Linear independence criteria for generalized polylogarithms with distinct shifts

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Abstract

For a given rational number $x$ and an integer $s \geq 1$, let us consider a generalized polylogarithmic function, often called the Lerch function, defined by

$$\Phi_s(x, z) = \sum_{k=0}^{\infty} \frac{x^{k+1}}{(k + x + 1)^s}.$$ 

We prove the linear independence over any number field $K$ of the numbers $1$ and $\Phi_{s_j}(x_j, \alpha_i)$ with any choice of distinct shifts $x_1, \ldots, x_d$ with $0 \leq x_1 < \ldots < x_d < 1$, as well as any choice of depths $1 \leq s_1 \leq r_1, \ldots, 1 \leq s_d \leq r_d$, at distinct algebraic numbers $\alpha_1, \ldots, \alpha_m \in K$ subject to a metric condition. As is usual in the theory, the points $\alpha_i$ need to be chosen sufficiently close to zero with respect to a given fixed place $v_0$ of $K$, Archimedean or finite.

This is the first linear independence result with distinct shifts $x_1, \ldots, x_d$ that allows values at different points for generalized polylogarithmic functions. Previous criteria were only for the functions with one fixed shift or at one point.

Further, we establish another linear independence criterion for values of the generalized polylogarithmic function with cyclic coefficients. Let $q \geq 1$ be an integer and $a = (a_1, \ldots, a_q) \in K^q$ be a $q$-tuple whose coordinates supposed to be cyclic with the period $q$. Consider the generalized polylogarithmic function with coefficients

$$\Phi_{a,s}(x, z) = \sum_{k=0}^{\infty} \frac{a_{k+1 \mod (q)} \cdot x^{k+1}}{(k + x + 1)^s}.$$ 

Under suitable condition, we show that the values of these functions are linearly independent over $K$.

Our key tool is a new non-vanishing property for a generalized Wronskian of Hermite type associated to our explicit constructions of Padé approximants for this family of generalized polylogarithmic function.

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1 Introduction

Padé approximation (confer [20, 21]) is a classical method in diophantine problems and has been specially efficient in proving irrationality of special values of locally holomorphic functions as well as providing sharp estimates for the associated irrationality measures in
question. With a rich history and contributions from a large number of authors after the pioneering work of Ch. Hermite and H. Padé, the method has been extensively studied.

For values of polylogarithmic functions or generalized ones, E. M. Nikišin in [17, 18], A. I. Galochkin in [9] and [10], G. V. Chudnovsky in [2], K. Väänänen in [27] and W. Zudilin in [30] gave several linear independence criteria, either over the field of rational numbers, or quadratic imaginary fields, albeit with only one fixed shift or at one point. Nikišin succeeded in proving the linear independence of polylogarithms in [17] by using Padé approximation of type I, and considered a variant of type II in a more general setup [18]. Nikišin’s results were with the single shift equal to zero, corresponding \( r = s, m = 1 \) of our case, over the base field being the rational number field. Our approach is inspired by these contributions, together with that by G. Rhin and P. Toffin showing Padé approximation of type II for logarithms at different points [22].

Considering one fixed shift but allowing all other parameters to be free, that was also one main purpose of our previous papers [3, 4]. See these articles for further reference on earlier works and a historical survey in this context. Note that our linear independence result of values for the functions in the current paper does not automatically follow from the criterion of G. V. Chudnovsky in [2].

Our aim in the present article is to explore a largely uncharted territory of families of generalized polylogarithmic functions with total freedom on the shifts, on the points, on cyclic coefficients, and parameters considered. Note that M. Hata [11] dealt with the special case of evaluation at a single rational point and proved the linear independence over the rational number field of values of generalized polylogarithmic functions with distinct shifts, at one rational point. Our construction of Padé approximants was applied in the forthcoming article [12] where the authors established, in addition, an effective version of the Poincaré-Perron theorem to prove new irrationality measures.

It is a classical remark that the Riemann \( \zeta \) function is expressed as \( \zeta(s) = \Phi_s(0, 1) \). Similarly, the Dirichlet \( L \)-function is viewed as values at \( z = 1 \) of a generalized polylogarithmic function with periodic coefficients and the shift \( x = 0 \), namely for any given sequence \( a = (a_{k+1})_{k \geq 0} \) of algebraic numbers, we consider the twisted polylogarithmic function by Hadamard product \( \ast \) by \( a \):

\[
a \ast \Phi_s(x, z) = \sum_{k=0}^{\infty} \frac{a_{k+1} \cdot z^{k+1}}{(k + x + 1)^s}.
\]

We may come back to this point at a later stage.

The special case when the sequence \( a \) is actually periodic (say of period \( q \)) is of particular interest (and more so if \( a \) is an arithmetic function such as a Dirichlet character). The values at \( z = 1 \) of such functions also encompass the Dirichlet \( L \)-functions as remarked above. We refer to previous important works due to T. Rivoal [23], R. Marcovecchio [13], S. Fischler [6], Fischler-J. Sprang-Zudilin [8], Li Lai-Pin Yu [13], where lower bounds for the dimension of the vector space are proven, for those spanned by polylogarithms or special values of the Riemann zeta functions, or those of the \( L \)-functions with Dirichlet characters (see also Nishimoto’s work [19]). Hence neither the irrationality nor the linear independence of the values follows from their works. On the other hand, in our work, the point \( z \) needs to be taken close to the origin, thus for the values of the Riemann zeta function or the Dirichlet \( L \)-function, our method does not directly give the linear independence result.
2 Notations and Main results

Throughout the article, we denote by \( \mathbb{N} \) the set of strictly positive integers. Let \( K \) be a number field. We denote the set of places of \( K \) by \( \mathfrak{M}_K \) (respectively by \( \mathfrak{M}_K^\infty \) for archimedean places, by \( \mathfrak{M}_K^f \) for finite places). For \( v \in \mathfrak{M}_K \), we denote by \( K_v \) the completion of \( K \) with respect to \( v \), and by \( C_v \) the completion of the algebraic closure \( \overline{K}_v \) of \( K_v \).

For \( v \in \mathfrak{M}_K \), we define the normalized absolute value \( | \cdot |_v \) as follows:

\[
|p|_v = p \frac{|K_v : Q_p|}{|K : Q|} \quad \text{if} \quad v \in \mathfrak{M}_K^f \quad \text{and} \quad v | p ,
\]

\[
|x|_v = |\epsilon_v(x)| \frac{|K_v : R|}{|K : Q|} \quad \text{if} \quad v \in \mathfrak{M}_K^\infty ,
\]

where \( p \) is a rational prime and \( \epsilon_v \) the embedding \( K \hookrightarrow \mathbb{C} \) corresponding to \( v \). The norm \( \| \cdot \|_v \) denotes the norm of the supremum in \( K_v^\ast \). With these normalizations, the product formula reads

\[
\prod_{v \in \mathfrak{M}_K} |\xi|_v = 1 \quad \text{for} \quad \xi \in K \setminus \{0\} .
\]

Let \( m \in \mathbb{N} \) and \( \beta := (\beta_0, \ldots, \beta_m) \in \mathbb{P}_m(K) \). We define the absolute Weil height of \( \beta \) by

\[
H(\beta) = \prod_{v \in \mathfrak{M}_K} \max\{|\beta_0|_v, \ldots, |\beta_m|_v\} ,
\]

and denote the logarithmic absolute Weil height by \( h(\beta) = \log H(\beta) \). For each \( v \in \mathfrak{M}_K \), put \( h_v(\beta) = \log \| \beta \|_v \). Then we have \( h(\beta) = \sum_{v \in \mathfrak{M}_K} h_v(\beta) \). Finally viewing \( \beta \in K \hookrightarrow \mathbb{P}_1(K) \) via the natural embedding \( \beta \longmapsto (1 : \beta) \), we have

\[
H(\beta) = \prod_{v \in \mathfrak{M}_K} \max\{1, |\beta|_v\} .
\]

Let \( S \subset \overline{\mathbb{Q}} \) be a finite set. Define the denominator of this finite set \( S \) by

\[
\text{den}(S) = \min\{1 \leq n \in \mathbb{Z} \mid \text{no is an algebraic integer for each} \ \alpha \in S\} .
\]

For a place \( v \in \mathfrak{M}_K \) and \( f(z) = \sum_{k=0}^\infty f_k/z^{k+1} \in (1/z) \cdot K[[1/z]] \), we denote the embedding corresponding to \( v \) by \( \epsilon_v : K \hookrightarrow K_v \subset C_v \) and write \( f_v(z) = \sum_{k=0}^\infty \epsilon_v(f_k)/z^{k+1} \in (1/z) \cdot C_v[[1/z]] \). We then consider \( f_v(z) \) as a \( v \)-adic analytic function inside its disk of convergence.

Consider the generalized polylogarithmic function

\[
\Phi_s(x, z) = \sum_{k=0}^\infty \frac{z^{k+1}}{(k + x + 1)^s} ,
\]

well defined for rational number \( x \) which is not negative integer and for any integer \( s \geq 1 \) inside its disk of convergence with the above conventions.

We are now in the position to introduce all the notations necessary to state our main result. Let \( m, d, r_1, \ldots, r_d \in \mathbb{N} \) and \( \mathbf{x} := (x_1, \ldots, x_d) \in \mathbb{Q}^d \) satisfying \( x_i - x_j \notin \mathbb{Z} \) for \( 1 \leq i < j \leq d \) where we may assume \( 0 \leq x_d < \cdots < x_1 < 1 \) without loss of generality. Let \( \alpha := (\alpha_1, \ldots, \alpha_m) \in (K \setminus \{0\})^m \) whose coordinates being pairwise distinct algebraic
numbers. Put \( \rho = \sum_{j=1}^{d} r_{j} \). Let \( v_{0} \) be a place of \( K \), \( \beta \in K \) with \( \| \alpha \|_{v_{0}} < \| \beta \|_{v_{0}} \). Put \( b = \max_{1 \leq j \leq d} \text{den}(x_{j}) \). For a rational number \( x \), let us define

\[
\mu(x) = \text{den}(x) \prod_{q \text{ prime}} q^{1/(q-1)},
\]

and write

\[
V(\alpha, x, \beta) = V_{v_{0}}(\alpha, x, \beta) = \log \| \beta \|_{v_{0}} - \rho m \text{h}(\alpha, \beta) - \rho m \log \| \alpha \|_{v_{0}} + \rho m \log \| (\alpha, \beta) \|_{v_{0}} - \sum_{j=1}^{d} r_{j} \log \mu(x_{j})
\]

\[
- \left[ \rho m \log(2) + \rho \left( \log(\rho m + 1) + \rho m \log \left( \frac{\rho m + 1}{\rho m} \right) \right) \right] - \max_{j} (r_{j}) \cdot b \rho m.
\]

Under the notations above, we give the following linear independence criterion first in a simple manner, for values of the generalized polylogarithmic functions with distinct shifts and several depths at different algebraic points.

**Theorem 2.1.** Assume \( V(\alpha, x, \beta) > 0 \). Then the \( \rho m + 1 \) numbers:

\[
1, \Phi_{s_j}(x_{j}, \alpha_{i}/\beta) \quad (1 \leq i \leq m, \ 1 \leq j \leq d, \ 1 \leq s_j \leq r_{j}),
\]

are linearly independent over \( K \).

**Remark 2.2.** Our approach is entirely effective, indeed, it is possible to give effective irrationality measures and effective linear independence measures. For details, see Theorem 5.5, the more complete, albeit more technical statement. We should also mention that our main theorem does not immediately follow from the statement nor the proof in [5], since the case with distinct shifts in our current setup cannot be covered by the argument in [5].

**Remark 2.3.** Fix \( \alpha, x \) and assume \( |\beta|_{v_{0}} > \max(1, \| \alpha \|_{v_{0}}) \). Regarding \( \beta \) as a variable, the number

\[
\log |\beta|_{v_{0}} - \rho m \text{h}(\alpha, \beta) + \rho m \log \| (\alpha, \beta) \|_{v_{0}} = \log |\beta|_{v_{0}} - \rho m \sum_{v \neq v_{0}} h_{v}(\alpha, \beta)
\]

is only the part of \( V(\alpha, x, \beta) \) depending on \( \beta \), and other terms do not depend on \( \beta \). Whenever the number

\[
\log |\beta|_{v_{0}} - \rho m \sum_{v \neq v_{0}} h_{v}(\alpha, \beta)
\]

is sufficiently large, then the condition \( V(\alpha, x, \beta) > 0 \) is verified. On the analytic side, we have tried to keep computations no too technical, at the cost here and there of optimality.

We should note that the condition \( V(\alpha, x, \beta) > 0 \) indeed depends on the field \( K \) (see the proof of Corollary below).

**Corollary 2.4.** Let \( K \) be an algebraic number field and \( v_{0} \) be a place of \( K \). Under the same notations as in Theorem 2.1, there exist infinitely many \( \beta \in K \) where the \( \rho m + 1 \) numbers:

\[
1, \Phi_{s_j}(x_{j}, \alpha_{i}/\beta) \quad (1 \leq i \leq m, \ 1 \leq j \leq d, \ 1 \leq s_j \leq r_{j}),
\]

are linearly independent over \( K \).
Proof.
We fix a finite set of places \( \mathfrak{M} \) of \( K \) of cardinality, say \( h \geq 2 \), with \( v_0 \notin \mathfrak{M} \). We denote \( \mathfrak{M} = \{v_1, \ldots, v_h\} \). Let \( \varepsilon \) be a positive real number with \( \varepsilon < 1/2 \). By strong approximation theorem (confer \[14, \text{Theorem 1.2}\]), there exists an element \( \beta = \beta(\varepsilon) \in K \) which depends on \( \varepsilon \) with
\[
|\beta - 1|_{v_1} < \varepsilon, \quad |\beta|_{v_i} < \varepsilon \quad \text{for} \quad 2 \leq i \leq h, \quad |\beta|_v \leq 1 \quad \text{for} \quad v \notin \mathfrak{M} \cup \{v_0\}.
\]
By the condition \( |\beta - 1|_{v_1} < \varepsilon \), we have \( \beta \neq 0 \) and \( |\beta|_{v_i} \leq 2 \). We use the product formula for \( \beta \), to obtain
\[
1 = |\beta|_{v_1} \cdot \prod_{i=2}^{h} |\beta|_{v_i} \cdot |\beta|_{v_0} \cdot \prod_{v \notin \mathfrak{M} \cup \{v_0\}} |\beta|_v \leq 2^{h-1}|\beta|_{v_0}.
\]
Thus we conclude
\[
1 < \frac{1}{2^{h-1}} \leq |\beta|_{v_0}.
\]
Using the inequalities above, we have
\[
\log |\beta|_{v_0} - \rho m \sum_{v \neq v_0} h_v(\alpha, \beta) \geq \log |\beta|_{v_0} - \rho m (h(\alpha) + \log(2))
\]
\[
\geq \log \left( \frac{1}{2^{h-1}} \right) - \rho m (h(\alpha) + \log(2))
\]
\[
\geq -(h-1)\log(\varepsilon) - (\rho m + 1)(h(\alpha) + \log(2)).
\]
Take a real number \( \varepsilon_1 \) with \( 0 < \varepsilon_1 < 1/2 \) to have
\[
(h-1)\log(\varepsilon_1) < -\rho m \log \|\alpha\|_{v_0} - \sum_{j=1}^{d} r_j \log x_j - \max_{j} (r_j) \rho m - (\rho m + 1)(h(\alpha) + \log(2))
\]
\[
- \left[ \rho m \log(2) + \rho \left( \log(\rho m + 1) + \rho m \log \left( \frac{\rho m + 1}{\rho m} \right) \right) \right],
\]
and \( \beta_1 := \beta(\varepsilon_1) \) satisfying the inequalities \((2.1)\) for \( \varepsilon_1 \). Then, by the definition of \( V(\alpha, x, \beta_1) \) and \((2.2)\), we have \( V(\alpha, x, \beta_1) > 0 \). Inductively, we take the sequences \( (\varepsilon_k) \in \mathbb{N} \) and \( (\beta_k) \in K^N \) with \( \varepsilon_{k+1} = \min(|\beta_k - 1|_{v_1}, |\beta_k|_{v_i}) \) for \( 2 \leq i \leq h \) and \( \beta_{k+1} = \beta(\varepsilon_{k+1}) \). Then we have \( \cdots < \varepsilon_{k+1} < \varepsilon_k < \cdots < \varepsilon_1, \beta_{k_1} \neq \beta_{k_2} \) for any \( k_1 \neq k_2 \) and \( V(\alpha, x, \beta_k) > 0 \) for any \( k \). By Theorem \((2.1)\) for any \( \beta_k \), the \( \rho m + 1 \) numbers:
\[
1, \Phi_{s_j}(x_j, \alpha_i/\beta_k) \quad (1 \leq i \leq m, \quad 1 \leq j \leq d, \quad 1 \leq s_j \leq r_j),
\]
are linearly independent over \( K \).

Now we turn to the situation of generalized polylogarithmic functions. It is indeed useful to put them inside the more general class of such functions of which they only happen to be a special case, and treat altogether the general case. Let \( q \geq 1 \) be an integer, \( b(z) \in K[z] \) be a polynomial of degree \( q \) and let finally \( w = w(z) \) be a polynomial of degree \( \leq q-1 \) in \( K[z] \). Let \( f_{b,w}(z) = \sum_{k=0}^{\infty} b_{w,k}/z^{k+1} \) be the power series expansion of the rational function \( w(z)/b(z) \).

