Strong Coupling Model for String Breaking on the Lattice.

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Abstract

A model for $SU(n)$ string breaking on the lattice is formulated using strong coupling ideas. Although necessarily rather crude, the model gives an explicit picture of string breaking in the presence of dynamical quarks as a mixing phenomenon that involves the string state and a two-meson state. A careful analysis, within the model, of the Wilson loop shows that the evolution of the mixing angle as a function of separation may obscure the expected crossover effect. If a sufficiently extensive mixing region exists then an appropriate combination of transition amplitudes can help in revealing the effect.

The sensitivity of the mixing region to the values of the meson energy and the dynamical quark mass is explored and an assessment made of the detectability of string breaking in a practical lattice simulation.
1 Introduction

Currently much effort in lattice gauge theory simulations is being devoted to dynamical quarks [1, 2, 3, 4, 5, 6]. For a review see S G"usken [7]. For computational reasons the quarks incorporated in the model are still relatively massive. This makes it difficult to detect dramatic differences in the computed hadron spectrum relative to results obtained in the quenched approximation. In order to highlight the effect of dynamical quarks it is important to investigate phenomena that cannot occur in their absence. Gauge string breaking is an example of this type of event. Here we show explicitly that indeed string breaking does occur on the lattice and shows up, as expected, as a mixing phenomenon.

The physical situation envisaged in lattice calculations comprises a static quark and anti-quark separated by a spatial distance of $R$ lattice units (we mostly set the lattice spacing to unity). The two static particles may either support a gauge string that runs between them or separately bind a quark and an anti-quark to create a two-meson state. When the string is stretched sufficiently, its energy coincides with the energy of the two meson state. In the neighbourhood of this critical separation the two different physical states mix permitting the occurrence of transitions between them.

There are two related issues of importance, namely the size of the range in $R$ over which there is substantial mixing and the magnitude of the energy split induced by mixing. As fractional effects we find that they are proportional to one another and therefore of essentially the same size. We show in terms of our model that, for the lowest quark masses and for meson excitations that are not too great, the mixing effects of string breaking may well be detectable. If the meson masses are too great and/or the quarks too heavy then the phenomenon will be undetectable on the lattice.

Because it is based on strong coupling ideas the results of our model are provisional and require interpretation if they are to be applied to actual simulations. Nevertheless the model is very suggestive and identifies crucial aspects of the mixing phenomenon to which the results are sensitive.

2 Strong Coupling Model

We formulate the model by first recalling the well known rules for evaluating simple graphs in the strong coupling limit of $SU(n)$ gauge theory [8, 9]. They are as follows:
1. A factor of
\[
\left( \frac{\beta}{2n^2} \right)
\]
for each plaquette.

2. A factor of
\[
2\kappa \left( \frac{1 + \gamma \cdot e}{2} \right)
\]
for each Wilson quark line in the direction of the unit vector \( e \). Here \( \kappa \) is the standard quark hopping parameter.

3. A factor of \((-1)\) for each internal quark loop.

4. A factor of \(1/n\) for each internal quark loop.

5. A trace over the spin matrix factors for each internal quark loop.

### 2.1 Simple String Model

In the absence of dynamical quarks our model for the correlation function of the string of length \( R \) over an imaginary time interval \( T \) is the standard \( R \times T \) Wilson loop. In leading strong coupling approximation the above rules give for this string-string propagator

\[
\mathcal{G}_{SS}(T) = \left[ \left( \frac{\beta}{2n^2} \right)^R \right]^T .
\]

This tells us immediately that the energy of the string state is \( V(R) \) where

\[
V(R) = \sigma R ,
\]

and \( \sigma \) is the dimensionless string tension given by

\[
\sigma = -\log \left( \frac{\beta}{2n^2} \right) .
\]

Note that we have omitted the spin degrees of freedom of the static quarks. They play no rôle in the model.
2.2 Model for Mesons

The simple model for mesons suggested by the strong coupling limit, is one in which a light quark propagates along a static (anti-)quark line. See Fig 1. Using the rules, we find that this meson propagator has the structure

\[ g(T) = \frac{1 + \gamma_0}{2} (2\kappa)^T . \]  

(4)

Of course we retain the spin degrees of freedom of the light quarks. They do play a rôle in the model.

