THE c-NILPOTENT SCHUR Lie-MULTIPLIER OF LEIBNIZ ALGEBRAS

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Abstract: We introduce the notion of $c$-nilpotent Schur Lie-multiplier of
Leibniz algebras. We obtain exact sequences and formulas of the dimensions of
the underlying vector spaces relating the $c$-nilpotent Schur Lie-multiplier of a
Leibniz algebra $q$ and its quotient by a two-sided ideal. These tools are used to
characterize Lie-nilpotency and $c$-Lie-stem covers of Leibniz algebras. We prove
the existence of $c$-Lie-stem covers for finite dimensional Leibniz algebras and the
non existence of $c$-covering on certain Lie-nilpotent Leibniz algebras with non
trivial $c$-nilpotent Schur Lie-multiplier, and we provide characterizations of $c$-Lie-
capability of Leibniz algebras by means of both their $c$-Lie-characteristic ideal
and $c$-nilpotent Schur Lie-multiplier.

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stem cover; $c$-Lie-capable; $c$-Lie-characteristic ideal.

1 Introduction

The general theory of central extensions relative to a chosen subcategory of a
base category was introduced in [20], where the simultaneous categorical and
Galois theoretic approach is based on, and generalizes, the work of the Fröhlich
school [16, 17, 25] which focused in varieties of $\Omega$-groups, was considered in the
context of semi-abelian categories [21] relative to a Birkhoff subcategory in [14
15]. Examples like groups vs. abelian groups, Lie algebras vs. vector spaces are
absolute, meaning that they fit in the general theory when the considered Birkhoff
subcategory is the subcategory of all abelian objects. A non-absolute example
is the category of Leibniz algebras together with the Birkhoff subcategory of Lie
algebras. The general theory provides the notions of relative central extension and relative commutator with respect to the Liezation functor \((-)_{\text{Lie}} : \text{Leib} \to \text{Lie}\) which assigns to a Leibniz algebra \(g\) the Lie algebra \(g_{\text{Lie}} = g / g^{\text{ann}}\), where \(g^{\text{ann}} = \langle \{[x, x] : x \in g\} \rangle\).

In the recent papers [9, 11, 12] authors approached the relative theory of Leibniz algebras with respect to the Liezation functor, yielding to the introduction of new notions of central extensions, capability, nilpotency, stem cover, isoclinism and Schur multiplier relative to the Liezation functor, the so-called Lie-central extensions, Lie-capability, Lie-nilpotency, Lie-stem cover, Lie-isoclinism and Schur Lie-multiplier.

It is well-known the interplay between the Schur multiplier [31] and other mathematical concepts like projective representations, central extensions, efficient presentations or homology. The Schur multiplier was generalized by Baer [5, 6, 7] to any variety of nilpotent groups. When \(V\) is the variety of nilpotent Lie algebras of class at most \(c \geq 1\), then the Baer-invariant \(M^{(c)}(L) = R^{\gamma_{c+1}(F)} / \gamma_{c+1}(F, R)\) (that is, the quotient doesn’t depend on the chosen free presentation [15]), where \(0 \to R \to F \to L \to 0\) is a free presentation of the Lie algebra \(L\) and \(\gamma_{c+1}(-)\) are the \((c + 1)\)-terms of the lower central series, was introduced in [29] and later developed, among others, in [1, 2, 3, 26, 27, 28, 30].

Our goal in this paper is to introduce the relative notion of \(c\)-nilpotent Schur Lie-multiplier, the Baer-invariant called \(c\)-nilpotent Schur Lie-multiplier, \(M^{(c)}_{\text{Lie}}(q) = \gamma^{\text{Lie}}_{c+1}(f) / \gamma^{\text{Lie}}_{c+1}(f, r)\), where \(\gamma^{\text{Lie}}_{c+1}(-)\) are the \((c+1)\)-terms of the Lie-lower central series of \(q\) [11] and \(0 \to r \to f \to q \to 0\) is a free presentation. Then we apply it to characterize Lie-nilpotency of Leibniz algebras and study \(c\)-Lie-stem covers and \(c\)-Lie-capability of Leibniz algebras.

The paper is organized as follows: in section 2 we provide preliminary results on Leibniz algebras from [11] which are needed through the paper. In section 3 we introduce the notion of \(c\)-nilpotent Schur Lie-multiplier \((c \geq 1)\) and we use it along with the Baer-invariant \(\gamma^{\text{Lie}}_{c+1}(q) = \gamma^{\text{Lie}}_{c+1}(f) / \gamma^{\text{Lie}}_{c+1}(f, r)\), where \(0 \to r \to f \to q \to 0\) is a free presentation, to characterize Lie-nilpotency of class \(c\) for a given Leibniz algebra. In section 4 we determine exact sequences providing several results on the dimension of the \(c\)-nilpotent Schur Lie-multiplier in the case when the Leibniz algebra is finite dimensional. In section 5 we use \(c\)-nilpotent Schur Lie-multiplier to provide a characterization of \(c\)-Lie-stem extensions, and prove the existence of \(c\)-Lie-stem covers for finite dimensional Leibniz algebras, and the non existence of \(c\)-Lie-covering on certain nilpotent Leibniz algebras with non trivial \(c\)-nilpotent Schur Lie-multiplier. Finally, in section 6 we study \(c\)-Lie-capability. In particular we provide characterizations of \(c\)-Lie-capability of Leibniz algebras by means of both their \(c\)-Lie-characteristic ideal and \(c\)-nilpotent Schur Lie-multiplier.
2 Preliminary results on Leibniz algebras

We fix $\mathbb{K}$ as a ground field such that $\frac{1}{2} \notin \mathbb{K}$. All vector spaces and tensor products are considered over $\mathbb{K}$.

2.1 Background on Leibniz algebras

A Leibniz algebra \cite{22, 23, 24} is a $\mathbb{K}$-vector space $\mathfrak{q}$ equipped with a linear map $[\cdot, \cdot] : \mathfrak{q} \otimes \mathfrak{q} \to \mathfrak{q}$, usually called the Leibniz bracket of $\mathfrak{q}$, satisfying the Leibniz identity:

$$[[x, y], z] - [[x, z], y] = [x, [y, z]], \quad x, y, z \in \mathfrak{q}.$$  

Leibniz algebras constitute a variety of $\Omega$-groups \cite{18}, hence it is a semi-abelian variety \cite{14, 21} denoted by $\text{Leib}$, whose morphisms are linear maps that preserve the Leibniz bracket.

A subalgebra $\mathfrak{h}$ of a Leibniz algebra $\mathfrak{q}$ is said to be left (resp. right) ideal of $\mathfrak{q}$ if $[\mathfrak{h}, \mathfrak{q}] \subseteq \mathfrak{h}$ (resp. $[\mathfrak{q}, \mathfrak{h}] \subseteq \mathfrak{h}$), for all $\mathfrak{h} \in \mathfrak{h}$, $\mathfrak{q} \in \mathfrak{q}$. If $\mathfrak{h}$ is both left and right ideal, then $\mathfrak{h}$ is called two-sided ideal of $\mathfrak{q}$. In this case $\mathfrak{q}/\mathfrak{h}$ naturally inherits a Leibniz algebra structure.

For a Leibniz algebra $\mathfrak{q}$, we denote by $\mathfrak{q}_{\text{ann}}$ the subspace of $\mathfrak{q}$ spanned by all elements of the form $[x, x], x \in \mathfrak{q}$.

Given a Leibniz algebra $\mathfrak{q}$, it is clear that the quotient $\mathfrak{q}_{\text{Lie}} = \mathfrak{q}/\mathfrak{q}_{\text{ann}}$ is a Lie algebra. This defines the so-called Liezation functor $(-)_{\text{Lie}} : \text{Leib} \to \text{Lie}$, which assigns to a Leibniz algebra $\mathfrak{q}$ the Lie algebra $\mathfrak{q}_{\text{Lie}}$. Moreover, the canonical surjective homomorphism $\mathfrak{q} \to \mathfrak{q}_{\text{Lie}}$ is universal among all homomorphisms from $\mathfrak{q}$ to a Lie algebra, implying that the Liezation functor is left adjoint to the inclusion functor $\text{Lie} \hookrightarrow \text{Leib}$.

Since $\text{Lie}$ is a subvariety of $\text{Leib}$, then it is a Birkhoff subcategory of $\text{Leib}$.

For a Leibniz algebra $\mathfrak{q}$ and two-sided ideals $\mathfrak{m}$ and $\mathfrak{n}$ of $\mathfrak{q}$, the Lie-centralizer of $\mathfrak{m}$ and $\mathfrak{n}$ over $\mathfrak{q}$ is

$$C^\text{Lie}_\mathfrak{q}(\mathfrak{m}, \mathfrak{n}) = \{ q \in \mathfrak{q} \mid [q, m] + [m, q] \in \mathfrak{n}, \text{ for all } m \in \mathfrak{m} \}.$$  

The Lie-commutator $[\mathfrak{m}, \mathfrak{n}]_{\text{Lie}}$ is the subspace of $\mathfrak{q}$ spanned by all elements of the form $[m, n] + [n, m], m \in \mathfrak{m}, n \in \mathfrak{n}$.

In particular, the two-sided ideal $C^\text{Lie}_\mathfrak{q}(\mathfrak{q}, 0)$ is the Lie-center of the Leibniz algebra $\mathfrak{q}$ and it will be denoted by $Z_{\text{Lie}}(\mathfrak{q})$, that is,

$$Z_{\text{Lie}}(\mathfrak{q}) = \{ z \in \mathfrak{q} \mid [q, z] + [z, q] = 0 \text{ for all } q \in \mathfrak{q} \}.$$  

Proposition 2.1 \cite{13, Example 1.9} Given an extension of Leibniz algebras $f : \mathfrak{g} \to \mathfrak{q}$ with $\mathfrak{n} = \text{Ker}(f)$, the following conditions are equivalent:

(a) $f : \mathfrak{g} \to \mathfrak{q}$ is Lie-central;
2.2 Lie-nilpotent Leibniz algebras

Definition 2.2 [11, Definition 5] Let \( m \) be a two-sided ideal of a Leibniz algebra \( q \). A series from \( m \) to \( q \) is a finite sequence of two-sided ideals \( m_i, 0 \leq i \leq k \), of \( q \) such that
\[
m = m_0 \subseteq m_1 \subseteq \cdots \subseteq m_{k-1} \subseteq m_k = q.
\]
k is called the length of this series.

A series from \( m \) to \( q \) of length \( k \) is said to be Lie-central (resp. Lie-abelian) if \([m_i, q]_{\text{Lie}} \subseteq m_{i-1}\) or equivalently \( m_i/m_{i-1} \subseteq Z_{\text{Lie}}(q/m_{i-1})\) (resp. if \([m_i, m_i]_{\text{Lie}} \subseteq m_{i-1}\) or equivalently \([m_i/m_{i-1}, m_i/m_{i-1}]_{\text{Lie}} = 0\)) for \( 1 \leq i \leq k \).

A series from \( 0 \) to \( q \) is called a series of the Leibniz algebra \( q \).

