BLOW-UP RESULTS FOR EFFECTIVELY DAMPED WAVE MODELS WITH NONLINEAR MEMORY

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Abstract. In this paper, we study the Cauchy problem for a special family of effectively damped wave models with nonlinear memory on the right-hand side. Our goal is to prove blow-up results for local (in time) Sobolev solutions. Due to the effective dissipation the model is parabolic like from the point of view of energy decay estimates of the corresponding linear Cauchy problem with vanishing right-hand side. For this reason we apply the test function method for proving our results.

1. Introduction. In this paper, let us consider the following Cauchy problem for a family of effectively damped wave models with a nonlinear memory on the right-hand side. The model we have in mind is

\begin{align}
&u_{tt} - \Delta u + (1 + t)^r u_t = \int_0^t (t - \tau)^{-\gamma}|u(\tau, \cdot)|^p d\tau, \quad (t,x) \in (0, \infty) \times \mathbb{R}^n, \\
&u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in \mathbb{R}^n, \tag{1.1}
\end{align}

where \( r \in (-1, 1) \) and \( \gamma \in (0, 1) \). The corresponding linear Cauchy problem with vanishing right-hand side is

\begin{align}
&u_{tt} - \Delta u + (1 + t)^r u_t = 0, \quad (t,x) \in (0, \infty) \times \mathbb{R}^n, \\
&u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in \mathbb{R}^n.
\end{align}

Due to the paper [8] this model is for \( r \in (-1, 1) \) parabolic like from the point of view of energy decay estimates. For this reason parabolic like tools are appropriate to derive existence or blow-up results.
At first, let us recall some recent results for (1.1) with $r = 0$, that is, for the model
\begin{equation}
\begin{cases}
  u_{tt} - \Delta u + u_t = \int_0^t (t-\tau)^{-\gamma}|u(\tau,\cdot)|^p d\tau, \quad (t,x) \in (0,\infty) \times \mathbb{R}^n, \\
  u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in \mathbb{R}^n.
\end{cases}
\tag{1.2}
\end{equation}

The basic question is the question for the critical exponent $p_{\text{crit}} = p_{\text{crit}}(n)$, dividing the range of admissible exponents $p$ in a set, where, in general, local (in time) small data Sobolev solutions blow-up and a set, where small data yield global (in time) Sobolev solutions. Moreover, in [6] the author studied blow-up of solutions, too, to $L^p$ in (1.3). Secondly, let us consider Sobolev solutions. It is clear that the exponent $p_{\text{crit}}$ in (1.3) and the regularity of the data influence the critical exponent. In [2] the author used decay estimates for Sobolev solutions and their partial derivatives to the corresponding linear Cauchy problem with vanishing right-hand side to prove global (in time) existence of small data Sobolev solutions and local (in time) existence of large data Sobolev solutions as well. Firstly let us consider $L^1 \cap L^2$ theory, that is, the data are from energy space without additional $L^1$ regularity. As critical exponent the author proposed
\begin{equation}
p(n,\gamma) = \max \left\{ p_\gamma(n); \frac{1}{\gamma} \right\}, \quad \text{where} \quad p_\gamma(n) = 1 + \frac{2(2-\gamma)}{(n-2(1-\gamma))^+},
\tag{1.3}
\end{equation}

which coincides with the critical exponent for the following classical heat equation with nonlinear memory:
\begin{equation}
u_t - \Delta u = \int_0^t (t-\tau)^{-\gamma}(|u|^{p-1}u)(\tau,x) d\tau, \quad (t,x) \in (0,\infty) \times \mathbb{R}^n.
\end{equation}

Here $(n - 2(1-\gamma))^+ = \max\{n - 2(1-\gamma); 0\}$. This was discovered in [1]. In this way the paper [2] filled some gaps of the paper [6], where the same model (1.2) was treated by using the method of weighted spaces to prove the existence of high regular solutions. Moreover, in [6] the author studied blow-up of solutions, too, to verify $p(n,\gamma)$ as the critical exponent. Let $\gamma$ tend to 1 in (1.2). Then the right-hand side tends in distributional sense to $|u|^p$. The critical exponent for
\begin{equation}
\begin{cases}
  u_{tt} - \Delta u + b(t)u_t = |u|^p, \quad (t,x) \in (0,\infty) \times \mathbb{R}^n, \\
  u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in \mathbb{R}^n,
\end{cases}
\tag{1.4}
\end{equation}
is the Fujita exponent $p_{\text{Fuj}} = p_{\text{Fuj}}(n) = 1 + \frac{2}{n}$, which appears after choosing $\gamma \to 1$ in (1.3). Secondly, let us consider $L^2$ theory, that is, the data are taken from some energy space without $L^1$ regularity. Then in [2] the author proposed the critical exponent
\begin{equation}
p(n,\gamma) = \max \left\{ \tilde{p}_\gamma(n); \frac{1}{\gamma} \right\}, \quad \text{where} \quad \tilde{p}_\gamma(n) = 1 + \frac{4(2-\gamma)}{(n-4(1-\gamma))^+},
\tag{1.5}
\end{equation}
If $\gamma$ tends to 1 in (1.5), then we get $p_{\text{crit}} = p_{\text{crit}}(n) = 1 + \frac{4}{n}$, the critical exponent in the $L^2$ theory for (1.4). Finally, the following Cauchy problem for the semilinear damped wave equation
\begin{equation}
\begin{cases}
  u_{tt} - \Delta u + b(t)u_t = |u|^p, \quad (t,x) \in (0,\infty) \times \mathbb{R}^n, \\
  u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in \mathbb{R}^n,
\end{cases}
\end{equation}
was studied in [3] and, more recently, in [5] under the assumption that the term $b(t)u_t$ is a general effective damping term in the sense of [8]. It is no surprise, that for this model the critical exponent is the Fujita exponent, too. Consequently, in the case $\gamma = 1$ the exponent $r \in (-1,1)$ in $b(t) = (1+t)^r$ has no influence on the...
critical exponent. But what happens in the case $\gamma \in (0, 1)$? Let us turn to answer this question in the following considerations. For this reason we come back to the model

$$
\begin{aligned}
&\left\{ u_{tt} - \Delta u + (1 + t)^r u_t = \int_0^t (t - \tau)^{-\gamma}|u(\tau, \cdot)|^p d\tau, \quad (t, x) \in (0, \infty) \times \mathbb{R}^n, \\
&u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \quad x \in \mathbb{R}^n.
\end{aligned}
$$

First let us introduce the notion of weak solutions to the last Cauchy problem. We recall for $\alpha \in (0, 1)$ the fractional derivative operator $D_{0+}^\alpha$ of order $\alpha$ in the sense of Riemann-Liouville defined by (see e.g. [7])

$$
D_{0+}^\alpha u = d_t I_{0+}^{1-\alpha} u
$$

and $I_{0+}^{1-\alpha}$ is the fractional integral of order $1 - \alpha$ defined as follows (see e.g. [7]):

$$
I_{0+}^{1-\alpha}v(t) = \frac{1}{\Gamma(1 - \alpha)} \int_0^t \frac{v(s)}{(t - s)^\alpha} ds \quad \text{for all} \quad v \in C(\mathbb{R}),
$$

where $\Gamma$ is the Gamma function.

**Definition 1.** Let $T > 0$, $\gamma \in (0, 1)$ and $b(t) = (1 + t)^r$ with $r \in (-1, 1)$. A weak solution for the Cauchy problem (1.1) on $(0, T) \times \mathbb{R}^n$ with the initial data $u_0, u_1 \in L^1_{loc}(\mathbb{R}^n)$ is a locally integrable function $u \in L^p((0, T), L^1_{loc}(\mathbb{R}^n))$ satisfying the relation

$$
\begin{aligned}
&\Gamma(\alpha) \int_0^T \int_{\mathbb{R}^n} I_{0+}^{\alpha}(|u|^p)\varphi(t, x) dx dt + \int_{\mathbb{R}^n} u_1(x)\varphi(0, x) dx - \int_{\mathbb{R}^n} u_0(x)\varphi_0(0, x) dx \\
&\quad + b(0) \int_{\mathbb{R}^n} u_0(x)\varphi(0, x) dx \\
&= \int_0^T \int_{\mathbb{R}^n} u(t, x)\varphi_{tt}(t, x) dx dt - \int_0^T \int_{\mathbb{R}^n} u(t, x)\Delta\varphi(t, x) dx dt \\
&\quad - \int_0^T \int_{\mathbb{R}^n} u(t, x)b(t)\varphi(t, x) dx dt - \int_0^T \int_{\mathbb{R}^n} u(t, x)b'(t)\varphi(t, x) dx dt
\end{aligned}
$$

(1.8)

for all nonnegative test functions $\varphi \in C^2([0, T] \times \mathbb{R}^n)$ such that $\varphi(T, \cdot) = \varphi_0(T, \cdot) = 0$ and $\alpha = 1 - \gamma$.

We propose the following result.

