Robust Linear Predictions: Analyses of Uniform Concentration, Fast Rates and Model Misspecification

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Abstract

The problem of linear predictions has been extensively studied for the past century under pretty generalized frameworks. Recent advances in the robust statistics literature allow us to analyze robust versions of classical linear models through the prism of Median of Means (MoM). Combining these approaches in a piecemeal way might lead to ad-hoc procedures, and the restricted theoretical conclusions that underpin each individual contribution may no longer be valid. To meet these challenges coherently, in this study, we offer a unified robust framework that includes a broad variety of linear prediction problems on a Hilbert space, coupled with a generic class of loss functions. Notably, we do not require any assumptions on the distribution of the outlying data points ($O$) nor the compactness of the support of the inlying ones ($I$). Under mild conditions on the dual norm, we show that for misspecification level $\epsilon$, these estimators achieve an error rate of $O(\max \{ |O|^{1/2} n^{-1/2}, |I|^{1/2} n^{-1} \} + \epsilon)$, matching the best-known rates in literature. This rate is slightly slower than the classical rates of $O(n^{-1/2})$, indicating that we need to pay a price in terms of error rates to obtain robust estimates. Additionally, we show that this rate can be improved to achieve so-called “fast rates” under additional assumptions.

1 Introduction

Linear prediction is the cornerstone of a significant group of statistical learning algorithms including linear regression, Support Vector Machines (SVM), regularized regressions (such as ridge, elastic net, lasso, and its variants), logistic regression, Poisson regression, probit models, single-layer perceptrons, and tensor regression, just to name a few. Thus, developing a deeper understanding of the pertinent linear prediction models and generalizing the methods to provide unified theoretical bounds is of critical importance to the machine learning community.

For the past few decades, researchers have unveiled different aspects of these linear models. Bartlett and Shawe-Taylor (1999) obtained high confidence generalization error bounds for SVMs and other learning algorithms such as boosting and Bayesian posterior classifier. Vapnik-Chervonenkis (VC) theory (Vapnik, 2013) and Rademacher complexity (Bartlett and Mendelson, 2001, 2002) have been instrumental in the machine learning literature to provide generalization bounds (Shalev-Shwartz and Ben-David, 2014). Theoretical properties of the multiple-instance extensions of SVM were analyzed by Doran and Ray (2014). Kakade et al. (2009b) presented a

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unified framework for these linear models under certain constraints. Contemporary analyses of linear regression and its variants especially in high-dimensional regimes can be found in (Wainwright, 2019) and the references therein. Recent results about tensor regression can be found in the works of Kossaifi et al. (2020); Rabusseau and Kadri (2016).

However, the matter is complicated when outliers are present in the data. To surpass the adverse effect due to the presence of outliers, various robust linear models have been proposed in the literature. Cantoni and Ronchetti (2001) puts forth the concept of robust deviances that can be used for step-wise model selection; McKean (2004) uses the robust analysis which is based on a fit based on norms other than \( \ell_2 \); Ló and Ronchetti (2009) proposes robust test statistic for hypothesis testing and variable selection in GLM; Valdora and Yohai (2014) robustifies generalized linear models by using M-estimators after applying variance stabilizing transformations; Wang and Blei (2018) and Gonçalves et al. (2020) take Bayesian approach. Some other related works in robustifying GLMs include (Lee and Nelder, 2003; Jearkpaporn et al., 2005; Ghosh and Basu, 2016; Cherapanamjeri et al., 2020; Bhatia et al., 2017; Mourtada et al., 2022). Yuan and Cai (2010) analyzed the theoretical properties of functional linear regression in a Reproducing Kernel Hilbert Space (RKHS).

Many of the aforementioned approaches, however, did not provide theoretical finite-sample guarantees which make them unreliable. The Median of Means (MoM) philosophy offers a viable and appealing paradigm for adapting linear model-based algorithms to become outlier-resistant, bridging this methodological gap. Even under the moderate condition of finite variance, MoM estimators are not only robust to anomalies in the data but also admit exponential concentration (Lugosi and Mendelson, 2019; Lecué et al., 2020; Bartlett et al., 2002; Lerasle, 2019; Laforgue et al., 2019). From this point of view, near-optimal analyses have recently been undertaken for mean estimation (Minsker, 2018), classification (Lecué et al., 2020), regression (Mathieu and Minsker, 2021; Lugosi and Mendelson, 2019), clustering (Klochkov et al., 2020; Brunet-Saumard et al., 2020), bandits (Bubeck et al., 2013), and optimal transport (Staerman et al., 2021).

Under the MoM framework, we propose a unified framework for analyzing robust linear prediction models under a general class of loss functions. To further generalize our framework, we consider any quantile of means (Klochkov et al., 2020) instead of the median and show that the properties of MoM transcend to this class of estimators as well. We analytically show that one can optimize the robust loss function through a general-purpose optimization algorithm such as gradient descent Newton’s method, which is popularly used in generalized linear models literature. We analytically show that under the convexity of the underlying loss function, the vanilla gradient descent converges to the global optima under our paradigm. Compared to the existing literature on MoM classification Lecué et al. (2020), we are not only able to relax the assumptions, but also able to generalize the MoM estimator to any quantile of means estimator. In addition to being able to match the learning rate with the existing literature, Lecué et al. (2020); Rodriguez and Valdora (2019); Paul et al. (2021b), compared to Lecué et al. (2020), we show the consistency of the proposed estimator under these relaxed assumptions, while deriving so-called “fast rates” and generalization bounds under model misspecifications.

Our paradigm allows us to divide the data into two categories: the set of inliers (\( I \)) and the set of outliers (\( O \)). We assume that inliers are independent and identically distributed (i.i.d.); whereas \( O \) requires no assumptions, enabling outliers to be unboundedly large, dependent on each other, drawn from a heavy-tailed distribution, and so forth. Under our framework, we find uniform concentration bounds of the risk of these classes of estimators under mild regularity conditions that apply to a large variety of popular linear supervised learning algorithms. The theoretical analysis undertaken in our framework appeals to Rademacher complexities (Bartlett and Mendelson, 2002; Bartlett et al., 2005) and symmetrization arguments (Vapnik, 2013). As opposed to the classical
literature on Rademacher complexity, we neither require the assumption of boundedness of the corresponding function nor do we need to assume a sub-gaussian behavior of the error terms; only finite variance suffices. Furthermore, we derive so-called fast rates, which although well-known for classical models, have rarely been analyzed in MoM literature (Tu et al., 2021). We prove for “simpler” learning problems one can achieve these fast rates under our robust framework, thus bridging the gap between the MoM and the classical approach from this perspective.

2 Robust Linear Models

Notations For any $n \in \mathbb{N}$, $[n]$ denotes the set $\{1, \ldots, n\}$. $\mathbb{R}_{\geq 0}$ denotes the set of all non-negative reals. For a measure $\mu$ and function $f$, $Pf = \int f d\mu$. $(x)_+ \triangleq \max(x, 0)$.

Problem Statement Suppose we have a prediction problem at hand: “Predict $Y$ given $X$”. Here $Y$ is our response random variable and $X$ is our predictor random variable. For example, in an image classification problem, $Y$ can be the class label of the image and $X$ is the feature vector of the image. We assume that $X \in \mathcal{X} \subseteq \mathbb{H}_x$, for some Hilbert space $\mathbb{H}_x$ and $Y \in \mathcal{Y} \subseteq \mathbb{H}_y$, for another Hilbert space $\mathbb{H}_y$. In classical machine learning, one approaches the problem by trying to estimate $Y$ by $f(X) (f: \mathbb{H}_x \to \mathbb{H}_y)$, with $f$ in some function class $\mathcal{F}$. The discrepancy between $Y$ and $f(X)$ is measured by some loss function, $L: \mathbb{H}_y \times \mathbb{H}_y \to \mathbb{R}_{\geq 0}$. One then casts this prediction problem as a minimization problem as follows:

$$\min_{f \in \mathcal{F}} \mathbb{E}L(f(X), Y).$$

Suppose we have access to training data $\mathcal{X}_n^{\text{train}} = \{(X_i, Y_i)\}_{1 \leq i \leq n}$. Since, we do not have access to the joint distribution of $(X, Y)$, one estimates $\mathbb{E}L(f(X), Y)$ by its empirical counterpart, $\frac{1}{n} \sum_{i=1}^{n} L(f(X_i), Y_i)$, assuming that the training data is independent and identically distributed. Thus, in practice, we consider the following Empirical Risk Minimization (ERM) problem:

$$\min_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^{n} L(f(X_i), Y_i).$$

Suppose $\langle \cdot, \cdot \rangle$ denote the inner product operation on $\mathbb{H}_x$. In the classical linear predictions, one takes this function class as $\mathcal{F} = \{ f(x) = \langle w, x \rangle : w \in \mathbb{H}_x \}$. To obtain better behaved estimates which possess desirable properties such as sparsity, uniqueness etc., we often add a regularizing term $F(w)$ on $w$ and pose the learning problem as minimization of the following objective function:

$$\frac{1}{n} \sum_{i=1}^{n} L(\langle w, X_i \rangle, Y_i) + \lambda F(w). \quad (1)$$

In literature, $F(\cdot)$ is assumed to be convex.

Examples In what follows, let us consider a few examples of linear models which is of the form of (1).

1. Linear Regression: Ridge, Lasso, and Elastic Net: Suppose we take $\langle a, b \rangle = a^\top b$ and $L(\cdot, \cdot)$ to be the squared error loss. Then objective (1) becomes $\sum_{i=1}^{n}(Y_i - w^\top X_i)^2 + \lambda F(w)$. Now, depending on whether the penalty $F(w)$ is $\ell_1$, $\ell_2$, or a convex combination of the two, we get lasso, ridge, and elastic net respectively. If $F(w) = 0 \forall w$, then it boils down to simple linear regression.
2. Generalized Linear Models (GLM): GLM is defined as follows: \( \mu_i = g^{-1}(\eta_i) = g^{-1}(w^T X_i) \)
where \( \mu_i = E(Y_i) \) and \( g^{-1} \) is the inverse link function. We minimize this w.r.t. squared error loss. In objective (1), if we take \( \langle a, b \rangle = a^T b \), then different choices of \( L \) leads to different special cases of GLM, e.g. squared error loss leads to linear regressions, cross-entropy loss leads to logistic regression and so on. Depending on the choice of \( F(\cdot) \) one can retrieve different choices of penalized GLMs (Park and Hastie, 2007; Xu et al., 2017).

3. Support Vector Machine (SVM): In objective (1), we take \( \langle a, b \rangle = a^T b \), the hinge loss, i.e. \( L(a, b) = \max\{0, (1 - ab)\} \) and \( F(w) = \frac{1}{2} \|w\|^2_2 \) to get the SVM objective function.

4. Functional Regression: In functional regression \( X = X(t) \) and \( Y = Y(t) \) are real-valued functions on some set \( T \). Here \( \langle w, x \rangle = \int_T w(t)X(t)dt \). See (Yuan and Cai, 2010) for a more detailed analysis of this model.

Extensions to kernel cases for both linear regressions and SVMs can be analyzed by mapping \( x \mapsto \phi(x) \) and considering the inner product in that RKHS.