Definition 2.3 [11, Definition 11] Let \( n \) be a two-sided ideal of a Leibniz algebra \( q \). The lower Lie-central series of \( q \) relative to \( n \) is the sequence
\[
\cdots \leq \gamma_i^\text{Lie}(q, n) \leq \cdots \leq \gamma_2^\text{Lie}(q, n) \leq \gamma_1^\text{Lie}(q, n)
\]
of two-sided ideals of \( q \) defined inductively by
\[
\gamma_1^\text{Lie}(q, n) = n \quad \text{and} \quad \gamma_i^\text{Lie}(q, n) = [\gamma_{i-1}^\text{Lie}(q, n), n]_{\text{Lie}}, \quad i \geq 2.
\]
The Leibniz algebra \( q \) is said to be Lie-nilpotent relative to \( n \) of class \( c \) if \( \gamma_c^\text{Lie}(q, n) = 0 \) and \( \gamma_c^\text{Lie}(q, n) \neq 0 \).

Note that \([\gamma_i^\text{Lie}(q, n)/\gamma_{i+1}^\text{Lie}(q, n), \gamma_i^\text{Lie}(q, n)/\gamma_{i+1}^\text{Lie}(q, n)]_{\text{Lie}} = 0\). When \( n = q \) we obtain Definition 9 in [11] and we will use the notation \( \gamma_c^\text{Lie}(q) \) instead of \( \gamma_i^\text{Lie}(q, q), 1 \leq i \leq n \). If \( \varphi : g \to q \) is a homomorphism of Leibniz algebras such that \( \varphi(m) \subseteq n \), where \( m \) is a two-sided ideal of \( g \) and \( n \) a two-sided ideal of \( q \), then \( \varphi(\gamma_i^\text{Lie}(g, m)) \subseteq \gamma_i^\text{Lie}(q, n), i \geq 1 \).

Definition 2.4 [11, Definition 10] The upper Lie-central series of a Leibniz algebra \( q \) is the sequence of two-sided ideals
\[
\cdots \leq Z_0^\text{Lie}(q) \leq Z_1^\text{Lie}(q) \leq \cdots \leq Z_i^\text{Lie}(q) \leq \cdots
\]
defined inductively by
\[
Z_0^\text{Lie}(q) = 0 \quad \text{and} \quad Z_i^\text{Lie}(q) = C^\text{Lie}_q(q, Z_{i-1}^\text{Lie}(q)), \quad i \geq 1.
\]

Theorem 2.5 [11, Theorem 4] A Leibniz algebra \( q \) is Lie-nilpotent with class of Lie-nilpotency \( k \) if and only if \( Z_k^\text{Lie}(q) = q \) and \( Z_{k-1}^\text{Lie}(q) \neq q \).
3 The c-nilpotent Schur Lie-multiplier

In this section we introduce the Baer-invariants $c$-nilpotent Schur Lie-multiplier and $\gamma_{c+1}^{\text{Lie}}(-)$ of a Leibniz algebra. Then we use them to characterize Lie-nilpotency of Leibniz algebras.

Let $0 \to r \to f \xrightarrow{\gamma} q \to 0$ be a free presentation of a Leibniz algebra $q$. We call $c$-nilpotent Schur Lie-multiplier of $q$ ($c \geq 1$) to the term

$$M_{\text{Lie}}^{(c)}(q) := \frac{r \cap \gamma_{c+1}^{\text{Lie}}(f)}{\gamma_{c+1}^{\text{Lie}}(f, r)} \quad (1)$$

Since $\text{Leib}$ is a category of interest (see [10]), hence is a category of $\Omega$-groups, and following Proposition 4.3.2 in [19] we can conclude that the collection of all nilpotent objects of class $\leq c$ in $\text{Leib}$ form a variety. Now following [15, Proposition 7.8], it can be showed that $M_{\text{Lie}}^{(c)}(q)$ and $\gamma_{c+1}^{\text{Lie}}(q) = \frac{\gamma_{c+1}^{\text{Lie}}(f)}{\gamma_{c+1}^{\text{Lie}}(f, r)}$ are Baer-invariants, which means that their definitions do not depend on the choice of the free presentation.

**Remark 3.1** $M_{\text{Lie}}^{(1)}(q)$ is the Schur Lie-multiplier of a Leibniz algebra $q$ (see [9, 12]).

**Definition 3.2** An exact sequence of Leibniz algebras $0 \to n \to g \xrightarrow{\pi} q \to 0$ is said to be a $c$-Lie-central extension if $\gamma_{c+1}^{\text{Lie}}(g, n) = 0$.

**Example 3.3**

(a) Let $g$ and $q$ be four and two-dimensional Leibniz algebra with respective $\mathbb{K}$-linear bases $\{a_1, a_2, a_3, a_4\}$ and $\{e_1, e_2\}$, with the Leibniz brackets given respectively by $[a_1, a_2] = a_3, [a_2, a_2] = a_4, [a_1, e_2] = a_3$ and zero elsewhere (g is a Lie-nilpotent Leibniz algebra of class 3 and q is a Lie-nilpotent Lie algebra of class 2 [12]). Then the surjective homomorphism of Leibniz algebras $f : g \to q$ defined by $f(a_1) = e_2$, $f(a_2) = e_1, f(a_3) = 0$ and $f(a_4) = 0$ is a 2-Lie-central extension.

(b) Let $g$ and $q$ be four and two-dimensional Leibniz algebra with respective $\mathbb{K}$-linear bases $\{a_1, a_2, a_3, a_4\}$ and $\{e_1, e_2\}$, with the Leibniz brackets given respectively by $[a_1, a_1] = a_2, [a_1, a_2] = a_3, [a_1, a_3] = a_4$ and $[e_2, e_2] = e_1$ and zero elsewhere (g is a Lie-nilpotent Leibniz algebra of class 4 and q is a Lie-nilpotent Lie algebra of class 2 [12]). Then the surjective homomorphism of Leibniz algebras $f : g \to q$ defined by $f(a_1) = e_2$, $f(a_2) = e_1, f(a_3) = 0$ and $f(a_4) = 0$ is a 2-Lie-central extension.

**Proposition 3.4** The exact sequence of Leibniz algebras $0 \to n \to g \xrightarrow{\pi} q \to 0$ is a $c$-Lie-central extension if and only if $n \subseteq Z_{c}^{\text{Lie}}(g)$.
Proof. Assume that the exact sequence of Leibniz algebras $0 \to n \to g \xrightarrow{\pi} q \to 0$ is a $c$-Lie-central extension, then $\gamma_{c+1}^\text{Lie}(g,n) = 0$. Let $t \in n$ with $t \notin Z_{c+1}^\text{Lie}(g)$. Then $t_1 := [g_1, t] + [t, g_1] \notin Z_{c+1}^\text{Lie}(g)$ for some $g_1 \in g$. So $t_2 := [g_2, t_1] + [t_1, g_2] \notin Z_{c+1}^\text{Lie}(g)$ for some $g_2 \in g$. Inductively, we have $t_c := [g_c, t_{c-1}] + [t_{c-1}, g_c] \notin Z_{c+1}^\text{Lie}(g) = 0$ for some $g_c \in g$. So $t_c \neq 0$. On the other hand, we have $t_c \in \gamma_{c+1}^\text{Lie}(g,n) = 0$ since $t \in n = \gamma_1^\text{Lie}(g,n)$. A contradiction.

Conversely, Assume that $n \subseteq Z_{c+1}^\text{Lie}(g)$ and let $z \in \gamma_{c+1}^\text{Lie}(g,n)$. Then $z := [z_1, g_1] + [g_1, z_1]$ for some $z_1 \in \gamma_{c+1}^\text{Lie}(g,n)$ and $g_1 \in g$. So $z_1 := [z_2, g_2] + [g_2, z_2]$ for some $z_2 \in \gamma_{c+1}^\text{Lie}(g,n)$ and $g_2 \in g$. Inductively we have $z_{c+1} := [z_c, g_c] + [g_c, z_c]$ for some $z_c \in \gamma_{c+1}^\text{Lie}(g,n)$ and $g_c \in g$. Since $\gamma_{c+1}^\text{Lie}(g,n) = n \subseteq Z_{c+1}^\text{Lie}(g)$, it follows that $z_c \in Z_{c+1}^\text{Lie}(g)$. Inductively, we have $z \in Z_{c+1}^\text{Lie}(g) = 0$. Thus $z = 0$. Therefore $\gamma_{c+1}^\text{Lie}(g,n) = 0$. ■

Example 3.5

(a) The short exact sequence $0 \to M_1^\text{Lie}(c)(q) \to \gamma_{c+1}^\text{Lie}(q) \xrightarrow{u} \gamma_{c+1}^\text{Lie}(q) \to 0$, where $u(x + \gamma_{c+1}^\text{Lie}(f,v)) = \rho(x)$, is a $c$-Lie-central extension.

(b) From the $3 \times 3$ Lemma [8, Theorem 4.2.7], the free presentation $0 \to r \to \overline{f} \xrightarrow{\beta} q \to 0$ induces the exact sequence $0 \to \overline{f} \xrightarrow{\beta} q \to 0$, where $\overline{f}$ is the natural epimorphism induced by $\rho$. Moreover this exact sequence is a $c$-Lie-central extension.

Lemma 3.6 Let $0 \to r \to \overline{f} \xrightarrow{\beta} q \to 0$ be a free presentation of a Leibniz algebra $q$ and let $0 \to m \to h \xrightarrow{\theta} p \to 0$ be a $c$-Lie-central extension of another Leibniz algebra $p$. Then for each homomorphism $\alpha : q \to p$, there exists a homomorphism $\beta : \overline{f} \xrightarrow{\beta} h$ such that $\beta \left( \overline{f} \gamma_{c+1}^\text{Lie}(f,v) \right) \subseteq m$ and the following diagram is commutative:

\[
\begin{array}{ccc}
0 & \xrightarrow{\gamma_{c+1}^\text{Lie}(f,v)} & \overline{f} \xrightarrow{\beta} q \xrightarrow{\alpha} 0 \\
\downarrow{\beta} & & \downarrow{\alpha} \\
0 & \xrightarrow{\gamma_{c+1}^\text{Lie}(f,v)} & h \xrightarrow{\theta} p \xrightarrow{0} 0
\end{array}
\]

Proof. Since $f$ is a free Leibniz algebra, then there exists $\omega : f \to h$ such that $\theta \circ \omega = \alpha \circ \rho$.

On the other hand, $\theta(\omega(r)) = \alpha(\rho(r)) = 0$, hence $\omega(r) \subseteq m$, which implies that $\omega(\gamma_{c+1}^\text{Lie}(f,v)) = 0$, hence $\omega$ induces $\beta : \overline{f} \gamma_{c+1}^\text{Lie}(f,v) \to h$ such that $\alpha \circ \beta = \theta \circ \beta$, and for any $r \in r$, $\beta(r + \gamma_{c+1}^\text{Lie}(f,v)) = \omega(r) \in m$. ■

Proposition 3.7 Let $q$ be a Leibniz algebra and $c \geq 1$. Then

(a) $\gamma_{c+1}^\text{Lie}(q) = 0$ if and only if $q$ is Lie-nilpotent of class $c$ and $M_1^\text{Lie}(c)(q) = 0$.

(b) If $\gamma_{c+1}^\text{Lie}(q) = 0$, then $\gamma_{c+1}^\text{Lie}(q/n) = 0$ for any two-sided ideal $n$ of $q$. 

