On two-dimensional finite-gap potential
Schrödinger and Dirac operators with singular
spectral curves

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1 Introduction

In the present paper we describe a wide class of two-dimensional potential
Schrödinger and Dirac operators which are finite-gap on the zero energy level
and whose spectral curves at this level are singular and, in particular, may have
n-multiple points with \( n \geq 3 \).

Dirac operators with such spectral curves are important for the Weierstrass
representation of tori in \( \mathbb{R}^3 \) \(^\dagger\) \( \ddagger \). A study of finite-gap operators with singular
spectral curves which usually were not especially considered because of the non-
generic situation is of a special interest for differential geometry where singular
curves may serve as the spectral curves of smoothly immersed tori. In particular,
the spectral curves of tori in \( \mathbb{R}^3 \) obtained by a rotation of circles lying in
the plane \( y = 0 \) around the \( x \) axis are rational curves with double points.

In the present paper the problem of describing such operators is reduced to
a problem which involves only nonsingular curves by using the normalization of
spectral curves. That makes this description to be effective.

2 Schrödinger and Dirac operators which are finite-gap on the zero energy level

2.1 Two-dimensional operators which are finite-gap the zero energy level

The notion of a two-dimensional operator which is finite-gap at one energy level
was introduced by Dubrovin, Krichever, and Novikov \( \S \) for the Schrödinger
operator.

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First we recall the definition of a Floquet eigenfunction (or a Bloch function) of a differential operator \( L \) with periodic coefficients. Let \( L \) act on functions on \( \mathbb{R}^n \) and let its coefficients be periodic with respect to a lattice \( \Lambda \) isomorphic to \( \mathbb{Z}^n \subset \mathbb{R}^n \). A solution of the equation
\[
L\psi = \lambda \psi, \quad \lambda \in \mathbb{C},
\]
is called a Floquet function (or a Bloch function) with the eigenvalue \( \lambda \), if for any vector \( \gamma \in \Lambda \) we have
\[
\psi(x + \gamma) = e^{2\pi i \langle k, \gamma \rangle} \psi(x),
\]
where \( \langle k, \gamma \rangle = \sum_{i=1}^n k_i \gamma_i \) is the standard scalar product. The components of the vector \( k = (k_1, \ldots, k_n) \) are called the quasimomenta of \( \psi \). We see that any Floquet function defines a homomorphism
\[
\mu : \Lambda \to \mathbb{C}^* = \mathbb{C} \setminus \{0\}, \quad \mu(\gamma) = e^{2\pi i \langle k, \gamma \rangle}.
\]

By using the Keldysh theorem and assuming that the coefficients of operators are bounded it has been proved that for the Schrödinger operator \( \Delta + u \), the heat operator \( \partial_t - \Delta \) and the two-dimensional Dirac operator, the quasimomenta and the eigenvalues of Floquet functions satisfy analytic relations (i.e. the dispersion laws \[4\]) and admissible tuples \( (k_1, \ldots, k_n, \lambda) \) form an analytic subset \( Q \) in \( \mathbb{C}^{n+1} \). Here it is essential that these operators are hypoelliptic. The set \( Q \) is invariant under translations by vectors from the dual lattice \( \Lambda^* = \{ \gamma^* : \langle \gamma^*, \gamma \rangle \in \mathbb{Z} \text{ for all } \gamma \in \Lambda \} \). Therefore it is easier to consider the quotient space \( Q/\Lambda^* \).

Let \( n = 2 \), i.e. the operators are two-dimensional. Then the intersection of \( Q/\Lambda^* \) with the plane \( \lambda = 0 \) is a complex curve (a Riemann surface) \( \Gamma' \), on which Floquet functions are glued into a function \( \psi(x, P), P \in \Gamma' \), which is meromorphic on the surface outside finitely many points. It is said that the operator \( L \) is finite-gap at the zero energy level \( \lambda = 0 \) if the curve \( \Gamma \), which is the normalization of the curve \( \Gamma' \), is a curve of finite genus, i.e. if \( \Gamma \) is an algebraic curve. This Riemann surface \( \Gamma \) is called the spectral curve of the operator \( L \) at the zero energy level.

