Metric perturbations of extremal surfaces

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Abstract
Motivated by the HRRT-formula for holographic entanglement entropy, we consider the following question: what are the position and the surface area of extremal surfaces in a perturbed geometry, given their anchor on the asymptotic boundary? We derive explicit expressions for the change in position and surface area, thereby providing a closed form expression for the canonical energy. We find that a perturbation governed by some small parameter $\lambda$ yields an expansion of the surface area in terms of a highly non-local expression involving multiple integrals of geometric quantities over the original extremal surface.

Keywords: holography, AdS/CFT correspondence, differential geometry, extremal surfaces, holographic entanglement entropy, relative entropy

1. Introduction

The AdS/CFT-correspondence is the conjecture that under certain conditions, a conformal field theory (CFT) on a $d$-dimensional spacetime (‘the boundary’) describes a theory of quantum gravity on a $(d + 1)$-dimensional asymptotically Anti de Sitter (AdS) spacetime (‘the bulk’) [1].

The ‘holographic dictionary’, the map between quantities in these dual theories, is a subject of continued study. An important identification was given by the HRRT-formula [2, 3], which states that the entanglement entropy of a sub-region of the CFT can be associated with the area of an extremal surface in the bulk.

In the AdS/CFT-correspondence, perturbations of the quantum state of the CFT can be associated with perturbations of the bulk geometry and bulk fields. The CFT-state is thought to describe a well defined classical bulk geometry only under certain conditions [4, 5]. The HRRT-formula was previously used to derive such a condition on the CFT [6]. A deeper understanding of the relation between entropy in the CFT and the area operator in the bulk is expected to reveal more information about the following questions:
which CFTs are ‘holographic’?
which CFT-states are dual to well defined classical bulk geometries?

In this article, we obtain explicit expressions for the change of the position and surface area of extremal surfaces, due to perturbations of the bulk metric.

A related geometric problem was previously discussed in the mathematics literature: what is the change of the area, when a surface is arbitrarily shifted away from extremality? At leading order in the ‘shift’, the change of the area is given by an integral involving the Jacobi or stability operator [7, 8]. Here, we consider a more complicated problem: first we perturb the metric, and subsequently we shift the surface such that it is extremal in the new, perturbed geometry.

Previous work in this direction includes [9], in which the metric perturbation of renormalized areas of extremal surfaces in $\mathbb{H}^3$ was studied (to second order). For certain black hole geometries, the corrections to holographic entanglement entropy were discussed in [10] and [11] (also up to second order). Shortly after this article appeared, [12] came out, of which the results partially overlap with this study. A different type of perturbations, those of the shape of the ‘anchoring surface’ at the asymptotic boundary, were discussed in [13, 14]. In [15], it was discussed how a number of problems can be addressed with the explicit perturbative expression for holographic entanglement entropy.

Our procedure allows for an explicit expansion of the area operator in terms of the metric perturbation, in a general gauge, and to arbitrary order.

We also provide an iterative procedure to construct a diffeomorphism that brings the metric in the Hollands–Wald (HW) gauge [16], a gauge that was found to be useful in the context of the AdS/CFT-correspondence [6, 17, 18].

2. Summary

2.1. Method

The HRRT-formula [2, 3] states that the entanglement entropy $S(B)$ of a (spatial) sub-region $B$ of the CFT, can be associated to the surface area of the bulk surface $\tilde{B}$, where $\tilde{B}$ has extremal area, and where $\tilde{B}$ is homological to the sub-region $B$ at the asymptotic boundary, by

\[ S(B) = \frac{A(\tilde{B})}{4G_N}. \tag{1} \]

There are corrections to this formula at sub-leading orders of $G_N$, that depend on other bulk fields, in addition to the bulk metric $g$ [19]. The HRRT-formula has been proved under mild assumptions [20–22].

The quantum state of the CFT can be described by a density matrix $\rho$. We consider perturbations of a reference state $\rho_0$, typically the vacuum, which are governed by a small parameter $\lambda^2$:

\[ \rho(\lambda) = \rho_0 + \Delta \rho = \rho_0 + \lambda \rho_1 + \lambda^2 \rho_2 + \ldots, \tag{2} \]

where $\rho_0$ is the density matrix associated with the background bulk geometry $g_0$. When the perturbation of the state (2) corresponds to a classical bulk geometry, the metric can be expanded as

\[ \text{At leading order in } G_N, \text{ and in Planck units } (\hbar = c = 1). \]

\[ \text{Where the } \rho_i \text{ must be traceless and hermitian, for } i > 0. \]
where $\Delta g$ vanishes near the asymptotic boundary. In this article we consider the extremal surface that ends on an arbitrary sub-region of the CFT on the asymptotic boundary and

- determine the location of the extremal surface in the perturbed geometry (3)
- derive a perturbative expansion for the surface area of the extremal surface in the perturbed geometry (3).

In section 3 we use a variational method, where we

(i) expand the embedding function $x^a(\alpha)$ as:

$$x^a(\alpha) = x^a_0(\alpha) + \Delta x^a(\alpha) = x^a_0(\alpha) + \lambda x^a_1(\alpha) + \lambda^2 x^a_2(\alpha) + \ldots$$  \hspace{1cm} (4)

where $\{\alpha\}$ is a set of $(d-1)$ parameters for the extremal surface, and $x^a_0(\alpha)$ is the embedding function of the original surface $\tilde{B}$, which is extremal for the unperturbed background geometry $g_0$.

(ii) expand the area functional using (3) and (4)

(iii) extremize the area functional with respect to $x_1, x_2, \ldots$

The extremization procedure yields 'equations of motion' for $x_1, x_2, \ldots$, which can be solved in certain cases. These 'equations of motion' correspond with the condition that the expansions of an extremal surface vanish [3]. The solutions for $x_1, x_2, \ldots$ can be substituted back into the area functional.

In sections 4 and 5 we also show that the extremization problem at any order in $\lambda$ can be reduced to that of order one, by using gauge transformations. Furthermore, in section 6 we derive solutions for a series of diffeomorphisms that bring the metric into the Hollands–Wald gauge.

### 2.2. Results

The extremization procedure described in section 3 yields 'equations of motion' for the 'shifts' $x_1, x_2, \ldots$ (4). These 'equations of motion' correspond to the requirement that the expansions 'K where $i = 0, 1$ labels the normal vectors 'N, vanish at all orders in $\lambda$:

$$\frac{d^i K}{d \lambda^i} \bigg|_{\lambda=0} = 0 \Leftrightarrow \text{e.o.m. for } x^a_i$$ \hspace{1cm} (5)

$$\frac{d^2 x^a}{d \lambda^2} \bigg|_{\lambda=0} = 0 \Leftrightarrow \text{e.o.m. for } x^a_2$$ \hspace{1cm} (6)

The first equation (5) can be simplified by choosing $x_1$ to be perpendicular to the original surface: $x^a_1 \parallel 0$. In this article, we focus mostly on metric perturbations of $\text{AdS}_{d+1}$, and ball-shaped sub-regions of the CFT\(^3\), for which the equation of motion of $x_1$ can be solved by using a Green's function:

$$x^a_1 = \sum_i \left( \int_{\tilde{B}} d^{d-1} \beta \sqrt{h} G_0(\alpha, \beta) \delta^a_i K(\beta) \right) N^a_i(\alpha)$$  \hspace{1cm} (7)

\(^3\)In this case, the HRRT-surfaces are totally geodesic; their extrinsic curvatures vanish completely.
where \( G_{\tilde{B}}(\alpha, \beta) \) is a Green’s function on the original HRRT-surface \( \tilde{B} \), which has hyperbolic geometry, satisfying

\[
(\Box_{\tilde{B}} - (d - 1)) G_{\tilde{B}}(\alpha, \beta) = \frac{1}{\sqrt{h_0}} \delta^{d-1}(\alpha - \beta),
\]

and \( \delta^1 \iota K \) is the first order metric perturbation of the expansions \( \iota K \), evaluated at \( \delta g_{ab} = g^1_{ab} \) (for an explicit expression, see equation (30)).

As described in sections 4 and 5, a series of diffeomorphisms can be used to reduce the extremization problem for \( x_2, x_3, \ldots \) to a problem of the same complexity as the extremization problem for \( x_1 \). For \( x_2 \), this diffeomorphism is generated by the vector field \( V^a = -x^a_1 \), which basically ‘reverses’ the shift at order \( \lambda \), and the solution for \( x_2 \) is given by

\[
x^a_2 = -\frac{1}{2} x^b_1 \partial_b x^a_1 + \sum_i \left( \int d^{d-1} \beta \sqrt{\tilde{h}} G_{\tilde{B}}(\alpha, \beta) \left( \delta^2 \iota K(\beta) + \delta^1 \iota K(\beta) \right) \right) x^a_0(\alpha)
\]

(10)

with \( \tilde{g}^1_{ab} = g^1_{ab} - L_{x_1} g^0_{ab} \) and \( \tilde{g}^2_{ab} = g^2_{ab} - L_{x_1} g^1_{ab} + \frac{1}{2} L_{x_1} L_{x_1} g^0_{ab} \).

Similar results can be obtained for \( x_3, x_4, \ldots \) et cetera.