A Robust Approach As intuitive and interpretable as the classical models are, whenever there are outliers in the data, \( \frac{1}{n} \sum_{i=1}^n L(\langle w, X_i \rangle, Y_i) \) might not be a good estimate of \( \mathbb{E}L(f(X), Y) \). Under the MoM framework, one begins by partitioning the data into \( K \) disjoint blocks \( B_1, \ldots, B_K \subset [n] \) of the same size \( b \) (possibly discarding few elements if \( n \) is not divisible by \( K \)). This partitioning is often constructed uniformly at random or can be shuffled throughout the algorithm. To simplify notations, we write \( \varphi_w(x, y) = L(\langle w, x \rangle, y) \) and let \( P_k \) denote the empirical distribution of \( \{(X_i, Y_i)\}_{i \in B_k} \). Thus, \( P_k \varphi_w = \frac{1}{b} \sum_{i \in B_k} L(\langle w, X_i \rangle, Y_i) \). Under the MoM framework, the goal is to minimize the following robust version of objective (1):

\[
\text{Median}(P_1 \varphi_w, \ldots, P_K \varphi_w) + \lambda F(w). \tag{2}
\]

To generalize the framework further, instead of taking the median, one can take any quantile, \( q \in (0, 1) \). Let \( Q_q(z) \) denote a \( q \)-th lower quantile of the vector \( z \in \mathbb{R}^m \). For concreteness, we take \( Q_q(z) = \inf\{z \in \mathbb{R} : m^{-1} \sum_{i=1}^m \mathbb{1}\{z \leq z_i\} \geq q\} \). To generalize the framework further, we take the \( q \)-th quantile instead of the median in (2) to obtain our objective function,

\[
g(w) = Q_q(P_1 \varphi_w, \ldots, P_K \varphi_w) + \lambda F(w). \tag{3}
\]

One should note that taking \( q = 1/2 \) recovers objective (2). Intuitively, these estimators are more robust than their ERM (Devroye et al., 2013) counterparts since under mild conditions, only a subset of the partitions is contaminated by outliers while others are outlier-free. Taking a quantile over partitions negates the negative influence of these spurious partitions, thus reducing the effect of outliers.

Optimization Moreover, optimizing (3) is computationally simple as one can implement gradient-based or second order methods. We note that \( \nabla_w g(w) = \frac{1}{b} \sum_{i \in B_{kq}(w)} \nabla_w L(\langle w, X_i \rangle, Y_i) + \lambda \nabla_w F(w) \),

where \( k_q(w) \) is such that \( g(w) = P_{k_q(w)} \varphi_w \) is the partition on which the empirical risk equals the \( q \)-th quantile. Similarly, for second order methods, the Hessian is given by, \( \nabla^2_w g(w) = \frac{1}{b} \sum_{i \in B_{kq}(w)} \nabla^2_w L(\langle w, X_i \rangle, Y_i) + \lambda \nabla^2_w F(w) \). The terms \( \nabla_w L(\langle w, X_i \rangle, Y_i) \) and \( \nabla^2_w L(\langle w, X_i \rangle, Y_i) \) are easy to compute. One can apply a simple first order or a second order method to optimize (3). For a pre-fixed step size sequence \( \{\epsilon_t\}_{t \in \mathbb{N}} \), a pseudo-code of the vanilla gradient descent is shown in Algorithm 1.
**Algorithm 1 Robust GLM via Newton’s Method**

**Input:** \(\{(X_i, Y_i)\}_{1 \leq i \leq n}, q, \lambda, K, L, \langle \cdot, \cdot \rangle, \{\varepsilon_t\}_{t \in \mathbb{N}}\) (step size).

**Output:** \(\hat{w}\). Initialization: Randomly partition \(\{1, \ldots, n\}\) into \(K\) many partitions of equal length. Initialize \(w_0\).

repeat

1. Find \(k_q^{(t)} \in \{1, \ldots, K\}\), such that \(g(w^{(t)}) = P_{k_q^{(t)}} f_{w^{(t)}}\).

2. Update \(w\) via, \(w^{(t+1)} \leftarrow w^{(t)} - \varepsilon_t \nabla_w g(w^{(t)})\)

until objective (3) converges

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**Convergence** If the underlying loss function is convex then, under standard regularity conditions Bottou and Le Cun (2005); Lecué et al. (2020), we show that the iterates of Algorithm 1 converges almost surely to the global optima \(\hat{w}\). The technical details are given in Appendix A.

**Theorem 2.1.** Grant the following assumptions:

1. The map \(w \mapsto L(\langle w, X \rangle, Y)\) is convex in \(w\) almost everywhere under \(P\).

2. For all \(\|\nabla_w L(\langle w, X \rangle, Y)\|_2 \leq C_1\) and \(\|\nabla_w F(w)\|_2 \leq C_2\), almost surely under \(P\), for some \(C_1, C_2 > 0\).

3. For any dataset, \(\{(X_i, Y_i)\}_{i \in [n]}\) and any \(w \in \mathbb{R}^p\), there exists an open ball \(B_r(w)\), centered at \(w\) and radius \(r > 0\) and \(k_{\text{med}} \in \{1, \ldots, K\}\), such that for all \(w' \in B_r(w)\), \(P_{k_{\text{med}}} w' = Q_q(P_1 \varphi w, \ldots, P_K \varphi w)\).

4. \(\sum_{t \geq 1} \varepsilon_t^2 < \infty\) and \(\sum_{t \geq 1} \varepsilon_t = \infty\).

5. For any dataset, \(\{(X_i, Y_i)\}_{i \in [n]}\) and any \(w \in \mathbb{R}^p\) and \(r > 0\), there exists \(\delta > 0\) such that, \(\inf_{w' \in B_r(w)} \langle w' - w, G_w \rangle > \delta\). Here \(G_w = \frac{1}{b} \sum_{i \in B_{k_{\text{med}}}} \nabla_w L\langle \langle w^{(t-1)}, X_i \rangle, Y_i \rangle + \lambda \nabla_w F(w^{(t-1)})\).

Then, \(w^{(t)}\) converges to \(w\), to the the global optima of (3), almost surely.

**Computational Complexity** For simplicity, let \(\mathbb{H}_x = \mathbb{R}^p\) and let \(T_1, T_2\) and \(T_3\) be the computational complexity of computing \(L(\langle w, x \rangle, y), \nabla_w L(\langle w, x \rangle, y)\) and \(\nabla_w L(\langle w, x \rangle, y)\), respectively. Also suppose \(T_4\) and \(T_5\) be the computational complexity of computing \(\nabla_w F(w)\) and \(\nabla_w F(w)\), respectively. Algorithm 1 takes \(O(nT_1 + n + K + bT_2 + T_4)\) per iteration. On the other hand, its ERM counterpart takes \(O(nT_2 + T_4)\). Usually, \(T_3 \gg T_1\), making the per iteration complexity of Algorithm 1 smaller than its ERM counterpart in general.

**Performance** To demonstrate the efficacy of the MoM paradigm we consider a simple simulation study in the context of logistic regression, with details appearing in Appendix E. In this simulation study, we plot the log of the \(\ell_2\) norm between the coefficient vector from the ground truth for both logistic regression with and without the MoM paradigm for a two class classification problem containing \(n^{0.3}\) many outliers. In Fig. 1, we plot this log error against log(\(T\)). It is easy to see that even in this simple case, though the classical logistic regression has access to an increasing number of samples, performs poorly compared to its MoM counterpart, which remains stable and consistently perform better than its ERM counterpart.
3 Theoretical Properties

This section discusses the theoretical properties of proposed framework defined in section 2, with complete proofs in the Appendix. We begin by assuming that the data index set, \([n]\) can be divided into two disjoint sets \(I\) (inliers) and \(O\) (outliers). We will assume that \(\{(X_i, Y_i)\}_{i \in I}\) are independent and identically distributed according to the distribution \(P\) on \(\mathbb{H}_x \times \mathbb{H}_y\). One should note that in order to estimate the \(q\)-th quantile, efficiently, one should have at least \(\min\{q, 1-q\}\) outlier-free partitions. This is to ensure that one can estimate both the tails efficiently. Thus, we will make the standard assumption that \(K\) is slightly larger than \(\min\{q, 1-q\}\). Formally, for \(\eta > 0\),

\[
A 1. K \geq \frac{(1+\eta)}{\min\{q, 1-q\}}|O|.
\]

We note that taking \(\eta = 1\) and \(q = \frac{1}{2}\) gives us “\(K \geq 4|O|\)”, which was assumed by Lecué et al. (2020) (see Theorem 2 therein). Taking \(q = \frac{1}{2}\) and replacing \(\eta\) with \(\frac{\eta}{2}\) boils down to assumption A6 of Paul et al. (2021c) and Assumption 3 of Paul et al. (2021b).

Let \(\|x\|\) denote the norm of \(x\) (this may not be the canonical norm induced by \(\langle \cdot, \cdot \rangle\), i.e. \(\sqrt{\langle x, x \rangle}\)). The dual \(\|\cdot\|_*\)-norm \(\mathbb{H}_x\) is defined as \(\|w\|_* \triangleq \sup_{\|x\| \leq 1} \langle w, x \rangle = \sup_{x \neq 0} \langle w, x \rangle / \|x\|\). Let \((X, Y)\) be distributed according to \(P\). We will assume the following moment condition on the dual norm:

\[
A 2. (\text{Finite Second Moment}) \mu_*^2 = \mathbb{E}\|X\|_*^2 < \infty.
\]

We observe that if we take \(\|x\| = \|x\|_{\mathbb{H}_x} = \sqrt{\langle x, x \rangle}\) to be the canonical norm on \(\mathbb{H}_x\), assumption A2 essentially implies that \(P\) admits a finite second moment in its first component \(X\). This assumption is quite standard in literature (Klochkov et al., 2021; Chakraborty and Das, 2019). We note that we do not impose any boundedness assumption (Chakraborty et al., 2020; Paul et al., 2021a; Chakraborty and Das, 2021) on the support of the underlying distribution.
Instead of dealing with the primal problem (3), we consider its dual counterpart,

$$\min_{w: F(w) \leq B} Q_q(P_1 \varphi_w, \ldots, P_K \varphi_w),$$

for some $B > 0$. For simplicity of notations, we write, $R_{n,K,q}(w) = Q_q(P_1 \varphi_w, \ldots, P_K \varphi_w)$. We denote $W = \{w : F(w) \leq B\}$. Let $\hat{w}$ be an empirical minimizer of (4), i.e. $Q_q(P_1 \varphi_{\hat{w}}, \ldots, P_K \varphi_{\hat{w}}) = \inf_{w \in W} Q_q(P_1 \varphi_w, \ldots, P_K \varphi_w)$. In this section, we will assume the models is realizable, i.e.

$$\inf_{f \in \mathcal{L}} L(f(x), Y) = \inf_{w \in W} P \varphi_w.$$

One can restrict $f$ to the set of all measurable functions from $H_x$ to $H_y$. The excess risk of the estimator $\hat{w}$ in this context is defined as

$$\mathcal{R}(\hat{w}) = P \varphi_{\hat{w}} - \inf_{f \in \mathcal{L}} L(f(x), Y).$$

Under the assumption that the model is specified, we observe that,

$$\inf_{f \in \mathcal{L}} L(f(x), Y) = \inf_{w \in W} P \varphi_w.$$

The goal of this section is to assert that $\mathcal{R}(\hat{w})$ becomes very small with a high probability, as we have access to more and more data. Formally, we state the assumption of realizability as follows:

A 3. (Realizability) $\inf_{f \in \mathcal{L}} L(f(x), Y) = \inf_{w \in W} P \varphi_w$.

