FULL ASYMPTOTIC EXPANSION OF THE HEAT TRACE FOR NON-Self-ADJOINT ELLIPTIC CONE OPERATORS

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ABSTRACT. The operator $e^{-tA}$ and its trace $\text{Tr} e^{-tA}$, for $t > 0$, are investigated in the case when $A$ is an elliptic differential operator on a manifold with conical singularities. Under a certain spectral condition (parameter-ellipticity) we obtain a full asymptotic expansion in $t$ of the heat trace as $t \to 0^+$. As in the smooth compact case, the problem is reduced to the investigation of the resolvent $(A-\lambda)^{-1}$. The main step consists in approximating this family by a parametrix of $A-\lambda$ constructed within a suitable parameter-dependent calculus.

1. INTRODUCTION

In this paper the operator $e^{-tA}$, $t > 0$, is investigated on manifolds with conical singularities. The operator $A$ is assumed to be an elliptic differential operator of arbitrary positive order, not necessarily self-adjoint, but satisfying an analog of Agmon’s condition (parameter-ellipticity) on a sector $\{ \lambda \in \mathbb{C} \mid \varphi \leq \arg \lambda \leq 2\pi - \varphi \}$ for some $0 < \varphi < \pi/2$. Our main aim is to describe the pseudodifferential structure of the resolvent $(A-\lambda)^{-1}$ as well as the asymptotic behavior, as $t \to 0^+$, of the operator $e^{-tA}$ and its trace $\text{Tr} e^{-tA}$ (heat trace).

From the analytic point of view a cone is a product $(0,c) \times X$ together with a metric of the form $dr^2 + r^2 g_X(r)$, where $g_X(r)$ is a smooth family of Riemannian metrics on the ‘cone base’ $X$. Here, $X$ is assumed to be a smooth compact manifold without boundary. For this reason, the analysis on a manifold with conical singularities takes place on a manifold with boundary $\mathcal{B}$ with the mentioned product structure near $\partial \mathcal{B} = X$. The natural differential operators appearing in this context are given, near the boundary, by the so-called operators of Fuchs type, cf. equation (2.1). These operators arise, for instance, when considering Laplace-Beltrami operators associated to the metrics given above. A cone differential operator is a differential operator on the manifold $\mathcal{B}$ which is of Fuchs type near the boundary $\partial \mathcal{B}$.

There is a large number of papers concerning asymptotic expansions for resolvents, heat kernels and heat traces on smooth compact manifolds with and without boundaries, and their applications to geometry, index theory, mathematical physics and other areas. A large collection of references related to this topic can be found in Gilkey’s book [17, Chapter 5]. Let us especially mention the important papers [3], [16], [18] and [14]. An elegant pseudodifferential method for the study of resolvents of elliptic differential operators on smooth compact manifolds was introduced by Seeley in [12], [13], [14]. The idea of his approach is to approximate the resolvents by means of parametrices that are constructed within a corresponding parameter-dependent pseudodifferential calculus, where the parameter (up to a natural anisotropy) is involved as an additional covariable. For genuine pseudodifferential operators the situation is more delicate and other methods are required, cf. [11], [19], [22], [17].
On manifolds with conical singularities the analysis becomes more complicated. However, for certain classes of self-adjoint operators on some singular manifolds there are many results concerning resolvent, heat kernel and heat trace asymptotics; in this context we want to refer to [4], [5], [7], [8], [9], [12], [26], [28] and [35]. In particular, Lesch [28] generalized the method introduced by Brüning and Seeley in [4] and obtained asymptotic expansions of the heat trace for self-adjoint differential operators with coefficients that are independent of the radial variable near the singularities. For the more general dependent case he obtained partial expansions. Let us finally mention that Loya [25], [30] has recently obtained similar results to those from this paper by using geometric blow-up techniques of Melrose.

The strategy of this work is to follow the resolvent approach mentioned above by introducing a suitable notion of parameter-dependent ellipticity (generalizing self-adjointness), and by giving an explicit construction of a parametrix within a natural class of operator-valued symbols. We also take advantage of a parameter-dependent symbol class introduced by Grubb and Seeley [22] in order to consider successfully operators whose coefficients do depend smoothly on the radial variable near 0 (i.e., near $\partial \mathbb{B}$), cf. equation (2.1).

In Section 3 we introduce the class of cone differential operators and discuss briefly some of the typical elements of Schulze’s cone calculus. For an extensive study of cone algebras we refer to [39], [40] and [36]. In the third section we analyze the operator family $A - \lambda$ for a cone differential operator $A$. The main idea is to consider $A - \lambda$ as an element of a certain class of operator-valued symbols, defined in an abstract setting by means of strongly continuous groups acting on Banach spaces. The parameter $\lambda$ plays the role of the covariable and is treated anisotropically with respect to the covariables of the local symbols. Moreover, by freezing the coefficients of $A$ at the boundary, cf. equation (3.3), we get a canonical object (the twisted homogeneous principal symbol) that together with the symbolic structure of $A$ characterizes the parameter-ellipticity of $A - \lambda$. Our concept of ellipticity turns out to be sufficient and necessary for the invertibility of $A - \lambda$ on the canonical weighted Sobolev spaces. Next, we construct a parametrix, that is, an element of the class that inverts $A - \lambda$ modulo operator-valued functions decreasing rapidly in the parameter and taking values in the Green cone operators. This parametrix construction essentially relies on techniques introduced by Schulze [18] for the analysis on manifolds with edges. In Section 3.3 we consider a modified version of the parameter-dependent pseudodifferential calculus developed in [22]. This class contains the holomorphic symbols arising from the Mellin pseudodifferential theory considered in this work. The main purpose is to achieve an asymptotic expansion of the local symbols in the parameter $\lambda$ as $|\lambda| \to \infty$.

The last section is devoted to the study of the heat operator associated to a cone differential operator $A$ of order $m > 0$ which is parameter-elliptic in a suitable sector $\Lambda \subset \mathbb{C}$. Using the resolvent estimates obtained here, we define the semigroup $\{e^{-tA}\}_{t \geq 0}$ by means of Dunford integrals. It is then proved that for each $t > 0$, the operator $e^{-tA}$ belongs to the class of Green elements in the cone algebra (see Appendix B). Green operators in the cone theory are actually integral operators whose kernels are smooth in the interior and behave in a particular way near the boundary. As a consequence, each operator $e^{-tA}$ ($t > 0$) is of trace class on the weighted Sobolev space $\mathcal{H}^{s,\gamma}(\mathbb{B})$ for all $s \in \mathbb{R}$, whenever $A$ is parameter-elliptic with respect to the weight $\gamma$. Moreover, we
show that the heat trace admits the expansion
\[ \text{Tr} e^{-tA} \sim \sum_{k=0}^{\infty} C_k t^{(k-n-1)/m} + \sum_{k=0}^{\infty} C'_k t^{k/m} \log t, \quad \text{as } t \to 0^+, \]
where \( n = \dim \partial \mathcal{B} \). The coefficients \( C_k \) and \( C'_k \) depend on the principal symbol of the cone operator \( A \), and on its boundary spectrum (cf. Section 2.1).

Roughly speaking, the construction relies on the use of three different pseudodifferential calculi of local and global nature. The problem is reduced to an asymptotic expansion of the resolvent power \((A - \lambda)^{-\ell}\), for \( \ell \in \mathbb{N} \) sufficiently large, which is viewed as an operator-valued symbol of negative order. Using the standard symbolic calculus we get a first approximation of \( (A - \lambda)^{-\ell} \) by means of a parametrix of \( (A - \lambda)^{\ell} \). The part of this parametrix localized in the interior of \( \mathcal{B} \) provides a ‘good’ approximation of the resolvent since the local symbols are homogeneous. In order to achieve a suitable approximation of the resolvent near the boundary, we refine the parametrix by decomposing it into twisted homogeneous operators, separating at the same time the smoothing Green terms from the nonsmoothing parameter-dependent Mellin operators with degenerate symbols. The twisted homogeneity, which induces scalability of the kernels, permits us to imitate in a global sense the usual homogeneity arguments, but it is not enough to obtain the expansion of the nonsmoothing Mellin components. Nevertheless, a complete asymptotic expansion of these operators can be achieved making use of the weakly parametric calculus from Section 3.3.

Finally, in order to reduce the search through the literature, some basic results and definitions are summarized in the Appendix.

2. Elements of the cone algebra

2.1. Cone differential operators. Let \( X \) be a smooth manifold of dimension \( n \). Denote by \( X^\wedge \) the space \( \mathbb{R}_+ \times X \). A differential operator in \( \text{Diff}^m(X^\wedge) \) is said to be of Fuchs type if, expressed in the coordinates \((r, x)\), it is of the form
\[ A = r^{-m} \sum_{k=0}^{\infty} a_k(r)(-r\partial_r)^k \]
with \( a_k \in C^\infty(\mathbb{R}_+, \text{Diff}^{m-k}(X)) \). \( \text{Diff}^\nu \) denotes the space of differential operators of order \( \nu \in \mathbb{N}_0 \) with smooth coefficients.

Example 2.1. Let \( X^\wedge \) be equipped with the cone metric \( dr^2 + r^2 g_x(r) \), where \( g_x(r) \) is a family of Riemannian metrics on \( X \), smooth in \( r \in \mathbb{R}_+ = [0, \infty) \). The Laplace-Beltrami operator \( \Delta_{cm} \) corresponding to this metric is an operator of Fuchs type. In fact,
\[ \Delta_{cm} = r^{-2} \left\{ \Delta_{g_x}(r) + \left\{ -n + 1 - rG^{-1}(\partial_r G) \right\} (-r\partial_r) + (-r\partial_r)^2 \right\}, \]
where, in local coordinates \((r, x_1, \ldots, x_n)\), \( G(r, x) = |\det(g_x(r)(\partial_{x_i}, \partial_{x_j}))|^{1/2} \).

On a smooth manifold with boundary \( M \) we may consider the so-called \( b \)-tangent bundle \( bTM \) (cf. Melrose [32 Section 2.2]). For a differential operator of Fuchs type \( A \) as in (2.1) there is a function \( \sigma^m_{\psi,b}(A) \) in \( C^\infty(bT^*X^\wedge \setminus \emptyset) \) such that, in local coordinates \((r, x) \in \mathbb{R}_+ \times U, U \subset X \), it takes the form
\[ \sigma^m_{\psi,b}(A)(r, x, \partial, \xi) = r^m \sigma^m_{\psi}(A)(r, x, \partial/r, \xi), \]
where $\sigma^m(A)$ is the usual homogeneous principal symbol of $A$ on $X^\land$. $bT^*X^\land$ denotes the dual of $bT^*X^\land$, and $(\rho, \xi) \in \mathbb{R} \times \mathbb{R}^n$ are the covariables to $(r, x)$.

Fredholm properties of Fuchs type operators are determined by a pair of symbols. The first one is the homogeneous principal symbol $\sigma^m_{\psi,b}(A)$ which characterizes the ellipticity in the interior. The second one is the so-called conormal symbol of $A$ which is an operator-valued symbol, living at the boundary, that can be described by means of the Mellin transform

$$\mathcal{M}_{r \to z} u(z) = \int_0^\infty r^{z-1} u(r) dr \text{ for } u \in C_0^\infty(\mathbb{R}_+).$$

The point is that the totally characteristic derivative ($-r \partial_r$) corresponds, in the image of the Mellin transform, to multiplication by the complex variable $z$. Therefore, if $a_k \in C^\infty_0(\overline{\mathbb{R}}_+, \text{Diff}^{-k}(X))$ for $k = 0, \ldots, m$, are the coefficients of the operator $A$ from (2.1), then the operator-valued polynomial

$$h(r, z) = \sum_{k=0}^m a_k(r) z^k \quad \text{with } (r, z) \in \overline{\mathbb{R}}_+ \times \mathbb{C},$$

may be interpreted as the corresponding Mellin symbol of $A$. In other words, for any real $\beta$ and $u \in C_0^\infty(X^\land) = C_0^\infty(\mathbb{R}_+, C^\infty(X))$ we have the relation

$$Au(r) = [\text{op}_M(h)u](r) := \frac{1}{2\pi i} \int_{\Gamma_\beta} r^{-z-m} h(r, z) [\mathcal{M}_{r \to z} u](z) dz,$$

where $\Gamma_\beta := \{z \in \mathbb{C} \mid \Re z = \beta\}$. Observe that $z \mapsto h(r, z) : \mathbb{C} \to \text{Diff}^m(X)$ is holomorphic for each $r \in \mathbb{R}_+$. Now, the conormal symbol $\sigma^m_M(A)$ of the Fuchs type operator $A$ is just the polynomial (2.3) evaluated at $r = 0$, i.e.,

$$\sigma^m_M(A)(z) := h(0, z) = \sum_{k=0}^m a_k(0) z^k.$$

The property of being Fuchs type is invariant under changes of coordinates, so (2.1) serves as model for a class of operators on manifolds with boundaries. After a suitable blow-up, a manifold with conical singularities becomes a smooth manifold with boundary that can be precisely described by operators of Fuchs type.

**Definition 2.2.** Let $B$ be a smooth compact manifold with boundary $\partial B = X$. The space $\text{Diff}^m(B)_{\text{cone}}$ of cone differential operators consists of all operators in $\text{Diff}^m(\text{int } B)$ which are of Fuchs type near the boundary.

Fix once and for all a defining function $r$ for the boundary of $B$, that is, a smooth function $r : B \to \overline{\mathbb{R}}_+$ such that $r$ is positive in the interior of $B$, it vanishes on $\partial B$, and $dr \neq 0$ on $\partial B$. We assume that $r^{-1}((0, 2))$ is a collar neighborhood of $\partial B$ in $B$.

The conormal symbol of a cone operator $A$ is defined as the operator family

$$u \mapsto t^{m-\frac{m}{r}} A(t^r \tilde{u})|_{t=0} : C^\infty(X) \to C^\infty(X),$$

where $\tilde{u}$ is some extension of $u$.

A function $\omega \in C_0^\infty(\overline{\mathbb{R}}_+)$ is called a (boundary) cut-off function if $\text{supp } \omega \subset [0, 2)$ and $\omega = 1$ near $r = 0$. Note that every $\omega$ can be viewed as a function on $[0, 2) \times X$ as well as a function on $B$, extending by zero. We say that two functions $\phi, \psi$ satisfy the relation $\phi \prec \psi$ if and only if $\phi \psi = \phi$. 


Remark 2.3. Let \( \omega_0, \omega_1 \) and \( \omega_2 \) be cut-off functions with \( \omega_2 < \omega_1 < \omega_0 \). Let \( \mathbb{B} \) be the double of \( \mathbb{B} \). Since differential operators are local, i.e., \( \text{supp } Au \subseteq \text{supp } u \), the elements of \( \text{Diff}^m(\mathbb{B}) \) can be written in the form
\[
A = \omega_1 A_0 \omega_1 + (1 - \omega_1) A_1 (1 - \omega_2),
\]
where \( A_0 \) is of Fuchs type on \( X^\wedge \) and \( A_1 \in \text{Diff}^m(\mathbb{B}) \). In this decomposition, \( A_0 \) is not canonical but its conormal symbols is that of the operator \( A \). Further, the interior principal symbol \( \sigma^m_\psi(A) \) of \( A \) induces a symbol \( \sigma^m_{\psi,b}(A) \in C^\infty(b^*T^*\mathbb{B} \setminus 0) \) that, in local coordinates near the boundary, satisfies the relation (2.2).

Proposition 2.4. The class of cone differential operators is closed under compositions. Moreover, the conormal symbols satisfy the relation
\[
(2.5) \quad \sigma^m_{\psi,b}(A_2 A_1)(z) = \sigma^m_{\psi,b}(A_2)(z + m_1) \sigma^m_{\psi,b}(A_1)(z).
\]

Definition 2.5. \( A \in \text{Diff}^m(\mathbb{B}) \) is called elliptic with respect to \( \gamma \in \mathbb{R} \) if
(i) \( \sigma^m_{\psi,b}(A) \neq 0 \) on \( b^*T^*\mathbb{B} \setminus 0 \),
(ii) \( \sigma^m_{\psi,b}(A)(z) : H^m(X) \to L^2(X) \) is an isomorphism for all \( z \in \Gamma_{\frac{n+1}{2} - \gamma} \).

As in \( \mathbb{R} \) denote
\[
\text{spec}_b(A) := \{ z \in \mathbb{C} \mid \sigma^m_{\psi,b}(A)(z) : H^m(X) \to L^2(X) \text{ is not an isomorphism} \}.
\]

This set is called the boundary spectrum of \( A \) and is known to be discrete, and finite on vertical strips.

Example 2.6. Let \( g \) be a metric on \( \mathbb{B} \) that coincides with a cone metric in a collar neighborhood of \( \partial \mathbb{B} = X \), cf. Example 2.1. Then the corresponding Laplace-Beltrami operator \( \Delta_g \) is an elliptic cone differential operator with conormal symbol
\[
\sigma^2_{\psi}(\Delta_g)(z) = \Delta_{g_X}(0) - (n - 1)z + z^2, \quad n = \dim X.
\]

In this case, \( z \in \text{spec}_b(\Delta_g) \) if and only if \( (n - 1)z - z^2 \in \text{spec}_{L^2(X)}(\Delta_{g_X}(0)) \).

2.2. Weighted Sobolev spaces. Let us introduce a scale of weighted (cone) Sobolev spaces on which the cone operators act continuously. These spaces are defined in a similar way as the usual Sobolev spaces but based on the Mellin (instead of the Fourier) transform in the singular direction. Recall that \( \Gamma_\beta = \{ z \in \mathbb{C} \mid \Re z = \beta \} \).

For \( s, \gamma \in \mathbb{R} \) let \( \mathcal{H}^{s,\gamma}(\mathbb{R}_+ \times \mathbb{R}^n) \) be the closure of \( C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^n) \) with respect to
\[
\| u \|_{\mathcal{H}^{s,\gamma}} = \frac{1}{2\pi i} \int_{\Gamma_{\frac{n+1}{2} - \gamma}} (1 + |s| + |\xi|^2)^s |(M_{r \rightarrow x} F_{x \rightarrow \xi} u)(z, \xi)|^2 dz d\xi.
\]

The transformation \( S_{\gamma,\frac{d}{dz}} : C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^n) \to C_0^\infty(\mathbb{R}^{1+n}) \) given by
\[
(S_{\gamma,\frac{d}{dz}} u)(r, x) = e^{-\gamma r} u(e^{-\gamma} r, x)
\]
extends to an isomorphism \( S_{\gamma,\frac{d}{dz}} : \mathcal{H}^{s,\gamma}(\mathbb{R}_+ \times \mathbb{R}^n) \to \mathcal{H}^{s}(\mathbb{R}^{1+n}) \), where \( \mathcal{H}^{s} \) denotes the standard Sobolev space. We get the equivalence of norms
\[
(2.6) \quad \| u \|_{\mathcal{H}^{s,\gamma}(\mathbb{R}_+ \times \mathbb{R}^n)} \sim \| S_{\gamma,\frac{d}{dz}} u \|_{\mathcal{H}^{s}(\mathbb{R}^{1+n})}.
\]

In order to define the suitable Sobolev spaces on the manifold \( X^\wedge \) we fix an open covering \( \{ U_1, \ldots, U_N \} \) of \( X \) with corresponding diffeomorphisms \( \chi_j : U_j \to \tilde{V}_j \subset \mathbb{R}^n \) and \( \tilde{r}_j : \tilde{V}_j \to V_j \subset S^n \), where \( S^n \) is the unit sphere in \( \mathbb{R}^{1+n} \). In addition, we define
\( \kappa_j : \mathbb{R}_+ \times U_j \to \mathbb{R}^{1+n} \setminus \{0\} \) by \( \kappa_j(r,x) := r\tilde{k}_j(x) \). Let finally \( \{\phi_1, \ldots, \phi_N\} \) be a subordinate partition of unity.

