Graphene in complex magnetic fields

David J. Fernández C.\textsuperscript{a}, Juan D. García-Muñoz\textsuperscript{b}

Physics Department, Cinvestav, P.O.B. 14-740, 07000 Mexico City, Mexico

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Abstract Exact analytic solutions for an electron in graphene interacting with external complex magnetic fields are found. The eigenvalue problem for the non-Hermitian Dirac–Weyl Hamiltonian leads to a pair of intertwined Schrödinger equations, which are solved by means of supersymmetric quantum mechanics. The Hamiltonian eigenstates turn out to be non-stationary, since their associated energy eigenvalues are complex, with the corresponding discrete spectrum thus defining the so-called Landau levels. By means of an analogy with non-uniform strained graphene, a prospective physical interpretation and a possible experimental situation, where the effects of the complex magnetic field could be detected, are given. The probability and current densities are explored and some remarkable differences as compared with the real case are discussed.

1 Introduction
At the beginning of the twenty-first century a 2D-material known as graphene was isolated for the first time by Geim and Novoselov [1]. After this discovery a lot of work trying to delve into its properties have been done. In particular, its electronic properties are of great interest, as the integer quantum Hall effect where the charge carriers behave as massless chiral quasiparticles with a linear dispersion relation [2, 3]. Within this wide field of study, the work of Kuru, Negro and Nieto is worth mentioned [4], since they show how to use supersymmetric quantum mechanics (SUSY QM) in order to find exact analytic solutions for a class of Hamiltonians describing the interaction of electrons in graphene with external magnetic fields. Other authors have explored further this technique, finding as well interesting results [5–13]. However, as far as we know a research about generalizing the method to complex magnetic fields has not been done yet.

Although complex magnetic fields are considered to be non-physical, in recent years it has been found that their effective action on the coherence of many-body quantum systems can be detected [14, 15]. Motivated by these results in this paper we shall study, from a theoretical viewpoint, the effects caused on the electron behavior by a complex magnetic field applied orthogonally to the graphene surface. It is worth noticing that we see the complex magnetic field amplitude as a quantity whose norm coincides with the amplitude of a real magnetic field applied to the graphene while its argument is a parameter allowing us to adjust the difference in magnitude between the real and imaginary parts, since the last can be considered as a perturbation term causing a Zeeman-like splitting in the charge carrier energies whose states are non-stationary [16].

In standard graphene theory the effective Hamiltonian describing a charge carrier in a magnetic field is Hermitian [4, 16]. However, in recent times a lot of works dealing with the possibility of using non-Hermitian quantum Hamiltonians have appeared (see, e.g., [17–24]). As a first approach to non-Hermitian Hamiltonian for monolayer graphene we will take the magnetic field to be complex, since this allows us to introduce an anti-Hermitian term in the Hamiltonian modifying the Landau energy levels, the probability and current densities and so on, which makes also easy to compare with the real case. A possible physical implication could be observed experimentally as a result of this comparison. Moreover, analogous non-Hermitian Hamiltonians appear in other areas of physics, as optics [25], and such exact analytical solutions could be of interest as well for calculating numerical solutions [26].

The paper has been organized as follows: In Sect. 2 we will describe how SUSY QM works to solve the eigenvalue problem for the graphene effective Hamiltonian in a complex magnetic field; Sect. 3 addresses some examples where different complex magnetic profiles are taken, and for which the algorithm of the previous section can be applied. A discussion about a possible physical implication of the complex magnetic fields, an important effect induced by their non-null imaginary parts, as well as a possible experimental situation in which these effects could be detected, are given in Sect. 4; finally, in Sect. 5 we present our conclusions.
## 2 Effective Hamiltonian for graphene in complex magnetic fields

A hexagonal structure of carbon atoms in a honeycomb 2D-arrangement is called monolayer graphene, or simply graphene (see Fig. 1). In the study of this material one usually works with an effective Hamiltonian describing the hopping of an electron from one atom to any of its nearest neighbors [16, 27–29]. Such Hamiltonian can be written as a $2 \times 2$ matrix operator of the form

$$H = v_0 \begin{pmatrix} 0 & \pi \\ \pi & 0 \end{pmatrix},$$

(1)

where $v_0 = \sqrt{3} \gamma_0 / 2 \hbar$ is the Fermi velocity and $\pi = p_x - ip_y$. The quantity $a \approx 2.46 \text{Å}$ is the intramolecular distance in the graphene layer [30], while $\gamma_0 \approx 3.033 \text{eV}$ is known as the in-plane hopping parameter, which is equal to the negative binding energy between two adjacent carbon atoms [29].

