A LOWER BOUND OF THE INTEGRATED CARATHÉODORY–REIFFEN METRIC AND INVARIANT METRICS ON COMPLETE NONCOMPACT KÄHLER MANIFOLDS

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ABSTRACT. In this paper, we attempt to make progress on the following long-standing conjecture in hyperbolic complex geometry: a simply connected complete Kähler manifold \((M, \omega)\) with negatively pinched sectional curvature is biholomorphic to a bounded domain in \(\mathbb{C}^n\) and the Carathéodory–Reiffen metric does not vanish everywhere. As a next development of the important recent results of D. Wu and S. T. Yau in obtaining uniform equivalence of the base Kähler metric with the Bergman metric, the Kobayashi–Royden metric, and the complete Kähler–Einstein metric in this class of Kähler manifolds, we note that the Carathéodory–Reiffen metric is missing, and provide an integrated gradient estimate of the bounded holomorphic function which becomes a quantitative lower bound of the integrated Carathéodory–Reiffen metric. Also, we establish the equivalence of the Bergman metric, the Kobayashi–Royden metric, and the complete Kähler–Einstein metric of negative scalar curvature under a bounded curvature condition of the Bergman metric with some other moderate conditions on \(n\)-dimensional complete noncompact Kähler manifolds. The equivalence of invariant metrics is useful in applications where the explicit description of the Bergman kernel is known. Through explicit computations, we examine the relations among invariant metrics on a nontrivial family of 3-dimensional bounded pseudoconvex domains for which the boundary limits of the holomorphic sectional curvature of the Bergman metric are not well defined.

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1. Introduction

It is fundamental that the complex projective space \( \mathbb{P}^n_{\mathbb{C}} \), the complex Euclidean space \( \mathbb{C}^n \), and the unit ball \( B_n \) in \( \mathbb{C}^n \) are model spaces in Kähler geometry for each \( n \in \mathbb{N} \). These are only three spaces that are simply-connected, complete Kähler manifolds with constant holomorphic sectional curvature by complete Kähler–Einstein metrics.

The complex geometric conditions that characterize model spaces for positive holomorphic sectional curvature and zero holomorphic sectional curvature are now well established and various studies have been carried out in each direction. The biholomorphic characterization of \( \mathbb{P}^n_{\mathbb{C}} \) among closed Kähler manifolds, starting from the Minimal Model program of S. Mori, was extended by S. T. Yau and Y. T. Siu to admitting the Kähler metric with positive bisectional curvature, and it was also proven by them that \( \mathbb{C}^n \) and its biholomorphic characterization are those where the sectional curvature of the complete Kähler metric of the complete simply-connected Kähler manifold has a faster decay than the quadratic decay \([34, 38, 39]\). Regarding the biholomorphic characterization of the Kähler model space with negative holomorphic sectional curvature, S. T. Yau proposed the following problem in 1971: if \((M, \omega)\) is an \( n \)-dimensional complete, noncompact, simply-connected Kähler manifold satisfying \(-B \leq \text{sectional curvature} \leq -A\) for two positive constants \( A \) and \( B \), then \( M \) is biholomorphic to a bounded domain in \( \mathbb{C}^n \) (see \([40], [38], p. 225], [46], p. 98], [43], P. 195, (1)], \([48], p. 47, c.)\). The basic example of this conjecture is the ball of radius \( r \) in \( \mathbb{C}^n \) based on the Poincaré metric whose holomorphic sectional curvature is a negative constant \( c \) and thereby the Riemannian sectional curvature is between \( 1/4c \) and \( c \) \([28], p. 167\]. Numerous studies have been made in relation to the above conjecture, among which D. Wu and S. T. Yau proved the following two important theorems based on the quasi-bounded geometry and Shi’s estimate \([37]\) with Kähler–Ricci flow. These results indicate that the bounded domain in \( \mathbb{C}^n \) is special under the premise that the above conjecture is true.

**Theorem 1** ([42], Corollary 7). Let \((M, \omega)\) be a complete simply-connected noncompact Kähler manifold whose Riemannian sectional curvature is negatively pinched. Then the base Kähler metric is uniformly equivalent to the Kobayashi–Royden metric, the Bergman metric and the complete Kähler–Einstein metric of negative scalar curvature.

**Theorem 2** ([42], Theorems 2, 3). Let \((M, \omega)\) be a complete Kähler manifold whose holomorphic sectional curvature is negatively pinched. Then the base Kähler metric is uniformly equivalent to the Kobayashi–Royden metric and the complete Kähler–Einstein metric of negative scalar curvature.

In order to study Yau’s problem, properties of the metrics in the Wu–Yau theorems are very important and they are interesting in their own rights. One classical result of N. Mok and S. T. Yau says that a bounded domain admits a complete Kähler–Einstein metric of negative scalar
curvature if and only if such a domain is biholomorphic to a bounded weakly pseudoconvex domain [33]. Hence if Yau’s classical conjecture is true, then $M$ is necessarily biholomorphic to a bounded (weakly but not necessarily strictly, see [8]) pseudoconvex domain in $\mathbb{C}^n$. In addition, since the Kobayashi–Royden metric, the complete Kähler–Einstein metric of negative scalar curvature, and the Bergman metric have the property that any automorphism becomes an isometry [40, 45], it is suitable for studying from the point of view of differential geometry. Hermitian metrics and Finsler metrics with this property are called invariant metrics.

Classical invariant metrics include the Carathéodory–Reiffen metric (its definition is given in the next section) in addition to the three metrics mentioned in the Wu–Yau theorems, and the Carathéodory–Reiffen metric is defined based on the existence of (non-constant) bounded holomorphic functions on noncompact complex manifolds, which is another big challenge of hyperbolic complex geometry. The four invariant metrics on a ball in $\mathbb{C}^n$ are the same as the Poincaré metric up to dimensional constants, and pseudoconvex domains that guarantee the equivalence of four invariant metrics including the Carathéodory–Reiffen metric are usually domains of very special classes (for example, see [3, 16, 22, 23, 25, 49]). Furthermore, invariant metrics are important objects in many different contexts not only from complex differential geometry or several complex variables but also from complex algebraic geometry and arithmetic geometry over number fields (for example, see [7, 12, 13, 17, 21, 26, 29, 30, 40, 41, 45] and references therein).

For a follow-up study after the Wu–Yau theorems, several directions can be pursued. Providing information about the Carathéodory–Reiffen metric in the settings of Theorem 1 is a natural next step. The Carathéodory–Reiffen metric is excluded from the implication of the Wu–Yau theorems, because we currently do not know whether a non-constant bounded holomorphic function exists in Kähler manifolds except some domains in $\mathbb{C}^n$. On the other hand, from Theorem 2, the weakest assumption that can be used to show the equivalence of the Bergman metric, Kobayashi-Royden metric, and Kähler–Einstein metric is that the holomorphic sectional curvature of the Bergman metric is negatively pinched. However, for general complex manifolds, the Bergman metric holomorphic sectional curvature has a value between $-\infty$ and $+2$ [14, 24], and there is an example [20] of a semi-finite type pseudoconvex domain in which the holomorphic sectional curvature of Bergman metric blows up to $-\infty$.

The upper bounds of the Carathéodory–Reiffen metric have been studied extensively. For the comparison between Carathéodory metric and the Bergman metric on the bounded domains, the first is provided by Qi-Keng Lu [31] and then on manifolds by K. T. Hahn [18, 19]. Further developments were made by T. Ahn, H. Gaussier and K. Kim [1]. Very recently, a comparison of Carathéodory distance and Kähler–Einstein distance of Ricci curvature $-1$ for certain weakly pseudoconvex domains was established by the first-named author of this paper [10].

In this paper, we first obtain a lower bound of the integrated Carathéodory–Reiffen metric on the setting of the Yau’s problem mentioned earlier. The positive lower bound of the Carathéodory–Reiffen metric is important in that it is the smallest invariant metric among invariant metrics [10, 21] and it provides quantitative information about non-constant bounded holomorphic functions.
We denote by $d$ the geodesic distance on $M$, and $\gamma_M$ the Carathéodory–Reiffen metric on $M$. Below are the first two main theorems of this paper.

**Theorem 3.** Let $(M, g)$ be a simply-connected complete noncompact $n$-dimensional Kähler manifold whose Riemannian sectional curvature $k$ of $g$ further satisfies $k \leq -a^2$ for some $a > 0$. For any $p \geq 2$,

A. Let $f$ be a holomorphic function from $M$ to the unit disk $D$ in $\mathbb{C}$. Then

$$\int_M \int_M G(x, y) |\nabla f|^2(y) dy dx \leq \left( \frac{p}{(2n-1)a} \right)^p \int_M |f|^p \gamma_M(x; \nabla f)^{\frac{p}{2}},$$

(1.1)

where $G(x, y)$ is minimal positive Green’s function on $M$.

B. If the Riemannian sectional curvature $k$ of $g$ further satisfies $-b^2 \leq k$ for some $b > 0$. Then there exists a constant $C(n) > 0$, which only depends on $n$, such that for any $f$ be a holomorphic function from $M$ to the unit disk $D$,

$$\int_0^\infty \int_M \left( \int_M t^{-n} \exp\left[ -\frac{d(x, y)^2}{2t} - \frac{(2n-1)^2 b^2 t}{8} - \frac{(2n-1)bd(x, y)}{2} \right] (1 + bd(x, y)) |\nabla f|^2(y) dy \right)^p dx dt \leq C(n) \left( \frac{2\pi p}{(2n-1)a} \right)^p \int_M |f(x)|^p \gamma_M(x; \nabla f(x))^{\frac{p}{2}} dx.$$

(1.2)

The inequalities (1.1), (1.2) can be interpreted as integrated gradient estimates of bounded holomorphic functions. Assuming a bounded domain in addition to the above theorem, $f$ can be selected as a globally defined coordinate function (with scaling by its diameter). Since an arbitrary vector field can be written as the gradient vectors of a linear combination of such $f$’s, the form of the Theorem 3-A in this case becomes the lower bound of the integrated Carathéodory–Reiffen metric. For example, In the case of unit disk $D$ in $\mathbb{C}$, (1.1) with $f(z) = z$ can be written as:

$$2\pi \int_0^1 \left( \frac{1}{6} - \frac{R^2}{2} \ln R - \frac{R^4}{8} (4 \ln R - 1) - \frac{R^6}{36} (6 \ln R - 1) \right)^p R dR \leq p^p \int_D |z|^p \gamma_D(z; \nabla z)^{\frac{p}{2}}, p \geq 2.$$

The details are provided in Proposition 11.

The next theorem examines the relationship between invariant metrics based on the condition of the Bergman metric, and presents a condition that implies the equivalence of invariant metrics while the Carathéodory–Reiffen metric does not vanish everywhere.

**Theorem 4.** Let $(M, \omega_B)$ be an $n$-dimensional complete noncompact Kähler manifold with a Bergman metric $\omega_B$ of a bounded curvature. Then the following statements hold:

A. There exists $C_0 > 0$, which only depends on $n$ and the curvature range of $\omega_B$, such that

$$\chi_M(p; v) \leq C_0 \sqrt{\omega_B(v, v)} \quad \text{for all } v \in T_p M, \ p \in M,$$

where $\chi_M$ is the Kobayashi–Royden metric on $M$. 
B. Assume that $\frac{B}{|v|^2}$ is a bounded function on $M$. Then there exist a complete Kähler–Einstein metric $\omega_{KE}$ of negative scalar curvature and a constant $C_1 > 0$ such that $\omega_{KE}$ is uniformly equivalent to $\omega_B$ by $C_1$, i.e.,
\[
\frac{1}{C_1} \omega_{KE}(v,v) \leq \omega_B(v,v) \leq C_1 \omega_{KE}(v,v)
\]
for all $v \in T'M$.

