UNIFORM ERROR ESTIMATES FOR THE LANCZOS METHOD

JOHN C. URSCHEL

ABSTRACT. The Lanczos method is one of the most powerful and fundamental techniques for solving an extremal symmetric eigenvalue problem. Convergence-based error estimates are well studied, with the estimate depending heavily on the eigenvalue gap. However, in practice, this gap is often relatively small, resulting in significant overestimates of error. One way to avoid this issue is through the use of uniform error estimates, namely, bounds that depend only on the dimension of the matrix and the number of iterations. In this work, we prove a number of upper and lower uniform error estimates for the Lanczos method. These results include the first known lower bounds for error in the Lanczos method and significantly improved upper bounds for error measured in the $p$-norm, $p > 1$. These lower bounds imply that the maximum error of $m$ iterations of the Lanczos method over all $n \times n$ symmetric matrices does indeed depend on the dimension $n$. In addition, we prove more specific results for matrices that possess some level of eigenvalue regularity or whose eigenvalues converge to some limiting empirical spectral distribution. Through numerical experiments, we show that the theoretical estimates of this paper do apply to practical computations for reasonably sized matrices.

1. Introduction

The computation of extremal eigenvalues of matrices is one of the most fundamental problems in numerical linear algebra. When a given matrix is large and sparse, methods such as the Jacobi eigenvalue algorithm and QR algorithm become computationally infeasible, and, therefore, techniques that take advantage of the sparsity of the matrix are required. Krylov subspace methods are a powerful class of techniques for approximating extremal eigenvalues, most notably the Arnoldi iteration for non-symmetric matrices and the Lanczos method for symmetric matrices.

The Lanczos method is a technique that, given a symmetric matrix $A \in \mathbb{R}^{n \times n}$ and an initial vector $b \in \mathbb{R}^n$, iteratively computes a tridiagonal matrix $T_m \in \mathbb{R}^{m \times m}$ that satisfies $T_m = Q_m^T A Q_m$, where $Q_m \in \mathbb{R}^{n \times m}$ is an orthonormal basis of the Krylov subspace

$$K_m(A, b) = \text{span}\{b, Ab, \ldots, A^{m-1}b\}.$$ 

The eigenvalues of $T_m$, denoted by $\lambda^{(m)}_1(A, b) \geq \ldots \geq \lambda^{(m)}_m(A, b)$, are the Rayleigh-Ritz approximations to the eigenvalues $\lambda_1(A) \geq \ldots \geq \lambda_n(A)$ of $A$ on $K_m(A, b)$, and, therefore, are given by

$$\lambda^{(m)}_i(A, b) = \min_{U \subseteq K_m(A, b)} \max_{\dim(U) = m+1-i, x \neq 0} \frac{x^T A x}{x^T x}, \quad i = 1, \ldots, m,$$

or, equivalently,

$$\lambda^{(m)}_i(A, b) = \max_{U \subseteq K_m(A, b)} \min_{\dim(U) = i, x \neq 0} \frac{x^T A x}{x^T x}, \quad i = 1, \ldots, m.$$

This description of the Lanczos method is sufficient for a theoretical analysis of error (i.e., without round-off error), but, for completeness, we provide a short description of the Lanczos method (when $K_m(A, b)$ is full-rank) in Algorithm 1 \cite{17}. For a more detailed discussion of the nuances of practical implementation and techniques to minimize the effects of round-off error, we refer the reader to \cite{7} Section 10.3. If $A$ has $\nu n$ non-zero entries, then Algorithm 1 outputs $T_m$ after approximately $(2\nu + 8)mn$ floating point operations. From here, the eigenvalues of a tridiagonal matrix can be easily computed using a number of different techniques, such as the divide-and-conquer algorithm and the fast multipole method. The complexity of this computation is typically negligible compared to the Lanczos tridiagonalization procedure, as, in practice,
Algorithm 1 Lanczos Method

Input: symmetric matrix $A \in \mathbb{R}^{n \times n}$, vector $b \in \mathbb{R}^n$, number of iterations $m$.
Output: symmetric tridiagonal matrix $T_m \in \mathbb{R}^{m \times m}$, $T_m(i, i) = \alpha_i$, $T_m(i, i+1) = \beta_i$,
satisfying $T_m = Q_m^T A Q_m$, where $Q_m = [q_1 \ldots q_m]$.

Set $\beta_0 = 0, q_0 = 0, q_1 = b/\|b\|$
For $i = 1, \ldots, m$,
$v = A q_i$
$\alpha_i = q_i^T v$
$v = v - \alpha_i q_i - \beta_{i-1} q_{i-1}$
$\beta_i = \|v\|$
$q_{i+1} = v/\beta_i$

when using an iterative method to estimate extreme eigenvalues of a large matrix, $m$ is orders of magnitude
less than $n$.

Equation (1) for the Ritz values $\lambda^{(m)}_1$ illustrates the significant improvement that the Lanczos method
provides over the power method for extremal eigenvalues. Whereas the power method uses only the iterate
$A^m b$ as an approximation of the largest magnitude eigenvalue, the Lanczos method uses the span of all of
the iterates of the power method (given by $K_m+1(A, b)$). However, the analysis of the Lanczos method is
significantly more complicated than that of the power method. While the majority of approximation results
for the power method are easily derived, a number of open questions regarding the analysis of the Lanczos
method remain. Error estimates for extremal eigenvalue approximation using the Lanczos method have been
well studied, most notably by Kaniel [8], Paige [12], Saad [14], and Kuczynski and Wozniakowski [9] (other
notable work includes [4, 10, 13, 18]). The work of Kaniel, Paige, and Saad focuses on the convergence of
the Lanczos method as $m$ increases, and, therefore, their results have strong dependence on the spectrum of
the matrix $A$ and the choice of initial vector $b$. For example, a standard result of this type is the estimate
\[
\frac{\lambda_1(A) - \lambda^{(m)}_1(A, b)}{\lambda_1(A) - \lambda_n(A)} \leq \left( \frac{\tan \angle(b, \varphi_1)}{T_{m-1}(1 + 2\gamma)} \right)^2, \quad \gamma = \frac{\lambda_1(A) - \lambda_2(A)}{\lambda_2(A) - \lambda_n(A)},
\]
where $\varphi_1$ is the eigenvector corresponding to $\lambda_1$, and $T_{m-1}$ is the $(m-1)^{th}$ degree Chebyshev polynomial
of the first kind [15, Theorem 6.4]. Kuczynski and Wozniakowski took a quite different approach, and estimated
the maximum expected relative error $(\lambda_1 - \lambda^{(m)}_1)/\lambda_1$ over all $n \times n$ symmetric positive definite matrices,
resulting in error estimates that depend only on the dimension $n$ and the number of iterations $m$. They
produced the estimate
\[
\sup_{\lambda \in S^{n+1}_+, \ b \sim U(S^{n-1})} \mathbb{E} \left[ \frac{\lambda_1(A) - \lambda^{(m)}_1(A, b)}{\lambda_1(A)} \right] \leq .103 \frac{\ln^2(n(m-1)^4)}{(m-1)^2},
\]
for all $n \geq 8$ and $m \geq 4$, where $S^{n+1}_+$ is the set of all $n \times n$ symmetric positive definite matrices, and
the expectation is with respect to the uniform probability measure on the hypersphere $S^{n-1} = \{ x \in \mathbb{R}^n \mid \|x\| = 1 \}$ [9, Theorem 3.2]. One can quickly verify that this equation also holds when the $\lambda_1(A)$ term in the denominator
is replaced by $\lambda_1(A) - \lambda_n(A)$, and the supremum over $S^{n+1}_+$ is replaced by the maximum over the set of all $n \times n$ symmetric matrices, denoted by $S^n$.

Both of these approaches have their benefits and drawbacks. If an extremal eigenvalue is known to have
a reasonably large eigenvalue gap (based on application or construction), then a distribution dependent
estimate provides a very good approximation of error, even for small $m$. However, if the eigenvalue gap is
not especially large, then distribution dependent estimates can significantly overestimate error, and estimates
that depend only on $n$ and $m$ are preferable. This is illustrated by the following elementary, yet enlightening,
example.

Example 1.1. Let $A \in S^{n+1}_+$ be the tridiagonal matrix resulting from the discretization of the Laplacian
operator on an interval with Dirichlet boundary conditions, namely, $A_{i,i} = 2$ for $i = 1, \ldots, n$ and $A_{i,i+1} =
A_{i+1,i} = -1$ for $i = 1, \ldots, n-1$. The eigenvalues of $A$ are given by $\lambda_i(A) = 2 + 2 \cos(i\pi/(n+1))$, $i = 1, \ldots, n$.
Consider the approximation of $\lambda_1(A)$ by $m$ iterations of the Lanczos method. For a random choice of $b$, the
expected value of \( \tan^2 \angle(b, \varphi_1) \) is \( (1 + o(1))n \). If \( \tan^2 \angle(b, \varphi_1) = Cn \) for some constant \( C \), then (3) produces the estimate

\[
\frac{\lambda_1(A) - \lambda_1^{(m)}(A, b)}{\lambda_1(A) - \lambda_n(A)} \leq Cn \frac{1}{T_{m-1}} \left( 1 + 2 \tan \left( \frac{\pi}{2(n+1)} \right) \tan \left( \frac{3\pi}{2(n+1)} \right) \right)^{-2} \approx n(1 + O(n^{-1}))^{-m} \approx n.
\]

In this instance, the estimate is a trivial one for all choices of \( m \), which varies greatly from the error estimate (4) of order \( \ln^2 n/m^2 \). The exact same estimate holds for the smallest eigenvalue \( \lambda_n(A) \) when the corresponding bounds are applied, since \( 4I - A \) is similar to \( A \).

Now, consider the approximation of the largest eigenvalue of \( B = A^{-1} \) by \( m \) iterations of the Lanczos method. The matrix \( B \) possesses a large gap between the largest and second-largest eigenvalue, which results in a value of \( \gamma \) for (3) that remains bounded below by a constant independent of \( n \), namely \( \gamma = (2 \cos(\pi/(n+1)) + 1)/(2 \cos(\pi/(n+1)) - 1) \). Therefore, in this instance, the estimate (3) illustrates a constant convergence rate, produces non-trivial bounds (for typical \( b \)) for \( m = \Theta(\ln n) \), and is preferable to the error estimate (4) of order \( \ln^2 n/m^2 \).

This example illustrates the benefits and drawbacks that both types of estimates possess. More generally, if a matrix \( A \) has eigenvalue gap \( \gamma \lesssim n^{-\alpha} \) and the initial vector \( b \) satisfies \( \tan^2 \angle(b, \varphi_1) \gtrsim n \), then the error estimate (3) is a trivial one for \( m \lesssim n^{\alpha/2} \). This implies that the estimate (3) is most useful when the eigenvalue gap is constant, or tends to zero very slowly (i.e., logarithmically). When the gap is not especially large (say, \( n^{-\alpha}, \alpha \) constant), then uniform error estimates are preferable for small values of \( m \).

In this work, we focus almost exclusively on uniform bounds, namely, error estimates that hold uniformly over some large set of matrices (typically \( S^m_{n+} \) or \( S^n \)). We begin by recalling some of the key existing uniform error estimates for the Lanczos method.

1.1. Related Work. Uniform error estimates for the Lanczos method have been produced almost exclusively for symmetric positive definite matrices, as error estimates for extremal eigenvalues of symmetric matrices can be produced from estimates for \( S^m_{n+} \) relatively easily. In addition, the majority of results apply only to either \( \lambda_1(A), \lambda_n(A) \), or some function of the two (i.e., condition number). All estimates are probabilistic in nature and take the initial vector \( b \) to be uniformly distributed on the hypersphere. Here we provide a short, non-exhaustive description of some of the key uniform error estimates previously produced for the Lanczos method.

In (4), Kuczynski and Wozniakowski produced a complete analysis of the power method and provided a number of upper bounds for the Lanczos method. Most notably, they produced error estimate (4) and provided the following upper bound for the probability that the relative error \( (\lambda_1 - \lambda_1^{(m)})/\lambda_1 \) is greater than some value \( \epsilon \):

\[
(5) \quad \sup_{A \in S^m_{n+}, \ b \sim \mathcal{U}(S^{n-1})} \mathbb{P} \left[ \frac{\lambda_1(A) - \lambda_1^{(m)}(A, b)}{\lambda_1(A)} > \epsilon \right] \leq 1.648\sqrt{n}e^{-\sqrt{\epsilon(2m-1)}}.
\]

However, the authors were unable to produce any lower bounds for (4) or (5), and stated that a more sophisticated analysis would most likely be required. In the same paper, numerical experiments were performed for the Lanczos method, and produced an error of the order \( m^{-2} \), leading the authors to suggest that the error estimate (4) may be an overestimate, and that the \( \ln^2 n \) term may be unnecessary.

In (10), Kuczynski and Wozniakowski noted that the above estimates immediately translate to relative error estimates for minimal eigenvalues when the error \( \lambda_1^{(m)} - \lambda_n \) is considered relative to \( \lambda_1 - \lambda_n \) (both normalizations can be shown to produce the same bound). However, it can be quickly verified that there exist sequences in \( S^m_{n+} \), for which the quantity \( \mathbb{E} \left[ (\lambda_1^{(m)} - \lambda_n)/\lambda_n \right] \) is unbounded. These results for minimal eigenvalues, combined with (4), led to error bounds for estimating the condition number of a matrix. Unfortunately, error estimates for the condition number face the same issue as the quantity \( (\lambda_1^{(m)} - \lambda_n)/\lambda_n \), and therefore, only estimates that depend on the value of the condition number can be produced.

The proof technique used to produce (4) works specifically for the quantity \( \mathbb{E} \left[ (\lambda_1 - \lambda_1^{(m)})/\lambda_1 \right] \) (i.e., the 1-norm), and does not carry over to more general p-norms of the form \( \mathbb{E} \left[ ((\lambda_1 - \lambda_1^{(m)})/\lambda_1)^p \right]^{1/p}, p \in [1, \infty) \). Later, in (11) Theorem 5.2, \( r = 1 \), Del Corso and Manzini produced an upper bound for arbitrary p-norms,
Theorem 1.2. \[ \sup_{A \in S^n \sim U(S^{n-1})} \mathbb{E} \left[ \left( \frac{\lambda_1(A) - \lambda_1^{(m)}(A, b)}{\lambda_1(A)} \right)^p \right]^{1/p} \leq \frac{1}{m^{1/p}} \left( \frac{\Gamma(p - 1/2)\Gamma(n/2)}{\Gamma(p)\Gamma((n - 1)/2)} \right)^{1/p} = \Theta \left( \left( \frac{\sqrt{n}}{m} \right)^{1/p} \right). \]

This bound is clearly worse than \[ \frac{1}{m^{1/p}} \left( \frac{\Gamma(p - 1/2)\Gamma(n/2)}{\Gamma(p)\Gamma((n - 1)/2)} \right)^{1/p} \], but it is the best known bound for \( p > 1 \). Again, the authors were unable to produce any lower bounds.

1.2. Contributions and Remainder of Paper. In what follows, we obtain improved upper bounds for \( \mathbb{E} \left[ (\lambda_1 - \lambda_1^{(m)})/\lambda_1 \right]^{1/p} \) for all \( p \in (1, \infty) \), and produce nearly-matching lower bounds. In particular, we prove the following theorem.

