Estimation of stress-strength probability in a multicomponent model based on geometric distribution

Milan Jovanović*, Bojana Milošević†, Marko Obradović‡

Faculty of Mathematics, University of Belgrade, Studenski trg 16, Belgrade, Serbia

Abstract

In this paper, the estimation of the stress-strength probability in a multicomponent model, in the case when all components follow the geometric distribution, is studied. This is the first time that multicomponent models with discrete probability distributions are considered. The MLE, UMVUE and Bayes point estimator, as well as asymptotic and bootstrap confidence intervals are presented. A simulation study is performed in order to compare the performance of various estimators. Finally, the methods are applied to real data examples from climatology and sport.

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1. Introduction

Stress-strength models have been popular for many years, mainly due to their applicability in engineering, meteorology, quality control, medicine, etc. The basic interpretation of the stress-strength probability is the reliability parameter of a system. In the simplest stress-strength model the system fails if the applied stress \(X\) is greater than strength \(Y\), so the reliability parameter \(R = P\{X \leq Y\}\) is a measure of system performance. In a broader interpretation, \(R\) can be viewed as a measure of difference between two populations. For example, in medicine, if \(Y\) represents the response of a treatment group, and \(X\) refers to a control group, \(R\) is a measure of the effect of the treatment.

Estimation of \(R\) is one of the main goals and it has been widely studied in statistical literature (see [17] for an excellent review of theory and applications in this area). The majority of papers deal with continuous probability distributions. However, there are some applications where stress and strength can have discrete distributions. For example, this is the case when the stress is the number of shocks the product undergoes and the strength is the number of shocks the product can withstand. The geometric distribution case was studied in [1,18], the negative binomial distribution was considered in [9,25], the Poisson distribution case was examined in [3], while the logarithmic distribution was considered...
in [19]. Recently, the case when the stress is from the geometric and the strength is from the Poisson distribution was investigated in [20], while the geometric-exponential model was studied in [10].

A system having more than one stress or strength component is called a multicomponent system. The most common is "s-out-of-k: G" system. In this model there is one stress $X$ and $k$ strength components $Y_1, Y_2, \ldots, Y_k$. A system functions if at least $s$, $1 \leq s \leq k$, strength components are greater or equal than the stress component. In this setting, the reliability parameter is $R_{s,k}$, where $R_{s,k} = P\{\text{at least } s \text{ of the } (Y_1, Y_2, \ldots, Y_k) \geq X\}$. Parallel and series circuits are examples of extreme 1-out-of-$k$ and $k$-out-of-$k$ multicomponent systems. A four-engine airplane which can operate when at least two of its engines work is an example of 2-out-of-4 system.

The reliability in a multicomponent stress-strength model was initially studied in [4]. Recently, this topic has gained popularity and quite a few papers have been published. The estimation of $R_{s,k}$ for the generalized exponential distribution was examined in [22]. The cases of Burr-XII, Weibull and two parameter exponentiated Weibull distribution were studied in [14,23,24], respectively. The Kumaraswamy distribution was considered in [5,15,16]. General classes of inverse exponentiated and exponentiated inverted exponential distributions were investigated in [7,13], respectively. The power Lindley distribution was examined in [21], the Chen distribution was considered in [11], while the Topp-Leone case was investigated in [2].

All mentioned papers are devoted to multicomponent models with continuous probability distributions. In this paper, for the first time, we focus on the case when both stress and strength components follow discrete distributions. In particular, we consider the geometric distribution.

A motivation for such model can be found, e.g. in hydrology and climatology, when modelling durations of various phenomena, such as droughts, floods, warm and cold spells, etc. There, the number of time intervals (days, months, etc.) during which the process continuously remains above or below a reference level is frequently modelled with the geometric distribution. See, for instance, [12] and [6].

With increasing popularity of the climate change issue, it is of interest to estimate the probability that a duration of some phenomenon will not exceed a certain threshold, set as a quantile of its distribution in the past. For example, the probability that the duration of the next warm spell will not exceed 80th percentile of duration of warm spell in the past is $R_{1,5}$. We illustrate this in a real data example in Section 7.

If a random variable $X$ has geometric $G(p)$ distribution, then its probability mass and distribution functions are given by

$$P\{X = x; p\} = (1 - p)^{x-1}p, \quad x \in \mathbb{N},$$

$$F_X(x) = 1 - (1 - p)^x, \quad x \in \mathbb{N}.$$  

Let the stress $X$ and the strengths $Y_i, i \in \{1, 2, \ldots, k\}$, be independent random variables such that $X$ follows $G(p_1)$ and each $Y_i$ follows $G(p_2)$ distribution. Then, the stress-strength probability is

$$R_{s,k} = P\{\text{at least } s \text{ of the } (Y_1, Y_2, \ldots, Y_k) \geq X\}$$

$$= \sum_{i=s}^{k} \sum_{x=1}^{\infty} P\{X = x, \text{ exactly } i \text{ of the } (Y_1, Y_2, \ldots, Y_k) \geq x\}$$

$$= \sum_{i=s}^{k} \sum_{x=1}^{\infty} (1 - p_1)^{x-1}p_1 \left(\frac{k}{i}\right) (F_Y(x-1))^{k-i} (1 - F_Y(x-1))^i$$

$$= \sum_{i=s}^{k} \sum_{x=1}^{\infty} (1 - p_1)^{x-1}p_1 \left(\frac{k}{i}\right) (1 - p_2)^{x-1} \sum_{j=0}^{k-i} \left(\frac{k-i}{j}\right) (-1)^j (1 - p_2)^{x-1}j$$
\[
R_{s,k} = \sum_{i=s}^{k} \sum_{j=0}^{k-i} (-1)^j \binom{k}{i} \binom{k-i}{j} p_1 \sum_{x=1}^{\infty} \left( (1 - p_1)(1 - p_2)^{i+j} \right)^{x-1}
\]

\[
= \sum_{i=s}^{k} \sum_{j=0}^{k-i} (-1)^j \binom{k}{i} \binom{k-i}{j} \frac{p_1}{1 - (1 - p_1)(1 - p_2)^{i+j}}.
\]

(1.1)

From this formula it is not easy to see how \(R_{s,k}\) depends on \(p_1\) and \(p_2\), so this dependence, for some choices of \(s\) and \(k\), is shown on Figure 1.

The rest of the paper is organized as follows. In section 2 the maximum likelihood estimator (MLE) of \(R_{s,k}\) and its asymptotic distribution are derived. Based on it, in section 3, the asymptotic and bootstrap-p confidence intervals are constructed. The uniformly minimum variance unbiased estimator (UMVUE) of \(R_{s,k}\) is obtained in section 4. Bayes estimator of \(R_{s,k}\) with respect to square loss function is derived in section 5. In section 6 we perform a simulation study and compare the obtained estimators, while in section 7 we present two real data examples.

