ROOTED TREES, NON-ROOTED TREES AND HAMILTONIAN B-SERIES

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Abstract. We explore the relationship between (non-planar) rooted trees and free trees, i.e. without root. We give in particular, for non-rooted trees, a substitute for the Lie bracket given by the antisymmetrization of the pre-Lie product.

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1. Introduction

A striking link between rooted trees and vector fields on an affine space $\mathbb{R}^n$ has been established by A. Cayley [8] as early as 1857. The interest for this correspondence has been renewed since J. Butcher showed the key role of rooted trees for understanding Runge-Kutta methods in numerical approximation [5, 4, 16]. The modern approach to this correspondence can be summarized as follows: the product on vector fields on $\mathbb{R}^n$ defined by:

$$ (\sum_{i=1}^{n} f_i \partial_i) \triangleright (\sum_{i=1}^{n} g_j \partial_j) := \sum_{j=1}^{n} \left( \sum_{i=1}^{n} f_i (\partial_i g_j) \right) \partial_j $$

is left pre-Lie, which means that for any vector fields $a, b, c$ the associator $a \triangleright (b \triangleright c) - (a \triangleright b) \triangleright c$ is symmetric with respect to $a$ and $b$. On the other hand, the free pre-Lie algebra with one generator (on some base field $k$) is the vector space $T$ spanned by the planar rooted trees [10, 15]. The generator is the one-vertex tree $\bullet$, and the pre-Lie product on rooted trees is given by grafting:

$$ s \rightarrow t = \sum_{v \in V(t)} s \rightarrow_v t, $$

where $s \rightarrow_v t$ is the rooted tree obtained by grafting the rooted tree $s$ on the vertex $v$ of the tree $t$. Hence for any vector field $a$ on $\mathbb{R}^n$ there exists a unique pre-Lie algebra morphism $F_a$ from $T$ to vector fields such that $F_a(\bullet) = a$. This can be generalized to an arbitrary number of generators, since the free pre-Lie algebra on a set $D$ of generators is the span of rooted trees with vertices coloured by $D$. In this case, for any collection $\underline{a} = (a_d)_{d \in D}$ of vector fields, there exists a unique pre-Lie algebra morphism $F_{\underline{a}}$ from the linear span $T_D$ of coloured trees to vector fields on $\mathbb{R}^n$, such that $F_{\underline{a}}(\bullet_d) = a_d$ for any $d \in D$.

The vector fields $F_{\underline{a}}(t)$ (or $F_{\underline{a}}(t)$ in the coloured case) are the elementary differentials, building blocks of the B-series [16] which are defined as follows: for any linear form $\alpha$ on $T_D \oplus \mathbb{R}1$ where $1$ is the empty tree, for any collection of vector fields $\underline{a}$ and for any initial point $y_0 \in \mathbb{R}^n$, the corresponding B-series is a formal series in the indeterminate $h$ given by:

$$ B_{\underline{a}}(\alpha, y_0) = \alpha(1)y_0 + \sum_{t \in T_D} h^{|t|} \frac{\alpha(t)}{\text{sym}(t)} F_{\underline{a}}(t)(y_0). $$

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1Such coloured B-series are sometimes called NB-series in the literature.
Here $|t|$ is the number of vertices of $t$, and $\text{sym}(t)$ is its symmetry factor, i.e. the cardinal of its automorphism group $\text{Aut} t$. For any vector field $a$, the exact solution of the differential equation:

$$\dot{y}(t) = a(y(t))$$

with initial condition $y(0) = y_0$ admits a (one-coloured) B-series expansion at time $t = h$, and its approximation by any Runge-Kutta method as well \[5, 6, 16\]. The formal transformation $y_0 \mapsto B_a(\alpha, y_0)$ is a formal series with coefficients in $C^\infty(\mathbb{R}^n, \mathbb{R}^n)$.

