Probabilistic whereabouts of the “quantum potential”

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Abstract. We review major appearances of the functional expression $\pm \Delta \rho^{1/2}/\rho^{1/2}$ in the theory of diffusion-type processes and in quantum mechanically supported dynamical scenarios. Attention is paid to various manifestations of “pressure” terms and their meaning.

1. From $\rho$ to $\pm \Delta \rho^{1/2}/\rho^{1/2}$

For clarity of presentation, we restrict considerations to probability density functions (pdfs) in one spatial dimension, with the time label suppressed for a while (albeit we have in mind not only random variables, but stochastic processes as well). Given a continuous, at least twice differentiable pdf $\rho(x)$, $\int_{R} \rho(x) \, dx = 1$, we infer a hierarchy of functions $-\ln \rho$, $-\nabla \ln \rho$ and $-\Delta \ln \rho$ whose meaning will soon become more transparent.

An information theoretic notion of the Shannon entropy of $\rho$ (with varied interpretations, among which the notions of disorder and/or uncertainty seem to prevail) reads

$$S(\rho) = -\langle \ln \rho \rangle = -\int \rho(x) \ln \rho(x) \, dx$$

(1)

thus giving $\ln \rho$ the meaning of (Shannon) entropy density. An affiliated measure of disorder/uncertainty named the Fisher information of $\rho$ has the form

$$F(\rho) \equiv \langle (\nabla \ln \rho)^2 \rangle = \int \frac{(\nabla \rho)^2}{\rho} \, dx$$

(2)

and in view of $\langle \nabla \ln \rho \rangle = 0$ stands for a mean square deviation of a function $\nabla \ln \rho(x)$ of the random variable $X$, with values $x \in R$.

These two information theoretic measures are interrelated, as for example can be seen through so-called isoperimetric inequality: $F \geq (2\pi e) \exp(2S)$. More than that, it is the Fisher information which is directly involved in the primordial form of the indeterminacy (uncertainty) principle. Namely, $\langle \nabla \ln x \rangle = 0$ and $Var(x) = \sigma^2 = \langle (x - \langle x \rangle)^2 \rangle$ imply an indeterminacy relationship (no quantum context as yet):

$$Var(\nabla \ln x) = F(\rho) \geq 1/\sigma^2 > 0.$$  

(3)

We can get a deeper insight into the indeterminacy rule, by noting that actually $\rho^{1/2}$ is a square integrable function. Therefore, standard $L^2(R)$ Fourier transform techniques $\psi \rightarrow \tilde{\psi}$ can...
be here adopted. In a slightly more general notation, which encompasses the previous case, 
\( \rho = |\psi|^2, \psi \in L^2 \implies \psi, \bar{\rho} = |\bar{\psi}|^2 \), one infers (in self-explanatory notation)

\[
\frac{1}{\sigma^2} \leq F \leq 16\pi^2 \bar{\sigma}^2
\]  

(4)

and

\[
\frac{4\pi}{\bar{\sigma}} \leq \left( \frac{1}{\sqrt{2\pi e}} \right) \exp[\mathcal{S}] \leq \sigma.
\]  

(5)

We can continue our discussion of \(- \ln \rho(x), -\nabla \ln \rho\) and \(-\Delta \ln \rho\) by simply playing with them, to reveal a number of emergent quantities, like e.g.

\[
-\Delta \ln \rho = -\frac{\Delta \rho}{\rho} + \frac{1}{2} \frac{(\nabla \rho)^2}{\rho^2} \implies -\langle \Delta \ln \rho \rangle = \frac{1}{2} \nabla \rho \Delta \ln \rho
\]  

(6)

At this point we introduce a potential \(\frac{\Delta \rho^{1/2}}{\rho^{1/2}}\) of a Newton-type “force field” with vanishing mean value \(\langle \nabla (\frac{\Delta \rho^{1/2}}{\rho^{1/2}}) \rangle = 0\)

\[
-\frac{\Delta \rho^{1/2}}{\rho^{1/2}} = \frac{1}{2} \frac{\Delta \rho}{\rho} + \frac{1}{2} \frac{(\nabla \rho)^2}{\rho^2} \implies +\nabla (\frac{\Delta \rho^{1/2}}{\rho^{1/2}}) = \frac{1}{2\rho} \nabla (\rho \Delta \ln \rho)
\]  

(7)

and make explicit its links with the Fisher information of the pdf \(\rho\), through

\[
-\langle \frac{\Delta \rho^{1/2}}{\rho^{1/2}} \rangle = -\frac{1}{4} \langle \Delta \ln \rho \rangle = \frac{1}{4} F(\rho) \geq \frac{1}{4\text{Var}(x)} > 0,
\]  

(8)

We emphasize a conspicuous absence of any specific physical context. Nonetheless, while accounting for a temporal evolution of \(\rho = \rho(x, t)\), a number of physically interesting quantities can be easily identified. They are omnipresent in local conservation laws for diffusion-type stochastic processes, as well as in the hydrodynamical formulation of the Schrödinger picture quantum dynamics.

### 2. Emergence of \(\pm \frac{\Delta \rho^{1/2}}{\rho^{1/2}}\) in hydrodynamical (local) conservation laws

#### 2.1. Quantum hydrodynamics

Taking as obvious the standard wisdom about a hydrodynamical representation of the Schrödinger picture quantum dynamics, we merely recall that the Schrödinger equation

\[
i\hbar \partial_t \psi = \left[ -\frac{\hbar^2}{2m} \Delta + V \right] \psi
\]  

(9)

involves a Hamiltonian \(\hat{H}\) that is a self-adjoint operator in a suitable Hilbert space domain. Since we shall be dealing with bounded from below operators, for later convenience we impose an additive renormalization of the Hamiltonian so that \(\hat{H} \geq 0\). Further notations are reproduced for the record: the pdf is \(\rho(x, t) = |\psi|^2(x, t), v = (\hbar/2m)(\nabla \psi)/\psi - (\nabla \psi^*/\psi^*)[\nabla \psi/\psi - (\nabla \psi^*/\psi^*)]\) stands for the current velocity field. With the polar (Madelung) decomposition of \(\psi\) being implicit, we get:

\[
\partial_t \rho = -\nabla (\rho v); \quad \partial_t s + \frac{1}{2m} (\nabla s)^2 + (V + Q) = 0 \implies \\
\partial_t v + (v \nabla v) = -\frac{1}{m} \nabla (V + Q)
\]  

(10)
where \( v = \frac{1}{m} \nabla s \) and \( Q = Q[\rho] = -\frac{\hbar^2}{2m} \frac{\Delta \rho^{1/2}}{\rho^{1/2}} \) has a folk name of a “quantum potential”.

