PLANARITY IS (ALMOST) LOCALLY CHECKABLE IN CONSTANT-TIME

GÁBOR ELEK

Abstract. Locally checkable proofs for graph properties were introduced by Göös and Suomela [4]. Roughly speaking, a graph property $\mathcal{P}$ is locally checkable in constant-time, if the vertices of a graph having the property can be convinced, in a short period of time not depending on the size of the graph, that they are indeed vertices of a graph having the given property. For a given $\varepsilon > 0$, we call a property $\mathcal{P}$ $\varepsilon$-locally checkable in constant-time if the vertices of a graph having the given property can be convinced at least that they are in a graph $\varepsilon$-close to the given property. We say that a property $\mathcal{P}$ is almost locally checkable in constant-time, if for all $\varepsilon > 0$, $\mathcal{P}$ is $\varepsilon$-locally checkable in constant-time. It is not hard to see that in the universe of bounded degree graphs planarity is not locally checkable in constant-time. However, the main result of this paper is that planarity of bounded degree graphs is almost locally checkable in constant-time. The proof is based on the surprising fact that although graphs cannot be convinced by their planarity or hyperfiniteness, planar graphs can be convinced by their own hyperfiniteness. The reason behind this fact is that the class of planar graphs are not only hyperfinite but possesses Property A of Yu.

Keywords. locally checkable proofs, planar graphs, Property A, hyperfiniteness

2010 Mathematics Subject Classification. 68M14, 05C85.

The author was partially supported by the ERC Consolidator Grant "Asymptotic invariants of discrete groups, sparse graphs and locally symmetric spaces" No. 648017.
1. Introduction

First, let us recall the notion of a locally checkable proof due to Göös and Suomela [4]. For an integer $d > 1$, let $G_d$ be the set of all finite, simple graphs of vertex degree bound $d$. Also, for the natural number $q$, let $[q]$ denote the set $\{0, 1, 2, \ldots , q\}$.

For a graph $G \in G_d$, a $q$-proof is a function $P : V(G) \rightarrow [q]$.

A $q$-verifier $A$ of local horizon $R$ is a subset of $B_{R,d}^{[q]}$, where $B_{R,d}^{[q]}$ is the set of all $[q]$-vertex labelled balls of radius $R$ and of vertex degree bound $d$.

A $q$-verifier $A$ of local horizon $R$ accepts a $q$-proof $P$ on the graph $G \in Gr_d$, if for all vertices $x \in V(G)$, $B_R(x, G, P) \in A$, where $B_R(x, G, P)$ is the ball of radius $R$ centered at $x$ with vertex labelling induced by $P$.

A $q$-verifier $A$ of local horizon $R$ rejects a $q$-proof $P$ on the graph $G \in Gr_d$, if for at least one vertex $x \in V(G)$, $B_R(x, G, P) \notin A$.

We refer to subsets of $G_d$ as ”properties” and we say that a property $\mathcal{P} \in G_d$ is locally checkable in constant-time (shortly: locally checkable properties) if there exists $q \geq 1$ and a verifier $A \subset B_{R,d}^{[q]}$ such that

- for any $G \in \mathcal{P}$ there exists a $q$-proof $P : V(G) \rightarrow [q]$ accepted by $A$,  
- for any $H \notin \mathcal{P}$ all the $q$-proofs on $H$ are rejected by $A$. 

If the $q$ can be chosen to be 0, that is the vertices are exploring a small un-labelled ball around themselves, we say that the property is locally checkable without labelling.

Local checkability in constant-time entails that the vertices of a graph $G \in \mathcal{P}$ can be convinced in a short period of time, that they are indeed vertices of a graph having the given property. Clearly, bipartiteness or being a union of cycles are locally checkable properties. On the other hand, being connected or planar are not locally checkable. In this paper we define a relaxation of the local checkability notion motivated by property testing of bounded-degree graphs [3].

Recall that if $\mathcal{P} \subset \mathcal{G}_d$ and $H \in \mathcal{G}_d$, then the edit distance of the property $\mathcal{P}$ and the graph $H$ is defined by

$$e(H, \mathcal{P}) := \inf_{G \in \mathcal{P}} \frac{|E(G) \triangle E(H)|}{|V(H)|}.$$  

Note that we realize the graphs as subgraphs of a common graph, which can be $H$ if $G$ and $H$ have the same amount of vertices.

**Definition 1.1.** For $\varepsilon > 0$, a property $\mathcal{P} \in \mathcal{G}_d$ is $\varepsilon$-locally checkable in constant-time if there exists $q \geq 0$ and a verifier $A \subset B^{[q]}_{R,d}$ such that

- for any $G \in \mathcal{P}$ there exists a $q$-proof $P$ on $G$ accepted by $A$,
- for any $H, e_d(H, \mathcal{P}) > \varepsilon$, all $[q]$-proofs on $H$ are rejected by $A$.

