SCHOTTKY GROUPS OVER VALUATION RINGS

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Abstract. Given a non-trivial complete valued field $K$, we construct a space of balls and a locally finite tree associated to a compact subset of $\mathbb{P}^1(K)$. We define hyperbolic matrices and Schottky groups over such fields. To any Schottky group $\Gamma$, we associate a compact set with an action of $\Gamma$, such that the quotient graph of the associated tree is a finite graph, and such that $\Gamma$ is naturally identified with its fundamental group. This results extend the classical ones for non-archimedean rank 1 valuations of Gerritzen and Van der Put [5].

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In 1972, David Mumford in his celebrated paper [6] constructed an algebraic curve from certain subgroups of $\text{PGL}_2(K)$, for $K$ the field of fractions of a complete noetherian local ring $\mathcal{O}$. This construction imitates the classical construction of Schottky over $\mathbb{C}$, so the groups he considered are called Schottky groups. Some years later, in 1980, Lothar Gerritzen and Marius van der Put in his seminal book [5] redid this construction but now for fields complete with respect to a non-archimedean absolute value. In this paper we consider Schottky groups but now for a field $K$ complete with respect to any valuation (which we may and do assume to be non-trivial, in order to obtain any interesting result, and written multiplicatively). The results we obtain generalize the ones obtained in the first chapter of [5]. However we use a construction inspired by Berkovich analytic geometric [1] introduced in 1990, and concretely the $\mathbb{R}$-tree structure of the analytification of $\mathbb{P}^1(K)$. See also [4] for some results on Mumford curves using these language. In our case we obtain a $\Lambda$-tree, which is a generalization of a tree and of $\mathbb{R}$-tree to a general totally ordered group $\Lambda$ (see [3]). This tree is the analogue of the classical Bruhat-Tits tree of $\text{PGL}_2(K)$ as introduced by Bruhat-Tits [2] and Serre [7].

The structure of the paper is as follows: in the first section we recall some well known results on valuations, and in the second we introduce the $\Lambda$-tree of balls,
which will substitute the Berkovich tree in our general case; it is shown then in
the next section that this tree is isomorphic to the (natural generalization of the)
Bruhat-Tits tree, obtained as a set as $\text{PGL}_2(\mathcal{O})/\text{PGL}_2(K)$. We show after that
how to associate a tree $T(\mathcal{L})$ (and not just a sub-$\Lambda$-tree) to a compact subset $\mathcal{L}$
of $\mathbb{P}^1(K)$, which we show to be locally finite. We also study how to recover the
compact set $\mathcal{L}$ from the tree $T(\mathcal{L})$, in case $\mathcal{L}$ is perfect, via the classical theory of
the ends of a tree. In the fifth section we introduce and study hyperbolic matrices
of $\text{PGL}_2(K)$. The main difference with the classical case is that we insist some
defining element to be topologically nilpotent, which in the classical rank 1 case
is equivalent to have absolute value (in pour notation, valuation) less than 1, but
in general is not. All these is combined in section 6 to the definition of Schottky
groups $\Gamma$ and its associated perfect and compact $\Gamma$-set $\mathcal{L}_\Gamma$. Finally, in section 7,
which can be considered the core of the paper, we show that the quotient of the
tree $T(\mathcal{L}_\Gamma)$ with respect to the natural action of $\Gamma$ is a finite graph, and that $\Gamma$ is
naturally identified with its fundamental group.

We tried to write the paper as self contained as possible, thus reproving some
results which are may be well known, but for which we did not found any reference.
Some of the proofs are directly inspired by the proofs of similar results in [5]; for
others, however, we tried to find more direct or more clear proof, even for the case
considered there.

Comparing with the already mentioned first chapter of [5] we do not study or
even define any analogous of what they call discontinuous groups, and hence we do
not prove any result concerning the existence of a normal Schottky subgroup of any
finitely generated discontinuous group with finite index. This type of results should
be not difficult to obtain, and they can be of independent interest in order to find
Schottky groups in nature, like for the case of the ones associated to Shimura or
Drinfeld modular curves.

Acknowledgements

This paper was in part as the research work of the first author in order to obtain
the masters degree in Mathematics, under the supervision of the second author and
Núria Vila. We thank Núria Vila for accepting to be the tutor of the first author,
and Angela Arenas for several comments and corrections.

1. Preliminaries on valuations and valuation rings

Recall that a totally ordered (abelian) group with $0 \neq \Lambda_0$ is an abelian group $\Lambda$
(which we will denote multiplicatively) together with an absorbent element $0 \notin \Lambda$
verifying $0 \cdot \rho = \rho \cdot 0 = 0$ for all $\rho \in \Lambda$, and with a total order $\leq$ such that

1. if $a \leq b$ and $c \in \Lambda$, then $a \cdot c \leq b \cdot c$.

2. $0 \leq a$ for all $a \in \Lambda$.

We say that $\Lambda_0$ is non-trivial if there exists $1 \neq \rho \in \Lambda$.

We say that a progression $\rho_n \in \Lambda$ for $n \geq 1$ has limit 0 if, for every $\epsilon \in \Lambda$, there
exists $n_0 \geq 1$ such that $\rho_n \leq \epsilon$ for all $n \geq n_0$. We say that $\rho \in \Lambda$ is topologically
nilpotent if the progression $\rho^n$ has limit 0. We will use the following fact throughout
the paper.

Remark 1.1. Since we are considering non-trivial ordered groups, there always exists
some topologically nilpotent $\rho \in \Lambda$. Hence, there exists an element $\rho \in \Lambda$ such that
for all $\epsilon \in \Lambda$, there exists some $n \geq 1$ such that $\rho^n \leq \epsilon$.

Given two elements $\rho_1 \leq \rho_2 \in \Lambda$, we denote the intervals as usual

$$[\rho_1, \rho_2] := \{\delta \in \Delta \mid \rho_1 \leq \delta \leq \rho_2\}$$
and \((\rho_1, \rho_2) := [\rho_1, \rho_2] \setminus \{\rho_1, \rho_2\}\) We denote also \([\rho_1, \infty) := \{\delta \in \Delta \mid \rho_1 \leq \delta\}\).

We define now valuations by using the multiplicative version, mimicking the notation for the non-archimedean absolute values.

**Definition 1.2.** A surjective map \(|\cdot| : K \to \Lambda \cup \{0\}\) from a field \(K\) to \(\Lambda \cup \{0\}\), where \(\Lambda\) is a non trivial ordered group, is called a \textit{valuation} of \(K\) if it satisfies

- \(|xy| = |x| \cdot |y| \forall x, y \in K\)
- \(|x + y| \leq \max\{|x|, |y|\} \forall x, y \in K\)
- \(|x| = 0 \iff x = 0\).

Recall that if \(|\cdot|\) is a valuation of a field \(K\) and if \(a, b \in K\) are such that \(|a| \neq |b|\) then \(|a + b| = \max\{|a|, |b|\}\), which implies that all triangles are isosceles. Moreover it is easy to see that \(|1| = 1\).

We say that \(|\cdot|\) is a \textit{non Archimedean absolute value} or that it has real rank 1 if it is a valuation and \(\Lambda_0 \hookrightarrow \mathbb{R}_{\geq 0}\) as ordered groups. By composing with \(-\log\) we get the usual notion of (additive) real valuation.

Given a field \(K\) with a valuation \(|\cdot|\), the ring of integers of \(K\) with respect to \(|\cdot|\) is

\[\mathcal{O}|_{\cdot|} = \{x \in K \mid |x| \leq 1\}\]

Note that \(\mathcal{O}|_{\cdot|}\) is a local domain whose field of fractions is \(K\), with maximal ideal \(\mathfrak{p}|_{\cdot|} := \{x \in K \mid |x| < 1\}\), and residue field \(k|_{\cdot|}\). For ease of simplicity, we will denote it by \(\mathcal{O}\) if there is no risk of confusion.

Recall that a domain \(A\) is a valuation ring if for all \(x \in Q(A)\) either \(x \in A\) or \(x^{-1} \in A\), where \(Q(A)\) denotes the field of fractions of \(A\). It is well known that there is an equivalence between fields with a valuation and valuation rings. The valuation associated to a valuation ring is

\[|\cdot| : K \to (\mathbb{K}^*/\mathbb{R}^*) \cup \{0\}\]

where \(\overline{a} \leq \overline{b}\) if and only if \(ab^{-1} \in R\).

Given a sequence \(\{a_n\}_n\) of elements in \(K\), and \(a \in K\), one says that \(\lim_{n \to \infty} a_n = a\) if and only if \(|a_n - a| \to 0\). One says it forms a Cauchy sequence if \(|a_{n+1} - a_n| \to 0\). Note that the notation can be misleading as these notions depend on the given valuation.

The field \(K\) is complete respect \(|\cdot|\) if every Cauchy sequence has limit. Any field with a valuation can be subsumed into a minimal field complete with respect to a valuation extending the given one, called its completion. Recall that any finite extension \(L\) of a field \(K\) complete with respect to a valuation has a natural valuation on it extending the one of \(K\), and moreover it is also complete.

We say that \(q \in \mathbb{K}^*\) is topologically nilpotent (with respect to \(|\cdot|\)) if \(|q| \in \Lambda\) it is, i.e. if \(\lim_{n \to \infty} q^n = 0\). If the image of the valuation is of rank 1, then it is equivalent to be topologically nilpotent than to have valuation strictly less than 1; this is never true if the rank is not 1.

**Lemma 1.3.** Take \(q \in \mathcal{O}, q \neq 0\). Then \(q\) is not topologically nilpotent if and only if there exists \(\rho, \rho' \in \Lambda\) such that \(\rho \leq |q^n| \leq \rho', \) for all \(n \in \mathbb{Z}\).

A valuation ring \(\mathcal{O}\) and its field of fractions \(K\) inherits a natural topology which makes them topological rings. It can be described as the I-adic topology for \(I = q\mathcal{O}\), for \(q\) a topologically nilpotent element, which is independent of \(q\). A basis of open sets is formed by the (closed) balls with radius \(\rho \in \Lambda\).

Given \(p \in K\) and \(\rho \in \Lambda_0\), the (closed) ball with center \(p\) and radius \(\rho\) is \(B(p, \rho) = \{y \in K \mid |y - p| \leq \rho\}\).
When we consider the projective line \( \mathbb{P}^1(K) = K \cup \{\infty\} \) with its inherited (analytic) topology, the closed balls don’t form a basis; one needs to include also the complements of the open balls

\[
B^c(p, \rho) := \{ z \in K \mid |z - p| \geq \rho \} \cup \{\infty\}
\]

for \( p \in K \) and \( \rho \in \Lambda \) to get a subbasis.

Given \( p \in K \), \( \rho_1 \in \Lambda_0 \) and \( \rho_2 \in \Lambda \cup \{\infty\} \), the (generalized) annulus with center \( p \) and radii \( \rho_1 \) and \( \rho_2 \) is

\[
C(p, \rho_1, \rho_2) := \{ z \in \mathbb{P}^1(K) \mid \rho_1 \leq |z - p| \leq \rho_2 \}.
\]

Note that the case \( \rho_1 = 0 \) are closed balls, and \( \rho_2 = \infty \) are complements of open balls. The set of all generalized annulus form a basis for the topology of \( \mathbb{P}^1(K) \).