The propagator for two mesons moving independently each bound to its static quark is

\[ g^{(1)}(T) \otimes g^{(2)}(T) = \left( \frac{1 + \gamma_0}{2} \right)^{(1)} \otimes \left( \frac{1 - \gamma_0}{2} \right)^{(2)} \left( (2\kappa)^2 \right)^T . \]  

(5)

In fact, as will become clear below, only a particular combination of quark-anti-quark spin wave functions enters the calculation. It is obtained by completing the light quark loop with the matrices

\[ \frac{1 \pm \gamma_1}{2} , \]  

and including a factor of \((-1)\). This two-meson propagator is

\[ G_{MM}(T) = -\text{Tr} \left( \frac{1 - \gamma_1}{2} \right) g^{(1)}(T) \left( \frac{1 + \gamma_1}{2} \right) g^{(2)}(T) = \frac{1}{2} \left( (2\kappa)^2 \right)^T . \]  

(7)

Finally we will give the light quarks, when bound in a meson, a hopping parameter \(\kappa'\) that may be distinct from the value \(\kappa\) that holds elsewhere in the diagrams of the model. Eq(4) should be modified therefore by the replacement \(\kappa \rightarrow \kappa'\). The essential point is that we are treating the energy, \(E_M\), of the mesons as a parameter. This is reasonable since the energy of the mesons is an important number in the string breaking process and a simple version of the hopping parameter expansion cannot reflect the full dynamics of the meson bound state. We assume then

\[ (2\kappa')^2 = e^{-E_M} , \]  

(8)

where \(E_M\) is the combined energy of the two bound mesons.

3 String Breaking

Our model for string breaking involves summing over planar graphs that incorporate transitions between the string and meson states described above. We arrive at the
rules for computing these graphs by considering the case of string-to-string propagation represented by Fig 2. From the rules we see that the corresponding contribution to the Green’s function is

\[
\left[ \left( \frac{\beta}{2n^2} \right)^R \right]^{T_1} \left[ (2\kappa')^2 \right]^{T_2} \left[ \left( \frac{\beta}{2n^2} \right)^R \right]^{T_3} \left[ (2\kappa')^2 \right]^{T_4} \left[ \left( \frac{\beta}{2n^2} \right)^R \right]^{T_1} \left( -\frac{t}{n} \right) \left[ (2\kappa)^2 \right]^R \left( -\frac{t}{n} \right) \left[ (2\kappa)^2 \right]^R ,
\]

where

\[
t = \text{Tr} \left( \frac{1 + \gamma_0}{2} \right) \left( \frac{1 + \gamma_1}{2} \right) \left( \frac{1 - \gamma_0}{2} \right) \left( \frac{1 - \gamma_1}{2} \right) = -\frac{1}{2} .
\]

The factors involving \( \kappa \) (as opposed to \( \kappa' \)) are associated with the vertical sides of the quark loops. Note that the trace of the internal quark loop is equivalent to the quark-anti-quark spin projection mentioned previously.

We reinterpret the structure of such diagrams as follows:

1. The factor

\[
\left( \frac{\beta}{2n^2} \right)^R
\]

propagates the string by one time step.

2. The factor

\[
(2\kappa')^2
\]

propagates the two-meson state by one time step.

3. The factor

\[
\frac{1}{\sqrt{2n}} (2\kappa)^R
\]

is associated with the transition from string to two-meson state and vice-versa.

At any stage we may view the system as being in either a string or a two-meson state. To describe the transition from an initial time to time \( T \) we need a \( 2 \times 2 \) matrix of transition amplitudes

\[
G(T) = \begin{pmatrix} G_{SS}(T) & G_{SM}(T) \\ G_{MS}(T) & G_{MM}(T) \end{pmatrix} .
\]

If for convenience, we introduce the parameters

\[
a = \left( \frac{\beta}{2n^2} \right)^R , \quad b = (2\kappa')^2 , \quad c = \frac{1}{\sqrt{2n}} (2\kappa)^R
\]

(12)
then the above stepping procedure can be represented by

\[ G(T + 1) = A \begin{pmatrix} G_{SS}(T) & G_{SM}(T) \\ G_{MS}(T) & G_{MM}(T) \end{pmatrix}, \]  

(13)

where the matrix \( A \) is given by

\[ A = \begin{pmatrix} a & ac \\ bc & b \end{pmatrix}. \]  

(14)

If for definiteness we set

\[ G(0) = 1, \]  

(15)

then

\[ G(T) = (A)^T. \]  

(16)

The matrix \( A \) is not symmetrical but can be expressed in terms of a symmetric matrix \( B \) in the form

\[ A = DBD^{-1}, \]  

