DESINGULARIZING POSITIVE SCALAR CURVATURE 4-MANIFOLDS

DEMETRE KAZARAS

Abstract. We show that the bordism group of closed 3-manifolds with positive scalar curvature (psc) metrics is trivial by explicit methods. Our constructions are derived from scalar-flat Kähler ALE surfaces discovered by Lock-Viaclovsky. Next, we study psc 4-manifolds with metric singularities along points and embedded circles. Our psc null-bordisms are essential tools in a desingularization process developed by Li-Mantoulidis. This allows us to prove a non-existence result for singular psc metrics on enlargeable 4-manifolds with uniformly Euclidean geometry. As a consequence, we obtain a positive mass theorem for asymptotically flat 4-manifolds with non-negative scalar curvature and low regularity.

Contents

1. Introduction and Main Results 2
2. Background 8
3. Proof of Theorem B 13
4. Proof of Theorem A 17
5. Proof of Corollaries A and B 21
Appendix A 23
Conflict of interest statement 24
References 25

Date: October 3, 2024.
2000 Mathematics Subject Classification. 53C21, 53D23, 53C80, 57R90.
Key words and phrases. Positive scalar curvature metrics, general relativity, cobordism.
1. Introduction and Main Results

It is a fundamental fact in differential geometry that the \( n \)-dimensional torus \( T^n \) cannot support a smooth Riemannian metric with positive scalar curvature (psc). In dimension 2 this result is a simple consequence of the Gauss-Bonnet formula and in higher dimensions it is known as the Geroch Conjecture. For \( 3 \leq n \leq 7 \) this was first proven by Schoen-Yau using a minimal hypersurface technique [SY79a] and Gromov-Lawson later ruled out psc tori in all dimensions using index theoretic methods [GL80b]. It is natural to wonder: does the non-existence of psc metrics on \( T^n \) still hold when one relaxes the smoothness assumption and allow for metric singularities? Without restricting one’s attention to a special class of singularities, the answer is of course “no” in general. Inspired by constructions arising in Gromov’s polyhedral comparison theory for psc metrics [Gro14], Li-Mantoulidis introduced a class of singular metrics well-suited for this non-existence question in [LM17], presented a general conjecture, and gave a complete answer in dimension 3. In the present paper, we consider the situation in dimension 4 where one encounters many new challenges.

Following [LM17], a metric \( g \) on a smooth manifold \( M \) is said to be uniformly Euclidean, denoted by \( g \in L^\infty_E(M) \), if it may be pointwise bounded above and below by some smooth metric. In order to state the relevant conjecture in sufficient generality, recall that the Yamabe invariant of a closed \( n \)-manifold \( M \) is given by

\[
\sigma(M) = \sup_C \inf_{\tilde{g} \in C} \frac{\int_M R_{\tilde{g}} d\text{vol}_{\tilde{g}}}{\text{Vol}_{\tilde{g}}(M)^{\frac{n-2}{n}}}
\]

where the supremum is taken over all conformal classes \( C \) of smooth Riemannian metrics on \( M \) and \( R_{\tilde{g}} \) denotes the scalar curvature of \( \tilde{g} \). This diffeomorphism invariant has a long and rich history. Its relevance to our present discussion: \( \sigma(M) > 0 \) if and only if \( M \) admits a psc metric. The following conjecture of Schoen originally appeared in [LM17].

**Conjecture 1.** [LM17] Let \( M^n \) be a closed manifold with \( \sigma(M) \leq 0 \) and let \( S \subset M \) be a smooth, closed, embedded submanifold of codimension at least 3. If \( g \) is an \( L^\infty_E(M) \cap C^\infty(M \setminus S) \) metric with \( R_g \geq 0 \) on \( M \setminus S \), then \( g \) extends smoothly to a Ricci-flat metric on \( M \).

Notice that a metric \( g \in L^\infty_E(M) \cap C^\infty(M \setminus S) \) may have unbounded curvature. Li-Mantoulidis have confirmed Conjecture [1] in dimension 3 [LM17, Theorem 1.4].
In Corollary A, we make additional progress on Conjecture 1 by applying our main desingularization result, described in Theorem A. For a 4-manifold $M^4$, we call a subset $S \subset M$ a tolerable singular set if it is the image of a finite collection of smooth and disjoint embeddings of circles and points. We emphasize that our interest is in metric singularities and the underlying manifolds we consider are assumed to be smooth.

**Theorem A.** Let $M^4$ be a closed oriented 4-manifold and let $S \subset M$ be a tolerable singular set. Suppose $g$ is an $L^\infty(E(M)) \cap C^\infty(M \setminus S)$ metric with $R_g > 0$ on $M \setminus S$. Then there exists a smooth closed oriented psc manifold $(\overline{M}, \overline{g})$ with a degree-1 map $F : \overline{M} \to M$. Moreover, there is a neighborhood $U \subset M$, containing and retracting onto $S$, so that $F|_{\overline{M \setminus F^{-1}(U)}}$ is a conformal diffeomorphism.

In Theorem A, we regard $(\overline{M}, \overline{g})$ as a desingularization of $(M, g)$, pictured in Figure 1. In general, $\overline{M}$ is not diffeomorphic to $M$, possibly having much larger second Betti number. This is in contrast to [LM17], where Li-Mantoulidis show that point singularities in 3-manifolds may be desingularized without changing the
underlying manifold’s topology. While we cannot show that this change in topology is necessary, one can regard this feature of our construction as a reflection of the added topological complexities one encounters in dimension 4.

We use Theorem A to establish a non-existence result for singular psc metrics of on a class of 4-manifolds with non-positive $\sigma$-invariant called enlargeable manifolds. Recall that a closed oriented $n$-manifold $M^n$ is said to be enlargeable if, for every Riemannian metric on $M$ and $\varepsilon > 0$, there is a cover of $M$ which admits an $\varepsilon$-contracting map of non-zero degree to the unit sphere. The prototypical example is the torus where one may consider sufficiently large coverings and produce highly compressive maps to the sphere. This notion was originally introduced by Gromov-Lawson in [GL80b] in order to relate the fundamental group to Dirac operator methods for spin manifolds, showing that enlargeable spin manifolds do not admit psc metrics. Cecchini-Schick have extended this result to non-spin manifolds in [CT18]. We review the relevant definitions and facts of enlargeable manifolds in Section 2.1.

**Corollary A.** Let $M^4$ be a closed oriented enlargeable 4-manifold and let $S \subset M$ be a tolerable singular set. Then $M$ cannot admit a metric $g$ in $L_{\mathcal{F}}^\infty (M) \cap C^\infty (M \setminus S)$ with $R_g > 0$ on $M \setminus S$. Moreover, if $g$ in $L_{\mathcal{F}}^\infty (M) \cap C^\infty (M \setminus S)$ satisfies $R_g \geq 0$ on $M \setminus S$, then $g$ is Ricci-flat on $M \setminus S$.

For context, we remark that while the class of enlargeable 4-manifolds is large, there is a comparable number of non-enlargeable 4-manifolds which do not admit psc metrics, due to the non-triviality of their Seiberg-Witten invariant. In Section 1.1 we discuss this and the status of Conjecture 1 in dimension 4. We expect that Ricci-flat metrics $g$ on $M \setminus S$ smoothly extend over the singular set.

As an application of Corollary A, we obtain a Riemannian positive mass theorem for singular metrics of non-negative scalar curvature on asymptotically flat 4-manifolds. See Section 2.4 for the relevant definitions.

**Corollary B.** Let $(M^4, g)$ be an asymptotically flat 4-manifold with $g \in L^\infty (M) \cap C^\infty (M \setminus S)$ where $S$ is a tolerable singular set. If $R_g \geq 0$, then $m(M, g) \geq 0$.

The proof of Corollary B is based on a well-known argument due to Lohkamp [Loh99] which reduces the Riemannian positive mass theorem to the non-existence of psc metrics on manifolds of the form $T^n \# N$ where $N$ is some closed oriented manifold.
We adapt this argument to our lower regularity setting. For other positive mass theorems in low-regularity settings, see the results of Lee-LeFloch [LL15].

Let us describe our second main result, Theorem B. When adapting the Li-Mantoulidis methods to the 4-dimensional setting, one is faced with the oriented bordism group of psc 3-manifolds. A pair of closed oriented psc \(n\)-manifolds \((M_0, g_0), (M_1, g_1)\) are said to be psc-bordant if there is a compact oriented \((n + 1)\)-manifold \(W\) equipped with a psc metric \(\bar{g}\) such that

1. \(W\) is an oriented cobordism from \(M_0\) to \(M_1\), i.e. \(\partial W = M_0 \sqcup -M_1\);
2. The metric \(\bar{g}\) takes the product form \(g_i + dt^2\) on a neighborhood of \(M_i\) for \(i = 0, 1\).

