Window Flow Control Systems with Random Service

Alireza Shekaramiz*, Jörg Liebeherr*, Almut Burchard**

* Department of ECE, University of Toronto, Canada.
** Department of Mathematics, University of Toronto, Canada.
E-mail: \{ashekaramiz, jorg\}@ece.utoronto.ca; almut@math.toronto.edu.

Abstract

We present an extension of the window flow control analysis by R. Agrawal et. al. (Reference [1]), C.-S. Chang (Reference [6]), and C.-S. Chang et. al. (Reference [8]) to a system with random service time and fixed feedback delay. We consider two network service models. In the first model, the network service process itself has no time correlations. The second model addresses a two-state Markov-modulated service.

I. Introduction

The discovery that deterministic feedback systems can be expressed in the network calculus as the solution of a fixed-point equation in the dioid algebra was exploited for an analysis of a general feedback system [1], [6], [8]. The analysis was conducted for deterministic systems, which cannot exploit statistical multiplexing of traffic flows. A detailed feedback model of the TCP Tahoe and TCP Reno algorithms as max-plus linear systems was presented in [3]. The analysis accounts for randomness in the feedback system, but assumes that the underlying service process is deterministic. A further simplification is that the derivations in [3] are conducted for an overloaded system, with a claim that the results extend to any load condition. Overcoming the limitations of a deterministic analysis has motivated the development of the stochastic network calculus, where traffic and service are characterized by random processes [13]. Whereas, over the last ten years, many key results of the network calculus have been shown to also hold
Fig. 1: Network with window flow control. Traffic is admitted to the network only if its backlog does not exceed $w$ bits. Feedback, with delay $d$, informs the network ingress about departures. Random processes $A, A', D,$ and $S$, respectively, describe the external arrivals, the admitted arrivals, the departures, and the network service.

under probabilistic assumptions, there has not been a successful treatment of stochastic feedback systems, see [11, p. 82], [12, p. 104], [13, pp. 215-216].

In this paper, we present an analysis of stochastic feedback systems with methods of the stochastic network calculus. Our paper extends the deterministic network calculus analysis of feedback systems in [1], [6], [8], [15]. Following prior works, we study a window flow control system as shown in Fig. 1, which enforces that only a limited number of $w$ bits, referred to as the window size, can be in transit at any time. The feedback consists of (acknowledgement) messages, which relay information on the amount of departing traffic back to the network entrance. The delay of the feedback is assumed to be $d$ time units. The feedback signal opens or closes a throttle that prevents traffic from entering the network.

Selecting a discrete-time model for the analysis, we derive upper and lower bounds on the service experienced by a traffic flow where the external arrivals and the available network service are characterized by bivariate random processes. We are interested in gaining insight into the service impediment caused by the feedback mechanism with given window size $w$ and feedback delay $d$. This is done by deriving an equivalent service process, which describes the available service when taking into account the window flow control constraints.

Our analysis applies the moment-generating function (MGF) network calculus from [9]. One
challenge for the analysis of a stochastic feedback system with the MGF network calculus is
that the standard method to compute the frequently encountered convolution operation is not
well-suited for feedback systems. We adapt the MGF network calculus to feedback systems by
deriving a sharpened version of the convolution estimate from [9].

It remains open to which degree our results can be applied to other feedback systems. Even
though our paper only presents single node results, we emphasize that available techniques of the
MGF network calculus permit an immediate extension to a multi-node network consisting of a
sequence of feedback systems. More complex network topologies, in particular, nested feedback
systems are not covered.

The rest of the paper is structured as follows. In Sec. II we provide background on network
calculus, analysis of feedback systems, and the MGF network calculus with bivariate processes.
In Sec. III we discuss the obstacles to a network calculus analysis of stochastic feedback systems,
and present the main results of this paper. In Sec. IV we consider a feedback system with an
i.i.d. service model. Sec. V extends the analysis to a service model with time correlations. We
present brief conclusions in Sec. VI.

II. BIVARIATE NETWORK CALCULUS

The derivations in this paper will be done in the context of a bivariate network calculus, which
has been used for deterministic as well as stochastic analyses of networks [7]. We consider a
discrete-time domain with \( t = 0, 1, 2, \ldots \) describing time slots.

The vast majority of network calculus research is done with univariate functions \( f(t) \) that
quantify events in a time interval \([0, t]\). A bivariate network calculus uses functions \( f(s, t) \) to
characterize events in the time interval \([s, t]\). The rationale for preferring the univariate network
calculus is that it has stronger algebraic properties. On the other hand, univariate functions do not
lend themselves easily to probabilistic extensions. For example, consider a deterministic system
that offers a time-invariant service \( S(s, t) \) for each time interval \([s, t]\). Since \( S(s, t) = S(0, t-s) \),
the service can be completely described by a univariate function \( S(\tau) \) expressing the service in
any time interval of length \( \tau \). In the corresponding probabilistic model, if the service is stationary,
\( S(s, t) \) and \( S(0, t-s) \) are equal in distribution, however, generally \( S(s, t) \neq S(0, t-s) \), thus
prohibiting the convenient reduction to univariate functions.
A. \((\wedge, \otimes)\)-dioid algebra for bivariate functions

We denote by \(\mathcal{F}\) the family of bivariate functions that are non-negative and non-decreasing in the second argument. We use \(\mathcal{F}_o\) to denote the set of functions \(f \in \mathcal{F}\) with \(f(t, t) = 0\), which we refer to as causal functions. For \(f, g \in \mathcal{F}\), the minimum \((\wedge)\) and convolution \((\otimes)\) operations are defined by

\[
\begin{align*}
    f \wedge g (s, t) &= \min \{ f(s, t), g(s, t) \}, \\
    f \otimes g (s, t) &= \min_{s \leq \tau \leq t} \{ f(s, \tau) + g(\tau, t) \}.
\end{align*}
\]

The two operations form a dioid algebra on the sets \(\mathcal{F}\) and \(\mathcal{F}_o\) \(^7\). The convolution operation in these dioids is associative, but not commutative. (In contrast, the convolution in the corresponding dioid algebra for univariate functions is commutative.) Also, the convolution distributes over the minimum, i.e., for three functions \(f, g, \) and \(h\) we have \(f \otimes (g \wedge h) = f \otimes g \wedge f \otimes h\).

The function \(\delta(s, t) = \begin{cases} 0 & s \geq t, \\ \infty & s < t, \end{cases}\) is the neutral element of the convolution operation, with \(f \otimes \delta = \delta \otimes f = f\) for any \(f \in \mathcal{F}\).

For the analysis of feedback systems, we need to convolve a function multiple times with itself. For that, we use the notation

\[
\begin{align*}
    f^{(0)} &= \delta, & f^{(1)} &= f, \\
    f^{(n+1)} &= f^{(n)} \otimes f, & (n \geq 1).
\end{align*}
\]

We define the subadditive closure of \(f \in \mathcal{F}\), denoted by \(f^*\), as

\[
f^* = \delta \wedge f \wedge f^{(2)} \wedge f^{(3)} \wedge \ldots = \bigwedge_{n=0}^{\infty} f^{(n)}.
\]

The attribute ‘subadditive’ derives from the property that \(f^*(s, t) \leq f^*(s, \tau) + f^*(\tau, t)\) for \(s \leq \tau \leq t\). Every subadditive function \(f\) satisfies \(f \otimes f = f\). If \(f \in \mathcal{F}_o\), then its subadditive closure is given by \(f^* = \lim_{n \to \infty} f^{(n)}\).

\(^1\)Statements that list functions without arguments hold for all pairs \((s, t)\) with \(s \leq t\). Also, we give the convolution operation precedence over the minimum, which allows us to omit some parentheses.
B. Bivariate arrival and service processes

The arrivals to a network element are characterized by a bivariate process $A$, where $A(s,t)$ describes the cumulative arrivals in the time interval $[s,t)$. We use $D(s,t)$ to describe the departures in the time interval $[s,t)$, subject to the causality condition $D(0,t) \leq A(0,t)$. Both $A$ and $D$ are causal functions ($A, D \in \mathcal{F}_o$). If $a_k$ and $d_k$ denote the arrivals and departures, respectively, at time $k$, we have

$$A(s,t) = \sum_{k=s}^{t-1} a_k \quad \text{and} \quad D(s,t) = \sum_{k=s}^{t-1} d_k.$$  

The available service at a network element is described by a bivariate service process $S(s,t) \in \mathcal{F}_o$ that satisfies the input-output relationship

$$D \geq A \otimes S$$

for any arrival process and corresponding departure process at that element. Such a process is called a dynamic server in [7, p. 178]. An exact service process satisfies the input-output relationship with equality. The service offered by a sequence of network elements with service processes $S_1, S_2, \ldots, S_N$ is given by their convolution $S_1 \otimes S_2 \otimes \ldots \otimes S_N$.

Given arrival and service processes, one can formulate bounds on backlog at a network element. The backlog $B$ of a network element denotes the arrivals that have not yet departed, given by $B(t) = A(0,t) - D(0,t)$. A bound on the backlog can be expressed in terms of the deconvolution operation ($\otimes$), which, for two functions $f, g \in \mathcal{F}$, is defined by

$$f \otimes g(s,t) = \max_{0\leq \tau \leq s} \{ f(\tau,t) - g(\tau,s) \} .$$

With this operation we have [9, Theorem 2]

$$B(t) \leq A \otimes S(t,t).$$  

(1)

Bounds on the delay and the burstiness of departures also involve the deconvolution operation.

