On factorisations of complete graphs into circulant graphs and the Oberwolfach Problem

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Abstract

Various results on factorisations of complete graphs into circulant graphs and on 2-factorisations of these circulant graphs are proved. As a consequence, a number of new results on the Oberwolfach Problem are obtained. For example, a complete solution to the Oberwolfach Problem is given for every 2-regular graph of order 2p where \( p \equiv 5 \, (\text{mod} \, 8) \) is prime.

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1 Introduction

The Oberwolfach problem was posed by Ringel in the 1960s and is first mentioned in [16]. It concerns graph factorisations. A factor of a graph is a spanning subgraph and a factorisation is a decomposition into edge-disjoint factors. A factor that is regular of degree \( k \) is called a \( k \)-factor. If each factor of a factorisation is a \( k \)-factor, then the factorisation is called a \( k \)-factorisation, and if each factor is isomorphic to a given graph \( F \), then we say it is a factorisation into \( F \).

Let \( F \) be an arbitrary 2-regular graph and let \( n \) be the order of \( F \). If \( n \) is odd, then the Oberwolfach Problem \( \text{OP}(F) \) asks for a 2-factorisation of \( K_n \) into \( F \), and if \( n \) is even, then \( \text{OP}(F) \)
asks for a 2-factorisation of $K_n - I$ into $F$, where $K_n - I$ denotes the graph obtained from $K_n$ by removing the edges of a 1-factor.

The Oberwolfach Problem has been solved completely when $F$ consists of isomorphic components \cite{1, 3, 18}, when $F$ has exactly two components \cite{29}, when $F$ is bipartite \cite{3, 17} and in numerous special cases. See \cite{7} for a survey of results up to 2006. It is known that there is no solution to OP($F$) for $F \in \{C_3 \cup C_3, C_4 \cup C_5, C_3 \cup C_3 \cup C_5, C_3 \cup C_3 \cup C_3 \cup C_3\}$, but a solution exists for every other 2-regular graph of order at most 40 \cite{13}.

In \cite{8}, it was shown that the Oberwolfach Problem has a solution for every 2-regular graph of order $2p$ where $p$ is any of the infinitely many primes congruent to 5 (mod 24), and for every 2-regular graph whose order is in an infinite family of primes congruent to 1 (mod 16). In this paper we extend these results as follows. We show that OP($F$) has a solution for every 2-regular graph of order $2p$ where $p$ is any prime congruent to 5 (mod 8) (see Theorem \ref{thm:2p}), and we obtain solutions to OP($F$) for broad classes of 2-regular graphs in many other cases (see Theorems \ref{thm:bipartite} and \ref{thm:other}). We also obtain results on the generalisation of the Oberwolfach Problem to factorisations of complete multigraphs into isomorphic 2-factors (see Theorem \ref{thm:multigraph}). Our results are obtained by constructing various factorisations of complete graphs into circulant graphs in Section \ref{sec:circulant} and then showing in Section \ref{sec:main_results} that these circulant graphs can themselves be factored into isomorphic 2-regular graphs in a wide variety of cases.

## 2 Factorising complete graphs into circulant graphs

Let $G = (G, \cdot)$ be a finite group with identity $e$ and let $S$ be a subset of $G$ such that $e \notin S$ and $s \in S$ implies $s^{-1} \in S$. The Cayley graph on $G$ with connection set $S$, denoted Cay($G; S$), has the elements of $G$ as its vertices and $g$ is adjacent to $g \cdot s$ for each $s \in S$ and each $g \in G$. A Cayley graph on a cyclic group is called a circulant graph. We use the standard notation of $\mathbb{Z}_n$ for the ring of integers modulo $n$, and we use $\mathbb{Z}_n^*$ for the multiplicative group of units modulo $n$.

In this section we consider factorisations of $K_n$ for $n$ odd (in Section \ref{subsec:odd}) and of $K_n - I$ for $n$ even (in Section \ref{subsec:even}) into circulant graphs. A 2-regular graph is a circulant if and only if its components are all isomorphic. Thus, for each 2-regular circulant graph $F$, there exists a factorisation of $K_n$ (if $F$ has odd order) or of $K_n - I$ (if $F$ has even order) into $F$; except that there is no such factorisation when $F \in \{C_3 \cup C_3, C_3 \cup C_3 \cup C_3 \cup C_3\}$. Considerably less is
known for factorisations into circulant graphs of degree greater than 2. Some factorisations into \( \text{Cay}(\mathbb{Z}_n; \pm\{1, 2\}) \) and \( \text{Cay}(\mathbb{Z}_n; \pm\{1, 2, 3, 4\}) \) are given in \cite{4} and \cite{8} respectively, and some further results, including results on self-complementary and almost self-complementary circulant graphs, appear in \cite{2, 14, 15, 26}.

2.1 Factorising complete graphs of odd order

In this subsection we will construct factorisations of complete graphs of odd order into isomorphic circulant graphs by finding certain partitions of cyclic groups. Problems concerning such partitions have been well-studied, for example see \cite{28}, and existing results overlap with some of the results in this subsection. In particular, Theorem 3 below is a consequence of Lemma 3.1 of \cite{24}.

**Lemma 1** Let \( s \) be an integer, let \( p \equiv 1 \pmod{2s} \) be prime, and let \( S = \pm\{d_1, d_2, \ldots, d_s\} \subseteq \mathbb{Z}_p^* \). Further, suppose \( a \) and \( b \) are integers such that \( 2abs = p - 1 \), let \( G = (\mathbb{Z}_p^*)^b \), and let \( H = (\mathbb{Z}_p^*)^{bs} \).

If \( d_1, d_2, \ldots, d_s \) represent the \( s \) distinct cosets of \( G/H \), then there exists a \( 2s \)-factorisation of \( K_p \) into \( \text{Cay}(\mathbb{Z}_p; S) \).

**Proof** For each \( x \in \mathbb{Z}_p \) let \( xS = \{xy : y \in S\} \). Since \( p \) is prime, \( \text{Cay}(\mathbb{Z}_p; xS) \cong \text{Cay}(\mathbb{Z}_p; S) \) for any \( x \in \mathbb{Z}_p \setminus \{0\} \). If there is a partition of \( \mathbb{Z}_p^* \) into sets \( x_1S, x_2S, \ldots, x_{ab}S \) where \( x_i \in \mathbb{Z}_p \setminus \{0\} \) for \( i = 1, 2, \ldots, ab \), then \( \{\text{Cay}(\mathbb{Z}_p; x_iS) : i = 1, 2, \ldots, ab\} \) is the required \( 2s \)-factorisation of \( K_p \). We now present such a partition.

Let \( \omega \) be a generator of \( \mathbb{Z}_p^* \). Thus, \( H = \omega_0, \omega^{bs}, \omega^{2bs}, \ldots, \omega^{(2a-1)bs} \), and \( \omega^{ab} = -1 \in H \). Let \( A = \omega_0, \omega^{bs}, \omega^{2bs}, \ldots, \omega^{(a-1)bs} \), so that \( H = A \cup -A \) (\( A \) is a set of representatives for the cosets in \( H \) of the order 2 subgroup of \( H \)). Since \( d_1, d_2, \ldots, d_s \) represent distinct cosets of \( G/H \), it is easy to see that \( \{xS : x \in A\} \) is a partition of \( G \). Thus, if \( B \) is a set of representatives for the cosets of \( \mathbb{Z}_p^*/G \), then \( \{xyS : x \in A, y \in B\} \) is the required partition of \( \mathbb{Z}_p^* \). \( \square \)

Note that upon putting \( s = 1 \) in Lemma 1 we obtain the Hamilton decomposition

\[
\{\text{Cay}(\mathbb{Z}_p; \{\pm 1\}), \text{Cay}(\mathbb{Z}_p; \{\pm 2\}), \ldots, \text{Cay}(\mathbb{Z}_p; \{\pm \frac{p-1}{2}\})\}
\]

of \( K_p \). We will be mostly interested in applications of Lemma 1 where the connection set \( S \) is \( \pm\{1, 2\}, \pm\{1, 2, 3\}, \pm\{1, 3, 4\} \) or \( \pm\{1, 2, 3, 4\} \). The factorisations given by Lemma 1 have the property that each factor is invariant under the action of \( \mathbb{Z}_p \). It is worth mentioning that for \( S \in \{\pm\{1, 2\}, \pm\{1, 2, 3\}, \pm\{1, 3, 4\}, \pm\{1, 2, 3, 4\}\} \), the construction given in Lemma 1 yields every
2s-factorisation of $K_p$ into $\text{Cay}(\mathbb{Z}_p; S)$ with this property. This follows from the results in \cite{21} and \cite{22}, together with Turner’s result \cite{30} that for $p$ prime $\text{Cay}(\mathbb{Z}_p; S) \cong \text{Cay}(\mathbb{Z}_p; S')$ if and only if there exists an $\alpha \in \mathbb{Z}_p^*$ such that $S' = \alpha S$.