Considering the class of all closed oriented psc \(n\)-manifolds modulo psc-bordism with the operation of disjoint union, one obtains an abelian group \(\Omega_{n,SO}^{SO,+}\) for \(n \geq 2\). One can also consider psc manifolds equipped with the extra data of a map to a fixed space \(X\) and additionally require that cobordisms \((W, \bar{g})\) admit extensions of this map. This yields an abelian group \(\Omega_{n,SO}^{SO,+}(X)\).

Very little is known about the groups \(\Omega_{n,SO}^{SO,+}\). In fact, the only previously computed psc-bordism group is \(\Omega_2^{SO,+}\), which is trivial. One could argue that, though these groups were not defined at the time, the triviality of \(\Omega_2^{SO,+}\) dates back to Weyl in 1916 where he showed the path-connectedness of the space of psc metrics on \(S^2\).

We mention a related result: in [MS15], Mantoulidis-Schoen show the connectedness of the space of metrics on \(S^2\) which are conformally psc.

Theorem B, which may be of independent interest, is a calculation of the groups \(\Omega_3^{SO,+}(S)\) where \(S\) is a finite 1-complex, i.e. \(S\) is a union of points and wedge-products of circles. It is an essential tool in our proof of Theorem A.

**Theorem B.** Let \(S\) be a finite 1-complex. Then the 3-dimensional oriented psc-bordism group \(\Omega_3^{SO,+}(S)\) is trivial.

Let us provide this result with some context. It has been known since the 1950’s, see Rohlin [Roh51] and Thom [Tho54], that every oriented 3-manifold is the boundary of some compact oriented 4-manifold, which we refer to as a null-cobordism. There are many proofs of this fact, both constructive and abstract, but it is far from clear whether or not null-cobordisms of psc 3-manifolds may themselves be equipped with psc metrics. However, for spherical space forms \(S^3/\Gamma\) where \(\Gamma \subset SO(4)\) acts
freely on $S^3$, algebraic geometry provides promising null-cobordisms for our purposes. Indeed, if $\Gamma \subset U(2)$, the minimal resolution of the flat cone $\mathbb{C}^2/\Gamma$,

$$X \to \mathbb{C}^2/\Gamma$$

often admits special geometries and large compact sets in $X$ are null-cobordisms of $S^3/\Gamma$. We adopt this strategy and rely on the remarkable constructions of Lock-Viaclovsky [LV19]. Another ingredient is the connectedness of the moduli space of psc metrics on 3-manifolds due to Marques [Mar12]. We remark that a generalization of this connectedness result has recently been obtained by Carlotto-Li [CL19].

1.1. Structure of the paper and further discussion. We begin in Section 2 by recalling the notions and tools we will require throughout the paper. These include conformal geometry on manifolds with boundary, the structure of 3-manifolds admitting psc metrics, and the asymptotically locally Euclidean (ALE) metrics we use. The scalar-flat Kähler ALE metrics constructed by Lock-Viaclovsky [LV19] are of fundamental importance to our work – large regions in these manifolds are the building blocks for the psc null-cobordisms we build to prove Theorem B in Section 3. Theorem A is proven in Section 4, though we relegate a technical discussion to the Appendix. We conclude by establishing Corollaries A and B in Section 5.

Now let us outline the proof of Theorem A and discuss the status of Conjecture 1 in dimension 4. In Section 4, we prove Theorem A by following a strategy developed in [LM17] and draw heavily from the analytical results there. Given a singular psc 4-manifold $(M,g)$, we geometrically “blow-up” the tolerable singular set $S$ in such a way that a neck appears near each component of $S$. In these necks, one finds minimal hypersurfaces. In analogy to the horizon of a black hole in General Relativity, these hypersurfaces act as shields, protecting the regular part of the manifold from the singular set. A classical argument of Schoen-Yau [SY79a] shows that these shields admit psc metrics. Next, we excise the singular regions behind the shields and cap-off the resulting boundary with the null psc-cobordisms we construct in Theorem B, producing a smooth closed psc manifold $(\overline{M}, \overline{g})$. Due to the nature of our psc null-cobordisms, $\overline{M}$ will generally not be diffeomorphic to $M$, having much larger second Betti number.

Let us describe this point in more detail. The building-block ALE manifolds we use have non-trivial second homology. Moreover, by considering the index of the Dirac operator, any (spin) null psc-cobordism of a non-trivial space form $S^3/\Gamma$ must
have non-zero signature, see [Dah97] for a nice description of this phenomenon. To consider an example, it may be possible that one of the shielding hypersurfaces is the connected sum of a lens space with itself – see [Don15] for work on the underlying embeddability question. In our desingularization procedure, we cap-off this hypersurface by taking the boundary connected sum of two copies of the null psc-cobordism of the lens space, introducing much new topology. That being said, we cannot rule out the existence of a more simple psc cap which would not increase topology. It is possible, therefore, that the change in topology is merely a defect of our methods.

Nevertheless, our desingularization $\overline{M}$ will admit a degree-1 map back to $M$ which collapses the added topology to the singular set. Because some components of $S$ may represent non-trivial elements of $\pi_1(M)$, care must be taken to arrange for this degree-1 map to exist. Indeed, if one is not deliberate, the process of excising and capping-off the singular set may have the effect of completely killing the fundamental group, potentially destroying the possibility of such a degree-1 map. It is for this reason we require the triviality of $\Omega_3^{SO,+}(S)$ and not just the smaller group $\Omega_3^{SO}$. We conclude this section with some informal remarks on Conjecture 1. In Theorem A we consider the class of enlargeable manifolds because the desingularization of $M$ always has a degree-1 map back to $M$ and thus does not leave the class of enlargeable manifolds. However, there are many 4-manifolds which are not enlargeable yet do not admit psc metrics: certain manifolds with non-trivial Seiberg-Witten invariant. Unlike enlargeable manifolds, this class is not closed under non-zero degree maps and so Theorem A has no immediate consequence for singular psc metrics on manifolds of this type. For instance, there is a homeomorphism

$$\#^3\mathbb{C}P^2 \#^{20}\mathbb{C}P^2 \rightarrow \mathbb{K} \# \mathbb{C}P^2.$$  

However, the one point blow up of $\mathbb{K}$ has non-trivial Seiberg-Witten invariant whereas the Gromov-Lawson construction [GL80] shows that the connected sum of $\mathbb{C}P^2$’s admits psc metrics. For this reason, we expect that either a much more refined version of our technique or entirely new ideas are needed to approach Conjecture 1 for general 4-manifolds.

The situation appears worse in higher dimensions since almost nothing is known about the groups $\Omega_n^{SO,+}$ for $n \geq 4$. For instance, in our proof of Theorem B, we make full use of the classification of psc 3-manifolds and the connectedness of the
moduli space of psc metrics on a fixed 3-manifold. Starting in dimension 4, not only is there no classification of psc manifolds, but the moduli space of psc metrics is known to be disconnected in general, see [Rub01] for dimension 4 and [BERW17] for higher dimensions.

1.2. Acknowledgments. I owe a debt of gratitude to Bradley Burdick for many enlightening conversations. I would also like to thank Professor Michael Anderson, Professor Christina Sormani, Professor Marcus Khuri, and Professor Blaine Lawson for their interest and helpful suggestions. For other helpful conversations, I thank Chao Li, Christos Mantoulidis, and Michael Albanese. The anonymous referee provided invaluable and helpful comments.

2. Background

2.1. Enlargeable manifolds. In this subsection, we will recall the enlargeability condition appearing in Theorem A and its fundamental properties. The notion of enlargeability was originally introduced by Gromov-Lawson in [GL80b] and substantially generalized in [GL83], which we refer readers to for a general discussion.

We follow the language of [GL83], but omit technical considerations related to non-compactness and spin structures since we will not require them. For $\varepsilon > 0$, a compact Riemannian $n$-dimensional manifold $(M, g)$ is said to be $\varepsilon$-hyperspherical if there exists a map $f : M \to S^n$ satisfying

$$||df(V)||_{g_{\text{rnd}}} \leq \varepsilon||V||_g$$

for all non-zero vectors $V \in TM$ where $g_{\text{rnd}}$ denotes the round metric of radius 1.

Definition. A compact manifold $M$ is called enlargeable if for all Riemannian metrics $g$ and $\varepsilon > 0$, there exists a cover $\overline{M} \to M$ which, equipping $\overline{M}$ with the lift of $g$, is $\varepsilon$-hyperspherical.

The prototypical examples of enlargeable manifolds are compact manifolds admitting non-positive sectional curvature metrics and compact solvmanifolds such as the torus $T^n$. Enlargeability is an invariant of the homotopy-type of the manifold. Even more, the class of enlargeable manifolds is closed under non-zero degree maps.

