Abstract

We consider gauge theories with scalar matter with and without supersymmetry at nonzero chemical potential. It is argued that a chemical potential plays a role similar to the FI term. We analyze theory at weak coupling regime at large chemical potential and argue that it supports nonabelian non-BPS strings. Worldsheet theory on the nonabelian string in a dense matter is briefly discussed.
1 Introduction

Recent studies in QCD provide a rich phase diagram involving color superconductivity \cite{1} and color-flavor locking phases at high density (see, for review \cite{2}). High density region due to the asymptotic freedom is treatable perturbatively and it is natural to use this lucky possibility to the full extent. In particular it is interesting to investigate if the extended objects like strings and domain walls exist in the theory with chemical potential. The example of such object whose very existence is ultimately related to the presence of a non-vanishing density has been found. Indeed the tension of the domain wall discussed in \cite{3} (see also \cite{4,5}) is proportional to the chemical potential hence it admits a quasiclassical description at high density.

On the other hand the gauge/string duality provides the description of the SUSY gauge theory at strong coupling regime. To discuss the dense matter along this way it is necessary to determine the background for the closed string which would encode a non-vanishing chemical potential. This background has been found in \cite{6} and it was demonstrated \cite{7,8} that the gravity description clearly indicates the deconfinement phase transition at large enough chemical potential. Similar studies have been done within the holographic description of QCD-like theory in Sakai-Sugimoto model \cite{9}. Some other aspects of the dense SUSY gauge theories has been discussed in $\mathcal{N} = 1$ \cite{10} and $\mathcal{N} = 2$ \cite{11} cases.

In this letter we shall search for the extended objects in a dense matter with and without SUSY. Our main goal shall be investigation of the nonabelian strings found recently in softly broken $\mathcal{N} = 2$ SYM theories with fundamental matter \cite{13,12}. They exist at weak coupling regime and therefore are under theoretical control. The nonabelian strings exist in $\mathcal{N} = 0$ \cite{14} gauge theories with the scalar matter as well (see \cite{17} for the reviews). Moreover there are strong indications that they provide the proper pattern for the explanation of the nonperturbative lattice QCD data \cite{18}. In $\mathcal{N} = 2$ case nonabelian strings are BPS objects keeping some amount of SUSY which protects their stability. In theories with less amount of SUSY this argument is lost nevertheless one can discuss the nonabelian strings at weak coupling regime on the firm ground. The theory on the string worldsheet can be analyzed explicitly, in particular the spectrum of the worldsheet theory involves kinks or kink-antikink pairs which can be identified with monopole or monopole-antimonopole pairs from the 4d bulk viewpoint \cite{15,16}.

The question under investigation in this letter concerns the existence of nonabelian strings if the chemical potential is switched on. It turns out that the classical nonabelian string solution does exist in a weakly coupled dense matter in the theory with large Fayet-Iliopoulos (FI) term $\xi$ and/or chemical potential $\mu$. Moreover strings still exist if we switch off FI term at all. We shall consider strings both in non-SUSY and SUSY theories and argue that the worldsheet theory on the nonabelian string is non-supersymmetric. Similar to the previous analysis \cite{15,16} it can be shown that at least at large $\xi$ and small $\mu$ there are kinks in the bosonic action on the nonabelian string which can be identified with the monopoles from the four-dimensional perspective. Since worldvolume theory is non-supersymmetric kinks are in the confining phase.

The paper is organized as follows. In Section 2 we discuss the non-SUSY gauge model with the scalar matter. It is shown that chemical potential $\mu$ for U(1) charge plays the role
similar to the FI term $\xi$ that is large $\mu$ limit supports the semiclassical nonabelian strings. In Section 3 we consider the softly broken SUSY models with the chemical potential. Section 4 concerns the comments on the worldsheet theory and the last Section is devoted to the conclusion.

