Several types of solvable groups as automorphism groups of compact Riemann surfaces

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Abstract

Let $X$ be a compact Riemann surface of genus $g \geq 2$. Let $\text{Aut}(X)$ be its group of automorphisms and $G \subseteq \text{Aut}(X)$ a subgroup. Sharp upper bounds for $|G|$ in terms of $g$ are known if $G$ belongs to certain classes of groups, e.g. solvable, supersolvable, nilpotent, metabelian, metacyclic, abelian, cyclic. We refine these results by finding similar bounds for groups of odd order that are of these types. We also add more types of solvable groups to that long list by establishing the optimal bounds for, among others, groups of order $p^m q^n$. Moreover, we show that Zomorrodian’s bound for $p$-groups $G$ with $p \geq 5$, namely $|G| \leq \frac{2p}{p-3}(g-1)$, actually holds for any group $G$ for which $p \geq 5$ is the smallest prime divisor of $|G|$.

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1. Introduction

We write $|G|$ for the order of a group $G$ and $G'$ for the commutator group. Also $C_n$ stands for a cyclic group of order $n$.

In this paper $X$ will always be a compact Riemann surface of genus $g \geq 2$. Its full group of conformal automorphisms is denoted by $\text{Aut}(X)$.

It is a classical theorem by Hurwitz that then $|\text{Aut}(X)| \leq 84(g-1)$. See for example [B, Theorem 3.17].

However, not every group whose order is divisible by 84 can occur in Hurwitz’s Theorem. For example, if $g = 7^n + 1$, a group of order $84(g-1) = 12 \cdot 7^{n+1}$ must be solvable by the Sylow Theorems; and for solvable groups there are stronger bounds (see Theorem 2.2 (a) below).

More generally, if $G$ is a (not necessarily proper) subgroup of $\text{Aut}(X)$ and $G$ belongs to a certain type of groups, one is interested in having a bound on $|G|$ in terms of $g$. Ideally one would like to have for any given type of groups a simple
function \( b(g) \) such that \(|G| \leq b(g)\) except for finitely many (explicitly known) \( g \), and that for infinitely many values of \( g \) there is a \( G \) of the desired type with \(|G| = b(g)\).

Such a function \( b(g) \) is of course not guaranteed to exist. But for many interesting types of groups it does. From the rich literature we summarize the results that are relevant to this paper.

**TABLE 1**

| class of \( G \)          | upper bound | exceptions           | source          |
|---------------------------|-------------|----------------------|-----------------|
| general                   | 84\((g - 1)\) | none                 | classical       |
| solvable                  | 48\((g - 1)\) | none                 | [Ch], [G1], [G2]|
| supersolvable             | 18\((g - 1)\) | for \( g = 2 \)      | [Z3], [GMI], [Z4]| |
| nilpotent                 | 16\((g - 1)\) | none                 | [Z1]            |
| metabelian                | 16\((g - 1)\) | for \( g = 2, 3, 5 \)| [ChP], [G2], [G3]| |
| metacyclic                | 12\((g - 1)\) | for \( g = 2 \)      | [Sch2]          |
| \( |G| \) is square-free    | 10\((g - 1)\) | for \( g = 3 \)      | [Sch2]          |
| abelian                   | 4\(g + 4\)  | none                 | classical, see also [Ml], [G1] |
| cyclic                    | 4\(g + 2\)  | none                 | classical, see also [Ha], [G1], [N] |

In Theorem 2.2 in the next section we will report more details on some of the groups. Here we only mention that the first proof for metabelian groups is in [ChP]. But the last theorem of [G2] points out that a metabelian group of order 24 for \( g = 2 \), covered by \( \Gamma(0; 2, 4, 6) \), had been overlooked. See also [G3] for a detailed description of the metabelian \( G \) with \(|G| = 16(g - 1)\) and some corrections concerning their possible orders.

Bounds and more information if \( G \) is a \( p \)-group were worked out in [Z2]; see Theorem 2.3. More recent results in [W2] and [MZ2] include the optimal bound \( b(g) \) for subgroups \( G \) of odd order in \( Aut(X) \), which we recall in Theorem 2.4.

In this paper we present several new results of a similar nature.

In Section 5 we refine the known results by determining for each type of group in Table 1 the sharp bound \( b(g) \) if \( G \subseteq Aut(X) \) is an odd order subgroup of that type. It turns out that odd order supersolvable \( G \) of maximal possible order are subject to much stronger restrictions than the other types (see Theorem 5.1).

The idea of considering groups of odd order can be generalized to prescribing the smallest prime divisor \( p \) of \(|G|\). We do this already in Section 4, before treating odd order groups, for reasons having to do with the logical dependence of the proofs. It
turns out that for \( p \geq 5 \) the bounds for the first 8 types of groups in Table 1 all collapse to the same bound, namely the one for \( p \)-groups from Theorem 2.3 (c).

For a prime number \( p \) denote by \( \mathcal{G}(p) \) the class of all finite groups whose orders are not divisible by any primes smaller than \( p \). So \( \mathcal{G}(2) \) denotes all finite groups and \( \mathcal{G}(3) \) all groups of odd order. Omitting some of the details, the following big picture emerges.

The bound for \( p \)-groups from Theorem 2.3 is also the bound for nilpotent groups inside the class \( \mathcal{G}(p) \), and any nilpotent group in \( \mathcal{G}(p) \) that reaches this bound must be a \( p \)-group. Moreover, with the exception of 6 individual groups (3 in \( \mathcal{G}(2) \) and 3 in \( \mathcal{G}(3) \)), this bound also is the sharp bound for metabelian groups in \( \mathcal{G}(p) \). If \( p \geq 3 \) and \( q \equiv 1 \mod p \) are primes, we can even find infinitely many metabelian, supersolvable \((p,q)\)-groups that attain this bound.

So in Section 6 we treat yet another type of groups that are known to be solvable, namely groups of order \( p^m q^n \) where \( p < q \) are primes. We use the results from the previous sections to get optimal bounds on \( |G| \) for \( G \subseteq \text{Aut}(X) \) with \( |G| = p^m q^n \), in general and if \( p \) and/or \( q \) is prescribed.

In Section 7 we get similar results for two (probably less important) types of groups that are lying strictly between the supersolvable and the solvable ones, namely groups with nilpotent commutator group, and groups in which the elements of odd order form a (necessarily normal) subgroup.

In Section 8 we derive sharp upper bounds on \( |G| \) for CLT groups of odd order. CLT groups are another type of groups between the supersolvable and the solvable ones. They are defined in elementary terms by the condition that for every divisor \( d \) of \( |G| \) there is a subgroup of order \( d \). But CLT groups are not at all well-behaved. See Section 8 for details. This makes them a very unwieldy object to handle and explains perhaps why we only get partial results for the size of CLT groups in general.

2. Known results

Let \( X \) be a compact Riemann surface of genus \( g \geq 2 \) and let \( G \) be a subgroup of \( \text{Aut}(X) \). The key tool to get information on \( G \) is that \( G \subseteq \text{Aut}(X) \) is covered by a Fuchsian group \( \Gamma = \Gamma(h; m_1, \ldots, m_r) \). This means that \( \Gamma \) is discretely embedded into \( \text{Aut}(U) \), where \( U \) is the complex upper halfplane, and that there is a torsion-free normal subgroup \( K \) of \( \Gamma \) with \( \Gamma/K \cong G \) such that \( U/K \cong X \) is the universal covering. Moreover, \( h \) is the genus of \( X/G \cong U/\Gamma \). See Section 3 in Chapter 1 of [B] for more background and references.

**Theorem 2.1.** If \( G \) is covered by a Fuchsian group \( \Gamma(h; m_1, m_2, \ldots, m_r) \), then

\[
|G| = \frac{2}{2h - 2 + \sum_{i=1}^{r}(1 - \frac{1}{m_i})}(g - 1).
\]

**Proof.** [B, Theorem 3.5 and p.15]. \( \Box \)
In particular, if $|G| > 4(g - 1)$, then $h = 0$. In practically all cases we are interested in, $\Gamma$ even is a triangle group, i.e. $h = 0$ and $r = 3$.

Since we are interested in solvable groups $G \subseteq \text{Aut}(X)$, the biggest abelian quotient $G/G'$ is very important for us. It must be a quotient of $\Gamma/\Gamma'$, the structure of which can be easily read off (at least if $h = 0$) from the generators and relations

$$\Gamma(0; m_1, \ldots, m_r) \cong \langle x_1, \ldots, x_r \mid x_1^{m_1} = x_2^{m_2} = \ldots = x_r^{m_r} = x_1 x_2 \cdots x_r = 1 \rangle.$$  

For frequent use throughout the paper we take from Table 4.1 in [GMl] all possible orders $|G| \geq 18(g - 1)$ for solvable groups $G \subseteq \text{Aut}(X)$ together with the corresponding triangle groups $\Gamma$.

