CONVOLUTION INVARIANT LINEAR FUNCTIONALS AND APPLICATIONS TO SUMMABILITY METHODS

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Abstract. Herein, we study continuous linear functionals on the set of essentially bounded functions on the real line that are invariant with respect to convolution operators. We prove the existence of the Riesz decomposition for these functionals and provide a characterization of normalized positive elements. It becomes clear that this notion has a close relation to a continuous analogue of the classical notion of Banach limits. We apply these results to summability methods and obtain a certain Tauberian theorem that can be viewed as a generalization of Wiener’s Tauberian theorem.

1. Introduction

Let $L^\infty(\mathbb{R})$ be the set of all essentially bounded functions on the real line $\mathbb{R}$ and $L^\infty_{0,+}(\mathbb{R}) = \{ \phi \in L^\infty(\mathbb{R}) : \lim_{x \to \infty} \phi(x) = 0 \}$ be the elements of $L^\infty(\mathbb{R})$ that vanish at $+\infty$. Let $L^1(\mathbb{R})$ be the group algebra of the additive group $\mathbb{R}$. In the present study, we consider exclusively real-valued functions. For any $f \in L^1(\mathbb{R})$, the symbol $F$ denotes the convolution operator on $L^\infty(\mathbb{R})$ defined as follows:

$F : L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R}), \quad (F \phi)(x) = (f \ast \phi)(x), \quad \phi \in L^\infty(\mathbb{R}),$

where $\ast$ is the convolution defined by

$(f \ast \phi)(x) = \int_{-\infty}^{\infty} \phi(x-t)f(t)dt = \int_{-\infty}^{\infty} \phi(t)f(x-t)dt, \quad x \in \mathbb{R}.$

Let $L^*_+(\mathbb{R})$ be the set of functions $f$ in $L^1(\mathbb{R})$ such that $\hat{f}(\xi) = 1$ only at point $\xi = 0$, where $\hat{f}$ is the Fourier transform of $f$ defined by

$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-ix\xi}dx, \quad \xi \in \mathbb{R}.$

Let $L^*_0(\mathbb{R})$ be the set of functions $f$ in $L^1(\mathbb{R})$ such that $f \geq 0$ and $\hat{f}(0) = 1$. Further, we define $\mathcal{L}^*_0(\mathbb{R})$ to be the set of functions $f$ in $L^1(\mathbb{R})$ such that $f \geq 0$, $\hat{f}(0) = 1$ and $|\hat{f}(\xi)| = 1$ only at point $\xi = 0$. It is noted that $F$ is a positive contraction with $\|F\| = 1$ if and only if $F$ is induced by an element $f$ of $\mathcal{L}^*_0(\mathbb{R})$. By the definitions, it is obvious that

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\[ L^1_b(\mathbb{R}) \subseteq L^1_1(\mathbb{R}) \] and \[ L^1_1(\mathbb{R}) \subseteq L^1_0(\mathbb{R}) \]. We will show in the next section that in fact, \[ L^1_0(\mathbb{R}) = L^1_3(\mathbb{R}) \] holds. Thus, it follows that \( f \in L^1_0(\mathbb{R}) \) if and only if \( f \in L^1_3(\mathbb{R}) \) and \( f \geq 0 \).

Let \( L^\infty(\mathbb{R})^* \) be the dual space of \( L^\infty(\mathbb{R}) \) and \( F^* \) be the adjoint operator of \( F \):

\[ F^*: L^\infty(\mathbb{R})^* \to L^\infty(\mathbb{R})^*, \quad (F^* \varphi)(\phi) = \varphi(F \phi). \]

Our main subjective of this paper is the convolution invariant linear functionals on \( L^\infty(\mathbb{R}) \) which vanish on \( L^\infty_{0,+}(\mathbb{R}) \). In other words, the continuous linear functionals \( \varphi \in L^\infty(\mathbb{R})^* \) such that

1. \( \varphi = F^* \varphi \),
2. \( \varphi = 0 \) on \( L^\infty_{0,+}(\mathbb{R}) \).

Let us denote the set of all \( F \)-invariant linear functionals on \( L^\infty(\mathbb{R}) \) by \( M_F \). \( M_F \) is obviously a closed subspace of \( L^\infty(\mathbb{R})^* \). We denote by \( M_F^* \) the subset of \( M_F \) that consists of all those members \( \varphi \) for which \( \varphi \geq 0 \) and \( \| \varphi \| = 1 \) holds. Note that \( M_F \) is a weak*-compact convex subset of \( L^\infty(\mathbb{R})^* \).

The first main result of this study is to provide an analytic expression for the sub-linear functional

\[ \overline{p}_F(\phi) = \sup_{\varphi \in M_F} \varphi(\phi), \quad \phi \in L^\infty(\mathbb{R}). \]

Note that the functional \( \overline{p}_F \) yields the maximal value of the elements of \( M_F \) for each fixed \( \phi \in L^\infty(\mathbb{R}) \). As it will be explained in the following section, this functional \( \overline{p}_F \) plays an important role in the study of \( M_F \). To be more precise, our result reads as follows. Let us define the sub-linear functional \( \overline{P} \) on \( L^\infty(\mathbb{R}) \) by

\[ \overline{P}(\phi) = \lim_{\theta \to \infty} \limsup_{x \to \infty} \frac{1}{\theta} \int_{x}^{x+\theta} \phi(t) dt, \quad \phi \in L^\infty(\mathbb{R}). \]

Then, for any \( F \) induced by \( f \in L^1_1(\mathbb{R}) \), it will be shown that

\[ \overline{p}_F(\phi) = \overline{P}(\phi) \]

holds for every \( \phi \in L^\infty(\mathbb{R}) \).

Furthermore, for an operator \( F \) induced by \( f \in L^1_1(\mathbb{R}) \) with \( f \geq 0 \), in other words, by \( f \in L^1_1(\mathbb{R}) \), we show that each \( \varphi \in M_F \) can be expressed uniquely as

\[ \varphi = \alpha \varphi_+ - \beta \varphi_-, \quad \| \varphi \| = \alpha \| \varphi_+ \| + \beta \| \varphi_- \|, \]

where \( \varphi_+, \varphi_- \in M_F \) and \( \alpha, \beta \geq 0 \). In other words, the elements of \( M_F \) have the Jordan decomposition.

Furthermore, we deal with an interesting representation of the sublinear functional \( \overline{P} \) as an infinite iteration of a sublinear functional related to \( F \). Let \( \overline{F}: L^\infty(\mathbb{R}) \to \mathbb{R} \) be defined by

\[ \overline{F}(\phi) = \limsup_{x \to \infty} (F\phi)(x) = \limsup_{x \to \infty} \int_{-\infty}^{\infty} \phi(x-t) f(t) dt, \quad \phi \in L^\infty(\mathbb{R}). \]
We now consider sublinear functionals $F_k, k = 1, 2, \ldots$ defined by the iteration of $F$. For $k \geq 1$, we define $F_k : L^\infty(\mathbb{R}) \to \mathbb{R}$ inductively as follows;

$$F_k(\phi) := F_{k-1}(F\phi) = \limsup_{x \to \infty} (F^k\phi)(x).$$

Note that one can write as $F_k(\phi) = \limsup_{x \to \infty} (f^*k \ast \phi)(x)$, where $f^*k$ is the k-th power of $f$ with respect to the convolution product. For an operator $F$ induced by $f \in L^1_0(\mathbb{R})$, we can easily verify that

$$F(\phi) = F_1(\phi) \geq F_2(\phi) \geq \cdots \geq F_k(\phi) \geq \cdots$$

for every $\phi \in L^\infty(\mathbb{R})$. Thus one can define the sublinear functional $F_\infty : L^\infty(\mathbb{R}) \to \mathbb{R}$ by

$$F_\infty(\phi) = \lim_{k \to \infty} F_k(\phi), \quad \phi \in L^\infty(\mathbb{R}).$$

Our second main result is the following equation. Let $F$ be an operator induced by $f \in L^1_0(\mathbb{R})$. Then, for every $\phi \in L^\infty(\mathbb{R})$, we have

$$F_\infty(\phi) = \overline{F}(\phi).$$

From the viewpoint of summability methods, this result can be interpreted as the representation of the relationship between a summability method defined by a convolution operator, and the summability method defined by a continuous version of Banach limits. As it will be described in Section 5, sublinear functionals $\overline{F}$ and $\overline{P}$ give rise to summability methods for functions on $\mathbb{R}$, which we call $F$ summability method and $P$ summability method, respectively. In particular, the class of $F$ summability methods includes most of the classical methods, including the Cesàro, Abel, and Lambert methods (see [14]). We show that for a function $\phi$ in $L^\infty(\mathbb{R})$, $F$ summability of $\phi$ implies $P$ summability of $\phi$, and provide a Tauberian condition under which the converse implication holds. This is our third main result.

We note that these results are closely related to the problems addressed in the literature. One is linear functionals on $l^\infty$ which are invariant with respect to the Cesàro operator ([2, 15, 18]), where $l^\infty$ is the set of all bounded functions on natural numbers $\mathbb{N}$. Another is its continuous version, that is, linear functionals on $L^\infty(\mathbb{R}^\times)$ of essentially bounded measurable functions on $\mathbb{R}^\times = (0, \infty)$ which are invariant with respect to the Hardy operator ([3, 15, 19]). Here, the Cesàro and Hardy operators $C$ and $H$ are defined respectively as follows:

$$C : l^\infty \to l^\infty, \quad (C\phi)(n) = \frac{1}{n} \sum_{i=1}^{n} \phi(i),$$

$$H : L^\infty(\mathbb{R}^\times) \to L^\infty(\mathbb{R}^\times), \quad (H\phi)(x) = \frac{1}{x} \int_{0}^{x} \phi(t)dt.$$

Hardy invariant means on $L^\infty(\mathbb{R}^\times)$ are defined as the elements $\varphi$ of $L^\infty(\mathbb{R}^\times)^*$ of the dual space of $L^\infty(\mathbb{R}^\times)$ that satisfy $\varphi \geq 0, \|\varphi\| = 1$, and $H^*\varphi = \varphi$, where $H^*$ is the adjoint
operator of $H$. Let $\mathcal{M}_H$ denote the set of all these linear functionals. Specifically, in [15], it has been proved that
\[
\sup_{\varphi \in \mathcal{M}_H} \varphi(\phi) = \overline{Q}(\phi)
\]
holds for every $\phi \in L^\infty(\mathbb{R}^\times)$, where $\overline{Q}$ is a sublinear functional on $L^\infty(\mathbb{R}^\times)$ defined by
\[
\overline{Q}(\phi) := \lim_{\theta \to \infty} \limsup_{x \to \infty} \frac{1}{\log \theta} \int_x^{\theta x} \frac{\phi(t)}{t} \, dt, \quad \phi \in L^\infty(\mathbb{R}^\times).
\]
We deduce this result in Section 6 based on our first main result presented above in conjunction with integration by substitution.

Similarly, Cesàro invariant means are defined as the elements $\varphi$ of $l^\infty_\ast$ of the dual space of $l^\infty$ that satisfy $\varphi \geq 0$, $\|\varphi\| = 1$, and $C^\ast \varphi = \varphi$, where $C^\ast$ is the adjoint operator of $C$. We also treat this object and show similar results to Hardy invariant means, which provide another proof of results in [15, 18].

The other subject involved with our study is a summability method which is a generalization of the Cesàro or Hölder summability methods.