Theorem 2 If $p \equiv 1 \pmod{4}$ is prime and 4 divides the order of $k$ in $\mathbb{Z}_p^*$, then there is a factorisation of $K_p$ into $\text{Cay}(\mathbb{Z}_p; \pm\{1, k\})$.

Proof Apply Lemma 1 with $S = \pm\{1, k\}$ taking $G$ to be the subgroup of $\mathbb{Z}_p^*$ generated by $k$, and $H$ to be the index 2 subgroup of $G$. \qed

Theorem 3 If $p \equiv 1 \pmod{6}$ is prime such that $2, 3 \notin (\mathbb{Z}_p^*)^3$ and $6 \in (\mathbb{Z}_p^*)^3$, then there is a factorisation of $K_p$ into $\text{Cay}(\mathbb{Z}_p; \pm\{1, 2, 3\})$.

Proof It follows from $2, 3 \notin (\mathbb{Z}_p^*)^3$ and $6 \in (\mathbb{Z}_p^*)^3$ that 1, 2 and 3 represent the three cosets of $\mathbb{Z}_p^*/(\mathbb{Z}_p^*)^3$. Thus, we obtain the required factorisation by applying Lemma 1 with $b = 1$. \qed

Theorem 4 If $p \equiv 1 \pmod{6}$ is prime such that $2, 3, 6 \notin (\mathbb{Z}_p^*)^3$, then there is a factorisation of $K_p$ into $\text{Cay}(\mathbb{Z}_p; \pm\{1, 3, 4\})$.

Proof It follows from $2, 3, 6 \notin (\mathbb{Z}_p^*)^3$ that 1, 3 and 4 represent the three cosets of $\mathbb{Z}_p^*/(\mathbb{Z}_p^*)^3$. Thus, we obtain the required factorisation by applying Lemma 1 with $b = 1$. \qed

The primes less than 1000 to which Theorem 3 applies are

$$7, 37, 139, 163, 181, 241, 313, 337, 349, 379, 409, 421, 541, 571, 607, 631, 751, 859, 877, 937,$$

and the primes less than 1000 to which Theorem 4 applies are

$$13, 19, 79, 97, 199, 211, 331, 373, 463, 487, 673, 709, 769, 823, 829, 883, 907.$$

In the next theorem we show that there are infinitely many primes to which Theorem 3 applies, and also infinitely many primes to which Theorem 4 applies.

Theorem 5 There are infinitely many values of $p$ such that $p$ is prime, $p \equiv 1 \pmod{6}$, $2, 3 \notin (\mathbb{Z}_p^*)^3$ and $6 \in (\mathbb{Z}_p^*)^3$, and there are infinitely many values of $p$ such that $p$ is prime, $p \equiv 1 \pmod{6}$ and $2, 3, 6 \notin (\mathbb{Z}_p^*)^3$.  

4
Proof Assume $p \equiv 1 \pmod{6}$. Let $\mathbb{F}_p$ be the field with $p$ elements. We use standard definitions and results from algebraic number theory, as found in [20]. The result essentially follows from the Chebotarev Density Theorem.

Let $\omega$ be a primitive cube root of unity, $\lambda = \sqrt[3]{2}$ be a cube root of 2 and $\rho = \sqrt[3]{3}$ a cube root of 3. Consider the following tower of fields:

$$M = \mathbb{Q}(\omega, \lambda, \rho) \supset L = \mathbb{Q}(\omega, \lambda) \supset K = \mathbb{Q}(\omega) \supset \mathbb{Q}.$$ 

Let $\mathcal{O}_K$, $\mathcal{O}_L$ denote the rings of integers of $K$ and $L$ respectively. We may ignore the finitely many ramified primes. Thus let $p$ be a prime number, sufficiently large that it is unramified in $M$, let $\mathfrak{p}$ be a prime in $K$ extending $p$ and $\mathfrak{q}$ a prime in $L$ extending $\mathfrak{p}$. Let $\mathbb{K} = \mathcal{O}_K/\mathfrak{p}$ and $\mathbb{L} = \mathcal{O}_L/\mathfrak{q}$ be the residue fields. We view $\mathbb{K}$ as embedded in $\mathbb{L}$ via the map $x + \mathfrak{p} \mapsto x + \mathfrak{q}$. As $p \equiv 1 \pmod{6}$, $p$ splits in $K$ and $\mathbb{K} = \mathcal{O}_K/\mathfrak{p} \simeq \mathbb{F}_p$.

Since $M$ and $L$ are splitting fields, $M/K$ and $L/K$ are Galois extensions. The Galois group of $M/K$ is $\text{Gal}(M/K) \simeq \mathbb{Z}_3 \times \mathbb{Z}_3$ generated by the maps $\alpha: \lambda \mapsto \lambda \omega$ and $\beta: \rho \mapsto \rho \omega$. The Frobenius map of $L/\mathbb{K}$ is the map $x \mapsto x^{[L]}$. The Frobenius element $\sigma_p^L$ is the element of $\text{Gal}(L/K)$ inducing the Frobenius map on $L/\mathbb{K}$. (A priori $\sigma_p^L$ could also depend on the choice of $\mathfrak{q}$ extending $\mathfrak{p}$, but this is not the case since $\text{Gal}(L/K)$ is abelian; see [20, III.2.1].) Define $\sigma_p^M \in \text{Gal}(M/K)$ analogously. Then $\sigma_p^L$ is the restriction of $\sigma_p^M$ to $L$ by [20, III.2.3].

By definition of $\mathbb{L}$, for all sufficiently large $p \equiv 1 \pmod{6}$, $2 \in (\mathbb{Z}_p^*)^3$ if and only if $\mathbb{L} = \mathbb{K}$. But $\mathbb{L} = \mathbb{K}$ if and only if $\sigma_p^L$ is the identity map, and it follows that $2 \in (\mathbb{Z}_p^*)^3$ if and only if $\sigma_p^M \in \langle \beta \rangle$. Similarly, $3 \in (\mathbb{Z}_p^*)^3$ if and only if $\sigma_p^M \in \langle \alpha \rangle$ and $6 \in (\mathbb{Z}_p^*)^3$ if and only if $\sigma_p^M \in \langle \alpha \beta \rangle$. In summary:

$$2, 3 \notin (\mathbb{Z}_p^*)^3, \ 6 \in (\mathbb{Z}_p^*)^3 \iff \sigma_p^M \in \{\alpha \beta, \alpha^2 \beta^2\}.$$ 

The Chebotarev Density Theorem [20, V.10.4] implies that for each $\theta \in \text{Gal}(M/K)$, the set of primes $\mathfrak{p}$ of $K$ (unramified in $M$) for which $\sigma_p^M = \theta$ is infinite. Thus each of the two conditions for $\sigma_p^M$ displayed above holds infinitely often. \hfill \square

It is possible to describe the primes in Theorem 5 more explicitly. Given $p \equiv 1 \pmod{6}$, factoring the ideal $p\mathcal{O}_K$ and taking norms, one shows there exist unique $c, d \in \mathbb{Z}$ with $d > 0$, $\gcd(c, d) = 1$, $c \equiv 2 \pmod{3}$ and $4p = (2c - 3d)^2 + 27d^2$. Let $t(p) = (c \pmod{6}, \ d \pmod{6})$. There are 9 possible values for $t(p)$: $(2, 1), (2, 3), (2, 5), (5, 0), (5, 1), (5, 2), (5, 3), (5, 4)$ and $(5, 5)$. The Chebotarev density theorem implies that each of the 9 possible $t(p)$ values occurs “equally often”
Lemma 6 Let \( p \) be prime, let \( H \) be the subgroup of \( \mathbb{Z}_p^* \) generated by \( \{-1, 6\} \), and let \( d \) be the order of \( 2H \) in \( \mathbb{Z}_p^*/H \). If there exist nonnegative integers \( \alpha \) and \( \beta \) such that \( d = 3\alpha + 4\beta \), then there is a factorisation of \( K_p \) into \( \frac{\alpha(p-1)}{2d} \) copies of \( \text{Cay}(\mathbb{Z}_p; \pm \{1, 2, 3, 4\}) \) and \( \frac{\beta(p-1)}{2d} \) copies of \( \text{Cay}(\mathbb{Z}_p; \pm \{1, 2, 3, 4\}) \).