2 Non-supersymmetric model

Here we discuss the simplest model which can be used to analyze nonabelian strings. The gauge group of the model is SU($N$)×U(1). The model contains SU($N$) and U(1) gauge bosons and $N$ scalar fields charged with respect to U(1) which form $N$ fundamental representations of SU($N$). It is convenient to write these fields in the form of $N \times N$ matrix $\Phi = \{\varphi^{kA}\}$ where $k$ is the SU($N$) gauge index while $A$ is the flavor index,

$$
\Phi = 
\begin{pmatrix}
\varphi^{11} & \varphi^{12} & \cdots & \varphi^{1N} \\
\varphi^{21} & \varphi^{22} & \cdots & \varphi^{2N} \\
\vdots & \vdots & \ddots & \vdots \\
\varphi^{N1} & \varphi^{N2} & \cdots & \varphi^{NN}
\end{pmatrix}.
$$

(1)

The action of the model reads as

$$
S = \int d^4x \left\{ -\frac{1}{4g_2^2} (F_{\mu\nu})^2 - \frac{1}{4g_1^2} (F_{\mu\nu})^2 \\
+ \text{Tr} (\nabla_{\mu} \Phi)^\dagger (\nabla^\mu \Phi) - \frac{g_2^2}{2} [\text{Tr} (\Phi^\dagger T^a \Phi)]^2 - \frac{g_1^2}{8} \left[ \text{Tr} (\Phi^\dagger \Phi) - N\xi \right]^2 \\
+ \frac{i \theta}{32 \pi^2} F_{\mu\nu}^{\alpha} \tilde{F}^{\alpha\mu\nu} \right\},
$$

(2)

where $T^a$ stands for the generator of the gauge SU($N$),

$$
\nabla_{\mu} \Phi \equiv \left( \partial_{\mu} - \frac{i}{\sqrt{2N}} A_{\mu} - iA_{\mu}^a T^a \right) \Phi,
$$

(3)

and $\theta$ is the vacuum angle. The last term in the second line forces $\Phi$ to develop a vacuum expectation value (VEV) while the last but one term forces the VEV to be diagonal,

$$
\Phi_{\text{vac}} = \sqrt{\xi} \text{diag} \{1, 1, ..., 1\}.
$$

(4)

We assume the FI parameter $\xi$ to be large,

$$
\sqrt{\xi} \gg \Lambda_4,
$$

(5)

where $\Lambda_4$ is the scale of the four-dimensional theory (2). This ensures the weak coupling regime as both couplings $g_2^2$ and $g_1^2$ are frozen at a large scale. The vacuum field (4) results
in the spontaneous breaking of both gauge and flavor SU($N$)'s. A diagonal global SU($N$) survives
\[ U(N)_{\text{gauge}} \times SU(N)_{\text{flavor}} \rightarrow SU(N)_{\text{diag}} \] (6)
yielding color-flavor locking phase in the vacuum.

The inclusion of the chemical potential into the relativistic invariant theory goes as follows [19]. The covariant derivative for scalars gets shifted by the term $i\mu\delta_{\nu0}$ which yields the total potential in the theory
\[
V = \frac{g_2}{2} \left[ \text{Tr} \left( \Phi^\dagger T^a \Phi \right) \right]^2 + \frac{g_2^2}{8} \left[ \text{Tr} \left( \Phi^\dagger \Phi \right) - N\xi \right]^2 - \mu^2 \text{Tr} \left( \Phi^\dagger \Phi \right)
\] (7)
It is clear that chemical potential provides the same symmetry breaking pattern as FI term and at large chemical potential theory is at weak coupling regime hence we get the effective FI term
\[
\xi_{\text{eff}} = \xi + \frac{4\mu^2}{Ng_f^2}
\] (8)

The topological argument providing the stability of the string involves the combination of the $Z_N$ center of SU($N$) with the elements $\exp(2\pi ik/N) \in U(1)$ A topologically stable string solution possesses both windings, in SU($N$) and U(1). In other words,
\[
\pi_1 \left( SU(N) \times U(1)/Z_N \right) \neq 0.
\] (9)
and this nontrivial topology amounts to winding of just one element of $\Phi_{\text{vac}}$, for instance,
\[
\Phi_{\text{string}} = \sqrt{\xi_{\text{eff}}} \text{diag}(1, 1, \ldots, e^{i\alpha(x)}), \quad x \rightarrow \infty.
\] (10)
Such strings can be called elementary, and the ANO string can be viewed as a bound state of $N$ elementary strings.

