EDT0L SOLUTIONS TO EQUATIONS IN GROUP EXTENSIONS

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ABSTRACT. We show that the class of groups where EDT0L languages can be used to describe solution sets to systems of equations is closed under direct products, wreath products with finite groups, and passing to finite index subgroups. We also add the class of groups that contain a direct product of hyperbolic groups as a finite index subgroup to the list of groups where solutions to systems of equations can be expressed as an EDT0L language. This includes dihedral Artin groups. We also show that the systems of equations with rational constraints in virtually abelian groups have EDT0L solutions, and the addition of recognisable constraints to any system preserves the property of having EDT0L solutions. These EDT0L solutions are expressed with respect to quasigeodesic normal forms. We discuss the space complexity in which EDT0L systems for these languages can be constructed.

1. Introduction

Formal languages have been successfully employed for the last 40 years to describe important sets in groups, in order to restrict the use of memory in algorithms describing these sets, and give the sets a clear structure. The use of languages to represent solutions to equations made a leap when Ciobanu, Diekert and Elder proved that the sets of solutions to systems of equations in free groups with rational constraints can be expressed as EDT0L languages [4]. Solutions to systems of equations in right-angled Artin groups were then shown to be EDT0L by Diekert, Jez and Kufleitner [11]. Virtually free groups [10], hyperbolic groups [5], and virtually abelian groups [16] followed later.

In the 1960s, Lindenmayer introduced a collection of classes of languages called L-systems. Originally used to study growth of organisms, L-systems saw significant interest in the 1970s and early 1980s, and Lindenmayer’s original classes inspired the definitions of many other L-systems, including Rozenberg’s EDT0L languages [29]. This class has recently had a variety of applications in and around group theory ([6], [3], [2], [9]). For a comprehensive introduction to L-systems, including EDT0L languages, we refer the reader to [30].

Theorem A collects the main results in this paper. The format used to express solutions as words is explained in the preliminaries (Section 2).

Theorem A. Let $G$ and $H$ be groups where solution languages to systems of equations are EDT0L, with respect to normal forms $\eta_G$ and $\eta_H$, respectively, and these EDT0L systems are constructible in $\text{NSPACE}(f)$, for some $f$. Then in the following groups, solutions to systems of equations are EDT0L, and an EDT0L system can be constructed in non-deterministic $f$-space:

1. $G \wr F$, for any finite group $F$ (Proposition 4.9);
2. $G \times H$ (Proposition 4.10);

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Any finite index subgroup of $G$ (Proposition 6.3);

In the following groups, solutions to systems of equations are EDT0L, and an EDT0L system can be constructed in $\text{NSPACE}(n^4 \log n)$:

(4) Any group that is virtually a direct product of hyperbolic groups (Corollary 7.9);
(5) Dihedral Artin groups (Corollary 7.10).

If $\eta_G$ and $\eta_H$ are both quasigeodesic or regular, then the same will be true for the normal forms used in (1), (2) and (3). It is possible to choose normal forms for the groups that are virtually direct products of hyperbolic groups (4), and dihedral Artin groups in (5) that are regular and quasigeodesic.

Whilst an understanding of the set of solutions to a system of equations in a direct product follows immediately from understanding the solutions to the projection onto each of the groups in the direct product, showing that the language can be expressed in the correct format requires more work, which we explore in Section 3. This format is also required to prove Theorem A(1).

The proof of Theorem A(4) is based on Ciobanu, Holt and Rees’ proof of the fact the satisfiability of systems of equations in these groups is decidable [7], in a work that also looks at recognisable constraints. We show that the addition of recognisable constraints to any system of equations preserves the property of having an EDT0L solution language, and use this to show that the class of groups where systems of equations have EDT0L solutions is closed under passing to finite index subgroups.

Equations with rational constraints have also attracted a large amount of attention; the addition of constraints allows for a certain level of control on what each of the variables can be. The fact that systems of equations in free and virtually free groups have EDT0L solutions was also shown to be true if rational constraints are added ([3], [10]). In hyperbolic groups, the addition of rational constraints was shown to preserve the fact that systems of equations have EDT0L solutions if the rational constraints were quasi-isometrically embedded [5]. We generalise the result on systems of equations in virtually abelian groups in [16] to include rational constraints. This result uses virtually abelian equation length, introduced in [16], which assigns a much smaller length to equations in virtually abelian groups than the standard length, and thus results using this length are stronger than equivalent results with standard length. This length is defined using the fact that equations in virtually abelian groups can be described using a number of integers, and the fact that integers can be stored in logarithmic space.

**Theorem B.** Solutions to a system of equations with rational constraints in a virtually abelian group are EDT0L in non-deterministic quadratic space, with respect to virtually abelian equation length, and with respect a regular and quasigeodesic normal form.

Group equations have seen significant interest as to the decidability of the satisfiability of equations in specific classes of groups, since Makanin showed in the 1980s that the satisfiability of systems in free groups was decidable ([21], [22], [23]). The satisfiability of systems of equations has been shown to be decidable or undecidable in a wide variety of other classes of groups ([25], [27], [8], [12], [7], [20]). Describing the set of solutions in any meaningful way has often proved difficult. The structure of sets of solutions in free groups were resolved by Razborov ([26], [25]), however the structure of solutions to systems of equations in soluble Baumslag-Solitar groups, and single
equations in the Heisenberg group are some of the many cases that are yet to be described. The recent use of EDT0L languages in equations has helped to describe a number of cases.

Section 2 covers the preliminaries of the topics used. In Section 3 we prove Lemma 3.5 on the parallel concatenation of words, which is an important part of the proofs of the stability of groups where systems of equations have EDT0L solution languages under direct products (Proposition 4.10), and wreath products with finite groups (Proposition 4.9). Section 4 covers the proofs of those propositions, along with Lemma 4.1 which allows us to understand equations with rational constraints in finite index overgroups of certain groups. We then use Lemma 4.1 to prove Theorem B in Section 5, which shows that the solution language to a system of equations with rational contraints in a virtually abelian group has an EDT0L solution language.

Section 6 includes the addition of recognisable constraints to equations with EDT0L solutions, and is used to prove that the property of systems of equations having EDT0L solution languages passes to finite index subgroups, with respect to the Schreier normal form, based on the normal form used in the finite index overgroup (Proposition 6.3). Section 7 concludes with the proof that systems of equations in groups that are virtually direct products of hyperbolic groups have EDT0L solution languages.

Notation 1.1. We introduce some notation to be used throughout.

- Functions will be written to the right of their arguments.
- Let $G$ be a group. We use $\text{FIN}(G)$ to denote the class of groups that contain $G$ as a finite index subgroup.
- If $S$ is a subset of a group, we define $S^\pm = S \cup S^{-1}$.
- We use $\varepsilon$ to denote the empty word.
- When defining endomorphisms of the free monoid $\Sigma^*$, we do this by defining the action of the endomorphism on some of the elements of $\Sigma$, and the remaining elements of $\Sigma$ are assumed to be fixed. The endomorphism is determined by its action on $\Sigma$.
- If $L$ is a language over an alphabet $\Sigma$, we use $L^c$ to denote the complement of $L$ within $\Sigma^*$.

2. Preliminaries

2.1. Rational and recognisable sets. We cover here the basic definitions of rational and recognisable sets. Both types are used as constraints for variables in equations, and we will use recognisable constraints to show that the class of groups where solutions to systems of equations form EDT0L languages is closed under passing to finite index subgroups.

Recall that a language over $\Sigma$ is any subset of $\Sigma^*$, where $\Sigma$ is a finite set, called an alphabet. Recall also that a regular language is any language accepted by a finite state automaton. We refer the reader to Chapter 2 of [19] for further details on languages and finite state automata.

Definition 2.1. Let $G$ be a group, and $\Sigma$ be a monoid generating set for $G$. Define $\pi : \Sigma^* \to G$ to be the natural homomorphism. We say a subset $A \subseteq G$ is

1. recognisable if $A\pi^{-1}$ is a regular language over $\Sigma$;
2. rational if there is a regular language $L$ over $\Sigma$, such that $A = L\pi$.

Remark 2.2. Recognisable sets are rational.

We give a few examples of recognisable and rational sets.
Lemma 2.5. Finite subsets of any group are rational. Finite subsets of a group $G$ are recognisable if and only if $G$ is finite \[1\]. Finite index subgroups of any group are recognisable, and hence rational.

The following result of Grunschlag relates the rational subsets of a finite index subgroup of a group $G$ to the rational subsets of $G$ itself.

Lemma 2.4 (Corollary 2.3.8 in \[17\]). Let $G$ be a group with finite generating set $\Sigma$, and $H$ be a finite index subgroup of $G$. Let $\Delta$ be a finite generating set for $H$, and $T$ be a right transversal for $H$ in $G$. For each rational subset $R \subseteq G$, such that $R \subseteq Ht$ for some $t \in T$, there exists a (computable) rational subset $S \subseteq H$ (with respect to $\Delta$), such that $R = St$.

Herbst and Thomas proved that recognisable sets in a group $G$ are always finite unions of cosets of a finite index normal subgroup of $G$ \[18\]. This can be used to prove many facts about recognisable sets, including the following lemma.

Lemma 2.5. Let $G$ be a finitely generated group with a finite index subgroup $H$, and let $S \subseteq H$. Then $S$ is recognisable in $G$ if and only if $S$ is recognisable in $H$.

2.2. Group equations. We define here a system of equations within a group, and certain generalisations including twisting and constraints. Twisted equations prove useful in proving that systems of equations with rational constraints in virtually abelian groups have EDT0L solutions.

Definition 2.6. Let $G$ be a group, and $X$ be a finite set of variables. A finite system of equations in $G$ with variables $X$ is a finite subset $\mathcal{E}$ of $G * F_X$, where $F_X$ is the free group on a finite set $X$. If $\mathcal{E} = \{w_1, \ldots, w_n\}$, we view $\mathcal{E}$ as a system by writing $w_1 = w_2 = \cdots = w_n = 1$. A solution to a system $w_1 = \cdots = w_n = 1$ is a homomorphism $\phi : F_X \to G$, and such that $w_1 \phi = \cdots = w_n \phi = 1_G$, where $\phi$ is the extension of $\phi$ to a homomorphism from $G * F_X \to G$, defined by $g \phi = g$ for all $g \in G$.

Let $\Omega \leq \text{Aut}(G)$. A finite system of $\Omega$-twisted equations in $G$ with variables $X$ is a finite subset $\mathcal{E}$ of the monoid $(G \cup F_X \times \Omega)^*$, and is again denoted $w_1 = \cdots = w_n = 1$. Define the function

$$p : G \times \text{Aut}(G) \to G$$

$$(g, \psi) \mapsto g \psi.$$ 

If $\phi : F_X \to G$ is a homomorphism, let $\bar{\phi}$ denote the (monoid) homomorphism from $(G \cup F_X \times \Omega)^*$ to $(G \times \Omega)^*$, defined by $(h, \psi) \bar{\phi} = (h \phi, \psi)$ for $(h, \psi) \in F_X \times \Omega$ and $g \bar{\phi} = g$ for all $g \in G$. A solution is a homomorphism $\phi : F_X \to G$, such that $w_1 \phi = \cdots = w_n \phi = 1_G$. When $\Omega = \text{Aut}(G)$, we omit the reference to $\Omega$, and call such a system a finite system of twisted equations.

For the purposes of decidability, in finitely generated groups, the elements of $G$ will be represented as words over a finite generating set, and in twisted equations, automorphisms will be represented by their action on the generators.

A finite system of (twisted) equations with rational (recognisable) constraints $\mathcal{E}$ in a group $G$ is a finite system of (twisted) equations $\mathcal{F}$ with variables $X_1, \ldots, X_n$, together with a tuple of rational (recognisable) subsets $R_1, \ldots, R_n$ of $G$. A solution to $\mathcal{E}$ is a solution $\phi$ to $\mathcal{F}$, such that $X_i \phi \in R_i$ for all $i$.

Remark 2.7. A solution to an equation with variables $X_1, \ldots, X_n$ will usually be represented as a tuple $(x_1, \ldots, x_n)$ of group elements, rather than a homomorphism. We can obtain the homomorphism from the tuple by defining $X_i \mapsto x_i$ for each $i$. 

Example 2.8. Equations in \( \mathbb{Z} \) are linear equations in integers, and elementary linear algebra can be used to determine satisfiability, and also describe solutions.

Example 2.9. The conjugacy problem in any group can be viewed as an equation \( X^{-1}gX = h \), where \( g \) and \( h \) are group elements, and \( X \) is a variable. For example, in the free group \( F(a, b) \), one could consider the equation \( X^{-1}abX = ba \). The set of solutions is \( \{(ab)^n b^{-1} \mid n \in \mathbb{Z}\} \).

The twisted conjugacy problem can similarly be viewed, using the equation \( X^{-1}gX = h \Phi \), for some automorphism \( \Phi \).

Example 2.10. Let \( \Phi = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \text{GL}_2(\mathbb{Z}) \). Consider the twisted equation in \( \mathbb{Z}^2 \), with the variables \( X \) and \( Y \):

\[
(X)\Phi = Y.
\]

This is just the automorphism problem in \( \mathbb{Z}^2 \), which can be solved using elementary linear algebra. In the free group \( F(a, b) \), an example of a twisted equation would be \( X(Y \Psi) = bX^{-1} \), with \( \Psi \in \text{Aut}(F(a, b)) \) defined by \( a\Psi = ab \) and \( b\Psi = b \). Computing solutions to this is more difficult, although an algorithm does exist to construct the set of solutions (see [10]).

2.3. Space complexity. We briefly define space complexity. We refer the reader to [24] for a comprehensive introduction to space complexity, or to [4] for the consideration of space complexity when constructing EDT0L systems.

Definition 2.11. Let \( f : \mathbb{Z}_0 \rightarrow \mathbb{Z}_0 \). An algorithm is said to run in \( \text{NSPACE}(f) \) (non-deterministic \( f \) space) if it can be performed by a non-deterministic Turing machine with the following:

1. A read-only input tape;
2. A write-only output tape;
3. A read-write work tape such that no computation path in the Turing machine uses more than \( O(nf) \) units of the work tape, for an input of length \( n \).

We will use space complexity to show that the EDT0L systems we will construct, can be done so with a clear bound on the amount of memory used. We will need the following standard result later on.

Lemma 2.12. Let \( L \) and \( M \) be regular languages over alphabets \( \Sigma_L \) and \( \Sigma_M \), that are constructible in \( \text{NSPACE}(f) \) for some \( f : \mathbb{Z}_0 \rightarrow \mathbb{Z}_0 \). Let \( \phi : \Sigma_L^* \rightarrow \Sigma_M^* \) be a free monoid homomorphism. Then the following languages are regular, and constructible in \( \text{NSPACE}(f) \).

1. \( L \cup M \) (union);
2. \( L \cap M \) (intersection);
3. \( L\phi \) (homomorphism);
4. \( M\phi^{-1} \) (inverse homomorphism).

Proof Let \( \mathcal{A}_L \) and \( \mathcal{A}_M \) be finite state automata accepting \( L \) and \( M \), respectively, that are both constructible in \( \text{NSPACE}(f) \). Let \( q_L \) and \( F_L \), and \( q_M \) and \( F_M \) be the start states, and sets of accept states of \( \mathcal{A}_L \) and \( \mathcal{A}_M \), respectively.