C. Bivariate feedback systems

A network element with feedback is one where the departures from the element influence the arrivals to the same element. Fig. [2] depicts a generic system-theoretic model of a closed-loop feedback system. There is a network element with service process $F$ whose output is re-combined
with the external input, so that the arrivals to the network element are the minimum of its output and the external arrivals. Such a system will be referred to as feedback server. The feedback server consists of a (non-causal) service element with service process $F(s, t)$ ($F \not\in \mathcal{F}_o$) and arrival and departure functions $D' \geq D \otimes F$. The service element labeled by ‘min’ in Fig. 2 represents a throttle, which enforces the minimum $D = A \wedge D'$. With this, $A$, $D$, and $F$ satisfy $D \geq A \wedge D \otimes F$. As shown in [1], [6] for univariate functions, the closed-loop system can be replaced by an equivalent system, which consists of a single network element without feedback with service process $F^*$, where $F^*$ is the subadditive closure of $F$. In [8], the result has been extended to bivariate functions, as expressed in the following lemma.

**Lemma 1:** (see [8, Lemma 2.2]) Given a feedback server as in Fig. 2 with bivariate functions $A$, $D$, and $F$. If $F \not\in \mathcal{F}_o$ and $D \geq A \wedge D \otimes F$, then $D \geq A \otimes F^*$.

Since the lemma applies when $A$, $D$, and $F$ are random processes, an analysis of stochastic feedback system appears readily available. However, computing the subadditive closure $F^*$ for a bivariate random service process presents considerable difficulties.

### D. MGF network calculus

The MGF network calculus [9] offers an analysis of network elements, when the arrivals and the offered service are bivariate random processes that are characterized in terms of their moment-generating functions. The MGF network calculus has been frequently applied to the analysis of wireless networks, e.g., [2], [10], [16], [18], since the random service model can capture the randomness of a wireless transmission system. We denote the moment-generating function of a random variable $X$ for any $\theta \in \mathbb{R}$ by $M_X(\theta) = E[e^{\theta X}]$. The MGF calculus exploits that for any two independent random variables $X$ and $Y$, the relation $M_{X+Y}(\theta) = M_X(\theta)M_Y(\theta)$ holds.
The stochastic analysis of feedback systems in this paper will take an MGF network calculus approach.

The moment-generating functions of an arrival process \( A \) and a service process \( S \) for \( \theta > 0 \) are denoted by

\[
M_A(\theta, s, t) = E[e^{\theta A(s, t)}] \quad \text{and} \quad M_S(-\theta, s, t) = E[e^{-\theta S(s, t)}].
\]

We assume that the arrival and the service are independent. For characterizing the random service of a feedback system, we will use a bivariate version of the statistical service curve from [5] and the effective capacity from [17]. A statistical service curve \( S^\varepsilon \) of a bivariate service process \( S \) for a given \( \varepsilon > 0 \) [9] is defined by the property that

\[
Pr\left( S(s, t) \leq S^\varepsilon(s, t) \right) \leq \varepsilon. \tag{2}
\]

The statistical service curve is a deterministic function giving a lower bound on the available service that is violated with a probability \( \varepsilon \) or less. Using the Chernoff bound, a statistical service curve can be computed from \( M_S \) as

\[
S^\varepsilon(s, t) = \max_{\theta > 0} \frac{1}{\theta} \left\{ \log \varepsilon - \log M_S(-\theta, s, t) \right\}. \tag{3}
\]

An alternative measure to describe a stochastic service in terms of \( M_S \) is the effective capacity \( \gamma_S(-\theta) \) [17], defined for \( \theta > 0 \) by

\[
\gamma_S(-\theta) = \lim_{t \to \infty} -\frac{1}{\theta t} \log M_S(-\theta, 0, t). \tag{4}
\]

Since it is defined as a time limit, the effective capacity is most useful when reasoning about long-term traffic rates and scaling properties. In particular, \( \gamma_S(0) = \lim_{\theta \to 0} \gamma_S(-\theta) \) equals the average service rate. In contrast, the statistical service curve provides a bound for finite values of \( t \).

Given the moment-generating functions of two bivariate processes \( f \) and \( g \), bounds on the convolution and deconvolution, given in [9], are

\[
M_{f \otimes g}(-\theta, s, t) \leq \sum_{\tau=s}^{t} M_f(-\theta, s, \tau) M_g(-\theta, \tau, t), \tag{5}
\]

\[
M_{f \oslash g}(\theta, s, t) \leq \sum_{\tau=0}^{s} M_f(\theta, \tau, t) M_g(-\theta, \tau, s). \tag{6}
\]
The convolution bound is used to estimate the moment-generating function of a sequence of service elements. The deconvolution bound plays a role when computing performance bounds. For example, with an application of the Chernoff bound, we can obtain from Eq. (1) a bound on the backlog distribution \( Pr(B(t) > b^*(t)) \leq \varepsilon \), where

\[
b^*(t) = \min_{\theta > 0} \frac{1}{\theta} \left\{ \log \left( \sum_{\tau=0}^{s} M_A(\theta, \tau, t)M_S(-\theta, \tau, s) \right) - \log \varepsilon \right\}.
\]

(7)

The delay can be treated in a similar fashion.

III. TOWARDS A STOCHASTIC FEEDBACK ANALYSIS

The generalization of an analysis of a deterministic feedback system to random arrival and service processes has remained open for considerable time. In this section, we present results that make such an analysis possible. We describe the issues that make a stochastic analysis of feedback systems within the framework of the network calculus hard, and then address how to resolve them.

A. Model Description

We will analyze the network with window flow control in Fig. 1. Traffic with arrival process \( A(s, t) \) is serviced by a network element with service process \( S(s, t) \), subject to the additional constraint that the total backlog in the element at any time may not exceed \( w > 0 \). Traffic in excess of that constraint is held in a FIFO buffer at the network entrance. The traffic leaving the network is expressed by a departure function \( D(s, t) \). The feedback information consists of the value of the departure function delayed by \( d \geq 0 \) time slots, i.e., \( D(s, t - d) \). If we use \( A' \) to denote the arrived traffic that is admitted into the network, the flow control system requires that

\[
A'(s, t) = \min \{ A(s, t), D(s, t - d) + w \}.
\]

(8)

This control ensures that admitted traffic that has not yet departed cannot exceed \( w \).

The system can be described by a feedback model as discussed in Subsec. II-C. Let the network service be given by a service process \( S \), i.e.,

\[
D \geq A' \otimes S.
\]

(9)
Further, for $w > 0$ we define a function $\delta^{+w}$ by

$$
\delta^{+w}(s, t) = \begin{cases} 
  w & s \geq t, \\
  \infty & s < t.
\end{cases}
$$

(10)

This allows us to write $f(s, t) + w = f \otimes \delta^{+w}(s, t)$. Note that $\delta^{+w}$ is not a causal function, i.e., $\delta^{+w} > 0$ for $t \leq s$. Even though the convolution of bivariate functions is generally not commutative, we have $f \otimes \delta^{+w} = \delta^{+w} \otimes f$ for a bivariate process $f$. Lastly, a service element offering a delay of $d \geq 0$ is given by a service process $\delta_d = \delta(s, t - d)$, such that $f(s, t - d) = f \otimes \delta_d(s, t)$ for every $f \in \mathcal{F}$. With these definitions we rewrite Eq. (8) as

$$
A' = A \land D \otimes \delta_d \otimes \delta^{+w}.
$$

(11)

In Fig. 3, we illustrate the feedback system as a concatenation of service elements. By inserting Eq. (9) into Eq. (11) we obtain

$$
A' \geq A \land A' \otimes S \otimes \delta_d \otimes \delta^{+w}.
$$

(12)

Note that $A'$ satisfies the conditions of Lemma 1 with $F = S \otimes \delta_d \otimes \delta^{+w}$. Applying the lemma and inserting the result into Eq. (9) yields

$$
D \geq A \otimes (S \otimes \delta_d \otimes \delta^{+w})^* \otimes S.
$$

Therefore, the flow control system can be represented by an equivalent service process $S_{\text{win}}$ given by

$$
S_{\text{win}} = (S \otimes \delta_d \otimes \delta^{+w})^* \otimes S.
$$

(13)

If $S$ is an exact service process, so is $S_{\text{win}}$. 

---

Fig. 3: Model with flow control with window size $w > 0$ and feedback delay $d \geq 0$. 

---
B. Challenges of stochastic feedback systems

The analysis of a feedback system with a bivariate random service process $S$ requires the computation of the subadditive closure in Eq. (13). Let us consider for the moment a system with no feedback delay, that is, $d = 0$. Since $(S \otimes \delta^+ w) \otimes (S \otimes \delta^+ w) = S \otimes S \otimes \delta^{2w}$, the computation of the subadditive closure involves the convolution of $S$ with itself. When the service process is subadditive, we have $S = S \otimes S$, and therefore obtain $(S \otimes \delta^+ w)^* \otimes S = S$. Hence, when a subadditive process $S$ is the service process of a closed-loop system without feedback delay, feedback has no impact on the overall service. On the other hand, a service process with a nonzero feedback delay is generally not subadditive and, therefore, the convolution of such a process will not yield a trivial result. To see that the concatenation of $S$ and $\delta_d$ with $d > 0$ is not subadditive, we compute

$$S \otimes \delta_d(s, \tau) + S \otimes \delta_d(\tau, t) = S(s, \tau - d) + S(\tau, t - d) < S(s, t - d) = S \otimes \delta_d(s, t).$$

To avoid trivial cases, we will henceforth consider feedback systems with $d > 0$.

