Automorphism Groups of Saturated Structures; A Review

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Abstract

We will review the main results concerning the automorphism groups of saturated structures which were obtained during the two last decades. The main themes are: the small index property in the countable and uncountable cases; the possibility of recovering a structure or a significant part of it from its automorphism group; the subgroup of strong automorphisms.

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1. Introduction

Saturated models play an important role in model theory. In fact, when studying the model theory of a complete theory $T$, one may work in a large saturated model of $T$ with its definable sets, and forget everything else about $T$. This large saturated structure is sometimes called the “universal domain”, sometimes the “monster model”.

A significant work has been done the last twenty years on the automorphism groups of saturated models. It is this work that I want to review here. There is a central question that I will use as a “main theme” to organize the paper: what information about $M$ and its theory are contained in its group of automorphisms? In the best case, $M$ itself is “encoded” in some way in this group; recovering $M$ from it is known as “the reconstruction problem”. A possible answer to this problem is a theorem of the form: If $M_1$ and $M_2$ are structures in a given class with isomorphic automorphism groups, then $M_1$ and $M_2$ are isomorphic.

Throughout this paper, $T$ is supposed to be a countable complete theory. The countability of $T$ is by no means an essential hypothesis. Its purpose is only to make the exposition smoother, and most of the results generalize without difficulty.

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to uncountable theories. We will denote by \( Aut(M) \) the group of automorphisms of the structure \( M \), and if \( A \) is a subset of \( M \), \( Aut_A(M) \) will be the pointwise stabilizer of \( A \):

\[
Aut_A(M) = \{ f \in Aut(M) : \forall a \in A f(a) = a \}.
\]

When we say “definable”, we mean “definable without parameters”.

2. The countable case

As a preliminary remark, let us say that the automorphism group of a saturated model is always very rich: if \( M \) has cardinality \( \lambda \), then its automorphism group has cardinality \( 2^{\lambda} \).

I do not know who was the first to introduce the small index property. As we will see, it is crucial in the subject.

**Definition 1** Let \( M \) be a countable structure. We say that \( M \) (or \( Aut(M) \)) has the small index property if for any subgroup \( H \) of \( Aut(M) \) of index less than \( 2^{\aleph_0} \), there exists a finite set \( A \subset M \) such that \( Aut_A(M) \subseteq H \).

Remark that the converse is true: any subgroup containing a subgroup of the form \( Aut_A(M) \) where \( A \) is finite, has a countable index in \( Aut(M) \). Moreover, the subgroups containing a subgroup of the form \( Aut_A(M) \) are precisely the open neighborhoods of the identity for the pointwise convergence topology. In other words, the small index property allows us to recover the topological structure of \( Aut(M) \) from its pure group structure.

The small index property has been proved for a number of countable saturated structures:

1. The infinite set without additional structure [28], [5].
2. The linear densely ordered sets [25].
3. The vector spaces over a finite field [6].
4. The random graph [10].
5. Various other classes of graphs [9].
6. Generic relational structures [8].
7. \( \omega \)-categorical \( \omega \)-stable structures [10].

The small index property has also been proved for some countable structures which are not saturated: for the free group with \( \omega \)-generators [2], for arithmetically saturated models of arithmetic [17].

There are examples of countable saturated structures which fail to have the small index property. The simplest may be an algebraically closed field of characteristic 0 of infinite countable transcendence degree: Let \( \overline{\mathbb{Q}} \) be the algebraic closure of the field of rational numbers. There is an obvious homomorphism \( \varphi \) from \( Aut(M) \) onto \( Aut(\overline{\mathbb{Q}}) \) (the restriction map). Now, it is well known that there is a subgroup \( H \) of \( Aut(\overline{\mathbb{Q}}) \) of countable index (in fact of finite index) which is not closed for the Krull topology, which is nothing else that the pointwise convergence topology. Then \( \varphi^{-1}(H) \) is not open, but of finite index in \( Aut(M) \).
As we will see later, the small index property is particularly relevant for $\omega$-categorical structures. Evans and Hewitt have produced an example of such a structure without the small index property \cite{7}.

With the pointwise convergence topology, $\text{Aut}(M)$ is a topological polish group. So, we may use the powerful tools of descriptive set theory. In many cases (for example for structures 1-6 above), it can be shown that there is a (necessarily unique) conjugacy class which is generic, that is, is the countable intersection of dense open subsets. The elements of this class are called generic automorphisms, and they play an important role in the proof of the small index property.

