Universality in the diffusion of knots

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(Dated: March 25, 2009)

We have evaluated a universal ratio between diffusion constants of the ring polymer with a given knot $K$ and a linear polymer with the same molecular weight in solution through the Brownian dynamics under hydrodynamic interaction. The ratio is found to be constant with respect to the number of monomers, $N$, and hence the estimate at some $N$ should be valid practically over a wide range of $N$ for various polymer models. Interestingly, the ratio is determined by the average crossing number ($N_{AC}$) of an ideal conformation of knotted curve $K$, i.e. that of the ideal knot. The $N_{AC}$ of ideal knots should therefore be fundamental in the dynamics of knots.

PACS numbers: 82.35.Lr,05.40.Fb,05.20.-y

I. INTRODUCTION

Novel knotted structures of polymers have recently been found in various research fields such as DNA, proteins and synthetic polymers. The topology of a ring polymer is conserved under thermal fluctuations in solution and represented by a knot. Topological constraints may lead to nontrivial statistical mechanical and dynamical properties of ring polymers.

Recent progress in experiments of ring polymers should be quite remarkable. Diffusion constants of linear, relaxed and supercoiled DNAs have been measured quite accurately. Here the DNA double helices are unknotted. Furthermore, hydrodynamic radius of circular DNA has also been measured. Ring polymers of large molecular weights are synthesized not only quite effectively but also with small dispersions and of high purity. Circular DNAs with various knot types are derived, and they are separated into knotted species by gel electrophoresis. We should remark that synthetic ring polymers with nontrivial knots have not been synthesized and separated experimentally, yet. However, it is highly expected that ring polymers of nontrivial knot types should be synthesized and their topological effects will be confirmed experimentally in near future.

In the paper we discuss diffusion constant $D_K$ of a ring polymer with fixed topology $K$ in good solution for various knot types. We evaluate it numerically via the Brownian dynamics with hydrodynamic interaction in which bond crossing is effectively prohibited through the finite extensible nonlinear elongational (FENE) potential. We evaluate diffusion constant $D_L$ of a linear polymer with the same molecular weight, and derive ratio $D_K/D_L$. The ratio should correspond to a universal amplitude ratio of critical phenomena and play a significant role in the dynamics of knotted ring polymers. According to the renormalization group arguments, ratio $D_K/D_L$ should be universal if the number of monomers, $N$, is large enough.

The ratio $D_K/D_L$ may have some experimental applications. Ring polymers of different knot types can be separated experimentally with respect to their topologies by making use of the difference among the sedimentation coefficients, which can be calculated from the diffusion constants. Here we remark that the diffusion constant of a ring polymer under no topological constraint, $D_R$, and that of the corresponding linear polymer has been numerically evaluated, and the ratio $C = D_R/D_L$ has been studied.

Through simulation we find that ratio $D_K/D_L$ is approximately constant with respect to $N$ for various knots. Thus, if we evaluate ratio $D_K/D_L$ at some value of $N$, it is practically valid for other values of $N$. We can therefore predict the diffusion constant $D_K$ of a polymer model at some value of $N$, multiplying the ratio $D_K/D_L$ by the estimate of $D_L$. Here we remark that the value of $D_L$ may depend on the number $N$ and on some details of the model.

Furthermore, we show numerically that ratio $D_K/D_L$ is a linear function of the average crossing number ($N_{AC}$) of the ideal knot of $K$, an ideal configuration of knotted curve $K$, which will be defined shortly. Since the ratio $D_K/D_L$ is almost independent of $N$, it follows that the linear fitting formula should be valid practically in a wide range of finite values of $N$. Thus, the ideal knot of a knotted curve $K$ should play a fundamental role in the dynamics of finite-size knotted ring polymers in solution.

Let us introduce the ideal knot, briefly. For a given knot $K$ it is given by the trajectory that allows maximal radial expansion of a virtual tube of uniform diameter centered around the axial trajectory of the knot $K$. We define the $N_{AC}$ of a knotted curve as follows: We take its projection onto a plane, and enumerate the number of crossings in the knot diagram on the plane. Then, we consider a large number of projections onto planes whose normal vectors are uniformly distributed on the sphere of unit radius, and take the average of the crossing number ($N_{AC}$) over all directions.

