Remarkable Classes of Almost 3-Contact Metric Manifolds

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Abstract: We introduce a new class of almost 3-contact metric manifolds, called $3-(0,\delta)$-Sasaki manifolds. We show fundamental geometric properties of these manifolds, analyzing analogies and differences with the known classes of $3-(\alpha,\delta)$-Sasaki ($\alpha \neq 0$) and $3$-$\delta$-cosymplectic manifolds.

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1. Introduction

An almost 3-contact metric manifold is a $(4n+3)$-dimensional differentiable manifold $M$ endowed with three almost contact metric structures $(\varphi_i, \xi_i, \eta_i, g)$, $i = 1, 2, 3$, sharing the same Riemannian metric $g$ and satisfying suitable compatibility conditions, equivalent to the existence of a sphere of almost contact metric structures. In the recent paper [1], new classes of almost 3-contact metric manifolds were introduced and studied. The first remarkable class is given by $3-(\alpha,\delta)$-Sasaki manifolds defined as almost 3-contact metric manifolds $(M, \varphi_i, \xi_i, \eta_i, g)$ such that

$$d\eta_i = 2\alpha \Phi_i + 2(\alpha - \delta)\eta_i \wedge \eta_j, \quad \alpha \in \mathbb{R}, \delta \in \mathbb{R},$$

for every even permutation $(i, j, k)$ of $(1, 2, 3)$. This is a generalization of 3-Sasakian manifolds, which correspond to the values $\alpha = \delta = 1$. A second class introduced in [1] is given by $3$-$\delta$-cosymplectic manifolds defined by the conditions

$$d\eta_i = -2\delta \eta_j \wedge \eta_k, \quad d\Phi_i = 0, \quad \delta \in \mathbb{R},$$

generalizing 3-cosymplectic manifolds which correspond to the value $\delta = 0$.

In the present paper we will introduce a third class of almost 3-contact metric manifolds, which is in fact a second (and alternative) generalization of 3-cosymplectic manifolds. We will consider almost 3-contact metric manifolds whose structure tensor fields satisfy

$$d\eta_i = -2\delta \eta_j \wedge \eta_k, \quad d\Phi_i = -2\delta (\eta_j \wedge \Phi_k - \eta_k \wedge \Phi_j), \quad \delta \in \mathbb{R},$$

for every even permutation $(i, j, k)$ of $(1, 2, 3)$. When $\delta = 0$ we recover a 3-cosymplectic manifold. We will call these manifolds $3-(0,\delta)$-Sasaki manifolds. The choice of name is due to the fact that for a $3-(\alpha,\delta)$-Sasaki manifold, Equation (1) implies

$$d\Phi_i = 2(\alpha - \delta)(\eta_j \wedge \Phi_k - \eta_k \wedge \Phi_j),$$

so that the two equations in (2) formally correspond to (1) and (3) with $\alpha = 0$, although in this case the second equation is no more a consequence of the first one. In fact the two conditions in (2) are not completely independent (see Remark 1). Examples of $3-(0,\delta)$-Sasaki structures can be defined on the semidirect products $SO(3) \ltimes \mathbb{R}^{4n}$. The structure on these Lie groups was introduced in [2] as an example of canonical abelian almost 3-contact...
metric structure. It is also shown in [2] that the Lie group \( \text{SO}(3) \ltimes \mathbb{R}^{3n} \) admits co-compact discrete subgroups, so that the corresponding compact quotients admit almost 3-contact metric structures of the same type.

One can show that for all the above three classes of manifolds, 3-(\(\alpha, \delta\))-Sasaki, 3-\(\delta\)-cosymplectic, and 3-(0, \(\delta\))-Sasaki manifolds, the structure is hypernormal, the characteristic vector fields \( \xi_i, i = 1, 2, 3 \), are Killing and they span an integrable distribution, called vertical, with totally geodesic leaves. Nevertheless, there are remarkable geometric differences between the three classes. In the 3-(\(\alpha, \delta\))-Sasaki case the 1-forms \( \eta \) are all contact forms, i.e., \( \eta \wedge (d\eta)^n \neq 0 \) everywhere on \( M \), while for the other two classes, the horizontal distribution defined by \( \eta_i = 0, i = 1, 2, 3 \), is integrable. Both 3-\(\delta\)-cosymplectic manifolds and 3-(0, \(\delta\))-Sasaki manifolds are locally isometric to the Riemannian product of a 3-dimensional Lie group, tangent to the vertical distribution, and a \( 4n \)-dimensional manifold tangent to the horizontal distribution. The Lie group is either isomorphic to \( \text{SO}(3) \) or flat depending on whether \( \delta \neq 0 \) or \( \delta = 0 \). Each horizontal leaf is endowed with a hyper-Kähler structure. The difference between 3-\(\delta\)-cosymplectic and 3-(0, \(\delta\))-Sasaki manifolds lies in the projectability of the structure tensor fields \( \phi_i, i = 1, 2, 3 \), with respect to the vertical foliation. They are always projectable for 3-\(\delta\)-cosymplectic manifolds, but not for 3-(0, \(\delta\))-Sasaki manifolds with \( \delta \neq 0 \). In this case one can project a transverse quaternionic structure, as it happens for 3-(\(\alpha, \delta\))-Sasaki manifolds. Finally, for the three classes of manifolds, we analyze the existence of a canonical metric connection with totally skew-symmetric torsion.

2. Almost Contact and Almost 3-Contact Metric Manifolds

An almost contact manifold is a smooth manifold \( M \) of dimension \( 2n + 1 \), endowed with a structure \( (\varphi, \xi, \eta) \), where \( \varphi \) is a \( (1,1) \)-tensor field, \( \xi \) a vector field, and \( \eta \) a 1-form such that

\[
\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1,
\]

implying that \( \varphi \xi = 0 \), \( \eta \circ \varphi = 0 \), and \( \varphi \) has rank \( 2n \). The tangent bundle of \( M \) splits as \( TM = \mathcal{H} \oplus \langle \xi \rangle \), where \( \mathcal{H} \) is the \( 2n \)-dimensional distribution defined by \( \mathcal{H} = \text{Im}(\varphi) = \text{Ker}(\eta) \). The vector field \( \xi \) is called the characteristic or Reeb vector field.

On the product manifold \( M \times \mathbb{R} \) one can define an almost complex structure \( J \) by

\[
J(X, f \frac{d}{dt}) = \left( \varphi X - f \xi, \eta(X) \frac{d}{dt} \right),
\]

where \( X \) is a vector field tangent to \( M \), \( t \) the coordinate of \( \mathbb{R} \) and \( f \) a \( C^\infty \) function on \( M \times \mathbb{R} \). If \( J \) is integrable, the almost contact structure is said to be normal and this is equivalent to the vanishing of the tensor field \( N_\varphi := [\varphi, \varphi] + d\eta \otimes \xi \), where \( [\varphi, \varphi] \) is the Nijenhuis torsion of \( \varphi \) [3]. More precisely, for any vector fields \( X \) and \( Y \), \( N_\varphi \) is given by

\[
N_\varphi(X, Y) = [\varphi X, \varphi Y] + \varphi^2 [X, Y] - \varphi [\varphi X, Y] - \varphi [X, \varphi Y] + d\eta(X, Y) \xi.
\]

It is known that any almost contact manifold admits a compatible metric, that is a Riemannian metric \( g \) such that \( g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y) \) for every \( X, Y \in \mathfrak{X}(M) \). Then \( \eta = g(\cdot, \xi) \) and \( \mathcal{H} = \langle \xi \rangle \). The manifold \((M, \varphi, \xi, \eta, g)\) is called an almost contact metric manifold. The associated fundamental 2-form is defined by \( \Phi(X, Y) = g(X, \varphi Y) \).

