UPPER BOUND FOR THE GROMOV WIDTH OF COADJOINT ORBITS OF COMPACT LIE GROUPS

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Abstract. We find an upper bound for the Gromov width of coadjoint orbits of compact Lie groups with respect to the Kirillov-Kostant-Souriau symplectic form by computing certain Gromov-Witten invariants. The approach presented here is closely related to the one used by Gromov in his celebrated non-squeezing theorem.

1. Introduction

The Darboux theorem in symplectic geometry states that around any point of a symplectic manifold, there is a system of local coordinates such that the symplectic manifold looks locally like \( \mathbb{R}^{2n} \) with its canonical symplectic form. A natural and fundamental problem in symplectic geometry is to know how far we can extend symplectically these coordinates in the symplectic manifold. This is how the concept of Gromov's width arises. The Gromov width of a symplectic manifold \((M,\omega)\) is defined as

\[
\text{Gwidth}(M,\omega) = \sup \\{ \pi r^2 : \exists \text{ a symplectic embedding } B_{2n}(r) \hookrightarrow M \}.
\]

The Gromov non-squeezing theorem gives us insights of how restrictive is the Gromov width from above. It says that if there is a symplectic embedding of the ball \( B_{2n}(r) \) of radius \( r \) into a cylinder \( B_2(\lambda) \times \mathbb{R}^{2n-2} \) of radius \( \lambda \), then \( r \leq \lambda \). In particular,

\[
\text{Gwidth}(B_2(\lambda) \times \mathbb{R}^{2n-2}) = \pi \lambda^2.
\]

Gromov's non-squeezing theorem was proven in [22], where the connection between \( J \)-holomorphic curves and symplectic geometry is established. Since then, several authors have used Gromov's method for bounding the Gromov width of other families of symplectic manifolds, such as Lu for symplectic toric manifolds in [40], Lu and Karshon-Tolman for complex Grassmannians manifolds in [39] and [33], respectively, and Zoghi for regular coadjoint orbits in [55] (see also McDuff-Polterovich [42], Biran [3]).

In this paper, we are particularly interested in finding upper bounds for the Gromov width of coadjoint orbits of compact Lie groups with respect to the Kostant-Kirillov-Souriau form. The main result obtained in this paper is the following theorem

**Main Theorem.** Let \( G \) be a compact connected simple Lie group with Lie algebra \( \mathfrak{g} \). Let \( T \subset G \) be a maximal torus and let \( \check{T} \) be the corresponding system of coroots. We identify the dual Lie algebra \( \mathfrak{t}^* \) with the fixed points of the coadjoint action of \( T \) on \( \mathfrak{g}^* \). Let \( \lambda \in \mathfrak{t}^* \subset \mathfrak{g}^* \), let \( O_\lambda \) be

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the coadjoint orbit passing through \( \lambda \) and let \( \omega_\lambda \) be the Kostant-Kirillov-Souriau form defined on \( O_\lambda \), then

\[
\text{Gwidth}(O_\lambda, \omega_\lambda) \leq \min_{\hat{\alpha} \in \hat{R}} |\langle \lambda, \hat{\alpha} \rangle| \\
\text{subject to } \langle \lambda, \hat{\alpha} \rangle \neq 0
\]

Zoghi, in his Ph.D thesis [55], has proved the same result for regular coadjoint orbits. Recall that a coadjoint orbit of a compact Lie group is regular if the stabilizer of any element of it under the coadjoint action is a maximal torus of the compact Lie group.

On the other hand, Pabiniak has considered the problem of determining lower bounds for the Gromov width of coadjoint orbits of compact Lie groups in [45], [46] and [47].

In [46] and [47], Pabiniak has proved that the upper bound appearing in the Main Theorem is indeed an equality for coadjoint orbits of \( U(n) \). Together with our result, this yields the following theorem:

**Theorem.** Let us identify the Lie algebra of \( U(n) \) with its dual via the ad-invariant inner product

\[
(A, B) \rightarrow \text{tr}(A \cdot B).
\]

For \((\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n\), let \( \lambda = i \text{diag}(\lambda_1, \cdots, \lambda_n) \in u(n) \cong u(n)^* \). Let \( O_\lambda \) be the coadjoint orbit of \( U(n) \) passing through \( \lambda \in u(n)^* \) and \( \omega_\lambda \) be the Kostant-Kirillov-Souriau form defined on the coadjoint orbit, then

\[
\text{Gwidth}(O_\lambda, \omega_\lambda) = \min_{\lambda_i \neq \lambda_j} |\lambda_i - \lambda_j|.
\]

This paper is organized as follows: in the second section, we review the \( J \)-holomorphic tools that we will use throughout the text, and then we explain how upper bounds for the Gromov width of symplectic manifolds can be obtained by a non-vanishing Gromov-Witten invariant.

In the third section, we recall background on the geometry of coadjoint orbits of compact Lie groups and homogeneous spaces.

In the fourth section, we show the upper bound appearing in the Main Theorem for Grassmannian manifolds coming from long simple roots.

In the later sections, the Main Theorem is reformulated for the classical Grassmannians. Finally, in the last section the Main Theorem is proven.

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## 2. \( J \)-holomorphic curves

In this section we give a short review of pseudoholomorphic theory and Gromov-Witten invariants, and we show how pseudoholomorphic curves are related with the Gromov width of a symplectic manifold. Most of the material presented here is adapted from [43].
2.1. **Pseudoholomorphic theory.** Let \((M^{2n}, \omega)\) be a symplectic manifold. An almost complex structure \(J\) of \((M, \omega)\) is a smooth operator \(J : TM \to TM\) such that \(J^2 = -I_d\). We say that an almost complex structure \(J\) is **compatible** with \(\omega\) if the formula
\[
g(v, w) := \omega(v, Jw)
\]
defines a Riemannian metric. We denote the space of \(\omega\)-compatible almost complex structures by \(\mathcal{J}(M, \omega)\).

Let \((\mathbb{C}P^1, j)\) be the Riemann sphere with its standard complex structure \(j\). Let \(J \in \mathcal{J}(M, \omega)\). A map \(u : \mathbb{C}P^1 \to M\) is called a **\(J\)-holomorphic curve** of genus zero or simply a **\(J\)-holomorphic curve** if
\[
J \circ du = du \circ j,
\]
or equivalently if \(\bar{\partial}_J u = 0\) where \(\bar{\partial}_J\) is the operator defined by
\[
\bar{\partial}_J u = (du + J \circ du \circ j).
\]

For a second homology class \(A \in H_2(M, \mathbb{Z})\), we define the **moduli space of simple \(J\)-holomorphic curves of degree \(A\)** as
\[
\mathcal{M}_A^s(M, J) = \{u : \mathbb{C}P^1 \to M : J \circ du = du \circ j, u_*[\mathbb{C}P^1] = A, u \text{ is simple}\}.
\]

Let \(\pi_1 : \mathbb{C}P^1 \times M \to \mathbb{C}P^1\) and \(\pi_2 : \mathbb{C}P^1 \times M \to M\) be the projections onto the first and the second factor, respectively. Let \(v : \pi_1^*(T \mathbb{C}P^1) \to \pi_2^*(TM)\) be a \(J\)-antilinear map, i.e., a map such \(\bar{\partial}_J \circ jv = -J \circ \bar{\partial}_J v\). We say that a map \(u : \mathbb{C}P^1 \to M\) is a **\(v\)-perturbed \(J\)-holomorphic curve** if it satisfies the equation
\[
\bar{\partial}_J (u)|_z = v_{(z, u(z))}
\]

For an antilinear map \(v : \pi_1^*(T \mathbb{C}P^1) \to \pi_2^*(TM)\), we denote by
\[
\mathcal{M}_A(M, J, v) := \{u : \mathbb{C}P^1 \to M : \bar{\partial}_J u = v, u_*[\mathbb{C}P^1] = A\}
\]
the **moduli space of \(v\)-perturbed \(J\)-holomorphic curves of degree \(A\)**.

A curve \(u : \mathbb{C}P^1 \to M\) is said to be **multiply covered** if it is the composite of a holomorphic branched covering map \((\mathbb{C}P^1, j) \to (\mathbb{C}P^1, j)\) of degree greater than one with a \(J\)-holomorphic map \(\mathbb{C}P^1 \to M\). It is **simple** if it is not multiply covered.

For generic \((J, v)\), the moduli space \(\mathcal{M}_A(M, J, v)\) is an oriented smooth manifold of dimension equal to
\[
\dim M + 2c_1(T_M)(A),
\]
where \(c_1\) denotes the first Chern class of the bundle \((T_M, J)\) (see e.g. Theorem 3.1.5)).

