COUNTABLE SETS VERSUS SETS THAT ARE COUNTABLE IN REVERSE MATHEMATICS

SAM SANDERS

Abstract. The program Reverse Mathematics (RM for short) seeks to identify the axioms necessary to prove theorems of ordinary mathematics, usually working in the language of second-order arithmetic $L_2$. A major theme in RM is therefore the study of structures that are countable or can be approximated by countable sets. Now, countable sets must be represented by sequences here, because the higher-order definition of ‘countable set’ involving injections/bijections to $\mathbb{N}$ cannot be directly expressed in $L_2$. Working in Kohlenbach’s higher-order RM, we investigate various central theorems, e.g. those due to König, Ramsey, Bolzano, Weierstrass, and Borel, in their (often original) formulation involving the definition of ‘countable set’ based on injections/bijections to $\mathbb{N}$. This study turns out to be closely related to the logical properties of the uncountably of $\mathbb{R}$, recently developed by the author and Dag Normann. Now, ‘being countable’ can be expressed by the existence of an injection to $\mathbb{N}$ (Kunen) or the existence of a bijection to $\mathbb{N}$ (Hrbacek-Jech). The former (and not the latter) choice yields ‘explosive’ theorems, i.e. relatively weak statements that become much stronger when combined with discontinuous functionals, even up to $\Pi^1_2$-CA$_0$. Nonetheless, replacing ‘sequence’ by ‘countable set’ seriously reduces the first-order strength of these theorems, whatever the notion of ‘set’ used. Finally, we obtain ‘splittings’ involving e.g. lemmas by König and theorems from the RM zoo, showing that the latter are ‘a lot more tame’ when formulated with countable sets.

1. Introduction

Concepts like ‘countable subset of $\mathbb{R}$’ and ‘the uncountability of $\mathbb{R}$’ involve arbitrary mappings with domain $\mathbb{R}$, and are therefore best studied in a language that has such objects as first-class citizens. Obviousness, much more than beauty, is however in the eye of the beholder. Lest we be misunderstood, we formulate a blanket caveat: all notions (computation, continuity, function, open set, etcetera) used in this paper are to be interpreted via their higher-order definitions, also listed below, unless explicitly stated otherwise.

1.1. Historical background and motivation. In a nutshell, this paper deals with the study of the logical and computational properties of theorems of ordinary mathematics formulated using the definition of ‘countable set’ based on injections/bijections to $\mathbb{N}$, in particular when this choice results in significant differences compared to the formulation involving sequences. A more detailed description is in

1Simpson describes ordinary mathematics in [85, I.1] as that body of mathematics that is prior to or independent of the introduction of abstract set theoretic concepts.
Section 1.2 while we now sketch the (historical) motivation for this paper, based on mathematical household names like Borel, König, Ramsey, and Cantor, as well as the mathematical mainstream.

First of all, the notion of ‘countable set’ can be defined in various ways. However, it is an empirical observation, witnessed by countless mainstream textbooks, that to show that a set is countable one often only constructs an injection (or bijection) to \( \mathbb{N} \). When given a countable set, one (additionally) assumes that this set can be enumerated, i.e. represented by some sequence. Hence, whatever one’s preferred definition of ‘countable set’ may be, implicit in much of mathematical practise is the following most basic principle about countable sets:

\begin{align*}
\text{a set that can be mapped to } \mathbb{N} \text{ via an injection (or bijection) can be enumerated.}
\end{align*}

This basic principle is formalised as \( \text{cocode}_i \) for \( i = 0, 1 \) in Section 3.2.3. We now provide some more historical and conceptual motivation for this study.

Secondly, Borel formulates the Heine-Borel theorem in [10] using countable collections of intervals (rather than sequences), i.e. the study of countable sets an sich has its roots in ordinary mathematics, namely as discussed in the following remark.

**Remark 1.1** (Borel’s Heine-Borel). Borel introduces the notion of ‘countable set’ (French: \textit{ensemble dénombrable}) via bijections to \( \mathbb{N} \) in [10, p. 6]. He then proceeds to explain the provenance of ‘countable’ (French: \textit{dénombrable}), namely that the elements of such sets can be enumerated, i.e. listed as a sequence. In this way, Borel makes use of the principle \( \text{cocode}_1 \) from Section 3.2.3 which state that certain countable sets can be enumerated.

Moreover, Borel’s formulation of the Heine-Borel theorem in [10, p. 42] involves \textit{une infinité dénombrable d’intervalles}, i.e. a countable infinity of intervals. Thus, Borel’s proof starts with the following (originally French):

\begin{quote}
Let us enumerate our intervals, one after the other, according to whatever law, but determined. [10, p. 42]
\end{quote}

This sentence constitutes another use of the aforementioned principle \( \text{cocode}_1 \). Borel then proceeds with the usual ‘interval-halving’ proof, similar to Cousin in [18]. Similar observations can be made for [9, p. 51], where Borel uses language similar to the previous quote.

Similar to the previous remark, Ramsey ([67, p. 264]) and König ([44]) used set theoretic jargon to formulate their eponymous theorem and lemma. König even explicitly studies countable sets in [44], while Ramsey only mentions the distinction between the finite and infinite. All these are well-studied in Reverse Mathematics (RM hereafter; see Section 2.1) with countable sets formulated using sequences, and it is therefore a natural question what happens if we work with countable sets involving injections or bijections to \( \mathbb{N} \) instead. In this paper, we provide a partial answer to this question, which constitutes a contribution to Kohlenbach’s higher-order RM (see Section 2.1). We note that the (second-order) concept of ‘countable set’ is introduced in [85, V.4.2] and used throughout [85].

Thirdly, more historical motivation is provided by the uncountability of \( \mathbb{R} \) which has an elegant formulation in terms of countable sets based on Cantor’s theorem, as follows. Now, Cantor’s first set theory paper [15], published in 1874 and boasting its own Wikipedia page ([91]), establishes the uncountability of \( \mathbb{R} \) as a corollary to:
for any sequence of reals, there is another real different from all reals in the sequence.

The logical and computational properties of this theorem, called Cantor's theorem, are well-known: it is provable in weak and constructive systems (\cite{S5} II.4.9 and \cite{6} p. 25) while there is an efficient computer program that computes the real in the conclusion from the sequence (\cite{32}). By contrast, the uncountability of \( \mathbb{R} \) has only recently been studied in detail (\cite{64}) in the guise of the following principles:

- **NIN**: there is no injection from \([0,1]\) to \( \mathbb{N} \),
- **NBI**: there is no bijection from \([0,1]\) to \( \mathbb{N} \),

Interpreting 'countable set' as 'there is an injection to \( \mathbb{N} \)' (see Definitions \cite{8.6} and \cite{3.4}), NIN is equivalent to the following reformulation of Cantor's theorem:

\[
\text{for countable } A \subset \mathbb{R}, \text{ there is a real } y \in [0,1] \text{ different from all reals in } A. \quad (A)
\]

Moreover, NIN follows from Cantor's (\cite{16}, §16, Theorem A*) restricted to \( \mathbb{R} \):

\[
\text{If a subset } A \subset [0,1] \text{ is countable, then } A \text{ cannot be perfect}. \quad (B)
\]

A second-order version of \((B)\) is provable in the base theory of RM (\cite{S5} II.5.9). The same results hold for bijections and NBI *mutatis mutandis* by Theorem \cite{8.15}.

In this light, the study of countable sets has its roots (implicitly and explicitly) in the work of Borel and Cantor (and others), and their (semi-)epoymous theorems.

Fourth, we provide some motivation based on mathematical logic. While *prima facie* quite similar, Cantor's theorem and \((A)\), have hugely different logical and computational properties. Indeed, as noted above, there are proofs of Cantor's theorem in weak and constructive systems, while the real claimed to exist can be computed efficiently. By contrast, NIN and NBI are hard to prove\(^2\) in that full second-order arithmetic comes to the fore. The real \( y \) from \((A)\) is similarly hard to compute in terms of the data, within Kleene's higher-order framework. Since \((A)\) is so elementary, these observations suggest that theorems about countable sets have rather extreme/interesting logical and computational properties.

Fifth, the aforementioned properties of NIN only serve to motivate the goal of this paper, namely the study of theorems of ordinary mathematics formulated using the definition of 'countable set' involving injections/bijections to \( \mathbb{N} \), in particular when this choice results in significant differences compared to the formulation involving sequences. The correct (or at least broader) interpretation of the logical and computational properties of NIN and NBI may be found in \cite{64} §1-2. We will say that, by \cite{64} Theorem 3.2, the principles NIN and NBI are provable *without using the Axiom of Choice* while the same holds for the theorems considered in this paper, by the below.

Finally, this paper (clearly) constitutes a spin-off from \cite{64}, which is part of our joint project with Dag Normann on the logical and computational properties of the uncountable. The interested reader may consult \cite{59,64} as an introduction.

\(^2\)The third-order statements NIN and NBI are not provable in \( \mathbb{Z}_2^3 \), a conservative and essentially third-order extension of second-order arithmetic \( \mathbb{Z}_2^3 \) going back to Sieg-Feferman (\cite{14} p. 129), i.e. we can say that NIN and \((A)\) are hard to prove. The reals claimed to exist by NIN are similarly hard to compute, in the sense of Kleene's S1-S9. All these results are proved in \cite{64} §3-4, while the associated definitions may (also) be found in Section 2.2 of this paper.
1.2. Countable infinity and Reverse Mathematics. We discuss the role of the countable in (Reverse) Mathematics and formulate four questions (Q0)-(Q3) that will be given (partial) answers in this paper.

Now, the word ‘countable’ and variations occur hundreds of times in Simpson’s excellent monograph [85], the unofficial bible of RM, and in [84], a sequel consisting of RM papers from 2001. The tally for Hirschfeldt’s monograph [35] on the RM zoo (see [21] for the latter) is about one hundred. Countable infinity does indeed take centre stage, as is clear from the following quotes by Hirschfeldt and Simpson.

We finish this section with an important remark: The approaches to analyzing the strength of theorems we will discuss here are tied to the countably infinite. ([35, p. 6])

Since in ordinary mathematics the objects studied are almost always countable or separable, it would seem appropriate to consider a language in which countable objects occupy center stage. ([85, p. 2])

As detailed in Section 2.1, RM studies ordinary mathematics and does so using the language of second-order arithmetic $L_2$. Now, the definition of ‘countable subset of $\mathbb{R}$’ based on injections/bijections to $\mathbb{N}$ cannot be expressed in $L_2$. Indeed, countable sets are given by *sequences* in RM, namely as in [85, V.4.2]. It is therefore a natural question what happens to the results of RM if we use the definition of countable set involving injections or bijections to $\mathbb{N}$ (see Definition 3.4), say working in Kohlenbach’s *higher-order* RM (see Section 2.1). This paper is devoted to the study of this question, as sketched in the rest of this section, right after the following quote providing ample motivation for this study.

This situation has prompted some authors, for example Bishop/Bridges [7, p. 38], to build a modulus of uniform continuity into their definitions of continuous function. Such a procedure may be appropriate for Bishop since his goal is to replace ordinary mathematical theorems by their “constructive” counterparts. However, as explained in chapter I, our goal is quite different. Namely, we seek to draw out the set existence assumptions which are implicit in the ordinary mathematical theorems *as they stand*. ([85, p. 137])

As noted above, Borel ([11]) and König ([14]), formulate their (semi-) eponymous theorems, namely the *Heine-Borel theorem* and *König’s (infinity) lemma*, using the definition of countable sets based on bijections to $\mathbb{N}$, while Ramsey formulates *Ramsey’s theorem* in [67, p. 264] in set theory lingo. Motivated by Simpson’s above quote, we will study versions of the aforementioned theorems formulated using the definition of ‘countable set’ based on injections/bijections to $\mathbb{N}$.

Now, there are (at least) two possible definitions of ‘countable set’, namely based on *injections to $\mathbb{N}$* (Kunen [50]) and *bijections to $\mathbb{N}$* (Hrbacek-Jech [37]). Hereafter, we shall always refer to the former as ‘countable’ and to the latter as ‘strongly countable’, as laid down in Definition 3.4. This naming is merely a matter of convenience: we do not claim that injections or bijections to $\mathbb{N}$ constitute the ‘standard’ or ‘mainstream’ definition of countable set. We discuss this and related topics in Section 3.6 where we also discuss the grand(er) scheme of things.

On a related note, we have studied the properties of *nets* in [76, 77]. Since nets are the generalisation of sequences to (possibly) uncountable index sets, it is a
natural question whether there is a difference between nets with countable index sets on one hand, and sequences on the other hand. As noted in [49], Dieudonné formulates a (rather abstract) theorem in [20] that is true for sequences but false for nets indexed by countable index sets.

Thus, we are led to the following motivating questions.

(Q0) Does replacing ‘sequence’ by ‘countable set’ make a (big) difference?
(Q1) Does replacing ‘countable’ by ‘strongly countable’ make a (big) difference?
(Q2) Are there (elementary) differences between sequences and nets with countable index sets?
(Q3) Are there natural splittings involving ‘countable’ and ‘strongly countable’?

Regarding (Q0), we exhibit numerous theorems for which the logical strength changes (sometimes dramatically) upon replacing ‘sequence’ by ‘countable set’. This includes theorems by Borel, Ramsey, and König that were originally formulated using ‘countable set’ (rather than ‘sequence’) or at least in set theory lingo. Moreover, we identify a number of theorems about countable sets that give rise to \( \Pi^1_2 \text{-CA}_0 \) when combined with higher-order \( \Pi^1_1 \text{-CA}_0 \), i.e. the Suslin functional. The first such example, namely the Bolzano-Weierstrass theorem for countable sets in \( 2^\mathbb{N} \), was identified in [64], and we obtain a number of interesting equivalences in Section 3.3. According to Rathjen ([68]), \( \Pi^1_2 \text{-CA}_0 \) dwarfs \( \Pi^1_1 \text{-CA}_0 \) and Martin-Löf talks of a chasm and abyss between these two systems in [53].

Regarding (Q1), the previous paragraph establishes that basic well-known theorems from second-order RM formulated with Kunen’s notion of countable are explosive, i.e. become much stronger when combined with discontinuous (comprehension) functionals. These theorems are not provable in \( \mathbb{Z}^2 \), while using the Hrbacek-Jech notion of countable set, the same theorems are no longer explosive, while still not provable in \( \mathbb{Z}^2 \). All these theorems are provable in \( \mathbb{Z}^2 \), \( \mathbb{Z}^2 \) and \( \mathbb{Z}^2 \) are the conservative extensions of \( \mathbb{Z}^2 \) with third and fourth-order comprehension from Section 2.2.

As to (Q2), nets with countable index sets give rise to explosive (convergence) theorems by Theorem 3.16, i.e. the power of nets does not (fully) depend on the cardinality of the index set. In particular, the aforementioned convergence theorems exist at the level of \( \text{RCA}_0 \), but yield \( \Pi^1_1 \text{-CA}_0 \) when combined with higher-order \( \Pi^1_1 \text{-CA}_0 \), i.e. the Suslin functional. Nets with uncountable index sets, namely \( \mathbb{N}^\mathbb{N} \), are similarly explosive when combined with the Suslin functional by [78, §3]. Hence, nets with countable index sets can behave quite differently compared to sequences.

Regarding (Q3), we obtain some nice splittings involving various theorems as follows: convergence theorems for nets with (strongly) countable index sets, lemmas by König ([44]), theorems from the RM zoo ([21]), and various coding principles that connect countable and strongly countable sets and/or sequences, including the Cantor-Bernstein theorem for \( \mathbb{N} \). As noted in Footnote 3, a ‘splitting’ is an equivalence \( A \leftrightarrow [B \land C] \) where a natural theorem \( A \) can be ‘split’ into two independent (somewhat) natural parts \( B \) and \( C \). These results suggest that principles from the RM zoo lose their ‘exceptional’ behaviour when formulated with countable sets instead of sequences.

\[ A \] A relatively rare phenomenon in second-order RM is when a natural theorem \( A \) can be ‘split’ as follows into two (somewhat) natural components \( B \) and \( C \) (say over \( \text{RCA}_0 \)): \( A \leftrightarrow [B \land C] \). As explored in [81], there is a plethora of splittings to be found in third-order RM.
We introduce Reverse Mathematics in Section 2.1, as well as Kohlenbach’s generalisation to higher-order arithmetic, and the associated base theory RCA_0^ω. We introduce some essential axioms in Section 2.2.

2. Preliminaries

2.1. Reverse Mathematics. Reverse Mathematics is a program in the foundations of mathematics initiated around 1975 by Friedman ([28, 29]) and developed extensively by Simpson ([85]). The aim of RM is to identify the minimal axioms needed to prove theorems of ordinary, i.e. non-set theoretical, mathematics.

We refer to [S7] for a basic introduction to RM and to [84, 85] for an overview of RM. We expect familiarity with RM, but do sketch some aspects of Kohlenbach’s higher-order RM ([46]) essential to this paper, including the base theory RCA_0^ω (Definition 2.1). As will become clear, the latter is officially a type theory but can accommodate (enough) set theory via Definitions 3.3 and 3.4.

