SOME GRADED IDENTITIES OF THE CAYLEY-DICKSON ALGEBRA

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Abstract. We work to find a basis of graded identities for the octonion algebra. We do so for the $\mathbb{Z}_2^2$ and $\mathbb{Z}_3$ gradings, both of them derived of the Cayley-Dickson process, the later grading being possible only when the characteristic of the scalars is not two.

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1. Definitions and Preliminary Results

Definition 1.1 (Graded Algebra). An algebra $A$ over the associative, commutative and unitary ring $R$ is said graded by the group $G$, or simply $G$-graded, if $A = \bigoplus_{a \in G} A_a$, as $R$-submodules and $A_aA_b \subseteq A_{ab} \ \forall a, b \in G$. We’ll denote by $a_h$ the projection of $a$ in $A_h$.

Notation 1.2 (Graded Polynomial). Let $X$ be a set, $G$ a group and $R$ an associative, commutative and unitary ring, we denote by $V[X_G]$ the free groupoid freely generated by $X_G := \{x^a | x \in X, a \in G\}$ (resp. $V[X_G]^\#$ for the unitary case) and $R_G\{X\} := RV[X_G]$ (resp. $R_G\{X\}^\# := RV[X_G]^\#$) the groupoid ring of $V[X_G]$ by $R$ (resp. $V[X_G]^\#$ by $R$). From now on we set $X = \{x_n | n \in \mathbb{N}\}$ and call $R_G\{X\}$ (resp. $R_G\{X\}^\#$) the $G$-graded non associative polynomial ring (resp. the unitary $G$-graded non associative polynomial ring) over $R$.

Set $g : V[X_G] \to G$ (resp. $g : V[X_G]^\# \to G$) recursively as $g(x^a) := a \ \forall x \in X$ and $g(uv) := g(u)g(v)$ (also let $g(1) = e$ the neutral element of $G$ for the $V[X_G]^\#$ case) and $R_G\{X\}_a := \text{lin.span} < g^{-1}(a) >$ (resp. $R_G\{X\}^\#_a := \text{lin.span} < g^{-1}(a) >$). Which makes $R_G\{X\}$ (resp. $R_G\{X\}^\#$) into the $G$-graded free algebra freely generated by $X$ (resp. the unitary $G$-graded free algebra freely generated by $X$). From now on we drop the superscript of the variables and refer to them by $g$. 

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Let $x, x_1, \ldots, x_n \in V[X_G]$ and $h, h_1, \ldots, h_n \in G$, set $\deg : V[X_G] \to G$ as $\deg u$ is the degree of $u$, $\deg_x : V[X_G] \to G$ as $\deg_x u$ is the degree of $u$ with respect to $x$ and $\deg_h : V[X_G] \to G$ as $\deg_h u$ is the degree of $u$ with respect to all $x$'s such that $g(x) = h$. Let $f \in R_G\{X\}\backslash\{0\}$ (resp. $f \in R_G\{X\}^\#\backslash\{0\}$) we shall denote by $\deg f$ as $\max\{\deg u|\text{ where } u \text{ is a monomial of } f\}$ and $\deg f$ as $\min\{\deg u|\text{ where } u \text{ is a monomial of } f\}$, analogue for $\deg_x$, $\deg_x$, $\deg_h$ and $\deg_h$.

We shall call $f$ homogeneous if $\deg f = \deg f$, homogeneous in $x_1, \ldots, x_n$ (resp. homogeneous in $h_1, \ldots, h_n$) if $\deg_{x_i} f = \deg_{x_i} f$ (resp. $\deg_{h_i} f = \deg_{h_i} f$) for $i = 1, \ldots, n$. Finally we shall say that $f$ is multihomogeneous (resp. multicomponent homogeneous) if it is homogeneous for every $x \in V[X_G]$ (resp. $h \in G$).

**Definition 1.3** (Graded Polynomial Identity). Let $f(x_1, \ldots, x_n) \in R_G\{X\}$ (resp. $f(x_1, \ldots, x_n) \in R_G\{X\}^\#$), $f$ is said a $G$-graded polynomial identity, polynomial identity, $G$-P.I. or simply an P.I. of the $G$-graded $R$-algebra $A$ if for any $a_i \in A_{g(x_i)}$ $i = 1, \ldots, n$ $f(a_1, \ldots, a_n) = 0$. The set of all $G$-graded polynomial identity of the $G$-graded $R$-algebra $A$ is called the $G$-graded $T$-ideal of $A$ and denoted by $T_G(A)$, it’s easy to see that $T_G(A)$ form an ideal Which is invariant under any $G$-graded endomorphism.

$T_G(A)$ is said homogeneous (resp. multihomogeneous, multicomponent homogeneous) if every homogeneous (resp. multihomogeneous, multicomponent homogeneous) of an polynomial in $T_G(A)$ lies in $T_G(A)$.

The next definition and theorem are due to Shirshov on his search for the answer of the Kurosh problem for alternate P.I. algebras and can be found in [Šir57b] and [Šir57a]. For a long time those articles were only available in russian and those proof could only be found in english on [ZSSS82]. However, recently several papers of Shirshov, including those two, received a translation to english in [Shi09].

**Definition 1.4** ($r$-words). Suppose that $X_G$ is ordered. Define recursively $< x_1 > := x_1$, $< x_1, \ldots, x_n, x_{n+1} > := < x_1, \ldots, x_n > \cdot x_{n+1}$ for $n \geq 1$. We shall call a non associative word of the form $< x_{i_1}, \ldots, x_{i_n} >$ an $r_1$-word. If the $r_1$-word $< x_{i_1}, \ldots, x_{i_n} >$ is such that $i_1 \leq \ldots \leq i_n$ then we shall call it an regular $r_1$-word. Furthermore, we shall call a non associative word of the form $< u_1, \ldots, u_n >$, where each $u_i$ is an $r_1$-word (resp. a regular $r_1$-word), an $r_2$-word (resp. a regular $r_2$-word).

**Theorem 1.5** (Shirshov). Let $A$ be an alternative algebra and $v(x_1, \ldots, x_n)$ a non associative word. Then for any elements $a_1, \ldots, a_n \in A$ the element $v(a_1, \ldots, a_n)$ is representable in the form of a linear combination of regular $r_2$-words from $a_1, \ldots, a_n$ with the same length as $v$.

The following two assertions are well known results in P.I. theory. An non graded proof of them can be found in [ZSSS82].

**Proposition 1.6.** Let $A$ be a $G$-graded algebra over an infinite domain $F$, torsion free as an $F$-module and $K$ an extension domain of $F$. Then $T_G(A)$ is multihomogeneous, furthermore if $A$ is free as an $F$-module then $T_G(A) = T_G(A \otimes K)$ as algebras over $F$. 

Lemma 1.7. Let $A$ be a $G$-graded algebra over an infinite domain $F$ and torsion free as an $F$-module. Suppose that we have $\mu : G^2 \to F$ and $\nu : G^3 \to F$ such that, $xy - \mu(g(x), g(y))yx = 0$ and $(xy)z - \nu(g(x), g(y), g(z))xz = 0$ are $G$-graded identities of $A$. Then $T_G(A)$ is generated by the two above “scheme” identities and possibly some nilpotent identities.