On the other hand, $p_j$ is the momentum operator in the $j$th direction, with $j = x, y$.

Let us suppose that a complex magnetic field orthogonal to the graphene layer is applied, which varies only along one direction, e.g., $B(x) = B(x)e_z$, $B(x) \in \mathbb{C}$. In the Landau gauge the associated vector potential can be written as $A(x) = A(x)e_y$, where $B(x) = dA(x)/dx$. Taking into account the minimal coupling rule, the effective Hamiltonian (1) becomes now

$$H = v_0 \begin{pmatrix} p_x - ip_y - i \epsilon c A(x) & 0 \\ 0 & p_x + ip_y + i \epsilon c A(x) \end{pmatrix}. $$

(2)

Since $H$ is invariant under translations along $y$-direction, its eigenvectors can be expressed as:

$$\Psi(x, y) = N e^{iky} \begin{pmatrix} \psi^+(x) \\ i \psi^-(x) \end{pmatrix},$$

(3)

with $N$ being a normalization factor and $k$ the wavenumber in $y$-direction. In the coordinates representation the momentum operator $p_j$ can be written as $-i \hbar \partial_j$, with $\partial_j = \partial / \partial_j$, $j = x, y$, thus the eigenvalue equation for $H$ looks like

$$H \Psi(x, y) = \hbar v_0 \begin{pmatrix} 0 & -i \partial_x - \partial_y - i \epsilon c A(x) \\ -i \partial_x + \partial_y + i \epsilon c A(x) & 0 \end{pmatrix} \Psi(x, y) = E \Psi(x, y).$$

(4)

Using expression (3), matrix Eq. (4) is reduced to a coupled system of equations:

$$L^- \psi^-(x) \equiv \begin{pmatrix} d / dx + k + e \epsilon c A(x) \end{pmatrix} \psi^-(x) = E \psi^+(x),$$

(5)

$$L^+ \psi^+(x) \equiv \begin{pmatrix} -d / dx + k + e \epsilon c A(x) \end{pmatrix} \psi^+(x) = E \psi^-(x),$$

with $E = E / \hbar v_0$. It is important to realize that $L^+$ is not the Hermitian conjugate of $L^-$, $(L^-)^\dagger = -d / dx + k + e \epsilon c \bar{A}(x) \neq L^+$, with $\bar{z}$ being the complex conjugate of $z$. However, all the previous expressions are similar to the ones appearing in [4], although in

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**Fig. 1** Graphene structure. The intramolecular distance $a$ is traced.
such work the magnetic field is real. In order to decouple the system (5), let us apply $L^+$ on the first equation and $L^-$ on the second, which leads to:

\[
L^+ L^- \psi^-(x) = \left[ -\frac{d^2}{dx^2} + \left( k + \frac{e}{\hbar} A(x) \right)^2 - \frac{e}{\hbar} A'(x) \right] \psi^-(x) = \varepsilon \psi^-(x),
\]

\[
L^- L^+ \psi^+(x) = \left[ -\frac{d^2}{dx^2} + \left( k + \frac{e}{\hbar} A(x) \right)^2 + \frac{e}{\hbar} A'(x) \right] \psi^+(x) = \varepsilon \psi^+(x),
\]

where $\varepsilon = \varepsilon^2$ and $dA(x)/dx \equiv A'(x)$. It is natural to identify now

\[
H^- = L^+ L^- , \quad H^+ = L^- L^+ ,
\]

with $H^\pm$ being two non-Hermitian Hamiltonians fulfilling

\[
H^+ L^- = L^- H^-. 
\]

The intertwining relation (8), together with the expressions (5) for $L^\pm$ and the factorizations in Eq. (6), are the basis of the so-called supersymmetric quantum mechanics (SUSY QM) [31–36]. In fact, it is standard to denote

\[
L^\pm = \mp \frac{d}{dx} + w(x),
\]

with the complex function

\[
w(x) = k + \frac{e}{\hbar} A(x) 
\]

being called superpotential. Thus, the non-Hermitian Hamiltonians $H^\pm$ take the form

\[
H^\pm = -\frac{d^2}{dx^2} + V^\pm(x),
\]

where the complex SUSY partner potentials $V^\pm$ are written in terms of the superpotential as follows:

\[
V^\pm = w^2(x) \pm w'(x).
\]

