Common greedy wiring and rewiring heuristics do not guarantee maximum assortative graphs of given degree

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Abstract

We examine two greedy heuristics — wiring and rewiring — for constructing maximum assortative graphs over all simple connected graphs with a target degree sequence. Counterexamples show that natural greedy rewiring heuristics do not necessarily return a maximum assortative graph, even though it is known that the meta-graph of all simple connected graphs with given degree is connected under rewiring. Counterexamples show an elegant greedy wiring heuristic from the literature may fail to achieve the target degree sequence or may fail to have the maximum assortativity.

Keywords: Assortativity, graph wiring, graph rewiring, graph algorithms

1. Introduction

1.1. Motivation

The assortativity of a graph (Newman [1]) is the correlation of the degrees of the endpoints of a randomly selected edge. High degree nodes tend to be connected to high (low) degree nodes in positively (negatively) assortative graphs.

One (of many) practical implication of assortativity is in graph search, e.g., searching a (often large order) graph for (one or all) vertices of maximum (or at least large) degree [2]: our prior work [3, 4] has studied the performance impact of assortativity on search heuristics such as sampling and random walks. Finding such nodes in large graphs has diverse applications, including viral marketing in social networks and network robustness analysis [5, 6], among numerous others.

Our motivation is the problem of identifying a collection of graphs, all from the class of graphs with a given degree sequence, with the assortativity of the graphs in the collection varying from the minimum to the maximum possible within that class. The performance impact of the assortativity on the search heuristic may be studied by running the heuristic on all graphs in the collection.

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Given this objective, the first step is to identify graphs with extremal assortativity within the class. This paper examines two greedy heuristics for finding maximum assortative graphs within a class: graph rewiring and wiring.

1.2. Related Work

There is an extensive literature on extremization of assortativity over different graph classes; we briefly cover the most pertinent points of this literature, focusing on the distinctions between our contribution and this prior work.

Assortativity. Newman [1] introduced (graph) assortativity, a statistic of the JDM, which is denoted $\alpha \in [-1, +1]$. Van Meighem [7] showed perfect assortativity ($\alpha = 1$) is only possible in regular graphs, while any complete bipartite graph $K_{m,n}$ ($m \neq n$) is perfectly disassortative ($\alpha = -1$). There is a large literature on network degree correlations and assortativity (e.g., [8]), and on graphs with extremal assortativity within a class (e.g., [9]).

Joint Degree Matrix (JDM). The generation of random graphs with a particular JDM (also called a 2K-series) has been the subject of a number of recent papers. Stanton [10] and Orsini [8] have proposed random edge rewiring as a method of sampling graphs with a given JDM while Gjorka [11] has introduced a random wiring methods for constructing these graphs. The maximization of assortativity over all graphs with a given JDM does not provide a direct or efficient means by which to maximize assortativity over all simple connected graphs with a given degree sequence.

Rewiring. The meta-graph for a degree sequence, with a vertex for each connected simple graph with that degree sequence and an edge connecting graphs related by rewiring a pair of edges, was studied by Taylor [12]; in particular, he showed this graph to be connected (Thm. 3.3) extending an earlier result by Rysler for simple graphs [13]. We use this fact in §2.

Following Rysler’s work, rewiring heuristics for sampling graphs with a particular degree sequence (e.g., [14], [15], [8]) have been introduced. Rewiring heuristics have also been proposed by Newman [16], Xuli-Burnet [17], Van Meighem [7], and Winterbach [18] along others for changing a graph’s assortativity. The first three of these algorithms, being purely stochastic, cannot efficiently maximize assortativity. Winterbach’s algorithm uses a guided rewiring technique to maximize assortativity. However, this technique does not maintain graph connectivity and, as its rewirings are a subset of those explored by rewiring heuristic A, Winterbach’s algorithm does not necessarily maximize assortativity, as shown in §2.

Wiring. Li and Alderson [19] introduced a greedy wiring heuristic for constructing a graph with maximum assortativity over the set of simple connected graphs with a target degree sequence. Kincaid [9] argues wiring a maximally assortative connected simple graph is NP-hard and proposes a heuristic which is shown numerically to perform near optimally in minimizing graph assortativity. Winterbach [18], Zhou [20], and Meghanathan [21] have also proposed methods unconstrained by connectivity of wiring maximally assortative graphs. We examine Li’s heuristic further in §3.
Graph enumeration and generation. We used geng, a tool in the nauty package by McKay [22], to generate all simple connected graphs of a given order.

