Mismatched Quantum Filtering and Entropic Information

Mankei Tsang*

* Department of Electrical & Computer Engineering, National University of Singapore
4 Engineering Drive 3, Singapore 117583

† Department of Physics, National University of Singapore
2 Science Drive 3, Singapore 117551
Email: eletmk@nus.edu.sg

Abstract—Quantum filtering is a signal processing technique that estimates the posterior state of a quantum system under continuous measurements and has become a standard tool in quantum information processing, with applications in quantum state preparation, quantum metrology, and quantum control. If the filter assumes a nominal model that differs from reality, however, the estimation accuracy is bound to suffer. Here I derive identities that relate the excess error caused by quantum filter mismatch to the relative entropy between the true and nominal observation probability measures, with one identity for Gaussian measurements, such as optical homodyne detection, and another for Poissonian measurements, such as photon counting. These identities generalize recent seminal results in classical information theory and provide new operational meanings to relative entropy, mutual information, and channel capacity in the context of quantum experiments.

I. INTRODUCTION

Long regarded as an afterthought in the development of quantum mechanics, the probabilistic nature of quantum measurements is now taking the center stage in theoretical and experimental physics [1], [2]. Quantum probability theory will inevitably play a more prominent role in not just fundamental science but also future technology, which will require increasingly precise estimation and control of physical devices in the quantum regime.

Most of the current quantum information processing technology relies on continuous electromagnetic fields to measure and control quantum devices. The Bayesian quantum filtering theory, pioneered by Belavkin [3], enables one to estimate the state of a quantum system from a continuous field measurement record and has therefore become a standard tool in the area. The theory is applicable to a wide range of current experiments, including those on atoms, mechanical oscillators, or superconducting circuits interacting with optical or microwave fields [4]. Foreseeable applications include, but are not limited to, quantum state preparation, quantum error correction, quantum metrology, and fundamental tests of quantum mechanics [4], [5], [6], [7].

From a decision-theoretic point of view, the Bayesian theory is optimal only if the model perfectly matches the reality. In practical situations, however, assumptions and approximations will introduce excess systematic errors. General theoretical results concerning mismatched estimation are highly desirable for practical filter design purposes but difficult to obtain, especially if the dynamics is nonlinear. In this regard, a few interesting identities that relate mismatched estimation to relative entropy for classical Gaussian or Poissonian channels have recently been discovered [8], [9], [10], [11], building upon earlier seminal work that relates estimation theory to Shannon mutual information [12], [13], [14]. These relations open up novel research directions and have already proved useful for deriving a variety of new results, as they enable a fresh attack on estimation problems using information-theoretic tools, and vice versa.

In this paper, I generalize two of these identities to the quantum regime and relate mismatched quantum filtering errors to relative entropy for continuous Gaussian or Poissonian measurements. Given the plethora of new results that have since been spawned from the classical relations, the quantum relations are envisioned to be similarly fruitful in both quantum estimation theory and quantum information theory and ultimately useful for quantum filter design.

II. MISMATCHED QUANTUM FILTERING

For Gaussian measurements, such as homodyne detection of an optical beam interacting with a quantum system, define

\[ Y_t \equiv \{ y_\tau, 0 \leq \tau \leq t \} \quad (1) \]

as the observation record up to time \( t \). The posterior statistics of the quantum system can be determined from the linear Belavkin equation [4], [15], [16], a quantum generalization of the Duncan-Mortensen-Zakai equation:

\[ df_t(Y_t) \equiv f_{t+dt}(Y_{t+dt}) - f_t(Y_t) \]
\[ = \mathcal{L}_t f_t(Y_t) dt + \frac{1}{2} \left[ a_t f_t(Y_t) + f_t(Y_t) a_t^\dagger \right] dy_t, \quad (2) \]

where \( f_t(Y_t) \) is the unnormalized posterior density operator in the Hilbert space for the quantum system, \( a_t \) is an operator that characterizes the interaction between the system and the probe, such that