Then, for \( x \in \mathbb{Q} \) which is not negative integer and for \( s \geq 0 \), the associated generalized polylogarithmic function with coefficients is defined by:
\[
f_{b,w,x,s}(z) = \sum_{k=0}^{\infty} \frac{b_{w,k} \cdot z^{-k-1}}{(k + x + 1)^s}.
\]
Note that for \( s = 0 \), \( f_{b,w,x,0} \) is just the rational function \( f_{b,w} \).

**Example 2.5.** In the special case where \( b(z) := z^q - \alpha^q \) and \( w(z) := \sum_{i=1}^q a_i \alpha^i z^q - t \), the function above has the form below with periodic coefficients:

\[
f_{b,w,x,s}(z) = \sum_{k=0}^{\infty} \frac{a_{k+1}}{(k + x + 1)^s} \alpha^{k+1} \text{ with } a_{k+1} = a_{t+1} \text{ if } k \equiv l \mod (q),
\]

and we indeed recover the statement of Theorem 2.1 for all such functions as \( f_{b,w,x,s}(z) \) with periodic coefficients, instead of \( \Phi_1(x,1/z) \).

Note that by the definition of \( f_{b,w,x,s}(z) \), we have

\[
b(z) \left( -z \frac{d}{dz} + x \right)^s (f_{b,w,x,s}(z)) = w(z) \in K[z].
\]

We now state our result for generalized polylogarithmic functions with periodic coefficients.

**Theorem 2.6.** Let \( m, d \geq 1 \). Assume that we are given \( r_1, \ldots, r_d \) integers \( \geq 1 \), put \( \rho = \sum_{j=1}^d r_j \). Let \( b(z) \in K[z] \) be a polynomial of degree \( m \) with simple roots \( \alpha_1, \ldots, \alpha_m \in K \). Let \( x_1, \ldots, x_d \in \mathbb{Q} \) such that for \( i \neq j \), \( x_i - x_j \notin \mathbb{Z} \). Let moreover \( w_0(z), \ldots, w_m-1(z) \) be \( K \)-linearly independent polynomials in \( K[z] \) of degree \( \leq m - 1 \). Let finally \( v_0 \) be a place of \( K \) and \( \beta \in K \) such that \( \|\alpha\|_{v_0} < |\beta|_{v_0} \).

Assume \( V(\alpha, x, \beta) > 0 \) where \( V(\alpha, x, \beta) \) is defined in Theorem 2.1. Then, the \( \rho m + 1 \) numbers, \( 1, f_{b,w_i,x_j,s}(\beta) \) with \( 0 \leq i \leq m - 1, 1 \leq j \leq d \) and \( 1 \leq s \leq r_j \) are linearly independent over \( K \).

We now state more precisely a criterion as a corollary for the class of functions of arithmetic interest: the generalized polylogarithmic functions with periodic coefficients.

**Corollary 2.7.** Let \( K \) be a number field and \( v_0 \) be a place of \( K \). Let \( m, d, q_1, \ldots, q_m, r_1, \ldots, r_d \in \mathbb{N} \). Put \( \rho = \sum_{j=1}^d r_j \). Let \( \alpha := (\alpha_1, \ldots, \alpha_m) \in (K \setminus \{0\})^m \) and \( \beta \in K \setminus \{0\} \) such that for \( i \neq j \), \( \alpha_i \neq \alpha_j \) and \( \|\alpha\|_{v_0} > |\beta|_{v_0} \). Let \( x_1, \ldots, x_d \in \mathbb{Q} \) such that for \( i \neq j \), \( x_i - x_j \notin \mathbb{Z} \). Let moreover \( b_i(z) := (z^{q_i} - \alpha^{q_i}) \) and \( w_{i,0}(z), \ldots, w_{i,q_m-1}(z) \) be \( K \)-linearly independent polynomials in \( K[z] \) of degree \( \leq q_i - 1 \) for \( 1 \leq i \leq m \). Assume \( V(\alpha', x, \beta) > 0 \) where \( V(\alpha', x, \beta) \) is defined in Theorem 2.1 for

\[
\alpha' := (\alpha_1, \alpha_1 \zeta_{q_1}, \ldots, \alpha_1 \zeta_{q_1}^{q_1-1}, \ldots, \alpha_m, \alpha_m \zeta_{q_m}, \ldots, \alpha_m \zeta_{q_m}^{q_m-1}),
\]

where \( \zeta_{q_i} \) is a primitive \( q_i \)-th root of unity. Then, the \( (q_1 + \cdots + q_m) \rho + 1 \) numbers, \( 1, f_{b,w_i,x_j,s}(\beta) \) with \( 1 \leq i \leq m, 0 \leq l_i \leq q_i - 1, 1 \leq j \leq d \) and \( 1 \leq s \leq r_j \) are linearly independent over \( K \).

**Remark 2.8.** By applying Theorem 2.6 with the field \( \mathbb{Q}(\alpha_1, \ldots, \alpha_m, \beta, \zeta_{q_1}, \ldots, \zeta_{q_m}) \) instead of \( \mathbb{Q}(\alpha_1, \ldots, \alpha_m, \beta) \subseteq K \), this corollary holds. This follows thanks to the fact that the crucial non-vanishing property of the determinant in the proof does not depend on the choice of \( w_i(z) \). Similar observations can be made (specially in the \( p \)-adic case) and the linear independence can be achieved over some larger but specific fields, however, it should be stressed that these methods do not lead to the linear independence over the algebraic closure, neither the transcendence.
The present article is organized as follows. In Section 3 we recall the construction of Padé approximants for the generalized polylogarithmic functions as well as various elementary facts that are going to come in the course of the proof. Section 4 contains the crucial non-vanishing result. Although the basic structure of the proof is very classical and dates back to A. T. Vandermonde (separation of variables), its proof is quite involved and extremely demanding. It is thus divided in various subsections and the main steps are outlined at the beginning of the section (see Subsection 4.1). Section 5 is devoted to the proof of our main results after recalling the analytic estimates pertaining to Padé approximation for the generalized polylogarithmic functions. Once non-vanishing proof is achieved, this part proceeds essentially along the lines of our previous paper devoted to a single shift; see [4]. Indeed, the presence of shifts is not significant in analytic estimates: they just marginally impact the value of the function $V(\alpha, x, \beta)$. Finally, Section 6 is devoted to providing concrete examples of Theorem 2.1.

3 Preliminaries

3.1 Padé approximants

In this subsection, we explicitly construct Padé approximants of the generalized polylogarithmic functions with distinct shifts. These can be found in our previous paper [4], and are reproduced here for the convenience of the reader. First we recall the definition of Padé approximants of formal Laurent series. We denote by $K$ a field of characteristic $0$. We define the order function at $z = \infty$, $\text{ord}_\infty$, by

$$\text{ord}_\infty : K((1/z)) \rightarrow \mathbb{Z} \cup \{\infty\}; \quad \sum a_k z^{-k} \mapsto \min\{k \in \mathbb{Z} | a_k \neq 0\}.$$ 

We first recall without proof the following elementary fact:

**Lemma 3.1.** Let $r$ be a positive integer, $f_1(z), \ldots, f_r(z) \in (1/z) \cdot K[[1/z]]$ and $n := (n_1, \ldots, n_r) \in \mathbb{N}^r$. Put $N = \sum_{i=1}^r n_i$. Let $M$ be a positive integer with $M \geq N$. Then, there exists a family of polynomials $(P_0(z), P_1(z), \ldots, P_r(z)) \in K[z]^{r+1} \setminus \{0\}$ satisfying the following conditions:

(i) $\deg P_0(z) \leq M$,

(ii) $\text{ord}_\infty (P_0(z)f_j(z) - P_j(z)) \geq n_j + 1$ for $1 \leq j \leq r$.

**Definition 3.2.** We say that a $(r + 1)$-tuple of polynomials $(P_0(z), P_1(z), \ldots, P_r(z)) \in K[z]^{r+1}$ satisfying the properties (i) and (ii) is a weight $n$ and degree $M$ Padé type II system of approximants of $(f_1, \ldots, f_r)$. For such approximants $(P_0(z), P_1(z), \ldots, P_r(z))$, of $(f_1, \ldots, f_r)$, we call the $(r+1)$-tuple of formal Laurent series $(P_0(z)f_j(z) - P_j(z))_{1 \leq j \leq r}$, id est the remainders, as weight $n$ degree $M$ Padé type approximations of $(f_1, \ldots, f_r)$.

In the following, we fix $x \in K$ which is not negative integer. We now introduce notations for formal primitive, derivation, and evaluation maps.

**Notation 3.3.** (i) For $\alpha \in K$, we denote by $\text{Eval}_\alpha$ the linear evaluation map $K[t] \rightarrow K$, $P \mapsto P(\alpha)$. At a later stage, when several variables are in play and there is a perceived ambiguity on which variable is being specialized, we shall denote the map by $\text{Eval}_{t \rightarrow \alpha}$.
(ii) For \( P \in K[t] \), we denote by \([P]\) the multiplication by \( P \) (the map \( Q \mapsto PQ \)).

(iii) We also denote by \( \text{Prim}_x \) the linear operator \( K[t] \to K[t] \), defined by \( P \mapsto \frac{1}{t^{x+1}} \int_0^1 \xi^x P(\xi) d\xi \) (formal primitive).

(iv) We denote by \( \text{Deri}_x \) the derivative map \( P \mapsto t^{-x} \frac{d}{dt}(t^{x+1} P(t)) \), and for \( n \geq 1 \), \( S_{0,x} = \text{Id} \), by \( S_{n,x} \) the map taking \( t^k \) to \( \frac{(k+x+1)}{n!} t^k \) where \( (k+x+1)_n := (k+x+1) \cdots (k+x+n) \) (the divided derivative

\[
P \mapsto \frac{1}{n!} t^{-x} \frac{d^n}{dt^n} (t^{n+x} P) = \frac{1}{n!} \left( \frac{d}{dt} + \frac{x}{t} \right)^n (t^n P),
\]

so that \( \text{Deri}_x = S_{1,x} \).

(v) If \( \varphi \) is a \( K \)-automorphism of a \( K \)-module \( M \) and \( k \) an integer, we denote

\[
\varphi^{(k)} = \begin{cases} 
\varphi \circ \cdots \circ \varphi & \text{if } k > 0 \\
\text{id}_M & \text{if } k = 0 \\
(\varphi^{-1} \circ \cdots \circ \varphi^{-1}) & \text{if } k < 0.
\end{cases}
\]

**Facts 3.4.** (i) The map \( \text{Prim}_x \) is an isomorphism and its inverse is \( \text{Deri}_x \) for \( x \in K \) which is not negative integer.

(ii) For any integers \( n_1, n_2 \geq 0 \) and \( x_1, x_2 \in K \setminus \mathbb{Z} \) which are not negative integers, the divided derivatives \( S_{n_1,x_1} \) and \( S_{n_2,x_2} \) commute, \( S_{n_1,x_1} \circ S_{n_2,x_2} = S_{n_2,x_2} \circ S_{n_1,x_1} \).

(iii) By continuity with respect to the natural valuation, all the above mentioned maps naturally extend to \( K[[t]] \).

We now quote properties of \( S_{n,x} \).

**Lemma 3.5.** [3] Lemma 3.6

Let \( n \) be a positive integer, \( k \) a non-negative integer and \( x \in K \) which is not negative integer, one has the following relations valid in \( \text{End}_K(K[[t]]) \):

(i) \( S_{n,x} = \frac{1}{n!} S_{1,x} \circ (S_{1,x} + \text{Id}) \circ \cdots \circ (S_{1,x} + (n-1)\text{Id}) \) and \( [t^k] \circ S_{1,x} = (S_{1,x} - k\text{Id}) \circ [t^k] \).

(ii) There exist rational numbers \( \{b_{n,m,l}\} \subset \mathbb{Q} \) with \( b_{n,m,0} = \frac{(-m)n}{n!} \) and, for every pair of integers \( 0 \leq m \leq n \),

\[
[t^m] \circ S_{n,x} = \sum_{l=0}^{m} b_{n,m,l} S_{1,x}^{(l)} \circ [t^m].
\]

**Definition 3.6.** The Padé kernel map associated to the generalized polylogarithmic function \( \Phi_s(x, \alpha/z) \) is defined as

\[
\varphi_{\alpha,x,s} = [\alpha] \circ \text{Eval}_x \circ \text{Prim}_x^{(s)}.
\]

Remark that we have

\[
\varphi_{\alpha,x,s} \circ S_{1,x} = \varphi_{\alpha,x,s-1}.
\]
Lemma 3.7. (confer [3], Fact 1 (v)) The kernel of the map \( \varphi_{\alpha,0} \) is the ideal \((t - \alpha)\).

We first concentrate on a few elementary properties of the maps \( \varphi \) which we regroup here and will be useful for the rest:

Lemma 3.8. (i) The morphisms \( \varphi_{\alpha,x_j,s_j} \circ \text{Prim}_{x_j}^{(l)} \), \( \varphi_{\beta,x_j,s_j} \circ \text{Prim}_{x_j}^{(l')} \) pairwise commute for \( 1 \leq j, j' \leq d, 1 \leq s_j \leq r_j, 1 \leq s_{j'} \leq r_{j'} \) and \( l, l' \in \mathbb{Z} \).

(ii) The operator \( \frac{\partial}{\partial a} \) commutes with any \( \varphi_{\beta,x_j,s_j} \).

(iii) For any \( a(t) \in K[t] \), we have \( \varphi_{\alpha,x_j,s_j}(a(t)) = [a] \circ \varphi_{1,x_j,s_j}(a(at)) \).

Proof.

The assertion (i) follows from the definition since both multiplication by a scalar, specialization of one variable or integration with respect to a given variable all pairwise commute. The assertion (ii) follows from commutation of integrals with respect to a parameter with differentiation with respect to that parameter.

We turn to (iii). Consider \( i = 1 \). Take \( k \geq 0 \), then

\[
\varphi_{\alpha,x,s}(t^k) = [a] \circ \text{Eval}_a \circ \text{Prim}_x^{(s)}(t^k) = [a] \circ \text{Eval}_a \left( \frac{t^k}{(k + x + 1)^s} \right) = \frac{\alpha^{k+1}}{(k + x + 1)^s}.
\]

Similarly, \([a] \circ \varphi_1(x,s)((a(t))^k) = \alpha^{k+1}/(k + x + 1)^s\). Since both coincide, for the basis \( \{t^k\}_{k \geq 0} \), they coincide on \( K[t] \) by linearity. This completes the proof of Lemma 3.8. \( \square \)

We now construct type II Padé approximants of the generalized polylogarithmic functions \( \Phi_{x_j}(x_j, \alpha_i/z) \) \( 1 \leq i \leq m \) \( 1 \leq j \leq d, 1 \leq s_j \leq r_j \). Let \( l \) be a non-negative integer with \( 0 \leq l \leq \rho m \) (recall \( \rho = \sum_{j=1}^d r_j \)). For a non-negative integer \( n \), we define a tuple of polynomials. Set

(3.2) \[ A_i(t) = t^d \prod_{i=1}^m (t - \alpha_i)^{m} \, , \]

(3.3) \[ P_i(z) = \text{Eval}_z \left( \alpha^{d}_{j=1} S_{n,x_j}^{(r_j)} (A_i(t)) \right) \, , \]

(3.4) \[ P_{i,i,j,s_j}(z) = P_{i,i,s_j}(z) = \varphi_{\alpha_i,x_j,s_j} \left( \frac{P_i(z) - P_i(t)}{z - t} \right) \, , \]

for \( 1 \leq i \leq m, 1 \leq j \leq d \). To ease notations, we have deleted the index \( j \) in the definition of \( P_{i,i,s_j} \). However, some ambiguity might arise when \( \alpha_i, x_j, s_j \) are specialized to some fixed values. To avoid it, we adopt the convention that the integer \( s_j \) always belongs to the tagged set of integers \( \{j, 2, \ldots, r_j\} \supseteq \{(j, 1), \ldots, (j, r_j)\} \).

Under the above notations, we obtain the following theorem. Note that these polynomials in fact also depend on \( n \), we suppressed the extra subscript to ease reading.

Theorem 3.9. The tuple of polynomials \( (P_i(z), P_{i,i,s_j}(z)) \) \( 1 \leq i \leq m \) \( 1 \leq j \leq d, 1 \leq s_j \leq r_j \) form a weight \((n, \ldots, n) \in \mathbb{N}^m \) and degree \( \rho mn + l \) Padé type II system of approximants of the generalized polylogarithmic functions

\[
\Phi_{x_j}(x_j, \alpha_i/z) \quad 1 \leq i \leq m, \quad 1 \leq j \leq d, 1 \leq s_j \leq r_j
\]

Proof.

This is a variation of Theorem 2 in [4]. By the definition of \( P_i(z) \), we have

\[
\deg P_i(z) = \rho mn + l \, ,
\]
so the degree condition is satisfied. We need only to check the valuation condition.