1.3. Notation

Let \( a \equiv b \) denote equal by definition. Let \([n]\) denote \(\{1, \ldots, n\}\) for \(n \in \mathbb{N}\).

A graph of order \(n\) is denoted \(G = (V, E)\), with vertices \(V = [n]\) and edges \(E\); size is denoted by \(m = |E|\). A directed edge between vertices \(i\) and \(j\) is denoted \((i, j)\), and an undirected edge is denoted \(ij\) or \(\{i, j\}\).\(^1\) Let \(d_i(G)\) denote the degree of vertex \(i\), and let \(d = (d_i, i \in V)\) denote the degree sequence of \(G\).

The collection of distinct unlabeled undirected simple connected graphs of order \(n \in \mathbb{N}\) is denoted \(\mathcal{V}(n)\). Let \(\mathcal{D}(n) \equiv \bigcup_{G \in \mathcal{V}(n)} d(G)\) be the set of degree sequences found in the collection, and let \(\mathcal{V}(n)(d) \equiv \{G \in \mathcal{V}(n) : d(G) = d\}\) be the graphs in \(\mathcal{V}(n)\) with degree sequence \(d\), henceforth referred to as the degree class \(d\). Let \((\mathcal{V}(n)(d), d \in \mathcal{D}(n))\) be the partition of \(\mathcal{V}(n)\) by degree sequence.

The \(S\)-metric and assortativity, for \(G = (V, E) \in \mathcal{V}(n)\), are defined below.

**Definition 1.** The \(S\)-metric \([19]\) is, for \(\{u, u'\} \sim \text{Uni}(E)\) an edge selected uniformly at random,

\[
s(G) \equiv E[d_u d_{u'}] = \frac{1}{|E|} \sum_{ij \in E} d_i d_j.
\]  

(1)

The assortativity \([1]\) is, for \(v \sim \text{Uni}(V)\) a vertex selected uniformly at random,

\[
\alpha(G) \equiv \text{Corr}(d_u, d_{u'}) = \frac{s(G) - E[d_v]^2}{\text{Var}(d_v)}.
\]  

(2)

It is evident that maximizing the \(S\)-metric is equivalent to maximizing assortativity over a degree class:

\[
\mathcal{V}_{\text{opt}}(n)(d) \equiv \arg\max_{G \in \mathcal{V}(n)(d)} s(G) = \arg\max_{G \in \mathcal{V}(n)(d)} \alpha(G).
\]  

(3)

Here, \(\mathcal{V}_{\text{opt}}(n)(d)\) denotes those graphs achieving maximum assortativity over \(\mathcal{V}(n)(d)\). If there is a unique such graph it is denoted \(G_{\text{opt}}(n)(d)\).

1.4. Contributions and outline

The rest of the paper is organized as follows. §2 studies several greedy rewiring heuristics, each with the goal of identifying a graph of maximum assortativity over the degree class. We present counterexamples showing each of the heuristics may fail to identify such a graph. §3 examines the greedy wiring heuristic of Li and Anderson [19] designed to identify a graph of maximum assortativity over the degree class. We present a counterexample showing the heuristic may fail to produce a graph in the degree class, and also present a

\(^1\)Except in Alg. 1 where undirected edges are listed as an ordered pair.
counterexample showing that the heuristic may produce a graph in the class that is not extremal. For both §2 and §3 we present tabulations of the number of counterexamples of the various types for graphs of order up to \( n = 9 \). §4 contains our concluding remarks.

2. Rewiring

For a degree class \( \mathcal{V}^{(n)}(d) \) and an initial graph \( G_0 \in \mathcal{V}^{(n)}(d) \), a rewiring heuristic produces a sequence of graphs \( (G_0, \ldots, G_T) \), each graph in \( \mathcal{V}^{(n)}(d) \), where \( G_{t+1} \) is obtained from \( G_t \) by selecting two edges (connecting four distinct vertices) from \( G_t \), say \((ij, kl)\), and forming \( G_{t+1} \) with \((ij, kl)\) replaced by either \((ik, jl)\) or \((il, jk)\). Any rewiring is invalid if the resulting graph is either disconnected or has multiple edges, i.e., not in \( \mathcal{V}^{(n)}(d) \).

