A Markovian production-inventory system with consideration of random quality disruption

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Received: 12 November 2019; Revised: 20 November 2019; Accepted: 9 April 2020

Abstract
In this paper, we develop a stochastic model for an imperfect production-inventory system that faces random quality disruption and has limited time for production. In each time planning horizon, the inventory level is affected by product quality disruption. When the production system switches to the ‘out-of-control’ state from the ‘in-control’ state, it starts producing some defective products along with defect-free products, and this switching time is stochastic. The process of the inventory levels at the start of the time horizon is expressed by using a finite and continuous state continuous-time Markov process. We construct mathematical expressions of the transition probability, the steady-state probability, and the long-run average cost. Through a numerical experiment, the near-optimal solution is achieved. The outcome shows that under the situation of production time constraint, the integration of safety stock in an interruption prone production-inventory system, assists in improving the average cost function. The results also show that the optimal safety stock is large when the defect percentage, or the production switching rate, or the shortage cost is very high. Conversely, the small safety stock is desirable when the inventory holding cost or the rework cost is high.

Keywords: Quality disruption, Imperfect production-inventory system, Safety stock, Rework, Production time constraint, Optimization

1. Introduction
A perfect production system is a system that produces a defect-free product, but in real life, it is almost impossible. The poor conditions of means of production, such as the poor condition of machines, unskilled workers, poor quality of raw materials, etc., are responsible for creating an imperfect production system. When the production system is affected by a quality disruption, it switches to the ‘out-of-control’ state from the ‘in-control’ state and starts producing some defective products along with defect-free products. In a practical situation, when multiple products are produced by using a single manufacturing facility a manufacturer needs to fix the production run time for each product while preparing a production plan and when any disruption occurs, the manufacturer cannot extend the production run time for a single product because it affects the production schedules of the next products. Sometimes it is very costly to revise the existing production schedule (Pujawran and Smart, 2012). Defective items are generated from an imperfect production system, and the manufacturer has to perform the reworking operations to restore the defective items to the perfect quality items. A small portion of the defective items can be reworked within the planning horizon because the reworking time is limited. Rework is essential when the produced product is too expensive, or the raw material is insufficient, and therefore it is not a wise decision to consider defective items as scraps. For example, defective metal cabinets and book-shelves are commonly renovated in the sheet metal processing industry, and the imperfect alignment of the steering wheels is adjusted to fix the steering column at a right-angle with the steering wheels in the automotive industry (Sarker et al., 2008).

To satisfy certain portions of customer demand, which can be lost due to the product’s quality failure, a manufacturer can use an emergency safety stock that can prevent a shortage caused by quality failure.

Large portions of research studies (Alfares et al.,2005; Chiu et al.,2007; Sarker et al., 2008; Pal et al., 2013; Pal and...
production-inventory system, and hence, several mathematical models are developed by taking production run-time, production lot size, base stock, and system reliability as decision variables. In a real-life manufacturing environment like the metallic books shelve and cabinet manufacturing, different batches of books shelve/cabinet are produced sequentially. As per the production schedule, a fixed production run time is allocated for every single batch. The production process also generates some defective items, and those defective items could not be considered as scraps because the produced items are very expensive, and therefore, must be reworked. In a multi-product manufacturing environment, revising a production schedule due to quality interruption is not a good decision. For example, if the production run time for the future batch (batch 2) is revised/reduced due to a certain extension of the production run time for the current batch (batch 1), then, due to the lower production compared to the planned production, the scarcity of demand will be created for the future batch. In the end, the production system constitutes a shortage cost. On the other hand, to avoid the cost of the shortage, if the amount of the shortage of batch 2 is produced during certain periods in the future, a new setup/changeover operation is required before starting a new production run for the overdue quantity. This also results in extra penalty costs due to the extra setup. Managing such kind of production-inventory system where the system has some resource constraint like the production time constraint is not an easy task.

The main contribution of our research is that we develop a production-inventory model for an imperfect production system by assuming a fixed production time for production and reworking activities. This modeling assumption has not been considered before. We also assume that the items produced are expensive, and therefore, all the defective items must be reprocessed within the fixed run time. Considering the beginning (initial) level of inventory in each fixed time planning horizon as a state, we formulate a continuous-time, continuous state Markov process. First, the mathematical expressions for the transition probability are constructed, and then with the help of the transition probability, the expressions for the steady-state probability are developed. Finally, the expression of the long-run average cost, consisting of a shortage cost, a cost of repair, a cost of a fixed inspection, and an inventory holding cost, is derived, after which it is minimized with respect to safety stock. The key objective is to develop a model that helps the manufacturer to make an optimal decision regarding the selection of the safety stock and to operate the production-inventory system effectively.

A complete literature review can be found in the next section. The model with important assumptions is discussed in section 3. A numerical example is presented in section 4. Finally, in section 5, we summarize our research and provide some directions for future research.

2. Literature review

E.W. Taft, in 1918, first developed an economic production quantity (EPQ) model for an integrated production-inventory system. The assumptions of the model, however, limit its applicability in many practical applications. As a result, researchers are now extending this model in different directions by relaxing the assumptions of the model. For example, Nobil et al. (2020) extend the traditional EPQ model by considering a multi-product single machine EPQ model where different products have different production and demand rates. Cárdenas-Barrón and Sana (2014) develop a production-inventory model for a supply chain system consisting of a manufacturer and a retailer. They extend the traditional model by taking up the backlog. The production inventory system of the supply chain is also being studied by Ueno et al. (2012). The authors introduce new terminology in their model, called advance demand information. One of the major extensions of the model is the consideration of the production of the defective item and the interruption of the machine, i.e., the assumption of an imperfect manufacturing system. Many researchers are currently studying a production inventory model for an imperfect production system.

