A Matrix Model for the Two Dimensional Black Hole

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We construct and study a matrix model that describes two dimensional string theory in the Euclidean black hole background. A conjecture of V. Fateev, A. and Al. Zamolodchikov, relating the black hole background to condensation of vortices (winding modes around Euclidean time) plays an important role in the construction. We use the matrix model to study quantum corrections to the thermodynamics of two dimensional black holes.

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1. Introduction

An interesting solution to the equations of motion of two dimensional string theory is the semi-infinite cigar \([1,2,3]\), described by the metric and dilaton

\[
ds^2 = k [dr^2 + \tanh^2 r \, d\theta^2] \\
\Phi - \Phi_0 = -2 \log \cosh r
\]  

where \(\theta\) is a periodic coordinate, \(\theta \sim \theta + 2\pi\), labeling the location around the cigar, and \(r \geq 0\) is the direction along the cigar, with \(r = 0\) corresponding to the tip (see fig. 1). The string coupling \(e^\Phi\) depends on \(r\) – it goes to zero far from the tip of the cigar, and attains its maximal value, \(g_s = e^{\Phi_0}\), at the tip. \(k\) is a free parameter which governs the overall size of the cigar.

![Fig.1. The semi-infinite cigar.](image)

Some of the contexts in which the geometry (1.1) appears in string theory are:

1. This geometry can be thought of as a Euclidean version of a 1+1 dimensional black hole \([3]\). The Minkowski black hole can be obtained from (1.1) by analytically continuing \(\theta \rightarrow it\). Furthermore, string propagation in the geometry (1.1) can be described by a coset conformal field theory; the Minkowski version corresponds to the coset \(SL(2, R)/U(1)\), while the Euclidean one is \(SL(2, \mathbb{C})/SU(2)/U(1)\). The parameter \(k\) in (1.1) is the level of the \(SL(2)\) current algebra that enters the construction; the mass of the black hole, \(M\), is encoded in the value of the string coupling at the tip of the cigar, \(M \propto \exp(-2\Phi_0)\).

2. The near-horizon geometry of non-extremal \(NS5\)-branes is a tensor product of the two dimensional black hole geometry (1.1), \(R^5\) (the spatial directions along the fivebranes) and \(S^3\) (the angular directions transverse to the fivebranes) \([4]\). The corresponding worldsheet theory is a supersymmetric version of the above coset CFT. In this case \(k\) is the number of \(NS5\)-branes, and the string coupling at the tip of the cigar measures the energy density of the fivebranes (see \([5]\) for a recent discussion).

\[\text{footnote}{The background (1.1) is in fact only valid for large } k. \text{ For finite } k \text{ there are corrections that were found in [4].}\]
String theory on a Calabi-Yau manifold near a point in moduli space at which the manifold develops an isolated singularity is described by a geometry that includes \((1.1)\) as a factor \([7,8]\). The level \(k\) depends in this case on the particular singularity, and the string coupling at the tip of the cigar corresponds to a non-perturbative energy scale in the theory – the energy of D-branes wrapped around cycles whose size vanishes at the singularity.

Further study of the geometry \((1.1)\) would thus be useful for understanding the physics associated with horizons and singularities in string theory. In particular, one would like to obtain a better understanding of the thermodynamics and statistical mechanics of the corresponding black holes (or black branes).

A promising framework in this regard is holography. In all three examples mentioned above, string theory in the geometry \((1.1)\) is believed to be holographically dual to a non-gravitational theory. In the case (1) the dual is the “old matrix model” – Matrix Quantum Mechanics (MQM) in the double scaling limit – while in the other two examples it is Little String Theory (LST).

In this paper we will focus on the first example, the 1 + 1 dimensional black hole. We will propose a solvable formulation of two dimensional string theory in the background \((1.1)\), based on MQM. This approach to two dimensional string theory in flat spacetime with a condensate of the tachyon field \([3]\) led to a solution of the theory to all orders in string perturbation theory \([1,2,3]\); see \([4,5,6]\) for reviews. The collective field theory of the singlet sector of MQM was identified with the dynamics of the tachyon field in this background \([7]\). Some early attempts to find a matrix realization (in the singlet sector of the matrix model) of two dimensional string theory in the background \((1.1)\) are described in \([8,9,20]\).

As mentioned above, one of the main motivations for looking for a holographic description is to study the statistical mechanics of these black holes. Standard black hole thermodynamics arguments imply that there is a Hagedorn density of black holes:

\[
\rho(M) \sim M^{s_1} e^{{\beta_H}M}. \tag{1.2}
\]

The inverse temperature \(\beta_H\) can be read off the geometry \((1.1)\), while \(s_1\) is given by a one loop calculation in string theory. The prediction \((1.2)\) is particularly interesting since, at least perturbatively, two dimensional string theory has very few states (one field theoretic degree of freedom in 1 + 1 dimensions). Thus, it is interesting to verify the thermodynamic prediction \((1.2)\) by a direct counting of states. In other examples, this proved easier to do in the holographically dual picture, and one might hope that the same will happen here.

The starting point of our analysis is an observation made by V. Fateev, A. Zamolodchikov and Al. Zamolodchikov \([21]\). These authors conjectured that the \(SL(2)/U(1)\) coset CFT is equivalent to the Sine-Liouville model, i.e. \(c = 1\) CFT coupled to a Liouville field,
with the cosmological constant tuned to zero and the scale set by a winding mode of the $c = 1$ field. This correspondence has not been proven but evidence for its validity has been presented\(^2\).\(^{21}\). Our main purpose here will be to define and study the Sine-Liouville background of two dimensional string theory using MQM\(^3\).

From the point of view of the usual Liouville background of two dimensional string theory, the Sine-Liouville vacuum corresponds to condensation of vortices (winding modes). We will treat the vortices as a perturbation, and will show that in the cases of interest, the process of taking the strength of the perturbation to infinity (corresponding to vortex condensation) can be studied using MQM. Some steps in this direction were taken in the past. In particular, G. Moore \(^{23}\) studied the Sine-Gordon model coupled to gravity by treating the Sine-Gordon term as a perturbation. The resulting structure was further discussed in \(^{24}\) where it was shown that the model describes a renormalization group flow from the $c = 1$ model coupled to gravity in the UV, to a set of decoupled $c = 0$ models coupled to gravity in the IR. As we will see below, in a certain region of the parameter space of the model, there is another possible critical point – the Sine-Liouville model.

In the matrix model formulation, it was conjectured \(^{25,26}\) and then demonstrated \(^{27}\) that vortices on the worldsheet correspond to $U(N)$ non-singlet states in MQM. Their wave functions are described by Young tableaux of $U(N)$; the vortex charge is equal to the number of boxes. It is also possible to study vortices, at least in the spherical approximation, using the dual matrix description based on discrete time \(^{11,26,28}\).

By using the connection between vortices and non-singlet states in MQM, we will construct an integrable system, the infinite Toda chain hierarchy, that interpolates between the usual $c = 1$ string and Sine-Liouville (or two dimensional black hole \(^{21}\)) backgrounds structure allows one to compute the partition sum and correlation functions of the theory to all orders in string perturbation theory. As an example we will give explicit expressions for the partition functions on the sphere and on the torus\(^4\).

The plan of the paper is as follows. In section 2 we review the precise statement of the correspondence between the $SL(2)/U(1)$ coset CFT and the Sine-Liouville model, and some of the evidence for it. In section 3 we use the results of \(^{23,24}\) to study the $c = 1$ string theory (at tree level) in the presence of the Sine-Liouville interaction. We show that in a certain range of the parameters defining the model, there is an obstruction

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\(^2\) The Sine-Liouville model is the massless limit of the “sausage model” considered in \(^{22}\).

\(^3\) A direct argument in favour of the description of the 2d black hole in terms of compact MQM, based on the similar conjecture relating thermal YM to the ADS black hole, was given to us by A. Polyakov in 1998.

\(^4\) Toda integrable structure of the $c = 1$ string theory perturbed by purely tachyon source has been first discovered by Dijgraaf, Moore and Plesser in \(^{29}\). Our description is related to their by T-duality.
to turning off the cosmological constant and using the Sine-Liouville interaction to set
the scale (associated with an instability of Sine-Liouville theory), but in the range of
parameters relevant for the correspondence of section 2 this obstruction disappears and
the Sine-Liouville/black hole phase is stable.

In section 4 we introduce the MQM describing vortices and reduce it to a one-matrix
model whose degrees of freedom correspond to the matrix holonomy factor around the
compactification circle. We find that the partition function of this model is a \( \tau \)-function
of the infinite Toda chain hierarchy\(^5\). We prove this statement for a more general partition
function, which can be thought of as the generating function of all (multi-)vortex correlators
in two dimensional string theory.

In Section 5 we use the Toda chain equations and the KPZ-DDK scaling properties
of the partition function to obtain ordinary differential equations for the genus \( h \) partition
functions. We solve these equations explicitly for the sphere \((h = 0)\), thus proving a
conjecture of [23], and for the torus \((h = 1)\) which is a new result.

In section 6 we discuss the implications of our results for black hole thermodynamics.
In particular we compute the coefficient \( a \) in (1.2) and show that the Hagedorn temperature
is associated with a phase transition. In section 7 we briefly describe the thermodynamics
of black holes in fermionic two dimensional string theory, which helps to clarify some
aspects of the bosonic analysis. In section 8 we comment on our results and describe some
open problems. Some of the technical details are contained in the appendices.

2. On the equivalence of the cigar and Sine-Liouville CFT’s

As mentioned above, the Euclidean black hole worldsheet CFT (1.1) corresponds to
the coset

\[
H_3^+ / U(1); \quad H_3^+ \equiv \frac{SL(2, C)}{SU(2)}.
\] (2.1)

It can be studied using standard coset CFT techniques, and is closely related to CFT on
Euclidean \( AdS_3 \) \((= H_3^+)\); see e.g. [30]. Below we list some properties of this theory.

The central charge of (2.1) is given in terms of the level of the underlying \( SL(2) \)
current algebra, \( k \), by

\[
c = \frac{3k}{k-2} - 1.
\] (2.2)

In particular, for \( k = 9/4 \), \( c = 26 \), and the coset is a good classical solution of two dimen-
sional bosonic string theory. An important set of observables corresponds to momentum
and winding modes on the cigar, \( V_{j;m,\bar{m}} \). These operators have scaling dimensions

\[
\Delta_{j;m,\bar{m}} = -\frac{j(j+1)}{k-2} + \frac{m^2}{k}; \quad \bar{\Delta}_{j;m,\bar{m}} = -\frac{j(j+1)}{k-2} + \frac{\bar{m}^2}{k}.
\] (2.3)

\(^5\) The fact that the generating function for the tachyon amplitudes in the \( c = 1 \) string theory
possesses Toda lattice symmetry has been established in [28].
where $m, \bar{m}$ are given by
\[
m = \frac{1}{2}(n_1 + n_2 k); \quad \bar{m} = -\frac{1}{2}(n_1 - n_2 k); \quad n_1, n_2 \in \mathbb{Z}.
\] (2.4)

One can think of the integers $n_1$ and $n_2$ as momentum and winding around the cigar (in the $\theta$ direction in figure 1). As is clear from the geometry of the cigar, the momentum, $m - \bar{m}$, is conserved in this CFT, while the winding, $m + \bar{m}$, is not.

Spherical two and three point functions of the observables $V_{j;m,\bar{m}}$ were computed in [21,30]. The two point function is (suppressing the dependence on the separation of the worldsheet locations of the vertex operators)
\[
\langle V_{j;m,\bar{m}} V_{j;-m,-\bar{m}} \rangle = (k - 2)[\nu(k)]^{2j+1} \frac{\Gamma(1 - \frac{2j+1}{k}) \Gamma(-2j - 1) \Gamma(j - m + 1) \Gamma(1 + j + \bar{m})}{\Gamma(\frac{2j+1}{k} \Gamma(2j + 2) \Gamma(-j - m) \Gamma(\bar{m} - j)}
\] (2.5)

where
\[
\nu(k) \equiv \frac{1}{\pi} \frac{\Gamma(1 + \frac{1}{k-2})}{\Gamma(1 - \frac{1}{k-2})}.
\] (2.6)

The result for the three point function is more involved and can be found in [30].

String perturbation theory is an expansion in $\exp(-2\Phi)$. On the cigar, the dilaton $\Phi$ varies as in (1.1), but string perturbation theory still makes sense – it can be thought of as an expansion in $\exp(2\Phi_0)$, the value of the dilaton at the tip (which is proportional to $1/M$). Thus, the genus expansion on the cigar is a $1/M$ expansion.

The dual Sine-Liouville theory [21] is described by the Lagrangian (we set $\alpha' = 1$)
\[
L = \frac{1}{4\pi} \left[ (\partial x)^2 + (\partial \phi)^2 + Q\dot{R}\phi + \lambda e^{b\phi} \cos R(x_L - x_R) \right].
\] (2.7)

The central charge of this theory is $c = 1 + 6Q^2 + 1$; comparing to (2.2) we find that
\[
Q^2 = \frac{1}{k - 2}.
\] (2.8)

The radius of $x$ is $R = \sqrt{k}$, the same as the radius of $\theta$ far from the tip of the cigar in (1.1). The cosine interaction in (2.7) is thus the lowest lying winding mode, with winding number equal to one. Requiring that the scaling dimension of the sine-Liouville interaction is equal to one, leads to
\[
\frac{1}{4}R^2 - \frac{1}{4}b(b + 2Q) = 1
\] (2.9)

with the solution
\[
b = -\frac{1}{Q} = -\sqrt{k - 2}.
\] (2.10)
Very far from the tip of the cigar, (1.1) describes a cylinder with a dilaton field that varies linearly along it. The same is true for the geometry corresponding to (2.7) far from the potential wall (i.e. at \( \phi \to \infty \)). The two cylinders can thus be identified (for large \( k \)) via

\[
\begin{align*}
  r &\sim -Q\phi \\
  \theta &\sim \frac{x}{\sqrt{k}}
\end{align*}
\] (2.11)

The non-trivial statement of [21] is that the two CFT’s also agree when one includes the interaction that cuts off the large string coupling region \( \phi \to -\infty \). This is far from obvious, since in one case the strong coupling region is eliminated by changing the topology of the cylinder to that of the cigar (1.1), while in the other this is achieved by turning on the potential in (2.7).

The relation between the coset and Sine-Liouville CFT’s is a strong-weak coupling duality on the worldsheet. The cigar CFT becomes weakly coupled in the limit \( k \to \infty \), where the geometry (1.1) becomes weakly curved. In that region the wavefunction of the Sine-Liouville potential, which has the large \( \phi \) behavior

\[
\Psi(\phi) \sim e^{(Q-\frac{1}{2})\phi}
\] (2.12)

goes rapidly to zero as \( \phi \to \infty \), i.e. \( \Psi \) is supported in the region of small \( \phi \) and the theory is strongly coupled. The semiclassical limit in Sine-Liouville is \( Q \to \infty \) (i.e. \( k \to 2 \) (2.8)). In this case the wavefunction (2.12) is supported at \( \phi \to \infty \), far from the potential wall (2.7) and the theory is weakly coupled. The cigar is highly curved in this limit; thus the coset CFT is strongly coupled.

The operators \( V_{j;m,\bar{m}} \) mentioned above are described in the Sine-Liouville model as

\[
V_{j;m,\bar{m}} \leftrightarrow e^{i p_L x_L + i p_R x_R + \beta \phi},
\] (2.13)

where

\[
\begin{align*}
  p_L &= \frac{n_1}{R} + n_2 R \\
  p_R &= \frac{n_1}{R} - n_2 R \\
  \beta &= 2Qj.
\end{align*}
\] (2.14)

Indeed, the scaling dimensions

\[
\begin{align*}
  \Delta = \frac{1}{4} p_L^2 - \frac{1}{4} \beta (\beta + 2Q); \\
  \bar{\Delta} = \frac{1}{4} p_R^2 - \frac{1}{4} \beta (\beta + 2Q)
\end{align*}
\] (2.15)

\footnote{More precisely, this is the form of the vertex operators as \( \phi \to \infty \).}
agree with (2.3). Eq. (2.14) also makes it clear that \( n_1 \) and \( n_2 \) are the momentum and winding around the cylinder (or cigar). Note also that like in the cigar CFT, in (2.7) the momentum \( p_L + p_R \) is conserved, while the winding number \( p_L − p_R \) is broken. In the cigar CFT the reason for that is that winding can slip off the tip of the cigar; in (2.7) the interaction breaks this symmetry explicitly.

The Sine-Liouville model (2.7) has similar scaling properties to Liouville theory. For example, the standard scaling analysis \[31\] implies that the partition sum has the following perturbative expansion:

\[
\mathcal{F}(\lambda, g_s) = \sum_{h=0}^{\infty} \mathcal{F}_h \left( g_s \lambda^{-\frac{1}{k-2}} \right)^{2(h-1)} ,
\]

(2.16)

where \( \mathcal{F}_h \) is the genus \( h \) partition sum, and \( g_s \) is the string coupling constant. As in Liouville theory, the physics depends only on the combination \( g_s \lambda^{-\frac{1}{k-2}} \), which can be thought of as the effective string coupling in the background (2.7). From now on, we will absorb \( g_s \) into \( \lambda \) (i.e. set \( g_s = 1 \)).

