THE DIMENSION OF SKEW SHIFTED YOUNG DIAGRAMS, AND PROJECTIVE CHARACTERS
OF THE INFINITE SYMMETRIC GROUP

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§0. Introduction

This article was originally published in Russian in "Representation Theory, Dynamical Systems, Combinatorial and Algorithmic Methods. Part 2" (A. M. Vershik, ed.), Zapiski Nauchnyh Seminarov POMI 240 (1997), 115–135 (this text in Russian is available via http://www.pdmi.ras.ru/znsl/1997/v240.html). As it was mentioned in the "Journal-Ref" field this English translation was published in Journal of Mathematical Sciences (New York) 96 (1999), no. 5, 3517–3530.

The dimension of a given skew shifted Young diagram is the number of standard labellings of this diagram. In §1 of this paper, we obtain a formula for the dimension of an arbitrary skew shifted Young diagram. For this purpose, we introduce polynomials \( P^*_{\mu} \) that are factorial analogues of a particular case of Hall–Littlewood polynomials \( P(\cdot,t) \) for \( t = -1 \) ([3, ch. III, §1]). The definition of the polynomials \( P^*_{\mu} \) is due to A. Yu. Okounkov.

As an application of the formula for the dimension of a skew shifted Young diagram, we obtain new proof of the classification of projective characters of the group \( S(\infty) \). The classic Thoma’s work [12] contains the description of characters (in von Neumann’s sense) of the infinite symmetric group \( S(\infty) \) that is the inductive limit of the chain of finite symmetric groups \( S(1) \subset S(2) \subset \ldots \) (characters in von Neumann’s sense correspond to finite factor–representations). In [5] M. L. Nazarov extended Thoma’s theorem to projective characters of the infinite symmetric group \( S(\infty) \). In §2 of this paper, we give new proofs of Nazarov’s ([5]) main results.

Earlier in [6] and [8] the analogous results in the ordinary (non-projective) case were obtained. In this work we follow the methods of [8].

The author is very grateful to G. I. Olshanski for setting the problem, constant attention to the work and remarks on projects of the manuscript, and to M. L. Nazarov for Remark 1.7 on the formula for the dimension of a skew shifted diagram.

§1. Formula for the dimension of a skew shifted Young diagram

A polynomial \( f(x_1, \ldots, x_n) \) is said to be supersymmetric if it satisfies the following conditions:

1. \( f \) is a symmetric polynomial in \( x_1, \ldots, x_n \);
2. for any integers \( i, j \) such that \( 1 \leq i < j \leq n \), the polynomial

\[
f(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_{j-1}, -t, x_{j+1}, \ldots, x_n)\]


Supersymmetric polynomials in $n$ variables form an algebra. We denote it by $\Omega(n)$. It is graded by degrees of polynomials, i.e.,

$$\Omega(n) = \bigoplus_{k \geq 0} \Omega^k(n),$$

where $\Omega^k(n)$ consists of homogeneous polynomials of degree $k$ (including the zero polynomial).

For $m \geq n$, we consider a homomorphism

$$p_{m,n} : \Omega(m) \to \Omega(n)$$

such that

$$(p_{m,n}f)(x_1, \ldots, x_n) = f(x_1, \ldots, x_n, 0, 0, \ldots, 0).$$

Restricting to $\Omega^k(m)$, we obtain linear transformations

$$p_{m,n}^k : \Omega^k(m) \to \Omega^k(n).$$

The projective limit of graded algebras $\Omega(n)$ in the category of graded algebras, taken with respect to the morphisms $p_{m,n}$, is an algebra and it is called the algebra of supersymmetric functions. We denote it by $\Omega$. By definition, an element $f$ of $\Omega$ is a sequence $(f_n)_{n \geq 1}$ satisfying the following conditions:

1) $f_n \in \Omega(n)$, $n = 1, 2, \ldots$,
2) $f_{n+1}(x_1, \ldots, x_n, 0) = f_n(x_1, \ldots, x_n)$ (stability condition),
3) $\sup_n \deg f_n < \infty$.

Now we consider examples of supersymmetric functions that are important for the sequel.

Let

$$p_{k|n}(x_1, \ldots, x_n) = x_1^k + \ldots + x_n^k \quad (k = 1, 2, \ldots).$$

If $\lambda = (\lambda_1, \ldots, \lambda_l)$ is a partition, then $p_{\lambda|n}$ is defined as

$$p_{\lambda|n} = p_{\lambda_1|n} \cdot p_{\lambda_2|n} \cdot \ldots \cdot p_{\lambda_l|n}.\$$

If $\lambda$ is a partition such that all its nonzero parts are odd, then

$$p_{\lambda|n} \in \Omega(n),$$

and the sequence $(p_{\lambda|n})_{n \geq 1}$ defines a supersymmetric function $p_\lambda$. Such functions $p_\lambda$ form a linear basis of the algebra $\Omega$. In other words, $p_1, p_3, p_5, \ldots$, which are called odd Newton sums, generate algebraically the algebra $\Omega$, see [9],[3].

A partition is called strict if all its nonzero parts are distinct. The set of strict partitions of $n$ is denoted by $DP_n$.

Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ be an arbitrary partition. We denote by $l(\lambda)$ the length of $\lambda$, i.e., the number of its nonzero parts. We denote by $|\lambda|$ the weight of $\lambda$,

$$|\lambda| = \lambda_1 + \ldots + \lambda_{l(\lambda)}.$$

We also use the notation $\lambda \vdash n$ if $n = |\lambda|$.

In what follows, $\mu$ and $\lambda$ denote strict partitions, unless otherwise specified.

