ON FRAMED QUANTUM PRINCIPAL BUNDLES

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ABSTRACT. A noncommutative-geometric formalism of framed principal bundles is sketched, in a special case of quantum bundles (over quantum spaces) possessing classical structure groups. Quantum counterparts of torsion operators and Levi-Civita type connections are analyzed. A construction of a natural differential calculus on framed bundles is described. Illustrative examples are presented.

1. Introduction

In classical differential geometry the formalism of principal bundles plays a central role. In particular, a distinguished role is played by framed principal bundles, characterized as covering bundles of subbundles of the bundle of linear frames of the base manifold. Framed principal bundles provide a natural conceptual framework for the study of fundamental classical differential-geometric structures.

In this paper classical idea of a framed principal bundle will be incorporated into an appropriate noncommutative-geometric [1] context. The main entities figuring in the game will be quantum principal bundles (over quantum spaces) possessing (compact) classical structure groups.

Conceptually, the paper is based on a general theory of quantum principal bundles, presented in [2].

The paper is organized as follows.

Section II is devoted to the definition and general properties of frame structures. As first, a very important class of integrable frame structures will be defined. Roughly speaking, integrable frame structures naturally induce “Levi-Civita” type connections on the bundle. On the other hand, such connections are compatible, in the appropriate sense, with internal geometrical structure of the bundle. This fact opens a possibility to construct, in a fully intrinsic manner, the complete differential calculus on the bundle, starting from a given integrable frame structure.

More precisely, starting from a quantum principal bundle $P$ and an appropriate representation $u$ of the structure group $G$ it is possible to construct a graded $*$-algebra $\mathfrak{hor}_P$ playing the role of horizontal forms on $P$ (together with the right (co)action of $G$ on this algebra). Then, the graded $*$-algebra $\Omega_M$ representing differential forms on the base manifold can be described as the subalgebra of $\mathfrak{hor}_P$, consisting of $G$-invariant elements. On the other hand, if an integrable frame structure on $P$ (with respect to $u$) is given, then it is possible to define, in a natural manner, a differential on the algebra $\Omega_M$. Finally, starting from $\mathfrak{hor}_P$ and $\Omega_M$ and applying ideas of [3] it is possible to construct (via the concept of a preconnection)
the whole graded differential \(\ast\)-algebra \(\Omega_P\), representing differential forms on the bundle \(P\).

In the framework of (integrable) frame structures, it is possible to define torsion operators associated to connection forms (as in the classical theory). These operators are analyzed at the end of Section II.

It is important to mention that connection forms will appear implicitly, represented by the corresponding covariant derivatives. Further, all connections appearing in this study will be regular and multiplicative (in the sense of [2]).

In Section III some examples of framed quantum principal bundles are presented.

Finally, in Section IV concluding remarks are made.

2. Frame Structures

Let \(M\) be a quantum space, represented by a \(*\)-algebra \(V\). Let \(G\) be an ordinary compact (matrix) Lie group. Concerning group entities, the notation of [6] will be followed (although in the classical context). In particular the group \(G\) will be described by a (commutative) Hopf \(*\)-algebra \(A\) consisting of polynomial functions on \(G\). The group structure is encoded in the coproduct \(\phi: A \to A \otimes A\), counit \(\epsilon: A \to \mathbb{C}\) and the antipode \(\kappa: A \to A\).

Let \(u\) be a real unitary representation of \(G\) in the standard \(n\)-dimensional unitary space \(\mathbb{C}^n\). We shall assume that the kernel of \(u\) is discret. Furthermore, \(u\) will be interpreted as a right comodule structure map \(u: \mathbb{C}^n \to \mathbb{C}^n \otimes A\), so that

\[
u(e_i) = \sum_{j=1}^{n} e_j \otimes u_{ji},
\]

where \(u_{ij}\) are matrix elements of \(u\) (and \(e_i\) are absolute basis vectors in \(\mathbb{C}^n\)).

Let \(\mathcal{P} = (\mathcal{B}, i, F)\) be a quantum principal \(G\)-bundle over \(M\). Here, \(\mathcal{B}\) is a (unital) \(*\)-algebra consisting of appropriate “functions” on the bundle, \(i: V \to \mathcal{B}\) is the dualized “projection” of \(P\) on \(M\), and \(F: \mathcal{B} \to \mathcal{B} \otimes A\) is the dualized “right action” of \(G\) on \(P\). The elements of \(V\) will be identified with their images from \(i(V)\). The algebra \(\mathcal{B}\) is understandable as a bimodule over \(\mathcal{V}\), in a natural manner.