2.1. Greedy rewiring heuristics

A stochastic rewiring heuristic involves selecting the two edges \((ij, kl)\) at random. While simple to implement, this heuristic has no guarantee on efficiency. We focus instead on greedy rewiring heuristics. Fix \( G \in \mathcal{V}^{(n)}(d) \) and four distinct vertices \( \{i, j, k, l\} \), such that \( G \) has edges \((ij, kl)\). Rewrite edges \((ij, kl)\) to produce either graph \( G' = G(ik, jl) \) or \( G' = G(il, jk) \); the arguments denote the two new edges replacing \((ij, kl)\). Rewiring induces a change in \( s \):

\[
\Delta_{G,G'} \equiv s(G') - s(G) = \begin{cases} 
(d_id_k + d_jd_l) - (d_id_j + d_kd_l), & G' = G(ik, jl) \\
(d_id_l + d_jd_k) - (d_id_j + d_kd_l), & G' = G(il, jk) 
\end{cases} 
\]  

(4)

Three greedy rewiring heuristics are developed using \( \Delta_{G,G'} \); each yields a neighborhood \( \mathcal{N}_G(d) \) of graphs in a meta-graph on \( \mathcal{V}^{(n)}(d) \) (defined below), where each \( G' \in \mathcal{N}_G(d) \) is reached by a heuristic-approved single rewiring of \( G \).

- **A**: Improve (or maintain) \( s(G) \): \( \mathcal{N}_G^{(A)}(d) \) holds all simple connected graphs \( G' \) obtainable by a single rewiring of \( G \) such that \( \Delta_{G,G'} \geq 0 \).
- **B**: Maximize \( \Delta \): \( \mathcal{N}_G^{(B)}(d) \) holds all simple connected graphs \( G' \) obtainable by a single rewiring of \( G \) such that \( \Delta_{G,G'} \) is maximum over all \( G' \).
- **C**: Improve and maximize: \( \mathcal{N}_G^{(C)}(d) \) holds all simple connected graphs \( G' \) obtainable by a single rewiring of \( G \) such that \( \Delta_{G,G'} \geq 0 \) and \( \Delta_{G,G'} \) is maximum over all \( G' \).

2.2. Meta-graphs for a degree class

Meta-graphs are graphs with vertices corresponding to the (simple and connected) non-isomorphic graphs in a degree class \( \mathcal{V}^{(n)}(d) \), for a given degree sequence \( d \in \mathcal{D}^{(n)} \). Taylor [12] defined the undirected meta-graph \( \mathcal{G}^{(n)}(d) = \langle \mathcal{V}^{(n)}(d), \tilde{E}(d) \rangle \), where edges are added between all pairs of graphs related by rewiring, i.e., \( \{G, G'\} \in \tilde{E}(d) \) iff \( G' = G(ik, jl) \) or \( G' = G(il, jk) \) for some pair of edges \((ij, kl)\). He proved (Thm. 3.3) that \( \mathcal{G}^{(n)}(d) \) is connected. Thus, any graph
in $\mathcal{V}^{(n)}(d)$ is obtainable, starting from any other graph in $\mathcal{V}^{(n)}(d)$, through a sequence of rewirings, where each graph in the sequence is simple and connected. Note $\hat{G}^{(n)}(d)$ may have self-loops as rewiring $G$ may yield $G'$ isomorphic to $G$.

Rewiring heuristics $A$, $B$, and $C$ each correspond to directed meta-graphs. First, label each graph $G \in \mathcal{V}^{(n)}(d)$ with its assortativity $\alpha(G)$ (alternately, $s(G)$). Next, for each heuristic $H \in \{A, B, C\}$, form the directed meta-graph $G_H^{(n)}(d) = (\mathcal{V}^{(n)}(d), \mathcal{E}_H(d))$, where $\mathcal{E}_H(d) \equiv \{(G, G') : G' \in \mathcal{N}_G(d)\}$. That is, each rewiring heuristic is represented by retaining (and orienting) the subset of edges in Taylor’s meta-graph $\hat{G}^{(n)}(d)$ that satisfy the heuristic.