We see that the string coupling expansion in Sine-Liouville theory is a large \( \lambda \) expansion, like in ordinary Liouville theory. In string theory on the cigar, as mentioned above, the string coupling at the tip of the cigar is related to the mass of the black hole via \( g_s^2 \sim 1/M \) where \( M \) is measured in Planck units. Therefore, in the correspondence of \[21\], \( \lambda \) is related to the mass of the black hole, \( M \),

\[
M \leftrightarrow \lambda^{\frac{k}{k-2}} .
\]

(2.17)

An important piece of evidence for the equivalence of the cigar and Sine-Liouville CFT’s is the agreement of a large class of two and three point functions on the sphere in the two models \[21\]. We will next review this agreement for the two point function; the three point functions can be discussed along the same lines.

The two point functions of \( V_{j;m,n} \) in the coset CFT, given by (2.3), exhibit a series of poles. As explained in \[3\] (for the supersymmetric case; a similar analysis applies to the bosonic case discussed here), the poles of the first \( \Gamma \) function in the numerator play a different role from those of the rest of the factors. These poles occur at values of \( j \) for which

\[
1 - \frac{2j + 1}{k - 2} = -n; \quad n = 0, 1, 2, \cdots .
\]

(2.18)

Near such a pole, the amplitude has the form

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7 defined as the value of \( \exp \Phi \) at some arbitrary point along the cylinder.
$$\langle V_{j;m,\bar{m}} V_{j; -m, -\bar{m}} \rangle \sim \frac{(k-2)[\nu(k)]^{2j+1}(-1)^n \Gamma(-2j-1)\Gamma(j-m+1)\Gamma(1+j+\bar{m})}{n!(n+1-\frac{2j+1}{k-2}) \Gamma(\frac{2j+1}{k-2})\Gamma(2j+2)\Gamma(-j-m)\Gamma(\bar{m}-j)}. \tag{2.19}$$

As observed in [21], the poles (2.18) are also obtained in Sine-Liouville theory (2.7), and the residues agree with (2.19). We next review this calculation.

The two point function (2.5) corresponds in Sine-Liouville to

$$A(p_L, p_R) \equiv \langle e^{ip_L x_L + ip_R x_R + \beta \phi} e^{-ip_L x_L - ip_R x_R + \beta \phi} \rangle \tag{2.20}$$

(see (2.13), (2.14)). The theory (2.7) is interacting, and in general such correlation functions are difficult to compute (although much progress has been made on this problem in the last ten years). However, in some cases one can compute the residues of certain poles (which will turn out to correspond precisely to (2.18)) as follows (see e.g. [32] for a review). Split the path integral over the Liouville field $\phi$ into a zero mode integral and a path integral over the non-zero modes. Performing the integral over the zero mode, one formally finds (after absorbing an overall constant into the definition of the path integral)

$$A(p_L, p_R) = \lambda^s \Gamma(-s) \langle e^{ip_L x_L + ip_R x_R + \beta \phi} e^{-ip_L x_L - ip_R x_R + \beta \phi} \left[ \int e^{b \phi \cos R(x_L - x_R)} \right]^s \rangle_{\lambda=0}. \tag{2.21}$$

The expectation value in (2.21) is understood to exclude the zero mode of $\phi$; note also that it is performed with the free action ($\lambda = 0$ in (2.7)). $s$ is the KPZ scaling parameter, which using (2.10), (2.14) is

$$s = \frac{2(2j+1)}{k-2}. \tag{2.22}$$

In general the representation (2.21) is highly formal since it is difficult to make sense of the non-integer power of the interaction in the correlator. In situations where $s$ is a non-negative integer, the correlator in (2.21) does seem to make sense but the prefactor diverges. The nature of the divergence is well understood [32]. Amplitudes with $s \in \mathbb{Z}_+$ are *bulk* amplitudes – they correspond to processes that can occur anywhere in the infinite region far from the Sine-Liouville wall and the divergence in question is nothing but the volume of that region.\footnote{Note that these amplitudes are in general sensitive to the structure of the wall, since they involve bulk interactions between the incoming quanta and those that make up the wall. Amplitudes with $s = 0$ are insensitive to the structure of the wall, a fact that will play a role later.}

Regularizing the theory by cutting off the region $\phi \to \infty$ (far from the wall (2.7)) amounts to replacing

$$\lambda^s \Gamma(-s) \to \frac{(-1)^{s+1}}{s!} \lambda^s \log \frac{\lambda}{\Lambda}. \tag{2.23}$$
where $\Lambda$ is a UV cutoff.

To summarize, the residues of the poles of $\Gamma(-s)$ in (2.21) are given by bulk amplitudes, which have an integral representation of Shapiro-Virasoro type. These can be computed by standard techniques.

When $s$ is odd, the bulk amplitude vanishes. The reason is that winding number, which is broken by the Sine-Liouville interaction, is actually a good symmetry in the bulk of space, and for $s \in 2\mathbb{Z} + 1$ the correlator (2.21) does not conserve winding. Thus, only poles with

$$s = 2(n + 1); \ n = 0, 1, 2, \cdots$$

(2.24)
survive (the amplitude with $s = 0$ is special and can be shown to vanish in this case). Comparing to (2.18) we see that the amplitude $A(p_L, p_R)$ (2.21) has poles at the same places as (2.5) (the rest of the poles of (2.5) do not correspond to bulk physics; they have to do with the existence of certain normalizable states living near the tip of the cigar).

Near the poles, (2.21) is given by

$$A(p_L, p_R) \sim \frac{\lambda^{2(n+1)}}{2^{n+1}(2n+2)!(n+1)!^2(n+1 - \frac{2j+1}{k-2})} \langle e^{ip_L x_L + ip_R x_R + \beta \phi} e^{-ip_L x_L - ip_R x_R + \beta \phi} \left[ \int e^{b \phi} e^{iR(x_L - x_R)} \right]^{n+1} \left[ \int e^{b \phi} e^{-iR(x_L - x_R)} \right]^{n+1} \rangle.$$  

(2.25)

As usual, all but three of the vertex operators in (2.25) are integrated over the complex plane. These integrals can be evaluated (they are generalizations of those discussed e.g. in [32]), and shown to agree with (2.19). We will not discuss the details of these calculations here.

One might worry that this agreement is coincidental since one can rescale the operators $V_{j;m,\bar{m}}$ by a $(j;m,\bar{m})$ dependent factor. However, the relative normalization of the operators entering the correspondence (2.13) is in fact fixed by requiring that their wave-functions coincide in the region very far from the tip of the cigar or Sine-Liouville potential (e.g. they can be taken to be incoming waves of unit strength). This makes the comparison of the two-point functions, which upon analytic continuation to Minkowski spacetime correspond to the scattering matrices off the tip of the cigar or Sine-Liouville potential, meaningful. The authors of [21] also showed that the three point functions of the operators (2.13) exhibit similar poles, and the residues again agree.

It is useful to mention for future reference that the correspondence between the cigar and Sine-Liouville conformal field theories has a (worldsheet) supersymmetric generalization proposed in [8]. It relates the $N = 1$ superconformal coset model $SL(2)/U(1)$ to the $N = 2$ Liouville theory (see e.g. [34] for a description of $N = 2$ Liouville).

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9 It might seem surprising that an $N = 1$ superconformal field theory can be equivalent to an $N = 2$ superconformal one. In fact, it turns out that the $N = 1$ supersymmetric $SL(2)/U(1)$ coset has an accidental $N = 2$ supersymmetry. It is a special case of the Kazama-Suzuki construction [33].
As discussed in section 2, one approach to studying the black hole background in two dimensional string theory is to study the dual Sine-Liouville theory (2.7). This is what we will do below.

In bosonic two dimensional string theory the central charge (2.2) is \( c = 26 \), which determines the different parameters that appeared in section 2 to be \( k = 9/4 \), \( Q = 2 \), \( R = 3/2 \). We would like to use the matrix model approach to analyze this model. An immediate problem that one runs into is that in the matrix model \( \lambda \) is set to zero, and instead the Lagrangian contains the Liouville potential \( \mu \phi \exp(-2\phi) \). One can attempt to define (2.7) by means of conformal perturbation theory, i.e. consider the theory with Lagrangian

\[
L = \frac{1}{4\pi} \left[ (\partial x)^2 + (\partial \phi)^2 + 2 \dot{R} \phi \right.
\]

\[+ \mu \phi e^{-2\phi} + \lambda e^{(R-2)\phi} \cos R(x_L - x_R) \]

and treat \( \lambda \) as a perturbation. General arguments lead one to expect that the radius of convergence of the resulting series is finite. Since we are interested in a continuation to \( \lambda \to \infty \) (or equivalently \( \mu \to 0 \)), we are not guaranteed that the series will converge. Indeed, we will see that in some cases there is an obstruction to taking this limit – one encounters a singularity at finite (real) \( \lambda \).

This is not unexpected; the classical Lagrangian (3.1) with \( \mu = 0 \) has a potential which is not bounded from below. Thus, at least for small \( R \), where the cosine is slowly varying and the semiclassical picture is reliable, we expect the limit \( \mu \to 0 \) not to exist. As \( R \) increases, quantum effects (on the worldsheet) become important, and the vacuum energy could in principle be renormalized in such a way that the limit exists. We will see below that this is indeed what happens. Fortunately, for \( R = 3/2 \), the value relevant for the correspondence with the cigar, there is no singularity and the Sine-Liouville model exists. Of course, this has to be the case if the conjecture described in section 2 is correct.

Consider, for example, the partition sum of (3.1) on the sphere. Define

\[
\mathcal{T}_p \equiv \frac{\Gamma(p)}{\Gamma(1-p)} \int e^{ip(x_L - x_R) + (|p|-2)\phi}.
\]

After rescaling \( \lambda \) appropriately, the partition sum corresponding to (3.1) becomes

\[
F_0(\lambda_{\pm}, \mu) = \langle e^{-\lambda_{\pm}T_R - \lambda_{-}T_{-R}} \rangle.
\]

We are interested in the case \( \lambda_{+} = \lambda_{-} = \frac{1}{2} \lambda \), but it will be convenient to keep \( \lambda_{\pm} \) independent. Expanding in \( \lambda_{\pm} \) and using winding number conservation, we find

\[
F_0(\lambda, \mu) = \sum_{n=0}^{\infty} \frac{(\lambda_{+}+\lambda_{-})^n}{n!^2} \langle T^n_R T_{-R}^n \rangle_0.
\]
The coefficients in this series are $2n$ point functions on the sphere, which can in principle be computed in the matrix model. G. Moore [23] conjectured a general form for these amplitudes based on an extrapolation of matrix model results from small $n$:

$$\langle T^n R T^n \rangle_0 = R \, n! \, \mu^2 [(1 - R) \mu^{R-2}]^n \frac{\Gamma(n(2 - R) - 2)}{\Gamma(n(1 - R) + 1)}; \ (n \geq 1). \quad (3.5)$$

The term with $n = 0$ is the usual $c = 1$ partition sum on the sphere,

$$F_0(\lambda = 0, \mu) = -\frac{R}{2} \mu^2 \log \frac{\mu}{\Lambda}. \quad (3.6)$$

Combining it with (3.5) we find

$$F_0(\lambda, \mu) = -\frac{R}{2} \mu^2 \log \frac{\mu}{\Lambda} + R \mu^2 \sum_{n=1}^{\infty} \frac{1}{n!} [(1 - R) \mu^{R-2} \lambda_+ \lambda_-] \frac{\Gamma(n(2 - R) - 2)}{\Gamma(n(1 - R) + 1)}. \quad (3.7)$$

Differentiating twice with respect to $\mu$ (and dropping a non-universal constant) we find

$$\chi_0(\lambda, \mu) \equiv \partial^2_\mu F_0(\lambda, \mu) = -R \log \mu + R \sum_{n=1}^{\infty} \frac{(-z)^n}{n!} \frac{\Gamma(n(2 - R))}{\Gamma(n(1 - R) + 1)}, \quad (3.8)$$

where

$$z \equiv (R - 1) \lambda_+ \lambda_- \mu^{R-2} \quad (3.9)$$

measures the “dimensionless strength” of the perturbation $\lambda$. The series in (3.8) can be summed using equation 5.2.13.30 in [35]:

$$\sum_{n=0}^{\infty} \frac{\Gamma(n(a + b - 1))}{n! \Gamma(n(a - 1) + b)} (-z)^n = \frac{(1 - s)^{b-1}}{b - 1}, \quad \frac{s}{(1 - s)^a} \equiv z. \quad (3.10)$$

Taking the limit $b \to 1$ and plugging in (3.8) (with $a = 2 - R$) leads to

$$\chi_0(\lambda, \mu) = -R \log \mu + R \log(1 - s), \quad (3.11)$$

where $s$ is given by

$$\frac{s}{(1 - s)^{2-R}} = z. \quad (3.12)$$

For small $\lambda$, expanding (3.12) one finds $s \approx z + O(z^2)$. The large $z$ behavior depends on $R$. Next we analyze this behavior as a function of $R$.

We restrict attention to $R \leq 2$. The reason is that for $R > 2$ the $\lambda$ perturbation in (3.1) qualitatively changes the behavior of the potential in the weak coupling (UV) region $\phi \to \infty$. Instead of dying off, it now fluctuates with an amplitude that diverges as $\phi \to \infty$. 
This is a reflection of the fact that in flat space (i.e. discarding $\phi$), the $\lambda$ perturbation is non-renormalizable in this case.

It turns out that there is also a difference between the behavior for $R < 1$ and $R > 1$. Consider first the case $R < 1$. The parameter $z$ (3.9) is negative in this case and we would like to analyze the behavior as $z \to -\infty$. However, looking at (3.12) we see that there is an obstruction: $\partial_s z$ is positive only for $s > s_c = -1/(1 - R)$. Thus, varying $s$ along the negative real axis we can not probe $z < z_c = -(1 - R)^{1-R}/(2 - R)^{2-R}$.

The physics of this obstruction was explained in [24]. It corresponds to $c = 0$ critical behavior which is expected to occur in the model (3.1). The $c = 1$ field $x$ (more precisely the T-dual field $\tilde{x} = x_L - x_R$) settles into one of the minima of the cosine potential and decouples, leaving behind a $c = 0$ system (coupled to gravity). While this is more clearly visible in the fixed area representation (see [24] for the details), it can also be seen directly from (3.11), (3.12). Near the critical point we have $z - z_c \sim (s - s_c)^2$. Thus the leading singular behavior of (3.11) as $z \to z_c$ is $\chi_0 \sim (z - z_c)^{1/2}$, which is the expected behavior for pure gravity.

Thus it appears that for $R < 1$ the Sine-Liouville theory (2.7) is unstable (as expected on general grounds – see the discussion after eq. (3.1)) and can not be reached from (3.1) by taking $\mu \to 0$.

We next turn to $1 < R < 2$. In [24] it was shown that quantum corrections generate in this case a finite positive contribution to the effective cosmological constant. Therefore, one expects the limit $\mu \to 0$ to be non-singular. This is indeed visible in the formulae above. In this case $z > 0$ (3.9) and the relation between $z$ and $s$ (3.12) is monotonous. Thus, as $z \to \infty$, $s \to 1$. This is the critical point we are interested in, in which $\mu$ has been effectively switched off. As a check, note that in this limit the dependence of the correlator $\chi_0$ on $\mu$ drops out, as it should. Indeed, combining (3.11) and (3.12) and using (3.9) we find the following algebraic equation for the susceptibility:

$$
\mu e^{\frac{\chi_0}{R}} + (R - 1)\lambda_+ \lambda_- e^{\frac{2-R}{R} \chi_0} = 1.
$$

From here we find that at large $\lambda$

$$
\chi_0 \simeq \frac{R}{R - 2} \log \left[ (R - 1)\lambda_+ \lambda_- \right].
$$

\[10\] We note for completeness that we could discuss the system with positive $z$, in which case there is no obstruction to taking $z \to \infty$. As is clear from (3.12), as $z \to \infty$, $s \to 1$, and we seem to find a new fixed point. Positive $z$ corresponds for $R < 1$ to imaginary $\lambda$ (see (3.9)). Thus, for $R < 1$ one can study an analog of Sine-Liouville, but with an imaginary potential. This theory is non-unitary. Also, for $R < 1$ the winding number two perturbation is relevant and one can study flows involving it.
This is easily seen to be the correct KPZ scaling for the two point function of $e^{-2\phi}$ in Sine-Liouville theory.

One can in fact go further and find the whole small $\mu$ (or large $\lambda$) expansion of the susceptibility. Namely, from (3.11) and (3.12) we obtain the following parametrization of the susceptibility

$$\chi_0(\lambda, \mu) = -R \log \mu + R \log t = -\frac{R}{2-R} \log ((R-1)\lambda_+ \lambda_-) + \frac{R}{2-R} \log (1-t),$$  \hspace{1cm} (3.15)

where $t(=1-s)$ is given by

$$\frac{t}{(1-t)^{1-R}} = y$$  \hspace{1cm} (3.16)

and

$$y \equiv z^{-\frac{1}{2-R}} = \frac{\mu}{((R-1)\lambda_+ \lambda_-)^{\frac{1}{2-R}}}.$$  \hspace{1cm} (3.17)

Using the formula (3.10) again, we arrive at the following small $\mu$ expansion:

$$\chi_0(\lambda, \mu) \equiv \partial^2_\mu F_0(\lambda, \mu) = -\frac{R}{2-R} \log ((R-1)\lambda_+ \lambda_-) + \frac{R}{2-R} \sum_{n=1}^{\infty} \frac{(-y)^n}{n!} \frac{\Gamma\left(\frac{n}{2-R}\right)}{\Gamma\left(\frac{n}{2-R} - n + 1\right)}.$$  \hspace{1cm} (3.18)

The coefficient of $\mu^n/n!$ in (3.18) is the $n$ point function of zero momentum tachyons in the Sine-Liouville background (2.7). According to the conjecture of [21], for $R = 3/2$ it should also be the $n$ point function of these tachyons in the Euclidean black hole geometry.

