A QUILLEN MODEL FOR CLASSICAL MORITA THEORY AND A TENSOR CATEGORIFICATION OF THE BRAUER GROUP

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Abstract. Let $K$ be a commutative ring. In this article we construct a symmetric monoidal Quillen model structure on the category of small $K$-categories which enhances classical Morita theory. We then use it in order to obtain a natural tensor categorification of the Brauer group and of its functoriality.

1. Introduction

Let $K$ be a commutative ring. In this article we address the following questions:

Question A: Can the classical Morita theory of $K$-algebras be viewed as a “homotopy theory”?

Question B: Does the Brauer group of $K$ admit a natural “tensor categorification”?

Morita theory. The classical Morita theory [12] defines an equivalence relation between $K$-algebras. Namely, two $K$-algebras $R$ and $S$ are called Morita equivalent if they have equivalent categories of representations, or equivalently, if there exist bimodules $RM_S$ and $SN_R$ and bimodule isomorphisms $RM_S \cong R$ and $SN_R \cong S$. In order to view this classical notion from a homotopical perspective we will consider $K$-categories, i.e. categories enriched over $K$-modules; see §2.

Brauer group. Based on the foundational work of Brauer and others, Auslander and Goldman [1] introduced the Brauer group $Br(K)$ of a commutative ring $K$ in their study of separable algebras. Concretely, $Br(K)$ consists of the Morita equivalence classes of Azumaya $K$-algebras with addition induced by the tensor product. Any homomorphism $K \to L$ of commutative rings gives naturally rise, by extension of scalars, to a map $r_{L/K} : Br(K) \to Br(L)$. When $L$ is a finite Galois extension of a field $K$, one moreover has a well-defined corestriction (or transfer, or Weil restriction) map $c_{L/K} : Br(L) \to Br(K)$ in the opposite direction; see [6].

Now, let $H$ be an arbitrary abelian group. Recall that a categorification of $H$ consists of an additive\(^1\) category $C_H$ with the property that its Grothendieck group...
$K_0(C_H)$ identifies with $H$. In this vein we define a tensor categorification of $H$ (or \$\otimes$-categorification) to be a symmetric monoidal category $C_H^\otimes$ with the property that its Picard group $\text{Pic}(C_H^\otimes)$ identifies with $H$. Our answer to Question B is then the following:

**Theorem 1.2.** The category $\text{Ho}(\text{Cat}_K)^\otimes$ of $\otimes$-invertible objects in the homotopy category $\text{Ho}(\text{Cat}_K)$ is a $\otimes$-categorification of $\text{Br}(K)$. Given any ring homomorphism $K \to L$, we have a base-change $\otimes$-functor $- \otimes_K L : \text{Ho}(\text{Cat}_K)^\otimes \to \text{Ho}(\text{Cat}_L)^\otimes$ such that $\text{Pic}(- \otimes_K L) \simeq r_{L/K}$. When $L$ is a finite Galois extension of a field $K$, we have moreover a corestriction $\otimes$-functor $C_{L/K} : \text{Ho}(\text{Cat}_L)^\otimes \to \text{Ho}(\text{Cat}_K)^\otimes$ such that $\text{Pic}(C_{L/K}) \simeq e_{L/K}$.

Note that Theorem 1.2 not only conceptually $\otimes$-categorifies the Brauer group but moreover its functorial behavior. Roughly speaking, it formalizes the equality

$$\text{Brauer} = \text{Picard}(\text{Morita invariance}).$$

2. $K$-LINEAR PRELIMINARIES

Let $K$ be a commutative and associative ring with unit, that we fix throughout. We denote by $\text{Mod} K$ the category of $K$-modules, and equip it with the usual closed symmetric monoidal structure $\otimes_K$. A $K$-category $A$ is a category where each Hom set $A(x,y)$ comes equipped with the structure of a $K$-module such that composition is $K$-bilinear. A $K$-linear functor $F : A \to B$, or $K$-functor, is a functor such that the structure maps $F : A(x,y) \to B(Fx,Fy)$ are $K$-linear for all objects $x,y \in \text{ob} A$. For $K$-categories $A$ and $B$, with $A$ small, we denote by $\text{Fun}_K(A,B)$ the $K$-category of $K$-linear functors $A \to B$ and natural transformations between them; $\text{Mod} A := \text{Fun}_K(A^{op}, \text{Mod} K)$ will denote the category of right $A$-modules. The usual Yoneda embedding $h_A : A \to \text{Mod} A$ sending $x \in \text{ob} A$ to $h_A(x) = A(-,x)$ is an example of a $K$-linear functor. An $A$-module is called representable if it is isomorphic to one of the form $h_A(x)$.

**Definition 2.1.** A Morita equivalence is a $K$-linear functor $F : A \to B$ with the property that extension of scalars $\text{Mod} A \to \text{Mod} B$ along $F$ (i.e. the left Kan extension of $h_B \circ F : A \to B \to \text{Mod} B$ along $h_A : A \to \text{Mod} A$, where $h_A,h_B$ are the Yoneda embeddings) is an equivalence of categories.

We write $\text{Cat}_K$ for the category of all small $K$-categories and $K$-linear functors between them. Note that $\text{Cat}_K$ contains the category $\text{Alg}_K$ of all $K$-algebras as a full subcategory: simply consider every $K$-algebra $A$ as a $K$-category with a single object having $A$ as its endomorphism algebra.

**Monoidal structure.** The category $\text{Cat}_K$ inherits from $\text{Mod} K$ a closed symmetric monoidal structure: the tensor product $A \otimes B$ of two $K$-categories has object set $\text{ob}(A \otimes B) = \text{ob} A \times \text{ob} B$, $\text{Hom}$ $K$-modules $(A \otimes B)((x,y),(x',y')) = A(x,x') \otimes_K B(y,y')$, and the evident induced composition; the tensor unit object is $K$, and the internal Hom is provided by $\text{Fun}_K(-,-)$.

**Direct sums.** Let $A \in \text{Cat}_K$. The additive hull $A_\oplus$ of $A$ is the small $K$-category defined as follows: its objects are formal words $x_1 \cdots x_n$ (also written $x_1 \oplus \cdots \oplus x_n$) on the set $\text{ob}(A)$ and the Hom $K$-modules are the spaces of matrices, written:

$$A(x_1 \cdots x_n, y_1 \cdots y_m) := \bigoplus_{j=1}^{n} \bigoplus_{i=1}^{m} A(x_j, y_i) \ni [a_{ij}].$$

Composition is the usual matrix multiplication, $[b_{ij}] \circ [a_{ij}] = \sum_k b_{ik} a_{kj}$. There is a canonical fully faithful functor $\sigma_A : A \to A_\oplus$, $x \mapsto x$. Given a $K$-linear functor
Let $A \in \text{Cat}_K$. We denote by $A^\mathbb{I}$ the idempotent completion (a.k.a. pseudo-abelian or Karoubian envelope) of $A$. Its objects are the pairs $(x, e)$ where $x \in \text{ob}(A)$ and $e = e^2$ is an idempotent endomorphism of $x$. Its Hom $K$-modules are $A^\mathbb{I}(x, e), (y, f)) = fA(x, y)e$, and the composition is induced by that of $A$. This defines a functor $(-)^\mathbb{I}: \text{Cat}_K \to \text{Cat}_K$ which comes equipped with a natural embedding $\tau_A: A \to A^\mathbb{I}$, $x \mapsto (x, 1_x)$.

**Saturation.** Let $A \in \text{Cat}_K$. We say that $A$ is saturated if it admits all finite direct sums and if all idempotents split. We define the saturation of $A$ to be $A^\mathbb{S}_0 := (A^\mathbb{I})^\mathbb{I}$. Note that $A^\mathbb{S}_0$ is indeed always saturated. We thus obtain a saturation functor $(-)^\mathbb{S}_0: \text{Cat}_K \to \text{Cat}_K$ and a natural $K$-linear embedding $\iota_A := \tau_A \circ \sigma_A: A \to A^\mathbb{S}_0$.

Let $X \subseteq A$ be a (full) subcategory of some $A \in \text{Cat}_K$. We say that $X$ additively generates $A$, or that $\text{ob}(X)$ is a set of additive generators for $A$, if the smallest (full) subcategory of $A$ containing $X$ and closed under taking direct sums and retracts is $A$ itself. We leave the easy proof of the next two lemmas as an exercise for the reader.

