ON ASYMPTOTIC RELATIONS BETWEEN SINGULAR AND CONSTRAINED CONTROL PROBLEMS OF ONE-DIMENSIONAL DIFFUSIONS

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Abstract. We study the asymptotic relations between certain singular and constrained control problems for one-dimensional diffusions with both discounted and ergodic objectives. By constrained we mean that controlling is allowed only at independent Poisson arrival times. We show that the solutions of the discounted problems converge in Abelian sense to those of their ergodic counterparts. Moreover, we show that the solutions of the constrained problems converge to those of their singular counterparts when the Poisson rate tends to infinity. We illustrate the results with drifted Brownian motion and quadratic cost.

1. INTRODUCTION

This paper is concerned with asymptotic relations between certain discounted and ergodic control problems for one-dimensional diffusions. More precisely, the following control problems are considered:

(A) Classical singular stochastic control problems with both discounted and ergodic criteria

(BG) Constrained bounded variation control problems where controlling is allowed only at the independent Poisson arrival times with both discounted and ergodic criteria

These control problems are expected to be linked to each other via certain limiting properties. For instance, it is often expected that in item (A), the values of the problems with discounted criterion are connected to the ergodic problems in an Abelian sense when the discounting factor vanishes. This relationship, often called the vanishing discount method and sometimes used in a heuristic manner, can be used to solve the ergodic problems [22, 20, 11, 7].

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Regarding item (B), the problems of this form have attracted attention in the recent years \([15, 16, 23, 18, 19]\), for related studies in optimal stopping see \([14, 10, 12]\). In these problems, it is reasonable to expect that the value functions of the constrained problems should converge to the values of their singular counterparts as the Poisson arrival rate of the control opportunities tends to infinity.

The main contribution of this paper is that we prove these expectations to be correct for time-homogeneous control problems with one-dimensional diffusion dynamics; our findings are summarized in Figure 1. These diffusion models are important in many applications. Furthermore, the time-homogeneous structure allows explicit calculations by which we can first solve the HJB-equations of both discounted and ergodic problems separately and then establish that the solutions satisfy the desired limiting properties. This is in contrast to the vanishing discount method, where the HJB-equation of the ergodic problem is solved using the solution of the discounted problem \([20]\).

The remainder of the paper is organized as follows. In section 2, we set up the diffusion dynamics. In section 3, we introduce the functionals appearing in the analysis of the control problems and study their properties. The control problems are introduced and their asymptotic relations are proved in section 4. Paper is concluded with an explicit example in section 5.

2. UNDERLYING DYNAMICS

Let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\) be a filtered probability space which satisfies the usual conditions. We consider an uncontrolled process \(X\) defined on \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\) which is modelled as a strong solution to regular Itô diffusion

\[
dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x,
\]

where \(W_t\) is the Wiener process and the functions \(\mu\) and \(\sigma\) are well-behaved (see \([13]\) chapter 5). For notional convenience we consider the case where the process evolves in \(\mathbb{R}_+\), even though all the results remain unchanged even if the state space would be replaced with any interval.
We define the second-order linear differential operator \(A\) which represents the infinitesimal generator of the diffusion \(X\) as

\[
A = \mu(x) \frac{d}{dx} + \frac{1}{2} \sigma^2(x) \frac{d^2}{dx^2}
\]

and for a given \(r > 0\) we respectively denote the increasing and decreasing solutions to the differential equation \((A - r)f = 0\) by \(\psi_r > 0\) and \(\varphi_r > 0\).

We denote by \(\tau\) the first exist time from \(\mathbb{R}_+\), i.e. \(\tau = \inf\{t \geq 0 \mid X_t \not\in \mathbb{R}_+\}\) and define a set \(\mathcal{L}_r^1\) of functions \(f\) that satisfy the integrability condition \(\mathbb{E}_x [\int_0^\tau e^{-rs} |f(X_s)| ds] < \infty\). Using this notation, we define the inverse of the differential operator \((r - A)\), called the resolvent \(R_r\), by

\[
(R_r f)(x) = \mathbb{E}_x \left[ \int_0^\tau e^{-rs} f(X_s) ds \right]
\]

for all \(x \in \mathbb{R}_+\) and functions \(f \in \mathcal{L}_r^1\). We also define the scale density of the diffusion by

\[
S'(x) = \exp \left( - \int_x^\infty \frac{2\mu(z)}{\sigma^2(z)} dz \right),
\]

which is the monotonic (and non-constant) solution to the differential equation \(Af = 0\).