Proposition 1. [GL80b] Suppose $M^n$ is a closed enlargeable $n$-manifold. If $N^n$ is a closed $n$-manifold and there exists a map $N \to M$ of non-zero degree, then $N$ is also enlargeable.
A basic application of Proposition 1: the connected sum $T^n \# M$ of a torus with a closed manifold $M$ is enlargeable by considering the map which collapses $M$ to a point.

By considering a twisted Dirac operator, Gromov-Lawson made the fundamental observation that enlargeable spin manifolds cannot admit psc metrics [GL83, Theorem A]. Recently, Cecchini-Schick have generalized this to the non-spin case.

**Theorem 1.** [CT18, Theorem A] A closed enlargeable $n$-dimensional manifold cannot admit a psc metric.

The proof of Theorem 1 involves a clever construction related to Gromov’s torical bands [Gro18] and involving the notion of minimal $k$-slicings introduced by Schoen-Yau in the preprint [SY17]. We remark that our applications of Theorem 1 are in dimension 4 and therefore do not require the full regularity theory developed in [SY17].

### 2.2. The conformal Laplacian with minimal boundary conditions.

In later sections we will require some basic facts about the Yamabe problem on compact manifolds. For brevity, we will treat the cases in which the manifold has empty and non-empty boundary simultaneously.

Now let $(W, \bar{h})$ be an $n$-dimensional manifold with possibly non-empty boundary $(\partial W, h)$ where $h = \bar{h}|_{\partial W}$. We consider the following pair of operators acting on $C^\infty(W)$:

\[
\begin{align*}
\mathcal{L}_{\bar{h}} &= -\Delta_{\bar{h}} + c_n R_{\bar{h}} \quad \text{in } W \\
B_{\bar{h}} &= \partial_\nu + 2c_n H_{\bar{h}} \quad \text{on } \partial W,
\end{align*}
\]

where $c_n = \frac{4(n-1)}{n-2}$ and $H_{\bar{h}}$ is the mean curvature of $\partial W$ with respect to $\nu$, the outward-pointing normal vector.

Recall that if $\phi \in C^\infty(W)$ is a positive function, then the scalar and boundary mean curvatures of the conformal metric $\tilde{h} = \phi^{\frac{4}{n-2}} \bar{h}$ are given by

\[
\begin{align*}
R_{\tilde{h}} &= c_n^{-1} \phi^{-\frac{n+2}{n-2}} \mathcal{L}_{\bar{h}} \phi \quad \text{in } W \\
H_{\tilde{h}} &= \frac{1}{2} c_n^{-1} \phi^{-\frac{n}{n-2}} B_{\bar{h}} \phi \quad \text{on } \partial W.
\end{align*}
\]

We consider the Rayleigh quotient coming from (4) and take the infimum:

\[
\lambda_1 = \inf_{\phi \neq 0 \in H^1(W)} \frac{\int_W (|\nabla \phi|^2 + c_n R_{\tilde{h}} \phi^2) \, d\mu + 2c_n \int_{\partial W} H_{\tilde{h}} \phi^2 \, d\sigma}{\int_W \phi^2 \, d\mu}.
\]
According to standard elliptic PDE theory, we obtain an elliptic boundary problem, denoted by \((L_\bar{h}, B_\bar{h})\), and \(\lambda_1\) is the principal eigenvalue of the minimal boundary problem \((L_\bar{h}, B_\bar{h})\). The corresponding Euler-Lagrange equations are the following:

\[
\begin{aligned}
L_\bar{h}\phi &= \lambda_1 \phi \quad \text{in } W \\
B_\bar{h}\phi &= 0 \quad \text{on } \partial W.
\end{aligned}
\]

This problem was first studied by Escobar [Esc92] in the context of the Yamabe problem on manifolds with boundary.

Let \(\phi\) be a solution of (7). It is well-known that the eigenfunction \(\phi\) is smooth and can be chosen to be positive. A straight-forward computation shows that the conformal metric \(\tilde{h} = \phi^{\frac{4}{n-2}}\bar{h}\) has the following scalar and mean curvatures:

\[
\begin{aligned}
R_{\tilde{h}} &= \lambda_1 \phi^{\frac{4}{n-2}} \quad \text{in } W \\
H_{\tilde{h}} &\equiv 0 \quad \text{on } \partial W.
\end{aligned}
\]

In particular, the sign of the eigenvalue \(\lambda_1\) is a conformal invariant, see [Esc92, Esc96]. We call a conformal manifold \((W, [h])\) Yamabe positive, null, or negative according to the sign of \(\lambda_1\).

The link between the above conformal considerations and psc bordisms is given by Akutagawa-Botvinnik in [AB02]. We will use the following approximation trick throughout the paper.

**Theorem 2.** [AB02, Corollary B] Let \(W^n\) be a manifold of dimension \(n \geq 3\) with non-empty boundary \(\partial W\). Let \(g\) be a psc metric on \(\partial W\). The following are equivalent:

1. There is a metric \(\bar{g}\) on \(W\) so that \(g \in [\bar{g}]_{\partial W}\) and \((W, [\bar{g}])\) is Yamabe positive;
2. There is a psc metric \(\tilde{g}\) on \(W\) which takes the product form \(\tilde{g} = g + dt^2\) on a neighborhood of \(\partial W\).

2.3. Psc 3-manifolds. In order to prove Theorem B, we will need the classification of closed oriented 3-manifolds admitting psc metrics. More precisely, the following result is a consequence of Gromov-Lawson [GL83, Theorem 8.1], Schoen-Yau [SY79b], and Perelman’s solution of the geometrization conjecture [Per02, Per03b, Per03a].

**Proposition 2.** Suppose \(M\) is a closed oriented 3-manifold. Then \(M\) admits a psc metric if and only if it is diffeomorphic to the connected sum \(\#^k_{i=1} M_i\) where each \(M_i\) is one of the following:
(1) $S^2 \times S^1$;
(2) $S^3/\Gamma$ where $\Gamma \subset SO(4)$ is a finite subgroup acting freely on $S^3$.

Another ingredient for the proof of Theorem B is the work of Marques in [Mar12] on the space of psc metrics on 3-manifolds. To state it, for a manifold $M$, let $\text{Riem}^+(M)$ denote the space of psc metrics on $M$, endowed with the Whitney topology. The group of diffeomorphisms of $M$, denoted $\text{Diff}(M)$, acts on $\text{Riem}^+(M)$ and one may form the moduli space of psc metrics

$$M_+(M) = \text{Riem}^+(M)/\text{Diff}(M).$$

By making careful use of Perelman’s Ricci flow with surgery and the Gromov-Lawson surgery construction, Marques obtained the following connectedness result.

**Theorem 3.** [Mar12 Main Theorem] Suppose $M$ is a closed oriented 3-manifold which admits a psc metric. Then the moduli space $M_+(M)$ is path connected.

### 2.4. ALE manifolds and mass.

The final ingredient we will require is a family of asymptotically locally Euclidean (ALE) metrics. We refer the reader to [Bar86] for a complete discussion of such spaces and their fundamental analytic properties.

**Definition.** Let $\Gamma \subset SO(n)$ be a finite subgroup acting freely on $S^{n-1}$ and let $\tau > 0$. A complete $n$-dimensional Riemannian manifold $(X, g)$ is said to be ALE of order $\tau$ with group at infinity $\Gamma$ if

1. There is a compact subset $K \subset X$ and a diffeomorphism called a chart at infinity
   $$\Psi : X \setminus K \to (\mathbb{R}^n \setminus B)/\Gamma$$
   where $B \subset \mathbb{R}^n$ is a ball centered at the origin;
2. Denoting the radial coordinate on $\mathbb{R}^n/\Gamma$ by $r$, we have
   $$|\partial^\alpha((\Psi^{-1})^*g_{ij} - \delta_{ij})| = O(r^{-|\alpha| - \tau})$$
   for any multi-index $\alpha$ with $|\alpha| = 0, 1, 2$ and $\delta$ is the Euclidean metric.

If an ALE manifold has trivial group at infinity, the manifold is said to be asymptotically flat.

If one fixes a base point $p$ in an ALE manifold $X$, notice that large geodesic spheres $S(p, R)$ become asymptotic to a spherical space form $S^{n-1}/\Gamma$. This space form is independent of $p$ and we call $S^{n-1}/\Gamma$ the sphere at infinity of $X$. 

We will be interested in 4-dimensional ALE manifolds with non-negative scalar curvature. In general, it is a difficult problem to find scalar non-negative ALE manifolds with a given group $\Gamma \subset SO(4)$ at infinity. Such constructions have a history dating back to [Haw77] too rich to cover here, so we will be brief. In [Kro89a, Kro89b], Kronheimer produced hyper-Kähler (and hence Ricci-flat) ALE manifolds for any finite $\Gamma \subset SU(2) \subset SO(4)$ at infinity. The negative-mass metrics constructed by LeBrun in [LeB88] provide examples of scalar-flat Kähler ALE metrics having cyclic groups at infinity not contained $SU(2)$. Calderbank-Singer [CS04] constructed Ricci-flat ALE metrics with any given 3-dimensional lens space as the sphere at infinity.