**Lemma 2.2.** If $F: A \to B$ is any $K$-linear functor, then $F^\mathbb{S}_B$ is an equivalence if and only if $F$ is fully faithful and the image of $\iota_B \circ F$ additively generates $B^\mathbb{S}_B$. \hfill $\square$

**Lemma 2.3.** Let $D$ be a (not necessarily small) saturated $K$-category. If a $K$-functor $F: A \to B$ is fully faithful and its image additively generates $B$, then every $K$-linear functor $G: A \to D$ extends, uniquely up to isomorphism, along $F$ to a $K$-linear functor $B \to D$. \hfill $\square$

### 3. Proof of Theorem 1.1

The construction of the Quillen model structure on $\text{Cat}_K$ is divided into two steps. First we construct a well-behaved “canonical” Quillen model structure on $\text{Cat}_K$; see Theorem 3.4. Then we localize it in order to obtain the desired Morita model structure; see Definition 3.15.

**Canonical model structure.** Note that we have a natural adjunction

\begin{equation}
\begin{array}{ccc}
\text{Cat}_K & \xleftarrow{\text{F}_K} & \text{Cat} \\
\downarrow \text{[-]} & & \\
\downarrow \text{[C]} & & \\
\text{Weq}_{\text{can}} & = & \{\text{(ordinary) equivalences of categories}\}
\end{array}
\end{equation}

where $[-]$ is the underlying category functor (which forgets the $K$-linear structure) and $\text{F}_K$ is the free $K$-category functor, given by the following construction: for a small category $C$, let $\text{F}_K C$ be the $K$-category with the same objects as $C$, with Hom $K$-modules given by $\text{F}_K C(x, y) = \prod_{C(x, y)} K$, and with composition induced by that of $C$. Recall, e.g. from [14], the definition of the well-known canonical (or folk) model structure on the category $\text{Cat}$ of small categories. It consists of the following three classes of functors:

- \text{Weq}_{\text{can}} = \{(\text{ordinary) equivalences of categories}\}
- \text{Cof}_{\text{can}} = \{\text{functors } F: A \to B \text{ such that } \text{ob}(F): \text{ob}(A) \to \text{ob}(B) \text{ is injective}\}
- \text{Fib}_{\text{can}} = \{\text{functors } F \text{ allowing the lift of isomorphisms of the form } Fx \xrightarrow{\sim} y\}.
The canonical model is cofibrantly generated, with the following sets of generating cofibrations and generating trivial cofibrations:
\[ I_{\text{can}} = \{ \emptyset \to \bullet, \bullet \sqcup \bullet \to 1, P \to 1 \} \quad J_{\text{can}} = \{ \bullet \to I \} . \]

Same explanations are in order. Here \( \emptyset \) denotes the initial (empty) category and \( \bullet \) the final category (consisting of precisely one object and its identity arrow). The categories \( 1, P \) and \( I \) all have precisely two objects \( 0 \) and \( 1 \), and, respectively, one non-identity arrow \( 0 \to 1 \), a pair of distinct arrows \( 0 \rightrightarrows 1 \), and one isomorphism \( u: 0 \cong 1 \). The functors are the evident ones; in particular, \( \bullet \sqcup \bullet \to 1 \) is the inclusion of the two endpoints, and \( 0: \bullet \to I \) is the inclusion of \( 0 \).

**Lemma 3.2.** For any \( K \)-functor \( F: A \to B \), consider the following pushout square

\[
\begin{array}{ccc}
A \otimes F_K(\bullet) = A & \xrightarrow{F} & B \\
\downarrow_{A \otimes F_K(0)} & & \downarrow^G \\
A \otimes F_K(I) & \xrightarrow{H} & B
\end{array}
\]

in \( \text{Cat}_K \). Then the \( K \)-functor \( G \) is a \( K \)-linear equivalence and the object-function of \( H \) is injective on the subset \( \{ (x, 1) \mid x \in \text{ob} A \} \subset \text{ob}(A \otimes F_K(I)) \).

**Proof.** To prove the claims it suffices to give an explicit description of \( \tilde{B} \), \( G \) and \( H \) with the required properties. Let \( \tilde{B} \) be the category with object set \( \text{ob}(\tilde{B}) = \text{ob}(B) \sqcup \text{ob}(A) \) and \( \text{Hom} \ K \)-modules given by

\[
\tilde{B}(x, y) := \begin{cases} 
B(Fx, Fy) & \text{if } x, y \in \text{ob}(A) \\
B(Fx, y) & \text{if } x \in \text{ob}(A), y \in \text{ob}(B) \\
B(x, Fy) & \text{if } x \in \text{ob}(B), y \in \text{ob}(A) \\
B(x, y) & \text{if } x, y \in \text{ob}(B).
\end{cases}
\]

The composition in \( \tilde{B} \) is induced by that of \( B \) in the evident way, and there is an obvious fully faithful inclusion \( G: B \to \tilde{B} \) defined by \( x \mapsto x \) \((x \in \text{ob} B)\). Moreover \( G \) is essentially surjective, because for any “new” object \( x \in \text{ob} (A) \subset \text{ob} (\tilde{B}) \) the arrow \( 1_{Fx} \in B(Fx, Fx) = \tilde{B}(x, Fx) = \tilde{B}(x, GFx) \) defines an isomorphism in \( \tilde{B} \) between \( x \) and an object in the image of \( G \); thus \( G \) is a \( K \)-equivalence. There is also a functor \( H: A \otimes F_K(I) \to \tilde{B} \) defined on objects by \( (x, 0) \mapsto Fx \) and \( (x, 1) \mapsto x \) and on arrows by the formula \( H(f \otimes 1_0) = H(f \otimes 1_1) = H(f \otimes u) = F(f) \). Clearly \( H \) satisfies the required injectivity, and the resulting square (3.3) is commutative.

It only remains to verify the pushout property. Consider a diagram of \( K \)-functors

![Diagram](image)

such that \( T_0F = T_1(A \otimes F_K(0)) \). In order to complete this to a commutative diagram, the \( K \)-functor \( T: \tilde{B} \to C \) must be defined on objects by

\[
T_x := \begin{cases} 
T_1(x, 1) & \text{if } x \in \text{ob}(A) \\
T_0x & \text{if } x \in \text{ob}(B)
\end{cases}
\]
and on arrows $f \in B(x,y)$ by

$$T(f) := \begin{cases} 
T_1(1_y \otimes u) \circ T_0(f) \circ T_1(1_x \otimes u^{-1}) & \text{if } x, y \in \text{ob}(A) \\
T_0(f) \circ T_1(1_x \otimes u^{-1}) & \text{if } x \in \text{ob}(A), y \in \text{ob}(B) \\
T_1(1_y \otimes u) \circ T_0(f) & \text{if } x \in \text{ob}(B), y \in \text{ob}(A) \\
T_0(f) & \text{if } x, y \in \text{ob}(B).
\end{cases}$$

It is straightforward (though mildly tedious) to verify that $T$ is well-defined, makes the diagram commute, and is the unique such $K$-functor. This shows that we have indeed constructed the required pushout. \hfill \Box

**Theorem 3.4.** The category $\text{Cat}_K$ carries a Quillen model structure where the weak equivalences are the $K$-equivalences, the fibrations are the $K$-linear functors $F$ such that $[F]$ is a fibration in $\text{Cat}$, and the cofibrations are the $K$-linear functors that are injective on objects. In particular, every object is fibrant and cofibrant and the adjunction (3.1) becomes a Quillen pair. Moreover, this Quillen model structure satisfies the following properties:

(i) It is cofibrantly generated and we may take the sets $F_K(I_{\text{can}})$ and $F_K(J_{\text{can}})$ as the generating cofibrations and trivial cofibrations.

(ii) It is symmetric monoidal in the sense of Mark Hovey (see [8, ch. 4]);

(iii) It is combinatorial in the sense of Jeff Smith (see [3]);

(iv) Every map $F: A \to B$ admits a natural “mapping cylinder” factorization (as a cofibration $J$ followed by a trivial fibration $Q$)

$$F : A \xrightarrow{J} \tilde{B} \xrightarrow{Q} B,$$

where $\tilde{B}$ is the pushout $(A \otimes F_K(I)) \cup_A B$ of $A \otimes F_K(0): A = A \otimes K \to A \otimes F_K(I)$ along $F$ and $J$ is the composite $A = A \otimes K \xrightarrow{A \otimes F_K(1)} A \otimes F_K(I) \to \tilde{B}$. The $K$-linear functor $Q$ is induced by the pushout property of $\tilde{B}$ by the two $K$-linear functors $\text{id}_B$ and $F \otimes F' : A \otimes F_K(I) \to B \otimes K = B$, where $F' := F_K(u \mapsto 1_K)$. (See (3.8) for a pictorial description.)