The paper consists of the following: In section II, the simulation method is explained. In section III, we present the estimates of the diffusion constant of a ring polymer in solution of knot type $K$ for various knot types. Then, we show numerically that the graph of $D_K/D_0$ is almost independent of $N$, and also that ratio $D_K/D_L$ is fitted by a linear function of $N_{AC}$ of the ideal knot of $K$. We also discuss the simulation result in terms of the ratio of equivalent radii, $\alpha_{AC}/\alpha_T$, which corresponds to the universal ratio of the radius of gyration to the hydrodynamic radius. We shall define the equivalent radii explicitly in section III. Finally, we give conclusion in section IV.

Throughout the paper, we employ the symbols of knots fol-
lowing Rolfsen’s textbook, as shown in Figure 1.

II. SIMULATION METHOD

The ring polymer is modeled as a cyclic bead-and-spring chain with $N$ beads connected by $N$ FENE springs with force given by

$$F(r) = -Hr/(1 - r^2/r_{\text{max}}^2) \quad (1)$$

where $r = |r|$. We denote by $b$ the unit of distance, which gives the average distance between neighboring monomers approximately. We set constants $H$ and $r_{\text{max}}$ by $H = 30k_BT/b^2$ and $r_{\text{max}} = 1.3b$. We assume the Lennard-Jones (LJ) potential by

$$V(r_{ij}) = 4\epsilon_{\text{LJ}}\{((\sigma_{\text{LJ}}/r_{ij})^{12} - (\sigma_{\text{LJ}}/r_{ij})^6)\} \quad (2)$$

Here $r_{ij}$ is the distance of beads $i$ and $j$, and $\epsilon_{\text{LJ}}$ and $\sigma_{\text{LJ}}$ denote the minimum energy and the zero energy distance, respectively. We set the Lennard-Jones parameters as $\sigma_{\text{LJ}} = 0.8b$ and $\epsilon_{\text{LJ}} = 0.1k_BT$ so that they give good solvent conditions. Here $k_B$ denotes the Boltzmann constant.

We employ the predictor-corrector version of the Ermak-McCammon algorithm for generating time evolution of a ring polymer in solution. The hydrodynamic interaction is taken into account through the Rotne-Prager-Yamakawa tensor, where the bead friction is given by $\zeta = 6\pi\eta_Ba$ with the bead radius $a = 0.257b$ and a dimensionless hydrodynamic interaction parameter $H^* = (\zeta/6\pi\eta_B)\sqrt{H/\pi k_BT} = 0.25$.

In the present simulation, physical quantities are given in dimensionless units such as in Ref. We divide length by $b$, energy by $k_BT$ and time by $\zeta b^2/k_BT$. Let us indicate dimensionless quantities by an asterisk as superscript. We have $H^* = 30$, $r_{\text{max}}^* = 1.3$. We take the simulation time step $\Delta t^* = 10^{-4}$.

We have set the FENE potential so that the topology of the ring polymer should be effectively conserved. However, bond crossing may occur with a very small probability. Calculating knot invariants, we have confirmed that the fraction of nontrivial knots is very small. If the initial knot type is the trivial knot it is given by $10^{-8} \sim 10^{-7}$, and if the initial knot type is a nontrivial knot, it is given by approximately $10^{-7}$.

III. SIMULATION RESULTS

We define the diffusion constant of a polymer by

$$D = \lim_{t\to\infty} \frac{1}{bt} \langle (r_C(t) - \bar{r}_C(0))^2 \rangle \quad (3)$$

Here $\bar{r}_C(t)$ denotes the position vector of the center of mass of the polymer at time $t$. Making use of (3) we have evaluated diffusion constants $D_L$ and $D_K$.

The estimates of diffusion constants $D_L$ and $D_K$ at $N = 45$ are listed in Table 1 together with those of the mean square radius of gyration $\langle R_G^2 \rangle$. The data of $D_L$ and $D_K$ are plotted against $N$ in Figure 2. The fitting curves to them are given by $D = aN^{-\nu}(1 + bN^{-\Delta})$. Here the errors of the diffusion constants are as small as $10^{-4}$.