However, it was not until recently that every finite $\Gamma \subset U(2) \subset SO(4)$ acting freely on $S^3$ was shown to arise as the group at infinity of a scalar-flat ALE manifold. Through a delicate assembly, Lock-Viaclovsky were able to piece together the known constructions to obtain the following:

**Theorem 4.** [LV19, Theorem 1.3] Let $\Gamma \subset U(2)$ be a finite group containing no complex reflections. Then there are scalar-flat Kähler ALE metrics on the minimal resolution of $\mathbb{C}^2/\Gamma$.

To explain the relevance to Theorem B, any finite subgroup of $SO(4)$ acting freely on $S^3$ is conjugate within $SO(4)$ to a subgroup of either the standard embedding $U(2) \subset SO(4)$ or its so-called orientation-reversed embedding, see [Sco83, Theorem 4.10]. It follows that the diffeomorphism type of any 3-dimensional spherical space form can be realized as the quotient of $S^3$ by a subgroup of $U(2)$. Using this observation with Theorem 4, one immediately obtains the following.

**Corollary 1.** Let $N^3$ be an orientable 3-dimensional spherical space form. Then there is a scalar-flat ALE manifold whose sphere at infinity is diffeomorphic to $N$.

We conclude this section by recalling the definition of the mass of an ALE manifold appearing in Corollary B. An ALE $n$-manifold $(M, g)$ can be viewed as a time-symmetric slice of an $(n+1)$-dimensional space-time and one may consider its ADM mass

$$m(M, g) = \lim_{R \to \infty} \frac{\Gamma(n)}{4(n-1)\pi^{n/2}} \int_{S_{R/\Gamma}} \sum_{i,j=1}^n (g_{ij,i} - g_{ii,j})\nu^j d\text{Vol}_3$$
where $S_R$ is the Euclidean coordinate sphere of radius $R$, the indices arise from co-ordinates given by the chart at infinity, $\nu$ is the outward-pointing Euclidean normal vector to $S_R/\Gamma \subset \mathbb{R}^n/\Gamma$, and $\Gamma(\frac{n}{2})$ is a value of the Gamma function (unrelated to the group $\Gamma$). If the order of decay $\tau$ satisfies $\tau > (n - 2)/2$ and $R_g \in L^1(M)$, then $m(M, g)$ is independent of the chart at infinity, due to [Bar86] and [Chr86].

If $(M, g)$ is asymptotically flat, of dimension less than 8, and has non-negative scalar curvature, then the celebrated positive mass theorem of Schoen-Yau [SY79c] states that $m(M, g) \geq 0$ and that the only such manifold with vanishing mass is Euclidean space. For the higher dimensional case, see [SY17] and [Loh16].

3. Proof of Theorem B

This section is devoted to the proof of Theorem B. Before beginning, we make some preparations. The following is an elementary observation about large sets in ALE manifolds of any dimension.

**Lemma 1.** Let $(X, g)$ be an $n$-dimensional ALE manifold with group at infinity $\Gamma \subset SO(n)$ and let $p \in X$. Given $\varepsilon > 0$, there exists a bounded open set $\Omega \subset X$ such that

1. $\partial \Omega$ is diffeomorphic to $S^{n-1}/\Gamma$;
2. the restriction metric $g|_{\partial \Omega}$ is $\varepsilon$-close to a metric of constant positive sectional curvature on $S^{n-1}/\Gamma$ in the $C^2$-topology;
3. $\partial \Omega$ has positive mean curvature with respect to the normal vector pointing towards infinity.

**Proof.** Choose a compact set $K$ and diffeomorphism $\Phi$ as in Definition 2.4. For $R >> 0$ let $B_R$ denote the coordinate ball in $(\mathbb{R} \setminus B)/\Gamma$. Considering the decay of $g$, $\frac{1}{R^n}(\Phi^{-1})^*g$ converges to the flat metric on the cone $\mathbb{R}^n/\Gamma$ as $R \to \infty$ in the $C^2$-topology. It follows that, for sufficiently large $R$, $\Phi^{-1}(\partial B_R)$ has positive mean curvature and the restriction of $g$ to $\Phi^{-1}(\partial B_R)$ is $\varepsilon$-close to the metric of constant curvature $\frac{1}{R}$. The set $\Omega = K \cup \Phi^{-1}(B_R)$ is the desired open set. \square

The following is a simple consequence of the connectedness result Theorem 3.

**Proposition 3.** Fix a closed oriented 3-dimensional manifold $M^3$ which admits a psc metric. The following are equivalent

1. For any psc metric $g$ on $M$, $(M, g)$ is psc null-bordant.
There exists some psc metric $g_0$ on $M$ so that $(M, g_0)$ is psc null-bordant.

Proof. To prove the non-trivial implication $(2) \Rightarrow (1)$, let $g_0$ be a psc metric on $M$ and suppose there exists a compact 4-dimensional psc manifold $(W^4, \bar{g})$ so that $\partial W = M$ and $\bar{g} = g_0 + dt^2$ near $M$. Let $g$ be a second psc metric on $M$.

By Theorem 2 there is a path of psc metrics $\{g(t)\}_{t \in [0, 1]}$ on $M$ so that $g(1) = g$ and $g(0) = \Phi^* g_0$ for some diffeomorphism $\Phi$ of $M$. By sufficiently slowing down the parameterization of this path, one can obtain a psc-bordism of the form

$$(M \times [0, L], g(s) + ds^2)$$

between $(M, \Phi^* g_0)$ and $(M, g)$, see [Ros07]. Using $\Phi$, one may glue this cylinder to the boundary of $W$ to obtain

$$(11) \quad \left( W \bigcup_{\Phi} (M \times [0, L]), \bar{g} \cup (g'(s) + ds^2) \right),$$

which is the promised psc null-cobordism of $(M, g)$. \hfill \square

Proof of Theorem B: First, we will show that $\Omega^{SO, +}_3$ is trivial. Let $M$ be a closed oriented 3-manifold which admits a psc metric. We first make some reductions. By Proposition 3 it suffices to construct a psc null-cobordism of $(M, g_0)$ for a single psc metric $g_0$ on $M$. In order to achieve this, by Theorem 2 it is sufficient to find a Yamabe positive conformal manifold $(W, \bar{g})$ whose boundary $(M, [\bar{g}]_M)$ is also Yamabe positive.

We will first consider the case where $M$ is one of the irreducible manifolds appearing in Proposition 2. For $M = S^2 \times S^1$, consider the manifold $W = S^3_+ \times S^1$ where $S^3_+$ denotes the upper hemisphere of the round 3-sphere. With the product metric $\bar{g}$, $S^3_+ \times S^1$ is psc with minimal boundary isometric to the product of the round $S^2$ and $S^1$. By inspecting the Rayleigh quotient \cite{6}, $(S^3_+ \times S^1, [\bar{g}])$ is Yamabe positive, competing our work in this case.

Now consider the case of a spherical space form $M = S^3/\Gamma$. By Corollary 1 there is a scalar-flat ALE manifold $(X, g_X)$ with sphere at infinity $S^3/\Gamma$. Applying Lemma 1 with some $\varepsilon > 0$, we obtain a large compact set $K_\varepsilon \subset X$ with metric $\bar{g}_\varepsilon = g_X|_{K_\varepsilon}$ which is scalar-flat with mean convex boundary $S^3/\Gamma$. By perhaps choosing smaller $\varepsilon$, we can ensure that the restriction metric $\bar{g}_\varepsilon|_{S^3/\Gamma}$ is sufficiently close to a round
metric to have positive scalar curvature. Due to the positive mean curvature of its boundary, \((K_c, [\bar{g}_c])\) is Yamabe positive, finishing our work with the space forms.

Finally, we consider general \(M\). By Proposition 2, \(M\) may be decomposed as \(M = \#_{i=1}^k M_i\) where each \(M_i\) is with \(S^2 \times S^1\) or a space form \(S^3/\Gamma\). By our work above, we may consider \((W_i, \bar{g}_i)\) which are psc null-cobordisms of some psc metrics \(g_i\) on \(M_i\) for each \(i = 1, \ldots, k\). Notice that the boundary connected sum

\[
W = \bigsqcup_{i=1}^k W_i
\]

is a null-cobordism of \(M\). To make \(W\) into a psc null-cobordism, we use a version of the classical Gromov-Lawson surgery construction [GL80a] adapted to the case of boundary connected sums by Carr [Car88, Lemma 10], c.f. [Wal14]. This produces a psc metric on \(W\) which is product near the boundary and resembles a Gromov-Lawson connected sum construction applied to the 3-manifolds \((M_i, \bar{g}_i|_{M_i})\) near the boundary of \(W\). It follows that there is a psc null-bordant psc metric on \(M\). This shows that \(\Omega_{SO, +}^3\) is trivial.