Let us assume that a system \(\tau = (\partial_1, \ldots, \partial_n)\) of \(\mathcal{B}\)-valued hermitian derivations \(\partial_i: \mathcal{V} \to \mathcal{B}\) is given such that

\[
F \partial_i(f) = \sum_{j=1}^{n} \partial_j(f) \otimes u_{ji}
\]

for each \(i \in \{1, \ldots, n\}\) and \(f \in V\). Finally, let us assume that the following completeness condition holds.

There exist a natural number \(d\) and elements \(b_{i\alpha} \in \mathcal{B}\) and \(v_{i\alpha} \in V\) (where \(\alpha \in \{1, \ldots, d\}\) and \(i \in \{1, \ldots, n\}\)) such that

\[
\sum_{\alpha} b_{i\alpha} \partial_j(v_{i\alpha}) = \delta_{ij} 1,
\]

for each \(i,j \in \{1, \ldots, n\}\).

**Definition 2.1.** Every system \(\tau\) satisfying the above conditions is called a frame structure on \(P\) (relative to \(u\)).
Definition 2.2. A frame structure $\tau$ is called integrable iff there exists a system $\hat{\tau} = (X_1, \ldots, X_n)$ of hermitian derivations $X_i: B \to B$ satisfying

\begin{align*}
FX_j &= \sum_{k=1}^{n} (X_k \otimes u_{kj})F \\
X_i|\mathcal{V} &= \partial_i \\
X_j \partial_j - X_j \partial_i &= 0
\end{align*}

for each $i, j \in \{1, \ldots, n\}$.

In the following, it will be assumed that the bundle $P$ is endowed with a fixed integrable frame structure $\tau$. Let us consider a graded $*$-algebra $\mathfrak{hor}_P = B \otimes \mathbb{C}_n^\wedge$, where $\mathbb{C}_n^\wedge$ is the corresponding external $*$-algebra. The elements of $\mathfrak{hor}_P$ will be interpreted as “horizontal forms” on $P$. Algebras $B$ and $\mathbb{C}_n^\wedge$ are naturally understandable as subalgebras of $\mathfrak{hor}_P$. We shall denote by $\theta_i \leftrightarrow 1 \otimes e_i$ special horizontal 1-forms corresponding to absolute basis vectors. Let $F^\wedge: \mathfrak{hor}_P \to \mathfrak{hor}_P \otimes A$ be the product of actions $F$ and $u^\wedge$, where $u^\wedge: \mathbb{C}_n^\wedge \to \mathbb{C}_n^\wedge \otimes A$ is the representation of $G$ in $\mathbb{C}_n^\wedge$ induced by $u$.

To each extension $\hat{\tau} = (X_1, \ldots, X_n)$ of $\tau$ it is possible to associate a first-order antiderivation $\nabla: \mathfrak{hor}_P \to \mathfrak{hor}_P$ such that

\begin{align*}
\nabla(b) &= \sum_{k=1}^{n} X_k(b)\theta_k \\
\nabla(\theta_i) &= 0
\end{align*}

for each $b \in B$ and $i \in \{1, \ldots, n\}$. Moreover,

\begin{align*}
\nabla^* &= \ast \nabla \\
F^\wedge \nabla &= (\nabla \otimes \text{id})F^\wedge
\end{align*}

Lemma 2.1. (i) There exists the common restriction $d_M: \Omega_M \to \Omega_M$ of all maps $\nabla \in \mathfrak{fr}(P)$.

(ii) The space $\Omega_M$ is linearly spanned by elements of the form

\begin{equation}
w = f_0d_Mf_1 \cdots d_Mf_k
\end{equation}

where $f_i \in \mathcal{V}$.

(iii) We have

\begin{equation}
d_M^2 = 0.
\end{equation}
Proof. Let us fix $\nabla \in \frak{f}(P)$ and let $d_M = \nabla|\Omega_M$. The completeness condition implies that each element $w \in \frak{hor}_P^k$ can be written in the form

$$w = \sum_i w_i d_M f_i$$

where $f_i \in \mathcal{V}$ and $w_i \in \frak{hor}_P^{k-1}$. This follows from the equality

$$\theta_i = \sum \alpha b_{i\alpha} d_M v_{i\alpha}.$$ (12)

Moreover, without a lack of generality we can assume that $w_i \in \Omega_{M}^{k-1}$. Therefore the statement follows by applying the principle of mathematical induction.

It is sufficient to check that (11) holds on elements of the form (10). Because of the graded Leibniz rule it is sufficient to check that $d^2_M(\mathcal{V}) = \{0\}$. However, this directly follows from (5) and from the definition of $\nabla$.

Finally, (i) follows from the graded Leibniz rule, (ii) and (iii), and from the fact that maps from $\frak{f}(P)$ act on elements from $\mathcal{V}$ in the same way (fixed by $\tau$). □

In other words $\Omega_M$, endowed with $d_M$, becomes a graded-differential *-algebra generated by $\mathcal{V}$.

