STRUCTURAL MATRIX ALGEBRAS, GENERALIZED FLAGS AND GRADINGS

F. BEŞLEAGĂ AND S. DĂSCĂLESCU

Abstract. We show that a structural matrix algebra $A$ is isomorphic to the endomorphism algebra of an algebraic-combinatorial object called a generalized flag. If the flag is equipped with a group grading, an algebra grading is induced on $A$. We classify the gradings obtained in this way as the orbits of the action of a double semidirect product on a certain set. Under some conditions on the associated graph, all good gradings on $A$ are of this type. As a bi-product, we obtain a new approach to compute the automorphism group of a structural matrix algebra.

1. Introduction and preliminaries

Flags and flag varieties play a key role in Algebraic Geometry, Representation Theory, Algebraic Groups and Combinatorics, see [6]. In this paper we consider a more general concept of flag and we give some applications. This new kind of flag arises as follows.

Let $k$ be a field. A structural matrix algebra is a subalgebra of a full matrix algebra over $k$, consisting of all matrices with zero entries on certain prescribed positions, and allowing anything on the remaining positions. An important number of examples and counterexamples use such algebras. These algebras were called structural matrix algebras in [10], but they had already been considered in [8]. Particular examples of structural matrix algebras are upper triangular matrix algebras, and more generally, upper block triangular matrix algebras, which are of fundamental importance in Linear Algebra, the study of linear groups, the study of numerical invariants of PI algebras, etc. A structural matrix subalgebra $A$ of $M_n(k)$ is associated with a preorder relation (i.e. reflexive and transitive) $\rho$ on the set $\{1, \ldots, n\}$; $A$ consists of all matrices $(a_{ij})_{1\leq i,j\leq n}$ such that $a_{ij} = 0$ whenever $(i,j) \notin \rho$. We denote $A = M(\rho, k)$; in other terminology, this is the incidence algebra over $k$ associated with $\rho$, see [9]. Automorphisms of structural matrix algebras are of interest, and the problem to understand them gets more complicated if we take $k$ to be just a ring. If $k$ is a field, the description of the automorphism group of $M(\rho, k)$ was done in [5].

A general problem in Ring Theory is to describe and classify all group gradings on various matrix algebras. In the case of full matrix algebras over a field $k$, the problem was solved in [1] and [2] for algebraically closed $k$, and descent theory was used in [4] to approach the case of an arbitrary $k$. For subalgebras of a full matrix algebra, in particular for structural matrix algebras, it is much more complicated to describe all gradings. The aim of our paper is to construct and classify a certain class of gradings on structural matrix algebras.

We show in Section 2 that in the same way the full matrix algebra $M_n(k)$ is the endomorphism group of a vector space of dimension $n$, a structural matrix algebra $M(\rho, k)$ is isomorphic to the endomorphism algebra of a certain algebraic-combinatorial structure $F$, which we call a $\rho$-flag. In Section 5 we explain that if $F$ is additionally equipped with a $G$-grading, where $G$ is a group, then its endomorphism algebra $\text{End}(F)$ gets an induced $G$-graded algebra structure; we denote by $\text{END}(F)$ the obtained $G$-graded algebra. This grading transfers to a $G$-grading on

2010 Mathematics Subject Classification. 16W50, 16W20, 16S50, 06A06.

Key words and phrases. structural matrix algebra, preorder relation, flag, group grading, automorphism group.
$M(\rho, k)$ via the isomorphism mentioned above. The gradings produced in this way on $M(\rho, k)$ are good gradings, i.e. all the matrix units present in $M(\rho, k)$ are homogeneous elements. It is an interesting question whether all good gradings are obtained like this. This is a problem of independent interest, and it can be formulated in simple terms related to the graph $\Gamma$ associated with $\rho$: if $G$ is a group, and on each arrow of $\Gamma$ we write an element of $G$ as a label, such that for any two paths starting from and terminating at the same points the product of the labels of the arrows is the same for both paths, does the set of labels arise from a set of weights on the vertices of $\Gamma$, in the sense that an arrow starting from $v_1$ and terminating at $v_2$ has label $g_1g_2^{-1}$, where $g_1$ and $g_2$ are the weights of $v_1$ and $v_2$? This problem was considered in \[5\] in the case where $G$ is abelian, and it was showed that the answer is positive if and only if the cohomology group $H^1(\Delta, G) = 0$, where $\Delta$ is a certain simplicial complex associated with $\rho$. Also, for a given $\rho$, the answer to the above question is positive for any abelian group $G$ if and only if the homology group $H_1(\Delta) = 0$. We show that the answer is positive for any arbitrary group $G$ if and only if the normal closure of two certain subgroups $A(\Gamma) \subseteq B(\Gamma)$ of the free group generated by the arrows of $\Gamma$ coincide; $A(\Gamma)$ and $B(\Gamma)$ are defined in terms of cycles of the un-directed graph obtained from $\Gamma$. This parallels the result in the abelian case, where $H_1(\Delta) = B/A$ for similar subgroups $A$ and $B$ in a free abelian group associated with $\Gamma$. In fact we use slightly different $A$ and $B$, by working with a different graph.

In order to classify $G$-gradings on the structural matrix algebra $M(\rho, k)$, we first look at the isomorphisms between the algebras $\text{End}(\mathcal{F})$ and $\text{End}(\mathcal{F}')$, where $\mathcal{F}$ and $\mathcal{F}'$ are $\rho$-flags under the vector spaces $V$ and $V'$. An equivalence relation $\sim$ on $\{1, \ldots, n\}$ arises from $\rho$, where $i \sim j$ whenever $i\rho j$ and $j\rho i$. Then $\rho$ induces a partial order on the set $\mathcal{C}$ of equivalence classes with respect to $\sim$. We show in Section \[5\] that the $\text{End}(\mathcal{F})$-submodules of $V$ are in a bijective correspondence with the antichains of $\mathcal{C}$; let $A(\mathcal{C})$ be the lattice structure on the set of all such antichains, induced via this bijection. Then an algebra isomorphism $\varphi: \text{End}(\mathcal{F}) \rightarrow \text{End}(\mathcal{F}')$ induces a linear isomorphism $\gamma: V \rightarrow V'$ which is a $\varphi'$-isomorphism for a certain deformation of $\varphi$. The new algebra isomorphism $\varphi'$ is obtained from $\varphi$ by using a transitive function on $\rho$ with values in $k^*$. Since $\varphi'$ is an algebra isomorphism, $\gamma$ induces an isomorphism between the lattices of $\text{End}(\mathcal{F})$-submodules of $V$ and of $\text{End}(\mathcal{F}')$-submodules of $V'$, and this lattice isomorphism reduces in fact to an automorphism of the lattice $A(\mathcal{C})$. Such an automorphism is completely determined by an automorphism $g$ of the poset $\mathcal{C}$. Moreover, we explain that $\varphi$ can be recovered from $g$, the deformation constants producing $\varphi'$ from $\varphi$, and a matrix of $\gamma$ in a fixed pair of bases. Thus we obtain that the set of algebra isomorphisms from $\text{End}(\mathcal{F})$ to $\text{End}(\mathcal{F}')$ is in a bijective correspondence with the equivalence classes of a set involving the invertible matrices of $M(\rho, k)$, the automorphisms of $\mathcal{C}$ preserving the cardinality of elements, and the transitive functions on $\rho$, with respect to an equivalence relation. In particular, if $\mathcal{F}' = \mathcal{F}$, the automorphism group of $\text{End}(\mathcal{F})$ is described as a factor group of a double semidirect product. As a bi-product, we obtain a descriptive presentation of the automorphism group of a structural matrix algebra. This automorphism group was computed in \[5\], and we show how the presentation in \[5\] can be derived from ours.

For classifying $G$-gradings arising from graded flags, we consider two $G$-graded $\rho$-flags $\mathcal{F}$ and $\mathcal{F}'$, and we look at the isomorphisms between the graded algebras $\text{END}(\mathcal{F})$ and $\text{END}(\mathcal{F}')$. Using the structure of isomorphisms between $\text{End}(\mathcal{F})$ and $\text{End}(\mathcal{F}')$, which we already know by now, and adding the additional information about gradings, we obtain in Section \[6\] that $\text{END}(\mathcal{F}) \simeq \text{END}(\mathcal{F}')$ if and only if the connected components of $\mathcal{F}$ and $\mathcal{F}'$ are pairwise isomorphic up to a permutation, some graded shifts and an automorphism of $\mathcal{C}$. Using this result, we show in Section \[7\] that the isomorphism types of graded algebras of the form $\text{END}(\mathcal{F})$ are classified by
the orbits of the action of a certain group, which is a double semidirect product of a Young subgroup of $S_n$, a certain subgroup of automorphisms of $C$, and $G^q$, where $q$ is the number of connected components of $C$, on the set $G^n$.

We use the standard terminology on gradings, see for example [7].

2. STRUCTURAL MATRIX ALGEBRAS AS ENDOMORPHISM ALGEBRAS

Let $k$ be a field, $n$ a positive integer and $\rho$ a preorder relation on $\{1, \ldots, n\}$. Let $M(\rho, k)$ be the structural matrix algebra associated with $\rho$.

Let $\sim$ be the equivalence relation on $\{1, \ldots, n\}$ associated with $\rho$, i.e. $i \sim j$ if and only if $i \rho j$ and $j \rho i$, and let $C$ be the set of equivalence classes. Then $\rho$ induces a partial order $\leq$ on $C$ defined by $\hat{i} \leq \hat{j}$ if and only if $i \rho j$, where $\hat{i}$ denotes the equivalence class of $i$.

For any $\alpha \in C$, let $m_\alpha$ be the number of elements of $\alpha$.

**Definition 2.1.** A $\rho$-flag is an $n$-dimensional vector space $V$ with a family $(V_\alpha)_{\alpha \in C}$ of subspaces such that there is a basis $B$ of $V$ and a partition $B = \bigcup_{\alpha \in C} B_\alpha$ with the property that $|B_\alpha| = m_\alpha$ and $\bigcup_{\beta \leq \alpha} B_\beta$ is a basis of $V_\alpha$ for any $\alpha \in C$. If $F = (V, (V_\alpha)_{\alpha \in C})$ and $F' = (V', (V'_\alpha)_{\alpha \in C})$ are $\rho$-flags, then a morphism of $\rho$-flags from $F$ to $F'$ is a linear map $f : V \to V'$ such that $f(V_\alpha) \subset V'_\alpha$ for any $\alpha \in C$.

If $\rho$ is just the usual ordering relation on $\{1, \ldots, n\}$ (which corresponds to $M(\rho, k)$ being the algebra of upper triangular matrices), then a $\rho$-flag is just an usual flag on an $n$-dimensional vector space.

More generally, if $\rho$ is such that $C = \{\alpha_1, \ldots, \alpha_r\}$ is totally ordered, say $\alpha_1 < \cdots < \alpha_r$, and $|\alpha_i| = m_i$ for any $1 \leq i \leq r$, then a $\rho$-flag is a flag of signature $(m_1, \ldots, m_r)$, and $M(\rho, k)$ is the algebra

$$
\begin{pmatrix}
M_{m_1}(k) & M_{m_1,m_2}(k) & \cdots & M_{m_1,m_r}(k) \\
0 & M_{m_2}(k) & \cdots & M_{m_2,m_r}(k) \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & M_{m_r}(k)
\end{pmatrix}
$$

of upper block triangular matrices, with diagonal blocks of size $m_1, \ldots, m_r$.

For any $i, j \in \{1, \ldots, n\}$ with $i \rho j$, let $e_{ij}$ be the matrix in $M_n(k)$ having 1 on the $(i,j)$-position and 0 elsewhere. The set of all such $e_{ij}$’s is a basis of $M(\rho, k)$. Now we present $M(\rho, k)$ as an algebra of endomorphisms.

**Proposition 2.2.** Let $F = (V, (V_\alpha)_{\alpha \in C})$ be a $\rho$-flag. Then the algebra $\text{End}(F)$ of endomorphisms of $F$ (with the map composition as multiplication) is isomorphic to $M(\rho, k)$.

**Proof.** Let $B$ be a basis of $V$ as in Definition 2.1 We can choose some set of indices such that $B = \{v_i\}_{1 \leq i \leq n}$ and for any $\alpha \in C$ the basis $B_\alpha$ of $V_\alpha$ is just $\{v_i | i \in \alpha\}$. If $i, j \in \{1, \ldots, n\}$, then $i \rho j$ if and only if $i \in \alpha$ and $j \in \beta$ for some $\alpha, \beta \in C$ with $\alpha \leq \beta$; in this case let $E_{ij} \in \text{End}(F)$ be defined by $E_{ij}(v_i) = \delta_{ij}v_1$ for any $t$ ($\delta_{ij}$ is Kronecker’s delta). Then it is easy to see that $E_{ij}E_{pq} = \delta_{jp}E_{iq}$ for any $i, j, p, q$ and $\{E_{ij} | i \rho j\}$ is a basis of $\text{End}(F)$. Hence the linear map $E_{ij} \mapsto e_{ij}$ for any $i, j$ with $i \rho j$, is an algebra isomorphism between $\text{End}(F)$ and $M(\rho, k)$.\[\square\]

We associate with $\rho$ another useful combinatorial object. Let $\Gamma = (\Gamma_0, \Gamma_1)$ be the graph whose set $\Gamma_0$ of vertices is the set $C$ of equivalence classes. The set $\Gamma_1$ of arrows is constructed as follows: if $\alpha, \beta \in C$, there is an arrow $a$ from $\alpha$ to $\beta$ (we write $s(a) = \alpha$, $t(a) = \beta$) if $\alpha < \beta$ and there is no $\gamma \in C$ with $\alpha < \gamma < \beta$. Clearly, if $\alpha, \beta \in C$, then $\alpha \leq \beta$ if and only if there is a
path in $\Gamma$ starting from $\alpha$ and ending at $\beta$ (recall that paths of length zero are just vertices of $\Gamma$). We denote by $\Gamma^u$ the undirected graph obtained from $\Gamma$ when we forget the orientation of arrows.