**Definition 1.2.** A property $\mathcal{P} \in \mathcal{G}_d$ is almost locally checkable in constant-time if for any $\varepsilon > 0$, $\mathcal{P}$ is $\varepsilon$-locally checkable in constant-time.

We will also use the notion of relative local checkability.

**Definition 1.3.** Let $\mathcal{P} \subset \mathcal{Q} \subset \mathcal{G}_d$ be properties. We say that $\mathcal{Q}$ is locally checkable in constant-time relative to $\mathcal{P}$ if there exists a verifier $A \subset B^{[q]}_{R,d}$ such that

- for every $G \in \mathcal{P}$ there exists a proof $P : V(G) \to [q]$ such that $A$ verifies $p$,
- for every $H \notin \mathcal{Q}$ all proofs $P : V(G) \to [q]$ are rejected by $A$.

The main result of this paper is the following theorem.

**Theorem 1.** Planarity is almost locally checkable in constant-time.

Our result hinges on the fact that although neither planarity nor $(\varepsilon, K)$-hyperfiniteness (see Section 2) are locally checkable properties, for every $\varepsilon > 0$ there exists $K > 0$ such that all planar graphs are $(\varepsilon, K)$-hyperfinitis and the property of being $(\varepsilon, K)$-hyperfinitis is locally checkable in constant-time relative to planarity.
We will also show (Proposition 3.2) that planarity is not almost locally checkable without labeling.

We actually prove that all hyperfinite monotone properties are almost locally checkable in constant-time (see Section 2 for definition). Note that the property testing analogue of this theorem is proved by Newman and Sohler [6]. Our proof uses the proposition, recently proved by Romero, Wrochna and Živný, that hyperfinite monotone graph classes are hyperfinite in a stronger than usual sense. Their result boils down to the fact that the class of planar graphs has Property A.

2. Hyperfiniteness and the Verification Distance

Let $G \in Gr_d$, $R > 1$, $q \geq 0$ and $A \subset B_{R,d}^q$. We say that $A$ verifies $G$ if there exists a proof $P : V(G) \to [q]$ such that $A$ accepts $P$.

Let $A \subset B_{R,d}^q$ be a verifier. We denote by $\mathcal{L}_A$ the set of graphs in $Gr_d$ verified by $A$. So, a property $\mathcal{P}$ is locally checkable in constant-time if there exists $A$ such that $\mathcal{P} = \mathcal{L}_A$. Observe that if $A_1 \subset B_{R,d}^q$ and $A_2 \subset B_{R,d}^q$, then there exists a verifier $A_3 \in B_{R,d}^{q_1+q_2}$ such that $\mathcal{L}_{A_3} = \mathcal{L}_{A_1} \cap \mathcal{L}_{A_2}$. If $\mathcal{P} \subset \mathcal{Q}$, then there exists $A$ such that $\mathcal{P} \subset \mathcal{L}_A \subset \mathcal{Q}$ if and only if $\mathcal{Q}$ is locally checkable in constant-time relative to $\mathcal{P}$. If $\mathcal{P}$ is a property and $\mathcal{P}_d$ is the set of graphs which are at most $\delta$ far from having the property $\mathcal{P}$, then $\mathcal{P}$ is almost locally-checkable in constant-time if for any $\delta > 0$ there exists a verifier $A_\delta$ such that $\mathcal{P} \subset \mathcal{L}_{A_\delta} \subset \mathcal{P}_\delta$.

If $G, H \in Gr_d$ and $A \subset B_{R,d}^q$, we say that $A$ distinguishes $G$ from $H$, if either $A$ verifies $G$ and does not verify $H$ or $A$ verifies $H$ and does not verify $G$.

Clearly, if $G$ and $H$ are nonisomorphic connected graphs then they can be distinguished by a verifier. Also, if $G$ and $H$ cannot be distinguished then they have isomorphic components.

**Definition 2.1.** Let $G, H \in Gr_d$. Then the verification distance of $G$ and $H$ is defined as $v(G, H) = 2^{-n}$, where $n$ is the smallest integer such that there exists $A \subset B_{R,d}^q$, $R \leq n$, $q \leq n$ distinguishing $G$ and $H$ ($v(G, H) = 0$ if $G$ and $H$ are undistinguishable).

It is not hard to see that $v$ defines a pseudo-metric on $Gr_d$ and from any sequence $\{G_n\}_{n=1}^\infty$ one can pick a convergent subsequence. Let $\mathcal{P} \subset Gr_d$ be a property. Then, $\mathcal{P}$ is not locally checkable in constant-time if and only if for any $\delta > 0$ there exists $G \notin \mathcal{P}$ such that $v(G, \mathcal{P}) < \delta$. Also, $\mathcal{P}$ is not almost locally checkable in constant-time if and only if there exists $\varepsilon > 0$ such that
for every $\delta > 0$ we have $G \notin P$ satisfying the inequalities $v(G, P) < \delta$ and $e(G, P) > \varepsilon$.