An imperfect production system is a system that deteriorates over time and produces certain defective products with defect-free products, and at the same time, can be confronted with disruption in production facilities/machines. Most previous research defines an imperfect system as a production system where defective items are produced at the start of production (Chiu et al., 2007; Sarker et al., 2008; Pal and Adhkari, 2019; Jain et al., 2018) or when it shifts from the ‘in-control’ state to the ‘out-of-control’ state (Alfares et al., 2005; Pal et al., 2013). Chiu et al. (2007) and Öztürk (2018) represent an imperfect system by assuming both random machine failure and the production of defective items. Table 1 provides research papers related to our research, composed of their assumption concerning the imperfect system, nature of production run time, and production policy.
Karim and Nakade, Journal of Advanced Mechanical Design, Systems, and Manufacturing, Vol.14, No.5 (2020)

Table 1   Literature review

| References               | Imperfect manufacturing system | Production run time/Period | Production policy                                                                 |
|--------------------------|--------------------------------|---------------------------|-----------------------------------------------------------------------------------|
| Time to generation of    | M/C failure                    |                           |                                                                                   |
| defects                  | Rework                         |                           |                                                                                   |
|                          | Scrap                           |                           |                                                                                   |
|                          | No scrap                        |                           |                                                                                   |
| Sarkar et al. (2010)     | ‘Out-of-control’ state          | ✓                         | Flexible                                                                          |
|                          |                                 |                           | Optimize machine reliability, safety stock, and production lot size                |
|                          |                                 |                           |                                                                                   |
| Sana (2010)              | Starting of production          | ✓                         | Flexible                                                                          |
|                          |                                 |                           | Optimize product reliability and production rate                                   |
|                          |                                 |                           |                                                                                   |
| Pal(2013)                | ‘Out –of –control’ state        | ✓                         | Flexible                                                                          |
|                          |                                 |                           | Optimize process reliability parameter, production lot size                        |
|                          |                                 |                           |                                                                                   |
| Taleizadeh et al. (2013) | Starting of production          | ✓                         | Flexible                                                                          |
|                          |                                 |                           | Optimize production cycle length                                                  |
|                          |                                 |                           |                                                                                   |
| Taleizadeh et al. (2014) | Starting of production          | ✓                         | Flexible                                                                          |
|                          |                                 |                           | Optimize production cycle length                                                  |
|                          |                                 |                           |                                                                                   |
| Paul et al. (2014)       | Starting of production          | ✓                         | Flexible                                                                          |
|                          |                                 |                           | Optimize production process reliability, lot size, and cycle length                |
|                          |                                 |                           |                                                                                   |
| Bhunia et al (2017)      | Starting of production          | ✓                         | Flexible                                                                          |
|                          |                                 |                           | Optimize production rate, and cycle length                                         |
|                          |                                 |                           |                                                                                   |
| Pal and Adhikari (2019)  | Starting of production          | ✓                         | Flexible                                                                          |
|                          |                                 |                           | Optimize production quantity, selling price, and back order level                  |
|                          |                                 |                           |                                                                                   |
| Aleem et al. (2018)      | Starting of production          | ✓                         | Flexible                                                                          |
|                          |                                 |                           | Optimize process reliability, production quantity, and demand rate                 |
|                          |                                 |                           |                                                                                   |
| Shah et al. (2018)       | Starting of production          | ✓                         | Flexible                                                                          |
|                          |                                 |                           | Optimize production process reliability                                           |
|                          |                                 |                           |                                                                                   |
| Nobil et al. (2018a)     | Starting of production          | ✓                         | Flexible                                                                          |
|                          |                                 |                           | Optimize production lot size, and no. of cycles for raw material                   |
|                          |                                 |                           |                                                                                   |
| Nobil et al. (2018b)     | Starting of production          | ✓                         | Flexible                                                                          |
|                          |                                 |                           | Optimize production cycle length, and shortage quantity                            |
|                          |                                 |                           |                                                                                   |
| Öztürk(2018)             | Starting of production          | ✓                         | Flexible                                                                          |
|                          |                                 |                           | Optimize production runtime                                                       |
|                          |                                 |                           |                                                                                   |
| Jain et al. (2018)       | Starting of production          | ✓                         | Flexible                                                                          |
|                          |                                 |                           | Optimize production runtime                                                       |
|                          |                                 |                           | and cycle length                                                                  |
|                          |                                 |                           |                                                                                   |
| Chiu et al. (2019)       | Starting of production          | ✓                         | Flexible                                                                          |
|                          |                                 |                           | Optimize production runtime                                                       |
|                          |                                 |                           |                                                                                   |
| This paper               | ‘Out-of-control’ state          | ✓                         | Fixed                                                                             |
|                          |                                 |                           | Optimize safety stock                                                              |

Sana (2010) formulates a mathematical model for an imperfect production-inventory system. According to him, the imperfect system produces perfect and imperfect items, and small quantities of nonconforming items are produced when the product reliability parameter is low. He also assumes that the development costs of production are a function of the reliability parameter of the product and that a lower value of the reliability parameter leads to an increase in the development costs of production. Shah et al. (2018) study an Economic Production Quantity model for perishable items. They represent the demand in terms of a quadratic function, and this demand depends on both the time and sales effort. They consider the reliability of the system as a decision variable. In their model, influenced by inflation, profit is optimized with respect to the reliability parameter. Aleem et al. (2018) study a manufacturing-inventory system, which
also takes into account the reliability of the production process. They believe that the quality of the manufactured products depends on the reliability of the production process. They use different non-traditional solution techniques such as Generalized Reduced Gradient (GRG) algorithm, Evolutionary Algorithm (EA), Monte Carlo non-deterministic method, etc. to obtain the optimum solution. Computational results are then compared and discussed. Considering the reliability of the production process and the production cycle length, Paul et al. (2014) study a manufacturing-inventory model for a defective production system. The optimal solution of their developed model is obtained through a meta-heuristic particle swarm optimization (PSO) algorithm. Taleizadeh et al. (2013) investigate an unreliable manufacturing system and propose an economic production quantity model for multiple items by considering partial back-ordering, rework, budget, and service level constraint. The authors consider the production lot size and the production cycle length as decision variables. They also prove the convexity of their proposed model. Taleizadeh et al. (2014) also investigate a production-inventory system that yields some defective products along with quality products. A single machine is utilized to produce multiple products. The faulty items are repaired, and it incurs a substantial cost. The defective products are repaired after the end of the regular manufacturing period. They also assume that there is no interruption during the recovery process, but the process is interrupted during normal operation. The cycle time is considered as a decision variable in their model, and henceforth, they recommend using an optimal cycle time that can decrease the total inventory cost.

Alfares et al. (2005) formulate an imperfect production-inventory model for deteriorating items by incorporating production and inspection decisions. According to their model, produced items and production process both are worsened over time. The system generates defective items when it switches to the ‘out-of-control’ state from the ‘in-control’ state. They assume that the planned inspection is carried out at equal intervals, and if defective products are identified during the inspection time, a remedial maintenance action is carried out until the system is transferred to the ‘in-control’ state from the ‘out-of-control’ state. Öztürk (2018) proposes a production-inventory model for an imperfect production system by including an inspection policy whereby the inspection is carried out both during the production period and after the end of production. He provides an analytical solution by considering production run-time as a decision variable. Chiu et al. (2019) study an unreliable production inventory system that takes into account product quality, machine reliability, and an adequate service level. Eventually, the authors determine the optimal policy of replenishment for the proposed model by incorporating the joint effects of random machine failure, back-order, rework, and scrap.

