Interacting modified Chaplygin gas in loop quantum cosmology

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Abstract We investigate the background dynamics when dark energy is coupled to dark matter in the universe described by loop quantum cosmology. We consider dark energy of the form modified Chaplygin gas. The dynamical system of equations is solved numerically and a stable scaling solution is obtained. It henceforth resolves the famous cosmic coincidence problem in modern cosmology. The statefinder parameters are also calculated to classify this dark energy model.

1 Introduction

Recent observations of type Ia Supernovae indicate that Universe is expanding with acceleration\textsuperscript{1} (Perlmutter et al. 1999; Riess et al. 1998) and lead to the search for a new type of matter which violates the strong energy condition, i.e., $\rho + 3p < 0$. In Einstein’s general relativity, an energy component with large negative pressure has to be introduced in the total energy density of the Universe in order to explain this cosmic acceleration. This energy component is known as dark energy (Sahni & Starobinsky 2000; Padmanabhan 2003). There are many candidates supporting this behavior (Copeland et al. 2006), scalar field or quintessence (Peebles & Ratra 2006) being one of the most favored candidates as it has a decaying potential term which dominates over the kinetic term thus generating enough pressure to drive acceleration.

Presently we live in an epoch where the densities of the dark energy and the dark matter are comparable. It becomes difficult to solve this coincidence problem without a suitable interaction. Generally interacting dark energy models are studied to explain the cosmic coincidence problem (Jamil et al. 2010; Jamil & Saridakis 2010; Jamil & Farooq 2010; Jamil et al. 2010; Jamil & Rahman 2009). Also the transition from matter domination to dark energy domination can be explained through an appropriate energy exchange rate. Therefore, to obtain a suitable evolution of the Universe an interaction is assumed and the decay rate should be proportional to the present value of the Hubble parameter for good fit to the expansion history of the Universe as determined by the Supernovae and CMB data (Jamil et al. 2010; Jamil & Saridakis 2010). A variety of interacting dark energy models have been proposed and studied for this purpose (Setare 2007, 2006; Hu & Ling 2006; Wu & Yu 2007; Jamil 2010; Setare 2006).

In recent years, the model of interacting dark energy has been explored in the framework of loop quantum cosmology (LQC) as well: It is shown in (Wu & Yu 2008) that for the quintessence model, the cosmological evolution in LQC is the same as that in classical Einstein cosmology, whereas for the phantom dark energy the loop quantum effect significantly reduce the parameter spacetime required by stability. In (Chen et al. 2008), the authors used a more general interaction term to study the interacting dark energy. They showed that in LQC, the parameter space for the existence of the accelerated scaling attractor is found to be smaller than that in Einstein cosmology. In another study (Fu et al. 2008), the authors studied the model with an interacting phantom scalar field with an exponential potential and deduced that the future singularity appearing in the standard FRW cosmology can be avoided by loop quantum effects.

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In this paper, we extend the model of interacting modified Chaplygin gas (MCG) from the framework of Einstein gravity to LQC. We construct a dynamical system of equations and solve them numerically. We obtain a stable scaling solution (which is also an ‘attractor’) of modified FRW equations. We discuss our results in the final section.

2 The model

The modified Friedmann equation for LQC is given by

\[ H^2 = \frac{\rho}{3} \left(1 - \frac{\rho_m}{\rho_1}\right). \]  

(1)

Here \( \rho_1 \equiv \sqrt{3\pi^2} \gamma^2 G^2 h \) is the critical loop quantum density and \( \gamma \) is the dimensionless Barbero-Immirzi parameter. We assume the interaction between dark energy and pressureless dark matter. Hence the energy balance equations for the interacting dark energy and dark matter can be expressed as

\[ \dot{\rho}_{mcg} + 3H(1 + \omega_{mcg})\rho_{mcg} = -Q, \]  

\[ \dot{\rho}_m + 3H\rho_m = Q, \]  

(2) \hspace{1cm} (3)

where \( Q = 3bH\rho \) is the interaction term, \( b \) is the coupling parameter (or transfer strength) and \( \rho = \rho_{mcg} + \rho_m \) is the total cosmic energy density which satisfies \( \dot{\rho} + 3H(\rho + p) = 0 \) \( \text{(Guo & Zhang 2005, Campo et al 2008)} \). Note that addition of the above two equations leads to the energy conservation. Due to unknown nature of both dark energy and dark matter, the interaction term cannot be derived from the first principles. It is worth noting that if \( Q < 0 \) then it will yield the energy density of dark energy to be negative at sufficiently early times, consequently the second law of thermodynamics can be violated \( \text{(Alcaniz & Lima 2003)} \) hence \( Q \) must be positive and small. From the observational data of 182 Gold type Ia supernova samples, CMB data from the three year WMAP survey and the baryonic acoustic oscillations from the Sloan Digital Sky Survey, it is estimated that the coupling parameter between dark matter and dark energy must be a small positive value (of the order unity), which satisfies the requirement for solving the cosmic coincidence problem and the second law of thermodynamics \( \text{(Feng et al 2008)} \). Because of the underlying interaction, the beginning of the accelerated expansion is shifted to higher redshifts.