**Definition 2.7.** Let \( \omega \) be a cut-off function. For \( s, \gamma \in \mathbb{R} \) let \( \mathcal{K}^{s,\gamma}(X^\wedge) \) denote the closure of \( C_0^\infty(X^\wedge) \) with respect to the norm

\[
\|\nu\|_{\mathcal{K}^{s,\gamma}}^2 = \sum_{j=1}^N \left\{ \|\chi_j(\phi_j \omega u)\|_{\mathcal{H}^{s,\gamma}(\mathbb{R}_+ \times \mathbb{R}^n)}^2 + \|\kappa_j \omega (1-\omega)u\|_{\mathcal{H}^{s,\gamma}(\mathbb{R}^{1+n})}^2 \right\}.
\]

Note that for every fixed localization data (partition of unity, diffeomorphisms \( \chi_j \) and \( \kappa_j \), and cut-off function \( \omega \)) the space \( \mathcal{K}^{s,\gamma}(X^\wedge) \) has a Hilbert space structure. Another choice of data yields a norm equivalent to (2.7). The spaces \( \mathcal{K}^{s,\gamma}(X^\wedge) \) are subspaces of \( \mathcal{H}_{0c}(s) \). In particular, we have \( \mathcal{K}^{0,0}(X^\wedge) = L^2(X^\wedge, r^s \, dr \, dx) \).

**Proposition 2.8.** Let \( A \in \text{Diff}^m(X^\wedge) \) be of Fuchs type with coefficients \( a_k(r) \) that are independent of \( r \) for large values of \( r \). Then the map 

\[ A : \mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{s-m,\gamma-m}(X^\wedge) \]

is continuous for all \( s, \gamma \in \mathbb{R} \).

**Definition 2.9.** For \( s, \gamma \in \mathbb{R} \) and \( \omega \) a cut-off function on \( \mathbb{B} \), let

\[ \mathcal{H}^{s,\gamma}(\mathbb{B}) := \{ u \in \mathcal{D}'(\text{int} \mathbb{B}) \mid \omega u \in \mathcal{K}^{s,\gamma}(X^\wedge) \text{ and } (1-\omega)u \in H^s(\mathbb{B}) \}. \]

We endow this space with the norm

\[ \|u\|_{\mathcal{H}^{s,\gamma}(\mathbb{B})} = \|\omega u\|_{\mathcal{K}^{s,\gamma}(X^\wedge)} + \|(1-\omega)u\|_{H^s(\mathbb{B})}. \]

Another cut-off function \( \omega \) leads to an equivalent norm. \( \mathcal{H}^{s,\gamma}(\mathbb{B}) \) is a subspace of \( \mathcal{H}_{0c}(\text{int} \mathbb{B}) \) and \( \mathcal{H}^{0,0}(\mathbb{B}) \cong r^{-n/2} L^2(\mathbb{B}) \) for any boundary defining function \( r \). Further, \( \mathcal{H}^{s,\gamma}(\mathbb{B}) = r^s \mathcal{H}^{0,0}(\mathbb{B}) \) for every \( s, \gamma \in \mathbb{R} \). Moreover, the space \( \mathcal{H}^{s,\gamma}(\mathbb{B}) \) is a Hilbert space.

**Lemma 2.10.** Let \( s, s', \gamma, \gamma' \in \mathbb{R} \), and let \( \omega_0, \omega_1 \) and \( \omega_2 \) be cut-off functions. Let \( A_0 \) and \( A_1 \) be operators on \( X^\wedge \) and \( \mathbb{B} \), respectively, such that

\[ \omega_1 A_0 \omega_0 \in \mathcal{L}(\mathcal{K}^{s,\gamma}, \mathcal{K}^{s',\gamma'}) \text{ and } (1-\omega_1)A_1(1-\omega_2) \in \mathcal{L}(H^s, H^{s'}). \]

If \( A = \omega_1 A_0 \omega_0 + (1-\omega_1)A_1(1-\omega_2) \), then there exists \( C > 0 \) such that

\[ \|A\|_{\mathcal{L}(\mathcal{H}^{s,\gamma}(\mathbb{B}), \mathcal{H}^{s',\gamma'}(\mathbb{B}))} \leq C \left( \|\omega_1 A_0 \omega_0\|_{\mathcal{L}(\mathcal{K}^{s,\gamma}, \mathcal{K}^{s',\gamma'})} + \|(1-\omega_1)A_1(1-\omega_2)\|_{\mathcal{L}(H^s, H^{s'})} \right). \]

**Proposition 2.11.** Any \( A \in \text{Diff}^m(\mathbb{B})_{\text{cone}} \) extends to a continuous operator

\[ A : \mathcal{H}^{s,\gamma}(\mathbb{B}) \to \mathcal{H}^{s-m,\gamma-m}(\mathbb{B}) \]

for all \( s, \gamma \in \mathbb{R} \).

**Fredholm property.** The concept of ellipticity introduced in Definition 2.7 leads to the Fredholm property of the elliptic operators on the weighted Sobolev spaces on \( \mathbb{B} \). More precisely, if \( A \in \text{Diff}^m(\mathbb{B})_{\text{cone}} \) is elliptic with respect to \( \gamma \), then the map

\[ A : \mathcal{H}^{s,\gamma}(\mathbb{B}) \to \mathcal{H}^{s-m,\gamma-m}(\mathbb{B}) \]

is Fredholm for all \( s \in \mathbb{R} \).

This is a consequence of the following

**Theorem 2.12.** To every \( A \in \text{Diff}^m(\mathbb{B})_{\text{cone}} \), elliptic with respect to \( \gamma \), there exists a cone pseudodifferential operator \( Q \) of order \(-m\) such that

\[ QA - 1 : \mathcal{H}^{s,\gamma}(\mathbb{B}) \to \mathcal{H}^{s,\gamma}(\mathbb{B}) \text{ and } AQ - 1 : \mathcal{H}^{s-m,\gamma-m}(\mathbb{B}) \to \mathcal{H}^{s-m,\gamma-m}(\mathbb{B}) \]

are smoothing operators of Green type (Appendix B). \( Q \) is called a parametrix of \( A \).
Remark 2.13. If $A$ is elliptic with respect to two different weights, say $\gamma_1$ and $\gamma_2$, then the difference between the Fredholm indices depends on the elements in the boundary spectrum of $A$ situated between the corresponding weight lines in $\mathbb{C}$. Moreover, there is an explicit formula for the difference $\text{ind}_{\gamma_1} A - \text{ind}_{\gamma_2} A$, cf. [33], [34].

Adjoint operators. The restriction to $C^\infty_0(X^\wedge) \times C^\infty_0(X^\wedge)$ of the sesquilinear pairing $(\cdot, \cdot)_{K^0,0}$ extends to a nondegenerate pairing

$$K^{s,\gamma}(X^\wedge) \times K^{-s,\gamma}(X^\wedge) \to \mathbb{C}$$

for any $s, \gamma \in \mathbb{R}$.

The formal adjoint of $A \in L(K^{s,\gamma}, K^{s-m,\gamma-m})$ with respect to $(\cdot, \cdot)_{K^0,0}$ is the unique operator $A^*_0$ satisfying $(Au, v) = (u, A^*_0v)$ for all $u, v \in C^\infty_0(X^\wedge)$. More precisely, if $A$ is a Fuchs type operator as in (2.1), and $u \in C^\infty_0(X^\wedge)$, then

$$A^*_0u(r) = r^{-m} \sum_{k=0}^m (n + 1 - m + r\partial_r)^k \{a_k(r)^*u(r)\},$$

where $a_k(r)^*$ denotes the pointwise formal adjoint in $L^2(X)$. For an arbitrary $\beta \in \mathbb{R}$ the adjoint of $A$ with respect to $(\cdot, \cdot)_{K^0,0}$ is given by $A^*_\beta = r^{2\beta} A^*_0 r^{-2\beta}$. Here $\tau$ is a defining function for the boundary of $X^\wedge$ with $\tau = 1$ near infinity.

3. Parameter-dependent calculus

In this section we basically concentrate our attention to the operator family $A - \lambda$ as the typical parameter-dependent cone operator. In fact, all the results exposed here can be generalized to wider classes of parameter-dependent cone pseudodifferential operators, cf. [33], [35], [36], [37], [38], [39].

Let $\Lambda \subset \mathbb{C}$ be a closed angle with vertex at the origin. Let $X$ be a smooth compact manifold, $\dim X = n$, and let $\mathbb{B}$ be the manifold with boundary $\partial \mathbb{B} = X$ introduced in Section 2.1. Recall that $\mathbb{B}$ denotes the double of $\mathbb{B}$. Again, let $X^\wedge = \mathbb{R}_+ \times X$.

3.1. Ellipticity condition for $A - \lambda$. If $A$ is a cone differential operator of order $m \in \mathbb{N}$, and $\lambda \in \Lambda$, then $A - \lambda$ can be written in the form

$$A - \lambda = \omega_1 \left(r^{-m} \sum_{k=0}^m a_k(r) (-r \partial_r)^k - \lambda\right) \omega_0 + (1 - \omega_1)(P - \lambda)(1 - \omega_2),$$

where $a_k \in C^\infty(\mathbb{R}_+, \text{Diff}^{m-k}(X))$, $P \in \text{Diff}^m(\mathbb{B})$, and $\omega_0$, $\omega_1$ and $\omega_2$ are cut-off functions with $\omega_2 \prec \omega_1 \prec \omega_0$. Denote $a(\lambda) = A - \lambda$ and

$$a(\lambda) = r^{-m} \sum_{k=0}^m a_k(r) (-r \partial_r)^k - \lambda \in L^m_m(X^\wedge; \Lambda).$$

Because of the presence of $\omega_0$ and $\omega_1$ in (3.1) we may assume that every $a_k(r)$ is independent of $r$ for $r > 2$. Then, for every $s, \gamma \in \mathbb{R}$ we have

$$a(\lambda) \in S^m_m(\Lambda; K^{s,\gamma}(X^\wedge), K^{s-m,\gamma-m}(X^\wedge))$$

(cf. Appendix C)

with the strongly continuous group of isomorphisms defined by

$$(\kappa_r u)(r, x) := \tau^{\frac{\omega_r}{\omega_0}} u(\tau r, x)$$

for $\tau > 0$ and $n = \dim X$. 


A proof of this statement can be found in [11] Prop. 3.1.1 or [10] Prop. 3.3.38. Finally, let

\[(3.3) \quad \sigma^m_{\lambda}(a)(\lambda) := r^{-m} \sum_{k=0}^{m} a_k(0)(-r\partial_r)^k - \lambda.\]

This operator-valued symbol satisfies the homogeneity relation

\[\sigma^m_{\tau\lambda}(a)(\tau^m \lambda) = \tau^m \kappa_{\tau} \sigma^m_{\lambda}(a)(\lambda) \kappa_{\tau}^{-1}\]

for all \(\tau > 0, \lambda \in \Lambda \setminus \{0\}\). For this reason, \((3.3)\) will be called the *twisted homogeneous principal symbol* of \(a(\lambda)\).

**Definition 3.1.** The family \(a(\lambda) = A - \lambda\) is said to be *parameter-elliptic* with respect to \(\gamma \in \mathbb{R}\) on \(\Lambda\) if and only if

(i) \(\sigma^m_{\nu,\nu}(A) - \lambda \neq 0\) on \(({^bT^*B} \times \Lambda) \setminus 0\),
(ii) \(\text{spec}_{\nu}(A) \cap \Gamma_{\nu,\nu} = \emptyset\), (cf. Definition 2.3)
(iii) \(\sigma^m_{\nu}(a(\lambda)) : \mathcal{K}^{s,\gamma}(X^\Lambda) \to \mathcal{K}^{s-m,\gamma-m}(X^\Lambda)\) is an isomorphism for some \(s \in \mathbb{R}\) and every \(\lambda \in \Lambda\) sufficiently large.

**Theorem 3.2.** If \(A - \lambda\) is parameter-elliptic with respect to \(\gamma\), then there is a constant \(R > 0\) such that \(A - \lambda : \mathcal{H}^{s,\gamma}(B) \to \mathcal{H}^{s-m,\gamma-m}(B)\) is invertible for every \(\lambda \in \Lambda\) with \(|\lambda| \geq R\), and all \(s \in \mathbb{R}\). Furthermore, there are constants \(C(s,\gamma,m) > 0\) and \(M(s,\gamma,m) \geq 0\) such that

\[\|(A - \lambda)^{-1}\|_{\mathcal{L}(\mathcal{H}^{s-m,\gamma-m})} \leq C(1 + |\lambda|)^{-1+M/m}.\]

**Proof.** In Theorem 3.4 we will construct a parametrix \(b(\lambda)\) of \(a(\lambda) = A - \lambda\) such that \(g = ab - 1 \in \mathcal{S}(\Lambda, C_G(B, (\gamma - m, \gamma - m)))\). Moreover, this construction will provide a family \(\{b(\lambda)\}_{\lambda \in \Lambda} \subset \mathcal{L}(\mathcal{H}^{s-m,\gamma-m}(B), \mathcal{H}^{s,\gamma}(B))\) that belongs to the class

\[S^{s-m+M,m}_{ct}(\Lambda; \mathcal{H}^{s-m,\gamma-m}(B), \mathcal{H}^{s-m,\gamma-m}(B))\]

for some \(M = M_{s,\gamma,m} > 0\), cf. Remark 3.14 and Appendix C.

Fix now \(s_0, \gamma, m > 0\). In \(\mathcal{L}(\mathcal{H}^{s_0,\gamma-m}(B))\) we have the estimate

\[\|a(\lambda)b(\lambda) - 1\| \leq C(1 + |\lambda|)^{-N} \text{ for every } N \in \mathbb{N}.\]

Hence for some \(R > 0\) we get \(|\|a(\lambda)b(\lambda) - 1\| \leq 1/2\) for all \(|\lambda| \geq R\). Therefore, \(a(\lambda)b(\lambda)\) is invertible in \(\mathcal{L}(\mathcal{H}^{s-m,\gamma-m}(B))\) for \(|\lambda| \geq R\), and \(\|a(\lambda)b(\lambda)\|^{-1} \leq 2\). In fact, the inverse of \(a(\lambda)\) is given by \(b(\lambda)[a(\lambda)b(\lambda)]^{-1}\). Furthermore,

\[\|a(\lambda)^{-1}\| = \|b(\lambda)[a(\lambda)b(\lambda)]^{-1}\| \leq 2\|b(\lambda)\| \leq C_{s_0,\gamma,m} (1 + |\lambda|)^{-1+M/m}\]

for some constant \(C_{s_0,\gamma,m}\). The last inequality is the symbol estimate of \(b(\lambda)\) as an operator-valued symbol of order \((-m + M,m)\). In order to see that for \(|\lambda| \geq R\) the operator \(a(\lambda)\) is invertible even for all \(s \in \mathbb{R}\), let \(\tilde{g} = ba - 1 \in \mathcal{S}(\Lambda, C_G(B, (\gamma, \gamma)))\). Then the inverse of \(a(\lambda)\) can also be written as

\[a(\lambda)^{-1} = b(\lambda) - b(\lambda)g(\lambda) + \tilde{g}(\lambda)a(\lambda)^{-1}g(\lambda)\]

which belongs to \(\mathcal{L}(\mathcal{H}^{s-m,\gamma-m}(B), \mathcal{H}^{s,\gamma}(B))\) for all \(s \in \mathbb{R}\) and \(|\lambda| \geq R\). Observe that the inverse of \(a(\lambda)\) appearing between \(\tilde{g}\) and \(g\) is the inverse with respect to \(s_0\).  \(\Box\)
Corollary 3.3. If $A - \lambda$ is parameter-elliptic with respect to $\gamma$ and $b(\lambda)$ is a corresponding parametrix, then there exists $R > 0$ such that

$$(A - \lambda)^{-1} - b(\lambda) \in S(\Lambda_R, C_C(\mathbb{B}, (\gamma - m, \gamma)))$$

where $\Lambda_R = \{ \lambda \in \Lambda \mid |\lambda| \geq R \}$.

Although the parameter-ellipticity introduced above seems to be a strong condition, it is indeed necessary for the invertibility of $A - \lambda$ on the weighted Sobolev spaces.

Theorem 3.4. Let $A \in \text{Diff}^m(\mathbb{B})_{\text{cone}}$ be such that $A - \lambda : \mathcal{H}^{s,\gamma}(\mathbb{B}) \to \mathcal{H}^{s-m,\gamma-m}(\mathbb{B})$ is invertible for all $\lambda \in \Lambda$ sufficiently large, and $\| (A - \lambda)^{-1} \|$ is uniformly bounded. Then $A - \lambda$ is parameter-elliptic with respect to $\gamma$ in the sense of Definition 3.1.

A proof of this theorem together with other results concerning resolvents of cone operators on arbitrary domains will be given in [12].

Example 3.5. Let $\beta \in C^\infty(X^\wedge)$ with $\beta(0) = 0$. Let $D(r)$ be a smooth family of elliptic selfadjoint differential operators on $X$ such that

$$\text{spec}_{L^2(X)}(D(0)) \subset (-\infty, \delta)$$

Then the Fuchs type operator $A_0 = r^{-2} \{ D(r) - (1 - n + \beta(r))(r\partial_r) + (r\partial_r)^2 \}$ is a positive self-adjoint operator on $K^{0,\gamma-2}(X^\wedge)$ with $K^{2,\gamma}(X^\wedge)$ as domain. Further let $A_1 \in \text{Diff}^2(\mathbb{B})$ be such that $A_1 - \lambda$ is parameter-elliptic on a conical set $\Lambda$ contained in the resolvent set of $A_0$. Then the operator family

$$(A - \lambda) := \omega_1(A_0 - \lambda)\omega_0 + (1 - \omega_1)(A_1 - \lambda)(1 - \omega_2), \quad \lambda \in \Lambda,$$

is parameter-elliptic with respect to $\gamma$ for any cut-off functions $\omega_2 < \omega_1 < \omega_0$.

Example 3.6. Let $A \in \text{Diff}^m(\mathbb{B})_{\text{cone}}$ be such that $A - \lambda$ is parameter-elliptic on $\Lambda$. If $B$ is a cone differential operator of order less than $m$, then for any sufficiently small $\varepsilon$ the family $A + \varepsilon B - \lambda$ is also parameter-elliptic.

3.2. Parametrix construction. Let us set again $a(\lambda) = A - \lambda$ for $\lambda \in \Lambda$. Using (3.1) and (3.2) we rewrite

$$(3.4) \quad a(\lambda) = \omega_1 a_\lambda(\lambda)\omega_0 + (1 - \omega_1)(P - \lambda)(1 - \omega_2).$$

Our strategy to find a parametrix of $a(\lambda)$ will be to construct a parametrix of $a_\lambda(\lambda)$ on $X^\wedge$ using techniques borrowed from the edge symbolic calculus introduced by Schulze, cf. [31], [46]. The parametrix construction for $P - \lambda$ is well-known and can be found, for instance, in [12, Section 11].

Definition 3.7. Let $\mu \in \mathbb{R}$, $d \in \mathbb{N}$ and let $\bar{d} := (1, d)$. Let $M_{C_G}^{\mu,d}(X; \Lambda)$ be the space of holomorphic functions

$$(z \mapsto h(z, \lambda) \in \mathcal{O}(\mathbb{C}, L^{\mu,d}(X; \Lambda)))$$

such that $h|_{\Gamma_\beta} \in L^{\mu,d}_{\text{eff}}(X; \Gamma_\beta \times \Lambda)$ for every $\beta \in \mathbb{R}$, uniformly for $\beta$ in compact intervals.

The elements of $C^\infty(\mathbb{R}_+, M_{C_G}^{\mu,d}(X; \Lambda))$ will be called parameter-dependent holomorphic Mellin symbols. The corresponding smoothing class is defined by replacing $\mu$ by $-\infty$ and omitting $d$. Recall that $\Gamma_\beta = \{ z \in \mathbb{C} \mid \Re z = \beta \}$.

\footnote{In this case, parameter-elliptic on $\Lambda$ just means $\sigma_{m}^{\mu}(A_1) - \lambda \neq 0$ on $T^*\mathbb{B} \times \Lambda \setminus 0$.}
Example 3.8. The operator-valued function $h(r, z, \lambda) = \sum_{k=0}^{m} a_k(r) z^k - r^m \lambda$ is clearly an element of $C^\infty(X^\land, M^{op}_{C^\infty}(X; \Lambda))$.