Often in computations it is useful to use the formula

\[
(R_r f)(x) = B_r^{-1} \varphi_r(x) \int_0^x \psi_r(y) f(y) m'(y) dy + B_r^{-1} \psi_r(x) \int_x^{\infty} \varphi_r(y) f(y) m'(y) dy,
\]

which connects the resolvent and the solutions \(\psi_r\) and \(\varphi_r\) in a rather convenient way. Here the positive constant (does not depend on \(x\))

\[
B_r = \frac{\psi_r'(x)}{S'(x)} \varphi_r(x) - \frac{\varphi_r'(x)}{S'(x)} \psi_r(x)
\]

is the Wronskian of the fundamental solutions and

\[
m'(x) = \frac{2}{\sigma^2(x) S'(x)}
\]

denotes the density of the speed measure. We also recall the resolvent equation

\[
R_q R_r = \frac{R_r - R_q}{q - r}.
\]
3. Properties of functionals $L$ and $K$

We study properties of the following functionals throughout this paper

$$L^r_f(x) = r \int_x^\infty \varphi_r(y)f(y)m'(y)dy + \frac{\varphi'_r(x)}{S'(x)}f(x),$$

$$K^r_f(x) = r \int_0^x \psi_r(y)f(y)m'(y)dy - \frac{\psi'_r(x)}{S'(x)}f(x).$$

Our main interest are the properties of these functionals when

$$f(x) = \theta_r(x) = \pi(x) + \gamma \rho(x),$$

where $\pi$ is the function measuring the payoff or cost and $\rho(x) = \mu(x) - rx$. In economical literature, the function $\theta_r$ can be understood as the net convenience yield of holding inventories [3, 9]. This function appears in wide range of control problems of one-dimensional diffusions when the criteria to be minimized includes discounting [15], [3], [17].

In addition, we note that in the absence of discounting $\theta_r$ reduces to

$$\pi_\mu(x) = \pi(x) + \gamma \mu(x),$$

which is in key role in many ergodic control problems of one dimensional diffusions [16, 6].

To setup the framework further we collect some assumptions below that are in accordance with most economical applications.

**Assumption 1.** We assume that:

1. the upper boundary $\infty$ and the lower boundary $0$ are natural,
2. the instantaneous payoff $\pi$ is continuous, non-negative and non-decreasing,
3. there is a unique state $x^* \geq 0$ such that $\theta_r$ is decreasing on $(0, x^*)$ and increasing on $(x^*, \infty)$,
4. the function $\theta_r$ satisfies the limiting condition $\lim_{x \to \infty} \theta_r(x) \geq 0$.

We make some remarks on these assumptions. First, we assume that the uncontrolled state variable $X$ cannot become infinitely large or small in finite time, see [8] pp. 18–20, for a characterization of the boundary behavior of diffusions. Second, we restrict our attention to the case where the function $\theta_r$ has a unique global minimum at $x^*$. In other words, $\theta_r$ is assumed to be negative for small values $x$ and to become positive for large values. Moreover, even though it is
not explicitly stated, we assume similar properties also for the limiting function \( \lim_{r \to 0} \theta_r(x) = \pi(x) \) as for \( \theta_\ast \).

In the next lemma we prove useful representations for functionals \( K_f^r \) and \( L_f^r \).

**Lemma 1.** The functions \( L_f^r \) and \( K_f^r \) have alternative representations

\[
L_f^r(x) = \frac{\sigma^2(x)}{2S'(x)} [\varphi''_r(x)(R_r f)'(x) - \varphi'_r(x)(R_r f)''(x)]
\]

\[
K_f^r(x) = \frac{\sigma^2(x)}{2S'(x)} [\psi'_r(x)(R_r f)''(x) - \psi''_r(x)(R_r f)'(x)]
\]

*Proof.* The proof for the claim on \( L_f \) is in [15] lemma 2 and the proof on \( K_f \) is completely analogous. \( \square \)

Under the assumption that the boundaries are natural, we also have that

\[
\varphi'_r(x) = -r \int_x^\infty \varphi_r(y)m'(y)dy, \quad \psi'_r(x) = \int_0^x \psi_r(y)m'(y)dy,
\]

and thus, we can further rewrite

\[
L_f^r(x) = r \int_x^\infty \varphi_r(y)(f(y) - f(x))m'(y)dy,
\]

\[
K_f^r(x) = r \int_0^x \psi_r(y)(f(y) - f(x))m'(y)dy.
\]

At this point it is worth to mention that the convexity of the minimal excessive functions \( \varphi_r \) and \( \psi_r \) are dependant on the monotonicity properties of \( \rho(x) = \mu(x) - rx \). This is because noting that \( (A - r)x = \rho(x) \) and using lemma [4] for \( \rho \) gives

\[
\varphi''_r(x) \frac{\sigma^2(x)}{2S'(x)} = L_\rho^r(x) = r \int_x^\infty \varphi_r(y)(\rho(y) - \rho(x))m'(y)dy.
\]

Similar calculations can be carried out for \( \psi_r(x) \). Hence, for example, in the regions where \( \rho(x) \) is increasing \( \varphi_r(x) \) is convex, we refer to [2] for details.