C. Assume that there exists a compact subset $K$ in $M$ such that the holomorphic sectional curvature of $\omega_B$ is negative outside of $K$, and that $M$ is biholomorphically and properly embedded into $B_N$, $N \geq n$, where $B_N$ is the unit ball in $\mathbb{C}^N$. Then the Carathéodory–Reiffen metric $\gamma_M$ is not essentially zero, and the Bergman metric is uniformly equivalent to the Kobayashi–Royden metric, i.e., there exists $C_2 > 0$ such that
\[
\frac{1}{C_2} \chi_M(p;v) \leq \sqrt{\omega_B(v,v)} \leq C_2 \chi_M(p;v)
\]
for all $v \in T'_p M$, $p \in M$.

Moreover, if $N = n$, the Bergman metric is uniformly equivalent to the complete Kähler–Einstein metric of negative scalar curvature.

We remark that by considering the complete Kähler metric with bounded curvature instead of the Bergman metric, the same conclusions hold for the corresponding metric in the first and third statements. The first statement of Theorem 4 follows from Wu–Yau’s quasi-bounded geometry and Shi’s estimate on Kähler–Ricci flow and the second statement of Theorem 4 corresponds to a consequence of the application of A. Chau’s Kähler–Ricci flow [4]. Our result will be very useful when an explicit description of the Bergman kernel near the boundary of a bounded domains is available.

The third statement of Theorem 4 differs from the Wu–Yau theorems in that the Bergman metric’s holomorphic sectional curvature is not required to be everywhere negative, but it still ensures the equivalence of invariant metrics. A biholomorphic embedding into a possibly higher dimensional ball is assumed to control the compact subset because Yau’s Schwarz lemma [35, 47] needs to be applied to obtain the Kähler–Einstein metric of negative scalar curvature by solving the complex Monge–Ampere equation [42, Lemma 31], and we do not know how to apply the maximum principle for moving Laplacian without assumption of the negative upper bound of the holomorphic sectional curvature. It is known that every bounded strictly pseudoconvex domain in $\mathbb{C}^n$ admits a proper holomorphic embedding into a ball. (for example, see [15, p. 11]). Moreover, the Carathéodory–Reiffen metric is non-vanishing because of this hypothesis. Hence this seems to be a sufficiently reasonable assumption for future studies on the presence of Carathéodory hyperbolicity in complex manifold settings.

We also note that, in the third statement of Theorem 4, the bounded curvature of the Bergman metric is assumed instead of the quasi-bounded geometry. This yields a generalization of the main theorem in [9] and greatly reduces the amount of computation required for verification when an explicit form of the Bergman kernel is given. This improvement is potentially quite important as it enables us to study non-trivial classes of pseudoconvex domains in $\mathbb{C}^n$, $n \geq 3$. In particular, our theorem provides new computational method, when a concrete formula of the Bergman kernel is
available, to prove the equivalence of invariant metrics for weakly pseudoconvex domains which may not have a nice boundary behavior of the holomorphic sectional curvature, since it is in fact sufficient to check only at most 4 derivatives of the Bergman kernel. Without this computational method, it seems very difficult to check the equivalence of invariant metrics by applying Wu–Yau theorems or by checking the quasi-bounded geometry of the Bergman metric from [9], even with computational tools from numerical analysis, except for simple examples. To demonstrate the usefulness of Theorem 4, we examine the concrete family of examples

\[ E_{p,\lambda} := \{(x, y, z) \in \mathbb{C}^3 : (|x|^{2p} + |y|^{2})^{1/\lambda} + |z|^2 < 1\}, \quad p, \lambda > 0, \]

which is interesting in its own right. Our computation shows that although the limit of curvature components does not exist on boundary points on \( E_{p,\lambda} \), we can still compare invariant metrics. We use a concrete formula for the Bergman kernel of \( E_{p,\lambda} \), which is obtained in [2].

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2. Basic settings and invariant metrics

Let \( M \) be an \( n \)-dimensional complex manifold equipped with a complex structure \( J \) and a Hermitian metric \( g \). The complex structure \( J : T_{\mathbb{R}}M \to T_{\mathbb{R}}M \) is a real linear endomorphism that satisfies for every \( x \in M \), and \( X,Y \in T_{\mathbb{R},x}M \), \( g_x(J_xX,Y) = -g_x(X,J_xY) \), and for every \( x \in M \), \( J_x^2 = -\text{Id}_{T_xM} \). We decompose the complexified tangent bundle \( T_{\mathbb{R}}M \otimes_{\mathbb{R}} \mathbb{C} = T'M \oplus \overline{T'M} \), where \( T'M \) is the eigenspace of \( J \) with respect to the eigenvalue \( \sqrt{-1} \) and \( \overline{T'M} \) is the eigenspace of \( J \) with respect to the eigenvalue \( -\sqrt{-1} \). We can identify \( v,w \) as real tangent vectors, and \( \eta,\xi \) as corresponding holomorphic \((1,0)\) tangent vectors under the \( \mathbb{R} \)-linear isomorphism \( T_{\mathbb{R}}M \to T'M \), i.e.

\[ \eta = \frac{1}{\sqrt{2}}(v - \sqrt{-1}Jv), \xi = \frac{1}{\sqrt{2}}(w - \sqrt{-1}Jw). \]

A Hermitian metric on \( M \) is a positive definite Hermitian inner product

\[ g_p : T'_{p}M \otimes \overline{T'_{p}M} \to \mathbb{C} \]

which varies smoothly for each \( p \in M \). The metric \( g \) can be decomposed into the real part denoted by \( \text{Re}(g) \), and the imaginary part, denoted by \( \text{Im}(g) \). The real part \( \text{Re}(g) \) induces an inner product called the induced Riemannian metric of \( g \), an alternating \( \mathbb{R} \)-differential 2-form. Define the \((1,1)\)-form \( \omega := -\frac{1}{2} \text{Im}(g) \), which is called the fundamental \((1,1)\)-form of \( g \). In local coordinates this form can written as

\[ \omega = \frac{\sqrt{-1}}{2} \sum_{i,j=1}^{n} g_{ij} dz_i \wedge d\overline{z_j}. \]
The components of the curvature 4-tensor of the Chern connection associated with the Hermitian metric $g$ are given by

\[
R_{ijkl} := R\left( \frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_l} \right) = g \left( \nabla^c_\partial \frac{\partial}{\partial z_i}, \nabla^c_\partial \frac{\partial}{\partial z_j}, \nabla^c_\partial \frac{\partial}{\partial z_k} \right)
\]

where $i, j, k, l \in \{1, \ldots, n\}$.

The holomorphic sectional curvature with the unit direction $\eta$ at $x \in M$ (i.e., $g_\omega(\eta, \eta) = 1$) is defined by

\[
H(g)(x, \eta) = R(\eta, \overline{\eta}, \eta, \overline{\eta}) = R(v, Jv, Jv, v),
\]

where $v$ is the real tangent vector corresponding to $\eta$. We will often write $H(g)(x, \eta) = H(g)(\eta) = H(\eta)$.

Given any complex manifold $M$, for each $p \in M$ and a tangent vector $v$ at $p$, define the Carathéodory–Reiffen metric and the Kobayashi–Royden metric by

\[
\gamma_M(p; v) := \sup \left\{ |df(p)(v)|; f : M \to D, f(p) = 0, f \text{ holomorphic} \right\},
\]

\[
\chi_M(p; v) := \inf \left\{ \frac{1}{R}; f : R^2 \to M, f(0) = p, df(\frac{\partial}{\partial z}|_{z=0}) = v, f \text{ holomorphic} \right\},
\]

respectively.

The Bergman metric is defined in terms of the Bergman kernel. Let $\Lambda^{(n,0)}M$ be the space of smooth complex differential $(n, 0)$ forms on $M$. For $\varphi, \psi \in \Lambda^{(n,0)}M$, define

\[
\langle \varphi, \psi \rangle = (-1)^{n^2/2} \int_M \varphi \wedge \overline{\psi},
\]

and

\[
||\varphi|| = \sqrt{\langle \varphi, \varphi \rangle}.
\]

Let $L^2_{(n,0)}$ be the completion of

\[
\left\{ \varphi \in \Lambda^{(n,0)}M; ||\varphi|| < +\infty \right\}
\]

with respect to $|| \cdot ||$. Then $L^2_{(n,0)}$ is a separable Hilbert space with respect to the inner product $\langle \cdot, \cdot \rangle$.

Define $\mathcal{H} = \left\{ \varphi \in L^2_{(n,0)}; \varphi \text{ is holomorphic} \right\}$. Suppose $\mathcal{H} \neq 0$. Let $\{e_j\}_{j \geq 0}$ be an orthonormal basis of $\mathcal{H}$ with respect to $\langle \cdot, \cdot \rangle$. Then the $2n$-form defined on $M \times M$ given by

\[
B(x, y) := \sum_{j \geq 0} e_j(x) \wedge \overline{e_j}(y), \quad x, y \in M,
\]
is called the Bergman kernel of $M$. Suppose for some point $p \in M$, we have $B(p, p) \neq 0$. Write $B(z, z) = b(z, z)dz_1 \wedge \cdots \wedge dz_n$. Define $\omega_B(z) := \sqrt{-1} \partial \bar{\partial} \log b(z, z)$.

If the real $(1, 1)$ form $\omega_B$ is positive definite, we call the corresponding Hermitian metric $g_M^B$ the Bergman metric. By definition, $g_M^B$ is Kähler.

Lastly, the Kähler–Einstein metric $\omega_{KE}$ means the Kähler metric which is also the Einstein metric, and the Kähler–Einstein metric of the negative scalar curvature becomes an invariant metric.

We will use the following lemma for Theorem 4:

**Lemma 5.** [42, Lemma 19] Let $(M, \omega)$ be a Hermitian manifold such that the holomorphic sectional curvature has the upper bound $-\kappa < 0$. Then the Kobayashi–Royden metric satisfies

$$\chi_M(x, v) \geq \sqrt{\kappa \frac{|v|}{2}} \omega,$$

for each $x \in M, v \in T_x^*M$.

### 3. Quasi-Bounded Geometry

In this section, we review some results from Section 2 in [42].

The notion of quasi-bounded geometry is introduced by S.T. Yau and S.Y. Cheng ([7]). Let $(M, \omega)$ be an $n$-dimensional complete Kähler manifold. For a point $p \in M$, let $B_\omega(p, \rho)$ be the open geodesic ball centered at $p$ in $M$ of radius $\rho$; we omit the subscript $\omega$ if there is no peril of confusion. Denote by $B_{C^n}(r)$ the open ball centered at the origin in $C^n$ of radius $r$ with respect to the standard metric $\omega_{C^n}$.

An $n$-dimensional Kähler manifold $(M, \omega)$ is said to have quasi-bounded geometry if there exist two constants $r_2 > r_1 > 0$ such that for each point $p \in M$, there is a domain $U \subset C^n$ and a nonsingular holomorphic map $\psi: U \rightarrow M$ satisfying

1. $B_{C^n}(r_1) \subset U \subset B_{C^n}(r_2)$ and $\psi(0) = p$;
2. there exists a constant $C > 0$ depending only on $r_1, r_2, n$ such that
   $$C^{-1} \omega_{C^n} \leq \psi^* (\omega) \leq C \omega_{C^n} \quad \text{on } U;$$
3. for each integer $l \geq 0$, there exists a constant $A_l$ depending only on $l, n, r_1, r_2$ such that
   $$\sup_{x \in U} \left| \frac{\partial^{\mu + |\mu|} g_{\bar{\nu} \bar{\nu}}}{\partial v^\mu \partial \bar{v}^\nu} \right| \leq A_l, \text{ for all } |\mu| + |\nu| \leq l,$$

where $g_{\bar{\nu} \bar{\nu}}$ are the components of $\psi^* \omega$ on $U$ in terms of the natural coordinates $(v^1, \ldots, v^n)$, and $\mu, \nu$ are multiple indices with $|\mu| = \mu_1 + \cdots + \mu_n$. We call $r_1$ a radius of quasi-bounded geometry.

