Abstract

We find logarithmic asymptotics of $L_2$-small deviation probabilities for weighted stationary Gaussian processes (both for real and complex-valued) having power-type discrete or continuous spectrum. Our results are based on the spectral theory of pseudo-differential operators developed by Birman and Solomyak.

Keywords: small deviations; spectral asymptotics; stationary processes.

AMS Classification: 60G15, 60G10, 60G22, 47G30.

1 Introduction

Let $(Y(t))_{t \in T}$ be a random process defined on some parametric measure space $(T, \mu)$. Many studies have been devoted to the asymptotic behavior of its $L_2$-small deviation probabilities

$$P\left(\|Y\|_2^2 = \int_T |Y(t)|^2 \mu(dt) \leq \varepsilon^2\right), \quad \text{as } \varepsilon \to 0,$$

see e.g. [7, 8, 9, 18, 19, 20, 21, 24], to mention just a small sample. The importance of small deviation probabilities in a broader context and a large number of their applications are described in the surveys [13, 15]; for an extensive up-to-date bibliography see [16].

In this work, we explore $L_2$-small deviation probabilities for weighted stationary Gaussian processes having power-type spectrum. Our goal is to relate the asymptotics of small deviation probabilities with that of the spectrum. From the historical point of view our results are closely related to those on fractional Brownian motion and its relatives, see e.g. [6, 10, 11, 17]. In terms of such processes with stationary increments our message is that the spectral asymptotics is relevant to the small deviation behavior but the self-similarity is not.

In Section 2 we consider periodic processes that correspond to discrete spectrum, while Section 3 handles continuous time processes with spectral density. The final results of two sections are quite similar, although intermediate technical details differ.

Our results are based on the spectral theory of pseudo-differential operators developed by Birman and Solomyak [3, 4]. This approach was initiated in [12], where a similar problem was considered in the discrete time setting. In passing, in Section 3 we prove a slightly stronger version of one result from [12].

\*St.Petersburg State University, Russia, St. Petersburg, Universitetskii pr. 28, email mikhail@lifshits.org.
\†MAI, Linköping University.
\‡St.Petersburg Department of Steklov Institute of Mathematics, email al.il.nazarov@gmail.com.
\§St.Petersburg State University, Russia, St. Petersburg, Universitetskii pr. 28.
The spectral results that we use are not sensible to the symmetry of the spectral measure. Therefore, it is very natural to apply them to the complex-valued processes. In this context proper Gaussian processes are particularly convenient because their distributions are determined by the spectra of the corresponding covariance operators. Our main results are logarithmic asymptotics of $L_2$-small deviation probabilities for weighted stationary Gaussian processes having power-type spectrum in Theorem 2.2 (real-valued periodic process), in Theorem 2.4 (complex-valued proper periodic process), in Theorem 4.2 (real-valued process with continuous spectrum), and in Theorem 4.3 (complex-valued proper process with continuous spectrum).

For the reader’s convenience, in Appendix we formulate some particular cases of deep results of [2]–[4] used in our proofs.

We denote by $F$ the Fourier transform

$$F u(\xi) = \int_{\mathbb{R}} \exp(-i\xi x) u(x) \, dx.$$ 

For any two sequences $a_k, b_k$ the standard notation $a_k \sim b_k$ means that $\lim_{k \to \infty} a_k b_k = 1$.

## 2 Periodic stationary processes

### 2.1 Spectral representations

We first recall the necessary information on the spectral representations of stationary periodic processes.

Let $X = \{X(t), t \in \mathbb{R}\}$ be a complex-valued $2\pi$-periodic centered second order mean-square continuous stationary process. Then its covariance function admits a spectral representation

$$K_X(s-t) := \text{cov}(X(s), X(t)) = \sum_{k \in \mathbb{Z}} \mu_k e^{ik(s-t)}, \quad s, t \in \mathbb{R},$$

where $\mu := (\mu_k)_{k \in \mathbb{Z}}$ is a finite non-negative measure on $\mathbb{Z}$ called the spectral measure of $X$.

The spectral representation of $X$ itself writes as

$$X(t) = \sum_{k \in \mathbb{Z}} \sqrt{\mu_k} \xi_k e^{ikt}, \quad (2.1)$$

where $\xi_k$ are centered uncorrelated complex random variables with $\mathbb{E}|\xi_k|^2 = 1$.

Just for completeness, recall a straightforward reformulation for real-valued processes. Let denote $\xi_k := \xi_k^{(re)} + i\xi_k^{(im)}$. The process $X$ is real-valued iff

- $\xi_0$ is real;
- $\mu_{-k} = \mu_k$ for all $k > 0$;
- $\xi_{-k} = \overline{\xi_k}$ for all $k > 0$;
- $\mathbb{E}|\xi_k^{(re)}|^2 = \mathbb{E}|\xi_k^{(im)}|^2 = 1/2$ for all $k \in \mathbb{Z}$;
- the real random variables $(\xi_0, (\xi_k^{(re)}, \xi_k^{(im)})_{k>0})$ are uncorrelated.
In this case (2.1) writes as

\[
X(t) = \sqrt{\mu_0} \xi_0 + \sum_{k=1}^\infty \sqrt{\mu_k} \left[ \sqrt{2} \xi_k^{(re)} [\sqrt{2} \cos(kt)] - \sqrt{2} \xi_k^{(im)} [\sqrt{2} \sin(kt)] \right],
\]

(2.2)

where the random variables $$\sqrt{2} \xi_k^{(re)}, \sqrt{2} \xi_k^{(im)}$$ have unit variance.

### 2.2 Covariance operators and their factorization

Let $$\nu(du) := \frac{du}{2\pi}$$ be the normalized Lebesgue measure on $$[0,2\pi]$$. In the following, we will consider $$X$$ as a random element of $$L_2([0,2\pi],\nu)$$. From this point of view, equations (2.1) and (2.2) represent the orthogonal expansions of $$X$$ with respect to the orthonormal bases $$(e^{ikt})_{k \in \mathbb{Z}}$$ and $$\{1, (\sqrt{2} \cos(kt), \sqrt{2} \sin(kt))_{k \geq 1}\}$$, respectively. The elements of these bases are eigenvectors of the corresponding covariance operator $$K_X$$ in $$L_2([0,2\pi],\nu)$$ and the corresponding eigenvalues are $$\mu_k$$.

The orthogonal expansions generate natural decompositions of $$K_X$$. Let $$e_k := \exp(ikt), k \in \mathbb{Z}$$. Then the operator square root of $$K_X$$ is defined by the formula $$D e_k := \sqrt{\mu_k} e_k, k \in \mathbb{Z}$$. Operator $$D$$ is bounded, self-adjoint, satisfies $$D D = D D^* = K_X$$, and can be interpreted as a convolution operator with the kernel

\[
D(s) := \sum_{k \in \mathbb{Z}} \sqrt{\mu_k} e_k(s).
\]

Indeed, for every $$k \in \mathbb{Z}$$ and $$s \in [0,2\pi]$$ we have

\[
\int_0^{2\pi} D(s-t) e_k(t) \nu(dt) = \sum_{\ell \in \mathbb{Z}} \sqrt{\mu_\ell} \int_0^{2\pi} e_\ell(s-t) e_k(t) \nu(dt) = \sum_{\ell \in \mathbb{Z}} \sqrt{\mu_\ell} e_\ell(s) \int_0^{2\pi} e_{k-\ell}(t) \nu(dt) = \sqrt{\mu_k} e_k(s).
\]

In the following we are interested in the small ball behavior of the weighted $$L_2$$-norm

\[
\int_0^{2\pi} q(t)|X(t)|^2 dt = 2\pi \|q X\|_{L_2,\nu}^2
\]

with some weight $$q \in L_1[0,2\pi]$$.

