A REVIEW OF \( W \) STRINGS

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Abstract

Recent progress on the physical states and scattering amplitudes of the \( W_3 \) string is reviewed with particular emphasis on the relation between this string theory and the Ising model.
1. Introduction

I carried out my doctorate studies at Imperial College and had the good fortune to have Abdus Salam as my supervisor. When I began research, the paper of Wess and Zumino [33] had induced many of Europe’s leading physicists to work on supersymmetry, and Abdus Salam and John Strathdee [34] had just written their classic paper discovering superspace and super-Feynman rules. With such rapid progress being made it was not easy for a graduate student to achieve anything of significance, but despite his many commitments Abdus Salam was always ready to give helpful advice and encouragement. It was impossible not to be infected by his great enthusiasm for new ideas and the enjoyment he derived from doing physics. One came away from his office feeling that all was possible and that failure was only a temporary phenomenon.

Although it had been understood [35] how to break supersymmetry using the classical potential, it was thought to be more desirable if it could be broken using radiative corrections. Abdus Salam characteristically encouraged Bob Delbourgo and myself to systematically examine every possibility. He also, however, advocated that if all else failed one could always tell the truth. In this case, as I eventually found, the truth was that if supersymmetry was preserved classically then the effective potential vanished [36]. It was this theorem which allowed others [37] to observe that supersymmetry solved the technical hierarchy problem.

The subject of my talk is $W_3$ strings. Paul Howe and myself, and also Bilal and Gervais, began studying this subject in 1988. The realisation that the $W_3$ algebra could be used to construct a new string theory was rather exciting, since we thought that a much stronger algebra than the Virasoro algebra must lead to a more exciting string than the bosonic string. After finding some interesting results, we became despondent. There were two reasons for this; firstly the only realisation known consisted of two scalars leaving no room for space-time. Secondly, the construction of the BRST charge by Thierry-Mieg had an intercept 4 which suggested the possibility of massless particles of spin greater than 2 in the theory. We realised, however, that tachyonic particles with spin necessarily have negative norm states, because we can choose our frame of reference such that the non-zero components carry a time-like index. We could not see how to eliminate such particles. No doubt, had we talked to Abdus Salam we would have continued,
but in the event we gave up.

We now know that $W_3$ strings do not involve massless higher spin particles and have available representations involving more than 2 scalars. The message of my talk is that despite their, perhaps, rather artificial appearance $W_3$ strings show all the magic of ordinary strings, they obey a no ghost theorem, have scattering amplitudes which obey duality and factorisation and they are modular invariant. Also one finds that the Ising model pervades all aspects of the theory.

The discovery, by Zamolodchikov [1], of two dimensional algebras that contain currents of spins greater than two has led to many interesting new developments. The simplest such algebra, $W_3$ is generated by the spin 2 energy-momentum tensor $T(z) = \sum L_n z^{-n-2}$ and the spin 3 current $W(z) = \sum W_n z^{-n-3}$; the algebra being

$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}$,

$[L_n, W_m] = (2n-m)W_{n+m}$,

$[W_n, W_m] = \frac{16}{22+5c}(n-m)\Lambda_{n+m}$

$+ (n-m)\left[\frac{1}{15}(n+m+2)(n+m+3) - \frac{1}{6}(n+2)(m+2)\right]L_{m+n}$

$+ \frac{c}{360}n(n^2-1)(n^2-4)\delta_{n+m,0}$,  \hspace{1cm} (1.1)

where

$\Lambda_n = \sum_m :L_{n+m}L_{-m}: - \frac{1}{20}\left[n^2 - 4 - \frac{5}{2}(1 - (-1)^n)\right]L_n \hspace{1cm} (1.2)$

and the normal ordering means that $:L_p L_q: = L_q L_p$ if $p > q$.

This algebra occurs naturally in the minimal model with central charge 4/5 since this theory contains a spin 3 primary field. The above algebra, however, is consistent for all values the central charge $c$. The most important new feature of these new algebras is that they are not Lie algebras; their commutators are expressed as a sum of terms some of which are quadratic in the generators.

An important consequence of this fact is that, unlike the Virasoro algebra, two representations of a $W_3$ algebra cannot, in general, be added to form a third. The earliest known realisation of the $W_3$ algebra is in terms of 2 scalars [2]; it is a generalisation of the classical Miura transformation that takes the mKdV equation into the KdV equation. It was found [3], however that one of the scalars
only occurred through its energy-momentum tensor and consequently, one could replace it by any energy momentum tensor with the same central charge. In particular, one could use any number of scalar fields, say \( x^\mu \) for \( \mu = 0, 1, ..., D - 1 \) with a suitable background charge. Denoting the other of the two original scalars by \( \varphi \), the \( W_3 \) generators, which carry a superscript \( m \), corresponding to matter, are given by

\[
T^m = T^\varphi + T^x
\]  

where

\[
T^\varphi = -\frac{1}{2} (\partial \varphi)^2 - Q \partial^2 \varphi,
\]

\[
T^x = -\frac{1}{2} \partial x^\mu \partial x_\mu - \alpha_\mu \partial^2 x^\mu,
\]

and

\[
W^m = -\frac{2i}{\sqrt{261}} \left[ \frac{1}{3} (\partial \varphi)^3 + Q \partial \varphi \partial^2 \varphi + \frac{1}{3} Q^2 \partial^3 \varphi + 2 \partial \varphi T^x + Q \partial T^x \right],
\]

It was natural given the existence of the \( W_3 \) algebra to try to build new string theories. Although there are, with hindsight, a number of ways of constructing the bosonic string, the path most suited to our present knowledge of the \( W_3 \) algebra is as follows: starting from the Virasoro algebra we construct the BRST charge \( Q \) and demand its square vanish. This condition implies that \( c = 26 \) and the intercept is one [4]. We then find a realization of the Virasoro algebra with \( c = 26 \), for example 26 free scalars \( x^\mu, \mu = 0, 1, ..., 25 \). The physical states are the non-trivial cohomology classes of \( Q \) which are also subject to a ghost constraint [4,5,15]. Given the physical states, one can then construct the scattering amplitudes.