Proof It is sufficient to partition \( \mathbb{Z}_p^* \) into \( \frac{\alpha(p-1)}{2d} \) 6-tuples of the form \( \pm \{x, 2x, 3x\} \) and \( \frac{\beta(p-1)}{2d} \) 8-tuples of the form \( \pm \{x, 2x, 3x, 4x\} \). Since \( d = 3\alpha + 4\beta \), there is a partition

\[
\{\{2^i-1 H, 2^i H, 2^i+1 H\} : i = 1, \ldots, \alpha\} \cup \{\{2^i-1 H, 2^i H, 2^i+1 H, 2^i+2 H\} : i = \alpha + 1, \ldots, \alpha + \beta\}
\]

of \( \{H, 2H, \ldots, 2^{d-1} H\} \). But \( 6 \in H \) implies \( 2^i-1 H = 3 \cdot 2^i H \) for \( i = 1, 2, \ldots, \alpha + \beta \). Thus, we can rewrite our partition of \( \{H, 2H, \ldots, 2^{d-1} H\} \) as

\[
\{\{H_i, 2H_i, 3H_i\} : i = 1, \ldots, \alpha\} \cup \{\{H_i, 2H_i, 3H_i, 4H_i\} : i = \alpha + 1, \ldots, \alpha + \beta\},
\]

where \( H_i = 2^i H \) for \( i = 1, \ldots, \alpha + \beta \).

Since \( -1 \in H \), for \( i = 1, \ldots, \alpha \), \( H_i \cup 2H_i \cup 3H_i \) can be partitioned into \( \frac{|H|}{2} \) 6-tuples of the form \( \pm \{x, 2x, 3x\} \), and for \( i = \alpha + 1, \ldots, \alpha + \beta \), \( H_i \cup 2H_i \cup 3H_i \cup 4H_i \) can be partitioned into \( \frac{|H|}{2} \) 8-tuples.
of the form \( \pm\{x, 2x, 3x, 4x\} \). If \( \mathcal{R} \) is the set of all \( \alpha \frac{|H|}{2} \) of these 6-tuples and \( \mathcal{S} \) is the set of all \( \beta \frac{|H|}{2} \) of these 8-tuples, then \( \mathcal{R} \cup \mathcal{S} \) is a partition of the subgroup \( G = H \cup 2H \cup \cdots \cup 2^{d-1}H \) of \( \mathbb{Z}_p^* \). Thus, if \( g_1, g_2, \ldots, g_t \) (\( t = \frac{p-1}{d|H|} \)) represent the cosets of \( \mathbb{Z}_p^*/G \), then

\[
\{g_iR : R \in \mathcal{R}, i = 1, \ldots, t\} \cup \{g_iS : S \in \mathcal{S}, i = 1, \ldots, t\}
\]

is a partition of \( \mathbb{Z}_p^* \) into \( t\alpha \frac{|H|}{2} \)-tuples of the form \( \pm\{x, 2x, 3x\} \) and \( t\beta \frac{|H|}{2} \)-tuples of the form \( \pm\{x, 2x, 3x, 4x\} \). This is the required partition of \( \mathbb{Z}_p^* \). □

Notice that any 6-factorisation of \( K_p \) into \( \text{Cay}(\mathbb{Z}_p; \pm\{1, 2, 3\}) \) given by Lemma 6 can also be obtained via Lemma 6. For if \( 1, 2, 3 \) represent the three distinct cosets of \( G/H \) (where \( G = (\mathbb{Z}_p^*)^6 \) and \( H = (\mathbb{Z}_p^*)^{3b} \), and \( p - 1 = 6ab \), then it follows that \( \{-1, 6\} \subseteq H \) and \( 2H \) has order 3 in \( G/H \). This means that if \( H' \) is the subgroup of \( \mathbb{Z}_p^* \) generated by \( \{-1, 6\} \), then \( H' \leq H \) and 3 divides the order \( d \) of \( 2H' \) in \( \mathbb{Z}_p^*/H' \). Thus, we can obtain our 6-factorisation of \( K_p \) into \( \text{Cay}(\mathbb{Z}_p; \pm\{1, 2, 3\}) \) by applying Lemma 6 with \( \alpha = \frac{d}{3} \) and \( \beta = 0 \). Similarly, any 8-factorisation of \( K_p \) into \( \text{Cay}(\mathbb{Z}_p; \pm\{1, 2, 3, 4\}) \) given by Lemma 6 can be obtained by applying Lemma 6 with \( \alpha = 0 \) and \( \beta = \frac{d}{4} \).

However, Lemma 6 gives us additional factorisations such as the following. When \( p = 101 \) we have \( H = \pm\{1, 6, 14, 17, 36\} \), and \( 2H \) has order \( d = 10 \) in \( \mathbb{Z}_p^*/H \). Taking \( \alpha = 2 \) and \( \beta = 1 \), we obtain a factorisation of \( K_{101} \) into 10 copies of \( \text{Cay}(\mathbb{Z}_p; \pm\{1, 2, 3\}) \) and 5 copies of \( \text{Cay}(\mathbb{Z}_p; \pm\{1, 2, 3, 4\}) \). Of course, 101 is neither 1 (mod 6) nor 1 (mod 8), so there is neither a 6-factorisation nor an 8-factorisation of \( K_{101} \).

### 2.2 Factorising complete graphs of even order

In this section we construct factorisations of \( K_{2p} - I \) where the factors are all isomorphic to \( \text{Cay}(\mathbb{Z}_{2p}; \pm\{1, 2\}) \) or all isomorphic to \( \text{Cay}(\mathbb{Z}_{2p}; \pm\{1, 2, 3, 4\}) \). We do this by considering \( K_{2p} - I \) as a Cayley graph on a dihedral group and partitioning its connection set to generate the factors. The dihedral group \( D_{2p} \) of order \( 2p \) has elements \( r_0, r_1, r_2, \ldots, r_{p-1}, s_0, s_1, s_2, \ldots, s_{p-1} \) and satisfies

\[
r_i \cdot r_j = r_{i+j}, \quad r_i \cdot s_j = s_{i+j}, \quad s_i \cdot r_j = s_{i-j}, \quad s_i \cdot s_j = r_{i-j}
\]

where arithmetic of subscripts is carried out modulo \( p \).

**Lemma 7** If \( p \geq 3 \) is prime, then

\[
\text{Cay}(D_{2p}; \{r_{\pm i}, s_j, s_{i+j}\}) \cong \text{Cay}(\mathbb{Z}_{2p}; \pm\{1, 2\})
\]
for all $i \in \mathbb{Z}_p \setminus \{0\}$ and all $j \in \mathbb{Z}_p$.

**Proof** An isomorphism is given by

$$
\begin{array}{cccccccccccc}
  & r_0 & r_i & r_{2i} & r_{3i} & \ldots & r_{(p-1)i} & s_j & s_{i+j} & s_{2i+j} & s_{3i+j} & \ldots & s_{(p-1)i+j} \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \ldots & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \ldots & \downarrow \\
0 & 2 & 4 & 6 & \ldots & 2p-2 & 2p-1 & 1 & 3 & 5 & \ldots & 2p-3 \\
\end{array}
$$

□

**Lemma 8** If $p \geq 5$ is prime, then

$$
\text{Cay}(D_{2p}; \{r_{\pm i}, r_{\pm 2i}, s_j, s_{i+j}, s_{2i+j}, s_{3i+j}\}) \cong \text{Cay}(\mathbb{Z}_{2p}; \{\pm\{1, 2, 3, 4\}\})
$$

for all $i \in \mathbb{Z}_p \setminus \{0\}$ and all $j \in \mathbb{Z}_p$.

**Proof** An isomorphism is given by

$$
\begin{array}{cccccccccccc}
  & r_0 & r_i & r_{2i} & r_{3i} & \ldots & r_{(p-1)i} & s_j & s_{i+j} & s_{2i+j} & s_{3i+j} & \ldots & s_{(p-1)i+j} \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \ldots & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \ldots & \downarrow \\
0 & 2 & 4 & 6 & \ldots & 2p-2 & 2p-3 & 2p-1 & 1 & 3 & \ldots & 2p-5 \\
\end{array}
$$

□

**Theorem 9** For each odd prime $p$, there is a factorisation of $K_{2p} - I$ into $\text{Cay}(\mathbb{Z}_{2p}; \{\pm\{1, 2\}\})$.

