A $P_{k+2}$ polynomial lifting operator on polygons and polyhedrons

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Abstract  A $P_{k+2}$ polynomial lifting operator is defined on polygons and polyhedrons. It lifts discontinuous polynomials inside the polygon/polyhedron and on the faces to a one-piece $P_{k+2}$ polynomial. With this lifting operator, we prove that the weak Galerkin finite element solution, after this lifting, converges at two orders higher than the optimal order, in both $L^2$ and $H^1$ norms. The theory is confirmed by numerical solutions of 2D and 3D Poisson equations.

Keywords  weak Galerkin, finite element methods, Poisson, polytopal meshes

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1 Introduction

In weak Galerkin finite element methods [12,13], discontinuous polynomials, $u_0$ defined inside each element and $u_b$ defined on each face of element, are employed to form an approximation space. In particular, on triangular/tetrahedral grids, the $P_k-P_{k+1}$ ($P_k$ inside a triangle, $P_{k+1}$ on an edge) weak Galerkin finite element solution is two-order superconvergent in both $L^2$ and $H^1$-like norms [2]. Further, with a careful construction of weak gradient, such $P_k$-$P_{k+1}$ weak Galerkin finite element is also two-order superconvergent.
on general polygonal and polyhedral meshes \[14\]. Here the super-convergence is defined for the difference between finite element solution \( u_0 \) and the local \( L^2 \) projection \( Q_h u \) of the exact solution.

In this paper, we construct a \( P_{k+2} \) polynomial lifting operator. It lifts an \((n+1)\)-piece polynomial, \( \{u_0, u_b\} \), on a \( n \)-polygon/polyhedron \( T \) to a one-piece \( P_{k+2} \) polynomial on \( T \). After such a lifting/post-processing, the weak Galerkin finite element solution is two-order super-convergent to the exact solution, i.e.,

\[
\|u - u_h\|_0 + h |u - u_h|_1,h \leq Ch^{k+1} |u|_{k+1},
\]

\[
\|u - L_h u_h\|_0 + h |u - L_h u_h|_{1,h} \leq Ch^{k+3} |u|_{k+3},
\]

where \( u_h \) and \( u \) are the finite element solution and the exact solution, respectively, and \( h \) is the mesh size.

This polynomial lifting operator is different from traditional polynomial lifting operators \[1,3–5,10\]. These operators only lift a polynomial trace on the boundary of an element to a polynomial inside the element, stably, i.e., subject to the minimum or a small energy. But here we lift both trace data and interior data to a polynomial, subject to the \( P_{k+2} \) accuracy. Additionally, even the trace (of boundary polynomials) is discontinuous here. Well, such a discontinuous-trace polynomial lifting is studied in \[7–9\], but for \( H(\text{curl}) \) and \( H(\text{div}) \) polynomial lifting.

### 2 Weak Galerkin finite element

For solving a model Poisson equation,

\[
-\Delta u = f \quad \text{in } \Omega, \tag{1}
\]

\[
u = 0 \quad \text{on } \partial \Omega, \tag{2}
\]

where \( \Omega \) is a polytopal domain in \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \), we subdivide the domain into shape-regular polygons/polyhedrons of size \( h, \mathcal{T}_h \). Here a polyhedron is Lipschitz (non-convex) and a union of finite shape-regular tetrahedra of size \( Ch \).

For \( k \geq 1 \), we define the weak Galerkin finite element spaces by

\[
V_h = \{v_h = \{v_0, v_b\} : v_0|_T \in P_k(T), \quad v_b|_e \in P_{k+1}(e), \quad e \subset \partial T, \quad T \in \mathcal{T}_h \}, \tag{3}
\]

\[
V^0_h = \{v_h : v_h \in V_h, \quad v_b = 0 \quad \text{on } e \subset \partial \Omega \}. \tag{4}
\]

The weak Galerkin finite element function assumes one \( d \)-dimensional \( P_k \) polynomial inside each element \( T \), and one \((d-1)\)-dimensional \( P_{k+1} \) polynomial on each face edge/polygon \( e \).