2. MLE of \(R_{s,k}\) and its asymptotic distribution

Let \(X = (X_1, X_2, \ldots, X_n)\) and \(Y = (Y_1, Y_2, \ldots, Y_m)\) be the samples from \(\mathcal{G}(p_1)\) and \(\mathcal{G}(p_2)\) distributions. The MLE’s of \(p_1\) and \(p_2\) are

\[
\hat{p}_1 = \frac{1}{X_n}, \quad \hat{p}_2 = \frac{1}{Y_m}.
\]

(2.1)

Using the invariance property of MLE, from equation (1.1) we get the MLE of \(R_{s,k}\)

\[
\hat{R}_{s,k} = \sum_{i=s}^{k} \sum_{j=0}^{k-i} (-1)^j \binom{k}{i} \binom{k-i}{j} \frac{\hat{p}_1}{1 - (1 - \hat{p}_1)(1 - \hat{p}_2)^{i+j}}.
\]

(2.2)

In the following theorem we derive the asymptotic distribution of \(\hat{R}_{s,k}\).
Theorem 2.1. Let the ratio $\frac{n}{m}$ converge to a positive number $\lambda$ when both $n$ and $m$ tend to infinity. Then

$$\sqrt{n} (\widetilde{R}_{s,k} - R_{s,k}) \xrightarrow{d} N(0, \sigma^2)$$

when both $n$ and $m$ tend to infinity, where

$$\sigma^2 = p_1^2 (1 - p_1) \left( \sum_{i=s}^{k-1} \sum_{j=0}^{i} (-1)^j \binom{k}{i} \binom{k-i}{j} \frac{1 - (1 - p_2)^{i+j}}{1 - (1 - p_1)(1 - p_2)^{i+j}} \right)^2$$

and

$$+ \lambda p_2^2 (1 - p_2) \left( \sum_{i=s}^{k-1} \sum_{j=0}^{i} (-1)^{j+1} \binom{k}{i} \binom{k-i}{j} \frac{(i+j)p_1(1-p_1)(1-p_2)^{i+j-1}}{1 - (1 - p_1)(1 - p_2)^{i+j}} \right)^2.$$

(2.3)

Proof. Since

$$- \frac{\partial^2 \ln P(X = x)}{\partial p_1^2} = \frac{x - 1}{(1 - p_1)^2} + \frac{1}{p_1^2}$$

and

$$E \left( \frac{X - 1}{(1 - p_1)^2} + \frac{1}{p_1^2} \right) = \frac{1}{p_1^2 (1 - p_1)},$$

from the asymptotic normality of maximum likelihood estimator (see [8]) it follows that

$$\sqrt{n}(\tilde{p}_1 - p_1) \xrightarrow{d} N(0, p_1^2 (1 - p_1))$$

when $n \rightarrow \infty$ and analogously

$$\sqrt{m}(\tilde{p}_2 - p_2) \xrightarrow{d} N(0, p_2^2 (1 - p_2))$$

when $m \rightarrow \infty$. Then

$$\sqrt{n}(\tilde{p}_2 - p_2) = \sqrt{\frac{n}{m}} \sqrt{m}(\tilde{p}_2 - p_2) \xrightarrow{d} N(0, \lambda p_2^2 (1 - p_2))$$

when both $n$ and $m$ tend to infinity. From the independence of $\tilde{p}_1$ and $\tilde{p}_2$ we get

$$(\sqrt{n}(\tilde{p}_1 - p_1), \sqrt{m}(\tilde{p}_2 - p_2)) \xrightarrow{d} N_2(0, J(p_1, p_2))$$

when both $n$ and $m$ tend to infinity, where

$$J(p_1, p_2) = \begin{bmatrix} p_1^2 (1 - p_1) & 0 \\ 0 & \lambda p_2^2 (1 - p_2) \end{bmatrix}.$$"
3. Confidence intervals

3.1. Asymptotic confidence interval

Using theorem 2.1 the asymptotic confidence interval for \( R_{s,k} \) can be constructed. From equation (2.3), using the invariance property, the MLE estimator of \( \sigma^2 \) is

\[
\hat{\sigma}^2 = \frac{1}{\hat{\rho}_1} \left( 1 - \hat{\rho}_1 \right) \sum_{i=0}^{k} \sum_{j=0}^{k-i} (-1)^j \binom{k-i}{j} \left( \frac{1 - (1 - \hat{\rho}_2)^{i+j}}{(1 - (1 - \hat{\rho}_1)(1 - \hat{\rho}_2)^{i+j})^2} \right) + \lambda \hat{\rho}_2^2 \left( \sum_{i=0}^{k} \sum_{j=0}^{k-i} (-1)^j \binom{k-i}{j} \left( \frac{1 - (1 - \hat{\rho}_1)(1 - \hat{\rho}_2)^{i+j-1}}{(1 - (1 - \hat{\rho}_1)(1 - \hat{\rho}_2)^{i+j})^2} \right)^2. \]

Then, the asymptotic interval of confidence level \( 1 - \alpha \) for \( R_{s,k} \) is given by

\[
I_{R_{s,k}}^{(ASYM)} = \left( \hat{R}_{s,k} - \frac{z_{\gamma/2}}{\sqrt{n}}, \hat{R}_{s,k} + \frac{z_{\gamma/2}}{\sqrt{n}} \right), \tag{3.1}
\]

where \( z_{\gamma} \) is the \( \gamma \)th quantile from standard normal distribution.

3.2. Bootstrap-p confidence interval

For small sample sizes asymptotic confidence intervals do not perform very well, so we propose a construction of the confidence interval based on parametric bootstrap-p method. The algorithm is illustrated below.

Step 1: From initial samples \( x = (x_1, x_2, ..., x_n) \) and \( y = (y_1, y_2, ..., y_m) \) calculate MLEs \( \hat{\rho}_1 \) and \( \hat{\rho}_2 \) using equations (2.1).

Step 2: Use those estimates to generate bootstrap sample \( x_1^* \) from geometric \( \mathcal{G}(\hat{\rho}_1) \) distribution and bootstrap sample \( y_1^* \) from geometric \( \mathcal{G}(\hat{\rho}_2) \) distribution. Based on these bootstrap samples compute estimates \( \hat{\rho}_1^* \) and \( \hat{\rho}_2^* \), using equations (2.1), and \( \hat{R}_{s,k}^* \) of \( R_{s,k} \) using equation (2.2).

Step 3: Repeat step 2, \( N \) boot times.

Step 4: Let \( \tilde{R}_{s,k}^{(\gamma)} \) be the \( \gamma \)th empirical quantile of the \( \hat{R}_{s,k}^* \) values obtained in step 3, that is, the \( N \gamma \)th value in the ordered list of the \( N \) replications of \( \hat{R}_{s,k}^* \). If \( \gamma \) is not an integer, assuming \( \gamma \leq 0.5 \), the largest integer less or equal \( (N + 1)\gamma \) should be used. The bootstrap-p interval of confidence level \( 1 - \alpha \) for \( R_{s,k} \) is given by

\[
I_{R_{s,k}}^{(BOOTp)} = \left( \tilde{R}_{s,k}^{(\gamma)}, \tilde{R}_{s,k}^{(1-\gamma)} \right).
\]

4. UMVUE of \( R_{s,k} \)

In this section we find the UMVUE of \( R_{s,k} \), denoted by \( \tilde{R}_{s,k} \). The complete sufficient statistics for \( p_1 \) and \( p_2 \) are \( T_X = \sum_{j=1}^{n} X_j \) and \( T_Y = \sum_{j=1}^{m} Y_j \). The statistic \( T_X \), as a sum of \( n \) independent identically distributed random variables with geometric distribution, has negative binomial distribution with parameters \( n \) and \( p_1 \), and the statistic \( T_Y \), analogously, has negative binomial distribution with parameters \( m \) and \( p_2 \). Their probability mass functions are

\[
P\{T_X = t_X\} = \binom{t_X - 1}{n - 1} p_1^n (1 - p_1)^{t_X - n}, \quad t_X \geq n;
\]

\[
P\{T_Y = t_Y\} = \binom{t_Y - 1}{m - 1} p_2^m (1 - p_2)^{t_Y - m}, \quad t_Y \geq m.
\]
An unbiased estimator for $\hat{R}_{s,k}$ is $I\{\text{at least s of the } (Y_1, Y_2, \ldots, Y_k) \geq X_1\}$. Because of this the sample size $m$ must be greater or equal to $k$. Then

$$E(I\{\text{at least s of the } (Y_1, Y_2, \ldots, Y_k) \geq X_1\}|T_X = t_X, T_Y = t_Y)$$

$$= \frac{k \cdot P\{\text{ exactly i of the } (Y_1, Y_2, \ldots, Y_k) \geq X_1, T_X = t_X, T_Y = t_Y\}}{P\{T_X = t_X, T_Y = t_Y\}}$$