**Theorem 1.1.** Let $0 < \gamma < 1$ and $p \in (1, \infty)$. Assume that the data $(u_0, u_1) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ and satisfies the conditions

$$
\int_{\mathbb{R}^n} u_0(x) dx > 0 \quad \text{and} \quad \int_{\mathbb{R}^n} u_1(x) dx > 0.
$$

(1.10)

Then, if $p$ satisfies for $r \in (-1, 0)$ or $r \in (0, 1)$ the condition

$$
\frac{p}{p - 1} > \inf_{d > 0} \max \left\{ \frac{\frac{n^d}{2} + 1}{1 - \gamma + d}, \sqrt{\frac{\frac{n^d}{2} + 1}{(1 - \gamma)(1 - r)} + \left( \frac{1 - \gamma - (1 - r) \frac{n^d}{2}}{2(1 - \gamma)(1 - r)} \right)^2}, \frac{1 - \gamma - (1 - r) \frac{n^d}{2}}{2(1 - \gamma)(1 - r)} \right\},
$$

(1.11)

or for $r = 0$ the condition

$$
\frac{p}{p - 1} > \frac{n^\frac{p}{2} + 1}{2 - \gamma},
$$

(1.12)
then weak solutions, in the sense of Definition 1, of the Cauchy problem (1.1) do not exist globally in time.

Throughout the present paper we write \( f \lesssim g \) when there exists a constant \( C > 0 \) such that \( f \leq C g \), and \( f \approx g \) when \( g \lesssim f \lesssim g \). Nonnegative constants \( C \) or \( C_l \), \( l \in \mathbb{N} \) are always supposed to be independent of \( T > 0 \).

2. A local (in time) existence result. The main goal of this section is the proof of a local (in time) well-posedness result for the Cauchy problem

\[
\begin{aligned}
& u_{tt} - \Delta u + (1 + t)^r u_t = \int_0^t (t - \tau)^{-\gamma} |u(\tau, \cdot)|^p \, d\tau, \quad (t, x) \in (0, T) \times \mathbb{R}^n, \\
& u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \quad x \in \mathbb{R}^n,
\end{aligned}
\]

(2.1)

where \( r \in (-1, 1) \), \( \gamma \in (0, 1) \) and the data \((u_0, u_1)\) is supposed to belong to the energy space \( H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \).

At first let us turn for \( r \in (-1, 1) \) and \( T > 0 \) to the linear Cauchy problem

\[
\begin{aligned}
& u_{tt} - \Delta u + (1 + t)^r u_t = F(t, x), \quad (t, x) \in (0, T) \times \mathbb{R}^n, \\
& u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \quad x \in \mathbb{R}^n.
\end{aligned}
\]

(2.2)

Applying results from strictly hyperbolic theory we are able to verify the following results.

Corollary 2.1. Let \((u_0, u_1) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \) and \( F \in C(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \), where \( T > 0 \) is given. Then there exists a unique energy solution \( u \in C([0, T]) \times H^1(\mathbb{R}^n) \cap C^1([0, T], L^2(\mathbb{R}^n)) \) of (2.2). Moreover, the energy solution \( u = u(t, x) \) satisfies the following estimates for all \( t \in (0, T) \):

\[
\begin{aligned}
& \|\nabla u(t, \cdot)\|_{L^2} + \|u_t(t, \cdot)\|_{L^2} \lesssim \|\nabla u_0\|_{L^2} + \|u_1\|_{L^2} + \int_0^t \|F(s, \cdot)\|_{L^2} \, ds, \\
& \|u(t, \cdot)\|_{L^2} \lesssim \|u_0\|_{L^2} + t\|\nabla u_0\|_{L^2} + \|u_1\|_{L^2} + \int_0^t \left( \int_0^s \|F(\tau, \cdot)\|_{L^2} \, d\tau \right) \, ds.
\end{aligned}
\]

Now let us turn to the Cauchy problem (2.1).

Theorem 2.1. Let \( r \in (-1, 1) \), \( \gamma \in (0, 1) \) and \((u_0, u_1) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \). If \( \frac{p}{p - 1} > 1 \) for \( n = 1, 2 \) or \( \frac{p}{p - 1} \geq \frac{n}{2} \) for \( n \geq 3 \), then the Cauchy problem

\[
\begin{aligned}
& u_{tt} - \Delta u + (1 + t)^r u_t = \int_0^t (t - \tau)^{-\gamma} |u(\tau, \cdot)|^p \, d\tau, \quad (t, x) \in (0, \infty) \times \mathbb{R}^n, \\
& u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \quad x \in \mathbb{R}^n,
\end{aligned}
\]

has a local (in time) energy solution

\[ u \in C\left([0, T_{\text{max}}], H^1(\mathbb{R}^n)\right) \cap C^1\left([0, T_{\text{max}}], L^2(\mathbb{R}^n)\right), \]

where \( T_{\text{max}} \leq \infty \).

Proof. Let us introduce for \( T > 0 \) the space of solutions

\[ X(T) = C\left([0, T], H^1(\mathbb{R}^n)\right) \cap C^1\left([0, T], L^2(\mathbb{R}^n)\right), \]

with the norm

\[ \|u\|_{X(T)} = \max_{t \in [0, T]} \left\{ \|u(t, \cdot)\|_{L^2} + \|\nabla u(t, \cdot)\|_{L^2} + \|u_t(t, \cdot)\|_{L^2} \right\}. \]

Let \( R > 0 \). Denoting \( B_R(T) \) the centered ball in \( X(T) \) defined by

\[ B_R(T) = \{ u \in X(T) : \|u\|_{X(T)} \leq R \}, \]
and defining a mapping $\Phi$ on $X(T)$ by $\Phi: v \rightarrow \Phi(v) = u$, where $u = u(t, x)$ is the energy solution to the Cauchy problem

$$
\begin{cases}
  u_{tt} - \Delta u + (1 + t)^s u_t = \int_0^t (t - \tau)^{-\gamma} |v(\tau, \cdot)|^p \, d\tau, & (t, x) \in (0, T) \times \mathbb{R}^n, \\
  u(0, x) = u_0(x), & \quad u_t(0, x) = u_1(x), \\
  x \in \mathbb{R}^n.
\end{cases}
$$

By Duhamel’s principle and with the fundamental solutions $E_0$ and $E_1$, $u = \Phi(v)$ is given by

$$
\Phi(v) = E_0(t, 0, x) * x u_0 + E_1(t, 0, x) * x u_1 + \int_0^t E_1(t, s, x) * g(s, v(s, x)) \, ds, 
$$

where

$$
g(s, v(s, x)) = \int_0^s (s - \tau)^{-\gamma} |v(\tau, x)|^p \, d\tau.
$$

In order to apply the estimates from Corollary 2.1, we need the following result which yields that $g = g(s, v) \in C([0, T], L^2(\mathbb{R}^n))$ as it is required.

**Lemma 2.1.** Let $g$ be the function defined as above for $p > 1$. Then, for all $t \in (0, T)$ the following estimate holds:

$$
\|g(t, v)\|_{L^2} \leq C t^{1-\gamma} \|v\|_{L^p((0, T), H^1)}^p.
$$

Moreover, $g \in C([0, T], L^2(\mathbb{R}^n))$.

**Proof.** We have for $p > 1$

$$
\|g(t, v)\|_{L^2} \lesssim \int_0^t (t - s)^{-\gamma} \|v(s, \cdot)\|_{L^{2p}}^p \, ds \lesssim \int_0^t (t - s)^{-\gamma} \|v(s, \cdot)\|_{L^p}^{p(1-\theta)} \|\nabla v(s, \cdot)\|_{L^2}^{p\theta} \, ds
$$

by Gagliardo-Nirenberg inequality with $\theta = \frac{1}{2} - \frac{1}{2p} \in (0, 1)$. Applying Cauchy Schwartz inequality and $0 < \theta < 1$ implies

$$
\lesssim \int_0^t (t - s)^{-\gamma} \left(\|v(s, \cdot)\|_{L^p}^p + \|\nabla v(s, \cdot)\|_{L^2}^p\right) \, ds \lesssim \int_0^t (t - s)^{-\gamma} \|v(s, \cdot)\|_{H^1}^p \, ds
$$

$$
\lesssim \sup_{t \in (0, T)} \|v(t, \cdot)\|_{H^1}^p \int_0^t (t - s)^{-\gamma} \, ds \lesssim t^{1-\gamma} \|v\|_{L^p((0, T), H^1)}^p.
$$

From the definition of $g$ we may conclude $g \in C([0, T], L^2(\mathbb{R}^n))$.

Let $\Phi$ be the mapping defined by (2.3). Then we will show that there exist positive $T$ and $R$ such that

1) for all $v \in B_R(T)$ we have $\Phi(v) \in B_R(T)$,

2) $\Phi$ is a contracting mapping from $B_R(T)$ to $B_R(T)$ satisfying

$$
\|\Phi(v_1) - \Phi(v_2)\|_{X(T)} \leq \frac{1}{2} \|v_1 - v_2\|_{X(T)} \text{ for all } v_1, v_2 \in B_R(T).
$$

Let $v \in B_R(T)$. Using the estimates from Corollary 2.1 we get the following estimates:

$$
\|\nabla u(t, \cdot)\|_{L^2} + \|u_t(t, \cdot)\|_{L^2} \leq C_0 I_0 + C_0 \int_0^t \|g(s, v(s, \cdot))\|_{L^2} \, ds,
$$

where

$$
I_0 = \|u_0\|_{L^2} + \|\nabla u_0\|_{L^2} + \|u_1\|_{L^2},
$$
Remark 2.1 provide a blow-up result for our Cauchy problem (2.1).