Pal and Adhikari (2019) develop a price-sensitive, unreliable manufacturing-inventory model considering back-logged situations at the beginning of the production cycle. They attempt to optimize the expected average profit for their proposed model by jointly optimize the selling price, order quantity, and back-order level. Sarkar et al. (2010) study an imperfect production inventory system. They consider the random machine breakdown, repair, and deterioration of the process in a single study. According to their model, when the process is deteriorated over time, it switches to the ‘out-of-control’ state from the ‘in-control’ state and producing defective items with a certain percentage. They consider machine reliability, safety stock, and production lot size as decision variables. Nobil et al. (2016) study a multi-machine, multi-product economic production quantity problem for an imperfect production system. They formulate the problem as a mixed-integer non-linear programming problem and solve the problem by using a hybrid genetic algorithm. The imperfect production-inventory system is also being studied by Nobil et al. (2018a), Nobil et al. (2018b), and Bhunia et al. (2017).

The literature study showed that the production time constraint in the context of an imperfect production inventory system is not discussed. Production time constraint is an important assumption that must be taken into account when modeling a real production system, but this assumption is absent in the existing literature. The motivation of the research comes from this significant research gap. We try to fill this research gap in this paper by considering an imperfect production-inventory system where 1. the production switching time is stochastic, and 2. the maximum production run time is fixed and the revision of a production plan/run time caused by the disruption is not possible.

3. Model development
3.1 Model description
An integrated production-inventory system is illustrated in Fig.1 where a product is manufactured at a rate of $P$ for a maximum production run time of $T_{max}$ because the system produces different products one after the other in accordance with the production plan, and the run time is fixed for each product, and this time is not extendable because
the revision of the production plan is costly. Production and inspection take place simultaneously. The fixed inspection cost is denoted by \( w \). The manufacturing system is imperfect, and after a certain time of \( X \), can switch to ‘out-of-control’ state from ‘in-control’ state. This time is a stochastic variable and follows an exponential distribution with a rate of \( \mu \).

The notations used in this paper are shown in Table 2.

| Notation | Description |
|----------|-------------|
| \( P \) | production rate |
| \( P_1 \) | reworking rate |
| \( D \) | demand rate |
| \( \alpha \) | the percentage of defective items produced in the ‘out-of-control’ state |
| \( H \) | the fixed planning horizon |
| \( T_{\text{max}} \) | maximum allowable production run time |
| \( X \) | production switching time: a continuous stochastic variable follows an exponential distribution. |
| \( x \) | the outcome production switching time |
| \( \mu \) | switching rate |
| \( z \) | the production runtime in ‘out-of-control’ state. |
| \( R \) | safety stock: a decision variable |
| \( R' \) | inventory level before the start of a time horizon |
| \( R'' \) | inventory level after the end of a time horizon |
| \( Q \) | defective item’s quantity |
| \( \bar{Q} \) | average value of defective item’s quantity |
| \( S \) | shortage quantity |
| \( \bar{S} \) | average value of shortage quantity |
| \( T \) | average value of inventory holding quantity |
| \( E[\bar{V} \mid R'] \) | expected value of \( \bar{V} \) for given \( R' \) |
| \( h \) | inventory holding cost per lot per unit time |
| \( s \) | shortage cost per lot |
| \( d \) | rework cost per lot |
| \( w \) | fixed inspection cost per unit time |
| \( C_t(R') \) | the expected total average cost in a planning horizon |
| \( C \) | a long-run average of total cost in a planning horizon |
| \( P(R' \mid R) \) | the probability of transition from state \( R' \) to state \( R'' \) where \( R'' = R, 0 \) |
| \( f_{R'}(R' \mid R) \) | the probability of transition from state \( R' \) to state \( R'' \) where \( 0 < R'' < R \) |
| \( \pi_s \) | the steady-state probability when \( R = R' \) |
| \( \pi_0 \) | the steady-state probability when \( R = 0 \) |
| \( f(r) \) | the steady-state density function when \( 0 < R = r < R \) |

The density function of \( X \) is defined as \( g(x) \). A safety stock of \( R \) is considered as a decision variable. If the production is carried out for a short period of time in the ‘out-of-control’ state, a safety stock will prevent demand shortage caused by a small production volume. The corrective maintenance operation is performed after the end of the production run time for each product (if necessary) because the production time for each product is limited, and it is not
Fig. 1 Production-inventory system for different case
possible to interrupt production by performing maintenance during production time. The defective products generated in the ‘out-of-control’ state are reworked at an expense just after the regular production time of \( x+z \), where \( z \) represents the production run time in ‘out-of-control’ state. If the system spends more time in the “out-of-control” state, it produces a large volume of defective products, which is not expected due to the high rework costs. Moreover, if the volume of defective items is high, it will then not be possible to repair all the items because of fixed production time. In this concern, the production run time in the ‘out-of-control’ state must be selected in such a way that the inventory level after \( H \) is maximized as much as possible, and also there are no defective items after the end of the planning horizon of \( H \). It means reworking quantity always equal to defective quantity. A significant proportion of defective items are produced in the ‘out-of-control’ state, and this proportion is constant and is defined as \( \alpha \). In every time horizon, the production system generates both quality and defective products over a period of \((x+z)\) if the production system undergoes in the ‘out-of-control’ state from the ‘in-control’ state. No scrap is produced during the period of \( T_{\text{max}} \). After the time period of \((x+z)\), the defective products are reworked at a normal rate of \( P_1 \) for a maximum time period of \( T_{\text{max}} - (x+z) \). The rework cost is denoted by \( d \). No further defective items are produced during the reworking time of \( T_{\text{max}} - (x+z) \). All defective items \((\alpha Pz)\) are reworked within the time of \( T_{\text{max}} - (x+z) \), i.e., \( x+z + \frac{\alpha Pz}{P_1} \leq T_{\text{max}} \). The same production facility is used for both production and reworking operations. The product is produced and sold within a planning horizon \((H)\), and this planning horizon is fixed. The product is consumed at a rate of \( D \), and this rate is also constant. The production and reworking rates are higher than the demand rate, i.e., \( P > D \), \((1-\alpha) P > D \), and \( P_1 > D \). Inventory is accumulated at a rate of \( P - D \) during normal production running conditions and at a rate of \((1-\alpha) P - D \) during ‘out-of-control’ state. As soon as the production and reworking are finished, inventory is depleted at a rate of \( D \). The inventory holding cost is \( h \). We also assume that \( PT_{\text{max}} - HD > R \), that is, the difference between the maximum production amount and the demand quantity in a planning horizon, is greater than the amount of safety stock.

To avoid loss of production, a safety stock of \( R \) is considered to prevent shortage or lost sales due to a random disruption of the quality of the product. The unsatisfied demand is considered as shortage or loss of sales. The shortage cost is denoted by \( s \).