We note that the ground state condition for \( \hat{H} \), with bottom eigenvalue 0, given \( |\psi| = \rho_s^{1/2} \), directly involves the “quantum potential”, presently evaluated with respect to the ground state pdf \( \rho_s \):

\[
V = + \frac{\hbar^2}{2m} \frac{\Delta \rho_s^{1/2}}{\rho_s^{1/2}} = -Q[\rho_s].
\]  

(11)

Denote \( u(x,t) = (\hbar/2m) \nabla \ln \rho \). It is well known that the dynamics arises via the \( \{ \rho, s \} \) extremum principle for

\[
I(\rho, s) = \int_{t_1}^{t_2} \left( \partial_t s + \frac{m}{2} (u^2 + v^2) + V \right) dt.
\]  

(12)

In terms of valid solutions \( \rho(x,t), s(x,t) \), we arrive at a strictly positive constant of motion:

\[-\langle \partial_t s \rangle = H = \langle \left[ \frac{m}{2} (u^2 + v^2) + V \right] \rangle > 0 \] (a finite energy condition).

2.2. Brownian hydrodynamics

The semigroup dynamics and the emergent generalized diffusion equation (note that by setting \( V = 0 \) we pass to the standard heat equation)

\[
\exp(-t\hat{H}/2mD)\Psi_0 = \Psi_t \implies \partial_t \Psi = \left[ D\Delta - \frac{V}{2mD} \right] \Psi
\]  

(13)

is a self-adjoint relative of the more familiar Fokker-Planck equation \( \partial_t \rho = D\Delta \rho - \nabla (b\rho) \) and likewise, although indirectly, determines the evolution of \( \rho(x,t) \). Here \( \hat{H} \) is self-adjoint, \( \hat{H} \geq 0, t \geq 0 \). (We keep in mind a re-definition \( \hbar = 2mD \).)

Let \( \Psi(x,t) \to \rho_s^{1/2} \) as \( t \to \infty \). Define \( \rho(x,t) = \Psi(x,t)\rho_s^{1/2}(x) \) with \( b = D\nabla \ln \rho_s, u = D\nabla \ln \rho \) and \( v = b - u = (1/m)\nabla s \). The connection between the Fokker-Planck and semigroup dynamics is being established, provided a compatibility condition

\[
\frac{V(x)}{2mD} = +D\frac{\Delta \rho_s^{1/2}}{\rho_s^{1/2}} = mD \left[ \frac{b^2}{2D} + \nabla b \right]
\]  

(14)

holds true. The (rescaled) “quantum potential” appears in the above (c.f. also (11)), as well as in Hamilton-Jacobi type equations of motion:

\[
\partial_t \rho = D\Delta \rho - \nabla (b\rho) \iff \partial_t \rho = -\nabla (v\rho)
\]  

(15)

\[
\partial_t s + (1/2m)(\nabla s)^2 - (V + Q) = 0 \implies \partial_t v + (v\nabla v) = +\frac{1}{m} \nabla (V + Q)
\]

The \( \{ \rho, s \} \) extremum principle for

\[
I(\rho, s) = \int_{t_1}^{t_2} \langle \left[ \partial_t s + (m/2)(v^2 - u^2) - V \right] \rangle
\]  

(16)

yields the previous Hamilton-Jacobi type dynamics. In terms of dynamically admitted fields \( \rho(x,t) \) and \( s(x,t) \), we have \( -\langle \partial_t s \rangle = H = \langle \left[ \frac{m}{2} (v^2 - u^2) - V \right] \rangle \equiv 0 \).
2.3. Time dependence of Shannon and Fisher functionals
The dynamics of $\rho(x,t)$ is dictated by the continuity equation and this equation alone sets the evolution rule for the Shannon entropy $S[\rho](t)$. Indeed, there holds (provided the fall-off of $\rho$ at spatial infinities ensures the vanishing of $v\rho$):

$$\frac{dS}{dt} = \langle \nabla v \rangle = -\frac{1}{D} \langle vu \rangle$$

where $D \equiv \hbar/2m$ corresponds to the quantum case. Obviously, there is an evolution of the velocity field $v(x,t)$ to be accounted for, c.f. the corresponding Hamilton-Jacobi type equations and their gradient versions.

By exploiting the Hamilton-Jacobi type equations, in case of time independent external potentials, we easily demonstrate that

$$\frac{dF}{dt} = \pm 2 \langle v \nabla P \rangle$$

where $P = \rho \Delta \ln \rho$. The minus sign in the above refers to diffusion processes, while the plus sign to the quantum motion.

A more detailed discussion of the pressure-type term $P$ and its functioning will be given later.

We note a clear parallel with a classical power release expression $\frac{dE}{dt} = vF$, where $F = -\nabla V$ is a standard Newtonian force.

3. Dynamical duality - illusion of “Euclidean time”
In the light of our previous discussion it appears quite persuasive to execute (at least formally) the Wick rotation in the complex time plane $it \to t \geq 0; \ h \to 2mD$

$$\exp(-i\hat{H}t/\hbar)\psi_0 = \psi_t \Rightarrow \exp(-t\hat{H}/2mD)\Psi_0 = \Psi_t$$

that maps between diffusion-type and quantum mechanical patterns of dynamical behaviour. Given the spectral solution for $\hat{H} = -\Delta + V$, the integral kernel of $\exp(-t\hat{H})$ reads

$$k(y, x, t) = \sum_j \exp(-\epsilon_j t) \Phi_j(y)\Phi_j^*(x).$$

Remember that $\epsilon_0 = 0$ and the sum may be replaced by an integral in case of a continuous spectrum, (with complex-valued generalized eigenfunctions). Set $V(x) = 0$ identically. Then we end up with a familiar heat kernel:

$$k(y, x, t) = [\exp(t\Delta)](y, x) = (2\pi)^{-1/2} \int \exp(-p^2 t) \exp(ip(y - x) dp =$$

$$(4\pi t)^{-1/2} \exp[-(y - x)^2/4t].$$

Consider $\hat{H} = (1/2)(-\Delta + x^2 - 1)$ (e.g. the rescaled harmonic oscillator Hamiltonian). The integral kernel of $\exp(-t\hat{H})$ is given by the classic Mehler formula:

$$k(y, x, t) = k(x, y, t) = [\exp(-t\hat{H})](y, x) =$$

$$[\pi(1 - \exp(-2t))^{-1/2} \exp[-(1/2)(x^2 - y^2)] - (1 - \exp(-2t))^{-1} (x \exp(-t) - y)^2].$$