In the discussion of the vacuum structure in the presence of chemical potential it was implicitly assumed that $A_\mu = 0$ in the vacuum. However this assumption would violate the Gauss law unless we add a source term $J_\mu A_\mu$ with constant background charge density $J_\mu = J_0 \delta_{\mu0}$ [19]. The necessary value of the background charge density $J_0$ is determined by requiring $A_\mu = 0$ in the vacuum. In the other way, we can induce a chemical potential without changing the covariant derivatives by adding a term $J_\mu A_\mu$ to the Lagrangian and considering the induced vev $A_0 = \mu N$ as a chemical potential. The value of $\mu$-induced is then determined from the Gauss law,
\[
\mu \left( \xi + \frac{4\mu^2}{Ng_f^2} \right) = -\sqrt{\frac{N}{2}}J_0.
\] (11)
In the following we shall use this second way of adding the chemical potential.

The nonabelian string in fact is the twisted $Z_N$ string hence let us first describe $Z_N$ solution which can be written as follows [13]:
\[
\Phi = \begin{pmatrix}
\phi(r) & 0 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots \\
0 & \ldots & \phi(r) & 0 \\
0 & 0 & \ldots & e^{i\alpha\phi_N(r)}
\end{pmatrix},
\]
\[ A_{i}^{SU(N)} = \frac{1}{N} \begin{pmatrix} 1 & \ldots & 0 & 0 \\ \vdots & \ldots & \ldots & \ldots \\ 0 & \ldots & 1 & 0 \\ 0 & 0 & \ldots & -(N-1) \end{pmatrix}(\partial_{i}\alpha)[-1 + f_{NA}(r)], \]

\[ A_{0}^{SU(N)} = -\frac{1}{N}g_{NA} \begin{pmatrix} 1 & \ldots & 0 & 0 \\ \vdots & \ldots & \ldots & \ldots \\ 0 & \ldots & 1 & 0 \\ 0 & 0 & \ldots & -(N-1) \end{pmatrix}, \]

\[ A_{i}^{U(1)} = \sqrt{\frac{2}{N}}(\partial_{i}\alpha)[1 - f(r)], \]

\[ A_{0}^{U(1)} = \sqrt{\frac{2}{N}}g, \] (12)

where \( i = 1, 2 \) labels coordinates in the plane orthogonal to the string axis and \( r \) and \( \alpha \) are the polar coordinates in this plane. The profile functions \( \phi(r) \) and \( \phi_{N}(r) \) determine the profiles of the scalar fields, while \( f_{NA}(r), g_{NA}(r) \) and \( f(r), g(r) \) determine the SU(\( N \)) and U(1) fields of the string solutions, respectively.

The equations for profile functions follow from the equations of motion,

\[ \frac{1}{g_{1}^{2}} \left( g'' + \frac{g'}{r} \right) = \frac{1}{N} \left( (N-1)(g - g_{NA})\phi^{2} + (g + (N-1)g_{NA})\phi_{N}^{2} \right) + \sqrt{\frac{N}{2}}J_{0} \]

\[ \frac{1}{g_{2}^{2}} \left( g_{NA}'' + \frac{g_{NA}'}{r} \right) = \frac{1}{N} \left( -(g - g_{NA})\phi^{2} + (g + (N-1)g_{NA})\phi_{N}^{2} \right) \]

\[ \frac{r}{g_{1}^{2}} \left( \frac{f'}{r} \right) = \frac{1}{N} \left( (N-1)(f - f_{NA})\phi^{2} + (f + (N-1)f_{NA})\phi_{N}^{2} \right) \]

\[ \frac{r}{g_{2}^{2}} \left( \frac{f_{NA}'}{r} \right) = \frac{1}{N} \left( -(f - f_{NA})\phi^{2} + (f + (N-1)f_{NA})\phi_{N}^{2} \right) \]

\[ \phi'' + \frac{1}{r}\phi' = \left[ \frac{1}{N^{2}} \left( \frac{1}{r^{2}}(f - f_{NA})^{2} - (g - g_{NA})^{2} \right) + \frac{g_{1}^{2}}{4}((N-1)\phi^{2} + \phi_{N}^{2} - N\xi) + \frac{g_{2}^{2}}{2} \frac{1}{N}(\phi^{2} - \phi_{N}^{2}) \right] \phi \]

\[ \phi_{N}'' + \frac{1}{r}\phi_{N}' = \left[ \frac{1}{N^{2}} \left( \frac{1}{r^{2}}(f + (N-1)f_{NA})^{2} - (g + (N-1)g_{NA})^{2} \right) + \frac{g_{1}^{2}}{4}((N-1)\phi^{2} + \phi_{N}^{2} - N\xi) + \frac{g_{2}^{2}}{2} \frac{1}{N}(\phi^{2} - \phi_{N}^{2}) \right] \phi_{N} \] (13)
These functions obey the following boundary conditions:

\[
\phi_N(0) = 0, \quad \phi'(0) = 0, \\
f_{NA}(0) = 1, \quad f(0) = 1, \\
g'(0) = 0, \quad g'_{NA}(0) = 0,
\]

at \( r = 0 \), and

\[
\phi_N(\infty) = \sqrt{\xi_{\text{eff}}}, \quad \phi(\infty) = \sqrt{\xi_{\text{eff}}}, \\
f_{NA}(\infty) = g_{NA}(\infty) = 0, \quad f(\infty) = 0, \quad g(\infty) = \mu N,
\]

at \( r = \infty \). The boundary conditions for the derivatives were added in order to exclude solutions that behave as logarithms at \( r = 0 \). The tension of this string can be calculated by substituting the expressions for the fields into the energy functional, and at large \( \xi_{\text{eff}} \) it behaves as \( \sqrt{\xi_{\text{eff}}} \).

To obtain the non-Abelian string solution from the \( Z_N \) string \([12]\) we apply the diagonal color-flavor rotation preserving the vacuum \([11]\). To this end it is convenient to pass to the singular gauge where the scalar fields have no winding at infinity, while the string flux comes from the vicinity of the origin. In this gauge we have

\[
\Phi = U \begin{pmatrix} 
\phi(r) & 0 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots \\
0 & \ldots & \phi(r) & 0 \\
0 & 0 & \ldots & \phi_N(r)
\end{pmatrix} U^{-1},
\]

\[
A^{SU(N)}_i = \frac{1}{N} U \begin{pmatrix} 1 & \ldots & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots \\
0 & \ldots & 1 & 0 \\
0 & 0 & \ldots & -(N-1)
\end{pmatrix} U^{-1} (\partial_i \alpha) f_{NA}(r),
\]

\[
A^{SU(N)}_0 = -\frac{1}{N} g_{NA} \begin{pmatrix} 1 & \ldots & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots \\
0 & \ldots & 1 & 0 \\
0 & 0 & \ldots & -(N-1)
\end{pmatrix},
\]

\[
A^{U(1)}_i = -\sqrt{\frac{2}{N}} (\partial_i \alpha) f(r),
\]

\[
A^{U(1)}_0 = \sqrt{\frac{2}{N}} g,
\]

at \( r = 0 \), and

\[
\phi_N(\infty) = \sqrt{\xi_{\text{eff}}}, \quad \phi'(\infty) = \sqrt{\xi_{\text{eff}}}, \\
f_{NA}(\infty) = g_{NA}(\infty) = 0, \quad f(\infty) = 0, \quad g(\infty) = \mu N,
\]

at \( r = \infty \). The boundary conditions for the derivatives were added in order to exclude solutions that behave as logarithms at \( r = 0 \). The tension of this string can be calculated by substituting the expressions for the fields into the energy functional, and at large \( \xi_{\text{eff}} \) it behaves as \( \sqrt{\xi_{\text{eff}}} \).
where $U$ is a matrix $\in SU(N)$. This matrix parameterizes orientational zero modes of the string associated with flux rotation in $SU(N)$. The orientational zero modes of a non-Abelian string were first observed in [13, 12]. Note that the nonabelian strings are not BPS objects and their stability is ensured by topological arguments. Moreover it is clear from the solutions to the equations of motion that they are charged with respect to $U(1)$ field.

3 SUSY model

In this Section we shall consider the vacuum structure of the SUSY theory with nontrivial chemical potential. In what follows we shall focus on the softly broken $\mathcal{N} = 2$ SUSY QCD with $N_f = N_c$ flavors. The gauge group is $SU(2) \times U(1)$ with different couplings for abelian and nonabelian parts and we add FI term for the $U(1)$ factor.