Writing the expression for the equivalent service process in Eq. (13) as

$$S_{\text{win}} = \bigwedge_{n=0}^{\infty} \left( (S \otimes \delta_d \otimes \delta^+ w)^{(n)} \otimes S \right)$$

makes apparent the need for tools to analyze the distribution of minima involving a bivariate random service process $S$. Note that the $n$-fold convolutions themselves involve minima as well as sums of random processes.

Consider, for comparison, the same formula in the case of an univariate service process $S(t)$. There, one can exploit the commutativity of the convolution to obtain for the $n$-th term in the minimum

$$(S \otimes \delta_d \otimes \delta^+ w)^{(n)} \otimes S(t) = S^{(n+1)} \otimes \delta^{nw} \otimes \delta_{nd}(t) = S^{(n+1)}(t - nd) + nw.$$

In particular, if $S$ itself is subadditive, then $S^{(n)} = S$, the $n$-th term is $S(t - nd) + nw$, and the entire minimum reduces to $S_{\text{win}} = S \otimes S_o$, where $S_o(t) = w \left\lfloor \frac{t}{d} \right\rfloor$. Since deterministic feedback systems can generally be expressed using univariate service processes, the computation of the
subadditive closure of the service process is much simplified. However, for bivariate service processes, the convolution is not commutative, and we must compute the $n$-th term as

$$\left(S \otimes \delta_d \otimes \delta^+ w \right)^{(n)} \otimes S(s, t) = \min_{\tau_n \leq \cdots \leq \tau_0} \left\{ \sum_{i=1}^{n} \left( S(\tau_{i-1}, \tau_i - d) + S(\tau_n, t) \right) \right\} + nw, \tag{15}$$

where the minimum ranges over all non-decreasing sequences $\tau_0, \ldots, \tau_n$ with $\tau_0 = s$ and $\tau_n \leq t$. One issue with this expression is that the number of terms grows rapidly with $n$ and $t - s$. The second difficulty is that the joint distribution of $S(\tau_{i-1}, \tau_i - d)$ for $i = 1, \ldots, n$ is not determined by the distribution of $S(s, t)$ alone, but requires information on time correlations. We now proceed to address these problems.

C. Service processes of stochastic feedback systems

Our first result provides an exact characterization of $S_{\text{win}}$ for the special case $d = 1$.

**Lemma 2:** Consider a feedback system as in Fig. 3 with an additive service process

$$S(s, t) = \sum_{k=s}^{t-1} c_k,$$

where $(c_k)_{k \geq 1}$ is an arbitrary sequence of non-negative random variables. For $d = 1$ and $w > 0$, the equivalent service process is given by

$$S_{\text{win}}(s, t) = \sum_{k=s}^{t-1} \min \{c_k, w\}.$$

**Proof:** Set $d = 1$. For $n = 1$, the convolution in Eq. (15) is given by

$$S \otimes \delta_1 \otimes \delta^+ w \otimes S(s, t) = \min_{s \leq \tau \leq t} \{ S(s, \tau - 1) + S(\tau, t) \} + w$$

$$= \min_{s \leq \tau \leq t} \left\{ \sum_{k=s}^{\tau-2} c_k + \sum_{k=\tau}^{t-1} c_k \right\} + w$$

$$= S(s, t) - \max_{k \in [s, t)} c_k + w.$$

Here, the convolution replaces the largest value in the sum representing $S(s, t)$ by $w$. Likewise, for $n > 1$, the minimum in Eq. (15) is obtained by replacing the $n$ largest values of $c_k$ on $[s, t)$ with $w$. In this case, the minimum in Eq. (14) is attained for $n = \# \{ k \in [s, t) \mid c_k > w \}$.
The following theorem shows that the exact result from Lemma 2 provides a lower bound on $S_{\text{win}}$ for $d > 1$ in terms of the ratio $w/d$. The theorem also provides a complementary upper bound on $S_{\text{win}}$.

**Theorem 1:** Given a service process $S \in F_o$. Let $S_{\text{win}}$ be the equivalent service process for a feedback system with parameters $d > 0$ and $w > 0$. Then

$$S'(s, t) \leq S_{\text{win}}(s, t) \leq \min \left\{ S(s, t), \left\lceil \frac{t-s}{d} \right\rceil w \right\} , \quad 0 \leq s \leq t ,$$

(16)

where $S'$ is the equivalent service process (given by Eq. (14)) with $d' = 1$ and $w' = w/d$.

**Proof:** For the lower bound, observe that

$$\delta_d \otimes \delta^{+w} = (\delta_1 \otimes \delta^{+w'})^{(d)} .$$

(17)

This implies

$$S \otimes \delta_d \otimes \delta^{+w}(s, t) \geq \min_{s = \tau_0 \leq \cdots \leq \tau_d = t} \sum_{i=1}^{d} \left( S(\tau_{i-1}, \tau_i - 1) + w' \right)$$

$$= \left( S \otimes \delta_1 \otimes \delta^{+w'} \right)^{(d)} (s, t) .$$

Indeed, the left-hand side appears as a term in the minimum on the right-hand side. Therefore, the $n$-th term in the expression for $S_{\text{win}}$ in Eq. (14) is bounded from below by the $(nd)$-th term in the expression for $S'$.

For the upper bound, we simply use just two terms from the minimum in Eq. (14) to bound $S_{\text{win}}$, namely $n = 0$ and $n = \left\lceil \frac{t-s}{d} \right\rceil$.

An implication of the theorem is that if $S$, $S_{\text{win}}$, and $S'_{\text{win}}$ have long-term average rates $C$, $C_{\text{win}}$, and $C'_{\text{win}}$, then

$$C'_{\text{win}} \leq C_{\text{win}} \leq \min \left\{ C, \frac{w}{d} \right\} .$$

Theorem 1 holds for general (deterministic or random) service processes. The bounds on the average rates depend only on the ratio $w/d$, not on the values of the individual parameters. Our main results, in Theorems 2 and 3, will strengthen the lower bound when $d > 1$ for two important classes of additive service processes. They rely on the next lemma, which reduces the number of terms that contribute to Eq. (15) and limits the range of the minimum in Eq. (14) to $n \leq \left\lceil \frac{t-s}{d} \right\rceil$.

12
Lemma 3: Given a feedback system with a service process $S \in \mathcal{F}_o$. Then, for any choice of $d > 0$ and $w > 0$,

$$S_{\text{win}}(s, t) = \bigwedge_{n=0}^{\lceil \frac{t-s}{d} \rceil} \left\{ \min_{\tau_0, \ldots, \tau_n \in C_n(s, t)} \left( \sum_{i=1}^{n} S(\tau_{i-1}, \tau_i - d) + S(\tau_n, t) \right) + nw \right\},$$

(18)

where the minimum in the braces ranges over

$$C_n(s, t) = \left\{ \tau_0, \tau_1, \ldots, \tau_n \mid \tau_0 = s, \tau_n \leq t, \text{ and } \tau_i - \tau_{i-1} \geq d \text{ for } i = 1, \ldots, n \right\},$$

if $nd \leq t - s$. If $nd > t - s$, then $C_n(s, t)$ contains the single sequence $\tau_i = (s + id) \wedge t$ for $i = 0, \ldots, n$.

The set $C_n(s, t)$ has a geometric interpretation, illustrated in Fig. 4. Each sequence $\tau_0, \ldots, \tau_n$ in $C_n(s, t)$ corresponds to a collection of $n$ disjoint subintervals of length $d$ in $[s, t]$, given by $(\tau_i - d, \tau_i]$. On each of these subintervals, the original service process $S(\tau_i - d, \tau_i]$ is interrupted by a delay of length $d$, followed by an addition of $w$. In the special case $d = 1$, the set $C_n(s, t)$ consists precisely of the $n$-element subsets of $[s, t)$, and we recover the statement of Lemma 2.

Proof: We need to prove that only sequences $\tau_0, \ldots, \tau_n$ in $C_n(s, t)$ contribute to the minimum in Eq. (15). If $nd > t - s$, the minimum value is achieved by subdividing $[s, t]$ into $n$ subintervals of length at most $d$, so that all terms involving $S$ vanish. Then the right-hand side of Eq. (15) equals $nw \geq \left\lceil \frac{t-s}{d} \right\rceil w$.

For $nd \leq t - s$, we proceed by induction over $n$. The $n = 0$ term in both Eq. (14) and Eq. (18) is given by $S(s, t)$ and there is nothing to show. The $n = 1$ term equals

$$S \otimes \delta_d \otimes \delta^+ w \otimes S(s, t) = \min_{s \leq \tau \leq t} \left\{ S(s, \tau - d) + S(\tau, t) \right\} + w.$$

(19)

The key observation is that the expression in the braces is non-increasing in $\tau$ for $s \leq \tau \leq d$, since the first term vanishes while the second term is always non-increasing. Therefore, the
minimum is achieved for some $\tau \geq s + d$. This means that the sequence $\tau_0 = s, \tau_1 = \tau$ lies in $C_1(s,t)$.