1. The finite state automaton that accepts \( L \cup M \) is obtained by taking the union \( \mathcal{A}_L \cup \mathcal{A}_M \), and adding an additional state, \( q_0 \). We attach an \( \varepsilon \)-labelled edge from \( q_0 \) to \( q_L \) and \( q_M \), and then set \( q_0 \) to be the start state. The accept states will be \( F_L \cup F_M \). Printing this can be
done using the memory required to print both of $A_L$ and $A_M$, plus a constant, and thus it is constructible in NSPACE($f$).

2. We can take $A_L \times A_M$, to be our finite state automaton for $L \cap M$, where the start state is $q_L \times q_M$, and the set of accept states is $F_L \times F_M$, and with additional $\varepsilon$-transitions from a state $(p_L, p_M)$ to $(p_L', p_M')$ whenever there is an $\varepsilon$-transition from $p_M$ to $p_M'$ in $A_M$ and the analogous $\varepsilon$-transitions for each $\varepsilon$-transition in $A_L$. To write this down, we proceed with the construction of $A_L$, but whenever we would normally output a state $q$, we instead output $(q) \times A_M$, and whenever we would add an edge between states $q_1$ and $q_2$, we instead add all edges between $(q_1) \times A_M$ and $(q_2) \times A_M$, by going through the construction of $A_M$. To do this, we never need to store more than the information required to write down both $A_M$ and $A_L$ plus a constant, and thus this can be completed in NSPACE($f$).

3. We can do this by constructing $A_L$, except whenever we would output an edge labelled with $a$, we instead output a path labelled with $a \phi$.

4. We adapt the construction in Proposition 3.3 in [5]. First let $\bar{\Sigma}_M = \{ \bar{a} \mid a \in \Sigma_M \}$ be a copy of $\Sigma_M$, disjoint with $\Sigma_L$, and let $M$ be the language obtained from $M$ by replacing every occurrence of $a \in \Sigma_M$ with $\bar{a}$. Now let

$$K = \{ y_0 \bar{x}_1 y_1 \cdots \bar{x}_n y_n \mid n \in \mathbb{Z}_{>0}, \ x_1 \cdots x_n \in M, y_1, \ldots, y_n \in \Sigma^*_L \}.$$ 

We can construct a finite state automaton accepting $K$ in NSPACE($f$), by constructing $A_M$, but replacing each occurrence of $a \in \Sigma_M$ with $\bar{a}$, and then for each $b \in \Sigma_L$, adding a loop in each vertex labelled with $b$. Now consider the regular language

$$S = \{ (y_1 \phi) \bar{y}_1 (y_2 \phi) \bar{y}_2 \cdots (y_n \phi) \bar{y}_n \mid n \in \mathbb{Z}_{>0}, y_1, \ldots, y_n \in \Sigma^*_L \}.$$ 

Note that the size of $S$ is constant; it depends only on $\phi$. Let $\tau : (\Sigma_L \cup \bar{\Sigma}_M)^* \to (\Sigma_L)^*$ be the free monoid homomorphism defined by $a \tau = a$ if $a \in \Sigma_L$, and $\bar{a} \tau = \varepsilon$, if $\bar{a} \in \Sigma_M$. By construction, $M \phi^{-1} = (K \cap S) \tau$. Using (2) and (3), it follows that $M \phi^{-1}$ is constructible in NSPACE($f$).

2.4. EDT0L languages. The class of languages we use to describe solution sets is the class of EDT0L languages. All EDT0L languages are ET0L, which are indexed languages, and hence context-sensitive, and all regular languages are EDT0L. However, there are EDT0L languages that are not context-free, and context-free languages that are not EDT0L [14]. Context-free languages do not work naturally with systems of equations, as most equations with 3 or more variables will not have context-free solutions; the system $XY^{-1} = XZ^{-1}$ = 1 in $\mathbb{Z}$, using the presentation $\langle a \rangle$, will have the solution language $\{ a^m \# a^m \# a^m \mid m \in \mathbb{Z} \}$, which is not context-free. For more information on EDT0L languages, we refer the reader to [30] and [31].

We can now define EDT0L languages. We base our definitions on [16], however there are a number of equivalent definitions used elsewhere.

Definition 2.13. An EDT0L system is a tuple $\mathcal{H} = (\Sigma, C, \omega, \mathcal{R})$, where

1. $\Sigma$ is an alphabet, called the (terminal) alphabet;
2. $C$ is a finite superset of $\Sigma$, called the extended alphabet of $\mathcal{H}$;
3. $\omega \in C^*$ is called the start word;
4. $\mathcal{R}$ is a regular (as a language) set of endomorphisms of $C^*$, called the rational control of $\mathcal{H}$.

The language accepted by $\mathcal{H}$ is $L(\mathcal{H}) = \{ \omega \phi \mid \phi \in \mathcal{R} \} \cap \Sigma^*$.
A language that is accepted by some EDT0L system is called an *EDT0L language*.

The following is a standard example of an EDT0L language that is not context-free.

**Example 2.14.** The language \( L = \{ a^n^2 \mid n \in \mathbb{Z}_{>0} \} \) is an EDT0L language over the alphabet \( \{a\} \). This can be seen by considering \( \omega = \bot \) and \( C = \{ \bot, s, t, u, a \} \) as the extended alphabet and start word, respectively, of an EDT0L system accepting \( L \), and using the finite state automaton from Figure 1 to define the rational control. Note that the rational control can also be written as \( \varphi_{\bot}(\varphi_1\varphi_2)^*\varphi_3 \).

This language is not context-free. This can be shown using the pumping lemma (Theorem 2.6.17 in [19]).

An *abstract family of languages* is one closed under the five operations in the following lemma, together with preimages under free monoid homomorphisms. The following lemma shows that even if EDT0L languages do not form a full abstract family of languages (Theorem V.2.17 in [30]) like regular (Proposition 2.5.10 and Proposition 2.5.14 in [19]), context-free (Proposition 2.6.27, Proposition 2.6.31 and Proposition 2.6.34 in [19]) and ET0L languages (Theorem V.1.7 in [30]), they are closed under most of the standard operations that are frequently used to manipulate languages.

**Lemma 2.15.** *The class of EDT0L languages is closed under the following operations:*

1. *Finite unions;*
2. *Intersection with regular languages;*
3. *Concatenation;*
4. *Kleene star closure;*
(5) Image under free monoid homomorphisms.

Moreover, if the EDT0L systems used in any of these operations can be constructed in $\text{NSPACE}(f)$, for some $f : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}_{\geq 0}$, then there is a computable EDT0L system accepting the resultant language that can also be constructed in $\text{NSPACE}(f)$.

**Proof** The proofs of (1), (2) and (5) can be found in Proposition 3.3 in [5]. We now show (3).

Let $\bar{\Sigma}$ be constructible in $\text{NSPACE}(f)$.

Let $\Sigma_L, C_L, \omega_L, \mathcal{R}_L$ and $(\Sigma_M, C_M, \omega_M, \mathcal{R}_M)$ be EDT0L systems that are constructible in $\text{NSPACE}(f)$, accepting languages $L$ and $M$, respectively.

(3) Let $\Sigma_M = \{ \bar{a} \mid a \in M \}$ be a copy of $\Sigma_M$ that is disjoint from $\Sigma_L$, and let $\bar{M} = \{ \bar{w} \mid w \in M \}$.

By modifying the EDT0L system for $M$ to replace occurrences of unbarred letters with barred ones, we have that there exists an EDT0L system $(\bar{\Sigma}_M, C_M, \omega_M, \mathcal{R}_M)$ accepting $\bar{M}$, that is constructible in $\text{NSPACE}(f)$. We can assume without loss of generality that $C_L \cap \Sigma_L$ and $C_M \setminus \Sigma_M$ are disjoint (and thus $C_L$ and $C_M$ are disjoint).

As $\mathcal{R}_L$ is regular, there is a finite set $\bar{B}_L \subseteq \text{End}(C^*_L)$ over which $\mathcal{R}_L$ is a regular language. For each $\psi \in \bar{B}_L$, define $\hat{\psi} \in \text{End}((C_L \cup \bar{C}_M)^*)$ by

$$a\hat{\psi} = \begin{cases} a\psi & a \in C_L \\ \bar{a} & a \in C_M. \end{cases}$$

Similarly define $\hat{\psi} \in \text{End}((C_L \cup \bar{C}_M)^*)$ for each $\psi$ in the finite set over which $\mathcal{R}_M$ is a regular language. Let $\bar{\mathcal{R}}_L$ and $\bar{\mathcal{R}}_M$ be the regular languages obtained from $\mathcal{R}_L$ and $\mathcal{R}_M$ by replacing each $\psi \in \bar{B}_L$ (or the equivalent for $M$) with their hatted versions.

By construction, the concatenation $LM$ is accepted by the EDT0L system $\mathcal{H} = (\Sigma_L \cup \bar{\Sigma}_M, C_L \cup C_M, \omega_L \omega_M, \mathcal{R}_L \cup \bar{\mathcal{R}}_L \cup \bar{\mathcal{R}}_M)$. Since $\mathcal{R}_L$ and $\mathcal{R}_M$ are constructible in $\text{NSPACE}(f)$, it follows that $\bar{\mathcal{R}}_L$ and $\bar{\mathcal{R}}_M$ are as well. Thus all of the sets used to define $\mathcal{H}$ are constructible in $\text{NSPACE}(f)$, and so $\mathcal{H}$ is constructible in $\text{NSPACE}(f)$.

Let $\theta : (\Sigma_L \cup \bar{\Sigma}_M) \rightarrow \Sigma_M$ be the free monoid homomorphism defined by $a\theta = a$ if $a \in \Sigma_L$ and $a\theta = \bar{b}$ if $a = b \in \Sigma_M$. It follows that $(LM)\theta = LM$, and so $LM$ is accepted by an EDT0L system that is constructible in $\text{NSPACE}(f)$ by part (5).

(4) Let $\Sigma_L = \{ \bar{a} \mid a \in \Sigma_L \}$ be a disjoint copy of $\Sigma_L$. Let $\bar{L} = \{ \bar{w} \mid w \in L \}$. Note that there is an EDT0L system $(\Sigma_L, C_L, \omega_L, \mathcal{R}_L)$ accepting $\bar{L}$ that is constructible in $\text{NSPACE}(f)$.

Let $\theta : \Sigma^*_L \rightarrow \Sigma^*_L$ be defined by $a\theta = a$. Thus $L$ is accepted by the EDT0L system $\mathcal{H} = (\Sigma_L, C_L \cup \Sigma_L, \omega_L, \mathcal{R}_L \theta)$ (where endomorphisms in $\mathcal{R}_L$ have been extended to act on $(\Sigma_L \cup C_L)^*$ as the identity function on $\Sigma_L$). We have that endomorphisms of the finite set over which the rational control of $\mathcal{H}$ is a regular language fix $\Sigma_L$. So we can assume without loss of generality that this is true in $(\Sigma_L, C_L, \omega_L, \mathcal{R}_L)$.

Let $\perp \notin C_L$. Define endomorphisms $\sigma$ and $\psi$ over $C \cup \{ \perp \}$ by $\perp \sigma = \{ \omega \mid \perp \}$ and $\perp \psi = \{ \varepsilon \}$. By construction, the EDT0L system $(\Sigma_L, C_L \cup \{ \perp \}, \perp, (\sigma \mathcal{R}_L)^* \psi)$ (where endomorphisms in $\mathcal{R}_L$ have been extended to act on $(C_L \cup \{ \perp \})^*$ as the identity function on $\perp$). Since $\psi$ and $\sigma$ can be constructed in constant space, this EDT0L system can be constructed in $\text{NSPACE}(f)$.

\[ \square \]

2.5. **Solution languages.** We now explain how we represent solution sets as languages. We start by defining a normal form.
Definition 2.16. Let $G$ be a group, and $\Sigma$ be a finite generating set for $G$. A normal form for $G$, with respect to $\Sigma$ (alternatively, a normal form for $(G, \Sigma)$), is a function $\eta: G \rightarrow (\Sigma^\pm)^*$ such that $g\eta$ represents $g$ for all $g \in G$.

A normal form $\eta$ is called

1. regular if $\text{im} \eta$ is a regular language over $\Sigma^\pm$;
2. geodesic if $\text{im} \eta$ comprises only geodesic words in $G$, with respect to $\Sigma$;
3. quasigeodesic if there exists $\lambda > 0$ such that $|g\eta| \leq \lambda|g|_{(G, \Sigma)} + \lambda$ for all $g \in G$.

Note that we are insisting our normal forms produce a unique representative for each element.

We are now in a position to represent solutions as languages, with respect to a given normal form.

Definition 2.17. Let $G$ be a group with a finite monoid generating set $\Sigma$, and let $\eta: G \rightarrow (\Sigma^\pm)^*$ be a normal form for $G$ with respect to $\Sigma$. Let $E$ be a system of equations in $G$ with a set $S$ of solutions. The solution language to $E$, with respect to $\eta$, is the language

$$\{(g_1\eta)\# \cdots \#(g_n\eta) \mid (g_1, \ldots, g_n) \in S\}$$

over $\Sigma^\pm \cup \{\#\}$.

We give an example of an equation in a group, with an EDT0L language of solutions.

Example 2.18. Consider the equation $XY^{-1} = 1$ in $\mathbb{Z}$ with the presentation $\langle a \mid \rangle$. The solution language with respect to the standard normal form is

$$L = \{a^n\#a^n \mid n \in \mathbb{Z}\},$$

over the alphabet $\{a, a^{-1}, \#\}$. The language $L$ is EDT0L; our system will have the extended alphabet $\{\bot, \#, a, a^{-1}\}$, start word $\bot \# \bot$, and rational control defined by Figure 2. Note that id denotes the identity function, and the rational control can also be expressed using the rational expression $\{\varphi^-, \varphi^*\}\phi$. 

**Figure 2.** Rational control for $L = \{a^n\#a^n \mid n \in \mathbb{Z}\}$ with start state $q_0$, and accept state $q_3$. 

\[\begin{array}{c}
\text{start} \quad q_0 \\
\downarrow \quad \downarrow \quad \downarrow \\
\text{id} \quad \text{id} \\
\varphi_-: \bot \rightarrow \bot \quad \varphi_+: \bot \rightarrow a \\
\phi: \bot \rightarrow \varepsilon \\
\end{array}\]

\[\begin{array}{c}
\phi: \bot \rightarrow \varepsilon \\
\varphi_-: \bot \rightarrow \bot a^{-1} \\
q_2 \quad q_1 \\
\end{array}\]

Note that we are insisting our normal forms produce a unique representative for each element.

We are now in a position to represent solutions as languages, with respect to a given normal form.

Definition 2.17. Let $G$ be a group with a finite monoid generating set $\Sigma$, and let $\eta: G \rightarrow (\Sigma^\pm)^*$ be a normal form for $G$ with respect to $\Sigma$. Let $E$ be a system of equations in $G$ with a set $S$ of solutions. The solution language to $E$, with respect to $\eta$, is the language

$$\{(g_1\eta)\# \cdots \#(g_n\eta) \mid (g_1, \ldots, g_n) \in S\}$$

over $\Sigma^\pm \cup \{\#\}$.