Suppose now that $\psi^\pm_n(x)$ are the eigenfunctions of $H^\pm$ with eigenvalues $\varepsilon^\pm_n$, the quantum number $n$ being a nonnegative integer. We choose $H^-$ as the Hamiltonian having as one of its eigenvalues the null energy, i.e., $\varepsilon^-_0 = 0$. This automatically fixes the superpotential, since

\[
H^- \psi^-_0 = \varepsilon^-_0 \psi^-_0 = 0 \Rightarrow L^- \psi^-_0 = 0 = (\psi^-_0)' + w(x) \psi^-_0 \Rightarrow w(x) = - (\psi^-_0)' \psi^-_0, \quad (13)
\]

where equation (7) was used. As $\psi^-_0$ is square-integrable, the solution to $H^+ \psi^+_0 = 0$, which as well satisfies $L^+ \psi^+_0 = 0 \Rightarrow \psi^+_0 = 1/\psi^-_0$, is not square-integrable, thus $\varepsilon^-_0 = 0$ is not in the spectrum of $H^+$. However, the intertwining relation (8) guarantees that any other non-null eigenvalue of $H^-$ ($\varepsilon^-_n , n = 1, 2, \ldots$) belongs to the spectrum of $H^+$. Proceeding by analogy with the real case, we will denote $\varepsilon^-_{n-1} = \varepsilon^-_n$, thus the corresponding eigenstates $\psi^\pm_n(x)$ are interrelated through $L^\pm$ as follows:

\[
\psi^+_n(x) = \frac{L^- \psi^-_n(x)}{\sqrt{\varepsilon^-_n}} , \quad \psi^-_n(x) = \frac{L^+ \psi^+_n(x)}{\sqrt{\varepsilon^+_n}} , \quad n = 1, 2, \ldots \quad (14)
\]

Note that, despite $L^+$ is not the Hermitian conjugate of $L^-$, the second equation in (14) is fulfilled since the factorizations (7) imply that $H^- L^+ = L^+ H^-$, then $L^+ \psi^+_n(x)$ is an eigenfunction of $H^-$ with eigenvalue $\varepsilon^-_n$.

It is important to stress that the potentials $V^\pm(x)$ are only auxiliary tools to solve the original problem, thus they do not have physical meaning. However, they are typically shape-invariant SUSY partner potentials, since the factorization energy involved in Eq. (7) is the null eigenvalue associated with a Hamiltonian $H^-$ chosen as to have such a symmetry [34].

Let us remark that the derivative of the superpotential is directly related to the magnetic field amplitude as follows

\[
w'(x) = \frac{e}{\hbar} B(x). \quad (15)
\]

Coming back to our initial problem, the eigenvectors and eigenvalues of the Hamiltonian (2) describing a graphene layer in the complex magnetic fields are given by:

\[
\Psi_0(x, y) = e^{iky} \left( \begin{array}{c} 0 \\ i \psi_0^- (x) \end{array} \right) , \quad E_0 = 0,
\]

\[
\Psi_n(x, y) = e^{iky} \sqrt{2} \left( \begin{array}{c} \psi^+_n (x) \\ i \psi^-_n (x) \end{array} \right) , \quad E_n = \pm \hbar v_0 \sqrt{\varepsilon^-_n} . \quad (16)
\]
with \( n \in \mathbb{N} \). Let us mention that the energies \( E_n \) with the plus sign are associated with electrons while the ones with the minus sign to holes. In our examples below only the electron energies will be considered.

Before addressing such examples, let us define first two physical quantities that will help us to describe the electron behavior ruled by the Hamiltonian (2). Since such Hamiltonian is a piece of a \( 2 \times 2 \) diagonal-block supermatrix, whose nonzero extra block is the transpose of \( H \), we are in fact dealing with a Dirac-like problem [16]. However, it is sufficient to solve \( H \) in order to obtain the whole solution of the Dirac-like Hamiltonian characterizing the monolayer graphene. The first physical quantity to be explored is the probability density defined by

\[
\rho = \Psi^\dagger \Psi, \tag{17}
\]

while the second one is the probability current written as

\[
J = v_0 \Psi^\dagger \sigma_3 \Psi. \tag{18}
\]