Now let us recall the notion of hyperfiniteness. For $\varepsilon > 0$ and $K \geq 1$, a graph $G$ is called $(\varepsilon, K)$-hyperfinite if there exists $W \subset V(G)$ such that

- $|W| \leq \varepsilon |V(G)|$,
- if we remove $W$ (and all the adjacent edges) from $G$, in the remaining graph all the components have size at most $K$.

A property $P \subset \text{Gr}_d$ is $(\varepsilon, K)$-hyperfinite if all $G \in P$ are $(\varepsilon, K)$-hyperfinite. The set of all $(\varepsilon, K)$-hyperfinite graphs is denoted by $\mathcal{H}_{\varepsilon,K}$.

A property $P$ is hyperfinite if for any $\varepsilon > 0$ there exists $K \geq 1$ such that $P \subset \mathcal{H}_{\varepsilon,K}$. Note that the class of planar graphs, or in general, graphs with an excluded minor, or graphs of subexponential growth are hyperfinite. On the other hand, expander sequences are very far from being a hyperfinite family.

**Proposition 2.1.** For any $\varepsilon > 0$ and $K \geq 1$, $\mathcal{H}_{\varepsilon,K}$ is not almost locally checkable in constant-time.

**Proof.** Let $G_n$ be the union of $n$ disjoint copies of a path of $K$ elements and let $\{H_n\}_{n=1}^\infty$ be an increasing expander sequence which is Cauchy in the verification pseudo-metric. Now, let $K_n$ be the disjoint union of $G_n$ and $H_{m_n}$, where

$$|V(H_{m_n})| < \varepsilon |G_n|.$$ 

Also, let $L_n$ be the disjoint union of $G_n$ and $H_{l_n}$, where

$$|V(H_{l_n})| > |G_n|.$$ 

Then, there exists $\delta > 0$ such that $e(L_n, \mathcal{H}_{\varepsilon,K}) > \delta$. On the other hand, $v(K_n, L_n) \rightarrow 0$. Hence, for any $A$, if $n$ is large enough either $A$ verifies both $K_n$ and $L_n$ or $A$ verifies none of them. Therefore, $\mathcal{H}_{\varepsilon,K}$ is not almost locally-checkable in constant-time.

### 3. Two Important Examples

The following proposition is certainly known, nevertheless we prove it for completeness.

**Proposition 3.1.** Let $P \subset \text{Gr}_d$ be the set of planar graphs. Then, $P$ is not locally-checkable in constant-time.

**Proof.** For $n \geq 1$, let $x \neq y$ and consider three paths connecting $x$ and $y$: $(x, a_1, a_2, \ldots, a_n, y), (x, b_1, b_2, \ldots, b_n, y), (x, c_1, c_2, \ldots, c_n, y)$ such that $a_i \neq b_j$ for $1 \leq i, j \leq n$. Let $L_n$ be the union of the three paths. Clearly, $L_n$ is a planar graph.
Again, for \( n \geq 1 \) let \( K_n \) be defined in the following way. For distinct vertices \( x_1, x_2, x_3, y_1, y_2, y_3 \) let \( K_n \) be the union of nine paths:

\[
\{ P_{i,j} := (x_i, g_{1}^{i,j}, g_{2}^{i,j}, \ldots, g_{n}^{i,j}, y_j) \}_{i,j=1}^{3}
\]

where \( g_{k}^{i,j} = g_{l}^{i,j} \) if and only if \( i_1 = i_2, \ j_1 = j_2, \ k = l \). By Kuratowski’s Theorem, \( K_n \) is not a planar graph. Let \( 2R < n, q \geq 0 \) and \( P : V(L_n) \to [q] \) be a proof. Then, there exists \( P' : V(K_n) \to [q] \) such that the set of \([q]\)-labeled \( R \)-balls (up to isomorphism) in \( L_n \) induced by \( P \) coincides with the set of \([q]\)-labeled \( R \)-balls (up to isomorphism) in \( K_n \) induced by \( P' \). Hence, all the verifiers of local horizon \( R \) which accepts \( L_n \) will accept \( K_n \), as well. Thus, our proposition follows. \( \square \)

Complementing our main theorem, the following proposition shows that in order to locally distinguish planar graphs from graphs that are far from being hyperfinite, we need labellings.

**Proposition 3.2.** Let \( d > 2 \). Then there exists a \( \delta > 0 \), a sequence of trees \( \{S_n\}_{n=1}^{\infty} \subset Gr_d \), a sequence of graphs \( \{G_n\}_{n=1}^{\infty} \subset Gr_d \) such that

- for all \( n \geq 1 \), \( e(G_n, \mathcal{P}) > \delta \), where \( \mathcal{P} \subset Gr_d \) is the class of planar graphs.
- for all \( R \geq 1 \) and \( n \geq R \), the sets of rooted balls of radius \( R \) (up to rooted isomorphism) in \( S_n \) and in \( G_n \) coincide.

