Gravitation and Cosmology in Generalized
(1+1)-dimensional dilaton gravity

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Abstract

The actions of the “$R = T$” and string-inspired theories of gravity in (1+1) dimensions are generalized into one single action which is characterized by two functions. We discuss differing interpretations of the matter stress-energy tensor, and show how two such different interpretations can yield two different sets of field equations from this action. The weak-field approximation, post-Newtonian expansion, hydrostatic equilibrium state of star and two-dimensional cosmology are studied separately by using the two sets of field equations. Some properties in the “$R = T$” and string-inspired theories are shown to be generic in the theory induced by the generalized action.
1 Introduction

In (1+1)-dimensional spacetime, the metric tensor $g_{\mu\nu}$ has only three different components, two of which may be eliminated by a choice of coordinates. This fact reduces the complexity of metric-related computations substantially. Moreover, the field equations are expected to be easier to solve than those in ordinary (3+1)-dimensional spacetime because all fields depend on at most two variables.

As a result, theorists have found two-dimensional gravity an attractive theoretical laboratory for gaining insight into issues in semi-classical and quantum gravity [1], [2]. Despite its relatively easy computability, the two-dimensional setting suffers a serious drawback: the Einstein field equations are no longer a feasible model of the spacetime because the Einstein tensor is identically zero, yielding a theory without any dynamical content. In order to cope with this difficulty, several theories of gravity in two dimensions have been proposed by relativists. Recently, two such theories, referred to as “$R = T$” theory [3] and string-inspired theory [4], have attracted attention for a variety of reasons, including the fact that they have interesting classical limits [5, 6] and admit black hole solutions.

The “$R = T$” theory can be derived from the action

$$S = \int d^2x \sqrt{-g} \left[ \frac{1}{2} g^{\mu\nu} \nabla_\mu \psi \nabla_\nu \psi + \psi R + \mathcal{L}_M \right], \quad (1)$$

where the scalar field $\psi$ is an auxiliary field, so called because the classical field equations may be rewritten in a form that removes the $\psi$-dependence from the equations governing the evolution of the metric and matter fields. These are

$$R = 8\pi G g^{\mu\nu} T_{\mu\nu}$$

and the equation for the covariant conservation of the stress-energy tensor.

The string-inspired theory arises from a noncritical string theory in (1+1) dimensions. By setting the one-loop $\beta$ function of the bosonic $\sigma$ model with two target spacetime dimensions to zero, the effective target space action becomes

$$S = \int d^2x \sqrt{-g} e^{-2\phi} \left\{ 4 \nabla_\lambda \phi \nabla^\lambda \phi + R + J \right\} + \mathcal{L}_M \right\}, \quad (2)$$
where $\phi$ is the dilaton field and the field equations of this action are

$$J = -R - 4\nabla^2 \phi + 4 (\nabla \phi)^2,$$

$$8\pi G T_{\mu\nu} = e^{-2\phi} (R_{\mu\nu} + 2 \nabla_\mu \nabla_\nu \phi).$$

The actions (1) and (2) are particular types of dilaton theories of gravity, theories in which a scalar field (the dilaton) couples non-minimally to gravity via the Ricci scalar. These theories have a number of interesting properties, both in terms of their classical conservation laws [7] and their quantum structure [8]. However (except in a few special cases [5, 6]) their detailed classical properties have never been systematically analyzed. In this paper we undertake this task.

A wide class of dilation gravity theories can be obtained if we generalize the actions for the "$R = T$" and string-inspired theories into the action

$$S = S_g - S_s,$$  \hspace{1cm} (3)

where

$$S_g := \int d^2x \sqrt{-g} \left[ H(\Phi) (\nabla \Phi)^2 + D(\Phi) R \right] \text{ and } S_s := \int d^2x \sqrt{-g} V$$

because the functions $H(\Phi)$ and $D(\Phi)$ can be used to characterize the theory [4]. For example, in "$R = T$" theory, $H(\Phi) = 1/2$ and $D(\Phi) = \Phi$, and in the string-inspired theory, $H(\Phi) = 4 D(\Phi) = 4 \exp(-2 \Phi)$. The term $V$, which is interpreted as the potential density for the matter and dilaton sources, may depend on $\Phi$, derivatives of $\Phi$, the metric tensor and other matter field. The variations of (3) with respect to the contravariant metric $g^{\mu\nu}$ and the dilaton field $\Phi$ give

$$\frac{\delta S_s}{\sqrt{-g} \delta g^{\mu\nu}} = H(\Phi) \left[ \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} (\nabla \phi)^2 g^{\mu\nu} \right] - \nabla_\mu \nabla_\nu D(\Phi) + \nabla^2 D(\Phi) g^{\mu\nu},$$  \hspace{1cm} (4)

$$\frac{\delta S_s}{\sqrt{-g} \delta \Phi} = D'(\Phi) R - 2H(\Phi) \nabla^2 \Phi - H'(\Phi) (\nabla \phi)^2.$$  \hspace{1cm} (5)

Since the Ricci scalar is the only measure of curvature in $(1+1)$ dimensions, we will require it to appear in at least one field equation. Consequently we assume the following throughout the paper:

$$D'(\Phi) \neq 0 \quad \forall \quad \Phi$$  \hspace{1cm} (6)
but otherwise consider $D(\Phi)$ to be an arbitrary function of $\Phi$.

Our analysis of the classical properties of dilaton gravity will therefore be based upon the action (3), with $H$ and $D$ arbitrary functions of the dilaton field $\Phi$ (modulo (6)) and $V$ an arbitrary function of the dilaton field and any other matter fields $\Psi$ in the system. This is the most general action linear in the curvature and quadratic in the derivatives of $\Phi$ and the matter fields. The action (3) actually only depends upon the function $V(\Phi; \Psi)$ since reparametrizations of $\Phi$ accompanied by $\Phi$-dependent Weyl rescalings of the metric allow one to relate models with different $H$’s and $D$’s [7]. Only the overall sign and critical points of $H$ and $D$ contain reparametrization invariant information [8]. The matter potential $V$ breaks Weyl invariance, and so the field equations (4) and (5) determine the evolution of the spacetime metric and matter fields. General coordinate invariance implies that locally the evolution of the metric is determined by the evolution of its conformal factor.

This arbitrariness in the potential $V$ yields some ambiguity in terms of how the left-hand-sides of (4) and (5) are interpreted. For example, if the variations of (4) and (5) equal

$$\frac{\delta S_s}{\sqrt{-g} \delta g^{\mu \nu}} := 8 \pi G \tilde{T}_{\mu \nu}$$
and
$$\frac{\delta S_s}{\sqrt{-g} \delta \Phi} := \tilde{J},$$

we obtain the field equations

$$\tilde{J} = D'(\Phi) R - 2 H(\Phi) \nabla^2 \Phi - H'(\Phi) (\nabla \Phi)^2,$$

(7)

$$8 \pi G \tilde{T}_{\mu \nu} = H(\Phi) \left[ \nabla_\mu \Phi \nabla_\nu \Phi - \frac{1}{2} (\nabla \Phi)^2 g_{\mu \nu} \right] - \nabla_\mu \nabla_\nu D(\Phi) + \nabla^2 D(\Phi) g_{\mu \nu}.$$  

(8)

We shall refer to the system (7) and (8) as the type-I field equations, and interpret $\tilde{T}_{\mu \nu}$ and $\tilde{J}$ as the stress-energy-momentum tensor of the matter field and the source of dilaton respectively. The divergence of (8) reads

$$8 \pi G \nabla^\nu \tilde{T}_{\mu \nu} = - \frac{1}{2} \tilde{J} \nabla_\mu \Phi.$$  

(9)

Note that the dilaton source $\tilde{J}$ vanishes if the tensor $\tilde{T}_{\mu \nu}$ is covariantly conserved.
Alternatively, consider the restriction

\[ H(\Phi) = kD'(\Phi) , \]  

(10)

where \( k \) is a non-zero constant. In this case equations (4) and (5) can be rearranged as

\[
\frac{\delta S_s}{\sqrt{-g} \delta \Phi} = D'(\Phi) R - 2k D'(\Phi) \nabla^2 \Phi - k D''(\Phi) (\nabla \Phi)^2 ,
\]

\[
\frac{\delta S_s}{\sqrt{-g} \delta g_{\mu\nu}} + \frac{g_{\mu\nu}}{2k} \frac{\delta S_s}{\sqrt{-g} \delta \Phi} = \frac{1}{k} D'(\Phi) R_{\mu\nu} - D'(\Phi) \nabla_{\mu} \nabla_{\nu} \Phi \\
+ [k D'(\Phi) - D''(\Phi)] \left[ \nabla_{\mu} \Phi \nabla_{\nu} \Phi - \frac{1}{2} (\nabla \Phi)^2 g_{\mu\nu} \right] .
\]  

(11)

If the left side of (11) is defined to be \( 8\pi G \tilde{T}_{\mu\nu} \), the equation will yield

\[
8\pi G \nabla^{\nu} \tilde{T}_{\mu\nu} = \frac{1}{k} \nabla_{\mu} \left[ \frac{1}{2} \frac{\delta S_s}{\sqrt{-g} \delta \Phi} e^{-k \Phi} \right] e^{k \Phi} .
\]

This result suggests that the term inside the square brackets can be identified as the dilaton source \( \tilde{J} \) such that the divergence of the stress-energy-momentum tensor becomes simply

\[
8\pi G \nabla^{\nu} \tilde{T}_{\mu\nu} = \frac{1}{k} e^{k \Phi} \nabla_{\mu} \tilde{J} .
\]

(12)