For a function $\phi$ on $\mathbb{N}$, Recall that its Cesàro mean $C(\phi)$ is defined by the limit
\[
C(\phi) := \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \phi(i),
\]
provided that the limit exists. This is a summability method of the simplest type. One of the simplest generalizations of Cesàro mean are its iteration, i.e., use the Cesàro operator, define the summability methods $C_1, C_2, \ldots, C_k, \ldots$ by
\[
C_1(\phi) := C(\phi), \quad C_2(\phi) := C(C(\phi)), \ldots, \quad C_k(\phi) := C_{k-1}(C(\phi)), \ldots.
\]
These classical methods are called the Hölder summability methods (see [10]). Furthermore, we can define the $C_\infty$ summability method, that was originally introduced by [19], by the limit of the sequence $\{C_k\}_{k \geq 1}$ of the Hölder summability methods. Let us define the sublinear functionals $\{\mathcal{C}_k\}_{k \geq 1}$ on $l_\infty$ as
\[
\overline{C}_1(\phi) := \overline{C}(\phi) := \limsup_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \phi(i),
\]
and for $k = 2, 3, \cdots$,
\[
\overline{C}_k(\phi) := \overline{C}_{k-1}(C(\phi)) = \overline{C}(C^{k-1}(\phi)).
\]
Further, we define the lower version $\underline{C}_k$ of $\overline{C}_k$ by
\[
\underline{C}_1(\phi) := \underline{C}(\phi) := \liminf_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \phi(i),
\]
and for $k = 2, 3, \cdots$,
\[
\underline{C}_k(\phi) := \underline{C}_{k-1}(C(\phi)) = \underline{C}(C^{k-1}(\phi)).
\]
It is obvious that for every $\phi \in l_\infty$,
\[
\overline{C}(\phi) \geq \overline{C}_2(\phi) \geq \ldots \geq \overline{C}_n(\phi) \geq \ldots \geq \underline{C}_n(\phi) \geq \ldots \geq \underline{C}_1(\phi) \geq \underline{C}(\phi).
\]
Thus, let us define the functionals $\overline{C}_\infty$ and $C_\infty$ by

$$\overline{C}_\infty(\phi) = \lim_{k \to \infty} \overline{C}_k(\phi), \quad C_\infty(\phi) = \lim_{k \to \infty} C_k(\phi),$$

respectively. The $C_\infty$ summability method is then defined as follows: for a function $\phi$ on $\mathbb{N}$, $C_\infty(\phi) = \alpha$ if and only if $\overline{C}_\infty(\phi) = C_\infty(\phi) = \alpha$.

This summability method was studied by several researchers [4, 5, 6, 7, 8, 9]. [18] also dealt with the sublinear functional $\overline{C}_\infty$, though not from the point of view of summability methods but that of Cesàro invariant linear functionals. As an application of our second main result, we obtain an analytic expression of the sublinear functional $\overline{C}_\infty$, from which we can deduce some of the properties of the $C_\infty$ summability method, including a necessary and sufficient condition that a given $\phi \in l_\infty$ is $C_\infty$ summable.

The paper is organized as follows: in Section 2, we present some preliminary results from the theory of topological linear spaces and Fourier analysis of the Banach algebra $L^1(\mathbb{R})$. A continuous analogue of Banach limits is introduced. Further, we establish elementary results concerning the Fourier transforms of elements in $\mathcal{L}_p^1(\mathbb{R})$. In Section 3, we prove the first main result. That is, we obtain a characterization of convolution invariant functionals. In Section 4, we prove the second main result concerning the infinite iteration of sublinear functionals induced by convolution operators. To do this, we use a theorem of Katznelson and Tzafrir from operator theory. In Section 5, we present an application to summability methods. Applying the results of Section 4, we present a Tauberian theorem involving $P$ and $F$ summability methods. In Section 6, we deal with a multiplicative version of the results in Sections 3, 4 and 5. Herein, everything goes as in the case of the additive case. Accordingly, sometimes we only present the results without their proofs. In Section 7, we consider the Cesàro invariant functionals. They can be viewed as a discrete analogue of the Hardy invariant functionals, from which similar results can be obtained by elementary arguments. We also present some results on the $C_\infty$ summability method.

2. Preliminaries

Since we will be concerned with weak*-compact convex subsets of dual spaces of $L^\infty$-spaces, the following version of the Krein-Milman theorem plays an important role (see [1]).

**Proposition 2.1.** Let $X$ be a Banach space and $X^*$ be its dual space. Let $\mathcal{C}$ be a weak*-compact convex subset of $X^*$ and $S \subseteq \mathcal{C}$. The following assertions are then equivalent:

1. $\sup_{\varphi \in S} \varphi(x) = \sup_{\varphi \in \mathcal{C}} \varphi(x)$ holds for each $x \in X$.
2. $\mathcal{C} = \overline{\text{co}}(S)$, where $\overline{\text{co}}(S)$ is the closed convex hull of $S$.
3. The closure $\overline{S}$ of $S$ contains all extreme points of $\mathcal{C}$.

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $L^\infty(\mu)$ be the set of essentially bounded functions on $\Omega$. Let $\mathcal{C}$ be the set of all weak*-compact convex subsets of the positive part of the unit sphere $S^+_\infty = \{ \varphi \in L^\infty(\mu)^*: \varphi \geq 0, \|\varphi\| = 1 \}$ of $L^\infty(\mu)^*$, the dual space
of \( L^\infty(\mu) \). Let \( S \) be the set of all sublinear functionals \( q \) on \( L^\infty(\mu) \) such that \( q \geq 0 \) and \( q(1) = 1 \). Consider the partially ordered sets \((C, \subseteq)\) and \((S, \leq)\), where \( \subseteq \) is inclusion of subsets and \( \leq \) is pointwise order of functionals. Based on the above proposition, we have the isomorphism between these partially ordered sets defined by

\[
C \ni C \rightarrow q(x) = \sup_{\varphi \in C} \varphi(x) \in S.
\]

The inverse mapping is given by

\[
S \ni q \rightarrow C := \{ \varphi : \varphi(x) \leq q(x) \text{ for each } x \in L^\infty(\mu) \} \in C.
\]

This fact illustrates the importance of the study of the sublinear functional \( \sup_{\varphi \in C} \varphi(x) \) for a given class \( C \) of linear functionals since it represents the size of \( C \) in \( S_{L^\infty(\mu)^*}^+ \). Furthermore, owing to the equivalence of (1) and (3), it can be used to obtain some information about the extreme points \( \text{ex}(C) \) of \( C \). Additionally, according to the equivalence of (1) and (2), one may obtain a set of functionals \( A \) in \( C \) with a simple form such that they generate all elements of \( C \) by taking its closed convex hull: \( C = \text{co}(A) \).

Note that from the maximal value functional \( q(x) = \sup_{\varphi \in C} \varphi(x) \) of \( C \), one can obtain minimal values of \( C \). In fact, since \( \varphi(-x) \leq q(-x) \), we have \( q(x) := -\overline{q}(-x) \leq \varphi(x) \). Hence, we obtain the range of \( \varphi \in C \) for each fixed \( x \in L^\infty(\mu) \):

\[
q(x) \leq \varphi(x) \leq \overline{q}(x).
\]

These inequalities are strict in the sense that for any real number \( \alpha \in [q(x), \overline{q}(x)] \), there exists some \( \varphi \in C \) such that \( \varphi(x) = \alpha \). This is a consequence of the Hahn-Banach theorem.

For maximal values of \( \mathcal{M}_F \), the following result can be easily verified. Let \( C_{bu}(\mathbb{R}) \) be the set of all bounded, uniformly continuous functions on \( \mathbb{R} \) and let \( C_{bu}(\mathbb{R})^* \) be its dual space. Obviously, \( C_{bu}(\mathbb{R}) \) is a closed subalgebra of \( L^\infty(\mathbb{R}) \).

**Lemma 2.1.** Let \( \varphi \in \mathcal{M}_F \). Then

\[
\varphi(\phi) \leq \overline{F}(\phi)
\]

holds for every \( \phi \in L^\infty(\mathbb{R}) \).

**Proof.** Let \( \varphi_0 \) be the restriction of \( \varphi \) to the closed subspace \( C_{bu}(\mathbb{R}) \) of \( L^\infty(\mathbb{R}) \). Then by the assumption that \( \varphi \) vanishes for \( \phi \in C_{bu}(\mathbb{R}) \) with \( \lim_{x \to \infty} \phi(x) = 0 \), we have

\[
\varphi_0(\phi) \leq \lim_{x \to \infty} \sup \phi(x)
\]

for every \( \phi \in C_{bu}(\mathbb{R}) \). Hence by the \( F \)-invariance of \( \varphi \), we obtain

\[
\varphi(\phi) = \varphi_0(F\phi) \leq \lim_{x \to \infty} \sup F\phi(x) = \overline{F}(\phi).
\]

Note that this result implies that \( \overline{F}(\phi) \leq \overline{F}(\phi) \) holds for each \( \phi \in L^\infty(\mathbb{R}) \).

We turn our attention to translation invariant linear functionals on \( C_{bu}(\mathbb{R}) \), which plays an important role in the study of the convolution invariant linear functionals
on $L^\infty(\mathbb{R})$. Let us introduce linear transformations \{${T}_s$\}$_{s \in \mathbb{R}}$ on $L^\infty(\mathbb{R})$ induced by the translations on $\mathbb{R}$:

\[ T_s : L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R}), \quad (T_s \phi)(x) = \phi(x + s) \quad (s \in \mathbb{R}). \]

For simplicity, for each $s \in \mathbb{R}$, we sometimes denote the translate $f(x+s)$ of a function $f(x)$ on $\mathbb{R}$ by $f_s(x)$.

Let $M_T$ be the set of all continuous linear functionals $\varphi$ on $C_{bu}(\mathbb{R})$ invariant under the translations on $\mathbb{R}$, i.e., $\varphi \in C_{bu}(\mathbb{R})^*$ for which $T_s^* \varphi = \varphi$ holds for every $s \in \mathbb{R}$, where $T_s^*$ is the adjoint operator of $T_s$. Let $\mathcal{M}_T$ be the subset of $M_T$ whose elements $\varphi$ satisfy the conditions $\varphi \geq 0$ and $\|\varphi\| = 1$. $\mathcal{M}_T$ is a weak*-compact convex subset of $C_{bu}(\mathbb{R})^*$. We then obtain the following result easily from the Banach lattice theory.

**Theorem 2.1.** Let $\varphi \in M_T$. There exist $\varphi_+$ and $\varphi_-$ in $\mathcal{M}_T$ such that

\[ \varphi = \alpha \varphi_+ - \beta \varphi_-, \quad \|\varphi\| = \alpha \|\varphi_+\| + \beta \|\varphi_-\| \]

holds for some constants $\alpha, \beta \geq 0$.

Hence, $M_T$ is generated by $\mathcal{M}_T$. Thus, it is sufficient to consider $\mathcal{M}_T$ for the study of $M_T$. The following result provides a necessary and sufficient condition that $\varphi \in C_{bu}(\mathbb{R})^*$ belongs to $\mathcal{M}_T$ (see [15] for the proof). Let the sublinear functional $\overline{\varphi} : L^\infty(\mathbb{R}) \to \mathbb{R}$ be the one defined in the introduction.

**Theorem 2.2.** For $\varphi \in C_{bu}(\mathbb{R})^*$, $\varphi \in \mathcal{M}_T$ if and only if

\[ \varphi(\phi) \leq \overline{\varphi}(\phi) \]

holds for every $\phi \in C_{bu}(\mathbb{R})$.

We mention that this class $\mathcal{M}_T$ of linear functionals can be viewed as a continuous analogue of the classical notion of Banach limits. Recall that a Banach limit is a continuous linear functional $\varphi$ on $l_\infty$ such that $\varphi \geq 0$, $\|\varphi\| = 1$, and invariant with respect to the translation operator $T : l_\infty \to l_\infty, T\phi(n) = \phi(n+1)$. Let $\overline{\varphi}$ be the sublinear functional on $l_\infty$ defined by

\[ \overline{\varphi}(\phi) = \lim_{k \to \infty} \limsup_{n \to \infty} \frac{1}{k} \sum_{i=n}^{n+k-1} \phi(i), \quad \phi \in l_\infty. \]

Then the following result was shown in [12]: for $\varphi \in l_\infty^*$, $\varphi$ is a Banach limit if and only if $\varphi(\phi) \leq \overline{\varphi}(\phi)$ holds for every $\phi \in l_\infty$. Notice that the functional $\overline{\varphi}$ is a continuous analogue of $\overline{\varphi}$ obtained by replacing the discrete summation with integration. For more detailed exposition of translation invariant linear functionals on $C_{bu}(\mathbb{R})$ (or $L^\infty(\mathbb{R})$), see [3, 15, 19].