This is proven by observing that the intersection of two generalized annulus is either empty or a generalized annulus.

We will denote also by \( C(p, \rho) := C(p, \rho_1, \rho) = B(p, \rho) \cap B^c(p, \rho) \) the circle with center \( p \) and radius \( \rho \).

### 2. The Tree of Balls \( \mathcal{T}_K \)

**Definition 2.1.** We define the space of balls \( \mathcal{T}_K = \{ B(p, \rho) \mid p \in K, \rho \in \Lambda \} \). We also define \( \mathcal{T}_K = \mathcal{T}_K \cup K \cup \infty \), which can be seen as the set of balls with radius \( \rho \in \Lambda_0 \cup \{\infty\} \), being \( B(p, \infty) := K \) for all \( p \in K \).

We will see in this section that \( \mathcal{T}_K \) has a natural structure of (oriented) \( \Lambda \)-tree, a generalization of (simplicial) trees and \( \mathbb{R} \)-trees. The order will be the inclusion relation (as a subsets of \( K \)). The main property is given by the following elementary result.

**Lemma 2.2.** Define \( \varrho : \mathcal{T}_K \to \Lambda \) by \( \varrho(B(p, \rho)) := \rho \). Then \( \varrho \) is well-defined and for any \( B(p, \rho) \in \mathcal{T}_K \) induces a bijection

\[
\{ B \in \mathcal{T}_K \mid B \subset B(p, \rho) \} \cong [\rho, \infty).
\]

**Proof.** We know that \( B(x_1, \rho_1) = B(x_2, \rho_2) \) if and only if \( \rho_1 = \rho_2 \) and \( x_2 \in B(x_1, \rho_1) \), which shows that \( \varrho \) is well defined, and that for any \( \delta \geq \varrho \),

\[
\{ B \in \mathcal{T}_K \mid B \subset B(p, \rho) \} \cap \varrho^{-1}(\delta) = \{ B(p, \delta) \}.
\]

\( \square \)

**Lemma 2.3.** Given two balls \( B_1 \) and \( B_2 \) in \( \mathcal{T}_K \), there exists a minimal ball \( B_1 \vee B_2 \) that contain both. Even more, the set

\[
\{ B \in \mathcal{T}_K \mid B \supset B_1 \text{ and } B \supset B_2 \}
\]

is totally ordered and with a minimal element with respect to the inclusion.

**Proof.** We only need to observe that

\[
B(x_1, \rho_1) \vee B(x_2, \rho_2) = B(x_1, \max\{|x_1 - x_2|, \rho_1, \rho_2\})
\]

verifies the properties. The second assertion is due to the well-known property \( B(x_1, \rho_1) \cap B(x_2, \rho_2) \) is either empty or the smallest of both. \( \square \)

We can give now a structure of \( \Lambda \)-metric space for \( \mathcal{T}_K \).

**Definition 2.4.** Given two balls \( B_1 \subset B_2 \) of \( \mathcal{T}_K \), we define the \( \Lambda \)-distance between them as \( d(B_1, B_2) := \varrho(B_2) \varrho(B_1)^{-1} \). Given any two balls \( B_1 \) and \( B_2 \) in \( \mathcal{T}_K \), we define

\[
d(B_1, B_2) := d(B_1, B_1 \vee B_2) d(B_2, B_1 \vee B_2).
\]

The following properties are elementary, and they define and show that \( \mathcal{T}_K \) is a \( \Lambda \)-metric space.
Properties 2.5. The map \( d: \mathcal{T}_K \times \mathcal{T}_K \rightarrow \Lambda \) verifies

1. \( d(B_1, B_2) \geq 1 \) for all \( B_1 \) and \( B_2 \in \mathcal{T}_K \).
2. \( d(B_1, B_2) = 1 \) if and only if \( B_1 = B_2 \).
3. \( d(B_1, B_2) = d(B_2, B_1) \) for all \( B_1 \) and \( B_2 \in \mathcal{T}_K \).
4. \( d(B_1, B_2) \leq d(B_1, B_3)d(B_3, B_1) \).

Note that we are considering the not so usual multiplicative notation. We can then define segments in \( \mathcal{T}_K \) and show it is geodesically linear: given any two balls there is a unique segment going from one to the other. This shows that \( \mathcal{T}_K \) is a \( \Lambda \)-tree as defined in [3].

Remark 2.6. In the case \(|.|\) is a non-archimedean valuation, the space \( \mathcal{T}_K \) is form by the so called type II points inside \( \mathbb{P}^{1,an}_K \), with its natural metric (see [1]). This form all the points of \( \mathbb{P}^{1,an}_K \setminus \mathbb{P}^1(K) \) if and only if \(|.|\) has image all \( \mathbb{R}_{\geq 0} \) and \( K \) is spherically complete.

Recall that a segment \([B_1, B_2]_\Lambda \) from \( B_1 \) to \( B_2 \) is an isometry \( \alpha: [p_1, p_2] \rightarrow \mathcal{T}_K \) where \( p_1 \) and \( p_2 \in \Lambda \) such that \( \alpha(p_1) = B_1 \) and \( \alpha(p_2) = B_2 \).

We define the path from \( p \in K \) to \( \infty \) inside \( \mathcal{T}_K \) as \( \pi(p, \infty) = \{ B(p, \rho) \mid \rho \in \Lambda \} \). In general, given any ball \( B(p, \rho) \) for \( \rho \in \Lambda \cup \{ \infty \} \), we define the path \( \pi(p, B(p, \rho)) = \{ B(p, \delta) \mid \delta \leq \rho \} \).

Note that \( \varrho \) induces an isometry \( \varrho: \pi(p, \infty) \cong \Lambda \). The intersection of two such paths is clearly \( \pi(p, \infty) \cap \pi(q, \infty) \) is the unique point \( \pi(p, \infty) \cap \pi(q, \infty) = \{ B(p, r) \mid r \geq |p - q| \} =: \pi(B(p, |p - q|, \infty)) \cong [|p - q|, \infty) \) and we define \( \pi(p, q) := \pi(p, B(p, |p - q|)) \cup \pi(q, B(p, |p - q|)) \).

Lemma 2.7. Let \( p_1, p_2 \) and \( p_3 \) three distinct points in \( \mathbb{P}^1(K) \). Then

\[ \pi(p_1, p_2) \cap \pi(p_2, p_3) \cap \pi(p_1, p_3) = \{ t(p_1, p_2, p_3) \} \]

is a unique point \( t(p_1, p_2, p_3) \) of \( \mathcal{T}_K \). Note that \( t(p_1, p_2, \infty) = B(p_1, |p_1 - p_2|) \). And in general, if \( |p_1 - p_3| = |p_2 - p_3| \geq |p_1 - p_2| \) then \( t(p_1, p_2, p_3) = B(p_1, |p_1 - p_2|) \).

Proof. First observe that the case one of the points is \( \infty \), say \( p_3 = \infty \), is clear from the definition. Second, if all points are in \( K \), we can suppose that \( |p_1 - p_3| = |p_2 - p_3| \geq |p_1 - p_2| \) after changing the order if necessarily. Then \( B(p_1, |p_1 - p_2|) \in \pi(p_1, p_2) \cap \pi(p_2, p_3) \cap \pi(p_1, p_3) \) clearly. On the other hand, a ball \( B(p_1, \delta) \in \pi(p_1, p_3) \) if \( \delta \leq |p_1 - p_3| \), and then \( B(p_1, \delta) \in \pi(p_2, p_3) \) since \( \delta \leq |p_2 - p_3| = |p_1 - p_3| \). But then \( B(p_2, \delta) = B(p_1, \delta) \) if \( \delta \geq |p_1 - p_2| \), and \( B(p_1, \delta) \in \pi(p_1, p_2) \) if \( \delta \leq |p_1 - p_2| \); so \( \delta = |p_1 - p_2| \) is the unique solution.

From the proof of the lemma we can see that for any three distinct points \( p_1, p_2 \), \( p_2 \) and \( p_3 \) in \( K \), if we order them such that \( |p_1 - p_3| = |p_2 - p_3| \geq |p_1 - p_2| \), then \( t(p_1, p_2, p_3) = t(p_1, p_2, \infty) \). We call such ordering a ball ordering.

Note that we have a natural bijection \( i_{p_1, p_2} \) from \( \pi(p_1, p_2) \) to \( \Lambda \), once fix an ordering of \( p_1 \) and \( p_2 \), and which sends the ball \( B(p, |p - q|) \) to 1.
We denote \( \mathbb{P}^1(K)^{<3>} := \{(p_1, p_2, p_3) \in \mathbb{P}^1(K)^3/p_1 \neq p_2 \neq p_3 \neq p_1\} = (\mathbb{P}^1(K)^3 \setminus \Delta), \)
where \( \Delta \) are the points \((p_1, p_2, p_3)\) such that \( p_1 \neq p_2 \neq p_3 \neq p_1 \).

So we have \( t: \mathbb{P}^1(K)^{<3>} \to T_K \) as \( t(p_1, p_2, p_3) := B(p_1, \rho) \) where \( \rho \) is the smallest distance between the three points and \( p_i \) is one of the two elements that gives this smallest distance. We also have \( t(p_1, p_2, \infty) := B(p_1, |p_1 - p_2|) \).

3. The tree of balls and the Bruhat-Tits tree

Consider the group of automorphisms \( \text{Aut}(\mathbb{P}^1_K) \cong \text{PGL}_2(K) \) of the projective line over \( K \), which we will identify with the projective linear group of matrices via the usual isomorphism. Recall that there is a natural bijection between \( \mathbb{P}^1(K)^{<3>} := \{(p_1, p_2, p_3) \in \mathbb{P}^1(K)^3/p_1 \neq p_2 \neq p_3 \neq p_1\} = \text{Aut}(\mathbb{P}^1_K), \)
and \( \text{Aut}(\mathbb{P}^1_K) \), given by sending \( \varphi \in \text{Aut}(\mathbb{P}^1_K) \) to the triple \((\varphi(0), \varphi(1), \varphi(\infty))\). The group \( \text{Aut}(\mathbb{P}^1_K) \) acts on \( \mathbb{P}^1(K)^{<3>} \) by the usual left action \( \phi(\varphi) = \varphi \circ \phi \), which is clearly transitive. Explicitly it is given by, for any \( \tau \in \text{Aut}(\mathbb{P}^1_K), \tau(t(p_1, p_2, p_3)) := t(\tau(p_1), \tau(p_2), \tau(p_3)) \). In this section we will show that this action descends to an action of \( \text{Aut}(\mathbb{P}^1_K) \) on \( T_K \) via the \( t \)-map, and that this action gives an identification of \( T_K \) as an analogous of the Bruhat-Tits (\( \Lambda \)-)tree of \( K \) (with respect to the valuation).