For simplicity, we make the standard assumption (Kakade et al., 2009b) that the regularizer $F$ is $\sigma$-strongly convex ($\sigma > 0$) with respect to the dual norm, $\| \cdot \|_*$, i.e. for any $w_1, w_2 \in H_x$,

$$F(\alpha w_1 + (1 - \alpha) w_2) \leq \alpha F(w_1) + (1 - \alpha) F(w_2) - \frac{\sigma}{2} \alpha(1 - \alpha) \| w_1 - w_2 \|^2_*.$$

One should note that the above strong convexity condition holds for most commonly used regularizers such as ridge, lasso, elastic-net penalty etc. We also make the assumption that $\inf_{w \in H_x} F(w) = 0$. Thus, to put it formally,

A 4. (Strong Convexity) $F$ is $\sigma$-strongly convex and $\inf_{w \in H_x} F(w) = 0$.

It is standard in literature (Lecué et al., 2020; Kakade et al., 2009b; Shalev-Shwartz and Ben-David, 2014) to assume that that $L(\cdot, y)$ is Lipschitz. One should note that although squared-error loss function is not Lipschitz on the entirety of a real finite-dimensional vector space, if one assumes that the support of $(X, Y)$ is bounded (this is a natural assumption since in real life one cannot observe unboundedly large data due to measurement limitations of machines), then the squared error loss function is Lipschitz on this bounded set.

A 5. (Lipschitzness) $L(\cdot, y)$ is $\tau$-Lipschitz.

In addition to the aforementioned five assumptions we assume that $\text{Var}(X,Y) \sim L(\langle w, X \rangle, Y)$ is uniformly bounded in $W$.

A 6. (Finite Variance) $V^2 = \sup_{w \in W} \text{VarL}(\langle w, X \rangle, Y) < \infty$.

This condition can be replaced by an alternative assumption that $F$ is symmetric and $X$ has a finite fourth moment in the dual norm. We discuss this in Section 3.1 in more details.

Before we proceed, we now state and discuss the implications of our main theorem (Theorem 3.1).
Theorem 3.1. (Main Theorem) Suppose A1–6 holds. Then, for any \( q \in (0,1) \),
\[
\mathcal{R}(\hat{w}) \leq 2V \sqrt{\frac{2(2 + \eta)}{q(1 - q)\eta b} + \frac{16B\tau\mu_s(2 + \eta)\sqrt{|I|}}{q(1 - q)\eta n_1\sqrt{\sigma}}},
\]
with probability at least \( 1 - e^{-2K\left(\frac{2a - |O|}{2n - |O|}\right)^2} - e^{-2K\left(\frac{2|O|}{2n - |O|}\right)^2} \).

Theorem 3.1 implies that \( \hat{w} \) admits a risk of the order at most \( O\left(\max\left\{K^{1/2}n^{-1/2}, n^{-1}\sqrt{|I|}\right\}\right) \) (since \( b = \frac{n}{K} \)). Note that as \( K \geq 1 \), \( \max\left\{K^{1/2}n^{-1/2}, n^{-1}\sqrt{|I|}\right\} = \Omega\left(n^{-1/2}\right) \). Thus, the convergence rates for \( \hat{w} \) in our framework are generally slower than its ERM counterpart, for which the rate is \( O(n^{-1/2}) \). This is unsurprising as MoM operates on outlier-contaminated data; there is “no free lunch” in trading off robustness for rate of convergence. However, if the number of partitions \( L \) grows slowly relative to \( n \) (say, \( L = O(\log n) \)) so that \( |O| = O(\log n) \), the convergence rates for MoM estimates become comparable to the ERM counterparts at \( O(n^{-1/2}) \).

We will state the following Lemma, which generalizes Lemma 4 of Kakade et al. (2009b). The proof is given in the supplement.

Lemma 3.1. Let \( \mathcal{W} \) be a closed convex subset of \( \mathcal{X} \) and let \( F: \mathcal{W} \to \mathbb{R} \) be \( \sigma \)-strongly convex w.r.t. \( \| \cdot \|_\star \). Let \( \{Z_i\}_{i \in \mathbb{N}} \) be mean zero, independent random vectors in \( \mathcal{H}_x \), such that \( E\|Z_i\|_\star^2 < \infty \). We define \( S_n = \sum_{i=1}^n Z_i \). Then, \( \{F^*(S_n) - \frac{1}{2\sigma} \sum_{j=1}^n E\|Z_j\|_\star^2\}_{n \in \mathbb{N}} \) is a supermartingale. Furthermore, if \( \inf_{w \in \mathcal{H}_x} F(w) = 0 \), then \( EF^*(S_n) \leq \frac{1}{2\sigma} \sum_{i=1}^n E\|Z_i\|_\star^2 \).

Before we proceed to Lemma 3.2, we recall that the population Rademacher and Gaussian complexities of a function class \( \mathcal{F} \) is defined as:
\[
\mathcal{R}_n(\mathcal{F}) \triangleq \frac{1}{n}E\sup_{f \in \mathcal{F}} \sum_{i=1}^n \epsilon_i f(X_i), \quad \text{and} \quad \mathcal{G}_n(\mathcal{F}) \triangleq \frac{1}{n}E\sup_{f \in \mathcal{F}} \sum_{i=1}^n g_i f(X_i),
\]
respectively. Here \( \{\epsilon_i\}_{i \in [n]} \) and \( \{g_i\}_{i \in [n]} \) are i.i.d. Rademacher and standard normal random variables, independent of \( \{X_i\}_{i \in [n]} \).

In the following Lemma, we will provide bounds for the complexity of the function class \( \mathcal{F}_\mathcal{W} \triangleq \{x \mapsto \langle w, x \rangle : w \in \mathcal{W}\} \). The key idea is to appeal to Lemma 3.1 and appeal to the strict convexity of \( F \).

Lemma 3.2. Let \( \{\epsilon_i\}_{i \in [n]} \) be i.i.d. such that \( E\epsilon_i = 0 \) and \( E\epsilon_i^2 = 1 \). Let \( \theta = \frac{1}{n} \sum_{i=1}^n \epsilon_i X_i \). Suppose \( \mathcal{W} \subseteq \{w : F(w) \leq B^2\} \). Then, \( E\epsilon \sup_{w \in \mathcal{W}} \langle w, \theta \rangle \leq \frac{2B}{n} \sqrt{\frac{1}{2\sigma} \sum_{i=1}^n \|X_i\|_\star^2} \).

One should note that if we take \( \epsilon_i \)'s to be i.i.d. Rademacher or Gaussian, one gets bounds on the sample Rademacher and Gaussian complexities respectively. Thus, we have a following immediate identities of Lemma 3.2.
\[
\hat{\mathcal{R}}_\mathcal{X}(\mathcal{F}_\mathcal{W}), \hat{\mathcal{G}}_\mathcal{X}(\mathcal{F}_\mathcal{W}) \leq \frac{2B}{n} \left( \frac{1}{2\sigma} \sum_{i=1}^n \|X_i\|_\star^2 \right)^{-1/2}.
\]

We now provide a recipe to provide bounds on the population Rademacher and Gaussian complexities in the following immediate corollary of Lemma 3.2.
Corollary 3.1. Let \( \{\epsilon_i\}_{i\in[n]} \) be i.i.d. such that \( \mathbb{E}\epsilon_i = 0 \) and \( \mathbb{E}\epsilon_i^2 = 1 \). Also let, \( \{X_i\}_{i\in[n]} \) be i.i.d. \( P \) and suppose \( \theta = \frac{1}{n} \sum_{i=1}^{n} \epsilon_i X_i \). Suppose \( W \subseteq \{w : F(w) \leq B^2\} \). Then, \( \mathbb{E}\sup_{w\in W} \|w, \theta\| \leq \frac{2B\mu^*}{\sqrt{2\sigma n}} \).

We are now ready to state a key result that plays an instrument role in proving our main Theorem (Theorem 3.1). Theorem 3.2 asserts that if \( \sup_{w\in W} |R_{n,K,q}(w) - P\varphi(w)| \leq \frac{1}{n^{1/2}} \), \( \sqrt{1-\delta} \), \( \mathbb{P} \) holds with high probability under standard assumptions. The proof of this result is rather technical and we refer the readers to the supplement.

Theorem 3.2. (Uniform Concentration) Under assumptions A1–6, with probability \( 1 - e^{-2K\left(\frac{2\eta}{2^{1-q}} - \frac{|Q|}{K}\right)^2} \), \( \sup_{w\in W}(P\varphi(w) - R_{n,K,q}(w)) \leq 2V\sqrt{\frac{(2+\eta)}{1-q}} + \frac{16B\mu^*(2+\eta)}{q\eta n} \), as well as, with probability at least \( 1 - e^{-2K\left(\frac{2(1-q)}{2^{1-q}} - \frac{|Q|}{K}\right)^2} \), \( \sup_{w\in W}(R_{n,K,q}(w) - P\varphi(w)) \leq 2V\sqrt{\frac{(2+\eta)}{(1-q)}} + \frac{16B\mu^*(2+\eta)}{(1-q)n} \).

We now give a proof sketch of Theorem 3.1, with details appearing in Appendix B.

Proof sketch of Theorem 3.1. The main idea is to first find \( w^*_m \) be such that, \( P\varphi(w^*_m) \leq \inf_{w\in W} P\varphi(w) + \frac{1}{m} \). Then one can upper bound \( \mathcal{R}(\hat{w}) \) by \( \sup_{w\in W}(P\varphi(w) - R_{n,K,q}(w)) + \sup_{w\in W}(R_{n,K,q}(w) - P\varphi(w)) \) (with some error) and control the individual terms by appealing to Theorem 3.2.

Remark 1. (How to choose \( q \)) From the bounds in Theorem 3.1, we observe that if \( q \) is too close to 0 or 1, the proposed bound on \( \mathcal{R}(\hat{w}) \) becomes meaningless. This result can also be intuitively observed that if \( q \) is very close to 0 or 1, we need more partitions (i.e. we need \( K \) to be large such that A1 is satisfied) to obtain enough outlier-free partitions to estimate the quantile of the empirical losses efficiently. The risk bound becomes the tightest if we choose \( q = 0.5 \), i.e. if we resort to the MoM approach.

3.1 Alternative Assumptions

In this section, we show that one can show that if one assumes that the fourth moment of \( X \) w.r.t the dual norm is finite then assumption A6 automatically follows. Thus, A6 can be replaced by the following alternate assumption.

A6* (Finite fourth moment) \( \mathbb{E}\|X\|^4 < \infty \).

A6* is more interpretable and easy to verify than A6. Its is also more commonly used in literature (Brownlees et al., 2015; Zhang et al., 2020; Paul et al., 2021a).

We begin by first proving that A2–5 and A6* implies A6 in the following Lemma.

Lemma 3.3. Under assumptions A2–5, \( F \) being symmetric with \( \mathbb{E}L^2(0, Y) < \infty \) then,

\[ V^2 \leq 4\gamma^2 \left( B^2 + 2\|\nabla F^*(0)\|^2 + 2\mathbb{E}\|X\|^2 + 2\mathbb{E}\|X\|^4 \right) + 2\mathbb{E}L^2(0, Y) < \infty. \]

Moreover, if \( L(0, y) \) is \( \gamma \)-Lipschitz in its second argument (w.r.t \( \|\cdot\|_{\mathbb{H}_y} \)) and \( L(0, 0) = 0 \), \( V^2 \leq 4\gamma^2 \left( B^2 + 2\|\nabla F^*(0)\|^2 + 2\mathbb{E}\|X\|^2 + 2\mathbb{E}\|X\|^4 \right) + 2\gamma^2 \mathbb{E}\|Y\|^2_{\mathbb{H}_y}. \) Here \( \mathbb{H}_y \) denotes the canonical norm on \( \mathbb{H}_y \).