We need to identify the optimum value of \( R \) for which the production-inventory system performs efficiently, and the total operating cost of the system is minimized. The range of \( z \) is \( 0 \leq z \leq T_{\text{max}} - x \). The inventory position before the start of a new production run is defined as \( R \) where \( 0 \leq R \leq R \). The inventory position after the end of the planning horizon is defined as \( R' \) where \( 0 \leq R' \leq R \). This \( R' \) also represents the initial inventory level \((R)\) for the next planning horizon. The value of \( R \) is denoted by \( r \).

Based on the different values of \( x \) and the level of \( R' \), we have four possible cases. All those cases are shown in Table 3. Sometimes the production switching time of \( x \) is such that for any value of \( R ' \) and without reworking, it is possible to reach the inventory level of \( R \) after the end of the time horizon of \( H \) (see also in Fig. 1). In this situation, the position of \( x \) can be defined as follows

\[
R' + P x - DH = R, \quad 0 \leq R' \leq R
\]

which implies \( x = \frac{DH + R - R'}{P} \).

Let \( x(R) = \frac{DH + R - R'}{P} \). The case where \( x(R) \leq x < T_{\text{max}} \) is defined as case 0.

Sometimes the position of \( x \) is such that for any value of \( R \), we can reach the inventory level of \( R \) after the end of the time horizon of \( H \), but some reworking is required. For this situation, \( x \) can be expressed as follows

\[
R' + P x + (1-\alpha)Pz + \alpha Pz - DH = R, \quad 0 \leq R' \leq R
\]

which implies \( z = \frac{DH + R - R'}{P} - x \).

By putting the value of \( z \) in the inequality equation of \( x + z + \frac{\alpha Pz}{P_1} \leq T_{\text{max}} \), we finally get

\[
x \geq \left( \frac{(P_1 + \alpha P)(DH + R - R')}{\alpha P^2} \right) - \frac{P_1}{\alpha P} T_{\text{max}}.
\]
Let \( x_1 = \frac{(p_1 + \alpha P)(DH - R - R')}{aP^2} - \frac{p_1}{aP} T_{\text{max}} \). The case where \( x(R') \leq x < x_1(R') \) is defined as case 1.

Again for any value of \( R \), the position of \( x \) leads to an inventory level of \( R' < R \) after the end of the planning horizon of \( H \). We need to select \( z \) in such a way that maximizes the value of \( R' \), and there will be no defective items after \( H \). Here, \( x \) satisfies the following inequalities.

\[
R + P x + (1 - \alpha) P z + a P z - DH < R \quad \text{and} \quad R + P x + (1 - \alpha) P z + a P z - DH > 0
\]

The maximum possible value of \( z \) that maximize \( R' \) is \( z = \frac{T_{\text{max}} - x}{1 + \frac{\alpha P}{P_1}} \). By putting the value of \( z \) in the inequality of \( R + P x + (1 - \alpha) P z + a P z - DH < R \) and \( R + P x + (1 - \alpha) P z + a P z - DH > 0 \), we finally get

\[
x < \left( \frac{(p_1 + \alpha P)(DH - R - R')}{aP^2} \right) - \frac{p_2}{aP} T_{\text{max}}, \quad x > \left( \frac{(p_1 + \alpha P)(DH - R')}{aP^2} \right) - \frac{p_2}{aP} T_{\text{max}}.
\]

Let \( x(R') = \frac{(p_1 + \alpha P)(DH - R')}{aP^2} - \frac{p_1}{aP} T_{\text{max}} \). The case where \( x(R') < x < x_1(R') \) is defined as case 2.

### Table 3 Boundary limits of \( x \) for different cases

| Case | The value of \( R' \) | \( x(R') \) | Boundary limit of \( x \) |
|------|---------------------|-------------|--------------------------|
| 0    | \( 0 \leq r \leq R \) | 0 < \( x_1(r) \) < \( x \) | \( x_1(r) = \frac{DH - r}{p} \) |
| 1    | \( 0 \leq r \leq R \) | 0 < \( x_1(r) \) < \( x_1(r) \) | \( x_1(r) = \frac{(p_1 + \alpha P)(DH - r)}{aP^2} - \frac{p_1}{aP} T_{\text{max}} \) |
| 2    | \( 0 < x < x_1(r) \) | \( 0 < x \leq x_1(r) \) | \( x_1(r) = \frac{(p_1 + \alpha P)(DH - r)}{aP^2} - \frac{p_1}{aP} T_{\text{max}} \) |
| 3    | \( 0 < x \leq x_1(r) \) | \( 0 < x \leq x_1(r) \) | \( x \leq \frac{(p_1 + \alpha P)(DH - R')}{aP^2} - \frac{p_1}{aP} T_{\text{max}}, \ i.e., \ x \leq x_1(R') \). |

Lastly, \( x \) falls in a position such that for any value of \( R' \) the production system may face shortage \( (R' \leq 0) \). Even though the maximum possible value of \( z = \frac{T_{\text{max}} - x}{1 + \frac{\alpha P}{P_1}} \) is utilized, but in the end, the system faces a shortage. Here, \( x \) can be defined as follows

\[
R + P x + (1 - \alpha) P z + a P z - DH \leq 0, \quad 0 \leq R \leq R
\]

which implies \( P x + P z \leq DH - R \).

By putting the value of \( z \) in the inequality of \( P x + P z \leq DH - R \), we finally get

\[
x \leq \frac{(p_1 + \alpha P)(DH - R')}{aP^2} - \frac{p_1}{aP} T_{\text{max}}, \ i.e., \ x \leq x_1(R').
\]

Finally, the case where \( 0 < x \leq x_1(R') \) is expressed as case 3.

### 3.2 Analysis

#### 3.2.1 Transition probability and steady state probability of transition

The steady-state probability for all the cases can be defined by the following equations (Feller, 1971; Miyazawa, 1993).

\[
\pi_R = \pi_0 P(R|R' = R) + \int_0^R f(r) dr P(R|R' = r)
\]

\[
f(r) = \pi_R f_R^*(r|R' = R) + \pi_0 f_R^*(r|R' = 0) + \int_0^R f(r') dr' f_R^*(r|R' = r'), 0 < r < R
\]

\[
\pi_R + \pi_0 + \int_0^R f(r) dr = 1
\]
X is a continuous random variable and assumed to follow an exponential distribution with a rate of $\mu$. Examples, where production switching time follows exponential distribution can be found in Alfares et al. (2005), and Pal et al. (2013). $X$ is truncated at $T_{\text{max}}$ and here we assume $\int_{0}^{T_{\text{max}}} g(x)dx + P(X = T_{\text{max}}) = 1$ and subsequently $P(X = T_{\text{max}}) = e^{-\mu T_{\text{max}}}$. It is not a problem because in case 1 under the assumption of $PT_{\text{max}} - DH > R$, and for any value $x_{1}(R')$ where $0 \leq R' \leq R$, it is possible to obtain the inventory level of $R$ after the end of planning horizon of $H$.

