Countable support iterations and large continuum

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Abstract

We prove that any countable support iteration formed with posets with \( \omega_2 \)-p.i.c. has \( \omega_2 \)-c.c., assuming CH in the ground model. This improves earlier results of Shelah by removing the restriction on the length of the iteration. Thus, we solve the problem of obtaining a large continuum via such forcing iterations.
Shelah [5, chapter VIII] introduces the notion of $\omega_2$-p.i.c. forcing. He shows that if $\langle P_\xi : \xi \leq \kappa \rangle$ is a countable support forcing iteration based on $\langle \dot{\mathcal{Q}}_\xi : \xi < \kappa \rangle$ such that $\dot{\mathcal{Q}}_\xi$ has $\omega_2$-p.i.c. in $V[G_{P_\xi}]$ for all $\xi < \kappa$, then if CH holds in $V$ and $\kappa \leq \omega_2$ then $P_\kappa$ has $\omega_2$-c.c. Many familiar forcings satisfy $\omega_2$-p.i.c., such as the forcings to add a Sacks real, a Mathias real, a Laver real, and so forth. Also, $\omega_2$-p.i.c. largely subsumes $\omega_2$-e.c.c., as demonstrated in [1, lemma 57], [2, lemma 24], and [3, lemma 15]. In this paper we eliminate the restriction on the length of the iteration. Also, the hypothesis of our theorem is that each $\dot{\mathcal{Q}}_\xi$ has weak $\omega_2$-p.i.c. and the iteration does not collapse $\omega_1$, where weak $\omega_2$-p.i.c. can be viewed as "$\omega_2$-p.i.c. minus properness."

The main idea which is required, aside from [5, chapter VIII] of course, is the notion of $\dot{\mathcal{P}}^{M}_{\eta, \kappa}$ from [4]. The statement $p \models_{P_\eta} "\dot{q} \in \dot{\mathcal{P}}^{M}_{\eta, \kappa} \cap M[G_{P_\eta}]"$ means that for every $p_1 \leq p$ there is $p_2 \leq p_1$ and $q_1 \in P_\kappa$ and $x \in M$ such that $x$ is a $P_\eta$-name and $p_2 \models "\dot{q} = x = q_1 \mathbf{1}[\eta, \kappa]"$. In contrast, the statement $p \models_{P_\eta} "\dot{q} \in \dot{\mathcal{P}}^{M}_{\eta, \kappa}"$ means that for every $p_1 \leq p$ there is $p_2 \leq p_1$ and $q_1 \in P_\kappa \cap M$ such that $p_2 \models "\dot{q} = q_1 \mathbf{1}[\eta, \kappa]"$. As always, the notation "$q_1 \mathbf{1}[\eta, \kappa]$" does not refer to the check (with respect to $P_\eta$) of the restriction of $q_1$ to the interval $[\eta, \kappa)$, but rather to the $P_\eta$-name which is forced to be a function with domain $[\eta, \kappa)$ such that for every $\gamma$ in the interval $[\eta, \kappa)$ we have that $q_1 \mathbf{1}[\eta, \kappa)(\gamma)$ is the $P_\eta$-name for the $\dot{P}_{\eta, \gamma}$-name corresponding to the $P_\gamma$-name $q_1(\gamma)$ (see [1, section 3] for greater detail on this point). The notion of $p \models "\dot{q} \in \dot{\mathcal{P}}^{M}_{\eta, \kappa}"$ is exploited in [4] to prove preservation of semi-properness under countable support (CS) iteration, preservation of hemi-properness under CS iteration, and a theorem giving a weak but sufficient condition for a CS iteration to
add no reals. The main property of $\hat{P}_\eta^M$ which is needed in [4] and in the present paper is the fact that $1 \models_{\eta} ((\forall \dot{q} \in \hat{P}_\eta^M)(\text{sup}(\dot{q}) \subseteq \check{M})).$ The fact that this holds is clear from the characterization of $p \models_{\eta} \dot{q} \in \hat{P}_\eta^M$ given above; in any case a detailed proof is given in [4, lemma 3].