First, we prove a proposition whose results we use in the sequel.
Proposition 1.1. Let $x_1, x_2, \ldots$ be variables, $r(x_1, \ldots, x_l)$ be a polynomial in $l$ variables. For $n \geq l$, let

$$R_n(x_1, \ldots, x_n) = r(x_1, \ldots, x_l) \prod_{i \leq l, i < j \leq n} \frac{x_i + x_j}{x_i - x_j}$$

and

$$\tilde{R}_n(x_1, \ldots, x_n) = \sum_{\omega \in S(n)} R_n(x_{\omega(1)}, \ldots, x_{\omega(n)}).$$

Then

a) $\tilde{R}_n$ is a polynomial, and

$$\deg \tilde{R}_n \leq \deg r;$$

b) $\tilde{R}_n$ is supersymmetric;

c) $\tilde{R}_n = 0$, if $r$ is symmetric with respect to at least two variables $x_i, x_j$, $i < j \leq l$;

d) if $x_1 \ldots x_l$ divides $r(x_1, \ldots, x_l)$, then

$$\tilde{R}_{n+1}(x_1, \ldots, x_n, 0) = (n + 1 - l)\tilde{R}_n(x_1, \ldots, x_n).$$

Proof. a) We will represent $\tilde{R}_n$ as a ratio of two polynomials. Denote by $V(x_1, \ldots, x_n)$ the Vandermonde determinant

$$V(x_1, \ldots, x_n) = \prod_{i < j} (x_i - x_j).$$

We also set

$$u_n(x_1, \ldots, x_n) = r(x_1, \ldots, x_l) \prod_{i \leq l, i < j \leq n} (x_i + x_j) \prod_{l < i < j \leq n} (x_i - x_j)$$

and

$$\tilde{u}_n(x_1, \ldots, x_n) = \sum_{\omega \in S(n)} \text{sgn}(\omega)u_n(x_{\omega(1)}, \ldots, x_{\omega(n)}). \quad (1.1)$$

Note the inequality

$$\deg \tilde{u}_n \leq \deg V + \deg r. \quad (1.2)$$

We have the following relation between the polynomials

$$\tilde{R}_n = \frac{\tilde{u}_n}{V}.$$

It follows from (1.1) that $\tilde{u}_n$ is a skew–symmetric polynomial in $x_1, \ldots, x_n$, hence $\tilde{R}_n$ is a polynomial in $x_1, \ldots, x_n$. It follows from (1.2) that

$$\deg \tilde{R}_n \leq \deg r.$$

b) Symmetry of $\tilde{R}_n$ follows from its definition.
Let \(x_i = t, x_j = -t\), for arbitrary integers \(i\) and \(j\) such that \(1 \leq i < j \leq n\). Then \(R_n(x_{\omega(1)}, \ldots, x_{\omega(n)})\) does not depend on \(t\) for any permutation \(\omega\) from the group \(S(n)\). Hence, the polynomial \(\tilde{R}_n\) also does not depend on \(t\).

Thus, the polynomial \(\tilde{R}_n\) is supersymmetric.

c) Let \(r\) be symmetric with respect to variables \(x_i\) and \(x_j\), \(i < j \leq l\). Then the above-defined polynomial \(u_n\) is also symmetric with respect to \(x_i\) and \(x_j\). It follows from (1.1) that in this case \(\tilde{u}_n = 0\), since the sum in (1.1) can be broken down into pairs of summands with equal absolute values which occur in this sum with different signs. Hence, \(\tilde{R}_n = 0\).

d) Let \(x_1 \ldots x_l\) divide \(r(x_1, \ldots, x_l)\), and \(x_{n+1} = 0\). Consider an arbitrary permutation \(\omega\) from the group \(S(n + 1)\). If \(\omega^{-1}(n + 1) \leq l\), then

\[R_{n+1}(x_{\omega(1)}, \ldots, x_{\omega(n)}, x_{\omega(n+1)}) = 0.\]

If \(\omega^{-1}(n + 1) > l\), then

\[R_{n+1}(x_{\omega(1)}, \ldots, x_{\omega(n)}, x_{\omega(n+1)}) = R_n(x_{\sigma(1)}, \ldots, x_{\sigma(n)}),\]

where the permutation \(\sigma\) from the subgroup \(S(n) \subset S(n + 1)\) is obtained from \(\omega\) by multiplying by the transposition \((n + 1, \omega(n + 1))\) from the left side.

This implies the desired equality

\[\tilde{R}_{n+1} = (n + 1 - l)\tilde{R}_n.\]

Proposition is proved. □

Let us consider two particular cases of Proposition 1.1.

Let \(\lambda\) be a partition, \(l(\lambda) = l < n\),

\[r(x_1, \ldots, x_l) = \frac{\prod_{i=1}^{l} x_i^{\lambda_i}}{(n-l)!}.\]

In this case, the polynomial \(\tilde{R}_n\), which occurs in the formulation of Proposition 1.1, is denoted by \(P_{\lambda|n}\). This is a particular case of Hall–Littlewood polynomial when parameter \(t\) equals \(-1\) ([3, ch. III, §1]). Put \(P_{\lambda|n} = 0\) if \(l(\lambda) > n\). It follows from Proposition 1.1 that the sequence \((P_{\lambda|n})_{n \geq 1}\) defines a supersymmetric function \(P_\lambda\).

The functions \(P_\lambda\) (\(\lambda\) denotes a strict partition), which are called Schur \(P\)-functions, form a linear basis of the algebra \(\Omega\), see [9]. Note that if \(\nu\) is a non–strict partition, then Proposition 1.1 c) implies \(P_\nu = 0\).

We define the \(k\)th decreasing factorial power of a variable \(x\) as

\[(x \downarrow k) = \prod_{i=1}^{k} (x - i + 1), \quad k = 1, 2, \ldots.\]

We also assume

\[(x \downarrow 0) = 1.\]

Now we introduce the polynomials that play an important role in §1.
**Definition 1.2** (A. Yu. Okounkov). Let \( l = l(\lambda) \leq n, x_1, \ldots, x_n \) be variables. Let

\[
F_{\lambda|n}(x_1, \ldots, x_n) = \prod_{i=1}^{l}(x_i \downarrow \lambda_i) \prod_{i \leq l, i < j \leq n} \frac{x_i + x_j}{x_i - x_j}.
\]

We introduce a polynomial \( P^*_\lambda|n \) by the formula

\[
P^*_\lambda|n = \frac{1}{(n-l)!} \sum_{\omega \in S(n)} F_{\lambda|n}(x_{\omega(1)}, \ldots, x_{\omega(n)}).
\]

**Proposition 1.3.** If \( l(\lambda) \leq n \), then

\[
P^*_\lambda|n(x_1, \ldots, x_n) = P_{\lambda|n}(x_1, \ldots, x_n) + g(x_1, \ldots, x_n),
\]

where \( g(x_1, \ldots, x_n) \) is a supersymmetric polynomial of degree less than \( |\lambda| \).

