BILINEAR DECOMPOSITIONS FOR THE PRODUCT SPACE 
\[ H_1^L \times BMO_L \]

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Abstract. In this paper, we improve a recent result by Li and Peng on products of functions in \( H_1^L(\mathbb{R}^d) \) and \( BMO_L(\mathbb{R}^d) \), where \( L = -\Delta + V \) is a Schrödinger operator with \( V \) satisfying an appropriate reverse Hölder inequality. More precisely, we prove that such products may be written as the sum of two continuous bilinear operators, one from \( H_1^L(\mathbb{R}^d) \times BMO_L(\mathbb{R}^d) \) into \( L^1(\mathbb{R}^d) \), the other one from \( H_1^L(\mathbb{R}^d) \times BMO_L(\mathbb{R}^d) \) into \( H_{\log}(\mathbb{R}^d) \), where the space \( H_{\log}(\mathbb{R}^d) \) is the set of distributions \( f \) whose grand maximal function \( Mf \) satisfies
\[
\int_{\mathbb{R}^d} \frac{|Mf(x)|}{\log(e + |Mf(x)|) + \log(e + |x|)} dx < \infty.
\]

1. Introduction

Products of functions in \( H^1 \) and \( BMO \) have been firstly considered by Bonami, Iwaniec, Jones and Zinsmeister in [2]. Such products make sense as distributions, and can be written as the sum of an integrable function and a function in a weighted Hardy-Orlicz space. To be more precise, for \( f \in H^1(\mathbb{R}^d) \) and \( g \in BMO(\mathbb{R}^d) \), we define the product (in the distribution sense) \( f \times g \) as the distribution whose action on the Schwartz function \( \varphi \in \mathcal{S}(\mathbb{R}^d) \) is given by
\[
\langle f \times g, \varphi \rangle := \langle \varphi g, f \rangle,
\]
where the second bracket stands for the duality bracket between \( H^1(\mathbb{R}^d) \) and its dual \( BMO(\mathbb{R}^d) \). It is then proven in [2] that
\[
f \times g \in L^1(\mathbb{R}^d) + H_{\Xi}^\sigma(\mathbb{R}^d).
\]
Here \( H_{\Xi}^\sigma(\mathbb{R}^d) \) is the weighted Hardy-Orlicz space related to the Orlicz function
\[
\Xi(t) := \frac{t}{\log(e + t)}
\]
and with weight \( \sigma(x) := \frac{1}{\log(e + |x|)} \).

Let \( L = -\Delta + V \) be a Schrödinger operator on \( \mathbb{R}^d \), \( d \geq 3 \), where \( V \) is a nonnegative potential, \( V \neq 0 \), and belongs to the reverse Hölder class \( RH_{d/2} \). In [3] and [4], Dziubański et al. introduced two kinds of function spaces associated with \( L \). One is

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Recall that the grand maximal operator

For each

Theorem 1. There are two bounded bilinear operators

and the uniform bound

where the second bracket stands for the duality bracket between

the dual space of

and functions

maybe not integrable. However, similarly to the classical setting, Li and Peng showed in [8] that such products can be defined in the sense of distributions which action on the Schwartz function

such that

More precisely, in [8], the authors proved the following.

Theorem 1. For each

there are two bounded linear operators

and the uniform bound

where

and its dual

Moreover, they proved that

can be written as the sum of two distributions, one in

the other in



Theorem 2. There are two bounded bilinear operators

such that for every

we have

and the uniform bound

where

and its weight

More precisely, in [8], the authors proved the following.

Theorem 2. There are two bounded bilinear operators

and

such that for every

we have

and the uniform bound

Here

is a new kind of Hardy-Orlicz space consisting of all distributions

such that

with the norm

Recall that the grand maximal operator

is defined by

(1.9)
where \( \mathcal{A} = \{ \phi \in \mathcal{S}(\mathbb{R}^d) : |\phi(x)| + |\nabla \phi(x)| \leq (1 + |x|^2)^{-(d+1)} \} \) and \( \phi_t(\cdot) := t^{-d} \phi(t^{-1} \cdot) \).

Note that \( H^{\text{log}}(\mathbb{R}^d) \subset H^\Xi_{L,\sigma}(\mathbb{R}^d) \) with continuous embedding, see Section 3. Compared with the main result of [8] (Theorem 1), our main result makes an essential improvement in two directions. The first one consists in proving that the space \( H^\Xi_{L,\sigma}(\mathbb{R}^d) \) can be replaced by a smaller space \( H^{\text{log}}(\mathbb{R}^d) \). Secondly, we give the bilinear decomposition (1.7) for the product space \( H^1_L(\mathbb{R}^d) \times \text{BMO}_L(\mathbb{R}^d) \) instead of the linear decomposition (1.5) depending on \( f \in H^1_L(\mathbb{R}^d) \). Moreover, we just need the \( \text{BMO}_L \)-norm (see (1.8)) instead of the \( \text{BMO}_L^+ \)-norm as in (1.6).

