BMS symmetry, soft particles and memory

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Abstract

In this work, we revisit unitary irreducible representations of the Bondi-Metzner-Sachs (BMS) group discovered by McCarthy. Representations are labelled by an infinite number of super-momenta in addition to four-momentum. Tensor products of these irreducible representations lead to particle-like states dressed by soft gravitational modes. Conservation of 4-momentum and supermomentum in the scattering of such states leads to a memory effect encoded in the outgoing soft modes. We note there exist irreducible representations corresponding to soft states with strictly vanishing four-momentum, which may nevertheless be produced by scattering of particle-like states. This fact has interesting implications for the S-matrix in gravitational theories.

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I. INTRODUCTION

One of the first breakthroughs in laying the foundation for an understanding of holography in Minkowski space was the work of Bondi-Metzner-Sachs [1, 2]. It revealed that asymptotic symmetry group of Minkowski space is a group of large diffeomorphisms called
the BMS group. Representations of the Poincare group [3] have played an important role in classifying elementary particles by their mass and spin. That motivates understanding the representations of the BMS group and its connection to elementary particles. In the 1970s McCarthy studied the positive energy unitary irreducible representations of BMS group [4–9]. But after this initial work, the subject has received little attention. The physical interpretation of the representations was not entirely clear at the time. In this work, we study from a physical viewpoint most of the interesting representations with the aim of identifying the interesting representations needed to construct a holographic dual. These include massive and massless particles and also soft particles with vanishing four-momentum. We find that in addition to zero momentum limit of massless particles there are many new soft modes predicted by BMS group which are related to gravitational memory [10, 11].

Recently, Strominger et. al. have discovered a relation between the BMS group, soft theorems and the memory effect [12–14]. They related supertranslations to memory effect [14] which led them to propose that black hole carries soft hair [15]. In this work we show that supertranslation charges indeed retain information about the initial states via a straightforward group theory construction. We consider a case where two particles collide and move away in different directions. Conservation of momenta (including supermomenta) reveals that final state has information about soft particles that stores information about the initial state. Another interesting discussion of the memory effect in electromagnetism appears in [16, 17].

Other recent papers on the BMS group include a realization [18, 19] on a scalar field, and more generally relation between the BMS group and elementary particles[20]. The connection between BMS group and non-relativistic conformal group, also known as Galilean group [21–24] has also been explored. The BMS charge algebra has been studied in [25] and BMS representations in three dimensions have been explored in the following papers [26–28].

In the present work we begin by reviewing the BMS group and establishing notation. We then revisit some of the most relevant results from McCarthy’s classification of unitary irreducible representations of the BMS group and connect the Bondi mass aspect to the function space on which BMS is realized. We try to highlight only the physically important representations and find all the massive and massless representations that appear in the usual Wigner classification of the Poincare group, as well as extra representations with differing supermomenta structures. The group invariant norms associated with these families of
representations are constructed, which is an essential step in any attempt at capturing the bulk dynamics via a holographic description. We then consider tensor products/scattering of these states which allows us to explore the extent to which gravitational memory allows the initial state to be reconstructed from a final state. We conclude with some comments on the relevance of the results to general gravitational S-matrix theories in asymptotically flat spacetime such as string theory, and the prospects for developing holographic models with BMS as a fundamental symmetry group.

II. REPRESENTATIONS OF THE BMS GROUP

Asymptotic flatness requires that the Weyl tensor of the metric must fall off like $O(r^{-3})$ for large $r$ [2] (for a recent review see [29]) which allows the choice of the following asymptotically flat coordinates at leading order at large $r$

$$ds^2 = -du^2 - 2dudr + 2r^2\gamma_{zz}dzd\bar{z} + 2\frac{m_B(u, \bar{z}, \bar{z})}{r}du^2 + rC_{zz}dz^2 + D^zC_{zz}dud\bar{z} + c.c + ...$$  \hspace{1cm} (1)

The function $m_B(u, z, \bar{z})$ is called the Bondi mass aspect and the other coefficients are functions only of $u, z$ and $\bar{z}$. The covariant derivative $D^z$ is defined with respect to the metric on the unit sphere $\gamma_{zz} = \frac{2}{(1 + z\bar{z})^2}$. In the next subsection we give a brief introduction to BMS group. Then we show that the invariant mass function introduced in [7] and the Bondi mass aspect are to be identified.