We give an example of an equation in a group, with an EDT0L language of solutions.

Example 2.18. Consider the equation $XY^{-1} = 1$ in $\mathbb{Z}$ with the presentation $\langle a \mid \rangle$. The solution language with respect to the standard normal form is

$$L = \{a^n\#a^n \mid n \in \mathbb{Z}\},$$

over the alphabet $\{a, a^{-1}, \#\}$. The language $L$ is EDT0L; our system will have the extended alphabet $\{\bot, \#, a, a^{-1}\}$, start word $\bot \# \bot$, and rational control defined by Figure 2. Note that id denotes the identity function, and the rational control can also be expressed using the rational expression $\{\varphi^-, \varphi^*\}\phi$. 

**Figure 2.** Rational control for $L = \{a^n\#a^n \mid n \in \mathbb{Z}\}$ with start state $q_0$, and accept state $q_3$. 

\[\begin{array}{c}
\text{start} \quad q_0 \\
\downarrow \quad \downarrow \quad \downarrow \\
\text{id} \quad \text{id} \\
\varphi_-: \bot \rightarrow \bot a^{-1} \\
\phi: \bot \rightarrow \varepsilon \\
\varphi_+: \bot \rightarrow a \\
q_1 \\
\end{array}\]

\[\begin{array}{c}
\phi: \bot \rightarrow \varepsilon \\
\varphi_-: \bot \rightarrow \bot a^{-1} \\
q_2 \quad q_1 \\
\end{array}\]
Remark 2.19. We now introduce space complexity to solution languages. We first need to define the ‘size’ of a system of equations, in order to measure our input. The definition of size can vary, as specific groups can have different ways of writing equations. For example, in [16], equations in virtually abelian groups were stored as tuples of integers, as this compressed the size of the equations, whilst storing all of the necessary information. This approach has not always been used in other cases when compression was possible. Since we deal with virtually abelian groups on their own, we will use this definition when referring to equations in virtually abelian groups.

When discussing equations in constructions based on other groups (such as direct products, finite index subgroups, wreath products) we will ‘inherit’ the input definition from the groups they are defined from. If these vary, we will use the general definition, which is less efficient than the specific virtually abelian case, and as a result, will still yield (at least) the same results. The general definition of equation size will also be used when talking about groups that are virtually direct products of hyperbolic groups.

We start with the general definition of equation length.

Definition 2.20. Let $G$ be a group, and $\omega = 1$ be an equation in $G$. Recall that $\omega \in F_V * G$, for some finite set $V$. Fix a generating set $\Sigma$ for $G$. We define the length of $\omega = 1$ to be the length of the group element $\omega \in F_V * G$, with respect to the generating set $\Sigma \cup V$.

Let $E$ be a finite system of equations in $G$. The length of $E$ is the sum of the lengths of all equations in $E$.

Before we define virtually abelian equation length, we must first consider the free abelian case. The compression is possible because we can store an integer $n$ with $\log n + c$ bits, for some constant $c$. This is covered in greater detail in Remark 3.6 and Remark 3.10 in [16].

Definition 2.21. Let $a_1, \ldots, a_k$ denote the standard generators of $\mathbb{Z}^k$. Let $\omega = 1$ be an equation in $\mathbb{Z}^k$ with a set $\{X_1, \ldots, X_n\}$ of variables. By reordering a given expression for $\omega$, we can assume $\omega = 1$ is in the form

$$X_1^{b_1} \cdots X_n^{b_n} a_1^{c_1} \cdots a_k^{c_k} = 1,$$

where $b_1, \ldots, b_n, c_1, \ldots, c_k \in \mathbb{Z} \setminus \{0\}$ (in the case when these values equal zero, we simply omit the relevant variables or generators from the equation). We can then define the free abelian length of $\omega = 1$ to be

$$\sum_{i=1}^n \log |b_i| + \sum_{j=1}^k \log |c_j| + Ckn.$$

Suppose now $\nu = 1$ is a twisted equation in $\mathbb{Z}^k$. By rearranging $\nu$, we can assume it is of the form

$$(X_1 B_1) \cdots (X_n B_n) a_1^{c_1} \cdots a_k^{c_k} = 1,$$

where each $B_r = [b_{rij}]$ is a $k \times k$ integer-valued matrix (not-necessarily invertible). These are described in more detail in the proof of Lemma 3.3 in [16]. The free abelian length of $\nu = 1$ is defined to be

$$\sum_{r=1}^n \sum_{i=1}^k \sum_{j=1}^k \log |b_{rij}| + C' nk^2 + \sum_{j=1}^k \log |c_j| + C' k,$$

where $C'$ is a constant.

From [16], any equation $\xi = 1$ in a virtually abelian group induces a twisted equation $\tilde{\xi} = 1$ in a free abelian group, which is unique up to the choice of transversal. We fix a choice of transversal, then define the virtually abelian length of $\xi = 1$ to be the free abelian length of $\tilde{\xi} = 1$. 
Let \( E \) be a finite system of equations in a virtually abelian group. The *virtually abelian length* of \( E \) is the sum of the virtually abelian lengths of all equations in \( E \). *Free abelian length* of a system of equations is defined analogously.

We now use these lengths as our input size. Unless we explicitly state that we are using virtually or free abelian equation length, we will assume we are using the general version of equation length.

**Definition 2.22.** Let \( \mathcal{C} \) be a class of languages, and fix a type of machine or grammar that constructs languages in \( \mathcal{C} \). Let \( f : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0} \). Let \( G \) be a group with a finite generating set \( \Sigma \), and let \( \eta \) be a normal form for \((G, \Sigma)\). We say that solutions to systems of equations in \( G \), with respect to \( \eta \), are \( \mathcal{C} \text{ in } \text{NSPACE}(f) \) if

1. The solution languages to systems of equations in \( G \) are \( \mathcal{C} \) with respect to \( \eta \);
2. Given a system of equations \( E \) in \( G \), a machine or grammar accepting the solution language can be constructed in \( \text{NSPACE}(f) \), with \( E \) as the input.

**Remark 2.23.** Since the only class of languages we will be using to describe solutions is the class of EDT0L languages, we will only be saying EDT0L in \( \text{NSPACE}(f) \), and the type of grammar we refer to when we say this is the EDT0L system.

### 2.6. Dihedral Artin groups.

We briefly define dihedral Artin groups. An application of Corollary [7.4] is that solution sets to systems of equations in these groups form EDT0L languages.

**Definition 2.24.** A *dihedral Artin group* \( \text{DA}_m \), where \( m \geq 2 \), is defined by the presentation

\[
\langle a, b \mid aba \cdots = bab \cdots \rangle.
\]

The following lemma is widely known. A brief sketch of the proof can be found in Section 2 of [7].

**Lemma 2.25.** A dihedral Artin group is virtually a direct product of free groups.

### 2.7. Schreier generators.

We use Schreier generators, along with the normal form they induce, in order to show that the class of groups where systems of equations have EDT0L languages of solutions is stable under passing to finite index subgroups. This subsection is based on Section 1.4 of [19].

We start with the definition of Schreier generators.

**Definition 2.26.** Let \( G \) be a group, generated by a finite set \( \Sigma \), \( H \) be a finite index subgroup of \( G \), and \( T \) be a right transversal of \( H \) in \( G \). For each \( g \in G \), let \( \bar{g} \) be the (unique) element of \( T \) that lies in the coset \( Hg \). The *Schreier generating set* for \( H \), with respect to \( T \) and \( \Sigma \), is defined to be

\[
Z = \{ t\bar{x}t^{-1} \mid t \in T, x \in \Sigma \}.
\]

Whilst the fact that the Schreier generating set is a finite generating set for \( H \) is widely known, we include a proof, as we later use ideas from the proof.

**Lemma 2.27.** Let \( G \) be a group, generated by a finite set \( \Sigma \), \( H \) be a finite index subgroup of \( G \), and \( T \) be a right transversal of \( H \) in \( G \). Let \( Z \) be the Schreier generating set for \( H \). Then \( Z \) is finite, and \( H = \langle Z \rangle \).
Proof We first show that
\[ Z^{-1} = \{tx^{-1}tx^{-1}^{-1} \mid t \in T, x \in \Sigma \}. \]
Let \( S = \{tx^{-1}tx^{-1}^{-1} \mid t \in T, x \in \Sigma \} \). Let \( g = \bar{tx}^{-1}t^{-1} = (tx\bar{tx}^{-1})^{-1} \in Z^{-1} \). Let \( v = \bar{tx} \). Note that \( \bar{vx}^{-1} = \bar{tx}^{-1}t^{-1} = t \). Then \( g = \bar{vx}^{-1}vx^{-1}^{-1} \in S \), and so \( Z^{-1} \subseteq S \).

Let \( g = tx^{-1}tx^{-1}^{-1} \in S \). Then \( g^{-1} = \bar{tx}^{-1}xt^{-1} \). Let \( v = \bar{tx}^{-1} \). Then \( \bar{vx} = t \), and so \( g^{-1} = \bar{vx}\bar{tx}^{-1} \in Z \). We can conclude that \( S \subseteq Z^{-1} \).

The fact that \( Z \) is finite follows from the fact that \( T \) and \( \Sigma \) are finite. Let \( t_0 \) be the unique element of \( T \cap H \). Let \( h \in H \) (this will usually be 1, but does not need to be). Then \( t_0^{-1}ht_0 = a_1 \cdots a_n \), for some \( a_1, \ldots, a_n \in \Sigma^\pm \). Let \( t_i = \bar{a}_i \cdots \bar{a}_1 \) for each \( i \in \{1, \ldots, n\} \), and note \( t_n = t_0 \). We have
\[ h = (t_0a_1t_1^{-1})(t_1a_2t_2^{-1})\cdots(t_{n-1}a_nt_n^{-1}). \]
Note that \( t_i\bar{a}_{i+1} = \bar{a}_1 \cdots \bar{a}_i \bar{a}_{i+1} = t_{i+1} \), and so
\[ h = (t_0a_1\bar{a}_1^{-1})(t_1a_2\bar{a}_2^{-1})\cdots(t_{n-1}a_n\bar{a}_n^{-1}). \]
Each of the parenthesised terms lie in \( Z \) if \( a_i \in \Sigma \), or \( S \) if \( a_i \in \Sigma^{-1} \). Since \( S = Z^{-1} \), we have \( h \in (Z) \). \( \square \)

The proof of Lemma \( \text{2.27} \) induced a normal form for the finite index subgroup, with respect to the Schreier generating set. We now give a formal definition of this normal form.

Definition 2.28. Let \( G \) be a group, generated by a finite set \( \Sigma \), \( H \) be a finite index subgroup of \( G \), and \( T \) be a right transversal of \( H \) in \( G \). Let \( Z \) be the Schreier generating set for \( H \). Fix a normal form \( \eta \) for \((G, \Sigma)\).

We define the Schreier normal form \( \zeta \) for \((H, Z)\), with respect to \( \eta \), as follows. Let \( h \in H \), and suppose \( h\eta = a_1 \cdots a_n \), where \( a_1, \ldots, a_n \in \Sigma^\pm \). Let \( t_0 \) be the unique element of \( T \cap H \), and define \( t_i = \bar{a}_1 \cdots \bar{a}_i \). Define \( h\zeta \) by
\[ h\zeta = (t_0\bar{a}_1t_0a_1^{-1})(t_1\bar{a}_2t_1a_2^{-1})\cdots(t_{n-1}a_nt_{n-1}a_n^{-1}). \]
The fact that this indeed defines an element of \( H \), and equals \( h \) is contained in the proof of Lemma \( \text{2.27} \).

If the normal form from the finite index overgroup is regular or quasigeodesic, then the Schreier normal form is regular or quasigeodesic, respectively. The latter requires an additional lemma that we prove later, however we can show that regularity is preserved without additional results.

Lemma 2.29. Let \( G \) be a group, generated by a finite set \( \Sigma \), \( H \) be a finite index subgroup of \( G \), and \( T \) be a right transversal of \( H \) in \( G \). Let \( Z \) be the Schreier generating set for \( H \). Fix a normal form \( \eta \) for \((G, \Sigma)\).

Let \( \zeta \) be the Schreier normal form with respect to \( \eta \), as in \( \text{1} \). If \( \eta \) is regular with respect to \( \Sigma \), then \( \zeta \) is regular with respect to \( Z \).

Proof We will extend \( \zeta \) to the whole of \( G \), with respect to the generating set \( Z \cup \{txu^{-1} \mid u, t \in T, x \in \Sigma \} \). Let \( g \in G \), and suppose \( t_0gt_0^{-1} = a_1 \cdots a_n \) where each \( a_i \in \Sigma^\pm \). Define \( \tilde{\zeta} : G \to ((Z \cup \{txu^{-1} \mid u, t \in T, x \in \Sigma \})^\pm)^* \) by
\[ g\tilde{\zeta} = (t_0\bar{a}_1\bar{a}_1^{-1})(t_1\bar{a}_2\bar{a}_2^{-1})\cdots(t_{n-1}a_nt_0^{-1}). \]
Note that $\tilde{\zeta}$ is an extension of $\zeta$. We will first show that $\tilde{\zeta}$ is regular, then use an intersection to show $\zeta$ is regular.

Consider a finite state automaton $A$ that accepts $\text{im } \eta$, with set of states $Q$, start state $q_0$, and set $F$ of accept states. We will construct a new finite state automaton $B$ to accept $\text{im } \tilde{\zeta}$. Our set of states will be $(Q \times T \times \{0, 1\}) \cup \{\lambda\}$, where $\lambda$ is a new state, our start state will be $(q_0, t_0, 0)$, and $\lambda$ will be our only accept state. For each transition $(p, a) \rightarrow q$ in $A$, and each $t \in T$, define the following transitions in $B$:

$$( (p, t, 0), a ) \rightarrow (q, ta, 1),$$

$$( (q, ta, 1), ta^{-1} ) \rightarrow (q, ta, 0).$$

For each $q \in Q$ and $t \in T$, we also have a transition

$$( (q, t, 1), t_0^{-1} ) \rightarrow \lambda.$$

By construction, whenever we read $ta$, we must follow with $ta^{-1}$, unless we are going to the accept state (at the end of the word), in which case we follow with $t_0^{-1}$. As a result, $B$ only accepts words in $\text{im } \tilde{\zeta}$. Conversely, $B$ accepts any word in $\text{im } \eta$ after its conversion into a word in $\text{im } \tilde{\zeta}$, and we can therefore conclude that $B$ accepts $\text{im } \tilde{\zeta}$.

We have that $\text{im } \zeta = \text{im } \tilde{\zeta} \cap (Z^\pm)^*$. As an intersection of regular languages, this is regular. \qed

3. EDT0L languages about a distinguished letter

Recall that we denote a solution $(g_1, \ldots, g_n)$ to a system of equations in a group $G$ using the word $(g_1\eta)\# \cdots \#(g_n\eta)$. In order to show that groups where systems of equations have EDT0L solution languages are closed under certain types of extension (such as direct products), we are required to prove Lemma 3.5, which allows us to concatenate in parallel two EDT0L languages where every word is of the form $u_0\# \cdots \# u_n$.

