Circular arcs are the only analytic Jordan curves with an exterior power point

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Abstract

In 1946, J. Rosenbaum proposed a family of problems asking how many power points are needed to ensure that the boundary $c$ of a given convex body is a disk. In this paper, we use Riordan matrices to show that, if this curve $c$ is analytic, then one single exterior power point is sufficient.

Keywords: Power point, Riordan matrix, Circle, Analytic curve, Geometric continuity

1. Introduction

For $k \geq 1$, a plane curve $c$ has geometric continuity of order $k$, and we write $c \in G^k$, if there exists a local regular parametrization of class $C^k$ of this curve in a neighborhood of each point of $c$. The definition also holds for $k = \infty$ and $k = \omega$ (analytic curves).

Geometric continuity is the natural notion of smoothness for a parametrized curve if we want the definition not to depend on the particular parametrization considered. Let $c$ be a plane curve which is defined piecewise. A classical problem in Applied Geometry (see [1, 2, 3]) is to find compatibility conditions between the parametrizations of each of those pieces to guarantee that $c \in G^k$. In [4] we can find a useful statement of these conditions in terms of Riordan matrices. This statement seems to be particularly useful to show uniqueness of plane a curve $c$ satisfying a given property (as shown in that article for the property “$c$ has two equichordal points”, among other things).

In this paper we use the conditions appearing in [4] in relation to a problem proposed by E. J. Rosenbaum in [5]. Let $B$ be a plane convex body with boundary $\partial B$. A point $P$, in the exterior (resp. interior) of $B$, is an exterior (resp. interior) power point of $B$ if there is a constant $k > 0$, called the power of $B$ at $P$, such that for each line $l$ intersecting $\partial B$ in two endpoints $A, B$ (if $l$ intersect $\partial B$ in a single point, we interpret that $A = B$) we have that $d(P, A) \cdot d(P, B) = k^2$. We have:

Rosenbaum’s Problem: Given some convex body $B$, how many distinct power points ensure that $B$ is a disk?
In Section 2 we provide some bibliography and include some known results concerning this problem to put our theorem in perspective. Our main result (Theorem 4) is stated and proved in Section 3 and it is related to the open Problem 18 in [4].

2. Some known results concerning Rosenbaum’s Problem

A classical theorem in plane Geometry, proved by J. Steiner in the 19th century could be stated, with the notation above, as follows. Sometimes, this theorem is also referred as the Intersecting Chords Theorem for the interior point case and as the Intersecting Secants Theorem for the exterior point case.

**Theorem 1 (Steiner’s Power of a Point Theorem).** Let $B$ be a disk. Then every point in the plane is a power point of $B$.

For more information about Rosenbaum’s Problem for interior power points we refer the reader to [6, 7]. For the rest of this section, we will focus in the exterior power points case.

A circular segment, is the convex region bounded by a circular arc and the chord joining its two endpoints. A **lune** is a plane (non-necessarily convex) body which boundary is the union of two circular arc. Let us call vertices to the points where the corresponding two circular arcs meet. If a lune is convex, it is called a lens. With this notation, in the article [7], we may find the following:

**Theorem 2 (Theorems 1, 2 and 5 in [7]).** Let $B$ be a convex body which admits two exterior power points $P_1, P_2$.

(i) If the line joining the power points intersects the interior of $B$. Then $B$ can be any lens, which vertices are collinear to $P_1, P_2$.

(ii) If the line joining the power points supports the boundary of $B$ in a non-degenerate segment, then $B$ can be any circular segment which chord is collinear to $P_1, P_2$.

(iii) If there exists a third exterior power point $P_3$ such that $P_1, P_2, P_3$ are not collinear, then $B$ is a disk and it is uniquely determined by its powers at these points.

If a convex body has a power point $P$ with power $k$, then it remains invariant under the inversion with respect to the circle of center $P$ and radius $k$. Inversions with respect to a circle map lines and circles to lines and circles. So, we can easily find lenses with a single power point $P$. The vertices of these lenses, are the points where they meet the circle. See Figure [1].

Since lenses are not, in general, smooth at their vertices, this family of examples motivates the **Smooth Version of Rosenbaum’s Problem** for which the boundary of the body $B$ is known to have geometric continuity of order $k$.

