Statistical Thermodynamics of Moving Bodies*

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In appreciation of Andrzej Kossakowski’s friendship and scientific achievements
on the occasion of his seventieth birthday

Abstract

We resolve the long standing question of temperature transformations of uniformly moving bodies by means of a quantum statistical treatment centred on the zeroth law of thermodynamics. The key to our treatment is the result, established by Kossakowski et al, that a macroscopic body behaves as a thermal reservoir with well-defined temperature, in the sense of the zeroth law, if and only if its state satisfies the Kubo-Martin-Schwinger (KMS) condition. In order to relate this result to the relativistic thermodynamics of moving bodies, we employ the Tomita-Takesaki modular theory to prove that a state cannot satisfy the KMS condition with respect to two different inertial frames whose relative velocity is non-zero. This implies that the concept of temperature stemming from the zeroth law is restricted to states of bodies in their rest frames and thus that there is no law of temperature transformations under Lorentz boosts. The corresponding results for nonrelativistic Galilean systems have also been established.

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1. Introduction

In the wake of Einstein’s theory of special relativity, Planck [1] and Einstein [2] proposed an extension of classical thermodynamics to bodies moving with uniform velocity $v$ relative to an inertial laboratory frame, $K_L$. This involved a supplementation of the usual set of thermodynamical variables of a body (pressure, volume, temperature, etc.) by this velocity $v$ and led to the result that its temperature $T_L$, as observed in $K_L$, is proportional to the Lorentz contraction factor $(1 - v^2/c^2)^{1/2}$; specifically that $T_L$ is related to the temperature $T_0$ of the body relative to a rest frame $K_0$ by the formula

$$T_L = (1 - v^2/c^2)^{1/2} T_0. \quad (1.1)$$

Evidently, this signifies that a uniformly moving body appears to be cooled by its motion relative to the inertial frame of observation.

This formula remained unchallenged for more than half a Century until Ott [3] proposed a different extension of classical thermodynamics to moving bodies, which led to the opposite result for the relationship between $T_L$ and $T_0$, namely

$$T_L = (1 - v^2/c^2)^{-1/2} T_0. \quad (1.2)$$

Subsequently, Landsberg [4] argued, on the basis of another extension of classical thermodynamics, that the temperature of the body should be a scalar invariant, i.e. that $T_L = T_0$.

These different approaches to the problem of extending classical thermodynamics to the relativistic domain led to further treatments and comments by a number of authors, e.g. [5-7]. In particular, Van Kampen [5] provided a very clear analysis of the underlying assumptions behind the works of [1-4] and proposed yet another, relativistically covariant extension of classical thermodynamics.

At this stage we note that all the above works [1-7] were based exclusively on the first and second laws of thermodynamics, without reference to either the zeroth law or the underlying statistical mechanics. A subsequent work by Landsberg and Matsas [8] invoked both of these latter items in a statistical mechanical treatment of a model comprising a two level atom coupled to black body radiation, the atom and radiation being at rest in the above described frames $K_L$ and $K_0$, respectively, and the radiation having a Planck spectrum relative to the latter frame. The result they obtained for this model was that the atom is not driven to a canonical equilibrium state, relative to $K_L$, unless $v = 0$. Thus, for this model, the temperature concept, as represented by the zeroth law, is applicable only to the radiation in its rest frame. This result accords with ideas expressed earlier by Landsberg [4].

In this note, as in a previous article [9], we address the question of the generality of this result by means of a model independent, quantum statistical treatment of the response of an arbitrary finite probe (thermometer!) $S$, at rest in $K_L$, to its coupling to a macroscopic, ideally infinite, system $\Sigma$, which is in thermal equilibrium in its rest frame $K_0$. For this
setup, we prove that, under very general conditions, it is only when $\Sigma$ is at rest relative to $K_L$ that it drives $S$ into a terminal canonical equilibrium state. This signifies that it is only then that $\Sigma$ has a well defined temperature relative to $K_L$, in the sense of the zeroth law. In other words, there is no law of temperature transformations under Lorentz boosts. Moreover, a similar argument has established the corresponding result for Galilei boosts of nonrelativistic systems [9].