![Figure 1: Fix $n = 7$ and set $d = (5, 5, 5, 4, 3, 2)$ and $\hat{d} = (4, 4, 3, 3, 2, 1, 1)$. The five meta-graphs are, from left to right: i) $\hat{G}_1^{(7)}(d)$, ii) $G_1^{(7)}(d)$, iii) $G_1^{(7)}(1,\hat{d})$, iv) $G_1^{(7)}(1,d)$, v) $G_1^{(7)}(d')$.](image)

Figure 1: Fix $n = 7$ and set $d = (5, 5, 5, 4, 3, 2)$ and $\hat{d} = (4, 4, 3, 3, 2, 1, 1)$. The five meta-graphs are, from left to right: i) $\hat{G}_1^{(7)}(d)$, ii) $G_1^{(7)}(d)$, iii) $G_1^{(7)}(1,\hat{d})$, iv) $G_1^{(7)}(1,d)$, v) $G_1^{(7)}(d')$.

![Figure 2: Fix $n = 7$ and set $d = (5, 5, 5, 4, 3, 2)$ and $\hat{d} = (4, 4, 3, 3, 2, 1, 1)$. The four graphs are, from left to right: i) initial graph $G_{0,A}$ and ii) target graph $G^{(7)}_{0,A}(d)$ in $\mathcal{V}^{(7)}(d)$; iii) initial graph $G_{0,B}$ and iv) target graph $G^{(7)}_{0,B}(d)$ in $\mathcal{V}^{(7)}(d)$.](image)

![Figure 2: Fix $n = 7$ and set $d = (5, 5, 5, 4, 3, 2)$ and $\hat{d} = (4, 4, 3, 3, 2, 1, 1)$. The four graphs are, from left to right: i) initial graph $G_{0,A}$ and ii) target graph $G^{(7)}_{0,A}(d)$ in $\mathcal{V}^{(7)}(d)$; iii) initial graph $G_{0,B}$ and iv) target graph $G^{(7)}_{0,B}(d)$ in $\mathcal{V}^{(7)}(d)$.](image)

2.3. Rewiring heuristics counterexamples

One might hope that (one or more of) the rewiring heuristics would provide a guarantee that, for any initial graph $G_0 \in \mathcal{V}^{(n)}(d)$, there exists a directed path, following the heuristic, from $G_0$ to one or more graphs in $\mathcal{V}^{(n)}_{opt}(d)$. Unfortunately, all three heuristics fail to achieve this goal, as shown by the counterexamples below. A counterexample for heuristic $H \in \{A, B, C\}$ identifies an $(n, d, G_0)$ triple, with $n \in \mathbb{N}$, $d \in \mathcal{D}^{(n)}$, and $G_0 \in \mathcal{V}^{(n)}(d)$, such that there is no path from $G_0$ to any graph in $\mathcal{V}^{(n)}_{opt}(d)$ under the directed meta-graph $G_H^{(n)}(d)$.

Note that our heuristics do not specify a particular rewiring, i.e., each heuristic identifies, in general, a collection of possible neighborhood graphs $\mathcal{N}_G$, each graph in the neighborhood consistent with the heuristic. Thus, a counterexample for the heuristic has the property that the heuristic would fail to achieve the target set for any possible choice of $G' \in \mathcal{N}_G$, for each $G$ “reachable” from $G_0$.

**Counterexample 1.** Fix order $n = 7$, degree sequence $d = (5, 5, 5, 4, 4, 3, 2)$, and initial graph $G_{0,A} \in \mathcal{V}^{(7)}(d)$ (Fig. 2). The (unique) graph with maximum
assortativity, \( G_{\text{opt}}^{(7)}(d) \), is also shown in Fig. 2. The meta-graph \( \hat{G}^{(7)}(d) \) and directed meta-graph under heuristic A, \( G_A^{(7)}(d) \) are both shown in Fig. 1. There is no path from \( G_0,A \) to \( G_{\text{opt}}^{(7)}(d) \) via \( G_A^{(7)}(d) \), and hence no path via \( G_C^{(7)}(d) \). Thus, \( (n,d,G_0,A) \) is a counterexample for heuristics A and C.

This counterexample asserts that graph \( G_0,A \) has locally (i.e., over graphs adjacent to \( G_0,A \) in \( \hat{G}^{(7)}(d) \)) maximal but not globally (i.e., over \( V^{(7)}(d) \)) maximum assortativity. To see that \( G_0,A \) is locally maximal, Table 1 lists possible pairs of edges from \( G_0,A \) which if rewired as \( G' = G_0,A(ik, jl) \) or \( G' = G_0,A(il, jk) \) maintain graph simplicity and connectivity: \( \Delta_{G_0,A,G'} < 0 \) for each possible \( G' \).