... for this reason, the statements of Theorems 1.1 and 2.1 and Theorem 2.1 are weak solutions in the sense of Definition 1. For this reason, the statements of Theorems 1.1 and 2.1 and Remark 2.1 provide a blow-up result for our Cauchy problem (2.1).
3. **Proof of Theorem 1.1 in the subcritical case.** The proof is divided into several steps. The model (1.1) is parabolic like as explained above. For this reason we apply the test function method. It is of importance how to choose the test function. Here we recall a proposal was already used in the papers [1, 4, 6]. For some $T > 0$ we choose the test function $\varphi_T = \varphi_T(t, x)$ as follows:

$$\varphi_T(t, x) = D_{t|T}^\alpha \psi_T(t, x) = \varphi_1(x)D_{t|T}^\alpha \varphi_2(t), \quad (t, x) \in (0, \infty) \times \mathbb{R}^n,$$

(3.1)

where $\psi_T = \psi_T(t, x) = \varphi_1(x)\varphi_2(t)$ with $\ell > 1$. Here $D_{t|T}^\alpha$ is the right fractional derivative operator of order $\alpha$ in the sense of Riemann-Liouville defined by (see e.g. [7])

$$D_{t|T}^\alpha v(t) = -\frac{1}{\Gamma(1-\alpha)} \int_t^T \frac{v(s)}{(s-t)\alpha} ds \text{ for all } v \in C[0, T].$$

The functions $\varphi_1 = \varphi_1(x)$ and $\varphi_2 = \varphi_2(t)$ are given by

$$\varphi_1(x) = \phi\left(\frac{|x|^2}{K}\right) \quad \text{and} \quad \varphi_2(t) = \left(1 - \frac{t}{T}\right)^\beta_+,$$

(3.2)

with $\beta \gg 1$, and $K > 0$ will be chosen suitably later. This choice of the test function bases on the nonlinear memory term. The function $\phi = \phi(s)$ is a cut-off non increasing function satisfying

$$\phi(s) = \begin{cases} 
1 & \text{if } 0 \leq s \leq 1, \\
0 & \text{if } s \geq 2,
\end{cases}$$

$0 \leq \phi \leq 1$ everywhere. Hence, for all $s \in \mathbb{R}$ it holds $|\phi'(s)| \leq \frac{C}{1+s^2}$ for some constant $C > 0$. We denote by $\Omega_K$ the support of $\varphi_1$, that is,

$$\Omega_K := \text{supp } \varphi_1 = \{x \in \mathbb{R}^N : |x|^2 \leq 2K\},$$

(3.3)

and by $\Delta_K$ the support of $\Delta \varphi_1$, that is,

$$\Delta_K = \{x \in \mathbb{R}^N : K \leq |x|^2 \leq 2K\}.$$

(3.4)

3.1. **Tools from fractional derivative calculus.**

**Proposition 3.1 ([7]).** For given $f, g \in C[0, T]$ such that $D_{0|t}^\alpha f(t)$ and $D_{0|T}^\alpha g(t)$ exist and belong to $C[0, T]$, one has for all $t \in (0, T)$ the following fractional version of the integration by parts formula:

$$\int_0^t f(s)D_{s|T}^\alpha g(s) \, ds = \int_0^t (D_{0|s}^\alpha f(s))g(s) \, ds.$$

**Proposition 3.2 ([7]).** For all $v \in L^q(0, T)$, $q \geq 1$, and $t \in (0, T)$ the following identity of composition holds:

$$(D_{0|t}^\alpha \circ I_{0|t}^\beta)(v) = v.$$

(3.5)

Moreover, for all $v \in C^1[0, T]$, $T > 0$, the following rule holds:

$$(-1)^l D_{t|T}^\alpha v(t) = D_{t|T}^{\alpha+l} v(t) \text{ for } l \in \mathbb{N}, \quad \alpha \in (0, 1),$$

(3.6)

where $D_{0|t}^\alpha$ and $I_{0|t}^\beta$ are the fractional derivative and fractional integral operators of order $\alpha$ in the Riemann-Liouville sense defined by (1.6) and (1.7), respectively.

Applying the above formula we may conclude the next statements.
Corollary 3.1. Given $\beta \gg 1$ and $T > 0$. Let $\varphi_2$ be the function defined by (3.2). Then, for all $\alpha \in (0, 1)$ we have
\[
D_{t|T}^\alpha \varphi_2(t) = \frac{\Gamma(\beta + 1)}{\Gamma(\beta - \alpha + 1)} T^{-\beta}(T-t)^{\beta-\alpha} = \frac{\Gamma(\beta + 1)}{\Gamma(\beta - \alpha + 1)} T^{-\alpha} \left(1 - \frac{t}{T}\right)^{\beta-\alpha},
\]
\[
D_{t|T}^{\alpha+1} \varphi_2(t) = \frac{\Gamma(\beta + 1)}{\Gamma(\beta - \alpha)} T^{-\beta}(T-t)^{\beta-\alpha-1} = \frac{\Gamma(\beta + 1)}{\Gamma(\beta - \alpha)} T^{-\alpha-1} \left(1 - \frac{t}{T}\right)^{\beta-\alpha-1},
\]
\[
D_{t|T}^{\alpha+2} \varphi_2(t) = \frac{\Gamma(\beta + 1)}{\Gamma(\beta - \alpha - 1)} T^{-\beta}(T-t)^{\beta-\alpha-2} = \frac{\Gamma(\beta + 1)}{\Gamma(\beta - \alpha - 1)} T^{-\alpha-2} \left(1 - \frac{t}{T}\right)^{\beta-\alpha-2}.
\]

Corollary 3.2. Let $T > 0$ and $\varphi_2$ be the function defined by (3.2). Then, for all $\alpha \in (0, 1)$, one has
\[
(D_{t|T}^\alpha \varphi_2)(0) = CT^{-\alpha}, \quad (D_{t|T}^{\alpha+1} \varphi_2)(0) = CT^{-\alpha-1} \quad \text{and} \quad (D_{t|T}^{\alpha+2} \varphi_2)(0) = CT^{-\alpha-2}.
\]

Proof. The proof of Corollary 3.1 bases on straight-forward calculations. For the first relation, we have
\[
D_{t|T}^\alpha \varphi_2(t) = -\frac{1}{\Gamma(1-\alpha)} d_t \int_t^T \frac{\varphi_2(s)}{(s-t)^{1-\alpha}} ds.
\]

Next, after the change of variables
\[
s \mapsto y = \frac{s-t}{T-t},
\]
and using the below introduced formula (3.7) we have
\[
D_{t|T}^\alpha \varphi_2(t) = -\frac{1}{\Gamma(1-\alpha)} d_t \int_t^T \frac{(1-y)^\beta}{y^{1-\alpha}(1-y)^\gamma} dy
\]
\[
= \frac{T^{-\beta}}{\Gamma(1-\alpha)} \left(1 - \frac{t}{T}\right)^{\beta-\alpha+1} \int_0^1 y^{-\alpha} (1-y)^\gamma dy
\]
\[
= \frac{(\beta - \alpha + 1) B(1-\alpha, \beta + 1)}{\Gamma(1-\alpha)} T^{-\beta} (T-t)^{\beta-\alpha} = \frac{\Gamma(\beta+1)}{\Gamma(\beta - \alpha + 1)} T^{-\beta} (T-t)^{\beta-\alpha},
\]
where $B$ is the Beta function defined by
\[
B(u, v) = \int_0^1 t^{u-1} (1-t)^{v-1} dt,
\]
and satisfies in particular
\[
B(u, v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u + v)}, \tag{3.7}
\]
For the second and the third relation we apply directly formula (3.6) with $l = 1$ and $l = 2$, respectively, to verify that
\[
D_{t|T}^{\alpha+1} \varphi_2(t) = \frac{d}{dt} D_{t|T}^\alpha \varphi_2(t) \quad \text{and} \quad D_{t|T}^{\alpha+2} \varphi_2(t) = \frac{d^2}{dt^2} D_{t|T}^\alpha \varphi_2(t) \quad \text{for all} \quad t \in (0, T).
\]

Hence, the desired relations are concluded from the first relation. This completes the proof. \qed
3.2. Treatment of the left-hand side (1.8) of the weak formulation of solutions. Including the test function defined by (3.1) and (3.2), we get after using the formula of integration by parts from Proposition 3.1 and the identity (3.5) the relation
\[
\int_0^T \int_{\mathbb{R}^n} I_{0|t}^\alpha(|u|^P) \varphi_T(t, x) \, dx \, dt = \int_0^T \int_{\mathbb{R}^n} D_{0|t}^\alpha \psi_T(t, x) \, dx \, dt
\]
and
\[
= \int_0^T \int_{\mathbb{R}^n} |u|^P \psi_T(t, x) \, dx \, dt.
\]  
(3.8)

For the second term of the left-hand side of (1.8), after using Corollary 3.2 we get
\[
\int_{\mathbb{R}^n} u_1(x) \varphi_T(0, x) \, dx
= \int_{\mathbb{R}^n} u_1(x) \varphi_1^\ell(x) D_{0|T}^{\alpha+1} \varphi_2(t)|_{t=0} \, dx = C_1 T^{-\alpha} \int_{\mathbb{R}^n} u_1(x) \varphi_1^\ell(x) \, dx,
\]  
(3.9)

since
\[
D_{0|T}^{\alpha+1} \varphi_2(t)|_{t=0} = \frac{\Gamma(\beta + 1)}{\Gamma(\beta - \alpha + 1)} T^{-\alpha} = C_1 T^{-\alpha}.
\]  
(3.10)