Equation (12) implies that if the stress-energy-momentum tensor \( \tilde{T}_{\mu\nu} \) obeys the local conservation laws, the dilaton source \( \tilde{J} \) must be a constant which is not necessarily zero, in contrast to the previous case. Consequently this interpretation, along with the restriction (10), yields another set of field equations as the following:

\[
\tilde{J} = \frac{1}{2} e^{-k \Phi} \left[ D'(\Phi) R - 2k D'(\Phi) \nabla^2 \Phi - k D''(\Phi) (\nabla \Phi)^2 \right] ,
\]

(13)

\[
8\pi G \tilde{T}_{\mu\nu} = \frac{1}{k} D'(\Phi) R_{\mu\nu} - D'(\Phi) \nabla_{\mu} \nabla_{\nu} \Phi \\
+ \left[ k D'(\Phi) - D''(\Phi) \right] \left[ \nabla_{\mu} \Phi \nabla_{\nu} \Phi - \frac{1}{2} (\nabla \Phi)^2 g_{\mu\nu} \right] .
\]

(14)
These new equations will henceforth be denoted as type-II field equations. As the proportionality constant $k$ in condition (10) is non-zero, the type-II field equations can be used only when $H \neq 0$. If $H$ is zero, only type-I field equations are available. It is worthwhile pointing out that the type-II field equations include not only the “$R = T$” theory but also the string-inspired theory as well. The ambiguity giving rise to the two types of field equations originates from lack of knowledge of the potential density $V$. Clearly many other interpretations of the metric variation of the matter action $S_s$ in terms of the stress-energy and dilation currents are possible. However, in the absence of additional information on $D$, $H$ and $V$, the type-I and type-II equations cover a large class of matter couplings (including both the $R = T$ and string-theoretic cases) and we shall henceforth consider only these two choices.

The outline of our paper is as follows. We first consider the weak-field approximations and post-Newtonian expansions of the field equations in sections 2 and 3 separately. In section 4, the interior structure of a ‘star’ in (1+1)-dimensional spacetime will be investigated. Cosmological solutions derived from the field equations will be considered in section 5 for dust-filled, radiation-filled and inflationary universes. Finally some conclusions will be drawn in section 6.

A given choice of the characteristic functions $D$, $H$, and $V$ corresponds to a given theory. All computations are carried out without imposing any restrictions upon these functions except for the constraint (6) (which guarantees that the metric has non-trivial evolution equations). Of course the freedom of choosing the characteristic functions permits the formulation of inverse problems. For example, what choice of characteristic functions is required for a desired spacetime evolution? An example of inverse problem will be shown in section 5.3.

2 Weak Field Approximation

We consider here a weak-field expansion of the metric about some fixed background for the type-I and type-II systems separately.
2.1 Type I Equations

Consider a weak-field expansion about some background metric. Under a conformal transformation

\[ g_{\mu\nu} = \exp\left( - \int \frac{H(u)}{D'(u)} \right) \hat{g}_{\mu\nu} , \]

the field equations (7) and (8) can be transformed into

\[ \tilde{J} = \exp\left( \int \frac{H(u)}{D'(u)} \right) \left[ D'\Phi \hat{R} - \frac{H(\Phi)}{D'(\Phi)} \hat{\nabla}^2 D(\Phi) \right] , \] (15)

\[ 8\pi G \hat{T}_{\mu\nu} = \hat{\nabla}^2 D(\Phi) \hat{g}_{\mu\nu} - \hat{\nabla}_\mu \hat{\nabla}_\nu D(\Phi) , \] (16)

where every term with the hat is interpreted with respect to the conformal metric \( \hat{g}_{\mu\nu} \). In carrying out the weak field calculation, it will be easier to use the transformed field equations (15) and (16) than to use the original equations (7) and (8) because the \( H \)-dependence in (8) is transformed away. This property of \( H \)-independence in (16) renders the vacuum calculation tractable without knowing any properties of the function \( D \).

Now we let the dilaton source \( \tilde{J} \) be

\[ \tilde{J} = \epsilon c + \mathcal{J} , \] (17)

where \( c \) is a non-zero constant and \( \epsilon \) is a parameter which may either be 0 or 1, depending on the choice of dilaton vacuum. As the term \( \mathcal{J} \) represents the non-vacuum part of \( \tilde{J} \), equation (17) indeed splits the dilaton source into “vacuum” and “non-vacuum” parts. Similarly, we expand the fields \( \hat{g}_{\mu\nu} \) and \( \Phi \) via

\[ \hat{g}_{\mu\nu} = \eta_{\mu\nu} + \hat{h}_{\mu\nu} \quad \text{and} \quad \Phi = \phi + \varphi , \]

where \( \hat{h}_{\mu\nu} \) and \( \varphi \) are perturbations of the fields from the vacuum fields \( \eta_{\mu\nu} \) and \( \phi \). As is standard in perturbation theory, we take \( |\hat{h}_{\mu\nu}| << 1 \) and \( |\varphi| << 1 \).

Since the flat metric \( \eta_{\mu\nu} \) and \( \phi \) are supposed to be the solutions in vacuum, they satisfy the equations

\[ \epsilon c = - \frac{H(\phi)}{D'(\phi)} \square D(\phi) \exp\left( \int \frac{H(u)}{D'(u)} \right) , \] (18)
\[ D(\phi) \eta_{\mu\nu} = \partial_{\mu} \partial_{\nu} D(\phi) \] (19)

where \( \Box \equiv \eta_{\mu\nu} \partial_{\mu} \partial_{\nu} \).

Equation (19) has the solution \( D(\phi) = At + Bx + C \), where \( A, B \) and \( C \) are integration constants. If we require \( D(\phi) = \text{constant} \) and set this constant to be \( D(0) \), it implies \( \phi = 0 \) such that the field equations (15) and (16) simply become

\[ 8\pi G \tilde{T}_{\mu\nu} = \hat{\nabla}^2 D(\varphi) \eta_{\mu\nu} - \hat{\nabla}_\mu \hat{\nabla}_\nu D(\varphi) \] (21)

If equations (20) and (21) are expanded about \( \varphi = 0 \) and \( \hat{g}_{\mu\nu} = \eta_{\mu\nu} \), the linear parts of the expansions are

\[ \mathcal{J} = \exp \left( \int^\varphi \frac{H(u)}{D'(u)} \, du \right) \left[ D'(\varphi) \hat{R} - \frac{H(\varphi)}{D'(\varphi)} \hat{\nabla}^2 D(\varphi) \right] \] (20)

\[ 8\pi G \tilde{T}_{\mu\nu} = \hat{\nabla}^2 D(\varphi) \eta_{\mu\nu} - \hat{\nabla}_\mu \hat{\nabla}_\nu D(\varphi) \] (21)

where the harmonic coordinate conditions

\[ g^{\mu\nu} \Gamma^\lambda_{\mu\nu} = \exp \left( \int^\Phi \frac{H(u)}{D'(u)} \, du \right) \left[ \eta^{\mu\nu} \eta^{\lambda\rho} \hat{\nabla}_{\mu\rho} + O(\hat{h}^2_{\mu\nu}) \right] = 0 \] (24)

are used when \( \hat{R} \) is expanded. In equation (22), the term \( \hat{h} \) is defined to be \( \hat{h} = \eta_{\mu\nu} \hat{h}_{\mu\nu} \). If equation (23) is contracted with the inverse flat metric \( \eta_{\mu\nu} \), the resultant scalar equation will be

\[ 8\pi G \left( \tilde{T}_{11} - \tilde{T}_{00} \right) = D'(0) \Box \varphi \] (25)

which has a general solution

\[ \varphi(t, x) = A_{\pm}(t \pm x) \pm \frac{4\pi G}{D'(0)} \int_{-\infty}^{\infty} dv \int_{-\infty}^{\infty} du \tilde{T}_{00}(v \pm |x - u|, u) \]
\[ \pm \frac{4\pi G}{D'(0)} \int_{-\infty}^{\infty} dv \int_{-\infty}^{\infty} du \tilde{T}_{11}(v \pm |x - u|, u) \] (26)

\(^1\text{This procedure only simplifies the forthcoming calculation but it is not necessary. Similar calculations without making use of this procedure will be done in the type-II case.}\)
where \(A_{\pm}\) are two arbitrary functions. On the other hand, by using (24), equation (22) implies that \(\hat{h}\) is given by

\[
\hat{h}(t,x) = B_{\pm}(t \pm x) \pm \frac{8 \pi G H(0)}{D'(0)^2} \int_{-\infty}^{\infty} du \int_{-\infty}^{t} dv \tilde{T}_{00}(v \pm |x - u|, u) \\
\pm \frac{8 \pi G H(0)}{D'(0)^2} \int_{-\infty}^{\infty} du \int_{-\infty}^{t} dv \tilde{T}_{11}(v \pm |x - u|, u) \\
\pm \frac{1}{D'(0)} \int_{-\infty}^{\infty} du \int_{-\infty}^{t} dv J(v \pm |x - u|, u),
\]

(27)

where \(B_{\pm}\) are another two arbitrary functions. The components of \(\hat{h}_{\mu\nu}\) can be expressed in terms of \(\hat{h}\) because the harmonic coordinate condition (24) is assumed. As a result, the original metric \(g_{\mu\nu}\) can be computed.

If equation (23) is expanded component-wise, we find that equation (26) is a general solution only when

\[
\partial_t \tilde{T}_{00} = \partial_x \tilde{T}_{10}, \quad \text{and} \quad \partial_t \tilde{T}_{10} = \partial_x \tilde{T}_{11}
\]

(28)

hold. However, we always have these conditions because one can show that (28) is just the first order expansion of the conservation laws (9). Note that the solution (27) forces \(c\epsilon = 0\) (as is easily seen by inserting the trace of (19) into (18)) regardless of whether the stress-energy-momentum tensor obeys the local conservation laws or not.