**Definition 1.** We say that $(P, M, N, i, j)$ is embryonic iff for some sufficiently large regular $\lambda$ we have that $M$ and $N$ are countable elementary substructures of $H_\lambda$ and $\omega_1 < i < j < \omega_2$ and $\text{cf}(i) = \text{cf}(j) = \omega_1$ and $P \in M \cap N$ and $i \in M$ and $j \in N$ and $\text{sup}(\omega_2 \cap M) < j$ and $i \cap M = j \cap N$.

**Definition 2.** We say that $(P, M, N, i, j, h)$ is passable iff for some sufficiently large regular cardinal $\lambda$ we have that $(P, M, N, i, j)$ is embryonic and $h$ is an isomorphism from $M$ onto $N$ and $h$ is the identity on $M \cap N$ and $h(i) = j$.

**Definition 3.** We say that $P$ has weak $\omega_2$-p.i.c. iff whenever $(P, M, N, i, j, h)$ is passable and $p \in P \cap M$ then there is some $q \leq p$ such that $q \leq h(p)$.

**Definition 4.** Suppose $\langle P_\xi : \xi \leq \kappa \rangle$ is a countable support iteration. We say that $(P_\kappa, M, N, i, j, \eta, h, p)$ is strictly passable iff $(P_\kappa, M, N, i, j)$ is embryonic and $\eta \in \kappa \cap M \cap N$ and $p \in P_\eta$ and $p \models \check{h}$ is an isomorphism from $M[G_{P_\eta}]$ onto $N[G_{P_\eta}]$ and the restriction of $h$ to $M[G_{P_\eta}] \cap N[G_{P_\eta}]$ is the identity and $h(i) = j$ and $\check{N}$ is the image of $\check{M}$ under $h$.

**Definition 5.** We say $P_\kappa$ is strictly weak $\omega_2$-p.i.c. iff whenever $(P_\kappa, M, N, i, j, \eta, h, p)$ is strictly passable and $p \models_{\eta} \dot{q} \in \hat{P}_\eta^M$ then there is $r \in P_\kappa$ such that $r \upharpoonright \eta = p$ and $p \models \check{r}[\eta, \kappa] \leq \dot{q}$ and $r \upharpoonright [\eta, \kappa] \leq h(\dot{q})$.
Lemma 6. Suppose $\langle P_\xi : \xi \leq \kappa \rangle$ is a countable support iteration based on
$\langle \dot{Q}_\xi : \xi < \kappa \rangle$ and $P_\xi$ is strictly weak $\omega_2$-p.i.c. whenever $\xi < \kappa$. Suppose that
if $\kappa = \gamma + 1$ then $1 \models \dot{q}_\gamma$ has weak $\omega_2$-p.i.c. and $\omega^V_1 = \omega_1^{V[G_{P_\gamma}]}$.
Then $P_\kappa$ is strictly weak $\omega_2$-p.i.c.

Proof: Suppose $(P_\kappa, M, N, i, j, \eta, h, p)$ is strictly passable and $p \models \dot{q} \in \dot{P}^M_{\eta, \kappa}$.

Case 1: $\kappa = \gamma + 1$.

We have that $(P_\gamma, M, N, i, j, \eta, h, p)$ is strictly passable, so we may take
$r_0 \in P_\gamma$ such that $r_0 \forces \eta = p$ and $p \models \neg \forall \gamma \forall \eta \gamma \leq \dot{q} \gamma$ and $r_0 \forces \eta, \gamma \leq h(\dot{q} \gamma)$.