\[ q_t \equiv \frac{1}{2} \left( a_t + a_t^\dagger \right) \quad (3) \]

is the system observable being measured, \( \mathcal{L}_t \) is a Lindblad superoperator that describes the system dynamics, including...
the effect of measurement backaction as a function of $a_t$, and $dy_t$ is the increment of the observation process defined as $dy_t \equiv y_{t+dt} - y_t$, with $dy_t^2 = dt$. The initial condition is given by the initial density operator $\rho_0$:

$$f_0 = \rho_0.$$  

(4)

Measurement-based feedback control can be modeled by making $(a_t, \mathcal{L}_t)$ depend on $Y_t$.

The expectation of a function $g(O_t, Y_t)$ in terms of any observable $O_t$ is given by

$$\mathbb{E} g(O_t, Y_t) = \int dP_0(Y_t) \, \text{tr} \, f_t(Y_t) g(O_t, Y_t),$$

(5)

where $dP_0(Y_t)$ is the probability measure for the standard Wiener process. The probability measure of an observation record is thus

$$dP(Y_t) = \mathbb{E} \, 1_{Y_t} = \int dP_0(Y_t) \, \text{tr} \, f_t(Y_t),$$

(6)

where $1_{Y_t}$ is the indicator function. The conditional expectation of an observable $O_t$ is given by

$$\mathbb{E}(O_t|Y_t) = \frac{\mathbb{E}(O_t 1_{Y_t})}{\mathbb{E} 1_{Y_t}} = \frac{\text{tr} \, f_t(Y_t) O_t}{\text{tr} \, f_t(Y_t)} = \rho_t(Y_t) O_t,$$

(7)

with the normalized posterior density operator given by

$$\rho_t(Y_t) = \frac{\text{tr} \, f_t(Y_t)}{\text{tr} \, f_t(Y_t)}.$$ 

(8)

Define a filtering estimator of the observable $q_t$ as $\hat{q}_t(Y_t)$. A common measure of the filtering error is

$$\text{cmse}_t \equiv \mathbb{E} \, (q_t - \hat{q}_t(Y_t))^2,$$

(9)

where cmse is short for causal mean-square error. It is not difficult to show that the quantum conditional expectation of $q_t$ minimizes cmse, analogous to the classical case:

$$\text{cmmse}_t \equiv \inf_{\hat{q}_t(Y_t)} \text{cmse}_t = \mathbb{E} \, (q_t - \mathbb{E}\{q_t|Y_t\})^2.$$ 

(10)

The amount of error in excess of the minimum value is called regret in decision theory. For mismatched quantum filtering with Gaussian measurements, I define a regret quantity as the excess mean-square error integrated over time:

$$\Pi \equiv \frac{1}{2} \int_0^T dt \, (\text{cmse}_t - \text{cmmse}_t),$$ 

(11)

where the factor of $1/2$ is for later technical convenience.

### III. Quantum Hypothesis Testing

Consider now a different statistical problem: the discrimination of two quantum models via continuous Gaussian measurements. A central quantity in this binary hypothesis testing problem is the likelihood ratio, defined as

$$\Lambda(Y_T) \equiv \frac{dP(Y_T)}{dP'(Y_T)},$$

(12)

assuming the second model. Eq. (6) enables one to relate $\Lambda(Y_t)$ to the quantum filters as

$$\Lambda(Y_T) = \frac{\text{tr} \, f_T(Y_T)}{\text{tr} \, f'_T(Y_T)},$$

(13)

where $f'$ obeys another linear Belavkin equation that assumes the second model:

$$df'_t(Y_t) = \mathcal{L}'_t f'_t(Y_t) dt + \frac{1}{2} \left[ a'_t f'_t(Y_t) + f'_t(Y_t) a'_t^\dagger \right] dy_t,$$