The normalization condition $\int k(y, x, t) \exp[(y^2 - x^2)/2] dy = 1$ actually defines the transition probability density of the Ornstein-Uhlenbeck process

$$p(y, x, t) = k(y, x, t) \rho^{1/2}_s(x)/\rho^{1/2}_s(y)$$
with \( \rho_*(x) = \pi^{-1/2} \exp(-x^2) \). A more familiar form of the kernel reads (note the presence of \( \exp(t/2) \) factor)

\[
k(y, x, t) = \frac{\exp(t/2)}{(2\pi \sinh t)^{1/2}} \exp \left[ -\frac{(x^2 + y^2) \cosh t - 2xy}{2 \sinh t} \right].
\]

To conform with the statistical physics lore of the 50-ies and 60-ties, we can easily pass to an integral kernel of the density operator, labeled by equilibrium values of the temperature. To this end one should set e.g. \( t \equiv \hbar \omega/k_B T \) for a harmonic oscillator with a proper frequency \( \omega \) and remember about evaluating the normalization factor \( 1/Z_T \) where \( Z_T \) stands for a partition function of the system.

Concerning the “Euclidean issue”, we note that by formally executing \( t \to it \) one arrives at the free Schrödinger propagator

\[
K(y, x, t) = \left[ \exp(it\Delta) \right] (y, x) = (2\pi)^{-1/2} \int \exp(-ip^2t) \exp(ip(y - x)) dp = (4\pi it)^{-1/2} \exp[i(y - x)^2/4t]
\]

and likewise, at that of (here \(-1\) renormalized) harmonic oscillator propagator

\[
K(y, x, t) = \frac{\exp(it/2)}{(2\pi i \sin t)^{1/2}} \exp \left[ +i \frac{(x^2 + y^2) \cos t - 2xy}{2 \sin t} \right]
\]

Learn a standard Euclidean (field) theory lesson concerning multi-time correlation functions. In the exemplary harmonic oscillator case, \( t > t' > 0; t \to it \) results in:

\[
E[X(t')X(t)] = \int \rho_*(x') x' p(x', t', x) x \, dx \, dx' = (1/2) \exp[-(t - t')] \implies (27)
\]

\[
W(t', t) = \langle \psi_0, \hat{q}_H(t') \hat{q}_H(t) \psi_0 \rangle = (1/2) \exp[-i(t - t')]
\]

where \( \hat{q}_H(t) \) stands for the position operator in the Heisenberg picture. In passing, we note that this appealing correspondence breaks down in \( R^n, n > 1 \), in the presence of electromagnetic fields.

The major message of our discussion is that we encounter two distinct dynamical patterns of behaviour that follow equally real (realistic) clocks. The Euclidean mapping (Wick rotation) is merely a mathematical artifice connecting the pertinent dynamical models, or rather transforming one model into another.

4. Comments on variational extremum principles

4.1. (Shannon) Entropy extremum principle

Given \( V = V(x) \), fix a priori \( \langle V \rangle = \zeta \). Extremize \( S = -\langle \ln \rho \rangle \) under this constraint, i.e. seek an extremum of

\[
S(\rho) + \alpha \langle V \rangle = \langle -\ln \rho + \alpha V \rangle
\]

where \( \alpha \) is a Lagrange multiplier. As an outcome we get the \( \alpha \)-family of pdfs \( \rho_\alpha = A_\alpha \exp[\alpha V(x)] \), provided \( (A_\alpha)^{-1} = \int \exp[\alpha V(x)] \, dx \) exists. The Lagrange multiplier \( \alpha \)-value must be inferred from the constraint \( \langle V \rangle_\alpha = \zeta \).
4.2. Fisher information extremum principle

Fix a priori \( \langle V \rangle = \zeta \). Extreme the Fisher information measure \( \mathcal{F}(\rho) \) under that constraint:

\[
\mathcal{F}(\rho) + \lambda\langle V \rangle = \langle (\nabla \ln \rho)^2 + \lambda V \rangle
\]

Remember that \(-\langle \Delta \rho^{1/2} \rangle = \frac{1}{4} \mathcal{F}(\rho)\). The extremizing pdf \( \rho(x) = \rho_s(x) \) comes out from

\[
V(x) = 2 \left( \frac{\Delta \rho}{\rho} - \frac{1}{2} \frac{(\nabla \rho)^2}{\rho^2} \right) + \frac{4 \Delta \rho^{1/2}}{\rho^{1/2}}.
\]

Outcome: \( \lambda \)-family of pdfs; \( \lambda \) gets fixed by \( \langle V \rangle_\lambda = \zeta \). Setting \( \lambda = 2/mD^2 \), we recover the Brownian framework; \( \lambda = 8m/h^2 \) is admitted as a special case.

4.3. Hamilton-Jacobi route

Think of a purely classical case \( H = p^2/2m + V(x) \), \( \{ \dot{q} = p/m, \dot{p} = -\nabla V(q) \} \). Next, assign random initial data (here, in space) \( \rho_0(x) \Rightarrow S(\rho) \) and \( \mathcal{F}(\rho) \). By extremizing the action functional we deduce the standard Hamilton-Jacobi description of an ensemble of classical systems:

\[
I_0(\rho, s) = \int_{t_1}^{t_2} \left( \partial_t s + \frac{1}{2m} (\nabla s)^2 + V \right) dt = \partial_t s + \frac{1}{2m} (\nabla s)^2 + V = 0 \tag{28}
\]

plus the continuity equation \( \partial_t \rho = -\nabla (v \rho) \). Here an assumption \( v = (1/m)\nabla s \) implies \( \partial_t v + (v \nabla v) = -\nabla V \).

4.4. Constrained Fisher information

Fix a priori \( \int_{t_1}^{t_2} \mathcal{F}(\rho)(t) \, dt = \zeta \). Extreme

\[
I_\gamma(\rho, s) = \int_{t_1}^{t_2} dt \left( \partial_t s + \frac{(\nabla s)^2}{m} \pm V + \gamma \frac{(\nabla \rho)^2}{\rho^2} \right) \implies \partial_t \rho = -\nabla (v \rho) \tag{29}
\]

\[
\partial_t s + \frac{(\nabla s)^2}{m} \pm V + 4\gamma \frac{\Delta \rho^{1/2}}{\rho^{1/2}} = 0
\]

where by denoting \( \pm V \) we intend to make a distinction between confining (generically bounded from below) and scattering potentials.

