Topological Spin Density Wave

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In this paper, we investigate the topological Hubbard model on honeycomb lattice. By considering the topological properties of the new states, we employ Chern-Simons-Hopf gauge field theories with different \( K \)-matrices. In the formulation of topological field theory, we found spin-charge separated charge-flux binding effect for A-TSDW and spin-charge synchronized charge-flux binding effect for B-TSDW. In addition, we studied the edge states and quantized Hall effect in different TSDWs.

PACS numbers: 75.30.Fv, 75.10.-b, 73.43.-f

Landau’s symmetry breaking paradigm has been very successful as a basis for understanding the physics of conventional solids including metals and (band) insulators. In Landau’s theory different orders are classified by symmetries. The phase transitions between one type of ordered phase and another one (ordered or disordered) are always accompanied by symmetry breaking. Taking spin-density-wave (SDW) as an example. To describe such ordered state with spontaneous spin rotation symmetry breaking, one can define a local order parameter that differs for different SDW states, (antiferromagnetic (AF) order, ferromagnetic order, ...).

As the first example beyond the Landau’s symmetry breaking paradigm, the integer quantum Hall (IQH) effect is a remarkable achievement in condensed matter physics. To describe the IQH state, the Chern number is or so called TKNN number,\(^1\) which is given by

\[
H = H_{\text{H}} + H' + U \sum_i \hat{n}_i \hat{n}_i - \mu \sum_{i,\sigma} \hat{c}_{i\sigma}^\dagger \hat{c}_{i\sigma}.
\]

Here \( H_{\text{H}} \) is the Hamiltonian of Haldane model\(^2\), which is given by

\[
H_{\text{H}} = -t \sum_{\langle i,j \rangle, \sigma} \left( \hat{c}_{i\sigma}^\dagger \hat{c}_{j\sigma} + h.c. \right) - t' \sum_{\langle(i,j)\rangle, \sigma} e^{i\phi_{ij}} \hat{c}_{i\sigma}^\dagger \hat{c}_{j\sigma}. \quad \text{and} \quad t' \text{ are the nearest neighbor and the next nearest neighbor hoppings, respectively.}
\]

We introduce a complex phase \( \phi_{ij} = \frac{e^{i\pi Q}}{2} \) to the next nearest neighbor hopping, of which the positive phase is set to be clockwise. \( H' = \varepsilon \sum_{i,\sigma} \hat{c}_{i\sigma}^\dagger \hat{c}_{i\sigma} - \varepsilon \sum_{i,\sigma} \hat{c}_{i\sigma}^\dagger \hat{c}_{i\sigma} \) denotes an on-site staggered energy : \( \varepsilon \) an order parameter for A site and \( -\varepsilon \) an order parameter for B site. \( U \) is the on-site Coulomb repulsion. \( \mu \) is the chemical potential and \( \mu = U/2 \) at half-filling for our concern in this paper.

When \( U \) is zero, the spectrum for free fermions is

\[
E_k = \pm \sqrt{\xi_k^2 + (\xi'_k + \varepsilon)^2}
\]

where \( |\xi_k| = t \sqrt{3 + 2 \cos(\sqrt{3}k_y) + 4 \cos(3k_x/2) \cos(\sqrt{3}k_y/2)} \) and \( \xi'_k = 2t' \left[ -2 \cos(3k_x/2) \sin(\sqrt{3}k_y/2) \right] \). According to this spectrum, we can see that there exist energy gaps \( \Delta_{f1}, \Delta_{f2} \) near points \( k_1 = -\frac{2\pi}{3}, 1/\sqrt{3} \) and \( k_2 = \frac{2\pi}{3}, 1/\sqrt{3} \) as \( \Delta_{f1} = |2\varepsilon - 6\sqrt{3}t'| \) and \( \Delta_{f2} = 2\varepsilon + 6\sqrt{3}t' \), respectively. In addition, there exist two phases for this case, the quantum anomalous Hall (QAH) state and the normal band insulator (NI) state with trivial topological properties. They are separated by the phase boundary \( \Delta_{f1} = 0 \). In QAH state, due to the nonzero TKNN number, \( Q = 2 \), there exists the IQH effect with a quantized (charge) Hall conductivity \( \sigma_H = 2e^2/h \).