**Counterexample 2.** Fix order \( n = 7 \), degree sequence \( \hat{d} = (4, 4, 3, 3, 2, 1, 1) \), and initial graph \( G_0,B \in V^{(7)}(\hat{d}) \) (Fig. 2). The (unique) graph with maximum assortativity, \( G_{\text{opt}}^{(7)}(\hat{d}) \), is also shown in Fig. 2. The meta-graph \( \hat{G}^{(7)}(\hat{d}) \) and directed meta-graph under heuristic B, \( G_B^{(7)}(\hat{d}) \) are both shown in Fig. 1. There is no path from \( G_0,B \) to \( G_{\text{opt}}^{(7)}(\hat{d}) \) via \( G_B^{(7)}(\hat{d}) \). Thus, \( (n, \hat{d}, G_0,B) \) is a counterexample for heuristic B.

This counterexample asserts that graph \( G_0,B \) has locally maximal but not globally maximum assortativity. This can be seen by enumerating all possible pairs of edges from \( G_0,B \) which if rewired maintain graph simplicity and connectivity, and showing the (unique) optimal choice to maximize \( \Delta_{G,G'} \) produces a new graph \( G' \) isomorphic to \( G_0,B \); this enumeration is omitted due to space constraints. This isomorphism between \( G_0,B \) and \( G' \) is why the only edge attached to \( G_0,B \in G^{(7)}(\hat{d}) \) in Fig. 1 is a self-loop.

| (ij, kl) | (ik, jl) | \( \Delta_{G_0,A,G'} \) | (ij, kl) | \( \Delta_{G_0,A,G'} \) |
|----------|----------|----------------|----------|----------------|
| (43, 57) | (45, 37) | -2             | (47, 35) | *              |
| (42, 57) | (45, 27) | -2             | (47, 25) | *              |
| (41, 57) | (45, 17) | -2             | (47, 15) | *              |
| (47, 53) | (45, 73) | -2             | (43, 75) | *              |
| (47, 52) | (45, 72) | -2             | (42, 75) | *              |
| (47, 51) | (45, 71) | -2             | (41, 75) | *              |
| (47, 63) | (46, 73) | -1             | (43, 67) | *              |
| (47, 62) | (46, 72) | -1             | (42, 76) | *              |
| (47, 61) | (46, 71) | -1             | (41, 76) | *              |
| (57, 63) | (56, 73) | -1             | (53, 67) | *              |
| (57, 62) | (56, 72) | -1             | (52, 76) | *              |
| (57, 61) | (56, 71) | -1             | (51, 76) | *              |

Table 1: Rewirings of edge pairs (ij, kl) (left) of \( G_0,A \), along with \( \Delta_{G_0,A,G'} \) for \( G' = G_0,A(ik, jl) \) (middle) or \( G' = G_0,A(il, jk) \) (right). Bold entries maximize \( \Delta_{G_0,A,G'} \); * indicates rewirings which violate graph simplicity or connectivity.

Finally, we close this section with Table 2, which counts the number of counterexamples for each of the three heuristics.
### 3. Wiring

If a sequence $d$ satisfies the Erdős Gallai theorem then there exists one or more simple connected graphs with that degree sequence $d$, i.e., $\mathcal{V}(n)(d) \neq \emptyset$ [23]. Given such a $d$, a wiring heuristic produces a sequence of graphs $(G_0, \ldots, G_T)$, with $G_0$ the empty graph, such that $G_{t+1}$ is formed from $G_t$ by adding one edge, subject to the constraint that no vertex $i \in V$ is ever assigned a degree exceeding its target $d_i$. It is typical to consider each vertex $i$ in graph $G_t$ as having $d_i$ “stubs” of which some number $\delta_i$ hold edges, and the remainder, $\delta_i = d_i - \delta_i$, are available for wiring. Our interest is in wiring heuristics to obtain a graph of maximum assortativity, i.e., in $\mathcal{V}^{\text{opt}}(n)(d)$.