Transition probabilities can be derived as follows.

$$P(R|R' = R) = P(X \geq x_{1}(R)) = e^{-\mu x_{1}(R)}, P(R|R' = 0) = P(X \geq x_{1}(0)) = e^{-\mu x_{1}(0)},$$

$$P(R|R' = r) = P(X \geq x_{1}(r)) = e^{-\mu x_{1}(r)}, 0 < r < R$$

Let $A = \frac{P_{1} + \alpha P}{\alpha P_{2}}$ and, $B = \mu x_{1}(R)$ then

$$\mu x_{1}(0) = B + AR, \quad \mu x_{1}(r) = B + AR - Ar, \quad \mu \left(\frac{P_{1} + \alpha P}{\alpha P_{2}} (r + DH - R) - \frac{P_{1} T_{\text{max}}}{\alpha P}\right) = B + AR - AR,$$

$$f_{R}(r|R' = R) = \frac{d}{dr} P(R' \leq r | R' = R).$$

Since $R + P_{X} + (1 - \alpha)Pz + \alpha P_{z} - DH = R'$, which implies $R + P_{X} (\frac{\alpha P}{P_{1} + \alpha P}) + \frac{P_{1} T_{\text{max}}}{P_{1} + \alpha P} - DH = R'$. So,

$$f_{R}(r|R' = R) = \frac{d}{dr} P \left( R + P_{X} (\frac{\alpha P}{P_{1} + \alpha P}) + \frac{P_{1} T_{\text{max}}}{P_{1} + \alpha P} - DH \leq r \right)$$

$$= \frac{d}{dr} P \left( x \leq \frac{P_{1} + \alpha P}{\alpha P_{2}} (r + DH - R) - \frac{P_{1} T_{\text{max}}}{\alpha P} \right).$$

Since $X$ is an exponential random variable and therefore, the above equation can now be simplified as follows.

$$f_{R}(r|R' = R) = \mu \frac{P_{1} + \alpha P}{\alpha P_{2}} e^{-\mu \left(\frac{P_{1} + \alpha P}{\alpha P_{2}} (r + DH - R) - \frac{P_{1} T_{\text{max}}}{\alpha P}\right)}$$

Similar way, we can compute

$$f_{R}(r|R' = r') = \mu \frac{P_{1} + \alpha P}{\alpha P_{2}} e^{-\mu \left(\frac{P_{1} + \alpha P}{\alpha P_{2}} (r + DH - r') - \frac{P_{1} T_{\text{max}}}{\alpha P}\right)},$$

$$f_{R}(r|R' = 0) = \mu \frac{P_{1} + \alpha P}{\alpha P_{2}} e^{-\mu \left(\frac{P_{1} + \alpha P}{\alpha P_{2}} (r + DH) - \frac{P_{1} T_{\text{max}}}{\alpha P}\right)}.$$

Simplifying Eq. (1) and Eq. (2) yield the following solutions,

$$\pi_{R} = \frac{e^{-B(AR + Ap)} \int_{0}^{R} f(r)(e^{-B(AR + Ap)} - e^{-B(AR + Ap)}) dr}{1 + e^{-B(AR + Ap)} - e^{-B}}, \quad 0 < r < R \quad (4)$$

$$f(r) = u(r) + \int_{0}^{R} K(r, r') f(r') dr', \quad 0 < r < R \quad (5)$$

where

$$u(r) = \frac{A e^{-\mu (Ar + B)}}{1 + e^{-(Ar + B)} - e^{-B}},$$

$$K(r, r') = \frac{e^{Ar} [2 A e^{-(Ar + 2B)}] + A e^{-(Ar + B)}}{1 + e^{-(Ar + B)} - e^{-B}} \cdot \frac{A e^{-(Ar + B)}}{1 + e^{-(Ar + B)} - e^{-B}}.$$

Equation (5) forms a Fredholm integral equation of the second kind. This equation can be solved by using the
Degenerate kernel method (Raisinghania, 2007). The kernel \( K(r, r') \) is degenerate or separate and we take \( K(r, r') = \sum_{i=1}^{m} w_i(r) g_i(r') \). In our problem, the value of \( m \) is 2, and \( K(r, r') \) is stated by the following equation where the function \( w_i(r) \) is assumed to be linearly independent.

\[
K(r, r') = w_1(r) g_1(r') + w_2(r) g_2(r') \tag{6}
\]

where

\[
w_1(r) = \frac{2Ae^{-(Ar+2B)} + Ae^{-(Ar+B)}}{1 + e^{-(AR+B)} - e^{-B}}, g_1(r') = e^{Ar'},
\]

\[
w_2(r) = -\frac{Ae^{-(Ar+B)}}{1 + e^{-(AR+B)} - e^{-B}} \quad \text{and} \quad g_2(r') = 1
\]

Replacing the value of \( K(r, r') \) from Eq. (6) to Eq. (5) yields the following expression.

\[
f(r) = u(r) + w_1(r) C_1 + w_2(r) C_2
\tag{7}
\]

where

\[
C_1 = \int_0^R g_1(r') f(r') dr', C_2 = \int_0^R g_2(r') f(r') dr'
\]

Putting the value of \( u(r) \), \( w_1(r) \), and \( w_2(r) \) in Eq. (7) and after rearrangement, we obtain the following equation.

\[
f(r) = \frac{Ae^{-(Ar+B)}}{1 + e^{-(AR+B)} - e^{-B}} (1 + C_1 - C_2) + \frac{2Ae^{-(Ar+2B)}}{1 + e^{-(AR+B)} - e^{-B}} C_1
\tag{8}
\]

Equation (8) is equivalent to Eq. (5). The solution of this equation yields the solution of \( f(r) \). Again, from Eq. (7) we can write

\[
f(r') = u(r') + w_1(r') C_1 + w_2(r') C_2 \tag{9}
\]

As \( w_1(r) \) and \( w_2(r) \) are linearly independent and now putting the values of \( C_1, C_2, f(r') \) in right-hand side of Eq. (7) and the value of \( f(r) \) itself in left-hand side of Eq. (7) and after rearrangement, we obtain the following system of linear equations.

