On the integrability of the octonionic Korteweg-de Vries equation

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Abstract. We introduce the octonionic Korteweg-de Vries (KdV) equation, starting with a Lagrangian formulation and presenting some of its symmetries. We introduce also the associated octonionic Gardner equation and deduce from it the infinite sequence of conserved quantities for octonionic KdV itself. We finally give a master Lagrangian from which we can get both the octonionic KdV and also the corresponding octonionic Miura KdV.

1. Introduction
Since the work of Miura and Gardner et al [1, 2, 3, 4, 5], carried out in the context of the Korteweg-de Vries (KdV) equation, a lot of interesting results about the so-called integrable systems have been obtained.

A class of extensions of KdV equation arises by introducing supersymmetry. Several integrable supersymmetric extensions were given in [7, 8, 9, 10, 11, 12].

In this work we consider the octonionic Korteweg-de Vries equation, that is, the defining field has values in the octonion algebra and the product being also the product in the same algebra.

A famous theorem by Hurwitz establishes that the only real normalized division algebras are the reals $\mathbb{R}$, the complex $\mathbb{C}$, the quaternions $\mathbb{H}$ and the octonions $\mathbb{O}$. In particular, these division algebras are directly related to the existence of Super Yang-Mills in several dimensions: three, four, six and ten dimensions [13]. The octonion algebra may be explicitly used in the formulation of superstring theory in ten dimensions and in the Supermembrane theory in eleven dimensions, relevant theories in the search for a unified theory of all the known fundamental forces in nature. The extension of the KdV equation to a partial differential equation for a field valued on a octonion algebra is then an interesting goal [14].

In [15] was showed that an extension of the KdV equation valued on a Clifford algebra gives rise to a coupled system with Liapunov stable soliton solutions but without an infinite sequence of local conserved quantities. In distinction, the system we will consider has an infinite sequence of conserved quantities.

The octonion algebra contains as subalgebras all other division algebras, hence our construction may be reduced to any of them.

The KdV equation on the octonion algebra can be seen as a coupled KdV system, as we will see it has some similarities to the construction in the previous sections. However, it is invariant under the exceptional Lie group $G_2$, the automorphisms of the octonions, and under the Galileo
transformations. Those symmetries characterize the octonionic system. In particular, they are not present on the Clifford algebra valued system in [15].

In section 2 we analyze the KdV extension where the field is valued on the octonion algebra. The system shares several properties of the original real KdV equation. It has soliton solutions and also has an infinite sequence of local conserved quantities derived from a Bäcklund transformation and a bi-Lagrangian and bi-Hamiltonian structure [14]. We also will show in section 2 a Lagrangian formulation for the octonion KdV equation.

In sections 3 and 4 we give the Gardner formulation and present the first conserved quantities (corresponding to the infinite sequence of conserved quantities) for the octonionic KdV, presenting also the master Lagrangian from which we can deduce both octonionic KdV and Miura equations.

In section 5 we give the conclusions.

2. Lagrangian formulation and symmetries for the octonionic Korteweg-de Vries equation

We denote \( u = u(x, t) \) a function with domain in \( \mathbb{R} \times \mathbb{R} \) valued on the octonion algebra. If we denote \( e_i, i = 1, \ldots, 7 \) the imaginary basis of the octonions, \( u \) can be expressed as

\[
u(x, t) = b(x, t) + \bar{B}(x, t)
\]

(1)

where \( b(x, t) \) is the real part and \( \bar{B} = \sum_{i=1}^{7} B_i(x, t)e_i \) its imaginary part.