We recall some remarkable classes of almost contact metric manifolds.

- An \( \alpha \)-contact metric manifold is defined as an almost contact metric manifold such that

\[
d\eta = 2\alpha \Phi, \quad \alpha \in \mathbb{R}^*,
\]

When \( \alpha = 1 \), it is called a contact metric manifold; the 1-form \( \eta \) is a contact form, that is \( \eta \wedge (d\eta)^n \neq 0 \) everywhere on \( M \). An \( \alpha \)-Sasaki manifold is a normal \( \alpha \)-contact metric manifold, and again such a manifold with \( \alpha = 1 \) is called a Sasaki manifold.

- An almost cosymplectic manifold is defined as an almost contact metric manifold such that

\[
d\eta = 0, \quad d\Phi = 0;
\]
A 3-contact metric manifold is cosymplectic if and only if 

$$\begin{align*}
\Phi & = A \gamma (X, Y) - \eta(X) Y, \\
\eta & = \eta_1 \circ \phi = - \eta_2 \circ \phi = - \eta_3 \circ \phi,
\end{align*}$$

for any even permutation $\{i, j, k\}$ of $(1, 2, 3)$ ([3]). The tangent bundle of $M$ splits as $TM = \mathcal{H} \oplus \mathcal{V}$, where

$$\mathcal{H} := \bigcap_{i=1}^{3} \ker(\eta_i), \quad \mathcal{V} := \langle \xi_1, \xi_2, \xi_3 \rangle.$$

In particular, $\mathcal{H}$ has rank $4n$. We call any vector belonging to the distribution $\mathcal{H}$ horizontal and any vector belonging to the distribution $\mathcal{V}$ vertical. The manifold is said to be hypernormal if each almost contact structure $(\phi_i, \xi_i, \eta_i)$ is normal. In [6] it was proved that if two of the almost contact structures are normal, then so is the third.

The existence of an almost 3-contact structure is equivalent to the existence of a sphere

$$\{ (\phi_x, \xi_x, \eta_x) \}_{x \in S^2}$$

of almost contact structures such that

$$\phi_x \circ \phi_y - \eta_y \otimes \xi_x = \phi_{x \times y} - (x \cdot y) 1, \quad \phi_x \eta_y = \xi_{x \times y}, \quad \eta_x \circ \phi_y = \eta_{x \times y},$$

for every $x, y \in S^2$, where $\cdot$ and $\times$ denote the standard inner product and cross product on $\mathbb{R}^3$. In fact, if the structure is hypernormal, then every structure in the sphere is normal ([7]).

An almost 3-contact manifold admits a Riemannian metric $g$ which is compatible with each of the three structures. Then $M$ is said to be an almost 3-contact metric manifold with structure $(\phi_i, \xi_i, \eta_i, g)$, $i = 1, 2, 3$. For ease of notation, we will denote an almost 3-contact metric manifold by $(M, \phi_i, \xi_i, \eta_i, g)$, omitting $i = 1, 2, 3$. The subbundles $\mathcal{H}$ and $\mathcal{V}$ are orthogonal with respect to $g$ and the three Reeb vector fields $\xi_1, \xi_2, \xi_3$ are orthonormal. In fact, the structure group of the tangent bundle is reducible to $\text{Sp}(n) \times \{1\}$ [8].

Given an almost 3-contact metric structure $(\phi_i, \xi_i, \eta_i, g)$, an $\mathcal{H}$-homothetic deformation is defined by

$$\begin{align*}
\eta'_i & = c \eta_i, \\
\xi'_i & = \frac{1}{c} \xi_i, \\
\phi'_i & = \phi_i, \\
g' & = ag + b \sum_{i=1}^{3} \eta_i \otimes \eta_i,
\end{align*}$$

where $a, b \in \mathbb{R}$.
where \(a, b, c\) are real numbers such that \(a > 0, c^2 = a + b > 0\), ensuring that \((\varphi'_r, \xi'_r, \eta'_r, g')\) is an almost 3-contact metric structure. In particular, the fundamental 2-forms \(\Phi_i\) and \(\Phi'_i\) associated to the structures are related by
\[
\Phi'_i = a\Phi_i - b\eta'_i \wedge \eta_k,
\]
where \((i, j, k)\) is an even permutation of \((1, 2, 3)\).

An almost 3-contact metric manifold is called

• 3-\(\alpha\)-Sasaki, with \(\alpha \in \mathbb{R}^+\), if \((\varphi_i, \xi_i, \eta_i, g)\) is \(\alpha\)-Sasaki for all \(i = 1, 2, 3\), i.e. the structure is hypernormal and
\[
d\eta_i = 2\alpha\Phi_i, \quad i = 1, 2, 3;
\]
when \(\alpha = 1\), it is a 3-Sasaki manifold;

• 3-cosymplectic if \((\varphi_i, \xi_i, \eta_i, g)\) is cosymplectic for all \(i = 1, 2, 3\), i.e. the structure is hypernormal and
\[
d\eta_i = 0, \quad d\Phi_i = 0, \quad i = 1, 2, 3;
\]

• 3-quasi-Sasaki manifold if each structure \((\varphi_i, \xi_i, \eta_i, g)\) is quasi-Sasaki; this class includes both 3-\(\alpha\)-Sasaki and 3-cosymplectic manifolds.

These classes were deeply investigated by various authors. See [5,9,10] and references therein for 3-Sasakian geometry, the papers [7,11,12] for 3-cosymplectic manifolds, and [13,14] for 3-quasi-Sasaki manifolds.

In fact, both for 3-Sasaki and 3-cosymplectic manifolds, the hypernormality is consequence of the structure Equations (7) and (8) respectively. This was proved by Kashiwada in [15] for 3-Sasaki manifolds, and in ([16], Theorem 4.13) for 3-cosymplectic manifolds.