Outcomes (an admissible case of \( \gamma = 0 \) is left aside):

(i) \( \gamma = -mD^2/2 \), eventually followed by setting \( D = h/2m \), leads to the \( D \)-labelled quantum hydrodynamics (before, we have referred to \(+V \) only)

\[
\partial_t s + \frac{1}{2m} (\nabla s)^2 \pm V + Q = 0
\]

(ii) \( \gamma = +mD^2/2 \), with the potential term \(-V \) only, leads to the Brownian hydrodynamics

\[
\partial_t s + (1/2m)(\nabla s)^2 - (V + Q) = 0
\]

Note: \( t \rightarrow it \) relationship can be secured for \(+V \), where \( V \) is a confining potential.

\[
\partial_t s + \frac{1}{2m} (\nabla s)^2 + (V + Q) = 0
\]

c.f. \( t \rightarrow it \implies \exp(-it\hat{H}/2mD)\Psi_0 = \Psi_t \implies \exp(-it\hat{H}/2mD)\psi_0 = \psi_t \) issue.

We demand \( \hat{H} \) to have a bottom eigenvalue equal zero (to yield a contractive semigroup). For a bounded from below Hamiltonian this can be always achieved, like e.g. in case of \( \hat{H} = (1/2)(-\Delta + x^2 - 1) \).
4.5. Hamilton-Jacobi route - a catalogue of “standards”
In what follows we emphasize a relevance of the sign of the external potential (positive - confinement, negative-scattering) in Lagrangian densities.
(i) \( \mathcal{L}^+ = -\rho \left[ \partial_t s + (m/2)(v^2 + u^2) + V \right] \Rightarrow \partial_t s + (1/2m)(\nabla s)^2 + (V + Q) = 0 \)
(ii) \( \mathcal{L}_\pm = -\rho \left[ \partial_t s + (m/2)v^2 \pm V \right] \Rightarrow \partial_t s + (1/2m)(\nabla s)^2 \pm V = 0 \)
(iii) \( \mathcal{L}^- = -\rho \left[ \partial_t s + (m/2)(v^2 - u^2) - V \right] \Rightarrow \partial_t s + (1/2m)(\nabla s)^2 - (V + Q) = 0 \).

A continuity equation \( \partial_t \rho = -\nabla (v \rho) \) is shared by all listed cases, provided we set \( v = (1/m)\nabla s \). On dynamically admitted fields \( \rho(t) \) and \( s(x,t) \), \( L(t) = \int dx \mathcal{L} \equiv 0 \), i.e. we have \( \partial_t s = -H \).

The respective Hamiltonian functionals have the form:
(i) \( H^+ = \int dx \rho \left[ (m/2)v^2 + V + (m/2)u^2 \right] > 0 \), is a (quantum) constant of motion
(ii) \( H_\pm = \int dx \rho \left[ (m/2)v^2 \pm V \right] = E, \quad E = (p^2/2m) \pm V(x) \), constant on each path
(iii) \( H^- = \int dx \rho \left[ (m/2)v^2 - V - (m/2)u^2 \right] = 0 \), identically in Brownian motion.

We emphasize that, from the start, \( V(x) \) is chosen to be confining. A class of continuous and bounded from below functions allows to secure \( \hat{H} \geq 0 \). Eventually, after subtracting the lowest eigenvalue of the bounded from below energy operator.

5. Kinetic theory lore: Brownian analogies and hints
Consider free phase-space Brownian motion in the large friction regime. \( W(x,u,t) \) stands for phase-space (velocity-position) probability distribution with suitable initial data at \( t = 0 \). Denote by \( w(u,t) \) and \( x(t) \), the marginal pdfs.

We set \( D = k_BT/m\beta \) and observe that actually, in the large friction regime, \( w(x,t) \) stays in the vicinity of (and ultimately converges to) the heat kernel solution \( w(x,t) \sim (4\pi D t)^{-1/2} \exp(-x^2/4Dt) \) of \( \partial_t w = D\Delta w \).

We introduce moments and local moments of the pdf \( w(x,t) \) in the large friction regime: \( \langle u \rangle = \int du u W(x,u,t) \rightarrow \langle u \rangle = \langle x \rangle + (2/4)\langle u \rangle w(x,t) \), \( \langle u \rangle_x = \langle u \rangle / w(x,t) = x/2t = -D(\nabla w)/w \), \( \langle u^2 \rangle_x = \langle u^2 \rangle / w(x,t) = (D\beta - D/2t) + \langle u \rangle_x^2 \).

The Kramers-Fokker-Planck equation
\[ \partial_t W + u \nabla_x W = \beta \nabla_u (W u) + q\Delta u W \] (30)
with \( q = D\beta^2 \), implies the local conservation laws
\[ \partial_t w + \nabla \langle u \rangle_x w = 0 \] (31)
\[ \partial_t \langle u \rangle_x w + \nabla_x \langle u^2 \rangle_x w = -\beta \langle u \rangle_x w. \]

Introducing the kinetic pressure notion \( P_{\text{kin}}(x,t) = [\langle u^2 \rangle_x - \langle u \rangle_x^2]w(x,t) \) we arrive at
\[ \partial_t + \langle u \rangle_x \nabla \langle u \rangle_x = -\beta \langle u \rangle_x - \nabla P_{\text{kin}}/w. \] (32)

In the large friction regime we have
\[ -\nabla P_{\text{kin}}/w = +\beta \langle u \rangle_x - \nabla P_{\text{osm}}/w \] (33)
where \( P_{\text{osm}} = D^2w\Delta \ln w \) denotes an osmotic pressure of the Brownian motion.

We note that \( \nabla P_{\text{osm}} = -w \nabla Q/m \) with \( Q = -2mD^2\Delta w^{1/2}/w^{1/2} \).

Actually \( -\nabla P_{\text{osm}} = (D/2t) \nabla w \). Thus, letting \( \langle u \rangle_x = v(x,t) \) we arrive at:
\[ (\partial_t + v \nabla)v = -\nabla P_{\text{osm}}/w = +1/m \nabla Q \] (34)
to be compared with the general Brownian hydrodynamics result
\[ \partial_t v + (v \nabla v) = \frac{1}{m} \nabla (V + Q) \] (35)

In the past (1992) I have named all that a “derivation of the quantum potential from realistic Brownian particle motions”.