In the next proposition we prove that the just introduced functionals, often appearing in bounded variation control problems of one-dimensional diffusion processes, satisfy asymptotic properties that are needed to establish useful relationships between different control problems, see section 3.
Proposition 1. Under the assumption 1, we have the following limiting properties

\[ \frac{L_{\theta_r}^r(x)}{r \varphi_r(x)} \xrightarrow{r \to 0} H(x, \infty), \quad \frac{K_{\theta_r}^r(x)}{r \psi_r(x)} \xrightarrow{r \to 0} H(0, x), \]

where

\[ H(x, y) = \int_x^y (\pi_\mu(z) - \pi_\mu(x))m'(z)dz. \]

In addition, we have

\[ \frac{L_{\theta_r}^{r+\lambda}(x)}{(r + \lambda) \varphi_{r+\lambda}(x)} \xrightarrow{\lambda \to \infty} 0, \quad \frac{K_{\theta_r}^{r+\lambda}(x)}{(r + \lambda) \psi_{r+\lambda}(x)} \xrightarrow{\lambda \to \infty} 0. \]

Proof. Let \( \tau_z = \inf\{t \geq 0 \mid X_t = z\} \). Then for all \( s > 0 \) we have

\[ \mathbb{E}_x[e^{-s \tau_z} \mid \tau_z < \infty] = \begin{cases} \psi_s(x) & x \leq z \\ \psi_s(z) & x > z, \end{cases} \]

therefore, by letting \( s \to 0+ \) we get by monotone convergence that

\[ \lim_{s \to 0^+} \psi_s(x) = \mathbb{P}_x[\tau_z < \infty] = 1, \]

\[ \lim_{s \to 0^+} \varphi_s(x) = \mathbb{P}_x[\tau_z < \infty] = 1, \]

under the assumption that the underlying diffusion is regular. In addition, again by (3), we find that

\[ \lim_{s \to \infty} \frac{\psi_s(x)}{\psi_s(z)} = 0, \quad \lim_{s \to \infty} \frac{\varphi_s(x)}{\varphi_s(z)} = 0. \]

Since \( \lim_{r \to 0+} \theta_r(x) = \pi_\mu(x) \) we see by using the above observations that

\[ \frac{L_{\theta_r}^r(x)}{r \varphi_r(x)} = \int_x^\infty \frac{\varphi_r(z)}{\varphi_r(x)}(\theta_r(z) - \theta_r(x))m'(z)dz \to H(x, \infty) \text{ as } r \to 0, \]

\[ \frac{K_{\theta_r}^r(x)}{r \psi_r(x)} = \int_0^x \frac{\psi_r(z)}{\psi_r(x)}(\theta_r(z) - \theta_r(x))m'(z)dz \to H(0, x) \text{ as } r \to 0. \]

Similarly, utilizing (8) we have

\[ \frac{L_{\theta_r}^{r+\lambda}(x)}{(r + \lambda) \varphi_{r+\lambda}(x)} = \int_x^\infty \frac{\varphi_{r+\lambda}(z)}{\varphi_{r+\lambda}(x)}(\theta_r(z) - \theta_r(x))m'(z)dz \to 0 \text{ as } \lambda \to \infty, \]

\[ \frac{K_{\theta_r}^{r+\lambda}(x)}{(r + \lambda) \psi_{r+\lambda}(x)} = \int_0^x \frac{\psi_{r+\lambda}(z)}{\psi_{r+\lambda}(x)}(\theta_r(z) - \theta_r(x))m'(z)dz \to 0 \text{ as } \lambda \to \infty. \]

\[ \square \]
It is worth mentioning that if the process evolves in the interval $(x,y)$ the functional $H(x, y)$ can be represented by using the stationary distribution of the diffusion. In other words, if $m(x,y) < \infty$, the limiting value $X_\infty$ is distributed according to the stationary measure which is characterized by the density (see \[8\] pp. 36-38)

$$p_{x,y}(z) = \frac{m'(z)}{m(x,y)}.$$ 

Thus, we find a representation

$$H(x, y) = \left[ \mathbb{E}[\pi_\mu(X_\infty)] - \pi_\mu(x) \right] m(x,y).$$

In the control problems, that we introduce in the next section, the optimal thresholds are solutions to equations that include the functionals $L_{\theta_r}(x)$, $K_{\theta_r}(x)$ and their ergodic counterpart $H(x, y)$. Thus, it is natural, that to ensure the existence and uniqueness of the control boundaries, we must study the shape of these functions. These are for completeness stated in the next lemma. The proofs can be found in \[15\] lemma 3.3 for $L$, \[4\] lemma 3.1 for $K$ and \[16\] lemma 2 for $H$.

**Lemma 2.** Let the assumption 1 hold and let $\lambda \geq 0$. Then there exists a unique $\hat{x}_\lambda < x^*$ such that

$$L_{\theta_r}^{r+\lambda}(x) \leq 0, \text{ when } x < \hat{x}_\lambda.$$ 

Also there exists a unique $\tilde{x}_\lambda > x^*$ such that

$$K_{\theta_r}^{r+\lambda}(x) \leq 0, \text{ when } x > \tilde{x}_\lambda.$$ 

Similarly, there exists a unique $\hat{x}$ and a unique $\tilde{x}$ such that

$$H(x, \infty) \leq 0, \text{ when } x < \hat{x},$$

$$H(0, x) \leq 0, \text{ when } x > \tilde{x}.$$ 

**Remark 1.** Define the functions $J_\lambda, I_\lambda : \mathbb{R}_+ \to \mathbb{R}$

$$J_\lambda(x) = \frac{(R_{r+\lambda} \pi_\gamma)'(x) - \gamma}{\varphi_{r+\lambda}'(x)}, \quad I_\lambda(x) = \frac{(R_{r+\lambda} \pi_\gamma)'(x) - \gamma}{\psi_{r+\lambda}'(x)}.$$

