ON VLASENKO’S FORMAL GROUP LAWS

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Abstract. Given a Laurent polynomial over a ring flat over \( \mathbb{Z} \), Vlasenko defines a formal group law. We identify this formal group law with a coordinate system of a formal group functor, prove its integrality. When the “Hasse–Witt matrix” of the Laurent polynomial is invertible, Vlasenko defines a matrix by taking a certain \( p \)-adic limit. We show that this matrix is the Frobenius of the Dieudonné module of this formal group modulo \( p \).

Introduction

On a flat \( \mathbb{Z} \)-algebra \( R \), given a Laurent polynomial \( f \) with coefficients in \( R \), Vlasenko [14, 15] introduced a formal group law

\[
F_f \quad (\text{a priori, } F_f \text{ is only defined over } R \otimes \mathbb{Q})
\]

using the coefficients of the Laurent polynomials \( f^n, n = 1, 2, \ldots \). While being very explicit and in some sense canonical, the definition of \( F_f \) constitutes some notation. Thus we invite the reader to consult §1.1 for the precise definition. In the two papers cited above, Vlasenko studied \( F_f \) and deduced many arithmetic consequences that can be stated without mentioning \( F_f \).

In this short note, we shall report our understanding to these formal group laws. Two topics will be discussed. Each topic constitutes a section in the main text. Let us give a brief overview of our results.

(a) Relation with geometry of hypersurfaces in toric varieties. The first task is to report the integrality of \( F_f \) and its relation with Artin–Mazur formal group functors [1]. Using toric geometry, we prove the following result.

Theorem 0.1. Let \( R \) be a flat \( \mathbb{Z} \)-algebra.

(1) (= Corollary 1.17) If \( R \) is noetherian, the formal group law \( F_f \) is a coordinate system of an Artin–Mazur type formal group functor (1.8) on \( R \).

(2) (= Theorem 1.4, also a theorem of Vlasenko) Without assuming \( R \) to be noetherian, the formal group law \( F_f \) is a power series with coefficients in \( R \).

The proof of Theorem 0.1(1) is a replica of Stienstra’s article [13] with some inputs from toric geometry. Item (2) will be a consequence of Item (1).

Some comments on Theorem 0.1(2) are in order.

- Beukers and Vlasenko [4] proved Theorem 0.1(2) when \( R \) is \( p \)-adically complete and the Hasse–Witt matrix (see (2.1)) of \( f \) is invertible.
• If $p$ is a prime number, and if $R$ admits a lift of the absolute Frobenius operator of $R/p$, Vlasenko [14] proves that $F_f$ has coefficients in $R(p)$. In particular, Vlasenko is able to prove Theorem 0.1.2 for polynomial rings $R = \mathbb{Z}[x_1, \ldots, x_N]$. In fact, this is already enough to prove Theorem 0.1.2 because we can prove the integrality of a “universal” case. We shall give an alternative treatment, and deduce Theorem 0.1.2 as a consequence of Theorem 0.1.1. Due to our use of toric geometry, the integrality result obtained is not as general as Vlasenko’s, see the footnote for (1.1).

The relation with Artin–Mazur functors was noted by Vlasenko [14] when the toric scheme defined by the Newton polytope of $f$ is a projective space. However, our result shows that $F_f$ should rather be thought as a coordinate system of the formal group functor associated with the ideal sheaf of the hypersurface defined by $f$, instead of the Artin–Mazur formal group functor of the hypersurface itself. See Remark 0.2 below.

**Remark 0.2** (We do not need the flatness of the hypersurface defined by $f$). The Laurent polynomial $f$ defines a hypersurface $X$ in a suitable toric scheme $\mathbb{P}$ over $R$. In [13], $f$ corresponds to a hypersurface in $\mathbb{P}^n$. Stienstra imposed a running flatness hypothesis on this hypersurface. In our result, we do not need the flatness of the hypersurface defined by $f$. The reason is that in our theorem we do not identify the formal group $F_f$ with the Artin–Mazur formal group of the hypersurface $X$, but with a variant of Artin–Mazur formal group attached to its ideal sheaf (which is an invertible sheaf, always flat over $R$). In fact, this point already occurred in Stienstra’s article [loc. cit.]. See Lemma 1.13 and §1.16.

When $X$ is flat over $R$, and when $R = W(k)$ is the ring of Witt vectors of a perfect field of characteristic $p > 0$, the isogeny class of the Cartier–Dieudonné module of the reduction of $F_f$ mod $p$ is the slope < 1 part of the rigid cohomology group (with proper support) $H^{\dim P_k}_{\rig,c}(P_k - X_k)_{<1}$. See Remark 2.15. Therefore the explicit formal group law allows us to extract the information of the slope < 1 part of the Newton polygon of the rigid cohomology.

At the end of §1, we revisit a theorem of Honda concerning formal group laws from hypergeometric equations. We explain why his formal group law is only integral over $\mathbb{Z}_{(p)}$ for large $p$, by relating his equations with Picard–Fuchs equations of “underdiagram deformations”. The formal integrals of some special power series solutions to the latter ordinary differential equations are logarithms of Vlasenko’s group laws.

(b) Higher Hasse–Witt matrices and Frobenius operators. Vlasenko [14] considered the matrices

$$(\alpha_s)_{u,v \in \Delta \cap \mathbb{Z}^d} = \text{the coefficient of } t^{p^u - v} \text{ in } f^{p^{u-1}}$$

over a $p$-adically complete torsion free ring $R$. These matrices were called “higher Hasse–Witt matrices” by Vlasenko. If $\alpha_s$ are all invertible modulo $p$, then Vlasenko proves the $p$-adic limit

$$\alpha = \lim_{s \to \infty} \alpha_{s+1}(\alpha_s^{-1})$$

exists. Vlasenko asked [loc. cit.] whether $\alpha$ is a Frobenius matrix acting on some crystal (which she did not specify). This question is confirmed by [4] Remark 5.4.
using what they call the “Dwork crystal” (with a very mild constraint on the coefficients of \(f\)). Huang–Lian–Yau–Yu \([10]\) also studied this question, and they are able to answer Vlasenko’s question assuming \(\Delta\) is a smooth, very ample polytope and \(f\) defines a smooth hypersurface in the toric variety defined by \(\Delta\).

In Section 2 we give an alternative answer to Vlasenko’s question (without constraints on \(\Delta\) or \(f\)). Let \(\Gamma_f\) be the mod \(p\) reduction of the formal group \(\Phi_f\) mentioned in Theorem 0.1. If \(\alpha_1\) is invertible mod \(p\), we shall show that the Dieudonné module of \(\Gamma_f\) is isoclinic of slope \(0\) (2.11), and the \(p\)-adic limit matrix \(\alpha\) is the Frobenius matrix of the Dieudonné module of \(\Gamma_f\).

**Theorem 0.3 (= Theorem 2.14).** Assume that \(R\) is \(p\)-adically complete flat \(\mathbb{Z}\)-algebra. Let \(D^\ast(\Gamma_f)\) be the (covariant) Dieudonné crystal of \(\Gamma_f\) on \(R/p\). Assume that the matrix \(\alpha_1\) (see above) is invertible. Then \(\alpha\) is a matrix of the Frobenius operation of \(D^\ast(\Gamma_f)_R\).

Here, we view \(\text{Spf}\ R\) as an inductive system divided power thickening of \(R/p\), i.e., an ind-object in the big crystalline site \(\text{CRIS}(\text{Spec}(\mathbb{Z}_p)/\mathbb{Z}_p)\), and \(D^\ast(\Gamma_f)_R\) is the Zariski sheaf on \(\text{Spf}\ R\) defined by the crystal \(D^\ast(\Gamma_f)\) (via taking limit).

When the hypersurface \(X\) defined by \(f\) in a toric scheme is flat over \(R\) (without assuming \(\alpha_1\) invertible), as mentioned in Remark 0.2 above, the Dieudonné module of \(\Gamma_f\) gives the slope < 1 part of the hypersurface defined by \(f\). See also Remark 2.15.

The significance of Theorem 0.3 is that when \(\alpha_1\) is invertible, we have a purely combinatorial way to extract the unit root part of (the rigid cohomology of) the reduction of \(X \to \text{Spec}(R)\), even when the general fibers are singular.

**Acknowledgments.** Professor Masha Vlasenko sent me a list of suggestions and corrections, and clarified some of my misconceptions. I would like to thank her for her invaluable help.

I am also grateful to Tsung-Ju Lee, for his suggestions on Example 1.22 to Shizhang Li, for pointing out how to use \(\delta\)-rings in the proof of Theorem 2.14 to Qixiao Ma and Luochen Zhao, for discussions on formal groups; to Chenglong Yu, for answering my questions about his paper; and to Jie Zhou, for discussions on GKZ systems.

1. **Integrality of Vlasenko’s formal group laws**

1.1. **Notation and conventions.** In this section we fix the following notation and conventions. Let \(R\) be a commutative ring flat over \(\mathbb{Z}\). Let \(f(t) = \sum_{u \in \mathbb{Z}^d} a_u t^u \in R[t_1, \ldots, t_d, (t_1 \cdots t_d)^{-1}]\) be a Laurent polynomial with coefficients in \(R\). Let \(\Delta\) be the Newton polytope of \(f\). Recall that \(\Delta\) is the convex hull in \(\mathbb{R}^d\) of \(\{w \in \mathbb{Z}^d : a_w \neq 0\}\).

We shall assume that the dimension of \(\Delta\) equals \(d\), and that the interior \(\Delta^\circ\) of \(\Delta\) contains at least one lattice point.