$$
\mathcal{L} = -\frac{1}{4g_2^2} (F^a_{\mu\nu})^2 - \frac{1}{4g_1^2} (F^0_{\mu\nu})^2 + \frac{1}{g_2^2} |D_\mu a^a|^2 + \frac{1}{g_1^2} |\partial_\mu a|^2 \\
+ |\nabla_\mu q^A|^2 + |\nabla_\mu \bar{q}^A|^2 \\
+ J_0 A_0 \\
- \frac{g_2^2}{2} \left( \frac{1}{g_2^2} \varepsilon^{abc} \bar{a}^b a^c + \bar{q}_A \frac{\tau^a}{2} q^A - \bar{q}_A \frac{\tau^a}{2} \bar{q}^A \right)^2 \\
- \frac{g_2^2}{8} |\bar{q}_A q^A - \bar{q}_A \bar{q}^A|^2 \\
- \frac{g_2^2}{2} |\bar{q}_A \tau^a q^A|^2 - \frac{g_1^2}{2} |\bar{q}_A q^A - \xi F|^2 \\
- \frac{1}{2} \sum_{A=1}^2 \left\{ |(a + \sqrt{2} m_A + \tau^a a^a) q^A|^2 + |(a + \sqrt{2} m_A + \tau^a a^a) \bar{q}_A|^2 \right\} + \text{fermions} \quad (17)
$$

The fields $A_\mu$, $a$ and $A^a_\mu$, $a^a$ belong to abelian and non-abelian $\mathcal{N} = 2$ gauge supermultiplets respectively, the fields $q^{kA}$ and $\bar{q}_{Ak}$ represent the matter hypermultiplets. Here $k = 1, 2$ is a color index and $A = 1, 2$ is a flavor index. The induced chemical potential for $U(1)$ charge is introduced by adding the source term $A_0 J_0$.

To determine the vacuum of the model consider the bosonic potential which in the absence of gauge fields looks as follows

$$
V = \frac{g_2^2}{2} \left( \frac{1}{g_2^2} \varepsilon^{abc} \bar{a}^b a^c + \bar{q}_A \frac{\tau^a}{2} q^A - \bar{q}_A \frac{\tau^a}{2} \bar{q}^A \right)^2 + \frac{g_2^2}{2} |\bar{q}_A \tau^a q^A|^2 \\
+ |(a + \sqrt{2} m_A + \tau^a a^a) q^A|^2 + |(a + \sqrt{2} m_A + \tau^a a^a) \bar{q}_A|^2 \\
+ \frac{g_2^2}{2} (\bar{q}_A q^A - \bar{q}_A \bar{q}^A)^2 + \frac{g_2^2}{2} |\bar{q}_A q^A - \xi|^2 - \frac{\mu^2}{4} (\bar{q}_A q^A + \bar{q}_A \bar{q}^A). \quad (18)
$$
Here we stated explicitly the term with vev of $A_0$ that comes from the kinetic terms for squarks. Let us minimize the expression in the third line $V_1$:

$$V_1 = \frac{g_1^2}{8} (\bar{q}_A q^A - \bar{q}_A \tilde{q}^A)^2 + \frac{g_1^2}{2} (\bar{q}_A q^A - \xi)^2 - \frac{\mu^2}{4} (\bar{q}_A q^A + \tilde{q}_A \tilde{q}^A).$$  \hspace{1cm} (19)$$

Upon differentiation over $q^{kB}$ and $\tilde{q}^{kB}$, we get

$$\left[ \frac{g_1^2}{4} (\bar{q}_A q^A - \bar{q}_A \tilde{q}^A) - \frac{\mu^2}{4} \right] \bar{q}_{Bk} = - \frac{g_1^2}{2} (\bar{q}_A \tilde{q}^A - \xi) \tilde{q}_{Bk}$$

$$\frac{g_1^2}{2} (\bar{q}_A q^A - \xi) \tilde{q}_{Bk} = \left[ \frac{g_1^2}{4} (\bar{q}_A q^A - \bar{q}_A \tilde{q}^A) + \frac{\mu^2}{4} \right] \bar{q}_{Bk}. \hspace{1cm} (20)$$