(17)

where \( D \) is the diagonal matrix

\[ D = \begin{pmatrix} \sqrt{a} & 0 \\ 0 & \sqrt{b} \end{pmatrix}. \]  

(18)

and

\[ B = \begin{pmatrix} a & \sqrt{ab} c \\ \sqrt{ab} c & b \end{pmatrix}. \]  

(19)

The eigenvalues of \( B \) are

\[ \lambda_{\pm} = \frac{1}{2} \left\{ (a + b) \pm \sqrt{(a - b)^2 + 4abc^2} \right\}. \]  

(20)

The corresponding eigenvectors are

\[ \chi_+ = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, \quad \chi_- = \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix}, \]  

(21)

where

\[ \tan \theta = \frac{-(a - b) + \sqrt{(a - b)^2 + 4abc^2}}{2 \sqrt{ab} c}. \]  

(22)

If we construct the orthogonal matrix

\[ O = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \]  

(23)
then

\[ B = O\Lambda O^{-1} \]  \hspace{1cm} (24)

where

\[ \Lambda = \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix} . \]  \hspace{1cm} (25)

It follows immediately that for any \( T \)

\[ A^T = DO \begin{pmatrix} \lambda_+^T & 0 \\ 0 & \lambda_-^T \end{pmatrix} O^{-1} D^{-1} . \]  \hspace{1cm} (26)

This shows clearly that the propagating eigenstates are mixtures of the string and two-meson states determined by the mixing angle \( \theta \). The corresponding eigen-energies are

\[ E_{\pm} = -\log \lambda_{\pm} . \]  \hspace{1cm} (27)

Note that \( E_+ \) is the lower of the two energies.

4 Wilson Loop

We can now use the above results to elucidate the behaviour of the Wilson loop provided we pay attention to end conditions. The Wilson loop begins with a factor \( a \). The sum of all such diagrams of length \( T \) are obtained as the components of

\[ A^{T-1} \begin{pmatrix} a \\ 0 \end{pmatrix} . \]  \hspace{1cm} (28)

The Wilson loop itself is obtained as

\[ W(R, T) = G_{ss}(T) = (1, 0) DO\Lambda^{T-1} O^{-1} D^{-1} \begin{pmatrix} a \\ 0 \end{pmatrix} . \]  \hspace{1cm} (29)

That is

\[ W(R, T) = a \left( \cos^2 \theta \lambda_+^{T-1} + \sin^2 \theta \lambda_-^{T-1} \right) . \]  \hspace{1cm} (30)

The Wilson loop, therefore, sees both exponentials

\[ e^{-E_{\pm}^T} = \lambda_{\pm}^T . \]  \hspace{1cm} (31)

In principle, the the lower energy exponential should dominate the asymptotic behaviour. In practice, what is observed will be influenced by the behaviour of the
coefficients of the exponential contributions through their dependence on the mixing angle.

Noting that
\[ \sin^2 \theta = \frac{-(a-b) + \sqrt{(a-b)^2 + 4abc^2}}{2 \sqrt{(a-b)^2 + 4abc^2}} , \quad (32) \]
we see that when \( V(R) \ll E_M \) then \( a \gg b \) and we have
\[ \theta \simeq 0 , \quad \lambda_+ \simeq a , \quad \lambda_- \simeq b , \quad (33) \]
with the result that \( E_+ \simeq V(R) \) and \( E_- \simeq E_M \). Under these circumstances the coupling to the state with energy \( E_- \) will vanish and only the exponential associated with \( E_+ \) will be observed as expected. When \( R \) is chosen so that \( V(R) \gg E_M \) then \( b \gg a \) and we have
\[ \theta \simeq \frac{\pi}{2} , \quad \lambda_+ \simeq b , \quad \lambda_- \simeq a , \quad (34) \]
with the result that \( E_+ \simeq E_M \) and \( E_- \simeq V(R) \). Under these circumstances the coupling to the state with energy \( E_+ \) will vanish and only the exponential associated with \( E_- \) will be observed. That is, both above and below the crossover point, only the original string behaviour
\[ e^{-V(R)T} , \]
will be observed. The movement of the mixing angle therefore obscures the crossover phenomenon in which the string energy is expected to be bounded by the energy of the two-meson state. As is clear from the above analysis this does not happen.