### 2.2 Type II Equations

We consider here the field equations (13) and (14). It is still convenient to perform the conformal transformation before carrying out the weak-field expansion, so we take

\[
g_{\mu\nu} = e^{-k\Phi} \hat{g}_{\mu\nu},
\]

and the field equations (13) and (14) give

\[
2 \tilde{J} = D'(\Phi) \hat{R} - k \hat{\nabla}^2 D(\Phi),
\]

(29)

\[
8 \pi G \hat{T}_{\mu\nu} = \frac{1}{k} D'(\Phi) \hat{R}_{\mu\nu} - \hat{\nabla}_\mu \hat{\nabla}_\nu D(\Phi) + \frac{1}{2} \hat{\nabla}^2 D(\Phi) \hat{g}_{\mu\nu}.
\]

(30)
Furthermore, the dilaton source $\mathcal{J}$, conformal metric $\hat{g}_{\mu\nu}$ and characteristic function $D(\Phi)$ are taken to have the form

$$\mathcal{J} = \epsilon c + J, \quad \hat{g}_{\mu\nu} = \eta_{\mu\nu} + \hat{h}_{\mu\nu}, \quad D(\Phi) = \sigma_o + \sigma,$$

where $\sigma_o$ and $\eta_{\mu\nu}$ are the vacuum solutions of (29) and (30). Therefore $\sigma_o$ has a general solution

$$\sigma_o(t, x) = \frac{\epsilon c}{2k} \left( t^2 - x^2 \right) + At + Bx + C,$$

where $A$, $B$ and $C$ are constants.

Expanding equations (29) and (30) about $D(\Phi) = \sigma_o$ and $\hat{g}_{\mu\nu} = \eta_{\mu\nu}$, and ignoring the non-linear and mixed terms of $\sigma$ and $\hat{h}_{\mu\nu}$, the two equations become

$$-2J = k \Box \sigma + k \hat{h}^{\mu\nu} \partial_\mu \partial_\nu \sigma + \frac{1}{2} D'(D^{-1}(\sigma_o)) \Box \hat{h},$$

$$8\pi G \hat{T}_{\mu\nu} = \frac{1}{4k} D'(D^{-1}(\sigma_o)) \Box \tilde{h}_{\mu\nu} \eta_{\mu\nu} - \partial_\mu \partial_\nu \sigma + \partial_\lambda \sigma_o \hat{\Gamma}_{\mu\nu}^{\rho} \eta^{\lambda\rho} + \frac{1}{2} \hat{h}_{\lambda\rho} \partial_\lambda \partial_\rho \sigma_o \eta_{\mu\nu} + \frac{1}{2} \Box \sigma_o \hat{h}_{\mu\nu} + \frac{1}{2} \Box \sigma \eta_{\mu\nu},$$

where the vacuum solutions are used and $\hat{h}_{\lambda\rho}$ is the first order term in the expansion of $\tilde{g}^{\lambda\rho}$ with respect to $\hat{h}_{\mu\nu}$. As a useful tool for simplifying the calculation, harmonic coordinate conditions (24) have again been used. Since we can write

$$\partial_\mu \partial_\nu \sigma_o = - \frac{\epsilon c}{k} \eta_{\mu\nu} \quad \text{and} \quad \hat{h}^{\mu\nu} = - \eta^{\mu\alpha} \eta^{\nu\beta} \hat{h}_{\alpha\beta},$$

equations (32) and (33) have the general solution

$$\hat{h}(t, x) = E_{\pm}(t \pm x) \pm 8\pi G k \int_{-\infty}^{\infty} du \int_{-\infty}^{t} dv \frac{\hat{T}_{00}(v \pm |x-u|, u)}{D'(D^{-1}(\sigma_o(v \pm |x-u|, u)))},$$

$$\mp 8\pi G k \int_{-\infty}^{\infty} du \int_{t}^{\infty} dv \frac{\hat{T}_{11}(v \pm |x-u|, u)}{D'(D^{-1}(\sigma_o(v \pm |x-u|, u)))},$$

$$\sigma(t, x) = F_{\pm}(t \pm x) \mp 4\pi G \int_{-\infty}^{\infty} du \int_{-\infty}^{t} dv \hat{T}_{00}(v \pm |x-u|, u)$$

$$\pm 4\pi G \int_{-\infty}^{\infty} du \int_{t}^{\infty} dv \hat{T}_{11}(v \pm |x-u|, u)$$
\[ \mp \frac{1}{k} \int_{-\infty}^{\infty} du \int_{-\infty}^{t} dv \mathcal{J}(v \pm |x-u|, u) \]
\[ \mp \frac{e \epsilon}{2k} \int_{-\infty}^{\infty} du \int_{-\infty}^{t} dv \hat{h}(v \pm |x-u|, u) , \]  

where \( E_\pm \) and \( F_\pm \) are four arbitrary functions. By using the harmonic coordinate conditions (24), equations (31), (34) and (35), the components of the original metric \( g_{\mu \nu} \) can be found.

Similar to the case we encountered in Section 2.1, the solutions (34) and (35) only satisfy the tensorial field equation (33) when the (weak-field) covariant conservation law (12) is satisfied.

### 3 Post-Newtonian Approximation

Consider a system of particles that are bound together by their mutual gravitational attraction. Let \( \bar{M}, \bar{r} \) and \( \bar{v} \) be the typical values of the masses, separations and velocities of these particles. Since the typical kinetic energy \( \bar{M} \bar{v}^2 / 2 \) is roughly of the same order of magnitude as the typical gravitational potential energy \( G \bar{M}^2 \bar{r} \) in Newtonian mechanics, we have \( \bar{v}^2 \sim G \bar{M} \bar{r} \).

If a Schwarzschild-like coordinates are employed, \( g_{\mu \nu} \) can be expressed as

\[
\begin{align*}
    g_{00} &= \frac{-1}{g_{11}} = -1 + \frac{2}{\bar{r}} g_{00} + \frac{4}{\bar{r}^2} g_{00} + \cdots \\
    g_{10} &= g_{10} + \frac{5}{\bar{r}^3} g_{10} + \cdots ,
\end{align*}
\]

where \( \tilde{g}_{\mu \nu} \) is of order \( \bar{v}^N \). Now if we write \( \Phi(t, x) \) as \( \Phi = \Phi + \tilde{\Phi} + \hat{\Phi} + \cdots \), where \( \Phi \) is of order \( \bar{v}^N \), the geometric parts of the field equations \( \mathcal{J} \) and \( \mathcal{T}_{\mu \nu} \) (i.e. the right sides of either the pair (7) and (8) of type-I equations or (13) and (14) of the type-II equations) can be expanded as

\[
\mathcal{J} = \mathcal{J}^0 + \mathcal{J}^1 + \mathcal{J}^2 + \cdots \quad \text{and} \quad \mathcal{T}_{\mu \nu} = \mathcal{T}_{\mu \nu}^0 + \mathcal{T}_{\mu \nu}^1 + \mathcal{T}_{\mu \nu}^2 + \cdots ,
\]

where the terms \( \mathcal{J}^N \) and \( \mathcal{T}_{\mu \nu}^N \) are of order \( \bar{v}^N / \bar{r}^2 \).

Simple arguments based on time-reversal invariance \( \mathcal{J}, \mathcal{T}_{\mu \nu}, \mathcal{J} \), however, imply that the stress-energy-momentum tensor \( T_{\mu \nu} \) and the dilaton source \( J \)

\[ \text{All the expansions in this section are done by using Maple V.} \]
have the expansions

\[ J = J^0 + J^1 + \cdots, \]
\[ T_{00} = T_{00}^0 + T_{00}^1 + \cdots, \quad T_{10} = T_{10}^1 + \cdots \quad \text{and} \quad T_{11} = T_{11}^1 + \cdots, \]

where \( J^\nu \) has an order \( \tilde{v}^N/\tilde{r}^2 \) and \( T^\mu_{\nu} \) denotes the term of order \( \tilde{v}^{N+2}/\tilde{r}^2 \). Therefore we obtain the following relationships by matching the order of individual terms:

\[ J^0 = J^1, \quad T_{00}^0 = T_{11}^1 = 0, \quad T_{00}^1 = 8 \pi G T_{00}^0, \quad \cdots \]

and so on.

### 3.1 Type-I Equations

Let us consider the zeroth order expansion of \( J \) and \( T_{\mu\nu} \) of the type-I field equations first. Expanding (7) and (8) we obtain

\[ J^0 = -2H(\Phi^0)\partial_{xx} \Phi^0 - H'(\Phi^0)\left(\partial_x \Phi^0\right)^2, \]
\[ T_{00}^0 = \frac{1}{2}H(\Phi^0)\left(\partial_x \Phi^0\right)^2 - D'(\Phi^0)\partial_{xx} \Phi^0 - D''(\Phi^0)\left(\partial_x \Phi^0\right)^2, \quad (36) \]
\[ T_{10}^0 = 0, \]
\[ T_{11}^0 = \frac{1}{2}H(\Phi^0)\left(\partial_x \Phi^0\right)^2. \quad (37) \]

Since \( T_{11}^0 \) is zero, there are two possible cases:

I \quad \partial_x \Phi^0 = 0,

II \quad \partial_x \Phi^0 \neq 0 \quad \text{and} \quad H(\Phi^0(t,x)) = 0.