All basic structural elements of the conceptual framework of [3] are now in the game. Let us recall that a preconnection on $P$ (relative to $\{\frak{hor}_P, F^\wedge, \Omega_M\}$) is a first-order hermitian antiderivation $D: \frak{hor}_P \rightarrow \frak{hor}_P$ satisfying

$$(13) \quad F^\wedge D = (D \otimes \text{id}) F^\wedge$$

$$(14) \quad D|\Omega_M = d_M$$

(according to (13) every $D$ is reduced in $\Omega_M$). Preconnections form a real affine space $\pi(P)$. Evidently, $\frak{f}(P) \subseteq \pi(P)$.

According to [3], for each $D \in \pi(P)$ and $E \in \frak{h}(P)$ (the vector space associated to $\pi(P)$) there exists the unique linear maps $\varphi_D, \chi_E: A \rightarrow \frak{hor}_P$ such that

$$(15) \quad D^2(\varphi) = -\sum_k \varphi_k \varphi_D(c_k),$$

$$(16) \quad E(\varphi) = -(-)^{\delta \varphi} \sum_k \varphi_k \chi_E^*(c_k)$$

for each $\varphi \in \frak{hor}_P$, where $\sum_k \varphi_k \otimes c_k = F^\wedge(\varphi)$. The maps $\varphi_D^*$ and $\chi_E^*$ determine, in a natural manner, a bicovariant first-order *-calculus $\Psi$ on $G$. This calculus is based (in the sense of [7]) on the right $A$-ideal $\mathcal{R} \subseteq \ker(\epsilon)$ consisting of elements annihilated by all $\varphi_D^*$ and $\chi_E^*$. These maps can be therefore factorized through $\mathcal{R}$. In such a way we obtain maps $\varphi_D, \chi_E: \Psi_{inv} \rightarrow \frak{hor}_P$, where $\Psi_{inv} = \ker(\epsilon)/\mathcal{R}$ is the space of left-invariant elements of $\Psi$. More precisely,

$$\varphi_D \pi = \varphi_D^*$$

$$\chi_E \pi = \chi_E^*$$

where $\pi: A \rightarrow \Psi_{inv}$ is the canonical projection map.
The following equalities hold

\[(17) \quad F^\vee \vartheta_D = (\vartheta_D \otimes \text{id}) \varpi\]
\[(18) \quad D \vartheta_D = 0\]
\[(19) \quad F^\vee \chi_E = (\chi_E \otimes \text{id}) \varpi\]

where \(\varpi: \Psi_{\text{inv}} \to \Psi_{\text{inv}} \otimes \mathcal{A}\) is the dualized (co)adjoint action, explicitly given by

\[\varpi \pi = (\pi \otimes \text{id}) \text{ad},\]

and \(\text{ad}: \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}\) is the dualized adjoint action of \(G\) on itself. Further,

\[(20) \quad \chi_E(\vartheta) \varphi = (-)^{\varphi} \sum_k \varphi_k \chi_E(\vartheta \circ c_k)\]
\[(21) \quad \vartheta_D(\vartheta) \varphi = \sum_k \varphi_k \vartheta_D(\vartheta \circ c_k)\]

for each \(\vartheta \in \Psi_{\text{inv}}\) and \(\varphi \in \mathfrak{hor}_P\), where \(\circ\) is the canonical right \(\mathcal{A}\)-module structure.

The map \(\vartheta_D\) is called the curvature of \(D\).

Every element \(\varphi \in \mathfrak{hor}_P\) can be written in the form

\[\varphi = \sum_k b_k d_M w_k,\]

where \(b_k \in \mathcal{B}\) and \(w_k \in \Omega_M\). This implies that each \(D \in \pi(P)\) is completely determined by its restriction \(D|B\). Explicitly, this restriction is described by

\[(22) \quad D(b) = \sum_{i=1}^{n} Y_i(b) \theta_i,\]

where \(Y_i: \mathcal{B} \to \mathcal{B}\) are hermitian derivations satisfying

\[(23) \quad Y_i|\mathcal{V} = \partial_i\]
\[(24) \quad FY_i(b) = \sum_{kj} Y_j(b_k) \otimes c_k u_{ji}\]

for each \(b \in \mathcal{B}\), where \(\sum b_k \otimes c_k = F(b)\). In terms of derivations \(Y_i\) the action of \(D\) on special horizontal forms \(\theta_j\) is given by

\[(25) \quad D(\theta_j) = \frac{1}{2} \sum_{akl} \left\{ Y_k(b_{j\alpha}) \partial_l(v_{j\alpha}) - Y_l(b_{j\alpha}) \partial_k(v_{j\alpha}) \right\} \theta_k \theta_l.\]

