Super 5-branes in $D = 10$ $N = 1$
Super Yang-Mills Theory

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Abstract

In $D = 10$ $N = 1$ super Yang-Mills theory we give the background breaking a half of supersymmetry. In the background there is a six-dimensional object so called $D = 10$ 5-brane.
1. Introduction

The possibility of partial supersymmetry breaking was discussed by Hughes and Polchinski [1]. They pointed out the way how to evade the existing no-go theorem of partial supersymmetry breaking based on the supersymmetry current algebra. They constructed a supersymmetric Nielsen-Olesen vortex [2] in a $D = 4$ $N = 2$ supersymmetric theory in which half of the supersymmetry is spontaneously broken. They also showed that the effective action for the vortex is the Green-Schwarz covariant action [3]. Moreover Hughes, Liu and Polchinski [4] gave a four-dimensional supermembrane solution of the six-dimensional Abelian $N = 1$ supersymmetric gauge theory. They showed that the effective action for the membrane is a generalization of the Green-Schwarz covariant action.

We give the background breaking a half of supersymmetry in $D = 10$ $N = 1$ super Yang-Mills theory. In the background there is a six-dimensional object so called $D = 10$ 5-brane. We also give the effective action for the 5-brane which is a generalization of the Green-Schwarz covariant action as same as the case of [4].

In sect.2 we first introduce $\Gamma$-matrices in $D = 10$ Minkowski space that will be important in the later sections. In sect.3 we consider dimensional reductions and review that $D = 4$ $N = 4$ super Yang-Mills Lagrangian can be derived from $D = 10$ $N = 1$ Lagrangian by trivial dimensional reduction [5]. In sect.4 we give a topological solution of $D = 10$ $N = 1$ super Yang-Mills theory which denotes a $D = 10$ 5-brane. This background effectively realizes $D = 4$ $N = 2$ super-Poincaré symmetry. Thus it has only half of the supersymmetry as compared with the case of trivial dimensional reduction. Finally we give the effective action for the 5-brane.

2. $\Gamma$-matrices in $D = 10$ Minkowski space

We introduce a particular representation of the $D = 10$ $\Gamma$-matrices:

$$\{\Gamma_M, \Gamma_N\} = 2\eta_{MN} \quad (1)$$

*We would now call it 3-brane.
†$P$-brane action was shown in [6][7]
‡We use the representation of the $D = 10$ $\Gamma$-matrices of [8].
where $\eta_{MN} = (1, -1, \ldots, -1)$. We choose $\Gamma_M (M = 0, 3, 5, \ldots, 10)$ as

$$
\Gamma_\mu = \gamma_\mu \otimes 1 \quad \text{for } \mu = 0, 3 \\
\Gamma_{4+m} = \gamma_5 \otimes \tilde{\Gamma}_m \quad \text{for } m = 1, 6
$$

(2)

where $\gamma_\mu, \gamma_5$ the $4 \times 4$ $\Gamma$ matrices in $D = 4$ Minkowski space with $\eta_{\mu\nu} = (1, -1, -1, -1)$ and $\tilde{\Gamma}_m$ the $8 \times 8$ $\Gamma$ matrices in $D = 6$ Euclidean space with $\eta_{mn} = \delta_{mn}$

$$
\gamma_\mu = \begin{pmatrix}
0 & (\sigma_\mu)_{\alpha\dot{\alpha}} \\
(\bar{\sigma}_\mu)^{\dot{a}a} & 0
\end{pmatrix} \quad \gamma_5 = \begin{pmatrix}
-i & 0 \\
0 & i
\end{pmatrix} \\
\tilde{\Gamma}_m = \begin{pmatrix}
0 & (\tilde{\sigma}_m)_{ij} \\
(\tilde{\sigma}_m^{-1})^{ij} & 0
\end{pmatrix}
$$