To summarize, we find that for $1 < R < 2$, there is no obstruction to turning on a large (real) Sine-Liouville coupling (3.1). There is a critical point corresponding to $\mu = 0$, $\lambda$ finite, which appears to be stable and according to the conjecture described in section 2 is equivalent (for $R = 3/2$) to the black hole background (2.1).

This raises an interesting question, what is the interpretation of the Sine-Liouville background with $R \neq 3/2$ in terms of the cigar geometry? Naively, changing the asymptotic radius of the cigar creates a conical singularity at the tip, and thus breaks conformal invariance. One natural proposal is that changing the radius $R$ in the Sine-Liouville description corresponds in the cigar theory to changing the asymptotic radius in the same way (so the asymptotic geometries agree) and turning on a potential like that in (3.1). This might lead to a change of the radius of $x$ as a function of $\phi$, such that the conical singularity at the tip is eliminated. It would be interesting to make this proposal more precise, but we will not attempt that here.

---

11 To do this it is useful to note that (3.13) is invariant under: $2-R \leftrightarrow \frac{1}{2-R}$; $\sqrt{2-R} \log \mu \leftrightarrow \frac{1}{\sqrt{2-R}} \log ((R-1)\lambda_+ \lambda_-)$, $\sqrt{\frac{R-2}{R}} \chi_0 \leftrightarrow \frac{1}{r \sqrt{2-R}} \chi_0$.

12 D.K. thanks S. Shenker and L. Susskind for a discussion of this issue.
In the next two sections we will construct a matrix model describing two-dimensional string theory with the worldsheet Lagrangian (3.1) and use it to compute the string partition function. We will in fact study a more general problem corresponding to two-dimensional string theory in an arbitrary winding background,

$$L = \frac{1}{4\pi}[(\partial x)^2 + (\partial \phi)^2 + 2R \phi + \mu e^{-2\phi} + \sum_{n \neq 0} t_n e^{i|n|R-2} \phi e^{in\pi x}]$$

The couplings $t_n$ carry winding number $n$; the model (3.1), (3.3) corresponds to the choice

$$t_n = \delta_{n,1} \lambda_+ + \delta_{n,-1} \lambda_-.$$  

Of course, most of the couplings $t_n$ are in general irrelevant, which is reflected in the fact that the corresponding perturbations in (3.19) are singular at $\phi \to \infty$. These couplings should be treated perturbatively, and the path integral (3.19) should be thought of as the generating functional of correlation functions of these operators, as a function of the relevant and marginal couplings.

4. A matrix model for the $c = 1$ string theory with vortices

In this section we will describe a matrix model for the $c = 1$ string theory in the presence of vortices of arbitrary vorticity, or in continuum language the theory described by the Lagrangian (3.19) with $t_n \neq 0$. We will show that the grand-canonical partition function of this matrix model is a $\tau$-function of the Toda chain hierarchy and, as such, it satisfies the Toda lattice equation. This equation can be used to compute the partition sum of the model to all orders in the string coupling (or equivalently to all orders in $1/M$ in the black hole background).

4.1. The role of twisted boundary conditions in MQM

The usual $c = 1$ string [4] corresponds to the fixed point in which all $t_n$ in (3.19) vanish and the scale is set by the cosmological constant $\mu$. The (perturbative) dynamics is in this case well described by the singlet sector of the $c = 1$ matrix model and, as mentioned above, contains very few degrees of freedom. The Euclidean problem with finite $R$ corresponds to finite temperature, $T = 1/2\pi R$. When the temperature is sufficiently low ($R > 2$ in our normalizations (2.14), (2.15)), one can neglect the non-singlet states [24]. Above this critical temperature ($R < 2$) the vortices become important; studying the Lagrangian (3.19) with $t_n \neq 0$ is a useful way to incorporate them.
To describe the c=1 string with vortex excitations (3.1) we will use the twisted MQM introduced in [27]. Its canonical (fixed $N$) partition function is defined as a functional integral with respect to the one-dimensional $N \times N$ Hermitian matrix field $M_{jk}(x)$

$$Z_N(\Omega) = \int_{M(2\pi R) = \Omega^\dagger M(0)\Omega} D M(x) e^{-\text{tr} \int_0^{2\pi R} dx \left[ \frac{1}{2} (\partial_x M)^2 + V(M) \right]} \quad (4.1)$$

where $\Omega$ is an arbitrary unitary matrix. Note that the path integral (4.1) depends on $\Omega$ only through its eigenvalues, $z_1 = e^{i\theta_1}, \ldots, z_N = e^{i\theta_N}$. The potential $V$ can be taken to be any function with the appropriate critical behavior, e.g.

$$V(M) = \frac{1}{2} M^2 - \frac{g}{3\sqrt{N}} M^3. \quad (4.2)$$

The projection onto the singlet sector is achieved by imposing the twisted boundary conditions (4.1) and then integrating with respect to the twist matrix $\Omega$. As we review in Appendix A, the singlet sector can be reduced to a system of free fermions corresponding to the eigenvalues $(y_1, \ldots, y_N)$ of the matrix $M$, each compactified on the circle, $y_j(x + 2\pi R) = y_j(x)$ and moving in the potential $V(y)$. The introduction of vortex excitations is achieved by including higher representations of $SU(N)$ [23,27]. The eigenvalue dynamics in these representations is more complicated.

To find the realization of the system (3.19) in MQM we next take a closer look at the role of $\Omega$ in the large $N$ limit. The $1/N$ expansion of the free energy

$$F_N(\Omega) = \log Z_N(\Omega)$$

of the $\Omega$-twisted matrix model (1.1) can be expressed in terms of connected planar graphs, embedded in the target-space circle. Namely

$$F_N(\Omega) = \sum_{h \geq 0} F_N^{(h)}(\Omega) \frac{N^{2-2h}}{2}, \quad (4.3)$$

where $F_N^{(h)}(\Omega)$ is the contribution of planar graphs with the topology of a sphere with $h$ handles. We will use here the usual connection between planar graphs in MQM and string worldsheets: each planar graph $\Sigma$ is viewed as a discretized worldsheet whose vertices, links and faces are labeled by $v$, $\ell$ and $f$, correspondingly, and the functional integral over the worldsheet fields $(x, \phi)$, where $\phi$ is the conformal (“Liouville”) factor of the worldsheet metric, is discretized as $\int D x D \phi \rightarrow \sum_{\Sigma} \int \prod_{v} dx_v$.

The planar diagram expansion is done with respect to the trivial (non-physical) vacuum, $M = 0$. The propagator of the matrix field $M$ with twisted boundary conditions is

$$\langle M_i^k(x) M_j^l(x') \rangle \equiv G_{ijkl}(x,x') = \sum_{m=-\infty}^{\infty} e^{-|x-x'|+2\pi R m} (\Omega^m)_{ij} (\Omega^{-m})^{kl}. \quad (4.4)$$
To each link $\ell = \langle vv' \rangle$ of the planar graph is associated a propagator

$$x_v^i \overset{m}{\overset{j}{\longrightarrow}} x_{v'}^j = G_{ij}^{kl}(x_v, x_{v'}; m),$$

which is given by the $m$-th term in the expansion (4.4). This gives rise to an integer-valued local field $m = m_{\langle vv' \rangle}$ on the graph. The weight of each planar graph (see Fig. 2) is given by

$$W(\Sigma) = N^{2-2h} g^{n_v} \prod_{f \in \Sigma} \frac{\text{tr} \Omega^w_f}{N} \exp \left(- \sum_{\langle vv' \rangle} |x_v - x_{v'} + 2\pi R m_{\langle vv' \rangle}| \right), \quad (4.5)$$

where $n_v$ is the total number of vertices in the graph; the integer field $w_f$ is related to the field $m_{\langle vv' \rangle}$ by

$$w_f = \sum_{\ell \in \partial f} m_{\ell}. \quad (4.6)$$

where the sum is taken along the links on the boundary of the face $f$.

Geometrically, $w_f$ is the winding number of the boundary $\partial f$ of the face $f$ on the discretized worldsheet around the circle. By definition this is the vorticity associated with this face. Following standard arguments \cite{34}, the sum over the field $m_{\ell}$ can be split into a sum over the vorticity field $w_f$ and its gradient (vorticity-free) part.\cite{13}

\cite{13} The latter sum, corresponding to $m_{vv'} = m_v - m_{v'}$ for some $m_v, m_{v'} \in \mathbb{Z}$, extends the integration over $x_v$, which was originally restricted to the interval $[0, 2\pi R]$, to the whole real axis.
The dependence of a given planar graph on $\Omega$ is through the factor $\prod_{f \in \Sigma} \text{tr}\Omega^{w_f}$; hence

To study the effects of vortices on the worldsheet, one can either analyze the dependence of the partition sum (4.1) on $\Omega$ (recall that in fact it depends only on the eigenvalues of $\Omega$), or integrate with respect to the twist matrix with an appropriate measure. We will use the second alternative, which is technically preferable.

We start by showing that the singlet sector of the model is obtained by integrating over $\Omega$ with the standard left/right invariant (Haar) measure $[d\Omega]$ normalized so that $\int_{U(N)} [d\Omega] = 1$. An elegant way to evaluate the integral $\int_{U(N)} [d\Omega] \text{tr}\Omega^{n_1} \cdots \text{tr}\Omega^{n_k}$ for $n_1 + \cdots + n_k \ll N$ was proposed in [37]. It uses a Fock space representation of the moments of the unitary matrix in terms of bosonic oscillator modes. Replacing $\text{tr}\Omega^n \rightarrow \alpha_{-n} + \bar{\alpha}_n$, where $\{\alpha_n\}$ and $\{\bar{\alpha}_n\}$ satisfy the canonical commutation relations $[\alpha_m, \alpha_n] = [\bar{\alpha}_m, \bar{\alpha}_n] = m\delta_{m+n,0}$, $[\alpha_m, \bar{\alpha}_n] = 0$, the above integral can be thought of as the expectation value with respect to the Fock vacuum $\alpha_n|0\rangle = \bar{\alpha}_n|0\rangle = 0$. This leads to:

$$\int_{U(N)} [d\Omega] \text{tr}\Omega^n = N\delta_{n,0},$$

(4.7)

$$\int_{U(N)} [d\Omega] \text{tr}\Omega^m \text{tr}\Omega^n = |m|\delta_{m+n,0}, \text{ etc.}$$

Using these equations it becomes clear that the sum over all Feynman graphs (4.5) with a given set of vortices $w_f$ corresponds in the continuum language to the string worldsheet path integral with an insertion of the corresponding winding modes, (3.2):

$$\text{tr}\Omega^n \leftrightarrow T_{nR}. \quad (4.8)$$

The $\delta$-functions in eq. (4.7) impose winding number conservation.

We would like to analyze the behavior of the partition sum

$$\int [d\Omega] Z_N(\Omega) \quad (4.9)$$

in the double scaling limit. By using (1.3) and the first line of (4.7) we see that each Feynman diagram with $w_f = 0$ for all faces contributes a term that goes like $N^{2-2h}$ (where $h$ is the genus of the discretized Riemann surface). The continuum limit is achieved in the matrix model by tuning the coupling $g$ in (4.2) to a critical point $g \rightarrow g_c$ where large Feynman diagrams dominate the path integral, and considering the double scaling limit $N \rightarrow \infty$, $g \rightarrow g_c$ with $N(g_c - g)$ held fixed.\(^{15}\) The sum over diagrams corresponding to large worldsheets with $w_f = 0$ scales like $[(g_c - g)N]^{2-2h} \sim \mu^{2-2h}$ in this limit.

\(^{14}\) This generalization of the arguments of [27] was suggested by P. Zinn-Justin.

\(^{15}\) up to certain logarithmic corrections, which are not essential here.
Feynman diagrams \((4.5)\) with \(w_f \neq 0\) do not contribute to the partition function in the double scaling (continuum) limit. In addition to the factor \(\mu^{2-2h}\) discussed above, each insertion of a vortex is suppressed by \(1/N\), and the sum over graphs gives rise to a factor \((g - g_c)^{-\alpha}\) where \(1 - \alpha = |w_f| R/2\) is the KPZ scaling dimension of the winding mode \(T_{w_f R} (3.2)\). Since \(\alpha < 1\), the contributions of surfaces with vortices go to zero in the double scaling limit.

Therefore, we conclude that the partition sum \((4.9)\) corresponds in the continuum limit to a sum over discretized worldsheets with no vortices, as claimed above.

4.2. Introducing vortex couplings in the matrix model

To enhance the contribution of vortices in the double scaling limit we replace the left-right invariant Haar measure \([d\Omega]\) on the \(U(N)\) group manifold by

\[
[D\Omega]_{\{\lambda\}} = [D\Omega] \exp \left( \sum_{n \in \mathbb{Z}} \lambda_n \text{tr} \Omega^n \right) \tag{4.10}
\]

which is invariant only with respect to internal automorphisms \(\Omega \rightarrow U^{-1}\Omega U\). The corresponding partition function is

\[
Z_N[\lambda] = \int [D\Omega]_{\{\lambda\}} Z_N(\Omega). \tag{4.11}
\]

Applying the integration rules \((4.7)\) we see that in addition to the contractions discussed above, which vanish in the double scaling limit, we now have new terms where we use the second line of \((4.7)\) to contract a \(\text{tr} \Omega^n\) from the exponent in \((4.10)\) with one of the factors \(\text{tr} \Omega^{w_f}\) in a Feynman graph \((4.3)\). This leads to a factor of \(\lambda_{w_f}\) for each face with vorticity \(w_f\). Therefore, the coupling constants \(\lambda_n\) can be thought of as fugacities of vortices with vorticity \(n\) \((n = \pm 1, \pm 2, ...)\).

One can also use the rules \((4.7)\) to contract the traces in the exponent in \((4.10)\) among themselves. This gives rise to a multiplicative factor \(\exp \sum_{n \geq 1} n \lambda_n \lambda_{-n}\) in the partition function, which is large but “non-universal” in the language of non-critical string theory.

To summarize, the free energy

\[
F_N[\lambda] = \log Z_N[\lambda] \tag{4.12}
\]

is given in the double scaling limit by a sum over all discretized worldsheets containing vortices, where the chemical potential of vortices with vorticity \(n\) is \(\lambda_n\) (up to a multiplicative factor which we will implicitly determine below). We conclude that the matrix model \((4.11)\) gives a discretization of the string theory \((3.19)\).
4.3. Integrating over the twist angles in the double scaling limit

The twisted partition function (4.11) can be calculated in the double scaling limit, where the matrix model simplifies drastically. As is familiar from the usual treatment of two dimensional string theory, in this limit the dynamics takes place near the top of the matrix potential $V(M) (4.2)$, and is dominated by large planar diagrams. Thus, to study the theory in the double scaling limit we can focus on the vicinity of the quadratic maximum of (4.2) and replace the potential by

$$V(M) = - \frac{M^2}{2} \quad \text{after shifting } M \to M - \sqrt{N/g}.$$

The behavior of the potential far from the top enters as a UV cutoff, which is usually taken to be a wall placed at a distance $\Lambda \sim \sqrt{N/g}$ from the top [10,11,12,13,14,15,16]. The twisted partition function (4.1) now reads

$$Z_N(\Omega) = \int_{M(2\pi R)=\Omega} \mathcal{D}M e^{\text{tr} \int_0^{2\pi R} dx \left( - \frac{1}{2} (\partial_x M)^2 + \frac{1}{2} M^2 \right)},$$

(4.13)

where the functional measure $\mathcal{D}M$ is regularized as discussed above.

For generic twist matrix $\Omega$ we actually do not need to regularize the measure because the matrix integral is convergent. In the compactified theory, the cutoff has no effect when the compactification radius is smaller than half the self-dual radius ($R < 1/2$ in our units); we review this argument of [27] in Appendix B. The Gaussian integration gives then a simple determinant depending only on the eigenvalues $z_1,...,z_N$ of the twist matrix $\Omega$, which are phases, $z_j = \exp(i\theta_j)$:

$$Z_N(\Omega) = \prod_{j,k=1}^{N} \frac{1}{|z_j q^{1/2} - z_k q^{-1/2}|},$$

(4.14)

where

$$q = e^{2\pi i R}.$$

(4.15)

Note that the partition sum (4.14) depends on $R$ only through $q$ (4.13), and is also symmetric under $q \to q^{-1}$; the integral over the twist matrix (4.11) preserves this property. Naively, this means that $R$ can be restricted to the interval $0 < R < \frac{1}{2}$. Later we will transform to the grand canonical ensemble, in which we will see that the symmetry is broken due to the appearance of certain divergences.