### 2.2 The Schrödinger operator

The two-dimensional Schrödinger operator with a magnetic field has the form
\[
L = \partial \bar{\partial} + A(z, \bar{z}) \bar{\partial} + u(z, \bar{z}),
\]
(1)
where
\[
\partial = \frac{\partial}{\partial x} - i \frac{\partial}{\partial y}, \quad \bar{\partial} = \frac{\partial}{\partial x} + i \frac{\partial}{\partial y}, \quad z = x + iy.
\]
Following \[3\], it is said that such an operator is finite-gap at the zero energy level \(^1\) if there exists

\(^1\)This definition as its analogue for Dirac operators (see §2.3) is given only for operators with nonsingular spectral curves. Operators with singular spectral curves appear from them in the limit under a degeneration which could be rather complicated as we show in this paper.
a) a nonsingular Riemann surface of finite genus $g$ with two marked points $\infty_\pm$ and local parameters $k_\pm^{-1}$ near these points such that $k_\pm^{-1}(\infty_\pm) = 0$;

b) an effective divisor (a formal sum of points on the surface) $D = P_1 + \ldots + P_g$ of degree $g$ and formed by points different from $\infty_\pm$ such that there is a function $\psi = \psi(x, y, P)$ on $\Gamma$ meeting the following conditions

1) this function is meromorphic with respect to $P$ on $\Gamma \setminus \{\infty_\pm\}$, has poles only at points from $D$ and the order of a pole is not greater than the number of appearances of this point in $D$: $(\psi) \geq -D$;

2) $\psi$ has the following asymptotics at $\infty_\pm$:

$$
\psi(x, y, P) \approx e^{k_\pm x}(1 + \xi(x, y)k_\pm^{-1} + O(k_\pm^{-2})) \quad \text{as} \quad P \to \infty_+,
$$

$$
\psi(x, y, P) \approx c(x, y)e^{k_\pm x}(1 + O(k_\pm^{-1})) \quad \text{as} \quad P \to \infty_-;
$$

3) $\psi$ satisfies the equation $L\psi = 0$ at every point $P \in \Gamma \setminus \{\infty_\pm\}$. It follows from the theory of Baker–Akhieser functions that

1) for a generic divisor $D$ the data $(\Gamma, \infty_\pm, k_\pm, D)$ determines a unique function $\psi$ meeting conditions 1 and 2;

2) given a function $\psi$, a unique Schrödinger operator of the form (1) can be constructed such that $L\psi = 0$. The explicit formulas take the form

$$
A = -\frac{\partial \log c}{\partial \bar{z}}, \quad u = -\frac{\partial \xi}{\partial z}.
$$

As it is shown in [6, 7], if there exists a holomorphic involution of $\Gamma$:

$$
\sigma : \Gamma \to \Gamma, \quad \sigma^2 = 1,
$$

such that $\sigma(\infty_\pm) = \infty_\pm, \sigma(k_\pm) = -k_\pm$ and there exists a meromorphic differential (i.e. a 1-form) $\omega$ on $\Gamma$ with poles of the first order in the points $\infty_+$ and $\infty_-$ and zeroes in the points from $D + \sigma(D)$:

$$
D + \sigma(D) - \infty_+ - \infty_- \sim K(\Gamma)
$$

(the divisor in the left-hand side is equivalent to the canonical divisor of the surface $\Gamma$), then this operator is potential: $c^2 = 1$ and, therefore, $A = 0$.

If in addition there exists an antiholomorphic involution

$$
\tau : \Gamma \to \Gamma, \quad \tau^2 = 1,
$$

such that

$$
\sigma \tau = \tau \sigma, \quad \tau(D) = D, \quad \tau(\infty_\pm) = \infty_\mp, \quad \tau(k_\pm) = \bar{k}_\mp,
$$

then the potential $u$ is real-valued.

It is easy to notice that the form $\omega$ is invariant under $\sigma$ and hence it descends to a form $\omega'$ on the quotient surface $\Gamma/\sigma$ and the form $\omega'$ has $g$ zeroes and two simple poles. Therefore the genus of $\Gamma/\sigma$ equals $g/2$ and the points $\infty_\pm$ are
exactly all fixed points of the involution $\sigma$ (notice that we assume that the surface $\Gamma$ is nonsingular). The natural covering $\Gamma \to \Gamma_0 = \Gamma/\sigma$ is two-sheeted and branched at the points $\infty_\pm$.

Generically the potential $u$ is quasi-periodic and if it is periodic then the function $\psi(x, y, P)$ is a Floquet function for every $P \in \Gamma \setminus \{\infty_\pm\}$ and the quasi-momenta are locally holomorphic functions on the surface $\Gamma$.

### 2.3 The Dirac operator

The Dirac operator (with potentials) has the form

$$D = \begin{pmatrix} 0 & \partial \\ -\bar{\partial} & 0 \end{pmatrix} + \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}.$$ 

It is said that it is finite-gap on the zero energy level if there exists

a) a nonsingular Riemann surface $\Gamma$ of finite genus $g$ with two marked points $\infty_\pm$ and local parameters $k_\pm^{-1}$ such that $k_\pm^{-1}(\infty_\pm) = 0$;

b) an effective divisor $D = P_1 + \ldots + P_{g+1}$ of degree $g + 1$ formed by points which differ from $\infty_\pm$ such that there is a vector function $\psi = (\psi_1, \psi_2)^\pm = \psi(x, y, P)$ meeting the following conditions:

1) the function $\psi$ is meromorphic in $P$ on $\Gamma \setminus \{\infty_\pm\}$ and $(\psi) \geq -D$;

2) there are the following asymptotics:

$$\psi(x, y, P) \approx e^{k_+ z} \left[ \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} \xi_1^+ \\ \xi_2^+ \end{pmatrix} k_+^{-1} + O(k_+^{-2}) \right] \text{ as } P \to \infty_+,$$

$$\psi(x, y, P) \approx e^{k_- z} \left[ \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} \xi_1^- \\ \xi_2^- \end{pmatrix} k_-^{-1} + O(k_-^{-2}) \right] \text{ as } P \to \infty_-;$$

3) the equation $D\psi = 0$ holds on $\Gamma \setminus \{\infty_\pm\}$.

As in the case of the Schrödinger operator for a generic divisor $D$ the data $(\Gamma, \infty_\pm, k_\pm, D)$ determines a function satisfying the conditions 1 and 2 uniquely and from this function one can construct a unique operator $D$ such that $D\psi = 0$:

$$U = -\xi_2^+, \quad V = \xi_1^-.$$ 

Again as in the case of the Schrödinger operator generically these potentials are quasi-periodic but when they are periodic the functions $\psi(x, y, P)$ are Floquet functions whose quasimomenta locally holomorphically depend on $P$.

If there exists a holomorphic involution $\sigma : \Gamma \to \Gamma$ such that

$$\sigma(\infty_\pm) = \infty_\pm, \quad \sigma(k_\pm) = -k_\pm,$$

and there exists a meromorphic differential $\omega$ with zeroes in $D + \sigma(D)$ and two poles in the marked points $\infty_\pm$ with the principal parts $\pm k_\pm^2 (1 + O(k_\pm^{-1})) dk_\pm^{-1}$, then the potentials $U$ and $V$ coincide: $U = V$ (2).
If there exists an antiholomorphic involution $\tau : \Gamma \to \Gamma$, such that

$$\tau(\infty_{\pm}) = \infty_{\mp}, \quad \tau(k_{\pm}) = -\bar{k}_{\mp}$$

and there exists a meromorphic differential $\omega'$ with zeroes in $D + \tau(D)$ and two poles in $\infty_{\pm}$ with the principal parts $k^2_\pm(1 + O(k^{-1}_{\pm}))dk^{-1}_{\pm}$, then the potentials $U$ and $V$ are real-valued: $U = \bar{U}, V = \bar{V}$ (\textit{2}).

For periodic operators these involutions are easily described on the language of quasimomenta:

$$\sigma(k_1, k_2) = (-k_1, -k_2), \quad \tau(k_1, k_2) = (\bar{k}_1, \bar{k}_2),$$

and the existence of them immediately follows from from the spectral properties of the Dirac operator $\mathbf{3}$, $\mathbf{2}$ Obviously in this case these involutions commute.

\textbf{Remark.} In papers $\mathbf{2}$, $\mathbf{8}$ we did some inaccuracy assuming rather strong conditions for the differentials $\omega$ and $\omega'$ by demanding that they have the following principal parts $(\pm k^2_\pm + O(k^{-1}_{\pm}))dk^{-1}_{\pm}$ at points $\infty_{\pm}$. But the exposed proofs work under weaker conditions mentioned above. Indeed,

a) the differential $\psi_1(P)\psi_2(\sigma(P))\omega$ has two poles of the first order at the points $\infty_+$ and $\infty_-$ and the sum of residues equals $-2\pi i(\xi^+_2 + \xi^-_1) = 0$ which implies that $U = V$;

b) the differentials $\psi_1(P)\bar{\psi}_1(\tau(P))\omega'$ and $\psi_2(P)\bar{\psi}_2(\tau(P))\omega'$ have first order poles at the points $\infty_+$ and $\infty_-$ and the sums of residues are equal to $2\pi i(\xi^-_1 - \xi^-_2) = 0$ and $2\pi i(\xi^+_2 - \xi^-_1) = 0$, respectively, which implies that $U = \bar{U}$ and $V = \bar{V}$.

Let us demonstrate the involutions $\sigma$ and $\tau$ by the following simple example.