The topological Hubbard model: The Hamiltonian of the topological Hubbard model on honeycomb lattice is given by\(^3\)

\[
H = H_{\text{H}} + H' + U \sum_i \hat{n}_i \hat{n}_i - \mu \sum_{i,\sigma} \hat{c}_{i\sigma}^\dagger \hat{c}_{i\sigma}.
\]

\( \hat{n}_i \) is the fermion annihilation operator for site \( i \), \( \hat{c}_{i\sigma} \) is the fermion creation operator for site \( i \) and spin \( \sigma \), and \( U \) is the on-site Coulomb repulsion. \( \mu \) is the chemical potential.

The topological spin density wave model: The Hamiltonian of the topological spin density wave model on honeycomb lattice is given by\(^4\)

\[
H = H_{\text{H}} + H' + U \sum_i \hat{n}_i \hat{n}_i - \mu \sum_{i,\sigma} \hat{c}_{i\sigma}^\dagger \hat{c}_{i\sigma}.
\]

\( \hat{n}_i \) is the fermion annihilation operator for site \( i \), \( \hat{c}_{i\sigma} \) is the fermion creation operator for site \( i \) and spin \( \sigma \), and \( U \) and \( \mu \) are the on-site Coulomb repulsion and the chemical potential, respectively.

\( \Delta_{f1}, \Delta_{f2} \) are the energy gaps near points \( k_1, k_2 \). \( k_1 = -\frac{2\pi}{3}, 1/\sqrt{3} \) and \( k_2 = \frac{2\pi}{3}, 1/\sqrt{3} \) as \( \Delta_{f1} = |2\varepsilon - 6\sqrt{3}t'| \) and \( \Delta_{f2} = 2\varepsilon + 6\sqrt{3}t'| \), respectively. In addition, there exist two phases for this case, the quantum anomalous Hall (QAH) state and the normal band insulator (NI) state with trivial topological properties. They are separated by the phase boundary \( \Delta_{f1} = 0 \). In QAH state, due to the nonzero TKNN number, \( Q = 2 \), there exists the IQH effect with a quantized (charge) Hall conductivity \( \sigma_H = 2e^2/h \).
written as $H = H_{M} + H' = \sum (-1)^{i} \Delta_{M} \hat{c}_{i\sigma}^{\dagger} \sigma_{2} \hat{c}_{i\sigma}$, with $\Delta_{M} = U M / 2$. By MF approach, we obtain the self-consistency equation for $M$ by minimizing the energy at zero temperature in the reduced BZ as

$$1 = \frac{1}{N_{s} M} \sum_{k} \left[ \frac{\xi_{k}^{I} + \Delta_{M} + \varepsilon}{2 E_{k_{1}}} - \frac{\xi_{k}^{I} - \Delta_{M} + \varepsilon}{2 E_{k_{2}}} \right]$$

where $N_{s}$ is the number of unit cells, $E_{k_{1}} = \sqrt{\left( \xi_{k}^{I} + \Delta_{M} + \varepsilon \right)^{2} + \left[ \xi_{k}^{I} \right]^{2}}$ and $E_{k_{2}} = \sqrt{\left( \xi_{k}^{I} - \Delta_{M} + \varepsilon \right)^{2} + \left[ \xi_{k}^{I} \right]^{2}}$.

To determine the phase diagram, there are two types of phase transitions: one is the magnetic phase transition [denoted by $(U/t)_{M}$] between a magnetic order state with $M \neq 0$ and a non-magnetic state with $M = 0$, the other one is the topological phase transition that is characterized by the condition of zero fermion’s energy gaps, $\Delta_{f1} = -6 \sqrt{3} t' + 2 \varepsilon + 2 \Delta_{M} = 0$ or $\Delta_{f2} = 6 \sqrt{3} t' + 2 \varepsilon - 2 \Delta_{M} = 0$. After determining the phase boundaries, we plot the phase diagram with five different quantum phases in Fig.1 for $\varepsilon = 0.15$: QAH state, NI, A-type topological SDW state (A-TSDW), B-type topological SDW state (B-TSDW), and trivial SDW state. In Fig.2, we also plot the staggered magnetization and the energy gaps $\Delta_{f1}$, $\Delta_{f2}$ for the same case.

FLG. 1: (Color online) The phase diagram for the case of $\varepsilon = 0.15$ : I is QAH state, II is A-TSDW, III is B-TSDW, IV is trivial SDW, V is NI. The black, red and blue lines are the critical lines of $(U/t)_{M}$, $(U/t)_{c1}$ and $(U/t)_{c2}$, respectively.