Consequently we obtain the modified Raychaudhuri equation

\[ \dot{H} = -\frac{1}{2}(\rho + p)\left(1 - \frac{2\rho_1}{\rho_1}\right), \]  

(4)

where \( p \) is the total pressure \( (p = p_{mcg}) \). We shall use modified Chaplygin gas as the dark energy. The MCG equation of state is given by

\[ p_{mcg} = A\rho_{mcg} - \frac{B}{\rho_{mcg}^{\alpha}}, \]  

(5)

where \( A, B \) and \( \alpha \) are constants. The MCG best fits with the 3-year WMAP and the SDSS data with the choice of parameters \( A = -0.085 \) and \( \alpha = 1.724 \) \( \text{(Lu et al 2008)} \) which are improved constraints than the previous ones \(-0.35 < A < 0.025 \) \( \text{(Jun & Zhou 2005)} \). Recently it is shown that the dynamical attractor for the MCG exists at \( \omega_{mcg} = -1 \), hence MCG crosses this value from either side \( \omega_{mcg} > -1 \) or \( \omega_{mcg} < -1 \), independent to the choice of model parameters \( \text{(Jing et al 2008)} \). Generalization of MCG is suggested in \( \text{(Dehmalt 2007)} \) by considering \( B \equiv B(a) = B_o a^n \), where \( n \) and \( B_o \) are constants. The MCG is the generalization of generalized Chaplygin gas \( \text{(Barreiro & Sei 2004, Carturan & Finelli 2003)} \) with the addition of a barotropic term. This special form also appears to be consistent with the WMAP 5-year data and henceforth the support the unified model with dark energy and matter based on generalized Chaplygin gas \( \text{(Barreiro et al 2008, Makler et al 2003, Setare 2009, 2007, a2007)} \). In the cosmological context, the Chaplygin gas was first suggested as an alternative to quintessence and demonstrated an increasing \( \Lambda \) behavior for the evolution of the universe \( \text{(Kamenshchik 2001)} \). Recent supernovae data also favors the two-fluid cosmological model with Chaplygin gas and matter \( \text{(Panotopoulos 2008)} \).

To analyze the dynamical system, we convert the physical parameters into dimensionless form as

\[ x = \ln a, \quad u = \frac{\rho_{mcg}}{3H^2}, \quad v = \frac{\rho_m}{3H^2}, \]  

(6)

where \( a_0 = 1 \) is assumed, where the subscript 0 refers to the present time. Making use of (1) to (6), we can write

\[ \frac{du}{dx} = -3b(u + v) - 3u(1 + \omega_{mcg}) \]  

\[ + 3u[1 + \omega_{mcg}] + v(1 + \frac{2}{u + v}), \]  

(7)

\[ \frac{dv}{dx} = 3b(u + v) - 3v + 3v[u(1 + \omega_{mcg}) + v] \]  

\[ \times \left(1 + \frac{2}{u + v}\right), \]  

(8)

where the state parameter of modified Chaplygin gas is

\[ \omega_{mcg} = \frac{p_{mcg}}{\rho_{mcg}} = A - \frac{B(u + v)^{2(\alpha+1)}}{\rho_{mcg}^{\alpha+1}u^{\alpha+1}(u + v - 1)^{\alpha+1}}. \]  

(9)
For the mathematical simplicity, we work out $\alpha = 1$ only. The critical points of the above system are obtained by putting $\frac{du}{dx} = 0 = \frac{dv}{dx}$ which yield

\[
\begin{align*}
{u_1}_c &= \left( \frac{1}{1 - b} + \frac{\sqrt{B}}{b\rho_1} \right)^{-1}, \\
{v_1}_c &= -\left[ (1 - \frac{A + (1 + b)}{1 + b}) \frac{1}{\rho_1} \right] \\
&\times \left[ (1 + A(1 - b))(1 + b) \rho_1 \right] \\
&+ \sqrt{B(1 - A(1 - b))(1 + b)}, \quad (10)
\end{align*}
\]

\[
\begin{align*}
{u_2}_c &= \left( \frac{1}{1 - b} - \frac{\sqrt{B}}{b\rho_1} \right)^{-1}, \\
{v_2}_c &= \left[ (1 + A(1 - b))(1 + b) \rho_1 \right] \\
&\times \left[ (1 + A(1 - b))(1 + b) \rho_1 \right] \\
&+ \sqrt{B(1 - A(1 - b))(1 + b)}.
\end{align*}
\]