By applying the $L^2$ estimate, the following theorem was proved.
Theorem 6 ([42], Theorem 9). Let \((M, \omega)\) be a complete Kähler manifold. Then the manifold \((M, \omega)\) has quasi-bounded geometry if and only if for each integer \(q \geq 0\), there exists a constant \(C_q > 0\) such that
\[
\sup_{p \in M} |\nabla^q R_m| \leq C_q,
\] (3.3)
where \(R_m = \{ R_{ijkl} \}\) denotes the curvature tensor of \(\omega\). In this case, the radius of quasi-bounded geometry depends only on \(C_0\) and the dimension of \(M\).

Also, we will use the following lemma:

Lemma 7. [42, Lemma 20] Suppose a complete Kähler manifold \((M, \omega)\) has quasi-bounded geometry. Then the Kobayashi–Royden metric satisfies
\[
\chi_M(x, v) \leq C|v|_\omega,
\]
for each \(x \in M, v \in T'_x M\), where \(C\) depends only on the radius of quasi-bounded geometry of \((M, \omega)\).

4. A lower bound of the integrated Carathéodory–Reiffen metric

Although the lemmas below are known, we prove them here for tracking explicit constants for the proof of Theorem 3.

Lemma 8. [36, Poincaré inequality] let \(M\) be an \(n\)-dimensional complete noncompact, simply connected Riemannian manifold with sectional curvature \(k \leq -a^2 < 0\). Then
\[
\int_M |u|^2 \leq \frac{4}{(n-1)^2 a^2} \int_M |\nabla u|^2, \quad u \in W^1_0(M).
\] (4.1)

Proof. Let \(L^2(M)\) be the space of \(L^2\) functions on \(M\). Denote by \(W^1(M)\) the Hilbert space consisting of \(L^2\) functions whose gradient are also \(L^2\), and by \(W^1_0(M)\) the subspace in \(W^1(M)\) which is the completion of the space \(C^\infty_0(M)\) under \(W^1(M)\)-norm. When \(M\) is complete, we have \(W^1(M) = W^1_0(M)\).

Let \(r(x) = d(p_0, x)\) be the distance function from a fixed point \(p_0 \in M\). From the Rauch comparison theorem, we have
\[
\Delta r \geq (n-1)a,
\] (4.2)
where \(a > 0\).

Let \(\Omega\) be the geodesic ball centered at \(p_0\) with radius \(R\) in \(M\), \(R > 0\). From the Green’s theorem, we have for every \(u \in C^\infty_0(\Omega),\)
\[
\int_\Omega |u|^2 \Delta r - \int_\Omega \nabla(|u|^2) \cdot \nabla r = \int_{\partial \Omega} |u|^2 d\sigma = 0,
\]
where \(d\sigma\) is the surface measure on \(\partial \Omega\). We remark that \(r\) may not be smooth at \(p_0\), but we can apply the Green’s theorem to \(\Omega\) minus a small ball of radius \(\epsilon > 0\) around \(p_0\) and let \(\epsilon \to 0\). From
(4.2) and $|\nabla r| = 1$, we have

$$(n - 1)a||u||^2 \leq \int_{\Omega} |u|^2 \Delta r = \int_{\Omega} \nabla(|u|^2) \cdot \nabla r \leq \int_{\Omega} |\nabla(|u|^2)| \leq 2||u||||\nabla u||.$$}

This gives

$$||u|| \leq \frac{2}{(n - 1)a}||\nabla u||, u \in C_0^\infty(\Omega).$$

Since $C_0^\infty(M)$ is dense in $W_0^1(M)$, we are done. □

We use Mckean’s estimate [32] on the first eigenvalue of the Laplace–Beltrami operator.

**Lemma 9.** [32, Mckean’s estimate] Let $\Delta_0$ denote the Laplace–Beltrami operator on functions. We have $\Delta_0 = -\Delta$. The Poincaré inequality immediately implies the following result on the smallest eigenvalue $\lambda_1$ of $\Delta_0$: Let $M$ be an $n$-dimensional complete noncompact, simply-connected Riemannian manifold with sectional curvature $k \leq -a^2 < 0$. Then

$$\lambda_1 \geq \frac{(n - 1)^2 a^2}{4}, \quad (4.3)$$

where $\lambda_1$ is the smallest eigenvalue of $\Delta_0$.

**Proof.** From Lemma 8, for every $u \in C_0^\infty(M)$,

$$(\Delta_0 u, u) = (du, du) = \int_{\Omega} |\nabla u|^2 \geq \frac{(n - 1)^2 a^2}{4} \int_{\Omega} |u|^2.$$}

The proof follows. □

**Lemma 10.** [6, Cheng] Let $M$ be an $n$-dimensional Riemannian manifold. Consider the first eigenvalue for the Dirichlet problem $\lambda_1(M) > 0$. Let $\Omega$ be a relatively compact domain of $M$ such that $b\Omega$ is smooth. Let $f \in C^\infty(M)$ and let $u$ be the solution of

$$\begin{cases}
\Delta u = \Delta f & \text{on } \Omega, \\
u = 0 & \text{on } b\Omega.
\end{cases}$$

Then for any $p \geq 2$,

$$\int_{\Omega} |u|^p \leq C_p \int_{\Omega} |\nabla f|^p, \quad (4.4)$$

where the constant $C_p$ depends only on $p$ and $\lambda_1(M)$.

**Proof.** Assume that $p$ is even. Multiplying the equation by $u^{p-1}$ and integrating it, we have

$$(p - 1) \int_{\Omega} |\nabla u|^2 u^{p-2} = (\nabla u, \nabla u^{p-1}) = (\nabla f, \nabla u^{p-1})$$

$$\leq (p - 1) \int_{\Omega} |\nabla f||u|^{p-2}$$

$$\leq (p - 1) \left( \int_{\Omega} |\nabla u|^2 u^{p-2} \right)^{1/2} \left( \int_{\Omega} |\nabla f|^2 u^{p-2} \right)^{1/2}.$$
Thus we have
\[ \frac{4}{p^2} \int_{\Omega} |\nabla u|^{p/2} |^{2} \leq \int_{\Omega} |\nabla f|^{2} u^{p-2} \leq \left( \int_{\Omega} |u|^p \right)^{\frac{p-2}{p}} \left( \int_{\Omega} |\nabla f|^p \right)^{\frac{2}{p}} . \]
From (4.1), we obtain
\[ \left( \frac{4\lambda_1}{p^2} \right)^{\frac{p}{2}} \int_{\Omega} |u|^p \leq \int_{\Omega} |\nabla f|^p . \]
The constant \( C_p \) depends only on \( p \) and \( \lambda_1 \). The general case can be proved similarly through multiplication by \( (\text{sgn} \, u) |u|^{p-1} \) and integration.

**Proof of Theorem 3.** From Lemma 9, \( M \) has the positive spectrum. It is a standard result that if the manifold has positive spectrum then there exists a positive symmetric Green’s function \( G \) on \( M \). Moreover, we can always take \( G(x,y) \) to be the minimal Green’s function constructed using exhaustion of compact subdomains. Hence
\[ G(x,y) = \lim_{i \to \infty} G_i(x,y) > 0, \]
where \( G_i \) is the Dirichlet Green’s function of a compact exhaustion \( \{ \Omega_i \} \) of \( M \), and the limit is uniform on compact subsets of \( M \).

Take any (bounded) holomorphic function \( f : M \to \mathbb{D} \). For any relatively compact subdomain \( \Omega \subset M \) with the smooth boundary \( b\Omega \), we use \( f^2 \) in Lemma 10 and solving the Dirichlet boundary problem with the inequality
\[ (g(\nabla f^2, \nabla f^2)(x))^{\frac{p}{2}} = (4|f(x)|^2 df(\nabla f)(x))^{\frac{p}{2}} \leq 2^p |f|^p(x) \gamma_M(x; \nabla f(x))^{\frac{p}{2}} \]
for any \( x \in M \) and \( p \geq 2 \) implies
\[ \int_{\Omega} |u|^p \leq \left( \frac{2p}{(2n-1)a} \right)^p \int_{\Omega} |f|^p \gamma_M(\cdot; \nabla f)^{\frac{p}{2}} \leq \left( \frac{2p}{(2n-1)a} \right)^p \int_{M} |f|^p \gamma_M(\cdot; \nabla f)^{\frac{p}{2}} , \]
where \( u \) is the solution of
\[ \begin{cases} \Delta u = 2|\nabla f|^2 & \text{on } \Omega, \\ u = 0 & \text{on } b\Omega, \end{cases} \]
and \( a > 0 \) is for the upper bound of the Riemannian sectional curvature \( \leq -a^2 < 0 \).

From the hypothesis \( |f|^p \gamma_M(\cdot; \nabla f)^{\frac{p}{2}} \in L^1(M) \) and from the exhaustion of compact subdomains, there exists \( u \in C^\infty(M, \mathbb{R}) \) such that
\[ \int_{M} |u|^p < \infty, \]
and \( \Delta u = 2|\nabla f|^2 \) on \( M \). Furthermore, \( \inf_{x \in \Omega} \text{Vol} B(x,r) > 0 \) for any \( r > 0 \) implies that \( u(x) \to 0 \) as \( d(p,x) \to \infty \) from some fixed point \( p \in M \). Thus the Dirichlet problem is solvable and \( u \) can be represented by
\[ u(x) = 2 \int_{M} G(x,y) |\nabla f|^2(y) dy, \]
which proves part (A).
For part (B), the positive minimal Green’s function satisfies
\[ G(x, y) = \int_0^\infty h_M(x, y, t) dt, \]
where we denote the heat kernel of the Laplace–Beltrami operator by \( h_M(x, y, t) \). Hence (4.8) becomes
\[ u(x) = 2 \int_0^\infty \int_M h_M(x, y, t) |\nabla f|^2(y) dy dt. \tag{4.9} \]

We use the Cheeger and Yau’s heat kernel comparison theorem [5]:
\[ h_M(x, y, t) \geq h_{M_k}(d(x, y)), \tag{4.10} \]
where \( M_k \) is the space form with constant sectional curvature equal to \( k \). From the two-sided estimate of Davies and Mandouvalos [11],
\[ c(n)^{-1} h(t, d(x, y)) \leq h_{M_k}(d(x, y)) \leq c(n) h(t, d(x, y)), \tag{4.11} \]
where \( c(n) \) depends only on \( n \) and
\[ h(t, r) = (2\pi t)^{-n} \exp\left[-\frac{r^2}{2t} - \frac{(2n - 1)^2 b^2 t}{8} - \frac{(2n - 1) b t}{2}\right] \left(1 + b r + \frac{b^2 t}{2}\right)^{\frac{2n-1}{2}} (1 + b r), \quad t, r > 0, \tag{4.12} \]
where \( b > 0 \) is for the lower bound of the Riemannian sectional curvature \( \geq -b^2 \).