**Definition 3.5.** We will call the model structure of Theorem 3.4 the canonical model structure on $\text{Cat}_K$ and we will denote it by $M_{\text{can}}$.

**Remark 3.6.** Note that $M_{\text{can}}$ is not cellular in the sense of Hirschhorn [7, Definition 11.1.1] since not all cofibrations are monomorphism. For instance from the adjunction (3.1) one observes that the generating cofibration $F_K(P \to 1)$ is not a monomorphism since it admits two evident (fully faithful) distinct sections $S_1, S_2 : F_K(1) \to F_K(P)$.

**Remark 3.7.** The analog of Theorem 3.4 holds (with the same proof) for $\mathcal{V}$-$\text{Cat}$, the category of small categories enriched over a bicomplete closed symmetric monoidal category $\mathcal{V}$, at least if one assumes that the tensor unit $1 \in \mathcal{V}$ is a finite object; see [8, 2.1.1]. Also, in order for the weak equivalences to coincide with the $\mathcal{V}$-equivalences one should assume that $[-] = \text{Hom}_\mathcal{V}(1, -)$ detects $\mathcal{V}$-equivalences, and in order for the model to be combinatorial one should assume $\mathcal{V}$ locally presentable.

**Proof of Theorem 3.4.** In order to establish the model structure it suffices to check conditions (1) and (2) of Kan’s lifting theorem [7, Theorem 11.3.2]. Condition (1) follows from the fact that the domains of the maps in $F_K(I_{\text{can}})$ and $F_K(J_{\text{can}})$ are small objects; note that $K$ is small (even finite) in $\text{Mod} K$. The functor $[-]$ preserves sequential colimits and a $K$-linear functor $F$ is a $K$-equivalence in $\text{Cat}_K$ if and only if $[F]$ is an equivalence in $\text{Cat}$. Hence, condition (2) follows from Lemma 3.2. This establishes the model structure with the described weak equivalences and fibrations, and moreover proves (i).
Since $\text{Mod}\,\mathbb{K}$, being a Grothendieck category, is locally presentable then so is $\text{Cat}\,\mathbb{K}$ by the main result of [9]. Hence the model structure on $\text{Cat}\,\mathbb{K}$ established above is not only cofibrantly generated but also combinatorial, as claimed in (iii). Denote by $\mathcal{C}$ the class of $\mathbb{K}$-functors which are injective on objects and by $\text{Cof}$ the class of cofibrations of the model structure. Then, the same argument as the one in the proof of [4, Lemma 4.10(ii)] shows us that $\mathcal{C} \subseteq \text{Cof}$.

Before proving the converse inclusion, let us establish (iv). For any $F : A \to B$, perform the construction described in (iv) (we identify $A$ with $A \otimes \mathbb{K}$ and $B$ with $B \otimes \mathbb{K}$):

\[
\begin{array}{ccc}
A & \xrightarrow{F} & B \\
\downarrow & & \downarrow 1_B \\
A \otimes F(I) & \xrightarrow{J} & B \\
\end{array}
\]

By Lemma 3.2, $H$ is injective on the objects of the form $(x, 1)$, hence $J$ is injective on objects and thus (by the inclusion we have already proved) is a cofibration. Also by Lemma 3.2 $G$ is a $\mathbb{K}$-linear equivalence, implying that $Q$ is. Since $Q$ is obviously surjective on objects, it is actually a trivial fibration. This proves (iv).

Now let $F : A \to B$ be any cofibration and factor it as $A \to \overline{B} \xrightarrow{\sim} B$, according to (iv). In particular $A \to \overline{B}$ is injective on objects. Since $F$ is a cofibration and $\overline{B} \xrightarrow{\sim} B$ is a trivial fibration, there exists a lifting in the following square

\[
\begin{array}{ccc}
A & \xrightarrow{\overline{B}} & B \\
\downarrow F & & \downarrow \sim \\
B & = & B \\
\end{array}
\]

which implies that $F$ must also be injective on objects. This shows $\text{Cof} \subseteq \mathcal{C}$ and thus $\text{Cof} = \mathcal{C}$, as claimed. Finally, the easy verification of (ii) proceeds exactly as in the proof of [4, Proposition 4.16], using the characterization of cofibrations we have just proved.

**Remark 3.9.** Applying (iv) of Theorem 3.4 we obtain the following cylinder

\[
\begin{array}{ccc}
\mathbb{K} \sqcup \mathbb{K} & \xrightarrow{(1_{\mathbb{K}}, 1_{\mathbb{K}})} & \mathbb{K} \\
\downarrow (J_1, J_2) & & \downarrow \sim \\
F\mathbb{K}I & = & Q \\
\end{array}
\]

on $\mathbb{K} \in \text{Cat}_{\mathbb{K}}$ (factor $F = (1_{\mathbb{K}}, 1_{\mathbb{K}})$). Here, $J_i$ is the unique $\mathbb{K}$-linear functor sending the unique object of $\mathbb{K}$ to $i \in \text{ob}(F\mathbb{K}I)$, and $Q$ the unique $\mathbb{K}$-linear functor sending the isomorphism $u$ to the identity. The corresponding canonical cylinder for any object $A \in \text{Cat}_{\mathbb{K}}$ can then be obtained by tensoring (3.10) with $A$.

**Corollary 3.11.** The category $\text{Ho}(\mathcal{M}_{\text{can}})$ is obtained from $\text{Cat}_{\mathbb{K}}$ simply by taking as morphisms the isomorphism classes of $\mathbb{K}$-functors.

**Proof.** Since every object is fibrant and cofibrant in the canonical model, we have a natural identification $\text{Hom}_{\text{Ho}(\mathcal{M}_{\text{can}})}(A, B) = \text{Hom}_{\text{Cat}_{\mathbb{K}}}(A, B)/\sim$, where the equivalence relation $\sim$ is the homotopy relation defined by the canonical cylinder objects of Remark 3.9. Now it suffices to notice that, for any pair of parallel $\mathbb{K}$-functors $F_0, F_1 : A \xrightarrow{\sim} B$, the homotopies $H : A \otimes F\mathbb{K}I \to B$ from $F_0$ to $F_1$ are in bijection with the isomorphisms $F_0 \simeq F_1$ of $\mathbb{K}$-functors. □
Morita model structure. Given a \( \mathbb{K} \)-category \( A \), let \( P(A) \) be the full subcategory of \( \text{Mod} A \) consisting of those \( A \)-modules \( M \) such that the represented functor \( \text{Hom}_A(M, -) : \text{Mod} A \to \mathbb{K} \) commutes with arbitrary colimits. Consider also the full subcategory \( \text{proj} A \subseteq \text{Mod} A \) of all finitely generated projective \( A \)-modules. As usual, an object \( M \in \text{Mod} A \) is projective if \( \text{Hom}_A(M, -) \) is exact, and it is finitely generated if there exists an epimorphism \( F \to M \) with \( F \) a finite coproduct of representable right \( A \)-modules.

**Lemma 3.12.** We have an equality \( P(A) = \text{proj} A \) and a natural \( \mathbb{K} \)-linear equivalence \( A^\oplus \xrightarrow{\sim} P(A) \).

**Proof.** This is well-known, but we reprove it for convenience. Recall (e.g. from [13, §4.11 Lemma 1]) that, in the Grothendieck category \( \text{Mod} A \), an object \( M \) is finitely generated projective if and only if it is projective and the functor \( \text{Hom}_A(M, -) \) preserves arbitrary coproducts. Thus \( M \in \text{proj} A \) if and only if \( \text{Hom}_A(M, -) \) preserves coproducts and cokernels. But since every colimit can be written as a cokernel of a map between two coproducts, this is the same as preserving arbitrary colimits. Thus \( P(A) = \text{proj} A \), as claimed. By the Yoneda lemma, the representable modules generate the abelian category \( \text{Mod} A \) and are projective. Thus, on the one hand, since \( \text{proj} A \) is saturated the Yoneda embedding \( h_A : A \to \text{Mod} A \) induces a \( \mathbb{K} \)-linear functor \( A^\oplus \to \text{proj} A \), which is unique up to isomorphism. On the other hand, if \( P \) is projective then every epimorphism from a coproduct of representables onto \( P \) must split; if \( P \) is moreover finitely generated, then the splitting factors through a finite summand. Therefore \( \text{proj} A \) consists precisely of the retracts of finite sums of representables. Hence the functor \( A^\oplus \to \text{proj} A \) is an equivalence. By uniqueness, it is also natural up to isomorphism. (Such naturality will suffice for all our purposes; for a stronger statement, one would have to choose canonical cokernels in \( \text{Mod} A \).) \( \square \)

**Proposition 3.13.** A \( \mathbb{K} \)-functor \( F : A \to B \) between small \( \mathbb{K} \)-categories is a Morita equivalence (Definition 2.1) if and only if \( F^\oplus : A^\oplus \to B^\oplus \) is a \( \mathbb{K} \)-equivalence. In other words, \( F \) is a Morita equivalence if and only if \( F \) is fully faithful and the image of \( \text{tr}_B \circ F \) additively generates \( B^\oplus \).