#### 3.1. Greedy wiring heuristic

Li and Alderson [19] developed the elegant greedy wiring heuristic in Alg. 1 which, given a degree sequence $d$ is intended to produce a graph $\tilde{G}$ that is i) feasible, i.e., in $\mathcal{V}(n)(d)$, and ii) optimal, i.e., in $\mathcal{V}^{\text{opt}}(n)(d)$. Although in our experience the heuristic performs well on most inputs, we will present counterexamples demonstrating neither property is guaranteed for all $d$.

Each potential edge, hereafter a “pedge”, is denoted by the ordered pair $(i,j)$ with $i < j$. The basic idea is to select from set of all pedges $O$ those with the largest endpoint degree product, $M$ (Line 4), removing from $O$ and $M$ pedges in $M$ without available (unwired) stubs $F$ (Line 6). If pedges remain then we further break ties by first (then second) selecting the pedge $(i,j)$ with the most unwired stubs $\delta_i$ (\delta_j). Vertices $[n]$ are partitioned into $A, B$, where $A (B)$ holds any vertex with one or more (no) edges. If the pedge $(i,j)$ has $i \in A$ and $j \in B$ then the edge is added and vertex $j$ moves from $B$ to $A$ (Line 11). Else $A$ holds both $i$ and $j$ and we must check the “tree condition” and “disconnected cluster condition” in Line 14 (see [19]) to ensure connectivity before adding the edge.

Alg. 1 is underspecified in Line 9, i.e., there may be multiple pedges after sorting $O$ by $d_id_j, \delta_i, \delta_j$, and no guidance is provided in [19] for selecting a pedge in such a case. Our software implementation selects all possible choices, via a breadth first search, returning all possible graphs $G$ that may result from making any possible selection in Line 9. We consider $d$ to be a counterexample for i) feasibility if none of the returned graphs is in $\mathcal{V}(n)(d)$, and ii) optimality if at least one returned graph is in $\mathcal{V}(n)(d)$, yet none are in $\mathcal{V}^{\text{opt}}(n)(d)$.

### Table 2: Rewiring heuristics counterexample counts: number of distinct graphs $|\mathcal{V}(n)|$ and degree sequences $|\mathcal{D}(n)|$, followed by number of distinct graphs (#$\mathcal{G}$) and degree sequences (#$\mathcal{d}$) that are counterexamples for heuristics A, B, C, for graphs of order 6, 7, 8, 9.

| $n$ | $|\mathcal{V}(n)|$ | $|\mathcal{D}(n)|$ | #G | #d | #G | #d | #G | #d |
|-----|-----------------|-----------------|-----|-----|-----|-----|-----|-----|
| 6   | 112             | 68              | 0   | 0   | 0   | 0   | 0   | 0   |
| 7   | 853             | 236             | 2   | 2   | 1   | 1   | 2   | 2   |
| 8   | 11,117          | 863             | 13  | 12  | 15  | 8   | 20  | 12  |
| 9   | 261,080         | 3,137           | 149 | 80  | 1045 | 67  | 1100 | 80  |

3. Wiring
Algorithm 1 Greedy wiring heuristic (adapted from [19])

1: require: \( d = (d_1, \ldots, d_n) \) with \( d_1 \geq \cdots \geq d_n \)
2: \( A := \{1\}, B := \{2, \ldots, n\}, \mathcal{E} := \{\}, \mathcal{G} := (A, \mathcal{E}), \mathcal{O} := \{(i, j) : 1 \leq i < j \leq n\} \)
3: while \( \mathcal{O} \neq \emptyset \) do
4: \( M := \arg\max_{(i, j) \in \mathcal{O}} (d_i - d_j) \)
5: \( \mathcal{F} := \{(i, j) \in M : \delta_i \delta_j = 0\} \)
6: \( \mathcal{O} := \mathcal{O} \setminus \mathcal{F}, M := M \setminus \mathcal{F} \)
7: if \( M \neq \emptyset \) then
8: \( M' := \arg\max_{(i, j) \in M} \delta_i \)
9: Select \((i, j) \in \mathcal{O} \setminus \mathcal{F}\)
10: else
11: \( d_B := \sum_{k \in B} d_k, \delta_A := \sum_{k \in A} \delta_k \)
12: if \( d_B \neq (2|B| - \delta_A) \land (\delta_A \neq 2) \) then
13: \( \mathcal{E} := \mathcal{E} \cup \{(i, j)\} \)
14: \( \mathcal{O} := \mathcal{O} \setminus \{(i, j)\} \)
15: return \( \tilde{G} \)

3.2. Wiring heuristic counterexamples

Counterexample 3. Fix \( n = 6 \) and \( d = (5, 4, 4, 4, 4, 3) \) (which satisfies the Erdős Gallai theorem). The graph \( \tilde{G} \) returned by Alg. 1 does not have the target degree sequence, i.e., \( d(\tilde{G}) \neq d \), and thus \( \tilde{G} \) is not feasible, i.e., \( \tilde{G} \not\in \mathcal{V}^\left(n^d\right) \).

