Emergent Electroweak Gravity

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We show that any massive cosmological relic particle with small self-interactions is a super-fluid today, due to the broadening of its wave packet, and lack of any elastic scattering. The WIMP dark matter picture is only consistent its mass $M \gg M_{pl}$ in order to maintain classicality. The dynamics of a super-fluid are given by the excitation spectrum of bound state quasi-particles, rather than the center of mass motion of constituent particles. If this relic is a fermion with a repulsive interaction mediated by a heavy boson, such as neutrinos interacting via the $Z^0$, the condensate has the same quantum numbers as the vierbein of General Relativity. Because there exists an enhanced global symmetry $SO(3,1)_{\text{space}} \times SO(3,1)_{\text{spin}}$ among the fermion’s self-interactions broken only by it’s kinetic term, the long wavelength fluctuation around this condensate is a Goldstone graviton. A gravitational theory exists in the low energy limit of the Standard Model’s Electroweak sector below the weak scale, with a strength that is parametrically similar to $G_N$.

INTRODUCTION

In the early universe, relics including photons, neutrinos and dark matter evolve out of thermal equilibrium as their interaction strength becomes small at low temperature in a process known as “freeze-out”. This calculation is essentially classical, assuming particles are point-like and using the Boltzmann equation [1, 2].

After freeze-out the number density of particles is fixed, and the temperature just evolves with Hubble expansion. Their time evolution is given only by the free particle kinetic term. It is usually assumed that the interaction strength is so weak that it can be neglected and that particles remain localized point particles forever. The free particle Hamiltonian propagates particles and also broadens their wave packets, described by their uncertainty $\Delta x$. This is due to the fact that the localization of particles causes them to not be an eigenstate of the Hamiltonian if they are massive.

There are two limits of interest for the particle uncertainty $\Delta x$ relative to the number density $n$. The classical gas limit is $\Delta x \ll n^{-1/3}$. Elastic scattering collisions and the Boltzmann equation describe this system. The opposite limit, $\Delta x \gg n^{-1/3}$ is a quantum liquid. Because particles have wave function overlap with their neighbors, one must take into account collective effects due to contact interactions. If there exists an attractive interaction in any partial wave, then the vacuum energy can be lowered by forming bound state quasi-particles. The system will undergo a phase transition to a super-fluid described by quasi-particles.

If the system contains global symmetries that are broken when the system becomes a super-fluid, then Goldstone bosons will emerge. As these are massless, their dynamics are extremely important.

The idea of gravity emerging from spinors is not new and fairly obvious, as one can construct a spin-2 particle as the direct product of spinors [3, 4]. However no workable theory has been yet constructed. The first idea of this type is due to Bjorken [5], who attempted to formulate the photon and graviton as a composite state. The most recent attempt and the most successful is due to Hebecker and Wetterich [6, 7]. Their theory can be regarded as a reformulation of gravity in terms of spinors, but they give no dynamics for the spinors which would lead to such a theory. This line of research was largely killed by the paper of Weinberg and Witten [8], which showed that a spin-2 particle could not couple to a covariant conserved current. Two ways out of this theorem are to quantize geometry (the approach of string theory), or to abandon diffeomorphism invariance as an exact symmetry. Sakharov originally suggested that the graviton could be emergent, and in such theories, diffeomorphism invariance can only be approximate [9].

QUANTUM LIQUID TRANSITION

The quantum liquid regime for a system occurs when the position uncertainty $\Delta x$ is larger than the inter-particle spacing

$$\Delta x \gg n^{-1/3}. \quad (1)$$

In this limit the system is not classical, and the condition of scattering theory that the impact parameter $b \gg \Delta x$ cannot be satisfied (often known as the “well-localized” assumption).

Particles in the classical gas limit will eventually time-evolve into a quantum liquid in the absence of interactions. The expansion of a free particle wave packet in time is

$$\Delta x(t)^2 = \Delta x_0^2 + \Delta n^2 t^2. \quad (2)$$

This can be intuitively understood because different momentum components may move with different velocities. The wave number at $p + \Delta p$ has a velocity $(p + \Delta p)/E$ while the wave number at $p - \Delta p$ has a smaller velocity $(p - \Delta p)/E$ and these two wave numbers will separate in space as they propagate if $E > p$. 