Put $R_{l,i,s_j}(z) = P_l(z)\Phi_{s_j}(x_j, \alpha_i/z) - P_{l,i,s_j}(z)$. Then we obtain

$$R_{l,i,s_j}(z) = P_l(z)\varphi_{\alpha_i,x_j,s_j} \left( \frac{1}{z-t} \right) - P_{l,i,s_j}(z) = \varphi_{\alpha_i,x_j,s_j} \left( \frac{P_l(t)}{z-t} \right) = \sum_{k=0}^{\infty} \varphi_{\alpha_i,x_j,s_j}(t^k P_l(t)) \cdot t^{k+1}.$$ 

We only need to show

$$\varphi_{\alpha_i,x_j,s_j}(t^k P_l(t)) = 0 \quad \text{for} \quad 1 \leq i \leq m, 1 \leq j \leq d, 1 \leq s_j \leq r_j \quad \text{and} \quad 0 \leq k \leq n - 1.$$ 

By Lemma 3.10, $[t^k] S_{n,x_j} = U(S_{1,x_j}) \circ [t^k]$ where $U \in \mathbb{Q}[X]$ is a degree $n$ and valuation $\geq 1$ polynomial, provided that $k \leq n - 1$. By iteration, $[t^k] S_{n,x_j} = S_{1,x_j} \circ V(S_{1,x_j}) \circ [t^k]$ where $V \in \mathbb{Q}[X]$ is now of degree $r_j (n - 1)$. Since the differential operators $S_{n,x_j}$ pairwise commute (see Fact 3.10(ii)), coming back to the definition of $P_l$, one gets

$$t^k P_l(t) = S_{1,x_j} \circ S_{1,x_j} \circ V(S_{1,x_j}) \circ [t^k] \circ \circ \circ (A_i(t)).$$

But $A_i(t)$ vanishes at $\alpha_i$ at order $\geq n$ and $S_{1,x_j} \circ V(S_{1,x_j}) \circ [t^k] \circ \circ \circ (A_i(t))$ involves differentials of order at most $n - s_j$, hence, by the Leibniz formula, $S_{1,x_j} \circ V(S_{1,x_j}) \circ [t^k] \circ \circ \circ (A_i(t))$ vanishes at $\alpha_i$ with multiplicity at least $s_j \geq 1$. Now, by definition, $\varphi_{\alpha_i,x_j,s_j} \circ S_{1,x_j} = \varphi_{i,x_j,0}$ whose kernel contains the ideal $(t - \alpha_i)$, and thus $\varphi_{\alpha_i,x_j,s_j}(t^k P_l(t)) = 0$ which is precisely what is claimed. 

\[\square\]

3.2 Valuations

We start with a couple of elementary linear algebra and valuations remarks.

**Facts 3.10.** Let $A = \sum_{i \geq k} A_i/z^i \in M_n(K[[1/z]])$ with $k \geq 0$. Then $\det(A)$ is of order $\text{ord}_\infty(\det(A)) \geq kn$ and the coefficient of order $kn$ is $\det(A_k)$.

**Lemma 3.11.** (i) Let $A \in M_n(K[z])$ a $n \times n$ matrix with polynomial entries. We assume

(a) The degree of the first row (maximum of the degrees of the entries) is at most $m \geq 0$.

(b) There exists a lower triangular unipotent element $S \in M_n(K[[1/z]])$ such that the i-th rows $L_i$ of $SA$ have an order $\text{ord}_\infty L_i \geq k$ for $2 \leq i \leq n$ (note that the first row $L_1$ of $SA$ is the first row of $A$).

(c) One has $k(n-1) \geq m+1$.

Then, $\det(A) = 0$.

(ii) Let $L_i(z) = \sum_{j \geq k} L_i^{(j)} z^{-j}$ the power series expansion of $L_i$ for $i \geq 2$ and $L_1 = \sum_{j=0}^{m} L_1^{(j)} z^j$. Assume (a), (b) and $k(n-1) = m$. Then, $\det(A) = \det(t^{L_1^{(m)}}, t_{L_2^{(k)}}, \ldots, t_{L_n^{(k)}})$. 

\[\square\]
Proof.
Denote by $D = (D_1, \ldots, D_n)$ the row vector where $D_i$ is the $(1, t)$-th cofactor of $SA$. Developing the determinant of $SA$ along the first row yields

$$\det(SA) = <L_1, D>,$$

where $< \cdots, \cdots >$ is the canonical bilinear form on $K[z]^n$. Thus, by hypothesis (a) and (b), the order $\text{ord}_\infty(\det(SA)) \geq (n - 1)k - m \geq 1$ by (c). On the other hand, by invariance of the determinant by similarity, $\det(A) = \det(SA)$ is a polynomial, hence if it is non-zero, its order is necessarily $\leq 0$; this yields (i). Point (ii) follows similarly by expanding the first row as a polynomial in $z$ and the other rows as power series in $1/z$. \hfill \square

Lemma 3.12. Let $A$ be a commutative unitary ring and $\{a_k\}_{k \geq 0} \subset A$ a sequence of elements of $A$. Let $f : A[x, y] \to A[x]$ be the $A[x]$-linear map defined by $f(y^k) = a_k \in A$ and, for $\lambda \in A$, $g$ the $A[y]$-linear map $A[x, y] \to A[y]$ defined by $g(x^k) = \lambda a_k$. Let $\tau$ be the transposition inverting $x$ and $y$. Then, for $P \in A[x, y]$ which is antisymmetric (i.e. $\tau(P) = -P$),

$$f \circ g(P) = 0.$$

Proof.
It is obvious. \hfill \square

Lemma 3.13. Let $A$ be a commutative unitary ring, $P(X, Y_1, \ldots, Y_\rho) \in A[\alpha, X, Y_1, \ldots, Y_\rho]$ be a polynomial with $\tau(P) = -P$ for any transposition $\tau$ interverting $X$ and one of the $Y_i$. Assume moreover that $(X - \alpha)^T \mid P$ for some $T \geq 1$. We further assume that we are given:

(i) A differential operator $\partial = c_0 + c_1 \frac{\partial}{\partial X} + \cdots + c_t \frac{\partial^t}{\partial X^t}$ of order $\leq T - 1$ with $c_i \in A$.

(ii) An $A[\alpha, X]$-linear map $f : A[\alpha, X, Y_1, \ldots, Y_\rho] \to A[\alpha, X, Y_1, \ldots, Y_\rho]$.

(iii) For each $1 \leq i \leq \rho$, a pair of linear maps $f_i, g_i$ satisfying the hypothesis of Lemma 3.12 for the pair of variables $(X, Y_i)$.

Then,

$$\text{Eval}_{X \to \alpha} \circ (f_1 \circ g_1 + \cdots + f_\rho \circ g_\rho + \partial \circ f)(P) = 0.$$

Proof.
Indeed, $\sum_{i=1}^\rho f_i \circ g_i(P) = 0$ by Lemma 3.12 and $f(P)$ is still divisible by $(X - \alpha)^T$ since $f$ treats both $X$ and $\alpha$ as scalars. Finally, since the order of $\partial$ is $\leq T - 1$, $X - \alpha$ divides $\partial(f(P))$, hence $\text{Eval}_{X \to \alpha} \circ \partial \circ f(P) = 0$. \hfill \square

The following lemma is a variant of derivation of a parameter integral. We write as a formal lemma that works for more general operators.

Lemma 3.14. The property holds for the differential operators:

(i) $\frac{\partial}{\partial \alpha} \circ [\alpha] = \text{Id} + [\alpha] \circ \frac{\partial}{\partial \alpha}$.

(ii) $\frac{\partial}{\partial \alpha} \circ \text{Eval}_{X_k \to \alpha} = \text{Eval}_{X_k \to \alpha} \circ \frac{\partial}{\partial \alpha} + \text{Eval}_{X_k \to \alpha} \circ \frac{\partial}{\partial X}$.

(iii) $\frac{\partial}{\partial \alpha} \circ \text{Prim}_{x, X_k} = \text{Prim}_{x, X_k} \circ \frac{\partial}{\partial \alpha}$.
(iv) \( S_{1,x,X_k} \circ [1/X_k] = [1/X_k] \circ S_{1,x,X_k} - [1/X_k] \).

Proof.
These identities follow from the definitions. \( \square \)

4 Padé approximation

4.1 Non-vanishing of the crucial determinants

For a positive integer \( n \) and non-negative integer \( l \) with \( 0 \leq l \leq \rho m \), recall the polynomials \( P_l(z), P_{i,s_j}(z) \) defined in (3.3) and (3.4) respectively. We define a column vector \( p_l(z) \in K[z]^{\rho m+1} \) by

\[
p_l(z) = \left( P_l(z), P_{l,i,s_j}(z) \right)_{1 \leq i \leq m, 1 \leq j \leq d, 1 \leq s_j \leq r_j}.
\]

The aim of this subsection is to prove the following proposition.

Proposition 4.1.

(4.1) \[
\Delta(z) = \det \left( p_0(z), \ldots, p_{\rho m}(z) \right) \in K \setminus \{0\}.
\]

In the next subsections we give the proof of Proposition 4.1 which is quite involved. However, it follows the broad lines of the non-vanishing proof in our previous paper [4] which we refer to as a template. The previously considered case (a single \( x \)) being much simpler, a few steps happened to be more straightforward.

The first reduction is performed in Subsection 4.1.1, where we prove that \( \Delta(z) \in K \) (i.e. \( \Delta(z) \) is a constant independent of \( z \)). This crucial step is summed up by Lemma 4.2. This is achieved by a valuation argument, by proving that \( \Delta(z) \) is a polynomial in \( z \), but of negative valuation with respect to \( z \) (thanks to Lemma 3.11).

Second, we move on to express \( \Delta = \Delta(z) \) as the image of a given polynomial by our base operators \( \varphi \). This is carried out in Subsection 4.1.2: the needed result is described by Lemma 4.3.

We then consider factor \( \Delta \) viewing the \( \alpha_i \) as variables: this is done in Section 4.1.4, the main result being stated as Proposition 4.4. To achieve this step, we introduce “extra” variables (where the variable \( t \) is split in many variables \( t_{i,s_j} \) as there are columns). Basically this follows the principle known since A. T. Vandermonde. After showing that the determinant is in fact the determinant of a given minor of the original matrix (all the others canceling out, this is just an extension of the previous valuation argument), we take advantage of the false variables and of the linearity of the operators \( \varphi \) to express the determinant as desired. It can be shown, using Leibniz formula and multilinearity of the determinant, that the few spurious terms which would not be nicely factored as desired, actually cancel out. After having checked homogeneity and found the trivial monomial factors in \( \alpha_i \) (Lemma 4.5), we need to show that \( \Delta \) also factors through the \( \alpha_i - \alpha_j \) at the appropriate power.

All these steps, up to minor adjustments are essentially the same as the template case of a single differential shift considered in [4].
To achieve the final factorization, one shows that the derivative of $\Delta$ with respect to any one of the variables $\alpha_i$ vanishes at the other $\alpha_j$ at the appropriate power. Unfortunately, derivation and operators $\varphi$ do not commute properly. We thus measure the defect of commutativity (Lemma 3.8), and proceed to note that a derivative of sufficiently high order will look like a derivative, because the operator $\varphi$ is essentially an iterated primitive. Since we start with a polynomial vanishing of a high order along $\alpha_i - \alpha_j$, these derivatives tend to produce a lot of vanishing (Lemmata 3.12 and 3.13). However, this argument is not quite enough since derivatives of lower order are not enough to remove all the primitivation, built in the operators $\varphi$. It is taken care via a symmetry argument (Lemma 3.12) and this finally shows the corresponding integrals all vanish.

Up to this stage, the argument is close to the principle of the proof of [4]. However, just replicating it would not provide enough multiplicities. One therefore performs an interpolation between the maps $\text{Prim}_{x_j}$ and $\text{Prim}_{x_j}'$ in order to recover the otherwise undetected multiplicities (Lemma 4.6). Indeed, it can be shown, again by symmetry, that the error term coming from the interpolation vanishes.

After having performed the required interpolation, one is ready to prove factorization: a combinatorics argument is enough to conclude. This is done in two steps: firstly to prepare the conditions of Lemma 4.10 by combinatorial argument, and secondly to clearly state the conclusion of Lemma 4.10 (it turns out that once an interpolation is done, one is actually reduced to the same combinatorial problem in the single shift case, which has already been laid out in our previous paper [4]).

Finally, it will remain to check that the last numerical constant (which depends on $x$ as well as of $r, m$ and $n$) does not vanish. This is spread in two subsections (4.1.4 and 4.1.5). Firstly, an induction step (Lemma 4.11) on the number $m$ of algebraic values $\alpha_1, \ldots, \alpha_m$ at which we evaluate the generalized polylogarithmic functions allows to reduce to the case $m = 1$. In this more simple situation, the remaining constant is shown to be an Hermite determinant in the variables $x_1, \ldots, x_d$: this is done in a relatively classical way in Lemmata 4.15, 4.17, 4.18, 4.19 and 4.20, to complete the proof of Proposition 4.1.

It should be mentioned that the whole construction is effective, and that the actual value of the determinant can be computed if needed.

4.1.1 First step

For a positive integer $n$ and non-negative integer $l$ with $0 \leq l \leq \rho m$, recall the polynomials $P_l(z), P_{l,i,s_j}(z)$ defined in (3.3) and (3.4) respectively.

**Lemma 4.2.** Define a column vector $u_l \in K^{\rho m}$ by

$$u_l = t^{l} \left( \varphi_{\alpha_i,x_j,s_j} (t^{n} P_l(t)) \right)_{1 \leq i \leq m, 1 \leq j \leq d, 1 \leq s_j \leq r_j}.$$

We use the same notations as in Proposition 4.1. Then, there exists a non-zero element $c \in K$ such that $\Delta(z) = c \cdot \det(u_0, \ldots, u_{\rho m - 1}) \in K$.

**Proof.**

Let

$$c = \frac{1}{(\rho m(n + 1))!} \left( \frac{d}{dz} \right)^{\rho m(n + 1)} P_{\rho m}(z),$$

where
be the coefficient of highest degree (= ρmn + ρm) of the polynomial $P_{ρm}(z)$. Consider

$$\begin{pmatrix}
1 & 0 & \cdots & 0 \\
Φ_1(x_1, α_1/z) & -1 & 0 & \cdots \\
\vdots & \ddots & \ddots & \vdots \\
Φ_{r_m}(x_d, α_m/z) & 0 & \cdots & -1
\end{pmatrix}
\begin{pmatrix}
P_0(z) \\
P_1(z) \\
\vdots \\
P_{ρm}(z)
\end{pmatrix} = 
\begin{pmatrix}
P_0(z) & \cdots & P_{ρm}(z) \\
R_{0,1,1}(z) & \cdots & R_{ρm,1,1}(z) \\
\vdots & \ddots & \vdots \\
R_{0,m,ρm}(z) & \cdots & R_{ρm,m,ρm}(z)
\end{pmatrix}.$$

Since the entries of the first row are (by definition on $P_i(z)$) polynomials of degree ≤ ρmn + ρm and entries of the other rows (by Theorem 4.9) are of valuation at least $n + 1$, we can apply Lemma 3.11 (ii). We need only to check the coefficients of highest degree (for the first row) and of minimal valuation (for all the other rows $1 ≤ s_j ≤ r_j$, $1 ≤ j ≤ d$). By construction, the vector of highest degree (= ρmn + ρm) for the first row is $(0, \ldots, 0, c)z^{ρmn+ρm}$ and

$$R_{l,i,s_j}(z) = \sum_{k=n}^{∞} \frac{\varphi_{α_i,x_j,s_j}(t^kP_i(t))}{z^{k+1}},$$

for $0 ≤ l ≤ ρm$, $1 ≤ i ≤ m$, $1 ≤ j ≤ d$ and $1 ≤ s_j ≤ r_j$. So, by Lemma 3.11 (ii)

$$\det(p_0(z), \ldots, p_{ρm}(z)) = ±c \cdot \det(\varphi_{α_i,x_j,s_j}(t^nP_i(t)))$$

$$\text{det}_{0≤i≤ρm,1≤j≤d,1≤s_j≤r_j},$$

as claimed.

4.1.2 Second step

We now define a column vector $w_l ∈ K^ρm$ by (recall the polynomial $A_l$ was defined in equation 3.2)

$$w_l = \left(\varphi_{α_i,x_j,s_j}(t^nA_l(t))\right)_{1≤i≤m,1≤j≤d,1≤s_j≤r_j}.$$

**Lemma 4.3.** We use the notations as above. Then there exists an effectively computable element $E ∈ Q[x_1, \ldots, x_d] \setminus \{0\}$ satisfying the following equality:

$$Θ = E \cdot \det\left(w_0, \ldots, w_{ρm-1}\right). \quad (4.2)$$

**Proof.**

We fix some $x ∈ Q$ which is not negative integer. Note that by definition, the linear operator $S_{1,x} = S_{1,x} + (x_j - x)1d$ and first look at $t^nP_i(t)$. By definition, this is $[t^n] \circ_{j=1}^d S_{n,x_j}(A_l(t))$.