To understand how the action descends it is natural to give an alternative description using balls. But it is clearly not true that \( \tau(B) \) is a ball for any ball \( B \) and for any \( \tau \in \text{Aut}(\mathbb{P}^1_K) \), as it shows the example of \( \tau(t) = 1/t \) and \( B = B(0, 1) \), since \( \infty \in \tau(B) \). The following shows that this is in fact the only obstruction.

If \( \gamma \in \text{Aut}(\mathbb{P}^1_K) \), we denote by \( \gamma'(p) := (bc - ad)(cp + d)^{-2} \) the derivative of \( \gamma(t) := (at + b)/(ct + d) \) with respect to \( t \) applied to \( p \). Denote also \( \gamma'(\infty) = (bc - ad)e^{-2} \).

**Lemma 3.1.** Consider \( \gamma \in \text{Aut}(\mathbb{P}^1_K), \gamma \neq \text{id}, p \in K \) and \( \delta \in \Lambda \). Suppose that \( \infty \notin \gamma(B(p, \delta)) \). Then
\[
|\gamma(p) - \gamma(q)| = |\gamma'(p)||p - q|
\]
for all \( q \in B(p, \delta) \).

**Proof.** First note that, if \( \gamma(t) = (at + b)/(ct + d) \), then
\[
\gamma(p) - \gamma(q) = (p - q)\gamma'(p)(\frac{CP + D}{CQ + D}).
\]
Now, the condition \( \infty \notin \gamma(B(p, \delta)) \) is equivalent to \( |CP + D| > \delta|C| \). But then \( |CP + D| = |CQ + D| \) since \( |(CP + D) - (CQ + D)| = |C||p - q| \leq |C|\delta < |CP + D| \).
\( \square \)

**Corollary 3.2.** Consider \( \gamma \in \text{Aut}(\mathbb{P}^1_K), \gamma \neq \text{id}, p \in K \) and \( \delta \in \Lambda \).

1. Suppose that \( \infty \notin \gamma(B(p, \delta)) \). Then \( \gamma(B(p, \delta)) = B(\gamma(p), |\gamma'(p)|\delta) \).
2. Suppose that \( \infty \in \gamma(B(p, \delta)) \). Then \( \gamma(B(p, \delta)) = B(\gamma(p), |\gamma'(\infty)|\delta^{-1}) \).

**Proof.** First of all, note that, if \( \gamma(t) = (at + b)/(ct + d) \), then \( \infty \notin \gamma(B(p, \delta)) \) is equivalent to \( |CP + D| > \delta|C| \).

Suppose first that \( \infty \notin \gamma(B(p, \delta)) \). If \( q \in B(p, \delta) \), then by lemma 3.1
\[
|\gamma(p) - \gamma(q)| = |\gamma'(p)||p - q| \leq |\gamma'(p)|\delta
\]
which shows that \( \gamma(q) \in B(\gamma(p), |\gamma'(p)|\delta) \). On the other side, we first show that \( \infty \notin \gamma^{-1}(B(\gamma(p), |\gamma'(p)|\delta)) \), or, equivalently, that \( a/d = \gamma(\infty) \notin B(\gamma(p), |\gamma'(p)|\delta) \).

But
\[
|\gamma(p) - \gamma(\infty)| = |\gamma'(p)|\frac{|CP + D|}{|C|} > \delta.
\]
Hence we can apply again lemma 3.1 and we get that for any \( q' \in B(\gamma(p), |\gamma'(p)|\delta) \),
\[
|\gamma^{-1}(q') - p| = |(\gamma^{-1})'(\gamma(p))||q' - \gamma(p)| \leq |(\gamma^{-1})'(\gamma(p))||\gamma'(p)|\delta = \delta
\]
by chain’s rule, which shows 1.
The second assertion is shown with analogous arguments. If $\infty \in \gamma(B(p, \delta))$, then
\[ |\gamma(p) - \gamma(\infty)| = |\gamma'(\infty)| \frac{|c|}{|cp + d|} \leq |\gamma'(\infty)| \delta^{-1} \]
and the same works for any $q \in B(p, \delta)$, so $\gamma(B(p, \delta)) \subset B(\gamma(\infty), |\gamma'(\infty)| \delta^{-1})$.
Now $\gamma^{-1}(\infty) \in B(p, \delta)$ since
\[ |\gamma^{-1}(\infty) - p| = \frac{|cp + d|}{|c|} \leq \delta \]
by hypothesis. Hence $B(p, \delta) = B(\gamma^{-1}(\infty), p)$. But for any $q \in B(\gamma(\infty), |\gamma'(\infty)| \delta^{-1})$, so $|cq - a| \geq |c| |\gamma'(\infty)| \delta^{-1}$, we get that
\[ |\gamma^{-1}(q) - \gamma^{-1}(\infty)| = |(\gamma^{-1})'(\infty)| \frac{|c|}{|cq - a|} \leq \frac{|(\gamma^{-1})'(\infty)|}{|\gamma'(\infty)| \delta^{-1}} = \delta \]
which so the reverse inclusion.

\textbf{Proposition 3.3.} For any $(p_1, p_2, p_3) \in \mathbb{P}^1(K)^{<3>}$ and for any $\gamma \in \text{Aut}(\mathbb{P}_K^1)$, let $B = B(p, \delta)$ be the ball representing $(p_1, p_2, p_3)$.

1. If $\infty \notin \gamma(B)$, then $\gamma(B)$ represents $t(\gamma(p_1), \gamma(p_2), \gamma(p_3))$.
2. If $\infty \in \gamma(B)$, then $B(\gamma(\infty), |\gamma'(\infty)| \delta^{-1})$ represents $t(\gamma(p_1), \gamma(p_2), \gamma(p_3))$.

As a consequence the action of $\text{Aut}(\mathbb{P}_K^1)$ on $\mathbb{P}^1(K)^{<3>}$ descents to an action on $T_K$.

\textbf{Proof.} We can and will suppose $(p_1, p_2, p_3)$ are in ball position, so $|p_1 - p_2| \leq |p_1 - p_3| = |p_2 - p_1|$, and $B = B(p_1, |p_1 - p_2|)$. We denote as above $\gamma(t) = (at + b)/(ct + d)$.

Suppose first $\infty \notin \gamma(B)$, so $\gamma(B) = B(\gamma(p_1), |\gamma'(p_1)| |p_1 - p_2|)$. Also we have by lemma \[3.1\] $|\gamma(p_1) - \gamma(p_2)| = |\gamma'(p_1)||p_1 - p_2|$.
But
\[ |\gamma(p_1) - \gamma(p_3)| = |p_1 - p_3||\gamma'(p_1)| \frac{|cp_1 + d|}{|cp_3 + d|} \]
is equal to $|\gamma(p_2) - \gamma(p_3)|$, since $|\gamma'(p_1)| = |\gamma'(p_2)|$ and $|cp_1 + d| = |cp_2 + d|$. Hence $\gamma(p_1), \gamma(p_2), \gamma(p_3)$ are in ball position and $B(\gamma(p_1), |\gamma(p_1) - \gamma(p_2)|) = \gamma(B)$.

Now, if $\infty \in \gamma(B)$, so $\gamma(B) = B(\gamma(\infty), |\gamma'(\infty)| \delta^{-1})$. Denote $p' := |\gamma'(\infty)| \delta^{-1}$, and we want to show that $B(p', p')$ represents $t(\gamma(p_1), \gamma(p_2), \gamma(p_3))$.
Note that we have
\[ |\gamma(p_i) - p'| \geq \delta' \]
for $i = 1, 2$. We divide the proof in two cases.

Suppose first that $p_3 \notin B(p_1, \delta)$, so $|p_3 - p_2| > |p_2 - p_1| = \delta$. Then
\[ |cp_3 + d| = |c| |p_3 - p_1| > |c| \delta \geq |cp_1 + d| \]
hence $|\gamma(p_3) - \gamma(\infty)| < \delta'$, thus $\gamma(p_3) \in B(p', \delta')$. Now, observe that either $|cp_1 + d| = |c| \delta$ or $|cp_2 + d| = |c| \delta$, since $|(cp_1 + d) - (cp_2 + d)| = |c| \delta$ and both are $\leq |c| \delta$. We can and will suppose it is $p_1$. But then
\[ |\gamma(p_3) - \gamma(p_1)| = |\gamma'(\infty)| \frac{|c|}{|cp_1 + d|} = \delta' \]
and $|\gamma(p_3) - \gamma(p_2)| \geq \delta'$, which shows that $\gamma(p_3), \gamma(p_1)$ and $\gamma(p_2)$ are in ball position and $B(\gamma(p_3), |\gamma(p_3) - \gamma(p_1)|) = B(p', \delta')$.

Now, suppose that $p_3 \in B(p_1, \delta)$, hence $|p_3 - p_2| = |p_1 - p_2| = |p_3 - p_1| = \delta$. Same arguments as in the previous case show that there exists $i$ such that $|cp_i + d| \leq \delta|c|$ and $|c_p + d| \leq \delta|c|$ for $j \neq i$. We can and will suppose that $i = 1$. Then one shows that $|\gamma(p_1) - \gamma(\infty)| = \delta'$, and the same for $p_2$, that $|\gamma(p_1) - \gamma(p_2)| = \delta'$, and finally that $|\gamma(p_2) - \gamma(p_1)| = |\gamma(p_2) - \gamma(p_1)| \geq \delta'$, which implies the result. \qed
or \( B^\gamma = B(\gamma(\infty), |\gamma'(\infty)|\delta(B)^{-1}) \) if it is not. The automorphisms of \( \text{Aut}(\mathbb{P}_K^1) \) preserve the distance between balls, using definition, hence they are isometries.

**Lemma 3.4.** For any pair of balls \( B \) and \( B' \), and an automorphism \( \gamma \in \text{Aut}(\mathbb{P}_K^1) \), we have \( d(B^\gamma, B'^\gamma) = d(B, B') \).

**Proof.** Suppose first that \( B' = B(p, \delta') \subset B = B(p, \delta) \). If \( \gamma(B) \) is a ball, then we have \( \gamma(B') \subset \gamma(B) \). Corollary 3.2 asserts that \( \varrho(\gamma(B)) = |\gamma'(p)|\delta \) and \( \varrho(\gamma(B')) = |\gamma'(p)|\delta' \). Then

\[
d(\gamma(B'), \gamma(B)) = \varrho(\gamma(B))\varrho(\gamma(B'))^{-1} = \frac{|\gamma'(p)|\delta}{|\gamma'(p)|\delta'} = \frac{\delta}{\delta'} = d(B, B').
\]

If \( \gamma(B') \) is not a ball, applying corollary 3.2 we get that \( B'^\gamma = B(\gamma(\infty), |\gamma'(\infty)|\delta'^{-1}) \subset B' = B(\gamma(\infty), |\gamma'(\infty)|\delta^{-1}) \), hence \( d(B', B'^\gamma) = \delta/\delta' = d(B, B') \) as above.

