BLACK HOLES AND NONPERTURBATIVE CANONICAL 2D DILATON GRAVITY

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ABSTRACT

We investigate nonperturbative canonical quantization of two dimensional dilaton gravity theories with an emphasis on the CGHS model. We use an approach where a canonical transformation is constructed such that the constraints take a quadratic form. The required canonical transformation is obtained by using a method based on the Bäcklund transformation from the Liouville theory. We quantize dilaton gravity in terms of the new variables, where it takes a form of a bosonic string theory with background charges. Unitarity is then established by going into a light-cone gauge. As a direct consequence, black holes in this theory do not violate unitarity, and there is no information loss. We argue that the information escapes during the evaporation process. We also discuss the implications of this quantization scheme for the quantum fate of real black holes. The main conclusion is that black holes do not have to violate quantum mechanics.

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1. Introduction

Since Hawking’s discovery that black holes evaporate due to quantum effects [1], the question of the quantum fate of a black hole has been a subject of a lot of debate and speculation [2]. The reason for this is a lack of a viable theory of quantum gravity, since everybody agrees that at the end-point of the black hole evaporation process quantum gravity effects will play a crucial role. In this situation the best one can do is to study toy models of gravitational collapse. The most realistic toy model studied so far is spherically symmetric scalar field collapse in 4d [3]. However, the resulting two-dimensional field theory is still too complicated to be useful, so one has to resort to even simpler models. Various toy models have been studied recently, including the spherical dust shell collapse [4], spherical dust cloud collapse [5], and most notably, two-dimensional dilaton gravity theories [2, 10]. In particular, the CGHS model of 2d dilaton gravity [6] has attracted a lot of attention, since it is classically exactly solvable, and the solution describes formation of a 2d black hole by massless scalar fields. Furthermore, there is a black hole evaporation effect and the model represents a renormalisable 2d field theory [6].

These nice features of the CGHS model have raised a hope that already in the one loop semi-classical approximation the effective equations of motion will be free of singular solutions [6]. However, very soon it was shown that singular solutions exist [8], and furthermore Hawking has argued that singularities exist in any semi-classical approximation [9]. All this indicates that the 2d metric has to be quantized if one wants to understand the quantum fate of the black hole, and hence a nonperturbative quantization of the model is necessary.

The nonperturbative approaches to 2d dilaton gravity which have been studied so far are path-integral and canonical. The idea of the path-integral approach is to perform the functional integral over the metric, dilaton and matter fields exactly, and then to study the corresponding effective action and the correlation functions (for a review and references see [2, 10]). Beside its own difficulties, it is not clear how to construct the physical Hilbert space within this approach, and how to address the corresponding conceptual questions.

In the canonical approach [11, 12, 13, 14, 15], the construction of the physical Hilbert space is the primary goal, from which all other questions are answered. This is achieved from the study of the constraints, which can be derived by using the ADM method [11]. Alternatively, in the covariant quantization method the space of the classical solutions defines the phase space and the constraints are derived from the components of the energy-momentum tensor [14, 15]. The advantage of the ADM...
method is its gauge independence, while in the covariant approach a gauge has to be chosen from the beginning. Either way, the idea is to simplify the constraints by constructing a canonical transformation which makes the constraints quadratic in the new canonical variables. Then one can derive the physical Hilbert space by using the standard BRST techniques from the string theory \cite{1, 4, 13}. Related to this is the issue of unitarity, which was first discussed in \cite{12}. There it was demonstrated that a physical gauge exists where the unitarity can be easily proven. Hence the black holes in this theory would not violate quantum mechanics. Subsequently, a unitary S-matrix was constructed in the covariant approach \cite{13}, which was in accordance with the result of ref. \cite{12}.

However, the Kac-Moody algebra method used in refs \cite{1, 12} to construct the required free-field canonical transformation is strictly valid only for the case of chiral matter. On the other hand, although the free-field transformations of refs \cite{14, 15} apply to the case of non-chiral matter, they do not have the canonical form and are valid only in the conformal gauge. In this paper we complete the program of canonical ADM quantization of dilaton gravity. We construct the free-field canonical transformation by using methods based on the Bäcklund transformation from the Liouville theory. Therefore we provide a strict proof of the unitarity of dilaton gravity based on the ideas of ref. \cite{12}.

In section 2 we describe the canonical ADM formulation of 2d dilaton gravity. In section 3 we introduce a Liouville dilaton gravity model, as a preparatory study for the CGHS model. We construct the free field canonical transformation after mapping the theory onto a usual Liouville theory and then using the Bäcklund transformation. We also find the Bäcklund transformation in the case when the Liouville energy is not positive definite. Then in section 4 we discuss the CGHS model by following the strategy from the Liouville model, i.e. we write the fields of the classical solution in the conformal gauge in terms of free fields and then look for a simple relation between the corresponding canonical variables. After obtaining such a relation, we show how the constraints can be mapped into constraints of a string theory with background charges. In section 5 we discuss the quantization of such a theory, and describe the BRST quantization in the compact case. In section 6 we describe the reduced phase space quantization, which also applies to the non-compact case. We find a physical gauge where all relevant dynamical variables can be expressed in terms of independent canonical variables. The physical Hilbert space is just a free-field Fock space for the matter, and the corresponding Hamiltonian is a free-field one. It can be promoted into a Hermitian operator, so that the quantum evolution is unitary.

We present our conclusions in sect. 7. We argue that the Hawking radiation is
present in spite of the trivial $S$-matrix. There is no information loss, and we argue that the information escapes during the evaporation. Implications for the real collapse are discussed and the problem of singularity in quantum theory is addressed.