**Proof** The required factorisation is $\mathcal{F} = \{X_i : i \in \mathbb{Z}_p \setminus \{0\}\}$ where

$$
X_i = \text{Cay}(D_{2p}; \{r_{\pm 2i}, s_i, s_{-i}\})
$$

for $i \in \mathbb{Z}_p \setminus \{0\}$. Note that $X_i = X_{-i}$ so $|\mathcal{F}| = \frac{p-1}{2}$ as required. Lemma 7 guarantees that $X_i \cong \text{Cay}(\mathbb{Z}_{2p}; \{\pm\{1, 2\}\})$ for each $i \in \mathbb{Z}_p \setminus \{0\}$. Also, $r_0$ is the identity of $D_{2p}$ and each element of $D_{2p} \setminus \{r_0, s_0\}$ occurs in exactly one $X_i$. Thus, $\mathcal{F}$ is a factorisation of $\text{Cay}(D_{2p}; D_{2p} \setminus \{r_0, s_0\}) \cong K_{2p} - I$ where the 1-factor $I$ is $\text{Cay}(D_{2p}; \{s_0\})$. □

Following work of Davenport [10, Theorem 5] and Weil, a special case of a result due to Moroz [23] yields the following. If $p \equiv 1 \pmod{4}$ is prime and $p > 8 \times 10^6$, then there exists an integer $x$ such that $x, x+1, x+2, x+3$ represent all four distinct cosets of $\mathbb{Z}_p^*/(\mathbb{Z}_p^*)^4$. A computer search using PARI/GP [25] verifies in a few minutes that such an $x$ also exists for all $p < 8 \times 10^6$ with $p \equiv 1 \pmod{4}$, with the exceptions $p = 13$ and $p = 17$. Thus, we have the following result.
Lemma 10 If \( p \equiv 1 \pmod{4} \) is prime with \( p \notin \{13, 17\} \), then there exists an \( x \in \mathbb{Z}_p^* \) such that \( x, x + 1, x + 2 \) and \( x + 3 \) represent all four distinct cosets of \( \mathbb{Z}_p^*/(\mathbb{Z}_p^*)^4 \).

Theorem 11 If \( p \equiv 5 \pmod{8} \) is prime, then there is a factorisation of \( K_{2p} - I \) into \( \text{Cay}(\mathbb{Z}_{2p}; \pm \{1, 2, 3, 4\}) \); except that there is no factorisation of \( K_{26} - I \) into \( \text{Cay}(\mathbb{Z}_{2p}; \pm \{1, 2, 3, 4\}) \).

Proof We first observe that there is no factorisation of \( K_{26} - I \) into \( \text{Cay}(\mathbb{Z}_{2p}; \pm \{1, 2, 3, 4\}) \). If such a factorisation exists, then we can assume without loss of generality that the vertex set is \( \mathbb{Z}_{26} \) and that \( \text{Cay}(\mathbb{Z}_{26}; \pm \{1, 2, 3, 4\}) \) is a factor. But no edge of \( \text{Cay}(\mathbb{Z}_{26}; \pm \{7\}) \) (for example) occurs in a complete subgraph of order 5 in \( \text{Cay}(\mathbb{Z}_{26}; \pm \{5, 6, 7, 8, 9, 10, 11, 12, 13\}) \). Since \( \text{Cay}(\mathbb{Z}_{26}; \pm \{1, 2, 3, 4\}) \) contains a complete subgraph of order 5, it follows that there is no factorisation of \( K_{26} - I \) into \( \text{Cay}(\mathbb{Z}_{2p}; \pm \{1, 2, 3, 4\}) \).

Let \( p \equiv 5 \pmod{8} \) be prime with \( p \neq 13 \). By Lemma 10 there exists an \( x \in \mathbb{Z}_p^* \) such that \( x, x + 1, x + 2 \) and \( x + 3 \) represent all four distinct cosets of \( \mathbb{Z}_p^*/(\mathbb{Z}_p^*)^4 \). By Lemma 8

\[
\text{Cay}(D_{2p}; \{r_{\pm 1}, r_{\pm 2}, s_x, s_{x+1}, s_{x+2}, s_{x+3}\}) \cong \text{Cay}(\mathbb{Z}_{2p}; \pm \{1, 2, 3, 4\}).
\]

Now let \( H = (\mathbb{Z}_p^*)^4 \) act on the subscripts of the connection set \( \{r_{\pm 1}, r_{\pm 2}, s_x, s_{x+1}, s_{x+2}, s_{x+3}\} \) and consider the collection \( S_1, S_2, \ldots, S_{2p-1} \) of subsets of \( D_{2p} \) thus formed.

We show that \( \{\text{Cay}(D_{2p}; S_i) : i = 1, 2, \ldots, 2p-1\} \) is a factorisation of \( K_{2p} - I \) into \( \text{Cay}(\mathbb{Z}_{2p}; \pm \{1, 2, 3, 4\}) \). If \( h \in H \), then

\[
\text{Cay}(D_{2p}; \{r_{\pm h}, r_{\pm 2h}, s_{hx}, s_{h(x+1)}, s_{h(x+2)}, s_{h(x+3)}\}) \cong \text{Cay}(\mathbb{Z}_{2p}; \pm \{1, 2, 3, 4\})
\]

by Lemma 8 (indeed this is true for any \( h \in \mathbb{Z}_p^* \)) so it remains only to verify that we have a decomposition of \( K_{2p} - I \). To do this we observe that \( S_1, S_2, \ldots, S_{2p-1} \) partitions \( D_{2p} \setminus \{r_0, s_0\} \) (\( r_0 \) is the identity in \( D_{2p} \) and \( \text{Cay}(D_{2p}; \{s_0\}) \) is a 1-factor in \( K_{2p} \)). We have \( Hx \cup H(x + 1) \cup H(x + 2) \cup H(x + 3) = \mathbb{Z}_p \setminus \{0\} \). Also, since \( p \equiv 5 \pmod{8} \) we have \( -1 \in (\mathbb{Z}^*_p)^2 \), \( -1 \notin (\mathbb{Z}^*_p)^4 \) and \( 2 \notin (\mathbb{Z}^*_p)^2 \) (by the law of quadratic reciprocity). Thus, \( \{\pm h : h \in H\} \cup \{\pm 2h : h \in H\} = \mathbb{Z}_p \setminus \{0\} \). So \( S_1, S_2, \ldots, S_{2p-1} \) does indeed partition \( D_{2p} \setminus \{r_0, s_0\} \) and we have the required decomposition. \( \square \)

3 2-factorisations of circulant graphs

In this section we present various results on 2-factorisations of circulant graphs, beginning with a couple of known results. Lemma 12 was proved independently in [4] and [27], and is a special case
Lemma 13 was proved in [8].

**Lemma 12 ([4, 27])** If \( n \geq 5 \) and \( F \) is any 2-regular graph of order \( n \), then there is a 2-factorisation of \( \text{Cay}(\mathbb{Z}_n; \pm \{1, 2\}) \) into a copy of \( F \) and a Hamilton cycle.

**Lemma 13 ([8])** If \( n \geq 9 \) and \( F \) is a 2-regular graph of order \( n \), then there is a 2-factorisation of \( \text{Cay}(\mathbb{Z}_n; \pm \{1, 2, 3, 4\}) \) into \( F \) with the definite exceptions of \( F = C_4 \cup C_5 \) and \( F = C_3 \cup C_3 \cup C_3 \cup C_3 \cup C_3 \), and the following possible exceptions.

1. \( F = C_3 \cup C_3 \cup \cdots \cup C_3 \) when \( n \equiv 3, 6 \) (mod 9), \( n \geq 21 \).
2. \( F = C_4 \cup C_4 \cup \cdots \cup C_4 \) when \( n \equiv 4 \) (mod 8), \( n \geq 20 \).
3. \( F = C_3 \cup C_3 \cup \cdots \cup C_3 \cup C_4 \) when \( n \equiv 1 \) (mod 3), \( n \geq 19 \).
4. \( F = C_3 \cup C_4 \cup C_4 \cup \cdots \cup C_4 \) when \( n \equiv 7 \) (mod 8), \( n \geq 23 \).

We now obtain results on 2-factorisations of \( \text{Cay}(\mathbb{Z}_n; \pm \{1, 2, 3\}) \), but first we need some definitions and notation. For each \( m \geq 1 \), the graph with vertex set \( \{0, 1, \ldots, m + 2\} \) and edge set \( \{(i, i + 1), (i + 1, i + 3), (i, i + 3) : i = 0, 1, \ldots, m - 1\} \) is denoted by \( J_{m}^{1,2,3} \). If \( F \) is a 2-regular graph of order \( m \), and there exists a decomposition \( \{H_1, H_2, H_3\} \) of \( J_{m}^{1,2,3} \) into \( F \) such that

1. \( V(H_1) = \{0, 1, \ldots, m + 2\} \setminus \{m, m + 1, m + 2\} \),
2. \( V(H_2) = \{0, 1, \ldots, m + 2\} \setminus \{0, 2, m + 1\} \), and
3. \( V(H_3) = \{0, 1, \ldots, m + 2\} \setminus \{0, 1, m + 2\} \),

then we shall write \( J_{m}^{1,2,3} \nrightarrow F \). Notice that for \( i = 1, 2, 3 \), the subgraph \( H_i \) of \( J_{m}^{1,2,3} \) contains exactly one vertex from each of \( \{0, m\}, \{1, m + 1\} \) and \( \{2, m + 2\} \).