\footnote{Vlasenko pointed out to me that her integrality proof does not require \(\Delta\) to be full dimensional as we assumed here. Thus our integrality proof is not as general as hers. Assuming her integrality theorem, the results in \((\text{2.11})\) can go through for an arbitrary \(\Delta\), except Remark 2.15 which requires the relation with algebraic geometry.}
Following Vlasenko, we define, for \( v, w \in \Delta^0 \cap \mathbb{Z}^d \),

\[
L_{v,w}(\tau) = \sum_{\nu} \beta_{v,w,\nu} \frac{\tau^\nu}{\nu!},
\]

where \( \beta_{v,w,\nu} \in R \) equals the coefficient of \( t^\nu w^{-v} \) in the expansion of \( f(t)^{v-1} \). We define

\[
L_v(\tau_w : w \in \Delta^0 \cap \mathbb{Z}^d) = \sum_{w \in \Delta^0 \cap \mathbb{Z}^d} L_{v,w}(\tau_w).
\]

Then

\[
L = (L_v : v \in \Delta^0 \cap \mathbb{Z}^d) \in (R \otimes \mathbb{Q})[\tau_w : w \in \Delta^0 \cap \mathbb{Z}^d]^N
\]

where \( N = \# \Delta^0 \cap \mathbb{Z}^d \). Finally, we define an \( N \)-dimensional formal group law on \( R \otimes \mathbb{Z} \mathbb{Q} \) by

\[
F_f(x, y) = L^{-1}(L(x) + L(y)) \in (R \otimes \mathbb{Q})[x, y],
\]

Here

\[
x = (x_w : w \in \Delta^0 \cap \mathbb{Z}^d), \quad y = (y_w : w \in \Delta^0 \cap \mathbb{Z}^d).
\]

M. Vlasenko \cite[Theorem 2]{14} proves that if the Frobenius operator of \( R/p \) lifts to \( R \), then \( F_f(x, y) \in R[x, y] \). In fact, one can use the argument in Remark \ref{rem:1.5} below to prove that \( F \in R[x, y] \) without assuming Frobenii can be lifted. However, we give a different argument without using Vlasenko’s theorem, and prove the integrality using a different argument based on formal group functors.

**Theorem 1.4.** Let notation and conventions be as in \ref{1.1}. Then \( F_f(x, y) \in R[x, y] \).

**Remark 1.5** (We can assume \( R \) is noetherian). The coefficients of \( f \) generates a finitely generated subring \( R' \) of \( R \). The coefficient expansions used in the definition are all contained in \( R' \), and the coefficients of the series \( L_{v,w}(\tau) \) are then in the power series \( (R' \otimes \mathbb{Q})[\tau] \), and \( F_f(x, y) \) lies in \( (R' \otimes \mathbb{Q})[x, y] \). If we replace \( R \) by \( R' \), and we can prove \( F_f \in R'[x, y] \), then we automatically get \( F_f \in R[x, y] \). Thus, it suffices to prove the theorem for \( R' \). The virtue of \( R' \) is that it is a quotient of a polynomial algebra over \( \mathbb{Z} \) with finitely many variables, hence is a noetherian ring.

As noted in \cite{14}, when the Laurent polynomial is of a special form, the formal group law \( F_f \) provides a coordinate system to the Artin–Mazur formal group of a hypersurface in a projective space. Our proof of Theorem \ref{thm:1.4} is based on this observation: using the method of J. Stienstra \cite{13}, we show \( F_f \) is a coordinate system of a formal group functor related to a hypersurface in a toric scheme.

The proof goes as follows.

1. Construct a toric scheme over \( R \) and a relatively ample divisor \((\bar{f} = 0)\) using \( \Delta \) and \( f \) \ref{1.10}.
2. Prove the formal group functor defined by the ideal sheaf of \((\bar{f} = 0)\) is a formal Lie group \ref{1.13}.
3. Using Čech cohomology, find an explicit logarithm of this formal Lie group over \( R \otimes \mathbb{Q} \) \ref{1.15}.
4. Prove that this formal logarithm agrees with the one in \ref{1.1} \ref{1.16}.
While Theorem 1.4 assumes $R$ to be a flat $\mathbb{Z}$-algebra, some of the results needed in the proof (e.g., the smoothness of a certain formal group functor) are valid over an arbitrary ring. Thus in the sequel we will be careful about the hypotheses.

1.6. We begin by recalling the notion of formal group functors. Let $R$ be a ring, $A$ (necessarily non-unital) $R$-algebra $A$ is a nil $R$-algebra if for any $a \in A$, there exists $r \geq 0$, such that $a^r = 0$. Let $\text{Nil}_R$ be the category of nil $R$-algebras. Let $A_i$ be a formal group on $A$ (necessarily non-unital) $R$-algebra with multiplication $(a_i : i \in I) \cdot (a'_i : i \in I) = (a_ia'_i : i \in I)$ ($a_i, a'_i \in A_i$, and only finitely many $a_i \neq 0$).

1.7. A commutative formal group functor, or simply a formal group functor, or simply a formal group (in this note, all formal groups are assumed to be commutative), on a ring $R$ is a functor $\Phi : \text{Nil}_R \to \text{Mod}_\mathbb{Z}$. Usually one imposes some exactness conditions such as the functor is asked to preserve direct sums, or to be “exact”. We shall not impose these conditions. In Lemma 1.9 we will establish an exactness property that we need later.

The simplest formal group functor is the formal additive group

$$\hat{G}_a : \text{Nil}_R \to \text{Mod}_\mathbb{Z}, \quad A \mapsto (A,+)$$

which simply forgets the multiplication on $A$.

We say a formal group $\Phi$ is a formal Lie group if it is naturally isomorphic, as set valued functors, to some $\hat{G}_a^n$. An isomorphism of set-valued functors $\hat{G}_a^n \to \Phi$ is called a coordinate system of $\Phi$. The group structure on $\Phi$ defines, by transport of structures, a group structure on the ideal $(x_1, \ldots, x_n)$ of the ring of power series $R[x_1, \ldots, x_n]$ as the ideal $(x_1, \ldots, x_n)$ is an inverse limit of nil $R$-algebras. This gives rise to a power series $F(x,y)$ subject to the axioms of a formal group law. Therefore, a formal group law is simply equivalent to a choice of a coordinate system on a formal Lie group.

One important example of a formal Lie group is the formal multiplicative group, notation $\hat{G}_m$, whose group law is given by the polynomial $L(x,y) = 1 - (1 - x)(1 - y) = x + y - xy$, which is the coordinate expansion of the usual multiplication at 1. The functorial definition of the formal multiplicative group is to send a nil algebra $A$ over $R$ to the multiplicative group $(A,*)$, where for $a,b \in A$, $a*b = a + b - ab$.

1.8. Let $\mathcal{I}$ be a sheaf of (possibly non-unital) $R$-algebras on an $R$-scheme $S$. Let $F$ be a formal group on $R$. Then for each nil algebra $A$, we can define a new sheaf of abelian groups by sheafifying the following presheaf

$$F(\mathcal{I}_R A) : U \mapsto F(\mathcal{I}(U)_R A).$$

Taking the $i$th cohomology of this sheaf yields a formal group. When $F = \hat{G}_m$, the above formal group is denoted by $\Phi^i(X,\mathcal{I})$, thus:

$$\Phi^i(X,\mathcal{I}) : A \mapsto H^i(X,\hat{G}_m(\mathcal{I}_R A)).$$

This is a variant of the deformation functor considered by Artin–Mazur [1]. We shall call $\Phi^i(X,\mathcal{I})$ the Artin–Mazur formal group functor associated with $\mathcal{I}$.

Each $R$-module $M$ defines a nil algebra subject to the condition $m_1 \cdot m_2 = 0$ for all $m_1, m_2 \in M$. Note that for such a nil algebra we have $\hat{G}_a(M) = \hat{G}_m(M)$. The
restriction of a formal functor to the subcategory of \( R \)-modules defines a functor called the \textit{tangent space} \( \Phi \). If \( M \) is an \( R \)-module viewed as a nil algebra, then

\[
\Phi^i(X, \mathcal{I})(M) = H^i(X, \hat{\mathcal{G}}_m(\mathcal{I} \otimes_R M)) = H^i(X, \mathcal{I} \otimes_R M).
\]

**Lemma 1.9.** Let \( R \) be a noetherian ring. Let \( X \) be a finite type scheme over \( R \). Let \( \mathcal{I} \) be a coherent ideal sheaf of \( \mathcal{O}_X \), flat over \( R \). Assume further that for any ideal \( J \) of \( R \), \( H^{i-1}(X \otimes_R R/J, \mathcal{I}) = 0 \). Then \( \Phi^i(X, \mathcal{I}) \) is a left exact functor. That is, for any exact sequence \( 0 \to N_1 \to N_2 \to N_3 \to 0 \) of nil algebras over \( R \) (exact as \( R \)-modules),

\[
0 \to \Phi^i(X, \mathcal{I})(N_1) \to \Phi^i(X, \mathcal{I})(N_2) \to \Phi^i(X, \mathcal{I})(N_3)
\]
is exact.

