Persistent current in a mesoscopic ring with diffuse surface scattering

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The persistent current in a clean mesoscopic ring with ballistic electron motion is calculated. The particle dynamics inside a ring is assumed to be chaotic due to scattering at the surface irregularities of atomic size. This allows one to use the so-called “ballistic” supersymmetric $\sigma$ model for calculation of the two-level correlation function in the presence of a nonzero magnetic flux.

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Since the pioneering work of Büttiker, Imry and Landauer [1], persistent currents in small metallic rings have been a subject of great theoretical and experimental interest. While earlier observations were made on disordered samples with diffusive electron dynamics [2], more recent experiments measured the persistent currents in high mobility semiconductor heterostructures with the elastic mean free path $l \simeq 1.3L$ ($L$ is the system size) [3].

The semiclassical electron dynamics in such clean samples is ballistic and can be either regular (integrable) or chaotic. The former possibility seems to be rather exceptional, since any deviation of the sample shape from a perfect annulus, however small, breaks down the integrability of a system. The purpose of this paper is to calculate the persistent current in a ballistic ring, in which a bulk disorder is absent but the electron dynamics is nevertheless chaotic due to the multiple surface scattering. The electron-electron interaction is neglected.

According to Ref. [4], the average current for a canonical ensemble (i.e., for a fixed number of electrons in a sample) is given by the following formula:

$$ I = -\frac{s}{2\Delta} \frac{\partial}{\partial \Phi} \int d\epsilon_1 d\epsilon_2 \rho(\epsilon_1, \epsilon_2) K(\epsilon_1, \epsilon_2; \Phi) $$

(1)

(we use the units in which $\hbar = c = 1$). Here $s$ is the spin degeneracy, $\Delta = (\rho_0 V)^{-1}$ is the mean level spacing in the system, $\rho_0$ is the average density of states, $V$ is the system volume, $\rho(\epsilon)$ is the Fermi distribution function, $K(\epsilon_1, \epsilon_2; \Phi)$ is the dimensionless two-level correlation function at a nonzero magnetic flux $\Phi$, which depends on the energy difference $\epsilon_2 - \epsilon_1 = \omega$:

$$ K(\omega, \Phi) = \frac{1}{\rho_0^2} (\delta \rho(E + \omega, \Phi) \delta \rho(E, \Phi)) $$

(2)

(\delta \rho(E, \Phi) = \rho(E, \Phi) - \rho_0$ is the deviation of the one-particle density of states from its average value). We restrict our analysis to the limit of $T \gg \Delta$, where the formula (1) is valid.

We assume that the sample has the shape of a planar coaxial ring with outer and inner radii $R_1$ and $R_2$, respectively, threaded by a solenoid carrying a flux $\Phi$ (see Fig. 1). We consider here only narrow rings with $d = R_1 - R_2 \ll R = (R_1 + R_2)/2$. For this geometry, the number of transverse channels $N = m v_F d/2\pi$, the average density of states $\rho_0 = m/2\pi$ and the mean level spacing $\Delta = (m R d)^{-1}$.

In order to calculate the two-level correlation function (2), we use the supersymmetric $\sigma$ model, which has become a powerful tool in the theory of disordered metals [4] and has been adapted to the description of the classically chaotic systems as well [4,5]. Here it is appropriate to emphasize that in a clean ring, where a natural ensemble is absent but the dynamics is classically chaotic, the averaging in Eq. (2) must be performed over a wide energy band [6], in contrast to the case of a disordered ring, where the angular brackets in Eq. (2) imply averaging over different realizations of the random potential. If the shape of a sample is highly symmetric and its surface is smooth on the atomic scale, then the specular boundary conditions commonly used in chaotic billiards give rise to integrability of the system, and the whole approach based on using the supersymmetric $\sigma$ model fails (this case was studied in Ref. [4] using the semiclassical trace formulas). In order to make the particle dynamics chaotic, one can, for instance, slightly deform the shape of the billiard to break its perfect rotational symmetry [6,7]. However, in this approach, the level correlations can be calculated only numerically. Another way to achieve the chaotic regime, at the same time preserving the macroscopic symmetry of a sample, is to assume that each act of the surface reflection is stochastic itself, i.e., the incident particle gets reflected in some random direction at the surface. This model is commonly referred to as the diffuse reflection and applies to the surfaces which are rough on the atomic scale, which seems to be quite reasonable physical assumption [8]. After several reflections at the walls, the dynamics of an electron becomes fully chaotic.