2. Canonical formulation

Two-dimensional dilaton gravity theories of interest can be described by an action

$$S = S_0 + S_m$$

$$S_0 = \int_M d^2x \sqrt{-g} e^{-\Phi} \left[ R + \gamma (\nabla \Phi)^2 + U(\Phi) \right]$$

$$S_m = -\frac{1}{2} \int_M d^2x \sqrt{-g} \sum_{i=1}^N (\nabla \phi_i)^2$$

(2.1)

where $\Phi$ and $\phi_i$ are scalar fields, $\gamma$ is a constant, $g$, $R$ and $\nabla$ are determinant, scalar curvature and covariant derivative respectively, associated with a metric on the 2d manifold $M$. For our purposes we will assume that $M = \Sigma \times \mathbb{R}$, so that $\Sigma = S^1$ (a circle) or $\Sigma = \mathbb{R}$ (a real line). We will refer to these two cases as compact and non-compact respectively. $S_0$ describes the coupling of the dilaton $\Phi$ to the metric, while $S_m$ represents conformally coupled scalar matter. Depending on the value of the constant $\gamma$ and the form of the potential $U$, one can get various dilaton gravity theories. For example, $\gamma = 2$ and $U = \kappa e^{2\Phi}$ corresponds to the spherically symmetric Einstein-Hilbert action, while $\gamma = 4$ and $U = 4\lambda^2$, where $\lambda$ is a constant, corresponds to the CGHS model.

Before explaining the canonical ADM formulation, we will briefly study field redefinitions, in order to arrive at the simplest possible form of the action. That in turn simplifies the constraints. Let $\psi^2 = e^{-2\Phi}$, then $S_0$ from the eq. (2.1) becomes

$$S_0 = \int_M d^2x \sqrt{-\tilde{g}} \left[ \frac{1}{2} (\nabla \psi)^2 + \frac{1}{2\gamma} R \psi^2 + \tilde{U}(\psi) \right] ,$$

(2.2)

where $\psi$ has been rescaled into $\frac{1}{\sqrt{2\gamma}} \psi$ ($\gamma \neq 0$). Then by performing a Russo-Tseytlin transformation [16]

$$\phi = \frac{1}{\gamma} \psi^2 , \quad \tilde{g}_{\mu\nu} = e^{-2\phi} g_{\mu\nu} , \quad 2\rho = \frac{1}{\gamma} \psi^2 - \frac{\gamma}{2} \ln \psi$$

(2.3)

we get

$$S_0 = \int_M d^2x \sqrt{-\tilde{g}} \left[ \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} \tilde{R} \phi + V(\phi) \right] ,$$

(2.4)

where $V(\phi) = \tilde{U} e^{2\rho}$. In the CGHS case $V = \frac{1}{2} \lambda^2 e^\phi$, and hence consider

$$S_0 = \int_M d^2x \sqrt{-g} \left[ \frac{1}{2} (\nabla \phi)^2 + \alpha R \phi + \Lambda e^{\beta \phi} \right] ,$$

(2.5)
where \( \alpha, \beta \) and \( \Lambda \) are constants. The action (2.5) represents a class of solvable dilaton gravity theories, which can be seen by redefining the metric as \( \tilde{g}_{\mu \nu} = e^{\beta \phi} g_{\mu \nu} \), so that

\[
S_0 = \int_M d^2x \sqrt{-\tilde{g}} \left[ \frac{\gamma}{2} (\nabla \phi)^2 + \frac{1}{2} R \phi + \Lambda \right] .
\] (2.6)

The action (2.6) is of the same form as the induced gravity action, and one can use the \( SL(2, \mathbb{R}) \) current algebra methods to construct the free-field canonical transformation [11]. In the case when \( \alpha \beta = \frac{1}{2} \), the \( SL(2, \mathbb{R}) \) current algebra degenerates into an extended 2d Poincare current algebra, and the analogous free-field construction exists [13]. However, when \( S_m \) is included, the current algebra method works only for the case of chiral matter.

Therefore consider the following action

\[
S = \int_M d^2x \sqrt{-g} \left[ \frac{\gamma}{2} (\nabla \phi)^2 + \alpha R \phi + V(\phi) - \frac{1}{2} \sum_{i=1}^{N} (\nabla \phi_i)^2 \right] ,
\] (2.7)

where \( \alpha \) and \( \gamma \) are constants. Note that the field redefinitions in eqs. (2.2-6) have always scaled the metric so that the form of the matter action is unchanged, because of its conformal invariance. Canonical reformulation simplifies if we use the ADM parametrization of the metric

\[
g_{\mu \nu} = \begin{pmatrix} -N^2 + gn^2 & gn \\ gn & g \end{pmatrix} ,
\] (2.8)

where \( N \) and \( n \) are the lapse function and the shift vector respectively, while \( g \) is a metric on \( \Sigma \). By defining the canonical momenta as

\[
p = \frac{\partial L}{\partial \dot{\phi}}, \quad \pi = \frac{\partial L}{\partial \phi}, \quad \pi_i = \frac{\partial L}{\partial \phi_i},
\] (2.9)

where \( L \) is the Lagrange density of (2.7) and dots stand for \( t \) derivatives, the action becomes

\[
S = \int dt dx \left( p \dot{g} + \pi \dot{\phi} + \pi_i \dot{\phi}_i - N_0 G_0 - N_1 G_1 \right) ,
\] (2.10)

where \( N_0 = \frac{N}{\sqrt{g}} \) and \( N_1 = n \). The constraints \( G_0 \) and \( G_1 \) are given as

\[
G_0(x) = \frac{\gamma}{2 \alpha^2} (gp)^2 - \frac{1}{\alpha} gp \pi - \frac{\gamma}{2} \frac{(\phi')^2}{g} - gV(\phi) + 2 \alpha \sqrt{g} \left( \frac{\phi'}{\sqrt{g}} \right)' + \frac{1}{2} \sum_{i=1}^{N} \left( \pi_i^2 + (\phi'_i)^2 \right)
\]