Based on the MF results, the TKNN numbers in A-TSDW, B-TSDW and the trivial SDW states are $Q = 2$, $Q = 1$, $Q = 0$, respectively. However, due to the quantum spin fluctuations, the classification of SDW states by the TKNN number is insufficient to give the final answer. Instead, a 2-by-2 matrix ($K$-matrix) plays a key role in the topological classification of the SDW orders with the same local order parameter, $M$.

**Induced CSH terms and K-matrices representation of TSDWs**: In this part we will derive the low energy effective theory of (T-)SDW states by considering quantum fluctuations of effective spin moments based on a formulation by keeping spin rotation symmetry, $\sigma_{a} \rightarrow n \cdot \sigma$ where $n$ is the SDW order parameter, $(\hat{c}_{1}^{\dagger} \sigma \hat{c}_{1}) = M n$. In this case, the Dirac-like effective Lagrangian with spin rotation symmetry in the continuum limit can be obtained as

$$\mathcal{L}_{f} = \sum_{a} \left[ i \bar{\psi}_{a} \gamma_{\mu} (\partial_{\mu} - i A_{\mu}) \psi_{a} + m_{a} \bar{\psi}_{a} \gamma_{5} - \delta \Delta_{M} \bar{\psi}_{a} \gamma_{5} \cdot n \psi_{a} \right]$$

which describes low energy charged fermionic modes $a = 1$ near $k_{1}$, $\psi_{1} = \psi_{1}^{1a}$ and $a = 2$ near $k_{2}$, $\psi_{2} = \psi_{2}^{2a}$. The masses of two-flavor fermions are $m_{1} = \varepsilon - 3 \sqrt{3} t'$ and $m_{2} = \varepsilon + 3 \sqrt{3} t'$. $\gamma_{\mu}$ is defined as $\gamma_{0} = \sigma_{0} \otimes \tau_{z}$, $\gamma_{1} = \sigma_{0} \otimes \tau_{y}$, $\gamma_{2} = \sigma_{0} \otimes \tau_{x}$ with $\sigma_{0} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. $\tau_{x}$, $\tau_{y}$, $\tau_{z}$ are Pauli matrices. $\delta = 1$ for $a = 1$ and $\delta = -1$ for $a = 2$. We have set the Fermi velocity to be unit $v_{F} = 1$.

In CP$^{1}$ representation, we may rewrite the effective Lagrangian of fermions in Eq.3 as

$$\mathcal{L}_{f} = \sum_{a} \bar{\psi}_{a} (i \gamma_{\mu} \partial_{\mu} + \gamma_{5} A_{\mu} - \gamma_{5} \sigma_{3} a_{\mu} + m_{a} - \delta \Delta_{M} \sigma_{3}) \psi_{a}$$

with $\psi_{a}(r, \tau) = U^{(1)}(r, \tau) \bar{\psi}_{a}(r, \tau)$, where $U(\tau, \tau)$ is a local and time-dependent spin SU(2) transformation defined by $U^{(1)}(r, \tau) n \cdot \sigma U(\tau, \tau) = \sigma_{3}$. And $a_{\mu}$ is introduced as an assistant gauge field as $i \sigma_{3} a_{\mu} \equiv U^{(1)}(r, \tau) \partial_{\mu} U(\tau, \tau)$.

An important property of above model in Eq.4 is the current anomaly. The vacuum expectation value of the fermionic current $J_{a\sigma}^{\mu} = i \langle \bar{\psi}_{a,\sigma} \gamma^{\mu} \psi_{a,\sigma} \rangle$ can be defined by
result, skyrmion’s spin is obtained by
\[ J_{a,\sigma}^\mu = i(\gamma^\mu)(i\hat{D} + im_{a,\sigma})^{-1}(i\hat{D} + im_{a,\sigma})^{-1} \]
where \( \hat{D} = \gamma^\mu(\partial_\mu - iA_\mu + i\sigma a_\mu) \) and the mass terms are \( m_{a,\sigma} = m_a - \delta M \sigma \). The topological current \( J_{a,\sigma}^\mu \) is given by
\[
J_{a,\sigma}^\mu = \frac{1}{4\pi} \frac{m_{a,\sigma}}{m_a} \epsilon^\mu\nu\lambda(\partial_\nu A_\lambda - \sigma \partial_\nu a_\lambda). \tag{5}
\]
Then we derive the CSH terms as \( \mathcal{L}_{CSH} = -i \sum_{a,\sigma}(A_\mu - \sigma a_\mu)J_{a,\sigma}^\mu \). \[ \text{Fig. 3:} \text{ (Color online)} \text{ Spin-charge separated charge-flux binding effect of A-TSDW : the induced quantum numbers on a half magnetic flux (a) and those on a } \frac{1}{2} \text{ skyrmion (b).} \]

example, by binding \( q_s = -2 \) "charge", the spin of \( \frac{1}{2} \) skyrmion is \( S = 1 \).