$$= \frac{1}{P\{T_X = t_X\}P\{T_Y = t_Y\}} \cdot \sum_{i=1}^{m} \sum_{x_1=1}^{k} \left[ \frac{\binom{n}{i} \cdot \binom{m}{i}}{(t_X - x_1 - 2)(1 - p_1)^{t_X - x_1 - 1}} \cdot \binom{t_X - x_1 - 1}{n - 2}(1 - p_1)^{t_X - x_1 - n + 1} \right] \cdot P\{\text{ exactly i of the } (y_1, y_2, \ldots, y_k) \geq x_1, \sum_{i=1}^{m} y_i = t_Y, Y_1 = y_1, Y_2 = y_2, \ldots, Y_m = y_m\}$$

$$= \frac{1}{(t_X - 1)(t_Y - 1)} \cdot \sum_{i=1}^{m} \sum_{x_1=1}^{k} \left[ \binom{t_X - x_1 - 1}{n - 2}(1 - p_1)^{t_X - x_1 - n + 1} \right] \cdot P\{\text{ exactly i of the } (y_1, y_2, \ldots, y_k) \geq x_1, \sum_{i=1}^{m} y_i = t_Y, Y_1 = y_1, Y_2 = y_2, \ldots, Y_m = y_m\}$$

$$= \frac{1}{(t_X - 1)(t_Y - 1)} \cdot \sum_{i=1}^{m} \sum_{x_1=1}^{k} \left[ \binom{t_X - x_1 - 1}{n - 2}(1 - p_1)^{t_X - x_1 - n + 1} \right] \cdot P\{\text{ exactly i of the } (y_1, y_2, \ldots, y_k) \geq x_1, \sum_{i=1}^{m} y_i = t_Y, Y_1 = y_1, Y_2 = y_2, \ldots, Y_m = y_m\}$$

$$= \sum_{i=1}^{m} \sum_{x_1=1}^{k} \left[ \binom{t_X - x_1 - 1}{n - 2}(1 - p_1)^{t_X - x_1 - n + 1} \right] \cdot \sum_{i=1}^{m} \sum_{x_1=1}^{k} A_i (1 - p_2)^{t_Y - m - p_2}$$

where $A_i$ is the number of solutions, on the set of natural numbers, of linear equation $y_1 + y_2 + \ldots + y_m = t_Y$ with constraints $y_1 \geq x_1, y_1 \geq x_1, y_{i+1} \leq x_1 - 1, \ldots, y_k \leq x_1 - 1$ (without loss of generality we can take the first $i$ variables to be constrained by $x_1$). Denote $z_1 = y_1 - (x_1 - 1), \ldots, z_i = y_i - (x_1 - 1), z_{i+1} = y_{i+1}, \ldots, z_m = y_m$. Then, $A_i$ is the number of solutions, on the set of natural numbers, of linear equation $z_1 + z_2 + \ldots + z_m = t_Y - i(x_1 - 1)$ with constraints $z_{i+1} \leq x_1 - 1, \ldots, z_k \leq x_1 - 1$. If we now denote with $\Omega$ the set of solutions of this linear equation without constraints, and with $B_j$ the set of solutions of this linear equation with constraint $z_j \geq x_1$, where $j \in \{i + 1, \ldots, k\}$, then $A_i = |\Omega \setminus (B_{i+1} \cup B_{i+2} \cup \ldots \cup B_k)|$.

A linear equation of the form $w_1 + w_2 + \ldots + w_r = d$, where $d \geq r > 1$ and $w_i, \quad i \in \{1,2,\ldots,r\}$, are natural numbers, has $\binom{d-1}{r-1}$ solutions. The same linear equation $w_1 + w_2 + \ldots + w_r = d$, but with a constraint $w_j \geq a$, for some $j \in \{1,\ldots,r\}$, where $a$ is a natural number, has the same number of solutions as the linear equation $w_1 + w_2 + \ldots + w_j - (a-1) + w_{j+1} + \ldots + w_r = d - (a-1)$, and that number is $\binom{d-1}{r-1}$.

Similarly, linear equation $w_1 + w_2 + \ldots + w_r = d$, with constraints $w_{j_1} \geq a, \ldots, w_{j_l} \geq a$, for some distinct $j_1, \ldots, j_l$ from $\{1,\ldots,r\}$, where $a$ is a natural number, has $\binom{d-1}{r-1}$ solutions.

Applying all this, we get that $|\Omega| = \binom{t_Y - (i(x_1 - 1) - 1)}{m - 1}$, $|B_j| = \binom{t_Y - (i(x_1 - 1) - 1)}{m - 1}$, where $j \in \{i + 1, \ldots, k\}$, and, for $l > 1$, $|B_{j_1} \cap \cdots \cap B_{j_l}| = \binom{t_Y - (i(x_1 - 1) - 1)}{m - 1}$, where $j_1, \ldots, j_l$ are
distinct numbers from \( \{i+1, \ldots, k\} \). Using the principle of inclusion and exclusion we get that

\[
A_i = \begin{cases} 
\binom{t_Y - (x_1 - 1)}{m-1} I\{t_Y - i(x_1 - 1) \geq m\} \\
- \sum_{l=1}^{k-i} \binom{-1}{l} \binom{t_Y - (i+l)(x_1 - 1) - 1}{m-1} I\{t_Y - (i+l)(x_1 - 1) \geq m\}, & s \leq i < k; \\
\binom{t_Y - k(x_1 - 1) - 1}{m-1} I\{t_Y - k(x_1 - 1) \geq m\}, & i = k.
\end{cases}
\]

Applying Rao-Blackwell and Lehmann-Sheffe theorems to equation (4.1), it follows that the UMVUE of \( R_{s,k} \) is

\[
\hat{R}_{s,k} = \frac{1}{(t_X - n - 1) (t_Y - 1)} \sum_{x_1=1}^{t_X - n + 1} \binom{t_X - x_1 - 1}{n - 2} \sum_{s=k}^{k-i} \binom{k-i}{s} \left[ \binom{t_Y - i(x_1 - 1) - 1}{m-1} ight] 
\]

\[
\cdot I\{t_Y - (x_1 - 1) \geq m\} - \sum_{l=1}^{k-i} \binom{-1}{l} \binom{t_Y - (i+l)(x_1 - 1) - 1}{m-1} 
\]

\[
\cdot I\{t_Y - (i+l)(x_1 - 1) \geq m\} \right) + \left[ \binom{t_Y - k(x_1 - 1) - 1}{m-1} I\{t_Y - k(x_1 - 1) \geq m\} \right].
\]

5. Bayes estimator of \( R_{s,k} \)

In this section we consider the Bayes estimator of \( R_{s,k} \) with respect to square loss function. Let us suppose that \( p_1 \) and \( p_2 \) have conjugate prior distributions, beta \( \mathcal{B}(a,b) \), \( a, b \in \mathbb{N} \), and beta \( \mathcal{B}(c,d) \), \( c, d \in \mathbb{N} \), with the following joint density:

\[
\pi(p_1, p_2) \propto p_1^{a-1} (1-p_1)^{b-1} p_2^{c-1} (1-p_2)^{d-1}, \quad p_1 \in (0,1), p_2 \in (0,1).
\]

Then, the joint posterior density given the sample \((x, y)\) is

\[
\pi(p_1, p_2 | x, y) = W p_1^{a-1+n} (1-p_1)^{b-1+t_x-n} p_2^{c-1+m} (1-p_2)^{d-1+ty-y \cdot m},
\]

\( p_1 \in (0,1), p_2 \in (0,1) \), where

\[
W = \left( \int_0^1 \int_0^{1} p_1^{a-1+n} (1-p_1)^{b-1+t_x-n} p_2^{c-1+m} (1-p_2)^{d-1+ty-y \cdot m} dp_1 dp_2 \right)^{-1}
\]

\[
= \frac{1}{B(a+n, b+ t_x-n) B(c+m, d+ty-m)}
\]

and \( B \) is beta function.