5.1. Functioning of pressure terms \( P_{\text{kin}} \) and \( P_{\text{osm}} \)
In view of \(-\langle \Delta \ln \rho \rangle = F(\rho) > 0\), the osmotic pressure \( P_{\text{osm}} \) is predominantly negative-definite. To the contrary, the kinetic pressure \( P_{\text{kin}} \) is positive definite. That imposes limitations on the validity of the large friction regime, to become operational after times \( t > (2\beta)^{-1} \).

Let us introduce the notion of kinetic temperature:
\[ 0 \leq \Theta_{\text{kin}} = \frac{m P_{\text{kin}}}{w} \sim (k_B T - \frac{mD^2}{2t}) < k_B T \] (36)
whose (large time limit) asymptotic value, \( k_B T \) actually is. Since \( P_{\text{osm}}/w = D^2 \Delta \ln w = -D/2t \), we learn that a (predominantly) positive-definite quantity
\[ \Theta_{\text{osm}} = -m \frac{P_{\text{osm}}}{w} = -mD^2 \Delta \ln w \implies \Theta_{\text{kin}} \sim (k_B T - \Theta_{\text{osm}}) \] (37)
gives account of the deviation from thermal equilibrium, in terms of the local “thermal energy” (agitation) \( \Theta_{\text{osm}} \).

One more useful identity (not an independent equation) is here valid. It expresses the “thermal energy” conservation law (observe that no thermal currents are hereby induced):
\[ (\partial_t + v \nabla) \Theta_{\text{osm}} = -2(\nabla v)\Theta_{\text{osm}} \implies \partial_t \Theta_{\text{osm}} = -2(\nabla v)\Theta_{\text{osm}} \] (38)

5.2. Meaning of the pressure term in Brownian hydrodynamics (\( P_{\text{osm}} \equiv P \))
We come back to local conservation laws of the Brownian “fluid”. This is an ensemble picture of the Brownian motion: imagine the Pablo Picasso art of placing one upon another hundreds of transparent foils, each carrying a drawing of one complete Brownian trajectory. All random paths are supposed to start from the same point and next allowed to run a pre-defined time period \([0, T]\), common for all repetitions. Brownian hydrodynamics is about statistical properties of such an ensemble:
\[ \partial_t v + (v \nabla v) = \frac{1}{m} \nabla (V + Q) = \frac{1}{m} F - \frac{\nabla P}{w} \] (39)
\[ -\frac{\nabla P}{w} = \frac{1}{m} \nabla Q; \; F \doteq -\nabla (-V) \]

In normal liquids the pressure is exerted upon any control volume (here, by an imagined small droplet), thus involving its compression. Just to the contrary, in case of Brownian motion we deal with a definite decompression.

Let us consider a reference volume (control interval, finite droplet) \([\alpha, \beta]\) in \( R^1 \) (or \( \Lambda \subset R^1 \)) which at time \( t > 0 \) comprises a certain fraction of particles (it is a loose term designating, whatever they would be, the Brownian “fluid” constituents).

The time rate of particles loss or gain by the volume \([\alpha, \beta]\) at time \( t \), is equal to the flow outgoing through the boundaries
\[ -\partial_t \int_{\alpha}^{\beta} \rho(x, t) dx = \rho(\beta, t)v(\beta, t) - \rho(\alpha, t)v(\alpha, t) \]
To analyse the momentum balance, let us slightly deform the boundaries $[\alpha, \beta]$ to compensate the mass imbalance: $[\alpha, \beta] \rightarrow [\alpha + v(\alpha, t) \Delta t, \beta + v(\beta, t) \Delta t]$. Effectively, we pass to a locally co-moving (droplet) frame; that is the Lagrangian picture.

(i) The mass balance has been thus established in the moving droplet:

$$ \lim_{\Delta t \downarrow 0} \frac{1}{\Delta t} \left[ \int_{\alpha + v_{\alpha} \Delta t}^{\beta + v_{\beta} \Delta t} \rho(x, t + \Delta t) dx - \int_{\alpha}^{\beta} \rho(x, t) dx \right] = 0 $$

(ii) For local matter flows $(\rho v)(x, t)$, in view of $\partial_t (\rho v) = -\nabla (\rho v^2) + (1/m) \rho \nabla (V + Q)$, the time rate of change of momentum (per unit of mass) of the droplet, reads

$$ \lim_{\Delta t \downarrow 0} \frac{1}{\Delta t} \left[ \int_{\alpha + v_{\alpha} \Delta t}^{\beta + v_{\beta} \Delta t} (\rho v)(x, t + \Delta t) dx - \int_{\alpha}^{\beta} (\rho v)(x, t) \right] = \int_{\alpha}^{\beta} \frac{1}{m} \rho \nabla (V + Q) dx $$

However, $\nabla Q/m = -\nabla \rho$ and $P = D^2 \rho \Delta \ln \rho$. Therefore:

$$ \int_{\alpha}^{\beta} \frac{1}{m} \rho \nabla (V + Q) dx = \int_{\alpha}^{\beta} \rho \nabla \Omega dx - \int_{\alpha}^{\beta} \nabla P dx = \frac{1}{m} E[\nabla V]_{\alpha} + P(\alpha, t) - P(\beta, t) $$

(iii) The time rate of change of the kinetic energy of the droplet is:

$$ \lim_{\Delta t \downarrow 0} \frac{1}{\Delta t} \left[ \int_{\alpha + v_{\alpha} \Delta t}^{\beta + v_{\beta} \Delta t} (\rho v^2)(x, t + \Delta t) dx - \int_{\alpha}^{\beta} (\rho v^2)(x, t) \right] = \int_{\alpha}^{\beta} \frac{1}{m} (\rho v) \nabla (V + Q) dx $$

Note that $\int_{\alpha}^{\beta} \rho v \nabla Q dx = -\int_{\alpha}^{\beta} \rho \nabla P dx$ (c.f. the standard notion of power release $\frac{dE}{dt} = F \cdot v$)

5.3. Meaning of the pressure term in quantum hydrodynamics ($-P_{osm} \equiv P$)

We do not care about a specific “trajectory” representation of the quantum motion and pass directly to local conservation laws:

$$ \partial_t v + (v \nabla v) = -\frac{1}{m} \nabla (V + Q) = \frac{F}{m} - \frac{\nabla P}{\rho} \implies $$

$$ -\frac{1}{m} \nabla Q = \frac{\nabla P_{osm}}{\rho} = -\frac{\nabla P}{\rho} $$

which enforce $-P_{osm} = -D^2 \rho \Delta \ln \rho \equiv P$, $D = \hbar/2m$, while $F = -\nabla V$. If compared to the Brownian hydrodynamics all $(V + Q)$ contributions come with an inverted sign. This carries over to the mass, momentum and kinetic energy rates.

