 + \nu_i \cdot \sigma \nu_i] |\nu_i| = 1 \). Therefore, substituting of the above to Eq. (4.24) and averaging over all directions of \( \nu_i \), we arrive at

\[
\sigma(k_i) = \frac{8\pi^2}{k^2} \sum_\gamma \sin^2 \delta_\gamma(E) \left| Y_\gamma(k_i) \right|^2. \tag{4.25}
\]

Finally, averaging the above scattering cross-section over all directions of incidence \( k_i \), again by virtue of Eq. (4.22), we get the total cross-section in the form

\[
\sigma_f(E) = \frac{2\pi}{k^2} \sum_\gamma \sin^2 \delta_\gamma(E). \tag{4.26}
\]

It should be emphasized that all the above considerations respecting scattering cross-sections may be repeated using the eigenchannel bispinor harmonics \( \{ Y_\gamma(k) \} \) instead of the eigenchannel spinor harmonics \( \{ Y_\gamma(k) \} \). The significant difference is that then the integrals over \( \hat{k}_f \) and \( \hat{k}_i \) need to be calculated using relation (4.17) instead of (4.22).

### V. EXAMPLE

We conclude our considerations providing here an illustrative example concerning the scattering from a spherical shell of radius \( R \), centered at the origin of the coordinate system. Due to the assumption of non-locality of potentials under consideration, we shall simulate this process by using a potential of the form

\[
V(r, r') = \omega v(r)v(r') \mathbb{1}_4, \quad v(r) = \frac{1}{\sqrt{4\pi}} \frac{\delta(r-R)}{R^2}, \tag{5.1}
\]

where \( \omega \neq 0 \). Notice that the potential defined above is the special case of that proposed recently by de Prunelé [15] (see also [16]). Scattering of the Dirac particles from delta-like potentials was also studied e.g. in Refs. [17, 18]. However, in these papers the authors considered only local potentials and not non-local ones.

At the very beginning, we need to bring the potential (5.1) to the previously postulated form (3.1). To this aim, let \( e_1 \) and \( e_2 \) constitute a standard basis in \( \mathbb{C}^2 \), i.e.,

\[
e_1 = (1 \, 0)^T \quad \text{and} \quad e_2 = (0 \, 1)^T.
\]

Moreover, let \( e_{ij} = e_i \otimes e_j \) and then by virtue of the fact that \( \mathbb{1}_4 = \sum_{i,j=1}^2 e_{ij} e_{ij} \), we may rewrite (5.1) as

\[
V(r, r') = \omega \sum_{i,j=1}^2 u_{ij}(r) u_{ij}^*(r), \quad u_{ij}(r) = v(r) e_{ij}.
\]

Now, we are in position to compute the matrix \( G \). Using Eqs. (2.7) and (5.1), after straightforward integrations we have

\[
G = ikj_0(kR) h_0^{(+)}(kR) \left( \begin{array}{cc} \eta_+ \mathbb{I}_2 & 0 \\ 0 & \eta_- \mathbb{I}_2 \end{array} \right), \tag{5.3}
\]

where \( \eta_{\pm} = (E \pm mc^2)/c^2 \hbar^2 \) and \( h_0^{(+)}(z) = j_0(z) + iy_0(z) \) is the spherical Hankel function of the first kind. Hence, by the definitions given in Eq. (4.2), we find that the explicit forms of matrices \( A \) and \( B \) are

\[
A = \left( \begin{array}{cc} [\omega - k j_0(kR) y_0(kR) \eta_+] \mathbb{I}_2 & 0 \\ 0 & [\omega - k j_0(kR) y_0(kR) \eta_-] \mathbb{I}_2 \end{array} \right), \tag{5.4}
\]

\[
B = \left( \begin{array}{cc} \eta_+ \mathbb{I}_2 & 0 \\ 0 & \eta_- \mathbb{I}_2 \end{array} \right).
\]
Then, using Eq. (4.16) and by virtue of the fact that

\[ \lambda_{\pm}(E) = \frac{\omega^{-1} - k j_0(kR) y_0(kR) \eta_{\pm}}{k j_0^2(kR) \eta_{\pm}} \]  

(5.5)

and respective eigenvectors

\[ X_{\pm}^{(1)(2)}(E) = \frac{1}{\sqrt{k \eta_{\pm} j_0(kR)}} e_1 \otimes e_{1(2)}, \]

\[ X_{\pm}^{(1)(2)}(E) = \frac{1}{\sqrt{k \eta_{\pm} j_0(kR)}} e_2 \otimes e_{1(2)}. \]