The solutions for \( x_1, x_2, \ldots \) can be substituted back into the area functional: this provides an expansion of the area functional in terms of integrals over the original HRRT-surface \( \tilde{B} \) (for ball-shaped boundary sub-regions and the AdS\(_{d+1}\) background geometry):

\[
A(\lambda) = A_0 + \lambda A_1 + \lambda^2 A_2 + \ldots,
\]

(12)

where the first non-trivial term, at order \( \lambda^2 \), is given by:

\[
A_2 = \frac{1}{2} \int d^{d-1} \alpha \sqrt{h_0} \left( \frac{\delta^2 \sqrt{h_0}}{\delta g^0_{ab}} + g^1_{ab} \frac{\delta^1 \sqrt{h_0}}{\delta g^1_{cd}} \frac{\delta \sqrt{h_0}}{\delta g^0_{ab}} \right)
\]

(13)

\[
+ \frac{1}{2} \sum_{i=0,1} \int d^{d-1} \alpha \sqrt{h_0} \int d^{d-1} \beta \sqrt{h_0} G_{\tilde{B}}(\alpha, \beta) \left( \delta^2 \iota K(\alpha) \right) \left( \delta^1 \iota K(\beta) \right).
\]

(14)

More generally, the term at order \( n \) in the expansion of the extremal surface area (12) has the structure

\[
A_n = \left( \int_{\tilde{B}} \left( \cdots \right) + \cdots + \int_{\tilde{B}} \cdots \int_{\tilde{B}} \left( \cdots \right) \cdots \right)^{n-1} \text{Green’s functions}.
\]

(15)

The methods detailed below can be used in an iterative procedure, in order to find a series of diffeomorphism generating vector fields that enforce the Hollands–Wald gauge. At first order, a solution for the diffeomorphism generating vector field is given by

\footnote{A Green’s function is generally not unique without specifying the appropriate boundary conditions; in this case we require the response function to vanish at the boundary. For a discussion of Green’s functions on hyperbolic space, see for example [23–25].}
\( V_1^a = S_1^a + K_1^a, \) \hspace{1cm} (16)

where \( S \) stands for ‘shift’ and \( K \) for ‘Killing’, as \( S_1^a = -x_1^a \) reverses the shift \( x_1 \), such that the coordinate position of the extremal surface is unchanged, and

\[
K_1^a = -\frac{1}{2n} \bar{g}_1^{ab} \xi^b + \frac{1}{8\pi^2} \epsilon_{ab} \bar{g}_1^{bc} \xi^c, \quad \bar{g}_1^{ab} = g_1^{ab} + L_S g_0^{ab}, \tag{17}
\]

enforces that the Killing equation is still satisfied at the extremal surface \( \bar{B} \). Here, \( \xi \) is the Killing vector, for which \( \bar{B} \) is the Killing horizon [26]. Note that \( K_1 \) vanishes at the extremal surface \( \bar{B} \), because the Killing vector \( \xi \) vanishes there. Details can be found in section 6, where we outline a procedure to construct explicit solutions for the \( V_i \) at higher orders of \( \lambda \).

### 3. Variational method

In this section, we first consider the area functional as a function of an arbitrary shift of the form (4)–this is generally not an extremal surface–, and expand it in orders of \( \lambda \). Extremization of the area functional with respect to \( x_1, x_2, \ldots \) (4) yields the ‘equations of motion’ (5) and (6), that can be solved in certain cases.

First, we construct a basis of normal vectors, that remains orthonormal for non-zero \( \lambda \), given an arbitrary shift of the position of the surface (4):

\[
iN_a(\lambda) = iN_a + \Delta iN_a + \lambda iN_1^a + \lambda^2 iN_2^a + \ldots \tag{18}
\]

where \( i = 0, 1 \) labels the two normal vectors (one timelike, one spacelike), \( a \) is a regular space-time index and the right superscript \((0,1,2,\ldots)\) represents the order in \( \lambda \) (see appendix A for notation). The area functional now depends on \( \lambda \) via:

(i) the perturbed metric (3)
(ii) the change of the position (4)
(iii) the normal vectors (18) associated to (3) and (4)

since

\[
A(\bar{B}) = \int_{\bar{B}} d^{d-1} \alpha \sqrt{h} \\
\text{where } h_{ab} = g_{ab} + \sum_i (-)^i iN^i_a N_b \text{ is the induced metric.} \tag{19}
\]

The HRRT-surface depends on \( \lambda \) via each point of the (original) surface:

\[
\frac{d}{d\lambda} = \int d^{d-1} \alpha \left\{ \frac{\partial \alpha^a}{\partial \lambda} \delta \frac{\partial g_{ab}}{\partial \alpha^a \delta x^b} + \frac{\partial g_{ab}}{\partial \alpha^a \delta x^b} \frac{\partial}{\partial \lambda} + \sum_{i=0,1} \frac{\partial \alpha^a}{\partial \lambda} \frac{\partial}{\partial \delta N_a} \delta N_a \right\}. \tag{20}
\]

In what follows, the geometric flow equation will provide a key simplification:

\[
\frac{\delta A}{\delta x^a} = \sum_i \sqrt{h} K^i N_a. \tag{21}
\]
3.1. Normal vectors

First, we construct the normal vectors (18) for the arbitrarily shifted surface (4), in the perturbed geometry (3). Here we state the result for the $iN^1$; for a full derivation see appendix B.

The tangent component $iN^1_{||a}$ (18) is determined by the orthogonality condition:

$$iN^1_{||a} = -h^0_{ab} \left( iN^0_{b} \nabla_{b} x^a_1 + x^a_1 \nabla_{a} iN^0_{b} \right).$$

(22)

The perpendicular component $iN^1_{\perp a}$ (18) is determined by requiring unit norm and orthogonality between the $iNa$, $i=0,1$:

$$iN^1_{\perp a} = \frac{1}{2} \sum_j \left( g^{bc} iN^0_{b} iN^0_{c} \right) jN^0_{a}.$$  

(23)

Note that the perpendicular component $iN^1_{\perp}$ (23) only depends on the metric perturbation $g^1$ and not on the shift $x^1_0$.

3.2. Expansion of the area functional to first order

The first $\lambda$-derivative (20) of the surface area $A$ (19) can be simplified by using the geometric flow equation (21):

$$\frac{dA}{d\lambda} = \int_B d^{d-1} \alpha \left( \frac{\partial g_{ab}}{\partial \lambda} \frac{\delta \sqrt{h}}{\delta g_{ab}} + \frac{\partial x^a}{\partial \lambda} \sum_i \sqrt{h} K^i N_a \right).$$

(24)

Evaluating this formula at $\lambda = 0$, using $\frac{\delta \sqrt{h}}{\delta g_{ab}} = \frac{1}{2} \sqrt{hh_{ab}}$, we find

$$\left. \frac{dA}{d\lambda} \right|_{\lambda=0} = \frac{1}{2} \int_B d^{d-1} \alpha \sqrt{hh_{ab}} g_{ab},$$

(25)

where we used that the expansions $iK$ vanish under the assumption that $x_0(\alpha)$ is an extremal surface for the background geometry $g^0$. We recovered a well-known result [3]: at first order in $\lambda$, the change of the surface area of an extremal surface does not depend on the shift of the surface, due to the fact that the expansions of an extremal surface vanish.

3.3. Expansion of the area functional to second order

We proceed by taking one more $\lambda$-derivative (using equation (20)) of (24) and evaluate at $\lambda = 0$:

$$\left. \frac{d^2A}{d\lambda^2} \right|_{\lambda=0} = \int_B d^{d-1} \alpha \left( 2g^2_{ab} \frac{\partial \sqrt{h_0}}{\partial g_{ab}} + g^1_{ab} g^1_{cd} \frac{\partial^2 \sqrt{h_0}}{\partial g_{ab} \partial g_{cd}} \right)$$

$$+ \int_B d^{d-1} \alpha \left( x^a_1 \nabla_a \left( g^1_{ab} \frac{\partial \sqrt{h_0}}{\partial g_{ab}} \right) + \sum_i iN^1_{||} \frac{\delta}{\delta N^0_i} \left( g_{ab} \frac{\delta \sqrt{h_0}}{\delta g_{ab}} \right) \right).$$

(26)

(27)

5 The normal vectors $iN^0$ are defined at the surface $B$, where they are orthogonal and have unit norm, and can—in principle—be extended away from $B$ in an arbitrary way. For convenience, we require that the $iN^0$ remain orthonormal in some neighborhood of $B$.

6 For this reason, we will include the terms involving $iN^1$ in the $iK^1$, the first order metric perturbation of the expansions $iK$. This is just a matter of “book keeping”.

7 Note that $\frac{\delta \sqrt{h}}{\delta g_{ab}} = \pm \sqrt{h} h_{ab}$, because the induced metric $h$ projects onto the tangent space.
\[ + \sum_i \int_B d^{d-1} \alpha \sqrt{h_0} x_i^N \left( \frac{dK}{d\lambda} \right)_{\lambda=0}. \tag{28} \]

Line (28) contains the first order correction to the extrinsic curvature (for arbitrary \(x_1\)). At first order,

\[
\frac{dK}{d\lambda} \bigg|_{\lambda=0} = \delta_i^{1i}K + \sum_j \delta_j^{1j}K
\]

\[= \delta_i^{1i}K + \sum_j \delta_j^{1j}K
\]

\[= \delta_i^{1i}K + x_i^N \partial_0^d N_i^a - h_0^{ab} \nabla_a \left(h_0^{cd} (\nabla_c x_i^N i^d) \right) - i K_{0b} h_0^{ac} \nabla_a x_i^b \tag{29}\]

where the \(\delta_i^{1i}K\) are the first order corrections to the expansions \(iK\) due to the shift (4), \(\delta_j^{1j}K\) is the first order correction to the expansions \(iK\) due to the change in the metric (3);

\[\delta_i^{1i}K = -h_0^{ab} \delta_i^{1j} \Gamma^j_{ab} i^N \delta_j \left( \delta_0^{cd} i^d \right) \tag{30}\]

\[\delta_j^{1j} \Gamma^c_{ab} = \frac{1}{2} g_0^{cd} \left( \nabla_a s_0^d + \nabla_b s_0^d - \nabla_d s_0^1 \right) \tag{31}\]

and \(\delta_j^{1j}K\) is the first order correction due to the tangent component of the change of the normal vector \(\gamma/N\) (for a derivation of (29), see appendix C).\(^8\)

Line (27) can be rewritten as

\[\sum_i \int_B d^{d-1} \alpha \sqrt{h_0} (x_i^N i^N) \left( \delta_i^{1i}K \right) + \sum_i \int_B d^{d-1} \alpha \sqrt{h_0} D_a \left( g_0^{ab} N_b^c x_i^N i^d \right), \tag{32}\]

where the second term of line (32) is a boundary term.

