ON THREE DIMENSIONAL AFFINE SZABÓ MANIFOLDS

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ABSTRACT. In this paper, we consider the cyclic parallel Ricci tensor condition, which is a necessary condition for an affine manifold to be Szabó. We show that, in dimension 3, there are affine manifolds which satisfy the cyclic parallel Ricci tensor but are not Szabó. Conversely, it is known that in dimension 2, the cyclic parallel Ricci tensor forces the affine manifold to be Szabó. Examples of 3-dimensional affine Szabo manifolds are also given. Finally, we give some properties of Riemannian extensions defined on the cotangent bundle over an affine Szabó manifold.

1. INTRODUCTION

The theory of connection is a classical topic in differential geometry. It was initially developed to solve pure geometrical problems. It provides an extremely important tool to study geometrical structures on manifolds and, as such, has been applied with great sources in many different settings. B. Opozda in ([14]) classified locally homogeneous connection on 2-dimensional manifolds equipped with torsion free affine connection. T. Arias-Marco and O. Kowalski [1] classify locally homogeneous connections with arbitrary torsion on 2-dimensional manifolds. E. García-Río et al. [7] introduced the notion of the affine Osserman connections. The affine Osserman connections are well understood in dimension two (see [4, 7] for more details).

A (Pseudo) Riemannian manifold \((M, g)\) is said to be Szabó if the eigenvalues of the Szabó operator given by

\[ S(X) : Y \rightarrow (\nabla_X R)(Y, X)X \]

are constants on the unit (Pseudo) sphere bundle, where \(R\) denoting the curvature tensor (see [2] and [3] for details). The Szabó operator is a self adjoint operator with \(S(X)X = 0\). It plays an important role in the study of totally isotropic manifolds [9]. Szabó in [16] used techniques from algebraic topology to show, in the Riemannian setting, that any such a metric is locally symmetric. He used this observation to prove that any two point homogeneous space is either flat or is a rank one symmetric space. Subsequently Gilkey and Stravrov [10] extended this result to show that any Szabó Lorentzian manifold has constant sectional curvature. However, for metrics of higher signature the situation is different. Indeed it was

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showed in [9] the existence of Szabó Pseudo-Riemannian manifolds endowed with metrics of signature \((p, q)\) with \(p \geq 2\) and \(q \geq 2\) which are not locally symmetric.

In [5], the authors introduced the so-called affine Szabó connections. They proved, in dimension 2, that an affine connection \(\nabla\) is affine Szabó if and only if the Ricci tensor of \(\nabla\) is cyclic parallel while in dimension 3 the concept seems to be very challenging by giving only partial results.

The aim of the present paper is to give an explicit form of two families of affine connections which are affine Szabó on 3-dimensional manifolds. Moreover, although both results provide examples of affine Szabó connections, they are essentially different in nature since, in the first family, the affine Szabó condition coincides with the cyclic parallelism of the Ricci tensor, whereas the second one is not. For any affine connection \(\nabla\) on \(M\), there exist a technique called Riemannian extension, which relates affine and pseudo-Riemannian geometries. This technique is very powerful in constructing new examples of pseudo-Riemannian metrics. The relation between affine Szabó manifolds and pseudo-Riemannian Szabó manifolds are investigated by using Riemannian extensions.

The paper is organized as follows. In section 2 we recall some basic definitions and geometric objects, namely, torsion, curvature tensor, Ricci tensor and affine Szabó operator on an affine manifold. In section 3 we study the cyclic parallelism of the Ricci tensor for two particular cases of affine connections in 3-dimensional affine manifolds. We establish geometrical configurations of affine manifolds admitting a cyclic parallel Ricci tensor (Propositions 3.2 and 3.4). In section 4 we study the Szabó condition on two particular affine connections (Theorems 4.5 and 4.7). Finally, we end the paper in section 5 by investigating the Riemannian extensions defined on the cotangent bundle over an affine Szabó manifold.

2. Preliminaries

Let \(M\) be an \(n\)-dimensional smooth manifold and \(\nabla\) be an affine connection on \(M\). We consider a system of coordinates \((x_1, x_2, \ldots, x_n)\) in a neighborhood \(U\) of a point \(p\) in \(M\). In \(U\) the affine connection is given by

\[
\nabla_\partial_i \partial_j = f^k_{ij} \partial_k
\]

(2.1)

where \(\{\partial_i = \frac{\partial}{\partial x_i}\}_{1 \leq i \leq n}\) is a basis of the tangent space \(T_p M\) and the functions \(f^k_{ij} (i, j, k = 1, 2, 3, \ldots, n)\) are called the Christoffel symbols of the affine connection. We shall call the pair \((M, \nabla)\) affine manifold. Some tensor fields associated with the given affine connection \(\nabla\) are defined below.

The torsion tensor field \(T^\nabla\) is defined by

\[
T^\nabla(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]
\]

(2.2)

for any vector fields \(X\) and \(Y\) on \(M\). The components of the torsion tensor \(T^\nabla\) in local coordinates are

\[
T^k_{ij} = f^k_{ij} - f^k_{ji}.
\]

(2.3)

If the torsion tensor of a given affine connection \(\nabla\) vanishes, we say that \(\nabla\) is torsion-free.
The curvature tensor field $R^\nabla$ is defined by
\[ R^\nabla(X, Y) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \] (2.4)
for any vector field $X, Y$ and $Z$ on $M$. The components in local coordinates are
\[ R^\nabla(\partial_k, \partial_l) = \sum_i R^i_{jkl} \partial_i. \] (2.5)
We shall assume that $\nabla$ is torsion-free. If $R^\nabla = 0$ on $M$, we say that $\nabla$ is flat affine connection. It is known that $\nabla$ is flat if and only if around a point $p$ there exist a local coordinate system such that $f^i_{jk} = 0$ for all $i, j, k$.

We define Ricci tensor $Ric^\nabla$ by
\[ Ric^\nabla(X, Y) = \text{trace}\{Z \mapsto R^\nabla(Z, X)Y\}. \] (2.6)
The components in local coordinates are given by
\[ Ric^\nabla(\partial_j, \partial_k) = \sum_i R^i_{kij}. \] (2.7)
It is known that in Riemannian geometry the Levi-Civita connection of a Riemannian metric has symmetric Ricci tensor, that is $Ric^\nabla(X, Y) = Ric^\nabla(Y, X)$. But this property is not true for an arbitrary torsion-free affine connection. In fact, the property is closely related to the concept of parallel volume element. (See [13] for more details).