**Proof.** For \( R \geq 1 \), let \( \{T_i\}_{i=1}^{t_{d,R}} \) be the set of rooted trees of diameter at most \( 3R \) and vertex degree bound \( d \) (up to rooted isomorphism). Let \( a_{d,R} = \sup_{1 \leq i \leq t_{d,R}} |V(T_i)| \). Now, let \( \{H_n\}_{n=1}^{\infty} \subset Gr_d \) be a sequence of \( d \)-regular graphs such that for any \( n \geq 1 \),

- \( |V(H_n)| \geq nt_{d,R}a_{d,R} \),
- the girth of \( H_n \) is at least \( n \).

Then, \( \liminf e(H_n, \mathcal{P}) > 0 \), since the graph sequence \( \{H_n\}_{n=1}^{\infty} \) does not have a hyperfinite subsequence.

Now, for each \( R \) and \( 1 \leq i \leq t_{d,R} \) we attach a copy of \( T_i \) to \( H_R \), by connecting a leaf of \( T_i \) to a midpoint of an edge of \( H_R \) (we need to add the midpoints to maintain the vertex degree bound). Let \( G_R \) be the resulting graph. Then,

- for any \( n \geq 1 \) and \( n \geq R \), all the rooted \( R \)-balls in \( G_n \) are trees,
- for any \( n \geq 1 \), \( n \geq R \) and for each rooted tree \( (S, p) \) of radius at most \( R \), there exists \( x \in V(G_n) \) such that the rooted ball \( B_R(x, G_n) \) is rooted-isomorphic to \( (S, p) \),
- there exists a \( \delta > 0 \) such that for any \( n \geq 1 \), \( e(G_n, \mathcal{P}) > \delta \).

Now, for \( R \geq 1 \), let \( L_R \) be a path of \( t_{d,R} \) vertices. For each \( 1 \leq i \leq t_{d,R} \), we attach a copy of \( T_i \) to \( L_R \), by connecting a leaf of \( T_i \) to a vertex of \( L_R \). The resulting graph \( S_R \) is a tree. Also, for any \( n \geq 1 \), \( n \geq R \), and for each rooted tree \( (S, p) \) of radius at most \( R \), there exists \( x \in V(S_n) \) such
that the rooted ball \( B_R(x, S_n) \) is rooted-isomorphic to \((S, p)\). Therefore, our proposition follows. \( \square \)

4. The Property A of Yu

In order to avoid confusion, in this section we use the phrase "graph class" instead of "graph property", since we will talk about the notion of Property A.

Let \( G \in Gr_d \) be a graph. Then, \( \text{Prob}(G) \) is the set of all probability measures on the vertices of \( G \). If \( f : V(G) \to \mathbb{R} \) and \( g : V(G) \to \mathbb{R} \), then their \( l_1 \)-distance is defined as \( \|f - g\|_1 := \sum_{x \in V(G)} |f(x) - g(x)| \). Now we are in the position to define Property A, an important notion introduced by Yu \cite{yu2004} in the context of group theory.

**Definition 4.1.** For \( \varepsilon > 0 \) and \( R \geq 1 \) a graph \( G \in Gr_d \) is called \((\varepsilon, R)\)-amenable if there exists a function \( f : V(G) \to \text{Prob}(G) \) such that

- for any adjacent pairs \( x, y \in V(G) \), \( \|f(x) - f(y)\|_1 < \varepsilon \),
- for any \( x \in V(G) \), we have that
  \[
  \text{Supp}(f(x)) \subset B_R(x, G)
  \]

We call a class of graphs \( \mathcal{P} \) \((\varepsilon, R)\)-amenable if all \( G \in \mathcal{P} \) are \((\varepsilon, R)\)-amenable and we denote the class of all \((\varepsilon, R)\)-amenable graphs by \( \mathcal{A}_{\varepsilon, R} \). We say that a graph class \( \mathcal{P} \) is of Property A if for every \( \varepsilon > 0 \), there exists some \( R \geq 1 \) such that \( \mathcal{P} \subset \mathcal{A}_{\varepsilon, R} \). We will see (Section 6) that Property A implies hyperfiniteness and the sequence \( \{K_n\}_{n=1}^{\infty} \) in Proposition 2.1 shows that hyperfiniteness does not imply Property A. However, we have the following proposition.

**Proposition 4.1.** For any \( \varepsilon' > 0 \) and \( R \geq 1 \), \( \mathcal{A}_{\varepsilon', R} \) is locally checkable in constant-time relative to \( \mathcal{A}_{\varepsilon, R} \).

**Proof.** We start with a technical lemma.

**Lemma 4.1.** Let \( G \in Gr_d \), \( R \geq 1 \), \( x \in V(G) \), \( f : B_R(x, G) \to \mathbb{R} \) be a nonnegative function such that \( \sum_{y \in B_R(x, G)} f(y) = 1 \). Let \( \alpha > \frac{3}{|\varepsilon|} \) be a positive integer. Then, there exists a function \( g : B_R(x, G) \to \mathbb{R} \) such that

- \( \sum_{y \in B_R(x, G)} g(y) = 1 \),
- \( \text{for any } y \in B_R(x, G), \ g(y) = \frac{i}{\alpha}, \text{ where } 0 \leq i \leq \alpha \text{ is an integer} \),
- \( \sum_{y \in B_R(x, G)} |f(y) - g(y)| < \frac{\varepsilon' - \varepsilon}{3} \).