The following lemma allows us to use different symbols for each $\#$ that delimits the group elements, rather than the same one each time. The proof is joint work with Alex Evetts. When used in conjunction with Lemma 3.3 and the fact that the class of EDT0L languages is closed under images under homomorphisms, we can use this to show that the solution language remains EDT0L when restricted to a subset of variables, rather than all of them.

Lemma 3.1. Let $n \in \mathbb{Z}_{>0}$, $\{\#, \#_1, \ldots, \#_n\}$ be a set of formal symbols, and $\Delta$ be an alphabet, such that $\#, \#_1, \ldots, \#_n \notin \Delta$. Let $A$ be a set of $n$-tuples of words over $\Delta$. Define languages $L$ and $M$ over $\Delta \cup \{\#\}$ and $\Delta \cup \{\#_1, \ldots, \#_n\}$, respectively, by

$$L = \{w_1\#w_2\# \cdots \#w_n \mid (w_1, \ldots, w_n) \in A\}$$

$$M = \{w_1\#_1w_2\#_2\cdots\#_{n-1}w_n\#_n \mid (w_1, \ldots, w_n) \in A\}.$$

Let $f : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}_{\geq 0}$. Then

1. The language $L$ is EDT0L if and only if $M$ is;
2. There exists an EDT0L system for $L$ that is constructible in $\text{NSPACE}(f)$ if and only if such an EDT0L system for $M$ exists.

Proof Applying the monoid homomorphism $\#_1, \ldots, \#_{n-1} \mapsto \#$, $\#_n \mapsto \varepsilon$ maps $M$ to $L$, so the backward directions of (1) and (2) follow by Lemma 2.15.
Suppose $L$ is EDT0L. We will first show that
$$N := \{ w_1\#_1 w_2\#_2 \cdots \#_{n-1} w_n \mid (w_1, \ldots, w_n) \in A \}$$
is EDT0L. Consider an EDT0L system $\mathcal{H}_L = (\Sigma \cup \{\#\}, C, \perp, \mathcal{R})$ that accepts $L$, and that is constructible in $\text{NSPACE}(f)$. Note that we can assume our start word is a single letter, instead of a word $\omega$ by adding an additional letter $\perp$, and preconcatenating the rational control with an endomorphism $\perp \mapsto \omega$. Let $B \subseteq \text{End}(C^*)$ be the (finite) set over which $\mathcal{R}$ is a regular language.

We will construct a new EDT0L system from $\mathcal{H}_L$ which will accept $M$. Let $C_{\text{ind}} = \{ c^{i,j+1}, \ldots, j \mid c \in C, i, j \in \{1, \ldots, n\} \}$ be the set of symbols obtained by indexing elements of $C$ with a section of the sequence $(1, \ldots, n)$, including the empty sequence (if $i > j$). By convention, we will consider a letter $c \in C$ indexed by the empty sequence to be equal to $c$, and so $C \subseteq C_{\text{ind}}$. Our extended alphabet will be $C_{\text{ind}}$. Let $\phi \in B$. Define $\Phi_{\phi} \subseteq \text{End}(C_{\text{ind}}^*)$ to be the set of all endomorphisms $\psi$ defined by
$$c^{i,j} \psi = x_{i_1}^{i_{1k_1}} x_{i_2}^{i_{2k_2}} \cdots x_{i_r}^{i_{rk_r}},$$
where $x_1 \cdots x_r = c\phi$, and $(i_{11}, \ldots, i_{rk_r}) = (i, \ldots, j)$. Note that some (or all) of the sequences may be empty. Let $\mathcal{R}$ be the regular language of endomorphisms of $C_{\text{ind}}^*$ obtained from $\mathcal{R}$ by replacing each $\phi \in B$ with $\Phi_{\phi}$. The EDT0L system $\mathcal{H}_M = (\Sigma \cup \{\#\}, \ldots, \#n, C_{\text{ind}}, \perp_{1, \ldots, n}, \mathcal{R})$ will only accept words of the form $a_1^{i_{11}} \cdots a_{rk_r}$, where $(i_{11}, \ldots, i_{rk_r}) = (1, \ldots, n)$, and $a_1 \cdots a_r \in L$. However, since our alphabet is $\Sigma \cup \{\#\}$, it can only accept words over that alphabet, which are precisely words of the form $w_0\#_1 \cdots \#_n w_n$, where $w_1\# \cdots \# w_n \in L$, and thus will accept $M$.

It now remains to show $\mathcal{H}_M$ is constructible in $\text{NSPACE}(f)$. It doesn’t require extra memory beyond a constant to add $\perp$ as the start symbol. To write down the new extended alphabet $C_{\text{ind}}$, we just proceed as we would when constructing $\mathcal{H}_L$, but whenever we write a symbol $c$, we also write all of the indexed versions. To do this we just need to record the letter $c$ we are on, along with the previous index written, so this is still possible in $\text{NSPACE}(f)$.

To output $\mathcal{R}$, we simply proceed with writing down the finite state automaton that accepts $\mathcal{R}$, and replace each edge labelled by $\phi \in B$ with a set of edges between the same states, labelled with each $\psi \in \Phi_{\phi}$. To do this, we can compute $\Phi_{\phi}$, store it, and remove each $\psi \in \Phi_{\phi}$ from the memory as we write it. This will require $n$ times as much memory as writing down $\mathcal{R}$, but since $n$ is a constant, it is constructible in $\text{NSPACE}(f)$.

We introduce the concept of a $(\#_1, \ldots, \#_n)$-separated EDT0L system, which is key in the proof of Lemma 3.5.

**Definition 3.2.** Let $\Sigma$ be an alphabet, and $\#_1, \ldots, \#_n \in \Sigma$. A $(\#_1, \ldots, \#_n)$-separated EDT0L system is an EDT0L system $\mathcal{H}$, with a start word of the form $\omega_0\#_1\omega_1\#_2 \cdots \#_n\omega_n$, where $\omega_i \in \langle \Sigma \setminus \{\#_1, \ldots, \#_n\} \rangle^*$ for all $i$, and such that $\#_i\phi^{-1} = \{\#_i\}$, for all $i$, and every $\phi$ in the rational control.

Most of the proof of Lemma 3.5 involves showing that EDT0L languages where every word contains precisely $n$ occurrences of the letter $\#$ are accepted by $(\#, \ldots, \#)$-separated EDT0L systems. To start this proof, we need the following lemma. The proof is very similar to the proof of Lemma 3.1.

**Lemma 3.3.** Let $L$ be an EDT0L language over an alphabet $\Sigma$, such that every word in $L$ contains precisely $n$ occurrences of the letter $\#$, where $n \in \mathbb{Z}_{\geq 0}$. Let $f : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$. Let $\Sigma^0 = \{a^0 \mid a \in \Sigma\}$, $\ldots$, $\Sigma^n = \{a^n \mid a \in \Sigma\}$ be pairwise disjoint copies of $\Sigma$, all disjoint from $\Sigma$. Let $\varphi^i : \Sigma^* \to (\Sigma^i)^*$ be the free monoid homomorphism defined by $aw^i = \bar{a}^i$ for all $a \in \Sigma$. Then
(1) The language
\[ M = \{ (w_0\varphi^0)\# \cdots \# (w_n\varphi^n) \mid w_0\# \cdots \# w_n \in L \} \]
is EDT0L.

(2) If there is an EDT0L system for \( L \) that is constructible in NSPACE(\( f \)), then there is an EDT0L system for \( M \) that is constructible in NSPACE(\( f \)), which has a single-letter start word.

**Proof** We will assume without loss of generality that \( \# \cdots \# \notin L \) (that is, the length of every word in \( L \) is at least \( n + 1 \)); if this is not the case, the fact that finite languages are EDT0L and the fact that EDT0L languages are closed under finite unions will give the result (Lemma 2.15).

Consider an EDT0L system \( \mathcal{H}_L = (\Sigma \cup \{\#\}, C, \bot, R) \) that accepts \( L \), and that is constructible in NSPACE(\( f \)). As in the proof of Lemma 3.1, if \( L \) is constructible in NSPACE(\( f \)), then there is an EDT0L system for \( L \) that is constructible in NSPACE(\( f \)).

We will construct a new EDT0L system from \( \mathcal{H}_L \) which will accept the language \( K \) of all words obtained from a word in \( L \) by replacing each letter \( a \in \Sigma \cup \{\#\} \) with \( \bar{a}^i \) for some \( i \), such that concatenating the indices of all letters in a word in \( K \) gives a non-decreasing sequence containing that numbers 1, \ldots, \( n \) (that is, of the form \( (0, \ldots, 0, 1, \ldots, 1, \ldots, n, \ldots, n) \)). Note that each letter in a word in \( K \) may have a different index \( i \). Let \( C_{\text{ind}} = \{ c_{i_1i_2\ldots i_j} \mid c \in C, i, j \in \{0, \ldots, n\} \} \) be the set of symbols obtained by indexing elements of \( C \) with a non-empty section of the sequence \( 0, \ldots, n \). Note that this differs from the proof of Lemma 3.1 where the empty sequence was permitted. Our extended alphabet will be \( C_{\text{ind}} \). Let \( \phi \in B \). Define \( \Phi_\phi \subseteq \text{End}(C^*) \) to be the set of all endomorphisms \( \psi \) defined by
\[ c^{i_1\ldots i_j} \psi = x_1^{i_{11}\ldots i_{1k_1}} x_2^{i_{21}\ldots i_{2k_2}} \cdots x_r^{i_{r1}\ldots i_{rk_r}}, \]
where \( x_1 \cdots x_r = c\phi \), and \( (i_{11}, \ldots, i_{rk_r}) \) is a non-decreasing sequence such that \( \{i_{11}, \ldots, i_{rk_r}\} = \{i, \ldots, j\} \), and such that each subsequence \( (i_{p1}, \ldots, i_{pk_p}) \) for some \( p \) is strictly increasing. That is, we split up the sequence \( (i, \ldots, j) \) across the word \( x_1 \cdots x_r \), potentially adding some repeats of integers across two letters, but never within the index of one letter. Note that some (or all) of the sequences may be empty. Let \( \mathcal{R} \) be the regular language of endomorphisms of \( C^*_{\text{ind}} \) obtained from \( \mathcal{R} \) by replacing each \( \phi \in B \) with \( \Phi_\phi \), and let \( B \) analogous set obtained from \( \mathcal{R} \).

Let \( \mathcal{H}_K = (\Sigma^0 \cup \cdots \cup \Sigma^n \cup \{\#^0, \ldots, \#^n\}, C_{\text{ind}}, \bot_{0,\ldots,n}, \mathcal{R}) \). Unlike the system created in the proof of Lemma 3.1 if \( \psi \in B^* \) is such that there exists \( \phi \in B^* \) such that \( \psi \phi \in \mathcal{R} \), then it is now possible for two letters in a word \( \bot \psi \) to have the same index (as we can have such a word of the form \( c^0c^1d^1c^1d^2 \) if \( n = 2 \)). This is not a problem as the only way two of the same letters can have the same index is if they have a single-number index (that is, there is a subword of the form \( c^\nu c^i \), for some \( \nu \in C^*_{\text{ind}} \)). In such a case, we will want to map both \( c^i \)'s to the same place (as all of their images should have only \( i \) as their index). In addition, we will not miss any words in \( L \) due to them being too short to fit all indices on; we start with \( n + 1 \) indices, and all words in \( L \) have at least length \( n + 1 \), by our assumption at the beginning of the proof.

The EDT0L system \( \mathcal{H}_K \) will only accept words of the form \( a_1^{i_1} \cdots a_r^{i_r} \), where \( (i_1, \ldots, i_r) \) is non-decreasing, \( \{i_1, \ldots, i_r\} \subseteq \{0, \ldots, n\} \) (we can have letters mapped to \( \varepsilon \), so we may lose some indices), and \( a_1 \cdots a_r \in L \). However, since our alphabet is \( \Sigma^0 \cup \cdots \cup \Sigma^n \cup \{\#^0, \ldots, \#^n\} \), it can only accept words in \( K \), and so \( K \) is EDT0L, accepted by the system \( \mathcal{H}_K \).

We will now show that \( \mathcal{H}_K \) is constructible in NSPACE(\( f \)). As in the proof of Lemma 3.1 we don't require extra memory beyond a constant to add \( \bot \) as the start symbol. To output the new extended
alphabet \( C_{\text{ind}} \), we just proceed as we would when constructing \( H_L \), but whenever we write a symbol \( c \), we also write all of the indexed versions. To do this we just need to record the letter \( c \) we are on, along with the previous index written, so this is still possible in \( \text{NSPACE}(f) \).

To output \( \mathcal{R} \), we simply proceed with writing down the finite state automaton that accepts \( \mathcal{R} \), and replace each edge labelled by \( \phi \in B \) with a set of edges between the same states, labelled with each \( \psi \in \Phi_\phi \). To do this, we can compute \( \Phi_\phi \), store it, and remove each \( \psi \in \Phi_\phi \) from the memory as we write it. This will require \( n \) times as much memory as writing down \( \mathcal{R} \), but since \( n \) is a constant, it is constructible in \( \text{NSPACE}(f) \).

Note that \( M \) obtained by intersecting \( K \) with the regular language \((\Sigma^0)^*\#^1 \cdots n(\Sigma^n)^*\), and then applying the free monoid homomorphism defined by \(#^i \mapsto \#\) for all \( i \). Thus by Lemma 2.15 \( M \) is accepted by an EDT0L system that is constructible in \( \text{NSPACE}(f) \).

Using Lemma 3.3, we can now show that EDT0L languages where every word contains \( n \) occurrences of the letter \( \# \) are always accepted by \((\#, \ldots, \#)\)-separated EDT0L systems.

**Lemma 3.4.** Let \( L \) be an EDT0L language, such that every word in \( L \) contains precisely \( n \) occurrences of the letter \( \# \), where \( n \in \mathbb{Z}_{\geq 0} \). Let \( f: \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0} \). Then

1. There is a \((\#, \ldots, \#)\)-separated EDT0L system \( H \) that accepts \( L \).
2. If an EDT0L system for \( L \) is constructible in \( \text{NSPACE}(f) \), then \( H \) is constructible in \( \text{NSPACE}(f) \).

**Proof** Let \( \Sigma^0, \ldots, \Sigma^n \) be the disjoint copies of \( \Sigma \) and let \( \varphi^0, \ldots, \varphi^n \) be the corresponding maps, as defined in Lemma 3.3. Let \( \Delta = (\Sigma^0)^* \cup \cdots \cup (\Sigma^n)^* \cup \{\#\} \). Suppose an EDT0L system for \( L \) is constructible in \( \text{NSPACE}(f) \). By Lemma 3.3

\[ M = \{(w_0\varphi^0)\# \cdots (w_n\varphi^n) \mid w_0\# \cdots \#w_n \in L\} \]

is EDT0L, and accepted by an EDT0L system \( H_M = (\Delta, C, \perp, \mathcal{R}) \) that is constructible in \( \text{NSPACE}(f) \). Note that we can assume that \( \# \) is fixed by endomorphisms in \( \mathcal{R} \) by adding an additional symbol \( \#' \) to the extended alphabet to replace \( \# \), and then post-composing the rational control with a map that sends \( \#' \) to \( \# \) and fixes everything else. This will not affect the space complexity in which \( H_M \) can be constructed.

