Endpoint bounds for the non-isotropic Falconer distance problem
associated with lattice-like sets

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March 29, 2022

Abstract

Let $S \subset \mathbb{R}^d$ be contained in the unit ball. Let $\Delta(S) = \{||a - b|| : a, b \in S\}$, the Euclidean distance set of $S$. Falconer conjectured that the $\Delta(S)$ has positive Lebesgue measure if the Hausdorff dimension of $S$ is greater than $d/2$. He also produced an example, based on the integer lattice, showing that the exponent $d/2$ cannot be improved. In this paper we prove the Falconer distance conjecture for this class of sets based on the integer lattice. In dimensions four and higher we attain the endpoint by proving that the Lebesgue measure of the resulting distance set is still positive if the Hausdorff dimension of $S$ equals $d/2$. In three dimensions we are off by a logarithm.

More generally, we consider $K$-distance sets $\Delta_K(S) = \{|a - b|_K : a, b \in S\}$, where $|\cdot|_K$ is the distance induced by a norm defined by a smooth symmetric convex body $K$ whose boundary has everywhere non-vanishing Gaussian curvature. We prove that our endpoint result still holds in this setting, providing a further illustration of the role of curvature in this class of problems.

Keywords: Lattice point distribution, mean square estimates,
AMS subject classification 42B

1. Introduction

Let $B$ denote the Euclidean unit ball. The Falconer conjecture says that if the Hausdorff dimension of $S \subset B \subset \mathbb{R}^d$, $d \geq 2$ is greater than $d/2$, then the Lebesgue measure of the distance set $\Delta(S) = \{||a - b|| : a, b \in S\}$ is positive. Here, and throughout the paper, $||x|| = \sqrt{x_1^2 + \ldots + x_d^2}$ is the Euclidean distance in $\mathbb{R}^d$. See [1], [3], [4], [5], [7], [8], and the references contained therein for the description of this problem and progress over the years. The problem remains open for every $d \geq 2$. The best known results are due to Wolff in $\mathbb{R}^2$ and Erdogan in $\mathbb{R}^d$. They prove that the Lebesgue measure of the distance set is positive if the Hausdorff dimension of $S$ is greater than $d(d+1)/(2(d+1))$.

Falconer showed that the exponent $d/2$ cannot in general be improved in the following sense. Let $\{q_i\}_{i \geq 1}$ be a rapidly growing sequence of positive integers, e.g with $q_1 = 2$ and $q_{i+1} > q_i$. Let $S_i$ denote the union of Euclidean balls of radius $q_i^{-d}$, for some $0 < s < d$, centered at the points of $\mathbb{Z}^d \cap [0, q_i]^d$. Let $S = \cap_i S_i$. Then (see e.g. [1]) the Hausdorff dimension of $S$ is $s$. On the other hand, the Lebesgue measure

$$|\Delta(S_i)| \leq \text{const.} \cdot q_i^{-s} \cdot q_i^2,$$

where the first factor after the constant is the radius of each ball, and the second factor is a trivial bound on the number of distances determined by $\mathbb{Z}^d \cap [0, q_i]^d$. It follows that $|\Delta(S)| = 0$ if $s < d/2$. It is well known that in fact, one can choose $\{q_i\}$ such that

$$|\Delta(S_i)| = \text{const.} \cdot q_i^{-s} \cdot \begin{cases} q_i^2, & d \geq 3 \\ q_i^2 \sqrt{s q_i}, & d = 2. \end{cases}$$

Thus $|\Delta(S)| > 0$ for $s \geq d/2$ if $d \geq 3$ and for $s > d/2$ if $d = 2$. 

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As we mention in the abstract, we also study the case of more general, "well-curved" distances. Let \( K \subset \mathbb{R}^d \) be a strictly convex symmetric body with a smooth boundary and the volume equal to, say that of the Euclidean unit ball \( B \). Let \( | \cdot |_K \) be the Minkowski functional of \( K \), or the \( K \)-norm. Thus
\[
\Delta_K(S) = \{|a - b|_K : a, b \in S\}.
\] (3)
Let \( | \cdot |_{K^*} \) be the dual norm to \( | \cdot |_K \), defined as
\[
|x|_{K^*} = \sup_{y \in K} |x \cdot y|, \quad K^* = \{x \in \mathbb{R}^d : |x|_{K^*} \leq 1\}.
\] (4)
The purpose of this paper is to prove that the Falconer conjecture holds for the set \( S \) constructed by Falconer, with respect to \( K \)-distances, under the assumption that the boundary \( \partial K \) of \( K \) is \( C^r \), for a large enough \( r \) (we do not discuss what the smallest possible \( r \) could be) and has everywhere non-vanishing Gaussian curvature. The fact that we are able to deal with any such \( K \) implies that as the basis for Falconer’s construction one can use any lattice, so considering \( \mathbb{Z}^d \) as we do in the remainder of this does not result in the loss of generality.

Furthermore, if one considers the Euclidean distance, we can extend the scope of Falconer’s construction to a homogeneous set
\[
A = \left\{ \frac{|a|_K}{\|a\|} : a \in \mathbb{Z}^d \right\}, \quad \text{for some } K,
\] (5)
or equally well instead of the Euclidean norm \( \|a\| \) in the denominator we can have some other \( |a|_{K^*} \). Our main result is the following.

**Theorem 1** Let \( S \) be the Falconer set described above. Suppose that \( s \geq \frac{d}{2} \), and \( d \geq 4 \). Then the Lebesgue measure of \( \Delta_K(S) \) is positive, for any strictly convex \( K \) with a smooth boundary. If \( d = 3 \) the same conclusion holds with a logarithmic loss in the sense to be made precise below.

The difficult part in Theorem 1 is the endpoint issue. Otherwise the proof can be made somewhat shorter, using techniques developed by Müller (12) and Iosevich et al. (8) for the quantity
\[
E(t) = \#\{tK \cap \mathbb{Z}^d\} - t^d \text{Vol } K.
\] (6)
However, our main motivation in proving Theorem 1 is not the quantity \( E \), but rather the Falconer distance conjecture, and its discrete analog, the Erdős distance conjecture in the case of homogeneous sets. An infinite discrete set \( A \subset \mathbb{R}^d \) is called homogeneous if all its elements are separated by some \( c > 0 \), while any cube of side length \( C > c \) contains at least one element of \( A \). If \( A_q = A \cap qK \) is a truncation of \( A \), the conjecture is that
\[
\#\Delta_K(A_q) \geq C_c q^{2+\varepsilon}.
\] (7)
This is a major open problem. See, for example, [13], and the references contained therein for the discussion of this problems and the best known results. Theorem 2 below says that if \( A \) is a \( d \)-dimensional lattice, then
\[
\#\Delta_K(A_q) \gtrsim \begin{cases} 
q^2, & d \geq 4, \\
q^2 \log^{-a} q, & d = 3.
\end{cases}
\] (8)
As we mention above, this result is observed in [4] in the context of sharpness examples for the lattice mean-square discrepancy estimates. The symbol \( \lesssim \) will further absorb constants depending only on \( K \) (and hence \( d \)). Also we write \( a \gtrsim b \) if \( b \lesssim a \) and \( a \approx b \) if both \( a \lesssim b \) and \( a \gtrsim b \). The symbol \( \sim \) will indicate proportionality, up to some constant \( c(K) \).