Denote, for simplicity,

\[
a^* = a + n, \quad b^* = b + t_x - n, \quad c^* = c + m, \quad d^* = d + ty - m.
\]

The Bayes estimator \( \hat{R}_{s,k} \) of \( R_{s,k} \) for the square loss function is the posterior mean of \( R_{s,k} \), hence we obtain

\[
\hat{R}_{s,k} = E(R_{s,k} | x, y) = \int_0^1 \int_0^1 R_{s,k}(p_1, p_2) \pi(p_1, p_2 | x, y) dp_1 dp_2
\]

\[
= \sum_{i=s}^{k} \sum_{j=d}^{k-i} \frac{(-1)^j \binom{k-i}{j}}{B(a^*, b^*) B(c^*, d^*)} \int_0^1 \int_0^{1} p_1^{a^*}(1-p_1)^{b^*-1} p_2^{c^*-1}(1-p_2)^{d^*-1} dp_1 dp_2
\]

\[
= \sum_{i=s}^{k} \sum_{j=0}^{k-i} \frac{(-1)^j \binom{k-i}{j}}{B(a^*, b^*) B(c^*, d^*)} \int_0^1 \int_0^{1} p_1^{a^*}(1-p_1)^{b^*+l-1} dp_1 \int_0^{1} p_2^{c^*-1}(1-p_2)^{d^*+(i+j)-1} dp_2
\]

\[
= \sum_{i=s}^{k} \sum_{j=0}^{k-i} \frac{(-1)^j \binom{k-i}{j}}{B(a^*, b^*) B(c^*, d^*)}
\]
\[ \sum_{i=0}^{s} \sum_{j=0}^{k} (-1)^j \binom{k}{i} \binom{k-i}{j} \sum_{l=0}^{\infty} \frac{B(a^*+1,b^*+l)B(c^*,d^*+l(i+j))}{B(a^*,b^*)B(c^*,d^*)} = \sum_{i=0}^{s} \sum_{j=0}^{k} (-1)^j \binom{k}{i} \binom{k-i}{j} \sum_{l=0}^{\infty} \frac{B(a^*+1,b^*+l)B(c^*,d^*+l(i+j))}{B(a^*,b^*)B(c^*,d^*)}. \] (5.2)

Next, we investigate the case when the prior is non-informative. We use Jeffreys non-informative prior, which is proportional to the square root of the determinant of the Fisher information matrix, and get
\[ \pi(p_1, p_2) \propto p_1^{-\frac{1}{2}}(1-p_1)^{-\frac{1}{2}} p_2^{-\frac{1}{2}}(1-p_2)^{-\frac{1}{2}}, \quad p_1 \in (0,1), p_2 \in (0,1). \] (5.3)

This prior has the form (5.1) with \( a = 0, b = \frac{1}{2}, c = 0, d = \frac{1}{2} \), so the Bayes estimator of \( R_{s,k} \) in this case is also given by equation (5.2), except that
\[ a^* = n, \quad b^* = \frac{1}{2} + t_X - n, \quad c^* = m, \quad d^* = \frac{1}{2} + t_Y - m. \]

6. Simulation study

In this section some Monte Carlo simulations are performed to compare different estimators for \( R_{s,k} \). We chose three multicomponent models: two extreme ones, 1-out-of-3 and 3-out-of-3, and "standard" one, 3-out-of-6. From Figure 1 it is visible that most interesting cases happen for smaller values of \( p_1 \) and \( p_2 \). Therefore, we consider \( p_1 \in \{0.1, 0.2, 0.5\} \) and \( p_2 \in \{0.1, 0.3, 0.7\} \). We study different sample sizes \( (n,m) \in \{(10,10), (10,20), (20,10), (20,20), (100,100)\} \).

For each model and each combination of \( n, m, p_1 \) and \( p_2 \) we generate one random sample from \( G(p_1) \) and one random sample from \( G(p_2) \) and calculate the MLE of \( R_{s,k} \) using equation (2.2), and the UMVUE of \( R_{s,k} \) using equation (4.2). We also calculate 95% asymptotic confidence interval using equation (3.1) and 95% bootstrap-p confidence interval using procedure described in subsection 3.2 with \( N = 1000 \) boot times. The Bayes estimates as well as 95% credible intervals are obtained from 5000 samples from two posterior distributions: one with Jeffreys prior given by expression (5.3) (noninformative case); and another with beta \( \mathcal{B}(10p_1, 10(1-p_1)) \) and beta \( \mathcal{B}(10p_2, 10(1-p_2)) \) priors (informative case – means of these prior distributions are equal to true parameter values). This procedure is repeated for 1000 times.

In Tables 1, 3 and 5 the averages of point estimates for \( R_{1,3}, R_{3,3} \) and \( R_{3,6} \), respectively, as well as their root-mean-square errors (denoted by \( \text{Er} \)), are presented. We can notice that in almost all cases the UMVUE has the value closest to \( R \) as expected due to its unbiasedness. Moreover, for smaller and larger values of \( R_{s,k} \), it is the estimator with the smallest root-mean-square error. For intermediate values of \( R_{s,k} \), the informative Bayes estimator is also competitive having the smallest root-mean-square error.

In Table 2, 4 and 6, for our three scenarios, the average lengths of asymptotic and bootstrap-p confidence intervals and both Bayes credible intervals are presented as well as the coverage percentages of these intervals (the percentage of intervals that contain true value of \( R \)). The noninformative Bayes credible interval has mostly the best coverage, with bootstrap-p slightly outperforming it for some higher values of \( R_{s,k} \). The smallest length almost uniformly has the informative Bayes credible interval.
### Table 1. Point estimates for $R_{1,3}$ and their root-mean-square errors

