A proposal to probe quantum non-locality of Majorana fermions in tunneling experiments

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Topological Majorana fermion (MF) quasiparticles have been recently suggested to exist in semiconductor quantum wires with proximity induced superconductivity and a Zeeman field. Although the experimentally observed zero bias tunneling peak and a fractional ac-Josephson effect can be taken as necessary signatures of MFs, neither of them constitutes a sufficient “smoking gun” experiment. Since one pair of Majorana fermions share a single conventional fermionic degree of freedom, MFs are in a sense fractionalized excitations. Based on this fractionalization we propose a tunneling experiment that furnishes a nearly unique signature of end state MFs in semiconductor quantum wires. In particular, we show that a “teleportation”-like experiment is not enough to distinguish MFs from pairs of MFs, which are equivalent to conventional zero energy states, but our proposed tunneling experiment, in principle, can make this distinction.

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Introduction: Majorana fermions \cite{1} are localized particle-like neutral zero energy states that occur at topological defects and boundaries in superconductors. Zero energy simply means that they occur at the chemical potential. A MF creation operator is a hermitian second quantized operator. A MF creation operator is a hermitian second quantized operator $\gamma^\dagger = \gamma$ which anti-commutes with other fermion operators. The hermiticity of MF operators implies that they can be construed as particles which are their own anti-particles \cite{1–4}. The key issues at this time in the condensed matter context are two fold, first, we must predict and characterize materials supporting Majorana fermions and second, we must detect them experimentally. In this paper we address the second issue of experimental detection by proposing a nearly sufficient experimental signature for Majorana fermions.

MFs have recently been proposed to exist in the topologically superconducting (TS) phase of a spin-orbit (SO) coupled cold atomic gases \cite{5}, semiconductor 2D thin film \cite{6} or 1D nanowire \cite{7} with proximity induced s-wave superconductivity and Zeeman splitting from a sufficiently large magnetic field. In principle, the MFs in such systems may be detected either by measuring the zero-bias conductance peak (ZBCP) from tunneling electrons into the end MFs \cite{7, 10–13}, by detecting the predicted fractional ac Josephson effect \cite{8, 9, 14–16}. The semiconductor Majorana wire structure, which will be the system of our focus, is of particular present interest since there is experimental evidence for both the ZBCP \cite{17, 20} and the fractional ac Josephson effect in the form of doubled Shapiro steps \cite{17}.

Despite their conceptual simplicity, neither the ZBCP nor the fractional ac-Josephson effect experiments constitute a sufficient proof of MFs at the ends of topological superconducting wires. A non-quantized $(2e^2/h)$ zero bias peak, such as that observed in the recent experiments \cite{18, 20}, can in principle arise even without end state MFs \cite{21, 23}. Similarly, while the observation of doubled Shapiro steps are consistent with the TS state \cite{24, 27}, a fractional ac-Josephson effect can exist even in Josephson junctions made of ordinary quasi-1D $p$-wave superconductors such as organic superconductors \cite{15} or the non-topological phase of the semiconductor nanowire \cite{28}. Given these caveats as well as the considerable complexity of existing experiments, there have been several alternative proposals to detect the presence of MFs \cite{29–32}. Based on the inherent quantum non-locality of MFs, in this work we propose an alternative tunneling experiment on semiconductor Majorana wires that furnishes a nearly sufficient signature of end-state MFs. We discuss in detail why only topological systems would show such quantum non-locality, which would even be absent for systems with a pair of MFs at each end.