We have a decomposition for covariance operator

\[
K_{\sqrt{\pi}X} = Q K_X Q = Q D D Q =: T^* T, \quad T = D Q,
\]

(2.4)

where $$Q$$ stands for the self-adjoint multiplication operator related to the function $$\sqrt{q} \in L_2[0,2\pi]$$. We claim that $$T$$ is the Hilbert–Schmidt operator although $$Q$$ need not be even bounded. Indeed, since $$\sqrt{q} \in L_2[0,2\pi]$$, it admits a Fourier series expansion

\[
\sqrt{q}(t) := \sum_{m \in \mathbb{Z}} q_m e^{int},
\]

with $$(q_m) \in \ell_2(\mathbb{Z})$$. Then we have

\[
T e_k = D \left[ \sum_{k \in \mathbb{Z}} q_{k-\ell} e_k \right] = \sum_{k \in \mathbb{Z}} \sqrt{\mu_k q_{k-\ell}} g(t) e_k,
\]
\[ \sum_{\ell \in \mathbb{Z}} ||Te_\ell||^2_{2,\nu} = \sum_{\ell \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \mu_k |q_{k-\ell}|^2 = \mu(\mathbb{Z}) ||q||_{1,\nu}. \]

For the study of logarithmic asymptotics of small deviation probabilities, we need to know the main term of the eigenvalue asymptotics for \( K_{\sqrt{q}X} \), see [18]. Since \( K_{\sqrt{q}X} \) is non-negative, its eigenvalues \( \lambda_n(K_{\sqrt{q}X}) \) coincide with its singular values \( s_n(K_{\sqrt{q}X}) \). We always label the eigenvalues and singular values in non-increasing order counting multiplicity.

We study the distribution function of singular values
\[ N(\lambda; K_{\sqrt{q}X}) := \# \{ n : s_n(K_{\sqrt{q}X}) \geq \lambda \}, \lambda > 0, \]
and its asymptotics as \( \lambda \to 0 \). For a compact operator \( T \) we introduce the notation
\[ \Delta_\theta(T) = \lim_{\lambda \to 0^+} \lambda^\theta N(\lambda; T). \]

The following relation is important in what follows:
\[ \Delta_\theta(T) = \Delta \iff \lim_{n \to \infty} n^{\frac{1}{\theta}} s_n(T) = \Delta^{\frac{1}{\theta}}. \tag{2.5} \]

### 2.3 Spectral asymptotics

From now on we assume that the spectral measure has a power-like decay
\[ \lim_{k \to \pm \infty} |k|^r \mu_k = M_{\pm}, \tag{2.6} \]
with some \( r > 1 \) and \( M_+ \geq 0, M_+ + M_- > 0 \). Assumption (2.6) is typical of the literature on small deviations of Gaussian processes; see for example [7].

**Lemma 2.1** Let the spectral measure of \( X \) satisfy (2.6), and let \( q \in L^1[0,2\pi] \). Then
\[ \lambda_n(K_{\sqrt{q}X}) = s_n(K_{\sqrt{q}X}) \sim \left( \frac{M_+^\frac{1}{r} + M_-^\frac{1}{r}}{2\pi} \right)^{\frac{2\pi}{r}} \int_0^{2\pi} q(t)^{\frac{1}{r}} dt \left( \frac{n}{2\pi} \right)^{-r}, \quad \text{as } n \to \infty. \tag{2.7} \]

**Proof:** We proceed similar to [12] where we used quite general results of Birman and Solomyak [3, 4].

We can consider \( K_{\sqrt{q}X} \) as an operator in \( L_2(\mathbb{R}) \)
\[ (K_{\sqrt{q}X}u)(s) = b(s) \int_{\mathbb{R}} \frac{1}{2\pi} K_X(s-t)b(t)u(t) dt, \]
where \( b = \sqrt{q} \cdot 1_{[0,2\pi]} \). Notice that since we are working on the interval of length \( 2\pi \), it is sufficient to consider only the restriction of our periodic function \( K_X \) to \([-2\pi,2\pi] \).

Let \( h \) be a smooth cut-off function such that \( h(t) = 1 \) if \( t \in \left[\frac{3\pi}{2},2\pi\right] \) and \( h(t) = 0 \) if \( t \in [-2\pi,\pi] \). Then it follows that the function \( h_0(s) := 1 - h(s) - h(-s) \) equals one on \([-\pi,\pi]\) and vanishes outside of the interval \([-\frac{3\pi}{2},\frac{3\pi}{2}]\). We decompose the kernel \( K_X \) as follows:
\[ \frac{1}{2\pi} K_X(s) = \frac{1}{2\pi} K_X(s)[h(s) + h(-s) + h_0(s)] =: K_+(s) + K_-(s) + K^{(0)}(s), \tag{2.8} \]
and claim that the function \( K^{(0)} \) satisfies
\[
\lim_{\xi \to \pm \infty} |\xi|^r \cdot F K^{(0)}(\xi) = M(\text{sgn}(\xi)),
\]
where \( M(\pm 1) = M_{\pm} \). Indeed, we have
\[
\frac{1}{2\pi} F(K_X \cdot h_0)(\xi) = \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} \mu_k F h_0(\xi - k),
\]
and then by splitting the series into two sums, we obtain
\[
F K^{(0)}(\xi) = \Sigma_1 + \Sigma_2 := \left( \sum_{|k-\xi| \leq \sqrt{\xi}} + \sum_{|k-\xi| > \sqrt{\xi}} \right) \frac{\mu_k}{2\pi} F h_0(\xi - k).
\]
Since \( F h_0 \) rapidly decays at infinity, we have \( \Sigma_2 = o(|\xi|^{-r}) \) as \( |\xi| \to \infty \). Furthermore, \ref{A.4} implies that
\[
\Sigma_1 = \frac{M(\text{sgn}(\xi))}{2\pi} |\xi|^{-r} \sum_{|k-\xi| \leq \sqrt{\xi}} F h_0(\xi - k) + o(|\xi|^{-r}) = \frac{M(\text{sgn}(\xi))}{2\pi} |\xi|^{-r} = M(\text{sgn}(\xi))|\xi|^{-r} + o(|\xi|^{-r})
\]
by the Poisson summation formula (see, e.g., \cite{27} Ch. II, Sect. 13), so that \ref{2.9} follows.