| $p_1$ | $p_2$ | $R_{1,3}$ | $n$ | $m$ | MLE $\hat{R}_{1,3}$ | $\text{SE}(\hat{R}_{1,3})$ | UMVUE $\tilde{R}_{1,3}$ | $\text{SE}(\tilde{R}_{1,3})$ | Bayes noninf. $\hat{R}_{1,3}$ | $\text{SE}(\hat{R}_{1,3})$ | Bayes inf. $\tilde{R}_{1,3}$ | $\text{SE}(\tilde{R}_{1,3})$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.10 | 0.10 | 0.763 | 10 | 10 | 0.748 | 0.117 | 0.760 | 0.124 | 0.738 | 0.113 | 0.741 | 0.103 |
| | | | 20 | 10 | 0.762 | 0.096 | 0.765 | 0.102 | 0.747 | 0.095 | 0.748 | 0.088 |
| | | | 20 | 20 | 0.749 | 0.100 | 0.765 | 0.103 | 0.747 | 0.095 | 0.749 | 0.088 |
| | | | 100 | 100 | 0.762 | 0.038 | 0.763 | 0.038 | 0.760 | 0.038 | 0.760 | 0.038 |
| 0.20 | 0.10 | 0.918 | 10 | 10 | 0.904 | 0.068 | 0.919 | 0.065 | 0.889 | 0.073 | 0.895 | 0.062 |
| | | | 20 | 10 | 0.911 | 0.055 | 0.919 | 0.054 | 0.895 | 0.061 | 0.899 | 0.052 |
| | | | 20 | 20 | 0.905 | 0.057 | 0.920 | 0.054 | 0.897 | 0.059 | 0.900 | 0.052 |
| | | | 100 | 100 | 0.916 | 0.020 | 0.918 | 0.020 | 0.915 | 0.020 | 0.915 | 0.020 |
| 0.50 | 0.10 | 0.993 | 10 | 10 | 0.989 | 0.015 | 0.993 | 0.011 | 0.984 | 0.020 | 0.987 | 0.013 |
| | | | 20 | 10 | 0.990 | 0.010 | 0.993 | 0.008 | 0.985 | 0.015 | 0.989 | 0.009 |
| | | | 20 | 20 | 0.990 | 0.011 | 0.993 | 0.008 | 0.987 | 0.014 | 0.988 | 0.010 |
| | | | 100 | 100 | 0.993 | 0.003 | 0.993 | 0.002 | 0.992 | 0.003 | 0.992 | 0.003 |
| 0.10 | 0.30 | 0.419 | 10 | 10 | 0.428 | 0.124 | 0.418 | 0.129 | 0.434 | 0.119 | 0.430 | 0.103 |
| | | | 20 | 10 | 0.440 | 0.110 | 0.425 | 0.111 | 0.438 | 0.105 | 0.435 | 0.094 |
| | | | 20 | 20 | 0.423 | 0.102 | 0.422 | 0.106 | 0.434 | 0.101 | 0.431 | 0.086 |
| | | | 100 | 100 | 0.420 | 0.039 | 0.418 | 0.039 | 0.420 | 0.039 | 0.420 | 0.038 |
| 0.20 | 0.30 | 0.652 | 10 | 10 | 0.646 | 0.126 | 0.649 | 0.136 | 0.641 | 0.119 | 0.643 | 0.098 |
| | | | 20 | 10 | 0.654 | 0.117 | 0.649 | 0.124 | 0.642 | 0.112 | 0.644 | 0.096 |
| | | | 20 | 20 | 0.643 | 0.109 | 0.652 | 0.114 | 0.646 | 0.103 | 0.647 | 0.086 |
| | | | 100 | 100 | 0.654 | 0.042 | 0.651 | 0.096 | 0.647 | 0.089 | 0.647 | 0.081 |
| 0.50 | 0.30 | 0.924 | 10 | 10 | 0.913 | 0.064 | 0.926 | 0.062 | 0.898 | 0.070 | 0.909 | 0.047 |
| | | | 20 | 10 | 0.918 | 0.055 | 0.926 | 0.055 | 0.902 | 0.061 | 0.912 | 0.043 |
| | | | 20 | 20 | 0.911 | 0.057 | 0.925 | 0.054 | 0.904 | 0.059 | 0.911 | 0.043 |
| | | | 100 | 100 | 0.922 | 0.018 | 0.924 | 0.044 | 0.909 | 0.048 | 0.913 | 0.039 |
| 0.10 | 0.70 | 0.187 | 10 | 10 | 0.199 | 0.076 | 0.187 | 0.073 | 0.209 | 0.079 | 0.199 | 0.060 |
| | | | 20 | 10 | 0.201 | 0.066 | 0.187 | 0.062 | 0.204 | 0.067 | 0.199 | 0.056 |
| | | | 20 | 20 | 0.196 | 0.056 | 0.192 | 0.055 | 0.208 | 0.060 | 0.198 | 0.044 |
| | | | 100 | 100 | 0.188 | 0.021 | 0.187 | 0.021 | 0.189 | 0.021 | 0.188 | 0.020 |
| 0.20 | 0.70 | 0.347 | 10 | 10 | 0.362 | 0.108 | 0.349 | 0.109 | 0.372 | 0.106 | 0.359 | 0.078 |
| | | | 20 | 10 | 0.368 | 0.103 | 0.351 | 0.101 | 0.369 | 0.099 | 0.361 | 0.079 |
| | | | 20 | 20 | 0.353 | 0.092 | 0.351 | 0.094 | 0.368 | 0.092 | 0.356 | 0.070 |
| | | | 100 | 100 | 0.349 | 0.033 | 0.348 | 0.033 | 0.350 | 0.033 | 0.349 | 0.032 |
| 0.50 | 0.70 | 0.701 | 10 | 10 | 0.692 | 0.124 | 0.692 | 0.132 | 0.686 | 0.115 | 0.690 | 0.079 |
| | | | 20 | 10 | 0.707 | 0.113 | 0.701 | 0.119 | 0.693 | 0.107 | 0.696 | 0.076 |
| | | | 20 | 20 | 0.694 | 0.104 | 0.701 | 0.109 | 0.698 | 0.096 | 0.696 | 0.071 |
| | | | 100 | 100 | 0.701 | 0.039 | 0.701 | 0.039 | 0.700 | 0.039 | 0.700 | 0.037 |
Table 2. Confidence intervals for $R_{1,3}$ and their coverages

| $p_1$ | $p_2$ | $R_{1,3}$ | $n$ | $m$ | asymptotic length | bootstrap-p length | Bayes noninf. length | Bayes inf. length |
|-------|-------|-----------|-----|-----|-------------------|---------------------|---------------------|------------------|
| 0.10  | 0.10  | 0.763     | 10  | 10  | 0.435             | 0.416              | 0.419               | 0.404            |
|       |       |           | 10  | 20  | 0.380             | 0.359              | 0.377               | 0.365            |
|       |       |           | 20  | 10  | 0.382             | 0.376              | 0.364               | 0.353            |
|       |       |           | 20  | 20  | 0.315             | 0.306              | 0.308               | 0.302            |
|       |       |           | 100 | 100 | 0.145             | 0.144              | 0.144               | 0.143            |
| 0.20  | 0.10  | 0.918     | 10  | 10  | 0.246             | 0.257              | 0.262               | 0.240            |
|       |       |           | 10  | 20  | 0.209             | 0.209              | 0.231               | 0.213            |
|       |       |           | 20  | 10  | 0.214             | 0.231              | 0.215               | 0.203            |
|       |       |           | 20  | 20  | 0.174             | 0.178              | 0.180               | 0.172            |
|       |       |           | 100 | 100 | 0.076             | 0.076              | 0.077               | 0.076            |
| 0.50  | 0.10  | 0.993     | 10  | 10  | 0.042             | 0.058              | 0.062               | 0.044            |
|       |       |           | 10  | 20  | 0.035             | 0.042              | 0.053               | 0.037            |
|       |       |           | 20  | 10  | 0.034             | 0.049              | 0.043               | 0.036            |
|       |       |           | 20  | 20  | 0.026             | 0.032              | 0.033               | 0.028            |
|       |       |           | 100 | 100 | 0.010             | 0.010              | 0.011               | 0.010            |
| 0.10  | 0.30  | 0.419     | 10  | 10  | 0.462             | 0.449              | 0.450               | 0.420            |
|       |       |           | 10  | 20  | 0.418             | 0.413              | 0.404               | 0.386            |
|       |       |           | 20  | 10  | 0.397             | 0.385              | 0.396               | 0.365            |
|       |       |           | 20  | 20  | 0.334             | 0.329              | 0.329               | 0.316            |
|       |       |           | 100 | 100 | 0.152             | 0.151              | 0.152               | 0.150            |
| 0.20  | 0.30  | 0.652     | 10  | 10  | 0.486             | 0.458              | 0.462               | 0.428            |
|       |       |           | 10  | 20  | 0.432             | 0.408              | 0.419               | 0.392            |
|       |       |           | 20  | 10  | 0.420             | 0.404              | 0.402               | 0.373            |
|       |       |           | 20  | 20  | 0.355             | 0.343              | 0.345               | 0.329            |
|       |       |           | 100 | 100 | 0.163             | 0.161              | 0.162               | 0.160            |
| 0.50  | 0.30  | 0.924     | 10  | 10  | 0.231             | 0.240              | 0.250               | 0.203            |
|       |       |           | 10  | 20  | 0.201             | 0.199              | 0.227               | 0.184            |
|       |       |           | 20  | 10  | 0.200             | 0.215              | 0.203               | 0.177            |
|       |       |           | 20  | 20  | 0.166             | 0.169              | 0.173               | 0.155            |
|       |       |           | 100 | 100 | 0.073             | 0.073              | 0.074               | 0.072            |
| 0.10  | 0.70  | 0.187     | 10  | 10  | 0.265             | 0.279              | 0.278               | 0.237            |
|       |       |           | 10  | 20  | 0.243             | 0.262              | 0.244               | 0.223            |
|       |       |           | 20  | 10  | 0.220             | 0.223              | 0.238               | 0.196            |
|       |       |           | 20  | 20  | 0.186             | 0.192              | 0.191               | 0.175            |
|       |       |           | 100 | 100 | 0.082             | 0.083              | 0.083               | 0.081            |
| 0.20  | 0.70  | 0.347     | 10  | 10  | 0.409             | 0.410              | 0.407               | 0.350            |
|       |       |           | 10  | 20  | 0.376             | 0.384              | 0.366               | 0.331            |
|       |       |           | 20  | 10  | 0.337             | 0.331              | 0.345               | 0.293            |
|       |       |           | 20  | 20  | 0.294             | 0.295              | 0.293               | 0.268            |
|       |       |           | 100 | 100 | 0.132             | 0.132              | 0.132               | 0.129            |
| 0.50  | 0.70  | 0.701     | 10  | 10  | 0.458             | 0.435              | 0.439               | 0.372            |
|       |       |           | 10  | 20  | 0.420             | 0.399              | 0.409               | 0.351            |
|       |       |           | 20  | 10  | 0.381             | 0.366              | 0.364               | 0.318            |
|       |       |           | 20  | 20  | 0.336             | 0.325              | 0.327               | 0.294            |
|       |       |           | 100 | 100 | 0.155             | 0.154              | 0.154               | 0.150            |
Table 3. Point estimates for $R_{3,3}$ and their root-mean-square errors