**Proof.** If we set, in Proposition 1.1,

\[
r(x_1, \ldots, x_{l(\lambda)}) = \prod_{i=1}^{l(\lambda)}(x_i \downarrow \lambda_i) - \prod_{i=1}^{l(\lambda)} x_i^{\lambda_i},
\]

then the polynomial \( \tilde{R}_n \) obtained in this Proposition coincides (up to a scalar factor) with the difference

\[
P^*_\lambda|n - P_{\lambda|n} = g.
\]

By Proposition 1.1 a), we have

\[
\deg g \leq \deg r < |\lambda|. \quad \square
\]

If \( l(\lambda) > n \), then put \( P^*_\lambda|n = 0 \). It follows from Proposition 1.1d) that the sequence \((P^*_\lambda|n)_{n \geq 1}\) defines the supersymmetric function \( P^*_\lambda \).

Proposition 1.3 yields the form of the highest term of \( P^*_\lambda \).

**Corollary 1.4.**

\[
P^*_\lambda = P_{\lambda} + g,
\]

where \( g \) is a supersymmetric function of degree less than \( |\lambda| \).

Let \( \mu \) be a strict partition. Denote

\[
H(\mu) = \prod_{i=1}^{l(\mu)} \mu_i! \prod_{i < j} \frac{\mu_i + \mu_j}{\mu_i - \mu_j}.
\]

Let \( \lambda \) be another partition. We write \( \mu \subset \lambda \) if \( \mu_i \leq \lambda_i \) for \( i = 1, 2, \ldots, \).

We now prove an important property of the functions \( P^*_\lambda \). Next, we write \( P^*_\mu(\lambda) \) instead of \( P^*_\mu(\lambda_1, \ldots, \lambda_l(\lambda)) \).
Theorem 1.5 (vanishing property).

a) If \( \mu \not\subset \lambda \), then \( P_{\mu}^*(\lambda) = 0 \);
b) \( P_{\mu}^*(\mu) = H(\mu) \).

(The statement of this Theorem is similar to the vanishing property for \( s^* \)-functions in [8] and [6].)

Proof. Note that \((a \upharpoonright b) = 0\), if \( a, b \in \mathbb{Z}_+ \) and \( b > a \). First, we prove a).

Let \( \mu \not\subset \lambda \), then \( \lambda_k < \mu_k \) for some natural \( k \). Let us choose an arbitrary \( n \geq \max(l(\mu), l(\lambda)) \).

By Definition 1.2,

\[
P_{\mu}^*(\lambda) = \frac{1}{(n - l(\mu))!} \sum_{\omega \in S(n)} F_{\mu|n}(\lambda(1), \ldots, \lambda(n)).
\]

For an arbitrary permutation \( \omega \) from the group \( S(n) \), consider the corresponding term in the sum

\[
F_{\mu|n}(\lambda(1), \ldots, \lambda(n)) = \prod_{i=1}^{l(\mu)} (\lambda(i) \upharpoonright \mu_i) \prod_{i \leq j \leq n \atop i < j} \frac{\lambda(i) + \lambda(j)}{\lambda(i) - \lambda(j)}.
\]

There exists a positive integer \( r \) such that \( r \leq k \) and \( \omega(r) \geq k \). Then we have a chain of inequalities

\[
\lambda(\omega(r)) \leq \lambda_k < \mu_k \leq \mu_r,
\]

therefore,

\[
(\lambda(\omega(r)) \upharpoonright \mu_r) = 0
\]

and

\[
F_{\mu|n}(\lambda(1), \ldots, \lambda(n)) = 0.
\]

Since the choice of \( \omega \) is arbitrary, \( P_{\mu}^*(\lambda) = 0 \).

b) Arguing as above, we see that

\[
P_{\mu}^*(\mu) = (\mu(1) \upharpoonright \mu_1) \ldots (\mu(l(\mu)) \upharpoonright \mu(l(\mu))) \cdot \prod_{\mu_i > \mu_j} \frac{\mu_i + \mu_j}{\mu_i - \mu_j} = H(\mu).
\]

Theorem is proved. \( \square \)

Let \( \nu \) be an arbitrary partition. We recall (for details, see [3]) that the Young diagram of a partition \( \nu \) is the set of points \((i, j) \in \mathbb{Z}^2 \) such that \( 1 \leq j \leq \lambda_i, 1 \leq i \leq l(\lambda) \). Let us replace each point with the unit square with the left upper vertex at this point. We assume that the first coordinate \( i \) (the row index) increases as one goes downwards, and the second coordinate \( j \) (the column index) increases as one goes from left to right.

For example, if \( \nu = (6, 5, 3, 1) \), then
If $\nu$ is a strict partition, then the shifted diagram $D'_\nu$ is obtained from the ordinary diagram $D_\nu$ by shifting the $i$th row $(i - 1)$ squares to the right, for all $i > 1$. For $\nu = (6, 5, 3, 1)$, we obtain the shifted diagram

If $\mu$ and $\lambda$ are strict partitions, $\mu \subset \lambda$, then the skew shifted diagram $D'_{\lambda/\mu}$ corresponding to the pair $(\lambda, \mu)$ is the difference of the shifted diagrams $D'_{\lambda}$ and $D'_{\mu}$.