We use some results from the theory of Fourier analysis on $L^1(\mathbb{R})$ and $L^\infty(\mathbb{R})$. We refer to the reader [17] for details. Recall that the group algebra $L^1(\mathbb{R})$ is a Banach algebra with the convolution $\ast$ as its product. Let $I$ be a closed ideal of $L^1(\mathbb{R})$. The zero set $Z(I)$ of an ideal $I$ is defined by $Z(I) = \{ \xi \in \mathbb{R} : \hat{f}(\xi) = 0 \ \forall f \in I \}$. $Z(I)$ is always closed and for each closed set $E$ of $\mathbb{R}$, there exists a closed ideal $I$ such that
$Z(I) = E$. In particular, a closed set $E$ which is the zero set of a unique ideal $I$ of $L^1(\mathbb{R})$ is referred to as a spectral synthesis set. In other words, $E \subset \mathbb{R}$ is a spectral synthesis set if and only if for any closed ideals $I$ and $J$ of $L^1(\mathbb{R})$ with $Z(I) = Z(J) = E$, $I = J$ holds. The following is a sufficient condition for a closed set $E$ of $\mathbb{R}$ to be a spectral synthesis set (see [17]):

**Theorem 2.3.** Let $E$ be a closed set of $\mathbb{R}$ whose boundary contains no perfect set. Then $E$ is a spectral synthesis set.

Note that the celebrated Wiener’s Tauberian theorem can be viewed as a special case of this result for $E = \emptyset$.

**Theorem 2.4.** Let $I$ be a closed ideal of $L^1(\mathbb{R})$ and $Z(I) = \emptyset$. Then $I = L^1(\mathbb{R})$ holds.

In this paper, we also need a special case of this result in which $E$ is a singleton $\{0\}$.

Now, recall that $L^\infty(\mathbb{R})$ is a dual space of $L^1(\mathbb{R})$. For any closed ideal $I$ of $L^1(\mathbb{R})$, its annihilator $I^\perp = \{\phi \in L^\infty(\mathbb{R}) : \langle f, \phi \rangle = 0 \text{ for every } f \in I\}$ in $L^\infty(\mathbb{R})$ is a weak*-closed translation invariant subspace. Conversely, the annihilator $I = \Phi^\perp$ of each weak*-closed translation subspace $\Phi$ is a closed ideal of $L^1(\mathbb{R})$. Using the duality between these classes of subspaces of $L^1(\mathbb{R})$ and $L^\infty(\mathbb{R})$, one can transfer the above result on closed ideals of $L^1(\mathbb{R})$ to the context of weak*-closed invariant subspaces of $L^\infty(\mathbb{R})$ that plays an important role in our study. For a weak*-closed invariant subspace $\Phi$ of $L^\infty(\mathbb{R})$, its spectrum $\sigma(\Phi)$ in the sense of spectral synthesis is defined by $\sigma(\Phi) = \{\lambda \in \mathbb{C} : e^{i\lambda x} \in \Phi\}$, i.e., the continuous characters of $\mathbb{R}$ contained in $\Phi$. $\sigma(\Phi)$ is always closed and for each closed set $E$ of $\mathbb{R}$, there exists a weak*-closed translation invariant subspace of $L^\infty(\mathbb{R})$ such that $\sigma(\Phi) = E$. A closed subset $E$ which is the spectrum $\sigma(\Phi)$ of a unique $\Phi$ is called a spectral synthesis set. This definition of spectral synthesis sets is equal to the above one, which fact can be verified by the relation $\sigma(\Phi) = Z(I)$ for $I = \Phi^\perp$. Suppose that $\sigma(\Phi)$ is a spectral synthesis set. Let us consider the weak*-closed translation invariant subspace $\Phi_1$ generated by $\sigma(\Phi)$. Then $\Phi = \Phi_1$ holds. These facts in conjunction with Theorem 2.3 imply the following result.

**Theorem 2.5.** Let $\Phi$ be a weak*-closed translation invariant subspace of $L^\infty(\mathbb{R})$ such that the boundary of its spectrum $\sigma(\Phi)$ contains no perfect set. Then $\Phi$ is equal to the weak*-closed translation invariant subspace generated by $\sigma(\Phi)$.

In particular, we need the following special case of the above theorem where the spectrum is a singleton $\{0\}$.

**Corollary 2.1.** Let $\Phi$ be a weak*-closed invariant subspace of $L^\infty(\mathbb{R})$ with $\sigma(\Phi) = \{0\}$. Then $\Phi$ is the subspace of $L^\infty(\mathbb{R})$ consisting of the constant functions.

Now we prove the assertion that $\mathcal{L}^1_1(\mathbb{R}) = \mathcal{L}^1_2(\mathbb{R})$.

**Theorem 2.6.** $\mathcal{L}^1_1(\mathbb{R}) = \mathcal{L}^1_2(\mathbb{R})$ holds.

**Proof.** It is obvious by the definitions that $\mathcal{L}^1_1(\mathbb{R}) \subset \mathcal{L}^1_2(\mathbb{R})$. Thus, it is sufficient to show the opposite inclusion. Suppose that $f \in \mathcal{L}^1_2(\mathbb{R})$. First we show that $\hat{f}(\xi) = 1$
only at point $\xi = 0$. In fact, suppose that for some $\xi \in \mathbb{R} \setminus \{0\}$

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-i\xi x}dx = \int_{-\infty}^{\infty} f(x)\cos(\xi x)dx - i\int_{-\infty}^{\infty} f(x)\sin(\xi x)dx = 1.$$  

It follows immediately that the first term is 1 and the second term is 0. By the assumption that $\hat{f}(0) = 1$ and $f \geq 0$, if

$$\int_{-\infty}^{\infty} f(x)\cos(\xi x)dx = 1$$

holds, then it is necessary that $f = 0$ on the subset of $\mathbb{R}$ at which $\cos(\xi x)$ is negative, i.e., $\{x \in \mathbb{R} : \cos(\xi x) < 0\} = \bigcup_{n=-\infty}^{\infty} \left(\frac{\pi}{2} + \frac{2n\pi}{\xi}, \frac{3\pi}{2} + \frac{2n\pi}{\xi}\right)$. Assume that this condition is satisfied, then we have

$$\int_{-\infty}^{\infty} f(x)\cos(\xi x)dx = \sum_{n=-\infty}^{\infty} \int_{-\frac{\pi}{2} + \frac{2n\pi}{\xi}}^{\frac{\pi}{2} + \frac{2n\pi}{\xi}} f(x)\cos(\xi x)dx < \sum_{n=-\infty}^{\infty} \int_{-\frac{\pi}{2} + \frac{2n\pi}{\xi}}^{\frac{\pi}{2} + \frac{2n\pi}{\xi}} f(x)dx = 1$$

since the Lebesgue measure of the set of points at which $\cos(\xi x) = 1$ is 0. This contradicts the assumption. From the fact that the space $\mathcal{L}_b^1(\mathbb{R})$ is translation invariant, we can generalize the above result to the case of absolute values of $\hat{f}(\xi)$ by the following argument. Suppose that for some $\xi_0 \in \mathbb{R} \setminus \{0\}$,

$$|\hat{f}(\xi_0)| = \left|\int_{-\infty}^{\infty} f(x)e^{-i\xi_0 x}dx\right| = 1.$$  

Let the argument of $\hat{f}(\xi_0)$ be $\theta_0$. Hence, we have that

$$e^{-i\theta_0} \hat{f}(\xi_0) = 1 \iff e^{-i\theta_0} \int_{-\infty}^{\infty} f(x)e^{-i\xi_0 x}dx = 1 \iff \int_{-\infty}^{\infty} f(x)e^{-i(\xi_0 x + \theta_0)}dx = 1.$$  

Put $x = y - \theta_0/\xi_0$ and through integration by substitution we have

$$\int_{-\infty}^{\infty} f(x)e^{-i(\xi_0 x + \theta_0)}dx = \int_{-\infty}^{\infty} f\left(y - \frac{\theta_0}{\xi_0}\right)e^{-i\xi_0 y}dy = 1.$$  

This means that $(f_{-\theta_0/\xi_0})^\wedge(\xi_0) = 1$. However, it is obvious that $f_{-\theta_0/\xi_0} \in \mathcal{L}_b^1(\mathbb{R})$. Since we have assumed $\xi_0 \neq 0$, it contradicts the result presented above. This completes the proof.
3. Convolution invariant functionals on $L^\infty(\mathbb{R})$

In this section, we characterize $F$-invariant functionals where the convolution operator $F$ is induced by elements $f$ in $L^1_1(\mathbb{R})$. First, we obtain a characterization of $F$-invariant functionals on $C_{bu}(\mathbb{R})$, which in turn is used to derive that of $F$-invariant functionals on $L^\infty(\mathbb{R})$.

**Theorem 3.1.** For any $\phi \in C_{bu}(\mathbb{R})$ and $\varphi \in L^\infty(\mathbb{R})^*$, it is valid that

$$(F^*\varphi)(\phi) = \int_{-\infty}^{\infty} \varphi(\phi_t)f(-t)dt.$$ 

**Proof.** Since the function $T_s\phi : (-\infty, \infty) \to C_{bu}(\mathbb{R})$ is continuous, it is $C_{bu}(\mathbb{R})$-valued Bochner $f(x)dx$-integrable. Thus, by Corollary 2 of [20, p.134], we have

$$(F^*\varphi)(\phi) = \varphi(F\phi) = \varphi\left(\int_{-\infty}^{\infty} \phi(x-t)f(t)dt\right) = \varphi\left(\int_{-\infty}^{\infty} \phi_{-t}(x)f(t)dt\right) = \int_{-\infty}^{\infty} \varphi(\phi_{-t})f(t)dt.$$ 

The proof is thus complete.

The following corollary follows immediately.

**Corollary 3.1.** For any $\phi \in C_{bu}(\mathbb{R})$ and $\varphi \in L^\infty(\mathbb{R})^*$, let $\psi(s) = \varphi(\phi_s)$ where $s \in \mathbb{R}$. Then we have

$$(F^*\varphi)(\phi_s) = \int_{-\infty}^{\infty} \varphi(\phi_t)f(s-t)dt = (\psi * f)(s).$$ 

Note that $\psi$ is in $C_{bu}(\mathbb{R})$. Hence, if $\varphi \in L^\infty(\mathbb{R})^*$ is $F$-invariant, then for any $\phi \in C_{bu}(\mathbb{R})$, we have

$$\psi(x) = F\psi(x),$$ 

where $\psi(x) = \varphi(\phi_x), x \in \mathbb{R}$. This means that $\psi$ is an eigenfunction of the convolution operator $F$ with an eigenvalue of one.

Now, we study some basic properties of convolution operators $F$ on $L^\infty(\mathbb{R})$:

$$F : L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R}), \quad (F\phi)(x) = (f * \phi)(x).$$ 

First, we show weak* -continuity of convolution operators.

**Theorem 3.2.** $F$ is a weak* -continuous linear operator on $L^\infty(\mathbb{R})$.

**Proof.** Suppose that $\phi_\alpha, \phi \in L^\infty(\mathbb{R})$ and $w^*-\lim_\alpha \phi_\alpha = \phi$, i.e., the net $\{\phi_\alpha\}$ converges to $\phi$ in the weak* sense. It is sufficient to show that $w^*-\lim_\alpha f * \phi_\alpha = f * \phi$. Namely, for every $g \in L^1(\mathbb{R})$ we show that

$$\lim_\alpha \int_{-\infty}^{\infty} (\phi_\alpha * f)(x)g(-x)dx = \int_{-\infty}^{\infty} (f * \phi)(x)g(-x)dx.$$
Notice that this equality is equal to

\[
\lim_{\alpha} \int_{-\infty}^{\infty} \phi_{\alpha}(x)(f \ast g)(-x) dx = \int_{-\infty}^{\infty} \phi(x)(f \ast g)(-x) dx
\]

and from the assumption that \( \phi_{\alpha} \) converges \( \phi \) in the weak* sense, we obtain the result.

We can now determine the spectrum of \( F \). From \([11]\) of Theorem 13.2, we have the following result.

**Theorem 3.3.** For any convolution operator \( F \), its spectrum \( \sigma(F) \) is \( \{0\} \cup \{\lambda \in \mathbb{C} : \hat{f}(x) = \lambda \text{ for some } x \in \mathbb{R}\} \). In particular, \( \sigma(F) \setminus \{0\} \) consists of point spectrums of \( F \).

We now consider the space of eigenfunctions of a convolution operator \( F \). Let \( \lambda \in \sigma_p(F) \) be an eigenvalue of \( F \), and \( E_\lambda(F) = \{ \phi \in L^\infty(\mathbb{R}) : F\phi = \lambda \phi \} \) be the eigenfunction space of \( F \) with respect to the eigenvalue \( \lambda \). The following results are now obtained concerning the space \( E_\lambda(F) \).