It is clear that $\bar{q}_Ak$ and $\tilde{q}_Ak$ are proportional at minimum. Let $q^{kA} = \frac{1}{\sqrt{2}} q \cdot A^{kA}$, $\tilde{q}^{kA} = \frac{1}{\sqrt{2}} \tilde{q} \cdot A^{kA}$, where $q$ and $\tilde{q}$ — complex variables and

$$\text{Tr} \ A^\dagger A = 2. \hspace{1cm} (21)$$

$$\left[ \frac{g_1^2}{4} (\bar{q} q - \bar{q} \tilde{q}) - \frac{\mu^2}{4} \right] \bar{q} = - \frac{g_1^2}{2} (\bar{q} \tilde{q} - \xi) \tilde{q}$$

$$\frac{g_1^2}{2} (\bar{q} q - \xi) \tilde{q} = \left[ \frac{g_1^2}{4} (\bar{q} q - \bar{q} \tilde{q}) + \frac{\mu^2}{4} \right] \bar{q}. \hspace{1cm} (22)$$

$$(q - \tilde{q}) \cdot (q^2 + \tilde{q}^2 + 2\xi - \frac{\mu^2}{g_1^2}) = 0. \hspace{1cm} (23)$$

At small $\mu$ there is unique real solution $q = \tilde{q}$ and we derive

$$q = \tilde{q} = \sqrt{\xi + \frac{\mu^2}{2g_1^2}}. \hspace{1cm} (24)$$

Another solution to (20) $q = \tilde{q} = 0$, does not provide minimum. If $m_1 = m_2$, with unitary $A$, and $a^a = 0$, $a = -\sqrt{2}m$, we get absolute minimum of $V$ while in the case $m_1 \neq m_2$,

$$A = 1, \ a^3 = \frac{m_2 - m_1}{\sqrt{2}}, \ a = -\frac{m_1 + m_2}{\sqrt{2}}. \hspace{1cm} (25)$$

When $\mu$ is large enough the second real solution to the extremum equation emerges. Namely $q = -\tilde{q}$ is possible however a quick inspection shows that it corresponds to the metastable state, provided that $\xi \neq 0$. The diagonalization of the matrix of the quadratic fluctuations yields all positive eigenvalues that is we consider the minimum of the potential indeed.

The case of $\xi = 0$ needs for a special care. The point is that for $\xi = 0$ the two branches of the solution join to a vacuum valley

$$q = e^{i\beta} \tilde{q} = \sqrt{\frac{\mu^2}{2g_1^2}}, \hspace{1cm} (26)$$
where $\beta$ is an arbitrary constant phase. If the $\xi \neq 0$ there is nontrivial sin-Gordon potential for this moduli field however at large $\xi$ it is frozen at the minimum of the potential and becomes non-dynamical. At small non-vanishing $\xi$ this field should be taken into account in the low-energy approximation.

Substituting the equation $q^{kA} = \tilde{q}^{kA}$ into the Lagrangian one can see that its bosonic part coincides with the Lagrangian of our non-SUSY model for $N = 2$ and $\Phi = \sqrt{2}q$ provided that the quark masses are equal. So the solution for string can be expressed in terms of the profile functions

\[
q^{kA} = \frac{1}{\sqrt{2}} U \begin{pmatrix} \phi(r) & 0 \\ 0 & \phi_N(r) \end{pmatrix} U^{-1},
\]

\[
A_{i}^{SU(2)} = \frac{1}{2} U \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} U^{-1} (\partial_i \alpha) f_{NA}(r),
\]

\[
A_{0}^{SU(2)} = -\frac{1}{2} g_{NA}(r) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

\[
A_{i}^{U(1)} = - (\partial_i \alpha) f(r),
\]

\[
A_{0}^{U(1)} = g(r),
\]

which satisfy the equations (13) for $N = 2$,

\[
\left( d_r^2 + \frac{1}{r} d_r - \frac{1}{4r^2} (f - f_{NA})^2 + \frac{1}{4} (g - g_{NA})^2 - \frac{g_1^2}{4} (\phi^2 + \phi_N^2 - 2\xi) - \frac{g_2^2}{4} (\phi^2 - \phi_N^2) \right) \phi = 0,
\]

\[
\left( d_r^2 + \frac{1}{r} d_r - \frac{1}{4r^2} (f + f_{NA})^2 + \frac{1}{4} (g + g_{NA})^2 - \frac{g_1^2}{4} (\phi^2 + \phi_N^2 - 2\xi) + \frac{g_2^2}{4} (\phi^2 - \phi_N^2) \right) \phi_N = 0,
\]

\[
d_r \left( \frac{f'}{r} \right) = \frac{g_1^2}{2r} \left( \phi^2 (f - f_{NA}) + \phi_N^2 (f + f_{NA}) \right),
\]

\[
d_r \left( \frac{f'_{NA}}{r} \right) = \frac{g_2^2}{2r} \left( -\phi^2 (f - f_{NA}) + \phi_N^2 (f + f_{NA}) \right),
\]

\[
\left( d_r^2 + \frac{1}{r} d_r \right) g_{NA} = \frac{g_2^2}{2} \left( -\phi^2 (g - g_{NA}) + \phi_N^2 (g + g_{NA}) \right),
\]

\[
\left( d_r^2 + \frac{1}{r} d_r \right) g = \frac{g_1^2}{2} \left( \phi^2 (g - g_{NA}) + \phi_N^2 (g + g_{NA}) \right) + g_1^2 J_0.
\]