We can show by a straightforward differentiation that for $\lambda \geq 0$

$$J_\lambda'(x) \leq 0, \text{ when } x < \hat{x}_\lambda, \quad I_\lambda'(x) \leq 0, \text{ when } x > \tilde{x}_\lambda.$$
4. The control problems

Before stating the control problems, we define the auxiliary functions $\pi_{\gamma} : \mathbb{R}_+ \to \mathbb{R}$ and $g : \mathbb{R}_+ \to \mathbb{R}$ as

$$g(x) = \gamma x - (R_r \pi)(x),$$

$$\pi_{\gamma}(x) = \lambda \gamma x + \pi(x).$$

The next lemma gives convenient relationships between these auxiliary functions. This lemma helps to rewrite the optimality condition of the discounted control problem with constraint, so that we can apply the results from section 2. The lemma can be proved by using the resolvent equation (4) and the harmonicity property $(A - r)(R_r \pi)(x) + \pi(x) = 0$.

**Lemma 3.** Let $r > 0$ and $g, \pi_{\gamma}, \theta_r \in L^r_1$. Then

\begin{align*}
(9) \quad (A - r)g(x) &= \theta_r(x), \\
(10) \quad (R_r + \lambda \pi_{\gamma})(x) &= \lambda (R_r + \lambda g)(x) + \pi(x), \\
(11) \quad \lambda (R_r + \lambda g)(x) &= (R_r + \lambda \theta_r)(x) + g(x).
\end{align*}

We now recall results on downward singular control of one-dimensional diffusions and on similar problems where controlling is allowed only at exogenously given Poisson arrival times. We also assume that the Poisson process is independent of the diffusion. We refer to this latter problem as a problem with constraint.

We assume in all of the following problems below that the controlled dynamics are given by the stochastic differential equation

$$X_t^D = \mu(X_t^D)dt + \sigma(X_t^D)dW_t - dD_t, \quad X_0^D = x \in \mathbb{R}_+, \quad D_t$$

where $D_t$ denotes the applied control policy. In the singular problems (theorems 1 and 2), we call a control policy admissible, if it is non-negative, non-decreasing, right-continuous, and $\{F_t\}_{t \geq 0}$-adapted, and denote the set of admissible controls by $D_s$. On the other hand, in the problems with constraint the set of admissible controls $D$ is given by those non-decreasing, left-continuous processes $D_{t \geq 0}$ that have the representation

$$D_t = \int_{[0, t]} \eta_s dN_s,$$

where $N$ is the signal process and the integrand $\eta$ is $\{F_t\}_{t \geq 0}$-predictable.
Under the assumptions 1, and in the just presented framework, the optimal policy in the singular control problems will be a local time barrier policy. In other words, when the process is below some boundary $y_s^*$ the process should be left uncontrolled but it should never be allowed to cross it, i.e. it is reflected at $y_s^*$. The situation in the problems with constraint is similar: when the process is below some threshold $y^*$ we do not act, but if the process crosses the boundary, and the Poisson process jumps, we immediately push it down to $y^*$ and start it anew. We will introduce these problem in more detail below but refer to [1, 4, 6, 5, 15, 16] for more details.

**Theorem 1** (Singular control with discounted criteria [1, 4]). Under the assumptions 1, the optimal control policy minimizing the objective

$$J(x, D^*) = \mathbb{E}_x \left[ \int_0^T e^{-rt}(\pi(X_t^{D^*})dt + \gamma dD_t) \right]$$

is characterized by the unique solution to the equation

$$K_{\theta}^r (y_s^*) = 0.$$  

Moreover, the value for the problem reads as

$$V_s(x) := \inf_{D \in D^*} J(x, D) = \begin{cases} 
\gamma x + \frac{\theta_r(y_s^*)}{r}, & x \geq y_s^* \\
(R_r \pi)(x) - \psi_r(x) \frac{(R_r \pi)'(y_s^*) - \gamma}{\psi_r'(y_s^*)}, & x < y_s^*.
\end{cases}$$