3. The lattice of $\text{End}(\mathcal{F})$-submodules of $V$

Let $\mathcal{F} = (V_\alpha)_{\alpha \in \mathcal{C}}$ be a $\rho$-flag on the space $V$. The aim of this section is to describe the lattice $\mathcal{L}_{\text{End}(\mathcal{F})}(V)$ of submodules of the $\text{End}(\mathcal{F})$-module $V$, where the action of $\text{End}(\mathcal{F})$ on $V$ is just the restriction of the usual $\text{End}(V)$-action on $V$. We also determine the automorphisms of this lattice.

If $\mathcal{D}$ is a subset of $\mathcal{C}$, we denote by $V_\mathcal{D} = \sum_{\alpha \in \mathcal{D}} V_\alpha$. By convention $V_\emptyset = 0$.

**Proposition 3.1.** The $\text{End}(\mathcal{F})$-submodules of $V$ are the subspaces of the form $V_\mathcal{D}$, where $\mathcal{D}$ is a subset of $\mathcal{C}$.

**Proof.** Since $V_\mathcal{D} = \sum_{\alpha \in \mathcal{D}} V_\alpha = \sum_{i=1}^{n} \{V_i \mid \text{there exists } \alpha \in \mathcal{D} \text{ such that } \hat{i} \leq \alpha \}$, and $E_{pq}v_i = \delta_{q,i}v_p$ for any $p, q$ with $\hat{p} \leq \hat{q}$, and any $i$, it is clear that $V_\mathcal{D}$ is an $\text{End}(\mathcal{F})$-submodule of $V$.

Conversely, let $X$ be an $\text{End}(\mathcal{F})$-submodule of $V$. If $v = \sum_{i=1}^{n} a_iv_i \in X - \{0\}$, and $a_{i_0} \neq 0$ for some $i_0$, then for any $j$ with $\hat{j} \leq \hat{i_0}$ we have $E_{ji_0}v = a_{i_0}v_j$, so $v_j \in X$, thus $V_{i_0} \subset X$. If $\mathcal{D} = \{i_0 \mid \text{there exists } v = \sum_{i=1}^{n} a_iv_i \in X - \{0\} \text{ with } a_{i_0} \neq 0\}$, we obtain that $V_\mathcal{D} = \sum_{\alpha \in \mathcal{D}} V_\alpha \subset X$. Obviously $X \subset V_\mathcal{D}$, so then $X = V_\mathcal{D}$.

\[\square\]

**Remark 3.2.** If $\mathcal{D} \subset \mathcal{C}$ and $\mathcal{D}_{\text{max}}$ is the set of maximal elements of $\mathcal{D}$ (with respect to the partial order of $\mathcal{C}$), it is clear that $V_\mathcal{D} = V_{\mathcal{D}_{\text{max}}}$. As $\mathcal{D}_{\text{max}}$ is an antichain in $\mathcal{C}$ (i.e. a subset of $\mathcal{C}$ whose any two different elements are not comparable with respect to $\leq$), and for any antichain $\mathcal{D}$ we have $\mathcal{D}_{\text{max}} = \mathcal{D}$, we conclude that the $\text{End}(\mathcal{F})$-submodules of $V$ are $V_\mathcal{D}$ with $\mathcal{D}$ an antichain in $\mathcal{C}$.

If we denote by $\mathcal{A}(\mathcal{C})$ the set of all antichains of $\mathcal{C}$, we have a bijection between $\mathcal{A}(\mathcal{C})$ and $\mathcal{L}_{\text{End}(\mathcal{F})}(V)$, given by $\mathcal{D} \mapsto V_\mathcal{D}$. Its inverse is $X \mapsto \mathcal{D}_\text{max}$, where $\mathcal{D}$ is a subset of $\mathcal{C}$ such that $X = V_\mathcal{D}$. This bijection induces a lattice structure on $\mathcal{A}(\mathcal{C})$ from the lattice $\mathcal{L}_{\text{End}(\mathcal{F})}(V)$. Since for $\mathcal{D}, \mathcal{E} \in \mathcal{A}(\mathcal{C})$ we have

$$V_\mathcal{D} \cap V_\mathcal{E} = \sum_{\alpha \in \mathcal{C}} \{V_\alpha \mid \text{there exist } \beta_1 \in \mathcal{D}, \beta_2 \in \mathcal{E} \text{ such that } \alpha \leq \beta_1 \text{ and } \alpha \leq \beta_2\}$$

and

$$V_\mathcal{D} + V_\mathcal{E} = V_{\mathcal{D} \cup \mathcal{E}},$$

we see that the infimum and the supremum in $\mathcal{A}(\mathcal{C})$ are given by

$$\mathcal{D} \land \mathcal{E} = \{\alpha \in \mathcal{C} \mid \text{there exist } \beta_1 \in \mathcal{D}, \beta_2 \in \mathcal{E} \text{ such that } \alpha \leq \beta_1 \text{ and } \alpha \leq \beta_2\}_{\text{max}},$$

$$\mathcal{D} \lor \mathcal{E} = (\mathcal{D} \cup \mathcal{E})_{\text{max}}$$

for any $\mathcal{D}, \mathcal{E} \in \mathcal{A}(\mathcal{C})$. Note that $(\mathcal{D} \cup \mathcal{E})_{\text{max}}$ may be strictly contained in $\mathcal{D} \cup \mathcal{E}$, since $\mathcal{D} \cup \mathcal{E}$ is not necessarily an antichain.

The partial order relation on $\mathcal{A}(\mathcal{C})$ is

$$\mathcal{D} \leq \mathcal{E} \iff V_\mathcal{D} \subset V_\mathcal{E} \iff \text{for any } \alpha \in \mathcal{D} \text{ there exists } \beta \in \mathcal{E} \text{ such that } \alpha \leq \beta.$$
Let $0$ denote the bottom element of a poset $(\mathcal{P}(\mathcal{C}), \subseteq)$.

The next result describes the automorphisms of the lattice $\mathcal{A}(\mathcal{C})$.

**Proposition 3.3.** If $g$ is an automorphism of the poset $(\mathcal{C}, \leq)$, then the map $f_g : \mathcal{A}(\mathcal{C}) \to \mathcal{A}(\mathcal{C})$, $f_g(\mathcal{D}) = g(\mathcal{D}) = \{g(\alpha) \mid \alpha \in \mathcal{D}\}$ is an automorphism of the lattice $\mathcal{A}(\mathcal{C})$. Moreover, for any lattice automorphism $f$ of $\mathcal{A}(\mathcal{C})$ there exists an automorphism $g$ of the poset $(\mathcal{C}, \leq)$ such that $f = f_g$.

**Proof.** It is straightforward to check the first part. Now let $f$ be a lattice automorphism of $\mathcal{A}(\mathcal{C})$. Let $\mathcal{L}_0(\mathcal{C})$ be the set of all minimal elements of $\mathcal{C}$, $\mathcal{L}_1(\mathcal{C})$ be the set of all minimal elements of $\mathcal{C} \setminus \mathcal{L}_0(\mathcal{C})$ (or equivalently, elements of height 1 in $\mathcal{C}$) and recurrently we define $\mathcal{L}_h(\mathcal{C})$ for any $h$, as the set of all minimal elements of $\mathcal{C} \setminus (\mathcal{L}_0(\mathcal{C}) \cup \cdots \cup \mathcal{L}_{h-1}(\mathcal{C}))$ (or equivalently, elements of height $h$ in $\mathcal{C}$).

The set $\mathcal{L}_0$ of minimal elements of $\mathcal{A}(\mathcal{C})$ consists of all singletons $\{\alpha\}$, with $\alpha \in \mathcal{L}_0(\mathcal{C})$. As $f$ is a lattice automorphism, we have $f(\mathcal{L}_0) = \mathcal{L}_0$, and this induces a bijection $g_0 : \mathcal{L}_0(\mathcal{C}) \to \mathcal{L}_0(\mathcal{C})$. We note that $g_0$ (or $f|_{\mathcal{L}_0}$) uniquely determines the value of $f$ at any non-empty $\mathcal{D} \subset \mathcal{L}_0$.

Next $\mathcal{A}(\mathcal{C}) \setminus \mathcal{P}(\mathcal{L}_0)$ is a poset with the order inherited from $\mathcal{A}(\mathcal{C})$, such that for any two elements their supremum exists. Moreover, $f$ induces by restriction an isomorphism of posets $f : \mathcal{A}(\mathcal{C}) \setminus \mathcal{P}(\mathcal{L}_0) \to \mathcal{A}(\mathcal{C}) \setminus \mathcal{P}(\mathcal{L}_0)$ (there is no harm if we also denote it by $f$), such that $f(x \lor y) = f(x) \lor f(y)$ for any $x, y$. The set $\mathcal{L}_1$ of minimal elements of $\mathcal{A}(\mathcal{C}) \setminus \mathcal{P}(\mathcal{L}_0)$ consists of all singletons $\{\alpha\}$, where $\alpha$ is minimal in $\mathcal{C} \setminus \mathcal{L}_0(\mathcal{C})$, i.e. $\alpha \in \mathcal{L}_1(\mathcal{C})$. Since $f(\mathcal{L}_1) = \mathcal{L}_1$, $f$ induces a bijection $g_1 : \mathcal{L}_1(\mathcal{C}) \to \mathcal{L}_1(\mathcal{C})$, and $g_0$ and $g_1$ uniquely determine the value of $f$ at any antichain $\mathcal{D}$ with $\mathcal{D} \subset \mathcal{L}_0(\mathcal{C}) \cup \mathcal{L}_1(\mathcal{C})$.

We continue recurrently, by considering for any $h$ the set $\mathcal{L}_h$ of minimal elements in $\mathcal{A}(\mathcal{C}) \setminus \mathcal{P}(\mathcal{L}_0 \cup \cdots \cup \mathcal{L}_{h-1})$. This consists of all singletons $\{\alpha\}$, where $\alpha \in \mathcal{L}_h(\mathcal{C})$. As above, $f$ induces an automorphism of the poset $\mathcal{A}(\mathcal{C}) \setminus \mathcal{P}(\mathcal{L}_0 \cup \cdots \cup \mathcal{L}_{h-1})$, hence a bijection $g_h : \mathcal{L}_h(\mathcal{C}) \to \mathcal{L}_h(\mathcal{C})$. As $\bigcup \mathcal{L}_h(\mathcal{C}) = \mathcal{C}$, the coproduct (i.e. disjoint union) $g$ of all $g_h$’s is an automorphism of the poset $\mathcal{C}$, and it is clear that $f = f_g$.

\[\square\]

### 4. Isomorphisms between endomorphism algebras of flags

We consider the set

$$\text{Aut}_0(\mathcal{C}, \leq) = \{g \in \text{Aut}(\mathcal{C}, \leq) \mid m_\alpha = m_{g(\alpha)} \text{ for any } \alpha \in \mathcal{C}\},$$

which is a subgroup of $\text{Aut}(\mathcal{C})$. For any $g \in \text{Aut}_0(\mathcal{C})$ we define a bijection $\tilde{g} : \{1, \ldots, n\} \to \{1, \ldots, n\}$ as follows: if $\alpha = \{i_1, \ldots, i_r\}$ with $i_1 < \ldots < i_r$, and $g(\alpha) = \{j_1, \ldots, j_r\}$ with $j_1 < \ldots < j_r$, then $\tilde{g}(i_1) = j_1, \ldots, \tilde{g}(i_r) = j_r$. If $g, h \in \text{Aut}_0(\mathcal{C})$, we clearly have $h \tilde{g} = \tilde{h} \tilde{g}$, thus $g \mapsto \tilde{g}$ is an embedding of $\text{Aut}_0(\mathcal{C})$ into the symmetric group $S_n$.

If $A \in M_n(k)$, let $A^g \in M_n(k)$ be the matrix whose $(i, j)$-entry is the $i, j$-entry of $A$, and let $^gA \in M_n(k)$ be the matrix whose $(i, j)$-entry is the $(\tilde{g}(i), j)$-entry of $A$. If $g, h \in \text{Aut}_0(\mathcal{C})$, then

$$(A^g)^h = A^{gh}, \quad ^h(A^g) = ^gA, \quad (^gA)^h = ^g(A^h)$$

for any $A \in M_n(k)$. Also

$$(AB)^g = AB^g, \quad ^g(AB) = (^gA)B, \quad (A^g)B = A(^{-1}B)$$
for any $A,B \in M_n(k)$ and any $g \in \Aut_0(\mathcal{C})$. It follows that if $A$ is invertible, then so is $A^g$, and its inverse is $g(A^{-1})$.

Let $\mathcal{T} = \{(a_{ij})_{i\rho j} \subset k^* \mid a_{ij}a_{jr} = a_{ir} \text{ for any } i,j,r \text{ with } i\rho j,j\rho r\}$. Using the terminology of Section 3, $\mathcal{T}$ can be identified with the set of transitive $k^*$-valued functions on $\rho$. Multiplication on positions (i.e. pointwise multiplication of functions) makes $\mathcal{T}$ a group.