Ratios $D_K/D_L$ should correspond to universal amplitude ratios in critical phenomena. Numerically we find that ratio $D_K/D_L$ of two different knots $K_1$ and $K_2$ is almost constant with respect to $N$, at least in the range investigated. For instance, the graph of ratio $D_{K_1}/D_0$ versus $N$ and that of ratio $D_{K_2}/D_0$ versus $N$ for the data are almost flat, as shown in Figures 3 and 4, respectively. Here 0, 31 and 41 denote the trivial, the trefoil and the figure-eight knot, respectively, as shown in Figure 1. The numerical values of $D_{K_1}/D_0$, $D_{K_2}/D_0$ are given from 1.14 to 1.17 in Figure 3, and those of $D_{K_1}/D_0$, $D_{K_2}/D_0$ are given from 1.14 to 1.21 in Figure 4. Thus, the estimate of $D_{K_1}/D_0$ evaluated at a value of $N$, say $N = 45$, for some knot $K$ should also be valid at other finite values of $N$, since it is almost independent of $N$.

For the diffusion constant of a ring polymer, $D_R$, the ratio $D_{R}/D_L$ should correspond to a universal amplitude ratio and should be universal if $N$ is large enough. For the diffusion constant $D_R$, there is no topological constraint in the ring polymer model and $D_R$ does not mean $D_K$ of a knot $K$. In the previous simulation it has been found that ratio $D_{K}/D_L$ is given by about 1.1 for the present polymer model and almost independent of $N$ within the range investigated.

From the numerical observations and the RG arguments, we have two conjectures: (A) $D_{B}/D_L$ should be given by 1.1 for some wide range of finite values of $N$ and also in the large $N$ limit; (B) ratio $D_{K}/D_L$ for a nontrivial knot $K$ should remain almost the same value in a wide range of finite values of $N$, i.e. the $N$-dependence should be very small.

Quite interestingly we find that ratio $D_{K}/D_L$ can be approximated by a linear function of the average crossing number ($N_{AC}$) of ideal knots, i.e. the ideal representations of the corresponding knots. In Figure 5 simulation data of $D_{K}/D_L$ are plotted against $N_{AC}$ of ideal knots. We find that the data points are fitted well by the following empirical formula:

$$D_{K}/D_L = a + bN_{AC}. \quad (4)$$

| knot type | $D$ | $\langle R_G^2 \rangle$ |
|-----------|-----|------------------|
| linear    | 0.12038 ± 0.00085 | 9.33029 ± 0.03219 |
| 0         | 0.13059 ± 0.00089 | 5.26539 ± 0.00913 |
| 31        | 0.14530 ± 0.00079 | 3.21052 ± 0.00505 |
| 41        | 0.14876 ± 0.00074 | 2.78817 ± 0.00160 |
| 51        | 0.15277 ± 0.00085 | 2.72300 ± 0.00261 |
| 52        | 0.15640 ± 0.00078 | 2.61427 ± 0.00132 |
| 61        | 0.15927 ± 0.00078 | 2.47449 ± 0.00137 |
| 62        | 0.15902 ± 0.00095 | 2.37272 ± 0.00126 |
| 71        | 0.16416 ± 0.00083 | 2.47162 ± 0.00109 |

TABLE I: Estimates of diffusion constants $D_L$ and $D_K$, and the mean square radius of gyration $\langle R_G^2 \rangle$ and for a linear polymer of $N = 45$ and ring polymers of $N = 45$ with various knot types.
Here, the estimates of $a$ and $b$ are given in the caption of Figure 5. Thus, the diffusion constant $D_K$ of a knot $K$ can be estimated in terms of the $N_{AC}$ of the ideal knot of $K$.

Let us discuss the $\chi^2$ values. We have $\chi^2 = 2$ for the fitting line of Figure 5, which is for the data of $N = 45$. For the data of $N = 36$ we have a good fitting line with $\chi^2 = 3$. The estimates of $a$ and $b$ for $N = 36$ are similar to those for $N = 45$. Thus, we may conclude that the graph of $D_K/D_L$ versus $N_{AC}$ is fitted by a linear line.

For a finite value of $N$, we can estimate the diffusion constant $D_K$ of a knot $K$ through formula (3) by the $N_{AC}$ of the ideal knot of $K$. Here we have assumed that coefficients $a$ and $b$ of (4) are independent of $N$ since the graphs of $D_K/D_0$ and $D_0/D_L$ are almost flat with respect to $N$. In fact, there is almost no numerical support for suggesting a possible $N$-dependence of $a$ and $b$, directly.