Let the potential $U = V = c$ equals a real nonzero constant $c$. Then the spectral curve is the complex projective line $\Gamma = \mathbb{C}P^1$ realized as the $\lambda$-plane completed by the infinity point $\lambda = \infty$. There two marked points $\infty_{\pm}$ on $\Gamma$ such that $\lambda = \infty$ at $\infty_+$ and $\lambda = 0$ at $\infty_-$. Let us define near these points local parameters $k^{-1}_{\pm}$ by the formulas

$$k_+ = \lambda, \quad k_- = -\frac{c^2}{\lambda}.$$ 

The function $\psi$ takes the form

$$\psi = \left( \begin{array}{c} \frac{1}{\lambda} \exp\left( \lambda z - \frac{c^2}{\lambda} \bar{z} \right) \\ \frac{c}{\lambda} \exp\left( \lambda z - \frac{c^2}{\lambda} \bar{z} \right) \end{array} \right),$$

the divisor $D$ is just the point $\lambda = c$:

$$D = c,$$

\textit{2} Notice that it needs to add the condition that the potentials $U$ and $V$ are real-valued to the part 2 of Proposition 3 in $\mathbf{3}$, where this condition is used in the proof.
and the involutions $\sigma$ and $\tau$ are defined by the formulas

$$\sigma(\lambda) = -\lambda, \quad \tau(\lambda) = \frac{c^2}{\lambda}.$$ 

The differentials $\omega$ and $\omega'$ have the form

$$\omega = \left(1 - \frac{c^2}{\lambda^2}\right) d\lambda, \quad \omega' = \frac{(\lambda - c)^2}{\lambda^2} d\lambda.$$

### 3 Some facts on singular algebraic curves

We expose some necessary facts on singular algebraic curves following mostly to the book by Serre [9].

In the following we refer to a (complex) algebraic curve as a curve. Assuming that algebraic varieties are embedded into $\mathbb{C}P^n$ we say that a mapping between them is regular if it is defined by polynomials in homogeneous coordinates.

If a curve $\Gamma'$ has singularities, then there is a normalization

$$\pi : \Gamma \to \Gamma',$$

where

1) $\Gamma$ is a nonsingular curve with a finite set $S$ of marked points on it and given an equivalence relation $\sim$ between these points;

2) the mapping $\pi$ maps the set $S$ exactly onto the singular locus $S'$ of the curve $\Gamma'$, and the preimage of every point from $S'$ consists of a class of all equivalent points;

3) the mapping $\pi : \Gamma \setminus S \to \Gamma' \setminus S'$ is a smooth one-to-one projection;

4) any regular mapping $F : X \to \Gamma'$ of a nonsingular variety $X$ with an everywhere dense image $F(X) \subset \Gamma'$ descends through $\Gamma$: $F = \pi G$ for some regular mapping $G : X \to \Gamma$.

We recall that for any point $P$ from an algebraic variety there is a corresponding local ring $\mathcal{O}_P$ defined as ring of functions on the variety which are induced by rational functions $f/g$ where $f$ and $g$ are homogeneous polynomials of the same degree and $g(P) \neq 0$ (here we assume that the variety is embedded into $\mathbb{C}P^n$). A point is nonsingular exactly when its local ring is integrally closed.

For a point $P \in \Gamma' \setminus S'$ its local ring $\mathcal{O}'_P$ is $\mathcal{O}_{\pi^{-1}(P)} = \mathcal{O}_P$. If $P \in S' \subset \Gamma'$, then $\mathcal{O}'_P$ is a subring of the ring

$$\mathcal{O}_P = \bigcap_{Q \to P} \mathcal{O}_Q$$

and moreover $\mathcal{O}'_P$ differs from $\mathcal{O}_P$ and for some integer $n$ we have the following inclusions

$$\mathbb{C} + R^n_P \subset \mathcal{O}'_P \subset \mathbb{C} + R_P \subset \mathcal{O}_P,$$

where $R_P$ is an ideal, of the ring $\mathcal{O}_P$, consisting of all functions vanishing at $\pi^{-1}(P)$. 
There is a particular case of constructing a singular curve $\Gamma_D$ from a nonsingular curve $\Gamma$ and an effective divisor $D = \sum n_P P$ with degree $\deg D = \sum n_P \geq 2$ on the curve $\Gamma$. Let us denote by $S$ the set of points from $\Gamma$ with $n_P > 0$ (the support of the divisor) and put $\Gamma_D = (\Gamma \setminus S) \cup \{pt\}$, i.e. contract all points from $S$ into one point which we denote by $Q$. Let us denote by $C_Q$ an ideal consisting of all functions $f$ which have at points $P \in S$ zeroes of order not less than $n_P$. Now put $\mathcal{O}_Q' = \mathcal{O}_Q + C_Q$. This is the set consisting of all functions which have the same value at all points $P \in S$ and whose first $(n_P - 1)$ derivatives vanish at such a point. The natural projection $\Gamma \to \Gamma_D$ is the normalization. For $D = P_1 + \ldots + P_n$ where all points $P_i$ are pair-wise different the curve $\Gamma$ has an $n$-multiple point $Q$ with different tangents.