(3)

where $i, j = 1, \ldots, 4$, Pauli-matrices $\sigma = (\sigma_1, \sigma_2, \sigma_3)$, $\sigma_\mu = (1, \sigma)$, $\bar{\sigma}_\mu = (1, -\bar{\sigma})$ and

$$
\bar{\sigma}_1 = \begin{pmatrix}
0 & -\sigma_3 \\
\sigma_3 & 0
\end{pmatrix} \quad \bar{\sigma}_2 = \begin{pmatrix}
0 & i \\
-i & 0
\end{pmatrix} \quad \bar{\sigma}_3 = \begin{pmatrix}
0 & \sigma_1 \\
-\sigma_1 & 0
\end{pmatrix} \\
\bar{\sigma}_4 = \begin{pmatrix}
0 & -\sigma_2 \\
\sigma_2 & 0
\end{pmatrix} \quad \bar{\sigma}_5 = \begin{pmatrix}
-\sigma_2 & 0 \\
0 & \sigma_2
\end{pmatrix} \quad \bar{\sigma}_6 = \begin{pmatrix}
i\sigma_2 & 0 \\
0 & i\sigma_2
\end{pmatrix}.
$$

(4)

These $\tilde{\sigma}_m$ satisfy the following relations:

$$
(\tilde{\sigma}_m)_{ij} = -(\tilde{\sigma}_m)_{ji}, \quad (\tilde{\sigma}_m^{-1})_{ij} = (\tilde{\sigma}_m^{-1})_{ji} \\
(\tilde{\sigma}_m)^*_{ij} \equiv -(\tilde{\sigma}_m^{-1})_{ij} = \frac{1}{2} \epsilon_{ijkl} (\tilde{\sigma}_m)_{kl} \\
(\tilde{\sigma}_m)_{ij} (\tilde{\sigma}_m^{-1})_{ji} = 4 \delta_{mn}, \quad (\tilde{\sigma}_m)_{ij} (\tilde{\sigma}_m^{-1})_{kl} = -2 (\delta^k_j \delta^l_i - \delta^k_i \delta^l_j) \\
(\tilde{\sigma}_m)_{ij} (\tilde{\sigma}_m)_{kl} = 2 \epsilon_{ijkl}, \quad (\tilde{\sigma}_m^{-1})_{ij} (\tilde{\sigma}_m^{-1})_{kl} = 2 \epsilon_{ijkl}
$$

(5)

$\Sigma$-matrices are defined by

$$
\Sigma_{MN} \equiv \frac{i}{2} [\Gamma_M, \Gamma_N] \\
\Sigma_{\mu\nu} = \frac{i}{2} [\Gamma_\mu, \Gamma_\nu] = \frac{i}{2} [\gamma_\mu, \gamma_\nu] \otimes 1 \\
\Sigma_{mn} = \frac{i}{2} [\Gamma_m, \Gamma_n] = -\frac{i}{2} 1 \otimes [\tilde{\Gamma}_m, \tilde{\Gamma}_n].
$$

(6)

### 3. Trivial dimensional reduction

Consider the $N = 1$ super Yang-Mills Lagrangian in $D = 10$ Minkowski space for a gauge field $A_M (M = 0, 3, 5, \ldots, 10)$ and a Majorana Weyl spinor $\lambda$ in the adjoint
representation of the gauge group:

\[ \mathcal{L} = -\frac{1}{4} F_{MN}^a F^{aMN} + i \frac{\bar{\lambda}^a \Gamma^M D_M \lambda^a}{2} \]  

(7)

where the \( D = 10 \) Majorana Weyl spinor \( \lambda \) can be written by \( D = 4 \) Weyl spinors as

\[ \lambda = (\lambda_{\alpha i}, 0, 0, \bar{\lambda}^{\dot{\alpha} i})^T \quad \text{for} \quad \alpha, \dot{\alpha} = 1, 2 \quad \text{and} \quad i = 1, \ldots 4 \]  