We now have an immediate corollary combining the results of Theorem 3.1 and Lemma 3.2.

Corollary 3.2. Suppose A1–5 and A6* hold. Then, if \( 0 < q \leq 1 \), \( \mathcal{R}(\hat{w}) \leq 2\gamma\sqrt{\frac{2(1-q)}{q}} + \frac{16B\mu^*(2+\eta)}{q(1-q)\eta n} \), with probability at least \( 1 - e^{-2K\left(\frac{2\eta}{2^{1-q}} - \frac{|Q|}{K}\right)^2} - e^{-2K\left(\frac{2(1-q)}{2^{1-q}} - \frac{|Q|}{K}\right)^2} \). Here, \( \gamma = \sqrt{2\mathbb{E}L^2(0, Y)} + \sqrt{2\tau} \left( B^2 + 2\|\nabla F^*(0)\|^2 + 2\mathbb{E}\|X\|^2 + 2\mathbb{E}\|X\|^4 \right)^{1/2} \).
Remark 2. We note that Corollary 3.2 and the Lipschitzness of $L(\cdot, \cdot)$ implies that $\mathfrak{R}(\hat{w}) \leq \max \left\{ \left( \sqrt{\mathbb{E}\|X\|^2 + \mathbb{E}\|X\|^4} + \sqrt{\mathbb{E}\|Y\|^2} \right) \sqrt{\frac{K}{n}}, \sqrt{\mathbb{E}\|X\|^2 \sqrt{\frac{d_q}{n}}} \right\}$, with high probability. Thus, it can be observed that if the moments of $X$ (in the dual norm) and $Y$ are small, then risk bound become tighter, while a large second and fourth moment in this form will lead to higher risk for the obtained estimates.

Remark 3. (Inference for finite-dimensional spaces) If we restrict ourselves in a finite-dimensional real vector space and assume that the elements of $X \in \mathbb{R}^{d_x}$ and $Y \in \mathbb{R}^{d_y}$ be independent and identically distributed then, the bounds in Corollary 3.2 take a more simpler form. Under the classical $\ell_2$ norm (i.e. $\| \cdot \| = \| \cdot \|_2$), $\| \cdot \|_* = \| \cdot \|_2$. In this case, $\mathbb{E}\|X\|^2 = O(d_x), \mathbb{E}\|X\|^4 = O(d_x^2)$ and $\mathbb{E}\|Y\|^2 = O(d_y)$. Thus, from Remark 2, $\mathfrak{R}(\hat{w}) \leq (d_x + \sqrt{d_y})\sqrt{\frac{K}{n} + \frac{d_xd_q}{n}}$. Instead of appealing to the Lipschitzness of $L(\cdot, \cdot)$ if we assume that the loss function is bounded (e.g. the 0-1 loss) then we can have $V = O(1)$ in Theorem 3.1, making, $\mathfrak{R}(\hat{w}) = O \left( \sqrt{\frac{K}{n} + \mu_+ \sqrt{d_xd_q} \frac{\sqrt{\|I\|}}{n}} \right)$. One should note that in this case the bound on $\mathfrak{R}(\hat{w})$ does not depend on $d_y$, the dimension of the response variable. Moreover, if $K \leq d_x, \mathfrak{R}(\hat{w}) \leq \sqrt{d_x/n}$, which is the classic VC parametric rate. Furthermore, if $d_x$ is kept fixed then $\mathfrak{R}(\hat{w}) = O \left( \sqrt{\frac{K}{n} + \sqrt{\|I\|/n}} \right)$, which is the bound found for the well-known cases of classification (Lecué et al., 2020) and clustering (Paul et al., 2021c).

3.2 Consistency

We now show that the excess risk $\mathfrak{R}(\hat{w})$ converges to 0 in probability. We state the required conditions as follows.

A 7. $K = o(n)$, and $K \to \infty$ as $n \to \infty$.

These conditions are natural: as $n$ grows, so too must $K$ to maintain a proportion of outlier-free partitions. On the other hand, $K$ must grow slowly relative to $n$ to ensure each partition can be assigned sufficient numbers of datapoints. We note that A7 implies $|O| = o(n)$, an intuitive and standard condition (Lecué et al., 2020; Lecué and Lerasle, 2020; Staerman et al., 2021; Paul et al., 2021b) as outliers should be few by definition.

Before we state our consistency result, we note the following corollary of Theorem 3.1. The corollary immediately follows by noting that $\frac{2\min\{q, 1-q\}}{2 + \eta} - \frac{|O|}{K} \geq \frac{\eta q}{(1+\eta)(2+\eta)}$.

Corollary 3.3. Under assumptions A1–6 (or A6* instead of A6), if $0 < q < 1$, with probability at least $1 - e^{-\frac{2\eta q^2}{(2+\eta)^2K}} - e^{-\frac{2\eta (1-q)q}{(2+\eta)^2K}}$, $\mathfrak{R}(\hat{w}) \leq \max \left\{ K^{1/2}n^{-1/2}, n^{-1}\sqrt{\|I\|} \right\}$.

Additionally, if we assume identifiability of the model, we can show that $\hat{w}$ converges to the population minimizer $w^*$. Formally, we state the assumption of identifiability as follows:

A 8. (Identifiability) There exists an $w^*$ such that $\forall \delta > 0$, there exists $\epsilon > 0$, such that $P_{\varphi_{w'}} > P_{f_{w^*}} + \epsilon$, whenever $\|w - w^*\| > \delta$.

Note that A8 implies $w^*$ is the unique minimizer of $P_{\varphi_{w}}$. We now state and prove our consistency result as follows:

Corollary 3.4. Under assumptions A1–7, $\mathfrak{R}(\hat{w}) = O_P \left( \max \left\{ K^{1/2}n^{-1/2}, n^{-1}\sqrt{\|I\|} \right\} \right)$. Moreover, $\mathfrak{R}(\hat{w}) \overset{P}{\to} 0$. Additionally, under A8, $\hat{w} \overset{P}{\to} w^*$. 

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Remark 4. (Comparison to classical bounds) We note that if $|O| \lesssim \log \left( \frac{1}{\delta} \right)$, we can choose $K \asymp \log \left( \frac{1}{\delta} \right)$, from Corollary 3.3, we observe that with probability at least $1 - \delta$,

$$\mathcal{R}(\hat{w}) \lesssim \sqrt{n^{-1} \log \left( \frac{1}{\delta} \right)} + n^{-1/2}$$

(6)

We note that the second term in the above inequality corresponds to the bound on the expectation of an empirical process (i.e. Rademacher complexity in this context; Rademacher complexity usually admits a $O\left(\frac{n}{\sqrt{n}}\right)$ in classical settings). The first term corresponds to the tail of a sub-gaussian process. We note that equation (6) is comparable to Theorems 5 of Bartlett and Mendelson (2002) and Talagrand’s inequality (Boucheron et al., 2013). As opposed to the classical literature on Rademacher complexities, we neither need the assumption of boundedness of the corresponding function nor do we need to assume a sub-gaussian behavior of the error terms (only finite variance suffices).

4 Fast Rates

Linear models, especially regularized ones are well known to admit faster rates of convergence than the ERM rate of $O\left(\frac{1}{\sqrt{n}}\right)$. For example, SVMs (Steinwart and Scovel, 2007), linear models (Sridharan et al., 2008) and more recently functional linear models (Zhang et al., 2020) are all known to achieve an error rate of $O\left(\frac{1}{n}\right)$. It should come as no surprise that MoM estimators also achieve this so-called “fast rate” under additional assumptions. Different conditions can be imposed on the learning problem to achieve such rates among which the popularly used conditions include strong convexity (Sridharan et al., 2008), exponential concavity (Cesa-Bianchi and Lugosi, 2006; Juditsky et al., 2008), Tsybakov margin conditions (Tsybakov, 2004), Bernstein condition (Bartlett and Mendelson, 2006). Such conditions are often referred to as “easiness” conditions because they, intuitively make the learning problem easier, allowing the learner to learn at a faster rate. See (Grünewald and Mehta, 2020) or (Cabannes et al., 2021) for a more detailed review on this topic.

In this paper we will assume that the population loss landscape is strongly convex. Formally, we assume,

A 9. $P\varphi_w$ is $\alpha$-strongly convex in $w$.

For notational simplicity, we write $QOM^0_{q,n,K}(h_w) = Qq(P_1h_w, \ldots, P_Kh_w)$. In this section we will assume that $\inf_w P\varphi_w$ is achieved at $w^*$. Following the proof of Theorem 3.2, one can show the following result.

Theorem 4.1. Suppose A1–6 holds. Then, for any non-negative function-class $\mathcal{G}$ and $q \in (0, 1)$, with probability at least $1 - e^{-2K(\frac{2\eta}{q\eta b} + \frac{|O|}{K})^2}$,

$$\sup_{f \in F} (Pf - QOM^0_{n,K}(f))$$

$$\leq 2\mathcal{V}(\mathcal{G}) \sqrt{\frac{2 + \eta}{q\eta b}} + \frac{8\sqrt{2}(2 + \eta)|J|}{q\eta n\sqrt{\sigma}}\mathcal{R}_{|J|}(\mathcal{G}),$$

(7)

where, $\mathcal{V}(\mathcal{G}) = \sup_{g \in \mathcal{G}}(Pg^2 - (Pg)^2)$ and $J = \{i \in [n] : i \in B_k, k \in [K] \mbox{ and } B_k \cap O = \emptyset\}$.

We consider the new function class $F^*_{W} = \{\varphi_w - \varphi^*_w : w \in W\}$. We also define $F^*_W = \{f_w = \frac{f_w}{\varphi^*_w} : f_w \in F^*_W \mbox{ and } \ell(w) = \min\{\ell \in \mathbb{N} : Pf_w \leq r4^\ell\}\}$ Also let, $\mathcal{V}(\mathcal{F}) = \sup_{\varphi \in \mathcal{F}} \text{Var}(\varphi)$. The following two lemmas put a bound on $\mathcal{V}$ and the Rademacher complexity of $F^*_W$.
Lemma 4.1. Under assumptions A2–5 and A9, we have, $\mathcal{V}(\mathcal{F}_{W,r}^*) \leq \tau \mu^* \sqrt{\frac{2r}{\alpha}}$.

Lemma 4.2. Under assumptions A2–5 and A9, $\mathcal{R}_m(\mathcal{F}_{W,r}^*) \leq 2\tau \mu^* \sqrt{\frac{2r}{\alpha m}}$.

Theorem 4.2 provides a uniform one-sided tail result which plays an instrumental role in deriving the fast rates of MoM estimates. We refer the reader to the supplement for detailed proof of this result.

Theorem 4.2. Under assumptions A1–6 and A9, for any $\beta > 0$ and $0 < q < 1$, the following holds at least with probability $1 - e^{-2K\left(\frac{2q}{\sqrt{n}} - \frac{1}{K}\right)^2}$.

$$\sup_{w \in \mathcal{W}} P(\varphi_w - \varphi_{w^*}) \leq (1 + \beta) \left(QOM^0_n, K(\varphi_w - \varphi_{w^*})\right) + 8\tau^2 \mu^2 \left(1 + \frac{1}{\beta}\right) \left(\frac{2(2 + \eta)}{\alpha q \eta b} + \frac{256(2 + \eta)^2 |\mathcal{I}|}{q^2 \eta^2 n^2 \alpha \sigma}\right).$$

We now focus on the special case of the MoM estimates, i.e. $q = 0$.

