The integrality of the Genocchi numbers obtained through a new identity and other results

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Abstract: In this note, we investigate some properties of the integer sequence of general term
\[ a_n := \sum_{k=0}^{n-1} k!(n - k - 1)! \quad (\forall n \geq 1) \]
and derive a new identity of the Genocchi numbers \( G_n \) \( (n \in \mathbb{N}) \), which immediately shows that \( G_n \in \mathbb{Z} \) for any \( n \in \mathbb{N} \). In another direction, we obtain nontrivial lower bounds for the 2-adic valuations of the rational numbers \( \sum_{k=1}^{n} \frac{2^k}{k} \).

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1 Introduction and Notation

Throughout this note, we let \( \mathbb{N} \) denote the set of positive integers and \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \) denote the set of non-negative integers. For \( x \in \mathbb{R} \), we let \( \lfloor x \rfloor \) denote the integer part of \( x \). Let \( s(n, k) \) and \( S(n, k) \) (with \( n, k \in \mathbb{N}_0, n \geq k \)) respectively denote the Stirling numbers of the first and second kinds, which can be defined as the integer coefficients appearing in the polynomial identities:

\[ X(X - 1) \cdots (X - n + 1) = \sum_{k=0}^{n} s(n, k)X^k, \quad \text{(for every } n \in \mathbb{N}_0). \]

\[ X^n = \sum_{k=0}^{n} S(n, k)X(X - 1) \cdots (X - k + 1) \]
This immediately implies the orthogonality relations (see, e.g., [1, 8]):

\[
\sum_{k \leq i \leq n} s(n, i) S(i, k) = \sum_{k \leq i \leq n} S(n, i) s(i, k) = \delta_{nk} \quad \text{(for every } n, k \in \mathbb{N}_0, n \geq k),
\]

where \( \delta_{nk} \) is the Kronecker delta. Among the many formulas related to the Stirling numbers, we mention the following result (see, e.g., [1, 6, 8]):

\[
\log^k(1 + x) = \sum_{n=0}^{\infty} s(n, k) \frac{x^n}{n!} \quad \text{(for every } k \in \mathbb{N}_0),
\]

which is needed later on. We let in addition \( B_n \) and \( G_n \) respectively denote the Bernoulli and the Genocchi numbers, which can be defined by their respective exponential generating functions (see, e.g., [1, 8]):

\[
x \log (1 + x) = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} \quad \text{and} \quad 2 x \log (1 + x) = \sum_{n=0}^{\infty} G_n \frac{x^n}{n!}.
\]

The famous Genocchi theorem [7] states that the \( G_n \)'s are all integers. There are at least two beautiful proofs of the Genocchi theorem: the first one uses the formula \( G_n = 2(1 - 2^n)B_n \) (see, e.g., [1]) together with the Fermat little theorem and the von Staudt-Clausen theorem, while the second one uses the remarkable Seidel formula [13]:

\[
\sum_{k=0}^{n} \binom{n}{k} G_{n+k} = 0 \quad \text{(for every } n \in \mathbb{N}_0).
\]

In this note, we give a new proof of the integrality of the \( G_n \)'s by expressing them in terms of the Stirling numbers of the second kind. The starting point of this research is the study of the integer sequence \( (a_n)_{n \in \mathbb{N}_0} \), defined by:

\[
a_0 = 0 \quad \text{and} \quad a_n := \sum_{k=0}^{n-1} k! (n - k - 1)! \quad \text{(for every } n \in \mathbb{N}).
\]

This sequence is closely related to the sum of the inverses of binomial coefficients, which is studied by several authors (see [10, 12, 16–18, 20, 21]). It must be noted that both Stirling numbers, Genocchi numbers, and the numbers \( a_n \) \( (n \in \mathbb{N}_0) \) have combinatorial interpretations (see, e.g., [1, 15] for the Stirling numbers, [4, 19] for the Genocchi numbers, and the sequence A003149 of [14] for the \( a_n \)'s).