Let $C_G(\mathbb{B}, (\cdot, \cdot))$ denote the class of Green cone operators as defined in Appendix [3] and let $S(\Lambda, C_G(\mathbb{B}, (\cdot, \cdot)))$ be the space of rapidly decreasing $C_G$-valued functions.

Theorem 3.9. (Parametrix) If $a(\lambda) = A - \lambda$ is parameter-elliptic with respect to $\gamma$, there is a family $b(\lambda)$ of cone pseudodifferential operators of order $-m$ such that

$$b a - 1 \in S(\Lambda, C_G(\mathbb{B}, (\gamma, \gamma))) \quad \text{and} \quad ab - 1 \in S(\Lambda, C_G(\mathbb{B}, (\gamma - m, \gamma - m))).$$

The family $b(\lambda)$ is a (parameter-dependent) parametrix of $a(\lambda)$.

To prove this theorem let us first construct a parametrix of $a_\lambda$ on $X^\land$, i.e., a pseudodifferential family $b_\lambda(\lambda)$ such that

$$(3.5) \quad b_\lambda a_\lambda - 1 \in R_G^{-\infty}(\Lambda, (\gamma, \gamma)) \quad \text{and} \quad a_\lambda b_\lambda - 1 \in R_G^{-\infty}(\Lambda, (\gamma - m, \gamma - m)), $$

where $R_G^{-\infty}(\Lambda, (\cdot, \cdot))$ is the class of Green operator-valued symbols from Appendix [3].

The construction of $b_\lambda(\lambda)$ will be carried out in several steps.

Lemma 3.10. (Step 1) There is a family $b_1(\lambda)$ of pseudodifferential operators of order $-m$ such that

1. $a_\lambda(\lambda)b_1(\lambda) - 1 \in L^{-\infty}(X^\land; \Lambda)$,
2. $\sigma_M^0(a_\lambda b_1)(z) - 1 \in M^{-\infty}_C(X),$
3. $a_\lambda b_1 - 1 \in S^0_{c, \ell}(\Lambda; K^{s,\gamma-m}(X^\land), K^{s',\gamma-m}(X^\land)')$ for all $s, s', \sigma \in \mathbb{R}$, where

$$K^{s,\gamma-m}(X^\land) := (1 + r)^{-\sigma} K^{s',\gamma-m}(X^\land).$$

Lemma 3.11. (Step 2) There is a meromorphic function $f$ such that if

$$v_1(\lambda) := \omega_1(r[\lambda]_m) r^m \omega_0(\omega_0(r[\lambda]_m)$$

with $[\lambda]_m$ as in (3.2), then $a_\lambda(b_1 + v_1) - 1 \in R_G^0(\Lambda, (\gamma - m, \gamma - m)).$

Lemma 3.12. (Step 3) There is a Green symbol $g_1 \in R_G^{0, m}(\Lambda, (\gamma - m, \gamma))$ such that $b_\lambda^0 := b_1 + v_1 + g_1$ satisfies $\sigma_M^0(a_\lambda)^{-1} = \sigma_M^{-m}(b_\lambda^0)$. In particular, this implies $a_\lambda b_\lambda^0 - 1 \in R_G^{-1, m}(\Lambda, (\gamma - m, \gamma - m)).$

Lemma 3.13. (Step 4) There is a parametrix $b_\lambda$ of $a_\lambda$ such that (3.5) holds.

Proof of Lemma 3.10. In the representation (3.1) we may choose the coefficients $a_k$ such that $a_k(r) = a_k(0)$ for $r \geq 2$. Thus $a_\lambda(\lambda)$ may be assumed to be parameter-dependent elliptic on $\Lambda$ as an operator family living on $X^\land$. Therefore, there exists a family of pseudodifferential operators $b_0(\lambda) \in L^{c, m}_{c, \ell}(X^\land; \Lambda)$ such that

$$(3.6) \quad a_\lambda(\lambda)b_0(\lambda) - 1 \in L^{-\infty}(X^\land; \Lambda).$$

Moreover, $b_0(\lambda)$ can be written as $r^m \omega_0(r \lambda) \lambda^m \omega_0(\omega_0(r \lambda))$ with a symbol $q$ such that

$$q(r, x, \xi, \lambda) = \tilde{q}(r, x, \xi, r \lambda) \text{ for some } \tilde{q} \in S^{c, m}_{c, \ell}(\mathbb{R}^n + \Omega \times \mathbb{R}^n; \Lambda).$$

The so-called Mellin quantization theorem, see e.g. [3, Theorem 3.2], states that there is a Mellin symbol $h(r, z, \lambda) = h(r, z, r^m \lambda)$ with $h \in C^\infty(X^\land, M^{op}_{C^\infty}(X; \Lambda))$ such that

$$\omega_2(\omega_1^m(\tilde{q} - 1) - \omega_1^m(\tilde{q} - 1)) = L^{c, m}(X^\land; \Lambda).$$

Let $\omega_2 < \omega_1 < \omega_0$ be cut-off functions,

$$b_1(\lambda) := \omega_1(r[\lambda]_m) r^m \omega_1(\omega_0(r[\lambda]_m) + (1 - \omega_1(\omega_0(r[\lambda]_m))) b_0(\lambda)(1 - \omega_2(r[\lambda]_m))).$$
Then \(3\) is satisfied because of (3.6), and because \(\text{op}_{\mathcal{M}}(h)(\lambda) - b_0(\lambda) \in L^{-\infty}(X^{\lambda}; \Lambda)\). Further, \(3\) is a consequence of the holomorphy of \(h\) and the fact that \(\tilde{g}(0, x, r_0, \xi, 0)\) is the symbol of a local parametrix of \(\sum_{k=0}^m a_k(0)(-r\partial_r)^k\). Finally, since \(b_1(\lambda)\) is a parameter-dependent parametrix of \(a_\lambda(\lambda)\) in the interior, and because its operator norm in \(\mathcal{L}(K^{s, \gamma-m}, K^{s+m, \gamma})\) can be estimated by semi-norms of its local symbols, we get \(b_1 \in S^{m-m}_{\mathcal{M}}(\Lambda; K^{s, \gamma-m}, K^{s+m, \gamma})\).

Recall that \(a_\lambda\) is itself an operator-valued symbol of order \((m, m)\), and the local symbols of \(b_1(\lambda)\) can be expressed in terms of those of \(a_\lambda(\lambda)\). Hence \(a_\lambda b_1 - 1 \in S^{m-m}_{\mathcal{M}}(\Lambda; K^{s, \gamma-m}, K^{s+m, \gamma})\) for every \(s \in \mathbb{R}\). Together with \(1\) this implies \(3\).

**Proof of Lemma 3.11.** A necessary condition for \(a_\lambda(b_1 + v_1) - 1\) to be of Green type is that its conormal symbol vanishes. So, we need \(\sigma^0_M(a_\lambda(b_1 + v_1))(z) = 1\) for \(z \in \Gamma_{\frac{m+1}{m} - \gamma}\). Because the relation (2.3) is also valid for cone pseudodifferential operators and because \(\sigma^m_M(b_1 + v_1)(z) = h(0, z, 0) + f(z)\), we want \(f\) to satisfy

\[
\sigma^m_M(a_\lambda)(z-m)\sigma^m_M(b_1 + v_1)(z) = \sigma^m_M(A)(z-m)(h(0, z, 0) + f(z)) = 1.
\]

Observe that the ellipticity of \(A - \lambda\) implies the invertibility of \(\sigma^m_M(A)(z)\) on \(\Gamma_{\frac{m+1}{m} - \gamma}\), so that for \(z \in \Gamma_{\frac{m+1}{m} - \gamma+m}\) we may set

\[
f(z) := \sigma^m_M(A)^{-1}(z-m)(1 - \sigma^m_M(A)(z-m)h(0, z, 0)).
\]

This function is clearly meromorphic with poles contained in the boundary spectrum of \(A\). Now, for \(|\lambda| \geq 1\) the operators of multiplication by \(\omega_0(r|\lambda|)\) and \(\omega_1(r|\lambda|)r^m\) are twisted homogeneous of degree \((0, m)\) and \((-m, m)\), respectively. Moreover, \(v_1(\lambda)\) is smoothing for every \(\lambda\). Thus

\[
v_1 \in S^{-m,m}_{\mathcal{M}}(\Lambda; K^{s, \gamma-m}(X^{\lambda}), K^{s', \gamma}(X^{\lambda})) \quad \text{for all} \ s, s', \sigma \in \mathbb{R}.
\]

Finally, the family \(a_\lambda(\lambda)(b_1(\lambda) + v_1(\lambda)) - 1\) is of Green type because it is a pointwise smoothing operator-valued symbol with \(\sigma^0_M(a_\lambda(b_1 + v_1) - 1) = 0\).

**Proof of Lemma 3.12.** Due to Lemma 3.11 there is \(g_r \in R^0_{G}(\Lambda, (\gamma - m, \gamma - m)\varepsilon_r)\) such that \(a_\lambda(b_1 + v_1) - 1 = g_r\). In a similar way, we get \((b_1 + v_1)a_\lambda - 1 = g_i\) for some \(g_i \in R^0_{G}(\Lambda, (\gamma, \gamma)\varepsilon_i)\). Because \(\sigma^m_M(a_\lambda) = \sigma^m_M(a)\) is invertible for \(\lambda \in \Lambda\) sufficiently large, we obtain there the elementary relation

\[
\sigma^m_M(a_\lambda)^{-1} = \sigma^m_M(b_1 + v_1) - \sigma^m_M((b_1 + v_1)g_r) + \sigma^m_M(g_i)\sigma^m_M(a_\lambda)^{-1}\sigma^m_M(g_r).
\]

Let \(\chi \in C^{\infty}(\Lambda)\) be such that \(\chi \equiv 0\) where \(\sigma^m_M(a_\lambda)\) is not invertible, and \(\chi \equiv 1\) for \(\lambda\) sufficiently large. Then \(\sigma^m_M(a_\lambda)^{-1} \in C^{\infty}(\Lambda, \mathcal{L}(K^{s, \gamma-m}, K^{s, \gamma}))\) is twisted homogeneous of degree \((-m, m)\) and so in \(S^{-m,m}_{\mathcal{M}}(\Lambda; K^{s, \gamma-m}, K^{s, \gamma})\), cf. Example 3.6. Now, for \(g_0 := g_i\chi\sigma^m_M(a_\lambda)^{-1}g_r\), Proposition C.3 yields

\[
g_0 \in S^{m,m}_{\mathcal{M}}(\Lambda; K^{s, \gamma-m}(X^{\lambda}), K^{s', \gamma+\varepsilon}(X^{\lambda})\sigma) \quad \text{and} \quad g_0^* \in S^{-m,m}_{\mathcal{M}}(\Lambda; K^{s, \gamma}(X^{\lambda}), K^{s, -\gamma+m+\varepsilon}(X^{\lambda})\sigma)
\]

for all \(s, s', \sigma \in \mathbb{R}\). Let \(\varepsilon = \min(\varepsilon_i, \varepsilon_r)\). The continuous embeddings

\[
K^{s, \gamma+\varepsilon}(X^{\lambda}) \hookrightarrow K^{s', \gamma+\varepsilon}(X^{\lambda}) \quad \text{and} \quad K^{s, -\gamma+m+\varepsilon}(X^{\lambda}) \hookrightarrow K^{s', -\gamma+m+\varepsilon}(X^{\lambda})
\]

together with Proposition C.3 imply \(g_0 \in R^0_{G}(\Lambda, (\gamma - m, \gamma)\varepsilon)\). Thus the proof is done by setting \(g_1 := -(b_1 + v_1)g_r + g_0\).
Lemma 3.12 implies \((1 - a_{\lambda}b_{\lambda})^N \in R^N_G(\Lambda, (\gamma - m, \gamma - m))\) so that
\[
(3.7) \quad b_{\lambda}^{(N)} := b_{\lambda}^0 + \sum_{j=0}^{N-1} (1 - a_{\lambda}b_{\lambda})^j = b_{\lambda}^0 + \sum_{j=1}^{N-1} b_{\lambda}^0 (1 - a_{\lambda}b_{\lambda})^j
\]
is a rough parametrix of \(a_{\lambda}\) with a better remainder than \(b_{\lambda}^0\). Moreover, we have \(b_{\lambda}^0 (1 - a_{\lambda}b_{\lambda})^j \in R^{-m-m}_G(\Lambda, (\gamma - m, \gamma))\) for every \(j \in \mathbb{N}\). Therefore, there is a Green symbol \(g_2 \in R^{-m-1}_G(\Lambda, (\gamma - m, \gamma))\) such that \(g_2 \sim \sum_{j=1}^{\infty} b_{\lambda}^0 (1 - a_{\lambda}b_{\lambda})^j\). Finally,
\[
(3.8) \quad b_{\lambda}(\lambda) := b_{\lambda}^0(\lambda) + g_2(\lambda) = b_1(\lambda) + v_1(\lambda) + g_1(\lambda) + g_2(\lambda)
\]
is a parameter-dependent parametrix of \(a_{\lambda}(\lambda)\).

**Construction of a global parametrix.** Let us write \(a(\lambda) = A - \lambda\) as in (3.3), let \(Q(\lambda) \in L^{-m,m}_{ce}(\mathbb{B}; \Lambda)\) be a parametrix of \(P - \lambda\). With \(b_{\lambda}\) from Lemma 3.13 define
\[
(3.9) \quad b(\lambda) = \omega_1 b_{\lambda}(\lambda) \omega_0 + (1 - \omega_1) Q(\lambda)(1 - \omega_2)
\]
for \(\lambda \in \Lambda\) and \(\omega_2 \prec \omega_1 \prec \omega_0\). This parameter-dependent pseudodifferential operator of order \((-m, m)\) is a compound of ‘local’ parametrices. Next we want to show that \(b(\lambda)\) is in fact a global parametrix of \(a(\lambda)\). To this end let \(\omega\) and \(\omega_3\) be cut-off functions such that \(\omega_0 \prec \omega\) and \(\omega_3 \prec \omega_2\). Then
\[
b(\lambda)a(\lambda) = \left[\omega_1 b_{\lambda}(\lambda) \omega_0 + (1 - \omega_1) Q(\lambda)(1 - \omega_2)\right] a(\lambda)
\]
\[
= \omega_1 b_{\lambda}(\lambda) \omega_0 a(\lambda) \omega + (1 - \omega_1) Q(\lambda)(1 - \omega_2) a(\lambda)(1 - \omega_3)
\]
\[
+ \omega_1 b_{\lambda}(\lambda) \left[\omega_0 a(\lambda)(1 - \omega)\right] + (1 - \omega_1) Q(\lambda)\left[(1 - \omega_2) a(\lambda)\omega_3\right]
\]
\[
= \omega_1 b_{\lambda}(\lambda) \omega_0 a(\lambda) \omega + (1 - \omega_1) Q(\lambda)(1 - \omega_2) a(\lambda)(1 - \omega_3)
\]
\[
= \omega_1 b_{\lambda}(\lambda) \omega_0 a(\lambda) \omega + (1 - \omega_1) Q(\lambda)(1 - \omega_2)(P - \lambda)(1 - \omega_3)
\]
\[
= \omega_1 b_{\lambda}(\lambda) a(\lambda) \omega + (1 - \omega_1) Q(\lambda)(P - \lambda)(1 - \omega_3)
\]
\[
= \omega_1 b_{\lambda}(\lambda) a(\lambda) \omega + (1 - \omega_1) Q(\lambda)(P - \lambda)(1 - \omega_3)
\]
\[
= \omega_1 b_{\lambda}(\lambda) \omega_0 a(\lambda) \omega + (1 - \omega_1) Q(\lambda)(1 - \omega_2)(P - \lambda)(1 - \omega_3)
\]
\[
= \omega_1 b_{\lambda}(\lambda) a(\lambda) \omega + (1 - \omega_1) Q(\lambda)(P - \lambda)(1 - \omega_3)
\]
\[
= \omega_1 b_{\lambda}(\lambda) \omega_0 a(\lambda) \omega - [(1 - \omega_1) Q(\lambda)\omega_2](P - \lambda)(1 - \omega_3)
\]
Here we have used the local property of \(a(\lambda)\). Now, the terms inside the brackets in the last line are smoothing elements in their classes because \(\text{supp}(\omega_1) \cap \text{supp}(1 - \omega_0)\) and \(\text{supp}(1 - \omega_1) \cap \text{supp}(\omega_2)\) are both empty. Because of the inclusions
\[
R^{-\infty}_G(\Lambda, (\gamma, \gamma))_{\epsilon} \subset S(\Lambda, \mathcal{L}(K^{s, \gamma - m}, K^{s', \gamma - m})) \quad \text{and} \quad L^{-\infty}(\mathbb{B}; \Lambda) \subset S(\Lambda, \mathcal{L}(H^s, H^{s'}))
\]
for all \(s, s' \in \mathbb{R}\), we can easily verify that \(ba - 1 \in S(\Lambda, C_G(\mathbb{B}, (\gamma, \gamma))_{\epsilon})\). Similarly, we obtain \(ab - 1 \in S(\Lambda, C_G(\mathbb{B}, (\gamma - m, \gamma - m))_{\epsilon})\) so Theorem 3.9 is proved.

**Remark 3.14.** The given construction of \(b_{\lambda}\) yields immediately
\[
b_{\lambda} \in S^{m, -m}_{ce}(\Lambda; K^{s-m, \gamma - m}(X^\wedge), K^{s', \gamma}(X^\wedge)) \quad \text{for all} \ s \in \mathbb{R}.
\]
This implies that \(b(\lambda)\) from (3.9) belongs to \(S^{m+M, m}(\Lambda; H^{s-m, \gamma - m}(\mathbb{B}), H^{s, \gamma}(\mathbb{B}))\) for some \(M\) depending on the norm estimates of \(Q(\lambda)\), and depending on the constants of growth corresponding to \(|\kappa(\lambda)|_{L(K^{s, \gamma})}\) and \(|\kappa^{-1}(\lambda)|_{L(K^{s-m, \gamma - m})}\), cf. Lemma 3.1.

The spaces on \(\mathbb{B}\) are here equipped with the trivial group action \(\kappa = \text{id}\).
Remark 3.15. The definition of $b_1(\lambda)$ in the proof of Lemma 3.10 by means of parameter-dependent cut-off functions, is just convenient for technical purposes. In that way it is easier to verify that $b_1(\lambda)$ is an operator-valued symbol. But in fact, there exists $g \in R_{\lambda}^{-m,m}(\Lambda, (\gamma - m, \gamma))$ (cf. Theorem 3.18) such that

\[ b(\lambda) = \omega_1(\tau^{m,0}_{\lambda}(h) - \lambda) + v(\lambda) + g(\lambda) + (1 - \omega_1) Q(\lambda) (1 - \omega_2) \]

is also a global parametrix of $A - \lambda$, where $h$ is the symbol found in the proof of Lemma 3.11, $v(\lambda)$ is like in Lemma 3.13, and $Q(\lambda)$ is as in (3.9).

3.3. Holomorphic weakly parametric symbols. Let $\Omega \subset \mathbb{R}^n$ be an open set and $\Lambda$ be a sector in $\mathbb{C}$. Let $\Gamma_\delta = \{ z \in \mathbb{C} \mid \Re z = \delta \}$.

Definition 3.16. For $\mu \in \mathbb{R}$ define $S^\mu(\Omega \times \mathbb{R}^n \times \mathbb{C})$ as the class of holomorphic functions $h \in \mathcal{O}(\mathbb{C}, S^\mu(\Omega \times \mathbb{R}^n))$ such that $h|_{\Gamma_\delta} \in S^\mu(\Omega \times \mathbb{R}^n; \Gamma_\delta)$ for each $\delta \in \mathbb{R}$, uniformly for $\delta$ in compact intervals.

This is a Fréchet space with the system of semi-norms given by

\[ (3.10) \sup_{\delta \in I} \{ \sup_{(\xi, \varphi) \in \mathbb{R}^{n+1}} | \delta^\alpha \partial_\xi^\beta \partial_\varphi^\ell h(x, \xi, \delta + i \varphi)| \} \]

for $\alpha, \beta \in \mathbb{N}_0^n$, $K \subset \subset \Omega$, $\ell \in \mathbb{N}_0$ and $I \subset \subset \mathbb{R}$.