We will be interested in canonical B-series \[7\], i.e. such that the formal transformation $B_a(\alpha, -)$ is a symplectomorphism for any collection of hamiltonian vector fields $a$. Here, the dimension $n = 2r$ is even, and $\mathbb{R}^{2r}$ is endowed with the standard symplectic structure:

$$\omega(x, y) = \sum_{i=1}^{r} x_i y_{r+i} - x_{r+i} y_i,$$

and a vector field $a = \sum_{i=1}^{2r} a_i \partial_i$ is hamiltonian if there exists a smooth map $H : \mathbb{R}^{2r} \to \mathbb{R}$ such that:

$$a_i = -\frac{\partial H}{\partial t_{i+r}} \quad \text{for } i = 1, \ldots, r,$$

$$a_i = \frac{\partial H}{\partial t_{i-r}} \quad \text{for } i = r + 1, \ldots, 2r.$$

Recall that the Poisson bracket of two smooth maps $f, g$ on $\mathbb{R}^{2r}$ is given by:

$$\{f, g\} = \sum_{i=1}^{r} \frac{\partial f}{\partial t_i} \frac{\partial g}{\partial t_{i+r}} - \frac{\partial g}{\partial t_i} \frac{\partial f}{\partial t_{i+r}}.$$

Hence hamiltonian vector fields are those vector fields $a$ which can be expressed as:

$$a = \{H, -\}$$

for some $H \in C^\infty(\mathbb{R}^{2r})$. A B-series turns out to be canonical if and only if the following condition holds for any rooted trees $s$ and $t$ \[3, \text{Theorem 2}\]:

$$\alpha(s \circ t) + \alpha(t \circ s) = \alpha(s)\alpha(t),$$

where $s \circ t$ is the right Butcher product, defined by grafting the tree $t$ on the root of the tree $s$. This result is also valid in the coloured case. The infinitesimal counterpart of this result expresses as follows \[16\], Theorem IX.9.10 for one-colour case): a B-series $B_a(\alpha, -)$ with $\alpha(1) = 0$ defines a hamiltonian vector field for any hamiltonian vector field $a$ if and only if:

$$\alpha(s \circ t) + \alpha(t \circ s) = 0.$$

Let us call the B-series of the type described above hamiltonian B-series. Our interest in non-rooted trees comes from the following elementary observation: the two rooted trees $s \circ t$ and $t \circ s$ are equal as non-rooted trees, and one is obtained from the other by shifting the root to a neighbouring vertex. As an easy consequence of \[8\], any hamiltonian B-series $B_a(\alpha, -)$ has to satisfy that if two rooted trees $s$ and $t$ are equal as non-rooted trees, then:

$$\alpha(s) = \pm\alpha(t).$$

This implies that, modulo a careful account of the signs involved, hamiltonian B-series are naturally indexed by non-rooted trees rather than by rooted ones. The sign is plus or minus according to the parity of the minimal number of "root shifts" $s_1 \circ s_2 \mapsto s_2 \circ s_1$ that are required to change $s$ into $t$. 


In the present paper we address the following question: what survives from the pre-Lie structure at the level of non-rooted trees? There is a natural linear map $\tilde{\mathcal{X}}$ from non-rooted trees to (the linear span of) rooted trees, sending a tree to the sum of all its rooted representatives, with alternating signs. Its precise definition involves a total order on rooted trees introduced by A. Murua [19]. We propose a binary product $\diamond$ on the linear span of non-rooted trees, which is roughly speaking an alternating sum of all trees obtained by linking a vertex of the first tree with a vertex of the second tree. Theorem 4 is the key result of the paper. It implies the fact that $\diamond$ is a Lie bracket and that $\tilde{\mathcal{X}}$ is a Lie algebra morphism, the Lie bracket on rooted trees being given by antisymmetrizing the pre-Lie product.