**TABLE 2**

| $|G|$      | $\Gamma$               | $\Gamma/\Gamma'$ | $\Gamma'$    |
|-----------|------------------------|------------------|--------------|
| $48(g - 1)$ | $\Gamma(0; 2, 3, 8)$   | $C_2$            | $\Gamma(0; 3, 3, 4)$ |
| $40(g - 1)$ | $\Gamma(0; 2, 4, 5)$   | $C_2$            | $\Gamma(0; 5, 5, 2)$ |
| $36(g - 1)$ | $\Gamma(0; 2, 3, 9)    | $C_3$            | $\Gamma(0; 2, 2, 3)$ |
| $30(g - 1)$ | $\Gamma(0; 2, 3, 10)   | $C_2$            | $\Gamma(0; 3, 3, 5)$ |
| $24(g - 1)$ | $\Gamma(0; 2, 3, 12)   | $C_6$            | $\Gamma(1; 2)$ |
| $24(g - 1)$ | $\Gamma(0; 2, 4, 6)$   | $C_2 \times C_2$ | $\Gamma(0; 2, 2, 3, 3)$ |
| $24(g - 1)$ | $\Gamma(0; 3, 3, 4)$   | $C_3$            | $\Gamma(0; 4, 4, 4)$ |
| $21(g - 1)$ | $\Gamma(0; 2, 3, 14)   | $C_2$            | $\Gamma(0; 3, 3, 7)$ |
| $20(g - 1)$ | $\Gamma(0; 2, 3, 15)   | $C_3$            | $\Gamma(0; 2, 2, 5)$ |
| $20(g - 1)$ | $\Gamma(0; 2, 5, 5)$   | $C_5$            | $\Gamma(0; 2, 2, 2, 2)$ |
| $\frac{5g}{3}(g - 1)$ | $\Gamma(0; 2, 3, 16)$ | $C_2$            | $\Gamma(0; 3, 3, 8)$ |
| $\frac{5g}{3}(g - 1)$ | $\Gamma(0; 2, 4, 7)$  | $C_2$            | $\Gamma(0; 7, 7, 2)$ |
| $18(g - 1)$ | $\Gamma(0; 2, 3, 18)   | $C_6$            | $\Gamma(1; 3)$ |

Actually, Table 4.1 in [GMl] contains four more triangle groups with $|G| \geq 18(g - 1)$, namely $\Gamma = \Gamma(0; 2, 3, p)$ with $p = 7, 11, 13, 17$. But for them $\Gamma' = \Gamma$ holds, and hence they cannot cover a solvable group. $\Gamma(0; 2, 3, 7)$ is of course the triangle group that covers Hurwitz groups.

Once one has a Riemann surface $X$ of genus $g \geq 2$ and a subgroup $G$ of $\text{Aut}(X)$ with desired properties and size, one can try to construct infinitely many more examples (with growing genus) from it. This is usually done by an approach that goes back to Macbeath [Mb] and is used, among others, in [Ch], [ChP], [G1], [MZ2] and [W2]. For every natural number $n$ there exists a compact Riemann surface $Y$
that is a totally unramified Galois cover of \( X \) with \( \text{Gal}(Y/X) \cong (C_n)^{2g} \). So the genus of \( Y \) is \( n^{2g}(g - 1) + 1 \). The key point is that \( G \) lifts to \( Y \) in the sense that \( \text{Aut}(Y) \) has a subgroup \( \tilde{G} \) with \( \text{Gal}(Y/X) \triangleleft \tilde{G} \) and \( \tilde{G}/\text{Gal}(Y/X) \cong G \).

By construction, a property like being solvable is obviously passed on from \( G \) to \( \tilde{G} \). But more special properties, for example being supersolvable, might get lost or only hold under certain circumstances.

Now we have most of the tools ready to find sharp bounds for \(|G|\) in terms of \( g \) when \( G \subseteq \text{Aut}(X) \) belongs to a certain class of groups. Table 1 in the Introduction gives a selective overview over the rich literature. Here we only spell out those cases of which we need details later on.

**Theorem 2.2.** Let \( X \) be a compact Riemann surface of genus \( g \geq 2 \), and let \( G \) be a subgroup of \( \text{Aut}(X) \).

(a) If \( G \) is solvable, then \(|G| \leq 48(g - 1)\). If \( G \) attains this bound, then necessarily \( G = \text{Aut}(X) \). There are infinitely many \( g \) for which there exists a Riemann surface \( X \) of genus \( g \) such that \( \text{Aut}(X) \) is solvable and of order \( 48(g - 1) \).

(b) If \( G \) is supersolvable, then for \( g \geq 3 \) we have \(|G| \leq 18(g - 1)\). Moreover, if \( g \geq 3 \) the necessary and sufficient condition for the existence of a Riemann surface \( X \) of genus \( g \) and a supersolvable \( G \subseteq \text{Aut}(X) \) with \(|G| = 18(g - 1)\) is that \( g - 1 \) is divisible by \( 9 \) and has no prime divisors that are congruent to \( 2 \) modulo \( 3 \).

(c) If \( G \) is nilpotent, then \(|G| \leq 16(g - 1)\). Any nilpotent \( G \) that reaches this bound must be a 2-group.

**Proof.** (a) From [B, Lemma 3.18] we get the following two facts: Groups \( G \) with \(|G| = 84(g - 1)\) have \( G' = G \), and hence they are not solvable. The next possible order is \( 48(g - 1) \). This also shows that \(|G| = 48(g - 1)\) implies \( G = \text{Aut}(X) \), because the only possible bigger order, \( 84(g - 1) \), is not a multiple. Note however, that \(|\text{Aut}(X)| = 48(g - 1)\) does not automatically imply solvability.

Chetiya [Ch] has constructed for each \( n > 0 \) a Riemann surface of genus \( 2n^6 + 1 \) whose automorphism group is solvable of order \( 96n^6 \). This is the special case \( m = 4 \) of the more general result in [Ch, Theorem 3.2]. The principal idea is to apply Macbeath’s construction to a Riemann surface of genus 3 whose automorphism group has order 96. By the same method, but starting with the unique Riemann surface of genus 2 with automorphism group of order 48, Gromadzki [G1, Section 5] for every \( n > 0 \) gets a Riemann surface of genus \( n^4 + 1 \) with a solvable automorphism group of order \( 48n^4 \).

See also [G2] for analogous results for groups of solvable length \( \leq 3 \).

(b) [Z3], [GMI] and [Z4]. In [Z3] the possibility that \( g - 1 \) might have prime divisors that are congruent to 1 modulo 3 is erroneously excluded.

(c) [Z1, Theorems 1.8.4 and 2.1.2]
To complete Theorem 2.2 (c), and for use in later sections, we state

**Theorem 2.3.** [Z2] Let $X$ be a compact Riemann surface of genus $g \geq 2$, and let $G$ be a subgroup of $\text{Aut}(X)$.

(a) If $G$ is a 2-group, then $|G| \leq 16(g-1)$. Moreover, for every $n \geq 4$ there exists a group $G$ of order $2^n$ that reaches this bound.

(b) If $G$ is a 3-group, then $|G| \leq 9(g-1)$. Moreover, for every $n \geq 4$ there exists a group $G$ of order $3^n$ that reaches this bound.

(c) If $G$ is a $p$-group with $p \geq 5$, then $|G| \leq \frac{2p}{p-3}(g-1)$. Moreover, for every $n \geq 1$ there exists a group $G$ of order $p^n$ that reaches this bound.

**Proof.** Theorems 1.1.2, 1.2.1, 1.3.1, and 2.0.1 in [Z2].

The four biggest possible orders if $G \subseteq \text{Aut}(X)$ has odd order are given in [MZ2, Proposition 1] together with the corresponding triangle groups. For further use throughout the paper we list them here.

**TABLE 3**

| $|G|$            | $\Gamma$          | $\Gamma/\Gamma'$ | $\Gamma'$ |
|-----------------|--------------------|-------------------|-----------|
| $15(g-1)$       | $\Gamma(0;3,3,5)$  | $C_3$             | $\Gamma(0;5,5,5)$ |
| $\frac{21}{2}(g-1)$ | $\Gamma(0;3,3,7)$  | $C_3$             | $\Gamma(0;7,7,7)$ |
| $9(g-1)$        | $\Gamma(0;3,3,9)$  | $C_3 \times C_3$ | $\Gamma(1;3,3,3)$ |
| $\frac{33}{4}(g-1)$ | $\Gamma(0;3,3,11)$ | $C_3$             | $\Gamma(0;11,11,11)$ |

See [MZ2] for more information. For example on page 328 of [MZ2] Macbeath’s method is used to show that there are $G$ with $|G| = 15(g-1)$ for every $g = 5n^{12} + 1$.

A more precise upper bound for $G$ of odd order is given in [W2, Main Theorem]. It also takes into account the highest power of 2 dividing $g-1$. If $g-1$ is odd, it coincides with $15(g-1)$, and otherwise it is smaller.

Correspondingly, [W2, Theorem 4.2] constructs infinite series of such groups. Specializing it to $l = 0$ we have $N_1 = 5$ and $h = 6$ and get the same infinite series with $|G| = 15(g-1)$.

We summarize the most important facts.

**Theorem 2.4.** [W2], [MZ2] Let $X$ be a compact Riemann surface of genus $g \geq 2$, and let $G$ be a subgroup of $\text{Aut}(X)$. If $|G|$ is odd, then $|G| \leq 15(g-1)$.

Conversely, for every odd integer $n$ there exists a compact Riemann surface $X$ of genus $g = 5n^{12} + 1$ such that $\text{Aut}(X)$ contains a subgroup $G$ of order $15(g-1)$. 
Of course, one should keep in mind throughout this paper that groups of odd order are always solvable. This is proved in the monumental article [FT]. So Theorem 2.4 should be compared to Theorem 2.2 (a) and not to the Hurwitz bound.

3. Some group theoretic tools

What makes supersolvable groups much more convenient to handle than the merely solvable ones is the following fact, that will be used repeatedly in the paper.

Theorem 3.1. (Zappa’s Theorem) Let $|G| = \prod_{i=1}^{s} p_i$ with prime numbers $p_1 \leq p_2 \leq \ldots \leq p_s$. If $G$ is supersolvable, then there exist normal subgroups $G_i$ of $G$ with

$$G = G_0 \triangleright G_1 \triangleright \ldots \triangleright G_{s-1} \triangleright G_s = I$$

and $[G_{i-1} : G_i] = p_i$.

See [H, Corollary 10.5.2] or [R, Theorem 5.4.8] for a proof. Strangely enough, this theorem is not mentioned in this form in the long survey article on supersolvable groups in [Wei].