The construction of a \( W_3 \) string along these lines was first discussed in reference [6] and the \( W_3 \) string with the field content given here was first formulated in reference [7]. Fortunately, the BRST charge for the \( W_3 \) algebra had previously been constructed [8] and found to square to zero if \( c = 100 \) and the intercept was 4. To build this operator one must introduce the usual reparameterization ghosts \( b \) and \( c \), and \( W_3 \) transformation ghosts \( d \) and \( e \) which have spins 3 and \(-2\) respectively. The total energy-momentum tensor \( T^{tot} \) and spin 3 current \( W^{tot} \) of the combined ghost and matter system are of the form

\[
T^{tot} = T^m + T^{gh}, \quad W^{tot} = W^m + W^{gh}
\]
where

\[
T^{gh} = -2b \partial c - \partial b c - 3d \partial e - 2\partial d e,
\]

\[
W^{gh} = -\partial d c - 3d \partial c - \frac{8}{9.27} [\partial (b e T^m) + b \partial e T^m]
+ \frac{25}{6.9.27} (2e \partial^3 b + 9d e \partial^2 b + 15\partial^2 e \partial b + 10\partial^3 e b).
\]

Here the background charge \(Q\) is given by \(Q^2 = 49/8\), and \(\alpha\) is such that \(T^x\) has central charge \(51/2\). The BRST charge \(Q\) is given by

\[
Q = \int dz j^{BRST},
\]

where

\[
j^{BRST} = c(T^m + \frac{1}{2} T^{gh}) + e(W^m + \frac{1}{2} W^{gh}).
\]

The reader will be aware of the difference between the background charge \(Q\) and the BRST charge \(Q\) from the different contexts in which they are used. Some useful relations are

\[
T^{tot}(z) = \{Q, b(z)\}, \quad W^{tot}(z) = \{Q, d(z)\},
\]

as a consequence of which

\[
[Q, T^{tot}(z)] = [Q, W^{tot}(z)] = 0.
\]

It will be useful to discuss the various possible vacua associated with the ghosts. The natural vacuum, \(|0\rangle\), of the ghost system is that for which \(e(z) = \sum_n e_n z^{n+2}\) and \(d(z) = \sum_n d_n z^{n-3}\) are well defined at \(z = 0\). This requires \(9\)

\[
e_n |0\rangle = 0, \quad n \geq 3, \quad d_n |0\rangle = 0, \quad n \geq -2
\]

\[
c_n |0\rangle = 0, \quad n \geq 2, \quad b_n |0\rangle = 0, \quad n \geq -1.
\]

We can construct other vacua by acting on \(|0\rangle\) with \(e_n\) for \(n = 0, \pm 1, \pm 2\) and with \(c_n\) for \(n = 0, \pm 1\). One of the most useful is

\[
c_1 e_1 e_2 |0\rangle \equiv | \downarrow \rangle,
\]
which is annihilated by \( e_n, c_n \) for \( n \geq 1 \) and \( b_n, d_n \) for \( n \geq 0 \). In terms of the conformal fields we may express the relation between the two states as

\[
c \partial e e |0\rangle = | \downarrow \rangle,
\]

where \( c \partial e e \) is understood to be evaluated at \( z = 0 \). Similar formulae hold for the other vacuum states.

In order to gain a non-zero vacuum expectation value with respect to the state \( |0\rangle \) we must have 3 factors of \( c \) and 5 of \( e \). We set

\[
\langle 0 | c_{-1} c_0 c_1 e_{-2} e_{-1} e_0 e_1 e_2 |0\rangle = \frac{1}{576} \langle 0 | \partial^2 c \partial c \partial^4 e \partial^3 e \partial^2 e \partial e |0\rangle = 1
\]

A correlator will also vanish unless the sum of the momenta of all the vertex operators in the correlator is \( 2i \) times background charge.

2. States in effective bosonic strings

In \( W_3 \) and probably \( W_N \) \( N \geq 4 \) strings one finds that the physical states belong to only a subspace of the full Fock space of the theory that is similar to the bosonic string. In this section, we will consider what are the physical states that can arise in such effective bosonic string sectors, but rather than base our discussion specifically within the context of \( W \) strings we will consider bosonic string-like sectors in the abstract, so making the discussion clearly applicable to any string theory in which they may arise. By an effective bosonic string sector we mean a string, or sector of a string theory, that has \( D \) scalar fields \( x^\mu \) which possess a background charge \( \alpha^\mu \). The physical states of the sector belong to the Fock space subspace \( \tilde{H} \) generated by \( \alpha_n^\mu \) and are taken to be subject to the Virasoro-like conditions

\[
\tilde{L}_n |\tilde{\psi}\rangle = 0, \ n \geq 1 \ , (\tilde{L}_0 - a) |\tilde{\psi}\rangle = 0
\]

where the intercept \( a \) is to be specified and

\[
\tilde{T} = -\frac{1}{2} \partial x^\mu \partial x^\nu \eta_{\mu\nu} - \alpha_\mu \partial^2 x^\mu
\]

\[
\tilde{T}(z) = \sum_n \tilde{L}_n z^{-n-2}, \quad i \partial x^\mu(z) = \sum_n \alpha_n^\mu z^{-n-1}
\]

We note that the background charge need not, be such that \( c = D + 12 \alpha^2 \) is equal to the critical value of 26.
The physical states in such a theory were found in reference [10] by using an extension of the methods of reference [11] and we now summarise these arguments. The first step is to introduce a set of operators that span $\tilde{H}$, but have sufficiently simple commutation relation properties with $\tilde{L}_n$ as to allow us to solve the constraints of equation (2.1). We begin by defining the operators

$$A_n^\mu = A_n^\mu - \frac{nk^\mu - 2i\alpha^\mu}{2} F_n \tag{2.4}$$

where

$$A_n^\mu = \oint dz : i\partial x^\mu e^{ink \cdot x} :$$

$$F_n = \oint dz : \frac{k \cdot \partial x}{k \cdot x} e^{ink \cdot x} : \tag{2.5}$$

These will commute with $\tilde{L}_n$ provided $k \cdot (k - 2i\alpha) = 0$. If we further take $k^2 = 0$, then they obey the algebra

$$[A_n^\mu, A_m^\nu] = (n\eta^{\mu\nu} + 2n^3k^\mu k^\nu) p.k\delta_{n+m,0} + mk^\mu A_n^\nu - nk^\nu A_n^\mu - i\alpha^\mu k^\nu m^2 \delta_{n+m,0} p \cdot k + i\alpha^\nu k^\mu m^2 \delta_{n+m,0} p \cdot k. \tag{2.6}$$

We next define the operators

$$C_n = -A_n^- - \left\{ \frac{1}{2} \sum_p \sum_i : C_{n-p}^i C_p^i : -i(n+1)\alpha \cdot C_n \right\} + 1$$

$$C_n^i = A_n^i + i\alpha^i \delta_{n,0} \tag{2.7}$$

which of course also commute with $\tilde{L}_n$. These operators obey the algebra

$$[C_n^i, C_m^j] = \delta_{n+m,0} \delta_{i,j}, \quad [C_n^i, C_m^j] = 0,$$

$$[C_n, C_m] = (n-m)C_{n+m} + (26-D-12\alpha^2) \frac{n(n^2-1)}{12} \delta_{n+m,0} \tag{2.8}$$

Since $C_n$ and $C_n^i$ commute with $\tilde{L}_n$, we conclude that they generate physical states which are of the form

$$C_{-m_1} \ldots C_{-m_q} C_{-n_1}^i \ldots C_{-n_p}^i \langle 0, p \rangle \tag{2.9}$$
where $|0,p\rangle$ is the tachyon state and is annihilated by $\alpha^\mu_n$, $n \geq 1$ and $p^\mu$ satisfies $p \cdot k = 1, \frac{1}{2}p \cdot (p - 2i\alpha) = a$.