First of all, in contrast to ‘classical’ RM based on second-order arithmetic \( \mathbb{Z}_2 \), higher-order RM uses \( L_\omega \), the richer language of higher-order arithmetic. Indeed, while the former is restricted to natural numbers and sets of natural numbers, higher-order arithmetic can accommodate sets of sets of natural numbers, sets of sets of sets of natural numbers, etc. To formalise this idea, we introduce the collection of all finite types \( T \), defined by the two clauses:

(i) \( 0 \in T \) and (ii) If \( \sigma, \tau \in T \) then \( (\sigma \to \tau) \in T \),

where \( 0 \) is the type of natural numbers, and \( \sigma \to \tau \) is the type of mappings from objects of type \( \sigma \) to objects of type \( \tau \). In this way, \( 1 \equiv 0 \to 0 \) is the type of functions from numbers to numbers, and \( n + 1 \equiv n \to 0 \). Viewing sets as given by characteristic functions, we note that \( \mathbb{Z}_2 \) only includes objects of type \( 0 \) and \( 1 \).

Secondly, the language \( L_\omega \) includes variables \( x^\rho, y^\rho, z^\rho, \ldots \) of any finite type \( \rho \in T \). Types may be omitted when they can be inferred from context. The constants of \( L_\omega \) include the type 0 objects \( 0, 1 \) and \( <_0, +_0, \times_0, =_0 \) which are intended to have their usual meaning as operations on \( \mathbb{N} \). Equality at higher types is defined in terms of ‘\( =_0 \)’ as follows: for any objects \( x^\tau, y^\tau \), we have

\[
[x =_\tau y] \equiv (\forall z^1 \ldots z^k)(xz_1 \ldots z_k =_0 yz_1 \ldots z_k),
\]

(2.1)

if the type \( \tau \) is composed as \( \tau \equiv (\tau_1 \to \ldots \to \tau_k \to 0) \). Furthermore, \( L_\omega \) also includes the recursor constant \( R_0 \) for any \( \sigma \in T \), which allows for iteration on type \( \sigma \)-objects as in the special case (2.2).

Formulas and terms are defined as usual. One obtains the sub-language \( L_{n+2} \) by restricting the above type formation rule to produce only type \( n + 1 \) objects (and related types of similar complexity).

**Definition 2.1.** The base theory RCA_0^ω consists of the following axioms.

(a) Basic axioms expressing that \( 0, 1, <_0, +_0, \times_0 \) form an ordered semi-ring with equality \( =_0 \).

(b) Basic axioms defining the well-known \( \Pi \) and \( \Sigma \) combinators (aka \( K \) and \( S \) in \([23]\)), which allow for the definition of \( \lambda \)-abstraction.

(c) The defining axiom of the recursor constant \( R_0 \): for \( m^0 \) and \( f^1 \):

\[
R_0(f, m, 0) := m \quad \text{and} \quad R_0(f, m, n + 1) := f(n, R_0(f, m, n)).
\]

(2.2)

(d) The axiom of extensionality: for all \( \rho, \tau \in T \), we have:

\[
(\forall x^\rho, y^\rho, \varphi^\rho{\to\tau})[x =_\rho y \to \varphi(x) =_\tau \varphi(y)].
\]

(E_{\rho, \tau})
(e) The induction axiom for quantifier-free formulas of \( L_\omega \).
(f) \( \text{QF-AC}^{1,0} \): the quantifier-free Axiom of Choice as in Definition 2.2.

Note that variables (of any finite type) are allowed in quantifier-free formulas of the language \( L_\omega \): only quantifiers are banned.

**Definition 2.2.** The axiom QF-AC consists of the following for all \( \sigma, \tau \in T \):

\[
(\forall x^\sigma)(\exists y^\tau) A(x, y) \rightarrow (\exists Y^{\sigma \rightarrow \tau})(\forall x^\sigma) A(x, Y(x)),
\]

(QF-AC\(^{\sigma \rightarrow \tau}_0\))

for any quantifier-free formula \( A \) in the language of \( L_\omega \).

As discussed in \( [16, \S 2] \), \( \text{RCA}_0^\omega \) and \( \text{RCA}_0 \) prove the same sentences ‘up to language’ as the latter is set-based and the former function-based. Recursion as in (2.2) is called primitive recursion; the class of functionals obtained from \( \text{RCA}_0 \) for all \( \rho \in T \) is called Gödel’s system \( T \) of (higher-order) primitive recursive functionals.

We use the usual notations for natural, rational, and real numbers, and the associated functions, as introduced in \( [16, \text{p. 288-289}] \).

**Definition 2.3** (Real numbers and related notions in \( \text{RCA}_0^\omega \)).

(a) Natural numbers correspond to type zero objects, and we use ‘\( n^0 \)’ and ‘\( n \in \mathbb{N} \)’ interchangeably. Rational numbers are defined as signed quotients of natural numbers, and ‘\( q \in \mathbb{Q} \)’ and ‘\( \lhd \mathbb{Q} \)’ have their usual meaning.

(b) Real numbers are coded by fast-converging Cauchy sequences \( q_{\langle \rangle} : \mathbb{N} \rightarrow \mathbb{Q} \), i.e. such that \( (\forall n^0, s^0)(|q_n - q_{n+1}| < q_{\langle s \rangle}) \). We use Kohlenbach’s ‘hat function’ from \( [16, \text{p. 289}] \) to guarantee that every \( q^1 \) defines a real number.

(c) We write ‘\( x \in \mathbb{R} \)’ to express that \( x^1 := (q_{\langle \rangle}) \) represents a real as in the previous item and write \( [x](k) := q_k \) for the \( k \)-th approximation of a real.

(d) Two reals \( x, y \) represented by \( q_{\langle \rangle} \) and \( r_{\langle \rangle} \) are equal, denoted \( x =_{\mathbb{R}} y \), if \( (\forall n^0)(|q_n - r_n| \leq 2^{-n+1}) \). Inequality ‘\( \lhd_{\mathbb{R}} \)’ is defined similarly. We sometimes omit the subscript ‘\( \mathbb{R} \)’ if it is clear from context.

(e) Functions \( F : \mathbb{R} \rightarrow \mathbb{R} \) are represented by \( \Phi^{1 \rightarrow 1} \) mapping equal reals to equal reals, i.e. extensionality as in \( (\forall x, y \in \mathbb{R})(x =_{\mathbb{R}} y \rightarrow \Phi(x) =_{\mathbb{R}} \Phi(y)) \).

(f) The relation ‘\( x \leq_{\mathbb{R}} y \)’ is defined as in (2.1) but with ‘\( \leq \)’ instead of ‘\( = \)’. Binary sequences are denoted ‘\( f^1, g^1 \leq_1 1 \)’, but also ‘\( f, g \in C' \)’ or ‘\( f, g \in \mathbb{N}^{\mathbb{N}} \)’.

Elements of Baire space are given by \( f^1, g^1 \), but also denoted ‘\( f, g \in \mathbb{N}^{\mathbb{N}} \)’.

(g) For a binary sequence \( f^1 \), the associated real in \([0, 1]\) is \( r(f) := \sum_{n=0}^{\infty} f(n) 2^{-n+1} \).

Next, we mention the highly useful ECF-interpretation.

**Remark 2.4** (The ECF-interpretation). The (rather) technical definition of ECF may be found in \( [18, \text{p. 138, \S 2.6}] \). Intuitively, the ECF-interpretation \( [A]_{\text{ECF}} \) of a formula \( A \in L_\omega \) is just \( A \) with all variables of type two and higher replaced by type one variables ranging over so-called ‘associates’ or ‘RM-codes’ (see \( [15, \S 4] \) and Remark 3.1); the latter are (countable) representations of continuous functionals. The ECF-interpretation connects \( \text{RCA}_0^\omega \) and \( \text{RCA}_0 \) (see \( [16, \text{Prop. 3.1}] \)) in that if \( \text{RCA}_0^\omega \) proves \( A \), then \( \text{RCA}_0 \) proves \( [A]_{\text{ECF}} \), again ‘up to language’, as \( \text{RCA}_0 \) is formulated using sets, and \( [A]_{\text{ECF}} \) is formulated using types, i.e. using type zero and one objects.

In light of the widespread use of codes in RM and the common practise of identifying codes with the objects being coded, it is no exaggeration to refer to
ECF as the canonical embedding of higher-order into second-order arithmetic. For completeness, we list the following notational convention for finite sequences.

**Notation 2.5** (Finite sequences). We assume a dedicated type for ‘finite sequences of objects of type ρ’, namely ρ∗, which we shall only use for ρ = 0, 1. Since the usual coding of pairs of numbers goes through in RCA0, we shall not always distinguish between 0 and 0∗. Similarly, we do not always distinguish between ‘sρ’ and ‘(sρ)∗’, where the former is ‘the object s of type ρ’, and the latter is ‘the sequence of type ρ∗ with only element sρ’. The empty sequence for the type ρ∗ is denoted by ‘⟨⟩ρ∗’, usually with the typing omitted.

Furthermore, we denote by ‘|s| = n’ the length of the finite sequence sρ∗ = ⟨sρ0, sρ1, . . . , sρn−1⟩, where |⟨⟩| = 0, i.e. the empty sequence has length zero. For sequences sρ∗, tρ∗, we denote by ‘s∗t’ the concatenation of s and t, i.e. (s∗t)(i) = s(i) for i < |s| and (s∗t)(j) = t(j−|s|) for |s| ≤ j < |s|+|t|. For a sequence sρ∗, we define \( πN := (s(0), s(1), . . . , s(N − 1)) \) for \( N < |s| \). For a sequence α0−ρ, we also write \( \piN = (\alpha(0), \alpha(1), . . . , \alpha(N − 1)) \) for any \( N^0 \). By way of shorthand, \((∀q^n ∈ Q^0)(A(q))\) abbreviates \((∀0 < |Q|)(A(Q(i)))\), which is (equivalent to) quantifier-free if A is.

### 2.2. Some axioms of higher-order Reverse Mathematics.

We introduce some axioms of higher-order RM which will be used below. In particular, we introduce some functionals which constitute the counterparts of second-order arithmetic \( \mathbb{Z}_2 \), and some of the Big Five systems, in higher-order RM. We use the ‘standard’ formulation from [46, 59].

First of all, ACA0 is readily derived from:

\[(∃μ^2)(∀f^1)((∃n)(f(n) = 0) → (f(μ(f)) = 0 ∧ (∀i < μ(f))(f(i) ≠ 0)) ∧ (∀n)(f(n) ≠ 0 → μ(f) = 0)), \]

and ACA0ω = RCA0ω + (μ2) proves the same sentences as ACA0 by [38, Theorem 2.5]. The (unique) functional μ2 in (μ2) is also called Feferman’s μ (3]), and is clearly discontinuous at f = 1, 11 . . . ; in fact, (μ2) is equivalent to the existence of F : \( \mathbb{R} \rightarrow \mathbb{R} \) such that F(x) = 1 if \( x > 0 \), and 0 otherwise ([46 §3]), and to

\[(∃φ^2 ≤ 2)(∀f^1)((∃n)(f(n) = 0) ↔ φ(f) = 0). \]

Secondly, Π1-CA0 is readily derived from the following sentence:

\[(∃S^2 ≤ 2)(∀f^1)((∃g^1)(∀n^0)(f(g^n0)) = 0) ↔ S(f) = 0), \]

and Π1-CA0ω = RCA0ω + (S2) proves the same \( Π^1 \)-sentences as Π1-CA0 by [74, Theorem 2.2]. The (unique) functional S2 in (S2) is also called the Suslin functional ([16]). By definition, the Suslin functional S2 can decide whether a Σ1-formula as in the left-hand side of (S2) is true or false.

We similarly define the functional \( S^2_k \) which decides the truth or falsity of \( Σ^1_k \)-formulas; we also define the system Π1-CA0ω as RCA0ω + (S2), where \( (S^2_k) \) expresses that \( S^2_k \) exists. Note that we allow formulas with function parameters, but not functionals here. In fact, Gandy’s Superjump ([30]) constitutes a way of extending Π1-CA0ω to parameters of type two. We identify the functionals \( \mathcal{P}^2 \) and \( S^2_0 \) and the systems ACA0ω and Π1-CA0ω for k = 0. We note that the operators \( ν_k \) from [13, p. 129] are essentially \( S^2_k \) strengthened to return a witness (if existant) to the Σ1-formula at hand.
Thirdly, full second-order arithmetic $\mathbb{Z}_2$ is readily derived from $\cup_k \Pi^1_k-CA^0_0$, or from:

$$(\exists E^3 \leq 3_1)(\forall Y^2)([(\exists f^1)(Y(f) = 0) \iff E(Y) = 0]], \tag{3.3}$$

and we therefore define $\mathbb{Z}^\omega_2 \equiv RCA^0_0 + (3.3)$ and $\mathbb{Z}^3_2 \equiv \cup_k \Pi^1_k-CA^0_0$, which are conservative over $\mathbb{Z}_2$ by [33 Cor. 2.6]. Despite this close connection, $\mathbb{Z}^\omega_2$ and $\mathbb{Z}^3_2$ can behave quite differently, as discussed in e.g. [59, §2.2]. The functional from (3.3) is also called ‘$3^\omega$', and we use the same convention for other functionals.

Fourth, a number of higher-order axioms are studied in [78] including the following comprehension axiom (see also Remark 2.7):

$$(\forall Y^2)(\exists X \subset \mathbb{N})(\forall n \in \mathbb{N})(n \in X \iff (\exists f \in \mathbb{N}^n)(Y(f, n) = 0)). \tag{BOOT}$$

We mention that this axiom is equivalent to e.g. the monotone convergence theorem for nets indexed by Baire space (see [78, §3]). The axiom BOOT$^-$ results from restricting BOOT to functionals $Y$ such that

$$(\forall n \in \mathbb{N})(\exists \text{ at most one } f \in \mathbb{N}^n)(Y(f, n) = 0). \tag{2.3}$$

The weaker BOOT$^-$ appears prominently in the RM-study of open sets given as (third-order) characteristic functions ([61]). In turn, BOOT$^-$ is BOOT$^-$ with ‘$\mathbb{N}^2$’ replaced by ‘$\mathbb{N}^\omega$’ everywhere; BOOT$^-$ was introduced in [23] §3.1 in the study of the Bolzano-Weierstrass theorem for countable sets in Cantor space.

Another weakening of BOOT is the following axiom, central to [70][82].

**Principle 2.6 (Δ-CA).** For $i = 0, 1, Y^2_i$, and $A_i(n) \equiv (\exists f \in \mathbb{N}^n)(Y_i(f, n) = 0)$:

$$(\forall n \in \mathbb{N})(A_0(n) \iff \neg A_1(n)) \rightarrow (\exists X \subset \mathbb{N})(\forall n \in \mathbb{N})(n \in X \iff A_0(n)).$$

Finally, we mention some historical remarks about BOOT.

**Remark 2.7** (Historical notes). First of all, BOOT is called the ‘bootstrap’ principle as it is rather weak in isolation (equivalent to ACA$_0$ under ECF, in fact), but becomes much stronger when combined with comprehension axioms: $\Pi^1_k-CA^0_0 + $BOOT readily proves $\Pi^1_{k+1}-CA^0_0$.

Secondly, BOOT is definable in Hilbert-Bernays’ system $H$ from the *Grundlagen der Mathematik* (see [31, Supplement IV]). In particular, one uses the functional $\nu$ from [31, p. 479] to define the set $X$ from BOOT. In this way, BOOT and subsystems of second-order arithmetic can be said to ‘go back’ to the *Grundlagen* in equal measure, although such claims may be controversial.

Thirdly, after the completion of [78], it was observed by the author that Feferman’s ‘projection’ axiom (Proj1) from [23] is similar to BOOT. The former is however formulated using sets (and set parameters), which makes it more ‘explosive’ than BOOT in that full $\mathbb{Z}_2$ follows when combined with $(\mu^2)$, as noted in [23 I-12].

### 3. Reverse Mathematics and Countable sets

**3.1. Introduction.** In this section, we obtain the results about countable sets sketched in Section 1.2, namely on limit point (and related notions of) compactness (Section 3.3), König’s lemma (Section 3.4), and theorems from the RM zoo (Section 3.5). In doing so, we shall provide interesting answers to (Q0)-(Q3) from Section 1.2. In Section 3.2, we introduce some necessary definitions and obtain preliminary results; the latter provide motivation for the choice of the former.
We single out the ‘explosive’ theorems from Section 3.3, i.e. the latter become much stronger when combined with discontinuous (comprehension) functionals. Our ‘previous best’ was the Lindelöf lemma for $\mathbb{N}^\mathbb{N}$ (see [62]), which is weak in isolation but yields $\Pi^1_1 \text{-CA}_0$ when combined with $(\exists^2)$. In this paper, we identify a number of theorems that yield $\Pi^1_2 \text{-CA}_0$ when combined with $\Pi^1_2 \text{-CA}_0^\omega$. In this light, formalising rather basic mathematics in weak (third-order) systems seems difficult.

Another important conceptual point is that results like Corollaries 3.6 and 4.23 establish that our results do not really depend on the exact notion of set used in this paper. In fact, these corollaries show that the ‘logical power’ (or lack thereof) of theorems about countable sets derives from the existence of injections (or bijections), and not from the particular notion of set used. Following the title of [35], our results can be said to be slicing the whole truth, and nothing but the truth.

Finally, we should mention the ‘excluded middle trick’ pioneered in [62], as well as two related results by Kohlenbach from [45, 46] essential to this paper.