Quite at variance with the Brownian $P = P_{osm}$, the quantum pressure term $P = -P_{osm}$ is predominantly positive. We recall that $-\langle \Delta \ln \rho \rangle (\langle \nabla \rho \rangle^2) = F(\rho) > 0$.

We note in passing that quantum mechanically derivable heat transfer equation

$$ (\partial_t + v \nabla) \Theta_{osm} = -2 \frac{\nabla q}{\rho} - 2(\nabla v) \Theta_{osm} $$

with $\Theta_{osm} = -m \frac{P_{osm}}{\rho} = -m D^2 \Delta \ln \rho$ and $q = -2mD^2 \rho \Delta v$, reproduces the Brownian form, at least for generic free Schrödinger wave packets with $\Delta v = 0$. We get $\partial_t \Theta_{osm} = -2(\nabla v) \Theta_{osm}$ as well. There is no heat current in such case.
6. Hamilton-Jacobi related hydrodynamics and (Bohmian) trajectory descriptions

At this point it seems instructive to make a comment on uses of the Eulerian picture (passive control) vs Lagrangian picture (active control in a co-moving frame) of hydrodynamical equations of motion. Let us simply give our droplet, previously considered as a co-moving control volume, an infinitesimal size. We readily identify the (fairly small sized) droplet dynamics: each droplet “looks” particle-like, while following Bohm-type trajectories.

Since, \( f(x, t) \rightarrow f(x(t + \Delta t), t + \Delta t) \sim [\partial_t f + (v \nabla) f] \Delta t; \ \dot{x} = v = v(x, t)|_{x(t)=x} \), with \( x(t + \Delta t) \sim v \Delta t, v = (1/m) \nabla s \) and \( \partial_t s = ds/dt - mv^2 \), we are in fact bound to work with:

(i) Classical hydrodynamics: (droplet) paths in the Lagrangian frame

\[
\frac{d\rho}{dt} = -\rho \nabla v \quad \Rightarrow \quad \rho(x(t + \Delta t), t + dt) \sim \exp[-(\nabla v) \Delta t] \rho(x, t)
\]  

\[
\frac{ds}{dt} = \frac{1}{2m} (\nabla s)^2 - (\pm V) \quad \Rightarrow \quad m \frac{dv}{dt} = -\nabla (\pm V)
\]

(ii) Brownian hydrodynamics: (droplet) paths in the Lagrangian frame

\[
\frac{d\rho}{dt} = -\rho \nabla v
\]  

\[
\frac{ds}{dt} = \frac{1}{2m} (\nabla s)^2 + (V + Q) \quad \Rightarrow \quad m \frac{dv}{dt} = +\nabla (V + Q)
\]

We need to recall a purely random (Wiener noise) background of the hydrodynamical formalism. We encounter here a primordial description in terms of random variables and paths. The latter may cross the droplet (e.g. enter from the outside, leave or simply stay within for a while):

\[
\frac{dX}{dt} = b(X(t)) dt + \sqrt{2D} dW(t) \quad \Rightarrow \quad \partial_t \rho = D \Delta \rho - \nabla (bp);
\]

\[
\frac{V(x)}{2md} = mD \left[ \frac{k^2}{2D} + \nabla b \right]
\]

(iii) Quantum hydrodynamics: (droplet) paths in the Lagrangian frame \( \Rightarrow \) Bohmian trajectories

\[
\frac{d\rho}{dt} = -\rho \nabla v
\]  

\[
\frac{ds}{dt} = \frac{1}{2m} (\nabla s)^2 - (V + Q) \quad \Rightarrow \quad m \frac{dv}{dt} = -\nabla (V + Q)
\]

7. Acceleration concept in random motion: 1st Newton law

Consider a Markovian diffusion process on \( R \), for times \( t \in [0, T] \)

\[
dX(t) = b(X(t), t) dt + \sqrt{2D} dW(t), \quad \text{where } W(t) \text{ stands for the Wiener noise and } X(t_0) = x_0.
\]

Given \( \rho(y, s, t), s \leq t \) and \( \rho_0(x) \), we can infer a statistical future of the process:

\[
\rho(x, t) = \int \rho(y, s)p(y, s, x, t) dy \quad \Rightarrow \quad \partial_t \rho = D \Delta - (\nabla bp)
\]

\[
b(x, t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int (y - x)p(y, s, t, y, t + \Delta t) dy = v(x, t) + (D \nabla \rho/\rho)(x, t)
\]

We can as well reproduce a statistical past of the process, by means of

\[
p_*(y, s, x, t) \triangleq p(y, s, x, t) \frac{\rho(y, s)}{\rho(x, t)} \quad \Rightarrow \quad \rho(y, s) = \int p_*(y, s, x, t) \rho(x, t) dx
\]

\[
b_*(y, s) = \lim_{\Delta s \to 0} \frac{1}{\Delta s} \int (y - s)p_*(x, s - \Delta s, y, s) dx = v(y, s) - (D \nabla \rho/\rho)(y, s)
\]
Making notice of \( v = (1/2)(b + b_\ast) \), we get:
\[
\partial_t \rho - \nabla (v \rho) = D \Delta \rho - (\nabla b \rho) = -D \Delta \rho - \nabla (b_\ast \rho).  \tag{47}
\]
Consider \( b = DX \) and \( b_\ast = D_\ast X \) as special cases of forward (predictive) and backward (retrodictive) time derivatives of functions of the random variable \( X(t) \):
\[
(D f)(X(t), t) = (\partial_t + b \nabla + D \Delta) f(X(t), t) \tag{48}
\]
\[
(D_\ast f)(X(t), t) = (\partial_t + b_\ast \nabla - D \Delta) f(X(t), t).
\]
Analyse acceleration formulas for diffusion-type processes, in terms of the time rate of change (forward and backwards) drift functions and their local averages. Their response to small time increments reads:

(i) \( b(x, t) \to b(X(t + \Delta t), t + \Delta t) \Rightarrow \langle b \rangle(x, t + \Delta t) \)

(ii) \( b(x, t + \Delta t) = \int p(x, t, z, t + \Delta t) b(z, t + \Delta t) dz \sim b(x, t) + (D^2 X)(t) \Delta t. \)