Let \(v : \pi_1^*(T \mathbb{C}P^1) \to \pi_2^*(TM)\) be a \(J\)-antilinear map that is equivariant with respect to the \(PSL(2, \mathbb{C})\) action, we denote by \(\mathcal{M}_{A,k}^s(M, J, v)\) the set of equivalence classes, under the action of the reparametrization group \(PSL(2, \mathbb{C})\), of simple \(v\)-perturbed \(J\)-holomorphic maps
\[
u : (\mathbb{C}P^1, z_1, \ldots, z_k) \to M
\]
of degree \(A\) with \(k\)-marked distinct points \(z_i \in \mathbb{C}P^1\). For generic \((J, v)\), the moduli space \(\mathcal{M}_{A,k}^s(M, J, v)\) is a smooth oriented manifold of dimension equal to
\[
\dim M + 2c_1(T_M)(A) + 2k - 6.
\]
We have an evaluation map
\[ \text{ev}_J = (\text{ev}_1, \ldots, \text{ev}_k) : \mathcal{M}_{A,k}^*(M, J, v) \to M^k \]
defined by
\[ \text{ev}_J(u, z_1, \cdots, z_k) = (u(z_1), \cdots, u(z_k)). \]

The moduli space \( \mathcal{M}_{A,k}(M, J, v) \) is usually not compact but it can be compactified by adding stable maps. A stable \( J \)-holomorphic map with \( k \)-marked distinct points
\[ u : (C, z_1, \cdots, z_k) \to M \]
is a tree \( C = \bigcup u_\alpha \) of \( J \)-holomorphic maps \( u_\alpha : \mathbb{C}P^1 \to M \) with at worst nodal singularities such that if a component \( u_\alpha : \mathbb{C}P^1 \to M \) is constant the number of marked and singular points that it contains is greater or equal to three. This implies that the automorphism group of \( u \) is finite. The degree of \( u \) is defined as
\[ \deg u = \sum_\alpha \deg u_\alpha \in H_2(M, \mathbb{Z}). \]

For \( A \in H_2(M, \mathbb{Z}) \), we denote by \( \overline{\mathcal{M}}_{A,k}(M, J, v) \) the compactified moduli space of \( v \)-perturbed \( J \)-holomorphic stable maps of degree \( A \) with \( k \)-marked points.

As stated in Li-Tian [37], Fukaya-Ono [17], Ruan [48], Siebert [50], and more recently in Chen-Li-Wang [8], Fukaya-Ohta-Oh-Ono [30], Hofer-Wysocki-Zehnder [26], [27], Cieliebak-Mohnke [12] and McDuff-Wehrheim [41], the moduli space \( \overline{\mathcal{M}}_{A,k}(M, J, v) \) carries a virtual fundamental class \( [\overline{\mathcal{M}}_{A,k}(M, J, v)]^\text{virt} \in H_*(\overline{\mathcal{M}}_{A,k}(M, J, v), \mathbb{Q}) \) that is used for defining the Gromov-Witten invariants.

2.1. Theorem. For generic almost complex structure \( J \) and perturbation \( v \) the moduli space \( \overline{\mathcal{M}}_{A,k}(M, J, v) \) carries a homology class \( [\overline{\mathcal{M}}_{A,k}(M, J, v)]^\text{virt} \in H_*(\overline{\mathcal{M}}_{A,k}(M, J, v), \mathbb{Q}) \). The pushforward of \( [\overline{\mathcal{M}}_{A,k}(M, J, v)]^\text{virt} \) under \( \text{ev}_J : \overline{\mathcal{M}}_{A,k}(M, J, v) \to M^k \) defines a homology class
\[ \text{GW}_{A,k}(M) \in H_{\dim}(M^k, \mathbb{Q}) \]
in dimension \( \dim = \dim \overline{\mathcal{M}}_{A,k}(M, J) = \dim M + 2c_1(T_M)(A) + 2k - 6 \).

The class \( \text{GW}_{A,k}(M) \) is independent of the perturbation \( v \) and it is invariant under smooth deformation of \( (\omega, J) \) through compatible structures and it is called the Gromov-Witten cycle of \( (M, \omega) \).

For \( \alpha_1, \cdots, \alpha_k \in H^*(M) \), the Gromov-Witten invariant is defined as
\[ \text{GW}_{A,k}(\alpha_1, \ldots, \alpha_k) := \langle \alpha_1 \times \cdots \times \alpha_k, \text{GW}_{A,k}(M) \rangle = \int_{[\overline{\mathcal{M}}_{A,k}(M, J, v)]^\text{virt}} \text{ev}_1^* \alpha_1 \cup \cdots \cup \text{ev}_k^* \alpha_k. \]

We fix geometric representatives \( A_i \subset M \) for the Poincaré duals of each cohomology class \( \alpha_i \), and assume that
\[ \dim \overline{\mathcal{M}}_{A,k}(M, J) = \dim M + 2c_1(T_M)(A) + 2k - 6 = \sum_i \deg \alpha_i. \]

For generic almost complex structure \( J \) and perturbation \( v \), the Gromov-Witten invariant \( \text{GW}_{A,k}(\alpha_1, \ldots, \alpha_k) \) can be interpreted, with appropriate sign and weight, as the number of
$J$-holomorphic perturbed maps of degree $A$ with $k$-marked points $u : (\mathbb{C}P^1, z_1, \cdots, z_k) \to M$ such that $u(z_i) \in A_i, i = 1, \cdots, k$.

2.3. Remark. We now enumerate some important properties of the Gromov-Witten invariants and how they are computed in homogeneous spaces $G_C/P$:

- Gromov-Witten invariants can be defined under simple assumptions, for example, if we assume that either the symplectic manifold $(M, \omega)$ is semipositive or the homology class $A \in H_2(M, \mathbb{Z})$ is $\omega$-indecomposable. Examples of semipositive symplectic manifolds are Grassmannian manifolds which are the quotients of complex Lie groups by maximal parabolic subgroups.

In these cases, for a regular almost complex structure $J$ of $(M, \omega)$, the evaluation map

$$ev_J : \overline{M}_{A,k}(M, J) \to M^k$$

represents a pseudocycle, meaning that its image can be compactified by adding a set of codimension at least two. A fundamental class can be associated to this pseudocycle; and this fundamental class can be used to define the Gromov-Witten invariant $GW_{A,k}$ [43, Theorem 7.1.1, Lemma 7.1.8].

- Gromov-Witten invariants are also defined in the algebraic-geometry category. A result due to Siebert states that when $M$ is a complex projective manifold its algebraic and symplectic Gromov-Witten invariants coincide [51].

- For a compact Lie group $G$, the complex quotient $G_C/P$ of the complexification $G_C$ of $G$ by a parabolic subgroup $P \subset G_C$ is endowed with an integrable and invariant almost complex structure $J$. For $A \in H_2(G_C/P, \mathbb{Z})$, the moduli space of stable maps of degree $A$ with $k$-marked point $\overline{M}_{A,k}(G_C/P, J)$ is a normal projective variety and its virtual fundamental class is the same as the fundamental class $[\overline{M}_{A,k}(G_C/P, J)] \in H^*(\overline{M}_{A,k}(G_C/P, J), \mathbb{Q})$ of the moduli space $\overline{M}_{A,k}(G_C/P, J)$ (see e.g. [18]).

- The Bertini-Kleiman transversality theorem implies that for irreducible subvarieties $\Gamma_1, \ldots, \Gamma_m \subset M = G_C/P$ Poincaré dual to cohomology class $\alpha_1, \ldots, \alpha_n \in H^*(G_C/P, \mathbb{Z})$ such that

$$\dim \overline{M}_{A,k}(M, J) = \sum_i \deg \alpha_i,$$

the intersection number

$$\sharp(ev_{1}^{-1}(g_1\Gamma_1) \cap \cdots \cap ev_{k}^{-1}(g_k\Gamma_k)),$$

that is the number of $J$-holomorphic curves of degree $A$ passing through $g_1\Gamma_1, \cdots, g_k\Gamma_k$, coincides with the Gromov-Witten invariant $GW_{A,k}(\alpha_1, \ldots, \alpha_k)$, for generic $g_1, \ldots, g_k \in G_C$ (see e.g. [18, Lemma 14]).

2.2. Gromov width. Given a symplectic manifold $(M^{2n}, \omega)$, its **Gromov’s width** is defined as

$$\text{Gwidth}(M, \omega) = \sup \{ \pi r^2 : \exists \text{ a symplectic embedding } B_{2n}(r) \hookrightarrow M \}.$$  

The Darboux theorem implies that the Gromov width of a symplectic manifold is always positive. Moreover, if the symplectic manifold is compact, its Gromov width is finite. The
following statement shows the relation between pseudoholomorphic curves and the Gromov width of symplectic manifolds:

2.4. **Theorem.** Let \((M^{2n}, \omega)\) be a compact symplectic manifold, and \(A \in H_2(M, \mathbb{Z}) \setminus \{0\}\) a second homology class. Suppose that for a dense subset of smooth \(\omega\)-compatible almost complex structures, the evaluation map

\[
ev_J : \mathcal{M}^\#_{A,1}(M, J) \to M
\]

has a dense image. Then for any symplectic embedding \(B_{2n}(r) \hookrightarrow M\), we have

\[
\pi r^2 \leq \omega(A),
\]

where \(\omega(A)\) denotes the symplectic area of \(A\). In particular,

\[
\textrm{Gwidth}(M, \omega) \leq \omega(A).
\]

2.5. **Remark.** This result goes back to Gromov [22] and it implies the non-squeezing theorem. A proof can be found for instance in [55, Proposition 3.6].