A. BMS group

The group of diffeomorphisms which preserve the form of the metric (1) is called the BMS group. It is given by

$$B = A \ltimes G$$

where $G = SL(2, \mathbb{C})$ and $A$ is the abelian group of pointwise addition of functions functions on a 2-sphere [9]. To make this statement well-defined we must specify more carefully the class of functions to be considered. We follow the definition of [7] and take these to be $C^\infty$ which implies that the representation of $G$ on $A$ is equivalent to the operator representation of $G$ on the space $D_{(2,2)}$ [30].
The space $D_{(2,2)}$ is characterized by a pair of functions $\xi(z)$ and $\hat{\xi}(z)$ on the complex plane, which may be thought of as functions on patches centered at the north and south poles of the sphere respectively. These functions are $C^\infty$ everywhere except at the origin and are related by the overlap condition

$$\hat{\xi}(z) = |z|^2 \xi(z^{-1})$$

The action of $SL(2, \mathbb{C})$ element

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

is given by

$$g\xi(z) = |\alpha + \gamma z|^2 \xi\left(\frac{\beta + \delta z}{\alpha + \gamma z}\right)$$

$$g\hat{\xi}(z) = |\beta + \delta z|^2 \hat{\xi}\left(\frac{\alpha + \gamma z}{\beta + \delta z}\right)$$

We will be mostly interested in the dual space of $A$. As we will see this corresponds most directly to the class of functions $m_B(u, z, \bar{z})$ that appear for some fixed value of $u$. The dual space corresponds to the space $D_{(-2, -2)}$ in the notation of [30] and, as we will see, is a space of distributions with a class of allowed singularities. It is specified again by a pair of functions satisfying the matching condition

$$\hat{\phi}(z) = |z|^{-6} \phi(z^{-1})$$

The action of $G$ is given by

$$g\phi(z) = |\alpha + \gamma z|^{-6} \phi\left(\frac{\beta + \delta z}{\alpha + \gamma z}\right)$$

$$g\hat{\phi}(z) = |\beta + \delta z|^{-6} \hat{\phi}\left(\frac{\alpha + \gamma z}{\beta + \delta z}\right)$$

The 4-momentum associated with the functions $\phi(z)$ may be extracted via the projector $\Pi$ expressed as the integral

$$\Pi\phi(z') = \frac{i}{\pi} \int dz d\bar{z} (z - z')(\bar{z} - \bar{z}') \phi(z)$$

$$= \frac{i}{\pi} \left( (p^0 + p^3) + (p^0 - p^3) z' \bar{z}' - (p^1 - ip^2) z' - (p^1 + ip^2) \bar{z}' \right)$$

which is a polynomial of weight 2 in $z'$, with coefficients corresponding to the 4-momenta $p^\mu$. For this to be well-defined, the regulator as $|z| \to \infty$ implicit in the definition of the
\(D_{(-2,-2)}\) distributions must be taken into account. This can therefore be rewritten in terms of convergent integrals as

\[
\Pi \phi(z') = \frac{i}{\pi} \int_{|z|<1} dz d\bar{z} \left( (z - z')(\bar{z} - \bar{z}')\phi(z) + (1 - zz')(1 - \bar{z}\bar{z}')\hat{\phi}(z) \right)
\]

(6)

The higher order terms in \(\phi(z)\) are labelled by the supermomenta. The supermomenta form a \(G\) invariant subspace, implying that an irreducible representation of the BMS group describes states with the same mass (i.e. 4-momentum squared). Equation (4) matches equation 72 in [2] which gives the Bondi 4-momentum in terms of an integral of the Bondi mass aspect \(m_B(u, z, \bar{z})\). Thus we may identify \(\phi(z)\) with \(m_B\) up to a rescaling factor, and the derived 4-momenta behave as expected under \(G\).

