STATE MORPHISM MV-ALGEBRAS

ANATOLIJ DVUREČENSKIJ1, TOMASZ KOWALSKI2 AND FRANCO MONTAGNA3

1 Mathematical Institute, Slovak Academy of Sciences
Štefánikova 49, SK-814 73 Bratislava, Slovakia
2 Department of Mathematics and Statistics, University of Melbourne
Parkville, VIC 3010, Australia
3 Università Degli Studi di Siena Dipartimento di Scienze Matematiche e Informatiche
“Roberto Magari”
Pian dei Mantellini 44, I-53100 Siena, Italy
E-mail: dvurecen@mat.savba.sk, kovatomasz@gmail.com
montagna@unisi.it

ABSTRACT. We present a complete characterization of subdirectly irreducible
MV-algebras with internal states (SMV-algebras). This allows us to classify
subdirectly irreducible state morphism MV-algebras (SMMV-algebras) and
describe single generators of the variety of SMMV-algebras, and show that we
have a continuum of varieties of SMMV-algebras.

1. Introduction

States on MV-algebras have been introduced by Mundici in [Mus]. A state on
an MV-algebra A is a map s from A into [0, 1] such that:
(a) s(1) = 1, and
(b) if x ⊙ y = 0, then s(x ⊕ y) = s(x) + s(y).

Special states are the so called [0, 1]-valuations on A, that is, the homomorphisms
from A into the standard MV-algebra [0, 1]MV on [0, 1].

States are related to [0, 1]-valuations by two important results. First of all,
[0, 1]-valuations are precisely the extremal states, that is, those states that cannot
be expressed as non-trivial convex combinations of other states. Moreover, by the
Krein-Milman Theorem, every state belongs to the convex closure of the set of all
[0, 1]-valuations with respect to the topology of uniform convergence. Finally, every
state coincides locally with a convex combination of [0, 1]-valuations (see [Mus],
[KM]). More precisely, given a state s on an MV-algebra A and given elements
a1, . . . , an of A, there are n + 1 extremal states s1, . . . , sn+1 and n + 1 elements
λ1, . . . , λn+1 of [0, 1] such that \(\sum_{h=1}^{n+1} \lambda_h = 1\) and for \(j = 1, \ldots, n, \sum_{i=1}^{n+1} \lambda_i s_i(a_j) = s(a_j)\).

Another important relation between states and [0, 1]-valuations is the following:
let XA be the set of [0, 1]-valuations on A. Then XA becomes a compact Hausdorff

1Keywords: MV-algebra, state MV-algebra, state morphism MV-algebra, varieties, subdirectly
irreducible algebra, cover variety.

AMS classification (2010): 06D35, 03B50.

AD thanks for the support by Center of Excellence SAS - Quantum Technologies -, ERDF
OP R&D Projects CE QUTE ITMS 26240120009 and meta-QUTE ITMS 26240120022, the grant
VEGA No. 2/0032/09 SAV, and by Slovak-Italian project SK-IT 0016-08.
subspace of $[0,1]^A$ equipped with the Tychonoff topology. To every element $a$ of $A$ we can associate its Gelfand transform $\hat{a}$ from $X_A$ into $[0,1]$, defined for all $v \in X_A$, by $\hat{a}(v) = v(a)$. Now Panti [P3] and Kroupa [K2] independently showed that to any state $s$ on $A$ it is possible to associate a (uniquely determined) Borel regular probability measure $\mu$ on $X_A$ such that for all $a \in A$ one has $s(a) = \int \hat{a} \ad \mu$. Hence, every state has an integral representation.

Yet another important result motivating the use of states, related to de Finetti’s interpretation of probability in terms of bets, is Mundici’s characterization of coherence [Muh]. That is, given an MV-algebra $A$, given $a_1, \ldots, a_n \in A$ and $\alpha_1, \ldots, \alpha_n \in [0,1]$, the following are equivalent:

1. There is a state $s$ on $A$ such that, for $i = 1, \ldots, n$, $s(a_i) = \alpha_i$.
2. For every choice of real numbers $\lambda_1, \ldots, \lambda_n$ there is a $[0,1]$-valuation $v$ such that $\sum_{i=1}^n \lambda_i (\alpha_i - v(a_i)) \geq 0$.

These results show that the notion of state on an MV-algebra is a very important notion and the first one shows an important connection between states and $[0,1]$-valuations. However, MV-algebras with a state are not universal algebras, and hence they do not provide for an algebraizalgean logic for reasoning on probability over many-valued events.

In [FM] the authors find an algebraizable logic for this purpose, whose equivalent algebraic semantics is the variety of SMV-algebras. An SMV-algebra (see the next section for a precise definition) is an MV-algebra $A$ equipped with an operator $\tau$ whose properties resemble the properties of a state, but, unlike a state, is an internal unary operation (called also an internal state) on $A$ and not a map from $A$ into $[0,1]$. The analogue for SMV-algebras of an extremal state (or equivalently of a $[0,1]$-valuation) is the concept of state morphism. By this terminology we mean an idempotent endomorphism from $A$ into $A$. MV-algebras equipped with a state morphism form a variety, namely, the variety of SMMV-algebras, which is a subvariety of the variety of SMV-algebras. Here below we mention some motivations for the study of SMMV-algebras:

1. Let $(A, \tau)$ be an SMV-algebra, and assume that $\tau(A)$, the image of $A$ under $\tau$, is simple. Then $\tau(A)$ is isomorphic to a subalgebra of $[0,1]_{SMV}$, and $\tau$ may be regarded as a state on $A$. Moreover, by Di Nola’s theorem [DN], $A$ is isomorphic to a subalgebra of $[0,1]^{*\tau}$ for some ultrapower $[0,1]^\ast$ of $[0,1]_{SMV}$ and for some index set $I$. Finally, using a result by Kroupa [K1] stating that any state on a subalgebra $A$ of an MV-algebra $B$ can be extended to a state on $B$, we obtain that $\tau$ can be extended to a state $\tau^\ast$ on $[0,1]^{*\tau}$. Note that, after identifying a real number $\alpha \in [0,1]$ with the function on $I$ which is constantly equal to $\alpha$, $\tau^\ast$ is also an internal state, and it makes $[0,1]^{*\tau}$ into an SMV-algebra. Moreover, by the Krein-Milman theorem, for every real number $\varepsilon > 0$ there is a convex combination $\sum_{i=1}^n \lambda_i v_i$ of $[0,1]$-valuations $v_1, \ldots, v_n$ such that for every $a \in A$, $|\tau(a) - \sum_{i=1}^n \lambda_i v_i(a)| < \varepsilon$. After identifying $v_i(a)$ with the function from $I$ into $[0,1]^\ast$ which is constantly equal to $v_i(a)$, these valuations can be regarded as idempotent endomorphisms on $[0,1]^{*\tau}$, and hence each of them makes $[0,1]^{*\tau}$ into an SMMV-algebra. Summing up, if $(A, \tau)$ is an SMV-algebra and $\tau(A)$ is simple, then $\tau$ can be approximated by convex combinations of state morphisms on (an extension of) $A$.

2. All subdirectly irreducible SMMV-algebras were described in [DiDv] [DDL2], but the description of all subdirectly irreducible SMV-algebras remains open, [FM].
As shown in [DDL1], if \((A, \tau)\) is an SMV-algebra and \(\tau(A)\) belongs to a finitely generated variety of MV-algebras, then \((A, \tau)\) is an SMMV-algebra. In particular, MV-algebras from a finitely generated variety only admit internal states which are state morphisms.

(4) A linearly ordered SMV-algebra is an SMMV-algebra, [DDL1]. Moreover, we will see that representable SMV-algebras form a variety which is a subvariety of the variety of SMMV-algebras.

The goal of the present paper is to continue in the algebraic investigations on SMMV-algebras which begun in [DDL1] and in [DiDv, DDL2].

The paper is organized as follows. After preliminaries in Section 2, we give in Section 3 a complete characterization of subdirectly irreducible SMV-algebras. This solves an open problem posed in [FM]. In Section 4 we present a classification of subdirectly irreducible SMMV-algebras introducing four types of subdirectly irreducible SMMV-algebras. In Section 5, we describe some prominent varieties of SMMV-algebras and their generators. In particular, we answer in positive to an open question from [DiDv] that the diagonalization of the real interval \([0, 1]\) generates the variety of SMMV-algebras. Section 6 shows that every subdirectly irreducible SMMV-algebra is subdiagonal. Finally, Section 7 describes an axiomatization of some varieties of SMMV-algebras, including a full characterization of representable SMMV-algebras. We show that in contrast of MV-algebras, there is a continuum of varieties of SMMV-algebras. In addition, some open problems are formulated.

2. Preliminaries

For all concepts of Universal Algebra we refer to [BS]. For concepts of many-valued logic, we refer to [Ha], and for MV-algebras in particular, we will also refer to [CDM].

**Definition 2.1.** An MV-algebra is an algebra \(A = (A, \oplus, \neg, 0)\), where \((A, \oplus, 0)\) is a commutative monoid, \(\neg\) is an involutive unary operation on \(A\), \(1 = \neg 0\) is an absorbing element, that is, \(x \oplus 1 = 1\), and letting \(x \rightarrow y = (\neg x) \oplus y\), the identity \((x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x\) holds.

In any MV-algebra \(A\), we further define \(x \odot y = (\neg x \oplus \neg y), x \odot y = (\neg x \oplus y), x \vee y = (x \rightarrow y) \rightarrow y\) and \(x \wedge y = x \odot (x \rightarrow y)\). With respect to \(\vee\) and \(\wedge\), \(A\) becomes a distributive lattice with top element 1 and bottom element 0.

We also define \(nx\) for \(x \in A\) and natural number \(n\) by induction as follows: \(0x = 0; (n+1)x = nx \oplus x\).

MV-algebras constitute the equivalent algebraic semantics of Lukasiewicz logic \(L\), cf. [Ha] for an axiomatization.

The standard MV-algebra is the MV-algebra \([0, 1]_{MV} = ([0, 1], \oplus, \neg, 0)\), where \(r \oplus s = \min\{r + s, 1\}\), \(-r = 1 - r\).