### 3.4. Extremization of the area functional with respect to \(x_1(\alpha)\)

Using some algebra and one partial integration (32), we rewrite the terms at second order, equations (26)–(28), as

\[\frac{d^2 A}{d\lambda^2} \bigg|_{\lambda=0} = \int d^{d-1} \alpha \left( 2 s_{ab} \delta \sqrt{h_0} \delta_0^b + s_{ab} \delta_0^c \delta \sqrt{h_0} \delta_0^d \right) \frac{\delta^2 \sqrt{h_0}}{\delta \sqrt{h_0} \delta \sqrt{h_0}} \]

\[+ \sum_i \int_B d^{d-1} \alpha \sqrt{h_0} (x_i^N \delta_i^{1i}K + K_{0b} \delta_0^{ac} \delta_0^{cd} i^d), \tag{33}\]

where only the second line contains terms involving \(x_1\).

Extremizing with respect to \(x_1\) gives an ‘equation of motion’ for \(x_1\), which in turn determines \(x_1\) in terms of the metric perturbation \(g_1\). Using (34) and (29) we find that the equation of motion for \(x_1\) is equivalent to the requirement that the expansions vanish at first order in \(\lambda\):

\[0 = \frac{dK}{d\lambda} \bigg|_{\lambda=0} \tag{35}\]

\(^8\) For totally geodesic surfaces \(K_{0b} = 0\).
\[ A_2 = \frac{1}{2} \int d^{d-1} \alpha \left( 2 g^{ab} \frac{\delta \sqrt{h_0}}{\delta g^{ab}} + g_0^{ab} \frac{\delta^2 \sqrt{h_0}}{\delta g^{ab} \delta g^{cd}} \right) + \frac{1}{2} \sum_i \int d^{d-1} \alpha \sqrt{h_0} (x_1 \cdot iN_0) \left( \delta^{ij} K \right). \]  

(37)

We can make connection to a problem that has been described in the mathematics literature: consider a minimal surface, and shift the position of that surface by \( \hat{n} f \), where \( \hat{n} \) is the normal vector and \( f \) a function; what is the change in the area? In our set-up, this question corresponds to setting \( \Delta g = g^1 = g^2 = \cdots = 0 \), and writing \( x_1^a = \sum_i f_1^i N_0^a \), which gives

\[ A_2 = \frac{1}{2} \sum_i \int d^{d-1} \alpha \sqrt{h_0} f_1^i (h_0^{ab} R_{abcd}^0 / N_0^j - \delta^{ij} \hat{B} - iK_{ab}^0 K^{ab}) f_1^j. \]  

(38)

The term in brackets is the Jacobi or stability operator\(^9\).

### 3.5. Solution for \( x_1(\alpha) \)

In this subsection, we will solve the equation of motion for \( x_1 \) (35), for perturbations of \( \text{AdS}_{d+1} \), and ball-shaped boundary sub-regions. The ‘shift’ \( x_1 \) can always be decomposed in a tangential part and a perpendicular part:

\[ x_1^a = h_0^{ab} x_b^a, \quad x_1^a = (g_0^{ab} - h_0^{ab}) x_b^a = \sum_i f_1^i N_0^a. \]  

(39)

which defines the set of functions \( f_1^i = (-)^i x_1 \cdot iN \). A tangential shift corresponds to a re-parametrization of the original surface, which does not affect the surface area, and for simplicity we will consider

\[ x_1^a = 0. \]  

(40)

We will restrict the discussion to the \( \text{AdS}_{d+1} \) background geometry, for which \( R_{abcd} = -(g_0^{ab} g_0^{cd} - g_0^{ad} g_0^{bc}) \), and HRRT-surfaces that end on ball-shaped boundary sub-regions. In \( \text{AdS}_{d+1} \), HRRT-surfaces that end on ball-shaped boundary sub-regions are totally geodesic \( (K_{ab}^0 = 0) \), and have hyperbolic geometry \( (B \sim H^{d-1}) \). In this case, the equations for the \( f_1^i \) decouple, and the equation of motion (35) can be written as an inhomogeneous wave equation:

\[ 0 = (\Box_B - (d-1)) f_1^i \pm \delta_0^{ij} K \]  

(41)

\(^9\) Usually, this formula is stated for codimension-one surfaces. Some background material can be found in [7] or [8], for example.
where the sign is ± for the timelike / spacelike normal vector respectively. Equation (41) can be solved with a Green’s function for $\mathbb{R}^{d-1}$, satisfying equation (8)\(^1\):

\[
 f_i^\alpha(x(\alpha)) = \int_B d^{d-1}\beta \sqrt{h} \, G_\beta(\alpha, \beta) \, \delta_i^\alpha K(\beta) 
\]

\[
 x_i^\alpha(\alpha) = \sum_i \left( \int_B d^{d-1}\beta \sqrt{h} \, G_\beta(\alpha, \beta) \, \delta_i^\alpha K(\beta) \right) J_0(\alpha). 
\]

The terms at second order in the area of the HRRT-surface (33) and (34) can thus be written as

\[
 \left. \frac{d^2A}{d\lambda^2} \right|_{\lambda=0} = \int_B d^{d-1}\alpha \left( 2g_{\alpha\beta} \frac{\delta \sqrt{h_0}}{\delta g^\alpha_0} + g_{\alpha\beta} \delta^2 \sqrt{h_0} \right) 
\]

\[
 + \sum_i \int_B d^{d-1}\alpha \sqrt{h_0} \int_B d^{d-1}\beta \sqrt{h_0} G_\beta(\alpha, \beta) \delta_i^\alpha K(\alpha) \delta_i^\beta K(\beta). 
\]

Note: in the Hollands–Wald gauge, the term in line (45) vanishes, but it’s non-zero in a general gauge.

### 3.6. An example: the planar black hole geometry

In this subsection we will briefly apply our formulas to a specific example, which was previously studied in [17]: the perturbation from the $\text{AdS}_{2+1}$ geometry to the planar black hole geometry, for which the metric is given by:

\[
 ds^2 = \frac{1}{z^2} \left( dz^2 + \left( 1 + \frac{1}{2} \lambda z^2 \right)^2 dx^2 - \left( 1 - \frac{1}{2} \lambda z^2 \right)^2 dt^2 \right).
\]

(46)

Note that one recovers the background $\text{AdS}_{2+1}$ metric as $\lambda \to 0$. The non-trivial components of $g^1$ and $g^2$ (see equation (3)) are given by

\[
 g^1_{xx} = 1, \quad g^1_{tt} = 1, \quad g^2_{xx} = \frac{z^2}{4} \quad \text{and} \quad g^2_{tt} = -\frac{z^2}{4}. 
\]

(47)

In appendix G we explicitly compute $x_1$ (see equation (42)) and $A_2$ (see equations (44) and (45)), by using the appropriate Green’s function for the HRRT-surface. The HRRT-surfaces in $\text{AdS}_{2+1}$ are space-like geodesics, and in terms of the proper distance $s$, the Green’s function is given by the solution to the Helmholtz equation with constant $c = -1$:

\[
 G(s, \tilde{s}) = -\frac{1}{2} e^{-|s-\tilde{s}|}. 
\]

(48)

Direct computation of $x_1$, further detailed in appendix G, gives

\[
 x_i^\alpha(\tilde{s}) = \int_{-\infty}^{\infty} ds \, G_\beta(s, \tilde{s}) \delta_i^\alpha \hat{K}(s) \cdot S^\alpha(\tilde{s}) 
\]

(49)

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10 A Green’s function is generally not unique without specifying the appropriate boundary conditions; in this case we require the response function to vanish at the boundary. For a discussion of Green’s functions on hyperbolic space, see for example [23–25]
\[
\frac{1}{6} \left( \frac{z(\hat{s})}{R} (2R^2 - z(\hat{s})^2) \right) \cdot \mathbf{S}^a(\hat{s}), \quad (50)
\]

where \( \mathbf{S} \) is the space-like normal vector of the HRRT-surface (in the \( t = 0 \) plane). In appendix G we show that the term \( A_2 \) in the expansion of the area (12) is equal to \( A_2 = -\frac{1}{8} R^4 \). These results are consistent with [17].

4. Knight diffeomorphisms

In this section, we briefly review the action of diffeomorphisms on the coordinates \( x^a \) and on two-tensors. In particular, we consider the action of a series of diffeomorphisms, at different orders of \( \lambda \), on the embedding coordinates (4) and the perturbed metric (3). The action of diffeomorphisms on tensor fields is described in [27], and we will briefly review some relevant results here.

We consider a series of diffeomorphisms generated by the vector fields

\[
\lambda V^a_1, \quad \lambda^2 V^a_2, \quad \lambda^3 V^a_3, \quad \ldots
\]

(51)

In other words, we consider the diffeomorphism which takes a point a distance \( \lambda \) along the integral curve of \( V_1 \), followed by a displacement \( \lambda^2 \) along the integral curve of \( V_2 \), et cetera. Such a diffeomorphism is called a ‘knight diffeomorphism’ [28].