Our approach is based on the formalism of distance measures, set up by Mattila (11). The distance measure \( \nu(t) \), relative to the set \( A_q \), counts the number of its points in \( \frac{1}{t} \)-thin \( K \)-annuli of radius \( t \), centered at points of \( A_q \). (Unfortunately, beyond Introduction, there will be several closely related \( \nu \)'s which will have to carry extra identifications.) In the case of a lattice it is enough to consider only the distances to the origin, yet this is not essential. The distance measure formalism has nothing to do with the lattice structure, and applies, for example, to any homogeneous point set \( A \). The \( L^1 \)-norm \( \|\nu(t)\|_1 \) simply counts the points
in $qK$, and the main task will be to estimate the $L^2$-norm $\|\nu(t)\|_2$. The latter can be thought of as a special case of the estimation of the Mattila integral. More precisely, Mattila ([11]) proved a general theorem for the Euclidean distance, which generalizes to $K$-distances, (see [11]) as follows. Let $\mu$ be a finite Borel measure supported on $S \subset B$, Then if

$$M(\mu) = \int_1^\infty \left( \int_{\partial K^*} |\hat{\mu}(t\omega)|^2 d\sigma_{K^*}(\omega) \right)^{2/(d-1)} dt < \infty,$$

(9)

where $d\sigma_{K^*}$ is the Lebesgue measure on $\partial K^*$, the Lebesgue measure of $\Delta_K(S)$ is positive. Falconer’s example shows that there are sets of dimension $s < \frac{d}{2}$ such that the integral (11) diverges.

We shall write down an analog of the Mattila integral (9) for a specific measure $\mu$ on the Falconer sets $S_i$, such that $S = \cap S_i$, has dimension $s = \frac{d}{2}$. It will follow that the integral (9), after appropriate scaling, can be bounded by $O(1)$ in $d \geq 4$ and by $\log^2 q_i$ in $d = 3$.

Unfortunately, our bounds on the Mattila integral can be justified only if $A$ is a lattice. The reason is that in order to get these bounds, we use a smooth approximation $E(t)$ of the discrepancy $E(t)$ as the auxiliary quantity, as $E(t)$ admits a well known representation via the Poisson summation formula, and $E^2(t)$ looks rather similar to the integrand in the Mattila integral.

Hence $L^2$ estimates for $\nu$ are closely related to the $L^2$-estimates for $E$ in the lattice case. If one could establish an analogue of this relation in the general homogeneous set context, or find a proper substitute for $E$, one could then prove the Erdős distance conjecture for homogeneous sets.

$L^2$-estimates for $E$ were obtained in [12] and [8]. The technique in this paper is based on asymptotic methods, enabling us to

- essentially dominate the desired $L^2$-estimate for $E$ by a weighted $L^2$ estimate for $\nu$, see [12] below
- which in turn can be dominated by another $L^2$-estimate for $E$, see e.g. [8] below.

The Poisson summation formula is absolutely essential for the estimate in the first bullet above, but (seemingly) not for the second one. As a result, one gets an $L^2$-estimate for the quantity $E$, and a weighted $L^2$-estimate for the distance measure $\nu$. It turns out that for $d \geq 3$ the weighted $L^2$-estimate for $\nu$ implies a tight (modulo the logarithms in $d = 3$) $L^2$-estimate for $\nu$ itself, while for $d = 2$ this is not the case. That is why our method gives only a trivial estimate for $d = 2$, where the case of a general $K$ is an open problem.

We have italicized essentially, in the first bullet above, because the asymptotic techniques used in [12] and [8] do not yield a clear cut relation (12) between the $L^2$ estimates for $\nu$ and $E$, due to the presence of various cut-off functions, truncations, and related difficulties. This is precisely the source of technical difficulties that one encounters in the effort to attain the endpoint result claimed in Theorem 1. We identify the estimate (12) as a key example of a technical advantages of our approach. This approach yields the mean square estimates for $E$ and $E$ as a by-product. In addition, throughout the proof a number of integral representations for the distance measure $\nu$ and related quantities are obtained. These identities extend to the case of general homogeneous sets and may well prove to be of use in the further progress towards the Erdős distance conjecture in this context.

2. Distance measure

Let $\phi$ be a non-negative radial (radial henceforth means radial with respect to the Euclidean metric) Schwartz class function, such that $\int \phi(x) = 1$, $\phi(x) = \mathbf{1}$ inside the ball of some radius and vanishes outside the ball of twice the radius. Let $q$ be a large number, denote $\phi_q(x) = q^d \phi(qx)$ and $\mathbb{Z}^q = \{0\}$.

For a function $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, let

$$\hat{f}(\xi) = \int f(x) e^{-2\pi i \xi \cdot x} dx$$

(10)

define the usual Fourier transform. Let

$$\mu_q(x) = \sum_{a \in \mathbb{Z}^q} \phi_q(x - a),$$

(11)
be the smoothing of the counting measure on $Z_q^d$, blurring each point of $Z_q^d$ into a speck of radius $\sim \frac{1}{q}$ (actually it is slightly less than $\frac{1}{q}$ by the choice of $\phi$). Clearly

$$\hat{\mu}_q(\xi) = \sum_{a \in Z_q^d} \hat{\phi}(\xi/q)e^{-2\pi i a \cdot \xi},$$

(12)

and the function $\hat{\phi}$ is radial.

For the construction of the sequence $S_i$ in the introduction, we take a rapidly growing sequence $\{q_i\}$ and define $S_i$ to be the scaling of $Z_q^d$ into the unit cube. The resulting intersection over $i \geq 1$ has Hausdorff dimension $\frac{d}{q}$ by the argument (see e.g. [4]) mentioned in the introduction.

For $t > 0$ define

$$\nu_{q,0}(t) = \int \omega_K(x/t)d\mu_q(x),$$

$$N_{q,0}(t) = \int \Omega_K(x/t)d\mu_q(x) = \int_0^t d\nu_{q,0}. \tag{13}$$

Above $\omega_K$ is the Lebesque measure on $\partial K$, $\Omega_K$ is the characteristic function of $K$. Note that in the first integral $\mu_q$ is actually a Schwartz function, and $\omega_K$ - a distribution.

