Regular obstructed categories and TQFT

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Abstract

A proposal of the concept of $n$-regular obstructed categories is given. The corresponding regularity conditions for mappings, morphisms and related structures in categories are considered. An $n$-regular TQFT is introduced. It is shown the connection of time reversibility with the regularity.

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Introduction

In the generalized histories approach to quantum theory the whole universe is represented by a class of 'histories'. In this approach the standard Hamiltonian time-evolution is replaced by a partial semigroup called a 'temporal support'. A possible realization of such program can be described in terms of cobordism manifolds and corresponding categories. The temporal support arises naturally as a cobordism $M$, where the boundary $\partial M$ of $M$ is a disjoint sum of the 'incoming' boundary manifold $\Sigma_0$ and the 'outgoing' one $\Sigma_1$. This means that the cobordism $M$ represents certain quantum process transforming $\Sigma_0$ into $\Sigma_1$. In other words, $\Sigma_1$ is a time consequence of $\Sigma_0$. Obviously, we have two opposite possibilities to declare which boundary is the initial one.

Let $N$ be a cobordism with the 'outgoing' boundary of $M$ as its 'incoming boundary' and $\Sigma_2$ as the 'outgoing boundary'. Then there is a cobordism $N \circ M$ whose incoming boundary is $\Sigma_0$, and the outgoing one is $\Sigma_3$. In this case we say that these two cobordisms are glued along $\Sigma_1$. Such gluing of cobordisms up to diffeomorphisms define a partial semigroup operation. One can consider cobordism with several incoming and outgoing boundary manifolds. The class of possible histories can be represented by gluing of cobordisms in several different ways. Hence there is the corresponding coherence problem for such description.

Let $\text{Cob}$ be a category of cobordisms, where the boundary $\partial M$ of $M \in \text{Cob}$ is a disjoint sum of the 'incoming' boundary manifold $\Sigma_0$ and the 'outgoing' one $\Sigma_1$. There is also the cylinder cobordism $\Sigma \times [0, 1]$ such that $\partial (\Sigma \times [0, 1]) = \Sigma \amalg \Sigma^*$. The class of boundary components is denoted by $\text{Cob}_0$. According to Atiyah, Baez and Dolan, the TQFT is a functor $\mathcal{F}$ from the category $\text{Cob}$ to the category $\text{Vect}$ of a finite-dimensional vector spaces. This means that $\mathcal{F}$ sends every manifold $\Sigma \in \text{Cob}_0$ into vector space $\mathcal{F}(\Sigma)$ such that

$$\mathcal{F}(\Sigma^*) = (\mathcal{F}(\Sigma))^*, \quad \mathcal{F}(\Sigma_0 \amalg \Sigma_1) = (\mathcal{F}\Sigma_0) \otimes (\mathcal{F}\Sigma_1), \quad \mathcal{F}(\emptyset) = I,$$

(1)

and a cobordism $M(\Sigma_0, \Sigma_1)$ to a mapping $\Phi(M) \in \text{lin}_I(\mathcal{F}\Sigma_0, \mathcal{F}\Sigma_1)$ such that $\mathcal{F}(\Sigma \times [0, 1]) = id_{\mathcal{F}\Sigma}$, where $I$ is a field, and $\Sigma^*$ is the same manifold $\Sigma$ but with the opposite orientation. Kerler found examples of categories formed by some classes of cobordism manifolds preserving some operations like the disjoint sum or surgery. It was discussed by Baez and Dolan that it is not easy to describe such categories in a coherent way. Crane applied the category theory to an algebraic structure of the quantum gravity.
The idea of regularity as a generalized inverse was firstly introduced by von Neumann [9] and applied by Penrose for matrices [10]. Let \( \mathbb{R} \) be a ring. If for an element \( a \in \mathbb{R} \) there is an element \( a^* \) such that

\[
aa^*a = a, \quad a^*aa^* = a^*,
\]

then \( a \) is said to be regular and \( a^* \) is called a generalized inverse of \( a \). Generalizing transition from inverses to regularity is a widely used method of abstract extension of various algebraic structures. The intensive study of such regularity and related directions was developed in many different fields, e.g. generalized inverses theory [11, 12, 13], semigroup theory [14, 15, 16, 17, 18], and [19, 20], supermanifold theory [21, 22, 23], Yang-Baxter equation in endomorphism semigroup and braided almost bialgebras [24, 25, 28], weak bialgebras, week Hopf algebras [26], category theory [29].

In this paper we are going to study certain class of categories which can be useful for the study of quantum histories with non-reversible time, quantum, gravity and field theory. The regularity concept for linear mappings and morphisms in categories are studied. Higher order regularity conditions are described. Commutative diagrams are replaced by ‘secommutative’ ones. The distinction between commutative and ‘secommutative’ cases is measured by a non-zero obstruction proportional to the difference of some self-mappings \( e^{(n)} \) from the identity. This allows to ‘regularize’ the notion of categories, functors and related algebraic structures. It is interesting that this procedure is unique up to an equivalence defined by invertible morphisms. Our regularity concept is nontrivial for equivalence classes of noninvertible morphisms. The regular version of TQFT is a natural application of the presented here formalism. In this case the n-regularity means that a time evolution is nonreversible, although repeated after n steps, but up to a classes of obstructions. Our considerations are based on the concepts of generalized inverse [13, 23], and semisupermanifolds [21].

