COMPACT SETS IN THE FREE TOPOLOGY

MERIC AUGAT, SRIRAM BALASUBRAMANIAN\textsuperscript{1}, AND SCOTT MCCULLOUGH\textsuperscript{2}

Abstract. Subsets of the set of \( g \)-tuples of matrices that are closed with respect to direct sums and compact in the free topology are characterized. They are, in a dilation theoretic sense, contained in the hull of a single point.

1. Introduction

Given positive integers \( n, g \), let \( M_n(\mathbb{C})^g \) denote the set of \( g \)-tuples of \( n \times n \) matrices. Let \( M(\mathbb{C})^g \) denote the sequence \( (M_n(\mathbb{C})^g)_n \). A subset \( E \) of \( M(\mathbb{C})^g \) is a sequence \( (E(n)) \) where \( E(n) \subset M_n(\mathbb{C})^g \). The free topology \cite{AM14} has as a basis free sets of the form \( G_\delta = (G_\delta(n)) \), where

\[ G_\delta(n) = \{ X \in M_n(\mathbb{C})^g : \| \delta(X) \| < 1 \} \]

and \( \delta \) is a (matrix-valued) free polynomial. Agler and McCarthy \cite{AM14} prove the remarkable result that a bounded free function on a basis set \( G_\delta \) is uniformly approximable by polynomials on each smaller set of the form

\[ K_{s\delta} = \{ X \in M(\mathbb{C})^g : \| \delta(X) \| \leq s \}, \quad 0 \leq s < 1. \]

For the definitive treatment of free function theory, see \cite{KV14}.

Sets \( E \subset M(\mathbb{C})^g \) naturally arising in free analysis (\cite{AM15 BMV BKP16 HKN14 KV KS Pas14 Voi10} is a sampling of the references) are typically closed with respect to direct sums in the sense that if \( X \in E(n) \) and \( Y \in E(m) \), then

\[ X \oplus Y = \left( \begin{pmatrix} X_1 & 0 \\ 0 & Y_1 \end{pmatrix}, \ldots, \begin{pmatrix} X_g & 0 \\ 0 & Y_g \end{pmatrix} \right) \in E(n + m). \]

Theorem \textsuperscript{1} below, characterizing free topology compact sets that are closed with respect to direct sums, is the main result of this article. A tuple \( Y \in M_n(\mathbb{C})^g \) \textbf{polynomially dilates} to a tuple \( X \in M_N(\mathbb{C})^g \) if there is an isometry \( V : \mathbb{C}^n \to \mathbb{C}^N \) such that for all free polynomials \( p \),

\[ p(Y) = V^* p(X) V. \]

\textsuperscript{1}Supported by the New Faculty Initiative Grant (MAT/15-16/836/NFIG/SRIM) of IIT Madras.

\textsuperscript{2}Research supported by the NSF grant DMS-1361501.
An ampliation of $X$ is a tuple of the form $I_k \otimes X$, for some positive integer $k$. The dilation hull of $X \in M(\mathbb{C})^g$ is the set of all $Y \in M(\mathbb{C})^g$ that dilate to an ampliation of $X$.

**Theorem 1.1.** A subset $E$ of $M(\mathbb{C})^g$ that is closed with respect to direct sums is compact if and only if it is contained in the polynomial dilation hull of an $X \in E$.

**Corollary 1.2.** If $E \subset M(\mathbb{C})^g$ is closed with respect to direct sums and is compact in the free topology, then there exists a non-zero free polynomial $p$ such that $E$ is a subset of the zero set of $p$; i.e., $p(Y) = 0$ for all $Y \in E$. In particular, there is an $N$ such that for $n \geq N$ the set $E(n)$ has empty interior.

