Bethe–Salpeter Equations with Instantaneous Confinement: Establishing Stability of Bound States

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Abstract. Analytic scrutinies of Salpeter equations with confining interactions may identify kernels that describe bound states free from the notorious instabilities encountered in numerical evaluations.

Keywords: Bethe–Salpeter formalism, three-dimensional reduction, instantaneous approximation, Salpeter equation, bound states within quantum field theory, spectral analysis, confining interactions

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TOO NAÏVELY IMPLEMENTED CONFINEMENT MIGHT ALLOW FOR UNSTABLE BOUND STATES

The perhaps most popular three-dimensional reduction of the Bethe–Salpeter framework for the description of bound states within quantum field theories is the Salpeter equation, found in the instantaneous limit of the Bethe–Salpeter formalism if assuming in addition free propagation of the bound-state constituents. Unfortunately, depending on the chosen Dirac structure of the Bethe–Salpeter kernel (which encodes all interactions between the bound-state constituents) even for Salpeter equations with confining interactions, arising, e.g., from quantum chromodynamics, the predicted — supposedly stable — bound states exhibit instabilities, probably of similar nature as those observed in Klein’s paradox. The observation of this kind of instability has been reported by numerous but to a large extent numerical studies investigating confining interaction potentials of in configuration space either strictly linear shape or a form interpolating between harmonic-oscillator and linear behaviour [1, 2]. Generally, for linearly confining Bethe–Salpeter kernels being mixtures of Lorentz scalar plus time-component Lorentz vector, stability can be assured only if the Dirac structure of this kernel is predominantly of time-component Lorentz-vector nature.

Building on experience acquired in previous analyses, focused to the simpler “reduced Salpeter equation” [3, 4] and its improvement [5, 6] by including dressed propagators for all bound-state constituents [7–9], this investigation aims at rigorous analytic proofs [10] of the stability of solutions of the (now) full Salpeter equation with confining interactions (such as harmonic-oscillator potentials) of frequently used Lorentz structures, in order to identify all those kernels for which bound-state stability can be taken as granted ab initio. We regard bound states as stable if their energy (or, in their center-of-momentum system, mass) eigenvalues belong to a real, discrete portion of the corresponding spectrum that is bounded from below. Such discussion might provide further insight into the reasons why, for Lorentz structures different from a time-component Lorentz vector, instabilities arise.
**FULL (IN CONTRAST TO REDUCED) SALPETER FORMALISM FOR RELATIVISTIC FERMION–ANTIFERMION BOUND STATES**

Assuming, as usual, the Lorentz structures of the effective couplings of both fermion and antifermion to be represented by identical Dirac matrices (generically called $\Gamma$ hereafter) and denoting the associated Lorentz-scalar interaction function by $V_\Gamma(p, q)$, our *Salpeter eigenvalue equation* governing a chosen fermion–antifermion bound state of mass $M$ and distribution $\Phi(p)$ of internal momenta $p$ (its “Salpeter amplitude”) reads in the rest frame

$$
\Phi(p) = \int \frac{d^3 q}{(2\pi)^3} \sum_\Gamma V_\Gamma(p, q) \left( \frac{\Lambda^+(p)\gamma_0 \Gamma \Phi(q) \Gamma \Lambda^-(p) \gamma_0}{M - 2E(p)} - \frac{\Lambda^-(p)\gamma_0 \Gamma \Phi(q) \Gamma \Lambda^+(p) \gamma_0}{M + 2E(p)} \right),
$$

