THE GENERALIZED KUREPA HYPOTHESIS AT SINGULAR CARDINALS

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Abstract. We discuss the generalized Kurepa hypothesis $KH_\lambda$ at singular cardinals $\lambda$. In particular, we answer questions of Erdős-Hajnal [1] and Todorcevic [6], [7] by showing that GCH does not imply $KH_{\aleph_\omega}$ nor the existence of a family $F \subseteq [\aleph_\omega]^{\aleph_0}$ of size $\aleph_{\omega+1}$ such that $F \upharpoonright X$ has size $\aleph_0$ for every $X \subseteq S, |X| = \aleph_0$.

1. INTRODUCTION

For an infinite cardinal $\lambda$ let the generalized Kurepa hypothesis at $\lambda$, denoted $KH_\lambda$, be the assertion: there exists a family $F \subseteq P(\lambda)$ such that $|F| > \lambda$ but $|F \upharpoonright X| \leq |X|$ for every infinite $X \subseteq \lambda, |X| < \lambda$, where $F \upharpoonright X = \{t \cap X : t \in F\}$.

By a theorem of Erdős-Hajnal-Milner [2], if $\lambda$ is a singular cardinal of uncountable cofinality, $\theta^{cf}(\lambda) < \lambda$ for all $\theta < \lambda$ and if $F \subseteq P(\lambda)$ is such that the set $\{\alpha < \lambda : |F \upharpoonright \alpha| \leq |\alpha|\}$ is stationary in $\lambda$, then $|F| \leq \lambda$. In particular, GCH implies $KH_\lambda$ fails for all singular cardinals $\lambda$ of uncountable cofinality. On the other hand, by an unpublished result of Prikry [5], $KH_\lambda$ holds in $L$, the Gödel’s constructible universe, for singular cardinals of countable cofinality (see [7]). Later, Todorcevic [6], [7] improved Prikry’s theorem by showing that if $\lambda$ is a singular cardinal of countable cofinality, then $\square_\lambda$ implies $KH_\lambda$. The following question is asked in [6] and [7].

Question 1.1. Does GCH imply $KH_{\aleph_\omega}$?

The question is also related to the following question of Erdős-Hajnal [1] (question 19/E)

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In personal communication, Stevo Todorcevic informed the author that Theorem 3.1 has been obtained by him before; for example can be found on page 231 of his book “Walks on ordinals and their characteristics”. However our proof is different from him.
Question 1.2. Assume GCH. Let $|S| = \aleph_\omega$. Does there exist a family $\mathcal{F}, |\mathcal{F}| = \aleph_{\omega+1}, \mathcal{F} \subseteq [S]^{\aleph_0}$ such that $\mathcal{F} \restriction X$ has size $\aleph_0$ for every $X \subseteq S, |X| = \aleph_0$?

We show that, relative to the existence of large cardinals, both of the above questions can consistently be false, and so they are independent of ZFC.

2. KH fails above a supercompact cardinal

In this section we prove the following.

Theorem 2.1. Suppose $\kappa$ is a supercompact cardinal and $\lambda \geq \kappa$. Then KH fails.

Proof. Let $\mathcal{F} \subseteq P(\lambda)$ be of size $\geq \lambda^+$. Let $j : V \to M$ be a $\lambda^+$-supercompactness embedding with $\text{crit}(j) = \kappa$. Also let $U$ be the normal measure on $P_\kappa(\lambda)$ derived from $j$, i.e.,

$$U = \{X \subseteq P_\kappa(\lambda) : j[\lambda] \in j(X)\}.$$ 

We have

- $M \models \text{"}j(\mathcal{F}) \subseteq P(j(\lambda))\text{" is of size $\geq j(\lambda)^+$.}$$
- $j''[\lambda] \in M$ and $M \models \text{"}j''[\lambda] = \lambda < j(\lambda)\text{".}$$
- $\mathcal{F} \in M$
- $a \neq b \in \mathcal{F} \implies j(a) \cap j''[\lambda] \neq j(b) \cap j''[\lambda]$.