The covariant derivative of the curvature tensor $R^\nabla$ is given by
\[ (\nabla_X R^\nabla)(Y, Z)W = \nabla_X \nabla^\nabla(Y, Z)W - \nabla^\nabla(\nabla_X Y, Z)W - \nabla^\nabla(Y, \nabla_X Z)W - \nabla^\nabla(Y, Z)\nabla_X W. \] (2.8)
The covariant derivative of the Ricci tensor $Ric^\nabla$ is defined by
\[ (\nabla_X Ric^\nabla)(Z, W) = X(\nabla^\nabla(Z, W)) - Ric^\nabla(\nabla_X Z, W) - Ric^\nabla(Z, \nabla_X W). \] (2.8)
For $X \in \Gamma(T_pM)$, we define the affine Szabó operator $S^\nabla(X) : T_pM \to T_pM$ with respect to $X$ by
\[ S^\nabla(X)Y := (\nabla_X R^\nabla)(Y, X) \] (2.9)
for any vector field $Y$. The affine Szabó operator satisfies $S^\nabla(X)X = 0$ and $S^\nabla(\beta X) = \beta S^\nabla(X)$ for $\beta \in \mathbb{R} - \{0\}$ and $X \in T_pM$. If $Y = \partial_m$, for $m = 1, 2, \cdots, n$ and $X = \sum \alpha_i \partial_i$ one get
\[ S^\nabla(X)\partial_m = \sum_{i,j,k=1}^n \alpha_i \alpha_j \alpha_k (\nabla_{\partial_k} R^\nabla)(\partial_m, \partial_j) \partial_k. \] (2.10)
Note that, by definition of the Ricci tensor, one has
\[ \text{trace}(Y \mapsto (\nabla_X R^\nabla)(Y, X)X) = (\nabla_X Ric^\nabla)(X, X). \] (2.11)
3. AFFINE CONNECTIONS WITH CYCLIC PARALLEL RICCI TENSOR

In this section, we investigate affine connections whose Ricci tensors are cyclic parallel. We shall consider two cases of 3-dimensional smooth manifolds with specific affine connections. We start with a formal definition.

**Definition 3.1.** The Ricci tensor $\text{Ric}^\nabla$ of an affine manifold $(M, \nabla)$ is cyclic parallel if

\[
(\nabla_X \text{Ric}^\nabla)(X, X) = 0,
\]

for any vector field $X$ tangent to $M$ or, equivalently, if

\[
\Phi_{X,Y,Z}(\nabla_X \text{Ric}^\nabla)(Y, Z) = 0,
\]

for any vector fields $X, Y,$ and $Z$ tangent to $M$ where $\Phi_{X,Y,Z}$ denotes the cyclic sum with respect to $X, Y,$ and $Z$.

Locally, the equation (3.1) takes the form

\[
(\nabla_{\partial_i} \text{Ric}^\nabla)_{jk} = 0
\]

or can be written out without the symmetrizing brackets

\[
(\nabla_{\partial_i} \text{Ric}^\nabla)_{jk} + (\nabla_{\partial_j} \text{Ric}^\nabla)_{ki} + (\nabla_{\partial_k} \text{Ric}^\nabla)_{ij} = 0.
\]

For $X = \sum \alpha_i \partial_i$, it is easy to show that

\[
(\nabla_X \text{Ric}^\nabla)(X, X) = \sum_{i,j,k} \alpha_i \alpha_j \alpha_k (\nabla_{\partial_i} \text{Ric}^\nabla)_{jk}.
\]

Now, we are going to present two cases of affine connections in which we investigate the cyclic parallelism of the Ricci tensor.

**Case 1:** Let $M$ be a 3-dimensional smooth manifold and $\nabla$ be an affine torsion-free connection. Suppose that the action of the affine connection $\nabla$ on the basis of the tangent space $\{\partial_i\}_{1 \leq i \leq 3}$ is given by

\[
\nabla_{\partial_1} \partial_1 = f_1 \partial_1, \quad \nabla_{\partial_2} \partial_1 = f_2 \partial_1 \quad \text{and} \quad \nabla_{\partial_1} \partial_3 = f_3 \partial_1,
\]

where the smooth functions $f_i = f_i(x_1, x_2, x_3)$ are Christoffel symbols. The non-zero components of the curvature tensor $\mathcal{R}^\nabla$ of the affine connection (3.5) are given by

\[
\mathcal{R}^\nabla(\partial_1, \partial_2)\partial_1 = (\partial_1 f_2 - \partial_2 f_1)\partial_1, \quad \mathcal{R}^\nabla(\partial_1, \partial_2)\partial_2 = -(\partial_2 f_1 + f_2^2)\partial_1, \\
\mathcal{R}^\nabla(\partial_1, \partial_2)\partial_3 = -(\partial_2 f_2 + f_2 f_3)\partial_1, \quad \mathcal{R}^\nabla(\partial_1, \partial_3)\partial_1 = (\partial_1 f_3 - \partial_3 f_1)\partial_1, \\
\mathcal{R}^\nabla(\partial_2, \partial_3)\partial_2 = -(\partial_3 f_2 + f_2 f_3)\partial_1, \quad \mathcal{R}^\nabla(\partial_3, \partial_1)\partial_3 = -(\partial_3 f_3 + f_3^2)\partial_1.
\]