In Figure 2 we depict the null-cobordism of a general psc \((M^3, g)\) we have obtained by combining all the constructions in this section.

Now we turn our attention to \(\Omega_{SO, +}^3(S)\) where \(S\) is some finite 1-complex. Let \((M, g)\) be a closed oriented psc 3-manifold equipped with a map \(F : M \to S\). We will assume that \(M\) and \(S\) are connected – if they are not, one may apply the proceeding argument to each component. By Proposition 2 we have the decomposition

\[
M \cong (\#_{i=1}^{k_1} S^3/\Gamma_i) \# (\#_{i=1}^{k_2} (S^2 \times S^1))
\]

where we allow \(k_1\) and \(k_2\) to possibly take the value of 0. Evidently, the bordism class \([M, g, F]\) is unchanged by homotopies of \(F\). As such, it suffices to consider the case where \(S\) is the wedge of \(l\) circles \(\wedge l S^1\).

An exercise in basic algebraic topology shows there is a bijection

\[
[M, \wedge l S^1] \to \text{Hom}(\pi_1(M), F_l) \cong \text{Hom}(\Gamma_1 \ast \cdots \ast \Gamma_{k_1} \ast F_{k_2}, F_l)
\]

where \([M, \wedge l S^1]\) is the set of homotopy classes of maps from \(M\) to \(\wedge l S^1\), \(F_l\) is the free group on \(l\) generators, and \(\ast\) is the free product. Since all \(\Gamma_i\) are finite and \(F_l\) has no torsion elements, it follows that \(F\) is homotopic to a map \(F' : M \to \wedge l S^1\) which maps the summand \((\#_{i=1}^{k_1} S^3/\Gamma_i)\) to a single point \(x_0 \in \wedge l S^1\).
Figure 2. The null-cobordism we construct of a psc \((M^3, g)\)

Now consider the psc null-cobordism of \((M, g)\) we constructed above:

\[
(W, \bar{g}) = \left( \bigoplus_{i=1}^{k_1} W_i, \bigoplus_{i=1}^{k_2} S^3_+ \times S^1, \bar{g} \right).
\]

We proceed by constructing an extension of \(F\) to a map \(\overline{F} : W \to S^1\) in two steps. First, declare \(\overline{F}|_{\partial W} = F\) and that \(\overline{F}\) map the summand \(\bigoplus_{i=1}^{k_1} W_i\) to the single point \(x_0\). Since \(\overline{F}\) is constant on the \(\bigoplus_{i=1}^{k_1} W_i\) summand, it suffices to extend \(\overline{F}\) over

\[
W \setminus \left( \bigoplus_{i=1}^{k_1} W_i \right) \cong \bigoplus_{i=1}^{k_2} S_+^3 \times S^1.
\]

Since the only 2-cells in \(\bigoplus_{i=1}^{k_2} S_+^3 \times S^1\) lie on its boundary where \(\overline{F}\) is already defined, all classes obstructing the extension of \(\overline{F}\) vanish. For readers unfamiliar with obstruction theory, see [Hat02, Section 4.3]. It follows that \(\overline{F}\) may be extended over
all of $W$ and so $(W, \bar{g}, \bar{F})$ is our desired null-cobordism. This finishes the proof of Theorem B.

4. Proof of Theorem A

Now that we have established the required bordism calculations, we are ready to prove Theorem A.

**Proof of Theorem A:** Let $M$, $S$, and $g$ be as in Theorem A. According to the condition $g \in L_\infty^e(M) \cap C^\infty(M \setminus S)$, there is a smooth metric $h$ and a positive constant $C$ so that $g$ is bounded above and below by $Ch$ and $h$, respectively. Fix some positive function $R$ on $M$ such that

$$0 < R \leq \min(1, R_g)$$

which will act as a bounded surrogate for the possibly unbounded scalar curvature of $g$. We may consider a positive Green’s-type function, $G$, of the modified conformal Laplacian

$$L_g = -\Delta_g + \frac{8}{3}R,$$

see [LSW63] Theorem 6.1]. As a distribution, $G$ solves the equation

$$\left(-\Delta_g + \frac{8}{3}R\right)G = \delta_S$$

where $\delta_S$ is the Dirac measure of $S$. Strictly speaking, [LSW63] considers only Green’s functions with singularities along points. However, generalizing this construction to find functions in the kernel of positive linear elliptic operators with singularities along smoothly embedded submanifolds of codimension greater than 2 is classical, see [SY79a].

Due to the uniform ellipticity of $L_g$, the asymptotics of $G$ near $S$ behave in a standard manner, see [LSW63, Theorem 7.1] and [SY79a, Appendix]. More precisely, there is a constant $C_0 > 0$ so that: near an isolated point singularity $x_0 \in S$,

$$C_0^{-1}d_h(x_0, x)^{-2} \leq G(x) \leq C_0d_h(x_0, x)^{-2}$$

and near a 1-dimensional component $L \subset S$

$$C_0^{-1}d_h(L, x)^{-1} \leq G(x) \leq C_0d_h(L, x)^{-1}.$$
Next, we study the family of metrics

\[ g_\varepsilon = (1 + \varepsilon G)^2 g \]

on \( M \setminus S \) where \( \varepsilon \) is a small positive parameter. A straight forward calculation using (13) shows \( \mathcal{L}_g(1 + \varepsilon G) > 0 \). Inspecting the conformal transformation formula (5), it follows that \( g_\varepsilon \) has positive scalar curvature.

Given a small distance \( \eta > 0 \), consider the tubular neighborhood \( U^\eta = \{ x \in M : d_g(x, S) < \eta \} \) and fix \( \eta_0 > 0 \) small enough so that there is a map \( f : U^{\eta_0} \to S \) deformation retracting \( U^{\eta_0} \) onto \( S \). Next, we consider the non-compact manifold \( Q = \overline{U^{\eta_0}} \setminus S \) with the restriction metric, which we continue denoting by \( g_\varepsilon \).

**Claim 1.** There is a \( \varepsilon_0 > 0 \) so that, for each \( \varepsilon \leq \varepsilon_0 \), there is a closed smoothly embedded hypersurface \( \Sigma_\varepsilon \subset U^{\eta_0} \) satisfying

1. \( \Sigma_\varepsilon \) is stable-minimal with respect to \( g_\varepsilon \);
2. \( \overline{U^{\eta_0}} \setminus \Sigma_\varepsilon \) consists of two components: \( A^{in}_\varepsilon \) containing \( S \) and \( A^{out}_\varepsilon \) containing \( \partial U^{\eta_0} \).

**Proof.** For each \( \varepsilon > 0 \), consider the minimization problem

\[ V_\varepsilon = \inf \{ \text{Vol}_{g_\varepsilon}(\partial \Omega) : \Omega \subset Q \text{ is open and contains } \partial U^{\eta_0} \} \]

The first step is to leverage Lemma 2 in the Appendix to study a solution of this minimization problem. In order to explain how Lemma 2 applies, we must take a moment to articulate (18) in a more amenable form.

By repeating the argument for each path component of \( S \), it suffices to consider the case where \( S \) is connected. The case where \( S \) is a point is treated in [LM17], so we will assume \( S \) is an embedded circle. Choosing a potentially smaller \( \eta_0 \), one may consider Fermi coordinates \( (t, y) \) on \( U^{\eta_0} \) where \( t \) parameterizes \( S \) and \( y \) ranges over the fibers of \( S \)'s normal bundle. Introduce polar coordinates \( (\rho, \theta, \varphi) \) in the \( y \) variables and consider a new radial coordinate \( r = -\log \rho \in (-\log(\eta_0), \infty) \). Using the asymptotics (15) and (16), there is a constant \( C_1 \) so that \( g_\varepsilon \) satisfies \( C_1^{-1}\hat{h}_\varepsilon \leq g_\varepsilon \leq C_1 \hat{h}_\varepsilon \) over \( U^{\eta_0} \) where \( \hat{h}_\varepsilon \) is given by

\[
\hat{h}_\varepsilon := (1 + \varepsilon/\rho)^2(dt^2 + d\rho^2 + \rho^2 g_{S^2}) = (1 + e^r)^2(dt^2 + e^{-2r} dr^2 + e^{-2r} g_{S^2}) = (e^{-r} + \varepsilon)^2(dr^2 + e^{2r} dt^2 + g_{S^2}).
\]
At this point we have compared the metrics $g_\epsilon$ and $h_\epsilon$ on the domain $Q \equiv [-\log \eta_0, \infty) \times S \times S^2 \cong (\mathbb{R}^2 \setminus B^2_{\eta_0}(0)) \times S^2$. Smoothly extend the restriction $g_\epsilon|_{U_{\eta_0}}$ over $\mathbb{R}^2 \times S^2$ to a metric $g'_\epsilon$. For each $\epsilon$, there is a constant $C_\epsilon$ so that

$$C_\epsilon^{-1}(g_{\mathbb{H}^2} + g_{S^2}) \leq g'_\epsilon \leq C_\epsilon(g_{\mathbb{H}^2} + g_{S^2})$$

on $\mathbb{R}^2 \times S^2$ where $g_{\mathbb{H}^2}$ denotes the metric on the hyperbolic plane with sectional curvature $-1$. To be clear, the constant $C_\epsilon$ depends heavily on $\epsilon$ and the choice of extension $g'_\epsilon$.