We thus summarize the simulation results so far as follows: ratio $D_K/D_0$ for a knot $K$ should be almost constant with respect to $N$ in a wide range of $N$ and can be expressed by the linear function of $N_{AC}$ of ideal knots. Eq. (4) should be useful in separating synthetic ring polymers into various knotted species by making use of the difference among sedimentation coefficients.

Ideal knots should play a fundamental role in the dynamics of knotted ring polymers in solution. In fact, we have shown it for the diffusion constants. In experiments of gel electrophoresis drift velocities of different knots formed on the same DNA molecules were shown to be simply related to the $N_{AC}$ of ideal knots. The two independent results suggest the importance of the $N_{AC}$ of ideal knots in the dynamics of knotted ring polymers, although the physical situations are different.

Let us now discuss the simulation results from the viewpoint of equivalent radii. The equivalent radius for any solution property is the radius of a spherical particle having the same value of solution property as that of the macromolecule under consideration. The ratio of equivalent radii should be universal, and it should play a similar role as the universal amplitude ratio such as the ratio of diffusion constants.

We define equivalent radii $a_G$ and $a_T$ explicitly by

$$a_G = \sqrt{\frac{5}{3}\langle R_G^2 \rangle},$$

$$a_T = \frac{k_BT}{6\pi \eta_D}.$$

Here $a_G$ and $a_T$ corresponds to the radius of gyration $R_G = \sqrt{\langle R_G^2 \rangle}$ and the translational friction coefficient $D$, respectively. The ratio $a_G/a_T$ corresponds to the ratio of the radius of gyration to the hydrodynamic radius, and should be universal.

The numerical estimates of $a_G/a_T$ for $N = 45$ for the present simulation are listed in Table 2 for linear and ring polymers with various knot types. In Figure 6, the ratio $a_G/a_T$ is plotted against the number of segments, $N$, for linear and ring polymers with various knot types. Interestingly, the graphs show a weak $N$-dependence. They are fitted by a function $a_G/a_T = a(1 - bN^{-c})$ with parameters $a$, $b$, and $c$.

| knot type | $a_G/a_T$          |
|-----------|--------------------|
| linear    | 1.8471 ± 0.00475   |
| 0         | 1.5053 ± 0.01869   |
| 3         | 1.3079 ± 0.01151   |
| 4         | 1.20824 ± 0.00055  |
| 5         | 1.26636 ± 0.00077  |
| 6         | 1.27029 ± 0.00052  |
| 7         | 1.25865 ± 0.00084  |
| 8         | 1.23046 ± 0.00058  |
| 9         | 1.29643 ± 0.00050  |

TABLE II: Estimates of ratio $a_G/a_T = \sqrt{\langle R_G^2 \rangle}/3D^*/a^*$ for a linear polymer of $N = 45$ and ring polymers of $N = 45$ with various knot types. Here, $\langle R_G^2 \rangle = \langle R_T^2 \rangle/b^2$, $D^* = 6\pi \eta_d aD/k_BT$ and $a^* = a/b$.

IV. CONCLUSION

We have evaluated universal ratios among the diffusion constants of knotted ring polymers in good solution for several knots, where bond crossing is effectively prohibited in the Brownian dynamics under hydrodynamic interaction. The universal ratio of diffusion constants $D_K/D_L$ is almost constant with respect to the number of polymer segments, $N$. Moreover, it is found that the ratio $D_K/D_L$ is determined by the $N_{AC}$ of the ideal knot of $K$. Through the linear relation, we can estimate the diffusion constant of a given knot.