(14)

with the measured observable defined as

$$q'_t \equiv \frac{1}{2} \left( a'_t + a'_t^\dagger \right),$$

(15)

and the initial condition given by

$$f'_0 = \rho'_0.$$ 

(16)

The conditional expectation assuming the second model becomes

$$\mathbb{E}'(O'_t|Y_t) = \frac{\text{tr} \, f'_t(Y_t) O'_t}{\text{tr} \, f'_t(Y_t)} = \rho'_t(Y_t) O'_t.$$

(17)

The following identity, first derived in Ref. [6] and generalizing a classical result by Duncan [18], will be useful:

**Lemma 1.** The log-likelihood ratio for two quantum models under continuous Gaussian measurements satisfies

$$\ln \Lambda(Y_T) = \int_0^T dy_t \left[ \mathbb{E}(q_t|Y_t) - \mathbb{E}'(q'_t|Y_t) \right] - \frac{1}{2} \int_0^T dt \left[ \mathbb{E}^2(q_t|Y_t) - \mathbb{E}'^2(q'_t|Y_t) \right],$$

(18)

where $\mathbb{E}(q_t|Y_t)$ and $\mathbb{E}'(q'_t|Y_t)$ are the filtering conditional expectations of the measured observable under the two models.

**Proof:** Tracing over Eqs. (2) and (14), one obtains $\text{tr} \, df_t = \text{tr} \, f_t = \mathbb{E}(q_t|Y_t) \, dy_t \, \text{tr} \, f_t$ and $\text{tr} \, df'_t = \mathbb{E}'(q'_t|Y_t) \, dy_t \, \text{tr} \, f'_t$, as the trace of a Lindblad superoperator on any operator is zero. Ito calculus can then be used to compute $d \ln \text{tr} \, f_t = d \ln \text{tr} \, f_t \, / \, \text{tr} \, f_t - (d \ln \text{tr} \, f_t \, / \, \text{tr} \, f_t)^2 / 2 = \mathbb{E}(q_t|Y_t) \, dy_t - \mathbb{E}^2(q_t|Y_t) \, dt / 2$, where the last step uses $dy_t^2 = dt$. A similar formula can be derived for $d \ln \text{tr} \, f'_t$. Integrating $d \ln \text{tr} \, f_t$ and $d \ln \text{tr} \, f'_t$ over time and plugging them into Eq. (13) results in Eq. (15).

The relative entropy between the two probability measures is defined as the expectation of the log-likelihood ratio $\ln \Lambda(Y_T)$ assuming the first model:

$$D(dP||dP') \equiv \mathbb{E} \ln \Lambda(Y_T),$$

(19)

which is a well known information quantity relevant to many statistical applications [19].
IV. Filter Regret and Relative Entropy

The first main result of this paper is the following theorem, generalizing a classical result by Weissman [9]:

**Theorem 1.** For continuous Gaussian measurements, the regret for mismatched quantum filtering is equal to the relative entropy between the true and nominal observation probability measures; viz.,

\[ \Pi = D(dP || dP^\prime). \]  

**Proof:** Substituting Eq. (18) in Lemma 1 into Eq. (19) and interchanging the order of integration and expectation,

\[ D(dP || dP^\prime) = \int_0^T E \left\{ dq \left[ E(q_t | Y_t) - E'(q'_t | Y_t) \right] \right\} - \frac{1}{2} \int_0^T dt E \left[ E^2(q_t | Y_t) - E^2(q'_t | Y_t) \right]. \]  

(21)

For the first expectation, one can use the orthogonality principle of the conditional expectation to write

\[ E \left\{ dq \left[ E(q_t | Y_t) - E'(q'_t | Y_t) \right] \right\} = E \left\{ E(dq_t | Y_t) [E(q_t | Y_t) - E'(q'_t | Y_t)] \right\} = E \left\{ E(q_t | Y_t) dt \left[ E(q_t | Y_t) - E'(q'_t | Y_t) \right] \right\}, \]  