**Proof.** The equivalence between the two descriptions of \( \mathbb{K} \)-equivalence follows from Lemma 2.2. To prove the first description, consider the following diagram of \( \mathbb{K} \)-linear functors:

\[
\begin{array}{ccc}
A & \xrightarrow{F} & B \\
\downarrow & & \downarrow \\
\text{Mod} A & \xrightarrow{P} & P(B) \\
\downarrow & & \downarrow \\
P(A) & \xrightarrow{F'} & P(B) \\
\end{array}
\]

The vertical arrows are the inclusions; \( F' \) is the unique extension of \( F \) commuting with direct sums and retracts, i.e., the functor identifying with \( F^\oplus \) under the natural equivalence of the previous lemma; and \( F'' \) is the left Kan extension of \( A \to \text{Mod} B \) along \( A \to \text{Mod} A \), and also of \( P(A) \to \text{Mod} B \) along \( P(A) \to \text{Mod} A \) (see [11, §X.3]). Since the functors \( A \to P(A) \) and \( P(A) \to \text{Mod} A \) are fully faithful, the diagram commutes up to isomorphism. We claim that \( F'' \) is an equivalence if and only if \( F' \) is; then the proposition will follow by Lemma 3.12. In one direction, if \( F'' \) is an equivalence then it must restrict to an equivalence \( P(A) \xrightarrow{\sim} P(B) \), because the properties of being finitely generated and projective are categorical; hence \( F' \) is an equivalence. In the other direction, if \( F' \) is an equivalence then it has a quasi-inverse \( G' \), which will induce its own left Kan extension \( G'' : \text{Mod} B \to \text{Mod} A \). The
Corollary 3.14. If \( D \) is a (not necessarily small) saturated \( \mathbb{K} \)-category, then the functor \( \text{Fun}_\mathbb{K}(\cdot, D) \) sends Morita equivalences to \( \mathbb{K} \)-equivalences.

Proof. Let \( F: A \to B \) be a Morita equivalence. By Proposition 3.13, \( F \) is fully faithful and the image of \( 1_B \circ F \) additively generates \( B^\mathbb{K}_2 \); a fortiori, the image of \( F \) additively generates \( B \). Since \( F \) is fully faithful, one checks easily that the \( \mathbb{K} \)-linear functor \( \text{Fun}_\mathbb{K}(F,D): \text{Fun}_\mathbb{K}(B,D) \to \text{Fun}_\mathbb{K}(A,D) \) is fully faithful. Since \( D \) is saturated and the image of \( F \) additively generates \( B \), it follows from Lemma 2.3 that \( \text{Fun}_\mathbb{K}(F,D) \) is essentially surjective. Therefore it is a \( \mathbb{K} \)-linear equivalence. \( \square \)

Definition 3.15. Define the Morita model structure on \( \text{Cat}_\mathbb{K} \) to be \( \mathcal{M}_{\text{Mor}} := L S \mathcal{M}_{\text{can}} \), the left Bousfield localization of the canonical model structure of Theorem 3.4 with respect to the set \( S := \{ R_0, R_1, S_2 \} \) consisting of the following three \( \mathbb{K} \)-linear functors:

(i) \( R_0: \emptyset \to 0 \) is the unique functor from the initial to the final object, i.e., from the empty to the zero \( \mathbb{K} \)-category.

(ii) \( R_1: E(1) \to R(1) \) is the universal addition of a retract. More precisely, let \( E(1) \) be the \( \mathbb{K} \)-category generated by one object \( o \) equipped with an idempotent endomorphism \( e = e^2: o \to o \), and let \( R(1) \) be the \( \mathbb{K} \)-category generated by two objects \( o \) and \( r \), two arrows \( p: o \to r \) and \( i: r \to o \), and the relation \( pi = 1_r \). Then \( R_1 \) is the unique (fully faithful) \( \mathbb{K} \)-linear functor sending \( e \) to the idempotent \( ip \).

(iii) \( S_2: \mathbb{K} \sqcup \mathbb{K} \to S(2) \) is the universal addition of a direct sum. More precisely, \( S(2) \) is generated by three objects \( o_1, o_2 \) and \( s \), arrows \( i_k: o_k \to s \) and \( p_k: s \to o_k \) \( (k = 1, 2) \), and relations \( p_ki_k = 1_{o_k} \) \( (k = 1, 2) \) and \( i_1p_1 + i_2p_2 = 1_s \). Then \( S_2 \) is the unique (fully faithful) \( \mathbb{K} \)-linear functor \( \mathbb{K} \sqcup \mathbb{K} \to S(2) \) sending the first copy of \( \mathbb{K} \) to \( o_1 \) and the second copy to \( o_2 \).

The left Bousfield localization, and therefore the Morita model structure, is well-defined because the canonical model \( \mathcal{M}_{\text{can}} \) is left proper (since all objects are cofibrant) and because by item (iii) of Theorem 3.4 it is combinatorial; see [2, 3]. Moreover, \( \mathcal{M}_{\text{Mor}} \) inherits the property of being combinatorial. Since \( \mathcal{M}_{\text{can}} \) is symmetric monoidal we can use here the \( \mathcal{M}_{\text{can}} \)-enriched version of Bousfield localization (see [2]) rather than the more common simplicial version (the result is the same).

Let us recall what this all means, in the situation at hand. An object \( D \in \text{Cat}_\mathbb{K} \) is \( S \)-local if for every \( (F: A \to B) \in S \), the induced map

\[
F^* = \text{Hom}_{\text{Ho}(\mathcal{M}_{\text{can}})}(F,D): \text{Hom}_{\text{Ho}(\mathcal{M}_{\text{can}})}(B,D) \to \text{Hom}_{\text{Ho}(\mathcal{M}_{\text{can}})}(A,D)
\]

is a bijection. A \( \mathbb{K} \)-linear functor \( F: A \to B \) is an \( S \)-local equivalence if, conversely, (3.16) is a bijection for every \( S \)-local object \( D \). By definition, the weak equivalences of \( \mathcal{M}_{\text{Mor}} \) are precisely the \( S \)-local equivalences, the cofibrations are the same as those of \( \mathcal{M}_{\text{can}} \), and the fibrations are determined as usual by the right lifting property with respect to the trivial cofibrations. By the theory, the Morita fibrant objects (i.e., those objects that are fibrant for the Morita model) are precisely the \( S \)-local ones.

Notation 3.17. By default, \( \text{Ho}(\text{Cat}_\mathbb{K}) \) always refers to \( \text{Ho}(\mathcal{M}_{\text{Mor}}) \).

Lemma 3.18. A \( \mathbb{K} \)-category is Morita fibrant if and only if it is saturated (see §2).
Proof. Consider the following three lifting problems for an object $D$ in $\text{Cat}_K$:

\[
\begin{array}{ccc}
\emptyset & \xrightarrow{G} & D \\
R_0 & \downarrow & \downarrow \text{R}_1 \\
0 & \xrightarrow{G} & \text{R}(1)
\end{array}
\quad
\begin{array}{ccc}
E(1) & \xrightarrow{G} & D \\
\downarrow & \downarrow & \downarrow s_2 \\
\text{K} \uplus \text{K} & \xrightarrow{H} & D
\end{array}
\]

Exactly as in the proof of [5, Proposition 4.24], it is easy to verify from the definitions that the unique functor $\emptyset \rightarrow D$ lifts along $R_0$ if and only if $D$ has a zero object; that every $G$ as above lifts along $R_1$ if and only if every idempotent of $D$ splits; and that every $H$ as above lifts along $S_2$ if and only if any two objects of $D$ have a direct sum. Hence, $D$ has the right lifting property with respect to the set $S$ precisely when it is saturated. Now we must show that $D$ has the right lifting property with respect to $S$ if and only if it is $S$-local. Since the three maps in $S$ are trivial fibration, of course $S$-locality (i.e. Morita fibrancy) implies the lifting property. So it only remains to prove the converse.