In both cases, \( J \) vanishes and so it is necessary for \( J^0 \) to be zero. In the next two parts, we shall consider these two cases separately.
3.1.1 Case I: \( \partial_x \Phi = 0 \)

After we impose the condition \( \partial_x \Phi = 0 \) on the first order expansions, the non-trivial expansions are

\[
\frac{\hat{J}}{J} = -2H(\Phi)\partial_{xx} \Phi \quad \text{and} \quad \frac{\hat{T}_{00}}{T} = -D'(\Phi)\partial_{xx} \Phi .
\]

As \( \frac{\hat{J}}{J} \) and \( \frac{\hat{T}_{00}}{T} \) must be zero, we conclude that

\[
\Phi(t,x) = \Phi_a(t)x .
\]

If we now expand to second order with the aid of the restrictions \( \partial_x \Phi = 0 \) and \( \partial_{xx} \Phi = 0 \), we find

\[
\frac{\hat{J}}{J} = 2H(\Phi)\left(\partial_{tt} \Phi - \partial_{xx} \Phi\right) + H'(\Phi)\left[(\partial_t \Phi)^2 - (\partial_x \Phi)^2\right] + D'(\Phi)\partial_{xx} \Phi \quad \text{and} \quad \frac{\hat{T}_{00}}{T} = -D'(\Phi)\partial_{xx} \Phi .
\]

(38)

Equations (38) and (39) can be integrated and give

\[
\Phi_a(t) = A \exp\left(2\Lambda(\Phi(t))\right) ,
\]

(42)

\[
\frac{d}{dt} D(\Phi(t)) = \pm \sqrt{A^2 \exp\left(4\Lambda(\Phi(t))\right) + B \exp\left(2\Lambda(\Phi(t))\right)} .
\]

(43)

if \( \partial_x \Phi \neq 0 \), where \( A \) and \( B \) are integration constants and \( \Lambda \) is defined as

\[
\Lambda(\Phi) := \frac{1}{2} \int \frac{H(u)}{D'(u)} du .
\]

(44)

Here the term \( D'(\Phi(t)) \) denotes the derivative of the function \( D(\zeta) \) with respect to its argument \( \zeta \) and is evaluated at \( \zeta = \Phi(t) \). Thus a term like
\( (1/D')'(\Phi^o) \), which we shall encounter later, denotes \(-D''(\Phi^o)/[D'(\Phi^o)]^2 \). Therefore if \( D \) and \( H \) and the initial conditions are given, \( \Phi^o \) can be solved by using the differential equation (13). Once \( \Phi^o \) is known, \( \Phi^1 \) can be found explicitly by using equation (12). However, it is not reasonable for \( \Phi^o \) and \( \Phi^1 \) depend solely on \( D \) and \( H \) rather than the source. That is to say even if the spacetime is sourceless, we shall obtain non-static \( \Phi^o \) and \( \Phi^1 \). Thus we restrict ourselves to the case

\[
\Phi^o = \Phi^1 = \text{constant} \quad \text{and} \quad \Phi^1(t) = 0 .
\]

After some manipulation of equations (38) and (39), \( \partial x x 2 \Phi \) and \( \partial x x g_{00} \) can be written down explicitly and they yield

\[
\begin{align*}
\Phi(t, x) &= -\frac{4 \pi G}{D'(\Phi^o)} \int_{-\infty}^{\infty} |x-u| \ T_{00}(t,u) \ du , \\
g_{00}(t, x) &= \frac{1}{2 \ D'(\Phi^o)} \int_{-\infty}^{\infty} |x-u| \ J(t,u) \ du \\
&\quad - 8 \pi G \ \frac{H(\Phi^o)}{[D'(\Phi^o)]^2} \int_{-\infty}^{\infty} |x-u| \ T_{00}(t,u) \ du .
\end{align*}
\]

Straightly speaking, there are arbitrary integration functions of \( t \) in the solutions above ; adopting static boundary conditions at large \( x \) renders them to be constants. Moreover, those integration terms linear in \( x \), which come from integration with respect to \( x \) twice, are ignored because when the spacetime is sourceless, such term with constant coefficients must be zero in order to achieve vacuum state.

The non-trivial third order field equations give

\[
- \ D'(\Phi^o) \partial x x \Phi(t, x) = 8 \pi G \ \tilde{T}_{10}(t, x) , \\
\partial x x x \Phi(t, x) = 0 .
\]

It is obvious that the last equation implies \( \tilde{T}(t, x) = 0 \) by the aforementioned boundary conditions and the first one is compatible with the \( t \)-component of the third order expansion of the conservation law

\[
\partial x \tilde{T}_{10}(t, x) = \partial_t \tilde{T}_{00}(t, x) .
\]
In the $n$th order expansions, the highest order second derivative terms will be $\partial_{xx}^n \Phi$, $\partial_{xx} g_{00}$, $\partial_{tx}^{n-1} \Phi$, $\partial_{tx} g_{10}$, $\partial_{tt}^{n-2} \Phi$ and $\partial_{tt} g_{00}$ if $n$ is even. When $n$ is odd, the derivatives are $\partial_{xx}^n \Phi$, $\partial_{xx} g_{10}$, $\partial_{tx}^{n-1} \Phi$, $\partial_{tx} g_{00}$, $\partial_{tt}^{n-2} \Phi$ and $\partial_{tt} g_{10}$. In either case, there are, at most, six unknown derivatives but only four equations from the field equations are available. As a result, we have to fix the gauge freedom in order to obtain a closed system. In our case, the metric $g_{\mu\nu}$ depends on two functions: $g_{00}$ and $g_{10}$. As a result, there is one more degree of freedom in the coordinate choice. If the $t$-component of the harmonic coordinate conditions is used here, that is to say

\[
\eta^{\mu\nu} \Gamma_{\mu\nu}^0 = \left[ \partial_t \dot{g}_{00} - \partial_x \dot{g}_{10} \right] + O(\bar{v}^5) = 0 \, ,
\]

there will be two more equations to describe the second derivatives of the metric because the harmonic coordinate conditions involve only first derivatives of the metric. Therefore after the use of the gauge above, one can find that the fourth order expansions of the $t$-x-component of the metric field equation is trivial but the dilaton and $t$-t-component of the metric equation give

\[
\partial_{xx} g_{00}(t, x) = \partial_{tt} \dot{g}_{00}(t, x) + \left( \frac{1}{D'} \right)'(\Phi_o) \frac{\dot{\Phi}(t, x)}{D'(\Phi_o)} \frac{\ddot{\Phi}(t, x)}{D'(\Phi_o)} + \frac{4}{D'(\Phi_o)} \left[ \frac{2}{D'(\Phi_o)} \right]^2 \left[ \dddot{T}_{11}(t, x) - \dddot{T}_{00}(t, x) - \dddot{g}_{00}(t, x) \dddot{T}_{00}(t, x) \right] - 16 \pi G \left( \frac{H}{D'[\Phi_o]^2} \right)'(\Phi_o) \frac{\dot{\Phi}(t, x)}{D'(\Phi_o)} \dot{T}_{00}(t, x) + \frac{H'(\Phi_o)D'(\Phi_o)}{D'[\Phi_o]^2} \left[ \frac{\dot{\Phi}(t, x)}{D'(\Phi_o)} \right]^2 \left[ \partial_x \dddot{\Phi}(t, x) \right]^2 ,
\]

\[
\partial_{xx} \Phi(t, x) = - \frac{8 \pi G}{D'(\Phi_o)} \dddot{T}_{00}(t, x) - \frac{16 \pi G}{D'(\Phi_o)} \frac{\dddot{g}_{00}(t, x)}{D'(\Phi_o)} \dddot{T}_{00}(t, x) + \frac{8 \pi G}{D'}(\Phi_o) \frac{\dot{\Phi}(t, x)}{D'(\Phi_o)} \dot{T}_{00}(t, x)
\]
\[ + \frac{H(\Phi_0) - 2 D'(\Phi_0)}{2 D'(\Phi_0)} \left[ \partial_x \frac{\Phi(t, x)}{x} \right]^2 \]
\[ + \frac{1}{2} \partial_x \frac{\Phi(t, x)}{x} \partial_x \frac{\Phi(t, x)}{x}. \]

Equations (45) and (46) can be used to find \( \dot{g}_{00}(t, x) \) and \( \Phi(t, x) \). There is one more equation from the expansions but one can show that it is equivalent to the \( x \)-component of the conservation laws.

3.1.2 Case II: \( \partial_x \Phi \neq 0 \)

When this is the case, \( H(\Phi) \) and \( \partial_{xx} D(\Phi) \) must be zero according to (36) and (37). Since \( H(\Phi) = 0 \), all the spatial derivatives of \( H(\Phi) \) must be zero and so \( H^{(n)}(\Phi) = 0 \) for all \( n \) because \( \partial_x \Phi \) is non-zero. Thus, the \( H \)-dependence of the expansions is gone because \( H(\Phi) \) is Taylor expanded about \( \Phi = \Phi_0 \).

The nontrivial first order expansions will be as follows if the restrictions above are imposed:

\[ T_{00}' = -\partial_{xx} \left[ \frac{\Phi}{x} D'(\Phi) \right] \quad \text{and} \quad T_{10}' = -\partial_{tx} D(\Phi). \]

Since \( T_{10}' = 0 \), the term \( \partial_x D(\Phi) \) has a constant value \( A \) because \( \partial_{xx} D(\Phi) = 0 \) from the zeroth order expansion. Therefore we have

\[ D(\Phi(t, x)) = Ax + A_a(t), \quad (47) \]

where \( A_a(t) \) is an arbitrary function of \( t \).