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The condition for the time-independent super-fluid transition can be derived by neglecting the second term of Eq. 2. In the non-relativistic limit one arrives at

\[ T < \frac{\lambda^2 n^{2/3}}{3mk_B}. \]  

(3)

The cross-section does not enter into this calculation, and the uncertainty \( \Delta x_0 \) is assumed to be proportional to the thermal de Broglie wavelength, \( \Delta x_0 = 1/\Delta p = \lambda/p = \lambda/\sqrt{3mkT} \), where \( \lambda \) is an \( O(1) \) parameter reflecting how “localized” the state is. This temperature may be further suppressed by elastic collisions, which must occur frequently enough to keep particles localized to their thermal de Broglie wavelength, but not so often that they destroy the condensate.

In the relativistic case, we also use Eq. 2 however the velocity uncertainty for relativistic states is

\[ \Delta v = \frac{\Delta p}{E} (1 - v^2) \]  

(4)

where \( v = p/E \). This correctly reflects the relativistic limit, \( v \rightarrow c \); massless wave packets do not broaden as each wave number propagates with the same velocity, \( v = c \).

The relevant time scale for wave packet broadening is the mean time between collisions \( \tau = 1/\sigma v \) in terms of the cross section \( \sigma \) since the uncertainty of a wave packet \( \Delta x_0 \) is set by the 3-momentum of an elastic scattering collision. The condition for a quantum liquid is then

\[ \frac{1}{p^2} + \frac{(1 - v^2)^2}{\sigma^2 n^2} > \frac{1}{\lambda^2 n^{2/3}}. \]  

(5)

In the limit that the first term on the left side is small compared to the second (e.g. for decoupled relics), the quantum liquid condition is:

\[ \sigma < \frac{\lambda(1 - v^2)}{n^{2/3}}. \]  

(6)

Thus, for any decoupled cosmological relic, it becomes a quantum liquid when its cross section is approximately less than the square of the inter-particle separation. This occurs faster for non-relativistic relics \( v \rightarrow 0 \) than relativistic ones \( v \rightarrow 1 \), and can be delayed if collisions are “well-localized” relative to the inter-particle separation (\( \lambda \rightarrow 0 \)).

This condition (Eq. 6) is extremely well satisfied for massive neutrinos and Weakly Interacting Massive Particle (WIMP) dark matter, so that today, WIMPs and at least two neutrino mass eigenstates are definitely quantum liquids.

An important implication of this result is that non-relativistic relics such as WIMP dark matter must be treated as quantum liquids. The phenomena currently attributed to dark matter can only be achieved by a classical gas of particles which must satisfy \( \Delta x(t) \ll n(t)^{-1/3} \) and the uncertainty \( \Delta x \) to the thermal de Broglie wavelength, \( \Delta x = \sqrt{3mkT} \), occurs faster for non-relativistic relics when its cross section is approximately \( \sigma_n \cdot n \) as compared to the second (e.g. for decoupled relics), the cross section \( \sigma < \lambda/\sqrt{3mkT} \).

In the relativistic case, we also use Eq. 2, however the condition \( \Delta x_0 \) is set by the 3-momentum of an elastic scattering collision. The relevant correction comes from an exchange (box) diagram and its contribution to the BCS potential \( V(x) \) in the \( \ell \)th partial wave is

\[ \delta V_\ell = (-1)^{\ell+1} \frac{n_{PF}}{4\pi^2} \frac{|V(\cos \theta = -1)|^2}{\ell^4} \]  

(7)

where \( n_{PF} = (3\pi n)^{1/3} \) and \( V(\cos \theta) \) is the tree-level potential evaluated on the Fermi surface. This is attractive for odd \( \ell \). The relevant infrared divergence occurs for \( \cos \theta = -1 \) and corresponds to an exchange of the propagating neutrino with a background neutrino. The divergence occurs at \( 2PF \) because it occurs in the internal loops, which contain two fermion propagators, both of which must lie on the Fermi surface.

This potential is parametrically \( O(p_F^2 G_F^2) \). Therefore this condensation is a much more important effect than
scattering, which is associated with the mean free path and is\(\mathcal{O}(p_f G_F^2)\). Note that \(\delta V_4\) is also parametrically the same order as Newton’s constant \(G_N\).

Therefore, an attractive self-interaction always exists in a neutrino or fermionic WIMP fluid, regardless of the sign of the fundamental interaction. If the mass is sufficiently small so that the conditions of the previous section are also satisfied, then such a cosmological relic is a super-fluid today. The two heavier neutrino species and WIMP dark matter are super-fluids today. Lighter species such the lightest neutrino (if sufficiently light) would require an early-universe analysis to determine if the conditions of the previous section can be satisfied.