**Theorem 2** (Singular control with ergodic criteria [6, 5]). Under the assumptions 1, the optimal control policy minimizing the objective

$$J_e(x, D^*) = \lim_{T \to \infty} \frac{1}{T} \mathbb{E}_x \left[ \int_0^T (\pi(X_t^{D^*})dt + \gamma dD_t) \right]$$

is characterized by the unique solution to the equation

$$H(0, b^*) = 0.$$  

Moreover, the long run average cumulative yield reads as

$$\beta_s := \inf_{D \in D^*} J_e(x, D) = \pi_\mu(b_s^*).$$

**Theorem 3** (Control with discounted criteria and constraint [15]). Under the assumptions 1, the optimal control policy minimizing the objective

$$J(x, D) = \mathbb{E}_x \left[ \int_0^T e^{-rt}(\pi(X_t^D)dt + \gamma dD_t) \right]$$

is characterized by the unique solution to the equation
\[ \psi_r(y^*)L_{\theta_r}^{\tau+\lambda}(y^*) = g'(y^*)L_{\psi_r}^{\lambda}(y^*), \]
which can be rewritten as
\[ \psi_r(y^*)L_{\theta_r}^{\tau+\lambda}(y^*) = -\varphi_{r+\lambda}(y^*)K_{\theta_r}^{\tau}(y^*). \]  

In addition, The value \( V(x) := \inf_{D \in D^x} J(x, D) \) for the problem reads as
\[ V(x) = \begin{cases} 
\gamma x + (R_{r+\lambda} \theta_r)(x) - \frac{(R_{r+\lambda} \theta_r)'(y^*)}{\varphi_{r+\lambda}^{\prime}(y^*)} \varphi_{r+\lambda}(x) + A(y^*), & x \geq y^* \\
\gamma x + (R_r \theta_r)(x) - \psi_r(x)\frac{(R_r \theta_r)'(y^*)}{\varphi_r^{\prime}(y^*)}, & x < y^* 
\end{cases} \]
where
\[ A(y^*) = \frac{\lambda}{r} \left[ (R_{r+\lambda} \theta_r)(y^*) - (R_{r+\lambda} \theta_r)'(y^*) \frac{\varphi_{r+\lambda}'(y^*)}{\varphi_{r+\lambda}^{\prime}(y^*)} \right]. \]

Proof: We only prove that the optimality condition can be rewritten as (14), and refer to [15] for the rest of the claim. To prove the representation, we first use the lemma 1 and then the formulas (9) and (11), to get
\[ \frac{2\lambda S'(y^*)}{\sigma^2(y^*)} \left[ \psi_r'(y^*)L_{\theta_r}^{\tau}(y^*) - g'(y^*)L_{\psi_r}^{\lambda}(y^*) \right] = \psi_r'(y^*)(\varphi_{r+\lambda}'(y^*)\lambda(R_{r+\lambda} g)'(y^*) - \varphi_{r+\lambda}'(y^*)\lambda(R_{r+\lambda} g)''(y^*)) - g'(y^*)(\varphi_{r+\lambda}'(y^*)\psi_r'(y^*) - \varphi_{r+\lambda}'(y^*)\varphi_r''(y^*)) = \psi_r'(y^*)(\varphi_{r+\lambda}'(y^*)(R_{r+\lambda} \theta_r)'(y^*) - \varphi_{r+\lambda}'(y^*)(R_{r+\lambda} \theta_r)''(y^*)) - \varphi_{r+\lambda}'(y^*)(\psi_r(y^*)(R_{r+\lambda} \theta_r)'(y^*) - \psi_r(y^*)(R_{r+\lambda} \theta_r)''(y^*)). \]

Utilizing the lemma 1 again, we see that the optimality condition has the form
\[ \psi_r'(y^*)L_{\theta_r}^{\tau+\lambda}(y^*) = -\varphi_{r+\lambda}'(y^*)K_{\theta_r}^{\tau}(y^*). \]

\[ \square \]

**Theorem 4** (Control with ergodic criteria and constraint [16]). Under the assumptions 1, the optimal control policy minimizing the objective
\[ J_\epsilon(x, D) = \lim_{T \to \infty} \frac{1}{T} \mathbb{E}_x \left[ \int_0^T (\pi(X_t^D)dt + \gamma dD_t) \right] \]
is characterized by the unique solution to the equation
\[ S'(b^*)m(0, b^*)L_{\pi^*}^{\lambda}(b^*) = -\varphi_\lambda(b^*)H(0, b^*). \]
Moreover, the long run average cumulative yield reads as

\[ \beta := \inf_{D \in D^*} J_e(x, D) = m(0, b^*)^{-1} \left[ \int_0^{b^*} \pi_\mu(z)m'(z)dz \right]. \]

**Remark 2.** The boundary classifications for the underlying diffusion can be relaxed in all of the above theorems. For example, in theorem 3 it can be shown that the results stays unchanged when the lower boundary is exit or killing, see [15] p. 5.

We are now ready to proof the main results of the paper.