For every \(D \in \pi(P)\) let \(\Theta_D: \mathbb{C}^n \to \mathfrak{hor}_P\) be a linear map given by

\[(26) \quad \Theta_D(e_i) = \Theta^i_D = D(\theta_i).\]

**Definition 2.3.** The map \(\Theta_D\) is called the torsion of \(D\).
Lemma 2.2. The map $\Theta_D$ is hermitian and satisfies

\begin{equation}
F^\wedge \Theta_D^i = \sum_{j=1}^n \Theta_D^j \otimes u_{ji}
\end{equation}

(27)

\begin{equation}
-D\Theta_D^i = \sum_{j=1}^n \vartheta_j \vartheta_D^* (u_{ji})
\end{equation}

(28)

for each $i \in \{1, \ldots, n\}$.

Proof. The statement follows directly from properties (13) and (15) and from the definition of $\Theta_D$ and $F^\wedge$. □

Identity (28) corresponds to the second Structure equation in classical differential geometry [5].

Lemma 2.3. The following equivalence holds

\begin{equation}
\Theta_D = 0 \iff D \in \mathfrak{fr}(P)
\end{equation}

(29)

for each $D \in \pi(P)$.

Proof. Let us assume that $D \in \mathfrak{fr}(P)$. From (2) and (25) it follows that the torsion vanishes. Conversely, if $\Theta_D = 0$ then

$$0 = d^2_M(f) = D^2(f) = \frac{1}{2} \sum_{ij} [Y_i, Y_j](f) \theta_i \theta_j$$

for each $f \in \mathcal{V}$. In other words, $D \in \mathfrak{fr}(P)$. □

Let us assume that the higher-order differential calculus on $G$ is described by the universal envelope $\Psi^\wedge$ of $\Psi$. Let $\Omega_P$ be the graded-differential *-algebra canonically associated, in the sense of [3], to algebras $\mathfrak{hor}_P$ and $\Psi^\wedge$, and to the system $\pi(P)$ of preconnections. The main property of this algebra is that each $D \in \pi(P)$ naturally induces a representation of the form

\begin{equation}
\Omega_P \leftrightarrow \mathfrak{vh}_P
\end{equation}

(30)

where $\mathfrak{vh}_P$ is a graded *-algebra representing “vertically-horizontally” decomposed forms [2] on $P$. At the level of graded vector spaces, we have

$$\mathfrak{vh}_P = \mathfrak{hor}_P \otimes \Psi^\wedge_{sw}$$

The *-algebra structure on $\mathfrak{vh}_P$ is specified by

$$(\psi \otimes \vartheta)(\varphi \otimes \eta) = \sum_k (-)^{\vartheta \partial \varphi} \varphi_k \otimes (\vartheta \circ c_k)\eta$$

$$(\varphi \otimes \vartheta)^* = \sum_k \varphi_k^* \otimes (\vartheta^* \circ c_k^*).$$
In terms of the identification (30) the differential structure on $\Omega_p$ is expressed via a $D$-dependent differential $\partial_D$ on $\mathfrak{h}(P)$, which is given by

$$\partial_D(\varphi) = D(\varphi) + (-)\partial_D \sum \varphi_k \pi(c_k)$$

$$\partial_D(\vartheta) = \vartheta_D(\vartheta) + d(\vartheta)$$

where $\vartheta \in \Psi_{inv}$ and $d: \Psi_{inv} \to \Psi_{inv}$ is the corresponding differential ($\partial_D$ is extended on $\mathfrak{h}(P)$ by the graded Leibniz rule).

**Lemma 2.4.** As a differential algebra, $\Omega_p$ is generated by $\mathcal{B} = \Omega_p^0$.

**Proof.** Because of (12), it is sufficient to check that elements of the form $\pi(a)$ belong to $\mathcal{B}\partial_D \{\mathcal{B}\}$. For a given $a \in \mathcal{A}$, let us choose elements $q_k, b_k \in \mathcal{B}$ such that $\sum q_k F(b_k) = 1 \otimes a$ (the group $G$ acts freely on $P$). Then the following equality holds

$$\sum q_k \partial_D(b_k) = \sum q_k D(b_k) + \pi(a).$$

This implies (together with (12) and (22)) that $\pi(a)$ is expressible in the desired way.

The map $F$ is uniquely extendible to the homomorphism $\tilde{F}: \Omega_p \to \Omega_p \otimes \Gamma^\wedge$ of graded differential *-algebras (corresponding to the “pull back” map of differential forms).