Corollary 4.1. Suppose $K$ is odd and $q = 0.5$. Denote $\text{MOM}_{n,K} = QOM^{0.5}_{n,K}$. Under assumptions A1–6 and A9, for any $\beta > 0$ and $0 < q < 1$, the following holds at least with probability $1 - e^{-2K\left(\frac{2q}{\sqrt{n}} - \frac{1}{K}\right)^2}$.

$$\sup_{w \in \mathcal{W}} P(\varphi_w - \varphi_{w^*}) \leq (1 + \beta) \left(\text{MOM}_{n,K}(\varphi_w - \varphi_{w^*})\right) + 8\tau^2 \mu^2 \left(1 + \frac{1}{\beta}\right) \left(\frac{2(2 + \eta)}{\alpha q \eta b} + \frac{256(2 + \eta)^2 |\mathcal{I}|}{q^2 \eta^2 n^2 \alpha \sigma}\right).$$

Corollary 4.1 immediately provides a risk bound for the MoM estimates $\hat{w}$. Note that this result is much stronger than the findings in Theorem 3.1, which only supports a bound of order $\max\left\{K^{1/2}n^{-1/2}, n^{-1}\sqrt{\mathcal{I}}\right\}$.

Corollary 4.2. Suppose $K$ is odd then under assumptions A1–6, and A9, for any $\beta > 0$, $0 < q < 1$, the following holds at least with probability $1 - e^{-2K\left(\frac{2q}{\sqrt{n}} - \frac{1}{K}\right)^2}$.

$$\mathcal{R}(\hat{w}) \leq 8\tau^2 \mu^2 \left(1 + \frac{1}{\beta}\right) \left(\frac{2(2 + \eta)}{\alpha q \eta b} + \frac{256(2 + \eta)^2 |\mathcal{I}|}{q^2 \eta^2 n^2 \alpha \sigma}\right).$$

Remark 5. One should note that similar to its ERM counterpart, MOM estimates also admit so-called fast rates under the mild condition of strong convexity of the population loss landscape. Additionally under A7, we observe that $\mathcal{R}(\hat{w}) = O_P(\max\left\{Kn^{-1}, |\mathcal{I}|n^{-2}\right\})$. Thus, if $|\mathcal{O}| \lesssim \log n$, one can choose $K \asymp \log n$, making the error rate at most $\tilde{O}(n^{-1})$. The assumption of strong convexity is perhaps not surprising. Indeed the connection between $1/n$ error-rates, variance bounds and strong convexity is well understood in the literature (Sridharan et al., 2008; Srebro et al., 2010; Mehta, 2017). It is interesting to note that we do not require any so-called “low noise” assumptions (Audibert and Tsybakov, 2007) on the generative model nor do we enforce the loss function $L(\cdot, \cdot)$ to be strictly convex.
of the estimates under our framework scales with the block size $b$ and the outliers. If the number of partitions overcomes the influence of the outliers, the performance and the effectiveness of the method depends on the intricate interaction between the partitions and $n$, so $|O|$ holds, it is critical that

$$\text{convergence rates than their ERM equivalents.}$$

We stress that there is a shortcoming can also be assessed through break-down point analysis as shown by Rodriguez and Valdora (2019). If $|O| = O(n^\beta)$, for some $0 < \beta < 1$, the error rate is $O(n^{(\beta-1)/2})$.

5 Model Misspecification

The analysis undertaken in the previous section assumes that the model is well specified, the existence of an optimal linear function $f^*(x) = \langle w^*, x \rangle$, such that $f^*$ minimizes $E_{(X,Y) \sim P} L(f(X), Y)$. One should note that if $L(a, b) = |a - b|^2$, $f(X) = \mathbb{E}[Y|X]$. If $L(a, b) = |a - b|$, $f(X)$ is the median of the conditional distribution of $Y|X$.

Thus, in practice the assumption of exact realizability, $f^* \in F_W$, typically does not hold in practice. Recent analyses in this direction, especially in the context of bandits (Zimmert and Seldin, 2019; Foster and Rakhlin, 2020; Lattimore et al., 2020) consider the assumption of uniform misspecification, with respect to $\ell_\infty$-norm, i.e. we say the model is $\epsilon$ uniformly misspecified if

$$\inf_{w \in W} \sup_{x \in \mathbb{R}^d} |\langle w^*, x \rangle - f^*(x)| \leq \epsilon. \quad (8)$$

In this paper, we consider a weaker notion of misspecification. We say that average misspecification of the model is

$$\epsilon(P) = \inf_{w \in W} E_{(X,Y) \sim P} |\langle w, X \rangle - f^*(X)| \quad (9)$$

Observe that this notion of misspecification is similar to that adopted by Foster et al. (2021). $\epsilon(P)$ measures the level of misspecification for the specific distribution $P$ and thus, offers tighter guarantees than uniform misspecification. Note that uniform misspecification in equation (8) implies $\epsilon(P) \leq \epsilon$; while, $\epsilon(P) = 0$ when the model is well-specified. The next theorem bounds the risk under this model of misspecification. Note that we can recover Theorem 3.1 from Theorem 5.1 if the model is well specified.

**Theorem 5.1.** Under assumptions A1-6, with probability at least $1 - e^{-2K} \left( \frac{2a}{n+q} \right)^2$, $\mathbb{R}(\hat{w}) \leq 2V \sqrt{\frac{2(2+\eta)}{q(1-q)\eta b}} + \frac{16B\tau\mu_\ast(2+\eta)\sqrt{f}}{q(1-q)n\sqrt{\sigma}} + \tau \epsilon(P)$.

6 Conclusion

This paper proposed a theoretical framework for robust linear prediction problems that offers a closure or unification of the suite of popular linear models used in the literature. Under our paradigm, we show that one can implement simple first or second-order algorithms that run with the same or lower per iteration complexity than their ERM counterparts. Under mild conditions, we derived uniform concentration bounds and thereby obtain bounds on the risk of our estimator.

As illustrated in our study, the robustness of the estimators comes at the expense of slower convergence rates than their ERM equivalents. We stress that there is no median of means magic, and the effectiveness of the method depends on the intricate interaction between the partitions and the outliers. If the number of partitions overcomes the influence of the outliers, the performance of the estimates under our framework scales with the block size $b$ as $1/\sqrt{b} = \sqrt{K/n}$. Since we can choose $K \propto |O|$ the obtained error rate is about $O(\sqrt{|O|/n})$. However, if $|O|$ grows proportionately with $n$, the error bound of $O(\sqrt{|O|/n})$ becomes meaningless. Thus, for the consistency results to hold, it is critical that $|O| = o(n)$, which then allows us to choose $L$ that satisfies A7. This shortcoming can also be assessed through break-down point analysis as shown by Rodriguez and Valdora (2019). If $|O| = O(n^\beta)$, for some $0 < \beta < 1$, the error rate is $O(n^{(\beta-1)/2})$. 

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Future studies in this area might lead to new avenues for further enhancing rates by identifying “super-fast” rates under further assumptions (Cabannes et al., 2021; Wainwright, 2019). An analysis of min-max lower bounds or high-dimensional sparse robust linear models in our paradigm can also render fruitful avenues.

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A Optimization Results

Before we prove the convergence of $w^{(t)}$ towards $\hat{w}$, we first recall the following standard assumptions from Theorem 2.1.

A 10. 1. The map $w \mapsto L(\langle w, X \rangle, Y)$ is convex in $w$ almost everywhere under $P$.

2. For all $\|\nabla_w L(\langle w, X \rangle, Y)\|_2 \leq C_1$ and $\|\nabla_w F(w)\|_2 \leq C_2$, almost surely under $P$, for some $C_1, C_2 > 0$.

3. For any dataset, $\{(X_i, Y_i)\}_{i \in [n]}$ and any $w \in \mathbb{R}^p$, there exists an open ball $B_r(w)$, centered at $w$ and radius $r > 0$ and $k_{\text{med}} \in \{1, \ldots, K\}$, such that for all $w' \in B_r(w)$, $P_{B_{k_{\text{med}}}} \varphi_{w'} = Q_q(P_1 \varphi_{w'}, \ldots, P_K \varphi_{w'})$.

4. $\sum_{t \geq 1} \epsilon_t^2 < \infty$.

5. $\sum_{t \geq 1} \epsilon_t = \infty$.

6. For any dataset, $\{(X_i, Y_i)\}_{i \in [n]}$ and any $w \in \mathbb{R}^p$ and $r > 0$, there exists $\delta > 0$ such that,

$$\inf_{w \in B_r(w)} \left( w - \hat{w}, \frac{1}{b} \sum_{i \in B_{k_{\text{med}}}} \nabla_w L(\langle w^{(t-1)}, X_i \rangle, Y_i) + \lambda \nabla_w F(w^{(t-1)}) \right) > \delta.$$ 

Assumptions 10(1) and 10(2) consider the convexity of the map and the smoothness of the gradient. For the squared error loss function, if one restricts the support to a compact set, 10(2) holds. Assumption 10(3) ensures that the median partition does not change on a neighborhood of the solutions. Assumptions 10(4) and 10(5) ensure that the step-sizes become small but not too small as to make the algorithm very slow. Assumption 10(6) imposes an identifiability condition to ensure the uniqueness of the global maxima $\hat{w}$ and regularity in a small ball around the same.

Theorem A.1. Under assumption 10, $w^{(t)}$, the iterates of the gradient descent in algorithm 1 converges to the the global optima $\hat{w}$, almost surely.

Proof. From assumption 10(3), we know that for any $t \geq 1$, there exists $r > 0$ such that $w' \in B_r(w^{(t-1)})$, $P_{B_{k_{\text{med}}}} \varphi_{w'} = Q_q(P_1 \varphi_{w'}, \ldots, P_K \varphi_{w'})$. In particular, we observe that for all $w' \in B_r(w^{(t-1)})$,

$$\frac{1}{b} \sum_{i \in B_{k_{\text{med}}}} \nabla_w L(\langle w', X_i \rangle, Y_i) = \nabla_w Q_q(P_1 \varphi_{w'}, \ldots, P_K \varphi_{w'}).$$
Then,
\[
\|w(t) - \hat{w}\|^2_2 = \left\|w(t-1) - \hat{w} - \epsilon_t \left( \frac{1}{b} \sum_{i \in B_{\text{med}}} \nabla wL(\langle w(t-1), X_i \rangle, Y_i) + \lambda \nabla w F(w(t-1)) \right) \right\|^2_2
\]
\[
= \left\|w(t-1) - \hat{w}\|^2_2 - 2\epsilon_t \left( w(t-1) - \hat{w}, \frac{1}{b} \sum_{i \in B_{\text{med}}} \nabla wL(\langle w(t-1), X_i \rangle, Y_i) + \lambda \nabla w F(w(t-1)) \right)
\]
\[
+ \epsilon_t^2 \left\| \frac{1}{b} \sum_{i \in B_{\text{med}}} \nabla wL(\langle w(t-1), X_i \rangle, Y_i) + \lambda \nabla w F(w(t-1)) \right\|^2_2
\]
(10)

We note that,
\[
\left\langle w(t-1) - \hat{w}, \frac{1}{b} \sum_{i \in B_{\text{med}}} \nabla wL(\langle w(t-1), X_i \rangle, Y_i) + \lambda \nabla w F(w(t-1)) \right\>
\]
\[
= \frac{1}{b} \sum_{i \in B_{\text{med}}} \left( \langle w(t-1) - \hat{w}, \nabla wL(\langle w(t-1), X_i \rangle, Y_i) + \lambda \nabla w F(w(t-1)) \rangle \right)
\]
\[
\geq \frac{1}{b} \sum_{i \in B_{\text{med}}} \left( L(\langle w(t-1), X_i \rangle, Y_i) + \lambda F(w(t-1)) - L(\langle \hat{w}, X_i \rangle, Y_i) + \lambda F(\hat{w}) \right)
\]
(11)
\[
\geq 0
\]