If the previous restrictions are imposed on the second order expansions, the expansions will give

\[ \mathcal{J} = \partial_{xx} \dot{g}_{00} D'(\Phi), \]
\[ \mathcal{T}_{00} = \frac{1}{2} A \partial_x \dot{g}_{00} - \partial_{xx} \left[ D'(\Phi) \frac{\Phi}{x} \right] - \frac{1}{2} \partial_{xx} \left[ D''(\Phi) \frac{\Phi^2}{x^2} \right], \]
\[ \mathcal{T}_{10} = -\partial_{tx} \left[ \frac{\Phi}{x} D'(\Phi) \right], \]
\[ \mathcal{T}_{11} = -\frac{A}{2} \partial_x \dot{g}_{00} - A_a''(t). \]

(48)
Because $\mathcal{T}_{00} = \mathcal{T}_{10} = 0$, $\partial_x [\phi^I D'(\Phi)]$ is constant and we may write

$$\phi(t, x) = \frac{[B_1 x + B_2]}{D'(\Phi(t, x))},$$

where $B_1$ and $B_2$ are constants of integration.

As $\partial_x \Phi \neq 0$, the constant $A$ in (47) cannot be zero. Thus (48) implies

$$\mathcal{J}_0(t, x) = -\frac{2}{A} \mathcal{J}_a(t) x + \mathcal{J}_{00b}$$

because $\mathcal{T}_{11} = 0$. As a result, we are able to conclude that $\mathcal{J} = 0$. In other words, when $\mathcal{J} \neq 0$, one is limited to the case $\partial_x \Phi(t, x) = 0$ as the only possible choice.

Now if we use the $x$-component of the harmonic gauge $\partial_x \mathcal{J}_0 = 0$, the second derivative of $A_a(t)$ can be removed. In other words, we have $A_a(t) = A_1 t + A_2$ where $A_1$ and $A_2$ are some constants and we obtain

$$\phi(t, x) = D^{-1}(A x + A_1 t + A_2),$$

$$\mathcal{J}_0(t, x) = -\frac{D''(\Phi(t, x))}{2 D'(\Phi(t, x))} \Phi(t, x) + \mathcal{J}_a x + \mathcal{J}_b$$

$$= \frac{4 \pi G}{D'(\Phi(t, x))} \int_{-\infty}^{\infty} |x - u| \mathcal{T}_{00}(t, u) du,$$

$$\mathcal{J}_{00}(t, x) = \mathcal{J}_{00b}.$$

The aforementioned choice of static boundary conditions for large $x$ will set $A_1 = 0$.

Higher order expansions will not be done here because this case, which is equivalent to the requirement $H = 0$, is not really interesting at all. (For example, the forthcoming calculation shows that $H = 0$ is not a proper model to describe a “star” in $(1 + 1)$ dimensions.

### 3.2 Type II Equations

If the type-II field equations (13) and (14) are expanded as before, the zeroth order expansions of the right sides of them become

$$\mathcal{J}_0 = -\frac{k}{2} \exp \left(-k \Phi \right) \left[ 2 D'(\Phi) \partial_x \Phi + D''(\Phi) \left( \partial_x \Phi \right)^2 \right],$$

(49)
\[
\begin{align*}
\mathcal{T}_{00} &= \frac{1}{2} \left[ k D'(\Phi) - D''(\Phi) \right] \left( \partial_x \Phi \right)^2, \\
\mathcal{T}_{10} &= 0, \\
\mathcal{T}_{11} &= -D'(\Phi) \partial_{xx} \Phi + \frac{1}{2} \left[ k D'(\Phi) - D''(\Phi) \right] \left( \partial_x \Phi \right)^2.
\end{align*}
\] (50)

As we require \( \mathcal{T}_{00} \) and \( \mathcal{T}_{11} \) be zero, this devolves into two cases:

I: \( \Phi(t,x) = \Phi(t) \)

II: \( \partial_x \Phi(t,x) \neq 0 \), \( k D'(\Phi) = D''(\Phi) \) and \( \partial_{xx} \Phi(t,x) = 0 \).

Let us consider these two possibilities separately.

### 3.2.1 Case I: \( \partial_x \Phi(t,x) = 0 \)

Because \( \Phi(t,x) = \Phi(t) \), the right sides of equations (49) to (50) will be zero and the zeroth order of the dilaton source \( J \) must be zero again in this case. Choosing boundary conditions so that \( \Phi \) is static at large \( x \) gives

\[
\Phi(t,x) = \Phi_o.
\]

If we consider the first order expansions, the non-trivial expansions will be

\[
0 = -k \exp \left( -k \Phi_o \right) D'(\Phi_o) \partial_{xx} \Phi \quad \text{and} \quad 0 = -D'(\Phi_o) \partial_{xx} \Phi
\]

because \( \mathcal{J} \) and \( \mathcal{T}_{11} \) must vanish. Since \( D' \) is assumed to be non-zero, we obtain

\[
\Phi(t,x) = \Phi_o x.
\]

As a result, the non-trivial second order field equations imply

\[
\mathcal{J} = -\frac{k^2}{2} D'(\Phi_o) \Phi_o^2 \exp \left( -k \Phi_o \right).
\]
\[ \partial_{xx} g_{00} = k \left[ k - \frac{D''(\Phi_o)}{D'(\Phi_o)} \right] \Phi_a - \frac{16 \pi G k}{D'(\Phi_o)} \tilde{T}_{00}(t, x), \]
\[ \partial_{xx} \Phi = - \frac{\tilde{J}\exp\left(k \Phi_o\right)}{k D'(\Phi_o)} + \left[ \frac{k}{2} - \frac{D''(\Phi_o)}{D'(\Phi_o)} \right] \Phi_a - \frac{8 \pi G}{D'(\Phi_o)} \tilde{T}_{00}(t, x). \]

because \( \tilde{T}_{11} = 0 \), \( \tilde{J} = \tilde{J} \) and \( \tilde{T}_{00} = 8 \pi G \tilde{T}_{00} \). Notice that in this case, the source \( \tilde{J} \) must be a constant. By using these results, we found that the third order expansions simply imply

\[ \frac{1}{2} \Phi_a = 0, \]
\[ \frac{3}{2} \Phi(t, x) = \frac{3}{2} \Phi_o x, \]
\[ 8 \pi G \frac{1}{2} \tilde{T}_{10}(t, x) = - D'(\Phi_o) \partial_{tx} \Phi(t, x). \]

The last equation yields nothing new because it is compatible with the \( t \)-component of the conservation laws (12). As \( \frac{1}{2} \Phi_a = 0 \), we have

\[ \tilde{J} = 0, \]
\[ \tilde{g}_{00}(t, x) = - \frac{8 \pi G k}{D'(\Phi_o)} \int_{-\infty}^{\infty} |x - u| \tilde{T}_{00}(t, u) du, \]
\[ \tilde{\Phi}(t, x) = - \frac{4 \pi G}{D'(\Phi_o)} \int_{-\infty}^{\infty} |x - u| \tilde{T}_{00}(t, u) du. \]

Finally, by using the harmonic gauge \( \partial_t \tilde{g}_{00} = \partial_x \tilde{g}_{10} \), one can show that in the fourth order expansions, the \( t-x \)-component of metric equation is trivial, the dilaton equation is compatible with the \( x \)-component of the conservation laws and the other two expansions give

\[ \partial_{xx} \tilde{g}_{00}(t, x) = \partial_{tt} \tilde{g}_{00}(t, x) - 2 \frac{\exp\left(k \Phi_o\right)}{D'(\Phi_o)} \tilde{J}(t, x) \]
\[ + \frac{16 \pi G k}{D'(\Phi_o)} \left[ \tilde{T}_{11}(t, x) - \tilde{T}_{00}(t, x) - \tilde{g}_{00}(t, x) \tilde{T}_{00}(t, x) \right] \]

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\[ + 16 \pi G k \frac{D''(\Phi_o)}{D'(\Phi_o)^2} \dot{\Phi}(t, x) \ddot{T}_{00}(t, x) \]

\[ - \frac{k D''(\Phi_o)}{D'(\Phi_o)} \left[ \partial_x \ddot{\Phi}(t, x) \right]^2 , \]

\[ \partial_{xx} \dddot{\Phi}(t, x) = - \frac{8 \pi G}{D'(\Phi_o)} \dddot{T}_{00}(t, x) - \frac{16 \pi G}{D'(\Phi_o)} \dddot{g}_{00}(t, x) \ddot{T}_{00}(t, x) \]

\[ + 8 \pi G \left[ \frac{D''(\Phi_o)}{D'(\Phi_o)^2} \right] \dddot{\Phi}(t, x) \ddot{T}_{00}(t, x) \]

\[ - \exp\left( k \frac{\Phi_o}{\Phi} \right) \frac{4}{k D'(\Phi_o)} \dddot{J}(t, x) + \left[ \frac{k}{2} - \frac{D''(\Phi_o)}{D'(\Phi_o)} \right] \left[ \partial_x \ddot{\Phi}(t, x) \right]^2 \]

\[ + \frac{1}{2} \partial_x \dddot{g}_{00}(t, x) \partial_x \dddot{\Phi}(t, x) . \]

Therefore \( \dddot{g}_{00}(t, x) \) and \( \dddot{\Phi}(t, x) \) are calculable in principle.