| $p_1$ | $p_2$ | $R_{3,3}$ | $n$ | $m$ | MLE $\hat{R}_{3,3}$ | Er($\hat{R}_{3,3}$) | UMVUE $\tilde{R}_{3,3}$ | Er($\tilde{R}_{3,3}$) | Bayes noninf. $\hat{R}_{3,3}$ | Er($\hat{R}_{3,3}$) | Bayes inf. $\tilde{R}_{3,3}$ | Er($\tilde{R}_{3,3}$) |
|-------|-------|-----------|-----|-----|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|
| 0.10  | 0.10  | 0.291     | 10  | 10  | 0.304 0.094          | 0.292 0.094          | 0.310 0.092          | 0.308 0.083          | 0.10 0.10            | 0.486 0.093          | 0.10 0.10            | 0.486 0.093          |
| 0.20  | 0.10  | 0.480     | 10  | 10  | 0.486 0.113          | 0.479 0.117          | 0.487 0.108          | 0.486 0.093          | 0.20 0.10            | 0.489 0.082          | 0.20 0.10            | 0.489 0.082          |
| 0.50  | 0.10  | 0.787     | 10  | 10  | 0.780 0.088          | 0.781 0.091          | 0.768 0.087          | 0.777 0.065          | 0.50 0.10            | 0.784 0.057          | 0.50 0.10            | 0.784 0.057          |
| 0.10  | 0.30  | 0.145     | 10  | 10  | 0.157 0.053          | 0.144 0.049          | 0.163 0.055          | 0.159 0.046          | 0.10 0.30            | 0.157 0.046          | 0.10 0.30            | 0.157 0.046          |
| 0.20  | 0.30  | 0.276     | 10  | 10  | 0.294 0.088          | 0.277 0.085          | 0.299 0.087          | 0.293 0.070          | 0.20 0.30            | 0.290 0.066          | 0.20 0.30            | 0.290 0.066          |
| 0.50  | 0.30  | 0.604     | 10  | 10  | 0.623 0.116          | 0.608 0.119          | 0.615 0.109          | 0.615 0.076          | 0.50 0.30            | 0.608 0.075          | 0.50 0.30            | 0.608 0.075          |
| 0.10  | 0.70  | 0.102     | 10  | 10  | 0.114 0.036          | 0.103 0.031          | 0.115 0.036          | 0.112 0.031          | 0.10 0.70            | 0.113 0.034          | 0.10 0.70            | 0.113 0.034          |
| 0.20  | 0.70  | 0.204     | 10  | 10  | 0.222 0.069          | 0.204 0.063          | 0.223 0.068          | 0.217 0.053          | 0.20 0.70            | 0.213 0.039          | 0.20 0.70            | 0.213 0.039          |
| 0.50  | 0.70  | 0.507     | 10  | 10  | 0.536 0.118          | 0.510 0.116          | 0.526 0.110          | 0.522 0.074          | 0.50 0.70            | 0.524 0.074          | 0.50 0.70            | 0.524 0.074          |
Table 4. Confidence intervals for $R_{3,3}$ and their coverages

| $p_1$ | $p_2$ | $R_{3,3}$ | $n$ | $m$ | asymptotic length cov | bootstrap-p length cov | Bayes noninf. length cov | Bayes inf. length cov |
|-------|-------|-----------|-----|-----|-----------------------|------------------------|------------------------|----------------------|
| 0.10  | 0.10  | 0.291     | 10  | 10  | 0.341 91.9            | 0.343 94.1             | 0.340 95.4            | 0.326 96.2           |
| 10    | 20    | 0.310 92.1| 20  | 10  | 0.280 91.1            | 0.317 92.7             | 0.304 93.7            | 0.292 94.3           |
| 20    | 20    | 0.243 94.3| 20  | 20  | 0.109 95.1            | 0.244 95.0             | 0.243 95.3            | 0.238 96.0           |
| 100   | 100   |           |     |     |                       |                        |                        |                      |
| 0.20  | 0.10  | 0.480     | 10  | 10  | 0.414 91.1            | 0.404 94.0             | 0.403 93.9            | 0.380 96.1           |
| 10    | 20    | 0.382 92.9| 20  | 10  | 0.346 93.3            | 0.376 94.0             | 0.371 95.6            | 0.348 96.9           |
| 20    | 20    | 0.301 93.1| 20  | 100 | 0.137 95.3            | 0.295 94.7             | 0.296 95.0            | 0.286 95.4           |
| 100   | 100   |           |     |     |                       |                        |                        |                      |
| 0.50  | 0.10  | 0.787     | 10  | 10  | 0.336 90.6            | 0.329 94.3             | 0.337 95.0            | 0.288 98.0           |
| 10    | 20    | 0.305 89.5| 20  | 10  | 0.269 92.3            | 0.297 91.0             | 0.313 95.1            | 0.264 98.1           |
| 20    | 20    | 0.241 94.5| 20  | 100 | 0.108 94.0            | 0.238 96.1             | 0.241 96.1            | 0.220 97.6           |
| 100   | 100   |           |     |     |                       |                        |                        |                      |
| 0.10  | 0.30  | 0.145     | 10  | 10  | 0.193 94.8            | 0.211 94.3             | 0.201 95.7            | 0.184 96.8           |
| 10    | 20    | 0.184 95.5| 20  | 10  | 0.148 93.8            | 0.204 92.3             | 0.184 95.2            | 0.173 96.4           |
| 20    | 20    | 0.134 95.1| 20  | 100 | 0.058 95.2            | 0.153 95.2             | 0.160 96.1            | 0.145 96.8           |
| 100   | 100   |           |     |     |                       |                        |                        |                      |
| 0.20  | 0.30  | 0.276     | 10  | 10  | 0.315 94.3            | 0.329 93.1             | 0.314 95.0            | 0.284 97.3           |
| 10    | 20    | 0.300 95.2| 20  | 10  | 0.238 93.1            | 0.318 92.1             | 0.294 95.9            | 0.270 98.2           |
| 20    | 20    | 0.222 93.0| 20  | 100 | 0.098 95.2            | 0.239 95.2             | 0.246 95.3            | 0.226 96.9           |
| 100   | 100   |           |     |     |                       |                        |                        |                      |
| 0.50  | 0.30  | 0.604     | 10  | 10  | 0.430 91.8            | 0.420 92.0             | 0.416 96.1            | 0.355 98.7           |
| 10    | 20    | 0.417 91.7| 20  | 10  | 0.329 92.8            | 0.410 92.5             | 0.405 93.9            | 0.344 97.8           |
| 20    | 20    | 0.312 93.1| 20  | 100 | 0.142 93.3            | 0.321 94.2             | 0.324 95.2            | 0.292 97.1           |
| 100   | 100   |           |     |     |                       |                        |                        |                      |
| 0.10  | 0.70  | 0.102     | 10  | 10  | 0.133 96.7            | 0.154 92.4             | 0.134 96.5            | 0.123 97.2           |
| 10    | 20    | 0.133 94.2| 20  | 10  | 0.091 96.3            | 0.154 90.7             | 0.132 94.3            | 0.124 96.0           |
| 20    | 20    | 0.089 95.5| 20  | 100 | 0.039 96.1            | 0.099 91.8             | 0.094 94.6            | 0.088 95.6           |
| 100   | 100   |           |     |     |                       |                        |                        |                      |
| 0.20  | 0.70  | 0.204     | 10  | 10  | 0.240 95.2            | 0.268 92.2             | 0.237 94.9            | 0.212 96.9           |
| 10    | 20    | 0.236 95.6| 20  | 10  | 0.167 95.2            | 0.263 93.8             | 0.230 95.0            | 0.209 96.8           |
| 20    | 20    | 0.167 95.4| 20  | 100 | 0.072 94.1            | 0.177 93.3             | 0.169 94.8            | 0.157 96.0           |
| 100   | 100   |           |     |     |                       |                        |                        |                      |
| 0.50  | 0.70  | 0.507     | 10  | 10  | 0.434 93.6            | 0.437 91.6             | 0.415 95.9            | 0.349 97.8           |
| 10    | 20    | 0.433 94.1| 20  | 10  | 0.311 93.0            | 0.437 90.4             | 0.414 96.7            | 0.349 99.7           |
| 20    | 20    | 0.310 93.4| 20  | 100 | 0.139 93.8            | 0.312 91.1             | 0.305 94.4            | 0.274 96.1           |
| 100   | 100   |           |     |     |                       |                        |                        |                      |
Table 5. Point estimates for $\hat{R}_{3,6}$ and their root-mean-square errors