The previous expression for \( J \) is the same for the real case given in [4] and for the free case in [37]; nevertheless, the continuity equation now is inhomogeneous, with the inhomogeneity term being given by

\[
\frac{2e v_0}{\hbar} \text{Im}[A(x)] \Psi^\dagger \sigma_3 \Psi. \tag{19}
\]

### 3 Exactly solvable cases

In these examples we shall take some magnetic profiles whose amplitude is the product of a complex constant times a real function of \( x \). We shall determine the corresponding superpotential, the auxiliary SUSY partner potentials and then the solutions to the original problem. It is worth noting that we shall solve first the potential \( V^-(x) \) and then, from its eigenfunctions and eigenvalues, the corresponding solutions of \( V^+(x) \) will be found. Moreover, any other parameter of the magnetic profile is supposed to be positive, unless otherwise specified.

#### 3.1 Constant magnetic field

The first magnetic profile we will consider is constant, i.e., \( B(x) = B e_z \), \( B \in \mathbb{C} \). In the Landau gauge the vector potential is \( A(x) = x B e_z \). Substituting this expression in Eq. (10) we get \( w(x) = k + \omega x / 2 \) with \( \omega = 2 e B / \hbar c \in \mathbb{C} \), and using Eq. (12) the auxiliary SUSY partner potentials become

\[
\begin{align*}
V^-(x) &= \frac{\omega^2}{4} \left( x + \frac{2 k}{\omega} \right)^2 - \frac{\omega^2}{2}, \\
V^+(x) &= \frac{\omega^2}{4} \left( x + \frac{2 k}{\omega} \right)^2 + \frac{\omega^2}{2}.
\end{align*} \tag{20}
\]

These are called complex harmonic oscillators [38], whose real and imaginary parts can be observed in Fig. 2. The corresponding eigenfunctions are given by

\[
\psi_n^\pm(x) = \begin{cases}
  c_n e^{-\frac{\xi^2}{2}} \mathcal{H}_n[\xi], & -\frac{\pi}{2} < \theta < \frac{\pi}{2}, \\
  c_n e^{-\frac{\xi^2}{2}} \mathcal{H}_n[\xi], & \frac{\pi}{2} < \theta < \frac{3\pi}{2},
\end{cases} \tag{21}
\]

with \( n \) being a nonnegative integer, \( \xi = \sqrt{\omega^2/2} (x + 2 k / \omega) \), \( \mathcal{H}_n[\xi] \) is a Hermite polynomial of degree \( n \) and complex argument [38]; we are denoting \( \sqrt{\omega} = \sqrt{|\omega|} e^{i \theta / 2} \) and \( \sqrt{\omega^2} = \sqrt{|\omega|} e^{i (\pi - \theta) / 2} \). The eigenvalues for the potentials (20) turn out to be

\[
\varepsilon^-_n = 0, \quad \varepsilon^+_n = \varepsilon^+_{n-1} = \pm n \omega, \tag{22}
\]

where \( n \) is a natural number, the upper sign + is taken for \( -\pi/2 < \theta < \pi/2 \) and the lower sign − for \( \pi/2 < \theta < 3\pi/2 \). Thus, the electron energies (16) for graphene in a constant complex magnetic field can be written as follows:

\[
E_n = \hbar v_0 \sqrt{\pm n \omega}, \tag{23}
\]

whose norms coincide with the result for the real case deduced in [4], but now they are rotated in the complex plane by an angle \( \theta/2 \) with respect to the positive real line (see Fig. 3a). In that plot it can be observed as well concentric circumferences of radius \( R \propto \sqrt{n |\omega|} \) centered at the origin, on which the energy \( E_n \) is located regardless of the angle \( \theta \). This leads us to conclude that despite its complex nature, for a fixed angle \( \theta \) the spectrum of \( H \) is ordered in the standard way. Moreover, \( \text{Sp}(H) \) is infinite discrete, and its energies do not depend on \( k \). In Fig. 3b it is shown the real and imaginary parts of the first energy eigenvalues as functions of \( k \). The square-integrability of \( \Psi_n(x, y) \) does not impose any constraint to the norm of \( \omega \), but it does on its argument \( \theta \), as it is shown.
Fig. 2 Real a and imaginary part b of the complex harmonic oscillator potentials $V^{\pm}$ and the associated magnetic field. The chosen parameters are $|\omega| = k = 1$ and $\theta = \pi/10$.