We add to our collection of oscillators the

$$
\phi_n = \oint dz z^{-1} e^{inkx}.
$$

(2.10)

These have the relations

$$
[\tilde{L}_p, \phi_m] = -p\phi_m, \quad [C_p, \phi_m] = m\phi_{m+p}
$$

$$
[\phi_n, \phi_m] = 0, \quad [C^i_n, \phi_p] = 0.
$$

(2.11)

One can show that the oscillators $C^i_n$, $C_n$ and $\phi_n$ do span the same Hilbert space as the original oscillators $\alpha^\mu_n$, $\mu = 0, 1, \ldots, D - 1$. It also follows, from the above commutation relations, that the physical states do not contain $\phi_n$ and so are none other than those of equation (2.9)

We now wish to analyse the physical states. The operators $C_{-n}$ obey a Virasoro algebra which has a central charge $\tilde{c} = 26 - D - 12\alpha^2$ and they act on highest weight states with weight $h = 1 - a$. The norms of these states is therefore controlled by the Kac determinant for the Virasoro-like operators $C_n$ with the above central charge and weight. If \( \tilde{c} \) is less than or equal to 1 then the states will only have positive norm if and only if \( \tilde{c} \) is a member of the minimal unitary series \( \tilde{c} = 1 - \frac{6}{(n+1)(n+2)} \) $n = 3, 4, \ldots$ and the intercepts $a_{r,s}$ are related to the weights of the corresponding minimal unitary series by $a_{r,s} = 1 - h_{r,s}$. The number of physical degrees of freedom is found by discarding those states of zero norm. This is the same as discarding those states which are descendant, but yet also highest weight with respect to the $C_n$. Consequently, the number of physical degrees of freedom $c_n$ at level $n$ is given by

$$
\sum c_n x^n = \prod_{n=1}^{\infty} \frac{1}{(1 - x^n)^{D-2} \hat{\chi}_h(x)}.
$$

(2.12)

The character $\chi_h$ is that corresponding to the above central charge and weight and is defined by

$$
\chi_h(z) = \sum_{n=0}^{\infty} z^{h - \frac{1}{D}} \dim V_{n+h} = z^{h - \frac{1}{D}} \hat{\chi}_h(z)
$$

(2.13)
where $h$ is the weight of the highest weight state and $V_q$ is the dimension of the space with weight $q$.

For a critical bosonic string $c = 26$ and $a = 1$, or equivalently $\tilde{c} = 0$ and $h = 0$, and in this case all the states involving $C_{-n}$ are null and so one finds that the count of states is a light-cone count in the sense that there are excitations for only $D - 2$ of the $D$ non-zero modes. When the effective bosonic string, or sector, is not critical one finds that the string does not lose 2 degrees of freedom and only some of the states involving $C_{-n}$ are null. When one is dealing with one of the minimal models, which must be the case if $\tilde{c} \leq 1$ and the states obey a no ghost theorem, the explicit form of the characters is known $[12]$ and so equation (2.12) gives the count of states. In this case, one can then consider the question of whether the string theory is modular invariant. The discussion of reference $[10]$ is for the Ising model case, but it is trivial to extend it to the general case. One finds that the cosmological constant is the product of a factor that is associated with the $x^\mu$ oscillators, which is modular invariant by itself and a factor associated with the minimal model. For the theory to be modular invariant this latter factor must also be invariant. Consequently, for every modular invariant of the minimal series $[13]$ one finds a corresponding modular invariant string.

3. The physical states of the $W_3$ string

Before we begin, it will be useful to recall what are the physical states in the ordinary critical bosonic string constructed from the 26 fields $x^{\mu}$. These states were found, by studying the original dual model $[14]$, to obey the Virasoro constraints $(L_0 - 1)|\psi^x\rangle = 0$, $L_m|\psi^x\rangle = 0$ $m \geq 1$ where $|\psi^x\rangle$ depends only on $x^{\mu}$. With the discovery of the free gauge covariant action, it became more common to consider the physical states as belonging to the cohomology of the BRST operator $Q$, subject to a certain ghost number constraint. We recall that the cohomology of $Q$ $[4,15]$ consists of the states $|\psi^x\rangle\downarrow$ and $|\psi^x\rangle c_0\downarrow$, where $\downarrow = c_1 |0\rangle$ and $|0\rangle$ is the $SL(2,\mathbb{R})$ invariant vacuum, as well as two further states with zero momentum, namely $|p^{\mu} = 0\rangle |0\rangle$ and $|p^{\mu} = 0\rangle c_{-1} c_0 c_1 |0\rangle$.

For a $W_3$ string, it is also natural to regard the physical states as being given by the cohomology of $Q$, subject to a suitable ghost number constraint. By analogy, we expect physical states to be contained in states of the form $|\psi^{x,\varphi}\rangle\downarrow$, where $\downarrow = c_1 e_1 e_2 |0\rangle$. Applying $Q$ to these states one finds $[6]$ that they belong to the
cohomology of $Q$ if

\begin{align}
(L_0^m - 4)|\psi^{x,\phi}\rangle &= 0, \quad W_0^m|\psi^{x,\phi}\rangle = 0, \\
L_n^m|\psi^{x,\phi}\rangle &= 0, \quad W_n^m|\psi^{x,\phi}\rangle = 0, \quad n \geq 1.
\end{align}

(3.1)

Included amongst the states satisfying these conditions are physical states having the form [7]

\[ |\psi^{x,\phi}\rangle = |\psi^x\rangle|0,\beta\rangle|\downarrow\rangle, \]

where $|0,\beta\rangle$ is a state with $\phi$ momentum equal to $\beta$ and no $\phi$ oscillators. Such states will satisfy the conditions (3.1) provided that the state $|\psi^x\rangle$, which depends on $x^\mu$ alone, satisfies the conditions

\[ L_n^x|\psi^x\rangle = 0, \quad n \geq 1, \quad (L_0^x - a)|\psi^x\rangle = 0, \]

(3.3)

where $a = 1$ for $\beta = 8iQ/7$ and $6iQ/7$, and $a = 15/16$ for $\beta = iQ$. The above values of $\beta$ are in fact the only ones allowed by the on-shell conditions of equation (3.1). These states exhibit what is a general phenomenon, namely the freezing of the $\phi$ momentum by the physical state conditions which was first noticed in reference [7]. The reader may observe that if one takes one minus the above effective intercepts then one gets 0 and 1/16 which remind one of the weights that occur in the Ising model. In references [7] and [16] this, and other phenomenological number matching, was shown to extend to a relation between higher $W_N$ strings and minimal models.