\[
G_1(x) = \pi \phi' - 2p' g - pg' + \sum_{i=1}^{N} \pi_i \phi'_i ,
\] (2.11)
where primes stand for $x$ derivatives. The $G$’s form the canonical diffeomorphism algebra with respect to the Poisson brackets. They also generate the diffeomorphisms of $M$, such that $G_1$ generates the diffeomorphisms of $\Sigma$, while $G_0$ generates time translations of $\Sigma$ and hence it is called the Hamiltonian constraint. A special feature of two dimensions is that

$$T_\pm = \frac{1}{2}(G_0 \pm G_1)$$

(2.12)

generate two commuting copies of the one-dimensional diffeomorphism algebra. When $\Sigma = S^1$ these two copies become two commuting Virasoro algebras.

3. Liouville dilaton gravity

As in the 4d canonical gravity, direct quantization of the constraints (2.11) is difficult due to their non-polynomial dependence on the canonical variables. One way around this problem is to follow the strategy introduced by Ashtekar [17], which is to find new canonical variables such that the constraints become polynomial. In the context of 2d gravity, this means that we will look for a canonical transformation which makes the constraints quadratic. We first study the model $\alpha = 1, \beta \neq \frac{1}{2}$, which we are going to call Liouville dilaton gravity, as a preparation for the CGHS model where $\alpha = 1, \beta = \frac{1}{2}$. The Liouville dilaton gravity corresponds to $U(\Phi) = 4\lambda^2 \exp[(\beta - \frac{1}{2}) e^{2\Phi}]$.

Since the matter part of the constraints is already quadratic and decoupled from the dilaton gravity sector, we need only to consider pure dilaton gravity. After trivial rescaling of the dilaton, eq. (2.6) becomes

$$S_0 = \int_M d^2x \sqrt{-g} \left[\frac{1}{2}(\nabla \phi)^2 + \alpha R \phi + \Lambda\right] ,$$

(3.1)

where $2^\alpha = \frac{1}{\sqrt{1-2\beta}}$. The constraints can be read off from eq. (2.11)

$$G_0 = \frac{1}{2\alpha^2}(gp)^2 - \frac{1}{\alpha} gp \pi - \frac{1}{2}(\phi')^2 - g\Lambda + 2\alpha \sqrt{g} \left(\frac{\phi'}{\sqrt{g}}\right)'$$

$$G_1 = \pi \phi' - 2p'g - pg' .$$

(3.2)

By going into the conformal gauge $N_0 = 1$, $N_1 = 0$, and analyzing the equations of motion for $g$ and $\phi$, one can see that a simplification occurs after the following canonical transformation

$$\rho = \alpha \ln g , \quad \pi_\rho = \frac{1}{\alpha} gp - \pi$$

$$\psi = \phi + \alpha \ln g , \quad \pi_\psi = \pi .$$

(3.3)
The constraints then become

\[
G_0 = -\frac{1}{2}\pi^2 - \frac{1}{2}(\psi')^2 + 2\alpha\psi'' + \frac{1}{2}\pi^2 + \frac{1}{2}(\rho')^2 - 2\alpha\rho'' - \Lambda \exp\left(\frac{\rho}{\alpha}\right)
\]
\[
G_1 = \pi\psi' - 2\alpha\pi\psi + \pi\rho' - 2\alpha\pi\rho .
\]  

(3.4)

Hence the new variables \(\rho\) and \(\psi\) are decoupled, and if \(\Lambda\) had been zero our task would have been already accomplished. The conformal factor \(\rho\) satisfies the Liouville equation in the conformal gauge

\[
4\partial_+\partial_-\rho - \frac{\Lambda}{\alpha} \exp\left(\frac{\rho}{\alpha}\right) = 0 ,
\]

(3.5)

where \(\partial_\pm = \partial/\partial x^\pm\) and \(x^\pm = t \pm x\). Since the \(\rho\)-part of the constraints is the same as the energy-momentum tensor of the Liouville theory, i.e.

\[
G_0 = -\frac{1}{2}\pi^2 - \frac{1}{2}(\psi')^2 + 2\alpha\psi'' + T_{00} , \quad G_1 = \pi\psi' - 2\alpha\pi\psi + T_{01} ,
\]

(3.6)

where

\[
T_{00} = \frac{1}{2}\pi^2 + \frac{1}{2}(\rho')^2 - 2\alpha\rho'' - \Lambda \exp\left(\frac{\rho}{\alpha}\right) , \quad T_{01} = \pi\rho' - 2\alpha\pi\rho ,
\]

(3.7)

one can use the canonical form of the Bäcklund transformation [18] to make \(T_{00}\) and \(T_{01}\) quadratic

\[
\pi_\rho = \omega' - \sqrt{-2\Lambda} \exp\frac{\rho}{2\alpha} \text{sh}\left(\frac{\omega}{2\alpha}\right) , \quad \rho' = \pi_\omega - \sqrt{-2\Lambda} \exp\frac{\rho}{2\alpha} \text{ch}\left(\frac{\omega}{2\alpha}\right) .
\]

(3.8)

When expressed in terms of the new variables \(\psi\) and \(\omega\), the constraints become quadratic

\[
G_0 = -\frac{1}{2}\pi^2 - \frac{1}{2}(\psi')^2 + 2\alpha\psi'' + \frac{1}{2}\pi^2 + \frac{1}{2}(\omega')^2 - 2\alpha\pi\omega
\]
\[
G_1 = \pi\psi' - 2\alpha\pi\psi + \pi\omega' - 2\alpha\omega'' ,
\]

(3.9)

which was our initial goal.