Let $D$ have the right lifting property with respect to each $F \in S$. Note that, by the uniqueness of zero objects, retracts, and direct sums, the resulting liftings in (3.19) are unique up to a canonical isomorphism of $K$-functors. This implies that for any $(F: A \rightarrow B) \in S$ the induced $K$-functor

\[ F^* = \text{Fun}_K(F, D): \text{Fun}_K(B, D) \rightarrow \text{Fun}_K(A, D) \]

is essentially surjective. Similarly, $F^*$ is easily seen to be fully faithful so it is an equivalence. By considering isomorphism classes of objects, we deduce with Corollary 3.11 that (3.16) is a bijection. Thus $D$ is $S$-local, as claimed.

Lemma 3.20. The weak equivalences in the Morita model structure are precisely the Morita equivalences in the sense of Definition 2.1.

Proof. Consider a $\mathbb{K}$-linear functor $F: A \rightarrow B$. It is a weak equivalence of the Morita model structure if and only if it is an $S$-local equivalence, i.e. by Corollary 3.11 and Lemma 3.18, if and only if the induced functor $F^*: \text{Fun}_K(B, D) \rightarrow \text{Fun}_K(A, D)$ is an equivalence of categories for all saturated $D$. Consider the following commutative diagram:

\[
\begin{array}{ccc}
\text{Fun}_K(B, D) & \xrightarrow{F^*} & \text{Fun}_K(A, D) \\
(\iota_B)^* & \simeq & (\iota_A)^* \\
\text{Fun}_K(B^\triangledown_B, D) & \xrightarrow{(F^\triangledown_B)^*} & \text{Fun}_K(A^\triangledown_B, D)
\end{array}
\]

By Corollary 3.14 and since $\iota_A$ and $\iota_B$ are Morita equivalences, $(\iota_A)^*$ and $(\iota_B)^*$ are equivalences. If $F$ is a Morita equivalence then $F^\triangledown_B$ is an equivalence (by Proposition 3.13 again), hence so is $(F^\triangledown_B)^*$, and it follows from the commutative diagram that $F^*$ is an equivalence. This shows that Morita equivalences are $S$-local equivalences. Together with Lemma 3.18, this also proves Corollary 3.21 below.

Conversely, assume that $F$ is an $S$-local equivalence; thus in the diagram $F^*$ is an equivalence for all saturated $D$. Hence $(F^\triangledown_B)^*$ is an equivalence too. By Corollary 3.21, and since every object is Morita cofibrant, this is the same as saying that $\text{Hom}_{\text{Ho}(\text{Cat}_K)}(F^\triangledown_B, C)$ is bijective for all $C \in \text{Cat}_K$. By Yoneda, $F^\triangledown_B$ is invertible in $\text{Ho}(\text{Cat}_K)$, i.e. it is an $S$-local equivalence. But its domain and codomain are Morita fibrant, hence $F^\triangledown_B$ is already a canonical weak equivalence, i.e. an equivalence of $K$-categories. In other words, $F$ is a Morita equivalence.

Corollary 3.21. The natural embedding $\iota_A: A \rightarrow A^\triangledown_B$ provides a functorial fibrant replacement for the Morita model structure.
We now obtain an explicit description of the Morita homotopy category.

**Proposition 3.22.** For any two \( A, B \in \text{Cat}_K \), there is a canonical bijection between maps \( \varphi: A \to B \) in \( \text{Ho}(\text{Cat}_K) \) and isomorphism classes of \( K \)-linear functors \( F: A \to B_{\mathbb{R}}, \) obtained by sending the isomorphism class of \( F \) to the equivalence class of the left fraction \( \iota_B^{-1} \circ F \). The map \( \varphi \) is invertible if and only if \( F \) is a Morita equivalence. If \( \varphi: A \to B \) and \( \psi: B \to C \) are represented by \( F \) and \( G \), respectively, then their composite \( \psi \circ \varphi \) is represented by \( G \circ F \), where \( G: B_{\mathbb{R}} \to C_{\mathbb{R}} \) is the (up to isomorphism, unique) \( K \)-linear extension of \( G \) along \( \iota_B \).

**Proof.** By Corollary 3.21, and since every object is cofibrant, every map \( \varphi: A \to B \) in the homotopy category is the equivalence class of some fraction \( A \to B_{\mathbb{R}} \hookrightarrow B \). Two \( K \)-linear functors \( F, F': A \to B_{\mathbb{R}} \) represent the same map precisely when they are isomorphic, as claimed, by Corollary 3.11. The other claims now follow immediately from the naturality of \( \iota \) and the fact that saturation is idempotent up to equivalence (cf. [5, Proposition 4.27]).

**Proposition 3.23.** The Morita model category, endowed with its closed symmetric monoidal structure (see \( \S 2 \)), is a symmetric monoidal model category. It follows in particular that \( \text{Ho}(\text{Cat}_K) \) is a closed symmetric monoidal category.

**Remark 3.24.** Note that, since every object is cofibrant, we do not need to derive the tensor product \( - \otimes - \). However, we need to derive the internal Homs \( \text{Fun}_K(-,-) \), and we can do this simply by saturating the target category.

**Proof.** Let \( F: A \to B \) be a \( K \)-linear functor. We show first that \( C \otimes F \) is a Morita equivalence whenever \( F \) is a Morita equivalence and \( C \in \text{Cat}_K \). Indeed, consider the commutative square in \( \text{Cat}_K \)

\[
\begin{array}{ccc}
\text{Fun}_K(C \otimes A, D) & \xrightarrow{F^*} & \text{Fun}_K(A, \text{Fun}_K(C, D)) \\
\downarrow & & \downarrow \\
\text{Fun}_K(C \otimes B, D) & \xrightarrow{F^*} & \text{Fun}_K(B, \text{Fun}_K(C, D))
\end{array}
\]

where \( D \) is any saturated \( K \)-category. Since \( D \) is saturated so is \( \text{Fun}_K(C, D) \); since \( F \) is a Morita equivalence, the rightmost \( F^* \) is then an equivalence. It follows that the leftmost \( F^* \) is also one. Since \( D \) is an arbitrary saturated \( K \)-category, this proves that \( C \otimes F \) is a Morita equivalence, as claimed (of course, for this argument we make implicit use of the characterizations of Morita fibrant objects and Morita local equivalences of Lemmas 3.18 and 3.20). Now we can directly verify the definition of a symmetric monoidal model, precisely as in the proof of [5, Proposition 6.3].

**Homotopy of \( K \)-algebras.** Let \( R, S \) and \( T \) be \( K \)-algebras, considered as objects in \( \text{Cat}_K \). A bimodule \( _RM_S \) which is finitely generated projective as a right \( S \)-module is the same data as a \( K \)-linear functor \( R \to \text{proj} S \). Two such bimodules \( _RM_S \) and \( _RM'_S \) are isomorphic as bimodules if and only if the corresponding \( K \)-linear functors are isomorphic. The \( K \)-linear functor corresponding to the tensor product \( _RM \otimes_S N_R \) identifies with the composition of \( R \to \text{proj} S \) with the canonical extension of \( S \to \text{proj} T \) along the embedding \( S \to \text{proj} S \). These facts combined with Proposition 3.22 furnish us with the following explicit description of the full subcategory of \( \text{Ho}(\text{Cat}_K) \) of \( K \)-algebras, which (with a slight abuse of notation) we will denote by \( \text{Ho}(\text{Alg}_K) \). Namely, its objects are the \( K \)-algebras. The Hom sets \( \text{Hom}_{\text{Ho}(\text{Alg}_K)}(R, S) \) are given by the isomorphism classes of the category \( \text{rep}(R, S) \) of those \( R-S \)-bimodules which are finitely generated projective as \( S \)-modules. Composition is given by

\[
\text{Iso rep}(S, T) \times \text{Iso rep}(R, S) \to \text{Iso rep}(R, T) \quad ([SN_T], [RM_S]) \mapsto [RM \otimes_S NT]
\]

and the tensor structure is induced by the tensor product of \( K \)-algebras.
4. Proof of Theorem 1.2

The proof of Theorem 1.2 is divided into three steps. First, we $\otimes$-categorify the Brauer group; see Corollary 4.2. Then, we establish the base-change and corestriction functoriality of this $\otimes$-categorification; see Corollary 4.5 and Proposition 4.24.