**Remark 3.1** (Two continuity results). First of all, combining [46, Prop. 3.7 and 3.12], the following are equivalent to $(\exists^2)$ over $\text{RCA}_0^\omega$:

- there is a function $f : \mathbb{R} \to \mathbb{R}$ that is not everywhere continuous,
- there is a function $g : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ that is not everywhere continuous,

where the usual ‘$\epsilon$-$\delta$-definition’ of continuity is used. In particular, we note that $\neg(\exists^2)$ is equivalent to ‘all functions on $\mathbb{R}$ (or $\mathbb{N}^\mathbb{N}$) are continuous’, i.e. a version of Brouwer’s (in)famous continuity theorem, over $\text{RCA}_0^\omega$.

Secondly, the standard ‘$\epsilon$-$\delta$-definition’ of continuity on $\mathbb{N}^\mathbb{N}$ coincides with the second-order definition involving ‘RM-codes’ from [85, II.6.1] in a weak system, as follows. Indeed, by combining [35, Prop. 4.4 and 4.10], $\text{RCA}_0^\omega + \text{WKL}$ proves:

if $F : \mathbb{N}^\mathbb{N} \to \mathbb{N}$ is $\epsilon$-$\delta$-continuous on $2^{\mathbb{N}}$, then there is an RM-code $\alpha : \mathbb{N} \to \mathbb{N}$ that coincides with $F$, i.e. we have $(\forall f \in 2^{\mathbb{N}})(\exists N \in \mathbb{N})(F(f) = \alpha(F(N)))$.

The final formula is usually described as ‘$\alpha(f)$ is defined and equals $F(f)$’. The same result holds *mutatis mutandis* for functions on $\mathbb{R}$ that are continuous on $[0, 1]$.

**Remark 3.2** (The excluded middle trick). The law of excluded middle as in $(\exists^2) \lor \neg(\exists^2)$ is quite useful as follows: suppose we are proving $T \to \text{NIN}$ over $\text{RCA}_0^\omega$. Now, in case $\neg(\exists^2)$, all functions on $\mathbb{R}$ are continuous by Remark 3.1 and $\text{NIN}$ holds trivially. Hence, what remains is to establish $T \to \text{NIN}$ in case we have $(\exists^2)$. However, the latter axiom e.g. implies $\text{ACA}_0$ and can uniformly convert reals to their binary representations. In this way, finding a proof in $\text{RCA}_0^\omega + (\exists^2)$ is ‘much easier’ than finding a proof in $\text{RCA}_0^\omega$. In a nutshell, we may wlog assume $(\exists^2)$ when proving theorems that are trivial (or readily proved) when all functions on $\mathbb{R}$ are continuous, like NIN. A considerable ‘extension’ of this trick is formulated in Remark 3.4.

**3.2. Countable sets in higher-order Reverse Mathematics.** We introduce our notion of ‘(strongly) countable set’ in $\text{RCA}_0^\omega$, namely Definitions 3.3 and 3.4 and formulate some associated principles, to be studied below. We also establish preliminary results that justify our choice of the aforementioned definitions. In particular, Corollary 3.6 suggests that our results do not depend on the notion of set used, while Theorem 3.5 shows that one cannot hope to study *in systems below* $\text{ACA}_0$ third-order injections and bijections to $\mathbb{N}$ for sets in the sense of second-order RM, as defined in [85, II.5] or (O) in Section 3.2.1 right below.
3.2.1. Definitions. In this paper, sets are given as in Definition 3.3, namely via characteristic functions as in [48, 60, 61]. In fact, Definition 3.3 is taken from [61] and is inspired by the RM-definition of open and closed sets ([85, II.5.6]), as follows.

As is well-known, in second-order RM, closed sets are the complements of open sets and an open set $U \subseteq \mathbb{R}$ is represented by some sequence of open balls $\bigcup_{n \in \mathbb{N}} B(a_n, r_n)$ where $a_n, r_n$ are rationals (see [85, I.4]). One then writes the following for any $x \in \mathbb{R}$:

$$x \in U \text{ if and only if } (\exists n \in \mathbb{N})(|x - a_n| < r_n).$$

We shall use the following definition of ‘countable set’ in $\text{RCA}_0^\omega$.

Definition 3.4. [Countable subset of $\mathbb{R}$] A set $A \subseteq \mathbb{R}$ is countable if there exists $Y : \mathbb{R} \to \mathbb{N}$ such that $(\forall x, y \in A)(Y(x) \neq Y(y) \rightarrow x \neq y)$. If $Y : \mathbb{R} \to \mathbb{N}$ is also surjective, i.e. $(\forall n \in \mathbb{N})(\exists x \in A)(Y(x) = n)$, we call $A$ strongly countable.

The first part of Definition 3.4 is from Kunen’s set theory textbook ([50, p. 63]) and the second part is taken from Hrbacek-Jech’s set theory textbook ([37] where the term ‘countable’ is used instead of ‘strongly countable’). For the rest of this paper, ‘strongly countable’ and ‘countable’ shall exclusively refer to Definition 3.4 except when explicitly stated otherwise.

Finally, it behoves us to briefly motivate our choice of definitions, as follows.

- The RM-definition of closed set and third-order injections/bijections to $\mathbb{N}$ readily give rise to $(\exists^2)$ by Theorem 3.5, i.e. this combination does not allow us to work in systems below $\text{ACA}_0$, suggesting the need for Definition 3.3.
- Some of our results do not really depend on the exact notion of set used by e.g. Corollaries 3.6 and 3.24. In particular, the ‘logical power’ (or lack thereof) of theorems about countable sets derives from the existence of injections (or bijections) to $\mathbb{N}$, and not from the particular notion of set.
- Definition 3.3 allows us to ‘split’ a given theorem, like item 1 in Theorem 3.26 into a well-known second-order part and a third-order ‘explosive’ part without first-order strength, namely as in item 1 in Theorem 3.26.

These results are obtained in the next sections as corollaries to our main results.

3.2.2. Basic properties of countable sets. It seems obvious that a sequence in $\mathbb{R}$ forms a countable set, but we show in this section that proving this fact requires non-trivial axioms. The same holds for seemingly trivial statements from topology, namely related to countability axioms and $\mathbb{N}_\infty$, which we introduce next. These
results motivate our choice of definitions made in the previous section. In particular, by Corollary 3.6 our results do not really depend on the exact definition of set.

First of all, the set $\mathbb{N}_\infty$ is the one-point compactification of $\mathbb{N}$, defined as follows:

\[ \mathbb{N}_\infty := \{ f \in 2^\mathbb{N} : \forall n \in \mathbb{N}(f(n) = 1 \rightarrow (\forall m < n)(f(m) = 1) \}. \quad (3.1) \]

We interpret ‘$f \in \mathbb{N}_\infty$’ as the underlined $\Pi^0_1$-formula from (3.1), i.e. $\mathbb{N}_\infty$ is an RM-closed set and items (i) and (v) in Theorem 3.5 express that $\mathbb{N}_\infty$ is (strongly) countable. Thus, Theorem 3.5 shows that one cannot hope to study bijections or injections on second-order RM-sets in systems below ACA0, which provides a modicum of motivation for our definitions in the previous section. Our study of $\mathbb{N}_\infty$ as in Theorem 3.5 was inspired by the decidability results pertaining to the latter in constructive mathematics from [22].

Secondly, we note that Munkres defines first-countable both in terms of countable sets ([58, p. 190]) and sequences ([58, p. 131]). By contrast, the notion second-countable is only defined based on countable sets by Munkres in [58]. We use the aforementioned standard definition4 based on countable sets. Similar to the previous paragraph, Theorem 3.5 provides motivation for our definitions from Section 3.2.1 and a partial answer to (Q0).

**Theorem 3.5.** The following are equivalent over RCA0:  
(i) $(\exists 2)$,  
(ii) any sequence $(x_n)_{n \in \mathbb{N}}$ of reals is a countable set,  
(iii) the unit interval is first-countable,  
(iv) $(\exists 2^2)(\forall f, g \in \mathbb{N}_\infty)(Y(f) = 0 \rightarrow Y(g) = f \neq g)$,  
(v) item ($\exists 1$) where $Y^2$ additionally satisfies $(\forall m)(\exists h \in \mathbb{N}_\infty)(Y(h) = n)$.

Over RCA0, $(\exists 2^2)$ follows from the statement: the unit interval is second-countable.

**Proof.** For the implication (i) $\rightarrow$ (v), note that $f \in \mathbb{N}_\infty$ means that either $f = f = 0, 1, 2, \ldots$ or $f = 1, 0, 1, 1, \ldots$. Using $\exists 2^2$ to distinguish between these cases, define $Y(f)$ as respectively 0, 1, $k + 1$ for $|\sigma| = k \geq 0$. Clearly, $Y$ satisfies the properties from item (v) as $Y(\sigma_k = 0, 0, 0, \ldots) = k + 1$ for $k \geq 1$ and $Y(0, 0, 0, \ldots) = 0$ and $Y(1, 1, 1, \ldots) = 1$.

For (i) $\rightarrow$ (iv) $\rightarrow$ (i), we use the following equivalence from [22]:

\[ (\forall f \in \mathbb{N}_\infty)(Y(f) > 0) \leftrightarrow Y(\epsilon(Y)) > 0, \quad (3.2) \]

where $\epsilon^{2^{-1}}$ is defined as follows: $\epsilon(Y)(n)$ is 1 if $Y(\sigma_k * 00, 00, \ldots) > 0$ for all $k \leq n$, and zero otherwise. The forward implication in (3.2) is obvious, while the reverse implication follows from the aforementioned case distinction. Indeed, assume $Y(\epsilon(Y)) > 0$ and note that if $Y(0, 0, 0, 0, \ldots) = 0$, then $\epsilon(Y) = 1, 0, 0, \ldots$ by definition, a contradiction. Similarly, assuming $\epsilon(Y) > 0$, if $Y(\sigma_k * 00, 00, \ldots) = 0$ for $k \geq 1$ the least such number, then $\epsilon(Y) = 1, 0, 0, \ldots$, which is impossible. Indeed, $\epsilon(Y) = 1, 0, 0, \ldots$ implies $0 < Y(\epsilon(Y)) = 1, 0, 0, \ldots$ by the extensionality of $Y$ and our assumption. However, $\epsilon(Y) = 1, 0, 0, \ldots$ also implies $\epsilon(Y)(k - 1) = 0$ and hence $Y(\sigma_k * 00, 00, \ldots) = 0$ by definition. Hence, we have that if $Y(\epsilon(Y)) > 0$, then $Y(\sigma_k * 00, 00, \ldots) > 0$ for any $k \in \mathbb{N}$. The consequent of the latter

---

4A topological space $X$ is first-countable if for any $x \in X$ there is a countable collection $\mathcal{B}$ of neighbourhoods of $x$ such that each neighbourhood of $x$ contains some $B \in \mathcal{B}$ ([58, p. 190]). A topological space $X$ is second-countable if there is a countable collection $\mathcal{U}$ of basic opens sets such that any open subset of can be written as a union of elements of $\mathcal{U}$ ([58, p. 78]).
observation also yields that \( \varepsilon(Y) = 1 \ldots \) by the definition of \( \varepsilon \). But then also \( Y(11 \ldots) = Y(\varepsilon(Y)) > 0 \) by the extensionality of \( Y \) and our assumption. In this way, we have established the left-hand side of \( (\exists 2) \), which tells us that quantifiers over \( \aleph_1 \) are decidable; this is proved in \( [22] \) for constructive mathematics, which is all the more impressive. Now let \( Y^2 \) be as in item \( (\exists 1) \) and consider:

\[
(\exists n)(f(n) = 0) \Leftrightarrow (\exists g \in \aleph_1)(f(Y(g)) = 0) \Leftrightarrow f(Y(\varepsilon(\lambda g.f(Y(g)))) = 0, \tag{3.3}
\]

which immediately provides us with \( (\exists 2) \). To obtain \( (\exists 1) \rightarrow (\exists 3) \), let \( Y^2 \) be as in the former and consider the following formula:

\[
(\forall f \in \aleph_1)(Y(f) \neq 0 Y(11 \ldots) \rightarrow (\exists k \in \aleph_1)(f(k) = 0)), \tag{3.4}
\]

which has the right format for applying \( \text{QF-AC}^{1,0} \) (included in \( \text{RCA}_0^2 \)) as ‘\( f \in \aleph_1 \)’ is \( \Pi_1^0 \). The ‘trick’ we use is that the \( \Sigma_1^0 \)-formula \( f \neq 1 1 \ldots \) can be replaced by the decidable formula \( Y(f) \neq 0 Y(11 \ldots) \), thanks to our assumptions on \( Y \). Thus, let \( G^2 \) be such that \( G(f) = k \) is the least \( k \in \aleph_1 \) such that \( f(k) = 0 \) for \( f \in \aleph_1 \) such that \( Y(f) \neq Y(11 \ldots) \). Now define \( Y_0^2 \) as follows:

\[
Y_0(f) := \begin{cases} 0 & Y(f) = Y(00 \ldots) \\ 1 & Y(f) = Y(11 \ldots) \\ G(f) + 1 & \text{otherwise} \end{cases}
\]

which has the right properties for item \( (\exists 1) \) by the definition of \( G \). Note that we could also use \( (\forall n \in \aleph_1)(f(n) > 0) \Leftrightarrow Y(f) = Y(11 \ldots) \) to obtain \( (\exists 2) \), where \( f(n) = 1 \) if \( (\forall i \leq n)(f(i) = 1) \), and zero otherwise. Note \( f \in \aleph_1 \) for all \( f \in 2^{\aleph_1} \).

For the implication \( (\exists 1) \rightarrow (\exists 3) \), let \( (x_n)_{n \in \aleph_1} \) be a sequence of reals and define \( x \in A \) as \( (\exists n \in \aleph_1)(x = x_n) \) using \( (\exists 2) \). Use \( \mu^x \) to define \( Y : \aleph \rightarrow \aleph_1 \) as follows: \( Y(x) \) is the least \( n \in \aleph_1 \) such that \( x = x_{n+1} \) if such exists, and 0 otherwise. Then \( Y \) is an injection on \( A \) by definition, and the latter set is therefore countable.

Next, assume \( (\exists 1) \) and suppose \( \neg (\exists 2) \). The latter implies that all functions on \( \aleph \) are continuous by Remark \( 5.1 \). Now fix some countable set \( A \subset \aleph \), i.e. there is \( Y : \aleph \rightarrow \aleph_1 \) that is injective on \( A \). By continuity, if \( x_0 \in A \), then for \( y \in \aleph \) close enough to \( x_0 \), we have \( y \in A \) by Definition \( 5.3 \). Again by continuity, if \( Y(x_0) = n_0 \) for \( x_0 \in A \), then for \( z \in \aleph \) close enough to \( x_0 \in A \), we have \( Y(z) = n_0 \), a contradiction. Hence, \( A \) must be empty and the same holds for all countable sets. However, item \( (\exists 1) \) applied to e.g. \( (1 + 1/n)_{n \in \aleph_1} \) gives rise to a non-empty set, a contradiction. Hence, we must have \( (\exists 2) \), and \( (\exists 1) \leftrightarrow (\exists 3) \) thus follows.

As to \( (\exists 1) \rightarrow (\exists 3) \), to show that \( [0,1] \) is first-countable, one uses \( (\exists 2) \) to define for \( x \in [0,1] \) the countable set \( A_x \) consisting of all intervals in \( [0,1] \) with rational end points and containing \( x \). For \( (\exists 1) \rightarrow (\exists 3) \), in the same way as in the previous paragraph, \( \neg (\exists 2) \) leads to a contradiction as the local basis as in the definition of first-countability cannot be empty. The final part follows in the same way as in the previous two paragraphs as a basis for a topology cannot be empty.

The previous theorem is important as follows: the topology of first-countable spaces can be described in terms of sequences, while other spaces require nets for this purpose. When \( (\exists 2) \) is not available, we are therefore more of less forced to use nets for describing the topology of \( [0,1] \), assuming coding is not an option. The (higher-order) RM-study of nets can be found in \( [75, 77] \). Moreover, modulo technical details, the previous theorem suggests we can derive \( (\exists 2) \) from the existence of a
countable set in many contexts, like the existence of a countable basis for vector spaces; the associated (second-order) RM study is in [85, III.4].

Next, the equivalence (1) ↔ (2) in Theorem 3.5 is partly due to our choice of the notion of set, namely as in Definition 3.3. Nonetheless, the next corollary shows that (1) → (2) goes through over RCA_0 + WKL no matter what notion of set is used. This ‘meta-result’ is not fully formal, but rigorous all the same.

**Corollary 3.6.** The implication (1) → (2) from the theorem goes through over RCA_0 + WKL regardless of the meaning of ‘x ∈ A’.

**Proof.** Assume (1) from the theorem and suppose ¬(∃^2). The latter implies that all functions on \( \mathbb{R} \) are continuous by Remark 3.1. Now fix some set \( A \subseteq [0, 1] \) (whatever it may be) that is countable, i.e. there is \( Y : \mathbb{R} \to \mathbb{N} \) that is injective on \( A \). Since \( Y \) is continuous on \([0, 1]\), it has an RM-code given WKL following Remark 3.4. By the RM of WKL in [85, IV], this RM-code, and hence \( Y \), is also uniformly continuous on \([0, 1]\). Let \( N_0 \in \mathbb{N} \) be such that \( |Y(x) - Y(y)| < 1 \) for any \( x, y \in [0, 1] \) with \( |x - y| < \frac{1}{2^N} \) and note that \( Y \) must be constant on \([0, 1]\). Hence, whatever ‘x ∈ A’ is, it can only be true for at most one \( x \) in \([0, 1]\), since \( Y \) is an injection. However, item (1) guarantees that the sequence \((\frac{x}{2^n})_{n \in \mathbb{N}}\) in \([0, 1]\) gives rise to a countable set with infinitely many distinct elements, a contradiction. □

The reader may verify that the below results satisfy ‘independence’ properties similar to Corollary 3.6. We have obtained such results in e.g. Corollary 3.24. We discuss the implications of these corollaries in Remark 3.14.