Non-local electron transfer Non-locality arises in MFs because they differ from conventional complex (Dirac) fermions in that they have no occupation number associated with them. To define a quantum state of a system with MFs we must consider a pair of MFs. The pair of MFs $\gamma_a$ and $\gamma_b$ at the ends $a$ and $b$ of a nanowire (NW) shown in Fig. 1 can be combined into a zero-energy complex fermion operator $d^\dagger = \frac{1}{2}[\gamma_a + i\gamma_b]$ associated with the pair of MFs $\gamma_a$ and $\gamma_b$ \cite{14}. The quantum state of the system is then determined by the eigenvalue of $n_d = d^\dagger d = 0, 1$. Since the fermion parity $F_P = (-1)^{n_d}$ is related to the MFs by

$$F_P = (-1)^{n_d} = i\gamma_a\gamma_b,$$

we see that the fermion parity of the whole system is determined by non-local correlations between the fractionalized MFs $\gamma_a$ and $\gamma_b$. In fact, the fractionalization of the $F_P$ into a pair of spatially separated operators $\gamma_{a,b}$
is the Heaviside step function and $\gamma_{a,b}$ are the Majorana fermion operators at the left and right end of the wire. The amplitude $G_R(\tau; ab)$ contains contributions from Majorana-assisted normal electron tunneling as well as Majorana-assisted non-local Andreev processes where the electron entering at the end $a$ exits as a hole at the end $b$. When the state $|g\rangle$ is a ground state of the TS system with a definite fermion-parity $F_{P}$, the detection of such a non-vanishing amplitude for a non-local Green function $G_R(\tau; ab)$ is a signature of the fractionalization associated with MFs. In equilibrium, the degeneracy of the different fermion parity states characteristic of a TS system lead to fluctuations in $F_{P}$, that would result in a vanishing average for the tunneling amplitude $G_R(\tau; ab)$. This is remedied [37] by introducing a charging energy $E_C$ on the superconducting island supporting the NW, which makes one of the fermion parities energetically favorable over the other.

**Coincidence probability**: The amplitude $G_R(\tau; ab)$ can lead to a so-called coincidence probability for the absence of an electron on the left lead and the presence of an electron on the right lead after the tunneling event.

The coincidence probability $P_c(\tau)$ maybe measured by using a joint measurement by two point contact detectors [40]. Alternatively, the non-local transfer of electron in the presence of charging energy [37], which fixes the fermion parity $F_{P}$, can also be measured by a non-local conductance or transconductance $\frac{dI}{dV}$ between the ends $a$ and $b$ in Fig. 1. This measurement does not require the loop SC in Fig. 1 and would require adding a lead to the end $a$. In such a set-up, a voltage $V_a$ applied to the left-lead $a$ (relative to the SC) results in a current $I_a$ in the right lead $b$. Using results of Ref. [34] for symmetric $t_a = t_b = t$ we find that

$$\frac{dI_b}{dV_a} = \frac{32V_a}{16t^2 + (\delta^2 - V_a^2)^2 + 8t^2(\delta^2 + V_a^2)},$$

which clearly vanishes for $\delta \rightarrow 0$ (energy separation between $F_{P} = \pm 1$). Here $\Gamma \propto t^2$ is the lead-induced broadening of the MFs.

**Topological versus non-topological systems**: However, a coincidence measurement does not directly imply a non-zero $G_R(\tau; ab)$ in more general situations. The amplitude $G_R(\tau; ab)$ in Eq. [2] reflects the amplitude for being able to transfer an electron from $a$ to $b$ while leaving the state $|g\rangle$ invariant. On the other hand, the measurement of the coincidence probability, $P_c$, does not keep track of the internal state of the system. For a general system (i.e. one that may be topological or non-topological), $P_c$ for an electron entering at $a$ and exiting at $b$ can be
written more generally as

$$P_c(\tau) = \sum_{g_1, g_2} |\langle g_2 | \gamma_n(0) \gamma_n^\dagger(\tau) | g_1 \rangle|^2, \quad (5)$$

where $g_1, g_2$ are the internal states of the wire, which are not necessarily identical. Here for a non-topological system $\gamma_{n,\delta}$ are conventional fermionic operators. While TS systems with MFs have a non-degenerate ground state in a given fermion parity sector, more general systems with zero-energy Dirac end states may have multiple allowed states for a given fermion parity sector, more general systems with zero-energy Dirac end states may have multiple allowed values for $g_1, g_2$. Therefore, the coincidence probability $P_c$ cannot be considered a unique signature for a topological system.

