THETA FUNCTIONS OF ARBITRARY ORDER AND THEIR DERIVATIVES

SAMUEL GRUSHEVSKY AND RICCARDO SALVATI MANNI

Abstract. In this paper we establish the relationships between theta functions of arbitrary order and their derivatives. We generalize our previous work [4] and prove that for any \( n > 1 \) the map sending an abelian variety to the set of Gauss images of its points of order \( 2n \) is an embedding into an appropriate Grassmannian (note that for \( n = 1 \) we only got generic injectivity in [4]). We further discuss the generalizations of Jacobi’s derivative formula for any dimension and any order.

1. Introduction and definitions

The study of theta functions of abelian varieties is a very classical subject that goes back to Jacobi, Riemann, Weierstrass, Fröbenius, Poincaré and many others. A purely algebraic modern treatment of the subject started with Weil [16]. In the 1960s Igusa [8] and Mumford [10] proved the fundamental theorem relating the values of theta functions at zero to injective maps from some modular varieties into the projective space, among other results. For a detailed history of the problem up till 1980 we refer to Igusa’s survey [9].

More precisely, let \( \mathcal{H}_g \) be the Siegel upper half-space — the set of symmetric \( g \times g \) complex matrices \( \tau \) with positive-definite imaginary part. For any \( z \in \mathbb{C}^g, \tau \in \mathcal{H}_g \) we define the theta function with characteristics \( \varepsilon, \delta \in \mathbb{R}^g \) to be

\[
\theta \left[ \begin{array}{c} \varepsilon \\ \delta \end{array} \right] (\tau, z) = \sum_{n \in \mathbb{Z}^g} e \left( \frac{1}{2} (n + \varepsilon)^t \tau (n + \varepsilon) + (n + \varepsilon)^t (z + \delta) \right),
\]

where \( e(t) := \exp(2\pi it) \), and \( A^t \) denotes the transpose of a matrix \( A \). The restriction of a theta function to \( z = 0 \) is called the associated theta constant.

If in the formula above we take \( \varepsilon \in \left( \frac{1}{n} \mathbb{Z}/\mathbb{Z} \right)^g, \delta = 0, \) and replace \( \tau \) and \( z \) by \( n\tau \) and \( nz \), the resulting theta function is called the theta function with characteristics of order \( n \). These theta functions form a
basis for the space of sections of $n\Theta$ — the $n$-th power of the symmetric line bundle inducing the principal polarization on the abelian variety with period matrix $\tau$. In this case the associated theta constants are modular forms with respect to a certain subgroup of $\text{Sp}(2g, \mathbb{Z})$. Let us define this.

The symplectic group $\text{Sp}(2g, \mathbb{Z})$ acts on $\mathcal{H}_g$. Let us write an element $\gamma \in \text{Sp}(2g, \mathbb{Z})$ as $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d$ being $g \times g$ integer matrices. Then the action is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau := (a\tau + b)(c\tau + d)^{-1},$$

and the quotient is the moduli space of principally polarized abelian varieties $A_g = \text{Sp}(2g, \mathbb{Z}) \backslash \mathcal{H}_g$. Let $\rho: \text{GL}(g, \mathbb{C}) \to \text{End} V$ be an irreducible rational representation with the highest weight $(k_1, k_2, \ldots, k_g)$, $k_1 \geq k_2 \geq \cdots \geq k_g$; then we call $k_g$ the weight of $\rho$. A representation $\rho_0$ is called reduced if its weight is equal to zero. Let us fix an integer $r$; we are interested in pairs $\rho = (\rho_0, r)$, with $\rho_0$ reduced. We call $r$ the weight of $\rho$ and use the notation

$$\rho(A) = \rho_0(A) \det A^{r/2}.$$

For a finite index subgroup $\Gamma \subset \text{Sp}(2g, \mathbb{Z})$ a multiplier system of weight $r/2$ is a map $v: \Gamma \to \mathbb{C}^*$, such that the map

$$\sigma \mapsto v(\sigma) \det(C\tau + D)^{r/2}$$

satisfies the cocycle condition for every $\sigma \in \Gamma$ and $\tau \in \mathcal{H}_g$ (note that the function $\det(C\tau + D)$ possesses a square root). Clearly a multiplier system of integral weight is a character. A map $f: \mathcal{H}_g \to V$ is called a $\rho$- or $V$-valued modular form, or simply a vector-valued modular form, if the choice of $\rho$ is clear, with multiplier $v$, with respect to a subgroup of finite index $\Gamma \subset \text{Sp}(2g, \mathbb{Z})$ if the transformation formula

$$(1) \quad f(\sigma \circ \tau) = v(\sigma)\rho(C\tau + D)f(\tau)$$

is satisfied for any $\sigma$ in $\Gamma$ and any $\tau$ in $\mathcal{H}_g$, and, for $g = 1$, if additionally $f$ is holomorphic at all cusps of $\Gamma \backslash \mathcal{H}_1$.