Now according to Lemma 2, for all sufficiently small $\epsilon > 0$ the infemum $V_\epsilon$ is realized by a bounded set $\Omega_\epsilon \subset \mathbb{R}^2 \times S^2$ containing $B_1(0) \times S^2$ whose boundary yields an embedded hypersurface $\Sigma_\epsilon$. According to the classical regularity theory in this dimension, $\Sigma_\epsilon$ is smooth away from the intersection $\Sigma_\epsilon \cap \partial U^{\eta_0}$. The rest of the proof is devoted to showing that the intersection $\Sigma_\epsilon \cap \partial U^{\eta_0}$ is empty for all sufficiently small $\epsilon$.

First observe that

$$\lim_{\epsilon \to 0} V_\epsilon = 0. \tag{20}$$

Indeed, for small distances $\eta$, one may consider the boundary of the neighborhood $U^{\eta}$ and compute

$$\text{Vol}_{g_\epsilon}(\partial U^{\eta}) \leq C_1 \eta^2 (1 + \epsilon \eta^{-1})^4 \tag{21}$$

for some constant $C_1 > 0$ depending only on $g$ and the background metric $h$. In particular, $\text{Vol}_{g_\epsilon}(\partial U^{\sqrt{\eta}}) \to 0$ as $\epsilon \to 0$ and (20) follows.

Now let $\eta_1 \in (0, \eta_0)$. We claim that there exists $\epsilon_0 > 0$ so that

$$\Sigma_\epsilon \cap (U^{\eta_0} \setminus U^{\eta_1}) = \emptyset \tag{22}$$

for all $\epsilon \leq \epsilon_0$. For sake of contradiction, assume there is a sequence $\epsilon_j \to 0$ so that $\Sigma_{\epsilon_j} \cap (U^{\eta_0} \setminus U^{\eta_1}) \neq \emptyset$ for all $j = 1, 2, \ldots$. For each $j$, choose a point

$$x_j \in \Sigma_{\epsilon_j} \cap (U^{(\eta_0 - \eta_1)/2} \setminus U^{\eta_1}).$$

Let $B^h_\eta(x)$ denote the metric ball about $x$ of radius $\eta$ using the metric $h$. Since $g_\epsilon \to g$ smoothly on the region $U^{\eta_0} \setminus U^{\eta_1}$ where $g$ is bounded in $C^k$, $B^h_\eta(x_j) \cap \Sigma_\epsilon$ are smooth minimal hypersurfaces with respect to metrics whose coefficients and their derivatives are uniformly bounded in, say, normal coordinates. In this context, the
monotonicity theorem for minimal submanifolds provides a lower bound

\[ \text{Vol}_{g_\varepsilon}(B^h_{(\varepsilon_0 - \varepsilon_1)/2}(x_j) \cap \Sigma_\varepsilon) \geq V_0 \]

where \( V_0 \) is a positive constant independent of sufficiently small \( \varepsilon \), contradicting (20). This finishes the proof of Claim 1. \( \square \)

During the remainder of the proof of Theorem A we fix some \( \varepsilon \) small enough to apply Claim 1. The parameter \( \varepsilon \) will no longer play a significant role and we only mention it where necessary. By Claim 1, we obtain a stable-minimal hypersurface \( \Sigma \) which separates \( M \) into two components: \( A^\text{in} \) containing \( S \) and \( A^\text{out} \) containing \( \partial U^{\eta_0} \). We mention that \( \Sigma \) will generally be disconnected, having as many components as \( S \). Let \( h' = g_\varepsilon|_\Sigma \) denote the restriction metric. More notation: let \( M_0 \) denote the compact manifold \( M \setminus A^\text{in} \). Abusing notation, we continue to denote the restriction of \( g_\varepsilon|_{M_0} \) by \( g_\varepsilon \).

Notice that \((M_0, g_\varepsilon)\) is a psc manifold whose boundary, \( \Sigma \), has vanishing mean curvature. It follows that \((M_0, g_\varepsilon)\) is Yamabe positive. Meanwhile, since \( \Sigma \) is a closed stable-minimal hypersurface with trivial normal bundle in a psc manifold, the classical observation of Schoen-Yau \cite{SY79a, Theorem 1} implies that \((\Sigma, h')\) is Yamabe positive. This allows us to find a new psc metric \( \tilde{h} \in [h'] \). By Theorem 2, we may find a new psc metric \( \tilde{g}_0 \) on \( M_0 \) which has the product structure

\[ \tilde{g}_0 = \tilde{h} + dt^2 \]

near \( \Sigma \). Now we are prepared to begin attaching psc null-cobordisms to the boundary of \((M_0, \tilde{g}_0)\).

Consider the closed psc manifold \((\Sigma, \tilde{h})\) and the restriction \( f|_\Sigma : \Sigma \to S \) where \( f \) is the retraction \( f : U^{\eta_0} \to S \). By Theorem B, we can find a psc null-cobordism \((W, g_W)\) of \((\Sigma, \tilde{h})\) and an extension \( \overline{f} : W \to S \). We will require \( \overline{f} \) later. For now, we can form

\[ \overline{M} = M_0 \bigcup_{\Sigma} W, \quad \overline{g} = \begin{cases} \tilde{g}_0 & \text{on } M_0 \\ g_W & \text{on } W. \end{cases} \]

Notice that \( \overline{M} \) is a smooth closed oriented manifold and \( \overline{g} \) is a smooth psc metric.

**Claim 2.** There is a degree-1 map \( F : \overline{M} \to M \).

**Proof.** Though this map is more easily understood with a visual aid, see Figure 1, we describe it in some detail. Consider the region \( W' = A^\text{out} \cup W \subset \overline{M} \). Also consider
the annular region \( U_{\eta_2} \setminus U_{\eta_0} \) surrounding \( W' \), where \( \eta_2 > \eta_0 \) has been chosen so that the retraction \( f : U_{\eta_0} \to S \) extends to a retraction \( U_{\eta_2} \to S \). It follows that we can find a map
\[
p : \overline{U_{\eta_2} \setminus U_{\eta_0}} \to U_{\eta_2}
\]
which equals the identity on \( \partial U_{\eta_2} \) and equals this retraction on \( \partial U_{\eta_0} \). Now consider the continuous map
\[
F : \overline{M} \to M,
\]
\[
P = \begin{cases} 
\text{Id} & \text{on } \overline{M} \setminus (W' \cup (U_{\eta_2} \setminus U_{\eta_0})) \\
p & \text{on } \overline{U_{\eta_2} \setminus U_{\eta_0}} \\
f & \text{on } A^\text{out} \\
f & \text{on } W.
\end{cases}
\]

Since \( F \) equals the identity far away from \( W \), \( F \) is degree 1.

We observe that, inspecting the proof of Theorem 2 found in [AB02], the above metric \( \tilde{g}_0 \) can be chosen to be conformal to \( g_{\epsilon} \) away from a neighborhood containing \( \Sigma \) of any desired size. From this and our construction in Claim 2, it follows that one can arrange for \( F \) to be a conformal diffeomorphism away from \( F^{-1}(U_{\eta_2}) \). This finishes the proof of Theorem A.

5. Proof of Corollaries A and B

We are now ready to prove Corollaries A and B. **Proof of Corollary A:** Let \( M \) and \( S \) be as in Corollary A. For sake of contradiction, suppose there exists \( g \), an \( L^\infty(M) \cap C^\infty(M \setminus S) \) Riemannian metric with \( R_g \geq 0 \) and \( \text{Ric}_g \neq 0 \) on \( M \setminus S \). By making use of the analysis contained in [LM17], it suffices to obtain a contradiction under the extra assumption that \( R_g > 0 \) on \( M \setminus S \). Let us briefly explain this reduction. Since \( \text{Ric}_g \neq 0 \), one can make a small non-conformal perturbation to obtain a metric \( g' \) with the same regularity as \( g \) such that \( R_{g'} \geq 0 \) and \( R_{g'}(p) > 0 \) for some point \( p \in M \setminus S \), see the proof of [LM17, Theorem 1.7]. Then one can find a conformal transformation of \( g' \) to a metric \( g'' \) of the same regularity which has positive scalar curvature on \( M \setminus S \), see [LM17, Corollary 4.2]. Therefore, we will assume \( R_g > 0 \) on \( M \setminus S \).