In turn, this provides a more physical justification for the choice of the space of functions \(D_{(-2,-2)}\). This space of distributions yield 4-momenta corresponding to finite center of mass energies, as well as finite supermomenta, and prescribed fall-off conditions [30] that guarantee integrals such as (4) are well-defined.

B. Little groups

As with Wigner’s classification of the irreducible representations of the Poincare group, the first step in understanding representations is to understand little groups. One may then construct the irreducible representations via the method of induced representations [3, 31], lifting representations of the subgroup to representations of the group.

In Wigner’s classification, one identifies classes of four-momenta invariant under Poincare subgroups. For BMS the goal is to find functions \(\phi(z)\) invariant under the little groups of BMS. McCarthy give a detailed list of most of the little groups [7]. Here we discuss some of them in detail.

1. \(SU(2)\)

The first important little group is \(SU(2)\). The class of functions invariant under this group is

\[
\phi(z) = \hat{\phi}(z) = m \left(1 + |z|^2\right)^{-3}
\]

(7)
This represents a particle of mass \( m \) at rest. One can check that the 4-momentum \((p_0, p_1, p_2, p_3)\) indeed transforms correctly under the action of Lorentz generators (3). As an example, let us look at the action of boost \( g_t = \begin{bmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{bmatrix} \). Acting on (7)

\[
g\phi(z) = me^{-3t} \left( 1 + |z|^2 e^{-2t} \right)^{-3} = m \left( e^{-t} |z|^2 + e^t \right)^{-3}
\]

Using equation (4) we get

\[
\Pi\phi(z') = \frac{i}{\pi} \int dzd\bar{z}(z - z')(\bar{z} - \bar{z}')m \left( e^{-t} |z|^2 + e^t \right)^{-3} = m \left( e^t + e^{-t} |z|^2 \right)
\]

which leads via (4) to \( p_0 = m \cosh t \) and \( p_3 = m \sinh t \) as expected for a boost.

Also note that the super-momenta get populated by the action of the boost due to the higher order terms present in (8) beyond order 2. Thus the simplest representation of BMS is that of a massive particle, matching what one expects of the Poincare group, but the representation traces out an orbit in the infinite dimensional space of supermomenta as one acts with Lorentz generators.

The representations of the little group may also carry spin \( \ell \) which is half-integer. As shown in [4] this yields a single spin \( \ell \) representation of the Poincare subgroup of BMS.

2. \( \Delta \)

The second important little group is \( \Delta \), in the notation of [7], or more commonly the Euclidean group in two dimensions \( E(2) \). It yields usual massless particles, and as above, Lorentz transformation fill out an orbit in supermomentum space. This corresponds to the invariant functions

\[
\phi(z) = K \\
\hat{\phi}(z) = K|z|^{-6} + A \frac{\partial^2}{\partial z^2} \frac{\partial^2}{\partial \bar{z}^2} \delta(z) + B \frac{\partial^2}{\partial z^2} \delta(z) + \bar{B} \frac{\partial^2}{\partial \bar{z}^2} \delta(z) + C \delta(z)
\]

Note here \( \delta(z) \equiv \delta(\text{Re}z)\delta(\text{Im}z) \), and likewise we suppress the \( \bar{z} \) dependence of \( \phi, \hat{\phi} \). Here \( A \) and \( C \) are real, and \( B \) is complex. This clearly illustrates the need for the \( D(-2,-2) \) space of generalized functions to correctly accommodate massless particles. These representations were not present in the earlier studies [4–6]. To evaluate four momentum on such a representation one must use the formula (6) to properly regulate the otherwise divergent expression.
Finite 4-momenta are obtained provided $K = 0$. In this case, $C$ is proportional to the light-like 4-momentum.

The spin of these representations has been studied in [7] and as expected one gets either a chiral massless representation with a single Poincare spin $s = 0, 1/2, \cdots$. Alternatively one may get one of the massless continuous spin representations of Wigner’s classification, whose physical significance remains unclear.