**Proof.** By Theorem 1.1, there is an $n$ and $X \in E(n)$ such that each $Y \in E$ polynomially dilates to an ampliation of $X$. Choose a nonzero scalar free polynomial $p$ such that $p(X) = 0$ (using the fact that the span of $\{w(X) : w$ is a word$\}$ is a subset of the finite dimensional vector space $M_n(\mathbb{C})$). It follows that $p(Y) = 0$ for all $Y$. Hence $E$ is a subset of the zero set of $p$. It is well known (see for instance the Amistur-Levitzki Theorem [Row80]) that the zero set $p$ in $M_n(\mathbb{C})^g$ must have empty interior for sufficiently large $n$. □

The authors thank Igor Klep for a fruitful correspondence which led to this article. The proof of Theorem 1.1 occupies the remainder of this article.

2. The proof of Theorem 1.1

**Proposition 2.1.** Suppose $E \subset M(\mathbb{C})^g$ is nonempty and closed with respect to direct sums. If for each $X \in E$ there is a matrix-valued free polynomial $\delta$ and a $Y \in E$ such that

$$\|\delta(X)\| < \|\delta(Y)\|,$$

then $E$ is not compact in the free topology.

**Proof.** By hypothesis, for each $X \in E$ there is a matrix-valued polynomial $\delta_X$ and $Y_X \in E$ such that $\|\delta_X(X)\| < 1 < \|\delta_X(Y_X)\|$. The collection $\mathcal{G} = \{G_{\delta_X} : X \in E\}$ is an open cover of $E$. Suppose $S \subset E$ is a finite. Observe that for each $X \in S$, $Y_X \in E \setminus G_{\delta_X}$. Since $E$ is closed with respect to direct sums, $Z = \oplus_{X \in S} Y_X \in E$. On the other hand, for a fixed $W \in S$,

$$\|\delta_W(Z)\| \geq \|\delta_W(Y_W)\| > 1.$$

Thus $Z \notin G_{\delta_W}$ and therefore $Z \in E$ but $Z \notin \cup_{X \in S} G_{\delta_X}$. Thus $\mathcal{G}$ admits no finite subcover of $E$ and therefore $E$ is not compact. □

The following lemma is a standard result.
Lemma 2.2. Suppose $X, Y \in M(C)^g$. The tuple $Y$ polynomially dilates to an ampliation of $X$ if and only if
\[ \| \delta(Y) \| \leq \| \delta(X) \| \]
for every free matrix-valued polynomial $\delta$.

Proof. Let $\mathcal{P}$ denote the set of scalar free polynomials in $g$ variables. Given a tuple $Z \in M_n(C)^g$, let $\mathcal{S}(Z) = \{ p(Z) : p \in \mathcal{P} \} \subset M_n(C)$. The set $\mathcal{S}(Z)$ is a unital operator algebra. Let $m$ and $n$ denote the sizes of $Y$ and $X$ respectively. The hypotheses thus imply that the unital homomorphism $\lambda : \mathcal{S}(X) \to \mathcal{S}(Y)$ given by $\lambda(p(X)) = p(Y)$ is well defined and completely contractive. Thus by Corollary 7.6 of [Pau02], it follows that there exists a completely positive map $\phi : M_n(C) \to M_m(C)$ extending $\lambda$. By Choi’s Theorem [Pau02], there exists an $M$ and, for $1 \leq j \leq M$, mappings $W_j : \mathbb{C}^m \to \mathbb{C}^n$ such that $\sum W_j^* W_j = I$ and
\[ \phi(T) = \sum W_j^* TW_j. \]
Let $W$ denote the column matrix with entries $W_i$. With this notation, $\phi(T) = W^*(I_M \otimes T)W$. In particular, $W$ is an isometry, since $I = \varphi(I) = W^*W$. Moreover, for polynomials $p$, $p(Y) = \varphi(p(X)) = W^*(I_M \otimes p(X))W$

and the proof of the reverse direction is complete.