(8)

This Lagrangian is invariant under supersymmetry transformations:

\[ \delta A_M^a = i \bar{\xi} \Gamma_M \lambda^a \]

\[ \delta \lambda^a = -i \frac{\bar{\lambda}^a}{2} \sum_{MN} \xi F_{MN}^a \]  

(9)

where \( \xi \) is a \( D = 10 \) Majorana Weyl spinor supersymmetry parameter and can be written by \( D = 4 \) Weyl spinors \((\xi_{\alpha i}, 0, 0, \bar{\xi}^{\dot{\alpha} i})^T\).

To obtain a \( D = 4 \) theory from this, we perform a trivial dimensional reduction:

\[ \partial_{4+m} = 0 \quad \text{for} \quad m = 1, \ldots 6 \]  

(10)

It gives us

\[ F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig [A_\mu, A_\nu] \]

\[ F_{\mu m} = \partial_\mu \phi_m - ig [A_\mu, \phi_m] \equiv D_\mu \phi_m \]

\[ F_{mn} = -i g [\phi_m, \phi_n] \]  

(11)

where \( \phi_m \equiv A_m \). The Lagrangian (7) now depends only on four coordinates \( x^\mu \) and reads

\[ -\frac{1}{4} F_{MN}^a F^{aMN} = -\frac{1}{4} F_{\mu \nu}^a F^{a\mu \nu} + \frac{1}{2} D_\mu \phi_m^a D^\mu \phi_m^a + \frac{1}{4} g^2 [\phi_m^a, \phi_n^a]^a [\phi_m, \phi_n]^a \]

\[ = -\frac{1}{4} F_{\mu \nu}^a F^{a\mu \nu} + \frac{1}{2} D_\mu \phi_{ij}^a D^\mu \phi_{aij}^a + \frac{1}{4} g^2 [\phi_{ij}, \phi_{kl}]^a [\phi_{ij}, \phi_{kl}]^a \]  

(12)

\[ \frac{i}{2} \bar{\lambda}^a \Gamma^M D_M \lambda^a = \frac{i}{2} \bar{\lambda}^a \Gamma^M D_\mu \lambda^a + \frac{1}{2} \bar{\lambda}^a \Gamma^{mn} [\phi_m, \lambda]^a \]

\[ = \frac{i}{2} (\lambda^a \sigma^a \bar{\lambda}^{ai} + \bar{\lambda}^a \bar{\sigma}^a D_\mu \lambda^a) + g f^{abc} (\phi^a_{bij} \lambda^c + \phi^b_{ij} \bar{\lambda}^{ai} \bar{\lambda}^{cj}) \]  

(13)

\(^\S\)Supersymmetric matter contents in higher dimensions are given in [9].
where $\phi_{ij}$ and $\phi^{ij}$ are real anti-symmetric tensors defined by $\phi_{ij} \equiv -\frac{1}{2}(\tilde{\sigma}^{-1} m)^{ij}\phi_m$, $\phi^{ij} \equiv \frac{1}{2}(\tilde{\sigma}^{-1} m)^{ij}\phi_m$ and $\phi^m = \frac{1}{2}(\tilde{\sigma}^{-1} m)^{ij}\phi_{ij} = -\frac{1}{2}(\tilde{\sigma} m)^{ij}\phi^{ij}$, and we have used the relations (5).