\[
(1 - \gamma_{11}) C_1 - \gamma_{12} C_2 = \beta_1 \tag{10}
\]

\[
-\gamma_{21} C_1 + (1 - \gamma_{22}) C_2 = \beta_2 \tag{11}
\]

where

\[
\gamma_{11} = \int_0^R g_1(r) w_1(r) dr = \frac{R(2Ae^{-2B} + Ae^{-B})}{1 + e^{-(AR+B)} - e^{-B}}
\]

\[
\gamma_{12} = \int_0^R g_1(r) w_2(r) dr = \frac{-RAe^{-B}}{1 + e^{-(AR+B)} - e^{-B}}
\]

\[
\gamma_{21} = \int_0^R g_2(r) w_1(r) dr = \frac{2(e^{-2B} - e^{-(AR+2B)}) + e^{-B} - e^{-(AR+B)}}{1 + e^{-(AR+B)} - e^{-B}}
\]

\[
\gamma_{22} = \int_0^R g_2(r) w_2(r) dr = \frac{e^{-(AR+B)} - e^{-B}}{1 + e^{-(AR+B)} - e^{-B}}
\]
\[ \beta_1 = \int_0^R g_1(r)u(r)dr = \frac{RAe^{-B}}{1 + e^{-(AR+B)} - e^{-B}} \]

\[ \beta_2 = \int_0^R g_2(r)u(r)dr = \frac{e^{-B} - e^{-(AR+B)}}{1 + e^{-(AR+B)} - e^{-B}} \]

When \((1 - y_{11})(1 - y_{22}) - y_{12}y_{21} \neq 0\), a unique non-zero solution of the system of equations of Eq. (10) and Eq. (11) exists, and hence we obtain a unique solution of \(f(r)\) from Eq. (8). Using the matrix inverse method, we solve the equations of (10) and (11) and obtain the following values of \(C_1\) and \(C_2\).

\[ C_1 = \frac{(1 - y_{22})\beta_1 + y_{12}\beta_2}{(1 - y_{11})(1 - y_{22}) - y_{12}y_{21}} \quad (12) \]

\[ C_2 = \frac{y_{22}\beta_1 + (1 - y_{11})\beta_2}{(1 - y_{11})(1 - y_{22}) - y_{12}y_{21}} \quad (13) \]

The values of \(C_1\) and \(C_2\) are used in Eq. (8) to achieve the steady-state probability distribution of \(f(r)\). Using the steady-state probability distribution of \(f(r)\) in Eq. (4) and Eq. (3), we obtain the following equations of \(\pi_R\) and \(\pi_0\), respectively.

\[ \pi_R = \frac{e^{-(AR+B)}}{1 + e^{-(AR+B)} - e^{-B}} \]

\[ + \{A(1 + C_1 - C_2)e^{-AR + 2B} + 2AC_1e^{-(AR + 3B)}\}R - (1 - e^{-AR})\{(1 + C_1 - C_2)e^{-(AR + 2B)} + 2C_1e^{-(AR + 3B)}\} \]

\[ \pi_0 = 1 - \frac{e^{-(AR+B)} + \{(1 + C_1 - C_2)e^{-B} + 2C_1e^{-2B}\}(1 - e^{-AR})}{1 + e^{-(AR+B)} - e^{-B}} \]

\[ - \{A(1 + C_1 - C_2)e^{-AR + 2B} + 2AC_1e^{-(AR + 3B)}\}R - (1 - e^{-AR})\{(1 + C_1 - C_2)e^{-(AR + 2B)} + 2C_1e^{-(AR + 3B)}\} \]

\[ (1 + e^{-(AR+B)} - e^{-B})^2 \]

### 3.2.2 Long run average of total cost

The long-run average total cost in a planning horizon for the derived model can be expressed by the following equation.

\[ C(R) = \pi_RC_f(R) + \int_0^R f(r)C_f(r)dr + \pi_0C_f(0) \quad (16) \]

\(C_f(R), C_f(r),\) and \(C_f(0)\) represent the expected total cost in a planning horizon when \(r = R, r < R\), and \(r = 0\), respectively. \(C_f(R),\) \(C_f(r),\) and \(C_f(0)\) can be defined by the following equations.

\[ C_f(R) = hE[ \tilde{I}[R' = R] + dE[ \tilde{Q}[R' = R] + sE[ \tilde{S}[R' = R]] + w \]s

\[ C_f(r) = hE[ \tilde{I}[R' = r] + dE[ \tilde{Q}[R' = r] + sE[ \tilde{S}[R' = r]] + w, 0 < r < R \]

\[ C_f(0) = hE[ \tilde{I}[R' = 0] + dE[ \tilde{Q}[R' = 0] + sE[ \tilde{S}[R' = 0]] + w \]

\(E[ \tilde{I}[R' = r], E[ \tilde{Q}[R' = r],\) and \(E[ \tilde{S}[R' = r]]\) represent the expected value of inventory holding, defective items, and shortage quantity, respectively for a certain value of \(r\) where \(0 \leq r \leq R\). These values can be defined as follows.
\[ E[\hat{I}'|R=r] = \left\{ \begin{array}{l} \int_{x_0(r)}^{x_0(r)} g(x)dx + P(X=T_{\text{max}}) \\ \int_{x_1(r)}^{x_1(r)} \overline{I}_1(x,r)g(x)dx + \int_{0}^{x_1(r)} \overline{I}_2(x,r)g(x)dx \\ + \int_{0}^{x_1(r)} \overline{I}_3(x,r)g(x)dx \end{array} \right. \]

where

\[ \overline{I}_0(r) = \frac{1}{H} \left\{ -\frac{r^2}{2P} + \frac{r(DH + R)}{P} + RH + \frac{DH^2}{2} - \frac{(DH + R)^2}{2P} \right\} \]

\[ \overline{I}_1(x,r) = \frac{1}{H} \left\{ \left(\frac{P_1 - D}{2}\right)\left(1 + \frac{AP}{P_1} \right)^2 - \left(P_1 + aP - D\right)\left(\frac{1}{2} + \frac{AP}{P_1} \right) + \frac{DP_1^2}{2PP_1} \right\} x^2 \]

\[ \overline{I}_2(x,r) = \frac{1}{H} \left\{ \left(\frac{aP^2(P_1 - D) + P_1(P_1 + 2aP)((1 - \alpha)P - D)}{2(P_1 + aP)^2} \right) + \left(P - D\right) - \frac{(P - D)^2}{2} \right\} x^2 \]

\[ E[Q'|R'] = \int_{x_1(r)}^{x_1(r)} \overline{Q}_1(x,r)g(x)dx + \int_{x_2(r)}^{x_2(r)} \overline{Q}_2(x)g(x)dx + \int_{0}^{x_2(r)} \overline{Q}_3(x)g(x)dx \]

where

\[ \overline{Q}_1(x,r) = \frac{Q_1(x,r)}{H}, \overline{Q}_2(x) = \frac{Q_{23}(x)}{H}, Q_1(x,r) = aP \left(\frac{DH + R - r}{P} - x\right), Q_{23}(x) = aP \left(\frac{T_{\text{max}} - x}{1 - \frac{P_1}{P}} \right) \]

\[ E[S'|R'] = \int_{0}^{x_2(r)} \overline{S}(x,r)g(x)dx \text{, } 0 \leq r \leq R \text{, where } \overline{S}(x,r) = \frac{S(x,r)}{H}, S(x,r) = \left\{ \begin{array}{l} H - \frac{(Q_1(x,r))}{D} \end{array} \right\} \]
4. Numerical example and sensitivity analysis