An important example of the inequivalence of $P_c(\tau)$ and $G_R(\tau; ab)$ is a non-topological superconductor with a complex fermion zero mode at each end (such a system could arise because we have an even number of weakly coupled Majorana fermions at each end). The quantum state is characterized by the occupancy $n_a, n_b$ of the two conventional zero energy end modes. We can easily have $P_c \neq 0$ in this non-topological setup. Suppose the initial state is $g_1 \equiv (n_a = n_b = 0)$, then the sum for $P_c$ in Eq. 5 would have a non-zero contribution from $g_2 \equiv (n_a = n_b = 1)$. The tunneling of an electron from the lead into the zero-mode at $a$ changes the occupation from $n_a = 0$ to $n_a = 1$. On the other hand, the electron required to change the occupation of the state $b$ from $n_b = 0$ to $n_b = 1$ comes from breaking of a Cooper pair. The other electron from the broken Cooper pair is emitted into the lead in the vicinity near $b$. Note that the process conserves the number of electrons within the system and cannot be eliminated even by the introduction of a finite charging energy [37]. Therefore in order to clearly distinguish this case from the process of Majorana-assisted electron tunneling (that also returns a non-zero $P_c$), we require $G_R(\tau; ab)$ given in Eq. 2 to be non-zero. In other words, we require that the system return to the same state $g$ after the tunneling process, so the same electron that enters at $a$ leaves at $b$. In this paper we focus on how $G_R(\tau; ab)$ can be directly measured to result in a unique signature of end state MFs [?].

Proposed set-up: As we now argue, the interference experiment shown in Fig. 1 can be used to measure the amplitude of coherent non-local quasiparticle transmission between the ends $a, b$ by measuring $G_R(\tau; ab)$. Unlike previous interference schemes [41] for MFs, this scheme will avoid the use of chiral edge states. The setup in Fig. 1 consists of an external superconducting electrode (SC) that is connected to the ends $a$ and $b$ through tunnel barriers. The superconducting electrode is assumed to be shorter than its coherence length $\xi_{SC} \equiv v_F/\Delta_{\text{loop}}$ so that quasiparticles with energies below the SC gap have a finite amplitude of tunneling from end $a$ to end $b$ through the superconducting wire. The coherence length $\xi_{SC}$ can be made large by weakly proximity coupling the loop to an external superconductor so that $\Delta_{SC}$ is small.

In the presence of a finite Majorana-assisted non-local electron transfer amplitude $G_R(\tau; ab)$ a quasiparticle tunneling from $a$ to $b$ through the loop (SC) can complete the circuit by tunneling back from $b$ to $a$ through the TS nanowire. The Berry phase associated with such a tunneling around the loop is sensitive to the flux $\Phi$ through the loop with a periodicity $2\Phi_0 = \hbar c/e$. Therefore, similar to the Aharonov-Bohm effect in mesoscopic rings [12], the energy-level of such a quasiparticle excitation spectrum $\varepsilon(\Phi)$ in the ring is expected to develop a $2\Phi_0$ periodic dependence on $\Phi$ for TS systems where there is a finite electron transfer amplitude across the NW. Other contributions to the Aharonov-Bohm effect that may result from either Cooper-pair transport around the ring and have a periodicity of the superconducting flux quantum $\Phi_0$. Therefore, the contribution from Majorana-assisted non-local quasiparticle transfer around the ring, in the topological case, can be extracted from the $2\Phi_0$ periodicity of the quasiparticle excitation spectrum $\varepsilon(\Phi)$.