The paper is organized as follows. In the Section II we consider linear mappings without requirement of ‘invertibility’. If \( f : X \to Y \) is a linear mapping, then instead of the inverse mapping \( f^{-1} : Y \to X \) we use less restricted ‘regular’ \( f^* \) one by extending ‘invertibility’ to ‘regularity’ according to the following relations

\[
f \circ f^* \circ f = f, \quad f^* \circ f \circ f^* = f^*.
\]

We also propose some higher regularity conditions. In Section III the higher regularity notion is extended to morphisms of categories. Commutative di-
agrams are replaced by semicommutative ones. The concept of regular cocycles of morphisms in an category is described. An existence theorem for these cocycles is given. The corresponding generalization of certain categorical structures as tensor operation, algebras and coalgebras etc... to our higher regularity case is given in the Section IV. Regular equivalence classes of cobordism manifolds and the corresponding structures are considered in the Section V. An \( n \)-regular TQFT is introduced as an \( n \)-regular obstructed category represented some special classes of cobordisms called 'interactions'. Our study is not complete, it is only a proposal for new algebraic structures related to topological quantum theories.

II Generalized invertibility and regularity

Let \( X \) and \( Y \) be two linear spaces over a field \( k \). We use the following notation. Denote by \( id_X \) and \( id_Y \) the identity mappings \( id_X : X \to X \) and \( id_Y : Y \to Y \). If \( f : X \to Y \) is a linear mapping, then the image of \( f \) is denoted by \( \text{Im} f \), and the kernel by \( \text{Ker} f \).

We are going to study here some generalizations of the standard concept of invertibility properties of mappings. Our considerations are based on the article of Nashed [13]. Let \( f : X \to Y \) be a linear mapping. If \( f \circ f^{-1} = id_Y \) for some \( f^{-1} : Y \to X \), then \( f \) is called a retraction, and \( f^{-1} \) is the right inverse. Similarly, if \( f^{-1} \circ f = id_X \), then it is called a coretraction, \( f^{-1} \) is the left inverse of \( f \). A mapping \( f^{-1} \) is called an inverse of \( f \) if and only if it is both, right and left inverse of \( f \).

This standard concept of inverses is in many cases too strong to be fulfilled. To obtain more weak conditions one has to introduced the following 'regularity' conditions

\[
f \circ f_{\text{in}}^* \circ f = f,
\]

where \( f_{\text{in}}^* : Y \to X \) is called an inner inverse, and such \( f \) is called regular. Similar “reflexive regularity” conditions

\[
f_{\text{out}}^* \circ f \circ f_{\text{out}}^* = f_{\text{out}}^*
\]

defines an outer inverse \( f_{\text{out}}^* \). Notice that in general \( f_{\text{in}}^* \neq f_{\text{out}}^* \neq f^{-1} \) or it can be that \( f^{-1} \) does not exist at all.
Definition 1. A mapping $f$ satisfying one of the condition (4) or (5) is said to be regular or 2-regular. A generalized inverse of a mapping $f$ is a mapping $f^*$ which is both inner and outer inverse $f^* = f_{in}^* = f_{out}^*$.

Lemma 2. If $f_{in}^*$ is an inner inverse of $f$, then a generalized inverse $f^*$ exists, but need not be unique.

Proof: If $f_{in}^*$ is an inner inverse, then
\[ f^* = f_{in}^* \circ f \circ f_{in}^* \] (6)
is always both inner and outer inverse i.e. generalized inverse. It follows from (6) that both regularity conditions (4) and (5) hold. □

Definition 3. Let us define two operators $P_f : Y \to Y$ and $P_{f^*} : X \to X$ by relations
\[ P_f := f \circ f^*, \quad P_{f^*} := f^* \circ f, \] (7)

Lemma 4. These operators satisfy
\[
\begin{align*}
P_f \circ P_f &= P_f, \quad P_f \circ f = f \circ P_f, = f \quad P_{f^*} \circ P_{f^*} = P_{f^*}, \quad P_{f^*} \circ f^* = f^* \circ P_{f^*} = f^*.
\end{align*}
\] (8)

Lemma 5. If $f^*$ is the generalized inverse of the mapping $f$, then the following properties are obvious
\[
\begin{align*}
\text{Im}(f) &= \text{Im}(f \circ f^*), \quad \text{Ker}(f \circ f^*) = \text{Ker} f^*, \\
\text{Im}(f^* \circ f) &= \text{Im} f^*, \quad \text{Ker}(f^* \circ f) = \text{Ker} f.
\end{align*}
\] (9)

In addition there are two decompositions
\[ X = \text{Im} f^* \bigoplus \text{Ker} f, \quad Y = \text{Im} f \bigoplus \text{Ker} f^*, \] (10)
The restriction $f \mid_{\text{Im} f^*} : \text{Im} f^* \to \text{Im} f$ is one to one mapping, and operators $P_{f}, P_{f^*}$ are projectors of $Y, X$ onto $\text{Im} f, \text{Im} f^*$, respectively.
Theorem 6. Let \( f : X \to Y \) be a linear mapping. If \( P \) and \( Q \) are projectors corresponding to the following two decompositions

\[
X = M \oplus \ker f, \quad Y = \text{Im} f \oplus N,
\]

respectively, then exist unique generalized inverse of \( f \), and

\[
f^* := i \circ \tilde{f}^{-1} \circ Q,
\]

where \( \tilde{f} := f \mid_M \), and \( i : M \hookrightarrow X \).