Next, the least common multiple of given positive integers \( u_1, u_2, \ldots, u_n \) \( (n \in \mathbb{N}) \) is denoted by \( \text{lcm}(u_1, u_2, \ldots, u_n) \) or by \( \text{lcm}\{u_1, u_2, \ldots, u_n\} \) if this is more convenient. For a given prime number \( p \) and a given positive integer \( n \), we let \( \varphi_p(n) \) and \( s_p(n) \) respectively denote the usual \( p \)-adic valuation of \( n \) (that is the greatest \( e \in \mathbb{N}_0 \) satisfying \( p^e \mid n \)) and the sum of base-\( p \) digits of \( n \). The function \( \varphi_p \) (\( p \) a prime) is naturally extended to \( \mathbb{Q}^* \) by defining \( \varphi_p(\pm \frac{a}{b}) = \varphi_p(a) - \varphi_p(b) \), for any positive integers \( a \) and \( b \). With this extension, the function \( \varphi_p \) (\( p \) a prime) satisfies several elementary properties; among them, we cite:
\[ \vartheta_p(rs) = \vartheta_p(r) + \vartheta_p(s) \quad \text{for every } r, s \in \mathbb{Q}^*, \]

\[ \vartheta_p \left( \frac{r}{s} \right) = \vartheta_p(r) - \vartheta_p(s) \quad \text{for every } r, s \in \mathbb{Q}^*, \]

\[ \vartheta_p(\text{lcm}(1, 2, \ldots, n)) = \left\lfloor \frac{\log n}{\log p} \right\rfloor \]

\[ \vartheta_p(r) \geq 0 \quad \text{for every } r \in \mathbb{Z}^*. \]

Furthermore, a well-known formula of Legendre (see, e.g., [9, Theorem 2.6.4, page 77]) states that for any prime number \( p \) and any positive integer \( n \), we have

\[ \vartheta_p(n!) = \frac{n - s_p(n)}{p - 1}. \]

In another direction, by leaning on Legendre’s formula (1.6), an identity due to Rockett [12], and another identity due to the author [5], we obtain nontrivial lower bounds for the 2-adic valuations of the rational numbers \( \sum_{k=1}^{n} \frac{2^k}{k} \) (\( n \in \mathbb{N} \)).

2 The results and the proofs

Our main result is the following:

Theorem 2.1. For all positive integer \( n \), we have

\[ G_n = \sum_{1 \leq \ell \leq k \leq n} (-1)^{k-1}(\ell - 1)!(k - \ell)!S(n, k). \]

In particular, \( G_n \) is an integer for any \( n \in \mathbb{N} \).

To prove this theorem, we need some intermediary results. The first one (Proposition 2.2 below) can be immediately derived from the following identity of Rockett [12]:

\[ \sum_{k=0}^{n} \frac{1}{\binom{n}{k}} = \frac{n + 1}{2^{n+1}} \sum_{k=1}^{n+1} \frac{2^k}{k} \quad \text{for every } n \in \mathbb{N}_0. \]

But for convenience, we prefer reproduce its proof here.

Proposition 2.2. For all positive integer \( n \), we have

\[ a_n = \frac{n!}{2^n} \sum_{k=1}^{n} \frac{2^k}{k}. \]

Proof. We begin by establishing a recurrent formula for the sequence \( (a_n)_n \). For any integer \( n \geq 2 \), we have:
\[ a_n := \sum_{k=0}^{n-1} k!(n-k-1)! \]
\[ = \sum_{k=0}^{n-2} k!(n-k-1)! + (n-1)! \]
\[ = \sum_{k=0}^{n-2} k!(n-k-2)!(n-k-1) + (n-1)! \]
\[ = n \sum_{k=0}^{n-2} k!(n-k-2)! - \sum_{k=0}^{n-2} (k+1)!(n-k-2)! + (n-1)! . \]