2.6. **Remark.** According to Theorem 2.4, in order to find upper bounds for the Gromov width of a symplectic manifold \((M, \omega)\), we want to prove that for generic almost complex structures \(J \in \mathcal{J}(M, \omega)\), the evaluation map

\[
ev_J : \mathcal{M}^\#_{A,1}(M, J) \to M
\]

is onto. One way to achieve the ontoness of the evaluation map is for example by proving that a Gromov-Witten invariant with one of its constraints being a point is different from zero. More precisely, if there exist cohomology classes \(a_1, \ldots, a_k\) such that \(\text{GW}_{A,k}(a_1, \ldots, a_k) \neq 0\) and \(a_1\) is Poincaré dual to the fundamental class of a point, then for a generic choice of almost complex structure \(J\), the evaluation map

\[
ev_J : \mathcal{M}^\#_{A,1}(M, J) \to M
\]

is onto in a dense subset of \(M\), which, by Theorem 2.4, implies that

\[
\textrm{Gwidth}(M, \omega) \leq \omega(A).
\]

3. **Coadjoint Orbits of Compact Lie groups**

In this section we recall some general statements about homogeneous spaces \(G_C/P\), coadjoint orbits and its geometry. Most of the material shown here can be found in the classical literature such as [3] and [33]. Most of the material presented in Section 3.3 is adapted from [19, Chapter 2, 3].

3.1. **Kostant-Kirillov-Souriau form.** Let \(G\) be a connected compact Lie group, \(g\) be its Lie algebra, and \(g^*\) be the dual of the Lie algebra \(g\). The compact Lie group \(G\) acts on its Lie algebra \(g^*\) by the coadjoint action. Let \(\lambda \in g^*\) and \(O_\lambda\) be the coadjoint orbit through \(\lambda\).

The coadjoint orbit \(O_\lambda\) carries a symplectic form defined as follows: for \(\lambda \in g^*\) we define a skew bilinear form on \(g\) by

\[
\omega_\lambda(X, Y) = \langle \lambda, [X, Y] \rangle.
\]
The kernel of \( \omega_\lambda \) is the Lie algebra \( g_\lambda \) of the stabilizer of \( \lambda \in g^* \) under the coadjoint action. In particular, \( \omega_\lambda \) defines a nondegenerate skew-symmetric bilinear form on \( g/g_\lambda \), a vector space that can be identified with \( T_\lambda(\mathcal{O}_\lambda) \subset g^* \). Using the coadjoint action, the bilinear form \( \omega_\lambda \) induces a closed, invariant, nondegenerate 2-form on the orbit \( \mathcal{O}_\lambda \), therefore defining a symplectic structure on \( \mathcal{O}_\lambda \). This symplectic form is known as the Kostant-Kirillov-Souriau form of the coadjoint orbit \( \mathcal{O}_\lambda \).

Let \( T \subset G \) be a maximal torus with Lie algebra equal to \( t \). The restricted action of \( T \subset G \) on \( \mathcal{O}_\lambda \) is Hamiltonian with momentum map

\[
\mu : \mathcal{O}_\lambda \hookrightarrow g^* \rightarrow t^*.
\]

The Kostant Convexity Theorem states that the image of the momentum map \( \mu : \mathcal{O}_\lambda \hookrightarrow t^* \) is the convex hull of the momentum images of the fixed points of the action of \( T \) on \( \mathcal{O}_\lambda \). This theorem is a special case of the Atiyah-Guillemin-Sternberg Convexity Theorem in symplectic geometry.

The compact Lie group \( G \) admits a complexification \( G_C \). Let \( L = \text{Stab}_G \lambda \subset G \) be the stabilizer of \( \lambda \in g^* \) with respect to the coadjoint action. Fix a positive Weyl chamber in \( t \) and hence a Borel subgroup \( B \) of \( G_C, B \supset T \), and a parabolic subgroup \( P \) of \( G_C, P \supset L \), so the homogeneous spaces \( G/L \) and \( G_C/P \) are diffeomorphic. One can induce a complex structure on \( G/L \cong \mathcal{O}_\lambda \) which is homogeneous under the \( G_C \)-action. Thus the coadjoint orbit \( \mathcal{O}_\lambda \) get a homogeneous complex structure \( J \). Together with the Kostant-Kirillov-Souriau form, this makes the coadjoint orbit \( \mathcal{O}_\lambda \) a Kähler manifold.

The homogeneous space \( G_C/P \) with the torus \( T \) action is a GKM space, i.e., the closure of every connected component of the set \( \{ x \in G_C/P : \dim C(T \cdot x) = 1 \} \) is a sphere. The closure of \( \{ x \in G_C/P : \dim C(T \cdot x) = 1 \} \) is called the 1-skeleton of \( G_C/P \). The moment graph or GKM graph of \( \mathcal{O}_\lambda \) is the graph whose vertices are the \( T \)-fixed points and the edges are the connected components of \( \{ x \in G_C/P : \dim C(T \cdot x) = 1 \} \).

### 3.2. Schubert varieties in \( G_C/P \)

Let \( R \subset t^* \) be the root system of \( T \) in \( G \) so

\[
g_C = t_C \oplus \bigoplus_{\alpha \in R} g_\alpha,
\]

where \( g_\alpha := \{ x \in g_C : [h,x] = \alpha(h)x \text{ for all } h \in t_C \} \) is the root space associated with the root \( \alpha \in R \).

Let \( R^+ \subset R \) be a choice of positive roots with simple roots \( S \subset R^+ \). Let \( W := N_G(T)/T \) be the Weyl group of \( G \). For every root \( \alpha \in R \), let \( s_\alpha \in W \) be the reflection associated to it. Let \( B \subset G_C \) be the Borel subgroup with Lie algebra

\[
b = t_C \oplus \bigoplus_{\alpha \in R^+} g_\alpha
\]

and \( P \subset G_C \) be a parabolic subgroup such that \( B \subset P \subset G_C \).

Let \( W_P = N_P(T)/T \) be the Weyl group of \( P \) and \( S_P := \{ \alpha \in S : s_\alpha \in W_P \} \subset S \) be the set of simple roots whose corresponding reflections are in \( W_P \). The group \( W_P \) is generated by the simple reflections \( s_\alpha \) for \( \alpha \in S_P \). The parabolic subgroup \( P \) is the group generated by the
Borel subgroup $B$ and $N_P(T)$. The map

$$\{ \text{Parabolic subgroups } P \subset G_C : B \subset P \} \rightarrow \{ \text{Subsets } S' \subset S \}$$

$$P \mapsto S_P$$

establishes a bijection between the parabolic subgroups of $G_C$ and subsets of the set of simple roots $S$ (see for instance [33, Chapter 5]).

Set $R_P = R \cap ZS_P$ and $R_P^+ = R^+ \cap ZS_P$, where $ZS_P = \text{span}_Z(S_P)$ is the Abelian group spanned by $S_P$ in $t^*$. The Lie algebra of $P$ is

$$p = b \oplus \bigoplus_{\alpha \in R_P^+} g_\alpha$$

For each $w \in W$, the length $l(w)$ of $w$ is defined as the minimum number of simple reflections $s_\alpha \in W, \alpha \in S$, whose product is $w$.

For $w', w \in W$, define $w' \rightarrow w$ if there exists simple reflections $s \in S$ such that

$$w = w' \cdot s$$

and $l(w) = l(w') + 1$. Then define $w' \leq_B w$ if there is a sequence

$$w' \rightarrow w_1 \rightarrow \ldots \rightarrow w_m = w.$$ 

The **Bruhat order** on $W$ is the partial ordering defined by the relation $\leq_B$.

Let $W^P \subset W$ be the set of all minimum length representatives for cosets in $W/W_P$. Each element $w \in W$ can be written uniquely as $w = w^P w_P$ where $w^P \in W^P$ and $w_P \in W_P$ (see e.g. [28]). The **Bruhat order** on $W^P$ is the restriction to $W^P$ of the Bruhat order in $W$. The **Bruhat order** on $W/W_P$ is defined by $w'w_P \leq_B w w_P$ if and only if $w'w_P \leq_B w w_P$ in $W^P$.

Let $w_0$ be the longest element in $W$ and let $B^{\text{op}} = w_0Bw_0 \subset G_C$ be the Borel subgroup opposite to $B$. For $w \in W^P$ we define the **Schubert cell**

$$C_P(w) := BwP/P \subset G_C/P$$

and the **opposite Schubert cell**

$$C^{\text{op}}_P(w) := B^{\text{op}}wP/P \subset G_C/P.$$ 

The **Schubert variety** $X_P(w)$ and its opposite $X^{\text{op}}_P(w)$ are by definition the closures of the Schubert cells $C_P(w)$ and $C^{\text{op}}_P(w)$, respectively. We have that for $w', w \in W^P$,

$$X_P(w') \subset X_P(w)$$

if and only if $w' \leq_B w$. Indeed,

$$X_P(w) = \bigsqcup_{w' \leq_B w} C_P(w')$$

For $w \in W^P$, the Schubert cell $C_w$ is isomorphic to an affine space of complex dimension equal to the length of $w$ that is the same as the complex codimension of the opposite Schubert cell $C^{\text{op}}_P(w)$. The Schubert cells $\{ X_P(w) \}_{w \in W^P}$ define a CW-complex for $G_C/P$ with cells occurring only in even dimension. Thus, the fundamental classes $[X_P(w)]$ of $X_P(w), w \in W^P$, are a free basis of $H_*(G_C/P, \mathbb{Z})$ as a $\mathbb{Z}$-module. Likewise, the Poincaré dual classes of $[X_P(w)], w \in W^P$, are a free basis of $H^*(G_C/P, \mathbb{Z})$ as a $\mathbb{Z}$-module.
Let $n = \dim_{\mathbb{C}}(G_{\mathbb{C}}/P, \mathbb{Z})$. For an integer $1 \leq d \leq n$, the intersection pairing is a bilinear map

$$H^{2d}(G_{\mathbb{C}}/P, \mathbb{Z}) \otimes H^{2n-2d}(G_{\mathbb{C}}/P, \mathbb{Z}) \to H^{2n}(G_{\mathbb{C}}/P, \mathbb{Z}) \cong \mathbb{Z}$$

$$\alpha \times \beta \to \langle \alpha, \beta \rangle$$

As $w', w \in W^P$ varies over the permutations of length $d$, we have

$$\langle \text{PD}[X^p_P(w')], \text{PD}[X_P(w)] \rangle = \delta_{ww'}$$

This shows that the classes $\{\text{PD}[X_P(w)] : l(w) = d\}$ form a basis for $H^{2n-2d}(G_{\mathbb{C}}/P, \mathbb{Z})$ and the classes $\{\text{PD}[X^p_P(w')] : l(w') = d\}$ form a dual basis for $H^{2d}(G_{\mathbb{C}}/P, \mathbb{Z})$.