The key to these results is the connection, established by Kossakowski, Frigerio, Gorini and Verri [10], between the zeroth law of thermodynamics and the Kubo-Martin-Schwinger (KMS) equilibrium condition. To explain this connection, we recall that the latter condition on the state of a conservative quantum system, $\Sigma$, is given formally be the equation [11]

$$\langle A(t)B \rangle = \langle BA(t + i\beta) \rangle,$$

(1.3)

where $\langle \cdot \rangle$ denotes expectation value for the state in question, $A$ and $B$ are arbitrary observables of the system, $A(t)$ is the evolute of $A$ at time $t$, and $\beta$ is the inverse temperature in units where $\hbar$ and $k_{Boltzmann}$ are equal to unity. The grounds for taking this condition to characterise equilibrium states are that

(a) it implies that the state is stationary;

(b) it comprises a generalisation of the canonical Gibbsian condition to infinite systems, which are the natural idealisations of macroscopic ones in the standard thermodynamic limit;

(c) it corresponds precisely to various dynamical and thermodynamical stability conditions [12-15] that are the natural desiderata for thermal equilibrium; and

(d) it is precisely the condition for which $\Sigma$ behaves as a thermal reservoir, in the sense of the zeroth law, in that it drives drives any finite test system (thermometer!) $S$ to which it is weakly and transitively* coupled into a terminal state that is the canonical equilibrium one of inverse temperature $\beta$ [10].

It follows from (d) that if a state of $\Sigma$ were thermal, in the sense of the zeroth law, from the standpoints of observers in both $K_0$ and $K_L$, then it would satisfy the versions of the KMS condition (1.3) relative to both those frames at some inverse temperatures $\beta_0$ and $\beta_L$, respectively. We shall prove, however, that this is not possible, by virtue of the mathematical constraints imposed by the KMS condition and the action of Lorentz transformations on the observables. Hence we conclude that there is no law of temperature transformations under Lorentz boosts and thus that the concept of temperature stemming from the zeroth law is restricted to states of bodies in their rest frames. The corresponding conclusion for the nonrelativistic setting, with the Lorentz boosts replaced by Galilean ones, has also been established [9].

We present our treatment of the statistical thermodynamics of moving bodies as follows. In Section 2, we formulate the generic operator algebraic model of a relativistic

* The transitivity condition is that the $\Sigma - S$ coupling induces transitions, whether direct or indirect, between all the eigenstates of $S$. 

macroscopic system, including a precise definition of the KMS condition and its relation to the Tomita-Takesaki modular theory. We then prove, in Section 3, that the model cannot support states that satisfy the KMS condition relative to two frames of reference whose relative velocity is non-zero: this establishes the conclusion described in the previous paragraph. In Section 4, we briefly summarise the basis of this conclusion and raise an open question concerning the thermodynamics of moving bodies.

2. The generic model

We take our model of a relativistic macroscopic system, \( \Sigma \), to be an infinitely extended one that occupies a Minkowski space \( X \), whose points we denote by \( x \). We formulate the model within the operator algebraic framework of Haag and Kastler [16], in which \( \Sigma \) is represented by a triple \((A, \mathcal{S}, \alpha)\), where \( A \) is a \( C^\ast \)-algebra of bounded observables, \( \mathcal{S} \) is the state space, comprising the positive normalised linear functionals on \( A \), and \( \alpha \) is a representation of the additive group \( X \) (the Minkowski space) in \( \text{Aut}(A) \), corresponding to space-time translations.

For a given inertial frame of reference, \( K \), we represent the points \( x \) of \( X \) by coordinates \( \{x^\mu | \mu = 0, 1, 2, 3\} \). Here \( x^0 \) is the time coordinate, in units for which \( c = 1 \), and the other \( x^\mu \)'s are the spatial ones. Thus, the unit vector along the time direction for \( K \) is \( u = (1, 0, 0, 0) \), and time translations of \( \Sigma \), relative to \( K \), are represented by the one parameter group \( \{\alpha(tu)|t \in \mathbb{R}\} \) of automorphisms of \( A \).

The KMS Condition and the Modular Automorphisms. The KMS condition, relative to \( K \), on a state \( \phi \) may be expressed in the following form [11]. For any \( A, B \in A \), the function \( F : t(\in \mathbb{R}) \rightarrow \langle \phi; \alpha((t+i\beta)u)A \rangle \) extends to the strip \( \{z \in \mathbb{C}|\text{Im}(z) \in [0, \beta]\} \), where it is analytic in the interior and continuous on the boundaries and where

\[
F(t + i\beta) = \langle \phi; [\alpha(tu)A]B \rangle \quad \text{and} \quad F(t) = \langle \phi; B\alpha(tu)A \rangle \quad \forall \ t \in \mathbb{R}.
\]

Thus, formally, the KMS condition is simply

\[
\langle \phi; [\alpha(tu)A]B \rangle = \langle \phi; B\alpha((t+i\beta)u)A \rangle \quad \forall \ A, B \in A, \ t \in \mathbb{R}.
\]

This condition is closely related to the Tomita-Takesaki theory [17] of modular automorphisms, which established that any faithful normal state \( \psi \) on a \( W^\ast \)-algebra \( \mathcal{M} \) induces a unique one parameter group, \( \{\tau(t)|t \in \mathbb{R}\} \), of automorphisms of \( \mathcal{M} \) that satisfies the KMS-like relation

\[
\langle \psi[\tau(t)M]N \rangle = \langle \psi N\tau(t+i)N \rangle \quad \forall \ M, N \in \mathcal{M}, \ t \in \mathbb{R}.
\]