At $\omega \gg \Delta$, the two-level correlation function (2) can be calculated perturbatively, using the “ballistic” version of the supersymmetric $\sigma$ model generalized to the presence of a nonzero magnetic flux [12]:

$$ K(\omega, \Phi) = \frac{\Delta^2}{2\pi^2} \Re \sum_i \frac{1}{|\omega - \lambda_i(\Phi)|^2} $$

(3)

where $\lambda_i(\Phi)$ are the eigenvalues of the (Cooperon) Liouville operator.
inside the Aharonov-Bohm billiard ($\mathbf{n}$ is the direction of momentum). The region of small frequencies $\omega \ll \Delta$, where the perturbative approach fails and the level correlations are described by the universal formulas of the random matrix theory [13], lies beyond the limits of applicability of the thermodynamic approach to the description of persistent currents. The expression (3) is a direct analog of the Altshuler-Shklovskii spectral function for diffusive systems [14]. Rewriting the Fermi distribution functions in Eq. (3) as Matsubara sums and integrating over energies, we end up with the following expression:

$$I = -\frac{2\pi s}{\Delta} T \sum_{n>0} \omega_n \frac{\partial K(i\omega_n, \Phi)}{\partial \Phi},$$

where $\omega_n = 2\pi n T$. The sum over $n$ on the right-hand side is convergent due to the presence of the differentiation over flux.

The spectrum of the Liouville operator is determined by the eigenvalue equation

$$v_F \mathbf{n}(\nabla_x - 2ie\mathbf{A})f(\mathbf{r}, \mathbf{n}) = \lambda f(\mathbf{r}, \mathbf{n}),$$

with some boundary conditions at the surfaces of the ring. Due to the similarity of Eq. (4) to the Boltzmann kinetic function, $f(\mathbf{r}, \mathbf{n})$ having the meaning of the classical distribution function, the boundary conditions at the diffusely reflecting surface $\mathbf{r} = \mathbf{R}$ with the outward normal $\mathbf{N}$ can be imposed by analogy with the classical kinetic theory. The distribution function of reflected particles can be represented as $f(\mathbf{R}, \mathbf{n}) = p f_0(\mathbf{R}) + (1 - p) f(\mathbf{R}, \mathbf{n})$ [13], where $0 \leq p \leq 1$ is “the diffuseness coefficient”, $f_0$ is an isotropic distribution function, and $\mathbf{n} = \mathbf{n} - 2(n\mathbf{N})\mathbf{N}$ is the direction of specular reflection. In this paper, we consider an isotropic diffuse scattering with $p = 1$, corresponding to the limit of “strong chaos” (for the discussion of applicability of this model to real experimental samples, see below) and the boundary condition, which follows from the particle number conservation, takes the form

$$\frac{1}{\pi} \int f(\mathbf{R}_i, \mathbf{n}) |(n\mathbf{N}_i)_{<0} = \int_{(n\mathbf{N}_i)_{>0}} d\mathbf{n}' (n'\mathbf{N}_i) f(\mathbf{R}_i, \mathbf{n}').$$

Here $\int d\mathbf{n} = \int d\mathbf{o}/2\pi$ ($\phi$ is the angle between the direction of momenta of incident particles and the outward normal $\mathbf{N}$), and $i = 1, 2$ correspond to the outer and inner surfaces of the ring. The distribution function of reflected particles on the left-hand side of Eq. (5) does not depend on $\mathbf{n}$. A similar approach was used in Ref. [14] for calculation of the corrections to the universal level correlations in a two-dimensional disk without magnetic flux.