Here we try to construct higher analogs of generalized inverses and regularity conditions (4)–(5). Let us consider two mappings \( f : X \to Y \) and \( f^* : Y \to X \) and introduce two additional mappings \( f^{**} : X \to Y \) and \( f^{***} : Y \to X \). We propose here the following higher regularity condition

\[
f \circ f^* \circ f^{**} \circ f^{***} \circ f = f,
\]

This equation define a 4-regularity condition. By cyclic permutations we obtain

\[
\begin{align*}
f^* \circ f^{**} \circ f^{***} \circ f \circ f^* & = f^*, \\
f^{**} \circ f^{***} \circ f \circ f^* \circ f^{**} & = f^{**}, \\
f^{***} \circ f \circ f^* \circ f^{**} \circ f^{***} & = f^{***}.
\end{align*}
\]

By recursive considerations we can propose the following formula of \( n \)-regularity

\[
f \circ f^* \circ f^{**} \cdots \circ f^{** \cdots **} \circ f = f,
\]

where \( n = 2k \), \( k = 1, 2, \ldots \) and their cyclic permutations. Observe that for a not unique **\cdots**-operation the following formula

\[
(g \circ f)^{** \cdots **} = f^{** \cdots **} \circ g^{** \cdots **}.
\]
leads to a difficulty. If the above operation is defined up to an equivalence, then the difficulty can be overcomes. We can introduce ‘higher projector’ by the relation

$$\mathcal{P}_f^{(n)} = f \circ f^* \circ f^{**} \ldots \circ f^{n-1}, \quad n = 2k. \quad (17)$$

**Lemma 7.** It is easy to check the following properties

$$\mathcal{P}_f^{(n)} \circ f = f. \quad (18)$$

and

$$\mathcal{P}_f^{(n)} \circ \mathcal{P}_f^{(n)} = \mathcal{P}_f^{(n)} \circ \mathcal{P}_f^{(n)}, n = 2k. \quad \Box$$

For a given $n = 2k$ all $f^*, f^{**}, \ldots f^{n-1}$ are different, and, for instance, $(f^*)^* \neq f^{**}$. The existence of analogous conditions for odd $n$ is a problem.

**Theorem 8.** Let $f : X \to Y$ be a linear mapping. If $P$ and $Q$ are projectors corresponding to the following two decompositions

$$X = M \oplus \text{Ker} f, \quad Y = \text{Im} f \oplus N, \quad (19)$$

respectively, and

$$f^* \big|_{\text{Im} f} = f^{***} \big|_{\text{Im} f}, \quad (20)$$

then the 4-regularity condition of $f$ can reduced to the two 2-regularity conditions

$$f \circ f^* \circ f = f, \quad f^* \circ f^{**} \circ f^* = f^*. \quad (21) \quad \Box$$

### III Semicommutative diagrams and regular obstructed categories

In the previous section we considered mappings and regularity properties for two given spaces $X$ and $Y$, because we studied various types of inverses. Now
we will extend these consideration to any number of spaces and introduce semicommutative diagrams (firstly introduced in [21]).

A directed graph \( \mathcal{C} \) is a pair \( \{ \mathcal{C}_0, \mathcal{C}_1 \} \) and a pair of functions

\[
s : \mathcal{C}_0 \leftrightarrow \mathcal{C}_1
\]

\[
t
\]

where elements of \( \mathcal{C}_0 \) are said to be objects, elements of \( \mathcal{C}_1 \) are said to be arrows or morphisms, \( sf \) is said to be a domain (or source) of \( f \), and \( tf \) is a codomain (or target) of \( f \in \mathcal{C}_1 \). If \( sf = X \in \mathcal{C}_0 \), and \( tf = Y \in \mathcal{C}_0 \), then we use the following notation \( X \xrightarrow{f} Y \) and

\[
\mathcal{C}(X, Y) := \{ f \in \mathcal{C}_1 : sf = X, tf = Y \}.
\]

We denote by \( \text{End}(X) \) the collection of all morphisms defined on \( X \) into itself, i.e. \( \text{End}(X) := \mathcal{C}(X, X) \), \( X \in \mathcal{C}_0 \).

Two arrows \( f, g \in \mathcal{C}_1 \) such that \( tf = sg \) are said to be composable. If in addition \( sf = X \), \( sg = tf = Y \), and \( tg = Z \), then we use the notation \( X \xrightarrow{f} Y \xrightarrow{g} Z \). In this case a composition \( g \circ f \) of two arrows \( f : X \to Y \) and \( g : Y \to Z \) can be defined as an arrow \( X \xrightarrow{g \circ f} Z \). The associativity means that \( h \circ (g \circ f) = (h \circ g) \circ f = h \circ g \circ f \). An identity ‘id’ in \( \mathcal{C} \) is an inclusion \( X \in \mathcal{C}_0 \hookrightarrow \text{id}_X \in \text{End}(X) \) such that

\[
f \circ \text{id}_X = \text{id}_Y \circ f = f.
\]

for every \( X, Y \in \mathcal{C}_1 \), and \( X \xrightarrow{f} Y \).

A directed graph \( \mathcal{C} \) equipped with associative composition of composable arrows and identity satisfying some natural axioms is said to be a category [32, 33]. If \( \mathcal{C} \) is a category, then right cancellative morphisms are epimorphisms which satisfy \( g_1 \circ f = g_2 \circ f \implies g_1 = g_2 \), where \( g_{1,2} : Y \to Z \) and left cancellative morphisms are monomorphisms which satisfy \( f \circ h_1 = f \circ h_2 \implies h_1 = h_2 \), where \( h_{1,2} : Z \to X \). A morphisms \( X \xrightarrow{f} Y \) is invertible means that there is a morphisms \( Y \xrightarrow{g} X \) such that \( f \circ g = \text{id}_Y \) and \( g \circ f = \text{id}_X \). Instead of such invertibility we can use the regularity condition [1], i.e. \( f \circ g \circ f = f \), where \( g \) plays the role of an inner inverse [8].
Invertible morphisms  Noninvertible (regular) morphisms