Now consider $n \geq 1$. Suppose we already know that

$$\left( S \otimes \delta_d \otimes \delta^{+w} \right)^{(n)} \otimes S(s,t) = \min_{\tau_0, \ldots, \tau_n \in C_n(s,t)} \left( S(\tau_{i-1}, \tau_i - d) + S(\tau_n, t) \right) + nw$$

for all $t \geq s$. Using the associativity of the convolution, we write

$$\left( S \otimes \delta_d \otimes \delta^{+w} \right)^{(n+1)} \otimes S = \left[ \left( S \otimes \delta_d \otimes \delta^{+w} \right)^{(n)} \otimes S \right] \otimes \delta_d \otimes \delta^{+w} \otimes S,$$

and then expand

$$\left( S \otimes \delta_d \otimes \delta^{+w} \right)^{(n+1)} \otimes S(s,t) = \min_{s \leq \tau \leq t} \left\{ \left( S \otimes \delta_d \otimes \delta^{+w} \right)^{(n)} \otimes S \right\} (s, \tau - d) + S(\tau, t) + w.$$

For $\tau - d \leq s + nd$, the term in the square brackets takes the constant value $nw$, while the second summand is non-increasing in $\tau$. Therefore, the minimum occurs at some $\tau \geq s + (n+1)d$. By the inductive assumption,

$$\left( S \otimes \delta_d \otimes \delta^{+w} \right)^{(n)} \otimes S(s,t) = \min_{\tau_0, \ldots, \tau_n \in C_n(s,\tau - d)} \left\{ \sum_{i=1}^{n} \left( S(\tau_{i-1}, \tau_i - d) + S(\tau_n, \tau - d) \right) + nw \right\}.$$

Since $\tau_0, \ldots, \tau_n, \tau \in C_{n+1}(s,t)$, the claim is proved.

### IV. Variable Bit Rate Service with Feedback

We next apply the results from the previous section to a specific random service process in a feedback system, consisting of a work-conserving FIFO buffer with a random time-variable service rate, which we refer to as variable bit rate (VBR) server. The feedback mechanism is as described earlier with window size $w > 0$ and feedback delay $d > 0$. The VBR server offers the service process $S(s,t) = \sum_{k=s}^{t-1} c_k$, where $c_k$ is a random amount of available service in the $k$-th time slot. We assume that the $c_k$’s are independent and identically distributed (i.i.d.) random variables, with moment-generating functions $M_c(\theta) = E[e^{\theta c_k}]$. The moment-generating function of $S$ is $M_S(\theta, s, t) = (M_c(\theta))^{t-s}$, and the effective capacity has the simple expression $\gamma_S(-\theta) = -\frac{1}{\theta} \log M_c(-\theta)$. 
A. Bounds for a VBR service with feedback

We now derive bounds on the equivalent service process $S_{\text{win}}$ of a VBR server in the feedback system of Fig. 3. The bounds will be expressed either in terms of a statistical service curve for a chosen violation probability $\varepsilon > 0$ (using Eq. (3)) or in terms of the effective capacity (from Eq. (4)), both of which require bounds on the moment-generating function of $S_{\text{win}}$, denoted by $M_{S_{\text{win}}}$. We use the expression for $S_{\text{win}}$ given in Eq. (14) as the starting point for the computation of $M_{S_{\text{win}}}$, where we exploit the following relationship of moment-generating functions.

**Lemma 4:** Given two bivariate random processes $f$ and $g$. Then, for every $\theta > 0$,

$$M_{f \wedge g}(-\theta, s, t) \leq M_f(-\theta, s, t) + M_g(-\theta, s, t).$$

Note that this lemma does not require $f$ and $g$ to be independent.

**Proof:**

$$M_{f \wedge g}(-\theta, s, t) = E[e^{-\theta \min\{f(s,t), g(s,t)\}}]$$

$$= E[\max\{e^{-\theta f(s,t)}, e^{-\theta g(s,t)}\}]$$

$$\leq E[e^{-\theta f(s,t)} + e^{-\theta g(s,t)}]$$

$$= M_f(-\theta, s, t) + M_g(-\theta, s, t).$$

Since the service guaranteed by $S$ is independent on disjoint intervals, that is $S(t_1, t_2)$ and $S(t_3, t_4)$ are independent for $t_1 < t_2 \leq t_3 < t_4$, standard techniques for bounding the moment-generating functions of sums and convolutions can be applied directly to Eq. (14) to obtain, for $\theta > 0$,

$$M_{S_{\text{win}}}(-\theta, s, t) \leq \sum_{n=0}^{\infty} \left\{ \sum_{s=\tau_0 \leq \ldots \leq \tau_n \leq t} \left( \prod_{i=1}^{n} M_S(-\theta, \tau_{i-1}, \tau_i - d) \right) \cdot M_S(-\theta, \tau_n, t)e^{-\theta nw} \right\}$$

$$\leq (M_c(-\theta))^{t-s} \sum_{n=0}^{\infty} \binom{t-s+1+n}{n} ((M_c(-\theta))^{d}e^{-\theta w})^n$$

$$= \frac{(M_c(-\theta))^{t-s}}{(1 - (M_c(-\theta))^{-d}e^{-\theta w})^{t-s+2}},$$

as long as the convergence condition

$$M_c(-\theta)^{-d}e^{-\theta w} < 1$$

15
holds. The first line follows by Lemma 4 and the independence of the service on disjoint intervals in each summand. In the second line, $M_S$ is expressed in terms of $M_c$; the binomial coefficient counts the number of non-decreasing sequences $\tau_0 \leq \ldots \leq \tau_n$ with $\tau_0 = s$ and $\tau_n \leq t$. The last line follows from the identity
\[
\sum_{n=0}^{\infty} \binom{t+n}{n} x^n = \frac{1}{(1-x)^{t+1}}, \quad (|x| < 1).
\]

Eq. (20) provides a useful bound only when $\theta$ is chosen so that the convergence condition is satisfied, and the right-hand side of Eq. (20) is less than one. We now improve the bound with the help of Lemma 3.

**Theorem 2:** Let $S(s, t)$ be a VBR server with feedback as described above. Then, for every $\theta > 0$,
\[
M_{S_{\text{win}}}(-\theta, s, t) \leq \left( (M_c(-\theta))^d + de^{-\theta w} \right)^{\left\lfloor \frac{t-s}{d} \right\rfloor}.
\]

**Proof:** Fix $\theta > 0$. Since $S_{\text{win}}(s, t)$ is non-decreasing in $t$, we can round down the length of the time interval to the nearest integer multiple of $d$. Thus, it suffices to consider the case where $t - s = Nd$ for some integer $N$. By Lemma 3,
\[
S_{\text{win}}(s, t) = \bigwedge_{n=0}^{N} \left\{ \min_{\tau_0, \ldots, \tau_n \in \mathcal{C}_n(s, t)} \sum_{i=1}^{n} \left( S(\tau_{i-1}, \tau_i - d) + S(\tau_n, t) \right) + nw \right\}.
\]

By Lemma 4 and using that $S$ is independent on disjoint time intervals, we estimate
\[
M_{S_{\text{win}}}(-\theta, s, t) \leq \sum_{n=0}^{N} \left\{ \sum_{\tau_0, \ldots, \tau_n \in \mathcal{C}_n(s, t)} \left( \prod_{i=1}^{n} M_S(-\theta, \tau_{i-1}, \tau_i - d) \right) M_S(-\theta, \tau_n, t)e^{-\theta nw} \right\}.
\]

The number of sequences in $\mathcal{C}_n(s, t)$ is bounded by
\[
|\mathcal{C}_n(s, t)| = \binom{(N-n)d+n}{n} = \prod_{j=1}^{n} \left( \frac{(N-n)d+j}{j} \right) \leq \prod_{j=1}^{n} \left( \frac{(N-n)d+dj}{j} \right) = \frac{d^n}{n!} \binom{N}{n}.
\]
Expressing $M_S$ in terms of $M_c$, we arrive at
\[
M_{S_{\text{win}}}(-\theta, s, t) \leq \sum_{n=0}^{N} |\mathcal{C}_n(s, t)| \left( (M_c(-\theta))^{(N-n)d} \right) e^{-\theta nw} \leq \sum_{n=0}^{N} \left( \frac{N}{n} \right) \left( (M_c(-\theta))^d \right)^{N-n} \left( de^{-\theta w} \right)^n = \left( (M_c(-\theta))^d + de^{-\theta w} \right)^N.
\]

16
Since \( N = \frac{t-s}{d} \), the claim is proved.

Theorem 2 (as well as Eq. (20)) can be directly inserted into expressions of the MGF calculus, e.g., for the backlog expression in Eq. (7). Further, via Eq. (5), the theorem also provides a statistical service curve for \( S_{\text{win}} \), which we will denote as \( S_{\text{win}}^c \). By taking the logarithm of \( M_{S_{\text{win}}} \), we can obtain bounds on the effective capacity \( \gamma_{\text{win}} \), as expressed in this corollary.

**Corollary 1:** Lower bounds on the effective capacity \( \gamma_{\text{win}} \) of a VBR process \( S(s,t) \) with feedback are given for \( \theta > 0 \) by

\[
\begin{align*}
\text{a)} \quad \gamma_{\text{win}}(-\theta) & \geq \gamma_S(-\theta) + \frac{1}{\theta} \log \left( 1 - e^{\theta(d\gamma_S(-\theta)-w)} \right), \\
\text{b)} \quad \gamma_{\text{win}}(-\theta) & \geq \gamma_S(-\theta) - \frac{1}{d\theta} \log \left( 1 + de^{\theta(d\gamma_S(-\theta)-w)} \right).
\end{align*}
\]

Eqs. (22) and (23) clearly express the service impediment due to the feedback process, by subtracting a positive term from the available service without feedback. In Subsec. [IV-C] we present numerical examples that evaluate both bounds.