The dual application

$$H^*(G_{\mathbb{C}}/P, \mathbb{Z}) \to H^*(G_{\mathbb{C}}/P, \mathbb{Z})$$

$$[X_P(w)] \mapsto [X^p_P(w)]$$

is compatible with the antiautomorphism of Bruhat order

$$W^P \to W^P$$

$$w \mapsto w^* := w_0 w w_p,$$

where $w_p$ denotes the longest element in $W_P$.

### 3.3. Chern classes and Stable curves.

Let $R \subset t^*$ be the root system of a maximal torus $T$ in $G$ with positive roots $R^+$ and simple roots $S \subset R^+$. Let $(\cdot, \cdot)$ denote an Ad-invariant inner product defined on $\text{Lie}(G) = g$. We identify the Lie algebra $g$ and its dual $g^*$ via this inner product. The inner product $(\cdot, \cdot)$ defines an inner product in $t^* = \mathbb{R} R$ that we will denote with the same notation $(\cdot, \cdot)$. Each root $\alpha \in R$ has a coroot $\check{\alpha} \in t$ that is identified with $\frac{2\alpha}{(\alpha, \alpha)}$ via the inner product $(\cdot, \cdot)$. The coroots form the dual root system $\check{R} = \{\check{\alpha} : \alpha \in R\}$, with basis of simple coroots $\check{S} = \{\check{\alpha} : \alpha \in S\}$. For $\alpha \in R$ we let $\omega_\alpha \in t^*$ denote the corresponding fundamental weight, defined by $(\omega_\alpha \check{\beta}) = \delta_{\alpha,\beta}$ for $\alpha \in R$. For a parabolic subgroup $T \subset P \subset G_{\mathbb{C}}$, we let $\check{S}_P := \{\check{\alpha} : \alpha \in S_P\} \subset \check{S}$.

The cohomology group $H^2(G_{\mathbb{C}}/P, \mathbb{Z})$ can be identified with the span

$$\mathbb{Z}\{\omega_\alpha : \alpha \in S \setminus S_P\}$$

and the homology group $H_2(G_{\mathbb{C}}/P, \mathbb{Z})$ with the quotient

$$\mathbb{Z}\check{S}/\mathbb{Z}\check{S}_P.$$

For each $\alpha \in S \setminus S_P$ we identify the class $[X_P(s_\alpha)] \in H_2(G_{\mathbb{C}}/P, \mathbb{Z})$ with $\check{\alpha} + \mathbb{Z}\check{S}_P \in \mathbb{Z}\check{S}/\mathbb{Z}\check{S}_P$ and we identify $\text{PD}[X^p_P(s_\beta)] \in H^2(G_{\mathbb{C}}/P, \mathbb{Z})$ with $\omega_\beta$. The Poincaré pairing $H^2(G_{\mathbb{C}}/P, \mathbb{Z}) \otimes H_2(G_{\mathbb{C}}/P, \mathbb{Z}) \to \mathbb{Z}$ is then given by the Ad-invariant inner product $(\cdot, \cdot)$ on $t$.

The following localization lemma, due to Bott [3], allows us to compute the first Chern classes of line bundles over $G_{\mathbb{C}}/P$:
3.2. **Lemma.** Suppose that a torus $T$ acts on a curve $C \cong \mathbb{C}P^1$, with fixed points $p \neq q$, and suppose $L$ is a $T$-equivariant line bundle on $C$. Let $\eta_p$ and $\eta_q$ be the weights of $T$ acting on the fibers $L_p$ and $L_q$, and let $\psi_p$ be the weight of $T$ acting on the tangent space to $C$ at $p$. Then

$$\eta_p - \eta_q = n \psi_p$$

where $n = \int_C c_1(L)$ is the degree of $L$.

The collection of points $wP$ for $w \in W^P$ is the set of all $T$-fixed points in $G_\mathbb{C}/P$. For each positive root $\alpha \in R^+ \setminus R^+_P$, there is a unique irreducible $T$-invariant curve $C_\alpha$ that contains $1 \cdot P$ and $s_\alpha \cdot P$. Indeed, $C_\alpha = SL(2, \mathbb{C})_\alpha \cdot P/P$ where $SL(2, \mathbb{C})_\alpha \subset G_\mathbb{C}$ is the subgroup of $G_\mathbb{C}$ with Lie algebra $g_\alpha \oplus g_{-\alpha} \oplus [g_\alpha, g_{-\alpha}]$. To see that $C_\alpha$ is unique, there is a neighborhood of $1 \cdot P/P$ that is $T$-equivariantly isomorphic to $g_\mathbb{C}/p$. The $T$-invariant curves in $g_\mathbb{C}/p$ correspond to weight spaces $g_{-\alpha}$, for $\alpha \in R^+ \setminus R^+_P$.

If $\lambda$ is a weight that vanishes on all $\beta$ in $S_P$, it determines a character on $P$, and so a line bundle $L(\lambda) = G_\mathbb{C} \times_P \mathbb{C}(\lambda)$ on $G_\mathbb{C}/P$.

The Chern class $c_1(L(\lambda)) \in H^2(G_\mathbb{C}/P, \mathbb{Z}) \cong \mathbb{Z}\{w_\alpha : \alpha \in S \setminus S_P\}$ is identified with the weight $\lambda$, and we have an isomorphism

$$\mathbb{Z}\{w_\alpha : \alpha \in S \setminus S_P\} \rightarrow H^2(G_\mathbb{C}/P, \mathbb{Z})$$

$$\lambda \mapsto c_1(L(\lambda))$$

Indeed, if $L$ is any holomorphic line holomorphic line bundle on $G_\mathbb{C}/P$, there exists a weight $\lambda \in \mathbb{Z}\{w_\alpha : \alpha \in S \setminus S_P\}$ such that $L = L(\lambda)$, and in particular $\text{Pic}(G_\mathbb{C}/P) \cong \mathbb{Z}\{w_\alpha : \alpha \in S \setminus S_P\}$ (see e.g. [13]).

The previous lemma implies that

$$\int_{C_\alpha} c_1(L(\lambda)) \cdot (-\alpha) = -\lambda - (-s_\alpha(\lambda)) = s_\alpha(\lambda) - \lambda$$

and thus

$$(\lambda, \check{\alpha}) = \int_{C_\alpha} c_1(L(\lambda))$$

As a consequence, we see that

$$[C_\alpha] = \check{\alpha} + \mathbb{Z}\check{S}_P \subset H_2(G_\mathbb{C}/P, \mathbb{Z}) \cong \mathbb{Z}\check{S}/\mathbb{Z}\check{S}_P$$

The tangent space of $G_\mathbb{C}/P$ at the point $1 \cdot P \in G_\mathbb{C}/P$ can be identified with

$$g_\mathbb{C}/p = \bigoplus_{\alpha \in R^+ \setminus R^+_P} g_{-\alpha}$$

The weight of the line bundle $\wedge^n T(G_\mathbb{C}/P)$, $n = \dim(G_\mathbb{C}/P)$, at the point $1 \cdot P \in G_\mathbb{C}/P$ is $-\sum_{\gamma \in R^+ \setminus R^+_P} \gamma$, and hence

$$c_1(T_{G_\mathbb{C}/P}) = c_1\left(\wedge^n T_{G_\mathbb{C}/P}\right) = c_1\left(L\left(\sum_{\gamma \in R^+ \setminus R^+_P} \gamma\right)\right)$$

$$= \sum_{\gamma \in R^+ \setminus R^+_P} \gamma \in H^2(G_\mathbb{C}/P, \mathbb{Z}) \cong \mathbb{Z}\{w_\alpha : \alpha \in S \setminus S_P\}.$$
3.3. Remark. If $\lambda \in \mathbb{R}\{w_\alpha : \beta \in S\setminus S_P\} \subset \mathfrak{t}^*$, the cohomology class of the Kostant-Kirillov form $[\omega_\lambda] \in H^2(\mathcal{O}_\lambda, \mathbb{R})$ of the coadjoint orbit $\mathcal{O}_\lambda$ passing through $\lambda$ is identified with $\lambda \in H^2(G_C/P, \mathbb{R}) \cong \mathbb{R}\{w_\alpha : \beta \in S\setminus S_P\} \subset \mathfrak{t}^*$, and for any positive root $\alpha \in R^+\setminus R_P^+$, the symplectic area

$$\omega_\lambda(C_\alpha) = \int_{C_\alpha} \omega_\lambda = \langle \lambda, \check{\alpha} \rangle$$

In particular, the coadjoint orbit $\mathcal{O}_\lambda$ is prequantizable if and only if $\lambda$ is integral, i.e., $\lambda \in \mathbb{Z}\{w_\alpha : \beta \in S\setminus S_P\}$.