Usually, for three objects $X, Y, Z$ and three morphisms $f : X \to Y$ and $g : Y \to Z$ and $h : Z \to X$ one can have the ‘invertible’ triangle commutative diagram $h \circ g \circ f = id_X$. Its regular extension has the form

$$ f \circ h \circ g \circ f = f. \quad (25) $$

Such a diagram

\[ \begin{array}{ccccc}
  & f & \text{‘Regularization’} & f \\
  h & \downarrow & & \downarrow & h \\
  g & \Rightarrow & & \Rightarrow & g \\
  & n = 3 & & & \\
\end{array} \]

Reversible morphisms  Noninvertible (regular) morphisms

can be called a \textit{semicommutative diagram}. By cyclic permutations of (25) we obtain

$$ h \circ g \circ f \circ h = h, \quad g \circ f \circ h \circ g = g. \quad (26) $$

These formulae define the concept of 3-regularity.

\textbf{Definition 9.} A mapping $f : X \to Y$ satisfying conditions (25) and (26) is said to be 3-regular. The mapping $h : Z \to X$ is called the first 3-inversion and the mapping $g : Y \to Z$ the second one.

The above concept can be expanded to any number of objects and morphisms.

\textbf{Definition 10.} Let $\mathcal{C} = (\mathcal{C}_0, \mathcal{C}_1)$ be a directed graph. An \textit{n-regular cocycle} $(X, f)$ in $\mathcal{C}$, $n = 1, 2, \ldots$, is a sequence of composable arrows in $\mathcal{C}$

$$ X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \ldots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} X_1, \quad (27) $$
such that
\begin{align*}
    f_1 & \circ f_n \circ \cdots \circ f_2 \circ f_1 = f_1, \\
    f_2 & \circ f_1 \circ \cdots \circ f_3 \circ f_2 = f_2, \\
    f_n & \circ f_{n-1} \circ \cdots \circ f_1 \circ f_n = f_n,
\end{align*}
and
\begin{align*}
    e_{X_1}^{(n)} & = f_n \circ \cdots \circ f_2 \circ f_1 \in \mathrm{End}(X_1), \\
    e_{X_2}^{(n)} & = f_1 \circ \cdots \circ f_3 \circ f_2 \in \mathrm{End}(X_2), \\
    e_{X_n}^{(n)} & = f_{n-1} \circ \cdots \circ f_1 \circ f_n \in \mathrm{End}(X_n).
\end{align*}

Definition 11. Let \((X, f)\) be an n-regular cocycle in \(\mathcal{C}\), then the correspondence \(e_X^{(n)} : X_i \in \mathcal{C}_0 \mapsto e_X^{(n)}(X_i) \in \mathrm{End}(X_i), i = 1, 2, \ldots, n,\) is called an n-regular cocycle obstruction structure on \((X, f)\) in \(\mathcal{C}\).

Lemma 12. We have the following relations
\begin{align*}
    f_i & \circ e_{X_i}^{(n)} = f_i, \\
    e_{X_{i+1}}^{(n)} & \circ f_i = f_i, \\
    e_{X_i}^{(n)} & \circ e_{X_i}^{(n)} = e_{X_i}^{(n)},
\end{align*}
for \(i = 1, 2, \ldots, n(\mod n).\)

Proof: The lemma simply follows from relations (28) and (29).

Definition 13. An n-regular obstructed category is a directed graph \(\mathcal{C}\) with an associative composition and such that every object is a component of an n-regular cocycle.

Example 1. If all obstruction are equal to the identity \(e_{X_i}^{(n)} = id_{X_i},\) and
\begin{align*}
    f_n \circ \cdots \circ f_2 \circ f_1 & = id_{X_1}, \\
    f_1 \circ \cdots \circ f_3 \circ f_2 & = id_{X_2}, \\
    f_{n-1} \circ \cdots \circ f_1 \circ f_n & = id_{X_n},
\end{align*}
then the sequence (27) is trivially n-regular. Observe that the trivial 2-regularity is just the usual invertibility, hence every groupoid \(G\) is a trivially 2-regular obstructed category. We are interested with obstructed categories equipped with some obstruction different from the identity.
**Definition 14.** The minimum number $n = n_{\text{obstr}}$ such that $e^{(n)}_X \neq id_X$ is called the obstruction degree.

**Example 2.** Every inverse semigroup $S$ is a nontrivial 2-regular obstructed category. It has only one object, morphisms are the elements of $S$.

**Theorem 15.** Let $\mathcal{C}$ be a category, and

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} X_1$$

(32)

be a sequence of morphisms of category $\mathcal{C}$. Assume that there is a sequence

$$Y_1 \xrightarrow{\tilde{f}_1} Y_2 \xrightarrow{\tilde{f}_2} \cdots \xrightarrow{\tilde{f}_{n-1}} Y_n \xrightarrow{\tilde{f}_n} Y_1,$$

(33)

where $Y_i$ is a subobject of $X_i$ such that there is a collection of mappings $\pi_i : X_i \to Y_i$ and $\iota : Y_i \to X_i$ satisfying the condition $\pi_i \circ \iota_i = id_{Y_i}$ for $i = 1, 2, \ldots, n$. If in addition

$$\tilde{f}_n \circ \cdots \tilde{f}_2 \circ \tilde{f}_1 = id_{Y_1},$$

$$\tilde{f}_1 \circ \cdots \tilde{f}_3 \circ \tilde{f}_2 = id_{Y_2},$$

$$\cdots$$

$$\tilde{f}_{n-1} \circ \cdots \tilde{f}_1 \circ \tilde{f}_n = id_{Y_n},$$

(34)

and

$$f_i := \iota_{i+1} \circ \tilde{f}_i \circ \pi_i$$

(35)

then the sequence (32) is an $n$-regular cocycle.