Inequality (11) follows from the convexity of \(L(\cdot, X), Y) + \lambda F(\cdot)\). Inequality (12) follows from the definition of \(\hat{w}\). We also note the following,
\[
\left\| \frac{1}{b} \sum_{i \in B_{\text{med}}} \nabla wL(\langle w(t-1), X_i \rangle, Y_i) + \lambda \nabla w F(w(t-1)) \right\|^2_2
\]
\[
\leq \frac{1}{b} \sum_{i \in B_{\text{med}}} \left( \|\nabla wL(\langle w(t-1), X_i \rangle, Y_i)\|_2 + \lambda \|\nabla w F(w(t-1))\|_2 \right)
\]
\[
\leq (C_1 + \lambda C_2).
\]
(13)

Combining equations (10), (12) and (13), we observe that,
\[
\|w(t) - \hat{w}\|^2_2 \leq \|w(t-1) - \hat{w}\|^2_2 + \epsilon_t^2 (C_1 + \lambda C_2)^2.
\]

Let \(a_t = \|w(t) - \hat{w}\|_2^2\), then this implies that \(a_t - a_{t-1} \leq \epsilon_t^2 (C_1 + \lambda C_2)^2\). Thus, for \(n > m\),
\[
a_n - a_m = (a_t - a_{t-1}) \leq (C_1 + \lambda C_2)^2 \sum_{t=m+1}^n \epsilon_t^2,
\]
which can be made small enough owing to assumption 10(4). Thus, the sequence \(\{a_t\} \in \mathbb{N}\) is Cauchy, hence converges to \(a_{\infty}\) (say). Noting that \(\|w(t) - \hat{w}\|_2^2 \geq 0\), we observe from equation (10) and (13),
\[
2\epsilon_t \left( w(t-1) - \hat{w} \right), \frac{1}{b} \sum_{i \in B_{\text{med}}} \nabla wL(\langle w(t-1), X_i \rangle, Y_i) + \lambda \nabla w F(w(t-1)) \right) \leq a_t - a_{t-1} + \epsilon_t^2 (C_1 + \lambda C_2)^2
\]
\[
\Rightarrow 2 \sum_{t=1}^m \epsilon_t \left( w(t-1) - \hat{w} \right), \frac{1}{b} \sum_{i \in B_{\text{med}}} \nabla wL(\langle w(t-1), X_i \rangle, Y_i) + \lambda \nabla w F(w(t-1)) \right) \leq a_m - a_0 + (C_1 + \lambda C_2)^2 \sum_{t=1}^m \epsilon_t^2
\]
Since, $\lim_{m \to \infty} a_m = a_\infty$, taking limit as $m \to \infty$, we get,
\[
2 \limsup_{m \to \infty} \sum_{t=1}^{m} \epsilon_t \left( w^{(t-1)} - \hat{w}, \frac{1}{b} \sum_{i \in B_{k_{med}}} \nabla w L(\langle w^{(t-1)}, X_i, Y_i \rangle + \lambda \nabla w F(w^{(t-1)})) \right) < \infty. \tag{14}
\]

Now suppose that $a_\infty > 0$. Then there exists $N \in \mathbb{N}$ such that $a_m \geq a_\infty/2$, for all $m \geq N$. Thus, for all $m \geq N$, $\|w_m - \hat{w}\|_2 \geq a_\infty/2$. Hence, by assumption 10(6), there exists a $\delta > 0$, such that,
\[
\left\langle w^{(t-1)} - \hat{w}, \frac{1}{b} \sum_{i \in B_{k_{med}}} \nabla w L(\langle w^{(t-1)}, X_i, Y_i \rangle + \lambda \nabla w F(w^{(t-1)})) \right\rangle > \delta.
\]
Hence,
\[
\sum_{m \geq N} \epsilon_t \left( w^{(t-1)} - \hat{w}, \frac{1}{b} \sum_{i \in B_{k_{med}}} \nabla w L(\langle w^{(t-1)}, X_i, Y_i \rangle + \lambda \nabla w F(w^{(t-1)})) \right) \geq \delta \sum_{m \geq N} \epsilon_t = \infty,
\]
by assumption 10(5). This gives us a contradiction to (14).

\[\Box\]

B Proofs from Section 3

Proof of Lemma 3.1

Proof. We begin by observing that $\inf_{w \in \mathcal{W}} F(w) = 0$ implies $F^*(0) = 0$.

Let $E_{n-1}[\cdot | Z_1, \ldots, Z_{n-1}]$ denote the conditional expectation w.r.t. $Z_1, \ldots, Z_{n-1}$. From inequality (5), we get
\[
F^*(S_{n-1} + Z_n)
\]
\[
\leq F^*(S_{n-1}) + \langle \nabla F^*(S_{n-1}), Z_n \rangle + \frac{1}{2\sigma} \|Z_n\|_s^2.
\]
Thus, taking expectation $E_{n-1}$ on both sides, we get,
\[
E_{n-1} F^*(S_{n-1} + Z_n)
\]
\[
\leq F^*(S_{n-1}) + \langle \nabla F^*(S_{n-1}), E Z_n \rangle + \frac{1}{2\sigma} E \|Z_n\|_s^2
\]
\[
= F^*(S_{n-1}) + \frac{1}{2\sigma} E \|Z_n\|_s^2, \tag{15}
\]
which implies that
\[
E_{n-1} \left( F^*(S_n) - \frac{1}{2\sigma} \sum_{j=1}^{n} E \|Z_j\|_s^2 \right) \leq F^*(S_{n-1}) - \frac{1}{2\sigma} \sum_{j=1}^{n-1} E \|Z_j\|_s^2.
\]
Hence, $\{F^*(S_n) - \frac{1}{2\sigma} \sum_{j=1}^{n} E \|Z_j\|_s^2\}_{n \in \mathbb{N}}$ is a supermartingale.

Now to bound $E F^*(S_n)$, we take expectation w.r.t. $Z_1, \ldots, Z_{n-1}$ on both sides of (15) and observe that,
\[
E F^*(S_n) \leq E F^*(S_{n-1}) + \frac{1}{2\sigma} E \|Z_n\|_s^2
\]
\[
\leq F^*(0) + \sum_{j=1}^{n} E \|Z_j\|_s^2
\]
\[
= \sum_{j=1}^{n} E \|Z_j\|_s^2.
\]
Hence the result. \[\Box\]
Proof of Lemma 3.2

Proof. Let $\lambda > 0$ (we will choose $\lambda$ later). By the definition of $F^*$, we observe that,

$$\langle w, \lambda \theta \rangle \leq F(w) + F^*(\lambda \theta).$$

Thus,

$$E_{\epsilon} \sup_{w \in W} \langle w, \theta \rangle \leq \frac{B^2}{\lambda} + \frac{1}{\lambda} E_{\epsilon} F^*(\lambda \theta) \tag{16}$$

We let $Z_i = \frac{\lambda}{n} \epsilon_i X_i$. In the terminology of Lemma 3.1, $S_n = \lambda \theta$. Moreover, $E_{\epsilon} \|Z_i\|^2 \leq \frac{\lambda^2}{n^2} \|X_i\|^2$. Thus, from Lemma 3.1, we get, $E_{\epsilon} F^*(\lambda \theta) = E_{\epsilon} F^*(S_n) \leq \frac{\lambda^2}{2\sigma^2 n^2} \sum_{i=1}^n \|X_i\|^2$. Thus, from equation (16), we get,

$$E_{\epsilon} \sup_{w \in W} \langle w, \theta \rangle \leq \frac{B^2}{\lambda} + \frac{\lambda^2}{2\sigma^2 n^2} \sum_{i=1}^n \|X_i\|^2. \tag{17}$$

Plugging in $\lambda = n \sqrt{\frac{2\sigma B}{\sum_{i=1}^n \|X_i\|^2}}$ in the above bound, we get,

$$E_{\epsilon} \sup_{w \in W} \langle w, \theta \rangle \leq \frac{2B}{n} \sqrt{\frac{1}{2\sigma} \sum_{i=1}^n \|X_i\|^2}. \tag{18}$$

Proof of Corollary 3.1

Proof. We observe that,

$$E \sup_{w \in W} \langle w, \theta \rangle = E E_{\epsilon} \sup_{w \in W} \langle w, \theta \rangle \leq \frac{2B}{n} \sqrt{\frac{1}{2\sigma} \sum_{i=1}^n \|X_i\|^2} \leq \frac{2B}{\sqrt{n}} \sqrt{\frac{1}{2\sigma} \|X_1\|^2}. \tag{17}$$

Here inequality (17) follows from applying Jensen’s inequality.

Proof of Theorem 3.2

Proof. For notational simplicity let $P_k$ denote the empirical distribution of $\{X_i\}_{i \in B_k}$. Suppose $\epsilon > 0$. We note that if

$$\sup_{w \in W} \sum_{k=1}^K \mathbb{1} \{(P - P_k)\varphi_w > \epsilon\} > qK,$$

then

$$\sup_{w \in W} \left( P \varphi_w - R_{n,K,\epsilon}(w) \right) > \epsilon.$$
Here again $\mathbb{1}\{\cdot\}$ denote the indicator function. Now let $\varphi(t) = (t-1)\mathbb{1}\{1 \leq t \leq 2\} + \mathbb{1}\{t > 2\}$. Clearly,

$$\mathbb{1}\{t \geq 2\} \leq \varphi(t) \leq \mathbb{1}\{t \geq 1\}.$$  \hfill (18)

Let $\mathcal{K} \subseteq [L]$ be the set of all partitions which do not contain an outlier. We observe that,

$$\sup_{w \in \mathcal{W}} \sum_{k=1}^{K} \mathbb{1}\{(P - P_k)\varphi_w > \epsilon\} \leq \sup_{w \in \mathcal{W}} \sum_{k \in \mathcal{K}} \mathbb{1}\{(P - P_k)\varphi_w > \epsilon\} + |\mathcal{O}|$$

$$\leq \sup_{w \in \mathcal{W}} \sum_{k \in \mathcal{K}} \varphi\left(\frac{2(P - P_k)\varphi_w}{\epsilon}\right) + |\mathcal{O}|$$

$$\leq \sup_{w \in \mathcal{W}} \sum_{k \in \mathcal{K}} \mathbb{E}[\varphi\left(\frac{2(P - P_k)\varphi_w}{\epsilon}\right)] + |\mathcal{O}|$$

$$+ \sup_{w \in \mathcal{W}} \sum_{k \in \mathcal{K}} \left[\varphi\left(\frac{2(P - P_k)\varphi_w}{\epsilon}\right) - \mathbb{E}[\varphi\left(\frac{2(P - P_k)\varphi_w}{\epsilon}\right)]\right]. \hfill (19)$$

To bound $\sup_{w \in \mathcal{W}} \sum_{k=1}^{K} \mathbb{1}\{(P - P_k)\varphi_w > \epsilon\}$, we will first bound the quantity $\mathbb{E}[\varphi\left(\frac{2(P - P_k)\varphi_w}{\epsilon}\right)]$.