**Proposition 2** (Asymptotics of the optimal thresholds). The optimal thresholds satisfy the following asymptotic results in terms of the intensity of the Poisson process

\[ y^* \xrightarrow{\lambda \to \infty} y^*_s, \quad b^* \xrightarrow{\lambda \to \infty} b^*_s, \]

and in terms of vanishing discount factor

\[ y^*_s \xrightarrow{r \to 0} b^*_s, \quad y^* \xrightarrow{r \to 0} b^*. \]

**Proof.** We prove the first and the last claim, as the proof of the second can be found in [10] proposition 3 and the proof of the third in [5] lemma 3.1. Define the functions

\[ G(x) = \psi'_r(x)L^{r+\lambda}_{\theta_r}(x) + \varphi'_{r+\lambda}(x)K^r_{\theta_r}(x), \]
\[ F(x) = S'(x)m(0, x)L^\lambda m(0, x) + \varphi'_r(x)H(0, x), \]

and let \( y^*_s, y^*, b^* \) be such that \( K(y^*_s) = 0, \ G(y^*) = 0 \) and \( F(b^*) = 0 \). Using these notations, the first claim can be re-expressed as

\[ \frac{G(y^*_s)}{\varphi_{r+\lambda}(y^*_s)} \to 0 \text{ as } \lambda \to \infty. \]

Now, utilizing the condition \( K(y^*_s) = 0 \), together with the lemma [1] we have that

\[ \frac{L^{r+\lambda}_{\theta_r}(y^*_s)}{\varphi_{r+\lambda}(y^*_s)} \to 0 \text{ as } \lambda \to \infty. \]

To prove the last claim, we first note as above that the claim is equivalent to

\[ \frac{G(b^*)}{\psi'_r(b^*)} \to 0 \text{ as } r \to 0. \]
Hence, utilizing (5), we get that
\[
G(b^*) = \lambda \int_{b^*}^{\infty} \psi_r(y^*) \varphi_{r+\lambda}(z) \theta_r(z) m'(z) dz
\]
\[
+ \frac{\varphi'_{r+\lambda}(b^*)}{S'(b^*)} \int_{b^*}^{\infty} \psi_r(z) \theta_r(z) m'(z) dz.
\]

By lemma 1 we have that
\[
\int_{b^*}^{\infty} \psi_r(z) \theta_r(z) m'(z) dz = K_{\theta_r}(b^*) + \theta_r(b^*) \int_{b^*}^{\infty} \psi_r(z) m'(z) dz
\]
\[
= \frac{K_{\theta_r}(b^*)}{\psi_r(b^*)} + \theta_r(b^*)
\]
\[
\overset{r \to 0}{\longrightarrow} \frac{H(0, b^*)}{m(0, b^*)} + \pi \mu(b^*).
\]

Thus, the claim follows from the assumption \( F(b^*) = 0 \). \( \square \)

Similar limiting results hold also for the corresponding values of the defined control problems. However, it is clear that in terms of the vanishing discounting factor, the results hold only in the following Abelian sense.

**Proposition 3** (Asymptotics of the values). The values of the control problems satisfy the following asymptotic results
\[
V(x) \overset{\lambda \to \infty}{\longrightarrow} V_s(x), \quad \beta \overset{\lambda \to \infty}{\longrightarrow} \beta_s.
\]

Also, we have the following Abelian limits
\[
rV(x) \overset{r \to 0}{\longrightarrow} \beta, \quad rV_s(x) \overset{r \to 0}{\longrightarrow} \beta_s.
\]

**Proof.** For the last claim see lemma 3.1 of [5]. To prove the third, we first re-write the value function (15) using lemma 3 as
\[
V(x) = \begin{cases} 
\gamma x + (R_{r+\lambda} \theta_r)(x) - \frac{(R_{r+\lambda} \theta_r)'(y^*)}{\varphi_{r+\lambda}(y^*)} \varphi_{r+\lambda}(x) + A(y^*), & x \geq y^*, \\
\gamma x + \psi_r(x) \left[ \frac{(R_{r} \theta_r)(x)}{\psi_r(x)} - \frac{(R_{r} \theta_r)'(y^*)}{\psi_r(y^*)} \right], & x < y^*.
\end{cases}
\]
where
\[
A(y^*) = \frac{\lambda}{r} \left[ (R_{r+\lambda} \theta_r)(y^*) - (R_{r+\lambda} \theta_r)'(y^*) \frac{\varphi_{r+\lambda}(y^*)}{\varphi'_{r+\lambda}(y^*)} \right].
\]
We notice that when \( x > y^* \) the value function \( rV(x) \) has convenient presentation in terms of the limit \( r \to 0 \). However, when \( x < y^* \) we have to proceed as follows. Because \( V(x) \) is continuous across the boundary \( y^* \), we find

\[
(r + \lambda)\psi_r'(y^*)(\varphi'_{r+\lambda}(y^*)(R_{r+\lambda}\theta_r)(y^*) - \varphi_{r+\lambda}(y^*)(R_{r+\lambda}\theta_r)'(y^*))
\]

\[
= r\varphi'_{r+\lambda}(y^*)(\psi_r'(y^*)(R_{r}\theta_r)(y^*) - \psi_r(y^*)(R_{r}\theta_r)'(y^*))
\]

which can be re-organized as

\[
- r\frac{(R_r\theta_r)'(y^*)}{\psi_r(y^*)} + \frac{(R_r\theta_r)(y^*)}{\psi_r(y^*)}
= (r + \lambda) \left( \frac{\varphi'_{r+\lambda}(y^*)(R_{r+\lambda}\theta_r)(y^*) - \varphi_{r+\lambda}(y^*)(R_{r+\lambda}\theta_r)'(y^*)}{\varphi'_{r+\lambda}(y^*)\psi_r(y^*)} \right).
\]