To every singular point $P \in \Gamma'$ there corresponds an integer-valued invariant

$$\delta_P = \dim_{\mathbb{C}} \mathcal{O}_P / \mathcal{O}_P' < \infty.$$  

It is obvious that for a singular point $Q$ of a curve $\Gamma_D$ we have

$$\delta_Q = \dim \mathcal{O}_Q / (\mathcal{O}_Q + C_Q) = \dim \mathcal{O}_Q / C_Q - 1 = \deg D - 1.$$  

The genus of the nonsingular curve $\Gamma$ which is the normalized curve is called the geometric genus of the curve $\Gamma'$ and it is denoted by $p_g(\Gamma')$, and the quantity

$$p_a(\Gamma') = p_g(\Gamma') + \sum_{P \in S} \delta_P$$

is called the arithmetic genus of the curve $\Gamma'$.

We notice that a meromorphic 1-form (a differential) $\omega$ on the curve $\Gamma$ is called a differential regular at a point $P \in \Gamma'$, if the equality

$$\sum_{Q \to P} \text{Res} (f\omega) = 0$$

holds for all $f \in \mathcal{O}_Q'$. It is evident that there are more regular differentials on $\Gamma'$ than regular differentials on $\Gamma$, since regular differentials on $\Gamma'$ may have poles in the preimages of singular points. For instance, for a curve $\Gamma_D$ forms which are regular at a singular point $Q$ are distinguished by the following conditions: a form $\omega$ may have poles only at $P \in D$ with their orders not greater than $n_P$ and

$$\sum_{Q \to P} \text{Res} \omega = 0.$$  

It is easy to see that the dimension of the space of regular differentials equals $p_a(\Gamma')$.

Let the support of an effective divisor $D$ on the curve $\Gamma'$ pairwise not intersect with the support of the divisor $S'$. We denote by $\dim L(D)$ the space of meromorphic functions on $\Gamma'$ with poles only at points from $D = \sum n_P P$ with orders not greater than $n_P$, and denote by $\Omega'(D)$ the space of regular differentials on $\Gamma'$ which have at every point $P \in S$ a zero whose order is not less than $n_P$. The Riemann–Roch theorem reads that

$$\dim L(D) - \dim \Omega'(D) = \deg D + 1 - p_a(\Gamma').$$

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For a generic divisor $D$ we have $\dim \Omega'(D) = 0$ and the Riemann–Roch theorem takes the form
$$\dim L(D) = \deg D + 1 - p_a(\Gamma').$$

### 4 Schrödinger and Dirac operators corresponding to singular spectral curves

We consider curves of the form $\Gamma_{B_1, \ldots, B_n}$ which are successively constructed from effective divisors $B_1, \ldots, B_n$ and a curve $\Gamma$ by the same procedure which constructs the curve $\Gamma_D$ from $\Gamma$ and $D$. Of course we assume that the supports of divisors $B_i, i = 1, \ldots, n$, are pair-wise nonintersecting.

**Theorem 1**

1) Let $\Gamma' = \Gamma_{B_1, \ldots, B_n}$ be a singular curve, let $\pi : \Gamma \to \Gamma'$ be its normalization, let $S_1, \ldots, S_n$ be the supports of divisors $B_1, \ldots, B_n$ and let $Q_1, \ldots, Q_n$ singular points of the curve $\Gamma'$: $\pi(S_i) = B_i, i = 1, \ldots, n$.

Let $\infty_+, \infty_-$ be a pair of different points from $\Gamma' \setminus \{Q_1, \ldots, Q_n\}$ with local parameters $k_{\pm}^{-1}$ near these points such that $k_{\pm}^{-1}(\infty_{\pm}) = 0$, and let $D = P_1 + \ldots + P_g$ be a generic effective divisor on $\Gamma \setminus \{Q_1, \ldots, Q_n, \infty_+, \infty_-\}$ of degree $g = p_a(\Gamma')$.

Then there exists an unique function $\psi(x, y, P), P \in \Gamma'$, such that it is meromorphic everywhere outside the points $\infty_+$ and $\infty_-$ where it has the following asymptotics

$$\psi(x, y, P) \approx e^{k_{+}z} \left(1 + \xi(x, y)k_{+}^{-1} + O(k_{+}^{-2})\right) \text{ as } P \to \infty_+,$$

$$\psi(x, y, P) \approx c(x, y)e^{k_{-}\bar{z}} \left(1 + O(k_{-}^{-1})\right) \text{ as } P \to \infty_-;$$

and it has poles only in points from $D = P_1 + \ldots + P_g = \sum n_P P$ of order not greater than $n_P$.

The function $\psi$ satisfies the equation $L\psi = 0$, where the operator $L = \partial\bar{\partial} + A(z, \bar{z})\partial + u(z, \bar{z})$ is uniquely reconstructed from $\psi$:

$$A = \frac{\partial \log c}{\partial z}, \quad u = -\frac{\partial \xi}{\partial \bar{z}}$$

2) Let there be a holomorphic involution $\sigma$ on $\Gamma'$ which preserves the marked points $\infty_{\pm}$ and inverts the local parameters near them: $\sigma(\infty_{\pm}) = \infty_{\mp}, \sigma(k_{\pm}^{-1}) = -k_{\mp}^{-1}$. Let all singular points of the curve be fixed points of the involution and let the involution preserve the branches of the curve in these points (i.e. the pullback of the involution on $\Gamma$ preserves all points from $S$).