Further analysis of low level states was carried out in references [17] and [18], but it was only in reference [19] that a systematic study of the physical states of the type of equation (3.1) at levels up to and including 2 was undertaken. It was found that any state that contained non-zero mode $\phi$ oscillators acting on the vacuum was null. By examining all other null states, the count of physical degrees of freedom at these levels was found. It thus became clear that the non-null physical states based on the ghost state $|\downarrow\rangle$ were of the form of equation (3.2) and that the open $W_3$ string had only one massless particle, a photon, namely the same massless states as the open bosonic string [19].

It will be useful to rewrite the physical states discussed above in terms of vertex operators acting on the vacuum $|0\rangle$ in the ghost sector as well as the usual vacua
for the bosonic oscillators. The states of equation (3.2) correspond to the vertex operators

\[ V(1, 0) = c \partial e e^{i \beta(1; 0)} \varphi V^x(1) \]  
\[ \bar{V}(1, 0) = c \partial e e^{i \bar{\beta}(1; 0)} \varphi V^x(1), \]

where, with further developments in mind, we have introduced the notation \( \beta(1; n) = (8 - 8n) \frac{iQ}{7} \) and \( \bar{\beta}(1, n) = (6 - 8n) \frac{iQ}{7} \), and

\[ V(15/16, 0) = c \partial e e^{i \beta(15/16, 0)} \varphi V^x(15/16) \]

where \( \beta(15/16, n) = (7 - 4n) iQ/7 \). Here \( V^x(a) \) is any vertex operator, constructed from \( x^\mu \) alone, that has conformal weight \( a \) with respect to \( T^x(z) \). The simplest is \( V^x(a) = e^{ip \cdot x} \), where \( \frac{1}{2} p \cdot (p - 2i\alpha) = a \).

For the bosonic string, all the physical states, except for those of fixed momentum, are built on a single state with given ghost number, but this is not the case for the W\(_3\) string. States in the cohomology of \( Q \) which are not of the above form were first found in the context of the 2 scalar W\(_3\) string in reference [16] and one such state was found in the 3 scalar W\(_3\) string. It was shown, as discussed below, that the consistency of the W\(_3\) scattering [21] and modular invariance [10] also required further states to those listed above and these additional states are given by [10]

\[ V(1/2, 0) = \left( ce - \frac{i}{\sqrt{522}} \partial e e \right) e^{i \beta(1/2, 0)} \varphi V^x(1/2) \]

where \( \beta(1/2, m) = (4 - 8m) \frac{iQ}{7} \).

These states were found by a vanishing null state argument which also led to states in the cohomology of \( Q \) that were of the form [10]

\[ V(15/16, 1) = \left( ce + \frac{i \partial e e}{\sqrt{522}} \right) e^{i \beta(15/16, 1)} \varphi V^x(15/16). \]

and

\[ \bar{V}(1/2, 0) = \left( -\frac{4}{3 \sqrt{58}} be \partial e e - \frac{4}{3 \sqrt{58}} \partial^2 e e + \frac{1}{\sqrt{29}} \partial \varphi \partial e e + i \sqrt{2} c e \partial \varphi - \frac{3i}{2} c \partial e \right) \]

\[ \times e^{i \bar{\beta}(1/2, 0)} \varphi V^x(1/2), \]
where $\bar{\beta}(1/2, m) = (2 - 8m)\frac{Q}{m}$. This argument also implied the presence of an infinite number of other such states. The above physical states of the $W_3$ string of equations (3.4), (3.6) and (3.7) are in effect contained in sectors which are effective bosonic strings with intercepts 1, 15/16 and 1/2 and central charge $c = 25 + 1/2$.

The states of equations (3.5), (3.8) and (3.9) are also of this form with effective intercepts 1, 15/16 and 1/2 respectively and so are in effect only copies of the previous states. Further such copies were found in references [22,26]

We can now analyse the physical content of these states, using the results of the previous section. These physical states are of the form of equation (2.9) with $\tilde{c} = 1/2$ and highest weights $h$ of 0, 1/16 and 1/2 and so the count of states of equation (2.12) has characters which are none other than those of the Ising model and further, these states obey a no ghost theorem in that they have positive norm [10]. We note that given $\tilde{c} = \frac{1}{2}$ these are the only values of the intercepts for which this is true. Thus, although the construction of the $W_3$ string did not seem to involve the Ising model, we find a deep connection between the states of the two theories.

We can now consider whether the above states are sufficient in number for the cosmological constant to be modular invariant. Since the partition function involves the Ising model characters one might suspect that one finds a modular invariant $W_3$ string for every modular invariant Ising model. In fact there is only one such Ising model and so only one modular invariant $W_3$ string. We refer the reader to reference [10] for a detailed discussion.

These facts, and the result, discussed below, that the factorisation of the tree level $W_3$ scattering amplitudes for these states leads to no new states strongly suggests [10] that the cohomology of $Q$ involves the states of equations (3.4), (3.6) and (3.7), copies of them, and possible discrete states.

Considerable further evidence for this conjecture was provided in reference [23] where it was shown how to generate infinite classes of states belonging to the cohomology of $Q$. We now summarize this work. The existence of copies suggests that there should be a screening charge of the form

$$\int dw \ e^{i\beta \varphi} f(b, c, d, e, \partial \varphi)$$

(3.10)

that commute with $Q$ and takes one copy into another. If we can find such charges
that do not involve the field $x^\mu$, then the insertion of these charges into a correlation function will not change the effective space-time interpretation of the correlation function. Indeed three such screening charges do exist [23] and are given by

$$\int dz\ e^{e^{i\beta\varphi}}$$

(3.11)

for $\beta = 6iQ/7$ or $\beta = 8iQ/7$. and, more importantly for our present purposes, the operator

$$S = \oint dz\left\{ d - \frac{5i}{3\sqrt{58}}\partial b - \frac{2}{3.87}\partial b\ e - \frac{4i}{3\sqrt{58}}d b\ e\right\}e^{i\beta^*\varphi}$$

(3.12)

where $\beta^* = -2iQ/7$. Since $S$ commutes with $Q$, it follows that whenever the action of $S$ on a physical state is well-defined and non-zero it will produce another physical state.