Now we introduce a model operator
\[
(A u)(s) = b(s) F^{-1}(a(\xi) F(b u)(\xi)),
\]
with
\[
a(\xi) = \zeta(\xi) M(\text{sgn}(\xi))|\xi|^{-r},
\]
where \( \zeta \) is a smooth cut-off function vanishing in a neighborhood of the origin. Since \( b \in L_2 \), Proposition \ref{A.1} can be applied to the operator \( A \). This gives
\[
\Delta_{(\mathcal{A})} = \frac{1}{2\pi} \int_{0}^{2\pi} \int_{\mathbb{R} \setminus \{0\}} 1_{\{|b(t)|^2 \cdot M(\text{sgn}(\xi))|\xi|^{-r} \geq 1\}} d\xi dt = \frac{1}{2\pi} \int_{0}^{2\pi} \left( M(-1)^{\frac{\tau}{2}} + M(1)^{\frac{\tau}{2}} \right) dt.
\]
Furthermore, the decomposition \ref{2.2} generates the corresponding operator decomposition
\[
\mathcal{K}_{\sqrt{\pi}X} = \mathcal{K}_+ + \mathcal{K}_- + \mathcal{K}^{(0)}.
\]
Since the relation \ref{2.9} implies \( F K^{(0)}(\xi) - a(\xi) = o(|\xi|^{-r}) \) as \( |\xi| \to \infty \), part 1 of Proposition \ref{A.2} gives \( \Delta_{(\mathcal{K}^{(0)} - \mathcal{A})} = 0 \). Moreover, since \( K_X \) is \( 2\pi \)-periodic, the singular values of \( \mathcal{K}_+ \) coincide with the singular values of the operator
\[
b(s + \pi) 1_{[0,\pi]}(s) \int_{\mathbb{R}} \frac{1}{2\pi} K_X(s-t) h(s+2\pi-t) b(t) 1_{[\pi,2\pi]}(t) u(t) dt.
\]
For this operator, the support of the “left” weight is \([0,\pi]\), and the support of the “right” weight is \([\pi,2\pi]\). Part 2 in Proposition \ref{A.2} gives \( \Delta_{(\mathcal{K}_+)} = 0 \). Similarly, \( \Delta_{(\mathcal{K}_-)} = 0 \). By Proposition \ref{A.4} we obtain
\[
\Delta_{(\mathcal{K}_{\sqrt{\pi}X})} = \Delta_{(\mathcal{K}^{(0)})} = \Delta_{(\mathcal{A})},
\]
and the equivalence in \ref{A.5} gives \ref{2.7}. □
2.4 Gaussian small deviations

Now we transform the information about the eigenvalues into that on small deviation asymptotic behavior. This can be done for real processes and also for an important class of complex processes. We handle two cases separately because the constants appearing in the results are slightly different.

2.4.1 Real processes

Recall that if we have a centered Gaussian random vector $Z$, in a real Hilbert space, and $K_Z$ stands for its covariance operator, then, by the Karhunen–Loève expansion (see [1, Section 1.4]),

$$||Z||^2 = \sum_{n=1}^{\infty} \lambda_n \xi_n^2,$$

where $(\xi_n)_{n \in \mathbb{N}}$ is a sequence of independent standard normal random variables and $(\lambda_n)_{n \in \mathbb{N}}$ are the eigenvalues of $K_Z$. Therefore, the sequence $(\lambda_n)_{n \in \mathbb{N}}$ determines the distribution of $||Z||$. In particular, if

$$\lambda_n \sim C n^{-r}, \quad \text{as } n \to \infty,$$

then it is well known from [7, p.67] or [26], that

$$\ln P(||Z|| \leq \varepsilon) \sim -B_r \left( C/\varepsilon^2 \right)^{r-1}$$

with $C$ from (2.10) and $B_r := \frac{1}{2} \left( \frac{\pi}{r \sin(\pi/r)} \right)^{\frac{1}{r-1}}$. If our process $X$ is real, we can apply the formula (2.11) to $\sqrt{q}X$ considered as an element of $L^2([0, 2\pi], \nu)$ and using eigenvalue asymptotics (2.7) as (2.10). Notice that for real processes the spectral measure is symmetric, i.e. we have $M_+ = M_- := M$. Taking into account (2.3) we immediately obtain the following result.

**Theorem 2.2** Let $\{X(t), t \in \mathbb{R}\}$ be a $2\pi$-periodic real centered mean-square continuous stationary Gaussian process. Assume that its spectral measure satisfies the asymptotic condition

$$\mu_k \sim M |k|^{-r}, \quad \text{as } |k| \to \infty,$$

with some $r > 1, M > 0$. Let $q$ be a summable weight.

Then we have, as $\varepsilon \to 0$,

$$\ln P \left( \int_0^{2\pi} q(t) |X(t)|^2 dt \leq \varepsilon^2 \right) \sim - \left( \frac{M^{\frac{1}{r}}}{r \sin(\pi/r)} \int_0^{2\pi} q(t)^{\frac{1}{r}} dt \right)^{\frac{1}{r-1}} \frac{(r-1)(2\pi)^{\frac{1}{r-1}}}{2 \varepsilon^{\frac{2}{r-1}}}.$$

2.4.2 Examples

Consider the Bogoliubov process [24, 25] – a 1-periodic centered stationary Gaussian process (with parameter $\omega > 0$) defined by

$$\beta(\omega)(s) := \sqrt{\mu_0} \xi_0 + \sum_{k=1}^{\infty} \sqrt{\mu_k} \left( \xi_k [\sqrt{2} \cos(2\pi ks)] + \zeta_k [\sqrt{2} \sin(2\pi ks)] \right), \quad s \in \mathbb{R},$$
with independent standard normal random variables \((\xi_k)_{k \geq 0}, (\zeta_k)_{k > 0}\) and \(\mu_k = \frac{1}{\omega^2 + (2\pi k)^2}\). Define a \(2\pi\)-periodic process \(X(t) := \beta(\omega)(t/2\pi)\), \(t \in \mathbb{R}\). In our notation, for the spectrum of \(X\) we have \(r = 2, M = \frac{1}{(2\pi)^2}\). By applying Theorem 2.2 we obtain for \(q \in L_1[0, 1]\)

\[
\ln \mathbb{P} \left( \int_0^1 q(s) |\beta(\omega)(s)|^2 ds \leq \varepsilon^2 \right) = \ln \mathbb{P} \left( \int_0^{2\pi} q(t/2\pi) |X(t)|^2 dt \leq (\varepsilon \sqrt{2\pi})^2 \right) \\
\sim - \frac{1}{8} \left( \int_0^1 \sqrt{q(s)} ds \right)^2 \varepsilon^{-2} .
\]

In the simplest case \(q(s) \equiv 1\) we have

\[
\ln \mathbb{P} \left( \int_0^1 |\beta(\omega)(s)|^2 ds \leq \varepsilon^2 \right) \sim - \frac{1}{8} \varepsilon^{-2} ,
\]

cf. [24] Theorem 1].