The KdV equation formulated on the algebra of octonions, or simply the octonion KdV equation, is given by

\[
u_t + \nu_{xxx} + \frac{1}{2}(\nu^2)_x = 0,
\]

(2)

when \( \bar{B} = 0 \) it reduces to the scalar KdV equation. In terms of \( b \) and \( \bar{B} \) the equation can be re-expressed as

\[
b_t + b_{xxx} + bb_x - \sum_{i=1}^{7} B_i B_{ix} = 0,
\]

(3)

\[(B_i)_t + (B_i)_{xxx} + (b B_i)_x = 0.
\]

(4)

A Lagrangian for (2) is given by

\[
L(w) = \int_{t_i}^{t_f} dt \int_{-\infty}^{+\infty} dx \Re \left[ -\frac{1}{2} w_x w_t - \frac{1}{6} (w_x)^3 + \frac{1}{2} (w_{xx})^2 \right],
\]

where \( w = w_x \). Independent variations with respect to \( w \) yields the octonionic KdV equation (2).

Equation (2) is invariant under the Galileo transformation given by

\[
\tilde{x} = x + ct, \\
\tilde{t} = t, \\
\tilde{u} = u + c
\]

where \( c \) is a real constant.

Additionally, equation (2) is invariant under the automorphisms of the octonions, that is, under the group \( G_2 \). If under an automorphism

\[
u \rightarrow \phi(\nu)
\]
then
\[ u_1 u_2 \to \phi(u_1 u_2) = \phi(u_1) \phi(u_2) \]

and consequently
\[ [\phi(u)]_t + [\phi(u)]_{xxx} + \frac{1}{2} ([\phi(u)]^2)_x = 0. \]

3. The Gardner formulation for the octonion valued algebra KdV equation

Associated to the real KdV equation there is a Gardner \( \varepsilon \)-transformation and a Gardner equation which allows to obtain in a direct way the corresponding infinite sequence of conserved quantities. There exists a generalization of this approach for the KdV valued on the octonion algebra. The generalized Gardner transformation, expressed in terms of a new field \( r(x,t) \) valued on the octonion is given by
\[ u = r + \varepsilon r_x - \frac{1}{6} \varepsilon^2 r_x^2. \]  

The generalized Gardner equation is then
\[ r_t + r_{xxx} + \frac{1}{2} (r r_x + r_x r) - \frac{1}{12} ((r^2) r_x + r_x (r^2)) \varepsilon^2 = 0 \]

where \( \varepsilon \) is a real parameter.

If \( r(x,t) \) is a solution of the generalized Gardner equation (6), then \( u(x,t) \) is a solution of the octonion algebra valued KdV equation (2).

It has been shown in [14] that \( \int_{-\infty}^{+\infty} \Re [r(x,t)] dx \) is a conserved quantity of (6). We note that this fact is a non-trivial one because of the non-associativity (and non-commutativity) of the algebra.

We can then invert (5), assuming a formal \( \varepsilon \)-expansion of the solution \( r(x,t) \), to obtain an infinite sequence of conserved quantities for the KdV equation valued on the octonion algebra. The first few of them are
\[ H_1 = \int_{-\infty}^{+\infty} \Re e(u) dx, \]
\[ H_2 = \int_{-\infty}^{+\infty} \left( (\Re e(u))^2 - \| m(u) \|^2 \right) dx, \]
\[ H_3 = \int_{-\infty}^{+\infty} \left( \frac{1}{3} (\Re e(u))^3 - (\Re e(u_x))^2 + \| m(u_x) \|^2 - \Re e(u) \| m(u) \| \right) dx. \]

4. The master Lagrangian for the KdV equation valued on the octonion algebra

We may now use the Helmholtz procedure to obtain a Lagrangian density for the generalized Gardner equation. The master Lagrangian formulated in terms of the casimir potential \( s(x,t) \),
\[ r(x,t) = s_x(x,t), \]
is
\[ L_\varepsilon(s) = \int_{t_i}^{t_f} dt \int_{-\infty}^{+\infty} L_\varepsilon(s) dx \]

where the Lagrangian density is given by
\[ L_\varepsilon(s) = \Re e \left[ -\frac{1}{2} s_x s_t - \frac{1}{6} (s_x)^3 + \frac{1}{2} (s_{xx})^2 + \frac{1}{72} \varepsilon^2 (s_x)^4 \right]. \]