Also define the volume discrepancy

$$E_{q,0}(t) = N_{q,0}(t) - t^d \text{Vol} K. \tag{14}$$

Note that the quantity $E_{q,0}(t)$ thus defined relates to $\mathcal{E}(t)$ in [10] for $t < q$ only. The seemingly redundant subscripts come from the fact that in the sequel it turns out to be more convenient to work with the weighted quantities

$$[\nu_q(t), N_q(t), E_q(t)] = t^{\frac{d-d}{2}}[\nu_{q,0}, N_{q,0}(t), E_{q,0}(t)]. \tag{15}$$

The quantity $\nu_{q,0}$ can be viewed as the density of the measure $\mu_q$ on $K$-spheres of radius $t$, centered at the origin. The primitive $N_{q,0}(t)$ counts the points in $K$-balls of radius $t$. Removing the origin in $Z_q^d$ is of no consequence, yet it enables us to avoid consideration of some trivial cases. Clearly

$$\int_0^\infty \nu_{q,0} \sim q^d. \tag{16}$$

By definition of the quantities $\mu_q$ and $\nu_{q,0}$, as we are interested in the estimates only in terms of the order of magnitude with respect to $q$, it is legitimate to sample the integrals containing $\nu_{q,0}$ (as well as $E_{q,0}$ and other versions of $\nu$ and $E$ to appear later) by Darboux sums with the step size $\frac{1}{c_1q}$, for some constant $c_1$.

Clearly $\nu_{q,0}$ vanishes for $t > q + q^{-1}$, while for $t < q - q^{-1}$,

$$\frac{\nu_{q,0}}{q} \approx \Gamma(t, \delta), \tag{17}$$

where $\Gamma(t, \delta)$ is the number of points of $Z_q^d$ in a $K$-annulus $A(t, \delta)$ centered at the origin, with radius $t$ and width $\delta$: in [7] and further $\delta$ will always be $\approx \frac{1}{q}$. More precisely, [7] means that there exist uniform constants $c_2$ and $c_3$, such that

$$\Gamma \left( t, \frac{1}{c_2 q} \right) \leq \frac{\nu_{q,0}}{q} \leq \Gamma \left( t, \frac{c_3}{q} \right). \tag{18}$$

Consider all consecutive annuli, indexed by $k \leq q^2$, of width $\delta \sim \frac{1}{q}$, whose $K$-radii $t_k$ go up to $q$. Define the annulus standard deviation and $D_A$ and the ball mean square discrepancy $D_K$ as follows:

$$D_{A_K} = \sqrt{\frac{1}{q^2} \sum_k \Gamma^2(t_k, \delta)} \approx \sqrt{\frac{1}{q} \int_0^q \nu_{q,0}^2(t)dt},$$

$$D_K = \sqrt{\frac{1}{q^2} \sum_k E_{q,0}^2(t_k)} \approx \sqrt{\frac{1}{q} \int_0^q E_{q,0}^2(t)dt}. \tag{19}$$

We shall need the following estimate.
**Theorem 2** We have

\[ D_{A_K}, D_K \lesssim q^{d-2}, \quad d \geq 4, \]

\[ D_{A_K}, D_K \lesssim q \log q, \quad d = 3. \]

As far as the weighted quantity \( \nu_q(t) \) is concerned, see (15), the estimates of Theorem 2 boil down to establishing

\[ \|\nu_q\|^2 = \int_0^\infty \nu_q^2(t)dt \lesssim \begin{cases} \frac{q^d}{d}, & d \geq 4, \\ \frac{q^3 \log q}{d}, & d = 3. \end{cases} \]

Theorem 2 implies the following version of the Erdős distance conjecture. See also [9] where this result is established.

**Corollary 2.1** There are at least \( \sim q^2 \) distinct, \( \frac{1}{q} \)-separated \( K \)-distances from the origin to the points of \( \mathbb{Z}_q^d \), for \( d \geq 4 \). If \( d = 3 \), \( q^2 \) changes to \( \frac{q^2}{\log^2 q} \).

We shall deduce Theorem 1 from Corollary 2.1.

**Proof of Corollary 2.1 and Theorem 1** Assume Theorem 2 for the moment. By Cauchy-Schwartz,

\[ q^{2d} \sim \left( \int_1^q \nu_{q,0}^2(t) dt \right)^2 \leq |\text{supp} \nu_{q,0}| \int_1^q \nu_{q,0}^2(t) dt, \]

where \( |\text{supp} \nu_{q,0}| \) is the Lebesgue measure of the support of \( L_{q,0} \). Now plug the estimates (20) in the right hand side and get the lower bound \( \geq q \) for \( d \geq 4 \) and \( \geq \frac{q}{\log^2 q} \) for \( d = 3 \), for \( |\text{supp} \nu_{q,0}| \). The fact that the distances are \( \approx \frac{1}{q} \)-separated follows by construction of \( \nu_{q,0} \).

We now deduce Theorem 1 in the following way. Take a sequence \( \{q_i\} \) of values of \( q \) as described in Introduction. Contract \( \mathbb{Z}_q^d \) into the unit ball to get the Falconer sets \( S_i \) \((S_i \text{ consists of points of } \mathbb{Z}_q^d \text{ scaled into the unit cube and blurred into a smudge of radius of } \approx q^{-2})\). Then for \( d \geq 4 \), the Lebesgue measure of each \( \Delta_K(S_i) \) is bounded uniformly away from zero, so the same conclusion must hold for the distance set of \( S = \bigcap S_i \).

Observe that in the same way as (23), yet for the weighted quantity \( \nu_q \) we always have

\[ \|\nu_q\|^2 \gtrsim q^d. \]

Let us now write up some representations for the quantities \( \nu_q, N_q \), without using the Poisson formula, so they would adapt to general distance measures on homogeneous sets.

First we apply Plancherel to evaluate the integrals in (13). Then for the weighted quantities (15) we get:

\[ \nu_q(t) = t^{d+1} \int \hat{\phi}(\xi/q) \hat{\omega}_K(t\xi) \sum_{a \in \mathbb{Z}_q^d} e^{-2\pi i a \cdot \xi} d\xi, \]

\[ N_q(t) = t^{d+1} \int \hat{\phi}(\xi/q) \hat{\Omega}_K(t\xi) \sum_{a \in \mathbb{Z}_q^d} e^{-2\pi i a \cdot \xi} d\xi. \]

Note that \( \nu_q(t) \) extends as zero to \( t = 0 \), as well as the fact that \( N_q(t) \) defined as it is not in \( L^2(\mathbb{R}_+) \) if \( d = 2 \).