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Theorem 3.9 A $c$-Lie-central extension $0 \to n \to g \xrightarrow{\pi} q \to 0$ of a class $c$ nilpotent Leibniz algebra $q$ is of class $c$ if and only if $\theta : M_{\text{Lie}}^c(q) \to n$ vanishes over $\text{Ker}(\tau)$, where $\tau : M_{\text{Lie}}^c(q) \to M_{\text{Lie}}^c(q/\gamma_c^\text{Lie}(q))$ is induced by the canonical projection $q \to q/\gamma_c^\text{Lie}(q)$.

Proof. Consider the following diagrams:

\[ \begin{array}{ccc}
0 & \to & s \\
\downarrow & & \downarrow \pi \circ \rho \\
0 & \to & n \\
\downarrow & & \downarrow \pi \\
0 & \to & g \\
\downarrow & & \downarrow \pi \\
0 & \to & q \\
\downarrow & & \downarrow \pi \\
0 & \to & 0 \\
\end{array} \quad \begin{array}{ccc}
0 & \to & t \\
\downarrow & & \downarrow \pi \circ \rho \\
0 & \to & f \\
\downarrow & & \downarrow \pi \circ \rho \\
0 & \to & g \\
\downarrow & & \downarrow \pi \circ \rho \\
0 & \to & q \\
\downarrow & & \downarrow \pi \circ \rho \\
0 & \to & q/\gamma_c^\text{Lie}(q) \\
\downarrow & & \downarrow \pi \circ \rho \\
0 & \to & 0 \\
\end{array} \]

where $s = \text{Ker}(\pi \circ \rho)$ and $t = \text{Ker}(pr \circ \pi \circ \rho)$, then $\theta : M_{\text{Lie}}^c(q) = \frac{s \cap \gamma_c^\text{Lie}(f)}{\gamma_c^\text{Lie}(f,s)} \to n$,

given by $\theta(x + \gamma_c^\text{Lie}(f,s)) = \rho(x)$, is well-defined and $\text{Ker}(\tau) \simeq \frac{\gamma_c^\text{Lie}(f,s)}{\gamma_c^\text{Lie}(f,s)}$.

Assume that $g$ is Lie-nilpotent of class $c$ and consider $x + \gamma_c^\text{Lie}(f,s) \in \text{Ker}(\tau)$. Then $\theta(x + \gamma_c^\text{Lie}(f,s)) = \rho(x) = 0$ since $\rho(x) \in \rho(\gamma_c^\text{Lie}(f,t)) \subseteq \gamma_c^\text{Lie}(g) + \gamma_c^\text{Lie}(g,n) = 0$. For the last inclusion, it is necessary to have in mind that $\pi(\rho(t)) \subseteq \gamma_c^\text{Lie}(q) = \pi(\gamma_c^\text{Lie}(g))$, and consequently $\rho(t) \subseteq \gamma_c^\text{Lie}(g) + n$. 

\[ \begin{array}{ccc}
\gamma_c^\text{Lie}(q) & \to & q \\
\downarrow & & \downarrow \pi \\
\gamma_c^\text{Lie}(q) & \to & q/\gamma_c^\text{Lie}(q) \\
\downarrow & & \downarrow \pi \\
\gamma_c^\text{Lie}(q) & \to & 0 \\
\end{array} \]

Definition 3.8 Let $q$ be a Lie-nilpotent Leibniz algebra of class $c$. An extension of Leibniz algebras $0 \to n \to g \xrightarrow{\pi} q \to 0$ is said to be of class $c$ if $g$ is nilpotent of class $c$. 

Proof. Consider the natural text for the proof.
Conversely, \( \gamma_{c+1}^{\text{Lie}}(g) = [\gamma_c^{\text{Lie}}(g), g]_{\text{Lie}} = [\rho(\gamma_c^{\text{Lie}}(f)), \rho(f)]_{\text{Lie}} \subseteq \rho(\gamma_{c+1}^{\text{Lie}}(f, t)) = 0 \) since \( \gamma_{c+1}^{\text{Lie}}(f, t) \subseteq \mathfrak{r} \) because \( \theta \) vanishes over \( \ker(\tau) \). For the last inclusion is necessary to have in mind that \( \pi(\rho(\gamma_c^{\text{Lie}}(f))) \subseteq \gamma_c^{\text{Lie}}(q) \), hence \( \gamma_c^{\text{Lie}}(f) \subseteq \mathfrak{t} \). □

**Proposition 3.10** Let \( g \) be a \( c \)-nilpotent Leibniz algebra of class \( c \) and \( f : g \rightarrow q \) be a surjective homomorphism of Leibniz algebras. If \( \ker(f) \subseteq \gamma_c^{\text{Lie}}(g) \) and \( M_c^{(c)}(q) = 0 \), then \( f \) is an isomorphism. In particular, if \( M_c^{(c)}(g/\gamma_c^{\text{Lie}}(g)) = 0 \), then \( M_c^{(c)}(g) = 0 \).

**Proof.** Let \( n = \ker(f) \), then \( M_c^{(c)}(g/n) = 0 \). From the exact sequence in Proposition 3.1 (a) we have that \( n \cap \gamma_c^{\text{Lie}}(g) \subseteq \gamma_c^{\text{Lie}}(g, n) \), then \( n \subseteq \gamma_c^{\text{Lie}}(g, n) \). Obviously \( \supseteq \) is true, then \( n = \gamma_c^{\text{Lie}}(g, n) \subseteq \gamma_c^{\text{Lie}}(g) = 0 \). Consequently, \( f \) is an isomorphism. □

### 4 On the dimension of the \( c \)-nilpotent Schur Lie-multiplier

This section is devoted to obtain some exact sequences involving the \( c \)-nilpotent Schur Lie-multiplier and some formulas concerning dimensions of the corresponding underlying vector spaces.

**Proposition 4.1** Let \( 0 \rightarrow \mathfrak{r} \rightarrow \mathfrak{f} \xrightarrow{\rho} q \rightarrow 0 \) be a free presentation of a Leibniz algebra \( q \). Let \( n \) a two-sided ideal of \( \mathfrak{f} \) and \( s \) a two-sided ideal of \( \mathfrak{f} \) such that \( n \cong \frac{\mathfrak{f}}{s} \) \( (s = \ker(\pi \circ \rho), \) where \( \pi : q \rightarrow \frac{\mathfrak{f}}{n} \) is the canonical projection). Then the following sequences are exact:

(a) \( 0 \rightarrow \frac{n \cap \gamma_{c+1}^{\text{Lie}}(f, s)}{\gamma_{c+1}^{\text{Lie}}(f, t)} \rightarrow M_c^{(c)}(q) \rightarrow M_c^{(c)}(\frac{\mathfrak{f}}{n}) \rightarrow \frac{n \cap \gamma_{c+1}^{\text{Lie}}(q, n)}{\gamma_{c+1}^{\text{Lie}}(q, n)} \rightarrow 0 \).

(b) \( M_c^{(c)}(q) \rightarrow M_c^{(c)}(\frac{\mathfrak{f}}{n}) \rightarrow \frac{n \cap \gamma_{c+1}^{\text{Lie}}(q, n)}{\gamma_{c+1}^{\text{Lie}}(q, n)} \rightarrow \frac{n}{\gamma_{c+1}^{\text{Lie}}(q, n)} \rightarrow \frac{q}{\gamma_{c+1}^{\text{Lie}}(q)} \rightarrow \frac{q}{n + \gamma_{c+1}^{\text{Lie}}(q)} \rightarrow 0 \).

(c) \( n \otimes^c q \rightarrow M_c^{(c)}(q) \rightarrow M_c^{(c)}(\frac{\mathfrak{f}}{n}) \rightarrow n \cap \gamma_{c+1}^{\text{Lie}}(q) \rightarrow 0 \), provided that \( 0 \rightarrow n \rightarrow q \rightarrow \frac{\mathfrak{f}}{n} \rightarrow 0 \) is a \( c \)-Lie-central extension. Here \( n \otimes^c q = n \otimes q \otimes \cdots \otimes q \).
**Proof.** (a) Consider the following diagram:

Define \( \Pi : \mathcal{M}^{(c)}_{\text{Lie}}(q) = \frac{\gamma_{c+1}^{\text{Lie}}(f)}{\gamma_{c+1}^{\text{Lie}}(f,v)} \to \frac{\eta^{\text{Lie}}_{c+1}(f,v)}{\gamma_{c+1}^{\text{Lie}}(f,v)} = \mathcal{M}^{(c)}_{\text{Lie}}(q,n) \) by \( \Pi(r + \gamma_{c+1}^{\text{Lie}}(f,v)) = r + \gamma_{c+1}^{\text{Lie}}(f,s) \). Obviously \( \text{Ker}(\Pi) = \frac{\gamma_{c+1}^{\text{Lie}}(f,s)}{\gamma_{c+1}^{\text{Lie}}(f,v)} \) and \( \text{Coker}(\Pi) = \frac{n \gamma_{c+1}^{\text{Lie}}(q,n)}{\gamma_{c+1}^{\text{Lie}}(q,n)} \) thanks to the following commutative diagram:

\[
\begin{array}{ccc}
\gamma_{c+1}^{\text{Lie}}(f,s) & \xrightarrow{\cap} & \gamma_{c+1}^{\text{Lie}}(f) \\
\downarrow & & \downarrow \\
\gamma_{c+1}^{\text{Lie}}(f,s) & \xrightarrow{\cap} & \gamma_{c+1}^{\text{Lie}}(f) \\
\downarrow & & \downarrow \\
\gamma_{c+1}^{\text{Lie}}(q,n) & \xrightarrow{\cap} & \gamma_{c+1}^{\text{Lie}}(q,n) \\
\end{array}
\]

(b) Combine statement (a) with the following diagram:
and have in mind the isomorphisms \( \frac{n+1}{c+1}(q) \cong \frac{n}{\gamma_{c+1}(q)} \cong \frac{n}{\gamma_{c+1}(q,n)} \).

(c) If \( n \) is \( c \)-Lie-central in \( q \), then (a) provides the exact sequence

\[
0 \rightarrow \gamma_{c+1}(f, g) \rightarrow M_{\text{Lie}}(q) \rightarrow \mathcal{M}_{\text{Lie}}(\frac{q}{n}) \rightarrow n \cap \gamma_{c+1}(q) \rightarrow 0
\]

(5)

Now consider \( \sigma : n \otimes q_{\text{Lie}} \rightarrow \gamma_{c+1}(f, s) \otimes q_{\text{Lie}} \) given by \( \sigma(n \otimes \gamma_{c+1}(f, r)) = \gamma_{c+1}(s, f)_{\text{Lie}} + \cdots + \gamma_{c+1}(f, r)_{\text{Lie}} \), where \( \rho(s) = n \) and \( \rho(f_i) = q_i \), \( 1 \leq i \leq c \). The condition \( \gamma_{c+1}(q, n) = 0 \) guarantees the well-definition of \( \sigma \). Moreover \( \sigma \) is a surjection, which completes the proof. ■  

**Corollary 4.2** Let \( n \) be a two-sided ideal of a finite-dimensional Leibniz algebra \( q \) together with the free presentations in diagram (5). Then

(a) \( \mathcal{M}_{\text{Lie}}(q) \) is finite-dimensional.

(b) \( \dim \left( \mathcal{M}_{\text{Lie}}(\frac{q}{n}) \right) \leq \dim \left( \mathcal{M}_{\text{Lie}}(q) \right) + \dim \left( \frac{n}{\gamma_{c+1}(q)} \right) \).