From (2.7), the non-zero components of the Ricci tensor $\text{Ric}^\nabla$ of the affine connection (3.5) are given by:

\[
\text{Ric}^\nabla(\partial_2, \partial_1) = \partial_1 f_2 - \partial_2 f_1, \quad \text{Ric}^\nabla(\partial_2, \partial_2) = -(\partial_2 f_2 + f_2^2), \\
\text{Ric}^\nabla(\partial_2, \partial_3) = -(\partial_2 f_2 + f_2 f_3), \quad \text{Ric}^\nabla(\partial_3, \partial_1) = \partial_1 f_3 - \partial_3 f_1, \\
\text{Ric}^\nabla(\partial_3, \partial_2) = -(\partial_3 f_2 + f_2 f_3), \quad \text{Ric}^\nabla(\partial_3, \partial_3) = -(\partial_3 f_3 + f_3^2).
\]
Proposition 3.2. On \( \mathbb{R}^3 \), the affine connection \( \nabla \) defined in (3.5) satisfies the relation (3.7) if the functions \( f_i = f_i(x_1, x_2, x_3) \), for \( i = 1, 2, 3 \), satisfy the following partial differential equations:

\[
\begin{align*}
\partial_2^2 f_3 &+ 2f_3 \partial_3 f_3 = 0 \\
\partial_2^2 f_2 &+ 2f_2 \partial_2 f_2 = 0 \\
\partial_2^2 f_1 &+ 4f_3 \partial_1 f_3 - 2f_3 \partial_3 f_1 = 0 \\
\partial_2^2 f_1 &+ 4f_2 \partial_1 f_2 - 2f_2 \partial_2 f_1 = 0 \\
\partial_2^2 f_3 - \partial_1 \partial_3 f_1 - f_1 \partial_1 f_3 + f_1 \partial_3 f_1 = 0 \\
\partial_2^2 f_2 - \partial_1 \partial_2 f_1 - f_1 \partial_1 f_2 + f_1 \partial_2 f_1 = 0 \\
\partial_2^2 f_3 + 2\partial_3 \partial_2 f_2 + 2f_2 \partial_3 f_2 + 2f_3 \partial_2 f_2 + 2f_2 \partial_2 f_3 = 0 \\
\partial_2^2 f_2 + 2\partial_3 \partial_2 f_3 + 2f_3 \partial_3 f_2 + 2f_2 \partial_3 f_3 = 0 \\
4f_3 \partial_1 f_2 + 4f_2 \partial_1 f_3 - 2f_3 \partial_2 f_1 - 2f_2 \partial_3 f_1 + 2\partial_3 \partial_2 f_1 = 0
\end{align*}
\] (3.6)

Proof: From a straightforward calculation, using (2.8) and (3.4), one obtains the result. \( \square \)

As an example to the Proposition 3.2, we have the following.

Example 3.3. The Ricci tensors of the affine connections defined in (3.5) on \( \mathbb{R}^3 \) with

\begin{itemize}
\item[(1)] \( f_1 = 0, f_2 = -x_3 \) and \( f_3 = x_2 \);
\item[(2)] \( f_1 = x_1, f_2 = 2x_3 \) and \( f_3 = -2x_2 \)
\end{itemize}

are cyclic parallel.

Case 2: Let \( M \) be a 3-dimensional smooth manifold and \( \nabla \) be an affine torsion-free connection. Suppose that the action of the affine connection \( \nabla \) on the basis of the tangent space \( \{ \partial_i \}_{1 \leq i \leq 3} \) is given by

\[
\begin{align*}
\nabla_{\partial_1} \partial_1 &= f_1 \partial_2, \quad \nabla_{\partial_2} \partial_2 = f_2 \partial_3 \quad \text{and} \quad \nabla_{\partial_3} \partial_3 = f_3 \partial_1, \\
\text{where smooth functions } f_i &= f_i(x_1, x_2, x_3) \text{ are Christoffel symbols}. \text{ The non-zero components of the curvature tensor } R^\nabla \text{ of the affine connection (3.7) are given by}
\end{align*}
\] (3.7)

\[
\begin{align*}
R^\nabla(\partial_1, \partial_2) \partial_1 &= -(\partial_2 f_1 \partial_2 + f_1 \partial_2 f_3), \quad R^\nabla(\partial_1, \partial_2) \partial_2 = \partial_1 f_2 \partial_3, \\
R^\nabla(\partial_1, \partial_3) \partial_1 &= -\partial_3 f_1 \partial_2, \quad R^\nabla(\partial_1, \partial_3) \partial_3 = \partial_1 f_3 \partial_1 + f_1 \partial_3 f_2, \\
R^\nabla(\partial_2, \partial_3) \partial_2 &= -(\partial_3 f_2 \partial_3 + f_3 f_2 \partial_1), \quad R^\nabla(\partial_2, \partial_3) \partial_3 = \partial_2 f_3 \partial_1.
\end{align*}
\]

From (3.7), the non-zero components of the Ricci tensor \( Ric^\nabla \) of the affine connection (3.7) are given by

\[
\text{If } \nabla \text{ defined on } \mathbb{R}^3 \text{ by (3.7) satisfies (3.1) if the functions } f_i = f_i(x_1, x_2, x_3), \text{ for } i = 1, 2, 3, \text{ has the following form:}
\]

\[
f_1 = f(x_1) + g(x_3), \quad f_2 = h(x_1) + u(x_2), \quad f_3 = v(x_2) + t(x_3),
\]

where \( f, g, h, u, v \) and \( t \) are smooth functions on \( \mathbb{R}^3 \).
Proof. From a straightforward calculation, using (2.5) and (3.4), one obtain the following partial differential equations:

\[
\frac{\partial_1}{\partial_2} f_1 = 0, \quad \frac{\partial_3}{\partial_2} f_1 = 0, \quad \frac{\partial_1}{\partial_3} f_2 = 0, \quad \frac{\partial_2}{\partial_3} f_2 = 0, \quad \frac{\partial_2}{\partial_1} f_3 = 0, \\
\frac{\partial_3}{\partial_1} f_3 = 0, \quad \frac{\partial_2}{\partial_3} f_1 - 2f_1\frac{\partial_3}{\partial_2} f_2 = 0, \quad \frac{\partial_1}{\partial_3} f_2 - 2f_2\frac{\partial_3}{\partial_1} f_3 = 0, \\
\frac{\partial_2}{\partial_1} f_3 - 2f_3\frac{\partial_1}{\partial_2} f_2 = 0, \quad \frac{\partial_1}{\partial_2} f_2 - 2f_2\frac{\partial_2}{\partial_1} f_3 = 0,
\]
and the result follows.