To prove the converse, suppose there is a $N$ and an isometry $V$ such that for all free scalar polynomials $p$,
\[ p(Y) = V^* p(I_N \otimes X)V = V^*[I_N \otimes p(X)]V. \]
Thus for all matrix free polynomials $\delta$, say of size $d \times d$ (without loss of generality $\delta$ can be assumed square),
\[ \delta(Y) = [V \otimes I_d]^* [I_N \otimes \delta(X)] [V \otimes I_d]. \]
It follows that $\| \delta(Y) \| \leq \| \delta(X) \|$. \hfill \Box

Proof of Theorem 1.1. If for each $X \in E$ there is a $Y \in E$ that does not polynomially dilate to an ampliation of $X$, then, by Lemma 2.2, for each $X \in E$ there is a $Y \in E$ and a matrix-valued polynomial $\delta_X$ such that $\| \delta_X(X) \| < \| \delta_X(Y) \|$. An application of Proposition 2.1 shows $E$ is not compact.

To prove the converse, suppose there exists $X \in E$ such that every $Y \in E$ polynomially dilates to an ampliation of $X$. Let $\mathcal{G}$ be an open cover of $E$. There is a $G \in \mathcal{G}$ and a matrix valued free polynomial $\delta$ such that $X \in G_\delta \subset G$. Since $Y$ polynomially dilates to an ampliation of $X$, it follows that $\| \delta(Y) \| \leq \| \delta(X) \| < 1$. Hence $Y \in G_\delta \subset G$ and therefore $E \subset G$. \hfill \Box
REFERENCES

[AM14] J. Agler, J. McCarthy: Global holomorphic functions in several non-commuting variables, Canad. J. Math. 67 (2015) 241–285.

[AM15] J. Agler, J. McCarthy: Pick interpolation for free holomorphic functions, Amer. J. Math. 137 (2015) 1685–1701.

[BMV] J.A. Ball, G. Marx, V. Vinnikov: Interpolation and transfer-function realization for the noncommutative Schur-Agler class, preprint http://arxiv.org/abs/1602.00762

[BKP16] Sabine Burgdorf, Igor Klep and Janez Povh: Optimization of polynomials in non-commuting variables, SpringerBriefs in Mathematics, Springer-Verlag, 2016.

[HKN14] J. William Helton, Igor Klep, Christopher S. Nelson: Noncommutative polynomials nonnegative on a variety intersect a convex set, J. Funct. Anal., 2014, vol. 266, pp. 6684-6752.

[KVV14] D. Kaluzhnyi-Verbovetskyi, V. Vinnikov: Foundations of Free Noncommutative Function Theory, Mathematical Surveys and Monographs 199, AMS, 2014.

[KV] Igor Klep, Jurij Volcic: Free loci of matrix pencils and domains of noncommutative rational functions, preprint http://arxiv.org/abs/1512.02648

[KŠ] I. Klep, Š. Špenko: Free function theory through matrix invariants, to appear in Canad. J. Math.

[Pas14] J.E. Pascoe: The inverse function theorem and the Jacobian conjecture for free analysis, Math. Z. 278 (2014) 987–994.

[Pau02] V. Paulsen: Completely bounded maps and operator algebras, Cambridge Univ. Press, 2002.

[Row80] L.H. Rowen: Polynomials identities in ring theory, Academic Press, New York, 1980.

[Voi10] D.-V. Voiculescu: Free analysis questions II: The Grassmannian completion and the series expansions at the origin, J. reine angew. Math. 645 (2010) 155–236.

MERIC AUGAT, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF FLORIDA, GAINESVILLE
E-mail address: mlaugat@ufl.edu

SRIRAM BALASUBRAMANIAN, DEPARTMENT OF MATHEMATICS, IIT MADRAS, CHENNAI - 600036, INDIA
E-mail address: bsriram@iitm.ac.in

SCOTT MCCULLOUGH, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF FLORIDA, GAINESVILLE
E-mail address: sam@math.ufl.edu