For an integral weight $\lambda = \sum_{\beta \in S\setminus S_P} l_\beta w_\beta \in \mathbb{Z}_{\geq 0}\{w_\alpha : \beta \in S\setminus S_P\}$, let $V_\lambda$ be the irreducible representation of $G_C$ with highest weight $\lambda$. The Borel-Weil-Bott Theorem states that the holomorphic sections $H^0(G_C/P, L(-\lambda))$ of the line bundle $L(-\lambda)$ is isomorphic as a $G_C$-representation to the irreducible representation $V_\lambda$.

Let $v_\lambda$ be a highest weight vector of $V_\lambda$. We can embed $G_C/P$ in the projective space $\mathbb{P}V_\lambda$ by the transformation

$$G_C/P \rightarrow \mathbb{P}V_\lambda$$

$$[g] \rightarrow [g \cdot v_\lambda]$$

(3.4)

A curve $u : \mathbb{C}P^1 \rightarrow G_C/P$ of degree $\check{\alpha} = \sum_{\beta \in S\setminus S_P} m_\beta \check{\beta} + Z\check{S}_P \in H_2(G_C/P, \mathbb{Z})$, $m_\beta \in \mathbb{Z}_{\geq 0}$, is push-forwarded by the embedding $G_C/P \rightarrow \mathbb{P}V_\lambda$ to a curve of degree

$$\int_u c_1(L_\lambda) = \sum_{\beta \in S\setminus S_P} m_\beta l_\beta \in H_2(\mathbb{P}V_\lambda, \mathbb{Z}).$$

(3.5)

4. Upper Bound for the Gromov Width of Grassmannian Manifolds

Let $G$ be a compact connected simple Lie group with Lie algebra $\mathfrak{g}$. Let $\lambda \in \mathfrak{t}^* \subset \mathfrak{g}^*$, let $\mathcal{O}_\lambda$ be the coadjoint orbit passing through $\lambda$ and let $\omega_\lambda$ be the Kostant-Kirillov-Souriau form defined on $\mathcal{O}_\lambda$. Let $S$ be a system of simple roots associated with a maximal torus $T \subset G$.

In this section we show that if there is a maximal parabolic subgroup $P \subset G_C$ associated with a long simple root $\alpha \in S$ such that $\mathcal{O}_\lambda \cong G_C/P$, then

$$\text{Gwidth}(\mathcal{O}_\lambda, \omega_\lambda) \leq \langle \lambda, \check{\alpha} \rangle$$

This upper bound would be obtained by computing a non-vanishing Gromov-Witten invariant with one of its constraints being Poincaré dual to a point. Having a explicit description of the moduli space of holomorphic lines for Grassmannian manifolds associated with long simple roots will allow us to compute such invariants.

In the later sections, we will describe the upper bound for the Gromov width of the Classical Grassmannian manifolds, such as Isotropic Grassmannians and Orthogonal Grassmannians.
4.1. **Fibrations.** Let $G$ be a compact simple Lie group. Let $T \subset G$ be a maximal torus, $B \subset G_C$ be a Borel subgroup with $T_C \subset B$ and $S$ be the corresponding system of simple roots. Let $W = N(T)/T$ be the associated Weyl group. For a parabolic subgroup $P \subset G_C, B \subset P$, let $W_P = N_P(T)/T$ be the Weyl group of $P$ and $S_P$ be the subset of simple roots of $S$ whose corresponding reflections are in $W_P$. Let $W^P \subset W$ be the set of all minimum length representatives for cosets in $W/W_P$. For $w \in W^P$, let $X_P(w) \subset G_C/P$ be the Schubert variety associated with $w \in W^P$.

For a pair of parabolic subgroups $P, Q \subset G_C$, such that $B \subset P \subset Q$, we have a quotient map $G_C/P \to G_C/Q$. We want to study the images and preimages of Schubert varieties under these quotient maps.

4.1. **Lemma (Stumbo).** For parabolic subgroups $P, Q \subset G_C$ such that $B \subset P \subset Q$ define

$$W^P_Q := \{ w \in W_Q : l(ws) > l(w) \text{ for } s \in S_P \}$$

as the corresponding quotient fibration. If we decompose $w$, we have a unique $w^Q \in W^P_Q$ and a unique $w^P_Q \in W^P$ such that $w = w^Q w^P_Q$. Their lengths satisfy $l(w) = l(w^Q) + l(w^P_Q)$.

4.2. **Lemma.** For parabolic subgroups $P, Q \subset G_C$ such that $B \subset P \subset Q$, let $w^0_p, w_p$ and $w_q$ be the longest elements in $W^P_Q, W_P$ and $W_Q$, respectively. Then, $w^0_p = w_q w_p$.

**Proof.** Let $w_0$ be the longest element in $W$. The quotient map $\pi : W \to W^P \cong W/W_P$ is order preserving and thus the longest element in $W^P$ is $\pi(w_0)$. By the previous lemma $w_0 = \pi(w_0) w_p$, so that $\pi^{-1}(w_0) = w_0 w_p^{-1}$. Similarly, for the quotient map $\pi' : W \to W^Q \cong W/W_Q$, we have that $\pi'(w_0) = w_0 w_q^{-1}$ is the longest element in $W^Q$. Using again the previous Lemma, we have that the permutation $\pi'(w_0) w^P_q$ is the longest permutation in $W^P$. So that $\pi'(w_0) w^P_q = \pi(w_0)$, and thus $w_0 w_q^{-1} w^P_q = w_q^{-1} w^P_q w_p^{-1}$. But the longest element $w_p$ in $W_P$ satisfies $w_p^2 = e$, and we are done. \qed

4.3. **Proposition.** For parabolic subgroups $P, Q \subset G_C$ such that $P \subset Q \subset G_C$, let $\pi : G_C/P \to G_C/Q$ be the corresponding quotient fibration. If we decompose $w \in W^P$ as $w^Q w^P_q$, where $w^Q \in W^Q$ and $w^P_q \in W^P$, then $\pi(X_P(w)) = X_Q(w^Q)$. On the other hand, if $w \in W^Q$, then $\pi^{-1}(X_Q(w)) = X_P(\tilde{w} w^P_q)$, where $w^P_q$ is the longest element in $W^P$.

**Proof.** The map $\pi : G_C/P \to G_C/Q$ is $B$-equivariant and closed (this is a consequence of for example the closed map lemma). This implies that Schubert cells, which are $B$-orbits, and Schubert varieties, which are their closures, in $G_C/P$ are mapped to Schubert cells and Schubert varieties in $G_C/Q$, respectively.

For $w \in W^P \subset W$, there exist unique $w^Q \in W^Q$ and $w^P \in W^P \subset W_Q$ such that $w = w^Q w^P_Q$ and $l(w) = l(w^Q) + l(w^P_Q)$. The Schubert cell $C_P(w) = Bw^P/P \subset G_C/P$ is mapped to the Schubert cell $C_Q(w^Q) = Bw^Q Q/Q \subset G_C/Q$ via $\pi$, and

$$\pi(X_P(w)) = \pi(C_P(w)) = \pi(C_P(w)) = C_Q(w^Q) = X_Q(w^Q).$$
On the other hand, if \( w \in W^Q \), then
\[
\pi^{-1}(C_Q(w)) = \bigcup_{v \in W^P \atop v^Q = w^Q} C_P(v).
\]

The maximum element, with respect to the Bruhat order defined on \( W^P \), in the set \( \{v \in W^P : v^Q = w^Q\} \) is \( w^P \), where \( w^P \) denotes the longest element in \( W^P_q \). Since \( \pi \) is a continuous map, we have that
\[
\pi^{-1}(X_Q(w)) = \pi^{-1}(C_Q(w)) = \bigcup_{v \in W^P \atop v^Q = w^Q} C_P(v) = \bigcup_{v \in W^P \atop v \leq w^P} C_P(v) = X_P(w^P).
\]

\[\square\]

The following two technical lemmas would be needed in the next section:

4.4. Lemma. Let \( \alpha \in S \) be a simple root and \( N(\alpha) \subset S \) be the neighbors of \( \alpha \) in the Dynkin diagram of \( G \), i.e., the simple roots connected to \( \alpha \) by an edge in the Dynkin diagram of \( G \). Let \( P, P'' \subset G_C \) be the parabolic subgroups such that \( S_P = S \setminus \{\alpha\}, S_{P''} = S \setminus (N(\alpha) \cup \{\alpha\}) \). Then
\[
W_{P''} \cdot s_\alpha \subset W^P.
\]

Proof. Let \( w \in W_{P''} \). We write \( w = s_1 \cdots s_r \), where \( s_1, \ldots, s_r \) are simple reflections in \( S_P \). Suppose that there exists a simple reflection \( t \in S_P \) such that \( l(ws_\alpha t) < l(ws_\alpha) \). By the Exchange Principle (see e.g. Humphreys [28]),
\[
ws_\alpha t = s_1 \cdots \hat{s}_i \cdots s_r s_\alpha
\]
for some \( i \), in particular \( s_\alpha t s_\alpha \in W_P \). We now consider two cases and see that this is not possible:

1. Suppose that \( s_\alpha t = ts_\alpha \). Thus \( t \notin N(\alpha) \) and
\[
l(ws_\alpha t) = l(wts_\alpha) = l(wt) + 1.
\]
But \( t \notin N(\alpha) \) and \( t \neq s_\alpha \), so \( t \in S \setminus (N(\alpha) \cup \{s_\alpha\}) = S_{P''} \). As \( w \in W_{P''} \)
\[
l(wt) > l(w),
\]
hence
\[
l(ws_\alpha t) = l(wts_\alpha) = l(wt) + 1 > l(w) + 1 = l(ws_\alpha),
\]
which contradicts our assumption of having \( l(ws_\alpha t) < l(ws_\alpha) \).