(1)

with one-particle kinetic energy $E(p)$ and energy projection operators $\Lambda^\pm(p)$, defined by

$$E(p) \equiv \sqrt{p^2 + m^2}, \quad p \equiv |p|, \quad \text{and} \quad \Lambda^\pm(p) \equiv \frac{E(p) \pm \gamma_0 (\gamma \cdot p + m)}{2E(p)};$$

here $m$ labels the common mass of the bound fermion and the associated antiparticle. The projector structure of Eq. (1) constrains all its solutions, the Salpeter amplitudes $\Phi(p)$, to

$$\Lambda^\pm(p) \Phi(p) \Lambda^\pm(-p) = 0. \quad (2)$$

**GENERAL FEATURES OF THE EIGENVALUE SPECTRA OF FULL SALPETER EQUATIONS**

The structural equivalence of Salpeter’s equation (1) to the random-phase-approximation equation [11], familiar from the investigation of collective excitations in nuclear physics, or direct inspection allow one to identify various characteristics common to all solutions:

- The random-phase-approximation structure of the Salpeter equation guarantees that the squares $M^2$ of the mass eigenvalues are real. In general, the spectrum itself is not necessarily real and, even where it is proven to be real, it is not bounded from below.
- The most important applications of the *instantaneous* Bethe–Salpeter formalism are those which adopt interaction kernels only composed of momentum-space potential functions $V_\Gamma(p, q)$ and Dirac couplings $\Gamma$ that satisfy $V_\Gamma^\dagger(p, q) = V_\Gamma(p, q) = V_\Gamma(q, p)$ and $\gamma_0 \Gamma^\dagger \gamma_0 = \pm \Gamma$. Their spectra of mass eigenvalues $M$ in the complex-$M$ plane are just unions of real opposite-sign pairs $(M, -M)$ and/or imaginary points $M = -M^*$. Since eigenvalues embedded in a continuous part of the spectrum may cause instabilities, the nature of the entire spectrum proves to be crucial: Apart from demanding eigenvalues to be real and bounded from below the bound-state stability sought is established if either eigenvalues and continuous spectrum are disjoint or the spectrum is purely discrete at all.

As a consequence of the constraint (2), any Salpeter amplitude $\Phi(p)$ may be expanded in terms of at most eight independent components called, say, $\phi_i(p), i = 1, \ldots, 8$. We start this quest for bound-state stability at a system requiring the least number of components.
HARMONIC-OSCILLATOR CONFINEMENT SIMPLIFIES THE 
SALPETER INTEGRAL EQUATION TO AN EASIER-TO-HANDLE 
SYSTEM OF RADIAL EIGENVALUE DIFFERENTIAL EQUATIONS

All the instabilities under consideration are expected to show up first in the pseudoscalar sector [1]. Consequently, the primary targets of all analyses of the present kind are bound states with spin-parity-charge conjugation assignment $J^{PC} = 0^{-+}$. A Salpeter amplitude $\Phi(p)$ describing such a state involves just two independent components $\phi_1(p)$ and $\phi_2(p)$:

$$ \Phi(p) = \left[ \phi_1(p) \frac{\gamma_0 (\gamma \cdot p + m)}{E(p)} + \phi_2(p) \right] \gamma_5 $$

is the unique form of any Salpeter amplitude $\Phi(p)$ for a fermion–antifermion bound state of total spin $J$, parity $P = (-1)^{J+1}$, and charge-conjugation quantum number $C = (-1)^J$.

Let our interaction kernel be of convolution type $[V_{\Gamma}(p, q) = V_{\Gamma}(p - q)]$, arising from a central potential $V(r)$, where $r \equiv |x|$, in configuration space. Using a harmonic-oscillator potential $V(r) = a r^2$, where $a \neq 0$ avoids triviality, to exemplify our line of reasoning, the Salpeter equation (1) reduces to a set of second-order differential equations utilizing only a single differential operator (which is just the Laplacian $\Delta \equiv \nabla \cdot \nabla$, acting on $\ell = 0$ states)

$$ D \equiv \frac{d^2}{dp^2} + \frac{2}{p} \frac{d}{dp} . $$

Assuming massless bound-state constituents, $m = 0$, facilitates our study of this problem.