We point out two immediate consequences that will also be used frequently in this paper. If $G$ is supersolvable, the Sylow $p$-subgroup for the biggest prime $p$ is normal. Also $G \cong N \rtimes P$ where $P$ is the Sylow $p$-subgroup for the smallest prime that divides $|G|$.

Another fact that we will frequently use is

Theorem 3.2. [H, Theorem 10.5.4], [R, Theorem 5.4.10] If the group $G$ is supersolvable, then its commutator group $G'$ is nilpotent.

In this paper we not only have to deal with groups which we assume to be supersolvable. For some of the results we also have to construct groups of a certain form and show that they are supersolvable. A very useful tool for this is the following criterion, which is not completely obvious.

Theorem 3.3. Let $p < q$ be primes.

(a) If $|G| = p \cdot q^n$ and $p|(q - 1)$, then $G$ is supersolvable.

(b) If $|G| = p^2 q^n$ and $p^2|(q - 1)$, then $G$ is supersolvable.

Proof. See [Wei p.6, Corollary 1.10] and the Exercise after it. The key point is that these groups have a normal Sylow $q$-subgroup $Q$ and that $G/Q$ is abelian of exponent dividing $q - 1$. Such groups, which in [Wei] are called strictly $q$-closed, are shown to be supersolvable in [Wei, p.5 Theorem 1.9]. □
A group $G$ is called **metacyclic** if it has a normal cyclic subgroup $N$ such that $G/N$ is also cyclic. Note that [H, p.146] uses a more restrictive definition for metacyclic than we (and most sources) do, namely that $G'$ and $G/G'$ are cyclic.

**Lemma 3.4.** Let $G$ be a metacyclic group and $p$ the smallest prime that divides $|G|$. Then $G$ has a normal cyclic subgroup $N$ such that $G/N$ is cyclic and $p$ divides $|G/N|$.

**Proof.** If not, then $G \triangleright N \cong C_{p^e}m$ and $G/N \cong C_n$ such that $m$ and $n$ are only divisible by primes that are bigger than $p$. In that case $C_m \triangleleft G$ and $G/C_m = H$ has a normal subgroup $C_{p^e}$ with quotient $C_n$. By the Schur-Zassenhaus Theorem [R, Theorem 9.1.2] we have $H \cong C_{p^e} \rtimes C_n$. Moreover, $|\text{Aut}(C_{p^e})| = p^{e-1}(p-1)$; so $C_n$ can only act trivially on $C_{p^e}$. Thus $H \cong C_{p^e}n$. □

A **Z-group** is a finite group whose Sylow subgroups are all cyclic. Such groups become important if one wants to bound the exponent of $G \subseteq \text{Aut}(X)$. Compare Theorem 5.8. The following relation with other types of groups is not completely obvious.

**Theorem 3.5.** *(Zassenhaus)* [H, Theorem 9.4.3], [R, Theorem 10.1.10] A Z-group that is not cyclic can be written as a semidirect product

$$C_m \rtimes C_n$$

where $(m, n) = 1$ and $m$ is odd. In particular, such a group is split metacyclic.

### 4. Bounds involving the smallest prime that divides $|G|$

Implicitly the bound from Theorem 2.3 (c) also occurs somewhere else in the literature for a different type of groups, namely:

**Example 4.1.** Let $p$, $q$ be primes with $p \geq 5$ and $p|(q-1)$. The paper [W1] determines every genus on which the (up to isomorphism unique) non-abelian group $C_q \rtimes C_p$ can act. By [W1, Corollary 4.2], the smallest such genus is $\mu = 1 + \frac{q^3}{2}$.

So if $C_q \rtimes C_p \subseteq \text{Aut}(X)$ we always have $|C_q \rtimes C_p| \leq \frac{2p}{p-3}(g-1)$, with equality if and only if $g = \mu$.

Since by Dirichlet’s Theorem on primes in arithmetic progressions for fixed $p$ there are infinitely many primes $q$ with $p|(q-1)$, this gives infinitely many $G$ that are not nilpotent with $|G| = \frac{2p}{p-3}(g-1)$.

We recall that $\mathcal{G}(p)$ denotes the class of all finite groups whose orders are not divisible by any prime smaller than $p$. Inspired by the coincidence of the bounds in Theorem 2.3 (c) and Example 4.1 we first tried to prove that the same bound holds
for supersolvable $G$ in $G(p)$ by an inductive process using Zappa’s Theorem. Then we realized that it actually holds for all $G$ in $G(p)$.

**Theorem 4.2.** Fix a prime $p \geq 5$. Let $X$ be a compact Riemann surface of genus $g \geq 2$, and let $G$ be a subgroup of $\text{Aut}(X)$ such that $|G|$ is not divisible by any primes that are smaller than $p$. Then

$$|G| \leq \frac{2p}{p-3}(g-1).$$

Moreover, if $|G| = \frac{2p}{p-3}(g-1)$, then $G$ is a quotient of $\Gamma(0; p, p, p)$.

**Proof.** If $G$ is covered by a Fuchsian group $\Gamma(h; m_1, m_2, \ldots, m_r)$, then from Theorem 2.1 we see that for $|G| \geq \frac{2p}{p-3}(g-1)$ we need $2h - 2 + \sum_{i=1}^{r} (1 - \frac{1}{m_i}) \leq 1 - \frac{2}{p}$. Moreover, the periods $m_i$ must divide the group order, so $m_i \geq p$. This implies $h = 0$, and since for $h = 0$ we have $r \geq 3$, we see that $\Gamma(0; p, p, p)$ is the only possibility. \qedsymbol

**Remark 4.3.** Theorem 2.3 (c) shows that there are infinitely many values of $g$ for which the bound in Theorem 4.2 is attained, even if we restrict to certain classes of groups, for example $p$-groups, nilpotent, or supersolvable.

And Example 4.1 provides infinitely many examples that reach the bound if we restrict to groups that are of square-free order or $Z$-groups or metacyclic or metabelian.

On the other hand, $\Gamma(0; p, p, p)$ shows that the only possible abelian groups that reach the bound are $C_p$ and $C_p \times C_p$.

If a group $G$ in Theorem 4.2 reaches the bound $|G| = \frac{2p}{p-3}(g-1)$, then $|G|$ obviously must be divisible by $p$. Now we formulate the analogues of Theorem 2.2 (b) and (c) for such groups.

**Proposition 4.4.** Fix a prime $p \geq 5$. Let $X$ be a compact Riemann surface of genus $g \geq 2$ and $G \subseteq \text{Aut}(X)$ such that $G$ is in $G(p)$ and reaches the bound from Theorem 4.2.

(a) If $G$ is supersolvable, then $|G|$ is only divisible by $p$ and by primes $q$ with $q \equiv 1 \mod p$.

(b) If $G$ is nilpotent, then $G$ must be a $p$-group.

**Proof.** (a) We know that $H = G/G'$ is isomorphic to $C_p$ or $C_p \times C_p$. If $G$ is supersolvable, then $G'$ is nilpotent by Theorem 3.2. Fix a prime $q \neq p$ that divides $|G'|$ and let $N$ be the product of all Sylow subgroups of $G'$ for primes different from $q$. Then $N$ is characteristic in $G'$ and hence normal in $G$. So if $Q$ is a Sylow $q$-subgroup of $G$, we get a quotient $G/N$ which is isomorphic to $Q \rtimes H$ and supersolvable (as a
quotient of a supersolvable group). By Zappa’s Theorem, this quotient has a normal subgroup which has index \( q \) in \( Q \). So we see that \( G \), and hence \( \Gamma(0; p, p, p) \) has a quotient of the form \( C_q \times H \). If \( q \) is not congruent to 1 modulo \( p \), this group is abelian, contradicting \( H = G/G' \).

(b) If \( G \) is nilpotent, then any Sylow \( q \)-subgroup of \( G \) is a quotient of \( G \) and hence of \( \Gamma(0; p, p, p) \), forcing \( q = p \).

Proposition 4.4 (b) is an anlogue of Theorem 2.2 (c). For the corresponding statement for nilpotent groups of odd order see Theorem 5.2. Example 4.1 shows that the statement ‘equality only possible for \( p \)-groups’ does not generalize to supersolvable \( G \).

We construct more examples of \( G \) that are metabelian and supersolvable and attain the bound from Theorem 4.2.

Example 4.5. Let \( p \geq 5 \) be a prime. Then there exists a Riemann surface of genus \( h = \frac{p-1}{2} \) with \( C_p \subseteq \text{Aut}(X) \). Fix any prime \( q > p \). By Macbeath’s construction, for every \( e > 0 \) there is a smooth Galois cover \( X \) of \( Y \) whose Galois group is a direct product of \( 2h \) copies of \( C_q^e \). So the genus of \( X \) is \( g = \frac{p-3}{2} q^{2he} + 1 \). The automorphism group of \( Y \) lifts to \( X \). So \( \text{Aut}(X) \) has a subgroup \( G \cong (C_q^e)^{2h} \times C_p \). Thus \( G \) is metabelian by construction and attains the bound from Theorem 4.2.

By Dirichlet’s Theorem on primes in arithmetic progressions there are infinitely many primes \( q \) with \( q \equiv 1 \mod p \). If we take one of those, \( G \) moreover is supersolvable by Theorem 3.3 (a).

Remark 4.6. From Theorems 2.2 (b) and Examples 4.1 and 4.5 one might get the impression that if \( G \subseteq \text{Aut}(X) \) is a supersolvable, non-nilpotent group of maximal possible order, then the smallest prime can divide \( |G| \) only once. Although this is true if the smallest prime is 2 or 3 (see Theorem 5.1 below), there are infinitely many counterexamples for every \( p \geq 5 \).