(22)

where the second step follows from the martingale property of the quantum innovation process \( E(dq_t - E(q_t | Y_t) dt | Y_t) = 0 \) [8, 4, 15, 16]. This results in

\[ D(dP || dP^\prime) = \frac{1}{2} \int_0^T dt E \left[ E(q_t | Y_t) - E'(q'_t | Y_t) \right]^2. \]  

(24)

The regret given by Eq. (11), on the other hand, is

\[ \Pi = \frac{1}{2} \int_0^T dt E \left\{ \left(q_t - E'(q'_t | Y_t) \right)^2 - \left(q_t - E(q_t | Y_t) \right)^2 \right\} \]  

(25)

\[ = \frac{1}{2} \int_0^T dt E \left\{ 2q_t \left[ E(q_t | Y_t) - E'(q'_t | Y_t) \right] + E^2(q'_t | Y_t) - E^2(q_t | Y_t) \right\} \]  

(26)

\[ = \frac{1}{2} \int_0^T dt E \left\{ 2E(q_t | Y_t) \left[ E(q_t | Y_t) - E'(q'_t | Y_t) \right] + E^2(q'_t | Y_t) - E^2(q_t | Y_t) \right\} \]  

(27)

\[ = \frac{1}{2} \int_0^T dt E \left[ E(q_t | Y_t) - E'(q'_t | Y_t) \right]^2 = D(dP || dP^\prime), \]  

(28)

where Eq. (27) uses the orthogonality principle for the quantum conditional expectation \( E \{ q_t - E(q_t | Y_t) | g(Y) \} = 0 \) [16], which is valid for any \( g(Y) \).

Apart from the assumption of continuous Gaussian measurements, Theorem 1 is applicable to arbitrary time \( T \) and rather general quantum Markov models, which shall hereafter be denoted by

\[ \mathcal{M} \equiv \{ \rho_0, a_t, \mathcal{L}_t; 0 \leq t \leq T \}, \]  

(29)

\[ \mathcal{M}' \equiv \{ \rho'_0, a'_t, \mathcal{L}'_t; 0 \leq t \leq T \}. \]  

(30)

The theorem is also applicable to adaptive models, if one makes \( (a_t, \mathcal{L}_t) \) and/or \( (a'_t, \mathcal{L}'_t) \) depend on \( Y_t \).

V. Implications

A. Bayes Quantum Filtering and Mutual Information

Suppose that the model \( \mathcal{M} \) is chosen from an ensemble \( \{ \pi(\theta), \mathcal{M}_\theta \} \) parametrized by \( \theta \). The prior probability measure for \( \theta \) is defined as \( d\pi(\theta) \), the expectation under which is denoted by \( E_\theta \). Assume that the true model has access to the exact \( \theta \), or \( \mathcal{M} = \mathcal{M}_\theta \), such that \( E(q_t | Y_t) = E(q_t | Y_t, \theta) \) and \( dP(Y_T) = dP_\theta(Y_T) \), but the nominal model does not. Theorem 1 can then be used to relate the expected regret for not knowing \( \theta \) to the cross-information:

\[ E_\theta \Pi = E_\theta D(dP_\theta || dP^\prime). \]  

(31)

If the nominal model has access to \( d\pi(\theta) \), the optimal filter should be a Bayes estimator, and \( \inf_{\mathcal{M}'} E_\theta \Pi \) is the Bayes regret. This turns out to be equal to the mutual information:

**Corollary 1.** The Bayes ignorance regret is equal to the mutual information; viz.,

\[ \inf_{\mathcal{M}'} E_\theta \Pi = I(\theta; Y) \equiv E_\theta D(dP_\theta || E_\theta dP_\theta). \]  

(32)

**Proof:** The Bayes filter that minimizes \( E_\theta \text{cmse} \) and therefore \( E_\theta \Pi \) is \( E'(q'_t | Y_t) = E_\theta [E(q_t | Y_t, \theta) | Y_t], \) with \( dP^\prime = E_\theta dP_\theta \). Substituting \( dP^\prime = E_\theta dP_\theta \) into Eq. (31) results in Eq. (32).