For the derived operations one has:

\[
\begin{align*}
    r \circ s &= \max\{r - s, 0\}, & r \odot s &= \max\{r + s - 1, 0\}, & r \rightarrow s &= \min\{1 - r + s, 1\}, \\
    r \vee s &= \max\{r, s\}, & r \wedge s &= \min\{r, s\}.
\end{align*}
\]

The variety of all MV-algebras is generated as a quasivariety by \([0, 1]_{MV}\). It follows that in order to check the validity of an equation or a quasi equation in all MV-algebras, it is sufficient to check it in \([0, 1]_{MV}\). We will tacitly use this fact in the sequel.
Definition 2.2. A filter of an MV-algebra $A$ is a subset $F$ of $A$ such that $1 \in F$ and if $a$ and $a \rightarrow b$ are in $F$, then $b \in F$.

Dually, an ideal of $A$ is a subset $J$ of $A$ such that $0 \in J$ and if $a$ and $b \oplus a$ are in $J$, then $b \in J$. A filter $F$ (an ideal $J$ respectively) of $A$ is called proper if $0 \notin F$ ($1 \notin J$ respectively) and maximal if it is proper and it is not properly contained in any proper filter (ideal respectively). The radical, $\text{Rad}(A)$, of $A$, is the intersection of all its maximal ideals, and the co-radical, $\text{Rad}_1(A)$, of $A$ is the intersection of all its maximal filters. An MV-algebra $A$ is called semisimple if $\text{Rad}(A) = \{0\}$, and is called local if it has exactly one maximal ideal.

It is well-known (and easy to prove) that an MV-algebra $A$ is semisimple iff $\text{Rad}_1(A) = \{1\}$, and it is local iff it has exactly one maximal filter.

Both the lattice of ideals and the lattice of filters of an MV-algebra $A$ are isomorphic to its congruence lattice via the isomorphisms $\theta \mapsto \{a \in A : (a,0) \in \theta\}$ and $\theta \mapsto \{a \in A : (a,1) \in \theta\}$, respectively. The inverses of these isomorphisms are:

$J \mapsto \{(a,b) \in A^2 : \neg(a \oplus b) \in J\}$ and $F \mapsto \{(a,b) \in A^2 : a \leftrightarrow b \in F\}$, respectively.

It follows that an MV-algebra is semisimple iff it has a subdirect embedding into a product of simple MV-algebras.

Definition 2.3. A Wajsberg hoop is a subreduct (subalgebra of a reduct) of an MV-algebra in the language $\{1, \ominus, \rightarrow\}$.

Definition 2.4. A lattice ordered abelian group is an algebra $G = (G, +, \cdot, 0, \vee, \wedge)$ such that $(G, +, 0)$ is an abelian group, $(G, \vee, \wedge)$ is a lattice, and for all $x, y, z \in G$, one has $x + (y \vee z) = (x + y) \vee (x + z)$.

A strong unit of a lattice ordered abelian group $G$ is an element $u \in G$ such that for all $g \in G$ there is $n \in \mathbb{N}$ such that $g \leq u + \cdots + u$.

If $G$ is a lattice-ordered abelian group and $u$ is a strong unit of $G$, then $\Gamma(G, u)$ denotes the algebra $A$ whose universe is $\{x \in G : 0 \leq x \leq u\}$, equipped with the constant 0 and with the operations $\oplus$ and $\neg$ defined by $x \oplus y = (x + y) \wedge u$ and $\neg x = u - x$. It is well-known (Musk06) that $\Gamma(G, u)$ is an MV-algebra, and every MV-algebra can be represented as $\Gamma(G, u)$ for some lattice ordered abelian group $G$ with strong unit $u$.

In the sequel, $\mathbb{Z} \times_{\text{lex}} \mathbb{Z}$ denotes the direct product of two copies of the group $\mathbb{Z}$ of integers, ordered lexicographically, i.e., $(a,b) \leq (c,d)$ if either $a < c$ or $a = c$ and $b \leq d$. For every positive natural number $n$, $S_n$ and $C_n$ denote $\Gamma(\mathbb{Z}, n)$ and $\Gamma(\mathbb{Z} \times_{\text{lex}} \mathbb{Z}, (n,0))$ respectively. The algebra $C_1$, that is $\Gamma(\mathbb{Z} \times_{\text{lex}} \mathbb{Z}, (1,0))$, is also referred to as Chang’s algebra.

Definition 2.5. A state on an MV-algebra $A$ (cf. Musk) is a map $s$ from $A$ into $[0,1]$ satisfying:

1. $s(1) = 1$.
2. $s(x \oplus y) = s(x) + s(y)$ for all $x, y \in A$ such that $x \ominus y = 0$.

Definition 2.6. An MV-algebra with an internal state (SMV-algebra in the sequel) is an algebra $(A, \tau)$ such that:

(a) $A$ is an MV-algebra.
(b) $\tau$ is a unary operation on $A$ satisfying the following equations:
Definition 3.1. A state morphism MV-algebra (SMMV-algebra for short) is an SMV-algebra further satisfying:

(c) \( \tau(x \oplus y) = \tau(x) \oplus \tau(y) \).

The following facts are easily provable:

Lemma 2.7. (see [FM, DDL1].) (1) In an SMV-algebra \((A, \tau)\), the following conditions hold:

(a) \( \tau(0) = 0 \).
(b) If \( x \circ y = 0 \), then \( \tau(x) \circ \tau(y) = 0 \) and \( \tau(x \oplus y) = \tau(x) \oplus \tau(y) \).
(c) \( \tau(\tau(x)) = \tau(x) \).
(d) \( \tau(A) \), the image of \( A \) under \( \tau \), is an MV-algebra, and \( \tau \) is the identity on it.

(2) The following conditions on SMMV-algebras hold:

(a) In an SMMV-algebra \((A, \tau)\), \( \tau(A) \) is a retract of \( A \), that is, \( \tau \) is a homomorphism from \( A \) onto \( \tau(A) \), the identity map is an embedding from \( \tau(A) \) into \( A \), and the composition \( \tau \circ \text{Id}_{\tau(A)} \), that is, the restriction of \( \tau \) to \( \tau(A) \) is the identity on \( \tau(A) \).
(b) An algebra \((A, \tau)\) is an SMMV-algebra iff \( A \) is an MV-algebra and \( \tau \) is an idempotent endomorphism on \( A \).
(c) An SMV-algebra \((A, \tau)\) is an SMMV-algebra iff it satisfies \( \tau(x \lor y) = \tau(x) \lor \tau(y) \) and satisfies \( \tau(x \land y) = \tau(x) \land \tau(y) \).
(d) Any linearly ordered SMV-algebra is an SMMV-algebra.

3. Subdirectly irreducible SMV-algebras

In this section we characterize and classify subdirectly irreducible SMV-algebras which answers to an open problem posed in [FM]. Our result also characterizes subdirectly irreducible SMV-algebras.

Definition 3.1. Let \((A, \tau)\) be any SMV-algebra. Any filter \( F \) of \( A \) such that \( \tau(F) \subseteq F \) is said to be a \( \tau \)-filter.

Let \( \tau(A) = \{ \tau(a) : a \in A \} \) and \( F_\tau(A) = \{ a \in A : \tau(a) = 1 \} \). Clearly, \( \tau(A) = (\tau(A), \oplus, \neg, 0) \) is a subalgebra of \( A \) and \( F_\tau(A) \) is a \( \tau \)-filter of \( A \), and hence \( F_\tau(A) = (F_\tau(A), \rightarrow, 0, 1) \) is a Wajsberg subhoop of \( A \). Say that two Wajsberg subhoops, \( B \) and \( C \), of an MV-algebra \( A \) have the disjunction property if for all \( x \in B \) and \( y \in C \), if \( x \lor y = 1 \), then either \( x = 1 \) or \( y = 1 \).

We recall that \( \tau \)-filters are in a bijection with SMV-congruences, and hence an SMV-algebra is subdirectly irreducible iff it has a minimum \( \tau \)-filter.

Lemma 3.2. Suppose that \((A, \tau)\) is a subdirectly irreducible SMV-algebra. Then:

1. If \( F_\tau(A) = \{ 1 \} \), then \( \tau(A) \) is subdirectly irreducible.
2. \( F_\tau(A) \) is (either trivial or) a subdirectly irreducible hoop.
3. \( F_\tau(A) \) and \( \tau(A) \) have the disjunction property.
Proof. Let $F$ denote the minimum filter of $A$. (1) Suppose $F(x) = \{1\}$. If $\tau(A) \cap F \neq \{1\}$, then $\tau(A) \cap F$ is the minimum non trivial filter of $\tau(A)$ and $\tau(A)$ is subdirectly irreducible. If $\tau(A) \cap F = \{1\}$, then for all $x \in F$, $\tau(x) = 1$ (because $\tau(x) \in \tau(A) \cap F$ and $F \subseteq F(A)$) is the trivial filter, a contradiction.

(2) Suppose that $F(A)$ is nontrivial. Then $F(A)$ is a non trivial $\tau$-filter. If $(A, \tau)$ is subdirectly irreducible, it has a minimum non trivial $\tau$-filter, $F$ say. So, $F \subseteq F(A)$, and hence $F$ is the minimum non trivial filter of $F(A)$. Hence, $F(A)$ is subdirectly irreducible.

(3) Suppose, by way of contradiction, that for some $x \in F(A)$ and $y = \tau(y) \in \tau(A)$ one has $x < 1$, $y < 1$ and $x \lor y = 1$. Then since the $\tau$-filters generated by $x$ and by $y$, respectively, are $\tau$-filters (easy to verify), they both contain $F$. Hence, the intersection of these filters contains $F$. Now let $c < 1$ be in $F$. Then there is a natural number $n$ such that $x^n \leq c$ and $y^n \leq c$. It follows that $1 = (x \lor y)^n = x^n \lor y^n \leq c$, a contradiction. □

Corollary 3.3. If $(A, \tau)$ is subdirectly irreducible, then $\tau(A)$ and $F(A)$ are linearly ordered.

Proof. That $\tau(A)$ is linearly ordered follows from [FM]. As regards to $F(A)$, by Lemma 3.2, $F(A)$ is a (possibly trivial) subdirectly irreducible Wajsberg hoop, and hence it is linearly ordered. □

Theorem 3.4. Suppose that $(A, \tau)$ is an SMV-algebra satisfying conditions (1), (2) and (3) in Lemma 3.2. Then $(A, \tau)$ is subdirectly irreducible, and hence, the above conditions constitute a characterization of subdirectly irreducible SMV-algebras.