Numerical calculation: To calculate the quasiparticle excitation gap $\varepsilon(\Phi)$ we consider the Hamiltonian of the system

$$H = E_C(i\partial_\phi + \tilde{\Delta}/2 - n_g)^2 + H_{BCS}(\phi, \Phi), \quad (6)$$

where $H_{BCS}(\phi, \Phi)$ is the BCS Hamiltonian for the system, $\phi$ is the phase of the superconducting island supporting the NW and $\tilde{\Delta}$ is the number of electrons in the NW. In the above $E_C$ is the charging energy of the superconducting island together with the NW and $2en_g$ is the gate charge [43][44]. In the limit $E_C \ll E_g$ (where $E_g$ is the gap of two-particle excitation), the lowest energy eigenstates of $H$ can be written in the form

$$|\Psi\rangle = \int d\phi \psi(\phi) |\xi(\phi)\rangle \otimes |\phi\rangle, \quad (7)$$

where $|\xi(\phi)\rangle$ is the instantaneous ground state of $H_{BCS}(\phi, \Phi)$ so that $H_{BCS}(\phi, \Phi) |\xi(\phi)\rangle = E_I(\phi) |\xi(\phi)\rangle$ and $\psi(\phi)$ is the probability amplitude for the superconducting phase to be at $\phi$. Computing the expectation value of $\langle \Psi | H | \Psi \rangle$ using Eq. 6 and Eq. 7 the variational Schrodinger equation satisfied by $\psi(\phi)$ is found to be

$$H_{eff} = E_C(i\partial_\phi - \Gamma + \langle \xi(\phi) | \tilde{\Delta} | \xi(\phi) \rangle/2 - n_g)^2 + E_I(\phi), \quad (8)$$

where

$$\Gamma = -i \int_0^{2\pi} \frac{d\phi}{2\pi} \langle \Phi(\phi) | \partial_\phi | \xi(\phi) \rangle, \quad (9)$$

is the Berry flux and can be computed numerically from $|\xi(\phi)\rangle$ [45]. As a model, we have considered a spin-orbit coupled semiconductor nanowire [8][9] for the NW. In order to model the contribution of conventional Andreev tunneling at the ends $a$ and $b$ we have added a term $-2E_{J,0} \cos(\pi\Phi/\Phi_0) \cos(\phi + \pi\Phi/\Phi_0)$, which ensures that the wave-function $\psi(\phi)$ is localized near $\phi \sim -(\pi\Phi/\Phi_0)$. The details of the model and the derivation of Eq. 8 are presented in the Supplementary material [35].
FIG. 2. Quasiparticle excitation spectrum $\varepsilon$ as a function of flux $\Phi/\Phi_0$ in the loop in Fig. 1 in both the topological phase and the nontopological phase. Each spectrum is plotted with an even and odd multiple of flux $\Phi_0 = h c/2e$ added to the loop. The excitation gap for even and odd flux quanta coincide with each other in the non-topological case, while the spectra in the topological case shows a dependence on the parity of the flux quantum near $\Phi \sim \Phi_0/2$. The change in the energy excitation spectrum from inserting a flux quantum $\Phi_0$ is shown to arise from the change in fermion parity, which is a unique character of the TS phase.

Result: Since a superconducting system conserves fermion parity, one can compute a ground state with energy $\varepsilon_{e,o}(\Phi)$ in each of the fermion-parity sectors i.e. even and odd. The quasiparticle excitation gap that can be measured using tunneling using the configuration in Fig. 1 is given by

$$\varepsilon(\Phi) = |\varepsilon_e(\Phi) - \varepsilon_o(\Phi)|$$

is plotted as a function of $\Phi$ in Fig. 2 for both the cases where NW is in the topological and non-topological phase. The result in Fig. 2 shows that despite the $\Phi_0$ periodicity of the excitation spectrum of $H_{BCS}(\phi, \Phi)$, the excitation spectrum including the charging energy shows a $2\Phi_0$ periodicity in the topological case, as expected from non-local Majorana-induced quasiparticle transfer across the wire. Since the presence of a finite tunneling amplitude $G_R(\tau; ab)$ depends on the charging energy $E_C$, which is competes by the effective Josephson energy $E_{J,eff}(\Phi) = 2E_{J,0}|\cos(\pi/\Phi_0)|$, the $2\Phi_0$ periodic modulation in the tunneling gap is substantial in the TS phase only when $E_{J,eff}(\Phi) \lesssim E_C$. Thus the set-up in Fig. 2 can be used to separate out Majorana-assisted electron transfer from direct transfer by tunneling of quasiparticles through NW since the former would be strongly peaked at $\Phi \sim \Phi_0/2$, where $E_C$ dominates over the Josephson energy.