Let us now define the level subgroups of the symplectic group to be

$$\Gamma_g(n) := \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_g \mid \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod n \right\},$$

$$\Gamma_g(n, 2n) := \left\{ \gamma \in \Gamma_g(n) \mid \text{diag}(a'b) \equiv \text{diag}(c'd) \equiv 0 \mod 2n \right\}.$$

We denote the corresponding level covers of $A_g$ by $A_g(n) := \Gamma_g(n) \backslash \mathcal{H}_g$ and $A_g(n, 2n) := \Gamma_g(n, 2n) \backslash \mathcal{H}_g$, respectively. It is known that theta
constants of order $n$ are modular forms of weight $1/2$ (and with $\rho_0 = \text{Id}$), for a suitable multiplier $v_n$, with respect to the group $\Gamma_g(n, 2n)$.

One of the main results proved by Igusa in [8] and Mumford in [10] is that the map

$$Th_n : A_g(n, 2n) \to \mathbb{P}^{n^g - 1}$$

sending a point to the set of values of all theta constants of a given order $n$ for any $n \geq 4$,

$$Th_n(\tau) := \left\{ \theta \begin{bmatrix} a \\ 0 \end{bmatrix}(n\tau, 0) \right\}_{\text{all } a \in (\frac{1}{n}Z/Z)^g},$$

defines an embedding of the level moduli space.

Recently in [4] we considered, in the case of both characteristics $\varepsilon, \delta$ of a theta function being half-integral (which is equivalent to the order 4 case, see [6]) the map sending a point in $\mathcal{H}_g$ to the $g \times 2^g - 1(2^g - 1)$ matrix of non-trivial gradients

$$\Phi_4 : \tau \to \left( \text{grad}_{z=0} \theta \begin{bmatrix} \varepsilon \\ \delta \end{bmatrix}(\tau, z) \right)_{\text{all odd } \varepsilon, \delta \in (\frac{1}{2}Z/Z)^g}$$

and showed that $\Phi_4$ induces a generically injective and immersive map of $A_g(4, 8)$ to the Grassmannian variety $G(g, 2^g - 1(2^g - 1))$ of $g$-planes in $\mathbb{C}^{2^g - 1(2^g - 1)}$. Here “odd” means that as a function of $z$ the theta function $\theta \begin{bmatrix} \varepsilon \\ \delta \end{bmatrix}(\tau, z)$ is odd, which is equivalent to the scalar product $2(\varepsilon, \delta)$ being zero in $(\frac{1}{2}Z/Z)^g$.

Passing to Plücker’s coordinates to embed the Grassmannian variety into a projective space, the image of the map $\Phi_4$ in the projective space produces some well-known modular forms, the so-called Jacobian determinants of theta functions. These are obtained as follows: for any set of $g$ odd characteristics $[\varepsilon_1, \delta_1], \ldots, [\varepsilon_g, \delta_g]$ we define their jacobian determinant to be

$$D([\varepsilon_1, \delta_1], \ldots, [\varepsilon_g, \delta_g])(\tau) := \pi^{-g} \text{grad} \theta \begin{bmatrix} \varepsilon_1 \\ \delta_1 \end{bmatrix}(\tau, 0) \wedge \text{grad} \theta \begin{bmatrix} \varepsilon_2 \\ \delta_2 \end{bmatrix}(\tau, 0) \wedge \cdots \wedge \text{grad} \theta \begin{bmatrix} \varepsilon_g \\ \delta_g \end{bmatrix}(\tau, 0).$$

The Jacobian determinants were also extensively studied in the nineteenth century, with special emphasis on their modular properties and relationship with theta constants. The first result in that direction was the famous Jacobi’s derivative formula

$$\theta \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}'(\tau, 0) = -\pi \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix}(\tau, 0) \theta \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}(\tau, 0) \theta \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}(\tau, 0),$$
which is the expression of the only non-zero Jacobian determinant for \( g = 1 \) in terms of even theta constants.

Generalizations of this formula were stated by Rosenhain in the case of genus two \([12]\), and mostly proved by Weber and Fröhbenius for genera up to four \([3]\). It seems that Riemann has also worked on this problem, and some generalizations can be found in \([11]\). Thomae \([15]\) then generalized the formula to the case of hyperelliptic curves of any genus, but the problem of completely generalizing Jacobi’s derivative formula to arbitrary abelian varieties remained open.

Recently in \([5]\) we found different generalizations of Jacobi’s derivative formula to higher genus, involving second order derivatives of theta functions at zero.

The aim of this paper is to present a general framework for deriving the generalizations of the results of \([4],[5]\) to arbitrary level. We will consider the map

\[
\Phi_{4n} : \mathcal{A}_g(4n, 8n) \to G(g, n^{2g})
\]
given by

\[
\Phi_{4n}(\tau) := \left\{ \text{grad}_{z=0} \theta \begin{bmatrix} a \\ 0 \end{bmatrix} (4n\tau, z) \right\}_{\text{all } a \in \left(\frac{1}{n} \mathbb{Z}/\mathbb{Z}\right)^g}
\]

and relate it to the theta constant maps \( Th_{2n} \) and \( Th_{4n} \). Note that the map \( \Phi_{4n} \) is well-defined because gradients of theta functions of order \( 4n \) are vector-valued modular forms with respect to \( \Gamma_g(4n, 8n) \) for the representation \( \text{std} \otimes \det^{1/2} \) (i.e. of weight 1/2 and with \( \rho_0(A) = A \)). Notice that unlike the \( n = 1 \) case, here for convenience we include the gradients of all theta functions irrespective of their parity, though of course since \( \theta \begin{bmatrix} a \\ 0 \end{bmatrix} (n, z) + \theta \begin{bmatrix} -a \\ 0 \end{bmatrix} (n, z) \) is even, there will be many identical columns in the \( g \times n^{2g} \) matrix, which is the image \( \Phi_{4n}(\tau) \).