Another possible nice property of these automorphism groups which is sometimes obtained as a bonus of the proof of the small index property, is the fact that its cofinality is not countable, that is, $\text{Aut}(M)$ is not the union of a countable chain of proper subgroups. This is proved in particular for the full permutation group of a countable set \cite{21}, for the random graph and for $\omega$-categorical $\omega$-stable structures \cite{10}.

I would like to mention here the work of Rubin \cite{24}. He has shown how to reconstruct a certain number of structures from their automorphism group using a somewhat different method. His methods apply essentially to “combinatorial structures” such as the random graph, the universal homogeneous poset, the generic tournament (a structure for which the small index property is not known), etc.

3. Subgroups and imaginary elements

Recall that an imaginary element of $M$ is a class of a tuple of $M^n$ modulo a definable equivalence relation on $M^n$. For instance, if $G$ is a group and $H$ a definable subgroup of $G^n$, then any coset of $H$ in $G^n$ is an imaginary element. When we add all these imaginary elements to a saturated structure $M$, we obtain the structure $M^{eq}$, and we can consider $M^{eq}$ as a saturated structure (in a larger language).

It is clear that $M$ and $M^{eq}$ have canonically the same automorphism group: every automorphism of $M$ extends uniquely to an automorphism of $M^{eq}$. This shows a limitation to the reconstruction problem: If $M$ and $N$ are two structures which are such that “$M^{eq}$ and $N^{eq}$ are isomorphic”, then $\text{Aut}(M)$ and $\text{Aut}(N)$ are isomorphic via a bicontinuous isomorphism. The condition “$M^{eq}$ and $N^{eq}$ are isomorphic” may seem weird, but in fact, it is natural. Roughly speaking, it means that $M$ can be interpreted in $N$, and conversely (a little more in fact, see \cite{1} for more details). In this case, we say that $M$ and $N$ are bi-interpretable.

Consider now the case of an $\omega$-categorical structure $M$. It is not difficult to see that any open subgroup of $\text{Aut}(M)$ is the stabilizer $\text{Aut}_\alpha(M)$ of an imaginary element $\alpha$. Moreover, $\text{Aut}(M)$ acts by conjugation on the set of its open subgroups, and this action is (almost) isomorphic to the action of $\text{Aut}(M)$ on $M^{eq}$ (almost because two different imaginary elements $\alpha$ and $\beta$ may have the same stabilizer).

So, from the topological group $\text{Aut}(M)$ we can (almost) reconstruct its action on $M^{eq}$. We can do better:

**Theorem 2** $\square$ Assume that $M$ and $N$ are countable $\omega$-categorical structures. Then the following two conditions are equivalent:
1. there is a bicontinuous isomorphism from $\text{Aut}(M)$ onto $\text{Aut}(N)$;
2. $M$ and $N$ are bi-interpretable.

In fact, these conditions are also equivalent to: there exists a continuous isomorphism from $\text{Aut}(M)$ onto $\text{Aut}(N)$ (see [15]). Thus, if one of the structure $M$ or $N$ has the small index property and $\text{Aut}(M)$ is isomorphic to $\text{Aut}(N)$ (as pure groups), then $M$ and $N$ are bi-interpretable.

Now, if $M$ is not necessarily $\omega$-categorical (but still saturated), the situation is a bit more complicated. We need to introduce new elements.

**Definition 3** 1. An ultra-imaginary element of $M$ is a class modulo $E$, where $E$ is an equivalence relation on $M^n$ ($n \leq \omega$) which is invariant under the action of $\text{Aut}(M)$. An ultra-imaginary element is finitary if $n < \omega$.
2. A hyperimaginary element of $M$ is a class modulo $E$, where $E$ is an equivalence relation on $M^n$ ($n \leq \omega$) which is defined by a (possibly infinite) conjunction of first order formulas.