Setting

$$H(T) = \frac{1}{m!} \prod_{l=1}^n (T - l), \quad H_x(T) = \prod_{k=1}^d H(T + x_k - x)^{r_k}.$$

By Lemma 3.13 we have

$$[t^n] \circ_{j=1}^d S_{n,x_j} = H(S_{1,x_j}) \circ [t^n], \quad [t^n] \circ_{j=1}^d S_{n,x_j}^{(r_j)} = H_x(S_{1,x}) \circ [t^n].$$
We apply this remark to a coordinate \( \varphi_{\alpha_i, x_j, s_j}(t^n P_l(t)) \) of the vector \( u_l \), (id est for some choice of \((i, s_j)\)), and get

\[
\varphi_{\alpha_i, x_j, s_j}(t^n P_l(t)) = \varphi_{\alpha_i, x_j, s_j} \circ H_{x_j}(S_{1,x_j}) \circ [t^n](A_l(t)).
\]

We now write \( H_{x_j}(T) = H_{x_j, s_j}(T) + T^{s_j} Q_{x_j, s_j}(T) \) where \( H_{x_j, s_j}, Q_{x_j, s_j} \in K[T] \) with \( \deg H_{x_j, s_j} < s_j \) via the euclidean division. Note that \( H_{x_j}(0) = H_{x_j, s_j}(0) \). By definition of \( \varphi_{\alpha_i, x_j, s_j} \), we thus have

\[
\varphi_{\alpha_i, x_j, s_j}(t^n P_l(t)) = \varphi_{\alpha_i, x_j, s_j} \circ H_{x_j, s_j}(S_{1,x_j}) \circ [t^n](A_l(t)) + \varphi_{\alpha_i, x_j, 0} \circ Q_{x_j, s_j}(S_{1,x_j}) \circ [t^n](A_l(t)),
\]

by the vanishing Lemma \( \square \) \( \varphi_{\alpha_i, x_j, 0} \circ Q_{x_j, s_j}(S_{1,x_j}) \circ [t^n](A_l(t)) = 0 \) since we differentiate at order \( \geq \rho m \) a polynomial \( = t^n A_l(t) \) vanishing at order \( \geq \rho m \) at \( \alpha_i \). Hence,

\[
\det(u_l)_{0 \leq l \leq \rho m - 1} = \det \left( \varphi_{\alpha_i, x_j, s_j} \circ H_{x_j, s_j}(S_{1,x_j}) \circ [t^n](A_l(t)) \right)_{0 \leq l \leq \rho m - 1}.
\]

By the properties of determinant, we get

\[
\det(u_l)_{0 \leq l \leq \rho m - 1} = \prod_{1 \leq j \leq d} H_{x_j, s_j}(0)^{-1} \det \left( \varphi_{\alpha_i, x_j, s_j} \circ [t^n](A_l(t)) \right)_{0 \leq l \leq \rho m - 1}.
\]

In order to complete the proof, we need only to prove

\[
E := \prod_{1 \leq j \leq d} H_{x_j, s_j}(0)^m \in \mathbb{Q}[x_1, \ldots, x_d] \setminus \{0\}.
\]

We thus just need to compute the constant term of the polynomials \( H_{x_j, s_j} \). Since the constant term of \( H_{x_j, s_j} \) is

\[
H_{x_j, s_j}(0) = H_{x_j}(0) = \prod_{k=1}^d H(x_k - x_j)^{x_k},
\]

and \( E \) is non-zero since \( x_l - x_k \notin \mathbb{Z} \) for \( 1 \leq k \neq l \leq d \). This concludes the proof of the lemma.

\[\square\]

4.1.3 Third step

Now, we consider the ring \( K[\{t_{i,s_j}\}_{1 \leq i \leq m, 1 \leq j \leq d, 1 \leq s_j \leq r_j}] \), the ring of polynomials in \( \rho m \) variables over \( K \). For each variable \( t_{i,s_j} \), any \( \alpha \in K \setminus \{0\} \), \( x_j \in K \) which is not negative integer, one has a well defined map \( \varphi_{\alpha_i, x_j, s_j} = \varphi_{\alpha_i, x_j, d_i, s_j} \) for \( 1 \leq i \leq m \):

\[
(4.3) \quad \varphi_{\alpha_i, x_j, s_j} : K[\{t_{i',s_j'}\}_{1 \leq i' \leq m, 1 \leq j \leq d, 1 \leq s_j \leq r_j}, t_{i,s_j}] \rightarrow K[\{t_{i',s_j'}\}_{1 \leq i' \leq m, 1 \leq j \leq d, 1 \leq s_j \leq r_j}] ; \quad t_{i,s_j} \mapsto \frac{\alpha_i^{k+1}}{(k + x + 1)^{s_j}},
\]

using the definition above where \( K[\{t_{i',s_j'}\}_{1 \leq i' \leq m, 1 \leq j \leq d, 1 \leq s_j \leq r_j}] \) is seen as the one variable polynomial ring \( K'[\{t_{i,s_j}\}] \). Then we have

\[
\mathbf{w}_l = \left( \varphi_{\alpha_i, x_j, s_j}(t^n_{i,s_j} A_l(t_{i,s_j})) \right)_{1 \leq i \leq m, 1 \leq j \leq d, 1 \leq s_j \leq r_j}.
\]

\[\text{Indeed, } \deg Q_{x_j} = \rho m - s_j < \rho m.\]

\[\text{Recall that the index } s_j \text{ runs through the tagged set of integers } \{j, 1, 2, \ldots\} \approx \{(j, 1), (j, 2), \ldots\}.
\]
We now define for non-negative integers $u, n$

\[
\hat{P}_{u,n}(t_{i,s}) = \hat{P}_n(t_{i,s}) := \prod_{i=1}^m \prod_{j=1}^d t_{i,s}^{r_j} \prod_{k=1}^m \left( t_{i,s} - \alpha_k \right)^{\rho n} \left( \prod_{(i_1,s_1) < (i_2,s_2)} (t_{i_2,s_2} - t_{i_1,s_1}) \right),
\]

where the order $(i_1,s_1) < (i_2,s_2)$ means lexicographical order and denote $(t_{i,s})$ by $t$.

If $k = (k_{i,s})$ is a $pm$-tuple of integers, we denote by $t^k$ the product $\prod_{i,s} t_{i,s}^{k_{i,s}}$. Also set (when no confusion is deemed to occur, we omit the subscripts $\alpha = (\alpha_1, \ldots, \alpha_m)$, $x = (x_1, \ldots, x_d)$):

\[
(4.4) \quad \psi = \psi_{\alpha,x} = \prod_{i=1}^m \prod_{j=1}^d t_{i,s}^{r_j} \varphi_{\alpha_i,x_i,s_j}.
\]

Note that, by the definition of $\det \left( w_0, \ldots, w_{pm-1} \right)$, we have

\[
(4.5) \quad \det \left( w_0, \ldots, w_{pm-1} \right) = \psi(\hat{P}_{n,n}(t)) .
\]

Indeed, since $\varphi_{\alpha_i,x_i,s_j}$ treats all variables $t_{i',s'j'}$ except $t_{i,s_j}$ as scalars,

\[
\det(w_0, \ldots, w_{pm-1}) = \psi \left( \prod_{i,s_j} t_{i,s_j}^{n} (t_{i,s_j} - \alpha_i)^{\rho n} \sum_{\sigma \in \mathfrak{S}_{pm}} \varphi(\sigma) t^{\sigma(t)} \right),
\]

and the last sum $\sum_{\sigma \in \mathfrak{S}_{pm}} \varphi(\sigma) t^{\sigma(t)}$ is nothing but the Vandermonde determinant in $t_{i,s_j}$.

In this subsection, we prove

**Proposition 4.4.** We use above notations. Then there exists a constant $c_{u,m} \in \mathbb{Q}[x_1, \ldots, x_d]$ with

\[
(4.6) \quad C_{u,m} := \psi(\hat{P}_u(t)) = c_{u,m} \left( \prod_{i=1}^m \alpha_i \right)^{u^2} \alpha_i^{(u+1) + \rho^2 n + \left( \sum_{1 \leq j_1 < j_2 \leq d} j_1 r_{j_1} + \sum_{j=1}^d \binom{r_j}{2} \right)} \prod_{1 \leq i_1 < i_2 \leq m} (\alpha_{i_2} - \alpha_{i_1})^{(2n+1)\rho^2}.
\]

It is also easy to see that since all the variables $t_{i,s_j}$ have been specialized, $C_{u,m} \in K$ is a polynomial in the $\alpha_i, x_j$. The statement is then about a factorization of this polynomial.

To prove Proposition 4.4, we are going to perform the following steps:

(i) Show that $C_{u,m}$ is homogeneous in $\alpha$ of degree

\[
\rho m(u + 1) + \rho^2 m^2 n + \rho^2 \binom{m}{2} + m \left( \sum_{1 \leq j_1 < j_2 \leq d} r_{j_1} r_{j_2} + \sum_{j=1}^d \binom{r_j}{2} \right).
\]

(ii) Show that $\prod_{i=1}^m \alpha_i^{(u+1) + \rho^2 n + \left( \sum_{1 \leq j_1 < j_2 \leq d} j_1 r_{j_1} + \sum_{j=1}^d \binom{r_j}{2} \right)}$ divides $C_{u,m}$.

(iii) Show that $\prod_{1 \leq i_1 < i_2 \leq m} (\alpha_{i_2} - \alpha_{i_1})^{(2n+1)\rho^2}$ divides $C_{u,m}$.

(iv) Show that the remaining $c_{u,m}$ (now a constant in $\alpha$) is a polynomial with rational coefficients in $x$.

We first prove (i) and (ii) and (iv).
Lemma 4.5. The number $C_{u,m}$ is homogeneous in $\alpha$ of degree

$$\rho m(u + 1) + \rho^2 m^2 n + \rho^2 \left(\frac{m}{2}\right) + m \left( \sum_{1 \leq j_1 < j_2 \leq d} r_{j_1} r_{j_2} + \sum_{j=1}^d \left(\frac{r_j}{2}\right) \right),$$

and is divisible by

$$\left( \prod_{i=1}^m \alpha_i \right)^{\rho(u+1)+\rho^2 n+\left(\sum_{1 \leq j_1 < j_2 \leq d} r_{j_1} r_{j_2} + \sum_{j=1}^d \left(\frac{r_j}{2}\right) \right)}.$$

Proof.

First the polynomial $P_u(t)$ is a homogeneous polynomial with respect to the variables $\alpha_i, t$ of degree

$$\rho m u + \rho^2 m^2 n + \rho^2 \left(\frac{m}{2}\right) + m \left( \sum_{1 \leq j_1 < j_2 \leq d} r_{j_1} r_{j_2} + \sum_{j=1}^d \left(\frac{r_j}{2}\right) \right).$$

Thus, by the definition of $\psi$ (which is degree increasing by $\rho m$), it is easy to see that $C_{u,m} = \psi(P_u(t))$ is a homogeneous polynomial with respect to the variables $\alpha_i$ of degree

$$\rho m(u + 1) + \rho^2 m^2 n + \rho^2 \left(\frac{m}{2}\right) + m \left( \sum_{1 \leq j_1 < j_2 \leq d} r_{j_1} r_{j_2} + \sum_{j=1}^d \left(\frac{r_j}{2}\right) \right).$$

Second we show the later assertion. By linear algebra, $\varphi_{\alpha_i,x,s_j}(Q(t)) = \alpha_i \text{Eval}_{\alpha_i \to 1} \circ \varphi_{1,x,s_j}(Q(\alpha_i t))$ (i.e. the variable $t$ specializes in 1, confer Lemma 3.8 (ii)) for any polynomial $Q(t) \in K[t]$. So, by composition, the same holds for $\psi$. So, putting $1 = (1, \ldots, 1) \in K^m$. We now compute

$$\hat{P}_u((\alpha_i t_{i,s_j})_{i,s_j}) = \left( \prod_{i=1}^m \alpha_i \right)^{\rho(u+1)+\rho^2 n+\left(\sum_{1 \leq j_1 < j_2 \leq d} r_{j_1} r_{j_2} + \sum_{j=1}^d \left(\frac{r_j}{2}\right) \right)} \cdot Q_u(t),$$

where

$$Q_u(t) = Q_{n,u,m}(t) = \left( \prod_{i=1}^m \prod_{j=1}^d \prod_{s_j} \left[ t_{i,s_j} \prod_{k \neq i} \left( \alpha_i t_{i,s_j} - \alpha_k \right)^{\rho n} (t_{i,s_j} - 1)^m \right] \right) \cdot \prod_{1 \leq i_1 < i_2 \leq m} \prod_{(j_1,s_{j_1}) < (j_2,s_{j_2})} \left( \alpha_{i_2} t_{i_2,s_{j_2}} - \alpha_{i_1} t_{i_1,s_{j_1}} \right) \cdot \prod_{i=1}^m \prod_{(j_1,s_{j_1}) < (j_2,s_{j_2})} (t_{i_2,s_{j_2}} - t_{i_1,s_{j_1}}),$$

by linearity,

$$C_{u,m} = \left( \prod_{i=1}^m \alpha_i \right)^{\rho(u+1)+\rho^2 n+\left(\sum_{1 \leq j_1 < j_2 \leq d} r_{j_1} r_{j_2} + \sum_{j=1}^d \left(\frac{r_j}{2}\right) \right)} \cdot \psi_{1,\alpha}(Q_u(t)).$$

This concludes the proof of the lemma.

Note, by definition we have $C_{u,m} \in \mathbb{Q}[\alpha_1, \ldots, \alpha_m, x_1, \ldots, x_d]$ and thanks to Lemma 4.5 we have $c_{u,m} \in \mathbb{Q}[x_1, \ldots, x_d]$ assuming step (iii).
Now we consider (iii). The case of \( m = 1 \) is trivial. Thus we assume \( m \geq 2 \).

We need to show that \((\alpha_{i_2} - \alpha_{i_1})^{(2n+1)n^2}\) divides \( C_{u,m} \). Without loss of generality, after renumbering, we can assume that \( i_2 = 2, i_1 = 1 \). To ease notations, we are going to take advantage of the fact that \( m \geq 2 \), and set \( X_s = t_1, Y_s = t_2 \), and \( \alpha_1 = \alpha, \alpha_2 = \beta \). We want to treat all the variables \( t_i \), for \( i \geq 3 \) as constants as they will play no subsequent role. So our polynomial \( \hat{P} \) rewrites as

\[
\hat{P}(X, Y) = \prod_{i=1}^{d} \prod_{j=1}^{r_i} (X_s Y_j)^u \left[(X_s - \alpha)(X_s - \beta)(Y_s - \alpha)(Y_s - \beta)\right]^{pn}
\]

\[
\cdot \prod_{(j_1, s_1) < (j_2, s_2)} (X_{s_2} - X_{s_1}) \prod_{(j_1, s_1) < (j_2, s_2)} (Y_{s_2} - Y_{s_1})
\]

\[
\cdot \prod_{(j_2, s_2) < (j_1, s_1)} (Y_{s_2} - X_{s_1}) \cdot c(t_{i,s}) \cdot \prod_{k \geq 3} \prod_{j=1}^{m} \prod_{s_1=1}^{d} \prod_{s_2=1}^{d} \prod_{(j_1, s_1) < (j_2, s_2)} (t_{k,s} - X_{s_1})(t_{k,s} - Y_{s_2})
\]

where \( c(t_{i,s}) \) is shortened for \( c(t_{i,s}){i \geq 3} \) and is defined by

\[
c(t_{i,s}) := \prod_{i=1}^{m} \prod_{j=1}^{d} \prod_{s_1=1}^{d} \prod_{s_2=1}^{d} \left[(t_{i,s} - \alpha)(t_{i,s} - \beta)\right]^{pn} \cdot \prod_{i_1, i_2 \geq 3} (t_{i_2,s_2} - t_{i_1,s_1})
\]

In order to differentiate computations between the set of variables associated to \( \alpha \) and \( \beta \) respectively, we set

\[
\psi_{\alpha} = \psi_{\alpha, \infty} = \circ_{j=1}^{d} \circ_{s_1=1}^{d} \varphi_{\alpha, \infty, s_1, s_2}, \quad \psi_{\beta} = \psi_{\beta, \infty} = \circ_{j=1}^{d} \circ_{s_1=1}^{d} \varphi_{\beta, \infty, s_1, s_2}.
\]

In other words, we have

\[
\psi = \psi_{\alpha, \infty} \circ \psi_{\beta, \infty} \circ \Xi,
\]

where \( \Xi = \circ_{i \geq 3} \circ_{j=1}^{d} \circ_{s_1=1}^{d} \varphi_{\alpha, \infty, s_1, s_2} = \circ_{i \geq 3} \psi_{\alpha, \infty} \).

We now write \( \hat{P} \) as a product of a symmetric and an antisymmetric polynomial:

\[
f_{u,n}(\alpha, \beta, X, Y) = f(\alpha, \beta, X, Y) := \prod_{j=1}^{d} \prod_{s_1=1}^{d} (X_{s_2} Y_s)^u \left[(X_s - \alpha)(X_s - \beta)(Y_s - \alpha)(Y_s - \beta)\right]^{pn}
\]

\[
\cdot c(t_{i,s}) \prod_{k \geq 3} \prod_{j=1}^{m} \prod_{s_1=1}^{d} \prod_{s_2=1}^{d} \prod_{(j_1, s_1) < (j_2, s_2)} (t_{k,s} - X_{s_1})(t_{k,s} - Y_{s_2})
\]

where \( c(t_{i,s}) \) is shortened for \( c(t_{i,s}){i \geq 3} \)

\[
g(X, Y) = g(\alpha, \beta, X, Y) := \prod_{(j_1, s_1) < (j_2, s_2)} (X_{s_2} - X_{s_1}) \prod_{(j_1, s_1) < (j_2, s_2)} (Y_{s_2} - Y_{s_1}) \cdot \prod_{(j_2, s_2) < (j_1, s_1)} (Y_{s_2} - X_{s_1}) \cdot \prod_{(j_2, s_2) < (j_1, s_1)} (X_{s_2} - X_{s_1})
\]

So that \( \hat{P} = \hat{f} = fg \) (for the rest of the proof, the indexes \( u, n \) will not play any role and may be conveniently left off to ease reading).