We start with the group of order \( p^2 \) acting on a surface \( Y \) of genus \( h = \frac{p(p-3)}{2} + 1 \) (see Theorem 2.3 (c)). Being a quotient of \( \Gamma(0; p, p, p) \), it can only be \( C_p \times C_p \). Then, by the same construction as in Example 4.5 for every prime \( q > p \) and every \( e > 0 \) we obtain a metabelian group \( G \cong (C_q^e)^{2h} \times (C_p \times C_p) \) acting on a surface of genus \( g = \frac{p(p-3)}{2} q^{2he} + 1 \).

If \( q \) is one of the infinitely many primes with \( q \equiv 1 \mod p^2 \), then moreover \( G \) is supersolvable by Theorem 3.3 (b).

Theorem 4.7. Let \( p \) be an odd prime. If \( G \subseteq \text{Aut}(X) \) is cyclic and in \( G(p) \), then

\[
|G| \leq \frac{2p}{p-1} q + p.
\]

This bound is attained for every \( g = \frac{p-1}{2} (m - 1) \) with a group \( C_{pm} \), provided all prime divisors of \( m > 1 \) are bigger than \( p \).
Proof. Let $G$ be a cyclic group of order $N$ and $p_1$ the smallest prime divisor of $N$. Then by [Ha, Theorem 6] the minimum genus on which $G$ can act is $\frac{p_1 - 1}{2} \cdot \frac{N}{p_1}$ if $N$ is prime or $p_1^2$ divides $N$, and $\frac{p_1 - 1}{2} \cdot \frac{(N/p_1 - 1)}{p_1}$ otherwise. This proves our theorem provided $p$ divides $|G|$.

Now assume that the smallest prime divisor of $|G|$ is $q > p$. If $|G|$ is bigger than the bound in our theorem, then necessarily $\frac{2p}{p-1}g + p < \frac{2q}{q-1}g + q$. An easy algebraic manipulation shows that this inequality is equivalent to $g < \frac{(p-1)(q-1)}{2}$. Plugging this into the bigger bound yields $|G| < pq$, leaving only the possibility $|G| = q$, for which the stronger $|G| \leq 2g + 1$ holds. □

Theorem 4.8. Let $p$ be an odd prime. If $G \subseteq \text{Aut}(X)$ is abelian and in $G(p)$, then

$$|G| \leq \frac{2p}{p-1}g + 2p,$$

except for the groups $C_q \times C_q$ where $q$ is a prime with $p < q < 2p$ acting on genus $g = \frac{(q-1)(q-2)}{2}$. The bound can be reached for every $g = \frac{p-1}{2}(m-2)$ for which $p$ is the smallest prime divisor of $m$ with a group $C_p \times C_m$.

Proof. Let $A \cong C_{m_1} \times C_{m_2}$ with $m_1|m_2$. Then by [Ml, Theorem 4] the minimum genus $g^*$ on which $A$ can act is given by $\frac{2(g^*-1)}{|A|} = 1 - \frac{1}{m_1} - \frac{2}{m_2}$. This transforms to $|A| \leq \frac{2m_1}{m_1-1}g + 2m_1$, and it also shows that for $m_1 = p$ the bound in Theorem 4.8 can be reached.

That bound obviously surpasses the bound from Theorem 4.7 for cyclic groups in $G(p)$. On the other hand, for abelian groups of odd order and rank $r > 2$ by [BMT, Theorem 2] we have $|A| \leq 2g + 6$, except for $C_3 \times C_3 \times C_3$, which can act on genus 10.

So if an abelian group $A$ in $G(p)$ tops the bound in the Theorem 4.8, it must be of the form $A \cong C_{m_1} \times C_{m_2}$ with $m_1|m_2$ and $m_1 > p$. Moreover, we then must have $\frac{2p}{p-1}g + 2p < \frac{2m_1}{m_1-1}g + 2m_1$, which is equivalent to $g < (p-1)(m_1-1)$. Then the bigger bound gives $|A| < 2pm_1$. Consequently, $A$ can only be $C_q \times C_q$ with a prime $q$ between $p$ and $2p$. By [Ml, Theorem 4] these groups can indeed act on genus $g^* = \frac{(q-1)(q-2)}{2}$ and hence surpass the bound in the theorem. The next bigger genus on which they can act is $g^* + q$, which is too big to compete with the bound from the theorem. Here we are using that $\frac{|A|}{\exp(A)} = q$ must divide $2(g - 1)$, a well-known fact that was reproved and elaborated on in [Sch1, Section 6]. □

5. Groups of odd order

In this section we refine Theorem 2.4 by working out the analogue of Table 1 for groups of odd order.
Theorem 5.1. Let $X$ be a compact Riemann surface of genus $g \geq 2$, and let $G$ be a subgroup of $\text{Aut}(X)$. If $G$ is supersolvable and of odd order, then $|G| \leq \frac{21}{2}(g-1)$.

Moreover, a supersolvable group $G$ of odd order that reaches this bound necessarily has $|G| = 3 \cdot 7^n$.

Conversely, for every $n \geq 1$ there exists a Riemann surface $X$ of genus $g = 2 \cdot 7^{n-1} + 1$ such that $\text{Aut}(X)$ contains a supersolvable subgroup $G$ of order $|G| = 3 \cdot 7^n$.

Proof. To show $|G| \leq \frac{21}{2}(g-1)$ by Table 3 we only have to exclude the possibility $|G| = 15(g-1)$. So suppose $|G| = 15(g-1)$, i.e., that $G$ is a quotient of $\Gamma(0; 3, 3, 3, 5)$. Then by Zappa’s Theorem $G$ has a quotient of order 9 (if 9 divides $|G|$) or a quotient of order 15 (if 9 does not divide $|G|$). Either one contradicts $G/G' \cong C_3$.

Now let $G \subseteq \text{Aut}(X)$ be a supersolvable group of odd order $\frac{21}{2}(g-1)$. Then $G$ is a quotient of $\Gamma(0; 3, 3, 7)$. By Table 3 we have $G/G' \cong C_3$ and $G'$ is a quotient of $\Gamma(0; 7, 7, 7)$. Moreover, $G'$, as the commutator group of a supersolvable group, must be nilpotent by Theorem 3.2. Thus every Sylow $p$-subgroup of $G'$ is a quotient of $G'$ and hence of $\Gamma(0; 7, 7, 7)$. But obviously this group has no quotients of order prime to 7. So $|G'| = 7^n$, and hence $|G| = 3 \cdot 7^n$.

There is a non-abelian group of order 21 acting on a Riemann surface $X$ of genus 3. More precisely, $X$ is the Klein quartic and $G$ is the normalizer of a Sylow 7-subgroup in $\text{Aut}(X) \cong PSL_2(\mathbb{F}_7)$.

By [W2, Theorem 4.2] for every $\mu \geq 0$ there is a group $G$ of order $3 \cdot 7^{30\mu+2}$ acting on a Riemann surface $X$ of genus $g = 2 \cdot 7^{30\mu+1} + 1$. These $G$ are supersolvable by Theorem 3.3. To fill in the groups of order $3 \cdot 7^n$ with the missing $n$, we start with such a group $G$ for a big enough $n = 30\mu + 2$ and inductively divide the group order by 7 as follows. By Zappa’s Theorem, $G$ has a normal subgroup $P \cong C_7$. Then $G/P$ acts on the Riemann surface $X/P$. If $h$ denotes the genus of $X/P$, then $h - 1 \leq \frac{1}{2}(g-1)$ by the Hurwitz formula. Because of the bound $|G/P| \leq \frac{21}{2}(h - 1)$, which we have already established in general, we actually must have $h - 1 = \frac{1}{7}(g-1)$, except if it unfortunately should happen that $h \leq 1$. But in that case $G'/P$, a group of order $7^{n-1}$ acting on a surface of genus 0 or 1, would have to be abelian, which for $n \geq 4$ contradicts the fact that $C_7 \times C_7$ is the biggest abelian quotient of $\Gamma(0; 7, 7, 7)$.

See also Proposition 7.3 and Theorem 8.2 for further characterizations of the groups in Theorem 5.1.

The following analogue of Theorem 2.2 (c) is at least implicitly already in [Z1] or [Z2].

Theorem 5.2. Let $X$ be a compact Riemann surface of genus $g \geq 2$, and let $G$ be a subgroup of $\text{Aut}(X)$. If $G$ is nilpotent and of odd order, then $|G| \leq 9(g-1)$.

Moreover, any nilpotent group of odd order that reaches this bound must be a 3-group.
Proof. If $G$ is nilpotent, then every Sylow $p$-subgroup of $G$ is a quotient of $G$. But the triangle groups $\Gamma(0; 3, 3, n)$ have no nontrivial quotients of order prime to 3. This rules out the possibilities $|G| = 15(g - 1)$ and $|G| = \frac{21}{2}(g - 1)$. The next one is $|G| = 9(g - 1)$ with $G$ being a quotient of $\Gamma(0; 3, 3, 9)$. By the same argument then $|G|$ is not divisible by any primes $p \neq 3$. □

Theorem 5.3. If $G \subseteq \text{Aut}(X)$ is a metabelian group of odd order, then, apart from 3 exceptions we have $|G| \leq 9(g - 1)$.

The exceptions are the non-abelian group of order 21 acting on a surface of genus 3, a group $(C_7 \times C_7) \rtimes C_3$ for $g = 15$, and a group $(C_5 \times C_5) \rtimes C_3$ for $g = 6$.

Moreover, the bound 9($g - 1$) is attained infinitely often. More precisely, if $m \in \mathbb{N}$ is only divisible by primes that are congruent to 1 modulo 3, then there exists a compact Riemann surface $X$ of genus $g = 9m + 1$ such that $\text{Aut}(X)$ contains a subgroup $G$ of order 9($g - 1$) that is supersolvable and metabelian.