All results in this section can be applied to a ‘leftover service’ model at a server offering a constant-rate service in the presence of cross-traffic with independent increments. The leftover service expresses the service available to a flow in terms of the capacity that is left unused by competing cross-traffic. The leftover model assumes that the analyzed traffic flow has lower priority than the cross-traffic. Explicitly, if the service rate is \( C \) and cross-traffic arrivals are given by

\[
A^c(s,t) = \sum_{k=s}^{t-1} a^c_k,
\]

where the cross-traffic arrivals in each time slot are given by an i.i.d. sequence of random variables \( a^c_k \), then the leftover service \( S^{\text{lo}} \) available to the flow satisfies

\[
S^{\text{lo}}(s,t) \geq \sum_{k=s}^{t-1} (C - a^c_k).
\]

Although the summands \( C - a^c_k \) may take negative values, Theorems 1 and 2 remain valid and provide non-trivial bounds on the service process with feedback, as long as the stability condition \( E[a^c_k] < C \) is satisfied.
B. Quality of the VBR bounds

We next use Theorem 1 to address the accuracy of the derived lower bounds from Subsec. IV-A. Applying the theorem to the VBR server gives the bounds

\[ \sum_{k=s}^{t-1} \min \{ c_k, \frac{w}{d} \} \leq S_{\text{win}}(s, t) \leq \min \left\{ \sum_{k=s}^{t-1} c_k, \left\lfloor \frac{t-s}{d} \right\rfloor w \right\}. \] (24)

Even though the upper bound is optimistic, its difference to the (lower) bounds on the service computed from Theorem 2 limits the deviation from the true value of \( S_{\text{win}} \). Moreover, the difference between the upper and lower bounds in Eq. (24) indicates the range of useful estimates.

Note that the lower bound in Eq. (24) is exact for the special case \( d = 1 \). In particular, any lower bound that falls below this bound can be replaced by the simpler estimate of Eq. (24).

Let us, for the moment, consider a deterministic server with \( c_k \equiv C \). Then, the lower and upper bounds are essentially equivalent, giving \( S_{\text{win}}(s, t) \approx \min \{ C, \frac{w}{d} \} (t-s) \). This corresponds to a well-known expression for the throughput of a window flow control system, e.g., [4, Eq. 6.1]. On the other hand, when \( S \) is random, the upper and lower bounds have different long-term average rates.

We will use Eq. (24) to obtain upper and lower bounds on \( S_{\text{win}}^\varepsilon \) and \( \gamma_{\text{win}} \). We obtain a lower bound on \( S_{\text{win}}^\varepsilon \) by taking the moment-generating function of the lower bound in Eq. (24), resulting for \( \theta > 0 \) in

\[ M_{S_{\text{win}}}(-\theta, s, t) \leq \left( E[e^{-\theta \min \{ c_k, \frac{w}{d} \}}] \right)^{t-s}. \] (25)

Note the change of direction of the inequality since we use \( -\theta \) as a function argument. Inserting this bound into Eq. (3) provides a lower bound on \( S_{\text{win}}^\varepsilon(s, t) \).

Obtaining an opposing upper bound on \( S_{\text{win}}^\varepsilon \) from Eq. (24) is not as straightforward, since statistical service curves express lower bounds on the available service. We exploit that, for some VBR servers, it is possible to get the exact distribution of \( S(s, t) = \sum_{k=s}^{t-1} c_k \), where \( S \) corresponds to the service process of the VBR system without feedback. For example, in Subsec. IV-C, where we use an exponentially distributed \( c_k \), the service \( S(s, t) \) has an Erlang distribution. In such cases, since the upper bound in Eq. (24) implies \( S_{\text{win}} \leq S \), the \( \varepsilon \)-quantile of \( S \) is an upper bound for any statistical service curve \( S_{\text{win}}^\varepsilon \) of the VBR server with feedback. Thus, Eq. (24) gives

\[ S_{\text{win}}^\varepsilon(s, t) \leq \min \{ \varepsilon \text{-quantile of } S(s, t), \left\lfloor \frac{t-s}{d} \right\rfloor w \}, \] (26)
where we also use the second (deterministic) term of the upper bound in Eq. (24).

Bounds for the effective capacity are directly obtained from Eq. (24) by first computing moment-generating functions and then taking the logarithm.

**Corollary 2:** Under the assumptions of Theorem 2, $\gamma_{\text{win}}(-\theta)$ is bounded for $\theta > 0$ by

$$-\frac{1}{\theta} \log E[e^{-\theta \min\{c_k, w/d\}}] \leq \gamma_{\text{win}}(-\theta) \leq \min\{\gamma_S(-\theta), \frac{w}{d}\}.$$  

The lower bound becomes an equality when $d = 1$. The upper bound is strictly larger than the lower bound for all values of $\theta > 0$. Most importantly, the upper bound becomes sharp if we take $d \to \infty$ while holding $w/d$ fixed. In the deterministic case, with $c_k \equiv C$, the upper and lower bounds both equal $\min\{C, \frac{w}{d}\}$, in accordance with our previous discussion. Moreover, in the limit $\theta \to \infty$, the right-hand side of Eq. (23) is asymptotic to the shown bounds. Therefore, the bounds on the effective capacity are sharp in the limit $\theta \to \infty$.

**C. Numerical evaluation of VBR bounds**

We now present a numerical evaluation of our bounds for the VBR server with feedback. We consider a VBR server with an exponential distribution with moment-generating function and effective capacity given by

$$M_S(-\theta, s, t) = (1 + C\theta)^{-(t-s)}, \quad \gamma_S(-\theta) = \frac{1}{\theta} \log(1 + C\theta).$$

We select $C = 1$ Mb for the available service in a time slot of length 1 ms, which gives an average service rate of 1 Gbps.

In Figs. 5(a) and 5(b) we plot statistical service curves $S_{\text{win}}^\varepsilon(0, t)$ as functions of time, where we use a violation probability of $\varepsilon = 10^{-6}$. We compute multiple service curves where we vary the delay $d$ and the window size $w$, but fix the ratio $w/d$. The statistical service curves use Eq. (3) with the bound on $M_{\text{win}}$ from Theorem 2. We compare these service curves to the upper and lower bounds computed from Eq. (24).

Fig. 5(a) shows statistical service curves, plotted as solid lines, using a ratio $w/d = 100$ Mbps. The upper and lower bounds are represented by dash-dotted lines. We also indicate the two terms in the upper bound of Eq. (26) by dotted lines. Note that the lower bound is in fact equal to $S_{\text{win}}^\varepsilon$ for the special case $d = 1$ ms, due to Lemma 2.

It is evident that the rates of the service curves match well to those of the upper and lower bounds. We observe that increasing the delay and the window size simultaneously improves the
Fig. 5: Statistical service curves $S_{\text{win}}^\varepsilon(0, t)$ for VBR server with feedback (Avg. rate: 1 Gbps, $\varepsilon = 10^{-6}$). The dotted lines correspond to the two terms of the upper bound in Eq. (26).

available service. For large values of $d$, the service curves appear to have a supremum well below the plotted upper bound. This is expected, since the rates of our bounds are exact for $d \to \infty$. In a deterministic feedback system with a fixed rate of 1 Gbps, varying $w$ and $d$ with
a fixed ratio $w/d$ results in all cases in an essentially constant rate service of 100 Mbps, with minuscule deviations.

Fig. 6(b) evaluates the same scenario for a different parameter selection, this time, fixing $w/d = 500$ Mbps. Note that the range of the y-axis is modified from Fig. 6(a). For the shown
range of time values, the rate of the statistical service curves are close to that of the lower bound, but noticeably smaller than that of the upper bound. This will change when we consider larger time intervals. The reason is that with this choice of \( w/d \), the upper bound of the service curves is dominated for a longer period of time by the first term of Eq. (26). Once the second term (with rate \( w/d \)) governs the bound, the rate of the statistical service curves will be close to \( w/d \) as well.

We now turn to the effective capacity. For the same set of parameters as before, we evaluate in Fig. 6 the lower bounds of the effective capacity \( \gamma_{\text{win}} \) from Corollary 1. Note that the corollary presents two bounds. Corollary 1(a) (Eq. (22)) does not take advantage of Theorem 2 whereas Corollary 1(b) (Eq. (23)) involves Theorem 2. We plot these bounds on \( \gamma_{\text{win}}(\theta) \) as a function of \( \theta > 0 \). Recall that the actual (not estimated) effective capacity \( \gamma_{\text{win}}(\theta) \) is a decreasing function of \( \theta \), and that its value for \( \theta \to 0 \) is the average service rate.

Fig. 6(a) depicts the bounds for \( w/d = 100 \) Mbps, and Fig. 6(b) those for \( w/d = 500 \) Mbps. Consider first the lower and upper bounds from Corollary 2, which are indicated by dash-dotted lines. As discussed in Sec. IV-B, the bounds converge for large values of \( \theta \). The figure indicates that the convergence occurs early. We observe that bounds from Corollary 1(a) are inferior to those of Corollary 1(b), which emphasizes the value of applying Theorem 2. For \( d > 1 \) ms, the lower bounds of Corollary 1(b) for different values of \( w \) and \( d \) are accurate when \( \theta > 4 \cdot 10^{-5} \) (Fig. 6(a)) and \( \theta > 0.5 \cdot 10^{-5} \) (Fig. 6(b)), but degrade for small values of \( \theta \). When this happens the lower bound from Corollary 2 should be used. For \( d = 1 \) ms, the bounds from Corollary 1(b) are pessimistic over a large range. Here, the lower bound from Corollary 2, which is exact when \( d = 1 \) ms, is the better result. Since the actual \( \gamma_{\text{win}}(\theta) \) increases when reducing \( \theta \), the best estimates of the average available service of \( S_{\text{win}} \) are obtained at the maximum of the curves. It is interesting to observe that Corollary 1 generates bounds which are not always monotonic when increasing \( d \).