But since
\[ \sum_{k=0}^{n-2} k!(n-k-2)! = a_{n-1} \]
and
\[ \sum_{k=0}^{n-2} (k+1)!(n-k-2)! = \sum_{\ell=1}^{n-1} \ell!(n-\ell-1)! \]
(by putting \( \ell = k + 1 \))
\[ = \sum_{\ell=0}^{n-1} \ell!(n-\ell-1)! - (n-1)! \]
\[ = a_n - (n-1)! , \]
it follows that:
\[ a_n = na_{n-1} - a_n + 2 \cdot (n-1)! . \]

Hence
\[ a_n = \frac{n}{2} a_{n-1} + (n-1)! . \]

(2.4)

Further, we remark that Formula (2.4) also holds for \( n = 1 \). Now, according to Formula (2.4), we have for any positive integer \( k \):
\[ \frac{2^k}{k!} a_k - \frac{2^{k-1}}{(k-1)!} a_{k-1} = \frac{2^k}{k} . \]

Then by summing both sides of the last equality from \( k = 1 \) to \( n \), we obtain (because the sum on the left is telescopic and \( a_0 = 0 \)) that:
\[ \frac{2^n}{n!} a_n = \sum_{k=1}^{n} \frac{2^k}{k} , \]

which gives the required formula. The proof is achieved. \( \square \)

**Corollary 2.3.** The exponential generating function of the sequence \((a_n)_n\) is given by:
\[ \sum_{n=0}^{\infty} a_n \frac{x^n}{n!} = \frac{-2 \log(1-x)}{2-x} . \]

(2.5)
Proof. Using Formula (2.3) of Proposition 2.2, we have
\[
\sum_{n=0}^{\infty} \frac{a_n x^n}{n!} = \sum_{n=1}^{\infty} \left( \frac{1}{2^n} \sum_{k=1}^{\infty} \frac{2^k}{k} \right) x^n = \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} \frac{1}{2^n} \frac{2^k}{k} x^n = \sum_{k=1}^{\infty} \frac{2^k}{k} \left( \sum_{n=k}^{\infty} \left( \frac{x}{2} \right)^n \right).
\]
But since \( \sum_{n=k}^{\infty} \left( \frac{x}{2} \right)^n = \left( \frac{x}{2} \right)^k \sum_{n=k}^{\infty} \frac{1}{2^n} = \frac{x^k}{2^k} \cdot \left( \frac{x}{2} \right)^{k-1} \cdot \frac{1}{1 - \frac{x}{2}} \), we get
\[
\sum_{n=0}^{\infty} a_n \frac{x^n}{n!} = \frac{2}{2-x} \sum_{k=1}^{\infty} x^k \frac{1}{k} = \frac{2}{2-x} (-\log(1-x)),
\]
as required. This achieves the proof. \(\square\)

Next, from Corollary 2.3 and Formula (1.2), we derive the following corollary:

**Corollary 2.4.** For any non-negative integer \( n \), we have
\[
a_n = (-1)^n \sum_{k=0}^{n} G_k s(n, k) \tag{2.6}
\]

Proof. Let us consider the following three functions (which are analytic on the neighborhood of zero):
\[
f(x) := \frac{-2 \log(1-x)}{2-x}, \quad g(x) := \frac{2x}{e^x + 1}, \quad \text{and} \quad h(x) := \log(1-x).
\]
We easily check that \( f = -g \circ h \). Since in addition \( h(0) = 0 \) then the power series expansion of \( f \) about the origin can be obtained by substituting \( h \) in the power series expansion of \( g \) about the origin (which is given by (1.3)) and multiplying by \( (-1) \). Doing so, we get
\[
f(x) = -\sum_{k=0}^{\infty} G_k \frac{(h(x))^k}{k!} = -\sum_{k=0}^{\infty} G_k \frac{\log(1-x)}{k!} \tag{2.7}
\]
Further, by substituting in (1.2) \( x \) by \( -x \), we have for any \( k \in \mathbb{N}_0 \):
\[
\frac{\log^k(1-x)}{k!} = \sum_{n=k}^{\infty} (-1)^n s(n, k) \frac{x^n}{n!}.
\]
So, by inserting this last expression into (2.7), we get
\[
f(x) = -\sum_{k=0}^{\infty} G_k \sum_{n=k}^{\infty} (-1)^n s(n, k) \frac{x^n}{n!}
\]
\[
= -\sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^n G_k s(n, k) \frac{x^n}{n!}
\]
\[
= \sum_{n=0}^{\infty} \left[ (-1)^{n-1} \sum_{k=0}^{n} G_k s(n, k) \right] \frac{x^n}{n!}.
\]
Comparing this last formula with Formula (2.5) of Corollary 2.3, we conclude that:
\[
a_n = (-1)^{n-1} \sum_{k=0}^{n} G_k s(n, k) \quad \text{(for every} \ n \in \mathbb{N}_0),
\]
as required. \(\square\)