In applications to nonlinear PDEs, the distribution \( f \times g \in \mathcal{S}'(\mathbb{R}^d) \) is used to justify weak continuity properties of the pointwise product \( fg \). It is therefore important to recover \( fg \) from the action of the distribution \( f \times g \) on the test functions. An idea that naturally comes to mind is to look at the mollified distributions

\[
(f \times g)_\epsilon = (f \times g) \ast \phi_\epsilon,
\]

and let \( \epsilon \to 0 \). Here \( \phi \in \mathcal{S}(\mathbb{R}^d) \) with \( \int_{\mathbb{R}^d} \phi(x)dx = 1 \).

In the classical setting of \( f \in H^1(\mathbb{R}^d) \) and \( g \in \text{BMO}(\mathbb{R}^d) \), Bonami et al. proved in [2] that the limit (1.10) exists and equals \( fg \) almost everywhere. An analogous result is also true for the Schrödinger setting. Namely, the following is true.

**Theorem 3.** Let \( f \in H^1_L(\mathbb{R}^d) \) and \( g \in \text{BMO}_L(\mathbb{R}^d) \). Then, for almost every \( x \in \mathbb{R}^d \),

\[
\lim_{\epsilon \to 0} (f \times g)_\epsilon(x) = f(x)g(x).
\]

Throughout the whole paper, \( C \) denotes a positive geometric constant which is independent of the main parameters, but may change from line to line.

The paper is organized as follows. In Section 2, we present some notations and preliminaries about Hardy type spaces associated with \( L \). Section 3 is devoted to prove that \( H^{\text{log}}(\mathbb{R}^d) \subset H^\Xi_{L,\sigma}(\mathbb{R}^d) \) with continuous embedding. Finally, the proofs of Theorem 2 and Theorem 3 are given in Section 4.

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## 2. Some preliminaries and notations

In this paper, we consider the Schrödinger differential operator

\[ L = -\Delta + V \]

on \( \mathbb{R}^d \), \( d \geq 3 \), where \( V \) is a nonnegative potential, \( V \neq 0 \). As in the works of Dziubański et al [3, 4], we always assume that \( V \) belongs to the reverse Hölder class \( RH_{d/2} \). Recall that a nonnegative locally integrable function \( V \) is said to belong to a reverse Hölder class \( RH_q \), \( 1 < q < \infty \), if there exists \( C > 0 \) such that

\[
\left( \frac{1}{|B|} \int_B (V(x))^q dx \right)^{1/q} \leq \frac{C}{|B|} \int_B V(x) dx
\]

holds for every balls \( B \) in \( \mathbb{R}^d \).
Let \( \{T_t\}_{t>0} \) be the semigroup generated by \( L \) and \( T_t(x, y) \) be their kernels. Namely,
\[
T_t f(x) = e^{-tL}f(x) = \int_{\mathbb{R}^d} T_t(x, y)f(y)dy, \quad f \in L^2(\mathbb{R}^d), \quad t > 0.
\]

We say that a function \( f \in L^2(\mathbb{R}^d) \) belongs to the space \( H_{1L}(\mathbb{R}^d) \) if
\[
\|f\|_{H_{1L}} := \|M_L f\|_{L^1} < \infty,
\]
where \( M_L f(x) := \sup_{t>0} |T_t f(x)| \) for all \( x \in \mathbb{R}^d \). The space \( H_{1L}(\mathbb{R}^d) \) is then defined as the completion of \( H_{1L}(\mathbb{R}^d) \) with respect to this norm.

In [3] it was shown that the dual of \( H_{1L}(\mathbb{R}^d) \) can be identified with the space \( BMO_L(\mathbb{R}^d) \) which consists of all functions \( f \in BMO(\mathbb{R}^d) \) with
\[
\|f\|_{BMO_L} := \|f\|_{BMO} + \sup_{\rho \leq 1} \frac{1}{|B(x, r)|} \int_{B(x, r)} |f(y)|dy < \infty,
\]
where \( \rho \) is the auxiliary function defined as in [9], that is,
\[
\rho(x) = \sup \left\{ r > 0 : \frac{1}{r^{d-2}} \int_{B(x, r)} V(y)dy \leq 1 \right\},
\]
for all \( x \in \mathbb{R}^d \). Clearly, \( 0 < \rho(x) < \infty \) for all \( x \in \mathbb{R}^d \), and thus \( \mathbb{R}^d = \bigcup_{n \in \mathbb{Z}} B_n \), where the sets \( B_n \) are defined by
\[
B_n = \{ x \in \mathbb{R}^d : 2^{-(n+1)/2} < \rho(x) \leq 2^{-n/2} \}.
\]

The following proposition is due to Shen [9].