In [1] the new classes of 3-(\(\alpha, \delta\))-Sasaki manifolds and 3-\(\delta\)-cosymplectic manifolds were introduced, generalizing the classes of 3-\(\alpha\)-Sasaki and 3-cosymplectic manifolds, respectively. We will review the definitions and the basic properties of these manifolds in the next section. For both these two classes the hypernormality is a consequence of the defining structure equations for the manifolds, thus generalizing the analogous results for 3-Sasaki and 3-cosymplectic manifolds. This is obtained by using the following Lemma:

**Lemma 1** ([1]). Let \((M, \varphi_i, \xi_i, \eta_i, g)\) be an almost 3-contact metric manifold. Then the following formula holds \(\forall X, Y, Z \in \mathfrak{X}(M)\):
\[
g(N_{\varphi_i}(X, Y), Z) =
\]
\[
= -d\Phi_j(X, Y, \varphi_j Z) + d\Phi_j(\varphi_i X, \varphi_j Y, \varphi_j Z) + d\Phi_i(X, \varphi_j Y, \varphi_j Z) + d\Phi_i(\varphi_i X, Y, \varphi_j Z) - \eta_i(X)[d\eta_j(\varphi_i Y, \varphi_j Z) + d\eta_k(\varphi_i Y, \varphi_i Z)] + \eta_i(Y)[d\eta_i(\varphi_i X, \varphi_j Z) + d\eta_k(\varphi_i X, \varphi_i Z)]
\]
\[
+ \eta_j(Z)[d\eta_i(X, Y) - d\eta_i(\varphi_i X, \varphi_i Y)] - \eta_j(Z)[d\eta_i(X, Y) + d\eta_i(\varphi_i X, Y)].
\]

In the following we will be concerned with various classes of almost 3-contact metric manifolds where the three Reeb vector fields are all Killing. In this case one can show that there exists a function \(\delta \in C^\infty(M)\) such that
\[
\eta_i([\xi_r, \xi_s]) = 2\delta \epsilon_{rst}, \quad r, s, t = 1, 2, 3
\]
where \(\epsilon_{rst}\) is the totally skew-symmetric symbol, or equivalently \(d\eta_i([\xi_r, \xi_s]) = -2\delta \epsilon_{rst}\). We call \(\delta\) a Reeb commutator function, we refer to [1] for more information on this notion.

### 3. 3-(\(\alpha, \delta\))-Sasaki Manifolds and 3-\(\delta\)-Cosymplectic Manifolds

This section is a short review of 3-(\(\alpha, \delta\))-Sasaki manifolds and 3-\(\delta\)-cosymplectic manifolds. These were discussed in detail in [1,17].
Definition 1. An almost 3-contact metric manifold \((M, \varphi_i, \zeta_i, \eta_i, g)\) is called a 3-(\(\alpha, \delta\))-Sasaki manifold if it satisfies  
\[ d\eta_i = 2\alpha \Phi_i + 2(\alpha - \delta)\eta_i \wedge \eta_k \]
for every even permutation \((i,j,k)\) of \((1,2,3)\), where \(\alpha \neq 0\) and \(\delta\) are real constants.

When \(\alpha = \delta = 1\), we have a 3-contact metric manifold, and hence a 3-Sasaki manifold by Kashiwada’s theorem [15].

Theorem 1 ([1], Theorem 2.2.1). Any 3-(\(\alpha, \delta\))-Sasaki manifold is hypernormal.

In particular, a 3-(\(\alpha, \delta\))-Sasaki manifold with \(\alpha = \delta\) is 3-a-Sasaki. It can be also shown that the vertical distribution of any 3-(\(\alpha, \delta\))-Sasaki manifold is integrable with totally geodesic leaves and each Reeb vector field \(\zeta_i\) is Killing.

We can distinguish three main classes of 3-(\(\alpha, \delta\))-Sasaki manifolds. A 3-(\(\alpha, \delta\))-Sasaki manifold is called degenerate if \(\delta = 0\) and non-degenerate otherwise. Quaternionic Heisenberg groups are examples of degenerate 3-(\(\alpha, \delta\))-Sasaki manifolds (see ([1], Example 2.3.2)). Considering an \(\mathcal{H}\)-homothetic deformation of a 3-(\(\alpha, \delta\))-Sasaki structure, as in (5), one can verify that the obtained structure \((\varphi', \zeta_i', \eta_i', g')\) is a 3-(\(\alpha', \delta'\))-Sasaki manifold with

\[ \alpha' = \alpha - c, \quad \delta' = \delta + c. \]

In particular, \(\mathcal{H}\)-homothetic deformations preserve the class of degenerate manifolds. In the nondegenerate case, one sees immediately that \(\alpha' \delta'\) has the same sign as \(\alpha \delta\). This justifies the distinction between positive 3-(\(\alpha, \delta\))-Sasaki manifolds, with \(\alpha \delta > 0\), and negative 3-(\(\alpha, \delta\))-Sasaki manifolds, with \(\alpha \delta < 0\). In fact, it can be shown that a 3-(\(\alpha, \delta\))-Sasaki manifold is positive if and only if it is \(\mathcal{H}\)-homothetic to a 3-Sasaki manifold, and negative if and only if it is \(\mathcal{H}\)-homothetic to a 3-(\(\alpha', \delta'\))-Sasaki manifold with \(\alpha' = -1\), \(\delta' = 1\).

Examples of negative 3-(\(\alpha, \delta\))-Sasaki manifolds can be obtained in the following way. It is known that quaternionic Kähler (not hyper-Kähler) manifolds with negative scalar curvature admit a canonically associated principal \(SO(3)\)-bundle \(P(M)\) which is endowed with a negative 3-Sasaki structure [18,19]. This is a 3-structure \((\varphi_i, \zeta_i, \eta_i, \check{g})\), \(i = 1, 2, 3\), where \((\varphi_i, \zeta_i, \eta_i)\) is a normal almost 3-contact structure, and \(\check{g}\) is a compatible semi-Riemannian metric, with signature \((3, 4n)\), where \(4n\) is the dimension of the base space, and \(d\eta_i(X, Y) = 2\check{g}(X, \varphi_i Y)\). Then, one can define the Riemannian metric

\[ g = -\check{g} + 2 \sum_{i=1}^{3} \eta_i \otimes \eta_i, \]

which is compatible with the structure \((\varphi_i, \zeta_i, \eta_i)\), and satisfies \(d\eta_i = -2\Phi_i - 4\eta_i \wedge \eta_k\), where \(\Phi_i(X, Y) = g(X, \varphi_i Y)\) (see also [19]). Therefore \((\varphi_i, \zeta_i, \eta_i, \check{g})\) is a 3-(\(\alpha, \delta\))-Sasaki structure with \(\alpha = -1\) and \(\delta = 1\).
The following Theorem regarding the transverse geometry with respect to the vertical foliation of a $3$-$(\alpha, \delta)$-Sasaki manifold is proved in [17]:

**Theorem 2.** Any $3$-$(\alpha, \delta)$-Sasaki manifold $M$ admits a locally defined Riemannian submersion $\pi: M \to N$ along its horizontal distribution $\mathcal{H}$ such that $N$ carries a quaternionic Kähler structure given by

$$\phi_i = \pi_* \circ \phi_i \circ s,$$

where $s: U \to M$ is any local smooth section of the Riemannian submersion. The covariant derivatives of the almost complex structures $\phi_i$ are given by

$$\nabla^\text{R} \phi_i = 2\delta(\eta_k(X)\phi_i - \eta_i(X)\phi_k)$$

where $\eta_i(X) = \eta_i(s_*X) \circ s$ for $i = 1, 2, 3$. The scalar curvature of the base space $N$ is $16n(n+2)a\delta$.