A shifted standard tableau of the form $\lambda/\mu$ is a labelling of the skew shifted diagram $D'_{\lambda/\mu}$ with the numbers $1, 2, \ldots, |\lambda| - |\mu|$ such that the numbers strictly increase from left to right along each row and down each column. The dimension $g_{\lambda/\mu}$ of a skew shifted diagram $D'_{\lambda/\mu}$ is the number of shifted standard tableaux of the form $\lambda/\mu$. We put $g_{\lambda/\mu} = 0$ if $\lambda$ does not contain $\mu$. Also put $g_{\lambda} = g_{\lambda/\{\emptyset\}}$, i.e. the number of shifted standard tableaux of the form $\lambda$. There is an explicit formula for $g_{\lambda}$ given in [3, ch. III, §8, example 12],

$$g_{\lambda} = \frac{|\lambda|!}{\lambda_1! \ldots \lambda_{l(\lambda)}!} \prod_{i<j} \frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j}.$$

Now we state the main result of this work which allows us to obtain an explicit formula for $g_{\lambda/\mu}$.

**Theorem 1.6.** Suppose $\mu$ and $\lambda$ be strict partitions. Let $m = |\mu|$, $k = |\lambda|$; then

$$g_{\lambda/\mu} = g_{\lambda} \cdot \frac{P_{\mu}^*(\lambda)}{(k \mid m)}.$$

**Proof.** Let us fix $\mu$ and consider $g_{\lambda/\mu}$ as a function of strict partition $\lambda$, where $|\lambda| \geq |\mu|$.

For two partitions $\mu$ and $\nu$, we write $\mu \nearrow \nu$ if $|\nu| = |\mu| + 1$ and $\mu \subset \nu$, or, in other words, $D_\nu$ is obtained from $D_\mu$ by adding one square.
As a function of $\lambda$, the expression $g_{\lambda/\mu}$ is defined by three properties:

(i) $g_{\mu/\mu} = 1$;
(ii) if $\lambda \nsubseteq \mu$, $|\lambda| \geq |\mu|$, then 
$$g_{\lambda/\mu} = 0;$$
(iii) $g_{\lambda/\mu} = \sum_{\nu \text{ is strict, } \nu \ntriangleright \lambda} g_{\nu/\mu}$, $|\lambda| \geq |\mu| + 1$.

Let us prove that these three properties are satisfied for

$$G(\lambda) = P^*_\mu(\lambda) \cdot \frac{g_{\lambda}}{(k \downarrow m)}.$$ 

(Since $k \geq m$, the denominator does not vanish.)

The property (i) is satisfied,

$$G(\mu) = P^*_\mu(\mu) \cdot \frac{g_{\mu}}{m!} = H(\mu) \cdot \prod_{i<j} \frac{\mu_i - \mu_j}{\mu_i + \mu_j} \frac{1}{\mu_i! \ldots \mu_{l(\mu)}!} = 1.$$ 

The property (ii) follows from the vanishing property (Theorem 1.5) for $P^*_\mu$. Let us prove (iii). In our case, $k \geq m + 1$.

If $\mu \nsubseteq \lambda$, then both parts of the desired equality are zero.

Now let $\mu \subset \lambda$. We set

$$F(\lambda_1, \ldots, \lambda_{l(\lambda)}) = \frac{(k - m)!}{\lambda_1! \ldots \lambda_{l(\lambda)}!} \prod_{t=1}^{l(\mu)} (\lambda_t \downarrow \mu_t) \prod_{l(\mu) < i < j} \frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j}. $$

It follows from Definition 1.2 and the formula for $g_\lambda$ that

$$G(\lambda) = \frac{1}{(l(\lambda) - l(\mu))!} \sum_{\omega \in S(n)} \text{sgn}(\omega) F(\lambda_{\omega(1)}, \ldots, \lambda_{\omega(l(\lambda)))}. \tag{1.3}$$

Let $\lambda^{(i)}$ denote the partition obtained from $\lambda$ by decreasing the $i$th part by 1,

$$\lambda^{(i)} = (\lambda_1, \ldots, \lambda_{i-1}, \lambda_i - 1, \lambda_{i+1}, \ldots, \lambda_n).$$

Let $\omega$ be an arbitrary permutation from the group $S(n)$. The equality

$$F(\lambda_{\omega(1)}, \ldots, \lambda_{\omega(n)}) = \sum_{i=1}^{n} F(\lambda^{(i)}_{\omega(1)}, \ldots, \lambda^{(i)}_{\omega(n)})$$

is equivalent to the identity

$$\sum_{i=l(\mu) + 1}^{l(\lambda)} \lambda_{\omega(i)} = \sum_{i=l(\mu) + 1}^{l(\lambda)} \lambda_{\omega(i)} \prod_{j \neq i, \ l(\mu) < j \leq l(\lambda)} \frac{\lambda_{\omega(i)} + \lambda_{\omega(j)} - 1}{\lambda_{\omega(i)} + \lambda_{\omega(j)}} \cdot \frac{\lambda_{\omega(i)} - \lambda_{\omega(j)} - 1}{\lambda_{\omega(i)} - \lambda_{\omega(j)}};$$

which is given (in another notation for variables) in [3, ch. III, §8, example 12] and [4].
Hence, we have for $G(\lambda)$

$$G(\lambda) = \sum_{\nu \triangleright \lambda} G(\nu).$$

Note that if $\nu$ is a non-strict partition, then

$$G(\nu) = 0,$$

because the sum in (1.3) breaks down into pairs of summands with equal absolute values which occur with different signs. Therefore, we obtain the relation

$$G(\lambda) = \sum_{\nu \triangleright \lambda, \nu \text{ is strict}} G(\nu).$$

Thus, all three properties are satisfied for $G(\lambda)$, and Theorem is proved. □

If $\nu$ and $\eta$ are ordinary partitions (not necessarily strict), $\nu \subset \eta$, then the skew diagram $D_{\eta/\nu}$ corresponding to the pair $(\nu, \eta)$ is the difference of diagrams $D_\eta$ and $D_\nu$.

A standard tableau of the form $\eta/\nu$ is a labelling of squares of the skew diagram $D_{\eta/\nu}$ with the numbers $1, 2, \ldots, |\eta| - |\nu|$ such that the numbers strictly increase from left to right along each row and down each column. The dimension $f_{\eta/\nu}$ of a skew diagram $D_{\eta/\nu}$ is the number of standard tableaux of the form $\eta/\nu$.