Now combining (4.6) with (4.9), (4.10), (4.11), and (4.12) gives the desired inequality (1.2). This completes the proof. □

**Proposition 11.** In the case of unit disk \( \mathbb{D} \) in \( \mathbb{C}^1 \), for each \( p \geq 2 \),
\[ 2\pi \int_0^1 \left(\frac{1}{6} - \frac{R^2}{2} \ln R - \frac{R^4}{8} (4 \ln R - 1) - \frac{R^6}{36} (6 \ln R - 1) \right)^p RdR \leq p^p \int_{\mathbb{D}} |z|^p \gamma_\mathbb{D}(z; \nabla z)^\frac{p}{2}. \]

**Proof.** The Green function in the unit disk \( \mathbb{D} \) in \( \mathbb{C}^1 \) has the following form:
\[ G(x, y) = \frac{1}{2\pi} \ln \frac{|x - y|}{|y - \overline{x}||}. \]

The function \( G \) satisfies \( \Delta_x G(x, y) = \delta_y \) at fixed \( y \in \mathbb{D} \) and \( G(x, y) = 0 \) when \( |x| = 1 \) and \( |y| < 1 \). Since for \( z \in \mathbb{D} \), the gradient vector of \( z \) with respect to the Poincaré metric is \( (1 - |z|^2) \frac{\partial}{\partial z} \), the integrand of the left-hand side of (1.1) is
\[ \int_{|y|<1} G(x, y)(1 - |y|^2)^2 dy. \tag{4.13} \]
Rewrite $G(x, y) = \frac{1}{4\pi} \ln \left( \frac{|x|^2 |y|^{-2}}{|x - y|^2} \right)$ and choose a coordinate $x = (R, 0)$ and $y = (r \cos \theta, r \sin \theta)$, then (4.13) becomes

$$\frac{1}{4\pi} \int_0^1 \int_0^{2\pi} \ln \left( \frac{1 + r^2 R^2 - 2r R \cos \theta}{R^2 + r^2 - 2r R \cos \theta} \right) r (1 - r^2)^2 d\theta dr = \frac{1}{4\pi} \int_0^1 r (1 - r^2)^2 (I(1, r R) - I(r, R)) dr,$$

where $I(a, b) := \int_0^{2\pi} \ln(a^2 + b^2 - 2ab \cos \theta) d\theta$. It is well-known that

$$I(a, b) = 4\pi \max\{\ln |a|, \ln |b|\}.$$

Since $0 \leq r, R \leq 1$, $I(1, r R) = 0$. Thus the integral becomes

$$- \int_0^1 r (1 - r^2)^2 \max\{\ln |r|, \ln |R|\} dr = - \ln R \int_0^R r (1 - r^2)^2 dr - \int_1^0 r (1 - r^2)^2 \ln r dr$$

$$= \frac{1}{6} \frac{R^2}{2} \ln R - \frac{R^4}{8} (4 \ln R - 1) - \frac{R^6}{36} (6 \ln R - 1).$$

Thus the left-hand side of (1.1) is

$$2\pi \int_0^1 \left( \frac{1}{6} \frac{R^2}{2} \ln R - \frac{R^4}{8} (4 \ln R - 1) - \frac{R^6}{36} (6 \ln R - 1) \right)^p R dR$$

□

5. The Maximum Principle and Shi’s estimate on Kähler–Ricci flow

Let $(M, \tilde{\omega})$ be an $n$-dimensional complete noncompact Kähler manifold. Suppose for some constant $T > 0$ there is a smooth solution $\omega(x, t) > 0$ for the evolution equation

$$\begin{cases}
\frac{\partial}{\partial t} g_{\alpha\overline{\beta}}(x, t) = -4R_{\alpha\overline{\beta}}(x, t) & \text{on } M \times [0, T], \\
g_{\alpha\overline{\beta}}(x, 0) = \tilde{g}_{\alpha\overline{\beta}}(x) & x \in M, 
\end{cases} \tag{5.1}$$

where $g_{\alpha\overline{\beta}}(x, t)$ and $\tilde{g}_{\alpha\overline{\beta}}$ are the metric components of $\omega(x, t)$ and $\tilde{\omega}$, respectively. Assume that the curvature $R_m(x, t) = \left\{ R_{\alpha\overline{\beta}\gamma\delta}(x, t) \right\}$ of $\omega(x, t)$ satisfies

$$\sup_{M \times [0, T]} |R_m(x, t)|^2 \leq k_0 \tag{5.2}$$

for some constant $k_0 > 0$.

The following lemma is an extension of Lemma 15 in [42] to the case of complement of compact subset. Though the proof is similar, we provide some details to indicate where modifications are needed for the compact subset’s complement.
Lemma 12. With the above assumptions, suppose a smooth tensor \( \{ W_{\alpha \beta \gamma \delta}(x,t) \} \) on \( M \) with complex conjugation \( W_{\alpha \beta \gamma \delta}(x,t) = W_{\beta \alpha \delta \gamma}(x,t) \) satisfies

\[
\left( \frac{\partial}{\partial t} W_{\alpha \beta \gamma \delta}(x,t) \right) \eta^\alpha \eta^\beta \eta^\gamma \eta^\delta \leq (\nabla W_{\alpha \beta \gamma \delta}) \eta^\alpha \eta^\beta \eta^\gamma \eta^\delta + C_1 |\eta|_\omega^4,
\]

for all \( x \in M, \eta \in T_x^* M, 0 \leq t \leq T \), where \( \Delta \equiv 2g^{\alpha \beta}(x,t)(\nabla \eta^\alpha \nabla \alpha + \nabla \alpha \nabla \eta^\beta) \) and \( C_1 \) is a constant. Let

\[
h(x,t) = \max \left\{ W_{\alpha \beta \gamma \delta}(x,t) \eta^\alpha \eta^\beta \eta^\gamma \eta^\delta ; \eta \in T^*_x M, |\eta|_\omega(x,t) = 1 \right\},
\]

for all \( x \in M \) and \( 0 \leq t \leq T \). For any compact subset \( K \) in \( M \), suppose

\[
\sup_{x \in M, 0 \leq t \leq T} |h(x,t)| \leq C_0,
\]

(5.4)

\[
\sup_{M \setminus K} h(x,0) \leq -\kappa,
\]

(5.5)

for some constants \( C_0 > 0 \) and \( \kappa \). Then,

\[
h(x,t) \leq (8C_0 \sqrt{n}k_0 + C_1)t - \kappa,
\]

for all \( x \in M \setminus K, 0 \leq t \leq T \).

Proof. Denote

\[
C = 8C_0 \sqrt{n}k_0 + C_1 > 0.
\]

(5.6)

Suppose

\[
h(x_1,t_1) - C t_1 + \kappa > 0,
\]

(5.7)

for some \( (x_1,t_1) \in M \setminus K \times [0,T] \). Then by (5.4) we have \( t_1 > 0 \). Under the conditions (5.1) and (5.2), from [37], there exists a function \( \theta \) such that

\[
0 < \theta(x,t) \leq 1, \text{ on } M \times [0,T],
\]

(5.8)

\[
\frac{\partial \theta}{\partial t} - \Delta_{\omega(x,t)} \theta + 2\theta^{-1} |\nabla \theta|_{\omega(x,t)} \leq -\theta \text{ on } M \times [0,T],
\]

(5.9)

\[
\frac{C_2}{1 + d_0(x_0,x)} \leq \theta(x,t) \leq \frac{C_2}{1 + d_0(x_0,x)} \text{ on } M \times [0,T],
\]

(5.10)

where \( x_0 \) is a fixed point in \( M \), \( d_0(x,y) \) is the geodesic distance between \( x \) and \( y \) with respect to \( \omega(x,0) \), and \( C_2 > 0 \) is a constant depending only on \( n, k_0 \), and \( T \). Let

\[
m_0 = \sup_{M \setminus K, 0 \leq t \leq T} ([h(x,t) - Ct + \kappa] \theta(x,t)).
\]

Then \( 0 < m_0 \leq C_0 + |\kappa| \) by (5.4),(5.7), and (5.8). Denote

\[
\Lambda = \frac{2C_2(C_0 + CT + |\kappa|)}{m_0} > 0.
\]

Then, for any \( x \in M \setminus K \) with \( d_0(x_0,x) \geq \Lambda \),

\[
|h(x,t) - Ct + \kappa| \theta(x,t) \leq \frac{C_2(C_0 + CT + |\kappa|)}{1 + d_0(x,x_0)} \leq \frac{m_0}{2}.
\]
It follows that the function \((h - Ct + \kappa)\theta\) must attain its supremum \(m_0\) on the compact set \(\overline{B(x_0; \Lambda)} \times (0, T] \subset M \setminus K \times [0, T]\), where \(\overline{B(x_0; r)}\) denotes the closure of the geodesic ball with respect to \(\omega(x, 0)\) centered at \(x_0\) of radius \(r\). Let 
\[
f(x, \eta, t) = \frac{W_{\omega(x,t)}}{|\eta|^4} - Ct + \kappa,
\]
for all \((x, t) \in M \setminus K \times [0, T], \eta \in T^*_x M \setminus \{0\}\). Then there exist \(x_s, \eta_s, t_s\) with \(x_s \in \overline{B(x_0; r)}, 0 \leq t_s \leq T, \eta_s \in T^*_x M\) and \(|\eta_s|_{\omega(x_s, t_s)} = 1\), such that 
\[
m_0 = f(x_s, \eta_s, t_s) = \max_{S_t \times [0, T]} (f\theta),
\]
where 
\[
S_t = \{(x, \eta) \in T^* M; x \in M, \eta \in T^*_x M, |\eta|_{\omega(x, t)} = 1\}.
\]
Since \(h(\cdot, 0)\) is a continuous function on \(M\), either \(x_s \in M \setminus K\) or \(x_s \in \partial K, t_s > 0\) by (5.5). Now from a standard process we extend \(\eta_s\) to a smooth vector field and from the same argument as in the proof of Lemma 15 in [42]. Since 
\[
f\theta = f(x, \eta(x), t)\theta(x, t)\]
attains its maximum at \((x_s, t_s)\), we have 
\[
\frac{\partial}{\partial t}(f\theta) \geq 0, \nabla (f\theta) = 0, \quad \Delta (f\theta) \leq 0 \quad \text{at } (x_s, t_s).
\]
From (5.11) and (5.9), one can see that at the point \((x_s, t_s)\), 
\[
0 \leq \frac{\partial}{\partial t}(f\theta) = -m_0 < 0
\]
(for details, see [42]). This yields a contradiction and the proof is completed. \(\square\)

The following lemma is an extension of Lemma 13 in [42] to the case of complement of a compact subset.

**Lemma 13.** Let \((M, \omega)\) be an \(n\)-dimensional complete noncompact Kähler manifold. Let \(K\) be a compact set in \(M\) such that 
\[
-\kappa_2 \leq H(\omega) \leq -\kappa_1 < 0 \quad \text{on } M \setminus K,
\]
where \(H(\omega)\) is the holomorphic sectional curvature and \(\kappa_1, \kappa_2\) are positive constants. Then there exists another Kähler metric \(\bar{\omega}\) such that 
\[
C^{-1} \omega \leq \bar{\omega} \leq C \omega \quad \text{on } M,
\]
\[
-\bar{\kappa}_2 \leq H(\bar{\omega}) \leq -\bar{\kappa}_1 < 0 \quad \text{on } M \setminus K,
\]
\[
\sup_{p \in M} |\nabla^q R_m| \leq C_q \quad \text{on } M,
\]
where \(\nabla^q\) denotes the \(q\)-th order covariant derivative of \(R_m\) of \(\bar{\omega}\) with respect to \(\bar{\omega}\), and the positive constants \(C = C(n), \bar{\kappa}_j = \bar{\kappa}_j(n, \kappa_1, \kappa_2), j = 1, 2, C_q = C_q(n, q, \kappa_1, \kappa_2)\) depend only on the parameters in their parentheses.

The conditions (5.13) and (5.15) are contained in [37, 42]. We provide below details for the pinching estimate.
Proof. From the short time existence of the Kähler–Ricci flow [37], the equation (5.1) admits a smooth solution \( \{g_{\alpha \overline{\beta}}(x, t)\} \) for all \( 0 \leq t \leq T \). The curvature \( R_m(x, t) \) satisfies that, for each nonnegative integer \( q \),
\[
\sup_{x \in M} |\nabla^q R_m(x, t)|^2 \leq \frac{C(q, n, K)(\kappa_2 - \kappa_1)^2}{t^q} \quad \text{for all } 0 < t \leq \frac{\theta_0(n, K)}{\kappa_2 - \kappa_1} \equiv T, \tag{5.16}
\]
where \( C(q, n, k) > 0 \) is a constant depending only on \( q \), the compact subset \( K \) and \( n \) and \( \theta_0(n, K) > 0 \) is a constant depending only on \( n \) and \( K \).