**Remark.** The classical relation between mutual information and filtering error [12] can be derived from Corollary 1 by setting \( \theta = \{ q_\tau, 0 \leq \tau \leq T \} \) and noting that \( E(q_t | Y_t, \theta) = q_t \) and \( \text{cmse} = 0 \). In the quantum case, the history of an observable has questionable decision-theoretic meaning unless it is a quantum nondemolition observable [21], but the more general Corollary 1 still holds.

Corollary 1 gives a new operational meaning to mutual information as a measure of parameter importance in quantum filtering: high \( I(\theta; Y) \) means more regret for not knowing \( \theta \) and \( \theta \) is thus worth knowing in the context of filtering, while low \( I(\theta; Y) \) means less regret for ignoring \( \theta \).

B. Minimax Quantum Filtering and Channel Capacity

If the prior \( d\pi(\theta) \) is not known except that it belongs to a certain set, one can consider the maximin regret \( \sup_{d\pi} \inf_{\mathcal{M}'} E_\theta \Pi \), which is the worst possible Bayes regret. This is related to the channel capacity as a direct result of Corollary 1

**Corollary 2.** The maximin ignorance regret is equal to the channel capacity; viz.,

\[ \sup_{d\pi} \inf_{\mathcal{M}'} E_\theta \Pi = C \equiv \sup_{d\pi} I(\theta; Y), \]  

(33)

and the least-favorable prior is equal to the capacity-attaining prior; viz.,

\[ \arg \sup_{d\pi} \inf_{\mathcal{M}'} E_\theta \Pi = d\pi^*(\theta) \equiv \arg \sup_{d\pi} I(\theta; Y). \]  

(34)
Consider also the minimax regret \( \inf_{M'} \sup_{d \pi} \mathbb{E}_\theta \Pi \), which uses a minimax filter that minimizes the worst possible regret. The channel-capacity connection can be exploited to prove the following, similar to the classical result \([11]\):

**Corollary 3.** The minimax and maximin ignorance regrets are equal and given by the channel capacity; viz.,

\[
\inf_{M'} \sup_{d \pi} \mathbb{E}_\theta \Pi = \sup_{d \pi} \inf_{M'} \mathbb{E}_\theta \Pi = C, \tag{35}
\]

and the minimax filter is equivalent to the Bayes filter with the least-favorable prior \( d\pi^*(\theta) \).

**Proof:** The proof may be done by applying the minimax theorem \([17]\) to quantum filtering, but here I shall use information theory instead. Let \( d\pi^*(\theta) = \sup_{d \pi} I(\theta; Y) \) be the capacity-attaining prior, the expectation under which is denoted by \( \mathbb{E}_\theta^\pi \). The redundancy-capacity theorem states that \([19], [22]\)

\[
C = \inf_{dP'} \sup_{d \pi} \mathbb{E}_\theta D(dP_\theta || dP'), \tag{36}
\]

and the minimax \( dP' \) is

\[
dP' = \inf_{dP'} \sup_{d \pi} \mathbb{E}_\theta D(dP_\theta || dP') = \mathbb{E}_\theta^\pi dP_\theta. \tag{37}
\]

A Bayes filter with model \( M' \) and \( dP' = \mathbb{E}_\theta^\pi dP_\theta \) exists, so \( \inf_{dP'} \) in Eq. (36) can be replaced with \( \inf_{dP'} \) as follows:

\[
C \equiv \inf_{dP'} \sup_{d \pi} \mathbb{E}_\theta D(dP_\theta || dP') = \inf_{dP'} \sup_{d \pi} \mathbb{E}_\theta \Pi, \tag{38}
\]

where the last step uses Corollary 1 Combining this with Corollary 2 leads to Corollary 3.