In particular,

$$M \models \text{"}j(\mathcal{F}) \upharpoonright j''[\lambda] \geq |\mathcal{F}| \geq \lambda^+\text{".}$$

This implies that

$$\{x \in P_\kappa(\lambda) : |\mathcal{F} \upharpoonright x| \geq |x|^+\} \in U.$$ 

In particular, $\mathcal{F}$ is not a KH-family. \qed

Remark 2.2. The above result is optimal in the sense that we can not in general find a set $x \subseteq \lambda$ of size in the interval $[\kappa, \lambda)$ such that $|\mathcal{F} \upharpoonright x| \geq |x|^+$. To see this assume $\kappa$ is supercompact and Laver indestructible. Then one can easily define a $\kappa$-directed closed forcing notion which adds a family $\mathcal{F} \subseteq P(\lambda)$ such that $|\mathcal{F}| \geq \lambda^+$, but $|\mathcal{F} \upharpoonright x| \leq |x|$ for any set $x$ with $\kappa \leq |x| < \lambda$. 

3. The Chang’s conjecture and \( KH_{\aleph_\omega} \)

In this section we prove our main theorem by showing a consistent negative answer to the questions of Erdős-Hajnal and Todorcevic. Recall from [4] that “GCH + the Chang’s conjecture \((\aleph_{\omega+1}, \aleph_\omega) \rightarrow (\aleph_1, \aleph_0)\)” is consistent, relative to the existence of a 2-huge cardinal. See also [3], where the large cardinal assumption is reduced to the existence of a \((+\omega + 1)\)-subcompact cardinal \( \kappa \).

**Theorem 3.1.** Assume \( GCH + \) Chang’s conjecture \((\aleph_{\omega+1}, \aleph_\omega) \rightarrow (\aleph_1, \aleph_0)\). Then \( KH_{\aleph_\omega} \) fails. Also, there does not exist a family \( F \subseteq [\aleph_\omega]^{\aleph_0}, |F| \geq \aleph_{\omega+1} \) such that \( F \restriction X \) has size \( \aleph_0 \) for every \( X \subseteq S, |X| = \aleph_0 \).

**Proof.** Suppose towards contradiction that there exists a family \( F \) which witnesses \( KH_{\aleph_\omega} \). Fix a bijection \( f : H_{\aleph_{\omega+1}} \leftrightarrow F \). Consider the structure

\[ A = (H_{\aleph_{\omega+1}}, \in, F, \aleph_\omega, f). \]

Let \( B = (B, \in, G, A, g) \prec A \) be such that \( |B| = \aleph_1 \) and \( |A| = \aleph_0 \).

Note that \( A \models \forall t \in F, t \subseteq \aleph_\omega \), and hence \( B \models \forall t \in G, t \subseteq A \), in particular, \( G \subseteq F \restriction A \).

On the other hand \( g : B \leftrightarrow G \) is a bijection, hence we have

\[ |F \restriction A| \geq |G| = |B| = \aleph_1 > \aleph_0. \]

We get a contradiction and the result follows.

Similar argument shows that there can not be a family \( F \subseteq [\aleph_\omega]^{\aleph_0} \) as stated above. \( \square \)

**References**

[1] Erdős, P.; Hajnal, A., Unsolved problems in set theory. 1971 Axiomatic Set Theory (Proc. Sympos. Pure Math., Vol. XIII, Part I, Univ. California, Los Angeles, Calif., 1967) pp. 17-48 Amer. Math. Soc., Providence, R.I.

[2] Erdős, P.; Hajnal, A.; Milner, E. C., On sets of almost disjoint subsets of a set. Acta Math. Acad. Sci. Hungar 19 1968 209-218.

[3] Hayut, Yair, Magidor-Malitz reflection. Arch. Math. Logic 56 (2017), no. 3–4, 253-272.

[4] Levinski, Jean-Pierre; Magidor, Menachem; Shelah, Saharon, Chang’s conjecture for \( \aleph_\omega \). Israel J. Math. 69 (1990), no. 2, 161-172.

[5] Prikry, Karel, Kurepa’s hypothesis for singular cardinals, unpublished note.
[6] Todorcevic, Stevo, Trees and linearly ordered sets. Handbook of set-theoretic topology, 235-293, North-Holland, Amsterdam, 1984.

[7] Todorcevic, Stevo, Aronszajn trees and partitions. Israel J. Math. 52 (1985), no. 1–2, 53-58.

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