In order to tackle the different shifts, we need to be able to interpolate between them. We have the following elementary observations.

Let \((j, s)\) as above, \( i.e. 1 \leq j \leq d, 1 \leq s \leq r_j \); we set \( k = \sum_{i=1}^{j-1} r_i + s \) (as usual, the empty sum is equal to zero), so that we have established a natural bijection between \{ \((j, s) \mid 1 \leq j \leq d, 1 \leq s \leq r_j \} \) and \( \{1, \ldots, r\} \).
Lemma 4.6. (i) Let $x, y \in \mathbb{Q}$ which are not negative integers and $n \in \mathbb{N}$. Then we have

$$\text{Prim}_y = \sum_{k=0}^{n-1} (x-y)^k \text{Prim}_x^{(k+1)} + (x-y)^n \text{Prim}_x^{(n)} \circ \text{Prim}_y.$$

(ii) For each $1 \leq j \leq d$, and any integer $k = \sum_{i=1}^{j-1} r_i + s_j$ with $1 \leq s_j \leq r_j$, we have

$$\text{Prim}_{x_j} = \left( c_{i=1}^{j-1} (x_i - x_j)^{r_i} \text{Prim}_x^{(r_i)} \right) \circ \text{Prim}_{x_j} + \left( \sum_{\mu \leq \sum_{i=1}^{j-1} r_i} \prod_{i=1}^{n-1} (x_i - x_j)^{r_i} (x_n - x_j)^{s_n-1} \text{Prim}_x^{(r_i)} \circ \text{Prim}_x^{(s_n)} \right),$$

where at each stage, one writes $\mu = \sum_{i=1}^{n-1} r_i + s_n$ in a unique fashion for some $s_n$ between 1 and $r_n$, where the primitive is always taken with respect to the same fixed variable $X_{x_j}$ (respectively $Y_{x_j}$ for specialization at $\beta$). As usual, empty product is 1, empty sum is zero and empty composition is identity.

In particular, we have

$$\varphi_{\alpha, x_j, s_j} \circ \text{Der}(s_j^{-1}) = [\alpha] \circ \text{Eval}_{x_j \rightarrow \alpha} \left( c_{i=1}^{j-1} (x_i - x_j)^{r_i} \text{Prim}_x^{(r_i)} \right) \circ \text{Prim}_{x_j} + [\alpha] \circ \text{Eval}_{x_j \rightarrow \alpha} \left( \sum_{\mu \leq \sum_{i=1}^{j-1} r_i} \prod_{i=1}^{n-1} (x_i - x_j)^{r_i} (x_n - x_j)^{s_n-1} \text{Prim}_x^{(r_i)} \circ \text{Prim}_x^{(s_n)} \right),$$

Proof.

The first property follow from the definition of the map $\text{Prim}_x$, by checking on the basis of the polynomial ring. We now prove (ii). We proceed by induction on $j$. If $j = 1$, there is nothing to prove since both compositions and sums are empty. Thus we may assume $j \geq 2$ and (ii) to be true for all $1 \leq l \leq j-1$. We now write, using Lemma 4.6 (i), with $y = x_j$, $x = x_1$ and $n = r_1$, we get:

$$\text{Prim}_{x_j} = \sum_{l=0}^{r_1-1} (x_1 - x_j)^l \text{Prim}_x^{(l+1)} + (x_1 - x_j)^{r_1} \text{Prim}_x^{(r_1)} \circ \text{Prim}_{x_j}.$$

We now apply the induction hypothesis with the set $\{x_2, \ldots, x_j\}$ of length $j-1$ and get:

$$\text{Prim}_{x_j} = c_{i=2}^{j-1} (x_i - x_j)^{r_i} \text{Prim}_x^{(r_i)} \circ \text{Prim}_{x_j} + \sum_{\mu \leq \sum_{i=2}^{j-1} r_i} \prod_{i=2}^{n-1} (x_i - x_j)^{r_i} (x_n - x_j)^{s_n-1} \text{Prim}_x^{(r_i)} \circ \text{Prim}_x^{(s_n)}.$$

Plugging in the above relation in the previous relation completes induction and the proof of Lemma 4.6.

For $\gamma \in K[\alpha, \beta, X, Y]$, we define the substitution morphism

$$\Delta_\gamma : K[\alpha, \beta, X, Y] \rightarrow K[\alpha, \beta, X, Y]; \quad \Delta_\gamma(\beta) = \gamma,$$

with $\Delta_\gamma|_{K[\alpha, X, Y]} = \text{Id}_{K[\alpha, X, Y]}$. We now rewrite our polynomial. Let $(j, s_j)$ as above, i.e. $1 \leq j \leq d, 1 \leq s_j \leq r_j$; we set $k = \sum_{i=1}^{j-1} r_i + s_j$ (as usual, the empty sum is equal to zero). We then rewrite our shifts setting $y_k = x_j$ (in other words, each $x_j$ is repeated $r_j$ times). Similarly, we rewrite the variable $X_{s_j}$ as $X_k$. In contrast to the first
part of the proof which consisted mainly along the same line as Hermite in separating
the variables, we shall now need to let both the shifts and the number of primitive to
vary with more flexibility. This forces to take into account: the number of primitivations
that are performed, the shifts associated to them and the variable with respect to which
primitivation is performed.

We put for any integer \( \nu \), and any integer \( 1 \leq \lambda \leq \rho \),

\[
\theta_{\alpha,\lambda,\nu,\kappa} = \varphi_{\alpha,x,\nu-\lambda,\kappa} \circ_{k=1}^{\lambda} \operatorname{Prim}_{y_k} X_k ,
\]

with as usual empty composition being the identity map, we define in the same way the
map theta with specialization at \( \beta \) and integration is taken with respect to the variable
\( Y_k = Y_{s_j} \).

We set \( \lambda = (\lambda_1, \ldots, \lambda_\rho), \nu = (\nu_1, \ldots, \nu_\rho) \in \mathbb{Z}^\rho \) and denote by

\[
(4.10) \quad \Theta_{\alpha,\lambda,\nu} = \circ_{k=1}^{\nu} \theta_{\alpha,\lambda,\nu_k,\kappa} ,
\]

and in a same fashion for specialization at \( \beta \).

**Lemma 4.7.** We have \( C_{u,m} = C(y) \Xi \circ \Theta_{\beta,\mu} \circ \Theta_{\alpha,\mu} (fg) \), where \( C(y) \) is some non-zero real number depending only on \( y \), where \( y = (y_1, \ldots, y_\rho) \) and \( \mu = (1, 2, \ldots, \rho) \in \mathbb{Z}^\rho \).

Proof.

We recall \( k := (1, \ldots, r_1, 1, \ldots, r_d) \in \mathbb{Z}^\rho \) and that \( C_{u,m} = \Xi \circ \psi_{\alpha,\mu} \circ \psi_{\beta,\mu} (fg) \). It is thus enough to show

\[
\psi_{\alpha,\mu} \circ \psi_{\beta,\mu} (fg) = C(y) \Theta_{\alpha,\mu} \circ \Theta_{\beta,\mu} (fg) .
\]

Let \( k \) be an integer with \( 1 \leq k \leq \rho \) and write \( k = \sum_{i=1}^{j-1} r_i + s_j \) for \( 1 \leq s_j \leq r_j \) (possible in a unique way). We now recall

\[
\varphi_{\alpha,x,s_j} X_{s_j} = [\alpha] \circ \operatorname{Eval}_{X_{s_j}} \circ \operatorname{Prim}_{X_{s_j}} \circ \operatorname{Prim}^{(s_j-1)} .
\]

We then apply Lemma 4.6 part (ii), and get

\[
\varphi_{\alpha,x,s_j} X_{s_j} = [\alpha] \circ \operatorname{Eval}_{X_{s_j}} \circ \operatorname{Prim}_{X_{s_j}} \circ \operatorname{Prim}^{(s_j-1)} .
\]

for some \( A_\mu \in \mathbb{Q}[y] \setminus \{0\} \). Plugging in the definition of \( \theta \) operators, one gets

\[
(4.11) \quad \varphi_{\alpha,x,s_j} X_{s_j} = \prod_{i=1}^{j-1} (x_i - x_j)^{r_i} \theta_{\alpha,k,k,X_k} + \sum_{\mu \leq \sum_{i=1}^{j-1} r_i} A_\mu \theta_{\alpha,\mu,\mu,X_k} \circ \operatorname{Prim}^{(s_j-1)} .
\]

We now claim \( \psi_{\alpha,\mu} (fg) = C(y) \Theta_{\alpha,\mu} (fg) \). Indeed, this can be proven easily by induction
on \( \rho \). If \( \rho = 1 \), there is nothing to prove since \( \Theta_{\alpha,1} = \psi_{\alpha,x,1} \). We now assume \( \sum_{i=1}^{j-1} r_i + s_j < \rho \) and

\[
\varphi_{\alpha,x,t}^s X_t \circ_{1 \leq i \leq j-1} \circ_{0 \leq s_i \leq r_i} \varphi_{\alpha,x,s_i} X_{s_i} (fg) = C(y) \circ_{i=1}^{\sum_{i=1}^{j-1} r_i + s_j} \theta_{\alpha,\mu,\mu,X_\mu} (fg) ,
\]
and prove the same relation for \( \sum_{i=1}^{j-1} r_i + s_j + 1 \). We separate the computation into two cases, firstly, \( s_j = r_j \). In this case, by (3.11),

\[
\varphi_{a,x_1+i,j+1,X_{j+1}} = \prod_{i=1}^{j}(x_i - x_{j+1})\theta_{a,k+1,k+1,X_{k+1}} + \sum_{\mu \leq \sum_{i=1}^{j} r_i} A_\mu \theta_{a,\mu,\mu,X_{k+1}}.
\]

By Lemma 3.12 \( \theta_{a,\mu,\mu,X_{k+1}} \circ \theta_{a,\mu,\mu,X_{k+1}}(fg) = 0 \), so, the induction is completed by expanding the composition. We now turn to the case where \( s_j + 1 \leq r_j \), and we look at \( \theta_{a,\mu,\mu,X_{k+1}} \circ \text{Prim}^{(s_j)} \) for some \( \mu \leq \sum_{i=1}^{j-1} r_i \). Since \( \text{Prim}_{x_j} = \text{Prim}_{y_{j+1}} + (y_{j+1} - x_j)\text{Prim}_{y_{j+1}} \) \( \circ \text{Prim}_{x_j} \), and

\[
\theta_{a,\mu,\mu,X_{k+1}} \circ \text{Prim}^{(s_j)} = \theta_{a,\mu,\mu,1+1,X_{k+1}} \circ \text{Prim}^{(s_j-1)}(y_{j+1} - x_j)\theta_{a,\mu,\mu,1+1,X_{k+1}} \circ \text{Prim}^{(s_j)}
\]

by induction, \( \theta_{a,\mu,\mu,X_{k+1}} \circ \text{Prim}^{(s_j)} \) is a linear combination of \( \theta_{a,\nu,\nu,X_{k+1}} \) for \( \nu \) varying between \( \mu + 1 \) and \( k \), but by Lemma 3.12 \( \theta_{a,\nu,\nu,X_{k+1}} \circ \theta_{a,\nu,\nu,X_{k}}(fg) = 0 \), hence the induction is also complete in this case and Lemma 4.7 is proved.

\[\Box\]

**Lemma 4.8.** Let \( \lambda = (\lambda_k)_{1 \leq k \leq \rho}, \nu = (\nu_k)_{1 \leq k \leq \rho} \) be two integral vectors in \( \mathbb{Z}^\rho \), with \( 1 \leq \lambda_k \leq k \) and \( 1 \leq k \leq \rho \). Then

\[
\frac{\partial}{\partial \alpha} \circ \Theta_{a,\lambda,\nu} = \Theta_{a,\lambda,\nu} \circ \left( \frac{\partial}{\partial \alpha} - \frac{r y_1}{\alpha} \right) + \sum_{j=1}^{\rho} \Theta_{a,\lambda,\nu - e_j} \circ \left[ \frac{1}{\alpha} \right],
\]

where \( e_j := (0,\ldots,0,1,0,\ldots,0) \in \mathbb{Z}^\rho \) and 1 is in the \( j \)-th spot.

**Proof.**

By definition, \( \Theta_{a,\lambda,\nu} = \circ_{k=1}^{\rho} \theta_{a,\lambda_k,\nu_k,X_k} \) and

\[
\theta_{a,\lambda,\nu,X_k} = [\alpha] \circ \text{Eval}_{X_k} \circ \text{Prim}^{(\nu - \lambda)}_{x_k,X_k} \circ_{k=1}^{\rho} \text{Prim}_{y_k,X_k}.
\]

Thus, using the elementary Lemma 3.13 above,

\[
\frac{\partial}{\partial \alpha} \circ \theta_{a,\lambda,\nu,X_k} = \frac{\partial}{\partial \alpha} \circ \theta_{a,\lambda,\nu,X_k} + [\alpha] \circ \frac{\partial}{\partial \alpha} \circ \text{Eval}_{X_k} \circ \text{Prim}^{(\nu - \lambda)}_{x_k,X_k} \circ_{k=1}^{\rho} \text{Prim}_{y_k,X_k}
\]

\[
= \frac{\partial}{\partial \alpha} \circ \theta_{a,\lambda,\nu,X_k} + [\alpha] \circ \frac{\partial}{\partial \alpha} \circ \text{Eval}_{X_k} \circ \text{Prim}^{(\nu - \lambda)}_{x_k,X_k} \circ_{k=1}^{\rho} \text{Prim}_{y_k,X_k}
\]

\[
= \frac{\partial}{\partial \alpha} \circ \theta_{a,\lambda,\nu,X_k} + [\alpha] \circ \text{Eval}_{X_k} \circ \text{Prim}^{(\nu - \lambda)}_{x_k,X_k} \circ_{k=1}^{\rho} \text{Prim}_{y_k,X_k}
\]

\[
= \frac{\partial}{\partial \alpha} \circ \theta_{a,\lambda,\nu,X_k} + [\alpha] \circ \text{Eval}_{X_k} \circ \text{Prim}^{(\nu - \lambda)}_{x_k,X_k} \circ_{k=1}^{\rho} \text{Prim}_{y_k,X_k}
\]

\[
= \frac{\partial}{\partial \alpha} \circ \theta_{a,\lambda,\nu,X_k} + [\alpha] \circ \text{Eval}_{X_k} \circ \text{Prim}^{(\nu - \lambda)}_{x_k,X_k} \circ_{k=1}^{\rho} \text{Prim}_{y_k,X_k}
\]

\[
= \frac{\partial}{\partial \alpha} \circ \theta_{a,\lambda,\nu,X_k} + [\alpha] \circ \text{Eval}_{X_k} \circ \text{Prim}^{(\nu - \lambda)}_{x_k,X_k} \circ_{k=1}^{\rho} \text{Prim}_{y_k,X_k}
\]

\[
= \frac{\partial}{\partial \alpha} \circ \theta_{a,\lambda,\nu,X_k} + [\alpha] \circ \text{Eval}_{X_k} \circ \text{Prim}^{(\nu - \lambda)}_{x_k,X_k} \circ_{k=1}^{\rho} \text{Prim}_{y_k,X_k}
\]

\[
= \frac{\partial}{\partial \alpha} \circ \theta_{a,\lambda,\nu,X_k} + [\alpha] \circ \text{Eval}_{X_k} \circ \text{Prim}^{(\nu - \lambda)}_{x_k,X_k} \circ_{k=1}^{\rho} \text{Prim}_{y_k,X_k}
\]

\[
= \frac{\partial}{\partial \alpha} \circ \theta_{a,\lambda,\nu,X_k} + [\alpha] \circ \text{Eval}_{X_k} \circ \text{Prim}^{(\nu - \lambda)}_{x_k,X_k} \circ_{k=1}^{\rho} \text{Prim}_{y_k,X_k}
\]

\[
= \frac{\partial}{\partial \alpha} \circ \theta_{a,\lambda,\nu,X_k} + [\alpha] \circ \text{Eval}_{X_k} \circ \text{Prim}^{(\nu - \lambda)}_{x_k,X_k} \circ_{k=1}^{\rho} \text{Prim}_{y_k,X_k}
\]

\[
= \frac{\partial}{\partial \alpha} \circ \theta_{a,\lambda,\nu,X_k} + [\alpha] \circ \text{Eval}_{X_k} \circ \text{Prim}^{(\nu - \lambda)}_{x_k,X_k} \circ_{k=1}^{\rho} \text{Prim}_{y_k,X_k}
\]

\[
= \frac{\partial}{\partial \alpha} \circ \theta_{a,\lambda,\nu,X_k} + [\alpha] \circ \text{Eval}_{X_k} \circ \text{Prim}^{(\nu - \lambda)}_{x_k,X_k} \circ_{k=1}^{\rho} \text{Prim}_{y_k,X_k}
\]

By inputting this in the definition of \( \Theta_{a,\lambda,\nu} \), one obtains the lemma. \[\Box\]
Lemma 4.9. Put $l = (1, \ldots, \rho) \in \mathbb{Z}^\rho$. Let $\lambda, l, k \in \mathbb{Z}$ with $1 \leq \lambda, k \leq \rho$ and $\lambda \leq l$. Let $P \in K[\alpha, \beta, X_1, \ldots, X_p, Y_1, \ldots, Y_p]$ be a polynomial such that $(X_k - \alpha)^{T_1}(X_k - \beta)^{T_2} | P$ for non-negative integers $T_1, T_2$ with either $T_1$ or $T_2$ is greater than 1 and $0 \leq l - \lambda \leq T_1 + T_2 - 1$. Assume the polynomial $P$ satisfies $\tau_k(P) = -P$ for transposition $\tau_k$ which inverts $X_k$ and $Y_k$. Then we have

$$\Delta_\alpha \circ \theta_{\alpha, \lambda, \lambda - l, X_k} \circ \Theta_{\beta, \lambda, l, t}(P) = 0 \ .$$

Proof.