For \(q(s) = e^{2as}, a \neq 0\), our result gives

\[
\ln \mathbb{P} \left( \int_0^1 e^{2as} |\beta(\omega)(s)|^2 ds \leq \varepsilon^2 \right) \sim - \frac{1}{8} \left[ \frac{e^a - 1}{a} \right]^2 \varepsilon^{-2} ,
\]

as proved in [24] Theorem 2].

Our next example is the so-called \(m\)-times integrated-centered Brownian bridge. Let \(B_0(\tau)\) be standard Brownian bridge on \([0, 1]\). We define the sequence of Gaussian processes

\[
B_{\{m\}}(s) = B_{m-1}(s) - \int_0^1 B_{m-1}(\tau) d\tau; \quad B_m(\tau) = \int_0^\tau B_{\{m\}}(s) ds, \quad m \in \mathbb{N} .
\]

It was shown in [19] Sec. 3 that

\[
B_{\{m\}}(s) = \sum_{k=1}^{\infty} (2\pi k)^{-m} \left( \xi_k [\sqrt{2} \cos(2\pi ks)] + \zeta_k [\sqrt{2} \sin(2\pi ks)] \right), \quad s \in [0, 1],
\]

with independent standard normal random variables \((\xi_k)_{k \geq 0}, (\zeta_k)_{k > 0}\). This formula obviously defines a \(1\)-periodic centered stationary Gaussian process on \(\mathbb{R}\). Define a \(2\pi\)-periodic process \(X_m(t) = B_{\{m\}}(t/2\pi)\). Then for the spectrum of \(X_m\) we have \(r = 2m, M = (2\pi k)^{-2m}\). By applying Theorem 2.2 we obtain for \(q \in L_1[0, 1]\)

\[
\ln \mathbb{P} \left( \int_0^1 q(s) |B_{\{m\}}(s)|^2 ds \leq \varepsilon^2 \right) = \ln \mathbb{P} \left( \int_0^{2\pi} q(t/2\pi) |X_m(t)|^2 dt \leq (\varepsilon \sqrt{2\pi})^2 \right) \\
\sim - \frac{2m - 1}{2} \left( \frac{1}{2m \sin(\pi/2m)} \int_0^1 q(s) \frac{1}{2m} ds \right) \left( \int_0^{2\pi} q(t/2\pi) |X_m(t)|^2 dt \right) \varepsilon^{-\frac{2m}{2m-1}} .
\]

For \(q(s) \equiv 1\) this result agrees with [19] Theorem 3.2].

7
Remark 2.3 In fact, the sharp small ball asymptotics for these processes were obtained in [19] and [24], see also [21] for more general weights. However, this is strongly connected with the fact that $\beta(\omega)$ and $B_{(m)}$ are the Green Gaussian processes i.e. their covariances are the Green functions for ordinary differential operators. In general case this seems to be a much harder problem.

2.4.3 Proper complex processes

If we have a centered Gaussian random vector $Z$ in a complex Hilbert space, and $K_Z$ stands for its covariance operator, then Karhunen–Loève expansion yields

$$Z = \sum_{n=1}^{\infty} \lambda_n \xi_n e_n,$$

where $(\xi_n)_{n \in \mathbb{N}}$ are uncorrelated complex jointly Gaussian random variables satisfying $E|\xi_n|^2 = 1$ and $(\lambda_n, e_n)_{n \in \mathbb{N}}$ are the eigenpairs of $K_Z$. We still have

$$||Z||^2 = \sum_{n=1}^{\infty} \lambda_n |\xi_n|^2,$$

but, unfortunately, unlike the real case, the variables $\xi_n$ need not be independent, although they are uncorrelated. Indeed, the independence of two centered complex Gaussian random variables $\eta_1$ and $\eta_2$ is equivalent to the pair of relations

$$\begin{cases} 
\text{cov}(\eta_1, \eta_2) = E\eta_1 \overline{\eta_2} = 0; \\
E\eta_1 \eta_2 = 0.
\end{cases}$$

Therefore, the sequence $(\lambda_n)_{n \in \mathbb{N}}$ does not determine the distribution of $||Z||$ in general case. For this reason, we need to restrict the consideration to an important subclass of the variables and processes where uncorrelated variables are independent, cf. [22, 23].

A complex-valued random process $(X(t))_{t \in T}$ is called centered proper (or circularly) Gaussian if

- For any $t_1, \ldots, t_n \in T$ the coordinate vector $(X^{(re)}(t_1), X^{(im)}(t_1), \ldots, X^{(im)}(t_n))$ is a centered Gaussian vector in $\mathbb{R}^{2n};$
- $E X(t_1)X(t_2) = 0$ for all $t_1, t_2 \in T.$

We clearly have $E X(t) = 0, \forall t \in T$. Moreover, the property $E X(t)^2 = 0$ yields that the distribution of $X(t)$ in the complex plane $\mathbb{C}$ is spherically symmetric.

These properties extend to the span of $X$. Let us denote $X := \text{span}\{X(t), t \in T\}$. For every $Y \in X$ we have $E Y = 0$, $E Y^2 = 0$, hence its distribution in $\mathbb{C}$ is spherically symmetric Gaussian. Moreover, for any $Y_1, Y_2 \in X$ we have $E Y_1 Y_2 = 0$ and $Y_1, Y_2$ are independent iff they are uncorrelated, i.e. $E Y_1 \overline{Y_2} = 0$. This can be easily verified by checking that their coordinates are uncorrelated.

By applying these facts to the expansion (2.12) of a proper Gaussian process $Z$, we see that the variables $(\xi_n)_{n \in \mathbb{N}}$ are independent and spherically symmetric. Therefore, (2.13) becomes

$$||Z||^2 = \sum_{n=1}^{\infty} \frac{\lambda_n}{2} (\xi_{n,1}^2 + \xi_{n,2}^2),$$
where \((\xi_{n,j})_{n\in\mathbb{N},j\in\{1,2\}}\) are independent real standard Gaussian random variables. This formula can be rewritten as
\[
||Z||^2 = \sum_{n=1}^{\infty} \lambda_n^2 \xi_n^2,
\]
where
\[
\lambda_{2n-1}^2 := \frac{\lambda_n}{2}, \quad \xi_{2n-1}^2 := \xi_{n,1},
\]
\[
\lambda_{2n}^2 := \frac{\lambda_n}{2}, \quad \xi_{2n}^2 := \xi_{n,2},
\]
for all \(n \geq 1\).

A straightforward calculation shows that \(\lambda_n \sim Cn^{-r}\) yields \(\lambda_n^* \sim 2^{r-1}Cn^{-r}\), as \(n \to \infty\). By applying (2.11) with \(2^{r-1}C\) instead of \(C\) we obtain the following result.

**Theorem 2.4** Let \(\{X(t), t \in \mathbb{R}\}\) be a \(2\pi\)-periodic complex centered mean-square continuous stationary proper Gaussian process. Assume that its spectral measure satisfies the asymptotic condition (2.6) with some \(r > 1\). Let \(q\) be a summable weight.