Proof. If $|G|$ is bigger than 9($g - 1$), then $G$ is covered by $\Gamma(0; 3, 3, p)$ with $p = 7$ or $p = 5$. So $G/G' \cong C_3$ and $G'$ is a quotient of $\Gamma(0; p, p, p)$. Since $G'$ is abelian, this leaves only the possibilities $G' \cong C_p$ or $G' \cong C_p \times C_p$. Moreover $G' \cong C_5$ is not possible, as then $G$ would be cyclic of order 15 on a surface of genus 2.

To prove the last statement, we note that by Theorem 2.2 (b) for the specified genus $g = 9m + 1$ there exists a surface $X$ with suppersolvable $H \subseteq \text{Aut}(X)$ of order 18($g - 1$). We take $G$ to be the index 2 subgroup in $H$. Then $G$ has the right order and is supersolvable. So there only remains to show that $G$ is metabelian.

Note that $G$ is a quotient of $\Gamma = \Gamma(0; 3, 3, 9)$. We have $\Gamma/\Gamma' \cong C_3 \times C_3$ and $\Gamma' = \Gamma(1; 3, 3, 3)$. By Zappa’s Theorem $\Gamma$ must have a quotient of order 9. So $G/G' \cong C_3 \times C_3$ and $G'$ is a quotient of $\Gamma(1; 3, 3, 3)$. Moreover, as $G$ is supersolvable, $G'$ must be nilpotent by Theorem 3.2. So $G'$ is a direct product of its Sylow subgroups. The Sylow 3-subgroup $N$ is of order 9 and in particular abelian. Actually, $G'/N$, being a quotient of $\Gamma(1; 3, 3, 3)$ of order prime to 3, must also be abelian. Thus $G'$ is abelian, which finishes the proof. □

For more information on the three exceptions see [MZ2].

Theorem 5.4.

(a) If $G \subseteq \text{Aut}(X)$ is a cyclic group of odd order, then

$$|G| \leq 3g + 3.$$  

This bound is attained for every $g \equiv 0$ or 4 mod 6.

(b) If $G \subseteq \text{Aut}(X)$ is an abelian group of odd order, then with the exception of $C_5 \times C_5$ acting on a Riemann surface of genus 6 we have

$$|G| \leq 3g + 6.$$  

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This bound can be attained for every \( g = 6k + 1 \) where \( k = 1, 2, \ldots \) with a group \( G \cong C_3 \times C_{6k+3} \).

**Proof.** (a) is a special case of Theorem 4.7. Alternatively, it follows directly from [N].

(b) is a special case of Theorem 4.8. \( \square \)

Next we prove the odd order analogues of the very recent results in [Sch2] concerning several types of metacyclic subgroups of \( \text{Aut}(X) \).

**Theorem 5.5.** If \( G \subseteq \text{Aut}(X) \) is a metacyclic group of odd order, then

\[
|G| \leq 6g + 3.
\]

This bound can be attained if and only if \( 2g + 1 \) is neither divisible by 9 nor by any prime \( p \) with \( p \equiv 2 \mod 3 \). The corresponding \( G \) then is of the form \( C_{2g+1} \times C_3 \).

**Proof.** If \( G \) is in \( G(5) \), then \( |G| \leq 5(g - 1) \) by Theorem 4.2. So we can assume that 3 divides \( |G| \). By Lemma 3.4 then \( G \) has a normal cyclic subgroup \( N \triangleleft G \) such that \( G/N \cong C_n \) with \( 3|n \). Actually, we must have \( n = 3 \). Bigger cyclic quotients are not possible, because then the periods would make \( |G| \) too small. For example, \( \Gamma(0; 3, 5, 15) \) gives \( |G| = 5(g - 1) \). Alternatively, for \( n > 1 \) one could invoke [MZ1, Theorem 1].

This also immediately shows that \( |N| \) cannot be divisible by any prime \( p \) with \( p \equiv 2 \mod 3 \), for then we would have \( G/H \cong C_{3p} \) where \( H \) is the unique subgroup of index \( p \) in \( N \).

So if \( |G| \geq 6g + 3 \), then \( G \) has a normal cyclic subgroup \( N \cong C_m \) with \( m \geq 2g + 1 \). Thus \( N \) is what is called a quasilarge abelian group (of automorphisms) in [PR]. By [PR, Proposition 4.1] the quotient surface \( X/N \) must have genus 0. More precisely, \( N \) can only be an instance of the last case (3 critical values) in [PR, Table 2]. So \( N \) is covered by some \( \Gamma(0; m_1, m_2, m_3) \).

Now by [BC, Theorem 4.2] the fact that \( N \) extends normally to \( G \subseteq \text{Aut}(X) \) with \( [G : N] = 3 \) is only possible if \( m_1 = m_2 = m_3 = t \geq 4 \) (see Case N6 in [BC, p.576]). In terms of [PR, Table 2] this means \( \alpha = 1, \delta_x = 1 \) and \( \beta = t = |N| \). Hence by [PR, Theorem 4.2] we have \( g = 1 + \frac{t}{3}(t - 3) \), i.e. \( t = 2g + 1 \). This establishes the bound \( |G| \leq 6g + 3 \).

Now let \( z_1, z_2, z_3 \in N \) be the images of the generators \( x_1, x_2, x_2 \) of \( \Gamma(0; t, t, t) \) where \( t = 2g + 1 \), and let \( u \) be a generator of \( G/N \cong C_3 \). Then \( u z_3 u^{-1} = z_3^b \) for some integer \( b \) that is relatively prime to \( t \). By [BC, Case N6] the necessary and sufficient condition for the existence of \( G \) is that conjugation with \( u \) permutes the \( z_i \) cyclically, i.e. that \( z_1 = z_3^b \) and \( z_2 = z_3^b \). Because of \( z_1 z_2 z_3 = 1 \) this forces \( 1 + b + b^2 \) to be divisible by \( t \). From this we see that \( t \) cannot be divisible by any prime \( p \equiv 2 \mod 3 \) (as we have already seen earlier) and that 9 cannot divide \( t \).

Conversely, if \( t = 2g + 1 \) is a product of primes that are congruent to 1 modulo 3, or 3 times such a number, then by the Chinese Remainder Theorem there exists
an integer \( b \), relatively prime to \( t \), such that \( t \) divides \( 1 + b + b^2 \). Then by [BC, Case N6] the desired \( G \) exists. Note that in the latter case \( b \equiv 1 \mod 3 \) and the Sylow 3-subgroup of \( G \) will then be \( C_3 \times C_3 \).

Finally, the fact that \( G \) cannot have a quotient \( C_9 \) guarantees that \( N \) has a complement, and hence \( G \) is a semidirect product. □

**Remark 5.6.** If \( p \) is a prime that is congruent to 1 modulo 3, then by [W1, Corollary 4.2] the smallest genus on which the (up to isomorphism unique) non-abelian group \( C_p \times C_3 \) can act is \( g = \frac{p - 1}{2} \). So the group order is \( 6g + 3 \). This was the initial inspiration for Theorem 5.5.

However, the more general construction with groups \( C_{2g+1} \times C_3 \) in [W2, Section 6] is not completely correct. For example, \( l = 4, d = 1 \) gives \( g = 17 \). So the group, which can only be \( C_5 \times (C_7 \times C_3) \), also has elements of order 15, contrary to what is claimed in [W2, p.219]. Indeed, by Theorem 5.5 this group cannot act on a surface of genus 17.

Coincidentally, [KS, Table 1], which allows to write down algebraic equations for examples as in Theorem 5.5, contains a similar inaccuracy. The conditions in Case C.1, namely that \( n \) in \( C_n \times C_3 \) must be bigger than 7, odd, and divisible by a prime that is congruent to 1 modulo 3, are at least misleading. If \( 1 < b < n \) and \( 1 + b + b^2 \) is divisible by \( n \) (which is the set-up in [KS]), this implies exactly the conditions from Theorem 5.5, namely \( 9 \nmid n \) and \( p \nmid n \) for every prime \( p \) with \( p \equiv 2 \mod 3 \).

We also point out that the approach in the proof of Theorem 5.5 can also be used to prove \( |G| \leq 12(g - 1) \) for general metacyclic groups independent of the line of argument in [Sch2].

First one uses [MZ1, Theorem 1] to show \( |G| \leq 12(g - 1) \) if \( G \) has a cyclic quotient \( C_q \) with \( q \geq 7 \). (Here \( q \) is an integer, not necessarily a prime.) So if \( G \) is metacyclic and \( |G| > 12(g - 1) \), then \( G \) has a normal cyclic subgroup \( N \) of order bigger than \( 2(g - 1) \). Thus \( N \) is quasilarge and must therefore show up in [PR, Table 2]. Then one compares with [BC]. Admittedly, some care with the details is required.

Theorem 5.5, when combined with Theorem 3.5, immediately yields more results.

**Corollary 5.7.** Let \( g \geq 2 \) and \( G \subseteq Aut(X) \).

(a) If \( G \) is a \( Z \)-group of odd order, then \( |G| \leq 6g + 3 \). This bound can be attained if and only if all prime divisors of \( 2g + 1 \) are congruent to 1 modulo 3.

(b) If \( |G| \) is square-free and odd, then \( |G| \leq 6g + 3 \). This bound can be attained if and only if \( 2g + 1 \) is a product of distinct primes that all are congruent to 1 modulo 3.

In both cases, if the bound is attained, then \( G \cong C_{2g+1} \times C_3 \).
Using the easy fact that the exponent $\exp(G)$ of a finite group $G$ equals $|G|$ if and only if $G$ is a $Z$-group, we also obtain the following result.

**Theorem 5.8.** If $G \subseteq \text{Aut}(X)$ has odd order, then

$$\exp(G) \leq 6g + 3.$$  

This bound can be attained if and only if all prime divisors of $2g + 1$ are congruent to $1$ modulo $3$ (with $G \cong C_{2g+1} \rtimes C_3$).