Note that the Bäcklund transformation (3.8) is defined only for \(\Lambda < 0\), which corresponds to positive definite \(T_{00}\). When \(\Lambda > 0\), the analytic continuation of eq. (3.8) does not work, and the corresponding expression can not be found in the Liouville literature, since it corresponds to the unphysical case of indefinite \(T_{00}\). However, the Bäcklund transformation still exists in that case, and can be found by considering first the zero-mode case (no \(x\) dependence) and by using the method of generating function. Then it is easy to see that the required transformation is

\[
\pi_\rho = \omega' + \sqrt{2\Lambda} \exp\frac{\rho}{2\alpha} \text{ch}\left(\frac{\omega}{2\alpha}\right) , \quad \rho' = \pi_\omega + \sqrt{2\Lambda} \exp\frac{\rho}{2\alpha} \text{sh}\left(\frac{\omega}{2\alpha}\right) ,
\]

(3.10)

which gives eq. (3.9) again.
4. CGHS model

As we have shown in section 2, the dilaton gravity sector of the CGHS model can be described by the action (2.6) with $\alpha = 1, \beta = \frac{1}{2}$, so that

$$S_0 = \int_M d^2 x \sqrt{-g} (R\phi + \Lambda) ,$$

where $\Lambda = 4\lambda^2$. The corresponding constraints are then

$$G_0 = -gp\pi - g\Lambda + 2\sqrt{g} \left( \frac{\phi'}{\sqrt{g}} \right)'$$
$$G_1 = \pi\phi' - 2p' g - pg' .$$

By introducing the conformal factor $\rho$ as $g = e^\rho$, so that $p = e^{-\rho}\pi\rho$ we get

$$G_0 = -\pi\rho\pi\phi - \Lambda e^\rho + 2\phi'' - \rho' \phi'$$
$$G_1 = \pi\phi' + \pi\rho\rho' - 2\pi'\rho .$$

Now in order to find the required canonical transformation, we adopt the strategy from the Liouville theory. We go into the conformal gauge and study the structure of the classical solution in order to find the analog of the Bäcklund transformation. In the conformal gauge $N_0 = 1$ and $N_1 = 0$ and the action (4.1) becomes

$$S_0 = \int dt dx \left( \pi\rho\dot{\rho} + \pi\phi\dot{\phi} - G_0 \right) .$$

The equations of motion are then

$$\dot{\rho} + \pi\phi = 0 , \quad \dot{\phi} + \pi\rho = 0$$

and

$$-\pi\rho - \phi'' + \Lambda e^\rho = 0 , \quad -\pi\phi - \rho'' = 0 .$$

By eliminating the momenta from eq. (4.5), eq. (4.6) becomes

$$4\partial_+ \partial_- \phi + \Lambda e^\rho = 0 , \quad \partial_+ \partial_- \rho = 0 ,$$

which can be solved as

$$e^\rho = \partial_+ p(x^+) \partial_- m(x^-) , \quad \phi = a(x^+) + b(x^-) - \lambda^2 p(x^+) m(x^-) ,$$

where $p$ and $m$ are arbitrary functions, $a = a_0 + a_1 p$ and $b = b_0 + b_1 m$, where $a_\mu$ and $b_\mu$ are arbitrary constants. When matter is present

$$a = a_0 + a_1 p - \frac{1}{2} \int dx^+ \partial_+ p \int dx^+ \frac{1}{\partial_+ p} \sum_i \partial_+ \phi_i \partial_+ \phi_i ,$$
\[ b = b_0 + b_1 m - \frac{1}{2} \int dx^- \partial_- m \int dx^- \frac{1}{\partial_- m} \sum_i \partial_- \phi_i \partial_- \phi_i . \]  
(4.9)

The task now is to define new fields in terms of \( a, b, p \) and \( m \), such that there is an explicit relation between the old and the new canonical variables, and that the constraints become quadratic. That this is no simple task, one can see from the example of the Liouville theory, where deriving the Bäcklund transformation from the classical solution is not straightforward [18]. The reason for this is that the canonical transformation is a relation between fields and their derivatives with respect to \( t \) and \( x \) coordinates, while the classical solution gives a relation between fields and their \( x^+ \) and \( x^- \) derivatives. As a result, going from one to another description is not trivial.

For example, the constraints (4.3) are non-polynomial in \( \rho \), but when expressed in the conformal gauge in terms of the classical solution, one gets an quadratic expression

\[ T_\pm = 2 \partial_\pm \rho \partial_\pm \phi - 2 \partial_\pm^2 \phi . \]  
(4.10)

In our case the obvious choice for the new variables is

\[ \chi = \ln \partial_+ p + \ln \partial_- m \quad , \quad \xi = a + b . \]  
(4.11)

Now we have to look for a simple relation between \( \phi, \rho, \chi \) and \( \xi \) and their \( \partial_\pm \) derivatives which follows from (4.11) and (4.8). The simplest two relations are

\[ \partial_\pm \rho \partial_\pm \phi - \partial_\pm^2 \phi = \partial_\pm \chi \partial_\pm \xi - \partial_\pm^2 \xi , \]  
(4.12)

which is nothing else but the statement that \( T_\pm \) stay quadratic in the new variables.

By using the relations between the canonical momenta and their velocities found in the equations of motion, the two eqs in (4.12) can be rearranged to become

\[
\begin{align*}
\pi_\rho \pi_\phi &+ \Lambda e^\phi - 2 \phi'' + \rho' \phi' = \pi_\chi \pi_\xi - 2 \xi'' + \chi' \xi' \\
\pi_\phi \phi' + \pi_\rho \rho' - 2 \pi_\rho' &= \pi_\xi \xi' + \pi_\chi \chi' - 2 \pi_\chi'.
\end{align*}
\]  
(4.13)

Eq. (4.13) implies that \( G_0 \) and \( G_1 \) have become quadratic when expressed in terms of the new variables. Also it can serve to determine \( (\phi, \pi_\phi) \) and \( (\rho, \pi_\rho) \) in terms of \( (\chi, \pi_\chi) \) and \( (\xi, \pi_\xi) \).