Let $\mathcal{F}$ be a $\rho$-flag on the space $V$, and $\mathcal{F}'$ be a $\rho'$-flag on the space $V'$. We keep the notation of Section 2 for $\mathcal{F}$, the basis of $V$, and the associated $E_{ij}$’s. Thus we fix a basis $(v_i)_i$ of $V$ such that \{v_i \mid i \leq \alpha\} is a basis of $V_\alpha$, and similarly a basis $(v'_i)_i$ of $V'$ such that \{v'_i \mid i \leq \alpha\} is a basis of $V'_\alpha$ for any $\alpha$; let $(E'_{ij})_{i\rho j}$ be the basis of $\End(\mathcal{F}')$ associated with $(v'_i)_i$.

Define $F : U(M(\rho,k)) \times \Aut_0(\mathcal{C}) \times \mathcal{T} \rightarrow \Iso_{\text{alg}}(\End(\mathcal{F}),\End(\mathcal{F}'))$ as follows. If $A \in U(M(\rho,k))$, $g \in \Aut_0(\mathcal{C})$ and $(a_{ij})_{i\rho j} \in \mathcal{T}$, let $(w_i)_{1 \leq i \leq n}$ be the basis of $V'$ defined by

$(\ldots, w_i, \ldots) = (\ldots, v'_i, \ldots) A^g$.

Then for any $i,j$ with $i\rho j$ let $F_{ij} \in \End(V')$ be such that $F_{ij}(w_j) = a_{ij} w_i$ for any $i\rho j$, and $F_{ij}(w_r) = 0$ for any $r \neq j$. Clearly $F_{ij}F_{jr} = F_{ir}$ for $i\rho j, j\rho r$.

**Lemma 4.1.** With the above notation, let $A = (\lambda_{pq})_{p,q}$ and $A^{-1} = (\overline{\lambda}_{pq})_{p,q}$. Then

$$F_{ij} = a_{ij} \sum_{s \rho j(i)} \lambda_{s\bar{g}(j)t} E'_{st}.$$

In particular $F_{ij} \in \End(\mathcal{F}')$.

**Proof.** Since $(\ldots, v'_i, \ldots) = (\ldots, w_i, \ldots) (A^g)^{-1} = (\ldots, w_i, \ldots) g(A^{-1})$, we see that $v'_t = \sum_p \overline{\lambda}_{\bar{g}(p)t} w_p$ for any $t$. Then

$$F_{ij}(v'_t) = \sum_p \overline{\lambda}_{\bar{g}(p)t} F_{ij}(w_p) = a_{ij} \overline{\lambda}_{\bar{g}(j)t} w_i = a_{ij} \sum_s \overline{\lambda}_{\bar{g}(j)t} \lambda_{s\bar{g}(i)} v'_s.$$

The coefficient of $v'_s$ in the last sum may be non-zero only if $s \leq \bar{g}(i) = g(i)$ and $g(j) = \bar{g}(j) \leq t$, thus it is zero unless $s \leq t$. This shows that

$$F_{ij} = a_{ij} \sum_{s \rho j(i)} \lambda_{s\bar{g}(j)t} E'_{st}.$$

$\square$

Now define $F(A,g,(a_{ij})_{i\rho j}) = \varphi$, where $\varphi : \End(\mathcal{F}) \rightarrow \End(\mathcal{F}')$ is the linear map such that $\varphi(E_{ij}) = F_{ij}$ for any $i\rho j$. Clearly $\varphi$ is an algebra isomorphism.

**Proposition 4.2.** $F$ is surjective.

**Proof.** Let $\varphi : \End(\mathcal{F}) \rightarrow \End(\mathcal{F}')$ be an algebra isomorphism. Denote $F_{ij} = \varphi(E_{ij})$ for any $i,j$ with $i\rho j$. Then $(F_{ij})_{i\rho j}$ is a basis of $\End(\mathcal{F}')$, $F_{ij}F_{jr} = F_{ir}$ for any $i,j,r$ with $i\rho j$ and $j\rho r$, and $(F_{ii})_i$ is a complete set of orthogonal idempotents of $\End(\mathcal{F}')$. 
It is easy to see that \( V' = \bigoplus_{i=1}^{n} Q_i \), where \( Q_i = \text{Im} F_{ii} \neq 0 \), so \( \dim Q_i = 1 \) for any \( i \). Choose some non-zero \( w_i \in Q_i \) for any \( i \). Then \( (w_i)_i \) is a basis of \( V' \). Since \( F_{ij} = \varphi(E_{ij}) \neq 0 \), \( F_{ij}(Q_r) = F_{ij}F_{ri}(V') = 0 \) for any \( r \neq j \), and \( F_{ij}(Q_j) = F_{ii}F_{ij}(Q_j) \subset Q_i \), we see that \( F_{ij}(w_j) = a_{ij}w_i \) for some \( a_{ij} \in k^* \). Since \( F_{ij}F_{ir} = F_{ir} \), we must have \( a_{ij}a_{jr} = a_{ir} \) for any \( i, j, r \) with \( ij \) and \( jpr \). In particular, \( a_{ii} = 1 \) for any \( i \).

Let \( \gamma : V \to V' \) be the linear isomorphism such that \( \gamma(v_i) = w_i \) for any \( i \). Regard \( V \) as a left \( \text{End}(F) \)-module, and \( V' \) as a left \( \text{End}(F') \)-module with the usual action of the endomorphism algebra. If \( ij \) we have

\[
\gamma(E_{ij}v_i) = \gamma(\delta_{jt}v_i) = \delta_{jt}w_i,
\]

and

\[
F_{ij}\gamma(v_i) = F_{ij}w_i = \delta_{jt}a_{ij}w_i
\]

for any \( t \). We get

\[
\varphi(E_{ij})\gamma(v_i) = a_{ij}\gamma(E_{ij}v_i)
\]

for any \( ij \) and any \( t \).

Thus in general \( \gamma \) is not a \( \varphi \)-isomorphism, the obstruction being the scalars \( a_{ij} \). However, \( \gamma \) is a \( \varphi' \)-isomorphism for a deformation \( \varphi' \) of \( \varphi \). Indeed, the linear map \( \theta : \text{End}(F) \to \text{End}(F) \) defined by \( \theta(E_{ij}) = a_{ij}^{-1}E_{ij} \) for any \( ij \), is an algebra automorphism, and \( 1 \) shows that

\[
(2)
\gamma(E_{ij}v_i) = (\varphi\theta)(E_{ij})\gamma(v_i)
\]

for any \( ij \) and any \( t \).

Thus \( \gamma \) is a \( \varphi' \)-isomorphism, where \( \varphi' = \varphi\theta : \text{End}(F) \to \text{End}(F') \) is also an algebra isomorphism. Then the lattice of \( \text{End}(F) \)-submodules of \( V \) is isomorphic to the lattice of \( \text{End}(F') \)-submodules of \( V' \) via the map

\[
\overline{\gamma} : \mathcal{L}_{\text{End}(F)}(V) \to \mathcal{L}_{\text{End}(F')}(V'),
\]

\[
\overline{\gamma}(X) = \gamma(X)
\]

for any \( \text{End}(F) \)-submodule \( X \) of \( V \).

By Remark 3.2, there is an isomorphism of lattices \( \Phi : \mathcal{L}_{\text{End}(F)}(V) \to \mathcal{A}(C) \), given by \( \Phi(X) = D_{\text{max}} \), where \( X = V_C \); its inverse is \( \Phi^{-1}(D) = V_D \) for any \( D \in \mathcal{A}(C) \). Similarly, there is an isomorphism of lattices \( \Phi' : \mathcal{L}_{\text{End}(F')}(V') \to \mathcal{A}(C) \). Let \( f : \mathcal{A}(C) \to \mathcal{A}(C) \) be the isomorphism of lattices such that the diagram

\[
\begin{array}{ccc}
\mathcal{L}_{\text{End}(F)}(V) & \xrightarrow{\overline{\gamma}} & \mathcal{L}_{\text{End}(F')}(V') \\
\Phi \downarrow & & \Phi' \downarrow \\
\mathcal{A}(C) & \xrightarrow{f} & \mathcal{A}(C)
\end{array}
\]

is commutative. By Proposition 3.3, \( f = f_g \) for some automorphism \( g \) of the poset \((C, \leq)\). Then for any \( \alpha \in C \)

\[
\gamma(V_{\alpha}) = \overline{\gamma}(V_{\alpha}) = (\Phi^{-1}f_g\Phi)(V_{\alpha}) = (\Phi'^{-1}f_g)(\{\alpha\}) = \Phi'^{-1}(g(\alpha)) = V_{g(\alpha)}.
\]

This shows that \( \dim V_{\alpha} = \dim V'_{g(\alpha)} = \dim V_{g(\alpha)} \) for any \( \alpha \in C \). Since \( \dim V_{\alpha} = \sum_{\beta \leq \alpha} m_{\beta} \), we see by induction on the length of \( \alpha \) that \( m_\alpha = m_{g(\alpha)} \) for any \( \alpha \in C \), so \( g \in \text{Aut}_0(C) \).
Let $M$ be the matrix of $\gamma$ in the bases $(v_i)_i$ and $(v'_i)_i$. Since $\gamma(v_i) = V'_g(a)$, $M$ may have non-zero entries only on positions $(i, j)$ with $i \leq g(j)$, i.e. on blocks $(a, \beta)$ with $\alpha \leq g(\beta)$. We see that $A = M^{g-1} \in M(\rho, k)$ and $M = A^g$. Moreover, $A$ is invertible and $A^{-1} = g^{-1}(M^{-1})$.

Now we have $\varphi = F(A, g, (a_{ij})_{i\rho j})$, and this ends the proof.

We consider the relation $\approx$ on $U(M(\rho, k)) \times \text{Aut}_0(C) \times T$ defined by $(A, g, (a_{ij})_{i\rho j}) \approx (B, h, (b_{ij})_{i\rho j})$ if and only if $g = h$ and there exist $d_1, \ldots, d_n \in k^*$ such that $a_{ij}b_{ij}^{-1} = d_id_j^{-1}$ for any $i\rho j$, and $B^g = A^g \text{diag}(d_1, \ldots, d_n)$ (or equivalently $AB^{-1} = g^{-1}(D^{-1})g^{-1}$ where $D = \text{diag}(d_1, \ldots, d_n)$); here $\text{diag}(d_1, \ldots, d_n)$ denotes the diagonal matrix with diagonal entries $d_1, \ldots, d_n$. This is clearly an equivalence relation.

**Theorem 4.3.** With the above notation, $F(A, g, (a_{ij})_{i\rho j}) = F(B, h, (b_{ij})_{i\rho j})$ if and only if $(A, g, (a_{ij})_{i\rho j}) \approx (B, h, (b_{ij})_{i\rho j})$. Thus $F$ induces a bijection

$$F : \frac{U(M(\rho, k)) \times \text{Aut}_0(C) \times T}{\approx} \rightarrow \text{Iso}_{alg}(\text{End}(F), \text{End}(F'))$$

**Proof.** Denote $F(A, g, (a_{ij})_{i\rho j}) = \varphi$ and $F(B, h, (b_{ij})_{i\rho j}) = \psi$. Thus $\varphi(E_{ij}) = F_{ij}$, where $F_{ij}(w_j) = a_{ij}w_i$; here $(\ldots, w_i, \ldots) = (\ldots, w'_i, \ldots)A^g$. Also $\psi(E_{ij}) = F'_{ij}$, where $F'_{ij}(w'_j) = b_{ij}w'_i$ and $(\ldots, w'_i, \ldots) = (\ldots, u'_i, \ldots)B^h$. Since $\varphi = \psi$, then $F_{ij} = F'_{ij}$ for any $i\rho j$. Now $\text{Im}F_{ij} = \langle w_j \rangle$ and $\text{Im}F'_{ij} = \langle w'_j \rangle$, so we must have $w'_j = d_jw_j$ for some $d_j \in k^*$, for any $j$. Thus $(\ldots, w'_i, \ldots) = (\ldots, w_i, \ldots)\text{diag}(d_1, \ldots, d_n)$, showing that $B^h = A^g \text{diag}(d_1, \ldots, d_n)$.

The $(\alpha, \beta)$-block of $B^h$ may be non-zero only if $\alpha \leq h(\beta)$, and the $(h(\beta), \beta)$-block of $B^h$ is non-zero (since the $(\beta, \beta)$-block of $B$ is non-zero). Similarly, the $(\alpha, \beta)$-block of $A^g$ may be non-zero only if $\alpha \leq g(\beta)$; the same holds for $A^g \text{diag}(d_1, \ldots, d_n)$. Since $B^h = A^g \text{diag}(d_1, \ldots, d_n)$, the $(h(\beta), \beta)$-block of $A^g$ must be non-zero, so then $h(\beta) \leq g(\beta)$. Similarly, since the $(g(\beta), \beta)$-block of $A^g$ is a non-zero, we obtain that $g(\beta) \leq h(\beta)$. Thus $g = h$.

On the other hand, $F_{ij}(w_j) = F'_{ij}(d_j^{-1}w'_j) = d_j^{-1}b_{ij}w'_i = d_id_j^{-1}b_{ij}w_i$, so $F_{ij} = F'_{ij}$ requires $a_{ij} = d_id_j^{-1}b_{ij}$ for any $i\rho j$. We conclude that $(A, g, (a_{ij})_{i\rho j}) \approx (B, h, (b_{ij})_{i\rho j})$.