| $p_1$ | $p_2$ | $R_{3,6}$ | MLE | UMVUE | Bayes noninf. | Bayes inf. |
|-------|-------|-----------|------|-------|--------------|------------|
|       |       | $\hat{R}_{3,6}$ | $\tilde{R}_{3,6}$ | $\tilde{R}_{3,6}$ | $\hat{R}_{3,6}$ | $\hat{R}_{3,6}$ |
| 0.10 | 0.10 | 0.593 | 0.10 | 0.70 | 0.50 | 0.30 |
| 0.20 | 0.10 | 0.822 | 0.10 | 0.30 | 0.20 | 0.10 |
| 0.50 | 0.10 | 0.982 | 0.10 | 0.30 | 0.20 | 0.10 |
| 0.10 | 0.30 | 0.276 | 0.10 | 0.30 | 0.20 | 0.10 |
| 0.20 | 0.30 | 0.484 | 0.10 | 0.30 | 0.20 | 0.10 |
| 0.50 | 0.30 | 0.839 | 0.10 | 0.30 | 0.20 | 0.10 |
| 0.10 | 0.70 | 0.124 | 0.10 | 0.30 | 0.20 | 0.10 |
| 0.20 | 0.70 | 0.242 | 0.10 | 0.30 | 0.20 | 0.10 |
| 0.50 | 0.70 | 0.565 | 0.10 | 0.30 | 0.20 | 0.10 |
| \(p_1\) | \(p_2\) | \(R_{3.6}\) | \(n\) | \(m\) | asymptotic \(p\) | bootstrap-p \(p\) | Bayes noninf. \(p\) | Bayes inf. \(p\) |
|---|---|---|---|---|---|---|---|---|
| 0.10 | 0.10 | 0.593 | 10 | 10 | 0.519 90.3 | 0.486 94.3 | 0.491 94.4 | 0.476 95.5 |
| | | | | | 0.463 90.7 | 0.438 94.1 | 0.444 94.6 | 0.430 95.9 |
| | | | | | 0.455 92.1 | 0.433 94.9 | 0.435 96.1 | 0.423 96.3 |
| | | | | | 0.382 93.4 | 0.368 95.3 | 0.370 95.3 | 0.362 96.0 |
| | | | | | 0.176 94.6 | 0.174 95.1 | 0.175 95.1 | 0.174 95.1 |
| 0.20 | 0.10 | 0.822 | 10 | 10 | 0.407 87.2 | 0.389 93.9 | 0.395 93.8 | 0.373 96.0 |
| | | | | | 0.352 87.4 | 0.331 93.4 | 0.356 95.6 | 0.334 97.3 |
| | | | | | 0.356 90.3 | 0.352 94.2 | 0.339 94.4 | 0.325 95.2 |
| | | | | | 0.293 90.0 | 0.286 94.3 | 0.289 94.6 | 0.279 95.5 |
| | | | | | 0.135 94.1 | 0.134 94.9 | 0.135 94.8 | 0.134 95.1 |
| 0.50 | 0.10 | 0.982 | 10 | 10 | 0.105 84.6 | 0.128 94.5 | 0.136 94.5 | 0.104 97.3 |
| | | | | | 0.079 82.6 | 0.090 92.8 | 0.110 94.9 | 0.082 97.4 |
| | | | | | 0.084 84.5 | 0.107 93.3 | 0.097 93.4 | 0.084 95.1 |
| | | | | | 0.063 86.8 | 0.072 93.5 | 0.075 94.5 | 0.065 95.9 |
| | | | | | 0.026 92.2 | 0.027 94.5 | 0.027 95.1 | 0.026 95.5 |
| 0.10 | 0.30 | 0.276 | 10 | 10 | 0.364 91.5 | 0.372 93.2 | 0.370 93.8 | 0.340 96.4 |
| | | | | | 0.332 94.0 | 0.346 95.0 | 0.327 95.5 | 0.311 96.8 |
| | | | | | 0.304 92.1 | 0.303 94.7 | 0.319 95.3 | 0.290 96.9 |
| | | | | | 0.260 93.2 | 0.263 94.8 | 0.263 95.1 | 0.251 95.6 |
| | | | | | 0.116 94.2 | 0.116 94.1 | 0.116 94.2 | 0.115 94.8 |
| 0.20 | 0.30 | 0.484 | 10 | 10 | 0.488 91.0 | 0.469 94.3 | 0.471 93.9 | 0.434 96.5 |
| | | | | | 0.445 90.9 | 0.434 92.9 | 0.428 94.2 | 0.400 96.4 |
| | | | | | 0.418 91.7 | 0.402 95.0 | 0.412 95.0 | 0.379 96.5 |
| | | | | | 0.356 91.9 | 0.348 93.8 | 0.349 94.2 | 0.332 95.4 |
| | | | | | 0.163 94.7 | 0.162 94.8 | 0.162 94.9 | 0.160 95.4 |
| 0.50 | 0.30 | 0.839 | 10 | 10 | 0.380 87.8 | 0.360 95.0 | 0.375 94.8 | 0.321 97.9 |
| | | | | | 0.343 88.1 | 0.322 92.5 | 0.349 93.2 | 0.297 96.5 |
| | | | | | 0.329 90.0 | 0.322 94.6 | 0.315 95.6 | 0.283 98.1 |
| | | | | | 0.278 92.2 | 0.270 95.3 | 0.276 95.0 | 0.251 97.1 |
| | | | | | 0.126 94.3 | 0.125 94.3 | 0.126 94.9 | 0.123 95.0 |
| 0.10 | 0.70 | 0.124 | 10 | 10 | 0.178 95.8 | 0.198 94.5 | 0.192 97.3 | 0.163 97.5 |
| | | | | | 0.165 94.8 | 0.186 93.1 | 0.168 95.4 | 0.154 96.5 |
| | | | | | 0.134 93.4 | 0.139 95.4 | 0.150 95.8 | 0.125 97.5 |
| | | | | | 0.121 93.9 | 0.127 94.0 | 0.126 94.8 | 0.116 95.8 |
| | | | | | 0.053 95.2 | 0.053 95.7 | 0.053 95.8 | 0.053 95.7 |
| 0.20 | 0.70 | 0.242 | 10 | 10 | 0.311 93.7 | 0.328 92.6 | 0.321 94.9 | 0.271 97.3 |
| | | | | | 0.293 94.0 | 0.314 91.7 | 0.292 94.3 | 0.259 97.3 |
| | | | | | 0.240 94.3 | 0.241 93.8 | 0.258 94.7 | 0.217 96.8 |
| | | | | | 0.214 94.7 | 0.220 94.5 | 0.219 95.4 | 0.200 97.0 |
| | | | | | 0.095 94.9 | 0.095 94.7 | 0.096 95.2 | 0.094 95.6 |
| 0.50 | 0.70 | 0.565 | 10 | 10 | 0.467 91.2 | 0.447 92.4 | 0.451 95.0 | 0.378 98.4 |
| | | | | | 0.446 89.9 | 0.434 90.0 | 0.429 94.4 | 0.367 97.8 |
| | | | | | 0.367 92.7 | 0.348 94.7 | 0.365 95.2 | 0.315 97.4 |
| | | | | | 0.341 94.3 | 0.332 95.5 | 0.335 96.3 | 0.300 98.2 |
| | | | | | 0.155 95.0 | 0.154 95.1 | 0.155 95.4 | 0.151 96.2 |
7. Real data application