Acknowledgments

The authors would like to thank Dr. A. Takano and Dr. K. Tsurusaki for valuable comments. The present study is partially supported by KAKENHI (Grant-in-Aid for Scientific Research) on Priority Area “Soft Matter Physics” from the Ministry of Education, Culture, Sports, Science and Technology of Japan, 19031007. We drew the figures of a linear and knotted ring polymers using OCTA([http://octa.jp]).
1. J. I. Sulkowska, P. Sulkowski, P. Szymczak and M. Cieplak, Phys. Rev. Lett. 100, 058106 (2008).
2. A. Y. Grosberg and Y. Rabin, Phys. Rev. Lett. 99, 217801 (2007).
3. M. Baiesi, E. Orlandini and A.L. Stella, Phys. Rev. Lett. 99, 058301 (2007).
4. E. Ercolini, F. Valle, J. Adamcik, G. Witz, R. Metzler, P. De Los Rios, J. Roca and G. Dietler, Phys. Rev. Lett. 98, 058102 (2007).
5. E. F. Casassa, J. Polym. Sci., Part A 3, 605 (1965).
6. A.V. Vologodskii, A.V. Lukashin, M.D. Frank-Kamenetskii, and V.V. Anshelevich, Sov. Phys. JETP 39, 1059 (1974).
7. J. R. Roovers and P. M. Toporowski, Macromolecules 16, 843 (1983).
8. Cyclic Polymers, edited by J.A. Semlyen (Elsevier Applied Science Publishers, London and New York, 1986); 2nd Edition (Kluwer Academic Publ., Dordrecht, 2000).
9. G. ten Brinke and G. Hadziioannou, Macromolecules 20, 480 (1987).
10. K. Koniaris and M. Muthukumar, Phys. Rev. Lett. 66, 2211–2214 (1991).
11. S. R. Quake, Phys. Rev. Lett. 73, 3317–3320 (1994).
12. T. Deguchi and K. Tsunusaki, Phys. Rev. E 55, 6245-6248 (1997).
13. A. Y. Grosberg, Phys. Rev. Lett. 85, 3858–3861 (2000).
14. P.-Y Lai, Phys. Rev. E 66, 021805 (2002).
15. M. K. Shimamura and T. Deguchi, Phys. Rev. E 65, 051802 (2002).
16. A. Dobay, J. Dubochet, K. Millett, P.-E Sottas and A. Stasiak, Proc. Natl. Acad. Sci. 100, 5611–5615 (2003).
17. B. Marcone, E. Orlandini, A. L. Stella and F. Zonta, J. Phys. A: Math. Gen. 38, L15-L21 (2005).
18. E. Orlandini, A. L. Stella, C. Vanderzande and F. Zonta J. Phys. A: Math. Theor. 41, 122002 (7pp) (2008)
19. E. Orlandini and S. G. Whittington, Rev. Mod. Phys. 79, 611 (2007).
20. R. M. Robertson, S. Laib and D. E. Smith, Proceedings of the National Academy of Science 103 7310-7314 (2006).
21. S. Araki, T. Nakai, K. Hizume, K. Takeyasu and K. Yoshikawa, Chem. Phys. Lett. 418, 255–259 (2006).
22. C. W. Bielawski, D. Benitez and R. H. Grubbs, Science 297, 2041–2044 (2002).
23. D. Cho, K. Masuoka, K. Koguchi, T. Asari, D. Kawaguchi, A. Takano and Y. Matsushita, Polymer Journal 37, 506–511 (2005).
24. A. Takano, Y. Kushida, K. Aoki, K. Masuoka, K. Hayashida, D. Cho, D. Kawaguchi and Y. Matsushita, Macromolecules 40, 679–681 (2007).
25. A.V. Vologodskii, N.J. Crisona, B. Laurie, P. Oieranski, V. Katritch, J. Dubochet and A. Stasiak, J. Mol. Biol. 278, 1–3 (1998).
26. N. Kanaeda and T. Deguchi, J. Phys. A: Math. Theor. 41, 145004 (11pp) (2008).
27. Y. Oono, Adv. Chem. Phys. 61, 301-437 (1985).
28. Y. Oono and M. Kohmoto, J. Chem. Phys. 78, 520 (1983).
29. B. Schaub and D. B. Creamer, Phys. Lett. A 121 435-442 (1987).
30. J.M. García Bernal and M.M. Tirado, J.J. Freire, and J. García de la Torre, Macromolecules 23 (1990) 3357–3362
31. J.M. García Bernal and M.M. Tirado, J.J. Freire, and J. García de la Torre, Macromolecules 24 (1991) 593–598
32. V. Bloomfield and B. H. Zimm, J. Chem. Phys. 44, 315–323 (1966).
33. B. Dünweg, D. Reith, M. Steinhauser and K. Kremer, J. Chem. Phys. 117, 914–924 (2002)
34. B. Liu and B. Dünweg, J. Chem. Phys. 118, 8061–8072(2003)
35. A. Y. Grosberg, A. Feigel and Y. Rabin, Phys. Rev. E 54 6618–6622 (1996).
36. V. Katritch, J. Bednar, D. Michoud, R.G. Scharein, J. Dubochet and A. Stasiak, Nature 384 142–145 (1996).
37. A. Ortega and J. Garcia de la Torre, Biomacromolecules 8, 2464–2475 (2007)
38. D. Rolfsen, Knots and Links (Publish or Perish, Wilmington DE, 1976).
39. J. G. Hernández Cifre, R. Pamics, M. C. López Martinez, J. García de la Torre, Polymer 46, 267–274 (2005).
40. A. Rey, J. J. Freire and J. García de la Torre, Macromolecules 20, 342–346 (1987).
41. A. Iniesta and J. Garcia de la Torre, J. Chem. Phys. 92, 2015-2018 (1990).
42. J. Rotne and S. Prager, J. Chem. Phys. 50, 4831–4837 (1969)
43. H. Yamakawa, J. Chem. Phys. 53, 207–246(1970)
FIG. 1: Figures of a linear polymer and knotted ring polymers with the symbols of knots given in Rolfsen’s textbook.
FIG. 2: Diffusion constants of linear and knotted ring chains with knots 0, 3_1, 4_1, 5_1, 6_1 and 7_1, versus $N$. Fitted by $D = aN^{-\nu}(1 + bN^{-\Delta})$ with the following best estimates: For a linear chain, $a = 0.90 \pm 0.23$, $\nu = 0.53 \pm 0.06$, $b = 0.51 \pm 0.93$, $\Delta = 1.14 \pm 2.39$, $\chi^2 = 17$; for the trivial knot (0), $a = 1.03 \pm 1.11$, $\nu = 0.55 \pm 0.18$, $b = 0.14 \pm 0.78$, $\Delta = 0.60 \pm 6.09$, $\chi^2 = 28$; for the trefoil knot (3_1) $a = 1.00 \pm 3.87$, $\nu = 0.52 \pm 0.67$, $b = 1.18 \pm 1.12$, $\Delta = 0.77 \pm 6.09$, $\chi^2 = 27$.