**Proposition 2.1** (see [9], Lemma 1.4). **There exist** \( C_0 > 1 \) and \( k_0 \geq 1 \) such that **for all** \( x, y \in \mathbb{R}^d \),
\[
C_0^{-1} \rho(x) \left( 1 + \frac{|x - y|}{\rho(x)} \right)^{-k_0} \leq \rho(y) \leq C_0 \rho(x) \left( 1 + \frac{|x - y|}{\rho(x)} \right)^{k_0 + 1}.
\]

Here and in what follows, we denote by \( C_L \) the \( L \)-constant
\[
C_L = 8.9^{k_0} C_0
\]
where \( k_0 \) and \( C_0 \) are defined as in Proposition 2.1.

**Definition 2.1.** **Given** \( 1 < q \leq \infty \). **A function** \( a \) is called a \( (H_{1L}, q) \)-atom related to the ball \( B(x_0, r) \) if \( r \leq C_L \rho(x_0) \) and
i) \( \text{supp} \ a \subset B(x_0, r) \),
ii) \( \|a\|_{L^q} \leq |B(x_0, r)|^{1/q - 1} \),
iii) **if** \( r \leq \frac{1}{C_L} \rho(x_0) \) **then** \( \int_{\mathbb{R}^d} a(x)dx = 0 \).

The following atomic characterization of \( H_{1L}(\mathbb{R}^d) \) is due to Dziubański and Zienkiewicz [4].
Theorem A. Let $1 < q \leq \infty$. A function $f$ is in $H^1_L(\mathbb{R}^d)$ if and only if it can be written as $f = \sum_j \lambda_j a_j$, where $a_j$ are $(H^1_L, q)$-atoms and $\sum_j |\lambda_j| < \infty$. Moreover, there exists $C > 1$ such that for every $f \in H^1_L(\mathbb{R}^d)$, we have

$$
C^{-1} \|f\|_{H^1_L} \leq \inf \left\{ \sum_j |\lambda_j| : f = \sum_j \lambda_j a_j \right\} \leq C \|f\|_{H^1_L}.
$$

Let $1 \leq q < \infty$. A nonnegative locally integrable function $w$ belongs to the Muckenhoupt class $A_q$, say $w \in A_q$, if there exists a positive constant $C$ so that

$$(2.4) \quad \frac{1}{|B|} \int_B w(x) dx \left( \frac{1}{|B|} \int_B (w(x))^{-1/(q-1)} dx \right)^{q-1} \leq C, \quad \text{if } 1 < q < \infty,$$

and

$$(2.5) \quad \frac{1}{|B|} \int_B w(x) dx \leq C \text{ess-inf}_{x \in B} w(x), \quad \text{if } q = 1,$$

for all balls $B$ in $\mathbb{R}^d$. We say that $w \in A_\infty$ if $w \in A_q$ for some $q \in [1, \infty)$.

Remark 2.1. The weight $\sigma(x) \equiv \frac{1}{\log(e+|x|)}$ belongs to the class $A_1$.

It is well known that $w \in A_p, 1 \leq p < \infty$, implies $w \in A_q$ for all $q > p$. For a measurable set $E$, we note $w(E) = \int_E w(x) dx$ its weighted measure.

Definition 2.2. Let $0 < p \leq 1$. A function $\Phi$ is called a growth function of order $p$ if it satisfies the following properties:

i) The function $\Phi$ is a Orlicz function, that is, $\Phi$ is a nondecreasing function with $\Phi(t) > 0, t > 0, \Phi(0) = 0$ and $\lim_{t \to \infty} \Phi(t) = \infty$.

ii) The function $\Phi$ is of lower type $p$, that is, there exists a constant $C > 0$ such that for every $s \in (0, 1]$ and $t > 0$,

$$
\Phi(st) \leq Cs^p \Phi(t).
$$

iii) The function $\Phi$ is of upper type $1$, that is, there exists a constant $C > 0$ such that for every $s \in [1, \infty)$ and $t > 0$,

$$
\Phi(st) \leq Cs \Phi(t).
$$

We will also say that $\Phi$ is a growth function whenever it is a growth function of some order $p < 1$.

Remark 2.2. i) Let $\Phi$ be a growth function. Then, there exists a constant $C > 0$ such that

$$
\Phi \left( \sum_{j=1}^\infty t_j \right) \leq C \sum_{j=1}^\infty \Phi(t_j)
$$

for every sequence $\{t_j\}_{j \geq 1}$ of nonnegative real numbers. See Lemma 4.1 of [1].

ii) The function $\Xi(t) \equiv \frac{1}{\log(e+t)}$ is a growth function of order $p$ for any $p \in (0, 1)$. 