**Lemma 14** If \( n \geq 7 \) and \( F \) is a 2-regular graph of order \( n \) such that there exists a decomposition \( J_{n}^{1,2,3} \nrightarrow F \), then there exists a 2-factorisation of \( \text{Cay}(\mathbb{Z}_n; \pm \{1, 2, 3\}) \) into \( F \).

**Proof** For each \( i \in \{0, 1, 2\} \), identify vertex \( i \) of \( J_{n}^{1,2,3} \) with vertex \( n + i \). The resulting graph is \( \text{Cay}(\mathbb{Z}_n; \pm \{1, 2, 3\}) \) and the 2-regular graphs in the decomposition \( J_{n}^{1,2,3} \nrightarrow F \) become the required 2-factors. \( \square \)
Lemma 15 If $F$ and $F'$ are vertex-disjoint 2-regular graphs and there exist decompositions $J^{1,2,3}_{V(F)} \mapsto F$ and $J^{1,2,3}_{V(F')} \mapsto F'$, then there exists a decomposition $J^{1,2,3}_{V(F)+V(F')} \mapsto F \cup F'$.

Proof Let $r$ and $s$ be the respective orders of $F$ and $F'$, let $\{H_1, H_2, H_3\}$ be a decomposition $J^{1,2,3}_r \mapsto F$ and let $\{H'_1, H'_2, H'_3\}$ be a decomposition $J^{1,2,3}_s \mapsto F'$. Apply the translation $x \mapsto x + r$ to the decomposition $\{H'_1, H'_2, H'_3\}$ to obtain a decomposition $\{H''_1, H''_2, H''_3\}$ of a copy of $J^{1,2,3}_s$ having vertex set $r, r+1, \ldots, r+s+2$ ($H''_i$ being the translation of $H'_i$ for $i \in \{1, 2, 3\}$). It is clear that $D = \{H_1 \cup H''_1, H_2 \cup H''_2, H_3 \cup H''_3\}$ is a decomposition $J^{1,2,3}_{r+s} \mapsto F \cup F'$. Properties (1)-(3) in the definition of $J^{1,2,3}_r \mapsto F$ ensure that $H_i$ and $H''_i$ are vertex-disjoint for $i \in \{1, 2, 3\}$, and that

1. $V(H_1 \cup H''_1) = \{0, 1, \ldots, r+s+2\} \setminus \{r+s, r+s+1, r+s+2\}$,
2. $V(H_2 \cup H''_2) = \{0, 1, \ldots, r+s+2\} \setminus \{0, 2, r+s+1\}$, and
3. $V(H_3 \cup H''_3) = \{0, 1, \ldots, r+s+2\} \setminus \{0, 1, r+s+2\}$.

\[\square\]

Lemma 16 For each $m \geq 4$, $J^{1,2,3}_m \mapsto C_m$.

Proof For $m \in \{4, 5, 6\}$, $H_1$, $H_2$, $H_3$ are as defined in the following table.

| $m$ | $H_1$  | $H_2$  | $H_3$  |
|-----|--------|--------|--------|
| 4   | (0, 1, 2, 3) | (1, 3, 6, 4) | (2, 4, 3, 5) |
| 5   | (0, 1, 2, 4, 3) | (1, 3, 5, 7, 4) | (2, 3, 6, 4, 5) |
| 6   | (0, 1, 2, 5, 4, 3) | (1, 3, 5, 8, 6, 4) | (2, 4, 7, 5, 6, 3) |

For $m \geq 7$ and odd

- $H_1$ contains the edges $\{0, 1\}$, $\{1, 2\}$, $\{0, 3\}$, $\{m-2, m-1\}$ and $\{i, i+2\}$ for $i \in \{2, 3, \ldots, m-3\}$,
- $H_2$ contains the edges $\{1, 3\}$, $\{m-2, m\}$, $\{m, m+2\}$, $\{m-1, m+2\}$, $\{i, i+1\}$ for $i \in \{4, 6, \ldots, m-3\}$ and $\{i, i+3\}$ for $i \in \{1, 3, \ldots, m-4\}$, and
- $H_3$ contains the edges $\{2, 3\}$, $\{m-2, m+1\}$, $\{m-1, m\}$, $\{m-1, m+1\}$, $\{i, i+1\}$ for $i \in \{3, 5, \ldots, m-4\}$ and $\{i, i+3\}$ for $i \in \{2, 4, \ldots, m-3\}$.

For $m \geq 8$ and even
• $H_1$ contains the edges $\{0, 1\}, \{1, 2\}, \{3, 4\}, \{0, 3\}, \{2, 5\}, \{m - 2, m - 1\}$ and $\{i, i + 2\}$ for $i \in \{4, 5, \ldots, m - 3\},$

• $H_2$ contains the edges $\{1, 3\}, \{1, 4\}, \{3, 5\}, \{m - 2, m\}, \{m, m + 2\}, \{m - 1, m + 2\}, \{i, i + 1\}$ for $i \in \{5, 7, \ldots, m - 3\}$ and $\{i, i + 3\}$ for $i \in \{4, 6, \ldots, m - 4\}$, and

• $H_3$ contains the edges $\{2, 4\}, \{m - 2, m + 1\}, \{m - 1, m\}, \{m - 1, m + 1\}, \{i, i + 1\}$ for $i \in \{2, 4, \ldots, m - 4\}$ and $\{i, i + 3\}$ for $i \in \{3, 5, \ldots, m - 3\}.

\[\square\]

**Lemma 17** For $m = 8$ and for each $m \geq 10$, $J^{1,2,3}_m \mapsto C_5 \cup C_{m-3}$.

**Proof** For $m \in \{8, 10, 11\}$, $H_1, H_2, H_3$ are as defined in the following table.

| $m$ | $H_1$ | $H_2$ | $H_3$ |
|-----|-------|-------|-------|
| 8   | $(4, 6, 7) \cup (0, 1, 2, 5, 3)$ | $(7, 8, 10) \cup (1, 3, 6, 5, 4)$ | $(2, 3, 4) \cup (5, 7, 9, 6, 8)$ |
| 10  | $(7, 8, 9) \cup (0, 1, 2, 4, 5, 6, 3)$ | $(1, 3, 4) \cup (5, 7, 6, 9, 12, 10, 8)$ | $(2, 3, 5) \cup (4, 6, 8, 11, 9, 10, 7)$ |
| 11  | $(8, 9, 10) \cup (0, 1, 2, 4, 5, 7, 6, 3)$ | $(1, 3, 4) \cup (5, 6, 9, 11, 13, 10, 7, 8)$ | $(2, 3, 5) \cup (4, 6, 8, 11, 10, 12, 9, 7)$ |

For $m \geq 12$ and even

• $H_1$ consists of the 3-cycle $(m - 3, m - 2, m - 1)$ and the $(m - 3)$-cycle with edges $\{0, 1\}, \{0, 3\}, \{1, 2\}, \{2, 4\}, \{m - 5, m - 4\}, \{i, i + 1\}$ for $i \in \{4, 6, \ldots, m - 6\}$ and $\{i, i + 3\}$ for $i \in \{3, 5, \ldots, m - 7\},$

• $H_2$ consists of the 3-cycle $(1, 3, 4)$ and the $(m - 3)$-cycle with edges $\{5, 7\}, \{m - 5, m - 2\}, \{m - 4, m - 3\}, \{m - 2, m\}, \{m, m + 2\}, \{m - 1, m + 2\}, \{i, i + 1\}$ for $i \in \{5, 7, \ldots, m - 7\}$ and $\{i, i + 3\}$ for $i \in \{6, 8, \ldots, m - 4\}$, and

• $H_3$ consists of the 3-cycle $(2, 3, 5)$ and the $(m - 3)$-cycle with edges $\{4, 6\}, \{4, 7\}, \{m - 2, m + 1\}, \{m - 3, m\}, \{m - 1, m\}, \{m - 1, m + 1\}$ and $\{i, i + 2\}$ for $i \in \{6, 7, \ldots, m - 4\}.$