Then, using Eq. (4.16) and by virtue of the fact that

\[ \langle k | \chi | u \rangle = \sqrt{\frac{4 \pi}{1 + \varepsilon^2}} j_0(kR) \]

\[ \times \left( \chi^1 e_1 \chi^1 e_2 \varepsilon \chi^1 \sigma \cdot \hat{k} e_1 \varepsilon \chi^1 \sigma \cdot \hat{k} e_2 \right), \]

(5.9)

we arrive at the four eigenchannel bispinor harmonics \{\gamma_i(k)\} in the form

\[ \gamma_{\pm}^{(1)(2)}(k) = \frac{1}{\sqrt{4 \pi (1 + \varepsilon^2)}} \left( \varepsilon \sigma \cdot \hat{k} e_{1(2)} \right) \]

(5.10)

and

\[ \gamma_{-}^{(1)(2)}(k) = \frac{1}{\sqrt{4 \pi (1 + \varepsilon^2)}} \left( \varepsilon \sigma \cdot \hat{k} e_{1(2)} \right). \]

(5.11)

Then, by virtue of Eq. (4.21), one obtains the eigenchannel spinor harmonics \{\Upsilon_{\gamma}(k)\} in the form

\[ \Upsilon_{\pm}^{(1)(2)}(k) = \frac{1}{\sqrt{4 \pi}} e_{1(2)}, \quad \Upsilon_{-}^{(1)(2)}(k) = \frac{1}{\sqrt{4 \pi}} \sigma \cdot \hat{k} e_{1(2)}. \]

(5.12)

The latter may be equivalently obtained combining Eqs. (4.23) and (5.12). Moreover, as one may easily verify, functions given by Eqs. (5.10) and (5.12) are orthonormal, respectively, in the sense (4.17) and (4.22).

Before we find an expression for total cross section, we compute the scattering amplitude. Since, as shown in Sec. II, the bispinor and both matrix scattering amplitudes are mutually related, we restrict our considerations to the \(2 \times 2\) scattering amplitude. Thus, combining Eqs. (4.20), (5.5), and (5.12) we obtain

\[ \mathcal{A}_{fi} = -j_0^2(kR) \left[ \frac{\mathbb{I}_2}{ik j_0(kR) h_0^{(-)}(kR) + (\omega_{\pm})^{-1}} + \frac{(\sigma \cdot \hat{k}) \cdot (\sigma \cdot \hat{k})}{ik j_0(kR) h_0^{(+)}(kR) + (\omega_{\pm})^{-1}} \right]. \]

(5.13)

Finally, substitution of Eqs. (5.8) and (5.12) to Eq. (4.24) with the aid of Eq. (4.19) yields

\[ \sigma(k_i, \nu_i) = \frac{4 \pi}{k^2} j_0^4(kR) \]

\[ \times \left\{ \frac{1}{[(k \omega_{\pm})^{-1} - j_0(kR)y_0(kR)]^2 + j_0^2(kR)} + \frac{1}{[(k \omega_{-})^{-1} - j_0(kR)y_0(kR)]^2 + j_0^2(kR)} \right\}. \]

(5.14)

Here it is evident that \( \sigma(k_i, \nu_i) = \sigma(k_i) = \sigma_{\pm}(E). \)

In order to illustrate the obtained results, the eigenphaseshifts for two different values of \( \omega \), derived from Eqs. (4.19) and (5.7), are plotted in Fig. 1 and 2. Figures 3 and 4 present partial \( \sigma_{\pm}(E) \) as well as total \( \sigma_{\pm}(E) \) cross sections.

It seems interesting to investigate the behavior of both eigenvalues \( \lambda_{\pm}(E) \) in the non-relativistic limit, i.e., when \( c \to \infty \). From (2.5) one concludes that

\[ \eta_{\pm} \xrightarrow{c \to \infty} \frac{2m}{\hbar^2}, \quad \eta_{\pm} \xrightarrow{c \to \infty} 0 \]

(5.15)

and therefore

\[ \lambda_{\mp}(E) \xrightarrow{c \to \infty} \frac{(\hbar^2/2m\omega) - k j_0(kR)y_0(kR)}{k j_0^2(kR)} \]

(5.16)

and

\[ \lambda_{-}(E) \xrightarrow{c \to \infty} \text{sgn}(\omega)\infty. \]

(5.17)

This means that \( \delta_{\pm}(E) \to n \pi \ (n \in \mathbb{Z}) \) in the limit of \( c \to \infty \). Therefore the cross section \( \sigma_{\pm}(E) \) vanishes in the non-relativistic limit and in this sense it has a purely relativistic character leading to the fact that the resonance appearing in Fig. 4 at about \( 1.25mc^2 \) is purely relativistic effect.