Instead of matrix $U$ it will be more convenient to use moduli $n^a$ defined as follows,

\[
U r^3 U^{-1} = \vec{n} \vec{\tau}, \quad \vec{n} \vec{\tau} \equiv \sum_a n^a \tau^a,
\]

\[
U r^3 U^{-1} = \vec{n} \vec{\tau}, \quad \vec{n} \vec{\tau} \equiv \sum_a n^a \tau^a,
\]
where $\tau^a$ are Pauli matrices. Then the string solution can be written as follows

$$q = \tilde{q} = \frac{1}{\sqrt{2}}(\phi(r) + \phi_N(r) + (\phi(r) - \phi_N(r))(\vec{n}\vec{\tau})),$$

$$A_i = \epsilon_{ij}\frac{x_j}{r^2}f(r),$$

$$A_i^a = -n^a\epsilon_{ij}\frac{x_j}{r^2}f_{NA}(r),$$

$$A_0 = g(r),$$

$$A_0^a = -n^ag_{NA}(r). \quad (30)$$

The equation for the profile functions can be solved numerically while the topological argument providing the stability is the same both for SUSY and non-SUSY cases. Let us comment on the central charges in the theory with the chemical potential. The BPS nonabelian string saturates the stringy central charge [20] in $N = 1$ theory with matter. If chemical potential is added the canonical momentum gets modified and the anticommutators of supercharges are modified as well. It can be shown that the nonabelian string in this theory does not saturate the modified central charge hence the equations of motion can not be reduced to the first order ones.

## 4 Worldsheet theory

In this Section we shall comment on the worldsheet theory of the nonabelian string in a dense matter. To get the worldsheet Lagrangian we consider the nonabelian moduli $n^a$ as functions of $x_k = (t, z)$ and substitute nonabelian string solution into the initial microscopic Lagrangian, following the standard procedure [16]. The substitution for fields looks like

$$q = \frac{1}{\sqrt{2}}(\phi_1(r) + \phi_2(r) + (\phi_1(r) - \phi_2(r))(\vec{n}\vec{\tau})),$$

$$\tilde{q} = e^{i\beta}q,$$

$$A_i = \epsilon_{ij}\frac{x_j}{r^2}f(r),$$

$$A_i^a = -n^a\epsilon_{ij}\frac{x_j}{r^2}f_{NA}(r),$$

$$A_0 = g(r),$$

$$A_0^a = -n^ag_{NA}(r) - \rho \epsilon^{abc}n^b\partial_0n^c, A_3^a = -\rho \epsilon^{abc}n^b\partial_3n^c. \quad (31)$$

Here we also want to take in account the quasimodulus $\beta(x_{\mu})$ which becomes a bona fide vacuum modulus at $\xi = 0$. The function $\rho(r)$ is an auxiliary field that should be later eliminated from the Lagrangian by its equation of motion.

After some calculations, we arrive at the $CP^1$ sigma model for $n^a$ fields perturbed by a term that is actually proportional to the chemical potential and a sine-Gordon theory for $\beta$. 