On an element \( T \in \mathcal{T}_h \), we define the weak gradient \( \nabla_w v_h \) of a weak function \( v_h = \{v_0, v_b\} \in V_h \) by the solution of polynomial equation on \( T \):

\[
\int_T \nabla_w v_h \cdot q \, dx = \int_{\partial T} v_b q \cdot n \, dS - \int_T v_0 \nabla \cdot q \, dx \quad \forall q \in A_k(T), \tag{5}
\]
where \( A_k(T) \) is a piece-wise polynomial space, but with one piece polynomial divergence and one piece polynomial trace on each face, on a sub-triangular/tetrahedral subdivision of \( T = \{ T_i, i = 1, \ldots, n \} \),

\[
A_k(T) = \{ q \in H(\text{div}, T) : q|_{T_i} \in P_{k+2}^d(T_i), T_i \subset T, \nabla \cdot q \in P_k(T), q \cdot n|_e \in P_{k+1}(e) \}. 
\]

Here \( n \) is a fixed normal vector on edge/polygon \( e \). To get a simplicial subdivision on \( T \), some face edges/polygons have to be subdivided. That is, in addition to \( T = \bigcup_i T_i, e = \bigcup_j e_j \), where \( \{ e_j \} \) is the set of face edges/triangles of \( \{ T_i \} \).

A weak Galerkin finite element approximation for (1)-(2) is defined by the unique solution \( u_h = \{ u_0, u_b \} \in V_0^h \) satisfying

\[
(\nabla w u_h, \nabla w v_h) = (f, v_0) \quad \forall v_h = \{ v_0, v_b \} \in V_0^h. \tag{6}
\]

In [14], both (5) and (6) are proved to have a unique solution.

**Theorem 1** ([14]) Let \( u \) and \( u_h \) be the solutions of (1) and (6), respectively. The following two-order superconvergence holds

\[
\| Q_h u - u_h \|_0 + h \| Q_h u - u_h \| \leq C h^{k+3} | u |_{k+3}, \tag{7}
\]

where \( Q_h u = \{ Q_0 u, Q_b u \} \in V_0^h \) (\( Q_0 \) and \( Q_b \) are local \( L^2 \)-projection on \( T \) and \( e \) respectively), and \( \| v_h \| = (\nabla w v_h, \nabla w v_h)^{1/2} \).

### 3 A \( P_{k+2} \) polynomial lifting operator

On an \( m \)-face polygon/polyhedron \( T \) we have \( (m + 1) \) pieces of polynomials from a weak Galerkin finite element function. We need to lift these polynomials to a one-piece \( P_{k+2} \) polynomial, preserving \( P_{k+2} \) polynomials in the sense that \( L_h Q_h u = u \) if \( u \) is a \( P_{k+2} \) polynomial.

**Theorem 2** The local \( L^2 \) projection \( Q_h : u \in P_{k+2}(T) \rightarrow u_h = \{ Q_0 u, Q_b u \} \in V_h \) is an injection, i.e.,

\[
Q_h u = 0 \text{ if and only if } u = 0.
\]

**Proof** Let \( u \in P_{k+2}(T) \) and \( Q_h u = 0 \). For any vector polynomial \( q_{k+1} \in [P_{k+1}(T)]^2 \), we have

\[
\int_T \nabla u \cdot q_{k+1} \, dx = \sum_{e \subset \partial T} \int_e u q_{k+1} \cdot n \, ds - \int_T u \nabla \cdot q_{k+1} \, dx
\]

\[
= \sum_{e \subset \partial T} \int_e Q_0 u q_{k+1} \cdot n \, ds - \int_T Q_0 u \nabla \cdot q_{k+1} \, dx
\]

\[
= 0.
\]

Thus \( \nabla u = 0 \) everywhere and \( u = C \). Since \( Q_0 u = 0 \), \( C = 0 \) and \( u = 0 \).
Theorem 3 The \( P_{k+2} \) polynomial lifting operator \( L_h \), defined in (10) below, is \( P_{k+2} \) polynomial preserving in the sense that
\[
L_h Q_h u = u, \quad \text{if } u \in P_{k+2}(T). \tag{8}
\]
Consequently we have
\[
\|u - L_h Q_h u\|_0 + h|u - L_h Q_h u|_{1,h} \leq C h^{k+3} |u|_{k+3}. \tag{9}
\]