In this section we apply the presented methods to real datasets. The data of the first example represent durations of warm spells in summer months (June, July and August) in Budapest, Hungary. For this purpose we define a warm spell as number of days where daily maximum temperature is at least 30 degrees Celsius.

We are interested in probabilities that the duration of a warm spell in the 21st century \( (X) \) does not exceed the 60th and the 80th percentile of the duration of warm spell in the 20th century \( (Y) \). Those probabilities then can be modelled as \( R_{2.5} \) and \( R_{1.5} \) respectively.

Two populations, one containing summer months from the 21st century (2001 till 2018) and other from 20th century (1901 till 2000), are obtained from the database of the European Climate Assessment & Dataset project available at https://www.ecad.eu/dailydata/predefinedseries.php.

Then, two simple random samples are drawn from each population. The samples, of sizes 20 (for the 21st century) and 100 (for the 20th century), are presented in Table 7.

Table 7. Warm spell duration

|         | 21st century |          | 20th century |
|---------|--------------|----------|--------------|
| duration| 1 2 3 4 5 7 8 9 11 | frequency| 5 3 3 3 1 2 1 1 1 |
| duration| 1 2 3 4 5 6 7 8 9 | frequency| 29 24 12 13 9 4 7 1 1 |

The point and interval estimates are presented in Table 8. Bayes estimates here are given for the noninformative case. From these results it follows that \( R_{2.5} \approx 0.61 \) and \( R_{1.5} \approx 0.78 \). Therefore, the probability that the duration of a warm spell in the 21st century will not exceed the 60th percentile of the duration of warm spell in the 20th century is estimated to be 0.61. In case of the 80th percentile the estimate of the corresponding probability is 0.78.

Table 8. Real data estimates for Budapest warm spells example

| \( R_{2,5} \) | \( \hat{R}_{2,5} \) | \( \hat{R}_{2,5} \) | asymptotic CI | bootstrap-p CI | Bayes CI |
|--------------|-----------------|-----------------|--------------|----------------|---------|
| 0.619        | 0.612           | 0.611           | (0.462,0.776) | (0.478,0.789) | (0.452,0.758) |

| \( R_{1,5} \) | \( \hat{R}_{1,5} \) | \( \hat{R}_{1,5} \) | asymptotic CI | bootstrap-p CI | Bayes CI |
|--------------|-----------------|-----------------|--------------|----------------|---------|
| 0.778        | 0.777           | 0.770           | (0.643,0.914) | (0.637,0.898) | (0.618,0.888) |

The data of the second example represent number of successive 2-point field goals made by a basketball player from the start of the game including his first miss. Anadolu Efes and Barcelona face each other in 2018/19 Euroleague playoffs. We choose two players with approximately equal regular season 2-point field goal percentage (Ante Tomić (62.11%) and Kevin Seraphin(62.50%)) from Barcelona and one player (Bryant Dunston(67.33%)) from Anadolu Efes. All three players play the center position. We are interested in estimating the probability that one (or both) of the Barcelona players will make great or equal number of consecutive field goals than B. Dunston. These probabilities can be utilized to set betting odds. This situation is an example of a multicomponent geometric model with two "strength" components, and underlying probabilities are \( R_{1,2} \) and \( R_{2,2} \).

Two simple random samples of the size ten were taken from the 2018/2019 Euroleague regular season games database available at https://www.euroleague.net. The first sample
is taken from 29 regular season games played by B. Dunston and the second sample is taken from 30+24 regular season games played by A. Tomić and K. Seraphin. The samples, accompanied with the names of corresponding adversary teams, are presented in Table 9.

Table 9. Number of consecutive field goals including first miss

|          | Bryant Dunston |              |              |              |          |              |              |          |              |          |              |          |
|----------|----------------|--------------|--------------|--------------|----------|--------------|--------------|----------|--------------|----------|--------------|----------|
|          | Khim. Bask. Arma. Olym. Barc. Real Arma. Olym. Zalg. Pana. | away | home | home | home | home | home | away | away | away | away | away | away |
| 1        | 1              | 2            | 3            | 2            | 1        | 1            | 4            | 1        | 4            | 1        | 4            | 2        |

Table 10. Real data estimates for Euroleague example

|          |          |          |          |          | R_{1,2} | R_{1,2} | R_{1,2} | asymptotic CI | bootstrap-p CI | Bayes CI |
|----------|----------|----------|----------|----------|---------|---------|---------|----------------|----------------|----------|
|          |          |          |          |          | 0.756   | 0.763   | 0.744   | (0.541,0.972)  | (0.523,0.932)  | (0.505, 0.919) |
|          |          |          |          |          | 0.556   | 0.536   | 0.552   | (0.331,0.781)  | (0.360,0.801)  | (0.332, 0.759) |

The point and interval estimates of $R_{1,2}$ and $R_{2,2}$ are given in Table 10. As in the previous example Bayes estimates are given for noninformative case. From these results it follows that $R_{1,2} \approx 0.75$ and $R_{2,2} \approx 0.54$. This means that the probability that at least one of the Barcelona centers will score at least as many consecutive field goals as Dunston is estimated to be 0.75, while the estimate of the probability of the corresponding event for both of them is 0.54.

8. Conclusion

In this paper we considered estimation of strength-stress probability in a multicomponent system, where both stress and strength come from the geometric distribution. This is the first study concerning such multicomponent system with discrete probability distributions.

We derived three point and three interval estimators and compared them in a simulation study. Based on both bias and root-mean-square error criteria, the UMVUE is shown to be most suitable among point estimators, with the informative Bayes estimator being also competitive for intermediate values of $R_{s,k}$. The Bayes credible intervals are mostly the best choices among interval estimators.

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