Let us now consider when two screening operators have a well defined action on a vertex operator with $\varphi$ momentum $\beta$. Since this has not, to our knowledge, been clearly discussed in the literature let us consider the more general case of two screening operators with $\varphi$ momentum $\beta_i^*, i = 1, 2$. Since the factors in front of the exponentials are made from either ghosts or $\partial\varphi$ they can only contribute integer powers of the coordinates in the operator product expansion, and consequently do not affect whether or not the action of the screening operators is well defined. Consequently, we must focus our attention on the factors

$$\int_{C_1} dw_1\ \int_{C_2} dw_2\ e^{i\beta_1^*\varphi(w_1)}\ e^{i\beta_2^*\varphi(w_2)}\ e^{\beta\varphi(z)}$$

$$= \int_{C_1} dw_1\ \int_{C_2} dw_2\ (w_1 - w_2)^{\beta_1^*\beta_2^*}(w_1 - z)^{\beta_1^*\beta}(w_2 - z)^{\beta_2^*\beta}e^{i\beta_1^*\varphi(w_1) + i\beta_2^*\varphi(w_2) + i\beta\varphi(z)}$$

(3.13)

Since $|w_1 - z| > |w_2 - z|$ we arrange the $w_1$ and $w_2$ contours to be around the point $z$ in such a way that the above condition is satisfied.

Let us consider the substitution $(w_1, w_2)$ to $(w_1, \tau)$ given by $(w_2 - z) = \tau(w_1 - z)$. The above condition implies that $|\tau| < 1$, but we also demand that $\tau = 1$ at one and only one point in other words the $w_1$ and $w_2$ contours touch one another once. This latter condition ensures that the value of the integral is dependent on only
the one place where the contours cross the branch cut. We are to regard $w_2$ as fixed and consider the $\tau$ integration. Substituting $dw_2 = d\tau(w_1 - z)$ we find the above integrals become

$$
\int_z \int d\tau \tau^{\beta_1^s \beta} (1 - \tau)^{\beta_1^s \beta_2^s} (w_1 - z)^P \exp(i\beta_1^s \varphi(z + (w_1 - z)) + i\beta_2^s \varphi(z + \tau(w_1 - z)) + i\beta \varphi(z))
$$

where $P = 1 + \beta_1^s \beta + \beta_1^s \beta_2^s + \beta_2^s \beta$. This integral is well defined if $P$ is an integer.

The generalization to $n$ screening charges with $\varphi$ momenta $\beta_i^s$ is straightforward. Their action contains the term

$$
\prod_{i=1}^n \left( \int dw_i e^{i\beta_i^s (w_i)} \right) \exp i \left\{ \sum_{j=1}^n \beta_j^s \varphi(z + \tau_j (w_n - z)) + \beta \varphi(z) \right\}
$$

(3.15)

We now replace $w_j, j = 2, \ldots, n$ by $w_1, \tau_j j = 2, \ldots, n$ using the formula $(w_j - z) = \tau_j (w_1 - z)$. The above expression becomes

$$
\int_z \int d\tau_i f(\tau_i) (w_1 - z)^{P'} \exp \left\{ \sum_{j=1}^n \beta_j^s \varphi(z + \tau_j (w_n - z)) + \beta \varphi(z) \right\}
$$

(3.16)

where $f(\tau_i)$ is a function of $\tau_i$ and

$$
P' = (n - 1) + \sum_{i,j=1}^N \beta_i^s \beta_j^s + \sum_{j=1}^N \beta_j^s \beta.
$$

(3.17)

Thus the integrals are well defined if $P'$ is an integer.

Let us apply this general discussion to the case of interest, namely the action of $n$ screening charges $S$ with momentum $\beta^s = -2iQ/7$. Taking $\beta = isQ/7$ with $s$ an integer, their action is well defined if

$$
\frac{n}{4} [-n - 1 + 1] \in \mathbb{Z},
$$

(3.18)

and then the momentum of the resulting vertex is

$$
\frac{iQ}{7} (s - 2n).
$$

(3.19)
For the action of screening charges on $V(15/16,0)$ we have $s = 7$. We then find that $n$ must be even and that the vertices so constructed have momenta 

$$\beta(15/16,m) = \frac{iQ}{7}(7 - 4m), \quad m \in \mathbb{Z}_+.$$ 

For the action on $V(1,0)$, for which $s = 8$, we find that $n = 4m$ or $4m + 1$ with $m \in \mathbb{Z}_+$, which leads to the vertices with momenta 

$$\beta(1,m) = (8 - 8m)\frac{iQ}{7}, \quad m \in \mathbb{Z}_+, \text{ and } \bar{\beta}(1,m) = (6 - 8m)\frac{iQ}{7}, \quad m \in \mathbb{Z}_+.$$ 

Finally, the action of $n$ screening charges on $V(1/2,0)$ is well defined if $n = 4m$ or $4m + 1$ with $m \in \mathbb{Z}_+$, which leads to vertices with momenta 

$$\beta(1/2,m) = \frac{iQ}{7}(4 - 8m) \quad \text{and} \quad \bar{\beta}(1/2,m) = \frac{iQ}{7}(2 - 8m).$$ 

It can, and does happen, that applying $S$’s alone leads to a vanishing result. This can be avoided, however, by the judicious use of the picture changing operator $P$ which is of the form $P = [a \cdot x + \varphi, Q]$; this operator was first used in the context of $W_3$ strings in reference [22].

The general pattern is as follows; in the intercept $15/16$ sector we have the vertices 

$$V(15/16,m) = (S^2P)^m V(15/16,0) \quad (3.20)$$

while for the intercept $a = 1$ and $1/2$ vertices, we have the vertices 

$$V(a,m) \quad \text{and} \quad \bar{V}(a,m) \quad (3.21)$$

These latter vertices are defined by the relations

$$\bar{V}(a,m) = SPV(a,m), \quad (3.22)$$

$$V(a,m) = S^3P\bar{V}(a,m-1). \quad (3.23)$$

To summarise, we have found [23] that given the basic vertices $V(a,0)$ for $a = 1, 1/2 \text{ and } 15/16$, we can use $S$ and $P$ to create the BRST invariant vertices $V(a,m)$; for $a = 1, 1/2$ we obtain in addition the vertices $\bar{V}(a,m)$. We can also obtain further vertices by the action of $P$ on these. Since the physical states for a given intercept have a spectrum generating algebra involving the operators $C_n^i, i = 1, \ldots D - 2$ and $C_n$ the states in the cohomology of $Q$ are generated by $C_n^i, C_n, S$ and $P$.

A number of examples of this procedure were given in reference [23], but here, we give only one example relating the first two copies in the intercept $1/2$ sector. A
short calculation shows that although $SV(1/2, 0)$ is well defined it vanishes. We therefore introduce further powers of the ghost $e$ into the vertex using the picture changing operator. We then have

$$PV(1/2, 0) = \left(5 \partial^2 e e - \frac{24Q}{7} \partial \phi \partial e e \right. $$

$$- \left. \frac{19i}{3\sqrt{58}} \partial^2 e \partial e e \right) e^{i\beta(1/2, 0)} \varphi V^x(1/2), \tag{3.24}$$

and acting on this with the screening operator we find the vertex $\bar{V}(1/2, 0)$,

$$SPV(1/2, 0) \propto \bar{V}(1/2, 0). \tag{3.25}$$

4. $W_3$ String Scattering

The scattering, at tree level, of $W_3$ strings was first found using the group theoretic method [24]. It was found that these scattering amplitudes contained within them the Ising model correlation functions, in particular the scattering amplitude of $N W_3$ string states is given by [21]

$$\int \prod_i' dz_i V f(z_i). \tag{4.1}$$

Here $V$ is the usual scattering vertex in the presence of a background charge, and $f$, the measure, is an Ising model correlation function that depends on the intercepts of the external states. To be specific, if the $N$ external states have effective intercepts $a_i, \quad i = 1, \ldots, N$, which can take only the values $1, 15/16$ or $1/2$, then $f = \langle \prod_{i=1}^{N} \varphi_i(z_i) \rangle$ where $\varphi_i$ is the Ising field of weight $h_i = 1 - a_i$.