$\otimes$-categorification. Let $K$ be a commutative ring. Recall that Azumaya $K$-algebras can be defined as those $K$-algebras $A$ for which there exists a $K$-algebra $B$, a faithful finitely generated projective $K$-module $P$, and an isomorphism $A \otimes B \simeq \text{End}_K(P)$ of $K$-algebras (see [10, Theorem 5.1]). Recall also (e.g. from [16, 18, 11]) that a finitely generated projective $K$-module $P \in \text{proj} K$ is faithful precisely when it is a generator of $\text{Mod} K$, or equivalently, when it additively generates $\text{proj} K$.

**Proposition 4.1.** The $\otimes$-invertible objects in $\text{Ho}(\text{Cat}_K)$ are precisely the $K$-categories which are Morita equivalent to Azumaya $K$-algebras.

*Proof.* If $P \in \text{proj} K \simeq K^*_2$, is an additive generator then $\text{Mod} \text{End}_K(P) \simeq \text{Mod} K$ by Corollary 3.14, and it follows immediately from the characterization recalled above that Azumaya $K$-algebras are $\otimes$-invertible objects in $\text{Ho}(\text{Cat}_K)$.

Now consider two $K$-categories $A$ and $B$ such that $A \otimes B \simeq K$ in $\text{Ho}(\text{Cat}_K)$. We may assume, without loss of generality, that $A$ and $B$ are saturated. By Proposition 3.22 this isomorphism can be realized by a Morita equivalence $F: A \otimes B \to K^*_2$. In particular there exist finitely many $(x_i, y_i) \in \text{ob } A \times \text{ob } B$ such that $\bigoplus_i F(x_i, y_i)$ is an additive generator for $K^*_2 \simeq \text{proj } K$. By setting $x := \bigoplus_i x_i$ and $y := \bigoplus_i y_i$, the object $P := F(x, y)$ is then also an additive generator of $K^*_2$, because it displays $\bigoplus_i F(x_i, y_i)$ as a retract; indeed we have $F(x, y) = \bigoplus_{i,j} F(x_i, y_j)$. It follows that the $K$-algebra $\text{End}_K(P)$ is Morita equivalent to $K$ and since $F$ is fully-faithful we obtain an isomorphism $A(x, x) \otimes B(y, y) \sim \text{End}_K(P)$. Therefore $A(x, x)$ is an Azumaya $K$-algebra. Let us now show that $A(x, x)$ and $A$ are isomorphic in $\text{Ho}(\text{Cat}_K)$, i.e. that they are Morita equivalent. Consider the composite functor

$$A(x, x) \otimes B \xrightarrow{J \otimes B} A \otimes B \xrightarrow{F} K^*_2$$

where $I: A(x, x) \to A$ denotes the inclusion. By construction, the composite is fully faithful and its image contains the additive generator $P$ of $K^*_2$. As a consequence, it is a Morita equivalence. By the 2-out-of-3 property, one concludes that $I \otimes B$ is also a Morita equivalence. Since $- \otimes B$ is an endo-equivalence of $\text{Ho}(\text{Cat}_K)$, the inclusion $I: A(x, x) \to A$ must also be a Morita equivalence, thus proving that $A(x, x)$ and $A$ are isomorphic in $\text{Ho}(\text{Cat}_K)$. This concludes the proof. \qed

**Corollary 4.2.** We obtain a natural isomorphism $\text{Pic}(\text{Ho}(\text{Cat}_K)^\otimes) \simeq \text{Br}(K)$. \qed

**Base-change.** Let $K \to L$ be a homomorphism of commutative rings. By base-change one obtains a $\otimes$-functor

$$(4.3) \quad - \otimes_K L: \text{Cat}_K \to \text{Cat}_L, \quad A \mapsto A \otimes_K L.$$

**Lemma 4.4.** The functor (4.3) preserves Morita equivalences.

*Proof.* Note first that, up to equivalence, the functor $- \otimes_K L$ commutes with the additive hull $(-)_\oplus$ but not with the idempotent completion $(-)^\oplus$, as in general we only get a fully faithful inclusion $A^\oplus \otimes_K L \subset (A \otimes_K L)^\oplus$. In order to prove this lemma we make use of the characterization of Proposition 3.13. Let $F: A \to B$ be a Morita equivalence in $\text{Cat}_K$. Thus $F$ is fully faithful, and the full image of $F$ additively generates $B^\oplus$. It follows that $F \otimes_K L$ is fully faithful, and that we have a fully faithful embedding

$$(B \otimes_K L)_\oplus = B_\oplus \otimes_K L \subset (\text{Im}(F)^\oplus_\oplus) \otimes_K L \subset (\text{Im}(F \otimes_K L)^\oplus_\oplus).$$
Showing that the full image of $F \otimes_{\mathbb{K}} \mathbb{L}$ additively generates $(B \otimes_{\mathbb{K}} \mathbb{L})_\mathbb{E}$. This allows us to conclude that the functor $F \otimes_{\mathbb{K}} \mathbb{L} : A \otimes_{\mathbb{K}} \mathbb{L} \to B \otimes_{\mathbb{K}} \mathbb{L}$ is a Morita equivalence. □

**Corollary 4.5.** The base-change functor (4.3) induces a well-defined $\otimes$-functor

$$- \otimes_{\mathbb{K}} \mathbb{L} : \text{Ho}(\text{Cat}_\mathbb{K})^\otimes \to \text{Ho}(\text{Cat}_\mathbb{L})^\otimes$$

such that $\text{Pic}(- \otimes_{\mathbb{K}} \mathbb{L}) \simeq \mathfrak{r}_{/\mathbb{K}}$ under the identification of Corollary 4.2. □

Thus Pic recovers the covariant functoriality of the Brauer group.

**Corestriction.** In this subsection we assume that $\mathbb{K}$ is a field and that $\mathbb{L}$ is a finite Galois extension of $\mathbb{K}$ with Galois group $G := \text{Gal}(\mathbb{L}/\mathbb{K})$ and degree $n := [\mathbb{L} : \mathbb{K}]$. We start by recalling some definitions and results from Galois descent theory; consult [6, §6] or [10, II.§5] for further details.

**Definition 4.6.** An $\mathbb{L}/\mathbb{K}$-Galois module is an $\mathbb{L}$-vector space $W$ endowed with a left $G$-action which is skew-linear, in the sense that $\sigma(x)\sigma(w) = \sigma(xw)$ for every $x \in \mathbb{L}$, $w \in W$ and $\sigma \in G$. Denote by $\text{GalMod}_{\mathbb{L}/\mathbb{K}}$ the category of $\mathbb{L}/\mathbb{K}$-Galois modules and $G$-equivariant $\mathbb{K}$-linear maps.

**Proposition 4.7** ([6, I.§6 Theorem 1]). Under the above hypotheses and notations, we have a natural $G$-equivariant $\mathbb{L}$-linear isomorphism

$$\mathbb{L} \otimes_{\mathbb{K}} W^G \sim \to W \quad \ell \otimes w \mapsto \ell w$$

for every $W \in \text{GalMod}_{\mathbb{L}/\mathbb{K}}$. Moreover, these isomorphisms form the counit of the following equivalence of categories:

$$\text{GalMod}_{\mathbb{L}/\mathbb{K}} \xrightarrow{\sim} \mathbb{L} \otimes_{\mathbb{K}} \mathbb{K} \text{Mod}_{\mathbb{K}} \left( \begin{array}{c} \mathbb{L} \otimes_{\mathbb{K}} \mathbb{K} \\ \text{Mod}_{\mathbb{K}} \end{array} \right) \right)^G.$$

**Definition 4.9.** If $V$ is an $\mathbb{L}$-vector space, denote by $^\sigma V$ the $\mathbb{L}$-vector space that coincides with $V$ as a group and whose $\mathbb{L}$-action is given by $x \cdot v := \sigma^{-1}(x)v$ for $x \in \mathbb{L}$ and $v \in V$. Then $\bigotimes_{\sigma \in G} {^\sigma V}$, where the tensor product is taken over $\mathbb{L}$, can be endowed with the skew-linear $G$-action $\tau(\bigotimes_{\sigma \in G} {^\sigma V}) := \bigotimes_{\sigma \in G} {^\tau \sigma V}$, for $\tau \in G$. By taking $G$-invariants, we obtain in this way a well-defined functor

$$\text{Cor}_{\mathbb{L}/\mathbb{K}} : \text{Mod}_{\mathbb{K}} \to \text{Mod}_{\mathbb{K}} \quad V \mapsto \left( \bigotimes_{\sigma \in G} {^\sigma V} \right)^G.$$

Let us now describe some properties of this functor.