There exists a natural bijective (affine) correspondence $D \leftrightarrow \omega$ between preconnections $D$ and regular connections $\omega$ on $P$. In terms of this correspondence,

$$R_\omega \leftrightarrow \vartheta_D$$

$$D_\omega \leftrightarrow D.$$

It is also possible to construct differential structures on $G$ and $P$ starting from a restricted set of preconnections forming an affine subspace of $\mathfrak{f}(P)$. In this case covariant derivatives of regular connections will (generally) form only an affine subspace of $\pi(P)$. In particular, every single element $\nabla \in \mathfrak{f}(P)$ determines a differential calculus on $P$. Let us assume that $\Psi$ is the minimal bicovariant first-order differential *-calculus over $G$ compatible with $\nabla$. Then the corresponding calculus on $P$ will be based on a graded-differential *-algebra $\Omega_p = (\mathfrak{h}_P, \partial_\nabla)$. It is also possible to vary this theme, and to choose for $\Psi$ an arbitrary (non-minimal) calculus satisfying the mentioned compatibility conditions.

Let $\pi(P)^\nabla \subseteq \pi(P)$ be the affine subspace consisting of preconnections interpretable as covariant derivatives of regular connections (relative to $\Omega_p$ constructed from $\{\nabla\}$).

A particularly interesting situation arises when $\nabla$ is compatible with the classical differential calculus on $G$.

**Definition 2.4.** An element $\nabla \in \mathfrak{f}(P)$ is called classical iff

$$\vartheta^\nabla(ab) = \epsilon(a) \vartheta^\nabla(b) + \vartheta^\nabla(a) \epsilon(b)$$

for each $a, b \in \mathcal{A}$. 
If $\nabla$ is classical then it is possible to assume that $\Psi$ is the classical differential calculus on $G$ (hence $\Psi_{\text{inv}} = \text{lie}(G)^\ast$). In this particular case,

$$
\begin{align*}
\varrho_D^\ast(ab) &= \epsilon(a)\varrho_D^\ast(b) + \varrho_D^\ast(a)\epsilon(b) \\
\chi_E^\ast(ab) &= \epsilon(a)\chi_E^\ast(b) + \chi_E^\ast(a)\epsilon(b) \\
\varrho_D(\vartheta)\varphi &= \varphi\varrho_D(\vartheta) \\
\chi_E(\vartheta)\varphi &= (-)^{\partial\varphi}\varphi\chi_E(\vartheta)
\end{align*}
$$

for each $D \in \pi(P_{\nabla})$ and $E \in \overrightarrow{\pi}(P_{\nabla})$ (the $\circ$-structure on $\Psi_{\text{inv}}$ is trivialized, because $\vartheta \circ a = \epsilon(a)\vartheta$).

**Lemma 2.5.** We have

$$
\pi(P_{\nabla}) \cap \mathfrak{fr}(P) = \{\nabla\}
$$

for each $\nabla \in \mathfrak{fr}(P)$.

**Proof.** Let us consider elements $D \in \pi(P_{\nabla})$ and $\nabla \in \mathfrak{fr}(P)$, and let us assume that

$$
\nabla \leftrightarrow (X_1, \ldots, X_n) \quad D \leftrightarrow (Y_1, \ldots, Y_n).
$$

Derivations $Z_i = X_i - Y_i$ possess the following properties

$$
\begin{align*}
FZ_i(b) &= \sum_{kj} Z_j(b_k) \otimes c_k u_{ji} \\
Z_i(f) &= 0
\end{align*}
$$

for each $b \in \mathcal{B}$ and $f \in \mathcal{V}$. Applying results of [3] we conclude that there exist linear maps $\lambda_i : \Psi_{\text{inv}} \to \mathcal{B}$ such that

$$
Z_i(b) = \sum_k b_k \lambda_i \pi(c_k)
$$

for each $b \in \mathcal{B}$ and $i \in \{1, \ldots, n\}$. In particular,

$$
Z_i \partial_j(f) = \sum_k \partial_k(f) \lambda_i^k
$$

where

$$
\lambda_i^k = \lambda_i(\pi(u_{kj})).
$$

Let us assume that $\Theta_D = 0$. This is equivalent to

$$
(Z_i \partial_j - Z_j \partial_i)(f) = 0
$$

for each $f \in \mathcal{V}$ and $i,j \in \{1, \ldots, n\}$. Identities (35)–(37), together with the completeness condition imply

$$
\begin{align*}
\lambda_i^k &= \lambda_j^i \\
\lambda_i^j &= -\lambda_j^i
\end{align*}
$$

On the other hand

$$
\lambda_i^k = -\lambda_j^k
$$
as easily follows from (36) and the hermicity of \( u_{ij} \). It follows that \( \lambda_i^k = 0 \), and hence \( \lambda_i = 0 \), because \( \Psi_{\text{inv}} \) is spanned by elements \( \pi(u_{ij}) \). Hence \( D = \nabla \). \( \square \)

The above equivalence corresponds to the classical characterization of the Levi-Civita connection, as the unique (metric) connection with vanishing torsion. Conceptually, we followed a classical proof [5] of the uniqueness of the Levi-Civita connection.