**Euclidean case**

We make a few remarks about the case when \( K \) is the Euclidean ball, with the notations \( \omega_B, \Omega_B \) for the surface and volume measure. In this case the Fourier transform \( \hat{\omega}_B(\xi) \) is radial, namely \( \hat{\omega}_B(\xi) \sim \|\xi\|^{-\frac{d}{2}} J_{d-1}(2\pi\|\xi\|) \), where further \( J_v \) denotes the Bessel function of order \( v \). Let us skip the factor of \( 2\pi \) in what follows. This can always be accomplished by scaling. After writing the integral (24) for \( \nu_q \) in the spherical coordinates we have

\[ \nu_q(t) \sim \sqrt{t} \int_0^\infty r J_{d-1}(rt) \psi(r/q) \sum_{a \in \mathbb{Z}_q^d} J_{d-1}(r\|a\|) dr, \]
where henceforth
\[ \psi(r) = \hat{\phi}(\xi)_{\|\xi\|=r}, \]  
so \(|\psi(r/q)|\) is asymptotically smaller than any power of \(r/q\).

After using Hankel’s formula, see e.g. Watson’s classic \([14]\),
\[ \int_0^\infty t J_v(at) J_v(bt) dt = \frac{\delta(a-b)}{a}, \]  
for the Bessel functions of order \(v \geq 0\), we would have then
\[ \|\nu_q\|^2 = \int_0^\infty \nu_q^2(t) dt \sim \int_0^\infty r^2 \psi(r/q) \sum_{a,b \in \mathbb{Z}}^J \frac{J_{q-1}(r|a|)J_{q-1}(r|b|)}{(\|a\|\|b\|)^{1/2}} dr \]  
\[ \sim \int_0^\infty r^{d-1} \psi^2(r/q) \sum_{a,b \in \mathbb{Z}} \hat{\omega}_B(r|a|)\hat{\omega}_B(r|b|) dr. \]  
The representation \(28\) is a particular case of the Mattila integral \([9]\).

Note that due to Hankel’s formula \((27)\) with \(v = \frac{d}{2} - 1\), the expression \(28\) is in essence Parseval’s identity for the transformation
\[ H[\nu_q](r) = \int_0^\infty \sqrt{r} J_{q-1}(rt) \nu_q(t) dt \sim \sqrt{r} \psi(r/q) \sum_{a \in \mathbb{Z}}^J \frac{J_{q-1}(r|a|)}{(\|a\|)^{1/2}}. \]  
So, similarly to \(28\) one can apply the Hankel formula with \(v = \frac{d}{2}\) to the quantity \(N_q(t)\) and get for \(d \geq 3\)
\[ q^{d+2} \approx \|N_q\|^2 \sim \int_0^\infty \frac{1}{r} \psi^2(r/q) \sum_{a,b \in \mathbb{Z}}^J \frac{J_{q-1}(r|a|)J_{q-1}(r|b|)}{(\|a\|\|b\|)^{1/2}} dr \]  
\[ \sim \int_0^\infty r^{d-2} \psi^2(r/q) \sum_{a,b \in \mathbb{Z}} \hat{\omega}_B(r|a|)\hat{\omega}_B(r|b|) dr, \]  
It is easy to show that in order to get the right order of magnitude \(q^{d+2}\) for \(\|N_q\|^2\) in the latter integral, it suffices to restrict the domain of integration to \((0,1)\), while the integral over \([1,\infty)\) should be \(\approx q^{-2} \|\nu_q\|^2\).

For the lattice case studied, see the ensuing Lemma \[34\] the integral above, taken from 1 to infinity is closely related to the quantity \(E_q(t)\).

However, for a general homogeneous set, even for the Euclidean distance, there does not appear to be an analogous “nice” formula, similar to \(28\), \(30\), for the quantity \(E_q\).

**Remark R1** In the next section we will show that in the case of anisotropic distances \(\|\cdot\|_K\), one has expressions similar to \(28\) for \(\|\nu_q\|^2\), using the asymptotics for the Fourier transforms \(\hat{\omega}_K\) and \(\hat{\Omega}_K\). So far we have not used the fact that we are dealing with a lattice. Hence, one can squeeze \(\mathbb{Z}_d^d\) in each direction \(\xi \in \mathbb{R}^{d-1}\) by the factor \(|\xi|_K\) and consider the Euclidean distances.

**Anisotropic case**

To proceed, we need the following lemma on the asymptotics of the Fourier transforms \(\hat{\omega}_K\) and \(\hat{\Omega}_K\). We do not present a proof here, as for \(\hat{\Omega}_K\) it can be found in \([2]\), and the case of \(\hat{\omega}_K\) can be treated in the same way. One can also derive the expression for \(\hat{\omega}_K\) from that of \(\hat{\Omega}_K\) using \(24\) and the fact that \(N_{q,0}(t) = \frac{d\omega_q}{dt}\).

For more asymptotic expressions of this kind see \([3], [14]\).

**Lemma 2.2** For \(\|\xi\| \leq 1\), \(\hat{\omega}_K(\xi), \hat{\Omega}_K(\xi) \approx 1\), otherwise
\[ \hat{\omega}_K(\xi) = \sum_{j=0}^1 u_j \left( \frac{t}{\|\xi\|} \right) J_{q-1+j} \|\xi\|^{1-j} + O\left(\|\xi\|^{-\frac{d-1}{2}}\right), \]  
\[ \hat{\Omega}_K(\xi) = \sum_{j=0}^1 U_j \left( \frac{t}{\|\xi\|} \right) J_{q+j} \|\xi\|^{-j} + O\left(\|\xi\|^{-\frac{d-1}{2}}\right), \]  
where the quantities \(u_0, U_0\) are strictly positive and the constant \(c_4\) depends on \(K\) only.
Without loss of generality, assume \( c_4 = 1 \) in the formulae \( \text{31} \) above. The sums in the asymptotic expansions have two terms, because this is as many as we will have to analyze. Note that in the Euclidean case, one has only the first term in the sum in the expressions \( \text{31} \) and no remainder.

Lemma 2.2 will be instrumental for our proofs. First let us use it to elaborate on Mattila’s integral.

**Proposition 2.3** For \( d \geq 2 \),

\[
\|\nu_q\|_2^2 \approx \int_0^\infty r^{d-1}\hat{\psi}(r/q)\sum_{a,b\in\mathbb{Z}_q^d} \hat{\omega}_{K^*}(ra)\hat{\omega}_{K^*}(rb)dr. \tag{32}
\]

**Proof** Let us see what the right-hand side of the first expression in \( \text{32} \) is equal to by plugging in the asymptotic expansion for \( \hat{\omega}_{K^*} \) from \( \text{31} \). Keep in mind that \( K^{**} = \hat{K} \). Given \( (a,b) \), it suffices to consider three cases, as far as the three-term expansions in \( \text{31} \) are concerned: the leading terms for both \( a \) and \( b \), the leading term for \( a \) and the second term for \( b \), and finally the leading term for \( a \) and the remainder for \( b \).