For the third term, noting that
\[
\partial_t \varphi_T(t, x) = -\varphi_1^\ell(x) D_{0|T}^{\alpha+1} \varphi_2(t),
\]
then by using Corollary 3.2 we get the relation
\[
\int_{\mathbb{R}^n} u_0(x) \partial_t \varphi_T(0, x) \, dx = -C_2 T^{-\alpha-1} \int_{\mathbb{R}^n} u_0(x) \varphi_1^\ell(x) \, dx,
\]  
(3.11)

since
\[
D_{0|T}^{\alpha+1} \varphi_2(t)|_{t=0} = \frac{\Gamma(\beta + 1)}{\Gamma(\beta - \alpha)} T^{-\alpha-1} = C_2 T^{-\alpha-1}.
\]  
(3.12)

For the fourth term after using (3.10) we find the relation
\[
\int_{\mathbb{R}^n} u_0(x) \varphi_T(0, x) \, dx = C_1 T^{-\alpha} \int_{\mathbb{R}^n} u_0(x) \varphi_1^\ell(x) \, dx.
\]  
(3.13)

3.3. Treatment of the right-hand side (1.9) of the weak formulation of solutions. Using (3.6) it follows
\[
\partial_t^2 \varphi_T(t, x) = \varphi_1^\ell(x) \frac{d^2}{dt^2} D_{0|T}^{\alpha+2} \varphi_2(t) = \varphi_1^\ell(x) D_{0|T}^{\alpha+2} \varphi_2(t).
\]

Therefore,
\[
\int_0^T \int_{\mathbb{R}^n} u(t, x) \partial^2_t \varphi_T(t, x) \, dx \, dt = \int_0^T \int_{\mathbb{R}^n} u(t, x) \varphi_1^\ell(x) D_{0|T}^{\alpha+2} \varphi_2(t) \, dx \, dt.
\]  
(3.13)

For the second term of the right hand side of (1.9), after using the identity
\[
\Delta(\varphi_1^\ell) = \ell \varphi_1^{\ell-1} \Delta \varphi_1 + \ell(\ell - 1) \varphi_1^{\ell-2} |\nabla \varphi_1|^2,
\]  
(3.14)

we get
\[
\int_0^T \int_{\mathbb{R}^n} u(t, x) \Delta \varphi_T(t, x) \, dx \, dt
\]
Next, after using (3.6) with $b(t) = (1 + t)^r$, $r \in (-1, 1)$ and $l = 1$ we find
\[
\int_0^T \int_{\mathbb{R}^n} u(t,x)b(t)\partial_t \varphi_T(t,x)\,dx\,dt = -\int_0^T \int_{\mathbb{R}^n} u(t,x)b(t)\varphi_1^\ell(x)D_{0,T}^{\alpha + 1}\varphi_2(t)\,dx\,dt, \quad (3.16)
\]
finally, for the last term we have directly
\[
\int_0^T \int_{\mathbb{R}^n} u(t,x)b'(t)\varphi_T(t,x)\,dx\,dt = \int_0^T \int_{\mathbb{R}^n} u(t,x)b'(t)\varphi_1^\ell(x)D_{0,T}^{\alpha + 1}\varphi_2(t)\,dx\,dt. \quad (3.17)
\]

### 3.4. Derivation of a basic estimate.
Inserting all formulas (3.8) up to (3.17) in the left-hand side (1.8) and the right-hand side (1.9) of the formulation of weak solutions we find
\[
\Gamma(\alpha) \int_0^T \int_{\mathbb{R}^n} |u|^p \psi_T(t,x)\,dx\,dt + C_1 T^{-\alpha} \int_{\mathbb{R}^n} (u_0(x) + u_1(x))\varphi_1^\ell(x)\,dx
\]
\[
+ b(0) C_2 T^{-\alpha - 1} \int_{\mathbb{R}^n} u_0(x)\varphi_1^\ell(x)\,dx
\]
\[
= \int_0^T \int_{\mathbb{R}^n} u(t,x)\varphi_1^\ell(x)D_{0,T}^{\alpha + 2}\varphi_2(t)\,dx\,dt
\]
\[
+ \int_0^T \int_{\mathbb{R}^n} u(t,x)b(t)\varphi_1^\ell(x)D_{0,T}^{\alpha + 1}\varphi_2(t)\,dx\,dt
\]
\[
- \int_0^T \int_{\mathbb{R}^n} u(t,x)b'(t)\varphi_1^\ell(x)D_{0,T}^{\alpha + 1}\varphi_2(t)\,dx\,dt
\]
\[
- \int_0^T \int_{\mathbb{R}^n} u(t,x)(\ell \varphi_1^{\ell - 1}\Delta \varphi_1 + \ell(\ell - 1)\varphi_1^{\ell - 2}|\nabla \varphi_1|^2)(x)D_{0,T}^{\alpha + 1}\varphi_2(t)\,dx\,dt, \quad (3.18)
\]
where $b(t) = (1 + t)^r$, $r \in (-1, 1)$. Taking account of $0 \leq \varphi_1 \leq 1$ and
\[
|\ell \varphi_1^{\ell - 1}\Delta \varphi_1 + \ell(\ell - 1)\varphi_1^{\ell - 2}|\nabla \varphi_1|^2| \leq C\varphi_1^{\ell - 2}(|\Delta \varphi_1| + |\nabla \varphi_1|^2), \quad (3.19)
\]
allows us to conclude from the formula (3.18) the following inequality:
\[
\Gamma(\alpha) \int_0^T \int_{\mathbb{R}^n} |u|^p \psi_T(t,x)\,dx\,dt + C_1 T^{-\alpha} \int_{\mathbb{R}^n} (u_0(x) + u_1(x))\varphi_1^\ell(x)\,dx
\]
\[
+ b(0) C_2 T^{-\alpha - 1} \int_{\mathbb{R}^n} u_0(x)\varphi_1^\ell(x)\,dx
\]
\[
\lesssim \int_0^T \int_{\mathbb{R}^n} |u(t,x)|\varphi_1^\ell(x)|D_{0,T}^{\alpha + 2}\varphi_2(t)|\,dx\,dt
\]
\[
+ \int_0^T \int_{\mathbb{R}^n} |u(t,x)||b(t)\varphi_1^\ell(x)|D_{0,T}^{\alpha + 1}\varphi_2(t)|\,dx\,dt
\]
\[
+ \int_0^T \int_{\mathbb{R}^n} |u(t,x)||b'(t)\varphi_1^\ell(x)||D_{0,T}^{\alpha + 1}\varphi_2(t)|\,dx\,dt
\]
\[
+ \int_0^T \int_{\mathbb{R}^n} |u(t,x)||\varphi_1^{\ell - 2}(x)(|\Delta \varphi_1| + |\nabla \varphi_1|^2)|D_{0,T}^{\alpha + 1}\varphi_2(t)|\,dx\,dt. \quad (3.20)
\]
Now, since $\Gamma(\alpha) > 0$, $T > 0$, $C_i > 0$, $i = 1, 2$, the integrals over the data are assumed to be positive for large $K$ according to (1.10) and $b(0) > 0$ we may conclude
Finally, for the fourth term we obtain the estimate:

\[
\int_0^T \int_{\mathbb{R}^n} |u|^p \psi_T(t, x) \, dx \, dt \\
\leq \int_0^T \int_{\mathbb{R}^n} |u(t, x)| |\varphi_1^e(x)| D_{\mathcal{H}^1T}^{\alpha+2} \varphi_2(t) | \, dx \, dt \\
+ \int_0^T \int_{\mathbb{R}^n} |u(t, x)| \bar{b}(t) \varphi_1^e(x) | D_{\mathcal{H}^1T}^{\alpha+1} \varphi_2(t) | \, dx \, dt \\
+ \int_0^T \int_{\mathbb{R}^n} |u(t, x)| |b'(t)| |\varphi_1^e(x)| | D_{\mathcal{H}^1T}^\alpha \varphi_2(t) | \, dx \, dt \\
+ \int_0^T \int_{\mathbb{R}^n} |u(t, x)| |\varphi_1^{e-2}(x)| (|\Delta \varphi_1| + |\nabla \varphi_1|^2) | D_{\mathcal{H}^1T}^\alpha \varphi_2(t) | \, dx \, dt. \quad (3.21)
\]

In order to estimate the terms of the right-hand side of (3.21), we apply for all \( A, B > 0 \) the \( \varepsilon \)-Young’s inequality

\[
AB \leq \varepsilon A^p + C(\varepsilon) B^q, \quad pq = p + q, \quad p, q > 1.
\]

Then we find for the first term the estimate

\[
\int_0^T \int_{\mathbb{R}^n} |u(t, x)| \varphi_1^e(x) | D_{\mathcal{H}^1T}^{\alpha+2} \varphi_2(t) | \, dx \, dt \\
= \int_0^T \int_{\mathbb{R}^n} |u(t, x)| \psi_T^{\frac{1}{2}} \psi_T^{-\frac{1}{2}} \varphi_1^e(x) | D_{\mathcal{H}^1T}^{\alpha+2} \varphi_2(t) | \, dx \, dt \\
\leq \varepsilon \int_0^T \int_{\mathbb{R}^n} |u|^p \psi_T(t, x) \, dx \, dt + C(\varepsilon) \int_0^T \int_{\mathbb{R}^n} \varphi_1^e(x) \varphi_2^{-\frac{p}{p-1}}(t) | D_{\mathcal{H}^1T}^{\alpha+2} \varphi_2(t) | \, dx \, dt. \quad (3.22)
\]