### 3.2.3. Basic principles about (un)countable sets.

We introduce some fundamental principles to be studied below.

First of all, the following coding principle is central to [64, 65] and this paper.

**Principle 3.7 (cocode_0).** For any non-empty countable set \( A \subseteq [0, 1] \), there is a sequence \((x_n)_{n \in \mathbb{N}}\) in \( A \) such that \((\forall x \in \mathbb{R})(x \in A \leftrightarrow (\exists n \in \mathbb{N})(x_n =_\mathbb{R} x))\).

We let fin_0 be cocode restricted as follows.

**Principle 3.8 (fin_0).** Let \( A \subseteq [0, 1] \) and \( Y : [0, 1] \to \mathbb{N} \) be such that the latter is injective and bounded on the former. Then there is a sequence \((x_n)_{n \in \mathbb{N}}\) in \( A \) such that \((\forall x \in \mathbb{R})(x \in A \leftrightarrow (\exists n \in \mathbb{N})(x_n =_\mathbb{R} x))\).

Intuitively speaking, fin_0 expresses that finite sets can be listed by sequences. Moreover, we let cocode_1 be cocode restricted to strongly countable sets. As will become clear in Section 3.3, there is a big difference between theorems formulated with ‘countable’ and with ‘strongly countable’ in that the former (but not the latter) are explosive, i.e. become much stronger when combined with a discontinuous (comprehension) functional, even up to \( \Pi^1_2 \)-CA_0, the current upper bound of RM, to the best of our knowledge.

Secondly, injections and bijections are connected via the Cantor-Bernstein theorem, formulated by Cantor in [177] and first proved by Bernstein (see [10], p. 104). Intuitively, if there is an injection from \( A \) to \( B \) and an injection from \( B \) to \( A \), then there is a bijection between them. We study the restriction to \( B = \mathbb{N} \) and \( A \subseteq \mathbb{R} \).

---

5It is a tedious-but-straightforward verification that the below proofs still go through if we replace the equivalence by a forward arrow in cocode. One ‘immediate’ example is that cocode → BOOT in the proof of Theorem 3.16.
Note that item (ii) in CBN provides the second injection (from \( \mathbb{N} \) to \( B \)) in the usual formulation of the Cantor-Bernstein theorem.

**Principle 3.9 (CBN).** For a set \( A \subset \mathbb{R} \), the following two conditions:

(i) there is \( Y : \mathbb{R} \to \mathbb{N} \) which is injective on \( A \), i.e. \( A \) is countable

(ii) there is a sequence \( (x_n)_{n \in \mathbb{N}} \) in \( A \) consisting of pairwise distinct reals, imply that there is \( Z : \mathbb{R} \to \mathbb{N} \) which is bijective on \( A \), i.e. \( A \) is strongly countable.

By Theorem 3.10 CBN and cocode\(_0\) are intimately connected, and both are ‘explosive’ by Corollary 3.14. By contrast, \( \mathbb{N}_\infty \) ‘trivially’ satisfies the Cantor-Bernstein theorem by (the proof of) Theorem 3.5, which provides another argument against using RM-closed sets rather than Definition 3.3. As it happens, our study of \( \mathbb{N}_\infty \) was motivated by the study of the Cantor-Bernstein theorem in constructive (reverse) mathematics via \( \mathbb{N}_\infty \) in [60].

Now, the usual formulation of the Cantor-Bernstein theorem involves two injections. Let CBN' be CBN with the ‘sequence’ condition replaced by ‘there is an injection from \( \mathbb{N} \) to \( A \)’, which is perhaps a rather naive formulation of the Cantor-Bernstein theorem. The following principle is the main subject of [63] and studied in [64], and connects these two versions.

**Principle 3.10 (NCC).** For \( Y^2 \) and \( A(n, m) \equiv (\exists f \in 2^\mathbb{N})(Y(f, m, n) = 0) \):

\[
(\forall n \in \mathbb{N}) (\exists m \in \mathbb{N}) A(n, m) \rightarrow (\exists g : \mathbb{N} \rightarrow \mathbb{N})(\forall n \in \mathbb{N}) A(n, g(n)). \tag{3.5}
\]

Intuitively speaking, NCC is a very weak version of countable choice as in QF-AC\(^{0,1}\), with interesting RM and computability properties, as studied in [63].

Finally, while \( \mathbb{R} \) and \([0, 1]\) are ‘obviously’ not countable, the following principles cannot be proved in \( \mathbb{Z}_2 \) (but \( \mathbb{Z}_2^3 \) can prove them), as shown in [64] §3.

**Principle 3.11 (NIN).** For \( Y : [0, 1] \rightarrow \mathbb{N} \), there are \( x, y \in [0, 1] \) such that \( x \not\approx \mathbb{R} y \) and \( Y(x) \approx_{\mathbb{N}} Y(y) \).

**Principle 3.12 (NBI).** For \( Y : [0, 1] \rightarrow \mathbb{N} \), either there are \( x, y \in [0, 1] \) such that \( x \not\approx \mathbb{R} y \) and \( Y(x) \approx_{\mathbb{N}} Y(y) \), or there is \( N \in \mathbb{N} \) such that \( (\forall x \in [0, 1])(Y(x) \neq N) \).

As is clear from [62] Figure 1], NIN and NBI follow from many elementary theorems of ordinary mathematics. As it turns out, deriving NIN (or NBI) often also allows us to derive cocode\(_0\) (or cocode\(_1\)), while the former proofs are conceptually simpler, in general. We shall provide examples of this phenomenon in this paper.

The following theorem connects NIN with (A) and (B) from Section 11. Note that the implication (B) \( \rightarrow \) NIN can be found in e.g. Rudin’s textbook [73] p. 41.

**Theorem 3.13.** The system \( \text{RCA}_0 \) proves NIN \( \leftrightarrow \) (A) and (B) \( \rightarrow \) NIN. The same holds for NBI and strongly countable sets.

*Proof.* For the equivalence, NIN and (A) are trivial in case \( \neg(\exists^2) \), by the proof of Theorem 3.3. In case \( (\exists^2) \), \([0, 1]\) is a set in the sense of Definition 3.3. The equivalence is now a mere manipulation of definitions using contraposition. The equivalence involving NBI and (A) restricted to strongly countable sets is proved in the same way. For (B) \( \rightarrow \) NIN, \( \neg\text{NIN} \) implies that \([0, 1]\) is countable and perfect. \( \square \)

Note that (B) \( \rightarrow \) NIN goes through for any definition of ‘perfect set’ that makes the unit interval a perfect set in \( \text{RCA}_0 \). Since NIN does not mention the notion
of ‘countable set’, the equivalence \(\text{NIN} \leftrightarrow \langle A \rangle\) suggests that our notion of set as in Definition 3.3 is relatively satisfactory.

Finally, we discuss the implications of Corollary 3.6 for principles concerning countable sets like \(\text{cocode}_0\).

**Remark 3.14** (Countable sets and continuity). The aim of Section 3.2.2 was to show that the exact definition of set does not matter much when dealing with countable sets, culminating in Corollary 3.6. In this light, we might as well chose the ‘most convenient’ definition of (countable) set, namely Definition 3.3. Indeed, given \(\neg (\exists^2)\), one readily\(^6\) shows that there are no non-empty countable subsets of \(\mathbb{R}\) or \(\mathbb{N}\), rendering the principles \(\text{cocode}_0\) and \(\text{NIN}\) trivial. Below, we shall often make use of this observation, which we have delayed until now for didactic reasons. The same observation holds for alternative representations of sets, which we will explore in a future publication.

### 3.3. Limit point versus sequential compactness

The main topic of this section is Section 3.3.1, namely the study of compactness via nets with countable index sets. These results give rise to improvements of results on nets with uncountable index sets from e.g. [77, 78], as explored in Section 3.3.2.

#### 3.3.1. Nets with countable index sets

In this section, we study limit point compactness and nets restricted to countable index sets, with an eye on the questions (Q0)-(Q3) from Section 1.2.

Now, the RM-study of sequential compactness is well-known ([58, III.2]); the RM-study of limit-point compactness and compactness based on nets is not as well-developed. In the latter case, [77] constitutes a first step and may be consulted for an introduction to nets in \(\text{RCA}_0\), including the required definitions, which are of course essentially the standard ones from the literature.

First of all, let \(\text{BW}^{0,1}_0\) be the following version of the Bolzano-Weierstrass theorem. We discuss the details of the formulation of this principle in Remark 3.28.

**Principle 3.15** (\(\text{BW}^{0,1}_0\)). For countable \(A \subset [0,1]\) and \(F : [0,1] \rightarrow [0,1]\), the supremum \(\sup_{x \in A} F(x)\) exists.

Using the usual interval-halving technique and working over \(\text{RCA}_0\), \(\text{ACA}_0\) is equivalent to the statement that for a sequence \((x_n)_{n \in \mathbb{N}}\) and any \(F : [0,1] \rightarrow [0,1]\), \(\sup_{n \in \mathbb{N}} F(x_n)\) exists. As is clear from Theorem 3.16 and its corollaries, the formulation using countable sets as in \(\text{BW}^{0,1}_0\) behaves quite differently.

We have studied the RM of nets in [77], and we refer to the latter for all relevant definitions related to nets in \(\text{RCA}_0\). While nets can have almost arbitrary index sets, Tukey shows in [59] that topology can be developed based on phalanxes, which are nets with index set consisting of the finite subsets of a given set, ordered by

\[\text{max} = \sup\text{-adherence}.\]

---

\(^6\)Let \(Y : [0,1] \rightarrow \mathbb{R}\) be injective on \(A \subset [0,1]\) where the latter is represented by \(Z : [0,1] \rightarrow \mathbb{N}\), i.e. \(x \in A \iff Z(x) = 1\) for \(x \in [0,1]\). Given \(\neg (\exists^2)\), \(Z, Y\) are continuous by the above. Hence, for \(x \in A\), we have \(y \in A\) for \(y \in B(x, 1/2^n)\), for some \(n \in \mathbb{N}\). Moreover, for some \(M \in \mathbb{N}\), \(Y(z) = Y(x)\) for all \(z \in B(x, 1/2^M)\). Considering \(\max(N, M)\), we observe that \(Y\) is not injective on \(A\).

\(^7\)A space is called *limit point compact* by Munkres in [58] p. 178 if every infinite subset has a limit (=adherence) point.
inclusion. Thus, let $\text{MCT}_0^{\text{set}}$ be the monotone convergence theorem for nets in $[0,1]$ with index set consisting of finite sequences without repetitions, as follows:

$$D = \{ w^j : (\forall i, j < |w|)(i \neq j \to w(i) \neq w(j)) \land (\forall i < |w|)(w(i) \in A) \},$$  \hspace{1cm} (3.6)

for countable $A \subset \mathbb{R}$. We order $D$ by inclusion, i.e. '$w \preceq_D v$' means that all elements of $w$ are included in $v$. We note that e.g. \[77\] §4 deals with nets that are not obviously phalanxes, in the context of sequential continuity and compactness.

Now, the monotone convergence theorem for sequences is equivalent to $\text{ACA}_0$ (\[85\] III.2), i.e. the formulation using countable sets as in $\text{MCT}_0^{\text{set}}$ behaves quite differently by the following theorem.

**Theorem 3.16.** The system $\text{ACA}_0^\omega$ proves that the following are equivalent:

(i) $\text{BW}_0^{[0,1]}$: Bolzano-Weierstrass theorem for countable sets in $[0,1]$,
(ii) $\text{MCT}_0^{\text{set}}$: monotone convergence theorem for nets with countable index sets,
(iii) $\text{BOOT}_C$,
(iv) $\text{cocode}_0$: a countable set in $[0,1]$ can be listed as a sequence,
(v) $\text{CBN} + \text{cocode}_1$.

Omitting item (\[\text{iii}\]), the equivalences go through over $\text{RCA}_0^\omega$.

**Proof.** First of all, while (3.2) is needed to state (3.6) and hence item (\[\text{iii}\]), all other items are vacuously true given $\neg(3.2)$, following the penultimate paragraph of the proof of Theorem 3.5 and Remark 3.14. Hence, the final sentence of the theorem involving $\text{RCA}_0^\omega$ follows from equivalences over $\text{ACA}_0^\omega$ by considering the law of excluded middle $(3.2) \lor \neg(3.2)$. For the rest of the proof, we assume (3.2) and hence $\text{ACA}_0$ and attendant RM-results from \[85\] III.2. In particular, we may use $\exists^2$ to freely convert between reals in $[0,1]$ and their binary representations. We shall therefore sometimes work over $2^\omega$ rather than $[0,1]$. Moreover, elementhood (as in Definition 3.3) is decidable.

Secondly, (\[\text{iv}\]) $\to$ (\[\text{iii}\]) is immediate as $\text{cocode}_0$ converts a countable set into a sequence, which has a supremum given $\text{ACA}_0$ by \[85\] III.2. Similarly, (\[\text{iv}\]) $\to$ (\[\text{iii}\]) follows from the observation that $\text{cocode}_0$ converts a countable index set into a sequence, after which the usual ‘interval-halving technique’ can be done using $\text{ACA}_0$.

The implication $\text{BOOT}_C$ $\to$ $\text{cocode}_0$ is proved in \[64\] Theorem 3.16; since $\text{cocode}_0$ deals with sets in $[0,1]$, the same proof also yields (\[\text{vi}\]) $\to$ (\[\text{v}\]) by coding real numbers as binary sequences using $\exists^2$. For completeness, we provide a sketch as follows. Fix $A \subset [0,1]$ and let $Y : [0,1] \to \mathbb{N}$ be an injection on $A$. Use $\text{BOOT}_C$ to define $X \subset \mathbb{N}^2 \times \mathbb{Q}$ such that for all $n, m \in \mathbb{N}$ and $q \in \mathbb{Q} \cap [0,1]$, we have:

$$(n, m, q) \in X \leftrightarrow (\exists x \in B(q, \frac{1}{2^m}) \land A)(Y(x) = n).$$ \hspace{1cm} (3.7)

By assumption, the following condition required for $\text{BOOT}_C$, is satisfied:

$$(\forall n, m \in \mathbb{N}, q \in \mathbb{Q} \cap [0,1])(\exists \text{ at most one } x \in B(q, \frac{1}{2^m}) \land A)(Y(x) = n).$$

We now use $x$ from (3.7) and the well-known interval-halving technique to create a sequence $(x_n)_{n \in \mathbb{N}}$. For fixed $n \in \mathbb{N}$, define $[x_n](0)$ as 0 if $(\exists x \in [0,1/2) \land A)(Y(x) = n)$, and 1/2 otherwise; define $[x_n](m + 1)$ as $[x_n](m)$ if $(\exists x \in [x_n](m), [x_n](m) + \frac{1}{2^m}) \land A)(Y(x) = n)$, and $[x_n](m) + \frac{1}{2^m}$ otherwise. By definition, we have:

$$(\forall n \in \mathbb{N})(\exists x \in A)(Y(x) = n) \leftrightarrow [x_n] \in A \land Y(x_n) = n],$$
which readily yields \( \text{cocode}_0 \) as \( \mu^2 \) can be used to removed \( x_n \) from the sequence in case \( x_n \not\in A \). This concludes the sketch based on the proof of [64, Theorem 3.16].

Next, we prove (iv) \( \rightarrow \) (iii). Fix \( Z \) such that \( (\forall n \in \mathbb{N})(\exists \text{ at most one } g \in 2^N)(Z(g, n) = 0) \) and define the set \( A := \{ g \in 2^N : (\exists n \in \mathbb{N})(Z(g, n) = 0) \} \). To show that \( A \) is countable, define \( Y(g) \) as the least \( n \in \mathbb{N} \) such that \( Z(g, n) = 0 \), if such there is, and zero otherwise; by assumption, \( Y \) is an injection on \( A \) and the latter is therefore countable. Let \( (h_m)_{m \in \mathbb{N}} \) be a sequence in \( 2^N \) such that \( g \in A \leftrightarrow (\exists m \in \mathbb{N})(g = h_m) \) for any \( g \in 2^N \). In this way, we have \( (\exists g \in 2^N)(Z(g, n) = 0) \leftrightarrow (\exists m \in \mathbb{N})(Z(h_m, n) = 0) \) for any \( n \in \mathbb{N} \), and \( \text{BOOT}_C \) now follows from \( \text{ACA}_0 \).