Conversely, the above computations show that the $F'_{ij}$’s associated to each of the two triples are the same. \qed

Let us fix some notation. If $G$ and $A$ are groups, a right action of $G$ on $A$ is a mapping $A \times G \rightarrow A$, $(a, g) \mapsto a \cdot g$, such that $(ab) \cdot g = (a \cdot g)(b \cdot g)$ and $(a \cdot g) \cdot h = a \cdot (gh)$ for any $a, b \in A$, $g, h \in G$. This is equivalent to giving a group morphism $\varphi : G \rightarrow \text{Aut}(A)$, where the multiplication of $\text{Aut}(A)$ is the opposite map composition. Indeed, one can take $\varphi(g)(a) = a \cdot g$.

In this case, the right crossed product $G \times A$ is the cartesian product $G \times A$ of sets, with multiplication $(g \times h)(x \times a) = hg \times (b \cdot g)a$ for any $g, h \in G, a, b \in A$; here we denote the pair $(g, a)$ by $g \times a$. The group $\text{Aut}_0(C)$ acts to the right on $T$ by $(a_{ij})_{i\rho j} \cdot g = (a_{i\rho j})_{i\rho j}$, thus we can form a right crossed product $\text{Aut}_0(C) \ltimes T$.

Similarly, a left action of $G$ on $A$ is a mapping $A \times G \rightarrow A$, $(a, g) \mapsto g \cdot a$, such that $g \cdot (ab) = (g \cdot a)(g \cdot b)$ and $g \cdot (h \cdot a) = (gh) \cdot a$; this is the same with giving a group morphism $G \rightarrow \text{Aut}(A)$, where this time the multiplication of $\text{Aut}(A)$ is just the map composition. The left crossed product $A \rtimes G$ is the set $A \times G$ with the multiplication $(a \rtimes g)(b \rtimes h) = a(g \cdot b) \rtimes gh$.

Now if $G \times A$ is a right crossed product, and $B$ is a group, then a left $G \times A$-action on $B$ is a pair consisting of a left $G$-action on $B$ and a left $A$-action on $B$, such that

$$a \cdot (g \cdot x) = g \cdot ((a \cdot g) \cdot x) \quad \text{for any } a \in A, \ g \in G, \ x \in B.$$
The action of $G \rtimes A$ on $B$ is $(g \rtimes a) \cdot x = g \cdot (a \cdot x)$. Indeed, this easily follows from the fact that $A$, respectively $G$, embeds into $G \rtimes A$ by $a \mapsto 1 \rtimes a$, respectively $g \mapsto g \rtimes 1$, and $(1 \rtimes a)(g \rtimes 1) = g \rtimes (a \cdot g) = (g \rtimes 1)(1 \rtimes (a \cdot g))$.

For later use (in Section 4), we note that if $G \rtimes A$ is a right crossed product, and $B$ is just a set, then a right action of $G \rtimes A$ on $B$ is a pair consisting of a right action of $G$ on $B$ and a right action of $A$ on $B$ satisfying the compatibility condition

$$(x \cdot a) \cdot g = (x \cdot g) \cdot (a \cdot g) \quad \text{for any } x \in B, a \in A, g \in G.$$  

Also for later use, we mention that if $A \rtimes G$ is a left semidirect product, and both $A$ and $G$ act to the right on a set $B$, then $x \cdot (a \rtimes g) = (x \cdot a) \cdot g$ defines a right action of $A \rtimes G$ on $B$, provided that

$$(x \cdot g) \cdot a = (x \cdot (g \cdot a)) \cdot g \quad \text{for any } x \in B, a \in A, g \in G.$$  

We use some of these facts in the following particular situation. The group $T$ acts to the left on the group $U(M(\rho,k))$ as follows: if $(a_{ij})_{i\rho j} \in T$ and $A = (\alpha_{ij})_{i\leq i,j \leq n} \in U(M(\rho,k))$, then $(a_{ij})_{i\rho j} \cdot A$ is the matrix whose $(i,j)$-spot is $\alpha_{ij}a_{ij}$ if $i \rho j$, and 0 elsewhere.

We note that $(a_{ij})_{i\rho j} \cdot A \in U(M(\rho,k))$, its inverse being just $(a_{ij})_{i\rho j}^{-1} \cdot A^{-1}$. The group $\text{Aut}_0(C)$ also acts to the left on $U(M(\rho,k))$ by $g \cdot A = g^{-1}A g^{-1}$. The two actions are compatible in the sense of (3), since

$$(a_{ij})_{i\rho j} \cdot g^{-1}A g^{-1} = g^{-1}((a_{g(i)g(j)})_{i\rho j} \cdot A)g^{-1}$$  

for any $(a_{ij})_{i\rho j} \in T$ and $g \in \text{Aut}_0(C)$. Indeed, if $A = (\alpha_{ij})_{i\leq i,j \leq n}$, it is easy to check that both sides in (6) have $a_{ij}(g_{g(i)g(j)})^{-1} \cdot (i,j)$ on the $(i,j)$-spot for any $i \rho j$.

We obtain that $\text{Aut}_0(C) \rtimes T$ acts to the left on $U(M(\rho,k))$ by

$$(g \rtimes (a_{ij})_{i\rho j}) \cdot A = g^{-1}((a_{ij})_{i\rho j} \cdot A)g^{-1},$$  

and we can form the left crossed product $U(M(\rho,k)) \rtimes (\text{Aut}_0(C) \rtimes T)$. Its multiplication is given by

$$(B \rtimes (h \rtimes (b_{ij})_{i\rho j})) \cdot (A \rtimes (g \rtimes (a_{ij})_{i\rho j})) = (B^{h^{-1}}((b_{ij})_{i\rho j} \cdot A)^{-1} \rtimes (h \rtimes (b_{g(i)g(j)}a_{ij})_{i\rho j})).$$

Now if we apply the construction of the map $F$ to the case where $F' = F$, we obtain a surjective map

$$F : U(M(\rho,k)) \rtimes (\text{Aut}_0(C) \rtimes T) \to \text{Aut}(\text{End}(F)).$$

**Theorem 4.4.** $F$ is a morphism of groups, and it induces a group isomorphism

$$U(M(\rho,k)) \rtimes (\text{Aut}_0(C) \rtimes T) \xrightarrow{D} \text{Aut}(\text{End}(F)),$$

where $D = \{\text{diag}(d_1,\ldots,d_n) \rtimes (\text{Id} \rtimes (d_i^{-1}d_j)_{i\rho j}) \mid d_1,\ldots,d_n \in k^*\}$.

**Proof.** Let $\varphi = F(A \rtimes (g \rtimes (a_{ij})_{i\rho j}))$, where $A = (\lambda_{ij})_{i\leq j}$ denote $A^{-1} = (\overline{\lambda}_{ij})_{i \leq j}$. Let also $\psi = F(B \rtimes (h \rtimes (b_{ij})_{i\rho j}))$, with $B = (\mu_{ij})_{i,j}$ and $B^{-1} = (\overline{\mu}_{ij})_{i,j}$. We will show that

$$(8) \quad \psi \varphi = F((B \rtimes (h \rtimes (b_{ij})_{i\rho j})) \cdot (A \rtimes (g \rtimes (a_{ij})_{i\rho j}))$$

and this will prove that $F$ is a group morphism. We know that

$$\varphi(E_{ij}) = a_{ij} \sum_{\iota \preceq (i) \preceq g(j)} \lambda_{\iota g(i)g(j)} \in E_{st},$$
so then
\[
\psi \varphi(E_{ij}) = a_{ij} \sum_{s \rho \tilde{g}(i)} \lambda_{s \tilde{g}(i)} \sum_{\tilde{g}(j) \in \mathbb{D}} \lambda_{\tilde{g}(j)} t \mu_{p \tilde{h}(s)} \pi_{h(t)q} E_{pq}
\]
\[
= a_{ij} \sum_{s \rho \tilde{g}(i)} \sum_{p \rho \tilde{h}(s)} b_{st} \lambda_{s \tilde{g}(i)} \lambda_{\tilde{g}(j)} t \mu_{p \tilde{h}(s)} \pi_{h(t)q} E_{pq}
\]
\[
= a_{ij} \sum_{p \rho \tilde{h}(s)} b_{\tilde{g}(i) \tilde{g}(j)} \sum_{\rho \tilde{h}(s)} b_{s \tilde{g}(i) \tilde{g}(j)} b_{\tilde{g}(j)} t \lambda_{s \tilde{g}(i)} \lambda_{\tilde{g}(j)} t \mu_{p \tilde{h}(s)} \pi_{h(t)q} E_{pq}
\]
\[
= a_{ij} b_{\tilde{g}(i) \tilde{g}(j)} \sum_{p \rho \tilde{h}(s)} b_{s \tilde{g}(i) \tilde{g}(j)} b_{\tilde{g}(j)} t \lambda_{s \tilde{g}(i)} \lambda_{\tilde{g}(j)} t \mu_{p \tilde{h}(s)} \pi_{h(t)q} E_{pq}.
\]

But the first bracket in the last row above is the product of the \(p\)-th row in \(B^h\) and the \(i\)-th column in \(((b_{ij})_{ipj} \cdot A)^g\), thus it is the \((p, i)\)-position in the matrix \(B^h((b_{ij})_{ipj} \cdot A)^g\), or equivalently, the \((p, \tilde{h}g(i))-position in

\[
(B^h((b_{ij})_{ipj} \cdot A)^g)_{pq} = B^h((b_{ij})_{ipj} \cdot A)^{h^{-1}} = B^{h^{-1}}((b_{ij})_{ipj} \cdot A)^{h^{-1}}.
\]

Denote \(B^{h^{-1}}((b_{ij})_{ipj} \cdot A)^{h^{-1}} = (\nu_{ij})_{i,j}\), and the inverse of this matrix by \((\varphi_{ij})_{i,j}\). A similar argument shows that the second bracket above is just \(\varphi_{ij}\).

\[
\psi \varphi(E_{ij}) = a_{ij} b_{\tilde{g}(i) \tilde{g}(j)} \sum_{p \rho \tilde{h}(s)} \nu_{p \tilde{h}(s)} \varphi_{\tilde{g}(j)} t \pi_{h(t)q} E_{pq},
\]

showing that \(\psi \varphi = F(B^{h^{-1}}((b_{ij})_{ipj} \cdot A)^{h^{-1}} \times (h g \times (b_{\tilde{g}(i) \tilde{g}(j)} a_{ij})))\), which using (7) is just (5).

Now \(z = A \rtimes (g \times (a_{ij} \cdot ipj))\) is in the kernel of \(F\) if and only if it is equivalent (via \(\approx\)) to the identity element. In view of Theorem 4.3 this is the same with \(z \in D\).

Now we explain how the description of the automorphism group of \(M(\rho, k)\) given in [5] can be deduced from Theorem 4.4 We recall a few basic things about semidirect products. We first note that if a group \(G\) acts to the right on a group \(A\), with action denoted by \(a \cdot g\) for \(a \in A\) and \(g \in G\), then there is also a left action of \(G\) on \(A\), defined by \(g \cdot a = a \cdot g^{-1}\), and the associated right and left semidirect products are isomorphic. Indeed, \(\phi : A \rtimes G \to G \rtimes A\), \(\phi(a \cdot g) = g \rtimes (g^{-1} \cdot a)\), is a group isomorphism, with inverse \(\phi^{-1}(g \rtimes a) = a \cdot g^{-1} \rtimes g\). Secondly, if we have a double semidirect product \(B \rtimes (A \rtimes G)\), then the left action of \(A \rtimes G\) on \(B\) induces (via the usual embeddings of \(A\) and \(G\)) left actions of \(A\) and \(G\) on \(B\). Moreover, \(G\) acts to the left on the group \(B \rtimes A\) by \(g \cdot (x \rtimes a) = g \cdot x \rtimes g \cdot a\) for any \(g \in G, x \in B, a \in A\), and the map \(\gamma : B \rtimes (A \rtimes G) \to (B \rtimes A) \rtimes G, \gamma(x \rtimes (a \cdot g)) = (x \rtimes a) \rtimes g\), is an isomorphism of groups. Finally, if \(A \rtimes G\) is a left semidirect product, and \(N\) is a normal subgroup of \(A\) which is invariant to the action of \(G\), then \(N \rtimes 1\) is a normal subgroup of \(A \rtimes G\) and \(\frac{A \rtimes G}{N \rtimes G} \approx \frac{A}{N} \rtimes G\), where the action of \(G\) on \(\frac{A}{N}\) is the one induced by the action of \(G\) on \(A\).