**Proof.** If $G$ is a $Z$-group, this follows from Corollary 5.7 (a). If $G$ is not a $Z$-group, we have $\exp(G) \leq \frac{1}{3} |G| \leq \frac{1}{3} 15(g - 1)$ by Theorem 2.4. $\square$

The analogue of Theorem 5.8 for groups of not necessarily odd order is [Sch1, Theorem 4.4]. The corresponding problem for solvable groups of not necessarily odd order is not completely solved. See [Sch2, Proposition 6.2 and Remark 6.3].

## 6. Groups of order $p^m q^n$

Several times we have seen in the preceding sections that one can often find $(p, q)$-groups that reach the sharp bound for a certain type of groups $G \subseteq \text{Aut}(X)$. Motivation enough for us to investigate this type of groups in their own right. Note that $(p, q)$-groups are always solvable, a famous theorem by Burnside [H, Theorem 9.3.2] or [R, Theorem 8.5.3].

In $|G| = p^m q^n$ we always choose the letter $p$ for the smaller one of the two primes $p$ and $q$. Moreover, to avoid certain trivialities, we implicitly assume that $m$ and $n$ are both positive. In accordance with Sections 4 and 5 we again refine the statements for odd group order and in terms of the smallest prime $p$.

**Theorem 6.1.** Let $X$ be a compact Riemann surface of genus $g \geq 2$, and let $G \subseteq \text{Aut}(X)$ be a subgroup. If $G$ is a $(p, q)$-group, then $|G| \leq 48(g - 1)$. Any $(p, q)$-group that reaches this bound necessarily is a $(2, 3)$-group.

Conversely, for every $g = 2^{6\mu+1}3^{6\nu} + 1$ and every $g = 2^{4\mu}3^{4\nu} + 1$ where $\mu, \nu \geq 0$ there exists a Riemann surface $X$ of genus $g$ such that $\text{Aut}(X)$ is a $(2, 3)$-group of order $48(g - 1)$.

**Proof.** Since $(p, q)$-groups are solvable, the first claim follows from part (a) of Theorem 2.2. The second claim is clear, because then 48 divides $|G|$. For the third claim we use the parametrizations from the proof of Theorem 2.2 (a) with $n = 2^\mu 3^\nu$. $\square$

**Theorem 6.2.** Let $X$ be a compact Riemann surface of genus $g \geq 2$, and let
Let $G \subseteq \text{Aut}(X)$ be a subgroup. If $G$ is a $(p,q)$-group of odd order, then $|G| \leq 15(g-1)$. Any $(p,q)$-group that reaches this bound necessarily is a $(3,5)$-group.

Conversely, for every $g = 3^{12\mu} 5^{12\nu+1} + 1$ with $\mu, \nu \geq 0$ there exists a Riemann surface $X$ of genus $g$ such that $\text{Aut}(X)$ contains a $(3,5)$-group $G$ of order $15(g-1)$.

**Proof.** The bound and the existence follow from Theorem 2.4 with $m = 3^\mu 5^\nu$. □

In Example 4.5 and Remark 4.6 we already showed the following result.

**Theorem 6.3.** Fix primes $p$, $q$ with $3 < p < q$. Then there are infinitely many metabelian $(p,q)$-groups $G \subseteq \text{Aut}(X)$ that reach the bound $|G| = \frac{2q}{p-3}(g-1)$ from Theorem 4.2.

Applying Macbeath’s construction to the $p$-groups in Theorem 2.3 (c), one can for every $n > 0$ (and every $q$) obtain $(p,q)$-groups $G$ with $\text{ord}_p|G| = n$ that attain the bound. But then in general $G$ will be neither metabelian nor supersolvable.

Finally, we address the problem what happens if we fix $p \leq 3$ and $q$. Actually, $(2,3)$-groups were already discussed in Theorem 6.1.

**Theorem 6.4.** Fix an odd prime $q$. There are infinitely many metabelian $(2,q)$-groups $G \subseteq \text{Aut}(X)$ with $|G| = 16(g-1)$. For $q \geq 11$ this is also the upper bound that can be achieved by a $(2,q)$-group.

For $(2,5)$-groups (resp. $(2,7)$-groups) $G \subseteq \text{Aut}(X)$ we have $|G| \leq 40(g-1)$ (resp. $|G| \leq \frac{64}{3}(g-1)$), and this bound is reached infinitely often.

**Proof.** Setting $\beta = 2$ and $k = 2^\mu q^\nu$ in [G3, Theorem 1.2] we get a metabelian group of order $2^{\mu+5}q^\nu$ acting on a surface of genus $g = 2^{\mu+1}q^\nu + 1$.

All orders in Table 2 are divisible by 3, 5 or 7. There are a handful more possible orders that are bigger than $16(g-1)$, to wit, for $\Gamma(0;2,3,m)$ with $19 \leq m \leq 23$, but they are all divisible by the wrong primes.

There are Riemann surfaces $X$ of genus 5 with $|\text{Aut}(X)| = 160$; they are called Humbert curves. By Macbeath’s construction, from them we can obtain for any $\mu, \nu \geq 0$ a group of order $2^{10\mu+5}5^{10\nu+1}$ acting on a Riemann surface of genus $g = 4 \cdot (2^\mu 5^\nu)^{10} + 1$. By [ChP] these groups cannot be metabelian.

Finally to $|G| = \frac{56}{3}(g-1)$. Let $\Gamma = \Gamma(0;2,4,7)$. Then $\Gamma/\Gamma' \cong C_2$ and $\Gamma' = \Gamma(0;7,7,2)$. So $\Gamma'/\Gamma'' \cong C_7$ and $\Gamma'' = \Gamma(0;2,2,2,2,2,2,2)$. Finally, $\Gamma''/\Gamma''' \cong (C_2)^6$ and $\Gamma''' = \Gamma(49;\cdot\cdot\cdot)$. So there exists a Riemann surface $X$ of genus $g = 49$ with a $G \subseteq \text{Aut}(X)$ with $|G| = 7 \cdot 2^7 = \frac{56}{3}(g-1)$. To that one we can apply Macbeath’s construction and get infinitely many more. However, the genus grows rapidly. The next one constructed by that method has genus $48 \cdot 2^{98} + 1$. □

**Theorem 6.5.** The bound on the order of a $(3,q)$-group $G$ in $\text{Aut}(X)$ is $15(g-1)$ if $q = 5$, $\frac{21}{2}(g-1)$ if $q = 7$, and $9(g-1)$ if $q \geq 11$. For any fixed $q$ this bound is
attained infinitely often.

Proof. This follows from the fact that $15(g - 1), \frac{21}{2}(g - 1)$ and $9(g - 1)$ are the three smallest possible odd orders together with Theorems 2.4 and 5.1 respectively applying Macbeath’s construction to the 3-groups from Theorem 2.3 (b).

If $q$ is congruent to 1 modulo 3, then Theorem 5.3 gives the much stronger result that for every $n \geq 1$ there exists a metabelian, supersolvable group $G \subseteq \text{Aut}(X)$ of order $3^4q^n = 9(g - 1)$.

Remark 6.6. If we only fix the bigger prime $q$, then we obtain from the previous results that the biggest order of a $(p, q)$-group $G \subseteq \text{Aut}(X)$ is always obtained with $(2, q)$-groups as given earlier in this section.

7. Between supersolvable and solvable

The converse of Theorem 3.2 is not true. Groups $G$ with nilpotent commutator group $G'$ are of course solvable, but not necessarily supersolvable, as can be seen from the standard counterexample $A_4$. On the other hand, $S_4$ shows that not every solvable $G$ has a nilpotent $G'$. So this type of groups lies properly between the supersolvable and the solvable ones. More or less by definition it also lies strictly between the metabelian and the solvable groups. It turns out that the optimal bound for $|G|$ in terms of $g$ for this type of groups also lies strictly between the bounds for supersolvable or metabelian and solvable $G$.

Theorem 7.1. Let $X$ be a compact Riemann surface of genus $g \geq 2$, and let $G \subseteq \text{Aut}(X)$ be a subgroup. If the commutator group $G'$ is nilpotent, then $|G| \leq 24(g - 1)$.

Conversely, for every $g = 2^{6n+1} + 1$ and for every $g = 2^{4n} + 1$ there exists an $X$ and $G \subseteq \text{Aut}(X)$ with $|G| = 24(g - 1)$ such that $G/G' \cong C_3$ and $G'$ is a 2-group.

Proof. By Table 2 there are only 4 possible orders bigger than $24(g - 1)$. Of these, $48(g - 1)$, coming from $\Gamma(0; 2, 3, 8)$, can only have $G/G' \cong C_2$; so $G'$, of order $24(g - 1)$, cannot be nilpotent by Theorem 2.2 (c). The case $|G| = 36(g - 1)$ corresponds to $\Gamma(0; 2, 3, 9)$. So if $G'$ is a proper subgroup, we have $G/G' \cong C_3$ and by Table 2 $G'$ is a quotient of $\Gamma(0; 2, 2, 2, 3)$. In particular, $G'$ cannot have a quotient that is a 3-group. But since $|G'| = 12(g - 1)$ is divisible by 3, this means that $G'$ cannot be nilpotent. The same proofs work for $40(g - 1)$ and $30(g - 1)$, respectively.

For the existence proof we start with a solvable group $H$ of order $48(g - 1)$ where $g = 2^{6n+1} + 1$ or $g = 2^{4n} + 1$. By the proof of Theorem 2.2 (a) such groups exist for every $n \geq 0$. Then $G = H'$ is the desired group. It has order $24(g - 1)$ and is a
quotient of $\Gamma(0; 3, 3, 4)$ (see Table 2). So $G/G' \cong C_3$, which for our choices ensures that $G'$ is a 2-group.