3.2.2 Case II: \( \partial_x \Phi(t, x) \neq 0 \)

When this is the case, we have

\[ D(\Phi) = A \exp\left( k \frac{\Phi}{\Phi} \right) + B , \quad (51) \]

\[ \Phi = \Phi_a x + \Phi_b , \quad (52) \]

where \( A \neq 0 \) and \( B \) are constants and \( \Phi_a \neq 0 \) and \( \Phi_b \) are arbitrary integration constants. As a result, if equations (51) and (52) hold, the only non-trivial zeroth order expansion gives \( \dddot{J} = - A k^3 \frac{\Phi_a^2}{2} / 2 \). In contrast to the other cases considered thus far, \( \dddot{J} \) is here non-zero. If we substitute equations (51) and (52) into the first order expansions, one will easily conclude that

\[ \dddot{\Phi}(t, x) = \dddot{\Phi}_a . \quad (53) \]
After we have imposed equations (51), (52) and (53) on the second order equation $\mathcal{T}_{11} = 0$, it yields

$$\partial_x \Phi = \frac{1}{2 k} \partial_x \varphi_{00} + \frac{\Phi_a}{2} \varphi_{00} + \Phi_a(t) ,$$  \hspace{1cm} (54)$$

where $\Phi_a$ is an arbitrary function of $t$ which can be determined by the other two equations $\mathcal{J} = \mathcal{J}$ and $\mathcal{T}_{00} = 8 \pi G \mathcal{T}_{00}$ as

$$\mathcal{J}(t) = \frac{k^3}{A} \Phi_a \Phi_a(t) ,$$

$$\partial_{xx} \varphi_{00}(t, x) = -k \Phi_a \partial_x \varphi_{00} - \frac{16\pi G A}{A} \exp \left( -k \Phi \right) \mathcal{T}_{00}(t, x) .$$

Finally, the remaining equation $\mathcal{T}_{10} = 0$ is just trivially satisfied. As a result, we can find the term $\varphi_{00}(t, x)$ if $\mathcal{T}_{00}(t, x)$ is known. Once we know $\varphi_{00}(t, x)$, $\Phi(t, x)$ can be obtained by using equation (54). Since the restriction (51) is similar to the string-inspired field theory, reader may consult [6] for similar third and fourth order expansions.

### 4 Stellar Structure

The existence of “stars” in two-dimensional spacetime is governed by the equation of hydrostatic equilibrium [9]

$$-\frac{d}{dx} p = (p + \rho) \frac{d}{dx} \ln(\sqrt{-g_{00}}) ,$$ \hspace{1cm} (55)$$

where $p$ is the pressure and $\rho$ is the density of the star. As we have two different sets of equations, we shall consider them separately. In this section, a static metric of the form

$$ds^2 = -B^2(x) dt^2 + dx^2$$ \hspace{1cm} (56)$$

with a perfect fluid stress tensor will be used to model the interior of a star in equilibrium state.
4.1 Type I Equations

If a perfect fluid is used, equation (55) implies that the dilaton source \( \tilde{J} \) is zero. Therefore the type-I field equations (7) and (8) can be written as

\[
-D'(\Phi) B''(x) = H(\Phi) \frac{d}{dx} \left[ B(x) \Phi'(x) \right] + \frac{H'(\Phi)}{2} B(x) \left[ \Phi'(x) \right]^2 ,
\]

(57)

\[
8 \pi G p(x) = \frac{1}{2} H(\Phi) \left[ \Phi'(x) \right]^2 + \frac{d}{dx} \ln(B) \frac{d}{dx} D(\Phi) ,
\]

(58)

\[
8 \pi G B(x) \left[ p(x) - \rho(x) \right] = \frac{d}{dx} \left[ B(x) \frac{d}{dx} D(\Phi) \right] ,
\]

(59)

because the static condition is assumed.

We first consider the sub-case when \( H(\Phi) \neq 0 \). Then the last two equations above can be rearranged algebraically such that

\[
\Phi'(x) = -\frac{D'(\Phi)}{H(\Phi)} \frac{d}{dx} \ln(B) \pm \sqrt{\left[ \frac{D'(\Phi)}{H(\Phi)} \frac{d}{dx} \ln(B) \right]^2 + \frac{16 \pi G p}{H(\Phi)}} ,
\]

\[
\Phi''(x) = 8 \pi G \frac{p - \rho}{D'(\Phi)} - \frac{16 \pi G D''(\Phi)p}{D'(\Phi) H(\Phi)} + \frac{2 D''(\Phi) - H(\Phi)}{H(\Phi)} \frac{d}{dx} \ln(B) \Phi'(x) .
\]

Given the metric (56), equation (55) implies that

\[
\frac{B'(x)}{B(x)} = \frac{-p'(x)}{p + \rho} \quad \text{and} \quad \frac{B''(x)}{B(x)} = \frac{2 p'(x) + \rho'(x)}{(p + \rho)^2} - \frac{p''(x)}{p + \rho} .
\]

(60)

Consequently, we obtain

\[
\Phi'(x) = \frac{D'(\Phi)}{H(\Phi)} \frac{p'(x)}{p + \rho} \pm \sqrt{\left[ \frac{D'(\Phi)}{H(\Phi)} \frac{p'(x)}{p + \rho} \right]^2 + \frac{16 \pi G p}{H(\Phi)}} ,
\]

(61)

\[
\Phi''(x) = 8 \pi G \frac{p - \rho}{D'(\Phi)} - \frac{16 \pi G D''(\Phi)p}{D'(\Phi) H(\Phi)} - \left[ \frac{2 D''(\Phi)}{H(\Phi)} - 1 \right] \frac{p'(x)}{p + \rho} \Phi'(x) .
\]

(62)

\[
\frac{B''(x)}{B(x)} = 8 \pi G \left[ 2 \frac{D''(\Phi)}{D'(\Phi)^2} - \frac{H'(\Phi)}{H(\Phi) D'(\Phi)} \right] p - 8 \pi G \frac{H(\Phi)}{[D'(\Phi)]^2} (p - \rho)
\]

\[- \frac{p'(x)}{p + \rho} \frac{d}{dx} \ln \left( H(\Phi)[D'(\Phi)]^2 \right) .
\]

(63)
If the equation of state $p = p(\rho)$ is given, (61) to (63) form a system of ordinary differential equations in $\Phi(x)$ and $\rho(x)$.

In the Newtonian limit $p \approx 0$, $|p'(x)| \ll 1$ and $\Phi \approx 0$, we obtain

$$\frac{B''(x)}{B(x)} \rho(x) \approx p'(x) \frac{d}{dx} \ln(\rho) - p''(x) \quad \text{and} \quad \frac{B''(x)}{B(x)} \approx \frac{8 \pi G H(0)}{[D'(0)]^2} \rho(x) .$$

As a result, if the fraction $|D'(0)|^2/H(0) = 2$, the results above will agree with Newton’s equation of stellar equilibrium

$$p'(x) \frac{d}{dx} \ln(\rho) - p''(x) = 4 \pi G \rho^2(x) \quad (64)$$

in two dimensions.

When $H(\Phi) = 0$, equations (57) to (59) imply that

$$xp'(x) + p(x) = -\rho(x) .$$

As $p(x)$ and $p'(x)$ tend to zero in the Newtonian limit, the density $\rho(x)$ goes to zero in this limit. So $H = 0$ is not appropriate to model one-dimensional stellar structure.

### 4.2 Type II Equations

Consider the stellar structure with the type-II equations (13) and (14). Here the static metric and the stress tensor for a perfect fluid are used as before. The conservation of stress-energy implies via (12) that $J$ is a constant. Therefore (61) still holds and the field equations can be expressed as

$$\Phi'(x) = \frac{p'(x)}{k(p + \rho)} \pm \frac{1}{k} \left[ \frac{p'(x)}{p + \rho} \right]^2 + \frac{2}{D'(\Phi)} \left( 8 \pi G kp - J \epsilon^{k \Phi} \right) , \quad (65)$$

$$\Phi''(x) = \left[ k - \frac{D''(\Phi)}{D'(\Phi)} \right] \left[ \Phi'(x) \right]^2 + \frac{p'(x)}{p + \rho} \Phi'(x) - \frac{8 \pi G}{D'(\Phi)} (p + \rho) , \quad (66)$$

$$\frac{2p' + \rho'}{(p + \rho)^2} p' - \frac{p''}{p + \rho} = k \left[ \frac{D''(\Phi)}{D'(\Phi)} - k \right] [\Phi']^2 + \frac{k p' \Phi'}{p + \rho} + \frac{8 \pi G k}{D'(\Phi)} \rho . \quad (67)$$

So that when the equation of state is given, equations (65) to (67) form a system of second order ordinary differential equations in $\Phi(x)$ and $\rho(x)$. 

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If this solution is compared with Newton’s equation of stellar equilibrium, in the Newtonian limit $\dot{J} \to 0$, $p \to 0$, $|p'(x)| \ll 1$ and $\Phi \to 0$, equation (67) will reduce to

$$ p'(x) \frac{d}{dx} \ln(\rho) - p''(x) \approx \frac{8 \pi G k}{D'(0)} \rho^2(x). $$

Thus this will agree with the Newtonian limit (64) if $D'(0)$ equals $2k$.

## 5 Cosmology

Let us consider the two-dimensional Robertson-Walker metric

$$ ds^2 = -dt^2 + a^2(t) \frac{dx^2}{1 - s x^2}. $$

Since the denominator of the $dx^2$ term is only $x$-dependent, it can be removed by redefining the spatial coordinate. Although there are still three different cosmological models (closed, open and flat) the evolution of the scale factor $a(t)$ does not depend upon which model is under consideration, and we can take $s = 0$ without loss of generality [5]. The metric then becomes

$$ ds^2 = -dt^2 + a^2(t) \, dx^2. \tag{68} $$

Moreover, we assume a perfect fluid stress tensor with the equation of state

$$ p = (\gamma - 1) \rho, \tag{69} $$

where $\gamma$ is a parameter.