The case $k = 1$ is already studied in [7] (at the beginning of Section 4): there exist convex bodies with differentiable boundary which admit an exterior power point and are not disks. More about this case is said in Lemma [6] in the previous section.
Figure 1: This lens has a power point at the origin $P$ with power 1. So $d(A, P)^2 = d(B, P)^2 = d(P, C) \cdot d(P, D)$.

3. Proof of the main Theorem

The definition of power point can be extended to a plane curve $c$ (which is not necessarily the boundary of a convex plane body) if $P$ satisfies one extra property: every line through $P$ meets $c$ at most twice.

If $c$ is a Jordan curve, we still have a natural notion of interior and exterior power points. If it is not, we say that the power point is interior if it always lies between the two intersection points (or coincide with them if they coincide) and exterior if it is never between the two intersection points.

As happened before, if $P$ is an exterior power point of a curve $c$ with power $k$, then it remains invariant under the inversion with respect to the circle of center $P$ and radius $k$. The points in which $c$ meets this circle will be called vertices (generalizing the notation introduced before for lunes).

**Remark 3.** If $c$ is a Jordan curve which is not the circle of center $P$ and radius $k$, then there are exactly two vertices. Similarly, if $c$ is a Jordan arc and it is not a circular arc of center $P$ and radius $k$, then it has exactly one vertex.

The main result in this paper is the following:

**Theorem 4.** Let $c$ be a Jordan arc (resp. a Jordan curve) and $P \notin c$ be an exterior power point of $c$ of power $k$. If $c$ contains exactly one vertex $V = (x_0, y_0)$ (resp. two vertices, one of them denoted by $V = (x_0, y_0)$) then:

(a) If $\gamma(t) = (x(t), y(t))$ is a parameterization of $c$ near $V$ with $\gamma(0) = V$, $x(t) = x_0 + t$, $c$ is in $G^n$ for some $n \geq 2$ and the (signed) curvature of $c$ at $V$ is given, then the Taylor polynomial of degree $n - 1$ of $y(t)$ is univocally determined.

(b) If $c$ is analytic, then $c$ must be a circular arc (resp. a circle).
To prove this theorem, we may assume that $P$ is the origin, $k = 1$ and $V = (1, 0)$. Moreover, if $c \in G^n$ for some $n \geq 2$, then there is a Jordan arc $\bar{c} \subset c$ corresponding to the graph of a differentiable real function of real variable.

In this setting, there is a parametrization $\gamma : [-1, 1] \to \bar{c}$ where:

$$
\gamma(t) = \begin{cases} 
\gamma_{\text{left}}(t) & \text{for } t \in [-1, 0) \\
V & \text{for } t = 0 \\
\gamma_{\text{right}}(t) & \text{for } t \in (0, 1] 
\end{cases},
$$

the function $\gamma_{\text{right}}$ is in $C^n([-1, 0])$ and

$$
\gamma_{\text{left}}(-t) = \frac{1}{\|\gamma_{\text{right}}(t)\|^2} \gamma_{\text{right}}(t) \quad (1)
$$

So, as discussed in [4], the Jordan arc $\bar{c}$ has geometric continuity of order $n$ if and only if there exists a regular $C^n$ reparametrization $u$ of $\gamma_{\text{left}}$ such that $u(0) = 0$, $u'(0) \neq 0$ and

$$
\tilde{\gamma}(t) = \begin{cases} 
\gamma_{\text{left}}(u(t)) & \text{for } t \in [-1, 0) \\
V & \text{for } t = 0 \\
\gamma_{\text{right}}(t) & \text{for } t \in (0, 1]
\end{cases}
$$

is of class $C^n$.

The condition that $\gamma_{\text{left}}, \gamma_{\text{right}}$ must satisfy, for the existence of this change of parameter are frequently expressed in the bibliography in terms of the so called connection matrices (see page 9 in [3]). The novelty of [4] is to express them in terms of Riordan matrices.

Riordan matrices are known to be a very useful tool for doing computations in Combinatorics and other branches of Mathematics. For more information about the Riordan group we refer the reader to the original paper [8], the old but recommendable survey [9], the article [10] which is fundamental for the ideas used in this construction, the recent preprint [11] that compiles much of the bibliography about this topic and the article [4] that presents a summary of the main definition, notation and ideas needed and is specifically oriented for a proof like the following.