We will show that \( \Phi_{4n} \) is an embedding for all \( n > 1 \) (recall that in \([4]\) we considered the case of \( n = 1 \) and were only able to prove generic injectivity), and will also obtain generalizations of Jacobi’s derivative formula for theta functions of arbitrary level. We think that similar results can also be obtained for other levels not divisible by 4, but dealing with those makes some computations much more technically involved, as working with theta functions of non-integral level is harder, and we will not treat such computations here.

We will work with theta functions of orders \( 2n \) and \( 4n \) (in \([4]\) and \([5]\) we worked with \( n = 1 \)). To try to avoid confusion, we will adhere to the following notations: Greek letters will stand for characteristics \( \varepsilon \in \).
\((\frac{1}{2}\mathbb{Z}/\mathbb{Z})^g\), which will play a special role, Latin letters at the end of the alphabet will be for vectors \(z \in \mathbb{C}^g\), and Latin letters at the beginning of the alphabet will be for characteristics \(a \in (\frac{1}{m}\mathbb{Z}/\mathbb{Z})^g\) for some even order \(m\), or sometimes for \(a \in (\mathbb{Q}/\mathbb{Z})^g\) for complete generality.

2. Addition theorem for theta functions

We work with a \(g\)-dimensional principally polarized abelian variety \(X = V/\Lambda\) with period matrix \(\tau\) and the polarization bundle \(\Theta\). We denote by \(X[2]\) the points of order two \(X\), i.e. points \(p \in X\) such that \(2p = 0 \in X\). For \(x = \tau \varepsilon + \delta\) in \(X[2]\) the shifted bundle \(t_x^*\Theta\) is still a symmetric line bundle. The theta function \(\theta_{\varepsilon, \delta}(\tau, z)\) is, up to a multiplicative constant, the unique section of \(t_x^*\Theta\). Note, however, that \(\theta_{\varepsilon, \delta}(\tau, 2z)\) is a section of \(\Theta^4\) due to the presence of the lower characteristic. In general a basis of \(H^0(X, \Theta^n)\) is given by the \(n\) theta functions \(\theta_{\varepsilon, 0}(n\tau, nz)\) with \(a \in (\frac{1}{n}\mathbb{Z}/\mathbb{Z})^g\). We now recall the formula in [6] at the top of p. 50:

\[
(2) \quad \theta_{\varepsilon, \delta}(\tau, z + \tau c + d) = e\left(-\frac{1}{2}c'(\tau c + z + d + b)\right) \theta_{\varepsilon + \frac{a}{2}, \varepsilon}(\tau, z).
\]

We will also need a slight generalization of the formula at the bottom of p. 171 in [6], relating theta functions of order twice larger and theta functions with a lower characteristic:

**Lemma 1.** For all \(\tau \in \mathcal{H}_g\), \(z \in \mathbb{C}^g\), \(a \in \mathbb{R}^g\) and \(\beta \in (\frac{1}{2}\mathbb{Z}/\mathbb{Z})^g\) we have

\[
\theta_{\varepsilon, \beta}(\tau, 2z) = \sum_{\varepsilon \in (\frac{1}{2}\mathbb{Z}/\mathbb{Z})^g} e(\beta'(2\varepsilon + a)) \theta_{\varepsilon + \frac{\beta}{2}, 0}(4\tau, 4z).
\]

One of the basic relations among theta functions is Riemann’s bilinear addition theorem, which essentially relates theta functions at \(\tau\) and \(2\tau\) or, if the characteristics are chosen appropriately, theta functions of order \(n\) and \(2n\). We will need to use it in two forms. The first form is the following

**Proposition 2** (specialization of Theorem 2, p. 139 in [6]). For all \(\tau \in \mathcal{H}_g\), \(z, w \in \mathbb{C}^g\), \(a, b \in \mathbb{R}^g\), and \(\varepsilon \in (\frac{1}{2}\mathbb{Z}/\mathbb{Z})^g\) the following holds:

\[
\theta_{\varepsilon, \delta}(4n\tau, 4nz) \theta_{\varepsilon, \delta}(4n\tau, 4nw) = \]
\[
\frac{1}{2^g} \sum_{\sigma} e(-2a^t \sigma) \theta \left[ \frac{a + b}{\sigma + \varepsilon} \right] (2n\tau, 2n(z + w)) \theta \left[ \frac{a - b}{\sigma} \right] (2n\tau, 2n(z - w)).
\]

We will also need another form of this addition theorem, which in some sense is the converse, expressing one term in the right-hand-side of the above as a combination of terms in the left-hand-side.