Proof Let \( P_h : u_h = \{u_0, u_b\} \in V_h \rightarrow \{(P_h u_h)_0, (P_h u_h)_b\} \in V_h \) be the local, discrete and mixed \( L^2 \) projection on to the image space \( Q_h P_{k+2}(T) \), i.e.,
\[
\int_T (P_h u_h)_0 Q_0 p_{k+2} + \sum_{e \subset \partial T} \int_e (P_h u_h)_b Q_b p_{k+2} + \sum_{e \subset \partial T} \int_e u_b Q_b p_{k+2} + \sum_{e \subset \partial T} \int_e u_b Q_b p_{k+2} dS = \int_T u_0 Q_0 p_{k+2} + \sum_{e \subset \partial T} \int_e u_b Q_b p_{k+2} dS \forall p_{k+2}(T).
\]

The above equation has a unique solution as the left hand side bilinear form is coercive. By last theorem, \( Q_h \) is one-to-one from \( P_{k+2}(T) \) on to the image space \( P_h V_h \). Its inverse defines an unique lifting operator:
\[
L_h u_h = Q_h^{-1}(P_h u_h) \in \prod_{T \in T_h} P_{k+2}(T). \tag{10}
\]
By definition, (8) holds. Further, because \( L_h Q_h \) is a stable (cf. (11) and (12) below), locally \( P_{k+2} \)-preserving operator (that is, \( L_h Q_h u = u \) if \( u \in P_{k+2}(T) \)), by [11], it is an optimal-order interpolation operator and (9) holds.

Theorem 4 Let \( u \) and \( u_h \) be the solutions of (1) and (6), respectively. Then
\[
\|u - L_h u_h\|_0 + h|u - L_h u_h|_{1,h} \leq C h^{k+3} |u|_{k+3},
\]
where \( |u|_{1,h}^2 = \sum_{T \in T_h} \langle \nabla u, \nabla u \rangle \).

Proof Noting the weak gradient of \( u_h - P_h u_h \) is a piece-wise higher order, \( \|\cdot\| \)-orthogonal polynomial over the polynomial \( \nabla L_h u_h \), we get the following triple-bar stability,
\[
|L_h u_h|_{1,h}^2 = \|P_h u_h\|^2 = \|u_h\|^2 - \|(I - P_h) u_h\|^2 \leq \|u_h\|^2. \tag{11}
\]
By the triangle inequality, (9) and (7),
\[
|u - L_h u_h|_{1,h} \leq |u - L_h Q_h u|_{1,h} + |L_h (Q_h u - u_h)|_{1,h} \leq C h^{k+3} |u|_{k+3} + \|Q_h u - u_h\| \leq C h^{k+2} |u|_{k+3}.
\]
By the linearity of $L_h$, Schwartz inequality, the trace inequality and the definition of weak gradient, we get the following $L^2$ stability,

$$
\|L_h u_h\|_0 \leq \sum_{T \in T_h} \left( \|Q_h^{-1} P_h \{0, u_0\}\|_T + \|Q_h^{-1} P_h \{0, u_0 - u_h\}\|_T \right)
$$

$$
\leq \sum_{T \in T_h} \left( \|u_0\|_T + Ch\|u_0 - u_h\|_{\partial T} \right)
$$

$$
\leq \|u_0\|_0 + Ch\|u_h\|.
$$

(12)

$$
L_h(Q_h u - u_h)\|_0 \leq \|Q_0 u - u_0\|_0 + Ch\|Q_h u - u_h\|.
$$

(13)

By the triangle inequality, (9), (13) and (7), we get

$$
\|u - L_h u_h\|_0 \leq \|u - L_h Q_h u\|_0 + \|L_h(Q_h u - u_h)\|_0 \leq Ch^{k+3}\|u\|_{k+3}.
$$

4 Numerical Experiments

![Fig. 1 The first three levels of quadrilateral grids, for Table 1.](image)

**Table 1** Errors and orders of convergence by the $P_1$-$P_2$ WG finite element on quadrilateral grids shown in Figure 1 for (14).

| level | $\|u - u_h\|_0$ | rate | $\|Q_h u - u_h\|_0$ | rate | $\|u - L_h u_h\|_0$ | rate |
|-------|----------------|------|----------------|------|----------------|------|
| 5     | 0.7356E-03     | 2.00 | 0.9360E-06     | 4.00 | 0.1308E-05     | 4.00 |
| 6     | 0.1838E-03     | 2.00 | 0.5851E-07     | 4.00 | 0.8178E-07     | 4.00 |
| 7     | 0.4595E-04     | 2.00 | 0.3663E-08     | 4.00 | 0.5116E-08     | 4.00 |
|       | $\|u - u_h\|_{1,h}$ | rate | $\|Q_h u - u_h\|_{1,h}$ | rate | $\|u - L_h u_h\|_{1,h}$ | rate |
| 5     | 0.5049E-01     | 1.00 | 0.2150E-03     | 3.00 | 0.2101E-03     | 3.00 |
| 6     | 0.2524E-01     | 1.00 | 0.2696E-04     | 3.00 | 0.2627E-04     | 3.00 |
| 7     | 0.1262E-01     | 1.00 | 0.3371E-05     | 3.00 | 0.3284E-05     | 3.00 |