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2. Structural facts about non-rooted trees

We denote by $T$ (resp. $FT$) the set of non-planar rooted (resp. non-rooted) trees. We denote by $\mathcal{T}$ (resp. $\mathcal{FT}$) the vector spaces freely generated by $T$ (resp. $FT$). The projection $\pi : T \rightarrow FT$ is defined by forgetting the root. It extends linearly to $\pi : \mathcal{T} \rightarrow \mathcal{FT}$. Rooted trees will be denoted by latin letters $s, t, \ldots$, non-rooted trees by greek letters $\sigma, \tau, \ldots$. We will also use "free tree" as a synonymous for "non-rooted tree". For any free tree $\tau$ and for any vertex $v$ of $\tau$, we denote by $\tau_v$ the unique rooted tree built from $\tau$ by putting the root at $v$.

2.1. A total order on rooted trees. Recall that any rooted tree $t$ is obtained by grafting rooted trees $t_1, \ldots, t_q$ on a common root:

$$t = B_+(t_1, \ldots, t_q).$$

The trees $t_j$ are called the branches of $t$. A. Murua defines in [19] a total order on the set of (one-colour) rooted trees in a recursive way as follows: the canonical decomposition of a tree $t$ is given by $t = t_L \diamond t_R$ where $t_R$ is the maximal branch of $t$. The maximality is to be understood with respect to the total order, supposed to be already defined for trees with number of vertices strictly smaller than $|t|$. Then $s < t$ if and only if:

- either $|s| < |t|$, 
- or $|s| = |t|$ and $s_L < t_L$, 
- or $|s| = |t|$, $s_L = t_L$ and $s_R < t_R$.

In the one-colour case, the total order of the first few trees is:

$$\bullet < \overset{1}{\triangle} < \overset{1}{\triangledown} < \overset{2}{\triangle} < \overset{2}{\triangledown} < \overset{3}{\triangle} < \overset{3}{\triangledown} < \overset{4}{\triangle} < \overset{4}{\triangledown} < \overset{5}{\triangle} < \overset{5}{\triangledown} < \overset{6}{\triangle} < \overset{6}{\triangledown} < \cdots$$

If we prescribe a total order on the set of colours $D$ and allow the set of one node coloured trees to inherit this order, incorporating this into the definition above gives a total order on the set of coloured rooted trees. Note that the structure of the one-colour order is not entirely preserved, as, for example, for two colours $\bullet < \circ$ we have $\overset{\bullet}{\circ} > \overset{\circ}{\circ}$ whereas $\overset{\bullet}{\circ} < \overset{\circ}{\bullet}$. 

2.2. **Superfluous trees.** This notion has been introduced in [1], where the authors describe order conditions for canonical B-series coming from Runge-Kutta approximation methods. Let $B_2(\alpha, -)$ be a hamiltonian B-series. According to [2], we have $\alpha(t \circ s) = 0$ for any rooted tree $t$. Any non-rooted tree $\tau$ such that there exists a rooted tree $s$ with $s \circ s \in \pi^{-1}(\tau)$ is called a superfluous tree, and a rooted tree $t$ is said to be superfluous if its underlying free tree $\pi(t)$ is. Such trees never appear in a hamiltonian B-series. For any free tree $\tau \in FT$, its canonical representative is the maximal element of the set $\pi^{-1}(\tau) \subset T$ for the total order above. The following lemma gives a characterization of superfluous trees:

**Lemma 1.** Let $\tau \in FT$ have two distinct vertices $v$ and $w$ such that $\tau_v = \tau_w$ is the canonical representative of $\tau$. Then:

1. $v$ and $w$ are the two ends of a common edge in $\tau$,
2. There exists $s \in T$ such that $\tau_v = \tau_w = s \circ s$.