We observe that,

$$\mathbb{E}[\varphi\left(\frac{2(P - P_k)\varphi_w}{\epsilon}\right)] \leq \mathbb{E}\left[\mathbb{1}\left\{\frac{2(P - P_k)\varphi_w}{\epsilon} > 1\right\}\right]$$

$$= \mathbb{P}\left[(P - P_k)\varphi_w > \frac{\epsilon}{2}\right]$$

$$\leq \frac{4V^2}{b\epsilon^2}. \hfill (20)$$

Here $V^2 = \sup_{w \in \mathcal{W}} \mathbb{V}(X, Y) \sim P\mathbb{L}(\langle w, X, Y \rangle)$ We now turn to bounding the term

$$\sup_{w \in \mathcal{W}} \sum_{k \in \mathcal{K}} \left[\varphi\left(\frac{2(P - P_k)\varphi_w}{\epsilon}\right) - \mathbb{E}[\varphi\left(\frac{2(P - P_k)\varphi_w}{\epsilon}\right)]\right].$$

Appealing to Theorem 26.5 of (Shalev-Shwartz and Ben-David, 2014) we observe that, with probability at least $1 - \beta$, for all $w \in \mathcal{W}$,

$$\frac{1}{K} \sum_{k \in \mathcal{K}} \varphi\left(\frac{2(P - P_k)\varphi_w}{\epsilon}\right)$$

$$\leq \mathbb{E}\left[\frac{1}{K} \sum_{k \in \mathcal{K}} \varphi\left(\frac{2(P - P_k)\varphi_w}{\epsilon}\right)\right] + 2\mathbb{E}\left[\sup_{w \in \mathcal{W}} \frac{1}{K} \sum_{k \in \mathcal{K}} \sigma_k \varphi\left(\frac{2(P - P_k)\varphi_w}{\epsilon}\right)\right] + \sqrt{\frac{\log(1/\beta)}{2|\mathcal{K}|}}$$

$$\leq \mathbb{E}\left[\frac{1}{K} \sum_{k \in \mathcal{K}} \varphi\left(\frac{2(P - P_k)\varphi_w}{\epsilon}\right)\right] + 2\mathbb{E}\left[\sup_{w \in \mathcal{W}} \frac{1}{K} \sum_{k \in \mathcal{K}} \sigma_k \varphi\left(\frac{2(P - P_k)\varphi_w}{\epsilon}\right)\right] + \sqrt{\frac{\log(1/\beta)}{2L}}. \hfill (21)$$

Here $\{\sigma_k\}_{k \in \mathcal{K}}$ are i.i.d. Rademacher random variables. Let $\{\xi_i\}_{i=1}^n$ be i.i.d. Rademacher random variables, independent form $\{\sigma_k\}_{k \in \mathcal{K}}$. We take $\delta = \sqrt{\frac{\log(1/\beta)}{2L}}$. Thus, $\beta = \exp\{-2L\delta^2\}$. From equation (21), we get,

$$\frac{1}{K} \sup_{w \in \mathcal{W}} \sum_{k \in \mathcal{K}} \left[\varphi\left(\frac{2(P - P_k)\varphi_w}{\epsilon}\right) - \mathbb{E}[\varphi\left(\frac{2(P - P_k)\varphi_w}{\epsilon}\right)]\right] \leq 2\mathbb{E}\left[\sup_{w \in \mathcal{W}} \frac{1}{K} \sum_{k \in \mathcal{K}} \sigma_k \varphi\left(\frac{2(P - P_k)\varphi_w}{\epsilon}\right)\right] + \delta$$
In equation (24),

\[
\mathbb{E} \left[ \sup_{w \in W} \sum_{k \in K} \sigma_k (P - P_k) \varphi_w \right] \leq \frac{4}{K \epsilon} \mathbb{E} \left[ \sup_{w \in W} \sum_{k \in K} \sigma_k (P - P_k) \varphi_w \right] + \delta. \tag{22}
\]

Equation (22) follows from the fact that \( \varphi(\cdot) \) is 1-Lipschitz and appealing to Lemma 26.9 of Shalev-Shwartz and Ben-David (2014). We now consider a “ghost” sample \( X' = \{(X'_i, Y'_i)\}_{i \in [n]} \), which are i.i.d. and follow the probability law \( P \). Thus, the first term of equation (22) can be further shown to give

\[
\frac{4}{K \epsilon} \mathbb{E} \left[ \sup_{w \in W} \sum_{k \in K} \sigma_k (P'_{B_k} - P_k) \varphi_w \right] \leq \frac{4}{K \epsilon} \mathbb{E} \left[ \sup_{w \in W} \sum_{k \in K} \sigma_k (P'_B - P_k) \varphi_w \right] \leq \frac{4}{K \epsilon} \mathbb{E} \left[ \sup_{w \in W} \sum_{k \in K} \sigma_k (P'_B - P_k) \varphi_w \right]
\]

Equation (23) follows from observing that \((\varphi_w(X'_i, Y'_i) - \varphi_w(X_i, Y_i)) \leq 4/K \). Thus, combining equations (19), (20), and (27), we conclude that, with probability of at least \( 1 - e^{-2Kd^2} \),

\[
\mathbb{E} \left[ \sup_{w \in W} \sum_{k \in K} \sigma_k (P - P_k) \varphi_w \right] \leq \frac{4}{K \epsilon} \mathbb{E} \left[ \sup_{w \in W} \sum_{k \in K} \sigma_k (P - P_k) \varphi_w \right] + \delta.
\]

Equation (23) follows from observing that \((\varphi_w(X'_i, Y'_i) - \varphi_w(X_i, Y_i)) \leq 4/K \). Thus, combining equations (19), (20), and (27), we conclude that, with probability of at least \( 1 - e^{-2Kd^2} \),

\[
\mathbb{E} \left[ \sup_{w \in W} \sum_{k \in K} \sigma_k (P - P_k) \varphi_w \right] \leq \frac{4}{K \epsilon} \mathbb{E} \left[ \sup_{w \in W} \sum_{k \in K} \sigma_k (P - P_k) \varphi_w \right] + \delta.
\]
We choose $\delta = \frac{2q}{2+q} - \frac{|O|}{\kappa}$ and
\[
\epsilon = \sqrt{2 \left( \frac{4V^2(2 + \eta)}{2q(1 - q)b} + \frac{16B\tau\mu*(2 + \eta)\sqrt{|I|}}{q(1 - q)\eta b} \right) - 2V \left( \frac{(2 + \eta)}{q(1 - q)\eta b} + \frac{16B\tau\mu*(2 + \eta)\sqrt{|I|}}{q(1 - q)\eta \sqrt{\sigma}} \right)}.
\]

Here we used the fact that $\frac{a}{a+b} + \frac{b^2}{(a+b)^2} \leq 1$. These choices of $\delta$ and $\epsilon$ makes the RHS of (28) smaller than $q$.

In essence, we have shown that $P\left( \sup_{w \in W} (P\varphi_w - R_{n,K,q}(w)) > \epsilon \right) \leq e^{-2K\delta^2}$.

Similarly, we can show (appealing to the fact that $Q_q(\xi) = Q_{1-q}(-\xi)$),
\[
P\left( \sup_{w \in W} (R_{n,K,q}(w) - P\varphi_w) > \epsilon' \right) \leq e^{-2K(\delta')^2},
\]
where, $\delta' = \frac{2(1-q)}{2+q} - \frac{|O|}{\kappa}$ and
\[
\epsilon' = 2 \sqrt{\frac{V^2(2 + \eta)}{(1 - q)\eta b} + \frac{16B\tau\mu*(2 + \eta)\sqrt{|I|}}{(1 - q)\eta \sqrt{\sigma}}}
\]

\[\square\]

**Proof of Theorem 3.1**

Proof. Let $w^*_m$ be such that, $P\varphi_{w^*_m} \leq \inf_{w \in W} P\varphi_w + \frac{1}{m}$. We begin by observing that
\[
\mathcal{R}(\bar{w}) = P\varphi_{\bar{w}} - \inf_{w \in W} P\varphi_w
\leq P\varphi_{\bar{w}} - P\varphi_{w^*_m} + m^{-1}
= P\varphi_{\bar{w}} - R_{n,K,q}(\bar{w}) + R_{n,K,q}(\bar{w}) - P\varphi_{w^*_m} + m^{-1}
\leq P\varphi_{\bar{w}} - R_{n,K,q}(\bar{w}) + R_{n,K,q}(w^*_m) - P\varphi_{w^*_m} + m^{-1}
\leq \sup_{w \in W} (P\varphi_w - R_{n,K,q}(w)) + \sup_{w \in W} (P\varphi_w - R_{n,K,q}(w^*_m)) + m^{-1}
\leq 2V \sqrt{\frac{2(2 + \eta)}{q(1 - q)b} + \frac{16B\tau\mu*(2 + \eta)\sqrt{|I|}}{q(1 - q)\eta \sqrt{\sigma}}} + m^{-1}
\]

with probability at least $1 - e^{-2K\left( \frac{2q}{2+q} - \frac{|O|}{\kappa} \right)^2} - e^{-2K\left( \frac{2(1-q)}{2+q} - \frac{|O|}{\kappa} \right)^2}$, by Theorem 3.2. Now taking $m \uparrow \infty$ gives us the desired result. \[\square\]

**Proof of Lemma 3.3**

Proof. We begin by noting that $|L(\langle w, X \rangle, Y) - L(0, 0)| \leq \tau |\langle w, X \rangle|$. Thus,
\[
\sup_{w \in W} \mathbb{E} \left( L(\langle w, X \rangle, Y) \right)^2 \leq 2\tau^2 \sup_{w \in W} \mathbb{E} |\langle w, X \rangle|^2 + 2\mathbb{E}L^2(0, Y)
\leq 2\tau^2 \mathbb{E} \sup_{w \in W} (\langle w, X \rangle)^2 + 2\mathbb{E}L^2(0, Y)
\]

(29)

24
\[\leq 2\tau^2 \mathbb{E} \left( \sup_{w \in \mathcal{W}} \langle w, X \rangle \right)^2 + 2\mathbb{E} L^2(0, Y) \tag{30}\]

\[\leq 2\tau^2 \mathbb{E} \left( \sup_{w \in \mathcal{W}} F(w) + F^*(X) \right)^2 + 2\mathbb{E} L^2(0, Y) \tag{31}\]

\[\leq 2\tau^2 \mathbb{E} \left( B + F^*(0) + \langle \nabla F^*(0), X \rangle + \|X\|_2^2 \right)^2 + 2\mathbb{E} L^2(0, Y) \tag{32}\]

\[\leq 2\tau^2 \mathbb{E} \left( B + \|\nabla F^*(0)\| \|X\|_* + \|X\|_2^2 \right)^2 + 2\mathbb{E} L^2(0, Y) \tag{33}\]

\[\leq 4\tau^2 \mathbb{E} \left( B^2 + \|\nabla F^*(0)\| \|X\|_* + \|X\|_2^2 \right)^2 + 2\mathbb{E} L^2(0, Y) \tag{34}\]

\[\leq 4\tau^2 \mathbb{E} \left( B^2 + 2\|\nabla F^*(0)\| \|X\|_* + 2\|X\|_2^2 \right) + 2\mathbb{E} L^2(0, Y) \tag{35}\]

\[= 4\tau^2 \left( B^2 + 2\|\nabla F^*(0)\| \|X\|_* + 2\|X\|_2^2 \right) + 2\mathbb{E} L^2(0, Y) \tag{36}\]

\[< \infty.\]