Furthermore, if there exists a tiny easy-plane anisotropic term (for example, the spin-orbital coupling term), we can define the skyrmion with fractional winding number. For example, by binding \( q_s = -1 \) "charge", the spin of \( \frac{1}{2} \) skyrmion is \( S = 1 \).

Thus, we get the identities \( \Phi_\sigma = \sigma \) and \( \Phi_\sigma = \sigma \) with \( K \)-matrix formulation that has been used to characterize FQH fluids successfully. \[ \text{Fig. 3:} \text{ (Color online)} \text{ Spin-charge separated charge-flux binding effect of A-TSDW : the induced quantum numbers on a half magnetic flux (a) and those on a } \frac{1}{2} \text{ skyrmion (b).} \]

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and \( A_\mu \) turn into
\[
\frac{1}{2\pi} \epsilon^{\mu\nu\lambda} \partial_\nu A_\lambda + \frac{1}{2\pi} \epsilon^{\mu\nu\lambda} \partial_\nu a_\lambda = -J_\mu \tag{9}
\]
and
\[
\frac{1}{2\pi} \epsilon^{\mu\nu\lambda} \partial_\nu A_\lambda + \frac{1}{2\pi} \epsilon^{\mu\nu\lambda} \partial_\nu a_\lambda = -J_{\mu\nu}. \tag{10}
\]

As a result, in B-TSDW, due to "spin-charge synchronized charge-flux binding" effect from the identities \( \Phi^e/2\pi + \Phi^e/2\pi = -q = -q_s \), the induced electric charge number is always equal to the induced "charge" number of \( a_\mu \) on a topological object.

\[ Q = 1 \text{ skyrmion} \]
\[ (a) \]
\[ q = -1 \quad q_s = -1 \]
\[ \text{half magnetic flux} \]
\[ (b) \]
\[ q = -1/2 \quad q_s = -1/2 \]

FIG. 4: (Color online) Spin-charge synchronized charge-flux binding effect of B-TSDW: the induced quantum numbers on a \( Q = 1 \) skyrmion (a) and those on a half magnetic flux (b).

On the one hand, due to the condition \( \Phi^s/2\pi = -q = -q_s \), \( Q \) skyrmion will carry \( q = -Q \) electric charge of gauge field \( A_\mu \) and \( q_s = -Q \) "charge" of gauge field \( a_\mu \). For example, \( Q = 1 \) skyrmion carries a unit electric charge \( q = -1 \) and a unit "charge" \( q_s = -1 \) (see FIG.4.(a)). With a unit "charge" \( q_s \), \( Q = 1 \) skyrmion gets half spin and becomes a charged \( S = 1/2 \) fermion; For a \( Q = 2 \) skyrmion, there exist two induced charge numbers on it, \( q = -2 \), \( q_s = -2 \). Then it becomes a charged \( S = 2 \) boson. In addition, for a SDW order with easy-plane anisotropic energy, we have the half skyrmion with fractional winding number \( Q = \pm 1/2 \), of which there exist fractional charge numbers, \( q_s = \mp 1/2 \).

Acknowledgments

This word is supported by SRFDP, NFSC Grant No. 10874017 and 10774015, National Basic Research Program of China (973 Program) under the grant No. 2011CB921803, 2011cb00102.

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A. The detailed calculations of the Chern-Simons-Hopf (CSH) terms in Eq.(6)

Firstly we calculate the induced Chern-Simons (CS) term of a one flavor fermionic-σ model. The Lagrangian of one flavor fermionic-σ model is written as

\[ \mathcal{L} = i \bar{\psi} \gamma \mu (\partial \mu - i b_\mu) \psi + m \bar{\psi} \psi \]

where \( m \) is a fermion mass. To obtain the induced CS term, we integrating over fermions and get

\[ S_{\text{eff}} = \ln Z \]

where

\[ Z = \int [d\bar{\psi}] [d\psi] \exp (-\int dx \mathcal{L}) \]