The Riemannian Ricci tensor of any $3$-$(\alpha, \delta)$-Sasaki manifold is computed in [1]:

$$\text{Ric}^g = 2\alpha (2\delta(n+2) - 3\alpha)g + 2(\alpha - \delta)(2\alpha + 3)\alpha - 3\delta \sum_{i=1}^3 \eta_i \otimes \eta_i.$$  \hfill (10)

In particular, a $3$-$(\alpha, \delta)$-Sasaki manifold is Riemannian Einstein if and only if $\delta = \alpha$, in which case the structure is $3$-$\alpha$-Sasaki, or $\delta = (2n+3)\alpha$.

Notice that, by Theorem 2, a non-degenerate $3$-$(\alpha, \delta)$-Sasaki manifold locally fibers over a quaternionic Kähler space of positive or negative scalar curvature, according to $\alpha\delta > 0$ or $\alpha\delta < 0$, respectively. In [17] a systematic study of homogeneous non-degenerate $3$-$(\alpha, \delta)$-Sasaki manifolds has been carried out, obtaining a complete classification in the positive case, where the base space of the homogeneous fibration turns out to be a symmetric Wolf space. In the case $\alpha\delta < 0$, one can provide a general construction of homogeneous $3$-$(\alpha, \delta)$-Sasaki manifolds fibering over nonsymmetric Alekseevsky spaces.

We recall now the definition and some basic facts on $3$-$\delta$-cosymplectic manifolds.

**Definition 2.** A $3$-$\delta$-cosymplectic manifold is an almost $3$-contact metric manifold satisfying

$$d\eta_i = -2\delta\eta_j \wedge \eta_k, \quad d\Phi_i = 0,$$

for some $\delta \in \mathbb{R}$ and for every even permutation $(i, j, k)$ of $(1, 2, 3)$.

When $\delta = 0$, the fact that the forms $\eta_i$ and $\Phi_i$ are all closed implies that the structure is hypernormal ([16], Theorem 4.13). In fact this immediately follows from (9). Therefore, a $3$-$\delta$-cosymplectic manifold with $\delta = 0$ is $3$-cosymplectic. In particular, it is $3$-quasi-Sasaki and the Reeb vector fields are all Killing. The local structure of these manifolds is described by the following:

**Proposition 1** ([12]). Any $3$-cosymplectic manifold of dimension $4n+3$ is locally the Riemannian product of a hyper-Kähler manifold of dimension $4n$ and a $3$-dimensional flat abelian Lie group.

As a consequence, since every hyper-Kähler manifold is Ricci flat, even the Riemannian Ricci tensor of any $3$-cosymplectic manifold vanishes.

As regards $3$-$\delta$-cosymplectic manifolds with $\delta \neq 0$, even in this case one can show that the structure is hypernormal, the Reeb vector fields are Killing, and the manifold locally decomposes as a Riemannian product [1]. In particular,

**Proposition 2.** Any $3$-$\delta$-cosymplectic manifold with $\delta \neq 0$ is locally the Riemannian product of a hyper-Kähler manifold and a $3$-dimensional Lie group isomorphic to $\text{SO}(3)$, with constant curvature $\delta^2$. Consequently, the Riemannian Ricci tensor is $\text{Ric}^g = 2\delta^2 \sum_{i=1}^3 \eta_i \otimes \eta_i$. 
In both cases, i.e., $\delta = 0$ or $\delta \neq 0$, the hyper-Kähler manifold is tangent to the horizontal distribution, while the 3-dimensional Lie group is tangent to the vertical distribution. In fact, examples of these manifolds can be obtained taking Riemannian products $N \times G$, where $(N, f_i, h)$, $i = 1, 2, 3$, is a hyper-Kähler manifold, and $G$ is a 3-dimensional Lie group, which is either abelian, or isomorphic to $\text{SO}(3)$. If $\xi_1$, $\xi_2$, $\xi_3$ are generators of the Lie algebra $\mathfrak{g}$ of $G$, satisfying $[\xi_i, \xi_j] = 2\delta_{ij} \xi_k$, $\delta \in \mathbb{R}$, then one can define in a natural manner an almost 3-contact metric structure $(\varphi_i, \xi_i, \eta_i, \mathfrak{g})$ on the product $N \times G$, setting

$$\varphi_i|_{TN} = f_i, \quad \varphi_i \xi_i = 0, \quad \varphi_i \xi_j = \xi_k, \quad \varphi_i \xi_k = -\xi_j,$$

$$\eta_j|_{TN} = 0, \quad \eta_j(\xi_i) = 1, \quad \eta_j(\xi_j) = \eta_j(\xi_k) = 0,$$

and $\mathfrak{g}$ the product metric of $h$ and the left invariant Riemannian metric on $G$ with respect to which $\xi_1$, $\xi_2$, $\xi_3$ are an orthonormal basis of $\mathfrak{g}$.

For a comparison with the class of 3-$(0, \delta)$-Sasaki manifolds, which will be introduced in the next section, it is worth remarking that for a 3-$\delta$-cosymplectic manifold $(M, \varphi_i, \xi_i, \eta_i, \mathfrak{g})$ the Lie derivatives of the structure tensor fields $\varphi_i$, $i = 1, 2, 3$ with respect to the Reeb vector fields are given by

$$\mathcal{L}_{\xi_i} \varphi_i = 0, \quad \mathcal{L}_{\xi_i} \varphi_j = 2\delta(\eta_i \otimes \xi_j - \eta_j \otimes \xi_i) = -\mathcal{L}_{\xi_j} \varphi_i \tag{11}$$

for every $i, j = 1, 2, 3$. Indeed, in a 3-$\delta$-cosymplectic manifold the Levi-Civita connection satisfies ([1], Proposition 2.1.1):

$$\nabla^g_{\xi_i} \varphi_i = 0,$$

$$(\nabla^g_{\xi_i} \varphi_j)X = \delta(\eta_i(X)\xi_j - \eta_j(X)\xi_i) = -(\nabla^g_{\xi_j} \varphi_i)X,$$

$$\nabla^g_{\xi_i} \xi_j = \delta(\eta_k(X)\xi_j - \eta_j(X)\xi_k),$$

where $(i, j, k)$ is an even permutation of $(1, 2, 3)$ and $X \in \mathfrak{X}(M)$. Therefore,

$$(\mathcal{L}_{\xi_i} \varphi_i)X = [\xi_i, \varphi_iX] - \varphi_i[\xi_i, X]$$

$$= \nabla^g_{\xi_i} \varphi_iX - \nabla^g_{\varphi_iX} \xi_i - \varphi_i(\nabla^g_{\xi_i} X) + \varphi_i(\nabla^g_{\xi_i} \xi_i)$$

$$= (\nabla^g_{\xi_i} \varphi_i)X - \nabla^g_{\varphi_iX} \xi_i + \varphi_i(\nabla^g_{\xi_i} \xi_i)$$

$$= -\delta(\eta_i(\varphi_iX)\xi_j - \eta_j(\varphi_iX)\xi_k) + \delta(\eta_k(\varphi_iX)\xi_j - \eta_j(\varphi_iX)\xi_k) = 0.$$

In the same way,

$$(\mathcal{L}_{\xi_i} \varphi_j)X = (\nabla^g_{\xi_i} \varphi_j)X - \nabla^g_{\varphi_jX} \xi_i + \varphi_j(\nabla^g_{\xi_i} \xi_i)$$

$$= \delta(\eta_i(\varphi_jX)\xi_j - \eta_j(\varphi_jX)\xi_i) - \delta \eta_k(\varphi_jX)\xi_j - \delta \eta_j(\varphi_jX)\xi_k$$

$$= 2\delta(\eta_i(X)\xi_j - \eta_j(X)\xi_i) = -\mathcal{L}_{\xi_j} \varphi_i X.$$

4. 3-$(0, \delta)$-Sasaki Manifolds

In this section we introduce the class of 3-$(0, \delta)$-Sasaki manifolds.