**Theorem 3.4.** \( E_\lambda(F) \) is a weak*-closed invariant subspace of \( L^\infty(\mathbb{R}) \).

**Proof.** Suppose that \( \phi \in E_\lambda(F) \), i.e., \( (f \ast \phi)(x) = \lambda \phi(x) \) for every \( x \in \mathbb{R} \). Then we have \( (f \ast \phi_s)(x) = (f \ast \phi)(x + s) = \lambda \phi(x + s) = \lambda \phi_s(x) \) for every \( s \in \mathbb{R} \). Thus \( E_\lambda(F) \) is translation invariant. The assertion that \( E_\lambda(F) \) is weak*-closed is obvious from Theorem 3.2.

**Theorem 3.5.** The spectrum of \( E_\lambda(F) \) in the sense of spectral synthesis is \( \{\xi \in \mathbb{R} : \hat{f}(\xi) = \lambda\} \).

**Proof.** For any \( \xi \in \mathbb{R} \), \( e^{i\xi x} \in E_\lambda(F) \) if and only if \( f \ast e_\xi = \lambda e_\xi \). This means that \( \hat{f}(\xi)e_\xi = \lambda e_\xi \), and we obtain the result immediately.

Based on these results, we can deduce the following characterization of \( F \)-invariant functionals on \( C_{bu}(\mathbb{R}) \).

**Theorem 3.6.** Let \( f \in L^1_\lambda(\mathbb{R}) \) and \( \varphi \in C_{bu}(\mathbb{R})^* \). Then \( \varphi \) is \( F \)-invariant if and only if \( \varphi \) is translation invariant.

**Proof.** Let us assume that \( \varphi \) is \( F \)-invariant. We show that \( \varphi(T_s\phi) = \varphi(\phi) \) for every \( \phi \in C_{bu}(\mathbb{R}) \) and \( s \in \mathbb{R} \). By corollary 3.1, for any \( \phi \in C_{bu}(\mathbb{R}) \) we have

\[
\varphi(\phi_s) = \int_{-\infty}^{\infty} \varphi(\phi_t)f(s - t) dt.
\]

Thus we obtain \( \psi(s) = \varphi(\phi_s) \in E_1(F) \). But \( \sigma(E_1(F)) = \{0\} \) by Theorem 3.5 and the assumption on \( f \). This implies that \( E_1(F) = \mathbb{R} \) by Corollary 2.1, i.e., that they are constant functions. Hence, we have \( \varphi(\phi_s) = \psi(s) = \psi(0) = \varphi(\phi) \) for every \( \phi \in C_{bu}(\mathbb{R}) \) and \( s \in \mathbb{R} \). Hence, \( \varphi \) is translation invariant on \( C_{bu}(\mathbb{R}) \).
Suppose that \( \varphi \) is in \( M_T \), that is, translation invariant. We show that \( \varphi \) is \( F \)-invariant. As indicated in Theorem 3.1, we have

\[
(F^* \varphi)(\phi) = \int_{-\infty}^{\infty} \varphi(\phi_t)f(-t)dt = \int_{-\infty}^{\infty} \varphi(\phi)f(-t)
\]

\[
= \varphi(\phi) \int_{-\infty}^{\infty} f(-t)dt = \varphi(\phi),
\]

which shows that \( \varphi \) is \( F \)-invariant. This completes the proof.

The lemma below is necessary to obtain the characterization of \( \mathcal{M}_F \).

**Lemma 3.1.** For any \( \phi \in L^\infty(\mathbb{R}) \) and \( f \in L^1(\mathbb{R}) \) with \( \hat{f}(0) = 1 \), \( \mathcal{P}(f \ast \phi - \phi) = 0 \) holds.

**Proof.** First, by direct computation, we can obtain

\[
\mathcal{P}(\phi - f \ast \phi) = \lim_{\theta \to \infty} \limsup_{x \to \infty} \frac{1}{\theta} \int_{x}^{x+\theta} (\phi - f \ast \phi)(t)dt
\]

\[
= \lim_{\theta \to \infty} \limsup_{x \to \infty} \frac{1}{\theta} \int_{x}^{x+\theta} \left\{ \phi(t) - \int_{-\infty}^{\infty} \phi(t-s)f(s)ds \right\} dt
\]

\[
= \lim_{\theta \to \infty} \limsup_{x \to \infty} \frac{1}{\theta} \int_{x}^{x+\theta} \int_{-\infty}^{\infty} \{\phi(t) - \phi(t-s)\} f(s)ds dt
\]

\[
= \lim_{\theta \to \infty} \limsup_{x \to \infty} \frac{1}{\theta} \int_{x}^{x+\theta} \{\phi(t) - \phi(t-s)\} dt \int_{-\infty}^{\infty} f(s)ds.
\]

For any \( \varepsilon > 0 \), we can choose \( R > 0 \) such that \( \int_{-R}^{R} f(s)ds = 1 - \varepsilon \). Observe that

\[
\left| \frac{1}{\theta} \int_{x}^{x+\theta} \{\phi(t) - \phi(t-s)\} dt \right| = \left| \frac{1}{\theta} \int_{x-s}^{x} \phi(t)dt + \frac{1}{\theta} \int_{x+\theta-s}^{x+\theta} \phi(t)dt \right|
\]

\[
\leq \frac{2\|\phi\|_{\infty} s}{\theta}
\]

and

\[
\left| \frac{1}{\theta} \int_{x}^{x+\theta} \{\phi(t) - \phi(t-s)\} dt \right| \leq \frac{1}{\theta} \cdot 2\|\phi\|_{\infty} = 2\|\phi\|_{\infty}.
\]
Hence, we have
\[
\left| \frac{1}{\theta} \int_{x}^{x+\theta} \{ \phi(t) - \phi(t-s) \} dt \int_{-\infty}^{\infty} f(s) ds \right|
\leq \left| \frac{1}{\theta} \int_{x}^{x+\theta} \{ \phi(t) - \phi(t-s) \} dt \int_{-R}^{R} f(s) ds \right|
+ \left| \frac{1}{\theta} \int_{x}^{x+\theta} \{ \phi(t) - \phi(t-s) \} dt \int_{(-\infty,-R]\cup[R,\infty)} f(s) ds \right|
\leq \frac{2\|\phi\|_{\infty} R}{\theta} + 2\|\phi\|_{\infty}\varepsilon,
\]
which tends to 0 as \( \theta \) tends to \( \infty \). We obtain the result.

**Corollary 3.2.** For any \( \phi \in L^\infty(\mathbb{R}) \) and \( f \in L^1(\mathbb{R}) \) with \( \hat{f}(0) = 1 \), \( \mathcal{P}(f \ast \phi) = \mathcal{P}(\phi) \) holds.

We provide a characterization of normalized positive \( F \)-invariant functionals \( \mathcal{M}_F \) for a convolution operator induced by an element in \( \mathcal{L}_1(\mathbb{R}) \).

**Theorem 3.7.** Let \( f \in \mathcal{L}_1(\mathbb{R}) \) and \( F \) be the induced convolution operator. For \( \varphi \in L^\infty(\mathbb{R})^\ast \), \( \varphi \in \mathcal{M}_F \) if and only if
\[
\varphi(\phi) \leq \mathcal{P}(\phi)
\]
holds for every \( \phi \in L^\infty(\mathbb{R}) \).

**Proof.** Suppose that \( \varphi \) is in \( \mathcal{M}_F \). By Theorem 3.6, \( \varphi \) is translation invariant on \( C_{bu}(\mathbb{R}) \), and thus in \( \mathcal{M}_T \). Then, by Theorem 2.2,
\[
\varphi(\phi) \leq \mathcal{P}(\phi)
\]
holds for every \( \phi \in C_{bu}(\mathbb{R}) \). For \( \phi \in L^\infty(\mathbb{R}) \), by Corollary 3.2 and \( F \)-invariance of \( \varphi \), we have
\[
\varphi(\phi) = \varphi(f \ast \phi) \leq \mathcal{P}(f \ast \phi) = \mathcal{P}(\phi),
\]
which proves the necessity. Conversely, if
\[
\varphi(\phi) \leq \mathcal{P}(\phi)
\]
holds for every \( \phi \in L^\infty(\mathbb{R}) \), then it is valid that
\[
\mathcal{P}(\phi - f \ast \phi) \leq \varphi(\phi - f \ast \phi) \leq \mathcal{P}(\phi - f \ast \phi).
\]
By the proof of Lemma 3.1, it is easy to see that \( \mathcal{P}(\phi - f \ast \phi) = 0 \) is also valid. Thus, we have
\[
\varphi(\phi - f \ast \phi) = 0.
\]
Hence,
\[
F^* \varphi(\phi) = \varphi(\phi)
\]
holds for every \( \phi \in L^\infty(\mathbb{R}) \), which implies that \( \varphi \) is \( F \)-invariant.
Moreover, under the additional assumption on \( f \) that \( f \geq 0 \), we have the following theorem concerning general \( F \)-invariant functionals \( M_F \).

**Theorem 3.8.** Let \( F \) be a convolution operator induced by an element \( f \) of \( \mathcal{L}_b^1(\mathbb{R}) \). Then, for any \( \varphi \in M_F \), \( \varphi \) have the Jordan decomposition in \( M_F \); i.e., there exist some positive elements \( \varphi_+ \) and \( \varphi_- \) in \( M_F \) such that

\[
\varphi = \varphi_+ - \varphi_-, \quad \|\varphi\| = \|\varphi_+\| + \|\varphi_-\|
\]

hold.

**Proof.** Let us denote by \( \varphi_0 \) the restriction of \( \varphi \) to \( C_{ba}(\mathbb{R}) \). Since \( \varphi_0 \) is translation invariant on \( C_{ba}(\mathbb{R}) \) by Theorem 3.6, \( \varphi_0 \) can be decomposed by Theorem 2.1 as

\[
\varphi_0 = \varphi_{0,+} - \varphi_{0,-},
\]

where \( \varphi_{0,+}, \varphi_{0,-} \in M_T \) are positive and \( \|\varphi_0\| = \|\varphi_{0,+}\| + \|\varphi_{0,-}\| \). We define continuous linear functionals \( \varphi_{0,+} \) and \( \varphi_{0,-} \) on \( L^\infty(\mathbb{R}) \), which are positive and \( F \)-invariant as

\[
\varphi_{0,+}(\phi) = \varphi_{0,+}(F\phi), \quad \varphi_{0,-}(\phi) = \varphi_{0,-}(F\phi).
\]

The positivity of \( \varphi_{0,+} \) and \( \varphi_{0,-} \) follows from the positivity of \( \varphi_{0,+}, \varphi_{0,-} \), and \( F \). \( F \)-invariance of \( \varphi_{0,+} \) and \( \varphi_{0,-} \) is straightforward from the definitions. Furthermore, note that both \( \varphi_{0,+} \) and \( \varphi_{0,-} \) are extensions of \( \varphi_{0,+} \) and \( \varphi_{0,-} \), respectively. In fact, since \( \varphi_{0,+} \) and \( \varphi_{0,-} \) are in \( M_T \), they are also \( F \)-invariant on \( C_{ba}(\mathbb{R}) \). Thus we have that, for any \( \phi \in C_{ba}(\mathbb{R}) \), \( \varphi_{0,+}(\phi) = \varphi_{0,+}(F\phi) = \varphi_{0,+}(\phi) \). In the same way, we can show that \( \varphi_{0,-}(\phi) = \varphi_{0,-}(\phi) \) for every \( \phi \in C_{ba}(\mathbb{R}) \). Observe that

\[
|\varphi_{0,+}(\phi)| = |\varphi_{0,+}(f \ast \phi)| \leq \|\varphi_{0,+}\| \cdot \|f \ast \phi\|_\infty \leq \|\varphi_{0,+}\| \cdot \|\phi\|_\infty,
\]

which implies that \( \|\varphi_{0,+}\| \leq \|\varphi_{0,+}\| \). Note that \( \|\varphi_{0,+}\| \geq \|\varphi_{0,+}\| \) follows by the fact that \( \varphi_{0,+} \) is an extension of \( \varphi_{0,+} \). Thus, we obtain \( \|\varphi_{0,+}\| = \|\varphi_{0,+}\| \). In the same way, we have \( \|\varphi_{0,-}\| = \|\varphi_{0,-}\| \).