The approach made in [4] (see Remark 6 in that article) leads to the following:

**Remark 5.** In the notation above, suppose that $\gamma_{\text{right}}(t) = (x(t), y(t))$, $\gamma_{\text{left}}(t) = (\bar{x}(t), \bar{y}(t))$. If $\bar{c}$ has geometric continuity of order $n$ then there exists a partial Riordan matrix $R_n(1, u)$ for some $u$ with Taylor polynomial $u_1 t + \ldots + u_n t^n$, $u_1 \neq 0$, such that

$$
R_n(1, u) \begin{bmatrix} 1 \\ x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_0 \\ \bar{x}_1 \\ \vdots \\ \bar{x}_n \end{bmatrix}, \\
R_n(1, u) \begin{bmatrix} 0 \\ y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} y_0 \\ \bar{y}_1 \\ \vdots \\ \bar{y}_n \end{bmatrix} \quad (2)
$$
where \(1 + x_1 t + \ldots + x_n, y_1 t + \ldots + y_n, \bar{x}_0 + \bar{x}_1 t + \ldots + \bar{x}_n\) and \(\bar{y}_0 + \bar{y}_1 t + \ldots + \bar{y}_n\) are the corresponding Taylor polynomials of \(x(t), y(t), \bar{x}(t), \bar{y}(t)\). Since, according to Equation (1):

\[
\begin{bmatrix}
\bar{x}_0 \\
\bar{x}_1 \\
\vdots \\
\bar{x}_n
\end{bmatrix}
= R\left(\frac{1}{x^2 + y^2}, u\right)
\begin{bmatrix}
x_1 \\
\vdots \\
x_n
\end{bmatrix},
\begin{bmatrix}
\bar{y}_0 \\
\bar{y}_1 \\
\vdots \\
\bar{y}_n
\end{bmatrix}
= R\left(\frac{1}{x^2 + y^2}, u\right)
\begin{bmatrix}
y_1 \\
\vdots \\
y_n
\end{bmatrix}
\]

we have that Equation (2) is equivalent to:

\[
R_n(x^2 + y^2, u)
\begin{bmatrix}
x_1 \\
\vdots \\
x_n
\end{bmatrix}
= R_n(x^2 + y^2, u)
\begin{bmatrix}
x_1 \\
\vdots \\
x_n
\end{bmatrix},
R_n(x^2 + y^2, u)
\begin{bmatrix}
y_1 \\
\vdots \\
y_n
\end{bmatrix}
= R_n(x^2 + y^2, u)
\begin{bmatrix}
y_1 \\
\vdots \\
y_n
\end{bmatrix}
\]

Before the proof of Theorem 4, we need the following:

**Lemma 6.** In the notation above, \(\bar{c}\) has geometric continuity of order 2 if and only if \(\gamma_{right} \in C^1([0, 1])\) and \(\gamma'_{right}(0)\) is parallel to the vector \(\bar{P}V\), that is, \(\bar{c}\) is perpendicular to the circle of center \(P\) and radius \(k\) at the vertex \(V\).

**Proof:** Expanding \(R_2(x^2 + y^2, u)\), Equation (3) for \(n = 2\) is:

\[
\begin{bmatrix}
1 \\
2x_1 + y_1^2 \\
2x_2 + x_1^2 + y_1^2 + u_1 \\
2x_2 + x_1^2 + y_1^2 + u_2
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_1^2 \\
x_2^2
\end{bmatrix}
= \begin{bmatrix}
1 \\
x_1 \\
x_2
\end{bmatrix}
\]

Each of those matricial equations lead to three linear equations (one corresponding to each row) in the indeterminates \(u_1, u_2\). The first equation in each matricial equation is trivial. The system of linear equations obtained from the second row in each matricial equation is the following:

\[
\begin{align*}
(1 + u_1)x_1 &= 0 \\
y_1(1 - u_1) &= 0
\end{align*}
\]

The solution \(u_1 = 1, x_1 = 0\) is discarded, since \(\bar{c}\) is the graph of a differentiable real function of real variable. So \(u_1 = -1, y_1 = 0\) and \(x_1\) is a free parameter (apart from being not equal to 0, since \(\gamma\) is a regular parametrization).