3. Examples

3.1. The classical case

Let \( P \) be a classical principal SO(\( k \))-bundle over a compact smooth \( n \)-dimensional manifold \( M \) (where \( k \leq n \)) and let \( \tau = (\partial_1, \ldots, \partial_k) \) be a frame structure on \( P \) (relative to the standard representation \( u \) of SO(\( k \)) in \( \mathbb{C}^k \)). Then every point \( p \in P \) naturally determines a \( k \)-tuple \((\xi_1, \ldots, \xi_k)\) on tangent vectors on \( M \) in the point \( x = \pi_M(p) \) as follows

\[
(40) \quad \xi_i(f) = [\partial_i(f)](p).
\]

Here \( \pi_M: P \to M \) is the projection map.

From the transformation property (1), it follows that the space \( \Sigma_x \subseteq T_x(M) \) spanned by \((\xi_1, \ldots, \xi_k)\) is independent of the choice of the point \( p \in \pi_M^{-1}(x) \). On the other hand, completeness condition (2) implies that \((\xi_1, \ldots, \xi_k)\) are linearly independent vectors.

In such a way an oriented \( k \)-dimensional subbundle \( \Sigma \) of \( T(M) \) is constructed. Fibers \((\Sigma_x)_{x \in M}\) possess a natural Euclidean structure defined by requiring that \((\xi_1, \ldots, \xi_k)\) are orthonormal vectors. In classical terms, \( P \) is identifiable with the bundle of oriented orthonormal frames of \( \Sigma \).

Let us assume that \( \tau \) is integrable. This implies that the space of smooth sections of \( \Sigma \) is closed with respect to the commutator of vector fields. In other words, \( \Sigma \) is integrable (according to Frobenius theorem).

Let \( N \subseteq M \) be an arbitrary leaf of the foliation \( \Sigma \), and \( P_N \) the portion of \( P \) over \( N \). This bundle coincides, in a natural manner, with the bundle of oriented orthonormal frames of \( N \). It is invariant under the action of fields \( X_i \). Restrictions of \( X_i \) on \( P_N \) determine the standard Levi-Civita connection.

Bundles \( P_N \) determine a foliation \( \Sigma^* \) of \( P \). The elements of \( \Omega_p \) are naturally interpretable as \( \Sigma^* \)-differential forms on \( P \). In this picture the algebra \( \Omega_M \) consists of \( \Sigma \)-differential forms on \( M \).

In particular, the case \( k = n \) is equivalent to the classical oriented Riemannian manifold structure on \( M \), so that \( P \) becomes the corresponding bundle of oriented orthonormal frames.

3.2. A Framed Quantum SO(2) Bundle

Let us assume that \( \mathcal{V} \) is endowed with a *-automorphism \( \gamma: \mathcal{V} \to \mathcal{V} \). Let us assume that \( G = \text{SO}(2) \). The Hopf *-algebra \( \mathcal{A} \) of polynomial functions on \( G \) is generated by a unitary element \( U = \cos + i \sin \) (and \{\cos, \sin\} are understood as
functions on $G$). We have $\phi(U) = U \otimes U$. The formulas
\begin{align}
(f \otimes U^m)(g \otimes U^n) &= f \gamma_m(g) \otimes U^{m+n} \\
(f \otimes U^m)^* &= \gamma^{-m}(F^\gamma) \otimes U^{-m}
\end{align}
(where $n, m \in \mathbb{Z}$) define a *-algebra structure on the vector space $B = \mathcal{V} \otimes A$. Let $i: \mathcal{V} \rightarrow B$ be the canonical inclusion map. The formula
\[ F(f \otimes U^m) = f \otimes U^m \otimes U^m \]
defines an action of $G$ by *-automorphisms of $B$, so that $P = (B, i, F)$ is a (quantum) principal $G$-bundle over $M$.