**Proof:** The corresponding obstruction structure is given by

$$e^{(n)}_{X_i} = \iota_i \circ \pi_i$$

(36)

If $x \in \text{Ker} f_1$, then the theorem is trivial, if $x \in X_i \setminus \text{Ker} f_1$, then we obtain

$$(f_1 \circ f_n \circ \cdots \circ f_2 \circ f_1)(x) = \iota_2 \circ \tilde{f}_1 \circ \pi_1 \circ \iota_1 \circ \tilde{f}_n \circ \cdots \circ \tilde{f}_2 \circ \tilde{f}_1 \circ \pi_1(x)$$

$$= \iota_2 \circ \tilde{f}_1 \circ \pi_1 = f_1(x),$$

where the condition (34) and (35) has been used. We can calculate all cyclic permutations in an similar way. □
Example 3. There is an $n$-regular obstructed category $\mathcal{C} = (\mathcal{C}_0, \mathcal{C}_1)$, where $\mathcal{C}_0 = \{X_i : i = 1, \ldots, n \pmod{n}\}$ and $\mathcal{C}_1 = \{f_i : i = 1, \ldots, n \pmod{n}\}$ are described in the above theorem.

Definition 16. Let $(X, f), (Y, g)$ be two $n$-regular cocycles in $\mathcal{C}$. An $n$-regular cocycle morphism $\alpha : (X, f) \to (Y, g)$ is a sequence of morphisms $\alpha := (\alpha_1, \ldots, \alpha_n)$ such that the diagram

$$
\begin{array}{cccccc}
X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} \cdots & \xrightarrow{f_{n-1}} & X_n & \xrightarrow{f_n} & X_1 \\
\downarrow \alpha_1 & & \downarrow \alpha_2 & & \cdots & & \downarrow \alpha_{n-1} & & \downarrow \alpha_n & & \downarrow \alpha_1 \\
Y_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} \cdots & \xrightarrow{g_{n-1}} & Y_n & \xrightarrow{g_n} & Y_1
\end{array}
$$

(37)

is commutative. If every component $\alpha_i$ of $\alpha$ is invertible, then $\alpha$ is said to be an $n$-regular cocycle equivalence.

It is obvious that the $n$-regular cocycle equivalence is an equivalence relation.

Definition 17. Let $\mathcal{C}$ be an $n$-regular obstructed category. A collection of all equivalence classes of $n$-regular cocycles in $\mathcal{C}$ and corresponding $n$-regular cocycle morphisms is denoted by $\mathcal{R}_{\text{reg}}(n)(\mathcal{C})$ and is said to be an $n$-regularization of $\mathcal{C}$.

Comment 18. It is obvious that the $n$-regular cocycle equivalence is an equivalence relation. Equivalence classes of this relation are just elements of $\mathcal{R}_{\text{reg}}(n)(\mathcal{C})$. Our $n$-regular cocycles and obstruction structures are unique up to an invertible $n$-regular cocycle morphisms. If $[(X, f)]$ is an equivalence class of $n$-regular cocycles, then there is the corresponding class of $n$-regular obstruction structures $e^{(n)}_X$ on it. The correspondence is a one to one.

IV Regularization of functors and related structures

We are going to introduce the concept of $n$-regular functors, natural transformations, involution, duality, and so on. All our definitions of are in general case the same like in the usual category theory [33], but the preservation of
the identity \( id_X \), is replaced by the requirement of preservation of obstructions \( e^{(n)}_X \) up to the n-regular cocycle equivalence.

It is known that for two usual categories \( \mathcal{C} \) and \( \mathcal{D} \) a functor \( F: \mathcal{C} \to \mathcal{D} \) is defined as a pair of mappings \( (F_0, F_1) \), where \( F_0 \) sends objects of \( \mathcal{C} \) into objects of \( \mathcal{D} \), and \( F_1 \) sends morphisms of \( \mathcal{C} \) into morphisms of \( \mathcal{D} \)

\[
F_1(f \circ g) = F_1(f) \circ F_1(g), \quad F_1(id_X) = id_{F_0(X)}.
\]

(38)

for \( X \in \mathcal{C}_0, F_X \in \mathcal{D}_0 \).

Let \( \mathcal{C} \) and \( \mathcal{D} \) be two n-regular obstructed categories. We postulate that all definitions are formulated on every n-regular cocycle \( (X, f) \) in \( \mathcal{C} \) up to the n-regular cocycle equivalence, and \( i = 1, 2, \ldots \) (mod n).

**Definition 19.** An or n-regular cocycle functor \( F^{(n)}: \mathcal{C} \to \mathcal{D} \) is a pair of mappings \( (F_0^{(n)}, F_1^{(n)}) \), where \( F_0^{(n)} \) sends objects of \( \mathcal{C} \) into objects of \( \mathcal{D} \), and \( F_1^{(n)} \) sends morphisms of \( \mathcal{C} \) into morphisms of \( \mathcal{D} \) such that

\[
F_1^{(n)}(f_i \circ f_{i+1}) = F_1^{(n)}(f_i) \circ F_1^{(n)}(f_{i+1}), \quad F_1^{(n)}(e^{(n)}_{X_i}) = e^{(n)}_{F_0(X_i)},
\]

(39)

where \( X \in \mathcal{C}_0 \).