The supersymmetry transformations (9) become

\begin{align*}
\delta A_a^\mu &= i\xi \Gamma_\mu \lambda_a^\mu = i(\xi_\mu \sigma_\mu \lambda^{ai} + \tilde{\xi}_\mu \sigma_\mu \lambda_i^a) \quad (14) \\
\delta \phi^a_m &= i\xi_\mu \sigma_\mu \lambda^a_m \\
\delta \phi^a_{ij} &= \xi_i \lambda_j^a - \xi_j \lambda_i^a + \epsilon_{ijkl}\tilde{\xi}^k \tilde{\lambda}^d \\
\delta \lambda^a &= -\frac{i}{2}\Sigma^{MN} \xi F^a_{MN} \quad (15) \\
\delta \lambda_{ai} &= -\frac{i}{2}(\sigma^{\mu \nu} \xi_i)_\alpha^a F^a_{\mu \nu} - 2i D_\mu \phi^a_{ij}(\sigma_\mu \tilde{\xi}^j)_\alpha - 2i g[\phi_{ij}, \phi^{jk}]_\alpha \xi^k \\
\delta \tilde{\lambda}^{a\dot{a}i} &= -\frac{i}{2}(\sigma^{\mu \nu} \tilde{\xi}^i)\dot{\alpha} F^a_{\mu \nu} - 2i D_\mu \phi^{aij}(\sigma_\mu \xi_j)\dot{\alpha} - 2i g[\phi^{ij}, \phi^{jk}]_\alpha \tilde{\xi}^{\dot{a}k} \quad (16)
\end{align*}

The background (10) has the global $SO(1,3) \times SO(6) \sim SO(1,3) \times SU(4)_R$ symmetry. Actually these Lagrangian and transformations are known as the $SU(4)$ covariant $D = 4$ $N = 4$ super Yang-Mills theory.

4. Partial supersymmetry breaking and super 5-branes

In this section we show that the super Yang-Mills Lagrangian in $D = 10$ Minkowski space have topological solutions which break half of the supersymmetries.

We choose the following topological solutions:

\begin{align*}
A_a^\mu &= 0 \quad \text{for } \mu = 0, \ldots, 3 \quad (M = 0, \ldots, 3) \\
A_a^9, A_a^{10} &= 0 \quad (17) \\
F^a_{\dot{m}\dot{n}k} = \frac{1}{2}\epsilon_{\dot{m}\dot{n}k}\dot{F}^a_{\dot{m}\dot{n}k} &= \tilde{F}^a_{\dot{m}\dot{n}} \\
F^a_{\dot{m}\dot{n}\dot{k}\dot{l}} &= \frac{1}{2}\epsilon_{\dot{m}\dot{n}\dot{k}\dot{l}}F^a_{\dot{m}\dot{n}\dot{k}\dot{l}} \equiv \tilde{F}^a_{\dot{m}\dot{n}\dot{k}\dot{l}} \quad (M = 5, \ldots, 8) \quad (18) \\
\chi^a &= 0 \quad (19) \\
\delta \chi^a &= \chi^a \quad (20)
\end{align*}

where $F^a_{\dot{m}\dot{n}}$ depend only on four coordinates $x^{\dot{m}}$. When we choose $SU(2)$ gauge symmetry, the topological background has the global $SO(1,5) \times SO(3)_{\text{diag}}$ symmetry. Thus these solutions denote a super 5-brane.

To see partial supersymmetry breaking clearly we assume that nothing depends on 2 dimensions:

\begin{align*}
\partial_9 = \partial_{10} = 0, \quad (21)
\end{align*}
this background has the global $SO(1, 3) \times SO(3)_{diag} \times SO(2) \sim SO(1, 3) \times SU(2)_R \times U(1)_R$ symmetry thus $D = 4$ $N = 2$ supersymmetry. We will expect that only half of the supersymmetries are unbroken in this background as compared with the case of trivial reduction of six dimensions. Actually we will show half of the supersymmetries are unbroken in the topological background while the other half are broken.