This can be done in the following way. The form of the classical solution (4.8) and the relations between the momenta and the velocities imply

\[
\begin{align*}
\rho &= \chi \quad , \quad \pi_\rho = \pi_\chi + \lambda^2 F(\chi, \pi_\xi) \\
\phi &= \xi + \lambda^2 G(\chi, \pi_\xi) \quad , \quad \pi_\phi = \pi_\xi ,
\end{align*}
\]  
(4.14)
where $F$ and $G$ are functionals of $\chi$ and $\pi_\xi$ to be determined. By inserting (4.14) into (4.13) we get

$$
-F\pi_\xi + 4e^\chi + 2G'' - \frac{1}{2}\chi'G' = 0
$$
$$
-\pi_\xi G' - F\chi' + 2F' = 0 .
$$

(4.15)

The system of eq. (4.15) can be rewritten as

$$
G' = -\frac{1}{\pi_\xi}(F\chi' - 2F')
$$

(4.16)

and

$$
F'' - \left(\chi' + \frac{\pi'_\xi}{\pi_\xi}\right)F' + \frac{1}{2}\left((\chi')^2 - 2\chi'' + 2\frac{\pi'_\xi}{\pi_\xi}\chi' - \pi_\xi^2\right)F + \pi_\xi e^\chi = 0 .
$$

(4.17)

The eq. (4.17) is a second order ordinary differential equation and hence a solution for $F$ exists. However, there is no explicit expression for $F$ since the coefficients in eq. (4.17) are arbitrary functions. In the zero-mode limit there is an explicit solution

$$
F = \frac{4e^\chi}{\pi_\xi} , \quad G = -\frac{4e^\chi}{\pi_\xi^2} , \quad
$$

(4.18)

and one can check then that the transformation (4.14) is canonical. This is in contrast to the Liouville case, where the implicit relations (3.8) give a first order ordinary differential equation for $u = \exp(-\frac{\rho}{\alpha})$, and hence an explicit expression for $(\rho, \pi_\rho)$ as a function of $(\omega, \pi_\omega)$ can be obtained. The lack of an explicit expression in the CGHS case may seem like a big drawback, but, as we are going to show in the next sections, there is enough useful information contained in the eq. (4.14) about the free-field canonical transformation. Hence the eq. (4.14) can be considered as the analog of the Bäcklund transformation from the Liouville theory.

Before discussing the quantization, we perform a further canonical transformation

$$
\chi = \frac{1}{\sqrt{2}}(\phi_0 + \phi_1) , \quad \xi = \frac{1}{\sqrt{2}}(\phi_0 - \phi_1)
$$
$$
\pi_\chi = \frac{1}{\sqrt{2}}(\pi_0 + \pi_1) , \quad \pi_\xi = \frac{1}{\sqrt{2}}(\pi_0 - \pi_1) .
$$

(4.19)

The constraints then take the form

$$
G_0 = \frac{1}{2}(\pi_\mu \pi^\mu + \phi'_\mu \phi'^\mu) + Q^\mu \phi''_\mu
$$
$$
G_1 = \pi'^\mu \phi'_\mu + Q^\mu \pi'_\mu .
$$

(4.20)
where $\mu = 0,1$, $\phi_\mu = (\phi_0, \phi_1)$, $\phi_\mu = (\pi_0, \pi_1)$ and $Q_\mu = -\sqrt{2}(1,1)$, while indices are raised with a metric $\eta_{\mu\nu} = \text{diag}(-1,1)$. The same can be done in the Liouville case, where

$$\psi' = \pi_0, \quad \pi_\psi = \phi'_0$$

$$\omega = \phi_1, \quad \pi_\omega = \pi_1$$

and $Q_\mu = -2\alpha(1,1)$. Note that in both cases $Q^2 = 0$, which is necessary in order for the expressions given by the eq. (4.20) to satisfy the 2d canonical diffeomorphism Poisson bracket algebra.

5. Quantization

In the canonical approach, there are two basic ways of quantizing a constrained system

1. quantize first and then solve the constraints (Dirac quantization),
2. solve the constraints first and then quantize (reduced phase space (RPS) quantization).

The Dirac quantization, and its variations (Gupta-Bleuler and BRST method), have an advantage over the RPS quantization because the symmetries of the theory are manifest. On the other hand, RPS quantization is easier to accomplish. In our case, we are going to study both approaches. This is possible because the constraints have the same form as the constraints of a $(N + 2)$-dimensional bosonic string with background charges, where many quantization techniques have been developed.

In order to accomplish the Dirac quantization, it is useful to introduce the left/right movers

$$P^\pm_I = \frac{1}{\sqrt{2}}(\pm \pi_I + \phi'_I)$$

which satisfy

$$\{P^\pm_I(x), P^\pm_J(y)\} = \pm \eta_{IJ}\delta'(x - y), \quad \{P^+_I(x), P^-_J(y)\} = 0$$

(5.2)

where $I = \mu, i$ and $\eta_{IJ} = \text{diag}(-1,1,...,1)$. Then the theory factorizes into two independent sectors, described by $P^+$ and $P^-$ variables, with the respective constraints

$$T_\pm = \sum_{I=1}^{N+2} \left( \frac{1}{2} P^\pm_I P^\pm_I + Q_I P^\pm_I \right) .$$

(5.3)

Hence it is sufficient to look for the physical Hilbert space in only one sector, since the total physical Hilbert space will be a tensor product of the left and the right sector.
Next we take $\Sigma$ to be compact, because not much is known about the representations of the 1d diffeomorphism algebra in the non-compact case. This creates a problem, since the black hole solutions strictly exist only in the non-compact case. This problem is usually resolved by putting the system into a large box, of length $L$, in the hope that when $L \to \infty$ a non-compact case is recovered.