Our last numerical example presents a backlog analysis of the VBR server with feedback, by applying Eq. (7). For the arrivals, we select a process similar to the service process, where arrivals in each time slot are i.i.d. with an exponential distribution that generates arrivals at an average rate of \( \lambda \) Mbps. The moment-generating function of this arrival process, denoted by
Fig. 7: Backlog bounds for VBR server with feedback ($C = 1$ Gbps).

$M_A$, is given by

$$M_A(\theta, s, t) = \frac{1}{(1 - \lambda\theta)^{t-s}}.$$  

The service is the same VBR server as evaluated before. By setting $d = 1$ ms, we can take advantage of the exact expression for $M_{S\text{win}}$ that follows from Lemma 2. Even though we have exact moment-generating functions for arrivals and service, the backlog bound of the MGF network calculus will be pessimistic, due to the deconvolution expression in Eq. (6). In Fig. 7 we depict backlog bounds for the ratios $w/d = 100$ Mbps and $500$ Mbps, for different violation probabilities $\varepsilon$, as functions of the arrival rate $\lambda$. We note that this presents the first probabilistic backlog bounds of a random feedback system using methods of the network calculus. We observe that for each choice of $w$, the system saturates at a well-defined rate. The saturation rates are close to the lower bounds on $\gamma_{\text{win}}(-\theta)$ for $\theta \to 0$ in Fig. 6. The plots indicate a low sensitivity of the bounds to $\varepsilon$. We also include simulation results for $\varepsilon = 10^{-6}$. A comparison with the simulations shows that the achieved backlog bounds are pessimistic, but track the blow up of the backlog at high utilizations well. The analytical bounds will be more pessimistic for $\varepsilon = 10^{-3}$ and less so for $\varepsilon = 10^{-9}$.

2The simulations represents runs of $10^9$ time slots, where the simulation is started with an empty system, and results from first $10^5$ time slots are discarded.
V. SERVICE PROCESSES WITH POSITIVE TIME CORRELATIONS

The VBR server from the previous section offered an i.i.d. service in each time slot, and the time correlations of the feedback system resulted exclusively from the feedback mechanisms. Nonetheless, the analysis of this simple feedback system proved to be substantial. This raises the question whether feedback systems with more complex service processes are at all tractable with our analysis approach. In this section, we provide a positive answer, by analyzing a feedback system for a service process with memory.

The server is represented by a Markov-modulated On-Off (MMOO) process [7], which operates in two states. In the ON state (state 1) the server is transmitting a constant amount of $P > 0$ units of traffic per time slot. In the OFF state (state 0), the server does not transmit any traffic. The state is selected at the beginning of each time slot using fixed transition probabilities, where $p_{ij}$ denotes the probability of moving from state $i$ to state $j$ ($i, j \in \{0, 1\}$). By definition, the service process is additive,

$$S(s, t) = \sum_{k=s}^{t-1} c_k,$$

where $c_k = P$ if the system is in the ON state at time $k$, and $c_k = 0$ otherwise. Since $p_{01} = 1 - p_{00}$ and $p_{10} = 1 - p_{11}$, the system is fully characterized by the three parameters, $p_{00}$, $p_{11}$, and $P$. The Markov chain is assumed to be in its steady-state, where the probability that the system is ON in any given time slot equals $Pr(ON) = \frac{p_{01}}{p_{01} + p_{00}}$. MMOO processes are frequently used in the literature for modelling bursty traffic or service [14]. The average rate of the MMOO process is

$$E[S(s, t)] = \frac{p_{01}}{p_{01} + p_{10}} P.$$

Its effective capacity is given in [7, Eq. (7.18)] as

$$\gamma_S(-\theta) = -\frac{1}{\theta} \log \left( \frac{1}{2} \left\{ (p_{00} + p_{11} e^{-\theta P}) + \sqrt{(p_{00} + p_{11} e^{-\theta P})^2 - 4(p_{00} + p_{11} - 1) e^{-\theta P}} \right\} \right) \tag{27}$$

for all $\theta \geq 0$.

We assume throughout this section that the transition probabilities satisfy $p_{01} + p_{10} < 1$. This condition ensures that the system does not alternative rapidly between the two states. Smaller values of $p_{01}$ cause the system to linger in the OFF state, while smaller values of $p_{10}$ cause extended bursts.
A. Time-correlation properties of MMOO processes

The analysis of an MMOO server with feedback requires a deeper inspection of the properties of an MMOO process. The following lemma concerns the time correlations of the underlying two-state Markov chain. In this subsection only, we allow the parameter $\theta$ to take both positive and negative values.

**Lemma 5:** Consider a two-state Markov chain with $p_{01} + p_{10} < 1$. For every strictly increasing sequence $\tau_1, \ldots, \tau_n$, the probability that the system is in the ON state at times $\tau_1, \ldots, \tau_n$ is a decreasing function of the differences $\tau_i - \tau_{i-1}$, $i = 2, \ldots, n$.

**Proof:** Consider the state variables

$$X_k = \begin{cases} 1 & \text{if the state is ON at time } k, \\ 0 & \text{else}. \end{cases}$$

The product $\prod_j X_{\tau_j}$ is the indicator of the event that the system is in the ON state at each time $\tau_i$, for $i = 1, \ldots, n$. By the Markov property,

$$Pr(\text{ON at time } \tau_i \text{ for all } i = 1, \ldots, n) = E\left[\prod_{i=1}^n X_{\tau_i}\right] = Pr(X_{\tau_1} = 1) \cdot \prod_{i=2}^n E\left[X_{\tau_i} \mid X_{\tau_{i-1}} = 1\right].$$

By stationarity, the leading factor is a constant determined by the steady state of the Markov chain, and the $i$-th factor in the product depends on $\tau_i - \tau_{i-1}$. We now verify that each of these factors decreases with $\tau_i - \tau_{i-1}$.

Let $p = Pr(X_k = 1)$ be the probability that the state is ON at time $k$, and let $\tau = \tau_i - \tau_{i-1}$. We compute the $\tau$-step transition matrix as

$$\left(\begin{array}{cc} p_{00} & p_{01} \\ p_{10} & p_{11} \end{array}\right)^\tau = \left(\begin{array}{cc} (1-p) & p \\ (1-p) & p \end{array}\right) + \mu^\tau \left(\begin{array}{cc} p & -p \\ -(1-p) & (1-p) \end{array}\right),$$

where the first matrix on the right-hand side is the spectral projection onto the steady state, $\mu = 1 - p_{01} - p_{10}$ is the non-trivial eigenvalue of the transition matrix, and the second matrix is the spectral projection onto the eigenstate corresponding to $\mu$. Since $0 < \mu < 1$,

$$E\left[X_{\tau_i} \mid X_{\tau_{i-1}} = 1\right] = p + (1-p)\mu^\tau.$$
is a decreasing function of $\tau$, proving the claim. □

The next lemma provides bounds on the moment-generating function of the MMOO service process.

**Lemma 6:** Let $S$ be an MMOO process as described above. If $p_{01} + p_{10} < 1$, then the following inequalities hold for all $\theta \in \mathbb{R}$:

1) For every strictly increasing sequence $\tau_1 < \cdots < \tau_n$,

$$
E \left[ e^{\theta \sum_{i=1}^n c_{\tau_i}} \right] \leq M_S(\theta, 0, n). \quad (29)
$$

2) $M_S(\theta, 0, t)$ is supermultiplicative in $t$,

$$
M_S(\theta, 0, s) \cdot M_S(\theta, 0, t) \leq M_S(\theta, 0, s + t) \quad (\forall s, t \geq 0). \quad (30)
$$

3) The moment-generating function is bounded by

$$
(M_c(\theta))^t \leq M_S(\theta, 0, t) \leq m_+(\theta)^t \quad (\forall t \geq 0), \quad (31)
$$

where $M_c(\theta) = (1-p) + pe^{\theta P}$ is the moment-generating function of the service in a single time slot, and where $m_+(\theta)$ is the larger eigenvalue of the matrix

$$
L(\theta) = \begin{pmatrix}
p_{00} & p_{01} \\
p_{10} & p_{11}
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
0 & e^{\theta P}
\end{pmatrix}.
$$

(32)

4) Furthermore,

$$
M_S(\theta, 0, t) \geq K(\theta)(m_+(\theta))^t, \quad (33)
$$

where $0 < K(\theta) < 1$ is an explicit constant (to be computed in the proof).