We now prove (iii) \( \rightarrow \) (ii). Recall that (iii) allows us to freely convert between real numbers and their binary representations, which means we can work over \( 2^N \) rather than \([0,1]\) without problems. Fix \( Z \) such that \( (\forall n \in \mathbb{N})(\exists \text{ at most one } g \in 2^N)(Z(g, n) = 0) \). Let \( A \subset 2^N \) be the countable set as in the previous paragraph and let \( Y : 2^N \rightarrow \mathbb{N} \) be the associated injection on \( A \). Let \( D \) be the set of finite sequences in \( A \) (without repetitions) and let \( \leq_D \) be the inclusion ordering, i.e. \( w \leq_D v \iff (\forall i < |w|)(\exists j < |v|)(w(i) = v(j)) \). Now define the net \( f_w : D \rightarrow 2^N \) as \( f_w := \lambda k. F(w, k) \) where \( F(w, k) = 1 \) if \( (\exists i < |w|)(Y(w(i)) = k) \), and zero otherwise. This net is increasing as \( w \leq_D v \rightarrow f_w \leq_{\text{lex}} f_v \) by definition. Let \( h \) be the limit of this net and consider the following equivalences, for any \( n \in \mathbb{N} \):

\[
 h(n) = 1 \leftrightarrow (\exists f \in A)(Y(f) = n) \leftrightarrow (\exists g \in 2^N)(Z(g, n) = 0), \tag{3.8}
\]

The second equivalence in (3.8) follows by the definition of \( Y \) in terms of \( Z \). The first equivalence in (3.8) follows from the definition of limit of a net. By (3.8), the set \( \{ n \in \mathbb{N} : (\exists g \in 2^N)(Z(g, n) = 0) \} \) exists, as required for \( \text{BOOT}_C \).

Next, (i) \( \rightarrow \) (iii) is obtained by modifying the previous paragraph as follows: the set \( B = \{ w^+ : (\forall i < |w|)(w(i) \in A) \} \) is countable as \( \tau(f_w) = \tau(f_v) \rightarrow f_w \neq f_v \), for \( w^+ \), \( v^+ \) finite sequences in \( A \). Modulo coding using \( \exists^2 \), \( B \) can be viewed as a subset of \( 2^N \). The supremum of \( \sup_{w \in B} F(w) \) for \( F(w) := f_w \) also yields (3.3) and hence \( \text{BOOT}_C \). Note that (i) \( \rightarrow \) (iii) is also proved in the proof of [64, Theorem 3.23] for \( \text{BW}_0[0,1] \) formulated using \( 2^N \) rather than \([0,1]\).

Finally, the implications (iv) \( \rightarrow \) (v) \( \rightarrow \) \( \text{cocode}_1 \) are clearly trivial. The remaining implication \( \text{cocode}_0 \rightarrow \text{cocode}_1 \) readily follows by using \( \exists^2 \) to ‘trim’ duplicate reals from the sequence provided by \( \text{cocode}_0 \).

For the next corollary, note that \( \text{cocode}_1 \) is weak and not explosive\(^8\), i.e. the principle \( \text{CBN} \) is (mostly) responsible for obtaining \( \Pi^1_3-\text{CA}_0 \).

**Corollary 3.17.** Let \( X \) be any item among (i)-(v) from the theorem. The system \( \Pi^1_3-\text{CA}_0 + X \) proves \( \Pi^1_2-\text{CA}_0 \).

**Proof.** Since (S\(^2\)) \( \rightarrow \) (S\(^2\)), we may use the latter to freely convert between reals in \([0,1]\) and their binary representations. Hence, \( \text{BW}_0[0,1] \) is readily seen to be equivalent to \( \text{BW}_0^C \), where the latter is the former but formulated for Cantor space \( 2^N \), namely as in [64, Def. 3.21]. By [64, Theorem 3.22], \( \Pi^1_2-\text{CA}_0 \) follows from \( \Pi^1_3-\text{CA}_0 + \text{BW}_0^C \), and we are done. \( \square \)

\(^8\)We have that QF-AC\(^{0,1} \rightarrow \) cocode\(_1 \) over RCA\(_0\) by [63, Theorem 3.24] and that \( \Pi^1_3-\text{CA}_0 \) + QF-AC\(^{0,1} \) is a \( \Pi^1_3 \)-conservative extension of \( \Pi^1_3-\text{CA}_0 \) by [43, Theorem 2.2].
Theorem 3.21. The system \( \text{QF-AC}^0 \) readily yields the choice function required by \( \text{QF-AC}^0 \). The system \( \text{MCT}^0 \) cannot prove \( \text{QF-AC}^0 \) in RCA0.

Proof. For the first part, by [63, §3], \( \text{QF-AC}^0 \) cannot prove NIN, while \( \text{cocode}_0 \rightarrow \text{NIN} \). For the remaining part, let \( X \in L_2 \) be a sentence not provable in RCA0. Then \( \text{RCA}_0^c + \text{BW}_0 \vdash X \) yields \( \text{RCA}_0 \vdash X \) under ECF as the latter makes \( \text{BW}_0 \) vacuously true, following the penultimate part of the proof of Theorem 3.5.

We could obtain similar results for the Ascoli-Arzelà theorem (see [85, III.2] for the associated RM results) for nets with countable index sets. The following corollary should be contrasted with [78, §3.2], where it is shown that the existence of a modulus of convergence for nets index by \( \mathbb{N}^\omega \), yields \( \text{QF-AC}^0 \).

Corollary 3.18. The system \( \text{Z}^\omega \) cannot prove \( \text{BW}_0 \). The system \( \text{RCA}_0^c + \text{BW}_0 \) cannot prove X.

Proof. For the first part, by [63, §3], \( \text{Z}^\omega \) cannot prove NIN, while \( \text{cocode}_0 \rightarrow \text{NIN} \). For the remaining part, let \( X \in L_2 \) be a sentence not provable in RCA0. Then \( \text{RCA}_0^c + \text{BW}_0 \vdash X \) yields \( \text{RCA}_0 \vdash X \) under ECF as the latter makes \( \text{BW}_0 \) vacuously true, following the penultimate part of the proof of Theorem 3.5.

We could obtain similar results for the Ascoli-Arzelà theorem (see [85, III.2] for the associated RM results) for nets with countable index sets. The following corollary should be contrasted with [78, §3.2], where it is shown that the existence of a modulus of convergence for nets index by \( \mathbb{N}^\omega \), yields \( \text{QF-AC}^0 \).

Corollary 3.19. The theorem remains valid if we require a modulus of convergence in the conclusion of \( \text{MCT}_0^\text{cf} \) or a convergent sub-sequence in \( \text{BW}_0 \).

Proof. Given \( \text{cocode}_0 \), a countable index set is given by a sequence. Hence, a modulus of convergence can be defined in the usual (arithmetical) way. The same argument works for \( \text{BW}_0 \).

Obviously, we would like to obtain equivalences like in Theorem 3.16 for strongly countable sets. Now, if we try to imitate the proof of e.g. [11] for strongly countable sets, we note that \( \text{MCT}^0 \) is trivial as \( Y \) is now a bijection, and hence \( g =_1 11 \ldots \) in \( \text{MCT}^0 \). Hence, we cannot obtain \( \text{BOO}_C \) in this way, which also follows from Footnote 8 and Corollary 3.17. Nonetheless, Corollary 3.19 suggest the following interesting version of item [11]: let \( \text{MCT}_1^\text{cf} \) be \( \text{MCT}_1^\text{cf} \) restricted to strongly countable index set, but ‘upgraded’ with the existence of a modulus of convergence. Similarly, item [11] can be reformulated based on Corollary 3.19 as:

Principle 3.20 (\( \text{BW}_1^{[0,1]} \)). For a strongly countable set \( A \subset [0,1] \), there is a sequence \( (y_n)_{n \in \mathbb{N}} \) in \( A \) that converges to the supremum of \( A \).

We could omit ‘to the supremum of \( A \)’ in \( \text{BW}_1^{[0,1]} \) if we also assume that the sequence is not eventually constant. We discuss the exact properties of \( \text{BW}_1^{[0,1]} \) and \( \text{MCT}_1^\text{cf} \), their relation to hyperarithmetical analysis in particular, in Remark 3.25.

Finally, let \( \text{QF-AC}^0 \) be \( \text{QF-AC}^0 \) restricted to unique existence.

Theorem 3.21. The system \( \text{ACA}_0^\omega \) proves \( \text{MCT}_1^\text{cf} \leftrightarrow \text{cocode}_1 \leftrightarrow \text{QF-AC}^0 \) and \( \text{MCT}_1^\text{cf} \leftrightarrow [\text{MCT}_1^\text{cf} + \text{CBN}] \).

Proof. First of all, \( \text{cocode}_1 \leftrightarrow \text{QF-AC}^0 \) follows by applying the latter to the second condition, i.e. surjection on \( \mathbb{N} \), of the definition of ‘bijection’. For the other implication, given \( Y \) such that \( (\forall n \in \mathbb{N})(\exists f \in 2^n)(Y(f, n) = 0) \), define \( A := \{g \in 2^n : (\exists n \in \mathbb{N})(Y(g, n) = 0)\} \) and define \( Z : 2^n \rightarrow \mathbb{N} \) as follows: \( Z(f) \) is the least \( n \) such that \( Y(f, n) = 0 \), if such there is, and zero otherwise. By assumption, \( Z \) is bijective on \( A \) and \( \text{cocode}_1 \) provides an enumeration of \( A \). The latter sequence readily yields the choice function required by \( \text{QF-AC}^0 \).
usual coding, we can replace a quantifier ‘(∃!f ∈ N^N)’ by a quantifier ‘(∃f ∈ 2^N)’, and !QF-AC^0,1 follows without restrictions.

Secondly, the equivalence MCT°^net ⇔ [MCT°^net + CBn] follows from MCT°^net ⇔ cocode_1 combined with Theorem 3.16. For the latter equivalence, the reverse implication is straightforward as the sequence provided by cocode_1 for the strongly countable index set from MCT°^net allows us to perform the usual ‘interval-halving’ proof using ACA_0. For the forward implication, let λw.f_w be the net as in the proof of Theorem 3.16 defined for a strongly countable set A and associated Y : 2^N → N which is bijective on A. By definition, we have lim_w f_w = 11... and let g^0→1* be a modulus of convergence, i.e.

(∀k ∈ N) (∀v ≥ g(k)) (∑k=0... k). Again by definition, we have (∀k ∈ N) (∃i < |g(k)|) (Y(g(k)(i)) = k), i.e. cocode_1 follows, and we are done.

Let IND_{Σ} be the induction axiom for Σ-formulas, i.e. of the form Φ(n) ≡ (∃f ∈ N^N)(Z(f, n) = 0). Applying ECF from Remark 2.4 shows that the base theory in Corollary 3.22 is a conservative extension of RCA_0.

Corollary 3.22. The system RCA°^0 + IND_{Σ} proves BW^0[0,1] ↔ cocode_1

Proof. The reverse implication follows as in the proof of the theorem. For the forward direction, fix A ⊆ [0, 1] and let Y : [0, 1] → N be a bijection on A. Now define B ⊆ [0, 1] as follows: x ∈ B if and only if the binary representation f_x ∈ 2^N of x (readily defined using \( \exists^2 \)) satisfies:

[\[ f_x = 11...11 * (0) * g_0 + g_1 + ... + g_k ] \land (∀i ≤ k)(Y(g_i) = i \land v(g_i) ∈ A). \] (3.9)

Note that induction as in IND_{Σ} is needed to establish that ‘A is strongly countable’ implies ‘B is strongly countable’. The sequence (y_n)_{n ∈ N} provided by BW^0[0,1] must converge to 1 by (3.9). This clearly yields a sequence listing the elements of A. □

Note that the previous proof does not go through if we omit the sequence from BW^0[0,1]. This sequence can of course be obtained from the supremum using QF-AC^0,1, but the latter already implies cocode_1. Moreover, by Footnote 5, MCT°^net and BW^0[0,1] are not explosive when combined with e.g. the Suslin functional, in contrast to MCT°^net by Corollary 3.17.

The following corollary deals with fin_0, CBn’, and NCC from Section 3.2.3.

Corollary 3.23. Over RCA°^0 + NCC, we have cocode_0 ↔ CBn’ ↔ [CBn’ + fin_0].

Proof. First of all, we may assume (\( \exists^2 \)) is given, as all principles from the corollary are vacuously true given ¬(\( \exists^2 \)), as in Remark 3.11. Now, given NCC, cocode_0 ↔ CBn follows from Theorem 3.16 as NCC → \( \Delta \)-CA → cocode_1 where the latter implications are from [90] [§3.1] and [64] Cor. 3.33.

Clearly, cocode_0 → [CBn’ + fin_0] while to obtain BOOT_C from CBn’ + fin_0, fix Z such that (∀n ∈ N)(∃ at most one \( g \) ∈ 2^N)(Z(g, n) = 0) and define the set A := \{ \( g \) ∈ 2^N : (∃n ∈ N)(Z(g, n) = 0) \}. This set is countable with an obvious injection defined in terms of Z. Now consider

(∀n ∈ N)(∃k ≥ n)(∃f ∈ 2^N)(Z(f, k) = 0), \] (3.10)
which intuitively expresses that \( A \) is infinite. Assuming (3.10) holds, apply NCC and let \( g : \mathbb{N} \to \mathbb{N} \) be the resulting sequence. Use primitive recursion to define \( h(0) := 0 \) and \( h(n + 1) := g(h(n)) \). Then \( (\forall n \in \mathbb{N})(\exists f \in 2^{\mathbb{N}})(Z(f, h(n)) = 0) \) and note that this existence is unique, by assumption on \( Z \). In the first paragraph, we mentioned \( \text{NCC} \to \text{cocode}_1 \), which yields \( \text{!QF-AC}^{0,1} \) by the theorem. Applying the latter to \( (\forall n \in \mathbb{N})(\exists f \in 2^{\mathbb{N}})(Z(f, h(n)) = 0) \) yields a sequence \((f_n)_{n \in \mathbb{N}}\) of distinct elements of \( A \). Now define the injection \( \mathbb{N} \to A \) as mapping \( n \in \mathbb{N} \) to \( f_n \) and note that \( A \) is strongly countable by \( \text{CB}^n \). Apply \( \text{!QF-AC}^{0,1} \) to the second part of the definition of ‘bijection to \( \mathbb{N} \)’ to obtain a sequence listing all elements of \( A \). Then \( \text{BOOT}_C \) readily follows. If (3.10) is false, then \( A \) is finite in the sense of the antecedent of \( \text{fin}_0 \), and the latter also readily yields \( \text{BOOT}_C \) in this case. \( \square \)

It is an interesting exercise to show that in the previous results, \( \text{CB}^n \) can be replaced by \( \text{CB}^n_\infty \), i.e. \( \text{CB}^n \) with ‘\( Y : \mathbb{R} \to \mathbb{N} \)’ replaced by ‘\( Y : \mathbb{R} \to \mathbb{N}_\infty \)’ in Definition 3.4. Another possible variation is to study bijections defined as mappings \( \mathbb{N}_\infty \to \mathbb{R} \), for which a version of (3.3) can be used to obtain \( \text{BOOT}_C \). In this light, the above results exhibit some robustness.

Next, we establish Corollary 3.24, which shows that \( \text{BW}^{[0,1]}_1 \) cannot imply \( \text{ACA}_0 \), even if we use a notion of set other than Definition 3.3. Here, MUC involves the intuitionistic fan functional:

\[
(\exists \Omega^3)(\forall Y^2)(\forall f, g \in 2^{\mathbb{N}})(f \Omega(Y) = g \Omega(Y) \Rightarrow Y(f) = 0 \vee Y(g))
\]

which formalises a version of Brouwer’s continuity theorem, i.e. that all \( Y : 2^{\mathbb{N}} \to \mathbb{N} \) are (uniformly) continuous. By Remark 3.4, MUC therefore implies \( \neg(\exists^2) \). As noted in [59, §3], \( \text{RCA}_0^\omega + \text{MUC} \) is a conservative extension of \( \text{WKL}_0 \) for \( \mathbb{L}_2 \)-sentences. As for Corollary 3.6, the following is actually a meta-result.

**Corollary 3.24.** Regardless of the meaning of ‘\( x \in A \)’, the system \( \text{RCA}_0^\omega + \text{MUC} \) proves that a countable set \( A \subset [0, 1] \) has a supremum.

**Proof.** Clearly, MUC implies \( \neg(\exists^2) \) and \( \text{WKL} \) (see [59, IV.2] for the latter). Now use the proof of Corollary 3.6 to conclude that a countable set \( A \subset [0, 1] \) has at most one element. Hence, the conclusion of the corollary is vacuously true. \( \square \)

The previous corollary suggests that theorems about (strongly) countable sets like \( \text{BW}^{[0,1]}_1 \) and \( \text{MCT}^{\text{net}}_1 \) must occupy a place between the base theory and \( \text{WKL}_0 \) in terms of second-order consequences, regardless of the representation of sets. This theme is explored in (much) more detail in [65].

In conclusion, the above results lead to the following answers to (Q0)-(Q3).

(A0) Our notion of ‘countable set’ yields explosive theorems, reaching up to \( \Pi^1_2\text{-CA}_0 \) when combined with \( \Pi^1_2\text{-CA}_0^\omega \) (Cor. 3.17). Replacing ‘sequence’ by ‘countable set’ can result in loss of logical strength (Cor. 3.15).

(A1) Our notion of ‘strongly countable set’ does not yield explosive theorems by Footnote 8 and Corollary 3.21, in contrast to (A0).

(A2) Nets with countable index sets can yield ‘explosive’ convergence theorems by Corollary 3.17 just like for uncountable index sets (see [78, §3]).

(A3) The notion of (strongly) countable set yields nice, but perhaps expected, splittings in light of Theorems 3.16 and 3.21.