**Proof.** Let \( g' \), \( g'' \) : \( B_R(x, G) \to \mathbb{R} \) be defined in the following way. \( g'(y) = \frac{1}{\alpha}, \ g''(y) = \frac{i+1}{\alpha}, \text{ where } \frac{1}{\alpha} \leq f(y) \leq \frac{i+1}{\alpha} \). Then, \( \sum_{y \in B_R(x, G)} (f(y) - g'(y)) < \frac{\varepsilon' - \varepsilon}{3} \) and \( \sum_{y \in B_R(x, G)} (g''(y) - f(y)) < \frac{\varepsilon' - \varepsilon}{3} \). So, by changing the value of \( g'(y) \) to
\[ g''(y) \text{ at some vertices } y, \text{ we can construct a function } g' \leq g \leq g'' \text{ satisfying the condition of our lemma.} \]

Now, we construct a set \( Q \) (which can be identified with the set \( |Q| - 1 \)) a verifier \( A \subset B_{R+2}^Q \) and for each \( G \in A_{\varepsilon,R} \) a proof \( P_G : V(G) \to Q \) such that

\begin{itemize}
  \item \( A \) accepts \( P_G \) for each \( G \in A_{\varepsilon,R} \),
  \item \( A \) rejects all \( P : V(H) \to Q \), if \( H \notin A_{\varepsilon,R} \).
\end{itemize}

So, let \( \beta \geq 1 \) be an integer, larger than the size of any ball of radius \( R + 2 \) with vertex degree bound \( d \). Let \( \Sigma \) be the set of all [\( \alpha \)]-valued functions on [\( \beta \)]-labelled balls of radius \( R \) with vertex degree bound \( d \). Finally, let \( Q = [\beta] \times \Sigma \).

Now let \( G \in A_{\varepsilon,R} \) and \( f : V(G) \to \text{Prob}(G) \) be a function witnessing the fact that \( G \in A_{\varepsilon,R} \). Let \( g_G : V(G) \to \text{Prob}(G) \) be a function witnessing the fact that \( G \in A_{\varepsilon,R} \) such that for each \( x, y \in V(G) \) the \( g_G(x)(y) = \frac{1}{\alpha} \), where \( 0 \leq i \leq \alpha \). Such function \( g_G \) exists by our previous lemma. For each \( x \in V(G) \) let \( h_G(x) : V(G) \to [\alpha] \) such that \( h_G(x)(y) = i \) if \( g_G(x)(y) = \frac{i}{\alpha} \).

We define \( P_G : V(G) \to Q \) in the following way. First, we consider a coloring \( c_G : V(G) \to [\beta] \) such that \( c_G(x) \neq c_G(y) \) if \( 0 < d_G(x,y) \leq R + 2 \). If \( x \in V(G) \), the first coordinate of \( P_G(x) \) equals to \( c_G(x) \). The second coordinate of \( P_G(x) \) equals to \( h_G(x) \in \Sigma \).

The verifier \( A \) accepts a proof \( P : V(G) \to Q \) if

\begin{itemize}
  \item the \( \beta \) coordinates defines a [\( \beta \)]-labeling \( l \) such that \( l(x) \neq l(y) \) if \( 0 < d_G(x,y) \leq R + 2 \),
  \item the \( \Sigma \) coordinates are compatible with a \( \alpha \)-labeling on balls of radius \( R \),
  \item the \( \Sigma \) coordinates are witnessing the fact that \( G \) is in \( A_{\varepsilon,R} \).
\end{itemize}

Then, \( A \) accepts all the proofs \( P_G \) above and rejects any proof \( P : V(H) \to Q \), if \( H \notin A_{\varepsilon,R} \).

\section{5. Strong hyperfiniteness}

As we have already seen, \( H_{\varepsilon,K} \) is not almost locally checkable in constant-time. The reason behind this fact was that some graph sequence \( \{G_n\} \subset H_{\varepsilon,K} \) contained a nonhyperfinite subgraph sequence. One can strengthen the definition of \( H_{\varepsilon,K} \)-hyperfiniteness to exclude the phenomenon above.

\textbf{Definition 5.1.} \( G \in Gr_d \) is uniformly \( (\varepsilon,K) \)-hyperfinit if for all induced subgraph \( F \subset G \), \( F \) is \( H_{\varepsilon,K} \)-hyperfinite as well.

We say that a graph property \( P \in Gr_d \) is uniformly \( H_{\varepsilon,K} \)-hyperfinit if all \( G \in P \) are uniformly \( H_{\varepsilon,K} \)-hyperfinit.