Using these remarks, we see that
\[
U(M(\rho, k)) \rtimes (\text{Aut}_0(C) \rtimes T) \cong U(M(\rho, k)) \rtimes (T \rtimes \text{Aut}_0(C))
\]
\[
\cong (U(M(\rho, k)) \rtimes T) \rtimes \text{Aut}_0(C)
\]
and in the third double semidirect product the action of $\text{Aut}_0(\mathcal{C})$ on $U(M(\rho, k)) \rtimes T$ is given by
\[
g \cdot (A \rtimes (a_{ij})_{ipj}) = g^{-1}A^g \rtimes (a_{ij})_{ipj} \cdot g^{-1}.
\]
The image of $D$ through these isomorphisms in $(U(M(\rho, k)) \rtimes T) \rtimes \text{Aut}_0(\mathcal{C})$ is $D_0 \rtimes 1$, where
\[
D_0 = \{\text{diag}(d_1, \ldots, d_n) \rtimes (d_i^{-1}d_j)_{ipj} \mid d_1, \ldots, d_n \in k^*\}.
\]
Clearly $D_0$ is a normal subgroup in $U(M(\rho, k)) \rtimes T$, since $D$ is a normal subgroup of $U(M(\rho, k)) \rtimes (\text{Aut}_0(\mathcal{C}) \rtimes T)$, and it is easy to check that $D_0$ is invariant under the action of $\text{Aut}_0(\mathcal{C})$. Thus we obtain that
\[
\frac{U(M(\rho, k)) \rtimes (\text{Aut}_0(\mathcal{C}) \rtimes T)}{D_0} \simeq \frac{(U(M(\rho, k)) \rtimes T) \rtimes \text{Aut}_0(\mathcal{C})}{D_0} \rtimes \text{Aut}_0(\mathcal{C}).
\]

Following [5], we consider an undirected graph $\Delta$, whose vertices are all elements $i \in \{1, \ldots, n\}$ such that the equivalence class $\hat{i}$ is not an isolated point in the poset $(\mathcal{C}, \leq)$; we say that $\alpha \in \mathcal{C}$ is an isolated point if any of $\alpha \leq \beta$ and $\beta \leq \alpha$ implies that $\beta = \alpha$. If $i$ and $j$ are vertices of $\Delta$, there is an edge connecting $i$ and $j$ if and only if $\hat{i}$ and $\hat{j}$ are not equal, but they are in relation $\leq$ (in any possible way). If $\Delta_1, \ldots, \Delta_z$ are the connected components of $\Delta$, choose a tree $T_\ell$ in each $\Delta_\ell$ (note that there are several possible such choices). Now let $\mathcal{G}$ be the subgroup of $T$ consisting of all $(a_{ij})_{ipj}$’s for which $a_{ij} = 1$ whenever $ipj$ and $i, j$ are vertices of $\Delta$, joined by an edge of some $T_\ell$, or when both $i$ and $j$ lie in an isolated equivalence class in $\mathcal{C}$.

Also, let $\mathcal{I}$ be the group of inner automorphisms of $M(\rho, k)$, i.e. $\mathcal{I} = \{C_{A} \mid A \in U(M(\rho, k))\}$, where $C_{A}(X) = AXA^{-1}$ for any $X \in M(\rho, k)$. Then there is a left action of $\mathcal{G}$ on $\mathcal{I}$ given by $(a_{ij})_{ipj} \cdot C_{A} = C_{(a_{ij})_{ipj} \cdot A}$. We prove that
\[
(9) \quad \frac{U(M(\rho, k)) \rtimes T}{D_0} \simeq \mathcal{I} \times \mathcal{G}
\]
and this will show that
\[
\text{Aut}(M(\rho, k)) \simeq \frac{U(M(\rho, k)) \rtimes (\text{Aut}_0(\mathcal{C}) \rtimes T)}{D_0} \simeq (\mathcal{I} \times \mathcal{G}) \rtimes \text{Aut}_0(\mathcal{C}),
\]
thus recovering the description of the automorphism group of the structural matrix algebra given in [5].

For proving (9), define
\[
\Psi : \mathcal{I} \times \mathcal{G} \to \frac{U(M(\rho, k)) \rtimes T}{D_0}, \quad \Psi(C_{A} \rtimes (a_{ij})_{ipj}) = A \rtimes (a_{ij})_{ipj}
\]
where $\overline{g}$ denotes the class of $y$ modulo $D_0$. We note that $\Psi$ is well defined since $C_{A} = C_{B}$ if and only if $A^{-1}B$ lies in the center of $M(\rho, k)$, and this is the set of diagonal matrices constant on all $i$th diagonal positions with $\hat{i}$ in the same connected component of $\mathcal{C}$. Then $B = A \text{diag}(d_1, \ldots, d_n)$ for such a central diagonal matrix $\text{diag}(d_1, \ldots, d_n)$. Since $d_i^{-1}d_j = 1$ for any $ipj$, we have that $\text{diag}(d_1, \ldots, d_n) \rtimes (1)_{ipj} \in D_0$. But $B \rtimes (a_{ij})_{ipj} = (\text{diag}(d_1, \ldots, d_n) \rtimes (1)_{ipj})(A \rtimes (a_{ij})_{ipj})$, so $B \rtimes (a_{ij})_{ipj} = A \rtimes (a_{ij})_{ipj}$.

We show that $\Psi$ is injective. Indeed, if $\Psi(C_{A} \rtimes (a_{ij})_{ipj})$ is trivial, then $A = \text{diag}(d_1, \ldots, d_n)$ and $a_{ij} = d_i^{-1}d_j$ for any $ipj$, where $d_1, \ldots, d_n$ are some non-zero scalars. Since $a_{ij} = 1$ for any $i, j$ joined by an edge of some $T_\ell$, and $T_\ell$ is a tree, we see that $d_i$ must be the same when $i$ runs through the vertices of a fixed $\Delta_\ell$. Also, $a_{ij} = 1$ if $i, j$ in the same isolated equivalence class $\alpha \in \mathcal{C}$, so $d_i$ is constant for $i$ in such a class. We conclude that $A$ is a central element, so $C_{A}$ is the identity, and all $a_{ij}$’s with $ipj$ are equal to 1.
To prove that $\Psi$ is surjective, it is enough to show that for any $A \times (a_{ij})_{i\rho j} \in U(M(\rho, k)) \times T$ there exists $B \times (b_{ij})_{i\rho j} \in \mathcal{I} \times G$ with $A \times (a_{ij})_{i\rho j} = B \times (b_{ij})_{i\rho j}$. We first show that there are $d_1, \ldots, d_n \in k^*$ such that $a_{ij} = d_id_j^{-1}$ for any $i,j$ joined by an edge of some $T_\ell$, and also for any $i,j$ in the same isolated equivalence class. Indeed, for the tree $T_\ell$ we can fix some vertex $i_0$, and then define $d_i$ for any other vertex in $T_\ell$ by induction on the distance from $i_0$ to $i$ in the tree $T_\ell$, using at each step the desired condition $a_{ij} = d_id_j^{-1}$. For an isolated equivalence class, say $\{i_1, \ldots, i_p\}$, set $d_{i_\rho} = 1$, and $a_{i_1,i_p} = d_{i_1}, \ldots, a_{i_{p-1},i_p} = d_{i_{p-1}}$. Then $a_{i_\rho,i_s} = a_{i_\rho,i_p}a_{i_p,i_s} = a_{i_\rho,i_p}a_{i_s,i_p} = d_{i_\rho}d_{i_s}^{-1}$. Now

$$(\text{diag}(d_1, \ldots, d_n) \times (d_i^{-1}d_j)_{i\rho j})(A \times (a_{ij})_{i\rho j}) = \text{diag}(d_1, \ldots, d_n)(d_i^{-1}d_j)_{i\rho j} \cdot A \times (d_i^{-1}d_ja_{ij})_{i\rho j}$$

and $(d_i^{-1}d_ja_{ij})_{i\rho j} \in G$, so we are done.

5. Graded flags and associated gradings of matrices

Let $G$ be a group. A $G$-graded vector space is a vector space $V$ with a decomposition $V = \bigoplus_{g \in G} V_g$, where each $V_g$ is a subspace. The elements of $\bigcup_{g \in G} V_g$ are called the homogeneous elements of $V$. Each $v \in V$ is uniquely written as $v = \sum_{g \in G} v_g$, $v_g \in V_g$.

**Definition 5.1.** A $G$-graded $\rho$-flag is a $\rho$-flag $(V, (V\alpha)_{\alpha \in \mathcal{C}})$ such that $V$ is a $G$-graded vector space, and the basis $B$ from Definition 2.1 consists of homogeneous elements.

If $F = (V, (V\alpha)_{\alpha \in \mathcal{C}})$ and $F' = (V', (V'\alpha)_{\alpha \in \mathcal{C}})$ are $G$-graded $\rho$-flags, then a morphism of graded flags from $F$ to $F'$ is a morphism $f : V \to V'$ of $\rho$-flags, which is also a morphism of graded vector spaces.

Note that in a graded flag any $V\alpha$ is a graded vector subspace, as it has a basis of homogeneous elements.

If $F = (V, (V\alpha)_{\alpha \in \mathcal{C}})$ is a $G$-graded $\rho$-flag and $\sigma \in G$, define

$$\text{End}(F)_\sigma = \{f \in \text{End}(F) \mid f(V\alpha) \subseteq V_{\sigma\alpha} \text{ for any } g \in G\}.$$ 

**Proposition 5.2.** $\text{End}(F) = \bigoplus_{\sigma \in G} \text{End}(F)_\sigma$, and this decomposition makes $\text{End}(F)$ a $G$-graded algebra.

**Proof.** It is clear that $\text{End}(F)_\sigma$ can be non-zero only for $\sigma \in (\text{supp } V) \cdot (\text{supp } V)^{-1}$, where $\text{supp } V = \{g \in G \mid V_g \neq 0\}$. Thus only finitely many $\text{End}(F)_\sigma$ are nonzero.

In order to see that $\sum_{\sigma \in G} \text{End}(F)_\sigma$ is a direct sum, choose some $f^\sigma \in \text{End}(F)_\sigma$ for each $\sigma \in G$.

If $\sum_{\sigma \in G} f^\sigma = 0$, then $0 = (\sum_{\sigma \in G} f^\sigma)(V_g) = \sum_{\sigma \in G} f^\sigma(V_g)$ for any $g \in G$. Since $f^\sigma(V_g) \subseteq V_{\sigma\alpha}$, this shows that $f^\sigma(V_g) = 0$ for any $\sigma$ and any $g$, and we get that $f^\sigma = 0$ for any $\sigma$.

Now let $f \in \text{End}(F)$. For any $\sigma \in G$ define the linear maps $f_\sigma : V \to V$ such that $(f_\sigma)(v_g) = f(v)_{\sigma\alpha}$ for any $g \in G$, $v_g \in V_g$. Then

$$f_\sigma(V\alpha) = \sum_{g \in G} f_\sigma((V\alpha)_g) \subseteq \sum_{g \in G} f(V\alpha)_{\sigma\alpha} \subseteq V\alpha$$

for any $\alpha \in \mathcal{C}$, so $f_\sigma \in \text{End}(F)_\sigma$. Moreover, it is clear that $f = \sum_{\sigma \in G} f_\sigma$, so $\text{End}(F) = \bigoplus_{\sigma \in G} \text{End}(F)_\sigma$. 

Obviously, \( \text{End}(\mathcal{F})_\sigma \text{End}(\mathcal{F})_\tau \subset \text{End}(\mathcal{F})_{\sigma\tau} \) for any \( \sigma, \tau \in G \), so \( \text{End}(\mathcal{F}) \) is a \( G \)-graded algebra. 

We will denote by \( \text{END}(\mathcal{F}) \) the algebra \( \text{End}(\mathcal{F}) \), regarded with the \( G \)-grading defined in Proposition 5.2. This grading transfers via the isomorphism defined in the proof of Proposition 2.2 to a \( G \)-grading on the structural matrix algebra \( \text{M}(\rho, k) \). If \( g_1, \ldots, g_n \) are the degrees of the basis elements \( v_1, \ldots, v_n \), then each matrix unit \( e_{ij} \) with \( i \rho j \) is a homogeneous element of degree \( g_i g_j^{-1} \) in this grading.

**Definition 5.3.** A \( G \)-grading on the algebra \( \text{M}(\rho, k) \) is called a good grading if \( e_{ij} \) is a homogeneous element for any \( i, j \) with \( i \rho j \).

Gradings on \( \text{M}(\rho, k) \) arising from graded flags are good gradings. We note that giving a good \( G \)-grading on \( \text{M}(\rho, k) \) is equivalent to giving a family \( (u_{ij})_{i \rho j} \) of elements of \( G \) such that \( u_{ij} u_{jr} = u_{ir} \) for any \( i, j, r \) with \( i \rho j \) and \( j \rho r \). If we regard this family as a function \( u : \rho \to G \), defined by \( u(i, j) = u_{ij} \) for any \( i, j \) with \( i \rho j \), then \( u \) is just a transitive function on \( \rho \) with values in \( G \), in the terminology of [8].

Examples of a transitive functions on \( \rho \) can be obtained as follows. Let \( g_1, \ldots, g_n \in G \), and let \( u_{ij} = g_i g_j^{-1} \) for any \( i, j \) with \( i \rho j \). Then \( (u_{ij})_{i \rho j} \) is a transitive function on \( \rho \). A transitive function on \( \rho \) is called trivial if it is obtained in this way. Clearly, the good \( G \)-gradings corresponding to trivial transitive functions on \( \rho \) are precisely the gradings obtained from graded flags as above. It is an interesting question whether all good gradings arise from graded flags, or equivalently

**Question.** Let \( \rho \) be a preorder relation. Is it true that for any group \( G \) all transitive functions \( u : \rho \to G \) are trivial?

Several variations of the above questions can be formulated, for example to determine for a fixed \( \rho \) all groups \( G \) such that any transitive function on \( \rho \) with values in \( G \) is trivial.

The following shows that the problem posed in the question above reduces to answering it for the associated poset \( (\mathcal{C}, \preceq) \). We also give an equivalent formulation involving the associated graph \( \Gamma \). The equivalence between (1) and (2) in the next Proposition was proved in [8].

**Proposition 5.4.** Let \( G \) be a group. The following are equivalent:

1. Any transitive function \( u : \rho \to G \) is trivial.
2. Any transitive function \( w : \leq \to G \) is trivial, where \( \leq \) is the partial order on \( \mathcal{C} \).
3. For any function \( v : \Gamma_1 \to G \) such that \( v(a_1) \cdots v(a_r) = v(b_1) \cdots v(b_s) \) for any paths \( a_1 \cdots a_r \) and \( b_1 \cdots b_s \) in \( \Gamma \) with \( s(a_1) = s(b_1) \) and \( t(a_r) = t(b_s) \), there exists a function \( f : \Gamma_0 \to G \) such that \( v(a) = f(s(a)) f(t(a))^{-1} \) for any \( a \in \Gamma_1 \).