### 5.1 Type-I Equations

In this case, the field equations (7) and (8) read

$$ \rho(t) = \rho_o a^{-\gamma}(t), \tag{70} $$

$$ a^{\gamma-1}(t) \frac{d}{dt} D(\Phi) = A e^{(2-\gamma)\Lambda(\Phi)}, \tag{71} $$

$$ 8 \pi G \rho_o e^{(\gamma-1)\Lambda(\Phi)} = A \frac{d}{dt} \left[ a(t) e^{\Lambda(\Phi)} \right]. \tag{72} $$
The functions $\Phi$ and $\rho$ are functions of time only, because the model is homogeneous. The constant $\rho_o$ comes from the conservation laws $\nabla_\nu T^{\mu\nu} = 0$ and the equation of state (59). The function $\Lambda$ is defined in a manner identical to that of (44) in the study of the post-Newtonian approximation and $A$ is a non-zero integration constant. Although solving these equations for general $\gamma$ is difficult, solutions may easily be found for two special cases, $\gamma = 1$ and $\gamma = 2$, corresponding to dust- and radiation-filled universes, respectively. When $\gamma = 1$, equations (71) and (72) give

$$a(t) = \left( \frac{8 \pi G \rho_o}{A} t + \alpha \right) e^{-\Lambda(\Phi)} ,$$
$$\Phi'(t) = A e^{\Lambda(\Phi)/D'(\Phi)} ,$$

where $\alpha$ is another integration constant. If the functions $D$ and $H$ are given, the dilaton field $\Phi(t)$ can be found from the first order (in general, nonlinear) ordinary differential equation (74). Once $\Phi$ is known, the density $\rho(t)$ and the metric component $a(t)$ can be written down immediately as a consequence of equations (70) and (73).

When $\gamma = 2$, the system (71) and (72) yields

$$a(t) = \frac{A}{\beta} \exp \left( \frac{8 \pi G \rho_o}{A^2} D(\Phi) - \Lambda(\Phi) \right) ,$$
$$\Phi'(t) = \frac{\beta}{D'(\Phi)} \exp \left( \Lambda(\Phi) - \frac{8 \pi G \rho_o}{A^2} D(\Phi) \right) ,$$

where $\beta$ is an integration constant. As a result, if the characteristic functions $D$ and $H$ are given, the dilaton field $\Phi(t)$ can be solved by using the first order differential equation (76). Thus the density and the metric component can be expressed as functions of $t$ by using equations (70) and (73). This procedure is similar to the one in the dust-filled universe case.

### 5.2 Type II Equations

For the metric (58) and the perfect fluid stress tensor with equation of state (59), the type-II equations (13) and (14) become

$$\rho(t) = \rho_o a^{-\gamma(t)} ,$$

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\[
8 \pi G \rho(t) = -\frac{\dot{J}}{k} e^{k \Phi} + D'(\Phi) \frac{d}{dt} \ln(a) \Phi'(t) + \frac{k}{2} D'(\Phi) \left[ \Phi'(t) \right]^2 , \quad (78)
\]
\[
D'(\Phi) \Phi''(t) = \gamma \frac{\dot{J}}{k} e^{k \Phi} - (\gamma - 1) D'(\Phi) \frac{d}{dt} \ln(a) \Phi'(t) + \left[ k \left( 1 - \frac{\gamma}{2} \right) D'(\Phi) - \Phi''(t) \right] \left[ \Phi'(t) \right]^2 . \quad (79)
\]

According to equation (12), the dilaton source \( \dot{J} \) is a constant. As in the last subsection, solutions for general \( \gamma \) are quite difficult to obtain, and we consider only \( \gamma = 1 \) and \( \gamma = 2 \).

### 5.2.1 Dust-filled universe

When the parameter \( \gamma = 1 \), the system (78) and (79) has a solution
\[
a(t) = 8 \pi G \rho_o \Sigma(\Phi(t)) e^{\frac{-k}{\alpha}} \int \Sigma^{-2}(\Phi(t)) \, dt ,
\]
\[
\frac{d}{dt} D(\Phi) = \Sigma(\Phi) e^{\frac{k}{2} \Phi} , \quad (80)
\]
where \( \alpha \) is a constant and \( \Sigma(\Phi) := \pm \sqrt{2 \dot{J} D(\Phi) / k + \alpha} \). If \( D(\Phi) \) is given, the ordinary differential equation (80) can be used to solve for \( \Phi \). Thus we are able to compute the density \( \rho(t) \) and the metric component \( a(t) \).

### 5.2.2 Radiation-filled universe

When \( \gamma = 2 \), the system (78) and (79) becomes
\[
\omega k a(t) \frac{d}{dt} D(\Phi) = \dot{J} a^2(t) e^{k \Phi} + \zeta , \quad (81)
\]
\[
\frac{d}{dt} \left[ a(t) e^{\frac{k}{2} \Phi} \right] = \omega e^{\frac{k}{2} \Phi} , \quad (82)
\]
where \( \zeta \) and \( \omega \) are integration constants. We shall divide the discussion up into two cases because equation (81) may either be linear or quadratic in \( a(t) \), depending on whether or not \( \dot{J} \) vanishes.

When \( \dot{J} = 0 \), equations (81) and (82) give
\[
a(t) = \frac{\zeta}{\omega \sigma k} e^{\frac{k}{2} \Phi} \exp \left( \frac{\omega^2 k}{\zeta} \int D(\Phi) - \frac{k}{2} \Phi \right) ,
\]

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\[
\frac{d}{dt} D(\Phi) = \sigma \exp \left( \frac{k}{2} \Phi - \frac{\omega^2 k}{\zeta} D(\Phi) \right), \tag{83}
\]

where \(\sigma\) is another integration constant.

On the other hand, if \(\bar{J} \neq 0\), the two equations (81) and (82) become

\[
a(t) = \frac{\omega k \frac{d}{dt} D(\Phi) \pm \sqrt{\left[ \omega k \frac{d}{dt} D(\Phi) \right]^2 - 4 \zeta \bar{J} e^{k \Phi}}}{2 \bar{J} e^{k \Phi}},
\]

\[
2 \omega \bar{J} e^{k \Phi} = \frac{\frac{d}{dt} \left[ \omega k \frac{d}{dt} D(\Phi) \pm \sqrt{\left[ \omega k \frac{d}{dt} D(\Phi) \right]^2 - 4 \zeta \bar{J} e^{k \Phi}} \right]}{2 \bar{J} e^{k \Phi}}. \tag{84}
\]

If the characteristic function \(D(\Phi)\) is given, the differential equations (83) and (84) can be used to solve for the field \(\Phi(t)\) in both cases. As a result, the density \(\rho(t)\) and the metric component \(a(t)\) can be found in principle.

### 5.3 Inflationary Universe

The inflationary universe paradigm was developed to address the flatness, horizon and monopole problems \([10]\) that beset the standard \((3 + 1)\)-dimensional model of cosmology. In \((1 + 1)\) dimensions the situation is somewhat different. Previous investigations of \((1 + 1)\)-dimensional cosmology within the context of the \(R=T\) theory \([11]\) indicated that (a) the standard problems of \((3 + 1)\)-dimensional cosmology were absent due to the lack of structure in the lower-dimensional universe and (b) the mechanisms for realizing an exponentially inflating universe involved rather unconventional assumptions within the context of \((1 + 1)\)-dimensional physics.

In this section we consider the circumstances under which inflation could occur for an arbitrary dilaton theory of gravity. In the last two sections we showed how to solve for the metric unknown \(a(t)\) given the functions \(D(\Phi)\) and \(H(\Phi)\). In this subsection we assume the function \(a(t)\) varies exponentially with \(t\), and we search for functions \(H(\Phi)\) and \(D(\Phi)\) which can realize this.

In order to allow the early universe to expand exponentially, there must be an outward pressure \([11]\)

\[
p = -\rho. \tag{85}
\]

This will happen when the universe is trapped in a false vacuum phase.
5.3.1 Type I Equations

When the equation of state becomes (85), the corresponding value of the parameter $\gamma$ is zero. Let us suppose that we know the characteristic function $H(\Phi)$ in advance; thus the other function $D(\Phi)$ is the object we are trying to find. If we suppose the metric component $a(t)$ has the form

$$a(t) = \frac{8 \pi G \rho_o}{\alpha \beta A} e^{\beta t},$$

where $\beta$ is a positive constant and $\alpha$ is a non-zero constant such that $\alpha A$ is positive, equations (70) to (72) will give

$$\rho(t) = \rho_o,$$

$$\beta t = \Sigma(\Phi),$$

$$D(\Phi) = \frac{8 \pi G \rho_o}{\alpha \beta} \left[ \frac{2 \alpha}{\beta} \Sigma(\Phi) - \frac{B}{\beta} e^{- \Sigma(\Phi)} \right] + C,$$  (87)

where $B$ and $C$ are integration constants and $\Sigma(\Phi)$ is defined as

$$\Sigma(\Phi) := \ln \left( \alpha e^{-2 \Lambda(\Phi)} \pm \sqrt{[\alpha e^{-2 \Lambda(\Phi)}]^2 + B e^{-2 \Lambda(\Phi)}} \right).$$  (88)

Although the left side of equation (87) is the characteristic function $D(\Phi)$, the equation is actually an integral equation because the function $\Sigma(\Phi)$ involves the integration of $D'(\Phi)$. Differentiating equation (88) yields, after some manipulation,

$$\frac{\alpha \beta^2}{4 \pi G \rho_o} = \pm \frac{H(\Phi)}{[D'(\Phi)]^2} \frac{2 \alpha + B e^{- \Sigma(\Phi)}}{\sqrt{\alpha^2 + B e^{2 \Lambda(\Phi)}}}.$$  (89)

It is now clear that the sign of the square root in equation (88) is determined by the sign of $\alpha$ and the sign of the function $H(\Phi)$ which is then restricted to be either positive or negative definite. If we define

$$\omega(\Phi) := \left[ \alpha^2 + B e^{2 \Lambda(\Phi)} \right]^{1/2},$$

the square root of equation (88) becomes

$$\epsilon \sqrt{\pm \frac{\alpha \beta^2}{4 \pi G \rho_o} H(\Phi)} = \left[ \frac{\alpha}{\omega(\Phi)} \pm \omega(\Phi) \right] \frac{H(\Phi)}{D'(\Phi)},$$  (90)
where $\epsilon$ is either 1 or -1, depending on the choice of the second square root operation on equation (89).