**Proposition 3** (a generalization of [6], Corollary, p. 141). For all \( \tau \in H, \ z, w \in \mathbb{C}, \ a, b \in \mathbb{R}, \) and \( \gamma, \sigma \in (\frac{1}{2}\mathbb{Z}/\mathbb{Z})^g \) the following holds

\[
\theta \left[ \frac{a}{\gamma + \sigma} \right] (2n\tau, 2nz) \theta \left[ \frac{b}{\gamma} \right] (2n\tau, 2nw) = \sum_{\varepsilon} e((a + b + 2\varepsilon)^t \gamma).
\]

Proof. This formula differs from the one in the previous proposition in that we are trying to pass to double argument rather than half the argument. We first apply formula (2) to the left-hand-side and then use proposition 2. Afterwards we use the formula in lemma 4

\[
\theta \left[ \frac{a}{\gamma + \sigma} \right] (2n\tau, 2nz) \theta \left[ \frac{b}{\gamma} \right] (2n\tau, 2nw) - \theta \left[ \frac{a + b}{\gamma + \sigma} \right] (2n\tau, 2n(z + w)) \theta \left[ \frac{a - b}{\gamma} \right] (2n\tau, 2n(z - w)).
\]

When we take the sum over \( \mu \) in this formula, this is just taking the sum \( \sum_{\mu} e(2(\varepsilon + \delta)^t \mu) \), which is zero unless \( \varepsilon = \delta \) and is equal to \( 2^g \) if \( \varepsilon = \delta \). Thus summing over \( \mu \) extracts \( 2^g \) times the \( \varepsilon = \delta \) terms of the above sum, and we end up with

\[
\sum_{\varepsilon} \theta \left[ \frac{\varepsilon + a + b}{0} \right] (4n\tau, 2n(z + w) + 2\gamma + \sigma) \theta \left[ \frac{\varepsilon - a - b}{0} \right] (4n\tau, 2n(z - w) + \sigma)
\]

\[
= \sum_{\varepsilon} e((a + b + 2\varepsilon)^t \gamma) \theta \left[ \frac{\varepsilon + a + b}{\sigma} \right] (4n\tau, 2n(z + w)) \theta \left[ \frac{\varepsilon - a - b}{\sigma} \right] (4n\tau, 2n(z - w))
\]
We end this section by recalling that as \( a \) varies in \( (\frac{1}{n}\mathbb{Z}/\mathbb{Z})^g \), by formula (2) the values of the theta functions \( \theta \begin{bmatrix} a \\ 0 \end{bmatrix} (n\tau, nz) \) at 0 are related to the values of the single theta function \( \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (n\tau, nz) \) at different points of order \( n \) on the abelian variety.

3. Injectivity of the gradient maps

In this section we follow, generalize and further advance the framework of establishing the relationships between gradients of theta functions and derivatives of theta constants that we have developed in [4] and [5]. We then use the general relationships between the maps \( \Phi \) and \( Th \) to show that the image of \( Th \) can be obtained uniquely from the image of \( \Phi \), thus eventually proving injectivity of \( \Phi_{4n} \) for \( n > 1 \). The improvement over the \( n = 1 \) case, where we could only get generic injectivity, is due to the fact that we can now preclude the massive vanishing of theta constants that plagued our computations in [4]; we are also aided by the knowledge that \( Th_{2n} \) is an embedding for \( n > 1 \), while it is still only a conjecture that \( Th_2 \) is injective.

For simplicity, we denote by \( \partial_i \theta \) the derivative of \( \theta \) with respect to \( z_i \), evaluated at \( z = 0 \). Similarly to [4] and [5], let us then define the \( g \times g \) matrices

\[
C_{ab} := \left( 2\partial_i \theta \begin{bmatrix} a \\ 0 \end{bmatrix} (4n\tau) \partial_j \theta \begin{bmatrix} b \\ 0 \end{bmatrix} (4n\tau) + 2\partial_j \theta \begin{bmatrix} a \\ 0 \end{bmatrix} (4n\tau) \partial_i \theta \begin{bmatrix} b \\ 0 \end{bmatrix} (4n\tau) \right)_{\text{all } i,j}
\]

for \( a, b \in (\mathbb{Q}/\mathbb{Z})^g \). We mainly shall use \( C \) with both indices \( a, b \in (\frac{1}{n}\mathbb{Z}/\mathbb{Z})^g \). Note that the \( C \)'s that we used in [4] and [5] are essentially the case \( n = 1 \) of the above, but here we used different indices for \( C \)'s, since we are using a different basis for theta functions of a given order.

Let us also define the \( g \times g \) matrices

\[
A_{\varepsilon}^{cd} := \left( \partial_i \partial_j \theta \begin{bmatrix} c \\ \varepsilon \end{bmatrix} (2n\tau) \partial_k \theta \begin{bmatrix} d \\ \varepsilon \end{bmatrix} (2n\tau) - \theta \begin{bmatrix} c \\ \varepsilon \end{bmatrix} (2n\tau) \partial_i \partial_j \theta \begin{bmatrix} d \\ \varepsilon \end{bmatrix} (2n\tau) \right)_{\text{all } i,j}
\]

for \( a, b \in (\mathbb{Q}/\mathbb{Z})^g \) and \( \varepsilon \in (\frac{1}{2}\mathbb{Z}/\mathbb{Z})^g \). Similarly to \( C \), the \( A \)'s we used in our previous works correspond to the case \( n = 1 \) of the definition we are now using, with some further restrictions on \( a \) and \( b \).