Now we make Fourier expansions

$$P_I(x) = \frac{1}{\sqrt{L}} \left( p_I + \sum_{n \neq 0} \alpha^I_n e^{in\pi x/L} \right)$$

(5.4)

where $p_i = 0^2$. Then

$$T(x) = \frac{1}{L} \sum_n L_n e^{in\pi x/L}$$

$$L_n = \frac{i}{2} \sum_m (n - m) \{ \alpha^I_{n-m} \alpha^J_m + i m Q_I \alpha^J_n \}$$

(5.5)

where $Q_i = 0$. The $L_n$’s are then promoted into operators acting on a Fock space $\mathcal{F}(\alpha^I_n)$ made out of the $\alpha_n$ modes in the standard way. The $L_n$’s form a Virasoro algebra classically, but in the quantum case there is an anomaly in the algebra, in the form of the central extension term with the central charge $c = 2 + N/12$. This type of situation is best handled in the BRST formalism $[11, 14]$. One enlarges the original Fock space $\mathcal{F}(\alpha^I_n)$ by introducing a canonical pair of ghost fields $(b, c)$, and constructs a nilpotent operator

$$\hat{Q} = \sum_n c_{-n}(L_n - a\delta_{n,0}) + \frac{i}{2} \sum_{n,m} (n - m) \{ c_{-n+m} b_{n+m} + \}$$

(5.6)

The nilpotency of $\hat{Q}$ requires

$$Q^I Q_I = -Q_0^2 + Q_1^2 = 2 - N/12 , \quad a = N/24$$

(5.7)

which is satisfied for $N = 24$ since $Q^I Q_I = 0$. The physical Hilbert space $\mathcal{H}^*$ is then determined as the cohomology of $\hat{Q}$

$$\mathcal{H}^* = \text{Ker} \hat{Q}/\text{Im} \hat{Q}$$

(5.8)

There is only a zero-ghost sector in the cohomology, since the intercept $a \neq 0$. The physical states satisfy

$$(L_0 - 1) \Psi = 0 , \quad L_n \Psi = 0 \quad n = 1, 2$$

(5.10)

2This condition is necessary because $\phi_i$ are matter fields and not the string coordinates, so that the Fock space vacuum does not carry any momentum

3Kuchar and Torre have shown that a new set of variables can be found such that there is no anomaly in the diffeomorphism algebra. However, the conformal symmetry then acquires the anomaly $[20]$
where $\Psi \in \mathcal{F}(\alpha_n^i)$. The conditions (5.10) are well known in the string theory, and they are satisfied by the transverse oscillator states, corresponding to the $\alpha_n^i$ modes. One can construct the corresponding observables (i.e. dynamical variables which commute with the constraints) by using the DDF construction \[15]\).

There are also discrete momentum states in the cohomology \[14]\). However, their meaning in the context of dilaton gravity is not clear since they do not have well defined scalar product. As far as the continuous momentum states are concerned we have been using the standard scalar product such that

$$
(a_n)^\dagger = a_{-n} \quad , \quad (L_n)^\dagger = L_{-n} .
$$

(5.11)

6. RPS Quantization

One can arrive at the same results much more easily by using the RPS approach. Since the constraint structure is the same as that of the bosonic string, one can use the light-cone gauge to solve the constraints \[19]\). Another advantage of the RPS approach is that it also works in the non-compact case, so that it avoids the problem of the previous section.

The light-cone gauge is defined in terms of the $\phi_\pm = \phi_0 \pm \phi_1$ variables, which in the CGHS case amounts to using the ($\xi, \chi$) variables. The standard light-cone gauge is

$$
\xi = pt \quad , \quad \pi_\chi = -p \quad ,
$$

(6.1)

where $p$ is $x$ independent. In the non-compact case $p$ is a numerical constant (i.e. $p = 1$), while in the compact case it is a dynamical variable, representing the remaining global degree of freedom of the dilaton gravity sector. By inserting the relations (6.1) into the constraints we get

$$
\pi_\xi = -\frac{1}{2p} \sum_{i=1}^{N} (\pi_i^2 + \phi_i^2)
$$

(6.2)

and

$$
\chi' = \frac{1}{p} \sum_{i=1}^{N} \pi_i \phi'_i .
$$

(6.3)

Hence the independent canonical variables are $(p, q)$ and $(\pi_i, \phi_i)$, which agrees with the Dirac quantization result that only the transverse mode states are physical.

The fact that the $G_0$ constraint can be put into the form (6.2) also means that $\xi$ is a time variable in the theory \[22]\). Hence a Hamiltonian can be associated with
the choice of time in (6.1), and it can be determined as

\[ H = -\int_\Sigma dx \pi_\xi = \frac{1}{2} \sum_n (\alpha_n^i \alpha_n^i + \tilde{\alpha}_n^i \tilde{\alpha}_n^i) + c_0 \]  

(6.4)

where \( \tilde{\alpha} \) are Fourier modes of \( P^- \). The constant \( c_0 \) is zero in the classical theory, although it can have a quantum contribution due to the normal ordering effects.

In the non-compact case, the Hamiltonian can be determined from a surface term analysis [21], but it is obvious from the equations of motion for \( \phi_i \) that it is a free-field hamiltonian (6.4).