Proof. Suppose first that $F(A) = \{1\}$ and that $\tau(A)$ is subdirectly irreducible. Let $F_0$ be the minimum nontrivial filter of $\tau(A)$ and let $F$ be the MV-filter of $A$ generated by $F_0$. Then $F$ is a $\tau$-filter. Indeed, if $x \in F$, then there is $\tau(a) \in F_0$ and a natural number $n$ such that $\tau(a)^n \leq x$. It follows that $\tau(x) \geq \tau(\tau(a)^n) = \tau(a)^n$, and $\tau(x) \in F$.

We claim that $F$ is the minimum non trivial $\tau$-filter of $(A, \tau)$. First of all, $\tau(A)$, being a subdirectly irreducible MV-algebra, is linearly ordered. Now in order to prove that $F$ is the minimum non trivial $\tau$-filter of $(A, \tau)$, it suffices to prove that every $\tau$-filter $G$ not containing $F$ is trivial. Now let $c < 1$ in $F \setminus G$. Then since $F(A) = \{1\}$, $\tau(c) < 1$. Next, let $d \in G$. Then $\tau(d) \in G$, and for every $n$ it cannot be $\tau(d)^n \leq \tau(c)$, otherwise $\tau(c) \in G$. Hence, for every $n$, $\tau(c) < \tau(d)^n$, and hence $\tau(c)$ does not belong to the $\tau$-filter of $\tau(A)$ generated by $\tau(d)$. By the minimality of $F$ in $\tau(A)$, $\tau(d) = 1$ and since $F(A) = \{1\}$, we conclude that $d = 1$ and $G$ is trivial, as desired.

Now suppose that $F(A)$ is nontrivial. By condition (2), $F(A)$ is subdirectly irreducible. Thus, let $F$ be the minimum filter of $F(A)$. Then $F$ is a non trivial $\tau$-filter, and it is left to prove that $F$ is the minimum non trivial $\tau$-filter of $(A, \tau)$. Let $G$ be any non trivial $\tau$-filter of $(A, \tau)$. If $G \subseteq F(A)$, then it contains the minimal filter, $F$, of $F(A)$, and $F \subseteq G$. Otherwise, $G$ contains some $x \notin F(A)$, and hence it contains $\tau(x) < 1$. Now by the disjunction property, for all $y < 1$ in $F(A)$, $\tau(x) \lor y < 1$ and $\tau(x) \lor y \in F(A) \cap G$. Thus, $G$ contains the filter generated by $\tau(x) \lor y$, which is a non trivial filter of $F(A)$, and hence it contains $F$, the minimum non trivial filter of $F(A)$. This settles the claim. □
Theorem 3.5. (1), (2) and (3) are independent conditions, and hence none of them is redundant in Theorem [3.4].

Proof. (1) Let $C_1$ be Chang’s MV-algebra, let $\tau_1$ be the identity on $C_1$ and $\tau_2$ be the function defined by $\tau_2(x) = 0$ if $x$ is an infinitesimal and $\tau_2(x) = 1$ otherwise. Clearly, both $(C_1, \tau_1)$ and $(C_1, \tau_2)$ are SMV-algebras, and so is their direct product $(B, \tau) = (C_1, \tau_1) \times (C_1, \tau_2)$. Let $(D, \tau)$ be the subalgebra of $(B, \tau)$ generating by all pairs $(x, y)$ such that $x$ is infinitesimal and all pairs $(y, 1)$ such that $y$ is not infinitesimal, and hence it is subdirectly irreducible (the minimum filter is the set of all $(y, 1)$ such that $y$ is not infinitesimal. Moreover, $F_\tau(D)$ consists of all elements of the form $(1, y)$ such that $y$ is not infinitesimal, and hence it is subdirectly irreducible, by the same argument. Clearly (3) does not hold (e.g., if $x$ is not infinitesimal and $x < 1$, then $(1, x) \in F_\tau(D)$, $(x, 1) \in \tau(D)$, and $(1, x) \vee (x, 1) = (1, 1)$, but $(x, 1) < (1, 1)$ and $(1, x) < (1, 1)$).

(2) Let $A$ be an ultrapower of $[0, 1]_{MV}$, and let $B$ be the subalgebra of $A$ generated by all the infinitesimals. Let $\tau$ be defined by $\tau(x) = 0$ if $x$ is an infinitesimal and $\tau(x) = 1$ otherwise. Then $\tau(A)$ is subdirectly irreducible, being the MV-algebra with two elements, and the disjunction property holds because $B$ is linearly ordered, but $F_\tau(B)$ consists of all infinitesimals and hence it is not subdirectly irreducible. (If $F$ is any nontrivial $\tau$-filter and $1 - \epsilon \in F$, with $\epsilon$ a positive infinitesimal, then the filter generated by $1 - \epsilon^2$ is a non trivial $\tau$-filter strictly contained in $F$).

(3) Let $B$ be as in (2) and let $\tau$ be the identity on $B$. Then $F_\tau(B)$ is subdirectly irreducible, being a trivial algebra, and the disjunction property holds because $B$ is linearly ordered, but $\tau(B) = B$ is not subdirectly irreducible. \qed

Subdirectly irreducible SMMV-algebras also enjoy another interesting property, namely:

Theorem 3.6. Let $(A, \tau)$ be a subdirectly irreducible SMMV-algebra, and let $a \in A$. Then there are uniquely determined $b \in \tau(A)$ and $c \in F_\tau(A)$ such that exactly one of the following two conditions holds:

(a) $a = b \circ c$, and $c$ is the greatest element with this property, or
(b) $a = c \rightarrow b$, and $b < c < 1$.

Proof. First of all, note that $\tau(a \rightarrow \tau(a)) = \tau(\tau(a) \rightarrow a) = \tau(a) \rightarrow \tau(a) = 1$, and hence, for every $a \in A$, $a \rightarrow \tau(a)$ and $\tau(a) \rightarrow a$ belong to $F_\tau(A)$. We now prove:

Lemma 3.7. If $(A, \tau)$ is a subdirectly irreducible SMMV-algebra, then for all $a \in A$, either $a \leq \tau(a)$ or $\tau(a) \leq a$.

Proof. Since $(A, \tau)$ is subdirectly irreducible, $F_\tau(A)$ is subdirectly irreducible and hence it is linearly ordered. Hence, 1 is join irreducible in $F_\tau(A)$. Now $(a \rightarrow \tau(a)) \lor (\tau(a) \rightarrow a) = 1$, and hence either $a \rightarrow \tau(a) = 1$ and $a \leq \tau(a)$, or $\tau(a) \rightarrow a = 1$ and $\tau(a) \leq a$. \qed

Continuing with the proof of Theorem [3.6] let $b = \tau(a)$ and let $c = b \rightarrow a$ if $a \leq b$, and $c = a \rightarrow b$ otherwise.

Suppose $a \leq b$. Then $a = a \land b = b \circ (b \rightarrow a) = b \circ c$. Finally, $c$ is the greatest element such that $b \circ c = a$, by the definition of residuum.
Now suppose \( b < a \). Then \( c \to b = (a \to b) \to b = a \lor b = a \). Moreover, \( c < 1 \), as \( b < a \). Finally, \( b < c \). Indeed, \( b \leq a \to b = c \), and it cannot be \( c = b \), as \( \tau(c) = 1 \) and \( \tau(b) = b < a \).

We now discuss uniqueness. If \( a = b \circ c \), with \( b \in \tau(A) \) and \( c \in F_r(A) \), then \( \tau(a) = \tau(b) \circ \tau(c) = b \circ 1 = b \). Thus \( b = \tau(a) \) is uniquely determined. Moreover, \( a \leq b \), \( b \circ c = a \) and \( c \) is the greatest element with this property. Hence, \( c = a \to b \).

If \( a = c \to b \) with \( c < 1 \), then \( b < a \). Moreover, \( \tau(a) = \tau(c) \to \tau(b) = 1 \to b = b \), and \( b \) is uniquely determined. Finally, in any MV-algebra, if \( z \leq x \), \( z \leq y \) and \( x \to z = y \to z \), then \( x = y \) (this property is expressed as a quasi equation and holds in \([0, 1]_{MV}\), and hence it holds in any MV-algebra). Now \( b < c < 1 \), \( b \leq (a \to b) \to b \), and \( c \to b = (a \to b) \to b \). It follows that \( c = a \to b \), and uniqueness of \( c \) is proved.

For all \( b \in \tau(A) \), the set \( M(b) = \{ x \in A : \exists c, d \in F_r(A), c \circ b \leq x \leq d \to b \} \) is called the monad of \( b \). Now by Theorem 3.6 if \((A, \tau)\) is subdirectly irreducible, then for all \( b \in \tau(A) \), \( M(b) \) is linearly ordered and \( \tau(A) \) is linearly ordered. Thus, although \( A \) need not be linearly ordered, it is close to be such. More precisely, let \( M = \{ \pm c : c \in F_r(A), c < 1 \} \). We define a poset \( M \) on \( M \) letting \(-c < -d\) iff \( d < c \), and \( c \leq 1 \leq -d \) for all \( c, d \in F_r(A) \setminus \{ 1 \} \). Then after identifying \( c \circ b \) with \( (b, c) \) and \( c \to b \) with \( (b, -c) \), we have that \( M \) is a subposet of \( M \times \{ \pm 1 \} \), and \( A \) may be identified with a subset of \( \tau(A) \times M \). Moreover, if \( b \leq b' \) in \( \tau(A) \) and \( \pm c \leq \pm c' \) in \( M \), then \( b, c \leq b', c' \) in \( A \). Hence, the order on \( A \) is an extension of a subposet of the product order on \( \tau(A) \times M \), that is, \( A \) as a poset is isomorphic to a quotient of a subposet of the product of two chains. This suggests that either \( A \) is a chain or a subalgebra of a product of two chains. This conjecture will be proved in Section 6. More precisely:

**Definition 3.8.** An SMMV-algebra \((A, \tau)\) is said to be diagonal if there are MV-chains \( B \) and \( C \) such that \( B \subseteq C \), \( A = B \times C \) and \( \tau \) is defined, for all \( b \in B \) and \( c \in C \), by \( \tau(b, c) = (b, b) \).

An SMMV-algebra is said to be subdiagonal if it is a subalgebra of a diagonal SMMV-algebra.

In Section 6 we will prove:

**Theorem 3.9.** Every subdirectly irreducible SMMV-algebra is subdiagonal.

4. A classification of subdirectly irreducible SMMV-algebras

We present a classification of SMMV-algebras introducing four types of subdirectly irreducible SMMV-algebras, type \( \mathcal{I} \), identity, type \( \mathcal{L} \), local, type \( \mathcal{D} \), diagonalization, and type \( \mathcal{K} \), killing infinitesimals.