Let \( C_0, \ldots, C_n \) be pairwise disjoint copies of \( C \), and let \( \Delta_0, \ldots, \Delta_n \) be the copies of \( \Delta \) sitting inside \( C_0, \ldots, C_i \), respectively. Let \( \theta_i: \Delta^* \to (\Delta_i)^* \) be the monoid homomorphism defined by mapping a letter in \( \Delta \) to its copy in \( \Delta_i \). We will show that

\[ K = (M\theta_0)\#(M\theta_1)\# \cdots (M\theta_n) \]

is EDT0L over the alphabet \( \Delta_0 \cup \cdots \cup \Delta_n \cup \{\#\} \), and accepted by a \((\#, \ldots, \#)\)-separated EDT0L system that is constructible in \( \text{NSPACE}(f) \). Most of this can be done by Lemma 2.15, however to show the stronger property that that a \((\#, \ldots, \#)\)-separated EDT0L system exists, we must do this directly. The proof is largely the same. For each \( i \), let \( H_{M\theta_i} = (\Delta_i, C_i, \perp, \mathcal{R}_i) \) be an EDT0L system for \( M\theta_i \) that is constructible in \( \text{NSPACE}(f) \) (such a system exists by replacing each \( c \) in \( C \) with the image in \( C_i \), and then updating the maps to act on \( C_i^* \) instead of \( C^* \)).

Our extended alphabet will be \( D = C_0 \cup \cdots \cup C_n \cup \{\#\} \). We now define the rational control for our EDT0L system for \( K \). Let \( B \subseteq \mathcal{R} \) be a finite set over which \( \mathcal{R} \) can be expressed as a regular
language. For each \( \phi \in B \), define \( \hat{\phi}_i \in \text{End}(D^*) \) by

\[
\begin{align*}
\hat{\phi}_i &= \begin{cases} 
    b\phi \theta_i & a \in C_i, \ a = b\theta_i \\
    a & a = \#.
\end{cases}
\end{align*}
\]

Let \( S \) be the regular language of endomorphisms of \( D^* \) obtained by replacing each occurrence of a label \( \phi \) within a finite state automaton for \( R \) with \( \hat{\phi} \). Our start word will be \( \perp_0 \# \perp_1 \# \cdots \# \perp_n \). By construction, \( S \) manipulates each \( \perp_i \) in parallel exactly the same way that \( R \) affects \( \perp \), except remaining in the correct copy \( C_i \).

Let \( \mathcal{H}_K = (\Delta_0 \cup \cdots \cup \Delta_n \cup \{\#\}, D, \perp_0 \# \perp_1 \# \cdots \# \perp_n, S) \). By construction, \( \mathcal{H}_K \) accepts \( K \) and is \((\#, \ldots, \#)\)-separated. Additionally, as each of \( R_1, \ldots, R_n \) is constructible in \( \text{NSPACE}(f) \), so is the rational control \( S \) of \( \mathcal{H}_K \). The extended alphabet \( D \) can be output using the same information required to construct each \( C_i \), which is just the same information needed to output \( C \), and so \( D \) is constructible in \( \text{NSPACE}(f) \).

Define the monoid homomorphism \( \Psi \) by its action on the letters in \( \Delta_0 \cup \cdots \cup \Delta_n \cup \{\#\} \). Note that \( \theta_i^{-1} \) and \( \varphi_i^{-1} \) are well-defined, because \( \theta_i \) and \( \varphi_i \) are bijections on the letters.

\[
\Psi: (\Delta_0 \cup \cdots \cup \Delta_n \cup \{\#\})^* \rightarrow \Sigma^*
\]

\[
a \mapsto \begin{cases} 
    a(\theta_i)^{-1}(\varphi_i)^{-1} & a \in M \theta_i \text{ and } a(\theta_i)^{-1} \in \Sigma^i \text{ for some } i \\
    a & a = \# \\
    \varepsilon & \text{otherwise}.
\end{cases}
\]

By construction, \( K\Psi = L \). Thus by post-composing the rational control of \( \mathcal{H}_K \) with \( \Psi \), we obtain a \((\#, \ldots, \#)\)-separated \( \text{EDT0L} \) system for \( L \). Moreover, this addition will not affect the space complexity of \( \mathcal{H}_K \), which is \( \text{NSPACE}(f) \).

We are now able to prove the main result of this section.

**Lemma 3.5.** Let \( L \) and \( M \) be \( \text{EDT0L} \) languages, such that every word in \( L \cup M \) contains precisely \( n \) occurrences of the letter \( \# \). Let \( f: \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}_{\geq 0} \). Then

1. The language

\[
N = \{ u_0v_0\# \cdots \#u_nv_n \mid u_0\# \cdots \#u_n \in L, \ v_0\# \cdots \#v_n \in M \},
\]

is \( \text{EDT0L} \);

2. If \( \text{EDT0L} \) systems for \( L \) and \( M \) are constructible in \( \text{NSPACE}(f) \), then an \( \text{EDT0L} \) system for \( N \) is constructible in \( \text{NSPACE}(f) \).

**Proof** By Lemma 3.4 and Lemma 3.1 we have that \( L \) and \( M \) are accepted by \( \text{EDT0L} \) systems \( \mathcal{H}_L \) and \( \mathcal{H}_M \), with start words \( \omega_0\#_1 \cdots \#_n\omega_n \) and \( \nu_0\#_1 \cdots \#_n\nu_n \), respectively, such that nothing other than \( \#_i \) is mapped to \( \#_i \) within both \( \mathcal{H}_L \) and \( \mathcal{H}_M \). Suppose also that these systems are constructible in \( \text{NSPACE}(f) \). Let \( C_L \) and \( C_M \) be the extended alphabets of \( \mathcal{H}_L \) and \( \mathcal{H}_M \), and let \( \Sigma_L \) and \( \Sigma_M \) be the terminal alphabets. Without loss of generality assume \( C_L \setminus \Sigma_L \) and \( C_M \setminus \Sigma_M \) are disjoint. Let \( \mathcal{R}_L \) and \( \mathcal{R}_M \) be the rational controls, and let \( B_L \) and \( B_M \) be the finite sets of endomorphisms over which \( \mathcal{R}_L \) and \( \mathcal{R}_M \) are regular languages.

Let \( \Sigma = \Sigma_L \cup \Sigma_M \), and let \( C = C_L \cup C_M \). For each \( \phi \in B_L \), define \( \tilde{\phi} \in \text{End}(C^*) \) by

\[
\tilde{c\phi} = \begin{cases} 
    c\phi & c \in C_L \\
    c & c \notin C_L.
\end{cases}
\]
Define $\bar{\phi}$ for each $\phi$ in $B_M$ analogously, and extend the bar notation to composition of functions, that is, $\bar{\phi}v = \phi \bar{v}$. Let $\mathcal{R} = \{ \phi \mid \phi \in R_L \cup R_M \}$, and note that $\mathcal{R}$ is a regular language over some alphabet of endomorphisms. Thus, $N$ is accepted by the EDT0L system $(\Sigma, C, \omega_0 \nu_0 \#_1 \cdots \#_n \omega_n \nu_n, \mathcal{R})$, as required.

Suppose there exist EDT0L systems for $L$ and $M$, which are constructible in $\text{NSPACE}(f)$. By Lemma 3.4 and Lemma 3.1, $\mathcal{H}_L$ and $\mathcal{H}_M$ are also constructible in $\text{NSPACE}(f)$. We can construct $C$ with the memory required to construct $C_M$ and $C_L$. The set $\{ \phi \mid \phi \in B_L \}$ is constructible in $\text{NSPACE}(f)$, by following the construction of $\mathcal{R}_L$, but writing a $\bar{\phi}$ instead of a $\phi$, for each occurrence of $\phi \in B_L$. By symmetry, $\{ \bar{\phi} \mid \phi \in B_M \}$ is constructible in $\text{NSPACE}(f)$. Since $\mathcal{R}$ is the union of these sets, we can construct $\mathcal{R}$ in $\text{NSPACE}(f)$ by Lemma 2.12. \hfill $\Box$

4. Equations in extensions

This section shows that the class of groups where systems of equations have EDT0L solution languages is closed under various extensions, including wreath products with finite groups and direct products. These facts are used in the proof of Theorem 7.8 on groups that are virtually a direct product of hyperbolic groups.

Furthermore, in Proposition 4.3 we deal with systems of equations with rational constraints in finite extensions, if twisted equations in a finite index normal subgroup have EDT0L solutions. This is used to show that systems of equations with rational constraints in virtually abelian groups are EDT0L.

The proof of the following is based on the proof of Lemma 3.9 in [16].

**Lemma 4.1.** Let $G$ be a group, and $T$ be a finite transversal of a normal subgroup $H$ of finite index. Let $\Omega$ be the group of automorphisms of $H$ induced by conjugating $H$ by elements of $G$. Let $S$ be the solution set to a finite system $E_G$ of equations with rational constraints in $n$ variables in $G$. Then there is a finite set $B \subseteq T^n$, and for each $t = (t_1, \ldots, t_n) \in B$, there is a solution set $A_t$ to a system $E_{H,t}$ of $\Omega$-twisted equations with rational constraints in $H$, such that

$$S = \bigcup_{(t_1, \ldots, t_n) \in B} \{(h_1 t_1, \ldots, h_n t_n) \mid (h_1, \ldots, h_n) \in A_{(t_1, \ldots, t_n)}\}.$$  

**Proof** Let

$$(2) \quad X_{ij}^{e_{ij}} g_{1j} \cdots X_{pq}^{e_{pq}} g_{pj} = 1$$

be a system $E_G$ of equations in $G$, with a set $\{R_{X_1}, \ldots, R_{X_n}\}$ of rational constraints, where $X_1, \ldots, X_n$ are the variables, and $j \in \{1, \ldots, k\}$. Let $S$ be the solution set. Note that we can assume that these equations start with variables by conjugating leading constants to the right. For each $X_i$, define new variables $Y_{i,j}$ over $H$, and $Z_{i}$ over $T$, such that $X_i = Y_{i,j} Z_i$. For each constant $g_i$, we have $\psi_i = h_i t_i$, for some $h_i \in H$ and $t_i \in T$, and so substituting these into (2) gives that $E_G$ is equivalent to

$$(3) \quad (Y_{i,j} Z_{i,j})^{e_{i,j}} h_{1j} t_{1j} \cdots (Y_{p,j} Z_{p,j})^{e_{p,j}} h_{pj} t_{pj} = 1.$$  

For all $g \in G$, define $\psi_g : G \to G$ by $h \psi_g = g h g^{-1}$. Note that $\psi_g \mid_{H} \in \Omega$ for all $g \in G$, by definition. By abusing notation, we can define $\psi_Z$ for each $i$. For all $i \in \{1, \ldots, n\}$, and $j \in \{1, \ldots, k\}$ define

$$\delta_{ij} = \begin{cases} 0 & \epsilon_{ij} = 1 \\ 1 & \epsilon_{ij} = -1. \end{cases}$$
We can use this notation to rearrange (3) into
\[
(Y^\varepsilon_{ij} \psi_{Z_{ij}}^\delta) \cdot Z_{ij}^{\varepsilon_{ij}} h_{ij} t_{ij} \cdots (Y^\varepsilon_{pj} \psi_{Z_{pj}}^\delta) Z_{pj}^{\varepsilon_{pj}} h_{pj} t_{pj} = 1.
\]
For \( l \in \{1, \ldots, p\} \), define
\[
W_l = (Y^\varepsilon_{ij} \psi_{Z_{ij}}^\delta) \cdot Z_{ij}^{\varepsilon_{ij}} h_{ij} t_{ij} \cdots (Y^\varepsilon_{lj} \psi_{Z_{lj}}^\delta) Z_{lj}^{\varepsilon_{lj}},
\]
\[
f_l = (h_{lj}) \psi_{Z_{lj}}^\delta \cdot Z_{lj}^{\varepsilon_{lj}} h_{lj} t_{lj} \cdots (h_{pj}) \psi_{Z_{pj}}^\delta Z_{pj}^{\varepsilon_{pj}}.
\]
By pushing all \( Y_i \)s and \( h_i \)s to the left within (4), we obtain
\[
W_1 f_1 \cdots W_p f_p Z_{ij}^{\varepsilon_{ij}} t_{ij} \cdots Z_{pj}^{\varepsilon_{pj}} t_{pj} = 1.
\]
As \( H \) is a finite index subgroup of \( G \), \( Ht \) is a recognisable subset of \( G \), for all \( t \in T \). For each coset \( Ht \) of \( H \), and each variable \( X_i \) let \( R_{ti} = R_{X_i} \cap (Ht) \). Note that each set \( R_{ti} \) is rational, since each \( R_{ti} \) is an intersection of a rational set with a recognisable set.

By Lemma 2.4, we have that for each \( t \in T \), \( R_{ti} = S_{ti} t \), for some rational subset \( S_{ti} \) of \( H \). For every \( (u_1, \ldots, u_n) \in T^n \) that forms a solution to the \( Z_i \)s within a solution to (5), we have \( u_1 t_1 \cdots u_n t_p \in H \). Let \( A \subseteq T^n \) be the set of all such \( n \)-tuples. If we plug a fixed choice of some \( (u_1, \ldots, u_n) \in T^n \) into (5), we obtain the following system of \( \Omega \)-twisted equations in \( H \):
\[
W_1 f_1 \cdots W_p f_p Z_{ij}^{\varepsilon_{ij}} t_{ij} \cdots Z_{pj}^{\varepsilon_{pj}} t_{pj} = 1,
\]
where
\[
W_l = (Y^\varepsilon_{ij} \psi_{u_{ij}}^\delta) \cdot Z_{ij}^{\varepsilon_{ij}} h_{ij} t_{ij} \cdots (Y^\varepsilon_{lj} \psi_{u_{lj}}^\delta) Z_{lj}^{\varepsilon_{lj}},
\]
is \( W_l \), with each \( Z_i \) being replaced by \( u_i \). We can now apply the rational constraint \( S_{ti} \) to the variable \( Y_i \), and we have a system of equations \( \mathcal{E}_{H,(u_1,\ldots,u_n)} \) with rational constraints in \( H \). Let \( B_{(u_1,\ldots,u_n)} \) be the solution set to \( \mathcal{E}_{H,(u_1,\ldots,u_n)} \). It follows that
\[
S = \bigcup_{(u_1,\ldots,u_n) \in A} \{(f_1 u_1, \ldots, f_n u_n) \mid (f_1, \ldots, f_n) \in B_{(u_1,\ldots,u_n)}\}.
\]

**Remark 4.2.** Let \( G \) be a finite index overgroup of a group \( H \). We will define a normal form for \( G \), induced by an existing normal form on \( H \). Let

- \( \Sigma_H \) be a finite generating set for \( H \);
- \( \eta_H \) be a normal form for \( H \), with respect to \( \Sigma_H \);
- \( T \) be a (finite) right transversal for \( H \) in \( G \).