(c) \( \dim \left( \mathcal{M}_{\text{Lie}}(q) \right) + \dim \left( n \cap \gamma_{c+1}(q) \right) = \dim \left( \mathcal{M}_{\text{Lie}}(\frac{q}{n}) \right) + \dim \left( \frac{n}{\gamma_{c+1}(q, n)} \right) \).

(d) \( \dim \left( \mathcal{M}_{\text{Lie}}(q) \right) + \dim \left( n \cap \gamma_{c+1}(q) \right) = \dim \left( \mathcal{M}_{\text{Lie}}(\frac{q}{n}) \right) + \dim \left( \frac{n}{\gamma_{c+1}(f, r)} \right) \).

(e) \( \dim \left( \mathcal{M}_{\text{Lie}}(q) \right) + \dim \left( \gamma_{c+1}(q) \right) = \dim \left( \gamma_{c+1}(\frac{q}{n}) \right) + \dim \left( \frac{n}{\gamma_{c+1}(f, r)} \right) \).

(f) If \( \mathcal{M}_{\text{Lie}}(q) = 0 \), then \( \mathcal{M}_{\text{Lie}}(\frac{q}{n}) \cong \frac{n}{\gamma_{c+1}(q, n)} \).

(g) \( \dim \left( \mathcal{M}_{\text{Lie}}(q) \right) + \dim \left( n \cap \gamma_{c+1}(q) \right) \leq \dim \left( \mathcal{M}_{\text{Lie}}(\frac{q}{n}) \right) + \dim \left( n \otimes q_{\text{Lie}} \right) \), provided that \( 0 \rightarrow n \rightarrow q \rightarrow \frac{q}{n} \rightarrow 0 \) is a \( c \)-Lie-central extension.

**Proof.** (a) Straightforward since \( q \) is a finite-dimensional Leibniz algebra.

(b) From Proposition 4.1 (a), the exact sequence yields

\[
\dim \left( \mathcal{M}_{\text{Lie}}(\frac{q}{n}) \right) + \dim \left( \frac{n}{\gamma_{c+1}(q, n)} \right) = \dim \left( \mathcal{M}_{\text{Lie}}(q) \right) + \dim \left( \frac{n}{\gamma_{c+1}(f, r)} \right) \).
\]

The result follows.

(c) The result follows from the equation (6) above combined with the equality \( \dim \left( \frac{n}{\gamma_{c+1}(q, n)} \right) = \dim \left( n \cap \gamma_{c+1}(q) \right) - \dim \left( \gamma_{c+1}(q, n) \right) \).
(d) The result holds from (c), since from diagram \(4\) and by the isomorphism theorem we have

\[
\gamma_{c+1}^{\text{Lie}}(q, n) \cong \frac{\gamma_{c+1}^{\text{Lie}}(f, s)}{r \cap \gamma_{c+1}^{\text{Lie}}(f, s)} \cong \frac{\gamma_{c+1}^{\text{Lie}}(f, t)}{r \cap \gamma_{c+1}^{\text{Lie}}(f, s)}
\]

(e) Apply statement (d) to the particular case \(n = q\).

(f) Straightforward from Proposition 4.1 (a).

(g) Straightforward from exact sequence in Proposition 4.1 (c).

**Definition 4.3** A Lie-nilpotent Leibniz algebra \(q\) of class \(c\) is said to be of maximal Lie-class \(c\) if \(\dim \left( \frac{\gamma_{c+1}^{\text{Lie}}(q)}{\gamma_{c+1}^{\text{Lie}}(q)} \right) = 1\), for \(j = 2, 3, \ldots, c\) and \(\dim \left( \frac{q}{\gamma_{1}^{\text{Lie}}(q)} \right) = 2\).

**Remark 4.4** The absolute case of Definition 4.3 is that when the Lie-alization functor is substituted by the abelianization functor, is equivalent to the notion of filiform Leibniz algebra \(4\).

**Proposition 4.5** Let \(q\) be a Lie-nilpotent Leibniz algebra of class \(c\). If \(q\) is of maximal Lie-class \(c\), then \(Z_{i+1}^{\text{Lie}}(q) = \gamma_{c-i+1}^{\text{Lie}}(q)\), for \(0 \leq i \leq c\).

**Proof.** The statement is true for \(i \in \{0, c\}\) since \(Z_{0}^{\text{Lie}}(q) = 0 = \gamma_{c+1}^{\text{Lie}}(q)\), and \(Z_{c}^{\text{Lie}}(q) = q = \gamma_{1}^{\text{Lie}}(q)\).

By induction, assume that \(Z_{i}^{\text{Lie}}(q) = \gamma_{c-i+2}^{\text{Lie}}(q)\), for \(1 \leq i < c\). Then \(Z_{i}^{\text{Lie}}(q) = C_{q}^{\text{Lie}}(q, Z_{i-1}^{\text{Lie}}(q)) = C_{q}^{\text{Lie}}(q, \gamma_{c-i+2}^{\text{Lie}}(q))\). So \(\gamma_{c-i+1}^{\text{Lie}}(q) \subseteq Z_{i}^{\text{Lie}}(q)\). Now, it is easy to check that \(\frac{\gamma_{c-i+1}^{\text{Lie}}(q)}{\gamma_{c-i+1}^{\text{Lie}}(q)} \not\subseteq \frac{\gamma_{c-i+1}^{\text{Lie}}(q)}{\gamma_{c-i+1}^{\text{Lie}}(q)}\). Indeed, since \(\dim \left( \frac{\gamma_{c-i+1}^{\text{Lie}}(q)}{\gamma_{c-i+1}^{\text{Lie}}(q)} \right) \neq 0\), it follows that \(\gamma_{c-i+2}^{\text{Lie}}(q) \not\subseteq \gamma_{c-i+1}^{\text{Lie}}(q)\). So let \(x \in \gamma_{c-i+1}^{\text{Lie}}(q) \setminus \gamma_{c-i+2}^{\text{Lie}}(q)\), then \(x = [x_{0}, m_{0}]_{\text{Lie}}\), for some \(x_{0} \in \gamma_{c-i}^{\text{Lie}}(q)\) and \(m_{0} \in q\). Then \(x_{0} \notin Z_{i}^{\text{Lie}}(q)\), otherwise \(x = [x_{0}, m_{0}]_{\text{Lie}} \in Z_{i}^{\text{Lie}}(q) = \gamma_{c-i+2}^{\text{Lie}}(q)\) which is a contradiction. Since \(\dim \left( \frac{\gamma_{c-i+1}^{\text{Lie}}(q)}{\gamma_{c-i+1}^{\text{Lie}}(q)} \right) = 1\), it follows that \(\frac{\gamma_{c-i+1}^{\text{Lie}}(q)}{\gamma_{c-i+1}^{\text{Lie}}(q)} = 0\), hence \(Z_{i}^{\text{Lie}}(q) = \gamma_{c-i+1}^{\text{Lie}}(q)\).

**Corollary 4.6** Let \(q\) be a finite dimensional Lie-nilpotent Leibniz algebra of maximal Lie-class \(c + 1\), then

\[
\dim \left( \mathcal{M}_{\text{Lie}}^{(c)}(q) \right) \leq \dim \left( \mathcal{M}_{\text{Lie}}^{(c)} \left( \frac{q}{Z_{\text{Lie}}^{(q)}} \right) \right) + 2^{c} - 1
\]

**Proof.** Letting \(n := Z_{\text{Lie}}(q)\) in Corollary 4.2 (g), we have

\[
\dim \left( \mathcal{M}_{\text{Lie}}^{(c)}(q) \right) + \dim \left( Z_{\text{Lie}}(q) \cap \gamma_{c+1}^{\text{Lie}}(q) \right) \leq \dim \left( \mathcal{M}_{\text{Lie}}^{(c)} \left( \frac{q}{Z_{\text{Lie}}^{(q)}} \right) \right) + \dim \left( n \otimes \gamma_{\text{Lie}}^{(c)}(q) \right).
\]

Note that since \(q\) is Lie-nilpotent of maximal Lie-class \(c + 1\), it follows that \(Z_{\text{Lie}}(q) = Z_{1}^{\text{Lie}}(q) = \gamma_{c+1}^{\text{Lie}}(q), \gamma_{c+2}^{\text{Lie}}(q) = 0\), then \(\dim \left( Z_{\text{Lie}}(q) \right) = \dim \left( \frac{\gamma_{c+1}^{\text{Lie}}(q)}{\gamma_{c+2}^{\text{Lie}}(q)} \right) \right) = 1\) and \(\dim \left( q_{\text{Lie}} \right) = \dim \left( \frac{q}{\gamma_{c+1}^{\text{Lie}}(q)} \right) = 2\). Therefore \(\dim \left( Z_{\text{Lie}}(q) \cap \gamma_{c+1}^{\text{Lie}}(q) \right) = \dim \left( Z_{\text{Lie}}(q) \right) = 1\) and \(\dim \left( Z_{\text{Lie}}(q) \otimes \gamma_{\text{Lie}}^{(c)}(q) \right) = 2^{c}\). Then the result follows.
5 \textit{c-Lie-stem covers}

In this section we analyze the interplay between \textit{c-Lie}-stem covers and the \textit{c-nilpotent Schur Lie-multiplier}.

**Definition 5.1** A \textit{c-Lie-central extension} \(0 \to n \to g \xrightarrow{\pi} q \to 0\) is said to be \textit{c-Lie-stem extension} whenever \(n \subseteq \gamma_{c+1}^{\text{Lie}}(g)\).

In addition, if \(n\) is isomorphic to \(M^{(c)}_{\text{Lie}}(q)\), then the \textit{c-Lie-stem extension} is called a \textit{c-Lie-stem cover} of \(q\). In this case \(g\) is said to be a \textit{c-Lie-covering} of \(q\).

A \textit{Leibniz algebra} \(q\) is said to be \textit{Hopfian} is every surjective homomorphism \(q \twoheadrightarrow q\) is an isomorphism.

**Proposition 5.2** For a \textit{c-Lie-central extension} \(\pi : g \to q\), with \(n = \ker(\pi)\), the following statements are equivalent:

(a) \(\pi : g \to q\) is a \textit{c-Lie-stem extension}.

(b) The induced map \(n \to \frac{g}{\gamma_{c+1}^{\text{Lie}}(g)}\) is the zero map.

(c) \(\theta : M^{(c)}_{\text{Lie}}(q) \to n\) is an epimorphism.

(d) The following sequence \(n \otimes_{c} g_{\text{Lie}} \to M^{(c)}_{\text{Lie}}(g) \to M^{(c)}_{\text{Lie}}(q) \xrightarrow{\theta} n \to 0\) is exact.

(e) \(\frac{g}{\gamma_{c+1}^{\text{Lie}}(g)} \cong \frac{q}{\gamma_{c+1}^{\text{Lie}}(q)}\).

**Proof.** The equivalences between (a), (b), (c) and (d) follow from exact sequences in Proposition 4.1. The equivalence between (a) and (e) is a consequence of the following \(3 \times 3\) diagram:

```
\begin{array}{ccc}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
0 & n \cap \gamma_{c+1}^{\text{Lie}}(g) & \gamma_{c+1}^{\text{Lie}}(g) \\
\downarrow & \downarrow & \downarrow \\
0 & n & g \\
\downarrow & \downarrow & \downarrow \\
0 & 0 & q \\
\downarrow & \downarrow & \downarrow \\
0 & 0 & 0 \\
\end{array}
```

\(\gamma_{c+1}^{\text{Lie}}(g)\) \(\gamma_{c+1}^{\text{Lie}}(q)\)
Proposition 5.3  For a c-Lie-central extension \( \pi : g \to q \) the following statements are equivalent:

(a) \( \pi : g \to q \) is a c-Lie-stem cover.