We set \( \varphi_0 = \varphi_{0,+} - \varphi_{0,-} \) and then \( \varphi_0 \) is obviously \( F \)-invariant. Note that \( \varphi_0 = \varphi \) on \( C_{ba}(\mathbb{R}) \), which implies that \( \varphi_0 = \varphi \) by the \( F \)-invariance of \( \varphi_0 \) and \( \varphi \). We also note that \( \|\varphi_0\| = \|\varphi_0\| \) can be proved as above. Thus, we obtain

\[
\|\varphi\| = \|\varphi_0\| = \|\varphi_0\| = \|\varphi_{0,+}\| + \|\varphi_{0,-}\| = \|\varphi_{0,+}\| + \|\varphi_{0,-}\|.
\]

Therefore, setting \( \varphi_+ := \varphi_{0,+} \) and \( \varphi_- := \varphi_{0,-} \), we obtain the desired decomposition of \( \varphi \).

4. **Iteration of sublinear functional \( \overline{F} \)**

In this section, we deal with sublinear functionals \( \overline{F}_\infty \) induced by a functional \( \overline{F} \) through infinite iteration. We need the following result from operator theory (see [13]): for any contraction \( U \) on a Banach space \( X \), we put

\[
\Gamma(U) = \sigma(U) \cap \Gamma,
\]

where \( \sigma(U) \) denotes the spectrum of \( U \) and \( \Gamma \) denotes the unit circle \( \{ z \in \mathbb{C} : |z| = 1 \} \). \( \Gamma(U) \) is called the peripheral spectrum of \( U \).
Theorem 4.1. Let $U$ be a linear contraction on a Banach space $X$. Then, \( \lim_{n \to \infty} \|U^n - U^{n+1}\| = 0 \) if and only if the peripheral spectrum $\Gamma(U)$ of $U$ consists of at most the point $z = 1$.

Using this theorem and results in Section 3, we can obtain the following result.

Theorem 4.2. Let $f \in \mathcal{L}^1_\flat(\mathbb{R})$ and $F$ be the induced convolution operator. Then
$$
F\infty(\phi) = \overline{P}(\phi)
$$
holds for every $\phi \in L^\infty(\mathbb{R})$.

Proof. First, note that the convolution operator $F$ induced by $f \in \mathcal{L}^1_\flat(\mathbb{R})$ satisfies $\|F\| = 1$ by the assumption that $\hat{f}(0) = 1$ and $f \geq 0$. Thus, $F$ is a contraction on $L^\infty(\mathbb{R})$. Furthermore, by Theorems 2.6 and 3.3, $\Gamma(F) = 1$ holds. Thus, consideration of Theorem 4.1 yields
$$
\lim_{n \to \infty} \|F^n - F^{n+1}\| = 0.
$$
Thus, for any $\phi \in L^\infty(\mathbb{R})$, we have
$$
\lim_{n \to \infty} \|F^n \phi - F^{n+1} \phi\|_{\infty} = 0.
$$
Additionally, for each $\phi \in L^\infty(\mathbb{R})$,
$$
F\infty(\phi - F\phi) = \lim_{n \to \infty} F_n(\phi - F\phi) = \lim_{n \to \infty} \limsup_{x \to \infty} (F^n \phi(x) - F\phi(x))
$$
$$
= \lim_{n \to \infty} \limsup_{x \to \infty} (F^n \phi(x) - F^{n+1} \phi(x))
$$
$$
\leq \lim_{n \to \infty} \|F^n \phi - F^{n+1} \phi\|_{\infty} = 0.
$$

Note that in the same way $F\infty(\phi - f * \phi) = 0$ can be also proved. Let $P$ and $F_\infty$ be the weak*-closed convex subsets of $L^\infty(\mathbb{R})^*$ defined by $\varphi \in P$ if and only if $\varphi(\phi) \leq \overline{P}(\phi)$ for all $\phi \in L^\infty(\mathbb{R})$ and $\varphi \in F_\infty$ if and only if $\varphi(\phi) \leq F_\infty(\phi)$ for all $\phi \in L^\infty(\mathbb{R})$, respectively. Since
$$
F_\infty(\phi - f * \phi) \leq \varphi(\phi - f * \phi) \leq \overline{P}(\phi - f * \phi)
$$
holds for each $\varphi \in F_\infty$, we have that for any $\varphi \in F_\infty$, $\varphi$ is $F$-invariant. Therefore, by Theorem 3.7, $\varphi \in P$ holds. Hence, we have
$$
F_\infty(\phi) = \sup_{\varphi \in F_\infty} \varphi(\phi) \leq \sup_{\varphi \in P} \varphi(\phi) = \overline{P}(\phi)
$$
for every $\phi \in L^\infty(\mathbb{R})$. Conversely, by Corollary 3.2, note that the sublinear functional $\overline{P}(\phi)$ is $F$-invariant: i.e., $\overline{P}(\phi) = \overline{P}(F\phi) = \overline{P}(F^2 \phi) = \cdots = \overline{P}(F^k \phi) = \cdots$ holds for every $\phi \in L^\infty(\mathbb{R})$. Hence, for each $k \geq 1$ we have
$$
\overline{P}(\phi) = \overline{P}(F^k \phi) \leq \limsup_{x \to \infty} F^k \phi(x) = F^k_\infty(\phi).
$$
Thus, we obtain
$$
\overline{P}(\phi) \leq \lim_{k \to \infty} F^k_\infty(\phi) = F_\infty(\phi).
$$
Hence, we have obtained the desired equation

\[ F_\infty(\phi) = \overline{P}(\phi) \]

for each \( \phi \in L^\infty(\mathbb{R}) \).

In conjunction with Theorems 3.7 and 3.8, we have the following:

**Theorem 4.3.** Let \( f \in L^1_0(\mathbb{R}) \) and \( F \) be the induced convolution operator. Then, for \( \varphi \in L^\infty(\mathbb{R})^* \), \( \varphi \in M_F \) if and only if

\[ \varphi(\phi) \leq \overline{P}(\phi) \]

holds for every \( \phi \in L^\infty(\mathbb{R}) \). For \( \varphi \in M_F \), there exists unique elements \( \varphi_+ \) and \( \varphi_- \) in \( M_F \) and nonnegative numbers \( \alpha \) and \( \beta \) such that

\[ \varphi = \alpha \varphi_+ - \beta \varphi_- \quad \| \varphi \| = \alpha \| \varphi_+ \| + \beta \| \varphi_- \|. \]

Furthermore, for every \( \phi \in L^\infty(\mathbb{R}) \), the formula

\[ \overline{F}_\infty(\phi) = \overline{P}(\phi) \]

holds.

5. **Relationship between Wiener’s Tauberian theorem and almost convergence**

In this section, we deal with the relationship between two summability methods on \( L^\infty(\mathbb{R}) \), i.e., summability methods defined via a convolution operator \( F \) and a continuous analogue of almost convergence. For any \( f \in L^1(\mathbb{R}) \), let us define a summability method \( F \) by

\[ F(\phi) = \lim_{x \to \infty} \int_{-\infty}^{\infty} \phi(x-t)f(t)dt, \]

provided that the limit exists. Note that the functional \( \underline{F}(\phi) := -\overline{F}(-\phi) \) on \( L^\infty(\mathbb{R}) \) can be expressed by

\[ \underline{F}(\phi) = \liminf_{x \to \infty} \int_{-\infty}^{\infty} \phi(x-t)f(t)dt. \]

It is then obvious that \( F(\phi) = \alpha \) if and only if \( \overline{F}(\phi) = \underline{F}(\phi) = \alpha \). The particularly important case is that when \( f \in L^1(\mathbb{R}) \) is a Wiener kernel, i.e., the zero set \( Z(f) \) of the Fourier transform of \( f \) is empty. The importance of Wiener kernels is illustrated by the following Wiener’s Tauberian theorem.

**Theorem 5.1.** Let \( f \in L^1(\mathbb{R}) \) be a Wiener kernel with \( \hat{f}(0) = 1 \). Let \( g \in L^1(\mathbb{R}) \) with \( \hat{g}(0) = 1 \). Then, for any \( \phi \in L^\infty(\mathbb{R}) \) if

\[ \lim_{x \to \infty} \int_{-\infty}^{\infty} \phi(x-t)f(t)dt = \alpha \]

then

\[ \lim_{x \to \infty} \int_{-\infty}^{\infty} \phi(x-t)g(t)dt = \alpha. \]
We now refer to the following simple observation that yields another simpler formulation of Wiener’s Tauberian theorem.

**Theorem 5.2.** Let \( f \in L^1(\mathbb{R}) \) be a Wiener kernel with \( \hat{f}(0) = 1 \). Then, for any \( \phi \in L^\infty(\mathbb{R}) \) if
\[
\lim_{x \to \infty} \int_{-\infty}^{\infty} \phi(x-t)f(t)dt = \alpha
\]
then
\[
w^*- \lim_{s} \phi_s(x) = \alpha,
\]
where the symbol \( w^*- \lim \) denotes the limit in the weak*-topology of \( L^\infty(\mathbb{R}) \).

**Proof.** Since limit is invariant with respect to translation, for any \( s \in \mathbb{R} \) we have
\[
\lim_{x \to \infty} \int_{-\infty}^{\infty} \phi(x-t)f(t)dt = 0 \iff \lim_{x \to \infty} \int_{-\infty}^{\infty} \phi_x(-t)f(t)dt = 0
\]
\[
\iff \lim_{x \to \infty} \int_{-\infty}^{\infty} \phi_x(t+s)f(-t)dt = 0
\]
\[
\iff \lim_{x \to \infty} \int_{-\infty}^{\infty} \phi_x(t)f(s-t)dt = 0.
\]
Hence, for any element \( h \) in the closed linear hull of the translates \( \{f_s(-t)\}_{s \in \mathbb{R}} \), we have
\[
\lim_{x \to \infty} \int_{-\infty}^{\infty} \phi_x(t)h(t)dt = 0.
\]
By the assumption that \( f \) is a Wiener kernel, these functions \( h \) consist of all functions in \( L^1(\mathbb{R}) \), and the result follows immediately.

This theorem implies that any summability method \( F \) induced by a Wiener kernel \( f \in L^1(\mathbb{R}) \) with \( \hat{f}(0) = 1 \) is equivalent to the summability method \( W \) defined by \( W(\phi) = \alpha \) if and only if \( w^*- \lim_s \phi_s(x) = \alpha \). Since the expression of \( W \) has the advantage of being independent of specific Wiener kernels, we use it occasionally thereafter.

Now we define a summability method defined via the functional \( P \). Note that the functional \( P(\phi) := -\overline{P}(-\phi) \) can be expressed by
\[
P(\phi) = \lim_{\theta \to \infty} \liminf_{x \to \infty} \frac{1}{\theta} \int_{x}^{x+\theta} \phi(t)dt.
\]
Then, for \( \phi \in L^\infty(\mathbb{R}) \) we define \( P(\phi) = \alpha \) if and only if \( \overline{P}(\phi) = P(\phi) = \alpha \). As the following theorems illustrate, this can be viewed as a continuous version of the classical notion of almost convergence introduced by Lorentz \([16]\).

**Theorem 5.3.** Let \( \phi \in L^\infty(\mathbb{R}) \). Then \( P(\phi) = \alpha \) if and only if for every \( \varepsilon > 0 \) there exists a constant \( R \geq 0 \) such that if \( \theta \geq R \), then
\[
\left| \frac{1}{\theta} \int_{x}^{x+\theta} \phi(t)dt - \alpha \right| \leq \varepsilon
\]
for sufficiently large \( x \geq 0 \).

**Proof.** First, we show the sufficiency. Suppose that the above assertion holds. We show that \( \overline{P}(\phi) = \underline{P}(\phi) = \alpha \). For any fixed \( \varepsilon > 0 \), there exists \( R \geq 0 \) such that if \( \theta \geq R \), then

\[
\alpha - \varepsilon \leq \frac{1}{\theta} \int_x^{x+\theta} \phi(t) dt \leq \alpha + \varepsilon
\]

for sufficiently large \( x \geq 0 \). This implies that

\[
\alpha - \varepsilon \leq \liminf_{x \to \infty} \frac{1}{\theta} \int_x^{x+\theta} \phi(t) dt \leq \limsup_{x \to \infty} \int_x^{x+\theta} \phi(t) dt \leq \alpha + \varepsilon
\]

whenever \( \theta \geq R \). Since \( \varepsilon > 0 \) is arbitrary, we have

\[
\lim \liminf_{\theta \to \infty} \frac{1}{\theta} \int_x^{x+\theta} \phi(t) dt = \lim \limsup_{\theta \to \infty} \int_x^{x+\theta} \phi(t) dt = \alpha.
\]

The desired result is thus obtained.