2. Suppose that \( s_\alpha t \neq ts_\alpha \). If \( l(s_\alpha t s_\alpha) \neq 3 \), by the Deletion Principle (see e.g. Humphreys [28]) either \( s_\alpha t s_\alpha = s_\alpha \), or \( s_\alpha t s_\alpha = t \), which are not possible. So \( l(s_\alpha t s_\alpha) = 3 \). Now, clearly \( l(s_\alpha t) = l(s_\alpha t s_\alpha s_\alpha) = 2 < l(s_\alpha t s_\alpha) \), so if \( s_\alpha t s_\alpha = s_1 s_2 s_3 \), for some simple reflections \( s_1, s_2, s_3 \in S_P \), by the Exchange Principle \( s_\alpha t \in W_P \) which would imply that \( s_\alpha \in W_P \), a contradiction.

\[\square\]
4.5. **Lemma.** Let $\alpha \in S$ be a simple root and $N(\alpha) \subset S$ be the neighbors of $\alpha$ in the Dynkin diagram of $G$. Let $P, P' \subset G_C$ be the parabolic subgroups such that $S_P = S \setminus \{\alpha\}, S_{P'} = S \setminus N(\alpha)$. Let $P'' = P \cap P'$, and, let $\pi : G_C/P'' \to G_C/P$ and $\pi' : G_C/P'' \to G_C/P'$ be the natural defined quotient maps, so we have the diagram of arrows

\[ G_C/P'' \xrightarrow{\pi'} G_C/P' \]

\[ \pi \]

\[ G_C/P \]

For any subset $X$ of $G_C/P$, we define

\[ \check{X} := \pi'(\pi^{-1}(X)) \subset G_C/P'. \]

There exists $w \in W_P$ such that the fundamental class $[\check{X}_P(w)] \in H_*(G_C/P', \mathbb{Z})$ is opposite to the fundamental class $[X_P(e)] \in H_*(G_C/P', \mathbb{Z})$, where $e$ denotes the identity of $W$.

**Proof.** For a permutation $w \in W_P$ and the corresponding Schubert variety $X_P(w) \subset G_C/P$, the set $\check{X}_P(w)$ is a $B$-stable Schubert variety in $G_C/P'$. We will denote by $\check{w}$ the permutation in $W_P$ such that

\[ \check{X}_P(w) = X_{P'}(\check{w}) \]

Let $w_p, w_p', w_p'', w_p''', w_p'''$ and $w_0$ be the longest elements in $W_P, W_{P'}, W_{P''}, W_{P'''}$ and $W$, respectively. We want to find a permutation $w \in W_P$ such that

\[ [X_{P'}(\check{w})] = [X_{P'}(\hat{e})]_{op} = [X_{P'}(w_0\check{w}')] \]

or equivalently a permutation $w \in W_P$ such that $\check{w} = w_0\hat{w}'$.

Let us find first an expression for $\hat{e}$: by the Proposition 4.3, we have that $\pi^{-1}(X_P(e)) = X_{P''}(w_p'')$, so

\[ X_{P'}(\hat{e}) = \pi'(\pi^{-1}(X_P(e))) = \pi'(X_{P''}(w_p'')) . \]

Note that $W_{P'''} = W_{P''} \cap W_P' \subset W_{P'}$, in particular $w_p''' \in W_{P'}$, and $X_{P'}(\hat{e}) = X_{P'}(w_p''')$, and as a consequence $\hat{e} = w_p'''$, or that is the same

\[ \pi'(\pi^{-1}(X_P(e))) = X_{P'}(w_p''') \]

(4.6)

Remember that we want to find $w \in W_P$ such that $\check{w} = w_0\hat{w}'$. If we find $w \in W_P$ such that $w'w_p''' \in W_{P'}$ and $w''w_p''' = w_0w_p'''w_p'$, then $\check{w} = w'w_p'''$ and $\check{w} = w_0\hat{w}'$. If such $w$ exists,
Notice first that it should be equal to $w_0 w_p^{p''} w_{p'} (w_p^{p''})^{-1}$. We have to verify that $w_0 w_p^{p''} w_{p'} (w_p^{p''})^{-1} \in W^P$ and $w_0 w_p^{p''} w_{p'} \in W^{P'}$.

Let

$$w = w_0 w_p^{p''} w_{p'} (w_p^{p''})^{-1}.$$ 

(4.7)

So $w \in W^P$ if and only if $w_p^{p''} s_\alpha \in W^P$, which we already know from the previous Lemma, and we are done. \hfill \Box

4.2. Upper Bound for the Gromov width of Grassmannian Manifolds: long root case.

Let $G$ be a compact Lie group, $\mathfrak{g}$ be its Lie algebra and $\mathfrak{g}^*$ be the dual of this Lie algebra. Let $\lambda \in \mathfrak{g}^*$ and $O_\lambda \subset \mathfrak{g}^*$ be the coadjoint orbit passing through $\lambda$. Let us assume that $O_\lambda \cong G_\mathfrak{C}/P$, where $P \subset G_\mathfrak{C}$ is a parabolic subgroup of $G_\mathfrak{C}$. Let $T \subset G$ be a maximal torus and let $B \subset G_\mathfrak{C}$ be a Borel subgroup with $T_\mathfrak{C} \subset B \subset P$. Let $W = N(T)/T$ be the associated Weyl group. Let $R$ be the corresponding set of roots and $S$ be the corresponding system of simple roots. Let $W_P$ be the Weyl group of $P$ and $S_P$ be the subset of simple roots whose corresponding reflections are in $W_P$.

If there exists a simple root $\alpha \in S$ such that $S_P = S \setminus \{\alpha\}$, we say that the parabolic subgroup $P \subset G_\mathfrak{C}$ is the maximal parabolic of $G_\mathfrak{C}$ associated with the simple root $\alpha \in S$ and we will call the corresponding homogeneous space $G_\mathfrak{C}/P$ a Grassmannian manifold.

We will assume from now in this section that $P \subset G_\mathfrak{C}$ is a maximal parabolic subgroup associated with a simple root $\alpha \in S$ and we will endow $G_\mathfrak{C}/P$ with a Kähler structure coming from its identification with $O_\lambda$. This Kähler structure and the one defined on $O_\lambda$ would be denoted indistinguishably by $(\omega, J)$.

The second homology group $H_2(G_\mathfrak{C}/P, \mathbb{Z})$ of a Grassmannian manifold $G_\mathfrak{C}/P$ is cyclic and is freely generated as a $\mathbb{Z}$-module by the fundamental class of the Schubert variety $X_P(s_\alpha)$. From now, we will denote this fundamental class by $A$.

Let $\mathcal{M}_A(G_\mathfrak{C}/P, J)$ be the moduli space of $J$-holomorphic curves of degree $A$. We will call elements of the moduli space $\mathcal{M}_A(G_\mathfrak{C}/P, J)$ holomorphic lines of $G_\mathfrak{C}/P$ or just lines. Let $\mathcal{M}_{A,k}(G_\mathfrak{C}/P, J)$ be the moduli space of $J$-holomorphic maps of degree $A$ with $k$-marked distinct points.

The complex group $G_\mathfrak{C}$ acts holomorphically on $G_\mathfrak{C}/P$ and trivially on $H_2(G_\mathfrak{C}/P, \mathbb{Z})$, as a consequence there is a group action of $G_\mathfrak{C}$. This action is transitive when $k = 0, 1$ and the simple root $\alpha$ is long:

4.8. Theorem (Manivel-Landsberg [34], Strickland [52]). Let $\alpha \in S$ be a long simple root and $N(\alpha)$ be the neighbors of $\alpha$ in the Dynkin diagram of $G$. Let $P' \subset G_\mathfrak{C}$ be the parabolic subgroup with $S_{P'} = S \setminus N(\alpha)$ and let $P'' = P' \cap P$. 