Comparison with the fractional Josephson effect: The signature of a TS phase in Fig. 2 appears as a $2\Phi_0$ periodicity of the tunneling spectrum. Formally, this resembles the $2\Phi_0$ periodicity of the ABS spectrum in the fractional Josephson effect in TS systems\(^\text{[11]}\)\(^\text{[14]}\). However, the ABS spectrum of the Josephson junction in a TS system is $2\Phi_0$-periodic as a result of zero-energy crossings in the ABS spectrum that are protected by fermion parity. The fermion parity protection is typically accomplished by requiring a non-equilibrium AC Josephson measurement\(^\text{[11]}\).

In contrast, the measurement proposed in this work is an equilibrium measurement. This is surprising given that the set-up in Fig. 1 might be viewed as a Josephson junction formed between the two MFs at the ends (a) and (b) of a TS nanowire and therefore as essentially identical to the set-up for the AC fractional Josephson effect. The additional ingredient that allows us to turn an otherwise non-equilibrium measurement into a more accessible DC measurement is the charging energy in the Hamiltonian in Eq. 6. The characterizing property of a TS system with periodic boundary conditions as in Fig. 1 which also is ultimately responsible for the AC fractional Josephson effect, is the $2\Phi_0$ periodic dependence of the ground state fermion parity\(^\text{[4]}\). The charging energy in Eq. 6 leads to an additional $\langle N \rangle$-dependence in the Hamiltonian that ultimately leads to a $2\Phi_0$-periodicity in the excitation spectrum that can be measured by tunneling even in equilibrium.

Summary and Conclusion: In this paper we have proposed a scheme for uniquely identifying the Majorana assisted non-local electron tunneling between two MFs at the ends of a wire in the TS phase. In principle, such a non-local transfer of electrons may be observable by a coincidence measurement\(^\text{[35]}\)\(^\text{[36]}\)\(^\text{[37]}\). However, as we have shown here that the Majorana assisted electron tunneling process using either a coincidence detection\(^\text{[35]}\)\(^\text{[40]}\) or by measuring the transconductance with a charging energy\(^\text{[37]}\), while interesting, cannot be taken as a definitive signature of MF modes because even conventional near-zero energy states trapped near the spatially separated leads can also produce such non-local signature. Instead we have proposed an interferometry experiment\(^\text{[37]}\) appropriately generalized to geometries without edge modes. We have shown that such a measurement can distinguish conventional and Majorana zero modes. Our proposed non-local correlation experiment in terms of tunneling, which requires the inclusion of charging energy to fix the fermion parity, provides a direct verification of the non-locality of MFs in TS wires. We emphasize that the non-locality of the end state MFs arises from the non-locality of the fermion parity, which is unique to topological systems and cannot be emulated by conventional systems\(^\text{[33]}\).

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Absence of non-local correlations in the non-topological case

We mention in the main text that the non-local correlation $G_R(\tau;ab)$ must vanish in the non-topological case. A particularly counter-intuitive case is the non-topological situation with two MFs at each end of a wire (as in Ref. \cite{32} where the wires are not coupled to each other. Such a system can in principle be prepared in a non-local initial state so that our current proposal gives a positive result by preparing each wire in a definite fermion parity state. However, as shown in the rest of the section, decoherence induced by coupling to the leads disfavors this non-local initial state and washes out any signature of non-locality in the present experiment. Hence, although a weakly coupled pair of MFs (or a conventional zero-energy state) at each end may still show a ZBCP at or near zero bias at finite temperature, our interference measurement will correctly demonstrate that the system is smoothly connected to a non-topological phase.