**Proof.** Denote \( \mathcal{C} = \{\alpha_1, \ldots, \alpha_h\} \) and pick \( i_1 \in \alpha_1, \ldots, i_h \in \alpha_h \).

1)\(\Rightarrow\)2) Let \( w : \leq \to G \) be a transitive function. Then \( u : \rho \to G \) defined by \( u(i, j) = w(i, j) \) for any \( i, j \) with \( i \rho j \), is a transitive function, so there exists a function \( f : \{1, \ldots, n\} \to G \) such that \( u(i, j) = f(i) f(j)^{-1} \) for any \( i, j \) with \( i \rho j \). Now define \( \overline{f} : \mathcal{C} \to G, \overline{f}(\alpha_q) = f(i_q) \) for any \( 1 \leq q \leq h \). Then if \( \alpha_q \leq \alpha_p \), we have \( w(\alpha_q, \alpha_p) = u(i_q, i_p) = f(i_q) f(i_p)^{-1} = \overline{f}(\alpha_q) \overline{f}(\alpha_p)^{-1} \), so \( w \) is trivial.

2)\(\Rightarrow\)3) Let \( v : \Gamma_1 \to G \) be as in (3). Define \( w : \leq \to G \) as follows. If \( \alpha < \beta \), let \( a_1 \cdots a_r \) be a path in \( \Gamma \) starting at \( \alpha \) and ending at \( \beta \); we define \( w(\alpha, \beta) = v(a_1) \cdots v(a_r) \), and we note that the definition does not depend on the path, taking into account the property satisfied by \( v \). We also define \( w(\alpha, \alpha) = e \), the neutral element of \( G \), for any \( \alpha \in \mathcal{C} \). Then \( w \) is a transitive
function, so there exists \( f : C \to G \) such that \( w(\alpha, \beta) = f(\alpha)f(\beta)^{-1} \) for any \( \alpha \leq \beta \). In particular \( v(a) = f(s(a))f(t(a))^{-1} \) for any \( a \in \Gamma_1 \).

(3)⇒(1) Let \( u : \rho \to G \) be a transitive function. Define \( v : \Gamma_1 \to G \) as follows: if \( a \in \Gamma_1 \) with \( s(a) = \alpha_p, t(a) = \alpha_q \), then \( v(a) = u(i_p, i_q) \). If \( a_1, \ldots, a_r \) and \( b_1, \ldots, b_s \) are paths in \( \Gamma \) with \( s(a_1) = s(b_1) = \alpha_p \) and \( t(a_r) = t(b_s) = \alpha_q \), it is clear that \( v(a_1) \ldots v(a_r) = v(b_1) \ldots v(b_s) = u(i_p, i_q) \).

Then there exist \( z_1, \ldots, z_n \in G \) such that \( v(a) = z_p z_q^{-1} \) for any arrow \( a \) with \( s(a) = \alpha_p, t(a) = \alpha_q \), \( 1 \leq p, q \leq h \).

Now define the family \( (g_i)_{1 \leq i \leq n} \) of elements of \( G \) by \( g_i = u(i, i_p)z_p \) for any \( i \), where \( p \) is such that \( i \in \alpha_p \) (note that \( u(i, i_p) \) makes sense since \( i, i_p \in \alpha_p \)). Then if \( i \) and \( j \) are such that \( ipj \), let \( i \in \alpha_p, j \in \alpha_q \). We know that \( g_i = u(i, i_p)z_p, g_j = u(j, i_q)z_q \) and \( u(i, i_q) = z_p z_q^{-1} \). Then

\[
\begin{align*}
    u(i, j) &= u(i, i_p)u(i_p, i_q)u(i_q, j) \\
    &= g_i z_p^{-1} z_q^{-1} u(j, i_q) \\
    &= g_i z_q^{-1} z_q g_j^{-1} \\
    &= g_i g_j^{-1}.
\end{align*}
\]

\( \square \)

The next result gives an easy way to check whether a function \( v \) as in Proposition 5.3 (3) arises from a function \( f : \Gamma_0 \to G \), by looking at the cycles of the undirected graph associated with \( \Gamma \). We consider the graph \( \tilde{\Gamma} \), constructed from \( \Gamma \) by ‘doubling the arrows’. It has the same vertices as \( \Gamma \) thus \( \tilde{\Gamma}_0 = \Gamma_0 \). For any arrow \( a \in \Gamma_1 \) from \( \alpha \) to \( \beta \), we consider an arrow \( \tilde{a} \) from \( \beta \) to \( \alpha \), and define \( \tilde{\Gamma}_1 = \Gamma_1 \cup \{ \tilde{a} | a \in \Gamma_1 \} \). If \( G \) is a group and \( v : \Gamma_1 \to G \) is a function, we denote by \( \tilde{v} : \tilde{\Gamma}_1 \to G \) the function whose restriction to \( \Gamma_1 \) is \( v \), and such that \( \tilde{v}(\tilde{a}) = v(a)^{-1} \) for any \( a \in \Gamma_1 \).

**Proposition 5.5.** Let \( G \) be a group, and let \( v : \Gamma_1 \to G \) such that \( v(a_1) \ldots v(a_r) = v(b_1) \ldots v(b_s) \) for any paths \( a_1, \ldots, a_r \) and \( b_1, \ldots, b_s \) in \( \Gamma \) with \( s(a_1) = s(b_1) \) and \( t(a_r) = t(b_s) \). Then the following are equivalent.

1. There exists a function \( f : \Gamma_0 \to G \) such that \( v(a) = f(s(a))f(t(a))^{-1} \) for any \( a \in \Gamma_1 \).
2. For any cycle \( z_1, \ldots, z_m \) in \( \tilde{\Gamma} \), with \( z_1, \ldots, z_m \in \tilde{\Gamma}_1 \) (this corresponds to a cycle in the undirected graph \( \Gamma^u \) associated with \( \Gamma \)) one has \( \tilde{v}(z_1) \ldots \tilde{v}(z_m) = 1 \).

**Proof.** (1)⇒(2) For any \( a \in \Gamma_1 \) starting from \( \alpha \) and ending at \( \beta \), we have \( \tilde{v}(a) = v(a) = f(\alpha)f(\beta)^{-1} \) and \( \tilde{v}(\tilde{a}) = f(\beta)f(\alpha)^{-1} \). Thus \( \tilde{v}(z) = f(s(z))f(t(z))^{-1} \) for any \( z \in \tilde{\Gamma}_1 \). Now it is clear that \( \tilde{v}(z_1) \ldots \tilde{v}(z_m) = 1 \) for any cycle \( z_1 \ldots z_m \) in \( \tilde{\Gamma} \).

(2)⇒(1) We construct a function \( f : \Gamma_0 \to G \) satisfying the desired property. It is clear that we can reduce to the case where \( \Gamma \) is connected (and putting together the functions constructed for the connected components). Choose some \( \omega \in \Gamma_0 \) and take \( f(\omega) \) to be an arbitrary element of \( G \). If \( \alpha \in \Gamma_0 \), let \( z_1, \ldots, z_d \) be a path from \( \alpha \) to \( \omega \) in \( \tilde{\Gamma} \). Define \( f(\alpha) = \tilde{v}(z_1) \ldots \tilde{v}(z_d) f(\omega) \); this does not depend on the path \( z_1, \ldots, z_d \), since for another path \( y_1, \ldots, y_h \) from \( \alpha \) to \( \omega \), \( z_1, \ldots, z_d y_h^{-1} \ldots y_1^{-1} \) is a cycle in \( \tilde{\Gamma} \), where \( y^{-1} \) denotes the arrow opposite to \( y \) (thus \( a^{-1} = \tilde{a} \), and \( \tilde{a}^{-1} = a \) for any \( a \in \Gamma_1 \)). It is clear that \( \tilde{v}(y^{-1}) = \tilde{v}(y)^{-1} \) for any \( y \in \tilde{\Gamma}_1 \). Then \( \tilde{v}(z_1) \ldots \tilde{v}(z_d) \tilde{v}(y_1^{-1}) \ldots \tilde{v}(y_h^{-1}) = 1 \), so \( \tilde{v}(z_1) \ldots \tilde{v}(z_d) = \tilde{v}(y_1) \ldots \tilde{v}(y_h) \). Now if \( a \in \Gamma_1 \) is an arrow from \( \alpha \) to \( \beta \), let \( z_1, \ldots, z_d \) be a path from \( \beta \) to \( \omega \) in \( \tilde{\Gamma} \). Then \( f(\beta) = \tilde{v}(z_1) \ldots \tilde{v}(z_d) f(\omega) \), and \( f(\alpha) = \tilde{v}(a) \tilde{v}(z_1) \ldots \tilde{v}(z_d) f(\omega) = \tilde{v}(a) f(\beta) \) for any \( a \in \Gamma_1 \).

\( \square \)

Let \( F(\Gamma) \) be the free group generated by the set \( \Gamma_1 \) of arrows of \( \Gamma \). Let \( A(\Gamma) \) be the subgroup of \( F(\Gamma) \) generated by all elements of the form \( a_1 \ldots a_r b_p^{-1} \ldots b_{p-1}^{-1} \), where \( a_1 \ldots a_r \) and \( b_1 \ldots b_p \) are
two paths (in $\Gamma$) starting from the same vertex and terminating at the same vertex. We also consider the subgroup $B(\Gamma)$ of $F(\Gamma)$ generated by all elements of the form $a_1a_2^{\varepsilon_2}\ldots a_m^{\varepsilon_m}$, where $a_1,\ldots,a_m$ are arrows forming in this order a cycle in the undirected graph obtained from $\Gamma$ when omitting the direction of arrows, and $\varepsilon_i = 1$ if $a_i$ is in the direction of the directed cycle given by $a_1$, and $\varepsilon_i = -1$ otherwise. Clearly $A(\Gamma) \subseteq B(\Gamma)$, since any generator of $A(\Gamma)$ lies in $B(\Gamma)$. Now we can give an answer to the question posed above in terms of these groups. We recall that for a group $X$ and a subgroup $Y$ of $X$, the normal closure $Y^N$ of $Y$ is the smallest normal subgroup of $X$ containing $Y$. The elements of $Y^N$ are all products of conjugates of elements of $Y$.

**Proposition 5.6.** With notation as above, the following are equivalent.

1. For any group $G$, any transitive function $u : \rho \to G$ is trivial.
2. $A(\Gamma)^N = B(\Gamma)^N$.
3. Any generator $b$ of $B(\Gamma)$ can be written in the form $b = g_1x_1g_1^{-1}\ldots g_mx_mg_m^{-1}$ for some positive integer $m$, some $g_1,\ldots,g_m \in F(\Gamma)$ and some $x_1,\ldots,x_m$ among the generators in the construction of $A(\Gamma)$.

**Proof.** By Propositions 5.4 and 5.5 (1) is equivalent to the fact that for any group $G$ and any group morphism $f : F(\Gamma) \to G$ such that $f(A(\Gamma)) = 1$, we also have $f(B(\Gamma)) = 1$. Indeed, giving a function $v : \Gamma_1 \to G$ is the same with giving a group morphism $f : F(\Gamma) \to G$; moreover, $v$ satisfies $v(a_1)\ldots v(a_r) = v(b_1)\ldots v(b_s)$ for any paths $a_1\ldots a_r$ and $b_1\ldots b_s$ in $\Gamma$ with $s(a_1) = s(b_1)$ and $t(a_r) = t(b_s)$, if and only if $f(A(\Gamma)) = 1$. On the other hand, such a $v$ is trivial if and only if $f(B(\Gamma)) = 1$.

Now if (1) holds, then the projection $F(\Gamma) \to F(\Gamma)/A(\Gamma)^N$ is trivial on $A(\Gamma)$, so it must be trivial on $B(\Gamma)$. This shows that $B(\Gamma) \subseteq A(\Gamma)^N$, or $A(\Gamma)^N = B(\Gamma)^N$. Conversely, if (2) holds, then any group morphism $f : F(\Gamma) \to G$ which is trivial on $A(\Gamma)$ is also trivial on $A(\Gamma)^N$, and then also on $B(\Gamma)$, so (1) holds.

The equivalence between (2) and (3) follows from the description of the normal closure of a subgroup, and the fact that the given set of generators of $A(\Gamma)$ is closed under inverse. □

**Example 5.7.** Assume that $\rho$ is a preorder relation such that the associated graph $\Gamma$ is of the form

for some integers $m \geq 3$ and $p \geq 1$. Thus there are two paths from 1 to $m$, these are $a_1\ldots a_{m-1}$ and $b_1\ldots b_{p+1}$. Then for any group $G$, any transitive function $u : \rho \to G$ is trivial. Indeed, if $v$ is a function as in Proposition 5.5, then $v(a_1)\ldots v(a_{m-1}) = v(b_1)\ldots v(b_{p+1})$, and then the condition in (2) of the mentioned Proposition is obviously satisfied, as it is clear that the associated undirected graph has just one cycle.
Alternatively, \( A(\Gamma) \) is the cyclic group generated by \( x = a_1 \ldots a_{m-1} b_{p+1} \ldots b_1^{-1} \), while the generators of \( B(\Gamma) \) are all conjugates of \( x \) or \( x^{-1} \). For example, \( a_2 \ldots a_{m-1} b_{p+1} \ldots b_1^{-1} a_1 = a_1^{-1} x a_1 \). Thus \( A(\Gamma)^N = B(\Gamma)^N \).