The next step in evaluating the partition sum is the integration over $\Omega$ in (4.11). Due to the symmetry of the measure with respect to internal automorphisms, the volume of the diagonal $U(N)$ group factors out and the $\Omega$ integral reduces to an integral over its eigenvalues $z_1,...,z_N$:

$$\int [D\Omega]_{\{\lambda\}} Z_N(\Omega) = \frac{1}{N!} \prod_{k=1}^{N} \frac{dz_k}{2\pi i} e^{2u(z_k)} |\Delta(z)|^2 Z_N(\Omega)$$

(4.16)
where $\Delta(z)$ is the Vandermonde determinant (see e.g. Appendix A) and we introduced the potential

$$u(z) = \frac{1}{2} \sum_n \lambda_n z^n.$$  \hfill (4.17)

Plugging (4.14) in (4.16), we find that

$$Z_N[\lambda] = \frac{1}{N!} \oint \prod_{k=1}^N \frac{dz_k}{2\pi i z_k} \frac{e^{2u(z_k)}}{(q^{1/2} - q^{-1/2})} \prod_{j \neq j'}^{N} \frac{z_j - z_{j'}}{q^{1/2}z_j - q^{-1/2}z_{j'}}.$$

Another useful representation of $Z_N$, obtained by using the Cauchy identity

$$\frac{\Delta(a)\Delta(b)}{\prod_{i,j}(a_i - b_j)} = \det_{(i,j)} \left( \frac{1}{a_i - b_j} \right)$$

is

$$Z_N[\lambda] = \prod_{k=1}^N \oint \frac{dz_k}{2\pi i} \det_{(i,j)} \left( \frac{\exp[u(z_i) + u(z_j)]}{z_i q^{1/2} - z_j q^{-1/2}} \right).$$

A natural way to avoid the ambiguity in the contour integration in (4.20) is to add a small imaginary part to the radius $R$ and integrate over the $\{z_k\}$ along the unit circle.\footnote{Note that with this prescription we do not obtain a doubling of the degrees of freedom, typical for matrix models with symmetric potentials.}

The partition function can be calculated by residues for any $N$ and $t_k$, but the formulae get more and more complicated when $N$ grows. Furthermore, the analytical structure of $Z_N$ makes it difficult to formulate a well defined $1/N$ expansion.

4.4. The grand canonical partition function as a $\tau$-function of the Toda chain hierarchy

The abovementioned problems can be overcome by considering instead of (4.18) the grand canonical partition function, where the cosmological constant $\mu$ becomes the chemical potential. Below we will show that the latter is a $\tau$-function of the infinite Toda chain hierarchy \cite{bms} and as such it satisfies the Toda equation (with appropriate boundary conditions), which plays the same role as the Painlevé equation of the KdV hierarchy in the $c=0$ string theory. It will be convenient to rescale the vortex couplings and define new couplings

$$t_n = \frac{\lambda_n}{q^{n/2} - q^{-n/2}},$$

which are commonly used in the literature on $\tau$-functions.

The grand canonical partition function is defined by

$$Z_{\mu}[\lambda] = \sum_{N=0}^{\infty} e^{2\pi R \mu N} Z_N[\lambda].$$

(4.22)
Using \((4.20)\), one can show that the grand canonical partition function \((4.22)\) can be represented as a Fredholm determinant
\[
\mathcal{Z}_\mu = \text{Det}(1 + e^{\mu 2\pi R \hat{K}}),
\]
where the integral operator \(\hat{K}\) is defined as
\[
(\hat{K} f)(z) = -\oint dz' \frac{e^{u(z)+u(z')}}{q^{1/2}z - q^{-1/2}z'} f(z').
\]

In Appendix \(C\), the partition function \((4.23)\) is identified as a solitonic \(\tau\)-function of the Toda chain hierarchy.\footnote{Usually a solitonic \(\tau\)-function is defined by a finite sum instead of an integral. Our partition function is a particular case of the continuum limit (infinitely many solitons) of the soliton solution referred to as “general solution” in ref. \[39\].}

Explicitly the \(\tau\)-function with charge \(l\) is defined as
\[
\tau_l[t] = e^{-\sum_n n t_n} \sum_{N=0}^\infty (q^l e^{2\pi R \mu})^N \mathcal{Z}_N[t] = e^{-\sum_n n t_n} \mathcal{Z}_{\mu+il}[t] \tag{4.24}
\]

The grand canonical partition function \((4.22)\) is obtained by taking \(l = 0\). The \(\tau\)-function with charge \(l\) can be obtained from \((4.22)\) by replacing \(\mu \to \mu + il\).

### 4.5. Toda equation for the free energy

The \(\tau\) functions \(\tau_l\) satisfy a hierarchy of Toda lattice equations which are differential equations in \(\{t_n\}\) and finite difference equations in \(l\). This infinite set of equations is generated by the corresponding Hirota identity, and reflects the \(Gl(\infty)\) symmetry of the system. This symmetry can be exhibited by representing the system in terms of chiral free fermions. The lowest equation from the Toda chain hierarchy is
\[
\tau_l(\partial_+ \partial_- \tau_l) - (\partial_+ \tau_l)(\partial_- \tau_l) + \tau_{l+1} \tau_{l-1} = 0, \tag{4.25}
\]
where we denoted \(\partial_\pm = \partial/\partial \lambda_\pm\) and \(\lambda_\pm = t_{\pm 1} = \pm \frac{\lambda_{\pm 1}}{2i \sin(\pi l/R)}\).

As mentioned above, the \(\tau\)-function depends on the cosmological constant \(\mu\) and the charge \(l\) via the combination \(\mu + il\)
\[
\tau_l(\mu) = \tau_0(\mu + il). \tag{4.26}
\]

We will restrict ourselves to the case \((5.20)\) (note that the constants \(\lambda_\pm\) are related to \(\lambda_{\pm 1}\) by \((4.21)\)), for which
\[
\tau_0(\mu) = e^{-\lambda_+ - \lambda_-} \mathcal{Z}_\mu(\lambda). \tag{4.27}
\]
Winding number conservation implies that the τ-function depends only on the product \( \lambda^2 = \lambda_+ \lambda_- \), so that

\[
\partial_+ \partial_- = \frac{1}{4} \lambda^{-1} \partial_\lambda \lambda \partial_\lambda, \quad \lambda = \sqrt{\lambda_+ \lambda_-}.
\]

The free energy \( F(\lambda, \mu) = \log Z_\mu(\lambda) \) of the matrix model satisfies

\[
\partial_+ \partial_- F(\lambda, \mu) + \exp \left[ \mathcal{F}(\lambda, \mu + i) + \mathcal{F}(\lambda, \mu - i) - 2\mathcal{F}(\lambda, \mu) \right] = 1. \tag{4.28}
\]

In the double scaling limit \( \mathcal{F}(\lambda, \mu) \) is assumed to be a smooth function of \( \mu \). Thus we can rewrite (4.28) as

\[
\partial_+ \partial_- F(\lambda, \mu) + \exp \left( -4 \sin^2 \left( \frac{1}{2} \partial_\mu \right) \mathcal{F}(\lambda, \mu) \right) = 1. \tag{4.29}
\]

Differentiating twice w.r.t. \( \mu \), we get an equation for the susceptibility \( \chi(\lambda, \mu) = \partial_\mu^2 F(\lambda, \mu) \):

\[
\partial_+ \partial_- \chi(\lambda, \mu) + \partial_\mu^2 \exp \left[ - \left( \frac{\sin \left( \frac{1}{2} \partial_\mu \right)}{\frac{1}{2} \partial_\mu} \right)^2 \chi(\lambda, \mu) \right] = 0. \tag{4.30}
\]

The operator in the exponent is to be understood as a power series in \( \partial_\mu \):

\[
\frac{\sin \left( \frac{1}{2} \partial_\mu \right)}{\frac{1}{2} \partial_\mu} = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{2^{-2n}}{(2n + 1)!} \partial_\mu^{2n}. \tag{4.31}
\]

Note that the differential equations (4.29), (4.30) do not depend on \( R \) explicitly. However, to solve them we need to supply boundary conditions. A convenient choice is to specify the free energy \( \mathcal{F} \) at \( \lambda = 0 \), and use the differential equations to solve for the \( \lambda \) dependence. Since \( \lambda = 0 \) corresponds to the standard \( c = 1 \) matrix model, \( \mathcal{F}(\mu) \equiv \mathcal{F}(\lambda, \mu)|_{\lambda=0} \) is known. We next review its form.

4.6. Boundary conditions

The \( c = 1 \) string theory without vortices corresponds to \( u(\Omega) = 0 \) (see (4.17)). The integral in (4.20) can be evaluated by residues and gives

\[
Z_N = q^{N^2/2} \prod_{n=1}^{N} \frac{1}{1 - q^n}. \tag{4.32}
\]

This formula makes sense for the usual matrix harmonic oscillator, but for the inverted oscillator it requires an analytic continuation. Fortunately, the corresponding grand canonical partition function (4.22) in the absence of vortices (all \( t_m = 0 \)) can be expressed as a partition function of fermions in terms of the density \( \rho(E) \) of the energy levels of the
inverted harmonic oscillator (see [25]). The (universal part of the) grand canonical free energy of the \( c = 1 \) string theory compactified on a circle of radius \( R \) has a well defined (asymptotic) \( 1/\mu \) expansion:

\[
\mathcal{F}(\mu) = \log Z_\mu[0] \\
= \frac{1}{2} \int_{-\infty}^{\infty} dE \rho(E) \log \left( 1 + e^{-2\pi R(\mu-E)} \right) \\
= \frac{1}{2\pi} \int_{-\infty}^{\infty} dE \sum_{k=0}^{\infty} \frac{k + \frac{1}{2}}{E^2 + (k + \frac{1}{2})^2} \log(1 + e^{-2\pi R(\mu-E)}) \\
= -\frac{R}{2\mu^2} \log \mu - \frac{1}{24} (R + \frac{1}{R}) \log \mu + R \sum_{k=2}^{\infty} \frac{\mu^{-2(k-1)} f_k(R)}{k} + O(e^{-2\pi \mu}),
\]

where we introduced the polynomials in \( \frac{1}{R} \)

\[
f_k(R) = (2k - 3)! \left( \frac{1}{R} \right)^{2n} \frac{(2^{2(k-n)} - 2)(2^{2n} - 2)B_{2(k-n)}B_{2n}}{[2(k-n)]![2n]!}
\]

and \( B_n \) are Bernoulli numbers. The partition function (4.33) has a T-duality symmetry, \( R \rightarrow \frac{1}{R}, \mu \rightarrow R\mu \). This symmetry is broken in the presence of vortices (i.e. for \( \lambda \neq 0 \)).

4.7. Canonical versus Grand Canonical partition sum

The sum over worldsheets in the continuum string theory is described in the matrix model by the \( 1/N \) expansion of the canonical free energy \( \mathcal{F}_N = \log Z_N \). The formalism described in the previous subsections allows one to compute the grand canonical free energy \( \mathcal{F}(\mu) = \log Z_\mu \), which is related to \( \mathcal{F}_N \) via

\[
\exp \mathcal{F}_N = \oint \frac{d\mu}{2\pi i} \exp[2\pi R \mu N + \mathcal{F}(\mu)]
\]

where the integration contour encircles the point \( e^{-2\pi R\mu} = 0 \).

As is well known [14,15,16], the \( 1/N \) expansion of \( \mathcal{F}_N(\lambda) \) can be rearranged as a \( 1/\mu_0 \) expansion, with \( \mu_0 \) defined by \( \mu_0 \log(\mu_0/\Lambda) = N(g_c - g) \). The grand canonical free energy \( \mathcal{F}(\lambda = 0, \mu) \) has a \( 1/\mu \) expansion given by (4.33). Similarly, in section 5 we will show that \( \mathcal{F}(\lambda, \mu) \) has a \( 1/\mu \) expansion which is obtained from (4.29). One can show that in the double scaling limit, the \( 1/\mu_0 \) expansion of \( \mathcal{F}_N \) coincides with the \( 1/\mu \) expansion of \( \mathcal{F}(\mu) \), up to a flip of the sign of the leading (spherical) contribution to \( \mathcal{F} \) (see [27], revised version, section 6).

In the next section we will construct the \( 1/\mu \) expansion of the grand canonical free energy, and use the above discussion to identify it with the genus expansion of the partition sum of the continuum theory.

\[\text{18} \] This sign flip is a standard feature of the Legendre transform.
5. Some results on the genus expansion of the free energy

The purpose of this section is to solve the differential equations (4.29), (4.30) with the boundary conditions (4.33). We have not found an exact solution of these equations, but as we will see below, one can solve them iteratively, in a genus expansion,

\[ F(\lambda, \mu) = \sum_{h=0}^{\infty} F_h(\lambda, \mu). \]  

(5.1)

The genus \( h \) partition sum \( F_h \) must, by KPZ-DDK scaling, have the form

\[ F_h(\lambda, \mu) = -\delta_{h,0} R^2 \mu^2 \log \mu - \frac{\delta_{h,1}}{24}(R + \frac{1}{R}) \log \mu + \mu^{2-2h} A_h(z), \]  

(5.2)

where \( z \) is defined in (3.9). The boundary conditions (4.33) determine \( A_h(z) \) at \( z = 0 \). Plugging the ansatz (5.2) into (4.29), (5.1) one can solve for \( A_h(z) \). It is remarkable that the Toda equations allow such an ansatz at all; this is due to their conformal symmetry.

At tree level one finds an ODE for \( A_0(z) \) which together with the boundary conditions fixes it uniquely. At genus \( h \) one finds a differential equation that mixes \( A_h(z) \) with \( A'_{h}(z) \) with \( h' < h \). These equations can be used to determine all \( A_h \) iteratively given \( A_h(0) \).

We will actually perform the calculations by transforming from the variables \((\mu, z)\) to \((\lambda, y)\), where \( y \) is defined in (3.17). In these variables the genus expansion is an expansion in inverse powers of \( \lambda^{2/R} \), and the boundary conditions are specified at \( y = \infty \). The genus \( h \) free energy in (5.1) takes the form

\[ F_0(\lambda, \mu) = (\eta \lambda^{2/R})^2 \left( -\frac{R}{2 - R} y^2 \log(\lambda \sqrt{R - 1}) + f_0(y) \right) \]

\[ F_1(\lambda, \mu) = C \log \lambda + f_1(y) \]

\[ F_h(\lambda, \mu) = (\eta \lambda^{2/R})^{2-2h} f_h(y) \quad (h \geq 2), \]  

(5.3)

where, as in (3.17),

\[ y = z^{\frac{1}{2-R}} = \frac{\mu}{\lambda^{2/(2-R)} \cdot \eta}, \quad \eta = (R - 1)^{1/(2-R)}. \]

The genus expansion corresponds to fixed \( y \) with \( \mu, \lambda \to \infty \). The logarithmic term in \( F_1 \) is a zero mode of the Toda equation (4.28). The constant \( C \) will be determined later.

We note in passing that \( C \) is a special case of a large class of zero modes of the Toda equation (4.28), whose general form is

\[ \Delta F = \sum_{n=0}^{\infty} (C_n + D_n \log \lambda) e^{-2n \pi \mu}. \]

19 The fact that the r.h.s. of (4.29) is one, leads to an additive contribution \( \lambda_+ \lambda_- \) to \( F \). This contribution is non-universal and will be dropped below.
For $n \neq 0$ these terms are non-perturbative in the genus expansion. The $\{C_n\}$ are nothing but the non-perturbative terms in the standard $c = 1$ string theory (4.33). Their presence is related to the $\sim (2h)!$ divergence of the coefficients $f_h$ in the perturbative expansion (5.3). The status of non-perturbative terms in two dimensional string theory is generally unclear. Here we are interested in the perturbative expansion of the free energy and will therefore drop all exponential corrections.

Differentiating (5.3) twice w.r.t $\mu$ we find the susceptibility,

$$\chi(\lambda, \mu) = \sum_h \chi_h(\lambda, \mu)$$

$$\chi_0(\lambda, \mu) = -\frac{2R}{2 - R} \log \left( \lambda \sqrt{R - 1} \right) + X_0(y), \quad (5.4)$$

$$\chi_h(\lambda, \mu) = \left( \eta \lambda \frac{R}{2 - R} \right)^{2h} X_h(y) \quad (h \geq 1),$$

with $f_h$ and $X_h$ related as

$$\partial_y^2 f_h(y) = X_h(y). \quad (5.5)$$

5.1. Partition function on the sphere

Plugging (5.4) into (4.30) and retaining in the spherical approximation only the $h = 0$ term (and only the leading term in (4.31)) we obtain the following ODE defining the string susceptibility $X_0(y)$

$$\alpha (y \partial_y)^2 X_0 + \partial_y^2 e^{-X_0} = 0 \quad (5.6)$$

where $\alpha = (R - 1)/(R - 2)^2$. The solution is

$$y = e^{-\frac{X_0}{R}} - e^{\frac{1 - R}{R} X_0} \quad (5.7)$$

reproducing Moore’s result (3.11)–(3.12). This expression can be checked by substitution or found by a direct solution of (5.6) (see Appendix D). The integration constant is fixed by comparing to the leading term in (4.33).

Using (5.7) and integrating $X_0$ twice we obtain the following parametrization of the partition function on the sphere:

$$\mathcal{F}_0(\lambda, \mu) = -\frac{1}{2} \mu^2 \left( \frac{2R}{2 - R} \log(\lambda \sqrt{R - 1}) - X_0 \right)$$

$$+ \eta^2 \lambda^{4/(2-R)} \left( \frac{3}{4} \frac{R}{R - 1} e^{-\frac{R - 1}{2R} X_0} + \frac{3}{4} (R - 1) e^{-\frac{R}{2(R - 1)} X_0} - R^2 - R + 1 e^{-X_0} \right) \quad (5.8)$$

with $X_0(y)$ defined from (5.7).
Consider the limits \( y \to 0, \infty \) in (5.8). In the limit \( y \to \infty (\mu \to \infty \) or \( \lambda \to 0 \) (5.8) reproduces the known asymptotics of the \( c = 1 \) string theory perturbed by vortices:

\[
\mathcal{F}_0 \simeq -\frac{1}{2} R \mu^2 \log \mu - \lambda^2 \mu^R + O(\lambda^4 \mu^{2R-2}), \quad \mu \to \infty
\]

(5.9)

whereas in the Sine Liouville limit \( \lambda \to \infty \) with \( \mu \) fixed we obtain the asymptotics

\[
\mathcal{F}_0 \simeq -A \lambda^{4/(2-R)} - B \mu \lambda^{2/(2-R)} + O(\mu^2), \quad \lambda \to \infty.
\]

(5.10)

where \( A = \frac{1}{4} (2 - R)^2 (R - 1)^{R/(2-R)} \) and \( B = (R - 1)^{1/(2-R)} (2-R)^3 \frac{R(R-1)}{2(2-R)} \).