Now, let us define weighted Hardy-Orlicz spaces associated with $L$.

**Definition 2.3.** Given $w \in A_{\infty}$ and $\Phi$ a growth function. We say that a function $f \in L^2(\mathbb{R}^d)$ belongs to $H^{\Phi}_{L,w}(\mathbb{R}^d)$ if $\int_{\mathbb{R}^d} \Phi(M_{L}f(x))w(x)dx < \infty$. The space $H^{\Phi}_{L,w}(\mathbb{R}^d)$ is defined as the completion of $H^{\Phi}_{L,w}(\mathbb{R}^d)$ with respect to the norm

$$\|f\|_{H^{\Phi}_{L,w}} := \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^d} \Phi \left( \frac{M_{L}f(x)}{\lambda} \right)w(x)dx \leq 1 \right\}.$$

Remark that when $w(x) \equiv 1$ and $\Phi(t) \equiv t$, the space $H^{\Phi}_{L,w}(\mathbb{R}^d)$ is just $H^1_L(\mathbb{R}^d)$.

We refer the reader to the recent work of D. Yang and S. Yang [10] for a complete study of the theory of weighted Hardy-Orlicz spaces associated with operators.

3. The inclusion $H^{\log}(\mathbb{R}^d) \subset H^{\Xi}_{L,\sigma}(\mathbb{R}^d)$

The purpose of this section is to establish the following embedding.

**Proposition 3.1.** $H^{\log}(\mathbb{R}^d) \subset H^{\Xi}_{L,\sigma}(\mathbb{R}^d)$ and the inclusion is continuous.

Recall (see [6]) that the weighted Hardy-Orlicz space $H^{\Xi}_{\sigma}(\mathbb{R}^d)$ is defined as the space of all distributions $f$ such that $\int_{\mathbb{R}^d} \frac{\mathcal{M}_{L}f(x)}{\log(e + \mathcal{M}_{L}f(x))} \frac{1}{\log(e + |x|)} dx < \infty$ with the norm

$$\|f\|_{H^{\Xi}_{\sigma}} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^d} \frac{\mathcal{M}_{L}f(x)}{\lambda} \frac{1}{\log \left( e + \frac{\mathcal{M}_{L}f(x)}{\lambda} \right) \log(e + |x|)} dx \leq 1 \right\}.$$

Clearly, $H^{\log}(\mathbb{R}^d) \subset H^{\Xi}_{\sigma}(\mathbb{R}^d)$ and the inclusion is continuous. Consequently, the proof of Proposition 3.1 can be reduced to showing that for every $f \in H^{\Xi}_{\sigma}(\mathbb{R}^d)$,

$$\|f\|_{H^{\log}_{L,\sigma}} \leq C \|f\|_{H^{\Xi}_{\sigma}}.$$

Let $1 < q \leq \infty$. Recall (see [6]) that a function $a$ is called a $(H^{\Xi}_{\sigma}, q)$-atom related to the ball $B$ if

i) $\text{supp } a \subset B$,

ii) $\|a\|_{L^q} \leq \sigma(B)^{1/q} \Xi^{-1}(\sigma(B)^{-1})$, where $\Xi^{-1}$ is the inverse function of $\Xi$;

iii) $\int_{\mathbb{R}^d} a(x)dx = 0$.

In order to prove Proposition 3.1, we need the following lemma.

**Lemma 3.1.** Let $1 < q < \infty$. Then,

$$\int_{\mathbb{R}^d} \Xi(\mathcal{M}_{L}f(x))\sigma(x)dx \leq C \sigma(B)\Xi(\sigma(B)^{-1/q} \|f\|_{L^q})$$

for every $f$ multiples of $(H^{\Xi}_{\sigma}, q)$-atom related to the ball $B = B(x_0, r)$.

To prove Lemma 3.1, let us recall the following.
Lemma 3.2 (see \cite{8}, Lemma 2). Let \( V \in RH_{d/2} \). Then, there exists \( \delta > 0 \) depends only on \( L \), such that for every \( |y - z| < |x - y|/2 \) and \( t > 0 \), we have

\[
|T_t(x, y) - T_t(x, z)| \leq C \left( \frac{|y - z|}{\sqrt{t}} \right)^d t^{-\frac{d}{q}} e^{-\frac{|x - y|^2}{4t}} \leq C \frac{|y - z|^d}{|x - y|^{d+\delta}}.
\]