Next we prove the necessity. Suppose that \( \overline{P}(\phi) = \underline{P}(\phi) = \alpha \), i.e., \( \overline{P}(\phi) = \underline{P}(\phi) = \alpha \). Then, for any \( \varepsilon > 0 \), there exists a constant \( R \geq 0 \) such that

\[
\liminf_{x \to \infty} \frac{1}{\theta} \int_x^{x+\theta} \phi(t) dt \geq \alpha - \frac{\varepsilon}{2}
\]

and

\[
\limsup_{x \to \infty} \frac{1}{\theta} \int_x^{x+\theta} \phi(t) dt \leq \alpha + \frac{\varepsilon}{2}
\]

whenever \( \theta \geq R \). Furthermore, we can choose a constant \( R_\theta \geq 0 \) such that

\[
\alpha - \varepsilon \leq \frac{1}{\theta} \int_x^{x+\theta} \phi(t) dt \leq \alpha + \varepsilon
\]

whenever \( x \geq R_\theta \). Hence, for any \( \varepsilon > 0 \), there exists \( R \geq 0 \) such that if \( \theta \geq R \) then

\[
\left| \frac{1}{\theta} \int_x^{x+\theta} \phi(t) dt - \alpha \right| \leq \varepsilon
\]

for \( x \geq R_\theta \). The proof is now complete.

We note that this assertion is equivalent to the following apparently stronger condition:

**Theorem 5.4.** Let \( \phi \in L^\infty(\mathbb{R}) \). Then \( P(\phi) = \alpha \) if and only if

\[
\lim \frac{1}{\theta} \int_x^{x+\theta} \phi(t) dt = \alpha
\]

holds uniformly in \( x \geq 0 \).
The proof is the same as the proof of Theorem 14 of [19], where the assertion is proved for the elements of \(C_{ba}(\mathbb{R})\).

Now, we direct attention to the relationship between summability methods \(F\) and \(P\). First, we show an Abelian theorem. The following result shows that almost convergence \(P\) is stronger than any summability method \(F\) induced by \(f \in L^1(\mathbb{R})\) with \(\hat{f}(0) = 1\).

**Theorem 5.5.** Let \(\phi \in L^\infty(\mathbb{R})\). Let \(f \in L^1(\mathbb{R})\) satisfy \(\hat{f}(0) = 1\). If
\[
\lim_{x \to \infty} \int_{-\infty}^{\infty} \phi(t)f(x-t)dt = \alpha
\]
then
\[
P(\phi) = \alpha.
\]

**Proof.** Let \(\phi \in L^\infty(\mathbb{R})\). By Corollary 3.2, we have
\[
P(\phi) = P(f \ast \phi) \leq \limsup_{x \to \infty} (f \ast \phi)(x) = F(\phi).
\]
Further, we have
\[
P(\phi) = -P(-\phi) = -P(-f \ast \phi) \geq \liminf_{x \to \infty} (f \ast \phi)(x) = F(\phi).
\]
Thus, we obtain the following relation
\[
F(\phi) \leq P(\phi) \leq P(\phi) \leq F(\phi).
\]
This implies the required result.

Next we consider a Tauberian theorem, i.e., we show that under a certain condition, \(P(\phi) = \alpha\) implies \(F(\phi) = \alpha\). We begin with the following lemma.

**Lemma 5.1.** Let \(\phi \in L^\infty(\mathbb{R})\). Then the following conditions are equivalent:

1. For every \(s \in \mathbb{R}\), \(w^*\lim_x (\phi_{x+s} - \phi_x) = 0\), i.e., \(W(\phi_s - \phi) = 0\);
2. \(\int_{-\infty}^{\infty} \phi(x-t)f(t)dt = 0\) for some \(f \in L^1(\mathbb{R})\) with \(Z(f) = \{0\}\).

**Proof.** (1) \(\Rightarrow\) (2): Let \(f \in L^1(\mathbb{R})\) be such that \(Z(f) = \emptyset\) and \(s\) be a real number. Then, by the assumption (1) we have
\[
\lim_{x \to \infty} \int_{-\infty}^{\infty} \{\phi_{x+s}(t) - \phi_x(t)\} f(-t)dt = 0
\]
\[
\iff \lim_{x \to \infty} \int_{-\infty}^{\infty} \phi_x(t)\{f(s-t) - f(-t)\} dt = 0
\]
\[
\iff \lim_{x \to \infty} \int_{-\infty}^{\infty} \phi(x-t)\{f_s(t) - f(t)\} dt = 0.
\]
Put \(g_s = f_s - f\) and then \(\hat{g}_s(\xi) = (e^{i\xi s} - 1)\hat{f}(\xi)\). Since we have assumed that \(Z(f) = \emptyset\) and \(s\) be an arbitrary, the zero set of the closed linear hull of the translates of functions \(\{g_s\}_{s \in \mathbb{R}}\) is \(\{0\}\). Since \(\{0\}\) is a spectral synthesis set, for any \(h \in L^1(\mathbb{R})\) with \(Z(h) = \{0\}\), we have \(H(\phi) = 0\) and thus (2) holds.

(2) \(\Rightarrow\) (1): This can be proved in the same way as above.
Theorem 5.6. Let $\phi \in L^\infty(\mathbb{R})$. Then $W(\phi) = \alpha$ if and only if $P(\phi) = \alpha$ and one of the two conditions in Lemma 5.1 holds. In other words, either condition of Lemma 5.1 is a Tauberian condition under which $P$ summability implies $F$ summability.

Proof. The necessity part is obvious. Thus, we prove the sufficiency. Note that by Theorem 5.1, without loss of generality, we can assume that the Wiener kernel $f$ is in $L^1_0(\mathbb{R})$. Observe that

$$\int_{-\infty}^{\infty} \{ \phi(t) - (F\phi)(t) \} f(x-t) dt = \int_{-\infty}^{\infty} \phi(t) \{ f(x-t) - f^{*2}(x-t) \} dt.$$ 

Put $g = f - f^{*2}$ and we have $\hat{g}(\xi) = \hat{f}(\xi)(1 - \hat{f}(\xi))$. Thus, we obtain $Z(g) = \{0\}$. Therefore, by the assumption, the right-hand side of the above equation is 0. Hence, we have

$$F(\phi - F\phi) = 0,$$

which implies that $F(F^k\phi - F^{k+1}\phi) = 0$ for every $k \geq 1$. Thus, by Theorem 4.2, we have

$$F(\phi) = \lim_{k \to \infty} F_k(\phi) = \mathcal{F}(\phi) = \alpha.$$ 

In the same way, we have

$$F(\phi) = \lim_{k \to \infty} F_k(\phi) = P(\phi) = \alpha.$$ 

Therefore, we obtain the result $F(\phi) = \mathcal{F}(\phi) = \mathbb{F}(\phi) = \alpha$.

6. Mellin convolution invariant functionals on $L^\infty(\mathbb{R}_\times)$

In this section, we deal with a multiplicative version of the preceding sections. Let $\mathbb{R}_\times = (0, \infty)$ be the positive part of the multiplicative group of the real field $\mathbb{R}$. Let $L^\infty(\mathbb{R}_\times)$ be the set of all essentially bounded functions on $\mathbb{R}$ and $L^\infty(\mathbb{R}_\times)^{0,+}$ be the subspace of $L^\infty(\mathbb{R}_\times)$ whose elements vanish at $+\infty$. Let $L^1(\mathbb{R}_\times)$ be the group algebra of $\mathbb{R}_\times$. Note that the Haar measure of the positive multiplicative group $\mathbb{R}_\times$ is $\frac{dt}{t}$. For any $g \in L^1(\mathbb{R}_\times)$, let the symbol $G$ denote the convolution operator as follows:

$$G : L^\infty(\mathbb{R}_\times) \to L^\infty(\mathbb{R}_\times), \quad (G\phi)(x) = (g * M\phi)(x), \quad \phi \in L^\infty(\mathbb{R}_\times),$$

where the Mellin convolution $M$ of $\phi \in L^\infty(\mathbb{R}_\times)$ and $g \in L^1(\mathbb{R}_\times)$ is defined by

$$(g * M\phi)(x) = \int_0^{\infty} \phi(x/t) g(t) \frac{dt}{t} = \int_0^{\infty} \phi(t) g(x/t) \frac{dt}{t}.$$
Let $\mathcal{L}_1^1(\mathbb{R}^\times)$ be the set of functions $g$ in $L^1(\mathbb{R}^\times)$ such that $\hat{g}(\xi) = 1$ only at point $\xi = 0$, where $\hat{g}$ is the Fourier transform of $g$ defined by

$$\hat{g}(x) = \int_0^\infty g(t)e^{ixt}dt,$$

where $t^{ix} = e^{ix\log t}$. Let $\mathcal{L}_g^1(\mathbb{R}^\times)$ be the set of functions $g$ in $L^1(\mathbb{R}^\times)$ such that $g \geq 0$ and $\hat{g}(0) = 1$. Further, we define $\mathcal{L}_g^1(\mathbb{R}^\times)$ be the set of functions $g$ in $L^1(\mathbb{R}^\times)$ such that $g \geq 0$, $\hat{g}(0) = 1$ and $|\hat{g}(\xi)| = 1$ only at point $\xi = 0$. The relation $\mathcal{L}_g^1(\mathbb{R}^\times) \subseteq \mathcal{L}_1^1(\mathbb{R}^\times)$ follows immediately from the definitions. Further, we have $\mathcal{L}_g^1(\mathbb{R}^\times) = \mathcal{L}_1^1(\mathbb{R}^\times)$, whose proof is the same as that of Theorem 2.6. Thus, we can also show that $g \in \mathcal{L}_1^1(\mathbb{R}^\times)$ if and only if $g \in \mathcal{L}_g^1(\mathbb{R}^\times)$ and $g \geq 0$.

Let $L^\infty(\mathbb{R}^\times)^*$ be the dual space of $L^\infty(\mathbb{R}^\times)$ and $G^*$ be the adjoint operator of $G$:

$$G^* : L^\infty(\mathbb{R}^\times)^* \rightarrow L^\infty(\mathbb{R}^\times)^*, \quad (G^* \varphi)(\phi) = \varphi(G^\phi).$$

Now we consider $G$-invariant linear functionals $\varphi \in L^\infty(\mathbb{R}^\times)^*$, i.e., $G^\ast \varphi = \varphi$, which vanish on $L^\infty(\mathbb{R}^\times)_{0,+}$. Let us denote by $M_G$ the set of all $G$-invariant functionals and by $M_{G'}$ the subset of $M_G$ whose elements satisfy the conditions that $\varphi \geq 0$ and $\|\varphi\| = 1$.

In particular, note that Hardy operator can be viewed as a convolution operator on $L^\infty(\mathbb{R}^\times)$ as follows:

$$(H\phi)(x) = \frac{1}{x} \int_0^x \phi(t)dt = \int_0^x \phi(t)\left(\frac{x}{t}\right)^{-1}dt = \int_0^\infty \phi(t)h(x/t)\frac{dt}{t} = (h^M \phi)(x),$$

where $h \in L^1(\mathbb{R}^\times)$ is defined as

$$h(x) = \begin{cases} x^{-1} & \text{if } x \geq 1, \\ 0 & \text{if } x < 1. \end{cases}$$

An important fact is that through the results on the additive group of $\mathbb{R}$, we can obtain analogous results for Sections 3, 4, and 5 in the multiplicative setting.

Let us define the isometry $W$ of $L^\infty(\mathbb{R}^\times)$ onto $L^\infty(\mathbb{R})$ as follows:

$$W : L^\infty(\mathbb{R}^\times) \rightarrow L^\infty(\mathbb{R}), \quad (W\phi)(x) = \phi(e^x).$$

Let $W^* : L^\infty(\mathbb{R})^* \rightarrow L^\infty(\mathbb{R}^\times)^*$ be its adjoint operator. For any $f \in L^1(\mathbb{R})$, let us define a function $g$ in $L^1(\mathbb{R}^\times)$ by $g(x) = f(\log x)$. The following commutative diagram is then obtained that can be proved easily through integration by substitution:

$$\begin{array}{ccc}
L^\infty(\mathbb{R}^\times) & \xrightarrow{G} & L^\infty(\mathbb{R}^\times) \\
W \downarrow & & \downarrow W \\
L^\infty(\mathbb{R}) & \xrightarrow{F} & L^\infty(\mathbb{R})
\end{array}$$
Notice that the mapping \( L^1(\mathbb{R}) \ni f(x) \mapsto g = f(\log x) \in L^1(\mathbb{R}^\times) \) preserves the order and spectral properties, that is, \( f \in \mathcal{L}^1(\mathbb{R}) \) if and only if \( g \in \mathcal{L}^1(\mathbb{R}^\times) \), \( f \in \mathcal{L}^1(\mathbb{R}) \) if and only if \( g \in \mathcal{L}^1(\mathbb{R}^\times) \), and \( f \in \mathcal{L}^1(\mathbb{R}) \) if and only if \( g \in \mathcal{L}^1(\mathbb{R}^\times) \).