In contrast, groups of order $3 \cdot 5^n$ obviously have a nilpotent commutator group, and we know from Theorem 2.4 that there are infinitely many among them that reach the bound $15(g - 1)$ for general (that is, solvable) groups of odd order. We now show that among the groups of odd order that reach this bound they are the only ones with nilpotent commutator group.

**Proposition 7.2.** Let $G$ be a group of odd order that reaches the bound $|G| = 15(g - 1)$ in Theorem 2.4. Then the commutator group $G'$ is nilpotent if and only if $|G| = 3 \cdot 5^n$.

**Proof.** If $G$ is of odd order with $|G| = 15(g - 1)$, then $G$ is a quotient of $\Gamma = \Gamma(0; 3, 3, 5)$. So $G/G' \cong C_3$ and $G'$ is a quotient of $\Gamma' = \Gamma(0; 5, 5, 5)$. This shows that $G'$ cannot have any quotients of order prime to 5. So $G'$, if nilpotent, must be a 5-group.

In the same vein we can show that the second statement of Theorem 5.1 holds under slightly weaker conditions.

**Proposition 7.3.** Let $X$ be a compact Riemann surface of genus $g \geq 2$. Let $G \subseteq \text{Aut}(X)$ be a subgroup of odd order with $|G| = \frac{3\cdot7^n}{2}(g - 1)$. Then the following are equivalent:

(a) $|G| = 3 \cdot 7^n$.

(b) $G$ is supersolvable.

(c) The commutator group $G'$ is nilpotent.

**Proof.** By Theorem 3.3, groups of order $3 \cdot 7^n$ are supersolvable. And supersolvable groups have a nilpotent commutator group by Theorem 3.2. So we only have to show that (c) implies (a), which is exactly the same argument as in the proof of Proposition 7.2.

See also Theorem 8.2 for yet another characterization of the groups in Theorem 5.1 and Proposition 7.3.

Another property of supersolvable groups is that the elements of odd order form a (normal) subgroup. This is an immediate consequence of Zappa’s Theorem. The converse is not true. For example it is well known that in a group $G$ with $|G| \equiv 2 \text{ mod } 4$ the elements of odd order form a normal subgroup of index 2. But such a group need of course not be supersolvable.
Now let $G$ be a finite group in which the elements of odd order form a subgroup $H$. Then, obviously, $H$ must be normal. Moreover, $H$ is solvable by the Theorem of Feit and Thompson [FT]. Since $G/H$, being a 2-group, is also solvable, we see that $G$ must be solvable. But of course not every solvable group has the property that the elements of odd order form a subgroup.

**Theorem 7.4.** Let $X$ be a compact Riemann surface of genus $g \geq 2$. Let $G \subseteq \text{Aut}(X)$ such that the elements of odd order in $G$ form a subgroup $N$. Then $|G| \leq 30(g - 1)$ and any $G$ that attains this bound must necessarily have $|G| \equiv 2 \mod 4$. Conversely, there are infinitely many $g$, necessarily with $g \equiv 6 \mod 10$, for which this bound is sharp.

**Proof.** We have $G/N \cong P$ where $P$ is the Sylow 2-subgroup. If 4 divides $|G|$, then $P$ and hence $G$ has an abelian quotient of order 4. This excludes the three biggest possibilities from Table 2.

If $|G| = 30(g - 1)$, for the same reason we must have $|G| \equiv 2 \mod 4$. In other words, $g - 1$ must be odd. The corresponding triangle group is $\Gamma = \Gamma(0; 2, 3, 10)$. Then $\Gamma' = \Gamma(0; 3, 3, 5)$ and $\Gamma'' = \Gamma(0; 5, 5, 5)$. So $G'/G'' \cong C_3$. Since $G'/G''$ cannot also be cyclic by [H, Theorem 9.4.2], we must have $G''/G''' \cong C_5 \times C_5$, and hence $5|(g - 1)$. So together $g \equiv 6 \mod 10$.

By the construction in [Ch, Theorem 3.2] for $m = 5$ and any odd $n$ we get $|G| = 2 \cdot 3 \cdot 5^2 \cdot n^{12}$ on a Riemann surface of genus $5n^{12} + 1$. □

**Corollary 7.5.** Let $X$ be a compact Riemann surface of genus $g \geq 2$ and $G \subseteq \text{Aut}(X)$. If $|G'|$ is odd, then $|G| \leq 30(g - 1)$. There are infinitely many $g$, necessarily with $g \equiv 6 \mod 10$, for which this bound is sharp.

**Proof.** In $G/G'$, which is abelian, the elements of odd order form a unique normal subgroup. So there is a normal subgroup $N$ of $G$ of odd order that contains all elements of odd order. □

**Corollary 7.6.** Let $X$ be a compact Riemann surface of genus $g \geq 2$. If $G \subseteq \text{Aut}(X)$ with $|G| \equiv 2 \mod 4$, then $|G| \leq 30(g - 1)$. There are infinitely many $g$, necessarily with $g \equiv 6 \mod 10$, for which this bound is sharp.

### 8. CLT groups

Another type of groups that has not been treated yet in the literature in their role as subgroups of $\text{Aut}(X)$ are CLT groups. The acronym stands for Converse of Lagrange’s Theorem.

The precise definition is: A finite group $G$ is CLT if for every divisor $d$ of $|G|$ there exists a subgroup $H$ of $G$ with $|H| = d$.  

□
Every CLT group is solvable, a nontrivial fact. Actually, much weaker conditions, for example the existence of complements of the Sylow subgroups, already guarantee solvability ([H, Theorem 9.3.3] or [R, Theorem 9.1.8]).

Every supersolvable group is CLT. Actually, more precisely, a finite group $G$ is supersolvable if and only if all its subgroups (including $G$ itself) are CLT [Wei, p.13 Theorem 4.1].

However, not every solvable group is CLT (smallest counterexample: $A_4$). And not every CLT group is supersolvable (example $S_4$).

But what makes CLT groups especially annoying to handle: A subgroup of a CLT group is not necessarily CLT (example $A_4$ in $S_4$). A quotient of a CLT group is not necessarily CLT (example $A_4 \times C_2$).

To put things into the context of this paper, we now show that the last four negative statements persist if one restricts to groups whose orders are not divisible by primes that are smaller than a given prime $p$.

**Example 8.1.** Fix an odd prime $p$. By Dirichlet’s theorem on primes in arithmetic progressions there are infinitely many primes $q$ such that $p$ divides $q + 1$. Then the finite field $\mathbb{F}_{q^2}$ contains the $p$-th roots of unity, but $\mathbb{F}_q$ doesn’t. So over $\mathbb{F}_q$ the $p$-th cyclotomic polynomial $\frac{x^p - 1}{x - 1}$ splits into irreducible quadratic factors. Let a generator of $C_p$ act on the 2-dimensional vector space $\mathbb{F}_q \oplus \mathbb{F}_q$ by a $2 \times 2$ matrix whose characteristic polynomial is one of these irreducible factors. Then there is no 1-dimensional $C_p$-invariant $\mathbb{F}_q$-subspace. Thus we obtain a group

$$H = (C_q \times C_q) \rtimes C_p$$

without subgroups of order $pq$. So $H$ is not CLT, but obviously it is solvable. Moreover, $H \times C_q$ is CLT, but not supersolvable.

We first treat CLT groups of odd order, because for them we can prove a definitive result.

**Theorem 8.2.** Let $X$ be a compact Riemann surface of genus $g \geq 2$. If $G \subseteq Aut(X)$ is a CLT group of odd order, then $|G| \leq \frac{21}{2}(g - 1)$. The CLT groups of odd order that reach this bound are exactly the supersolvable groups described in Theorem 5.1 and Proposition 7.3.

**Proof.** To establish the bound and its sharpness, in view of Theorems 2.4 and 5.1 it suffices to show that $|G|$ cannot be $15(g - 1)$.

So let’s assume $|G| = 15(g - 1)$, i.e., that $G$ is a quotient of $\Gamma(0; 3, 3, 5)$. Being a CLT group, $G$ then has a subgroup $H$ of index 5. Left multiplication of $G$ on the cosets $G/H$ gives a group homomorphism onto a transitive subgroup $U$ of $S_5$. Since $U$ has odd order, the only possibility is $U \cong C_5$, contradicting $G/G' \cong C_3$.

Now let $G$ be a CLT group of odd order $|G| = \frac{21}{2}(g - 1)$. By Proposition 7.3 we still have to show $|G| = 3 \cdot 7^n$. By exactly the same proof as above we see that 5
cannot divide $|G|$.  

Now assume that 9 divides $|G|$. Then $G$ has a subgroup of index 9. As above, we get a homomorphism from $G$ onto a transitive subgroup $U$ of $S_9$ of order $3^\mu 7^\nu$ with $2 \leq \mu \leq 4$ and $\nu \leq 1$. In any case, the Sylow 7-subgroup of $U$ is either trivial or normal, leading to a quotient of $G$ of order $3^\mu$, and ultimately to a quotient of $\Gamma(0; 3, 3, 7)$ of order 9. This shows that 9 cannot divide $|G|$.  

Finally, let $p$ be the smallest prime divisor of $|G|$ that is bigger than 7. The following proof that $p$ cannot exist uses the same trick that was used in the proof of [Sch1, Proposition 5.3], following a very helpful suggestion by the referee of that paper. 