One sees that in the non-relativistic limit the cross section (5.14) reduces to

\[ \sigma_{\pm}(E) \xrightarrow{c \to \infty} \frac{4 \pi}{k^2} j_0^4(kR) \]

\[ \times \left\{ \frac{1}{[(\hbar^2/2mk\omega) - j_0(kR)y_0(kR)]^2 + j_0^2(kR)} \right\}. \]

(5.18)

The above cross section may also be obtained using non-relativistic formulation of the present method given in Ref. [12].

**VI. CONCLUSIONS**

In this work, an application of the recently proposed eigenchannel method [1] to the scattering of Dirac particles from non-local separable potentials has been presented. Application of such a particular case of the non-local potentials reduces naturally the general weighted eigenvalue problem stated in Ref. [1] to its matrix counterpart given by Eq. (4.7) leading to the definition of
FIG. 1: Behavior of eigenphaseshifts $\delta_+(E)$ (solid curve) and $\delta_-(E)$ (dashed curve) as functions of energy $E$ (in units of $mc^2$) for $\omega = -\hbar^3/m^2c$ and $R = \hbar/mc$. The eigenphaseshift $\delta_+(E)$ has been constrained to the range $[-\pi/2, \pi/2]$.

FIG. 2: Behavior of eigenphaseshifts $\delta_+(E)$ (solid curve) and $\delta_-(E)$ (dashed curve) as functions of energy $E$ (in units of $mc^2$) for $\omega = -5\hbar^3/m^2c$ and $R = \hbar/mc$. Both eigenphaseshifts have been constrained to the range $[-\pi/2, \pi/2]$.

FIG. 3: Partial $\sigma_+(E)$ (dashed curve), $\sigma_-(E)$ (dotted curve), and total $\sigma_\tau(E)$ (solid curve) cross sections (all in units of $R^2$) as functions of energy $E$ (in units of $mc^2$) for $\omega = -\hbar^3/m^2c$ and $R = \hbar/mc$.

FIG. 4: Partial $\sigma_+(E)$ (dashed curve), $\sigma_-(E)$ (dotted curve), and total $\sigma_\tau(E)$ (solid curve) cross sections (all in units of $R^2$) as functions of energy $E$ (in units of $mc^2$) for $\omega = -5\hbar^3/m^2c$ and $R = \hbar/mc$.

Eigenchannel vectors. Using the notion of the eigenchannel vectors the definitions of eigenchannel spinor as well as bispinor harmonics have been given. The latter provide us with the formulas for scattering amplitudes similar to that well-known for central potentials generalizing them at the same time to the case of non-local separable potentials.

The general considerations have been extended with an illustrative example in which the Dirac particles are scattered from non-local, delta-like potential. In this particular case, the general eigenvalue problem (4.7) become just a $4 \times 4$ matrix equation and therefore is easily solvable (notice that in the case of non-relativistic scattering it would be just a one-dimensional problem). The eigenvalues of this problem are two-fold degenerated and therefore give two different eigenphaseshifts from which one has a purely relativistic character in the sense that it tends to $n\pi$ $(n \in \mathbb{Z})$ whenever $c \to \infty$ giving no contribution to total cross sections in non-relativistic limit. One sees also that even such a simple example of non-local potentials may lead to some resonances (see Fig. 4).

The next step in our considerations will be to investigate the applicability of the new formulation of the eigenchannel method in the case of inelastic scattering from separable potentials. Moreover it seems also interesting to investigate the applicability of the method to the other, more complicated examples of separable potentials.

Acknowledgments

I am grateful to R. Szmytkowski for very useful discussions, suggestions and commenting on the manuscript. Discussions with M. Czachor are also acknowledged.

APPENDIX A: POSITIVE SEMIDEFINITENESS OF THE MATRIX $\mathbb{B}$

The proof follows the suggestions of Szmytkowski [19]. Positive semidefiniteness of the matrix $\mathbb{B}$ means that the inequality

$$X_\gamma(E)\mathbb{B}X_\gamma(E) \geq 0$$

(A.1)
is satisfied. To prove the above statement let us notice that

$$\int_{4\pi} d^2k \ e^{i\mathbf{k}\cdot\mathbf{\theta}} (\epsilon \mathbf{\alpha} \cdot \mathbf{k} + \beta mc^2 + E \mathbf{1}_4) = 4\pi$$