More generally, under a diffeomorphism generated by a vector field \( V \), moving a parameter distance \( \beta \) along its integral curves, we have the following transformation rule for a tensor field \( T \) [27]:

\[
T \rightarrow \tilde{T} = \sum_k \frac{\beta^k}{k!} \mathcal{L}_V^k T.
\]

(52)

Now considering the series of diffeomorphisms generated by the vector fields \( \lambda V_1, \lambda^2 V_2, \ldots \) (51); for the coordinate functions \( x^a \) we have:

\[
x^a \rightarrow \tilde{x}^a = x^a + \lambda V^a_1 + \lambda^2 \left( V^a_2 + \frac{1}{2} V^a_1 \partial_b V^b_1 \right) + \ldots
\]

(53)

and a symmetric two-tensor \( W \) transforms as:

\[
W_{ab} \rightarrow \tilde{W}_{ab} = W_{ab} + \lambda \mathcal{L}_{V_1} W_{ab} + \lambda^2 \left( \mathcal{L}_{V_2} W_{ab} + \frac{1}{2} \mathcal{L}_{V_1} \mathcal{L}_{V_1} W_{ab} \right) + \ldots
\]

(54)

Using the transformation rule for symmetric two-tensors (54), and sorting by orders of \( \lambda \), we find that the metric perturbations \( g^0, g^1, \ldots \) (3) transform as:

\[
g^0_{ab} \rightarrow \tilde{g}^0_{ab} = g^0_{ab}
\]

(55)

\[
g^1_{ab} \rightarrow \tilde{g}^1_{ab} = g^1_{ab} + \mathcal{L}_{V_1} g^0_{ab}
\]

(56)

\[
g^2_{ab} \rightarrow \tilde{g}^2_{ab} = g^2_{ab} + \mathcal{L}_{V_1} g^1_{ab} + \mathcal{L}_{V_1} g^0_{ab} + \frac{1}{2} \mathcal{L}_{V_1} \mathcal{L}_{V_1} g^0_{ab}.
\]

(57)

Similarly, the embedding function (4), defined by \( x_0, x_1, \ldots \), transforms as:

\[
x^a_0 \rightarrow \tilde{x}^a_0 = x^a_0
\]

(58)
\[ x_1^a \mapsto \tilde{x}_1^a = x_1^a + V_1^a \]  
(59)

\[ x_2^a \mapsto \tilde{x}_2^a = x_2^a + V_2^a + \frac{1}{2} V_1^b \partial_b V_1^a + [V_1, x_1]^a. \]  
(60)

We will use these transformation rules in section 5, to simplify the extremization problem at higher orders of \( \lambda \).

5. Higher order corrections

In this section we will show how a knight diffeomorphism (see section 4) can be used to reduce the extremization problem for \( x_2 \) to a problem of similar complexity as the extremization problem for \( x_1 \). The generalization of this procedure to higher orders is trivial, but we will briefly comment on this generalization in appendix D.

The ‘equation of motion’ for \( x_2 \) -equivalent to the requirement that the expansions \( iK \) vanish at order \( \lambda^2 \) - involves terms linear and quadratic in \( x_1 \). Now consider the diffeomorphism generated by \( V_1 = -x_1 \), where \( x_1 \) is given by (42). In the new coordinates \( \tilde{x} \), we have: \( \tilde{x}_1 = 0 \) (see equation (59))\(^{11} \); and \( iN_1 \parallel = 0 \) (see equation (22)). In these coordinates, the embedding function (4) starts –by construction– at order \( \lambda^2 \):\(^ {12} \)

\[ \tilde{x}_a(\alpha) = \tilde{x}_0^a(\alpha) + \lambda^2 \tilde{x}_2^a(\alpha) \ldots \]  
(61)

The techniques that were previously used to compute \( iN_1 \parallel \) (see sections 3.1 and appendix B) can now be used to compute \( iN_2 \parallel \). The equation of motion for \( x_2 \) is relatively simple in these coordinates:

\[
0 = \left. iK \right|_{\partial(\lambda^2)} = \delta_{\tilde{g}^1}^1 iK + \delta_{\tilde{g}^2}^1 iK + \sum_j \delta_{\tilde{x}_2^a}^1 iK + \sum_j \delta_{\tilde{x}_2^a}^2 \frac{1}{2} iK
\]  
(62)

\[
= \delta_{\tilde{g}^1}^1 iK + \delta_{\tilde{g}^2}^1 iK + \tilde{x}_2^a h_{0b}^{ab} R_{b}^0 + iN_{0a}^0 - h_{0a}^b \nabla_{\tilde{g}} (h_{0b}^{bc} \tilde{x}_2^c) + iN^0_a^0
\]  
(63)

where \( \tilde{g}^1 \) and \( \tilde{g}^2 \) are given by the transformation rules (56) and (57)\(^ {13} \). We require \( \tilde{x}_2 \) to be perpendicular to the original surface (similar to condition (40) on \( x_1 \)), \( \tilde{x}_2 \parallel = 0 \), so that we can write \( \tilde{x}_2 \parallel = \sum f^2_i N_{0}^i \). Restricting the discussion to the AdS\(_{d+1} \) background geometry and ball-shaped boundary sub-regions, the ‘equation of motion’ for \( \tilde{x}_2 \) becomes

\[ 0 = (\Box_B - (d-1)) f_i^2 = \left( \delta_{\tilde{g}}^1 iK + \delta_{\tilde{g}}^2 iK \right). \]  
(64)

We can solve (6) with a Green’s function on \( \mathbb{H}^{d-1} \), similar to (42):

\[ f_i^2 = \int_B d^{d-1} \beta \sqrt{h} G_B(\alpha, \beta) \left( \delta_{\tilde{g}}^1 iK(\beta) + \delta_{\tilde{g}}^2 iK(\beta) \right). \]  
(65)

Using the relation between \( \tilde{x}_2 \) and \( x_1 \) and \( x_2 \) (see equation (60)), we find

\(^{11} \) Equivalently, \( \delta_{\tilde{g}}^1 iK = 0 \), where \( \tilde{g}^1 \) is given by equation (56)

\(^{12} \) Note: \( x_0 \parallel \) is unchanged by the diffeomorphism generated by \( V_1 \); \( x_0 = \tilde{x}_0 \).

\(^{13} \) For completeness: \( \tilde{x}_a^1 = g_{ab} x_b^1 + \nabla_a x_b^0 \parallel , \tilde{x}_a^2 = g_{ab} + \nabla_a x_b^0 \parallel + \frac{1}{2} \nabla_b x_c^0 \parallel \nabla_c x_d^0 \parallel \). Note: at this step, we only consider the diffeomorphism generated by \( V_1 \).
\[
\begin{align*}
  x''_a &= -\frac{1}{2} \kappa^{\rho}_a \partial_\rho x'_a \\
  &\quad + \sum \left( \int d^{d-1} \beta \sqrt{h} \ G_B(\alpha, \beta) \left( \delta_{\beta_\rho}^a K(\beta) + \delta_{\beta_\rho}^a K(\beta) \right) \right) N''_0(\alpha).
\end{align*}
\]  

This procedure can be repeated for \( x_3, x_4 \) et cetera. We will briefly comment on this generalization in appendix D.

### 6. The Hollands–Wald gauge

The Hollands–Wald gauge [16] allows one to exploit the Wald formalism [29, 30], which greatly simplifies the analysis of HRRT-surfaces, and has been used to derive (from the CFT) the gravitational equations of motion up to second order in the metric perturbation [6, 17, 18, 26]. However, for some applications, it is more convenient to work in a different gauge\(^{14}\), and it can be cumbersome to find a diffeomorphism that brings the metric into the HW-gauge. In this section, we devise a method to derive a solution for this problem.

The entire section is limited to the AdS\(^d+1\) background geometry and HRRT-surfaces that end on ball-shaped boundary sub-regions. In AdS\(^d+1\), the HRRT-surface \( B \) for a ball-shaped boundary sub-region is a Killing horizon for a Killing vector \( \xi_B \), which vanishes at the Killing horizon: \( \xi_B|_B = 0 \)\(^{15}\).

The Hollands–Wald gauge is a gauge in which

(i) the coordinate position of the surface does not change: \( \Delta x(\alpha) = 0 \)

(ii) the Killing equation is still satisfied at the extremal surface \( \tilde{B} \) in the perturbed geometry:

\[
\mathcal{L}_{\xi} g_{ab}|_{\tilde{B}} = 0.
\]  

We will slightly modify the steps of section 5, in order to compute a series of diffeomorphism generating vector fields \( \lambda V_1, \lambda V_2, \ldots \) that bring the metric into the Hollands–Wald gauge. The idea is that we reverse the shift of the coordinate position of the surface with a diffeomorphism, as in section 5; but in addition, we must also enforce the Killing equation (67) at the extremal surface \( \tilde{B} \).

At leading order, the shift of the coordinate position of the surface, \( x^1 \), can be reversed by the diffeomorphism generated by\(^ {16} \)

\[
S^a_1 = -x^a_1
\]  

where \( x_1 \) is given in terms of \( g^1 \) by equation (42). Note that \( x_1 \) can be extended away from the original HRRT-surface in an arbitrary way. Under such a diffeomorphism, \( g^1_{ab} \) transforms as (56)

\[
g^1_{ab} \rightarrow \bar{g}^1_{ab} = g^1_{ab} + \mathcal{L}_S g^0_{ab} = g^1_{ab} + \nabla_a S^1_b + \nabla_b S^1_a.
\]

An additional gauge transformation is necessary to enforce that at first order in \( \lambda \), the Killing equation (67) is satisfied at \( \tilde{B} \).