In [8] there is an explicit formula for $f_{\eta/\nu}$ in terms of so-called shifted Schur polynomials $s^*_\mu$. If $l(\mu) \leq n$, then

$$s^*_\mu(x_1, \ldots, x_n) = \frac{\det[((x_i + n - i)(\mu_j + n - j))]_{i,j=1}^n}{\prod_{i<j}((x_i - x_j) + j - i)}.$$

(We note that the polynomial $s^*_\mu$, like ordinary Schur polynomials, possesses the stability property as $n \to \infty$). The formula for $f_{\eta/\nu}$ takes the form

$$f_{\eta/\nu} = f_\eta \cdot \frac{s^*_\nu(\eta)}{|\eta| \downarrow |\nu|}.$$

There is an explicit formula for $f_\eta$

$$f_\eta = \frac{|\eta|!}{\prod_{i=1}^{l(\eta)}(\eta_i + l(\eta) - i)!} \cdot \prod_{1 \leq i < j \leq l(\eta)}(\eta_i - \eta_j + j - i).$$

**Remark 1.7.** If $\mu$ and $\lambda$ are strict partitions, $\mu \subset \lambda$, then the skew shifted diagram $D'_{\lambda/\mu}$ coincides with an ordinary skew diagram $D_{\eta/\nu}$ for some partitions $\eta$ and $\nu$ if and only if one of the following conditions holds:

$$l(\lambda) = l(\mu) \quad \text{or} \quad l(\lambda) = l(\mu) + 1.$$

This means that $D'_{\mu}$ completely contains the part of the diagram $D'_{\lambda}$ that lies to the left of the vertical line $j = l(\lambda) - 1$. 

9
If \( l(\lambda) = l(\mu) \) or \( l(\lambda) = l(\mu) + 1 \), then we define \( \eta \) and \( \nu \) as

\[
\eta_i = \lambda_i + i - 1, \\
\nu_i = \mu_i + i - 1, \quad i = 1, 2, \ldots, l(\lambda).
\]

Then \( D'_{\lambda/\mu} = D_{\eta/\nu} \). In this case, the number of standard tableaux \( f_{\eta/\nu} \) equals the number of shifted standard tableaux \( g_{\lambda/\mu} \).

This implies the following identity for \( P^*_{\mu}(\lambda) \) and \( s^*_{\nu}(\eta) \):

\[
P^*_{\mu}(\lambda) \cdot \frac{g_{\lambda}}{|\lambda| \downarrow |\mu|} = g_{\lambda/\mu} = f_{\eta/\nu} = s^*_{\nu}(\eta) \cdot \frac{f_{\eta}}{|\eta| \downarrow |\nu|}.
\]

One can check this identity directly from definitions of \( P^*_{\mu} \) and \( s^*_{\nu} \).

One can compare this identity with a well–known fact from the theory of Schur superfunctions, the Berele-Regev formula, see [3, ch. I, §3, example 23,(4)].

**§2. PROOF OF THE FORMULA FOR CHARACTERS OF THE INFINITE SPIN–SYMMETRIC GROUP**

The symmetric group \( S(n) \) (the group of all permutations of the numbers \( 1, \ldots, n \)) is generated by the permutations \( S_k \) of the numbers \( k \) and \( k + 1 \) \( (k = 1, \ldots, n - 1) \) with relations

\[
S_k^2 = e; \quad (S_k \cdot S_{k+1})^3 = e; \quad (S_k S_{k'})^2 = e, \quad k - k' > 1.
\]

The group \( S(\infty) \) (the group of all finite permutations of natural numbers) is the inductive limit of the sequence \( S(1) \subset S(2) \subset \ldots \).

The spin–symmetric group \( \tilde{S}(n) \) is a non–trivial central \( \mathbb{Z}_2 \)–extension of the group \( S(n) \). It is defined as the group with generators \( c, t_1, t_2, \ldots, t_{n-1} \) and relations

\[
c^2 = e; \quad c t_k = t_k c; \quad t_k^2 = e; \quad (t_k t_{k+1})^3 = e; \quad (t_k t_{k'})^2 = e, \quad k' - k > 1.
\]

Projective characters of the group \( S(n) \) are linearized by the group \( \tilde{S}(n) \), see [4], [10], [11].

Define \( \tilde{S}(\infty) \) as the inductive limit of the chain \( \tilde{S}(1) \subset \tilde{S}(2) \subset \ldots \). The group \( \tilde{S}(\infty) \) is a non–trivial central \( \mathbb{Z}_2 \)–extension of the group \( S(\infty) \).

Next, we consider irreducible representations of the group \( \tilde{S}(n) \) that send the element \( c \) to \( -1 \). They may be indexed by strict partitions of the number \( n \) [10]. If a strict partition \( \lambda \vdash n \) is such that \( n - l(\lambda) \) is even, then there is one irreducible representation that corresponds to this partition; we denote the character of this representation by \( \chi_{\lambda}^\lambda \). If a strict partition \( \lambda \vdash n \) is such that \( n - l(\lambda) \) is odd, then there are two irreducible representations that correspond to this partition; we denote their characters by \( \chi_{\lambda}^\lambda \) and \( \chi_{\lambda}^- \).

If \( \rho \) is a partition of \( n \), then we set

\[
t_\rho = (t_1 t_2 \ldots t_{\rho_1 - 1})(t_{\rho_1 + 1} \ldots t_{\rho_1 + \rho_2 - 1}) \ldots (t_{\rho_1 + \ldots + \rho_{(\rho)} - 1 + 1} \ldots t_{n-1}).
\]
Clearly, it suffices to define characters of irreducible representations on the elements \( t_\rho \) for their complete definition on the whole group \( \tilde{S}(n) \).

In the sequel, \( \lambda, \mu, \nu \) always denote strict partitions.