If \( \gamma(B) \) is not a ball, but \( \gamma(B') \) it is, then \( B^\gamma = B(\gamma(\infty), \delta') \), where \( \delta' = |(\gamma')'(\infty)|\delta^{-1} \) by corollary 3.2. Now, \( \gamma(B') \cap B'^\gamma = B(\gamma(p), |\gamma(p) - \gamma(\infty)|) \), and

\[
d(\gamma(B'), B'^\gamma) = d(B(\gamma(p), |\gamma(p) - \gamma(\infty)|), \gamma(B'))d(B(\gamma(p), |\gamma(p) - \gamma(\infty)|), B'^\gamma) = \frac{|\gamma(p) - \gamma(\infty)|^2}{|\gamma'(p)||\gamma'(\infty)||\delta'|}.
\]

But one easily shows that \( |\gamma(p) - \gamma(\infty)|^2 = |\gamma'(p)||\gamma'(\infty)| \). \( \square \)

We denote by \( \varpi(\varphi) = t(\varphi(0), \varphi(1), \varphi(\infty)) \). The equivalence relation determined by \( \varphi \sim \varphi' \) when \( \varpi(\varphi) = \varpi(\varphi') \) is thus determined by the stabilizer of an element, say \( t_0 := t(0,1,\infty) = O \).

**Theorem 3.5.** An automorphism \( \varphi \in \text{Aut}(\mathbb{P}_K^1) \) stabilizes \( t_0 \) if and only if \( \varphi \in \text{Aut}(O) \).

**Proof.** Denote by \( \Gamma_0 \subset \text{Aut}(\mathbb{P}_K^1) \) the stabilizer of \( t_0 \). Observe that an automorphism \( \varphi \in \text{Aut}(\mathbb{P}_K^1) \) is in fact \( \varphi \in \text{Aut}(O) \) if and only if it can be written as \( \varphi(t) = (at + b)/(ct + d) \) for \( a, b, c, d \in O \) with \( |ad - bc| = 1 \).

First of all, observe that the automorphisms \( \tau \) that fix the set \( \{0,1,\infty\} \) are both in \( \Gamma_0 \) and also in \( \text{Aut}(O) \). If we compose an automorphism \( \psi \) with one such \( \tau \) in order to obtain an automorphism \( \gamma = \tau \circ \psi \) that \( \gamma(0), \gamma(1) \) and \( \gamma(\infty) \) are ball ordered, then \( \gamma(0), \gamma(1), \gamma(\infty) = t(\gamma(0), \gamma(1), \gamma(\infty)) \), and \( \gamma \in \Gamma_0 \) (respectively \( \psi \in \text{Aut}(O) \)) if and only if \( \gamma \in \Gamma_0 \) (respectively \( \gamma(0) \in \text{Aut}(O) \)).

So we are reduced to consider only the case that \( \gamma \) verifies that \( \gamma(0), \gamma(1) \) and \( \gamma(\infty) \) are ball ordered, which we will say that \( \gamma \) is ball suited.

We will show first that \( \Gamma_0 \subset \text{Aut}(O) \). We decompose a ball suited \( \gamma \) as composition of two automorphisms: the automorphism \( \gamma_0 \) which sends \( \gamma_0(0) = \gamma(0), \gamma_0(1) = \gamma(1) \) and \( \gamma_0(\infty) = \gamma(\infty) \), and the diagonalizable automorphism \( \gamma_1 \), sending \( \gamma_1(\gamma_0(0)) = \gamma_0(0), \gamma_1(\gamma_0(1)) = \gamma_1(1) \) and \( \gamma_1(\gamma_0(\infty)) = \gamma_1(\infty) \). Hence \( \gamma_1 \) has two fixed points, \( \gamma_1(0) \) and \( \gamma_1(1) \).

We will see that the \( \gamma_1 \in \Gamma_0 \) if \( \gamma_1 \in \Gamma_0 \) for \( i = 1, 2, \) and that \( \gamma \in \Gamma_0 \) if and only if \( \gamma_1 \in \Gamma_0 \) for \( i = 0 \) and 1. Note that \( \Gamma_0 \) and \( \text{Aut}(O) \) are subgroups, hence, if \( \gamma_1 \) are in one of them for \( i = 0 \) and 1, so it is their composition.

Now, if \( \gamma \in \Gamma_0 \), then \( \gamma_0(t_0) = t(\gamma(0), \gamma(1), \gamma(\infty)) = t(\gamma(0), \gamma(1), \gamma(\infty)) = t_0 \), being \( \gamma \) ball suited. Hence \( \gamma_0 \in \Gamma_0 \), and thus \( \gamma_1 \in \Gamma_0 \).

Therefore we are reduced to show the result for the automorphisms of the type \( \gamma_0 \) and \( \gamma_1 \). The first case is easy; one has \( \gamma_0(t) = (at + b)/(ct + d) \) and \( b = \gamma(0) \). Since \( \gamma_0 \) is ball suited, \( t(\gamma(0), \gamma(1), \gamma(\infty)) = t_0 \) if and only if \( |a| = 1 \) and \( |b| \leq 1 \), which happens exactly when \( \gamma_0 \in \text{Aut}(O) \).

We consider now the second case of diagonalizable automorphisms of the type \( \gamma_1 \), with fixed points \( p_0 \) and \( p_1 \), and \( \gamma_1 \in \Gamma_0 \), hence \( t(p_0, p_1, \infty) = t_0 \). Therefore \( |p_0| \leq 1, |p_1| = 1 \) and \( |p_0 - p_1| = 1 \). Take \( \tau \in \text{Aut}(O) \) such that \( \tau(0) = p_0 \), \( \tau(\infty) = p_1 \) and \( \tau(1) = \infty \). Explicitly \( \tau(t) = (p_0 t - p_1)/(t - 1) \), which is clearly
in $\text{Aut}(\mathbb{P}^1_K)$ since $|p_1 - p_0| = 1$. Moreover $\tau \in \Gamma_0$ since $\tau(t_0) = t(p_0, \infty, p_1) = t_0$. Then $\psi := \tau^{-1}\gamma\tau \in \Gamma_0$ verifies $\psi(0) = 0$ and $\psi(\infty) = \infty$, so $\psi(t) = qt$ for some $q \in K^*$. But then $\psi \in \Gamma_0$ if and only if $1 = |\psi(1)| = |q|$. Hence $\psi \in \text{Aut}(\mathbb{P}^1_O)$, so $\gamma_1 = \tau\psi\tau^{-1}$ also.

Finally, suppose $\gamma(t) = (at + b)/(ct + d) \in \text{Aut}(\mathbb{P}^1)$ with $a, b, c$ and $d \in \mathcal{O}$ and $|ad - bc| = 1$ and we want to show $\gamma(t_0) = t_0$. If $\infty \notin \gamma(B(0,1))$, equivalently $|d| > |c|$, so $|d| = |a| = 1$, then $|\gamma(0)| = |c/d| < 1$ and $|\gamma'(0)| = |d|^{-2} = 1$. Therefore $\gamma(B(0,1)) = B(\gamma(0), 1) = B(0, 1)$ by corollary 3.2. If, on the contrary, $\infty \in \gamma(B(0,1))$, i.e. $|d| \leq |c|$, so $|c| = 1$, then $\gamma(\infty) = a/c \in \mathcal{O}$ and $|\gamma'(\infty)| = |c|^{-2} = 1$. Therefore, using again the same corollary, $\gamma(B(0,1)) = B^c(0,1)$. Proposition 3.3 implies the result.

The following corollary is a well-known consequence of the transitivity of the action of $\text{Aut}(\mathbb{P}^1_K)$ on $T_K$ and the previous theorem.

**Corollary 3.6.** The map $\varpi : \text{Aut}(\mathbb{P}^1_K) \to T_K$ determines a bijection $\text{Aut}(\mathbb{P}^1_K) \setminus \text{Aut}(\mathbb{P}^1_K) \cong T_K$.

4. The tree associated to a compact set

Recall a compact subset $\mathcal{L}$ of a topological space is one such that any covering by open subsets has a finite subcovering. In our case we have an easier criterium.

**Lemma 4.1.** A subset $\mathcal{L} \subset \mathbb{P}^1(K)$ is compact if given a covering $\mathcal{L} \subset \bigcup_{j \in J} \mathcal{C}(p_i, \rho_j, \rho_{j'})$, of annulus of $\mathbb{P}^1(K)$, exists finite subset $J \subset I$ with $\mathcal{L} \subset \bigcup_{j \in J} \mathcal{C}(p_i, \rho_j, \rho_{j'})$.

**Definition 4.2.** Let $\mathcal{L} \subset \mathbb{P}^1(K)$ be with at least three elements. We define the $\Lambda$-subtree associated to $\mathcal{L}$ as

$$\mathcal{T}(\mathcal{L}) = \bigcup_{p_1, p_2 \in \mathcal{L}} \pi(p_1, p_2)$$

Given the $\Lambda$-tree $\mathcal{T}(\mathcal{L})$ we will construct a (simplicial) graph as follows: the set of vertices is

$$\mathcal{V}(\mathcal{L}) = \{t(p_1, p_2, p_3) \mid (p_1, p_2, p_3) \in \mathcal{L}^3 \cap \mathbb{P}^1(K)^{<3}>\} \subset \mathcal{T}(\mathcal{L})$$

Given $v_1, v_2 \in \mathcal{V}(\mathcal{T}(\mathcal{L}))$, we say that they determine an edge $[v_1, v_2]$ with ends $v_1$ and $v_2$ if $[v_1, v_2] \cap \mathcal{V}(\mathcal{L}) = \{v_1, v_2\}$.

We denote the set of edges as $\mathcal{E}(\mathcal{L})$. The (simplicial) graph they determine will be denote by $T(\mathcal{L})$.

Note that, if $\mathcal{L}$ is finite, then the graph $T(\mathcal{L})$ is a tree. This is because, given any two vertices $v_1$ and $v_2$ of the graph, the segment $[v_1, v_2]_{\Lambda}$ can be subdivided in a finite number of edges.

Note that, if $\mathcal{L}$ contains exactly three points, then $T(\mathcal{L})$ has only one vertex and no edge, whereas if it has four points, then it has either one vertices or two vertices and one edge. In general the number of vertices is bounded by $\#\mathcal{L} - 2$ and the number of edges by $\#\mathcal{L} - 3$. The following lemma shows this result by induction and it will be useful later.

**Lemma 4.3.** Given $\mathcal{L} \subset \mathbb{P}^1(K)$ any subset, the inclusion $\mathcal{L} \subset \overline{\mathcal{L}}$ into the closure gives natural bijections $\mathcal{T}(\mathcal{L}) = \mathcal{T}(\overline{\mathcal{L}})$ and $T(\mathcal{L}) = T(\overline{\mathcal{L}})$.