**Proof.** First of all, the maximal branch of $\tau_v$ contains $w$ (and vice-versa). Indeed, Suppose the maximal branch of $\tau_v$ does not contain $w$ (and hence vice-versa). Let

$$\tau_v = B^+(t_1, t_2, \ldots, t_n, t_w, t_{\text{max}}), \quad \tau_w = B^+(t'_1, t'_2, \ldots, t'_n, t'_v, t'_{\text{max}}),$$

where $t_w$ is the branch of $\tau_v$ containing $w$ and $t'_v$ similarly. It is clear that $t'_v$ contains all branches of $\tau_v$ except $t_w$. Hence $|t'_v| > |t_1| + \ldots + |t_n| + |t_{\text{max}}|$ and as $|t_{\text{max}}| = |t'_{\text{max}}|$ we have $|t'_v| > |t'_{\text{max}}|$, a contradiction. Now suppose that $v$ and $w$ are not neighbours, and choose a vertex $x$ between $v$ and $w$, i.e. such that there is a path from $v$ to $w$ of meeting $x$. The maximal branch of $\tau_x$ cannot contain both $v$ and $w$; suppose it does not contain $v$. Then it is a subtree of the maximal branch $\tau_v$ and hence contains strictly less vertices. Looking at the canonical decompositions:

$$t := \tau_v = \tau_w = t_L \circ t_R, \quad t' := \tau_x = t'_L \circ t'_R,$$

we have then $|t'_L| > |t_L|$, which immediately yields $\tau_x > \tau_v$, which is a contradiction. This proves the first assertion, and the second assertion follows immediately. \hfill \Box

There are four superfluous free trees with six vertices or less. The corresponding superfluous rooted trees are:

![Superfluous trees](image)

We denote by $S$ the set of superfluous free trees and by $FT'$ the set of non-superfluous trees, hence $FT = FT' \sqcup S$. The corresponding linear spans will be denoted by $S$ and $FT'$. We have $FT = S \oplus FT'$, which leads to a linear isomorphism:

$$FT' \sim FT / S.$$

2.3. **Symmetries.** We keep the notations of the previous subsection. For any non-superfluous tree $\tau \in FT'$ we denote by $\star$ the unique vertex such that $\tau_\star$ is the canonical representative of $\tau$. The group of automorphisms of $\tau$ is the subgroup $\text{Aut } \tau$ of the group of permutations $\varphi$ of $\mathcal{V}(\tau)$ which respect the tree structure, i.e. such that, for any $v, w \in \mathcal{V}(\tau)$, there is an edge between $v$ and $w$ if and only if there is an edge between $\varphi(v)$ and $\varphi(w)$.

For any rooted tree $t$ we also denote by $\text{Aut } t$ its group of automorphisms, i.e. the subgroup of the group of permutations $\varphi$ of $\mathcal{V}(t)$ which respect the rooted tree structure. It obviously coincides with the stabilizer of the root in $\text{Aut } \pi(t)$. Now for any non-superfluous free tree $\tau$ it is obvious from Lemma [1] that $\text{Aut } \tau$ fixes the vertex $\star$, hence $\text{Aut } \tau = \text{Aut } \tau_\star$.\hfill \Box
Now $\text{Aut } \tau$ acts on the set of vertices $V(\tau)$. Moreover, for any vertex $v$ this group acts transitively on the subset of possible roots for $\tau_v$, namely:
\[ R_v(\tau) := \{ w \in V(\tau), \tau_w \sim \tau_v \}. \]

Hence $R_v(\tau)$ identifies itself with the homogeneous space:
\[ R_v(\tau) \sim \text{Aut } \tau_v / \text{Aut } \tau_v. \]

This immediately leads to the following proposition, which is implicit in the proof of Lemma IX.9.7 in \cite{16}:

**Proposition 2.** Let $\tau$ be a non-superfluous free tree, let $t$ be a rooted tree such that $\pi(t) = \tau$, and let $N(t, \tau)$ be the number of vertices $v \in V(\tau)$ such that $\tau_v = t$. Then:
\[ N(t, \tau) = \frac{\text{sym}(\tau_v)}{\text{sym}(t)}. \]