Unitarity of the theory follows from the fact that the Hamiltonian (6.4) can be promoted into a Hermitian operator acting on the physical Hilbert space

\[ \mathcal{H}^* = \mathcal{F}(\alpha_n^i) \otimes \mathcal{F}(\tilde{\alpha}_n^i) = \mathcal{F}(\alpha_k^i) \]  

(6.5)

which is the usual free-field Fock space with \( \alpha_k = \alpha_n \) for \( k > 0 \) and \( \alpha_k = \tilde{\alpha}_n \) for \( k < 0 \). Therefore one has a unitary evolution described by a Schrödinger equation

\[ i \frac{\partial}{\partial t} \Psi(t) = \hat{H} \Psi(t) \]  

(6.6)

where \( \Psi(t) \in \mathcal{H}^* \), and hence no transitions from pure into mixed states occur in this theory.

Although the gauge (6.1) is suitable for deriving the RPS Hamiltonian and demonstrating the unitarity of the theory, it is not useful for studying other relevant dynamical variables, like metric and curvature. The CGHS conformal factor \( \tilde{\rho} \) is given by

\[ e^{\tilde{\rho}} = e^\rho \]  

(6.7)

and by using the eq. (4.14), \( \tilde{\rho} \) can be expressed in the gauge (6.1). However, that expression is not very useful since we do not know the explicit expression for \( F \). On the other hand, when choosing the light-cone gauge, instead of using \( (\phi_-, \pi_-) \) variables one can use \( (\phi_+, \pi_-) \) variables, or equivalently \( (\chi, \pi_\xi) \). The advantage of doing this is evident from the eq. (4.14), since it implies \( \chi = \rho \) and \( \pi_\xi = \pi_\phi \). Then from the constraint equations one can obtain an explicit expression for \( \phi \). Hence the gauge

\[ \chi = 0 \quad , \quad \pi_\xi = 0 \]  

(6.8)

is equivalent to

\[ \rho = 0 \quad , \quad \pi_\phi = 0 \]  

(6.9)

so that the constraint equations become

\[ G_0 = -4\lambda^2 + 2\phi'' + \frac{1}{2} \sum_i (\pi_i^2 + \phi_i^2) = 0 \]  

(6.10)
and
\[ G_1 = -2\pi' + \sum_i \pi_i \phi_i' = 0 \] \hspace{2em} (6.11)

From eq. (6.10) we get
\[ \phi = A + Bx + \lambda^2 x^2 - \frac{1}{4} \int dx \int dx' \sum_i (\pi_i^2 + \phi_i'^2) \] \hspace{2em} (6.12)

When compared to the solution (4.8-9), we see that the gauge (6.8) corresponds to \( p = x^+ \) and \( m = x^- \) and
\[ A = a_0 + b_0 - \lambda^2 t^2 \quad , \quad B = -a_1 = b_1 \] \hspace{2em} (6.13)

This illustrates the fact that choosing a canonical gauge is the same as choosing coordinates on \( M \). The corresponding Hamiltonian is again the free-field Hamiltonian (6.4).

7. Conclusions

The immediate conclusion is that a unitary quantum theory of 2d dilaton gravity can be constructed. In turn that implies that the 2d black holes in such a theory do not destroy information, and a unitary S-matrix exists. Furthermore, the S-matrix is trivial since the matter is described by a free-field Hamiltonian. This makes one suspicious whether black holes in such a theory have semi-classical properties we wanted to study in the first place, most notably do they evaporate. That black hole evaporation can occur in the theory can be seen from the following argument [12].

Let \( \Psi_0 \) be a physical state at \( t = 0 \) such that
\[ < \Psi_0|\hat{g}(x)|\Psi_0 > , \quad < \Psi_0|\hat{R}(x)|\Psi_0 > \] \hspace{2em} (7.1)

are regular functions for every \( x \in \Sigma \). \( g(x) \) is a spatial metric, given by eq. (6.7), and in the gauge (6.8) \( \phi \) is given by eq. (6.12). \( R(x) \) is the scalar curvature, which can be also expressed in terms of the free fields, via the formula \( R = e^{-\tilde{\rho}} \partial_+ \partial_- \tilde{\rho} \). Promoting these variables into well defined Hermitian operators is not an easy task, but the results of the covariant approach [23] imply that it can be done. The time evolution of \( \Psi_0 \) is then given by
\[ \Psi(t) = e^{-i\hat{H}t}\Psi_0 \] \hspace{2em} (7.2)

and when the apparent horizon forms in the effective metric \( < \Psi(t)|\hat{g}(x)|\Psi(t) > \), one can split the modes of \( \phi_i \) into those which are inside the horizon and those which are outside the horizon. Then a density matrix \( \hat{\rho}(t) \) can be associated with \( \Psi(t) \)
by tracing out the states corresponding to the modes inside the horizon. Given the density matrix $\hat{\rho}(t)$, one could find out when it takes approximately the thermal form

$$\hat{\rho} \approx \frac{1}{Z} e^{-\beta H_+} ,$$

(7.3)

where $H_+$ is the Hamiltonian of the modes outside the horizon. More precisely, let $\Psi_0$ be a semiclassical state such that the effective quantum metric

$$g_{\text{eff}}(t,x) = \langle \Psi_0 | \hat{g}_H(t,x) | \Psi_0 \rangle = g_{\text{b.h.}}(t,x) \left( 1 + \epsilon \delta g_1(t,x) + \epsilon^2 \delta g_2(t,x) + \ldots \right) ,$$