Thus, we get that

\[
r\gamma x + r\psi_r(x) \left[ \frac{(R_r\theta_r)(x)}{\psi_r(x)} - \frac{(R_r\theta_r)'(y^*)}{\psi_r(y^*)} \right] = r\gamma x + r(R_r\theta_r)(x) - r\frac{\psi_r(x)}{\psi_r(y^*)}(R_r\theta_r)(y^*)
\]

\[
+ (r + \lambda) \frac{\psi_r(x)}{\psi_r(y^*)} \left( \frac{\varphi'_{r+\lambda}(y^*)(R_{r+\lambda}\theta_r)(y^*) - \varphi_{r+\lambda}(y^*)(R_{r+\lambda}\theta_r)'(y^*)}{\varphi'_{r+\lambda}(y^*)} \right).
\]

Using the formula (3) and (7), we see that

\[
r(R_r\theta_r)(x) = r\varphi_r(x) \int_0^x \psi_r(z)\theta_r(z)m'(z)dz + \psi_r(x) \int_x^\infty \varphi_r(z)\theta_r(z)m'(z)dz
\]

\[
= \varphi_r(x) \int_0^x \psi_r(z)\theta_r(z)m'(z)dz + \varphi_r(x) \int_x^\infty \varphi_r(z)\theta_r(z)m'(z)dz
\]

\[
= \int_0^x \psi_r(z)\theta_r(z)m'(z)dz + \int_x^\infty \varphi_r(z)\theta_r(z)m'(z)dz
\]

\[
r \to 0, \int_0^\infty \varphi_r(z)\theta_r(z)m'(z)dz
\]

and thus, by (7) we have

\[
r(R_r\theta_r)(x) - r\frac{\psi_r(x)}{\psi_r(y^*)}(R_r\theta_r)(y^*) \to 0
\]

Therefore, by continuity and (2), the value function satisfies

\[
rV(x) \to 0 \begin{cases} 
\lambda \frac{\varphi'_x(b^*)(R_x\pi_x)(b^*) - \varphi_x(b^*)(R_x\pi_x)'(b^*)}{\varphi'_x(b^*)}, & x \geq b^* \\
\lambda \frac{\varphi'_x(b^*)(R_x\pi_x)(b^*) - \varphi_x(b^*)(R_x\pi_x)'(b^*)}{\varphi'_x(b^*)}, & x < b^*. 
\end{cases}
\]
Finally, utilizing (3), the limiting value reads as
\[-\frac{\lambda S'(b^*)}{\varphi'(b^*)} \int_{b^*}^{\infty} \pi_\mu(z) \varphi_\lambda(z)m'(z)dz,
\]
which completes the proof on the third claim.

To prove the second claim, we notice that the value function $V(x)$ is independent of $\lambda$ when $x < y^*$. Thus, we focus this time on the region $x > y^*$. We re-organize the terms $V(x)$ in the upper region as
\[
\gamma x + (R_{r+\lambda} \theta_r)(x) - \frac{(R_{r+\lambda} \theta_r)'(y^*)}{\varphi_{r+\lambda}(y^*)} \varphi_{r+\lambda}(x) + A(y^*)
\]
where
\[
A(y^*) = \frac{\lambda}{r} \left[ (R_{r+\lambda} \theta_r)(y^*) - (R_{r+\lambda} \theta_r)'(y^*) \frac{\varphi_{r+\lambda}(y^*)}{\varphi_{r+\lambda}'(y^*)} \right].
\]

Because diffusions are Feller-processes, we know that $\lambda(R_{r+\lambda} \theta_r) \to \theta_r$ as $\lambda \to \infty$ (in sup-norm), see [21] pp. 235. Thus,
\[
\gamma x + \frac{\lambda(R_{r+\lambda} \theta_r)(x)}{\lambda} + \lambda r \frac{(R_{r+\lambda} \theta_r)(y^*)}{r} \xrightarrow{\lambda \to \infty} \gamma x + \frac{\theta_r(y^*)}{r}.
\]

To deal with the remaining terms in (17), we note that by (18)
\[
\frac{(R_{r+\lambda} \theta_r)'(y^*)}{\varphi_{r+\lambda}(y^*)} = \frac{(R_{r+\lambda} \theta_r)(y^*)}{\varphi_{r+\lambda}(y^*)} - \frac{r}{r + \lambda} \frac{(R_{r} \theta_r)(y^*)}{\varphi_{r+\lambda}(y^*)} + \frac{r}{r + \lambda} \psi_r(y^*) \frac{(R_{r} \theta_r)'(y^*)}{\varphi_{r+\lambda}(y^*)}.
\]