**Definition 3.** An almost 3-contact metric manifold $(M, \varphi_i, \xi_i, \eta_i, \mathfrak{g})$ will be called 3-$(0, \delta)$-Sasaki manifold if

$$d\eta_i = -2\delta \eta_j \wedge \eta_k, \quad d\Phi_i = -2\delta(\eta_i \wedge \Phi_k - \eta_k \wedge \Phi_i) \tag{12}$$

for every even permutation $(i, j, k)$ of $(1, 2, 3)$, and for some real constant $\delta \in \mathbb{R}$.

In particular, the structure is not 3-quasi-Sasaki when $\delta \neq 0$, and we have the following basic properties for a 3-$(0, \delta)$-Sasaki manifold:

1. The horizontal distribution $\mathcal{H}$ is integrable;
2. Each $\xi_i$ is an infinitesimal automorphism of the distribution $\mathcal{H}$, i.e.
\[ d\eta_i(X, \xi_s) = 0 \quad X \in \Gamma(\mathcal{H}), \quad r, s = 1, 2, 3; \]

3. The constant $\delta$ is the Reeb commutator function.

**Remark 1.** In case $\delta \neq 0$, the two equations in (12) are not completely independent. Indeed, if one assumes $d\Phi_k = -2\gamma(\eta_1 \rightleftharpoons \Phi_k - \eta_1 \rightleftharpoons \Phi_1)$, $\gamma \in \mathbb{R}^*$, differentiating this equation, and combining with $d\eta_i = -2\delta \eta_i + \eta_i$, a straightforward computation gives $\gamma = \delta$. Thus, there is no freedom for the choice of constant in the second equation.

If $(\phi_i, \xi_i, \eta_i, g)$ is a 3-(0, $\delta$)-Sasaki structure, applying an $\mathcal{H}$-homothetic deformation as in (5), an easy computation using (6) shows that the new structure $(\phi'_i, \eta'_i, \xi'_i, g')$ is again 3-(0, $\delta'$)-Sasaki, with $\delta' = \frac{\delta}{\xi}$.

**Example 1.** Consider the abelian Lie algebra $\mathbb{R}^{4n}$ spanned by vectors $v_r, v_{n+r}, v_{2n+r}, v_{3n+r}$, $r = 1, \ldots, n$, and endowed with the hypercomplex structure $\{J_1, J_2, J_3\}$ defined by $J_1(v_r) = v_{n+r}$, $J_2(v_r) = -v_r$, $J_3(v_r) = v_{2n+r}$, $J_4(v_r) = -v_{3n+r}$, for every even permutation $(i, j, k)$ of $(1, 2, 3)$. Let us consider also the Lie algebra $\mathfrak{so}(3)$ spanned by $\xi_1, \xi_2, \xi_3$ with Lie brackets given by $[\xi_i, \xi_j] = 2\delta \xi_k$, $\delta \neq 0$. Let $\rho$ be the representation of $\mathfrak{so}(3)$ on $\mathbb{R}^{4n}$ given by $\rho : \mathfrak{so}(3) \rightarrow \mathfrak{gl}(4n, \mathbb{R})$, $\rho(\xi_i) = \delta J_i$, $i = 1, 2, 3$.

On the Lie algebra $g = \mathfrak{so}(3) \rtimes_\rho \mathbb{R}^{4n}$ on can define in a natural way an almost 3-contact metric structure $(\phi_i, \xi_i, \eta_i, g)$, with
\[ \phi_i|_{\mathbb{R}^{4n}} = J_i, \quad \phi(\xi_j) = 0, \quad \phi(\xi_j) = \xi_j = -\phi(\xi_j), \]
\[ \eta_i|_{\mathbb{R}^{4n}} = 0, \quad \eta_i(\xi_j) = 1, \quad \eta_i(\xi_j) = \eta_i(\xi_j) = 0, \]
and where $g$ is the inner product such that the vectors $\xi_i, v_i, i = 1, 2, 3$, $l = 1, \ldots, 4n$ are orthonormal. In particular, the non zero brackets on $g$ are given by $[\xi_i, \xi_j] = 2\delta \xi_k$, $[\xi_i, X] = \delta \phi_i(X)$, $X \in \mathbb{R}^{4n}$.

The representation $\rho : \mathfrak{so}(3) \rightarrow \mathfrak{gl}(4n, \mathbb{R})$ can be integrated to a representation $\hat{\rho} : \text{SO}(3) \rightarrow \text{GL}(4n, \mathbb{R})$. Therefore, identifying $\mathbb{R}^{4n}$ with $\mathbb{H}^n$ in a natural way, the simply connected Lie group $G = \text{SO}(3) \rtimes_\hat{\rho} \mathbb{H}^n$, with Lie algebra $g$, admits a left invariant almost 3-contact metric structure $(\phi_i, \xi_i, \eta_i, g)$. One can easily verify that this structure satisfies (12).

**Remark 2.** For more details on the above example we refer to [2], where $g$ is described as a remarkable example of a Lie algebra endowed with an abelian almost 3-contact metric structure. In fact, the structure defined on $g$ belongs to the class of canonical abelian structures, so that the Lie group $G$ admits a unique metric connection with totally skew symmetric torsion $\nabla$ such that
\[ \nabla_X \phi_i = 2\delta (\eta_k(X)\phi_j - \eta_j(X)\phi_k) \]
for every vector field $X$ and for every even permutation $(i, j, k)$ of $(1, 2, 3)$. The torsion of the canonical connection $\nabla$ is $T = 2\delta\eta_1 \wedge \eta_2 \wedge \eta_3$, which satisfies $\nabla T = 0$.

It is also shown in [2] that the Lie group $G$ admits co-compact discrete subgroups, so that the corresponding compact quotients admit almost 3-contact metric structures of the same type.

**Proposition 3.** Let $(M, \phi_i, \xi_i, \eta_i, g)$ be a 3-(0, $\delta$)-Sasaki manifold. Then the structure is hypernormal.
Proof. In order to compute the tensor fields $N_{\psi_i}$, we apply Lemma 1. We always denote by $X, Y, Z$ horizontal vector fields and by $(i, j, k)$ an even permutation of $(1, 2, 3)$.