Definition 3.17. For $d \in \mathbb{N}$ define $S^\mu_d(\Omega \times \mathbb{R}^n \times \mathbb{C}; \Lambda)$ as the class of parameter-dependent symbols consisting of all $h(x, \xi, z, \lambda)$ such that

(i) $z \mapsto h(\cdot, \cdot, z, \cdot) \in \mathcal{O}(\mathbb{C}, S^\mu_d(\Omega \times \mathbb{R}^n; \Lambda))$;
(ii) $h(\cdot, \cdot, \delta + i \varphi, \lambda) \in S^\mu_d(\Omega \times \mathbb{R}^n; \Gamma_\delta \times \Lambda)$ for each $\delta \in \mathbb{R}$, uniformly for $\delta$ in compact intervals, and with $d = (1, d)$ (cf. Appendix A).

Furthermore, a symbol $h \in S^\mu_d(\Omega \times \mathbb{R}^n \times \mathbb{C}; \Lambda)$ is said to depend holomorphically on $\lambda$ if the map

\[ \lambda \mapsto h(\cdot, \cdot, \cdot, \lambda) : \Lambda \to S^\mu(\Omega \times \mathbb{R}^n; \Gamma_\delta) \]

is holomorphic for every $\delta \in \mathbb{R}$. Observe that (ii) implies $h(\cdot, \cdot, \cdot, \lambda) \in S^\mu(\Omega \times \mathbb{R}^n; \Gamma_\delta)$. Finally, if for every $\delta \in \mathbb{R}$ the parameter-dependent symbols in (ii) are asked to be classical, then we will write $h \in S^\mu_d(\Omega \times \mathbb{R}^n \times \mathbb{C}; \Lambda)$. In the case when $h$ is classical there is, for every $\delta \in \mathbb{R}$, an expansion

\[ h \sim \sum h_{\mu-j} \text{ in } S^\mu_d(\Omega \times \mathbb{R}^n; \Gamma_\delta \times \Lambda) \]

such that for $|\xi| + |\varphi| + |\lambda|^{1/d} \geq 1$,

\[ h_{\mu-j}(x, \tau \xi, \delta + i \tau \varphi, \tau^d \lambda) = \tau^{\mu-j} h_{\mu-j}(x, \xi, \delta + i \varphi, \lambda) \text{ for every } \tau \geq 1. \]

When $\delta$ is fixed will we identify $\Gamma_\delta \cong \mathbb{R}$ and then replace in the argument of the symbols $\delta + i \varphi$ by $\varphi$.

Lemma 3.18. If $h \in S^\mu_d(\Omega \times \mathbb{R}^n \times \mathbb{C}; \Lambda)$ depends holomorphically on $\lambda$, then for any fixed $\delta \in \mathbb{R}$ every homogeneous component $h_{\mu-j} \in S^\mu_{\mu-j, d}(\Omega \times \mathbb{R}^n; \Gamma_\delta \times \Lambda)$ is holomorphic on $\Lambda \cap \{|\lambda| > 1\}$, as a function taking values in $S^\mu_{\mu-j}(\Omega \times \mathbb{R}^n; \Gamma_\delta)$. 

Proof. For simplicity we omit the variable $x$ and assume $d = 1$; the general case is completely analogous. We first show the holomorphy of the principal symbol $h_\mu$ making use of the relation

$$h_\mu(\xi, \varrho, \lambda) = \lim_{\tau \to \infty} \tau^{-\mu} h(\tau \xi, \tau \varrho, \tau \lambda).$$

If the convergence in (3.11) is compact, that is, uniformly on any compact subset of $\Lambda \cap \{ |\lambda| > 1 \}$, then $h_\mu$ is holomorphic there since so is $h$; consequently, $h - h_\mu$ is also holomorphic and we can proceed as above to assert the same for $h_{\mu - 1}$. Induction then yields the holomorphy of every $h_{\mu - j}$.

To prove the compact convergence of (3.11) in the topology of $S^\mu(\mathbb{R}^n; \Gamma_\delta)$ let $\alpha \in \mathbb{N}^n_0$ and $\ell \in \mathbb{N}_0$ be given. For $\tau \geq 1$ and $|\lambda| \geq 1$ we have

$$h_\mu(\xi, \varrho, \lambda) - \tau^{-\mu} h(\tau \xi, \tau \varrho, \tau \lambda) = \tau^{-\mu} \left( h_\mu(\tau \xi, \tau \varrho, \tau \lambda) - h(\tau \xi, \tau \varrho, \tau \lambda) \right).$$

Denoting $r_{\mu - 1} := h_\mu - h \in S^{{\mu - 1},d}(\mathbb{R}^n; \Gamma_\delta \times \Lambda)$ then

$$\tau^{-\mu} \left| \partial^\alpha \varrho^\ell r_{\mu - 1}(\tau \xi, \tau \varrho, \tau \lambda) \right| \leq C_{\alpha, \ell} \tau^{-\mu+|\alpha|+\ell} \left| \partial^\alpha \varrho^\ell r_{\mu - 1}(\tau \xi, \tau \varrho, \tau \lambda) \right| \leq C'_{\alpha, \ell} \tau^{-|\alpha|-|\lambda|} \tau^{-1},$$

because $\langle \tau \eta, \tau \lambda \rangle \leq 2 \tau \langle \eta \rangle \langle \lambda \rangle$ and $\langle \tau \eta, \tau \lambda \rangle^{-1} \leq \tau^{-1} \langle \eta \rangle^{-1}$ for $|\lambda| \geq 1$. So,

$$\sup_{\xi, \varrho} \left| \partial^\alpha \varrho^\ell (h_{\mu}(\xi, \varrho, \lambda) - \tau^{-\mu} h(\tau \xi, \tau \varrho, \tau \lambda)) \right| \leq C \langle \lambda \rangle^{-|\alpha|-|\lambda|} \tau^{-1}.$$

This implies that the convergence in (3.11) is compact for $|\lambda| > 1$.

**Weakly parametric symbols.** For our purposes we need an anisotropic version of a symbol class introduced by Grubb and Seeley in [22]. Let $d$ denote the anisotropy.

**Definition 3.19.** For $d \in \mathbb{N}$ and $\nu \in \mathbb{R}$ define $S^\nu_w(\Omega \times \mathbb{R}^n \times \mathbb{R}; \Lambda)$ as the space of functions $h \in C^\infty(\Omega \times \mathbb{R}^n \times \mathbb{R} \times \Lambda)$ that are holomorphic in $\lambda$ for $|\lambda| > 1$, and such that for every $k$,

$$\partial^k h(\cdot, \cdot, \frac{1}{w}) \in S^{\nu + k}(\Omega \times \mathbb{R}^n \times \mathbb{R}) \quad \text{for } \frac{1}{w^d} \in \Lambda, \text{ uniformly for } |w| \leq 1.$$

Note that there are $d$ convex subsets of $\{ w \in \mathbb{C} | \frac{1}{w} \in \Lambda \text{ and } |w| \leq 1 \}$ such that $w \mapsto \frac{1}{w^d}$ maps each one onto $\Lambda \cap \{ |\lambda| \geq 1 \}$. We fix one of these subsets and denote it by $D(\Lambda)$. The requirement for $h$ means that for every $k \in \mathbb{N}_0$,

$$\pi_{w,k}(h) := \sup_{(\xi, \varrho) \in \mathbb{R}^{n+1}} \langle \xi, \varrho \rangle^{-\nu-k+|\alpha|+\ell} \left| \partial^\alpha \varrho^\ell \partial^k h(x, \xi, \varrho, \frac{1}{w}) \right|$$

must be uniformly bounded in $D(\Lambda)$ for every $\alpha, \beta \in \mathbb{N}^n_0$, $K \subset \subset \Omega$ and $\ell \in \mathbb{N}_0$.

**Theorem 3.20.** Let $h \in S^\nu_w(\Omega \times \mathbb{R}^n \times \mathbb{R}; \Lambda)$, $d \in \mathbb{N}$ and $\nu \in \mathbb{R}$. For any $N \in \mathbb{N}$ there are symbols $h_k \in S^{\nu + k}(\Omega \times \mathbb{R}^n \times \mathbb{R})$ such that

$$\Lambda^{N/d} \left\{ h(x, \xi, \varrho, \lambda) - \sum_{k=0}^{N-1} \lambda^{-k/d} h_k(x, \xi, \varrho) \right\} \in S^\nu_w(\Omega \times \mathbb{R}^n \times \mathbb{R}; \Lambda).$$
Proof. The variable $x$ will be omitted again since it does not play any role along the proof. Let us set

$$f(\xi, \varrho, w) := h(\xi, \varrho, \frac{1}{w^\nu}) \text{ for } w \in D(\Lambda).$$

For every $k \in \mathbb{N}_0$ the mapping $w \mapsto \partial_{w}^{k+1} f : D(\Lambda) \to S^{\nu+k+1}$ is bounded by definition, thus $w \mapsto \partial_{w}^{k} f : D(\Lambda) \to S^{\nu+k+1}$ is uniformly continuous (fundamental theorem of calculus) implying that $h_k := \frac{1}{n!} \lim_{w \to 0} \partial_{w}^{k} f(w)$ exists in $S^{\nu+k+1}(\mathbb{R}^n \times \mathbb{R})$; note that $0 \in \partial D(\Lambda)$. In particular, for every $(\xi, \varrho) \in \mathbb{R}^n \times \mathbb{R}$ and $\alpha \in \mathbb{N}_0^{n+1}$ we have the pointwise convergence $\partial_{\xi, \varrho}^{\alpha} \partial_{w}^{k} f(\xi, \varrho, w) \to \partial_{\xi, \varrho}^{\alpha} h_k(\xi, \varrho)$ as $w \to 0$, and so

$$\langle \xi, \varrho \rangle^{-\nu-k+|\alpha|} \partial_{\xi, \varrho}^{\alpha} \partial_{w}^{k} f(\xi, \varrho, w) \to \langle \xi, \varrho \rangle^{-\nu-k+|\alpha|} \partial_{\xi, \varrho}^{\alpha} h_k(\xi, \varrho) \text{ as } w \to 0.$$ 

Therefore, $h_k \in S^{\nu+k}(\mathbb{R}^n \times \mathbb{R})$ since $w \mapsto \partial_{w}^{k} f$ is bounded in $S^{\nu+k}(\mathbb{R}^n \times \mathbb{R})$. Now, for every $N \in \mathbb{N}$ a Taylor expansion yields

$$w^{-N} \left\{ f(\xi, \varrho, w) - \sum_{k=0}^{N-1} w^k h_k(\xi, \varrho) \right\} = \frac{1}{(N-1)!} \int_0^1 (1-t)^{N-1} \left( \partial_{w}^{N} f(\xi, \varrho, tw) \right) dt.$$ 

The left side, written in terms of $\lambda = \frac{1}{w^\nu}$, is exactly the expression in (3.13). Denoting the integral above by $r_N(\xi, \varrho, \frac{1}{w^\nu})$ we have for every $k$,

$$\partial_{w}^{k} r_N(\xi, \varrho, \frac{1}{w^\nu}) = \int_0^1 (1-t)^{N-1} t^k \left( \partial_{w}^{N} f(\xi, \varrho, tw) \right) dt.$$ 

We want to show that $\partial_{w}^{k} r_N(\cdot, \cdot, \frac{1}{w^\nu}) \in S^{\nu+N+k}(\mathbb{R}^n \times \mathbb{R})$ uniformly for $w \in D(\Lambda)$, but this follows from the estimates

$$\sup_{w \in D(\Lambda)} \pi_{w, k}(r_N) \leq \int_0^1 (1-t)^{N-1} t^k \sup_{w \in D(\Lambda)} \pi_{tw, N+k}(h) dt \leq \int_0^1 (1-t)^{N-1} t^k \sup_{tw \in D(\Lambda)} \pi_{tw, N+k}(h) dt$$

taking into account that $h \in S^{\nu,d}_W$ and so, in particular, the supremum in the latter integral is uniformly bounded.

The expansion (3.13) is the main motivation for the consideration of the class $S^{\nu,d}_W$. As it was done in [22] this kind of expansions can be used to obtain a complete asymptotic expansion of the resolvent of certain pseudodifferential operators on smooth compact manifolds (cf. [23]). In particular, Dirac-type operators with nonlocal boundary conditions in the spirit of Atiyah, Patodi and Singer [3] are considered in [23]. Further related results can be found in [20], [21] and [23].

Lemma 3.21. Let $\nu \leq 0$ and $d \in \mathbb{R}$. Let $h \in S^{\nu,d}(\Omega \times \mathbb{R}^n; \Gamma \delta \times \Lambda)$ be homogeneous of degree $(\nu, d)$ for $|\xi| + |\varrho| + |\lambda|^{1/d} \geq 1$, and such that it depends holomorphically on $\lambda$ for $|\lambda| > 1$. Then $h \in S^{\nu,d}_W(\Omega \times \mathbb{R}^n \times \mathbb{R}; \Lambda)$, identifying $\Gamma \delta \cong \mathbb{R}$.

Proof. Without loss of generality we omit the variable $x$. It is true that

$$h(\xi, \varrho, \frac{1}{w}) = |w|^{-\nu} h(|w||\xi|, |\varrho||w|^d \frac{1}{w}) \text{ for } w \in D(\Lambda).$$

Let $w = re^{i\theta} \in D(\Lambda)$, i.e., $r \leq 1$ and $e^{-i\theta} \in \Lambda$. Hence

$$h(\xi, \varrho, \frac{1}{w}) = r^{-\nu} h(r\xi, r\varrho, e^{-i\theta}) = : f(\xi, \varrho, r)$$
leading to the relation (because $h$ is holomorphic in $w$)

$$\partial^k_w h(\xi, \varrho, \frac{1}{w}) = (\partial^k_w f)(\xi, \varrho, r)e^{-ik\theta} \text{ for every } k \in \mathbb{N}.$$ 

To prove the assertion we need $\sup_{w \in D(\Lambda)} \pi_{w,k}(h) < \infty$ for every semi-norm as in (3.12). Now, setting $\lambda_0 = e^{-i\theta}$ we have

$$(\partial^k_w f)(\xi, \varrho, r) = \partial^k_w \left(r^{-\nu} h(r\xi, r\varrho, \lambda_0)\right) = \sum_{j+j' = k} C_{jj'} r^{-\nu-j'} \partial^j_w h(r\xi, r\varrho, \lambda_0) = \sum_{j+j' = k} C_{jj'} r^{-\nu-j'} \sum_{|\beta|+k' = j} C_{\beta k'} \xi^\beta r^{k'} \partial^{\beta}_{\xi} (\partial^{k'}_{\varrho} h)(r\xi, r\varrho, \lambda_0).$$

Because $|\xi^\beta r^{k'}| \leq \langle \xi, r \rangle$, and since $\partial^{\beta}_{\xi} h$ is a symbol of order $\nu - j$, then for $\alpha \in \mathbb{N}^n_0$ and $\ell \in \mathbb{N}$ the derivatives $\partial^{\beta}_{\xi} \partial^{k'}_{\varrho} h(\xi, \varrho, \frac{1}{w})$ can be estimated by terms of the form

$$\sum_{\nu+j' \leq 0} C_{jj'} r^{-\nu-j'} \sum_{\alpha_1 + \alpha_2 = \alpha} \sum_{\ell_1 + \ell_2 = \ell} C_{\alpha \ell} \langle \xi, \varrho \rangle^{j-j-|\alpha_1|-\ell_1} \langle r\xi, r\varrho \rangle^{\nu-j-|\alpha_2|+\ell_2} r^{\alpha_2+\ell_2}$$

Using now the inequality $r^s (r\xi, r\varrho)^{-s} \leq \langle \xi, \varrho \rangle^{-s}$ for $r \leq 1$ and $s \geq 0$, we get

$$|\partial^k_w f^\ell (\xi, \varrho, \frac{1}{w})| \leq C \langle \xi, \varrho \rangle^{\nu+k-|\alpha|-\ell}$$

with a constant $C$ depending uniformly on $\theta = \arg w$ and being independent of the variable $r$. Hence $\sup_{w \in D(\Lambda)} \pi_{w,k}(h)$ is finite.

**Lemma 3.22.** Let $\nu \leq 0$ and $h \in S^{\nu,d}_{0,\ell}(\Omega \times \mathbb{R}^n \times \mathbb{C}; \Lambda)$. If $h$ depends holomorphically on $\lambda$, its restriction from $\mathbb{C}$ to any $\Gamma_\delta$ induces a symbol in $S^{\nu,d}_{w',\ell}(\Omega \times \mathbb{R}^n \times \mathbb{R}; \Lambda)$.

**Proof.** As it was done before we ignore for a moment the variable $z$. For any $\delta \in \mathbb{R}$ the symbol $h \in S^{\nu,d}_{0,\ell}(\mathbb{R}^n \times \mathbb{C}; \Lambda)$ admits an asymptotic expansion

$$r_N = h - \sum_{j=0}^{N-1} h_{\nu-j} \in S^{\nu-N,\ell}_{0,\ell}(\mathbb{R}^n \times \mathbb{R}; \Lambda) \text{ for any } N \in \mathbb{N}$$

with symbols $h_{\nu-j} \in S^{\nu-j,\ell}$ that are homogeneous for $|\xi| + |\varrho| + |\lambda|^{1/d} \geq 1$, and are holomorphic in $\lambda$ for $|\lambda| > 1$ due to Lemma 3.18. In order to prove the claim we have to investigate $\sup_{w \in D(\Lambda)} \pi_{w,k}(h)$ for any semi-norm in $S^{\nu,d}_{w',\ell}(\mathbb{R}^n \times \mathbb{R}; \Lambda)$. To this end let $k \in \mathbb{N}_0$ be given. For any homogeneous component each semi-norm is bounded due to Lemma 3.22, so we only need to show $\sup_{w \in D(\Lambda)} \pi_{w,k}(r_N) < \infty$ for some $N$. In fact,
choosing \(N \geq \nu + k\) we achieve
\[
\left| \partial^\nu_w \partial^k_w r_N(\xi, \theta, \frac{1}{w^n}) \right| \leq \sum_{k'=1}^k C_{k'} |w|^{-k-dk'} \left| (\partial^\nu_w \partial^k_w r_N)(\xi, \theta, \frac{1}{w^n}) \right|
\]
\[
\leq \sum_{k'=1}^k C_{k'} |w|^{-k-dk'} \langle \xi, \theta, \frac{1}{w^n} \rangle^{\nu-N-|\alpha|} \langle \xi, \theta, \frac{1}{w^n} \rangle^{-\ell} \langle \xi, \theta, \frac{1}{w^n} \rangle^{\nu-N+k-|\alpha|} - \ell
\]
\[
\leq C \langle \xi, \theta \rangle^{\nu+k-|\alpha|-\ell} \text{ with } C > 0 \text{ independent of } w.
\]

In the last estimate one uses the relations \(|w|^s \langle \xi, \theta, \frac{1}{w^n} \rangle^{-s} \leq C_s \) for \(s \geq 0\), \(|w| \leq 1\), and \(\langle \xi, \theta, \frac{1}{w^n} \rangle^{\nu+N+k} \leq \langle \xi, \theta \rangle^{\nu+k} \) for \(N \geq \nu + k\).