We will use \( \Sigma = \Sigma_H \sqcup T \) as our generating set for \( G \). Each \( g \in G \) can be written uniquely in the form \( g = h_g t_g \) for some \( h_g \in H \) and \( t_g \in T \). Define \( \eta: G \to (\Sigma^\pm)^* \) by
\[
\eta h = (h_g \eta_H) t_g.
\]
Note that if \( \eta_H \) is regular, then \( \eta \) is regular, as the concatenation of \( \text{im} \eta_H \) with a finite language.

As the following lemma shows, this construction also preserves the property of being quasigeodesic.

**Lemma 4.3.** Let \( G, H, \Sigma, \Sigma_H, T, \eta \) and \( \eta_H \) be defined as in Remark 4.2. Then \( \eta_H \) is quasigeodesic if and only if \( \eta \) is quasigeodesic.
Proof ($\Rightarrow$): Suppose $\eta_H$ is quasigeodesic. Then there exists $\lambda > 0$, such that $|h\eta_H| \leq \lambda|h|(H, \Sigma_H) + \lambda$ for all $h \in H$.

For each $t \in T$ and $a \in \Sigma^\pm$, $ta = \nu_{t,a}$, for some $\nu_{t,a} \in \im \eta$. For all $t, t' \in T$, we have $tt' = \rho_{t,t'}$, for some $\rho_{t,t'} \in \im \eta$. For each $t^{-1} \in T^{-1}$, we have that $t^{-1} = \varepsilon \cdot x_{t^{-1}}$, where $x_{t^{-1}} \in (\Sigma^+_H \cup T)^\ast$. Let

$$
\mu = \max_{t^{-1} \in T^{-1}} |x_{t^{-1}}| + \max_{t^{-1}, t' \in T} |\rho_{t,t'}| + \max_{a \in \Sigma_H} |\nu_{t,a}|
$$

Let $w \in (\Sigma^\pm)^\ast$ be a geodesic. We will convert $w \in (\Sigma^\pm)^\ast$ into a word $u$, such that $u = G w$ and $u \in \im \eta$, and we will show that $|u| \leq \mu^2 \lambda |w| + \mu^2 \lambda$.

We first replace each occurrence of $t^{-1} \in T^{-1}$ with the word $x_{t^{-1}}$ within $w$. Since $|x_{t^{-1}}| \leq \mu$ for all $t^{-1} \in T$, doing this will result in a new word $w_1 \in (\Sigma^+_H \cup T)^\ast$, such that $w_1 = G w$, and $|w_1| \leq \mu |w|$.

We now modify $w_1$ into a word $w_2$ such that $w_1 = G w_2$, and $w_2$ contains no subword of the form $ta$ or $tt'$, where $t, t' \in T$ and $a \in \Sigma^+_H$. For each subword $ta$ of $w$, we can replace $ta$ with $\nu_{t,a}$, and for every occurrence of $tt'$, we can replace this with $\rho_{t,t'}$. Each time we do this, we increase the length of the word by at most $\mu$. Repeating this process until no subwords of the form $ta$ remain, will yield $w_2$.

To ensure we don’t need to do too many of these replacements to satisfy linear bound of the length of $w_2$ in terms of $w_1$, we will always apply the leftmost substitution possible. As every replacement involves a letter $t \in T$ at the beginning of a two-letter word, and results in a word with exactly one two-letter in $T$ at the end, one ‘sweep’ along $w_1$ will be sufficient to reach a word where no substitutions are possible. It follows that we can make at most $|w_1|$ replacements, and since each substitution increases the length by at most $\mu$, we have that $|w_2| \leq \mu |w_1|$.

We have that $w_2 = vt$, for some $v \in (\Sigma^+_H)^\ast$, and some $t \in T$. To convert $w_3$ into $u$, it remains to replace $v$ with an equivalent word $q \in \im \eta_H$. As $\eta_H$ is quasigeodesic with the constant $\lambda$, $|q| \leq \lambda |v| + \lambda$. If we take $u = qt$, then $u$ is equivalent in $G$ to $w$, and $u \in \im \eta$. Note also that $|v| \leq \lambda |w_2| + \lambda$. Therefore

$$
|u| = \lambda |w_2| + \lambda \leq \mu \lambda |w_1| + \mu \lambda \leq \mu^2 \lambda |w| + \mu^2 \lambda.
$$

It follows that $\eta_H$ is quasigeodesic, with respect to the constant $\lambda \mu^2$.

($\Leftarrow$): Suppose $\eta$ is quasigeodesic, with respect to a constant $\lambda > 0$. Let $w \in (\Sigma^+_H)^\ast$ be a geodesic, $u \in \im \eta_H$ be such that $u = H w$, and $v \in (\Sigma^\pm)^\ast$ be a geodesic in $G$, such that $v = G w$. Note that $u \in \im \eta$. As $\eta$ is quasigeodesic, $|u| \leq \lambda |v| + \lambda$. Moreover, since $|w|$ and $|v|$ are both geodesic words representing elements that lie in $H$, but $v$ is over the generating set $\Sigma_G$ that contains the generating set $\Sigma_H$ for $w$, $|v| \leq |w|$. Thus $|u| \leq \lambda |w| + \lambda$, as required. \qed

We can use Lemma 4.3 to show that passing to the Schreier normal form also preserves the property of being quasigeodesic.

Lemma 4.4. Let $G$ be a group, generated by a finite set $\Sigma$, $H$ be a finite index subgroup of $G$, and $T$ be a right transversal of $H$ in $G$, containing 1. Let $Z$ be the Schreier generating set for $H$. Fix a normal form $\eta$ for $(G, \Sigma)$. If $\eta$ is quasigeodesic with respect to $\Sigma$, then the Schreier normal form with respect to $\eta$ is quasigeodesic with respect to the Schreier generators.

Proof Let $\zeta$ be the Schreier normal form for $H$, with respect to $\eta$. We will show that the normal form from Remark 4.2 inherited from $\zeta$, is quasigeodesic. The result will then follow by
the backward direction of Lemma 4.3. Since \( \eta \) is quasigeodesic, there exists \( \lambda > 0 \), such that 
\[
|\eta t| \leq \lambda |gt|^G + \lambda \text{ for all } g \in G.
\]

Let \( \xi \) denote the normal form from Remark 4.2, inherited from \( \zeta \), with respect to the transversal \( T \). Let \( w \in (\Sigma^\pm)^* \) be geodesic. We have that there exists \( v \in \text{im} \eta \), such that \( v = G w \), and 
\[
|v| \leq \lambda |w| + \lambda.
\]
We also have that there exists \( t_0 \in T \) such that \( vt_0 \) represents an element of \( H \). We can then convert this into Schreier normal form to give a word \( u \). Note that 
\[
|u| \leq |vt_0|.
\]

We also have that there exists \( t_1 \in T \), such that \( ut_1 = G w \). Note that \( ut_1 \in \text{im} \xi \). Combining our inequalities that relate \( u \), \( v \) and \( w \), gives:
\[
|ut_1| \leq |vt_0t_1| = |v| + 2 \leq \lambda |w| + 2\lambda.
\]

So \( \xi \) is quasigeodesic, with respect to a constant \( 2\lambda \). The result now follows by Lemma 4.3. \( \square \)

The following result is not new; a slightly different version of it is used implicitly to show systems of equations with rational constraints in virtually abelian groups have EDT0L solution languages. We use it here to show the same is true for virtually abelian groups.

**Proposition 4.5.** Let \( G \) be a group with a finite index normal subgroup \( H \), and let \( \Omega \) be the group of automorphisms of \( H \) induced by conjugation by elements of \( G \). Suppose that solutions to systems of \( \Omega \)-twisted equations in \( H \) with rational constraints are EDT0L in \( \text{NSPACE}(f) \), where \( f: \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0} \). Then solutions to systems of equations with rational constraints in \( G \) are EDT0L in \( \text{NSPACE}(f) \) with respect to the normal form from Remark 4.2.

**Proof** We can use the same proof that is used in Proposition 3.9 and Lemma 3.11 in \[16\], using Lemma 4.1 in place of the Lemma 3.8 used in \[16\]. \( \square \)

In order to prove our results about wreath products and direct products, we need some normal forms on groups made using these constructions.

**Remark 4.6.** Let \( H_1, \ldots, H_k \) be groups, with finite generating sets \( \Sigma_{H_1}, \ldots, \Sigma_{H_k} \), and normal forms \( \eta_{H_1}, \ldots, \eta_{H_k} \), respectively. Let \( G = \prod_{i=1}^k H_i \). We will use \( \Sigma = \Sigma_{H_1} \sqcup \cdots \sqcup \Sigma_{H_k} \) as a generating set for \( G \). Define the \( \eta: G \to (\Sigma^\pm)^* \) by
\[
(h_1, \ldots, h_k)\eta = (h_1\eta_{H_1}) \cdots (h_k\eta_{H_k}).
\]
Since concatenations of regular languages are regular, if every \( \eta_{H_i} \) is regular, then \( \eta \) is a regular normal form.

In addition, the length of any element \( g \in G \) with respect to \( \Sigma \) is just the sum of the lengths of the projection of \( g \) to each \( H_i \), and from this it follows that if every \( \eta_{H_i} \) is (quasi)geodesic, then so is \( \eta \).

**Remark 4.7.** Let \( H \) be a group, and \( K \) be a finite group. Let \( \Sigma_H \) be a generating set for \( H \), and \( \eta_H \) be a normal form with respect to \( \Sigma_H \). We define a generating set and normal form for \( H \wr K \), using \( \Sigma_H \) and \( \eta_H \). Note that \( H \wr K \) contains \( \prod_{i=1}^n H_i \) as a finite index subgroup, where \( n \in \mathbb{Z}_{>0} \), and \( H_i \cong H \) for all \( i \). We endow \( \prod_{i=1}^n H_i \) with a generating set and normal form using Remark 4.6. After this, we can use the generating set and normal form from Remark 4.2 for \( H \wr K \), with respect the generating set and normal form of \( \prod_{i=1}^n H_i \).

Since the two constructions we have used to produce a normal form for \( H \wr K \) preserve the properties of regular and quasigeodesic, if \( \eta_H \) is regular or quasigeodesic, then so is the normal form on \( H \wr K \).
We show that the class of groups with EDT0L solutions to systems of equations is closed under direct products, and wreath products with finite groups. We start with the latter. We refer the reader to [19] for the definition of a wreath product.

We first consider the properties of the normal forms we will be using.

**Lemma 4.8.** Let $H$ and $\eta_H$ be as in Remark 4.7. If $\eta_H$ is regular or quasigeodesic, then the normal form on $H \wr K$ from Remark 4.7 will be regular or quasigeodesic, respectively.

**Proof** Recall that the normal form in Remark 4.7 is created by using the normal form for direct products (Remark 4.6), followed by the normal form for finite extensions 4.2. Since both of these constructions preserve the properties regular and quasigeodesic, the result follows.

We can now show that equations in wreath products have the desired properties.

**Proposition 4.9.** Let $H$ be a group such that solutions to systems of equations with respect to a normal form $\eta_H$ are EDT0L in $\text{NSPACE}(f)$, where $f: \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}_{\geq 0}$. Let $K$ be a finite group. Then

1. Solutions to systems of equations in $H \wr K$ are EDT0L in $\text{NSPACE}(f)$, with respect to the normal form from Remark 4.7.
2. If $\eta_H$ is regular or quasigeodesic, then the normal form on $H \wr K$ will be regular or quasigeodesic, respectively.

**Proof** First note that (2) follows from Lemma 4.8.

Let $A$ be the finite set that $K$ acts on, and define $H \wr K$ with respect to this action. Let $H_1, \ldots, H_{|A|}$ be the isomorphic copies of $H$. Using Proposition 4.5, it suffices to show that solutions to systems of $\Omega$-twisted equations in $G := \prod_{i=1}^{|A|} H_i$ are EDT0L in $\text{NSPACE}(f)$, with respect to the normal form from Remark 4.6, where $\Omega$ is the set of automorphisms defined by permuting the $H_i$s.

Consider a system $\mathcal{E}$ of $\Omega$-twisted equations in $G$ in $n$ variables. As every element of $G$ can be written in the form $h_1 \cdots h_{|A|}$, where $h_i \in H_i$ for all $i$, for each variable $X$ in $\mathcal{E}$, we can define new variables $X_i$ over $H_i$ for each $i$, by $X = X_1 \cdots X_{|A|}$. As the elements of $H_i$ commute with the elements of $H_j$ for each $i \neq j$, we can view any (untwisted) equation in $G$ as a system of $|A|$ equations in $H$, each with disjoint sets of variables, by projecting the original equation onto $H_i$. The fact that these sets are disjoint follows from the fact that the $i$th equation in the system will be the projection to $H_i$, whose variables will be of the form $X_i$, for some original variable $X$.

Let $\Phi \in \Omega$, and let $\sigma \in S_n$ be the permutation induced by the action of $\Phi$. Then $X\Phi = (X_1 \cdots X_{|A|})\Phi = X_{1\sigma} \cdots X_{(|A|)\sigma}$. It follows that any twisted equation in $G$ can be viewed as a system of $|A|$ equations in $H$, again using projections to each $H_i$. The variables of each of the equations will no longer be disjoint, however. It follows that a system of twisted equations in $G$ projects to a system of equations in $H$. Thus, there exists a system $\mathcal{F}$ of equations in $H$ with solution set $S_\mathcal{F}$, such that $\mathcal{F}$ has $|A|n$ variables, and each variable is assigned an index in $\{1, \ldots, |A|\}$, such that precisely $n$ variables have each index, and such that the solution language of $\mathcal{E}$ is equal to

$$\{x_{11} \cdots x_{1|A|} \# \cdots \# x_{ni} \cdots x_{n|A|} \mid (x_{1i}, \ldots, x_{ni}) \in S_\mathcal{F} \text{ with each variable indexed by } i \text{ for all } i\}.$$

From our assumptions, we have that the solution language to $\mathcal{F}$ is EDT0L, and can be constructed in $\text{NSPACE}(f)$. It follows that the language

$$L_i = \{x_{1i} \# \cdots \# x_{ni} \mid (x_{1i}, \ldots, x_{ni}) \in S_\mathcal{F} \text{ with each variable indexed by } i\}$$
is EDT0L for each choice of \(i\), and constructible in \(\text{NSPACE}(f)\), using Lemma\ref{thm:edt0l_construction}, and then taking the image under an appropriate free monoid endomorphism with Lemma\ref{thm:edt0l_image}. Lemma\ref{thm:edt0l_solution} then shows that the solution language to \(\mathcal{E}\) is EDT0L in \(\text{NSPACE}(f)\).

We conclude this section with the proof that direct products also preserve the property of having EDT0L solution languages.

**Proposition 4.10.** Let \(f: \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}_{\geq 0}\). Let \(G\) and \(H\) be finitely generated groups where solutions to systems of equations are EDT0L in \(\text{NSPACE}(f)\). Then

1. The same holds in \(G \times H\), with respect to the normal form from Remark\ref{rem:normal_form};
2. If the normal forms on \(G\) and \(H\) are regular or quasigeodesic, then the normal form on \(G \times H\) will be regular or quasigeodesic, respectively, with respect to the union of the generating sets for \(G\) and \(H\).