We finally derive our main result from Corollary 2.4 above by applying the well-known inversion formula recalled in the following proposition.
Proposition 1. Let \((u_n)_{n \in \mathbb{N}_0}\) and \((v_n)_{n \in \mathbb{N}_0}\) be two real sequences. Then the two following identities 
(I) and (II) are equivalent:

\[
\begin{align*}
    u_n &= \sum_{k=0}^{n} v_k s(n, k) \quad \text{(for every } n \in \mathbb{N}_0\), \\
    v_n &= \sum_{k=0}^{n} u_k S(n, k) \quad \text{(for every } n \in \mathbb{N}_0). 
\end{align*}
\]

Proof. Use the orthogonality relations (1.1) (see, e.g., [1] or [11] for the details).

Proof of Theorem 2.1. It suffices to apply Proposition 1 for \(u_n = (-1)^{n-1} a_n\) and \(v_n = G_n\), for every \(n \in \mathbb{N}_0\). In view of (2.6), Identity (I) holds; so (II) also, that is

\[G_n = \sum_{k=0}^{n} (-1)^{k-1} a_k S(n, k) \quad \text{(for every } n \in \mathbb{N}_0).\]

Finally, by substituting in this last equality \(a_k\) by its expression given by (1.4), we get for any \(n \in \mathbb{N}\):

\[
G_n = \sum_{k=1}^{n} (-1)^{k-1} \left( \sum_{i=0}^{k-1} i! (k - i - 1)! \right) S(n, k)
= \sum_{k=1}^{n} (-1)^{k-1} \left( \sum_{\ell=1}^{k} (\ell - 1)! (k - \ell)! \right) S(n, k) \quad \text{(by putting } \ell = i + 1) 
= \sum_{1 \leq \ell \leq k \leq n} (-1)^{k-1} (\ell - 1)! (k - \ell)! S(n, k),
\]

as required.

Remark 2.5. In the relatively recent literature, there are several ways to explain the integrality of the Genocchi numbers. For example, it is shown (see [3]) that the Genocchi numbers are (up to a sign) the values of the Gandhi polynomials (lying in \(\mathbb{Z}[X]\)) at 1. On the other hand, the combinatorial interpretation of the Genocchi numbers discovered by Dumont (see, e.g., [4, 19]) immediately explains the integrality of the \(G_n\’s\).

Now, we turn to present another type of results providing nontrivial lower bounds for the 2-adic valuations of the rational numbers \(\sum_{k=1}^{n} \frac{2^k}{k}\) \((n \in \mathbb{N})\).