From the evolution equation of the curvature tensor (see [37, 42]), we have
\[
\frac{\partial}{\partial t} R_{\alpha \overline{\beta} \gamma \overline{\delta}} = 4\Delta R_{\alpha \overline{\beta} \gamma \overline{\delta}} + 4 \rho_{\alpha \overline{\beta} \gamma \overline{\delta}} \left( R_{\alpha \overline{\beta} \mu \nu} R_{\gamma \overline{\delta} \mu \nu} + R_{\alpha \overline{\beta} \mu \nu} R_{\gamma \overline{\delta} \mu \nu} - R_{\alpha \overline{\beta} \mu \nu} R_{\gamma \overline{\delta} \mu \nu} \right) - 2 \rho_{\alpha \overline{\beta} \gamma \overline{\delta}} \left( R_{\alpha \overline{\beta} \mu} R_{\gamma \overline{\delta} \mu} + R_{\alpha \overline{\beta} \mu} R_{\gamma \overline{\delta} \mu} + R_{\alpha \overline{\beta} \mu} R_{\gamma \overline{\delta} \mu} + R_{\alpha \overline{\beta} \mu} R_{\gamma \overline{\delta} \mu} \right),
\]
where \( \Delta \equiv \Delta_{\omega(x, t)} = \frac{1}{2} \frac{\partial}{\partial t} \left( |\nabla g_{\alpha \overline{\beta}}| \right)(x, t) \). It follows that
\[
\frac{\partial}{\partial t} R_{\alpha \overline{\beta} \gamma \overline{\delta}} \eta^\alpha \eta^\beta \eta^\gamma \eta^\delta \leq 4(\Delta R_{\alpha \overline{\beta} \gamma \overline{\delta}}) \eta^\alpha \eta^\beta \eta^\gamma \eta^\delta + C_1(n) |\eta|^4 d_{\omega(x, t)} |R_m(x, t)|^2 \tag{5.17}
\]
by (5.16) with \( q = 0 \). Let
\[
H(x, \eta, t) = \frac{R_{\alpha \overline{\beta} \gamma \overline{\delta}} \eta^\alpha \eta^\beta \eta^\gamma \eta^\delta}{|\eta|^4_{\omega(x, t)}}.
\]
Then by (5.12) and (5.16),
\[
H(\bar{\omega}) \leq -\bar{\kappa}_1 < 0 \text{ on } M \setminus K,
\]
\[
|H(x, \eta, t)| \leq |R_m(x, t)|_{\omega(x, t)} \leq C_0(n, K)(\kappa_2 - \kappa_1).
\]
To apply the maximum principle, let us denote
\[
h(x, t) = \max \left\{ H(x, \eta, t); |\eta|_{\omega(x, t)} = 1 \right\},
\]
for all \( x \in M \) and \( 0 \leq t \leq \frac{\theta_0(n, K)}{\kappa_2 - \kappa_1} \). Then \( h \) with (5.17) satisfies the three conditions in Lemma 12. Then
\[
H(x, \eta, t) \leq h(x, t) \leq -\frac{\kappa_1}{2} < 0,
\]
for all \( 0 < t \leq t_0 := \min \left\{ \frac{\kappa_1}{2C_1(n, K)(\kappa_2 - \kappa_1)^2}, \frac{\theta_0(n, K)}{\kappa_2 - \kappa_1} \right\} \). Since the curvature tensor is bounded by (5.16) with \( q = 0 \), the complete Kähler metric \( \omega(x, t) = \sqrt{\frac{1}{2} g_{\alpha \overline{\beta}}(x, t)dz^\alpha \wedge d\overline{z}^\beta} \) is a desired metric for an arbitrary \( t \in (0, t_0] \).

Remark 14. If one replaces (5.12) by
\[
-\kappa_2 \leq H(\omega) \leq -\kappa_1 \text{ on } M, \kappa_1 \in \mathbb{R},
\]
then still (5.13) and (5.15) follow from the original Shi’s argument. Combining it with Lemma 7, we obtain the first statement of Theorem 4.
6. Generation of Kähler metrics with negative holomorphic sectional curvature

**Proposition 15.** Given an $n$-dimensional Kähler manifold $(M, \omega)$, assume that there exists a compact subset $K$ in $M$ such that the holomorphic sectional curvature of $\omega_B$ is negative outside of $K$, and $M$ is biholomorphically embedded into $B_N, N \geq n$, where $B_N$ is the unit ball in $\mathbb{C}^N$. Then there exists a complete Kähler metric $\tilde{\omega}$ whose holomorphic sectional curvature has a negative upper bound and $\tilde{\omega} \geq \omega$.

**Proof.** From the holomorphic embedding $M \hookrightarrow B_N$, consider a Kähler metric of the form

$$\omega_m := m\omega_P + \omega, \quad m > 0,$$

where $\omega_P$ is the Poincaré metric of the unit ball $B_N$ in $\mathbb{C}^N$. It is clear that $\omega_m \geq \omega$ for each $m > 0$. From the decreasing property of the holomorphic sectional curvature, $\omega_P$ restricting to $M$ has a negative holomorphic sectional curvature [44]. From Lemma 4 of [44], we may assume that the holomorphic sectional curvature of $\omega_m := m\omega_P + \omega$ is the Gaussian curvature on some embedded Riemann surfaces in $M$. Recall that for a Hermitian metric $G$ on a Riemann surface, the holomorphic sectional curvature of $G$ is the Gaussian curvature $H(g) = -\frac{1}{2} \frac{\partial^2 \log g}{\partial z \partial \overline{z}}$ of $G$ for some positive smooth function $g = g(z, \overline{z})$. In this case, the holomorphic sectional curvature $H(G, t)$ becomes a real-valued function independent of the unit vector $t$. Thus we write $H(G)$ instead of $H(G, t)$.

From [27, Proposition 3.1], for any positive functions $f$ and $g$ with $m > 0$,

$$H(f + mg) \leq \frac{f^2}{(f + mg)^2} H(f) + \frac{m^2 g^2}{(f + mg)^2} H(mg) = \frac{f^2}{(f + mg)^2} H(f) + \frac{m^2 g^2}{(f + mg)^2} H(g).$$

From here, we can deduce that $H(\omega_m)$ becomes negative on $K$ by taking sufficiently large $m$. Since $H(\omega_m)$ is negative on $M \setminus K$, we are done. \hfill \Box

7. Proof of Theorem 4

**Proof of Theorem 4.** The first statement follows from Lemma 7, Lemma 13, and Remark 14. The second statement corresponds to a direct consequence of the main theorem in [4]. The third statement follows from Lemma 5, Lemma 7, Proposition 15, with the relation that for each $m > 0$,

$$\omega_B \leq \tilde{\omega},$$

where $\tilde{\omega}$ is defined in Proposition 15. For the last statement when $N = n$, the metric $\tilde{\omega}$ has the bounded curvature. Then one can solve the complex Monge–Ampere equation by following Wu–Yau’s approach (see Lemma 31 and Theorem 3 in [42]). \hfill \Box

**Remark 16.** When $N > n$, the lower bound of the holomorphic sectional curvature $\tilde{\omega}$ does not need to be bounded because of the presence of the second fundamental form (see [44]).
8. Domain $E_{p, \lambda}$

In this section, we consider the domain

$$E_{p, \lambda} = \{(x, y, z) \in \mathbb{C}^3; (|x|^{2p} + |y|^{2p})^{1/\lambda} + |z|^2 < 1\}, \quad p, \lambda > 0,$$

and perform necessary computations to examine the comparisons of invariant metrics based on the verification of hypothesis in Theorem 4.

First, we take a suitable compact set $K \subset E_{p, \lambda} \cup \partial E_{p, \lambda}$ that satisfies the conditions in Theorem 4. Since any point $(x, y, z) \in \mathbb{C}^3$ can be realized as

$$|x| < r(z, y) = \left( (1 - |z|^2)^{\lambda} - |y|^2 \right)^{\frac{1}{2p}},$$

with fixed pair $(y, z)$, the point $(x, y, z)$ can be mapped biholomorphically onto the form $(0, y, z)$ through the automorphism of one-dimensional disc with the radius $r(y, z)$ centered at the origin. Then using rotations, we can make the other two entries to have non-negative real-values. Since all these transformations are automorphisms of $E_{p, \lambda}$, we take the compact set:

$$K_1 = \{(0, y, z) \in E_{p, \lambda}; 0 \leq x, y < 1\},$$

where the closure is taken with respect to the usual topology of $\mathbb{C}^3$.

An explicit formula of Bergman kernel $B$ on $E_{p, \lambda}$ is computed in [2]:

$$B((x, y, z), (x, y, z)) = \frac{(1 - \nu_3)^{\lambda} - \nu_2^{\lambda-3} \nu_1^2(p - 1)(\lambda(p - 1) + p)}{(1 - \nu_3)^{2-2\lambda} \pi^3 p^2 \left( \nu_1 - ((1 - \nu_3)^{\lambda} - \nu_2)^{1/p} \right)^4} \quad (8.1)$$

$$+ \frac{(1 - \nu_3)^{\lambda-2} ((1 - \nu_3)^{\lambda} - \nu_2^{\lambda-3} \nu_1^2(p - 1)(\lambda - 1)\nu_2^p)}{\pi^p \left( \nu_1 - ((1 - \nu_3)^{\lambda} - \nu_2)^{1/p} \right)^4}$$

$$+ \frac{((1 - \nu_3)^{\lambda} - \nu_2^{\lambda-3} (p + 1) ((1 - \nu_3)^{\lambda}(\lambda + \lambda p + p) + (\lambda - 1)\nu_2^p)}{(1 - \nu_3)^{2-2\lambda} \pi^3 p^2 \left( \nu_1 - ((1 - \nu_3)^{\lambda} - \nu_2)^{1/p} \right)^4}$$

$$- \frac{((1 - \nu_3)^{\lambda} - \nu_2^{\lambda-3} 2\nu_1 ((1 - \nu_3)^{\lambda}(p^2 - 2) + p^2) + (\lambda - 1)\nu_2^p)}{(1 - \nu_3)^{2-2\lambda} \pi^3 p^2 \left( \nu_1 - ((1 - \nu_3)^{\lambda} - \nu_2)^{1/p} \right)^4},$$

where we set $\nu_1 := x\overline{x}, \nu_2 := y\overline{y}$ and $\nu_3 := z\overline{z}$.