Utilizing the above we get by (7)
\[
\frac{(R_{r+\lambda} \theta_r)'(y^*)}{\varphi_{r+\lambda}(y^*)} - \varphi_{r+\lambda}(x)
\]
\[
= \frac{\varphi_{r+\lambda}(x)}{\varphi_{r+\lambda}(y^*)} (R_{r+\lambda} \theta_r)(y^*) - \frac{r}{r + \lambda} \varphi_{r+\lambda}(x) (R_{r} \theta_r)(y^*) + \frac{r}{r + \lambda} \psi_r(y^*) \varphi_{r+\lambda}(x) (R_{r} \theta_r)'(y^*)
\]
\[
\xrightarrow{\lambda \to \infty} 0
\]
Figure 1. Relations between the control problems. These relations hold for the optimal thresholds and also for the values, in the sense of propositions 1 and 2.

and

$$\frac{\lambda (R_r + \lambda \theta_r)'(y^*)}{r} \varphi_r + \lambda (y^*) = \lambda \theta_r(y^*) - \frac{\lambda}{r + \lambda} (R_r \theta_r)(y^*) + \frac{\lambda}{r + \lambda} \psi_r(y^*) (R_r \theta_r)'(y^*)$$

As the value function $V_s(x)$ is continuous over the boundary $y_s^*$, we further find that

$$\frac{\theta_r(y_s^*)}{r} - \frac{\psi_r(y_s^*)}{\psi_r'(y_s^*)} (R_r \theta_r)'(y_s^*) = 0.$$

Combining the above limits the result follows by continuity and proposition 2.

Lastly, the second claim of the proposition follows by continuity and 2 as $\beta_s$ can also be represented as (see [6], pp. 17)

$$\beta_s = m(0, b_s^*)^{-1} \left[ \int_0^{b_s^*} \pi_\mu(z) m'(z) dz \right].$$

5. Illustration

5.1. Brownian motion with drift. Let the underlying process $X_t$ be defined by

$$dX_t = \mu dt + dW_t, \quad X_0 = x,$$

where $\mu > 0$. Also, we let the process evolve in $\mathbb{R}$ and choose a quadratic running cost $\pi(x) = x^2$. The minimal excessive functions are in
this case known to be
\[ \varphi_\lambda(x) = e^{-\left(\sqrt{\mu^2 + 2\lambda + \mu}\right)x}, \quad \psi_\lambda(x) = e^{\left(\sqrt{\mu^2 + 2\lambda - \mu}\right)x}, \]

and the scale density and speed measure read as
\[ S'(x) = \exp(-2\mu x), \quad m'(x) = 2 \exp(2\mu x), \]
respectively. The net convenience yield now takes the form \( \theta(x) = x^2 + \gamma(\mu - rx) \). Therefore, we notice immediately that our assumptions hold and so the results apply.

To illustrate the results of proposition 2, we solve the optimality conditions (12), (13), (14) and (16). Conveniently the solution to all of the equations can be represented explicitly. To solve the equations we need to find the functions \( H(0, x), m(0, x), K^r_{\theta_r}(x), L^r+\lambda_{\theta_r}(x) \) and \( L^\lambda_{\pi_r}(x) \). Elementary integration yield
\[
H(0, x) = \frac{e^{2\mu(1 - 2x\mu)}}{2\mu^3}, \\
m(0, x) = \frac{1}{\mu} e^{2x\mu}, \\
K^r_{\theta_r}(x) = \frac{2e^{x\alpha_+^+} (2 + (-2x + r\gamma)\alpha_+^+)}{(\alpha_+^+)^3}, \\
L^r+\lambda_{\theta_r}(x) = \frac{2e^{x\alpha_-^-} (2 + (2x - r\gamma)\alpha_-^-)}{(\alpha_-^-)^3}, \\
L^\lambda_{\pi_r}(x) = \frac{4e^{x\alpha_-^-} (1 + x\alpha_-^-)}{(\alpha_-^-)^3},
\]

where \( \alpha_+^+ = \mu + \sqrt{2r + \mu^2} \) and \( \alpha_-^- = \mu - \sqrt{2r + \mu^2} \). Plugging these presentations to the equations (12), (13), (14) and (16), a simplification yields as solutions the thresholds
\[
y^*_s = \frac{r\gamma}{2} + \frac{1}{\alpha_+^+}, \quad b^*_s = \frac{1}{2\mu}, \\
y^* = \frac{r\gamma}{2} + \frac{1}{\alpha_+^+} + \frac{1}{\alpha_+^+ - \alpha_-^-}, \quad b^* = \frac{1}{2\mu} - \frac{\alpha_+^+}{2\mu}. 
\]
A direct calculation now shows that \( \alpha_+^+ \xrightarrow{r \to 0} 2\mu, \alpha_+^+ \xrightarrow{\lambda \to \infty} \infty \) and \( \alpha_+^+ / 2\lambda \xrightarrow{\lambda \to \infty} 0 \). Using these auxiliary limits, we get similar limits as in proposition 2. These threshold are further illustrated in the figures 2 and 3.
Figure 2. Threshold boundaries as a function of the discounting factor with the parameters $\gamma = 0.001$, $\mu = 0.1$ and $\lambda = 10$.

Figure 3. Threshold boundaries as a function of the intensity of the Poisson process with the parameters $\gamma = 0.001$, $\mu = 0.1$ and $r = 0.001$.

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