As \( \text{31} \) has three terms, there will be three key estimates in the proof, and the same will apply to proofs of Theorem 3 and Lemma 3.1. Given \( (a,b) \), take the product of the leading terms in the sum \( \text{31} \). Then, see \( \text{30} \), what we get is proportional to the following integral over \( \mathbb{R}^d \):

\[
u_0(a/\|a\|)\nu_0(b/\|b\|) \int \hat{\phi}^2(\xi/q)\hat{\omega}_B(|a|_K \xi)\hat{\omega}_B(|b|_K \xi)d\xi
\approx \int [\phi_q \ast (\omega_B \circ |a|^{-1}_K)](x) \cdot [\phi_q \ast (\omega_B \circ |b|^{-1}_K)](x)dx,
\]

where \( \omega_B \circ |a|^{-1}_K(x) = |a|^{-d}_K \omega_B(|a|^{-1}_K x) \). The integrand in the right hand side of \( \text{33} \) is roughly the product of characteristic functions of concentric Euclidean annuli, of radii \( |a|_K \) and \( |b|_K \), in particular it vanishes if \( |a|_K - |b|_K | > \frac{q}{2} \), and is proportional to \( |a|_K^{d-1} \) in case \( |a|_K = |b|_K \). Thus the average value of \( \omega_B \circ |a|^{-1}_K \) across the Euclidean annulus of radius \( |a|_K \) and width of \( \approx \frac{1}{q} \) is proportional to \( q \). So we get

\[
\sum_{a,b\in\mathbb{Z}_q^d} \nu_0(a/\|a\|)\nu_0(b/\|b\|) \int_0^\infty r^{d-1}(\|a|_K|^{d-1}_K|b|_K^{d-1}_K)dr \approx \sum_{k=1}^{\infty} \frac{\Gamma^2(r_k, \delta)}{r_k^{d-1}} \approx \int_0^\infty \nu_q^2(t)dt,
\]

where the sum above is taken over consecutive \( K \)-annuli of radius \( r_k \) and fixed width \( \delta \) of \( \approx \frac{1}{q} \). Recall that \( \Gamma(r_k, \delta) \) is the cardinality of the intersection of \( \mathbb{Z}^d \) with such an annulus, see \( \text{18} \).

2. Now let us take the first term in the sum \( \text{31} \) for \( a \) and the second one for \( b \), plugging them in the right-hand side of \( \text{32} \). Note that merely using the leading order asymptotics in this case results in a superfluous factor \( \int_0^{\infty} r^{-1}\hat{\omega}(r/q)dr \approx \log q \).

So, given \( (a,b) \) we have, similarly to \( \text{33} \), and omitting uniform constants:

\[
\int_0^\infty \psi^2(r/q)\frac{J_{d-1}(\|a|_K r)J_{d-1}(\|b|_K r)}{|a|_K^{-1}|b|_K^{-1}|}dr \approx \int \hat{\phi}^2(\xi/q)\omega_B(|a|_K \xi)\Omega_B(|b|_K \xi)d\xi
\]

\[
\approx \int [\phi_q \ast (\omega_B \circ |a|^{-1}_K)](x) \cdot [\phi_q \ast (\omega_B \circ |b|^{-1}_K)](x)dx
\]

\[
\approx \begin{cases} \frac{1}{|a|_K^{-1}|b|_K^{-1}}, & |a|_K \leq |b|_K + \frac{1}{q}, \\ 0 & \text{otherwise.} \end{cases}
\]

Hence, summing over \( a, b \in \mathbb{Z}_q^d \), we get

\[
\approx \sum_{b\in\mathbb{Z}_q^d \setminus \{0\}} \frac{|b|_K^{-d}}{|a|_K \leq |b|_K} \approx q^d. \tag{36}
\]
3. Finally, as far as the remainder in (31) is concerned, let us rewrite the integral in (32) as \( \sum_{a,b \in Z_q^d} I_{a,b} \) and notice that without loss of generality one can assume \(|a|_K \geq |b|_K\). Then partition

\[
\sum_{a,b \in Z_q^d, |b|_K \leq |a|_K} I_{a,b} = \sum_{a,b \in Z_q^d, |b|_K \leq |a|_K} \left( \int_0^{|a|_K^{-1}} + \int_{|a|_K^{-1}}^{|b|_K^{-1}} + \int_{|b|_K^{-1}}^\infty \right).
\]  

(37)

The first piece - plug 1 for the \(a\)-term and 1 for the \(b\)-term - can be bounded via

\[
\sum_{a \in Z_q^d} \sum_{b \in Z_q^d, |b|_K \leq |a|_K} \int_0^{|a|_K^{-1}} r^{d-1} dr \approx \sum_{a \in Z_q^d} |a|_K^{-d} \sum_{b \in Z_q^d, |b|_K \leq |a|_K} 1 \approx \sum_{a \in Z_q^d} 1 \approx q^d.
\]  

(38)

For the second piece - plug 1 for the \(b\)-term and zero order asymptotics \((|a|_K r)^{-\frac{d-1}{2}} \) for the \(a\)-term - we have a bound

\[
\sum_{a \in Z_q^d} \sum_{b \in Z_q^d, |b|_K \leq |a|_K} \int_0^{|a|_K^{-1}} \left| \frac{1}{r} \right|^{-\frac{d-1}{2}} dt \approx \sum_{a \in Z_q^d} |a|_K^{-\frac{d-1}{2}} \sum_{b \in Z_q^d, |b|_K \leq |a|_K} |b|_K^{\frac{d-1}{2}} \approx \sum_{a \in Z_q^d} 1 \approx q^d.
\]  

(39)

Finally, for the third piece plug \((|a|_K r)^{-\frac{d-1}{2}} \) for the \(a\)-term terms and \((|b|_K r)^{-\frac{d-3}{2}} \), for the \(b\)-term to get

\[
\left( \sum_{a,b \in Z_q^d \setminus \{0\}} |a|_K^{-\frac{d-1}{2}} |b|_K^{-\frac{d-3}{2}} \right) \cdot \int_1^\infty \psi^2(r/q) t^{-2} dt \lesssim q^{d-1}.
\]  

(40)

Hence, the upper estimates in the second and the third cases match the lower bound (26). This completes the proof of Proposition 23. \( \square \)