Theorem 1.2.

\[
\max_{A \in S^n \sim U(S^{n-1})} \mathbb{E} \left[ \frac{\lambda_1(A) - \lambda_1^{(m)}(A, b)}{\lambda_1(A) - \lambda_n(A)} \right] \geq (1 - o(1)) \times \begin{cases} 
1 & \text{for } m = o(\ln n) \\
1 - \alpha & \text{for } m \leq \frac{\ln n - \ln(\ln^2 n)}{4 \ln \left( \frac{1 + \sqrt{2}}{\sqrt{\alpha}} \right)}, \; \alpha \in (0, 1] \\
\frac{249 \ln n}{m^2 \ln n} & \text{for } m = O(\ln n) \\
\frac{637}{m^2} & \text{for } m = o \left( \frac{n^{1/2}}{\ln^{1/2} n} \right) 
\end{cases}
\]

and

\[
\max_{A \in S^n \sim U(S^{n-1})} \mathbb{E} \left[ \left( \frac{\lambda_1(A) - \lambda_1^{(m)}(A, b)}{\lambda_1(A) - \lambda_n(A)} \right)^p \right]^{1/p} \leq 0.068 \frac{(n(m - 1)^{8p})}{(m - 1)^2}, \quad \text{for } n \geq 100, \; m \geq 10, \; p \geq 1.
\]

**Proof.** The result follows from Lemmas 3.4, 3.7, 3.10 and 4.1. \( \square \)

By Hölder’s inequality, the lower bounds in Theorem 1.2 also hold for arbitrary \( p \)-norms, \( p \in [1, \infty) \), as well. Through the course of producing the above estimates, we can make even stronger statements regarding bounds that occur with high probability. In particular, we also prove the following theorem.

Theorem 1.3.

\[
\sup_{A \in S^n \sim U(S^{n-1})} \mathbb{P} \left[ \frac{\lambda_1(A) - \lambda_1^{(m)}(A, b)}{\lambda_1(A) - \lambda_n(A)} \geq 1 - o(1) \right] \geq 1 - o(1/n)
\]

for \( m = o(\ln n) \),

\[
\sup_{A \in S^n \sim U(S^{n-1})} \mathbb{P} \left[ \frac{\lambda_1(A) - \lambda_1^{(m)}(A, b)}{\lambda_1(A) - \lambda_n(A)} \geq (1 - o(1))(1 - \alpha) \right] \geq 1 - o(1/n)
\]

for \( m \leq \frac{\ln n - \ln(\ln^2 n)}{4 \ln \left( \frac{1 + \sqrt{2}}{\sqrt{\alpha}} \right)}, \; \alpha \in (0, 1] \),

\[
\sup_{A \in S^n \sim U(S^{n-1})} \mathbb{P} \left[ \frac{\lambda_1(A) - \lambda_1^{(m)}(A, b)}{\lambda_1(A) - \lambda_n(A)} \geq \frac{249 \ln n}{m^2 \ln n} \right] \geq 1 - o(1/n)
\]

for \( m = O(\ln n) \),

\[
\sup_{A \in S^n \sim U(S^{n-1})} \mathbb{P} \left[ \frac{\lambda_1(A) - \lambda_1^{(m)}(A, b)}{\lambda_1(A) - \lambda_n(A)} \geq \frac{637}{m^2} \right] \geq 1 - o(1/n)
\]

for \( m = o \left( \frac{n^{1/2}}{\ln^{1/2} n} \right) \), and

\[
\sup_{A \in S^n \sim U(S^{n-1})} \mathbb{P} \left[ \frac{\lambda_1(A) - \lambda_1^{(m)}(A, b)}{\lambda_1(A) - \lambda_n(A)} \leq \frac{0.571 \ln^2 (n(m - 1)^{2/3})}{(m - 1)^2} \right] \geq 1 - o(1/n)
\]

for all \( m > 1 \).
Proof. The result follows from Lemmas 3.4, 3.7, 3.10 and 4.2.

These two results mark a major breakthrough in the understanding of the Lanczos algorithm and its error, as the above estimates are the first lower bounds produced for the Lanczos algorithm. Of course, the above (and all following) results are equally applicable to $\lambda_n$, as the Krylov subspace (and, therefore, the output of the Lanczos algorithm) is unaffected by shifting and scaling, namely $K_m(A, b) = K_m(\alpha A + \beta I, b)$ for all $\alpha, \beta \in \mathbb{R}$, $\alpha \neq 0$. To complement these two results, we also provide error analysis for matrices that have a certain structure. We produce improved dimension-free upper bounds for matrices that have some level of "regularity" near $\lambda_1$, and prove a powerful theorem that can be used to determine the asymptotic relative error for any sequence of matrices $X_n \in S^n$, $n = 1, 2, \ldots$, that exhibits suitable convergence of its empirical spectral distribution. As an example, we use this theorem to obtain a tight bound of order $m^{-2}$ for Wigner matrices. This example illustrates that the inverse quadratic dependence on the number of iterations $m$ in the estimates produced in this paper does not simply illustrate the worst case, but is actually indicative of the average case in some sense. We also produce results similar to Theorems 1.2 and 1.3 for arbitrary eigenvalues $\lambda_i$. The lower bounds follow relatively quickly from the estimates for $\lambda_1$, but the upper bounds require some mild assumptions on the eigenvalue gaps of the matrix. These results mark the first uniform-type bounds for arbitrary eigenvalues. Finally, we present numerical experiments that support the bounds require some mild assumptions on the eigenvalue gaps of the matrix. These results mark the first uniform-type bounds for arbitrary eigenvalues. Finally, we present numerical experiments that support the accuracy and practical usefulness of the theoretical estimates in this work.

The remainder of the paper is as follows. In Section 2, we prove basic results regarding relative error and make a number of fairly straightforward observations. In Section 3, we prove asymptotic lower bounds for relative error. In Section 4, we produce improved upper bounds for the relative error in an arbitrary $p$-norm. In Section 5, we produce a dimension-free error estimate for a large class of matrices and prove a theorem that can be used to determine the asymptotic relative error for any sequence of matrices $X_n \in S^n$, $n = 1, 2, \ldots$, with suitable convergence of its empirical spectral distribution. We apply this theorem to Wigner matrices and obtain a tight estimate of relative error. In Section 6, under some mild additional assumptions, we prove versions of Theorems 1.2 and 1.3 for arbitrary eigenvalues. Finally, in Section 7, we perform a number of numerical experiments and discuss how these numerics compare to the theoretical estimates in this work.

2. Preliminary Results

The Lanczos method applies only to symmetric matrices, and, therefore, all matrices in this paper are assumed to belong to $S^n$. In fact, we may restrict ourselves to $S^n_{++}$, since results for $S^n_{++}$ are easily extended to $S^n$ by proper shifting and scaling. In particular, for any matrix $A \in S^n_{++}$ and $0 < \epsilon < \lambda_n(A)$, we have $A - \epsilon I \in S^n_{++}$ and

$$\frac{\lambda_1(A - \epsilon I) - \lambda_1^{(m)}(A - \epsilon I, b)}{\lambda_1(A - \epsilon I)} = \frac{\lambda_1(A) - \lambda_1^{(m)}(A)}{\lambda_1(A)} - \epsilon \frac{\lambda_1(A) - \lambda_1^{(m)}(A)}{\lambda_1(A)},$$

The quantity $(\lambda_1(A) - \lambda_1^{(m)}(A, b)) / (\lambda_1(A) - \lambda_n(A))$ is unaffected by shifts, and so

$$\sup_{A \in S^n_{++}} \mathbb{E}_{b \sim U(S^n_{++})}\left[\left(\frac{\lambda_1(A) - \lambda_1^{(m)}(A, b)}{\lambda_1(A)}\right)^{p_1}\right]^{1/p} = \max_{A \in S^n_{++}} \mathbb{E}_{b \sim U(S^n_{++})}\left[\left(\frac{\lambda_1(A) - \lambda_1^{(m)}(A, b)}{\lambda_1(A)}\right)^{p_1}\right]^{1/p}.$$

Therefore, for the remainder of the paper, all matrices are also assumed to be positive definite. We begin by rewriting expression (2) for $\lambda_1^{(m)}$ in terms of polynomials. The Krylov subspace $K_m(A, b)$ can be alternatively defined as

$$K_m(A, b) = \{ P(A) b \mid P \in \mathcal{P}_{m-1} \},$$

where $\mathcal{P}_{m-1}$ is the set of all real-valued polynomials of degree at most $m - 1$. Suppose $A \in S^n_{++}$ has eigendecomposition $A = Q \Lambda Q^T$, where $Q \in \mathbb{R}^{n \times n}$ is an orthogonal matrix and $\Lambda \in \mathbb{R}^{n \times n}$ is the diagonal matrix satisfying $\Lambda(i, i) = \lambda_i(A)$, $i = 1, \ldots, n$. Then we have the relation

$$\lambda_1^{(m)} = \max_{x \in K_m(A, b), x \neq 0} \frac{x^T A x}{x^T x} = \max_{P \in \mathcal{P}_{m-1}, P \neq 0} \frac{\tilde{b}^T P^2(A) \tilde{b}}{\tilde{b}^T P^2(A) b} = \max_{P \in \mathcal{P}_{m-1}, P \neq 0} \frac{\tilde{b}^T P^2(A) \Lambda \tilde{b}}{\tilde{b}^T P^2(A) b}.$$
where \( \tilde{b} = Q^T b \). We may assume \( \lambda_1(A) = 1 \), since

\[
\frac{\lambda_1(A) - \lambda_1^{(m)}(A, b)}{\lambda_1(A)} = 1 - \max_{x \neq 0} \frac{x^T [A/\lambda_1] x}{x^T x}.
\]

Assuming \( \lambda_1(A) = 1 \), the relative error is given by

\[
\frac{\lambda_1(A) - \lambda_1^{(m)}(A, b)}{\lambda_1(A)} = \min_{P \in \mathcal{P}_{m-1}} \frac{\sum_{i=2}^n \tilde{b}_i^2 P^2(\lambda_i)(1 - \lambda_i)}{\sum_{i=1}^n \tilde{b}_i^2 P^2(\lambda_i)}.
\]

and the expected \( p^{th} \) moment of the relative error is given by

\[
E_p \left[ \left( \frac{\lambda_1(A) - \lambda_1^{(m)}(A, b)}{\lambda_1(A)} \right)^p \right] = \int_{S^{m-1}} \min_{P \neq 0} \left[ \frac{\sum_{i=2}^n \tilde{b}_i^2 P^2(\lambda_i)(1 - \lambda_i)}{\sum_{i=1}^n \tilde{b}_i^2 P^2(\lambda_i)} \right]^p d\sigma(\tilde{b}),
\]

where \( \sigma \) is the uniform probability measure on \( S^{m-1} \). Because the relative error does not depend on the norm of \( \tilde{b} \) or the sign of any entry, we can replace the integral over \( S^{m-1} \) by an integral of \( y = (y_1, ..., y_n) \) over \( [0, \infty)^n \) with respect to the joint chi-square probability density function

\[
f_Y(y) = \frac{1}{(2\pi)^{n/2}} \exp \left\{ -\frac{1}{2} \sum_{i=1}^n y_i \right\} \prod_{i=1}^n y_i^{-1/2}
\]

of \( n \) independent chi-square random variables \( Y_1, ..., Y_n \sim \chi_1^2 \) with one degree of freedom each. In particular, we have

\[
E_p \left[ \left( \frac{\lambda_1(A) - \lambda_1^{(m)}(A, b)}{\lambda_1(A)} \right)^p \right] = \int_{[0, \infty)^n} \min_{P \neq 0} \left[ \frac{\sum_{i=2}^n y_i P^2(\lambda_i)(1 - \lambda_i)}{\sum_{i=1}^n y_i P^2(\lambda_i)} \right]^p f_Y(y) dy.
\]

Similarly, probabilistic estimates with respect to \( b \sim \mathcal{U}(S^{m-1}) \) can be replaced by estimates with respect to \( Y_1, ..., Y_n \sim \chi_1^2 \), as

\[
P \left[ \frac{\lambda_1(A) - \lambda_1^{(m)}(A, b)}{\lambda_1(A)} \geq \epsilon \right] = P \left[ \min_{P \neq 0} \frac{\sum_{i=2}^n Y_i P^2(\lambda_i)(1 - \lambda_i)}{\sum_{i=1}^n Y_i P^2(\lambda_i)} \geq \epsilon \right].
\]

For the remainder of the paper, we almost exclusively use equation (8) for expected relative error and equation (9) for probabilistic bounds for relative error. If \( P \) minimizes the expression in equation (8) or (9) for a given \( y \), then any polynomial of the form \( \alpha P, \alpha \in \mathbb{R}, \{0\} \), is also a minimizer. Therefore, without any loss of generality, we can alternatively minimize over the set \( \mathcal{P}_{m-1} = \{Q \in \mathcal{P}_{m-1} : Q(1) = 1\} \).

3. **Asymptotic Lower Bounds**

In this section, we produce asymptotic lower bounds for the maximum expected relative error of \( m \) iterations of the Lanczos method for estimating the largest eigenvalue of a matrix \( A \in S^+_n \). By Hölder’s inequality, lower bounds for arbitrary \( p \)-norms follow immediately from bounds for the 1-norm. The general technique we use to produce lower bounds is as follows. We choose an infinite sequence of matrices \( \{A_n\}_{n=1}^\infty \), \( A_n \in S^+_n \), treat \( m \) as a function of \( n \), and show that, as \( n \) tends to infinity, for “most” choices of an initial vector \( \bar{b} \), the relative error of this sequence of matrices is well approximated by an integral polynomial minimization problem for which relatively tight bounds can be obtained.

The first goal of this section is to show that, contrary to experimental results and assertions of [9], the maximum expected relative error does indeed depend on \( n \). We first consider the regime in which \( m = O(\ln n) \), a region in which no non-constant upper bounds have been produced. We show that, as \( n \) increases, the maximum expected relative error becomes arbitrarily close to one for any \( m \) that is \( o(\ln n) \), and that, in general, one cannot guarantee an expected relative error less than .99 until \( m \geq 1/13 \ln n \).

In order to do so, we must first produce a lower bound for the \( L^2[0, \alpha] \) norm, \( \alpha \in (0, 1] \), of all polynomials \( Q \in \mathcal{P}_{m-1} \). We prove the following lemma.
Lemma 3.1. Let \( m \in \mathbb{N} \) and \( \alpha \in (0,1] \). Then
\[
\min_{Q \in P_{m-1}(1)} \int_0^\alpha Q^2(y) \, dy \geq (1 + o(1)) \frac{2^{15/4} \pi^{3/2} \sqrt{m - 1} \alpha^{2m-1}}{m(1 + \sqrt{2})^{4m}}.
\]

Proof. Let \( Q(y) = \sum_{i=0}^{m-1} a_i y^i \), \( a_i \in \mathbb{R} \), \( i = 0, \ldots, m - 1 \). Then
\[
\int_0^\alpha Q^2(y) \, dy = \int_0^\alpha \left[ \sum_{i=0}^{m-2} y^i \sum_{p+q=i} a_p a_q \right] \, dy = \sum_{i=0}^{m-2} \sum_{p+q=i} a_p a_q = a a^T DHDa,
\]
where \( a = (a_0, a_1, \ldots, a_{m-1})^T \in \mathbb{R}^m \), \( D \in \mathbb{R}^{m \times m} \) is the diagonal matrix satisfying \( D_{\alpha}(i,i) = \alpha^i \), \( i = 0, \ldots, m - 1 \), and \( H \in \mathbb{R}^{m \times m} \) is the Hilbert matrix \( H_{i,j} = (i+j+1)^{-1}, i,j = 0, \ldots, m - 1 \). We have converted our minimization problem into the minimization of the quadratic form \( a a^T DHDa \) subject to the condition \( a^T 1 = 1 \), where \( 1 = (1, \ldots, 1)^T \in \mathbb{R}^m \).

The Hilbert matrix \( H \) is positive definite and, by [19, Equation 3.35], equals
\[
\lambda_{\min}(H) = (1 + o(1)) \pi^{3/2} (8 + 6\sqrt{2})^{5/2} \sqrt{m - 1} (1 + \sqrt{2})^{-4m-5}.
\]
The condition \( a^T 1 = 1 \) implies that \( \|a\|_1 \geq 1 \). Therefore,
\[
\|Da\|_2 \geq \frac{\|Da\|_1}{\sqrt{m}} \geq \frac{\alpha^{m-1}}{\sqrt{m}}
\]
and
\[
a a^T DHDa \geq a \lambda_{\min}(H) \|a\|_2^2 \geq (1 + o(1)) \frac{\pi^{3/2} (8 + 6\sqrt{2})^{5/2} \sqrt{m - 1} \alpha^{2m-1}}{(1 + \sqrt{2})^5} \frac{m(1 + \sqrt{2})^{4m}}{m(1 + \sqrt{2})^{4m}}.
\]

This lower bound is only as tight as our purposes require, and tighter results can be obtained at the cost of brevity. In order to convert Lemma 3.1 into an asymptotic lower bound for the error of the Lanczos method, we require the following two propositions regarding Gaussian quadrature and Chernoff bounds for chi square random variables.