Proof. Let $u$ by a monomial, $J$ the $T$-ideal generated by $xy - \mu(g(x), g(y))yx$ and $(xy)z - \nu(g(x), g(y), g(z))xz$. We are now going to show that $u \equiv \lambda w \pmod{J}$, where $w$ is a regular $r_1$-word and $\lambda \in F$, for any order we put on the variables. Which proves the lemma, by proposition 1.6.

We shall do so by induction on the degree of $u$. Every monomial of degree one is a regular $r_1$-word and the identity $xy - \mu(g(x), g(y))yx = 0$ takes care of the degree two, this proves the initial case. Suppose that we have already proved the assertion for all words of degree less than $n$ ($n > 2$), then it is true for all words of degree up to $n$, with effect:

Let $u$, be a monomial of degree $n$ then $u = v_1s_1$ for some $v_1$, $s_1$ monomials of lesser degree. By the induction hypothesis we have that $v_1 \equiv \lambda_1 v \pmod{J}$ and $s_1 \equiv \lambda_2 s \pmod{J}$, $v$, $s$ regular $r_1$-words. $v = v'x$ and $s = s'y$, where $x$ (resp. $y$) is the greatest element of $v$ (resp. $s$), by definition. If $x > y$ we have that $\lambda_1v'x \cdot \lambda_2s'y \equiv \lambda_1\lambda_2\nu(g(v'x), g(s'y))\mu(g(x), g(s'y))(v' \cdot s'y)x \pmod{J}$, if $x \leq y$ we have that $\lambda_1v'x \cdot \lambda_2s'y \equiv \lambda_1\lambda_2\mu(g(v's), g(s'y))\nu(g(s'), g(y), g(v'x))\mu(g(y), g(v'x))(s' \cdot v'x)y \pmod{J}$.

In any case we have that $u \equiv \gamma lz \pmod{J}$, where $\gamma \in F$, $l$ is a monomial of degree $n - 1$ and $z$ is the greatest element of $u$. Finally, by the induction hypothesis, $l \equiv \sigma w \pmod{J}$, where $\sigma \in F$ and $w$ is a regular $r_1$-word, Which proves the assertion and therefore the lemma.

Definition 1.8 (Composition Algebra). A function $n$, from the $F$-vector space $A$ to the field $F$, is called a quadratic form if $n(\lambda x) = \lambda^2 n(x)$ and $f(x, y) := n(x + y) - n(x) - n(y)$ is a bilinear form, $\lambda \in F$, $x, y \in A$. Furthermore if $A$ is an algebra then $A$ is said a composition algebra if:

- $n(xy) = n(x)n(y) \forall x, y \in A$;
- the form $n$ is strictly non degenerate, i.e., $f$ is non degenerated;
- $A$ is unitary.

Hurwitz was the first to obtain a classification of finite dimension composition algebras for the case of the field of complex numbers in [Hur89], later Dickson gave another proof that carried over to any algebraically closed field of characteristic not two in [Dic19], finally in [Alb12] Albert obtained a proof for any field. Further Albert weakened the non degeneracy of $f$ and obtained a new class of solutions when the field has characteristic two.

The first to study infinite dimensional composition algebras was Kaplasky in [Kap53], and proved that it has to be finite dimensional, if the non degeneracy of $f$ is weakened then the composition algebra can also be a purely inseparable quadratic extension of the field, being of characteristic two and the form $f(x) = x^2$. Finally Jacobson in [Jac58], study the automorphisms of composition algebras and, beside other things, narrowed down the isomorphisms classes of composition algebras.

For the reminder of this section we’ll recall some results of those articles. The treatment we use is the same one found in [ZSSS82].
Notation 1.9. Denote $\bar{a} := f(1, a) - a$, $t(a) := a + \bar{a}$ and $n(a) := a\bar{a}$.

- Every composition algebra is alternative, that is, they satisfy the identity $(x, x, y) = (x, y, y) = 0$ where $(x, y, z) := (xy)z - x(yz)$ is the associator;
- The map $a \rightarrow \bar{a}$ is an involution which leaves the elements of $F$ fixed;
- The elements $t(a) = a + \bar{a}$ and $n(a) = a\bar{a}$ lie in $F$;
- Every composition algebra satisfy the equality $a^2 - t(a)a + n(a) = 0$.

Definition 1.10 (Cayley-Dickson process). Let $A$ be an unitary $F$-algebra with an involution $a \rightarrow \bar{a}$, where $a + \bar{a}$, $a\bar{a} \in F \forall a \in A$ and $\alpha \in F \setminus \{0\}$. We shall now construct a new algebra $(A, \alpha)$ which involution satisfying the same conditions of $A$, therefore we can apply the Cayley-Dickson process on $(A, \alpha)$. Moreover it contains an isomorphic copy of $A$.

$(A, \alpha) := A \oplus A$ as vector spaces, $(a_1, a_2)(a_3, a_4) := (a_1a_3 + \alpha a_4, a_1a_4 + a_3a_2)$ as the multiplication and $(a_1, a_2) := (\bar{a}_1, -a_2)$ as the involution, clearly $(1, 0)$ is the identity element of $(A, \alpha)$. We also denote $(1, 0)$ and $(0, 1)$ simply by 1 and $v$ respectively, so $(a_1, a_2)$ is also denoted by $a_1 + va_2$.

If the quadratic form $n(a) = a\bar{a}$ is strictly non degenerate on $A$ then $n(x) = x\bar{x}$ in strictly non degenerate on $(A, \alpha)$. Moreover if $A$ is a composition algebra, then $(A, \alpha)$ is a composition algebra if and only if $A$ is associative. Finally if $A$ is $G$-graded (every algebra is graded by the trivial group), then $(A, \alpha)$ is $(G \times \mathbb{Z}_2)$-graded, as follow: $(A, \alpha)(_{h, 0}) := A_h$ and $(A, \alpha)(_{h, 1}) := vA_h$.

We now give four examples of composition algebras:

1. The field $F$ with $n(x) = x^2$ if char$F \neq 2$, otherwise $f(x, y) = 0$.
2. $K(\mu) := F \oplus Fv_1$ as vector spaces, $(a + bv_1)(c + dv_1) := ac + \mu bd + (ad + bc + bd)v_1$ as multiplication and $a + bv_1 = (a + b) - bv_1$, where $4\mu + 1 \neq 0$. If char$F \neq 2$ then $K(\mu) = F \oplus vF = (A, \alpha)$ where $v = v_1 - 2^{-1}$ and $\alpha = \mu - 4^{-1} \neq 0$, Conversely, if char$F \neq 2$ then $(A, \alpha) = F \oplus vF = K(\mu)$, where $v_1 = v + 2^{-1}$ and $\mu = 4^{-1}$ also $4\mu + 1 \neq 0$.
3. $\mathbb{Q}(\mu, \beta) := (K(\mu), \beta)$ with $\beta \neq 0$, this is the algebra of generalized quaternions. It’s easy to see that $\mathbb{Q}(\mu, \beta)$ is associative but not commutative.
4. $C(\mu, \beta, \gamma) := (\mathbb{Q}(\mu, \beta), \gamma)$ with $\gamma \neq 0$ is the Cayley-Dickson algebra or simply the Octonions and it is also denoted by $\mathbb{O}$. It’s easy to see that the octonions are not associative, therefore we cannot continue the Cayley-Dickson process to produce other composition algebras.