3. $SL(2, C)$

In general one may take the entire group of Lorentz transformations to be a little group, in which case the invariant functions take the form

$$\phi(z) = \dot{\phi}(z) = 0$$

which implies vanishing of the 4-momentum and of all the supermomentum. Nevertheless, one may pick a unitary representation of the little group and lift it to a representation of BMS. It is natural to think of such representations as arising from a unitary irreducible representation corresponding to a massive (or massless) field on an internal three-dimensional de Sitter spacetime $dS_3$ [32]. Such representations are infinite-dimensional. In any case, the situation here is unchanged from the usual Poincare group. The standard procedure is to throw out all but the trivial representation, leaving the Poincare invariant vacuum as the unique state with vanishing 4-momentum. Lifting to BMS, we obtain a unique state with vanishing 4-momentum and supermomentum. Since the other infinite-dimensional families of states are not generated from tensor products of the other states we will consider with the vacuum, we can safely ignore these exotic infinite dimensional representations with vanishing momentum.
4. \( SL(2, R) \)

The situation is more interesting for this maximal little group. In this case the invariant functions take the form

\[
\phi(z) = K \left( \frac{z - \bar{z}}{i} \right)^{-3} + A\delta^2 \left( \frac{z - \bar{z}}{i} \right)
\]

\[
\hat{\phi}(z) = K \left( \frac{\bar{z} - z}{i} \right)^{-3} + A\delta^2 \left( \frac{\bar{z} - z}{i} \right)
\]

where \( K \) and \( A \) are real parameters. For the Poincare group, this little group would usually give rise to the tachyonic representations where \( p_\mu p^\mu < 0 \). Here the nuclear topology restricts the class of distributions to those with vanishing 4-momentum when inserted into (6). Nevertheless, the higher order terms present in the invariant functions generate a nontrivial orbit corresponding to nonvanishing supermomentum.

As with the case of \( SL(2, C) \) one can assign such representations a nontrivial representation of the little group. In this case it would correspond to a massive or massless field on an internal two-dimensional de Sitter spacetime, which has the isometry group \( SL(2, R) \). However again such representations are infinite dimensional, and will not arise from tensor products of the elementary representations we will consider. These representations arise already in Wigner’s classification of the representations of the Poincare group, and are likewise not thought to be physically relevant, because one can construct self-consistent theories where they do not appear.

An exception is the trivial representation of the little group \( SL(2, R) \). Under the usual Poincare classification, these would be invariant under a larger little group \( SL(2, C) \) and so would be equivalent to the \( SL(2, C) \) invariant vacuum state. However under BMS such modes carry nontrivial supermomentum. This leads to a class of “soft modes” which in general will be produced in the scattering of particle-like states, and are in general necessary to enforce conservation of supermomentum.

5. \( \Gamma \)

For the Poincare group, the maximal little groups exhaust the set of little groups. However for BMS it is also necessary to consider the group \( \Gamma \) which is a subgroup of all the above
little groups corresponding to rotations in a plane \( \begin{pmatrix} \omega & 0 \\ 0 & \bar{\omega} \end{pmatrix} \) with \( \omega \) a complex number of unit modulus. While the 4-momenta invariant with respect to this little group are actually invariant under a larger little group, this is no longer the case when the supermomenta are included. The invariant function takes the form

\[
\phi(z) = \beta(r) \\
\hat{\phi}(z) = r^{-6} \beta(1/r)
\]

where \( z = re^{i\phi} \) with \( \phi = [0, 2\pi) \) and \( r \geq 0 \). Here \( \beta \) is a distribution satisfying the conditions above. The 4-momenta corresponding to these representations may have \( m^2 = 0, m^2 > 0 \) or \( m^2 < 0 \).

For \( m^2 > 0 \) the representation corresponds [4] to an infinite tower of Poincare spins labelled by some integer/half-integer \( j \) with the tower corresponding to all spins \( s = j, j + 1, \cdots \).

For \( m^2 = 0 \) and \( m^2 < 0 \) the Poincare representations are more exotic, with integrals over continuous spins needed to generate the BMS representation.

6. **Indecomposable**

In the present work we are restricting our consideration to unitary irreducible representations of the BMS group. It is possible this is too restrictive a class of representations to build a useful holographic description of asymptotically flat space. Because the BMS group is non-compact, representations that may be decomposed into irreducible representations are actually rather special, and more generally one should consider indecomposable representations. As far as we are aware, the classification of such representations for non-compact groups is still relatively undeveloped.