We finish this section with an important remark on \( \text{BW}^{[0,1]}_1 \) and \( \text{MCT}^{\text{net}}_1 \).
Remark 3.25 (Hyperarithmetical analysis). First of all, the notion of hyperarithmetical set ([55, VIII.3]) gives rise to the (second-order) definition of system statement of hyperarithmetical analysis (see e.g. [55] for the exact definition), which includes \( \Sigma^1_1 \)-CA \( \subseteq \) ACA. Montalbán claims in [55] that INDEC, a special case of [41, IV.3.3], is the first ‘mathematical’ statement of hyperarithmetical analysis. The latter theorem by Jullien can be found in [27, 6.3.4.3] and [72, Lemma 10.3]. Secondly, a classical result by Kleene (see e.g. [52, Theorem 5.4.1]) establishes that the following classes coincide: the hyperarithmetical sets, the \( \Delta^1_1 \)-sets, and the subsets of \( \mathbb{N} \) computable (S1-S9) in \( \mathbb{P}^2 \). Hence, one expects a connection between hyperarithmetical analysis and extensions of ACA\( \omega^0_0 \). By way of example, ACA\( \omega^0_0 \) + QF-AC\( ^{0,1}_0 \) is a conservative extension of ACA\( \omega^0_1 \) by [55, Cor. 2.7]. The latter is \( \Pi^2_2 \)-conservative over ACA\( \omega^0_0 \) (see [55, IX.4.4]).

Thirdly, the monographs [27, 41, 72] are all ‘rather logical’ in nature and INDEC is the restriction of a higher-order statement to countable linear orders in the sense of RM ([55 V.1.1]), i.e. such orders are represented by subsets of \( \mathbb{N} \). In our opinion, the statements MCT\( \omega^{0,1}_1 \), cocode\( 1_1 \), and BW\( \omega^{0,1}_1 \) are (much) more natural than INDEC as they are obtained from theorems of mainstream mathematics by a (similar to the case of INDEC) restriction, namely to strongly countable sets. Moreover, the system ACA\( \omega^0_0 \) + X where X is either MCT\( \omega^{0,1}_1 \), cocode\( 1_1 \), or BW\( \omega^{0,1}_1 \) + IND\( \Sigma_1 \), proves weak-\( \Sigma^1_1 \)-CA \( \subseteq \) ACA\( \omega^0_2 \), with a uniqueness condition, by Theorem 3.21. Hence, ACA\( \omega^0_2 \) + MCT\( \omega^{0,1}_1 \) is a rather natural system in the range of hyperarithmetical analysis, as it sits between RCA\( \omega^0_0 \) + weak-\( \Sigma^1_1 \)-CA \( \subseteq \) ACA\( \omega^0_2 \) + QF-AC\( ^{0,1}_0 \).

3.3.2. Nets with uncountable index sets. The results from Section 3.3.1 are interesting in their own right, but also lead to improvements to results on nets with uncountable index sets from e.g. [77, 78], as explored in this section.

With the gift of hindsight, we can now formulate the following result for BOOT and ‘obvious’ generalisations of BW\( \omega^{0,1}_1 \) and MCT\( \omega^{0,1}_1 \) from Theorem 3.10. The equivalence MCT\( \omega^{0,1}_1 \) \( \Leftrightarrow \) BOOT was first proved in [78 §3]; we refer to the latter or [77] for definitions of nets with index set \( \mathbb{N}^\mathbb{N} \) in RCA\( \omega^0_0 \).

Theorem 3.26. The system RCA\( \omega^0_0 \) proves that the following are equivalent:

(i) ACA\( \omega^0_0 \) plus: for any \( A \subset [0, 1] \) and \( F : [0, 1] \to [0, 1] \), sup\( _{x \in A} F(x) \) exists,

(ii) MCT\( \omega^{0,1}_1 \), monotone convergence theorem for nets with index set \( \mathbb{N}^\mathbb{N} \),

(iii) BOOT,

(iv) For any RM-closed set \( C \subset [0, 1] \) and \( F : [0, 1] \to [0, 1] \), sup\( _{x \in C} F(x) \) exists.

Proof. First of all, (ii) \( \Leftrightarrow \) (i) immediately follows from [78 Theorem 3.7]. Note that in case \( \neg (\exists^2 \omega) \), all functions on \( \mathbb{R} \) are continuous by Remark 3.1. Hence, BOOT reduces to ACA\( \omega^0_0 \) while the second part of item (ii) becomes provable given ACA\( \omega^0_0 \) using the usual interval-halving technique. Thus, (ii) \( \Leftrightarrow \) (ii) holds in this case.

Secondly, in case \( (\exists^2 \omega) \), fix some \( A \subset [0, 1] \) and \( F : [0, 1] \to [0, 1] \). Note that \( \exists x \in A \) is decidable in light of Definition 3.3. Now fix \( (q_n)_{n \in \mathbb{N}} \), an enumeration of all rationals in \( \mathbb{Q} \), and use BOOT to obtain a set \( X_0 \subset \mathbb{N} \) such that

\[ n \in X_0 \Leftrightarrow (\exists x \in \mathbb{R} \exists y \in A \cap F(x) \supset q_n), \]

where the underlined formula is decidable modulo \( \exists^2 \omega \). To define the supremum \( y = \sup_{x \in A} F(x) \), define \( X \subset \mathbb{Q} \) as the set of those \( q_n \) with \( n \in X_0 \). Define \([y](0)\)
as \( \frac{1}{2} \) if \( \frac{1}{2} \in X \), and 0 otherwise. For the general case, \([y](n + 1) = [y](n) + \frac{1}{n+1}\) if the latter is in \( X \), and \([y](n)\) otherwise. Thus, we have \((i) \leftrightarrow (iii)\) in general.

Thirdly, by the proof of [78, Theorem 3.7], \( \text{BOOT} \) follows from the monotone convergence theorem for nets indexed by \( 2^N \) or \([0, 1] \), assuming \( \text{ACA}_0 \). However, a net in \([0, 1] \) with index set \( D \subset [0, 1] \) has a supremum by item \( (i) \), i.e. we obtain \( (i) \rightarrow (iii) \). Similarly, item \( (iv) \) guarantees the existence of the supremum of such a net, i.e. we also have \( (iv) \rightarrow \text{BOOT} \). To obtain \( \text{ACA}_0 \) from item \( (iv) \), note that for \( F(x) := x \), we obtain the supremum sup \( C \) of an RM-closed set \( C \subset [0, 1] \). The latter property yields \( \text{ACA}_0 \) by (the proof of) [21, Theorem 3.8] or [85, IV.2.11]. The proof of \( \text{BOOT} \rightarrow (iv) \) follows in the same way as for \( \text{BOOT} \rightarrow (i) \). □

We note that item \( (iv) \), formulated with RM-codes for continuous functions \( F \), exists in the RM-literature, namely [55, IV.2.11.2]. Thus, the ‘small’ change to arbitrary third-order objects has a massive effect. Of course, applying \( \text{ECF} \) to \( (iv) \leftrightarrow \text{BOOT} \), one obtains the equivalence between \( \text{ACA}_0 \) and [55, IV.2.11.2] in accordance with the theme of [78]. Thus, the least we can say is that our definition of set as in Definition 3.15 states: \( \text{ACA}_0 \) indexed by \( \mathbb{N}^N \) has a monotone sub-net with a countable index set.

Corollary 3.27. \( \text{ACA}_0 \) proves \( \{\text{BOOT} + \text{QF-AC}^{0,1}\} \leftrightarrow \{\text{BOOT}^C + \text{SUBNET}_0\} \).

Proof. The forward direction follows from [77, Cor. 4.6]. Indeed, the (proof of) the latter yields that, assuming \( \text{QF-AC}^{0,1} \), a monotone net in \([0, 1] \) indexed by \( \mathbb{N}^N \) and converging to \( x \), has a sub-sequence that also converges to \( x \). Using \( (\exists^2) \), this sequence is readily seen to be a net with a countable index set.

For the reverse direction, to obtain item \( (i) \) from Theorem 3.26 apply \( \text{SUBNET}_0 \) and note that \( \text{BOOT}^C \) is equivalent to item \( (i) \) from Theorem 3.16. Moreover Corollary 3.15 supplies a modulus of convergence, which in turn yields \( \text{QF-AC}^{0,1} \) by [78, Cor. 3.17]. □

On a historical note, Root, a student of E.H. Moore, already studied when limits from Moore’s General Analysis [57] can be replaced by limits given by sequences [71]. Thus, the idea of replacing nets by sequences goes back more than a century, but generally requires the Axiom of Choice by [77, Cor. 4.6].

We finish this section with a remark on the formulation of \( \text{BW}_0^{0,1} \).

Remark 3.28 (On formalisation). A valid critical question is whether \( \text{BW}_0^{0,1} \) does really formalise an instance of the ‘Bolzano-Weierstrass theorem for countable sets’. As always, this kind of question is complicated and we therefore spend some time and effort on a detailed answer. Recall that \( \text{BW}_0^{0,1} \) as in Definition 3.15 states:

for countable \( A \subset [0, 1] \) and \( F : [0, 1] \rightarrow [0, 1], \) the supremum sup \( x \in A \) \( F(x) \) exists.

First of all, \( \text{BW}_0^{0,1} \) states intuitively speaking that the supremum exists for the set \( F(A) = \{y \in [0, 1] : (\exists x \in A)(F(x) = y)\} \) for countable \( A \subset [0, 1] \) and \( F : [0, 1] \rightarrow [0, 1] \). However, it is a common theme in RM that in weak systems certain sets or functions do not exist as mathematical objects, but only in a certain ‘virtual’ or ‘comparative’ sense (see e.g. [55, p. 392]) or represented by a formula (like open sets in e.g. [Q] above). Now, \( \text{BW}_0^{0,1} \) is equivalent to the following:
for countable $A \subset [0, 1]$ and $Y : [0, 1] \to \mathbb{N}$ injective on $A$, the set
\[ \{ n \in \mathbb{N} : (\exists x \in A)(Y(x) = n) \} \]
exists.

Hence, there is no hope that in general $F(A)$ as above exists as a set in $\mathbb{Z}_2^\omega$. In fact, this existence would yield $\text{BW}^{[0,1]}_0$ by the previous. For this reason, the formulation of $\text{BW}^{[0,1]}_0$ avoids mentioning the set $F(A)$. Moreover, since $F(A)$ does not exist as a set in $\text{RCA}^+_{\omega_1}$, the question whether it is countable (in the sense of Definitions 3.3 and 3.4) does not really make sense.

Secondly, the previous paragraph notwithstanding, we can give meaning to statements like ‘the set $X_\varphi = \{ x \in [0, 1] : \varphi(x) \}$ is countable’, namely as
\[ (\exists Y : [0, 1] \to \mathbb{N})(\forall x, y \in X_\varphi)(Y(x) =_0 Y(y) \to x =_R y), \tag{3.11} \]
where ‘$x \in X_\varphi$’ is just short for ‘$\varphi(x)$’. Now consider $B$ and $F(w) := f_w$ from the proof of Theorem 3.16. One readily proves that ‘the set $F(B) = \{ y \in [0, 1] : (\exists w^+)(F(w) = y) \}$ is countable’ in the sense of (3.11). Indeed, let $(\sigma_n)_{n \in \mathbb{N}}$ be an enumeration of the finite binary sequences and define $Y_0(f)$ as $n + 1$ if $n$ is the unique number such that $f =_1 \sigma_n * 00 \ldots$, and 0 otherwise. Clearly, we have
\[ (\forall f, g \in F(B))(Y_0(f) =_0 Y(g) \to f =_1 g), \tag{3.12} \]
i.e. we can restrict $\text{BW}^{[0,1]}_0$ to countable sets $A \subset [0, 1]$ and $F : [0, 1] \to [0, 1]$ where $F(A)$ is countable in the sense of (3.11). The resulting restriction is however less elegant than $\text{BW}^{[0,1]}_0$. Moreover, Theorem 3.5 explains why one wants to avoid ‘sets’ like $X_\varphi$: even in the basic case of $\mathbb{N}_\infty$, one obtains $(\exists^2)$, which implies $\text{ACA}_0$.

Thirdly, we can view $F(B)$ from the proof of Theorem 3.16 as a subset of a countable set, where ‘inclusion’ is interpreted in the ‘comparative’ second-order sense. Indeed, one readily proves ‘$F(B) \subset D$', where $D = \{ f \in 2^{\mathbb{N}} : (\exists n \in \mathbb{N})(f =_1 \sigma_n * 00 \ldots ) \}$ and where $\sigma_n$ is the $n$-th finite binary sequence. The set $D$ exists and is countable (following Definition 3.4) given $\text{ACA}^+_{\omega_1}$. To be absolutely clear, ‘$F(B) \subset D$’ means the following:
\[ (\forall f \in 2^{\mathbb{N}})([\exists w^+ \in B](F(w) =_1 f) \to f \in D). \tag{3.13} \]
In light of the above, we can restrict $\text{BW}^{[0,1]}_0$ to $A$ and $F$ such that $F(A) \subset D$ for some countable $D \subset [0, 1]$. This formulation is however again far less elegant.

In conclusion, while $F(B)$ may not exist in weak systems, we can express the countability of $F(B)$ via (3.12) or (3.13). Restricting to such sets does not change the equivalences in Theorem 3.16. Thus, $\text{BW}^{[0,1]}_0$ does deal with the supremum of countable sets, but the technical details can be treacherous.

3.4. König’s Lemma versus Lemmas by König. We study well-known eponymous lemmas by König from [43, 44] for (strongly) countable sets (of reals) following Definition 3.4. We show that these lemmas imply NBI or $\text{cocode}_0$ and are hence unprovable in $\mathbb{Z}_2^\omega$, but provable in $\mathbb{Z}_2^\mathbb{N}$, i.e. without additional choice axioms. In contrast to the (second-order) version involving trees, our lemmas based on Definition 3.4 do not imply $\text{ACA}_0$. Nonetheless, we obtain some interesting reversals, including $\text{splittings} \; \text{cocode}_0 \leftrightarrow [\text{Körk}_0 + \text{CBN}]$ over $\text{RCA}^+_{\omega_1}$, where $\text{Körk}_0$ is a lemma due to König formulated using Definition 3.4. We also discuss the history of these lemmas, as this would seem illuminating in light of [85, p. 125, Notes for §III.3].

First of all, let König’s tree lemma be the statement ‘every infinite finitely branching tree has a path’. When formulated in $L_2$, the latter is equivalent to $\text{ACA}_0$ by
In [S5, III.7.2], Simpson refers to [44] as the original source for König’s tree lemma in [S5, p. 125], but [44] does not mention the word ‘tree’ (i.e. the word ‘Baum’ in German). In fact, König’s original (graph theoretic) lemma from [44] is as follows.

**Principle 3.29** (König’s infinity lemma for graphs). If a countably infinite graph \( G \) can be written as countably many non-empty finite sets \( E_1, E_2, \ldots \) such that each point in \( E_{n+1} \) is connected to a point in \( E_n \) via an edge, then \( G \) has an infinite path \( a_1a_2\ldots \) such that \( a_n \in E_n \) for all \( n \in \mathbb{N} \).

The original version, introduced a year earlier in [43], is formulated in the language of set theory as follows, in both [43, 44].

**Principle 3.30** (König’s infinity lemma for sets). Given a sequence \( E_0, E_1, \ldots \) of finite non-empty sets and a binary relation \( R \) such that for any \( x \in E_{n+1} \), there is at least one \( y \in E_n \) such that \( yRx \). Then there is an infinite sequence \((x_n)_{n \in \mathbb{N}}\) such that for all \( n \in \mathbb{N} \), \( x_n \in E_n \) and \( x_nRx_{n+1} \).

The names König’s infinity lemma and König’s tree lemma are used in [90] which contains a historical account of these lemmas, as well as the observation that they are equivalent; the formulation involving trees apparently goes back to Beth around 1955 in [5], as also discussed in detail in [90]. When formulated in set theory, König’s infinity lemma is equivalent to a fragment of the Axiom of Choice ([51, p. 298]) over \( ZF \), and is (strictly) implied by Ramsey’s theorem ([26]).

Now, since ‘countable subset of \( \mathbb{R} \)’ is a third-order notion, it makes sense for Simpson to study the second-order König’s tree lemma, although this begs the question what the logical strength of König’s infinity lemma(s) is. By the below, König’s infinity lemmas formulated for countable subsets of \( \mathbb{R} \) as in Definition 3.4 are not provable in \( Z_{\omega}^2 \), in contrast to the \( L_2 \)-version of König’s tree lemma of course is (provable in \( ACA_0 \)). Nonetheless, we can obtain nice equivalences for certain versions of these lemmas, showing in particular that the Axiom of Choice is not needed (as everything is provable in \( Z_{\omega}^{11} \)).

Next, we discuss some conceptual motivations for our study of the infinity lemmas. The following quote by König constitutes motivation and evidence that graph theory was intended to be infinitary.

Diese Bemerkung ist wichtig, da, wenn man sie einmal angenommen hat, nichts im Wege steht die “Sprache der Graphen” auch dann zu nutzen, wenn die Mengen […] nicht endlich, ja sogar von beliebig grosser Machtigkeit sind. ([12, p. 460])

The final sentence states that graphs of any cardinality can be studied in graph theory. Moreover, we note that König’s infinity lemma is introduced in [44] as a graph-theoretic formulation of another theorem from [43]. In both the French ([43, §3]) and German formulation ([44 §1]), the word ‘sequence’ is used in the conclusion, i.e. an infinite path is a sequence of elements. By contrast, the antecedent is always formulated using countable sets. Thus, the above König’s infinity lemmas are close to the original theorems ‘as they stand’, i.e. without enrichment or modification.