Note that \( A \) and \( C \) are vector-valued modular forms with respect to \( \Gamma_g(4n, 8n) \) and the representation

\[ \rho = \text{Sym}^2(\text{std}) \otimes \text{det} \].
Theorem 4. The matrices $A$ and $C$ can be expressed in terms of each other as follows:

a) $C_{ab}^{\sigma} = \frac{1}{2g} \sum_{\sigma} e(-2a't \sigma) A_{a+b,a-b}^{\sigma}.$

b) $A_{b}^{ab} = 2 \sum_{\varepsilon} e((a + b + 2\varepsilon)^t \delta) C_{\varepsilon + \frac{a+b}{2},\varepsilon + \frac{a-b}{2}}.$

Proof. Indeed, to get part a) let us take the derivative $\partial z_i \partial w_j + \partial w_i \partial z_j$ of the formula in proposition 2 and then evaluate at $z = w = 0.$ Differentiating the left-hand-side is easy. On the right-hand-side we notice that the terms where each factor is differentiated once will cancel because of the minus sign for $w$ in the argument of the second theta function. Thus we arrive at

$$2\partial_i \theta \left[ \begin{array}{c} a \\ \varepsilon \end{array} \right] (4n\tau) \partial_j \theta \left[ \begin{array}{c} b \\ \varepsilon \end{array} \right] (4n\tau) + 2\partial_j \theta \left[ \begin{array}{c} a \\ \varepsilon \end{array} \right] (4n\tau) \partial_i \theta \left[ \begin{array}{c} b \\ \varepsilon \end{array} \right] (4n\tau)$$

$$= \frac{1}{2g} \sum_{\sigma} e(-2a'\sigma) \left( \partial_i \partial_j \theta \left[ \begin{array}{c} a + b \\ \sigma + \varepsilon \end{array} \right] (2n\tau) \theta \left[ \begin{array}{c} a - b \\ \sigma \end{array} \right] (2n\tau) \right.$$ 

$$- \theta \left[ \begin{array}{c} a + b \\ \sigma + \varepsilon \end{array} \right] (2n\tau) \partial_i \partial_j \theta \left[ \begin{array}{c} a - b \\ \sigma \end{array} \right] (2n\tau) \right)$$

which, when written in terms of $A$ and $C,$ gives us part a) of the theorem.

For the proof of part b) let us take the derivative $\partial z_i \partial w_j - \partial w_i \partial z_j |_{z = w = 0}$ of the formula in proposition 3. Differentiating the left-hand-side is easy; on the right-hand-side we notice that the terms that do not cancel are the ones where each of the factors is differentiated once, and thus we end up with

$$\partial_i \partial_j \theta \left[ \begin{array}{c} a \\ \gamma \end{array} \right] (2n\tau) \theta \left[ \begin{array}{c} b \\ \delta \end{array} \right] (2n\tau) - \theta \left[ \begin{array}{c} a \\ \gamma \end{array} \right] (2n\tau) \partial_i \partial_j \theta \left[ \begin{array}{c} b \\ \delta \end{array} \right] (2n\tau) =$$

$$2 \sum_{\varepsilon} e((a + b + 2\varepsilon)^t \gamma) \cdot \left( \partial_j \theta \left[ \begin{array}{c} \varepsilon + \frac{a+b}{2} \\ \gamma + \frac{\Delta}{2} \end{array} \right] (4n\tau) \partial_i \theta \left[ \begin{array}{c} \varepsilon + \frac{a-b}{2} \\ \gamma + \frac{\Delta}{2} \end{array} \right] (4n\tau) \right) +$$

$$2 \sum_{\varepsilon} e((a + b + 2\varepsilon)^t \gamma) \cdot \left( \partial_i \theta \left[ \begin{array}{c} \varepsilon + \frac{a+b}{2} \\ \gamma + \frac{\Delta}{2} \end{array} \right] (4n\tau) \partial_j \theta \left[ \begin{array}{c} \varepsilon + \frac{a-b}{2} \\ \gamma + \frac{\Delta}{2} \end{array} \right] (4n\tau) \right) ,$$

which in terms of $A$ and $C$ is exactly part b) of the theorem. □

In the following we will only use this theorem for the case when the indices of $C$ lie in $\left( \frac{1}{4n} \mathbb{Z} / \mathbb{Z} \right)^g$ and the upper indices of $A$ lie in $\left( \frac{1}{4n} \mathbb{Z} / \mathbb{Z} \right)^g$ with the extra condition that the indices of the corresponding $C$ appearing in part a) of theorem 4 are all in $\left( \frac{1}{4n} \mathbb{Z} / \mathbb{Z} \right)^g,$ i.e. with the condition that $a + b \in \left( \frac{1}{2n} \mathbb{Z} / \mathbb{Z} \right)^g.$
We will now proceed to show the injectivity of the gradient theta map at all levels — this is done similarly to the computations in \[4\] while taking advantage of the more general $A$ and $C$, so we now streamline the argument.

**Lemma 5.** The following identity holds:
\[
A_{\varepsilon}^{ab} \theta \left[ \varepsilon \right] (2n\tau) + A_{\varepsilon}^{bc} \theta \left[ \varepsilon \right] (2n\tau) + A_{\varepsilon}^{ca} \theta \left[ \varepsilon \right] (2n\tau) = 0.
\]

**Proof.** This is a trivial computation with all the six terms canceling pairwise. \(\square\)

We observe that the above lemma in particular holds for $a, b, c \in (\frac{1}{2n}\mathbb{Z}/\mathbb{Z})^g$ with $a + b \in (\frac{1}{2n}\mathbb{Z}/\mathbb{Z})^g$ and $a + c \in (\frac{1}{2n}\mathbb{Z}/\mathbb{Z})^g$

In an improvement over the $n = 1$ case, where we had trouble proving non-degeneracy, we can now prove

**Lemma 6.** For any $n > 1$ the rank of the $(2n)^g \times (\frac{g(a+1)}{2} + 1)$ matrix with columns
\[
\left( \theta \left[ a + \frac{\delta}{2n} \right] (2n\tau), \partial_i \partial_j \theta \left[ a + \frac{\delta}{2n} \right] (2n\tau) \right) \text{ all } a \in (\frac{1}{2n}\mathbb{Z}/\mathbb{Z})^g, \text{ all } (i,j)
\]
for any fixed $\varepsilon$ and $\delta$ is maximal, for all $g \geq 1$.