The following theorem was proved in [DiDv, DDL2, Dvu].

**Theorem 4.1.** Let \((A, \tau)\) be a subdirectly irreducible SMMV-algebra. Then \((A, \tau)\) belongs to exactly one of the following classes:

(i) \( A \) is linearly ordered, \( \tau \) is the identity on \( A \) and the MV-reduct of \( A \) is a subdirectly irreducible MV-algebra.

(ii) The state morphism operator \( \tau \) is not faithful, \( A \) has no nontrivial Boolean elements and is a local MV-algebra. Moreover, \( A \) is linearly ordered if and only if \( \text{Rad}_1(A) \) is linearly ordered, and in such a case, \( A \) is a subdirectly
irreducible MV-algebra such that the smallest nontrivial \( \tau \)-filter of \((A, \tau)\), and the smallest nontrivial MV-filter for \(A\) coincide.

(iii) The state morphism operator \(\tau\) is not faithful, \(A\) has a nontrivial Boolean element. There are a linearly ordered MV-algebra \(B\), a subdirectly irreducible MV-algebra \(C\), and an injective MV-homomorphism \(h : B \to C\) such that \((A, \tau)\) is isomorphic to \((B \times C, \tau_h)\), where \(\tau_h(x, y) = (x, h(x))\) for any \((x, y) \in B \times C\).

Note that while every SMMV-algebra satisfying (i) or (iii) is subdirectly irreducible, the same is not true of SMMV-algebras satisfying (ii). A full classification of subdirectly irreducible SMMV-algebras is obtained by combining Theorem 4.1, Theorem 4.3, and Theorem 3.4.

Let us consider the following classes of SMMV-algebras:

**Definition 4.2.** Type \(\mathcal{I}\) (identity). The MV-reduct, \(A\), of \((A, \tau)\) is a subdirectly irreducible MV-algebra and \(\tau\) is the identity function on \(A\).

**Type \(\mathcal{L}\) (local).** \((A, \tau)\) is subdiagonal, the MV-reduct, \(A\), of \((A, \tau)\) is a local MV-algebra (hence it has no Boolean nontrivial elements), \(F_\tau(A)\) is a nontrivial subdirectly irreducible hoop, \(F_\tau(A)\) and \(\tau(A)\) have the disjunction property.

**Type \(\mathcal{D}\) (diagonalization).** The MV-reduct, \(A\), of \((A, \tau)\) is of the form \(B \times C\), where \(C\) is a subdirectly irreducible MV-algebra and \(B\) is a subalgebra of \(C\). Moreover, \(\tau\) is defined by \(\tau(b, c) = (b, b)\).

**Theorem 4.3.** An SMMV-algebra is subdirectly irreducible if and only if it is of one of the types \(\mathcal{I}\), \(\mathcal{L}\) and \(\mathcal{D}\). Moreover, these types are mutually disjoint.

**Proof.** We first prove, using Theorem 4.1, that all members of \(\mathcal{I} \cup \mathcal{L} \cup \mathcal{D}\) are subdirectly irreducible. For type \(\mathcal{I}\), the claim is easy and for type \(\mathcal{L}\) the claim follows from the definition of type \(\mathcal{L}\) and from Theorem 3.4. For type \(\mathcal{D}\), if \((A, \tau)\) is diagonal, say, \(A = B \times C\) with \(B \subseteq C\), \(C\) is subdirectly irreducible and \(\tau\) is diagonal, we have that \(F_\tau(A)\) consists of all pairs \((1, c)\) with \(c \in C\), and hence it is isomorphic (as a Wajsberg hoop) to \(C\). Since \(C\) is subdirectly irreducible, so is \(F_\tau(A)\). Finally, \(\tau(A)\) consists of all pairs of the form \((b, b)\) with \(b \in B\). Now if \((b, b) \vee (1, c) = (1, 1)\), then either \((b, b) = (1, 1)\) or \((1, c) = (1, 1)\). Hence, \(\tau(A)\) and \(F_\tau(A)\) have the disjunction property, and by Theorem 3.4 \((A, \tau)\) is subdirectly irreducible.

For the converse, we use Theorem 4.1. It is clear that condition (i) in Theorem 4.1 corresponds to type \(\mathcal{I}\). For case (ii) the additional conditions that \(F_\tau(A)\) is subdirectly irreducible and \(F_\tau(A)\) and \(\tau(A)\) have the disjunction property follows from Theorem 3.4 and the additional condition that \((A, \tau)\) is subdiagonal follows from Theorem 4.3.

Now, suppose (iii) is the case. Identifying \(B\) with its isomorphic copy \(h(B)\), we can rephrase the definition of \(\tau\) as \(\tau(b, c) = (b, b)\), and hence \((A, \tau)\) is of type \(\mathcal{D}\).

Finally, types \(\mathcal{I}\), \(\mathcal{L}\) and \(\mathcal{D}\) are mutually disjoint, because if \((A, \tau)\) is of type \(\mathcal{I}\), then \(F_\tau(A)\) is trivial, while if \((A, \tau)\) is of type \(\mathcal{L}\) or \(\mathcal{D}\), then \(F_\tau(A)\) is non-trivial. Moreover, the MV-reduct of a diagonal SMMV-algebra has two maximal filters, and hence it cannot be a local MV-algebra. This finishes the proof.

There is yet another type of subdirectly irreducible SMMV-algebras, namely, type \(\mathcal{K}\) (killing infinitesimals), which is described as follows:
Definition 4.4. An SMMV-algebra \((A, \tau)\) is said to be of type \(\mathcal{K}\) if \(A\) is of type \(\mathcal{L}\) and is linearly ordered.

The next example shows that the class of SMMV-algebras of type \(\mathcal{K}\) is properly contained in the class of SMMV-algebras of type \(\mathcal{L}\).

Example 4.5. Let \(C_1\) be the Chang MV-algebra. Let \(A\) be the subalgebra of \(C_1 \times C_1\) generated by \(\text{Rad}(C_1) \times \text{Rad}(C_1)\), i.e., \(A = (\text{Rad}(C_1) \times \text{Rad}(C_1)) \cup (\text{Rad}_1(C_1) \times \text{Rad}_1(C_1))\). We define \(\tau : A \to A\) via \(\tau(x, y) = (x, x)\). Then \(\tau\) is a state morphism operator on \(A\) such that \((A, \tau)\) is a subdirectly irreducible SMMV-algebra. \(\text{F}_*(A) = \{1\} \times \text{Rad}_1(C_1)\), \(\tau\) is not faithful, \(A\) has no nontrivial Boolean elements, but it is not linearly ordered. We note that \(\text{Rad}_1(A) = \text{Rad}_1(C_1) \times \text{Rad}_1(C_1)\) is the unique maximal filter.

5. Varieties of SMMV-algebras and their generators

We describe the varieties of SMMV-algebras and their generators. In particular, we answer in positive to an open question from [DiDv] that the diagonalization of the real interval \([0, 1]\) generates the variety of SMMV-algebras.

Given a variety \(\mathcal{V}\) of MV-algebras, \(\mathcal{V}_{SMMV}\) will denote the class of SMMV-algebras whose MV-reduct is in \(\mathcal{V}\). Clearly, \(\mathcal{V}_{SMMV}\) is a variety.

Definition 5.1. For every MV-algebra \(A\) we set \(D(A) = (A \times A, \tau_A)\), where \(\tau_A\) is defined, for all \(a, b \in A\), by \(\tau_A(a, b) = (a, a)\). For every class \(\mathcal{K}\) of MV-algebras, we set \(D(\mathcal{K}) = \{D(A) : A \in \mathcal{K}\}\).

As usual, given a class \(\mathcal{K}\) of algebras of the same type, \(I(\mathcal{K})\), \(H(\mathcal{K})\), \(S(\mathcal{K})\) and \(P(\mathcal{K})\) and \(P_0(\mathcal{K})\) will denote the class of isomorphic images, of homomorphic images, of subalgebras, of direct products and of ultraproducts of algebras from \(\mathcal{K}\), respectively. Moreover, \(V(\mathcal{K})\) will denote the variety generated by \(\mathcal{K}\).

Lemma 5.2. (1) Let \(\mathcal{K}\) be a class of MV-algebras. Then \(VD(\mathcal{K}) \subseteq V(\mathcal{K})_{SMMV}\).

(2) Let \(\mathcal{V}\) be any variety of MV-algebras. Then \(\mathcal{V}_{SMMV} = ISD(\mathcal{V})\).

Proof. (1) We have to prove that every MV-reduct of an algebra in \(VD(\mathcal{K})\) is in \(V(\mathcal{K})\). Let \(\mathcal{K}_0\) be the class of all MV-reducts of algebras in \(D(\mathcal{K})\). Then since the MV-reduct of \(D(A)\) is \(A \times A\), and since \(A\) is a homomorphic image (under the projection map) of \(A \times A\), \(\mathcal{K}_0 \subseteq P(\mathcal{K})\) and \(\mathcal{K} \subseteq H(\mathcal{K}_0)\). Hence, \(\mathcal{K}_0\) and \(\mathcal{K}\) generate the same variety. Moreover, MV-reducts of subalgebras (homomorphic images, direct products respectively) of algebras from \(D(\mathcal{K})\) are subalgebras (homomorphic images, direct products respectively) of the corresponding MV-reducts. Therefore, the MV-reduct of any algebra in \(VD(\mathcal{K})\) is in \(HSP(\mathcal{K}_0) = HSP(\mathcal{K}) = V(\mathcal{K})\), and claim (1) is proved.

(2) Let \((A, \tau) \in \mathcal{V}_{SMMV}\). We claim that the map \(\Phi : a \mapsto (\tau(a), a)\) is an embedding of \((A, \tau)\) into \(D(A)\). Clearly, \(\Phi\) is one-one. Moreover, since \(\tau\) is an MV-endomorphism, \(\Phi\) is an MV-homomorphism. Finally, \(\Phi(\tau(a)) = (\tau(\tau(a)), \tau(a)) = (\tau(a), \tau(a)) = \tau_A((\tau(a), a)) = \tau_A(\Phi(a))\). Hence, \(\Phi\) is compatible with \(\tau\), and \((A, \tau) \in ISD(\mathcal{V})\). Conversely, the MV-reduct of any algebra in \(D(\mathcal{V})\) is in \(\mathcal{V}\), (being a direct product of algebras in \(\mathcal{V}\)), and hence the MV-reduct of any member of \(ISD(\mathcal{V})\) is in \(IS(\mathcal{V}) = \mathcal{V}\). Hence, any member of \(ISD(\mathcal{V})\) is in \(\mathcal{V}_{SMMV}\). \(\Box\)

Lemma 5.3. Let \(\mathcal{K}\) be a class of MV-algebras. Then:

(1) \(DH(\mathcal{K}) \subseteq HD(\mathcal{K})\).
(2) DS(\mathcal{K}) \subseteq ISD(\mathcal{K}) .
(3) DP(\mathcal{K}) \subseteq IPD(\mathcal{K}).
(4) VD(\mathcal{K}) = ISD(V(\mathcal{K})).