Let $(\eta_i)_{1 \leq i \leq l}$ be a sequence of elements of $K$ with $\eta_1 = y_\lambda, \eta_2 = y_{\lambda - 1}, \ldots, \eta_\lambda = \cdots = \eta_l = y_1$. Then there exists a sequence of elements $(b_h)_{0 \leq h \leq l}$ with\n
$$S_{1, \eta_1, X_k}^{(l)} = \sum_{h=0}^l b_h \circ_{i=1}^h S_{1, \eta_i, X_k} \ ,$$

where we mean $\circ_{i=1}^h S_{1, \eta_i, X_k} = \text{id}$ if $h = 0$. Thus we have\n
$$\theta_{\alpha, \lambda, \lambda - l, X_k} = \varphi_{\alpha, \lambda, \lambda - l, X_k} \circ_{u=1}^\lambda \text{Prim}_{y_u, X_k} \circ \text{Eval}_{Y_k \to \alpha} \circ S_{1, \alpha, X_k}^{(l)} \circ_{u=1}^\lambda \text{Prim}_{y_u, X_k} \circ \text{Eval}_{Y_k \to \alpha} \circ \text{Eval}_{Y_k \to \alpha} \circ S_{1, \alpha, X_k}^{(l-h)} \ .$$

We remark that, in the above equalities, we use $S_{1, y, X_k} \circ \text{Prim}_{y, X_k} = \text{Id}_K[X_k]$ for any $y \in K$ which is not negative integer. We apply Lemma 3.13 for

$$\Delta_\alpha \circ \theta_{\alpha, \lambda, \lambda - l, X_k} \circ \Theta_{\beta, \lambda, l, t}(P) = \theta_{\alpha, \lambda, \lambda - l, X_k} \circ \Theta_{\beta, \lambda, l, t} \circ \Delta_\alpha(P) \ .$$

using (4.12), we obtain the assertion. \hfill $\square$

Let $l, k \in \mathbb{Z}$ with $0 \leq l \leq k$. We define a set of differential operators

$$X_{l,k} = \{V = \partial_1 \cdots \partial_l \mid \partial_i \in \{1/\alpha, \partial_{\theta_{l_i}} - \rho y_1/\alpha\}, \# \{1 \leq i \leq l, \partial_i = 1/\alpha\} = k\} \ .$$

Let $\lambda = (\lambda_i)_{1 \leq i \leq \rho}, \nu = (\nu_i)_{1 \leq i \leq \rho}$ be two integral vectors in $\mathbb{Z}^\rho$. By Lemma 4.8 one gets that

$$\frac{\partial^j}{\partial \alpha^l}(\Theta_{\alpha, \lambda, \nu}(\hat{P})) = \sum_{1 \leq |I| \leq l} \sum_{V \in X_{l,k}} \Theta_{\alpha, \lambda, \nu - 1}(V(\hat{P})) \ .$$

By the Leibnitz formula, $V(\hat{P})$ is a linear combination (over $K[1/\alpha]$) of the derivatives $\frac{\partial^u}{\partial \alpha^v}(\hat{P})$ for $0 \leq u \leq l - |I|$. Since $\hat{P} = fg$, it is a linear combination of $g^u f^v$, for $0 \leq u \leq l - |I|$. Relying on Lemma 3.13 and Lemma 4.9 together with the same argument of [4] Lemma 9, we obtain
Lemma 4.10. (confer [3] Lemma 9) Let $l := (1, 2, \ldots, \rho) \in \mathbb{Z}^\rho$ and $I = (a_1, \ldots, a_\rho) \in \mathbb{Z}^\rho$ with $a_i \geq 0$. Let $0 \leq l$ be an integer with $|I| \leq l$. Assume further either of these two to be true:

(a) The $2\rho$ dimensional vector $(I, l-I)$ has two coordinates in common (in other words, either $i - a_i = j - a_j$ for some pair $i \neq j$ or for some $i$, $1 \leq i - a_i \leq \rho$, that is one of the coordinates of $I$ is equal to one of the coordinates of $l-I$).

(b) There exists an index $1 \leq s \leq \rho$ such that $0 \leq a_s - s < 2\rho - l + |I|$.

Then, $\Delta_n \circ \Theta_{\beta, l,l} \circ \Theta_{a,l-1}(g_{\rho, l}) = 0$ for all $0 \leq u \leq l - |I|$. Moreover, the smallest integer $l$ for which there exists $I = (a_1, \ldots, a_\rho) \in \mathbb{Z}^\rho$ satisfying $a_i \geq 0$ and $|I| \leq l$ where neither (a) nor (b) holds, is indeed $(2n+1)^2$.

To completes the proof of Proposition 4.4 it remains to show (iii). Since Lemma 4.10 ensures that

$$\frac{\partial^l}{\partial t^l}(C_{u,m}) \bigg|_{\alpha=\beta} = 0$$

for all $0 \leq l \leq (2n+1)^2 - 1$,

it follows that $C_{u,m}$ is divisible by $(\alpha - \beta)(2n+1)^2$, which is (iii). Combining this result and Lemma 4.5 we obtain Proposition 4.4.

4.1.4 Fourth step

We study the non-vanishing of the constant $c_{u,m} = c_{u,m}$ in Proposition 4.4. At this stage we shall need to consider specializing the variables $t_i, s_j$ to 1 instead of $\alpha_i$. Since we are going to proceed by induction on $m$, only one set of variables $t_{m,s_j}$ will be considered at a time, the others acting essentially as scalars. We shall thus denote by $\varphi_{1,x,y,s_j}$ the map taking $t_{1,y} \mapsto 1/(k+y)$ (compare with (4.3)). Similarly, $\psi_{y,x} = \varphi_{y,x}^{-1} \circ \varphi_{1,x,y,s_j}$ so that $\psi_{y,x} = \varphi_{1,x}^{-1} \circ \psi_{1,x,y,s_j}$.

We shall write $\Xi = \varphi_{1,x}^{-1} \circ \psi_{1,x,y,s_j}$ (compare with (4.2)).

Lemma 4.11. Set

$$B_m(t_m) = B(t) = \prod_{j=1}^{d} \prod_{s_j=1}^{r_j} \left[ t_{m,s_j}^m (t_{m,s_j} - 1)^{\rho_n} \right] \prod_{(j_1,s_1) < (j_2,s_2)} (t_{m,s_{j_2}} - t_{m,s_{j_1}}).$$

Then, we have

$$c_{u,m} = (-1)^{\rho^2n(m-1)} c_{u,u+\rho(n+1),m-1} \cdot \psi_{v,x}(B(t)).$$

Proof.

Recall that by (4.3),

$$D_{n,m} := \prod_{i=1}^{m} \gamma_i^{\rho(u+1)+\rho^2n+(\sum_{1 \leq j_1 < j_2 \leq d} r_{j_1}r_{j_2} + \sum_{j=1}^{d} (\frac{j}{2}))}.$$

We are going to evaluate $D_{n,m}$ at $\alpha_m = 0$ and thus separate the variables in $Q_{n,m}$ (recall that $Q_{n,m}$ is defined in equation (4.7)) first. By definition, one has

$$Q_{n,m}(t) = Q_{n,m-1}(t) \cdot B(t) C(t),$$

with
where
\[
C(t) = \prod_{j=1}^{d} \prod_{s_j=1}^{r_j} \prod_{k=1}^{m-1} (\alpha_m t_{m,s_j} - \alpha_k)_{\text{on}} \cdot \prod_{i=1}^{m-1} \prod_{j=1}^{r_j} (\alpha_i t_{i,s_j} - \alpha_m)_{\text{on}} \cdot \prod_{i=1}^{m-1} \prod_{j=1}^{r_j (1 \leq j_1 < j_2 \leq d \atop 1 \leq s_{j_1} \leq r_{j_1} \atop 1 \leq s_{j_2} \leq r_{j_2})} (\alpha_m t_{m,s_{j_2}} - \alpha_i t_{i,s_{j_1}}).
\]

Note that \(Q_{n,u,m-1,B}\) do not depend on \(\alpha_m\), and \(\psi_{t,x}\) treats \(\alpha_m\) as a scalar. Hence,
\[
(4.13) \quad \text{Eval}_{\alpha_m \to 0} (D_{n,u,m}) = c_{n,u,m} \prod_{i=1}^{m-1} (-\alpha_i)^{(2n+1)\rho^2} \prod_{1 \leq i < j \leq m-1} (\alpha_j - \alpha_i)^{(2n+1)\rho^2} = \psi_1 (Q_{n,u,m-1}(t)B(t)\text{Eval}_{\alpha_m \to 0} (C(t))).
\]

But
\[
\text{Eval}_{\alpha_m \to 0} (C(t)) = \prod_{i=1}^{m-1} (-\alpha_i)^{\rho^2} \prod_{i=1}^{m-1} \prod_{j=1}^{r_j} (\alpha_i t_{i,s_j})_{\text{on}} \prod_{i=1}^{m-1} \prod_{j=1}^{r_j} (-\alpha_i t_{i,s_j})^\rho = (-1)^{\rho^2 (m-1)(n+1)} \alpha_i^{(2n+1)\rho^2} \prod_{i=1}^{m-1} \prod_{j=1}^{r_j} t_{i,s_j}^{\rho (n+1)}.
\]

We now note that \(\psi_{m,x}\) treats the variables \(t_{i,s_j}, 1 \leq i \leq m-1\) as scalars and \(\Xi\) treats variables \(t_{m,s_j}\) as scalars and remark
\[
Q_{n,u,m-1}(t)\text{Eval}_{\alpha_m \to 0} (C(t)) = (-1)^{\rho^2 (m-1)(n+1)} \prod_{i=1}^{m-1} \alpha_i^{(2n+1)\rho^2} Q_{n,u+\rho(n+1),m-1}(t).
\]

Thus
\[
\psi_{t,x} (Q_{n,u,m-1}(t)B(t)\text{Eval}_{\alpha_m \to 0} (C(t))) = (-1)^{\rho^2 (m-1)(n+1)} \prod_{i=1}^{m-1} \alpha_i^{(2n+1)\rho^2} \Xi (Q_{n,u+\rho(n+1),m-1}(t)) \psi_{m,x} B(t).
\]

Using the relation (4.13), taking into account \(D_{n,u+\rho(n+1),m-1} = \psi_{t,x} (Q_{n,u+\rho(n+1),m-1}(t))\) and simplifying,
\[
c_{n,u,m} = (-1)^{\rho^2 n(m-1)} c_{n,u+\rho(n+1),m-1} \cdot \psi_{m,x} (B(t)).
\]

This completes the proof of the lemma. \(\Box\)

By Lemma (4.11) we have
\[
(4.14) \quad c_{n,u,m} = (-1)^{\rho^2 n(m-1)/2} c_{n,u+(m-1)\rho(n+1),1} \prod_{i=2}^{m} \psi_{i,x} (B_i(t_i)).
\]

By the definition of \(c_{n,u+(m-1)\rho(n+1),1}\) (see Proposition (4.4)), we have
\[
(4.15) \quad c_{n,u+(m-1)\rho(n+1),1} = \prod_{j=1}^{d} \prod_{s_j=1}^{r_j} \left[ (u+(m-1)\rho(n+1))(t_{1,s_j} - \alpha_1)\right]_{\text{on}} \prod_{(j_1,s_{j_1}) < (j_2,s_{j_2})} (t_{1,s_{j_2}} - t_{1,s_{j_1}}),
\]

where
\[
B_1(\alpha_1, t_1) = \prod_{j=1}^{d} \prod_{s_j=1}^{r_j} \left[ (u+(m-1)\rho(n+1))(t_{1,s_j} - \alpha_1)\right]_{\text{on}} \prod_{(j_1,s_{j_1}) < (j_2,s_{j_2})} (t_{1,s_{j_2}} - t_{1,s_{j_1}}).
\]
4.1.5 Last step

Let \( u \) be a non-negative integer. By (4.14) and (4.15), to complete the proof of non-vanishing of \( c_{n,u,m} \), it is enough to prove

\[
\phi_{j=1}^{d} \varphi_{s_j=1}^{r_j} \varphi_{1,x_j,s_j}(B(t)) \neq 0,
\]
where

\[
B(t) = \prod_{j=1}^{d} \prod_{s_j=1}^{r_j} \left[ t^{u_{j,s_j}(t_{j,s_j} - 1)^{\rho n}} \right] \prod_{(j_1,s_1) < (j_2,s_2)} (t_{j_2,s_2} - t_{j_1,s_1}).
\]

To prove (4.16), we introduce the column vectors

\[
M_h := \left( \int_{1}^{0} t^{x_j + h - 1} (t_1 - 1)^{\rho n} \right)_{1 \leq h \leq \rho}.
\]

We now prove \( \det(M) \neq 0 \). By changing \( x_j \) to \( x_j + u \) for \( 1 \leq j \leq d \), we may assume \( u = 0 \).

We are going to prove the following key proposition.

**Proposition 4.12.** Suppose \( x_i \neq x_j \) for all \( i \neq j \) (1 \( \leq i \leq d \)). Then we have \( \det(M) \neq 0 \).

We now study the matrix \( M \). For this purpose, we denote by \( C(x) \) the row vector:

\[
C(x) = \left( \int_{1}^{0} t^{x_j + h - 1} (t_1 - 1)^{\rho n} \right)_{1 \leq h \leq \rho}.
\]

We have the following lemma.

**Lemma 4.13.**

\[
M = \left( \frac{(-1)^{s_j - 1} d^{s_j - 1} C(x_j)}{dx_j^{s_j - 1}} \right)_{1 \leq j \leq d, 1 \leq s_j \leq r_j}.
\]

Proof.

By simply noting that the integration with respect to \( t \) and the differentiation with respect to \( x \) commutes, one gets:

\[
\frac{d^k C(x)}{dx^k} = \left( \int_{1}^{0} \frac{d^k [t^{x+h-1}(t-1)^{\rho n}]}{dx^k} dt \right) = \left( \int_{1}^{0} t^{x+h-1}(t-1)^{\rho n} \log(t)^k dt \right).
\]

The lemma follows by adjusting the sign. \( \square \)

**Lemma 4.14.** Let \( y > 0 \) be a real number, and \( m \geq 0 \) be an integer. We have

\[
F(y,m) := \int_{1}^{0} t^y (t-1)^m dt = \frac{(-1)^m m!}{\prod_{j=0}^{m} (y + j + 1)}.
\]
Proof.
The proof is simply done by induction on \( m \) (confer Euler’s Beta function in \[1, 29\]),
since
\[
F(y, m) = \left[ \frac{t^{y+1}}{(y+1)\cdot (t-1)^m} \right]_0^1 = -\frac{m}{y+1} \cdot F(y+1, m-1)
\]

\( \square \)

**Lemma 4.15.** Set \( \tilde{Q}(x) = \prod_{j=1}^{r^m} (x + j) \) and, for \( 1 \leq h \leq \rho \),
\( \tilde{P}_h(x) = \prod_{j=1}^{\rho-1}(x + j) \prod_{j'=h+1}^{h}(x + j' + mn) \) (although these functions actually depend on \( \rho, n \),
we omit the subscripts, except in the proof below as they will be fixed and play no role).
Then we have
\[
\tilde{C}(x) = (-1)^{\rho n}(\rho n)! \cdot \left( \frac{\tilde{P}_h(x)}{\tilde{Q}(x)} \right)_{1 \leq h \leq \rho}.
\]

Proof.
The lemma follows from Lemma \[4.13\] applied for \( m = \rho n \) and \( y = x + h - 1 \),
\( 1 \leq h \leq \rho \).