**Proof:**

1) Consider first the case $\theta \geq 0$, and let $X_k$ be the indicator function that the system is in the ON state at time $k$. Writing

$$
c_k = P X_k, \quad e^{\theta c_k} = 1 + (e^{\theta P} - 1)X_k,
$$

we expand

$$
e^{\theta \sum_{i=1}^n c_{\tau_i}} = \prod_{i=1}^n (1 + (e^{\theta P} - 1)X_{\tau_i}) = \sum_{J \subseteq \{\tau_1, \ldots, \tau_n\}} (e^{\theta P} - 1)^{|J|} \prod_{j \in J} X_j.
$$
For each subset $J \subset \{\tau_1, \ldots, \tau_n\}$, the distance between consecutive elements increases with the distances $\tau_i - \tau_{i-1}$ for $i = 2, \ldots, n$, and all coefficients are positive. Therefore, we can apply Lemma 5 to see that
\[
E \left[ e^{\theta \sum_{i=1}^n c_{\tau_i}} \right] = \sum_{J \subset \{\tau_1, \ldots, \tau_n\}} \left( e^{\theta P} - 1 \right)^{|J|} E \left[ \prod_{j \in J} X_j \right]
\]
is a decreasing function of $\tau_i - \tau_{i-1}$. Since these differences take the smallest possible value when $\{\tau_1, \ldots, \tau_n\} = \{0, \ldots, n - 1\}$, this proves the claim for $\theta \geq 0$.

For $\theta < 0$, let $Y_k = 1 - X_k$ be the indicator function that the state is OFF at time $k$, and set $\phi = -\theta > 0$. We write
\[
e^{\theta c_k} = e^{-\phi P(1-Y_k)} = e^{-\phi P} \left( 1 + (e^{\phi P} - 1)Y_k \right),
\]
and argue as in the other case.

2) Fix $\theta \in \mathbb{R}$. For integers $\ell \geq 0$, let $f(\ell) = E \left[ e^{\theta(S(0,s)+S(s+\ell,t+s+\ell))} \right]$. Factoring the exponential as in the proof of Eq. (29), it follows from Lemma 5 that $f(\ell)$ decreases with $\ell$. Taking $\ell \to \infty$ and $\ell = 0$ yields Eq. (30).

3) By Part 2, the function $g(t) = \log M_S(\theta,0,t)$ is superadditive. Therefore, the ratio $\frac{1}{t} g(t)$ decreases monotonically from $g(1) = \log M_c(\theta)$ to $\lim_{t \to \infty} \frac{1}{t} g(t) = \log m(\theta)$, and Eq. (31) follows.

4) We start from [7, Eq. (7.15)], which states that
\[
M_S(\theta,0,t) = \left( (1-p), p \right) \left( L(\theta) \right)^t \left( \begin{array}{c} 1 \\ 1 \end{array} \right).
\]
Let $m_+(\theta)$ and $m_- (\theta)$ be the larger and smaller eigenvalues of $L(\theta)$, respectively. Inserting the spectral decomposition of $L(\theta)$ with respect to its eigenvalues $m_+(\theta)$ and $m_-(\theta)$ into the expression for $M_S(\theta,0,t)$, we obtain a constant $K(\theta)$ such that
\[
M_S(\theta,0,t) = K(\theta)(m_+(\theta))^t + (1 - K(\theta))(m_- (\theta))^t \quad (\forall t \geq 0, \theta \in \mathbb{R}). \tag{34}
\]
Using that $M_S(\theta,0,1) = M_c(\theta)$, we see that
\[
K(\theta) = \frac{M_c(\theta) - m_- (\theta)}{m_+(\theta) - m_-(\theta)}.
\]
Since \( m_+ (\theta) > M_c (\theta) > m_- (\theta) \), it follows that \( 0 < K (\theta) < 1 \), and we can drop the second summand in Eq. (34) to obtain Eq. (33).

\hspace{1cm} \blacksquare

Remark. The representation for \( M_S (\theta) \) in Eq. (34) implies in particular that

\[
\gamma_S (\theta) = \lim_{t \to \infty} \frac{1}{\theta t} \log M_S (\theta, 0, t) = \frac{1}{\theta} \log m_+ (\theta),
\]

in agreement with Eq. (27).

The validity of the bounds on the moment-generating function and the effective capacity extends to a broader class of time-homogeneous two-state Markov-modulated processes, as described in [7, Example 7.2.7]. The Markov-modulated process is given by a two-state Markov chain in the steady-state and two independent sequences of i.i.d. random variables, \( c_0^k \) and \( c_1^k \). The service rate in the \( k \)-th time slot is given by

\[
c_k = (1 - X_k)c_0^k + X_k c_1^k,
\]

where \( X_k \) is the state variable of the system. The service process \( S (s, t) \) is defined by \( S (s, t) = \sum_{k=s}^{t-1} c_k \).

Lemma 7: Let \( S \) be the two-state Markov-modulated process described above. Assume that the random variables \( c_0^k \) and \( c_1^k \) have moment-generating functions \( M_{c,0} (\theta) = e^{\theta c_0^k} \) and \( M_{c,1} (\theta) = e^{\theta c_1^k} \). If \( p_01 + p_{10} < 1 \), then the conclusions of Lemma 6 hold, with \( M_{c,0} (\theta) = (1 - p) M_{c,0} (\theta) + p M_{c,1} (\theta) \), and

\[
L (\theta) = \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix} \begin{pmatrix} M_{c,0} (\theta) & 0 \\ 0 & M_{c,1} (\theta) \end{pmatrix}. \tag{36}
\]

Proof: Fix \( \theta \in \mathbb{R} \), and assume without loss of generality that \( M_{c,0} (\theta) \leq M_{c,1} (\theta) \). We follow the proof of Lemma 6.

For the first claim, we condition the left hand side of Eq. (29) on the state of the system at the times \( \tau_1, \ldots, \tau_n \). In a single time slot, we obtain from Eq. (35) that

\[
E \left[ e^{\theta c_k} \bigg| X_k \right] = M_{c,0} (\theta) + (M_{c,1} (\theta) - M_{c,0} (\theta)) X_k.
\]

Since the random variables \( c_0^k, c_1^k \) and \( X_k \) are all independent, it follows that

\[
E \left[ e^{\theta \sum_{i=1}^n c_{\tau_i}} \right] = \prod_{i=1}^n M_{c,0} (\theta) + (M_{c,1} (\theta) - M_{c,0} (\theta)) X_{\tau_i}
\]

\[
= \sum_{J \subset \{\tau_1, \ldots, \tau_n\}} (M_{c,0} (\theta))^{n-|J|} (M_{c,1} (\theta) - M_{c,0} (\theta))^{|J|} \prod_{j \in J} X_j.
\]
Taking expectations, we obtain

\[
E \left[ e^{\theta \sum_{i=1}^{n} \epsilon_i} \right] = \sum_{J \subset \{\tau_1, \ldots, \tau_n\}} (M_{c,\theta}(\theta))^{n-|J|} (M_{c,1}(\theta) - M_{c,\theta}(\theta))^{|J|} E \left[ \prod_{j \in J} X_j \right].
\]

Since all coefficients are non-negative, Eq. (29) now follows from Lemma 5.

A similar application of Lemma 5 yields Eq. (30), which directly implies Eq. (31). Finally, Eq. (33) follows as before from [7, Eq. (7.15)].

B. Bounds for an MMOO server with feedback

The results in the previous subsection imply bounds on the equivalent service of a feedback system in Fig. 3 containing an MMOO server. We assume \( d > 0 \) and \( w > 0 \) as parameters of the system. Our first result is analogous to Theorem 2.

**Theorem 3:** Let \( S(s, t) \) be an MMOO service process with feedback. If the transition probabilities satisfy \( p_{01} + p_{10} < 1 \), then, for every \( \theta > 0 \),

\[
M_{\text{win}}(-\theta, s, t) \leq \left( (m_+(-\theta)^d + de^{\theta w}) \right)^{\frac{1-w}{d}},
\]

where \( m_+(-\theta) \) is the larger eigenvalue of the matrix \( L(-\theta) \) defined by Eq. (32).

**Proof:** By Lemma 6 (Parts 1 and 3), we have for each choice of \( \tau_0, \ldots, \tau_n \) in \( C_n(s, t) \) the bound

\[
E \left[ e^{\theta \sum_{i=1}^{n} (S(\tau_{i-1}, \tau_i-d) + S(\tau_n, t))} \right] \leq M_S(-\theta, 0, t-s-nd) \leq (m(-\theta))^{1-s-nd}.
\]

Inserting this estimate into the proof of Theorem 2 gives the result.

The theorem implies the following lower bound on the effective capacity of the service process with feedback.

**Corollary 3:** Under the assumptions of Theorem 3, the effective capacity \( \gamma_{\text{win}} \) of the MMOO process \( S(s, t) \) with feedback satisfies for \( \theta > 0 \)

\[
\gamma_{\text{win}}(-\theta) \geq \gamma_S(-\theta) - \frac{1}{d\theta} \log \left( 1 + de^{\theta(d\gamma_S(-\theta) - w)} \right),
\]

where \( \gamma_S(-\theta) \) is given by Eq. (27).

Both results extend to more general two-state Markov-modulated service processes.

**Corollary 4:** Let \( S(s, t) \) the two-state Markov-modulated process at the end of Subsection IV-A. If \( p_{01} + p_{10} < 1 \), then the moment-generating function of the service process with feedback
satisfies Eq. (37), where where $m_+(-\theta)$ is the larger eigenvalue of the matrix $L(-\theta)$ defined by Eq. (36). Its effective capacity $\gamma_{\text{win}}(-\theta)$ satisfies Eq. (38), where $\gamma_S(-\theta) = -\frac{1}{\theta} \log m_+(-\theta)$ is the effective capacity of the service process.