Now, the restriction of $h$ to $M[G_{P_\gamma}]$, which by notational convention is
the set of all $\dot{P}_{\eta, \gamma}$-names in $M[G_{P_\gamma}]$, induces an isomorphism $h^* \in V[G_{P_\gamma}]$ from
$M[G_{P_\gamma}]$ onto $N[G_{P_\gamma}]$ such that $h^*(i) = j$ and the restriction of $h^*$ to
$M[G_{P_\gamma}] \cap N[G_{P_\gamma}]$ is the identity and $\dot{N}$ is the image of $M$. Hence we
may use the fact that $\dot{Q}_\gamma$ has weak $\omega_2$-p.i.c. to take $\dot{r}_1 \in \dot{Q}_\gamma$ such that
$r_0 \forces \dot{r}_1 \leq \dot{q}(\gamma)$ and $\dot{r}_1 \leq h^*(\dot{q}(\gamma))$. Then let $r = (r_0, \dot{r}_1) \in P_\kappa$. We have
that $r$ is as required.

Case 2: $\kappa$ is a limit ordinal.

Let $\alpha = \sup(\kappa \cap M \cap N)$ and take $\langle \alpha_n : n \in \omega \rangle$ an increasing sequence of
ordinals from $\alpha \cap M \cap N$ cofinal in $\alpha$ such that $\alpha_0 = \eta$. Build $\langle p_n : n \in \omega \rangle$
such that $p_0 = p$ and for every $n \in \omega$ we have $p_{n+1} \forces \alpha_n = p_n$ and $p_n \models \neg \forall \alpha_n \alpha_{n+1}
\forces \dot{q} \alpha_n, \alpha_{n+1} \leq \dot{h} \alpha_n, \alpha_{n+1}$ and $p_{n+1} \forces \dot{h} \alpha_n, \alpha_{n+1} \leq h_n^*(\dot{q} \alpha_n, \alpha_{n+1})$ where
$h_n^*$ is the isomorphism from $M[G_{P_{\alpha_n}}]$ onto $N[G_{P_{\alpha_n}}]$ induced by $h$ as
in the successor case above. This is possible by the induction hypothesis
and [4, lemma 5]. Take $r \in P_\kappa$ such that $r \forces \alpha_n = p_n$ for every $n \in \omega$, and

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\[ r(\xi) = \dot{q}(\xi) \text{ for all } \xi \in \kappa \cap M \text{ such that } \alpha \leq \xi, \text{ and } r(\xi) = h(\dot{q})(\xi) \text{ for all } \xi \in \kappa \cap N \text{ such that } \alpha \leq \xi, \text{ and } r(\xi) = 1_{\dot{Q}_\xi} \text{ in all other cases. Then } r \text{ is as required. As noted earlier, the main point is that } p \models " \text{supt}(\dot{q}) \subseteq \bar{M}."

We repeat [5, VIII.2.3] (see also [1, lemma 42] but note by the way that the proof of [1, lemma 43] is incorrect; the present paper is in part a repair of this deficiency).

**Lemma 7.** Suppose CH holds and P has weak \( \omega_2 \)-p.i.c. Then P has \( \omega_2 \)-c.c.

**Proof:** Take \( \lambda \) a sufficiently large regular cardinal. Given \( \langle p_i : \text{cf}(i) = \omega_1 \text{ and } i < \omega_2 \rangle \) (potentially, a counterexample to \( \omega_2 \)-p.i.c.), take for each such \( i \) a countable elementary submodel \( N_i \) of \( H_\lambda \) such that \( \{p_i, i, P\} \subseteq N_i \). It suffices to show that there are \( \eta < \xi < \omega_2 \) and \( h \) such that \( \eta \cap N_\eta = \xi \cap N_\xi \) and \( \text{sup}(\omega_2 \cap N_\eta) < \xi \) and \( h \) is an isomorphism from \( N_\eta \) onto \( N_\xi \) and \( h(\eta) = \xi \) and the restriction of \( h \) to \( N_\eta \cap N_\xi \) is the identity. Take \( f(i) = \text{sup}(i \cap N_i) \) for all \( i < \omega_2 \) such that \( \text{cf}(i) = \omega_1 \). Take \( S_0 \subseteq \{i < \omega_2 : f(i) = \omega_1\} \) stationary and \( \gamma < \omega_2 \) such that \( (\forall i \in S_0)(f(i) = \gamma) \). By CH we may take \( S_1 \subseteq S_0 \) such that \( |S_1| = \aleph_2 \) and whenever \( i < j \) are both in \( S_1 \) then \( i \cap N_i = j \cap N_j \).