Note that the dependence of the partition sum on \( \lambda \) is in agreement with the KPZ-DDK analysis in Sine-Liouville theory. In particular, for the black hole radius \( R = 3/2 \), we find \( \mathcal{F}_0 \sim \lambda^8 \), in agreement with (2.16). It is interesting that the partition sum on the sphere goes like an even integer power of \( \lambda \). This is a non-universal contribution to the partition sum of Sine Liouville theory, which corresponds to condensation of vortices. Recall that \( \lambda \) is the fugacity of vortices, and a finite integer power corresponds to a contribution of a finite number of vortices and antivortices. Hence it should be dropped from the physical answer. As we shall see in section 6, this has an important physical consequence: the free energy of the two dimensional black hole vanishes in the tree approximation, in agreement with the results of [11].

Not also that the coefficient \( A \) is complex for \( R < 1 \), in agreement with our earlier analysis – the imaginary part for \( R < 1 \) signals the instability of Sine-Liouville (with real \( \lambda \)) in this range.

5.2. Linear equations for any genus

Substituting (5.3) into (4.30) we obtain the following ODE for \( f_h \)

\[
\alpha(y \partial_y + 2h - 2)^2 f_h - e^{-X_0(y)} \partial_y^2 f_h = \mathcal{H}(f_0, \cdots, f_{h-1}).
\]

(5.11)

Here \( X_0(y) \) is defined by (5.7) and the function \( \mathcal{H}(f_0, \cdots, f_{h-1}) \) is defined as

\[
\mathcal{H}(f_0, \cdots, f_{h-1}) =
\]

\[
- \left[ (\sqrt{R - 1} \lambda)^{-\frac{4-R}{2-R}} \exp \left( -4 \sin^2 \left( \frac{1}{2} \partial_y \right) \sum_{k=0}^{h-1} (\sqrt{R - 1} \lambda)^{-4k/(2-R)} f_k \right) \right]_0,
\]

(5.12)

where by \([\cdots]_0\) we denoted the constant term in the limit \( \lambda, \mu \to \infty \) with \( y \) fixed.

We see that the equations for \( \{ f_h \} \) form a triangular system of linear second order ODE’s with \( y \) dependent coefficients. We do not know how to solve them in general; in the next subsection we will describe the solution for the case of genus one (the torus), where the equation simplifies.

\footnote{Note that a more natural parameter would be \( \tilde{\lambda} = \sqrt{R - 1} \lambda \); indeed, looking back at (5.8), (3.3) we see that the dependence of the partition sum on \( \lambda \) is through \( \tilde{\lambda} \). In terms of \( \tilde{\lambda} \) one finds that \( \mathcal{F}_0 \simeq -\frac{1}{4\alpha} \tilde{\lambda}^{4/(2-R)} \).}
5.3. Partition function on the torus

Eqs. (5.11) and (5.12) lead to the following first order linear inhomogeneous ODE for the torus partition function \( f_1(y) \)

\[
(\alpha y^2 - e^{-X_0}) f''_1 + \alpha y f'_1 = -\frac{1}{12} X''_0 e^{-X_0}.
\]  

(5.13)

Note that (5.13) is a first order differential equation for \( f'_1 \). Integrating this equation and using the boundary condition at large \( \mu \) (4.33) to fix the integration constant (see Appendix D) we obtain (up to a non-universal constant)

\[
F_1(\lambda, \mu) = \frac{R + R^{-1}}{24} \left( \frac{R^{-1} X_0 - \frac{2}{2 - R} \log(\lambda \sqrt{R - 1})}{2 - R} \right) - \frac{1}{24} \log \left( 1 - (R - 1)e^{\frac{2\alpha}{\mu^2} X_0} \right)
\]  

(5.14)

with \( X_0(y) \) defined by (5.7). In the limit \( \mu \to \infty \) with \( \lambda \) fixed we reproduce from (5.14) the known asymptotics of the \( c = 1 \) string theory perturbed by vortices:

\[
F_1 \simeq -\frac{R + R^{-1}}{24} \log \mu + \frac{R^2 - 1}{24R} \lambda^2 \mu^{R-2} + \cdots, \quad (\mu \to \infty)
\]  

(5.15)

whereas in the Sine-Liouville limit \( \lambda \to \infty \) and \( \mu \) fixed we obtain the asymptotics

\[
F_1 \simeq -\frac{R + R^{-1}}{12(2 - R)} \log(\lambda \sqrt{R - 1}) - \frac{R - 1}{24(2 - R)} \frac{\mu}{((R - 1)\lambda^2)^{1/(2 - R)}} + \cdots, \quad (\lambda \to \infty).
\]  

(5.16)

The expansion (5.15) of the solution (5.14) can be compared with the explicit expressions for genus one tachyon correlation functions in \( c = 1 \) string theory obtained in [24]. Since [24] studied momentum modes, to compare one has to perform a T-duality transformation \( \tilde{R} = 1/R, \tilde{\mu} = R\mu \). After this redefinition, the results of [24] read:

\[
\langle T^n R^n \rangle_{\tilde{h} = 1, \tilde{R}} = -\frac{1}{\tilde{R}} \frac{n!}{24} [(1 - R)\mu^{R-2}]^n ((-)^n \tilde{f}_n(R) + g_n \tilde{R}^2)
\]  

(5.17)

where

\[
g_n(R) = \frac{\Gamma(n(2 - R))}{\Gamma(n(1 - R) + 1)}
\]

and

\[
f_1(R) = R^2 - R - 1,
\]

\[
f_2(R) = 3R^3 - 8R^2 + 3R + 3,
\]

\[
f_3(R) = 17R^4 - 72R^3 + 90R^2 - 17R - 20,
\]

etc.

For low \( n \) one finds

\[
\langle T^1 T^- \rangle_{h = 1} = \frac{R^2 - 1}{24R} \mu^{R-2}, \quad \langle T^2 T^- \rangle_{h = 1} = -\frac{1}{12R} (1 - R)^2 (R^3 - 5R^2 + 3R + 3) \mu^{2R-4}.
\]  

(5.18)

This is exactly what one obtains by expanding the solution (5.14).
5.4. Comparison to continuum calculations

Most of the matrix model results described above are difficult to derive using continuum methods. In this subsection we will discuss two cases in which the matrix model predictions can actually be obtained directly in the continuum formulation; the non-vanishing of the partition sum on the sphere and the precise value of the partition sum on the torus.

Consider first the genus zero case. In the matrix model we found a non-zero partition sum given by (5.10) (with \( R = 3/2 \) for the Euclidean black hole case). In the continuum analysis on the cigar\(^{21}\) this can be reproduced as follows.

In string theory the partition sum on the sphere is usually said to vanish, due to the volume of the Conformal Killing Group (CKG) of the sphere, \( SL(2, C) \). If the target space is non-compact, the partition sum is actually proportional to \( V/\text{vol}(SL(2, C)) \) where \( V \) is the divergent volume of spacetime. Thus, at first sight the partition sum is \( \infty/\infty \), i.e. it seems to be ill defined. However, in most situations one is actually interested in the partition sum per unit volume. E.g. if the vacuum is translationally invariant in the non-compact directions, the partition sum per unit volume is the Lagrangian density in this vacuum (the classical cosmological constant), and it vanishes due to the volume of the CKG.

In the Euclidean cigar background \( H^+_3/U(1) \), the above discussion has to be reexamined. There is no reason to divide by the volume of the cigar since the background is not translationally invariant in \( \phi \), and in any case for comparison with the matrix model we are interested in the total partition sum and not the partition sum per unit volume. The ratio of the volume of the cigar to the volume of the CKG is finite in this case. The volume of \( H^+_3 \) contains precisely the same kind of divergence as that of the CKG. Since the volume of the \( U(1) \) in (2.11) is finite, we conclude that the partition sum of string theory in the cigar background is non-zero (and proportional to the mass of the black hole \( M \)). We will not attempt to compute its precise value.

Moving on to the torus, we would like to derive the result (5.16) directly in the continuum formalism. We will describe the calculation in the Sine-Liouville language, to study the \( R \) dependence. Similar considerations can be used to perform the calculation on the cigar.

The torus partition sum in the background (2.7) is an example of a bulk amplitude in the language of section 2. By performing the integral over the zero mode of \( \phi \) as in eq. (2.21) and using (2.23) one finds

\[
Z_{\text{torus}} = -\frac{1}{2 - R} \log \frac{\Lambda}{\lambda} \int \frac{d^2\tau}{\tau_2^2} Z_1(\tau, \bar{\tau}),
\]  

(5.19)

\(^{21}\) A similar analysis can be performed directly in Sine-Liouville theory.
where $Z_1(\tau, \bar{\tau})$ is the torus partition sum of the theory with $\lambda = 0$, i.e. string theory on an infinite cylinder of radius $R$. $Z_1$ can be computed by using free field techniques; the linear dilaton term in (2.7) can be ignored in this case since the curvature of the torus vanishes. Equation (5.19) has a simple interpretation: the bulk of the contribution to the torus partition sum comes from the region far from the Sine-Liouville wall. It can be computed by cutting off the infinite region very far from the wall, by restricting to $\phi \leq \phi_{UV}$. As indicated by the notation, this can be thought of as adding a UV cutoff to the theory.

The Sine-Liouville wall acts as an effective IR cutoff, restricting
\[
\lambda e^{-(2-R)\phi} \leq 1, \quad \text{or} \quad \phi \geq \frac{1}{2-R} \log \lambda. \quad (5.20)
\]

Thus, the length of the cylinder is
\[
L_\phi \simeq \phi_{UV} - \frac{1}{2-R} \log \lambda. \quad (5.21)
\]

The prefactor of the modular integral in (5.19) is hence nothing but the (universal part of the) length of the cylinder, and the integral computes the free energy of perturbative string modes (tachyons) living on the cylinder.

This makes it clear that the coefficient of $-\frac{1}{2-R} \log \lambda$ in (5.19) must be the same as the coefficient of $-\frac{1}{2} \log \mu$ in the torus partition sum in standard $c = 1$ string theory, which has been computed by performing the integral (5.19) (see e.g. [15]). As we see in (4.33), this coefficient is $(R + \frac{1}{R})/12$, in agreement with (5.16).

6. Thermodynamics of two dimensional string theory

Our results from the previous sections can be used to study the thermodynamics of two dimensional string theory. We start by reviewing the low temperature thermodynamics (corresponding to $R > 2$ in the notations of the previous sections), and then describe the physics for temperatures slightly below and slightly above the Hagedorn temperature, which corresponds to $R = 3/2$. We also comment on the behavior in the intermediate region $3/2 < R < 2$.

6.1. Low temperature thermodynamics, $R > 2$

The canonical partition sum $Z(\beta)$ is obtained by computing the string path integral with Euclidean time compactified on a circle of circumference $\beta = 2\pi R$. The precise relation is
\[
\log Z(\beta) \equiv -\beta F(\beta) = Z_{\text{string}}(R), \quad (6.1)
\]
where $F$ is the free energy and $Z_{\text{string}}$ is the string partition sum, which is given by a sum over connected Riemann surfaces. For two dimensional string theory, below the Kosterlitz-Thouless temperature (i.e. for $R > 2$ in our units) one can ignore winding modes, i.e. set $t_n = 0$ in (3.19), and the free energy (6.1) is given by the familiar formula (4.33)

$$
Z_{\text{string}}(R) = -\frac{R}{2} \mu^2 \log \frac{\mu}{\Lambda} - \frac{1}{24} \left( R + \frac{1}{R} \right) \log \frac{\mu}{\Lambda} + \cdots. \quad (6.2)
$$

We have written the first two terms in the $1/\mu$ (or genus) expansion. $\Lambda$ is a large scale $\Lambda >> \mu$ corresponding to a UV cutoff.

Comparing to (6.1), we see that the free energy $F(\beta)$ is given by

$$
F(\beta) = \frac{1}{2\pi} \left( \frac{1}{2} \mu^2 \log \frac{\mu}{\Lambda} + \frac{1}{24} \left( 1 + \frac{1}{R^2} \right) \log \frac{\mu}{\Lambda} + \cdots \right). \quad (6.3)
$$

The temperature independent terms in (6.3) can be attributed to vacuum energy. Indeed, the energy

$$
E = \frac{\partial (\beta F)}{\partial \beta} = \frac{1}{2\pi} \left( \frac{1}{2} \mu^2 \log \frac{\mu}{\Lambda} + \frac{1}{24} \left( 1 - \frac{1}{R^2} \right) \log \frac{\mu}{\Lambda} + \cdots \right) \quad (6.4)
$$

goes as $R \to \infty$ to the ground state energy

$$
E_0 = \frac{1}{2\pi} \left( \frac{1}{2} \mu^2 \log \frac{\mu}{\Lambda} + \frac{1}{24} \log \frac{\mu}{\Lambda} + \cdots \right) \quad (6.5)
$$

and it is natural to subtract it from both $E$ and $F$. This leads to

$$
E = -\frac{1}{2\pi} \frac{1}{24R^2} \log \frac{\mu}{\Lambda} + \cdots \quad (6.6)
$$
$$
F = +\frac{1}{2\pi} \frac{1}{24R^2} \log \frac{\mu}{\Lambda} + \cdots.
$$

We can now also use the thermodynamic relation

$$
-\beta F = S - \beta E \quad (6.7)
$$

to determine the entropy,

$$
S = \beta (E - F) = -\frac{1}{12R} \log \frac{\mu}{\Lambda} + \cdots. \quad (6.8)
$$

Eqs. (6.6), (6.8) have an obvious interpretation: they correspond to thermodynamics of a single massless $1 + 1$ dimensional field which lives in a spatial volume

$$
V_L = -\log \frac{\mu}{\Lambda}. \quad (6.9)
$$

Indeed, solving (6.6) for $R$ in terms of $E$, $V_L$, and substituting in (6.8), one finds

$$
S = \sqrt{\frac{2\pi EV_L}{6}} \quad (6.10)
$$

which is the standard result for a single massless scalar field in $1+1$ dimensions. The scalar field in question is of course the massless “tachyon” of $1 + 1$ dimensional string theory, the only perturbative field theoretic degree of freedom. Thus, we conclude that the low temperature thermodynamics corresponds to a gas of tachyons.
6.2. Near-Hagedorn thermodynamics, $A$: $R > 3/2$

As discussed in the introduction, the high energy density of states of two dimensional string theory has the Hagedorn form (1.2). Thus, it is interesting to consider the thermodynamics in the vicinity of the Hagedorn temperature $T_H = 1/\beta_H$. As we show below, $\beta_H = 2\pi R_H$, with $R_H = 3/2$. The thermodynamics in the vicinity of the Hagedorn temperature is quite sensitive to the value of the constant $s_1$ in (1.2). We will see below that

$$-2 < s_1 < -1.$$  \hspace{1cm} (6.11)

For now we will take (6.11) for granted and study its consequences for the thermodynamics at temperatures slightly below $T_H$. In the next subsection we discuss the situation for temperatures slightly above $T_H$, and in the process determine both $T_H$ and $s_1$.

Consider the canonical partition sum

$$Z(\beta) = \int_0^\infty dM \rho(M) e^{-\beta M}. \hspace{1cm} (6.12)$$

The high energy behavior of $\rho(M)$, given by (1.2), implies that the integral converges for $T < T_H$, but the exponential Boltzmann suppression disappears as $\beta \to \beta_H$. Naively, one might expect that in this limit the integral (6.12) becomes dominated by high energy states, so that

$$Z(\beta) \sim (\beta - \beta_H)^{-s_1-1}, \hspace{1cm} (6.13)$$

but this is not quite the case. The contribution of the high energy part of the spectrum to $Z$ is given in (6.13). The rest of the states give an additive contribution $Z_{\text{low}}(\beta)$ which is analytic as $\beta \to \beta_H$ and approaches a non-zero constant in this limit. The partition sum (6.12) can thus be written for $\beta$ slightly larger than $\beta_H$ as

$$Z(\beta) = Z_{\text{low}}(\beta) + c(\beta - \beta_H)^{-s_1-1}, \hspace{1cm} (6.14)$$

where $c$ is a constant. Since $s_1 + 1$ is negative (see (6.11)), the contribution of the high energy states (the last term in (6.14)) goes to zero, and the partition sum (6.12) is in fact dominated by states with moderate energies.

Naively, the Hagedorn temperature appears to be a limiting temperature, since the energy as measured in the canonical ensemble diverges:

$$E = -\frac{\partial \log Z}{\partial \beta} \sim (\beta - \beta_H)^{-s_1-2}. \hspace{1cm} (6.15)$$

Note that $-s_1 - 2$ is negative (see (6.11)); thus the derivative of the second term on the r.h.s. of (6.14) is much larger than that of the first. Hence, it appears that one needs to supply an infinite amount of energy to heat up the system to the Hagedorn temperature.
However, one has to be careful with this conclusion, since the whole notion of the canonical ensemble breaks down as one approaches the Hagedorn temperature from below, since the fluctuations around the mean energy are large:

$$\frac{\langle E^2 \rangle - \langle E \rangle^2}{\langle E \rangle^2} \sim (\beta - \beta_H)^{s_1+1}. \quad (6.16)$$

In such situations the system should be studied in the microcanonical ensemble, where the energy is fixed, and the temperature is given by

$$\beta = \frac{\partial S}{\partial M}. \quad (6.17)$$

Using the entropy (1.2) we find

$$\beta = \beta_H + \frac{s_1}{M} + O\left(\frac{1}{M^2}\right). \quad (6.18)$$

Since $s_1$ is negative (see (6.11)), we find that at high energies the temperature is above the Hagedorn temperature. Clearly, this does not mean that temperatures below the Hagedorn one are not achievable in two dimensional string theory – as we reviewed in the previous subsection, the low temperature thermodynamics is perfectly sensible. Instead, (6.18) implies that the microcanonical thermodynamics slightly below $T_H$ corresponds to a finite energy $M$, obtained by solving (6.17).