As an application to this proposition, we have:

Example 3.5. The Ricci tensor of the following affine connection defined on \( \mathbb{R}^3 \) by

\[
\nabla_{\partial_1} \partial_1 = x_1^2 \partial_2, \quad \nabla_{\partial_2} \partial_2 = (x_1 + x_2) \partial_3, \quad \text{and} \quad \nabla_{\partial_3} \partial_3 = (x_2 + x_3^2) \partial_1,
\]
is cyclic parallel.

The manifolds with cyclic parallel Ricci tensor, known as \( L_3 \)-spaces, are well-developed in Riemannian geometry. The cyclic parallelism of the Ricci tensor is sometime called the “First Ledger condition” [15]. In [17], for instance, the author proved that a smooth Riemannian manifold satisfying the first Ledger condition is real analytic. These Riemannian manifolds were introduced by A. Gray in [11] as a special subclass of (connected) Riemannian manifolds \((M, g)\), called Einstein-like spaces, all of which have constant scalar curvature. Also, Riemannian manifolds of dimension 3 with cyclic parallel Ricci tensor are locally homogeneous naturally reductive ( [15]). Tod in [18] used the same condition to characterize the 4-dimensional Kähler manifolds which are not Einstein. It has also enriched the D’Atri spaces (see [12, 15] for more details).

4. The affine Szabó manifolds

Let \((M, \nabla)\) be an \( n \)-dimensional affine manifold, i.e., \( \nabla \) is a torsion free connection on the tangent bundle of a smooth manifold \( M \) of dimension \( n \). Let \( \mathcal{R}^\nabla \) be the associated curvature operator. We define the affine Szabó operator \( S^\nabla(X) : T_pM \to T_pM \) with respect to a vector \( X \in T_pM \) by

\[
S^\nabla(X)Y := (\nabla_X \mathcal{R}^\nabla)(Y, X)X.
\]

Definition 4.1. Let \((M, \nabla)\) be a smooth affine manifold and \( p \in M \).

1. \((M, \nabla)\) is called affine Szabó at \( p \in M \) if the affine Szabó operator \( S^\nabla(X) \) has the same characteristic polynomial for every vector field \( X \) on \( M \).

2. Also, \((M, \nabla)\) is called affine Szabó if \((M, \nabla)\) is affine Szabó at each point \( p \in M \).

Theorem 4.2. Let \((M, \nabla)\) be an \( n \)-dimensional affine manifold and \( p \in M \). Then \((M, \nabla)\) is affine Szabó at \( p \in M \) if and only if the characteristic polynomial of the affine Szabó operator \( S^\nabla(X) \) is

\[
P_\lambda(S^\nabla(X)) = \lambda^n,
\]
for every \( X \in T_pM \).
Corollary 4.3. We say that \((M, \nabla)\) is affine Szabó if the affine Szabó operators are nilpotent, i.e., 0 is the eigenvalue of \(S^\nabla(X)\) on the tangent bundle.

Corollary 4.4. If \((M, \nabla)\) is affine Szabó at \(p \in M\), then the Ricci tensor is cyclic parallel.

Affine Szabó connections are well-understood in 2-dimension, due to the fact that an affine connection is Szabó if and only if its Ricci tensor is cyclic parallel \cite{5}. The situation is however more involved in higher dimensions where the cyclic parallelism is a necessary but not sufficient condition for an affine connection to be Szabó.

Let \(X = \sum_{i=1}^{3} \alpha_i \partial_i\) be a vector field on a 3-dimensional affine manifold \(M\). Then the affine Szabó operator is given by

\[
S^\nabla(X)(\partial_m) = \sum_{i,j,k=1}^{3} \alpha_i \alpha_j \alpha_k (\nabla^\nabla \nabla^\nabla) (\partial_m, \partial_j) \partial_k, \quad m = 1, 2, 3.
\]

### 4.1. First Family of affine Szabó connection

Next, we give an example of a family of affine Szabó connection on a 3-dimensional manifold. Let us consider the affine connection defined in (3.5), i.e.,

\[
\nabla_{\partial_i} \partial_1 = f_1 \partial_1, \quad \nabla_{\partial_i} \partial_2 = f_2 \partial_1 \quad \text{and} \quad \nabla_{\partial_1} \partial_3 = f_3 \partial_1,
\]

where the smooth functions \(f_i = f_i(x_1, x_2, x_3)\) (\(i = 1, 2, 3\)) are Christoffel symbols. For \(X = \sum_{i=1}^{3} \alpha_i \partial_i\), the affine Szabó operator is given by

\[
S^\nabla(X)(\partial_1) = \alpha_{11} \partial_1, \quad S^\nabla(X)(\partial_2) = \alpha_{12} \partial_1 \quad \text{and} \quad S^\nabla(X)(\partial_3) = \alpha_{13} \partial_1
\]

with

\[
\alpha_{11} = \alpha_3^2 \{\partial_3^2 f_3 + 2 f_3 \partial_3 f_3\} + \alpha_2^3 \{\partial_2^2 f_2 + 2 f_2 \partial_2 f_2\} \\
+ \alpha_2^3 \alpha_1 \{\partial_2^2 f_1 + 4 f_2 \partial_1 f_3 - 2 f_3 \partial_3 f_1\} \\
+ \alpha_3^2 \alpha_1 \{\partial_3^2 f_1 + 4 f_3 \partial_1 f_2 - 2 f_2 \partial_2 f_1\} \\
+ \alpha_1^3 \alpha_3 \{\partial_1^2 f_3 - \partial_1 \partial_3 f_1 - f_1 \partial_1 f_3 + f_1 \partial_3 f_1\} \\
+ \alpha_2^3 \alpha_2 \{\partial_2^2 f_2 - \partial_2 \partial_2 f_1 - f_1 \partial_2 f_1 + f_1 \partial_2 f_1\} \\
+ \alpha_3^2 \alpha_3 \{\partial_3^2 f_3 + 2 \partial_3 \partial_3 f_2 + 2 f_2 \partial_3 f_2 + 2 f_2 \partial_2 f_3 + 2 f_2 \partial_2 f_3\} \\
+ \alpha_3^2 \alpha_2 \{\partial_3^2 f_2 + 2 \partial_3 \partial_2 f_3 + 2 f_3 \partial_2 f_3 + 2 f_3 \partial_2 f_3\} \\
+ \alpha_3 \alpha_2 \alpha_3 \{4 f_3 \partial_1 f_3 + 4 f_2 \partial_1 f_3 - 2 f_3 \partial_2 f_1 - 2 f_2 \partial_2 f_1 + 2 \partial_3 \partial_2 f_1\},
\]