Then we have, as \(\varepsilon \to 0\),
\[
\ln \mathbb{P} \left( \int_0^{2\pi} q(t)|X(t)|^2 dt \leq \varepsilon^2 \right) \sim - \left( \frac{M_q^{\frac{1}{2}} + M_q^{\frac{1}{2}}}{2r \sin(\pi r)} \int_0^{2\pi} |q(t)|^{\frac{1}{r}} dt \right)^{-1} \left( \frac{r - 1}{2} \frac{2^p}{\varepsilon^{\frac{2}{2p-1}}} \right). \]

### 3 Stationary sequences

Let a real stationary centered Gaussian sequence \((U_k)_{k \in \mathbb{Z}}\) admit a representation
\[
U_k = \sum_{m=-\infty}^{\infty} a_m X_{k-m}, \quad (3.1)
\]
where \((a_m) \in \ell_2(\mathbb{Z})\), and \(X_m\) are independent standard Gaussian random variables (this representation exists iff \((U_k)\) has a spectral density).

The following result was essentially obtained in [12].

**Theorem 3.1** Let a real stationary centered Gaussian sequence \((U_k)_{k \in \mathbb{Z}}\) admit a representation (3.1) and let the coefficients \((d_k)_{k \in \mathbb{Z}}\) have the asymptotics
\[
\lim_{k \to \pm \infty} |k|^p d_k = d_{\pm}, \quad \text{for some } p > \frac{1}{2},
\]
where at least one of the numbers \(d_{\pm}\) is strictly positive. Then, as \(\varepsilon \to 0\),
\[
\ln \mathbb{P} \left( \sum_{k \in \mathbb{Z}} d_k^2 U_k^2 \leq \varepsilon^2 \right) \sim - \left( \frac{d_{-}^{\frac{1}{p}} + d_{+}^{\frac{1}{p}}}{4 \sin \left( \frac{\pi}{2p} \right)} \int_0^{2\pi} |a(t)|^{\frac{1}{r}} dt \right)^{\frac{2p}{2p-1}} \frac{2p - 1}{2 \varepsilon^{\frac{2}{2p-1}}},
\]
where \(a(t) = \sum_{k \in \mathbb{Z}} a_k e^{ikt}\).
However, in [12], for \( p < 1 \) an additional assumption was imposed. Now we show that it was not necessary, answering the question raised in [12, Remark 1.2].

**Sketch of the proof:** We have to study the norm of the random vector \( Z \in \ell_2(\mathbb{Z}) \) defined by its coordinates \( Z_k = d_k U_k, \ k \in \mathbb{Z} \). It was proved in [12] that the corresponding covariance operator \( K_Z \) admits a representation

\[
K_Z = DA^*D , \tag{3.2}
\]

where \( D \) is the convolution operator with the kernel \( \sum_{k \in \mathbb{Z}} d_k e^{ikt} \) while \( A \) is the multiplication operator related to the function \( a(t) \).

We see that the elements of decomposition in (3.2) are the same as in (2.4) but the order of use of operators is different. However, a well-known theorem in operator theory, see, e.g., [5, Sec. 2.10, Theorem 5], implies the coincidence of non-zero eigenvalues for operators \( TT^* \) and \( T^*T \) for any bounded linear operator \( T \). Thus, Lemma 2.1 implies that spectral asymptotics of (2.7) type holds for the operator \( K_Z \) (with the natural replacement \( r \to 2p, M_\pm \to d_\pm^2, \sqrt{q} \to a \)). Using formula (2.11) we obtain the claimed small deviation asymptotics. \( \square \)

### 4 Stationary processes with continuous spectra

#### 4.1 Spectral representations

Now we consider general aperiodic stationary processes. Let \( X(t), t \in \mathbb{R} \), be a centered second order complex stationary process on \( \mathbb{R} \).

The analogue of spectral representation (2.1) is more involved and writes as follows:

\[
X(t) = \int e^{iu} \xi(du), \quad t \in \mathbb{R},
\]

where \( \xi(du) \) is an uncorrelated white noise with a control measure \( \mu \) called spectral measure of \( X \).

The only information about white noise integrals, that we need here is that the random variable \( \int g(u) \xi(du) \) is well defined and centered iff \( g \in L_2(\mathbb{R}, \mu) \), while for the covariances we have the expression

\[
\text{cov} \left( \int g_1(u) \xi(du), \int g_2(u) \xi(du) \right) = \int g_1(u) \overline{g_2(u)} \mu(du).
\]

In particular,

\[
K_X(s-t) := \text{cov}(X(s), X(t)) = \int e^{iu(s-t)} \mu(du), \quad s, t \in \mathbb{R}.
\]

We are interested in the small ball behavior of the weighted \( L_2 \)-norm

\[
\int_{\mathbb{R}} q(t) |X(t)|^2 dt = ||\sqrt{q} X||_2^2,
\]

where \( q \in L_1(\mathbb{R}) \) is a non-negative weight.
Assume that the spectral measure $\mu$ has a density $m \in L_1(\mathbb{R})$. Then it is easy to see that
\[
(K_{\sqrt{q}X}u)(s) = \sqrt{q(s)} \int_{\mathbb{R}} K_X(s-t)\sqrt{q(t)}u(t)\,dt = 2\pi \sqrt{q(s)} \mathcal{F}^{-1}(m(\xi)\mathcal{F}(\sqrt{q}u)(\xi))
\]
(we recall that $\mathcal{F}$ stands for the Fourier transform).

4.2 Spectral asymptotics

From now on we assume that the spectral density $m$ has a power-like decay analogous to (2.6),
\[
\lim_{u \to \pm\infty} |u|^r m(u) = M_{\pm},
\]
with some $r > 1$ and $M_+ \geq 0$, $M_+ + M_- > 0$.

**Lemma 4.1** Let the spectral density of $X$ satisfy (4.1). Assume that $q \in L_1(\mathbb{R})$, and
\[
|q|^r := \sum_{j \in \mathbb{Z}} \|q\|_{L_1(j,j+1)} < \infty.
\]
Then
\[
\lambda_n(K_{\sqrt{q}X}) = s_n(K_{\sqrt{q}X}) \sim 2\pi \left( \frac{M_-^r + M_+^r}{2\pi} \int_{\mathbb{R}} q(t)^{\frac{r}{2}} \,dt \right)^{\frac{1}{r}} n^{-r}, \quad \text{as } n \to \infty. \tag{4.3}
\]

**Proof:** We cannot apply Proposition A.1 directly since it requires boundedness of the weights supports. Therefore, we use subtle estimates of [2, Sec. 5], see Proposition A.3. We introduce a decomposition similar to (2.4):
\[
K_{\sqrt{q}X} = \tilde{T}^* \tilde{T}, \quad \tilde{T} = \mathcal{M} \mathcal{Q},
\]
where $\mathcal{M}$ and $\mathcal{Q}$ stand for the multiplication by $\sqrt{m} \in L_2(\mathbb{R})$ and $\sqrt{q} \in L_2(\mathbb{R})$, respectively.