The set of all uniformly \( H_{\varepsilon,K} \)-hyperfinit graphs will be denoted by \( UH_{\varepsilon,K} \). We call a graph property \( P \) uniformly \textbf{hyperfinit} if for any \( \varepsilon > 0 \), there exists \( K \geq 1 \) such that \( P \subset UH_{\varepsilon,K} \).
By monotonicity, the class of planar graphs in uniformly hyperfinite. In our paper, we will use another strengthening of hyperfiniteness introduced by Romero, Wrochna and Živný [7] under the name of fractional-cc-fragility.

First, we need a definition. For a graph $G \in Gr_d$ we call $Y \subset V(G)$ a $K$-separator if by removing $Y$ (and all the adjacent edges) the components of the remaining graph are of size at most $K$.

**Definition 5.2.** $G \in Gr_d$ is strongly $(\varepsilon, K)$-hyperfinite if there exists a probability measure $\mu$ on the $K$-separators of $G$ such that for any $x \in Y$

$$\mu(Y \mid x \in Y) < \varepsilon.$$ 

We say that a graph class $\mathcal{P} \subset Gr_d$ is strongly $H_{\varepsilon,K}$-hyperfinite if all $G \in \mathcal{P}$ are strongly $H_{\varepsilon,K}$-hyperfinite. The set of all strongly $H_{\varepsilon,K}$-hyperfinite graphs will be denoted by $\mathcal{SH}_{\varepsilon,K}$. We call a graph class $\mathcal{P}$ **strongly hyperfinite** if for any $\varepsilon > 0$, there exists $K \geq 1$ such that $\mathcal{P} \subset \mathcal{SH}_{\varepsilon,K}$.

It is not hard to see that a strongly $H_{\varepsilon,K}$-hyperfinite graph is $H_{\varepsilon,K}$-hyperfinite as well. However, the main result of [2] is that the notions of Property A, uniform hyperfiniteness and strong hyperfiniteness are, in fact, coincide. However, it was first proved by Romero, Wrochna and Živný [7] that monotone hyperfinite properties are strongly hyperfinite. This result plays an important role in the proof of Theorem 1.

**6. Amenability implies hyperfiniteness**

First, let us fix some notation, which will be used in the section. Let $G \in Gr_d$ and $A \subset V(G)$. Then, $\partial G(A)$ is the set of vertices $x \in A$ such that there exists $y \notin A$, $x \sim y$. Also, $\partial e G(A)$ is the set of edges $e = (x, y)$, where $x \in \partial G(A)$ and $y \notin A$.

**Proposition 6.1.** For any $\varepsilon > 0$ and $R \geq 1$,

$$\mathcal{A}_{\varepsilon,R} \subset H_{\varepsilon, N^d_{2R}},$$

where $N^d_{2R}$ is the maximum number of vertices in a rooted ball of radius $R$ and vertex degree bound $d$.

**Proof.** First we need a technical lemma, which is very similar to Proposition 4.2 in [8]. Let $G \in Gr_d$ and $F \subset G$ be an induced subgraph. We say that $F$ is $(\varepsilon, R)$-amenable relative to $G$ if there exists a function $f : V(F) \rightarrow \text{Prob}(F)$ such that for any pair of adjacent vertices $x, y \in V(F)$,

$$\|f(x) - f(y)\|_1 \leq \varepsilon$$

and for any $x \in V(F)$,

$$\text{Supp}(f(x)) \subset B_R(x, G)$$

**Lemma 6.1.** If $G \in \mathcal{A}_{\varepsilon,R}$ and $F \subset G$ is an induced subgraph, then $F$ is $(\varepsilon, 2R)$-amenable relative to $G$. 


Proof. For \( x \in V(G) \), let \( \tau(x) \in V(F) \) such that \( d_G(x, \tau(x)) = d_G(x, F) \). Let \( g : V(G) \rightarrow \text{Prob}(G) \) be a function witnessing the fact that \( G \in \mathcal{A}_{\varepsilon, R} \). For \( x \in V(F) \), let \( f(x) = \sum_{t \in \tau^{-1}(x)} g(x)(t) \). Clearly, \( f : V(F) \rightarrow \text{Prob}(F) \) and if \( x, y \) are adjacent vertices, then

\[
\|f(x) - f(y)\|_1 \leq \|g(x) - g(y)\|_1 \leq \varepsilon.
\]

Also, \( \text{Supp}(f(x)) \subseteq B_{2R}(x, G) \). Indeed, if \( f(x)(z) \neq 0 \), then there exists \( t \in \tau^{-1}(z) \) such that \( g(x)(t) \neq 0 \). Hence, \( d_G(x, t) \leq R \), so \( d_G(t, z) \leq R \), that is, \( d_G(x, z) \leq 2R \). Therefore, our lemma follows.

\[\Box\]

Lemma 6.2. Let \( \varepsilon \in \mathcal{A}_{\varepsilon, R}, F \subseteq G \) be an induced subgraph. Then, there exists a subset \( L \subseteq V(F) \) such that \( |\partial_F(L)| \leq \frac{\varepsilon}{2}|L| \) and \( |L| \leq N^d_{2R} \).