In our analysis interaction kernels of time-component Lorentz-vector, $\Gamma \otimes \Gamma = \gamma^0 \otimes \gamma^0$, and Lorentz-scalar, $\Gamma \otimes \Gamma = 1 \otimes 1$, nature may be discussed simultaneously if introducing

$$ \sigma = \begin{cases} 
+1 & \text{for } \Gamma \otimes \Gamma = \gamma^0 \otimes \gamma^0 \quad \text{(time-component Lorentz-vector interactions)} \\
-1 & \text{for } \Gamma \otimes \Gamma = 1 \otimes 1 \quad \text{(Lorentz-scalar interactions)}
\end{cases} $$

as a discriminating parameter. By factorizing off all dependence on angular variables, the Salpeter equation for harmonic-oscillator interactions of time-component Lorentz-vector or Lorentz-scalar Dirac structure reduces to a system of two radial differential equations:

$$ \begin{align*}
(2p - a \sigma D) \phi_2(p) &= M \phi_1(p)
\end{align*} $$

$$ \begin{align*}
\left[ 2p - a \left( D - \frac{2}{p^2} \right) \right] \phi_1(p) &= M \phi_2(p).
\end{align*} $$

Adding the sets for $\sigma = +1$ and $\sigma = -1$, we find the Salpeter equation for Lorentz-scalar plus time-component Lorentz-vector mixing, $\Gamma \otimes \Gamma = \xi \gamma^0 \otimes \gamma^0 + \eta 1 \otimes 1$, where $\xi, \eta \in \mathbb{R}$:

$$ \begin{align*}
\left[ 2p - a (\xi - \eta) D \right] \phi_2(p) &= M \phi_1(p)
\end{align*} $$

$$ \begin{align*}
\left[ 2p - a (\xi + \eta) \left( D - \frac{2}{p^2} \right) \right] \phi_1(p) &= M \phi_2(p).
\end{align*} $$

The units of the momentum $p$ can be chosen, without loss of generality, such that $|a| = 1$.

However, in spite of the considerable technical simplification achieved by reduction of the Salpeter integral equation to a set of differential equations, for the kernels of different Lorentz structure we shall perform the spectral analyses still due on a case-by-case basis.
CONFINING INTERACTION KERNELS OF TIME-COMPONENT
LORENTZ-VECTOR DIRAC STRUCTURE (\(\Gamma \otimes \Gamma = \gamma^0 \otimes \gamma^0\))

For the time-component Lorentz-vector Dirac couplings (\(\sigma = +1\)), the mere existence of bound states requires the coupling \(a\) to be positive: \(a > 0\). In this case, an entirely analytic proof of the (so far basically numerically established) stability of the bound states may be constructed. Expressed by the positive self-adjoint operators on the Hilbert space \(L^2(\mathbb{R}^3)\)

\[ A \equiv -\Delta + 2r = A^\dagger \geq 0, \quad B \equiv -\Delta + 2r + \frac{2}{r^2} = B^\dagger \geq 0, \quad r \equiv |x|, \]

our radial Salpeter equation (3) with harmonic-oscillator confining interactions is clearly equivalent to the (vanishing angular momentum sector of the) matrix eigenvalue problem

\[
\begin{align*}
A f_2 &= M f_1, \\
B f_1 &= M f_2
\end{align*} \iff \left( \begin{array}{cc} A & B \\ 0 & 0 \end{array} \right) \left( \begin{array}{c} f_1 \\ f_2 \end{array} \right) = M \left( \begin{array}{c} f_1 \\ f_2 \end{array} \right), \quad f_1, f_2 \in L^2(\mathbb{R}^3).
\]

Reiteration of the L.H.S. operation yields an equivalent problem for real eigenvalues \(M^2\):

\[
\left( \begin{array}{cc} AB & 0 \\ 0 & BA \end{array} \right) \left( \begin{array}{c} f_1 \\ f_2 \end{array} \right) = M^2 \left( \begin{array}{c} f_1 \\ f_2 \end{array} \right) \iff \begin{array}{c} AB f_1 = M^2 f_1, \\
BA f_2 = M^2 f_2. \end{array}
\]