In order to connect this precisely to the \( C^\ast \)-algebraic KMS condition (2.2)' , we introduce the GNS triple \((\mathcal{H}, \pi, \Phi)\) of the state \( \phi \) and note that, as this state is stationary, the
automorphisms $\alpha(tu)$ are implemented by the one-parameter group $\{U(t)|t\in\mathbb{R}\}$ of unitary transformations of $\mathcal{H}$ defined by the formula \[2.4\]

\[U(t)\pi(A)\Phi = \pi(\alpha(tu)A)\Phi \quad \forall \, A\in\mathcal{A}, \, t\in\mathbb{R}.\]

We then define the canonical extensions, $\tilde{\phi}$ and $\tilde{\alpha}_u(t)$, of $\phi$ and $\alpha(tu)$, respectively, to $\pi(A)^\prime\prime$ by the formulae

\[2.5\]

\[\tilde{\phi}(F) = (\Phi, F\Phi) \quad \text{and} \quad \tilde{\alpha}_u(t) = U(t)FU(-t) \forall \, F\in\pi(A)^\prime\prime, \, t\in\mathbb{R}.\]

In particular,

\[2.6\]

\[\tilde{\alpha}_u(t)\pi(A) = \pi(\alpha(tu)A) \quad \forall \, A\in\mathcal{A}, \, t\in\mathbb{R}.\]

It follows from the last two formulae that the KMS condition \(2.2\)' for $\phi$ extends to $\tilde{\phi}$ in the form

\[2.7\]

\[\langle \tilde{\phi}; [\tilde{\alpha}_u(t)F]G \rangle = \langle \tilde{\phi}; F\tilde{\alpha}_u(t+i\beta) \rangle \quad \forall \, F,G\in\pi(A)^\prime\prime, \, t\in\mathbb{R}.\]

Moreover, the state $\tilde{\phi}$ is faithful \[11\]. Consequently, it follows from a comparison of Eqs. \(2.3\) and \(2.7\), with $\mathcal{M} = \pi(A)^\prime\prime$ and $\psi = \tilde{\phi}$, that the automorphisms $\tilde{\alpha}_u$ are related to the modulars $\tau$ by the formula

\[2.8\]

\[\tilde{\alpha}_u(t/\beta) = \tau(t).\]

### 3. Incompatibility of KMS conditions relative to different inertial frames

**Definition 3.1.** We say that space-time translations act non-trivially in a representation $\pi$ of $\mathcal{A}$ if, for any non-zero $a\in X$, there exists a pair $(A,s)$ in $\mathcal{A}\times\mathbb{R}$ such that $\pi(\alpha(sa)A) \neq \pi(A)$.

**Proposition 3.1.** Assume that space-time translations act non-trivially in the GNS representation of a state $\phi$ on $\mathcal{A}$. Then $\phi$ cannot satisfy KMS conditions with respect to two inertial frames whose relative velocity is non-zero.

**Proof.** Let $K$ and $K'$ be inertial frames and let $v$ be the velocity of $K'$ relative to $K$. We choose the spatial coordinate axes so that those of $K'$ are parallel to the corresponding ones of $K$ and the velocity $v$ is directed along $Ox^1$. Then the unit time translational vector of $K'$, as represented in the $K$ coordinate system, is

\[3.1\]

\[u' = ((1 - v^2)^{-1/2}, -v(1 - v^2)^{-1/2}, 0, 0).\]

Suppose now that $\phi$ satisfies the KMS conditions relative to both $K$ and $K'$ for inverse temperatures $\beta$ and $\beta'$, respectively. Then it follows from Eqs. \(2.6\) and \(2.8\) that both $\pi(\alpha(tu/\beta)A)$ and $\pi(\alpha(tu'/\beta')A)$ are equal to $\tau(t)\pi(A)$. Thus

\[3.2\]

\[\pi(\alpha(tu/\beta)A) = \pi(\alpha(tu'/\beta')A) \quad \forall \, A\in\mathcal{A}, \, t\in\mathbb{R}.\]
On replacing $A$ by $\alpha(-tu/\beta)A$ in this formula and invoking the abelian character of the space-time translation group, we see that

$$\pi\left(\alpha\left(t(\beta\beta')^{-1}(\beta'u' - \beta'u)\right)A\right) = \pi(A) \forall A \in A, \ t \in \mathbb{R}. \quad (3.3)$$

Hence, as space translations are assumed to be non-trivial in the representation $\pi$,

$$\beta u' = \beta'u, \quad (3.4)$$

which, by Eqs. (2.1) and (3.1), implies that

$$\beta(1-v^2)^{-1/2} = \beta' \text{ and } v\beta(1-v^2)^{-1/2} = 0. \quad (3.5)$$

In view of the finiteness of $\beta$ and the subluminal condition that $|v| < 1$, these equations cannot be satisfied for non-zero $v$. This completes the proof of the Proposition.