(b) \( \frac{g}{\gamma_{c+1}(g)} \cong \frac{q}{\gamma_{c+1}(q)} \) and the induced map \( \mathcal{M}^{(c)}_{\text{Lie}}(g) \to \mathcal{M}^{(c)}_{\text{Lie}}(q) \) is the zero map.

Proof. This is a direct consequence of Proposition 5.2 (e) and Proposition 4.1. \( \blacksquare \)

Example 5.4

(a) The 2-Lie-central extension given in Example 3.3 (b) is a 2-Lie-stem extension, but the 2-Lie-central extension in Example 3.3 (a) is not a c-Lie-stem extension.

(b) Let \( 0 \to r \to f \xrightarrow{\rho} q \to 0 \) be a free presentation of a Leibniz algebra \( q \). Then \( \gamma_{c+1}^{\text{Lie}}(f, r) \) is a two-sided ideal of \( f \), \( \gamma_{c+1}^{\text{Lie}}(f, r) \subseteq r \) and the sequence

\[
0 \to \frac{r}{\gamma_{c+1}^{\text{Lie}}(f, r)} \to \frac{f}{\gamma_{c+1}^{\text{Lie}}(f, r)} \xrightarrow{\bar{\rho}} q \to 0
\]  

is a c-Lie-central extension (see Example 3.2 (b)). It has the property that the induced map \( \mathcal{M}^{(c)}_{\text{Lie}}(f/\gamma_{c+1}^{\text{Lie}}(f, r)) \to \mathcal{M}^{(c)}_{\text{Lie}}(q) \) is the zero map. This can be readily checked by using the isomorphism (7) for the given free presentation of \( q \) and the free presentation \( 0 \to \gamma_{c+1}^{\text{Lie}}(f, r) \to f \to f/\gamma_{c+1}^{\text{Lie}}(f, r) \to 0 \) of the Leibniz algebra \( f/\gamma_{c+1}^{\text{Lie}}(f, r) \). Moreover, we have the short exact sequence (c.f. the last row of 3 \times 3 diagram in the proof of Proposition 5.2)

\[
0 \to \frac{r}{r \cap \gamma_{c+1}^{\text{Lie}}(f)} \to \frac{f}{\gamma_{c+1}^{\text{Lie}}(f, r)} \to \frac{q}{\gamma_{c+1}^{\text{Lie}}(q)} \to 0.
\]  

(c) As a particular case of (b), consider the Leibniz algebra \( q = f/\gamma_{c+1}^{\text{Lie}}(f) \) for a free Leibniz algebra \( f \). Then \( r = \gamma_{c+1}^{\text{Lie}}(f) \) and the sequence (7) turns to

\[
0 \to \frac{\gamma_{c+1}^{\text{Lie}}(f)}{\gamma_{c+1}^{\text{Lie}}(f, r)} \to \frac{f}{\gamma_{c+1}^{\text{Lie}}(f, r)} \to \frac{f}{\gamma_{c+1}^{\text{Lie}}(f)} \to 0,
\]  

which, by (8), is a c-Lie-stem cover of the Leibniz algebra \( f/\gamma_{c+1}^{\text{Lie}}(f) \).

Lemma 5.5  Let \( 0 \to r \to f \xrightarrow{\rho} q \to 0 \) be a free presentation of a Leibniz algebra \( q \). Then the extension \( 0 \to m \to q^* \xrightarrow{\psi} q \to 0 \) is a c-Lie-stem cover of \( q \) if and only if there exists a two-sided ideal \( s \) of \( f \) such that

(a) \( q^* \cong f/s \) and \( m \cong r/s \);
(b) $r/\gamma_{c+1}(f, r) \cong M^{(c)}_{\text{Lie}}(q) \oplus s/\gamma_{c+1}(f, r)$.

**Proof.** Let $0 \to m \to q^* \xrightarrow{\psi} q \to 0$ be a $c$-Lie-stem cover of $q$. Then by Lemma 3.6, the identity map $q \to q$ induces a homomorphism $\beta : \frac{f}{\gamma_{c+1}(f, r)} \to q^*$ such that $\beta \left( \frac{r}{\gamma_{c+1}(f, r)} \right) \subseteq m$ and $\psi \circ \beta = \tilde{\rho}$ (see diagram (2)). Since $q^* = \text{Im} (\beta) + m$ and $m \subseteq \gamma_{c+1}(q^*) = \gamma_{c+1} \left( \text{Im} (\beta) \right) \subseteq \text{Im} (\beta)$, hence $\beta$ is a surjective homomorphism and $\beta \left( \frac{r}{\gamma_{c+1}(f, r)} \right) = m$.

Now, let $s$ be a two-sided ideal of $f$ such that $\text{Ker} (\beta) = \frac{s}{\gamma_{c+1}(f, r)}$. Then we have the exact sequence $0 \to \frac{s}{\gamma_{c+1}(f, r)} \to \frac{f}{\gamma_{c+1}(f, r)} \xrightarrow{\beta} q^* \to 0$ which induces the short exact sequence $0 \to \frac{s}{\gamma_{c+1}(f, r)} \to \frac{r}{\gamma_{c+1}(f, r)} \to m \to 0$. It follows from these two exact sequences and the third isomorphism theorem that $q^* \cong \frac{f}{\gamma_{c+1}(f, r)}/\frac{s}{\gamma_{c+1}(f, r)} \cong \frac{r}{s}$ and $m \cong \frac{r}{\gamma_{c+1}(f, r)}/\frac{s}{\gamma_{c+1}(f, r)} \cong \frac{r}{s}$. Moreover, $\frac{r}{\gamma_{c+1}(f, r)} \cong m \oplus \frac{s}{\gamma_{c+1}(f, r)}$ as $\mathbb{K}$-vector spaces, and thus $\frac{r}{\gamma_{c+1}(f, r)} \cong M^{(c)}_{\text{Lie}}(q) \oplus \frac{s}{\gamma_{c+1}(f, r)}$, since $0 \to m \to q^* \xrightarrow{\psi} q \to 0$ is a $c$-Lie-stem cover of $q$.

Conversely, suppose the existence of a two-sided ideal $s$ of $f$ satisfying (a) and (b). Then, $\frac{r}{m} \cong \frac{r}{s} \cong \frac{r}{t} \cong q$, and $M^{(c)}_{\text{Lie}}(q) \cong \frac{r}{\gamma_{c+1}(f, r)}/\frac{s}{\gamma_{c+1}(f, r)} \cong \frac{r}{s} \cong m$. Moreover $m \cong \frac{r}{s} \subseteq \frac{s}{\gamma_{c+1}(f, r)} \cong \frac{r}{\gamma_{c+1}(f, r)} \subseteq \gamma_{c+1}(q^*)$. Therefore the extension $0 \to m \to q^* \xrightarrow{\psi} q \to 0$ is a $c$-Lie-stem cover of $q$.

**Corollary 5.6** Any finite-dimensional Leibniz algebra has at least one $c$-Lie-covering.

**Proof.** Let $q$ be a finite dimensional Leibniz algebra, and let $0 \to r \to f \xrightarrow{\rho} q \to 0$ be a free presentation of $q$. Following the proof of Lemma 5.5, choose a two-sided ideal $s$ of $f$ such that $\frac{s}{\gamma_{c+1}(f, r)}$ is the complement of $M^{(c)}_{\text{Lie}}(q)$ in $\frac{r}{\gamma_{c+1}(f, r)}$. Then the extension $0 \to r/s \to f/s \to q \to 0$ is a $c$-Lie-stem cover of $q$.

**Theorem 5.7** Let $0 \to m \to h \xrightarrow{\theta} q \to 0$ be a $c$-Lie-stem extension of a finite-dimensional Leibniz algebra $q$. Then there exists a $c$-Lie-covering $q^*$ of $q$ such that $h$ is a quotient of $q^*$.

**Proof.** Let $0 \to r \to f \xrightarrow{\rho} q \to 0$ be a free presentation of $q$. Using Lemma 3.6, choose a two-sided ideal $s_0$ of $f$ such that $\text{Ker} (\beta) = \frac{s_0}{\gamma_{c+1}(f, r)}$. Since the surjective homomorphism $\beta$ is induced by the identity map, it follows that $\text{Ker} (\beta) = \text{Ker} (\beta_0)$, hence $s_0$ in fact is a two-sided ideal of $r$. Following the proof of Lemma 5.5 we therefore have that $\frac{s_0}{\gamma_{c+1}(f, r)}$ is the complement of $M^{(c)}_{\text{Lie}}(q)$ in $\frac{r}{\gamma_{c+1}(f, r)}$. So
(r \cap \gamma_{c+1}(f)) + s_0 = r and (r \cap \gamma_{c+1}(f)) \cap s_0 = \gamma_{c+1}(f, r). Now let s be a two-sided ideal of s_0 such that \frac{s}{\gamma_{c+1}(f, r)} is the complement of \frac{s_0 \cap \gamma_{c+1}(f)}{\gamma_{c+1}(f, r)}. So (s_0 \cap \gamma_{c+1}(f)) + s = s_0 and (s_0 \cap \gamma_{c+1}(f)) \cap s = \gamma_{c+1}(f, r). This implies that (r \cap \gamma_{c+1}(f)) + s = r and (r \cap \gamma_{c+1}(f)) \cap s = \gamma_{c+1}(f, r) since s \subseteq s_0 \subseteq r. Therefore r/\gamma_{c+1}(f, r) = M_{\text{Lie}}(q) \oplus s/\gamma_{c+1}(f, r). Hence q^* := f/s is a c-Lie-stem cover of q by Lemma 5.5. Clearly

\[
\frac{q^*}{s_0/s} \cong \frac{f/s_0}{\gamma_{c+1}(f, r)}/\frac{s_0}{\gamma_{c+1}(f, r)} = \frac{f}{\gamma_{c+1}(f, r)}/\text{Ker}(\beta) \cong \mathfrak{h}.
\]

Hence \mathfrak{h} is a quotient of q^*.

Lemma 5.8 Let q be a Leibniz algebra and

\[
\begin{array}{ccccccc}
0 & \to & n_1 & \to & g_1 & \to & q & \to 0 \\
\alpha \downarrow & & \beta \downarrow & & \gamma \downarrow & & \\
0 & \to & n_2 & \to & g_2 & \to & q & \to 0
\end{array}
\]

be a commutative diagram of short exact sequences of Leibniz algebras such that the bottom row is a c-Lie-stem extension. If the homomorphism \gamma is surjective, then \beta is a surjective homomorphism as well.

Proof. Obviously g_2 = \text{Im}(\beta) + n_2. Since n_2 \subseteq \gamma_{c+1}(g_2) and \gamma_{c+1}(g_2, n_2) = 0, then n_2 \subseteq \beta(\gamma_{c+1}(g_1)). Therefore g_2 \subseteq \text{Im}(\beta) + \beta(\gamma_{c+1}(g_1)), i.e. \beta is surjective.