In the present section of the appendix, we show that coupling to a Fermionic bath leads to such destruction of non-local correlations. We accomplish this in two subsections. In the first part, we show that when placed in contact with a fermion bath in thermal equilibrium, the system reaches a unique thermal equilibrium state. In the second part, we show that the system in thermal equilibrium cannot have any non-local correlations of the
where $G_\gamma$ diagonal between system (system density matrix ten as interacting fermion problem can be solved in terms of $\rho_{ab}(t) = i(\gamma_a(t)\gamma_b(t))$. (11)

The Hamiltonian in terms of Majorana operators is written as

$$H = i \sum_{a,b} h_{a,b} \gamma_a \gamma_b.$$ (12)

The equation of motion is written as

$$\partial_t \gamma_a = h_{a,b} \gamma_b.$$ (13)

with $h_{a,b}$ being anti-symmetric. The time-evolution of $\gamma_a(t)$ is written as

$$\gamma(t) = G^{(R)}(t)\gamma(0),$$ (14)

where $G^{(R)}(t) = e^{ht}$ for $t > 0$. Expanding $G^{(R)}(t)$ in eigenstates

$$G^{(R)}(t) = \sum_n \psi_n^\dagger \psi_n e^{i\varepsilon_n t}$$

$$= \sum_n \psi_n^\dagger \int_{-\infty}^\infty \frac{e^{i\omega t}}{\omega - \varepsilon_n - i\delta} = \int_{-\infty}^\infty \frac{e^{i\omega t}}{\omega - ih - i\delta}. (15)$$

The time evolution of the density matrix is written as

$$\rho(t) = G^{(R)}(t)\rho(0)G^{(A)}(t).$$ (16)

Assuming that we start with a density matrix that is diagonal between system ($s$) and bath ($b$) (i.e. lead), the system density matrix

$$\rho_s(t) = G^{(R)}_s(t)\rho_s(0)G^{(A)}_s(t) + G^{(R)}_s(t)\rho_b(0)G^{(A)}_b(t). (17)$$

Assuming the system-bath structure for the Hamiltonian and defining $g^{(R,A)}_s(\omega) = (\omega \mp i\delta - H_b)^{-1}$, one can write the final system Green function as

$$G^{(R,A)}_s(\omega) = (\omega \mp i\delta - H_b g^{(R,A)}_b(\omega) H_b)^{-1}. (18)$$

The system bath interaction Green function can be written as

$$G^{(R,A)}_{sb}(\omega) = G^{(R,A)}_s(\omega)H_b g^{(R,A)}_b(\omega)$$

$$G^{(R,A)}_{bs}(\omega) = g^{(R,A)}_b(\omega)H_b G^{(R,A)}_s(\omega). (19)$$

Provided the anti-hermitian part of the fermion-self-energy

$$\Sigma^{(R,A)}_s(\omega) = Im[H_b g^{(R,A)}_b(\omega)H_b]$$ (20)

does not have a null-space (i.e. a space of zero-eigenvalues) the Green function $G^{(R,A)}_{s}(t)$ can be assumed to decay exponentially in time. Therefore it follows from Eq. 17 that the first term must vanish. To evaluate the second term in Eq. 17 we note that from Eq. 19 it is clear that

$$G^{(R,A)}_{sb}(t) = \int dt G_s(t - \tau)H_b g_b(\tau). (21)$$

Since $G_s(t - \tau)$ is exponentially decaying, the $\Theta(\tau)$ component of $g_b$ is not relevant and one can approximate

$$G^{(R,A)}_{sb}(t) \approx \int dt G_s(t - \tau)H_b e^{-h_b t}$$

$$= \int_0^\infty dt G_s(t - \tau)H_b e^{-h_b (t-\tau)} (22)$$

up to exponential factors. Applying this identity to Eq. 17 we obtain

$$\rho_s(t) \approx \int_{-\infty}^\infty d\omega f(\omega)G^{(R)}_s(\omega)H_{sb}\rho_b(0)H_b G^{(A)}_s(\omega), (24)$$

which is a $t$-independent asymptotic value. Transforming to frequency space,

$$\rho_s(t) \approx \int_{-\infty}^\infty d\omega f(\omega)G^{(R)}_s(\omega)H_{sb}\rho_b(0)H_b G^{(A)}_s(\omega).$$