Proposition 3.2. ([3] Chapter 1.13, [5]) Let \( P \in P_{2k-1} \), \( \{x_i\}_{i=1}^k \) be the zeros of the \( k \)th degree Legendre polynomial \( P_k(x) \), \( P_k(1) = 1 \), in descending order \( (x_1 > \ldots > x_k) \), and \( w_i = 2(1-x_i^2)^{-1} |P_k(x_i)|^{-2} \), \( i = 1, \ldots, k \). Then
1. \( \int_{-1}^1 P(x) \, dx = \sum_{i=1}^k w_i P(x_i) \),
2. \( x_i = \left(1 - \frac{1}{8k^2}\right) \cos \left(\frac{4i-1}{4k+2}\pi\right) + O(k^{-3}) \), \( i = 1, \ldots, k \),
3. \( \frac{\pi}{k+1/2} \sqrt{1-x_i^2} \left(1 - \frac{1}{8(k+1/2)^2(1-x_i^2)}\right) \leq w_1 < w_2 < \ldots < w_\left[k+1\atop 2\right] \leq \frac{\pi}{k+1/2} \).

Proposition 3.3. Let \( Z \sim \chi_k^2 \). Then
\[
P[Z \leq x] \leq \left[ \frac{x}{k} \exp \left\{1 - \frac{x}{k}\right\}\right]^{k/2} \quad \text{for } x \leq k
\]
and
\[
P[Z \geq x] \leq \left[ \frac{x}{k} \exp \left\{1 - \frac{x}{k}\right\}\right]^{k/2} \quad \text{for } x \geq k.
\]

Proof. The result follows from taking [2] Lemma 2.2 and letting \( d \to \infty \).

We are now prepared to prove our desired lower bound.
Lemma 3.4. Let $\alpha \in (0, 1]$. If

$$m \leq \frac{\ln n - \ln(\ln^2 n)}{4 \ln \left( \frac{1 + \sqrt{2}}{\sqrt{\alpha}} \right)},$$

then

$$\sup_{A \in S^n_{++}, b \sim U(S^{n-1})} \mathbb{P} \left[ \frac{\lambda_1(A) - \lambda_1^{(m)}(A, b)}{\lambda_1(A)} \geq (1 - o(1))(1 - \alpha) \right] \geq 1 - o(1/n),$$

and, therefore,

$$\sup_{A \in S^n_{++}, b \sim U(S^{n-1})} \mathbb{E} \left[ \frac{\lambda_1(A) - \lambda_1^{(m)}(A, b)}{\lambda_1(A)} \right] \geq (1 - o(1))(1 - \alpha).$$

In addition, if $m = o(\ln n)$, then the maximum expected error is $1 - o(1)$.

Proof: The main idea of the proof is to choose a matrix with eigenvalues that partition $[0, \alpha]$, use equation [9] for probabilistic estimates, and show that a large subset of $[0, \infty)^n$ (with respect to $f_Y(y)$) satisfies certain conditions that allow us to lower bound relative error using the integral minimization problem of Lemma 3.1. Throughout the proof, we assume $m = O(\ln n)$.

Let $A \in \mathbb{R}^{n \times n}$ have eigenvalue $\lambda_1 = 1$ with multiplicity one, and the remaining eigenvalues given by $\alpha(x_j + 1)/2$, $j = 1, \ldots, m$, where $x_j$ is the $j^{th}$ zero of the $m^{th}$ degree Legendre polynomial, and each eigenvalue has multiplicity at least $\lfloor (n - 1)/m \rfloor$. By equation [9],

$$\mathbb{P} \left[ b \sim U(S^{n-1}) \right] \left[ \frac{\lambda_1(A) - \lambda_1^{(m)}(A, b)}{\lambda_1(A)} \geq \epsilon \right] = \mathbb{P} \left[ \min_{Y_i \sim \lambda_i^2} \sum_{i=2}^n Y_i Q^2(\lambda_i)(1 - \lambda_i) \geq \epsilon \right],$$

We have

$$\min_{Q \in \mathcal{P}_{m-1}(1)} \frac{\sum_{i=2}^n Y_i Q^2(\lambda_i)(1 - \lambda_i)}{\sum_{i=1}^n Y_i Q^2(\lambda_i)} \geq (1 - \alpha) \min_{Q \in \mathcal{P}_{m-1}(1)} \frac{\sum_{j=1}^m \tilde{Y}_j Q^2 \left( \frac{\alpha(x_j + 1)}{2} \right)}{\sum_{j=1}^m \tilde{Y}_j Q^2 \left( \frac{\alpha(x_j + 1)}{2} \right)},$$

where $Y_1, \ldots, Y_n \sim \lambda_1^2$, and $\tilde{Y}_j$ is the sum of the $Y_i$’s that satisfy $\lambda_i = \alpha(x_j + 1)/2$, and, therefore, each $\tilde{Y}_j$ has at least $\lfloor (n - 1)/m \rfloor$ degrees of freedom. Then, by Proposition 3.3

$$\mathbb{P}[Y_1 \geq 3 \ln n] \leq \left( \frac{3e \ln n}{n^3} \right)^{1/2} = o \left( 1/n \right),$$

and

$$\mathbb{P} \left[ \tilde{Y}_j \geq \frac{1}{2} \left[ \frac{n - 1}{m} \right], j = 1, \ldots, m \right] \geq \left[ 1 - \left( \frac{e^{1/2}}{2} \right)^{\frac{1}{n-1}} \right]^m = 1 - o(1/n).$$

This implies that, with probability $1 - o(1/n)$, we can upper bound $Y_1$ by $3 \ln n$ and lower bound $\tilde{Y}_j$ by $\lfloor (n - 1)/m \rfloor/2$ for all $j = 1, \ldots, m$, which produces the lower bound

$$\min_{Q \in \mathcal{P}_{m-1}(1)} \frac{\sum_{i=2}^n Y_i Q^2(\lambda_i)(1 - \lambda_i)}{\sum_{i=1}^n Y_i Q^2(\lambda_i)} \geq (1 - \alpha) \min_{Q \in \mathcal{P}_{m-1}(1)} \frac{\sum_{j=1}^m \tilde{Y}_j Q^2 \left( \frac{\alpha(x_j + 1)}{2} \right)}{3 \ln n + \frac{(n-1)/m}{2} \sum_{j=1}^m Q^2 \left( \frac{\alpha(x_j + 1)}{2} \right)}.$$
where \( w_j \) are the corresponding weights, given by \( w_j = 2(1-x_j^2)^{-1}[P'_m(x_j)]^{-2} \). By Proposition 3.2, \( 0 < w_j \leq \pi/(m+1/2) \). By Lemma 3.1,

\[
\min_{Q \in \mathcal{P}_{m-1}(1)} \sum_{i=1}^{n} y_i Q^2(\lambda_i)(1-\lambda_i) \geq (1-\alpha) \min_{Q \in \mathcal{P}_{m-1}(1)} \frac{m+1/2}{2\pi} \sum_{j=1}^{m} w_j Q^2(\frac{\alpha(x_j+1)}{2}) \]

\[
= (1-\alpha) \min_{Q \in \mathcal{P}_{m-1}(1)} \frac{m+1/2}{2\pi} \sum_{j=1}^{m} w_j Q^2(\frac{\alpha(x_j+1)}{2}) \]

\[
\geq (1-\alpha) \frac{M(n,m,\alpha)}{3\ln n + M(n,m,\alpha)},
\]

where

\[
M(n,m,\alpha) = (1 + o(1))2^{15/4}n^{1/2}\left(\frac{m+1/2}{\alpha} (m-1)^{1/2}\right)^{4m} \frac{1}{\alpha^2 m} \left(\frac{m-1}{m}\right) \left(\frac{\sqrt{\alpha}}{1+\sqrt{2}}\right)^{4m}.
\]

Let \( m \) and \( \alpha \) satisfy \( m = O(\ln n) \), \( 1/\alpha = O(\ln n) \), and

\[
\left(\frac{\sqrt{\alpha}}{1+\sqrt{2}}\right)^{4m} \geq \frac{\ln^2 n}{\ln \left(\frac{1+\sqrt{2}}{\alpha}\right)}.
\]

The last statement follows from noting that if \( m = o(\ln n) \), then \( [\ln^2 n]/\ln n \) is incorrect.

This result proves that the maximum expected error does depend on the dimension \( n \) of the matrix, as no non-trivial error estimates can be produced when \( m = o(\ln n) \). One sample estimate that follows immediately from the previous lemma is that, if \( m \leq 1/13 \ln n \), then the maximum expected error is at least \( (1-o(1))(1-10^{-2}) \). The bound in Lemma 3.4 worsens exponentially as \( m \) increases, and gives no indication whether the estimate \( m^{-2} \) is the correct dependence on \( m \).

Next, we provide a lower bound of order \( m^{-2} \). Again, we begin by estimating a related integral polynomial minimization problem. However, for this minimization problem, rather than represent \( \mathcal{P}_{m-1} \) in the standard basis \( \{y^i\}_{i=0}^{m-1} \), we consider a basis consisting of shifted and scaled Jacobi polynomials. We first recall a number of basic facts regarding Jacobi polynomials. For a more detailed introduction, we refer the reader to [16].

Proposition 3.5 ([16], Chapter 3.2). Let \( \{J_k^{\alpha,\beta}(x)\}_{k=0}^{\infty} \), \( \alpha, \beta > -1 \), be the orthogonal system of Jacobi polynomials over \([-1,1]\) with respect to weight function \( \omega^{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta \), namely,

\[
J_k^{\alpha,\beta}(x) = \frac{\Gamma(k+\alpha+1)}{k! \Gamma(k+\alpha+\beta+1)} \sum_{i=0}^{k} \binom{k}{i} \frac{\Gamma(k+i+\alpha+\beta+1)}{\Gamma(i+\alpha+1)} \left(\frac{x-1}{2}\right)^i.
\]

Then

(i) \( \int_{-1}^{1} [J_k^{\alpha,\beta}(x)]^2 \omega^{\alpha,\beta}(x) \, dx = \frac{2^{2\alpha+\beta+1}\Gamma(k+\alpha+1)\Gamma(k+\beta+1)}{(2k+\alpha+\beta+1)k! \Gamma(k+\alpha+\beta+1)} \),

(ii) \( J_k^{\alpha+1,\beta}(x) = \frac{\Gamma(k+\beta+1)}{\Gamma(k+\alpha+\beta+2)} \sum_{i=0}^{k} \binom{2\ell+\alpha+\beta+1}{\ell+\alpha+\beta+1} \frac{\Gamma(k+\alpha+\beta+1)}{\Gamma(\ell+\beta+1)} J_k^{\alpha,\beta}(x) \),

(iii) \( \max_{x \in [-1,1]} |J_k^{\alpha,\beta}(x)| = \max_{x \in [-1,1]} |J_k^{\alpha,\beta}(x)| = \max \left\{ \frac{\Gamma(k+\alpha+1)}{k! \Gamma(\alpha+1)}, (-1)^k \frac{\Gamma(k+\beta+1)}{k! \Gamma(\beta+1)} \right\} \),

(iv) \( \frac{d}{dx} J_k^{\alpha,\beta}(x) = \frac{k+\alpha+\beta+1}{2} J_k^{\alpha+1,\beta+1}(x) \)
(v) \( J_k^{α,β}(-x) = (-1)^k J_k^{β,α}(x) \).

For the sake of brevity, the inner product and norm on the interval \([-1, 1]\) with respect to the weight \(w^{α,β}(x) = (1 - x)^α(1 + x)^β\) will be denoted by \(\langle \cdot, \cdot \rangle_{α,β}\) and \(\| \cdot \|_{α,β}\), namely

\[
\langle P, Q \rangle_{α,β} = \int_{-1}^{1} P(x)Q(x)(1 - x)^α(1 + x)^β \, dx
\]

and

\[
\| P \|_{α,β} = \left( \int_{-1}^{1} P^2(x)(1 - x)^α(1 + x)^β \, dx \right)^{1/2}.
\]

Using the alternate basis of Jacobi polynomials for \(P_{m-1}\) and Proposition 3.5, we can lower bound the following integral polynomial minimization problem relatively easily.

**Lemma 3.6.** Let \(\ell, m \in \mathbb{N}\). Then

\[
\min_{\substack{P ∈ P_{m-1} \setminus \{0\} \atop P \neq 0}} \frac{\int_{0}^{1} P^2(y)(1 - y)^\ell \, dy}{\int_{0}^{1} P^2(y)(1 - y)^{\ell-1} \, dy} \geq \frac{\ell}{m(m + \ell)}. \]

**Proof.** Let \(P(y) = \sum_{i=0}^{m-1} a_i J_i^{0,0}(2y - 1)\), \(a_i \in \mathbb{R}\), \(i = 0, \ldots, m - 1\). By a change of variables and equation (i) of Proposition 3.5,

\[
\int_{0}^{1} P^2(y)(1 - y)^\ell \, dy = \sum_{i,j=0}^{m-1} a_i a_j \int_{0}^{1} J_i^{0,0}(2y - 1)J_j^{0,0}(2y - 1)(1 - y)^\ell \, dy
\]

\[
= \frac{1}{2^\ell + 1} \sum_{i,j=0}^{m-1} a_i a_j \langle J_i^{0,0}, J_j^{0,0} \rangle_{\ell,0}
\]

\[
= \sum_{i=0}^{m-1} \frac{a_i^2}{2i + \ell + 1}.
\]

Using equation (ii) of Proposition 3.5, we can write the inner product between Jacobi polynomials \(J_i^{0,0}(x)\) and \(J_j^{0,0}(x)\) with respect to weight function \(ω^{−1,0}(x) = (1 − x)^{−1}\) as

\[
\langle J_i^{0,0}, J_j^{0,0} \rangle_{−1,0} = \frac{i!}{(i + \ell)!} \frac{j!}{(j + \ell)!} \sum_{r=0}^{\min\{i,j\}} \frac{(2r + \ell)(r + \ell - 1)!}{r!} \frac{(2s + \ell)(s + \ell - 1)!}{s!} \langle J_r^{−1,0}, J_s^{−1,0} \rangle_{−1,0}
\]

\[
= 2^\ell \frac{i!}{(i + \ell)!} \frac{j!}{(j + \ell)!} \sum_{r=0}^{\min\{i,j\}} \left[ \frac{(r + \ell - 1)!}{r!} \right]^2 (2r + \ell).
\]

Therefore,

\[
\int_{0}^{1} P^2(y)(1 - y)^{\ell-1} \, dy = \sum_{i,j=0}^{m-1} a_i a_j \int_{0}^{1} J_i^{0,0}(2y - 1)J_j^{0,0}(2y - 1)(1 - y)^{\ell-1} \, dy
\]

\[
= \frac{1}{2^\ell} \sum_{i,j=0}^{m-1} a_i a_j \langle J_i^{0,0}, J_j^{0,0} \rangle_{−1,0}
\]

\[
= \sum_{i,j=0}^{m-1} a_i a_j \frac{i!}{(i + \ell)!} \frac{j!}{(j + \ell)!} \sum_{r=0}^{\min\{i,j\}} \left[ \frac{(r + \ell - 1)!}{r!} \right]^2 (2r + \ell).
\]
Now, consider the change of variable $\tilde{a}_i = \frac{a_i}{\sqrt{2i+\ell+1}}$, $i = 0, \ldots, m-1$, and let $\tilde{a} = (\tilde{a}_0, \ldots, \tilde{a}_{m-1})^T \in \mathbb{R}^m$.