**Example 5.8.** Assume that \( \rho \) is a preorder relation such that the associated graph \( \Gamma \) is of the form

Thus the un-directed graph \( \Gamma^u \) associated to \( \Gamma \) is cyclic, and in \( \Gamma \) there are at least two vertices where both adjacent arrows terminate (equivalently, \( \Gamma^u \) is cyclic and \( \Gamma \) is not of the type in Example [5.7]). Then for any non-trivial group \( G \), there exist transitive functions \( u : \rho \rightarrow G \) that are not trivial. Indeed, it is enough to show that the condition (3) in Proposition 5.4 does not hold. Since \( \Gamma \) does not have two different paths starting from the same vertex and ending at the same vertex, we only have to see that not any function \( v : \Gamma_1 \rightarrow G \) is given by \( v(a) = f(s(a)) f(t(a))^{-1}, \ a \in \Gamma_1 \), for some function \( f : \Gamma_0 \rightarrow G \). This is clear, for instance we can take \( v \) to be the identity of \( G \) on all but one arrows.

We note that \( A(\Gamma) \) is trivial, since there are no two different paths starting from the same vertex and terminating at the same vertex. On the other hand, \( B(\Gamma) \) is not trivial, since \( \Gamma^u \) has a cycle.

The simplest example of such a graph is

and the corresponding structural matrix algebra, whose not all good gradings arise from graded flags, is

\[
\begin{pmatrix}
  k & 0 & k & k \\
  0 & k & k & k \\
  0 & 0 & k & 0 \\
  0 & 0 & 0 & k \\
\end{pmatrix}.
\]

**Example 5.9.** If we construct a graph \( \Delta \) by taking a graph \( \Gamma \) as in Example 5.8, adding a vertex \( v \), and adding an arrow from each vertex in \( \Gamma \) where both adjacent arrows terminate to \( v \), as in the picture below
then all transitive functions (on the corresponding preordered set) are trivial. For simplicity, we explain this in the case where $\Delta$ is

but the argument is the same in general. In this case, $A(\Delta)$ is generated by $g = axy^{-1}b^{-1}$ and $h = cxy^{-1}d^{-1}$, while $B(\Delta)$ is generated by certain conjugates of $g$ and $h$ (as explained in Example 5.8), $ac^{-1}db^{-1}$, and certain conjugates of the latter. But $ac^{-1}db^{-1} = g(db^{-1})^{-1}h^{-1}(db^{-1})$, showing that $A(\Delta)^N = B(\Delta)^N$.

6. Isomorphisms between graded endomorphism algebras

If $V = \bigoplus_{g \in G} V_g$ is a $G$-graded vector space, then for any $\sigma \in G$ the right $\sigma$-suspension of $V$ is the $G$-graded vector space $V(\sigma)$ which is just $V$ as a vector space, with the grading shifted by $\sigma$ as follows: $V(\sigma)_g = V_{g\sigma}$ for any $g \in G$. Also, the left $\sigma$-suspension of $V$, denoted by $(\sigma)V$, is the vector space $V$ with the grading given by $((\sigma)V)_g = V_{g\sigma}$. For any $\sigma, \tau \in G$ one has $(V(\sigma))(\tau) = V(\tau\sigma)$, $(\sigma)((\tau)V) = (\tau\sigma)V$ and $((\sigma)V)(\tau) = (\sigma)(V(\tau))$. The fact that there are two types of suspensions for a graded vector space $V$ can be explained by the fact that $V$ is an object in the category of left graded modules, and also an object in the category of right graded modules over the algebra $k$, regarded with the trivial $G$-grading. Then $V(\sigma)$ and $(\sigma)V$ are just the suspensions when $V$ is regarded in these categories.

If $V$ and $W$ are $G$-graded vector spaces and $\sigma \in G$, we say that a linear map $f : V \to W$ is a morphism of left degree $\sigma$ if $f(V_g) \subseteq W_{\sigma g}$ for any $g \in G$; this means that $f$ is a morphism of graded vector spaces when regarded as $f : V \to (\sigma)W$. Similarly, $f$ is a morphism of right degree $\sigma$ if $f(V_g) \subseteq W_{g\sigma}$ for any $g \in G$.

If $F = (V, (V_\alpha)_{\alpha \in \mathcal{C}})$ is a $G$-graded $\rho$-flag and $\sigma \in G$, then the right suspension of $F$ is $F(\sigma) = (V(\sigma), (V_\alpha)_{\alpha \in \mathcal{C}})$, and the left suspension of $F$ is $(\sigma)F = ((\sigma)V, (V_\alpha)_{\alpha \in \mathcal{C}})$. It is clear that $\text{END}(F)_\sigma$ is just the space of morphisms of graded flags from $F$ to $(\sigma)F$.

Let $F = (V, (V_\alpha)_{\alpha \in \mathcal{C}})$ be a $G$-graded $\rho$-flag, with a homogeneous basis $B = \bigcup_{\alpha \in \mathcal{C}} B_\alpha$ of $V$ providing the flag structure.
Let $\mathcal{C} = \mathcal{C}^1 \cup \ldots \cup \mathcal{C}^q$ be the decomposition of $\mathcal{C}$ in disjoint connected components; these correspond to the connected components of the undirected graph $\Gamma^u$. For each $1 \leq t \leq q$, let $\rho_t$ be the preorder relation on the set $\bigcup_{\alpha \in \mathcal{C}^t} \alpha$, by restricting $\rho$.

If $V^t = \sum_{\alpha \in \mathcal{C}^t} V_\alpha$, then $F^t = (V^t, (V_\alpha)_{\alpha \in \mathcal{C}^t})$ is a $G$-graded $\rho_t$-flag with basis $\bigcup_{\alpha \in \mathcal{C}^t} B_\alpha$. Obviously, $V = \bigoplus_{1 \leq t \leq q} V^t$. In a formal way we can write $F = F^1 \oplus \ldots \oplus F^q$, where $F$ is a $G$-graded $\rho$-flag, and $F^t$ is a $G$-graded $\rho_t$-flag for each $1 \leq t \leq q$.

**Definition 6.1.** Let $\rho$ and $\mu$ be isomorphic preorder relations (i.e., the preordered sets on which $\rho$ and $\mu$ are defined are isomorphic). Let $\mathcal{C}$ and $\mathcal{D}$ be the posets associated with $\rho$ and $\mu$, and let $g : \mathcal{C} \to \mathcal{D}$ be an isomorphism of posets. We say that a $\rho$-flag $F = (V, (V_\alpha)_{\alpha \in \mathcal{C}})$ is $g$-isomorphic to a $\mu$-flag $G = (W, (W_\beta)_{\beta \in \mathcal{D}})$ if there is a linear isomorphism $u : V \to W$ such that $u(V_\alpha) = W_{g(\alpha)}$ for any $\alpha \in \mathcal{C}$. If $F$ and $G$ are $G$-graded flags, we say that they are $g$-isomorphic as graded flags if there is such an $u$ which is a morphism of graded vector spaces.

**Lemma 6.2.** With notation as in Definition 6.1, if the $G$-graded flags $F$ and $G$ are $g$-isomorphic, then $\text{END}(F)$ and $\text{END}(G)$ are isomorphic as $G$-graded algebras.

**Proof.** If $u : V \to W$ is a $g$-isomorphism between the graded flags $F$ and $G$, then $\Phi : \text{END}(F) \to \text{END}(G)$, $\Phi(\phi) = u\phi u^{-1}$ is an isomorphism of $G$-graded algebras. □

Now we will consider another $G$-graded $\rho$-flag $F' = (V', (V'_\alpha)_{\alpha \in \mathcal{C}})$. As we did for $F$, we also have $V' = \bigoplus_{1 \leq t \leq q} V'^t$ and $F' = F'^1 \oplus \ldots \oplus F'^q$, where $F'^t$ is a $G$-graded $\rho_t$-flag for each $1 \leq t \leq q$.

**Theorem 6.3.** Let $F = (V, (V_\alpha)_{\alpha \in \mathcal{C}})$ and $F' = (V', (V'_\alpha)_{\alpha \in \mathcal{C}})$ be $G$-graded $\rho$-flags. Then the following assertions are equivalent:

1. $\text{END}(F)$ and $\text{END}(F')$ are isomorphic as $G$-graded algebras.
2. There exist $g \in \text{Aut}_0(\mathcal{C})$, $\sigma_1, \ldots, \sigma_q \in G$ and a $g$-isomorphism $\gamma : V \to V'$ between the (ungraded) $\rho$-flags $F$ and $F'$, such that $\gamma_{|V^t} : V^t \to V'^{q}$ is a linear isomorphism of left degree $\sigma_t$ for any $1 \leq t \leq q$, where $\mathcal{S} \in \mathcal{S}_q$ is the permutation induced by $g$, i.e. $g(\mathcal{C}^t) = \mathcal{C}^{\mathcal{S}(t)}$.
3. There exists a permutation $\tau \in \mathcal{S}_q$, an isomorphism $g_\tau : \mathcal{C}^t \to \mathcal{C}^{\mathcal{S}(t)}$ for each $1 \leq t \leq q$, and $\sigma_1, \ldots, \sigma_q \in G$, such that $F(t) = g_\tau$-isomorphic to $F'^{\mathcal{S}(t)}$ for any $1 \leq t \leq q$.

**Proof.** (1) $\Rightarrow$ (2) Let $\phi : \text{END}(F) \to \text{END}(F')$ be an isomorphism of $G$-graded algebras. We follow the line of proof of Proposition 4.2 and its notation, adding the additional information related to the graded structure. Let $g_i = \deg v_i$ for any $i$. Then $E_{ij}$ is a homogeneous element of degree $g_i g_j^{-1}$ of $\text{END}(F)$, and $F_{ij} = \phi(E_{ij})$ also has degree $g_i g_j^{-1}$ in $\text{END}(F')$.

Now $Q_i = \text{Im} F_{ii}$ is a graded subspace of $V'$, $V' = \bigoplus_{1 \leq i \leq n} Q_i$ and each $Q_i$ is 1-dimensional. For each $i$ pick $w_i \in Q_i \setminus \{0\}$, which is a homogeneous element. Then for any $i, j$ with $i \rho j$ we have $F_{ij}(w_j)_{i} = a_{ij} w_i$ for some $a_{ij} \in k^*$, and $a_{ij} a_{jr} = a_{ir}$ for any $i, j, r$ with $i \rho j$ and $j \rho r$.

Since $(F_{ij})_{Q_j}^Q_j : Q_j \to Q_i$ is a linear isomorphism of degree $g_i g_j^{-1}$ (with inverse $(F_{ji})_{Q_i}^Q_i : Q_i \to Q_j$), we obtain that $Q_i \simeq (g_j g_i^{-1}) Q_j \simeq (g_i^{-1}) (g_j) Q_j$ for any $i, j$ with $i \rho j$, so then $(g_i) Q_i \simeq (g_j) Q_j$ as graded vector spaces. This implies that $(g_i) Q_i$ has the same isomorphism type for all $i \in \alpha$ with $\alpha$ lying in a connected component of $\mathcal{C}$. 

On the other hand, if $R_i = kv_i$ for any $i$, then $(g_i)R_i$ and $(g_j)R_j$ are isomorphic graded vector spaces for any $i, j$. Then there are $\sigma_1, \ldots, \sigma_q \in G$ such that for any $1 \leq t \leq q$ we have $Q_i \simeq R_i(\sigma_t)$ for any $i \in \alpha$ with $\alpha \in C^t$. Indeed, fix some $t$ and pick $\alpha_0 \in C^t$ and $t_0 \in \alpha_0$. Since $Q_i$ and $R_i$ are 1-dimensional graded vector spaces, there exists $\sigma_t \in G$ such that $Q_i(\sigma_t) \simeq R_i$. Then for any $\alpha \in C^t$ and any $i \in \alpha$ one has

$$Q_i(\sigma_t) \simeq (g_i g_i^{-1} Q_i)(\sigma_t) = (g_i g_i^{-1})(Q_i(\sigma_t)) \simeq (g_i g_i^{-1}) R_i \simeq R_i$$

Thus we obtain that

$$\text{deg} v_i = (\text{deg} w_i)\sigma_t \quad \text{for any } i \in \alpha \text{ with } \alpha \in C^t$$

As in the proof of Proposition 4.2, the linear map $\gamma : V \to V'$ with $\gamma(v_i) = w_i$ for any $i$, is a $\phi'$-isomorphism for a certain algebra isomorphism $\phi' : \text{END}(F) \to \text{END}(F')$, and then there exists $g \in \text{Aut}_0(C)$ with $\gamma(V_\alpha) = V'_g(\alpha)$ for any $\alpha \in C$. By (10), $\gamma|_{V^t}$ is a linear morphism of left degree $\sigma_t$, and we are done.

(2)$\Rightarrow$(3) Take $\tau = \overline{g}$, and let $g_t : C^t \to C^{\tau(t)}$ be the restriction and corestriction of $g$ for each $t$. Then the restriction and corestriction of $\gamma$ to $V^t$ and $V'^{\tau(t)}$ gives a $g_t$-isomorphism of graded flags $F^t(\sigma_t) \simeq F'^{\tau(t)}$.

(3)$\Rightarrow$(1) It is clear that $\text{END}(F) \simeq \text{END}(F^1) \times \ldots \times \text{END}(F^q)$ and $\text{END}(F') \simeq \text{END}(F'^1) \times \ldots \times \text{END}(F'^q)$ as $G$-graded algebras. By Lemma 6.2, we see that $\text{END}(F^t(\sigma_t)) \simeq \text{END}(F'^{\tau(t)})$ for any $1 \leq t \leq q$. As it is obvious that $\text{END}(F^t(\sigma_t)) = \text{END}(F^t)$, we get $\text{END}(F^t) \simeq \text{END}(F'^{\tau(t)})$ for any $1 \leq t \leq q$. We conclude that $\text{END}(F) \simeq \text{END}(F')$.

\[ \square \]

7. Classification of gradings arising from graded flags

The aim of this section is to classify $G$-gradings on $M(p, k)$ arising from graded flags by the orbits of a certain group action. We keep all the notations of Section 6. We first consider three group actions on the set $G^n$.