The essential steps, using the group theoretic method to calculate a scattering amplitude are first the computation of the vertex $V$ using overlap relations, and then the determination of the measure $f$ by demanding that null states decouple. Using this technique it was possible to work with the reduced subspace of the full $W_3$ Fock space in which the physical states sit. It is important to understand that the properties of the vertices and null states used in this reduced Hilbert space are those inherited from the full Fock space of the $W_3$ string. We found that the decoupling of the null states of the $W_3$ string implied that $f$ obeyed the differential equations satisfied by the Ising model. The reader is referred to reference [21] for the details of this derivation.
It is straightforward to evaluate the $W_3$ string scattering given by equation (3.1) whenever the Ising model correlation functions are known. The vertex $V$ can be found in reference [21] for any states of the $W_3$ string, but for tachyons it has the simple expression

$$
\prod_{i<j}(z^i - z^j)^{2\alpha' p_i \cdot p_j}
$$

(4.2)

For four tachyon scattering, with the choice of Koba-Nielsen coordinates $z_1 = \infty$, $z_2 = 1$, $z_3 = x$ and $z_4 = 0$, this reduces to

$$
x^{-\alpha' s - a_3 - a_4} (1 - x)^{-\alpha' t - a_2 - a_3},
$$

(4.3)

where $s = -(p_1 + p_2) \cdot (p_1 + p_2 - 2i\alpha)$, $t = -(p_2 + p_3) \cdot (p_2 + p_3 - 2i\alpha)$ and where a factor of $(z_1)^{-2a_1}$, which is cancelled by other such factors in the amplitude, has been removed.

Let us give one example of the evaluation of such amplitudes, namely the case of four intercept $15/16$ tachyonic $W_3$ strings, whose subsequent study will be instructive.

$$
F(s, t) = \int_0^1 dx x^{-\alpha' s - 15/8 (1 - x)^{-\alpha' t - 15/8} f(x)} z_1^{1/8}
$$

(4.4)

The function $f(x)$ is the Ising correlation function for 4 weight $1/16$ fields which is of the form [25]

$$
f(x) = \left[(z_1 - z_3)(z_2 - z_4)\right]^{-1/8} Y(x)
$$

(4.5),

where

$$
x = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)}
$$

(4.6)

and

$$
Y(x) = \frac{1}{[x(1 - x)]^{1/8}} (a \cos \theta + b \sin \theta),
$$

(4.7)

with $x = \sin^2 \theta$. The two constants $a$ and $b$ correspond to the two solutions to the second order differential equation that this correlation function satisfies due to the existence of null states in the Ising model. Taking the above choice of $z$’s we find that

$$
F(s, t) = \int_0^1 dx x^{-\alpha' s - 2} (1 - x)^{-\alpha' t - 2} (a \cos \theta + b \sin \theta).
$$

(4.8)
In the above we have introduced the slope $\alpha'$, which is usually taken to be $1/2$ for the open string.

The values of the constants $a$ and $b$ are to be chosen by the physical requirement of crossing. The open string amplitude $T^{(4)}(p_i)$ is as usual the sum of three terms

$$T^{(4)}(p_i) = F(s, t) + F(t, u) + F(u, s). \quad (4.8)$$

Crossing for four identical particles means that $T^{(4)}(p_i)$ should be symmetric under the exchange of any two legs or equivalently momenta. This means, for example, that it should be symmetric under $s \leftrightarrow t$, which follows provided that $F$ itself is a symmetric function of its arguments. This property is in turn guaranteed if the integrand is symmetric under $x \leftrightarrow 1-x$, provided, at the same time, we interchange $s$ and $t$, or $p_2$ and $p_4$. The transformation $x \to 1-x$ can be written as $\theta \to \pi/2 - \theta$, whereupon it is obvious that we should take

$$F(s, t) = \int_0^1 dx x^{-\alpha's-2}(1-x)^{-\alpha't-2} \cos(\theta/2 - \pi/8)$$

$$= \frac{1}{\sqrt{2}} \int_0^1 dx x^{-\alpha's-2}(1-x)^{-\alpha't-2}\left\{ \cos \frac{\pi}{8} \sqrt{1 + \sqrt{1 + x}} + \sin \frac{\pi}{8} \sqrt{1 - \sqrt{1 - x}} \right\}. \quad (4.9)$$

Having found the expression for four tachyon scattering we can examine the particles exchanged in a given channel. It is clear from its integral representation that the amplitude obeys duality, that is $F(s, t)$ can be expressed as a sum of poles in either the s-channel or the t-channel. Let us consider the s-channel. Expanding the factor $(1 - x)^{-\alpha't-2}$ we find that

$$F(s, t) = \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \cos \frac{\pi}{8} \sqrt{2} a_p \frac{(\alpha't + 2)(\alpha't + 3) \ldots, \alpha't + n + 1}{(-\alpha's - 1 + p + n)}$$

$$+ \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \sin \frac{\pi}{8} \sqrt{2} b_p \frac{(\alpha't + 2)(\alpha't + 3) \ldots, \alpha't + n + 1}{(-\alpha's - 1 + p + n + 1/2)}, \quad (4.10)$$

where

$$\sqrt{1 + \sqrt{1 + x}} = \sqrt{2} \sum_{p=0}^{\infty} a_p x^p \quad (4.11)$$
and

\[ \sqrt{1 - \sqrt{1 - x}} = \sqrt{\frac{x}{2} \sum_{p=0}^{\infty} b_p x^p}. \]  

The states exchanged in the first and second terms have masses which satisfy \( \alpha' m^2 = p + n - 1 \) and \( \alpha' m^2 = p + n - 1/2 \) respectively. While the former states are contained in the intercept 1 sector the latter are the intercept 1/2 sector. Consequently, we find, as discussed above, that factorisation of the \( W_3 \) string scattering requires the existence of intercept 1/2 states in its spectrum in addition to those of intercept 1 and 15/16 sectors. The fact that the \( W_3 \) scattering involves Ising correlations includes the 3 point function. The form of the 3 point functions of the Ising model, as for any conformal model, are well known to be equivalent to the fusion rules. Consequently, the fusion rule \( \sigma \sigma = 1 + \epsilon \), where \( \sigma \) and \( \epsilon \) are the Ising fields of weight 1/16 and 1/2 respectively implies the existence of intercept 1/2 states in the \( W_3 \) string.

