Hadron Masses and Quark Condensate from Overlap Fermions

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We present results on hadron masses and quark condensate from Neuberger’s overlap fermion. The scaling and chiral properties and finite volume effects from this new Dirac operator are studied. We find that the generalized Gell-Mann-Oakes-Renner relation is well satisfied down to the physical u and d quark mass range. We find that in the range of the lattice spacing we consider, the $\pi$ and $\rho$ masses at a fixed $m_\pi/m_\rho$ ratio have weak $O(a^2)$ dependence.

The recent advance in chiral fermion formulation which satisfies Ginsparg-Wilson relation has a great promise in implementing chiral fermion for lattice QCD at finite lattice spacing. It is shown to have exact chiral symmetry on the lattice \cite{1} and it has no order $a$ artifacts \cite{2}. Neuberger’s Dirac operator \cite{3} derived from the overlap formalism has a compact form in four dimensions which involves a matrix sign function

$$D = \frac{1}{2} [1 + \mu + (1 - \mu)\gamma_5 \epsilon(H)].$$

In this talk, I present some preliminary results from our numerical implementation of the Neuberger fermion. We adopt the optimal rational approximation of the matrix sign function \cite{4} with 12 terms in the polynomials. The smallest 10 to 20 eigenvalues of $H^2$ are projected out for exact evaluation of the sign function for these eigenstates. We use multi-mass conjugate gradient as the matrix solver for both the inner and outer loops. With residuals at $10^{-7}$, the inner loop takes $\sim 200$ iterations and the outer loop takes $\sim 100$ iterations. We check the unitarity of the matrix $V = \gamma_5 \epsilon(H)$. For $Vx = b$, we find $|x^\dagger x - b^\dagger b| \sim 10^{-9}$. Even for topological sectors with $Q \neq 0$, we find the critical slowing down is much milder than that of the Wilson fermion and there are no exceptional configurations. The critical slowing down sets in quite abruptly after $\mu a = 0.003$ which is already at the physical u and d masses.

It is shown \cite{5} that the generalized Gell-Mann-Oakes-Renner (GOR) relation

$$\mu \int d^4x (\pi(x)\pi(0)) = 2\langle \bar{\Psi}\Psi \rangle,$$

with $\pi(x)$ being the pion interpolation field, is satisfied for each quark mass and volume, configuration by configuration. We utilize this relation as a check of our numerical implementation of the Neuberger operator. We find that for the lattices we considered ($6^3 \times 12, \beta = 5.7, 5.85; 8^3 \times 16, \beta = 5.85$, and $10^3 \times 20, \beta = 5.85, 6.0$) the GOR relation is satisfied very well (to within 1%) all the way down to the smallest mass $\mu a = 0.0001$ for the $Q = 0$ sector. For the $Q \neq 0$ sector, the presence of zero modes demands higher precision for the approximation of $\epsilon(H)$. For example, we show in Fig. \textsuperscript{3} the ratio of the right to left side of Eq. (2) for a configuration with topology on the $6^3 \times 12$ lattice at $\beta = 5.7$ as a function of the quark mass. When only 10 smallest eigenmodes are projected, we see that the ratio deviates from

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one for small quark masses. The situation is con-
siderably improved when 20 smallest eigenmodes
are included. The situation is better than the
domain-wall fermion case when the size of the
fifth dimension is limited to $L_s = 10$ to 48\[6\].

Figure 1. Ratio of the right to left side of Eq.
(2) for a configuration with topology. $\bigcirc$/●
indicates the case with projection of 10/20 smallest
eigenmodes.

We also calculate the quark condensate
$\langle \bar{\psi} \psi \rangle$ with 3 to 6 $Z_2$
oises for each configuration. For
small quark mass, it has the form

$$\langle \bar{\psi} \psi \rangle = \frac{\langle |Q| \rangle}{\mu V} + c_0 + c_1 \mu. \quad (3)$$

The singular term which is due to the zero modes
in the configurations with topology ($Q \neq 0$) is
specific to the quenched approximation. It will
be suppressed when the determinant is included
in the dynamical fermion case. We see this clearly
in the following figure which is first seen with the
domain-wall fermion [6]. A fit to the formula in
Eq. (3) is given in Fig. 2. We see that $c_0$ is non-
zero. The standard definition of the quark chiral
condensate entails the extrapolation of $c_0$ to the
infinite volume before taking the massless limit.
Another way is to consider the finite-size scaling
[6]. When the size of the lattice is much smaller
than the pion Compton wavelength, i. e. $L \ll
1/m_{\pi}$, the $\langle \bar{\psi} \psi \rangle$ is proportional to $\mu \Sigma^2 V$ for small
masses besides the $\frac{\langle |Q| \rangle}{\mu V}$ term due to quenching.
From this, the infinite volume condensate $\Sigma$ can
be extracted. We plot in Fig. 3 $\langle \bar{\psi} \psi \rangle a^3/\mu a$ vs $\mu a$
in the $Q = 0$ sector for 3 lattice volumes ($6^3 \times
12$, $8^3 \times 16$, and $10^3 \times 20$) at $\beta = 5.85$. We see
that they are quite flat which indicates that the
condensate is indeed proportional to $\mu$ and we
also see that they increase with volume.

We have calculated the $\pi, \rho$ and nucleon
masses. A typical result on the $8^3 \times 16$ lattice
at $\beta = 5.85$ is given in Fig. 4. We see the fi-
nite volume effect on the nucleon mass when
$\mu a$ is smaller than $\sim 0.15$. To see the behavior
of pion masses near the chiral limit, we plot $m_{\pi}^2 a^2$
as a function of $\mu a$ in Fig. 5 for three lattices with
about the same physical volume. It appears that
there might be a $\sqrt{\mu a}$ behavior in the very small
$\mu a$ region which we will explore further. When
we project only 10 smallest eigenmodes in the ap-
proximation for the sign function in the $6^3 \times 12$
case, we see that $m_{\pi}^2 a^2$ tends to a finite value as
$\mu a \to 0$. This implies a residual mass due to the
poor approximation of $\epsilon(H)$, a behavior similar
to that observed in the domain-wall fermion with

![Figure 2. Quark condensate as a function of the quark mass. This is done on 50 $6^3 \times 12$ lattices at $\beta = 5.7$.](image)

![Figure 3. $\langle \bar{\psi} \psi \rangle a^3/\mu a$ vs $\mu a$ in the $Q = 0$ sector for 3 lattice volumes at $\beta = 5.85$.](image)
Figure 4. Masses of $\pi$, $\rho$ and $N$ on the $8^3 \times 16$ lattice at $\beta = 5.85$ are plotted vs $\mu a$.

finite $L_s$. Finally, we check scaling. We plot in Fig. 4 $\pi/\sqrt{\sigma}$ and $\rho/\sqrt{\sigma}$ vs $\sigma a^2$ where $\sigma$ is the string tension from which the lattice spacings are determined. It is known that the overlap operator does not have $O(a)$ artifacts. Now it appears that the $O(a^2)$ errors are small.

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Figure 5. $m_\pi^2 a^2$ are plotted as a function of $\mu a$ for three lattices with about the same physical volume.

Figure 6. $\pi/\sqrt{\sigma}$ and $\rho/\sqrt{\sigma}$ are plotted vs $\sigma a^2$ for three lattices with similar physical volumes.