**Proof.** Lemma 11, p. 188 in \[6\] proves this result for $\delta = \varepsilon = 0$ and for any order $m$ divisible by 4. The proof given there clearly works for any even $m = 2n \geq 4$ as well. Using formula (2), we can then obtain a proof of the lemma by evaluating the theta functions $\theta \left[ a \right] (2n\tau, 2nz)$ at the point $z = \tau \frac{\delta}{2n} + \frac{\varepsilon}{2n}$.

The reason why the $n = 1$ case would not work for the lemma above is that all even theta functions vanish at odd points. We would also like to remark that this result is closely related to the injectivity of certain higher order embeddings of abelian varieties — obtained by using theta functions, not their derivatives — which were studied in \[4\].

Now similarly to proposition 12 in \[4\] we can reconstruct the (projectivized) values of theta constants from the knowledge of $A$’s and thus, by theorem \[4\] from the $C$’s, i.e. from $\Phi_{4n}(\tau)$.

**Proposition 7.** The value of $\Phi_{4n}(\tau)$ uniquely determines for any fixed $\gamma, \delta \in (\frac{1}{2}\mathbb{Z}/\mathbb{Z})^g$ the projective point
\[
\left\{ \theta \left[ a + \frac{\delta}{2n} \right] \right\}_{a \in (\frac{1}{2n}\mathbb{Z}/\mathbb{Z})^g}.
\]
For $\gamma = 0$ this point is simply the value $Th_{2n}(\tau)$. Since we know that $Th_{2n}$ is an embedding of $A_g(2n, 4n)$ for $n > 1$, this means that $\Phi_{4n}(\tau)$ determines the class of $\tau$ in $A_g(2n, 4n)$ uniquely. Since the cover $A_g(4n, 8n) \rightarrow A_g(2n, 4n)$ is finite, it follows immediately that the map $\Phi_{4n}$ on $A_g(4n, 8n)$ is at most finite-to-one. We would now like to show that $\Phi_{4n}$ is in fact injective by showing that from the knowledge of $\Phi_{4n}(\tau)$ we can determine uniquely the class of $\tau$ in $A_g(4n, 8n)$ and not only in $A_g(2n, 4n)$. The first step in this direction is the following

**Theorem 8.** For any fixed $\sigma, \delta \in \left(\frac{1}{2}\mathbb{Z}/\mathbb{Z}\right)^g$ and fixed $a, b \in \left(\frac{1}{2}\mathbb{Z}/\mathbb{Z}\right)^g$ with $a + b - \delta n \in \left(\frac{1}{2n}\mathbb{Z}/\mathbb{Z}\right)^g$, the value of $\Phi_{4n}(\tau)$ uniquely determines the projective point

$$\left\{ \theta \left[ \begin{array}{c} a \\ \gamma + \sigma \end{array} \right] (n\tau) \theta \left[ \begin{array}{c} b \\ \gamma \end{array} \right] (n\tau) \right\}_{\gamma}.$$ 

**Proof.** Indeed, let us use the addition formula from proposition 3 with $2n$ and $4n$ replaced by $n$ and $2n$. Then in the right-hand-side we will have a linear combination of terms appearing in proposition 7, which are uniquely determined by $\Phi_{4n}(\tau)$, while in the left-hand-side we will be getting products of two theta functions at $n\tau$ of the kind described. \hfill \square

The problem we had in [4] in trying to prove injectivity was due in large part to the possibility of many theta constants vanishing simultaneously, so that we were unable to determine certain signs uniquely. For $n > 1$ we can deal with this.

**Lemma 9.** For all $n > 1$ and for any fixed $\gamma, \sigma, \delta \in \left(\frac{1}{2}\mathbb{Z}/\mathbb{Z}\right)^g$ there always exist some $a, b \in \left(\frac{1}{2n}\mathbb{Z}/\mathbb{Z}\right)^g$ with $a + b - \frac{\delta}{n} \in \left(\frac{1}{n}\mathbb{Z}/\mathbb{Z}\right)^g$, such that

$$\theta \left[ \begin{array}{c} a \\ \gamma + \sigma \end{array} \right] (n\tau) \theta \left[ \begin{array}{c} b \\ \gamma \end{array} \right] (n\tau) \neq 0.$$ 

**Proof.** First note that for any fixed $\gamma + \sigma$ there is at least one among $\theta \left[ \begin{array}{c} a \\ \gamma + \sigma \end{array} \right] (n\tau)$ that does not vanish: indeed, these are the values of all theta functions of order $2n$ at the point $\gamma + \sigma$. Thus let us pick some $a$ such that $\theta \left[ \begin{array}{c} a \\ \gamma + \sigma \end{array} \right] (n\tau) \neq 0$.