For any (symmetric) tensor field \( W_{ab} \) we can construct a diffeomorphism generating vector field \( K \), which satisfies:

---

\(^{14}\) For example, the Fefferman-Graham gauge [31, 32] is often used when expressing bulk fields in terms of boundary operators (e.g. [33, 34]).  

\(^{15}\) Note: \( \xi_B \) is non-zero away from \( B \).  

\(^{16}\) \( S \) stands for ‘shift’ here.
\( \mathcal{L}_\xi W_{ab}\big|_{\tilde{B}} + \mathcal{L}_\xi \mathcal{L}_K S_{ab}\big|_{\tilde{B}} = 0 \) and \( K\big|_{\tilde{B}} = 0. \) \hfill (70)

This equation is solved by (see appendix E for a derivation):

\[
K_a = -W_{ab}\xi^b + \frac{1}{8\pi^2} \epsilon_{abc} W^{bc}\xi_c.
\hfill (71)
\]

where \( \epsilon_{abc} \) is the antisymmetric bi-normal of the surface \( \tilde{B} \).

Combining (68), (69) and (71), we find a diffeomorphism generating vector field that enforces the Hollands–Wald gauge at order one in \( \lambda \):

\[
V_1^a = S_1^a + K_1^a, \quad \text{with}
\hfill (72)
\]

\[
S_1^a = -x_1^a \quad \text{and}
\hfill (73)
\]

\[
K_1^a = -\frac{1}{2\pi} g_{ab}^1 \epsilon^b + \frac{1}{8\pi^2} \epsilon_{abc} g_{bc}^1 \xi_c, \quad g_{ab}^1 = g_{ab}^0 + \mathcal{L}_S g_{ab},
\hfill (74)
\]

where \( x_1 \) is given by (42). At higher orders in \( \lambda \), we can also use (71) to find the diffeomorphisms that bring the metric in the HW-gauge (see appendix D).

Returning to the example of section 3.6, we can compute the Hollands–Wald vector \( V_1 \) evaluated on the HRRT-surface \( \tilde{B} \), where \( K_1 \) vanishes and only \( S_1 = -x_1 \) gives a non-zero contribution. In appendix G we rewrite the result for \( V|_{\tilde{B}} \) in ‘spherical coordinates’ \( z = r \cos \theta \) and \( x = r \sin \theta \), which allows comparison with [17]:

\[
V|_{\tilde{B}} = \frac{R}{6} (\cos^2 \theta - 2) \partial_r.
\hfill (75)
\]

This is consistent with the result found in [17]\(^{17}\).

7. Discussion

7.1. Relative entropy at higher orders

The quantum relative entropy is closely related to the entanglement entropy, whose bulk dual is given by the HRRT-formula. In this section, we will briefly review the concept of relative entropy, and we will compare our expression for the change in the extremal surface area (13) and (14) with previous results [17].

The relative entropy between two states described by density matrices \( \rho \) and \( \rho_0 \) is defined as

\[
S(\rho|\rho_0) = \text{Tr}(\rho \log \rho) - \text{Tr}(\rho \log \rho_0) = \Delta(H_0) - \Delta S
\hfill (76)
\]

with \( \Delta S = S(\rho) - S(\rho_0), H_0 = -\log \rho_0 \) and \( \Delta(H_0) = \text{Tr}[\Delta\rho H_0] \).

In [17], the Wald formalism [29] was used to establish an expression for the holographic dual of the second order terms of the relative entropy of a ball-shaped boundary sub-region \( B \), by using the equations of motion:

\[
S_B(\rho|\rho_0) = \lambda^2 \left( \int_{\Sigma(B)} \omega(g^1, \mathcal{L}_\xi g^1) + \int_{\tilde{B}} d\rho(g^1, V) \right) + O(\lambda^3),
\hfill (77)
\]

\(^{17}\) See equation (63) of [17].
where \( \omega \) is the symplectic current, and \( \rho(g^1, V) \) is a (d-1)-form given in [17]. The ‘Hollands–Wald vector’ \( V \) must satisfy [17]

\[
0 = \left( g^1_A + \nabla_A V_i + \nabla_i V_A \right) |_{\tilde{B}}
\]

\[
0 = \left( \nabla_i V \nabla^i V_A + [\nabla_i, \nabla_A] V^i + \nabla^i g^1_{A \beta} - \frac{1}{2} \nabla_A g^1_{i \beta} \right) |_{\tilde{B}}
\]

where \( i \) indexes the longitudinal directions and \( A \) the transverse directions\(^{18,19}\). The second order contribution to the holographic dual of relative entropy (77) is also called the ‘canonical energy’ [16, 17].

In [6], it was shown—a by means of a CFT-computation— that the relative entropy at second order matches the holographic result (for Einstein gravity) at second order if the two central charges \( a^* \) and \( C_T \) of the CFT are equal\(^20\).

In this article, we presented a closed form expression for ‘Hollands–Wald vector’ \( V \), which solves (78) and (79): see equation (72). One finds a closed form expression for the canonical energy upon substitution of (72) into equation (77). Note: in the example of sections 3.6 and appendix G, the canonical energy is given directly by equation (G.19), as the modular Hamiltonian is of order one in \( \lambda \) [17]. Assuming the HRRT-formula, the vectors \( \lambda V_1, \lambda^2 V_2, \ldots \) that generate the knight diffeomorphism that brings the metric into the Hollands–Wald gauge can also be explicitly computed (see sections 5 and appendix D). It would be interesting to compute the (relative) entropy (in the CFT) at higher orders in the perturbation, and to match these results to geometric results. Matching these formulas at higher orders will most certainly provide conditions—for the CFT and its quantum states to be ‘holographic’.

### 7.2. Perturbations of non-AdS geometries

In appendix F we briefly discuss the application of our techniques to perturbations of more general asymptotically-AdS geometries. For the ‘shift’ \( x_i^a = \sum_i f_i^a N_0^i \), the \( f_i^a \) must solve\(^21\)

\[
0 = - \sum_{j=0,1} f_i^a R_{ab}^i N_0^a N_0^b + (-)^i f_i^a N_0^a N_0^b N_0^c N_0^d \delta_{abc} R_{dbac}
\]

\[
- h_{ab}^i \Gamma_{c}^{ab} N_c^i - g^1_{1b} K_{ab} - \Box_{kl} f_i^1 K_{ab}^i K_{ab}^j - \sum_j f_j^1 K_{ab}^j K_{ab}^i.
\]

It is not obvious that these equations \((i = 0, 1)\) can always be solved, but (80) simplifies under some assumptions. Assuming that the background metric solves the vacuum Einstein equations\(^22\), and assuming that the original extremal surface lies on a constant-time slice of a static background geometry, the equations of motion for the \( f_i^a, i = 0, 1 \) decouple. If we can solve equation (80), then the second order correction to the area functional (see equation (12)) is given by

\(^{18}\) These equations (78) and (79) correspond to equations (70) and (35) and were solved here by taking \( V \) to be perpendicular (40), by invoking a Green’s function on the HRRT-surface \( \tilde{B} \) (42).

\(^{19}\) One of these equations was also re-casted in a different form in [17].

\(^{20}\) In fact, the central charges \( a^* \) and \( C_T \) must be equal, as defined in [6]. These central charges can be related to the central charges \( a \) and \( c \) of the CFT.

\(^{21}\) HRRT-surfaces are codimension-two surfaces with a two-dimensional normal space, spanned by some set of orthonormal vector fields \( N \). Given the label \( i \), the label \( \neq \) denotes the normal direction orthogonal to \( N \).

\(^{22}\) Or by considering only the terms at leading order in \( G_{\omega} \).
\[ A_2 = \frac{1}{2} \int_{\tilde{B}} d^{d-1} \alpha \left( 2 g_{ab}^2 \frac{\delta \sqrt{h_0}}{\delta g_{ab}} + g_{ab}^1 \frac{\delta^2 \sqrt{h_0}}{\delta g_{ab}^1 \delta g_{cd}} \right) \] (82)

\[ + \frac{1}{2} \sum_{i=0,1} \int_{\tilde{B}} d^{d-1} \alpha_i (\alpha) \delta_i \delta^1 \cdot K. \] (83)

For simple cases, such as the AdS_{d+1} and AdS-Schwarzschild background geometries, equation (80) is an inhomogeneous wave equation.

### 7.3. Quantum extremal surfaces

At leading order in \( G_N \), the bulk dual of boundary entanglement entropy is given by the HRRT-formula (1); this formula receives corrections at sub-leading orders [19, 22]:

\[ S = \frac{A}{4 G_N} + S_{\text{bulk}}, \] (84)

where \( S_{\text{bulk}} \) is the bulk entanglement entropy across the surface that extremizes (84). This surface is called the ‘quantum extremal surface’ [35]. In addition to the expansion in \( \lambda \), the parameter that governs the perturbation of the state (2), one can consider an expansion \( G_N \) as well. In particular, for the quantum extremal surface, the shifts \( x_1, x_2, \ldots \) (42) and (66) will receive corrections at sub-leading orders of \( G_N \). If there is no asymmetry between the ‘inside’ and ‘outside’ regions of the extremal surface, there are no corrections [22] at sub-leading order of \( G_N \). Our results thus hold at subleading order in \( G_N \) for (coherent) states for which only the stress tensor is sourced.

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### Appendix A. Notation and conventions

#### Surfaces:

- \( B \) boundary sub-region
- \( \tilde{B} \) bulk HRRT surface for \( B \), parametrized by \( \{ \alpha \} \).