By applying Theorem A, we obtain a smooth oriented psc manifold \((\overline{M}, \overline{g})\) with a degree-1 map to \( M \). According to Proposition 1, \( \overline{M} \) is enlargeable. However,
the existence of the psc metric $\bar{g}$ contradicts Theorem 1, completing the proof of Corollary A.

**Proof of Corollary B:** Let $(M, g)$ and $S$ be as in Corollary B. Notice that, as a smooth manifold, we may regard $M$ as a closed oriented manifold $M'$ with a point removed i.e. $M \cong M' \setminus \{x_0\}$.

For sake of contradiction, suppose that $m(M, g) < 0$. We will obtain a contradiction to Theorem A by adapting an argument first established by Lohkamp in [Loh99] to our low-regularity situation using some tools from [LM17]. We proceed in several steps. Our argument is very similar to the proof of [Loh99, Proposition 6.1], but we present it in order to point out the modifications necessary in our setting. Fix a compact set $K \subset M$ which contains the singular set $S$ and so that $M \setminus K$ is diffeomorphic to $\mathbb{R}^4 \setminus \overline{B_1^4}$ with the metric decay as in Definition 2.4. Let $r$ denote the radial coordinate on $\mathbb{R}^4 \setminus \overline{B_1^4}$.

The first step is to conformally change $g$ to a scalar-flat metric with negative mass. To begin, we may find a $W^{1,2}_{\text{loc}}(M)$ solution $u > 0$ satisfying

\[
\begin{cases}
  \mathcal{L}_g u = 0 \\
  \lim_{r \to \infty} u = 1,
\end{cases}
\]

see the proof of [LM17, Theorem 1.9]. It follows that $0 < u < 1$ and $u \in C^{0,\alpha}(M) \cap C^\infty(M \setminus S)$ by the maximum principle and standard elliptic estimates. According to [Bar86], $u$ has the asymptotic expansion

\[u(x) = 1 + Ar^{-2} + O(r^{-3})\]

where $A \leq 0$ is a constant. Assembling the above facts, the conformal metric $g_1 = u^2 g$ has the same regularity as $g$, is scalar-flat on $M \setminus S$, is still asymptotically flat, and has mass $m(M, g_1) = m(M, g) + A < 0$.

Next, we perturb $g_1$ to make the end conformally Euclidean while retaining the negativity of the mass. Consider a function $\psi : M \to \mathbb{R}$ such that $\psi \equiv 1$ on $K$, $\psi \equiv 0$ for $r$ sufficiently large in the asymptotic region, and $0 \leq \psi \leq 1$. Set $g_\psi = \psi g_1 + (1 - \psi)\delta$ where $\delta$ is the flat Euclidean metric on $M \setminus K$. By [Sch89, Proposition 4.1], $\psi$ can be chosen so that, upon finding a solution $u' > 0$ to

\[
\begin{cases}
  \mathcal{L}_{g_\psi} u' = 0 \\
  \lim_{r \to \infty} u' = 1,
\end{cases}
\]
the metric $g_2 = (u')^2 g_\psi$ is scalar-flat, asymptotically Euclidean, and still has negative mass. The metric $g_2$ has the same regularity as $g$ for the same reason $g_1$ had this regularity. Notice that $g_2$ is conformally flat on $\{r \geq R_0\} \subset (M \setminus K)$ for sufficiently large $R_0$. It follows that $g_2 = \phi^2 \delta$ on $\{r \geq R_0\}$ for some function $\phi : \mathbb{R}^4 \setminus B_{R_0}^1 \to \mathbb{R}$.

Now we will perturb $g_2$ to make it flat at infinity while at the same time making scalar curvature positive somewhere. Since $g_2$ is scalar-flat, $\phi$ is harmonic with respect to $\delta$ and tends to 1 at infinity. It follows that we may write

$$\phi = 1 + \frac{m(M, g_2)}{12r^2} + f$$

where $f$ is a function satisfying $|f| \leq Cr^{-3}$ for some constant $C$. By [Loh99, Lemma 6.2], in this setting one can explicitly construct a conformal factor $\phi' > 0$ so that

$$\begin{cases} 
\phi' \equiv \phi & \text{on } \{R_1 > 0\}^C \\
\phi' \equiv \text{const.} & \text{on } \{R_2 > 0\} \\
\Delta_\delta \phi' \leq 0 & \text{on } M \\
\Delta_\delta \phi(x) < 0 & \text{for some point } x \in M,
\end{cases}$$

where $R_1 < R_2$ are two radii larger than $R_0$.

Now, the metric $g_3 = (\phi')^2 g_\psi$ is flat on $\{R_2 > 0\}$, scalar non-negative everywhere, and not scalar-flat. The metric $g_3$ has the same regularity as $g_2$ since we have not altered it within $K$. By appropriately identifying opposing faces of the cubical complex $\partial([-R_2, R_2]^4)$ in $M \setminus K$, $g_3$ descends to a metric on $T^4 \#M'$. Since the map $T^4 \#M' \to T^4$ collapsing the $M'$ factor to a point has degree 1, Proposition 1 implies that $T^4 \#M'$ is enlargeable. Since $g_3$ is not scalar-flat, this contradicts Corollary A, completing the proof of Corollary B.

**Appendix A.**

In this section we will provide some details on the minimization problem found in Claim 1 during the proof of Theorem A. The argument we present here is only a slight modification of the proof of [LM17, Lemma 6.1].

Some notation: if $(M, g)$ is a Riemannian manifold and $k$ is some whole number, we write $\mathcal{H}^k_g$ for the $k$-dimensional Hausdorff measure associated with $g$.

**Lemma 2.** Suppose $g'$ is a smooth metric on $\mathbb{R}^2 \times S^2$ which satisfies

$$\lambda^{-1}(g_{\mathbb{R}^2} + g_{S^2}) \leq g' \leq \lambda(g_{\mathbb{R}^2} + g_{S^2})$$

(26)
for a positive constant $\lambda$. Then there exists a radius $R$ so that the minimization problem

$$V = \inf \{ \mathcal{H}_{g'}^3(\partial \Omega) : \Omega \subset \mathbb{R}^2 \times S^2 \text{ open, containing } B_1(0) \times S^2 \}$$

is solved by an open set lying within $B_R(0) \times S^2 \subset \mathbb{R}^2 \times S^2$.

**Proof.** A solution to problem (27) is known as an *outer minimizing hull* of the region $B_1(0) \times S^2$. According to a result of Fogagnolo-Mazzieri [FM22, Theorem 1.1], such an outward minimizing hull exists and is given by a bounded subset of $\mathbb{R}^2 \times S^2$ so long as $(\mathbb{R}^2 \times S^2, g')$ satisfies the following Euclidean isoperimetric inequality: there is a constant $C > 0$ so that

$$\mathcal{H}_g^4(\Omega) \leq C \mathcal{H}_{g'}^3(\partial \Omega)^{\frac{4}{3}}$$

for all bounded regions $\Omega \subset \mathbb{R}^2 \times S^2$ with smooth boundary. As such, we aim to establish (28).

Leveraging the uniformity assumption (26), it suffices to establish an Euclidean isoperimetric inequality for the product manifold $M_* := (\mathbb{R}^2 \times S^2, g_{\mathbb{R}^2} + g_{S^2})$. To this end, suppose we are given an open set $\Omega \subset M_*$ with smooth boundary. When $\Omega$ has sufficiently small volume, the inequality (28) follows from a classical argument using the fact that $M_*$ satisfies a Ricci curvature lower bound $\text{Ric}^{M_*} \geq -g$ and a noncollapsing condition $\text{Vol}_{M_*}(B_1(p)) \geq 1$. Namely, the result [Heb99, Lemma 3.2] implies there is an $\eta > 0$ and constant $C$ so that any $\Omega \subset M_*$ with $\mathcal{H}^4(\Omega) \leq \eta$ satisfies inequality (28). Now assume $\mathcal{H}^4(\Omega) \geq \eta$. The result [Gro99, Theorem 6.19] states that such $\Omega$ satisfy the same isoperimetric inequality as the one satisfied by the fundamental group of any manifold covered by $M_*$. Since $M_*$ covers manifolds with hyperbolic fundamental group, we conclude the existence of a $C' > 0$ such that the linear inequality $\mathcal{H}^4(\Omega) \leq C' \mathcal{H}^3(\partial \Omega)$ holds, from which the Euclidean inequality (28) quickly follows.

\[\square\]

**Conflict of interest statement**

The author states that there is no conflict of interest.
REFERENCES

[AB02] Kazuo Akutagawa and Boris Botvinnik. Manifolds of positive scalar curvature and conformal cobordism theory. *Math. Ann.*, 324(4):817–840, 2002.