In light of these observations, we define \( K_{\omega_0} \) as follows where we interpret ‘finite set’ as a set \( A \subset \mathbb{R} \) for which there exists \( Y : \mathbb{R} \to \mathbb{N} \) which is injective and bounded on \( A \), i.e. just like for \( \text{fin}_0 \) introduced in Section 3.2.3. As is clear from its proof, Theorem 3.32 (and Corollary 3.34) still goes through if we replace item (b) in Principle 3.31 by ‘for all \( n \in \mathbb{N} \) there exists a non-empty finite sequence listing
the elements of $V_n$. This endows the below results with a certain robustness, an important concept in RM according to Montalbán ([56]).

**Principle 3.31 (Korg$_0$).** Let $G = (V, E)$ be a graph where $V = \bigcup_{n \in \mathbb{N}} V_n \subset \mathbb{R}$ and

(a) the vertex set $V$ is countable,
(b) each $V_n$ is non-empty and finite,
(c) each vertex in $V_{n+1}$ is connected to a vertex in $V_n$ via an edge in $E$.

Then there is a sequence $(v_n)_{n \in \mathbb{N}}$ such that $v_n \in V_n$ and $(v_n, v_{n+1}) \in E$ for $n \in \mathbb{N}$.

We let Korg$_1$ be Korg$_0$ restricted to strongly countable sets $V$. In our opinion, the principles Korg$_i$ for $i = 0, 1$ are therefore close to König’s original interpretation/meaning. We now have the following theorem.

**Theorem 3.32.** The system RCA$^\omega_0$ proves Korg$_1 \rightarrow$ NBI.

*Proof.* Suppose $\neg$NBI and note that $[0, 1]$ is strongly countable. The graph $G = (V, E)$ is defined as $V = [0, 1]$ with incidence relation $xEy$ defined as $Y(y) = Y(x) + 1 \lor Y(x) = Y(y) + 1$, where $Y : [0, 1] \rightarrow \mathbb{N}$ is the bijection provided by $\neg$NBI. Define $V_n := \{x \in [0, 1] : Y(x) = n\}$ and note that this collection satisfies all the requirements from Korg$_1$, in particular for $y \in V_{n+1}$, we have $xEy$ for $x \in V_n$. Hence, there is a path $(x_n)_{n \in \mathbb{N}}$ through $G$. Note that this sequence enumerates $[0, 1]$ and use [33, II.4.9] to obtain a contradiction. $\square$

The following corollary provides a nice answer to (Q0). We believe the use of CBN to be essential in the second part.

**Corollary 3.33.** The system RCA$^\omega_0$ proves Korg$_1 \leftrightarrow$ cocode$_1$ and cocode$_0 \leftrightarrow$ [Korg$_0$ + CBN]. The system RCA$^\omega_0$ + NCC proves cocode$_0 \leftrightarrow$ [Korg$_0$ + CBN + fin].

*Proof.* For the first part, similar to the proof of Theorems 3.3 and 3.10 we may assume $(\exists^2)$, as the equivalence trivially holds in case of $(\exists^2)$, following Remark 3.14. The implication Korg$_1 \rightarrow$ cocode$_1$ follows by repeating the proof of the theorem for $[0, 1]$ replaced by a strongly countable set $A \subseteq \mathbb{R}$. For the reverse implication, let $G$ be as in Korg$_1$ and let $(y_n)_{n \in \mathbb{N}}$ be a sequence such that $x \in G \leftrightarrow (\exists n \in \mathbb{N}) (x =_R y_n)$ for any $x \in \mathbb{R}$, as provided by cocode$_1$. We can now carry out the ‘usual’ proof of König’s tree lemma as $(\exists x \in G) A(x)$ is equivalent to $(\exists n \in \mathbb{N}) A(y_n)$. Indeed, expressions like ‘an element of $G$ has infinitely many successors in $G$’ are now arithmetical, which ACA$_0$ can handle as in the usual proof of König’s tree lemma (from $L_2$).

For the second part, we may assume $(\exists^2)$ as in the previous paragraph. Since Korg$_0 \rightarrow$ Korg$_1 \rightarrow$ cocode$_1$, the reverse implication is immediate. For the forward implication, CBN readily follows by using $\exists^2$ to ‘trim’ duplicate reals from the sequence provided by cocode$_0$. The usual proof of König’s lemma (in ACA$_0$) yields Korg$_0$ after converting the countable set into a sequence using cocode$_0$. The third part is now immediate by Corollary 3.24. $\square$

By the following, Korg$_1$ is provable without using countable choice. Note that (the proof of) Corollary 3.24 implies that the exact definition of set also does not matter for the second part.

**Corollary 3.34.** The system RCA$^\omega_0 + \Delta$-CA proves Korg$_1$, while RCA$^\omega_0 +$ Korg$_1$ cannot prove ACA$_0$. 


Proof: It is shown in [64] §3 that QF-AC\(^0,1\) → Δ-CA → cocode\(_1\) over RCA\(^ω_0\). Since ECF converts QF-AC\(^0,1\) to QF-AC\(^0,0\), the former does not imply ACA\(_0\). □

We now turn to Principle 3.30 formalised as follows. A binary relation \( R \) on reals is given by a characteristic function \( F_R : \mathbb{R}^2 \to \{0, 1\} \), i.e. \( xRy \equiv F_R(x, y) = 1 \).

**Principle 3.35 (Korg).** Let \((E_n)_{n \in \mathbb{N}}\) be a sequence of sets in \( \mathbb{R} \) and let \( R \) a binary relation on reals such that for all \( n \in \mathbb{N} \) we have:

- the set \( E_n \) is finite and non-empty,
- for any \( x \in E_{n+1} \), there is at least one \( y \in E_n \) such that \( yRx \).

Then there is a sequence \((x_n)_{n \in \mathbb{N}}\) such that for all \( n \in \mathbb{N} \), \( x_n \in E_n \) and \( x_nR_{n+1} \).

As expected, we have Korg\(_2 \) → cocode\(_1\) but no reversal is known. Moreover, the use of extra induction in the proof of Korg\(_2 \) seems necessary. Let \( \Pi^1_2\)-IND\(^ω\) be the induction axiom for \( \Pi^1_2 \)-formulas with arbitrary parameters.

**Theorem 3.36.** RCA\(^ω_0\) proves Korg\(_2 \) → cocode\(_1\) and RCA\(^ω_0\) + ACA\(_0\) + \( \Pi^1_2\)-IND\(^ω\) + QF-AC\(^0,1\) proves Korg\(_2 \).

Proof. To obtain Korg\(_2 \) → cocode\(_1\), let \( A \subset \mathbb{R} \) be strongly countable as witnessed by the bijection \( Y : A \to \mathbb{N} \). Define \( E_n := \{ x \in A : Y(x) = n \} \) and note that this set is non-empty and finite for any \( n \in \mathbb{N} \). In fact, \( E_n \) has exactly one element by definition. Define the binary relation \( yRx \) as \( Y(x) + 1 =_0 Y(y) \) and note that for \( y \in E_{n+1} \), there is (exactly one) \( x \in E_n \) such that \( yRx \). Hence, Korg\(_2 \) provides \((x_n)_{n \in \mathbb{N}}\) such that \( x_n \in E_n \) and \( x_{n+1}R_{n+1} \) for any \( n \in \mathbb{N} \). By definition, \( Y(x_n) = n \) for any \( n \in \mathbb{N} \) and cocode\(_1\) follows.

For the second part, we shall prove the following for all \( n \in \mathbb{N} \):

\[
(\exists w^{1^+})(|w| = n \land (\forall i < |w|)(w(i) \in E_i) \land (\forall j < |w| - 1)(w(j)Rw(j + 1))) \tag{3.14}
\]

Now apply QF-AC\(^0,1\) to obtain \((w_n)_{n \in \mathbb{N}}\). With the latter sequence, one readily builds a finitely branching tree (in the sense of second-order RM). Now use König’s tree lemma (provided by ACA\(_0\); see [225] III.7.2) to obtain the path required by Korg\(_2 \). To prove (3.14) for all \( n \in \mathbb{N} \), consider the formula \( \varphi(n) \) defined as

\[
(\forall x \in E_n)(\exists w^{1^+})(|w| = n + 1 \land (\forall i < |w|)(w(i) \in E_i) \land x = w(n) \\
\land (\forall j < |w| - 1)(w(j)Rw(j + 1)))
\]

Now use \( \Pi^1_2\)-IND\(^ω\) to prove \( (\forall n \in \mathbb{N})\varphi(n) \), which implies (3.14) for all \( n \in \mathbb{N} \). □

By [64] Theorem 3.1, Z\(^ω_0\) + QF-AC\(^0,1\) cannot prove \( \text{NIN} \), which is established via a model \( P \) of the former in which the latter is false. It can be shown that this model (or a variation thereof) satisfies the induction axiom as in the previous theorem (and more). As a result, Korg\(_2 \) cannot imply \( \text{NIN} \) (or cocode\(_0\) for that matter).

As a preliminary conclusion, our results concerning König’s various lemmas are somewhat more complicated than those in Section 3.33. On one hand, Corollary 3.33 provides the nice equivalence Korg\(_1 \) ↔ cocode\(_1\), but we do not know how to obtain an equivalence between Korg\(_0 \) and cocode\(_0\). On the other hand, the splitting in Corollary 3.33 and the results in Theorem 3.36 are of course quite nice.

Finally, the restriction of König’s tree lemma to binary trees yields the second Big Five system, called WK\(_L_0\). We briefly discuss the latter and associated equivalences.

---

\(^9\)Let \( \sigma \in T \) hold in case \( \sigma \) is an initial segment of \( w_n \) for some \( n \in \mathbb{N} \).
Now, \( \text{WKL}_0 \) is equivalent to the Heine-Borel theorem for \emph{sequences} of intervals covering \([0, 1]\) \cite[IV.1]{55}. The following version is closer to Borel’s original version from \cite{10}, as discussed in Remark 1.1.

**Principle 3.37 (HBC\(_0\)).** For countable \( A \subset \mathbb{R}^2 \) with \((\forall x \in [0, 1])(\exists (a, b) \in A) (x \in (a, b))\), there are \((a_0, b_0), \ldots (a_k, b_k) \in A\) with \((\forall x \in [0, 1])(\exists i \leq k) (x \in (a_i, b_i))\).

We now have the following theorem. Similar results hold for e.g. Vitali’s covering theorem (see e.g. \cite[X.1]{85}).

**Theorem 3.38.** The system \( \text{Z}_2^\omega + \text{QF-AC}^{0,1} \) cannot prove \( \text{HBC}_0 \), the latter does not imply \( \text{WKL}_0 \) over \( \text{RCA}_0^\omega \), and the same for any sentence not provable in \( \text{RCA}_0^\omega \).

**Proof.** For the first part, it is shown in \cite{64} that \( \text{HBC}_0 \rightarrow \text{NIN} \) and \( \text{Z}_2^\omega + \text{QF-AC}^{0,1} \not\vdash \text{NIN} \). For the second part, note that \( \text{RCA}_0^\omega + \text{HBC}_0 \vdash \text{WKL}_0 \) is converted to \( \text{RCA}_0 \vdash \text{WKL}_0 \) as \( \text{HBC}_0 \) is vacuously true under \( \text{ECF} \) following the proof of Corollary 3.6. Indeed, \( \text{ECF} \) replaces all third-order objects by (continuous by definition) \( \text{RM} \)-codes, meaning that countable sets are interpreted as finite sets.

We finish this section with a conceptual remark.

**Remark 3.39 (Similar results).** Since \( \text{RCA}_0^\omega + \text{MUC} \) is a conservative extension of \( \text{WKL}_0 \), there is not much sense in proving a result like Corollary 3.24 here. However, note that \( \text{ACA}_0 \) follows from the statement that \emph{any separably closed set in \([0, 1]\) has the Heine-Borel property} (see \cite{36} for details). Similar to the proof of Corollary 3.24 we have that regardless of the meaning of ‘\( x \in A \)’, the system \( \text{RCA}_0^\omega + \text{MUC} \) proves that any separably closed set in \([0, 1]\) has the Heine-Borel property \emph{formulated with countable collections of intervals} \( A \subset \mathbb{R}^2 \). Hence, \( \text{HBC}_0 \) generalised to separably closed sets does not imply \( \text{ACA}_0 \).

3.5. On theorems from the RM zoo. We study theorems from the RM zoo formulated using (strongly) countable set (of reals) as in Definition 3.4. We provide detailed results for \( \text{ADS} \) and sketch the results for \( \text{CAC} \) and \( \text{RT}_2^2 \). We assume basic familiarity with the RM zoo and the aforementioned principles, although \( \text{ADS} \) is introduced below in Principle 3.40.

In particular, we show that \( \text{Z}_2^\omega \) cannot prove the higher-order versions of \( \text{ADS} \), \( \text{CAC} \), and \( \text{RT}_2^2 \) formulated using Definition 3.4, while \( \text{Z}_2^\omega \) of course can prove these higher-order versions, i.e. the Axiom of Choice is not needed. Similar to the previous, we have the splitting \( \text{cocode}_0 \leftrightarrow [\text{ADS}_0 + \text{CBN}] \) over \( \text{RCA}_0^\omega \), where \( \text{ADS}_0 \) is \( \text{ADS} \) formulated using Definition 3.4 as in Principle 3.41. The same holds for the higher-order versions of \( \text{CAC} \) and \( \text{RT}_2^2 \) based on Definition 3.4, i.e. the RM zoo is a lot more ‘tame’ formulated in third-order arithmetic.

First of all, as to motivation, the word ‘countable’ and variations appears about one hundred times in \cite{35}, Hirschfeldt’s monograph that provides a partial overview of the RM zoo. Countable infinity does indeed take centre stage, as is clear from Hirschfeldt’s quote in Section 1.2. As it happens, this quote is preceded in \cite{35} by:

The work of Gödel and others has shown that mathematics, like everything else, is built on sand. As Borges reminds us, this fact should not keep us from building, and building boldly. However, it also behooves us to understand the nature of our sand.
While we do not agree with Hirschfeldt’s foundational claims regarding Gödel, we share his sentiment regarding the necessary nature of the study of the foundations of mathematics. Thus, it behooves us to study the logical strength of theorems from the RM zoo formulated using the ‘real’ definition of countable set.

The previous paragraph constitutes general motivation, but particular theorems come with ‘extra’ motivation. We single out fragments of Ramsey’s theorem as Ramsey himself in [67], the original source of ‘Ramsey’s theorem’, formulates the infinite version of Ramsey’s theorem using ‘infinite sets’ and not using sequences. Moreover, versions of the Rival-Sands theorem from [70] are apparently equivalent to ADS and RT$^2_2$ (see [24][25]). The following quote by Rival-Sands strongly suggests their work is also formulated using ‘infinite (countable) sets’ and not sequences.

Recently, M. Gavalec and P. Vojtas have pointed out to us that the natural analogue of our Theorem 1 holds for graphs of regular cardinality $\kappa$. ([70, p. 396])

We could obtain a version of Theorem 3.42 and corollaries for the various second-order versions of the Rival-Sands theorem. On a related note, the topic of [2] is the (Weihrauch degree) study of ADS and variations involving (second-order) sets rather than sequences in the consequent. Thus, our idea of studying ADS based on Definition 3.4 is definitely in the same spirit.

Secondly, we now formulate the ascending-descending sequence principle from [35, Def. 9.1], which is the following $L_2$-sentence.

**Principle 3.40 (ADS).** Every infinite linear order has an infinite ascending or descending sequence.

Countable linear orders are represented by subsets of $\mathbb{N}$ (see e.g. [85, V.1.1]) in RM, and we now study what happens if we adopt the definition of (strongly) ‘countable set’ as in Definition 3.4.

Of course, we use the usual definition of ‘linear order’ $(X, \preceq_X)$; we shall assume that $X \subseteq \mathbb{R}$ or $X \subseteq \mathbb{N}^\omega$, as this already guarantees that the associated third-order version of ADS is not provable in $Z^2_2$. If the set $X$ is (strongly) countable, then we say that $(X, \preceq_X)$ is (strongly) countable. An infinite ascending sequence $(x_n)_{n \in \mathbb{N}}$ in $X$ satisfies $x_n \prec x_{n+1}$ for any $n \in \mathbb{N}$, and similar for descending sequences. We say that $(X, \prec_X)$ is infinite if for any $n \in \mathbb{N}$ there are pairwise distinct $x_0, \ldots, x_n \in X$.

With the above in mind, Definition 3.4 in particular, we make the following definition in $\text{RCA}_0^\omega$. We note that ADS$_0$ is provable in $Z^2_2$ by [64, Figure 1], where the latter system does not involve the Axiom of Choice.

**Principle 3.41 (ADS$_0$).** Every countable and infinite linear order has an infinite ascending or descending sequence.

Let ADS$_1$ be ADS restricted to strongly countable sets. Recall that IND$_\Sigma$ is the induction axiom for $\Sigma$-formulas, i.e. of the form $\varphi(n) \equiv (\exists f \in \mathbb{N}^\omega)(Z(f, n) = 0)$.

Thirdly, we obtain some results about ADS$_0$, beginning with the following.