\[
\alpha_{12} = \alpha_3^2 \{\partial_3 \partial_2 f_1 - \partial_3^2 f_2 - f_1 \partial_2 f_1\} \\
+ 2 f_1 \partial_1 f_2 + \alpha_2^3 \alpha_1 \{\partial_2^2 f_2 + 2 f_2 \partial_2 f_2 - \partial_3^2 f_2\} \\
+ \alpha_3^2 \alpha_1 \{\partial_3^2 f_3 + 2 \partial_3 \partial_3 f_3 + 2 f_2 \partial_3 f_3 + 2 f_2 \partial_3 f_3\} \\
+ \alpha_2^3 \alpha_3 \{2 \partial_3 \partial_3 f_3 - \partial_3 \partial_3 f_3 + \partial_3 \partial_3 f_3 + 2 f_3 \partial_2 f_1 + 4 f_3 \partial_1 f_2\} \\
+ \alpha_1^2 \alpha_2 \{\partial_1^2 f_1 - 2 f_2 \partial_2 f_1 + 4 f_2 \partial_1 f_2\} + \alpha_1 \alpha_2 \alpha_3 \{4 f_3 \partial_3 f_3 + 2 \partial_3^2 f_3\}.
\]
\[ a_{13} = \alpha_1^3 \{ \partial_1 \partial_3 f_1 - \partial_1^2 f_3 - f_1 \partial_3 f_1 + f_1 \partial_1 f_3 \} \\
+ \alpha_1^2 \alpha_3 \{ \partial_3^2 f_1 + 3 f_3 \partial_1 f_3 - f_3 \partial_3 f_1 \} \\
+ \alpha_1^2 \alpha_2 \{ \partial_2 \partial_3 f_1 + 4 f_2 \partial_1 f_3 - 2 \partial_2 \partial_1 f_3 - 3 f_2 \partial_3 f_1 + f_3 \partial_2 f_1 \} \\
+ \alpha_2^2 \alpha_1 \{ 2 f_3 \partial_2 f_2 + 2 f_2 \partial_2 f_3 + \partial_2 \partial_3 f_2 \} \\
+ \alpha_3^2 \alpha_1 \{ \partial_3^2 f_3 - f_3 \partial_3 f_3 - f_3^3 + 2 f_3 \partial_3 f_3 \} \\
+ \alpha_1^2 \alpha_3 \{ 4 f_3 \partial_2 f_3 - 2 f_2 \partial_3 f_3 - f_3 \partial_3 f_2 + 2 \partial_3^2 f_2 - f_3^2 f_2 \}. \]

Since the Ricci tensor of any affine Szabó connection is cyclic parallel, it follows that \( a_{11} = 0 \). Thus the characteristic polynomial of the matrix associated to \( S^\nabla (X) \) with respect to the basis \( \{ \partial_1, \partial_2, \partial_3 \} \) is equal to:

\[ P_\lambda (S^\nabla (X)) = -\lambda^3. \]

We have the following result.

**Theorem 4.5.** Let \( M = \mathbb{R}^3 \) and \( \nabla \) be the torsion free affine connection, whose nonzero coefficients of the connection are given by

\[ \nabla_{\partial_i} \partial_1 = f_1 \partial_1, \quad \nabla_{\partial_i} \partial_2 = f_2 \partial_1 \quad \text{and} \quad \nabla_{\partial_i} \partial_3 = f_3 \partial_1. \]

Then \( (M, \nabla) \) is affine Szabó if and only if the Ricci tensor of \( (M, \nabla) \) is cyclic parallel.

From Theorem 4.5, one can construct examples of affine Szabó connections.

**Example 4.6.** The following affine connections on \( \mathbb{R}^3 \) whose non-zero Christoffel symbols are given by:

1. \( \nabla_{\partial_3} \partial_1 = 0, \quad \nabla_{\partial_3} \partial_2 = x_3 \partial_1, \quad \nabla_{\partial_3} \partial_3 = x_2 \partial_1; \)
2. \( \nabla_{\partial_3} \partial_1 = x_1 \partial_1, \quad \nabla_{\partial_3} \partial_2 = 2 x_3 \partial_1, \quad \nabla_{\partial_3} \partial_3 = -2 x_2 \partial_1; \)

are affine Szabó.

Note that the result in Theorem 4.5 remains the same if the affine connection \( \nabla \) has non-zero components \( \nabla_{\partial_i} \partial_1, \nabla_{\partial_i} \partial_2 \) and \( \nabla_{\partial_i} \partial_3 \) in the same direction of the element of the basis \( \{ \partial_i \}_{i=1,2,3} \).

The affine manifolds in Theorem 4.5 are also called \( L_3 \)-spaces, and Therefore, are d’Atri spaces. We refer to [12] for a further discussion of D’Atri spaces.

4.2. **Second Family of affine Szabó connection.** Let us consider the affine connection defined in (3.7), i.e.,

\[ \nabla_{\partial_1} \partial_1 = f_1 \partial_2, \quad \nabla_{\partial_2} \partial_2 = f_2 \partial_3 \quad \text{and} \quad \nabla_{\partial_3} \partial_3 = f_3 \partial_1 \]

where the smooth functions \( f_i = f_i(x_1, x_2, x_3) \), for \( i = 1, 2, 3 \), are Christoffel symbols. Since the Ricci tensor of any affine Szabó connection is cyclic parallel, it follows from the Proposition 3.4 that the matrix associated to the affine Szabó operator with respect to the basis \( \{ \partial_1, \partial_2, \partial_3 \} \) is reduced to