7. **Non-connected subgroups**

There are a variety of non-connected little groups that can appear as subgroups of the BMS group [7]. For simplicity we do not consider these in the present work.
III. HOLOGRAPHY

One of the main motivations for considering the irreducible representations of the BMS group, is to get a better understanding of the basic ingredients needed to build a holographic description of the theory on null infinity $I$. The same considerations also apply when considering the allowed set of asymptotic states in an S-matrix description of a gravitational theory. As such, we now turn our attention to defining BMS invariant norms for the representations of interest, and see that these may be realized as integrals on $I$.

In the general case the norm is defined using the group invariant measure on the coset space $G/H$ where $G = SL(2, C)$ and $H$ is the little group \[6\]

$$
\int f(g) d\mu(g) = \int_{G/H} \left( \int_{H} f(gh) d\mu(h) \right) d\mu_{G/H}.
$$

### A. $SU(2)$ and $\Delta$

It is perhaps simplest to begin in momentum space. As we have seen for the $SU(2)$ little group, we have representations of BMS that essentially coincide with ordinary massive particle representations of the Poincare group. The same is true for massless particles and the little group $\Delta$. Wigner has given the invariant norm for these two subgroups as

$$(\psi_1, \psi_2) = \int_0^{\infty} \psi_1(p) \psi_2(p) \frac{dp_1 dp_2 dp_3}{p_4}.$$  

As we see, this integral may be viewed as an on-shell integral in the bulk $p_4^2 = m^2 + \sum_i p_i^2$, or as an off-shell integral over the holographic boundary $I$.

### B. $SL(2, R)$

Again the little group is three-dimensional, but now the invariant norm can be interpreted as an integral over three-dimensional de Sitter spacetime which corresponds to the coset $SL(2, C)/SL(2, R)$

$$(\psi_1, \psi_2) = \int_0^{\infty} \psi_1(p) \psi_2(p) \frac{dp_1 dp_2 dp_3}{\sqrt{\sum_i p_i^2 - 1}}.$$
Figure 1. The figure depicts two particles of mass $M$ moving in opposite directions forming a bound state.

C. $\Gamma$

Since $\Gamma$ is only one-dimensional the coset space will be five-dimensional and may be written as an integral over on-shell 4-momenta ($p_4^2 = \sum_i p_i^2 + m^2$) supplemented by a pair of angles

$$ (\psi_1, \psi_2) = \int_0^\infty \psi_1(p, \theta)\psi_2(p, \theta) \frac{dp_1 dp_2 dp_3 d\theta_1 d\theta_2}{p_4} \quad (10) $$

which may be interpreted as an integral over $I$ and two internal degrees of freedom $\theta$.

IV. SCATTERING EXAMPLES

By taking tensor products of the above representations of BMS we can gain insight into how the symmetry constrains the scattering of particle-like representations and study what BMS representations appear when ordinary particles undergo scattering.
A. Particles forming bound state

Consider a representations of the $SU(2)$ little group corresponding to two particles with mass $M$. One is moving in angular direction $(\theta, \phi) = (\alpha, \beta)$ and the other in the opposite direction $(\alpha - \pi, \beta)$. After sometime they collide and form a bound state as shown in figure(1) The mass aspect functions of a particle can be found by boosting (7) along $z$-axis and then rotating by $\alpha$ around $y$-axis followed by rotation around $z$-axis by $\beta$. That is

$$\phi_{+\alpha} = g_{\alpha} \phi_{z+}$$

where

$$\phi_{+\alpha}(\theta, \phi) = \left[ \begin{array}{c}
\cos \frac{\alpha}{2} \\
-\sin \frac{\alpha}{2}
\end{array} \right] \phi_{z+} = \phi_{z+} \left( \frac{z \cos \frac{\alpha}{2} - \sin \frac{\alpha}{2}}{z \sin \frac{\alpha}{2} + \cos \frac{\alpha}{2}} \right) = \left( e^{t \left| \frac{z \cos \frac{\alpha}{2} - \sin \frac{\alpha}{2}}{z \sin \frac{\alpha}{2} + \cos \frac{\alpha}{2}} \right|^2} \right)^{\frac{3}{2}} M$$

and rotating by $\beta$ around $z$-axis gives

$$\phi_{+\alpha,\beta}(\theta, \phi) = \frac{M}{(\cosh t - \sinh t \cos \theta \cos \alpha - \sinh t \sin \theta \cos \phi \sin \alpha)^3}$$