Following [2], for $f \in L_2(\mathbb{R})$ we define the numerical sequence
\[
v(f) = \{v_j(f)\}_{j \in \mathbb{Z}}; \quad v_j(f) := \|f\|_{L_2(j,j+1)}.
\]
Using the notation in Appendix we can write the assumption (4.1) as follows:
\[
\|v(\sqrt{m})\|_{2,\infty} < \infty.
\]
Further, the assumption (4.2) is equivalent to $v(\sqrt{q}) \in \ell_2^2$, and the (quasi)-norm $\|v(\sqrt{q})\|_{2,\infty}$ coincides with $|q|^r$.

Now we consider the sequence of operators $\tilde{T}_k = \mathcal{M} \mathcal{Q}_k$, $k \in \mathbb{N}$, where $\mathcal{Q}_k$ is multiplication by compactly supported weight
\[
b_k(t) = \sqrt{q(t)} \cdot 1_{[-k,k]}(t).
\]
Obviously, $v(b_k) \to v(\sqrt{q})$ in $\ell_2$.

Since $\frac{2}{r} < 2$, we can apply Proposition A.3 to the operator $\tilde{T}^* - \tilde{T}_k^*$. This gives
\[
\sup_n \left( n^\frac{2}{r} s_n(\tilde{T}^* - \tilde{T}_k^*) \right) \leq \text{const} \cdot \|v(\sqrt{m})\|_{2,\infty} \cdot \|v(\sqrt{q}) - v(b_k)\|_2 \to 0 \quad \text{as } k \to \infty.
\]
By (2.5) and Proposition A.4 we infer
\[
\Delta_2^r(\tilde{T}^*) \to \Delta_2^r(\tilde{T}^*) \quad \text{as } k \to \infty.
\]
Since \(\lambda_n(\mathcal{K}_{\sqrt{q}} X) = s_n^2(\tilde{T}^*)\), this implies
\[
\Delta_1^r(\mathcal{K}_k) \to \Delta_1^r(\mathcal{K}_{\sqrt{q}} X) \quad \text{as } k \to \infty,
\]
where
\[
(\mathcal{K}_k u)(s) = (\tilde{T}_k^* \tilde{T}_k u)(s) = b_k(s) \int K_X(s - t)b_k(t)u(t) \, dt.
\]
The weights \(b_k\) satisfy the assumptions of Proposition A.1. Using Proposition A.1, part 1 in Proposition A.2 and the last statement in Proposition A.4, we obtain
\[
\Delta_1^r(\mathcal{K}_k) = \frac{1}{2\pi} \int \int_{\mathbb{R} \setminus \{0\}} 1_{\{2\pi |b_k(t)|^2 \leq M |\operatorname{sgn}(\xi)| |\xi|^{-r} \geq 1\}} \, d\xi \, dt
\]
\[
= \frac{1}{2\pi} \int_{-k}^{k} (2\pi q(t))^{\frac{1}{r}} \, dt \left( M(-1)^{\frac{1}{r}} + M(1)^{\frac{1}{r}} \right)
\]
(recall that \(M(\pm 1) = M_{\pm}\)). We pass to the limit as \(k \to \infty\), and the equivalence in (2.5) yields (4.3).

\[\square\]

### 4.3 Gaussian small deviations

#### 4.3.1 Real processes

By combining spectral asymptotics (4.3) with small deviation asymptotics (2.11) we immediately obtain the following result.

**Theorem 4.2** Let \(\{X(t), t \in \mathbb{R}\}\) be a real centered mean-square continuous stationary Gaussian process. Assume that it has a spectral density satisfying asymptotical condition

\[
m(u) \sim M|u|^{-r}, \quad \text{as } |u| \to \infty,
\]

with some \(r > 1, M > 0\). Let \(q\) be a summable weight satisfying condition (4.2).

Then we have, as \(\varepsilon \to 0\),

\[
\ln P \left( \int_{\mathbb{R}} q(t)|X(t)|^2 \, dt \leq \varepsilon^2 \right) \sim - \left( \frac{M^\frac{1}{r}}{r \sin(\pi/r)} \int_{\mathbb{R}} q(t)^\frac{r}{2} \, dt \right) \cdot \frac{(r - 1)(2\pi)^{\frac{r-1}{2}}}{2\varepsilon^{\frac{r}{2}}}
\]

Apart from the weight integration domain, the constant in the limit is exactly the same as in Theorem 2.2.

This result has an intersection with that of S. Gengembre [10] who considered the non-weighted \(L_p\)-norm, \(1 \leq p \leq +\infty\), on a bounded interval and the range \(1 < r < 3\) that enables comparison with fractional Ornstein–Uhlenbeck processes and thus a reduction to the small deviation results on fractional Brownian motion, cf. [17]. We illustrate this connection in the next sub-section.
4.3.2 Basic example

Let $H \in (0, 1)$ be the fractionality parameter. Let $W_H$ be a fractional Brownian motion and let $U_H(t) = e^{-Ht/2}W_H(e^t), t \in \mathbb{R}$, be a fractional Ornstein-Uhlenbeck (OU) process. (There are several other ways to extend the classical OU-process to the fractional case. We refer to [13] for alternative definitions and further references.) In other words, it is a real centered Gaussian stationary process with covariance

$$K_H(t) = \frac{1}{2} \left( e^{Ht} + e^{-Ht} - \left| e^{t/2} - e^{-t/2} \right|^{2H} \right).$$

The asymptotic behavior of the corresponding spectral density $m_H(u)$ is well known, see e.g. [10, Proposition 1],

$$m_H(u) \sim \frac{\Gamma(2H + 1) \sin(\pi H)}{2\pi |u|^{-2H-1}} =: M_H |u|^{-2H-1}, \quad \text{as } u \to \infty. \quad (4.5)$$

This is essentially due to the behavior of the covariance at the origin,

$$K_H(t) = 1 - \frac{|t|^{2H}}{2} + O(|t|^{1+\min\{1,2H\}}), \quad \text{as } t \to 0.$$

It will be also useful for us to consider integrated versions of fractional Brownian motion and their stationary versions. Let us denote $V_h := W_h$ for $h \in (0, 1)$, and define processes $V_h$ for all non-integer positive $h > 1$ inductively, by

$$V_{h+1}(t) := \int_0^t V_h(s)ds, \quad t \geq 0.$$

It is easy to see that the process $V_h$ is $h$-self-similar. Therefore, $\{U_h(t) = e^{-ht}V_h(e^t), t \in \mathbb{R}\}$ is a stationary process with the covariance function

$$K_h(t) = e^{-ht}E \left( V_h(1)V_h(e^t) \right), \quad t \in \mathbb{R},$$

for all positive non-integer values of the parameter $h$. We can also easily find the inductive formula for the spectral measures of $U_h$. Indeed, for any $h > 1, t \in \mathbb{R}$, we have

$$U_h'(t) = -he^{-ht}V_h(e^t) + e^{-ht}e^tV_{h-1}(e^t) = -hU_h(t) + U_{h-1}(t).$$

Rewrite this identity as

$$U_{h-1}(t) = U_h'(t) + hU_h(t)$$

and translate it in the language of spectral measures. Let $\mu_h$ denote the spectral measure of $U_h$. Recall that $U_h$ has a spectral representation

$$U_h = \int_{\mathbb{R}} e^{itu}Z_h(du)$$

where $Z_h$ is a centered random measure with orthogonal values on $\mathbb{R}$ such that $E |Z_h(\cdot)|^2 = \mu_h(\cdot)$. Since

$$U_h'(t) = \int_{\mathbb{R}} e^{itu} iu Z_h(du),$$
we get
\[ U_{h-1}(t) = \int_{\mathbb{R}} e^{itu}(iu + h)Z_h(du). \]

By the uniqueness of the spectral representation, it follows that \((iu + h)Z_h(du) = Z_{h-1}(du)\) and we finally obtain
\[ \mu_h(du) = \frac{\mu_{h-1}(du)}{|iu + h|^2} = \frac{\mu_{h-1}(du)}{u^2 + h^2}. \]

It follows from (4.3) that \(\mu_h\) has a spectral density \(m_h\) satisfying
\[ m_h(u) \sim M_H |u|^{-2h-1}, \quad \text{as } u \to \infty. \]

(Here and elsewhere \(H := \{h\}\) is the fractional part of \(h\)).