Lemma 1.11. Let $B$ be a subalgebra with 1 of the composition algebra $A$ and $a, b \in B$, $v \in B^\perp$. Then we have the following relations:

(A) $\bar{v} = -v$, $av = v\bar{a}$;
(B) $a \cdot vb = v \cdot \bar{a}b$, $vb \cdot a = v \cdot ab$;
(C) $va \cdot vb = v^2 \cdot b\bar{a}$.

Theorem 1.12 (Generalized Hurwitz). Let $A$ be a composition algebra. Then $A$ is isomorphic to one of the four mentioned composition algebras above.
Lemma 1.13. For a composition algebra $A$ the following conditions are equivalent:

- $n(x) = 0$ for some $0 \neq x \in A$;
- there are zero divisors in $A$;
- $A$ contains an idempotent $e \neq 0, 1$.

Such a composition algebra is said split.

Theorem 1.14. Any two split composition algebra of the same dimension over a field $F$ are isomorphic. Furthermore every composition algebra over an algebraically closed field is split.

2. Some Identities

Our goal here is to encounter all the $\mathbb{Z}_2^2$-graded identities, here the grading is given by the Cayley-Dickson process. For that we first look at the $\mathbb{Z}_2^2$-graded identities (obviously for that the field cannot have characteristic two). There are two great things about the $\mathbb{Z}_3^2$ grading, first all the non zero $\mathbb{Z}_3^2$ homogeneous elements are invertible, second soon we’ll know all it’s $\mathbb{Z}_2^2$-graded identities.

We can digest a good part of lemma 1.11 relations into graded identities. We first note that $g(v) \notin H := \langle g(a), g(b) \rangle$ imply that $1 \in B = \bigoplus_{h \in H} \mathbb{O}_h$ and $v \in B^1$, $B$ is clearly a subalgebra. With that we’ll slash 1.11 hypotheses. There is still the involution, but it can be overcame in virtue of (A), as follow:

Proposition 2.1. Let $F$ be an infinite field whose characteristic is not two. Then $T_{\mathbb{Z}_3^2} \mathbb{O}$ is generated by:

1. $[x_1, x_2] = 0$, $\langle g(x_1), g(x_2) \rangle \leq 2$;
2. $x_1 \circ x_2 = 0$, $\langle g(x_1), g(x_2) \rangle \geq 4$;
3. $(x_1, x_2, x_3) = 0$, $\langle g(x_1), g(x_2), g(x_3) \rangle \leq 4$;
4. $(x_1 x_2) x_3 + x_1 (x_2 x_3) = 0$, $\langle g(x_1), g(x_2), g(x_3) \rangle = \mathbb{Z}_3^2$.

Proof. By lemma 1.11 we have that $\mathbb{O}$ satisfies the above identities therefore it’s under the conditions of the proposition 1.7 for the $\mathbb{Z}_2^2$ grading. Furthermore it cannot have any nilpotent identity, since every homogeneous element is invertible, which proves the proposition.

Corollary 2.2. Let $D$ be an infinite domain whose characteristic is not two and form the “Cayley-Dickson” algebra over $D$, $\mathbb{O}$. Then $T_{\mathbb{Z}_3^2} \mathbb{O}$ is generated by identities (1)-(4).

Proof. A direct application of 1.6.

Now we will enter the $\mathbb{Z}_2^2$ realm. The identities bellow are obtained in the same way that we used to obtain the $\mathbb{Z}_2^2$ identities.
Proof. We have that
\[ ab \cdot v = v \cdot ba, \quad g(v) \neq 0 \neq g(a) = g(b); \]
\[ (ax \cdot b)v = v(ba \cdot x), \quad g(v) \neq 0 = g(x) \neq g(a) = g(b); \]
\[ v(ax \cdot b) = (ba \cdot x)v, \quad g(v) \neq 0 = g(x) \neq g(a) = g(b); \]
\[ x \circ y = 0 \quad \langle g(x), g(y) \rangle = \mathbb{Z}_2^2; \]
\[ v(b \cdot a) = v \cdot ab, \quad g(v) \notin \langle g(a), g(b) \rangle; \]
\[ va \cdot w + wa \cdot v = -(v \circ w)a, \quad g(v), g(w) \notin \langle g(a) \neq 0, g(b) \rangle. \]

Beside these we also have:
\[ (x, y, z) = 0, \quad |\langle g(x), g(y), g(z) \rangle| \leq 2; \]
\[ [x, y] = 0, \quad g(x) = g(y) = 0; \]
\[ (x, x, y) = (x, y, y) = 0; \]
\[ v \cdot wb + w \cdot vb = (v \circ w)b, \quad g(v), g(w) \notin \langle g(b) \rangle. \]

Now let \( I \) be the \( T_{\mathbb{Z}_2^2} \)-ideal generated by (5)-(16), it’s easy to see that (16) and (13) are consequences of (5)-(14). Our goal is to prove that \( I = T_{\mathbb{Z}_2^2} \mathbb{O} \). Here on forward we will simply say that \( a \) is equivalent to \( b \) or \( a \equiv b \) instead of \( a \) is equivalent to \( b \) modulo \( I \) or \( a \equiv b \) (mod \( I \)).

The basic idea of the proof is to assume, by contradiction, that \( I \neq T_{\mathbb{Z}_2^2} \mathbb{O} \), then there is a \( f \in T_{\mathbb{Z}_2^2} \mathbb{O} \setminus I \) of minimal degree. Following with an appropriate substitutions slice up \( f \) in several identities \( f_i \), each being a consequence of some identity \( g_i \) of lesser degree, therefore in \( I \) contradicting that \( f \notin I \).

The first step is to reduce every monomial to a normal form, in virtue of Shirshov’s Theorem, it’s enough to consider \( u \) a regular \( r_2 \)-word (\( x_\nu^\mu < x_\nu^\nu \) if \( \nu < \mu \) or \( \nu = \mu, n < m \), for now we will only say that 0 is the greatest element of \( \mathbb{Z}_2^2 \)). Strictly speaking we don’t need to use Shirshov’s Theorem, however it will save us half the work, so we’ll gladly use it.

It is worth noting that all the identities that generate \( I \) are multilinear therefore \( I \) is multihomogeneous and the equivalence preserves multidegree.

3. The Zero Component Variables

Lemma 3.1. Let \( u \) be a regular \( r_2 \)-word and \( x \) the greatest element that \( u \) depends on, suppose that \( g(x) = 0 \). Then we have the following possibilities:
- \( u \equiv \pm xy; \)
- \( u \equiv \pm xy, g(u) \neq 0; \)
- \( u \equiv \pm yx \cdot z, g(y) = g(z) \neq 0; \)
where \( y, z \) are monomials.