Proof. (1) Let D(\mathcal{C}) \in DH(\mathcal{K}) . Then there are A \in \mathcal{K} and a homomorphism h from A onto C . Let for all a, b \in A, h^*(a, b) = (h(a), h(b)) . We claim that h^* is a homomorphism from D(A) onto D(C) . That h^* is an MV-homomorphism is clear. We verify that h^* is compatible with \tau_A . We have h^*(\tau_A(a, b)) = h^*(a, a) = (h(a), h(a)) = \tau_C(h(a), h(b)) = \tau_C(h^*(a, b)) . Finally, since h is onto, given (c, d) \in C \times C , there are a, b \in A such that h(a) = c and h(b) = d . Hence, h^*(a, b) = (c, d) , h^* is onto, and D(C) \in HD(\mathcal{K}).

(2) Almost trivial.

(3) Let A = \prod_{i \in I} (A_i) \in P(\mathcal{K}) , where each A_i is in \mathcal{K} . Then the map
\Phi : ((a_i : i \in I), (b_i : i \in I)) \mapsto ((a_i, b_i : i \in I)

is an isomorphism from D(A) onto \prod_{i \in I} D(A_i) . Indeed, it is clear that \Phi is an MV-isomorphism. Moreover, denoting the state morphism of \prod_{i \in I} D(A_i) by \tau^*, we get:
\Phi(\tau_A(a_i : i \in I), (b_i : i \in I)) = \Phi((a_i : i \in I), (a_i : i \in I)) =

= ((a_i, a_i : i \in I) = (\tau_A(a_i, b_i : i \in I) = \tau^*(\Phi((a_i : i \in I), (b_i : i \in I))),

and hence \Phi is an SMMV-isomorphism.

(4) By (1), (2) and (3), DV(\mathcal{K}) = DHSP(\mathcal{K}) \subseteq HSPD(\mathcal{K}) = VD(\mathcal{K}) , and hence ISDV(\mathcal{K}) \subseteq ISVD(\mathcal{K}) = VD(\mathcal{K}) . Conversely, by Lemma 5.2(1), VD(\mathcal{K}) \subseteq V(\mathcal{K})_{\text{SMMV}} , and by Lemma 5.2(2), V(\mathcal{K})_{\text{SMMV}} = ISDV(\mathcal{K}). This settles the claim. \qed

Theorem 5.4. (1) For every MV-algebra A , V(D(A)) = V(A)_{\text{SMMV}} .
(2) Let A and B be MV-algebras. Then V(D(A)) = V(D(B)) \iff V(A) = V(B) .
(3) The variety of all SMMV-algebras is generated by D([0, 1]_{\text{MV}}) as well as by any D(A) such that A generates the variety of MV-algebras.
(4) Let C_1 be Chang’s algebra and let C be the variety generated by it. Then C_{\text{SMMV}} is generated by D(C_1) .

Proof. (1) By Lemma 5.2(4), VD(A) = V(D(A)) = ISD(V(A)) . Moreover, by Lemma 5.2(2), V(A)_{\text{SMMV}} = ISDV(A) . Hence, V(D(A)) = V(A)_{\text{SMMV}} .

(2) We have V(D(A)) = V(A)_{\text{SMMV}} and V(D(B)) = V(B)_{\text{SMMV}} . Clearly, V(A) = V(B) implies V(A)_{\text{SMMV}} = V(B)_{\text{SMMV}} , and hence V(D(A)) = V(D(B)) . Conversely, V(D(A)) = V(D(B)) implies V(A)_{\text{SMMV}} = V(B)_{\text{SMMV}} . But any algebra C \in V(A) is the MV-reduct of an algebra in V(A)_{\text{SMMV}} , namely, of (C, Id_C) , where Id_C is the identity on C .

It follows that, if V(A)_{\text{SMMV}} = V(B)_{\text{SMMV}} , then the classes of MV-reducts of V(A)_{\text{SMMV}} and of V(B)_{\text{SMMV}} coincide, and hence V(A) = V(B) .

(3) Since V([0, 1]_{\text{MV}}) is the variety \mathcal{MV} of all MV-algebras, V(D([0, 1]_{\text{MV}})) is \mathcal{MV}_{\text{SMMV}} , that is, the variety of all SMMV-algebras. The same argument holds if we replace [0, 1]_{\text{MV}} by any MV-algebra which generates the whole variety \mathcal{MV} .

(4) Completely parallel to (3). \qed

Another consequence is the decidability of the variety \mathcal{SMMV} of all SMMV-algebras.

Theorem 5.5. \mathcal{SMMV} is decidable.
Proof. We associate to every term \( t(x_1, \ldots, x_n) \) of SMMV-algebras a pair of terms \( t^1, t^2 \) whose variables are among \( x_1^1, x_1^2, \ldots, x_n^1, x_n^2 \) by induction as follows: If \( t \) is a variable, say, \( t = x_i \), then \( t^1 = x_i^1 \) and \( t^2 = x_i^2 \); if \( t = 0 \), then \( t^1 = t^2 = 0 \). If \( t = \neg s \), then \( t^1 = \neg s^1 \) and \( t^2 = \neg s^2 \); if \( t = s \oplus u \), then \( t^1 = s^1 \oplus u^1 \) and \( t^2 = s^2 \oplus u^2 \). Finally, if \( t = \tau(s) \), then \( t^1 = t^2 = s^1 \). The following lemma is straightforward.

**Lemma 5.6.** Let \( a_1^1, a_1^2, \ldots, a_n^1, a_n^2, b_1^1, b_2^2 \in [0, 1] \) and let \( t(x_1, \ldots, x_n) \) be a term. Then the following are equivalent:

1. \( t((a_1^1, a_1^2), \ldots, (a_n^1, a_n^2)) = (b_1^1, b_2^2) \) holds in \( D([0, 1]_{MV}) \).
2. \( t^1(a_1^1, a_1^2, \ldots, a_n^1, a_n^2) = b_i^1 \), for \( i = 1, 2 \) holds in \( [0, 1]_{MV} \).

As a consequence, we obtain that an equation \( t = s \) holds identically in \( D([0, 1]_{MV}) \) iff \( t^1 = s^1 \) and \( t^2 = s^2 \) hold identically in \( [0, 1]_{MV} \). Since validity of an equation in \( [0, 1]_{MV} \) is decidable, the equational logic of \( D([0, 1]_{MV}) \) is decidable, and since \( D([0, 1]_{MV}) \) generates the whole variety of SMMV-algebras, the claim follows. \( \square \)

6. Every subdirectly irreducible SMMV-algebra is subdiagonal

We are in a position to prove Theorem 3.9 stating that every subdirectly irreducible SMMV-algebra is subdiagonal (subalgebra of a diagonal SMMV-algebra). We start from some easy facts.

First of all, any linearly ordered SMMV-algebra \( (A, \tau) \) is subdiagonal, being isomorphic to a subalgebra of \( (\tau(A) \times A, \tau^*) \), with \( \tau^*(\tau(a), a) = (\tau(a), \tau(a)) \). Next we prove that the variety of SMMV-algebras has CEP.

**Lemma 6.1.** The variety of SMMV-algebras has Congruence Extension Property.

**Proof.** Let \( (A, \tau) \subseteq (B, \tau) \) be SMMV-algebras and \( \theta \) a congruence on \( (A, \tau) \). Thus, \( 1/\theta \) is a \( \tau \)-filter of \( (A, \tau) \). By monotonicity of \( \tau \) the upward closure (in \( B \)) of \( 1/\theta \) is a \( \tau \)-filter of \( (B, \tau) \), which restricts to \( 1/\theta \) on \( (A, \tau) \). This proves the claim. \( \square \)

The next lemma is also easy:

**Lemma 6.2.** The class of subdiagonal SMMV-algebras is closed under subalgebras and ultraproducts.

**Proof.** Closure under \( S \) is definitional. Closure under \( P_U \) follows from the following facts:

1. For every class \( K \) of algebras of the same type \( P_U S(K) \subseteq SP_U(K) \) (this is a well-known fact of Universal Algebra).

2. Every ultraproduct \( \prod_{i \in I} (B_i \times C_i, \tau_i) / U \) of diagonal SMMV-algebras is isomorphic to the diagonal SMMV-algebra \( \prod_{i \in I} (B_i / U) \times (C_i / U, \tau_{U_i}) \), where \( \tau_{U_i}((b_i : i \in I)/U, (c_i : i \in I)/U) = ((b_i : i \in I)/U, (b_i : i \in I)/U) \), with respect to the isomorphism \( (b_i, c_i) : i \in I)/U \mapsto (b_i : i \in I)/U, (c_i : i \in I)/U) \). \( \square \)

To deal with homomorphic images we need the following definition:

**Definition 6.3.** An SMMV-algebra \( (A, \tau) \) is said to be \( skew \) \( diagonal \) if it has the form \( (B \times C/\phi, \tau) \), where \( B \) and \( C \) are MV-chains, \( B \) is a subalgebra of \( C, \phi \) is a congruence of \( C \) and \( \tau \) is defined \( \tau(b, c/\phi) = (b, b/\phi) \) for all \( b \in B \) and \( c \in C \).

The projection onto the first coordinate is a homomorphism from the skew-diagonal algebra \( (B \times C/\phi, \tau) \) onto \( (B, Id_B) \). Compatibility with \( \tau \) is proved as follows: \( \pi_1 \tau(b, c) = \pi_1(b, b) = b = Id_B \pi_1(b, c) \).
Lemma 6.4. Let \((A, \tau)\) be a subdiagonal algebra with \(A \subseteq B \times C\), and \(\theta\) a congruence on \((A, \tau)\). Then there are MV-chains \(D \subseteq E\), and a congruence \(\varphi\) on \(E\) such that \((A, \tau)/\theta\) is subdirectly embedded into a skew-diagonal algebra \((D \times E)/\varphi, \tau)\).