**Proof.** Since \( \mathcal{I} \) is flat over \( R \), the sequence

\[
0 \to \mathcal{I} \otimes_R N_1 \to \mathcal{I} \otimes_R N_2 \to \mathcal{I} \otimes_R N_3 \to 0
\]
is exact. Applying \( \hat{\mathcal{G}}_m \), we get the exact sequence

\[
0 \to \hat{\mathcal{G}}_m(\mathcal{I} \otimes_R N_1) \to \hat{\mathcal{G}}_m(\mathcal{I} \otimes_R N_2) \to \hat{\mathcal{G}}_m(\mathcal{I} \otimes_R N_3) \to 0.
\]

Applying cohomology groups, we get an exact sequence

\[
\Phi^{i-1}(X, \mathcal{I})(N_3) \to \Phi^i(X, \mathcal{I})(N_1) \to \Phi^i(X, \mathcal{I})(N_2) \to \Phi^i(X, \mathcal{I})(N_3).
\]

Therefore, it suffices to prove \( \Phi^{i-1}(X, \mathcal{I})(N) = 0 \) for any nil algebra \( N \). Since \( N \) is a filtered colimit of finitely generated nil algebras, and since (on a noetherian topological space) taking the Zariski cohomology group of sheaves of abelian groups commutes with filtered colimits, it suffices to assume that \( N \) is a finitely generated nil algebra. Each such algebra \( N \) fits into a sequence

\[
N = N_0 \to N_1 \to \cdots \to N_r = 0
\]
in which \( N_j \to N_{j+1} \) is surjective, and the kernel is generated by a single element \( \epsilon \), such that \( \epsilon^2 = 0 \). Therefore, by using induction on \( r \), and using the exact sequences of cohomology, the vanishing of \( \Phi^{i-1}(X, \mathcal{I})(N) \) follows from the vanishing of \( \Phi^{i-1}(X, \mathcal{I})(Re) \), where \( \epsilon^2 = 0 \). Let \( J \) be the annihilator of \( \epsilon \). Then (recall (1.8) that \( \hat{\mathcal{G}}_m(M) = \hat{\mathcal{G}}_n(M) \) if \( M \) is an \( R \)-module viewed as a nil algebra)

\[
\Phi^{i-1}(X, \mathcal{I})(Re) = H^{i-1}(X, \mathcal{I} \otimes_R Re) = H^{i-1}(X \otimes_R R/J, \mathcal{I}) = 0.
\]

This completes the proof. \( \square \)

Next, we recall some toric geometry that we need. Our reference is \([7]\). This reference treats only complex toric varieties. But the parts related to fans, polytopes, and combinatorial description of sheaf cohomology groups are also valid over \( \mathbb{Z} \) and over any ring. The part on vanishing theorems work for any algebraically closed field, and hence the vanishing over a ring follows from an easy base change argument.

**1.10.** Let \( R \) be an arbitrary ring. Let \( f \in R[t_1, \ldots, t_d, (t_1 \cdots t_d)^{-1}] \) be a Laurent polynomial. Let \( \Delta \subset \mathbb{Z}^d := M \) be the Newton polytope of \( f \). Let \( \Sigma \subset N = M^\vee \) be the normal fan of \( \Delta \). Let \( \Sigma(1) \) be the set of 1-cones of \( \Sigma \). We set up the following notation (see \([7]\), Chapter 5) for more about the Cox ring and homogeneous coordinates).
(1) \( z_\rho : A^{\Sigma(1)} \to A^1 \) is the coordinate function with respect to the 1-cone \( \rho \).

(2) For a cone \( \sigma \in \Sigma \), \( \hat{z}_\sigma = \prod_{\rho \in \sigma} z_\rho \).

(3) \( Z(\Sigma) = \text{Zeros}\{ \hat{z}_\sigma : \sigma \in \Sigma \} \subset A^{\Sigma(1)} \).

(4) \( U(\Sigma) = A^{\Sigma(1)} \setminus Z(\Sigma) \).

(5) \( D(\Sigma) \) is the algebraic torus associated with \( \text{Cl}(\Sigma) \) defined by the exact sequence
\[
0 \to M \to \mathbb{Z}^{\Sigma(1)} \to \text{Cl}(\Sigma) \to 0.
\]

(6) \( P_\Sigma \) is the toric variety associated with the fan \( \Sigma \), thus we have a \( D(\Sigma) \)-torsor \( \pi : U(\Sigma) \to P_\Sigma \).

(7) \( S = S(\Sigma) = R[z_\rho : \rho \in \Sigma(1)] \) is the “Cox ring” of \( P_\Sigma \).

It is a well-known result in toric geometry. One can obtain it by applying Serre duality to [7, Proposition 5.4.1]. Since its proof is needed in the proof of Lemma 1.13 below, we feel obliged to sketch the proof.

**Lemma 1.11.** Notation be as in \[1.10\]. Let \( Y \) be an effective Cartier divisor of \( P_\Sigma \) whose divisor class is \( \beta \in \text{Cl}(\Sigma) \). Then there is an isomorphism
\[
\left( \prod_{\rho \in \Sigma(1)} R[z_\rho^{-1} : \rho \in \Sigma(1)] \right)_{-\beta} \cong H^d(P_\Sigma; \mathcal{O}_{P_\Sigma}(-Y)).
\]

where \( \tilde{f} \in S \) is the defining equation of \( Y \).

**Proof.** Let \( U_\sigma = \{ z \in A^{\Sigma(1)} : \hat{z}_\sigma \neq 0 \} \). Then \( U_\sigma \) is stable under the action of \( D^*_\sigma \), and \( U_{\sigma \cap \sigma'} = U_\sigma \cap U_{\sigma'} \). Let \( V_\sigma \) the image of \( U_\sigma \) in \( P_\Sigma \). Then
\[
(1.12) \quad U = \{ V_\sigma : \sigma \text{ maximal cone in } \Sigma \}
\]
is an open covering of \( P_\sigma \).

Let \( \mathcal{F} \) be a quasi-coherent sheaf on \( P_\Sigma \). Then the cohomology of \( \mathcal{F} \) can be computed using the (alternating) Čech cohomology with respect to the covering \( U \).

The Čech complex is
\[
\prod_{\sigma \text{ maximal}} \mathcal{F}(V_\sigma) \to \prod_{\sigma \text{ codim 1}} \mathcal{F}(V_\sigma) \to \cdots \to \prod_{\rho \in \Sigma(1)} \mathcal{F}(V_\rho) \to \mathcal{F}(T)
\]
where \( T \) is the embedded torus.

In our case, \( \mathcal{F} = \mathcal{O}_{P_\Sigma}(-\beta) \). Then \( \mathcal{F}(V_\sigma) = (S[\hat{z}_\sigma^{-1}])_{-\beta} \) (the homogeneous piece of degree \( -\beta \)). The proposition then follows from a straightforward computation.

□

Now we construct an explicit coordinate system for \( \Phi^d(P_\Sigma; \mathcal{O}_{P_\Sigma}(-X)) \).

**Lemma 1.13.** Notation be as in \[1.10\]. Assume that \( R \) is noetherian. The formal group \( \Phi^d(P_\Sigma; \mathcal{O}_{P_\Sigma}(-X)) \) is a formal Lie group.

**Proof.** Following Stienstra, we prove this by constructing a coordinate. Let \( \tilde{f} \) be the homogeneous equation that cuts out \( X \).
We shall use the Čech cohomology to define an isomorphism (as functor of sets) between $\Phi^d(P_\Sigma, \mathcal{O}_{P_\Sigma}(-X))$ and the $N$-fold self-product of the formal additive group, where $N$ is the $R$-rank of

$$\left(\prod_{\rho \in \Sigma(1)} z_{\rho}^{-1} : \rho \in \Sigma(1)\right)^{-\beta}.$$  

To organize the combinatorics, for each function $m : \Sigma(1) \to \mathbb{Z}_{\geq 1}$, write

$$z^m = \prod_{\rho \in \Sigma(1)} z_{\rho}^{m(\rho)}.$$

Then an element in (1.14) is written as

$$\sum_{m} \lambda_m z^m.$$

Thus for each nil $R$-algebra $A$, we identify $A^N$ with the set of $N$-uples $(a_m)$. Then we define a map

$$A^N \to \widehat{G}(\mathcal{O}_{P_\Sigma}(-X) \otimes_R A)(T)$$

sending $(a_m)$ to

$$\sum_{m} \frac{\widehat{f}}{z^m} \otimes a_m$$

the summation being taken using the group structure of

$$\widehat{G}(\mathcal{O}_{P_\Sigma}(-X) \otimes_R A)(T).$$

We claim the composition

$$A^N \to \widehat{G}(\mathcal{O}_{P_\Sigma}(-X) \otimes_R A)(T) \to \Phi^d(P_\Sigma, \mathcal{O}_{P_\Sigma}(-X))(A)$$

is an isomorphism. Here, the second arrow is to take the cohomology class via the Čech complex used in the proof of Lemma 1.11. The theorem then follows from this claim.

Indeed, Lemma 1.11 shows that this morphism (of set-valued functors on nil algebras) induces an isomorphism on the tangent space. Since the functor $A \mapsto A^N$ is smooth and exact, the claim then follows from [16, Theorem 2.30] in view of Lemma 1.9. The vanishing needed in Lemma 1.9 is ensured by the Batyrev–Borisov vanishing theorem [7, Theorem 9.2.7]. Since we have started with a polytope $\Delta$, and the divisor $X$ is linearly equivalent to the divisor $D_{\Delta}$ described in [7, (4.2.7)], the amplitude needed in the vanishing is ensured by [7, Proposition 6.1.10(a)]. Although [7, Theorem 9.2.7] as stated requires the toric variety to be defined over the field of complex numbers, its proof is combinatorial and works for any field. The result over a base ring then follows from the theorem on cohomology and base change.

1.15. Construction of logarithm. In this paragraph, we provide an explicit isomorphism between $\Phi^d(P_\Sigma, \mathcal{O}_{P_\Sigma}(-X))$ and a product of additive groups over $R \otimes \mathbb{Q}$, following the method of Stienstra. For this purpose, we could replace $R$ by the $\mathbb{Q}$-algebra $R \otimes \mathbb{Q}$. Thus, in this paragraph, we will assume $R$ is a noetherian $\mathbb{Q}$-algebra.
Recall that the usual logarithm defines, for each nil algebra $A$ over $R$, an isomorphism of abelian groups

$$\ell (A) : \hat{\mathbb{G}}_m(A) \to \hat{\mathbb{G}}_a(A), \quad a \mapsto \sum_{n=1}^{\infty} \frac{1}{n} a^n$$

We shall use $\ell_A$ to define an explicit isomorphism between $\Phi^d(P_{\Sigma}, \mathcal{O}_{P_{\Sigma}}(-X))$ and a product of formal additive group.