For $m \geq 13$ and odd

• $H_1$ consists of the 3-cycle $(m - 3, m - 2, m - 1)$ and the $(m - 3)$-cycle with edges $\{0, 1\}, \{0, 3\}, \{1, 2\}, \{2, 4\}, \{3, 6\}, \{4, 5\}, \{5, 7\}, \{m - 5, m - 4\}, \{i, i + 1\}$ for $i \in \{7, 9, \ldots, m - 6\}$ and $\{i, i + 3\}$ for $i \in \{6, 8, \ldots, m - 7\},$
• $H_2$ consists of the 3-cycle $(1, 3, 4)$ and the $(m - 3)$-cycle with edges \{5, 6\}, \{m - 5, m - 2\}, 
\{m - 4, m - 3\}, \{m - 2, m\}, \{m, m + 2\}, \{m - 1, m + 2\}, \{i, i + 1\} for $i \in \{6, 8, \ldots, m - 7\}$ 
and \{i, i + 3\} for $i \in \{5, 7, \ldots, m - 4\}$, and

• $H_3$ consists of the 3-cycle $(2, 3, 5)$ and the $(m - 3)$-cycle with edges \{4, 6\}, \{4, 7\}, \{m - 2, m + 1\}, 
\{m - 3, m\}, \{m - 1, m\}, \{m - 1, m + 1\} and \{i, i + 2\} for $i \in \{6, 7, \ldots, m - 4\}$.

\[ \square \]

**Lemma 18** Let $n \geq 7$ and let $F$ be a 2-regular graph of order $n$. If $\nu_3(F) \leq \nu_5(F) + \sum_{i=7}^{n} \nu_i(F)$ 
where $\nu_n(F)$ denotes the number of $m$-cycles in $F$, then there exists a 2-factorisation of $\text{Cay}(\mathbb{Z}_n; \pm\{1, 2, 3\})$ into $F$.

**Proof** If $n \geq 7$ and $F$ is a 2-regular graph of order $n$ such that $\nu_3(F) \leq \nu_5(F) + \sum_{i=7}^{n} \nu_i(F)$,
then $F$ can be written as a vertex-disjoint union of 2-regular graphs $G_1, G_2, \ldots, G_t$ where each $G_i$ 
is isomorphic to either

• $C_m$ with $m \geq 4$, or

• $C_3 \cup C_{m-3}$ with $m = 8$ or $m \geq 10$.

By Lemmas [16] and [17] we have a decomposition $J_{\{V(G_i)\}}^{1,2,3} \mapsto G_i$ for $i = 1, 2, \ldots, t$. Applying Lemma [15] we obtain a decomposition $J_n^{1,2,3} \mapsto F$, 
and from this we obtain the required 2-factorisation of $\text{Cay}(\mathbb{Z}_n; \pm\{1, 2, 3\})$ into $F$ by applying Lemma [14] \[ \square \]

We can obtain an analogue of Lemma 18 for $\text{Cay}(\mathbb{Z}_n; \pm\{1, 3, 4\})$ by using using similar methods, 
but we will require $F$ to have girth at least 6. The graph with vertex set \{0, 1, \ldots, m + 3\} and 
edge set \{\{i, i + 1\}, \{i + 1, i + 4\}, \{i, i + 4\} : i = 0, 1, \ldots, m - 1\} is denoted by $J_m^{1,3,4}$. We write 
$J_m^{1,3,4} \mapsto F$ when there exists a decomposition \{\$H_1, H_2, H_3\$\} of $J_m^{1,3,4}$ into a 2-regular graph $F$ such that

1. $V(H_1) = \{0, 1, \ldots, m + 3\} \setminus \{m, m + 1, m + 2, m + 3\}$,
2. $V(H_2) = \{0, 1, \ldots, m + 3\} \setminus \{0, 3, m + 1, m + 2\}$, and
3. $V(H_3) = \{0, 1, \ldots, m + 3\} \setminus \{0, 1, 2, m + 3\}$.
Notice that for \( i = 1, 2, 3 \), the subgraph \( H_i \) of \( J^{1,3,4}_m \) contains exactly one vertex from each of \( \{0, m\}, \{1, m+1\}, \{2, m+2\} \) and \( \{3, m+3\} \). It is clear that the proofs of Lemmas 14 and 15 can be easily modified to give the following two results.

**Lemma 19** If \( n \geq 9 \) and \( F \) is a 2-regular graph of order \( n \) such that there exists a decomposition \( J^{1,3,4}_n \mapsto F \), then there exists a 2-factorisation of \( \text{Cay}(\mathbb{Z}_n; \pm\{1,3,4\}) \) into \( F \).

**Lemma 20** If \( F \) and \( F' \) are vertex-disjoint 2-regular graphs and there exist decompositions \( J^{1,3,4}_{|V(F)|} \mapsto F \) and \( J^{1,3,4}_{|V(F')|} \mapsto F' \), then there exists a decomposition \( J^{1,3,4}_{|V(F)|+|V(F')|} \mapsto F \cup F' \).

Lemmas 19 and 20 allow us to obtain 2-factorisations of \( \text{Cay}(\mathbb{Z}_n; \pm\{1,3,4\}) \) via the same method we used in the case of \( \text{Cay}(\mathbb{Z}_n; \pm\{1,2,3\}) \), providing we can find appropriate decompositions of \( J^{1,3,4}_m \). We now do this.

**Lemma 21** For \( m = 6, m = 7 \) and each \( m \geq 9 \), \( J^{1,3,4}_m \mapsto C_m \).

**Proof** For \( m \in \{6,7,9,10\} \), \( H_1, H_2, H_3 \) are as defined in the following table.

| \( m \) | \( H_1 \) | \( H_2 \) | \( H_3 \) |
|---|---|---|---|
| 6 | \((0,1,5,2,3,4)\) | \((1,2,6,9,5,4)\) | \((3,6,5,8,4,7)\) |
| 7 | \((0,1,2,3,6,5,4)\) | \((1,4,7,10,6,2,5)\) | \((3,4,8,5,9,6,7)\) |
| 9 | \((0,1,2,3,7,6,5,8,4)\) | \((1,4,7,8,12,9,6,2,5)\) | \((3,4,5,9,8,11,7,10,6)\) |
| 10 | \((0,1,2,3,6,9,5,8,7,4)\) | \((1,4,8,9,13,10,7,6,2,5)\) | \((3,4,5,6,10,9,12,8,11,7)\) |

For \( m \geq 11 \) and odd

- \( H_1 \) contains the edges \( \{0,1\}, \{0,4\}, \{1,2\}, \{2,3\}, \{3,7\}, \{5,6\}, \{m-3,m-2\}, \{m-5,m-1\}, \{m-4,m-1\} \) and \( \{i,i+4\} \) for \( i \in \{4,5,\ldots,m-6\} \),

- \( H_2 \) contains the edges \( \{1,4\}, \{1,5\}, \{2,5\}, \{2,6\}, \{4,7\}, \{m,m+3\}, \{m-1,m+3\}, \{m-2,m-1\}, \{m-3,m\}, \{i,i+1\} \) for \( i \in \{7,9,\ldots,m-4\} \) and \( \{i,i+3\} \) for \( i \in \{6,8,\ldots,m-5\} \), and

- \( H_3 \) contains the edges \( \{3,4\}, \{3,6\}, \{4,5\}, \{m-1,m\}, \{m-2,m+1\}, \{m-1,m+2\}, \{m-4,m\}, \{m-3,m+1\}, \{m-2,m+2\}, \{i,i+1\} \) for \( i \in \{6,8,\ldots,m-5\} \) and \( \{i,i+3\} \) for \( i \in \{5,7,\ldots,m-6\} \).
For \( m \geq 12 \) and even

- \( H_1 \) contains the edges \( \{0,1\}, \{0,4\}, \{1,2\}, \{2,3\}, \{3,6\}, \{4,7\}, \{5,6\}, \{5,9\}, \{m-5, m-2\}, \{m-4, m-3\}, \{m-4, m-1\}, \{m-2, m-1\}, \{i, i+1\} \) for \( i \in \{7,9,\ldots, m-7\} \) and \( \{i, i+3\} \) for \( i \in \{8,10,\ldots, m-6\} \),

- \( H_2 \) contains the edges \( \{1,4\}, \{1,5\}, \{2,5\}, \{2,6\}, \{4,8\}, \{m-6, m-2\}, \{m-5, m-4\}, \{m-5, m-1\}, \{m-3, m-2\}, \{m-3, m\}, \{m-1, m+3\}, \{m, m+3\}, \{i, i+1\} \) for \( i \in \{6,8,\ldots, m-8\} \) and \( \{i, i+3\} \) for \( i \in \{7,9,\ldots, m-7\} \), and