The one fermion loop effective action becomes

\[ S_{\text{eff}} = \ln \det (i \gamma \mu \partial \mu + \gamma \mu b_\mu + m) \]

\[ = \text{tr} \log (i \gamma \mu \partial \mu + \gamma \mu b_\mu + m) + \frac{1}{2} \text{tr} \left( \frac{1}{i \gamma \mu \partial \mu + \gamma \mu b_\mu + m} \right) + \ldots \]

Then the quadratic term of \( b_\mu \) in the effective action is

\[ S_{\text{quad}} = \frac{1}{2} \int \frac{d^3 p}{(2\pi)^3} \left[ \theta(p') D^{\mu\nu} b^\nu(p) \right] \]

where \( D^{\mu\nu} \) is

\[ D^{\mu\nu} = \int \frac{d^3 k}{(2\pi)^3} \text{tr} \left[ \gamma^\mu p^\nu + k^\nu \gamma^\mu - m \frac{k^\nu k_\nu}{k^2 + m^2} \right] \]

Under \( \text{tr}(\gamma^\mu \gamma^\nu \gamma^\lambda) = -2 \epsilon^{\mu\nu\lambda} \), we obtain

\[ D^{\lambda\nu}(p, m) = \epsilon^{\mu\nu\lambda} p_\lambda 2m \int \frac{d^3 k}{(2\pi)^3} \left[ \frac{1}{[(p + k)^2 + m^2][k^2 + m^2]} \right] \]

\[ = \epsilon^{\mu\nu\lambda} p_\lambda \frac{m}{2\pi} \arcsin \left( \frac{|p|}{\sqrt{p^2 + 4m^2}} \right) \]

In the long wavelength limit, \( (\frac{p}{m} \to 0) \), due to \( \Theta \sim \frac{1}{4\pi} \frac{m}{|m|} \), we get an induced CS term as

\[ \mathcal{L}_{\text{eff}} = -\frac{i}{4\pi} \frac{m}{|m|} \epsilon^{\mu\nu\lambda} b_\mu \partial_\nu b_\lambda \]

Next we calculate the induced CSH term in Eq.(6) in the paper that denotes a four-component fermionic model

\[ \mathcal{L}_{11} = \bar{\psi} \gamma^\mu \partial_\mu + \gamma^\mu A_\mu - \gamma^\mu a_\mu + m_1 - \Delta_M \psi \]

\[ \mathcal{L}_{14} = \bar{\psi} \gamma^\mu \partial_\mu + \gamma^\mu A_\mu + \gamma^\mu a_\mu + m_1 + \Delta_M \psi \]

\[ \mathcal{L}_{21} = \bar{\psi} \gamma^\mu \partial_\mu + \gamma^\mu A_\mu - \gamma^\mu a_\mu + m_2 + \Delta_M \psi \]

\[ \mathcal{L}_{24} = \bar{\psi} \gamma^\mu \partial_\mu + \gamma^\mu A_\mu + \gamma^\mu a_\mu + m_2 - \Delta_M \psi \]

Integrating \( \psi'_{11} \), we get the induced CS term

\[ \mathcal{L}_{11}(A_\mu) = -\frac{i}{8\pi} \frac{m_1 - \Delta_M}{m_1 - \Delta_M} \epsilon^{\mu\nu\lambda} (A_\mu - a_\mu) \partial_\nu (A_\lambda - a_\lambda) \]

Integrating \( \psi'_{14} \), we get the induced CS term

\[ \mathcal{L}_{14}(A_\mu) = -\frac{i}{8\pi} \frac{m_1 + \Delta_M}{m_1 + \Delta_M} \epsilon^{\mu\nu\lambda} (A_\mu + a_\mu) \partial_\nu (A_\lambda + a_\lambda) \]

Integrating \( \psi'_{21} \), we get the induced CS term

\[ \mathcal{L}_{21}(A_\mu) = -\frac{i}{8\pi} \frac{m_2 + \Delta_M}{m_2 + \Delta_M} \epsilon^{\mu\nu\lambda} (A_\mu - a_\mu) \partial_\nu (A_\lambda - a_\lambda) \]

Integrating \( \psi'_{24} \), we get the induced CS term

\[ \mathcal{L}_{24}(A_\mu) = -\frac{i}{8\pi} \frac{m_2 - \Delta_M}{m_2 - \Delta_M} \epsilon^{\mu\nu\lambda} (A_\mu + a_\mu) \partial_\nu (A_\lambda + a_\lambda) \]