$$\times \left[ i\epsilon k j_1(k|\mathbf{g}|) \alpha \cdot \frac{\mathbf{g}}{|\mathbf{g}|} + (\beta mc^2 + E \mathbf{1}_4) j_0(k|\mathbf{g}|) \right] \quad (A.2)$$

where $\mathbf{g} = \mathbf{r} - \mathbf{r}'$. Then using Eq. (2.10), we may rewrite Eq. (4.5) in the form

$$B_{\nu\mu} = \frac{E k}{8\pi^2 c^2 \hbar^2} \int_{4\pi} d^2k \int_{\mathbb{R}^3} d^3r e^{i\mathbf{k}\cdot\mathbf{r}} u_{\nu}^*(\mathbf{r}) \mathcal{P}(\mathbf{k})$$

$$\times \int_{\mathbb{R}^3} d^3r' e^{-i\mathbf{k}\cdot\mathbf{r}'} u_{\mu}(\mathbf{r}') \quad (A.3)$$

which after application to Eq. (A.1) yields

$$X_{\gamma}(E) \mathcal{B} X_{\gamma}(E) = \frac{mk}{8\pi^2 \hbar^2} \int_{4\pi} d^2k \left| \sum_{\nu} X_{\gamma\nu}^*(E) \int_{\mathbb{R}^3} d^3r e^{i\mathbf{k}\cdot\mathbf{r}} u_{\nu}^*(\mathbf{r}) \mathcal{P}(\mathbf{k}) \right| ^2 \geq 0 \quad (A.4)$$

finishing obviously the proof. Here $X_{\gamma\nu}(E)$ denotes the $\nu$th element of the eigenchannel vector $X_{\gamma}(E)$ and $||\Omega|| = \sqrt{\Omega^\dagger \Omega}$.

**APPENDIX B: ORTHONORMALITY OF THE ANGULAR FUNCTIONS $\mathcal{Y}_{\gamma}(\mathbf{k})$ AND $\Upsilon_{\gamma}(\mathbf{k})$**

We begin with proof for the functions $\mathcal{Y}_{\gamma}(\mathbf{k})$. Application of Eq. (4.16) to Eq. (4.17) with the aid of Eq. (2.11) and the fact that $\mathcal{P}(\mathbf{k})$ is a projector, we can deduce that

$$\int_{4\pi} d^2k \mathcal{Y}_{\gamma\nu}^*(\mathbf{k}) \mathcal{Y}_{\gamma}(\mathbf{k}) = \frac{E \epsilon k}{8\pi^2 c^2 \hbar^2} \sum_{\nu \mu} X_{\gamma\nu}^*(E)$$

$$\times \left[ \int_{\mathbb{R}^3} d^3r \int_{\mathbb{R}^3} d^3r' u_{\nu}^*(\mathbf{r}) \right] \times \int_{4\pi} d^2k e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} \mathcal{P}(\mathbf{k}) u_{\mu}(\mathbf{r}') \quad (B.1)$$

Comparison with Eq. (A.3) shows that the square brackets in the above contain the expression proportional to certain element of the matrix $\mathcal{B}$. Therefore, we may rewrite Eq. (B.1) as

$$\int_{4\pi} d^2k \mathcal{Y}_{\gamma\nu}^*(\mathbf{k}) \mathcal{Y}_{\gamma}(\mathbf{k}) = X_{\gamma\nu}(E) \mathcal{B} X_{\gamma}(E) \quad (B.2)$$

Finally, substitution of Eq. (4.9) to Eq. (B.2) leads us directly to Eq. (4.17), finishing the proof. To prove the orthonormality relation for the functions $\Upsilon_{\gamma}(\mathbf{k})$, it suffices to combine Eq. (4.23) with Eq. (B.2).
[12] R. Augusiak, Ann. Phys. (Leipzig) 14, 398 (2005).
[13] B. Thaller, J. Phys. A 14, 3067 (1981) and references therein;
[14] H. A. Antosiewicz, in Handbook of Mathematical Functions, edited by M. Abramowitz and I. A. Stegun (Dover, New York, 1965), Chap. 10.
[15] E. de Prunelé, J. Phys. A 30, 7831 (1997); Yad. Fiz. 61, 2090 (1998).
[16] X. Bouju and E. de Prunelé, phys. stat. sol. (b) 217, 819 (2000); E. de Prunelé and X. Bouju, ibid. 225, 95 (2001); E. de Prunelé, Phys. Rev. B 66, 094202 (2002).
[17] N. Dombey and P. Kennedy, J. Phys. A 35, 6645 (2002).
[18] M. Loewe and S. Mendizabal, quant-ph/0402030.
[19] R. Szmytkowski (private communication).