Let us define derivations $X_\pm: B \rightarrow B$ by
\begin{align}
X_+(b) &= (\alpha \otimes U)b - b(\alpha \otimes U) \\
X_-(b) &= (\beta \otimes \bar{U})b - b(\beta \otimes \bar{U})
\end{align}
where $\alpha, \beta \in \mathcal{V}$ are such that $\beta = -\gamma^{-1}(\alpha^*)$. We have
\[ [X_+, X_-](b) = vb - bv \]
where $v = \alpha \gamma(\beta) - \beta \gamma^{-1}(\alpha)$. Let $X_1, X_2: B \rightarrow B$ be (hermitian) derivations given by $X_\pm = X_1 \mp iX_2$. Let $\partial_1, \partial_2 = X_1, X_2|\mathcal{V}$ be the corresponding restrictions. If the completeness condition (2) holds then $\tau = (\partial_1, \partial_2)$ is a frame structure on $P$, relative to the standard representation
\[ u = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \]
of $\text{SO}(2)$ in $\mathbb{C}^2$. If $v$ is a central element of $\mathcal{V}$ then $\tau$ is integrable, and we can write $\hat{\tau} = (X_1, X_2)$. The corresponding curvature is given by
\[ \vartheta^\tau(U^m) = \frac{1}{4t}(v - \gamma^{-m}(v))\theta_1\theta_2. \]
In general, such frame structures induce nonstandard differential calculi on $G$. Let us assume that $v$ is non-trivial, and that $\gamma(v) = tv$, for some $t \in \mathbb{N} \setminus \{-1, 0, 1\}$. This naturally induces a 1-dimensional calculus on $G$.

The space $\Psi_{\text{inv}}$ is spanned by $\zeta = \pi(U - \bar{U})$. The corresponding ideal $\mathcal{R}$ is generated by $tU + \bar{U} - (1+t)1$. The o-structure on $\Psi_{\text{inv}}$ is specified by $\zeta \circ U^m = t^{-m}\zeta$.

### 3.3. Free Actions of Simple Lie Groups on Quantum Spaces

Let us assume that a compact simple Lie group $H$ acts freely on a quantum space $P$ (determined by a *-algebra $B$). Let $G$ be a compact subgroup of $H$. Let $F^*: B \otimes A^*$ be the dualized action of $H$ on $P$ (where $A^*$ is the *-algebra of polynomial functions on $H$) and let $F = (\text{id} \otimes j)F^*$ be the restriction of the action of $H$ on $G$. Here $j: A^* \rightarrow A$ is the restriction map. In what follows the entities endowed with $\ast$ will refer to $H$. Both groups will be endowed with standard differential structures.

The triplet $P = (B, i, F)$ is a quantum principal $G$-bundle over $M$, where $M$ is the quantum space based on the $F$-fixed-point *-subalgebra $\mathcal{V}$, and $i: \mathcal{V} \hookrightarrow B$ is the inclusion map.
Let $\mathfrak{g}^*$ be the (complex) Lie algebra of $H$. Let $u$ be the adjoint representation of $G$ in the space $\mathfrak{g}^* \subseteq \mathfrak{g}^*$. Here, $\mathfrak{g} \subseteq \mathfrak{g}^*$ is the Lie algebra of $G$, and it is assumed that $\mathfrak{g}^*$ is endowed with the (positive) Killing scalar product. Let us assume that the kernel of $u$ is discrete. Further, let us assume that $\mathfrak{g}^\perp$ is identified with $\mathbb{C}^n$, with the help of a real orthonormal basis $(\xi_1, \ldots, \xi_n)$ in $\mathfrak{g}^\perp$.

For each $i \in \{1, \ldots, n\}$ let $X_i = (\text{id} \otimes \xi_i \pi^*)(F^*)$ be the hermitian derivation on $B$ corresponding to $\xi_i$ (here we have identified $\mathfrak{g}^* = (\Psi^{\ast \ast})^*$).

Let $\partial_i : \mathcal{V} \to \mathcal{B}$ be the restrictions of $X_i$ on $\mathcal{V}$.

Lemma 3.1. Under the above assumptions $\tau = (\partial_1, \ldots, \partial_n)$ is a frame structure on $P$. This structure is integrable if

\[ [\mathfrak{g}^\perp, \mathfrak{g}^\perp] \subseteq \mathfrak{g}. \]

In this case

\[ \hat{\tau} = (X_1, \ldots, X_n). \]

Proof. A direct computation gives

\[
FX_i(b) = \sum_k F(b_k \xi_i \pi^*(d_k)) = \sum_k b_k \xi_i \pi^*(d_k^{(2)}) \otimes j \left[ d_k^{(1)} \kappa^*(d_k^{(3)}) d_k^{(4)} \right] = \sum_{jk} X_j(b_k) \otimes u_{ji} j(d_k). 
\]

Here, $F^*(b) = \sum_k b_k \otimes d_k$ and we have applied the identity

\[
\xi_i \pi^*(a^{(2)}) \otimes j (a^{(1)} \kappa^*(a^{(3)})) = \sum_j \xi_j \pi^*(a) \otimes u_{ji}.
\]

Hence, derivations $X_i$ transform in the appropriate way.