Then we get the total induced CSH term as

\[ \mathcal{L}(A_\mu) = \mathcal{L}_{11}(A_\mu) + \mathcal{L}_{14}(A_\mu) + \mathcal{L}_{21}(A_\mu) + \mathcal{L}_{24}(A_\mu) \]

(i) For the case of \( m_1, m_2 > \Delta_M \), we get

\[ \mathcal{L}(A_\mu) = \mathcal{L}_{11}(A_\mu) + \mathcal{L}_{14}(A_\mu) + \mathcal{L}_{21}(A_\mu) + \mathcal{L}_{24}(A_\mu) \]

\[ = -\frac{i}{8\pi} \epsilon^{\mu\nu\lambda} (A_\mu + a_\mu) \partial_\nu (A_\lambda + a_\lambda) \]

\[ - \frac{i}{8\pi} \epsilon^{\mu\nu\lambda} (A_\mu - a_\mu) \partial_\nu (A_\lambda - a_\lambda) \]

\[ - \frac{i}{8\pi} \epsilon^{\mu\nu\lambda} (A_\mu - a_\mu) \partial_\nu (A_\lambda + a_\lambda) \]

\[ - \frac{i}{8\pi} \epsilon^{\mu\nu\lambda} (A_\mu + a_\mu) \partial_\nu (A_\lambda + a_\lambda) \]

\[ = -\frac{i}{2\pi} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda - \frac{i}{2\pi} \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda \]

(ii) For the case of \( m_2 > \Delta_M > m_1 \), we get

\[ \mathcal{L}(A_\mu) = \mathcal{L}_{11}(A_\mu) + \mathcal{L}_{14}(A_\mu) + \mathcal{L}_{21}(A_\mu) + \mathcal{L}_{24}(A_\mu) \]

\[ = -\frac{i}{8\pi} \epsilon^{\mu\nu\lambda} (A_\mu + a_\mu) \partial_\nu (A_\lambda + a_\lambda) \]

\[ + \frac{i}{8\pi} \epsilon^{\mu\nu\lambda} (A_\mu - a_\mu) \partial_\nu (A_\lambda - a_\lambda) \]

\[ - \frac{i}{8\pi} \epsilon^{\mu\nu\lambda} (A_\mu - a_\mu) \partial_\nu (A_\lambda + a_\lambda) \]

\[ - \frac{i}{8\pi} \epsilon^{\mu\nu\lambda} (A_\mu + a_\mu) \partial_\nu (A_\lambda + a_\lambda) \]

\[ = -\frac{i}{4\pi} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda - \frac{i}{2\pi} \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda - \frac{i}{4\pi} \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda \]
(iii) For the case of \( m_1, m_2 < \Delta_M \), we get

\[
\mathcal{L}(A_\mu) = \mathcal{L}_{11}(A_\mu) + \mathcal{L}_{12}(A_\mu) + \mathcal{L}_{22}(A_\mu)
\]

\[
= -\frac{i}{8\pi} \varepsilon^{\mu\nu\lambda}(A_\mu + a_\mu) \partial_\nu (A_\lambda + a_\lambda) \\
+ \frac{i}{8\pi} \varepsilon^{\mu\nu\lambda}(A_\mu - a_\mu) \partial_\nu (A_\lambda - a_\lambda) \\
- \frac{i}{8\pi} \varepsilon^{\mu\nu\lambda}(A_\mu - a_\mu) \partial_\nu (A_\lambda - a_\lambda) \\
+ \frac{i}{8\pi} \varepsilon^{\mu\nu\lambda}(A_\mu + a_\mu) \partial_\nu (A_\lambda + a_\lambda)
\]

\[= 0.\]

Finally, by the \( \mathcal{K} \)-matrix, the CSH term is written as

\[
\mathcal{L}_{CSH} = -i \sum_{I,J} \frac{K_{IJ}}{4\pi} \varepsilon^{\mu\nu\lambda} a_I^\mu \partial_\nu a_J^\lambda
\]  

(16)

where \( \mathcal{K} \) is 2-by-2 matrix, \( a_I^\mu = A_\mu \) and \( a^\mu = a_\mu \). Thus for different SDW orders with the same order parameter \( M \), we have different \( \mathcal{K} \)-matrices: for \( m_1, m_2 > \Delta_M \), \( \mathcal{K} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \); for \( m_2 > \Delta_M > m_1 \), \( \mathcal{K} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \); for \( m_1, m_2 < \Delta_M, \mathcal{K} = 0 \).