### CONDENSATE QUANTUM NUMBERS

A condensate will break Lorentz invariance, but if the underlying theory is invariant, we can classify the condensates by their Lorentz representation. A Weyl fermion condenses as \((\frac{1}{2}, 0) \otimes (\frac{1}{2}, 0) = (0, 0) \oplus (1, 0)\) according to its representation under the spin Lorentz group. A p-wave condensate must contain a derivative, giving

\[
A_\mu(x, y) = \frac{i}{2} \partial_\mu \chi \xi - \chi \partial_\mu \xi ;
\]

\[
E_\mu^\alpha(x, y) = \frac{i}{2} (\partial_\mu \chi \tilde{\sigma}^\alpha \xi - \chi \tilde{\sigma}^\alpha \partial_\mu \xi),
\]

where \(\partial_\mu\) represents the deviation in momentum from the Fermi surface, \(p_0 = 0\), \(|\vec{p}| = 2p_F\), and we abbreviate \(\chi = \chi(x)\) and \(\xi = \chi(y)\). In condensed matter nomenclature, these excitations are “zero-sound”.

The four-point operator for these two condensates is the same since they are related by a Fierz transformation, therefore we may write it as

\[
\frac{g_2^2 m_F}{4 \pi^2 M_Z^4} \int_{xy} \left[ (1 - \eta_\nu) E_\mu^\alpha E_\nu^\beta + \eta_\nu A_\mu A_\nu \right],
\]

where

\[
\eta_\nu = \frac{n_\nu - \bar{n}_\nu}{n_\nu + \bar{n}_\nu}
\]

is the asymmetry between neutrinos and anti-neutrinos. After the phase transition (Eq 8) has occurred, the original Fermi gas is described by momentum distribution functions for \(A_\mu\) and \(E_\mu^\alpha\), rather than original one for free fermions.

The condensate \(E_\mu^\alpha\) contains both particles and antiparticles, while \(A_\mu\) contains only particles (or antiparticles). Therefore, \(A_\mu\) only condenses among the unpaired particles that don’t have an antiparticle partner. The Cosmic Neutrino Background (CNB) is expected to contain very nearly equal numbers of neutrinos and anti-neutrinos. The asymmetry \(\eta_\nu\) is proportional to the baryon to photon ratio, \(\eta_\nu \sim 6 \times 10^{-10}\). Therefore \(E_\mu^\alpha\) is the dominant condensate and the dynamics of \(A_\mu\) are sub-leading so we will neglect them. A right-handed neutrino state (if they are Dirac) has interactions that are much weaker than the left-handed state, and can be ignored. Likewise, repulsive Majorana dark matter such as a bino is usually not assumed to have any matter/antimatter asymmetry and again can be treated as a single Weyl spinor super-fluid which condenses into \(E_\mu^\alpha\).

### LORENTZ BREAKING

The condensation of \(A_\mu\) and \(E_\mu^\alpha\) breaks Poincaré invariance, since both fields have Lorentz indices, and the neutrinos should have a spatially varying density distribution. This symmetry breaking is dynamical and spontaneous, due to the condensation of a physical background; the SM is Poincaré and Lorentz invariant. As a consequence of the symmetry breaking, both have corresponding Goldstone bosons, which are long wavelength fluctuations about the expectation values for \(A_\mu\) and \(E_\mu^\alpha\).

Neutrino self-interactions are mediated by the \(Z^0\) boson. In the Feynman gauge we may write the tree level effective 4-point operator as

\[
- \frac{g_2^2}{2M_Z} \int_{xy} \{ \chi^\dagger \tilde{\sigma}^\alpha \chi \xi^\dagger \tilde{\sigma}^\alpha \xi \}.
\]

This interaction has the enhanced symmetry \(SO(3, 1) \times SO(3, 1)\). The only term that breaks this enhanced symmetry is the fermion’s kinetic term, which ties together a derivative and a gamma or sigma matrix of the spin Lorentz group:

\[
i \int_x \chi^\dagger \tilde{\sigma}^\alpha \partial_\mu \chi = \int_{xy} \delta_\mu^\alpha \delta^4(x - y).
\]

However this term is a tadpole for the condensate \(E_\mu^\alpha\). As such, when \(E_\mu^\alpha\) condenses, the field must be shifted \(E_\mu^\alpha \rightarrow \hat{E}_\mu^\alpha + \delta_\mu^\alpha \delta^4(x - y)\) to remove this tadpole, and \(\hat{E}_\mu^\alpha\) is the order parameter of the \(SO(3, 1) \times SO(3, 1)\) symmetry breaking. In the limit that \(E_\mu^\alpha \rightarrow 0\), the effective action has this enhanced symmetry (and the fermion has no kinetic energy).