**Theorem 3.23.** For \(\nu \leq 0\) and \(\mu \in \mathbb{N}_0\) every symbol \(h \in S^{\nu-\mu, d}_{\mathcal{O}, cf} (\Omega \times \mathbb{R}^n \times \mathbb{C}; \Lambda)\) depending holomorphically on \(\lambda\) admits, for any \(N \in \mathbb{N}\), the expansion
\[
(3.14) \quad \lambda^{(N+\mu)/d} \left\{ h(x, \xi, \delta + ig, \lambda) - \sum_{k=0}^{N-1} \lambda^{(-k-\mu)/d} h_k(x, \xi, \delta + ig) \right\} \in S^{\nu+N,d}_W
\]
with \(h_k \in S^{\nu+k}(\Omega \times \mathbb{R}^n \times \mathbb{C})\) given by
\[
(3.15) \quad h_k(x, \xi, z) = \frac{1}{k!} \lim_{w \to 0} \partial^k_w \left( w^{-\mu} h(x, \xi, z, \frac{1}{w^n}) \right), \quad w \in D(\Lambda).
\]

**Proof.** Due to the inclusion \(\lambda^{\mu/d} S^{\nu-\mu,d}_{\mathcal{O}, cf} \subset S^{\nu,d}_{\mathcal{O}, cf}\) it is enough to prove the statement for \(\mu = 0\). For simplicity, let us drop the variable \(x\). Lemma 3.22 assures the existence of an expansion (3.13) of \(h|_{z \in \Gamma_\delta}\) for every \(\delta \in \mathbb{R}\). As proven in Theorem 3.20 the coefficients \(h_k\) are indeed given by (3.15) on every line \(\Gamma_\delta\). It only remains to prove that \(h_k \in S^{\nu+k}(\mathbb{R}^n \times \mathbb{C})\). In other words, we have to verify:

(i) \(h_k \in S^{\nu+k}(\mathbb{R}^n \times \Gamma_\delta)\) uniformly for \(\delta\) in compact intervals,

(ii) \(z \to h_k(\cdot, z) \in \mathcal{O}(\mathbb{C}, S^{\nu+k}(\mathbb{R}^n))\).

To this end let \(I \subset \subset \mathbb{R}\) be a compact interval. Since every semi-norm \(\pi_{w,\lambda}(h|_{\Gamma_\delta})\) is uniformly bounded for \(w \in D(\Lambda)\) and \(\delta \in I\) then (i) follows by a pointwise consideration as in the proof of Theorem 3.20. Furthermore, the convergence in (3.15), taking place in \(S^{\nu+k+1}\), is uniform for \(\delta \in I\) hence compact in \(\mathbb{C}\). Thus \(h_k\) is holomorphic with values in \(S^{\nu+k+1}(\mathbb{R}^n)\). By using now (i) and the fundamental theorem of calculus we obtain for neighboring \(z\) and \(\tilde{z}\) the estimates
\[
|\partial^\nu_w (h_k(\xi, z) - h_k(\xi, \tilde{z})| \leq |z - \tilde{z}| \int_0^1 |\partial^\nu_w \partial_t h_k(\xi, \tilde{z} + t(z - \tilde{z}))| \, dt
\]
\[
\leq |z - \tilde{z}| C \langle \xi \rangle^{\nu+k-|\alpha|} \int_0^1 \langle \partial_t \rangle^{\nu+k-|\alpha|} \, dt
\]
with \(\exists \tilde{z} \leq \partial_t \leq \exists z = \vartheta\). That is, the mapping \(z \to h_k(\cdot, z) : \mathbb{C} \to S^{\nu+k}(\mathbb{R}^n)\) is continuous which implies, in particular, that the Cauchy integral \(\frac{1}{2\pi i} \int_B \frac{h_k(\xi, \zeta)}{\zeta - z} \, d\zeta\) exists in \(S^{\nu+k}(\mathbb{R}^n)\) for every disk \(B\). Hence (ii) holds and the proof is done.

**Remark 3.24.** This theorem will be applied to the homogeneous components of the pseudodifferential Mellin symbols appearing in the parametrix construction of \(A - \lambda\), cf. Lemma 3.10. These homogeneous symbols clearly depend holomorphically on \(\lambda\).
4. Heat Trace Asymptotics

Along this section $A$ will always denote an elliptic cone differential operator of order $m > 0$ on the manifold $\mathbb{B}$, cf. Section B.2.

4.1. The operator $e^{-tA}$ in the cone algebra. Fix $\delta > 0$ and $0 < \varphi < \frac{\pi}{2}$. We consider the contour $\Upsilon = \Upsilon_1 \cup \Upsilon_2 \cup \Upsilon_3$ in $\mathbb{C}$, where

$$
\lambda = re^{i\varphi} \quad (+\infty > r \geq \delta) \quad \text{on} \ \Upsilon_1,
$$

$$
\lambda = \delta e^{i\theta} \quad (\varphi \leq \theta \leq 2\pi - \varphi) \quad \text{on} \ \Upsilon_2,
$$

$$
\lambda = re^{-i\varphi} \quad (\delta \leq r < +\infty) \quad \text{on} \ \Upsilon_3.
$$

Let $\Lambda$ be the sector $\{ \lambda \in \mathbb{C} \mid \varphi \leq \arg \lambda \leq 2\pi - \varphi \}$ and let $\Lambda_\delta = \Lambda \cap \{|\lambda| \geq \delta\}$. We assume that $A - \lambda$ is invertible on an open neighborhood $U(\Lambda_\delta)$ of $\Lambda_\delta$. Using the identity $(A - \lambda)^{-1}(1 - (A - \lambda_0)(A - \lambda_0)^{-1}) = (A - \lambda_0)^{-1}$ and the embedding properties of our weighted Sobolev spaces, it can be easily proven (via Neumann series) that the mapping $\lambda \mapsto (A - \lambda)^{-1} : U(\Lambda_\delta) \to L(H^{s-m,\gamma-m}(\mathbb{B}), H^{s,\gamma}(\mathbb{B}))$ is holomorphic.

For $t > 0$ we now define, as usual,

$$
e^{-tA} = \frac{i}{2\pi} \int_{\Upsilon} e^{-t\lambda}(A - \lambda)^{-1}d\lambda.
$$

This integral converges absolutely in $L(H^{s-m,\gamma-m}(\mathbb{B}), H^{s,\gamma}(\mathbb{B}))$ for any real $s$, and it does not depend on $\Upsilon$ provided that $\Upsilon$ is a path in $\Lambda_\delta$ such that $\Re \lambda \to \infty$ as $|\lambda| \to \infty$. In particular, for $0 < t < 1$ the path $t^{-1}\Upsilon$ stays inside $\Lambda_\delta$ and we can write

$$
e^{-tA} = \frac{i}{2\pi} \int_{t^{-1}\Upsilon} e^{-t\lambda}(A - \lambda)^{-1}d\lambda.
$$

This representation will be more convenient for some computations later on.

**Theorem 4.1.** Let $A - \lambda$ be parameter-elliptic with respect to $\gamma$. Then, for every $t > 0$ the operator $e^{-tA}$ belongs to $C_G(\mathbb{B}, (\gamma - m, \gamma))$.

**Proof.** Let $t > 0$ be fixed. We need to show certain mapping properties according to Definition B.2. First of all, observe that (integration by parts)

$$
e^{-tA} = \frac{i}{2\pi} \int (A - \lambda)^{-1} d\lambda.
$$

for every $\ell \in \mathbb{N}$. Thus $e^{-tA}$ and its formal adjoint $(e^{-tA})^*$ are for every $s, s' \in \mathbb{R}$ bounded operators in $L(H^{s,\gamma-m}, H^{s',\gamma})$ and $L(H^{s',\gamma-m}, H^{s,\gamma})$, respectively. Hence they are smoothing operators in the interior, but in order to be in $C_G(\mathbb{B}, (\gamma - m, \gamma))$ these operators have to improve the weights by some $\varepsilon > 0$. We prove this by making use of the parametrix $b(\lambda)$ of $A - \lambda$ given in (3.9). Then $b(\lambda)^\ell$ is a parametrix of $(A - \lambda)^\ell$ and $(A - \lambda)^{-\ell} - b(\lambda)^\ell$ belongs to $\mathcal{S}(\Lambda_\delta, C_G(\mathbb{B}, (\gamma - m, \gamma)))$. Moreover, for $\ell > 1$ we can write

$$
b(\lambda)^\ell = \omega_1 (r^{m} \text{op}_M(h_\ell)(\lambda) + g_\ell(\lambda)) \omega_0 + (1 - \omega_1) Q_\ell(\lambda) - \omega_2 + G_\ell(\lambda),
$$

where $h_\ell \in C^\infty(\mathbb{R}_+, M^{m,\ell,m}_G(X; \Lambda))$, $g_\ell \in R^{m,\ell,m}_G(\Lambda, (\gamma - m, \gamma))$, $Q_\ell \in L^{m,\ell,m}_c(\mathbb{B}; \Lambda)$, and $G_\ell \in \mathcal{S}(\Lambda, C_G(\mathbb{B}, (\gamma - m, \gamma)))$. Notice that all the contributions involving the smoothing Mellin element $v(\lambda)$ from (3.3) are now contained in $g_\ell(\lambda)$ or in $G_\ell(\lambda)$ since they are either supported in the interior of $\mathbb{B}$ or multiplied by a Green element or multiplied by a factor $r^m$ which improves the weight whenever $\ell > 1$. Clearly, the nonsmoothing parts of $b(\lambda)^\ell$ improve the weight $\gamma$ at least by $m(\ell - 1)$ while the families $g_\ell(\lambda)$ and $G_\ell(\lambda)$ has the same gain $\varepsilon > 0$ as the smoothing part of $b(\lambda)$. \qed
4.2. **Approximation of the resolvent.** Following Seeley’s ideas [12] we want to approximate the resolvent \((A - \lambda)^{-1}\) by means of a suitable parameter-dependent parametrix of \(A - \lambda\). More generally, we will approximate \((A - \lambda)^{-\ell}\) for any \(\ell \in \mathbb{N}\). Our aim is to extract a finite part of the parametrix having homogeneous components and such that the remainder decreases fast enough in \(\lambda\). To this end, we make use of the operator-valued symbolic calculus. For simplicity, we will mostly work with the case \(\ell = 1\) omitting it from the notation; corresponding comments about the general case will be given when necessary.

As given in (3.9) consider the parametrix

\[
\tilde{b}(\lambda) = \omega_1 \left( r^m \text{op}_M(h)\lambda + v(\lambda) + g(\lambda) \right) \omega_0 + (1 - \omega_1)Q(\lambda)(1 - \omega_2)
\]

of \(A - \lambda\), where \(h(r, z, \lambda) = \tilde{h}(r, z, r^m \lambda)\) with \(\tilde{h} \in C^\infty(\mathbb{R}^+, M\mathcal{C}^{-m,m}(X; \Lambda))\), \(v(\lambda)\) is a family of smoothing operators as in Lemma 3.11, and \(g \in \mathcal{R}^{-m,m}(\Lambda,(\gamma - m, \gamma))\). Recall that \(Q(\lambda)\) is defined as an asymptotic summation \(Q(\lambda) \sim \sum_j Q_j(\lambda)\), where each \(Q_j(\lambda)\) is a parameter-dependent pseudodifferential operator of order \(-m - j\) with anisotropic homogeneous local symbols. Moreover, since \(v + g\) is a classical operator-valued symbol of order \((-m, m)\), it admits an expansion of the form

\[
v(\lambda) + g(\lambda) = \sum_{j=0}^{N-1} g_j(\lambda) + g_{[N]}(\lambda)
\]

such that every \(g_j\) is twisted homogeneous of degree \((-m - j, m)\), and

\[g_{[N]} \in S^{-m-N,m}(\Lambda; \mathcal{K}^{s,\gamma-m}(\Omega^1), \mathcal{K}^{s',\gamma}(\Omega^1))\]

for every \(s, s' \in \mathbb{R}\).

Thus the parametrix \(b(\lambda)\) can be expanded as

\[
b(\lambda) = b(\lambda_1) + r_N(\lambda)
\]

\[
:= \omega_1 b^{(N)}(\lambda) \omega_0 + (1 - \omega_1)Q^{(N)}(\lambda)(1 - \omega_2) + r_N(\lambda),
\]

where \(r_N(\lambda)\) is a remainder of order \(-m - N\), and

\[b^{(N)}(\lambda) = r^m \text{op}_M(h) + \sum_{j=0}^{N-1} g_j(\lambda) \quad \text{and} \quad Q^{(N)}(\lambda) = \sum_{j=0}^{N-1} Q_j(\lambda).
\]

Now, by means of the Taylor expansion

\[
\tilde{h}(r, z, \lambda) = \sum_{j=0}^{N-1} \frac{1}{j!} (\partial^j \tilde{h})(0, z, \lambda) r^j + r^N \tilde{h}_{[N]}(r, z, \lambda)
\]

with

\[
\tilde{h}_{[N]}(r, z, \lambda) = \frac{1}{(N - 1)!} \int_0^1 (1 - \theta)^{N-1} (\partial^N \tilde{h})(\theta r, z, \lambda) d\theta.
\]

Set \(h_j(r, z, \lambda) := (\partial^j \tilde{h})(0, z, r^m \lambda)\) for \(j < N\), \(h_{[N]}(r, z, \lambda) := \tilde{h}_{[N]}(r, z, r^m \lambda)\), and define

\[
b^{(N)}(\lambda) := r^{m+j} \text{op}_M(h_j)(\lambda) + g_j(\lambda) \quad \text{(cf. (4.3))}
\]

Then every \(b^{(N)}(\lambda)\) is twisted homogeneous of degree \((-m - j, m)\), and we have

\[
b^{(N)}(\lambda) = \sum_{j=0}^{N-1} b^{(N)}_{\lambda,j}(\lambda) + r^{m+N} \text{op}_M(h_{[N]})(\lambda).
\]
Similarly, for $\ell \in \mathbb{N}$, $b(\lambda)^\ell$ admits an expansion of the form
\begin{equation}
(4.7) \quad b(\lambda)^\ell = b(\lambda)^N(\lambda) + r_{\ell,N}(\lambda)
\end{equation}
which can be used in order to approximate $(A - \lambda)^{-\ell}$.

**Theorem 4.2.** Given $k \in \mathbb{N}_0$ there exist $N(k) \in \mathbb{N}$ and $C_k > 0$ such that
\begin{equation*}
\left\| (A - \lambda)^{-\ell} - b(\lambda)^N(\lambda) \right\|_{\mathcal{L}(\mathcal{H}^{-k,\gamma-m}(\mathbb{B}), \mathcal{H}^k(\mathbb{B}))} \leq C_k (1 + |\lambda|)^{-k+\ell}
\end{equation*}
for every $N \geq N(k)$ and $\lambda \in \Delta_\delta$.

**Proof.** First of all, we have
\begin{equation*}
(A - \lambda)^{-\ell} - b(\lambda)^N(\lambda) = \left\{ (A - \lambda)^{-\ell} - b(\lambda)^\ell \right\} + \left\{ b(\lambda)^\ell - b(\lambda)^N(\lambda) \right\}
\end{equation*}
\begin{equation*}
= \left\{ (A - \lambda)^{-\ell} - b(\lambda)^\ell \right\} + r_{\ell,N}(\lambda).
\end{equation*}

Since $(A - \lambda)^{-\ell} - b(\lambda)^\ell$ belongs to $\mathcal{S}(\Delta_\delta, \mathcal{C}(\mathbb{B}, \gamma-m, \gamma))$, it is for every integer $k$ an element of $\mathcal{L}(\mathcal{H}^{-k,\gamma-m}(\mathbb{B}), \mathcal{H}^k(\mathbb{B}))$, and its norm is $O(|\lambda|^p)$ for every $p \in \mathbb{Z}$. Thus it remains to verify the norm estimate for $r_{\ell,N}(\lambda)$. For simplicity of notation we will check it explicitly only for $\ell = 1$. In this case, $r_{\ell,N}(\lambda)$ is just
\begin{equation*}
b - b(\lambda)^N = \omega_1 \left( b_\lambda - b(\lambda)^N \right) \omega_0 + (1 - \omega_1) (Q - Q(\lambda))(1 - \omega_2),
\end{equation*}
where $b_\lambda = \text{r}^m \text{op}_M(h)(\lambda) + v(\lambda) + g(\lambda)$. From the standard parameter-dependent calculus (cf. [45, Section 9]) we get the norm estimate
\begin{equation}
(4.8) \quad \left\| (1 - \omega_1)(Q(\lambda) - Q(\lambda))(1 - \omega_2) \right\|_{\mathcal{L}(\mathcal{H}^{-k}(\mathbb{B}), \mathcal{H}^k(\mathbb{B}))} \leq C_k [\lambda]^{-m-N+2k}
\end{equation}
for $N \geq -m + 2k$. On the other hand,
\begin{equation*}
\omega_1 \left( b_\lambda - b(\lambda)^N \right) \omega_0 \in S_{\ell\in M,m}^{-m-N,m} (\Lambda; \mathcal{H}^{-k,\gamma-m}(\mathbb{B}), \mathcal{H}^k(\mathbb{B}))
\end{equation*}
so that
\begin{equation}
(4.9) \quad \left\| \omega_1 \left( b_\lambda - b(\lambda)^N \right) \omega_0 \right\|_{\mathcal{L}(\mathcal{H}^{-k,\gamma-m}(\mathbb{B}), \mathcal{H}^k(\mathbb{B})))} \leq C_{k}(\lambda)^{-m-N+M(k)}
\end{equation}
with $M(k) = M_1 + M_2$, where $M_1$ and $M_2$ are the constants of growth corresponding to $\|\kappa(\lambda)\|_{\mathcal{L}(\mathcal{K},\mathcal{K}^\prime)}$ and $\|\kappa^{-1}(\lambda)\|_{\mathcal{L}(\mathcal{K},\mathcal{K}^\prime)}$, respectively (cf. Lemma [5.1]). By means of the estimates (4.8) and (4.9) together with Lemma 2.10 we get
\begin{equation*}
\left\| b(\lambda)^\ell - b(\lambda)^N(\lambda) \right\|_{\mathcal{L}(\mathcal{H}^{-k,\gamma-m}, \mathcal{H}^k(\mathbb{B})))} \leq \tilde{C}_k [\lambda]^{-m-N+M(k)} \leq C_k (1 + |\lambda|)^{-k+1}
\end{equation*}
for $-m - N + \max(2k, M(k)) \leq -(k + 1)$. Set $N(k) = km + \max(2k, M(k))$.

For an arbitrary $\ell \in \mathbb{N}$ the proof is quite the same once $b(\lambda)^\ell$ is split as $b(\lambda)$ in (4.4). This can be done by expanding $g_\ell(\lambda)$ and $Q_\ell(\lambda)$ in (4.2) inside their corresponding pseudodifferential calculi. Notice that $G_\ell(\lambda)$ in (4.2) is rapidly decreasing. 

The finite parametrix $b(\lambda)^N(\lambda)$ from (4.7) can also be used to approximate the operator $e^{-tA}$ for small $t$. For $\ell, N \in \mathbb{N}$ and $0 < t < 1$ we define
\begin{equation}
(4.10) \quad E^{(N)}_\ell(t) := \frac{i(t-1)^{\ell}}{2\pi} t^{-\ell+1} \int_{t^{-1}\mathbb{T}} e^{-\lambda t} b(\lambda)^N(\lambda) d\lambda,
\end{equation}
 cf. (4.1) and (4.1). As a consequence of Theorem 4.2 we get the following
Corollary 4.3. Given \( k \in \mathbb{N} \) there exist \( N(k) \in \mathbb{N} \) and \( C_{k,t} > 0 \) such that
\[
\left\| e^{-tA} - E_t^{(N)}(t) \right\|_{\mathcal{L}(\mathcal{H}^{-k,\gamma-m}(\mathbb{B}),\mathcal{H}^{k,\gamma}(\mathbb{B}))} \leq C_{k,t} t^k
\]
for every \( N \geq N(k) \) and \( 0 < t < 1 \).