(7.4)

where $\epsilon$ is a small dimensionless parameter constructed from $\lambda$, Planck’s constant and parameters of the matter state $\Psi_0$. $\hat{g}_H(t,x) = e^{iHt} \hat{g}(x)e^{-iHt}$, while $g_{\text{b.h.}}$ is a black hole metric corresponding to the matter distribution described by the semiclassical state $\Psi_0$. The corrections $\delta g_n$ describe the backreaction effects of the quantum matter on the classical metric, and can be calculated from the identity

$$\hat{g}_H = (-\lambda^2 x^+ x^- - \hat{F})^{-1} = (-\lambda^2 x^+ x^- - \langle \hat{F} \rangle)^{-1}(1 - \delta \hat{F})^{-1}$$

(7.5)

where

$$\delta \hat{F} = (-\lambda^2 x^+ x^- - \langle \hat{F} \rangle)^{-1}(\hat{F} - \langle \hat{F} \rangle) = g_{\text{b.h.}}(\hat{F} - \langle \hat{F} \rangle) ,$$

(7.6)

so that

$$\epsilon^n \delta g_n = \langle \delta \hat{F}^n \rangle .$$

(7.7)

All expectation values are with respect to $\Psi_0$, and

$$\hat{F} = \frac{1}{2} \int dx^+ \int dx^+ : \partial_+ \hat{\phi}_i \partial_+ \hat{\phi}_i : + \frac{1}{2} \int dx^- \int dx^- : \partial_- \hat{\phi}_i \partial_- \hat{\phi}_i : ,$$

(7.8)

where the normal ordering is with respect to the in vacuum $|0_{in} \rangle$, which is defined with respect to the $g_{\text{b.h.}}$ metric at $t = -\infty$. In the semiclassical approximation the backreaction effects are neglected, and therefore one gets a quantum free field propagation on the black hole background. Furthermore if $\Psi_0$ is chosen such that $\Psi(t)$ is close to the $|0_{in} \rangle$ for late times, then by the standard argument [24], the density matrix will be thermal for late times with the temperature equal to the Hawking temperature

$$T = \frac{\lambda}{2\pi} .$$

(7.9)

This program will involve some non-trivial calculations, and still has to be tested, but we do not see anything in principle which could spoil this scenario.

If we accept the argument that the theory has a correct semiclassical limit together with the fact that the theory is unitary, we are then in position to say something about
the information loss problem. Since the matter comes in and out of the black hole without any hindrance, we can say that there are no remnants, because they would correspond to a formation of a bound state. Hence the information which falls in has to escape during the evaporation process. This scenario is usually criticized on the grounds that it violates locality and causality. However, in a quantum theory, non-locality is unavoidable since quantum mechanics is not local, which was confirmed by numerous EPR-type experiments. Moreover, as demonstrated in the case of the EPR experiment, this non-locality may be such that information cannot be transmitted faster than light. Another point is that the usual notion of locality is defined for a field theory on a fixed spacetime background, while in quantum gravity the metric is a hermitian operator, so that the definite spacetime background only emerges for special states.

One can also study the unitarity of the theory in the S-matrix formalism. However, in order to obtain a non-trivial scattering matrix, the authors of had to modify the theory by introducing a reflecting boundary condition. Such a theory is locally the same as 2d dilaton gravity, and the S-matrix is unitary, which is in agreement with our results. It is interesting that the studies of other toy models of gravitational collapse have given the same result, i.e. that a unitary quantum theory of such models exists. This is a very good sign for believing in the existence of a unitary theory of a real collapse, because if we had not been able to make the toy models unitary, than it would not have been any hope for the full theory. On the other hand, demonstrating the unitarity of a real collapse will be an extraordinary task. Even in the spherically symmetric case, the equations of motion are not integrable, which means that the technique of constructing free-field canonical transformation is not going to work. Most probably one would have to adapt techniques developed in the context of full general relativity with matter, like loop variables. The problem of time in quantum gravity may seem like an immediate obstacle for a unitary theory, but this is really the problem of quantum cosmology, while in the gravitational collapse we are dealing with the asymptotically flat boundary conditions, hence a global time is always available.

We must also stress that in the framework of canonical quantum gravity one deals only with a single universe, and spacetime topology changes where a baby universe or a wormhole is created cannot be addressed by the formalism. Such processes can be used to explain the information loss, but the problem with this is that they require a third quantized theory of gravity, which is even less understood than the canonical quantum gravity.

Beside the question of unitarity of gravitational collapse, the question of the fate
of the classical singularity in the quantum theory requires an explanation. Although we have managed to find observables in our toy model which are well defined at the singularity, there will be other observables, those associated with the scalar curvature, which will not be well defined at the singularity. One way of resolving this problem is to study an effective scalar curvature \[ R_{\text{eff}}(x,t) = \langle \Psi_0 | e^{i\hat{H}t}\hat{R}(x)e^{-i\hat{H}t}|\Psi_0 \rangle . \] (7.10)

If it stays a regular function for every \( x \) and \( t \geq 0 \) and for every \( \Psi_0 \) that satisfies the conditions of eq. (7.1), then we could say that the singularity has been removed from the quantum theory. A very similar idea has been proposed in [27]. However, there is no a priori reason for something like this to happen, and the result will depend on the dynamics of the theory. Note that such approach has been already tried in the context of mini-superspace cosmological models [28]. In analogy with the 2d dilaton gravity, a canonical transformation was constructed which maps the Hamiltonian constraint into a quadratic form. As a consequence the quantum evolution is unitary, but the classical singularity is not removed, since the expectation value of the Weyl curvature scalar is infinite at \( t = 0 \) for every physical state.

One may hope that something like this will not happen in the case of dilaton gravity models, because of the extra smearing due to presence of a spatial coordinate. However, judging from the results of the covariant approach analysis [23], it is very likely that the effective scalar curvature (7.10) will not stay finite for all regular initial states. Still, it is not clear what does this mean, since one can invoke the argument that the initial state has evolved into a state which does not have a spacetime interpretation. Clearly, further studies are necessary in order to clarify this issue.

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