Now consider a particle moving in opposite direction. That is angular coordinates $(\alpha - \pi, \beta)$.

$$\phi_{-\alpha,\beta}(\theta, \phi) = \frac{M}{(\cosh t + \sinh t \cos \theta \cos \alpha + \sinh t \sin \theta \cos \phi \sin \alpha)^3}$$

At the linearized level, the mass aspect function of the whole system is

$$\phi_{\alpha,\beta}(\theta, \phi) = \phi_{+\alpha,\beta}(\theta, \phi) + \phi_{-\alpha,\beta}(\theta, \phi) = \frac{M}{(\cosh t - \sinh t \cos \theta \cos \alpha - \sinh t \sin \theta \cos (\phi + \beta) \sin \alpha)^3} + \frac{M}{(\cosh t + \sinh t \cos \theta \cos \alpha + \sinh t \sin \theta \cos (\phi + \beta) \sin \alpha)^3} \quad (11)$$

$$\quad (12)$$

Since two particles are moving in opposite directions, in no frame will both the particles be at rest together. Neither boosts nor rotations can transform the above function into a constant. The 4-momenta may be evaluated using (4) and are given by $p_0 = 2M \cosh t, p_1 = p_2 = p_3 = 0$. The higher momentums corresponding to supermomenta are nontrivial, and are functions of $\alpha, \beta$. Performing a rotation of $-\beta$ around $z$-axis followed by $-\alpha$ around $y$-axis transforms (12) to a function of $\cos \theta$ only. This implies the function is invariant under $\Gamma$ little group and no bigger subgroup of Lorentz group.
This construction also provides insight into the invariant norm for the $\Gamma$ representations (10). While boosts fill out three dimensions of the associated states as usual, one needs an extra integral over the angular directions corresponding to $(\alpha, \beta)$ to generate the complete set of associated states, yielding the five-dimensional integral in (10).

So we come to an interesting conclusion. The mass aspect functions of $\Gamma$ can be viewed as tensor product of reps of $SU(2)$. One may perform essentially the same computation for the massless representations associated with the little group $\Delta$ replacing those of $SU(2)$. BMS representation of the final system retains memory about the direction of the incoming particles. In this case of two-body scattering, the supermomenta allow all the information about the initial state of the system to be retrieved from the final bound state. This is in line with the soft hair proposal of Strominger et al [15].

B. Soft modes in scattering

Extending the above considerations, we now consider $2 \rightarrow 2$ scattering. Consider an initial state is $\phi_z$ and a final state $\phi_x$ accompanied by soft modes. Figure (2) shows the process.

$$\phi_{\text{initial}}(\theta, \phi) = \phi_z(\theta, \phi) = (2M, 0, 0, 0, \{p_{l,m}\}_z) = \frac{M}{(\cosh t - \sinh t \cos \theta)^3} + \frac{M}{(\cosh t + \sinh t \cos \theta)^3} \in \Gamma$$

Part of the final state is two particles going along the $x$-axis

$$\phi_x(\theta, \phi) = \frac{M}{(\cosh t - \sinh t \sin \theta \cos \phi)^3} + \frac{M}{(\cosh t + \sinh t \sin \theta \cos \phi)^3}$$

By conservation of supermomenta, the initial mass aspect should match the final mass aspect

$$\phi_{\text{initial}} = \phi_{\text{final}}$$

$$\phi_z(\theta, \phi) = \phi_x(\theta, \phi) + \phi_{\text{soft}}$$

$$\phi_z(\theta, \phi) = (2M, 0, 0, 0, \{p_{l,m}\}_z) = (2M, 0, 0, 0, \{p_{l,m}\}_x) + (0, 0, 0, 0, \{p_{l,m}\}_z - \{p_{l,m}\}_x)$$