Theorem 5.9 Let q be a Leibniz algebra and let 0 \to m_i \to h_i \to q \to 0, i = 1, 2, be two c-Lie-stem covers of q. If \eta : h_1 \to h_2 is a surjective homomorphism such that \eta(m_1) \subseteq m_2, then \eta is an isomorphism.

Proof. Let 0 \to r \to f \to q \to 0 be a free presentation of q. Then by Lemma 5.5 there exist two-sided ideals s_i of f, i = 1, 2, such that h_i \cong f/s_i, m_i \cong r/s_i and r/\gamma_{c+1}(f, r) \cong M_{\text{Lie}}(q) \oplus s_i/\gamma_{c+1}(f, r). It is therefore enough to prove that the surjective homomorphism \tilde{\eta} : f/s_1 \to f/s_2 induced by \eta is an isomorphism. Following the proof of Lemma 5.5 we have a surjective homomorphism \beta_2 : f/\gamma_{c+1}(f, r) \to h_2 such that \beta_2(f/s_1) = m_2 and \text{Ker}(\beta_2) = f/s_1. Since f is a free Leibniz algebra, there exists a homomorphism \tilde{\delta} : f \to f/s_1 such that the
The following diagram is commutative,

\[
\begin{array}{ccc}
\gamma_{c+1}(f) & \xrightarrow{\text{nat}} & f \\
\downarrow \delta & & \downarrow \eta \\
\delta & \xrightarrow{\beta_2} & f/\gamma_1 \\
\end{array}
\]

where \( \delta \) is a homomorphism induced by \( \bar{\delta} \). Since \( \theta_1 \circ \delta = \tilde{\gamma} \), then \( \delta \) is a surjective homomorphism by Lemma 5.8.

In addition, if \( s \) is a two-sided ideal of \( f \) such that \( \text{Ker}(\delta) = \gamma_{c+1}(f) \), then \( \gamma_{c+1}(f)/s \) is the complement of \( \mathcal{M}_{\text{Lie}}^c(q) \) in \( \gamma_{c+1}(f)/t \), which implies that \( (r \cap \gamma_{c+1}(f)) + s = r \). Now, one easily shows that \( s \subseteq s_2 \). Therefore \( s = s_2 \) by Lemma 5.5. Hence \( \text{Ker}(\delta) = \text{Ker}(\beta_2) \), which implies that \( \bar{\eta} \) is injective.

**Corollary 5.10** Let \( q \) be a nilpotent Leibniz algebra of class \( c \geq 1 \), then every c-Lie-covering of \( q \) is Hopfian.

**Proof.** Let \( 0 \to m \to q^* \xrightarrow{\psi} q \to 0 \) is a c-Lie-stem cover of \( q \). Then by Proposition 5.8, \( \gamma_{c+1}(q^*) \simeq \gamma_{c+1}(q) \simeq q \) since \( q \) is a nilpotent Leibniz algebra of class \( c \), hence \( 0 \to \gamma_{c+1}(q^*) \to q^* \to q \to 0 \) is a c-Lie-stem cover. Now let \( \eta : q^* \to q^* \) be a surjective homomorphism, then \( \eta(m) \subseteq \gamma_{c+1}(q^*) \) since \( m \subseteq \gamma_{c+1}(q^*) \). So \( \eta([q^*, q^*]_{\text{Lie}}) = [\eta(q^*), \eta(q^*)]_{\text{Lie}} = [q^*, q^*]_{\text{Lie}} \), and thus \( \eta(\gamma_{c+1}(q^*)) = \gamma_{c+1}(q^*) \). It follows now by Theorem 5.9 that \( \eta \) is an isomorphism, hence \( q^* \) is Hopfian.

**Theorem 5.11** Let \( 0 \to m_i \to h_i \xrightarrow{\theta_i} q \to 0 \), \( i = 1, 2 \), be two c-Lie-stem covers of a finite-dimensional Leibniz algebra \( q \). Then \( Z^L_{c+1}(h_1)/m_1 \simeq Z^L_{c+1}(h_2)/m_2 \).

**Proof.** Let \( 0 \to r \to \gamma_{c+1}(q) \to 0 \) be a free presentation of \( q \), and \( q^* \) a c-Lie-covering of \( q \), which exists by Corollary 5.9. Then by Lemma 5.5 there exists a two-sided ideal \( s \) of \( f \) such that \( r \simeq f/s \) and \( m \simeq r/s \), and \( r/\gamma_{c+1}(f, r) \simeq \mathcal{M}_{\text{Lie}}^c(q) \oplus s/\gamma_{c+1}(f, r) \). Now let \( t \) be a two-sided ideal of \( f \) such that \( Z^L_{c+1}(f/\gamma_{c+1}(f, r)) = t/\gamma_{c+1}(f, r). \) This implies that for all \( t \in t \) and \( f_i \in f \), \( 1 \leq i \leq c \), we have \( [[[t+s, f_i + s]_{\text{Lie}}, f_1 + s]_{\text{Lie}}, \ldots, f_c + s]_{\text{Lie}} = [[[t, f_1]_{\text{Lie}}, f_2]_{\text{Lie}}, \ldots, f_c]_{\text{Lie}} \subseteq s \subseteq \gamma_{c+1}(f, r) \). Indeed, let \( x+s \in Z^L_{c+1}(f/s) \). Then for all \( f_i \in f \), \( 1 \leq i \leq c \), we have \( [[[x, f_1]_{\text{Lie}}, f_2]_{\text{Lie}}, \ldots, f_c]_{\text{Lie}} \subseteq s \cap \gamma_{c+1}(f, r) = \gamma_{c+1}(f, r). \) So \( x \in Z^L_{c+1}(f, r) \) in \( Z^L_{c+1}(f/\gamma_{c+1}(f, r)) = t/\gamma_{c+1}(f, r), \) implying that \( x \in t. \)
Hence \( Z_{c+1}^{\text{Lie}}(f/s) = t/s \). So \( Z_{c+1}^{\text{Lie}}(q^*)/m \cong \frac{Z_{c+1}^{\text{Lie}}(f/s)}{t/s} \cong \frac{t/s}{r/s} \cong t/r \), and is therefore uniquely determined by the free presentation \( 0 \to r \to f \xrightarrow{f} q \to 0 \). □

**Theorem 5.12** Let \( q \) be a Lie-nilpotent Leibniz algebra of class at most \( k \geq 1 \) such that \( M_{\text{Lie}}^{(c)}(q) \neq 0 \), for some \( c > k \). Then \( q \) has not \( c \)-covering.

**Proof.** We proceed by contradiction. Let \( 0 \to m \to q^* \xrightarrow{q^*} q \to 0 \) be a \( c \)-Lie-stem cover of \( q \). Then \( m \subseteq \gamma_{c+1}^{\text{Lie}}(q^*) \) and \( m \cong M_{\text{Lie}}^{(c)}(q) \). Now since \( q \) is Lie-nilpotent of class \( k \), it follows by the proof of Corollary 5.10 that \( \gamma_{k+1}^{\text{Lie}}(q^*) \cong m \), and thus \( \gamma_{c+1}^{\text{Lie}}(q^*) \subseteq \gamma_{k+1}^{\text{Lie}}(q^*) \cong m \) since \( c > k \). So \( \gamma_{c+1}^{\text{Lie}}(q^*) = \gamma_{k+1}^{\text{Lie}}(q^*) = m \). We claim that \( \gamma_{c+1}^{\text{Lie}}(q^*) = 0 \). Indeed, we have by Proposition 3.4 that \( m \subseteq Z_{c}^{\text{Lie}}(q^*) \). Therefore

\[
[[[m, q^*]_{\text{Lie}}, q^*]_{\text{Lie}}, \ldots, q^*]_{\text{Lie}} = 0.
\]

So if \( c \geq 2k \), we have \( \gamma_{c+1}^{\text{Lie}}(q^*) \subseteq \gamma_{2k+1}^{\text{Lie}}(q^*) = 0 \). Otherwise, \( c < 2k \) and we have

\[
\gamma_{3k-c+1}^{\text{Lie}}(q^*) = [[[\gamma_{k+1}^{\text{Lie}}(q^*), q^*]_{\text{Lie}}, q^*]_{\text{Lie}}, \ldots, q^*]_{\text{Lie}}
\]

\[
= [[[\gamma_{c+1}^{\text{Lie}}(q^*), q^*]_{\text{Lie}}, q^*]_{\text{Lie}}, \ldots, q^*]_{\text{Lie}}
\]

\[
= \gamma_{2k+1}^{\text{Lie}}(q^*) = 0.
\]

Similarly, if \( c \geq \frac{3}{2}k \), we have \( \gamma_{c+1}^{\text{Lie}}(q^*) \subseteq \gamma_{3k-c+1}^{\text{Lie}}(q^*) = 0 \). Otherwise, \( c < \frac{3}{2}k \) and we have

\[
\gamma_{4k-2c+1}^{\text{Lie}}(q^*) = [[[\gamma_{k+1}^{\text{Lie}}(q^*), q^*]_{\text{Lie}}, q^*]_{\text{Lie}}, \ldots, q^*]_{\text{Lie}}
\]

\[
= [[[\gamma_{c+1}^{\text{Lie}}(q^*), q^*]_{\text{Lie}}, q^*]_{\text{Lie}}, \ldots, q^*]_{\text{Lie}}
\]

\[
= \gamma_{3k-c+1}^{\text{Lie}}(q^*) = 0.
\]

This process continues and stops when \( c \leq k \), yielding \( \gamma_{c+1}^{\text{Lie}}(q^*) = 0 \). This implies \( M_{\text{Lie}}^{(c)}(q) \cong m = 0 \). A contradiction. □
6 \: c\text{-Lie}-capability

In this section we introduce the \( c \)-Lie-characteristic ideal of a Leibniz algebra, which is used to study \( c \)-Lie-capability of Leibniz algebras.

**Definition 6.1** A Leibniz algebra \( q \) is said to be \( c \)-Lie-capable if there exists a Leibniz algebra \( h \) such that \( q \cong h/\mathcal{Z}_c^{\text{Lie}}(h) \).

We call \( c \)-Lie-characteristic ideal of a Leibniz algebra \( q \), denoted by \( \mathcal{Z}^*(q) \), to the smallest two-sided ideal \( s \) of \( q \) such that \( q/s \) is \( c \)-Lie-capable.

**Proposition 6.2** Let \( q \) be a Leibniz algebra, then \( \mathcal{Z}^*(q) \) is a two-sided ideal of \( q \) contained in \( \mathcal{Z}_c^{\text{Lie}}(q) \), and \( \mathcal{Z}^*(q)/\mathcal{Z}^*(q) = 0 \).

**Proof.** Since \( q/\mathcal{Z}_c^{\text{Lie}}(q) \) is \( c \)-Lie-capable and \( \mathcal{Z}^*(q) \) is the smallest two-sided ideal such that \( q/\mathcal{Z}^*(q) \) is \( c \)-Lie-capable, then \( \mathcal{Z}^*(q) \subseteq \mathcal{Z}_c^{\text{Lie}}(q) \).