Proof. We have that \( u = (\ldots((u_1 u_2) u_3) \ldots u_{n-1}) u_n \) where each \( u_i \) is a regular \( r_1 \)-word. We’ll prove the lemma by induction on \( n \). The initial case is exactly \( yx \). If \( x \) appears on \( u_n \), \( n \neq 1 \) then we need only to agglutinate what’s to the left of \( x \), that is, we have \( z \cdot yx \) and want to obtain \( wx \) or
$xw$ (just to make things crystal clear, $z = ((u_1u_2)u_3) \ldots u_{n-2}u_{n-1}$ and $u_n = yx$). This part of the proof (as many more to come) is divided into cases. Each case is one, or more, possibilities of the homogeneous component of each variable.

- $\langle g(z), g(y) \rangle \neq \mathbb{Z}_2^2$
  - $z \cdot yx \equiv 13 zy \cdot x$,
  - $\langle g(z), g(y) \rangle = \mathbb{Z}_2^2$
  - $z \cdot yx \equiv 10 x \cdot zy$.

Now if $x$ doesn’t appear in $u_n$ we have, by the induction hypothesis, that $u \equiv \pm xy \cdot z$, $u \equiv \pm xy \cdot z$ or $(tx \cdot w)z$, where $g(t) = g(w) \neq 0$. We are going to divide those three cases into sub cases. Each sub case is one, or more, possibilities of the homogeneous component of each variable. We begin which the first two cases:

- $\langle g(z), g(y) \rangle = \mathbb{Z}_2^2$
  1. $xy \cdot z \equiv 13 -z \cdot xy \equiv 10 -zy \cdot x$, 2. $yx \cdot z \equiv 13 -z \cdot yx \equiv 10 -x \cdot zy$;
- $g(y) = 0$
  1. $xy \cdot z \equiv 13 x \cdot yz$, 2. $yx \cdot z \equiv 13 x \cdot yz$;
- $g(z) = 0$
  1. $xy \cdot z \equiv 13 x \cdot yz$, 2. $yx \cdot z \equiv 13 y \cdot xz \equiv 13 yz \cdot x$;
- $g(y) = g(z) \neq 0$
  1. $xy \cdot z \equiv 13 x \cdot yz$, 2. $yx \cdot z$.

We now proceed to the last case:

- $g(z) = 0$
  - $(tx \cdot w)z \equiv 13 tx \cdot wz$;
- $g(z) = g(t)$
  - $(tx \cdot w)z \equiv 13 t(x \cdot wz) \equiv 13 t(wz) \equiv 13 (t \cdot wz)x$;
- $\langle g(t), g(z) \rangle = \mathbb{Z}_2^2$
  - $(tx \cdot w)z \equiv 13 z(wt) \cdot x \equiv 13 (z \cdot wt)x$.

\[ \square \]

**Corollary 3.2.** Let $f$ be a multihomogeneous polynomial, $x$ the greatest element that $f$ depends on and $n = \deg_x f$, where $g(x) = 0$. Then we have one of the following:

- $f = \sum_{i=0}^{n} x^i y_i$, if $g(f) \neq 0$ therefore $g(y_i) = g(f) \neq 0$;
- $f = \sum_{i=0}^{n} \Sigma_j y_{ij} x^i \cdot z_{ij} x^{n-i}$ if $g(f) = 0$, where $g(y_{ij}) = g(z_{ij}) \neq 0$.

**Proof.** We shall prove the corollary by induction on $n$, by Shirshov’s theorem we may assume that all of $f$ monomials are regular $r_2$-words, the initial case is just the lemma 3.1 which is already proved. Suppose that the assertion is valid for polynomials of degree $n$ then it is valid for polynomials of degree $n + 1$, with effect, let $\deg_x f = n + 1$. Suppose that $g(f) \neq 0$ then $f \equiv px + xh$ by lemma 3.1 and by the induction hypothesis we have that $px \equiv (\sum_{i=0}^{n} x^i p_i x^{n-i})x$ and $xh \equiv x(\sum_{i=0}^{n} x^i h_i x^{n-i})$, which proves this case.

Suppose that $g(f) = 0$ then $f \equiv px + \sum_j y_j x \cdot z_j$ by lemma 3.1 and by the induction hypothesis we have that $p \equiv \sum_{i=0}^{n} \Sigma_j u_{ij} x^i \cdot w_{ij} x^{n-i}$, $y_j \equiv \sum_{i=0}^{n} x^i p_{ij} x^{n_j-i}$ and $z_j \equiv \sum_{i=0}^{n} x^i h_{ij} x^{n_j-i}$, s.t., $m_j + n_j = n$. Which proves this case and with that the corollary. \[ \square \]
Proposition 3.3. Let $f$, $x$ and $n$ be as in 3.3, suppose that $f \in T_{p}^{\mathbb{Z}}(\mathbb{Z})$ and $F$ is an infinite field. Then $y_i \in T_{p}^{\mathbb{Z}}(\mathbb{Z})$ for $i = 1, \ldots, n$ in the first case of 3.2 or $\sum_j y_i, j \cdot z_{i,j} \in T_{p}^{\mathbb{Z}}(\mathbb{Z})$ for $i = 1, \ldots, n$ in the second case of 3.2.

Proof. If $g(f) \neq 0$ (resp. $g(f) = 0$) we have, by 3.2, that $f \equiv \sum_{i=0}^{n} x^i y_i x^{n-i}$ where $g(y_i) = g(f) \neq 0$ (resp. $f \equiv \sum_{i=0}^{n} \sum_j y_{i,j} x^i \cdot z_{i,j} x^{n-i}$, where $g(y_{i,j}) = g(z_{i,j}) \neq 0$). By 1.6 we can assume that $F$ is algebraically closed, therefore $\mathbb{Z}_0 \cong F \oplus F$ where $(a, b) = (b, a)$.

Under any evaluation of $f$ we have that $f = y_i \bar{x}^i x^{n-i}$ (resp. $f = p_i \bar{x}^i x^{n-i}$) where $p_i = \sum_j y_i, j z_{i,j}$). Let $x = (x_1, x_2)$ and $y_i = v(y_i, y_i')$ where $v$ is given by the Cayley-Dickson process (resp. $p_i = (p_i', p_i'')$). Then $f = v(\sum_i y_i x^{n-i} x^i, \sum_i y_i x^{n-i} x^i)$ (resp. $f = (\sum_i p_i x^{n-i} x^i, \sum_i p_i x^{n-i} x^i)$). Let $x_1, x_2$ be algebraically independent variables over $F$ then $y_i = y_i'' = 0$ (resp. $p_i = p_i'' = 0$, $0 = p_i = x p_i = x \sum_j y_{i,j} z_{i,j} \equiv \sum_j y_{i,j} x \cdot z_{i,j}$) under any substitution in $F$. □

Proposition 3.4. Let $f \in T_{p}^{\mathbb{Z}}(\mathbb{Z}) \setminus I$ multihomogeneous of minimal degree. Then $\deg_{0} f \leq 1$.