We have the following commutative diagram

$$\begin{array}{ccc}
\hat{\mathbb{G}}_m(\mathcal{O}_{P_{\Sigma}}(-X) \otimes_R A)(T) & \xrightarrow{\ell(A)} & \hat{\mathbb{G}}_a(\mathcal{O}_{P_{\Sigma}}(-X) \otimes_R A)(T) \\
A^N \xrightarrow{\ell_A} \Phi^d(P_{\Sigma}, \mathcal{O}_{P_{\Sigma}}(-X))(A) & \xrightarrow{\ell_X(A)} & \hat{\mathbb{G}}_a^N(A)
\end{array}$$

In the diagram, the right square is a diagram of abelian groups, whereas the left triangle is a diagram of sets. The vertical maps are “taking the cohomology class” of Čech cocycles. The map $\ell_X(A)$ is induced by the logarithms on the Čech cochains with respect to the covering $\{(1.12)$ after taking cohomology. It is an isomorphism since $\ell(A)$ induces chain level isomorphisms. The map $u$ is the coordinate we constructed in the proof of Lemma 1.13. The composition $\ell_X \circ u$ is then the “coordinate representation” of the formal logarithm of $\Phi^d(P_{\Sigma}, \mathcal{O}_{P_{\Sigma}}(-X))$.

Let $L_X = \ell_X \circ u$. By chasing the diagram, for each $(a_m)$ as in the proof of Lemma 1.13 we have

$$L_X(a_m) = \text{the cohomology class of } \sum_{\nu : \Sigma(1) \to \mathbb{Z}_{\geq 1}} \sum_{\nu, w} \beta_{\nu, w, \nu} a_{w}^{\nu}.$$ 

To spell out the “cohomology class”, we use Lemma 1.11. We should only look at those monomials in the expansions $\tilde{f}^{\nu-1}$ which has the following properties

- has degree $\beta(\nu - 1)$, thus $w$ must be of degree $\beta$,
- part of the monomial “cancels” the denominator $z^{\nu w}$, and
- the rest part of the monomial produces a monomial $z^{-\nu}$ for some $v$ of degree $\beta$.

Thus, to get the terms with contributions, we define, for an integer $\nu$, and for maps $v, w : \Sigma(1) \to \mathbb{Z}_{\geq 1}$, $\deg(v) = \deg(w) = \beta$.

$$\beta_{v, w, \nu} = \text{coefficient of } z^{\nu w - \nu} \text{ in the expansion of } \tilde{f}^{\nu-1}.$$ 

Then for each $v$ of degree $\beta$, the contribution of $f/z^v$ is given by $\sum_{\nu, w} \beta_{v, w, \nu} a_{w}^{\nu} / \nu$. Therefore we can write the logarithm as

$$L_X (a_w) = (L_{v}(a_w)),$$

where

$$L_{v}(a_w : v : \Sigma(1) \to \mathbb{Z}_{\geq 1}) = \sum_{\nu=1}^{\infty} \beta_{v, w, \nu} a_{w}^{\nu} / \nu$$

The formal group law for the formal group we constructed is therefore

$$F(X, Y) = L_X^{-1}(L_X(X_v) + L_X(Y_w))$$

which is a matrix of formal power series in $X_v, Y_w$. 

1.16. In this paragraph we finish the proof of Theorem 1.4. By Remark 1.5, we can assume \( R \) is a noetherian ring flat over \( \mathbb{Z} \). In this case, we shall prove the formal logarithms defined in 1.15 agrees with the series (1.2).

Recall that we begin with the Newton polytope \( \Delta \) of the Laurent polynomial \( f \). The normal fan \( \Sigma \) of \( \Delta \) defines \( P_\Sigma \) and the lattice gives rise to a torus invariant relative Cartier divisor \( D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho \) \([7, \text{Equation (4.2.7)}]\). The lattice \( \Delta \) can be recovered from the numbers \( a_\rho \), \( \Delta = \{ w \in \mathbb{R}^d : \langle w, u_\rho \rangle \geq -a_\rho, \forall \rho \in \Sigma(1) \} \), by \([7, \text{Proposition 6.1.10, Theorem 6.2.1, Exercise 4.3.1}]\), and the Laurent polynomial can be viewed as a section of the Cartier divisor \( O_{P_\Sigma}(D) \). Moreover, \( H^0(P_\Sigma, O_{\Sigma}(D)) \cong R^{\Delta \cap \mathbb{Z}^d} \), by \([7, \text{Example 4.3.7}]\). Thus the information about \( \Delta \) is equivalent to the information about \( \Sigma \) and the relative Cartier divisor \( D \).

Next, write the Laurent polynomial \( f \) into \( \sum_{u \in \Delta \cap \mathbb{Z}^d} x_u t^u \). Then the Laurent monomial \( t^u \) corresponds to the monomial \( z^{\langle u, D \rangle} = \prod_{\rho \in \Sigma(1)} z^{\langle u, u_\rho \rangle + a_\rho} \) where \( u_\rho \) is the smallest generator of the ray \( \rho \in \Sigma(1) \) in the dual space of \( \mathbb{R}^d \). Thus \( f \) gives rise to an element \( \bar{f}(z) = \sum x_u z^{\langle u, D \rangle} \) in the Cox ring \( S \). It is homogeneous of degree \( \beta = |D| \) in \( S \), and defines an effective Cartier divisor \( X \) on \( P_\Sigma \). Moreover, the correspondence \( f \leftrightarrow \bar{f} \) establishes a bijection \( H^0(P_\Sigma, O_{\Sigma}(D)) \cong S_\beta \), see \([7, \text{Proposition 5.4.1(b)}]\). Under this correspondence, the expansion coefficients in \( \bar{f}^m \) are the same as the expansion coefficients of \( f^m \) since for the monomial \( w : \Sigma(1) \to \mathbb{Z}_{\geq 1} \), homogeneous of degree \( \beta \), \( z^w \) corresponds to \( t^w \) for some \( w \in \Delta \cap \mathbb{Z}^d \). This finishes the proof of Theorem 1.4. \( \square \)

**Corollary 1.17** (of the proof of Theorem 1.4). Assume that \( R \) is noetherian. The formal group law \( F_f \) considered by Vlasenko \([13]\) is a coordinate system of the formal group functor \( \Phi^{d}(P_{\Sigma}, G_{\mathbb{m}}(O_{P_{\Sigma}}(\mathbb{S}(-X)))) \).

From now on, we shall use \( \Phi_f \) to denote the formal Lie group over a ring \( R \) determined by the formal group law \( F_f \).

The integrality of \( \Phi_f \) could be used to explain the integrality of some formal group laws considered by Professor T. Honda \([9]\). Let \( N \) be an integer. Honda considered the generalized hypergeometric ordinary differential equation

\[
(\tau^N \prod_{\theta \in S} (\delta + N\theta) - \delta[S]) g(\tau) = 0.
\]
where \( \delta = \tau \partial_{\tau} \), \( S \) is a subset of \( \{ 1/N, \ldots, (N-1)/N \} \). Let \( g(\tau) = \sum_{n \geq 0} A(n) \tau^{Nn} \) be the generalized hypergeometric function which is the only solution to (1.18) at \( 0 \). Set \( f(x) = \int_0^x g(\tau) d\tau \).

**Theorem 1.19 (Honda [9])**. Suppose that \( \{ N \theta : \theta \in S \} \) contains all the reduced residues mod \( N \), then \( F(x,y) = f^{-1}(f(x) + f(y)) \), a priori a rational power series, actually lies in \( \mathbb{Z}_p[x,y] \), for every \( p > N \).

Let us indicate how Honda’s integrality is related to the integrality of \( F_f \).

1.20. In the sequel, we assume \( R \) is flat over \( \mathbb{Z} \), and assume we have fixed an embedding of \( R \otimes \mathbb{Q} \) into \( \mathbb{C} \). Let \( f \) be a Laurent polynomial with Newton polytope \( \Delta \).

For each \( w \in \Delta^\circ \cap \mathbb{Z}^d \), consider the 1-parameter family of hypersurfaces in the \( P_\Sigma \)

\[ X_w(\tau) = \text{Zeros}(t^w + \tau f(t)) \subset P_\Sigma, \]

called an “underdiagram deformation”. This deformation then determines a “Picard–Fuchs system” on \( A^\Sigma_1 \). By definition, a differential operator is said to be a Picard–Fuchs operator if it annihilates the cohomology class of a differential form on the generic \( X_w \). The Picard–Fuchs system is the cyclic \( \mathcal{O} \)-module obtained by dividing the left ideal generated by Picard–Fuchs operators.

The relation between solutions of differential equations and the formal group \( \Phi_f \) is based on the following trivial observation.

**Lemma 1.21.** In the situation above, the formal power series

\[ \sum_{\nu=1}^{\infty} \beta_{v,w,\nu} \tau^{\nu-1} \]

(the derivative of a series appears in the formal logarithms of \( F_f \) considered in [11, 18] is a formal power series solution to the Picard–Fuchs system around \( \tau = 0 \).)

**Proof.** For each \( v \in \Delta^\circ \cap \mathbb{Z}^d \), we have a standard volume form on the torus given by

\[ \Theta_v = t^v \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_d}{t_d}. \]

Then we know the cohomology class (in the logarithmic cohomology of the complement of \( t^w + \tau f = 0 \))

\[ \Theta_v / t^w + \tau f(t) \]

(or the residue of it) satisfies the Picard–Fuchs equation by fiat. We then integrate the form along the standard top homology cycle of \( T \) to get an expansion with respect to \( \tau \). This is equivalent to taking the degree 0 term of the fraction

\[
t^{v-w}(1 + \tau t^{-w} f(t))^{-1} = t^{v-w} \sum_{\nu=1}^{\infty} \tau^{\nu-1} t^{-(\nu-1)w} f(t)^{\nu-1} \]

\[
= t^{v-w} \sum_{\nu=1}^{\infty} \tau^{\nu-1} t^{-(\nu-1)w} \sum_{w_1, \ldots, w_{\nu-1}} x_{w_1} \cdots x_{w_{\nu-1}} t^{w_1 + \cdots + w_{\nu-1}}. 
\]
Hence the constant term is
\[
\sum_{\nu} \sum_{w_1, \ldots, w_{\nu-1}} \frac{x_{w_1} \cdots x_{w_{\nu-1}} \tau^{\nu-1}}{\sum_{w_i=1}^{\nu w - w}} = \sum_{\nu=1}^{\infty} \beta_{\nu, w, \nu} \tau^{\nu-1}.
\]
as desired. \(\square\)

It is not very easy to produce the precise ordinary differential equations directly from the Laurent polynomial \(f\) we start with. Relatively easier is to produce ordinary differential equations whose solutions contain the “period integrals” using the so-called GKZ systems \([8]\).