Assume that \( \varphi \in L^\infty(\mathbb{R}^\times)^* \) is \( F \)-invariant. Then, \( W^\ast \varphi \) is \( G \)-invariant. In fact, since \( G = W^{-1}FW \) by the above diagram, we have

\[
(G^*(W^\ast \varphi))(\phi) = (W^\ast \varphi)(GW) = \varphi(WW) = \varphi(W) = (W^\ast \varphi)(\phi)
\]

for every \( \phi \in L^\infty(\mathbb{R}^\times) \). Thus, we have that \( G^*W^\ast \varphi = W^\ast \varphi \) and \( W^\ast \varphi \) is \( G \)-invariant. Therefore, we have proved the following result: The restriction of \( W^\ast \) to \( M_F \)

\[
W^\ast : M_F \to M_G
\]

is a linear isomorphism between \( M_F \) and \( M_G \). It is also obvious that the restriction of \( W^* \) to \( M_F \)

\[
W^*: M_F \to M_G
\]

is an affine isomorphism between \( M_F \) and \( M_G \).

We can easily confirm the following result by direct computation (through integration by substitution).

**Lemma 6.1.** For every \( \phi \in L^\infty(\mathbb{R}^\times) \), it holds that

\[
\overline{P}(W\phi) = \overline{Q}(\phi).
\]

We provide a multiplicative version of Theorems 3.7, 3.8, and 4.2 as follows:

**Theorem 6.1.** Let \( g \in \mathcal{L}^1(\mathbb{R}^\times) \) and \( G \) be the induced operator. Then, for \( \varphi \in L^\infty(\mathbb{R}^\times)^* \), \( \varphi \in M_G \) if and only if

\[
\varphi(\phi) \leq \overline{Q}(\phi)
\]

holds for every \( \phi \in L^\infty(\mathbb{R}^\times) \).

**Proof.** It is sufficient to prove that \( \sup_{\psi \in M_G} \psi(\phi) = \overline{Q}(\phi) \) for every \( \phi \in L^\infty(\mathbb{R}^\times) \). Note that for any \( \psi \in M_G \), there exists some \( \varphi \in M_F \) such that \( W^\ast \varphi = \psi \) by the fact mentioned above. Thus, by Theorem 3.7 and Lemma 6.1, for any \( \phi \in L^\infty(\mathbb{R}^\times) \) we have

\[
\sup_{\psi \in M_G} \psi(\phi) = \sup_{\varphi \in M_F} (W^\ast \varphi)(\phi) = \sup_{\varphi \in M_F} \varphi(W\phi) = \overline{P}(W\phi) = \overline{Q}(\phi).
\]

The following is a corresponding expression for Theorem 3.8 that can be easily deduced via the isomorphism \( W^\ast \) of \( M_F \) onto \( M_G \).

**Theorem 6.2.** Let \( G \) be a convolution operator induced by an element \( g \) of \( \mathcal{L}^1(\mathbb{R}^\times) \). Then, for any \( \varphi \in M_G \), \( \varphi \) have the Jordan decomposition in \( M_G \); i.e., there exist some positive elements \( \varphi_+ \) and \( \varphi_- \) in \( M_G \) such that

\[
\varphi = \varphi_+ - \varphi_- \quad \|\varphi\| = \|\varphi_+\| + \|\varphi_-\|
\]

hold.

The corresponding result of Theorem 4.2 is as follows.
Theorem 6.3. Let $g \in L^1_b(\mathbb{R}^\times)$ and $G$ be the induced convolution operator. Then
$$\overline{G}_\infty(\phi) = \overline{Q}(\phi)$$
holds for every $\phi \in L^\infty(\mathbb{R}^\times)$.

Proof. We only show that $\overline{G}_\infty(\phi - G\phi) = 0$ for every $\phi \in L^\infty(\mathbb{R}^\times)$. The rest of the proof is similar to that for Theorem 4.2 and is left to the reader. Observe that
$$\overline{G}_\infty(\phi - G\phi) = \lim_{n \to \infty} \overline{G}_n(\phi - G\phi) = \lim_{n \to \infty} \limsup_{x \to \infty} (G^n\phi(x) - G^{n+1}\phi(x))$$
$$= \lim_{n \to \infty} \limsup_{x \to \infty} ((W^{-1}F^nW\phi)(x) - (W^{-1}F^{n+1}W\phi)(x))$$
$$= \lim_{n \to \infty} \limsup_{x \to \infty} ((F^nW\phi)(\log x) - (F^{n+1}W\phi)(\log x))$$
$$\leq \lim_{n \to \infty} \|F^nW\phi - F^{n+1}W\phi\|_\infty = 0.$$ Note that in the last equation we use the fact from the proof of Theorem 4.2.

In conjunction with Theorems 6.1 and 6.2, we have the following.

Theorem 6.4. Let $g \in L^1_b(\mathbb{R}^\times)$ and $G$ be the induced convolution operator. Then, for $\varphi \in L^\infty(\mathbb{R}^\times)^*$, $\varphi \in M_G$ if and only if
$$\varphi(\phi) \leq \overline{Q}(\phi)$$
holds for every $\phi \in L^\infty(\mathbb{R}^\times)$. For $\varphi \in M_G$, there exists unique elements $\varphi_+$ and $\varphi_-$ in $M_G$ and nonnegative numbers $\alpha$ and $\beta$ such that
$$\varphi = \alpha \varphi_+ - \beta \varphi_-, \quad \|\varphi\| = \alpha\|\varphi_+\| + \beta\|\varphi_-\|.$$ Further, for every $\phi \in L^\infty(\mathbb{R}^\times)$, the formula
$$\overline{G}_\infty(\phi) = \overline{Q}(\phi)$$
holds.

Now let us consider an example. For $r > 0$, let us define $g_r(x) \in L^1_b(\mathbb{R}^\times)$ by
$$g_r(x) = \begin{cases} rx^{-r} & \text{if } x \geq 1, \\ 0 & \text{if } x < 1. \end{cases}$$
The corresponding convolution operator $G_r : L^\infty(\mathbb{R}^\times) \to L^\infty(\mathbb{R}^\times)$ is given as follows:
$$(G_r\phi)(x) = \int_0^\infty f(t)r \left( \frac{x}{t} \right)^{-r} \frac{dt}{t} = \frac{r}{x^r} \int_0^x f(t)t^{r-1}dt,$$
where $\phi \in L^\infty(\mathbb{R}^\times)$. In particular, for $r = 1$, we have the Hardy operator $G_1 = H$. By Theorem 6.4, we obtain the following result, in which the main theorems of [15] is contained as a special case.
Theorem 6.5. For $\varphi \in L^\infty(\mathbb{R}^\times)^*$, $\varphi$ is $G_r$-invariant mean if and only if $$\varphi(\phi) \leq Q(\phi)$$ holds for every $\phi \in L^\infty(\mathbb{R}^\times)$. Further, we have the following equation $$\overline{G}_{r,\infty}(\phi) = Q(\phi)$$ for every $\phi \in L^\infty(\mathbb{R}^\times)$.

Finally, we present an application to summability methods. In what follows, we present only assertions without the proofs since they are similar to those in Section 5. For any $g \in L^1(\mathbb{R}^\times)$, let us define the summability method $G$ by $$G(\phi) = \lim_{x \to \infty} \int_0^\infty \phi(x/t)g(t)\frac{dt}{t}$$ provided that the limit exists. For $\phi \in L^\infty(\mathbb{R}^\times)$ and $r > 0$, let us define $\phi^*_r(x) = \phi(rx)$. The multiplicative analogue of Theorem 5.2 is as follows.

Theorem 6.6. Let $g \in L^1(\mathbb{R}^\times)$ satisfy $Z(g) = \emptyset$ and $\hat{g}(0) = 1$. Then for any $\phi \in L^\infty(\mathbb{R}^\times)$ if $$\lim_{x \to \infty} \int_0^\infty \phi(x/t)g(t)\frac{dt}{t} = \alpha$$ then $$w^*-\lim_r \phi^*_r(x) = \alpha,$$ where the symbol $w^*-\lim$ denotes the limit in the weak*-topology of $L^\infty(\mathbb{R}^\times)$.

Let $W^*$ be the summability method on $L^\infty(\mathbb{R}^\times)$ defined by $W^*(\phi) = \alpha$ if and only if $w^*-\lim_r \phi^*_r(x) = \alpha$. Further, let us define the summability method $Q$ on $L^\infty(\mathbb{R}^\times)$ by $Q(\phi) = Q(\phi) = Q(\phi)$, where $Q(\phi) = -\overline{Q}(-\phi)$. The following result is obtained that correspond to Theorem 5.4.

Theorem 6.7. Let $\phi \in L^\infty(\mathbb{R}^\times)$. Then $Q(\phi) = \alpha$ if and only if $$\lim_{\theta \to \infty} \frac{1}{\log \theta} \int_x^{\theta x} \phi(t)\frac{dt}{t} = \alpha$$ holds uniformly in $x \geq 1$.

Subsequently, we address Abelian and Tauberian theorems between summability methods $G$ and $Q$. Note that the following results are counterparts of Theorem 5.5, Lemma 5.1, and Theorem 5.6, respectively.

Theorem 6.8. Let $\phi \in L^\infty(\mathbb{R}^\times)$. Let $g \in L^1(\mathbb{R}^\times)$ satisfy $\hat{g}(0) = 1$. If $$\lim_{x \to \infty} \int_0^\infty \phi(t)g(x/t)\frac{dt}{t} = \alpha$$ then $$Q(\phi) = \alpha.$$
Lemma 6.2. Let $\phi \in L^\infty(\mathbb{R}^\times)$. Then the following conditions are equivalent:

(1) For every $r > 0$, $w^*-\lim_{x}(\phi^*_x - \phi^*_x) = 0$, i.e., $W^*(\phi^*_r - \phi) = 0$;

(2) $\int_0^\infty \phi(x/t)g(t)dt = 0$ for some $g \in L^1(\mathbb{R}^\times)$ with $Z(g) = \{0\}$.

Theorem 6.9. Let $\phi \in L^\infty(\mathbb{R}^\times)$. Then $W^*(\phi) = \alpha$ if and only if $Q(\phi) = \alpha$ and one of the two conditions in Lemma 6.2 holds. In other words, Either condition of Lemma 6.2 is a Tauberian condition under which $Q$ summability implies $G$ summability.

7. Applications to Cesàro operators

In this section, we consider the relationship between Cesàro invariant functionals $M_C$ and Hardy invariant functionals $M_H$. Specifically, we show that $M_C$ and $M_H$ are isomorphic. Based on this result, we obtain some results pertaining to the $C_\infty$ summability method. In what follows, the terms $H$-invariant and $C$-invariant mean Hardy invariant and Cesàro invariant, respectively.

Let us define the linear operator $V$ as follows:

$V : l_\infty \to L^\infty(\mathbb{R}^\times), \quad (V\phi)(x) = \phi([x + 1]), \quad x > 0.\]

Let us consider its adjoint operator $V^* : L^\infty(\mathbb{R}^\times)^* \to l^*_\infty$ and show that $V^*$ is a one-to-one and onto mapping from $M_H$ to $M_C$.

First, we show that if $\varphi \in L^\infty(\mathbb{R}^\times)^*$ is $H$-invariant, then $V^*\varphi$ is $C$-invariant. Suppose that $\varphi \in M_H$. We show that $C^*V^*\varphi(\phi) = V^*\varphi(\phi)$ for every $\phi \in l_\infty$. Note that $C^*V^*\varphi(\phi) = V^*\varphi(C\phi) = \varphi(VC\phi)$ and

$$(VC\phi)(x) = (C\phi)([x + 1]) = \frac{1}{[x + 1]} \sum_{i=1}^{[x+1]} \phi(i)$$

$$= \frac{1}{[x + 1]} \int_0^{[x+1]} (V\phi)(t)dt = (HV\phi)([x + 1])$$

holds. Observe that

$$|(HV\phi)(x) - (HV\phi)([x + 1])| \to 0 \text{ as } x \to \infty.$$  

Then, by the assumption that $\varphi$ vanishes on $L^\infty_{0,+}(\mathbb{R})$, we have

$$C^*V^*\varphi(\phi) = \varphi(VC\phi) = \varphi(HV\phi) = \varphi(V\phi) = V^*\varphi(\phi).$$

Thus, we have proved that $V^*$ maps $M_H$ into $M_C$.