We estimate the second term as follows:

\[
\int_0^T \int_{\mathbb{R}^n} |u(t, x)| \bar{b}(t) \varphi_1^e(x) | D_{\mathcal{H}^1T}^{\alpha+1} \varphi_2(t) | \, dx \, dt \\
\leq \varepsilon \int_0^T \int_{\mathbb{R}^n} |u(t, x)|^p \psi_T(t, x) \, dx \, dt \\
+ C(\varepsilon) \int_0^T \int_{\mathbb{R}^n} \varphi_1^e(x) \bar{b}(t)^{-\frac{p}{p-1}} \varphi_2(t) \frac{p}{p-1} | D_{\mathcal{H}^1T}^{\alpha+1} \varphi_2(t) | \, dx \, dt. \quad (3.23)
\]

For the third term we have

\[
\int_0^T \int_{\mathbb{R}^n} |u(t, x)| |b'(t)| |\varphi_1^e(x)| | D_{\mathcal{H}^1T}^\alpha \varphi_2(t) | \, dx \, dt \\
\leq \varepsilon \int_0^T \int_{\mathbb{R}^n} |u(t, x)|^p \psi_T(t, x) \, dx \, dt \\
+ C(\varepsilon) \int_0^T \int_{\mathbb{R}^n} \varphi_1^e(x) |b'(t)|^{\frac{p}{p+1}} \varphi_2(t)^{-\frac{1}{p+1}} | D_{\mathcal{H}^1T}^\alpha \varphi_2(t) | \, dx \, dt. \quad (3.24)
\]

Finally, for the fourth term we obtain the estimate

\[
\int_0^T \int_{\mathbb{R}^n} |u| \varphi_1^{e-2}(x) (|\Delta \varphi_1| + |\nabla \varphi_1|^2) | D_{\mathcal{H}^1T}^\alpha \varphi_2(t) | \, dx \, dt \\
\leq \varepsilon \int_0^T \int_{\mathbb{R}^n} |u|^p \psi_T(t, x) \, dx \, dt
\]
using Fubini’s theorem, we get for
\( I \beta \phi \) and noting that \( \text{supp} \) the integral:

\[
12 \text{TAYEB HADJ KADDOUR AND MICHAEL REISSIG}
\]

\[ I = 1 \]

\( \epsilon > 0 \) conclude from (3.21), for \( \epsilon > 0 \) small enough, the estimate

\[
\int_0^T \int_{\mathbb{R}^n} |u|^p \psi_T(t, x) \, dx \, dt
\]

\[ \lesssim \int_0^T \int_{\mathbb{R}^n} \phi_1(x) \phi_2(t) - \frac{1}{p-1} |D_{t/T}^{\alpha+2} \phi_2(t)|^\frac{p}{p-1} \, dx \, dt
\]

\[ + \int_0^T \int_{\mathbb{R}^n} \phi_1(x) b(t) \phi_2(t) - \frac{1}{p-1} |D_{t/T}^{\alpha+1} \phi_2(t)|^\frac{p}{p-1} \, dx \, dt
\]

\[ + \int_0^T \int_{\mathbb{R}^n} \phi_1(x) |b'(t)| \phi_2(t) - \frac{1}{p-1} |D_{t/T}^{\alpha} \phi_2(t)|^\frac{p}{p-1} \, dx \, dt
\]

\[ + \int_0^T \int_{\mathbb{R}^n} \Lambda(\phi_1) \phi_2(t) - \frac{1}{p-1} |D_{t/T}^{\alpha} \phi_2(t)|^\frac{p}{p-1} \, dx \, dt \lesssim I_1 + I_2 + I_3 + I_4, \quad (3.26)
\]

where

\[ I_1 = \int_0^T \int_{\mathbb{R}^n} \phi_1(x) \phi_2(t) - \frac{1}{p-1} |D_{t/T}^{\alpha+2} \phi_2(t)|^\frac{p}{p-1} \, dx \, dt,
\]

\[ I_2 = \int_0^T \int_{\mathbb{R}^n} \phi_1(x) b(t) \phi_2(t) - \frac{1}{p-1} |D_{t/T}^{\alpha+1} \phi_2(t)|^\frac{p}{p-1} \, dx \, dt,
\]

\[ I_3 = \int_0^T \int_{\mathbb{R}^n} \phi_1(x) |b'(t)| \phi_2(t) - \frac{1}{p-1} |D_{t/T}^{\alpha} \phi_2(t)|^\frac{p}{p-1} \, dx \, dt,
\]

\[ I_4 = \int_0^T \int_{\mathbb{R}^n} \Lambda(\phi_1) \phi_2(t) - \frac{1}{p-1} |D_{t/T}^{\alpha} \phi_2(t)|^\frac{p}{p-1} \, dx \, dt.
\]

3.5. Verification of the subcritical case. We choose in (3.2) the scaling parameter \( K = T^d \), \( T \) large, where the positive constant \( d \) will be determined later. Then, to estimate the integrals \( I_1, I_2, I_3 \) and \( I_4 \) we choose the scaled variables

\[ x = T^{\frac{d}{2}} y \quad \text{and} \quad t = T \tau, \quad (3.27)
\]

and noting that \( \text{supp} \phi_1 = \Omega_{T^d} \) (this notation was introduced in (3.3)). Then, by using Fubini’s theorem, we get for \( I_1 \) the estimate

\[
I_1 = \left( \int_{\Omega_{T^d}} \phi_1(x)^\ell \, dx \right) \left( \int_0^T \phi_2(t)^{-\frac{1}{p-1}} |D_{t/T}^{\alpha+2} \phi_2(t)|^\frac{p}{p-1} \, dt \right) =: I_{11} I_{12}. \quad (3.28)
\]

First, we have

\[
I_{11} = \int_{\Omega_{T^d}} \phi_1(x)^\ell \, dx = T^{\frac{d\ell}{2}} \int_{|y| \leq 2} \phi(y)^\ell \, dy = CT^{\frac{d\ell}{2}} \quad \text{with some} \quad C > 0. \quad (3.29)
\]

Next, we will use Corollary 3.1 with \( \beta \gg 1 \) to ensure that \( \beta - (\alpha + 2) \frac{p}{p-1} > 0 \) (with \( \alpha = 1 - \gamma \)). This results from the following relation and the assumed existence of the integral:

\[
I_{12} = \int_0^T \phi_2(t)^{-\frac{1}{p-1}} |D_{t/T}^{\alpha+2} \phi_2(t)|^\frac{p}{p-1} \, dt
\]
Taking into account (3.34) we may estimate
\[ I = \int_0^1 (1 - \tau)^{-\frac{\beta}{\alpha+2}} (1 - \tau)^{\beta-\alpha-2} \frac{\partial}{\partial \tau} dt. \]
\[ = T^{-1-\frac{(\alpha+2)\beta}{\alpha+2}} \int_0^1 (1 - \tau)^{-\frac{(\alpha+2)\beta}{\alpha+2}} \frac{\partial}{\partial \tau} d\tau = CT^{-1-\frac{(\alpha+2)\beta}{\alpha+2}}. \] (3.30)

Putting (3.29) and (3.30) into (3.28) we find
\[ I_1 = \int_0^T \int_{\Omega_T} \varphi_1(x)^T \varphi_2(t) \left( \int_0^T b(t) \frac{\partial x}{\partial \tau} \partial_t D_\alpha(t^{\alpha} \varphi_2(t)) \right) dx dt = CT^{-1-\frac{(\alpha+2)\beta}{\alpha+2}}. \] (3.31)

Next, we estimate the terms \( I_2 \) and \( I_3 \). Again, by Fubini’s theorem we have
\[ I_2 = \left( \int_{\Omega_T} \varphi_1(x)^T \varphi_2(t) \left( \int_0^T b(t) \frac{\partial x}{\partial \tau} \partial_t D_\alpha(t^{\alpha} \varphi_2(t)) \right) dx \right) : I_{21} I_{22}. \] (3.32)
\[ I_3 = \left( \int_{\Omega_T} \varphi_1(x)^T \varphi_2(t) \left( \int_0^T b(t) \frac{\partial x}{\partial \tau} \partial_t D_\alpha(t^{\alpha} \varphi_2(t)) \right) dx \right) : I_{21} I_{32}. \] (3.33)

The estimates for \( I_{21} \) and \( I_{31} \) coincide with the above estimate for \( I_{11} \), that is, with (3.29). Now we estimate \( I_{22} \) and \( I_{32} \). For this reason, we distinguish between the three cases \( r \in (0, 1) \), \( r = 0 \) and \( r \in (-1, 0) \) in \( b(t) = (1 + t)^r \).