- \( H_3 \) contains the edges \( \{3,4\}, \{3,7\}, \{4,5\}, \{5,8\}, \{6,9\}, \{m-6, m-5\}, \{m-4, m\}, \{m-3, m+1\}, \{m-2, m+1\}, \{m-2, m+2\}, \{m-1, m\}, \{m-1, m+2\} \) and \( \{i, i+4\} \) for \( i \in \{6,7,\ldots, m-7\} \).

\[\Box\]

**Lemma 22** For each \( m \geq 14 \), \( J_m^{1,3,4} \mapsto C_8 \cup C_{m-8} \).

**Proof** For \( m \in \{14,15,16,17\} \), \( H_1, H_2, H_3 \) are as defined in the following table.

| \( m \) | \( H_1 \) | \( H_2 \) | \( H_3 \) |
|---|---|---|---|
| 14 | \( (0,1,2,3,7,8,5,4) \cup (6,9,13,12,11,10) \) | \( (8,11,14,17,13,10,9,12) \cup (1,4,7,6,2,5) \) | \( (7,10,14,13,16,12,15,11) \cup (3,4,8,9,5,6) \) |
| 15 | \( (0,1,2,3,6,5,8,4) \cup (7,10,14,13,9,12,11) \) | \( (1,4,7,8,9,6,2,5) \cup (10,11,14,18,15,12,13) \) | \( (8,11,15,14,17,13,16,12) \cup (3,4,5,9,10,6,7) \) |
| 16 | \( (0,1,5,6,2,3,7,4) \cup (8,9,10,11,15,14,13,12) \) | \( (1,2,5,9,6,7,8,4) \cup (10,13,16,19,15,12,11,14) \) | \( (3,4,5,8,11,7,10,6) \cup (9,12,16,15,18,14,17,13) \) |
| 17 | \( (0,1,2,3,7,6,5,4) \cup (8,9,13,16,12,15,14,10,11) \) | \( (1,4,8,12,9,6,2,5) \cup (7,10,13,14,17,20,16,15,11) \) | \( (3,4,7,8,5,9,10,6) \cup (11,12,13,17,16,19,15,18,14) \) |

For \( m \geq 18 \) and even
• $H_1$ consists of the 8-cycle $(0, 1, 5, 6, 2, 3, 7, 4)$ and the $(m - 8)$-cycle with edges $\{8, 9\}, \{9, 10\}, \{10, 11\}, \{8, 12\}, \{m - 5, m - 1\}, \{m - 4, m - 3\}, \{m - 3, m - 2\}, \{m - 2, m - 1\} \{i, i + 1\}$ for $i \in \{12, 14, \ldots, m - 6\}$ and $\{i, i + 3\}$ for $i \in \{11, 13, \ldots, m - 7\}$,

• $H_2$ consists of the 8-cycle $(1, 2, 5, 9, 6, 7, 8, 4)$ and the $(m - 8)$-cycle with edges $\{10, 13\}, \{11, 12\}, \{m - 6, m - 2\}, \{m - 5, m - 2\}, \{m - 4, m - 1\}, \{m - 3, m\}\{m - 1, m + 3\}, \{m, m + 3\}$ and $\{i, i + 4\}$ for $i \in \{10, 11, \ldots, m - 7\}$, and

• $H_3$ consists of the 8-cycle $(3, 4, 5, 8, 11, 7, 10, 6)$ and the $(m - 8)$-cycle with edges $\{9, 12\}, \{9, 13\}, \{m - 4, m\}, \{m - 3, m + 1\}, \{m - 2, m + 1\}, \{m - 2, m + 2\}, \{m - 1, m\}, \{m - 1, m + 2\}, \{i, i + 1\}$ for $i \in \{13, 15, \ldots, m - 5\}$ and $\{i, i + 3\}$ for $i \in \{12, 14, \ldots, m - 6\}$.

For $m \geq 19$ and odd

• $H_1$ consists of the 8-cycle $(0, 1, 2, 3, 7, 6, 5, 4)$ and the $(m - 8)$-cycle with edges $\{8, 9\}, \{8, 11\}, \{9, 13\}, \{10, 11\}, \{10, 14\}, \{12, 15\}, \{12, 16\}, \{m - 4, m - 1\}, \{m - 3, m - 2\}$ and $\{i, i + 4\}$ for $i \in \{13, 14, \ldots, m - 5\}$,

• $H_2$ consists of the 8-cycle $(1, 4, 8, 12, 9, 6, 2, 5)$ and the $(m - 8)$-cycle with edges $\{7, 10\}, \{7, 11\}, \{10, 13\}, \{11, 15\}, \{m - 4, m - 3\}, \{m - 3, m\}, \{m - 2, m - 1\}, \{m - 1, m + 3\}, \{m, m + 3\}, \{i, i + 1\}$ for $i \in \{13, 15, \ldots, m - 6\}$ and $\{i, i + 3\}$ for $i \in \{14, 16, \ldots, m - 5\}$, and

• $H_3$ consists of the 8-cycle $(3, 4, 7, 8, 5, 9, 10, 6)$ and the $(m - 8)$-cycle with edges $\{11, 12\}, \{11, 14\}, \{12, 13\}, \{m - 4, m\}, \{m - 3, m + 1\}, \{m - 2, m + 1\}, \{m - 2, m + 2\}, \{m - 1, m\}, \{m - 1, m + 2\}, \{i, i + 1\}$ for $i \in \{14, 16, \ldots, m - 5\}$ and $\{i, i + 3\}$ for $i \in \{13, 15, \ldots, m - 6\}$.

$\square$

Lemma 23 $J_{24}^{1,3,4} \rightarrow C_8 \cup C_8 \cup C_8$.

Proof Take

$H_1 = (0, 1, 2, 3, 6, 5, 8, 4) \cup (7, 10, 9, 12, 13, 14, 15, 11) \cup (16, 17, 18, 19, 23, 22, 21, 20)$,

$H_2 = (1, 4, 7, 8, 9, 6, 2, 5) \cup (10, 11, 12, 15, 16, 13, 17, 14) \cup (18, 21, 24, 27, 23, 20, 19, 22)$, and

$H_3 = (3, 4, 5, 9, 13, 10, 6, 7) \cup (8, 11, 14, 18, 15, 19, 16, 12) \cup (17, 20, 24, 23, 26, 22, 25, 21)$. 

16
The following result is an analogue of Lemma 18 for 2-factorisations of Cay($\mathbb{Z}_n; \pm\{1, 3, 4\}$).

**Lemma 24** If $n \geq 9$ and $F$ is a 2-regular graph of order $n$ with girth at least 6, then there exists a 2-factorisation of Cay($\mathbb{Z}_n; \pm\{1, 3, 4\}$) into $F$.

**Proof** If $n \geq 9$ and $F$ is a 2-regular graph of order $n$ with girth at least 6, then $F$ can be written as a vertex-disjoint union of 2-regular graphs $G_1, G_2, \ldots, G_t$ where each $G_i$ is isomorphic to either

- $C_m$ with $m = 6, 7$ or $m \geq 9$,
- $C_8 \cup C_{m-8}$ with $m \geq 14$, or
- $C_8 \cup C_8 \cup C_8$.

By Lemmas 21, 22 and 23 we have a decomposition $J_{\phi_i}$ $\mapsto$ $G_i$ for $i = 1, 2, \ldots, t$. Applying Lemma 20 we obtain a decomposition $J_n^{1,3,4}$ $\mapsto$ $F$, and from this we obtain the required 2-factorisation of Cay($\mathbb{Z}_n; \pm\{1, 3, 4\}$) into $F$ by applying Lemma 19. □

## 4 2-factorisations and the Oberwolfach Problem

In this section we use results from the preceding sections to obtain results on the Oberwolfach Problem (and an additional result on 2-factorisations of $K_n - I$ into a number of specified 2-factors and Hamilton cycles). We will also use the following corollary of Lemma 13 which was proved in [8].

**Lemma 25** ([8]) If there exists a factorisation of $K_n$ or of $K_n - I$ into Cay($\mathbb{Z}_n; \pm\{1, 2, 3, 4\}$), then OP($F$) has a solution for each 2-regular graph $F$ of order $n$, with the exception that there is no solution to OP($C_4 \cup C_5$).

**Theorem 26** If $p \equiv 5 \pmod{8}$ is prime, then OP($F$) has a solution for every 2-regular graph $F$ of order $2p$.