\[
(S^\nabla)(X) = \begin{pmatrix}
0 & b_{12} & b_{13} \\
 b_{21} & 0 & b_{23} \\
 b_{31} & b_{32} & 0
\end{pmatrix}
\]
with
\[ b_{12} = \alpha_1^2\alpha_2(-\partial_1 \partial_3 f_1) + \alpha_1\alpha_2^2(f_2 \partial_3 f_1) + \alpha_1\alpha_3^2(f_3 \partial_1 f_1 - \partial_3 f_1) \]
\[ + \alpha_2^2\alpha_3(-2f_2 \partial_3 f_1) + \alpha_2\alpha_3^2(f_1 \partial_2 f_3) + \alpha_3^3(2f_3 \partial_3 f_1 + f_1 \partial_3 f_3); \]
\[ b_{13} = \alpha_1^2\alpha_2(-2f_1 \partial_1 f_2 - f_2 \partial_1 f_1) + \alpha_1\alpha_2^2(\partial_1^2 f_2 - f_1 \partial_2 f_2) \]
\[ + \alpha_1\alpha_2\alpha_3(-2f_2 \partial_3 f_1 + \partial_3 f_2 f_1); \]
\[ b_{21} = \alpha_1^2\alpha_3(2f_3 \partial_2 f_1) + \alpha_1\alpha_2\alpha_3(-2f_3 \partial_1 f_2) + \alpha_2^2\alpha_3(-2f_2 \partial_2 f_3 - f_3 \partial_2 f_2) \]
\[ + \alpha_2^2\alpha_3^2(\partial_2^2 f_3 - f_2 \partial_3 f_3) + \alpha_3^3(\partial_3 \partial_2 f_3); \]
\[ b_{23} = \alpha_1^2(2f_1 \partial_3 f_2 + f_2 \partial_3 f_1) + \alpha_1^2\alpha_2(-\partial_1^2 f_2 + f_1 \partial_2 f_2) + \alpha_1^2\alpha_3(f_2 \partial_3 f_1) \]
\[ + \alpha_1\alpha_2^2(-\partial_2 \partial_1 f_2) + \alpha_1\alpha_3^2(-2f_2 \partial_3 f_1 + \alpha_2\alpha_3^2(f_3 \partial_1 f_2); \]
\[ b_{31} = \alpha_1^2\alpha_2(-2f_1 \partial_3 f_3) + \alpha_1^2\alpha_3(f_1 \partial_2 f_3) + \alpha_1\alpha_2^2(f_3 \partial_1 f_2) \]
\[ + \alpha_1^2\alpha_3(-\partial_2^2 f_3 + f_2 \partial_3 f_3) + \alpha_2\alpha_3^2(-\partial_3 \partial_2 f_3); \]
\[ b_{32} = \alpha_1^2(\partial_1 \partial_3 f_1) + \alpha_1^2\alpha_3(-f_3 \partial_1 f_1 + \partial_3 f_1) + \alpha_1\alpha_2^2(2f_1 \partial_3 f_2) \]
\[ + \alpha_1\alpha_2\alpha_3(-2f_1 \partial_3 f_3) + \alpha_1\alpha_3^2(-2f_3 \partial_3 f_1 - f_1 \partial_3 f_3). \]

The characteristic polynomial of the affine Szabó operator is now seen to be:
\[ P[S^\nabla(X)](\lambda) = -\lambda^3 + (b_{12}b_{21} + b_{23}b_{32} + b_{13}b_{31})\lambda + (b_{12}b_{23}b_{31} + b_{13}b_{21}b_{32}). \]

It follows that the affine connection given by (3.7) is affine Szabó if and only if:
\[ b_{12}b_{21} + b_{23}b_{32} + b_{13}b_{31} = 0 \quad \text{and} \quad b_{12}b_{23}b_{31} + b_{13}b_{21}b_{32} = 0. \]

A straightforward calculation shows that:
\[ b_{12}b_{23}b_{31} + b_{13}b_{21}b_{32} = 0. \]
Then \( S^\nabla(X) \) has eigenvalue zero if and only if:
\[ b_{12}b_{21} + b_{23}b_{32} + b_{13}b_{31} = 0. \quad (4.1) \]

(1) Assume \( f_1 = 0 \). Then, the relation (4.1) reduces to:
\[ b_{13}b_{31} = 0. \]

(a) If \( \partial_1 f_2 = 0 \), then \( f_2 = u(x_2) \) and \( f_3 = v(x_2) + t(x_3). \)
(b) If \( \partial_1 f_2 \neq 0 \), then \( f_3 = 0 \).

(2) Assume \( f_2 = 0 \), then we have
\[ b_{12}b_{21} = 0. \]

(a) If \( \partial_2 f_3 = 0 \), then \( f_3 = t(x_3) \) and \( f_1 = f(x_1) + g(x_3). \)
(b) If \( \partial_2 f_3 \neq 0 \), then \( f_1 = 0 \).

(3) Assume \( f_3 = 0 \), then we have
\[ b_{23}b_{32} = 0. \]

(a) If \( \partial_3 f_1 = 0 \), then \( f_1 = f(x_1) \) and \( f_2 = h(x_1) + k(x_2). \)
(b) If \( \partial_3 f_1 \neq 0 \), then \( f_2 = 0 \).

We have the following result.
Theorem 4.7. Let $M = \mathbb{R}^3$ and $\nabla$ be the torsion free affine connection, whose non-zero coefficients of the connection are given by

$$\nabla_{\partial_1} \partial_1 = f_1 \partial_2, \quad \nabla_{\partial_2} \partial_2 = f_2 \partial_3 \quad \text{and} \quad \nabla_{\partial_3} \partial_3 = f_3 \partial_1.$$ 

Then $(M, \nabla)$ is affine Szabó if at least one of the following conditions holds:

1. $f_1 = 0$, $f_2 = u(x_2)$ and $f_3 = v(x_2) + t(x_3)$.
2. $f_2 = 0$, $f_3 = t(x_3)$ and $f_1 = f(x_1) + g(x_3)$.
3. $f_3 = 0$, $f_1 = f(x_1)$ and $f_2 = h(x_1) + u(x_2)$.

Or at least one of the following conditions holds:

1. $f_1 = 0$, $f_2 = f(x_1) + g(x_2)$ and $f_3 = 0$.
2. $f_2 = 0$, $f_3 = v(x_2) + t(x_3)$ and $f_1 = 0$.
3. $f_3 = 0$, $f_1 = f(x_1) + g(x_3)$ and $f_2 = 0$.

From Theorem 4.7 one can construct examples of affine Szabó connections. As an example, we have the following.