Proof of Lemma 3.1. First, note that \( \sigma \in A_1 \) and \( \Xi \) is a growth function of order \( p \) for any \( p \in (0, 1) \), see Remark 2.1 and Remark 2.2. Denote by \( \mathcal{M} \) the classical Hardy-Littlewood maximal operator. Then, the estimate \( \mathcal{M}_L f \leq C \mathcal{M} f \), the \( L^q_\sigma \)-boundedness of \( \mathcal{M} \) and Hölder inequality give

\[
\int_{B(x_0, 2r)} \Xi(\mathcal{M}_L f(x)) \sigma(x) dx 
\leq C \int_{B(x_0, 2r)} \Xi(\mathcal{M} f(x) + \sigma(B)^{-1/q} \|f\|_{L^q_\sigma}) \sigma(x) dx 
\leq C \int_{B(x_0, 2r)} \left( \frac{\mathcal{M} f(x) + \sigma(B)^{-1/q} \|f\|_{L^q_\sigma}}{\sigma(B)^{-1/q} \|f\|_{L^q_\sigma}} \right) \Xi(\sigma(B)^{-1/q} \|f\|_{L^q_\sigma}) \sigma(x) dx 
\leq C \sigma(B) \Xi(\sigma(B)^{-1/q} \|f\|_{L^q_\sigma}),
\]

where we used the facts that \( t \mapsto \Xi(t) \) is nonincreasing and \( \sigma(B(x_0, 2r)) \leq C \sigma(B) \).

Let \( x \notin B(x_0, 2r) \) and \( t > 0 \). By Lemma 3.2 and (2.4),

\[
|T_t f(x)| = \left| \int_{\mathbb{R}^d} T_t(x, y) f(y) dy \right| = \left| \int_{B} (T_t(x, y) - T_t(x, x_0)) f(y) dy \right| 
\leq C \int_{B} \frac{|y - x_0|^\delta}{|x - x_0|^{d+\delta}} |f(y)| dy 
\leq C \sigma(B)^{-1/q} \|f\|_{L^q_\sigma} \frac{r^{d+\delta}}{|x - x_0|^{d+\delta}}.
\]

Therefore, as \( \Xi \) is of lower type \( \frac{2d+\delta}{2(d+\delta)} < 1 \),

\[
\int_{(B(x_0, 2r))^c} \Xi(\mathcal{M}_L f(x)) \sigma(x) dx 
\leq C \Xi(\sigma(B)^{-1/q} \|f\|_{L^q_\sigma}) \int_{(B(x_0, 2r))^c} \left( \frac{r^{d+\delta}}{|x - x_0|^{d+\delta}} \right)^{2d+\delta/(d+\delta)} \sigma(x) dx 
\leq C \sigma(B) \Xi(\sigma(B)^{-1/q} \|f\|_{L^q_\sigma}),
\]

(3.4)
where we used (see [5], page 412)
\[ \int_{(B(x_0, 2r))^c} \frac{r^{d+\delta/2}}{|x - x_0|^{d+\delta/2}} \sigma(x) dx \leq C \sigma(B(x_0, 2r)) \leq C \sigma(B). \]

Then, (3.2) follows from (3.3) and (3.4). This completes the proof. \(\square\)

**Proof of Proposition 3.1.** As mentioned above, it is sufficient to show that
\[ \|f\|_{H^\Xi_{L, \sigma}} \leq C \|f\|_{H^\Xi_{\sigma}} \]
for every \( f \in H^\Xi_{\sigma}(\mathbb{R}^d) \). By Theorem 3.1 of [6], there are multiples of \((H^\Xi_{\sigma}, 2)-\)atoms \( b_j, j = 1, 2, ... \), related to balls \( B_j \) such that \( f = \sum_{j=1}^\infty b_j \) and
\[ \Lambda_2(\{b_j\}) \leq C \|f\|_{H^\Xi_{\sigma}}, \]
where
\[ \Lambda_2(\{b_j\}) := \inf \left\{ \lambda > 0 : \sum_{j=1}^\infty \sigma(B_j) \Xi \left( \frac{\sigma(B_j)^{-1/2} \|b_j\|_{L^2}}{\lambda} \right) \leq 1 \right\}. \]

On the other hand, the estimate \( M_L f \leq \sum_{j=1}^\infty M_L(b_j) \), Remark 2.2 and Lemma 3.1 give
\[ \int_{\mathbb{R}^d} \Xi \left( \frac{M_L f(x)}{\Lambda_2(\{b_j\})} \right) \sigma(x) dx \leq C \sum_{j=1}^\infty \int_{\mathbb{R}^d} \Xi \left( \frac{M_L(b_j)(x)}{\Lambda_2(\{b_j\})} \right) \sigma(x) dx \]
\[ \leq C \sum_{j=1}^\infty \sigma(B_j) \Xi \left( \frac{\sigma(B_j)^{-1/2} \|b_j\|_{L^2}}{\Lambda_2(\{b_j\})} \right) \]
\[ \leq C, \]
which implies that \( \|f\|_{H^\Xi_{L, \sigma}} \leq C \Lambda_2(\{b_j\}) \). Therefore, (3.5) yields
\[ \|f\|_{H^\Xi_{L, \sigma}} \leq C \|f\|_{H^\Xi_{\sigma}}, \]
which completes the proof of Proposition 3.1. \(\square\)