If \( n - l(\lambda) \) is even, \( \lambda \vdash n \), then it follows from \( \chi^\lambda(t_\rho) \neq 0 \) that \( \rho \) is a partition of \( n \) into odd parts. If \( n - l(\lambda) \) is odd, \( \lambda \vdash n \), then irreducible characters \( \chi^\lambda_+ \) and \( \chi^\lambda_- \) are such that \( \chi^\lambda_+(t_\rho) \) or \( \chi^\lambda_-(t_\rho) \) can be nonzero only in the following cases: either all parts of \( \rho \) are odd or \( \rho = \lambda \). In the first case, \( \chi^\lambda_+(t_\rho) = \chi^\lambda(t_\rho) \), and, in the second case, \( \chi^\lambda_-(t_\lambda) = -\chi^\lambda(t_\lambda) \).

Our purpose is to describe indecomposable characters of the group \( \tilde{S}(\infty) \) in the sense of the following definition.

**Definition 2.1.** Let \( G \) be an abstract group. A function \( \chi : G \to \mathbb{C} \) is said to be an indecomposable character if the following conditions hold:
1) \( \chi(e) = 1 \);
2) \( \chi(g_1g_2) = \chi(g_2g_1), \forall g_1, g_2 \in G \);
3) \( \sum_{k,l} \chi(g_1^{-1}g_l)\delta_{k,l} \geq 0, \forall g_1, \ldots, g_n \in G, \forall a_1, \ldots, a_n \in \mathbb{C} \);
4) if \( \chi, \chi_1 \) and \( \chi_2 \) satisfy conditions 1)–3), and there exists a number \( a \) such that \( 0 < a < 1 \) and \( \chi = a\chi_1 + (1 - a)\chi_2 \), then \( \chi = \chi_1 = \chi_2 \).

If \( G \) is a finite group, then its indecomposable characters (in the sense of Definition 2.1) coincide with normalized irreducible characters.

It follows from Vershik–Kerov theorem [1, 2, ch. I, §1] that every indecomposable character of the group \( \tilde{S}(\infty) \) is a pointwise limit of normalized irreducible characters of the groups \( \tilde{S}(n) \) as \( n \to \infty \).

Consider a sequence \( (\lambda(n) \vdash n) \) of strict partitions, \( n = 1, 2, \ldots \). For every \( n \), we denote by \( \xi_n \) any of the following normalized irreducible characters,

\[
\frac{\chi^\lambda(n)}{\chi^{\lambda(n)}(e)} , \quad \frac{\chi^\lambda_+(n)}{\chi^{\lambda(n)}_+(e)} , \quad \frac{\chi^\lambda_-(n)}{\chi^{\lambda(n)}_-(e)} . \tag{2.1}
\]

**Theorem 2.2** (M. L. Nazarov). The pointwise limit \( \lim_{n \to \infty} \xi_n \) exists if and only if the limits \( \lim_{n \to \infty} \frac{\lambda_i(n)}{n} = \gamma_i, i = 1, 2, \ldots, \) exist.

**Proof.** A projection \( \pi : \tilde{S}(n) \to S(n) \) is defined as

\[
\pi(e) = e, \quad \pi(t_k) = S_k \quad (k = 1, \ldots, n - 1).
\]

If at least one cycle in \( \pi(t) \) has even length, then \( \xi_n(t) = 0 \) for all sufficiently large \( n \). It follows that the existence of the limit \( \lim_{n \to \infty} \xi_n \) and its value depend only on the sequence \( \lambda(n) \), but not on the choice of signs “+” or “−” in (2.1).

Thus, we introduce the following notation.

\[
\chi^\lambda_\pm = \begin{cases} 
\chi^\lambda, & \text{if } |\lambda| - l(\lambda) \text{ is even,} \\
\frac{\chi^\lambda_+ + \chi^\lambda_-}{\sqrt{2}}, & \text{if } |\lambda| - l(\lambda) \text{ is odd.}
\end{cases}
\]

In the space of functions \( f(t) \) on the group \( \tilde{S}(k) \) such that \( f(ct) = -f(t) \), we introduce a scalar product

\[
\langle f, g \rangle_k = \frac{1}{k!} \sum_{s \in S(k)} (f\tilde{g})(s) .
\]
Denote by $\text{Res}_k$ the operator of restriction to the subgroup $\tilde{S}(k) \subset \tilde{S}(n)$, $n > k$.

The pointwise convergence of $\xi_n$ is equivalent to the following statement. For every $k$ and every strict partition $\mu \vdash k$, there exists the limit

$$\lim_{n \to \infty} \left\langle \frac{\text{Res}_k \chi^\lambda(n)}{\chi^\lambda(e)}, \chi^\mu \right\rangle_k.$$ 

We denote

$$\epsilon(n) = \begin{cases} 1, & \text{if } n \text{ is odd}, \\ \sqrt{2}, & \text{if } n \text{ is even}. \end{cases}$$

The branching rule for $\chi^\nu$ takes the following form [4, Theorem 10.2]

$$\text{Res}_{|\nu|-1} \chi^\nu = \sum_{\mu \vdash \nu} \chi^\mu \cdot \epsilon(l(\nu) - l(\mu)),$$

where we assume that $\mu$ is a strict partition; the notation $\mu \not\vdash \nu$ was introduced in the proof of the formula for the dimension of a skew shifted diagram.

Let

$$\chi^\lambda_0 = \begin{cases} 2^{(|\lambda| - l(\lambda))} \chi^\lambda, & \text{if } |\lambda| - l(\lambda) \text{ is even}, \\ 2^{(|\lambda| - l(\lambda) - 1)} (\chi^\lambda_1 + \chi^\lambda_2), & \text{if } |\lambda| - l(\lambda) \text{ is odd}. \end{cases}$$

In this normalization, the branching rule takes the simplest form,

$$\text{Res}_{|\nu|-1} \chi^\nu_0 = \sum_{\mu \vdash \nu} \chi^\mu_0.$$

This implies the following fact. Suppose $k \leq n$, and $\lambda \vdash n$ be a strict partition. Then

$$\text{Res}_k \chi^\lambda_0 = \sum_{\mu \vdash k} g_{\lambda/\mu} \chi^\mu_0,$$

where $g_{\lambda/\mu}$ denotes the dimension of the skew shifted diagram $D'_{\lambda/\mu}$.