In the following example, the GKZ system is simple enough so that we can easily get the Picard–Fuchs equations out of them.

**Example 1.22.** We consider the polytope \(\Delta\) generated by (minimal) lattice points \(u_1, \ldots, u_n\) in \(\mathbb{Z}^n\) subject to the only relation
\[
\sum_{i=1}^{n} q_i u_i = 0.
\]
Note that \(\Delta\) is not the polytope of a weighted projective space. It is its face fan that defines the weighted projective space \(\mathbb{P}(q_1, \ldots, q_n)\). Let \(N = \sum q_i\). We assume that \(q_i | N\) so that \(\Delta\) is a reflexive polytope. We shall consider the GKZ system associated with the matrix
\[
A = \begin{bmatrix}
1 & 1 & 1 & \cdots & 1 \\
0 & u_1 & u_1 & \cdots & u_n \\
\end{bmatrix}.
\]
The GKZ system is a \(\mathcal{A}\)-module on the “space of coefficients” which is the affine space \(\mathbb{A}^{n+1}\) with coordinate system \((a_0, a_1, \ldots, a_n)\). Each point \((a_0, \ldots, a_n)\) corresponds to a Laurent polynomial \(a_0 + \sum_{i=1}^{n} a_i t^{u_i}\). To get the desired Picard–Fuchs equation, we shall descend the “modified box operator”
\[
\square a_0^{-1} = \partial_0^N a_0^{-1} - \prod_{i=1}^{n} \partial_i^{q_i} a_0^{-1}
\]
via \(\tau^N = \prod_{i=1}^{n} a_i^{q_i} / a_0^N\), to the underdiagram deformation \(1 + \tau \sum_{i=1}^{n} t^{u_i}\) \((a_0^{-1} \tau^N\) is killed by all the “Euler operators” in the GKZ system with parameter \((-1, 0, \ldots, 0)\)). By computation we get
\[
\mathcal{L}_{\text{GKZ}} = (-\tau)^N \prod_{i=1}^{n} (\delta + i) - \prod_{i=1}^{n} q_i \prod_{j=0}^{q_i-1} \left( \frac{q_i}{N} \delta - j \right) \quad (\delta = \tau \partial_\tau).
\]
Using the commuting relation \(\tau^N \delta = \delta \tau^N - N \tau^N\), we have a factorization \(\mathcal{L}_{\text{GKZ}} = P(\delta) \mathcal{L}_{\text{PF}}\).

We thus get an ordinary differential equation \(\mathcal{L}_{\text{PF}} g(\tau) = 0\). Sometimes, after scaling the variable \(\tau \to \pm N \tau\), \(\mathcal{L}_{\text{PF}}\) changes to a differential operator considered by Honda. In these situations, by Lemma 1.21 above, the unique power series solution of \(\mathcal{L}_{\text{PF}} g(\tau) = 0\) should give rise to an integral formal group law (Theorem 1.4), and
will imply Honda’s integrality, while Honda’s formal group law is only integral in \( \mathbb{Z}_{(p)} \) for \( p \nmid N \) due to the scaling of \( \tau \).

We give two special cases illustrating our point. The first is to consider the Laurent polynomial \( f(t) = \sum_{i=1}^{d-1} t_i + \frac{1}{t_1 \cdots t_{d-1}} \), which corresponds to the case \((q_1, \ldots, q_d) = (1, 1, \ldots, 1)\). Then the only underdiagram deformation is given by \( 1 + \tau f(t) \). From the GKZ system we can infer an ordinary differential equation

\[
\mathcal{L}_{\text{GKZ}} = (-d)^d \tau^d \prod_{i=1}^{d} (\delta + i) - \delta^d = \delta \cdot ((-d)^d(\delta - 1) \cdots (\delta - d + 1)\tau^d - \delta^{d-1}).
\]

This differential operator has a factor

\[
\mathcal{L}_{PF} = (-d)^d \tau^d(\delta + 1) \cdots (\delta + d - 1) - \delta^{d-1}.
\]

This is almost of Honda’s type. The difference is a constant factor. The related equation considered by Honda is obtained from \( \mathcal{L}_{PF} g(\tau) = 0 \) by making a substitution \( \tau \leftrightarrow (-d)\tau \). Therefore the related Honda’s formal group law is only \( \mathbb{Z}_{(p)} \)-integral for those \( p \nmid d \).

As another example, consider \( N = 4 \), and \((q_1, q_2, q_3) = (1, 1, 2)\). The above method produces the operator

\[
\mathcal{L}_{PF} = 4^4 \tau^4(\delta + 1)(\delta + 3) - \delta^2.
\]

which is related to Honda’s under the correspondence \( \tau \leftrightarrow 4\tau \).

2. Recovering unit-root Frobenius when Hasse–Witt is invertible

Let \( p \) be a prime number. Let \( R, f, \Delta, \text{ and } \Sigma \) be as in \([1, 10]\). We assume in addition that \( R \) admits a lift \( \sigma \) of the absolute Frobenius of \( R/p \). Let us consider

\[
(\alpha_s)_{u,v} \in \Delta_{\text{red}} \mathbb{Z}^d = \text{the coefficient of } t^{p^u} u \text{ in } f^{p^v-1}.
\]

In \([14]\), these matrices were called “higher Hasse–Witt matrices” of \( f \). Vlasenko proved the following result concerning these matrices.

**Proposition 2.2** (Vlasenko [loc. cit.]). Let notation be as above. Then we have

1. For every \( s \geq 1 \), \( \alpha_s \equiv \alpha_1 \cdot \alpha_1^{\sigma} \cdots \alpha_1^{\sigma^{s-1}} \mod p \).
2. Assume \( \alpha_1 \) (hence any of the \( \alpha_s \)) is invertible over \( R \), and \( R \) is \( p \)-adically complete. Then \( \alpha_{s+1}(\alpha_s^\sigma)^{-1} \equiv \alpha_s(\alpha_{s-1}^\sigma)^{-1} \mod p^s \).

In the situation of Proposition 2.2(2), the \( p \)-adic limit

\[
\alpha = \lim_{s \to \infty} \alpha_{s+1}(\alpha_s^\sigma)^{-1}
\]

exists. Vlasenko conjectured in [loc. cit.] that \( \alpha \) is a matrix of some Frobenius operation on some \( F \)-crystal. Under a very mild assumption on the coefficients of \( f \), Beukers–Vlasenko [4] Remark 2.5] identifies \( \alpha \) with a matrix of the Frobenius operation of their “Dwork crystal”. Assuming the hypersurface \( X \) we mentioned in Section 2 is smooth over the base, the ambient toric variety is smooth, and a technical condition on \( \Delta \), the paper of Huang–Lian–Yau–Yu [10] identifies \( \alpha \) with the Frobenius of the unit root part of the relative crystalline cohomology of the family.

The purpose of this section is to prove that, under the assumption of Proposition 2.2(2), \( \alpha \) is a matrix of the Frobenius operation of the (covariant) Dieudonné
crystal of the formal group $\Gamma_f$, the reduction of $\Phi_f$ modulo $p$. At the end of this section (Remark 2.15) we explain how to relate the Dieudonné crystal with the geometric isocrystal associated with the hypersurfaces in the toric variety defined by $\Delta$.

2.4 (An overview of Cartier’s theory of curves). We need the some basic facts in Cartier’s curve theory. A thorough reference is Lazard’s book [12]. We shall follow Lazard’s conventions and notation, and explain some part of the theory we find necessary to understand the rest of the note.

1. A curve on a formal Lie group $G$ is simply a morphism (of set-valued functors) $\gamma$ from $\hat{G}$ into $G$. If $R$ is a ring of characteristic 0, so that $R$ embeds in to $R \otimes \mathbb{Q}$, then we can represent a curve using the formal logarithm $\log_G : G_{R \otimes \mathbb{Q}} \to \mathbb{G}_a^r$, thus identify a curve with a power series $\log_G \gamma(t) = \sum_{i=1}^{\infty} a_i t^i$ where $a_i \in R^r \otimes \mathbb{Q}$.

2. The tangent space of $G$ could be regarded as a free $R$-module. A basic set of curves of $G$ is a collection of curves $\gamma_1, \ldots, \gamma_r$ on $G$ such that their tangent vectors form a basis of the tangent space of $G$. If $\ast$ is the group operation of $G$, the morphism (of set valued functors) $\hat{G}^r \to G$, $(t_1, \ldots, t_r) \mapsto \gamma_1(t_1) \ast \cdots \ast \gamma_r(t_r)$ is an isomorphism, and it provides a coordinate system of $G$. The group law under this coordinate system is called a curvilinear formal group law.

Suppose that $R$ is a $\mathbb{Q}$-algebra. Then in terms of the notation above, we have

$$(\log_G \circ \sigma)(t_1, \ldots, t_r) = \sum_{i=1}^{r} \log_G \gamma_i(t_i)$$

is literally the sum of formal power series.

3. Vlasenko’s formal group law $F_f$ for $\Phi_f$ is an example of a curvilinear formal group law, defined by the curves $\ell_w$, $w \in \Delta^o \cap \mathbb{Z}^d$, where

$$(\log_{\Phi_f} \circ \ell_w)(\tau) = \sum_{\nu=1}^{\infty} \beta_{w, \nu} \frac{\tau^\nu}{\nu}$$

in which $\beta_{w, \nu} = (\beta_{v, w, \nu})$, see [11].