**Proposition 4.11.** The functor (4.10) is symmetric monoidal.

**Proof.** The canonical $\mathbb{K}$-linear embedding

$$\mathbb{K} \to \left( \bigotimes_{\sigma \in G} {^\sigma \mathbb{L}} \right)^G \quad 1 \mapsto 1 \otimes \cdots \otimes 1$$

and the natural $\mathbb{K}$-linear homomorphisms $(V, W \in \text{Mod}_{\mathbb{L}})$

$$(\bigotimes_{\sigma \in G} {^\sigma V})^G \otimes_{\mathbb{K}} (\bigotimes_{\sigma \in G} {^\sigma W})^G \to (\bigotimes_{\sigma \in G} {^\sigma (V \otimes_{\mathbb{L}} W)})^G,$$

sending $(\bigotimes_{\sigma \in G} {^\sigma V}) \otimes (\bigotimes_{\sigma \in G} {^\sigma W})$ to $\bigotimes_{\sigma \in G} (v_{\sigma} \otimes w_{\sigma})$, are easily seen to equip (4.10) with a lax symmetric monoidal structure. To see that they are invertible, by Proposition 4.7 it suffices to show that their images under $\mathbb{L} \otimes_{\mathbb{K}} -$ : $\text{Mod}_{\mathbb{K}} \to \text{GalMod}_{\mathbb{L}/\mathbb{K}}$ are invertible. Under the identification (4.8) these images correspond to the evident isomorphisms $\mathbb{L} \simeq \mathbb{L} \otimes_{\mathbb{K}} \mathbb{K} \simeq \bigotimes_{\sigma \in G} {^\sigma \mathbb{L}} \simeq \mathbb{L}$ and

$$(\bigotimes_{\sigma \in G} {^\sigma V}) \otimes_{\mathbb{L}} (\bigotimes_{\sigma \in G} {^\sigma W}) \overset{\sim}{\to} \bigotimes_{\sigma \in G} {^\sigma (V \otimes_{\mathbb{L}} W)}.$$ 

This achieves the proof. □
Lemma 4.12. There is a natural isomorphism of \( \mathbb{K} \)-vector spaces
\[
\text{Cor}_{L/\mathbb{K}}(\bigoplus_{i=1}^{m} V) \simeq \bigoplus_{i=1}^{m} \text{Cor}_{L/\mathbb{K}}(V)
\]
for every \( V \in \text{Mod}\ L \) and integer \( m \geq 1 \).

Proof. One sees easily that there is a natural isomorphism of \( \mathbb{L} \)-vector spaces
\[
\bigotimes_{\sigma \in G} \sigma^*(\bigoplus_{i=1}^{m} V) \rightarrow \bigoplus_{f: G \rightarrow \{1, \ldots, m\}} \bigotimes_{\sigma \in G} \sigma^* V \quad \oplus_{\sigma}(v_{\sigma, i})_i \rightarrow (\oplus_{\sigma} v_{\sigma, f(\sigma)})_f.
\]
If one equips the left hand side with the skew-linear action of Definition 4.9, then (4.13) becomes \( G \)-equivariant provided we equip the right hand side with the following skew-linear action: \( \tau \cdot (\oplus_{\sigma} w_{\sigma, f})_f := (\oplus_{\sigma} w_{\sigma^{-1} f})_f, \) \( \tau \in G \), where \( \tau^{-1} f: G \rightarrow \{1, \ldots, m\} \) is the function \( (\tau^{-1} f)(t) := f(\tau t), \) \( t \in G \). There is also a natural \( G \)-equivariant isomorphism
\[
\bigotimes_{f: G \rightarrow \{1, \ldots, m\}} \bigotimes_{\sigma \in G} \sigma^* V \rightarrow \bigoplus_{\sigma \in G} \bigotimes_{f: G \rightarrow \{1, \ldots, m\}} \sigma^* V \quad \oplus_{\sigma}(w_{\sigma, f})_f \rightarrow (\oplus_{\sigma} w_{\sigma}^*(f))_f,
\]
where \( \sigma \cdot f: G \rightarrow \{1, \ldots, m\} \) is the function \( (\sigma \cdot f)(t) := f(t\sigma), \) \( t \in G \), and the right hand side is equipped with the diagonal \( G \)-action. By applying \((\cdot) G\) to the composite (4.14) \( (\cdot) \), and after fixing an ordering for the elements of \( G \), we obtain the claimed natural isomorphism. \( \square \)

Remark 4.15. It follows from Proposition 4.11 that \( \text{Cor}_{L/\mathbb{K}} \) maps \( \mathbb{L} \)-algebras \( S \) to \( \mathbb{K} \)-algebras \( \text{Cor}_{L/\mathbb{K}}(S) \) and left (resp. right) \( S \)-actions to left (resp. right) \( \text{Cor}_{L/\mathbb{K}(S)} \)-actions.

Proposition 4.16. Let \( R, S \) and \( T \) be \( \mathbb{L} \)-algebras and \( M \) an \( R-S \)-bimodule which is finitely generated and projective as a right \( S \)-module. Then, there is a canonical isomorphism of \( \text{Cor}_{L/\mathbb{K}(R)} \)-\( \text{Cor}_{L/\mathbb{K}(T)} \)-bimodules
\[
\text{Cor}_{L/\mathbb{K}}(M) \otimes_{\text{Cor}_{L/\mathbb{K}(S)}} \text{Cor}_{L/\mathbb{K}}(N) \simeq \text{Cor}_{L/\mathbb{K}}(M \otimes S N)
\]
for every \( S-T \)-bimodule \( N \).

Proof. In order to simplify the exposition we will simply write \( C \) instead of \( \text{Cor}_{L/\mathbb{K}} \). Recall that \( M \otimes S N \) is defined as the coequalizer
\[
M \otimes S \otimes N \xrightarrow{\mu_M \otimes id_N} M \otimes N \xrightarrow{id_M \otimes \mu_N} M \otimes S N,
\]
where \( \mu_M \) (resp. \( \mu_N \)) denotes the right (resp. left) action of \( S \) on \( M \) (resp. on \( N \)). Note that \( M \otimes S N \) comes equipped with an \( R-T \)-bimodule structure induced by the left action of \( R \) on \( M \) and by the right action of \( T \) on \( N \). Since by Proposition 4.11 the functor \( C \) is symmetric monoidal it is enough to show that the induced diagram
\[
\xymatrix{ C(M \otimes S \otimes N) \ar[rr]^{C(\mu_M \otimes id_N)} \ar[rrr]^{C(id_M \otimes \mu_N)} & & & C(M \otimes S \otimes N) \ar[llll]_{C(\mu_M \otimes id_N)} \ar[llll]_{C(id_M \otimes \mu_N)} }
\]
is a coequalizer in \( \text{Mod}\ \mathbb{K} \). The proof is now divided into three different cases:

Case 1: Assume that \( M \simeq S \). In this case our claim is clear since \( S \otimes S N \simeq N \).

Case 2: Assume that \( M \simeq S^{\otimes m} \) for some integer \( m > 1 \). In this case the above coequalizer (4.17) identifies with
\[
\bigoplus_{i=1}^{m} (S \otimes S \otimes N) \xrightarrow{\mu_S \otimes id_N} S \otimes N \xrightarrow{id_S \otimes \mu_N} S \otimes S N.
\]
and hence by Lemma 4.12 the diagram (4.18) identifies with the following direct sum of diagrams

\[(4.19) \quad \bigoplus_{i=1}^{m|C|} \left( C(S \otimes S \otimes N) \xrightarrow{C(\mu_S \otimes \id_N)} C(S \otimes N) \xrightarrow{C(\id_S \otimes \mu_N)} C(M \otimes S N) \right). \]

Now, since by Case 1 the diagram inside the brackets is a coequalizer in \(\text{Mod}_K\) one deduces that (4.19) is also a coequalizer. This proves Case 2.