Theorem 2.6. For any positive integer \(n\), we have

\[\vartheta_2 \left( \sum_{k=1}^{n} \frac{2^k}{k} \right) \geq s_2(n) \quad (2.8)\]

and (more strongly):

\[\vartheta_2 \left( \sum_{k=1}^{n} \frac{2^k}{k} \right) \geq n - \left\lfloor \frac{\log n}{\log 2} \right\rfloor. \quad (2.9)\]
Proof. Let $n$ be a fixed positive integer. Since $a_n \in \mathbb{Z}$ then we have $\vartheta_2(a_n) \geq 0$. But, by using Formula (2.3) of Proposition 2.2 together with the properties of (1.5), this is equivalent to:

$$\vartheta_2(n!) - n + \vartheta_2 \left( \sum_{k=1}^{n} \frac{2^k}{k} \right) \geq 0.$$

Then, using the Legendre formula (1.6) for the prime number $p = 2$, which says that $\vartheta_2(n!) = \frac{n - s_2(n)}{2 - 1} = n - s_2(n)$, we get

$$\vartheta_2 \left( \sum_{k=1}^{n} \frac{2^k}{k} \right) \geq s_2(n),$$

confirming (2.8). To establish the stronger lower bound (2.9), we use the Rockett formula (2.2) together with the identity:

$$\text{lcm} \left\{ \left( m \over 0 \right), \left( m \over 1 \right), \ldots, \left( m \over m \right) \right\} = \frac{\text{lcm}(1, 2, \ldots, m, m + 1)}{m + 1} \quad \text{(for every } m \in \mathbb{N}_0), \quad (2.10)$$

established by the author in [5]. According to (2.2) and (2.10), we have that

$$\sum_{k=1}^{n} \frac{2^k}{k} = \frac{2^n}{n} \sum_{k=0}^{n-1} \left( \frac{n-1}{k} \right) \quad \text{and} \quad 1 = \frac{n}{\text{lcm}(1, 2, \ldots, n)} \cdot \text{lcm} \left\{ \left( n-1 \over 0 \right), \left( n-1 \over 1 \right), \ldots, \left( n-1 \over n-1 \right) \right\}. \quad$$

By multiplying side by side these last equalities, we get

$$\sum_{k=1}^{n} \frac{2^k}{k} = \frac{2^n}{\text{lcm}(1, 2, \ldots, n)} \cdot \text{lcm} \left\{ \left( n-1 \over 0 \right), \left( n-1 \over 1 \right), \ldots, \left( n-1 \over n-1 \right) \right\} \sum_{k=0}^{n-1} \frac{1}{\left( \frac{n-1}{k} \right)}. \quad$$

But since the rational number $\text{lcm}\left\{ \left( n-1 \over 0 \right), \left( n-1 \over 1 \right), \ldots, \left( n-1 \over n-1 \right) \right\} \sum_{k=0}^{n-1} \frac{1}{\left( \frac{n-1}{k} \right)}$ is obviously a positive integer, then it has a nonnegative 2-adic valuation; so it follows (according to the properties of the functions $\vartheta_p$ given in (1.5)) that:

$$\vartheta_2 \left( \sum_{k=1}^{n} \frac{2^k}{k} \right) \geq \vartheta_2 \left( \frac{2^n}{\text{lcm}(1, 2, \ldots, n)} \right) = n - \left\lfloor \frac{\log n}{\log 2} \right\rfloor,$$

confirming (2.9) and completes the proof. \hfill \Box

Remark 2.7. Very recently, Dubickas [2] has shown that the lower bound (2.9) of Theorem 2.6 is essentially optimal (it is attained if $n$ has the form $2^k - 1, k \in \mathbb{N}$).

3 Two open problems

Open problem 1. Find a generalization of Theorem 2.6 to other prime numbers $p$ other than $p = 2$. Notice that the generalization that might immediately come to mind:

$$\vartheta_p \left( \sum_{k=1}^{n} \frac{p^k}{k} \right) \geq s_p(n)$$

is false for $p > 2$ (take for example $n = 2$).
Open problem 2. Since every term in Formula (2.1) of Theorem 2.1 has an easily understood combinatorial meaning (see [1] for the factorials and the Stirling numbers and [4, 19] for the Genocchi numbers), it is natural to ask whether there exists a combinatorial proof of that formula (permitting us to understand it intuitively). Note that one of the classic references providing an enormous amount of combinatorial proofs is the Stanley book [15].

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