Example 4.8. The following connections on $\mathbb{R}^3$ whose non-zero Christoffel symbols are given by:

1. $\nabla_{\partial_1} \partial_1 = 0, \quad \nabla_{\partial_2} \partial_2 = x_2 \partial_3, \quad \nabla_{\partial_3} \partial_3 = (x_2 + x_3^2) \partial_1$;
2. $\nabla_{\partial_1} \partial_1 = x_1^2 \partial_2, \quad \nabla_{\partial_2} \partial_2 = (x_1 + x_2) \partial_3, \quad \nabla_{\partial_3} \partial_3 = 0$;

are affine Szabó.

Remark 4.9. The affine connection defined in Example 4.5 has a Ricci tensor which is cyclic parallel but it is not affine Szabó. This means that the manifold defined in Example 4.5 is an $L_3$-space but not an affine Szabó manifold.

One has also the following observation.

Theorem 4.10. Let $(M_1, \nabla_1)$ be an affine Szabó at $p_1 \in M_1$ and $(M_2, \nabla_2)$ be an affine Szabó at $p_2 \in M_2$. Then the product manifold $(M, \nabla) := (M_1 \times M_2, \nabla = \nabla_1 \oplus \nabla_2)$ is affine Szabó at $p = (p_1, p_2)$.

Proof. Let $X = (X_1, X_2) \in T_{(p_1, p_2)}(M_1 \times M_2)$ with $X_1 \in T_{p_1}M_1$ and $X_2 \in T_{p_2}M_2$. Then we have $S^\nabla(X_1) = S^{\nabla_1}(X_1) \oplus S^{\nabla_2}(X_2)$. So $\text{Spect}\{S^\nabla(X)\} = \text{Spect}\{S^{\nabla_1}(X_1)\} \cup \text{Spect}\{S^{\nabla_2}(X_2)\} = \{0\} \cup \{0\} = \{0\}$. 

Affine Szabó connections are of interest not only in affine geometry, but also in the study of Pseudo-Riemannian Szabó metrics since they provide some examples without Riemannian analogue by means of the Riemannian extensions.

5. RIEMANNIAN EXTENSION CONSTRUCTION

Let $\nabla$ be a torsion free affine connection on an $n$-dimensional affine manifold $M$ and $T^*M$ be the cotangent bundle of $(M, \nabla)$. In the locally induced coordinates $(u_j, u^*_i)$ on $\pi^{-1}(U) \subset T^*M$, the Riemannian extension $g^r$ is the pseudo-Riemannian metric on $T^*M$ of neutral signature $(n, n)$ defined by

$$g^r = \left( \begin{array}{cc}
-2u_kT^k_{ij} & \delta^j_i \\
\delta^j_i & 0
\end{array} \right),$$

(5.1)
with respect to $\partial_1, \ldots, \partial_n, \partial_{i'}, \ldots, \partial_{n'} (i, j, k = 1, \ldots, n; k' = k + n)$, where $\Gamma^k_{ij}$ are the Christoffel symbols of the Riemann curvature tensor $\nabla$.

Let $(M, \nabla)$ be an 3-dimensional affine manifold. Let $(x_1, x_2, x_3)$ be local coordinates on $M$. We expand $\nabla_{\partial_i} \partial_j = \sum_k \Gamma^k_{ij} \partial_k$ for $i, j, k = 1, 2, 3$ to define the Christoffel symbols $\Gamma^k_{ij}$ of $\nabla$. If $\omega \in T^* M$, we expand $\omega = x_i dx_i + x_j dx_j + x_k dx_k$ to define the dual fiber coordinates $(x_4, x_5, x_6)$ thereby obtain canonical coordinates $(x_1, x_2, x_3, x_4, x_5, x_6)$ on $T^* M$. The Riemannian extension in the metric of neutral signature $(3, 3)$ on the cotangent bundle $T^* M$ given locally by

$$\tilde{g}_\nabla(\partial_1, \partial_1) = \tilde{g}_\nabla(\partial_2, \partial_2) = \tilde{g}_\nabla(\partial_3, \partial_3) = 1,$$

$$\tilde{g}_\nabla(\partial_1, \partial_1) = -2x_4 \Gamma^1_{11} - 2x_5 \Gamma^2_{11} - 2x_6 \Gamma^3_{11},$$

$$\tilde{g}_\nabla(\partial_1, \partial_2) = -2x_4 \Gamma^1_{12} - 2x_5 \Gamma^2_{12} - 2x_6 \Gamma^3_{12},$$

$$\tilde{g}_\nabla(\partial_1, \partial_3) = -2x_4 \Gamma^1_{13} - 2x_5 \Gamma^2_{13} - 2x_6 \Gamma^3_{13},$$

$$\tilde{g}_\nabla(\partial_2, \partial_2) = -2x_4 \Gamma^2_{22} - 2x_5 \Gamma^2_{22} - 2x_6 \Gamma^3_{22},$$

$$\tilde{g}_\nabla(\partial_2, \partial_3) = -2x_4 \Gamma^2_{23} - 2x_5 \Gamma^2_{23} - 2x_6 \Gamma^3_{23},$$

$$\tilde{g}_\nabla(\partial_3, \partial_3) = -2x_4 \Gamma^3_{33} - 2x_5 \Gamma^3_{33} - 2x_6 \Gamma^3_{33}.$$

**Lemma 5.1.** Let $(M, \nabla)$ be an $n$-dimensional affine manifold and $(T^* M, \tilde{g}_\nabla)$ be the cotangent bundle with the twisted Riemannian extension. Then, we have

$$\text{Spect}\{\tilde{S}(\bar{X})\} = \text{Spect}\{\tilde{S}^\nabla(X)\}.$$

**Proof.** Let $\Gamma^k_{ij}$ be the Christoffel symbols of $\nabla$. The non-zero Christoffel symbols $\tilde{\Gamma}^k_{ij}$ of the Levi-Civita connection of $\tilde{g}_\nabla$ are given by:

$$\tilde{\Gamma}^k_{ij} = \Gamma^k_{ij}, \quad \tilde{\Gamma}^k_{ij'} = -\Gamma^j_{ik}, \quad \tilde{\Gamma}^k_{ij'} = -\Gamma^j_{ik},$$

$$\tilde{\Gamma}^k_{ij'} = \sum_r x_r \left( \partial_r \Gamma^r_{ij} - \partial_i \Gamma^r_{jk} - \partial_j \Gamma^r_{ik} + 2 \sum_l \Gamma^r_{lk} \Gamma^l_{ij} \right),$$

where $(i, j, k, l, r = 1, \ldots, n)$ and $(i' = i + n, j' = j + n, k' = k + n, r' = r + n)$. The non-zero components of the curvature tensor of $(T^* M, \tilde{g}_\nabla)$ up to the usual symmetries are given as follows: we omit $\tilde{R}^h_{ij;i'}$, as it plays no role in our considerations.