Proof. Let \( f : V(F) \rightarrow \text{Prob}(F) \) satisfying (1) and (2). That is,

\[
\sum_{x \in V(F)} \sum_{y \sim x} \|f(x) - f(y)\|_1 \leq \varepsilon \sum_{x \in V(F)} \|f(x)\|_1.
\]

Hence,

\[
\sum_{z \in V(F)} \sum_{x \in V(F)} \sum_{y \sim x} |f(x)(z) - f(y)(z)| \leq \varepsilon \sum_{z \in V(F)} \sum_{x \in V(F)} f(x)(z).
\]

Hence, there exists \( z \in V(F) \) such that

\[
\sum_{x \in V(F)} \sum_{y \sim x} |f(x)(z) - f(y)(z)| \leq \varepsilon \sum_{x \in V(F)} f(x)(z).
\]

That is, if we define the function \( \zeta : V(F) \rightarrow \mathbb{R} \) by \( \zeta(x) = f(x)(z) \),

(3)

\[
\sum_{x \in V(F)} \sum_{y \sim x} |\zeta(x) - \zeta(y)| \leq \varepsilon \sum_{x \in V(F)} \zeta(x).
\]

So far, we followed the proof of Proposition 3.2 in [1], however, in order to avoid some heavy machinery now we choose a different path. Let us recall the area and coarea formulas (Lemma 3.6 and Lemma 3.7) from [5]. If \( G \in Gr_d \) and \( f : V(G) \rightarrow [0, 1] \),

(4)

\[
\frac{1}{2} \sum_{x, y, x \sim y} |f(x) - f(y)| = \int_0^1 |\partial_F^x(\Omega_t(f))| \, dt.
\]

(5)

\[
\sum_x f(x) = \int_0^1 |\Omega_t(f)| \, dt,
\]

where

\[
\Omega_t(f) = \{x \in V(G) \mid f(x) > t\}.
\]

So, by 3

\[
2 \int_0^1 |\partial_F^x(\Omega_t(\zeta))| \, dt \leq \varepsilon \int_0^1 |\Omega_t(\zeta)| \, dt.
\]

That is, for some \( t \geq 0 \) we have

(6)

\[
\partial_F(\Omega_t(\zeta)) \leq |\partial_F^x(\Omega_t(\zeta))|.
\]
Hence, taking $\Omega_1(\zeta) = L$, our lemma follows.

Now we finish the proof of our proposition. Let $F_1 = G$ and $L_1 \subset V(F_1)$ be a set of size at most $N_{2R}^d$ such that $|\partial F_1(L_1)| \leq \varepsilon |L_1|$ and remove $L_1$ together with all the vertices adjacent to $L_1$ from $V(F_1)$. Let $F_2$ be the subgraph of $G$ induced on the remaining vertices. Let $L_2 \subset V(F_2)$ be a set of size at most $N_{2R}^d$ such that $|\partial F_2(L_2)| \leq \varepsilon |L_2|$ and remove $L_2$ together with all the vertices adjacent to $L_2$ from $V(F_2)$. Inductively, we construct disjoint sets $L_1, L_2, \ldots, L_n$ of size at most $N_{2R}^d$ such that

$$|V(G) \setminus \bigcup_{i=1}^n L_i| \leq \varepsilon d |\bigcup_{i=1}^n L_i|.$$ 

Hence, our proposition follows. □

7. STRONG HYPERFINITENESS IMPLIES AMENABILITY

**Proposition 7.1.** For any $\varepsilon > 0$ and $K \geq 1$,

$$\mathcal{SH}_{\varepsilon, K} \subset \mathcal{A}_{2\varepsilon, K}.$$ 

**Proof.** Let $G \in \mathcal{SH}_{\varepsilon, K}$ and $\mu$ be a probability measure on the set of $K$-separators of $G$ such that for any $x \in V(G)$ we have

$$\mu(Y \mid x \in Y) \leq \varepsilon.$$ 

For a $K$-separator $Y$ and $x \in V(G)$, let $f_{Y,x} : V(G) \to \mathbb{R}$ be defined in the following way.

- If $x \in Y$, let $f_{Y,x}(x) = \mu(Y)$ and if $x \neq y$ let $f_{Y,x}(x) = 0$.
- If $x \notin Y$, let $f_{Y,x}(y) = \frac{\mu(Y)}{|C_{Y,x}|}$ provided that $y \in C_{Y,x}$, where $C_{Y,x}$ is the component of $G \setminus Y$ containing the vertex $x$.

Now, let $f(x) \in \text{Prob}(G)$ be defined as

$$f(x) := \sum_Y f_{Y,x}.$$ 

Then,

- for any $x \in V(G)$, $\text{Supp}(f(x)) \subset B_G(x, K)$,
- for any pair of adjacent vertices $x \sim y$, we have

$$\|f(x) - f(y)\|_1 = \mu(Y \mid x \in Y) + \mu(Y \mid y \in Y) \leq 2\varepsilon.$$ 

Hence, $G \in \mathcal{A}_{2\varepsilon, K}$. □

**Lemma 7.1.** For any $\varepsilon > 0$, there exists $R > 0$ such that $\mathcal{P} \subset \mathcal{A}_{\varepsilon, R}$, where $\mathcal{P} \subset \text{Gr}_d$ is the class of planar graphs.