4. PROOF OF THEOREM 2 AND THEOREM 3

Let \( P(x) = (4\pi)^{-d/2} e^{-|x|^2/4} \) be the Gauss function. For \( n \in \mathbb{Z} \), following [4], the space \( h^1_n(\mathbb{R}^d) \) denotes the space of all integrable functions \( f \) such that
\[ M_n f(x) = \sup_{0 < t < 2^{-n}} |P_{\sqrt{t}} \ast f(x)| = \sup_{0 < t < 2^{-n}} \left| \int_{\mathbb{R}^d} p_t(x, y) f(y) dy \right| \in L^1(\mathbb{R}^d), \]
where the kernel \( p_t \) is given by \( p_t(x, y) = (4\pi t)^{-d/2} e^{-|x-y|^2/4t} \). We equipped this space with the norm \( \|f\|_{h^1_n} := \|M_n f\|_{L^1} \).

For convenience of the reader, we list here some lemmas used in our proofs.
Lemma 4.1 (see [4], Lemma 2.3). There exists a constant $C > 0$ and a collection of balls $B_{n,k} = B(x_{n,k}, 2^{-n/2})$, $n \in \mathbb{Z}$, $k = 1, 2, \ldots$, such that $x_{n,k} \in B_n$, $B_n \subset \bigcup_k B_{n,k}$, and
\[
\text{card } \{ (n', k') : B(x_{n,k}, R2^{-n/2}) \cap B(x_{n',k'}, R2^{-n'/2}) \neq \emptyset \} \leq R^C
\]
for all $n, k$ and $R \geq 2$.

Lemma 4.2 (see [4], Lemma 2.5). There are non-negative $C^\infty$-functions $\psi_{n,k}$, $n \in \mathbb{Z}$, $k = 1, 2, \ldots$, supported in the balls $B(x_{n,k}, 2^{1-n/2})$ such that
\[
\sum_{n,k} \psi_{n,k} = 1 \quad \text{and} \quad \|\nabla \psi_{n,k}\|_{L^\infty} \leq C2^{n/2}.
\]

Lemma 4.3 (see (4.7) in [4]). For every $f \in H^1_0(\mathbb{R}^d)$, we have
\[
\sum_{n,k} \|\psi_{n,k}f\|_{h_k} \leq C\|f\|_{H^1_0}.
\]

In this section, we fix a non-negative function $\varphi \in \mathcal{S}(\mathbb{R}^d)$ with supp $\varphi \subset B(0,1)$ and $\int_{\mathbb{R}^d} \varphi(x)dx = 1$. Then, we define the linear operator $\mathcal{S}$ by
\[
\mathcal{S}(f) = \sum_{n,k} \left( \psi_{n,k}f - \varphi_{2-n/2} * (\psi_{n,k}f) \right).
\]

In order to prove Theorem 2, we need two key lemmas.

Lemma 4.4. The operator $\mathcal{S}$ maps continuously $H^1_0(\mathbb{R}^d)$ into $H^1(\mathbb{R}^d)$.

The proof of Lemma 4.4 can be found in [7] (see Lemma 5.1 of [7]).

Lemma 4.5. There exists a constant $C = C(\varphi, d) > 0$ such that for all $(n,k) \in \mathbb{Z} \times \mathbb{Z}^+$, $g \in BMO_L(\mathbb{R}^d)$ and $f \in h^1_n(\mathbb{R}^d)$ with supp $f \subset B(x_{n,k}, 2^{1-n/2})$, we have
\[
\left\| (\varphi_{2-n/2} * f)g \right\|_{H^1_L} \leq C\|f\|_{h_k} \|g\|_{BMO_L}.
\]

To prove Lemma 4.5 we need the following.

Lemma 4.6 (see [7], Lemma 6.5). Let $1 < q \leq \infty$, $n \in \mathbb{Z}$ and $x \in B_n$. Suppose that $f \in h^1_n(\mathbb{R}^d)$ with supp $f \subset B(x, 2^{1-n/2})$. Then, there are $(H^1_L, q)$-atoms $a_j$ related to the balls $B(x_j, r_j)$ such that $B(x_j, r_j) \subset B(x, 2^{2-n/2})$ and
\[
f = \sum_j \lambda_j a_j, \quad \sum_j |\lambda_j| \leq C\|f\|_{h_k}
\]
with a positive constant $C$ independent of $n$ and $f$.

Here and in what follows, for any $B$ a ball in $\mathbb{R}^d$ and $f$ a locally integrable function, we denote by $f_B$ the average of $f$ on $B$. 