$$\tilde{R}^h_{kji} = R^h_{kji}, \quad \tilde{R}^h_{kji'} = -R^h_{kji},$$

where $R^h_{kji}$ are the components of the curvature tensor of $(M, \nabla)$. Let $\bar{X} = \alpha_i \partial_i + \alpha_{i'} \partial_{i'}$ and $\bar{Y} = \beta_i \partial_i + \beta_{i'} \partial_{i'}$ be vectors fields on $T^* M$. Let $X = \alpha_i \partial_i$ and $\dot{Y} = \beta_i \partial_i$ be the corresponding vectors fields on $M$. Let $S^\nabla(X)$ be the matrix of the affine Szabó operator on $M$ relative to the basis $\{\partial_i\}$. Then the matrix of the Szabó operator $\tilde{S}(\bar{X})$ with respect to the basis $\{\partial_i, \partial_{i'}\}$ have the form

$$\tilde{S}(\bar{X}) = \left( \begin{array}{cc} S^\nabla(X) & 0 \\ tS^\nabla(X) & \end{array} \right).$$
We have the following results.

**Theorem 5.2.** Let \((M, \nabla)\) be a smooth torsion-free affine manifold. Then the following statements are equivalent:

(i) \((M, \nabla)\) is affine Szabó.

(ii) The Riemannian extension \((T^*M, g_\nabla)\) of \((M, \nabla)\) is a pseudo-Riemannian Szabó manifold.

**Proof.** Let \(\tilde{X} = \alpha_i \partial_i + \alpha' \partial'\) be a vector field on \(T^*M\). Then the matrix of the Szabó operator \(\tilde{S}(\tilde{X})\) with respect to the basis \(\{\partial_i, \partial'_i\}\) is of the form

\[
\tilde{S}(\tilde{X}) = \begin{pmatrix}
S^\nabla(X) & 0 \\
* & S^\nabla(X)
\end{pmatrix}.
\]

(5.2)

where \(S^\nabla(X)\) is the matrix of the affine Szabó operator on \(M\) relative to the basis \(\{\partial_i\}\). Note that the characteristic polynomial \(P_\lambda[\tilde{S}(\tilde{X})]\) of \(\tilde{S}(\tilde{X})\) and \(P_\lambda[S^\nabla(X)]\) of \(S^\nabla(X)\) are related by

\[
P_\lambda[\tilde{S}(\tilde{X})] = P_\lambda[S^\nabla(X)] \cdot P_\lambda[\tilde{S}^\nabla(X)].
\]

(5.3)

Now, if the affine manifold \((M, \nabla)\) is assumed to be affine Szabó, then \(S^\nabla(X)\) has zero eigenvalues for each vector field \(X\) on \(M\). Therefore, it follows from (5.2) that the eigenvalues of \(\tilde{S}(\tilde{X})\) vanish for every vector field \(\tilde{X}\) on \(T^*M\). Thus \((T^*M, g_\nabla)\) is pseudo-Riemannian Szabó manifold.

Conversely, assume that \((T^*M, g_\nabla)\) is a pseudo-Riemannian Szabó manifold. If \(X = \alpha_i \partial_i\) with \(\alpha_i \neq 0\), for any \(i\), is a vector field on \(M\), then \(\tilde{X} = \alpha_i \partial_i + \frac{1}{2\alpha_i} \partial_i\) is an unit vector field at every point of the zero section on \(T^*M\). Then from (5.2), we see that, the characteristic polynomial \(P_\lambda[\tilde{S}(\tilde{X})]\) of \(\tilde{S}(\tilde{X})\) is the square of the characteristic polynomial \(P_\lambda[S^\nabla(X)]\) of \(S^\nabla(X)\). Since for every unit vector field \(\tilde{X}\) on \(T^*M\) the characteristic polynomial \(P_\lambda[\tilde{S}(\tilde{X})]\) would be the same, it follows that for every vector field \(X\) on \(M\) the characteristic polynomial \(P_\lambda[S^\nabla(X)]\) is the same. Hence \((M, \nabla)\) is affine Szabó.

\(\square\)

As an example, we have the following. The Riemannian extension of the affine Szabó connection on \(\mathbb{R}^3\) defined by

\[
\nabla_{\partial_i} \partial_1 = x_1 \partial_1, \quad \nabla_{\partial_i} \partial_2 = 2x_3 \partial_1, \quad \nabla_{\partial_i} \partial_3 = -2x_2 \partial_1
\]

is the pseudo-Riemannian metric of signature (3,3) given by

\[
g_\nabla = 2dx_1 \otimes dx_4 + 2dx_2 \otimes dx_5 + 2dx_3 \otimes dx_6
\]

\[\
- 2x_1 x_4 dx_1 \otimes dx_1 - 4x_3 x_4 dx_1 \otimes dx_2 + 4x_2 x_4 dx_1 \otimes dx_3.
\]

After, a straightforward calculation, it easy to see that this metric is Szabó.

The Riemannian extensions provide a link between affine and pseudo-Riemannian geometries. Some properties of the affine connection \(\nabla\) can be investigated by means of the corresponding properties of the Riemannian extension \(g_\nabla\). For more details and information about Riemannian extensions, see [2, 3, 6, 7] and references therein. For instance, it is known, in [2, 3] and references therein, that a
Walker metric is a triple \((M, g, D)\), where \(M\) is an \(n\)-dimensional manifold, \(g\) is a pseudo-Riemannian metric on \(M\) and \(D\) is an \(r\)-dimensional parallel null distribution \((r > 0)\). In [3], the authors showed that any four-dimensional Riemannian extension is necessarily a self-dual Walker manifold, but for some particular cases, they proved that the converse holds.

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