The spectral theorem for self-adjoint operators allows one to define the unique positive self-adjoint square root of \(A\): \(A^{1/2} = (A^{1/2})^\dagger \geq 0\). Setting \(g \equiv A^{1/2} f_2\) converts the relation \(BA f_2 = M^2 f_2\) to an eigenvalue equation, \(Q g = M^2 g\), of the positive self-adjoint operator \(Q \equiv A^{1/2} BA^{1/2} = Q^\dagger \geq 0\). Obviously, our operators \(A\) and \(B\) satisfy the inequality \(A \leq B\); left and right multiplication by \(A^{1/2}\) transforms this inequality into \(A^2 \leq A^{1/2} BA^{1/2} \equiv Q\). A theorem about the spectrum of a Hamiltonian with potential increasing beyond bounds claims that a Schrödinger operator \(H \equiv -\Delta + V\), defined as sum of quadratic forms, with locally bounded, positive, infinitely rising potential \(V(V(x) \to +\infty \text{ for } r \to \infty)\) has purely discrete spectrum; application of this theorem guarantees that the spectrum of \(A\) is purely discrete. The spectral theorem for self-adjoint operators allows for a representation of \(A^2\) in terms of the same projection-valued spectral measure as \(A\); consequently, the spectrum of \(A^2\) is also entirely discrete. Trivially, the positive operators \(A^2\) and \(Q\) are bounded from below. Combining [12–17] the characterization of all discrete eigenvalues of an arbitrary self-adjoint operator bounded from below by the minimum–maximum principle with the operator inequality \(A^2 \leq Q\) shows that the spectrum of our real squared mass eigenvalues \(M^2\) of \(Q\) must be purely discrete and therefore the spectrum of bound-state masses \(M\) too.

This stability of full-Salpeter bound states, proven for time-component Lorentz-vector harmonic-oscillator interaction with positive strength \(a > 0\), excludes the possibility that, in the case \(\sigma = +1\), instabilities of bound states are induced by energy eigenvalues being embedded in a continuous spectrum of the Salpeter operator controlling the bound states, instead of belonging to its discrete spectrum, as expected for true confinement. The proof cannot be transferred to \(a < 0\) as in this case the counterparts of our operators \(A\) and \(B\) are

\[ \tilde{A} \equiv \Delta + 2r, \quad \tilde{B} \equiv \Delta + 2r - \frac{2}{r^2}. \]

For both operators \(\tilde{A}\) and \(\tilde{B}\), positivity is lost. This fact invalidates most steps of our proof.
CONFINING INTERACTION KERNELS OF LORENTZ-SCALAR \((\Gamma \otimes \Gamma = 1 \otimes 1)\) OR LORENTZ-PSEUDOSCALAR \((\Gamma \otimes \Gamma = \gamma_5 \otimes \gamma_5)\)

DIRAC STRUCTURE

In the limit of the bound-state constituents being exactly massless, the Salpeter equations for Lorentz-scalar [i.e., \(\sigma = -1\) in Eq. (3)] and Lorentz-pseudoscalar interaction kernels (must) become identical, as the anticipated manifestation of chiral symmetry. The matrix eigenvalue problem is formulated in terms of the operator pairs \((\tilde{A}, B)\) for \(a > 0\) and \((A, \tilde{B})\) for \(a < 0\). In both cases, one of these operators is not positive, which spoils our reasoning.