Thus, if we imagine a process where two dimensional string theory is gradually heated up by supplying energy to the system in the microcanonical ensemble, the temperature $T_H$ is reached at a finite value of the energy $M$. This suggests that the Hagedorn temperature is associated with a phase transition (as opposed to being a limiting temperature).

6.3. Near-Hagedorn thermodynamics, B: $R < 3/2$

In the previous subsection we saw that just below the Hagedorn temperature the thermodynamics is dominated by states with moderate energies. Since two dimensional black holes should only dominate the thermodynamics at large energies, it is clear that the low temperature phase is not well described by perturbative string theory in the Euclidean black hole background (1.1), even as $T \rightarrow T_H$. Instead, as we show in this subsection, string theory in the background (1.1) describes the thermodynamics just above the Hagedorn temperature.

For very large $M$, the thermodynamics is obtained by studying classical string theory on the cigar. The mass $M$ is related to the value of the string coupling at the tip of the cigar, while the temperature is determined by the asymptotic radius of the cigar. Since the latter is independent of the former, we conclude [41] that

$$\beta = \frac{\partial S}{\partial M} = \beta_H = 2\pi \frac{3}{2} \quad (6.19)$$
is constant and the entropy-energy relation is

\[ S = \beta_H M \] (6.20)
i.e. the density of states exhibits Hagedorn growth, \( \rho(M) \sim \exp(\beta_H M) \) at large \( M \). The free energy (6.7) should thus vanish in the classical approximation.

Quantum string theory on the cigar leads to \( 1/M \) corrections to (6.19), (6.20), whose general form is (see (1.2))

\[
S = \beta_H M + s_1 \log M + O \left( \frac{1}{M} \right), \\
\beta = \beta_H + \frac{\alpha}{M} + O \left( \frac{1}{M^2} \right),
\] (6.21)

where \( s_1, \alpha \) are constants to be determined.

For \( \beta < \beta_H \), the integral (6.12) defining the canonical partition sum diverges. At first sight this looks like a problem, but in fact it is standard in black hole theory. For example, black holes in flat spacetime generally have an entropy that grows like \( M^a \) with \( a > 1 \), so the analog of (6.12) diverges for all temperatures. The Gibbons-Hawking prescription [42], whose generalization to string theory we are studying here, defines the canonical partition sum by a formal saddle point evaluation of the integral (6.12). The imaginary part of the resulting partition sum is associated with the thermodynamic instability of these black holes (their negative specific heat), and is related to the appearance of an unstable mode (a tachyon) in the Euclidean black hole background.

We will see that something very similar happens here. Two dimensional string theory is thermodynamically unstable for \( T > T_H \). Nevertheless, just like in [42], we can compute the free energy by a formal saddle point evaluation of (6.12). The Euclidean black hole background has a tachyon, whose condensation needs to be understood in order to study the high temperature phase of two dimensional string theory.

It is not difficult to show that a saddle point evaluation of the integral (6.12) with the density of states (1.2) gives the result (6.13). The saddle point is located at \( M \) given by (6.18), and assuming that \( s_1 \) is indeed negative (6.14), as we will find soon, we see that for \( \beta \) slightly below \( \beta_H \) it is located at a large positive value of \( M \), which is consistent with our use of the asymptotic density of states (1.2) to find it.

Since the partition sum is given by (6.13), the free energy is

\[ -\beta F = -(s_1 + 1) \log(\beta - \beta_H) + O(1). \] (6.22)

The energy-temperature relation (in the canonical ensemble) is obtained from (6.22) by using

\[ M = \frac{\partial(\beta F)}{\partial \beta} = \frac{s_1 + 1}{\beta - \beta_H}. \] (6.23)
Note that it is different from the relation (6.18), which is valid in the microcanonical ensemble. The two are not the same here because of the large energy fluctuations in the canonical ensemble.

Substituting (6.23) in (6.22) we conclude that
\[-\beta F = (s_1 + 1) \log M + O(1). \quad (6.24)\]

Thus, by computing the string partition sum (6.1) we can deduce the entropy-energy relation characterizing the black hole, (6.21).

An immediate puzzle is that while we expect the tree level contribution to the free energy (the term that goes like \(M\) in (6.24)) to vanish, our explicit calculation in section 5 gave in fact a finite answer for this coefficient. This apparent paradox is resolved by noting [41] that the thermodynamic relation (6.7) is modified in this case by the presence of a massless scalar field, the dilaton, that mediates a long range force. The black hole spacetime is in fact charged under the dilaton current \(J^a = \epsilon^b_a \nabla_b e^\phi\), and therefore the thermodynamic relation (6.7) contains a correction that is proportional to the dilaton charge of spacetime. It was shown in [41] that this gives rise to a term in the free energy that goes like \(M\) (times a constant that is ambiguous in the gravity approximation), and should be interpreted as a chemical potential for the dilaton charge.

Therefore, we believe that the correct procedure for studying the thermodynamics of two dimensional string theory is to subtract this term from \(F\). Note that in the Sine-Liouville description, this is a term that goes like \(\lambda^8\) (see (5.10)). In general in non-critical string theory, such terms are due to physics far from the Liouville (or in this case Sine-Liouville) wall and are “non-universal” (as noted at the end of section 5.1). In our particular case one could alter the coefficient of this term by adding a constant (independent of \(\mu\)) to the eight point function \(\langle T^4_R T^4_{-R} \rangle\) given in (3.3); such a constant is non-universal in Liouville theory (or the double scaled matrix model).

As further justification for the validity of this procedure we discuss in the next section a generalization of two dimensional string theory to the fermionic case (with worldsheet supersymmetry but no spacetime supersymmetry). The physics of this theory is very similar to that of the bosonic one, but the tree level partition sum vanishes.

The one loop correction to the free energy discussed in section 5.3 (see (5.16)) implies that
\[s_1 + 1 = -(R + R^{-1})/48 = -13/288. \quad (6.25)\]
which is in agreement with (5.11).

The high temperature thermodynamics described by the black hole is unstable. As is clear from (6.23), (6.18), the specific heat is negative: as the energy increases, the temperature decreases towards \(T_H\). This is of course closely related to the fact that in the formal saddle point evaluation of the partition sum (6.12), we are in fact expanding
around a maximum of \( f(M) = \beta M - S(M) \) (rather than a minimum). This thermodynamic instability is associated with the presence of a negative mode in the Euclidean path integral, \textit{i.e.} one expects to find a \textit{tachyon} in the cigar background \((\text{[-I]}\)).

Because the instability of the thermodynamics is a one loop effect in the cigar background, one actually expects to find a zero mode in the classical theory, which becomes a negative mode after one loop effects are included. There indeed is a very natural candidate for precisely such a mode which we will describe next. This issue was recently discussed in the context of LST in \([\text{[-II]}]\), so we will be brief.

It is known (see \textit{e.g.} \([\text{[-III]}]\)) that in the cigar geometry there are normalizable states localized near the tip, corresponding to the principal discrete series representations of \(SL(2), \text{i.e. for} \)

\[
|m| = j + l; \quad l = 1, 2, 3, \cdots.
\]

One way to find these states is to study the correlation functions of the non-normalizable observables \(V_{j,m,\bar{m}}\) (see section 2); the states \((6.26)\) give rise to poles in these correlation functions. The classical zero mode (or "massless state") referred to above corresponds to\footnote{Since the result is general, we are writing it as a function of \(k\). For our case, as usual, \(k = 9/4\).}

\[
m = \bar{m} = \frac{k}{2}; \quad j = \frac{k}{2} - 2.
\]

It is easy to check that this combination of \(j\) and \(m\) satisfies \(\Delta_{j,m,\bar{m}} = \bar{\Delta}_{j,m,\bar{m}} = 1\) (see \((2.3)\)), \textit{i.e.} this is an on shell massless state (a zero mode). As discussed in \([\text{[-III]}]\), it is likely that including one loop effects turns it into a negative mode. It would be interesting to prove this by an explicit one loop calculation. This should be possible using the results of section 5.

Thus, we conclude that our perturbative analysis of the black hole background describes the high energy phase of two dimensional string theory. The thermodynamics is unstable, and there is a negative mode (or more precisely a classical zero mode that is conjectured to become unstable when one loop effects are taken into account). The high temperature phase presumably involves condensation of this mode and would be interesting to study using the techniques of sections 4, 5.

\textit{6.4. The intermediate region} \(3/2 < R < 2\)

So far we have discussed the thermodynamics below the Kosterlitz-Thouless temperature, \textit{i.e.} for \(R > R_{KT} = 2\), where it is dominated by a gas of perturbative string states (see \((6.6) - (6.10))\), and at temperatures near the Hagedorn temperature. This leaves an unexplored intermediate region, \(3/2 < R < 2\). We will not study this region in detail here, but would like to make a few comments on it.
As the temperature is raised above the Kosterlitz-Thouless one, the system is expected to undergo a phase transition to a phase where the non-singlet degrees of freedom contribute to the thermodynamics \[24\]. In this phase, the free energy and entropy are expected to receive tree level contributions (in contrast to (6.6), (6.8)), associated with the large density of non-singlet states.

In the continuum description, one expects the winding modes (3.19) to condense. For \(1 < R < 2\), only the winding number one mode is relevant, and thus we expect \(t_{\pm 1} = \lambda/2\) in (3.19) to be the only non-zero couplings, in addition to \(\mu\). An important question for analyzing the thermodynamics is how does \(\lambda\) depend on \(R\). It is not clear to us how one can determine \(\lambda(R)\), but it appears likely that it is such that the parameter \(z\) defined in (3.9) varies smoothly between zero and some finite value, as \(R\) decreases from \(R = 2\) to \(R = 3/2\) (the Hagedorn temperature). This variation is related to the dependence of the energy on the temperature. As \(R \to 3/2\) the energy should approach a finite value (as discussed above), while as \(R \to 2\) we should connect to the low temperature thermodynamics, for which \(z = 0\).

The qualitative picture proposed in the previous paragraph is in agreement with the phase diagram of the Sine-Gordon model (or, in statistical mechanics language, the Kosterlitz-Thouless phase diagram). In the region \(R < 2\) the phase diagram is such that as the distance scale increases (i.e. as one goes from the UV to the IR), \(\lambda\) grows and \(R\) decreases. String thermodynamics for \(3/2 < R < 2\) can be thought of as the Sine-Gordon model coupled to worldsheet gravity. It is well known that the qualitative structure of the RG in two dimensional QFT coupled to gravity is the same as in the theory without gravity, with \(\mu\) playing the role of the RG (energy) scale. \(\mu \to \infty\) corresponds to the UV, while small \(\mu\) is the IR region. Thus, in the Sine-Gordon model coupled to gravity one expects to find \(\lambda = \lambda(\mu)\) and \(R = R(\mu)\), i.e. \(\lambda = \lambda(R)\). It is not difficult to see that the qualitative properties obtained from this picture are in agreement with the proposal in the previous paragraph. A more precise analysis is left for future work.

7. Fermionic two dimensional string theory

In this section we briefly describe a natural generalization of two dimensional string theory to the fermionic (type 0) case. Our main motivations for discussing this theory are the following.

\[23\] Naively, \(\lambda, R\) are two independent parameters that specify superselection sectors in the theory. However, as we saw above and discuss further in section 8, it is important here to keep the UV cutoff \(\Lambda\) finite. This makes \(\lambda, R\) fluctuating fields, and one must extremize with respect to them. See \[13\] for a related discussion.
In section 6 we argued that in the bosonic 2d string, the tree level free energy that goes like the mass of the black hole, corresponds to a vacuum energy type contribution, and should be subtracted. We will see that most of the physical properties of the type 0 theory are very similar to the bosonic one, but the tree level free energy vanishes. This provides further evidence for the validity of the procedure of section 6.

The second reason for studying this generalization is that it provides a bridge to higher dimensional physics. In particular, the two dimensional worldsheet supersymmetric black hole plays an important role in the study of the thermodynamics of Little String Theory (see [6] for a recent discussion).

Two dimensional type 0 string theory contains two worldsheet superfields, \( X \), with central charge \( c = \frac{3}{2} \) and \( \phi \), with \( c = \frac{3}{2} + 6Q^2 \). The fact that the total central charge is \( c = 15 \) determines \( Q = \sqrt{2} \). The zero temperature physics of the theory is described in [32]. The perturbative spectrum includes in this case two massless scalars, a NS-NS field \( T(X) \), and a R-R field \( V(X) \). As in the bosonic case, there are no perturbative massive string states (for generic momenta). The bulk amplitudes of \( T \) and \( V \) are known; they are completely determined by an infinite dimensional symmetry algebra, which is very similar to that found in the bosonic string, and are themselves very similar to the bosonic ones.

Our main interest here is in the high temperature thermodynamics of the model. Just like in the bosonic case, it should be dominated by states associated with 1+1 dimensional black holes. The black hole background corresponds in this case to an \( N = 1 \) superconformal version of the coset (2.1). Algebraically, it can be thought of as follows.

One starts with the \( N = 1 \) superconformal theory on \( H^+_3 \), which contains an \( SL(2,R)_L \times SL(2,R)_R \) current algebra with total level \( k \). The level \( k \) receives a contribution of \( k + 2 \) from a bosonic WZW on \( H^+_3 \), and \(-2\) from three free (left and right moving) fermions in the adjoint of \( SL(2) \). The total central charge of the theory is

\[
c_H = \frac{3(k+2)}{k} + \frac{3}{2}.
\]

One then gauges a \( U(1) \) super-Kac-Moody algebra, thereby decreasing the central charge to

\[
c_{\text{coset}} = \frac{3(k+2)}{k}.
\]

The requirement that the total central charge is equal to 15 determines

\[
k = \frac{1}{Q^2} = \frac{1}{2}.
\]

Much of the discussion of section 6 goes through here. The classical analysis gives

\[
S = \beta_H M; \quad \beta_H = 2\pi \cdot \frac{1}{\sqrt{2}}.
\]

\[24\] We work in the same units as in the bosonic theory. The Kosterlitz-Thouless radius is \( R = \sqrt{2} \) in the fermionic case.
Therefore, the classical free energy (6.7) should vanish. Unlike the bosonic case, here this can indeed be shown to be the case, as we briefly review next (see [3] for further discussion).

As discussed in section 5, in the bosonic Euclidean cigar background, the spherical partition sum (which is related to the classical free energy via (5.1)) is in general non-zero. The volume of the cigar cancels the volume of the Conformal Killing Group of the sphere and leaves behind a finite residue.

In the superstring one has to take into account fermionic zero modes. The CKG is replaced in this case by a supergroup, SCKG, but the associated fermionic zero modes are cancelled by fermionic zero modes of the \( N = 1 \) SCFT on the cigar. Thus, it appears that in the fermionic string, the partition sum in the background (2.1) is non-zero as well.

This conclusion is incorrect because of the fact, mentioned at the end of section 2, that the SCFT on the cigar has in fact a larger symmetry, an accidental \( N = 2 \) superconformal symmetry. Thus, this SCFT has twice as many fermionic zero modes as one would guess, and integrating over them leads to a vanishing spherical partition sum [34].

The vanishing of the free energy in this case is due to \( N = 2 \) worldsheet supersymmetry. It does not require spacetime supersymmetry, which in fact is absent in the type 0 theory under consideration. It is also interesting to note that the free energy only vanishes at the Hagedorn temperature. This can be seen by using the duality of [8] relating the Euclidean cigar SCFT to \( N = 2 \) Liouville theory. As discussed in [8] for \( k < 1 \), which is the case here (7.3), the \( N = 2 \) Liouville is more weakly coupled and thus more appropriate.

As one changes the radius of Euclidean time, the \( N = 2 \) Liouville superpotential changes to one that only preserves an \( N = 1 \) subalgebra of the \( N = 2 \) superconformal symmetry. Thus, the arguments above no longer imply that the free energy vanishes, and one expects that it does not. This is in agreement with the thermodynamic picture that one expects: the free energy goes to zero as the energy goes to infinity, or \( \beta \to \beta_H \). At finite energy the free energy is non-zero.

One can also repeat for this case the calculation of the one loop partition sum in section 5.4, and the discussion of section 6, with very similar conclusions. To go beyond one loop probably requires finding a matrix model that describes the dynamics. This is an interesting open problem.

8. Some open problems and future directions

In this paper we discussed the thermodynamics of two dimensional string theory above the temperature corresponding to the Kosterlitz-Thouless phase transition. In the Euclidean time formulation of the problem, this regime is characterized by the condensation of winding modes around Euclidean time. We showed that the theory undergoes a finite temperature phase transition around the Hagedorn temperature \( T_H \). The high temperature phase near \( T \approx T_H \) is well described by the Euclidean two dimensional black hole.
Furthermore, by using the conjectured correspondence \[21\] of string theory in the black hole and Sine-Liouville backgrounds, we constructed a matrix model which provides a powerful tool for studying the two dimensional black hole to all orders in string perturbation theory.