Proof. We combine (4.1) and (4.10) to write
\[
e^{-tA} = \frac{\ii (\ell-1)!}{2\pi} t^{-\ell+1} \int_{t^{-1} \mathcal{Y}} e^{-t\lambda}(A - \lambda)^{-\ell} d\lambda.
\]
Let \( k \in \mathbb{N} \). In \( \mathcal{L}(\mathcal{H}^{-k,\gamma-m}(\mathbb{B}),\mathcal{H}^{k,\gamma}(\mathbb{B})) \) we get
\[
\left\| e^{-tA} - E_t^{(N)}(t) \right\| \leq \frac{(\ell-1)!}{2\pi} C_k t^{-\ell} \int_{t^{-1} \mathcal{Y}} e^{-\Re \lambda} \left| (A - \frac{\lambda}{t})^{-\ell} - b_t^{(N)}(\frac{\lambda}{t}) \right| d\lambda \\
\leq \frac{(\ell-1)!}{2\pi} C_k t^{-\ell} \int_{t^{-1} \mathcal{Y}} e^{-\Re \lambda} (1 + t^{-1}|\lambda|)^{-k+\ell} |d\lambda| \\
\leq \frac{(\ell-1)!}{2\pi} C_k t^k \int_{t^{-1} \mathcal{Y}} e^{-\Re \lambda} |\lambda|^{-k+\ell} |d\lambda| \leq C_{k,t} t^k
\]
after making the change of variables \( \lambda \rightarrow t^{-1} \lambda \), and applying Theorem 4.2. \( \square \)

4.3. Asymptotic expansion of the heat trace. As proved in Theorem 4.1 the operator \( e^{-tA} \) belongs to \( \mathcal{C}_G(\mathbb{B},(\gamma-m,\gamma)) \) for \( t > 0 \), so it is an operator of trace class with kernel in \( \mathcal{H}^{\infty,(-\gamma+m+\varepsilon)}(\mathbb{B}) \otimes \pi \mathcal{H}^{\infty,(-\gamma+\varepsilon)}(\mathbb{B}) \) for some \( \varepsilon > 0 \), cf. Appendix B. The kernel \( K(t,y,y') \) of \( e^{-tA} \) is commonly called the heat kernel. We further call the trace of \( e^{-tA} \) the heat trace for \( A \). In this section we will obtain an asymptotic expansion, as \( t \rightarrow 0^+ \), of the heat trace for a cone differential operator \( A \) such that \( A - \lambda \) is parameter-elliptic on the sector \( \Gamma \) defined in Section 4.1. The complete expansion will be given in Theorem 4.9.

In order to expand the heat trace we will use the approximation \( E_t^{(N)}(t) \) from (4.10). Imitating the steps around (4.11) we get in (4.11)
\[
b_t^{(N)}(\lambda) = \sum_{j=0}^{N-1} \left[ \omega_1 b_{\lambda,\ell,j}(\lambda) \omega_0 + (1 - \omega_1) Q_{\ell,j}(\lambda)(1 - \omega_2) \right] \\
+ \omega_1 r^{m\ell+N} \mathop{\text{op}}_{\mathcal{M}}(h_{\ell,[N]})(\lambda) \omega_0,
\]

where every \( Q_{\ell,j} \) is a parameter-dependent operator of order \( -m\ell - j \) with anisotropic homogeneous local symbols, \( b_{\lambda,\ell,j} = r^{m\ell+j} \mathop{\text{op}}_{\mathcal{M}}(h_{\ell,j}) + g_{\ell,j} \) is twisted homogeneous of degree \( -m\ell - j, m \), and \( h_{\ell,[N]} \) is essentially the remainder of the Taylor expansion of the parameter-dependent Mellin symbol \( h_{\ell,j} \) from (1.2).

Using now (4.11), the integral (4.10) becomes
\[
E_t^{(N)}(t) = \sum_{j=0}^{N-1} E_{\ell,j}(t) + R_N(t)
\]
with
\[
E_{\ell,j}(t) = \frac{\ii (\ell-1)!}{2\pi} t^{-\ell+1} \int_{t^{-1} \mathcal{Y}} e^{-t\lambda} \left[ \omega_1 b_{\lambda,\ell,j}(\lambda) \omega_0 + (1 - \omega_1) Q_{\ell,j}(\lambda)(1 - \omega_2) \right] d\lambda,
\]
\[
R_N(t) = \frac{\ii (\ell-1)!}{2\pi} t^{-\ell+1} \int_{t^{-1} \mathcal{Y}} e^{-t\lambda} \omega_1 r^{m\ell+N} \mathop{\text{op}}_{\mathcal{M}}(h_{\ell,[N]})(\lambda) \omega_0 d\lambda.
\]
Every component supported in the interior of $\mathcal{B}$
\[ \frac{\text{i}(\ell-1)\text{i}}{2\pi} t^{-\ell+1} \int_{t^{-1}\Gamma} e^{-t\lambda}(1 - \omega_1)Q_{\ell,j}(\lambda)(1 - \omega_2) d\lambda \]
can be treated on the closed manifold $\mathcal{B}$ as in [17]. The components near the boundary
\[ \frac{\text{i}(\ell-1)\text{i}}{2\pi} t^{-\ell+1} \int_{t^{-1}\Gamma} e^{-t\lambda} \omega_1 b_{\Lambda,\ell,j}(\lambda) \omega_0 d\lambda \]
and $R_N(t)$ require, however, more sophisticated calculations. More precisely, in order to expand in $t$ these boundary components, we will use the twisted homogeneity of every $b_{\Lambda,\ell,j}$, and the results from Section 3.3. Notice that every $b_{\Lambda,\ell,j}$ is a sum of a parameter-dependent Mellin operator and an operator-valued Green symbol. While Mellin operators can be described through its local symbols, the operator-valued Green symbols are of global nature. For this reason, we will discuss these cases independently.

If $a(\lambda)$ is an operator family on $X^\epsilon = \mathbb{R}_+ \times X$ with Schwartz kernels $k_a(\lambda, r, x, r', x')$, then for $0 < t < 1$ and $x \in X$ we define formally
\[ s[a](t, x) := t^{-\ell+1} \int_0^\infty \int_{t^{-1}\Gamma} e^{-t\lambda} \omega_1(r)k_a(\lambda, r, x, r) d\lambda dr \]
with $d\lambda := \frac{1}{t} d\lambda$. Without loss of generality we may assume $\omega_1(r) = 1$ for $r \leq 1$, and $\delta = 1$ in the definition of $\Gamma$, cf. Section [3.3].

Expansion of the Green elements

**Lemma 4.4.** Let $g \in R_G^{-m\ell-j,m}(\Lambda, (\gamma - m, \gamma))$ be a twisted homogeneous Green symbol of degree $(-m\ell-j,m)$, and let $k_g$ be its integral kernel. Then for every $x \in X$, $0 < t < 1$ and $N \in \mathbb{N}$ with $N > j$, we have
\[ s[g](t, x) = t^{j/m} \left\{ \int_0^\infty \int_{t^{1/m}} \left[ \int e^{-t\lambda} k_g(\lambda, r, x, r) d\lambda dr \right] \right\} + O(t^{N/m}). \]

**Proof.** Let $0 < t < 1$. Split the integral $\int_0^\infty = \int_0^{t^{1/m}} + \int_{t^{1/m}}^\infty$ in $s[g]$, and denote the components by $s'[g]$ and $s''[g]$, respectively. Because $\omega_1(r) = 1$ on $[0, t^{1/m}]$ we have
\[ s'[g](t, x) = t^{-\ell+1} \int_0^{t^{1/m}} \int_{t^{-1}\Gamma} e^{-t\lambda} k_g(\lambda, r, x, r) d\lambda dr. \]
With the change $\lambda \to t^{-1}\lambda$ and Lemma 3.4 we get
\[ \int_{t^{-1}\Gamma} e^{-t\lambda} k_g(\lambda, r, x, r) d\lambda = t^{-1} \int_{t^{-1}\Gamma} e^{-t\lambda} k_g(t^{-1}\lambda, r, x, r) d\lambda = t^{-1+\ell+j-1/m} \int_{t^{-1}\Gamma} e^{-t\lambda} k_g(\lambda, t^{-1/m}r, x, t^{-1/m}r, x) d\lambda. \]
Therefore, the change of variables $r \to t^{1/m}r$ yields
\[ s'[g](t, x) = t^{(j-1)/m} \int_0^{t^{1/m}} \int_{t^{-1}\Gamma} e^{-t\lambda} k_g(\lambda, t^{-1/m}r, x, t^{-1/m}r, x) d\lambda dr = t^{j/m} \left\{ \int_0^1 \int_{t^{-1}\Gamma} e^{-t\lambda} k_g(\lambda, r, x, r) d\lambda dr \right\}. \]
With the same calculations we also get
\[ s''[g](t, x) = t^{j/m} \int_1^\infty \int_{t^{-1}\Gamma} e^{-t\lambda} \omega_1(t^{1/m}r) k_g(\lambda, r, x, r) d\lambda dr. \]
A Taylor expansion at \( r = 0 \) yields \( \omega_1(r) = 1 + r^M \omega_{1,M}(r) \) with
\[
\omega_{1,M}(r) = \frac{1}{(M-1)!} \int_0^1 (1 - \theta)^{M-1} \omega_1^{(M)}(\theta r) d\theta \quad \text{and} \quad \omega_1^{(M)} = \partial_r^M \omega_1.
\]
In particular, for \( M = N - j \) we get
\[
s''[g](t, x) = t^{j/m} \left\{ \int_1^\infty \int_T e^{-\lambda} k_g(\lambda, r, x, x) d\lambda dr \right\}
+ t^{N/m} \int_1^\infty e^{-\lambda} \omega_{1,N-j}(t^{1/m} r) r^{N-j} k_g(\lambda, r, x, x) d\lambda dr.
\]
(4.13)

Since \( \omega_{1,M}(t^{1/m} r) \) is for any \( M \) uniformly bounded, and because \( k_g \) decays rapidly as \( r \to \infty \), the last integral is uniformly bounded as desired.

**Expansion of parameter-dependent Mellin operators**

First of all, we fix \( \ell \) large enough such that every \( E_{\ell,j}(t) \) and \( R_N(t) \) from (1.12) have continuous Schwartz kernels. For simplicity we will often drop \( \ell \) from the notation.

Let \( \hat{h} \in M_{\mathcal{D}^{-m\ell,m}}(X; \Lambda) \) and set \( h(r, z, \lambda) = \hat{h}(z, r^m \lambda) \). Fix \( j \in \mathbb{N}_0 \) and define
\[
a(\lambda) := r^{m\ell+j} \text{op}_M(h)(\lambda)
\]
which is twisted homogeneous of degree \((-m\ell - j, m)\). Observe that the Mellin component of \( b_{t,\ell,j} \) in (1.11) is of this form. Let \( k_a(\lambda, r, x, x') \) denote the continuous Schwartz kernel of \( a(\lambda) \), taken with respect to the measure \( dr'dx' \). As in Lemma 4.4 we conveniently split the function \( s[a](t, x) \) in two components \( s'[a](t, x) \) and \( s''[a](t, x) \), and treat them separately.

**Lemma 4.5.** For \( 0 < t < 1 \) and \( x \in X \) we have
\[
s'[a](t, x) = t^{j/m} \left\{ \int_0^1 \int_T e^{-\lambda} k_a(\lambda, r, x, x) d\lambda dr \right\}.
\]

**Proof.** Noting that \( k_a(\lambda, r, x, x) \) satisfies the relation (4.3) from Lemma C.4 with \( d = m \) and \( \mu = -m\ell - j \) (\( a(\lambda) \) is twisted homogeneous), the assertion follows making the same calculations as in the proof of Lemma 4.4.

The second component of \( s[a] \) is more delicate and cannot be treated as in Lemma 4.4 since the kernels \( k_a(\lambda, r, x, x) \) increase as \( r \to \infty \), so the integrals without a cut-off function do not exist. However, we can get an expansion in \( t \) of \( s''[a](t, x) \) by means of the results from Section 3.3 (cf. Theorem 3.25). Recall that
\[
s''[a](t, x) = t^{-\ell+1} \int_{t^{1/m}}^\infty \int_T e^{-t\lambda} \omega_1(r) k_a(\lambda, r, x, x) d\lambda dr.
\]
Since \( r \geq t^{1/m} \) and \( |\lambda| \geq t^{-1} \), we may assume \( r^m|\lambda| \geq 1 \).

**Lemma 4.6.** For every \( x \in X \), \( 0 < t < 1 \) and \( N \in \mathbb{N} \) with \( N > j \), we get
\[
s''[a](t, x) = \sum_{k=0}^{N+n} c_k(x) t^{k(n-1)/m} + \sum_{k=0}^{N-1} c'_k(x) t^k \log t + O(t^{N/m}),
\]
where \( n = \dim X \). The coefficients \( c_k \) and \( c'_k \) depending also on \( \ell \) and \( j \).
In local coordinates the function $K$ is anisotropic homogeneous in $(\eta, r, x)$, where $\eta = (\xi, z) \in \mathbb{R}^n \times \Gamma_{\beta} \cong \mathbb{R}^{n+1}$, $d\eta = \frac{1}{(2\pi)^{n+1}} d\eta$, $p(r, x, \eta, \lambda)$, and $\tilde{p} \in S_{\alpha, c, \ell}^{-m, \ell, m}(\Omega \times \mathbb{R}^n \times \mathbb{C}; \Lambda)$ is a local symbol of $h(z, \lambda)$. Thus there are symbols $\tilde{p}_k$, anisotropic homogeneous in $(\eta, \lambda)$ of degree $-m\ell - k, m$, for $|\eta| \geq 1$, such that for every $J \in \mathbb{N}$,

$$\tilde{p} = \sum_{k=0}^{J-1} \tilde{p}_k + \tilde{g}_J \in S_{\alpha, c, \ell}^{-m, \ell, J, m}(\Omega \times \mathbb{R}^n \times \mathbb{C}; \Lambda).$$

This expansion induces a decomposition of $K(\lambda, r, x)$, say $K = K_{p_0} + K_{p_1} + \cdots + K_{g_J}$, and a decomposition of $S(t, x)$, say $S = S_{p_0} + \cdots + S_{p_{J-1}} + S_{g_j}$. Our purpose is to expand in $t$ all these components.

**Step 1:** (Expansion of $S_{g_j}$). First of all, let $g_j(r, x, \eta, \lambda) = \tilde{g}_j(x, \eta, r, \lambda)$. Following the proof of Lemma 4.4 we can write, exactly as in (4.13),

$$S_{g_j}(t, x) = t^{j/m} \left\{ \int_1^\infty e^{-\lambda k_j(\lambda, r, x, x)} d\lambda \right\} + t^{N/m} \left\{ \int_1^\infty e^{-\lambda \omega_{1, N-j}(t^{1/m} r) N-j k_j(\lambda, r, x, x)} d\lambda \right\},$$

where $k_j = r^{m\ell - j} K_g(\lambda, r, x)$. For $J > N$ the kernel $k_j$ decays fast enough as $r \to \infty$ so that the integrals exist, and the last one is uniformly bounded. Thus

$$(4.14) \quad S_{g_j}(t, x) = c(x) t^{j/m} + O(t^{N/m})$$

with $c(x)$ depending on $\ell, j, J$.

**Step 2:** (Splitting of $S_{p_k}$). For $k < J$ let us split (recall that $r^m|\lambda| \geq 1$)

$$(4.15) \quad \int p_k \, d\eta = \int_{|\eta| \geq r|\lambda|^{1/m}} p_k \, d\eta + \int_{|\eta| \leq 1} p_k \, d\eta + \int_{r|\lambda|^{1/m} \geq |\eta| \geq 1} p_k \, d\eta.$$

This equality split likewise $S_{p_k} = S_{k,1} + S_{k,2} + S_{k,3}$ with

$$(4.16) \quad S_{k,\delta} = t^{-\ell+1} \int_{t^{-1/\ell}} e^{-t^\lambda \omega_1(r) r^{m\ell - j-1} \left( \int_{V_\delta} \tilde{p}_k(r, x, \eta, r^m \lambda) d\eta \right) d\lambda dr$$

for $\delta \in \{1, 2, 3\}$, where $V_\delta$ denotes the corresponding domain of integration.

**Step 3:** (Expansion of $S_{k,1}$). The change of variables $\eta \to r|\lambda|^{1/m} \eta$ yields

$$\int_{|\eta| \geq r|\lambda|^{1/m}} \tilde{p}_k(x, \eta, r^m \lambda) d\eta = \int_{|\eta| \geq 1} (r|\lambda|^{1/m})^n \tilde{p}_k(x, r|\lambda|^{1/m} \eta, r^m |\lambda|^{1/\lambda} \eta) d\eta$$

$$= (r|\lambda|^{1/m})^{-m\ell - k + n+1} \int_{|\eta| \geq 1} \tilde{p}_k(x, \eta, \frac{\lambda}{\lambda}) d\eta$$
due to the homogeneity of $\tilde{p}_k$. If we denote $\hat{c}_k(x, \lambda/|\lambda|) = \int_{|\eta| \geq 1} \tilde{p}_k(x, \eta, \lambda/|\lambda|) \, d\eta$, and set $\overline{m}_k := -m\ell - k + n + 1$, then for $\beta = 1$ the integral (4.16) becomes

$$S_{k,1} = t^{-\ell+1} \int_{t^{\ell-1}/m}^\infty \int_{t^{\ell-1}/\gamma} e^{-t\lambda} \omega_1(r) r^{m\ell} \alpha_k(x, \lambda/|\lambda|) \, d\lambda \, dr$$

(4.17)

$$= t^{(k-n-1)/m} \int_{t^{\ell-1}/m}^\infty \int_{t^{\ell-1}/\gamma} e^{-t\lambda} \omega_1(r) r^{j-k+n} |\lambda|^{m_k/m} \hat{c}_k(x, \lambda/|\lambda|) \, d\lambda \, dr$$

Finally, Lemma 4.7 leads to the expansion

$$S_{k,1}(t, x) = \hat{c}(x) t^{(k-n-1)/m} + c(x) t^{j/m} + c'(x) t^{j/m} \log t$$

with coefficients depending on $\ell, j, k$. In particular, $c' = 0$ for $k \neq j + n + 1$.

Step 4: (Expansion of $S_{k,2}$). Since $\tilde{p}_k \in S^{-m\ell-k, m}_0(\Omega \times \mathbb{R}^n \times \mathbb{C}; \Lambda)$ depends holomorphically on $\lambda$, we can apply Theorem 3.23 with $d = m$, $\nu = -k$ and $\mu = m\ell$ in order to expand $\tilde{p}_k$ as

$$\tilde{p}_k(x, \eta, \lambda) = \sum_{\alpha=0}^{M-1} \lambda^{-\ell-\alpha/m} q_{k\alpha}(x, \eta) + \lambda^{-\ell-M/m} \tilde{p}_k(x, \eta, \lambda)$$

for any $M \in \mathbb{N}$, with $q_{k\alpha} \in S^{-k+\alpha}_0(\Omega \times \mathbb{R}^n \times \mathbb{C})$ and $\tilde{p}_k(x, \eta, \lambda) \in S^{-k-M}_0(\Omega \times \mathbb{R}^n \times \mathbb{C})$ uniformly for $\lambda \in \Lambda$ with $|\lambda| \geq 1$. For $r^m |\lambda| \geq 1$, the degenerate symbol $p_k(x, \eta, \lambda) = \tilde{p}_k(x, \eta, r^m \lambda)$ can then be written as

$$p_k = \sum_{\alpha=0}^{M-1} (r^m \lambda)^{-\ell-\alpha/m} q_{k\alpha}(x, \eta) + (r^m \lambda)^{-\ell-M/m} \tilde{p}_k(x, \eta, r^m \lambda),$$

and so

$$\int_{V_2} p_k \, d\eta = \sum_{\alpha=0}^{M-1} (r^m \lambda)^{-\ell-\alpha/m} \int_{V_2} q_{k\alpha}(x, \eta) \, d\eta$$

+ $(r^m \lambda)^{-\ell-M/m} \int_{V_2} \tilde{p}_k(x, \eta, r^m \lambda) \, d\eta.$

Set $\hat{c}_{k,\lambda}(x, r^m \lambda) = \int_{V_2} \tilde{p}_k(x, \eta, r^m \lambda) \, d\eta$ which is finite as $V_2 = \{|\eta| \leq 1\}$ is compact. Now, since $\int_{\gamma} e^{-t\lambda} \lambda^{-\ell-\alpha/m} \, d\lambda = 0$, we get

$$S_{k,2} = t^{M/m} \int_{t^{\ell-1}/\gamma}^\infty \int_{t^{\ell-1}/\gamma} e^{-t\lambda} \omega_1(r) r^{j-M-1} \lambda^{-\ell-M/m} \hat{c}_{k,\lambda}(x, r^m \lambda) \, d\lambda \, dr$$

(4.20)

due to the change of variables $\lambda \to t^{-1} \lambda$ and $r \to t^{1/m} r$. Similarly as in the proof of Lemma 4.4 we expand $\omega_1(r) = 1 + r^{M-j+1} \omega_{1,\lambda}(r)$. Thus

$$S_{k,2} = t^{j/m} \int_{t^{\ell-1}/\gamma}^\infty \int_{t^{\ell-1}/\gamma} e^{-t\lambda} \omega_1(t^{1/m} r) r^{j-M-1} \lambda^{-\ell-M/m} \hat{c}_{k,\lambda}(x, r^m \lambda) \, d\lambda \, dr$$

$$+ t^{(M-1)/m} \int_{t^{\ell-1}/\gamma}^\infty \int_{t^{\ell-1}/\gamma} e^{-t\lambda} \omega_{1,\lambda}(t^{1/m} r) \lambda^{-\ell-M/m} \hat{c}_{k,\lambda}(x, r^m \lambda) \, d\lambda \, dr.$$
Since \( \omega_{1,M}(t^{1/m}r) \) and \( \tilde{c}_{k,M}(x, r^m \lambda) \) are uniformly bounded for \( 0 < t < 1 \), \( r \geq 1 \) and \( r^m |\lambda| \geq 1 \), then the integral at \( t^{(M-1)/m} \) exists (presence of \( r^{-2} \)) and is also uniformly bounded. Choosing \( M \geq N + 1 \) we finally get

\[
S_{k,2}(t, x) = c(x)t^{j/m} + O(t^{N/m})
\]

with \( c(x) \) depending on \( \ell, j, k \).