2.4. **Grafting and linking.** Let $\sigma$ and $\tau$ be two non-rooted trees, and let us choose a vertex $v$ of $\sigma$ and a vertex $w$ of $\tau$. We will denote by $\sigma_{v \rightarrow w} \tau$ the non-rooted tree obtained by taking $\sigma$ and $\tau$ together and adding a new edge between $v$ and $w$. This linking operation is related to grafting of rooted trees as follows: for any other choice of vertices $x$ of $\sigma$ and $y$ of $\tau$ we have:
\[ (\sigma_{v \rightarrow w} \tau)_y = \sigma_v \rightarrow_w \tau_y, \]
\[ (\sigma_{v \rightarrow w} \tau)_x = \tau_w \rightarrow_v \sigma_x. \]

3. **A binary operation on non-rooted trees**

The linear map $\widetilde{X} : T \rightarrow T$ is defined for any non-rooted tree $\tau$ by:
\[ \widetilde{X}(\tau) = \sum_{v \in V(\tau)} \varepsilon(v, \tau) \tau_v, \]

and extended linearly. Here $\varepsilon(v, \tau)$ is equal to 0 if $\tau$ is superfluous, and is equal to 1 (resp. $-1$) if $\tau$ is not superfluous and if the number of requested root shifts to change $\tau_v$ into the canonical representative of $\tau$ is even (resp. odd). This number, which we denote by $\kappa(v, \tau)$, is indeed unambiguous for non-superfluous trees according to Lemma \cite{11}. We obviously have:
\[ \varepsilon(v, \tau) = \varepsilon(\varphi(v), \tau) \]
for any $\varphi \in \text{Aut } \tau$. The introduction of the map $\widetilde{X}$ is justified by the fact that, according to \cite{5}, \cite{15} and Proposition 2 rooted trees involved in hamiltonian B-series do group themselves under terms $\tilde{X}(\tau)$ with $\tau \in FT$. Indeed,

**Proposition 3.**
\[ B_{\text{sym}}(\alpha, -) = \sum_{\tau \in FT} h^{v|} \frac{\alpha(\tau_v)}{\text{sym}(\tau_v)} F_{\text{sym}}(\widetilde{X}(\tau)). \]

Now let us define a binary product on $FT$ by the formula:
\[ \sigma \circ \tau = \sum_{v \in V(\sigma), w \in V(\tau)} \delta(v, w) \sigma_{v \rightarrow w} \tau, \]

with $\delta(v, w) := \varepsilon(w, \sigma_{v \rightarrow w} \tau) \varepsilon(v, \sigma) \varepsilon(w, \tau)$.

**Theorem 4.** We have $\sigma \circ \tau \in FT'$ for any $\sigma, \tau \in FT$, and $\sigma \circ \tau = 0$ if $\sigma$ or $\tau$ is superfluous. The product $\circ$ is antisymmetric, and the following relation holds:
\[ \tilde{X}(\sigma \circ \tau) = \tilde{X}(\sigma) \rightarrow \tilde{X}(\tau) \rightarrow \tilde{X}(\sigma) = [\tilde{X}(\sigma), \tilde{X}(\tau)]. \]
Proof. A computation of the left-hand side gives:
\[
\tilde{X}(\sigma \circ \tau) = \sum_{v, x \in V(\sigma), w \in V(\tau)} \varepsilon(x, \sigma_{v \rightarrow w}) \varepsilon(w, \sigma_{v \rightarrow w}) \varepsilon(v, \sigma) \varepsilon(w, \tau)(\sigma_{v \rightarrow w})_x \\
+ \sum_{v \in V(\sigma), w, y \in V(\tau)} \varepsilon(y, \sigma_{v \rightarrow w}) \varepsilon(w, \sigma_{v \rightarrow w}) \varepsilon(v, \sigma) \varepsilon(w, \tau)(\sigma_{v \rightarrow w})_y,
\]
and computing the right-hand side gives:
\[
[\tilde{X}(\sigma), \tilde{X}(\tau)] = - \sum_{v, x \in V(\sigma), w \in V(\tau)} \varepsilon(v, \sigma) \varepsilon(w, \tau) \tau_w \rightarrow_x \sigma_v \\
+ \sum_{v \in V(\sigma), w, y \in V(\tau)} \varepsilon(v, \sigma) \varepsilon(w, \tau) \tau_v \rightarrow_y \tau_w.
\]
Exchanging \(x\) and \(v\) in the first sum, and \(y\) and \(w\) in the second, we get:
\[
[\tilde{X}(\sigma), \tilde{X}(\tau)] = - \sum_{v, x \in V(\sigma), w \in V(\tau)} \varepsilon(x, \sigma) \varepsilon(w, \tau) \tau_w \rightarrow_v \sigma_x \\
+ \sum_{v \in V(\sigma), w, y \in V(\tau)} \varepsilon(v, \sigma) \varepsilon(y, \tau) \tau_v \rightarrow_y \tau_y.
\]
The first assertion is immediate since \(\varepsilon(w, \sigma_{v \rightarrow w})\) vanishes if \(\sigma_{v \rightarrow w} \tau\) is superfluous. The second assertion is also immediate, since \(\delta(v, w)\) vanishes if \(\sigma\) or \(\tau\) is superfluous. The antisymmetry comes from the fact that \(v\) and \(w\) are neighbours in \(\sigma_{v \rightarrow w} \tau\).