Similarly let us consider theta functions of order $n$, $\theta \left[ \begin{array}{c} c \\ 0 \end{array} \right] (n\tau, nz)$ for $c \in \left(\frac{1}{n}\mathbb{Z}/\mathbb{Z}\right)^g$. Among these there is at least one not vanishing...
at \( z = \tau(-a + \frac{\delta}{n}) + \frac{\gamma}{n}; \) let us choose such a \( c. \) We then finally set \( b := c - a + \frac{\delta}{n}, \) and by formula 2 this implies that \( \theta_{\left[ \begin{array}{c} b \\ \gamma \end{array} \right]}(n\tau, 0) \neq 0. \) □

Now we are ready to prove the main result.

**Theorem 10.** The map \( \Phi_{4n} \) is injective on \( \mathcal{A}_g(4n, 8n) \) for all \( n > 1 \) and all \( g \geq 2. \)

**Proof.** Recall that \( \Gamma_g(2n, 4n)/\Gamma_g(4n, 8n) \) acts on theta constants of order \( 4n \) by multiplying them by \( \pm 1, \) depending on characteristics. Thus to finish reconstructing \( Th_{4n}(\tau) \) from \( \Phi_{4n}(\tau) \) (and thus also knowing \( Th_{2n}(\tau') \)) we need to deal with the “projectivization” happening in theorem 8 to recover the necessary signs. By the formulas on page 171 of [6] (see also section 2 of this paper), instead of considering the theta constants \( \theta_{\left[ \begin{array}{c} c \\ 0 \end{array} \right]}(4n\tau) \) with \( c \in \left( \frac{1}{4n}\mathbb{Z}/\mathbb{Z} \right)^g \), we can consider the theta constants \( \theta_{\left[ \begin{array}{c} c \\ \mu \end{array} \right]}(n\tau) \) with \( c \in \left( \frac{1}{2n}\mathbb{Z}/\mathbb{Z} \right)^g. \)

Indeed, suppose that \( \Phi_{4n}(\tau) = \Phi_{4n}(\tau'). \) The previous lemma states that for fixed \( \sigma, \delta, \gamma \) we can always find \( a \) and \( b \) with \( a + b - \frac{\delta}{n} \in \left( \frac{1}{n}\mathbb{Z}/\mathbb{Z} \right)^g \) such that

\[
\theta_{\left[ \begin{array}{c} a \\ \gamma + \sigma \end{array} \right]}(n\tau)\theta_{\left[ \begin{array}{c} b \\ \gamma \end{array} \right]}(n\tau) \neq 0.
\]

Since such products are projectively unique by theorem 8 we have

\[
\theta_{\left[ \begin{array}{c} a \\ \gamma + \sigma \end{array} \right]}(n\tau)\theta_{\left[ \begin{array}{c} b \\ \gamma \end{array} \right]}(n\tau) = t_{\sigma, \delta} \theta_{\left[ \begin{array}{c} a \\ \gamma + \sigma \end{array} \right]}(n\tau')\theta_{\left[ \begin{array}{c} b \\ \gamma \end{array} \right]}(n\tau')
\]

for some (unique, since both sides are non-zero — this is crucial!) constant \( t_{\sigma, \delta} \) independent of \( \gamma. \)

Squaring the above formula we get

\[
t_{\sigma, \delta}^2 = t_{0,0}^2
\]

We claim then that the map

\[
X : \left( \frac{1}{2}\mathbb{Z}/\mathbb{Z} \right)^{2g} \to \pm 1, \quad X(\sigma, \delta) := t_{\sigma, \delta}/t_{0,0}
\]

is a group morphism. In fact for fixed \( \sigma \) we can always find \( a, b \in \left( \frac{1}{n}\mathbb{Z}/\mathbb{Z} \right)^g \) with

\[
\theta_{\left[ \begin{array}{c} a \\ \sigma \end{array} \right]}(n\tau)\theta_{\left[ \begin{array}{c} b \\ 0 \end{array} \right]}(n\tau) \neq 0
\]
and for fixed $\delta$ we can always find $b_1$ with $b_1 - \frac{\delta}{n} \in \left(\frac{1}{n}\mathbb{Z}/\mathbb{Z}\right)^g$ satisfying

$$\theta \left[ \begin{array}{c} b \\ 0 \end{array} \right] (n\tau) \theta \left[ \begin{array}{c} b_1 \\ 0 \end{array} \right] (n\tau) \neq 0.$$ 

This then shows that $X(\sigma, \delta) = X(\sigma, 0)X(0, \delta)$. Similarly one proves that $X(\sigma + \rho, 0) = X(\sigma, 0)X(\rho, 0)$ and $X(0, \delta + \varepsilon) = X(0, \delta)X(0, \varepsilon)$, and thus we see that $X$ is indeed a morphism.

To show that $Th_{4n}(\tau) = Th_{4n}(\tau')$, we need to show that $X$ is identically equal to $+1$. Since $X$ is a morphism, we only need to check that a basis gets mapped to $+1$. If this is not the case, then we have some $X(\sigma, \delta) = -1$. Then we can find an element $M \in \Gamma_g(2n, 4n)/\Gamma_g(4n, 8n)$ (in fact such an element can be found in $\Gamma_g(4n)/\Gamma_g(4n, 8n)$), the action of which on theta constants of level $4n$ would change precisely the appropriate signs — the argument for $n > 1$ is identical to the one given in [4] for $n = 1$.