We have
\[ \max_{P \notin \mathbb{P}_{m-1}} \int_0^1 P^2(y)(1-y)^{\ell-1} \, dy = \max_{\tilde{a} \notin \mathbb{R}^m} \frac{\sum_{i,j=0}^{m-1} \tilde{a}_i \tilde{a}_j \frac{\partial^2 P_{y+j}}{(y+j)!} + \sum_{r=0}^{\min(i,j)} \frac{(r+\ell+1)!}{r!}}{\sum_{i=0}^{m-1} \tilde{a}_i^2} (2r+\ell). \]

The matrix corresponding to this Rayleigh quotient is positive definite, as the left hand side is always positive. Therefore, we can upper bound this Rayleigh quotient by the trace of the associated matrix. This produces the bound
\[ \max_{P \notin \mathbb{P}_{m-1}} \int_0^1 P^2(y)(1-y)^{\ell-1} \, dy \leq \frac{m-1}{\ell} \sum_{i=0}^{m-1} \frac{2i+\ell+1}{\ell} = \frac{m(m+\ell)}{\ell}. \]

Taking the reciprocal of the left and right hand sides completes the proof.

We are now prepared to prove a lower bound of order $m^{-2}$ for maximum expected error.

**Lemma 3.7.** If $m = \Theta \left( \frac{n^{1/2}}{\ln^{1/2} n} \right)$, then
\[ \sup_{A \in S_{++}^{n \times n} \text{ b-}\mathcal{U}(S^{n-1})} \mathbb{P} \left[ \frac{\lambda_1(A) - \lambda_1^{(m)}(A,b)}{\lambda_1(A)} \geq \frac{.637}{m^2} \right] \geq 1 - o(1/n), \]

and, therefore,
\[ \sup_{A \in S_{++}^{n \times n} \text{ b-}\mathcal{U}(S^{n-1})} \mathbb{E} \left[ \frac{\lambda_1(A) - \lambda_1^{(m)}(A,b)}{\lambda_1(A)} \right] \geq (1 - o(1)) \frac{.637}{m^2}. \]

**Proof.** This proof follows a similar structure to that of Lemma 3.4. We choose a matrix with eigenvalues based on the zeros of a Legendre polynomial, and show that a large subset of $[0, \infty)^n$ (with respect to $f_Y(y)$) satisfies conditions that allow us to lower bound the relative error using the integral minimization problem of Lemma 3.6. For simplicity, we do not scale our matrix to have spectral radius one (i.e., $\lambda_1(A) \neq 1$).

Let $x_1 > \ldots > x_{2m}$ be the zeros of the $(2m)$th degree Legendre polynomial. Let $A \in \mathbb{R}^{n \times n}$ have eigenvalue $(x_j+1)/2$ with multiplicity one, and remaining eigenvalues given by $(x_j+1)/2$, $j = 2, \ldots, 2m$, each with multiplicity at least $[(n-1)/(2m-1)]$. By equation (9),
\[ \mathbb{P}_{b-\mathcal{U}(S^{n-1})} \left[ \frac{\lambda_1(A) - \lambda_1^{(m)}(A,b)}{\lambda_1(A)} \geq \epsilon \right] = \min_{P \notin \mathbb{P}_{m-1}} \mathbb{P} \left[ \frac{\sum_{i=1}^{2m} Y_i P^2(\lambda_i) - \lambda_i}{\sum_{i=1}^{2m} Y_i P^2(\lambda_i)} \geq \epsilon \right]. \]

We have
\[ \frac{1}{\lambda_1} \min_{P \notin \mathbb{P}_{m-1}} \sum_{j=1}^{2m} \frac{\lambda_j P^2(\lambda_j)}{\lambda_j - \lambda_i} = \frac{2}{x_1 + 1} \min_{P \notin \mathbb{P}_{m-1}} \sum_{j=1}^{2m} \frac{\hat{Y}_j P^2(\lambda_j)}{\hat{Y_j} P^2(\lambda_j)} \left( \frac{x_j+1}{2} - \frac{x_j+1}{2} \right), \]

where $Y_1, \ldots, Y_n \sim \lambda_1^2$, and $\hat{Y}_j$ is the sum of the $Y_i$ that satisfy $\lambda_i = (x_j+1)/2$. $\hat{Y}_1$ has one degree of freedom, and $\hat{Y}_j$, $j = 2, \ldots, 2m$, each have at least $[(n-1)/(2m-1)]$ degrees of freedom. Let $w_1, \ldots, w_{2m}$ be the weights of Gaussian quadrature associated with $x_1, \ldots, x_{2m}$. By Proposition 3.2
\[ x_1 = \left( 1 - \frac{1}{32m^2} \right) \cos \left( \frac{3\pi}{8m+2} \right) + O(m^{-3}) = 1 - \frac{4 + 9\pi^2}{128m^2} + O(m^{-3}), \]
\[ x_2 = \left( 1 - \frac{1}{32m^2} \right) \cos \left( \frac{7\pi}{8m+2} \right) + O(m^{-3}) = 1 - \frac{4 + 49\pi^2}{128m^2} + O(m^{-3}). \]
and, therefore, \(1 - x_1^2 = \frac{4 + 9\pi^2}{64m^2} + O(m^{-3})\). Again, by Proposition 3.2 we can lower bound the smallest ratio between weights by

\[
\frac{w_1}{w_m} \geq \sqrt{1 - x_1^2} \left(1 - \frac{1}{8(2m + 1/2)^2(1 - x_1^2)}\right) = \left(\frac{\sqrt{4 + 9\pi^2}}{8m} + O(m^{-2})\right) \left(1 - \frac{1}{(4 + 9\pi^2)/2 + O(m^{-1})}\right) = \frac{2 + 9\pi^2}{8\sqrt{4 + 9\pi^2}m} + O(m^{-2}).
\]

Therefore, by Proposition 3.3

\[
\mathbb{P} \left[ \min_{j=2, \ldots, 2m} \hat{Y}_j \geq \frac{w_m}{w_1} Y_1 \right] \geq \mathbb{P} \left[ \hat{Y}_j \geq \frac{1}{3} \left(\frac{n - 1}{2m - 1}\right), \ j = 2, \ldots, 2m \right] \times \mathbb{P} \left[ \hat{Y}_1 \leq \frac{w_1}{3w_m} \left(\frac{n - 1}{2m - 1}\right) e^{-\frac{w_1}{3w_m} \left(\frac{n - 1}{2m - 1}\right)} \right]^{1/2} = 1 - o(1/n).
\]

We now restrict our attention to values of \(Y = (Y_1, \ldots, Y_n) \in [0, \infty)^n\) that satisfy \(\min_{j=2, \ldots, 2m} \hat{Y}_j \geq \frac{w_m}{w_1} Y_1\). If, for some fixed choice of \(Y\),

\[
\sum_{j=2}^{2m} \hat{Y}_j P^2 \left(\frac{x_{j+1}}{2}\right) \left(\frac{x_{j+1}}{2} - \frac{x_j}{2}\right) \leq \frac{x_1 + 1}{2} - \frac{x_2 + 1}{2},
\]

then, by Proposition 3.2 and Lemma 3.6

\[
\min_{\substack{P \in P_{m-1} \\setminus \{0\}}} \frac{\sum_{j=2}^{2m} \hat{Y}_j P^2 \left(\frac{x_{j+1}}{2}\right) \left(\frac{x_{j+1}}{2} - \frac{x_j}{2}\right)}{\sum_{j=1}^{2m} \hat{Y}_j P^2 \left(\frac{x_{j+1}}{2}\right)} \geq \frac{\sum_{j=2}^{2m} w_j P^2 \left(\frac{x_{j+1}}{2}\right) \left(\frac{x_{j+1}}{2} - \frac{x_j}{2}\right)}{\sum_{j=1}^{2m} w_j P^2 \left(\frac{x_{j+1}}{2}\right)} = \frac{\sum_{j=1}^{2m} w_j P^2 \left(\frac{x_{j+1}}{2}\right) \left(1 - \frac{x_{j+1}}{2}\right)}{\sum_{j=1}^{2m} w_j P^2 \left(\frac{x_{j+1}}{2}\right)} - \left(1 - \frac{x_1 + 1}{2}\right) = \frac{\int_0^1 P^2(y)(1 - y) \ dy}{\int_0^1 P^2(y) \ dy} - \frac{4 + 9\pi^2}{256m^2} + O(m^{-3}).
\]

Alternatively, if equation (10) does not hold, then we can lower bound the left hand side of (10) by

\[
\frac{x_1 + 1}{2} - \frac{x_2 + 1}{2} = \frac{5\pi^2}{32m^2} + O(m^{-3}).
\]

This completes the proof. \(\square\)

The previous lemma shows that the \(m^{-2}\) dependence on the number of iterations in upper bounds for relative error is in fact necessary. We now present a lower bound of the form \(\ln n/(m^2 \ln \ln n)\), which is, in essence, a combination of the two previous lower bounds. However, due to the distribution of eigenvalues required to produce this bound, Gaussian quadrature is no longer exact, and we must make use of estimates for basic composite quadrature. We recall the following basic proposition.
Proposition 3.8. If \( f \in C^1[a, b] \), then for \( a = x_0 < \ldots < x_n = b \),
\[
\left| \int_a^b f(x) \, dx - \sum_{i=1}^n (x_i - x_{i-1}) f(x_i) \right| \leq \frac{b-a}{2} \max_{x \in [a,b]} |f'(x)| \max_{i=1,\ldots,n} (x_i - x_{i-1}),
\]
where \( x_i^* \in [x_{i-1}, x_i], i = 1, \ldots, n \).

However, often we will be using quadrature for functions that, as a result of our eigenvalue distribution, do not belong to \( f \in C^1[0,1] \). In order to avoid this issue, we will make use of monotonicity properties, which requires estimating the largest zero of Jacobi polynomials. We present the following proposition, which is a restatement of an upper bound for zeros of Gegenbauer polynomials, combined with a fairly standard interlacing theorem.

Proposition 3.9. Let \( \alpha \geq \beta > -11/12 \). Then the largest zero \( x_1 \) of \( J_k^{\alpha,\beta}(x) \), \( k \geq 1 \), is bounded above by
\[
x_1 \leq \sqrt{1 - \left( \frac{\alpha + 3/2}{\bar{k} + \alpha + 1/2} \right)^2}.
\]

Proof. By [11] Lemma 3.5, the largest zero \( x_1 \) of the \( k^{th} \) degree Gegenbauer polynomial (i.e., \( J_k^{\lambda-1/2,\lambda-1/2}(x) \)) with \( \lambda > -5/12 \), satisfies
\[
x_1^2 \leq \frac{(k - 1)(k + 2 + 1)}{(k + \lambda)^2 + 3\lambda + 5/4 + 3(\lambda + 1/2)^2/(k - 1)} \leq \frac{(k - 1)(k + 2 + 1)}{(k + \lambda)^2} = 1 - \left( \frac{\lambda + 1}{k + \lambda} \right)^2.
\]

By [5] Theorem 2.1, the largest zero of \( J_k^{\alpha,\beta}(x) \) is strictly greater than the largest zero of \( J_k^{\alpha,\beta}(x) \) for any \( t > 0 \). As \( \alpha \geq \beta \), combining these two facts provides our desired result.

We are now prepared to prove the following lower bound.

Lemma 3.10. If \( m = O(\ln n) \), then
\[
\sup_{A \in S^{n^2}_+} \mathbb{P}_{b \sim U(S^{n-1})} \left[ \frac{\lambda_1(A) - \lambda_1^{(m)}(A, b)}{\lambda_1(A)} \geq \frac{249 \ln n}{m^2 \ln \ln n} \right] \geq 1 - o(1/n),
\]
and, therefore,
\[
\sup_{A \in S^{n^2}_+} \mathbb{E}_{b \sim U(S^{n-1})} \left[ \frac{\lambda_1(A) - \lambda_1^{(m)}(A, b)}{\lambda_1(A)} \right] \geq (1 - o(1)) \frac{249 \ln n}{m^2 \ln \ln n}.
\]

Proof. The structure of the proof is similar in concept to that of Lemma 3.7 but requires much more work. As in the proof of Lemma 3.7, we choose a matrix with eigenvalues based on a function corresponding to the integral minimization problem of Lemma 3.6 and show that a large subset of \([0, \infty)^n\) (with respect to \( f_y(y) \)) satisfies conditions that allows us to lower bound the relative error using the aforementioned integral minimization problem. The main difficulty faced in this proof can be summarized as follows. In the proof of Lemma 3.7, we made use of Lemma 3.6 with \( \ell = 1 \). However, to obtain the improved bounds of this lemma, we use \( \ell > 1 \), which requires quadrature for functions that are no longer polynomials, and, therefore, cannot be represented exactly using Gaussian quadrature. In addition, the required quadrature is for functions whose derivative has a singularity at \( x = 1 \), and so Proposition 3.8 is not immediately applicable. Instead, we perform a two part error analysis. In particular, if a function is \( C^1[0, \alpha] \), \( \alpha < 1 \), and monotonic on \([\alpha, 1]\), then using Proposition 3.8 on \([0, \alpha]\) and a monotonicity argument on \([\alpha, 1]\) results in a bound for the error due to quadrature.

Let \( \ell = \left\lfloor \frac{2495 \ln n}{\ln \ln n} \right\rfloor \), \( k = \lfloor m^{0.044} \rfloor \), and \( m = O(\ln n) \). Consider a matrix \( A \in S^{n^2}_+ \) with eigenvalues given by \( f(x_j), j = 1, \ldots, k \), each with multiplicity either \( \lfloor n/k \rfloor \) or \( \lceil n/k \rceil \), where \( f(x) = 1 - (1 - x)^k \), and
We now replace each quadrature $x_j = j/k$, $j = 1, \ldots, k$. Then

$$\mathbb{P}_{b \sim \mathcal{U}(S^{n-1})} \left[ \frac{\lambda_1(A) - \lambda_1^{(m)}(A, b)}{\lambda_1(A)} \geq \epsilon \right] = \mathbb{P}_{\gamma_i \sim \chi^2_n} \left[ \min_{\gamma_i \neq 0} \frac{\sum_{i=1}^{n} Y_i P^2(\lambda_i) (1 - \lambda_i)}{\sum_{i=1}^{n} Y_i P^2(\lambda_i)} \geq \epsilon \right] = \mathbb{P}_{\gamma_i \sim \chi^2_n} \left[ \min_{\gamma_i \neq 0} \frac{\sum_{j=1}^{k} \hat{Y}_j P^2(f(x_j)) (1 - f(x_j))}{\sum_{j=1}^{k} \hat{Y}_j P^2(f(x_j))} \geq \epsilon \right],$$

where $Y_1, \ldots, Y_n \sim \chi^2_1$, and $\hat{Y}_j$ is the sum of the $Y_i$’s that satisfy $\lambda_i = f(x_j)$. Each $\hat{Y}_j$, $j = 1, \ldots, k$, has either $[n/k]$ or $[n/k]$ degrees of freedom. Because $m = O(\ln n)$, we have $k = o(n/\epsilon)$ and, by Proposition 3.3, 

$$\mathbb{P} \left[ .999 [n/k] \leq \hat{Y}_j \leq 1.001 [n/k], j = 1, \ldots, k \right] \geq \left( 1 - (0.999e^{-001})^{n/k} - (1.001e^{-001})^{[n/k]} \right)^k = 1 - o(1/n).$$

Therefore, with probability $1 - o(1/n)$,

$$\min_{\gamma_i \neq 0} \frac{\sum_{j=1}^{k} \hat{Y}_j P^2(f(x_j)) (1 - f(x_j))}{\sum_{j=1}^{k} \hat{Y}_j P^2(f(x_j))} \geq .998 \min_{\gamma_i \neq 0} \frac{\sum_{j=1}^{k} P^2(f(x_j))(1 - f(x_j))}{\sum_{j=1}^{k} P^2(f(x_j))}.$$ 

Let $P(y) = \sum_{r=0}^{m-1} a_r J_r^f(2y - 1)$, $a_r \in \mathbb{R}$, $r = 0, \ldots, m - 1$, and define

\[ g_{r,s}(x) = J_r^f(2f(x) - 1) J_s^f(2f(x) - 1), \]

\[ \hat{g}_{r,s}(x) = J_r^f(2f(x) - 1) J_s^f(2f(x) - 1)(1 - f(x)), \]

$r, s = 0, \ldots, m - 1$. Then we have

$$\frac{\sum_{j=1}^{k} P^2(f(x_j))(1 - f(x_j))}{\sum_{j=1}^{k} P^2(f(x_j))} = \frac{\sum_{r,s=0}^{m-1} a_r a_s \sum_{j=1}^{k} g_{r,s}(x_j)}{\sum_{r,s=0}^{m-1} a_r a_s \sum_{j=1}^{k} \hat{g}_{r,s}(x_j)}.$$ 

We now replace each quadrature $\sum_{j=1}^{k} \hat{g}_{r,s}(x_j)$ or $\sum_{j=1}^{k} g_{r,s}(x_j)$ in the previous expression by its corresponding integral, plus a small error term. However, the functions $g_{r,s}$ and $\hat{g}_{r,s}$ are not elements of $C^1[0,1]$, as $f'(x)$ has a singularity at $x = 1$, and, therefore, we cannot use Proposition 3.8 directly. Instead we break the error analysis of the quadrature into two components. We have $g_{r,s}, \hat{g}_{r,s} \in C[0,1]$ for any $0 < \alpha < 1$, and, if $\alpha$ is chosen to be large enough, both $g_{r,s}$ and $\hat{g}_{r,s}$ will be monotonic on the interval $[\alpha, 1]$. In that case, we can apply Proposition 3.8 to bound the error over the interval $[0, \alpha]$ and use basic properties of Riemann sums of monotonic functions to bound the error over the interval $[\alpha, 1]$.