Following the general tree-level calculations of \( W_3 \) string scattering amplitudes given in [21], covariant approaches to \( W_3 \) string scattering have been considered. The first such attempt [22,26] involved calculations of correlation functions using only vertex and picture-changing operators. The resulting scattering amplitudes disagreed with those of reference [21] and did not share the connection with Ising model correlation functions found in [21]; in addition they violated general principles of S-matrix theory and string theory. The problem was most readily apparent when considering the scattering of 4 intercept 15/16 strings considered above. It is easy to see that the vertices with this intercept can never be combined in a correlator so as to balance the \( \varphi \) background charge \( 2iQ \). As a consequence, the authors of references [22,26] concluded that this amplitude and many like it vanished. The intercept 15/16 amplitude given above, did however factorise properly [21] and it is clear that this amplitude cannot vanish and be consistent with factorisation i.e duality [29]. At a more elementary level it is clear that the vanishing of such a four point function is inconsistent with the optical theorem and its generalisations[27,28]

The use of only vertex and picture-changing operators to calculate \( W_3 \) covariant scattering amplitudes was a straightforward generalization of previous covariant string scattering calculations for the bosonic and superstrings [9]; in view of the expected role of \( W \)-moduli it is perhaps not surprising that this simple generalization
is not adequate for $W$-strings.

In reference [23], a general covariant procedure for calculating $W_3$-scattering was given. It involved using not only the vertices $V(a,0)$ and the picture changing operator $P$, but also the screening charge $S$ in the computation of scattering amplitudes. This method led to amplitudes in agreement with those found earlier [21] and which satisfied the general principles expected. We now summarize this approach: the $W_3$ string scattering amplitudes are to be constructed from the building blocks

$$V(a,0) ; \ a = 1, 15/16, 1/2 \ ; \ S, P$$  \hspace{1cm} (4.13)

and the operation

$$\int dz \oint \frac{dv}{z} b(v).$$  \hspace{1cm} (4.14)

The latter is a standard operation used for the $b-c$ ghost system. We must, however, assemble the building blocks so that the $\phi$ momentum sums to $2iQ$. This tells us the required number $N_s$ of screening charges. We must also have, after carrying out all the operator product expansions, 3 $c$ ghosts and 5 $e$ ghosts, otherwise the correlator will vanish. As usual, we require for an $N$ string amplitude $N-3$ of the operators of equation (4.14). The blocks $V(a,0)$ with $a = 1, 15/16$, $V(1/2,0)$, $S$ and $P$ have ghosts number 3, 2, $-1$ and 1 respectively. The ghost number requirement gives us the number $N_p$ of picture changing operators $P$. To be precise if we have a scattering of $N_1$ intercept 1, $N_{15/16}$ intercept 15/16 and $N_{1/2}$ intercept 1/2 strings, then $\phi$ momentum conservation demands that

$$8N_1 + 7N_{15/16} + 4N_{1/2} - 2N_P = 14$$  \hspace{1cm} (4.15)

while the ghost number count yields the relations

$$3N_1 + 3N_{15/16} + 2N_{1/2} - N_S + N_P - (N_1 + N_{15/16} + N_{1/2} - 3) = 8$$  \hspace{1cm} (4.16)

These equations imply that

$$N_S = 4N_1 + \frac{7}{2}N_{15/16} + 2N_{1/2} - 7$$  \hspace{1cm} (4.17)

$$N_P = 2N_1 + \frac{3}{2}N_{15/16} + N_{1/2} - 2$$
To be concrete let us consider the scattering of 4 intercept 15/16 tachyonic strings. The above equations tell us that we require 7 factors of $S$ and 4 factors of $P$. One way to distribute these factors in the correlator is as follows

$$\langle 0 | PV(15/16, 0)(z_1)V(15/16, 1)(z_2) \int dz_3 \int dv b(v)V(15/16, 1)(z_3)V(15/16, 1)(z_4)S|0 \rangle = \int dz_3 \int dw \langle 0 | \left( c \partial^2 e \partial e e^{i\beta(15/16,0)}\varphi V^x(15/16) \right) (z_1) \left( c e^{i\beta(15/16,1)}\varphi V^x(15/16) \right) (z_2) \left( e^{i\beta(15/16,1)}\varphi V^x(15/16) \right) (z_3) \left( c e^{i\beta(15/16,1)}\varphi V^x(15/16) \right) (z_4) \left( d - \frac{5i}{3\sqrt{58}} \partial b - \frac{2}{3.87} \partial bb e - \frac{4i}{3} \frac{1}{\sqrt{58}} db e e^{i\beta e} \varphi \right) (w)|0 \rangle$$

$$= - \int dz_3 \int dw \langle 0 | \left( \prod_{i=1, i \neq 3}^{4} c(z_i) \right) (\partial^2 e \partial e e)(z_1)e(z_2)e(z_3)e(z_4)d(w)$$

$$e^{i\beta(15/16,0)\varphi}(z_1) \prod_{i=2}^{4} e^{i\beta(15/16,1)\varphi}(z_i) e^{i\beta e \varphi}(w) \prod_{i=1}^{4} V^x(15/16)(z_i)|0 \rangle$$

By bosonizing the ghosts we find the $c$ ghosts give a factor

$$(z_1 - z_2)(z_1 - z_3)(z_2 - z_3)$$

and the $e-d$ ghosts a factor

$$(z_1 - w) \prod_{i=2}^{4} (z_1 - z_i)^3(z_i - w)^{-1} \prod_{i,j=2}^{4} \prod_{i<j} (z_i - z_j).$$

Evaluating the exponential factors in the usual way we find that the amplitude for four intercept 15/16 tachyonic states is proportional to

$$\int dw \int dx x(1 - x)^{-1/8}[w(1 - w)(x - w)]^{-1/4}(1 - x)^{p_2p_3x^{p_3p_4}} = \int dw \int dx x^{s/2-2}(1 - x)^{t/2-2}[w(1 - w)(x - w)w]^{-1/4}$$

where we have chosen $z_1 = \infty$, $z_2 = 1$, $z_3 = x$ and $z_4 = 0$. This expression can be shown to agree with that given in equation(4.8), thus agreeing with the result of reference [21].
Let us now consider the general scattering amplitude and how we must distribute all the factors of $P$ and $S$, whose number is specified in equation (4.17), among the vertices so as to gain a non-zero result. To be concrete, let us consider the scattering of $N_{15/16} = 2n \geq 6$ intercept $15/16$ states, in which case $N_S = 7(n - 1)$ and $N_P = 3n - 2$. One way to do this is to take $n - 3$ of the vertices $V(15/16, 2) = (S^2P)^2V(15/16, 0)$ and $n + 3$ of the vertices $V(15/16, 1) = S^2PV(15/16, 0)$. This leaves over one $P$ factor and $n - 1$ $S$ factors and so leads to the correlator

$$\langle 0 | \prod_{i=4}^{2n} \left\{ \oint_{z_i} dz_i \oint_{v_i} dv_i \ b(v_i) \right\} \prod_{i=1}^{n+2} V(15/16, 1)(z_i) \prod_{j=n+4}^{2n} V(15/16, 1)(z_{j}) S^{n-1} | 0 \rangle$$