**Lemma 20.** Let \( \mathcal{C} \) and \( \mathcal{D} \) be n-regular obstructed categories, and let

\[
X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots X_n \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} X_1
\]

be an n-regular cocycle in \( \mathcal{C} \). If \( F^{(n)}: \mathcal{C} \to \mathcal{D} \) is n-regular cocycle functor, then

\[
F^{(n)}(f_i) \circ e^{(n)}_{X_i} = F^{(n)}(f_i).
\]

(41)

**Proof:** It is a simple calculation

\[
F^{(n)}(f_i) = F^{(n)}(f \circ e^{(n)}_{X_i}) = F^{(n)}(f) \circ F^{(n)}(e^{(n)}_{X_i}) = F^{(n)}(f_i) \circ e^{(n)}_{F_0(X_i)}.
\]

(42)

Multifuncors can be regularized in a similar way.

Let \( F^{(n)} \) and \( G^{(n)} \) be two n-regular cocycle morphisms of the category \( \mathcal{C} \) into the category \( \mathcal{D} \).
Definition 21. An \( n \)-regular natural transformation \( s : F^{(n)} \to G^{(n)} \) of \( F^{(n)} \) into \( G^{(n)} \) is a collection of functors \( s = \{ s_{X_i} : F_0(X_i) \to G_0(X_i) \} \) such that
\[
s_{X_{i+1}} \circ F_1^{(n)}(f_i) = G_1^{(n)}(f_i) \circ s_{X_i},
\]
for \( f_i : X_i \to X_{i+1} \).

Definition 22. An \( n \)-regular obstructed monoidal category \( \mathcal{C} \equiv \mathcal{C}(\otimes, I) \) can be defined as usual, but we must remember that instead of the identity \( id_X \otimes id_Y = id_{X \otimes Y} \) we have an obstruction structure \( e_X^{(n)} = \{ e_{X_i}^{(n)} \in \text{End}(X_i); n = 1, 2, ... \} \) satisfying the condition
\[
e_X^{(n)} \otimes Y_i = e_X^{(n)} \otimes e_Y^{(n)}
\]
for every two \( n \)-regular cocycles \( (X, f) \) and \( (Y, f') \).

Let \( \mathcal{C} \) be an \( n \)-regular obstructed monoidal category. We introduce an \(*\)-operation in \( \mathcal{C} \) as a function which send every object \( X_i \) into object \( X_i^* \) called the dual of \( X_i \),
\[
X_i^{**} = X_i, \quad (X_i \otimes Y_i)^* = X_i^* \otimes Y_i^*,
\]
reverse all arrows
\[
(f \circ g)^* = g^* \circ f^*.
\]
The category \( \mathcal{C} \) equipped with such \(*\)-operation is called an \( n \)-regular obstructed monoidal category with duals.

Lemma 23. Let \( \mathcal{C} \) be an \( n \)-regular obstructed monoidal category with duals. If \( (X, f) \) is an \( n \)-regular cocycle in \( \mathcal{C} \), then there is a corresponding \( n \)-regular cocycle \( (X^*, f^*) \) in \( \mathcal{C}^* \), called the dual of \( (X, f) \).

Proof: If we reverse all arrows in \( (X, f) \) and replace all objects by the corresponding duals, then we obtain \( (X^*, f^*) \), where
\[
X_1^* \xrightarrow{f_1^*} X_n^* \xrightarrow{f_n^{*\cdot1}} \cdots \xrightarrow{f_2^*} X_2^* \xrightarrow{f_1^*} X_1^*
\]
is a sequence such that
\[
f_1^* \circ f_n^* \circ \cdots \circ f_2^* \circ f_1^* = f_1^*, \quad e_{X_1^*}^{(n)} := f_n^* \circ \cdots \circ f_2^* \circ f_1^*.
\]
where \( f_i^* : X_{i+1}^* \to X_i^*, i = 1, \ldots, n \), and \( X_{n+1}^* \equiv X_1^* \) is the dual. We have corresponding relations for all cyclic permutations. \( \square \)
**Definition 24.** An n-regular pairing $g_{\mathcal{C}}$ in an n-regular obstructed monoidal category $\mathcal{C}$ can be defined in an analogy to the usual case as a collection of mappings

$$
g_{\mathcal{C}} = \{g_{X_i} \equiv \langle -| - \rangle_{X_i} : X_i^* \otimes X_i \rightarrow I \} \tag{49}
$$

satisfying some natural consistency conditions and in addition the following regularity relations

$$
g_{X_{i+1}} \circ (f_i^* \otimes f_i) = g_{X_i}, \tag{50}
$$

and

$$
\langle e^{(n)}_{X_i^*} | X_{i} \rangle_{X_i} = \langle X_i^* | e^{(n)}_{X_i} \rangle_{X_i}, \tag{51}
$$

where $(X, f)$ is a regular $n$-cocycle in $\mathcal{C}$, and let $(X^*, f^*)$ is the corresponding duals.

It is known that an associative algebra in an ordinary category is an object $A$ of this category such that there is a multiplication $m : A \otimes A \rightarrow A$ which is also a morphism of this category satisfying some axioms like the associativity, the existence of the unity.

**Definition 25.** Let $\mathcal{C}$ be an $n$-regular obstructed monoidal category. An $n$-regular cocycle algebra $A$ in the category $\mathcal{C}$ is an object of this category equipped with an associative multiplication $m : A \otimes A \rightarrow A$ such that

$$
m \circ (e^{(n)}_A \otimes e^{(n)}_A) = e^{(n)}_A \circ m. \tag{52}
$$

Obviously such multiplication not need to be unique.