**Comments.** (1) Assuming that the laws of thermodynamics are valid in rest frames, it follows from this Proposition and Ref. [10] that a state $\phi$ that satisfies the zeroth law relative to these frames does not satisfy that law relative to moving ones. Hence the very concept of temperature is restricted to rest frames and so there is no law of temperature transformation under Lorentz boosts.

(2) The corresponding result for the non-relativistic setting, with Lorentz boosts replaced by Galilean ones, has also been established [9] on a similar basis.

(3) In view of the commutativity of the space-time translation group, the KMS condition (2.2)' is equivalent to the formula [19, 20]

$$\langle \phi; [\alpha(x)A]B \rangle = \langle \phi; B\alpha(x + i\beta u)A \rangle \forall A, B \in A, \ x \in X. \quad (3.6)$$

This formula may be referred to any inertial frame $\tilde{K}$, with $u$ represented by coordinates $(\tilde{u}^0, \tilde{u}^1, \tilde{u}^2, \tilde{u}^3)$. The time component of $\beta u$ there is then $\beta\tilde{u}^0$. However, this should not be taken to be the inverse temperature relative to $\tilde{K}$ if this is not a rest frame, since, by Comment (1), $\phi$ does not then satisfy the zeroth law for this frame.

(4) Since the frames $K$ and $K'$ of Prop. 3.1 are both inertial, this Proposition has nothing to say about temperatures in accelerating frames or, equivalently, in gravitational fields. Consequently it has no bearing on phenomena such as the Hawking and Unruh effects [21-23].

**4. Concluding remarks.**

We have established that the concept of temperature, which ensues from the zeroth law of thermodynamics, is restricted to equilibrium states of systems in their rest frames. The essential ingredients in the proof of this result were the relationships of the KMS condition to the zeroth law of thermodynamics [10] and to the Tomita-Takesaki modular theory [19].
Granted the validity of classical thermodynamics for systems in their rest frames, the question naturally arises whether this discipline may be canonically extended to heterotachic processes comprising exchanges of energy and momentum between systems in relative motion. In fact, Van Kampen [5] has initiated an approach to this question via a thermodynamical argument to the effect that, at least in certain natural model situations, the sum of the entropies of these systems, as defined relative to their rest frames, increases in such processes. It would be interesting to have a model independent statistical thermodynamical generalisation of this result.

References.

[1] M. Planck: Sitzber. K1. Preuss. Akad. Wiss. P. 542, 1907
[2] A. Einstein: Jahrb. Radioaktivitaet Elektronik 4, 411, 1907
[3] H. Ott: Zeits. Phys. 175, 70, 1963
[4] P. T. Landsberg: Nature 212, 571, 1966; Nature 214, 903, 1967
[5] N. G. van Kampen: Phys. Rev. 173, 295, 1968
[6] T. W. B. Kibble: Nuov. Cim. 41B, 72, 1966
[7] H. Kallen and G. Horowitz: Amer. J. Phys. 39, 938, 1971
[8] P. T. Landsberg and G. E. A. Matsas: Phys. Lett. A 223, 401, 1996
[9] G. L. Sewell: J. Phys. A 41, 382003, 2008
[10] A. Kosakowski, A. Frigerio, V. Gorini and M. Verri: Commun. Math. Phys. 57, 97, 1977
[11] R. Haag, N. M. Hugenholtz and M. Winnink: Commun. Math. Phys. 5, 215, 1967
[12] R. Haag, D. Kastler and E. B. Trych-Pohlmeyer: Commun. Math. Phys. 56, 214, 1977
[13] W. Pusz and S. L. Woronowicz: Commun. Math. Phys. 58, 273, 1978
[14] H. Araki and G. L. Sewell: Commun. Math. Phys. 52, 103, 1977
[15] G. L. Sewell: Commun. Math. Phys. 55, 53, 1977
[16] R. Haag and D. Kastler: J. Math. Phys. 5, 848, 1964
[17] M. Takesaki: *Tomita’s Theory of Modular Hilbert Algebras and its Applications*, Lec. Notes in Maths. Vol. 128, Springer, Berlin, Heidelberg, New York, 1970
[18] I. E. Segal: Ann. Math. 48, 930, 1947
[19] I. Ojima: Lett. Math. Phys. 11, 73, 1986
[20] J. Bros and D. Buchholz: Nucl. Phys. B 429, 291, 1994
[21] S. Hawking: Commun. Math. Phys. 43, 199, 1975
[22] W. Unruh: Phys. Rev. D 14, 870, 1976

[23] G. L. Sewell: Ann. Phys. 141, 201, 1982