**Proof** Part (2) follows from Remark\ref{rem:normal_form}.

Let \(\Sigma_G\) be a finite generating set for \(G\), and \(\Sigma_H\) be a finite generating set for \(H\). We will use \(\Sigma = \Sigma_G \cup \Sigma_H\) as our generating set for \(G \times H\). Consider an equation \(\omega = 1\) in \(G \times H\). Let \(X\) be the set of variables in \(\omega\). We have that every element of \(G \times H\) can be expressed in the form \(gh\) for some \(g \in G\) and \(h \in H\). We can reflect this in the variables as well, by defining new variables \(X_G\) over \(G\) and \(X_H\) over \(H\), for each \(X \in X\), such that \(X = X_G X_H\).

As elements of \(G\) commute with elements of \(H\), we can rearrange \(\omega = 1\) into the form \(\nu \zeta = 1\), where \(\nu \in (\Sigma^+_G \cup X^+_G)^*\) and \(\zeta \in (\Sigma^+_H \cup X^+_H)^*\). Consider a potential solution \((g_1 h_1, \ldots, g_n h_n)\) to \(\omega = 1\), where each \(g_i \in G\) and each \(h_i \in H\). We have that this is a solution if and only if \((g_1, \ldots, g_n)\) is a solution to the equation \(\nu = 1\), and \((h_1, \ldots, h_n)\) is a solution to the equation \(\zeta = 1\). Note that these are equations in \(G\) and \(H\), respectively.

Let \(\mathcal{E}\) be a system of equations in \(G \times H\). It follows that there exist systems of equations in \(G\) and \(H\) with solution sets \(S_G\) and \(S_H\), such that the solution set to \(\mathcal{E}\) equals

\[
\{(g_1 h_1, \ldots, g_n h_n) | (g_1, \ldots, g_n) \in S_G, (h_1, \ldots, h_n) \in S_H\}.
\]

If \(\mathcal{L}_G\) and \(\mathcal{L}_H\) are EDT0L solution languages corresponding to these systems in \(G\) and \(H\), respectively, it follows that the solution language to \(\mathcal{E}\) equals

\[
\{\omega_0 \nu_0 \# \cdots \# \omega_n \nu_n | \omega_0 \# \cdots \# \omega_n \in \mathcal{L}_G, \nu_0 \# \cdots \# \nu_n \in \mathcal{L}_H\}.
\]

The result now follows by Lemma\ref{thm:edt0l_solution}.\qed

## 5. Virtually abelian groups

In this section we expand the work of Evetts and the author\cite{Evetts2012}, to show that systems of equations with rational constraints in virtually abelian groups have EDT0L solution languages. We start by looking at rational sets in free abelian groups and show that systems of twisted equations in free abelian groups have EDT0L solution languages. After this, we use Proposition\ref{thm:edt0l_solution} to complete the proof. Grunschlag’s result about rational sets in finite index subgroups (Lemma\ref{thm:edt0l loadImage}) is used to allow this generalisation.

We start with the definition of semilinear sets, used to give a description of rational subsets of free abelian groups.
Definition 5.1. Let $k \in \mathbb{Z}_{>0}$. A subset of $\mathbb{Z}^k$ that can be written in the form
\[
\{c_1n_1 + \cdots + c_rn_r + d \mid n_1, \ldots, n_r \in \mathbb{Z}_{\geq 0}\},
\]
where $c_i, d \in \mathbb{Z}^k$ for all $i$, is called linear. A finite union of linear sets is called semilinear.

Showing that semilinear sets are rational is immediate from the definition. The converse is also true, thus giving a full classification of rational sets in free abelian groups.

Lemma 5.2 ([15]). A subset of a free abelian group is rational if and only if it is semilinear.

Since semilinear sets are defined in terms of equations and inequalities, we can use this to describe sets of solutions to systems of twisted equations with rational constraints in free abelian groups.

Lemma 5.3. Let $S_E$ be the solution set of a finite system $E$ of twisted equations in $\mathbb{Z}^k$ in $n$ variables with rational constraints. Then there is a finite disjunction $F$ of finite systems of equations, and inequalities of the form $X \geq 0$, for some variable $X$, in $\mathbb{Z}$ with $kn$ variables and solution set $S_F$ such that
\[
S_E = \{(x_1, \ldots, x_k), \ldots, (x_{(k-1)n+1}, \ldots, x_{kn}) \mid (x_1, \ldots, x_{kn}, y_1, \ldots, y_r) \in S_F\}.
\]

Proof Converting the twisted system into a system over $\mathbb{Z}$ can be done by replacing each variable $X$ over $\mathbb{Z}$ with $k$ variables $X_1, \ldots, X_k$ over $\mathbb{Z}$, and considering the system that results from looking at each coordinate individually. A full proof of this can be found in [16]. Now consider the membership problem of a variable $X$ into a linear set $R = \{c_1n_1 + \cdots + c_rn_r + d \mid n_1, \ldots, n_r \in \mathbb{Z}_{\geq 0}\}$ (we will then generalise to semilinear).

Write $c_i = (c_{i1}, \ldots, c_{ik})$ and $d = (d_1, \ldots, d_k)$. Consider the following system of equations and inequalities over $\mathbb{Z}$.
\[Y_i \geq 0, \quad X_j = c_{ij}Y_1 + \cdots c_{rj}Y_r + d_j\]
for all $i \in \{1, \ldots, r\}$, and $j \in \{1, \ldots, k\}$, where $Y_1, \ldots, Y_r$ are new variables over $\mathbb{Z}$. We have that $(x_1, \ldots, x_k) \in \mathbb{Z}^k$ occurs within a solution $(x_1, \ldots, x_k, y_1, \ldots, y_k)$ to the above system, if and only if $(x_1, \ldots, x_k) \in R$.

The result follows from the fact that the solution set to a disjunction of systems is just the union of the solution sets, so if we take the disjunction of the systems obtained from each linear set used in the finite union of a semilinear set, we obtain the desired disjunction. □

We are now in a position to describe the solution language to a system of twisted equations with constraints in a free abelian group, using an EDT0L system.

Lemma 5.4. Solutions to systems of twisted equations with rational constraints in a free abelian group are EDT0L in non-deterministic quadratic space, with respect to free abelian equation length, and the standard normal form.

Proof We will use $\Sigma = \{a_1, \ldots, a_k\}$ to denote the standard generating set for $\mathbb{Z}^k$. Let $E$ be a system of equations in $\mathbb{Z}^k$ with solution language $L$. By Lemma 5.3, there is a disjunction $F$ of systems of equations, and inequalities of the form $X \geq 0$, in $\mathbb{Z}$, with set of solutions $S_F$, such that
\[
L = \{a_1^{x_1} \cdots a_k^{x_k} \# \cdots \# a_1^{x_{(k-1)n+1}} \cdots a_k^{x_{kn}} \mid (x_1, \ldots, x_{kn}, y_1, \ldots, y_r) \in S_F\}.
\]

Consider the following language
\[
M = \{a_1^{x_1} \# \cdots a_k^{x_k} \# \cdots a_1^{x_{(k-1)n+1}} \cdots a_k^{x_{kn}} b_1^{r_1} \# \cdots \# b_r^{r_r} \mid (x_1, \ldots, x_{kn}, y_1, \ldots, y_r) \in S_F\}.
\]
We will start by showing that $M$ is EDT0L. First note that as finite unions of EDT0L languages are EDT0L, we can assume $F$ is a single system of equations and inequalities, rather than a disjunction of systems. Let $m$ be the number of inequalities of the form $X \geq 0$ within $F$.

We will proceed by induction on $m$. If $m = 0$, then $F$ is a system of equations in $\mathbb{Z}$, and thus the solutions are EDT0L in $\text{NSPACE}(n^2)$, by \cite{16}. Inductively suppose $M$ is EDT0L, and an EDT0L system is constructible in $\text{NSPACE}(n^2)$, when $m = k$, where $k \in \mathbb{Z}_{\geq 0}$. If $m = k + 1$, then $F$ can be obtained from a system of equations and inequalities $G$, with the addition of a single inequality $X \geq 0$. By our inductive hypothesis, the solution language of $G$ is EDT0L, and an EDT0L system is constructible in $\text{NSPACE}(n^2)$. The fact that the solution $X \geq 0$, can be achieved by intersecting the solution language of $G$ with the regular language

$$(\Sigma^\pm)^* (#(\Sigma^\pm)^* \cdots #)(a_1^\pm) \cdots (a_j^\pm) a_j^\pm \cdots (a_{j+1}^\pm) a_{j+1}^\pm \cdots (a_k^\pm) #(\Sigma^\pm)^* \cdots #(\Sigma^\pm)^*,$$

where $j$ is the free abelian generator in the correct position corresponding to $X$. The fact that $M$ is EDT0L, and an EDT0L system is constructible in $\text{NSPACE}(n^2)$ now follows from Lemma 2.15.

For each $i \in \{0, \ldots, n - 1\}$, let

$$u_i = a_1^x a_{ik+1} \cdots a_{ik+k-1} a_k^x #ik+k.$$

Since $M$ is EDT0L, it follows by Lemma 5.1 that

$$M' = \{ u_1 \cdots u_{n-1} b_1^n \# b_{n+1} \cdots \# b_{n-r} + 1 b_r^n \# \# b_{kn+1} \cdots \# b_{kn+r} - 1 b_r^n \# \# b_{kn+1} \cdots \# b_{kn+r} \mid (x_1, \ldots, x_{kn}, y_1, \ldots, y_r) \in S_F \}$$

is EDT0L, and a system is constructible in $\text{NSPACE}(n^2)$. In order to show that $L$ is EDT0L, it suffices apply a sequence of free monoid homomorphisms to $M'$ to obtain $L$. Firstly, apply the homomorphism defined by mapping each of $b_1, \ldots, b_r$ and $\#$ to $\#k, \ldots, k_{n-r}$ to $\epsilon$ to $M'$ to obtain

$$M'' = \{ a_1^x \# \cdots \# b_k \# a_k^x \# \cdots \# b_k \# a_k^x \# \cdots \# b_k \# a_k^x \# \cdots \# b_k \# a_k^x \mid (x_1, \ldots, x_{kn}, y_1, \ldots, y_r) \in S_F \}.$$

Now apply the homomorphism which maps each $\#$ to $\epsilon$, with the exception of $\#k, \ldots, \#k_{n-1}$. These will instead be mapped to $\#$. The image of $M''$ under this homomorphism is $L$. The result now follows from Lemma 2.15.

We now have everything needed to show the following.

**Theorem B.** Solutions to a system of equations with rational constraints in a virtually abelian group are EDT0L in non-deterministic quadratic space, with respect to virtually abelian equation length, and with respect to the regular quasigeodesic normal form from Remark 4.2, induced by the standard normal form on free abelian groups.

**Proof.** This fact that the solutions are EDT0L in $\text{NSPACE}(n^2)$ follows from Lemma 5.4 and Proposition 4.5. The fact that the normal form is regular and quasigeodesic follows from Remark 4.2 and Lemma 4.3, respectively, together with the fact that the standard normal form on a free abelian group is regular and quasigeodesic.

6. Recognisable constraints and finite index subgroups

This section is used to show Proposition 6.3, that is, that the class of groups where systems of equations have EDT0L solutions is closed under passing to finite index subgroups. We use recognisable constraints to show this fact, by first proving that the addition of recognisable constraints to a system of equations with an EDT0L solution set does not change the fact that the solution
set is EDT0L with respect to the ambient normal form of the group. We can then use the fact that finite index subgroups are recognisable, however the resulting language will be expressed as words over the generators for the ambient group. Expressing solutions to the finite index subgroup as words over one of its own generating sets, such as the Schreier generators, requires additional arguments.

We start by showing that the addition of recognisable constraints to systems of equations in a group preserves the property that all such systems have EDT0L solution languages.

**Proposition 6.1.** Let $G$ be a finitely generated group such that solutions to systems of equations are EDT0L in $\text{NSPACE}(f)$ with respect to some normal form $\eta$, where $f : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$. Then solutions to systems of equations in $G$ with recognisable constraints are EDT0L in $\text{NSPACE}(f)$, with respect to $\eta$.

**Proof** Let $\Sigma$ be a finite generating set for $G$, and fix a normal form $\eta$ for $(G, \Sigma)$ such that solution languages to systems of equations are EDT0L. Consider a system of equations $\mathcal{E}$ with recognisable constraints in $G$ with $n$ variables. Let $R_1, \ldots, R_n$ denote the constraints. Let $L$ be the solution language to $\mathcal{E}$ with the constraints removed. Let $\pi : \Sigma^* \to G$ be the natural homomorphism. Note that $S = (R_1\pi^{-1})\#(R_2\pi^{-1})\# \cdots \#(R_n\pi^{-1})$ is a regular language. By Lemma 2.15, $L \cap S$ is EDT0L, and if an EDT0L system for $L$ is constructible in $\text{NSPACE}(f)$, then one for $L \cap S$ is also constructible in $\text{NSPACE}(f)$. As $L \cap S$ is the solution language to $\mathcal{E}$, the results follow. □

Since finite index subgroups are examples of recognisable sets, we can show the following.

**Lemma 6.2.** Let $f : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$. Let $G$ be a finitely generated group where solutions to systems of equations are EDT0L in $\text{NSPACE}(f)$, with respect to some normal form $\eta$, and let $H$ be a finite index subgroup of $G$. Let $\mathcal{E}$ be a system of equations in $G$. Then

1. the language of all solutions to $\mathcal{E}$ that lie in $H$ forms an EDT0L language, with respect to the normal form $\eta$ restricted to $H$;
2. The EDT0L system for this language is constructible in $\text{NSPACE}(f)$.

**Proof** In order to restrict our solutions to $H$, we add the constraint that every variable lies in $H$, which is a recognisable subset of $G$. The results now follow from Proposition 6.1. □

**Proposition 6.3.** Let $f : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$. Let $G$ be a group where solutions to systems of equations are EDT0L in $\text{NSPACE}(f)$, with respect to a normal form $\eta$. Then the same holds in any finite index subgroup of $G$ with respect to the Schreier normal form, inherited from $\eta$.

**Proof** Let $X$ be a finite generating set for $G$, $T$ be a right transversal for $H$, and $Z$ be the Schreier generating set for $H$. Let $\zeta$ be the Schreier normal form for $H$.

Fix a system $\mathcal{E}$ of equations in $H$. This can be considered as a system of equations in $G$, with the restriction that the solutions must lie in $H$. Let $L$ be the solution language to $\mathcal{E}$ when expressed as words over $G$ using the normal form $\eta$; that is

$$L = \{(g_1\eta)\# \cdots \#(g_n\eta) \mid (g_1, \ldots, g_n) \text{ is a solution to } \mathcal{E}\}.$$ 

Note that we require that solutions lie in $H$, as $\mathcal{E}$ is a system over $H$. By Lemma 6.2, $L$ is an EDT0L language over an alphabet $\Sigma$. Let $\mathcal{H} = (\Sigma, C, \omega, \mathcal{R})$ be an EDT0L system for $L$ that is constructible in $\text{NSPACE}(f)$. Let $B \subseteq \text{End}(C^*)$ be the (finite) set over which $\mathcal{R}$ is a regular language.
By Lemma 3.4, we can assume our start word is of the form \( \omega_1 \# \cdots \# \omega_n \). By adding new letters \( \perp_1 \cdots \perp_n \) to \( C \), and preconcatenating the rational control by the endomorphism defined by \( \perp_i \mapsto \omega_i \) for all \( i \), we can assume our start word is of the form \( \perp_1 \# \cdots \# \perp_n \). Note that as we can easily construct our new start word from our existing one, this will not affect space complexity.