Finally we look at the system of equations obtained from the third row of each matrix:

\[
\begin{align*}
(2x_2 + x_1^2) + (-2x_1 + u_2)x_1 + x_2 &= x_2 \\
y_2 &= y_2
\end{align*}
\]
The second equation is trivial and for every choice of $y_2, x_2 \in \mathbb{R}$, we can obtain a value for $u_2$ to solve the first one:

$$u_2 = \frac{x_1^2 - 2x_2}{x_1}.$$ 

Note that in the statement of the Lemma no restriction is imposed for the curvature of $\tilde{c}$ at $V$ (the parameters $x_2, y_2$ are free).

Using the lemma above, we can easily find examples of curves $\tilde{c}$ in $G^2$ that have a power point at the origin (see Figure 2 and Figure 3).

We are finally ready for the proof of the main theorem:

**Proof of Theorem 4:**

(a) We are going to prove the first point by induction. The case $n = 2$ has been studied in Lemma 6 (note that the signed curvature determines $y_2$) and Equation (3) determines $u_1 = -1, u_2 = 1$.

Let us consider the statement for $n = 4$. In this case, Equation (3) is:

$$\begin{bmatrix}
1 & 2 & -1 \\
1 & -1 & 1 \\
0 & 1 + u_3 & 0 & -1 \\
y_2^2 & 1 + 2u_3 + u_4 & -2 - 2u_3 & 1
\end{bmatrix} \begin{bmatrix}
1 \\
1 \\
0 \\
0
\end{bmatrix} = \begin{bmatrix}
1 \\
1 \\
0 \\
0
\end{bmatrix}.$$
where \( y_2 t^2 + y_3 t^3 y_4 t^4 \) is the Taylor polynomial of \( y(t) \). We only have four new equations (corresponding to the last row in each matricial equation) with respect to the case \( n = 2 \). These equations imply \( u_3 = -1, u_4 = 1 - y_2^2, y_3 = 0 \).

Let us assume that the result is true for some \( n \geq 4 \) even. For the case \( n + 2 \), let \( y_2 t^2 + \ldots + y_{n+2} t^{n+2} \) be the Taylor polynomial of degree \( n + 2 \) of \( y \). Now we will study:

\[
R_{n+2}(x^2 + y^2, u) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}, \quad R_{n+2}(x^2 + y^2, u) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ y_{n+2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ y_{n+2} \end{bmatrix}
\]

(4)

Since:

\[
R_{n+2}(x^2 + y^2, u) = \begin{bmatrix} R_n(x^2 + y^2, u) \\ a_{n+1,0} & \ldots & a_{n+1,n} \\ a_{n+2,0} & \ldots & a_{n+2,n} \end{bmatrix} = \begin{bmatrix} -1 & \ldots & a_{n+2,n+1} \\ a_{n+1,0} & \ldots & a_{n+1,n} \\ a_{n+2,0} & \ldots & a_{n+2,n} \end{bmatrix}
\]

from Equation (3) to Equation (4) only four new equations appear, corresponding to the last two rows in each matricial equation:

\[
\begin{align*}
a_{n+1,0} x_0 + \ldots + a_{n+1,n} x_n - x_{n+1} &= x_{n+1} \\
a_{n+2,0} x_0 + \ldots + a_{n+2,n} x_n + a_{n+2,n+1} x_{n+1} + x_{n+2} &= x_{n+2} \\
a_{n+1,0} y_0 + \ldots + a_{n+1,n} y_n - y_{n+1} &= y_{n+1} \\
a_{n+2,0} y_0 + \ldots + a_{n+2,n} y_n + a_{n+2,n+1} y_{n+1} + y_{n+2} &= y_{n+2}
\end{align*}
\]

and they must determine univocally \( y_n, y_{n+1} \) and \( u_{n+1}, u_{n+2} \).