The function $f(x)$ is monotonically increasing on $[0,1]$, and, by equation (iv) of Proposition 3.8 and Proposition 3.9, the function $J_r^f(0)(z)$, $r = 2, \ldots, m - 1$, is monotonically increasing on the interval

$$\left[ \sqrt{1 - \left( \frac{\ell + 5/2}{m + \ell - 1/2} \right)^2}, 1 \right].$$

The functions $J_r^f(0)(z) = 1$ and $J_r^f(0)(z) = \ell + 1 + (\ell + 2)(z - 1)/2$ are clearly monotonically non-decreasing over interval $[11]$ as well. Using the identity $\sqrt{1 - x} \leq 1 - x/2$, $x \in [0,1]$, the function $J_r^f(0)(2f(x) - 1)$, $r = 0, \ldots, m - 1$, is monotonically non-decreasing on the interval

$$\left[ 1 - \left( \frac{\ell + 5/2}{2m + 2\ell - 1} \right)^{2\ell}, 1 \right].$$

Therefore, the functions $g_{r,s}(x)$ are monotonically non-decreasing and $\hat{g}_{r,s}(x)$ are monotonically non-increasing on the interval $[12]$ for all $r, s = 0, \ldots, m - 1$. The term $\left( \frac{\ell + 5/2}{2m + 2\ell - 1} \right)^{2\ell}$ is $\omega(1/k)$, and so there exists an index $j^* \in \{1, \ldots, k\}$ such that

$$1 - \left( \frac{\ell + 5/2}{2m + 2\ell - 1} \right)^{2\ell} \leq x_{j^*} < 1 - \left( \frac{\ell + 5/2}{2m + 2\ell - 1} \right)^{2\ell} + \frac{1}{k} < 1.$$
We now upper bound the derivatives of $g_{r,s}$ and $\hat{g}_{r,s}$ on the interval $[0, x_j^*]$. By equation (iv) of Proposition 3.5, the first derivatives of $g_{r,s}$ and $\hat{g}_{r,s}$ are given by

$$
g'_{r,s}(x) = 2f'(x) \left( \frac{d}{dz} J_{r,s}^0(z) \right)_{z=2f(x)-1} J_{r,s}^0(2f(x) - 1) + J_{r,s}^0(2f(x) - 1) \left( \frac{d}{dz} J_{r,s}^0(z) \right)_{z=2f(x)-1}
$$

$$
= \frac{(1 - x)^{-\ell - 1}}{\ell} \left( (r + \ell + 1) J_{r+1}^0(2f(x) - 1) J_{s}^0(2f(x) - 1) + (s + \ell + 1) J_{r}^0(2f(x) - 1) J_{s+1}^0(2f(x) - 1) \right),
$$

and

$$
\hat{g}'_{r,s}(x) = g'_{r,s}(x)(1 - x)^{-\ell} - g_{r,s}(x) \frac{(1 - x)^{-\ell - 1}}{\ell}.
$$

By Proposition 3.5 and the identity \( \left( \frac{x}{y} \right)^y \leq \left( \frac{e x}{y} \right)^y \), \( x, y \in \mathbb{N}, y \leq x \),

$$
\max_{x \in [0, x_j^*]} |g'_{r,s}(x)| \leq \frac{(1 - x_j^*)^{-\ell - 1}}{\ell} \left( (r + \ell + 1) \left( \begin{array}{c} r + \ell \\ \ell + 1 \end{array} \right) \left( \begin{array}{c} s + \ell \\ \ell \end{array} \right) + (s + \ell + 1) \left( \begin{array}{c} r + \ell \\ \ell \end{array} \right) \left( \begin{array}{c} s + \ell \\ \ell + 1 \end{array} \right) \right)
$$

$$
\leq 2 \frac{m + \ell}{\ell} \left( \left( \frac{\ell + 5/2}{2m + 2\ell - 1} \right)^{2\ell} - \frac{1}{k} \right) \left( \frac{e(m + \ell - 1)}{\ell} \right)^{2\ell + 1},
$$

and

$$
\max_{x \in [0, x_j^*]} |\hat{g}'_{r,s}(x)| \leq \max_{x \in [0, x_j^*]} |g'_{r,s}(x)| + \frac{(1 - x_j^*)^{-\ell - 1}}{\ell} \left( \begin{array}{c} r + \ell \\ \ell \end{array} \right) \left( \begin{array}{c} s + \ell \\ \ell \end{array} \right)
$$

$$
\leq 3 \frac{m + \ell}{\ell} \left( \left( \frac{\ell + 5/2}{2m + 2\ell - 1} \right)^{2\ell} - \frac{1}{k} \right) \left( \frac{e(m + \ell - 1)}{\ell} \right)^{2\ell + 1}.
$$

Therefore, both $\max_{x \in [0, x_j^*]} |g'_{r,s}(x)|$ and $\max_{x \in [0, x_j^*]} |\hat{g}'_{r,s}(x)|$ are $o(m^{4.002\ell})$. Then, by Propositions 3.5 and 3.8 and monotonicity on the interval $[x_j^*, 1]$, we have

$$
\left| \frac{1}{k} \sum_{j=1}^{k} g_{r,s}(x_j) - \int_0^1 g_{r,s}(y) dy \right| \leq \left| \frac{1}{k} \sum_{j=1}^{k} g_{r,s}(x_j) - \int_0^{x_j^*} g_{r,s}(y) dy \right| + \left| \frac{1}{k} \sum_{j=1}^{k} g_{r,s}(x_j) - \int_0^1 g_{r,s}(y) dy \right|
$$

$$
\leq \frac{1 + o(1)}{2km^{4.002\ell}} + \frac{g_{r,s}(1)}{k}
$$

$$
\leq \frac{1 + o(1)}{2m^{4.002\ell}} + \frac{1}{k} \left( \frac{e(m + \ell - 1)}{\ell + 1} \right)^{2(\ell + 1)}
$$

$$
\leq \frac{1 + o(1)}{2m^{4.002\ell}},
$$

and similarly,

$$
\left| \frac{1}{k} \sum_{j=1}^{k} \hat{g}_{r,s}(x_j) - \int_0^1 \hat{g}_{r,s}(y) dy \right| \leq \left| \frac{1}{k} \sum_{j=1}^{k} \hat{g}_{r,s}(x_j) - \int_0^{x_j^*} \hat{g}_{r,s}(y) dy \right| + \left| \frac{1}{k} \sum_{j=1}^{k} \hat{g}_{r,s}(x_j) - \int_0^1 \hat{g}_{r,s}(y) dy \right|
$$

$$
\leq \frac{1 + o(1)}{2m^{4.002\ell}} + \frac{\hat{g}_{r,s}(x_j^*)}{k}
$$

$$
\leq \frac{1 + o(1)}{2m^{4.002\ell}} + \frac{g_{r,s}(1)}{k}
$$

$$
\leq \frac{1 + o(1)}{2m^{4.002\ell}}.$$
Let us denote this upper bound by $M = (1 + o(1))/(2m^{0.02}\ell)$. By using the substitution $y = 1 - (1 - z)^\ell$, followed by a linear change of variable, we have
\[
\int_0^1 g_{r,s}(y) \, dy = \ell \int_0^1 J_r^{f,0}(2z - 1)J_s^{f,0}(2z - 1)(1 - z)^\ell \, dz
\]
and, similarly,
\[
\int_0^1 g_{r,s}(y) \, dy = \ell \int_0^1 J_r^{f,0}(2z - 1)J_s^{f,0}(2z - 1)(1 - z)^{\ell - 1} \, dz
\]
Then
\[
\max_{P \in P_{m-1}} \frac{\sum_{j=1}^k P^2(f(x_j))}{\sum_{j=1}^k P^2(f(x_j))} \leq \max_{a \in \mathbb{R}^m} \frac{\sum_{r,s=0}^{m-1} a_r a_s \left[ \ell \left(J_r^{f,0}, J_s^{f,0}\right)_{\ell,0} + \epsilon_{r,s} \right]}{\sum_{r,s=0}^{m-1} a_r a_s \left(\ell \left(J_r^{f,0}, J_s^{f,0}\right)_{\ell,0} + \epsilon_{r,s} \right)}.
\]

Letting $\tilde{a}_r = a_r \|J_r^{f,0}\|_{\ell,0}$, $r = 0, \ldots, m-1$, and noting that, by Proposition 3.5, $\|J_r^{f,0}\|_{\ell,0} = \left(2\ell + 1 + (2r + \ell + 1)^{1/2}(2r + \ell + 1)^{1/2}\right)^{1/2}$, we obtain the bound
\[
\max_{P \in P_{m-1}} \frac{\sum_{j=1}^k P^2(f(x_j))}{\sum_{j=1}^k P^2(f(x_j))} \leq 2 \max_{\tilde{a} \in \mathbb{R}^m} \frac{\sum_{r,s=0}^{m-1} \tilde{a}_r \tilde{a}_s \left(\ell \left(J_r^{f,0}, J_s^{f,0}\right)_{\ell,0} \right)}{\sum_{r,s=0}^{m-1} \tilde{a}_r \tilde{a}_s \left(\ell \left(J_r^{f,0}, J_s^{f,0}\right)_{\ell,0} \right) + \epsilon_{r,s}}.
\]

Let $B \in \mathbb{R}^{m \times m}$ satisfy $B(r,s) = \langle J_r^{f,0}, J_s^{f,0}\rangle_{\ell,0} \|J_r^{f,0}\|_{\ell,0}^{-1} \|J_s^{f,0}\|_{\ell,0}^{-1}$, $r, s = 0, \ldots, m - 1$. From the proof of Lemma 3.6, we have that
\[
2 \max_{\tilde{a} \neq 0} \frac{\tilde{a}^T B \tilde{a}}{\tilde{a}^T \tilde{a}} = \max_{\tilde{a} \neq 0} \frac{\sum_{r,s=0}^{m-1} \tilde{a}_r \tilde{a}_s \ell \left(J_r^{f,0}, J_s^{f,0}\right)_{\ell,0}}{\sum_{r,s=0}^{m-1} \tilde{a}_r \tilde{a}_s \ell \left(J_r^{f,0}, J_s^{f,0}\right)_{\ell,0} + \epsilon_{r,s}} \leq \frac{m(m + \ell)}{\ell}.
\]
This implies that
\[
\max_{P \in P_{m-1}} \frac{\sum_{j=1}^k P^2(f(x_j))}{\sum_{j=1}^k P^2(f(x_j))} \leq \max_{\tilde{a} \neq 0} \frac{\ell \left(B + \epsilon I I^T\right) \tilde{a}}{\tilde{a}^T \left(I - \epsilon I I^T\right) \tilde{a}} \leq \frac{2}{1 - \epsilon m} \frac{\max_{\tilde{a} \neq 0} \langle \tilde{a}^T B \tilde{a} \rangle}{\tilde{a}^T \tilde{a}} \leq \frac{1}{1 - \epsilon m} \frac{m(m + \ell)}{\ell} + \epsilon m,
\]
where
\[
\epsilon = \frac{M(2m + \ell - 1)}{\ell} = (1 + o(1)) \frac{2m + \ell - 1}{2m^{0.02}\ell} = o(1/m).
\]
This implies that, with probability $1 - o(1/n)$,
\[
\min_{P \neq 0} \frac{\sum_{j=1}^k Y_j^2 P^2(f(x_j))}{\sum_{j=1}^k Y_j P^2(f(x_j))} \geq (1 - o(1)) \frac{0.998 \ell}{m(m + \ell)} \geq 0.249 \ln n.
\]
This completes the analysis of asymptotic lower bounds. In the
following section, we produce improved upper bounds. The lower bounds of this section, paired with the upper bounds in the following section, will complete the proofs of Theorems 1.2 and 1.3, the main results of the paper.

4. Improved Upper Bounds

To complement the asymptotic lower bounds of the previous section, we prove upper bounds for relative error. Our estimate for relative error in the one-norm is of the same order as the estimate (1), but with an improved constant. However, our technique is much more robust, and results in significantly improved estimates over [6] for all $p$-norms. Our technique for obtaining these estimates differs from the technique of [9] in one key way. Rather than integrating first by $b_1$ and using properties of the arctan function, we replace the ball $B^n$ by $n$ chi-square random variables on $[0, \infty)^n$, and iteratively apply Cauchy-Schwarz to our relative error until we obtain an exponent $c$ for which the inverse chi-square distribution with one degree of freedom has a convergent $c^{th}$ moment. We prove the following lemma.