Considering the integrand at $\omega$ away from a pole of $g_s(\omega)$,

$$G^{(R)}_s(\omega)H_{sb}\rho_b(0)H_b G^{(A)}_s(\omega)$$

$$= i G^{(R)}_s(\omega)H_{sb}\{g^{(R)}_b(\omega) - g^{(A)}_b(\omega)\}H_b G^{(A)}_s(\omega)$$

$$= i G^{(R)}_s(\omega)\{\Sigma^{(R)}_s(\omega) - \Sigma^{(A)}_s(\omega)\}G^{(A)}_s(\omega)$$

$$= i \{G^{(R)}_s(\omega) - G^{(A)}_s(\omega)\}, (25)$$

which can be checked using Dyson’s equation. Therefore, in the absence of a protected null-space in the dissipative part of the self-energy $\Sigma_s(\omega)$, the density matrix equilibrates to the grand-canonical thermal equilibrium value

$$\rho_s(t) \approx i \int_{-\infty}^\infty d\omega f(\omega)\{G^{(R)}_s(\omega) - G^{(A)}_s(\omega)\}. (26)$$

Let us now discuss correlations in the grand-canonical thermal state. First we prove a general lemma: Consider the grand-canonical thermal partition function of a system of Majorana fermions (or fermions) which can be partitioned into two parts $L$ and $R$, so that the Hamiltonian has a symmetry $\gamma_{L,a} \rightarrow -\gamma_{L,a}$ for all MFs $\gamma_{L,a}$ on the left half of the system $L$. Then all correlation
functions $\langle \gamma_{L,a} \gamma_{R,b} \rangle = 0$. To prove this simply apply the transformation $\gamma_{L,a} \rightarrow -U \gamma_{L,a} U^\dagger$ to

$$
\langle \gamma_{L,a} \gamma_{R,b} \rangle \propto \text{Tr}[\gamma_{L,a} \gamma_{R,b} e^{-\beta H}] = \text{Tr}[U \gamma_{L,a} U^\dagger \gamma_{R,b} e^{-\beta H} U^\dagger]
= -\text{Tr}[\gamma_{L,a} \gamma_{R,b} e^{-\beta H}] = 0.
$$

Therefore all such non-local fermionic correlations must vanish in the thermal state.

As discussed before, in the topological state, the charging energy gives rise to a fermion parity dependent term in the Hamiltonian

$$
H_{nl} \propto i \gamma_a \gamma_b,
$$

which violates the conditions for the above lemma and leaves a non-zero value for this non-local correlator. Therefore the presence of the Majorana fermion term is crucial for the appearance of a non-local correlation function that can contribute to the interference. Such an interference will not happen even if the conventional end zero mode is made of a pair of Majorana fermions. This is because, as discussed in the previous sub-section, coupling Majorana fermions to leads causes them to thermalize into the grand-canonical ensemble. Once they have thermalized in this ensemble, $H_{nl}$ violates the conditions of the lemma and generates a non-local fermion correlation. The long-range Coulomb interaction for conventional fermionic modes does not lead to the violation of the conditions of the lemma and does not lead to any long range correlation. Following the rest of the argument in the text, it is clear that once non-local Green functions are absent, there can be no non-local correlation.

**Technical details of calculating the excitation spectrum**

The quasiparticle degrees of freedom in the Hamiltonian in Eq. [6] is taken to be the BCS Hamiltonian for a superconducting proximity coupled semiconductor nanowire given by

$$
H_{BCS}(\phi, \Phi) = \sum_\sigma \int dx \frac{\hbar^2}{2m^*} [(i\partial_x - a(x))c_\sigma(x)]^2 + (V(x) - \mu)c_\sigma^\dagger(x)c_\sigma(x) + V_{Z,x}\sigma c_{\sigma}^\dagger(x)c_{\sigma}(x) + V_{Z,x}\sigma c_{\sigma}^\dagger(x)c_{\sigma}(x) + \alpha c_{\sigma}^\dagger(x)(i\partial_x - a(x))c_\sigma(x) + (\Delta(x)\sigma c^\dagger_{\sigma}(x)c^\dagger_{-\sigma}(x) + h.c),
$$