THE PRODUCT SPACE $H^1_L \times BMO_L$
Proof of Lemma 4.5. As $x_{n,k} \in B_n$, it follows from Lemma 4.6 that there are $(H^1_L, 2)$-atoms $a_{j,n,k}^n$ related to the balls $B(x_{n,k}, 2^{-n/2}) \subset B(x_{n,k}, 2^{-n/2})$ such that

$$f = \sum_j \lambda_j a_{j,n,k}^n$$

and $\sum_j |\lambda_j| \leq C\|f\|_{L^1}$, where the positive constant $C$ is independent of $f, n, k$.

Now, let us establish that $\varphi_{2^{-n/2}} \ast a_{j,n,k}^n$ is $C$ times a $(H^1_L, 2)$-atom related to the ball $B(x_{n,k}, 5.2^{-n/2})$. Indeed, it is clear that $\frac{1}{CL}\rho(x_{n,k}) < 5.2^{-n/2} < CL\rho(x_{n,k})$ since $x_{n,k} \in B_n$; and $\text{supp} \varphi_{2^{-n/2}} \ast a_{j,n,k}^n \subset B(x_{n,k}, 5.2^{-n/2})$ since $\text{supp} \varphi \subset B(0, 1)$ and $a_{j,n,k}^n \subset B(x_{n,k}, 2^{-n/2})$. In addition,

$$\|\varphi_{2^{-n/2}} \ast a_{j,n,k}^n\|_{L^2} \leq \|\varphi_{2^{-n/2}}\|_{L^2} \|a_{j,n,k}^n\|_{L^2} \leq (2^{-n/2})^{-d/2} \|\varphi\|_{L^2} \leq C|B(x_{n,k}, 5.2^{-n/2})|^{-1/2}.$$

These prove that $\varphi_{2^{-n/2}} \ast a_{j,n,k}^n$ is $C$ times a $(H^1_L, 2)$-atom related to $B(x_{n,k}, 5.2^{-n/2})$.

By an analogous argument, it is easy to check that $(\varphi_{2^{-n/2}} \ast a_{j,n,k}^n)(g - g_B(x_{n,k}, 5.2^{-n/2}))$ is $C\|g\|_{BMO}$ times a $(H^1_L, 3/2)$-atom related to $B(x_{n,k}, 5.2^{-n/2})$.

Therefore, (4.1) yields

$$\left\| (\varphi_{2^{-n/2}} \ast f)g \right\|_{H^1_L} \leq C \sum_j |\lambda_j| \| (\varphi_{2^{-n/2}} \ast a_j)(g - g_B(x_{n,k}, 5.2^{-n/2})) \|_{H^1_L}$$

$$+ C \sum_j |\lambda_j| \| \varphi_{2^{-n/2}} \ast a_j \|_{H^1_L} \| g_B(x_{n,k}, 5.2^{-n/2}) \|$$

$$\leq C \|f\|_{H^s_L} \|g\|_{BMO},$$

where we used $|g_B(x_{n,k}, 5.2^{-n/2})| \leq \|g\|_{BMO}$ since $\rho(x_{n,k}) \leq 5.2^{-n/2}$.

Our main results are strongly related to the recent result of Bonami, Grellier and Ky [1]. In [1], the authors proved the following.

**Theorem 4.** There exists two continuous bilinear operators on the product space $H^1(\mathbb{R}^d) \times BMO(\mathbb{R}^d)$, respectively $S : H^1(\mathbb{R}^d) \times BMO(\mathbb{R}^d) \mapsto L^1(\mathbb{R}^d)$ and $T : H^1(\mathbb{R}^d) \times BMO(\mathbb{R}^d) \mapsto H^{log}(\mathbb{R}^d)$ such that

$$f \times g = S(f, g) + T(f, g).$$

Before giving the proof of the main theorems, we should point out that the bilinear operator $T$ in Theorem 4 satisfies

$$\|T(f, g)\|_{H^{log}} \leq C\|f\|_{H^1}(\|g\|_{BMO} + \|g_Q\|)$$

where $Q := [0, 1)^d$ is the unit cube. To prove this, the authors in [1] used the generalized Hölder inequality (see also [2])

$$\|fg\|_{L^{log}} \leq C\|f\|_{L^1} \|g\|_{exp}$$
and the fact that \( \|g - g_Q\|_{\text{Exp}} \leq C \|g\|_{BMO} \). Here, \( L^\log(\mathbb{R}^d) \) denotes the space of all measurable functions \( f \) such that \( \int_{\mathbb{R}^d} \frac{|f(x)|}{\lambda} \log(e + |f(x)| + \log(e + |x|)) \, dx < \infty \) with the norm

\[
\|f\|_{L^\log} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^d} \frac{|f(x)|}{\lambda} \log(e + |f(x)| + \log(e + |x|)) \, dx \leq 1 \right\}
\]

and \( \text{Exp}(\mathbb{R}^d) \) denotes the space of all measurable functions \( f \) such that \( \int_{\mathbb{R}^d} \frac{1}{1 + |x|^{2d}} \, dx < \infty \) with the norm

\[
\|f\|_{\text{Exp}} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^d} \left( \frac{e^{|f(x)|}/\lambda - 1}{(1 + |x|)^2} \right) \, dx \leq 1 \right\}.
\]