Being $d\Phi_{i}(X, Y, Z) = 0$, then $N_{\psi_i}(X, Y, Z) = 0$ for every $i = 1, 2, 3$. Furthermore, since the horizontal distribution is integrable, by the definition of the tensor field $N_{\psi_i}$ (see (4)), one has $N_{\psi_i}(X, Y, \zeta_i) = 0$ for all $i = 1, 2, 3$. Notice that, since

$$
\zeta_{i,j} \Phi_i = 0, \quad \zeta_{j,i} \Phi_i = -\eta_i, \quad \zeta_{k,j} \Phi_i = \eta_j,
$$

from the second equation in (12), we have,

$$
\zeta_{i,j} \Phi_i = 0, \quad \zeta_{j,i} \Phi_i = -2\delta(\Phi_k + \eta_i), \quad \zeta_{k,j} \Phi_i = 2\delta(\Phi_j + \eta_k). \tag{13}
$$

Therefore, from Lemma 1, applying (12) and (13), we have

$$
N_{\psi_i}(X, \zeta_i, Z) = -d\Phi_i(X, \zeta_i, \psi_j Z) + d\Phi_i(\psi_j X, \zeta_i, \psi_j Z) + d\eta_i(\psi_j X, \psi_j Z) + d\eta_i(X, \psi_j Z)
$$

$$
= -2\delta\Phi_i(\psi_j Z, X) = -2\delta\Phi_i(Z, \psi_j X) = -2\delta\Phi_i(\psi_j X, Z) = -2\delta\Phi_i(X, \psi_j Z) = 0,
$$

$$
N_{\psi_i}(X, \zeta_j, Z) = d\Phi_i(\psi_j Z, \zeta_k, \psi_j Z) + d\Phi_i(\psi_j X, \zeta_k, \psi_j Z) + d\eta_i(\psi_j Z, \psi_j X) + d\eta_i(\psi_j X, \psi_j Z) = 0,
$$

$$
N_{\psi_i}(X, \zeta_k, Z) = -d\Phi_i(X, \zeta_k, \psi_j Z) - d\Phi_i(X, \zeta_k, \psi_k X) = -2\delta\Phi_i(\psi_j Z, X) = 0.
$$

Equations (13) imply $d\Phi_i(X, \zeta_s, \zeta_t) = 0$ for every $r, s, t = 1, 2, 3$ and $X \in \Gamma(\mathcal{H})$. Taking also into account that $d\eta_i(X, \zeta_i) = 0$, we deduce from (9) that

$$
N_{\psi_i}(X, \zeta_s, \zeta_t) = N_{\psi_i}(\zeta_s, \zeta_t, X) = 0.
$$

Finally, (9) implies together with $d\eta_i(\zeta_s, \zeta_t) = -2\delta\epsilon_{rst}$ that

$$
N_{\psi_i}(\zeta_i, \zeta_j, \zeta_k) = N_{\psi_i}(\zeta_i, \zeta_k, \zeta_j) = N_{\psi_i}(\zeta_j, \zeta_k, \zeta_i) = 0,
$$

completing the proof that $M$ is hypernormal. \hfill \Box

Proposition 4. Let $(M, \psi_i, \zeta_i, \eta_j, g)$ be a 3-$(0, \delta)$-Sasaki manifold. Then the Levi-Civita connection satisfies for all $X, Y \in \mathfrak{X}(M)$ and any cyclic permutation $(i, j, k)$ of $(1, 2, 3)$:

$$
(\nabla_X^g \psi_i) Y = 2\delta \left[ \eta_j(X) \psi_j Y - \eta_j(X) \psi_k Y \right] - \delta \left[ \eta_j(X) \eta_j(Y) + \eta_j(X) \eta_k(Y) \right] \zeta_i + \delta \eta_i(Y) \left[ \eta_j(X) \zeta_j + \eta_k(X) \zeta_k \right] \tag{14}
$$

and

$$
\nabla^g_{\zeta_i} \zeta_i = 0, \quad \nabla^g_{\zeta_i} \zeta_i = -\nabla^g_{\zeta_i} \zeta_i = \delta \zeta_k. \tag{15}
$$

In particular, each $\zeta_i$ is a Killing vector field.

Proof. Since the structure is hypernormal, by ([3], Lemma 6.1), the Levi-Civita connection satisfies

$$
2g((\nabla_X^g \psi_i) Y, Z) = d\Phi_i(X, \psi_j Y, \psi_k Z) - d\Phi_i(X, Y, Z)
$$

$$
+ d\eta_i(\psi_j Y, X) \eta_j(Z) - d\eta_i(\psi_k Z, X) \eta_k(Y). \tag{17}
$$

Further, an easy computation (see [11]) shows that for every cyclic permutation $(i, j, k)$ of $(1, 2, 3)$,
\[ \Phi_j(\varphi_i X, \varphi_i Y) = -\Phi_j(X, Y) - (\eta_k \wedge \eta_i)(X, Y), \]
\[ \Phi_k(\varphi_i X, \varphi_i Y) = -\Phi_k(X, Y) - (\eta_i \wedge \eta_i)(X, Y), \]
\[ \Phi_j(\varphi_i X, Y) = -\Phi_j(X, Y) - \eta_i(X)\eta_i(Y), \]
\[ \Phi_k(\varphi_i X, Y) = \Phi_j(X, Y) - \eta_i(X)\eta_i(Y). \]

Then, using the second equation in (12) and the above equations, we have
\[
d\Phi_i(X, \varphi_i Y, \varphi_i Z) =
\begin{align*}
&= -2\delta [\eta_i(X)\Phi_k(\varphi_i Y, \varphi_i Z) + \eta_i(\varphi_i Y)\Phi_k(\varphi_i Z, X) + \eta_i(\varphi_i Z)\Phi_k(X, \varphi_i Y)] \\
&\quad - \eta_i(X)\Phi_j(\varphi_i Y, \varphi_i Z) - \eta_i(\varphi_i Y)\Phi_j(\varphi_i Z, X) - \eta_i(\varphi_i Z)\Phi_j(X, \varphi_i Y) \\
&\quad - \eta_i(X)\Phi_k(Y, Z) - \eta_i(X)(\eta_i \wedge \eta_i)(Y, Z) \\
&\quad - \eta_i(Y)\Phi_k(Z, X) + \eta_i(Z)\eta_i(X) - \eta_i(Z)\eta_i(Y)\eta_i(X) \\
&\quad + \eta_i(X)\Phi_j(Y, Z) + \eta_i(X)(\eta_i \wedge \eta_i)(Y, Z) \\
&\quad + \eta_i(Y)\Phi_k(Z, X) - \eta_i(Z)\eta_i(X) - \eta_i(Z)\eta_i(Y)\eta_i(X)] \\
&= d\Phi_i(X, Y, Z) + 4\delta [\eta_i(X)\Phi_k(Y, Z) - \eta_i(X)\Phi_j(Y, Z)] \\
&\quad + 4\delta \eta_i(X)(\eta_i Y)\eta_i(Z) - \eta_i(Y)\eta_i(Z)] \\
&\quad + 4\delta \eta_i(X)(\eta_i Y)(Z) - \eta_i(Y)\eta_i(Z)].
\end{align*}
\]

On the other hand, again using the first equation in (12), we obtain
\[
d\eta_i(\varphi_i Y, X)\eta_i(Z) - d\eta_i(\varphi_i Z, X)\eta_i(Y) =
\begin{align*}
&= -2\delta \eta_i(\varphi_i Y, X)\eta_i(Z) + 2\delta \eta_i(\varphi_i Z, X)\eta_i(Y) \\
&= -2\delta \eta_i(Z) - \eta_i(X)\eta_i(Y) + 2\delta \eta_i(Y)[-\eta_i(Z)\eta_i(X) - \eta_i(X)\eta_i(Z)].
\end{align*}
\]