An imaginary element is hyperimaginary, and a hyperimaginary element is ultra-imaginary. A hyperimaginary element is a class modulo an equivalence relation $E$ defined by a formula of the form

$$\bigwedge_{i \in I} \varphi_i$$

where the $\varphi_i$ are first-order formulas (without parameters) and whose free variables are among the $x_k$ for $k < n$. An ultra-imaginary element is a class modulo an equivalence relation $E$ defined by a formula of the form

$$\bigvee_{j \in J} \bigwedge_{i \in I} \varphi_{i,j}$$

where the $\varphi_{i,j}$ are first-order-formulas (without parameters) and whose free variables are among the $x_k$ for $k < n$.

If $M$ is a countable saturated structure, the stabilizer of a finitary ultra-imaginary element is clearly an open subgroup, and it is not difficult to see that if $H$ an open subgroup of $\text{Aut}(M)$, then there exists a finitary ultra-imaginary element $\alpha$ such that $H$ is the stabilizer of $\alpha$. In the $\omega$-categorical case, any finitary ultra-imaginary is in fact imaginary, and this explain why this case is so simple.

In some cases, for example for $\omega$-stable theories (see [18]), it is possible to characterize, among all open subgroups, those which are of the form $\text{Aut}_\alpha(M)$ with $\alpha$ imaginary. Something similar has been done for countable arithmeticaly saturated models of arithmetic in [11], and in [13], it is proved that if two such models have isomorphic automorphism groups, then they are isomorphic.

### 4. Strong automorphisms

It is now time to introduce the group of strong automorphisms.
Definition 4 [14] The group of strong automorphisms of $M$ is the group generated by the set

$$\bigcup \{ \text{Aut}_N(M) : N \preceq M \}$$

and is denoted $\text{Aut}^f(M)$.

It is easy to see that $\text{Aut}^f(M)$ is a normal subgroup of $\text{Aut}(M)$. Its index is at most $2^{\aleph_0}$. Moreover, the quotient group $\text{Aut}(M)/\text{Aut}^f(M)$ depend only on $T$: if $M$ and $M'$ are two saturated models, $M \preceq M'$, then there is a natural isomorphism from $\text{Aut}(M)/\text{Aut}^f(M)$ onto $\text{Aut}(M')/\text{Aut}^f(M')$. $\text{Aut}(M)/\text{Aut}^f(M)$ will be denoted $\text{Gal}(T)$ (of course, $\text{Gal}$ stands for Galois). For example, if $T$ is the theory of algebraically closed fields of characteristic 0, $\text{Aut}(M) = \text{Aut}_\bar{Q}(M)$ and $\text{Gal}(T)$ is (isomorphic to) the group of automorphisms of $\bar{Q}$.

In fact this interpretation is general. Assume first that $M$ is of cardinality bigger than $2^{\aleph_0}$. Let $\alpha$ be an ultra-imaginary element of $M$. It can be shown that the following conditions are equivalent:

1. $\text{card}\{ f(\alpha) : f \in \text{Aut}(M) \} < \text{card}(M)$;
2. $\text{card}\{ f(\alpha) : f \in \text{Aut}(M) \} \leq 2^{\aleph_0}$;
3. $\alpha$ (as a set) is the class modulo an invariant bounded equivalence relation.

If these conditions are satisfied, we say that $\alpha$ is bounded. It should be remarked that an imaginary element is bounded if and only if it is algebraic, if and only if its orbit is finite.

We will denote by $\text{Bdd}(M)$ the set of bounded ultra-imaginary elements of $M$. This set does not really depend on $M$ (as soon as its cardinality is big enough) but only on its theory: any invariant bounded equivalence relation has a representative in any uncountable saturated model. We will allow ourself to write $\text{Bdd}(T)$ when convenient. Moreover $\text{Aut}^f(M)$ is exactly the pointwise stabilizer of $\text{Bdd}(M)$ so that $\text{Gal}(T)$ can be identified with the group of elementary permutations of $\text{Bdd}(T)$.

With some care, we can generalize this interpretation to models of small cardinality: for example, assume $M$ countable, and let $M'$ be a large saturated extension of $M$. Then any automorphism $f$ of $M$ extends to an automorphism of $M'$, and if $f_1$ and $f_2$ are two such extensions, then their action on $\text{Bdd}(M')$ are equal: $\text{Aut}^f(M)$ is exactly the set of automorphisms whose extensions to $M'$ act trivially on $\text{Bdd}(M')$.