- $\text{Aut}_0(C)$ acts to the right on $G^n$ by
  $$ (h_i)_{1 \leq i \leq n} \leftarrow g = (h_{g(i)})_{1 \leq i \leq n}$$
  for any $(h_i)_{1 \leq i \leq n} \in G^n$ and $g \in \text{Aut}_0(C)$.

- $G^q$ acts to the right on $G^n$ by
  $$ (h_i)_{1 \leq i \leq n} \leftarrow (\sigma_t)_{1 \leq t \leq q} = (h'_i)_{1 \leq i \leq n}$$
  where for each $i$ we define $h'_i = h_i \sigma_p$, where $p$ is such that $i \in C^p$.

- For each $\alpha \in C$ let $S(\alpha)$ be the symmetric group of $\alpha$ (regarded as a subset of $\{1, \ldots, n\}$. We consider the group $\prod_{\alpha \in C} S(\alpha)$, which is a Young subgroup of $S_n$ (isomorphic to $\prod_{\alpha \in C} S_m(\alpha)$). Then $\prod_{\alpha \in C} S(\alpha)$ acts to the right on $G^n$ by
  $$ (h_i)_{1 \leq i \leq n} \leftarrow (\psi_\alpha)_{\alpha \in C} = (h'_i)_{1 \leq i \leq n}$$
  with $h'_i$ defined by $h'_i = h_{\psi_\alpha(i)}$, where $\alpha = \widehat{i}$, for each $i$. 

Now there is a right action of the group $\text{Aut}_0(\mathcal{C})$ on the group $G^q$ defined by

$$(\sigma_t)_{1 \leq t \leq q} \overset{g}{\rightarrow} (\sigma_t)_{1 \leq t \leq q}$$

where $\tau \in S_q$ is the permutation induced by $g$. Then we have a right semidirect product $\text{Aut}_0(\mathcal{C}) \rtimes G^q$. Moreover, the compatibility condition holds for these actions, i.e.

$$(11) \quad ((h_i)_{1 \leq i \leq n} \overset{g}{\rightarrow} (\sigma_t)_{1 \leq t \leq q} \overset{g}{\rightarrow} (h_i)_{1 \leq i \leq n}) \overset{g}{\rightarrow} ((\sigma_t)_{1 \leq t \leq q} \overset{g}{\rightarrow} (\sigma_t)_{1 \leq t \leq q})$$

Indeed, it is easy to see that both sides of equation $(11)$ have on the $i$th position $h_i g(i) \sigma_{\tau(p)}$, where $p$ is such that $i \in \mathcal{C}^p$. We conclude that $\text{Aut}_0(\mathcal{C}) \rtimes G^q$ acts to the right on the set $G^n$ by

$$(12) \quad ((h_i)_{1 \leq i \leq n} \overset{g}{\rightarrow} (\sigma_t)_{1 \leq t \leq q} \overset{g}{\rightarrow} (h_i)_{1 \leq i \leq n}) \overset{g}{\rightarrow} ((\sigma_t)_{1 \leq t \leq q} \overset{g}{\rightarrow} (\sigma_t)_{1 \leq t \leq q})$$

On the other hand, it is straightforward to check that $\text{Aut}_0(\mathcal{C})$ acts to the left on the group $\prod_{\alpha \in \mathcal{C}} S(\alpha)$ by $g \rightarrow (\psi_\alpha)_{\alpha \in \mathcal{C}} = (\psi'_\alpha)_{\alpha \in \mathcal{C}}$, where for any $\alpha \in \mathcal{C}$, $\psi'_\alpha$ is defined by

$$\psi'_\alpha(i) = \tilde{g}(\psi_{g^{-1}(\alpha)}(\tilde{g}^{-1}(i)))$$

for any $i \in \alpha$. Moreover, one can check that

$$(13) \quad ((h_i)_{1 \leq i \leq n} \overset{g}{\rightarrow} (\sigma_t)_{1 \leq t \leq q} \overset{g}{\rightarrow} (h_i)_{1 \leq i \leq n}) \overset{g}{\rightarrow} ((\sigma_t)_{1 \leq t \leq q} \overset{g}{\rightarrow} (\sigma_t)_{1 \leq t \leq q})$$

Indeed, it is easily checked that on the $i$th position of each side one finds the element $h_i \psi_{\alpha(i)} \sigma_p$, where $\alpha = i \in \mathcal{C}^p$. Now $(12)$ and $(13)$ show that the compatibility relation is satisfied for the right actions of $\text{Aut}_0(\mathcal{C}) \rtimes G^q$ and $\prod_{\alpha \in \mathcal{C}} S(\alpha)$ on $G^n$. In conclusion, the group $\prod_{\alpha \in \mathcal{C}} S(\alpha) \rtimes (\text{Aut}_0(\mathcal{C}) \rtimes G^q)$ acts to the right on the set $G^n$ by

$$(14) \quad ((h_i)_{1 \leq i \leq n} \overset{g \times (\sigma_t)}{\rightarrow} ((h_i)_{1 \leq i \leq n} \overset{g}{\rightarrow} (\sigma_t)_{1 \leq t \leq q})) \overset{g}{\rightarrow} ((\sigma_t)_{1 \leq t \leq q} \overset{g}{\rightarrow} (\sigma_t)_{1 \leq t \leq q})$$

Now we can prove the following.

Theorem 7.1. The isomorphism types of $G$-gradings of the type $\text{END}(\mathcal{F})$, where $\mathcal{F}$ is a $G$-graded $\rho$-flag, are classified by the orbits of the right action of the group $\prod_{\alpha \in \mathcal{C}} S(\alpha) \rtimes (\text{Aut}_0(\mathcal{C}) \rtimes G^q)$ on the set $G^n$.

Proof. We first need some simple remarks. If $V$ and $V'$ are $G$-graded vector spaces with homogeneous bases $\{v_1, \ldots, v_n\}$, respectively $\{v'_1, \ldots, v'_n\}$, then $V$ and $V'$ are isomorphic as $G$-graded vector spaces if and only if $\dim V_g = \dim V'_g$ for any $g \in G$, and this is also equivalent to the fact that the $n$-tuple $(\deg v_1, \ldots, \deg v_n)$ of elements of $G$ is obtained from $(\deg v'_1, \ldots, \deg v'_n)$ by a permutation.

Also, if $\sigma \in G$, then $V(\sigma) \simeq V'$ if and only if $((\deg v_1)\sigma^{-1}, \ldots, (\deg v_n)\sigma^{-1})$ is obtained from $(\deg v'_1, \ldots, \deg v'_n)$ by a permutation; this follows from the fact that a homogeneous element of degree $g$ of $V$ has degree $g\sigma^{-1}$ in $V(\sigma)$.

Let $\mathcal{F}$ be a $\rho$-flag with basis $B = \{v_1, \ldots, v_n\}$ as in Definition 2.1. A $G$-graded structure on $\mathcal{F}$ is given by assigning arbitrary degrees $h_1, \ldots, h_n \in G$ to $v_1, \ldots, v_n$. Thus $G$-graded structures on $\mathcal{F}$ are given by elements $(h_1, \ldots, h_n)$ of $G^n$. 
Let $\mathcal{F}$ and $\mathcal{F}'$ be $G$-graded $\rho$-flags given by $n$-tuples $(h_1, \ldots, h_n)$ and $(h'_1, \ldots, h'_n)$ as above. The homogeneous basis of $V$ and $V'$ are denoted by $B = \bigcup_{\alpha \in \mathcal{C}} B_\alpha$ and $B' = \bigcup_{\alpha \in \mathcal{C}} B'_\alpha$.

We show that $\text{END}(\mathcal{F}) \simeq \text{END}(\mathcal{F}')$ as $G$-graded algebras if and only if $(h_1, \ldots, h_n)$ and $(h'_1, \ldots, h'_n)$ are in the same orbit of $G^n$ with respect to the right action of $\prod_{\alpha \in \mathcal{C}} S(\alpha) \times (\text{Aut}_0(\mathcal{C}) \times G^q)$, and this will finish the proof.

By Theorem 6.3, $\text{END}(\mathcal{F}) \simeq \text{END}(\mathcal{F}')$ if and only if there exists $g \in \text{Aut}_0(\mathcal{C})$ such that $\mathcal{F}'(\sigma_t) = g(\mathcal{F})(t)$ for any $1 \leq t \leq q$, where $\sigma_t$ is the permutation such that for any $t$, $g(C^t) = C'^t$, and $g_t : C^t \rightarrow C'^t$ is the isomorphism of posets induced by $g$ via restriction and corestriction. Using the considerations above, we obtain that $\text{END}(\mathcal{F}) \simeq \text{END}(\mathcal{F}')$ if and only if there exists $g \in \text{Aut}_0(\mathcal{C})$ such that for each $1 \leq t \leq q$ and any $\alpha \in \mathcal{C}_t$, the degrees of the elements of $B_\alpha$ multiplied to the right by $\sigma_t^{-1}$ are obtained by a permutation from the elements of $B'_g(\alpha)$. But this is equivalent to

\[(h_i)_{1 \leq i \leq n} \leftrightarrow (\sigma_t)_{1 \leq t \leq q} \leftrightarrow (\psi)_{\alpha \in \mathcal{C}} = (h'_i)_{1 \leq i \leq n} \leftrightarrow g\]

for some $(\psi)_{\alpha \in \mathcal{C}} \in \prod_{\alpha \in \mathcal{C}} S(\alpha)$. Since the right actions of $G^n$ and $\prod_{\alpha \in \mathcal{C}} S(\alpha)$ on $G^n$ commute, this is the same with

\[(h_i)_{1 \leq i \leq n} = ((h'_i)_{1 \leq i \leq n} \leftrightarrow g) \leftrightarrow (\sigma_t)_{1 \leq t \leq q} \leftrightarrow (\psi)_{\alpha \in \mathcal{C}}^{-1}\]

which is the same to $(h_i)_{1 \leq i \leq n}$ and $(h'_i)_{1 \leq i \leq n}$ lying in the same orbit of the right action of $\prod_{\alpha \in \mathcal{C}} S(\alpha) \times (\text{Aut}_0(\mathcal{C}) \times G^q)$ on $G^n$. $\square$

Example 7.2. As particular cases of our results we obtain the following.

1. Let $A = M_n(k)$, the full matrix algebra. If $G$ is a group, then the good $G$-gradings on $A$ are all isomorphic to gradings of the form $\text{END}(V)$, where $V$ is a $G$-graded vector space of dimension $n$. These gradings are classified by the orbits of the biaction of the groups $S_n$ (by usual permutations of the elements) and $G$ (by right translations) on $G^n$. Indeed, in this case $\mathcal{C}$ is a singleton, so obviously $\text{Aut}_0(\mathcal{C})$ is trivial. This result appears in [3].

2. Let $A$ be the algebra of upper block triangular matrices of type $m_1, \ldots, m_r$, where $n = m_1 + \ldots + m_r$. Then $\mathcal{C}$ is isomorphic to the poset $\{1, \ldots, r\}$ with the usual order, so any good grading on $A$ is of the type $\text{END}(\mathcal{F})$, where $\mathcal{F}$ is a graded (usual) flag of signature $(m_1, \ldots, m_r)$. Again, $\text{Aut}_0(\mathcal{C})$ is trivial and $\mathcal{C}$ is connected, so the isomorphism types of good $G$-gradings on $A$ are classified by the orbits of the biaction of a Young subgroup $S_{m_1} \times \ldots \times S_{m_r}$ (by permutations) and $G$ (by translations) on $G^n$. This result appears in [3].

References

[1] Yu. A. Bahturin, S. K. Sehgal, M. V. Zaicev, Group gradings on associative algebras, J. Algebra 241 (2001), 677–698.
[2] Yu. A. Bahturin, M. V. Zaicev, Group gradings on matrix algebras, Canad. Math. Bull. 45 (2002), 499–508.
[3] M. Bărăşcu, S. Dăscălescu, Good gradings on upper block triangular matrix algebras, Comm. Algebra 41 (2013), 4290–4298.
[4] S. Caenepeel, S. Dăscălescu, C. Năstăsescu, On Gradings of Matrix Algebras and Descent Theory, Comm. Algebra 30 (2002), 5901–5920.
[5] S. P. Coelho, The automorphism group of a structural matrix algebra, Linear Alg. Appl. 195 (1993), 35-58.
[6] V. Lakshmibai and J. Brown, Flag varieties. An interplay of geometry, combinatorics and representation theory, Hindustan Book Agency, 2009.
[7] C. Năstăsescu and F. van Oystaeyen, Methods of graded rings, Lecture Notes in Math., vol. 1836 (2004), Springer-Verlag.
[8] A. Nowicki, Derivations of special subrings of matrix rings and regular graphs, Tsukuba J. Math. 7 (1983), 281-297.
[9] E. Spiegel and C. J. O’Donnell, Incidence algebras, Pure and Appl. Math. 206 (1997), Marcel Dekker, New York.

[10] L. Van Wyk, Maximal left ideals in structural matrix rings, Comm. Algebra 16 (1988), 399-419.

University of Bucharest, Faculty of Mathematics, Str. Academiei 14, Bucharest 1, RO-010014, Romania

E-mail: filoteia_besleaga@yahoo.com, sdascal@fmi.unibuc.ro