We now set \( P(x) = (\tilde{P}_h(x))_{1 \leq h \leq \rho} \) and define the matrix \( N \)
\[
N = \left( \frac{d^{s_j-1}P(x_j)}{dx_j^{s_j-1}} \right)_{1 \leq j \leq d, 1 \leq s_j \leq r_j}.
\]
We have

**Lemma 4.16.**
\[
\det(M) = \frac{(-1)^{\rho^2 n-\rho/2+\frac{1}{2}(\sum_{j=1}^{d} r_j^2)/(\rho n)!\rho}}{\prod_{j=1}^{d} Q(x_j)^{r_j}} \det(N).
\]

Proof.
Taking into account Lemma \[4.13\] we have on the one hand :
\[
\det(M) = (-1)^{\sum_{j=1}^{d} \sum_{s_j=1}^{r_j} (s_j-1)} \det \left( \frac{d^{s_j-1}C(x_j)}{dx_j^{s_j-1}} \right)_{1 \leq j \leq d, 1 \leq s_j \leq r_j}.
\]
On the other hand,
\[
\det \left( \frac{d^{s_j-1}C(x_j)}{dx_j^{s_j-1}} \right)_{1 \leq j \leq d, 1 \leq s_j \leq r_j} = \det \left( \frac{(\rho n)!d^{s_j-1}P(x_j)}{dx_j^{s_j-1}} \right)_{1 \leq j \leq d, 1 \leq s_j \leq r_j}.
\]
Using the multi-linearity of determinant again,
\[
\det \left( \frac{d^{s_j-1}\tilde{C}(x_j)}{dx_j^{s_j-1}} \right)_{1 \leq j \leq d, 1 \leq s_j \leq r_j} = (-1)^{\rho^2 n}(\rho n)!\rho \det \left( \frac{d^{s_j-1}\tilde{P}_h(x_j)}{dx_j^{s_j-1}} \right)_{1 \leq j \leq d, 1 \leq s_j \leq r_j}.
\]
However, since we have
\[
\frac{d}{dx_j} \left( \frac{\tilde{P}_h(x)}{Q(x)} \right) = \frac{\frac{d}{dx_j} \tilde{P}_h(x_j)}{Q(x_j)} - \frac{\tilde{P}_h(x_j)}{Q(x)} \frac{d}{dx_j} \tilde{Q}(x_j),
\]

\[
\frac{d}{dx_j} \left( \frac{\tilde{P}_h(x)}{Q(x)} \right) = \frac{\frac{d}{dx_j} \tilde{P}_h(x_j)}{Q(x_j)} - \frac{\tilde{P}_h(x_j)}{Q(x)} \frac{d}{dx_j} \tilde{Q}(x_j).
\]
the Leibniz formula and the multi-linearity of the determinant yields:

\[
\det(M) = \frac{(-1)^{\sum_{j=1}^{d} \sum'_{j=1}^{r_j}(s_j-1)} \cdot (-1)^{\rho^2 n}(\rho n)!^\rho}{\prod_{j=1}^{d} Q(x_j)^{r_j}} \det(N).
\]

The lemma follows up to the computation of the normalizing combinatorial factor. To ease reading, we provide a couple of details:

\[
\sum_{j=1}^{d} \sum_{s_j=1}^{r_j} (s_j - 1) = \sum_{j=1}^{d} \frac{r_j^2 - r_j}{2} - \frac{\rho}{2},
\]

and thus

\[
(-1)^{\sum_{j=1}^{d} \sum'_{j=1}^{r_j}(s_j-1)} \cdot (-1)^{\rho^2 n}(\rho n)!^\rho = (-1)^{\rho^2 n - \rho/2 + (\sum_{j=1}^{d} r_j^2)/2}(\rho n)!^\rho.
\]

We now compute \(\det(N)\). For this, we introduce a few notations. We first define, for a set of variables \(w_1, \ldots, w_\rho\),

\[
V = V(w_1, \ldots, w_\rho) = \det \left( w_j^i \right)_{0 \leq i \leq \rho-1, 1 \leq j \leq \rho} = \prod_{1 \leq i < j \leq \rho} (w_j - w_i),
\]

the usual Vandermonde determinant and the degenerate version corresponding to Hermit interpolation problem:

\[
H = H(x_1, \ldots, x_d) = \left( \frac{d^{r_j-1} x_j^i}{dx_j} \right)_{0 \leq i \leq \rho-1, 1 \leq j \leq d, 1 \leq s_j \leq r_j}.
\]

The evaluation of the Hermite determinant is indeed classical, found in T. Muir’s encyclopedia, [16]. It is attributed to L. Schendel, [24] and recent appearances in the literature can be found for instance in [28] by A. van der Poorten and by Loring W. Tu [26]. We rely on this evaluation in the following lemma.

**Lemma 4.17.** One has

\[
\det(H) = (-1)^{\sum_{j=1}^{d} r_j(r_j-1)/2} \cdot \left( \prod_{i=1}^{\rho-1} \prod_{s_i=0}^{r_i} s_i! \right) \cdot \prod_{1 \leq i < j \leq d} (x_j - x_i)^{r_i r_j}.
\]

We now compute special values of the particular interpolation determinant we are interested in.

**Lemma 4.18.** Define the non-degenerate analog of \(N\) as

\[
\tilde{N}(w_1, \ldots, w_\rho) = (P(w_j))_{1 \leq j \leq \rho}.
\]

Then,

\[
\det(\tilde{N}(-1, \ldots, -\rho)) = (-1)^{\rho(\rho-1)/2} \prod_{i=1}^{\rho-1} (d!)^2 \prod_{i=1}^{\rho-1} \binom{i + \rho n}{\rho n}.
\]

Proof. Recall that \(\tilde{P}_1(x) = \prod_{j=1}^{\rho-1} (x + j) \prod_{j=i+1}^{\rho} (x + j + \rho n)\), so, if \(-\rho \leq -i \leq -1\), one has

\[
P(-i) = \mathcal{P}(\cdot, \ldots, \cdot, \tilde{P}_1(-i), 0, \ldots, 0).
\]
It follows that

$$\det \left( \tilde{N}(-1, \ldots, -\rho) \right) = \prod_{i=1}^{\rho} \tilde{P}_i(-i) = (-1)^{\rho(\rho-1)/2} \prod_{i=1}^{\rho} (i-1)! \prod_{i=1}^{\rho-1} \prod_{i=1}^{\rho-1} \left( i + \frac{\rho n}{\rho} \right) .$$

We are now in a position to compare determinants.

**Lemma 4.19.** We have the following equality between determinants. Let $R$ be the matrix representing the polynomials $(\tilde{P}_i(x))_{1 \leq i \leq \rho}$ in the standard basis $t(1, \ldots, x^{\rho-1})$. This is a square $\rho \times \rho$ matrix. Then,

$$\det \left( \tilde{N}(w_1, \ldots, w_\rho) \right) = \det(R) \cdot V ,$$

and

$$\det (N(x_1, \ldots, x_d)) = \det(R) \cdot \det(H) .$$

**Proof.**

The matrix $N$ is obtained from the matrix $H$ by multiplication by $R$ and similarly, $\tilde{N}$ is obtained by multiplying the Vandermonde matrix by $R$. $\square$

**Lemma 4.20.** One has

$$\det (N(x_1, \ldots, x_d)) = (-1)^{\sum_{i=1}^{d} r_i(r_i-1)/2} \prod_{1 \leq i < j \leq \rho} (j-i) \prod_{i=1}^{d} \prod_{s_j=0}^{r_j-1} \prod_{1 \leq i < j \leq d} (x_j - x_i)^{r_i r_j} .$$

**Proof.**

It is enough to use Lemma 4.19 with the Vandermonde equality at $(-1, \ldots, -\rho)$ to compute $\det(R)$ and then plug in the constant in the equality linking $\det(N)$ with the determinant of the Hermite matrix of Lemma 4.19. One finds

$$\det(R) = \frac{\det(\tilde{N}(-1, \ldots, -\rho))}{V(-1, \ldots, -\rho)} = \frac{(-1)^{\rho(\rho-1)/2} \prod_{i=1}^{\rho} (i!)^2 \prod_{i=1}^{\rho} (i+\rho n)}{\prod_{1 \leq i < j \leq \rho} (j-i)} = \frac{\prod_{i=1}^{\rho} (i!)^2 \prod_{i=1}^{\rho} (i+\rho n)}{\prod_{1 \leq i < j \leq \rho} (j-i)} .$$

Thus, by above equality and Lemma 4.17 we obtain the desire equality. $\square$

## 5 Proof of Theorems

### 5.1 Analytic estimates in Padé approximation

In this section, we use the following notations. Let $K$ be a number field and $v$ a place of $K$. Denote by $K_v$ the completion of $K$ at $v$ and $| \cdot |_v$, by the absolute value corresponding to $v$. Throughout the section, the small o-symbol $o(1)$ refers when $n$ tends to infinity. Put $\varepsilon_v = 1$ if $v|\infty$ and 0 otherwise.

Let $I$ be a non-empty finite set of indices, $A = K[\alpha_i]_{i \in I}[z, t]$ be a polynomial ring in indeterminate $\alpha_i, z, t$. We set $\|P\|_v = \max\{|a|_v\}$ where $a$ runs in the coefficients of $P$. Thus the ring $A$ is endowed with a structure of normed vector space. If $\phi$ is an endomorphism of $A$, we denote by $\|\phi\|_v$ the endomorphism norm defined in a standard way $\|\phi\|_v = \inf\{M \in \mathbb{R}, \forall x \in A, \|\phi(x)\|_v \leq M\|x\|_v = \sup\{\|\phi(x)\|_v : 0 \neq x \in A\}$. This norm is well defined provided $\phi$ is continuous. Unfortunately, we will have to deal also
with non-continuous morphisms. In such a situation, we restrict the source space to some appropriate sub-vector space $E$ of $A$ and talk of $\|\phi\|_v$ with $\phi$ seen as $\phi|_E : E \to A$ on which $\phi$ is continuous. In case of perceived ambiguity, it will be denoted by $\|\phi|_E,v$. The degree of an element of $A$ is as usual the total degree with respect to all the $\alpha_i, t$ and $z$.

**Lemma 5.1** (confer [4], Lemma 13). Let $E_N$ be the subspace of $A$ consisting of polynomials of degree at most $N$ in $t$. Let $x = a/b \in \mathbb{Q} \cap [0, 1)$ with $a, b$ are coprime integers. Then $\text{Prim}_x : E_N \to A$ as defined in Notation 5.3 satisfies

$$\|\text{Prim}_x\|_{E_N,v} \leq \max\{|1/(k + x + 1)|_v, 0 \leq k \leq N\} \leq |d_{N+1}(a, b)|^{-1}_v,$$

with $d_N(a, b) = \text{l.c.m.}(a, a+b, \ldots, a+bN)$ where l.c.m. denotes the least common multiple.

From the preceding lemma, we deduce the following estimates.

**Lemma 5.2** (confer [4], Lemma 14). Let $x_j = a_j/b_j \in \mathbb{Q} \cap [0, 1)$ for $1 \leq j \leq d$ as in Theorem 2.1. We denote by $v$ the place of $\mathbb{Q}$ such that $v|w$. Then, one has:

(i) **The polynomial** $P_i(z) = P_{n,l}(\alpha, x|z)$ **satisfies**

$$\log \|P_i(z)\|_v \leq \begin{cases} \frac{n[K_v : \mathbb{Q}_v]}{|K : \mathbb{Q}|} \left( \rho m \log 2 + \rho \left( \log(\rho m + 1) + \rho m \log \left( \frac{\rho m + 1}{\rho m} \right) \right) + o(1) \right) & \text{if } v|\infty \\ \sum_{j=1}^d r_j \log \|\mu_n(x_j)\|_v^{-1} & \text{otherwise} \end{cases}$$

Recall that $P_i(z)$ is of degree $\rho n$ in each variable $\alpha_i$, of degree $\rho mn + l$ in $z$ and constant in $t$.

(ii) **The polynomial** $P_{i,i,j,s_j}(z) = P_{n,l,i,j,s_j}(\alpha, x|z)$ **satisfies**

$$\log \|P_{i,i,j,s_j}(z)\|_v \leq \begin{cases} \frac{n[K_v : \mathbb{Q}_v]}{|K : \mathbb{Q}|} \left( \rho m \log 2 + \rho \left( \log(\rho m + 1) + \rho m \log \left( \frac{\rho m + 1}{\rho m} \right) \right) + o(1) \right) & \text{if } v|\infty \\ \sum_{j=1}^d r_j \log \|\mu_n(x_j)\|_v^{-1} + \log \|d_{\rho m(n+1)+1}(a_j, b_j)\|_v^{-s_j} & \text{otherwise} \end{cases}$$

Also, $P_{i,i,j,s_j}(z)$ is of degree $\leq \rho mn + l$ in $z$, of degree $\rho m$ in each of the variables $\alpha_j$ except for the index $i$ where it is of degree $\rho n + 1$ (recall that $\varphi_{\alpha_i, x_j, s_j}$ involves multiplication by $[\alpha_i]$).

(iii) **For any integer** $k \geq 0$, the polynomial $\varphi_{\alpha_i, x_j, s_j} \circ [t^{k+n}](P_i(t))$ **satisfies**

$$\log \|\varphi_{\alpha_i, x_j, s_j} \circ [t^{k+n}](P_i(t))\|_v \leq \begin{cases} \frac{n[K_v : \mathbb{Q}_v]}{|K : \mathbb{Q}|} \left( \rho m \log 2 + \rho \left( \log(\rho m + 1) + \rho m \log \left( \frac{\rho m + 1}{\rho m} \right) \right) + o(1) \right) & \text{if } v|\infty \\ \sum_{j=1}^d r_j \log \|\mu_n(x_j)\|_v^{-1} + \log \|d_{\rho m(n+1)+1}(a_j, b_j)\|_v^{-s_j} & \text{otherwise} \end{cases}$$

By definition, it is a homogeneous polynomial in just the variables $\alpha$ of degree $\leq \rho mn + l + k + n + 1$.

Proof.

For a non-negative integer $N$, we denote $E_N$ the sub-vector space of $K[y_1, \ldots, y_m, z, t]$ consisting of polynomials of degree at most $N$ in the variables $y_i$ and define a morphism

$$\Gamma : E_N \to K[\alpha_1, \ldots, \alpha_m, z, t]; \ P(y_1, \ldots, y_m, z, t) \mapsto P(t - \alpha_1, \ldots, t - \alpha_m, z, t).$$
Then, by \cite{4} Lemma 5.3 (iv)], we have \[\|\Gamma\|_{E_{N,v}} \leq (2^{N^m(N+1)^m})^{\varepsilon_v[K_v:Q_w]/[K:Q]} .\]

Set \( B_{n,l}(y,t) = t^l \prod_{i=1}^m y_i^{\rho_{mn}} \), since \( B_{n,l} \) is a monomial, its norm \( \|B_{n,l}\|_v = 1 \). By definition, \( P(z) = \text{Eval}_{t \rightarrow z} \circ \delta_{j=1}^d S_{n,x_j}^{(r_j)} \circ \Gamma(B_{n,l}) \), and thus, by sub-multiplicativity of the endomorphism norm,

\[\|P(z)\|_v \leq 2^{\varepsilon_v \rho_{mn}[K_v:Q_w]/[K:Q]} \prod_{j=1}^d \|S_{n,x_j}\|_v^{r_j} e^{\varepsilon_v \alpha(1)} ,\]

(one can use \cite{4} Lemma 5.2 (iv)] and \( N = \rho m(n+1) + l \) for property \cite{4} Lemma 5.2 (v)] using \( 0 \leq l \leq \rho m \), and note that the original polynomial is a constant in \( z \) so the evaluation map is an isometry).

Now, by definition, \( P_{l,i,j,s_j}(z) = \varphi_{\alpha_i,z_j} \circ \Theta(P(z)) \) where

\[\Theta : K[\alpha_1, \ldots, \alpha_m, z, t] \rightarrow K[\alpha_1, \ldots, \alpha_m, z, t]; P \mapsto \frac{P(\alpha_i, z) - P(\alpha_i, t)}{z - t} .\]

By definition, \( \varphi_{\alpha_i,z_j} = [\alpha_i] \circ \text{Eval}_{t \rightarrow \alpha_i} \circ \text{Prim}_{s_j} \). Using again \cite{4} Lemma 5.2 (iii), (vi)] with \( N = \rho m(n+1) \) and \cite{4} Lemma 5.2 (i), (ii)] with \( N = \rho m(n+1) \), and since \( \rho mn = n \exp(\alpha(1)) \), one gets (ii).

Finally, we have \( \varphi_{\alpha_i,z_j} \circ [t^{k+n}](P(t)) = [\alpha_i] \circ \text{Eval}_{t \rightarrow \alpha_i} \circ \text{Prim}_{s_j} \circ [t^{k+n}](P(t)). \)

Again, using \cite{4} Lemma 5.2, one gets (iii).\( \square \)

Recall that if \( P \) is a homogeneous polynomial in some variables \( y_i, i \in I \), for any point \( \alpha = (\alpha_i)_{i \in I} \in K^{\text{Card}(I)} \) where \( I \) is any finite set, and \( \| \cdot \|_v \) stands for the sup norm in \( K^{\text{Card}(I)} \), with \( C_v(P) = (\deg(P) + 1)^{\frac{\varepsilon_v[K_v:Q_w]/[K:Q]}{\text{Card}(I)}} \),

one has

\[\|P(\alpha)\|_v \leq C_v(P)\|P\|_v \cdot \|\alpha\|_v^{\deg(P)} .\]

So, the preceding lemma yields trivially estimates for the \( v \)-adic norm of the above given polynomials.

**Remark 5.3.** It may be worthwhile to note that it is also possible to estimate with respect to \( h_v(\alpha_i - \beta) \) instead of \( h_v(\alpha_i) \) which can be useful in certain circumstances (saving of the archimedean error term \( \rho mn \log(2) \), useful if the local heights of \( \beta, \alpha_i \) are not too far apart).