**Proof** The case $p = 13$ is covered in [13]. For $p \neq 13$, Theorem 11 gives us a factorisation of $K_{2p} - I$ into Cay($\mathbb{Z}_{2p}; \pm\{1, 2, 3, 4\}$) and the result then follows by Lemma 25. □
Theorem 27 Let \( \mathcal{P} \) be the set of primes given by \( p \in \mathcal{P} \) if and only if \( p \equiv 1 \pmod{6} \) and neither 4 nor 32 is in the subgroup of \( \mathbb{Z}_p^* \) generated by \( \{-1, 6\} \). Then \( \mathcal{P} \) is infinite and if \( p \in \mathcal{P} \), then \( \text{OP}(F) \) has a solution for every 2-regular graph \( F \) of order \( p \) satisfying \( \nu_3(F) \leq \nu_5(F) + \sum_{i=7}^{n} \nu_i(F) \) where \( \nu_m(F) \) denotes the number of \( m \)-cycles in \( F \).

**Proof** Let \( p \) be prime such \( p \equiv 1 \pmod{6} \), \( 2, 3 \notin (\mathbb{Z}_p^*)^3 \) and \( 6 \in (\mathbb{Z}_p^*)^3 \). Theorem 5 says that there are infinitely many such \( p \). We shall show that \( p \in \mathcal{P} \), which shows that \( \mathcal{P} \) is also infinite. We have \( -1 \in (\mathbb{Z}_p^*)^3 \), and this together with the fact that \( 6 \in (\mathbb{Z}_p^*)^3 \) implies that the subgroup of \( \mathbb{Z}_p^* \) generated by \( \{-1, 6\} \) is a subgroup of \( (\mathbb{Z}_p^*)^3 \). Since it follows from \( 2 \notin (\mathbb{Z}_p^*)^3 \) that \( 4, 32 \notin (\mathbb{Z}_p^*)^3 \), neither 4 nor 32 is in the subgroup of \( \mathbb{Z}_p^* \) generated by \( \{-1, 6\} \). That is, \( p \in \mathcal{P} \).

Now let \( p \) be an arbitrary element of \( \mathcal{P} \) and let \( G \) be the subgroup of \( \mathbb{Z}_p^* \) generated by \( \{-1, 6\} \). The condition that neither 4 nor 32 is in \( G \) implies that the order \( d \) of \( 2G \) in \( \mathbb{Z}_p^*/G \) is neither 1, 2 nor 5, and so there exist non-negative integers \( \alpha \) and \( \beta \) such that \( d = 3\alpha + 4\beta \). Thus, by Lemma 6 there is a factorisation of \( K_p \) in which each factor is either Cay\((\mathbb{Z}_p; \pm\{1,2,3\})\) or Cay\((\mathbb{Z}_p; \pm\{1,2,3,4\})\).

Let \( F \) be a 2-regular graph of order \( p \) satisfying \( \nu_3(F) \leq \nu_5(F) + \sum_{i=7}^{n} \nu_i(F) \). Lemma 18 gives us a 2-factorisation of Cay\((\mathbb{Z}_p; \pm\{1,2,3\})\) into \( F \), and Lemma 13 gives us a 2-factorisation of Cay\((\mathbb{Z}_p; \pm\{1,2,3,4\})\) (the facts that \( p \) is prime and that \( \nu_3(F) \leq \nu_5(F) + \sum_{i=7}^{n} \nu_i(F) \) imply that \( F \) is not amongst the possible exceptions listed in Lemma 13). The result follows. \( \square \)

Theorem 28 Let \( \mathcal{P} \) be the set of primes such that \( p \in \mathcal{P} \) if and only if \( p \equiv 1 \pmod{6} \) and \( 2, 3, 6 \notin (\mathbb{Z}_p^*)^3 \). Then \( \mathcal{P} \) is infinite and if \( p \in \mathcal{P} \), then \( \text{OP}(F) \) has a solution for every 2-regular graph \( F \) of order \( p \) with girth at least 6.

**Proof** By Theorem 5 \( \mathcal{P} \) is infinite. If \( p \in \mathcal{P} \), then Theorem 4 gives us a factorisation of \( K_p \) into Cay\((\mathbb{Z}_p; \pm\{1,3,4\})\) and the result then follows by applying Lemma 24 to each factor (7 \( \notin \mathcal{P} \) so Lemma 24 can indeed be applied). \( \square \)

For each odd prime \( p \), the following theorem states there is a 2-factorisation of \( K_{2p} - I \) into \( \frac{p-1}{2} \) prescribed 2-factors and \( \frac{p-1}{2} \) Hamilton cycles.

Theorem 29 If \( p \) is an odd prime and \( G_1, G_2, \ldots, G_{\frac{p-1}{2}} \) are 2-regular graphs of order \( 2p \), then there is a 2-factorisation \( \{F_1, F_2, \ldots, F_{p-1}\} \) of \( K_{2p} - I \) such that \( F_i \cong G_i \) for \( i = 1, 2, \ldots, \frac{p-1}{2} \) and \( F_i \) is a Hamilton cycle for \( i = \frac{p+1}{2}, \frac{p+3}{2}, \ldots, p-1 \).
Proof  By Theorem 9 there is a factorisation of $K_{2p} - I$ into Cay($\mathbb{Z}_p; \pm\{1, 2\}$). By Lemma 12 each copy of Cay($\mathbb{Z}_p; \pm\{1, 2\}$) can be factored into any specified 2-regular graph of order 2p and a Hamilton cycle. The result follows. \qed

5 Isomorphic 2-factorisations of complete multigraphs

The complete multigraph of order $n$ and multiplicity $s$ is denoted by $sK_n$. It has $s$ distinct edges joining each pair of distinct vertices.

Lemma 30 If $p$ is an odd prime and $S = \pm\{d_1, d_2, \ldots, d_s\} \subseteq \mathbb{Z}_p^*$, then there exists a 2s-factorisation of $sK_p$ into Cay($\mathbb{Z}_p; S$).

Proof The required factorisation is given by \{Cay($\mathbb{Z}_p; \omega^iS$) : $i = 0, 1, \ldots, \frac{p-1}{2}$\} where $\omega$ is primitive in $\mathbb{Z}_p$ and $\omega^iS = \{\omega^i s : s \in S\}$. \qed

Theorem 31 If $p$ is an odd prime and $F$ is any 2-regular graph of order $p$ satisfying $\nu_3(F) \leq \nu_5(F) + \sum_{i=7}^n \nu_i(F)$, where $\nu_m(F)$ denotes the number of $m$-cycles in $F$, then there exists a 2-factorisation of $3K_p$ into $F$.

Proof The cases $p = 3$ and $p = 5$ are trivial so assume $p \geq 7$. By Lemma 30 there exists a 6-factorisation of $3K_p$ into Cay($\mathbb{Z}_p; \pm\{1, 2, 3\}$), and by Lemma 18 each such 6-factor has a 2-factorisation into $F$. \qed

Theorem 32 If $p$ is an odd prime and $F$ is any 2-regular graph of order $p$, then there exists a 2-factorisation of $4K_p$ into $F$.

Proof The cases $p = 3$ and $p = 5$ are trivial. Since solutions to OP($C_7$) and OP($C_3 \cup C_4$) exist, the case $p = 7$ can be dealt with by taking four copies of these 2-factorisations of $K_7$. So we may assume $p \geq 11$. By Lemma 30 there exists an 8-factorisation of $4K_p$ into Cay($\mathbb{Z}_p; \pm\{1, 2, 3, 4\}$), and by Lemma 13 each such 8-factor has a 2-factorisation into $F$; except in the case where $F$ is one of the listed exceptions or possible exceptions in Lemma 13. These are easily dealt with as follows. Since $p$ is prime the only relevant exceptions are $F = C_3 \cup C_3 \cup \cdots \cup C_3 \cup C_4$ where the number of copies of $C_3$ is at least 5, and $F = C_3 \cup C_4 \cup C_4 \cup \cdots \cup C_4$ where the number of copies of $C_4$ is odd and at least 5. However, it is known that for each such $F$, there is a 2-factorisation.
of $K_p$ into $F$; the former case is covered in [11], and the latter case is covered in [21]. Thus, by taking four copies of these 2-factorisations of $K_p$, we obtain the required 2-factorisations of $4K_p$. □

**Theorem 33** Let $p$ be an odd prime and let $F$ be a 2-regular graph of order $p$. If $\lambda \equiv 0 \pmod{4}$, then there exists a 2-factorisation of $\lambda K_p$ into $F$. Moreover, if $F$ satisfies $\nu_3(F) \leq \nu_5(F) + \sum_{i=7}^{n} \nu_i(F)$, where $\nu_m(F)$ denotes the number of $m$-cycles in $F$, then the result also holds for $\lambda = 3$ and for all $\lambda \geq 6$.

**Proof** For the given values of $\lambda$, it is trivial to factorise $\lambda K_p$ such that each factor is either $3K_p$ or $4K_p$, and with each factor being $4K_p$ when $\lambda \equiv 0 \pmod{4}$. Thus, the result follows by Theorems 31 and 32. □

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