\( q/\mathcal{Z}^*(q) \) is \( c \)-Lie-capable by definition, then \( q/\mathcal{Z}^*(q) \cong h/\mathcal{Z}_c^{\text{Lie}}(h) \) for some Leibniz algebra \( h \). Since \( \mathcal{Z}^*(q/\mathcal{Z}^*(q)) \) is the smallest two-sided ideal \( s \) such that \( q/\mathcal{Z}^*(q) \) is \( c \)-Lie-capable, this \( s \) should be the trivial one thanks to the above isomorphism.

\( \square \)

**Proposition 6.3** The two-sided ideal \( \mathcal{Z}^*(q) \) of a Leibniz algebra \( q \) is the intersection of all two-sided ideals \( f(Z_h^{\text{Lie}}(g)) \), where \( f : g \to q \) is a \( c \)-Lie-central extension.

**Proof.** Let \( A = \bigcap \{ f(Z_h^{\text{Lie}}(g)) \mid f : g \to q \text{ is a } c\text{-Lie-central extension}\} \) be. By Definition 6.1 \( q/\mathcal{Z}^*(q) \) is \( c \)-Lie-capable, i.e. there exists a Leibniz algebra \( h \) such that \( 0 \to \mathcal{Z}_c^{\text{Lie}}(h) \to h \xrightarrow{\theta} q/\mathcal{Z}^*(q) \to 0 \). Obviously this sequence is a \( c \)-Lie-central extension by Proposition 3.4.

Consider the Leibniz algebra \( m = \{(q, h) \in q \times h \mid \theta(h) = q + \mathcal{Z}^*(q)\} \) and let \( \phi : m \to q \) given by \( \phi(q, h) = q \). We claim that \( \phi \) is a surjective homomorphism with \( \text{Ker} (\phi) \subseteq Z_c^{\text{Lie}}(m) \) and \( \phi(Z_c^{\text{Lie}}(m)) \subseteq Z^*(q) \).

Indeed, for any \( q \in q \), consider \( q + Z^*(q) \), then there exists \( h \in h \) such that \( \theta(h) = q + Z^*(q) \). Hence \( \phi(q, h) = q \).

On the other hand, \( \text{Ker}(\phi) = \{(0, h) \in q \times h \mid \theta(h) \in Z^*(q)\} \). In order to show that \( \text{Ker}(\phi) \subseteq Z_c^{\text{Lie}}(m) \), it is enough to show that \( \gamma_{i+1}(m, \text{Ker}(\phi)) = 0 \) thanks to Proposition 3.4. And this fact holds because for any \((0, h) \in \text{Ker}(\phi), (q_i, h_i), 1 \leq i \leq c\), we have \( [(0, h), (q_1, h_1)]_\text{Lie}, (q_2, h_2)]_\text{Lie}, \ldots, (q_c, h_c)]_\text{Lie} = (0, [[h, h_1]]_\text{Lie}, h_2)_\text{Lie}, \ldots, h_c)_\text{Lie} \in (0, \text{Ker}(\theta)) = (0, Z_c^{\text{Lie}}(h)) \subseteq Z_c^{\text{Lie}}(q) \times Z_c^{\text{Lie}}(h) = Z_c^{\text{Lie}}(m).

Finally, for any \((q, h) \in Z_c^{\text{Lie}}(m)\), then \( \phi(q, h) = q \) with \( \theta(h) = q + Z^*(q) \), but \( h \in Z_c^{\text{Lie}}(h) = \text{Ker}(\theta) \), hence \( q \in Z^*(q) \).

Therefore we have showed that \( A \subseteq Z^*(q) \).
For the converse inclusion, consider the following commutative diagram associated to some $c$-Lie-central extension $0 \to n \to g \xrightarrow{f} q \to 0$:

\[
\begin{array}{c}
\xymatrix{
 n & Z_c^{\text{Lie}}(g) & Z_c^{\text{Lie}}(q) \\
 0 & g/Z_c^{\text{Lie}}(g) & q/Z_c^{\text{Lie}}(q)
}
\end{array}
\]

where $n \subseteq Z_c^{\text{Lie}}(g)$ by Proposition 3.4 and $f(Z_c^{\text{Lie}}(g)) = Z_c^{\text{Lie}}(q)$ by a standard induction.

Hence $q/f(Z_c^{\text{Lie}}(g)) \cong g/Z_c^{\text{Lie}}(g)$ means that $0 \to Z_c^{\text{Lie}}(g) \to g \to q/f(Z_c^{\text{Lie}}(g)) \to 0$ is a $c$-Lie-central extension, i.e. $q/f(Z_c^{\text{Lie}}(g))$ is $c$-Lie-capable. Consequently $Z^*(q) \subseteq f(Z_c^{\text{Lie}}(g))$, since $Z^*(q)$ is the smallest two-sided ideal satisfying this property.

So $Z^*(q) \subseteq f(Z_c^{\text{Lie}}(g))$ for any $c$-Lie-central extension $f : g \to q$, then $Z^*(q) \subseteq \bigcap \{ f(Z_c^{\text{Lie}}(g)) \mid f : g \to q \text{ is a } c\text{-Lie-central extension} \}$. ■

**Lemma 6.4** Let $0 \to r \to f \xrightarrow{\rho} q \to 0$ be a free presentation and $0 \to m \to h \xrightarrow{\theta} q \to 0$ be a $c$-Lie-central extension of a Leibniz algebra $q$, then $\overline{\rho}(Z_c^{\text{Lie}}(f/\gamma_{c+1}^{\text{Lie}}(f, r))) \subseteq \theta(Z_c^{\text{Lie}}(h))$, where $\overline{\rho}$ is the natural surjective homomorphism induced by $\rho$.

**Proof.** Since $f$ is a free Leibniz algebra, then there exists a homomorphism $\beta : f \to h$ such that $\theta \circ \beta = \rho$. Obviously $\beta(r) \subseteq m$ and $\beta(\gamma_{c+1}^{\text{Lie}}(f, r)) = 0$. Then we have the following commutative diagram:

\[
\begin{array}{c}
\xymatrix{
 \gamma_{c+1}^{\text{Lie}}(f) & \gamma_{c+1}^{\text{Lie}}(f, r) & 0 \\
 r & f & q \\
m & h & q
}
\end{array}
\]

where $\pi$ is induced by $\beta$. Having in mind that $h = \Ker(\theta) + \text{Im}(\beta)$, then it is an easy task to verify that $\pi(Z_c^{\text{Lie}}(f/\gamma_{c+1}^{\text{Lie}}(f, r))) \subseteq Z_c^{\text{Lie}}(h)$.

Since $\overline{\rho} \circ \rho r = \rho = \theta \circ \beta = \theta \circ \pi \circ \rho r$, then $\overline{\rho}(Z_c^{\text{Lie}}(f/\gamma_{c+1}^{\text{Lie}}(f, r))) = \theta(Z_c^{\text{Lie}}(f/\gamma_{c+1}^{\text{Lie}}(f, r))) \subseteq \theta(Z_c^{\text{Lie}}(h))$. ■
Corollary 6.5 Let \( 0 \to r \to f \xrightarrow{\rho} q \to 0 \) be a free presentation of a Leibniz algebra \( q \), then \( \overline{\rho}(Z_c^{\text{Lie}}(f/\gamma_{c+1}(f,r))) = Z^*(q) \).

Proof. From the \( c \)-Lie-central extension \( 0 \to \frac{r}{\gamma_{c+1}(f,r)} \to \frac{f}{\gamma_{c+1}(f,r)} \xrightarrow{\overline{\rho}} q \to 0 \) in Example 5.4 (b), it follows that \( Z^*(q) \subseteq \overline{\rho}(Z_c^{\text{Lie}}(f/\gamma_{c+1}(f,r))) \) by Proposition 6.3.

By Lemma 6.3, \( \overline{\rho}(Z_c^{\text{Lie}}(f/\gamma_{c+1}(f,r))) \subseteq \overline{\theta}(Z_c^{\text{Lie}}(h)) \), for any \( c \)-Lie-central extension \( \theta : h \to q \), then \( \overline{\rho}(Z_c^{\text{Lie}}(f/\gamma_{c+1}(f,r))) \subseteq \bigcap \overline{\theta}(Z_c^{\text{Lie}}(h)) = Z^*(q) \).

Corollary 6.6 \( Z^*(q) = 0 \) if and only if \( q \) is a \( c \)-Lie-capable Leibniz algebra.

Proof. If \( q \) is a \( c \)-Lie-capable Leibniz algebra, then there exists a \( c \)-Lie-central extension \( 0 \to Z_c^{\text{Lie}}(h) \to h \xrightarrow{\rho} q \to 0 \); now Proposition 6.3 implies that \( Z^*(q) \subseteq f(Z_c^{\text{Lie}}(h)) = 0 \).

Conversely, if \( Z^*(q) = 0 \), for any free presentation \( 0 \to r \to f \xrightarrow{\rho} q \to 0 \), Corollary 6.5 implies that \( \overline{\rho}(Z_c^{\text{Lie}}(f/\gamma_{c+1}(f,r))) = 0 \), i.e. \( Z_c^{\text{Lie}}(f/\gamma_{c+1}(f,r)) \subseteq \text{Ker}(\overline{\rho}) = r/\gamma_{c+1}(f,r) \).

Moreover \( 0 \to \frac{r}{\gamma_{c+1}(f,r)} \to \frac{f}{\gamma_{c+1}(f,r)} \xrightarrow{\overline{\rho}} q \to 0 \) is a \( c \)-Lie-central extension, then \( r/\gamma_{c+1}(f,r) \subseteq Z_c^{\text{Lie}}(f/\gamma_{c+1}(f,r)) \) by Proposition 3.3.

Thus \( 0 \to Z_c^{\text{Lie}}(f/\gamma_{c+1}(f,r)) \to \frac{f}{\gamma_{c+1}(f,r)} \xrightarrow{\overline{\rho}} q \to 0 \) is a \( c \)-Lie-central extension, i.e. \( q \) is \( c \)-Lie-capable.

Theorem 6.7 Let \( 0 \to m \to h \xrightarrow{\psi} q \to 0 \) be a \( c \)-Lie-stem cover of a Leibniz algebra \( q \). Then \( \psi(Z_c^{\text{Lie}}(h)) = Z^*(q) \).

Proof. Let \( 0 \to r \to f \xrightarrow{\rho} q \to 0 \) be a free presentation of \( q \). Then by Lemma 6.3 there exists a two-sided ideal \( s \) of \( f \) such that \( h \cong f/s \) and \( m \cong r/s \), and \( r/\gamma_{c+1}(f,r) \cong \mathcal{M}_{\gamma_{c+1}(f,r)}^c(q) \oplus s/\gamma_{c+1}(f,r) \).

Now let \( t \) be a two-sided ideal of \( f \) such that \( Z_c^{\text{Lie}}(h) = Z_c^{\text{Lie}}(f/s) = t/s \). We claim that \( Z_c^{\text{Lie}}(\frac{f}{\gamma_{c+1}(f,r)}) = \frac{t}{\gamma_{c+1}(f,r)} \).

Indeed, let \( x + \gamma_{c+1}^{\text{Lie}}(f,r) \in Z_c^{\text{Lie}}(\frac{f}{\gamma_{c+1}^{\text{Lie}}(f,r)}) \). Then for all \( f_i \in f, 1 \leq i \leq c \), we have \( [[[x,f_i]^{\text{Lie}}, f_2^{\text{Lie}}, \ldots, f_c^{\text{Lie}}]^{\text{Lie}}(f,r) \subseteq s \), implying that \( x + s \in Z_c^{\text{Lie}}(f/s) = t/s \). So \( x \in t \), and thus \( x + \gamma_{c+1}^{\text{Lie}}(f,r) \in t/\gamma_{c+1}^{\text{Lie}}(f,r) \).