A free fermion \(\psi(x)\) transforms with two Lorentz symmetries. The first is defined on the coordinates of spacetime, with the generators

\[
L_{\mu\nu} = i (x_\mu \partial_\nu - x_\nu \partial_\mu).
\]

Under this symmetry \(\psi\) transforms as a scalar. The second Lorentz symmetry is defined with the generators

\[
S_{ab} = i \frac{1}{2} (\gamma_a \gamma_b - \gamma_b \gamma_a),
\]

under which \(\psi\) transforms in the 1/2 (spinor) representation. Normally we consider these to be two different representations of the same \(SO(3, 1)\) Lorentz symmetry. The SM Lagrangian is not symmetric under both groups
separately. We write Greek indices for the space-time Lorentz group, and Roman indices for the spinor Lorentz group to indicate the difference. Since both groups contain the Minkowski metric $\eta_{\mu\nu}$ and $\eta_{ab}$, we will use this to raise and lower indices. We can define the mixed generators

$$M_{\mu\nu} = L_{\mu\nu} + S_{ab} e_a^\mu e_b^\nu, \quad N_{\mu\nu} = L_{\mu\nu} - S_{ab} e_a^\mu e_b^\nu \quad (16)$$

where $e_a^\mu = \langle \hat{E}_a^\mu \rangle \simeq \delta_a^\mu$. The new operator $N_{\mu\nu}$ is the broken generator, and corresponds for a massless fermion to local violations of being in a helicity eigenstate. A plane wave could be a helicity eigenstate, but a localized state is not an energy or momentum eigenstate, and therefore is also cannot be a helicity eigenstate unless it is completely delocalized. Thus $e_a^\mu$ is the order parameter of the $SO(3,1) \times SO(3,1) \rightarrow SO(3,1)$ symmetry breaking.

By Goldstone’s theorem, a vacuum expectation value for $\hat{E}_a^\mu$ not only breaks this symmetry but also generates Goldstone bosons from the broken symmetry generators. Here care must be taken because the number of Goldstones is not the same as the number of broken generators, because the broken symmetry is a space-time symmetry. 4, 13, 16.

The Goldstones carry a representation of the unbroken group $M_{\mu\nu}$. The field $\hat{E}_a^\mu$ however carries an index of both the original groups. The propagating Goldstone is

$$g_{\mu\nu} = \hat{E}_a^\mu E_a^\nu \eta_{ab} \quad (17)$$

which we identify as spin-2 graviton under $M_{\mu\nu}$. This should be familiar from the Palatini formalism for quantizing gravity, if we identify $\hat{E}_a^\mu$ as the vierbein (tetrad).

The gravitational theory arising here does not conflict with the Weinberg-Witten Theorem because of the presence of a physical background, and consequently this emergent gravitational theory isn’t diffeomorphism invariant. There are many ways to see this, but in particular, the Lorentz symmetry is not exact in the gravitational theory, spatial variations of $\rho_F$ lead to a spatially varying interaction strength (Eq.10), and the emergent vierbein (Eq.9) is nonlocal.

From here one can almost directly follow the program of “Spinor Gravity” 6, 7, with the exception that due to the Lorentz symmetry breaking, we have the metric $\eta_{\mu\nu}$ with which to tie up spacetime indices, which gives rise to a spin connection which was absent in “Spinor Gravity”. The existence of $\eta_{\mu\nu}$ implies more invariants as well.

CONCLUSIONS

We have shown that massive cosmological relics are not classical gasses. If they have attractive interactions or are fermions, they instead are a super-fluid. This implies that WIMP dark matter scenarios are inconsistent: WIMPs cannot both be decoupled and localized for the age of the universe.

Cosmic background neutrinos must exist. They are a super-fluid, and their self-interactions are a gravitational theory. These dynamics arise in the SM, which is a renormalizable quantum field theory. We suggest that this may actually be the gravity that we observe.

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