3.5.1. The function \( b = b(t) = (1 + t)^r \) is strictly increasing, so \( r \in (0, 1) \). In this case we note that
\[ b(t) \leq b(T) = (1 + T)^r, \] (3.34)
Taking into account (3.34) we may estimate \( I_{22} \) as follows:
\[ I_{22} \lesssim (1 + T)^{\frac{\alpha+1}{\alpha+2}} T^{1-\frac{(\alpha+1)r}{\alpha+2}} \lesssim T^{-(\alpha+1-r)} \frac{\partial x}{\partial \tau} + 1, \] (3.35)
for \( T \) sufficiently large. Consequently, taking into account the fact that the estimates for \( I_{21} \) and \( I_{11} \) coincide we may estimate \( I_2 \) after using (3.28) and (3.35) as follows:
\[ I_2 \leq CT^{-(\alpha+1-r)} \frac{\partial x}{\partial \tau} + \frac{\alpha+2}{\alpha+1} + 1. \] (3.36)

To estimate \( I_{32} \) after using \( b(t) \sim (1 + t)^{r-1} \), \( r \in (0, 1) \), we have to understand how to estimate (put \( \tau = \frac{t}{T} \))
\[ T^{1-\alpha} \frac{\partial x}{\partial \tau} \int_0^1 (1 + \tau T)^{(r-1)\frac{\alpha+1}{\alpha+1}} (1 - \tau)^{-\frac{\alpha}{\alpha+1}} (1 - \tau)^{(\beta-\alpha)\frac{\alpha+1}{\alpha+1}} d\tau. \]

For this reason we split the integral
\[ \int_0^1 = \int_0^{T^-\kappa} + \int_{T^-\kappa}^1 \]
into two integrals. Let us choose \( \kappa \in (0, 1) \). If \( \beta \) is large, then we may estimate the first integral (the integrand is bounded) as follows:
\[ \int_0^{T^-\kappa} (1 + \tau T)^{(r-1)\frac{\alpha+1}{\alpha+1}} (1 - \tau)^{-\frac{\alpha}{\alpha+1}} (1 - \tau)^{(\beta-\alpha)\frac{\alpha+1}{\alpha+1}} d\tau \leq CT^{-\kappa}. \]

In the second integral we use in the integrand the estimate
\[ (1 + \tau T)^{(r-1)\frac{\alpha+1}{\alpha+1}} \leq (1 + T^{1-\kappa})(r-1)\frac{\alpha+1}{\alpha+1}. \]
If $\beta$ is large, then we may estimate
\[
\int_{T^{-\kappa}}^1 (1 + \tau T)^{(r-1)\frac{p}{p-\tau}} (1 - \tau)^{-\frac{\beta}{p-\tau}}(1 - \tau)^{(\beta-\alpha)\frac{p}{p-\tau}} d\tau \\
\leq (1 + T^{1-\kappa})^{(r-1)\frac{p}{p-\tau}} \int_{T^{-\kappa}}^1 (1 - \tau)^{-\frac{\beta}{p-\tau}}(1 - \tau)^{(\beta-\alpha)\frac{p}{p-\tau}} d\tau \leq C(1 + T^{1-\kappa})^{(r-1)\frac{p}{p-\tau}}.
\]
The optimal choice for $\kappa$ is given by the condition (for large $T$)
\[ T^{-\kappa} \sim T^{(1-\kappa)(r-1)\frac{p}{p-\tau}}, \]
that is,
\[ \kappa = \frac{(1-r)p}{p-1 + (1-r)p}. \]
Therefore, taking into account the fact that the estimate for $I_{31}$ coincides with the above estimate for $I_{11}$, that is, with (3.29), we get the estimate
\[ I_3 \leq C T^{1-\alpha} \frac{p}{p-\tau} - \frac{(1-r)p}{p-1 + (1-r)p} + \frac{n\ell}{4}. \]  
(3.37)

Now we deal with $I_4$. We have
\[
I_4 = \left( \int_{\Omega_x^d} \Lambda(\varphi_1(x))\varphi_1^{\ell-2q}(x) dx \right) \left( \int_0^T \varphi_2^{\frac{1}{p}}(t) \left| D_{lT}^\alpha \varphi_2(t) \right|^q dt \right) =: I_{41}I_{42}. \]  
(3.38)

Then, after replacing $q$ by its value $\frac{p}{p-\tau}$ we get
\[
I_{41} = \int_{\Omega_x^d} \left( \left| \Delta \varphi_1(x) \right| \frac{p}{p-\tau} + \left| \nabla \varphi_1(x) \right|^{\frac{2q}{p-\tau}} \right) \varphi_1^{\ell-2q} dx \leq CT^{-d} \frac{p}{p-\tau} + \frac{n\ell}{4}. \]  
(3.39)

To get the last estimate (3.39), first we choose $\ell$ such that $\ell - \frac{2p}{p-1} > 0$. This guarantees that $\varphi_1^{\ell-2q}$ is bounded since $\varphi_1$ is bounded. Next we have for all $j = 1, \cdots, n$ the relations
\[
\partial_{x_j} \varphi_1 = \partial_{x_j} \phi \left( \frac{|x|^2}{K} \right) \frac{2}{K} \phi' \left( \frac{|x|^2}{K} \right) x_j, \quad \left| \nabla \varphi_1 \right|^{2q} = \left( \frac{2}{K} \right)^{2q} \phi' \left( \frac{|x|^2}{K} \right)^{2q} |x|^{2q}.
\]

Using the change of variables (3.27), the facts that $K = T^d$ and $|\phi'(s)| \leq \frac{C}{T^{\frac{d}{2}}}$, we arrive at
\[
\int_{\mathbb{R}^d} \left| \nabla \varphi_1(x) \right|^{2q} dx = \left( \frac{2}{Td} \right)^{2q} T^{-\frac{n\ell}{2}} \int_{|y|^2 \leq 2} \phi' \left( |y|^2 \right)^{2q} (T^d |y|^2)^q dy \\
\leq CT^{-d} \phi' \left( 2 \right)^{\frac{n\ell}{2}} \int_{|y|^2 \leq 2} \frac{|y|^{2q}}{(1 + |y|^2)^{2q}} dy \leq C_0 T^{-d} + \frac{n\ell}{4}. \]  
(3.40)

On the other hand, we have for all $j = 1, \cdots, n$ the following relations:
\[
\partial_{x_j}^2 \varphi_1 = \frac{2}{K} \partial_{x_j} \left( \phi' \left( \frac{|x|^2}{K} \right) x_j \right) = \frac{4}{K^2} \phi'' \left( \frac{|x|^2}{K} \right) x_j^2 + \frac{2}{K} \phi' \left( \frac{|x|^2}{K} \right),
\]
\[
|\Delta \varphi_1(x)|^q = \left( \frac{4}{K^2} \phi'' \left( \frac{|x|^2}{K} \right) |x|^2 + 2n \frac{K}{K} \phi' \left( \frac{|x|^2}{K} \right) \right)^q.
\]

Then,
\[
\int_{\mathbb{R}^d} \left| \Delta \varphi_1(x) \right|^q dx = \int_{\mathbb{R}^d} \left[ \frac{4}{K^2} \phi'' \left( \frac{|x|^2}{K} \right) |x|^2 + 2n \frac{K}{K} \phi' \left( \frac{|x|^2}{K} \right) \right]^q dx.
\]
(3.41)

We are going to prove the estimate
\[
\int_{\mathbb{R}^d} \left| \Delta \varphi_1(x) \right|^q dx \leq CT^{-d} + \frac{n\ell}{4}.
\]
Applying Minkowski’s inequality in (3.41) we get, after using the indicated change of variables, the estimate

\[
\left( \int_{\mathbb{R}^n} |\Delta \varphi_1(x)|^q \, dx \right)^{\frac{1}{q}} \leq \left( \int_{\mathbb{R}^n} \frac{4}{K^2} \phi''(\frac{|x|^2}{K}) |x|^2 |x|^q \, dx \right)^{\frac{1}{q}} + \left( \int_{\mathbb{R}^n} \frac{2n}{K} \phi'(\frac{|x|^2}{K}) |x|^q \, dy \right)^{\frac{1}{q}} \leq \left( 4qK^\frac{q}{2} - 2q \right) \int_{|y|^2 \leq 2} |\phi''(|y|^2)(K|y|^2)|^q \, dy \right)^{\frac{1}{q}} + \left( (2n)^q K^\frac{q}{2} - q \right) \int_{|y|^2 \leq 2} |\phi'(|y|^2)|^q \, dy \right)^{\frac{1}{q}} \leq (C_1 4qK^\frac{q}{2} - q)^{\frac{1}{q}} + ((2n)^q C_2 K^\frac{q}{2} - q)^{\frac{1}{q}} \leq CK^\frac{q}{2} - 1, \tag{3.42}
\]

where

\[
C_1 = \int_{|y|^2 \leq 2} |\phi''(|y|^2)(|y|^2)|^q \, dy, \text{ and } C_2 = \int_{|y|^2 \leq 2} |\phi'(|y|^2)|^q \, dy.
\]

Hence, setting \( K = T^d \) and \( q = \frac{p}{p-1} \) we find from (3.42) the desired estimate

\[
\left( \int_{\mathbb{R}^n} |\Delta \varphi_1(x)|^q \, dx \right)^{\frac{1}{q}} \leq CT^{\frac{q}{2} - d} \frac{p}{p-1} T ^d. \tag{3.43}
\]

Finally the estimate (3.39) is concluded from (3.43) and (3.40). Moreover, we have the relations

\[
I_{42} = \int_0^T \varphi_2(t) - \int_0^T |D^\alpha_{|T|} \varphi_2(t)|^\frac{p}{p-1} \, dt = \int_0^1 (1 - \tau)^{-\frac{p}{p-1}} (T^{-\alpha}(1 - \tau)^\beta - \alpha)^\frac{p}{p-1} T \, d\tau = T^{-\alpha} \frac{p}{p-1} + 1 \tag{3.44}
\]