To see mixing effects directly in the Wilson loop we must fix \( R \) near the critical crossover value \( R_c \), for which \( V(R) = E_M \) and hence \( a = b \). That is
\[ R_c = \frac{E_M}{\sigma} . \quad (35) \]
At this point \( \theta = \pi/4 \) and
\[ W(R,T) = \frac{a}{2} \left( \lambda_+^{T-1} + \lambda_-^{T-1} \right) , \quad (36) \]
so the two exponentials appear with equal strength. In order to separate them in a measurement it must be possible to measure their difference. We find
\[ \lambda_\pm = b(1 \pm c) . \quad (37) \]
So approximately for small $c$

$$E_\pm = -\log \lambda_\pm \simeq E_M(1 \mp c) \ , \quad (38)$$

that is

$$\frac{\Delta E}{E_M} = \frac{2c}{\sigma R_c} \ . \quad (39)$$

We can define the mixing region as the range of $R$ for which both $\sin^2 \theta$ and $\cos^2 \theta$ are significantly different from unity. The range of the mixing region in $R$ can be estimated as

$$\frac{\Delta R}{R_c} = \left( \frac{\pi}{2} \frac{dR}{d\theta} \right)_{R=R_c} \ . \quad (40)$$

It is closely related to the mixing energy we find

$$\frac{\Delta R}{R_c} = \pi \frac{\Delta E}{E_M} \ . \quad (41)$$

We estimate the numerical significance of these results below.

5 String-Meson Transition

The string-meson transition amplitude can computed from eq(28) as

$$G_{MS}(T) = (0, 1)A \left( \begin{array}{c} a \\ 0 \end{array} \right) \ . \quad (42)$$

This gives

$$G_{MS}(T) = \sqrt{ab} \ \sin \theta \cos \theta \left( \lambda_+^{T-1} - \lambda_-^{T-1} \right) \ . \quad (43)$$

Because of the presence of the factor $\sin \theta \cos \theta$ this transition amplitude vanishes outside the mixing region. In this region, however, it is possible to detect the presence of the dominant energy provided the two energies are sufficiently well separated. However if the two eigen-energies are only weakly separated then the amplitude will experience a further suppression because the two contributions have opposite signs that give rise to a cancellation. This is consistent with the results of the previous section on the Wilson loop and fits in with the obvious idea that if the primitive transition process vanishes then the transition amplitude itself vanishes.
6 Meson-Meson Transition

The meson-meson transition involves a sum over amplitudes that begin and end with a factor \( b \). They can be evaluated as

\[
G_{MM}(T) = (0, 1)A\begin{pmatrix} 0 \\ b \end{pmatrix}.
\] (44)

The result is

\[
G_{MM}(T) = b\left(\sin^2 \theta \lambda^T_+ - \cos^2 \theta \lambda^T_-\right)
\] (45)

Clearly this amplitude shows properties similar to that for string-string transitions in that on both sides of the mixing region in \( R \) only the meson energy is detected in the \( T \)-dependence of the correlator. However the result suggests that when \( R \) is within the mixing region, the combination \( b \ G_{SS}(T) + a \ G_{MM}(T) \) will show a reduced influence from the mixing angle and will allow the best chance of tracking both \( E_+ \) and \( E_- \) individually from the resulting \( T \)-dependence. Of course this requires favourable circumstances in which the mixing region is sufficiently large and the energy separation of the mixed states is sufficiently great.

7 Interpretation of Results

In order to use the results of our analysis we will make the assumption that the parameters of the model can be interpreted directly in terms of the appropriate physical parameters of an actual simulation rather than as bare lattice parameters. We have already anticipated this approach when we introduced \( \kappa' \) as a way of parametrising the meson energy \( E_M \). In the same spirit we will interpret the other hopping parameter \( \kappa \) in terms of the light quark mass, \( m_q \), and require

\[
2\kappa = e^{-m_q}.
\] (46)

We therefore have the identifications

\[
a = e^{-\sigma R}, \quad b = e^{-E_M}, \quad c = \frac{1}{\sqrt{R}}e^{-m_q R}.
\] (47)

With these identifications we then see that the fractional energy shift due to mixing is

\[
\frac{\Delta E}{E_M} = \sqrt{\frac{2}{3} e^{-m_q R_c \kappa}}.
\] (48)
Table 1: These results are for $a^{-1} = 1.5$ GeV, $\sigma^{1/2} = 0.427$ GeV, $E_M = 1.0$ GeV. The critical distance is $R_c = 8.22$ lattice spacings.