Proof. It’s enough to consider the case $f = \sum_i y_i (x_1 \cdots x_n) \cdot z_i$ where $g(x_j) = 0 \neq g(y_i) = g(z_i)$, $j = 1, \ldots, n$ and $\deg_{0} f = n$, by induction and 3.3. Substituting $x_2, \ldots, x_n$ for 1 we see that $\sum_i y_i x \cdot z_i$ is an identity.

Suppose, by contradiction, that $n > 1$ therefore $\sum_i y_i x_1 \cdot z_i \in I$, by the minimality of $f$’s degree. If we let $x_1$ go to $x_1 \cdots x_n$ we see that $f$ is a consequence of $\sum_i y_i x_1 \cdot z_i$, which is a contradiction. □

4. THE STRICTLY NON ZERO COMPONENT VARIABLES

Definition 4.1. Let $U$ be the polynomial sub-algebra generated by all variables that aren’t from the zero component and $*: U \to U$ linear defined on monomials by induction on the degree as follow: $u^* := -u$ if deg $u = 0$ and $(uv)^* := w^* v^*$.

Lemma 4.2. Let $f \in U$ and $x$ a non zero component variable. Then we have the following:

1. * is an involution of $U$;
2. if $f_0 = 0$ then $f^* = -f$;
3. if $f = v \cdot w$, $g(w) = g(v) \neq 0$ then $f^* \equiv w \cdot v$;
4. if $f_0(x) = 0$ then $f x \equiv f x^*$;
5. $f^*$ goes to $\tilde{f}$ under any evaluation;

Proof. By linearity it’s enough to prove the lemma only for the case where $f$ is a monomial.

1. We’ll start proving by induction that * has order 2. If $\deg f = 1$ then $f^{**} = (-f)^* = -(-f) = u$ if deg $f = 1$ then $f = v \cdot w$ and $f^{**} = (v \cdot w)^* = v^* \cdot w^* = v \cdot w = f$. Clearly * is a anti-homomorphism.

Both (2) and (4) are valid when deg $f = 1$. Suppose they are valid for all monomials of degree less than $n$, then they are valid for monomials of degree $n$, with effect:

2. $f = v \cdot w$, we have the following cases:
   a. $\langle g(w), g(v) \rangle = \mathbb{Z}_2^{2}$
   b. $f^* = w^* \cdot v^* = w^* \cdot v^* \equiv (-w) \cdot (-v)$ by the induction hypothesis for...
We have that $y, z$ agglutinate what's to the left of $g$

Let Lemma 4.3.

(4) $f = v \cdot w$, we have the following cases:

- $g(v) = 0 \neq g(w)$

$g(v) = 0 \neq g(w)$

- $g(w) = 0 \neq g(v)$

$f^* = w^* \cdot v^* \equiv w^* \cdot (v)$ by the induction hypothesis for (2), $w^* \cdot (v) \equiv -v \cdot w = -f$ by the induction hypothesis for (4);

Lemma 4.3. Let $u$ be a regular $r_2$-word and $x$ the greatest element that $u$ depends on, suppose that $g(x) \neq 0$. Then we have one of the following:

- $u \equiv \pm xy$,
- $u \equiv \pm z \cdot xy$ if $g(x) = g(y)$,
- $u \equiv \pm z \cdot x y$ if $g(x) = g(y)$ and $\langle g(x), g(z) \rangle = Z^2_2$ or
- $u \equiv \pm z \cdot x y$ if $g(x) = g(y) = g(z)$;

where $y, z$ are monomials.

Proof. We have that $u = (\ldots((u_1 u_2)\ldots u_{n-1})u_n$ where each $u_i$ is a regular $r_1$-word. We’ll prove the lemma by induction on $n$. The initial case is exactly $yx$. If $x$ appears on $u_n$ and $n \neq 1$ then we need only to agglutinate what’s to the left of $x$, that is, we have $z \cdot yx$, where $z = \ldots((u_1 u_2)\ldots u_{n-1}, yx = u_n$ and want to obtain $w x$ or $t \cdot x y$ with $g(s) = g(x)$ and $\langle g(x), g(t) \rangle = Z^2_2$:

- $\langle g(x), g(y), (g(z) \rangle \neq Z^2_2$

$z \cdot y x \equiv (4) z y \cdot x;$

- $\langle g(x), (g(y), g(z) \rangle = Z^2_2$

$z \cdot y x \equiv (5) z y \cdot x = 0$

- $\langle g(x), g(y) \rangle = Z^2_2$, $g(z) = 0$

$z \cdot y x \equiv (6) z y \cdot x = 0$

- $\langle g(x), g(y) \rangle = Z^2_2$, $g(y) = g(z)$

$z \cdot y x \equiv (7) z y \cdot x = 0$

- $\langle g(x), g(y) \rangle = Z^2_2$, $g(y) = g(z)$

$z \cdot y x \equiv (8) z y \cdot x = 0$

- $\langle g(x), g(z) \rangle = Z^2_2$, $g(y) = g(y)$

$z \cdot y x$, nothing to see here, move along;

- $\langle g(x), g(y), g(z) \rangle = Z^2_2$, $g(x) + g(y) + g(z) = 0$

$z \cdot y x \equiv (9) z y \cdot x = 0$

If $x$ doesn’t appear on $u_n$, then we have that $u \equiv \pm y x \cdot z$, $u \equiv \pm z \cdot x y \cdot z$, $u \equiv \pm (t \cdot x y) w$ or $(z \cdot x y) w$ with $g(s) = g(x)$ and $\langle g(x), g(t) \rangle = Z^2_2$ in the third case or $g(x) = g(y) = g(z)$ in the last case, by the induction hypothesis, and
want to obtain \( \pm wx, \pm xw \) or \( \pm t \cdot sx \), with the same component restriction.

Let’s begin with the case \( \pm (t \cdot sx)w \).

- \( g(w) = 0 \)
  \[
  (t \cdot sx)w \equiv 13 \quad t(s \cdot xw) \equiv 12 \quad t(s \cdot w^*x) \equiv 13 \quad t(sw^* \cdot x);
  \]
- \( g(t) = g(w) \)
  \[
  (t \cdot sx)w \equiv 12 \quad (sx \cdot t)w \equiv 13 \quad x(s \cdot tw);
  \]
- \( g(x) = g(w) \)
  \[
  (t \cdot sx)w \equiv 13 \quad t(ws \cdot x) \equiv 12 \quad t(x \cdot sw) \equiv 13 \quad (t \cdot sw)x;
  \]
- \( g(w) = g(x) + g(t) \)
  \[
  (t \cdot sx)w \equiv 13 \quad (tx \cdot s)w \equiv 13 \quad (s \cdot xt)w \equiv 13 \quad s(w \cdot xt) \equiv 10 \quad -s(x \cdot wt) \equiv 12 \quad -sx \cdot wt \equiv 12 \quad -wt \cdot xs.
  \]

Moving to \( \pm xy \cdot z \) and \( \pm xy \cdot z \):