Let us prove the completeness condition. It is sufficient to prove that for each $a \in \mathcal{A}^*$, invariant under the right action of $G$ on $H$, there exist elements $b_k \in \mathcal{B}$ and $v_k \in \mathcal{V}$ such that

\[ \sum_k b_k F^*(v_k) = 1 \otimes a. \]

Indeed, for a given $G$-invariant element $a \in \mathcal{A}^*$ there exist elements $q_k, b_k \in \mathcal{B}$ such that

\[ \sum_k b_k F^*(q_k) = 1 \otimes a \]

(this is the place where the freeness assumption enters the game). This implies

\[ \sum_{kl} b_k q_{kl} \otimes a_{kl}^{(1)} h j(a_{kl}^{(2)}) = 1 \otimes a \]

where $F^*(q_k) = \sum_l q_{kl} \otimes a_{kl}$ and $h : \mathcal{A} \to \mathbb{C}$ is the Haar measure on $G$. If we define

\[ v_k = \sum_l q_{kl} h j(a_{kl}), \]

then $v_k \in \mathcal{V}$ and (48) holds.
Finally, let us assume that (46) holds. Then $[X_i, X_j](v) = 0$ for each $v \in \mathcal{V}$ and $i, j \in \{1, \ldots, n\}$, according to the definition of $\mathcal{V}$. Hence, $\tau$ is integrable and (47) holds.

The corresponding graded-differential $\ast$-algebra $\Omega_P$ can be naturally realized as

$$\Omega_P = [\Psi_{\text{inv}}]^\ast \otimes \mathcal{B},$$

in other words, $\mathcal{B}$-valued forms on $g^\ast$ (with the standard algebraic structure).

The curvature map of the Levi-Civita connection $\nabla$ is given by

$$\vartheta^\ast_{\nabla}(a) = -\frac{1}{2} \sum_{ij} [\xi_i, \xi_j] \pi(a) \theta_i \theta_j.$$

4. Concluding remarks

Quantum counterparts of various important differential-geometric structures can be introduced in the framework of the concept of the frame structure. In particular, frame structures on quantum $\text{SO}(n)$-bundles provide a natural framework for a noncommutative-geometric version of (oriented) Riemannian geometry (the Xodge $\ast$-operator and the Laplace operator, for example).

All essential elements of the algebraic structure appearing in the theory of Kahler manifolds are preserved in its noncommutative-geometric version, dealing with framed quantum $\text{U}(n)$-bundles. In particular, quantum spaces $M$ considered in the previous section can be naturally endowed with Kahler manifold structures, in accordance with the analogy $M \leftrightarrow \text{CP}(n)$.

In this paper, we have assumed that the structure quantum group is compact. This assumption is not essential. The whole formalism can be directly incorporated in a more general conceptual framework, including non-compact structure groups and non-unitary representations $u$. However in this case (as in classical geometry) $\nabla$ is generally not uniquely determined by its class $\pi(P)\nabla$, even if the calculus on the structure group is classical.

In terms of the constructed differential calculus on $P$, frame structures are completely represented by $n$-tuples $\theta = (\theta_1, \ldots, \theta_n)$ of special horizontal 1-forms $\theta_i$ (counterparts of classical frame forms [5]). These forms transform covariantly, according to the representation $u$.

In this sense, integrable frame structures can be viewed as a special case of frame structures introduced in [4]-Appendix B.

The presented formalism admits a natural generalization to the fully quantum context (in which the structure group $G$ is a quantum object). The only essentially new phenomena naturally appearing in the game is the presence of a non-trivial right $\mathcal{A}$-module structure $\circ$ on the representation space $\mathbb{C}^n$, which is compatible with the right comodule structure $u$ in such a way that $\mathbb{C}^n$ together with $u$ and $\circ$ becomes a “left-invariant part” of a bicovariant $\ast$-bimodule $\Gamma$ over $G$. This requires the following compatibility condition

$$u(e_i \circ a) = \sum_j (e_j \circ a^{(2)}) \otimes \kappa(a^{(1)}) u_{ji} a^{(3)}$$

for each $i \in \{1, \ldots, n\}$ and $a \in \mathcal{A}$. 
The external algebra $\mathbb{C}_n^\wedge$ should be replaced by the quantum external algebra, associated to the “left-invariant part” $\sigma: \mathbb{C}^n \otimes \mathbb{C}^n \to \mathbb{C}^n \otimes \mathbb{C}^n$ of the corresponding canonical flip-over [7] operator (or its scalar multiple). Also, the construction of the horizontal algebra incorporates elements of the bicovariant bimodule structure. Maps $X_i$ are not derivations, although their restrictions $\partial_i = X_i|\mathcal{V}$ still satisfy the Leibniz rule.

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