Inequality (29) follows from A2. Equation (30) follows from the fact that \( \mathcal{W} \) is symmetric (since \( F(\cdot) \) is symmetric). Inequality (31) follows from appealing the identity \( 0 \leq \sup_{w \in \mathcal{W}} \langle w, x \rangle \leq F(w) + F^*(x) \) (this is due to the definition of \( F^* \)). Equation (32) follows from (5). Equations (33) follows from observing that \( F^*(0) = 0 \) and the rest follows from the fact that \( (a + b)^2 \leq 2(a^2 + b^2) \).

\[\text{Proof of Corollary 3.3}\]

\[\text{Proof.} \quad \text{We note that } \frac{2\eta}{2+\eta} - \frac{|\mathcal{O}|}{K} \geq \frac{\eta}{(1+\eta)(2+\eta)} \text{ and appealing to Theorem 3.1.} \]

\[\text{Proof of Corollary 3.4}\]

\[\text{Proof.} \quad \text{We note that under A7, } \exp \left\{ -\frac{2\eta^2 a^2 K}{(1+\eta)^2(2+\eta)^2} \right\}, \exp \left\{ -\frac{2\eta^2 (1-\eta)^2 K}{(1+\eta)^2(2+\eta)^2} \right\} = o(1). \text{ Thus, from Corollary 3.2, } \mathbb{P} \left( \mathfrak{N}(\hat{w}) = O \left( \max \left\{ K^{1/2} n^{-1/2}, n^{-1} \sqrt{|I|} \right\} \right) \right) = 1 - o(1). \text{ Hence, }\]

\[\mathfrak{R}(\hat{w}) = O_P \left( \max \left\{ K^{1/2} n^{-1/2}, n^{-1} \sqrt{|I|} \right\} \right). \]

Moreover, from Corollary 3.3, \( \mathfrak{N}(\hat{w}) \leq K \max \left\{ \sqrt{K} n, \frac{\sqrt{|I|}}{n} \right\} \leq \max \left\{ \sqrt{K} n, \frac{1}{\sqrt{n}} \right\} \to 0, \text{ by A7.} \]

Thus, \( \mathfrak{R}(\hat{w}) \xrightarrow{P} 0. \) Now, for any \( \epsilon, \delta > 0, \mathbb{P}(P \varphi_{\hat{w}} \leq P \varphi_w + \epsilon) \geq 1 - \delta, \) if \( n \) is large. From assumption A8, \( \mathbb{P}(\|\hat{w} - w^*\| \leq \epsilon) \geq 1 - \delta \text{ for any prefixed } \eta > 0, \text{ and } n \text{ large.} \) Thus, \( \|\hat{w} - w^*\| \xrightarrow{P} 0, \) which proves the result.

\[\text{Proofs from Section 4}\]

\[\text{Proof of Theorem 4.1} \quad \text{Theorem 4.1 essentially follows from the proof of Theorem 3.2 and noticing in this case, } V \text{ is replaced by } V(G). \text{ In that proof one also needs to keep the expression involving the Rademacher complexity as opposed to plugging it in as done in the proof of Theorem 3.2.}\]

\[\text{Proof of Lemma 4.1}\]
Proof. Consider the new function class $\mathcal{F}_W^* = \{ \varphi_w - \varphi_w^* : w \in W \}$. We also define $\mathcal{F}_{W,r}^* = \{ f_w = \frac{f_w}{\ell(w)} : f_w \in \mathcal{F}_W^* \}$ and $\ell(w) = \min \{ \ell \in \mathbb{N} : P f_w \leq r^4 \}$ and $H(b) = \{ f_w \in \mathcal{F}_W^* : P f_w \leq b \}$.

\[
V^2(\mathcal{F}_{W,r}^*) \leq \frac{1}{16 \ell(w)^2} \sup_w \mathbb{E}(\varphi_w(X,Y) - \varphi_w^*(X,Y))^2
\]

\[
\leq \frac{\tau^2}{16 \ell(w)^2} \sup_w \mathbb{E}((\mathbf{w} - \mathbf{w}^*, \mathbf{X})^2)
\]

\[
\leq \frac{\tau^2}{16 \ell(w)^2} \sup_w \|\mathbf{w} - \mathbf{w}^*\|^2 \mathbb{E}\|\mathbf{X}\|^2
\]

\[
\leq \frac{2\tau^2}{\alpha 16 \ell(w)} \sup_w P(\varphi_w - \varphi_w^*) \mu_*^2
\]

\[
= \frac{2\tau^2 \mu_*^2}{\alpha 16 \ell(w)} \sup_{f_w \in H(4\ell(w)r)} Pf_w
\]

\[
= \frac{2\tau^2 \mu_*^2}{\alpha 16 \ell(w)} 4\ell(w)r
\]

\[
\leq \frac{2\tau^2 \mu_*^2}{\alpha r}
\]

Proof of Lemma 4.2

Proof. Recall from Lemma 3.2 that

\[
\mathcal{R}_m(\{ x \mapsto \langle \mathbf{w}, \mathbf{x} \rangle : t(\mathbf{w}) \leq b \}) \leq \frac{2\sqrt{b} \mu_*}{\sqrt{2\alpha m}},
\]

for any $\alpha$-strongly convex function $t(\cdot)$. We take $t(\mathbf{w}) = P(\varphi_w - \varphi_w^*)$, which is clearly $\alpha$-strongly convex. From the Lipschitz composition property of Rademacher complexity, $\mathcal{R}_m(H(b)) \leq \frac{2\tau^2 \mu_*}{\sqrt{2\alpha m}}$.

Now observe the following:

\[
\mathcal{R}_m(\mathcal{F}_{W,r}^*) = \mathcal{R}_m(\bigcup_{j=0}^{\infty} 4^{-j} H(4^j r))
\]

\[
\leq \sum_{j=0}^{\infty} 4^{-j} \mathcal{R}_m(H(4^j r))
\]

\[
\leq \sum_{j=0}^{\infty} 4^{-j} \frac{2\tau 4^{j/2} \sqrt{r} \mu_*}{\sqrt{2\alpha m}}
\]

\[
= 2\tau \mu_* \sqrt{\frac{2r}{\alpha m}}.
\]

Proof of Theorem 4.2

Proof. Let $f_w = \varphi_w - \varphi_w^*$. From Lemmas 4.1 and 4.2, we observe that at least with probability $1 - e^{-2K (\frac{2}{2 + \eta} - \frac{\eta}{K})^2}$, the following holds for all $f_w \in \mathcal{F}_W^*$

\[
Pf_w - \text{QOM}_n^q(f_w) \leq 4\ell(w) (P f_w - \text{QOM}_n^q(f_w))
\]

\[
\leq 4\ell(w) \sqrt{r} D,
\]

(41)
where \( D = 2\tau\mu_* \left( \sqrt{\frac{2(2+\eta)}{\alpha q q_b}} + \frac{16(2+\eta)\sqrt{|J|}}{q\eta\sqrt{\alpha\sigma}} \right) \leq 2\tau\mu_* \left( \sqrt{\frac{2(2+\eta)}{\alpha q q_b}} + \frac{16(2+\eta)\sqrt{|J|}}{q\eta\sqrt{\alpha\sigma}} \right) \). We now consider two possibilities.

If \( \ell(w) = 0 \), from equation (41), we get,

\[
P_{f_w} \leq QOM^q_{n,K}(f_w) + \sqrt{\tau}D.
\]  

(42)

Otherwise if \( \ell(w) > 0 \), we know that \( 4\ell(w) - 1 < P_{f_w} \). Substituting this into (41), we get,

\[
P_{f_w} - QOM^q_{n,K}(f_w) \leq \frac{4}{\tau} P_{f_w},
\]

which in turn gives us,

\[
P_{f_w} \leq \frac{1}{1 - 4D/\sqrt{\tau}} (QOM^q_{n,K}(f_w))_+ + \sqrt{\tau}D.
\]  

(43)

One requires \( r > (4D)^2 \). Now combining equations (42) and (43), we observe,

\[
P_{f_w} \leq \frac{1}{1 - 4D/\sqrt{\tau}} (QOM^q_{n,K}(f_w))_+ + \sqrt{\tau}D.
\]  

(44)

Now taking \( r = \left( 1 + \frac{1}{\beta} \right)^2 (4D)^2 \) gives us the desired bound. \( \square \)

**Proof of Corollary 4.1**

Proof. Let \( A(\hat{w}, w^*) = \{ k \in [K] : P_k(\hat{w}) \leq P_k(w^*) \} \). By definition of \( \hat{w} \), \( |A(\hat{w}, w^*)| > \frac{K}{2} \). Thus, on \( A(\hat{w}, w^*) \), \( P_k(\hat{w} - w^*) \geq P_k(\varphi_w - \varphi_{w^*}) \). Hence, \( \text{MOM}_{n,K}(\varphi_w - \varphi_{w^*}) \leq \text{MOM}_{n,K}(\varphi_{w} - \varphi_{w^*}) \). The result now immediately follows from Theorem 4.2. \( \square \)

**D Proofs from Section 5**

**Proof of Theorem 5.1**

Proof. We observe that,

\[
P_{\varphi_{\hat{w}}} - PL(f^*(x), y) = P_{\varphi_{\hat{w}}} - \inf_{w \in W} P_{\varphi_{w}} + \inf_{w \in W} P_{\varphi_{w}} - PL(f^*(x), y)
\]

\[
\leq P_{\varphi_{\hat{w}}} - \inf_{w \in W} P_{\varphi_{w}}
\]

\[
+ \inf_{w \in W} P(L(\langle w, x \rangle, y) - L(f^*(x), y))
\]

\[
\leq P_{\varphi_{\hat{w}}} - \inf_{w \in W} P_{\varphi_{w}} + \tau \inf_{w \in W} P(\langle w, x \rangle - f^*(x))
\]

\[
\leq P_{\varphi_{\hat{w}}} - \inf_{w \in W} P_{\varphi_{w}} + \tau \epsilon(P)
\]

\[
\leq 2V \sqrt{\frac{2(2+\eta)}{q(1-q)\eta b}} + \frac{16B\tau\mu_*(2+\eta)\sqrt{|J|}}{q(1-q)\eta}) + \tau \epsilon(P),
\]

with probability at least \( 1 - e^{-2K \left( \frac{2}{2+\eta} - \frac{|O|}{\pi} \right)^2} - e^{-2K \left( \frac{2(1-q)}{2+\eta} - \frac{|O|}{\pi} \right)^2} \), by Theorem 3.1. Inequality (45) follows from A5. \( \square \)
E A Simulation Study

In this section, we validate our algorithmic performance as well as the bound via simulation studies. In this particular study, we show this result for logistic regression.

The data generation procedure is as follows. For the inliers, first, we generate 2 classes with an equal number of points from Gaussian random variables with mean all 1’s and all −1’s and we took the variance-covariance matrix to be 0.1 on each diagonal entry and 0 on the off-diagonals. Now we took β₀, the true coefficients to be the vector of all 1’s. Now, for the outliers, we generate \( n^β \) many points from Gaussian distribution with mean 0.5 and variance 100, where \( β \) is taken to be 0.3. The outlying observations are assigned to any of the two classes at random. Now we take \( n \) as \( 2^m \) and vary \( m \) from 5 to 16, so that \( n \) varies from 32 to 65536.

Now we run both Median-of-Means logistic regression and the vanilla logistic regression on each of the generated datasets and repeat the experiment 20 times for each \( n \). We take the average value of the squared Euclidean distance between the obtained and the true coefficients.

The average training error corresponding to the number of outliers both in logarithmic scales are shown in figure 1. It can be observed that the error has a linearly decreasing trend with an increasing number of datapoints.