- \( g(y) = 0, \quad \langle g(x), g(z) \rangle \neq \mathbb{Z}_2^2 \)
  \[
  yx \cdot z \equiv 4 \quad xy^* \cdot z \equiv 13 \quad x \cdot y^* \cdot z;
  \]
- \( g(y) = 0, \quad \langle g(x), g(z) \rangle = \mathbb{Z}_2^2 \)
  \[
  yx \cdot z \equiv 4 \quad xy^* \cdot z \equiv 13 \quad x \cdot zy \equiv 13 \quad -zy^* \cdot x;
  \]
- \( \langle g(x), g(y) \rangle = \mathbb{Z}_2^2, \quad g(z) = 0 \)
  \[
  yx \cdot z \equiv 13 \quad y \cdot zx;
  \]
- \( \langle g(x), g(y) \rangle = \mathbb{Z}_2^2, \quad g(z) = g(y) \)
  \[
  yx \cdot z \equiv 13 \quad -xy \cdot z \equiv 13 \quad -x \cdot zy \equiv 13 \quad yz \cdot x;
  \]
- \( \langle g(x), g(y) \rangle = \mathbb{Z}_2^2, \quad g(z) = g(x) + g(y) \)
  \[
  yx \cdot z \equiv 13 \quad -xy \cdot z \equiv 13 \quad zy \cdot x;
  \]
- \( g(x) = g(y), \quad g(z) = 0 \)
  \[
  1. \quad yx \cdot z \equiv 13 \quad z \cdot yx \equiv 13 \quad zy \cdot x, \quad 2. \quad xy \cdot z \equiv 13 \quad x \cdot yz;
  \]
- \( g(x) = g(y), \quad \langle g(x), g(z) \rangle = \mathbb{Z}_2^2 \)
  \[
  1. \quad yx \cdot z \equiv 13 \quad z \cdot xy \equiv 13 \quad zy \cdot x, \quad 2. \quad xy \cdot z \equiv 13 \quad z \cdot yx;
  \]
- \( g(x) = g(y), \quad g(z) = g(x) \)
  \[
  1. \quad yx \cdot z \equiv 13 \quad y \cdot xz, \quad 2. \quad xy \cdot z \equiv 13 \quad x \cdot yz \equiv 13 \quad zy \cdot x.
  \]

We now proceed to the last case:

- \( g(w) = 0 \)
  \[
  (z \cdot xy)w \equiv 13 \quad z(x \cdot yw);
  \]
- \( g(x) = g(w) \)
  \[
  (z \cdot xy)w \equiv 13 \quad z(x \cdot yw) \equiv 13 \quad z(zy \cdot x) \equiv 13 \quad (z \cdot wy)x;
  \]
- \( \langle g(x), g(y) \rangle = \mathbb{Z}_2^2 \)
  \[
  (z \cdot xy)w \equiv 13 \quad -w(z \cdot xy) \equiv 13 \quad -xy \cdot wz \equiv 12 \quad -wz \cdot yx.
  \]

\[ \square \]

**Corollary 4.4.** Let \( f \) be a multihomogeneous polynomial, \( x \) the greatest element that \( f \) depends on and \( n = \deg_x f \), where \( g(x) \neq 0 \). Then we have one of the following:

- \( f \equiv px^n + \sum_{i=1}^n \sum_{j=1}^i p_{i,j}^j x^{n-i} + \sum_{i=1}^n \sum_{j=1}^i h_{i,j}^j x_i \cdot x^j \cdot x^{n-i} \)
- if \( g(f) \in \langle g(x) \rangle \), where \( g(p_{i,j}^j) = g(h_{i,j}^j) = g(x) \), \( \forall i, i', j, l, \quad (n + i + 1)g(x) = g(f) \) and \( (n + i')g(x) = g(f) \);
• \( f = zn^m + \sum_{i=1}^n \sum_{j=1}^n x^{i,j}p_{i,j+1}^1 x \cdots p_{i,1}^1 x^{n-i} + \sum_{i=1}^n \sum_{j=1}^n x^{i,j}h_{i,j}^1 x \cdots h_{i,j}^1 x^{n-i}, \)

if \( (g(f), g(x)) = Z_2^d, \) where \( g(p_{i,j}^1) = g(h_{i,j}^1) = g(x), \) \( (g(z_j), g(x)) = Z_2^d, \forall i, i', j, l, \) and \( n - i \equiv n - i' \equiv 1 \pmod{2} \)

\textbf{Proof.} We shall prove the corollary by induction on \( n, \) by Shirshov’s theorem we may assume that all of \( f \)'s monomials are regular \( r_2 \)-words, the initial case is just the lemma \( \text{[4.3]} \) which is already proved. Suppose that the assertion is valid for polynomials of degree \( n \) then it is valid for polynomials of degree \( n+1, \) with effect, let \( \deg f = n+1. \)

If \( g(f) \in (g(x)) \) then we have \( f = qz + xh + \sum_{j=1}^n x^{j}y_{j}^r, \) where \( g(p_j), g(h_l) \)

\( g(y_{j}^r) \in (g(x)), \) by \( \text{[4.3]} \) We have that \( q = p^r x^n + \sum_{i=1}^n \sum_{j=1}^n x^{i,j}p_{i,j+1}^1 x \cdots p_{i,1}^1 x^{n-i} + \sum_{i=1}^n \sum_{j=1}^n x^{i,j}h_{i,j}^1 x \cdots h_{i,j}^1 x^{n-i}, \) by the induction hypothesis, so \( xh = p^r x^{n+1} + \sum_{i=1}^n \sum_{j=1}^n x^{i,j}h_{i,j}^1 x \cdots h_{i,j}^1 x^{n+1-i} + \sum_{i=1}^n \sum_{j=1}^n xh_{i,j}^1 x \cdots h_{i,j}^1 x^{n+1-i}. \) Analogously, \( xq = px^{n+1} + \sum_{i=1}^n \sum_{j=1}^n x^{i,j}p_{i,j+1}^1 x \cdots p_{i,1}^1 x^{n-i} + \sum_{i=1}^n \sum_{j=1}^n x^{i,j}h_{i,j}^1 x \cdots h_{i,j}^1 x^{n-i+2}. \) The last summand is analogous.

Clearly \( f = p^r x^n + \sum_{i=1}^n \sum_{j=1}^n x^{i,j}h_{i,j}^1 x \cdots h_{i,j}^1 x^{n-i} + \sum_{i=1}^n \sum_{j=1}^n x^{i,j}p_{i,j+1}^1 x \cdots p_{i,1}^1 x^{n-i}, \)

\( g(p_{i,j}^1), g(h_{i,j}^1) \in (g(x)). \) If \( g(p_{i,j}^1) = 0 \) then \( p_{i,j}^1 x \cdots p_{i,1}^1 x^{n-i} = p_{i,j}^1 x \cdots x \)

\( p_{i,j+1}^1 x \cdots p_{i,1}^1 x^{n-i+2} \) if \( l \neq i + 1 \) and \( p_{i,j+1}^1 x \cdots p_{i,1}^1 x^{n-i} = xh_{i,j}^1 x \cdots h_{i,j}^1 x \cdots h_{i,j}^1 x \cdots h_{i,j}^1 x + xh_{i,j}^1 x \cdots h_{i,j}^1 x. \) Therefore \( g(p_{i,j}^1) = g(h_{i,j}^1) = g(x), \forall i, i', j, l. \) Calculating \( g \) on both sides of the equivalence we obtain that \( (n+i+1)g(x) = g(f) \) and \( (n+i')g(x) = g(f). \)

The case \( (g(f), g(x)) = Z_2^d, \) is analogous.