| $m_q$ (GeV) | $\Delta E/E_M$ | $\Delta R$ |
|------------|----------------|-----------|
| .1         | .7077          | 18.3      |
| .2         | .4089          | 10.6      |
| .3         | .2363          | 6.1       |
| .4         | .1365          | 3.5       |
| .5         | .0789          | 2.0       |

Table 2: These results are for $a^{-1} = 1.5$ GeV, $\sigma^{1/2} = 0.427$ GeV, $E_M = 2.0$ GeV. The critical distance is $R_c = 16.45$ lattice spacings.

| $m_q$ (GeV) | $\Delta E/E_M$ | $\Delta R$ |
|------------|----------------|-----------|
| .1         | .2044          | 10.6      |
| .2         | .0682          | 3.5       |
| .3         | .02295         | 1.2       |
| .4         | .0076          | 0.4       |
| .5         | .0025          | 0.1       |

and the mixing range in $R$ is

$$\Delta R = \pi \sqrt{\frac{2}{3} \frac{e^{-m_q R_c}}{E_M}} R_c .$$  \hspace{1cm} (49)$$

We have now specialised to $SU(3)$. If we recall that $R_c = E_M/\sigma$ then we see from this formula, through its dependence on $R_c$, that the mixing range is very sensitive to both the meson energy, $E_M$ and to the quark mass, $m_q$. In fact

$$\Delta R = \pi \sqrt{\frac{2}{3} \frac{e^{-m_q E_M/\sigma}}{\sigma}} .$$  \hspace{1cm} (50)$$

In order to obtain numerical estimates we consider a range of reasonable values appropriate to practical simulations for these parameters. The results are shown in Tables 1& 2. These results show directly the sensitivity of the physical picture to the energy of the two-meson state and to the quark mass. For an inverse lattice spacing of 1.5 GeV and a two-meson energy of 1 GeV, we obtain a critical separation $R_c$ of roughly 8 lattice spacings. At the rather light value of 100 MeV for the quark mass we see that the mixing region is large and covers a range of more than twice this critical distance.
For such circumstances we might expect to see the effects of mixing clearly. As the quark mass increases in size the mixing region shrinks rapidly so that by about 400 MeV it has reduced to a barely observable range of just over 1 lattice spacing either side of the critical separation. When the two-meson energy is increased to 2 GeV only the lightest quark mass of 100 MeV yields a mixing region of any size. At the lower two-meson energy the results suggest that there would be a reasonably detectable energy difference for the lower quark masses. At the larger two-meson energy only the lightest quark mass yields a substantial mass difference in the mixing region.

8 Conclusions

We have formulated a simple model of string breaking in terms of strong coupling ideas. The model shows explicitly string breaking as a mixing phenomenon between string and two-meson states. Because of the nature of the dependence of the Wilson loop on the angle of mixing between these two states, we do not expect to see the crossover phenomenon for the string tension in this amplitude. Instead the original string tension should be observed on both sides of the mixing region in $R$. In a similar way the meson-meson correlator will show only the meson energy on both sides of the mixing region. The string-meson transition amplitude vanishes outside the mixing region and suffers a suppression inside the region because the eigen-energy exponentials enter with opposite signs. This amplitude should therefore show at best only a weak signal in a simulation unless the mixing region is sufficiently large and the eigen-energies sufficiently distinct. The model suggests that a particular combination of the string-string and meson-meson amplitudes would show a reduced dependence on the mixing angle and give the best chance of tracking the two eigen-energy exponentials through the mixing region. In a practical simulation this combination would have to be sought empirically.

To observe string breaking it is necessary to be able to measure amplitudes in the mixing region with sufficient accuracy. For this to be possible a simple interpretation of the model in terms of realistic simulation parameters suggests that it will only be possible if the meson energy is not too high and the quark mass is sufficiently low. More precisely we would expect that at the critical value of the static quark separation, $R_c$, we have $V(R_c) \leq 1.0$ GeV and $m_q \leq 200$ MeV. The sensitivity of the mixing region to the value of the light quark mass, predicted by the model, may yield interesting results on string breaking if exploited systematically in practical simulations.
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Figure 1: A dynamical quark (light line) bound to a static quark (heavy line). There is closure at the ends of the lines and an implied trace for the gauge degrees of freedom but not for the light quark spin degrees of freedom.

Figure 2: Wilson loop (heavy line) containing internal quark loops (light lines).