If now $\mathcal{L}$ is closed and $p \in \mathbb{P}^1(K) \setminus \mathcal{L}$, then $\mathcal{V}(\mathcal{L}) \cup \{v_p\} = \mathcal{V}(\mathcal{L} \cup \{p\})$ for a vertex $v_p$ (which may or may not be in $\mathcal{V}(\mathcal{L})$).
Proof. Given $p_1$ and $p_2 \in \overline{L}$, first we will suppose that $[B_1, B_2]_A \subset T(\mathcal{L})$. We can and will suppose that $B_1 = B(p_1, \delta_1)$ and $B_2 = B(p_2, \delta_2)$, with $\delta_1$ and $\delta_2 \leq |p_1 - p_2|$. Then there exists $p'_1 \in \mathcal{L} \cap B(p_1, \delta_1)$ and $p'_2 \in \mathcal{L} \cap B(p_2, \delta_2)$, and then $[B_1, B_2]_A \subset \pi(p'_1, p'_2) \subset T(\mathcal{L})$.

Now, suppose moreover than $B_1 \subset V(\mathcal{L})$, so $B_1 = t(p_1, p_2, p_3)$ for some $p_1, p_2$ and $p_3 \in \overline{L}$. Taking as before three points $p'_1 \in \mathcal{L}$ sufficiently close to $p_i$ for all $i = 1, 2, 3$, we have $t(p_1, p_2, p_3) = t(p'_1, p'_2, p'_3) \in V(\mathcal{L})$.

If one of the points is equal to $\infty$, one adapts the argument by using complements of open balls.

Finally, to show the last assertion, suppose $p \neq \infty$. Then $v_p = B(p, \delta)$, where $\delta := \sup \{ \epsilon \in \Lambda \mid B(p, \epsilon) \cap \mathcal{L} = \emptyset \}$.

We will show that $T(\mathcal{L})$ is a tree for any compact subset.

**Theorem 4.4.** Let $\mathcal{L} \in \mathbb{P}^1(K)$ be a compact subset. Then $[v_1, v_2]_A \cap V(\mathcal{L})$ is finite for any $v_1$ and $v_2 \in V(T(\mathcal{L}))$.

**Proof.** We will suppose $\infty \in \mathcal{L}$, since if it is not then $\mathcal{L} \cup \{ \infty \}$ would be also compact, and $V(\mathcal{L}) \subset V(\mathcal{L} \cup \{ \infty \})$ (and it fact it contains at most one more vertex). Given two vertices $v_1$ and $v_2$, we denote by $v_1 \vee v_2$ the element in $T_K$ corresponding to the minimal ball containing both. Since $\infty \in \mathcal{L}$, $v_1 \vee v_2 \in V(\mathcal{L})$.

Then $[v_1, v_2]_A = [v_1, v_1 \vee v_2]_A \cup [v_1 \vee v_2, v_2]_A$

hence we are reduced to show only the case that $v_1 \leq v_2$ with respect to the partial order of $T_K$.

Then $v_1 = B(p, \rho_1) \subset B(p, \rho_2) = v_2$ and

$$[v_1, v_2]_A \cap V(T(\mathcal{L})) = \{ B(p, \rho) \mid \rho_1 \leq \rho \leq \rho_2 \}$$

where $B(p, \rho) = t(p, q, \infty)$ for some $q \in \mathcal{L}$. We want to see that there are a finite number of such $q$. Each $q$ is in the circle $C(p, \rho)$ for $\rho_1 \leq \rho \leq \rho_2$. We have

$$\mathcal{L} \subset B(p, \rho_1) \cup B^c(p, \rho_2) \cup \bigcup_{\rho_1 < \rho < \rho_2} C(p, \rho)$$

with $C(p, \rho) \cap \mathcal{L} \neq \emptyset$. The sets are disjoint and open, so there is a finite number of them.

**Corollary 4.5.** The graph $T(\mathcal{L})$ is a tree.

**Proof.** We need to show it is connected. But given two vertices $v$ and $v'$, we have

$$[v, v']_A \cap V(\mathcal{L}) = \{ v = v_0, v_1, \ldots, v_n, v_{n+1} = v' \} = [v, v_1] \cup \cdots \cup [v_n, v']$$

for some $n \geq 0$, and any of these $[v_i, v_{i+1}]$ are edges. Clearly this is the unique path from $v$ to $v'$, hence it is a tree.

Recall that the star of a vertex $v \in V(\mathcal{L})$ is $\text{Star}_{T(\mathcal{L})}(v) = \{ [v, w] \in E(T(\mathcal{L})) \}$. A graph is called **locally finite** if $\text{Star}_{T(\mathcal{L})}(v)$ is finite for all $v \in V(T)$.

**Theorem 4.6.** The tree $T(\mathcal{L})$ is locally finite.

**Proof.** As in the proof of theorem 4.3, we will construct a covering of our compact set $\mathcal{L}$ by non-empty and disjoint open sets indexed by $\text{Star}_{T(\mathcal{L})}(v) = \{ [v, w] \in E(\mathcal{L}) \}$, at least when $\mathcal{L}$ has no isolated points. Let $\mathcal{L}'$ be the set of isolated points of $\mathcal{L}$; since $\mathcal{L}$ is compact, $\mathcal{L}'$ is finite. Consider $\mathcal{L}' := \mathcal{L} \setminus \mathcal{L}'$, which is also compact. Then Lemma 4.3 allows us to show that $T(\mathcal{L})$ is locally finite if and only if $T(\mathcal{L}')$ is locally finite. So we can and will suppose that $\mathcal{L}$ has no isolated points (it is perfect).
Given a vertex \( v \in V(\mathcal{L}) \), denote by \( B_v \) the corresponding closed ball. Fix a vertex \( v \), and consider the set of balls \( B_w \) corresponding to the vertices \( w \in S_v \), where \( \text{Star}_{T}(v) = \{ [v, w] : w \in S_v \} \). Then, either \( B_w \) are all disjoint or there exists a \( w_0 \) such that \( B_w \subset B_{w_0} \) for all \( w \) and the rest of \( B_w \) are disjoint. In fact, if \( B_w \) and \( B_{w'} \) are not disjoint and both are contained in \( B_v \), then one is inside the other, and hence one is inside the path from \( B_v \) to the other, so it does not form and edge. If \( B_w \) and \( B_{w'} \) contain both \( B_v \), then one is contained in the other, and the same argument applies. Note that the first case happens exactly when \( B_w \) contains \( \mathcal{L} \).

Now, in the first case, we consider the set \( \bigcup_{w \in S_v} B_w \), while in the second case we take
\[
\bigcup_{w \in S_v, w \neq w_0} B_w \cup B_{w_0}^c.
\]
We are going to see that they are coverings of \( \mathcal{L} \).

Given any point \( p \in \mathcal{L} \cap B_v \), consider a ball \( B(p, \delta) \not\subset B_v \). This ball contains infinite points of \( \mathcal{L} \) (since \( \mathcal{L} \) has no isolated points), hence we can take three of them, \( p_1 = p, p_2 \) and \( p_3 \in \mathcal{L} \cap B(p, \delta) \). The corresponding vertex \( v' = (p, p_2, p_3) \) is in \( V(\mathcal{L}) \) and hence \( [v, v'] \) contains a vertex \( w \in S_v \), and then \( p \in B_w \). Now, if \( p \in \mathcal{L} \setminus B_v \), repeat the same argument with the complement of an open ball \( B(p, \delta) \) such that \( B(p, \delta) \cap B_v = \emptyset \). □

Finally, we will show that, if \( \mathcal{L} \) is compact and perfect, then \( \mathcal{L} \) can be identified as the set of ends of \( T(\mathcal{L}) \). In fact, the bijection is an homeomorphism when considering the ends with the natural topology.

Recall that a ray on a tree \( T = (V, E) \) is an infinite sequence \( v_0, v_1, \ldots \) of vertices such that \( [v_i, v_{i+1}] \) is an edge and \( v_i \neq v_j \) \( \forall i \not= j \). Given a progression \( v_0, v_1, \ldots, v_n, \ldots \) of distinct vertices we say they generate a ray if the progression formed by the ordered set \( \bigcup_{i \geq 0} V \setminus [v_i, v_{i+1}] \) is a ray, which we call the ray generated by the \( v_n \)'s. We denote \( \text{Rays}(T) \) the set of rays of \( T \). Now the ends of a tree \( T \) is the set of equivalence classes of rays with respect the equivalence relation \( \sim \), where \( r = (v_0) \sim s = (w_0) \) if and only if \( r \cap s \) is given ray.

The set of ends \( \text{Ends}(T) := \text{Rays}(T) / \sim \) has a natural topology which has as a subbasis the following sets: for any oriented (i.e. ordered) edge \( e = [v_0, v_1] \), we denote
\[
\mathcal{B}(e) := \{ r \in \text{Rays}(T) | e \subset r \} \sim \subset \text{Ends}(T)
\]
where \( e \subset r \) is as ordered sets. Note that \( \mathcal{B}(\bar{e}) = \mathcal{B}(e)^c \) for \( \bar{e} = [v_1, v_0] \) is the opposed edge.

**Lemma 4.7.** Consider \( [v_0, v_1] \) and \( [v_1, v_2] \) edges of \( T(\mathcal{L}) \), with \( v_0 \neq v_2 \). Each vertex corresponds to a ball denoted \( B_0, B_1 \) and \( B_2 \) respectively. Then if \( B_0 \cap B_2 = \emptyset \) we have that \( B_1 \) contains \( B_0 \) and \( B_2 \); and if \( B_0 \cap B_2 \neq \emptyset \) either \( B_2 \subset B_1 \subset B_0 \) or \( B_0 \subset B_1 \subset B_2 \).

**Proof.** If \( B_0 \) and \( B_2 \) are disjoint but they are linked to the same vertex then they must be in the ball corresponding to this vertex. If \( B_0 \cap B_2 \neq \emptyset \), let \( p \) be in the intersection. Since they are linked to \( B_1 \), \( p \) is also in \( B_1 \), so the possibilities are either \( B_2 \subset B_1 \subset B_0 \) or \( B_0 \subset B_1 \subset B_2 \).

A direct consequence of this fact is that if \( (B_0, B_1, \ldots) \) is a ray then either there exists \( m \geq 0 \) such that \( B_i \subset B_{i+1} \) for all \( i \geq m \), or \( B_{i+1} \subset B_i \) for all \( i \geq 0 \). In fact, once you find two balls in the sequence such that \( B_j \subset B_i \) then \( B_{i+1} \subset B_i \) for all \( i \geq j \).
Proposition 4.8. If, given m ≥ 0, Bi+1 ⊂ Bi for all i ≥ m, where the balls corresponds to the vertex of a ray defined as above then

\( \bigcap_{i \geq m} (B_i \cap \mathcal{L}) = \{p\} \), where \( p \in \mathcal{L} \).