We will construct a new EDT0L system from \( H \). Our extended alphabet will be letters in \( C \) transversal element \( t \in T \). Define

\[
C_{\text{ind}} = \{ c^{t,a} \mid c \in C, \ t \in T, \ a \in X^\pm \} \cup \{ \perp_1, \ldots, \perp_n \}.
\]

Our alphabet will be \( \Sigma_{\text{ind}} \cup \{ \# \} = \{ a^{t,a} \mid a \in X^\pm, \ t \in T \} \cup \{ \# \} \). Our start word will be \( \perp_1 \# \cdots \# \perp_n \). We define our rational control as follows. For each \( \phi \in B \), define \( \Phi_\phi \) to be the set of all \( \psi \in \text{End}(C_{\text{ind}}^*) \) such that

\[
c^{t,a}_i \psi = x_1^{t_1,b_1} \cdots x_k^{t_k,b_k},
\]

where \( c \psi = x_1 \cdots x_k \), with every \( x_i \in C \), each \( t_i \in T \), \( t_1 = t \), and \( b_i b_i = t_{i+1} \). Let \( R_1 \) be the regular language obtained by replacing each occurence of \( \phi \in B \) with the finite set \( \Phi_\phi \). Let \( t_0 \) be the unique element in \( T \cap H \). Let \( \Psi \subseteq \text{End}(C_{\text{ind}}^*) \) be the set of all \( \psi \) defined by

\[
\perp_i \psi = \perp_i^{t_0, a_i},
\]

for some \( a_1, \ldots, a_n \in X^\pm \). Define \( G = (\Sigma_{\text{ind}}, C_{\text{ind}}, \perp_1 \# \cdots \# \perp_n, \Psi R_1) \). By construction, \( G \) accepts words in \( L \), where each letter, excluding \( \# \), has an index \( (t,a) \in T \times X^\pm \), and such that for each indexed word \( w = a_1^{t_1,a_1} \cdots a_k^{t_k,a_k} \), the following hold:

1. \( t_1 = t_0 \);
2. \( t_i a_i = t_{i+1} \) for all \( i \).

To show that the solution language to \( E \) is EDT0L with respect to \( \zeta \), it remains to apply the free monoid homomorphism \( \theta : \Sigma_{\text{ind}}^* \rightarrow (Z^\pm \cup \{ \# \})^* \) to \( L(G) \), defined by

\[
a^{t,a} \mapsto a \theta t^{-1}.
\]

It now remains to show that this EDT0L system can be constructed in \( \text{NSPACE}(f) \). By Lemma 2.15, applying the homomorphism \( \theta \) does not affect the space complexity, so it is sufficient to show that \( G \) is constructible in \( \text{NSPACE}(f) \). The number of indices we use is \( 2|X||T| \), which is constant, as it is based only on the group \( H \). It follows that we can write down \( C_{\text{ind}} \) and \( \Sigma_{\text{ind}} \) in \( \text{NSPACE}(f) \).

The set \( \Psi \) is again only based on \( |X| \), and so to show our rational control is constructible in \( \text{NSPACE}(f) \), it suffices to prove that \( R_1 \) is.

Note that \( |\Phi_\phi| \) is again only based on \( |X||T| \), and so is constant. We construct \( R_1 \) by proceeding with the procedure we used to construct \( R \), except whenever we would add an edge labelled \( \phi \in B \) between two states, we add edges labelled with all of \( \Phi_\phi \) between the same states. We can compute \( \Phi_\phi \) each time we need it, so we need only record the information we used to construct \( R \). We can conclude that \( G \) is constructible in \( \text{NSPACE}(f) \), and so the language of solutions to \( E \) is EDT0L in \( \text{NSPACE}(f) \).

\[\square\]

7. Virtually direct products of hyperbolic groups

In this section, we show that solution languages to systems of equations in groups that are virtually direct products of hyperbolic groups are EDT0L. We adapt the method that Ciobanu, Holt and Rees use to show that the satisfiability of systems of equations in these groups is decidable [7]. For an introduction to hyperbolic groups, we refer the reader to Chapter 6 of [19].
We start with some lemmas needed to prove this result. The following lemma gives an embedding as a finite index subgroup of a group that is virtually a direct product of hyperbolic groups, into a direct product of groups where equations are better understood.

**Lemma 7.1** (Lemma 3.5 in [7]). Let $G$ be a group that contains a group of the form $K_1 \times \cdots \times K_n$ as a finite index normal subgroup, such that every conjugate of each of the subgroups $K_i$ lies in the set \{ $K_1$, \ldots, $K_n$ \}. Then

1. If the groups $K_i$ are all conjugate to each other, then $G$ is isomorphic to a finite index subgroup of $J \wr P$, where $J \cong N_G(K_1)/(K_2 \times \cdots \times K_n)$ contains a finite index subgroup isomorphic to $K_1$, and $P$ is finite;
2. Suppose $K_1$, \ldots, $K_k$ are representatives of the conjugacy classes of $K_1$, \ldots, $K_n$ within $G$. Then $G$ is isomorphic to a finite index subgroup of a direct product $W_1 \times \cdots \times W_k$, where $W_i = J_i \wr P_i$, $J_i$ contains $K_i$ as a finite index subgroup, and $P_i$ is finite, for all $i$.

We define a normal form for groups that are virtually direct products.

**Remark 7.2.** Let $G$ be a group that has a finite index subgroup of the form $K_1 \times \cdots \times K_n$. Fix a finite generating set $\Sigma_{K_i}$, and normal form $\eta_{K_i}$ for each $K_i$. Using Lemma 7.1, $G$ embeds as a finite index subgroup of $W_1 \times \cdots \times W_k$, where $W_i = J_i \wr P_i$, $K_i$ embeds as a finite index subgroup of $J_i$, and $P_i$ is finite.

- We start by defining a generating set and normal form for each $J_i$. Since $J_i$ contains $K_i$ as a finite index subgroup, we can use the generating set and normal form from Remark 1.2 induced by $\Sigma_{K_i}$ and $\eta_{K_i}$. We will denote this generating set and normal form using $\Sigma_{J_i}$ and $\eta_{J_i}$, respectively;
- Using $\Sigma_{J_i}$ and $\eta_{J_i}$, we can use the generating set and normal form defined in Remark 1.2 to define a normal form for each $W_i = J_i \wr P_i$. Using these generating sets and normal forms, Remark 1.6 gives us a generating set $\Delta$ and a normal form $\mu$ for $W_1 \times \cdots \times W_k$;
- As $G$ embeds as a finite index subgroup of $W_1 \times \cdots \times W_k$, we can use the Schreier generating set $Z$ and normal form $\zeta$ on $G$, induced by $\Delta$ and $\mu$.

**Lemma 7.3.** Let $G$ be a group that has a finite index subgroup of the form $K_1 \times \cdots \times K_n$, and let $\eta_{K_i}$ be defined as in Remark 7.2. Let $\zeta$ be the normal form on $G$ from Remark 7.2. If each $\eta_{K_i}$ is regular or quasigeodesic, then $\zeta$ is regular or quasigeodesic, respectively.

**Proof.** Since each of the constructions we have used to create $\zeta$ preserve the properties of being regular and quasigeodesic (Lemma 2.29 Remark 1.6 Lemma 1.8 Lemma 1.3 Lemma 1.4), if every $\eta_{K_i}$ is regular or every $\eta_{K_i}$ is quasigeodesic, then $\zeta$ will be regular or quasigeodesic, respectively. □

We now use Lemma 7.1 to show that the group that is virtually a direct product has an EDT0L solution language, subject to conditions on the groups it is virtually a direct product of.

**Proposition 7.4.** Let $G$ be a group that contains a group of the form $K_1 \times \cdots \times K_n$ as a finite index normal subgroup, such that every conjugate of each of the subgroups $K_i$ lies in the set \{ $K_1$, \ldots, $K_n$ \}. Let $f : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}_{\geq 0}$.

1. If solutions to systems of equations are EDT0L in NSPACE($f$) in each group in FIN($K_i$), with respect to a normal form $\eta_{K_i}$, then solutions systems of equations in $G$ are EDT0L NSPACE($f$), with respect to the normal form $\zeta$ from Remark 7.2.
2. If all the normal forms used in the groups in FIN($K_i$) are regular or quasigeodesic, then $\zeta$ will be regular or quasigeodesic, respectively.
Proof By Lemma 7.1 we have that $G$ embeds as a finite index subgroup into $W_1 \times \cdots \times W_k$, where $W_i = J_i \wr P_i$ for finite index overgroups $J_i$ of $K_i$, and finite groups $P_i$. By Lemma 6.3 it suffices to show that solutions to systems of equations are EDT0L in $\text{NSPACE}(f)$ in $W_1 \times \cdots \times W_k$. The fact that solutions to systems of equations are EDT0L in $\text{NSPACE}(f)$ in each of the groups $W_i$ follows by our assumptions, together with Proposition 4.9. We can then use Proposition 4.10 to show that the same holds in $W_1 \times \cdots \times W_k$. Part (2) follows from Lemma 7.3. □

We now apply Proposition 7.4 to the specific case when the groups in the direct product comprise one virtually abelian group, and other non-elementary hyperbolic groups.

Lemma 7.5 (Proposition 4.4 in [7]). Let $A$ be a virtually abelian group, and let $H_1, \ldots, H_n$ be non-elementary hyperbolic groups. Let $G$ be a group with a finite index subgroup $H$ that is isomorphic to $A \times H_1 \times \cdots \times H_n$. Then $G$ has a finite index normal subgroup isomorphic to $B \times K_1 \times \cdots \times K_n$, where $B$ is a finite index subgroup of $A$, and each $K_i$ is a finite index subgroup of $H_i$, such that every conjugate of each of the subgroups $K_i$ lies in the set $\{K_1, \ldots, K_n\}$.

We finally need the fact that languages of solutions to systems of equations in hyperbolic groups are EDT0L.

Lemma 7.6 ([5]). Solutions to a system of equations in any hyperbolic group are EDT0L in $\text{NSPACE}(n^4 \log n)$, with respect to any finite generating set, and any quasigeodesic normal form. If the hyperbolic group is torsion-free, the solutions are EDT0L in $\text{NSPACE}(n^2 \log n)$.

We are now in a position to show that groups that are virtually direct products of hyperbolic groups have EDT0L languages of solutions. Since every hyperbolic group admits a regular geodesic normal form, if these normal forms are used to induce the normal forms in the hyperbolic groups, then the normal form in the virtually direct product will be quasigeodesic and regular.

Remark 7.7. In the following theorem, our groups are constructed from virtually abelian groups and other groups. As such, we are measuring our input size using equation length, not virtually abelian equation length. However, we will continue to use space complexity results from [10] and Section 5 that use virtually abelian length. This is okay, since virtually abelian equation length is approximately the log of equation length, and so the actual space complexity will be at least as small. It is possible that the space complexity will be a smaller than stated, but it will still be polynomial.

Theorem 7.8. Let $G$ be a group that is virtually $A \times H_1 \times \cdots \times H_n$, where $A$ is virtually abelian, and $H_1, \ldots, H_n$ are non-elementary hyperbolic. Then

1. Solutions to systems of equations in $G$ are EDT0L in $\text{NSPACE}(n^4 \log n)$, with respect to the normal form $\zeta$ from Remark 7.2.
2. If, in addition, all of the groups $H_i$ are torsion-free, then the solutions are EDT0L in $\text{NSPACE}(n^2 \log n)$.
3. The normal form $\zeta$ can be chosen to be quasigeodesic and regular.

Proof We have from Lemma 7.3 that $G$ has a finite index subgroup isomorphic to $B \times K_1 \times \cdots \times K_n$, where $B$ is a finite index subgroup of $A$, and each $K_i$ is a finite index subgroup of $H_i$, such that every conjugate of each of the subgroups $K_i$ lies in the set $\{K_1, \ldots, K_n\}$. We have that $B$ is virtually abelian and the groups $K_i$ are non-elementary hyperbolic. Thus, all groups in $\text{FIN}(B)$ are virtually abelian, and all groups in $\text{FIN}(K_i)$ are hyperbolic for each $i$. We can equip each of
these with a regular quasigeodesic normal form. Theorem B and Lemma 7.6 imply that solutions to systems of equations are EDT0L in \( \text{NSPACE}(n^4 \log n) \) in all of these groups. The result then follows from Proposition 7.4.

We can reformulate Theorem 7.8 in the following way.

**Corollary 7.9.** Let \( G \) be a group that is virtually a direct product of hyperbolic groups (resp. torsion-free hyperbolic groups). Then the solutions to systems of equations in \( G \) are EDT0L in \( \text{NSPACE}(n^4 \log n) \) (resp. \( \text{NSPACE}(n^2 \log n) \)), with respect to the normal form from Remark 7.2, which can be constructed to be quasigeodesic and regular.

As dihedral Artin groups are virtually a direct product of free groups, we have the following result. Note that the generating set and normal form will not be the standard Artin group ones; they are derived by taking the Schreier generators with respect to some finite index overgroup. As with Theorem 7.8 we can choose the regular geodesic normal forms for the free groups that dihedral Artin groups are virtually a direct product of, to give a regular quasigeodesic normal form for these dihedral Artin groups.

**Corollary 7.10.** The solutions to systems of equations in dihedral Artin groups are EDT0L in \( \text{NSPACE}(n^2 \log n) \), with respect to the normal form from Remark 7.2, which can be constructed to be quasigeodesic and regular.

**Proof** This follows from Corollary 7.9, together with the fact that dihedral Artin groups are virtually direct products of free groups (Lemma 2.25).

**Remark 7.11.** The generating set and normal form from Remark 7.2 will be the Schreier generating set and normal form inherited from some finite index overgroup. This will not (necessarily) be a ‘sensible’ generating set and normal form for groups that are virtually a direct product of hyperbolic groups, or any of the standard normal forms used in dihedral Artin groups.

It is easy to change the generating set whilst preserving the property of EDT0L solutions. To add a (redundant) generator \( a \), one can use the existing normal form, which never uses \( a \), and so the solution language will be unchanged. To remove a redundant generator \( b \), one can apply the free monoid homomorphism that maps \( b \) to some word \( w_b \) over the remaining generators and inverses, that represents \( b \), to the solution language to remove all occurrences of \( b \). Applying the free monoid homomorphism that maps \( b^{-1} \) to \( w_b^{-1} \) after this, will give a new solution language, with \( b \) removed from the generating set. As images of EDT0L languages under free monoid homomorphisms are EDT0L, this new solution language will also be EDT0L.

Changing the normal form is more difficult. Section 5 of [5] contains a successful attempt at this for hyperbolic groups, which uses Ehrenfeucht and Rozenberg’s Copying Lemma [13]; a common tool used to show preimages of EDT0L languages under free monoid homomorphisms are EDT0L in certain cases, along with a result about languages of quasigeodesics in hyperbolic groups. Languages of quasigeodesics in virtually abelian groups are not so well behaved, and any attempt to show that alternative normal forms work in many of the groups considered here will need an alternative approach.

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