Fig. 3 a First energy eigenvalues in the complex plane for the constant magnetic profile with three different angles $\theta$. The ground state is the same for all these $\theta$-values and it is drawn as a red circle at the origin; the other potential parameters were taken as $|\omega|= k = 1$. b Real (top) and imaginary (bottom) part of the first five energy eigenvalues as functions of $k$ for $|\omega|= 1$ and $\theta = \pi/10$.

in Eq. (21). Furthermore, when $\theta = \pm \pi/2$ the eigenfunctions $\psi_n^{\pm}(x)$ are not square-integrable, since in this case $V^{\pm}(x)$ in Eq. (20) become inverted oscillator potentials [39], displaced by some imaginary quantities in the coordinate $x$ as well as in the energy origin, and thus the Hamiltonian $H$ does not have bound state solutions at all. The probability and current densities are drawn in Fig. 4 for
Fig. 4 Probability density (top), current density in the x-direction (middle) and in the y-direction (bottom) for the constant magnetic field. The potential parameters taken are $|\omega| = k = 1$ and $\theta = \pi/10$

Fig. 5 Real (top) and imaginary part (bottom) of the complex trigonometric Rosen–Morse potentials and the corresponding magnetic field. The chosen potential parameters are $|D| = 4, \theta = \pi/10, k = -2, \mu = 1$

the first four bound states. Note that the ”ground state”, for $n = 0$, does not have associated any current density, since its upper entry is zero, as it is seen in Eq. (16).

3.2 Trigonometric singular well

In this case a complex magnetic field of trigonometric form is taken, $B(x) = B \csc^2(\mu x)e_z$, $B \in \mathbb{C}$, $\mu \in \mathbb{R}^+$. The vector potential is given by $A(x) = (-B/\mu) \cot(\mu x)e_y$, thus it is straightforward to obtain the superpotential as $w(x) = k - D\cot(\mu x)$, with $D = eB/c\hbar \mu$. Hence, the auxiliary potentials now acquire the form

$$V^-(x) = D(D - \mu)\csc^2(\mu x) - 2Dk\cot(\mu x) + k^2 - D^2,$$

$$V^+(x) = D(D + \mu)\csc^2(\mu x) - 2Dk\cot(\mu x) + k^2 - D^2.$$  \hspace{1cm} (24)

These expressions suggest to call the previous $V^\pm(x)$ as complex trigonometric Rosen–Morse potentials. Their real and imaginary parts are plotted in Fig. 5. The corresponding eigenfunctions are given in terms of Jacobi polynomials $P_n^{(\alpha, \beta)}(\xi)$ with complex argument and indexes, namely

$$\psi_n^j(x) = c_n(-1)^{-(s_j+n)/2}(1 + \xi^2)^{-(s_j+n)/2}e^{s_j\arccot(\xi)}P_n^{(-s_j-n-ir_j, -s_j-n+ir_j)}(i\xi), \hspace{1cm} j = \pm.$$  \hspace{1cm} (25)
Using now the polar form $D$ out to be infinite discrete, as it can be seen in Fig. 6b. Lastly, plots of the probability and current densities are displayed in Fig. 7 for expression in Eq. (12), we get the auxiliary SUSY partner potentials $E_n$ parameters. The first electron energies on the complex plane are shown in Fig. 6a. It can be observed as well concentric ellipses belonging to the ellipse whose semimajor axis coincides with the $\theta$ case regardless of the value of $\theta$. However, this happens only in the interval $(-\pi/2 < \theta < \pi/2)$. The spectra of the Hamiltonians $H^\mathbb{R}$ consist of the complex eigenvalues

$$
\epsilon_n = \epsilon_{n-1} = k^2 - D^2 + (D + n\mu)^2 - \frac{k^2 D^2}{(D + n\mu)^2}, \quad n \in \mathbb{N}.
$$

Substituting these expression in Eq. (16) the electron energies for the complex trigonometric singular magnetic field turn out to be

$$
E_n = \hbar v_0 \sqrt{k^2 - D^2 + (D + n\mu)^2 - \frac{k^2 D^2}{(D + n\mu)^2}}.
$$