In any case, $\text{Aut}(M)$ leaves fixed the set of bounded imaginary elements and the set of bounded hyperimaginary elements. In some cases (for example for algebraically closed fields), $\text{Gal}(T)$ acts faithfully on the set of bounded imaginary elements. It is the case if $T$ is stable ([14]). It is not known if it is always true for simple theories, but it is true for the so-called low simple theories ([3]) and in particular for supersimple theories. For simple theories, $\text{Gal}(T)$ acts faithfully on the set of bounded hyperimaginary elements ([12]). In [4] there is an example of a theory where the action of $\text{Gal}(T)$ on the set of hyperimaginary elements is not faithful.
There is a natural topology on $Gal(T)$ (see [19] for details). It can be defined in two different ways.

My favorite one is via the ultraproduct construction. Let $(\gamma_i ; i \in I)$ be a family of elements of $Gal(T)$ and $\mathcal{U}$ an ultrafilter on $I$. Choose a saturated model $M$ and, for each $i \in I$ an automorphism $f_i \in Aut(M)$ lifting $\gamma_i$. Consider the saturated model $M'$ and, for each $i \in I$ an automorphism $f_i \in Aut(M')$ lifting $\gamma_i$. Choose a saturated model $M'$ and, for each $i \in I$ an automorphism $f_i \in Aut(M')$ lifting $\gamma_i$. Consider the ultrapower $M' = \prod_{i \in \mathcal{U}} M_i$. We can define the automorphism $\prod_{i \in \mathcal{U}} f_i$ on $M'$. This automorphism acts on $Bdd(M') = Bdd(T)$, so defines an element of $Gal(T)$, say $\beta$. This element $\beta$ should be considered as a limit of the family $(\gamma_i ; i \in I)$ along $\mathcal{U}$. A subset $X$ of $Gal(T)$ is closed for the topology we are defining if it is closed for this limit operation. You should be aware that the element $\beta$ may depend on the choices of the $f_i$'s, because the topology we are defining is not Hausdorff in general.

The other way to define a topological structure on $Gal(T)$ is to define a topology on $Bdd(T)$. If, as it is the case when $T$ is stable, $Gal(T)$ can be identified with a group of permutation on the set of imaginary elements, then we just endow $Gal(T)$ with the pointwise convergence topology (that is we consider the set of imaginary elements with the discrete topology). Otherwise, it is more complicated, and here is what should be done in general:

For each $n \leq \omega$ and $E$ invariant bounded equivalence relation on $M^n$, consider the canonical mapping $\varphi_E$ from $M^n$ onto $M^n/E$. By definition, a subset $X$ of $M^n/E$ is closed if and only if $\varphi_E^{-1}(X)$ is the intersection of a family of subsets definable with parameters. $Gal(T)$ acts on $M^n/E$ and the topology on $Gal(T)$ is defined as the coarsest topology which makes all these actions (with various $n$ and $E$) continuous.

Now, we can prove:

**Theorem 5** 1. $Gal(T)$ is a topological compact group.

2. It is Hausdorff if and only if it acts faithfully on the set of bounded hyperimaginary elements, if and only if it acts faithfully on the set of finitary bounded hyperimaginary elements.

3. It is profinite if and only if it acts faithfully on the set of bounded imaginary elements.

There is a Galois correspondence between the subgroups of $Gal(T)$ and the bounded ultra-imaginary elements: every subgroup of $Gal(T)$ is the stabilizer of an ultra-imaginary element. The hyperimaginary elements correspond to the closed subgroups and the imaginary elements correspond to the clopen subgroups of $Gal(T)$.

Let $H_0$ be the topological closure of the identity. Then $H_0$ is a normal subgroup of $Gal(T)$. If we consider $Gal(T)$ as a permutation group on $Bdd(T)$, $H_0$ is exactly the pointwise stabilizer of the set of bounded hyperimaginary elements. So, if we set $Gal_0(T) = Gal(T)/H_0$, $Gal_0(T)$ acts faithfully on the set of bounded hyperimaginary elements. As a quotient group, $Gal_0(T)$ is canonically endowed with a topology. This way, we get a compact Hausdorff group.

Recently, L. Newelski ([22]) has proved that $H_0$ is either trivial or of cardinality $2^{\aleph_0}$.