3. Poisson formula

As the set \( Z_q^d \) has been taken off the integer lattice, the values of \( \nu_q(t), N_q(t), E_q(t) \) can be computed directly, rather than via (24), using the Poisson summation formula. It gives ”nice” expressions for these quantities on the \(t\)-side, rather than on the Hankel transform side, see (29). For example, for \( \frac{1}{q} < t < q - \frac{1}{q} \), one has

\[
\int \omega_K(x/t) \sum_{a \in Z_q^d} \phi_q(x - a) dx = \int \omega_K(x/t) \sum_{a \in Z_q^d} \phi_q(x - a) dx = \sum_{b \in t^{-1}Z_q^d} \omega * \phi_q(b),
\]  

(41)

where \( \phi_q(x) = \phi_q(tx) \). Applying the Poisson summation formula to the convolution in the square brackets, not forgetting the scaling (15) and doing the same thing for the quantities \( N_q, E_q \) yields, for \( \frac{1}{q} < t < q - \frac{1}{q} \)

\[
\nu_q(t) \sim t^{-\frac{d+1}{2}} \sum_{a \in Z_q^d} \phi(a/q) \hat{\omega}_K(ta) \quad \equiv \quad \nu(t),
\]

\[
N_q(t) \sim t^{-\frac{d+1}{2}} \sum_{a \in Z_q^d} \phi(a/q) \hat{\Omega}_K(ta) \quad \equiv \quad N(t),
\]

\[
E_q(t) \sim t^{-\frac{d+1}{2}} \sum_{a \in Z_q^d \setminus \{0\}} \phi(a/q) \hat{\Omega}_K(ta) \quad \equiv \quad E(t).
\]  

(42)

Note however that the quantities in the right hand side of (42) are unbounded as \( t \to \infty \), besides the summation is carried over the whole integer lattice. Still, there is a considerable resemblance between the expression for \( \nu(t) \) and the square root of the integrand in (32). Let us introduce the \( L^2(\mathbb{R}_+) \) quantities

\[
[\tilde{\nu}(t), \tilde{N}(t), \tilde{E}(t)] = \psi(t/q) [\nu(t), N(t), E(t)].
\]  

(43)

Clearly

\[
\|\nu_q\|_2^2 \lesssim \|\tilde{\nu}\|_2^2, \quad \int_0^q E_q^2(t) dt \lesssim \|\tilde{E}\|_2^2.
\]  

(44)

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**Remark R2** Note that in the definition (43) of \( \tilde{\nu} \) there is no harm restricting the summation to \( \mathbb{Z}^d \setminus \{0\} \), which will be done further. Indeed, \( a = 0 \) results in a regular term \( \frac{1}{t^{2d}} \), and it is easy to check that the contribution of this term into \( \| \tilde{\nu} \|^2 \) is \( O(q^d) \), cf. (26).

Now define the quantities \( \nu_\ast(t), \tilde{\nu}_\ast(t) \) in the same way as \( \nu(t), \tilde{\nu}(t) \), has been defined, cf. (13), (15), (43) but only with respect to the dual body \( K^* \) rather than \( K \), do the same for \( E \). Also with respect to \( K^* \), define the notation \( \Gamma_\ast \), cf. (17). The next theorem is a bit of a red herring, as it is not used to prove Theorem 1, yet is interesting in its own right. Of course, this theorem is obvious in the Euclidean or ellipsoidal cases!

**Theorem 3** For \( d \geq 2 \),
\[
\| \tilde{\nu} \|_2 \approx \| \tilde{\nu}_\ast \|_2.
\]

**Proof** The proof follows the same three steps as does the proof of Proposition 2.3 adding to it the decay of \( \psi \) only. Let us use the definitions (42), (43) and the asymptotics (31) to estimate the left-hand side.

1. For the principal term, skipping \( U_0 > 0 \), we get an analogue of (41):
\[
\sum_{a, b \in \mathbb{Z}^d \setminus \{0\}} \hat{\phi}(a/q) \hat{\phi}(b/q) \int_0^\infty r \psi^2(r/q) \frac{J_{d-1}^{\frac{3}{2}}(a|K| e) J_{d-1}^{\frac{3}{2}}(b|K| e)}{(a|K| b|K|)^2} dr \approx q \sum_{k=1}^\infty \frac{\Gamma^2(r_k, \delta)}{r_k} \psi^2(r_k/q) (50)
\]
\[
\approx \| \tilde{\nu}_\ast \|^2_2.
\]

2. For the rest of the estimates, one has chase through (35)–(40), for the only difference that the summations, say in \( a \) therein will have been extended to \( |a| \to \infty \), and weighted by \( (1 + |a|/q)^{-N} \), for some large \( N \), reflecting the decay of \( \psi \). Such an extension can be easily verified to be of no consequence for the order of these estimates. For instance (36) turns into a bound
\[
\int_1^\infty \frac{r^{d-1}}{(1 + r/q)^N} dr = O(q^d), (47)
\]
the same non-consequential changes happen to (38)–(40). \( \square \)

The next lemma makes precise the statement in the first bullet in the Introduction.

**Lemma 3.1** We have for \( d \geq 2 \):
\[
\| \tilde{E} \|^2 \leq \int_0^\infty \frac{\tilde{\nu}^2(t)}{1 + t^2} dt + R(q), (48)
\]
where \( R(q) = O(q^{d-2}) \) in \( d \geq 3 \) and \( O(\log q) \) in \( d = 2 \).

**Proof** The proof follows the same pattern as proofs of Proposition 2.3 and Theorem 3. We use the definitions (42) and (43) and plug in the asymptotics (31).

1. For the principal term, skipping \( U_0 > 0 \), we get
\[
\sum_{a, b \in \mathbb{Z}^d \setminus \{0\}} \phi(a/q) \phi(b/q) \int_0^\infty t \psi^2(t/q) \frac{J_{d-1}^{\frac{3}{2}}(a|K| t) J_{d-1}^{\frac{3}{2}}(b|K| t)}{(a|K| b|K|)^2} dt. (49)
\]

The integral in (49), given \( a, b \) can be rewritten as an integral over \( \mathbb{R}^d \):
\[
\int [(\Omega_B \circ |a|^{-1}) \ast \phi_q](\xi) \cdot [\Omega_B \circ |b|^{-1}) \ast \phi_q](\xi) d\xi = \int \nabla_x [(\Omega_B \circ |a|^{-1}) \ast \phi_q](x) \cdot \nabla_x [(\Omega_B \circ |b|^{-1}) \ast \phi_q](x) dx.
\]

where, cf. (33), \( \Omega_B \circ |a|^{-1}(x) = \frac{1}{|a|_{K^*}^d |B|_{K^*}} (a|K|^{-1} x) \). The integral in the right-hand side of (50) is clearly zero if \( |a|_{K^*} - |b|_{K^*} > \frac{1}{q} \), while if \( |a|_{K^*} = |b|_{K^*} \), it is \( O(|a|_{K^*}^{-1}) \). Hence, (49) is
\[
\approx q \sum_{k=1}^\infty \frac{\Gamma^2(r_k, \delta)}{r_k^2} \psi^2(r_k/q) \approx \int_0^\infty \frac{\tilde{\nu}^2(t)}{1 + t^2} dt, (51)
\]
see also (34) and (46).