**Proof.** By Theorem 1.6 of [7], for any $\varepsilon > 0$, there exists $K > 0$ such that $\mathcal{P} \subset \mathcal{SH}_{\varepsilon, K}$. Hence our lemma follows, from Proposition 6.1. □
8. The Proof of the Main Theorem

In this section, we prove Theorem 1. Let $\delta > 0$, $\mathcal{P} \subset Gr_d$ be the set of planar graphs and $\mathcal{P}_\delta \subset Gr_d$ be the set of graphs that are not further from $\mathcal{P}$ in edit distance than $\delta$. We need to prove that there exists a verifier $\mathcal{A}$ such that (using the notation of Section 2)

$$\mathcal{P} \subset L_{\mathcal{A}} \subset \mathcal{P}_\delta.$$  

(7)

For $K > 0$, let $L_{\mathcal{P}_K} \subset \mathcal{P}_K$ denote the class of graphs such that if $G \in L_{\mathcal{P}_K}$ all of the induced subgraph of $G$ which has diameter at most $K$ is planar. Clearly, there exists a verifier $B_K$ such that $L_{\mathcal{P}_K} = L_{B_K}$.

**Lemma 8.1.** For any $\varepsilon > 0$, there exists $M_\varepsilon > 0$ such that if $K \geq M_\varepsilon$, then

$$\mathcal{P} \subset H_{\varepsilon,K} \subset L_{\mathcal{P}_K} \subset \mathcal{P}_{\delta}.$$  

Proof. Pick $M_\varepsilon > 0$ such that $\mathcal{P} \subset H_{\varepsilon,M_\varepsilon}$, let $K \geq M_\varepsilon$ and let $G \in H_{\varepsilon,K} \cap L_{\mathcal{P}_K}$. Then, we can remove at most $\varepsilon d |V(G)|$ edges from $G$ to obtain a graph having components of size at most $M_\varepsilon$ and all these components are planar. \qed

**Lemma 8.2.** For any $\varepsilon > 0$, there exists $N_\varepsilon > 0$ and a verifier $C_\varepsilon$ such that if $K \geq N_\varepsilon$, then

$$\mathcal{P} \subset L_{C_\varepsilon} \subset H_{\varepsilon,K}.$$  

Proof. Let $R \geq 1$ be an integer such that $\mathcal{P} \subset A_{\frac{4}{2R},R}$. Such $R$ exists by Lemma 7.1. Let $N_\varepsilon = N_{\frac{d}{2R}}$, then our lemma follows directly from Proposition 4.1 and Proposition 6.1. \qed

Now we can prove Theorem 1. Let $K_\varepsilon = \max N_\varepsilon, M_\varepsilon$ and $D_\varepsilon = B_{K_\varepsilon}$, then by our previous lemmas,

$$\mathcal{P} \subset L_{C_\varepsilon} \cap L_{D_\varepsilon} \subset \mathcal{P}_{\delta}.$$  

Hence (see Section 2), we have a verifier $\mathcal{A}$ satisfying (7). \qed

**References**

[1] J. Brodzki, G. Niblo, J. Špakula, R. Willett and N. Wright, Uniform local amenability. J. Noncommut. Geom. 7 (2013), no. 2, 583-603.

[2] G. Elek, Uniform local amenability implies Property A.
preprint https://arxiv.org/pdf/1912.00806.pdf

[3] O. Goldreich and D. Ron, Property testing in bounded degree graphs. Algorithmica 32 (2002), no. 2, 302-343.

[4] M.Göös and J. Suomela, Locally checkable proofs in distributed computing. Theory Comput. 12 Paper No. 19 (2016)

[5] M. Keller and D. Mugnolo, General Cheeger inequalities for $p$-Laplacians on graphs. Nonlinear Analysis 147 (2016), 80-95.

[6] I. Newman and C. Sohler, Every property of hyperfinite graphs is testable. SIAM J. Comput. 42 (2013), no. 3, 1095-1112.

[7] M. Romero, M. Wrochna and S. Živný, Treewidth-Pliability and PTAS for Max-CSP’s. preprint https://arxiv.org/pdf/1911.05204.pdf

[8] J-L. Tu, Remarks on Yu’s"property A" for discrete metric spaces and groups. Bull. Soc. Math. France 129 (2001), no. 1, 115–139.
[9] G. Yu, The coarse Baum-Connes conjecture for spaces which admit a uniform embedding into Hilbert space. *Invent. Math.* **139** (2000), no. 1, 201-240.

Department of Mathematics and Statistics, Fylde College, Lancaster University, Lancaster, LA1 4YF, United Kingdom

E-mail address: g.elek@lancaster.ac.uk