4.1 Numerical example

In this section, we conduct an experiment by considering the following parameters. We assume production is done as a lot, and the production rate of $P$ means the number of lot produced per unit time is $P$, and similarly, the

| Table 4 Numerical example |
|---------------------------|
| **Input Parameter**       | **Production rate($P$)** | 7 |  
|                           | **Production switching rate($\mu$)** | 0.1 |
|                           | **Reworking rate($P_1$)** | 9  |
|                           | **Demand rate($D$)** | 2  |
|                           | **Defect percentage($\alpha$)** | 0.3 |
|                           | **Fixed planning horizon($H$)** | 35 |
|                           | **Maximum allowable production run time($T_{max}$)** | 11 |
|                           | **Safety stock: a decision variable($R$)** | 0.1 |
|                           | **Inventory holding cost per lot per unit time($h$)** | 10 |
|                           | **Shortage cost per lot($s$)** | 900 |
|                           | **Rework cost per lot($d$)** | 200 |
|                           | **Fixed inspection cost per unit time($w$)** | 30 |

Fig. 2 Convexity of long run average cost function

First, input parameters and all the equations from Eq. (17) to Eq. (19) are used to calculate the expected total average cost $C_T$ consisting of the expected value of average inventory holding cost, average rework cost, average shortage cost, and fixed inspection cost. Thereafter using Eq. (8), Eq. (14), and Eq. (15), we calculate the steady-state probability ($\pi_R, \pi_0, f(r)$). Finally using Equation (16), the long-run average cost ($C(R)$) is computed. For the assumed input parameters, the long-run average cost is $359.2092$. We repeat the computations for different values of $R$. It is observable that when the safety stock is increased from 0.1 lots to 3.7 lots, the long-run average cost is decreased from $359.2092$ to $356.1124$, and afterward, this cost starts to increase with the additional increment of safety stock ($R$). From this experimental setup, we can reach a conclusion that the near-optimum value of safety stock is 3.7 lots or 3700 units, and consequently, the near-optimal long-run average cost is $356.1124$. In Figure 2, we show the convexity of the long-run average cost function.

4.2 Sensitivity analysis

Sensitivity analysis shows how safety stock and the long-run average cost respond with respect to change of different input parameters. First, we change the inventory holding cost from 1% to 10% (all other costs keep unchanged) and see its effect towards the near-optimal safety stock and the near-optimal long-run average cost. Afterward, we repeat the same things for the shortage cost and the rework cost. The detail findings are provided in Table 5. From Table 5, it is obvious that the inventory holding cost has a significant impact on safety stock. When the
holding cost of inventory increases, the safety stock decreases to minimize the long-run average cost. On the other hand, the opposite situation occurs for the shortage cost because when the shortage cost increases, the safety stock also increases to minimize the long-run average value of shortage quantity and the long-run average cost as well. Rework cost does not have a significant impact on the safety stock as near-optimal safety stock remains unchanged when the rework cost decreases. Whatsoever, when the rework cost is increased by 5%, the safety stock slightly decreases to minimize the long-run average cost. From Table 5, we see that the near-optimal safety stock slightly declines from 3.7 lots to 3.6 lots when the rework cost is increased by 5% ($210). However, if the rework cost is further increased by 10%, 20%, 30%, and so on, then the near-optimal safety stock remains unchanged (3.6 lots). The main reason for having the same safety stock of 3.6 lots even though the rework cost is very high, is that it avoids the high shortage cost. If the safety stock is further decreased from 3.6 lots to some lower value because of high increment of rework cost, then even though it reduces the rework cost but at the same, it increases the shortage cost, and this high shortage cost is responsible for the increment of the long-run average cost. In the end, keeping a small inventory as safety stock is the right choice when the inventory holding cost is high, or the rework cost is very high, or the shortage cost is low and vice versa.

Table 5 Sensitivity analysis with respect to cost parameters

| % change of parameters | 10% | 5% | 1% | Base value | -1% | -5% | -10% |
|------------------------|-----|----|----|------------|-----|-----|------|
| Inventory holding cost/Lot($) | 11.0 | 10.5 | 10.1 | 10.0 | 9.90 | 9.50 | 9.00 |
| Optimal safety stock($R^*$) | 3.0 | 3.0 | 3.6 | 3.7 | 3.7 | 3.9 | 4.1 |
| Change in optimal long-run average cost | 382.0525 | 369.1842 | 358.7205 | 356.1124 | 353.4984 | 343.0176 | 329.8526 |
| Shortage cost/Lot($) | 990 | 945 | 909 | 900 | 891 | 855 | 810 |
| Optimal safety stock($R^*$) | 4.1 | 3.9 | 3.7 | 3.7 | 3.6 | 3.3 | 2.9 |
| Change in optimal long-run average cost | 358.3352 | 357.2546 | 356.3455 | 356.1124 | 355.8734 | 354.8885 | 353.5564 |
| Rework cost/Lot($) | 220 | 210 | 202 | 200 | 198 | 190 | 180 |
| Optimal safety stock($R^*$) | 3.6 | 3.6 | 3.7 | 3.7 | 3.7 | 3.7 | 3.7 |
| Change in optimal long-run average cost | 360.2522 | 358.1823 | 356.5264 | 356.1124 | 355.6984 | 354.0423 | 351.9723 |

The impact of the defect percentage on the long-run average cost is being investigated. In this concern, we change the defect percentage ($\alpha$) from 0.25 to 0.35 (Table 6), and we study its effect on the cost function. From Table 6, we observe that when the percentage of defects is high, we need to keep a large safety stock to minimize the long-run average cost. For example, when the defect percentage is increased from 0.3 to 0.35, the optimal safety stock increases from 3.7 lots to 3.9 lots (new near-optimal), and consequently, the long-run average cost slightly increases from $356.1124 to $356.5264. However, if we keep the same safety stock level of 3.7 lots, even if the defect percentage is increased from 0.30 to 0.35, the long-run average cost becomes $366.7200. As a result, for the increment of safety stock by 1.8 lots, the cost is minimized by $1.6576. The main reason behind keeping a small inventory as safety stock is the right choice when the inventory holding cost is high, or the rework cost is very high, or the shortage cost is low and vice versa.