In this case, while the outgoing massive particles transform under the standard $SU(2)$ little groups, there is an additional soft mode with vanishing 4-momentum but non–vanishing supermomentum. In this case the soft mode transforms under the $\Gamma$ little group and represents the gravitational memory effect.
Figure 2. The left figure represents two particles of mass $m$ moving along $z$-axis. They collide and move out along $x$-axis. The figure on the right represents the final state. Subtracting blue patch from the red patch on the celestial sphere gives the soft mode in the final state.

V. CONCLUSION

Many of the results we have discussed appear in McCarthy’s original works but have been passed over in much of the subsequent literature, and our goal was to cast the most relevant selection of these results in a modern context, where they may be of use to researchers attempting holographic formulations of asymptotically flat spacetime, or simply trying to understand gravitational memory from the perspective of the BMS group. We started with a brief introduction to the BMS group and identified 4-momenta and the supermomenta. Representations of $\Delta, SU(2)$ represent massless and massive particles respectively corresponding directly to Wigner’s original classification of the Poincare group. Then we derive the invariant measure and invariant norm for some of the little groups. This revealed that invariant norm of little groups other than $SU(2), \Delta$ involves integrating over a larger phase space. Specifically for $\Gamma$ one encounters integrals over 5 dimensions. Starting with a representative state of $\Gamma$, both rotation and boosts are required to traverse complete orbit inside $\Gamma$. This implies that rotations produce states which cannot be obtained just by boosts. This is related to the fact that representations of $\Gamma$ can be expressed as bound state of rep of
$SU(2), \Delta$. To explore this point we considered two particles moving in opposite directions forming a bound state. Momenta of final state depend on the direction of initial particles. In other words, BMS representations store not just the total 4-momenta of the system but also retain information about the individual 4-momenta of the initial state. This is in contrast to Poincare representations where the final state just depends on total energy.

These results should have important implications for any $S-$matrix theory of gravity in asymptotically flat spacetime. In string theory, for example, these $S-$matrix elements are built using vertex operators corresponding to representations of the Poincare group. For such a description to be consistent it is implicit that the scattering states of such particles form a complete set. According to our analysis of the BMS group, that is not the case. For example, there exist unitary irreducible representations of the BMS group with vanishing 4-momenta but non-vanishing supermomenta that are not limits of massless particles (with non-vanishing light-like 4-momentum) such as the soft mode representations of the $SL(2, R)$ little group that we discussed. One also has irreducible representations of the little group $\Gamma$ that can also generate soft modes with vanishing 4-momenta, but non-vanishing supermomenta. On the other hand, it is clear there is a unique vacuum state, the trivial representation of the BMS group, which is of course invariant under all the asymptotic symmetries. There has been some preliminary discussion of some of these issues in the bosonic string [33] but we believe the present results warrant further study of the spectrum of string theory to obtain a more complete understanding of the soft modes.

From the perspective of holography the present work shows what irreducible representations of the BMS group are needed to formulate the elementary ingredients of such a description. There is some commonality with the AdS/CFT approach, namely a holographic “operator” transforming as an irrep of BMS in one-to-one correspondence with bulk fields with fixed mass and spin. Such operators naturally live in a three-dimensional space according to the norms described in section III. However the existence of the more exotic representations discussed above suggest this picture in not complete in the case of BMS. For example if representations of the little group $\Gamma$ must be introduced as elementary operators in the holographic description, they naturally live in a five-dimensional space. Furthermore the operators corresponding to the $SL(2, R)$ representations will serve to generate states with nontrivial supermomenta, with no cost in 4-momentum. These representations appear to live in an auxiliary three-dimensional de Sitter spacetime. From the usual perspective,
this would imply the vacuum is highly degenerate, making it difficult to construct a reasonable interacting theory based on such operators at the quantum level. In any case, we hope the present work goes some way to highlighting the obstacles that need to be addressed in formulating holography in asymptotically flat spacetime.

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