Since $G$ is a CLT group, it has a subgroup of index $p$. Let $N$ be its core. Then $[G : N]$ can only be divisible by $p$ (once), by 3 (once), and by a power of 7. Let $U$ be the quotient of $G/N$ by its biggest normal 7-subgroup. Since the only abelian quotient of $\Gamma(0; 3, 3, 7)$ is $C_3$, we have $[U : U'] = 3$ and $U'$ is a quotient of $\Gamma(0; 7, 7, 7)$. So we obtain a sequence of normal subgroups

$$U \triangleright U' \triangleright P \triangleright I$$

where $P (\cong C_p)$ is the Sylow $p$-subgroup (which is normal in $U$) and $U'/P$ is isomorphic to the not necessarily abelian Sylow 7-subgroup $S$ of $U$. Thus $U' \cong P \times S$. Now let $C$ be the centralizer of $P$ in $U$. As $U/C$ can be embedded in the automorphism group of $P$, which is cyclic, we see that $C$ contains $U'$, and hence that $P$ is central in $U'$. So $U'$ is a direct product of $P$ and $S$. In particular, $P$ is a quotient of $U'$ and hence of $\Gamma(0; 7, 7, 7)$, giving the desired contradiction. $\square$

As CLT groups are solvable, in general we have the bound $|G| \leq 48(g - 1)$. We refine this a little bit. For the proof we recall that a Hall subgroup $H$ of a finite group $G$ is a subgroup whose order $|H|$ is relatively prime to its index $[G : H]$. 

**Lemma 8.3.** A CLT group $G$ that reaches the bound $|G| = 48(g - 1)$ must necessarily be a $(2, 3)$-group. 

**Proof.** If $G \subseteq Aut(X)$ is a CLT group of order $48(g - 1)$, let $p$ be the smallest prime divisor of $|G|$ that is bigger than 3. The following proof that $p$ cannot exist uses the same trick as in the proof of Theorem 8.2, due to the referee of the paper [Sch1]. 

Since $G$ is a CLT group, it has a subgroup of index $p$. Let $N$ be its core. Then $[G : N]$ can only be divisible by $p$ (once) and by powers of 2 and 3. Let $U$ be the quotient of $G/N$ by its biggest normal subgroup of order prime to $p$. Since the only abelian quotient of $\Gamma(0; 2, 3, 8)$ is $C_2$, we have $[U : U'] = 2$ and $U'$ is a quotient of $\Gamma(0; 3, 3, 4)$. So we obtain a sequence of normal subgroups

$$U \triangleright U' \triangleright P \triangleright I$$
where $P$ (isomorphic to $C_p$) is the Sylow $p$-subgroup (which is normal in $U$) and $U''/P$ is isomorphic to a not necessarily abelian Hall subgroup $H$ of $U'$ (see [H, Theorem 9.3.1]). Thus $U'' \cong P \times H$. Now let $C$ be the centralizer of $P$ in $U$. As $U/C$ can be embedded in the automorphism group of $P$, which is cyclic, we see that $C$ contains $U'$, and hence that $P$ is central in $U'$. So $U'$ is a direct product of $P$ and $H$. In particular, $P$ is a quotient of $U'$ and hence of $\Gamma(0; 3, 3, 4)$, giving the desired contradiction. \[\square\]

**Example 8.4.** The group of order 48 acting on a Riemann surface $X$ of genus 2 is

$$G \cong GL_2(\mathbb{F}_3) \cong Q_8 \rtimes S_3$$

where $Q_8$ is the quaternion group of order 8. Let $Z(\cong C_2)$ be the center of $Q_8$ and $C_3$ the subgroup of $S_3$. Then it is easy to see that $G$ is a CLT group. Subgroups of order 6, 12 and 24 are $Z \times C_3$, $Z \times S_3$ and $Q_8 \rtimes C_3$. For the other orders the existence of a subgroup is clear from the Sylow Theorems.

By similar arguments one can show that the group of order 96 that acts on a Riemann surface of genus 3, namely

$$G \cong (C_4 \times C_4) \rtimes S_3$$

also is a CLT group.

**Remark 8.5.** Originally, our intention was to prove the existence of infinitely many CLT groups of order $48(g - 1)$ by applying Macbeath’s approach to one of the groups from Example 8.4.

Let for example $G$ be the group of order 48 acting on a Riemann surface $X$ of genus 2. For every $\mu \geq 1$ there exists a totally unramified Galois covering $Y$ of $X$ with $Gal(Y/X) \cong (C_{2\mu})^4$. Then $Aut(Y)$ contains a subgroup $\widetilde{G}$ with $Gal(Y/X) \triangleleft \widetilde{G}$ and $\widetilde{G}/Gal(Y/X) \cong G$.

If one manages to show that

$$\widetilde{G} \cong Gal(Y/X) \rtimes G,$$

then $\widetilde{G}$ is easily seen to be a CLT group by the following argument. Let $d$ be a divisor of $|G| = 3 \cdot 2^{4\nu + 4}$. Choose the biggest $\nu \leq \mu$ such that $2^{4\nu}$ divides $d$. Then $d = 2^{4\nu}d'$ with $d'|48$. We pick the subgroup $(C_{2\nu})^4$ of $Gal(Y/X)$, which is characteristic in $Gal(Y/X)$ and hence stable under the action of $G$. And we also pick a subgroup $U$ of order $d'$ of the CLT group $G$. Then $(C_{2\nu})^4 \rtimes U$ is the desired subgroup of $\widetilde{G}$ of order $d$.

However, we have not been able to show the semi-direct product structure or something comparable that would make our attempt work.

On the other hand, it might equally well be that the two groups in Example 8.4 are indeed the only CLT groups of order $48(g - 1)$. 

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As Theorem 8.2 and Lemma 8.3 are probably the first results in the literature that deal with CLT groups as automorphism groups of Riemann surfaces, we elaborate a bit more on this encounter. For that we need the following tools.

**Theorem 8.6.** Let $G$ be a CLT group. Then

(a) Every Hall subgroup $H$ of $G$ is also CLT.

(b) $[G : G']$ is divisible by the smallest prime divisor of $|G|$.

(c) If 4 divides $|G|$, then $G$ has a quotient $C_4$ or $C_2 \times C_2$ or $C_6$ or $S_4$.

**Proof.** (a) Let $d$ be a divisor of $m = |H|$. Being a CLT group, $G$ has a subgroup $U$ of order $d$. But by [H, Theorem 9.3.1] any subgroup of order $d$ is contained in a Hall subgroup of order $m$, and all Hall subgroups of order $m$ are conjugate. So $H$ contains a conjugate of $U$.

(b) Let $p$ be the smallest prime dividing $|G|$. Being a CLT group, $G$ has a subgroup $U$ of index $p$, and since $p$ is the smallest prime dividing $|G|$, this subgroup is normal; so $G' \subseteq U$.

(c) Being a CLT group, $G$ has a subgroup $U$ of index 4. Let $N$ be its core. Then $G/N$ is isomorphic to one of $C_4$, $C_2 \times C_2$, $D_4$, $A_4$, $S_4$. The first three imply the existence of a quotient of order 4. And $A_4$ has a quotient of order 3, which together with part (b) shows that $G/G'$ has a quotient $C_6$. □

**Lemma 8.7.** There are no CLT groups among the groups of orders $40(g - 1)$, $36(g - 1)$, $30(g - 1)$, $21(g - 1)$, $20(g - 1)$, $\frac{56}{3}(g - 1)$.

**Proof.** Compare Table 2.

The case $|G| = 21(g - 1)$ is the most tricky one. Then $G$ is a quotient of $\Gamma(0; 2, 3, 14)$. Let $G$ also be a CLT group. Then 4 cannot divide $|G|$, because $\Gamma(0; 2, 3, 14)$ has none of the quotients listed in Theorem 8.6 (c). On the other hand, $C_2$ is the only abelian quotient of $\Gamma(0; 2, 3, 14)$. So $G'$ is a Hall subgroup of $G$, and hence must also be CLT by Theorem 8.6 (a). Consequently, $|G'| = 3 \cdot 7^a$ by Theorem 8.2. Being a CLT group, $G$ has a subgroup of index 7. Its core is a normal subgroup $N$ of $G$ of index 42. Here we are using again (twice) that if $G$ is solvable and $[G : G']$ a prime, then $G'$ contains all normal subgroups of $G$. Denote $G/N$ by $H$. Since the only abelian quotients of $\Gamma(0; 2, 3, 14)$ and of $\Gamma(0; 3, 3, 7)$ are $C_2$ resp. $C_3$, we get $H/H' \cong C_2$, $H'/H'' \cong C_3$ and $H''/H''' \cong C_7$. But this contradicts [H, Theorem 9.4.2], which says that $H'/H''$ and $H''/H'''$ cannot both be cyclic.

The case $\Gamma(0; 2, 3, 10)$, i.e., $|G| = 30(g - 1)$ is discarded similarly. Theorem 8.6 (c) shows that 4 cannot divide $|G|$; then Theorem 8.6 (a) implies the existence of an odd order CLT group $H$ with $|H| = 15(g - 1)$, in contradiction to Theorem 8.2.

The other cases are also quickly excluded by Theorem 8.6. The only quotient of $\Gamma(0; 2, 4, 5)$ of order prime to 5 is $C_2$; and $C_2$ is also the only quotient of $\Gamma(0; 2, 4, 7)$
of order prime to 7. The groups $\Gamma(0; 2, 3, 9), \Gamma(0; 2, 3, 15)$ and $\Gamma(0; 2, 5, 5)$ do not even have a quotient $C_2$. □

For the same reasons the triangle group $\Gamma(0; 3, 3, 4)$ can never furnish a CLT group. However, although $\Gamma(0; 3, 3, 4)$ is the commutator group of $\Gamma(0; 2, 3, 8)$, this does not exclude the possibility of CLT groups of order $48(g – 1)$. The reason simply is that the commutator group of a CLT group, even if it has index 2, is not necessarily a CLT group, as is already shown by the standard example $S_4$.

**Corollary 8.8.** The sharp bound for CLT groups $G$ whose order is not divisible by 8 is the same as for supersolvable groups, namely $|G| \leq 18(g – 1)$ if $g \geq 3$.

**Proof.** From Table 2 we see that the orders bigger than $18(g – 1)$ that were not already excluded by Lemma 8.7 are all divisible by 8. □

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