(4.22)

We have chosen to assign the final $P$ factor to the $n + 3$ vertex, but any other vertex is just as good. We could also use two of the final screening charges on the $PV(15/16, 1)$ vertex to yield the correlator

$$\langle 0 | \prod_{i=4}^{2n} \left\{ \oint_{z_i} dz_i \oint_{v_i} dv_i \ b(v_i) \right\} \prod_{i=1}^{n+2} V(15/16, 1)(z_i) \prod_{j=n+3}^{2n} V(15/16, 2)(z_{j}) S^{n-3} | 0 \rangle$$

(4.23)

Clearly there are many ways to write such a correlation using also the vertices $V(15/16, m), m > 2$.

For the scattering of $N_{1/2} = 2n$ intercept $1/2$ states, $N_S = 4n - 7$ and $N_P = 2(n - 1)$, we can assign all the factors of $S$ to the vertices, for example by the choice of $n - 2$ of the vertices $V(1/2, 1)$, one of $\bar{V}(1/2, 0)$ and $n + 1$ of the vertices $V(1/2, 0)$ and then place on one of these vertices the required extra $P$. The correlator is then

$$\langle 0 | \prod_{i=4}^{2n} \left\{ \int dz_i \oint_{v_i} dv_i \ b(v_i) \right\} \prod_{i=1}^{n+1} V(1/2, 0)(z_i) \prod_{j=n+2}^{2n-1} V(1/2, 1)(z_{j}) \ P\bar{V}(1/2, 0)(z_{2n}) | 0 \rangle$$

(4.24)
Finally for $N_1$ intercept 1 states, $N_S = 4N_1 - 7$ and $N_P = 2N_1 - 2$, which we can assign as $N_1 - 2$ vertices $V(1, 1)$, one of $V(1, 0)$ and one of $\bar{V}(1, 0)$, with one additional factor of $P$ to give the correlator

$$\langle 0 | \prod_{i=4}^{N_1} \int dz_i \oint_{z_i}^{N_1-2} dv_i \prod_{j=1}^{N_1} V(1; 1)(z_j) \cdot P V(1, 0)(z_{N_1-1}) \bar{V}(1, 0)(z_{N_1}) | 0 \rangle \quad (4.25)$$

The many ways of constructing the above correlators lead to the same results. The correlator is independent of the place where the picture changing operator is applied, since

$$P(z_1) - P(z_2) = [Q, \varphi(z_1) - \varphi(z_2)] = [Q, \int_{z_1}^{z_2} \partial \varphi], \quad (4.26)$$

and this is a BRST trivial operator as it contains $\partial \varphi$ and not $\varphi$. Further, we can change on which vertex the screening charges act by deforming the $w$-contours. There are also different choices for the contours of the residual screening charges and these lead to different results. As explained above, in the context of the 4 intercept $15/16$ scattering, we require these different solutions since only a particular combination of the contours gives a crossing symmetric amplitude.

5. Discussion

Let us begin by summarising some of the main results. The physical states of the $W_3$ string belong to three effective bosonic string sub-sectors of the theory which have intercepts $a = 1, 1/2$ and $15/16$ and an effective central charge of $25 + \frac{1}{2}$. The count of physical degrees of freedom in the sub-sector with intercept $a$ involves the Ising model character, $\chi_a$ where $a = 1 - h$. The cohomology of the BRST charge $Q$ contains an infinite number of copies of the above states at different ghost numbers and $\varphi$ momenta. There also exists strong evidence that, apart from possible discrete states, these are the only states in the cohomology of $Q$.

The scattering of $W_3$ strings, at tree level, has been found to contain, as part of its integrand, a factor that is none other than the correlation functions of the Ising model. This result first emerged from an application [21] of the group theoretic approach to string theory, but it can also be found using a gauge covariant approach [23].
It would be interesting to give a path-integral derivation of these scattering results. This would require a knowledge of $W$-moduli. Any such derivation would, however, have to reproduce the above results, and this could provide a clue to our understanding of $W$-moduli. We observe that the number of $W$-moduli is $2N - 5$ for the scattering of $N$ strings, and this number emerges from our results in the guise of $N_S - N_P + N_{1/2}$.

The $W_3$ string shows much of the same magic as the bosonic and superstrings; it obeys a no ghost theorem, it is modular invariant, and the scattering amplitudes satisfy duality and factorisation. Perhaps the most spectacular property of the $W_3$ string is its connection with the Ising model. This connection was first noticed on the basis of phenomenological number matching \cite{7,16}, but it also occurs in the count of physical degrees of freedom \cite{10} and the scattering amplitudes \cite{21,23} and as such it pervade all aspects of the $W_3$ strings. One can wonder how the $W_3$ string differs from a non-critical string constructed from the Ising model and D-1 scalar fields $x^\mu$. In the latter theory, the Liouville field will dress the vertices and, as first noticed in reference \cite{7}, can be identified with the remaining scalar. If we adopt the currently perceived wisdom \cite{30}, then the scattering amplitudes for such a theory will be constructed from such dressed vertices and so will agree with those found for the $W_3$ string.

Given this correspondence, one is entitled to ask what is the role of the $W$ symmetry when the model can seemingly be formulated without it. Presumably, although one finds the Ising model in the $W_3$ string one cannot find all possible tensored Ising models by starting in this way, while for higher $W_N$ strings this statement would extend to tensor products of the unitary minimal models. One intriguing possibility is that only those strings constructed from minimal models that can also be found from $W$, or some other larger symmetry, are really consistent.

It is interesting to note that the non-perturbative results that emerge from the matrix model approach to two dimensional quantum gravity do involve $W$ type constraints on the square root of the partition function \cite{38}. Indeed, the Ising model arises from a two matrix model which involves $W_3$ constraints and the unitary minimal models which should occur in $W_N$ strings arises from an $N - 1$ matrix model that involves $W_N$ constraints. It would be good to understand the connections between these results.
The Ising model is usually realised by a Feigin-Fuchs construction [31] involving only one scalar field, but the $W_3$ string involves an Ising model constructed from two scalars, the field $\varphi$ and the bosonised $d, e$ ghosts. After this talk was given, it was shown [32] that this realisation of the Ising model involves parafermions, which play a crucial role in determining the properties of the $W_3$ string.

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