There are many open problems associated with our work, some of which are listed below.

While the differential equation \((4.29)\) in principle determines the partition sum \(F(\lambda, \mu)\) to all orders in string perturbation theory (as described in section 5), we have so far not been able to solve the resulting equations \((5.11)\) beyond genus one. Furthermore, \((4.29)\) is just the lowest equation in an infinite hierarchy which should allow the calculation of correlators of winding modes with arbitrary winding number at any order in \(g_s\). Some low genus correlation functions, like \((2.3)\), have been computed in the continuum formalism, and it would be nice to reproduce them using the Hirota equations (see appendix C). These calculations remain an interesting open problem.

There are observables in two dimensional string theory which seem to be outside of the framework of the Toda equations. Examples include modes carrying momentum around Euclidean time, and discrete states. It would be interesting to generalize the formalism to include these couplings.

In the discussion of black hole thermodynamics in section 6, we assumed that the mass of the black hole is related to the string coupling at the tip of the cigar via the classical relation \(M \sim \exp(-2\Phi_0)\). This relation is expected to be corrected quantum mechanically, and such corrections will influence the thermodynamic relations at two loop level and beyond. Thus, this issue will need to be addressed when the higher order results mentioned above become available.

It would be interesting to understand better the thermodynamics of two dimensional string theory in the intermediate region \(3/2 < R < 2\), as well as in the high temperature phase \(R < 3/2\), and in particular to explore further the relation to the Kosterlitz-Thouless phase diagram mentioned in section 6.

The FZZ correspondence \([21]\) described in section 2 played an important role in our discussion, since it allowed us to replace the curved geometry of the cigar, which is difficult to realize in MQM, by the winding mode condensate which is easier to study in the matrix model. This correspondence remains very mysterious. It is not clear how the metric of the cigar is encoded in the Sine-Liouville potential. The phenomenon is probably much more general (e.g., as we mentioned, it can be extended to the fermionic string) and should be very interesting to understand better.

String thermodynamics seems to suggest that two dimensional string theory\[25\] has a Hagedorn spectrum of states at high energies, given by eq. \((1.2)\). This is surprising, since

\[25\] Note that here we mean the standard \(c = 1\) string theory at zero temperature and with the scale set by the cosmological constant \(\mu\).
the perturbative spectrum consists of one field theoretic degree of freedom. Thus, it must be that most of the states described by (1.2) are non-perturbative, and it would be very interesting to understand what they are. A natural proposal is that these states belong to the non-singlet sector of the matrix model. The degeneracy of non-singlet states is very large, but their masses are proportional to $\log(\Lambda/\mu)$ (with $\Lambda$ a UV cutoff) and thus diverge in the limit $\Lambda \rightarrow \infty$. It appears that one can define two dimensional string theory in two different ways.

If one sends $\Lambda$ to infinity, all the non-singlet states are removed from the spectrum. The matrix dynamics reduces to that of just the singlets (i.e. the eigenvalues of $M$), which in spacetime means that the massless “tachyon” is the only physical degree of freedom. The low temperature thermodynamics reviewed in the beginning of section 6 applies then for all temperatures, and all the vortex couplings $t_n$ in (3.19) are set to zero. We believe that the resulting theory does not admit two dimensional black holes (1.1) and does not have a Hagedorn density of states.

A second possibility is to keep the UV cutoff $\Lambda$ large but fixed. Then the energy gap to the non-singlet states remains finite, and the physics is much richer. In particular, the model does contain black holes, and the corresponding Hagedorn spectrum of states (1.2) should be reproduced by counting non-singlet states in MQM.

It may seem puzzling that there is more than one way to define two dimensional string theory, but in fact this situation is standard in holographic theories that were studied in recent years. Consider, for example string theory on $AdS_3$. As is well known, three dimensional gravity with negative cosmological constant can be quantized as a Chern-Simons theory. According to the AdS-CFT correspondence, this theory is equivalent to a two dimensional CFT with a large central charge (in the semiclassical limit), which in this particular case contains only the conformal block of the identity. The high energy density of states of this theory is much smaller than that given by (BTZ) black hole thermodynamics; thus this theory does not contain $2+1$ dimensional black holes.

An alternative definition of gravity on $AdS_3$ is obtained by studying string theory on $AdS_3$ and including all the perturbative and non-perturbative string excitations. This gives rise to a much richer theory, and in particular one can show that it does have black

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26 We note that a very similar situation occurs in weakly coupled Little String Theory in the double scaling limit defined in [8]. The perturbative spectrum has a Hagedorn density of states of the form (1.2), but thermodynamics suggests that the full theory has a richer spectrum. In particular, $\beta_H$ is predicted to be larger than the perturbative one. Thus, most of the states are non-perturbative.

27 Similar comments can be made for $d > 3$, but the case $d = 3$ is special in that three dimensional gravity without matter exists as a quantum theory, and that the theory has an infinite dimensional symmetry, like two dimensional string theory.
hole states and their entropy can be computed microscopically, by studying the CFT dual to string theory on $AdS_3$. This CFT can be formulated (roughly) as the extreme infrared limit of a matrix field theory; the bulk of the black hole entropy comes from off-diagonal components of the matrices. Moreover, just like in our case, the mass gap to the lowest lying black holes is non-perturbative in the string coupling.

In two dimensional string theory, the above analogy suggests that we should keep $\Lambda$ large but fixed in the double scaling limit. This is in fact very natural since $\Lambda \sim N$, so we are proposing that the non-perturbative theory has large but finite $N$. This is again similar to string theory on AdS space, where the analogs of $N, \mu$ are kept large but fixed.\

The large entropy (1.2) should then be due to non-singlet states. Not much is known about the spectra of such states at present, especially in the large $N$ limit of interest here. One approach to this problem is to rewrite the partition function (4.11) as a sum of Gibbs partition functions over representations $r$ of $SU(N)$:

$$Z_N(\beta, \lambda) = \sum_r g_r[\lambda] T_r e^{-\beta \hat{H}_r},$$

where

$$g_r[\lambda] = \int [D\Omega] \chi_r(\Omega^\dagger) \exp \left( \sum_{n \in \mathbb{Z}} \lambda_n \text{tr} \Omega^n \right),$$

($\chi_r(\Omega^\dagger)$ is the Weyl character in the representation $r$) and

$$\hat{H}_r = P_r \sum_{k=1}^N \left[ -\frac{1}{2} \frac{\partial^2}{\partial x_k^2} - \frac{1}{2} x_k^2 \right] + \frac{1}{2} \sum_{i \neq j} \frac{\hat{\tau}_{ij}^r \hat{\tau}_{ij}^r}{(x_i - x_j)^2}$$

is the Hamiltonian in representation $r$ in terms of the eigenvalues $x_i$ of the matrix $M$ with the inverted oscillator potential. This Hamiltonian is a matrix in the representation space: $P_r$ is a projector to this space and $\hat{\tau}_{ij}^r$ are $SU(N)$ generators on $r$ (see [27,45] for details). These Hamiltonians define, in principle, in a natural way the time evolution of the system. Therefore to count the states in a given energy interval one can try to diagonalize them for representations corresponding to large Young tableaux. This appears to be difficult due to the Calogero type interaction of the eigenvalues with an $SU(N)$ spin structure (see [26,27] for a discussion of the adjoint representation).

If the non-singlet states indeed account for the entropy of black hole (1.2), it is natural to ask whether one can think of non-singlets as excitations of black holes directly. A possible direction for making the relation more precise is to note that the non-singlet states can roughly be thought of as open strings. In black hole physics it has been proposed in

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28 These analogs are the radius of curvature of the AdS space in string units, and in Planck units.
the past that the black hole entropy is due to open strings whose ends lie on the horizon of the black hole. It would be interesting to make this relation more precise.

An interesting set of issues is associated with the Minkowski continuation of our results. The description of the black hole in terms of the Sine-Liouville model involves winding modes around Euclidean time and thus seems inherently Euclidean. Yet, many of the interesting questions about black hole physics can only be discussed in the Minkowski version. Examples are the unitarity of scattering off black holes and the understanding of the dynamics of black hole formation. In principle, the Hamiltonians (8.3) provide a direct way for the analytical continuation to Minkowski space. Thus, to understand the formation and Minkowski evolution of black holes one needs a better understanding of the dynamics of the non-singlet states. This will have to be left for future work. However, we would like to comment on one point related to black hole formation in two dimensional string theory.

It is sometimes stated in the old matrix model literature that the black holes (1.1) are not good toy models of higher dimensional black holes, since they cannot be formed in two dimensional string theory because of the large symmetry present in this theory.\(^{29}\) This \(W_\infty\) symmetry (see \([15,16]\)) is closely related to the interpretation of tachyon dynamics in terms of free fermions, with the tachyon describing ripples on the Fermi surface. The fact that the underlying fermions are free does not allow black holes to form in the evolution of tachyon pulses, even if they are large.

Our discussion in this section clarifies the extent to which this statement is expected to be correct. In the theory with \(\Lambda \rightarrow \infty\), which corresponds to free fermions and describes the dynamics of tachyons as perturbations of the Fermi surface, one indeed does not expect to be able to make black holes since the theory does not contain black holes. However, in the theory with finite \(\Lambda\), the free fermions are just one sector of the Hilbert space (corresponding to singlet dynamics) and black hole formation is expected to occur in the collapse of matter made out of non-singlet states.

Note that we are not proposing that the \(W_\infty\) symmetry of perturbative two dimensional string theory is broken non-perturbatively. Rather, we are saying that it does not prevent the appearance of non-trivial dynamics in the non-singlet sectors. As mentioned above, the situation is analogous to string theory on \(AdS_3\). There, the infinite dimensional spacetime Virasoro algebra (an analog of the \(W_\infty\) symmetry of two dimensional string theory) is a symmetry of the full theory, but it does not prevent non-trivial dynamics and black hole formation in general, although the conformal block of the identity indeed does not contain black hole states (just like the singlet sector in two dimensional string theory).

Finally, we would like to mention another aspect of our results, related to the Hagedorn spectrum (1.2). In perturbative string theory, a spectrum of this sort would lead

\(^{29}\) For a review of gravitational physics in two dimensional string theory see \([19]\).
to an infrared instability (a tachyon), unless there was an almost complete cancellation between bosons and fermions [46,47]. It is an interesting open question whether this relation between the density of states and IR stability is more general. Assuming that it is, it would be interesting to understand how to reconcile it with the behavior found here.

One possibility is that the theory actually has fermionic excitations. Perturbative two dimensional string theory has no spacetime fermions; however, the statistics of the non-perturbative states is not obviously bosonic. Recall that in the singlet sector, the underlying degrees of freedom are free fermions. Another possibility is that the actual high energy density of states is in fact much smaller than that suggested by (1.2). Further discussion of this possibility requires a better understanding of the high temperature phase of the theory (associated with tachyon condensation - see section 6). These issues deserve a better understanding.

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Appendix A. Free compactified fermions from the MQM averaged over twist variable

To show how free fermions emerge from the MQM partition function (1.1) let us use the following formula:

\[ Z^{\text{single}}_N = \int [d\Omega]_{SU(N)} Z_N(\Omega) = \int \mathcal{D}M \int [d\Omega]_{SU(N)} K_{2\pi R}(M, \Omega^+M\Omega), \] (A.1)

where \( K_{2\pi R}(M, M') \) is the propagation kernel for the matrix variable from the initial value \( M \) to the final value \( M' \) during the time \( 2\pi R \).

Using the discrete time version of (1.1) with the action given by

\[ S = \sum_k \text{Tr} \left( \frac{1}{2a} (M_k - M_{k-1})^2 + aV(M_k) \right), \] (A.2)

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diagonalizing the matrices and applying consecutively the Itzykson-Zuber formula to integrate out the angular variables \[14\] we obtain in the limit of a small lattice spacing \(a \to 0\) the following representation for the propagation kernel of the eigenvalues \(u_1, \ldots, u_N\):

\[
K_{2\pi R}(\{u\}, \{u'\}) = \frac{1}{\Delta(u)\Delta(w)} \text{det}_{kj} K_{2\pi R}(u_k, u'_j)
\]

(A.3)

where \(\Delta(u) = \prod_{k>j}(u_k - u_j)\) is the Vandermonde determinant. We ignored in (A.3) the overall normalization constant. Finally putting \(M = M'\) (which means \(u_k = u'_k\)) and integrating with respect to the eigenvalues of the final matrix \(M\) with the Dyson measure \(\Delta^2(u) \prod_{k=1}^N du_k\) we obtain from (A.1) and (A.3) the following representation for the singlet partition function:

\[
Z^{\text{(singlet)}} = \int \prod_{i=1}^N du_i \text{det}_{kj} K_{2\pi R}(u_k, u_j),
\]

(A.4)

or, returning to the functional integral,

\[
Z_{N}^{\text{(singlet)}} = \sum_{\mathcal{P}} (-1)^\mathcal{P} \int_{u(2\pi R) = \mathcal{P}^{-1} u(0)} du \prod_{k=1}^N \mathcal{D} u(x) e^{-\int_0^{2\pi R} dx \left[\frac{1}{2} (\partial_x u_k)^2 + V(u_k)\right]}
\]

(A.5)

where the sum goes over permutations \(\mathcal{P}\) of the eigenvalues \((u_1, \ldots, u_N)\) in the process of propagation around the circle.

The last formula gives the explicit free fermion representation of the singlet sector of the compactified MQM.

**Appendix B. Evaluation of the twisted partition function in the scaling limit**

If we forget for a moment about the wall, the problem decomposes into \(N^2\) independent inverted oscillators. They are related only by common twist angles. For each complex matrix element \(M_{kj}\), the Schrödinger equation on the wave function \(\Psi(M_{kj})\) looks as

\[
-\frac{1}{2} \left( \frac{\partial^2}{\partial M_{kj} \partial M^*_{kj}} + M_{kj} M^*_{kj} \right) \Psi(M_{kj}) = E \Psi(M_{kj})
\]

(B.1)

where there are no summations over the repeated indices. We have to calculate from here the density of states in the presence of twists. In the lagrangian formulation we have the periodic boundary conditions \(M_{kj}(2\pi R) = e^{i\theta_{kj}} M_{kj}(0)\), where \(\theta_{kj} = \theta_k - \theta_j\). To express them in the hamiltonian language we have to go to the polar coordinates \(M_{kj} = r_{kj} e^{i\theta_{kj}}\), where we fixed the angular dependence by the difference of fixed twist angles, and look for the wave function in the form

\[
\Psi(r_{kj}, \theta_{kj}) = \sum_{m=-\infty}^{\infty} \chi_m(r_{kj}) e^{im\theta_{kj}}.
\]

(B.2)
The density of energy will be given in terms of the sum of densities $\rho_m(E)$ of the states with a given angular momentum $m$ weighted by $e^{im\theta_{kj}} = e^{im(\theta_k - \theta_j)}$ (which can be calculated from the analysis of the quasicalssical wave function, see [27] for the details):

$$\rho(\theta, E) = \sum_{m=-\infty}^{\infty} e^{im\theta} \rho_m(E).$$

The explicit calculation gives

$$\rho(\theta, E) = \frac{\sinh[(\pi - \theta)E]}{\sinh[\pi E] \sin \theta} + \delta(\theta) \log \Lambda^2. \quad (B.3)$$

Note that this result does not depend on the familiar cutoff of the $c = 1$ string theory for the nonequal twist angles $\theta_{kj} = \theta_k - \theta_j \neq 0$. The system becomes stabilized due to the twisting!

Now we can calculate the partition function of the individual matrix element in the fixed temperature $T = 1/2\pi R$:

$$Z_1(\theta_{kj}, R) = \int_{-\infty}^{\infty} dE e^{-2\pi R E} \rho(\theta_{kj}, E) = \frac{1/2}{\cos(2\pi R) - \cos \theta_{kj}}. \quad (B.4)$$

Strictly speaking, this formula makes sense only for $R < 1/2$ and the distribution of $\theta$’s in the interval $|\theta_{\text{max}} - \theta_{\text{min}}| < \pi R$. These limitations are inherited from the canonical (fixed $N$) ensemble used so far. We will see that it is not present in the grand canonical ensemble.

From this formula, by taking the product of partition functions of the individual matrix elements, we derive the partition function of the twisted inverted oscillator

$$Z_N(\theta_1, \cdots, \theta_N, R) = (2 \sin \pi R)^{-N} \prod_{i>j} \frac{1}{\cos(2\pi R) - \cos(\theta_i - \theta_j)} \quad (B.5)$$

where the first factor is the $\theta$- independent contribution of the diagonal matrix elements.