Since the magnetic field is absent inside the ring, the trajectory of a particle between collisions with the walls is a straight line. Equation (5) can be solved along the trajectory

$$f(l) = f(0) \exp\left(\frac{\lambda f}{v_F} l\right) \exp\left(2ie \int_0^l A d\ell\right).$$

This expression allows one to establish a relation between $f(\mathbf{R}_1, \mathbf{n}) |(n\mathbf{N}_1)_{>0}$ and $f(\mathbf{R}_1, \mathbf{n}) |(n\mathbf{N}_1)_{<0}$ or $f(\mathbf{R}_2, \mathbf{n}) |(n\mathbf{N}_2)_{<0}$, and also between $f(\mathbf{R}_2, \mathbf{n}) |(n\mathbf{N}_2)_{>0}$ and $f(\mathbf{R}_1, \mathbf{n}) |(n\mathbf{N}_1)_{<0}$ in Eq. (6) and obtain a rather cumbersome algebraic equation for the eigenvalues of the Liouville operator in a ring of arbitrary width. Fortunately, in the case of a narrow ring, the problem can be considerably simplified, since we can replace our annular billiard by a strip of length $L = 2\pi R$ and width $d$ such that $\delta = d/L \ll 1$ (see Fig. 2). In addition, the vector potential can be put constant inside the sample: $A_0 = \Phi/L$ ($\Phi$ is the polar angle in real space). This simplification implies that the contribution from the trajectories connecting two points at the outer wall is neglected (it can be checked that this contribution is indeed small at $\delta \ll 1$). However, there is an important property of the annular geometry which should be taken into account, namely the finiteness of the flight length $l(\phi)$ between successive collisions with the walls ($l \ll L$). This feature can be restored in the strip billiard if to assume that there exists the maximum scattering angle $\phi_0$ such that $\sin \phi_0 = R_2/R_1 \approx 1 - 2\pi \delta$.

Let $x = R\theta$, then we obtain, from Eq. (5) and Fig. 2:

$$f_{1,>}(x, \phi) = f_{2,>}(x - d \tan \phi) \times \exp\left(\frac{\lambda f}{v_F \cos \phi}\right) \exp\left(-2\pi i \frac{2\Phi d \tan \phi}{L}\right).$$

$$f_{2,>}(x, \phi) = f_{1,>}(x + d \tan \phi) \times \exp\left(\frac{\lambda f}{v_F \cos \phi}\right) \exp\left(-2\pi i \frac{2\Phi d \tan \phi}{L}\right).$$

Here $f_{i,>}(x, \phi) = f(\mathbf{R}_i, \mathbf{n}) |(n\mathbf{N}_i)_{>0}$ and $\Phi_0 = 2\pi \hbar c/e$ is the flux quantum. It is convenient to expand the functions $f_i$ in the Fourier series: $f_{i,>}(x) = \sum_q f_{i,q} e^{iqx}$. Since $f(x + L, \mathbf{n}) = f(x, \mathbf{n})$, the wave number is quantized: $q = m(2\pi/L)$, where $m = 0, \pm 1, \ldots$. Substitution of the expressions (7) in Eq. (6) results in the following expression for the eigenvalues $\lambda_{m,k}(\Phi) = (v_F/d)z_{m,k}(\Phi)$ of the Liouville operator (8):

$$F_m(z, \nu) \equiv A_{m,1}^2(z, \nu) - 1 = 0.$$

Here $\nu = 2\Phi/\Phi_0$ and

$$A_m = \frac{1}{\sin \phi_0} \int_0^{\phi_0} d\phi \cos \phi \exp(z \sec \phi) \times \cos(2\pi \delta (m - \nu) \tan \phi).$$

To guarantee correct normalization of the distribution function of reflected particles, the prefactor $\sin^{-1} \phi_0$ has been included in the definition of $A_m$.

The eigenvalues of the Liouville operator at fixed $m$ are labeled by $k = 0, 1, 2, \ldots$, and have complex conjugated partners for all $(m, k)$. At $\Phi = 0$, Eq. (10) has the
solution \( z_{0,0} = 0 \) corresponding to the equilibrium distribution function, which does not depend on \( n \) and \( r \). It is clear from Eq. (11) that the eigenvalues \( \lambda_{m,k}(\Phi) \) have the following property: \( \lambda_{m,k}(\Phi + \Phi_0/2) = \lambda_{m+1,k}(\Phi) \) so that any quantity which can be represented in the form \( \sum_{m,k} h(\lambda_{m,k}(\Phi)) \), where \( h(\lambda) \) is some function, must be periodic function of \( \Phi \) with the period of half the flux quantum \( \Phi_0 \). Note that, since the eigenvalues of the Liouville operator physically correspond to the relaxation rates of different harmonics of a nonequilibrium classical distribution function, the real parts of all \( \lambda_{m,k} \) are positive.