Conversely, as \( Z_c^{\text{Lie}}(f/s) = t/s \), we have for all \( t \in t \) and \( f_i \in f, 1 \leq i \leq c \), that \( [[[t + s, f_1 + s]^{\text{Lie}}, f_2 + s]^{\text{Lie}}, \ldots, f_c + s]^{\text{Lie}} = 0 \), then \( [[[t, f_1]^{\text{Lie}}, f_2]^{\text{Lie}}, \ldots, f_c]^{\text{Lie}} \in s \).

So \( \gamma_{c+1}^{\text{Lie}}(f,t) \subseteq s/\gamma_{c+1}^{\text{Lie}}(f,r) \subseteq \gamma_{c+1}^{\text{Lie}}(f,r) \), implying that \( \frac{1}{\gamma_{c+1}^{\text{Lie}}(f,r)} \subseteq Z_c^{\text{Lie}}(\frac{f}{\gamma_{c+1}^{\text{Lie}}(f,r)}) \).

We now have (here we use similar notations to diagram 10)

\[
\psi(Z_c^{\text{Lie}}(h)) = \psi(t/s) = \rho(t) = \overline{\rho}(t/\gamma_{c+1}^{\text{Lie}}(f,r)) = \overline{\rho}(Z_c^{\text{Lie}}(\frac{f}{\gamma_{c+1}^{\text{Lie}}(f,r)})) = Z^*(q).
\]

The last equality holds thanks to Corollary 6.5.
Theorem 6.8 Let \( n \) be a c-Lie-central two-sided ideal of a Leibniz algebra \( q \). Then the following statements are equivalent:

(a) \( n \cap \gamma_{c+1}^{\text{Lie}}(q) \cong \frac{\mathcal{M}_{\text{Lie}}^{(c)}(q/n)}{\mathcal{M}_{\text{Lie}}^{(c)}(q)} \).

(b) \( n \subseteq Z^*(q) \).

(c) The natural map \( \mathcal{M}_{\text{Lie}}^{(c)}(q) \to \mathcal{M}_{\text{Lie}}^{(c)}(q/n) \) is injective.

Proof. The equivalence between statements (a) and (c) directly follows from Proposition 4.1 (c).

For the equivalence between statements (b) and (c), consider the free presentations in diagram [3]. By exact sequence [3], we only need to prove that \( \gamma_{c+1}^{\text{Lie}}(f, s) = \gamma_{c+1}^{\text{Lie}}(f, r) \) if and only if \( n \subseteq Z^*(q) \).

Set \( f = \frac{r}{\gamma_{c+1}^{\text{Lie}}(f, s)} \), then \( \gamma_{c+1}^{\text{Lie}}(f, s) = \gamma_{c+1}^{\text{Lie}}(f, r) \) if and only if \( \bar{s} \subseteq Z_c^{\text{Lie}}(f) \).

By Corollary 6.5, \( Z^*(q) = \mathcal{F}(Z_c^{\text{Lie}}(f)) \), consequently, we obtain that \( \mathcal{F}(\bar{s}) \subseteq Z^*(q) \) if and only if \( \bar{s} \subseteq Z_c^{\text{Lie}}(f) \), but \( \mathcal{F}(\bar{s}) = \rho(\bar{s}) = n \), which completes the proof.

Corollary 6.9 Any Leibniz algebra \( q \) is c-Lie-capable if and only if the natural map \( \mathcal{M}_{\text{Lie}}^{(c)}(q) \to \mathcal{M}_{\text{Lie}}^{(c)}(q/\langle x \rangle) \) has non-trivial kernel for all non-zero element \( x \in Z_c^{\text{Lie}}(q) \).

Proof. Assume that Ker \( \sigma_x : \mathcal{M}_{\text{Lie}}^{(c)}(q) \to \mathcal{M}_{\text{Lie}}^{(c)}(q/\langle x \rangle) \) = 0 for any non-zero element \( x \in Z_c^{\text{Lie}}(q) \). By Theorem 6.8 \( \sigma_x \) injective if and only if \( \langle x \rangle \subseteq Z^*(q) \), so \( Z^*(q) \neq 0 \), i.e. \( q \) is not c-Lie-capable by Corollary 6.6.

For every non-zero element \( x \in Z_c^{\text{Lie}}(q) \), we have \( 0 \neq \langle x \rangle \nsubseteq Z^*(q) = 0 \), then \( \sigma_x \) cannot be an injective homomorphism.

Proposition 6.10 Let \( n \) be a c-Lie-central two-sided ideal of a Leibniz algebra \( q \). Then \( n \subseteq Z^*(q) \) if and only if the natural surjection \( q \to q/n \) induces an isomorphism \( \gamma_{c+1}^{\text{Lie}}(q) \cong \gamma_{c+1}^{\text{Lie}}(q/n) \).

Proof. By the proof of Theorem 6.8 \( \gamma_{c+1}^{\text{Lie}}(f, s) = \gamma_{c+1}^{\text{Lie}}(f, r) \) if and only if \( n \subseteq Z^*(q) \).

From diagram [3], the kernel of the induced surjective homomorphism \( \gamma_{c+1}^{\text{Lie}}(f) = \gamma_{c+1}^{\text{Lie}}(q/n) \) is \( \gamma_{c+1}^{\text{Lie}}(f, s) \), thus the proposition is obvious.

Corollary 6.11 Let \( n \) be a two-sided ideal of a finite-dimensional Leibniz algebra \( q \) such that \( n \subseteq Z_c^{\text{Lie}}(q) \). Then \( n \subseteq Z^*(q) \) if and only dim \( (\mathcal{M}_{\text{Lie}}^{(c)}(q/n)) = \end{align*} \)

\[ \dim (\mathcal{M}_{\text{Lie}}^{(c)}(q)) + \dim (n \cap \gamma_{c+1}^{\text{Lie}}(q)). \]

Proof. By the proof of Theorem 6.8 and Corollary 4.2 (d).
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References

[1] M. Araskhan: *The dimension of the c-nilpotent multiplier* J. Algebra 386 (2013), 105–112.

[2] M. Araskhan: *On the c-covers and a special ideal of Lie algebras*, Iran. J. Sci. Technol. Trans. A Sci. 40 (3) (2016), 165–169.

[3] M. Araskhan and M. R. Rismanchian: *Dimension of the c-nilpotent multiplier of Lie algebras*, Proc. Indian Acad. Sci. Math. Sci. 126 (3) (2016), 353–357.

[4] Sh.A. Ayupov and B. A. Omirov: *On some classes of nilpotent Leibniz algebras*, (Russian) Sibirsk. Mat. Zh., 42 (1) (2001), 18–29; translation in Siberian Math. J., 42 (1) (2001), 15–24.

[5] R. Baer: *Representations of groups as quotient groups. I*, Trans. Amer. Math. Soc. 58 (1945), 295–347.

[6] R. Baer: *Representations of groups as quotient groups. II. Minimal central chains of a group*, Trans. Amer. Math. Soc. 58 (1945), 348–389.

[7] R. Baer: *Representations of groups as quotient groups. III. Invariants of classes of related representations*, Trans. Amer. Math. Soc. 58 (1945), 390–419.

[8] F. Borceaux and D. Bourn: *Mal’cev, protomodular, homological and semi-abelian categories*, Mathematics and its Applications vol. 566, Kluwer Academic Publishers, 2004.

[9] G. R. Biyogmam and J. M. Casas: *On Lie-isoclinic Leibniz algebras*, J. Algebra (2017) (in press). DOI: [http://dx.doi.org/10.1016/j.jalgebra.2017.01.034](http://dx.doi.org/10.1016/j.jalgebra.2017.01.034)

[10] J. M. Casas, T. Datuashvili and M. Ladra: *Universal strict general actors and actors in categories of interest*, Appl. Categor. Struct. 18 (1) (2010), 85–114.

[11] J. M. Casas and E. Khmaladze: *On Lie-central extensions of Leibniz algebras*, RACSAM 111 (1) (2017), 39–56.

[12] J. M. Casas and M. A. Insua: *The Schur Lie-multiplier of Leibniz algebras*, arXiv: 1703.07148 (2017).
[13] J. M. Casas and T. Van der Linden: A relative theory of universal central extensions, Pré-Publicações do Departamento de Matemática, Universidade de Coimbra Preprint Number 09- (2009).

[14] J. M. Casas and T. Van der Linden: Universal central extensions in semi-abelian categories, Appl. Categor. Struct. 22 (1) (2014), 253–268.

[15] T. Everaert and T. Van der Linden: Baer invariants in semi-abelian categories I: General theory, Theory Appl. Categ. 12 (1) (2004), 1–33.

[16] A. Fröhlich: Baer-invariants of algebras, Trans. Amer. Math. Soc. 109 (1963), 221–244.

[17] J. Furtado-Coelho: Homology and generalized Baer invariants, J. Algebra 40 (1976), 596–609.

[18] P. J. Higgins: Groups with multiple operators, Proc. London Math. Soc. (3) 6 (1956), 366–416.

[19] S. A. Huq: Commutator, nilpotency, and solvability in categories, Quart. J. Math. Oxford Ser. (2) 19 (1968), 363–389.

[20] G. Janelidze and G. M. Kelly: Galois theory and a general notion of central extension, J. Pure Appl. Algebra 97 (1994), 135–161.

[21] G. Janelidze, L. Márki and W. Tholen: Semi-abelian categories, J. Pure Appl. Algebra 168 (2002), 367–386.

[22] J.-L. Loday: Cyclic homology, Grundl. Math. Wiss. Bd. 301, Springer (1992).

[23] J.-L. Loday: Une version non commutative des algèbres de Lie: les algèbres de Leibniz, L’Enseignement Mathématique 39 (1993), 269–292.

[24] J.-L. Loday and T. Pirashvili: Universal enveloping algebras of Leibniz algebras and (co)homology, Math. Ann. 296 (1993), 139–158.

[25] A. S.-T. Lue: Baer-invariants and extensions relative to a variety, Math. Proc. Cambridge Philos. Soc. 63 (1967), 569–578.

[26] M. R. Rismanchian and M. Araskan: Some properties of the c-nilpotent multiplier and c-covers of Lie algebras, Algebra Colloq. 21 (4) (2014), 421–426.

[27] Z. Riyahi and A. R. Salemkar: A remark on the Schur multiplier of nilpotent Lie algebras, J. Algebra 438 (2015), 1–6.
[28] A. Sadeghieh and M. Araskhan: Verification of some properties of the c-nilpotent multiplier in Lie algebras, J. Gen. Lie Theory Appl. 9 (2) (2015), 3 pp.

[29] A. R. Salemkar, B. Edalatzadeh and M. Araskhan: Some inequalities for the dimension of the c-nilpotent multiplier of Lie algebras, J. Algebra 322 (5) (2009), 1575–1585.

[30] A. R. Salemkar and Z. Riyahi: Some properties of the c-nilpotent multiplier of Lie algebras, J. Algebra 370 (2012), 320–325.

[31] I. Schur: Über die darstellung der endlichen gruppen durch gebrochen lineare Substitutionen, J. Reine Angew. Math. 127 (1904), 20–50.