When $B = 0$, the function $\omega(\Phi)$ equals $\sqrt{|\alpha|}$. Since the choice of $\epsilon$ is arbitrary, we can conclude that

$$D(\Phi) = \pm \sqrt{\frac{16 \pi G \rho_o}{\beta^2}} \int \sqrt{-H(u)} \, du + C$$  \hspace{1cm} (91)

when $\alpha$ is either positive or negative. Furthermore, we can also conclude that $H(\Phi)$ must be strictly negative for any non-zero arbitrary integration constant $A$, in order that the type-I equation possess an exponential solution. Finally, the dilaton field $\Phi$ can be solved as a function of $t$ by using equation (91) for $D(\Phi)$ together with equation (86).

When $B \neq 0$, we obtain the equation $H(\Phi) = G(\omega)$, where

$$H(\Phi) := \epsilon \sqrt{\frac{\beta^2}{4 \pi G \rho_o}} \int \sqrt{\pm \alpha H(\Phi)} \, d\Phi \quad \text{and} \quad G(\omega) := \pm 4 \omega + 4 \alpha \int \frac{d\omega}{\omega^2 \mp \alpha}.$$  

Therefore equation (91) implies

$$D'(\Phi) = \pm \frac{H(\Phi)}{H'(\Phi)} \left[ \frac{G^{-1}(H(\Phi))}{G^{-1}(H(\Phi))} \right]^2 \pm \alpha.$$  \hspace{1cm} (92)

If the function $H(\Phi)$ is strictly positive,

$$G(\omega) = \pm 4 \omega + \frac{4 \alpha}{\sqrt{|\alpha|}} \arctan \left( \frac{\omega}{\sqrt{|\alpha|}} \right)$$  \hspace{1cm} (93)

but when $H(\Phi)$ is strictly negative,

$$G(\omega) = \pm 4 \omega + \frac{2 \alpha}{\sqrt{|\alpha|}} \ln \left( \frac{\omega - \sqrt{|\alpha|}}{\omega + \sqrt{|\alpha|}} \right).$$

In the case when $H(\Phi)$ is strictly negative, equation (92) will have no zero because $\pm \alpha = |\alpha|$ but when $H(\Phi)$ is strictly positive, (92) will vanish if

$$H(\Phi) = G(\pm \sqrt{|\alpha|}).$$  \hspace{1cm} (94)
As $G(\omega)$ in (93) is well-defined at the points $\pm \sqrt{|\alpha|}$, there always exist a point $\Phi = \Phi_0$ such that condition (94) holds (because we have an arbitrary integration constant in $H(\Phi)$). Thus, the assumption (6) is violated in this case. Therefore $H(\Phi)$ can only be strictly negative and we have

$$D(\Phi) = \mathrm{sgn}(\alpha) \int \frac{H(\Phi)}{X'(\Phi)} \left[ \frac{Y^{-1}(X(\Phi))}{Y^{-1}(X(\Phi))} \right]^2 + |\alpha| \, d\Phi ,$$  

(95)

where the functions $X$ and $Y$ are defined as

$$X(\Phi) := \pm \sqrt{\frac{\beta^2}{4 \pi G \rho_0}} \int \sqrt{-|\alpha| H(\Phi)} \, d\Phi ,$$

$$Y(\omega) := \mathrm{sgn}(\alpha) 4 \omega + \frac{2 \alpha}{\sqrt{|\alpha|}} \ln \left( \frac{\omega - \sqrt{|\alpha|}}{\omega + \sqrt{|\alpha|}} \right) .$$

Finally, we can find the dilaton field $\Phi$ as a function of $t$ by using equation (95) for $D(\Phi)$ together with equation (86).

### 5.3.2 Using the type-II equations

Suppose

$$a(t) = \alpha e^{\beta t}$$

when $\gamma = 0$, where $\alpha$ and $\beta$ are positive constants, the system (77) to (79) gives

$$\rho(t) = \rho_0 ,$$

$$\frac{d}{dt}D(\Phi) = \frac{2}{k \beta A} e^{\beta t + k \Phi} ,$$

$$8 \pi G \rho_0 k \beta A e^{\beta t + \chi(t)} = \frac{d}{dt} \left[ e^{2 \beta t + k \Phi} e^{\chi(t)} \right] ,$$

(96)

where $A$ is an integration constant and $\chi(t)$ is defined as

$$\chi(t) := J A e^{-\beta t} .$$
When the dilaton source $\dot{J}$ equals zero, the function $\chi(t)$ vanishes. This simplification yields

$$\Phi(t) = \frac{1}{k} \ln \left( 8 \pi G \rho_o k A e^{-\beta t} + B_1 e^{-2\beta t} \right),$$

$$D(\Phi) = C_1 + \frac{16 \pi G \rho_o}{\beta^2} \Sigma(\Phi) - \frac{2 B_1}{k \beta^2 A} e^{-\Sigma(\Phi)},$$

(97)

where $B_1$ and $C_1$ are integration constants and $\Sigma(\Phi)$ is defined as

$$\Sigma(\Phi) := \ln \left( 4 \pi G \rho_o k A e^{-\Phi} \pm \sqrt{\left[ 4 \pi G \rho_o k A e^{-\Phi} \right]^2 + B_1 e^{-\Phi}} \right).$$

In the case when $\dot{J}$ is not zero, equation (96) becomes

$$\frac{\dot{J}}{8 \pi G \rho_o k} e^{\Phi} = \chi(t) - \chi^2(t) \Pi(\Phi) e^{-\chi(t)} + B_2 \chi^2(t) e^{-\chi(t)},$$

where $B_2$ is another integration constant and the function $\Pi(\Phi)$ is defined as

$$\Pi(u) := \int_{u}^{1} \frac{1}{s} e^s ds = \ln(u) + \sum_{n=1}^{\infty} \frac{u^n}{n n!}.$$ 

Thus, we can write the function $D$ as

$$D(\Phi) = \frac{16 \pi G \rho_o}{\beta^2} \left[ C_2 + B_2 e^{-\theta(\Phi)} - \Pi(\theta(\Phi)) e^{-\theta(\Phi)} \right],$$

(98)

where $C_2$ is a constant and $\theta(\Phi)$ is implicitly defined as

$$\frac{\dot{J}}{8 \pi G \rho_o k} e^{\Phi} = \theta + B_2 \theta^2 e^{-\theta} - \Pi(\theta) \theta^2 e^{-\theta}.$$

The ordinary differential equations (80), (83) or (84) can be solved completely in principle if the function $D$ is given. However, when the function $D$ in either equation (97) or (98) is used, all the differential equations mentioned above appear to be analytically intractable, rendering numerical solutions the only practical option.
6 Conclusions

We have investigated many of the basic properties of a wide class of \((1 + 1)\)-dimensional dilaton theories of gravity coupled to matter. In this section we recapitulate our results.

In Section 1, the action (3) was introduced as a generalization of the actions (1) and (2). This generalized action is characterized by two functions \(D\) and \(H\), along with a matter Lagrangian described by a general ‘potential’ \(V\). The ambiguity in the definition of material sources relative to the potential \(V\) can yield a variety of field equations, of which we explored two types, denoted I and II. The former set was obtained by taking the stress-energy tensor density \(T_{\mu\nu}\) to be the variation of the matter action with respect to the metric, whereas the latter incorporated the variation of the matter action with respect to the dilaton field into its definition (as in (11)). The actual type-II equations we employed incorporated the constraint \((10)\) \(H(\Phi) = k D'(\Phi)\). It is important to note that the restriction \((10)\) put on the type-II field equations is not the reason why the two sets of field equations have different dynamical properties. Even if type-I equations satisfy condition \((11)\), they will not in general have the same dynamical properties as those induced by type-II equations. The two sets will have exactly the same dynamical properties only if condition \((11)\) holds and the dilaton source vanishes because they have different interpretation of the sources. This confusion in interpretation arises as a consequence of the lack of knowledge of the potential density \(V\) which appears in the action. As a result, the ambiguity can only be clarified once the potential density is explicitly specified.

For both types of field equations, the weak-field approximation was calculated. It is interesting to notice that the roles of the dilaton and matter sources seem to be interchanged in the two sets of calculations. In the type-I calculations, the perturbation of the dilaton field depends only on the matter source and the trace of the metric perturbation depends upon both dilaton and matter sources. The reverse situation happens in the type-II calculation.

The post-Newtonian expansion, as in General Relativity, can be carried out to an arbitrary high order for both types of field equations. It is quite clear in Section 4 that the stellar equilibrium equations for both types of field equations can be reduced into Newtonian equation in the Newtonian limit.

If we compare the solutions of the dust-filled and radiation-filled cases,
we will find that the solutions in the radiation-filled case are generally more complicated than those in the dust-filled case. This is the result we expected because there is no interaction between the particles, that is galaxies, in dust-filled model. During the course of developing a model for inflationary universe, even though the parametric functions \( D \) can be found for each type of field equations, these functions are mathematically so complicated that the differential equations cannot be solved analytically when dust-filled or radiation-filled universes are considered.

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