Proof. Clearly, we may assume that the natural identity embedding \(A \subseteq B \times C\) is subdirect. By CEP, the congruence \(\theta\) extends to a congruence \(\psi\) on \((B \times C, \tau)\). Of course, \(\psi\) is also a congruence on the MV-reduct \(B \times C\). By congruence distributivity, all congruences of finite products are product congruences, so \(\psi = \psi_B \times \psi_C\) for some congruences \(\psi_B\) on \(B\) and \(\psi_C\) on \(C\).

The congruences \(\psi_B\) and \(\psi_C\) are defined as follows: \((b, b') \in \psi_B\) iff there are \(c, c' \in C\) such that \(((b, c), (b', c')) \in \psi\), and \((c, c') \in \psi_C\) iff there are \(b, b' \in B\) such that \(((b, c), (b', c')) \in \psi\). Denoting by \(\theta_1\) and \(\theta_2\) the congruences associated to the projection maps, and using congruence distributivity, we have: \(((b, c), (b', c')) \in \psi\) iff \(((b, c), (b', c')) \in (\psi \lor \theta_1) \land (\psi \lor \theta_2)\) iff \((b, b') \in \psi_B\) and \((c, c') \in \psi_C\), and \(\psi = \psi_B \times \psi_C\).

It follows:

\[
(B \times C)/\psi = B/\psi_B \times C/\psi_C
\]

and moreover, since \(\psi\) is compatible with \(\tau\) we obtain

\[
\tau(b, c)/\psi = (b, b)/\psi = (b/\psi_B, b/\psi_C).
\]

Furthermore, \(((b, 1), (1, 1)) \in \psi\) implies \((\tau(b, 1), \tau(1, 1)) = ((b, b), (1, 1)) \in \psi\). It follows that \((b, 1) \in \psi_B\) implies \((b, 1) \in \psi_C\). Let \(\chi\) be the congruence of \(C\) generated by \(\psi_B\). Then \(\chi \subseteq \psi_C\), and by the CEP, \(\psi_B = \chi \cap \psi_C\). Now let \(D = B/\psi_B, E = \chi/\psi_C\). Note that \(D\) and \(E\) are MV-chains. Moreover, by construction we have \(\psi_B \subseteq E\), and hence

\[
A/\theta \subseteq (B \times C)/\psi = B/\psi_B \times C/\psi_C = D \times E/\varphi
\]

proving the claim for the MV-reducts of the appropriate algebras. In particular, the embedding is subdirect. Furthermore,

\[
\tau(b, c)/\psi = (b/\psi_B, b/\psi_C) = (b/\psi_B, (b/\chi)/\varphi)
\]

and the embedding lifts to the full type of SMMV.

Lemma 6.5. Let \((A, \tau)\) be a subdirectly irreducible SMMV-algebra, and suppose that \((A, \tau)\) is a subalgebra of a skew-diagonal SMMV-algebra \((B \times C)/\varphi, \tau^*)\), and that the identity MV-embedding of \(A\) into \((B \times C)/\varphi\) is subdirect. Then \((A, \tau)\) is subdiagonal.

Proof. If for all \(b \in B\), \((b, 1) \in \varphi\) implies \(b = 1\), then the map \(b \mapsto b/\varphi\) is one-one and \(B\) is (isomorphic to) a subalgebra of \(C/\varphi\). Hence, \(C/\varphi\) is an MV-chain and \(B\) is a subchain of \(C/\varphi\). It follows that \((B \times C)/\varphi, \tau^*)\) is diagonal and \((A, \tau)\) is subdiagonal. Now suppose that \((b, 1) \in \varphi\) for some \(b \in B \setminus \{1\}\). Since \(A\) is a subdirect product of \(B \times C/\varphi\), there is \(c \in C\) such that \((b, c/\varphi) \in A\). Moreover, \(\tau(b, c)/\varphi = (b, b)/\varphi = (b, 1/\varphi) \in \tau(A)\).

Now if \((1, c/\varphi) \in A\), then \(\tau(1, c/\varphi) = (1, 1/\varphi)\) and hence \((1, c/\varphi) \in F_\tau(A)\). Clearly, \((1, c/\varphi) \lor (b, 1/\varphi) = (1, 1/\varphi)\), and since \(\tau(A)\) and \(F_\tau(A)\) have the disjunction property, we must have \(c/\varphi = 1/\varphi\). Now \(F_\tau(A)\) consists of all elements of the form \((1, c/\varphi)\), and hence it is the singleton of \((1, 1/\varphi)\). On the other hand, \(F_\tau(A)\) is the filter associated to the homomorphism \(\tau\), and hence \(\tau\) is an embedding and \(A\) is isomorphic to \(\tau(A)\), which is in turn isomorphic to \(B\) via the map \(b \mapsto (b, b/\varphi)\).

Since \(B\) is linearly ordered, \(A\) is linearly ordered and hence subdiagonal. □
We can conclude the proof of Theorem 3.9.

Proof. Let $A$ be subdirectly irreducible. Since the variety of SMMV-algebras is generated by $D([0,1]_{MV})$, and since SMMV-algebras are congruence distributive, by Jónsson’s lemma $A$ belongs to $\text{HSP}_U(D([0,1]_{MV}))$. Thus, $A$ is a homomorphic image of some $B \in \text{SP}_U(D([0,1]_{MV}))$.

Now $D([0,1]_{MV})$ is subdiagonal, and by Lemma 6.4 subdiagonal SMMV-algebras are closed under $S$ and $P_0$, so $B$ is subdiagonal as well. Then, since $A$ is subdirectly irreducible, Lemma 6.5 applies, and we conclude that $A$ is subdiagonal. Hence, every subdirectly irreducible SMMV-algebra is subdiagonal.

We end this section with an example showing that the class of subdiagonal SMMV-algebras is not closed under homomorphic images. Indeed, our example shows that not even the class of subdirectly irreducible subdiagonal SMMV-algebras is closed under homomorphic images.

Consider the diagonal algebra $A = \langle C_1 \times C_1, \tau_{C_1} \rangle$, where $C_1$ stands for Chang’s algebra. The set $F = \{1\} \times \text{Rad}_1(C_1)$ is a $\tau$-filter of $A$. It is easy to see that the congruence $\theta_F$ corresponding to $F$ is the smallest nontrivial congruence on $A$, so $A$ is subdirectly irreducible. It is not difficult to see that the MV-reduct of the quotient algebra $A/\theta_F$ is isomorphic to $C_1 \times 2$, where 2 is the two-element Boolean algebra. The operation $\tau$ on this algebra is given by

$$\tau(c,1) = \tau(c,0) = \begin{cases} (c,1) & \text{if } c \in \text{Rad}_1(C_1) \\ (c,0) & \text{if } c \notin \text{Rad}_1(C_1). \end{cases}$$

Lemma 6.6. The algebra $A/\theta_F$ is not subdiagonal.

Proof. If $A/\theta_F$ is subdiagonal then there exist linearly ordered MV-algebras $D$ and $E$ such that $C_1 \subseteq D$, $2 \subseteq E$ and either $(D \times E, \tau)$ is diagonal, or $(E \times D, \tau)$ is diagonal. Now, if $(D \times E, \tau)$ is diagonal, we have $\tau(d,e) = (d,d)$ for all $(d,e) \in D \times E$. In particular, $(c,z) = (c,c)$ for any $(c,z) \in C_1 \times 2$. This fails for any $c \notin \{0,1\}$. Then, if $(E \times D, \tau)$ is diagonal, we have $\tau(e,d) = (e,e)$ for all $(e,d) \in E \times D$. In particular, $(z,c) = (z,z)$ for any $(z,c) \in 2 \times C_1$. This again fails for any $c \notin \{0,1\}$. Thus, $A/\theta_F$ is not subdiagonal.

7. Varieties of SMMV-algebras

When studying a variety of universal algebras, an interesting problem is the investigation of the lattice of its subvarieties. In the case of SMMV-algebras, we have a unique atom (above the trivial variety), namely, the variety $\mathcal{B}L$ of Boolean algebras equipped with the identical endomorphism. This variety is generated by the two element Boolean algebra equipped with the identity map. Since this algebra is a subalgebra of any non-trivial SMMV-algebra, $\mathcal{B}L$ is contained in any non-trivial variety of SMMV-algebras.

Other varieties of SMMV-algebras are obtained as follows: let $\mathcal{V}$ be a variety of MV-algebras, let $\mathcal{V}_{SMMV}$ denote the class of algebras whose MV-reduct is in $\mathcal{V}$, and $\mathcal{V}_I$ denote the class of SMMV-algebras $(A, \text{Id}_A)$, where $\text{Id}_A$ is the identity on $A$ and $A \in \mathcal{V}$. The following problem arises: given a variety $\mathcal{V}$ of MV-algebras, investigate the varieties of SMMV-algebras between $\mathcal{V}_I$ and $\mathcal{V}_{SMMV}$. To begin with, besides $\mathcal{V}_I$ and $\mathcal{V}_{SMMV}$, we will discuss two more kinds of subvarieties, namely, the subvariety generated by all SMMV-chains in $\mathcal{V}_{SMMV}$ (representable SMMV-algebras).
and the subvariety generated by all algebras in \( \mathcal{V}_{SMMV} \) whose MV-reduct is a local MV-algebra. The above classes will be denoted by \( \mathcal{V}_R \) and \( \mathcal{V}_L \) respectively. We consider \( \mathcal{V}_{SMMV} \) and \( \mathcal{V}_I \) first. The following result is straightforward.

**Theorem 7.1.** (1) \( \mathcal{V}_{SMMV} \) is axiomatized over the axioms of SMMV-algebras by the defining equations of \( \mathcal{V} \).

(2) \( \mathcal{V}_I \) is axiomatized over \( \mathcal{V}_{SMMV} \) by the identity \( \tau(x) = x \).

(3) \( \mathcal{V}_I \subseteq \mathcal{V}_R \), and the inclusion is proper if and only if \( \mathcal{V} \) is not finitely generated.

(4) The maps \( \mathcal{V} \mapsto \mathcal{V}_I \) and \( \mathcal{V} \mapsto \mathcal{V}_{SMMV} \) are embeddings of the lattice of MV-varieties into the lattice of SMMV-varieties.