(iii) \( b_\ast(x, t) \to b_\ast(X(t + \Delta t), t + \Delta t) \Rightarrow \langle b_\ast \rangle(x, t + \Delta t). \)

(iv) \( b_\ast(X(t + \Delta t), t + \Delta t) \sim b_\ast(x, t + (D_\ast X)(t) \Delta t \to b_\ast(x, t) \langle b_\ast \rangle(x, t - \Delta t) \sim b_\ast(x, t) - (D^2_\ast X)(t) \Delta t. \)

Accordingly, we can associate various acceleration formulas with general diffusion-type processes. They may be interpreted as stochastic analogues of the \( II^{nd} \) Newton law in the (local) mean: accelerations are related to conservative volume forces of external origin. We indicate that by over-emphasizing the Newtonian viewpoint, we get somewhat blurred an important intrinsic acceleration input, due to the background random motion. It is encoded in the \( \pm 1/m \nabla Q \) contribution to the current velocity time rate of change. Following the standard association of accelerations with external (volume) forces, one is inclined to attribute the second Newton law meaning to the formula
\[
(D^2 X)(t) = (\partial_t + v \nabla) v - \frac{1}{m} \nabla Q = (D^2_\ast X)(t) = \pm \frac{1}{m} \nabla V \tag{49}
\]
in case of the Brownian motion, while
\[
\frac{1}{2}[(DD_\ast + D_\ast D)X](t) = (\partial_t + v \nabla) v + \frac{1}{m} \nabla Q = \mp \frac{1}{m} \nabla V \tag{50}
\]
in case of Nelson’s stochastic mechanics, i.e. the diffusion-type probabilistic counterpart of the Schrödinger picture quantum motion.

It is the mid-term in the formulas (49) and (50), where the role of \( \mp \nabla Q \) must be strongly emphasized. Set \( \nabla V = 0 \); the mid-term still accounts for definite acceleration phenomena, that are intrinsic to the random motion proper (c.f. a discussion of Section 5):

(i) \( \frac{dv}{dt} - \frac{1}{m} \nabla Q = 0 \), (ii) \( \frac{dv}{dt} + \frac{1}{m} \nabla Q = 0 \)
\[
\tag{51}
\]
Therefore, as appropriate candidates for the \( II^{nd} \) Newton law, we promote not (47) or (48), but rather the (hydrodynamical) local conservation laws:
\[
(iii) (\partial_t + v \nabla) v = \pm \frac{1}{m} \nabla (Q + V) \tag{52}
\]
where the (seemingly minor) sign difference of the right-hand-side terms is hereby crucial and will set grounds below to the \( III^{rd} \) Newton law in the local mean.
8. **III**\textsuperscript{rd} **Newton law**

8.1. **Impulse-momentum change law for for small times**

Both the acceleration and impulse-momentum change concepts have been borrowed directly from classical mechanics. Nonetheless, an exploitation of properly tailored local mean values allows to extend the meaning of purely mechanical concepts to the theory of random motion (e.g. diffusion-type processes, with continuous, but generically non-differentiable sample paths). That was the case in connection with the **II**\textsuperscript{nd} Newton law. It could have been explicitly rooted in the impulsive (short time increments) behaviour of drifts in the Brownian motion:

\[ b^\ast(x, t) - \langle b^\ast\rangle(x, t - \Delta t) \sim \langle b\rangle(x, t + \Delta t) - b(x, t) \sim \frac{1}{m} \nabla V \Delta t \]  

and/or an impulsive behaviour of drifts in stochastic mechanics (probabilistic counterpart of the Schrödinger picture evolution)

\[ b^\ast(x, t) - \langle b^\ast\rangle(x, t - \Delta t) \sim \langle b\rangle(x, t + \Delta t) - b(x, t) \sim \frac{1}{m} \nabla (V + 2Q) \Delta t. \]

However, there is no other than aesthetic reason to associate the Brownian acceleration with the first and not the second (stochastic mechanics) formulas above. Indeed, more careful examination proves that the Brownian motion can be characterized on an equal footing by both acceleration definitions:

\[ (D^2 X)(t) = (D^2_\ast X)(t) = + \frac{1}{m} \nabla V \iff \frac{1}{2}[(DD_\ast + D_\ast D)X](t) = \frac{1}{m} \nabla (V + 2Q) \]  

and likewise, in the stochastic description of quantum motion (e.g. stochastic mechanics):

\[ (D^2 X)(t) = (D^2_\ast X)(t) = - \frac{1}{m} \nabla (V + 2Q) \iff \frac{1}{2}[(DD_\ast + D_\ast D)X](t) = - \frac{1}{m} \nabla V \]

It is now clear that the Brownian motion and the stochastic transcription of quantum motion (stochastic mechanics) do differ fundamentally in their response to both random noise and external forces. We may stay happy with that observation, e.g. a clearly identified difference between two types of random motion. Accordingly, they can be interpreted as totally divorced from each other, except for incidental formal connections (c.f. the Euclidean time and the Wick transformation issue).

8.2. **Action-reaction rule**

There is still another possibility, that may be given a physical status of relevance, albeit on sufficiently small time-scales only. Considering the impulse-momentum change law as a valid property of random motion, we may as well interpret the two considered dynamical patterns of behaviour as being involved in a perpetual Brownian – anti-Brownian acceleration intertwine. That is, via the **III**\textsuperscript{rd} Newton law in the mean. In the past (1992, 1999) we have heuristically formalized this idea in the concept of the Brownian recoil principle.

Indeed, all acceleration expressions in Eqs. (55) can be mapped into those of Eqs. (56), and in reverse, by addition/subtraction of the force term:

\[ \downarrow \pm \frac{2}{m} \nabla (V + Q) \uparrow \]  

provided, at a given time instant we per force attribute the same values of \( \rho(x, t) \) and \( v(x, t) \) to both the Brownian and quantum hydrodynamical fields.
If the two motions are coupled by the action-reaction principle (IIIrd Newton law) at time \( t \), then an impulse - momentum (here, velocity field) change law can be used to predict +\( \Delta t \) updated values of \( \rho \) and \( v \). Thus e.g. the Brownian impulse in a co-moving frame (given \( \rho \) and \( v \)) induces

\[
\Delta \rho = -[(\nabla v) \Delta t] \rho \quad m \Delta v = +\nabla (V + Q) \Delta t \quad (58)
\]

while an accompanying anti-Brownian impulse in a co-moving frame (given \( \rho \) and \( v \)) reads

\[
\Delta \rho = -[(\nabla v) \Delta t] \rho \quad m \Delta v = -\nabla (V + Q) \Delta t . \quad (59)
\]

We do not attempt to address the celebrated “egg-before-hen” dilemma. The formulas (58) and (59) are regarded as a statistically relevant record of the recoil effect where an “anti-Brownian” impulse associated with a quantum particle (we do not bother what is actually meant under this notion) induces, and in turn gets induced, by the Brownian motion pulse excited in a dissipative random medium (any conceivable notion of a surrounding medium, like e.g. the “vacuum”, zero point radiation field etc.). The dissipation assumption destroys the symmetry of the action-reaction picture, since the Brownian pulse should quickly decay.