\[
\mathcal{L}_2 = \frac{1}{2}((\partial_0 n)^2 - (\partial_3 n)^2) \cdot \left[ \frac{2\pi}{g_2^2} \int r \, dr \left( \frac{r^2 N}{r^2} (1 - \rho)^2 + (\rho')^2 + g_2^2((\phi^2 + \phi_N^2) \frac{\rho^2}{2} + (\phi - \phi_N)(1 - \rho)) \right) \right] \\
+ \frac{1}{2}(\partial_3 n)^2 \cdot \frac{2\pi}{g_2^2} \left[ \int r \, dr g_N^2 (1 - \rho)^2 \right] \\
+ \frac{1}{2} \int 2\pi r \, dr \left[ (\partial_\nu \beta)^2 - g_1^2 g_N \xi \cos \beta \right] \cdot \left[ (\phi_1^2 + \phi_2^2) \right] \\
+ \frac{i}{2} \int 2\pi r \, dr \, \partial_0 \beta \cdot \left[ g(\phi_1^2 + \phi_2^2) - g_N(\phi_1^2 - \phi_2^2) \right] \\
- \frac{i}{2} \int d^2 x \, \partial_i \beta \cdot \epsilon_{ij} \frac{x_j}{r^2} \left[ f(\phi_1^2 + \phi_2^2) - f_N(\phi_1^2 - \phi_2^2) \right],
\]

(32)

where \(\rho(r)\) should be substituted from its equation of motion.

Let us emphasize the difference between the orientational moduli in the worldsheet action and \(\beta\)-dependent terms. The orientational moduli are purely two-dimensional fields while \(\beta\)-field is essentially four-dimensional. We have written this field in the worldsheet action implying that it is projected on the worldsheet but in general situation 2d integral should be substituted by 4d space-time integral. In other words the quanta of \(\beta\) field which have mass of order \(g_1 \sqrt{\xi}\) can escape string worldsheet and propagate in the bulk. They have the natural interpretation as the superpartners of the Higgsed photon.

The two-dimensional \(CP(N-1)\) model is an effective low-energy theory relevant for the description of internal string dynamics at low energies, much lower than the inverse thickness of the string which, in turn, is given by \(g_2 \sqrt{\xi_{\text{eff}}}\). Thus, \(g_2 \sqrt{\xi_{\text{eff}}}\) plays the role of a physical ultraviolet cutoff. and

\[
\Lambda_{CP(N-1)}^N = g_2^N \frac{\xi_{\text{eff}}^{N/2}}{\sqrt{\xi}} e^{-\frac{\Lambda^2}{3}}.
\]

(33)

Note that in the bulk theory, due to the VEV’s of the squark fields, the coupling constant is frozen at \(g_2 \sqrt{\xi_{\text{eff}}}\).

The worldsheet theory is non-supersymmetric \(\sigma\)-model which has single vacuum state and which spectrum consists of kink-antikink bound states [21]. This claim is certainly true at small chemical potential when we obtain the perturbed \(\sigma\) -model. In this limit we can consider the chemical potential as a small perturbation yielding the corrections of the type \(\mu\) which are assumed to be small and can not strongly modify the vacuum structure. The bound states in this limit can be identified with the monopole-antimonopole bound states from the four-dimensional viewpoint. At larger values of \(\mu\) such naive arguments can not be justified and the quantum behavior of the worldsheet theory of the nonabelian string deserves for separate investigation.

If we start with softly broken supersymmetric theory we should discuss the impact of fermions. The fermionic degrees of freedom in the worldsheet theory follow from the normalized fermion modes on the nonabelian string solution. Generically there could be
translational zero modes which are superpartners of the bosonic coordinates of the string as well of orientational modes. From the general index arguments we can expect that the number of fermionic modes does not change when we switch on the chemical potential. At small chemical potential the explicit expressions for these modes can be found in perturbation theory.

5 Conclusion

In this letter we have discussed the nonabelian strings in the gauge theories with scalars supplemented by the chemical potential. It is argued that chemical potential works as FI term and at large $\mu$ the theory supports the semiclassical nonabelian strings at weak coupling regime. The worldsheet theory at small chemical potential corresponds to the perturbed nonsupersymmetric $CP(N-1)$ model and the bound states of kink and antikinks are the excitations in this theory. From the four-dimensional viewpoint these are monopole-antimonopole pairs. At large chemical potential theory supports the nonabelian strings however the worldsheet theory is essentially different. In particular it is unclear if monopole-antimonopole bound states survive in this regime.

It would be interesting to extend the analysis to the vacua with broken orientational symmetry and with non-vanishing gluon condensate found recently in [22]. Another interesting problem concerns the behavior of the nonabelian strings at nonzero temperature. One could ask if monopoles in the Higgs phase could escape string worldsheet above some critical temperature. In particular one could consider the case when $N_F > N_C$ and semilocal string solutions are present. The chemical potential could be introduced for some flavour global charges and no complications related with the Gauss law shall emerge.

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