**Case 3**: In full generality now, we can assume that \(M\) is a retract of \(S^{\oplus m}\) for some integer \(m \geq 1\). By definition, there exist maps of right \(S\)-modules

\[
\iota_M : M \to S^{\oplus m} \quad \pi_M : S^{\oplus m} \to M
\]

verifying the equality \(\pi_M \circ \iota_M = \id_M\). This allows us to construct the following diagram in \(\text{Mod}_L\):

\[
(4.20) \quad \begin{array}{ccc}
M \otimes S \otimes N & \xrightarrow{\mu_M \otimes \id_N} & M \otimes N \\
\pi_M \otimes \id_S \otimes \id_N & & \id_M \otimes \mu_N \\
S^{\oplus m} \otimes S \otimes N & \xrightarrow{\mu_{S^{\oplus m}} \otimes \id_N} & S^{\oplus m} \otimes N \\
\iota_M \otimes \id_S \otimes \id_N & & \id_{S^{\oplus m}} \otimes \mu_N \\
M \otimes S \otimes N & \xrightarrow{\mu_M \otimes \id_N} & M \otimes N \\
\pi_M \otimes \id_S \otimes \id_N & & \id_M \otimes \mu_N
\end{array}
\]

where \(\iota\) and \(\pi\) are induced by the universal property of the coequalizer. Note that the three vertical compositions are identities. By applying the functor \(C\) to (4.20) we obtain, thanks to Case 2, an analogous diagram where the middle row is an equalizer. The proof now follows from the general Lemma 4.21 below.

**Lemma 4.21.** Consider the following commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\pi_X & \downarrow{g} & \pi_Y \\
X & \xrightarrow{t' \circ \iota_X} & Y \\
\pi_X & \downarrow{g} & \pi_Y \\
X & \xrightarrow{t} & Y
\end{array}
\]

where the middle row is a coequalizer and each vertical composite is the identity. By the commutativity of the left hand side squares we mean that \(f \circ \pi_X = \pi_Y \circ f\), \(g \circ \pi_X = \pi_Y \circ g\), \(f \circ \iota_X = \iota_Y \circ f\), and \(g \circ \iota_X = \iota_Y \circ g\). Under these assumptions, the first (and last) row is also a coequalizer.

**Proof.** Let \(t : Y \to T\) be such that \(f \circ \pi_X = \pi_Y \circ f\), \(g \circ \pi_X = \pi_Y \circ g\), hence there exists a unique \(t' : Z \to T\) such that \(t'h = t\pi_Y\). Now set \(t'' := t' \iota_Z\). Then \(t'' = t' \iota_Z\). This proves that the first row is a coequalizer, as claimed. \(\square\)

**Definition 4.22.** The \(\otimes\)-functor (4.10) induces, by “base-change” of enriched categories, a well-defined \(\otimes\)-functor

\[
(4.23) \quad C_{L/K} : \text{Cat}_L \to \text{Cat}_K.
\]
as follows. For $A \in \text{Cat}_L$, one obtains the $\mathbb{K}$-category $C_{L/\mathbb{K}}(A)$ with the same object set as $A$, and with composition maps $\text{Cor} A(y, z) \otimes_{\mathbb{K}} \text{Cor} A(x, y) \simeq \text{Cor} A(y, z \otimes_{L} A(x, y)) \rightarrow \text{Cor} A(x, z)$ and identities $\mathbb{K} \simeq \text{Cor} L \rightarrow \text{Cor} A(x, x)$, obtained by applying the functor $\text{Cor} = \text{Cor}_{L/\mathbb{K}}$ to the compositions and identities of $A$ and by composing with its monoidal structure maps, in the evident way. If $F : A \rightarrow B$ is an $\mathbb{L}$-functor with components $F(x, y)$, then $C_{L/\mathbb{K}}(F)$ is the $\mathbb{K}$-functor $C_{L/\mathbb{K}}(A) \rightarrow C_{L/\mathbb{K}}(B)$ with components $\text{Cor} F(x, y)$.

**Proposition 4.24.** The corestriction functor (4.23) induces a well-defined $\otimes$-functor $C_{L/\mathbb{K}} : \text{Ho}(\text{Cat}_{L})^\otimes \rightarrow \text{Ho}(\text{Cat}_{\mathbb{K}})^\otimes$ such that $\text{Pic}(C_{L/\mathbb{K}}) \simeq c_{L/\mathbb{K}}$ under the identification of Corollary 4.2.

**Proof.** By a result of Richm [15, Theorem 11] (see also [6, §8-9]), in the case of finite Galois extensions the corestriction map of Brauer groups: $[C_{L/\mathbb{K}}(A)] = c_{L/\mathbb{K}}(A)$ in $\text{Br}(\mathbb{K})$ for each Azumaya $\mathbb{L}$-algebra $A$. So the only remaining question is whether $C_{L/\mathbb{K}}$ is well-defined at the level of the Morita homotopy categories. As proved in Lemma 4.25 below, the functor (4.10) induces a well-defined $\otimes$-functor $C_{L/\mathbb{K}} : \text{Ho}(\text{Alg}_{L}) \rightarrow \text{Ho}(\text{Alg}_{\mathbb{K}})$. Hence, consider the diagram

$$
\begin{array}{ccc}
\text{Ho}(\text{Alg}_{L})^\otimes & \xrightarrow{C_{L/\mathbb{K}}} & \text{Ho}(\text{Alg}_{\mathbb{K}})^\otimes \\
\cong & & \cong \\
\text{Ho}(\text{Cat}_{L})^\otimes & \xrightarrow{\sim} & \text{Ho}(\text{Cat}_{\mathbb{K}})^\otimes,
\end{array}
$$

where the vertical arrows are the symmetric monoidal fully faithful inclusions. Since these latter functors are equivalences by Proposition 4.1, we conclude that $C_{L/\mathbb{K}}$ has an (essentially) unique extension to a symmetric monoidal functor

$$
C_{L/\mathbb{K}} : \text{Ho}(\text{Cat}_{L})^\otimes \rightarrow \text{Ho}(\text{Cat}_{\mathbb{K}})^\otimes
$$
as claimed. This ends the proof of Proposition 4.24. \qed

**Lemma 4.25.** The functor (4.10) induces a $\otimes$-functor $C_{L/\mathbb{K}} : \text{Ho}(\text{Alg}_{L}) \rightarrow \text{Ho}(\text{Alg}_{\mathbb{K}})$.

**Proof.** In order to simplify the exposition we will simply write $C$ instead of $\text{Cor}_{L/\mathbb{K}}$. Being monoidal, as we have already remarked $C$ maps $\mathbb{L}$-algebras to $\mathbb{K}$-algebras and bimodules to bimodules. Let $R$ and $S$ be two $\mathbb{L}$-algebras and $R M_S$ a $R$-$S$-bimodule which is finitely generated and projective as a right $S$-module. Consider the description of $\text{Ho}(\text{Alg})$ given at the end of Section 3 in terms of bimodules and tensor products. If $sN_T$ is a $S$-$T$-bimodule which is finitely generated and projective as a right $T$-module, we see thanks to Proposition 4.16 that $C$ preserves their composition: $[C(M \otimes_S N)] = [C(M) \otimes_{C(S)} C(N)]$. Similarly, it preserves identities: $[C(sS_T)] = [C(s)C(S)C(sT)]$. It remains only to show that the $C(R)$-$C(S)$-bimodule $C(R M_S)$ obtained is also finitely generated and projective as a right $C(S)$-module. From Lemma 3.12 one knows that $M_S$ belongs to $S_{(S)}^\circ$. By functoriality of $C$ idempotents are mapped to idempotents and so one can assume without loss of generality that $M_S$ belongs to $S_{(S)}^\circ$. It remains then to show that $C(M)_{C(S)}$ belongs to $C(S)_{(S)}$. (Note that this is not automatic since $C$ is not additive.) By combining the canonical isomorphism $C(\mathbb{L}) \simeq \mathbb{K}$ with Lemma 4.12 we obtain, for every integer $m \geq 1$, an isomorphism

$$
C(\mathbb{L}^{\otimes m}) \simeq \bigoplus_{i=1}^{m^{[G]}} \mathbb{K}.
$$
Hence, since $M_S \simeq S^{\oplus m}$ the following isomorphisms of right $C(S)$-modules hold:

\[
C(M_S) \simeq C(S^{\oplus m}) \\
\simeq C(L^{\oplus m}_S \otimes L S) \\
\simeq C(L^{\oplus m}_S) \otimes_K C(S) \quad \text{by Prop. 4.11} \\
\simeq \bigoplus_{i=1}^{m(c)} K \otimes_K C(S) \quad \text{by (4.26)} \\
\simeq \bigoplus_{i=1}^{m(c)} C(S).
\]

This shows that $C(M_S)$ belongs to $C(S)\otimes$ and hence we conclude that $C$ descends to a well-defined functor $C_{L/K} : \text{Ho}(\text{Alg}_{L}) \to \text{Ho}(\text{Alg}_K)$. Finally, the fact that this functor is symmetric monoidal is inherited from the corresponding property of (4.10).

This concludes the proof of Theorem 1.2.

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