In fact, Inequality (4.2) also holds when we replace the unit cube \( Q \) by \( B(0, r) \) for every \( r > 0 \) since \( \|g - g_{B(0, r)}\|_{\text{Exp}} \leq C \|g\|_{BMO} \). More precisely, there exists a constant \( C > 0 \) such that

\[
\|f\|_{L^\log} \leq C \|f\|_{L^1} (\|g\|_{BMO} + |g_{B(0, \rho(0))}|) \leq C \|f\|_{L^1} \|g\|_{BMO_L}
\]

for all \( f \in L^1(\mathbb{R}^d) \) and \( g \in BMO_L(\mathbb{R}^d) \). As a consequence, we obtain

\[
\|T(f, g)\|_{H^\log} \leq C \|f\|_{H^1} \|g\|_{BMO_L}
\]

for all \( f \in H^1(\mathbb{R}^d) \) and \( g \in BMO_L(\mathbb{R}^d) \).

Now, we are ready to give the proof of the main theorems.

**Proof of Theorem**

We define two bilinear operators \( S_L \) and \( T_L \) by

\[
S_L(f, g) = S(\mathcal{F}(f), g) + \sum_{n,k} (\varphi_{2^{-n/2}} \ast (\psi_{n,k} f)) g
\]

and

\[
T_L(f, g) = T(\mathcal{F}(f), g)
\]

for all \( (f, g) \in H^1_L(\mathbb{R}^d) \times BMO_L(\mathbb{R}^d) \). Then, it follows from Theorem 4.3 and Lemma 4.4 that

\[
\|S_L(f, g)\|_{L^1} \leq \|S(\mathcal{F}(f), g)\|_{L^1} + C \sum_{n,k} \left( \varphi_{2^{-n/2}} \ast (\psi_{n,k} f) \right) g \|_{H^1_L}
\]

\[
\leq C \|g\|_{BMO} \|\mathcal{F}(f)\|_{H^1} + C \|g\|_{BMO_L} \sum_{n,k} \|\psi_{n,k} f\|_{H^1_L}
\]

\[
\leq C \|f\|_{H^1_L} \|g\|_{BMO_L},
\]

and as 4.4,

\[
\|T_L(f, g)\|_{H^\log} = \|T(\mathcal{F}(f), g)\|_{H^\log} \leq C \|\mathcal{F}(f)\|_{H^1} \|g\|_{BMO_L} \leq C \|f\|_{H^1_L} \|g\|_{BMO_L}.
\]
Furthermore, in the sense of distributions, we have
\[
S_L(f, g) + T_L(f, g)
= \left( \sum_{n,k} \left( \psi_{n,k} f - \varphi_{2^{-n/2}} (\psi_{n,k} f) \right) \right) \times g + \sum_{n,k} \left( \varphi_{2^{-n/2}} (\psi_{n,k} f) \right) g
= \left( \sum_{n,k} \psi_{n,k} f \right) \times g = f \times g,
\]
which ends the proof of Theorem 2.

\[ \square \]

Proof of Theorem. By the proof of Theorem 2, the function \( \sum_{n,k} (\varphi_{2^{-n/2}} (\psi_{n,k} f)) g \) belongs to \( H^1_L(\mathbb{R}^d) \subset L^1(\mathbb{R}^d) \). This implies that \( \left( \sum_{n,k} (\varphi_{2^{-n/2}} (\psi_{n,k} f)) g \right) \ast \phi_\epsilon \) tends to \( \sum_{n,k} (\varphi_{2^{-n/2}} (\psi_{n,k} f)) g \) almost everywhere, as \( \epsilon \to 0 \). Therefore, applying Theorem 1.8 of [2], we get
\[
\lim_{\epsilon \to 0} (f \times g)_{\epsilon}(x) = \lim_{\epsilon \to 0} (S_L(f) \times g)_{\epsilon}(x) + \lim_{\epsilon \to 0} \left( \sum_{n,k} (\varphi_{2^{-n/2}} (\psi_{n,k} f)) g \right) \ast \phi_\epsilon(x)
= S_L(f)(x)g(x) + \left( \sum_{n,k} (\varphi_{2^{-n/2}} (\psi_{n,k} f))(x) \right) g(x)
= f(x)g(x)
\]
for almost every \( x \in \mathbb{R}^d \), which completes the proof of Theorem 3.

\[ \square \]

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