Inserting the above computations in (17), we conclude that
\[
g((\nabla^Y_\varphi_i)Y, Z) = 2\delta [\eta_i(X)g(\varphi_i Y, Z) - \eta_i(X)g(\varphi_i Y, Z)] \\
- \delta \eta_i(Z)\eta_i(X) + \eta_i(X)\eta_i(Y) + \delta \eta_i(Y)\eta_i(Z)\eta_i(X) + \eta_i(X)\eta_i(Z) \\
= -\delta(\eta_i(X)\xi_j + \eta_i(X)\xi_k).
\]

Applying \(\varphi_i\) on both hand-sides, we obtain (15). Equations (16) are immediate consequences of (15). Furthermore, we also get
\[ g(\nabla^Y_\varphi_i Y) = -\delta(\eta_i \wedge \eta_k)(X, Y) \]
for every \(X, Y \in \mathfrak{X}(M)\). Since \(\nabla^Y_\varphi_i\) is skew-symmetric, \(\xi_i\) is Killing. \(\square\)

**Corollary 1.** Let \((M, \varphi_i, \xi_i, \eta_i, g)\) be a 3-(0, \(\delta\))-Sasaki manifold. Then for every even permutation \((i, j, k)\) of \((1, 2, 3)\) we have
\[
\mathcal{L}_{\xi_i}\varphi_i = 0, \quad \mathcal{L}_{\xi_i}\varphi_j = -\mathcal{L}_{\xi_j}\varphi_i = 2\delta \varphi_k.
\]

**Proof.** For the first Lie derivative, notice that by (14) we have \(\nabla^Y_\varphi_i \varphi_i = 0\). Then, applying also (15), for every vector field \(X\) we have
\[
(\mathcal{L}_{\xi_i}\varphi_i)X = (\nabla^Y_\varphi_i \varphi_i)X - \nabla^Y_{\varphi_i X}\xi_i + \varphi_i(\nabla^Y_\varphi_i \xi_i) \\
= -\delta(\eta_k(\varphi_i X)\xi_j - \eta_j(\varphi_i X)\xi_k) + \delta(\eta_k(X)\varphi_i \xi_j - \eta_j(X)\varphi_i \xi_k) = 0.
\]
Now, using (14) for the covariant derivative $\nabla^g \varphi$, for every vector field $Y$, we have

$$(\nabla^g \varphi_i)_Y = 2\delta \varphi_k Y - \delta (\eta_i (Y) \xi_j - \eta_j (Y) \xi_i).$$

Therefore, applying also (15), we get

$$(L_{\varphi_i} \varphi_j)X = (\nabla^g \varphi_i)X - \nabla^g \varphi_j \xi_i + \varphi_j (\nabla^g \xi_i)$$

$$= 2\delta \varphi_k X - \delta (\eta_i (X) \xi_j - \eta_j (X) \xi_i) - \delta \eta_k (\varphi_j X) \xi_i - \delta \eta_i (X) \varphi_j \xi_k$$

$$= 2\delta \varphi_k X.$$

Analogously, $L_{\varphi_i} \varphi_i = -2\delta \varphi_k$. □

**Theorem 3.** Let $(M, \varphi_i, \xi_i, \eta_i, g)$ be a $3-(0, \delta)$-Sasaki manifold. Then both the horizontal and the vertical distribution are integrable with totally geodesic leaves. Each leaf of the vertical distribution is locally isomorphic to the Lie group $SO(3)$, with constant sectional curvature $\delta^2$; each leaf of the horizontal distribution is endowed with a hyper-Kähler structure. Consequently, the Riemannian Ricci tensor of $M$ is given by

$$\text{Ric}^g = 2\delta^2 \sum_{i=1}^{3} \eta_i \otimes \eta_i.$$  \hspace{1cm} (19)

**Proof.** We already know that the horizontal distribution $\mathcal{H}$ is integrable. From (15), for every $X, Y \in \Gamma(\mathcal{H})$ and $i = 1, 2, 3$, we have

$$g(\nabla^g_X \xi_i, Y) = -g(\nabla^g_{\xi_i} X, Y) = 0,$$ so that the distribution $\mathcal{H}$ has totally geodesic leaves. Furthermore, Equation (16) implies that the vertical distribution $\mathcal{V}$ is also integrable with totally geodesic leaves. In particular $[\xi_i, \xi_j] = 2\delta \xi_k$ for an even permutation $(i, j, k)$ of $(1, 2, 3)$, so that the leaves of $\mathcal{V}$ are locally isomorphic to the Lie group $SO(3)$ and have constant sectional curvature $\delta^2$. On each leaf of the horizontal distribution $\mathcal{H}$ one can consider the almost hyper-Hermitian structure defined by $(f_i := \varphi_i|_{\mathcal{H}}, g)$, which turns out to be hyper-Kähler due to (14). Consequently, $M$ is locally the Riemannian product of 3-dimensional sphere of curvature $\delta^2$ and a $4n$-dimensional manifold $M'$, which is endowed with a hyper-Kähler structure. Since any hyper-Kähler manifold is Ricci flat, we obtain that the Riemannian Ricci tensor of $M$ is given by (19). □

**Remark 3.** From Theorem 3 it follows that any $3-(0, \delta)$-Sasaki manifold is locally isometric to the Riemannian product of 3-dimensional sphere and a $4n$-dimensional manifold $M'$, which is endowed with a hyper-Kähler structure. We recall that 3-\delta-cosymplectic manifolds are also locally isometric to the Riemannian product of a 3-dimensional sphere of constant curvature $\delta^2$ and a hyper-Kähler manifold. Nevertheless, there is a difference between the two geometries. Looking at the transverse geometry of the foliation defined by the vertical distribution $\mathcal{V}$, in both cases the Riemannian metric $g$ is projectable, being the vector fields $\xi_i$, $i = 1, 2, 3$, all Killing. In the case of 3-\delta-cosymplectic manifolds, each tensor field $\varphi_i$ is also projectable, as by (11), the Lie derivatives with respect to the Reeb vector fields satisfy $(L_{\varphi_i} \varphi_j)X = 0$ for every $i, j = 1, 2, 3$ and for every horizontal vector field $X$. In the case of $3-(0, \delta)$-Sasaki manifolds, owing to (18), the tensor fields are not projectable. Nevertheless, taking into account the horizontal parts $\Phi_i^H := \Phi_i + \eta \wedge \eta \wedge e_k$ of the fundamental 2-forms $\Phi_i$, one can verify that horizontal 4-form \[
\Phi_1^H \wedge \Phi_2^H + \Phi_2^H \wedge \Phi_3^H + \Phi_3^H \wedge \Phi_1^H \]
is projectable and defines a transversal quaternionic structure, which turns out to be locally hyper-Kähler.
5. Connections with Totally Skew-Symmetric Torsion

In this section we will show that every 3-(0, δ)-Sasaki manifold is canonical in the sense of the definition given in [1], thus admitting a special metric connection with totally skew-symmetric torsion, called canonical. Recall that a metric connection \( \nabla \) with torsion \( T \) on a Riemannian manifold \( (M, g) \) is said to have totally skew-symmetric torsion, or skew torsion for short, if the \( (0,3) \)-tensor field \( T \) defined by \( T(X, Y, Z) := g(T(X, Y), Z) \) is a 3-form. The relation between \( \nabla \) and the Levi-Civita connection \( \nabla^g \) is then given by

\[
\nabla_X Y = \nabla^g_X Y + \frac{1}{2} T(X, Y).
\]

For more details we refer to [20]. We recall now the definition and the characterization of canonical almost 3-contact metric manifolds.