We must mention that the norm of $E_n$ is different in general from the result for the real case shown in [4], except in the case with $\theta = 0$. In fact, the argument of $E_n$ depends in a non-trivial way of $\theta$, and it shows also a strong dependence on the potential parameters. The first electron energies on the complex plane are shown in Fig. 6a. It can be observed as well well concentric ellipses centered at the origin, with the energy $E_n$ belonging to the ellipse whose semimajor axis coincides with the $n$th energy in the real case regardless of the value of $\theta$. As in the previous case this fact implies that, for a fixed angle $\theta$, $\text{Sp}(H)$ is ordered in the standard way. However, this happens only in the interval $(-k_0, k_0)$, where $k_0$ is such that $\text{Im}(E_1(k_0)) = 0$. Hence, the spectrum of $H$ turns out to be infinite discrete, as it can be seen in Fig. 6b. Lastly, plots of the probability and current densities are displayed in Fig. 7 for the first four bound states.

### 3.3 Exponentially decaying magnetic field

Our last example is an exponentially decaying complex magnetic field $B(x) = Be^{-\mu x}e_z$, $B \in \mathbb{C}$, $\mu \in \mathbb{R}^+$, whose vector potential is $A(x) = -(B/\mu)e^{-\mu x}e_y$. According to Eq. (10), the superpotential is given by $w(x) = k - D e^{-\mu x}$, $D = eB/\hbar\mu$. Inserting this expression in Eq. (12), we get the auxiliary SUSY partner potentials

$$
V^- = k^2 + D^2 e^{-2\mu x} - 2D(k + \frac{\mu}{2}) e^{-\mu x},
$$

$$
V^+ = k^2 + D^2 e^{-2\mu x} - 2D(k - \frac{\mu}{2}) e^{-\mu x}.
$$

which are the Morse potentials, but with the parameter $D$ being now complex. Their real and imaginary parts are shown in Fig. 8. These potentials are as well exactly solvable, the corresponding eigenfunctions are given by

$$
\psi_j^\pm(n) = c(n)(\xi)^{j-n} e^{-\frac{\mu}{2}} \mathcal{L}_{\frac{n}{2}}^2(\xi), \quad j = \pm,
$$

where $s_- = D/\mu$, $s_+ = s_- + 1$, $r_- = -kD/\mu(D + n\mu)$, $r_+ = -kD/\mu(D + \mu + n\mu)$, $\zeta = \cot(\mu x)$ and $n$ is a nonnegative integer.

Fig. 6 a First electron energies $E_n$ for the trigonometric singular well in the complex plane with three different angles. The ground state is marked as a red circle at the origin, and the other potential parameters are $|D| = 4$, $k = -2$, $\mu = 1$. b Real (top) and imaginary part (bottom) of $E_n$ as functions of $k$ for $|D| = 4$, $\theta = \pi/10$, $\mu = 1$.
Fig. 7 Probability density (top), current density in x-direction (middle) and y-direction (bottom) in the case of a trigonometric singular well. The potential parameters taken are $|D| = 4, \theta = \pi/10, k = -2$ and $\mu = 1$

where $s_+ = k/\mu$, $s_- = s_+ - 1$, $\zeta = (2D/\mu)e^{-\mu x}$, $n$ is a nonnegative integer and $L_\xi^\nu(\zeta)$ is an associated Laguerre polynomial of complex argument. The polar form $D = |D|e^{i\theta}$ allows us to deduce the square-integrability conditions: $-\pi/2 < \theta < \pi/2$ and $k > n\mu$. The corresponding eigenvalues are

$$\varepsilon^-_n = 0, \quad \varepsilon^+_n = \frac{\varepsilon^-_{n-1}}{k^2 - (k - n\mu)^2},$$

with $n$ being a natural number. It is worth stressing that the spectra of $H^\pm$ are real, since unlike the previous cases now these Hamiltonians are pseudo-Hermitian \[23, 24\]. Hence, the energy eigenvalues for the graphene electron in the exponentially decaying complex magnetic field turn out to be

$$E_n = \hbar v_0\sqrt{k^2 - (k - n\mu)^2}.$$  \(31\)

Note that, in this example, $\mathrm{Sp}(H)$ coincides exactly with the spectrum for the real case addressed in \[4\]. Such spectrum is finite discrete, since once the parameters $k$ and $\mu$ are fixed, the condition $k > n\mu$ limits the number of square-integrable eigenfunctions and hence the number of allowed electron energies (see Fig. 9). Moreover, there is an enveloping line, also shown in Fig. 9b, whose slope (equal to $v_0$) represents the average y-velocity. This line separates the $k$-domain into two subsets, one where there are bound states and the other where there are just scattering states. Finally, the probability and current densities are plotted in Fig. 10.