None of the coefficients \( a_{n+1,i}, a_{n+2,i} \) depend on \( y_n, y_{n+1} \) and only \( a_{n+1,1}, a_{n+2,1} \) depend on \( u_{n+1}, u_{n+2} \). Taking also into account that \( x_0 = x_1 = 1, y_0 = y_1 = 0 \) and \( x_i = 0 \) for \( i \geq 2 \), we can write the system above as:

\[
\begin{align*}
a_{n+1,1} &= -a_{n+1,0} \\
a_{n+2,1} &= -a_{n+2,0} \\
a_{n+1,n} y_n - 2y_{n+1} &= -a_{n+1,2} y_2 - \ldots - a_{a_{n+1,n-1}} y_{n-1} \\
a_{n+2,2} y_2 + a_{n+2,n} y_n + a_{n+2,n+1} y_{n+1} &= -a_{n+2,3} y_3 - \ldots - a_{n+2,n-1} y_{n-1}
\end{align*}
\]

(5)
where everything in the right-hand side of the equations above is determined by induction hypothesis. Since \( u_1 = -1, \ u_2 = 1, \ u_3 = -1 \), we have:

\[
\begin{align*}
\{ a_{n+1,1} &= u_{n+1} + \sum_{i=1}^{n} a_{0,i} a_{n+1-i} \\
\{ a_{n+2,1} &= u_{n+2} + a_{10} u_{n+1} + \left( \sum_{i=2}^{n+1} a_{i0} u_{n+1-i} \right) = u_{n+2} + 2u_{n+1} + \left( \sum_{i=2}^{n+1} a_{i0} u_{n+1-i} \right) \\
\{ a_{n+1,n} &= 2 - n \cdot u_1 = -(n - 2) \\
\{ a_{n+2,n} &= \sum_{i=1}^{n+1} a_{i1} u_{n+2-i} = -u_{n+1} + (\sum_{i=2}^{n} a_{i1} u_{n+2-i}) - a_{n+1,1} = -2u_{n+1} + (\sum_{i=2}^{n} a_{i1} u_{n+2-i}) - (\sum_{i=1}^{n} a_{i0} u_{n+1-i}) \\
\{ a_{n+2,n+1} &= (t^{n+2})((x^2 + y^2)u^n) = \left( t^2 \right) (x^2 + y^2) (\frac{x}{y})^n = \left( t^2 \right) ((1 + 2t + t^2)(-1 + t - t^2)^n) = \left( t^2 \right) (1 + 2t + t^2)(-1)^n + (-1)^{n-1}nt + (-1)^{n-2}n(n-1) + (-1)^n n) = (-1)^n + (-1)^{n-1}2n + ((-1)^{n-2}n(n-1) + (-1)^n n) = (-1)^n(n + 1) + (-1)^{n-1}2n + (-1)^{n-2}n(n-1) \\
\{ a_{n+2,n+1} &= -2 + (n + 1)u_1 = n - 1 \\
\end{align*}
\]

and so system \([5]\) can be written as:

\[
\begin{align*}
u_{n+1} &= \lambda_1 \\
u_{n+2} &= \lambda_2 \\
-(n - 2)y_n &= \lambda_3 \\
-2y_{2n+1} &= \lambda_4 \\
\end{align*}
\]

where \( \lambda_1, \lambda_2, \lambda_3, \lambda_4 \) are numbers already determined (they do not depend on \( u_{n+1}, u_{n+2}, y_n, y_{n+1} \)). There is a unique solution for \( u_{n+1}, u_{n+2} \) satisfying the first two equations. So, to ensure that the system above has a unique solution we only need to see that:

\[
\begin{vmatrix}
-(n-2) & \lambda_1 \\
(n-1)^n(n+1) + (-1)^{n-1}2n + (-1)^{n-2}n(n-1) & \lambda_2 \\
-2 & \lambda_3 \\
-2 & \lambda_4 \\
\end{vmatrix} = \frac{-2}{n-1}
\]

\[
= \begin{cases} 
-1 + 2(-1)^n \cdot n^2 + [3 + 4(-1)^{n-1}]n + [-2 + 2(-1)^n] \\
\end{cases} = \begin{cases} 
\frac{n^2 - n}{3n^2 + 7n - 4} & n \text{ even} \\
\text{odd} & n \text{ odd}
\end{cases}
\]

is not 0 for \( n \geq 2 \). That completes the proof of statement (a).

Statement (b), follows from analytic continuation. First, note that if \( c \) is analytic, we have an analytic parametrization \( \alpha(t) = (x(t), y(t)) \) near \( V \). We may assume that \( x(t) = 1 + t \) and so, according to (a), \( y(t) \) is totally determined. The Principle of Analytic Continuation ensures that the entire \( c \) is a circular arc.

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