Proof. For any \( i, B_i \cap \mathcal{L} \neq \emptyset \), where \( B_i = t(p, p', p'') \) for \( p, p', p'' \in \mathcal{L} \) and two of them are in \( B_i \). In fact \( (B_i \cap \mathcal{L}) \setminus (B_{i+1} \cap \mathcal{L}) \neq \emptyset \), so since \( B_i \cap \mathcal{L} \) are closed in \( \mathcal{L} \) and non empty

\[ \bigcap_{i \geq m} B_i \cap \mathcal{L} \neq \emptyset. \]

Now we have to see that in the intersection there is a unique point. Suppose \( p_1, p_2 \) different in \( \bigcap_{i \geq m} B_i \cap \mathcal{L} \). Then we take \( p_3 \in B_m \cap \mathcal{L} \) different from \( p_1 \) and \( p_2 \). Therefore \( v = t(p_1, p_2, p_3) \in V(\mathcal{L}) \). But note that \( v_i \in [v_m, v] \) for any \( i \geq m \) because \( B_m \supset B_i \supset B(p_1, [p_1, p_2]) \). But this is not possible because we know that \( \#[v_m, v] \cap V(\mathcal{L}) < \infty \). So \( \bigcap_{i \geq m} B_i \cap \mathcal{L} = \{p\}. \)

Conversely, a sequence of nested balls \( B_i \supset B_{i+1} \) for all \( i > 0 \) generate a ray if they intersection \( \bigcap (B_i \cap \mathcal{L}) \) is a point.

Theorem 4.9. Let \( \mathcal{L} \) be a compact subset of \( P^1(K) \). Then there is a well defined map

\[ \Psi : \text{Rays}(T(\mathcal{L})) \rightarrow \mathcal{L} \]

whose image is the set of non isolated points \( \mathcal{L}' \subset \mathcal{L} \). The map \( \Psi \) determines an homeomorphism between the space of ends and \( \mathcal{L}' \).

Proof. If \( r = (v_1, v_2, \ldots) \) with corresponding balls verifying \( B_{i+1} \subset B_i \) for some \( i \), then \( B_{i+1} \subset B_i \) for all \( i > m \) and Proposition\[^1\] implies that \( \bigcap_{i \geq m} (B_i \cap \mathcal{L}) = \{p\} \). We define then \( \Psi(r) = p \). In the other case we have \( B_{i+1} \supset B_i \) for all \( i \) and we define \( \Psi(r) = \infty \).

If \( p \neq \infty \) is in \( \text{Im}(\Psi) \), so \( p = \Psi(r) \) with \( r = (v_1, v_2, \ldots) \), then \( v_i = B_i = B(p_{v_0}, p_v) \) for some \( p_{v_0} \neq p \) and \( p - p_{v_0} = p_v \). Since \( B_j \subset B_i \) for all \( j > i > m \), for some \( m \), we have \( p_{v_0} \neq p_v \) and \( |p - p_{v_0}| = p_v \) for some \( m \) and \( \Psi(p) \) tends to \( \infty \) by Proposition\[^1\].

Moreover, any non-isolated point \( x \in \mathcal{L} \) is in the image of \( \Psi \), since, if \( x_i \in \mathcal{L} \), \( x_i \neq x_j \) for \( i \neq j \) and \( \lim x_i = x \), then \( v_i := t(x_i, x, x) \) for \( i > 1 \) large enough generate a ray. To show this, suppose \( x \neq \infty \) (the case \( x = \infty \) is done by an analogous argument). Then for \( i \) large enough, \( v_i \) corresponds to a ball \( B_i \) around \( x \) and \( B_i \supset B_{i+1} \) since \( x \) converge to \( x \).

Now, it is clear that two rays \( r_1 \) and \( r_2 \) have the same image if they are equivalent, since \( \Psi \) only depends of a tail of the ray. Moreover, if two rays have image \( p \), this means that \( p \) is inside the balls of both rays (for large enough index), so they must be equivalent. That the map \( \Psi \) determines an homeomorphism is clear from the given description.

Corollary 4.10. If \( \mathcal{L} \) is compact and perfect, then \( T(\mathcal{L}) \) is a locally finite tree with all vertices of valence strictly bigger than 2 and \( \Psi \) is surjective.

Proof. Only the assertion on the valence of any vertex needs a comment. Let \( v \) be a vertex of \( T(\mathcal{L}) \), corresponding to a ball \( B(p, \Lambda) = t(p, p', p'') \) for some \( p, p', p'' \in \mathcal{L} \) with \( \delta = |p-p'| \). Take \( \epsilon < \delta \) in \( \Lambda \). Then, since \( \mathcal{L} \) is perfect, \( B(p, \epsilon) \cap \mathcal{L} \) contains another point \( r \in \mathcal{L} \), and \( t(p, r, p'') \neq t(p, p', p'') \). Similarly, there exists \( p' \neq r' \in \mathcal{L} \cap B(p', \epsilon) \) and moreover \( B(p', \epsilon) \cap B(p, \epsilon) = \emptyset \), and even with \( p'' \) (with some minor changes in the case that \( p'' = \infty \)). So we have vertices \( v', v'' \) and \( v''' \) connected with disjoint paths to \( v \), which means \( v \) has valence 3 or larger.
5. Hyperbolic matrices

Given any matrix $A \in \text{GL}_2(K)$, we denote by $\varpi(A) := \frac{\text{Tr}(A)^2}{\det(A)} \in K$. It is easily shown that $\varpi(A)$ does not depend of the class in $\text{PGL}_2(K)$, so it gives a well defined map $\varpi : \text{PGL}_2(K) \to K$. Using the natural isomorphism $\text{Aut}(\mathbb{P}^1_K) \cong \text{PGL}_2(K)$, we will use also $\varpi(\gamma)$ for a given $\gamma \in \text{Aut}(\mathbb{P}^1_K)$.

**Definition 5.1.** Given $\gamma \in \text{Aut}(\mathbb{P}^1_K)$, we say that $\gamma$ is **hyperbolic** if $\varpi(\gamma) \in K^*$ and $\varpi(\gamma)^{-1}$ is topologically nilpotent.

Given any $q \in K^*$, we denote by $\mu_q \in \text{Aut}(\mathbb{P}^1_K)$ the automorphism given by $\mu_q(x) = qx$ for all $x \in K$.

**Proposition 5.2.** Let $\gamma \in \text{Aut}(\mathbb{P}^1_K)$ be any automorphism. Then $\gamma$ is hyperbolic if and only if there exists $\tau \in \text{Aut}(\mathbb{P}^1_K)$ such that $\tau \gamma \tau^{-1} = \mu_q$, where $q \in K$ is topologically nilpotent.

**Proof.** Suppose first that $\gamma = \mu_q$ with $q$ topological nilpotent. Let $A$ be the corresponding matrix

$$A = \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}.$$ 

Then $\text{Tr}(A) = q + 1$, $\det(A) = q$, $\varpi(A) = \frac{(q+1)^2}{q}$, so $|\varpi(A)| = |q|^{-1}$ since $|q+1| = 1$.

Now suppose $\gamma$ is hyperbolic, with associated matrix $A \in \text{GL}_2(K)$. We take a representative $A$ with coefficients in $\mathcal{O}_K$. Let $f(x) = x^2 - ax + b$ be the characteristic polynomial of $A$, so $\varpi(A) = \frac{a^2}{b}$. By hypothesis $t = \frac{|f(0)|}{|f'(0)|^2}$ is topologically nilpotent. Using Hensel’s Lemma we see that there exists $\alpha \in \mathcal{O}_K$ with $f(\alpha) = 0$. Therefore there exist also $\beta \in \mathcal{O}_K$ such that $f(x) = (x-\alpha)(x-\beta)$. Note that $\beta \in \mathcal{O}_K$ because $\alpha$ and $\alpha + \beta \in \mathcal{O}_K$.

Moreover $|\alpha| \neq |\beta|$, since by hypothesis

$$\left| \frac{\alpha \beta}{(\alpha + \beta)^2} \right| < 1,$$

so $|\alpha| = |\beta|$ implies $|\alpha|^2 < |2\alpha|^2 \leq |\alpha|^2$ which is a contradiction. Summarizing we have that or $|\frac{\alpha}{\beta}| < 1$ or $|\frac{\beta}{\alpha}| < 1$, hence one of these is equal to $|\varpi(A)^{-1}|$, therefore topologically nilpotent.

Given an hyperbolic automorphism $\gamma \neq \text{id}$, we denote by $q_\gamma \in K^*$ the unique topologically nilpotent element such that $\gamma$ is equal to $\mu_{q_\gamma}$ modulo conjugation. We denote also $\gamma(\gamma) = |q_\gamma| \in \Lambda$.

Hyperbolic automorphisms are specially interesting for us since they don’t fix any element of $T_K$.

**Lemma 5.3.** Let $\gamma \neq \text{id}$ be hyperbolic automorphism. Then $t(p_1, p_2, p_3) \neq t(\gamma p_1, \gamma p_2, \gamma p_3)$.

**Proof.** By conjugation, we can and will suppose that $\gamma = \mu_q$, for $q$ topologically nilpotent. So $t(p_1, p_2, p_3)$ corresponds (reordering if is necessary) to $B(p_1, |p_1 - p_2|)$ but also $t(qp_1, qp_2, qp_3)$ corresponds (reordering if is necessary) to $B(qp_1, |q||p_1 - p_2|)$ and since $|q| < 1$ one has $|q||p_1 - p_2| < |p_1 - p_2|$ so $B(p_1, |p_1 - p_2|) \supset B(qp_1, |q||p_1 - p_2|)$.

In the following we will show some properties that characterize hyperbolic matrices. Recall that a $\gamma \in \text{Aut}(K)$, with $\gamma \neq \text{id}$, either has one or two fixed points over the algebraic closure $\bar{K}$. The hyperbolic automorphisms have two fixed points and defined over $K$. 
Theorem 5.4. Suppose $\gamma \in \text{Aut}(K)$ has two fixed points over $\overline{K}$. Let $\Gamma = \langle \gamma \rangle$ be the subgroup generated by $\gamma$. For any $p \in \mathbb{P}^1(K)$, let $\Gamma_p = \{ \gamma^n p \mid n \in \mathbb{Z} \}$ be the orbit of $p$. Then

$$\overline{\langle \gamma \rangle p} \text{ is compact for all } p \in \mathbb{P}^1(K) \iff \begin{cases} \gamma \text{ is hyperbolic}, \\ \gamma \text{ is of finite order}. \end{cases}$$

Proof. The reverse implication is easy. First note that if $\gamma$ is of finite order it means that $\langle \gamma \rangle p$ is finite, so it is compact. It remains to show that if $\gamma$ is hyperbolic then $\overline{\langle \gamma \rangle p}$ is compact. Since an hyperbolic matrix $\gamma$ is conjugated to $\mu_q$ given by $\mu_q(x) = qx$ with $q$ topologically nilpotent, the closure of the orbit of $\gamma$ will have the same structure as $\mu_q$. Note that

$$\langle \mu_q \rangle p = \begin{cases} \{ p \} \text{ if } p = 0 \text{ or } p = \infty, \\ \{ q^n p \mid n \in \mathbb{Z} \text{ if } p \neq 0, \infty \}. \end{cases}$$

We have to see that the second case is also compact. First note that $\overline{\langle \gamma \rangle p} = \{ q^n p \mid n \in \mathbb{Z} \} \cup \{0, \infty\}$ because $\lim_{n \to \infty} q^n p = 0$ and $\lim_{n \to -\infty} q^n p = \infty$. Let $\Gamma_1 = \bigcup_{i \in \mathbb{Z}} B_i$ be a covering of open balls. Since $0$ is in the covering we have a ball that contains it, which must be of the form $B = B(0, \rho)$. By the same reasoning there is a ball that contains $\infty$ like $B' = B'(0, \rho')^c$. But then $\overline{\langle \gamma \rangle p \cup B \cup B'}$ is finite, since there exists $n_0, n_1 \geq 1$ such that $q^n p \in B(0, \rho)$ for all $n \geq n_0$ and $q^n p \in B'(0, \rho')^c$ for all $n \geq n_1$.