[Bar86] Robert Bartnik. The mass of an asymptotically flat manifold. *Comm. Pure Appl. Math.*, 39(5):661–693, 1986.

[BERW17] Boris Botvinnik, Johannes Ebert, and Oscar Randal-Williams. Infinite loop spaces and positive scalar curvature. *Invent. Math.*, 209(3):749–835, 2017.

[Car88] Rodney Carr. Construction of manifolds of positive scalar curvature. *Trans. Amer. Math. Soc.*, 307(1):63–74, 1988.

[Chr86] Piotr Chruściel. Boundary conditions at spatial infinity from a Hamiltonian point of view. In *Topological properties and global structure of space-time (Erice, 1985)*, volume 138 of *NATO Adv. Sci. Inst. Ser. B Phys.*, pages 49–59. Plenum, New York, 1986.

[CL19] Alessandro Carlotto and Chao Li. Constrained deformations of positive scalar curvature metrics. arXiv:1903.11772, 2019.

[CS04] David M. J. Calderbank and Michael A. Singer. Einstein metrics and complex singularities. *Invent. Math.*, 156(2):405–443, 2004.

[CT18] S. Cecchini and Schick T. Enlargeable metrics on nonspin manifolds. arXiv:1810.02116, 2018.

[Dah97] Mattias Dahl. The positive mass theorem for ALE manifolds. In *Mathematics of gravitation, Part I (Warsaw, 1996)*, volume 41 of *Banach Center Publ.*, pages 133–142. Polish Acad. Sci. Inst. Math., Warsaw, 1997.

[Don15] Andrew Donald. Embedding Seifert manifolds in S^4. *Trans. Amer. Math. Soc.*, 367(1):559–595, 2015.

[Esc92] José F. Escobar. Conformal deformation of a Riemannian metric to a scalar flat metric with constant mean curvature on the boundary. *Ann. of Math. (2)*, 136(1):1–50, 1992.

[Esc96] José F. Escobar. Conformal deformation of a Riemannian metric to a constant scalar curvature metric with constant mean curvature on the boundary. *Indiana Univ. Math. J.*, 45(4):917–943, 1996.

[Fed69] Herbert Federer. *Geometric measure theory*. Die Grundlehren der mathematischen Wissenschaften, Band 153. Springer-Verlag New York Inc., New York, 1969.

[FM22] Mattia Fogagnolo and Lorenzo Mazzieri. Minimising hulls, p-capacity and isoperimetric inequality on complete Riemannian manifolds. *J. Funct. Anal.*, 283(9):Paper No. 109638, 49, 2022.

[GL80a] Mikhael Gromov and H. Blaine Lawson, Jr. The classification of simply connected manifolds of positive scalar curvature. *Ann. of Math. (2)*, 111(3):423–434, 1980.

[GL80b] Mikhael Gromov and H. Blaine Lawson, Jr. Spin and scalar curvature in the presence of a fundamental group. I. *Ann. of Math. (2)*, 111(2):209–230, 1980.

[GL83] Mikhael Gromov and H. Blaine Lawson, Jr. Positive scalar curvature and the Dirac operator on complete Riemannian manifolds. *Inst. Hautes Études Sci. Publ. Math.*, (58):83–196 (1984), 1983.
[Gro99] Misha Gromov. Metric structures for Riemannian and non-Riemannian spaces, volume 152 of Progress in Mathematics. Birkhäuser Boston, Inc., Boston, MA, 1999. Based on the 1981 French original [MR0682063 (85e:53051)], With appendices by M. Katz, P. Pansu and S. Semmes, Translated from the French by Sean Michael Bates.

[Gro14] Misha Gromov. Dirac and Plateau billiards in domains with corners. Cent. Eur. J. Math., 12(8):1109–1156, 2014.

[Gro18] Misha Gromov. Metric inequalities with scalar curvature. Geom. Funct. Anal., 28(3):645–726, 2018.

[Hat02] Allen Hatcher. Algebraic topology. Cambridge University Press, Cambridge, 2002.

[Haw77] S. W. Hawking. Gravitational instantons. Phys. Lett. A, 60(2):81–83, 1977.

[Heb99] Emmanuel Hebey. Nonlinear analysis on manifolds: Sobolev spaces and inequalities, volume 5 of Courant Lecture Notes in Mathematics. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 1999.

[Kro89a] P. B. Kronheimer. The construction of ALE spaces as hyper-Kähler quotients. J. Differential Geom., 29(3):665–683, 1989.

[Kro89b] P. B. Kronheimer. A Torelli-type theorem for gravitational instantons. J. Differential Geom., 29(3):685–697, 1989.

[LeB88] Claude LeBrun. Counter-examples to the generalized positive action conjecture. Comm. Math. Phys., 118(4):591–596, 1988.

[LL15] Dan A. Lee and Philippe G. LeFloch. The positive mass theorem for manifolds with distributional curvature. Comm. Math. Phys., 339(1):99–120, 2015.

[LM17] Chao Li and Christos Mantoulidis. Positive scalar curvature with skeleton singularities. 2017.

[Loh99] Joachim Lohkamp. Scalar curvature and hammocks. Math. Ann., 313(3):385–407, 1999.

[Loh16] J. Lohkamp. The higher dimensional positive mass theorem ii. arXiv:1612.07505, 2016.

[LSW63] W. Littman, G. Stampacchia, and H. F. Weinberger. Regular points for elliptic equations with discontinuous coefficients. Ann. Scuola Norm. Sup. Pisa (3), 17:43–77, 1963.

[LV19] Michael T. Lock and Jeff A. Viaclovsky. A smörgåsbord of scalar-flat Kähler ALE surfaces. J. Reine Angew. Math., 746:171–208, 2019.

[Mar12] Fernando Codá Marques. Deforming three-manifolds with positive scalar curvature. Ann. of Math. (2), 176(2):815–863, 2012.

[MS15] Christos Mantoulidis and Richard Schoen. On the Bartnik mass of apparent horizons. Classical Quantum Gravity, 32(20):205002, 16, 2015.

[Per02] Grisha Perelman. The entropy formula for the ricci flow and its geometric applications. arXiv:math/0211159, 2002.

[Per03a] Grisha Perelman. Finite extinction time for the solutions to the ricci flow on certain three-manifolds. arXiv:math/0307245, 2003.

[Per03b] Grisha Perelman. Ricci flow with surgery on three-manifolds. arXiv:math/0303109, 2003.
[Roh51] V. A. Rohlin. A three-dimensional manifold is the boundary of a four-dimensional one. 
*Doklady Akad. Nauk SSSR (N.S.),* 81:355–357, 1951.

[Ros07] Jonathan Rosenberg. Manifolds of positive scalar curvature: a progress report. In *Surveys in differential geometry. Vol. XI,* volume 11 of *Surv. Differ. Geom.*, pages 259–294. Int. Press, Somerville, MA, 2007.

[Rub01] Daniel Ruberman. Positive scalar curvature, diffeomorphisms and the Seiberg-Witten invariants. *Geom. Topol.*, 5:895–924, 2001.

[Sch89] Richard M. Schoen. Variational theory for the total scalar curvature functional for Riemannian metrics and related topics. In *Topics in calculus of variations (Montecatini Terme, 1987),* volume 1365 of *Lecture Notes in Math.*, pages 120–154. Springer, Berlin, 1989.

[Sco83] Peter Scott. The geometries of 3-manifolds. *Bull. London Math. Soc.*, 15(5):401–487, 1983.

[SY79a] R. Schoen and S. T. Yau. On the structure of manifolds with positive scalar curvature. *Manuscripta Math.*, 28(1-3):159–183, 1979.

[SY79b] R. Schoen and Shing Tung Yau. Existence of incompressible minimal surfaces and the topology of three-dimensional manifolds with nonnegative scalar curvature. *Ann. of Math. (2),* 110(1):127–142, 1979.

[SY79c] Richard Schoen and Shing Tung Yau. On the proof of the positive mass conjecture in general relativity. *Comm. Math. Phys.*, 65(1):45–76, 1979.

[SY17] Richard Schoen and Shing-Tung Yau. Positive scalar curvature and minimal hypersurface singularities. arXiv:1704.05490, 2017.

[Tho54] René Thom. Quelques propriétés globales des variétés différentiables. *Comment. Math. Helv.*, 28:17–86, 1954.

[Wal14] Mark Walsh. The space of positive scalar curvature metrics on a manifold with boundary. arXiv:1411.2423, 2014.

*Department of Mathematics, Duke University, Durham, NC, 27708, USA*

*Email address: demetre.kazaras@duke.edu*