Appendix C. $\tau$-function and Hirota equations for the Toda chain hierarchy

C.1. Vertex operator representation

The simplest way (at least for a physicist) to obtain the fermionic representation is through the vertex operator construction. It is easy to see that this is the Coulomb gas partition function describing a chiral bosonic field with exponential interaction. Introduce the bosonic field $\varphi(z)$ with mode expansion

$$\varphi(z) = \hat{q} + \hat{p} \log z + \sum_{n \neq 0} \frac{H_n}{n} z^{-n}, \quad (C.1)$$
\[ [H_n, H_m] = n \delta_{m+n,0}; \quad [\hat{p}, \hat{q}] = 1. \]  
(C.2)

and the vacuum state \(|l\rangle\) defined by

\[ H_n|l\rangle = 0, \quad (n > 0); \quad \hat{p}|l\rangle = l|l\rangle. \]  
(C.3)

The associated normal ordering is defined by putting \(H_n, n > 0\) to the right. Now define the vertex operators

\[ V_q(z) =: e^{\varphi(q^{1/2}z)} :: e^{-\varphi(q^{-1/2}z)} : \]  
(C.4)

satisfying the OPE

\[ V_q(z)V_q(z') = \frac{(z - z')^2}{(q^{1/2}z - q^{-1/2}z')(q^{-1/2}z - q^{1/2}z')} : V_q(z)V_q(z') :. \]  
(C.5)

Then the \(\tau\)-function (4.24) can be written as the expectation value

\[ \tau_l[t] = \langle l \bigg| \exp \left( \sum_{n>0} t_n H_n \right) g \exp \left( - \sum_{n<0} t_n H_n \right) \bigg| l \rangle \]  
(C.6)

where the operator \(g\) is defined as

\[ g = \exp \left( e^{2\pi R\mu} \oint \frac{dz}{2\pi} V_q(z) \right). \]  
(C.7)

C.2. Fermionic representation

The fermionic representation of the \(\tau\)-function now follows from the bosonization formulas

\[ \psi(z) =: e^{-\varphi(z)} :, \quad \psi^*(z) =: e^{\varphi(z)} :, \quad \partial \varphi(z) =: \psi^*(z)\psi(z) : \]  
(C.8)

where

\[ \psi(z) = \sum_{r \in \mathbb{Z} + \frac{1}{2}} \psi_r z^{-r-\frac{1}{2}}, \quad \psi^*(z) = \sum_{r \in \mathbb{Z} + \frac{1}{2}} \psi^*_r z^{-r-\frac{1}{2}} \]  
(C.9)

is a chiral complex Ramond fermion field, whose modes satisfy the canonical anticommutation relations

\[ [\psi_r, \psi^*_s]_+ = \delta_{rs}. \]  
(C.10)

The Hamiltonians \(H_n\) and the vertex operator (C.4) are represented as fermion bilinears

\[ H_n = \sum_{r \in \mathbb{Z} + \frac{1}{2}} : \psi^*_{r-n} \psi_r : \quad (n \in \mathbb{Z}) \]  
(C.11)

\[ V_q(z) = \psi(q^{1/2}z)\psi^*(q^{-1/2}z). \]  
(C.12)
and the vacuum states with given electric charge \( l \) satisfy
\[
\langle l | \psi_r \rangle = \langle l | \psi_r^* \rangle = 0 \quad (r > l) \\
\psi_r | l \rangle = \psi_r^* | l \rangle = 0 \quad (r > l).
\] (C.13)

C.3. Hirota bilinear equations

The ensemble of \( \tau \)-functions with different charge satisfy a set of bilinear equations [48]. Below we give a sketch of their derivation.

The operator \( g \) defined by (C.7) and (C.12) is an exponential of a fermionic bilinear and as such can be thought of as a \( GL(\infty) \) rotation. The corresponding Lie algebra \( gl(\infty) \) is generated by all bilinears \( \sum_{rs} a_{rs} \psi_r \psi_s^* \). The operator
\[
S_{12} = \sum_{r \in \mathbb{Z}+\frac{1}{2}} \psi_r \otimes \psi_r^* = \oint \frac{dz}{2\pi i} \psi(z) \otimes \psi^*(z)
\]
is a fermionic analog of the tensor Casimir in the sense that it commutes with the tensor product of two copies of the operator \( g \):
\[
S_{12} g \otimes g = g \otimes g S_{12}.
\] (C.14)

Further, the definition of the vacuum implies that \( \psi_r | l \rangle \otimes \psi_r^* | l' \rangle = 0 \) (\( l > l' \)), since either \( \psi_r | l \rangle = 0 \) or \( \psi_r^* | l' \rangle = 0 \). Therefore
\[
S_{12} | l \rangle \otimes | l' \rangle = 0
\]
and similarly for the left vacuum.

Now we multiply eq. (C.14) from the left by \( \langle l+1 | e^{\sum_{n>0} t_n H_n} \otimes | l' - 1 \rangle \) and from the right by \( e^{-\sum_{n<0} t_n H_n} | l \rangle \otimes e^{-\sum_{n<0} t_n H_n'} | l' \rangle \) and commute the fermion operators with the exponents until they hit the left (right) vacuum and disappear. As a result we obtain the following identities known as Hirota bilinear equations [48]
\[
\oint_{C_\infty} \frac{dz}{2\pi i} z^{l-l'} \exp \left( \sum_{n>0} (t_n - t_n') z^n \right) \tau_l(t - \tilde{\zeta}_+^+) \tau_{l'}(t' + \tilde{\zeta}_+^+) = \\
\oint_{C_0} \frac{dz}{2\pi i} z^{l-l'} \exp \left( \sum_{n<0} (t_n - t_n') z^{-n} \right) \tau_{l+1}(t - \tilde{\zeta}_-) \tau_{l'-1}(t' + \tilde{\zeta}_-),
\] (C.15)

where
\[
\tilde{\zeta}_+ = (\ldots, 0, 0, z^{-1}, z^{-2}/2, z^{-3}/3, \ldots), \quad \tilde{\zeta}_- = (\ldots, z^3/3, z^2/2, z, 0, 0 \ldots).
\]
Expanding in $y_n = t'_n - t_n$ we obtain an infinite hierarchy of soliton partial differential equations.

$$
\sum_{j=0}^{\infty} P_{j+m}(-2y_+)P_j(\bar{D}_+) \exp \left( \sum_{j \neq 0} y_j D_j \right) \tau_{l+m+1}[t] \cdot \tau_l[t]
$$

$$
= \sum_{j=0}^{\infty} P_{j-m}(-2y_-)P_j(\bar{D}_-) \exp \left( \sum_{j \neq 0} y_j D_j \right) \tau_{l-m}[t] \cdot \tau_{l+1}[t]
$$

(C.16)

where

$$
t_{\pm} = (t_{\pm1}, t_{\pm2}, \ldots), \quad y_{\pm} = (y_{\pm1}, y_{\pm2}, \ldots),
$$

$$
\bar{D}_{\pm} = (D_{\pm1}, D_{\pm2}/2, D_{\pm3}/3, \ldots)
$$

represent the Hirots’s bilinear operators

$$
D_n f[t] \cdot g[t] = \frac{\partial}{\partial x} f(t_n + x) g(t_n - x) \bigg|_{x=0},
$$

(C.17)

and finally $P_j$ are the Schur polynomials defined by

$$
\sum_{j=1}^{\infty} P_j[t] \lambda^j = \exp \left( \sum_{n=1}^{\infty} t_n \lambda^n \right).
$$

(C.18)

The lowest, Toda equation, which is what we will actually need, is obtained by equating to zero the coefficient in front of $y_{-1}$:

$$
\tau_l \partial_1 \partial_{-1} \tau_l - \partial_1 \tau_l \partial_{-1} \tau_l + \tau_{l+1} \tau_{l-1} = 0.
$$

(C.19)

Appendix D. Calculation of partition functions on torus and sphere

D.1. Solution of the equation for the sphere

Changing the variable $y = e^u$ and the function $X_0 = -\phi - 2u$ we bring the equation (5.6) to the form

$$
\alpha \partial_u^2 \phi = (\partial_u^2 \phi + (\partial_u \phi)^2 + 3 \partial_u \phi + 2)e^\phi
$$

where it does not contain explicitly $u$.

By denoting $\psi(\phi) = \partial_u \phi$ we obtain the first order ODE

$$
(\alpha e^{-\phi} - 1) \psi' \psi = (\psi + 1)(\psi + 2)
$$

with the solution

$$
\frac{(\partial_u \phi + 1)}{(\partial_u \phi + 2)^2(e^\phi - \alpha)} = -C
$$

where C is the integration constant. Solving the quadratic equation for $\partial_u \phi$ and integrating it we obtain:

$$
\phi + 2u = \int \frac{d\phi}{\sqrt{1 - 4C\alpha - 4Ce^\phi}} = \frac{2}{\sqrt{1 - 4C\alpha}} \text{arcsinh} \left( \frac{\sqrt{1 - 4C\alpha}}{4C} e^{-\phi/2} \right) + C_3.
$$

Going back to original variable $y$ we see that fixing the couplings $C = (2-R/R)^2$ and $C_3 = 0$ we reproduce (5.7) generating the series in Moore’s formula (3.10).
D.2. Solution of the equation for the torus

The easiest is to change the variable from $y$ to $t = e^{2-R}x_0$, according to the spherical solution (5.7). After a tedious but direct calculation (5.13) takes the form

$$[t - (R - 1)]\partial_t f'_1 + \frac{R - 1}{R - 2}(1 - 1/t)f'_1 = \frac{R(R - 2)}{12} t^{1/(2-R)} \frac{(R - 1)^2 t - 1}{[(R - 1)t - 1]^3}. $$

It is a first order inhomogeneous ODE with the explicit solution

$$f'_1 = \frac{R(R - 2)}{12} \int^t ds e^{B(s) - B(t)} s^{1/(2-R)} \frac{(R - 1)^2 s - 1}{[(R - 1)s - 1]^3[s - (R - 1)]}$$

with

$$B(t) = \frac{R - 1}{R - 2} \int^t dt \frac{t - 1}{t[t - (R - 1)]} = \log \left( t^{1/(R-2)}[t - (R - 1)] \right).$$

Fortunately (and may be due to the integrability of the original Toda equation) the resulting integral for $f_1$ can be explicitly calculated:

$$f'_1 = \frac{R(R - 2)}{12} \frac{t^{1/(2-R)}}{t - (R - 1)} \left( \frac{2 - R}{2(R - 1)} \frac{1}{[(R - 1)t - 1]^2} - \frac{1}{(R - 1)t - 1} + D \right)$$

where $D$ is an integration constant.

Using $f_1(y) = \int dy f'_1$ and integrating once more we get for the partition function

$$f_1(y) = -\frac{1}{24} \left( \frac{2DR}{R - 1} + \frac{R^2}{(R - 1)^2} \right) \log t$$

$$- \frac{1}{24} \left( \frac{2DR^2(R - 2)}{R - 1} - \frac{2R^2 - 2R + 1}{(R - 1)^2} \right) \log[(R - 1) - t] - \frac{1}{24} \log[1 - t(R - 1)] + C \log \lambda$$

where the last term is the zero mode of Toda equation with yet unknown coefficient.

To fix $D$ and $C$ let us use the boundary condition (4.33) for $\lambda = 0$

$$f_1(\mu, \lambda) = -\frac{R + R^{-1}}{24} \log \mu, \quad \lambda = 0.$$  

(Note that we always drop a constant, cutoff dependent, term from the partition function.) Since $y \to \infty$ we have $t = e^{(1-2/R)x} \to y^{R-2}$ and comparing the solution with the boundary conditions (4.33) we conclude:

$$2D \frac{R(R - 2)}{R - 1} + \frac{R^2(R - 2)}{(R - 1)^2} = R + R^{-1}.$$  

Note that, “miraculously”, the second logarithmic term disappears from (D.1) for this value of $D$.

Since $F(\mu, \lambda)$ is regular at $\lambda = 0$ the constant $C$ should be chosen as

$$C = -\frac{R + R^{-1}}{12(2 - R)}.$$  

With these values of $C$ and $D$ we finally obtain the result (5.14).
References

[1] S. Elitzur, A. Forge and E. Rabinovici, Nucl. Phys. B359 (1991) 581.
[2] G. Mandal, A. Sengupta, and S. Wadia, Mod. Phys. Lett. A6 (1991) 1685.
[3] E. Witten, Phys. Rev. D44 (1991) 314.
[4] R. Dijkgraaf, E. Verlinde and H. Verlinde, Nucl. Phys. B371 (1992) 269.
[5] G. Horowitz and A. Strominger, Nucl. Phys. B360 (1991) 197; J. Maldacena and A. Strominger, hep-th/9710014.
[6] D. Kutasov and D. A. Sahakyan, hep-th/0012258.
[7] H. Ooguri and C. Vafa, hep-th/9511164, Nucl. Phys. B463 (1996) 55.
[8] A. Giveon, D. Kutasov and O. Pelc, hep-th/9907178, JHEP 9910 (1999) 035; A. Giveon and D. Kutasov, hep-th/9909110, JHEP 9910 (1999) 034; hep-th/9911039, JHEP 0001 (2000) 023.
[9] V. Kazakov and A. A. Migdal, Nucl. Phys. B311 (1988) 171.
[10] E. Brezin, V. Kazakov and Al. Zamolodchikov, Nucl. Phys. B338(1990) 673.
[11] G. Parisi, Phys. Lett. B238 (1990) 209, 213.
[12] D. Gross and N. Miljkovic, Phys. Lett. B238 (1990) 217.
[13] P. Ginsparg and J. Zinn-Justin, Phys. Lett. B240 (1990) 333.
[14] V. Kazakov, “Bosonic strings and string field theories in one-dimensional target space”, proceedings of Cargese workshop on Random Surfaces, Quantum Gravity and Strings, 1990.
[15] I. Klebanov, proceedings of the ICTP Spring School on String Theory and Quantum Gravity, Trieste, April 1991, hep-th/9108019.
[16] P. Ginsparg and G. Moore, proceedings of TASI 1992, hep-th/9304011.
[17] S. Das and A. Jevicki, Mod. Phys. Lett. A5 (1990) 1639.
[18] S. R. Das, Mod. Phys. Lett. A8 (1993) 69, 1331; A. Dhar, G. Mandal and S. Wadia, Mod. Phys. Lett. A7 (1992) 3703; A8 (1993) 1701; A. Dhar Nucl. Phys. B507 (1997) 277.
[19] J. Polchinski, “What is String Theory?”, Lectures presented at the 1994 Les Houches Summer School “Fluctuating Geometries in Statistical Mechanics and Field Theory”, hep-th/9411028.
[20] A. Jevicki and D. Yoneya, Nucl. Phys. B411 (1994) 64.
[21] V. Fateev, A. Zamolodchikov and Al. Zamolodchikov, unpublished.
[22] Al. B. Zamolodchikov, unpublished; V.A. Fateev, Phys. Lett. B357 (1995) 397.
[23] G. Moore, hep-th/9203057.
[24] E. Hsu and D. Kutasov, hep-th/9212023, Nucl. Phys. B396 (1993) 693.
[25] D. Gross and I. Klebanov, Nucl. Phys. B344 (1990) 475.
[26] D. Gross and I. Klebanov, Nucl. Phys. B354 (1990) 459.
[27] D. Boulatov and V. Kazakov, preprint LPTENS 91/24 (1991), Int. J. Mod. Phys. A8 (1993) 809, revised version: hep-th/0012228.

[28] A. Matytsin and P. Zaugg, hep-th/9611170, Nucl. Phys. B497 (1997) 658; hep-th/9701148, Nucl. Phys. B497 (1997) 699.

[29] R. Dijkgraaf, G. Moore, R. Plesser, Nucl. Phys. B394 (1993) 356.

[30] J. Teschner, hep-th/9712256, Nucl. Phys. B497 (1997) 390; hep-th/9712258, Nucl. Phys. B497 (1997) 699.

[31] V. Knizhnik, A. Polyakov and A. Zamolodchikov, Mod. Phys. Lett. A3 (1988) 819; F. David, Mod. Phys. Lett. A3 (1988) 1651; J. Distler and H. Kawai, Nucl. Phys. B571 (1999) 509.

[32] P. Di Francesco and D. Kutasov, hep-th/9109005, Nucl. Phys. B375 (1992) 119.

[33] Y. Kazama and H. Suzuki, Nucl. Phys. B321 (1989) 232.

[34] D. Kutasov and N. Seiberg, Phys. Lett. B251 (1990) 67; for a review, see D. Kutasov, hep-th/9110041.

[35] A. Prudnikov, Yu. Brichkov, O. Marychev, “Integrals and Series”, Nauka, Moscow, 1981.

[36] V. L. Berezinski, JETP 34 (1972) 610; J. M. Kosterlitz and D. J. Thouless, J. Phys. C6 (1973) 1181; J. Villain, J. Phys. C36 (1975) 581.

[37] M. R. Douglas, “Conformal theory techniques in Large N Yang-Mills Theory”, talk at the 1993 Cargèse meeting, hep-th/9311130.

[38] K. Ueno and K. Takasaki, Adv. Stud. Pure Math. 4 (1984) 1; K. Takasaki, Adv. Stud. Pure Math. 4 (1984) 139.

[39] M. Jimbo and T. Miwa, Publ. RIMS, Kyoto Univ. 19, No. 3 (1983) 943.

[40] S. Shenker, Proceedings of Cargese workshop on Random Surfaces, Quantum Gravity and Strings, 1990.

[41] G. Gibbons and M. Perry, hep-th/9204090, Int. J. Mod. Phys. D1 (1992) 335; C. R. Nappi and A. Pasquinucci, hep-th/9208002, Mod. Phys. Lett. A7 (1992) 3337.

[42] G. W. Gibbons and S. W. Hawking, Phys. Rev. D15 (1977) 2752.

[43] N. Seiberg and S. Shenker, hep-th/9201017, Phys. Rev. D45 (1992) 4581.

[44] J. Maldacena, hep-th/9711200, Adv. Theor. Math. Phys. 2 (1998) 231.

[45] V. A. Kazakov, Solvable Matrix Models, proceedings of the MSRI Workshop “Matrix Models and Painlevé Equations”, Berkeley (USA) 1999; hep-th/0003064.

[46] D. Kutasov and N. Seiberg, Nucl. Phys. B358 (1991) 600.

[47] V. Niarchos, hep-th/0010154.

[48] R. Hirota, Direct Method in Soliton Theory, Solitons, Ed. by R. K. Bullogh and R. J. Caudrey, Springer, 1980.