\textbf{Definition 4.5.} We shall define \( v_{1,0} = (0,1) \in \mathbb{Q}(\mu, \beta), \) \( v_{0,1} = (0,1) \in \mathbb{C}(\mu, \beta, \gamma) \) given by the Cayley-Dickson process, \( v_{1,1} = v_{0,1}v_{0,0} \) and \( v_{0,0} = 1. \) \( a, v_{(i)} = v_{h}a, \) \( a, v_{(0)} = a \) if \( n \equiv 0 (\pmod{2}) \) or \( a^a = a \pmod{2} \).

\textbf{Proposition 4.6.} Let \( f, x \) and \( n \) be as in \( \text{[4.4]}, \) suppose that \( f \in T_{Z_2^d}(A) \) and \( F \) is an infinite field. Then \( p, \sum_{j=1}^n x^{j}p_{j+1}^1 x \cdots p_{1}^1 x^{n-j}, \sum_{j=1}^n x^{j}h_{j}^1 x \cdots h_{j}^1 x^{n-j}, \)

\( \sum_{j=1}^n x^{j}h_{j}^1 x \cdots h_{j}^1 x \in T_{Z_2^d}(A) \) for \( i, i' = 1, \ldots, n \) in the first case of \( \text{[4.4]} \) or \( z, \sum_{j=1}^n x^{j}p_{j+1}^1 x \cdots p_{1}^1 x^{n-j}, \sum_{j=1}^n x^{j}h_{j}^1 x \cdots h_{j}^1 x \in T_{Z_2^d}(A) \) for \( i, i' = 1, \ldots, n \) in the second case of \( \text{[4.4]} \).

\textbf{Proof.} For the first case we have that under any evaluation on \( Z_2^d \) \( f = px^n + \sum_{j=1}^n x^{j}a_{j}x^{n-j} + \sum_{j=1}^n x^{j}b_{j}x^{n-j} \) where \( a_{j} = \sum_{j=1}^n P_{j+1}^1 x \cdots P_{1}^1 x, \sum_{j=1}^n x^{j}h_{j}^1 x \cdots h_{j}^1 x \in T_{Z_2^d}(A) \) for \( i, i' = 1, \ldots, n \) in the second case of \( \text{[4.4]} \).

\( P_{j+1}^1 = Ph_{j+1}^1 \) if \( l \) is odd or \( P_{j+1}^1 = Ph_{j+1}^1 \) if \( l \) is even and \( H_{j+1}^1 = h_{j+1}^1 \) if \( l \) is odd or \( H_{j+1}^1 = h_{j+1}^1 \) if \( l \) is even. Now we have four sub-cases:

• \( n \) is odd \( g(f) = g(x), \) therefore \( i \) is odd and \( i' \) is even, then \( f = \)

\( x^n p + \sum_{j=1}^n x^{j}a_{j}x^{n-j} + \sum_{j=1}^n x^{j}b_{j}x^{n-j} \)

• \( n \) is even \( g(f) = 0, \) therefore \( i \) is odd and \( i' \) is even, then \( f = \)

\( x^n p + \sum_{j=1}^n x^{j}a_{j}x^{n-j} + \sum_{j=1}^n x^{j}b_{j}x^{n-j} \)

• \( n \) is odd \( g(f) = 0, \) therefore \( i \) is even and \( i' \) is odd, then \( f = \)

\( \sum_{j=1}^n x^{j}a_{j}x^{n-j} + \sum_{j=1}^n x^{j}b_{j}x^{n-j} \)
Proof. We will prove the proposition by induction on $P = 1_n$, $n = p$. In any case, we can use generic elements as was done in 3.3 that results in following forms:

**Proposition 4.7.** Let $g$ be a multihomogeneous polynomial, $h \in \mathbb{Z}_2^3 \setminus \{0\}$ the greatest component that depends on $f$ with $x_i$, $i = 1, \ldots, n$ the variables from the $h$ component that $f$ depends on and $m_i = \deg_{x_i} f$, $m = \sum_{i=1}^n m_i = \deg_h f$. Suppose that $f \in T_{\mathbb{Z}_2^3} \setminus I$ of minimal degree. Then $f$ is of one of the following forms:

- $f \equiv \Sigma_j P^j_1 \cdots P^j_n p^j$,
- $f \equiv \Sigma_j z^j \cdot P^j_1 \cdots P^j_n$,

where $P^j_i = p^j_{i1}x_1 \cdots p^j_{im_i}x_i$, $g(p^j_i) = g(p^j) = g(x)$, $\langle g(z^j), g(x) \rangle = \mathbb{Z}_2^3$.

Proof. We will prove the proposition by induction on $n$, the initial case, $n = 1$, is just 4.3. Suppose that the proposition’s assertion is true up to $n$, then it is true for $n + 1$, with effect:

Ignoring $x_{n+1}$ we obtain that $f$ is either $f \equiv \Sigma_j P^j_1 \cdots P^j_n p^j$ or $f \equiv \Sigma_j z^j \cdot P^j_1 \cdots P^j_n$. If an $x_{n+1}$ appear in a $p$ then we have that $p \equiv p_1 x_{n+1}$ or $p \equiv p_1 x_{m+1} p_2$ where $g(p_1) = g(p_2) = g(x_{n+1})$ by 4.3. Using the same substitution arguments we may assume, without loss of generality, that $p \equiv p_1 x_{n+1} p_2$. If an $x_{n+1}$ appear in a $z$ then we have that $p \equiv z_1 x_{n+1}$ or $p \equiv z_1 \cdot p_1 x_{n+1}$ where $g(p_1) = g(x_{n+1})$ and $\langle g(z_1), g(x_{n+1}) \rangle = \mathbb{Z}_2^3$, by 4.3. Using the same substitution arguments we may assume, without loss of generality, that $p \equiv z_1 \cdot p_1 x_{n+1}$. The proposition now follows from induction in $m_{n+1}$, the degree of $f$ with respect to $x_{n+1}$. □

5. Coup de Grâce

**Remark 5.1.** Let $u$ be a monomial that depends only on two components, both of them non-zero. Then $u \equiv \pm w v$ where $w$ is a monomial that depends only on one component and $v$ is a monomial that depends only on the other component.

Proof. A simple proof by induction on the degree of the monomial. □

**Lemma 5.2.** Let $f$ be a multihomogeneous polynomial, suppose that $f \in T_{\mathbb{Z}_2^3} \setminus I$ of minimal degree and $F$ is an infinite field. Then $f$ depends on a zero component variable.