The two critical points correspond to the era dominated by dark matter and MCG type dark energy and exist for $A > \frac{1}{2\pi}$. For the two critical points, the state parameter (9) of the interacting dark energy takes the form

\[
u_{\text{mcg}}^i = \frac{B(u_{ic} + v_{ic})^{2(\alpha + 1)}}{\rho_1^{\alpha + 1}u_{ic}^{\alpha + 1}}v_{ic}^{-1}, \quad i = 1, 2
\]

which holds only when $u_{ic} + v_{ic} \neq 1$.

We further check the stability of the dynamical system (Eqs. (7) and (8)) about the critical point. To do this, we linearize the governing equations about the critical point i.e. $u = u_c + \delta u$ and $v = v_c + \delta v$, we obtain

\[
\delta \left( \frac{du}{dx} \right) = \left[ -3(1 + b + 2u + v) + A \left( -3 + 6u \left( -1 + \frac{u + 2v}{u + v} \right) \right) + \frac{3B(u + v)^2}{u^2(1 + u + v)^3\rho_1^2} \times \left\{ 2u^4 - (1 + 1)v^2 - uv(2 + v) + u^2(-1 + v)(-3 + 2v) + u^3(-5 + 4v) \right\} \right] \delta u
\]

\[
\delta \left( \frac{dv}{dx} \right) = \left[ 3b - 3v \left( 1 + \frac{A(-2v + (u + v)^2)}{u + v} \right) + \frac{3B(v + u)^2}{u^2(1 + u + v)^3\rho_1^2} \left\{ u^3 + u^2(-3 + v) + (-2 + v)v + u(1 + (-1 + v)u) \right\} \right] \delta v.
\]

The subscript $c$ refers to quantities evaluated at the critical point of the dynamical system. We also calculate the deceleration parameter $q = -1 - (H/H^2)$, in this model as

\[
q = -1 + \frac{3}{2} \left( 1 + \frac{\rho_{\text{mcg}}}{\rho} \right) \left( \frac{1 - 2\rho/\rho_1}{1 - \rho/\rho_1} \right), \quad (12)
\]

which can be written in terms of dimensionless density parameter $\Omega_{\text{mcg}} = \rho_{\text{mcg}}/\rho$:

\[
q_{\text{LQC}} = -1 + \frac{3}{2} \left( 1 + \frac{\rho_{\text{mcg}}\Omega_{\text{mcg}}}{\rho} \right) \left( \frac{1 - 2\rho/\rho_1}{1 - \rho/\rho_1} \right),
\]

Clearly in the limit of $\rho_1 \rightarrow \infty$, we retrieve the result for the Einstein’s gravity as

\[
q_{\text{EG}} = -1 + \frac{3}{2} \left( 1 + \frac{\rho_{\text{mcg}}\Omega_{\text{mcg}}}{\rho} \right).
\]

Assuming $\rho/\rho_1 = \epsilon \sim O(1)$ and using (10), we obtain

\[
q = -1 + \frac{3}{2} \left( 1 + \frac{\rho_{\text{mcg}}}{\rho} \right) \left( \frac{1 - 2\epsilon}{1 - \epsilon} \right), \quad (13)
\]

Since the only physically acceptable solution corresponds to the first stable critical point, such that $(u, v) \rightarrow (u_{1c}, v_{1c})$. Hence (13) gives

\[
q_c = -1 + \frac{3}{2} \left( 1 + \frac{\rho_{\text{mcg}}u_{1c}}{u_{1c} + v_{1c}} \right) \left( \frac{1 - 2\epsilon}{1 - \epsilon} \right). \quad (14)
\]

As special cases, observe that for $\epsilon = 1/2$, we have

\[
q = -1 \text{ while } \epsilon = 1 \text{ yields } q \rightarrow -\infty \text{ if } \rho_{\text{mcg}} \geq -1/u_{1c}.
\]

Moreover the Hubble parameter varies as

\[
H = \frac{2}{3\lambda}, \quad (15)
\]

where we have ignored the integration constant. Integration of (15) yields

\[
a(t) = a_0 t^{\frac{3}{\lambda}}, \quad (16)
\]
which gives a power law form of the expansion.