Before proceeding this for preparation we will investigate Σ-matrices. We define $\tilde{\Sigma}_{\hat{m}\hat{n}}$ as

$$[\tilde{\Gamma}_{\hat{m}}, \tilde{\Gamma}_{\hat{n}}] \equiv 2i \begin{pmatrix} \tilde{\Sigma}_{\hat{m}\hat{n}}^1 \\ \tilde{\Sigma}_{\hat{m}\hat{n}}^2 \\ \tilde{\Sigma}_{\hat{m}\hat{n}}^3 \\ \tilde{\Sigma}_{\hat{m}\hat{n}}^4 \end{pmatrix}$$

and find the following relation:

$$\frac{1}{2} \epsilon_{\hat{k}\hat{l}\hat{m}\hat{n}} [\tilde{\Gamma}_{\hat{m}}, \tilde{\Gamma}_{\hat{n}}] = \frac{1}{2} \epsilon_{\hat{k}\hat{l}\hat{m}\hat{n}} 2i \begin{pmatrix} \tilde{\Sigma}_{\hat{m}\hat{n}}^1 \\ \tilde{\Sigma}_{\hat{m}\hat{n}}^2 \\ \tilde{\Sigma}_{\hat{m}\hat{n}}^3 \\ \tilde{\Sigma}_{\hat{m}\hat{n}}^4 \end{pmatrix} = 2i \begin{pmatrix} -\tilde{\Sigma}_{\hat{m}\hat{n}}^1 \\ \tilde{\Sigma}_{\hat{m}\hat{n}}^2 \\ -\tilde{\Sigma}_{\hat{m}\hat{n}}^3 \\ \tilde{\Sigma}_{\hat{m}\hat{n}}^4 \end{pmatrix}.$$  \hfill (23)

In the last step we have used the representations of (3) and (4).

Now we will see the supersymmetry transformations in the topological background. The supersymmetry variation of bosonic fields is zero. The supersymmetry variation of fermion field is

$$\delta \lambda^a = -\frac{i}{2} \Sigma_{\hat{m}\hat{n}} \xi F_{\hat{m}\hat{n}} = -\frac{i}{2} \{ -\frac{i}{2} \bigotimes [\tilde{\Gamma}_{\hat{m}}, \tilde{\Gamma}_{\hat{n}}] \} \xi F_{\hat{m}\hat{n}}$$

$$= -\frac{i}{2} F_{\hat{m}\hat{n}} \bigotimes \begin{pmatrix} \tilde{\Sigma}_{\hat{m}\hat{n}}^1 \\ \tilde{\Sigma}_{\hat{m}\hat{n}}^2 \\ \tilde{\Sigma}_{\hat{m}\hat{n}}^3 \\ \tilde{\Sigma}_{\hat{m}\hat{n}}^4 \end{pmatrix} \begin{pmatrix} \xi_{\alpha_1} \\ 0 \\ 0 \\ \xi_{\bar{\alpha}_4} \end{pmatrix}$$

$$= -\frac{i}{2} F_{\hat{m}\hat{n}} \begin{pmatrix} (\tilde{\Sigma}_{\hat{m}\hat{n}}^1 \xi_{\alpha_1}, \tilde{\Sigma}_{\hat{m}\hat{n}}^2 \xi_{\alpha_2}, \tilde{\Sigma}_{\hat{m}\hat{n}}^3 \xi_{\alpha_3}, \tilde{\Sigma}_{\hat{m}\hat{n}}^4 \xi_{\alpha_4})^T \\ 0 \\ 0 \\ (\tilde{\Sigma}_{\hat{m}\hat{n}}^1 \bar{\xi}_{\bar{\alpha}_1}, \tilde{\Sigma}_{\hat{m}\hat{n}}^2 \bar{\xi}_{\bar{\alpha}_2}, \tilde{\Sigma}_{\hat{m}\hat{n}}^3 \bar{\xi}_{\bar{\alpha}_3}, \tilde{\Sigma}_{\hat{m}\hat{n}}^4 \bar{\xi}_{\bar{\alpha}_4})^T \end{pmatrix}.$$  \hfill (24)
On the other hand using the self-dual equation (19)

\[ \delta \lambda^a = -\frac{i}{2} \sum_{\hat{m}\hat{n}} \xi_{\hat{m}\hat{n}} \tilde{F}_{\hat{m}\hat{n}} \]