4. Among all curves on a formal Lie group $G$ on a $\mathbb{Z}(p)$-algebra there is a special class that is most relevant to our discussion. These are the $p$-typical curves on $G$. We shall not give the precise definition of $p$-typical curves. For our discussion, it is useful to know that if $R$ has characteristic 0, then a curve $\gamma$ on $G$ is $p$-typical if and only if the power series $\log_G \circ \gamma$ is of the form $\sum_{i=1}^{\infty} a_i t^i$, i.e., in the power series expansion, only $t$ to some $p$-power has possibly nonzero coefficients. The abelian group of all $p$-typical curves on $G$ is denoted by $\mathcal{C}(G)$.

5. For each ring $R$, Cartier defined a noncommutative ring $E(R)$. The ring $E(R)$ consists of “operators” on the $p$-typical curves. Therefore, for each formal Lie group $G$ over $R$, $\mathcal{C}(G)$ is a left $E(R)$-module. (see [12, IV §2]). When $R$ is a perfect field of characteristic $p > 0$, the $E(R)$-module $\mathcal{C}(G)$ consisting of $p$-typical curves on $G$ is also called the (covariant) Cartier–Dieudonné module (or simply the Dieudonné module) of $G$. 

(6) The curves $\ell_v$ described in Item (3) are not $p$-typical. The $p$-typical component $\gamma_v$ of $\ell_v$ is determined by

$$ (\log_{\Phi_f} \circ \gamma_v)(\tau) = \sum_{s=0}^{\infty} \alpha_v,s \frac{\tau^p^s}{p^s} $$

(see (2.1) and Item (4) above). Clearly they form a basic set of $p$-typical curves on $\Phi_f$. For formal Lie groups over a $\mathbb{Z}(p)$-algebra, using $p$-typical curves is sufficient to determine the formal group (see [12, Chapter IV]).

Example 2.5. (Homothety, Verschebung, and Frobenius) There are three operators in $E(R)$. For each $a \in R$, we can define an element $[a] \in E(R)$: $([a] \gamma)(t) = \gamma(at)$, called the homothety operator of $a$. There is also the $p$-typical Verschebung operator $V$ and the $p$-typical Frobenius operator $F$. For our purposes, we only need to know the effect of the operators $V$ and $[a]$ on additive curves, summarized below.

When $G$ is the additive group $\hat{G}_a^r$, a $p$-typical curve on $G$ is given by a power series

$$ \gamma(t) = \sum_{i=0}^{\infty} a_i t^{p^i} \quad (a_i \in R^r), $$

and the abelian group structure of $C(\hat{G}_a^r)$ is simply the addition of power series. We have

$$ (F\gamma)(t) = \sum_{i=0}^{\infty} pa_{i+1} t^{p^i} $$

$$ (V\gamma)(t) = \sum_{i=1}^{\infty} a_{i-1} t^{p^i}, $$

$$ ([c] \gamma)(t) = \sum_{i=0}^{\infty} c^{p^i} a_i t^{p^i}. $$

Lemma 2.6. Let $G$ be an $r$-dimensional formal Lie group over a $\mathbb{Z}(p)$-algebra $R$.

1. Let $\gamma_1, \ldots, \gamma_r$ be a basic set of $p$-typical curves on $G$. Then every $p$-typical curve $\gamma$ can be uniquely written as

$$ \gamma = \sum_{n=0}^{\infty} V^n [x_n, i] \tilde{\gamma}_i. $$

2. Let $\varphi : R \to R'$ be a ring homomorphism. Let $\varphi_* : E(R) \to E(R')$ be the base-change homomorphism of the Cartier rings. Let $x = \sum_{i,j} V^i [x_{ij}] F^j$ be an element in $E(R)$. Then $\varphi_*(x) = \sum_{i,j} V^i [\varphi(x_{ij})] F^j$.

Proof. Item (1) is [12, IV 5.15, IV 5.17]. Item (2) is [12, IV 2.5].

2.7. Lemma 2.8 below will give a tool to produce congruence relations from the theory of curves. To state it, we make some hypotheses and set up some notation.

1. Let $R$ be a flat $\mathbb{Z}$-algebra. Let $\varphi : R \to R/p$ be the reduction mod $p$ map. Let $G$ be an $r$-dimensional formal Lie group over $R$.

2. Let $\gamma_1, \ldots, \gamma_r$ and $\gamma_1^*, \ldots, \gamma_r^*$ be two basic sets of $p$-typical curves on $G$ such that the tangent vectors satisfy $\tilde{\gamma}_i = \tilde{\gamma}_i^*$. Assume further that $\varphi_* \gamma_i = \varphi_* \gamma_i^*$. 


(3) Let \( \log_G \) be the formal logarithm of \( G \) (after changing the base ring to \( R \otimes Q \)). Then we can write
\[
(\log_G \circ \gamma_j)(t) = \sum_{s=0}^{\infty} a_{j,s} t^{p^s}/p^s, a_j \in R^e,
\]
and similarly
\[
(\log_G \circ \gamma_j^*)(t) = \sum_{s=0}^{\infty} a_{j,s}^* t^{p^s}/p^s, a_j^* \in R^e,
\]
see \cite[8.19]{[12]}.

(4) Finally let \( a_s \) to be the matrix \((a_{1,s}, \ldots, a_{r,s})\) (similarly define \( a_s^* \)).

**Lemma 2.8.** Let notation and conventions be as in \( \text{(2.7)} \). We have
\[
a_s \equiv a_s^* \mod p^s.
\]

**Proof.** By Lemma \( \text{(2.6)}(1) \), we can write
\[
(2.9) \quad \gamma_j = \sum V^n [x_{n,i}^{(j)}] \gamma_i^*.
\]
Since \( \varphi \gamma_j = \varphi \gamma_j^* \), we have by Lemma \( \text{(2.6)}(2) \) and Hypothesis \( \text{(2.7)}(2) \) that
\[
\varphi \gamma_j = \varphi \gamma_j^* = \sum V^n [\varphi(x_{n,i}^{(j)})] \varphi \gamma_i^*.
\]
Applying the uniqueness part of Lemma \( \text{(2.6)}(1) \) (for the ring \( E(R/p) \)), we conclude that \( \varphi(x_{n,i}^{(j)}) = 0 \) in \( R/p \) unless \( i = j \) and \( n = 0 \). In other words, \( p \mid x_{n,i}^{(j)} \) for all \( (n, i, j) \neq (0, i, i) \). In view of Hypothesis \( \text{(2.7)}(2) \), \( x_{0,i}^{(j)} = 1 \).

Thanks to Lemma \( \text{(2.8)}(2) \), if \( \psi : R \rightarrow R \otimes Q \) is the inclusion of \( R \) in \( R \otimes Q \), then \( \text{(2.9)} \) remains valid (by abuse of notation, we identify \( \gamma_j \) with \( \psi \gamma_j \)). Now we apply \( \log_G \) to the equality \( \text{(2.9)} \). By the basic rules of \( V \) and \([a]\) described in Example \( \text{(2.5)} \) we get
\[
\sum_{s=0}^{\infty} a_{j,s} t^{p^s}/p^s = \sum_{s=0}^{\infty} V^n [x_{n,i}^{(j)}] \left( \sum_{s=0}^{\infty} a_{i,s}^* t^{p^s}/p^s \right)
\]
\[
= \sum_{s=0}^{\infty} V^n \left( \sum_{s=0}^{\infty} (x_{n,i}^{(j)})^{p^s} a_{i,s}^* /p^s \right)
\]
\[
= \sum_{s=0}^{\infty} \sum_{n=0}^{s} (x_{n,i}^{(j)})^{p^s-n} a_{i,s-n}^* /p^{s-n}
\]
It follows that in the ring \( R \otimes Q \), we have
\[
a_{j,s} = \sum_{n=0}^{s} p^n (x_{n,i}^{(j)})^{p^s-n} a_{i,s-n}^*
\]
As both sides fall in \( R \), the displayed equality is valid in \( R \) as well. Since \( p \mid x_{n,i}^{(j)} \) unless \( n = 0, i = j \), we see (remember that \( p^{s-n} > s - n \)) \( p^s \mid p^n (x_{n,i}^{(j)})^{p^s-n} a_{i,s-n}^* \). This implies that \( a_{j,s} \equiv a_{i,s}^* \mod p^s \). \( \square \)
We will also need the following simple fact.

**Lemma 2.10.** Let $R$ be a ring. Let $a$ be an element in $R$. Assume that $R$ is $a$-torsion free and $a$-adically complete. Let $\varphi : R^n \to R^n$ be a morphism of free $R$-modules. Then $\varphi$ is invertible if and only if the reduction of $\varphi$ modulo $a$ is invertible.

**Proof.** Since $\varphi$ is given by a linear map, it is continuous. The “only if” part is obvious. Let us prove the “if” part. Let $x \in R^n$ be an element. Write $x = x_0$. Inductively define $x_i$ and $y_i$ so that $x_i - \varphi(y_i)$ is zero modulo $a$, write $x_i - \varphi(y_i) = ax_i+1$, and we define $y_{i+1}$ such that $\varphi(y_{i+1}) \equiv x_{i+1} \mod a$. Then the image of

$$y = \sum_{i=0}^{\infty} a^i y_i,$$

satisfies

$$\varphi(y) = \sum_{i=0}^{\infty} a^i \varphi(y_i) = \sum_{i=0}^{\infty} (a^i x_i - a^{i+1} x_{i+1}) = x_0 = x.$$

This proves $\varphi$ is surjective.