**C. Quantum Information Bounds**

Perhaps the most remarkable property of Theorem 1 is that it relates the regret for mismatched quantum filtering to the amount of information for binary hypothesis testing, such that a limitation on one application implies a guaranteed performance for the other. Upper bounds on the filter regrets should be particularly useful for robust quantum estimation and control design \([23], [24], [25], [26]\) and proving the stability of quantum filters \([20], [27]\).

For example, suppose that the two models share identical dynamics and measurements and differ only in the initial conditions \( \rho_0 \) and \( \rho'_0 \). The observation probability measures can then be expressed with respect to the same positive operator-valued measure (POVM) \( d\mu(Y) \) \([4]\):

\[
dP(Y_T) = \text{tr} \rho_0 d\mu(Y_T), \quad dP'(Y_T) = \text{tr} \rho'_0 d\mu(Y_T), \tag{39}
\]

and quantum upper bounds on the regrets can be obtained as follows:

**Corollary 4.** If the two models differ only in the initial conditions, the filter regret is bounded by the quantum relative entropy between the two initial density operators; viz.,

\[
\Pi \leq D(\rho_0 || \rho'_0) \equiv \text{tr} \rho_0 (\ln \rho_0 - \ln \rho'_0). \tag{40}
\]

**Proof:** \( \Pi = D(dP || dP') \) from Theorem 1 and it is known from quantum information theory that \( D(dP || dP') \leq D(\rho_0 || \rho'_0) \) for any \( d\mu(Y_T) \) \([28]\).

Corollary 4 proves that the time-averaged regret \( \Pi/T \) due to a mismatched initial condition is guaranteed to decrease inversely with time if \( D(\rho_0 || \rho'_0) < \infty \).

Regrets due to ignorance can also be bounded by quantum information quantities as follows:

**Corollary 5.** If \( dP(Y_T) = \text{tr} \rho_0 d\mu(Y_T) \) and \( dP'(Y_T) = \text{tr} \rho'_0 d\mu(Y_T) \), the Bayes and minimax ignorance regrets are bounded by the Holevo information \( \chi \); viz.,

\[
\inf_{M'} \mathbb{E}_\theta \Pi \leq \chi \{ d\pi(\theta), \rho_0 \} \equiv \mathbb{E}_\theta D(\rho_0 || \mathbb{E}_\theta \rho_0), \tag{41}
\]

\[
\inf_{dP'} \sup_{d \pi} \mathbb{E}_\theta \Pi \leq \sup \chi \{ d\pi(\theta), \rho_0 \}. \tag{42}
\]

**Proof:** \( \inf_{dP'} \sup_{d \pi} \mathbb{E}_\theta \Pi = I(\theta; Y) \) from Corollary 1 and the Holevo bound states that \( I(\theta; Y) \leq \chi \) for any POVM \([28]\). Eq. (42) follows from Corollary 3 and Eq. (41).

### VI. POISSONIAN MEASUREMENTS

The quantum filter for continuous Poissonian measurements, such as photon counting of the optical probe beam, is similar to the Gaussian case, except that the unnormalized filtering equation now reads \([3], [16]\):

\[
df_t(Y_t) = L_t f_t(Y_t) dt + \left[ a_t f_t(Y_t) a_t^\dagger - f_t(Y_t) \right] (dy_t - dt), \tag{43}
\]

and the measured observable is now \( q_t \equiv a_t^\dagger a_t \). It is not difficult to show that \( dP(Y_t) = dP_0(Y_t) \text{tr} f_t(Y_t) \), where \( dP_0(Y_t) \) is the probability measure of a standard Poisson process with \( \mathbb{E}_\theta(dy_t | Y_T) = dt \). The log-likelihood ratio satisfies the following identity, similar to Lemma 1 (see the Supplemental Material of Ref. [6] for a proof):