One can define an $n$-regular cocycle coalgebra or bialgebra in a similar way. A comultiplication $\Delta : A \rightarrow A \otimes A$ can be regularized according to the relation

$$
\Delta \circ e^{(n)}_A = (e^{(n)}_A \otimes e^{(n)}_A) \circ \Delta. \tag{53}
$$

**Definition 26.** Let $A$ be an $n$-regular cocycle algebra. If $A$ is also regular coalgebra such that $\Delta (ab) = \Delta (a) \Delta (b)$, then it is said to be an $n$-regular cocycle almost bialgebra.
If $\mathcal{A}$ is an $n$-regular cocycle algebra, then we denote by $hom_m(\mathcal{A}, \mathcal{A})$ the set of morphisms $s \in hom_c(\mathcal{A}, \mathcal{A})$ satisfying the condition
\[ s \circ m = m \circ (s \otimes s). \tag{54} \]

Let $\mathcal{A}$ be an $n$-regular cocycle almost bialgebra. We define the convolution product
\[ s \ast t := m \circ (s \otimes t) \circ \Delta, \tag{55} \]
where $s, t \in hom_m(\mathcal{A}, \mathcal{A})$. If $\mathcal{A}$ is a regular $n$-cocycle almost bialgebra, then the convolution product is regular.

**Definition 27.** An $2$-regular cocycle almost bialgebra $\mathcal{H}$ equipped with an element $S \in hom_m(\mathcal{H}, \mathcal{H})$ such that
\[ S \ast id_{\mathcal{H}} \ast S = S, \quad id_{\mathcal{H}} \ast S \ast id_{\mathcal{H}} = id_{\mathcal{H}}. \tag{56} \]
is said to be an $2$-regular cocycle almost Hopf algebra $\mathcal{H}$.

The above definition is a regular analogy of week Hopf algebras considered in [26]. Similar algebras has been also considered in [27].

**Lemma 28.** If $\mathcal{A}$ is an $n$-regular cocycle algebra, then there is an $n$-regular cocycle coalgebra $\mathcal{A}^\ast$ such that
\[ \langle \Delta(\xi), x_1 \otimes x_2 \rangle = \langle \xi, m(x_1 \otimes x_2) \rangle, \tag{57} \]
where $x_1, x_2 \in \mathcal{A}, \xi \in \mathcal{A}^\ast$.

**Proof:** Let us apply the regularity condition (52) to the above duality condition (57). Then the lemma follows from relations (14), (53), and (71). □

**Lemma 29.** Let $\mathcal{A}$ be an $n$-regular cocycle almost bialgebra. Then the dual $\mathcal{A}^\ast$ is also $n$-regular cocycle almost bialgebra
\[ \langle \Delta(\xi), x_1 \otimes x_2 \rangle = \langle \xi, m(x_1 \otimes x_2) \rangle, \]
\[ \langle \hat{m}(\xi \otimes \zeta), x_1 \otimes x_2 \rangle = \langle \xi \otimes \zeta, \Delta x \rangle. \tag{58} \]
Let $\mathcal{A}$ be an $n$-regular cocycle algebra. Then we can define a left $n$-regular cocycle $\mathcal{A}$-module as an object equipped with a $\mathcal{A}$-module action $\rho_M : \mathcal{A} \otimes M \to M$ such that

$$\rho_M \circ (m \otimes id_M) = \rho_M \circ (id_A \otimes \rho_M),$$

$$\rho_M \circ (e^{(n)}_A \otimes e^{(n)}_M) = e^{(n)}_M \circ \rho_M.$$  

(59)

If $\mathcal{A}$ is an $n$-regular cocycle coalgebra, then one can define an $n$-regular cocycle comodule $M$ in a similar way. For a coaction $\delta_M : \mathcal{A} \to \mathcal{A} \otimes M$ of $\mathcal{A}$ on $M$ we have the following regularity condition

$$\delta_M \circ (e^{(n)}_A \otimes e^{(n)}_M) = e^{(n)}_M \circ \rho_M,$$  

(60)

**Remark 1.** Observe that we have the following duality between $\mathcal{A}$-module action $\rho_M : \mathcal{A} \otimes M \to M$ and $\mathcal{A}^*$-comodule coactions $\delta_{M^*} : \mathcal{A}^* \to M^* \otimes \mathcal{A}^*$

$$\langle \delta_{M^*}(\xi), a \otimes x \rangle = \langle \xi, \rho_M(a \otimes x) \rangle,$$  

(61)

where $a \in \mathcal{A}, x \in M, \xi \in \mathcal{A}^*$.

\section{Regular cobordisms and TQFT}

Let $\text{Cob}$ be a directed graph of cobordisms whose objects $\text{Cob}_0$ are $d$-dimensional compact smooth and oriented manifolds without boundary and whose arrows are classes of cobordism manifolds with boundaries. We would like to discuss the corresponding $n$-regular cocycles and their meaning. For this goal we use here a parametrization such that the boundary $\partial M$ is a multicorrelated space, a disjoint sum of the ‘incoming’ boundary manifold $\Sigma_{in}$ and the ‘outgoing’ one $\Sigma_{out}$. We call them ‘physical’. The empty boundary component is also admissible. Let $\Sigma_0, \Sigma_1 \in \text{Cob}_0$, then the disjoint sum is denoted by $\Sigma_0 \amalg \Sigma_1$. For a manifold $\Sigma \in \text{Cob}_0$ there is the corresponding manifold $\Sigma^*$ with the opposite orientation.