**Theorem 3.42.** The system $\text{RCA}_0^\omega + \text{IND}_\Sigma$ proves ADS$_0 \rightarrow \text{NBI}$.

**Proof.** Let $Y : [0, 1] \rightarrow \mathbb{N}$ be a bijection, i.e. we have $\sim \text{NBI}$. Note that we have access to $(\exists^2)$, which allows us to convert reals into binary representation. Define the set $X = \{ w^1^* : (\forall i < |w|)(Y(w(i)) = i \land w(i) \in [0, 1]) \}$, which is readily
seen to be countable. Define the binary relation $\leq_X$ on $X$ as follows: $w \leq_X v$ if $|w| \leq |v|$. Trivially, this relation is transitive and connex. Since $Y$ is an injection, this relation is also antisymmetric, and hence $(X, \leq_X)$ is a linear order. Since $Y$ is also a bijection, we have $(\forall n \in \mathbb{N})(\exists w^I)(|w| = n + 1 \wedge w \in X)$, which has an obvious proof using $\text{IND}_\Sigma$. Hence, $(X, \leq_X)$ is an infinite linear order. Applying $\text{ADS}_0$, there is an infinite ascending or descending sequence. Now, this sequence cannot be descending and let $(w_n)_{n \in \mathbb{N}}$ be such that $w_n \prec w_{n+1}$ for all $n \in \mathbb{N}$. By definition, we have $Y(w_n(n)) = n$, i.e. the sequence $(w_n(n))_{n \in \mathbb{N}}$ lists all reals in $[0, 1]$. By [85] II.4.9, there is $y \in [0, 1]$ not in this sequence. But then for $n_0 := Y(y)$, we have $Y(w_{n_0}(n_0)) = n_0 = Y(y)$, a contradiction, and $\text{NBI}$ follows. □

**Corollary 3.43.** The system $\text{RCA}_0^\omega + \text{IND}_\Sigma$ proves $\text{ADS}_0 \rightarrow \text{cocode}_1$, while $\text{ADS}_0$ does not imply $\text{ADS}$ over $\text{RCA}_0^\omega$.

**Proof.** For the first part, repeat the proof of the theorem with $[0, 1]$ replaced by a strongly countable set $A \subset \mathbb{R}$. For the second part, $\text{ADS}_0$ is vacuously true under $\text{ECF}$, i.e. applying the latter to $\text{RCA}_0^\omega + \text{ADS}_0 \vdash \text{ADS}$ leads to a contradiction. □

We do not know how to prove $\text{ADS}_1 \rightarrow \text{cocode}_1$ or $\text{ADS}_0 \rightarrow \text{cocode}_0$. We do have the following corollary.

**Corollary 3.44.** The system $\text{RCA}_0^\omega + \text{IND}_\Sigma$ proves $\text{cocode}_0 \leftrightarrow [\text{ADS}_0 + \text{CBN}]$. The system $\text{RCA}_0^\omega + \text{IND}_\Sigma + \text{NCC}$ proves $\text{cocode}_0 \leftrightarrow [\text{ADS}_0 + \text{CBN} + \text{fin}_0]$.

**Proof.** The first part follows from Corollary 3.43 and (the proof of) Corollary 3.33. Note in particular that $(\exists^2) \rightarrow \text{ACA}_0 \rightarrow \text{ADS}$. The second part follows from Corollary 3.26. □

For the following corollary, a linear order $(X, \leq_X)$ is stable if every element either has only finitely many $\prec_X$-predecessors or has only finitely many $\prec_X$-successors. The linear order from Theorem 3.42 is clearly stable.

**Corollary 3.45.** Theorem 3.42 and corollaries also hold for $\text{SADS}_0$, i.e. $\text{ADS}_0$ restricted to stable linear orders.

The following corollary should be contrasted to the fact that $\text{WKL}_0 \not\vdash \text{ADS}$ ([35]). Recall that $\text{RCA}_0^\omega + \text{MUC}$ is a conservative extension of $\text{WKL}_0$ (see [46] §3).

**Corollary 3.46.** Regardless of the meaning of ‘$x \in A$’, $\text{RCA}_0^\omega + \text{MUC}$ proves that a countable infinite linear order in $[0, 1]$ has an infinite ascending or descending sequence.

**Proof.** As in the proof of Corollary 3.24. □

Fourth, we discuss similar results for related theorems from the RM-zoo. Such results hold for e.g. the chain-antichain principle $\text{CAC}$ which expresses that every infinite partial order has an infinite chain or antichain in $L_2$ ([35] Def. 9.2]). Indeed, $(X, \leq_X)$ from the proof of Theorem 3.42 is a countable (in the sense of Definition 3.4) and infinite partial order without infinite antichains. Moreover, $\text{RT}_2^\omega$ implies $\text{CAC}$ via an elementary argument ([35] p. 144]) that carries over to the third-order versions. We repeat that Ramsey formulates the infinite version of Ramsey’s theorem in [67] p. 264] using ‘infinite sets’ (Ramsey uses ‘classes’ rather than ‘sets’). Hence, at the very least, we should study Ramsey’s theorem formulated using (strongly) countable sets rather than sequences.
In conclusion, RT\textsuperscript{2} and CAC formulated with the definition of ‘countable set’ as in Definition 3.4 is not provable in Z\textsuperscript{2}. Nonetheless, since they are both provable in ACA\textsubscript{0}, CAC and RT\textsuperscript{2} yield splittings as in Corollary 3.4. i.e. the RM zoo is easily tamed by introducing Definition 3.4.

3.6. Countable sets in mathematics and logic. As noted in Section 1.2, we do not claim that the definition of countable sets via injections/bijections to \mathbb{N} constitutes the ‘standard’ or ‘mainstream’ one. We have studied these notions in higher-order RM since they yield interesting results. In this section, we discuss some related results in the grand(er) scheme of things, as well as an argument for the study of Kunen’s definition of countable set based on injections to \mathbb{N}. We believe the results in this section to provide some context for the results in this paper.

First of all, we list textbooks in which ‘countable sets’ are defined via sequences.

- The textbook *Introductory Real Analysis* by Kolmogorov and Fomin ([47]).
- The textbook *Calculus* by Spivak ([86]).
- Bishop’s textbook on *Constructive Analysis* ([6]).

In particular, the definition of countable set based via sequences appears to be the usual definition in the setting of elementary calculus and real analysis where the general notion of ‘cardinality’ is not needed or developed.

Secondly, it is well-known that ‘disasters’ can happen in the absence of the Axiom of Choice (AC for short), by which it is meant that many beautiful theorems are no longer provable in the absence of AC, as discussed in [33, Preface]. We point out that such disasters already happen for rather ‘mundane’ topics like finite or countable sets, like e.g. the fact that \( \mathbb{R} \) is not the countable union of countable sets, or basic cardinal arithmetic (see [33, §4]). Nonetheless, the principles studied in this paper, especially cocode, for \( i = 0, 1 \), are all provable in Z\textsuperscript{2} and weaker systems, i.e. without AC. As it happens, the author and Dag Normann study the relationship between the *countable union theorem* and cocode, for \( i = 0, 1 \) in [65, §3.2].

Thirdly, constructive mathematics is usually qualified as mathematics based on intuitionistic logic with some appropriate extra ‘semi-constructive’ axioms (see e.g. [8]). For instance, Bishop’s aforementioned constructive analysis additionally assumes the axiom of countable choice (and other axioms). The field constructive RM seeks to develop RM over a constructive base theory (see e.g. [19, 39]). A result relevant to this paper may be found in [4], which essentially shows that NIN can be false in certain approaches to constructive mathematics. Another related result is in [66], showing that the Cantor-Bernstein theorem implies the law of excluded middle. A well-known aspect of constructive mathematics (which also shows up in classical RM) is that classically equivalent notions are no longer equivalent in a constructive setting. A relevant example pertaining to countability is subcountability, but we will content ourselves with pointing the interested reader to e.g. [54, 69, 83] and the references therein.

Fourth, we provide an argument for the study of Kunen’s definition of countable set based on injections to \( \mathbb{N} \). In [64], Dag Normann and the author study the RM of the principles NIN and NBI, also introduced in Section 1.1. We identify a number of third-order principles that do not mention the notion ‘countable set’ (based on bijections, injections, or enumerations) explicitly, yet all imply NIN. In fact, it is quite hard to find a natural principle that does not mention ‘countable set’ explicitly, implies NBI, and does not imply NIN; a somewhat natural example can
We list two examples of natural third-order theorems of ordinary mathematics that do not mention ‘countable set’ in any way, but do imply NIN.

- Arzelà’s convergence theorem for the Riemann integral (1885; [11]).
- Jordan’s decomposition theorem (1881; [40]).

The proof that the first item implies NIN may be found in [64, §3.1.2]. That the second item implies NIN has not been published yet.

Acknowledgement 3.47. I thank Anil Nerode and Dag Normann for their helpful suggestions. My research was supported by the John Templeton Foundation via the grant a new dawn of intuitionism with ID 60842 and by the Deutsche Forschungsgemeinschaft via the DFG grant SA3418/1-1. Opinions expressed in this paper do not necessarily reflect those of the John Templeton Foundation. The results in Section 3.5 go back to the stimulating BIRS workshop (19w5111) on Reverse Mathematics at CMO, Oaxaca, Mexico in Sept. 2019. I thank the anonymous referees for their many suggestions that have greatly improved this paper, esp. Section 3.6.

References
[1] Cesaro Arzelà, Sulla integrazione per serie, Atti Acc. Lincei Rend., Rome 1 (1885), 532–537.
[2] Eric P. Astor, Damir D. Dzhafarov, Reed Solomon, and Jacob Suggs, The uniform content of partial and linear orders, Ann. Pure Appl. Logic 168 (2017), no. 6, 1153–1171.
[3] Jeremy Avigad and Solomon Feferman, Gödel’s functional (“Dialectica”) interpretation, Handbook of proof theory, Stud. Logic Found. Math., vol. 137, 1998, pp. 337–405.
[4] Andrej Bauer, An injection from the Baire space to natural numbers, Math. Structures Comput. Sci. 25 (2015), no. 7, 1484–1489.
[5] E. W. Beth, Semantic entailment and formal derivability, Mededelingen der koninklijke Nederlandse Akademie van Wetenschappen, afd. Letterkunde. Nieuwe Reeks, Deel 18, No. 13, N. V. Noord-Hollandsche Uitgevers Maatschappij, Amsterdam, 1955.
[6] Errett Bishop, Foundations of constructive analysis, McGraw-Hill, 1967.
[7] Errett Bishop and Douglas S. Bridges, Constructive analysis, Grundlehren der Mathematischen Wissenschaften, vol. 279, Springer-Verlag, Berlin, 1985.
[8] Douglas Bridges and Fred Richman, Varieties of constructive mathematics, London Mathematical Society Lecture Note Series, vol. 97, Cambridge University Press, Cambridge, 1987.
[9] Emil Borel, Sur quelques points de la théorie des fonctions, Ann. Sci. École Norm. Sup. (3) 12 (1895), 9–55.
[10] , Leçons sur la théorie des fonctions, Gauthier-Villars, Paris, 1898.
[11] Douglas K. Brown, Functional analysis in weak subsystems of second-order arithmetic, PhD Thesis, The Pennsylvania State University, ProQuest LLC, 1987.
[12] , Notions of closed subsets of a complete separable metric space in weak subsystems of second-order arithmetic, Logic and computation (Pittsburgh, PA, 1987), Contemp. Math., vol. 106, Amer. Math. Soc., Providence, RI, 1990, pp. 39–50.
[13] Wilfried Buchholz, Solomon Feferman, Wolfram Pohlers, and Wilfried Sieg, Iterated inductive definitions and subsystems of analysis, LNM 897, Springer, 1981.
[14] Georg Cantor, Ueber eine Eigenschaft des Inbegriffs aller reellen algebraischen Zahlen, J. Reine Angew. Math. 77 (1874), 258–262.
[15] , Ueber unendliche, lineare Punktmannichfaltigkeiten, Mathematische Annalen 17-23. Published in parts: 1879-1884.
[16] , Mitteilungen zur Lehre von den Transfinitten, Zeitschrift für Philosophie und philosophische Kritik 91 (1887), 81–125.
[17] Pierre Cousin, Sur les fonction de n variables complexes, Acta Math. 19 (1895), 1–61.
[18] Hannes Diener, Constructive Reverse Mathematics, Habilitationsschrift, University of Siegen, Germany, https://nbn-resolving.org/urn:nbn:de:hbz:467-13084 (2018).
[19] Jean Dieudonné, Sur un théorème de Jessen, Fund. Math. 37 (1950), 242–248 (French).
[20] Damir D. Dzhafarov, Reverse Mathematics Zoo. http://rmzoo.uconn.edu/
COUNTABLE SETS VERSUS SETS THAT ARE COUNTABLE

[22] Martín H. Escardó, Infinite sets that satisfy the principle of omniscience in any variety of constructive mathematics, J. Symbolic Logic 78 (2013), no. 3, 764–784.

[23] Solomon Feferman, How a Little Bit goes a Long Way: Predicative Foundations of Analysis, 2013. Unpublished notes from 1977-1981 with an updated introduction, https://math.stanford.edu/~feferman/papers/pfa.pdf.

[24] Marta Fiori-Carones, Paul Shafer, and Giovanni Solla, An inside/outside Ramsey theorem and recursion theory, Preprint arxiv https://arxiv.org/abs/2006.16969 (2020), pp. 34.

[25] Marta Fiori-Carones, (Extra)ordinary equivalences with ADS, Booklet of Abstracts, 2020 North American Annual Meeting Of The Association For Symbolic Logic (2020), p. 28.

[26] Thomas E. Forster and John K. Truss, Ramsey’s theorem and König’s lemma, Arch. Math. Logic 46 (2007), no. 1, 37–42.

[27] Roland Fraïssé, Theory of relations, Studies in Logic and the Foundations of Mathematics, vol. 145, North-Holland, 2000. With an appendix by Norbert Sauer.

[28] Harvey Friedman, Systems of second order arithmetic and their use, Proceedings of the International Congress of Mathematicians (Vancouver, B. C., 1974), Vol. 1, 1975, pp. 235–242.

[29] Denis R. Hirschfeldt, Slicing the truth, Lecture Notes Series, Institute for Mathematical Sciences, National University of Singapore, vol. 28, Word Scientific Publishing, 2015.

[30] Karel Hrbacek and Thomas Jech, Introduction to set theory, 3rd ed., Monographs and Textbooks in Pure and Applied Mathematics, vol. 220, Marcel Dekker, Inc., New York, 1999.

[31] Hajime Ishihara, Reverse mathematics in Bishop’s constructive mathematics, Philosophia Scientiae (Cahier Spécial) 6 (2006), 43-59.

[32] Camillie Jordan, Sur la série de Fourier, Comptes rendus de l’Académie des Sciences, Paris, Gauthier-Villars 92 (1881), 228–230.

[33] Pierre Julieen, Contribution à l’étude des types d’ordres dispersés, PhD thesis, University of Mariles, 1969.

[34] Ulrich Kohlenbach, Foundational and mathematical uses of higher types, Reflections on the foundations of mathematics, Lect. Notes Log., vol. 15, Assoc. Symbol. Logic, La Jolla, CA, 2002, pp. 92–116.

[35] A. N. Kolmogorov and S. V. Fomin, Introductory real analysis, Revised English edition. Translated from the Russian and edited by Richard A. Silverman, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1970.

[36] Alexander P. Kreuzer, Measure theory and higher order arithmetic, Proc. Amer. Math. Soc. 143 (2015), no. 12, 5411–5425.

[37] K. Krzicek, My encounters with martingales [reprint of MR2520673], Izv. Nats. Akad. Nauk Armenii Mat. 44 (2009), no. 1, 17–24.
[80] ______, Reverse Mathematics of the uncountability of \( \mathbb{R} \): Baire classes, metric spaces, and unordered sums, Submitted, arXiv: https://arxiv.org/abs/2011.02915 (2020), pp. 15.
[81] ______, Splittings and disjunctions in reverse mathematics, Notre Dame J. Form. Log. 61 (2020), no. 1, 51–74.
[82] ______, Lifting recursive counterexamples to higher-order arithmetic, Proceedings of LFCS2020, Lecture Notes in Computer Science 11972, Springer (2020), 249-267.
[83] Philip Scowcroft, A model of intuitionistic analysis in which \( \emptyset \)-definable discrete sets are subcountable, MLQ Math. Log. Q. 62 (2016), no. 3, 258–277.
[84] Stephen G. Simpson (ed.), Reverse mathematics 2001, Lecture Notes in Logic, vol. 21, Assoc. Symbol. Logic, La Jolla, CA, La Jolla, CA, 2005.
[85] ______, Subsystems of second order arithmetic, 2nd ed., Perspectives in Logic, Cambridge University Press, 2009.
[86] Michael Spivak, Calculus, Publish or Perish, Inc., 1994 (4th edition: 2008).
[87] J. Stillwell, Reverse mathematics, proofs from the inside out, Princeton Univ. Press, 2018.
[88] Anne Sjerp Troelstra, Metamathematical investigation of intuitionistic arithmetic and analysis, Springer Berlin, 1973. Lecture Notes in Mathematics, Vol. 344.
[89] John W. Tukey, Convergence and Uniformity in Topology, Annals of Mathematics Studies, no. 2, Princeton University Press, Princeton, N. J., 1940.
[90] George Weaver, König’s Infinity Lemma and Beth’s tree theorem, Hist. Philos. Logic 38 (2017), no. 1, 48–56.
[91] Wikipedia contributors, Cantor’s first set theory article, Wikipedia, The Free Encyclopedia, https://en.wikipedia.org/wiki/Cantor%27s_first_set_theory_article (2020).