8.3. A concept of the Brownian recoil principle

Consider \( \Delta t \ll 1 \). Within \([t, t + \Delta t]\), let the action-reaction coupling between the “vacuum” (whatever that may be) and matter particles sets rules of the game \( \implies \langle \Delta p \rangle_{\text{vacuum}} + \langle \Delta p \rangle_{\text{matter}} = 0 \).

The “vacuum turbulence” propels matter particles by transferring them an anti-Brownian recoil impulse (set \( D = \hbar/2m \)), whose “vacuum” trace (and reason) is the Brownian impulse (may quickly die out due to dissipation, we track the matter data).

Step I. Given the matter data \( \rho(x, t) \) and \( v(x, t) \). At \( t + \Delta t \) we have \( \rho + \Delta \rho = \exp[-(\nabla v) \Delta t] \rho \) and \( v \to v + \Delta v \), where the action (“vacuum” impulse)

\[
\Delta v = + \frac{1}{m} \nabla (V + Q) \Delta t \quad (\text{Brownian}) \quad (60)
\]

is paralleled by the reaction (matter impulse): (\( \downarrow \) - subtract; \( \uparrow \) - add: \( \frac{2}{m} \nabla (V + Q) \))

\[
\Delta v = - \frac{1}{m} \nabla (V + Q) \Delta t \quad (\text{anti – Brownian, e.g. quantum}) \quad (61)
\]

Step II. Update the matter data to \( \rho(x, t + \Delta t), v(x, t + \Delta t) \), leave aside those referring to the “vacuum” and to the preceding Brownian impulse, turn to the next \( \Delta t \) episode when both impulses are excited anew. The new (updated) values of \( \rho \) and \( v \) at time \( t + \Delta t \) are presumed to be determined by the anti-Brownian impulse again.

Any physical justification of the Brownian recoil principle needs a double-medium picture:

(i) an active “vacuum” (background random field, non-equilibrium reservoir, zero-point fluctuations) that is generating and supporting Brownian pulses. These may be interpreted in terms virtual particles

(ii) matter particles, whose dynamics is governed by the IIIrd Newton law and the resultant recoil effect.

A detailed theory of the “vacuum”-particle coupling is obviously necessary to go beyond heuristics.

With an inspiration coming from Yves Couder’s lecture on particle-wave associations, and from deepened studies of the role of trajectory descriptions in diffraction/interference phenomena, as a residual subfield of quantum chemistry, we end up with a simple statement
that there is plenty of room down there. Indeed, before going into any sophisticated formal arguments, one should always keep in mind the quantum mechanical scales of interest: atomic nucleus size \( \sim 10^{-15} - 10^{-14} \text{m} \), atom size \( \sim 10^{-10} - 10^{-9} \text{m} \), electron size (whatever that means for inspired theoreticians) \( \sim 10^{-15} \text{m} \), possibly down to \( \sim 10^{-18} \text{m} \).

Presumably it is not devastatingly naive to address an issue of the (Schrödinger’s wave function) \( \psi \)-ness of the electron “cloud” in the atom, while realizing that, with or without the second quantization and with or without quantum electro- or chromodynamics, the “vacuum” (not an empty void) functioning in quantum physics is still an open territory.

9. Bibliographic notes

There is no way to give justice to all contributors in the field of quantum hydrodynamics or stochastic mechanics. Our selection of references will be less then modest and in addition to Nelson’s, Holland’s and Wyatt’s monographs, will mainly concentrate on a sample of my own research in this area. More references (with a bibliography of the subject) can be found and retrieved from my personal Web page: http://www.fiz.uni.opole.pl/pgar/.

My own hunch is that the Schrödinger picture quantum motion admits a consistent representation in terms of diffusion-type processes. That was the main idea of Nelson’s stochastic mechanics. However, in search for “reasons of randomness” we have found unavoidable to admit a coupled double-medium picture.

Its qualitative features seem to be not distant from the particle-wave association picture (with all reservations raised in the original paper due to Y. Couder and E. Fort, (2006)), in which a physically “real” particle induces a physically relevant “realistic” wave, and that wave in turn is capable of affecting further motion of the particle. A conceptual input of diffusion waves (c.f. A. Mandelis, (2001)) might be useful at this point.

References

[1] Sonogo S 1991 Found. Phys. 21 1135
[2] Holland P 1993 Quantum Theory of motion (Cambridge University Press, Cambridge)
[3] Wyatt R E 2005 Quantum Dynamics with Trajectories (Springer-Verlag, Berlin)
[4] Coffey T M, Wyatt R E and Schieve W C 2011 Phys. Rev. Lett. 107 240403
[5] Couder Y and Fort E 2006 Phys. Rev. Lett. 97 154101
[6] Eddit A, Sultan E, Mouktar J, Fort E, Rossi M and Couder Y 2011 J.Fluid Mech. 647 433
[7] Genovese M 2005 Physics Reports 413 319
[8] Nelson E 1985 Quantum Fluctuations (Princeton University Press, Princeton)
[9] Faris W G (ed.) 2006 Diffusion, Quantum Theory and Radically Elementary Mathematics (Princeton University Press, Princeton)
[10] Garbaczewski P and Vigier J P 1992 Phys. Rev. A 46 4634
[11] Garbaczewski P 1999 Phys. Rev. E 59 1498
[12] Garbaczewski P 2008 Phys. Rev. E 78 031101
[13] Czopnik R and Garbaczewski P 2001 Phys. Rev. E 63 021105
[14] Garbaczewski P 2006 J. Stat. Phys. 123 315
[15] Frisch L and Haugk M 2003 Ann. Physik. (Leipzig) 12 371
[16] Grässing G, Mesa Pascasio J and Schwabl H 2011 Found. Phys. 41 1437
[17] Mandelis A, 2001 Diffusion-Wave Fields (Springer-Verlag, Berlin)