Thus we know that

$$\theta \left[ \begin{array}{c} a \\ \gamma + \sigma \end{array} \right] (n\tau) \theta \left[ \begin{array}{c} b \\ \gamma \end{array} \right] (n\tau) = \theta \left[ \begin{array}{c} a \\ \gamma + \sigma \end{array} \right] (nM\tau') \theta \left[ \begin{array}{c} b \\ \gamma \end{array} \right] (nM\tau'),$$

from which it follows that $Th_{4n}(\tau) = Th_{4n}(M\tau')$ — by fixing some $b, \gamma$ such that $\theta \left[ \begin{array}{c} b \\ \gamma \end{array} \right] (n\tau) \neq 0$ and varying $a$ and $\gamma$, so that we get

$$\theta \left[ \begin{array}{c} a \\ \gamma + \sigma \end{array} \right] (n\tau)/\theta \left[ \begin{array}{c} a \\ \gamma + \sigma \end{array} \right] (nM\tau') = \text{const independent of } a \text{ and } \sigma.$$ 

Since $Th_{4n}$ is injective, it means that $\tau$ and $M\tau'$ represent the same point in $A_g(4n, 8n)$. This then implies that $\Phi_{4n}(\tau') = \Phi_{4n}(\tau) = \Phi_{4n}(M\tau')$. However, there cannot be an $M \not\in \Gamma_g(4n, 8n)$ such that its action does not change the image under the map $\Phi_{4n}$ (see [4]). Hence we must have $M \in \Gamma_g(4n, 8n)$, so $\tau = \tau'$ in $A_g(4n, 8n)$, and thus the injectivity of $\Phi_{4n}$ is proved. \hfill \Box

Remark 11. We observe that the assumption $n > 1$ has been used to prove that $X(\cdot, \cdot)$ is a homomorphism. In fact, for $n = 1$ we could not show that $X$ is indeed defined, as we did not have the non-vanishing results and thus some of $t_{\sigma, \delta}$ could be undefined if many theta constants vanished.

Remark 12. The injectivity of $\Phi_{4n}$ on the tangent spaces follows from lemma 17 in [4] and the result in [14].

4. Generalized Jacobi’s derivative formulas

In the same spirit as above, the results of [5] can be generalized to higher level. The relationship between $A$ and $C$ provides us with a
way to express vector-valued modular forms constructed using theta constants and their \( \tau \)-derivatives (which, by the heat equation, are the same as the second \( z \)-derivatives) in terms of the gradients of theta functions. These can be used to deduce relations among scalar modular forms involving Jacobian determinants of theta functions. In fact both formulas from [5] can be generalized to higher level. Below we give the appropriate version of Theorem 5 from that paper.

We recall the matrix differential operator

\[
D := \begin{pmatrix}
\frac{\partial}{\partial \tau_{11}} & \frac{1}{2} \frac{\partial}{\partial \tau_{12}} & \cdots & \frac{1}{2} \frac{\partial}{\partial \tau_{1g}} \\
\frac{1}{2} \frac{\partial}{\partial \tau_{21}} & \frac{\partial}{\partial \tau_{22}} & \cdots & \frac{1}{2} \frac{\partial}{\partial \tau_{2g}} \\
\cdots & \cdots & \cdots & \cdots \\
\frac{1}{2} \frac{\partial}{\partial \tau_{g1}} & \cdots & \cdots & \frac{\partial}{\partial \tau_{gg}}
\end{pmatrix}.
\]

Then we have

**Theorem 13.** For any \( a \in \left( \frac{1}{2n} \mathbb{Z}/\mathbb{Z} \right)^g \), \( \delta \in \left( \frac{1}{2} \mathbb{Z}/\mathbb{Z} \right)^g \) the following holds:

\[
\text{const } \left( \theta \begin{bmatrix} 0 \\ \delta \end{bmatrix} (2n\tau) \right)^{2g} \det D \left( \theta \begin{bmatrix} a \\ \delta \end{bmatrix} (2n\tau) / \theta \begin{bmatrix} 0 \\ \delta \end{bmatrix} (2n\tau) \right) = \sum_{\varepsilon_1, \ldots, \varepsilon_g \in \left( \frac{1}{2} \mathbb{Z}/\mathbb{Z} \right)^g} e(2\delta^t(\varepsilon_1 + \ldots + \varepsilon_g)) D([a/2 + \varepsilon_1, 0], \ldots, [a/2 + \varepsilon_g, 0]) (4n\tau)
\]

for some computable constant \( c \).

**Proof.** This follows by linear algebra arguments from the expression of \( A \) in terms of \( C \) and applying the Binet’s formula to the matrix \( C^a \), which has rank one, being equal to the product of a vector and a covector. The proof is the same as in [5]. \( \square \)

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MATHMATICS DEPARTMENT, PRINCETON UNIVERSITY, FINE HALL, WASHINGTON ROAD, PRINCETON, NJ 08544, USA
E-mail address: sam@math.princeton.edu

DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI ROMA “LA SAPIENZA”, PIAZZALE ALDO MORO, 2, I-00185 ROMA, ITALY
E-mail address: salvati@mat.uniroma1.it