For ball-shaped boundary sub-regions and the AdS background geometry we have

\[ \tilde{B} \sim \| \|^{d-1}. \] (A.3)

The normal vectors of the bulk HRRT-surface \( \tilde{B} \):

\[ i \bar{N}_a. \] (A.4)

---

23 For example, \( S_{\text{bulk}} \) depends linearly on \( x_1 [22] \), and the equation of motion (35) receives an additional ‘source term’ at sub-leading order of \( G_N \). This additional term involves a two-point function of the bulk modular Hamiltonian and another operator defined in [22].
1 labels the normal vectors: for a codimension-k surface we have \( i = 0, \ldots, k - 1 \)
a is the normal bulk space-time index.

We consider normal vectors that remain orthonormal in some neighborhood of \( \tilde{B} \); otherwise, there are no restrictions away from \( \tilde{B} \).

- Numeric sub or superscripts denote the order in the small parameter \( \lambda \)
- \( h_{ab} \) denotes the induced metric of \( \tilde{B} \):
  \[
  h_{\mu \nu} = \frac{\partial x^a}{\partial \alpha^\mu} \frac{\partial x^b}{\partial \alpha^\nu} g_{ab}, \quad h_{ab} = g_{ab} + \sum_i (-)^i N_i^a N_i^b
  \] (A.5)

- The expansions \( iK \) are given by:
  \[
  iK_{ab} = h_{ac} \nabla^c N_b, \quad iK^{ab} = h^{ac} \nabla^c N_b
  \] (A.6)

- Indices are always raised and lowered with the background metric \( g_{ab} \)
- the covariant derivative \( \nabla_a \) always refers to the covariant derivative for the background metric \( g_{ab} \).

**Variations:** \( \delta^i \) refers to the \( i \)-th order variation of a quantity with respect to the metric, evaluated at \( g_{ab} = g^i_{ab} \). Example,

\[
\delta^1 \Gamma_{ab} = -h^i_{ab} \Gamma_{i cd} N_c^0 - g^i_{ab} \Gamma_{i cd} + \frac{1}{2} h^i_{ab} (\nabla_a g^i_{cd} + \nabla_b g^i_{cd} - \nabla_d g^i_{ab}).
\]

**Appendix B. Change of the normal vector**

The vectors tangent to the surface given by the embedding equation (4) are given by

\[
\frac{dx^a}{d\alpha^\mu} = \frac{dx^a_0}{d\alpha^\mu} + \lambda \frac{dx^a_1}{d\alpha^\mu} + \ldots
\] (B.1)

At the new position (4), the \( N (18) \) can be expanded as:

\[
\begin{align*}
  iN_a(x(\alpha)) &= iN^0_a(x(\alpha)) + \lambda iN^1_a(x(\alpha)) + \ldots \\
  &= iN^0_a(x(\alpha)) + \lambda \left( \frac{1}{2} \partial_a \Gamma_{i cd} N^i_c(x_0(\alpha)) \right) + \ldots \\
\end{align*}
\] (B.2)

**Orthogonality**

\[
\begin{align*}
  0 &= \frac{dx^a}{d\xi^0} iN_a(x(\alpha)) \\
  &= \left( \frac{dx^a_0}{d\alpha^\mu} + \lambda \frac{dx^a_1}{d\alpha^\mu} + \ldots \right) \left( iN^0_a(x_0(\alpha)) + \lambda \left( \frac{1}{2} \partial_a \Gamma_{i cd} N^i_c(x_0(\alpha)) \right) + \ldots \right) \\
  &= \frac{dx^a_0}{d\alpha^\mu} iN^0_a(x_0(\alpha)) + \lambda \left\{ \frac{dx^a_0}{d\alpha^\mu} \left( \frac{1}{2} \partial_a \Gamma_{i cd} N^i_c(x_0(\alpha)) \right) + \frac{dx^a_1}{d\alpha^\mu} iN^0_a(x_0(\alpha)) \right\} + \ldots
\end{align*}
\] (B.4)

The term at order \( O(\lambda^0) \) vanishes by requiring the \( iN_0 \) to be normal to the original surface. At order \( O(\lambda) \), the term in brackets must vanish. We replace the tangent vector \( \frac{dx^a}{d\alpha^\mu} \) by the induced metric \( h^i_{ab} \), since the term in brackets must vanish for all the tangent vectors. This yields
\[ 0 = h^{ab}_{0} \left( i_{N}^{a} + x_{1} \partial_{c} i_{N}^{c} \right) + h^{bc}_{0} \left( \partial_{c} x_{1}^{b} \right) i_{N}^{0} \]  
(B.5)

\[ = h^{ab}_{0} \left( i_{N}^{a} \right) + x_{1} \nabla_{c} i_{N}^{c} + h^{bc}_{0} \left( \nabla_{c} x_{1}^{b} \right) i_{N}^{0}. \]  
(B.6)

This equation determines the tangent component of the \( i_{N}^{a} \):

\[ i_{N}^{a}_{1} = -h^{ab}_{0} \left( x_{1} \nabla_{c} i_{N}^{c} + \left( \nabla_{b} x_{1}^{b} \right) i_{N}^{0} \right). \]  
(B.7)

For \( x_{1}^{a} = \sum_{j} f_{j}^{a} i_{N}^{0} \)

\[ i_{N}^{a}_{1} = - \sum_{j} h^{ab}_{0} f_{j}^{a} i_{N}^{0} \nabla_{c} i_{N}^{c} - h^{ab}_{0} \nabla_{b} f_{j}^{a}. \]  
(B.8)

**Normality** of \( i_{N}^{a} \) completely determines the normal component of \( i_{N}^{a} \):

\[ i_{N}^{a}_{j} = i_{N}^{a}_{j} + \lambda x_{1}^{a} \partial_{c} \left( i_{N}^{a} \nabla_{0} i_{N}^{0} \right) + \lambda x_{1}^{a} \partial_{b} \left( i_{N}^{b} \nabla_{0} i_{N}^{0} \right) + \ldots. \]  
(B.9)

\[ = 1 + \lambda \left\{ \left( i_{N}^{a}_{j} \nabla_{0} i_{N}^{0} \right) + \left( i_{N}^{b} \nabla_{0} i_{N}^{0} \right) - \left( i_{N}^{a} \nabla_{0} i_{N}^{0} \right) \right\} + \ldots. \]  
(B.10)

The quantity between brackets must vanish for \( i = j \) and for \( i \neq j \); note that the tangent component of \( i_{N}^{1} \) does not affect the normalization at order one in \( \lambda \). A solution is given by\(^{24}\):

\[ i_{N}^{1}_{a} = \frac{1}{2} \sum_{j} \left( i_{N}^{a}_{j} \nabla_{0} i_{N}^{0} \right) i_{N}^{0}. \]  
(B.12)

Now \( i_{N}^{1} \) is fully determined in terms of \( x_{1} \) and \( g^{1} \):

\[ i_{N}^{1}_{a} = i_{N}^{1}_{a} + i_{N}^{1}_{b} \]  
(B.13)

\[ = \frac{1}{2} \sum_{j} \left( i_{N}^{a}_{j} \nabla_{0} i_{N}^{0} \right) i_{N}^{0} - h^{ab}_{0} \left( x_{1} \nabla_{c} i_{N}^{c} + \left( \nabla_{b} x_{1}^{b} \right) i_{N}^{0} \right). \]  
(B.14)

**Appendix C. Change of the extrinsic curvature**

The expansions of the surface \( x_{0} \) in the background metric \( g^{0} \) vanish by assumption. In the perturbed geometry \( g_{ab} (\lambda) \) (3) the surface with embedding function \( x_{0}^{a} \) (4) has a non-vanishing extrinsic curvatures, which can be expanded in orders of \( \lambda \). Taking into account the change in the metric, the change in position and the change of the normal vectors, we have at first order:

\[ \frac{d}{d\lambda} \bigg|_{\lambda=0} = \delta_{i}^{1} i_{K} + \delta_{i}^{1} i_{K} + \sum_{j} \delta_{i}^{1} i_{K} \]  
(C.1)

with

\[ \delta_{i}^{1} i_{K} = x_{1} \nabla_{c} i_{K} \]  
(C.2)

\(^{24}\)This is a solution, as one can add an arbitrary rotation in the normal space.
\[ x_i h_0^{ab} \nabla_a \nabla_b N_0 + x_i (\nabla_c N_0^a) N_0^{ab} = 0 + x_i (\nabla_c N_0^a) N_0^{ab} \quad (C.3) \]

\[ \delta^1_{g} iK = -h_0^{ab} \delta^1_{g} \Gamma^c_{ab} N_0^c - g_1 h_0^{ab} K_{ab} \]
\[ = (N_0^a h_0^{bc} (\nabla_a g_{bc} + \nabla_b g_{ac} - \nabla_c g_{ab}) - g_1 h_0^{ab} K_{ab} \quad (C.4) \]

and

\[ \sum_j \delta^1_{h_0^j} iK = \mp x_i (\nabla_c N_0^a) N_0^{ab} \nabla_a iN_0^b - h_0^{ab} \nabla_a (h_0^{bc} (\nabla_c x_i^j N_0^d) ) \quad (C.5) \]

Combining all these terms, we find

\[ \frac{dK}{d\lambda} \bigg|_{\lambda=0} = \delta^1_{g} iK + x_i h_0^{ab} R_{dabc} N_0^d \]
\[ = -h_0^{ab} \nabla_a (h_0^{bc} (\nabla_c x_i^j N_0^d) ) - iK_{ab} h_0^{cd} \nabla_c x_i^j. \quad (C.6) \]

If we take \( x_i = \sum f_j N_0^j \), we find:

\[ \frac{dK}{d\lambda} \bigg|_{\lambda=0} = \delta^1_{g} iK + \sum_j f_j (N_0^a h_0^{bc} R_{dabc} N_0^d - \Box g_1 f_j - \sum_j f_j (K_{ab}^0 iK_{ab}^0). \quad (C.9) \]

**Appendix D. An iterative procedure for the computation of \( V_n \)**

In this appendix we comment on the generalization of section 5 to arbitrary order.