We write

$$a = 1 - \nu_3, \quad b = (1 - \nu_3)^{\lambda} - \nu_2, \quad c = ((1 - \nu_3)^{\lambda} - \nu_2)^{1/p}.$$  

Then

$$B = \frac{b^{\lambda-3} \nu_1^2(p - 1)(\lambda(p - 1) + p)}{a^{2-2\lambda} \pi^3 p^2 c^4} + \frac{a^{\lambda-2} b^{\lambda-3} \nu_1^2(p - 1)(\lambda - 1)\nu_2^p}{\pi^3 p^2 c^4}$$

$$+ \frac{b^{\lambda-3}(p + 1) \left( a^{\lambda}(\lambda + \lambda p + p) + (\lambda - 1)\nu_2^p \right)}{a^{2-\lambda} \pi^3 p^2 c^4} - \frac{b^{\lambda-3} 2\nu_1 \left( a^{\lambda}(p^2 - 2) + p^2 + (\lambda - 1)\nu_2^p \right)}{a^{2-\lambda} \pi^3 p^2 c^4}.$$
Write $D = a^2 c^4$ and

$$N = a^{2\lambda} b^{-3} (p - 1)(\lambda(p - 1) + p) + a^\lambda b^{-3} (p - 1)(\lambda - 1)p + a^\lambda b^{-3} (p + 1)\left( a^\lambda (\lambda + \lambda p + p) + (\lambda - 1)\nu_2 p \right)$$

$$- a^\lambda b^{-3} \nu_1 \left( a^\lambda (\lambda(p^2 - 2) + p^2) + (\lambda - 1)\nu_2 p^2 \right).$$

Then

$$B = \frac{N}{\pi^3 p^2 D}.$$ (8.3)

Write

$$N_1 = a^{2\lambda} b^{-3} \nu_1^2, \quad N_2 = a^{\lambda} b^{-3} \nu_1 \nu_2, \quad N_3 = a^{2\lambda} b^{-3},$$

$$N_4 = a^{\lambda} b^{-3} \nu_2, \quad N_5 = a^{2\lambda} b^{-3} \nu_1, \quad N_6 = a^{\lambda} b^{-3} \nu_1 \nu_2,$$

$$u_1 = (p - 1)(\lambda(p - 1) + p), \quad u_2 = p(p - 1)(\lambda - 1), \quad u_3 = (p + 1)(\lambda + \lambda p + p),$$

$$u_4 = p(p + 1)(\lambda - 1), \quad u_5 = -2(\lambda(p^2 - 2) + p^2), \quad u_6 = -2(\lambda - 1)p^2.$$

Then

$$N = \sum_{i=1}^{6} u_i N_i.$$

Note that we have

$$u_1 + u_3 + u_5 = 6\lambda \quad \text{and} \quad u_2 + u_4 + u_6 = 0.$$

From the description of the Bergman kernel, we can check the pseudoconvexity of $E_{p,\lambda}$ for each $p, \lambda > 0$.

**Proposition 17.** $E_{p,\lambda}$ is a pseudoconvex domain for each $p, \lambda > 0$.

**Proof.** To show that $u = u_{p,\lambda} := \left( |x|^{2p} + |y|^2 \right)^{'\lambda} + |z|^2$ is a (bounded) plurisubharmonic exhaustion function of $E_{p,\lambda}$, it suffices to show that $v = v_{p,\lambda} := \left( |x|^{2p} + |y|^2 \right)^{12}$ is plurisubharmonic. To this end,

$$\log v = \frac{1}{\lambda} \log \left( e^{\psi_1} + e^{\psi_2} \right), \quad \text{where} \quad \psi_1 := 2p \log |x|, \ \psi_2 := 2 \log |y|,$$

the plurisubharmonicity of $\log v$ follows from the elementary fact that $\log \left( e^{\psi_1} + e^{\psi_2} \right)$ is always plurisubharmonic whenever $\psi_1$ and $\psi_2$ are plurisubharmonic, as a consequence of the ensuing formula of one complex variable:

$$\frac{\partial^2}{\partial z \partial \bar{z}} \log \left( e^{\psi_1} + e^{\psi_2} \right)$$

$$= \frac{1}{\left( e^{\psi_1} + e^{\psi_2} \right)^2} \left( e^{\psi_1 + \psi_2} \left( \frac{\partial \psi_1}{\partial z} - \frac{\partial \psi_2}{\partial \bar{z}} \right)^2 + e^{\psi_1} \frac{\partial^2 \psi_1}{\partial z \partial \bar{z}} + e^{\psi_2} \frac{\partial^2 \psi_2}{\partial z \partial \bar{z}} \right) \geq 0.$$  

From the plurisubharmonicity of $\log v$ it follows that $v = e^{\log v}$ is plurisubharmonic, as desired. □
We are interested in behaviours of the metric and curvature components on the compact set
\(K_1 = \{(0, y, z) \in E_{\lambda,p}; 0 \leq y, z < 1\}\). In what follows, we compute those components.

Recall the formula for the components of the Bergman metric
\[
g_{ij} = \frac{\partial^2 \log B}{\partial z_i \partial \bar{z}_j}, \quad i, j = 1, 2, 3,
\]
where we set \((z_1, z_2, z_3) = (x, y, z)\). For \(i = 1, 2, 3\), we write
\[
\partial_i = \frac{\partial}{\partial z_i} \quad \text{and} \quad \bar{\partial}_i = \frac{\partial}{\partial \bar{z}_i}.
\]

**Proposition 18.** Each component of the Bergman metric \(g_{ij}\) at \((0, y, z) \in E_{p,\lambda}, 0 \leq y, z < 1\), is given as follows:

\[
\begin{align*}
g_{11} &= \frac{1}{c} \cdot \frac{u_5 + u_6 \delta}{u_3 + u_4 \delta} + \frac{4}{c}, \\
g_{22} &= \frac{a^\lambda}{b^2} \left( \frac{1}{p} + 3 \right) + \frac{a^\lambda}{b^2} \cdot \frac{u_3 u_4 (1 - \delta)^2}{(u_3 + u_4 \delta)^2}, \\
g_{33} &= \frac{1 + \delta(\lambda z^2 - 1)}{a^2 - 2 \lambda b^2} \cdot \frac{\lambda}{\lambda + \delta(2 - 2 \lambda) + \delta(2 \lambda z^2 - 4) + \lambda + 2} \\
&\quad + \frac{\lambda \delta}{a^2 - 2 \lambda b^2} \cdot \frac{u_3 u_4 (1 + \delta^2)(1 + \lambda z^2) + u_3^2 \delta(1 + (\lambda z^2 - 1) \delta + \delta^2) + u_3^2(1 + \lambda z^2)}{(u_3 + u_4 \delta)^2}, \\
g_{ij} &= 0 \quad \text{otherwise},
\end{align*}
\]

where we write \(\delta := y^2/a^\lambda = y^2/(1 - z^2)^\lambda\).

**Proof.** All the formulas for \(g_{ij}\) are obtained from direct computations. For example, since

\[
\begin{align*}
\overline{\partial}_1 D &= -4a^2 c^3 x, \quad \overline{\partial}_1 N_1 = 2 a^{2 \lambda} b^\frac{1}{p^3} \nu_1 x, \quad \overline{\partial}_1 N_2 = 2 a^{2 \lambda} b^\frac{1}{p^3} \nu_1 x \nu_2, \\
\overline{\partial}_1 N_3 &= 0, \quad \overline{\partial}_1 N_4 = 0, \quad \overline{\partial}_1 N_5 = a^{2 \lambda} b^\frac{1}{p^3} x, \quad \overline{\partial}_1 N_6 = a^{2 \lambda} b^\frac{1}{p^3} x \nu_2,
\end{align*}
\]

and

\[
\begin{align*}
\partial_i \overline{\partial}_1 D &= -4a^2 c^3 + 12a^2 c^2 \nu_1, \quad \partial_i \overline{\partial}_1 N_1 = 4 a^{2 \lambda} b^\frac{1}{p^3} \nu_1, \quad \partial_i \overline{\partial}_1 N_2 = 4 a^{2 \lambda} b^\frac{1}{p^3} \nu_1 \nu_2, \\
\partial_i \overline{\partial}_1 N_3 &= 0, \quad \partial_i \overline{\partial}_1 N_4 = 0, \quad \partial_i \overline{\partial}_1 N_5 = a^{2 \lambda} b^\frac{2}{p^3}, \quad \partial_i \overline{\partial}_1 N_6 = a^{2 \lambda} b^\frac{2}{p^3} \nu_2,
\end{align*}
\]
we have
\[ g_{11} = \frac{N(\partial_{1}\bar{\partial}_{1}N) - (\partial_{1}N)(\bar{\partial}_{1}N)}{N^2} - \frac{D(\partial_{1}\bar{\partial}_{1}D) - (\partial_{1}D)(\bar{\partial}_{1}D)}{D^2}, \]
\[ \frac{(0,y,z)}{\partial_{1}\bar{\partial}_{1}N - \partial_{1}\bar{\partial}_{1}D} = \frac{u_{5}a^{2}\lambda b_{p}^{2} - 3 + u_{6}a\lambda b_{p}^{2} - 3 y^{2} + 4a^{2}c^{3}}{u_{3}a^{2}\lambda b_{p}^{2} - 3 + u_{4}a\lambda b_{p}^{2} - 3 y^{2}}, \]
\[ = \frac{1}{c} \cdot \frac{u_{5} + u_{6} \delta}{u_{3} + u_{4} \delta} + \frac{4}{c}, \]
where we use \( c = b_{p}^{2} \) at \((0,y,z)\).

The other \( g_{ij} \) can be computed similarly, and we omit the details. \( \Box \)

**Remark 19.** When \((0,y,z)\) approaches the boundary of \( K_{1} \), we find that the limits of the metric components and those of curvature components cannot be determined. However, using \( \delta \) introduced in the above proposition, we will be able to control the limit behaviors.

Write
\[ g_{11} = \frac{1}{c} \cdot A_{1}, \quad g_{22} = \frac{a^{2}}{b^{2}} \cdot A_{2}, \quad g_{33} = \frac{\lambda yz}{a^{2} - \lambda b^{2}} \cdot A_{2}, \quad g_{33} = \frac{1}{a^{2} - 2\lambda b^{2}} \cdot A_{3}, \quad (8.4) \]

where
\[ A_{1} = \frac{u_{5} + u_{6} \delta}{u_{3} + u_{4} \delta} + 4, \quad A_{2} = \frac{1}{p} + 3 + \frac{u_{3}u_{4}(1 - \delta)^{2}}{(u_{3} + u_{4} \delta)^{2}}, \]
\[ A_{3} = (1 + \delta(\lambda z^{2} - 1)) \cdot \frac{\lambda}{p} + \delta^{2}(2 - 2\lambda) + \delta(2\lambda^{2}z^{2} - 4) + \lambda + 2 \]
\[ + \lambda \delta \cdot \frac{u_{3}u_{4}(1 + \delta^{2})(1 + \lambda z^{2}) + u_{3}^{2}\delta(1 + (\lambda z^{2} - 1)\delta + \delta^{2}) + u_{3}^{2}(1 + \lambda z^{2})}{(u_{3} + u_{4} \delta)^{2}}. \]

Then
\[ g_{22}g_{33} - g_{23}g_{32} = \frac{1}{a^{2} - 3\lambda b^{4}} \cdot A_{2}(A_{3} - \lambda^{2} \delta z^{2} A_{2}) = \frac{1 - \delta}{a^{2} - 3\lambda b^{4}} \cdot A_{2}A_{4} = \frac{A_{2}A_{4}}{a^{2} - 2\lambda b^{2}}, \quad (8.5) \]

where we put \( A_{4} := (A_{3} - \lambda^{2} \delta z^{2} A_{2})/(1 - \delta) \) and use \( 1 - \delta = b/a^{\lambda} \). More explicitly, we have
\[ A_{4} = \frac{\delta^{2} p^{2} (r - 2)(r - 1) + \delta p (r - 1)(4pr + 4p + 3r) + p^{2} r^{2} + 3p^{2} r + 2p^{2} + 2pr^{2} + 3pr + r^{2}}{p^{2} (p r - 1) + p r + p + r}. \]

Note that \( 0 \leq \delta < 1 \). Furthermore, as \((0,y,z) \in E_{p,\lambda} \) approaches the boundary, we have \( \delta \to 1^{-} \). One sees that
\[ \lim_{\delta \to 1^{-}} A_{1} = \frac{4(2 + p)}{1 + 2p}, \quad \lim_{\delta \to 1^{-}} A_{2} = \frac{3 + \frac{1}{p}}{p} \quad \text{and} \quad \lim_{\delta \to 1^{-}} A_{4} = \lambda \left( 3 + \frac{1}{p} \right). \quad (8.6) \]