(1) If \(\sigma\) or \(\tau\) is superfluous, any individual term in both sides vanishes.

(2) If \(\sigma\) and \(\tau\) are not superfluous, it may happen that \(\sigma_{v \rightarrow w} \tau\) is superfluous for some \(v \in V(\sigma)\) and \(w \in V(\tau)\). The corresponding term \(\tilde{X}(\sigma_{v \rightarrow w} \tau)\) in \(\tilde{X}(\sigma \circ \tau)\) vanishes. On the other hand, the sum of all terms in \([\tilde{X}(\sigma), \tilde{X}(\tau)]\) corresponding to the couple \((v, w)\) chosen above writes down as:
\[
T_{v, w} := - \sum_{x \in V(\sigma)} (-1)^{\kappa(x, \sigma) + \kappa(w, \tau)} \tau_w \rightarrow_v \sigma_x \\
+ \sum_{y \in V(\tau)} (-1)^{\kappa(v, \sigma) + \kappa(y, \tau)} \sigma_v \rightarrow_w \tau_y.
\]
The distance \(d(x, v)\) between \(x\) and \(v\) in \(\sigma\) is defined as the length of the (unique) path joining \(x\) and \(v\) in \(\sigma\). It is clearly equal modulo 2 to the sum \(\kappa(x, \sigma) + \kappa(v, \sigma)\). Similarly, \(d(y, w) = \kappa(y, \tau) + \kappa(w, \tau)\) modulo 2. Hence, using (12) and (13) we get:
\[
T_{v, w} = (-1)^{\kappa(v, \sigma) + \kappa(w, \tau)} \left( - \sum_{x \in V(\sigma)} (-1)^{d(x, v)} (\sigma_{v \rightarrow w} \tau)_x + \sum_{y \in V(\tau)} (-1)^{d(y, w)} (\sigma_{v \rightarrow w} \tau)_y \right).
\]
Now the distance \(d(x, v)\) is the same if we compute it in \(\sigma\) or in \(\sigma_{v \rightarrow w} \tau\), and similarly for \(d(y, w)\). Finally, using the fact that \(v\) and \(w\) are neighbours in \(\sigma_{v \rightarrow w} \tau\), we have \(d(x, w) = d(x, v) + 1\) for any \(x \in V(\sigma)\), the distance being computed in \(\sigma_{v \rightarrow w} \tau\). This finally gives:
\[
T_{v, w} = (-1)^{\kappa(v, \sigma) + \kappa(w, \tau)} \sum_{z \in V(\sigma_{v \rightarrow w} \tau)} (-1)^{d(z, w)} (\sigma_{v \rightarrow w} \tau)_z,
\]
which vanishes since \(\sigma_{v \rightarrow w} \tau\) is superfluous.
(3) Finally, if \( \sigma, \tau \) and \( \sigma v \ldots w \tau \) are not superfluous, using \([12]\) and \([13]\), both sides will be equal if we have:

\[
\begin{align*}
\kappa(x, \sigma v \ldots w \tau) + \kappa(w, \sigma v \ldots w \tau) + \kappa(v, \sigma) &= \kappa(x, \sigma) + 1 \mod 2, \\
\kappa(y, \sigma v \ldots w \tau) + \kappa(w, \sigma v \ldots w \tau) + \kappa(w, \tau) &= \kappa(y, \tau) \mod 2.
\end{align*}
\]

Using the fact that \( v \) and \( w \) are neighbours, it rewrites as:

\[
\begin{align*}
\kappa(x, \sigma v \ldots w \tau) + \kappa(x, \sigma) &= \kappa(v, \sigma v \ldots w \tau) + \kappa(v, \sigma) \mod 2, \\
\kappa(y, \sigma v \ldots w \tau) + \kappa(y, \tau) &= \kappa(w, \sigma v \ldots w \tau) + \kappa(w, \tau) \mod 2.
\end{align*}
\]

These two last identities are always verified: looking for example at the right-hand side of the first one, moving vertex \( v \) to a neighbour will change both \( \kappa \)'s by \( \pm 1 \). It remains then to jump from neighbour to neighbour up to \( x \). The proof of the second identity is completely similar.

Using the identification of \( \mathcal{FT}/S \) with \( \mathcal{FT}' \), a straightforward consequence of Theorem 4 is the following:

**Corollary 5.** The linear map \( \bar{X} \) is an injection of \( \mathcal{FT}' \) into \( \mathcal{T} \), and the product \( \circ : \mathcal{FT}' \times \mathcal{FT}' \to \mathcal{FT}' \) verifies:

\[ \bar{X}(\sigma \circ \tau) = [\bar{X}(\sigma), \bar{X}(\tau)]. \]

As a consequence, the product \( \circ \) satisfies the Jacobi identity, and \( \bar{X} \) is an embedding of Lie algebras.

### 4. Application to elementary hamiltonians

Keeping the previous notations, the vector field \( \mathcal{F}_{\underline{\mathcal{A}}} (\bar{X}(\tau)) \) is hamiltonian for any (decorated) non-rooted tree \( \tau \). Hence it can be uniquely written as \( \{H_{\underline{\mathcal{A}}} (\tau), -\} \) for some \( H_{\underline{\mathcal{A}}} (\tau) \in C^\infty(\mathbb{R}^2r) \), called the elementary hamiltonian associated with \( \tau \).

**Proposition 6.** For any free trees \( \sigma, \tau \) we have:

\[ \{H_{\underline{\mathcal{A}}} (\sigma), H_{\underline{\mathcal{A}}} (\tau)\} = H_{\underline{\mathcal{A}}} (\sigma \circ \tau). \]

**Proof.** We compute:

\[
\begin{align*}
\{H_{\underline{\mathcal{A}}} (\sigma), H_{\underline{\mathcal{A}}} (\tau)\} &= \{H_{\underline{\mathcal{A}}} (\sigma), -\}, \{H_{\underline{\mathcal{A}}} (\tau), -\}\} \\
&= \mathcal{F}_{\underline{\mathcal{A}}} (\bar{X}(\sigma), \bar{X}(\tau)) \\
&= \mathcal{F}_{\underline{\mathcal{A}}} (\bar{X}(\sigma \circ \tau)) \\
&= \mathcal{F}_{\underline{\mathcal{A}}} \circ \bar{X}(\sigma \circ \tau) \\
&= \{H_{\underline{\mathcal{A}}} (\sigma \circ \tau), -\}.
\end{align*}
\]

One concludes by using the uniqueness of the hamiltonian representation of a hamiltonian vector field.

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