**Proof.** Claims (1) and (2) are immediate.

As regards to (3), since subdirectly irreducible algebras of type \( I \) are linearly ordered we have that \( \mathcal{V}_I \subseteq \mathcal{V}_R \). If \( \mathcal{V} \) is finitely generated, then \( \mathcal{V}_I = \mathcal{V}_R \), because every MV-chain in \( \mathcal{V} \) is finite, and its only endomorphism is the identity. Finally, if \( \mathcal{V} \) is not finitely generated, then it contains Chang’s algebra, \( C_1 \). Let \( \tau \) be defined for all \( x \in C_1 \), by \( \tau(x) = 0 \) if \( x \) is infinitesimal and \( \tau(x) = 1 \) otherwise. Then \( (C_1, \tau) \in \mathcal{V}_R \setminus \mathcal{V}_I \), and the inclusion \( \mathcal{V}_I \subseteq \mathcal{V}_R \) is proper.

Finally, claim (4) is almost immediate (using Theorem 5.4).

We now concentrate ourselves on \( \mathcal{V}_R \).

**Theorem 7.2.** Representable SMMV-algebras constitute a proper subvariety of the variety of SMMV-algebras, which is characterized by the equation

\[
(\text{lin}_\tau) \quad \tau(x) \lor (x \to (\tau(y) \leftrightarrow y)) = 1.
\]

**Proof.** We have to prove that a subdirectly irreducible SMMV-algebra \((A, \tau)\) satisfies \((\text{lin}_\tau)\) if it is linearly ordered. Thus, let \((A, \tau)\) be a subdirectly irreducible SMMV-algebra.

Suppose first that \((A, \tau)\) satisfies \((\text{lin}_\tau)\). We start from the following observation. Let \( z, u \in A \). Then \( z \to (\tau(u) \leftrightarrow u) \in F_\tau(A) \). Since \( \tau(A) \) and \( F_\tau(A) \) have the disjunction property, we have that either \( \tau(z) = 1 \) or \( z \leq \tau(u) \leftrightarrow u \). Now every element of \( u \in F_\tau(A) \) is equal to \( \tau(u) \leftrightarrow u \), and vice versa every element of the form \( \tau(u) \leftrightarrow u \) is in \( F_\tau(A) \). It follows that if \( \tau(z) < 1 \), then \( z \) is a lower bound of \( F_\tau(A) \).

Now assume, by way of contradiction, that \( x, y \in A \) are incomparable with respect to the order. We distinguish three cases.

(i) If \( x \to y \in F_\tau(A) \) and \( y \to x \in F_\tau(A) \), then since \( F_\tau(A) \) is linearly ordered and \((x \to y) \lor (y \to x) = 1\), we must have either \( x \to y = 1 \) or \( y \to x = 1 \), a contradiction.

(ii) If \( x \to y \notin F_\tau(A) \) and \( y \to x \notin F_\tau(A) \), then they are both lower bounds of \( F_\tau(A) \), and hence \( 1 = (x \to y) \lor (y \to x) \) is a lower bound of \( F_\tau(A) \). But then \( A \) would be isomorphic to \( \tau(A) \), and hence it would be linearly ordered, a contradiction.

(iii) Finally, suppose \( x \to y \in F_\tau(A) \) and \( y \to x \notin F_\tau(A) \) (or vice versa). Then \( y \to x \) is a lower bound of \( F_\tau(A) \), and hence \( y \to x \leq x \to y \). But in any MV-algebra this is the case iff \( x \leq y \), and again a contradiction has been obtained.

Hence, \((A, \tau)\) is linearly ordered. Conversely, if \((A, \tau)\) is linearly ordered, then for all \( x, z \) such that \( \tau(x) < 1 \) and \( \tau(z) = 1 \), we cannot have \( z < x \), and hence we must have \( x \leq z \). Taking \( z = \tau(y) \leftrightarrow y \), we obtain that for all \( x \) either \( \tau(x) = 1 \) or \( x \leq z \), and \((\text{lin}_\tau)\) holds.
Finally, representable SMMV-algebras constitute a proper subvariety of the variety of SMMV-algebras, because any subdirectly irreducible SMMV-algebra of type \( D \) is not linearly ordered. \( \square \)

**Remark 7.3.** According to [DDL1, Prop. 3.6], if \((A, \tau)\) is an SMV-algebra such that \( A \) is a chain, then \((A, \tau)\) is an SMMV-algebra. Hence, the class of all representable SMV-algebras satisfies \((\text{lin}_\tau)\). We do not know whether every subdirectly irreducible SMV-algebra satisfying \((\text{lin}_\tau)\) has a linearly ordered MV-reduct.

**Theorem 7.4.** \( VR \subseteq VL \), and the inclusion is proper if and only if \( V \) is not finitely generated.

*Proof.* Since every linearly ordered SMMV-algebra is local, the inclusion follows. Moreover, every local and finite MV-algebra is linearly ordered, and hence for finitely generated MV-varieties the opposite inclusion also holds. On the other hand, if \( V \) is not finitely generated, then it contains Chang’s algebra \( C_1 \), and the subalgebra of \( D(C_1) \) described in Example 4.5 is a local subdirectly irreducible SMMV-algebra in \( V_{\text{SMMV}} \) which is not linearly ordered. Hence, the inclusion \( VR \subseteq VL \) is proper. \( \square \)

Next, we discuss varieties of the form \( VL \).

**Theorem 7.5.**

1. The variety \( VL \) is axiomatized over \( V_{\text{SMMV}} \) by the equation
   \[
   -((\tau(x) \leftrightarrow x) \leq (\tau(x) \leftrightarrow x)).
   \]
2. For any non-trivial variety \( V \) of MV-algebras, \( VL \) is a proper subvariety of \( V_{\text{SMMV}} \).

*Proof.* We start from the following lemma:

**Lemma 7.6.** Let \( A \) be a local MV-algebra and \( M \) be its only maximal filter. Then for every \( m \in M \), \( -m \leq m \).

*Proof.* Since \( m^2 \in M \), \( -(m^2) \notin M \). Since \( M \) is the only maximal filter, then \( -(m^2) \) generates a degenerate filter, and hence there is a natural number \( n \) such that \( -(m^2)^n = 0 \). Now let us decompose \( A \) into a subdirect product \( \prod_{i \in I} A_i \) of MV-chains. If for some index \( i \) we had \( m_i < -m_i \), then we would get \( m_i^2 = 0 \). But this would imply \( -(m_i^2) = 1 \), and hence \( -(m^2)^n > 0 \) for every \( n \), a contradiction. \( \square \)

We continue the proof of Theorem 7.5. In order to prove claim (1), it suffices to prove that an SMMV-algebra is subdirectly irreducible iff it satisfies \((\text{loc}_\tau)\). Now in every SMMV-algebra we have \( \tau(\tau(x) \leftrightarrow x) = 1 \), and hence \( \tau(x) \leftrightarrow x \in F_*(A) \subseteq M \), where \( M \) denotes the unique maximal filter of \( A \). Then Lemma 7.6 implies that every subdirectly irreducible local SMMV-algebra satisfies \((\text{loc}_\tau)\). Before proving the converse, we prove claim (2).

Let \( A \) be a non-trivial chain in \( V \). Then \((\text{loc}_\tau)\) is invalidated in \( D(A) \), taking \( x = (1,0) \). We have \( \tau(x) = (1,1), \tau(x) \leftrightarrow x = (1,0), \) and

\[
-(\tau(x) \leftrightarrow x) = (0,1) \leq (1,0) = -((\tau(x) \leftrightarrow x)).
\]

This settles the claim.

In order to prove the opposite direction of claim (1), note that every subdirectly irreducible SMMV-algebra is either of type \( I \) (in which case it is local) or of type \( L \) (in which case, once again it is local) or of type \( D \). In the last case the proof of (2) shows that it does not satisfy \((\text{loc}_\tau)\). Hence if a subdirectly irreducible SMMV-algebra satisfies \((\text{loc}_\tau)\) it is local. \( \square \)
Another interesting problem in the study of the lattice of subvarieties of a variety is the investigation of covers of a given subvariety (if any). For instance, one may wonder what are the covers of $BL$. A partial answer to this question is provided by the following theorem:

**Theorem 7.7.** Let $V$ and $W$ be varieties of $MV$-algebras. If $W$ is a cover of $V$, then $WI$ is a cover of $VI$. Hence, if $W$ is generated either by $S_p$ for some prime number $p$ or by Chang’s algebra $C_1$, then $WI$ is a cover of $BL$.

**Proof.** If $(A, \tau) \in WI \setminus VI$, then since $\tau$ is forced to be the identity, we must have $A \in V \setminus V$, and since $W$ is a cover of $V$, the variety generated by $\{A\} \cup V$ is $W$, and hence the variety generated by $\{(A, \tau)\} \cup VI$ is $WI$, and the claim follows. \qed

**Remark 7.8.** Varieties $VI$, where $V$ is a cover of the Boolean variety $B$, do not exhaust the covers of $BL$. Another cover is $B_{SMMV}$. Indeed, any subdirectly irreducible SMMV-algebra $(A, \tau)$ in $B_{SMMV} \setminus BL$ must have a Boolean reduct and cannot be of type $I$ or $L$, otherwise $\tau$ would be identical. Hence, it must be of type $D$ and $D(S_1)$ is a subalgebra of $(A, \tau)$. Therefore, $(A, \tau)$ generates the whole variety $B_{SMMV}$.

Theorem 7.7 suggests the following problem:

**Problem 2.** Let $V$ be a variety of $MV$-algebras and let $V'$ be a cover of $V$. Is it true that $V''_{SMMV}$ is a cover of $V'_{SMMV}$? Or, equivalently, is $VD(V)$ a cover of $VD(V')$?

The answer to these questions is no, in general. Here is a sample of counter-examples.

(1) Let $V$ be the variety of Boolean algebras and $V'$ be the variety generated by Chang’s algebra. Then $V'$ is a cover of $V$. However, there is an intermediate variety between $V'_{SMMV}$ and $V''_{SMMV}$, namely, the subvariety $V''_{SMMV}$ axiomatized by the equation

$$\tau(x) \lor \tau(\neg x) = 1. \quad (*)$$

Indeed, clearly the equation $(*)$ holds in any Boolean SMMV-algebra. Moreover, there is an algebra in $V'_{SMMV}$ which satisfies $(*)$ and its reduct is not a Boolean algebra, namely, Chang’s algebra $C_1$ with $\tau$ defined by $\tau(x) = 0$ if $x \in Rad(C_1)$ and $\tau(x) = 1$ otherwise.