Next, we show that $V^* : M_H \to M_C$ is surjective. Let us define the linear operator $V_1$ as

$$V_1 : L^\infty(\mathbb{R}^\times) \to l_\infty, \quad (V_1\phi)(n) = \int_{n-1}^{n} \phi(x)dx, \quad n \geq 1.$$  

Let $V_1^* : l^*_\infty \to L^\infty(\mathbb{R}^\times)^*$ be its adjoint operator. We show that if $\psi \in l^*_\infty$ is $C$-invariant, then $V_1^*\psi$ is $H$-invariant. Note that $H^*V_1^*\psi(\phi) = V_1^*\psi(H\phi) = \psi(V_1H\phi)$. First, observe
that
\[
CV_1\phi(n) = \frac{1}{n} \sum_{i=1}^{n} (V_1\phi)(i) = \frac{1}{n} \sum_{i=1}^{n} \int_{i-1}^{i} \phi(x) dx
= \frac{1}{n} \int_{0}^{n} \phi(t) dt = (H\phi)(n).
\]
Note that
\[
|(V_1 H\phi)(n) - (H\phi)(n)| = \left| \int_{n-1}^{n} (H\phi)(t) dt - (H\phi)(n) \right|
= \left| \int_{n-1}^{n} ((H\phi)(t) - (H\phi)(n)) dt \right|
\leq \sup_{x \in [n-1, n]} |(H\phi)(x) - (H\phi)(n)|
\leq \sup_{x \in [n-1, n]} \frac{1}{n} |H\phi(x)| + \frac{1}{n} \|\phi\|_{\infty},
\]
which tends to 0 as \( n \to \infty \). Thus, we have
\[
|(CV_1\phi)(n) - (V_1 H\phi)(n)| \leq |(CV_1\phi)(n) - (H\phi)(n)| + |(H\phi)(n) - (V_1 H\phi)(n)|
\leq \sup_{x \in [n-1, n]} \frac{1}{n} |H\phi(x)| + \frac{1}{n} \|\phi\|_{\infty}.
\]
Since the last term tends to 0 as \( n \to \infty \), we obtain \( CV_1\phi(n) - V_1 H\phi(n) \) tends to 0 as \( n \to \infty \). Therefore, we have
\[
H^* V_1^* \psi(\phi) = \psi(V_1 H\phi) = \psi(CV_1\phi) = \psi(V_1\phi) = (V_1^* \psi)(\phi).
\]
We have shown that \( V_1^* \psi \) is \( H \)-invariant if \( \psi \) is \( C \)-invariant. Moreover, notice that \( V^*(V_1^* \psi) = \psi \) for every \( \psi \in L^*_\infty \). In fact, for any \( \phi \in L^*_\infty \) we have
\[
(V_1 V\phi)(n) = \int_{n-1}^{n} (V\phi)(t) dt = \int_{n-1}^{n} \phi([t+1]) dt
= \int_{n-1}^{n} \phi(n) dt = \phi(n).
\]
This implies the required result immediately.

Now, given any \( \psi \in M_C \), \( \varphi = V_1^* \psi \) is in \( M_H \) and \( V^* \varphi = V^* V_1^* \psi = \psi \). Thus, \( V^* : M_H \to M_C \) is surjective.

Finally, we show that \( V^* : M_H \to M_C \) is injective. Suppose that \( V^* \varphi = V^* \varphi_1 \) for some \( \varphi, \varphi_1 \in M_H \). It is sufficient to show that for any \( \phi \in L^\infty(\mathbb{R}^\times) \) there exists some \( \psi \in L^\infty \) such that \( \lim_{t \to \infty} |(H\phi)(x) - (HV\psi)(x)| = 0 \). In fact, if this holds, we have \( \varphi(\phi) = \varphi(H\phi) = \varphi(HV\psi) = \varphi(V\psi) = V^* \varphi(\psi) \) and similarly \( \varphi_1(\phi) = V^* \varphi_1(\psi) \). By the assumption, we have \( \varphi(\phi) = \varphi_1(\phi) \) for any \( \phi \in L^\infty(\mathbb{R}^\times) \), and we obtain \( \varphi = \varphi_1 \).
If we set
\[
\psi(n) := \int_{n-1}^{n} \phi(t) dt, \quad n \geq 1,
\]
we have
\[
(HV \psi)(x) = \frac{1}{x} \int_{0}^{x} (V \psi)(t) dt = \frac{1}{x} \sum_{i=1}^{[x]} \int_{i-1}^{i} (V \psi)(t) dt + \frac{1}{x} \int_{[x]}^{x} (V \psi)(t) dt
\]
\[
= \frac{1}{x} \sum_{i=1}^{[x]} \psi(i) + \frac{1}{x} \int_{[x]}^{x} \psi([x] + 1) dt
\]
\[
= \frac{1}{x} \int_{0}^{[x]} \phi(t) dt + \frac{x - [x]}{x} \cdot \psi([x] + 1)
\]
\[
= (H \phi)(x) - \frac{1}{x} \int_{[x]}^{x} \phi(t) dt + \frac{x - [x]}{x} \cdot \psi([x] + 1).
\]

Since the second and third terms of the last expression tend to 0 as \( x \to \infty \), \( \psi \) is the desired function. This completes the proof.

In particular, the restriction of \( V^* \) to \( \mathcal{M}_H \) is an affine isomorphism between \( \mathcal{M}_H \) and \( \mathcal{M}_C \), where \( \mathcal{M}_H \) and \( \mathcal{M}_C \) are the sets of normalized positive elements in \( M_H \) and \( M_C \), respectively. Now we consider the maximal value attained by \( \mathcal{M}_C \) for any fixed \( \phi \in l_\infty \):
\[
\overline{p}_C(\phi) := \sup_{\psi \in \mathcal{M}_C} \psi(\phi).
\]
According to the above observation and Theorem 6.5, we have
\[
\sup_{\psi \in \mathcal{M}_C} \psi(\phi) = \sup_{\varphi \in \mathcal{M}_H} (V^* \varphi)(\phi) = \sup_{\varphi \in \mathcal{M}_H} \varphi(V \phi)
\]
\[
= \lim_{\theta \to \infty} \limsup_{x \to \infty} \frac{1}{\log \theta} \int_{x}^{\theta x} \frac{(V \phi)(t) dt}{t}
\]
\[
= \lim_{\theta \to \infty} \limsup_{n \to \infty} \frac{1}{\log \theta} \int_{n}^{\theta n} \frac{(V \phi)(t) dt}{t}.
\]
Notice that \( (V \phi)(t) = \phi([t + 1]) \) and we have
\[
\int_{n}^{\theta n} (V \phi)(t) \frac{dt}{t} = \sum_{i=n}^{[\theta n]-1} \phi(i + 1) \int_{i}^{i+1} \frac{dt}{t} + \int_{[\theta n]}^{\theta n} \phi([\theta n] + 1) \frac{dt}{t}.
\]
It is obvious that the second term tends to 0 as \( n \to \infty \). Further, observe that

\[
\left| \sum_{i=n}^{\lfloor \theta n \rfloor} \phi(i+1) \int_i^{i+1} \frac{dt}{t} - \sum_{i \in [n, \theta n]} \frac{\phi(i+1)}{i} \right|
\]

\[
\leq \left| \phi(n+1) \cdot \left( \frac{1}{n} - \log \left( 1 + \frac{1}{n} \right) \right) + \phi(n+2) \cdot \left( \frac{1}{n+1} - \log \left( 1 + \frac{1}{n+1} \right) \right) + \cdots \right|
\]

\[
+ \phi(\lfloor \theta n \rfloor) \cdot \left( \frac{1}{\lfloor \theta n \rfloor - 1} - \log \left( 1 + \frac{1}{\lfloor \theta n \rfloor - 1} \right) \right) + \left| \frac{\phi(\lfloor \theta n \rfloor + 1)}{\lfloor \theta n \rfloor} \right|
\]

\[
\leq |\phi(n+1)| \cdot \frac{1}{2} \cdot \frac{1}{n^2} + |\phi(n+2)| \cdot \frac{1}{2} \cdot \frac{1}{(n+1)^2} + \cdots
\]

\[
+ |\phi(\lfloor \theta n \rfloor)| \cdot \frac{1}{2} \cdot \frac{1}{(\lfloor \theta n \rfloor - 1)^2} + \left| \frac{\phi(\lfloor \theta n \rfloor + 1)}{\lfloor \theta n \rfloor} \right|
\]

\[
\leq \frac{\|\phi\|_{\infty}}{2} \left( \frac{1}{n^2} + \frac{1}{(n+1)^2} + \cdots \frac{1}{(\lfloor \theta n \rfloor - 1)^2} \right) + \left| \frac{\phi}{\theta n} \right|.
\]

The last expression tends to 0 as \( n \to \infty \). Furthermore, observe that the difference

\[
\left| \sum_{i \in [n, \theta n]} \frac{\phi(i)}{i} - \sum_{i \in [n, \theta n]} \frac{\phi(i+1)}{i} \right|
\]

tends to 0 as \( n \to \infty \). Hence, we obtain the following.

\[
\overline{\Lambda}_C(\phi) = \sup_{\psi \in \mathcal{M}_C} \psi(\phi) = \lim_{\theta \to \infty} \limsup_{n \to \infty} \frac{1}{\log \theta} \sum_{i \in [n, \theta n]} \frac{\phi(i)}{i}.
\]

We now consider the functional \( \overline{\mathcal{C}} \). Note that for \( \phi \in l_{\infty} \)

\[
C_{\phi}(n) = \frac{1}{n} \sum_{i=1}^{n} \phi(i) = \frac{1}{n} \int_0^n (V\phi)(t)dt = (HV\phi)(n).
\]

Then, we have

\[
\overline{\mathcal{C}}(\phi) = \limsup_{n \to \infty}(C_{\phi})(n) = \limsup_{n \to \infty}(HV\phi)(n) = \overline{H}(V\phi).
\]

Hence, for each \( k \geq 2 \), we have

\[
\overline{\mathcal{C}}_k(\phi) = \overline{\mathcal{C}}_{k-1}(C_{\phi}) = \overline{H}_{k-1}(HV\phi) = \overline{H}_k(V\phi).
\]

Therefore, we obtain

\[
\overline{\mathcal{C}}_{\infty}(\phi) = \lim_{k \to \infty} \overline{\mathcal{C}}_k(\phi) = \lim_{k \to \infty} \overline{H}_k(V\phi) = \overline{Q}(V\phi) = \overline{\Lambda}_C(\phi).
\]
by the above computation. We have obtained the analytic expression of $C_\infty(\phi)$:

$$C_\infty(\phi) = \lim_{\theta \to \infty} \limsup_{n \to \infty} \frac{1}{\log \theta} \sum_{i \in [n, \theta n]} \frac{\phi(i)}{i}, \quad \phi \in l_\infty.$$ 

The proof of the following characterization of $C_\infty$ summability is similar to that of Theorem 6.7. Accordingly, the proof is omitted.

**Theorem 7.1.** For $\phi \in l_\infty$, $\phi$ is $C_\infty$ summable to the number $\alpha$ if and only if

$$\lim_{\theta \to \infty} \frac{1}{\log \theta} \sum_{i \in [n, \theta n]} \frac{\phi(i)}{i} = \alpha$$

uniformly in $n \in \mathbb{N}$.

It is apparent that this summability method has relation to the logarithmic method which is defined as follows. For a function $\phi$ on $\mathbb{N}$, we say that $\phi$ is summable to $\alpha$ by logarithemic method if

$$\lim_{n \to \infty} \frac{1}{\log n} \sum_{i=1}^{n} \frac{\phi(i)}{i} = \alpha.$$ 

The following result, which was given in [5], is an immediate consequence of Theorem 7.1.

**Theorem 7.2.** For $\phi \in l_\infty$, if $\phi$ is $C_\infty$ summable, then $\phi$ is logarithmic summable.

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