Assuming condition (4.2) on the weight to hold and applying Theorem 4.2 with \(r = 2h + 1\), \(M = M_H\) we obtain as \(\varepsilon \to 0\),
\[ \ln \mathbb{P} \left( \int_{\mathbb{R}} q(t)|U_h(t)|^2 dt \leq \varepsilon^2 \right) \sim - \frac{1}{(2h + 1) \sin(\frac{\pi}{2n+1})} \int_{\mathbb{R}} q(t)^{\frac{1}{2n+1}} dt \cdot \frac{2h+1}{\varepsilon^2} \cdot \frac{h(\Gamma(2H+1) \sin(\pi H))^{\frac{1}{2n+1}}}{\varepsilon^\frac{2n+1}{2n+2}}. \] (4.6)

In view of the identity
\[ \int_{\mathbb{R}} q(t)|U_h(t)|^2 dt = \int_{0}^{\infty} \rho(t)|V_h(t)|^2 dt \]
with the weight
\[ \rho(t) := \frac{q(ln t)}{t^{2n+1}}, \quad t > 0, \]
formula (4.6) immediately yields an equivalent result for the weighted \(L_2\)-norm of \(V_h\). The small ball asymptotics for the weighted \(L_2\)-norm of \(V_h\) and \(U_h\) was obtained in [20, Theorems 3.1, 3.3 and 4.2] but only for the weights with bounded support.

One should also mention [11, 17] where small deviations of more general weighted \(L_p\)-norms, \(1 \leq p \leq +\infty\) were studied for fractional Brownian motions and for Riemann–Liouville processes.

**4.3.3 Proper complex processes**

In all the previous examples the spectral density \(m\) satisfied (4.1) with \(M_+ = M_-\). For complex-valued processes this condition may be violated. By repeating the proof of Theorem 2.4 and using asymptotics (4.3) we obtain the following analogue of Theorem 2.4 for complex-valued processes with continuous spectra.

**Theorem 4.3** Let \(\{X(t), t \in \mathbb{R}\}\) be a complex centered mean-square continuous stationary proper Gaussian process. Assume that it has a spectral density satisfying the asymptotic condition (4.1) with some \(r > 1\). Let \(q\) be a summable weight on \(\mathbb{R}\) satisfying (4.2).

Then we have, as \(\varepsilon \to 0\),
\[ \ln \mathbb{P} \left( \int_{0}^{2\pi} q(t)|X(t)|^2 dt \leq \varepsilon^2 \right) \sim - \left( \frac{M_+^\frac{1}{r} + M_-^\frac{1}{r}}{2r \sin(\pi/r)} \int_{\mathbb{R}} q(t)^{\frac{1}{r}} dt \right) \cdot (r-1) (2\pi)^{\frac{r-1}{r}} \cdot \frac{1}{\varepsilon^{\frac{2}{r-1}}}. \]
Apart from the weight integration domain, the constant in the limit is exactly the same as in Theorem 2.4.

Appendix

Here we collect some statements from [2]–[5]. Recall that $\mathcal{F} : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ is standard Fourier transform. For a compact operator $T$ in $L_2(\mathbb{R})$ we denote by $s_n(T)$ its singular values and by

$$N(\lambda; T) := \#\{n : s_n(T) \geq \lambda\}$$

the distribution function of $s_n(T)$. Define

$$\Delta(\theta) = \lim_{\lambda \to 0^+} \lambda^{\theta} N(\lambda; T).$$

Denote by $\ell_\delta$ and $\ell_{\delta,\infty}$ the spaces of sequences $(x_j)$ (with $j \in \mathbb{N}$ or $j \in \mathbb{Z}$) such that, respectively,

$$\|x_j\|_\delta := \left( \sum_j |x_j|^\delta \right)^{\frac{1}{\delta}} < \infty; \quad \|x_j\|_\delta,\infty := \sup_j (|j|^{\frac{1}{\delta}}|x_j|) < \infty.$$

**Proposition A.1** (a particular case of [3, Theorem 1 (b) and Theorem 2]). Let

$$(Au)(s) = b(s)\mathcal{F}^{-1}(a(\xi)\mathcal{F}(cu)(\xi)), \quad (A.1)$$

where functions $b, c \in L_2(\mathbb{R})$ have compact supports while $a$ has the form

$$a(\xi) = \zeta(\xi)M(\text{sgn}(\xi))|\xi|^{-r},$$

here $r > 1, M : \{-1, 1\} \to [0, +\infty)$ and $\zeta$ is a smooth cut-off function vanishing in a neighborhood of the origin. Then

$$\Delta_{\frac{1}{2}}(A) = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}\setminus\{0\}} 1_{\{|b(t)||c(t)||M(\text{sgn}(\xi))||\xi|^{-r}\geq 1\}} d\xi dt.$$

**Proposition A.2** (a particular case of [4, Corollary 4 and Lemma 3]). Let operator $\mathcal{A}$ have the form $A.1$.

1. Suppose that weight functions $b$ and $c$ satisfy the assumptions of Proposition $A.1$ while $a(\xi) = \zeta(\xi)\psi(\xi)$, where $\zeta$ is a smooth cut-off function vanishing in a neighborhood of the origin and $\psi \in L_{\infty}(\mathbb{R}), \psi(\xi) = o(|\xi|^{-r})$ as $|\xi| \to \infty, r > 1$. Then $\Delta_{\frac{1}{2}}(A) = 0$.

2. Suppose that functions $a, b$ and $c$ satisfy the assumptions of Proposition $A.1$. Let $\text{supp}(b) \subset I_1, \text{supp}(c) \subset I_2$, where $I_1$ and $I_2$ are closed bounded segments with non-overlapping interiors. Then $\Delta_{\frac{1}{2}}(A) = 0$.

**Proposition A.3** (a particular case of [3, Subsection 5.7]). Let

$$(\tilde{A}u)(t) = b(t)\mathcal{F}^{-1}(a(\xi)v(\xi)), \quad (A.4)$$

where $a, b \in L_2(\mathbb{R})$. Define sequences $v(a)$ and $v(b)$ according to $A.4$.