If there is a differential $\omega$ on $\Gamma_{2B_1, \ldots, 2B_n}$ such that it is regular everywhere outside the points $\infty_+$ and $\infty_-$ in which it has first order poles with residues $\pm 1$ and it has zeroes exactly in the points of $D + \sigma(D)$, then $L$ is a potential operator:

$$L = \partial\bar{\partial} + u.$$

3) If in addition there is an antiholomorphic involution $\tau : \Gamma' \to \Gamma'$ such that $\sigma\tau = \tau\sigma, \tau(D) = D, \tau(\infty_{\pm}) = \infty_{\mp}, \tau(k_{\pm}) = \bar{k}_{\pm}$, then the potential $u$ is a real-valued function.
We obtain the functions \( \psi \) proportional to \( B \) and for \( \psi \). Together with conditions for \( \psi \) there therefore curve \( \Gamma \) and the divisors \( L \psi \) therefore \( \psi \). Let us apply this construction to the normalized points \( \infty \) expands in a series in even degrees of a local parameter \( k \). Since the form \( \omega \) is regular on \( \Gamma \) and moreover, since \( \psi \) descends to a function on \( \Gamma \), \( \psi \) descends to a function on \( \Gamma \). For instance, given \( B = Q_1 + \ldots + Q_{g-l+1} \), the first condition is written as

\[
\psi(Q_1) = \psi(Q_2), \quad \psi(Q_1) = \psi(Q_3), \ldots, \quad \psi(Q_1) = \psi(Q_{g-l+1}),
\]

and for \( B = mQ \) it takes the form

\[
\frac{\partial \psi(Q)}{\partial w} = \ldots = \frac{\partial^{n-1-1} \psi(Q)}{\partial^{n-1} w} = 0,
\]

where \( w \) is a local parameter on \( \Gamma \) near \( Q \). The second condition looks the same in both cases:

\[
c_1 + \ldots + c_{g-l+1} = 1.
\]

The uniqueness of the functions \( \psi \) follows from the Riemann–Roch theorem \( \text{R} \). Together with conditions for \( c_i \) this implies the uniqueness of the function \( \psi \) for a generic divisor \( D \). For given functions \( A \) and \( u \) the function \( L \psi \) is proportional to \( \psi \), but its asymptotics as \( P \to \infty_+ \) are \( \alpha(z, \bar{z})e^{k_*}k_+^{-1} \), and therefore \( L \psi \) vanishes everywhere \( \text{R} \).

2) Let us consider the form \( \psi(P)\psi(\sigma(P))\psi(P) \). It is meromorphic and the points \( \infty_+ \) and \( \infty_- \) have residues 1 and \( -c^2 \), respectively. Let \( Q_i \) be a singular point, of \( \Gamma' \), corresponding to the divisor \( B_i = \sum n_i Q \). Since the function \( \psi(P)\psi(\sigma(P)) \) is invariant under the involution, near a fixed point \( Q \) this function expands in a series in even degrees of a local parameter \( k \) where \( k(Q) = 0, \sigma(k) = -k \):

\[
\psi(P)\psi(\sigma(P)) = \psi(Q)^2 + a_4 k^2 + \ldots + a_{2n} k^{2n} + \ldots, \quad k(P) = k,
\]

and moreover, since \( \psi \) descends to a function on \( \Gamma' \), we have \( a_j = 0 \) for \( j < n \). Since the form \( \omega \) is regular on \( \Gamma' \), \( \omega \) has a pole of order not greater than \( 2n \) at the point \( P \):

\[
\omega = b_{2n} \frac{dA}{k^{2n}} + \ldots + b_1 \frac{dA}{k} + \text{the regular terms}.
\]

Therefore, there is the following formula for the residue:

\[
\text{Res} [\psi(P)\psi(\sigma(P))\omega] \bigg|_{P=Q} = b_1(Q)\psi(Q)^2.
\]
The regularity of $\omega$ at $Q_i \in \Gamma'$ implies that the sum of the residues of $\omega$ over the preimage of this point vanishes:

$$\psi(Q_i)^2 \sum_{Q \in S_i} b_1(Q) = 0.$$ 

Therefore every singular point $Q_i$ does not contribute to the sum of the residues of the differential $\psi(P)\psi(\sigma(P))\omega$ and, since this sum equals

$$1 - c^2 = 0,$$

we have $c^2 = 0$ and $A = -\frac{\partial \log \epsilon}{\partial z} = 0$.