We also calculate the statefinder parameters. Sahni et al. (Sahni et al. 2003) introduced a pair of cosmological diagnostic pair \( \{r, s\} \) which they termed as Statefinder. The two parameters are dimensionless and are geometrical since they are derived from the cosmic scale factor alone, though one can rewrite them in terms of the parameters of dark energy and matter. Additionally, the pair gives information about dark energy in a model independent way i.e. it categorizes dark energy in the context of background geometry only which is not dependent on the theory of gravity. Hence geometrical variables are universal. Also this pair generalizes the well-known geometrical parameters like the Hubble parameter and the deceleration parameter. This pair is algebraically related to the equation of state of dark energy and its first time derivative.

\[
r \equiv \frac{\ddot{a}}{aH^2}, \quad s \equiv \frac{r - 1}{3(q - 1/2)}. \tag{17}
\]

In the present model, \( r_{\text{LQC}} \) gives

\[
r_{\text{LQC}} = \left(1 - \frac{3X}{2}\right)(1 - 3X), \tag{18}
\]

\[
s_{\text{LQC}} = 2X. \tag{19}
\]

It is interesting to note that the pair \( \{r_{\text{LQC}}, s_{\text{LQC}}\} \) yields the ΛCDM (cosmological constant-cold dark matter model) \( \{r_{\text{EG}}, s_{\text{EG}}\} = \{1, 0\} \) when \( X = 0 \) (or \( \epsilon = 1/2 \)).

### 3 Discussion

In this work, we considered modified Friedmann model in loop quantum cosmology. We assumed dark energy of the form modified Chaplygin gas. The interaction between dark matter and MCG has been investigated in LQC. The dynamical system of equations is solved numerically and a stable scaling solution is obtained. It henceforth resolves the famous cosmic coincidence problem in modern cosmology. The deceleration parameter and statefinder parameters are also calculated to classify this dark energy model. The dimensionally density parameters \( v \) and \( u \) are drawn in figures 1 and 2. We see that \( v \) decreases and \( u \) increases during evolution of the universe. From figure 4, we also see that the ratio of the above parameters decreases during time. The phase space diagram (figure 3) shows the attractor solution hence the present state and the future evolution of the universe is independent to the choice of initial conditions. Moreover the expansion of the universe is governed by a power-law form, rather than exponential or oscillatory. Hence the expansion will go on forever with an ever increasing rate. The variations of \( q_{\text{LQC}}, r_{\text{LQC}} \) and \( s_{\text{LQC}} \) are shown in figure 5-7 respectively against \( \omega_{\text{mcg}} \). It is observed that the more negative the state-parameter of MCG, the more negative values will be taken by the deceleration parameter. Finally our results also reduce to those for Einstein’s gravity under suitable limits of parameters.

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Fig. 1 The dimensionless density parameters are plotted against e-folding time. The initial condition is \( v(0) = 0.9, u(0) = 0.2 \). Other parameters are fixed at \( b = 0.2, A = 0.3, B = 0.5 \) and \( \alpha = 1 \).

Fig. 2 The dimensionless density parameters are plotted against e-folding time. The initial condition is \( v(0) = 0.6, u(0) = 0.6 \). Other parameters are fixed at \( b = 0.03, A = 0.3, B = 0.5 \) and \( \alpha = 0.5 \).
Fig. 3 The phase space diagram of parameters depicting an attractor solution. The initial conditions chosen are $v(0) = 0.5$, $u(0) = 0.6$ (green); $v(0) = 0.6$, $u(0) = 0.6$ (blue); $v(0) = 0.7$, $u(0) = 0.6$ (red); $v(0) = 0.8$, $u(0) = 0.6$ (brown). Other parameters are fixed at $b = 0.3$, $A = 0.3$, $B = 0.5$ and $\alpha = 1$.

Fig. 4 The ratio of density parameters is shown against e-folding time. The initial condition chosen is $v(0) = 0.6$, $u(0) = 0.6$. Other parameters are fixed at $b = 0.03$, $A = 0.3$, $B = 0.5$ and $\alpha = 0.5$.

Fig. 5 The deceleration parameter is plotted against the state parameter. Other parameters are fixed at $b = 0.3$, $\epsilon = 0.4$, $A = 0.4$ and $B = 0.5$.

Fig. 6 The statefinder parameter $r$ is plotted against the state parameter. Other parameters are fixed at $b = 0.3$, $\epsilon = 0.4$, $A = 0.4$ and $B = 0.5$. 
Fig. 7  The statefinder parameter $s$ is plotted against the state parameter. Other parameters are fixed at $b = 0.3$, $c = 0.4$, $A = 0.4$ and $B = 0.5$. 