To see the direct implication, observe first that, by extending the field, we can reduce to the case that $\gamma$ has two fixed points defined over $K$. Hence we are reduced to show that the automorphism $\mu_q$, with $|q| \leq 1$ and $q$ not topologically nilpotent, does have a non compact orbit. We will consider the orbit of $1$, i.e. $\Gamma 1 = \{ q^n \mid n \in \mathbb{Z} \}$. Since $q$ is not topologically nilpotent, by Lemma 4.3 there exists $\lambda \in \Lambda$ such that $\lambda^{-1} > |q^n| > \lambda > 0$ for all $n \in \mathbb{Z}$. So, for $n > m$,

$$|q^n - q^m| = |q^m| |q^{n-m} - 1| = |q^m| |1| > \lambda.$$ 

So $\Gamma 1$ can be covered by $\bigcup_{n \in \mathbb{Z}} B(q^n, \lambda)$ which satisfies

$$B(q^n, \lambda) \cap \overline{\langle \gamma^n(1) \mid n \in \mathbb{Z} \rangle} = \{ q^n \}$$

and are pairwise disjoint, so we cannot remove any ball which means that $\Gamma 1$ is compact, and it is closed. Since $|q|^n > \lambda$ for all $n \in \mathbb{Z}$ then $|q|^{-n} < \lambda^{-1}$ for all $n \in \mathbb{Z}$, so $\overline{\Gamma 1} \subset B(0, \lambda^{-1})$. \hfill $\Box$

Note that in the proof of the theorem we have shown the following result.

Corollary 5.5. Let $\gamma \in \text{Aut}(\mathbb{P}^1_K)$ be an hyperbolic automorphism. Let $p \in \mathbb{P}^1(K)$ be such that $\gamma(p) \neq p$. Then $\lim_{n \to \infty} \gamma^n(p)$ and $\lim_{n \to -\infty} \gamma^n(p)$ exist and are the two fixed points by $\gamma$.

Proof. By conjugation, we reduce to the case $\gamma = \mu_q$ with $q$ topologically nilpotent. In this case $\lim_{n \to \infty} |q^n||p| = |0|$ so $\lim_{n \to \infty} \gamma^n p = 0$ and the fixed points are $0$ and $\infty$. \hfill $\Box$

Theorem 5.6. Let $K$ be a complete field and algebraically closed and either $\text{char}(K) = p > 0$ or $\text{char}(\mathcal{O}_K)/\mathfrak{m}_K = 0$. Then for $\gamma \in \text{PGL}_2(K)$

$$\overline{\langle \gamma \rangle p} \text{ is compact for all } p \in \mathbb{P}^1(K) \iff \begin{cases} \gamma \text{ is hyperbolic}, \\ \gamma \text{ is of finite order}. \end{cases}$$

Proof. Since $K$ is algebraically closed we can suppose that either $\gamma$ is diagonalizable, a case already done in theorem 5.4, or it is conjugate to $\psi(t) = t + a$ for some $a \in K \setminus \{0\}$. We can and will suppose $\phi = \psi$.  

Consider then the orbit of 0, \( (\gamma)0 = \{na \mid n \in \mathbb{Z} \} \). If \( \text{char}(K) = p > 0 \), then \( \gamma \) has finite order equal to \( p \). Now, if \( \text{char}(K) = 0 \) and \( \text{char}(\mathcal{O}_K/m_K) = 0 \), then \( |n| = 1 \) for all \( n \in \mathbb{Z} \) so \( |na| = |a| \) for all \( n \in \mathbb{Z} \). Moreover if \( n \neq m \) then \( |na - ma| = |n - m||a| = |a| \). Then all points in the set \( (\gamma)0 \) are isolated, so the set is closed, but it has an infinite number of points, so it is not compact. \( \square \)

**Remark 5.7.** The result is false if \( \text{char}(K) = 0 \) and \( \text{char}(\mathcal{O}_K/m_K) = p > 0 \), because in this case \( |.|_Q = |.|_p \), where \( |.|_p \) denotes the \( p \)-adic absolute value. Then the closure of the orbit of 0 for \( \phi(t) = t + 1 \) is \( \mathbb{Z}_p \), the \( p \)-adic integers, which is compact. For any \( p \in \mathbb{P}^1(K) \) the closure of the orbit is a translate of \( \mathbb{Z}_p \), hence compact as well.

### 6. Schottky Groups

**Definition 6.1.** For \( \Gamma \subseteq \text{Aut}(\mathbb{P}_K^1) \) a subgroup, we define

\[
\text{Fix}(\Gamma) = \{p \in \mathbb{P}^1(K) \mid \exists \gamma \in \Gamma, \gamma \neq \text{Id} \text{ with } \gamma(p) = p\}
\]

and \( \mathcal{L}_\Gamma = \overline{\text{Fix}(\Gamma)} \) its closure.

**Definition 6.2.** Let \( \Gamma \subseteq PGL_2(K) \) a subgroup. We say that it is a **Schottky group** if

- \( \Gamma \) is finitely generated
- every element of \( \Gamma \) different from the identity is hyperbolic
- \( \Gamma_p \) (the closure of the orbit of \( p \)) is compact for all \( p \in \mathbb{P}^1(K) \).
- \( \Gamma \) is not cyclic, i.e. has rank bigger or equal than 2.

Note that a Schottky group is torsion free. Note also that a finitely generated but not cyclic subgroup of a Schottky group is a Schottky group.

We will show that for any Schottky group \( \Gamma \), the set \( \mathcal{L}_\Gamma \) is compact and perfect.

**Lemma 6.3.** If \( q \) and \( r \in K^* \) topologically nilpotent, the subgroup \( \Gamma \) that they generate does not have torsion elements and \( q^m = r^n \) for some \( n \) and \( m \in \mathbb{Z} \setminus \{0\} \), then \( \Gamma \) is cyclic.

**Proof.** We can suppose \( m \) and \( n \) are coprime, since, if \( s = \text{gcd}(n, m) \), \( n = sn', m = sm' \), then \( (q^n r^{-m'})^s = q^{ns} r^{-sn'} = 1 \) so \( q^{s} r^{-sn'} = 1 \) since \( \Gamma \) has no elements of finite order. But now, there exists \( a, b \in \mathbb{Z} \) with \( am + bn = 1 \) and we have \( (q^{sn} r^{-s})^m = q^n r^{-mn} = r \), so \( q \) and \( r \) belong to the subgroup generated by \( q^{s} r^{-sn} \).

**Lemma 6.4.** If \( q \) and \( r \in K^* \) are topologically nilpotent, the subgroup \( \Gamma \) that they generate does not have torsion elements and \( |q|^m \neq |r|^n \) for all \( n \) and \( m \in \mathbb{Z} \setminus \{0\} \), then \( W := \{q^n r^m : (n, m) \in \mathbb{Z}^2\} \) is not compact.

**Proof.** We will suppose that \( |r| > |q| \). We will show that \( W \) contains infinitely many isolated points. Consider

\[
W := W \cap \{x \in K \mid 1 \geq |x| > |q|\}
\]

Now, for any \( x \in W \), take the ball \( B(x, |q|) \). Observe that for any \( y \in B(x, |q|) \), \( |y| = |(y-x)+x| = \max(|y-x|, |x|) = |x| \), since \( |y-x| \leq q < |x| \). But by hypothesis no two elements in \( W \) have the same valuation, hence \( W \cap B(x, |q|) = \{x\} \) for any \( x \in W \).

But the set \( W \) contains an infinite number of points, since, that for any \( m \geq 1 \), there exists \( f(m) \in \mathbb{Z} \) such that \( |r|^{-f(m)} > |q|^m \leq |r|^{-f(m)+1} \), hence \( x_m := q^n r^{-f(m)} \) is in \( W \) for all \( m \), since \( 1 > |q|^m |r|^{-f(m)} \geq |r| > |q| \).
Then

\[ W = \{ q^m r^n : (n,m) \in \mathbb{Z}^2 \} \subseteq \left( \bigcup_{w \in W} B(w,|q|) \right) \cup B(0,|q|) \cup B^{\delta}(0,1)^c \]

and no ball can be removed. \( \square \)

**Lemma 6.5.** Let \( \Gamma \) be a Schottky group. Then, for any \( \text{id} \neq \gamma \in \Gamma \), there exists \( \tau \in \Gamma \) such that there exists \( p \in \mathbb{P}^1(K) \) with \( \tau(p) = p \) and \( \gamma(p) \neq p \).

**Proof.** Suppose it is false for some \( \gamma \in \Gamma \). This means there exists \( \tau \in \Gamma \) such that \( \tau^m = \gamma^n \) for all \( n,m \in \mathbb{Z} \), but with the same fixed points.

To see this, observe that if \( \tau^m = \gamma^n \) for some \( n \) and \( m \in \mathbb{Z} \setminus \{0\} \), then they have the same fixed points. These is because the fixed points of \( \alpha \) and of \( \gamma^n \) are the same for any \( n \in \mathbb{Z} \setminus \{0\} \).

So we can suppose \( \gamma(x) = qx \) and \( \tau(x) = rx \) for some \( q \) and \( r \in K^* \) topologically nilpotent, and \( q^m = r^n \).

So, by Lemma 6.3 there should exists \( \gamma \) and \( \tau \) that they do not belong to a cyclic subgroup of \( \Gamma \) but with the same fixed points. We take the subgroup generated by \( \gamma \) and \( \tau \) described as above, which must be a Schottky group, and we will find a contradiction.

Now, if \( q^m \neq r^n \) for all \( n \) and \( m \in \mathbb{Z} \setminus \{0\} \), but \( |q|^m = |r|^m \) for some \( n \) and \( m \in \mathbb{Z} \setminus \{0\} \), then \( q^n r^{-n} \) is not 1 but has valuation 1. Hence \( \gamma^m \tau^{-n} \in \Gamma \) and it is not hyperbolic. Hence we can suppose \( |q|^m \neq |r|^m \) for all \( n \) and \( m \in \mathbb{Z} \setminus \{0\} \). But Lemma 6.4 gives us a contradiction. \( \square \)

**Lemma 6.6.** Suppose \( \Gamma \) is a Schottky group. Consider \( p \in L_{\Gamma} \). Then \( \overline{p \Gamma} = L_{\Gamma} \).