Proof. Table 3 gives the sequence of wirings satisfying \((i, j) \in \arg\max_{(i, j) \in \mathcal{O}} (d_i - d_j)\), illustrated in Fig. 3. The first four edges added, namely \((1, 2), (1, 3), (1, 4), (1, 5)\), have identical priority as \(d_i - d_j\), \(\delta_i\), and \(\delta_j\) are equal for each. These four may be added in any order without affecting the resulting graph. The next edges added will be \((2, 3), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5)\). Finally, \((1, 6)\) will be added, leaving the only two stubs in the graph on vertex 6, which can only be wired via a self-loop, thereby violating the requirement that \( \tilde{G} \) be simple. \( \square \)

| \((i, j)\) | \(d_i, d_j\) | \(\delta_i\) | \(\delta_j\) |
|-------------|--------------|-------------|-------------|
| \((1, 2)\)  | 20, 5        | 5           | 4           |
| \((1, 3)\)  | 20, 5        | 5           | 4           |
| \((1, 4)\)  | 20, 5        | 5           | 4           |
| \((1, 5)\)  | 20, 5        | 5           | 4           |
| \((2, 3)\)  | 16, 3        | 3           | 3           |
| \((2, 4)\)  | 16, 3        | 3           | 3           |
| \((2, 5)\)  | 16, 3        | 3           | 3           |
| \((3, 4)\)  | 16, 3        | 3           | 3           |
| \((3, 5)\)  | 16, 3        | 3           | 3           |
| \((4, 5)\)  | 16, 3        | 3           | 3           |
| \((1, 6)\)  | 15, 1        | 1           | 3           |

Table 3. Subset of edge wirings for C.E. 3. The first set of rows correspond to wirings which are optimal at wiring step 1. The second set of rows are optimal wirings at wiring step 5. The final row is the only legal at wiring at step 11.
Counterexample 4. Fix \( n = 8 \) and \( d = (6, 4, 4, 4, 3, 2, 1) \) (which satisfies the Erdős Gallai theorem). The graph \( \tilde{G} \) returned by Alg. 1 is feasible, but its assortativity is not maximum and thus \( \tilde{G} \) is not optimal, i.e., \( \tilde{G} \not\in V_{\text{opt}}^{(n)}(d) \).

Proof. The proof is similar to C.E. 3. The partially wired graphs at steps 5, 11, 12, and 14 are shown in Fig. 4. The returned graph \( \tilde{G} = \tilde{G}_{14} \) achieves the target degree sequence \( d \), however its assortativity is not optimal. Namely, \( \alpha(\tilde{G}_{14}) = -0.04886 \) while \( \alpha(G_{\text{opt}}^{(n)}(d)) = -0.00326 \).

Finally, we close this section with Table 4, which counts the number of counterexamples for both feasibility and optimality.

| \( n \) | \( |D^{(n)}| \) | feasibility | optimality |
|-------|-------------|-------------|-------------|
| 5     | 19          | 0           | 0           |
| 6     | 68          | 2           | 0           |
| 7     | 236         | 16          | 0           |
| 8     | 863         | 91          | 4           |
| 9     | 3,137       | 443         | 36          |

Table 4: Wiring heuristic counterexample counts: number of degree sequences \( |D^{(n)}| \), number of degree sequences for which the returned graph is not feasible, and (if feasible) is not optimal, for graphs of order 5, 6, 7, 8, 9.

4. Conclusion

The main point of this letter is to demonstrate the failure of natural greedy heuristics, for both graph rewiring and wiring, to produce connected simple graphs with maximum assortativity over the target degree class. Many open
questions remain, such as how the relative prevalence of the various classes of counterexamples scales with \( n \). One possible direction for future work is to seek to characterize common structural properties of the degree sequences \( d \in D^{(n)} \) comprising the four types of counterexamples.

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