**Lemma 20.** At \((0,y,z) \in E_{p,\lambda}, 0 \leq y, z < 1\), the ratio \( \frac{\det g_{ij}}{B} \) is bounded.
Proof. From (8.3), (8.4) and (8.5), we obtain
\[
\frac{\det g_B}{B} = \frac{1}{\zeta N} A_1 A_2 A_4 \left( \frac{\pi^3 p^2 A_1 A_2 A_4 a^2 c^4}{ca^2 - 2\lambda b^3 \cdot a^\lambda b^{p + 3} (p + 1) (a^\lambda (\lambda + \lambda p + p) + (\lambda - 1) y^2 p)} \right) \left( p + 1 \right) \left( (\lambda + \lambda p + p) + (\lambda - 1) \delta \right),
\]
which is bounded. \qed

**Proposition 21.** The inverse metric of the Bergman metric \( g_\gamma \) at \((0, y, z) \in E_{p, \lambda}, 0 \leq y, z < 1\), are given as follows:
\[
g_\gamma^{\bar{\theta}} = \frac{\delta}{A_1}, \quad g_\gamma^{\bar{\xi}} = \frac{\partial_2 g_{\bar{\eta}}} {A_4}, \quad g_\gamma^{\bar{y}} = \delta = 0 \text{ otherwise}.
\]

Proof. The formulas are obtained by taking the inverse matrix of the \(3 \times 3\) matrix \((g_\gamma)_{i,j=1,2,3}\) calculated in Proposition 18. In particular, the determinant of the \(2 \times 2\) block \((g_\gamma)_{i,j=2,3}\) is computed in (8.5). Also recall \( 1 - \delta = b/a^\lambda \). \qed

Through direct computations, we obtain the following for \((0, y, z) \in K_1:\n
\[
\begin{align*}
\partial_1 g_\gamma^{\bar{\theta}} &= \partial_2 g_\gamma^{\bar{\xi}} = \partial_3 g_\gamma^{\bar{y}} = \frac{y}{bc} G_1, \\
\partial_1 g_\gamma^{\bar{\eta}} &= \partial_3 g_\gamma^{\bar{\xi}} = \partial_3 g_\gamma^{\bar{y}} = \frac{z}{a^{1 - \lambda} b c} G_2, \\
\partial_2 g_\gamma^{\bar{\eta}} &= \partial_2 g_\gamma^{\bar{\xi}} = \frac{ya^\lambda}{b^3} G_3, \\
\partial_2 g_\gamma^{\bar{y}} &= \partial_2 g_\gamma^{\bar{\xi}} = \frac{y^2 z}{a^{1 - \lambda} b^3} G_4, \\
\partial_2 g_\gamma^{\bar{\xi}} &= \partial_3 g_\gamma^{\bar{\eta}} = \frac{y^2 z}{a^{1 - \lambda} b^3} G_5, \\
\partial_3 g_\gamma^{\bar{\eta}} &= \partial_3 g_\gamma^{\bar{\xi}} = \frac{yz^2}{a^{2 - 2\lambda} b^3} G_6, \\
\partial_3 g_\gamma^{\bar{\xi}} &= \partial_3 g_\gamma^{\bar{y}} = \frac{yz^2}{a^{2 - 2\lambda} b^3} G_7, \\
\partial_3 g_\gamma^{\bar{y}} &= \partial_3 g_\gamma^{\bar{\xi}} = \frac{z}{a^{3 - 3\lambda} b^3} G_8, \\
\partial_i g_\gamma^{\bar{k}} &= \partial_i g_\gamma^{\bar{k}} = 0 \text{ otherwise}.
\end{align*}
\]

**Table 1:** Formulas for \( \partial_i g_\gamma^{\bar{k}} \)
Here $G_i$ are set to be the remaining factors after pulling out the factors involving $a,b,c,y,z$. Explicitly, we have

$$G_1 = \frac{4}{p} - \frac{(u_5 + u_6\delta)((2p - 3)u_4\delta + 3(p - 1)u_3 + pu_4)}{p(u_3 + \delta u_4)^2} + \frac{2(p - 1)u_6\delta + (3p - 2)u_5 + pu_6}{p(u_3 + \delta u_4)},$$

$$G_2 = \frac{4\lambda}{p} + \frac{\lambda}{p} \cdot \frac{u_5 + u_6\delta}{u_3 + u_4\delta} - \frac{\lambda\delta(1 - \delta)(u_4u_5 - u_3u_6)}{(u_3 + u_4\delta)^2},$$

For simplicity, we do not present expressions for the other $G_i$’s. Since $u_3 + u_4\delta > 0$, one can see that $G_i$ are bounded for $i = 1, 2, \ldots, 8$ as $\delta \to 1^-$. 

**Lemma 22.** We have

$$G_4 = \lambda G_3.$$

If we define $F_1$ and $F_2$ by

$$F_1 := \frac{z^2}{1 - \delta} (G_6 - \lambda\delta G_5) \quad \text{and} \quad F_2 := \frac{1}{1 - \delta} (G_8 - \lambda\delta z^2 G_7),$$

then

$$\lim_{\delta \to 1^-} F_1 = \lambda \left(3 + \frac{1}{p}\right) \quad \text{and} \quad \lim_{\delta \to 1^-} F_2 = \frac{2\lambda^2(1 + 3p)}{p}.$$

**Proof.** We verify the identities through direct computations with help of a computer algebra system. \qed

Similarly, we obtain
$\partial_1 \overline{\partial}_1 g_{1\overline{1}} = \frac{1}{c^2} H_1,$

$\partial_1 \overline{\partial}_2 g_{2\overline{2}} = \partial_2 \overline{\partial}_1 g_{1\overline{2}} = \partial_2 \overline{\partial}_2 g_{1\overline{1}} = \frac{a^\lambda}{b^c} H_2,$

$\partial_1 \overline{\partial}_3 g_{3\overline{3}} = \partial_1 \overline{\partial}_1 g_{1\overline{3}} = \partial_2 \overline{\partial}_1 g_{1\overline{3}} = \partial_1 \overline{\partial}_2 g_{3\overline{2}} = \partial_3 \overline{\partial}_1 g_{1\overline{3}} = \partial_3 \overline{\partial}_2 g_{1\overline{1}} = \frac{y z}{a^2 - \lambda b^2} H_3,$

$\partial_1 \overline{\partial}_3 g_{3\overline{3}} = \partial_1 \overline{\partial}_1 g_{1\overline{3}} = \partial_3 \overline{\partial}_1 g_{1\overline{3}} = \partial_3 \overline{\partial}_3 g_{1\overline{1}} = \frac{1}{a^2 - 2\lambda b^2 c} H_4,$

$\partial_2 \overline{\partial}_2 g_{2\overline{2}} = \frac{a^{2\lambda}}{b^4} H_5,$

$\partial_2 \overline{\partial}_2 g_{2\overline{2}} = \partial_3 \overline{\partial}_3 g_{3\overline{3}} = \partial_2 \overline{\partial}_2 g_{2\overline{2}} = \partial_3 \overline{\partial}_3 g_{3\overline{3}} = \frac{y z}{a^1 - 2\lambda b^4} H_6,$

$\partial_2 \overline{\partial}_2 g_{2\overline{2}} = \partial_3 \overline{\partial}_3 g_{3\overline{3}} = \partial_2 \overline{\partial}_2 g_{2\overline{2}} = \partial_3 \overline{\partial}_3 g_{3\overline{3}} = \frac{1}{a^2 - 3\lambda b^4} H_7,$

$\partial_3 \overline{\partial}_2 g_{2\overline{2}} = \partial_3 \overline{\partial}_2 g_{2\overline{2}} = \frac{y z^2}{a^2 - 2\lambda b^4} H_8,$

$\partial_2 \overline{\partial}_3 g_{3\overline{3}} = \partial_3 \overline{\partial}_3 g_{3\overline{3}} = \partial_2 \overline{\partial}_2 g_{2\overline{2}} = \partial_3 \overline{\partial}_3 g_{3\overline{3}} = \frac{y z}{a^1 - 3\lambda b^4} H_9,$

$\partial_3 \overline{\partial}_3 g_{3\overline{3}} = \frac{1}{a^4 - 4\lambda b^4} H_{10},$

$\partial_3 \overline{\partial}_j g_{k\ell} = 0$ otherwise.

Table 2: Formulas for $\partial_i \overline{\partial}_j g_{k\ell}$

Here $H_i$ are the remaining factors; in particular, we have

$$H_1 = 8 + 4 \cdot \frac{u_1 + u_2 \delta}{u_3 + u_4 \delta} - 2 \cdot \frac{(u_5 + u_6 \delta)^2}{(u_3 + u_4 \delta)^2}.$$  

We do not present explicit expressions for the other $H_i$’s. Using $0 \leq \delta < 1$ and $u_3 + u_4 \delta > 0$, one can check that $H_i$ are bounded for $i = 1, 2, \ldots, 10$ as $\delta \rightarrow 1^-$. 

Proposition 23. Each curvature components of the Bergman metric at \((0, y, z) \in E_{p, \lambda}, 0 \leq y, z < 1\), is given by

\[
R_{1111} = \frac{1}{c^2}(-H_1) = \frac{1}{c^2} \cdot \tilde{H}_1,
\]

\[
R_{1122} = R_{2112} = R_{1133} = R_{3211} = \frac{a^\lambda}{b^2 c} \cdot \left(-H_2 + \frac{\delta G_1^2}{A_1}\right) = \frac{a^\lambda}{b^2 c} \cdot \tilde{H}_2,
\]

\[
R_{1132} = R_{1212} = R_{2211} = R_{1321} = R_{3212} = R_{3311} = \frac{a^{2\lambda-2}}{b^4} \cdot \left(-H_4 + \frac{z^2 G_2^2}{A_1}\right) = \frac{a^{2\lambda-2}}{b^4} \cdot \tilde{H}_4,
\]

\[
R_{1133} = R_{1333} = R_{3313} = R_{3333} = \frac{a^{3\lambda-2}}{b^4} \cdot \left(-H_5 + \frac{\delta G_3^2}{A_2}\right) = \frac{a^{3\lambda-2}}{b^4} \cdot \tilde{H}_5,
\]

\[
R_{1222} = R_{2222} = R_{2322} = R_{3222} = \frac{y z a^{\lambda-1}}{b^4} \cdot \left(-H_6 + \frac{G_4 G_5}{A_2}\right) = \frac{y z a^{\lambda-1}}{b^4} \cdot \tilde{H}_6,
\]

\[
R_{2333} = R_{2333} = R_{3333} = R_{3333} = \frac{a^{3\lambda-2} y z}{b^4} \cdot \left(-H_7 + \frac{\delta^2 G_2^2}{A_2} + \frac{\delta(1 - \delta) F_1^2}{A_2}\right) = \frac{a^{3\lambda-2} y z}{b^4} \cdot \tilde{H}_7,
\]

\[
R_{3333} = R_{3333} = R_{3333} = R_{3333} = \frac{a^{3\lambda-3} y z}{b^4} \cdot \left(-H_8 + \frac{G_5 G_7}{A_2} + \frac{(1 - \delta) F_1 F_2}{A_2}\right) = \frac{a^{3\lambda-3} y z}{b^4} \cdot \tilde{H}_8,
\]

\[
R_{3333} = R_{3333} = R_{3333} = R_{3333} = \frac{a^{4\lambda-4}}{b^4} \cdot \left(-H_9 + \frac{\delta^4 G_2^2}{A_2} + \frac{z^2 (1 - \delta) F_2^2}{A_2}\right) = \frac{a^{4\lambda-4}}{b^4} \cdot \tilde{H}_9,
\]

where we define \(\tilde{H}_i\) for \(i = 1, 2, \ldots, 10\) for later use.