I would like to conclude this section by a conjecture. In all the known examples of countable saturated structures where the small index property is false, there is a
non open subgroup of $Gal(T)$ of countable index (and, its preimage by the canonical homomorphism from $Aut(M)$ onto $Gal(T)$ is a non open subgroup of $Aut(M)$ of countable index). If $A$ has a cardinality strictly less than $\text{card}(M)$, define $Aut_f(A)$ as the subgroup of $Aut_A(M)$ generated by

$$\bigcup\{Aut_N(M) : A \subseteq N \prec M\}.$$ 

The following conjecture is open, even in the $\omega$-categorical case:

**Conjecture 6** Assume that $M$ is a countable saturated structure and let $H$ be a subgroup of $Aut_f(M)$ of index strictly less than $2^{\aleph_0}$. Then, there exists a finite subset $A \subset M$ such that $Aut_f(A) \subseteq H$.

In [16], this conjecture is proved for almost strongly minimal sets (so, in particular for algebraically closed fields).

5. The uncountable case

We are now given a saturated structure $M$ of cardinality $\lambda > \aleph_0$. The small index property has a natural generalization. If we assume that $\lambda^{<\lambda} = \lambda$ (i.e. there is exactly $\lambda$ subsets of $M$ of cardinality less than $\lambda$) then any subgroup of $Aut(M)$ containing a subgroup of the form $Aut_A(M)$ with $\text{card}(A) < \lambda$ has index at most $\lambda$. The converse is true:

**Theorem 7** [20] Assume that $M$ is a saturated structure of uncountable cardinality $\lambda = \lambda^{<\lambda}$, and let $H$ be a subgroup of $Aut(M)$ of cardinality at most $\lambda$. Then, there exists a subset $A$ of $M$ of cardinality less than $\lambda$ such that $Aut_A(M) \subseteq H$.

Here again, we may introduce a topological structure on $Aut(M)$: if $\mu$ is an infinite cardinal, let $T_{\mu}$ be the group topology on $Aut(M)$ for which a basis of open neighborhoods of the identity is

$$\{Aut_A(M) : A \subseteq M \text{ and } \text{card}(A) < \mu\}.$$ 

To complete this definition, let $T_a(M)$ be the group topology on $Aut(M)$ for which a basis of open neighborhoods of the unit is

$$\{Aut_f(A) : A \subseteq M \text{ and } A \text{ finite}\}.$$ 

The above theorem just says that the subgroups of $Aut(M)$ of index at most $\lambda$ are exactly the open subgroups for $T_{\lambda}(M)$, and consequently, the topology $T_{\lambda}(M)$ can be reconstructed from the pure group structure. It is also clear that the open subsets for $T_{\lambda}$ are just the intersections of less than $\lambda$ $T_a$-open subsets. So, if one knows $T_a(M)$, one knows $T_{\lambda}(M)$.

With a few cardinality hypotheses, we can reconstruct one topological group from another: (see [15] for details):
1. Let $M$ and $M'$ be two saturated models of the same theory. Then we can reconstruct $(\Aut(M'), T_\lambda)$ from $(\Aut(M), T_\mu)$.

2. Let $M$ and $M'$ be two models of the same theory, and assume $\text{card}(M) = \lambda < \mu = \text{card}(M')$. Then we can reconstruct $\Aut(M')$ from $(\Aut(M), T_\lambda)$ (and from $\Aut(M)$ alone if $\lambda = \lambda^{<\lambda}$). In fact we can reconstruct $(\Aut(M'), T_\upsilon(M'))$ for every cardinal $\upsilon$, $\lambda \leq \upsilon \leq \mu$.

3. Let $M$ be a saturated structure of uncountable cardinality $\lambda = \mu^+ = 2^\mu$ and assume that $T$ has a saturated model of cardinality $\mu$. Then $(\Aut(M), T_\lambda)$ can be reconstructed from $\Aut(M)$.

Let us give an example of a theorem which can be proved using the above facts: Assume GCH and let $T_1$ and $T_2$ be two theories with saturated models $M_1$ and $M_2$ of cardinality $\mu^+$. Assume that $\Aut(M_1)$ and $\Aut(M_2)$ are isomorphic. Then, for all cardinal $\lambda$, if $T_1$ has a saturated model of cardinality $\lambda$, then $T_2$ has also a saturated model in cardinality $\lambda$, and the automorphism groups of these two models are isomorphic.

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