After replacing (3.39) and (3.44) into (3.38) we find

\[
I_4 = \int_0^T \int_{|y|^d} \Lambda(\varphi_1) \varphi_1^{2q} \varphi_2^{2q} |D^\alpha_{|T|} \varphi_2|^q \, dxdt \leq CT^{-(\alpha + d)} \frac{p}{p-1} + \frac{d}{2} + 1, \tag{3.45}
\]

with some constant \( C > 0 \). Including all the estimates (3.31), (3.36), (3.37) and (3.45) into (3.26) we may conclude

\[
\int_0^T \int_{|y|^d} |u|^p \psi_T(t, x) \, dxdt \leq C \left(T^{-(\alpha + 2)} \frac{p}{p-1} + \frac{d}{2} + 1 + T^{-(\alpha + 1) - r} \frac{p}{p-1} + \frac{d}{2} + 1 + T^{-(\alpha + d)} \frac{p}{p-1} + \frac{d}{2} + 1 \right).
\]

To have that all exponents of \( T \) on the right-hand side are negative it is sufficient to guarantee

\[-(\alpha + d) \frac{p}{p-1} + \frac{nd}{2} + 1 < 0 \text{ and } - \alpha \frac{p}{p-1} - \frac{(1 - r)p}{p - 1 + (1 - r)p} + \frac{nd}{2} + 1 < 0, \quad d > 0.\]

Introducing \( \rho = \frac{nd}{2} \) and \( q = \frac{p}{p-1} \) the first inequality implies the relation

\[
q > \frac{\rho + 1}{\alpha + \frac{2p}{n}}.
\]
From the second inequality we verify
\[ q > \sqrt{\frac{\rho + 1}{\alpha(1 - r)}} + \left(\frac{\alpha - (1 - r)\rho}{2\alpha(1 - r)}\right)^2 - \frac{\alpha - (1 - r)\rho}{2\alpha(1 - r)}. \]

Consequently,
\[ q > \max\left\{ \frac{\rho + 1}{\alpha + \frac{2p}{n}}, \sqrt{\frac{\rho + 1}{\alpha(1 - r)}} + \left(\frac{\alpha - (1 - r)\rho}{2\alpha(1 - r)}\right)^2 - \frac{\alpha - (1 - r)\rho}{2\alpha(1 - r)} \right\}. \]

So, we derived the condition
\[ \frac{p}{p-1} > \inf_{d>0} \max\left\{ \frac{nd + 1}{\alpha + d}, \sqrt{\frac{nd + 1}{\alpha(1 - r)}} + \left(\frac{\alpha - (1 - r)nd}{2\alpha(1 - r)}\right)^2 - \frac{\alpha - (1 - r)nd}{2\alpha(1 - r)} \right\}, \quad (3.47) \]

that is condition (1.11) with \( \alpha = 1 - \gamma \). We find from (3.46) and (3.47) the estimate
\[ \int_0^T \int_{\Omega_{x,d}} |u|^p \psi_T(t, x) \, dx \, dt \lesssim T^{-\delta}, \quad (3.48) \]

where \( \delta = \delta(p, n, \gamma, r) > 0 \). Then after passing to the limit \( T \to \infty \) in (3.48) and using the dominated convergence theorem and the fact that
\[ \lim_{T \to \infty} \psi_T(t, x) = 1 \quad \text{for all} \quad (t, x) \in (0, T) \times \mathbb{R}^n \]
it follows
\[ \int_0^\infty \int_{\mathbb{R}^n} |u|^p \, dx \, dt = 0. \]

This gives immediately \( u \equiv 0 \) and this is a contradiction to (1.10).

3.5.2. The function \( b \equiv 1 \), so \( r = 0 \). Repeating the estimates from the previous subsection we arrive at \((I_3 = 0)\)
\[ \int_0^T \int_{\Omega_{x,d}} |u|^p \psi(t, x) \, dx \, dt \]
\[ \leq C \left( T^{-(\alpha+2)-\alpha} \right) + T^{-(\beta+1)} + T^{-(\alpha+1)} + T^{-(\alpha+d)\frac{nd}{d} + 1} + T^{-(\alpha+d)\frac{nd}{d} + 1} \].

As above we derive the condition
\[ \frac{p}{p-1} > \inf_{d>0} \max\left\{ \frac{nd + 1}{\alpha + d}, \frac{nd + 1}{\alpha + d} \right\} = \frac{\alpha + 1}{\alpha + 1}, \]

that is condition (1.12) with \( \alpha = 1 - \gamma \). In the same way we conclude \( u \equiv 0 \) and this is a contradiction to (1.10).

3.5.3. The function \( b = b(t) = (1 + t)^r \) is decreasing, so \( r \in (-1, 0) \). Remember that our strategy is to estimate in an effective way the integrals \( I_{(2+j)2}, j = 0, 1 \), which are given by
\[ I_{(2+j)2} = \int_0^T \left| b^{(j)}(t) \right|^{\frac{p}{p-1}} \varphi_2(t) - \frac{1}{T^{\frac{1}{p}}} |D_t^{\alpha+1-j}\varphi_2(t)|^{\frac{p}{p-1}} \, dt \]
(see (3.32) and (3.33)). If we write \( b(t) = (1 + t)^{-s} \), \( s \in (0, 1) \), then we have to understand how to estimate (put \( \tau = \frac{t}{T} \))
\[ T^{1-(\alpha+1-j)} \int_0^1 (1 + \tau T)^{-(s+j)} \frac{1}{\tau^{\frac{1}{p}}} (1 - \tau)^{-\frac{1}{p}} (1 - \tau)^{(\beta-\alpha-1+j)} \frac{1}{\tau^{\frac{1}{p}}} \, dt. \]
For this reason we split the integral
\[ \int_0^1 = \int_0^{T^{-\kappa_j}} + \int_1^{T^{-\kappa_j}} \]
into two integrals. Let us choose \( \kappa_j \in (0, 1) \). If \( \beta \) is large, then we may estimate the first integral (the integrand is bounded) as follows:
\[ \int_0^{T^{-\kappa_j}} (1 + \tau T)^{-\frac{s}{p+1}} (1 - \tau)^{-\frac{1}{p+1}} (1 - \tau)^{(\beta - \alpha - 1)} \frac{d\tau}{p+1} \leq CT^{-\kappa_j}, \ j = 0, 1. \]

In the second integral we use in the integrand the estimate
\[ (1 + \tau T)^{-\frac{s}{p+1}} \leq (1 + T^{1-\kappa_j})^{-\frac{s}{p+1}}, \ j = 0, 1. \]

If \( \beta \) is large, then for \( j = 0, 1 \), we may estimate
\[ \int_1^{T^{-\kappa_j}} (1 + \tau T)^{-\frac{s}{p+1}} (1 - \tau)^{-\frac{1}{p+1}} (1 - \tau)^{(\beta - \alpha - 1)} \frac{d\tau}{p+1} \leq (1 + T^{1-\kappa_j})^{-\frac{s}{p+1}} \int_1^{T^{-\kappa_j}} (1 - \tau)^{-\frac{1}{p+1}} (1 - \tau)^{(\beta - \alpha - 1)} \frac{d\tau}{p+1} \leq C(1 + T^{1-\kappa_j})^{-\frac{s}{p+1}}. \]

The optimal choice for \( \kappa_j \) is given by the condition (for large \( T \))
\[ T^{-\kappa_j} \sim T^{-(1-\kappa_j)\frac{s}{p+1}}, \]
that is,
\[ \kappa_j = (1 - \kappa_j)^{\frac{s}{p+1}}. \]

We find
\[ \kappa_j = \frac{p(s + j)}{p - 1 + (s + j)p}. \]

Therefore, taking into account the fact that the estimates of \( I_{21} \) and \( I_{31} \) coincide with the above estimate for \( I_{11} \), that is, with (3.29), we get the estimate
\[ I_2 + I_3 \lesssim T^{-(\alpha + 1)} \frac{p}{p+1} + T^{-(\alpha + d)} \frac{p}{p+1} + \frac{nd}{2} + 1 + T^{-\alpha} \frac{p}{p+1} + \frac{p(1-r)}{p+1} + \frac{nd}{2} + 1. \]

Next, due to (3.28) and (3.45) we conclude the following estimate:
\[ \int_0^T \int_{\Omega_T} |u|^p \psi_T(t, x) \, dx \, dt \leq C(T^{-(\alpha + 1)} \frac{p}{p+1} + \frac{nd}{2} + 1 + T^{-(\alpha + d)} \frac{p}{p+1} + \frac{nd}{2} + 1 + T^{-\alpha} \frac{p}{p+1} - \frac{p(1-r)}{p+1} + \frac{nd}{2} + 1). \]

(3.49)

To have that all exponents of \( T \) on the right-hand side are negative it is sufficient to guarantee
\[ -(\alpha + d) \frac{p}{p-1} + \frac{nd}{2} + 1 < 0 \text{ and } -\alpha \frac{p}{p-1} - \frac{p(1-r)}{p-1 + p(1-r)} + \frac{nd}{2} + 1 < 0. \]