Let $v(a) \in \ell_\delta,\infty, v(b) \in \ell_\delta$ for some $\delta \in (0, 2)$. Then $(s_j(\tilde{A})) \in \ell_{\delta,\infty}$, and

$$\|(s_j(\tilde{A}))\|_{\delta,\infty} \leq \text{const} \cdot \|v(a)\|_{\delta,\infty} \cdot \|v(b)\|_\delta,$$

where constant depends on $\delta$. 

15
Proposition A.4 (Corollary 5 in [5, Sec. 11.6]). If $\Delta_\theta(T_1), \Delta_\theta(T_2)$ are finite then
\[
|\left(\Delta_\theta(T_1)\right)^{\frac{1}{\sigma+1}} - \left(\Delta_\theta(T_2)\right)^{\frac{1}{\sigma+1}}| \leq \left(\Delta_\theta(T_1 - T_2)\right)^{\frac{1}{\sigma+1}}.
\]
In particular, if $\Delta_\theta(T_1 - T_2) = 0$ then
\[
\Delta_\theta(T_1) = \Delta_\theta(T_2).
\]

Acknowledgments

We are grateful to V. A. Sloushch who provided us with reference [2]. We are also grateful to the anonymous referee and to the Editor for their careful reading and for the help with our work on the manuscript.

The work was supported by SPbSU-DFG grant 6.65.37.2017 and by RFBR grant 16-01-00258.

References

[1] R.B. Ash, M.F. Gardner, Topics in Stochastic Processes. Academic Press, New York, 1975.
[2] M.Sh. Birman, G.E. Karadzhov, M.Z. Solomyak, Boundedness conditions and spectrum estimates for the operators $b(X)a(D)$ and their analogs. In: Estimates and asymptotics for discrete spectra of integral and differential equations (Leningrad, 1989-90), Adv. Soviet Math., 7, AMS, Providence, R.I., 1991, 85–106.
[3] M.Š. Birman, M.Z. Solomjak, Asymptotics of the spectrum of pseudodifferential operators with anisotropic-homogeneous symbols, Vestnik LGU (1977), no 13, 13–21 (Russian); English transl.: Vestnik Leningrad Univ. Math. 10 (1982), 237–247.
[4] M.Š. Birman, M.Z. Solomjak, Asymptotics of the spectrum of pseudodifferential operators with anisotropic-homogeneous symbols. II, Vestnik LGU (1979), no 3, 5–10 (Russian). English transl.: Vestnik Leningrad Univ. Math. 12 (1980), 155–161.
[5] M.S. Birman, M.Z. Solomjak, Spectral Theory of Self-Adjoint Operators in Hilbert Space, 2nd edition, revised and extended. Lan’, St.Petersburg, 2010 (Russian). English transl. of the 1st ed.: Mathematics and Its Applications. Soviet Series. 5, Kluwer, Dordrecht etc. 1987.
[6] J.C. Bronski, Small ball constants and tight eigenvalue asymptotics for fractional Brownian motions, J. Theoret. Probab. 16 (2003), no. 1, 87–100.
[7] T. Dunker, M.A. Lifshits, W. Linde, Small deviations of sums of independent variables. In: Proc. Conf. High Dimensional Probability; Ser. Progress in Probability, 43, Birkhäuser, 1998, 59–74.
[8] Gao, F., Hannig, J., Lee, T.-Y., and Torcaso, F.: Laplace transforms via Hadamard factorization with applications to small ball probabilities. Electronic J. Probab. 8, (2003), paper 13.
[9] Gao, F., Hannig, J., Lee, T.-Y., and Torcaso, F.: Exact $L_2$-small balls of Gaussian processes, J. Theoret. Probab. 17, (2004), no. 2, 503–520.
S. Gengembre, Probabilités de petites déviations pour les processus stationnaires gaussiens. 
Publ. IRMA Lille 60 (2003), no. X, 1–24.

S. Gengembre, Petites déviations pour les processus fractionnaires. Memoire de D.E.A. 
Université Lille I, 2002, 19 pp.

S.Y. Hong, M.Lifshits, A.Nazarov, Small deviations in $L_2$-norm for Gaussian dependent 
sequences, Electronic Comm. Probab. 21 (2016), no. 41, 1–9.

T. Kaarakka, P. Salminen, On fractional Ornstein–Uhlenbeck processes, Comm. Stoch. Anal. 
5 (2011), no. 1, 121–133.

Li, W.V. and Shao, Q.-M.: Gaussian processes: inequalities, small ball probabilities and 
applications, In: Stochastic Processes: Theory and Methods, Handbook of Statistics (C.R. 
Rao and D. Shanbhag, eds.), 19, North-Holland/Elsevier, Amsterdam, 2001, pp. 533–597.

Lifshits, M.A.: Asymptotic behavior of small ball probabilities, In: Probab. Theory and 
Math. Statist. Proc. VII International Vilnius Conference (1998) (B. Grigelionis, ed.), 
VSP/TEV. Vilnius, 1999, pp. 453–468.

Lifshits, M.A.: Bibliography of small deviation probabilities, On the small deviation web-
site http://www.proba.jussieu.fr/pageperso/smalldev/biblio.pdf

M.A. Lifshits, W. Linde, Small deviations of weighted fractional processes and average non-
linear approximation, Trans. Amer. Math. Soc. 357 (2005), 2059–2079.

A.I. Nazarov, Log-level comparison principle for small ball probabilities. Statist. & Probab. 
Letters 79 (2009), no. 4, 481–486.

A.I. Nazarov, Exact $L_2$-small ball asymptotics of Gaussian processes and the spectrum of 
boundary-value problems, J. Theor. Probab. 22 (2009), no. 3, 640–665.

A.I. Nazarov, Ya.Yu. Nikitin, Logarithmic $L_2$-small ball asymptotics for some fractional 
Gaussian processes, Theory Probab. Appl. 49 (2004), no. 4, 645–658.

A.I. Nazarov, R.S. Pusev, Comparison theorems for the small ball probabilities of Gaussian 
processes in weighted $L_2$-norms. Algebra & Analysis 25 (2013), no. 3, 131–146 (Russian); 
English transl.: St. Petersburg Math. J. 25 (2014), no. 3, 455–466.

F. Neeser and J. Massey, Proper complex random processes with applications to information 
theory, IEEE Transactions on Information Theory 39 (1993), no. 4, 1293–1302.

E. Ollila, On the circularity of a complex random variable, IEEE Signal Processing Letters 
15 (2008), 841–844.

R.S. Pusev, Asymptotics of small deviations of the Bogoliubov processes with respect to a 
quadratic norm, Theoret. and Math. Phys. 165 (2010), no. 1, 1348–1357.

D.P. Sankovich, Some properties of functional integrals with respect to the Bogoliubov mea-
sure, Theoret. and Math. Phys. 126 (2001), no. 1, 121–135.

V.M. Zolotarev, Asymptotic behavior of Gaussian measure in $\ell_2$, J. Sov. Math. 35 (1986), 
2330–2334.

A. Zygmund. Trigonometrical Series, Vol.1, Cambridge University Press, Cambridge, 1959.