**Lemma 4.1.** Let $n \geq 100$, $m \geq 10$, and $p \geq 1$. Then

$$
\max_{A \in S^{n-1}_+} \mathbb{E}_{b \sim \mathcal{U}(S^{n-1})} \left[ \left( \frac{\lambda_1(A) - \lambda_1^{(m)}(A, b)}{\lambda_1(A)} \right)^p \right]^{1/p} \leq 0.068 \frac{(n - 1)^{6p}}{(m - 1)^2}.
$$

**Proof.** By repeated application of Cauchy-Schwarz,

$$
\sum_{i=1}^n y_i Q^2(\lambda_i)(1 - \lambda_i) \leq \left( \sum_{i=1}^n y_i Q^2(\lambda_i) \right)^{2p} \leq \left( \sum_{i=1}^n y_i Q^2(\lambda_i)(1 - \lambda_i)^{2p} \right)^{1/p}
$$

for $q \in \mathbb{N}$. Choosing $q$ to satisfy $2p < 2^q \leq 4p$, and using equation (8) with polynomial normalization $Q(1) = 1$, we have

$$
\mathbb{E}_{b \sim \mathcal{U}(S^{n-1})} \left[ \left( \frac{\lambda_1(A) - \lambda_1^{(m)}(A, b)}{\lambda_1(A)} \right)^p \right]^{1/p} \leq \min_{Q \in \mathcal{P}_{m-1}(1)} \left[ \int_{[0, \infty)^n} \left( \sum_{i=1}^n y_i Q^2(\lambda_i)(1 - \lambda_i)^{2p} \right)^{\frac{p}{2}} f_Y(y) \, dy \right]^{\frac{1}{p}}
$$

$$
\leq \min_{Q \in \mathcal{P}_{m-1}(1)} \left[ \int_{[0, \infty)^n} \left( \sum_{i=1}^n y_i Q^2(\lambda_i)(1 - \lambda_i)^{2p} \right)^{\frac{p}{2}} f_Y(y) \, dy \right]^{\frac{1}{p}}
$$

$$
+ \left[ \int_{[0, \infty)^n} \left( \sum_{i=1}^n y_i Q^2(\lambda_i)(1 - \lambda_i)^{2p} \right)^{\frac{p}{2}} f_Y(y) \, dy \right]^{\frac{1}{p}}
$$

for any $\beta \in (0, 1)$. We treat the two terms on the right hand side separately, beginning with the second. The second term is always bounded above by

$$
\left[ \int_{[0, \infty)^n} \left( \sum_{i=1}^n y_i Q^2(\lambda_i)(1 - \lambda_i)^{2p} \right)^{\frac{p}{2}} f_Y(y) \, dy \right]^{\frac{1}{p}} \leq 1 - \beta.
$$

The integrand of the first term satisfies

$$
\left( \sum_{i=1}^n y_i Q^2(\lambda_i)(1 - \lambda_i)^{2p} \right)^{\frac{p}{4}} \leq \left( \sum_{i=1}^n y_i Q^2(\lambda_i)(1 - \lambda_i)^{2p} \right)^{\frac{1}{4}}
$$

$$
\leq \max_{x \in [0, 1]} |Q(x)|^{1/2} (1 - x)^{2p-2} \left( \sum_{i=1}^n y_i \right)^{\frac{1}{4}}.
$$

We replace the minimizing polynomial in $\mathcal{P}_{m-1}(1)$ by $\hat{Q}(x) = \frac{T_{m-1}(\frac{2}{\beta}x - 1)}{T_{m-1}(\frac{2}{\beta} - 1)}$, where $T_{m-1}()$ is the Chebyshev polynomial of the first kind. The Chebyshev polynomials $T_{m-1}()$ are bounded by one in magnitude on
the interval \([-1, 1]\), and this bound is tight at the endpoints. Therefore, our maximum is achieved at \(x = 0\), and

\[
\max_{x \in [0, \beta]} |\tilde{Q}(x)|^{1/2}(1 - x)^{2\gamma - 2} = |T_{m-1} \left( \frac{2}{\beta} - 1 \right)|^{-1/2}.
\]

By the definition \(T_{m-1}(x) = 1/2 \left( (x - \sqrt{x^2 - 1})^{m-1} + (x + \sqrt{x^2 - 1})^{m-1} \right)\), \(|x| \geq 1\), and the standard identity \(e^{2x} \leq \frac{1 + x}{1 - x}, x \in [0, 1]\)

\[
T_{m-1} \left( \frac{2}{\beta} - 1 \right) \geq \frac{1}{2} \left( \frac{2}{\beta} - 1 + \sqrt{\left( \frac{2}{\beta} - 1 \right)^2} \right)^{m-1}
= \frac{1}{2} \left( 1 + \sqrt{1 - \beta} \right)^{m-1}
\geq \frac{1}{2} \exp \left\{ 2\sqrt{1 - \beta} (m - 1) \right\}.
\]

In addition,

\[
\int_{[0, \infty)^n} \left( \sum_{i=2}^n y_i \right)^{1/4} f_Y(y) \, dy = \frac{\Gamma(n/2 - 1/4) \Gamma(1/4)}{\Gamma(n/2 - 1/2) \Gamma(1/2)} \leq \frac{\Gamma(1/4)}{2^{1/4} \Gamma(1/2)} n^{1/4},
\]

which gives us

\[
\mathbb{E}_{b \sim \mathcal{U}(S^{n-1})} \left[ \left( \frac{\lambda_1(A) - \lambda_1^{(m)}(A, b)}{\lambda_1(A)} \right)^p \right]^{1/p} \leq \left[ \frac{2^{1/4} \Gamma(1/4)}{\Gamma(1/2)} n^{1/4} \right]^{1/p} e^{-\gamma (m - 1)/p} + \gamma^2,
\]

where \(\gamma = \sqrt{1 - \beta}\). Setting \(\gamma = \frac{p}{m-1} \ln \left( n^{1/4p} (m-1)^2 \right) \) (assuming \(\gamma < 1\), otherwise our bound is already greater than one, and trivially holds), we obtain

\[
\mathbb{E}_{b \sim \mathcal{U}(S^{n-1})} \left[ \left( \frac{\lambda_1(A) - \lambda_1^{(m)}(A, b)}{\lambda_1(A)} \right)^p \right]^{1/p} \leq \frac{\left( \frac{2^{1/4} \Gamma(1/4)}{\Gamma(1/2)} n^{1/4} \right)^{1/p} + \frac{1}{m} \ln^2 (n(m-1)^{8p})}{(m-1)^2}
\]

for \(m \geq 10, n \geq 100\). This completes the proof.

\(\square\)

A very similar proof, paired with probabilistic bounds on the quantity \(\sum_{i=2}^n Y_i / Y_1\), where \(Y_1, ..., Y_n \sim \chi_1^2\), gives the following probabilistic estimate.

**Lemma 4.2.**

\[
\sup_{A \in S^n_+} \mathbb{E}_{b \sim \mathcal{U}(S^{n-1})} \left[ \frac{\lambda_1(A) - \lambda_1^{(m)}(A, b)}{\lambda_1(A)} \right] \leq 0.571 \frac{\ln^2 (n(m-1)^{2/3})}{(m-1)^2} \geq 1 - o(1/n).
\]
Proof. By the proof of Lemma 4.1, we have

\[ \frac{\lambda_1(A) - \lambda_1^{(m)}(A,b)}{\lambda_1(A)} = \min_{Q \in F_{m-1}(1)} \sum_{i=2}^{n} Y_i Q^2(\lambda_i)(1 - \lambda_i) \]

\[ \leq \min_{Q \in F_{m-1}(1)} \max_{x \in [0,\beta]} Q^2(x)(1 - x) \frac{\sum_{i=2}^{n} Y_i}{Y_1} + (1 - \beta) \]

\[ \leq T_{m-1}^2 \left( \frac{2}{\beta} - 1 \right) \frac{\sum_{i=2}^{n} Y_i}{Y_1} + (1 - \beta) \]

\[ \leq 4 \exp\{-4\sqrt{1 - \beta} (m - 1)\} \frac{\sum_{i=2}^{n} Y_i}{Y_1} + (1 - \beta). \]

By Proposition 3.3

\[ P \left[ \frac{\sum_{i=2}^{n} Y_i}{Y_1} \geq n^{3.02} \right] \leq P \left[ Y_1 \leq n^{-2.01} \right] + P \left[ \sum_{i=2}^{n} Y_i \geq n^{1.01} \right] \]

\[ \leq (e/n^{2.01})^{1/2} + \left( n^{1.01} e^{-n^{0.1}} \right)^{(n-1)/2} = o(1/n). \]

Let \( \sqrt{1 - \beta} = \frac{\ln (n^{3.02} (m - 1)^2)}{4(m - 1)} \). Then, with probability \( 1 - o(1/n) \),

\[ \frac{\lambda_1(A) - \lambda_1^{(m)}(A)}{\lambda_1(A)} \leq \frac{4}{(m - 1)^2} + \frac{\ln^2 (n^{3.02} (m - 1)^2)}{16(m - 1)^2}. \]

The \( 4/(m - 1)^2 \) term is dominated by the log term as \( n \) increases, and, therefore, with probability \( 1 - o(1/n) \),

\[ \frac{\lambda_1(A) - \lambda_1^{(m)}(A)}{\lambda_1(A)} \leq \frac{0.571 \ln^2 (n(m - 1)^2/3)}{(m - 1)^2}. \]

\( \square \)

Combining the lower bounds in Lemmas 3.4, 5.7, 5.10, the upper bounds in Lemmas 4.1 and 4.2, and the observations from Section 2 regarding the extension from \( S^n_{++} \) to \( S^n \), completes the proofs of Theorems 1.2 and 1.3 the main results of the paper.

5. Distribution Dependent Bounds

In this section, we consider improved estimates for matrices with certain specific properties. First, we show that if a matrix \( A \) has a reasonable number of eigenvalues near its spectral radius, then we can produce an error estimate that no longer depends on the dimension of the matrix, and depends only on the number of iterations \( m \). In particular, we suppose that the eigenvalues of our matrix \( A \) are such that, once scaled, there are at least \( n/(m - 1)^{\alpha} \) eigenvalues in the range \( [\beta, 1] \), for a specific value of \( \beta \) satisfying \( 1 - \beta = O (m^{-2} \ln^2 m) \). For a large number of matrices for which the Lanczos method is used, this assumption holds true. Under this assumption, we prove the following error estimate for \( A \).

Theorem 5.1. Let \( A \in S^n, m \geq 10, p \geq 1, \alpha > 0, \) and \( n \geq m(m - 1)^{\alpha} \). If

\[ \# \left\{ \lambda_i(A) \mid \frac{\lambda_1 - \lambda_i}{\lambda_1 - \lambda_n} \leq \left( \frac{2p + \alpha/4}{m - 1} \right)^2 \right\} \geq \frac{n}{(m - 1)^{\alpha}}, \]

then

\[ E_{b \sim U(S^{n-1})} \left[ \left( \frac{\lambda_1(A) - \lambda_1^{(m)}(A,b)}{\lambda_1(A) - \lambda_n(A)} \right)^{1/p} \right] \leq 0.077 \left( \frac{2p + \alpha/4}{m - 1} \right)^2 \ln^2 (m - 1). \]

In addition, if

\[ \# \left\{ \lambda_i(A) \mid \frac{\lambda_1 - \lambda_i}{\lambda_1 - \lambda_n} \leq \left( \frac{\alpha + 2}{4(m - 1)} \right)^2 \right\} \geq \frac{n}{(m - 1)^{\alpha}}, \]

then
then
\[
\mathbb{P}_{b \sim U(S^{n-1})} \left[ \frac{\lambda_1(A) - \lambda_1^{(m)}(A,b)}{\lambda_1(A) - \lambda_n(A)} \right] \leq \frac{(\alpha + 2)^2 \ln^2(m-1)}{(m-1)^2} \geq 1 - O(e^{-m}).
\]

**Proof.** We begin by bounding expected relative error. The main idea is to proceed as in the proof of Lemma 4.1 but take advantage of the number of eigenvalues near \(\lambda_1(A)\). For simplicity, let \(\lambda_1 = 1\) and \(\lambda_n = 0\). We consider eigenvalues in the ranges \([0, 2\beta - 1]\) and \([2\beta - 1, 1]\) separately, \(1/2 < \beta < 1\), and then make use of the lower bound for the number of eigenvalues in \([\beta, 1]\). From the proof of Lemma 4.1 we have

\[
\mathbb{E}_{b \sim U(S^{n-1})} \left[ \left( \frac{\lambda_1(A) - \lambda_1^{(m)}(A,b)}{\lambda_1(A) - \lambda_n(A)} \right)^{p/2^\beta} \right] \leq \min_{Q \in \mathcal{P}_{m-1}(1)} \left[ \int_{(0,\infty)^n} \left( \frac{\sum_{i: \lambda_i < 2\beta - 1} y_i Q^2(\lambda_i)(1 - \lambda_i)^{2^\beta}}{\sum_{i=1}^n y_i Q^2(\lambda_i)} \right)^{\frac{1}{p}} f_Y(y) \, dy \right]^{\frac{p}{2^\beta}}
\]

where \(q \in \mathbb{N}, 2p < 2^\beta \leq 4p\), and \(\beta \in (1/2, 1)\). The second term is at most \(2(1 - \beta)\), and the integrand of the first term is bounded above by

\[
\left( \frac{\sum_{i: \lambda_i < 2\beta - 1} y_i Q^2(\lambda_i)(1 - \lambda_i)^{2^\beta}}{\sum_{i=1}^n y_i Q^2(\lambda_i)} \right)^{p/2^\beta} \leq \left( \frac{\sum_{i: \lambda_i < 2\beta - 1} y_i Q^2(\lambda_i)(1 - \lambda_i)^{2^\beta}}{\sum_{i=1}^n y_i Q^2(\lambda_i)} \right)^{1/4}
\]

Let \(\sqrt{1 - \beta} = \frac{(2p + \alpha/4) \ln(m-1)}{m-1} < 1/4\). By the condition of the theorem, there are at least \(n/(m - 1)^\alpha\) eigenvalues in the interval \([\beta, 1]\). Therefore,

\[
\int_{(0,\infty)^n} \left( \frac{\sum_{i: \lambda_i < 2\beta - 1} y_i}{\sum_{i: \lambda_i \geq \beta} y_i} \right)^{1/4} f_Y(y) \, dy \leq \mathbb{E}_{Y \sim \chi^2_k} \int_{(0,\infty)^n} \mathbb{E}_{\tilde{Y} \sim \chi^2_{n/(m - 1)^\alpha}} \tilde{Y}^{1/4} \, dy \\
= \frac{\Gamma(n/2 + 1/4) \Gamma([n/(m - 1)^\alpha]/2 - 1/4)}{\Gamma(n/2) \Gamma([n/(m - 1)^\alpha]/2)} \leq 1.04 (m - 1)^{\alpha/4}
\]

for \(n \geq m(m - 1)^\alpha\) and \(m \geq 10\). Replacing the minimizing polynomial by \(\tilde{Q}(x) = \frac{T_{m-1}(\frac{2x}{2\beta - 1} - 1)}{T_{m-1}(\frac{2}{2\beta - 1} - 1)}\), we obtain

\[
\frac{\max_{x \in [0, 2\beta - 1]} \left| \tilde{Q}(x) \right|^{1/2} (1 - x)^{2^\beta - 2}}{\min_{x \in [\beta, 1]} \left| \tilde{Q}(x) \right|^{1/2}} = \frac{1}{T_{m-1}^{1/2} (\frac{2\beta}{2\beta - 1} - 1)} \leq \frac{1}{T_{m-1}^{1/2} (\frac{2}{\beta} - 1)} \leq \frac{\sqrt{2}}{\exp\{\sqrt{1 - \beta}(m - 1)\}}.
\]
Combining our estimates results in the bound
\[
\mathbb{E}_{b \sim 	ext{i.i.d.}(S^{n-1})} \left[ \left( \frac{\lambda_1(A) - \lambda_1^{(m)}(A,b)}{\lambda_1(A) - \lambda_n(A)} \right)^{\frac{1}{p}} \right]^p \leq \left( 1 + \sqrt{2} (m - 1)^{\alpha/4} \right)^{1/p} e^{-\sqrt{1 - \beta}(m - 1)/p} + 2(1 - \beta)
\]
\[
= \frac{(1 + \sqrt{2})^{1/p}}{(m - 1)^2} + \frac{(2p + \alpha/4)^2 \ln^2(m - 1)}{(m - 1)^2}
\]
\[
\leq 0.077 \frac{(2p + \alpha/4)^2 \ln^2(m - 1)}{(m - 1)^2}
\]
for \( m \geq 10, p \geq 1, \) and \( \alpha > 0. \) This completes the bound for expected relative error.