LINEAR COMBINATIONS OF BOTH TIME-COMPONENT LORENTZ-VECTOR \((\Gamma \otimes \Gamma = \gamma^0 \otimes \gamma^0)\) AND LORENTZ-SCALAR \((\Gamma \otimes \Gamma = 1 \otimes 1)\) CONFINING INTERACTION KERNELS

Introducing the abbreviations \(\alpha \equiv a \xi \in \mathbb{R}, \beta \equiv a \eta \in \mathbb{R}\), the two operators entering in the Salpeter equation (4) for some mixture of scalar and time-component vector kernels read

\[
\mathcal{A} \equiv (\alpha - \beta) (-\Delta) + 2r, \quad \mathcal{B} \equiv (\alpha + \beta) \left(-\Delta + \frac{2}{r^2}\right) + 2r, \quad r \equiv |\mathbf{x}|.
\]

Positivity of both (symmetric) operators \(\mathcal{A}, \mathcal{B}\) and presence of both derivatives demands \(\alpha - \beta > 0\) and \(\alpha + \beta > 0\). These two relations restrain the couplings \(\alpha\) and \(\beta\) to the range

\[
\alpha = |\alpha| > |\beta| \geq 0 \quad \iff -1 < \frac{\beta}{\alpha} < +1 \quad \text{and} \quad \alpha > 0.
\]

A proof similar to the above for a pure time-component Lorentz-vector kernel establishes stability irrespective of the relative sign of the two contributions of unequal Lorentz type: the dominance of the time-component Lorentz-vector kernel guarantees stability. Table 1 illustrates these findings for various parameter ratios \(\beta/\alpha\) within the tolerable region (5).

**TABLE 1.** Lowest-lying positive mass eigenvalues (in units of \(3\sqrt{a}\)) of the Salpeter equation (4) for several values \(\beta/\alpha\) of our couplings in the stability-compatible range \(-1 \leq \beta/\alpha < +1\).

| Level | \(\frac{\beta}{\alpha} = \frac{\eta}{\xi}\) |
|-------|----------------------------------|
|       | +0.99  | +0.75  | +0.50  | +0.25  | 0.00   | -0.25  | -0.50  | -0.75  | -1.00  |
| 0     | 4.53   | 4.69   | 4.71   | 4.67   | 4.60   | 4.47   | 4.28   | 3.97   | 2.93   |
| 1     | 6.15   | 6.84   | 7.07   | 7.16   | 7.15   | 7.06   | 6.87   | 6.49   | 4.68   |
| 2     | 7.63   | 8.78   | 9.14   | 9.30   | 9.33   | 9.24   | 9.01   | 8.55   | 6.14   |
| 3     | 9.01   | 10.54  | 11.02  | 11.23  | 11.28  | 11.18  | 10.92  | 10.38  | 7.45   |
| 4     | 10.31  | 12.18  | 12.75  | 13.01  | 13.07  | 12.97  | 12.68  | 12.06  | 8.65   |
| 5     | 11.55  | 13.73  | 14.38  | 14.67  | 14.75  | 14.65  | 14.32  | 13.63  | 9.77   |
| 6     | 12.74  | 15.19  | 15.92  | 16.25  | 16.35  | 16.23  | 15.87  | 15.11  | 10.83  |
| 7     | 13.88  | 16.59  | 17.39  | 17.76  | 17.86  | 17.74  | 17.35  | 16.52  | 11.84  |
| 8     | 14.98  | 17.93  | 18.81  | 19.21  | 19.33  | 19.19  | 18.77  | 17.87  | 12.81  |
| 9     | 16.04  | 19.23  | 20.19  | 20.64  | 20.78  | 20.63  | 20.16  | 19.18  | 13.74  |
SUMMARY, FINDINGS, CONCLUSIONS, AND PERSPECTIVES

Salpeter equations with potential functions rising to infinity in configuration space do not automatically predict stable bound states: for this to happen, the Lorentz behaviour of the involved Bethe–Salpeter kernels too is crucial. Truly confining interaction kernels can be singled out rather systematically by requiring the emerging bound-state energy spectra to be both real and discrete. This task is comparatively easy if the Salpeter equation reduces to a differential equation (as for the harmonic oscillator) but might be conducted for more general (say, linearly confining) potentials if all integral-equation issues can be mastered.

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