**Step 5: (Expansion of \( S_{k,3} \)).** According to (4.19) let

\[
I_\alpha = (r^m \lambda)^{-\ell-\alpha/m} \int_{V_3} q_{k\alpha}(x, \eta) d\eta,
\]

\[
\tilde{I}_M = (r^m \lambda)^{-\ell-M/m} \int_{V_3} \tilde{p}_{k,M}(x, \eta, r^m \lambda) d\eta,
\]

so that

\[
S_{k,3} = t^{-\ell+1} \int_{\lambda^{-1}T}^{\infty} \int_{\lambda^{-1}T}^{\infty} e^{-t\lambda \omega_1(r)} r^{m\lambda+j-1} \left( \sum_{\alpha=0}^{M-1} I_\alpha + \tilde{I}_M \right) d\lambda dr.
\]

Recall that in (4.19) each \( q_{k\alpha}(x, \eta) = \frac{1}{\lambda} \lim_{\omega \to 0} \partial_{\omega} \left( w^{-m/\lambda} \tilde{p}_{k}(x, \eta, \frac{w}{\lambda}) \right) \) is homogeneous in \( \eta \) of degree \( -k + \alpha \) and consequently \( \tilde{p}_{k,M}(x, r^m \lambda) \) is anisotropic homogeneous of degree \( -(k + M, m) \) for \( |\eta| \geq 1 \). Then

\[
\int_{V_3} \tilde{p}_{k,M}(x, \eta, r^m \lambda) d\eta = \int_{V_3} \eta^{-k+M} \tilde{p}_{k,M}(x, \eta, r^m \lambda) d\eta.
\]

Taking now \( M \) such that \( -k + M > -(n + 1) \) the latter integral can be split in two integrals \( \int_{r|\lambda|^{1/m} \geq |\eta|} - \int_{|\eta| \leq 1} \). We denote the second integral again by \( \tilde{c}_{k,M}(x, r^m \lambda) \), and make the change of variables \( \eta \to r|\lambda|^{1/m} \eta \) in the first one. Thus

\[
\int_{r|\lambda|^{1/m} \geq |\eta|} \tilde{p}_{k,M}(x,\eta, r^m \lambda) d\eta = (r|\lambda|^{1/m})^{-k+M+n+1} \int_{|\eta| \leq 1} |\eta|^{-k+M} \tilde{p}_{k,M}(x, \frac{\eta}{|\eta|}, \frac{\lambda}{|\eta|^{1/m}}) d\eta
\]

\[
= (r|\lambda|^{1/m})^{-k+M+n+1} \tilde{c}_{k,M}(x, \lambda/|\lambda|).
\]

Now, with the usual change of variables of \( \lambda \) and \( r \),

\[
t^{-\ell+1} \int_{\lambda^{-1}T}^{\infty} \int_{\lambda^{-1}T}^{\infty} e^{-t\lambda \omega_1(r)} r^{m\lambda+j-1} \tilde{I}_M(r, x, \lambda) d\lambda dr
\]

\[
= t^{(k-n-1)/m} \int_{\lambda^{-1}T}^{\infty} \omega_1(r) r^{j-k-n} dr \left( \int_{\lambda^{-1}T}^{\infty} e^{-\lambda |\lambda|^{m_k/m} \tilde{c}_{k,M}(x, \frac{\lambda}{|\lambda|})} d\lambda \right)
\]

\[
- t^{j/m} \int_{\lambda^{-1}T}^{\infty} e^{-\lambda \omega_1(t^{1/m}r)} r^{j-M-1} \lambda^{-\ell-M/m} \tilde{c}_{k,M}(x, r^m \lambda) d\lambda dr
\]

with \( m_k = -m\ell - k + n + 1 \) and \( \tilde{c}_{k,M} = (\lambda/|\lambda|)^{-\ell-M/m} \tilde{c}_{k,M}(x, \lambda/|\lambda|) \). These integrals are of the same type as (4.17) and (4.20), so (4.22) is of the form

\[
\tilde{c}(x)t^{(k-n-1)/m} + c(x)t^{j/m} + c'(x)t^{j/m} \log t + O(t^{N/m})
\]

with coefficients depending on \( \ell, j, k \). In particular, \( c' = 0 \) for \( k \neq j + n + 1 \).
In order to complete the expansion of $S_{k,3}$ we next treat those terms involving $I_\alpha$. By means of polar coordinates and the homogeneity in $\eta$ of $q_{k\alpha}$,
\[
\int_V q_{k\alpha}(x,\eta) d\eta = d_{k\alpha}(x) \int_1^{\rho|\lambda|^{1/m}} \rho^{-k+n} d\rho
\]
\[
= d_{k\alpha}(x) \left\{ \begin{array}{ll}
\frac{1}{\alpha+n+1-k} \left( (r|\lambda|^{1/m})^{-k+n+1} - 1 \right) & \text{for } \alpha \neq k - n - 1 \\
\log(r|\lambda|^{1/m}) & \text{for } \alpha = k - n - 1
\end{array} \right.
\]
which we denote by $\tilde{c}_{k\alpha}(x, r|\lambda|^{1/m})$. Then, for $\alpha < M$, the corresponding component of $S_{k,3}$ becomes
\[
t^{-t+1} \int_{t^1/m}^\infty \int_{t^{-1}T} e^{-t\lambda} \omega_1(r) r^{mj+j-1} I_\alpha(r, x, \lambda) d\lambda dr
\]
(4.24)
\[
= t^{\alpha/m} \int_{t^1/m}^\infty \int_{t^{-1}T} e^{-t\lambda} \omega_1(r) r^{j-\alpha-1} \lambda^{-\alpha/m} \tilde{c}_{k\alpha}(x, r|t^{-1}|^{1/m}) d\lambda dr.
\]
If $\alpha \neq k - n - 1$, then (4.24) equals
\[
t^{(k-n-1)/m} \int_{t^1/m}^\infty \omega_1(r) r^{j-k+n} dr \left( d_{k\alpha}(x) \int_T e^{-t\lambda} \lambda^{-\alpha/m} |\lambda|^{(\alpha+n+1-k)/m} d\lambda \right)
\]
since $\int_T e^{-t\lambda} \lambda^{-\alpha/m} d\lambda = 0$ for every $\alpha$. Making use of Lemma 4.7 this expression can be written as a linear combination of $t^{(k-n-1)/m}$, $t^j/m$ and $t^{j/m} \log t$. Moreover, the term $t^{j/m} \log t$ disappears when $k \neq j + n + 1$. If $\alpha = k - n - 1$, then (4.24) becomes
\[
t^{(k-n-1)/m} \int_{t^1/m}^\infty \omega_1(r) r^{j-k+n} dr \left( d_{k\alpha}(x) \int_T e^{-t\lambda} \lambda^{-\alpha/m} |\lambda|^{(\alpha+n+1-k)/m} d\lambda \right)
\]
since the integral $\int_{t^1/m}^\infty \omega_1(r) r^{j-k+n} \log(t^{-1/m}r) dr$ is finite and $\int_T e^{-t\lambda} \lambda^{-\alpha/m} d\lambda = 0$. Applying Lemma 4.7 once more we finally obtain
(4.25) $S_{k,3}(t, x) = \tilde{c}(x) t^{(k-n-1)/m} + c(x) t^j/m + c'(x) t^{j/m} \log t + O(t^{N/m})$, with coefficients depending on $t, j, k$. In particular, $c' = 0$ for $k \neq j + n + 1$.

Summary: By virtue of the relation
\[
s''[a](t, x) = \sum_{k=0}^{J-1} (S_{k,1}(t, x) + S_{k,2}(t, x) + S_{k,3}(t, x)) + S_{g,3}(t, x),
\]
the assertion follows summing up (4.18), (4.21), (4.25) and (4.14).

Lemma 4.7. Let $0 < \tau < 1$ and let $\omega$ be a cut-off function with $\omega(r) = 1$ for $r < 1$. For every $j, \nu \in \mathbb{R}$ there is a constant $C_{j,\nu} > 0$ such that
\[
\tau^\nu \int_\tau^\infty \omega(r) r^{j-\nu-1} dr = C_{j,\nu} \tau^\nu + \left\{ \begin{array}{ll}
\frac{-1}{j-\nu} \tau^j & \text{for } \nu \neq j \\
-\tau^j \log \tau & \text{for } \nu = j
\end{array} \right.
\]
Proof. Since $\omega(r) = 1$ on $[0, 1]$,
\[
\tau^\nu \int_\tau^\infty \omega(r) r^{j-\nu-1} dr = \tau^\nu \left( \int_1^\infty \omega(r) r^{j-\nu-1} dr + \int_\tau^1 r^{j-\nu-1} dr \right)
\]
\[
= \tau^\nu \left( \int_1^\infty \omega(r) r^{j-\nu-1} dr + \left\{ \begin{array}{ll}
\tau^\nu & \text{for } \nu \neq j \\
\frac{\tau^j}{j-\nu} - \frac{\tau^j}{j-\nu} \log \tau & \text{for } \nu = j
\end{array} \right. \right).
\]
In order to analyze the contribution of $R_N(t)$ from (4.12), observe that the Mellin operator involved in its definition can be written as $a_N(\lambda) = r^{mT+N} \mathcal{O}_M(h_N)(\lambda)$ with

$$h_N(r, z, \lambda) = \frac{1}{(N-1)!} \int_0^1 (1 - \theta)^{N-1} \tilde{h}(\theta r, z, r^m \lambda) d\theta,$$

where $\tilde{h} \in C^\infty(\mathbb{R}_+, M_{\mathcal{O}}^{-m, \mathcal{O}}(X; \Lambda))$. As seen in the previous calculations, there is no loss of generality if we drop the variable $x \in X$. Let $k_{a_N}(\lambda, r, r')$ be the continuous Schwartz kernel of $a_N(\lambda)$ and let

$$s[a_N](t) = t^{-\ell+1} \int_0^\infty \int_{T^{-1} \mathbb{Y}} e^{-t\lambda} \omega_1(r) k_{a_N}(\lambda, r) d\lambda dr.$$

Split $s[a_N](t) = s'[a_N](t) + s''[a_N](t)$ as done before. As a matter of fact, the proof of Lemma 4.6 also works for Lemma 4.8.

The main difference between $a_N(\lambda)$ and the Mellin operators considered at the beginning of this section, is the additional dependence on $r$ of the nondegenerate symbol $\tilde{h}(r, z, \lambda)$. The family $a_N(\lambda)$ is unfortunately not twisted homogeneous and Lemma 4.5 cannot be applied. However, we have the following

**Lemma 4.8.** Let $0 < t < 1$ and $N \in \mathbb{N}$. Then, $s'[a_N](t) = O(t^{N/m})$.

**Proof.** We can write $k_{a_N}(\lambda, r, r') = r^{mT+N} K_N(\lambda, r)$ with

$$K_N(\lambda, r) = \frac{1}{(N-1)!} r^{-1} \int_0^1 (1 - \theta)^{N-1} \tilde{h}(\theta r, \tau, r^m \lambda) d\theta d\tau.$$

Since $(1 - \theta)^{N-1} = \sum_{k=0}^{N-1} \binom{N-1}{k} (-1)^k \theta^k$, the function $K_N(\lambda, r)$ can be split, inducing a corresponding decomposition $s'[a_N](t) = \sum_{k=0}^{N-1} c_{N,k} S_k(t)$ with

$$S_k(t) = t^{-\ell+1} \int_0^{t^{1/m}} \int_{T^{-1} \mathbb{Y}} e^{-t\lambda} r^{mT+N-1} \left( \int_0^1 \theta^k \tilde{h}(\theta r, \tau, r^m \lambda) d\theta d\tau \right) d\lambda dr$$

after the change of variables $\lambda \to t^{-1} \lambda$, $r \to t^{1/m} r$, and $\theta \to t^{-1/m} \theta$. Now,

$$\int_0^{t^{1/m}} \theta^k \tilde{h}(\theta r, \tau, r^m \lambda) d\theta = \frac{1}{k+1} t^{(k+1)/m} \tilde{h}(\xi r, \tau, r^m \lambda)$$

for some $\xi_t \in [0, t^{1/m}]$, and so

$$S_k(t) = t^{N/m} \frac{1}{k+1} \int_0^1 \int_{T^{-1} \mathbb{Y}} e^{-\lambda} r^{mT+N-1} \left( \int_{T} \tilde{h}(\xi_t r, \tau, r^m \lambda) d\tau \right) d\lambda dr$$

which is $O(t^{N/m})$ since the integral is uniformly bounded. □
Theorem 4.9. Let $A$ be a cone differential operator of order $m > 0$ such that $A - \lambda$ is parameter-elliptic with respect to some $\gamma \in \mathbb{R}$ on a sector $\Lambda$ as given in Section 4. Then, the heat trace admits the asymptotic expansion

$$\text{Tr} e^{-tA} \sim \sum_{k=0}^{\infty} C_k t^{(k-n-1)/m} + \sum_{k=0}^{\infty} C'_k t^{k/m} \log t, \quad \text{as } t \to 0^+,$$

where $C_k$ and $C'_k$ are constants depending on the symbolic structure of $A$.

Proof. Actually, we only need to put together all the single expansions obtained in this section. The asymptotic summation here means:

For a given $K \in \mathbb{N}$ there exists $N(K) \in \mathbb{N}$ and $C_K > 0$ such that

$$\left| \text{Tr} e^{-tA} - \sum_{k=0}^{N-1} \tau_k(t) \right| \leq C_K t^K$$

for every $N \geq N(K)$ and $0 < t < 1$, where $\tau_k(t)$ is some expression of 'degree' $k$ from the right-hand side above, that is, $\tau_k(t)$ is a linear combination of terms like $t^{(k-n)/m} \log^\beta t$ with $\alpha = 0$ or $n + 1$, and $\beta = 0$ or $1$.

First of all, we choose $\ell \in \mathbb{N}$ large enough such that $(A - \lambda)^{-\ell}$ is of trace class. Hence the family $E^{(N)}_\ell(t)$ from (1.10) is of trace class too, and Corollary 4.3 together with Theorem B.6 imply: for a given $K \in \mathbb{N}$ there are $N(K)$ and $C_K > 0$ such that

$$\left| \text{Tr} e^{-tA} - \text{Tr} E^{(N)}_\ell(t) \right| \leq C_K t^K$$

for $0 < t < 1$. Recall that $E^{(N)}_\ell(t) = \sum_{k=0}^{N-1} E_{\ell,k}(t) + R_N(t)$ with $E_{\ell,k}(t)$ and $R_N(t)$ as in (1.12). By definition, every $E_{\ell,k}(t)$ splits into a component supported in the interior of $B$ and another one supported near $\partial B$. It is known (cf. [17, Sec. 1.8.1]) that the trace of the interior part is of the desired form due to the homogeneity of the local symbols. On the other hand, all the components supported near the boundary are either operator-valued Green symbols or parameter-dependent Mellin operators with degenerate symbols. By using Lemma 4.4 for the integrals of Green type and Lemma 4.5, Lemma 4.6 and Lemma 4.8 for the rest, we obtain the desired expansion integrating over the manifold $X$ the local expansions obtained there. \qed

5. Appendix

A. Parameter-dependent pseudodifferential operators. In this section we give the basic definitions of the local parameter-dependent symbols that we use in this paper. For further information the reader is referred, for instance, to the book of Shubin [46, Section 9].

Let $\Omega$ be an open set in $\subset \mathbb{R}^n$, and let $\Lambda$ be a closed angle in $\mathbb{C}$ with vertex at the origin. Let $\mu \in \mathbb{R}$, $d \in \mathbb{N}$. A function $p(x, \xi, \lambda) \in C^\infty(\Omega \times \mathbb{R}^n \times \Lambda)$ is said to be in the class of symbols $S^{\mu,d}(\Omega \times \mathbb{R}^n; \Lambda)$ if for any multi-indices $\alpha, \beta \in \mathbb{N}_0^d$, $\gamma \in \mathbb{N}_0^n$, and any compact set $K \subset \Omega$ there is a positive constant $C_{\alpha,\beta,\gamma,K}$ such that

$$|D^\alpha_\xi D^\beta_\lambda D^\gamma_\lambda p(x, \xi, \lambda)| \leq C_{\alpha,\beta,\gamma,K} (1 + |\xi| + |\lambda|^{1/d})^{\mu - |\alpha| - d|\gamma|}$$

for $x \in K$, $\xi \in \mathbb{R}^n$, $\lambda \in \Lambda$. The smoothing elements

$$S^{-\infty}(\Omega \times \mathbb{R}^n; \Lambda) := \bigcap_{\mu \in \mathbb{R}} S^{\mu,d}(\Omega \times \mathbb{R}^n; \Lambda)$$
are independent of $d$. To every symbol $p \in S^{\mu,d}(\Omega \times \mathbb{R}^n; \Lambda)$ we associate the operator family $P(\lambda) = op_\Lambda(p)(\lambda)$ given by

$$[P(\lambda)u](x) := \int e^{i(x-y)\xi} p(x,\xi,\lambda) u(y) \, dy \, d\xi \quad \text{for} \ u \in C_0^\infty(\Omega).$$

It is a pseudodifferential operator of order $\mu$ depending smoothly on the parameter $\lambda \in \Lambda$ with anisotropy $d$; we write $P(\lambda) \in L^{\mu,d}(\Omega; \Lambda)$. When $d = 1$ we omit it from the notation.

A function $f(x,\xi,\lambda)$ on $\Omega \times \mathbb{R}^n \times \Lambda$ is called (anisotropic) homogeneous in $(\xi,\lambda)$ of degree $(\nu,d)$ if

$$f(x,\tau\xi,\tau^d\lambda) = \tau^{\nu} f(x,\xi,\lambda) \quad \text{for every} \ \tau \geq 1.$$ 

The class $S_{\text{cel}}^{\mu,d}(\Omega \times \mathbb{R}^n; \Lambda)$ of classical pseudodifferential symbols is defined as the space of symbols $p \in S^{\mu,d}(\Omega \times \mathbb{R}^n; \Lambda)$ that admit an asymptotic expansion of the form

$$p(x,\xi,\lambda) \sim \sum_{j=0}^\infty p_{\nu-j}(x,\xi,\lambda),$$

where each $p_{\nu-j}$ is homogeneous of degree $(\mu-j,d)$ for $|\xi| + |\lambda|^{1/d} \geq 1$.

For $d = (d_1,d_2) \in \mathbb{N}^2$ a smooth function $p$ is said to be in $S^{\mu,d}(\Omega \times \mathbb{R}^n; \mathbb{R}^\ell \times \Lambda)$, if for any $\alpha,\beta,\gamma \in \mathbb{N}_0^n$, $\gamma \in \mathbb{N}_0^{\ell+1}$, and any compact $K \subset \Omega$ there is a positive constant $C = C(\alpha,\beta,\gamma,K)$ such that

$$|D^\alpha_x D^\beta_\xi D^\gamma_{\vec{\nu}} p(x,\xi,\lambda)| \leq C (1 + |\xi| + |\lambda|^{1/d_1} + |\lambda|^{1/d_2})^{\mu - |\alpha| - d_1|\gamma_1| - d_2|\gamma_2|}$$

for $x \in K$, $\xi \in \mathbb{R}^n$ and $(\vec{\nu},\lambda) \in \mathbb{R}^\ell \times \Lambda$.