We now produce a probabilistic bound for relative error in a similar fashion to the proof of Lemma 4.2.
Let \( \sqrt{1 - \beta} = \frac{(\alpha + 2) \ln(m - 1)}{4(m - 1)}. \) We have
\[
\frac{\lambda_1(A) - \lambda_1^{(m)}(A,b)}{\lambda_1(A) - \lambda_n(A)} = \min_{Q \in \mathcal{P}_{m-1}(1)} \sum_{i=2}^{m} Y_i Q^2(\lambda_i)(1 - \lambda_i) \sum_{i=1}^{m} Y_i Q^2(\lambda_i)
\]
\[
\leq \min_{Q \in \mathcal{P}_{m-1}(1)} \max_{x \in [0,1]} Q^2(x)(1 - x) \sum_{i: \lambda_i < 2\beta - 1} Y_i \sum_{i: \lambda_i \geq \beta} Y_i + 2(1 - \beta)
\]
\[
\leq T_{m-1} \left( \frac{2}{\beta} - 1 \right) \sum_{i: \lambda_i < 2\beta - 1} Y_i \sum_{i: \lambda_i \geq \beta} Y_i + 2(1 - \beta)
\]
\[
\leq 4 \exp(-4 \sqrt{1 - \beta}(m - 1)) \frac{\sum_{i: \lambda_i < 2\beta - 1} Y_i \sum_{i: \lambda_i \geq \beta} Y_i}{\sum_{i: \lambda_i \geq \beta} Y_i} + 2(1 - \beta).
\]
By Proposition 3.3,
\[
P \left[ \sum_{i: \lambda_i < 2\beta - 1} Y_i \sum_{i: \lambda_i \geq \beta} Y_i \geq 4(m - 1)^{\alpha} \right] \leq P \left[ \sum_{i: \lambda_i < 2\beta - 1} Y_i \geq 2n \right] + P \left[ \sum_{i: \lambda_i \geq \beta} Y_i \leq n \frac{2(1 - \beta)}{2(m - 1)^{\alpha}} \right]
\]
\[
\leq (2/e)^{n/2} + (\sqrt{e}/2)^n (m - 1)^{\alpha} = O(e^{-m}).
\]

Then, with probability \( 1 - O(e^{-m}), \)
\[
\frac{\lambda_1(A) - \lambda_1^{(m)}(A)}{\lambda_1(A) - \lambda_n(A)} \leq 16(m - 1)^{\alpha} e^{-4 \sqrt{1 - \beta}(m - 1)} + 2(1 - \beta)
\]
\[
= \frac{16}{(m - 1)^2} + \frac{(\alpha + 2)^2 \ln^2(m - 1)}{8(m - 1)^2}.
\]
The \( 16/(m - 1)^2 \) term is dominated by the log term as \( m \) increases, and, therefore, with probability \( 1 - O(e^{-m}), \)
\[
\frac{\lambda_1(A) - \lambda_1^{(m)}(A)}{\lambda_1(A)} \leq 0.126 \frac{(\alpha + 2)^2 \ln^2(m - 1)}{(m - 1)^2}.
\]

The above theorem shows that, for matrices whose distribution of eigenvalues is independent of \( n, \) we can obtain dimension-free estimates. For example, the above theorem holds for the matrix \( A \) from Example 1.1 for \( \alpha = 2. \)

When a matrix \( A \) has eigenvalues known to converge to a limiting distribution as dimension increases, or a random matrix \( X_n \) exhibits suitable convergence of its empirical spectral distribution \( L_{X_n} := 1/n \sum_{i=1}^{n} \delta_{\lambda_i(A_n)}, \)

improved estimates can be obtained by simply estimating the corresponding integral polynomial minimization problem. However, to do so, we first require a law of large numbers for weighted sums of independent identically distributed (i.i.d.) random variables. We recall the following result.

**Proposition 5.2.** Let \( a_1, a_2, \ldots \in [a,b] \) and \( X_1, X_2, \ldots \) be i.i.d. random variables, with \( \mathbb{E}[X_1] = 0 \) and \( \mathbb{E}[X_1^2] < \infty. \) Then \( \frac{1}{n} \sum_{i=1}^{n} a_i X_i \to 0 \) almost surely.
We are present the following theorem regarding the error of random matrices that exhibit suitable convergence.

**Theorem 5.3.** Let $X_n \in S^n$, $n = 1, 2, \ldots$ be a sequence of random matrices, such that $L_{X_n}$ converges in probability to $\sigma(x) \, dx$, where $\sigma(x) \in C_b(\mathbb{R})$ has support $[a, b]$. Then, for all $m \in \mathbb{N}$ and $\epsilon > 0$,

$$
\lim_{n \to \infty} \mathbb{P} \left( \left| \frac{\lambda_1(X_n) - \lambda_1^{(m)}(X_n, b)}{\lambda_1(X_n) - \lambda_n(X_n)} \right| - \min_{P \in \mathcal{P}_{m-1}} \sum_{a \neq 0} \int_a^b P^2(x) \frac{b - x}{b - a} \sigma(x) \, dx \right) > \epsilon = 0.
$$

**Proof.** The main idea of the proof is to use Proposition [5,2] to control the behavior of $Y$, and the convergence of $L_{X_n}$ to $\sigma(x) \, dx$ to show convergence to our integral minimization problem. We first write our polynomial $P \in \mathcal{P}_{m-1}$ as $P(x) = \sum_{j=0}^{m-1} \alpha_j x^j$ and our unnormalized error as

$$
\lambda_1(X_n) - \lambda_1^{(m)}(X_n, b) = \min_{P \neq 0} \sum_{i=1}^{m-1} \lambda_i P^2(\lambda_i) \sum_{i=1}^{m-1} \lambda_i P^2(\lambda_i) = \min_{\alpha \in \mathbb{R}^m} \sum_{j=1}^{m-1} \alpha_j \lambda_i^j, j = 0, \ldots, m-1,
$$

where $\alpha = (\alpha_0, \ldots, \alpha_{m-1})^T$ and $Y_1, \ldots, Y_n$ are i.i.d. chi-square random variables with one degree of freedom each. The functions $x^j$, $j = 0, \ldots, 2m - 2$, are bounded on $[a, b]$, and so, by Proposition [5,2] for any $\epsilon_1, \epsilon_2 > 0$,

$$
\left| \frac{1}{n} \sum_{i=2}^{n} Y_i \lambda_i^j (\lambda_1 - \lambda_i) - \frac{1}{n} \sum_{i=1}^{n} \lambda_i (\lambda_1 - \lambda_i) \right| < \epsilon_1, \quad j = 0, \ldots, 2m - 2,
$$

and

$$
\left| \frac{1}{n} \sum_{i=1}^{n} Y_i \lambda_i^j - \frac{1}{n} \sum_{i=1}^{n} \lambda_i^j \right| < \epsilon_2, \quad j = 0, \ldots, 2m - 2,
$$

with probability $1 - o(1)$. $L_{X_n}$ converges in probability to $\sigma(x) \, dx$, and so, for any $\epsilon_3, \epsilon_4 > 0$,

$$
\left| \frac{1}{n} \sum_{i=2}^{n} \lambda_i^j (\lambda_1 - \lambda_i) - \int_a^b x^j (b - x) \sigma(x) \, dx \right| < \epsilon_3, \quad j = 0, \ldots, 2m - 2,
$$

and

$$
\left| \frac{1}{n} \sum_{i=1}^{n} \lambda_i^j - \int_a^b x^j \sigma(x) \, dx \right| < \epsilon_3, \quad j = 0, 1, \ldots, m - 1,
$$

with probability $1 - o(1)$. This implies that

$$
\lambda_1(X_n) - \lambda_1^{(m)}(X_n, b) = \min_{\alpha \in \mathbb{R}^m} \sum_{j=1}^{m-1} \alpha_j \lambda_i^j \left( \int_a^b x^{j_1 + j_2} (b - x) \sigma(x) \, dx + \hat{E}(j_1, j_2) \right),
$$

where $|\hat{E}(j_1, j_2)| < \epsilon_4$ and $|E(j_1, j_2)| < \epsilon_2 + \epsilon_4$, $j_1, j_2 = 0, \ldots, m - 1$, with probability $1 - o(1)$.

The minimization problem

$$
\min_{P \neq 0} \sum_{a \neq 0} \int_a^b P^2(x) \frac{b - x}{b - a} \sigma(x) \, dx \int_a^b P^2(x) \sigma(x) \, dx
$$
corresponds to a generalized Rayleigh quotient \( \alpha^T A \alpha \), where \( A, B \in \mathcal{S}^m_{++} \) and \( \lambda_{\max}(A), \lambda_{\max}(B) \), and \( \lambda_{\min}(B) \) are all constants independent of \( n \). By choosing \( \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 \) sufficiently small,

\[
\left| \frac{\alpha^T (A + \hat{E}) \alpha}{\alpha^T (B + \hat{E}) \alpha} - \frac{\alpha^T A \alpha}{\alpha^T B \alpha} \right| \leq \frac{\left| (\alpha^T \hat{E} \alpha) \alpha^T B \alpha - (\alpha^T E \alpha) \alpha^T A \alpha \right|}{(\alpha^T B \alpha + \alpha^T E \alpha) \alpha^T B \alpha} \leq \frac{(\epsilon_1 + \epsilon_4) m \lambda_{\max}(B) + (\epsilon_2 + \epsilon_4) m \lambda_{\max}(A)}{\lambda_{\min}(B) - \epsilon} \leq \epsilon
\]

for all \( \alpha \in \mathbb{R}^m \) with probability \( 1 - o(1) \). This completes the proof. \( \square \)

The above theorem is a powerful tool for explicitly computing the error in the Lanczos method for certain types of matrices, as the computation of the extremal eigenvalue of an \( m \times m \) matrix is a nominal computation compared to one application of an \( n \) dimensional matrix. Of course, Theorem 5.3 is equally applicable for estimating \( \lambda_n \). As an example, we apply the above theorem to the class of Wigner matrices.

**Example 5.4.** A random matrix \( X_n = n^{-1/2} Y_n \in \mathcal{S}^n \) is said to be a Wigner matrix with second moment \( t \) if \( Y_n(i, j), 1 \leq i < j \leq n \), are i.i.d. random variables with mean zero and variance \( t \), and \( Y_n(i, i), i = 1, \ldots, n \), are i.i.d. random variables with mean zero and finite variance. The empirical spectral distribution \( L_n(X_n) \) converges in probability to the semi-circle distribution \( \frac{1}{2\pi t} \sqrt{4t - x^2} dx \). By Theorem 5.3, the error approaches

\[
\min_{P \in \mathcal{P}_{m-1} \setminus \{0\}} \int_{-2\sqrt{t}}^{2\sqrt{t}} P^2(x) \left( 2\sqrt{t} - x \right) \sqrt{4t - x^2} dx = \frac{1}{2} \int_{-1}^{1} P^2(y) (1 - y)^{3/2} (1 + y)^{1/2} dy
\]

By representing \( P(y) \) in the basis \( \{ P_{3/2}(y) \}_{r=0}^{m-1} \) and using Proposition 3.5, it can be shown that this minimum is given by \( \lambda_{\min}(M^{-1}) \), where

\[
M(i, j) = \frac{i! j! (i + 1) (j + 1)}{(i + 3)! (j + 3)!} \sum_{\ell=0}^{\min(i, j)} (2\ell + 3)^2 (\ell + 2)^2, \quad i, j = 0, \ldots, m - 1.
\]

The simple lower and upper bounds of \( \text{tr}(M)^{-1} \) and the Rayleigh quotient of \( \mathbb{1} \) with respect to \( M^{-1} \), which agree in order, illustrate that Wigner matrices exhibit an order \( m^{-2} \) error.

Example 5.4 provides further evidence that the dependence of \( m^{-2} \) is not just a worst case bound for the Lanczos method, but actually a precise estimate of typical error for a large number of matrices. In addition, we note that the choice of Wigner matrices in Example 5.4 is a fairly arbitrary one, as the above technique is applicable to a large class of matrices.

### 6. Estimates for Arbitrary Eigenvalues

Up to this point, we have concerned ourselves almost exclusively with the extremal eigenvalues \( \lambda_1 \) and \( \lambda_n \) of a matrix. In this section, we extend the techniques of this paper to arbitrary eigenvalues. The results of this section provide the first uniform error estimates for arbitrary eigenvalues. Lower bounds for arbitrary eigenvalues follow relatively quickly from our previous work. However, our proof technique for upper bounds requires some mild assumptions regarding the eigenvalue gaps of the matrix. We begin with asymptotic lower bounds for an arbitrary eigenvalue, and present the following corollary of Theorems 1.2 and 1.3.

**Corollary 6.1.**

\[
\sup_{A \in \mathcal{S}^n} \mathbb{P} \left[ \frac{\lambda_i(A) - \lambda_{\min}(A, b)}{\lambda_{\max}(A) - \lambda_{\min}(A)} \geq 1 - o(1) \right] \geq 1 - o(1/n)
\]

for \( m = o(\ln n) \),

\[
\sup_{A \in \mathcal{S}^n} \mathbb{P} \left[ \frac{\lambda_i(A) - \lambda_{\min}(A, b)}{\lambda_{\max}(A) - \lambda_{\min}(A)} \geq (1 - o(1))(1 - \alpha) \right] \geq 1 - o(1/n)
\]
Theorem 6.2. Let $\lambda_i(A)$ be the $i$th largest eigenvalue of a matrix $A$. We present the following theorem.

**Theorem 6.2.**

Let $\lambda_i(A)$ be the $i$th largest eigenvalue of a matrix $A$. Then

$$
\sup_{A \in \mathcal{S}^n \sim \mathcal{U}(\mathcal{S}^{n-1})} \frac{\sum_{i=1}^{\lfloor n/2 \rfloor} \lambda_i(A) - \lambda_i^{(m)}(A,b)}{\lambda_i(A) - \lambda_n(A)} \geq \begin{cases} 
1 & \text{for } m = o(1/n) \\
1 - \frac{\ln n - \ln(\ln^2 n)}{4 \ln \left(1 + \frac{\sqrt{2}}{\sqrt{\alpha}}\right)} & \text{for } m = O(\ln n) \\
\frac{.249 \ln n}{m^2 \ln \ln n} & \text{for } m = o\left(\frac{n^{1/2}}{\ln^{1/2} n}\right), \text{ and, therefore,}
\end{cases}
$$

**Proof.** By the inequalities

$$
\lambda_i^{(m)}(A,b) \leq \max_{x \in \mathcal{K}_{m}(A,b), x \neq 0, j=1,\ldots,i-1} \frac{x^T A x}{x^T x} \leq \max_{x \in \mathcal{K}_{m}(A,b) - \sum_{j=1}^{i-1} (\varphi_j b)} \frac{x^T A x}{x^T x} = \lambda_i^{(m)}\left(A, b - \sum_{j=1}^{i-1} (\varphi_j b)\right),
$$

the relative error

$$
\frac{\lambda_i(A) - \lambda_i^{(m)}(A,b)}{\lambda_i(A) - \lambda_n(A)} \geq \frac{\lambda_i(A) - \lambda_i^{(m)}(A,b - \sum_{j=1}^{i-1} (\varphi_j b))}{\lambda_i(A) - \lambda_n(A)}.
$$

The right hand side corresponds to an extremal eigenvalue problem of dimension $n - i + 1$. Setting the largest eigenvalue of $A$ to have multiplicity $i$, and defining the eigenvalues $\lambda_i, \ldots, \lambda_n$ based on the eigenvalues (corresponding to dimension $n - i + 1$) used in the proofs of Lemmas 3.3, 3.7 and 3.10 completes the proof.

In general, the index $i$ in Corollary 6.1 should be thought of as a fixed constant, but the result also holds when $i$ is a function of $m$, though the conclusion is much less useful. Next, we provide an upper bound for the relative error in approximating $\lambda_i$ under the assumption of non-zero gaps between eigenvalues $\lambda_1, \ldots, \lambda_i$. We present the following theorem.