**Lemma 2.** The log-likelihood ratio for two quantum models under continuous Poissonian measurements satisfies

\[
\ln \Lambda(Y_T) = \int_0^T dy_t \ln \frac{\mathbb{E}(q_t | Y_t)}{\mathbb{E}'(q_t | Y_t)} - \int_0^T dt \left[ \mathbb{E}(q_t | Y_t) - \mathbb{E}'(q_t | Y_t) \right]. \tag{44}
\]

To obtain a result analogous to Theorem 1 I follow Atar and Weissman \([10]\) and define the following loss function instead of the quadratic criterion:

\[
l(q, \tilde{q}) = q \ln \frac{\tilde{q}}{q} - q + \tilde{q}. \tag{45}
\]

The mean-loss error of a causal estimate \( \tilde{q}_t(Y_t) \) at time \( t \) becomes

\[
c_{\text{MLE}} = \mathbb{E} l(q_t, \tilde{q}_t(Y_t)). \tag{46}
\]

It is easy to show that the conditional expectation \( \mathbb{E}(q_t | Y_t) \) minimizes this error as well, such that the minimum error is

\[
c_{\text{MLE}} = \mathbb{E} l(q_t, \mathbb{E}(q_t | Y_t)). \tag{47}
\]
and the regret can then be defined as
\[ \Pi_l = \int_0^T dt \left( \text{cmle}_t - \text{cmmle}_t \right). \tag{48} \]

The second main result of this paper thus follows naturally as a generalization of a classical result by Atar and Weissman \[10\]:

**Theorem 2.** For continuous Poissonian measurements,
\[ \Pi_l = D(dP||dP'). \tag{49} \]

Proof: The proof is similar to that for Theorem 1. Taking the expectation of Eq. (44) in Lemma 2 and noting the martingale property for Poissonian measurements given by \( \mathbb{E}[dY_t - \mathbb{E}(q_t|Y_t) dt|Y_t] = 0 \) \[29\], the relative entropy can be written as
\[ D(dP||dP') = \int_0^T dt \mathbb{E}[\ln \frac{q_t}{\mathbb{E}(q_t|Y_t)}] \tag{50} \]
\[ = \int_0^T dt \mathbb{E}[\frac{q_t}{\mathbb{E}(q_t|Y_t)} - q_t + \mathbb{E}(q_t|Y_t) + \mathbb{E}'(q_t|Y_t)] \tag{51} \]
\[ = \int_0^T dt \mathbb{E}[\ln \frac{q_t}{\mathbb{E}(q_t|Y_t)} - \mathbb{E}(q_t|Y_t) + \mathbb{E}'(q_t|Y_t)] + \mathbb{E}'(q_t|Y_t) \tag{52} \]
\[ = \int_0^T dt \mathbb{E}[\ln \frac{q_t}{\mathbb{E}(q_t|Y_t)}] + \mathbb{E}'(q_t|Y_t) \tag{53} \]
\[ = \int_0^T dt \mathbb{E}[\ln \mathbb{E}(q_t|Y_t)] + \mathbb{E}'(q_t|Y_t)] = D(dP||dP'), \tag{54} \]
where Eq. (53) again follows from the orthogonality principle for quantum conditional expectations.

One direct consequence of Theorem 2 is that Corollaries 1-5 are also applicable to Poissonian measurements, if we consider \( \Pi_l \) instead of \( \Pi \).

**VII. CONCLUSION**

With Theorems 1 and 2 I have taken the first step towards a quantum generalization of the fascinating connections between estimation theory and Shannon information theory for Gaussian and Poissonian channels. The presented results are envisioned to aid the study of quantum estimation and control techniques for complex systems, as they enable one to analyze and design quantum filters using techniques borrowed from information theory. Regardless of the potential applications, these new relations between central quantities in quantum estimation and information theory are bound to bring fresh insights to both areas.

**ACKNOWLEDGMENT**

This work is supported by the Singapore National Research Foundation under NRF Grant No. NRF-NRFF2011-07.

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