Suppose the vectors \( V_1, \ldots, V_{n-1} \) are already constructed. The action of the knight diffeomorphism generated by these vector fields on the embedding function (4) and the metric (3) involves a number of combinatorial factors, as described below. We will now describe how to construct \( V_n \).

(i) Define \( g_{ab}^m \), with \( 0 \leq m \leq n \) as the sum of all combinations

\[ L_{V_1} \ldots L_{V_n} g_{ab}^m, \quad (D.1) \]

where \( q_1 \geq q_2 \geq \cdots \geq q_r, q_1 + \cdots + q_r + l = m \), weighted by appropriate combinatorial pre-factors (following equation (52)); if \( N_j \) is the number of times that \( V_j \) appears in (D.1), then one should multiply (D.1) by \( (N_1! \ldots N_{n-1}!)^{-1} \).

(ii) Define \( \Delta_n^1 K \) as the sum of all combinations of the variations of the expansions \( iK \),

\[ \delta^1_{g_1} \ldots \delta^1_{g_n} iK, \quad (D.2) \]

where \( \sum_{i=1}^n q_i = n \), weighted by appropriate combinatorial factors: if \( \tilde{N}_j \) is the number of times that \( \tilde{g}_j \) appears in (D.2), then one should multiply by \( \tilde{N}_1! \ldots \tilde{N}_n! \).

(iii) Define \( \Delta x_{n}^m \), with \( 0 \leq m \leq n \), as the sum of all combinations

\[ L_{V_1} \ldots L_{V_n} x_i^j, \quad (D.3) \]
where \( q_1 \geq q_2 \geq \cdots \geq q_r \), \( q_1 + \cdots + q_r + l = m \), weighted by appropriate combinatorial pre-factors: if \( N_j \) is the number of times that \( V_j \) appears in (D.1), then one should multiply (D.1) by \( (N_i! \cdots N_{n-l}!)^{-1} \).

(iv) The diffeomorphism generating vector field \( V_n \) is now given by

\[
V_n^a = -\Delta X_n^a - x_n^a(\alpha)
= -\Delta X_n^a - \sum_i \left( \int_B d^{d-1} \beta \sqrt{h} G_B(\alpha, \beta) \Delta_i^a K(\beta) \right) N_0^i(\alpha). \quad (D.4)
\]

For the Hollands–Wald gauge, some steps need to be modified. Let’s assume that we have the vectors \( V_1, \ldots, V_{n-1} \) that bring the metric into the HW-gauge up to order \( \lambda^{n-1} \). Steps 1–3 are unchanged, but step 4 is replaced by

(v) The vector \( V_n \) is given by \( V_n = S_n^a + K_n^a \), with:

\[
S_n^a = -\Delta X_n^a - x_n^a(\alpha) \quad (D.5)
\]

\[
= -\Delta X_n^a - \sum_i \left( \int_B d^{d-1} \beta \sqrt{h} G_B(\alpha, \beta) \Delta_i^a K(\beta) \right) N_0^i(\alpha) \quad (D.6)
\]

and

\[
K_n^a = \frac{1}{2\pi} \tilde{s}_n^{ab} \xi^b + \frac{1}{8\pi^2} \epsilon_{abc} \tilde{s}_n^c \xi^b = \tilde{s}_n^a + \mathcal{L}_{\xi} \tilde{s}_n^0. \quad (E.3)
\]

This can be checked by using:

\[
\nabla_a \xi_b = 2\pi \epsilon_{ab} \text{on } \tilde{B}, \quad \text{where } \epsilon_{ab} \text{ is the antisymmetric binormal} \quad (E.4)
\]

\[
\nabla_a \nabla_b \xi_c = R^0_{cabc} \xi^d \quad \text{using that } \xi \text{ is Killing w.r.t. } g^0 \quad (E.5)
\]

\[
\nabla_a \epsilon_{bc} = 0 \quad \text{using (E.5) and using that } \xi_{\tilde{B}} = 0. \quad (E.6)
\]

Note that the vector (E.3) vanishes at \( \tilde{B} \). This solution can be used to establish the gauge in which \( \mathcal{L}_{\xi} g(\lambda) = 0 \) at all orders in \( \lambda \).

Appendix E. Solving for the Hollands–Wald gauge

The problem is to find a vector field, that generates a diffeomorphism, such that one obtains a gauge in which the perturbed metric (3) satisfies the Killing equation at \( \tilde{B} \):

\[
\mathcal{L}_\xi g = 0. \quad (E.1)
\]

In other words, for any (symmetric) tensor field \( W_{ab} \) we can need to construct a diffeomorphism generating vector field \( K \) such that

\[
\mathcal{L}_\xi W_{ab}|_{\tilde{B}} + \mathcal{L}_\xi \mathcal{L}_K g^0_{ab}|_{\tilde{B}} = \mathcal{L}_\xi W_{ab}|_{\tilde{B}} + \mathcal{L}_\xi (\nabla_a K_b + \nabla_b K_a)|_{\tilde{B}} = 0. \quad (E.2)
\]

This equation is solved by

\[
K_a = -\frac{1}{2\pi} W_{ab} \epsilon^b + \frac{1}{8\pi^2} \epsilon_{abc} W^{bc} \xi^c. \quad (E.3)
\]

This can be checked by using:

\[
\nabla_a \xi_b = 2\pi \epsilon_{ab} \text{on } \tilde{B}, \quad \text{where } \epsilon_{ab} \text{ is the antisymmetric binormal} \quad (E.4)
\]

\[
\nabla_a \nabla_b \xi_c = R^0_{cabc} \xi^d \quad \text{using that } \xi \text{ is Killing w.r.t. } g^0 \quad (E.5)
\]

\[
\nabla_a \epsilon_{bc} = 0 \quad \text{using (E.5) and using that } \xi_{\tilde{B}} = 0. \quad (E.6)
\]
E.1. Proof of solution (E.3) to equation (E.2)

\[ \mathcal{L}_\xi W_{ab} |_B + \mathcal{L}_\xi \mathcal{L}_K \mathcal{K}_{ab} |_B = \mathcal{L}_\xi W_{ab} |_B + \mathcal{L}_\xi (\nabla_a K_b + \nabla_b K_a) |_B \quad \text{(E.7)} \]

\[ = W_{ab} \nabla_c \xi^c + W_{bc} \nabla_d \xi^d + (\nabla_a K_b) \nabla_c \xi^c + (\nabla_c K_a) \nabla_b \xi^b \]

\[ + (\nabla_b K_a) \nabla_d \xi^d + (\nabla_c K_a) \nabla_b \xi^b \]

\[ = 2\pi W_{ab} \xi^b + 2\pi W_{bc} \xi^d \]

\[ + 2\pi (\nabla_a K_b) \nabla_a \xi^b + (\nabla_c K_a) \nabla_b \xi^a + (\nabla_b K_a) \nabla_c \xi^a \]

(\text{E.8})

where still everything is evaluated at \( \tilde{B} \). Now plug in solution (E.3), using (E.6) and using that on \( \tilde{B} \):

\[ \nabla_a (W_{bc} \xi^c) = 2\pi W_{bc} \xi^c \quad \text{(E.9)} \]

\[ \nabla_a (\epsilon_{bc} W^{cd} \xi_d) = 2\pi \epsilon_{bc} W^{cd} \epsilon_{ad} \quad \text{(E.10)} \]

we find:

\[ \mathcal{L}_\xi W_{ab} |_B + \mathcal{L}_\xi \mathcal{L}_K \mathcal{K}_{ab} |_B = 2\pi W_{bc} \xi^c + 2\pi W_{bc} \xi^d \]

\[ - 2\pi (W_{cd} \epsilon_{bc} \epsilon_{ad} + W_{da} \epsilon_{bc} \epsilon_{bc} + W_{ab} \epsilon_{bc} \epsilon_{bc}) \]

\[ + \pi (\epsilon_{bc} W^{de} \epsilon_{ac} \epsilon_{cd} + \epsilon_{bc} W^{de} \epsilon_{bc} \epsilon_{cd} + \epsilon_{bc} W^{de} \epsilon_{bc} \epsilon_{cd}) \]

\[ = 2\pi W_{bc} \xi^c + 2\pi W_{bc} \xi^d - 2\pi (W_{cd} \epsilon_{bc} \epsilon_{ad} + W_{da} \epsilon_{bc} \epsilon_{bc} + W_{ab} \epsilon_{bc} \epsilon_{bc}) \]

\[ + \pi (\epsilon_{bc} W^{de} \epsilon_{ac} \epsilon_{cd} + \epsilon_{bc} W^{de} \epsilon_{bc} \epsilon_{cd} + \epsilon_{bc} W^{de} \epsilon_{bc} \epsilon_{cd}) \]

\[ = 0 \quad \text{on } \tilde{B}. \quad \text{(E.11)} \]

Appendix F. Non-AdS backgrounds

In section 3 we established that the extremization of the area functional yields the same equation of motion for \( x_i^1 \) as the condition that the expansions vanish at first order in \( \lambda \):

\[ \frac{dK}{d\lambda} \bigg|_{\lambda=0} = \delta_i^1 K + x_i^1 h_{ab0} R_{dab} N_0^d - h_{ab} (\nabla_x x^1) |^0_0 \]

\[ - \nabla_b x_i^1. \]