**Theorem 6.2.** Let $n \geq 100$, $m \geq 9 + i$, $p \geq 1$, and $A \in \mathcal{S}^n$. Then

$$
\mathbb{E}_{b \sim \mathcal{U}((\mathcal{S}^{n-1}) \left[ \left( \frac{\lambda_i(A) - \lambda_i^{(m)}(A,b)}{\lambda_i(A) - \lambda_n(A)} \right)^p \right]^{1/p} \leq 0.68 \frac{n^{2-(i-1)/p} n(m-i)^{3p}}{(m-i)^2}
$$

and

$$
\mathbb{P}_{b \sim \mathcal{U}((\mathcal{S}^{n-1}) \left[ \frac{\lambda_i(A) - \lambda_i^{(m)}(A,b)}{\lambda_i(A) - \lambda_n(A)} \leq 0.57 \frac{n^{2-(i-1)/3} n(m-1)^{2/3}}{(m-1)^2} \geq 1 - o(1/n),
$$

where

$$
\delta = \frac{1}{2} \min_{k=2,\ldots,i} \frac{\lambda_i(A) - \lambda_k(A)}{\lambda_i(A) - \lambda_n(A)}.
$$

**Proof.** As in previous cases, it suffices to prove the theorem for matrices $A$ with $\lambda_1(A) = 1$ and $\lambda_n(A) = 0$ (if $\lambda_i = \lambda_n$, we are done). Similar to the polynomial representation of $\lambda_i^{(m)}(A,b)$, the Ritz value $\lambda_i^{(m)}(A,b)$
corresponds to finding the polynomial in $\mathcal{P}_{m-1}$ with zeros $\lambda_k^{(m)}$, $k = 1, \ldots, i - 1$, that maximizes the corresponding Rayleigh quotient. For the sake of brevity, let $\phi_i(x) = \prod_{k=1}^{i-1} (\lambda_k^{(m)} - x)^2$. Then $\lambda_i^{(m)}(A, b)$ can be written as

$$
\lambda_i^{(m)}(A, b) = \max_{P \in \mathcal{P}_{m-i}} \frac{\sum_{j=1}^{n} b_j^2 P^2(\lambda_j) \phi_i(\lambda_j)}{\sum_{j=1}^{n} b_j^2 P^2(\lambda_j) \phi_i(\lambda_j)},
$$

and, therefore, the error is bounded above by

$$
\lambda_i(A) - \lambda_i^{(m)}(A, b) \leq \max_{P \in \mathcal{P}_{m-i}} \frac{\sum_{j=i+1}^{n} b_j^2 P^2(\lambda_j)(1-\lambda_j) \phi_i(\lambda_j)}{\sum_{j=1}^{n} b_j^2 P^2(\lambda_j) \phi_i(\lambda_j)}.
$$

The main idea of the proof is very similar to that of Lemma 4.1 and 4.2, paired with a pigeonhole principle. The intervals $[\lambda_j(A) - \delta \lambda_1, \lambda_j(A) + \delta \lambda_1]$, $j = 1, \ldots, i$, are disjoint, and so there exists some index $j^*$ such that the corresponding interval does not contain any of the Ritz values $\lambda_k^{(m)}(A, b)$, $k = 1, \ldots, i - 1$.

We begin by bounding expected relative error. As in the proof of Lemma 4.1, by integrating over chi-square random variables and using Cauchy-Schwarz, we have

$$
\mathbb{E}_{b \sim \mathcal{U}(\mathcal{S}^{n-1})} \left[ (\lambda_i(A) - \lambda_i^{(m)}(A, b))^p \right]^{1/p} \leq \min_{P \in \mathcal{P}_{m-i}} \left[ \int_{[0,\infty)^n} \left( \frac{\sum_{j=i+1}^{n} b_j^2 P^2(\lambda_j)(1-\lambda_j) \phi_i(\lambda_j)}{\sum_{j=1}^{n} b_j^2 P^2(\lambda_j) \phi_i(\lambda_j)} \right)^{\frac{p}{q}} f_Y(y) \ dy \right]^{\frac{1}{p}}
$$

$$
\leq \min_{P \in \mathcal{P}_{m-i}} \left[ \int_{[0,\infty)^n} \left( \frac{\sum_{j:j \neq j^*} y_j^2 P^2(\lambda_j)(1-\lambda_j) \phi_i(\lambda_j)}{\sum_{j=1}^{n} y_j^2 P^2(\lambda_j) \phi_i(\lambda_j)} \right)^{\frac{p}{q}} f_Y(y) \ dy \right]^{\frac{1}{p}}
$$

$$
+ \left[ \int_{[0,\infty)^n} \left( \frac{\sum_{j:j = j^*} y_j P^2(\lambda_j)(1-\lambda_j) \phi_i(\lambda_j)}{\sum_{j=1}^{n} y_j P^2(\lambda_j) \phi_i(\lambda_j)} \right)^{\frac{p}{q}} f_Y(y) \ dy \right]^{\frac{1}{p}}
$$

for $q \in \mathbb{N}$, $2p < 2^{q' \leq 4p}$, and any $\beta \in (0, 1)$. The second term on the right hand side is bounded above by $1 - \beta$, and the integrand of the first term is bounded above by

$$
\left( \frac{\sum_{j:j \neq j^*} y_j P^2(\lambda_j)(1-\lambda_j) \phi_i(\lambda_j)}{\sum_{j=1}^{n} y_j P^2(\lambda_j) \phi_i(\lambda_j)} \right)^{\frac{p}{q}} \leq \left( \frac{\sum_{j:j \neq j^*} y_j P^2(\lambda_j)(1-\lambda_j) \phi_i(\lambda_j)}{\sum_{j=1}^{n} y_j P^2(\lambda_j) \phi_i(\lambda_j)} \right)^{\frac{p}{q}} \leq \max_{x \in [0,\beta]} \left( \frac{P(x)}{P(\lambda_{j^*})} \right)^{1/2} \phi_i^{1/4}(x) \frac{1/4}{\phi_i^{1/4}(\lambda_{j^*})} \left( \frac{\sum_{j:j \neq j^*} y_j}{y_{j^*}} \right)^{1/4}.
$$

By replacing the minimizing polynomial in $\mathcal{P}_{m-i}$ by $T_{m-i}(2/\beta x - 1)$, the maximum is achieved at $x = 0$, and, by monotonicity of $T_{m-i}$ on $[1, \infty)$,

$$
T_{m-i}(2/\beta \lambda_j - 1) \geq T_{m-i}(2/\beta - 1) \geq \frac{1}{2} \exp \left\{ 2 \sqrt{1 - \beta (m - i)} \right\}.
$$

In addition,

$$
\frac{\phi_i^{1/4}(0)}{\phi_i^{1/4}(\lambda_{j^*})} = \prod_{k=1}^{i-1} \left( \frac{\lambda_k^{(m)}}{\lambda_k^{(m)} - \lambda_{j^*}} \right)^{1/2} \leq \delta^{-(i-1)/2}.
$$

This gives us

$$
\mathbb{E}_{b \sim \mathcal{U}(\mathcal{S}^{n-1})} \left[ (\lambda_i(A) - \lambda_i^{(m)}(A, b))^p \right]^{1/p} \leq \left[ \frac{2^{1/4} \Gamma(1/4)}{\Gamma(1/2)} \frac{n^{1/4}}{\delta^{(i-1)/2}} \right]^{1/p} e^{-\gamma(m-i)/p} + \gamma^2.
$$
where \( \gamma = \sqrt{1 - \beta} \). Setting \( \gamma = \frac{p}{m - i} \ln \left( \delta^{-(i-1)/2} \right) \) (assuming \( \gamma < 1 \), otherwise our bound is already greater than one, and trivially holds), we obtain

\[
\mathbb{E}_{b \sim \mathcal{U}(S^{-1})} \left[ \left( \lambda_i(A) - \lambda_i^{(m)}(A, b) \right)^p \right]^{1/p} \leq \frac{\left( \frac{2^{1/4} \Gamma(1/4)}{\Gamma(1/2)} \right)^{1/p} + \frac{1}{16} \ln^2 \left( \delta^{-(i-1)} n (m - i)^8 \right)}{(m - i)^2} \leq 0.568 \frac{\ln^2 \left( \delta^{-(i-1)} n (m - i)^8 \right)}{(m - i)^2},
\]

for \( m \geq 9 + i, n \geq 100 \). This completes the proof of the expected error estimate. The probabilistic estimate follows relatively quickly, and is very similar in nature to the proof of Lemma 4.2. We have

\[
\lambda_i(A) - \lambda_i^{(m)}(A, b) \leq \min_{P \in P_{m-i}} \frac{\sum_{j=i+1}^{n} Y_j P^2(\lambda_j) (1 - \lambda_j) \phi_i(\lambda_j)}{\sum_{j=1}^{n} Y_j P^2(\lambda_j) \phi_i(\lambda_j)} \leq \min_{P \in P_{m-i}} \max_{x \in [0, \beta]} \frac{P^2(x) (1 - x) \phi_i(x)}{P^2(\lambda_j^*) \phi_i(\lambda_j^*)} \sum_{j:\lambda_j < \beta} \frac{Y_j}{Y_j^*} + (1 - \beta) \leq \delta^{-2(i-1)} T_{m-i}^{-2} \left( \frac{2}{\beta} - 1 \right) \sum_{j:\lambda_j < \beta} \frac{Y_j}{Y_j^*} + (1 - \beta).\]

By Proposition 3.3

\[
\mathbb{P} \left[ \sum_{j:\lambda_j < \beta} \frac{Y_j}{Y_j^*} \geq n^{0.02} \right] \leq \mathbb{P} \left[ Y_{j^*} \leq n^{-2.01} \right] + \mathbb{P} \left[ \sum_{j \neq j^*} Y_j \geq n^{1.01} \right] \leq \left( e/n^{2.01} \right)^{1/2} + \left( n^{0.01} e^{-n^{1.01}} \right)^{(n-1)/2} = o(1/n).
\]

Let \( \sqrt{1 - \beta} = \frac{\ln \left( \delta^{-(i-1)} n^{3.02} (m - 1)^2 \right)}{4(m - 1)} \). Then, with probability \( 1 - o(1/n) \),

\[
\lambda_i(A) - \lambda_i^{(m)}(A) \leq \frac{4}{(m - 1)^2} + \frac{\ln^2 \left( \delta^{-(i-1)} n^{3.02} (m - 1)^2 \right)}{16(m - 1)^2}.
\]

The \( 4/(m - 1)^2 \) term is dominated by the log term as \( n \) increases, and, therefore, with probability \( 1 - o(1/n) \),

\[
\lambda_i(A) - \lambda_i^{(m)}(A) \leq 0.571 \frac{\ln^2 \left( \delta^{-(i-1)/3} n (m - 1)^{2/3} \right)}{(m - 1)^2}.
\]

For typical matrices with no repeated eigenvalues, \( \delta \) is usually a very low degree polynomial in \( n \), and, for \( i \) constant, the estimates for \( \lambda_i \) are not much worse than that of \( \lambda_1 \). In addition, given any matrix \( A \), a small random perturbation of \( A \) before the application of the Lanczos method will satisfy the condition of the theorem with high probability, and change the eigenvalues by a negligible amount. However, it is very likely that, with a different proof, the same (or improved) estimates can be shown without any assumptions on eigenvalue gaps. Of course, the bounds from Theorem 6.2 for maximal eigenvalues \( \lambda_i \) also apply to minimal eigenvalues \( \lambda_{n-i} \).

\section{Experimental Results}

In this section, we present a number of experimental results that illustrate error estimates of the Lanczos method in practice. We consider four different spectra:

- a uniform partition of \([0, 1] \), \( \Lambda_{unif} = \{(n + 1 - i)/n \}_{i=1}^{n} \),
- nodes of Gaussian quadrature, \( \Lambda_{gauss} = \{ x_i \}_{i=1}^{n} \),
- eigenvalues of 1D Laplacian with Dirichlet boundary conditions \( \Lambda_{lap} = \{ 2 + 2 \cos(i\pi/(n + 1)) \}_{i=1}^{n} \),
- eigenvalues corresponding to Lemma 3.10, \( \Lambda_{log} = \{ 1 - [(n + 1 - i)/n] \}_{i=1}^{n} \).
Figure 1. Plot and box plot of relative error times $m^2$ vs iteration number $m$ for $\Lambda_{\text{unif}}$, $\Lambda_{\text{gauss}}$, and $\Lambda_{\text{lap}}$. The plot contains curves for each dimension $n$ tested. Each curve represents the empirical average relative error for each value of $m$, averaged over 100 random initializations. The box plot illustrates the variability of relative error for $n = 10^3$. For a given $m$, the 25th and 75th percentile of the values, denoted by $q_1$ and $q_3$, are the bottom and top of the corresponding box, and the red line in the box is the median. The whiskers extend to the most extreme points in the interval $[q_1 - 1.5(q_3 - q_1), q_3 + 1.5(q_3 - q_1)]$, and outliers not in this interval correspond to the ‘+’ symbol.
Figure 2. Plot and box plot of relative error times $m^2$ vs iteration number $m$ for $\Lambda_{\text{log}}$. The plot contains curves for each dimension $n$ tested. Each curve represents the empirical average relative error for each value of $m$, averaged over 100 random initializations. The box plot illustrates the variability of relative error for $n = 10^8$. For a given $m$, the 25th and 75th percentile of the values, denoted by $q_1$ and $q_3$, are the bottom and top of the corresponding box, and the red line in the box is the median. The whiskers extend to the most extreme points in the interval $[q_1 - 1.5(q_3 - q_1), q_3 + 1.5(q_3 - q_1)]$, and outliers not in this interval correspond to the '+' symbol.

Acknowledgments

The author would like to thank Alan Edelman, Michel Goemans and Steven Johnson for interesting conversations on the subject, and Louisa Thomas for improving the style of presentation. The author was supported in part by ONR Research Contract N00014-17-1-2177.
References

[1] Jack Cuzick. A strong law for weighted sums of iid random variables. Journal of Theoretical Probability, 8(3):625–641, 1995.
[2] Sanjoy Dasgupta and Anupam Gupta. An elementary proof of a theorem of johnson and lindenstrauss. Random Structures & Algorithms, 22(1):60–65, 2003.
[3] Philip J Davis and Philip Rabinowitz. Methods of numerical integration. Courier Corporation, 2007.
[4] Gianna M Del Corso and Giovanni Manzini. On the randomized error of polynomial methods for eigenvector and eigenvalue estimates. Journal of Complexity, 13(4):419–456, 1997.
[5] Kathy Driver, Kerstin Jordaan, and Norbert Mbuyi. Interlacings of the zeros of jacobi polynomials with different parameters. Numerical Algorithms, 49(1-4):143, 2008.
[6] Klaus-Jürgen Förster and Knut Petras. On estimates for the weights in gaussian quadrature in the ultraspherical case. Mathematics of computation, 55(191):243–264, 1990.
[7] Gene H Golub and Charles F Van Loan. Matrix computations, volume 3. JHU press, 2012.
[8] Shmuel Kaniel. Estimates for some computational techniques in linear algebra. Mathematics of Computation, 20(95):369–378, 1966.
[9] Jacek Kuczyński and Henryk Woźniakowski. Estimating the largest eigenvalue by the power and lanczos algorithms with a random start. SIAM journal on matrix analysis and applications, 13(4):1094–1122, 1992.
[10] Jacek Kuczyński and Henryk Woźniakowski. Probabilistic bounds on the extremal eigenvalues and condition number by the lanczos algorithm. SIAM Journal on Matrix Analysis and Applications, 15(2):672–691, 1994.
[11] Geno Nikolov. Inequalities of duffin-schaeffer type ii. Institute of Mathematics and Informatics at the Bulgarian Academy of Sciences, 2003.
[12] Christopher Conway Paige. The computation of eigenvalues and eigenvectors of very large sparse matrices. PhD thesis, University of London, 1971.
[13] Beresford N Parlett, H Simon, and LM Stringer. On estimating the largest eigenvalue with the lanczos algorithm. Mathematics of computation, 38(157):153–165, 1982.
[14] Yousef Saad. On the rates of convergence of the lanczos and the block-lanczos methods. SIAM Journal on Numerical Analysis, 17(5):687–706, 1980.
[15] Yousef Saad. Numerical methods for large eigenvalue problems: revised edition, volume 66. Siam, 2011.
[16] Jie Shen, Tao Tang, and Li-Lian Wang. Spectral methods: algorithms, analysis and applications, volume 41. Springer Science & Business Media, 2011.
[17] Lloyd N Trefethen and David Bau III. Numerical linear algebra, volume 50. Siam, 1997.
[18] Jos LM Van Dorselaer, Michiel E Hochstenbach, and Henk A Van Der Vorst. Computing probabilistic bounds for extreme eigenvalues of symmetric matrices with the lanczos method. SIAM Journal on Matrix Analysis and Applications, 22(3):837–852, 2001.
[19] Herbert S Wilf. Finite sections of some classical inequalities, volume 52. Springer Science & Business Media, 2012.

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA, USA.
E-mail address: urschel@mit.edu