To show $\varphi$ is injective, suppose $\varphi(x) = 0$. Since the reduction of $\varphi$ is injective, we must have $a \mid x$. Then $x = ax_1$ for some $x_1$. Since $R$ has no $a$-torsion, it follows that $\varphi(x_1) = 0$ as well. Continuing this way we see $x \in \bigcap_{i=1}^{\infty} a^i R^n$. Since $R$ is $a$-adically complete, in particular it is $a$-adically separated. Therefore $x = 0$ as desired. □

The last basic lemma we need concerning curve theory is a criterion for a formal group to have an “ordinary” reduction modulo $p$. Recall that a formal Lie group over a perfect field of characteristic $p$ is called ordinary, or of codimension $0$, or isoclinic of slope $0$, if the semilinear Frobenius map on the Dieudonné module is an isomorphism. Isoclinic modules of slope $0$ are the simplest in the spectrum of the Dieudonné–Manin classifications of crystals over a perfect field. See [12, VI §6, §8].

**Lemma 2.11.** Let $R$ be complete discrete valuation ring of characteristic $0$. Assume that its residue field $k$ is perfect of characteristic $p$. Let $G$ be a formal group on $R$ of dimension $r$. Let $\gamma_1, \ldots, \gamma_r$ be a basic set of $p$-typical curves on $G$. Let $\Gamma$ be the reduction of $G$ modulo the maximal ideal of $R$. Write

$$\log_G \gamma_i = \sum_{i=0}^{\infty} a_{i,s} \frac{tp^s}{p^s}, \quad a_{i,s} \in \text{ the tangent space of } G.$$

Assume that the matrix $a_1 = (a_{1,1}, \ldots, a_{r,s})$ is a basis of the tangent space modulo the maximal ideal of $R$. Then $\Gamma$ is isoclinic of slope $0$.

**Proof.** Fix a basis of the tangent space of $G$, we can regard $a_{i,s}$ as column vectors and $a_1$ as a matrix. The condition is then $a_1$ is an invertible matrix modulo the maximal ideal of $R$.

Let $F$ be the $p$-typical Frobenius operator. We need to show that $C(\Gamma) = FC(\Gamma)$ by [12] VI 7.5]. Since $a_1$ is invertible modulo the maximal ideal, it is invertible as a matrix with entries in $R$. See Lemma 2.10. Since

$$(\log_G \circ F \gamma_i)(t) = \sum_{i=0}^{\infty} a_{i+1,s} \frac{tp^s}{p^s},$$
and since $a_1$ is invertible, we see the tangent vectors of $F\gamma_1, \ldots, F\gamma_r$ still generate the tangent space of $G$. Therefore $F\gamma_1, \ldots, F\gamma_r$ is also a basic set of curves. This means that $\varphi_* F\gamma_1, \ldots, \varphi_* F\gamma_r$ is a basic curve of $\Gamma$. To finish the proof, we recall that in $E(k)$ the following relations hold:

$$FV = VF, \quad F[a] = [a^p]F \quad (\forall a \in k).$$

Since $F\gamma_1, \ldots, F\gamma_r$ is a basic set of curves, we can write any curve in $C(\Gamma)$ as

$$\gamma = \sum_{n,i} V^n [x_{n,i}] F\gamma_i \quad (x_{n,i} \in k)$$

$$= \sum_{n,i} FV^n [x_{n,i}]^{1/p} F\gamma_i \quad \in FC(\Gamma)$$

by Lemma 2.6(1) (the second equality holds thanks to the perfectness of $k$). This implies the desired equality $FC(\Gamma) = C(\Gamma)$. □

**Example 2.12.** Set $R = \mathbb{Z}_p$, whose Frobenius operation is the identity. Let us consider the Laurent polynomial $f(t_1, t_2) = t_1 + t_2 + (t_1 t_2)^{-2}$. Its Newton polytope consists of two interior points $u = (0, 0)$ and $v = (-1, -1)$. The following table summarizes the higher Hasse–Witt matrix $\alpha_1$ of $f$ with respect to $u, v$.

| $p^s$     | $\alpha_s$ |
|-----------|-------------|
| $p^s = 5k + 1$ | $egin{bmatrix} \frac{(5k)!}{(5k)!} & 0 \\ \frac{(5k)!}{(5k)!} & 0 \end{bmatrix}$ |
| $p^s = 5k + 2$ | $egin{bmatrix} \frac{(5k)!}{(5k-1)!} & 0 \\ \frac{(5k)!}{(5k-1)!} & 0 \end{bmatrix}$ |
| $p^s = 5k + 3$ | $egin{bmatrix} \frac{(5k)!}{(5k+2)!} & 0 \\ \frac{(5k)!}{(5k+2)!} & 0 \end{bmatrix}$ |
| $p^s = 5k + 4$ | $egin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ |

Thus, the mod $p$ reduction of $\Phi_f$ is isoclinic of slope 0 if and only if $p \equiv 1 \mod 5$, and it is a direct sum of two multiplicative groups.

Now we can turn back to the group $\Phi_f$. In the sequel, we denote the reduction of $\Phi_f$ modulo $p$ by $\Gamma_f$.

**Lemma 2.13.** Let $R = W(k)$ be the ring of Witt vectors of a perfect field $k$ of characteristic $p$. Let $\Gamma_f$ be the reduction of $\Phi_f$ modulo $p$. Assume that the matrix $\alpha_1$ (2.1) is invertible. Then the $p$-adic limit

$$\lim_{s \to \infty} \alpha_{s+1} \cdot (\alpha_s^p)^{-1}$$

(see (2.1)) exists [12], and is a matrix of the Frobenius operator $\eta$ on the Cartier–Dieudonné module $M = C(\Gamma_f)$.

**Proof.** We consider the generalized Lubin–Tate group $LT(M, \eta)$ à la Cartier [6]. In general, for any formal group $\Gamma_f$ of finite height over $k$, the generalized Lubin–Tate group $LT(M, \eta)$ ($M$ being the Cartier–Dieudonné module of $p$-typical curves on $\Gamma_f$) is the universal extended lift of $\Gamma_f$ over $W(k)$, see [12] VII 7.17. In our case,
Γ_f is isoclinic of slope 0 (by Lemma 2.11), LT(M, η) is therefore without additive kernels. Thus, for any lift Φ of Γ_f over W(k), we always have an isomorphism LT(M, η) ∼= Φ. In particular, there is an isomorphism LT(M, η) ∼= Φ_f.

The idea of the proof is as follows. The generalized Lubin–Tate group admits an explicit set of basic p-typical curves defined by the Frobenius operation, and our formal group Φ_f admits an explicit set of basic p-typical curves defined by α_s. The above mentioned isomorphism will then provide a rule transforming the Frobenius related curve set to the expansion-coefficients related curve set. The limit formula will then be a consequence of Lemma 2.8.

Let us carry out the above scheme. Define, for v ∈ Δ^o ∩ Z^d a curve γ_v ∈ C(Φ_f) by

\[ \log_{\Phi_f} \gamma_v(x) = \sum_{s=0}^{\infty} \alpha_v, s \frac{x^{p^s}}{p^s}, \]

where \(\alpha_v, s = (\alpha_s)_{u,v} : u ∈ Δ^o ∩ Z^d\) (2.1). In view of 1.15, each γ_v is a p-typical curve of the formal group functor Φ_f. Since \(\alpha_0\) is the identity matrix, the \(\gamma_v\)'s form a basic set of p-typical curves. Since \(M = C(\Gamma_f) = E(k) ∼= E(W) \subset C(\Phi_f)\) [12, VII 6.8], the images \(e_v\) of the curves \(γ_v\) form a basic set of curves of \(Γ_f\). In particular, \(\{e_v : v ∈ Δ^o ∩ Z^d\}\) is a basis of the Cartier–Dieudonné module \(M\). For each \(e_v\), let \(γ_v^*\) be the curve in \(C(LT(M, η))\) given by

\[ \log_{\Phi_f} \gamma_v^*(x) = \sum_{s=0}^{\infty} \eta^s(e_v) \frac{x^{p^s}}{p^s} = \sum_{s=0}^{\infty} b_{v,s} \frac{x^{p^s}}{p^s}. \]

Then \(γ_v^*\) form a basic set of curves in \(Φ_f ∼= LT(M, η)\), see [13, (8), (17) above, (19c)]. By construction, \(\{γ_v\} \) and \(\{γ_v^*\}\) are two basic sets of curves on \(Φ_f\) which restrict to the same set of curves \(\{e_v\}\) of \(Φ_{f_{\bar{u}}}\). This enables us to apply Lemma 2.8 and we get a rather strong congruence relation:

\[ \alpha_s ≡ b_s \mod p^s. \]

Write \(\alpha_s = b_s + p^s c_s\). By definition and by that \(η\) is semilinear, \(b_1\) is the matrix of η with respect to the basis \(\{e_v\}\), and the matrix \(b_s\) is \(b_1 b_1^s \cdots b_1^{s-1}\). Note in particular we have \(b_{s+1}(b_s^*)^{-1} = b_1\). It follows that

\[ \alpha_{s+1}(\alpha_s^*)^{-1} = (b_{s+1} + p^{s+1} c_{s+1})(b_s^*)^{-1}(Id + p^s c_s (b_s^*)^{-1})^{-1} \]

\[ ≡ b_{s+1}(b_s^*)^{-1} \mod p^s \]

\[ ≡ b_1 \mod p^s. \]

Therefore, \(p^s | (b_1 - \alpha_{s+1}(\alpha_s^*)^{-1})\). Thus the limit \(\alpha = \lim \alpha_{s+1}(\alpha_s^*)^{-1}\) exists, and equals \(b_1\), which is the matrix of the Frobenius operation on \(M\) with respect to the basis \(\{e_v\}\).

We have explained that, when the base is \(W(k)\), the limit matrix \(\alpha\) is related to the Frobenius action on the Cartier–Dieudonné module of \(Γ_f\). Now if \(R/p\) is not a perfect field, the analogue of the Cartier–Dieudonné module of the formal group \(Γ_f\) is its (covariant) Dieudonné crystal \(D^*(Γ_f)\) (which is the contravariant Dieudonné crystal of the Cartier dual of \(Γ_f\)). The basic reference for Dieudonné crystal is [3]. We shall not review the theory of Dieudonné crystals. It suffices to know that the
value of $D^*(\Gamma_f)$ on a “perfect point” $x : \text{Spec}(k) \to \text{Spec}(R/p)$ ($k$ is a perfect field) of $R/p$ is given by the Cartier–Dieudonné module of the fiber of $\Gamma_f$ over $x$.