Let us proceed by choosing \( x_1 \) to be transverse, \( x_{1\parallel} = 0 \), and by expanding \( x_1 \) on a basis of normal vectors \( \{N_0^a, i = 0, 1\} \): \( x_i^1 = \sum_{j=0,1} f_{ij} N_0^j \). The equation of motion for \( x_i^1 \), which holds for any boundary region and any background solution \( g^0 \), can now be simplified by using the symmetries of the Riemann tensor\(^{25}\):

\[ 0 = - \sum_{j=0,1} f_{ij} R_{aj} N_0^j + (\pm) f_{ij} \nabla_a (h_{ij} K_{ab} |_{K_0^a} \nabla_a x^1) + \delta_i^1 K \]

\[ - \square f_{ij} - \sum_{j} f_{ij} K_{ab} |_{K_0^a}. \quad \text{(F.1)} \]

If we are willing to assume that the background metric \( g^0 \) is a solution of the vacuum Einstein equations, and if the original extremal surface lies on some constant-time slice of a static geometry, then the equations for the \( f_{ij}^1, i = 0, 1 \) decouple, and we have\(^{26}\)

\(^{25}\) Using that we have a codimension-two surface

\(^{26}\) Here we also use that the \( N_0 \) are perpendicular everywhere on \( \tilde{B} \).
\[ 0 = -f_0 R - \frac{\Lambda}{2} - f_0^1 N_0^0 \Lambda_0^0 N_0^0 R_{dbac} + \delta_{g_0}^1 K - \Box f_0^1, \quad \text{(F.3)} \]
\[ 0 = +f_1 R - \frac{\Lambda}{2} + f_1^1 N_0^1 \Lambda_0^0 N_0^0 R_{dbac} + \delta_{g_0}^1 K + \Box f_1^1 - f_1^1 K ab K_{ab}, \quad \text{(F.4)} \]

Equation (F.3) holds for any boundary sub-region on the constant-time slice.

**Appendix G. An example: from \( \text{AdS}_{2+1} \) to the planar black hole geometry**

In this appendix we will put our formalism to work in an explicit example: the perturbation from \( \text{AdS}_{2+1} \) to the planar black hole geometry. This example was also considered in [17], and our method reproduces their results.

In Poincaré coordinates, the \( \text{AdS}_{2+1} \) metric \( g^0 \) is given by:

\[ ds^2 = \frac{1}{z^2} \left( dz^2 + dx^2 - dt^2 \right). \quad \text{(G.1)} \]

The non-zero Christoffel symbols are given by:

\[ \Gamma^{\mu}_{\nu z} = -\frac{1}{z} \delta^\mu_\nu, \quad \Gamma^{\nu}_{\mu z} = \frac{1}{z} \eta_{\mu\nu}, \quad \Gamma^{z}_{z z} = -\frac{1}{z}. \quad \text{(G.2)} \]

In 'spherical coordinates' \( x = r \sin \theta \) and \( z = r \cos \theta \) this metric can be written as

\[ ds^2 = \frac{1}{r^2 \cos^2 \theta} \left( dr^2 + r^2 d\theta^2 - dt^2 \right). \quad \text{(G.3)} \]

The 'spherical coordinates' are used in [17], as they simplify certain steps in their computations.

For a ball-shaped boundary sub-region, one can choose the coordinates such that the sub-region corresponds with the boundary interval \(-R \leq x \leq R\) and \( t = 0 \). Next, following [17], we consider a perturbation towards the planar black hole metric:

\[ ds^2 = \frac{1}{z^2} \left( dz^2 + \left( 1 + \frac{1}{2} \lambda z^2 \right)^2 dx^2 - \left( 1 - \frac{1}{2} \lambda z^2 \right)^2 dt^2 \right). \quad \text{(G.4)} \]

From this metric we can determine the non-zero components of \( g^1 \) and \( g^2 \) (3):

\[ g^1_{xx} = 1, \quad g^1_{tt} = 1 \]

\[ g^2_{xx} = -\frac{z^2}{4}, \quad g^2_{tt} = -\frac{z^2}{4}. \quad \text{(G.6)} \]

The HRRT-surface \( \tilde{B} \) is given by \( R^2 = z^2 + x^2 \) and \( t = 0 \), and a set of normal vectors is given by:

\[ T = \begin{bmatrix} T^r \\ T^x \\ T^z \end{bmatrix} = z \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad S = \begin{bmatrix} S^r \\ S^x \\ S^z \end{bmatrix} = \frac{z}{\sqrt{z^2 + x^2}} \begin{bmatrix} 0 \\ x \\ z \end{bmatrix}. \quad \text{(G.7)} \]

We proceed with the computation of \( x_1 \) (see equation (42)) and \( A_2 \) (see equations (44) and (45)). First we compute the first order metric correction \( \delta^1 K \) to the expansions \( ^1 K, i = T, S \) (see equation (30)). One can check that for \( g^1 \) with constant components,
\[
\delta_1^1 e^{1c} = -g_1^{cd} \delta_0^d e^{0b}. \tag{G.8}
\]

Using equations (G.1), (G.2), (G.5)–(G.8) and (30), we find

\[
\delta_1^1 \Gamma = 0, \quad \delta_1^1 \tilde{\Gamma} = -2z^3 R^2 x^2. \tag{G.9}
\]

Let \( s \) be the affine parameter (proper distance) along \( \tilde{B} \) defined by \( x = R \tanh s \) and \( z = R \sech s \), such that \( -\infty < s < \infty \). The Green’s function on the geodesic \( \tilde{B} \), satisfying
\[
\left( \frac{d^2}{ds^2} - 1 \right) G(s, \tilde{s}) = \delta(s - \tilde{s}), \quad \lim_{s, \tilde{s} \to \pm \infty} G(s, \tilde{s}) = 0, \tag{G.10}
\]
is given by

\[
G(s, \tilde{s}) = -\frac{1}{2} e^{-|s-\tilde{s}|}. \tag{G.11}
\]

Now we can compute \( x_1 \) (see equation (42))

\[
x_1^i(\tilde{s}) = \int_{-\infty}^{\infty} ds \ G_\tilde{B}(s, \tilde{s}) \ \delta_1^1 \tilde{\Gamma}(s) \cdot S^i(\tilde{s}). \tag{G.12}
\]

Evaluation of this integral, using equations (G.9) and (G.11) yields:

\[
x_1^i(\tilde{s}) = \left( \frac{R^2}{6} \cosh(2\tilde{s}) \sech^3 \tilde{s} \right) \cdot S^i(\tilde{s}), \tag{G.13}
\]
which can also be expressed in Poincaré coordinates as

\[
x_1^i(z) = \frac{1}{6} \left( \frac{z}{R} (2R^2 - z^2) \right) \cdot S^i(z). \tag{G.14}
\]

Similarly, we can compute the ‘shift dependent’ term in \( A_2 \) (see equation (45)):

\[
\frac{1}{2} \int ds \ G(s, \tilde{s}) \ \delta_1^1 \tilde{\Gamma}(s) \cdot \delta_1^1 \tilde{\Gamma}(\tilde{s}) = -\frac{4}{63} R^4. \tag{G.15}
\]

We proceed by computing the ‘shift-independent’ contribution to \( A_2 \) (see equation (44)), so that we can compare our result with the ‘brute force’ computation of [17, 36]. The ‘shift-independent’ term is simply the pull-back of the perturbed metric onto the original HRRT-surface:

\[
\int ds \sqrt{\frac{dx_1^0}{ds} \frac{dx_1^0}{ds} \left( g_0^{ab} + \lambda g_1^{ab} + \lambda^2 g_2^{ab} + \ldots \right)}. \tag{G.16}
\]

Expanding the square-root, and keeping only terms at order \( \lambda^2 \), we get

\[
\lambda^2 \int ds \left( \frac{1}{2} \delta_0^{ab} \frac{dx_0^a}{ds} \frac{dx_0^b}{ds} - \frac{1}{8} \left( \delta_0^{ab} \frac{dx_0^a}{ds} \frac{dx_0^b}{ds} \right)^2 \right). \tag{G.17}
\]

Using the embedding equation of the HRRT-surface, and the expressions for \( g_1 \) and \( g_2 \) (see equations (G.5) and (G.6)) we perform the integration: the first term in (G.17) is equal to \(-\frac{1}{15} R^4\) and the second term equates to \(+\frac{2}{45} R^4\). Combining these result with equation (G.15) we find

\[\text{27 Equation (G.11) is a special case of the Helmholtz equation, of which the solution is well-known.}\]
\[ A_2 = \left( -\frac{4}{63} - \frac{4}{35} + \frac{2}{15} \right) R^4 = -\frac{2}{45} R^4, \]  
(G.18)

or equivalently

\[ S_2 = \frac{A_2}{4G_N} = -\frac{R^4}{90G_N}, \]  
(G.19)

which is consistent with [17]^{28}.

Finally, we discuss the vector field that generates the diffeomorphism that brings the metric in the Hollands–Wald gauge (see equations (72)–(74)), and compare our results with [17] once more.

The vector field \( V_1 \) evaluated at \( \tilde{B} \) is completely determined by \( S_1 = -x_1 \) (see equation (73)), as the additional contribution \( K \) (see equation (74)) only affects \( V_1 \)’s derivatives; \( K \) vanishes on \( \tilde{B} \). Rewriting \( x_1 \) in ‘spherical coordinates’ (see equation (G.3)), we find

\[ V_{|\tilde{B}} = S_{|\tilde{B}} = -x_1(\theta) \]  
(G.20)

\[ = -\frac{R^2}{6} \left( \cos \theta (2 - \cos^2 \theta) \right) \cdot S(\theta) \]  
(G.21)

\[ = \frac{R}{6} \left( \cos^2 \theta - 2 \right) \partial_r \]  
(G.22)

which is consistent with [17]^{29}. The derivatives of \( V \) are determined by equation (74).

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^{28} See page 20, paragraph ‘Comparison with relative entropy’ of [17].  
^{29} See equation (63) of [17].
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