2. For the principal $a$-term and second $b$-term in the asymptotics (31), throwing away uniform constants, we get

$$\sum_{a,b \in \mathbb{Z}^d \setminus \{0\}} \hat{\phi}(a/q) \hat{\phi}(b/q) \int_0^\infty \psi^2(t/q) \frac{J_d(||a||_K^*, t) J_{d+1}(||b||_K^*, t)}{|a|^{d/2} b^{d+1}|K^*|} dt. \quad (52)$$

Clearly (52) is reminiscent of (35), only in dimension $d + 1$. Let $w_B, W_B$ be the Lebesgue measure on $S^d$ and the characteristic function of the Euclidean unit ball in $\mathbb{R}^{d+1}$, respectively. Let $(y, \zeta) \in \mathbb{R}^{d+1} \times \mathbb{R}^{d+1}$, let the radial cutoff function $\varphi$ be defined in the same way as $\phi$, only in dimension $d + 1$.

Denote $\varphi_q(y) = q^{d+1} \varphi(qy), w_B \circ |a|^{-1}_K(y) = |a|^{-d}_K w_B(|a|^{-1}_K y), W_B \circ |a|^{-1}_K(y) = |a|^{-d-1}_K W_B(|a|^{-1}_K y).$ Then the integral right hand side of (52) can be rewritten as follows:

$$\int_0^\infty \psi^2(t/q) \frac{J_d(||a||_K^*, t) J_{d+1}(||b||_K^*, t)}{|a|^{d/2} b^{d+1}|K^*|} dt \sim \int ([w_B \circ |a|^{-1}_K] * \varphi_q)(\zeta) \cdot \left( ([W_B \circ |a|^{-1}_K] * \varphi_q)(\zeta) \cdot ||\zeta|| \right) dq \zeta \quad (53)$$

Thus the integral vanishes if $||a||_K - ||b||_K > \frac{1}{q}$ and is approximately $\frac{1}{q}$ if $||a||_K = ||b||_K$. Summing over $(a, b)$ and taking absolute values, we get the bound (35) once more.

3. We deal with the remainder in (31) in the same way as it was done in Proposition 2.3 and Theorem 3 to show that its contribution is $O(q^d)$. The demonstration is routine. One simply changes through (37) (40), modifying the exponents involved in the obvious way and extending summations to infinity, yet weighting them by $(1 + |a||K^*/q)^{-N}, (1 + |b||K^*/q)^{-N}$, which is not consequential for the order of magnitude of these estimates in q. Note that an extra log $q$ gets picked up in $d = 2$ in (35) where $|a||K^*/q$ changes to $|a||K^*/q - 2$. □

Remark R3 It is clear that estimating the right-hand-side of (18) which is essentially an $L^2$-estimate for $\frac{\nu}{t}$ does not suffice to get a sharp estimate for $\nu$, when it grows in average slower than $\sqrt{t}$ as $t \to \infty$. That is why we cannot prove Theorem 1 for $d = 2$. The necessity of the logarithmic factor in $d = 3$ in the estimate for $\nu$ is also questionable.

4. Proof of Theorem 2

Theorem 2 will follow immediately from the bound (18) of Lemma 3.1 and the following lemma, which somewhat generalizes the results of [8].

Lemma 4.2 We have the following bound:

$$\|\tilde{E}\|_2^2 \leq b_d(q), \text{ where } b_d(q) = \begin{cases} q^{d-2}, & d \geq 4, \\ q \log^2 q, & d = 3, \\ q, & d = 2. \end{cases} \quad (54)$$

Proof There is no harm changing in (18) the lower limit of integration to 1 and $1 + t^2$ in the denominator to $t^2$.

By definition of $\nu$, for any $t_l, t_u$, with $t_u - t_l \gg \frac{1}{q}$ and a small enough $\delta \sim \frac{1}{q}$, we have the representation of the integral as a Darboux sum:

$$\int_{t_l}^{t_u} \frac{\nu^2(t)}{t^2} dt \approx \sum_k \frac{\nu^2(t_k)}{t_k^2} \psi(t_k/q) \delta, \quad (55)$$

where the intervals $I_k = [t_k, t_{k+1})$ of length $\delta$ partition $[t_l, t_u)$ and choice of $t \in [t_k, t_{k+1})$ is arbitrary.

Then one can always choose $t$ inside each interval $I_k$ in such a way that

$$\nu(t) \lesssim \max \left[ t^{\frac{d-1}{2}}, q E_\nu(t) \right]. \quad (56)$$
This claim quantizes the second bullet assertion in Introduction.

Indeed, the first term inside the above maximum corresponds to the case of the existence of \( t \in I_k \) such that \( \nu_*(t) \lesssim \frac{1}{\nu_*(t)} \). Otherwise, let us use the fact that \( \frac{dE_*(t)}{dt} \approx \nu_*(t) + O\left(\frac{1}{\nu_*(t)}\right) \), and if \( \nu_*(t) \gtrsim \frac{1}{\nu_*(t)} \), the \( O\left(\frac{1}{\nu_*(t)}\right) \) term can be omitted. Then \( |E_*(t)| \gtrsim \frac{1}{\nu_*(t)} \), where at \( t_0 \), \( |E_*| \) has its absolute minimum in \( I_k \). Which implies that \( \sup_{I_k} |E_*(t)| \gtrsim \inf_{I_k} \nu_*(t) \) in this case.

Note that due to (23) all the “regular” terms \( O\left(\frac{1}{\nu_*(t)}\right) \) that pop up further would a-priori result in (54), and in fact stronger inequalities for \( d = 2, 3 \).

Furthermore by (56) the integral of \( \frac{dE_*}{dt} \) has its absolute minimum in \( I_k \). Which implies that \( c_5 q |E_*(t)| \) for \( d \geq 3 \) and \( \log q \) for \( d = 2 \).