4 Discussion

Since the Hamiltonian (2) is non-Hermitian, its eigenvalues are not necessarily real. In fact, it can be written as $H/v_0 = \sigma_x p_x + \sigma_y [p_y + (e/c)A(x)]$. Now, if the vector potential is expressed in polar form it turns out that

$$H = H_R + iv_0(e/c)\sigma_y |A(x)| \sin \theta,$$

where $H_R = v_0[\sigma_x p_x + \sigma_y (p_y + (e/c)|A(x)| \cos \theta)]$ is a Hermitian operator whose eigenvalues are real, which is similar to the Hamiltonian addressed in \[4\]. The second term of Eq. (32) is an anti-Hermitian operator whose eigenvalues are purely imaginary. In order to understand the nature of this term, let us remember that the Dirac–Weyl equation in graphene describes a massless pseudo-spin \(1/2\) particle, where pseudo-spin ‘up’ means that the electron is in the sublattice B and ‘down’ in the sublattice A. In terms of the pseudo-spin ladder operators $S_{\pm} = S_x \pm iS_y$ it follows that the second term can be written as $(ev_0/c\hbar)(S_+ - S_-)|A(x)| \sin \theta$. It induces pseudo-spin rotations, and it is analogous to the corresponding term that appears in the Hamiltonian describing non-uniformly strained graphene (see \[40\]). By sticking to this analogy, this anti-Hermitian term can be associated with the layer curvature induced by strain \[41\]. However, the issue is actually to find a measurable phenomenon that could be associated with the anti-Hermitian term.
Fig. 8 Real (top) and imaginary part (bottom) of the complex Morse potentials and the corresponding magnetic field. The potential parameters taken are $|D|=1, \theta = \pi/10, k = 6, \mu = 1$.

Fig. 9 a First electron energies $E_n$ for the exponentially decaying magnetic field in the complex plane for $k = 6, \mu = 1$. b Electron energies $E_n$ as functions of $k$ for the same $\mu$ value as in a.

in (32). Then, since such Hamiltonian is time-independent, the time evolution of the total probability associated with any eigenstate acquires an exponential factor which depends on the imaginary part of its associated eigenvalue $E_n$, namely

$$\mathcal{P}_T(t) = \langle \Psi_n(t) | \Psi_n(t) \rangle = e^{\frac{\text{Im}E_n}{\hbar} t} \langle \Psi_n(0) | \Psi_n(0) \rangle.$$  

Note that a small probability increase (decrease) appears for $2t\text{Im}E_n/\hbar \ll 1$, which occurs for approximate times inversely proportional to the imaginary part of the eigenvalue. Using the polar form $E_n = |E_n|e^{i\phi_n}$ for the first bound states, it turns out that for $\phi_n \ll 1$ we obtain long enough times for guaranteeing the probability conservation, except for a small perturbation term. To clarify this idea, let us raise a physical situation where in principle we could observe the effects of the non-Hermitian term in (32), such as a non-null current density in $x$-direction, differences in the energy eigenvalues as compared with the Hermitian case, and so on. Thus, let us consider a monolayer graphene placed in a constant homogeneous magnetic field, an experimental situation similar to the one used in [2, 3] for addressing the quantum Hall effect in graphene to observe the corresponding Landau levels (LL). Let us suppose that this system is described by the non-Hermitian Hamiltonian (2), then the electrons in graphene would be in one of the LL given by Eq. (23). We must mention that these LL do not correspond anymore to stationary states, due to the non-null imaginary part of $E_n$. In nature it is more common to observe decaying systems, thus let us choose the imaginary parts of $E_n$ as negative.
numbers. As discussed previously the energy argument is $\phi \approx \theta/2$, then the maximum time at which the probability density is approximately conserved fulfills

$$ T \ll \frac{1}{v_0 \theta \sqrt{|w|}}. \quad (34) $$

In particular, for the first excited state with a magnetic induction of 5 T and $\theta = \pi/36$ it turns out that $T \ll 95$ fs. After such a $T$ the first excited state has decayed noticeably, e.g., at 1 ps its probability density will be 26 millionths of the initial density, namely 38461 times smaller than at the beginning. Note that another eigenstates decay in a similar way. Consequently, some physical quantities as the probability and current densities will quickly vanish in time. However, this does not mean that for any time $t \gg T$ the system finish without bound states, since as $\theta \ll 1$ the anti-Hermitian term in (32) is a perturbation producing a Zeeman-like splitting of the electron energies in the complex plane (see [16] and Fig. 3a). Therefore, the quantum states corresponding to the graphene complex LL are approximately stationary states of the Hamiltonian (2). Thus, if the devices used in the experimental situation described above are not sensitive enough to make a quick measurement (in a small time lapse of the order of femtoseconds) we will not be able to detect the effects of the anti-Hermitian term of the Hamiltonian (32), but only the effects of the corresponding standard graphene LL.