Taking $\chi^\lambda_0$ instead of $\chi^\lambda_0$, we obtain

$$\text{Res}_k \chi^\lambda_0 = 2^{(|\lambda| - l(\lambda))} \sum_{\mu \vdash k} g_{\lambda/\mu} \chi^\mu_0 = \sum_{\mu \vdash k} 2^{(|\lambda| - l(\lambda) - |\mu| + l(\mu))} g_{\lambda/\mu} \chi^\mu_0.$$

Note also that

$$\chi^\lambda_0(e) = 2^{(|\lambda| - l(\lambda))} \chi^\lambda_0(e) = 2^{(|\lambda| - l(\lambda))} g_\lambda.$$ 

Since $\langle \chi^\mu, \chi^\nu \rangle_k = 1$, we have

$$\langle \frac{\text{Res}_k \chi^\lambda(e)}{\chi^\lambda_0(e)}, \chi^\nu \rangle_k = \frac{g_{\lambda/\mu}}{g_\lambda} \cdot 2^{(|\lambda| - |\mu|)} = 2^{(|\lambda| - |\mu|)} \cdot \frac{P^*_{\mu}(\lambda)}{(|\lambda| \downarrow |\mu|)},$$

where the last equality follows from Theorem 1.6.

Note that Corollary 1.4 implies

$$P^*_{\mu}(\lambda(n)) = P_{\mu}(\lambda(n)) + O(|\lambda(n)|^{||\mu||-1})$$

12
as \( n \to \infty \).

Also we have
\[
(n \downarrow |\mu|) = n^{|\mu|} + O(n^{|\mu|-1})
\]
as \( n \to \infty \).

Hence, the existence of the pointwise limit \( \lim_{n \to \infty} \xi_n \) is equivalent to the existence of the limits
\[
\lim_{n \to \infty} \frac{P_\mu(\lambda(n))}{n^{|\mu|}}
\]
for all \( \mu \).

The functions \( P_\mu \) form a linear basis of the algebra of supersymmetric functions \( \Omega \). Since odd Newton sums \( p_1, p_3, p_5, \ldots \) generate algebraically \( \Omega \), the existence of the limits (2.2) is equivalent to the existence of the limits
\[
\lim_{n \to \infty} \frac{p_m(\lambda(n))}{n^m}, \quad m = 1, 3, 5, \ldots
\]

We will prove that if there exist the limits
\[
\lim_{n \to \infty} \frac{\lambda_i(n)}{n} = \gamma_i \quad (i = 1, 2, \ldots),
\]
then there exist the limits
\[
\lim_{n \to \infty} \frac{p_m(\lambda(n))}{n^m} = \sum_{k=1}^{\infty} \gamma_k^m = p_m(\gamma_1, \gamma_2, \ldots)
\]
for \( m = 3, 5, 7, \ldots \) (Note that \( \frac{p_1(\lambda(n))}{n} = 1 \)).

If \( l(\lambda(n)) \) is bounded as \( n \to \infty \), then this statement is trivial.

Hence, we may assume that \( \lim_{n \to \infty} l(\lambda(n)) = \infty \). Then the desired fact is evident from the following estimation. Suppose \( N \leq l(\lambda), m \geq 3 \). Then
\[
\frac{p_m(\lambda_1, \ldots, \lambda_i)}{n^m} = \frac{p_m(\lambda_1, \ldots, \lambda_N)}{n^m} + \sum_{i=N+1}^{l(\lambda)} \frac{\lambda_i^m}{n^m} \leq \frac{p_m(\lambda_1, \ldots, \lambda_N)}{n^m} + \frac{\lambda_{N+1}}{n}.
\]

In the last inequality we used the fact that
\[
\sum_{i=N+1}^{l(\lambda)} \lambda_i^{m-1} = \sum_{i=1}^{l(\lambda)} \lambda_i^{m-1} \leq \left( \sum_{i=1}^{l(\lambda)} \lambda_i \right)^{m-1} = n^{m-1}.
\]

On the other hand,
\[
\frac{p_m(\lambda_1, \ldots, \lambda_i)}{n^m} \geq \frac{p_m(\lambda_1, \ldots, \lambda_N)}{n^m}.
\]
Taking into account that
\[
\lim_{r \to \infty} \sum_{k=r+1}^{\infty} \gamma_k^m = 0, \quad m = 3, 5, 7, \ldots,
\]
and
\[
\lim_{N \to \infty} \left( \lim_{n \to \infty} \frac{\lambda_{N+1}(n)}{n} \right) = \lim_{N \to \infty} \gamma_{N+1} = 0,
\]
we obtain the desired statement.

Thus, the existence of the limits
\[
\lim_{n \to \infty} \frac{\lambda_i(n)}{n} \quad (i = 1, 2, \ldots)
\]
implies the existence of the pointwise limit \( \lim_{n \to \infty} \xi_n \).

Now we will prove the inverse statement. We still consider a sequence of strict partitions \((\lambda(n))_{n \geq 1}\) such that \( |\lambda(n)| = n \).