**Proof.** If \( p \) is fixed by \( \gamma \neq 1 \in \Gamma \), then \( \alpha(p) \) is fixed by \( \alpha^{-1} \gamma \alpha \) for any \( \alpha \in \Gamma \). So \( p \Gamma \subseteq L_{\Gamma} \).

Now, if \( p' \) is another point in \( L_{\Gamma} \), with \( \gamma(p') \neq p' \), and fixed by some \( \alpha \in \Gamma \), by the previous lemma, then \( \alpha(p) \neq p \) and hence \( \alpha^n(p) \to p' \) for \( n \to \pm \infty \). So \( p' \in \overline{p \Gamma} \).

So all points fixed by some \( \alpha \in \Gamma \), except may be the other point different from \( p \) fixed by \( \gamma \), are in \( \overline{p \Gamma} \), which imply that its closure, which is \( L_{\Gamma} \), is contained in \( \overline{p \Gamma} \).

Finally, if \( p \in L_{\Gamma} \) is the limit of points \( p_n \) fixed by some \( \gamma_n \in \Gamma \), then any point in \( \overline{p \Gamma} \) is limit of points in \( \overline{p_n \Gamma} = L_{\Gamma} \), so it is in \( L_{\Gamma} \). The reverse inclusion is also clear. \( \square \)

**Theorem 6.7.** Suppose \( \Gamma \) is a Schottky group. Then the set \( L_{\Gamma} \) is perfect and compact.

**Proof.** It is compact since, by the previous lemma, \( L_{\Gamma} = \overline{p \Gamma} \) for some \( p \in L_{\Gamma} \), and \( \overline{p \Gamma} \) is compact by definition of Schottky group.

Let \( p \) be fixed by \( \gamma \in \Gamma \). Take \( p' \in L \) not fixed by \( \gamma \) (for example, fixed by some \( \gamma' \) not contained in the subgroup generated by \( \gamma \), that it exists because \( \Gamma \) is not cyclic). Then \( \gamma^n(p') \to p \) when \( n \to \infty \) or when \( n \to -\infty \). Hence no point fixed by some \( \gamma \neq 1 \) in \( \Gamma \) is isolated, so the same is true for the points in the closure. \( \square \)

7. The finite graph associated to a Schottky group.

The main aim of this section is to show that the quotient by \( \Gamma \) of the tree associated to \( L_{\Gamma} \) for a Schottky group \( \Gamma \) is finite, and that the quotient map is the universal cover, hence identifying \( \Gamma \) with the fundamental group. We will denote \( T_{\Gamma} := T(L_{\Gamma}) \).

**Theorem 7.1.** Let \( \Gamma \) be a Schottky group on a field complete with respect to a valuation. Then the tree \( T_{\Gamma} \) is locally finite, the group \( \Gamma \) acts freely on \( T_{\Gamma} \) and the quotient \( G_{\Gamma} := T_{\Gamma}/\Gamma \) is a finite graph.
We will prove the theorem along the section. The first part of the result is a consequence of Lemma 6.6 and the results of Section 2. The group acts freely because of Lemma 5.3 which says that for all \( \gamma \in \Gamma \) different from the identity and for all \( v \in V(\mathcal{T}_\Gamma) \), \( \gamma(v) \neq v \).

So we can take the quotient \( G_\Gamma = \mathcal{T}_\Gamma / \Gamma \) and the quotient map \( \mathcal{T}_\Gamma \to G_\Gamma \) is the universal cover. We only need to show that the graph \( G_\Gamma \) is finite.

**Definition 7.2.** Let \( B_\Gamma \subset \Gamma \) be a finite set of generators verifying that, if \( \gamma \in B_\Gamma \), then \( \gamma^{-1} \in B_\Gamma \), and id \( \in B_\Gamma \). For a fixed vertex \( \omega \in \mathcal{T}_\Gamma \) we consider \( S_\omega = \{ \gamma \omega \mid \gamma \in B_\Gamma \} \), which is a finite set of vertices. We denote \( T_{S_\omega} = \bigcup_{v_1, v_2 \in S_\omega}[v_1, v_2] = \bigcup_{\gamma \in B_\Gamma}[\omega, \gamma \omega] \), the minimal finite subtree that contains \( S_\omega \). Finally we denote

\[
T_{B_\Gamma, \omega} = \bigcup_{\gamma \in \Gamma} \gamma(T_{S_\omega}).
\]

Our aim will be to show in a series of lemmata that \( T_{B_\Gamma, \omega} = \mathcal{T}_\Gamma \), and the finiteness of \( G_\Gamma = T_{B_\Gamma, \omega} / \Gamma \) will be inferred.

This is because \( T_{B_\Gamma, \omega} / \Gamma \) has a finite number of vertices, since

\[
V(T_{S_\omega}) \to V(T_{B_\Gamma, \omega} / \Gamma) = V(\mathcal{T}_\Gamma / \Gamma)
\]

and \( V(T_{S_\omega}) \) is finite, and the tree \( \mathcal{T}_\Gamma \) is locally finite, hence also \( G_\Gamma \).

**Lemma 7.3.**

1. \( \forall \gamma \in \Gamma \) [\( \omega, \gamma \omega \) \( \in \mathcal{T}_{B_\Gamma, \omega} \).
2. \( \forall \gamma \neq \gamma' \in \Gamma \) [\( \gamma \omega, \gamma' \omega \) \( \in \mathcal{T}_{B_\Gamma, \omega} \).

**Proof.** Since \( \gamma \in \Gamma \), then \( \gamma = \gamma_1 \gamma_2 \ldots \gamma_n \), where \( \gamma_i \in B_\Gamma \). Then

\[
[\omega, \gamma \omega] \subset [\omega, \gamma_1 \omega] \cup [\gamma_1 \omega, \gamma_1 \gamma_2 \omega] \cup \cdots \cup [\gamma_1 \gamma_2 \ldots \gamma_{n-1} \omega, \gamma_1 \gamma_2 \ldots \gamma_n \omega]
\]

and also each

\[
[\gamma_1 \ldots \gamma_i \omega, \gamma_1 \ldots \gamma_{i+1} \omega] \subset (\gamma_1 \ldots \gamma_i)T_{B_\Gamma, \omega} = \mathcal{T}_{B_\Gamma, \omega}.
\]

To show the second part, we can divide the path as follows [\( \gamma \omega, \gamma' \omega \) \( \subset [\gamma \omega, \omega] \cup [\omega, \gamma' \omega] \subset \mathcal{T}_{B_\Gamma, \omega} \), or we also can argue that [\( \gamma \omega, \gamma' \omega = \gamma [\omega, \gamma^{-1} \gamma' \omega] \subset \gamma(\mathcal{T}_{B_\Gamma, \omega}) = \mathcal{T}_{B_\Gamma, \omega}. \]

**Lemma 7.4.** The graph \( \mathcal{T}_{B_\Gamma, \omega} \) is connected, hence it is a subtree.

**Proof.** Consider two vertices \( v_1 \) and \( v_2 \) \( \in \mathcal{T}_{B_\Gamma, \omega} \). Then \( v_1 = \gamma_1(\omega_1) \) and \( v_2 = \gamma_2(\omega_2) \) for some \( \omega_1, \omega_2 \) \( \in \mathcal{T}_{B_\Gamma, \omega} \) and some \( \gamma_i \in \Gamma \) for \( i = 1, 2 \). Since \( \mathcal{T}_{B_\Gamma, \omega} \) is connected, there exist paths \( [\omega_1, \omega] \) and \( [\omega_2, \omega] \) in \( \mathcal{T}_{B_\Gamma, \omega} \) and from this one has that \( \gamma_1[\omega_1, \omega] = [v_1, \gamma_1(\omega)] \) and \( \gamma_2[\omega_2, \omega] = [v_2, \gamma_2(\omega)] \) are contained in \( \mathcal{T}_{B_\Gamma, \omega} \). By Lemma 7.3 we have \( [\gamma_1(\omega), \gamma_2(\omega)] \subset \mathcal{T}_{B_\Gamma, \omega} \), so

\[
[v_1, v_2] \subset [v_1, \gamma_1(\omega)] \cup [\gamma_1(\omega), \gamma_2(\omega)] \cup [\gamma_2(\omega), v_2] \subset \mathcal{T}_{B_\Gamma, \omega}.
\]

**Lemma 7.5.** Let \( \Gamma \) be a Schottky group. Let \( \mathcal{T}' \subset \mathcal{T}_\Gamma \) be a non-empty subtree which is invariant by \( \Gamma \). Then \( \mathcal{T}' = \mathcal{T}_\Gamma \).

**Proof.** First, \( \mathcal{T}' \) is infinite since it contains infinite vertices: the ones of the form \( \gamma(v) \), for some \( v \in \mathcal{T}' \) and \( \gamma \in \Gamma \).

Let \( \mathcal{L}' \) be the image of \( \mathcal{T}' \) with respect to the map

\[
\Psi : \text{Rays}(\mathcal{T}(\mathcal{L}_\Gamma)) \to \mathcal{L}_\Gamma.
\]

Clearly \( \mathcal{L}' \) is invariant by \( \Gamma \), and non-empty since \( \mathcal{T}' \) is infinite, so it contains some ray. Take \( p \in \mathcal{L}' \). Then \( \Gamma p \subset \mathcal{L}' \). By lemma 6.6 we have \( \mathcal{L}_\Gamma = \Gamma p \subset \mathcal{L}' \subset \mathcal{L}_\Gamma \), thus \( \mathcal{L}' = \mathcal{L}_\Gamma \).
First, observe that for any \( x \) and \( y \in \mathcal{L}' \), all the points of the form \( t(x, y, z) \), for \( z \in \mathcal{L} \), are in fact in \( \mathcal{T}' \). To show this, observe that \( x \in \mathcal{L}' \) implies that the ray \([t(x, y, z), x]\) contains some vertex \( v_x \) of \( \mathcal{T}' \) (in fact, infinitely many). The same happens for \( y \), so \([t(x, y, z), y]\) contains a vertex \( v_y \) of \( \mathcal{T}' \). But \( t(x, y, z) \in [v_x, v_y] \subset \mathcal{T}' \) since \( \mathcal{T}' \) is a tree, hence connected.

But \( \mathcal{L}' \) is closed. Effectively, suppose we have a progression of distinct points \( p_n \in \mathcal{L}' \) such that \( p_n \to p \in \mathcal{L} \) when \( n \to \infty \). Then the vertices \( v_i := t(p_1, p_2, p_i) \) for \( i > 2 \) are in \( \mathcal{T}' \), and they generated a ray \( r \). Then \( \Psi(r) = p \), and hence \( p \in \mathcal{L}' \). So \( \mathcal{L}_\Gamma = \mathcal{L}' = \mathcal{L}' \), and hence \( c\mathcal{T}' = \mathcal{T}_\Gamma \).

As a consequence, we can finish the proof of the theorem. We have \( \mathcal{T}_{B_\Gamma, \omega} \) is invariant by \( \Gamma \) by definition and it is a subtree by Corollary \( \mathcal{T}_{B_\Gamma, \omega} = \mathcal{T}_\Gamma \).

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