FIG. 3: Ratio $D_{3_1}/D_0$ of diffusion constants for the trefoil knot (3_1) and the trivial knot (0) versus the number of segments $N$. Fitting curve is given by $D_{3_1}/D_0 = a(1 + bN^{-c})$, where $a = 1.07 \pm 0.64$, $b = 0.25 \pm 0.39$, and $c = 0.39 \pm 3.29$ with $\chi^2 = 6$. 
FIG. 4: Ratio $D_{4_1}/D_0$ of diffusion constants for the figure-eight knot ($4_1$) and the trivial knot ($0$) versus the number of segments $N$. Fitting curve is given by $D_{4_1}/D_0 = a(1+bN^c)$ where $a = 1.02 \pm 0.56$, $b = 1.76 \pm 8.26$, and $c = 0.70 \pm 2.58$ with $\chi^2 = 0.03$.

FIG. 5: $D_K/D_L$ versus the average crossing number ($N_{AC}$) of ideal knot $K$ for $N = 45$: The data are approximated by $D_K/D_L = a + bN_{AC}$ where $a = 1.11 \pm 0.02$ and $b = 0.0215 \pm 0.0003$ with $\chi^2 = 2$. 
FIG. 6: $a_G/a_T$ of linear and knotted ring chains with knots $0$, $3_1$, $4_1$, $5_1$, $6_1$ and $7_1$, versus $N$. Fitted by $a_G/a_T = a(1 - bN^{-c})$ with the following best estimates: For a linear chain, $a = 2.37 \pm 0.29$, $b = 0.52 \pm 0.03$, $c = 0.22 \pm 0.10$, $\chi^2 = 55$; for the trivial knot ($0$), $a = 1.51 \pm 0.03$, $b = 0.72 \pm 0.23$, $c = 0.89 \pm 0.22$, $\chi^2 = 14$; for the trefoil knot ($3_1$), $a = 1.30 \pm 0.03$, $b = 2.10 \pm 3.71$, $c = 1.27 \pm 0.76$, $\chi^2 = 20$. 