Let us turn to the second integral in (57). Evaluation of this integral goes along the lines of [12], [8]. Clearly, in order to get the upper bound, the integral can be extended from \( I \) to \( \mathbb{R}_+ \), under the assumption that \( \nu_*(t) \leq c_5 q |E_*(t)| \) everywhere (note that \( I \) can be represented as the union of intervals of length not smaller than \( \approx \frac{1}{\nu} \) each). Under this assumption, we write out a dyadic decomposition:

\[
\int_{2^k}^{2^{k+1}} |E_*|^2 \nu_*(t) dt \lesssim |\psi(2^k/q)| \left( |E_*(2^k)|^3 + |E_*(2^{k+1})|^3 + 2^{k+1} \int_{2^k}^{2^{k+1}} E_*^2(t) dt \right). 
\]

The cubic terms in brackets are bounded as

\[
O\left[2^{3k} \left(\frac{4}{d-2}\right)^3 + 2^{3k+1} \frac{d-2}{d-1} q^{-3}\right],
\]

which follows from the well known, see e.g. Landau’s classic ([10]), \( L^\infty \) estimate

\[
|E_0(t)| \lesssim t^{d-2} + \frac{1}{\pi^2} + q^{-1} t^{d-1},
\]

where \( E_0(t) = t^{\frac{d-2}{2}} E(t) \), in view of the scaling ([15]). It is a routine calculation to show using the decay of \( \psi \) that the contribution of these terms into (59) is well in compliance with (60).

Hence we are left with

\[
\int_{2^k}^{2^{k+1}} E^2(t) dt \lesssim \sum_{k=0}^\infty 2^{k} |\psi(2^k/q)| \sqrt{\int_{2^k}^{2^{k+1}} \tilde{E}_*^2(t) dt}. 
\]

Assuming that the sum above is \( \gtrsim \sum_{k=0}^\infty 2^{k} |\psi(2^k/q)| \), for \( d \geq 4 \), then, as clearly

\[
\sum_{k=0}^\infty 2^{k} |\psi(2^k/q)| \lesssim q^{d-2}, \text{ for } d \geq 4,
\]

\[
\int_0^\infty \tilde{E}^2(t) dt \lesssim b_d(q) + q \sum_{k=0}^\infty 2^{k} |\psi(2^k/q)| \sqrt{\int_{2^k}^{2^{k+1}} \tilde{E}_*^2(t) dt}. 
\]

### Solutions to Exercises

1. **Exercise 1:**
   - Let \( f(x) = \frac{1}{x^2 + 1} \) be a function defined on \( x \geq 0 \).
   - Find the derivative of \( f(x) \).
   - Calculate the definite integral of \( f(x) \) from 0 to 1.

2. **Exercise 2:**
   - Consider the function \( g(x) = e^{-x^2} \) for \( x \geq 0 \).
   - Determine the area under the curve of \( g(x) \) from 0 to \( \infty \).
   - Compute the indefinite integral of \( g(x) \).

3. **Exercise 3:**
   - Let \( h(x) = \sin(x) \cos(x) \) for \( x \in [0, \pi] \).
   - Find the antiderivative of \( h(x) \).
   - Evaluate \( h(x) \) at \( x = \pi/2 \).

### Additional Notes

- **Note 1:** For Exercise 1, the integral can be evaluated using the substitution technique, setting \( u = x^2 + 1 \).
- **Note 2:** In Exercise 2, the area under the curve can be expressed as an improper integral, requiring limits to be taken at infinity.
- **Note 3:** Exercise 3 involves trigonometric identities and the product rule for differentiation.
we have
\[ \int_0^\infty \tilde{E}^2(t) dt \lesssim q^{2r-1} \sqrt{\int_0^\infty \tilde{E}^2(t) dt}, \]  
(65)
and it follows that \( \| \tilde{E} \|_2, \| \tilde{E}_* \|_2^2 \lesssim b_d(q) = q^{d-2}, d \geq 4, \) as one can certainly swap the subscript \( \ast \) to the left-hand side.

The case \( d = 2, 3 \) requires some extra consideration, see [8], which we have adopted from the latter reference for the sake of completeness. Recall that the quantity \( E \) has been defined with respect to the parameter \( q \), where \( \frac{1}{q} \) is the characteristic size of a speck, each point has been blown up to. To reflect this fact, let us further write \( E = E(q) \), \( \tilde{E} = \tilde{E}(q) \). It is easy to verify by definition of \( E \) that for \( t \lesssim \tilde{q} \lesssim q \), one has
\[ |E(q)(t)| \lesssim |E(\tilde{q})(t)| + O(t^{\frac{d-1}{2} \tilde{q}^{-1}}). \]  
(66)
Qualitatively this means that as the radial density of lattice points increases with the radius, so the points near the origin can be blown up into larger specks, without a risk of miscounting.

We rewrite (63) as follows:
\[ \int_0^\infty |\tilde{E}(q)|^2(t) dt \lesssim b_d(q) + q \sup_k \left( \int_0^\infty \frac{|E(q)|^2(t) dt}{b_d(2^{k+1})} \right) \sum_{k=0}^\infty 2^k (\frac{d}{2} - 2) |\psi(2^k/q)| \sqrt{b_d(2^{k+1})}, \]  
(67)
therefore, evaluating the sum we get
\[ \int_0^\infty \frac{|\tilde{E}(q)|^2 dt}{b_d(q)} \lesssim 1 + \sup_k \left( \int_0^\infty \frac{|E(q)|^2 dt}{b_d(2^{k+1})} \right). \]  
(68)
The supremum above is at most some power of \( q \). It should also be achieved for some finite \( k \), because of the decay, built into the quantity \( \tilde{E} \), due to the presence of the cutoff \( \psi \). Consider then such \( k \), when
\[ m(q) = \max \sup_k \left( \int_0^\infty \frac{|E(q)|^2 dt}{b_d(2^{k+1})}, \int_0^\infty \frac{|\tilde{E}(q)|^2 dt}{b_d(2^{k+1})} \right) \]  
(69)
is achieved. Suppose the maximum will be effected by the first entry in (68).

Then clearly
\[ \int_0^\infty \frac{|\tilde{E}(q)|^2 dt}{b_d(q)} \lesssim \int_0^\infty \frac{|E(q)|^2 dt}{b_d(2^{k+1})}, \]  
(70)
that is \( 2^{k+1} \lesssim q \). Then let \( \tilde{q} = 2^{k+1} \), and now look back at (68) with \( \tilde{q} \) instead of \( q \). Then by (66) the terms in the right-hand side of (68) corresponding to \( k \lesssim \tilde{k} \) would change by at most a constant factor. On the other hand once more \( m(\tilde{q}) \) should be achieved for some \( k \lesssim \tilde{k} \). That means that \( m(\tilde{q}) \approx m(q) \), that is the supremum in (68) with \( \tilde{q} \) substituting \( q \) is of the same order as the left-hand side, hence it is \( O(1) \). By definition of \( \tilde{q} \) this is thence the case for (68) per se as well. This completes the proof of Lemma 4.2 and Theorem 3. □

Acknowledgements: Research has been partially supported by the NSF Grant DMS02-45369, EPSRC Grant GR/S13682/01 and the Bristol Institute for Advanced Study.

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