The Lagrangian density \( L_\varepsilon(s) \) is invariant under the action of the exceptional Lie group \( G_2 \).
Independent variations with respect to $s$ yields
$$\delta L_e(s) = \text{Re} \left[ -\frac{1}{2} (s)_x s_t - \frac{1}{2} s_x (\delta s)_t - \frac{1}{6} \left( (\delta s)_x (s_x)^2 + s_x (\delta s)_x s_x + (s_x)^2 (\delta s)_x \right) \right] +$$
$$+ \text{Re} \left[ \frac{1}{2} ((\delta s)_{xx} s_{xx} + (\delta s)_{xx} s_{xx}) + \frac{1}{72} \epsilon^2 \left( (\delta s)_x (s_x)^3 + s_x (\delta s)_x (s_x)^2 + (s_x)^2 (\delta s)_x s_x + (s_x)^3 (\delta s)_x \right) \right].$$

Using properties of the octonion algebra we obtain from the stationary requirement $\delta L_e(s) = 0$ the generalized Gardner equation (6).

In the calculation the property to be a division algebra of the octonions is explicitly used.

If we take the limit $\epsilon \to 0$, we obtain a first Lagrangian for the KdV equation valued on the octonion algebra,
$$L(w) = \int_{t_i}^{t_f} dt \int_{-\infty}^{+\infty} dx \text{Re} \left[ -\frac{1}{2} w_x w_t - \frac{1}{6} (w_x)^3 + \frac{1}{2} (w_{xx})^2 \right].$$

If we consider the following redefinition
$$s \to \hat{s} = \epsilon s$$
$$L_e(s) \to \epsilon^2 L_e(\hat{s})$$

and take the limit $\epsilon \to \infty$ we obtain
$$\lim_{\epsilon \to \infty} \epsilon^2 L_e(\hat{s}) = L^M(\hat{s}),$$

where
$$L^M(\hat{s}) = \text{Re} \left[ -\frac{1}{2} \hat{s}_x \hat{s}_t + \frac{1}{2} (\hat{s}_{xx})^2 + \frac{1}{72} (\hat{s}_x)^4 \right].$$

We get in this limit the generalized Miura Lagrangian
$$L^M(\hat{s}) = \int_{t_i}^{t_f} dt \int_{-\infty}^{+\infty} dx L^M(\hat{s}).$$

The Miura equation is then obtained by taking variations with respect to $\hat{s}$, we get
$$\dot{\hat{r}}_t + \hat{r}_{xxx} - \frac{1}{18} (\hat{r}_x)^3 = 0, \quad \hat{r} \equiv \hat{s}_x,$$

while the Miura transformation arises after the redefinition process, it is $u = \hat{r}_x - \frac{1}{6} \hat{r}^2$.

Any solution of the Miura equation, through the Miura transformation, yields a solution of the KdV equation valued on the octonion algebra. Since $L_e(s)$ is invariant under $G_2$, the same occurs for $L(w)$ and $L^M(\hat{s})$ and consequently for the equations arising from variations of them.

The Lagrangian formulation of the octonionic KdV equation may be used as the starting step to obtain the Hamiltonian structure of the octonion algebra valued KdV equation.

5. Conclusions
We introduced the octonionic Korteweg-de Vries equation, starting with a Lagrangian formulation and presenting some of its symmetries. We analyzed the integrable properties of this equation using a generalized Gardner equation. With this tool we obtained the infinite conserved quantities for the octonionic KdV equation. We obtained also a master Lagrangian from which we can get both the octonionic KdV and the octonionic Miura KdV. We expect, in view of the connection of the octonion algebra with string theories, to consider a supersymmetric formulation for the octonionic KdV equation. We will report on this extension elsewhere.

Acknowledgments
A. R. and A. S. are partially supported by Project Fondecyt 1161192, Chile.
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