**Lemma 5.4** (confer \cite{4}, Lemma 15). Let \( n \) be a positive integer, \( x_j = a_j/b_j \in Q \cap [0,1) \) and \( \beta \in K \) with \( \|x_i\|_v \leq |\beta|_v \). We denote by \( w \) the place of \( Q \) such that \( v|w \). Then we have for all \( 0 \leq l \leq \rho m, 1 \leq i \leq m, 1 \leq j \leq d \) and \( 1 \leq s_j \leq r_j \),

\[\log |R_{l,i,j,s_j}(\beta)|_v \leq \rho m(n+1) \log \|\alpha\|_v + (n+1) \log \left( \frac{\|\alpha\|_v}{|\beta|_v} \right) + \log \left( \frac{\varepsilon_v|\beta|_v}{|\beta|_v - \|\alpha\|_v} + (1 - \varepsilon_v) \right) \]

\[+ \begin{cases} n\frac{[K_v:Q_w]}{[K:Q]} (\rho m \log(2) + \rho (\log(\rho m + 1) + \rho m \log \left( \frac{\rho m + 1}{\rho m} \right) + o(1)) \quad \text{if } v \nmid \infty \\ \sum_{j=1}^d r_j \log |\mu_i(x_j)|_v^{-1} + \log \left| \delta_{\rho m(n+1)+1}(a_j, b_j) \right|_v^{-1} \quad \text{otherwise} \end{cases} .\]
Proof.
By the definition of $P_1(z)$, as formal power series, we have
$$R_{t, i, j, s}(z) = \sum_{k=0}^{\infty} \varphi_{\alpha, x, s}(t^{k+n} P_1(t)) z^{k+n+1},$$
using the triangle inequality, the fact that $0 \leq l \leq pm$ and Lemma 5.2 (iii), together with inequality (5.1)
$$|R_{t, i, j, s}(\beta)| \leq \|\alpha\|_{v}^{pm(n+1)} \sum_{k=0}^{\infty} \left(\frac{\|\alpha\|_{v}}{\|\beta\|_{v}}\right)^{n+1+k} 2^{p_{n} m}[K, \mathbb{Q}] [K : \mathbb{Q}] \prod_{j=1}^{d} \|S_{n, x, j}\|_{v}^{r_{j}} \left(e^{n_{0}(1)} [d_{pm(n+1)+1}(a_{j}, b_{j})]^{-s_{j}} v \right) \text{ if } v \mid \infty \right. \right.$$
$$\text{and the lemma follows using geometric series summation.} \quad \square$$

5.2 Proof of Theorem 2.1
To prove Theorem 2.1 we show the following theorem.

Theorem 5.5. We use the same notations as in Theorem 2.1 Denote $x_{j} = a_{j}/b_{j}$ with $a_{j}, b_{j} \in \mathbb{Z}$ coprime. For a strictly positive integer $n$, we put
$$D_{n}(m, x) = D_{n} = l.c.m.(a_{j}, a_{j} + b_{j}, \ldots, a_{j} + b_{j}(pm(n+1) + 1)| j = 1, \ldots, d).$$
For any place $v \in \mathcal{M}_{\mathcal{K}}$ we define the constants
$$c(x, v) = \varepsilon_{v} [K : \mathbb{Q}] \left[\log \left(2 \log(pm) + \rho \left(\log(pm) + \rho \log \left(pm + 1 \rho \log \left(pm + 1 \rho m\right)\right)\right)\right] \right.$$n
where $\varepsilon_{v} = 1$ if $v \mid \infty$ and 0 otherwise. We also define
$$A_{n}(\alpha, x, \beta) = \log(\beta)_{v_{0}} - (pm + 1) \log \|\alpha\|_{v_{0}} - c(x, v_{0}) + (1 - \varepsilon_{v}) \cdot \max_{1 \leq j \leq d} (r_{j}) \lim_{n \to \infty} \frac{\log |D_{n}|_{v_{0}}}{n},$$
$$U(\alpha, x, \beta) = pmh_{v_{0}}(\alpha, \beta) + c(x, v_{0}),$$
and finally
$$V(\alpha, x, \beta) = \log(\beta)_{v_{0}} - pmh(\alpha, \beta) - pm \log \|\alpha\|_{v_{0}} + pm \log \|\alpha, \beta\|_{v_{0}} - \sum_{j=1}^{d} r_{j} \log \mu(x_{j})$$
$$- \left[pm \log(2) + \rho \left(\log(pm) + \rho \log \left(pm + 1 \rho \log \left(pm + 1 \rho m\right)\right)\right)\right] \right.$n
$$\max_{j}(r_{j}) \lim_{n \to \infty} \sum_{v \in \mathcal{M}_{\mathcal{K}}} \log |D_{n}|_{v} / n.$$ Assume $V(\alpha, x, \beta) > 0$. Then for any positive number $\varepsilon$ with $\varepsilon < V(\alpha, x, \beta)$, there exists an effectively computable positive number $H_{0}$ depending on $\varepsilon$ and the given data such that the following property holds. For any $\lambda := (\lambda_{0} \lambda_{i, j, s})_{1 \leq i \leq m, 1 \leq j \leq d, 1 \leq s_{i} \leq s_{j}} \in K^{pm+1} \setminus \{0\}$ satisfying $H_{0} \leq H(\lambda)$, then we have
$$\left|\lambda_{0} + \sum_{i=1}^{m} \sum_{j=1}^{d} \sum_{s_{i}=1}^{s_{j}} \lambda_{i, j, s} \Phi_{s_{j}}(x_{j}, \alpha_{i}/\beta)\right|_{v_{0}} > C(\alpha, x, \beta, \varepsilon)H_{v_{0}}(\lambda)H(\lambda)^{-\mu(\alpha, x, \beta, \varepsilon)},$$

where
$$\mu(\alpha, x, \beta, \varepsilon) := \frac{A(\alpha, x, \beta) + U(\alpha, x, \beta)}{V(\alpha, x, \beta) - \varepsilon},$$
$$C(\alpha, x, \beta, \varepsilon) := \exp \left(-\left(\frac{\log(2)}{V(\alpha, x, \beta) - \varepsilon} + 1\right)(A(\alpha, x, \beta) + U(\alpha, x, \beta))\right).$$
Proof.

By Proposition 4.1, the matrix $M_n = \begin{pmatrix} P_l(\beta) \\ P_{l,i,j,s_j}(\beta) \end{pmatrix}$ with entries in $K$ is invertible. We apply a linear independence criterion in [4, Proposition 5.6]. We have by Lemma 5.2 (i) together with inequality \[5.1\]
\[
\log \| P_l(\beta) \|_v \leq \varepsilon_v \left( \frac{n p[K_v : Q_w]}{[K : Q]} \left[ \rho m \log(2) + \rho \left( \log(\rho m + 1) + \rho m \log \left( \frac{\rho m + 1}{\rho m} \right) \right) \right] + o(1) \right)
\]
\[
+ (1 - \varepsilon_v) \sum_{j=1}^d r_j \log |\mu_n(x_j)|_v^{-1} + (\rho mn + l)h_v(\alpha, \beta)
\]
\[
\leq n (\rho mh_v(\alpha, \beta) + c(x, v)) + o(1) = nU(\alpha, x, \beta) + o(1).
\]

Similarly, using this time Lemma 5.2 (ii) and inequality \[5.1\],
\[
\log \| P_{l,i,j,s_j}(\beta) \|_v \leq \varepsilon_v \left( \frac{n p[K_v : Q_w]}{[K : Q]} \left[ \rho m \log(2) + \rho \left( \log(\rho m + 1) + \rho m \log \left( \frac{\rho m + 1}{\rho m} \right) \right) \right] + o(1) \right)
\]
\[
+ (1 - \varepsilon_v) \sum_{j=1}^d r_j \log |\mu_n(x_j)|_v^{-1} + (\rho mn + l)h_v(\alpha, \beta) - (1 - \varepsilon_v)s_j \log |D_n|_v
\]
\[
\leq n (\rho mh_v(\alpha, \beta) + c(x, v)) + f_v(n),
\]

where
\[
f_v : \mathbb{N} \to \mathbb{R}_{\geq 0}; \ n \mapsto \rho mh_v(\alpha, \beta) - (1 - \varepsilon_v) \max_{1 \leq j \leq d} (r_j) \cdot \log |D_n|_v.
\]

We define
\[
F_v(\alpha, x, \beta) : \mathbb{N} \to \mathbb{R}_{\geq 0}; \ n \mapsto n (\rho mh_v(\alpha, \beta) + c(x, v)) + f_v(n).
\]

By Lemma 5.4 ensures
\[
\log |R_{l,i,j,s_j}(\beta)|_{v_0} \leq n (-\mathcal{A}(\alpha, x, \beta) + o(1)),
\]
by a linear independence criterion in [4] Proposition 5.6 for $\{\theta_{l,i,j,s_j} := \Phi_{s_j}(x_j, \alpha_i / \beta)\}_{1 \leq i \leq m, 1 \leq j \leq d, 1 \leq s_j \leq r_j}$ and the above data, we obtain the assertions of Theorem 5.5. Using the prime number theorem, we have
\[
\lim_{n \to \infty} \sum_{v \in \mathfrak{m}_K} f_v(n) / n = \max_{1 \leq j \leq d} (r_j) \lim_{n \to \infty} \sum_{v \in \mathfrak{m}_K} \frac{\log |D_n|_v}{n} \leq \max_{1 \leq j \leq d} (r_j) b \rho m,
\]
and
\[
\sum_{v \in \mathfrak{m}_K} c(x, v) = \rho m \log(2) + \rho \left( \log(\rho m + 1) + \rho m \log \left( \frac{\rho m + 1}{\rho m} \right) \right) + \sum_{j=1}^d r_j \log \mu(x_j),
\]
we conclude
\[
\mathcal{A}(\alpha, x, \beta) - \lim_{n \to \infty} \frac{1}{n} \sum_{v \neq v_0} F_v(\alpha, x, \beta)(n) \geq \log |\beta|_{v_0} - \rho mh(\alpha, \beta) + \rho m \log \| (\alpha, \beta) \|_{v_0} - (\rho m + 1) \log \| \alpha \|_{v_0}
\]
\[
- \sum_{v \in \mathfrak{m}_K} c(x, v) - \max_{1 \leq j \leq d} (r_j) \lim_{n \to \infty} \sum_{v \in \mathfrak{m}_K} \frac{\log |D_n|_v}{n} \geq V(\alpha, x, \beta). \]
5.3 Proof of Theorem 2.6

We show that the periodic case (Theorem 2.6) is a corollary of Theorem 2.1.

Lemma 5.6. In the situation of Theorem 2.6, for \( x \in \mathbb{Q} \) which is not a negative integer and any integer \( s \geq 1 \), the \( K \)-subspace of \( K_{b_0} \) generated by \( \{ f_{b,w,x,s}(\beta) \mid 0 \leq i \leq m-1 \} \) is the one generated by \( \{ \Phi_s(x, \alpha_i/\beta) \mid 0 \leq i \leq m-1 \} \).

Proof. We denote the former \( K \)-subspace by \( V_1 \) and the latter by \( V_2 \). Let \( w(z) \in K[z] \) and assume \( \deg w(z) \leq q-1 \). Then the rational function \( w(z)/b(z) \) can be written as

\[
\frac{w(z)}{b(z)} = \sum_{i=1}^{m} \frac{\gamma_i}{z - \alpha_i},
\]

for some \( \gamma_i \in K \). Hence,

\[
\frac{w(z)}{b(z)} = \sum_{i=1}^{m} \gamma_i \sum_{k=0}^{\infty} \frac{\alpha_i^k}{z^{k+1}}.
\]

By definition,

\[
f_{b,w,x,s}(z) = \sum_{i=1}^{m} \gamma_i \Phi_s(x, \alpha_i/z).
\]

Above identity yields \( V_1 \subseteq V_2 \). Conversely, since \( w_0, \ldots, w_{m-1} \) are linearly independent, the rational functions \( 1/(z - \alpha_i) \) are linear combinations of \( w_0(z)/b(z), \ldots, w_{m-1}(z)/b(z) \). By the same argument as above, we obtain \( V_2 \subseteq V_1 \). \( \square \)

Now, using Theorem 2.1 we know that provided \( V(\alpha, x, \beta) > 0 \), the vector space

\[
K \cdot 1 + \sum_{j=1}^{d} \sum_{s_j=1}^{r_j} \sum_{i=1}^{m} K \cdot \Phi_{s_j}(x_j, \alpha_i/\beta)
\]

is of dimension \( \rho m + 1 \) and is equal to

\[
K \cdot 1 + \sum_{j=1}^{d} \sum_{s_j=1}^{r_j} \sum_{i=1}^{m} K \cdot f_{b,w,x_j,s_j}(\beta)
\]

by Lemma 5.6. This concludes the proof of Theorem 2.6.

Remark 5.7. This result shows that considering the periodic case does not produce new numbers. The special case of purely periodic generalized polylogarithmic functions corresponds to \( b(z) = \prod_{l=1}^{l} (z^q - \alpha_l^q) \), that is \( m = lq \) and roots \( \zeta_l^q \alpha_i \). To prove Corollary 2.7 we apply Theorem 2.6 for \( b(z) := \prod_{i=1}^{m} (z^{q_i} - \alpha_i^{q_i}) \) and

\[
w_i(z) := w_{i,i}(z) \prod_{j \neq i} (z^{q_j} - \alpha_i^{q_j})
\]

for \( 1 \leq i \leq m \) and \( 0 \leq l_i \leq q_i - 1 \).

It is worthwhile noting that restricting to the case where \( b(z) \) has only simple roots is necessary. Indeed, if \( b(z) \) does have multiple roots, an argument similar to Lemma 5.6 shows that one gets linearly dependent numbers.
6 Examples

Example 6.1. Let \( p \) be a prime number, \( q \) a positive integer. We take \( d = p \), \( r_1 = \cdots = r_p = 10 \), \( m = 10 \) and

\[
\alpha := \left( 1, \ldots, \frac{1}{10} \right), \quad x := \left( 0, \frac{1}{p}, \ldots, \frac{p-1}{p} \right).
\]

Let \( g, M \in \mathbb{N} \) with \( g, M \geq 2 \). Define \( f_{M,g}(X) \in \mathbb{Q}[X] \) by

\[
f_{M,g}(X) = \left( 2 + \frac{1}{M} \right) X^g - \frac{2}{M} X^{g-1} - 2X + \frac{2}{M}.
\]

Using the Eisenstein’s irreducibility criterion of polynomial, we obtain that \( f_{M,g}(X) \) is an irreducible polynomial in \( \mathbb{Q}[X] \). Let \( \frac{1}{\beta M} \) be a root of \( f_{M,g}(X) \) and put \( K = \mathbb{Q}(\beta M) \).

Let \( \alpha, x, w_{i,l}(z) \) and \( K \).

Let \( M_0 \) be the minimal positive integer with \( |\beta_M^{(h)}| \leq 2 \) for \( 2 \leq h \leq g \) and \( M \) a positive integer with \( M \geq M_0 \). Since \( \text{den}(\beta_M) \leq 2 \) and \( \log |\beta_M|, \leq [K_v : \mathbb{Q}] \log(2)/g \) for \( v|\infty \), we have the following tables showing values of \( \log(M_0) \) such that the right-hand of the inequality below becomes positive whenever \( M \geq M_0 \) for each tuple of \( g, p, q \):

\[
V(\alpha, x, \beta_M) \geq \frac{1}{g} \log |\beta_M| - \frac{100pq}{g} \left( (g - 1) \log(2) + \log(2520) \right) - 10p \log(p)
\]

\[
- \left[ 100pq \log(2) + 100 \left( \log(100pq + 1) + 100pq \log \left( \frac{100pq + 1}{100pq} \right) \right) \right] - 1000pq.
\]

For example, the case \( g = 2 \) is described as follows:

\[
\begin{array}{cccc}
p & 2 & 3 & 4 \\
q & 3158 & 5816 & 8449 & 11072 \\
& 4509 & 8466 & 12398 & 16320 \\
& 7192 & 13748 & 20278 & 26798 \\
& 9868 & 19021 & 28150 & 37268 \\
\end{array}
\]

the case \( g = 3 \):

\[
\begin{array}{cccc}
p & 2 & 3 & 4 \\
q & 4427 & 8104 & 11744 & 15368 \\
& 6298 & 11769 & 17202 & 22620 \\
& 10013 & 19071 & 28092 & 37097 \\
& 13717 & 26362 & 38969 & 51562 \\
\end{array}
\]

the case \( g = 4 \):

\[
\begin{array}{cccc}
p & 2 & 3 & 4 \\
q & 5695 & 10391 & 15038 & 19664 \\
& 8087 & 15071 & 22006 & 28920 \\
& 12834 & 24394 & 35905 & 47395 \\
& 17565 & 33702 & 49789 & 65855 \\
\end{array}
\]
the case $g = 5$:

\[
\begin{array}{cccc}
p \setminus q & 1 & 2 & 3 & 4 \\
2 & 6964 & 12679 & 18332 & 23960 \\
3 & 9876 & 18374 & 26809 & 35219 \\
5 & 15655 & 29718 & 43719 & 57694 \\
7 & 21414 & 41042 & 60608 & 80148 \\
\end{array}
\]

and the case $g = 6$:

\[
\begin{array}{cccc}
p \setminus q & 1 & 2 & 3 & 4 \\
2 & 8233 & 14967 & 21627 & 28256 \\
3 & 11665 & 21676 & 31613 & 41519 \\
5 & 18476 & 35041 & 51532 & 67992 \\
7 & 25263 & 48382 & 71427 & 94441 \\
\end{array}
\]

demonstrated.

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