The group action of $G_C$ on the moduli spaces $\mathcal{M}_{A,0}(G_C/P,J), \mathcal{M}_{A,1}(G_C/P,J)$ is transitive; and, the moduli spaces $\mathcal{M}_{A,0}(G_C/P,J), \mathcal{M}_{A,1}(G_C/P,J)$ are isomorphic to the homogeneous spaces $G_C/P', G_C/P''$, respectively.

Recall that we want to prove that there exist a cycle $X \subset G_C/P$ so the Gromov-Witten invariant $GW_{A,2}(PD[p], PD[X])$ is different from zero. If so, by Remark (2.6) we will have that

$$G\text{width}(G_C/P, \omega_\lambda) \leq \omega_\lambda(A)$$

4.9. **Theorem.** Let $G$ be a compact connected simple Lie group with Lie algebra $\mathfrak{g}$. Let $\lambda \in \mathfrak{t}^* \subset \mathfrak{g}^*$, let $O_\lambda$ be the coadjoint orbit passing through $\lambda$ and let $\omega_\lambda$ be the Kostant-Kirillov-Souriau form defined on $O_\lambda$. Assume that there is a long simple root $\alpha \in S$ such that $O_\lambda \cong G_C/P$, then

$$G\text{width}(O_\lambda, \omega_\lambda) \leq \langle \lambda, \check{\alpha} \rangle$$

**Proof.** Let $f : \mathcal{M}_{A,1}(G_C/P,J) \to \mathcal{M}_{A,0}(G_C/P,J)$ be the forgetful map that maps a pair $[u,z]$ to $[u]$ and $ev_J : \mathcal{M}_{A,1}(G_C/P,J) \to G_C/P$ be the evaluation map that maps a pair $[u,z]$ to $u(z)$. We have a diagram of arrows

$$\begin{array}{ccc}
\mathcal{M}_{A,1}(G_C/P,J) & \xrightarrow{f} & \mathcal{M}_{A,0}(G_C/P,J) \\
\downarrow_{ev_J} & & \\
G_C/P & & \\
\end{array}$$

Let $N(\alpha)$ be the neighbors of $\alpha$ in the Dynkin diagram of $G$. Let $P' \subset G_C$ be the parabolic subgroup with $S_{P'} = S \setminus N(\alpha)$ and let $P'' = P' \cap P$. By Theorem 4.8, we have that $\mathcal{M}_{A,0}(G_C/P,J) \cong G_C/P'$ and $\mathcal{M}_{A,0}(G_C/P,J) \cong G_C/P''$. The diagram of arrows shown above can be identified with the diagram.
\[ \begin{array}{c} G_{C}/P'' \xrightarrow{\pi'} G_{C}/P' \\ \downarrow \pi \\ G_{C}/P \end{array} \]

where \( \pi \) and \( \pi' \) denote the projection quotient maps (see e.g. [2, Theorem 1]).

For any subset \( X \) of \( G_{C}/P \), we define \( \hat{X} := f(\text{ev}_{J}^{-1}(X)) = \{ u \in M_{A,0}(G_{C}/P) : u \text{ is incident to } X \} \subset M_{A,0}(G_{C}/P) \)

The set \( \hat{X} = f(\text{ev}_{J}^{-1}(X)) \subset M_{A,0}(G_{C}/P) \) can be identified with the set \( \pi'(\pi^{-1}(X)) \subset G_{C}/P' \).

By Lemma 4.5, there exists a permutation \( w \in W_{P} \) such that the fundamental class \( [\hat{X}_{P}(w)] \in H_{*}(G_{C}/P'; \mathbb{Z}) \) is dual to the fundamental class of \( [\hat{X}_{P}(e)] \in H_{*}(G_{C}/P', \mathbb{Z}) \). This implies that for generic \( g \in G_{C} \), the Schubert variety \( \hat{X}_{P}(w) \) intersects transversally \( g \hat{X}_{P}(e) \) at one point in \( M_{A,0}(G_{C}/P, J) \cong G_{C}/P' \). In other words, for generic \( g \in G_{C} \), there is one holomorphic line in \( G_{C}/P \) passing through \( g \cdot P \in G_{C}/P \) and \( X_{P}(w) \subset G_{C}/P \).

Bertini-Kleiman’s Theorem implies that for generic \( g \in G_{C} \) the evaluation map

\[ \text{ev}_{J} : M_{A,2}(G_{C}/P, J) \to (G_{C}/P)^{2} \]

is transverse to \( \{g \cdot P\} \times X_{P}(w) \subset (G_{C}/P)^{2} \).

Note that the Schubert variety \( X_{P}(w) \subset G_{C}/P \) satisfies the dimensional constraint

\[ \dim_{C} X_{P}(w) + \dim_{C} M_{A,2}(G_{C}/P) = 2 \dim_{C} G_{C}/P : \]

The permutation \( w \in W_{P} \) is given by Equation 4.7, so by construction of \( w \) we have that

\[ \dim_{C} X_{P}(w) = \dim_{C}(G_{C}/P) - l(w_{p''}^{p'} s_{\alpha}) \]

where \( w_{p''}^{p'} \) is the longest element in \( W_{P} \). Moreover,

\[ l(w_{p''}^{p'} s_{\alpha}) = l(w_{p''}^{p'}) + 1 = \dim_{C}(P/P''') + 1 = \dim_{C}(G_{C}/P') + 1 = \dim_{C} M_{A,1}(G_{C}/P) - \dim_{C}(G_{C}/P) + 1 \]

So in conclusion,

\[ \dim_{C} X_{P}(w) = 2 \dim_{C} G_{C}/P - \dim_{C} M_{A,1}(G_{C}/P) - 1 \]

\[ = 2 \dim_{C} G_{C}/P - \dim_{C} M_{A,2}(G_{C}/P). \]
By Proposition 7.4.5 of [43], the Gromov-Witten invariant
\[ GW_{A,2}(PD[X_P(\epsilon)], PD[X_F(w)]) \]
is positive, and thus by Remark 2.6
\[ \text{Gwidth}(O_\lambda, \omega_\lambda) \leq \omega_\lambda(A) = \langle \lambda, \bar{\alpha} \rangle. \]

5. Upper bounds for the Gromov width of isotropic Grassmannians

In this section we are going to find upper bounds for the Gromov width of Grassmannian manifolds of type $C$ better known as **Isotropic Grassmannians**.

Let $(\mathbb{C}^{2n}, \Omega)$ be the standard complex symplectic vector space with complex coordinates $(z_1, \ldots, z_n, w_1, \ldots, w_n)$ and with complex bilinear skew-symmetric form
\[ \Omega = \sum dz_i \wedge dw_i. \]

Let $Sp(n, \mathbb{C})$ be the complex Lie group of linear transformation on $\mathbb{C}^{2n}$ that preserves $\Omega$. Let $Sp(n) = Sp(2n, \mathbb{C}) \cap U(2n)$ be the symplectic group of quaternionic unitary transformations. The group $Sp(n)$ is a compact form of $Sp(n, \mathbb{C})$.

The set of diagonal matrices $T$ in $Sp(n)$ forms a maximal torus in $Sp(n)$ with Lie algebra equals to $t = \{ i \text{diag}(\lambda, -\lambda) \in M_{2n}(\mathbb{C}) : \lambda \in \mathbb{R}^n \} \cong i\mathbb{R}^n$.

Let $e_i : t^* \to \mathbb{R}$ be the projection that maps a matrix $i \text{diag}(\lambda, -\lambda) \in t$ to the $i$-th component $\lambda_i$ of $\lambda \in \mathbb{R}^n$. The root system of $Sp(n)$ with respect to the maximal torus $T$ is the set $R = \{ \pm e_i \pm e_j \ (i \neq j), \pm 2e_i \}_{1 \leq i,j \leq n}$, with a choice of simple roots given by $S = \{ \alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \ldots, \alpha_{n-1} = e_{n-1} - e_n, \alpha_n = 2e_n \}$ and Dynkin diagram
\[ \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\end{array} \begin{array}{c}
\longrightarrow \\
\longrightarrow \\
\longrightarrow \\
\end{array} \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\end{array} \begin{array}{c}
e_1 \ne_2 \ne_3 \ne_{n-1} \ne_n \end{array} \]

Let $\lambda \in \mathbb{R}^n$ and
\[ C_\lambda := \{ A \in M_{2n}(\mathbb{C}) : A = QA^TQ, A^* = -A, \ \text{spectrum} \ A = i(\lambda, -\lambda) \} \]

where $Q$ is the matrix
\[ \begin{pmatrix}
0 & -I_n \\
I_n & 0
\end{pmatrix} \in M_{2n}(\mathbb{C}) \]

The compact group $Sp(n)$ acts transitively on $C_\lambda$ by conjugation. Indeed, the set of matrices $C_\lambda$ corresponds to an adjoint orbit of $Sp(n)$, and it can be identified with a coadjoint orbit of $Sp(n)$ via an Ad-invariant product defined on the Lie algebra $\mathfrak{sp}(n)$. Let $(\omega_\lambda, J)$ be the Kähler structure defined on $C_\lambda$ by identifying it with a coadjoint orbit of $Sp(n)$ via an Ad-invariant inner product.