As proved above, the existence of the pointwise limit \( \lim_{n \to \infty} \xi_n \) implies the existence of the limits
\[
\lim_{n \to \infty} \frac{p_m(\lambda(n))}{n^m}, \quad m = 1, 3, 5, \ldots.
\]
The sequences \( \left( \frac{\lambda_i(n)}{n} \right)_{n \geq 1} \) are bounded for all \( i \). We can choose a subsequence \((\lambda(n_k))_{k \geq 1}\) such that the limits
\[
\lim_{k \to \infty} \frac{\lambda_i(n_k)}{n_k}
\]
exist for all \( i \). Let
\[
\lim_{k \to \infty} \frac{\lambda_i(n_k)}{n_k} = \gamma_i, \quad i = 1, 2, \ldots.
\]
Then, arguing as above, we obtain the equalities
\[
\lim_{n \to \infty} \frac{p_m(\lambda(n))}{n^m} = \lim_{k \to \infty} \frac{p_m(\lambda(n_k))}{n_k^m} = p_m(\gamma_1, \gamma_2, \ldots)
\]
for \( m = 3, 5, 7, \ldots \). Note that every sequence \( \gamma = (\gamma_i)_{i \geq 1} \) such that \( \gamma_1 \geq \gamma_2 \geq \ldots \geq 0, \sum_i \gamma_i \leq 1 \) is uniquely defined by the values
\[
p_3(\gamma), p_5(\gamma), \ldots.
\]
In fact, \( \gamma_1 \) is uniquely defined by the condition that
\[
\lim_{n \to \infty} \frac{p_{2n+1}(\gamma)}{\gamma_1^{2n+1}}
\]
exists and is not zero. In order to define \( \gamma_{i+1} \), given \( \gamma_1, \gamma_2, \ldots, \gamma_i \), it suffices to consider the values
\[
p_3(\gamma) - \sum_{k=1}^{i} \gamma_3^k, p_5(\gamma) - \sum_{k=1}^{i} \gamma_5^k, \ldots.
\]
It follows that for all \( i \), the sequence \( \left( \frac{\lambda_i(n)}{n} \right)_{n \geq 1} \) has only one limit point, and thus it converges. Theorem follows. □

Now we will find the pointwise limit of the sequence \( \xi_n \) as
\[
\lim_{n \to \infty} \frac{\lambda_i(n)}{n} = \gamma_i, \quad i = 1, 2, \ldots
\]
Theorem 2.3 (M. L. Nazarov [5]). Suppose

\[ \gamma = (\gamma_1 \geq \gamma_2 \geq \ldots \geq 0) \]

and

\[ \lim_{n \to \infty} \frac{\lambda_i(n)}{n} = \gamma_i, \quad i = 1, 2, \ldots, \lambda(n) \vdash n. \]

We denote by \( \psi_\gamma \) the pointwise limit of the sequence \( \xi_n \) defined by (2.1). Then

\[ \psi_\gamma(t_\rho) = \begin{cases} 
\prod_{i \geq 2} p_i(\gamma)^{m_i(\rho)} \cdot 2^{\frac{(|\rho| - |\rho|)}{2}}, & \text{if } \rho \text{ is a partition into odd parts,} \\
0, & \text{otherwise,}
\end{cases} \]

where \( m_i(\rho) \) is the number of parts of the partition \( \rho \) equal to \( i \), and \( t_\rho \) is the element of the group \( \widetilde{S}(\infty) \) that was introduced above.

Proof. Let \( \rho \) be a partition of \( k \) into odd parts. Then

\[ \psi_\gamma(t_\rho) = \lim_{n \to \infty} \frac{\chi_\gamma^{\lambda(n)}(t_\rho)}{\chi_\gamma^{\lambda(n)}(e)} = \lim_{n \to \infty} \sum_{\mu \vdash k} \left\langle \text{Res}_k \chi_\gamma^{\lambda(n)}, \chi_\gamma^{\mu} \right\rangle_k \chi_\gamma^{\mu}(t_\rho). \quad (2.3) \]

Taking into account the above–obtained equality

\[ \left\langle \text{Res}_k \chi_\gamma^{\lambda}, \chi_\gamma^{\mu} \right\rangle_k = \frac{2^{\frac{(|\mu| - k)}{2}} P_\mu(\lambda)}{|\lambda(n)|^k}, \]

we rewrite (2.3) as

\[ \lim_{n \to \infty} \sum_{\mu \vdash k} \frac{2^{\frac{(|\mu| - k)}{2}} P_\mu(\lambda(n))}{(|\lambda(n)|^k)} \cdot \chi_\gamma^{\mu}(t_\rho) = \lim_{n \to \infty} \sum_{\mu \vdash k} \frac{2^{\frac{(|\mu| - k)}{2}} \cdot P_\mu(\lambda(n))}{|\lambda(n)|^k} \cdot \chi_\gamma^{\mu}(t_\rho) \quad (2.4) \]

Now we use an equality from [11, §7] which in our notation takes the form

\[ \sum_{\mu \vdash k} 2^{\frac{(|\mu| - k)}{2}} P_\mu(\lambda) \cdot \chi_\gamma^{\mu}(t_\rho) = 2^{\frac{|\rho| - k}{2}} P_\rho(\lambda). \]

Hence (2.4) takes a simple form

\[ \lim_{n \to \infty} \frac{p_\rho(\lambda(n))}{|\lambda(n)|^k} \cdot 2^{\frac{|\rho| - k}{2}} = 2^{\frac{|\rho| - |\rho|}{2}} \prod_{i \geq 2} p_i(\gamma)^{m_i(\rho)}. \]

If at least one part of the partition \( \rho \) is even, then

\[ \psi_\gamma(t_\rho) = 0, \]

since then

\[ \xi_n(t_\rho) = 0 \]

for all sufficiently large \( n \). This completes the proof. \( \square \)
Proposition 2.4. The functions $\psi_\gamma$ obtained in the previous Theorem are indeed indecomposable characters of the group $\tilde{S}(\infty)$.

Proof. We check Definition 2.1 for the functions $\psi_\gamma$. Properties 1)–3) are satisfied since they are satisfied for normalized characters whose limit is $\psi_\gamma$. For $\psi_\gamma$, the multiplicativity property is satisfied, i.e., if $\rho$ and $\sigma$ are arbitrary partitions, and $\rho \cup \sigma$ denotes their disjoint union, then

$$\psi_\gamma(t_{\rho \cup \sigma}) = \psi_\gamma(t_\rho) \cdot \psi_\gamma(t_\sigma).$$

This implies the indecomposability of $\psi_\gamma$ (property 4 of Definition 2.1) [13, 7].

Supported by Soros International Educational Program, grant 2093s. Translated by N. V. Tsilevich.

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