Take \( S_2 \subseteq S_1 \) of size \( \aleph_2 \) such that whenever \( i < j \) are both in \( S_2 \) then \( \text{sup}(\omega_2 \cap N_i) < \omega_j \). By \( \Delta \)-system, take \( S_3 \subseteq S_2 \) of size \( \aleph_2 \) and a countable \( N \) such that whenever \( i \) and \( j \) are distinct elements of \( S_3 \) then \( N_i \cap N_j = N \). Fix an enumeration \( \langle c_k : k \in \omega \rangle \) of \( N \). Let \( N_i^+ = \langle N_i; \in, i, c_0, c_1, \ldots \rangle \).

Up to isomorphism there are only \( \aleph_1 \)-many possible \( N_i^+ \), so the lemma is established.

Thus we have proved the following:

**Theorem 8.** Suppose \( \langle P_\xi : \xi \leq \kappa \rangle \) is a countable support iteration based
on \( \langle \dot{Q}_\xi : \xi < \kappa \rangle \) and for each \( \xi < \kappa \) we have that \( \dot{Q}_\xi \) has weak \( \omega_2 \)-p.i.c. in \( V[G_{P_\xi}] \). Suppose also that \( \omega_1^V = \omega_1^{V[G_{P_\kappa}]} \) and that CH holds in \( V \). Then \( P_\kappa \) has \( \omega_2 \)-c.c.

We also have the following:

**Fact 9.** Suppose \( \langle P_\xi : \xi \leq \kappa \rangle \) is a countable support forcing iteration. Then no reals are added at limit stages of uncountable cofinality.

Proof: Suppose \( \alpha \leq \kappa \) and \( \text{cf}(\alpha) > \omega \) and \( q \Vdash_{P_\alpha} \text{"} \dot{r} \in \omega \text{"} \). Take \( M \) a countable model containing all relevant data. Let \( \beta = \text{sup}(\alpha \cap M) \) and let \( \langle \beta_n : n \in \omega \rangle \) be an increasing sequence of ordinals from \( \beta \cap M \) cofinal in \( \beta \) with \( \beta_0 = 0 \). Build \( \langle p_n, q_n : n \in \omega \rangle \) such that \( q = q_0 \) and \( p_n \Vdash_{P_\beta_n} \text{"} q_n \in \dot{P}^M_{\beta_n, \alpha} \text{ and } q_n \leq q_{n-1} \downarrow [\beta_n, \alpha] \text{ and } q_n \text{ decides the value of } r(n-1) \text{"} \) and \( p_{n+1} \Vdash_{\beta_n} \beta_n = p_n \) and \( p_n \Vdash \text{"} p_{n+1} \downarrow [\beta_n, \beta_{n+1}] \leq q_n \downarrow [\beta_{n+1}] \text{"} \). Then take \( q' \in P_\alpha \) such that \( \text{supt}(q') \subseteq \beta \) and \( q' \downarrow [\beta_n] = p_n \) for all \( n \in \omega \). We have that \( q' \leq q \) and \( q' \downarrow [\beta] \Vdash \text{"} 1 \downarrow [\beta_{\alpha}] \text{"} \) for some \( s \in V[G_{P_\kappa}] \).

**Fact 10.** The \( \omega \)-bounding property, the Sacks property, the Laver property, etc., are preserved by countable support iteration (without the assumption of properness).

This follows from Fact 9, together with the arguments of [5, section VI.2]; the only place [5] uses the properness assumption is to handle the uncountable cofinality case.
References

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