The following theorem shows that we can identify the limit matrix $\alpha$ with the Frobenius action on the value of the Dieudonné crystal $D^*(\Gamma_f)$ on $R$. It turns out we can reduce the general case to the special case treated before, by some standard yoga.

**Theorem 2.14.** Let notation be as in [1.10]. Assume further that

1. $R$ is $p$-adically complete, $p$-torsion free ring,
2. $R$ has a lifting $\sigma$ of the absolute Frobenius of $R/p$.

Let $D^*(\Gamma_f)$ be the (covariant) Dieudonné crystal of the reduction $\Gamma_f$ of $\Phi_f$ mod $p$. Assume $\alpha_1$ is invertible in $R/p$. Then the $p$-adic limit $\alpha$ is the Frobenius of the the $R$-module $D^*(\Gamma_f)_R$.

**Proof.** By Lemma 2.10 $\alpha_1$ itself is invertible in $R$. As a first step, we assume both $R$ and $R/p$ are integral domains. In this step, we repeat a construction used by N. Katz [11]. Let $A$ be the perfection of $R/p$. Then there is a unique lifting of the inclusion $R/p \to A$ to an inclusion $R \to W(A)$ which sits in a commutative diagram

$$
\begin{array}{ccc}
R & \to & W(A) \\
\sigma \downarrow & & \sigma \downarrow \\
R & \to & W(A)
\end{array}
$$

where the right vertical arrow, still denote by $\sigma$, is the canonical Frobenius of $W(A)$. Let $K_0$ be the field of fractions of $R/p$. In [11] §7, it is shown that we have an injection $R \to W(K_0^{\text{perf}})$. We have two matrices $\alpha$ and $\beta_1$, both have entries in $R$. In order to show $\alpha = \beta_1$, it suffices to prove it in a larger ring $W(K_0^{\text{perf}})$. The result for this ring has been established in Lemma 2.13 above.

We proceed to prove the theorem in its full generality. We shall use some simple properties about $\delta$-rings (see for example [5, §2]). The upshot is that on a $p$-torsion free ring, having a $\delta$-ring structure is equivalent to fixing a lifting of the mod $p$ Frobenius, and a $\delta$-ring homomorphism between $p$-torsion free rings is equivalent to a map preserving Frobenius.

Set $R_0 = Z_{[p]}[x_w : w \in \Delta \cap \mathbb{Z}^d]$. The free $\delta$-ring with variables $x_w$ is denoted by $R_1 = Z_{[p]}\{x_w : w \in \Delta \cap \mathbb{Z}^d\}$ (see [5, 2.11]). Abstractly, this is a polynomial ring with infinitely many variables. We use $\phi$ to denote the Frobenius of $R_1$.

Let $R$ be the $p$-adic completion of the localization $R_1[\phi^m(\det A_1)^{-1}, m \geq 0]$, where $A_1$ is the Hasse–Witt matrix for $f(t) = \sum_{w \in \Delta \cap \mathbb{Z}^d} x_w t^w$. Since we are localizing a system stable under the Frobenius, $R$ is a $\delta$-ring.

**Claim 1.** The $\delta$-ring $R$ has the following universal property: suppose that we are given a ring homomorphism $\varphi : R_0 \to R$, where

- $R$ is a $p$-adically complete, $p$-torsion free $\delta$-ring,
- $\varphi(\det A_1)$ is invertible on $R$,

there is a unique $\delta$-ring map $R \to R$ compatible with $\varphi$.

**Claim 2.** Let $R$ be as in Claim 1. Let $\sigma$ be the Frobenius lift of $R$. then $x \in R$ is invertible if and only if $\sigma(x)$ is.
Proof of Claim 2. Since \( \sigma(x) = x^p + p\delta(x) \), and \( R \) is \( p \)-adicaly complete, \( \sigma(x) \) is invertible if and only if \( x^p \) is invertible. But \( x^p \) is invertible if and only if \( x \) is. \( \square \)

Applying Claim 2 to \((R, \sigma) = (\mathcal{R}, \phi)\), we have \( \mathcal{R} = \mathcal{R}[(\det A_1)^{-1}] \).

Proof of Claim 1. The universal property of \( \mathcal{R}_1 \) implies that \( \phi \) canonically factors through \( \mathcal{R}_1 \) as a homomorphism of \( \delta \)-rings. By Claim 2, as \( \phi(\det A_1) \) is invertible in \( R \), \( \sigma^m(\phi(\det A_1)) \) are all invertible in \( R \). Thus \( \phi \) canonically factors through a \( \delta \)-ring homomorphism \( \mathcal{R}_1[\phi^m(\det A_1)^{-1} : m \geq 0] \to R \). Passing to the completion finishes the argument. \( \square \)

Since \( \mathcal{R} \) is a completion of a localization of a polynomial ring (with infinitely many variables), \( \mathcal{R} \) is an integral domain. Therefore the theorem holds for the Laurent polynomial \( \mathcal{f}(t) = \sum_{w \in \Delta_X} x_w t^w \) with coefficients in \( \mathcal{R} \).

Now let \( f \) be as in the statement of the theorem. By construction, there is a homomorphism \( R_0 \to R \) sending \( f \) to \( f \). Since \( R \) is equipped with a \( p \)-Frobenius and satisfies the hypotheses of Claim 1, we get a ring homomorphism \( \Psi : \mathcal{R} \to R \) compatible with the Frobenii on \( \mathcal{R} \) and \( R \). As the formal group laws \( F_f, F_r \) are defined by coefficients of expansions, we have \( \Psi \circ \Phi_f = \Phi_f \). Since \( \mathcal{R} \) is the completion of a localization of a polynomial ring, we know \( \mathcal{R} \) and \( \mathcal{R}/p \) are domains. Let \( D^\ast(\Gamma_f) \) be the covariant Dieudonné crystal of the reduction of \( \Phi_F \). This is a special sheaf on the big crystalline site \( \text{CRIS}((\mathcal{R}/p)/\mathbb{Z}_p) \).

We have a commutative diagram

\[
\begin{array}{ccc}
\mathcal{R} & \xrightarrow{\psi} & R \\
\downarrow & & \downarrow \\
\mathcal{R}/p & \xrightarrow{\psi} & R/p \\
\end{array}
\]

Since \( \Phi_F \) is formally smooth over \( R \), its formation commutes with base change. We have \( \Gamma_f \otimes_{\mathcal{R}/p} R/p = \Phi_{\mathcal{R}} \otimes_{\mathcal{R}} R/p = \Gamma_f \). Therefore \( D^\ast(\Gamma_f) = \text{Spec}(\psi)^\ast D^\ast(\Gamma_f) \) by [3 (1.3.3.4)] and the definition of the pull-back functor [3 p. 30]. Since the Frobenius action on \( D^\ast(\Gamma_f) \) is induced from that of \( D^\ast(\Gamma_f) \), and we have checked that in the “universal case” \( \beta_1 = \alpha \), the theorem for \( R \) and \( f \) follows from the theorem for \( \mathcal{R} \) and \( f \) by base change. \( \square \)

We finish with a remark on the relation between the Dieudonné module and rigid cohomology.

**Remark 2.15** (Relation with rigid cohomology). So far we have been completely ignoring the geometric meaning of \( \mathcal{C}(\Gamma_f) \). In this remark we explain how to relate \( \mathcal{C}(\Gamma) \) to quantities with geometric meaning. In addition to the hypotheses above we assume further that \( R \) is a noetherian ring. Then by Lemma [1.13] we can identify \( \Gamma_f \) with an Artin–Mazur type formal group functor.

Let \( X \) be closure of \( f = 0 \) in \( \mathbb{P} := P_{S,R} \). Let \( U = \mathbb{P} - X \). Let \( P_0, X_0, U_0 \) be the reduction of \( \mathbb{P}, X, U \) modulo \( p \), respectively. Assume that \( X \) is flat (so the formation of its ideal sheaf commutes with base change). Then \( \Gamma_f \) is the Artin–Mazur formal group functor associated with the ideal sheaf of \( X \). The value of \( D^\ast(\Gamma_f) \) at a perfect point \( x : R \to k \) is the Witt vector cohomology

\[
H^d(P_0 \otimes_R k, \text{Ker}\{W^i \mathcal{O}_{P_0 \otimes_R k} \to W^i \mathcal{O}_{X_0 \otimes_R k}\}).
\]
The “isogeny class” of \(C(\Gamma_f \otimes_R k)\) is then the slope < 1 part of the rigid cohomology group \(H^d_{\text{rig},c}(U_0 \otimes_R k)\) with proper support \([2]\) Theorem 1.2.

Note that this is also the slope < 1 part of the \((d-1)\)th rigid cohomology of the hypersurface \(X_0 \otimes_{R,x} k\), since we have an exact sequence of vector spaces over \(W(k)[1/p]\): (assuming \(d \geq 2\) to avoid the trivial case)

\[
H^{d-1}_{\text{rig}}(P_0 \otimes_R k) \to H^{d-1}_{\text{rig}}(X_0 \otimes_R k) \to H^d_{\text{rig},c}(U_0 \otimes_R k) \to H^d_{\text{rig}}(P_0 \otimes_R k)
\]

and since \(H^{d-1}_{\text{rig}}(P_0 \otimes_R k)\) and \(H^d_{\text{rig}}(P_0 \otimes_R k)\) are isoclinic of slope \(d-1, d\) respectively.

Under the hypothesis that \(\alpha_1\) is invertible, the limit \(\alpha\) then gives a way to construct a formula for the “unit-roots” of \(X_0\). Assuming \(k = F_q\) is a finite field, Theorem 2.14 then gives a way to extract the unit roots of the zeta functions of a flat family of (even singular) hypersurfaces in a possibly singular toric variety over \(k\).

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