Introducing \( \rho = \frac{nd}{2} \) and \( q = \frac{p}{p-1} \) the first inequality implies the relation
\[ q > \frac{\rho + 1}{\alpha + \frac{2p}{n}}. \]
From the second inequality we verify
\[ q > \sqrt{\frac{\rho + 1}{\alpha(1-r)}} + \left(\frac{\alpha - (1-r)\rho}{2\alpha(1-r)}\right)^2 - \frac{\alpha - (1-r)\rho}{2\alpha(1-r)}. \]
Consequently,
\[ q > \max \left\{ \frac{\rho + 1}{\alpha + \frac{n}{n}} \sqrt{\frac{\rho + 1}{\alpha(1-r)}} + \left(\frac{\alpha - (1-r)\rho}{2\alpha(1-r)}\right)^2 - \frac{\alpha - (1-r)\rho}{2\alpha(1-r)} \right\}. \]
So, we derived the condition
\[ \frac{p}{p-1} > \inf_{d>0} \max \left\{ \frac{nd + 1}{\alpha + d}; \sqrt{\frac{nd + 1}{\alpha(1-r)}} + \left(\frac{\alpha - (1-r)\frac{nd}{2}}{2\alpha(1-r)}\right)^2 - \frac{\alpha - (1-r)\frac{nd}{2}}{2\alpha(1-r)} \right\}. \]
that is condition (1.11) with \( \alpha = 1 - \gamma. \) We find from (3.49) and (3.50) the estimate
\[ \int_0^T \int_{\Omega_T} |u|^\rho \phi_T(t,x) \, dx \, dt \leq T^{-\delta}, \]
where \( \delta = \delta(p, n, \gamma, r) > 0. \) Then after passing to the limit \( T \to \infty \) in (3.51) and using the dominated convergence theorem and the fact that
\[ \lim_{T \to \infty} \psi_T(t,x) = 1 \text{ for all } (t,x) \in (0,T) \times \mathbb{R}^n \]
it follows
\[ \int_0^\infty \int_{\mathbb{R}^n} |u|^p \, dx \, dt = 0. \]
This gives immediately \( u \equiv 0 \) and this is a contradiction to (1.10).

4. Discussion of condition (1.11). As before we set \( \alpha = 1 - \gamma. \) If we choose \( d \) from the set \( \mathbb{R}^+, \)
then
\[ \frac{nd + 1}{\alpha + d} = \sqrt{\frac{nd + 1}{\alpha(1-r)}} + \left(\frac{\alpha - (1-r)\frac{nd}{2}}{2\alpha(1-r)}\right)^2 - \frac{\alpha - (1-r)\frac{nd}{2}}{2\alpha(1-r)} \]
if and only if
\[ d = d_{eq} = -\frac{\alpha}{2((1-r)\frac{n}{2} + 1)} + \sqrt{\frac{\alpha^2}{4((1-r)\frac{n}{2} + 1)^2} + \frac{\alpha(1-r)}{(1-r)^\frac{n}{2} + 1}}. \]
Consequently,
\[ \inf_{d>0} \max \left\{ \frac{nd + 1}{\alpha + d}; \sqrt{\frac{nd + 1}{\alpha(1-r)}} + \left(\frac{\alpha - (1-r)\frac{nd}{2}}{2\alpha(1-r)}\right)^2 - \frac{\alpha - (1-r)\frac{nd}{2}}{2\alpha(1-r)} \right\} \]
\[ = \min \left\{ \inf_{d \in (0,d_{eq})} \frac{nd + 1}{\alpha + d}; \inf_{d \in [d_{eq},\infty)} \sqrt{\frac{nd + 1}{\alpha(1-r)}} + \left(\frac{\alpha - (1-r)\frac{nd}{2}}{2\alpha(1-r)}\right)^2 - \frac{\alpha - (1-r)\frac{nd}{2}}{2\alpha(1-r)} \right\}. \]
For the first value we obtain
\[ \inf_{d \in (0,d_{eq})} \frac{nd + 1}{\alpha + d} = \frac{1}{\alpha} \text{ for } n > \frac{2}{\alpha}, \]
and
\[ \inf_{d \in (0,d_{eq})} \frac{nd + 1}{\alpha + d} = \frac{nd_{eq} + 1}{\alpha + d_{eq}} \text{ for } n \leq \frac{2}{\alpha}. \]
For the second value one can show that the function
\[ h : d \in [d_{eq}, \infty) \to h(d) = \sqrt{\frac{nd}{2} + 1} + \frac{(\alpha - (1 - r) \frac{nd}{2})^2}{2\alpha(1 - r)} - \frac{\alpha - (1 - r) \frac{nd}{2}}{2\alpha(1 - r)} \]
is increasing. For this reason we get
\[ \inf_{d \in [d_{eq}, \infty]} \sqrt{\frac{nd}{2} + 1} + \frac{(\alpha - (1 - r) \frac{nd}{2})^2}{2\alpha(1 - r)} - \frac{\alpha - (1 - r) \frac{nd}{2}}{2\alpha(1 - r)} = h(d_{eq}). \tag{4.3} \]
The conditions (4.1) to (4.3) imply together with Theorem 1.1 the following corollaries.

**Corollary 4.1.** Let \( 0 < \gamma < 1 \), \( p \in (1, \infty) \) and let the dimension \( n \) satisfy the condition \( n \leq \frac{2}{1 - \gamma} \). Assume that the data \((u_0, u_1) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)\) and satisfies the conditions
\[ \int_{\mathbb{R}^n} u_0(x) \, dx > 0 \text{ and } \int_{\mathbb{R}^n} u_1(x) \, dx > 0. \]
Then, if \( p \) satisfies for \( r \in (-1, 0) \) or \( r \in (0, 1) \) the condition
\[ \frac{p}{p - 1} > \frac{nd_{eq} + 1}{1 - \gamma + d_{eq}}, \]
where
\[ d_{eq} = \frac{1 - \gamma}{2((1 - r) \frac{n}{2} + 1)} + \sqrt{\frac{(1 - \gamma)^2}{4((1 - r) \frac{n}{2} + 1)^2 + (1 - \gamma)(1 - r)}} \]

then weak solutions, in the sense of Definition 1, of the Cauchy problem (1.1) do not exist globally in time.

**Corollary 4.2.** Let \( 0 < \gamma < 1 \), \( p \in (1, \infty) \) and let the dimension \( n \) satisfy the condition \( n \geq \frac{2}{1 - \gamma} \). Assume that the data \((u_0, u_1) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)\) and satisfies the conditions
\[ \int_{\mathbb{R}^n} u_0(x) \, dx > 0 \text{ and } \int_{\mathbb{R}^n} u_1(x) \, dx > 0. \]
Then, if \( p \) satisfies for \( r \in (-1, 0) \) or \( r \in (0, 1) \) the condition
\[ \frac{p}{p - 1} > \frac{1}{1 - \gamma}, \]
then weak solutions, in the sense of Definition 1, of the Cauchy problem (1.1) do not exist globally in time.

5. **Treatment of the critical case in low dimensions.** In this section we are interested to study the critical case in low dimensions, that is,
\[ \frac{p}{p - 1} = \frac{nd_{eq} + 1}{1 - \gamma + d_{eq}} \text{ if } n \leq \frac{2}{1 - \gamma}. \]
To consider this case we choose with a given positive \( A > 0 \) the time \( T > A \) and \( T \) large in comparison with \( A \). Moreover, we choose the constant \( K = K(T, A) := (\frac{T}{4})^{d_{eq}} \) (see also the paper [6]). Then following the calculations of Sections 3.5.1 to 3.5.3 we arrive at the estimate
\[ \int_{0}^{T} \int_{\Omega_{K(T, A)}} |u|^p \psi_T(t, x) \, dx \, dt \leq C(T^{-\delta} A^{-\frac{nd}{2}} + A^{-\frac{nd}{2}}) \]
with a suitable positive \( \delta \) and a constant \( C \) which is independent of \( A \) and \( T \). Let \( T > A \) tend to infinity we get
\[
\int_0^\infty \int_{\mathbb{R}^n} |u|^p \, dx \, dt \leq CA^{-\frac{n\gamma}{2}}.
\]
But, we can choose at the beginning \( A \) arbitrarily large. So,
\[
\int_0^\infty \int_{\mathbb{R}^n} |u|^p \, dx \, dt = 0.
\]
This gives immediately \( u \equiv 0 \) and this is a contradiction to (1.10).

**Corollary 5.1.** Let \( 0 < \gamma < 1 \), \( p \in (1, \infty) \) and let the dimension \( n \) satisfy the condition \( n \leq \frac{2}{1-\gamma} \). Assume that the data \( (u_0, u_1) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \) and satisfies the conditions
\[
\int_{\mathbb{R}^n} u_0(x) \, dx > 0 \quad \text{and} \quad \int_{\mathbb{R}^n} u_1(x) \, dx > 0.
\]
Then, if \( p \) satisfies for \( r \in (-1, 0) \) or \( r \in (0, 1) \) the condition
\[
\frac{p}{p-1} = \frac{nd_{eq}}{2} + 1,
\]
where
\[
d_{eq} = -\frac{1-\gamma}{2((1-r)\frac{n}{2} + 1)} + \sqrt{\frac{(1-\gamma)^2}{4((1-r)\frac{n}{2} + 1)^2} + \frac{(1-\gamma)(1-r)}{(1-r)\frac{n}{2} + 1} - \gamma},
\]
then weak solutions, in the sense of Definition 1, of the Cauchy problem (1.1) do not exist globally in time.

**Remark 5.1.** In the same way we can prove a blow-up result for \( r = 0 \) in the critical case
\[
\frac{p}{p-1} = \frac{n}{2} + 1 - \gamma.
\]

**Remark 5.2.** The treatment of the critical case in large dimensions, that is,
\[
\frac{p}{p-1} = \frac{1}{1-\gamma} \quad \text{if} \quad n > \frac{2}{1-\gamma}
\]
remains as an open problem.

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