where $c_\sigma^\dagger(x)$ is the electron creation operator in the nanowire with spin $\sigma$, $m^*$ is the effective mass, $a(x)$ is the vector potential from the flux in the loop and $V(x)$ is the gate controlled potential. The chemical potential of the electrons is represented by $\mu$, $V_Z$ is the Zeeman potential set by a magnetic field, $\alpha$ is the Rashba spin-orbit coupling strength and $\Delta(x)$ is the proximity-induced superconducting pair potential. To represent the geometry in Fig. [1] the electrons are chosen to have periodic boundary conditions with the end $a$ represented by $x = 0$ and the superconducting pair potential $\Delta(x) = \Delta_0 e^{i\phi}$ in NW and $\Delta(x) = \Delta_1$ in the SC loop. The vector potential $\alpha(x) = 2\pi \delta(x) \Phi / \Phi_0$ is determined by the flux $\Phi$ in the loop. For the result in Fig. [2] we have chosen $\mu = 2 K$, $V(x) = 0 K$ in the NW, $V(x) = 1 K$ in the SC loop, the spin-orbit coupling strength is $\alpha = 0.5 eV - A$, the pair potential in the NW is $\Delta_0 = 3 K$, and the pair potential in the SC loop is $\Delta_1 = 0.2 K$. The Zeeman potentials are taken to be $V_{Z,x} = 4.5 K$ and $V_{Z,x} = 0.6 K$. The effective mass is taken to be $m^* = 0.015 m_e$ for InSb, and the length of the NW and the SC loop is taken to be $L = 5 \mu m$. The charging energy $E_C$ in the Hamiltonian $H$ in Eq. [6] is taken to be $E_C = 0.7 K$, while the $\phi$-dependence of the other electrons in the quasi one-dimensional system is modelled as a Josephson energy of amplitude $E_{J,0} = 0.4 K$.

The Hamiltonian in Eq. [6] is complicated by the term $\tilde{N}/2$ in the charging energy term proportional to $E_C$. This term can be eliminated by a unitary transformation $U = \int d\phi |\phi\rangle \langle \phi| e^{i\phi \tilde{N}/2}$. Such a unitary transformation transforms the BCS Hamiltonian $H_{BCS}(\phi)$ to $\tilde{H}_{BCS}(\phi) = U H_{BCS}(\phi) U^\dagger$ and the eigenstate $|\xi(\phi)\rangle$ to $|\tilde{x}(\phi)\rangle = U |\xi(\phi)\rangle$. Defining a variational ansatz for the eigenstate for the transformed Hamiltonian $\tilde{H} = U H U^\dagger$ as

$$
|\tilde{\Psi}\rangle = \int d\phi \psi(\phi) \langle \xi(\phi) \rangle \otimes |\phi\rangle,
$$

we obtain an effective Hamiltonian for $\tilde{\psi}$ which is written as

$$
H_{eff} = E_C (i\partial_\phi - \tilde{\Gamma} - n_g)^2 + \tilde{E}_J(\phi),
$$

where $\tilde{\Gamma} = -i\langle \xi(\phi)| \partial_\phi |\xi(\phi)\rangle = \Gamma + \langle \xi(\phi)| \tilde{N}|\xi(\phi)\rangle / 2$, and $\Gamma = -i\langle \xi(\phi)| \partial_\phi |\xi(\phi)\rangle$. The eigenstate of the original Hamiltonian $H$ is defined in terms of $\psi(\phi)$ as

$$
|\Psi\rangle = U |\tilde{\Psi}\rangle = \int d\phi \psi(\phi) \langle \xi(\phi) \rangle \otimes |\phi\rangle,
$$

where $\psi(\phi)$ satisfies Eq. [8]