**Definition 4 ([1]).** An almost 3-contact metric manifold \( (M, \varphi, \xi, \eta, g) \) is called canonical if the following conditions are satisfied:

(i) each \( N_\varphi \) is totally skew-symmetric on \( \mathcal{H} \),

(ii) each \( \xi_i \) is a Killing vector field,

(iii) for any \( X, Y, Z \in \Gamma(\mathcal{H}) \) and any \( i, j = 1, 2, 3 \),

\[
N_\varphi(X, Y, Z) = \frac{1}{2} \beta_i(X, Y, Z) - \frac{1}{2} \beta_j(X, Y, Z) = \frac{1}{2} \beta_k(X, Y, Z),
\]

(iv) \( M \) admits a Reeb Killing function \( \beta \in C^\infty(M) \), that is the tensor fields \( A_{ij} \) defined on \( \mathcal{H} \) by

\[
A_{ij}(X, Y) := g(\mathcal{L}_{\xi_i} \varphi_j, X, Y) + d\eta_j(X, \varphi_i Y) + d\eta_i(X, \varphi_j Y),
\]

satisfy

\[
A_{ij}(X, Y) = 0, \quad A_{ij}(X, Y) = -A_{ji}(X, Y) = \beta \Phi_k(X, Y),
\]

for every \( X, Y \in \Gamma(\mathcal{H}) \) and every even permutation \( i, j, k \) of \( (1, 2, 3) \).

Here \( N_\varphi \) also denotes the \((0,3)\)-tensor field defined by \( N_\varphi(X, Y, Z) := g(N_\varphi(X, Y), Z) \) and we say that \( N_\varphi \) is totally skew-symmetric on \( \mathcal{H} \) if the \((0,3)\)-tensor is a 3-form on \( \mathcal{H} \).

**Theorem 4 ([1]).** An almost 3-contact metric manifold \( (M, \varphi, \xi, \eta, g) \) is canonical, with Reeb Killing function \( \beta \), if and only if it admits a metric connection \( \nabla \) with skew torsion such that

\[
\nabla_X \varphi_i = \beta(\eta_k(X) \varphi_i - \eta_j(X) \varphi_k)
\]

for every vector field \( X \) on \( M \) and for every even permutation \( i, j, k \) of \( (1, 2, 3) \). If such a connection \( \nabla \) exists, it is unique and its torsion is given by

\[
T(X, Y, Z) = N_\varphi(X, Y, Z) - \frac{1}{2} \beta_k(X, Y, Z),
\]

\[
T(X, Y, \xi_i) = d\eta_i(X, Y),
\]

\[
T(X, \xi_i, \xi_j) = -g([\xi_i, \xi_j], X),
\]

\[
T(\xi_1, \xi_2, \xi_3) = 2(\beta - \delta),
\]

for every \( X, Y, Z \in \Gamma(\mathcal{H}) \), and \( i, j = 1, 2, 3 \), and where \( \delta \) is the Reeb commutator function.

The connection \( \nabla \) is called the canonical connection of \( M \), and also satisfies

\[
\nabla_X \xi_i = \beta(\eta_k(X) \xi_i - \eta_j(X) \xi_k), \quad \nabla_X \eta_i = \beta(\eta_k(X) \eta_i - \eta_j(X) \eta_k)
\]

for every vector field \( X \) on \( M \). Therefore, when \( \beta = 0 \) the canonical connection parallelizes all the structure tensor fields, in which case we call the almost 3-contact metric manifold parallel.
Both 3-(α, δ)-Sasaki manifolds and 3-δ-cosymplectic manifolds turn out to be canonical. In particular,

**Theorem 5** ([1]). Every 3-(α, δ)-Sasaki manifold is a canonical almost 3-contact metric manifold, with constant Reeb Killing function \( \beta = 2(\delta - 2\alpha) \). The torsion \( T \) of the canonical connection \( \nabla \) is given by

\[
T = 3 \sum_{i=1}^{3} \eta_i \wedge d\eta_i + 8(\delta - \alpha) \eta_{123} = 2\alpha \sum_{i=1}^{3} \eta_i \wedge \Phi_i^H + 2(\delta - 4\alpha) \eta_{123}
\]

and satisfies \( \nabla T = 0 \).

We denote by \( \eta_{123} \) the 3-form \( \eta_1 \wedge \eta_2 \wedge \eta_3 \). From the above theorem, it follows that any 3-(α, δ)-Sasaki manifold is a parallel canonical manifold if and only if \( \delta = 2\alpha \), in which case the 3-(α, δ)-Sasaki structure is positive (\( \alpha \delta > 0 \)).

Regarding 3-δ-cosymplectic manifolds, we have:

**Proposition 5** ([1]). Any 3-δ-cosymplectic manifold is a parallel canonical almost 3-contact metric manifold. The torsion of the canonical connection is given by

\[
T = -2\delta \eta_{123}.
\]

For the class of 3-(0, δ)-Sasaki manifolds, we have the following

**Proposition 6.** Every 3-(0, δ)-Sasaki manifold is a canonical almost 3-contact metric manifold, with constant Reeb Killing function \( \beta = 2\delta \). The torsion \( T \) of the canonical connection \( \nabla \) is given by

\[
T = 2\delta \eta_{123},
\]

which satisfies \( \nabla T = 0 \).

**Proof.** Let \((M, \varphi, \xi, \eta, g)\) be a 3-(0, δ)-Sasaki manifold. We showed that the structure is hypernormal and the Reeb vector fields are Killing. Furthermore, by the second equation in (12), \( d\Phi(X, Y, Z) = 0 \) for every \( X, Y, Z \in \Gamma(\mathcal{H}) \). Therefore, conditions (i), (ii) and (iii) in Definition 4 are easily verified. As regards condition (iv), applying the first equation in (4) and Corollary 1, for every \( X, Y \in \Gamma(\mathcal{H}) \) we have

\[
A_{ij}(X, Y) = 0, \quad A_{ij}(X, Y) = -A_{ji}(X, Y) = 2\delta \Phi_k(X, Y).
\]

Hence, the structure is canonical with Reeb commutator function \( \beta = 2\delta \). Now, by Theorem 4, taking also into account the fact that the vertical distribution is integrable, the only non-vanishing term of the canonical connection is \( T(\xi_1, \xi_2, \xi_3) = 2\delta \), so that \( T = 2\delta \eta_{123} \). Although the structure is not parallel when \( \delta \neq 0 \), the torsion satisfies \( \nabla T = 0 \), as by (20), the 3-form \( \eta_{123} \) is parallel with respect to \( \nabla \). \( \square \)

The above result generalizes the result obtained in [2] for the Lie group described in Example 1 (see also Remark 2).

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