We wish to represent quantum processes of certain physical system by cobordism manifolds $M$ with the ‘incoming’ boundary manifold $\Sigma_0$ (an ‘input’), and the ‘outgoing’ one $\Sigma_0$, (an ‘output’). The ‘incoming’ boundary manifold $\Sigma_0$ represents an initial conditions of the system, the ‘outgoing’ boundary represents the final configuration, and the cobordism manifolds
represent possible interaction of the system. Note that the same cobordism manifold $M$ but with different boundary parametrization represent different physical processes!

**Definition 30.** An 'interaction' is a triple $Σ_0 M Σ_1$, where the 'incoming' boundary manifold $Σ_0$ is multiconnected space with $m$ components and the 'outgoing' one $Σ_1$ is equipped with $n$ components, and $M$ is a class of cobordism manifolds up to parametrization preserving diffeomorfisms, $Σ_0, Σ_1 ∈ Cob_0, M ∈ Cob_1$.

**Definition 31.** The 'opposite interaction' of $Σ_0 M Σ_1$ is the 'interaction' $Σ_1 M^o_0 Σ_0$ with reversed boundary parametrization, i. e. the 'incoming' boundary of $M$ is the 'outgoing' boundary of $M^o$ and vice versa.

**Example 4.** A 'collapsion' of $Σ ∈ Cob_0$ is an arbitrary 'interaction' of the forms $Σ M Σ_0$, this means the 'incoming' boundary is $Σ$ and the 'outgoing' boundary is empty. The corresponding 'expansion' of $Σ$ is the opposite of the collapsion.

**Definition 32.** Let us denote by $Cob = (Cob_0, Cob_1)$ a directed graph whose objects are $Cob_0 ≡ Cob_0$ and arrows $Cob_1$ are 'interactions'. A composition of two 'interactions' $Σ_1 M_1 Σ_2$ and $Σ_2 M_2 Σ_3$ is an 'interaction' $Σ_1 (M_1 Σ_2 M_2) Σ_3$, where $M_1 Σ_2 M_2$ is a result of gluing $M_1$ and $M_2$ along $Σ_2$.

The trivial gluing along the empty boundary component is also admissible. For instance we can glue a 'collapsion' of $Σ$ and the corresponding 'expansion' in the trivial way. In this way we obtain an 'interaction' $Σ (M M^o) Σ$. If we glue the 'expansion' of $Σ$ and the 'collapsion' of $Σ$ along $Σ$, then we obtain a class of manifolds with empty boundaries.

**Example 5.** Classes of two dimensional surfaces with holes provide examples of string interactions.

We wish to built the temporal support semigroup as an arbitrary sequence

\[
X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_n
\]

(62)

of objects and arrows of a directed graph $C$ indexed by a discrete time. We wish to represent an 'interaction' $Σ_1 M Σ_2$ as an arrow $X_1 \xrightarrow{f} X_2$ of $C$. Obviously composable arrows $X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3$ should represent the gluing
Two 'interactions' $\sigma_1 \mathcal{M}_{\Sigma_2}$ and $\sigma_2 \mathcal{M}_{\Sigma_2}$ should be represented by the same arrow $X_1 \xrightarrow{f} X_2$ if and only if both 'interactions' are 'parallel (simultaneous) in the time'.

Let us assume that the directed graph $\mathcal{C}$ is an $n$-regular monoidal category with duals. Let $X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_n} X_n$ be an $n$-regular cocycle. If there is an equivalence $\cong$ in $\text{Ob} \mathcal{C}$ such that objects of the $n$-regular cocycle represent equivalence classes of $\cong$ and arrows represent time consequences, then we say that we have an $n$-regular TQFT.

What means here the $n$-regularity? It is natural to assume that the opposite $\mathcal{M}_{\Sigma_2}^{op}$ of $\sigma_1 \mathcal{M}_{\Sigma_2}$ should be representing by a reversed arrow $X_1 \xleftarrow{f} X_2$. The trivial 2-regularity is clear, it means that the time is reversible. We postulate that the time is directed and always run further, never back, never stop. In other words, 'our time' is not reversible in general, but it can be $n$-regular, where the regularity is nontrivial.

**Example 6.** Let us consider for instance the 2-regular 'interactions'. Let $\sigma_1 \mathcal{M}_{\Sigma_2}$ and $\sigma_2 \mathcal{M}_{\Sigma_1}$ be two interactions, then $\sigma_1 (\mathcal{M}_{\Sigma_2} \mathcal{M}_{\Sigma_1}) \Sigma_3$ and $\sigma_2 (\mathcal{M}_{\Sigma_1} \mathcal{M}_{\Sigma_2}) \Sigma_3$ can be represented as arrows $X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_1$, and $X_2 \xrightarrow{f_3} X_1 \xrightarrow{f_1} X_2$, respectively. Interactions $\mathcal{M}_{\Sigma_2} \mathcal{M}_{\Sigma_1} \mathcal{M}_{\Sigma_2} \mathcal{M}_{\Sigma_1}$ and $\mathcal{M}_{\Sigma_1} \mathcal{M}_{\Sigma_2} \mathcal{M}_{\Sigma_2} \mathcal{M}_{\Sigma_1}$ should be represented by $X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_2$, and $X_2 \xrightarrow{f_3} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_3} X_1$, respectively. Now the 2-regularity conditions are clear.

Observe that the regularity concept can be useful for the construction of quantum theory of the whole universe with nonreversible time evolution. In fact the nontrivial $n$-regularity conditions mean that all processes always go further, never back, never stop, but are cyclically repeating after $n$-steps up to an equivalence.

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