3) For nonsingular curves this statement was proved by Novikov and Veselov and in the nonsingular case the proof works without changes as follows. Let us consider the expansion for the function $\psi$ at $\infty$:

$$\psi \sim c e^{k z} (1 + \eta k^{-1} + O(k^{-2})).$$

Since $L = \bar{\partial} + u$ and $c^2 = 1$, we obtain

$$L\psi = c e^{k z} ((u + \partial \eta) + O(k^{-1})) = 0,$$

which implies the formula for the potential in terms of the asymptotics of $\psi$ near $\infty$:

$$u = -\partial \eta.$$

Since the function $\psi$ is uniquely reconstructed from the data $\Gamma', \infty, k, D$, this uniqueness theorem implies the equality

$$\overline{\psi(\tau(P))} = \bar{c}\psi(P) = c\psi(P).$$

In particular, $\xi = \bar{\eta}$, and comparing the two formulas for $u$ we have

$$u = -\bar{\partial} \xi = -\partial \eta,$$

We conclude that the potential $u$ is real-valued: $u = \bar{u}$.

This proves the theorem.

The essential part of this theorem is the condition for the differential $\omega$ which has to be regular not on $\Gamma_{B_1, \ldots, B_n}$ but on $\Gamma_{2B_1, \ldots, 2B_n}$. Together with that the proof of the following theorem on Dirac operators is obtained by a modification of the proof for a nonsingular case (see [2]).

**Theorem 2** 1) Let $\Gamma' = \Gamma_{B_1, \ldots, B_n}$ be a singular curve, let $\pi : \Gamma \rightarrow \Gamma'$ be its normalization, let $S_1, \ldots, S_n$ be the supports of the divisors $B_1, \ldots, B_n$ and let $Q_1, \ldots, Q_n$ be singular points of $\Gamma'$: $\pi(S_i) = B_i, i = 1, \ldots, n$.

Let $\infty_+, \infty_-$ be a pair of different points from $\Gamma' \setminus \{Q_1, \ldots, Q_n\}$ with local parameters $k_z^{\pm 1}$ near these points such that $k^{\pm 1}_z(\infty_\pm) = 0$, and let $D = P_1 + \ldots + P_n$ be a divisor on $\Gamma'$.

3Although it is not difficult to formulate the following theorem using Theorem 1 and the results exposed in §2.3 we do that for the completeness of exposition.
\[ \ldots + P_g + P_{g+1} \text{ be a generic effective divisor on } \Gamma \setminus \{Q_1, \ldots, Q_n, \infty_+, \infty_-\} \text{ of degree } g + 1 = p_a(\Gamma') + 1. \]

Then there exists a unique vector function \( \psi(x, y, P), P \in \Gamma' \), such that it is meromorphic everywhere outside the points \( \infty_+ \) and \( \infty_- \) where it has the following asymptotics

\[
\psi(x, y, P) \approx e^{k + z} \left[ \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} \xi^+ \\ \xi^- \end{pmatrix} k^{-1} + O(k^{-2}) \right] \quad \text{as } P \to \infty_+, \\
\psi(x, y, P) \approx e^{k - z} \left[ \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} \xi^- \\ \xi^+ \end{pmatrix} k^{-1} + O(k^{-2}) \right] \quad \text{as } P \to \infty_-;
\]

and it has poles only in points from \( D = P_1 + \ldots + P_{g+1} = \sum n_P P \) of order not greater than \( n_P \).

The function \( \psi \) satisfies the equation \( D \psi = 0 \), where the operator

\[
D = \begin{pmatrix} 0 & \partial \\ -\bar{\partial} & 0 \end{pmatrix} + \begin{pmatrix} U \\ 0 \end{pmatrix} \begin{pmatrix} 0 & V \\ 0 & V \end{pmatrix}
\]

is uniquely reconstructed from \( \psi \):

\[
U = -\xi^+_{2}, \quad V = \xi^-_{1}.
\]

2) Let there be a holomorphic involution \( \sigma : \Gamma' \to \Gamma' \) which preserves the marked points \( \infty_\pm \) and inverts the local parameters near them: \( \sigma(\infty_\pm) = \infty_\pm, \quad \sigma(k_{\pm}^{-1}) = -k_{\pm}^{-1} \). Let all singular points of the curve be fixed points of the involution and the involution preserves the branches of the curve at these points (i.e. the pullback of the involution on \( \Gamma \) preserves all points from \( S \)).

If there is a differential \( \omega \) on \( \Gamma_{2B_1, \ldots, 2B_n} \) such that it is regular everywhere outside the points \( \infty_+ \) and \( \infty_- \) in which it has second order poles with the principal parts \( \pm k_{\pm}^2 (1 + O(k_{\pm}^{-1})) dk_{\pm}^{-1} \) and its has zeroes exactly at points from \( D + \sigma(D) \), then the potentials \( U \) and \( V \) coincide:

\[
U = V.
\]

3) Let there be an antiholomorphic involution \( \tau : \Gamma' \to \Gamma' \) such that it interchanges the points \( \infty_+ \) and \( \infty_- \):

\[
\tau(\infty_\pm) = \infty_\mp, \quad \tau(k_{\pm}) = -k_{\mp},
\]

and its pullback onto \( \Gamma \) preserves all points from \( S_1 \cup \ldots \cup S_n \), changing local parameters \( k \) by the formula \( \tau(k) = -\bar{k} \).