Proof. Recall that the components of curvature tensor \(R\) associated with \(g\) is given by

\[
R_{\gamma\delta\gamma\delta} = -\partial_{k} \partial_{l} g_{\gamma\delta} + \sum_{p, q=1}^{3} g^{\sigma\gamma}(\partial_{k} g_{\sigma\delta})(\partial_{l} g_{\gamma\delta}).
\]

Thus the results follow from Tables 1 and 2 and Proposition 21.

\[
\tilde{H}_3 = \lambda \tilde{H}_2, \quad \tilde{H}_6 = \lambda \tilde{H}_5, \quad \tilde{H}_8 = \lambda \tilde{H}_6 \quad \text{and} \quad \tilde{H}_9 = 2 \lambda \tilde{H}_7 - \lambda^2 \delta z^2 \tilde{H}_6.
\]

Lemma 24. We have

\[
\tilde{H}_3 = \lambda \tilde{H}_2, \quad \tilde{H}_6 = \lambda \tilde{H}_5, \quad \tilde{H}_8 = \lambda \tilde{H}_6 \quad \text{and} \quad \tilde{H}_9 = 2 \lambda \tilde{H}_7 - \lambda^2 \delta z^2 \tilde{H}_6.
\]
If we define
\[
\tilde{F}_1 := \frac{1}{1 - \delta} \left( \tilde{H}_4 - \lambda \delta z^2 \tilde{H}_3 \right), \quad \tilde{F}_2 := \frac{1}{1 - \delta} \left( \tilde{H}_7 - \lambda \delta z^2 \tilde{H}_6 \right),
\]
\[
\tilde{F}_3 = \frac{1}{(1 - \delta)^2} \left( \tilde{H}_{10} - 4\lambda^2 \delta z^2 \tilde{H}_7 + 3\lambda^3 \delta^2 z^4 \tilde{H}_6 \right),
\]
then
\[
\lim_{\delta \to 1^-} \tilde{F}_1 = -\frac{4\lambda(2 + p)}{p(1 + 2p)}, \quad \lim_{\delta \to 1^-} \tilde{F}_2 = -\lambda \left( 3 + \frac{1}{p} \right) \quad \text{and} \quad \lim_{\delta \to 1^-} \tilde{F}_3 = -2\lambda^2 \left( 3 + \frac{1}{p} \right). \quad (8.7)
\]

**Proof.** The identities are verified through direct computations and can be checked by a computer algebra system. \(\square\)

In order to see cancellations of factors involving \(a, b, c\) in the holomorphic sectional curvature, we apply the Gram–Schmidt process to determine an orthonormal frame \(X, Y, Z\) instead of using the global coordinate vector fields \(\frac{\partial}{\partial z_i}, i = 1, 2, 3\). Indeed, let \(g\) be any Hermitian metric, and take the first unit vector field
\[
X = \frac{\partial_1}{\sqrt{g_{11}}}. \quad (8.8)
\]
Write \(k_1 := \frac{1}{\sqrt{g_{11}}} \) so that \(X = k_1 \partial_1\). Then a vector field \(\tilde{Y}\) which is orthogonal to \(X\) is given by
\[
\tilde{Y} = \frac{\partial_2}{\sqrt{g_{22}}} - g \left( \frac{\partial_2}{\sqrt{g_{22}}}, X \right) X = a_1 \partial_1 + a_2 \partial_2,
\]
where we put
\[
a_1 := -\frac{g_{21}}{g_{11} \sqrt{g_{22}}} \quad \text{and} \quad a_2 := \frac{1}{\sqrt{g_{22}}}.
\]
Since \(g(\tilde{Y}, \tilde{Y}) = a_1 \overline{a_1} g_{11} + a_1 \overline{a_2} g_{12} + a_2 \overline{a_1} g_{21} + a_2 \overline{a_2} g_{22}\), we take
\[
Y = \frac{\tilde{Y}}{\sqrt{g(\tilde{Y}, \tilde{Y})}} = \frac{a_1 \partial_1 + a_2 \partial_2}{\sqrt{a_1 \overline{a_1} g_{11} + a_1 \overline{a_2} g_{12} + a_2 \overline{a_1} g_{21} + a_2 \overline{a_2} g_{22}}} = t_1 \partial_1 + t_2 \partial_2, \quad (8.9)
\]
where we put
\[
t_i := \frac{a_i}{\sqrt{a_1 \overline{a_1} g_{11} + a_1 \overline{a_2} g_{12} + a_2 \overline{a_1} g_{21} + a_2 \overline{a_2} g_{22}}}, \quad i = 1, 2. \quad (8.10)
\]

Similarly, consider
\[
\tilde{Z} = p_1 \partial_1 + p_2 \partial_2 + p_3 \partial_3,
\]
where
\[
p_1 := -\frac{g_{33}}{g_{11} \sqrt{g_{33}}} - \frac{t_1}{\sqrt{g_{33}}} (t_1 g_{31} + t_2 g_{32}),
\]
\[
p_2 := -\frac{t_2}{\sqrt{g_{33}}} (t_1 g_{31} + t_2 g_{32}), \quad p_3 := \frac{1}{\sqrt{g_{33}}}
\]
Normalizing \( \tilde{Z} \) yields

\[
Z = s_1 \partial_1 + s_2 \partial_2 + s_3 \partial_3, \tag{8.11}
\]

where

\[
s_i := \frac{p_i}{\sqrt{\sum_{k,l=1}^{3} p_k p_l g_{k\bar{l}}}}, \quad i = 1, 2, 3.
\]

These \( X, Y, Z \) are used in the following proposition which is the main result of this section.

**Proposition 25.** At \( (0, y, z) \in E_{p,\lambda}, 0 \leq y, z < 1 \), the components of the holomorphic sectional curvature \( R \) are given by as follows.

\[
H(X) = R(X, \bar{X}, X, \bar{X}) = \frac{\bar{H}_1}{A_1^2}, \quad B(Y, Z) = R(X, \bar{X}, \bar{Y}, Y) = \frac{\bar{H}_2}{A_1 A_2},
\]

\[
H(Y) = R(Y, \bar{Y}, Y, \bar{Y}) = \frac{\bar{H}_5}{A_2^2}, \quad B(X, Z) = R(X, \bar{X}, Z, \bar{Z}) = \frac{\bar{F}_1}{A_1 A_4},
\]

\[
H(Z) = R(Z, \bar{Z}, Z, \bar{Z}) = \frac{\bar{F}_3}{A_4^2}, \quad B(Y, Z) = R(Y, \bar{Y}, Z, \bar{Z}) = \frac{\bar{F}_2}{A_2 A_4},
\]

\[
R(X, \bar{X}, X, \bar{Y}) = R(Y, \bar{Y}, Y, \bar{X}) = R(Z, \bar{Z}, Z, \bar{Y}) = R(Y, \bar{X}, \bar{X}, \bar{X}) = 0,
\]

\[
R(X, \bar{X}, Z, \bar{Z}) = R(Y, \bar{Y}, Y, \bar{Z}) = R(Z, \bar{Z}, Z, \bar{X}) = R(Z, \bar{X}, \bar{X}, \bar{X}) = 0,
\]

\[
R(X, \bar{X}, Y, \bar{Z}) = R(Y, \bar{Y}, X, \bar{Z}) = R(Z, \bar{Z}, X, \bar{Y}) = R(Z, \bar{Y}, Y, \bar{Y}) = 0.
\]

**Proof.** All the identities follow from Proposition 23 and Lemma 24. To illustrate the process, we compute \( H(X) \), \( B(X, Y) \) and \( R(Y, \bar{Y}, Y, \bar{Z}) \). Computations of the other components are similar.

Since \( g_{\bar{Z}Z} = 0 \) and \( g_{\bar{Y}Y} = 0 \), we have \( a_1 = 0, t_1 = 0, p_1 = 0 \) and \( s_1 = 0 \) on \( (0, y, z) \). On the other hand,

\[
t_2 = \frac{a_2}{\sqrt{a_2^2 b_2^2 g_{\bar{Z}Z}}} = \frac{1}{\sqrt{g_{\bar{Z}Z}}}.
\]

Thus, using (8.4), we obtain

\[
H(Y) = t_2^4 R_{\bar{Z}Z} = \frac{b^4}{a^{2\lambda} A_2^2} \cdot \frac{a^{2\lambda} \bar{H}_5}{b^4} = \frac{\bar{H}_5}{A_2^2}.
\]

Similarly,

\[
B(X, Y) = k_1^2 t_2^4 R_{\bar{Y}Y} = \frac{1}{g_{\bar{Y}Y} g_{\bar{Z}Z}} \cdot \frac{1}{a^{\lambda} b^2 c} \cdot \bar{H}_2 = \frac{c}{A_1 a^\lambda A_2 b^2 c} \bar{H}_2 = \frac{1}{A_1 A_2} \bar{H}_2.
\]

To compute \( R(Y, \bar{Y}, Y, \bar{Z}) \), first observe

\[
s_2 = -s_3 t_2^2 g_{\bar{Z}Z} = -s_3 \frac{g_{\bar{Z}Z}}{g_{\bar{Z}Z}} = -s_3 \frac{\lambda y z}{a}.
\]
Thus it follows from Proposition 23 and Lemma 24 that

\[
R(Y, \bar{Y}, Y, \bar{Z}) = t_2^3 s_2 R_{2\bar{2}2\bar{2}} + t_2^3 s_3 R_{2\bar{2}3\bar{2}} = t_2^3 \left( -s_3 \frac{\lambda yz}{a} \right) a^2 b^4 \bar{H}_5 + t_2^3 s_3 \frac{yz a^{2\lambda - 1}}{b^4} \bar{H}_6
\]

\[
= \frac{t_2^3 s_3 a^{2\lambda - 1} yz}{b^4} (\lambda \bar{H}_5 + \bar{H}_6) = 0.
\]

\[\square\]

**Corollary 26.** The holomorphic sectional curvature near \(\partial K_1\) is bounded for any \(p, \lambda > 0\).

**Proof.** The assertion follows from (8.6) and (8.7) and the fact that \(G_i\) and \(H_i\) are bounded as \(\delta \to 1^−\).

\[\square\]

It is known [8] that the curvature tensor of the Bergman metric is bounded for \(\lambda = 1\) and \(p > 0\). The following proposition tells us that the same is true for any \(p, \lambda > 0\).

**Proposition 27.** The curvature tensor of the Bergman metric on \(E_{p,\lambda}\) is bounded for any \(p, \lambda > 0\).

**Proof.** The curvature tensor can be explicitly expressed in terms of the holomorphic sectional curvature \(H_{gB}\). Using the invariance of the Bergman metric, it suffices to show \(H_{gB} \leq C\) on \(\partial K_1\) by some constant \(C \in \mathbb{R}\). By Corollary 26, we are done.

\[\square\]

**Corollary 28.** For any \(p, \lambda > 0\), there exist \(C_0 > 0\) such that

\[
\chi_{E_{p,\lambda}}(p; v) \leq C_0 \sqrt{\omega_B(v, v)} \quad \text{for all } v \in T'_p E_{p,\lambda}, \ p \in M,
\]

and \(C_1 > 0\) such that

\[
\frac{1}{C_1} \omega_{KE}(v, v) \leq \omega_B(v, v) \leq C_1 \omega_{KE}(v, v) \quad \text{for all } v \in T'_p E_{p,\lambda}.
\]

**Proof.** The consequences immediately follow from Proposition 27 and Lemma 20.

\[\square\]

**Remark 29.** For the third statement of Theorem 4, in general, the holomorphic sectional curvature is not negatively pinched for \(E_{p,\lambda}\). For example, when \(\lambda = 1\) and \(p = 1/5\), we have \(\lim_{\delta \to 1^-} H(X) \approx 0.033 > 0\).

Lastly, we obtain interesting rigidity from the direct computation of the Ricci curvature of the Bergman metric.

**Proposition 30.** The Bergman metric \(g_B\) on \(E_{p,\lambda}\) is a Kähler–Einstein metric if and only if \(\lambda = p = 1\).
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