Proof. Assume, by contradiction, that $f$ does not depend on the zero component variable. Let $h \in G \setminus \{0\}$ s.t. $\forall y \in G \setminus \{0\}$, $\deg_h f \geq \deg_y f$. Then $f \equiv \Sigma_j P^j_1 \cdots P^j_n p^j$ or $f \equiv \Sigma_j z^j \cdot P^j_1 \cdots P^j_n$, where $P^j_i = p^j_{i1}x_1 \cdots p^j_{im_i}x_i$, $g(p^j_i) = g(p^j) = h$, $\langle g(z^j), g(x) \rangle = \mathbb{Z}_2^3$, $g(x_i) = h$, $i = 1, \ldots, m$, $\langle g(z^j), h \rangle = \mathbb{Z}_2^3$ and $m = \sum_{i=1}^n m_i = \deg_h f$, by 4.7.

Furthermore we have that $p^j_{i1} \equiv \alpha^j_{i1} z^j_{i1} w^j_{i1}$ and $z^j \equiv \alpha^j z^j w^j$ where $\alpha^j_{i1}$, $\alpha^j \in F$ and all the $z^j_{i1}$’s and $z^j$’s are product of variables from the same
non zero component that is not $h$ and the $u'_{i,l}$'s are products of variables from the third non zero component. Therefore $\deg_{g(z)} f > \deg_h f$ which is a contradiction.

Proposition 5.3. Let $f$ be a multihomogeneous polynomial, assume that $f \in T_{\mathbb{Z}}(\mathbb{O})$ and $F$ is an infinite field. Then $f \in I$.

Proof. Assume, by contradiction, that $f \notin I$ so we can assume without loss of generality that $f$ is of minimal degree.

We have that $\deg_0 f = 1$ by [3.3] and [5.2] so $f \equiv \Sigma_{i} y_{i} x z_{i}$ where $g(x) = 0 \neq g(y_{i}) = g(z_{i})$ and the $y$'s, $z$'s are free from the zero component by [3.3]. Let $h \in \mathbb{Z}_{2} \setminus \{0\}$ such that $\forall \alpha \in \mathbb{Z}_{2} \setminus \{0\}$, $\deg_h f > \deg_{\alpha} f$ and $\omega$ a variable s.t. $g(\omega) = 1$, $\deg_{\omega} f > 0$. Applying [1.3], [1.4], [1.5], [1.6], [1.7], [1.14] and [1.15] on the $\omega$'s and $\omega$'s we obtain that $f$ is of the form:

$$f \equiv wx^{\Sigma_{i=1}^{n-1} \Sigma_{j=1}^{i} w_{j} \cdot \cdot \cdot p_{i,j} w_{j} w_{i} + \Sigma_{j=1}^{n} wp_{j} w_{j} w_{j} + \Sigma_{j=1}^{n} wp_{j} w_{j} w_{j} + \Sigma_{j=1}^{n} wp_{j} w_{j} w_{j} + \Sigma_{j=1}^{n} wp_{j} w_{j} w_{j}$$

where the $p_j$'s are from the $h$ component and the $k$'s are either zero or from the $h$ component.

Applying the usual substitution argument and the counting argument from [1.14] we obtain that $\Sigma_{j} p_{j} w_{j} \cdot \cdot \cdot p_{j,j} w_{j} + \Sigma_{j} w_{j} + \Sigma_{j} wp_{j} w_{j} + \Sigma_{j} wp_{j} w_{j} + \Sigma_{j} wp_{j} w_{j} + \Sigma_{j} wp_{j} w_{j} + \Sigma_{j} wp_{j} w_{j} + \Sigma_{j} wp_{j} w_{j} + \Sigma_{j} wp_{j} w_{j} + \Sigma_{j} wp_{j} w_{j} + \Sigma_{j} wp_{j} w_{j} + \Sigma_{j} wp_{j} w_{j} + \Sigma_{j} wp_{j} w_{j}$.

Which is a contradiction.

Theorem 5.4. If $F$ is an infinite field, then $I = T_{\mathbb{Z}}(\mathbb{O})$.

Theorem 5.5. Let $D$ be an infinite domain, and form the “Cayley-Dickson” algebra over $D$, $\mathbb{O}$. Then $I = T_{\mathbb{Z}}(\mathbb{O})$.

Proof. A direct application of [1.6] □

Remembering that two split composition algebra of the same dimension are isomorphic we see that $M_{2}(F) \cong \mathbb{Q}(0, 1)$. If one pushes the $\mathbb{Z}_{2}$ grading over the isomorphism he gets that the zero component is formed by the diagonal matrices, and the unitary component by the anti-diagonal matrices. More generally, let $M_{n}(F)_{\alpha} := \text{lin.span}\{e_{i,j} | j - i = \alpha \text{ (mod n)}\}$. For that we have the following:

Theorem 5.6. Let $F$ be a field of characteristic zero. Then $T_{\mathbb{Z}}(M_{n}(F))$ is generated as a $T$-ideal by associativity and the following identities:

$$xy - yx = 0 \quad g(x) = g(y) = 0;$$

$$x_1 x_2 - x_2 x_1 = 0 \quad g(x_2) = g(x_1) = -g(x).$$

This theorem was first proved by Di Vincenzo in [DV92] for two by two matrices, former Vaslovsky extended the proof for matrices of any order in [Vas99]. Afterwards Koshlykov and Azevedo proved the theorem for two by two matrices over an infinite field of characteristic grater then two in [KdA02]. Finally in [BRK09] Brandao, Koshlykov and Krasilnikov remarked that the proof in [KdA02] is still valid for an infinite integral domain, that is:
Theorem 5.7. Let $D$ be an infinite domain. Then $T_{\mathbb{Z}_2}(M_2(D))$ is generated as a $T$-ideal by the identities $(17), (18)$ and associativity.

Which we now re-obtain:

Proof. Let’s write the identities that generate $I$, but restricted to $\mathbb{Z}_2$:

(5*) $ab \cdot v = v \cdot ba$, $g(v) = g(a) = g(b) = \bar{1}$;
(6*) $(ax \cdot b)v = v(ba \cdot x)$, $g(x) = 0$, $g(a) = g(b) = g(v) = \bar{1}$;
(7*) $v(ax \cdot b) = (ba \cdot x)v$, $g(x) = 0$, $g(a) = g(b) = g(v) = \bar{1}$;
(9*) $vb \cdot a = v \cdot ab$, $g(v) = \bar{1}$, $g(a) = g(b) = 0$;
(13*) $(x, y, z) = 0$, $g(x) = g(y) = g(z) = 0$;
(14*) $[x, y, z] = 0$.

Equations (8), (10), (11) and (12) do not intersect the $\mathbb{Z}_2$ realm. (13*) is associativity, (14*) and (5*) are respectively (17) and (18). Substituting $a$ for $ax$ in (5*) and using associativity we obtain (6*), simultaneously substituting $a$ for $v$, $b$ for $ax$ and $v$ for $b$ in (5*) and using associativity we obtain (7*), finally multiplying (14*) by $v$ and using associativity we obtain (9*). □

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