**Remark:** The formula $\gamma(-\theta) = -\frac{1}{\theta} \log m_+(-\theta)$ agrees with [7, Eq. (7.17)].

**Proof:** In the proof Theorem 3, replace Lemma 6 by Lemma 7.

An important application of Corollary 4 is a leftover service model with a VBR server (as analyzed in Sec. IV), where cross-traffic arrivals are governed by a two-state Markov-modulated process. Let $c_k$ denote the total service available to all traffic flows in time slot $k$, and let $a_k^c$ denote the cross-traffic arrivals in that time slot. Then the leftover service available to the through flow can be bounded from below by

$$S_{\text{lo}}(s,t) = \left( t - \sum_{k=s}^{t-1} (c_k - a_k^c) \right)^+ \geq \sum_{k=s}^{t-1} (c_k - a_k^c),$$

where we use $[x]^+ = \max\{x, 0\}$. Corollary 4 applies since the sum on the right-hand side is a two-state Markov-modulated process. It provides a non-trivial lower bound on the service of the feedback system, provided that the stability condition $E[a_k^c] < E[c_k]$ is met.

**C. Quality of the MMOO bounds**

As with the VBR server, we can use the bounds of Theorem 1 to test the accuracy of the results on the MMOO process. Inserting the parameters of the MMOO process in Theorem 1, $S_{\text{win}}$ is bounded by

$$S_{\text{win}}(s,t) \leq \min\left\{ P, \frac{w}{d} \right\} \sum_{k=s}^{t-1} X_k \leq S_{\text{win}}(s,t) \leq \min\left\{ S(s,t), \left\lceil \frac{t-s}{d} \right\rceil w \right\}.$$  \hspace{1cm} (39)

The lower bound on the left corresponds to a service process of a MMOO process with ON rate $P' = \min\{ P, \frac{w}{d} \}$. The lower bound is exact for when $d = 1$. Its moment-generating function can be bounded with the help of Lemma 6 (Part 3). For the upper, we can only exploit $S_{\text{win}}(s,t) \leq \left\lceil \frac{t-s}{d} \right\rceil w$, since we do not have a simple upper bound on the $\varepsilon$-quantiles of the MMOO process.

We also apply Lemma 6 (Part 4) to the upper bound. Using these expressions in Eq. (4) yields the following bounds for the effective capacity:

**Corollary 5:** Under the assumptions of Theorems 3, $\gamma_{\text{win}}(-\theta)$ is bounded for $\theta > 0$ by

$$\gamma'_{\text{win}}(-\theta) \leq \gamma_{\text{win}}(-\theta) \leq \min\left\{ \gamma_S(-\theta), \frac{w}{d} \right\} \text{, } (\theta > 0).$$  \hspace{1cm} (40)
Here, $\gamma_{\text{win}}'(-\theta)$ is the effective capacity of an MMOO process with peak rate $P' = \min\{P, \frac{w}{d}\}$. Note that the average rate of this process, given by $\gamma_{\text{win}}'(0)$, lies strictly below the upper bound, $\min\{\gamma_S'(0, \frac{w}{d})\}$. In the limit $\theta \to \infty$, the three bounds in Eqs. (38) and (40) become sharp.

D. Numerical evaluation of MMOO bounds

We present numerical examples for an MMOO server with window flow control, proceeding in a similar fashion as in the evaluation of the VBR server. The parameters of the MMOO process are selected as

$$p_{00} = 0.2, \quad p_{11} = 0.9, \quad P = 1.125 \text{ Mb}.$$

With time slots of length 1 ms, the server has an average rate of 1 Gbps, which is the same rate as that of the VBR service evaluated in Subsec. IV-C.

In Fig. 8 we present the statistical service curve $S_{\text{win}}^e(0, t)$ as a function of time where we fix $w/d = 100$ Mbps in Fig. 8(a) and $w/d = 500$ Mbps in Fig. 8(b). We set $\varepsilon = 10^{-6}$. The statistical service curves, plotted as solid lines, have been constructed with the bound on $M_{S_{\text{win}}}$ from Theorem 3. We include for comparison, upper and lower bounds obtained from Theorem 1 via Eq. (39) represented by dash-dotted lines. Note that the lower bounds become positive only for $t > 10$ ms, and that this is well matched by the (solid line) statistical service curves. The initial latency is a property of the MMOO service process, which may reside for extended time periods in the OFF state. As seen in the VBR service in Fig. 5, the statistical service curves increase when increasing $w$ and $d$ proportionally. Recall that the lower bound is also a statistical service curve when $d = 1$ ms. As another comparison with the VBR service, we note that the upper and lower bounds in Fig. 8 are separated by a wider margin than in Fig. 5. This is due to the simpler upper bound, since we have not derived an upper bound for the $\varepsilon$-quantiles of the MMOO process.

Fig. 9 shows the bounds on the effective capacity $\gamma_{\text{win}}'(-\theta)$ as a function of $\theta$, where we use $w/d = 100$ Mbps in Fig. 9(a) and $w/d = 500$ Mbps in Fig. 9(b). The bounds from Corollary 3, for different values of $w$ and $d$, are shown as solid lines. The upper and lower bounds obtained from Corollary 5 are depicted as dash-dotted lines. When the bounds from Corollary 3 fall below those of Corollary 5, the better bound should be used. Except for small values of $\theta$, the lower bounds of Corollary 3 are close to the upper bound from Corollary 5, indicating that the effective capacity can be accurately computed.
VI. CONCLUSIONS

We have approached a well-known open problem in the stochastic network calculus, i.e., an extension of the analysis of feedback systems. Our analysis addressed a window flow control system with stochastic service, where we considered a service with and without time correlations.
We analyzed and then addressed the difficulty of accounting for the time correlations introduced by feedback mechanisms. Our analysis revealed major differences between deterministic and random window flow control systems, in particular, an additional dependency between the characteristic time scales of the feedback delay and the service process. We provided lower as well as upper bounds on the available service of the feedback system, which enabled us
to discuss the accuracy of our results, and discovered special cases where exact expressions for the service can be obtained. The results in this paper can be extended in many directions. Obvious generalizations are to consider random feedback delays and time-variable window sizes. We chose a window flow control network since a corresponding deterministic network calculus analysis exists in the literature. There are numerous other feedback systems that await an analysis of their backlog and/or delay properties. Applying our analysis to the detailed dioid algebraic models of TCP feedback in [3] is a logical first step.

REFERENCES

[1] R. Agrawal, R. L. Cruz, C. Okino, and R. Rajan. Performance bounds for flow control protocols. *IEEE/ACM Transactions on Networking*, 7(3):310–323, June 1999.

[2] H. Al-Zubaidy, J. Liebeherr, and A. Burchard. A (min,×) network calculus for multi-hop fading channels. In *Proc. IEEE Infocom*, pages 1833–1841, April 2013.

[3] F. Baccelli and D. Hong. TCP is max-plus linear and what it tells us on its throughput. In *Proc. ACM Sigcomm*, pages 219–230, August 2000.

[4] D. P. Bertsekas and R. G. Gallager. *Data Networks, (2nd edition)*. Prentice Hall, 1992.

[5] A. Burchard, J. Liebeherr, and S. D. Patek. A min-plus calculus for end-to-end statistical service guarantees. *IEEE Transactions on Information Theory*, 52(9):4105 – 4114, September 2006.

[6] C.-S. Chang. On deterministic traffic regulation and service guarantees: a systematic approach by filtering. *IEEE/ACM Transactions on Information Theory*, 44(3):1097–1110, May 1998.

[7] C.-S. Chang. *Performance Guarantees in Communication Networks*. Springer Verlag, 2000.

[8] C.-S. Chang, R. L. Cruz, J.-Y. Le Boudec, and P. Thiran. A (min, +) system theory for constrained traffic regulation and dynamic service guarantees. *IEEE/ACM Transactions on Networking*, 10(6):805–817, 2002.

[9] M. Fidler. An end-to-end probabilistic network calculus with moment generating functions. In *IEEE 14th International Workshop on Quality of Service (IWQoS)*, pages 261–270, June 2006.

[10] M. Fidler. A network calculus approach to probabilistic quality of service analysis of fading channels. In *Proc. IEEE Globecom*, pages 1–6, Nov. 2006.

[11] M. Fidler. Survey of deterministic and stochastic service curve models in the network calculus. *IEEE Communications Surveys & Tutorials*, 12(1):59–86, 2010.

[12] M. Fidler and A. Rizk. A guide to the stochastic network calculus. *IEEE Communications Surveys & Tutorials*, 17(1):92–105, 2015.

[13] Y. Jiang and Y. Liu. *Stochastic Network Calculus*. Springer, 2008.

[14] G. Kesidis, J. Walrand, and C. Chang. Effective bandwidths for multiclass Markov fluids and other ATM sources. *IEEE/ACM Transactions on Networking*, 1(4):424–428, August 1993.

[15] J. Y. Le Boudec and P. Thiran. *Network Calculus*. Springer Verlag, Lecture Notes in Computer Science, LNCS 2050, 2001.

[16] K. Mahmood, A. Rizk, and Y. Jiang. On the flow-level delay of a spatial multiplexing MIMO wireless channel. In *Proc. IEEE ICC*, pages 1–6, June 2011.

34
[17] D. Wu and R. Negi. Effective capacity: a wireless link model for support of quality of service. *IEEE Transactions on Wireless Communications*, 2(4):630–643, 2003.

[18] K. Zheng, F. Liu, L. Lei, C. Lin, and Y. Jiang. Stochastic performance analysis of a wireless finite–state Markov channel. *IEEE Transactions on Wireless Communications*, 12(2):782–793, 2012.