Lastly, the impact of the production switching rate on the long-run average cost is also investigated. In this connection, we change the switching rate ($\mu$) from 0.09 to 0.11 (Table 7), and we observe its effect on the cost function. From Table 7, we observe that when the switching rate ($\mu$) is high, we need to keep a large safety stock to minimize the long-run average cost. For example, when the switching rate is increased from 0.1 to 0.11 but the optimal safety stock remains the same ($R^*$ = 3.7), and as a result, the long-run average cost is $358.7312. Nonetheless, if we increase the safety stock level from 3.7 lots to 4.3 lots (new near-optimal), then the long-run average cost decreases from $358.7312 to $358.4616. The explanation behind this result is that a small amount of defect-free items is produced when the switching rate is high, a large amount of defective items is produced, and a long time is required to rework these defective items. In the end, the production and reworking operations are not sufficient to meet the customer demand.
and therefore, to avoid the high shortage cost, excessive items are required as safety stock.

Table 6 Sensitivity analysis with respect to the defect percentage ($\alpha$)

| Change of parameter ($\alpha$) | Optimal safety stock ($R^*$) | Long run average cost ($C(R^*)$) |
|-------------------------------|-----------------------------|----------------------------------|
| 0.250                         | 0.8                         | 346.1594                         |
| 0.300                         | 3.7                         | 356.1124                         |
| 0.303                         | 3.8                         | 356.6669                         |
| 0.330                         | 4.8                         | 361.5389                         |
| 0.350                         | 5.5                         | 365.0144                         |

Table 7 Sensitivity analysis with respect to the production switching rate ($\mu$)

| Change of parameter ($\mu$) | Optimal safety stock ($R^*$) | Long run average cost ($C(R^*)$) |
|-------------------------------|-----------------------------|----------------------------------|
| 0.090                         | 2.5                         | 352.8766                         |
| 0.100                         | 3.7                         | 356.1124                         |
| 0.101                         | 3.7                         | 356.3793                         |
| 0.110                         | 4.3                         | 358.4616                         |

The sensitivity of all observed parameters, in particular $P, D, P_1, \alpha, \mu, h, s,$ and $d$ in relation to the long-run average cost, is illustrated in the following Fig. 3. From Figure 3, we see that in terms of cost increment, the long-run average cost is most sensitive towards demand rate, and after demand rate, it is second most sensitive towards per lot inventory holding cost. On the other side, in terms of cost decrement, this long-run average cost is more sensitive towards the reworking rate than the production rate. When all the observed parameters are increased by 1%, the increment of the long-run average cost is highest for the demand rate, and this cost is lowest for the shortage cost. On the other side, for the reworking rate the long-run average cost is lower than the production rate.

However, sometimes based on the input parameters (For example, when $\alpha$ is very small), $x_2(R)$ or $x_2(R)$ is negative or both are negative for any value of $R$. When $x_2(R)$ is negative, we have no case 3, and the boundary limit for case 2 is from 0 to $x_2(R)$. At the same time, there is no transition from 0 to $R$ and $r(<R)$, as if case 3 does not exist, the inventory level never reaches 0 after the horizon of $H$. In addition, there is no steady-state probability of $\pi_0$. $\pi_0$ will be eliminated from Eq. (1), Eq. (2), and Eq. (3), respectively, and there will be no shortage. We will get different forms for Eq. (4), Eq. (5), Eq. (8), and Eq. (14). Equation (15), Equation (19), and the expressions of its components never hold if case 3 does not exist. The Equation (16) will be modified as $C(R) = \pi_0 C_f(R) + \int_0^R f(r) C_f(r) dr$. When both $x_3(R)$ and $x_3(R)$ are negative, there will be no case 2 and case 3, only case 1 and case 0 exist, and the boundary limit of case 1 starts from 0 to $x_3(R)$. There will be no transition from 0 to $R$, and $r(<R)$, also no transition from $r (<R)$ to $R$, and $r' (<R)$. If case 2 does not exist, Eq. (2), Eq. (4), Eq. (5), Eq. (18) and the expressions of its components never hold. Both $\pi_0$ and $f(r)$ terms will be eliminated from Eq. (1) and Eq. (3). The Equation (1) and Equation (3) will be reduced as $C(R) = \pi_0 P(R|R'=R)$ and $\pi_0 = 1$, and one of the equations become redundant. The remaining all equations (from (6) to (15)) correspond to $\pi_0$ and $f(r)$ will not be utilized for the determination of the long-run average cost of $C(R)$. The Equation (16) will be modified as $C(R) = \pi_0 C_f(R)$. 

[DOI: 10.1299/jamdsm.2020jamdsm0071] © 2020 The Japan Society of Mechanical Engineers
5. Conclusion

The results show that, if the defect percentage, or the production switching rate, or the shortage cost is very high, a decision-maker then should keep a large safety stock to minimize the system’s long-run average cost. On the other hand, if the inventory holding cost is high, a decision-maker must hold a small safety stock. Similarly, when the rework cost is very high, a decision-maker should maintain a small safety stock to prevent the generation of a large number of defective items in the ‘out-of-control’ state caused by a large safety stock. As a large number of defective items increase the long-run average cost caused by the rework cost. Based on the parameters, a decision-maker should make a trade-off between the inventory holding, shortage, and reworking quantity that eventually minimizes the long-run average cost.

The proposed model of an imperfect production-inventory system reflects many real-world production systems where the manufacturer is confronted with random disruptions such as quality failure. In order to provide an effective solution, many models have been developed so far by addressing a production-inventory system that is imperfect.

In reality, however, the manufacturer sometimes has a resource constraint, and limited production time is one of them. The manufacturer has to fulfill the customer demand using limited available time of the production. Sometimes customers are not ready to wait for the desired product and switch to another product, and in the end, the manufacturer loses its revenue and goodwill as well. Considering the situation mentioned above, we develop a mathematical model for a production-inventory system where production run time is fixed. We determine the near-optimal value of safety stock that ultimately minimizes the long-run average cost. This model will act as a decision support tool and will help managers to optimize their resources in light of production time constraints.

We derive the model by assuming an exponential distribution of the production switching time. The analysis can be performed by considering other distributions like a uniform distribution. Studying the model by incorporating random machine failure would be a challenging topic. However, our model can further be expanded by redefining the imperfect production system, whereby the production process initially generates a very small percentage of defective items, and when it goes to the ‘out-of-control’ state, it generates a large percentage of defective items. This can be considered in the future.

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