Let there exist a differential \( \omega' \), on \( \Gamma_{2B_1, \ldots, 2B_n} \), which is regular everywhere outside the points \( \infty_+ \) and \( \infty_- \) in which it has second order poles with the principal parts \( k_{\pm}^2 (1 + O(k_{\pm}^{-1})) dk_{\pm}^{-1} \) and which has zeroes exactly at points from \( D + \tau(D) \). Then the potentials \( U \) and \( V \) are real-valued:

\[
U = \bar{U}, \quad V = \bar{V}.
\]
Remarks.

1) Potential Schrödinger operators whose spectral curves have only double points were described in the initial paper by Novikov and Veselov [6] as obtained by contracting invariant cycles on nonsingular curves into points (see pic. 1 where it is demonstrated by a deformation of an elliptic curve. Here by an involution we mean a rotation by π around the horizontal line which lies in the plane of the picture and the right arrow denotes the normalization mapping).

![Figure 1: A double point with preserved branches.](image)

These potentials are described in terms of the Prym theta functions of double-sheeted coverings of singular curves. The curve Γ has double points which are fixed points of the involution and moreover the branches are not permuted by the involution (i.e. on the normalized curve the preimages of such fixed points are fixed by the involution, see pic. 1). In this case a complete Prym variety is defined as a principally-polarized Abelian variety in a limit under the degeneration of nonsingular curves Γ. In this limit the Jacobian variety of the curve Γ/σ is a non-complete Abelian variety, i.e. it has a form \( \mathbb{C}^g/Z \) where the rank of a lattice \( Z \) is less than \( 2g \).

If the branches in a double point are permuted by the involution (see pic. 2), then the limiting Prym variety is not complete but the limiting Jacobian variety of the curve Γ/σ is complete.

![Figure 2: A double point with permuted branches.](image)

2) As we already pointed out above (in § 2), for a nonsingular curve corresponding to a finite-gap potential Schrödinger operator the following equality holds

\[
p_a(\Gamma) = 2p_a(\Gamma/\sigma),
\]

which relates the arithmetic genera of the curve and its quotient under the involution σ. It also holds for curves with double points. Let \( \Gamma_B \) be a curve such that Γ is a hyperelliptic curve of genus two, σ is the hyperelliptic involution, and \( \infty_+ \cup B \) are six fixed points of σ: \( B = Q_1 + \ldots + Q_4 \). We have \( p_a(\Gamma_B) = p_g(\Gamma) + 3 = 5 \) and \( \Gamma_B/\sigma \) is a rational curve (a sphere) with a quadruple point.
Therefore, for this example we have

\[ p_a(\Gamma') = 5, \quad p_a(\Gamma'/\sigma) = 3. \]

In this case the potential of the Schrödinger operator is written in terms of the Prym variety of the double-sheeted covering \( \Gamma' \to \Gamma'/\sigma \). This Prym variety is isomorphic to the Jacobian variety of the curve \( \Gamma \).

3) Notice, that, by the Krichever theorem [10], all smooth real potentials of the Schrödinger operator are approximated by the potentials which are finite-gap on the zero energy level to arbitrary precision. Moreover this approximation is generated by approximations of their Floquet spectra by nonsingular spectral curves of finite genera. For the Dirac operator the analogue of this theorem is not proved but it is clear that its proof can be obtained by some modification of Krichever’s reasonings.

4) One-dimensional Schrödinger operators \( L = \partial_x^2 + u \) have hyperelliptic spectral curves which parameterize Bloch functions for all values of the energy \([11]\). Degenerations of such curves (inside the class of hyperelliptic curves) lead to soliton potentials on the background of finite-gap potentials. \([12]\).

For instance, on the language of § 3 the rational potential \( u(x) = 2x^{-2} \) is constructed from the curve \( \Gamma : w^2 = E \) and the point \( P \) with \( E = w = 0 \). Its spectral curve is \( \Gamma_{2P} \), and the spectral curve of the potential \( u(x) = 2/\cosh^2(x) \) has the form \( \Gamma_{Q+\sigma(Q)} \), where \( \sigma \) is the hyperelliptic involution and \( Q \neq \sigma(Q) \).

5) Notice that the operators \( L = i\partial_y - \partial_x^2 + u \) whose spectral curves on the zero energy level are singular and normalized by a rational curve were described in the paper \([13]\).

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