Finally, there is an algebra in $V''_{SMMV}$ which does not satisfy $(*)$, namely, the diagonalization, $D(C_1)$, of Chang’s algebra. Indeed, if $c \in Rad(C_1) \setminus \{0\}$, then $\tau(c, c) = (c, c)$ and $\tau(\neg(c, c)) = (\neg c, \neg c)$. Hence, $\tau(c, c) \lor \tau(\neg(c, c)) = (\neg c, \neg c) < 1$.

(2) Let $V = V(S_{i_1}, \ldots, S_{i_n})$ and $V' = V(S_{i_1}, \ldots, S_{i_n}, C_1)$ for some integers $1 \leq i_1 < \cdots < i_n$. Then $V'$ is a cover variety of $V$. Define $V''_{SMMV}$ as the class of all $(A, \tau) \in V'$ such that $\tau(A) \in V$.

Then $V_{SMMV} \subseteq V''_{SMMV} \subseteq V'$. But if $\tau$ is as in (1), then $(C_1, \tau) \in V''_{SMMV}$ and $D(C_1) \in V''_{SMMV} \setminus V'_{SMMV}$.

(3) Define on $C_n \times C_n$ a map $\tau_n(i, j) = (i, 0)$ for all $(i, j) \in C_n$, then $(C_n, \tau_n)$ is an SMMV-algebra.

Let $1 = i_1 < \cdots < i_n$ and $1 = j_1 < \cdots < j_k$ with $k \geq 2$ be finite sets of integers such that every $j_s$ does not divide any $j_t$ with $1 < j_s < j_t$ and fix an index $j_0 \in J := \{j_1, \ldots, j_k\}$ with $j_0 \geq 2$ such that $j_0 \in I := \{i_1, \ldots, i_k\}$. 
Let \( V' = V(\{S_i, C_j : i \in I, j \in J\}) \) and \( V = V(\{S_i, C_j : i \in I, j \in J \setminus \{j_0\}\}) \). Set \( \mathcal{V}_{SMMV}' \) as the class of \( (A, \tau) \in \mathcal{V}_{SMMV} \) such that \( \tau(A) \in V \). Then \( (C_{j_0}, \tau_{j_0}) \in \mathcal{V}_{SMMV}' \setminus \mathcal{V}_{SMMV} \) and \( D(C_{j_0}) \in \mathcal{V}_{SMMV}' \setminus \mathcal{V}_{SMMV} \).

(4) Let \( V' = V(S_{i_1}, \ldots, S_{i_n}) \), where \( 1 = i_1 < \cdots < i_n, n \geq 2 \) and every \( i_s \) does not divide any \( i_t \) with \( 1 < i_s < i_t \). Let \( i_0 \in \{i_2, \ldots, i_n\} \) be fixed and let \( V = V(S_i : i \in \{i_1, \ldots, i_n\} \setminus \{i_0\}) \). Then \( V' \) is a cover of \( V \). Let \( \mathcal{V}' \) be the variety generated by \( \mathcal{V}_{SMMV} \) and \( (S_{i_0}, \text{Id}_{S_{i_0}}) \). Then \( \mathcal{V}_{SMMV} \subset \mathcal{V}' \subset \mathcal{V}_{SMMV}' \) because \( (S_{i_0}, \text{Id}_{S_{i_0}}) \in \mathcal{V}' \setminus \mathcal{V}_{SMMV} \) and \( D(S_{i_0}) \in \mathcal{V}_{SMMV}' \setminus \mathcal{V}' \).

(5) Let \( V' = V(C_{j_1}, \ldots, S_{j_k}) \), where \( 1 = i_1 < \cdots < i_k, k \geq 2 \) and every \( j_s \) does not divide any \( j_t \) with \( 1 < j_s < j_t \). Let \( j_0 \in \{j_2, \ldots, j_n\} \) be fixed and let \( V = V(C_j : j \in \{j_1, \ldots, j_k\} \setminus \{j_0\}) \). Let \( \mathcal{V}' \) be the variety generated by \( \mathcal{V}_{SMMV} \) and \( (S_{j_0}, \tau) \). Then \( \mathcal{V}_{SMMV} \subset \mathcal{V}' \subset \mathcal{V}_{SMMV}' \) because \( (C_{j_0}, \text{Id}_{C_{j_0}}) \in \mathcal{V}' \setminus \mathcal{V}_{SMMV} \) and \( D(C_{j_0}) \in \mathcal{V}_{SMMV}' \setminus \mathcal{V}' \).

The above examples offer several interesting methods for obtaining intermediate varieties. But the fact that if \( W \) is an MV-cover of \( V \), then \( W_{SMMV} \) need not be a cover of \( \mathcal{V}_{SMMV} \) can be strengthened:

**Theorem 7.9.** If \( W \) properly contains \( V \), then the join, \( V_{SMMV} \vee WI \), of \( V_{SMMV} \) and \( WI \), is a proper extension of \( V_{SMMV} \) and a proper subvariety of \( W_{SMMV} \). Hence, \( W_{SMMV} \) can never be a cover of \( V_{SMMV} \).

**Proof.** Inclusions are clear. Moreover, if \( A \in W \setminus V \), then \( (A, \text{Id}_A) \in (W \vee V_{SMMV}) \setminus V_{SMMV} \), and hence the first inclusion is proper. In order to prove that also the inclusion \( (W \vee V_{SMMV}) \subseteq W_{SMMV} \), consider an MV-identity \( \eta(x) = 1 \) which axiomatizes \( V \) over \( W \), and set \( (\epsilon_V) \eta(x) \vee (\tau(y) \leftrightarrow y) = 1 \).

Clearly, \( (\epsilon_V) \) holds both in \( V_{SMMV} \) and in \( WI \), and hence it holds in \( V_{SMMV} \vee WI \). Now take a subdirectly irreducible MV-algebra \( A \in W \setminus V \). Then \( D(A) \in W_{SMMV} \), but it is readily seen that \( (\epsilon_V) \) is not valid in \( D(A) \), and also the inclusion \( (V_{SMMV} \vee WI) \subseteq W_{SMMV} \) is proper. \( \square \)

It follows that Problem 2 should be replaced by the following:

**Problem 3.** Suppose that \( W \) is an MV-cover of \( V \). Is it true that \( WI \vee V_{SMMV} \) is a cover of \( V_{SMMV} \)?

We now investigate the number of varieties of SMMV-algebras, and we prove that there are uncountably many of them. Let \([0, 1]^*\) be an ultrapower of the MV-algebra on \([0, 1]\), and let us fix a positive infinitesimal \( \varepsilon \in [0, 1]^* \). For every set \( X \) of prime numbers, we denote by \( A(X) \) the subalgebra of \([0, 1]^*\) generated by \( \varepsilon \) and by the set of all rational numbers \( \frac{n}{m} \) with \( 0 \leq n \leq m, m > 0 \) such that:

1. \( \text{either } n = 0 \text{ or } \text{gcd}(n, m) = 1 \);
2. \( \text{for all } p \in X, \text{ } p \text{ does not divide } m \).

Note that for all \( x \in A(X) \), the standard part of \( x \) is a rational number \( \frac{n}{m} \) satisfying (1) and (2). Indeed the set of rational numbers satisfying (1) and (2) is closed under all MV-operations.

On \( A(X) \) we define \( \tau(x) \) to be the standard part of \( x \). Note that \( \tau \) is an idempotent homomorphism from \( A(X) \) into itself, and hence \( (A(X), \tau) \) is a linearly ordered SMMV-algebra.
Corollary 7.12. There are varieties of representable SMMV-algebras which are not recursively axiomatizable, and hence not finitely axiomatizable.

References

[BP] W. Blok and D. Pigozzi, “Algebraizable Logics”, Mem. Amer. Math. Soc. 396(77) Amer. Math. Soc., Providence, 1989.

[BS] S. Burris and H.P. Sankappanavar, “A Course in Universal Algebra”, Springer Verlag, New York 1981.
[Ch] C.C. Chang, *A new proof of the completeness of Łukasiewicz axioms*, Trans. Amer. Math. Soc. **93** (1989), 74–80.

[CDM] R. Cignoli, I. D’Ottaviano and D. Mundici, “*Algebraic Foundations of Many-valued Reasoning*”, Kluwer Academic Publishers, Dordrecht 2000.

[DN] A. Di Nola, *Representation and reticulation by quotients of MV-algebras*, Ricerche Matem. **40** (1991), 291–297.

[DiDv] A. Di Nola, A. Dvurečenskij, *State-morphism MV-algebras*, Ann. Pure Appl. Logic **161** (2009), 161–173.

[DDL1] A. Di Nola, A. Dvurečenskij, A. Lettieri, *On varieties of MV-algebras with internal states*, Inter. J. Approx. Reasoning **51** (2010), 680–694.

[DDL2] A. Di Nola, A. Dvurečenskij, A. Lettieri, *Erratum “State-morphism MV-algebras” [Ann. Pure Appl. Logic 161 (2009) 161-173]*, Ann. Pure Appl. Logic **161** (2010), 1605–1607.

[Dvu] A. Dvurečenskij, *Subdirectly irreducible state-morphism BL-algebras*, Archive Math. Logic **50** (2011), 145–160. DOI:10.1007/s00153-010-0206-7

[FM] T. Flaminio and F. Montagna, *MV-algebras with internal states and probabilistic fuzzy logics*, Inter. J. Approx. Reasoning **50** (2009), 138–152.

[Ha] P. Hájek, “*Metamathematics of Fuzzy Logic*”, Kluwer Academic Publishers, Dordrecht 1998.

[Kr] T. Kroupa, “*Reasoning about Uncertainty*”, MIT Press, 2003.

[Kr1] T. Kroupa, *Every state on a semisimple MV algebra is integral*, Fuzzy Sets and Systems, **157** (2006), 2771–2787.

[KM] J. Kühr and D. Mundici, *De Finetti theorem and Borel states in [0,1]-valued algebraic logic*, Inter. J. Approx. Reasoning **46** (2007), 605–616.

[Mu86] D. Mundici, *Interpretations of AF C*-algebras in Łukasiewicz sentential calculus*, J. Funct. Analysis **65** (1986), 15–63.

[Mus] D. Mundici, *Averaging the truth value in Łukasiewicz logic*, Studia Logica **55** (1995), 113–127.

[Mub] D. Mundici, *Bookmaking over infinite-valued events*, Inter. J. Approx. Reasoning **46** (2006), 223–240.

[Pa] G. Panti, *Invariant measures in free MV-algebras*, Comm. Algebra **36** (2008), 2849–2861.