We solve the 2D Poisson equation (1) on the unit square domain. The exact solution is chosen as

$$
u = \sin(\pi x) \sin(\pi y).$$

(14)
We compute the solution (14) on a perturbed quadrilateral grids, shown in Figure 1. We have two orders of superconvergence in $L^2$-norm and in $H^1$-like norm, shown in Tables 1-2. In particular, the error after lifting is two orders higher than that of the original error.

Next we solve again the 2D Poisson equation (1) on the unit square domain with exact solution (14). We use quadrilateral-pentagon-hexagon hybrid grids, shown in Figure 2. Again the error after lifting is two orders higher, shown in Table 3.

![Fig. 2 The first three levels of mixed-polygon grids, for Tables 3.](image)

### Table 2 Errors and orders of convergence by the $P_2$-$P_3$ WG finite element on quadrilateral grids shown in Figure 1 for (14).

| level | $\| u - u_h \|_0$ | rate | $\| Q_h u - u_h \|_0$ | rate | $\| u - L_h u_h \|_0$ | rate |
|-------|-----------------|------|-----------------|------|-----------------|------|
| 4     | 0.2229E-03      | 3.00 | 0.7659E-06      | 4.98 | 0.8555E-06      | 4.98 |
| 5     | 0.2787E-04      | 3.00 | 0.2404E-07      | 4.99 | 0.2682E-07      | 5.00 |
| 6     | 0.3484E-05      | 3.00 | 0.7521E-09      | 5.00 | 0.8390E-09      | 5.00 |

### Table 3 Errors and orders of convergence, by the $P_1$-$P_2$ WG finite element on mixed-polygon grids shown in Figure 2 for (14).

| level | $\| u - u_h \|_0$ | rate | $\| Q_h u - u_h \|_0$ | rate | $\| u - L_h u_h \|_{1,h}$ | rate |
|-------|-----------------|------|-----------------|------|-----------------|------|
| 5     | 0.1293E-01      | 2.00 | 0.1487E-03      | 3.99 | 0.9441E-04      | 3.99 |
| 6     | 0.3233E-02      | 2.00 | 0.9307E-05      | 4.00 | 0.5911E-05      | 4.00 |
| 7     | 0.8084E-03      | 2.00 | 0.5819E-06      | 4.00 | 0.3696E-06      | 4.00 |

Finally we solve the 3D Poisson equation (1) on the unit cube, with exact solution

$$u = \sin(\pi x) \sin(\pi y) \sin(\pi z).$$ (15)
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We use a wedge-type grids shown in Figure 3. The lifted finite element solution has two orders of superconvergence, shown in Table 4.

![Wedge grids](image)

**Fig. 3** The first three levels of wedge grids used in Table 4.

| level | $\|u - u_h\|_0$ rate | $\|Q_h u - u_h\|_0$ rate | $\|u - L_h u_h\|_0$ rate |
|-------|----------------|----------------|----------------|
| 4     | 0.9655E-02 2.0 | 0.1608E-03 3.9 | 0.2626E-03 3.9 |
| 5     | 0.2398E-02 2.0 | 0.1022E-04 4.0 | 0.1658E-04 4.0 |
| 6     | 0.5987E-03 2.0 | 0.6419E-06 4.0 | 0.1039E-05 4.0 |

| level | $\|u - u_h\|_{1,h}$ rate | $\|Q_h u - u_h\|_{1,h}$ rate | $\|u - L_h u_h\|_{1,h}$ rate |
|-------|----------------|----------------|----------------|
| 4     | 0.2289E+00 1.0 | 0.2500E-01 3.0 | 0.1269E-01 3.0 |
| 5     | 0.1145E+00 1.0 | 0.3136E-02 3.0 | 0.1595E-02 3.0 |
| 6     | 0.5724E-01 1.0 | 0.3923E-03 3.0 | 0.1997E-03 3.0 |

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