It is possible that in the future some experiments could check these predictions for the complex LL associated with non-stationary eigenstates. Nevertheless, the theoretic model of graphene in complex magnetic fields described by the effective Hamiltonian (2) could find analogies in some optical systems, such as optical fibers or nanocrystals. In particular, in [25] it is addressed the so-called photonic graphene, in which a non-Hermitian term similar to the one in (32) has been used for modeling the gain or loss in optical fibers. In addition, the exact analytical solutions obtained in this work could be used as well to test some numerical algorithms for generating solutions in graphene and other physical systems [26].

Now, it is interesting to observe that there are some $x$-points for which the non-trivial imaginary parts of $eB(x)/c\hbar$ and $V(x)$ are equal, as it is shown in Figs. 2, 5 and 8. If we denote as $\chi$ one of these points in order to fulfil Eq. (12), it turns out that $\text{Im}[w(\chi)] = 0$, and this implies that $\text{Re}[w(\chi)] = 0$. Let us recall now a classical quantity, the ‘kinematical’ momentum along $y$-direction, which is given by $p_y = p_y + (e/c)A$. Since the canonical momentum $p_y$ is a constant of motion, it follows that $\text{Re}[p_y(\chi)] = \hbar \text{Re}[w(\chi)] = 0$. Note that the maximum of the ground state probability density appears at the point $\chi$, and the latter depends on the angle $\theta$ (see Figs. 4, 7 and 10). It is worth noticing that despite the Hamiltonian is non-Hermitian, this does not ensure its eigenvalues to be complex, as it was seen in the third example of the previous section where the eigenvalues were real, and thus the total probability of its associated eigenstates was conserved. We must mention as well an interesting case where the anti-Hermitian term in the Hamiltonian (32) cannot be considered as a perturbation: in the limit $\theta \to \pi/2$ our Hamiltonian describes the free graphene [37] plus an interaction term between the pseudo-spin electron and a purely imaginary magnetic field. All the magnetic profiles of section 3 lead us to auxiliary potentials without bound states in this limit. However, if we choose an argument
$\theta \approx \pi/2$ we will get potentials with “weak” bound states, whose probability densities have a pronounced maximum that diverges in the limit when $\theta$ tends to $\pi/2$.

5 Conclusions

Exact analytic solutions for non-Hermitian Hamiltonian $H$ describing an electron in graphene interacting with external complex magnetic fields were found. It is worth noticing that $H$ can be expressed as the sum of a Hermitian operator plus a non-Hermitian term, the last causing pseudo-spin rotations. Although we have solved the eigenvalue problem for the Hamiltonian (2) assuming that complex magnetic fields in principle can be produced, a possible physical interpretation could be that a real magnetic field, with the same amplitude as the complex one, is applied to graphene. The angle $\theta$ is just a parameter allowing us to introduce in the Hamiltonian a perturbation term inducing a Zeeman-like splitting of the Landau levels, while the corresponding eigenstates in the complex plane turn out to be non-stationary (decaying in times of the order of femtoseconds). Such effect could be observed, if the quantum Hall effect would be detected for times small enough to avoid the decay of the corresponding eigenstates. It is worth mentioning the possible application of non-Hermitian Hamiltonian of kind (32) in other areas of physics, as optics where it has direct analogies with the so-called photonic graphene, as well as in numerical models for calculating solutions in 2D materials. Due to the complex nature of the problem, there are important differences in some physical quantities with respect to the real case, as the probability and current densities. In particular, the current density along $x$-direction turns out to be non-null, in contrast with the result found for the real case. Furthermore, the ground state probability density acquires its maximum now at the point where the imaginary parts of $eB/\hbar c$ and $V^*$ become equal.

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Data Availability All data generated or analyzed during this study are included in this published article.

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