B. Green cone operators. In this section we only discuss a special class of Green operators that describes sufficiently well the smoothing elements appearing in our context. For recent and more general results concerning Green cone operators we refer to [27].

Let us begin by pointing out some embedding properties of the weighted Sobolev spaces on the manifold $\mathbb{B}$ (cf. Section 2.2).

Lemma B.1. For $s \geq s', \gamma \geq \gamma'$ the embedding $\mathcal{H}^{s,\gamma}(\mathbb{B}) \hookrightarrow \mathcal{H}^{s',\gamma'}(\mathbb{B})$ is continuous. If $\gamma > \gamma'$, then it is compact if $s > s'$, Hilbert-Schmidt if $s > s' + \dim \mathbb{B}/2$, and trace class when $s > s' + \dim \mathbb{B}$.

The intersection over $s \in \mathbb{R}$ of all these spaces will be denoted by $\mathcal{H}^{\infty,\gamma}(\mathbb{B})$, and will be topologized as the projective limit $\mathcal{H}^{\infty,\gamma}(\mathbb{B}) = \text{proj lim}_{s \in \mathbb{R}} \mathcal{H}^{s,\gamma}(\mathbb{B})$ for every $\gamma \in \mathbb{R}$. Finally, we introduce the space

$$\mathcal{H}^{\infty,\gamma,\gamma^{-\frac{1}{2}}}(\mathbb{B}) := \text{proj lim}_{k \in \mathbb{N}} \mathcal{H}^{k,\gamma^{-\frac{1}{2}}}(\mathbb{B}).$$

Observe that $\mathcal{H}^{\infty,\gamma}(\mathbb{B}) \hookrightarrow \mathcal{H}^{\infty,\gamma^{-\frac{1}{2}}}(\mathbb{B})$. Moreover, $\mathcal{H}^{\infty,\gamma^{-\frac{1}{2}}}(\mathbb{B})$ is nuclear since the embedding $\mathcal{H}^{k,\gamma^{-\frac{1}{2}}}(\mathbb{B}) \hookrightarrow \mathcal{H}^{k,\gamma^{-\frac{1}{2}}}(\mathbb{B})$ is Hilbert-Schmidt for $k > k + \frac{n+1}{2}$.

Definition B.2. Let $\gamma, \delta \in \mathbb{R}$. An operator $G \in \bigcap_{s,s' \in \mathbb{R}} \mathcal{L}(\mathcal{H}^{s,\gamma}(\mathbb{B}), \mathcal{H}^{s',\delta}(\mathbb{B}))$ is called a Green operator if there is an $\varepsilon = \varepsilon(G) > 0$ such that

$$G : \mathcal{H}^{s,\gamma}(\mathbb{B}) \to \mathcal{H}^{s'+\varepsilon,\delta}(\mathbb{B}) \quad \text{and} \quad G^* : \mathcal{H}^{s',\delta}(\mathbb{B}) \to \mathcal{H}^{s,-\gamma+\varepsilon}(\mathbb{B})$$

are continuous maps for all $s,s' \in \mathbb{R}$, where $G^*$ is the formal adjoint of $G$ with respect to $(\cdot,\cdot)_{H^{0,0}}$. The space of Green operators with asymptotic data $(\gamma,\delta)$ is denoted by
C_G(\mathbb{B}, (\gamma, \delta)), or C_G(\mathbb{B}, (\gamma, \delta))_\varepsilon if we fix the value of \varepsilon > 0. The latter is a Fréchet space with the system of (semi)norms \(|\cdot|_{\Sigma}, |\cdot|_{\delta}^*\) for \(\Sigma = (s, s') \in \mathbb{Z}^2\) given by

\[|G|_\Sigma = \|G\|_{L(H^{s,\gamma}, H^{s',\delta+\varepsilon})}, \quad |G^*|_\Sigma = \|G^*\|_{L(H^{s,-\gamma}, H^{s',-\gamma+\varepsilon})}.
\]

**Example B.3.** If \(\omega_1\) and \(\omega_2\) are cut-off functions on \(\mathbb{B}\), then

\[\left(1 - \omega_1\right)L^{-\infty}(\text{int } \mathbb{B})\left(1 - \omega_2\right) \subset C_G(\mathbb{B}, (\gamma, \delta))_\varepsilon\]

for every \(\gamma, \delta \in \mathbb{R}\) and \(\varepsilon > 0\).

**Example B.4.** Let \(K \in H^{\infty,-\gamma+\varepsilon}(\mathbb{B}) \hat{\otimes}_\pi H^{\infty,\delta+\varepsilon}(\mathbb{B})\) for \(\gamma, \delta \in \mathbb{R}\) and \(\varepsilon > 0\), where \(\hat{\otimes}_\pi\) denotes the projective tensor product. The operator \(G\) defined by

\[G u(y) = \int_\mathbb{B} K(y, y') u(y') dy'\]

for \(u \in C_0^{\infty}(\mathbb{B})\), is an element of \(C_G(\mathbb{B}, (\gamma, \delta))_\varepsilon\). Here, \(dy'\) denotes the measure induced by the fixed metric on \(\mathbb{B}\).

**Theorem B.5.** Let \(\gamma, \delta, \varepsilon \in \mathbb{R}\) and \(G \in C_G(\mathbb{B}, (\gamma, \delta))_\varepsilon\). For every \(s, s' \in \mathbb{R}\) the operator \(G\) belongs to \(L_2(H^{s,\gamma}(\mathbb{B}), H^{s',\delta}(\mathbb{B}))\), the space of Hilbert-Schmidt operators. Moreover, its kernel satisfies

\[K_G \in H^{\infty,(-\gamma+\varepsilon)^-}(\mathbb{B}) \hat{\otimes}_\pi H^{\infty,(-\delta+\varepsilon)^-}(\mathbb{B}).\]

In fact, \(G\) is in \(L_2(H^{s,\gamma}, H^{s',\delta})\) because it maps \(H^{s,\gamma} \to H^{s',\delta+\varepsilon}\) continuously, and the embedding \(H^{\infty,\delta+\varepsilon} \hookrightarrow H^{s',\delta}\) is in \(L_2\) for every \(s' \in \mathbb{R}\), see Lemma [3.1]. Observe that because of the relation

\[H^{\infty,(-\gamma+\varepsilon)^-}(\mathbb{B}) \hat{\otimes}_\pi H^{\infty,(-\delta+\varepsilon)^-}(\mathbb{B}) \hookrightarrow C^{\infty}(\text{int } \mathbb{B}) \hat{\otimes}_\pi C^{\infty}(\text{int } \mathbb{B}) \cong C^{\infty}(\text{int } \mathbb{B} \times \text{int } \mathbb{B}),\]

we clearly have \(C_G(\mathbb{B}, (\gamma, \delta)) \hookrightarrow L^{-\infty}(\text{int } \mathbb{B})\) for all \(\gamma, \delta\) and \(\varepsilon\).

For our purposes it is convenient to describe the kernels of Green operators with the help of the Hilbert tensor product. If \(H_1\) and \(H_2\) are Hilbert spaces then the tensor product \(H_1 \hat{\otimes}_H H_2\) is defined as the space of finite dimensional operators \(H_1 \to H_2\) endowed with the topology of \(L_2(H_1', H_2)\). Hence

\[H_1 \hat{\otimes}_H H_2 = L_2(H_1', H_2)\quad \text{(completion)}\]

This concept of tensor product extends to the so-called hilbertizable locally convex spaces, i.e., spaces whose completion can be written as a reduced projective limit of Hilbert spaces. Nuclear spaces are hilbertizable. For further details we refer to [11], [23], [31]. Let us just point out some basic properties. If \(E\) and \(F\) are hilbertizable spaces then \(E \hat{\otimes}_H F\) is hilbertizable. Furthermore, \(E \hat{\otimes}_\pi F \hookrightarrow E \hat{\otimes}_H F \hookrightarrow E \hat{\otimes}_\varepsilon F\), so if \(E\) or \(F\) is a nuclear space, then

\[E \hat{\otimes}_\pi F \cong E \hat{\otimes}_\varepsilon F \cong E \hat{\otimes}_H F.\]

Furthermore, the Hilbert tensor product commutes with projective limits, e.g.,

\[H^{\infty,\gamma} \hat{\otimes}_H H^{\infty,\delta} = \text{proj lim}_{k, k' \in \mathbb{N}} H^{k,\gamma-k'} \hat{\otimes}_H H^{k',\delta-k'}\quad \text{for } \gamma, \delta \in \mathbb{R}.
\]

According to Theorem [3.5] the kernel of \(G \in C_G(\mathbb{B}, (\gamma, \delta))_\varepsilon\) belongs to

\[H^{\infty,(-\gamma+\varepsilon)^-} \hat{\otimes}_\pi H^{\infty,(-\delta+\varepsilon)^-} \hookrightarrow H^{\infty,(-\gamma+\varepsilon)^-} \hat{\otimes}_H H^{\infty,(-\delta+\varepsilon)^-} \hookrightarrow H^{\infty,(-\gamma+\varepsilon)^-} \hat{\otimes}_\varepsilon H^{\infty,\delta_\varepsilon}\]

for every \(\delta_\varepsilon < \delta + \varepsilon\) and \(-\gamma_\varepsilon < -\gamma + \varepsilon\).
The embedding properties from Lemma B.1 together with some standard results from functional analysis, cf. [46, Appendix 3], imply the following

**Theorem B.6.** Let G be an operator of order \(-\mu < -\dim \mathbb{R}\). Assume further that \(G : \mathcal{H}^{s,\gamma}(\mathbb{R}) \rightarrow \mathcal{H}^{s+\mu,\gamma+\varepsilon}(\mathbb{R})\) is bounded for some \(\gamma \in \mathbb{R}, \varepsilon > 0\), and all \(s \in \mathbb{R}\). Then G is of trace class on \(\mathcal{H}^{s,\gamma}(\mathbb{R})\) and, for any \(k > s + \dim \mathbb{R}\) and \(\gamma \leq \gamma_\varepsilon < \gamma + \varepsilon\),

\[
|\text{Tr} \, G| \leq C \|G\|_{L(\mathcal{H}^{-k,\gamma}, \mathcal{H}^{k,\gamma})}
\]

for some \(C > 0\). Notice that every \(G \in C_G(\mathbb{R}, (\gamma, \gamma)_\varepsilon)\) is of this form.

**C. A class of operator-valued symbols.** One of the main ideas of Schulze’s edge calculus is to consider the operators (near the edge) as operator-valued symbols acting on Banach spaces together with groups of isomorphisms. In this section we introduce analogously a class of anisotropic symbols similarly to [6] but here on conical subsets of the complex plane. In this context, every Banach space \(E\) will be considered together with a strongly continuous group action \(\{\kappa_t\}_{t \in \mathbb{R}_+}\), that is, a group of isomorphisms on \(E\) satisfying

1. \(\tau \mapsto \kappa_\tau e : \mathbb{R}_+ \rightarrow E\) is continuous for each \(e \in E\),
2. \(\kappa_{\tau+\sigma} = \kappa_\tau \circ \kappa_\sigma\) for all \(\tau, \sigma > 0\).

The following estimate can be obtained by means of the Banach-Steinhaus theorem. For a proof see, e.g., Remark 2.2 in [24].

**Lemma C.1.** There are constants \(C, M \geq 0\) such that

\[
\|\kappa_t\|_{L(E)} \leq C \max(\tau, \tau^{-1})^M.
\]

Let us fix a positive smooth function \([\cdot] : \mathbb{C} \rightarrow \mathbb{R}_+\) with \([\lambda] = |\lambda|\) for \(|\lambda| \geq 1\). We also fix \(d \in \mathbb{N}\) to describe the anisotropy. Further, let us set

\[
(5.2) \quad [\lambda]_d := |\lambda|^{1/d} \quad \text{and} \quad \kappa(\lambda) := \kappa_{[\lambda]}_d \quad \text{for some group action} \ \{\kappa_t\}_{t \in \mathbb{R}_+}.
\]

For the rest of this section let \(\Lambda \subset \mathbb{C}\) be a conical set. Let also \(E_0\) and \(E_1\) be Banach spaces with group actions \(\kappa_0 = \{\kappa_{0t}\}_{t \in \mathbb{R}_+}\) and \(\kappa_1 = \{\kappa_{1t}\}_{t \in \mathbb{R}_+}\), respectively.

**Definition C.2.** Let \(\mu \in \mathbb{R}\). A function \(a \in C^\infty(\Lambda, \mathcal{L}(E_0, E_1))\) is said to be in the space \(S^{\mu, d}(\Lambda; E_0, E_1)\) of operator-valued symbols, if for every \(\alpha \in N_2^d\) there is a constant \(C_\alpha > 0\) such that

\[
(5.3) \quad \left\|\kappa_{-1}^{-1}(\lambda) \left\{\partial_\alpha^\Lambda a(\lambda)\right\} \kappa_0(\lambda)\right\|_{\mathcal{L}(E_0, E_1)} \leq C_\alpha |\lambda|^{-d|\alpha|}
\]

for all \(\lambda \in \Lambda\). Furthermore, we define

\[
S^{-\infty}(\Lambda; E_0, E_1) := \bigcap_{\mu \in \mathbb{R}} S^{\mu, d}(\Lambda; E_0, E_1) = S(\Lambda, \mathcal{L}(E_0, E_1))
\]

which is independent of \(d\) and the group actions \(\kappa_0, \kappa_1\).

Most of the usual symbol properties (e.g. embedding, composition, asymptotic summation, etc.) that are known for the scalar-valued symbols, can also be formulated in the operator-valued case (see e.g. [5]).

**Proposition C.3.** Let \(\mu, \tilde{\mu} \in \mathbb{R}\) and \(E_2, E_3\) be further Banach spaces. Then

1. \(S^{\mu, d}(\Lambda; E_1, E_2) \hookrightarrow S^{\tilde{\mu}, d}(\Lambda; E_0, E_3)\) if \(E_0 \hookrightarrow E_1\) and \(E_2 \hookrightarrow E_3\) with \(\kappa_1 = \kappa_0\) on \(E_0\) and \(\kappa_3 = \kappa_2\) on \(E_2\).
2. \(S^{\mu, d}(\Lambda; E_0, E_1) \hookrightarrow S^{\mu+M_0+M_1, d}(\Lambda; E_0, E_1)_{(\text{id})}\), where \(M_0\) and \(M_1\) are associated to \(\kappa_0\) and \(\kappa_1\) as in Lemma C.1, and the subscript (id) means that \(E_0\) and \(E_1\) are considered with the trivial group action \(\kappa = \text{id}\),
3. \( \partial^*_\Lambda S^{\mu,d}(\Lambda; E_0, E_1) \subset S^{\mu-|\alpha|,d}(\Lambda; E_0, E_1) \) for every \( \alpha \in \mathbb{N}^2 \),
4. \( S^{\mu,d}(\Lambda; E_1, E_2) \cdot S^{\tilde{\mu},d}(\Lambda; E_0, E_1) \subset S^{\mu+\tilde{\mu},d}(\Lambda; E_0, E_2) \).

An operator-valued function \( a \in C^\infty(\Lambda, \mathcal{L}(E_0, E_1)) \) is called **twisted homogeneous of degree** \((\mu, d)\) (with respect to the group actions \( \kappa_0 \) and \( \kappa_1 \)) if

\[
a(\tau^d \lambda) = \tau^\mu \kappa_1 \tau^1 a(\lambda) \kappa_0^{-1}_0, \tag{5.4}
\]

for all \( \tau \geq 1 \) and \( \lambda \in \Lambda \) sufficiently large.

**Lemma C.4.** Let \( E_0 \) and \( E_1 \) be Banach spaces over \( X^\wedge = \mathbb{R}_+ \times X \) with the same group action. Let \( a \in C^\infty(\Lambda, \mathcal{L}(E_0, E_1)) \) be twisted homogeneous of degree \((\mu, d)\) and assume that every \( a(\lambda) \) is an integral operator with kernel \( k_a(\lambda, r, x, r', x') \). Then

\[
k_a(\tau^d \lambda, r, x, r', x') = \tau^\mu \kappa_1 \tau^1 k_a(\lambda, \tau r, x, \tau r', x') \tag{5.5}
\]

for every \( \tau \geq 1 \) and \( \lambda \in \Lambda \) sufficiently large.

The concept of twisted homogeneity allows to define classical symbols:

**Definition C.5.** The space \( S^{\mu,d}_{cl}(\Lambda; E_0, E_1) \) is the subspace of \( S^{\mu,d}(\Lambda; E_0, E_1) \) consisting of symbols \( a(\lambda) \) with an expansion \( a(\lambda) \sim \sum_{j=0}^\infty a_{\mu-j}(\lambda) \) such that every \( a_{\mu-j} \) is twisted homogeneous of degree \((\mu - j, d)\). In this case we call \( a(\lambda) \) the **principal part** of \( a(\lambda) \).

**Example C.6.** If \( a \in C^\infty(\Lambda, \mathcal{L}(E_0, E_1)) \) is twisted homogeneous of degree \((\mu, d)\), then \( a \in S^{\mu,d}_{cl}(\Lambda; E_0, E_1) \).

**Proposition C.7.** Let the symbol \( a \in S^{\mu,d}(\Lambda; E_0, E_1) \) be such that \( a(\lambda) : E_0 \to E_1 \) is an invertible operator for every \( \lambda \in \Lambda \). Then \( a^{-1} \in S^{-\mu,d}(\Lambda; E_1, E_0) \) if and only if \( \| \kappa_0^{-1}(\lambda) a(\lambda)^{-1} \kappa_1(\lambda) \|_{\mathcal{L}(E_1, E_0)} \leq C|\lambda|^{-\mu}_d \) for some \( C > 0 \).

**Green operator-valued symbols**

Let \( \mu, \gamma, \delta \in \mathbb{R}, d \in \mathbb{N} \) and \( \varepsilon > 0 \). The function

\[
g \in \bigcap_{\Sigma \in \mathbb{R}^4} C^\infty(\Lambda, \mathcal{L}(K^s, \gamma(X^\wedge)^\sigma, K^{s', \delta}(X^\wedge)^{\sigma'})),
\]

where \( \Sigma = (s, s', \sigma, \sigma') \), is said to be in the class \( R^\mu_G(\Lambda, (\gamma, \delta))_\varepsilon \) if and only if

\[
g \in \bigcap_{\Sigma \in \mathbb{R}^4} S^{\mu,d}_{cl}(\Lambda; K^s, \gamma(X^\wedge)^\sigma, K^{s', \delta+\varepsilon}(X^\wedge)^{\sigma'}) \quad \text{and} \quad g^* \in \bigcap_{\Sigma \in \mathbb{R}^4} S^{\mu,d}_{cl}(\Lambda; K^{s, -\delta}(X^\wedge)^\sigma, K^{s', -\gamma+\varepsilon}(X^\wedge)^{\sigma'}),
\]

where \( g^*(\lambda) \) is the pointwise formal adjoint of \( g(\lambda) \) in \( K^{0,0} \). Furthermore, we denote by \( R^\mu_G(\Lambda, (\gamma, \delta)) \) the union for \( \varepsilon > 0 \) of all these classes. Every element \( g \in R^\mu_G(\Lambda, (\gamma, \delta)) \) is called a **Green symbol** of order \((\mu, d)\) with weight data \((\gamma, \delta)\). As a classical operator-valued symbol such a Green symbol \( g \) has a twisted homogeneous principal component which we denote \( \sigma^\mu_g(\lambda) \). Finally, we set

\[
R^\infty_G(\Lambda, (\gamma, \delta)) := \bigcap_{\mu \in \mathbb{R}} R^\mu_G(\Lambda, (\gamma, \delta))
\]

which is independent of \( d \) and the group actions involved.
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