\[ = -\frac{i}{2} F^a_{\hat{m}\hat{n}} 1 \otimes \left( \begin{array}{cc}
-\tilde{\Sigma}^2_{\hat{m}\hat{n}} & -\tilde{\Sigma}^3_{\hat{m}\hat{n}} \\
& \ldots \\
-\tilde{\Sigma}^4_{\hat{m}\hat{n}} & \ldots 
\end{array} \right) \left( \begin{array}{c}
\xi_{\alpha 1} \\
0 \\
0 \\
\tilde{\xi}^{\dot{\alpha} 1} \\
\ldots \\
\tilde{\xi}^{\dot{\alpha} 4} 
\end{array} \right) \]

Thus the \((\xi_{\alpha 1}, \tilde{\xi}^{\dot{\alpha} 1})\) and \((\xi_{\alpha 3}, \tilde{\xi}^{\dot{\alpha} 3})\) supersymmetry transformations leave the topological background invariant while the \((\xi_{\alpha 2}, \tilde{\xi}^{\dot{\alpha} 2})\) and \((\xi_{\alpha 4}, \tilde{\xi}^{\dot{\alpha} 4})\) do not. We easily notice when anti-self dual equation is satisfied instead of (19) the unbroken supersymmetries and the broken ones exchange.

We have shown that there is a six-dimensional object so called super 5-brane which breaks half of the supersymmetries in the background (17)~(20). Finally we would like to obtain the effective action for the 5-brane using the method of nonliner realization. In the low energy theory, there are four Nambu-Goldstone bosons of the broken translatons and a \(D = 6\) Nambu-Goldstone Weyl spinor of the breaking supersymmetry. The Nambu-Goldstone bosons of \(SU(2)\) gauge symmetry are absorbed by the \(D = 6\) gauge fields and the \(D = 6\) gauge fields are massive. Fortunately we know that the action for the Nambu-Goldstone fields is just the generalized Green-Schwarz covariant superstring one [4][6][7][10]:

\[ S = S_1 + S_2 \]

where \(S_1\) is a supersymmetric Nambu-Goto action, and \(S_2\) is a Wess-Zumino action.

\[ S_1 = \int d^6 \sigma ( - \sqrt{-h} ) \]

where \(h = \det h_{nm}, \ h_{mn} = \Pi^M_m \Pi^M_n, \ \Pi^M_m = \partial_m x^M - i \theta \Gamma^M \partial_m \theta \). Here \(m, n = 1, \ldots 6\) are world sheet indices and \(M = 1, \ldots 10\) is a spacetime index. The Fermi field \(\theta\) is a scalar on the world sheet and a Majorana-Weyl spinor in \(D = 10\).

\[ S_2 = \int_{W_6} \Omega_6 \]

(28)
where $\Omega_6$ is given by $\Omega_7$ as $d\Omega_6 = \Omega_7$. The closed supersymmetric seven-form $\Omega_7$ is unique up to normalization and given by

$$
\Omega_7 = \Pi^I \Pi^J \Pi^K \Pi^L \Pi^M d\bar{\theta} \Gamma_{IJJKLM} d\theta.
$$  \hspace{1cm} (29)

The action (26) is invariant under the nonlinear supersymmetry

$$
\delta \theta = \xi, \quad \delta x^M = i\bar{\xi} \Gamma_M \theta.
$$  \hspace{1cm} (30)

However there is a local fermionic symmetry, this symmetry involves half of the nonlinear action of $\xi$, so only half of the supersymmetries (30) act $\theta$ nonlinearly. Similarly all the translations act nonlinearly for Bose fields $x^M$. However six of the translations are equivariant to coordinate translations on the membrane ground state. As the same way, we shall be able to get $D = 9$ 4-brane, $D = 8$ 3-brane, ... and $D = 6$ 1-brane.

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