Using the representation of the sum over \( i = (m,k) \) in Eqs. (3) and (4) as an integral over a contour \( C \) enclosing all zeros of the function \( F_m(\nu, z) \) in the complex plane of \( z \):

\[
\sum_{m,k} \frac{1}{\omega_n + \lambda_{m,k}(\Phi)} = \left( \frac{d}{v_F} \right)^2 \frac{1}{2\pi i} \sum_m \int_C \frac{dz}{2\pi i (z + \omega_n d/v_F)^2} \frac{\partial}{\partial z} \ln F_m(z, \nu)
\]

\[
= - \left( \frac{d}{v_F} \right)^2 \frac{\partial^2}{\partial z^2} \ln F_m(z, \nu) \bigg|_{z = -\omega_n d/v_F},
\]

we finally obtain

\[
\frac{I}{I_0} = \frac{2s\delta^2}{\pi N^{3/2}} \left( \frac{T}{\Delta} \right)^2 \Re \sum_{n>0} \sum_m \nu
\]

\[
\times \frac{\partial}{\partial \nu} \frac{\partial^2}{\partial z^2} \ln F_m(z, \nu) \bigg|_{z = -2\pi \delta t/N \Delta},
\]

where \( I_0 = e v_F / L \) is the current carried by a single electron in an ideal one-dimensional ring, and \( F_m(z, \nu) \) is given by Eq. (4).

Due to the existence of different energy scales in the system, the temperature dependence of the persistent current is characterized by several distinct regimes. The smallest energy scale is the mean level spacing \( \Delta \), which also limits the applicability of the thermodynamic approach itself. Two other scales are given by the inverse times \( t_{\nu}^{-1} = v_F / L = N\Delta \) and \( t_{\delta}^{-1} = v_F / d = N\Delta / \delta \). It follows from Eqs. (12) and (13) that at \( T \gg t_{\nu}^{-1} \) the persistent current is exponentially small: \( I \sim I_0 \exp(-T/N\Delta) \).

In a multichannel ring, there also exists yet another energy scale \( \Delta \ll E_c \ll N\Delta \), whose origin can be most easily understood if to return to Eq. (4). Using the identity \( e^{-z^2} = \int_0^\infty dy e^{-\nu y} \) and calculating the sum over \( n \), we end up with the following expression

\[
\frac{I}{I_0} = \frac{s}{2\pi^3 T} \int_0^\infty \frac{d\xi}{\sinh^2 \xi} \varphi(\xi; T, \Phi),
\]

where

\[
\varphi = \frac{\Phi_0 d}{2v_F} \Re \sum_{m,k} \frac{d\lambda_{m,k}(\Phi)}{d\Phi} \exp \left( -\frac{\lambda_{m,k}(\Phi) \xi}{\pi T} \right).
\]
than the inverse typical flight length \((v_F t_f)^{-1}\). This condition, rewritten as \(\Phi/\Phi_0 \ll N\delta^{-7/4}\), is always satisfied in narrow rings.

In order to facilitate comparison of our results with the experimental data, let us rewrite Eq. (15) in a different form, using the identity \(\Delta/T = N^{-1}(L_T/L)\), where \(L_T = v_F/T\) is the length scale associated with temperature. In the experiment of Ref. [3], \(\delta \approx 0.1\), \(N \approx 4\), \(L_T/L \approx 5\), so that \(T \sim \Delta\). Due to the presence of the factor \(\delta \ln \delta \ll 1\) in Eq. (15), the predicted current turns out to be smaller than the experimentally observed (and also than the theoretically calculated for the case of specular reflection [8]). This discrepancy can be attributed to the fact that, because of the low density of carriers, the Fermi wavelength greatly exceeds the size of the surface irregularities, so that a considerable fraction of particles gets reflected specularly rather than diffusely (i.e., \(p < 1\)), and one should describe the semiclassical dynamics in the experimental conditions of Ref. [3] as “weakly chaotic”.

To summarize, we calculated the persistent current in a small clean metal ring, in which the electron dynamics is chaotic due to the stochastic surface scattering. A general analytical expression for the persistent current is derived in the limit of “strong chaos”, and a Curie-type orbital magnetic response on a small external flux is predicted at \(\Delta \ll T \ll E_c\).

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