The set of matrices $C_\lambda$ is isomorphic to a quotient of the form $Sp(n, \mathbb{C})/P$, where $P$ is a parabolic subgroup of $Sp(n, \mathbb{C})$. Also, there exists a sequence of positive integers $a_1 \leq a_2 \leq \cdots \leq a_k \leq n$ such that the homogeneous space $Sp(n, \mathbb{C})/P$ is isomorphic to the variety of isotropic flags
\[ \{ V^{a_1} \subset V^{a_2} \subset \cdots \subset V^{a_k} \subset \mathbb{C}^{2n} : \Omega|_{V^{a_i}} = 0, 1 \leq i \leq k \}, \]
(see e.g. Section 23.3 of Fulton-Harris [23]).
For an integer 0 < k ≤ n, let \( IG(k, 2n) \) denote the space of \( k \)-dimensional isotropic subspaces of \( \mathbb{C}^{2n} \), i.e.,
\[
IG(k, 2n) := \{ V^k \in G(k, 2n) : \Omega|_{V^k} = 0 \}.
\]
When \( k = n \), the isotropic Grassmannian \( IG(n, 2n) \) is the space of Lagrangian subspaces of \( \mathbb{C}^{2n} \). The isotropic Grassmannian manifold \( IG(k, 2n) \) has dimension equal to
\[
2k(n - k) + \frac{k(k + 1)}{2}
\]
and is isomorphic to the complex quotient \( Sp(2n, \mathbb{C})/P_{\alpha_k} \), where \( P_{\alpha_k} \subset Sp(n, \mathbb{C}) \) is the maximal parabolic subgroup associated with the simple roots \( \alpha_k \in S \).

When the components of \( \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n \) are of the form
\[
\lambda_1 = \cdots = \lambda_k > \lambda_{k+1} = \cdots = \lambda_n = 0,
\]
the set of matrices \( C_\lambda \) is isomorphic to the isotropic Grassmannian manifold \( IG(k, n) \).

Let \( A \) be the effective cyclic generator of the second homology group \( H_2(IG(k, 2n), \mathbb{Z}) \). The moduli space of (unparameterized) \( J \)-holomorphic curves of degree \( A \) in \( IG(k, 2n) \) is given by
\[
\mathcal{M}_{A,0}(IG(k, 2n), J) \cong \{(V^{k-1}, V^{k+1}) : V^{k-1} \subset V^{k+1} \subset (V^{k-1})^\perp \subset \mathbb{C}^{2n}\}
\]
We will call the elements of this moduli space holomorphic lines in \( IG(k, 2n) \). The holomorphic line associated with a pair \( (V^{k-1}, V^{k+1}) \in \mathcal{M}_{A,0}(IG(k, 2n), J) \) consists of all the isotropic subspaces \( V^k \in IG(k, 2n) \) with \( V^{k-1} \subset V^k \subset V^{k+1} \).

If a pair of isotropic subspaces \( V^k_1, V^k_2 \in IG(k, 2n) \) are collinear, i.e., there exists \( (V^{k-1}, V^{k+1}) \in \mathcal{M}_{A,0}(IG(k, 2n), J) \) such that
\[
V^{k-1} \subset V^k_1, \quad V^k_2 \subset V^{k+1},
\]
then \( V^{k-1} = V^k_1 \cap V^k_2 \) and \( V^{k+1} = V^k_1 + V^k_2 \) (see e.g. Cohen-Cooperstein [13, Section 3.4], Landsberg-Manivel [30, Remark 5.7], Strickland [32, Proposition 3] for more details).

5.1. Remark. The moduli space of (unparameterized) homolomorphic lines in the Grassmannian manifold \( G(k, 2n) \) of \( k \)-dimensional vector subspaces of \( \mathbb{C}^n \) is isomorphic to the two-step flag manifold
\[
Fl(k-1, k+1; 2n) := \{(V^{k-1}, V^{k+1}) : V^{k-1} \subset V^{k+1} \subset \mathbb{C}^{2n}\}
\]
We have a natural defined embedding of the isotropic Grassmannian \( IG(k, 2n) \) into the standard Grassmannian \( G(k, 2n) \). With respect to this embedding, a holomorphic line in the isotropic Grassmannian \( IG(k, 2n) \) is a holomorphic line in the standard Grassmannian \( G(k, 2n) \) that is totally contained in the isotropic Grassmannian \( IG(k, 2n) \). Also, a line in \( G(k, 2n) \) that contains two isotropic vector subspaces in \( IG(k, 2n) \) is totally contained in \( IG(k, 2n) \).

The dimension of \( \mathcal{M}_{A,0}(IG(k, 2n)) \) can be computed by considering the fibration
\[
\pi : \mathcal{M}_{A,0}(IG(k, 2n)) \to IG(k-1, 2n)
\]
\[
(V^{k-1}, V^{k+1}) \to V^{k-1}.
\]
This fibration has fiber isomorphic to $G(2, 2n - 2k + 2)$, so that
\[
\dim_{\mathbb{C}} \mathcal{M}_{A, 0}(IG(k, 2n)) = \dim_{\mathbb{C}} IG(k - 1, 2n) + \dim_{\mathbb{C}} G(2, 2n - 2k + 2)
\]
\[
= \frac{k(k - 1)}{2} + 2(k - 1)(n - k + 1) + 2(2n - 2k)
\]
\[
= \frac{-3k^2}{2} + 2kn - \frac{k}{2} + 2n - 2
\]
\[
= \dim_{\mathbb{C}} IG(k, 2n) - k + 2n - 2
\]

Just as before, we want to find a cycle $X \subset IG(k, 2n)$ such that for a generic isotropic subspace $V^k \subset IG(k, 2n)$, the Gromov-Witten invariant $GW_{A, 2}(PD[p], PD[X])$ would be different from zero; if so, we will have that
\[
Gwidth(IG(k, 2n), \omega) \leq \omega(A).
\]

We claim that the Grassmannian submanifold
\[
X = \{ \Sigma^k \in IG(k, 2n) : \mathbb{C} \subset \Sigma^k \subset \mathbb{C}^\Omega \cong \mathbb{C}^{2n-1} \} \subset IG(k, 2n)
\]
satisfies this condition. Note that $X$ can be identified with the isotropic Grassmannian $IG(k - 1, 2(n - 1))$.

5.2. **Theorem.** Let $X = \{ \Sigma^k \in IG(k, 2n) : \mathbb{C} \subset \Sigma^k \subset \mathbb{C}^\Omega \cong \mathbb{C}^{2n-1} \} \subset IG(k, 2n)$, and $p \in IG(k, 2n)$. Then
\[
GW_{A, 2}(PD[p], PD[X]) \neq 0.
\]

**Proof.** We check first that $X$ satisfies the dimensional constraint
\[
\dim_{\mathbb{C}} X = 2 \dim_{\mathbb{C}} IG(k, 2n) - \dim_{\mathbb{C}} \mathcal{M}_{A, 0}(IG(k, 2n), J) - 2:
\]
\[
\dim_{\mathbb{C}} X = \dim_{\mathbb{C}} IG(k - 1, 2(n - 1))
\]
\[
= 2(k - 1)(n - k) + \frac{(k - 1)k}{2}
\]
\[
= 2k(n - k) + \frac{k(k + 1)}{2} + k - 2n
\]
\[
= \dim_{\mathbb{C}} IG(k, 2n) + k - 2n
\]
\[
= 2 \dim_{\mathbb{C}} IG(k, 2n) - \dim_{\mathbb{C}} \mathcal{M}_{A, 0}(IG(k, 2n), J) - 2
\]

Now we prove that the Gromov-Witten invariant $GW_{A, 2}(PD[p], PD[X])$ is non-zero by considering the embedding of the isotropic Grassmannian $IG(k, 2n)$ in the standard Grassmannian $G(k, 2n)$:

Let $\Sigma^k \in IG(k, 2n) \hookrightarrow G(k, 2n)$. There is a unique holomorphic line in $G(k, 2n)$ passing through $\Sigma^k \in G(k, 2n)$ and
\[
Y = \{ V^k \in G(k, 2n) : \mathbb{C} \subset V^k \subset \mathbb{C}^\Omega \} \subset G(k, 2n),
\]
this line intersects $Y$ at $\Gamma^k = \mathbb{C} \oplus (\Sigma^k \cap \mathbb{C}^\Omega)$ in $G(k, 2n)$ (see e.g. proof of Lemma 4.2 in [11]).

Note that $\Gamma^k$ is an isotropic subspace of $\mathbb{C}^n$ because $\Sigma^k$ is isotropic. This implies that the line in
G(k, 2n) passing through Σ^k and Y ⊂ G(k, 2n) is totally contained in IG(k, 2n) and intersects X = Y ∩ IG(k, 2n). Thus,

\[ GW_{A, 2}(PD[p], PD[IG(k - 1, 2n - 2)]) \neq 0 \]

Now we state our upper bound for the Gromov width of Isotropic Grassmannian manifolds:

5.3. Theorem. For \( \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n \) such that

\[ \lambda_1 = \cdots = \lambda_k > \lambda_{k+1} = \cdots = \lambda_n = 0, \]

let

\[ \mathcal{C}_\lambda = \{ A \in M_{2n}(\mathbb{C}) : A = QA^TQ, A^* = -A, \text{ spectrum } A = i(\lambda, -\lambda) \}, \]

where \( Q \) is the matrix

\[
\begin{pmatrix}
0 & -I_n \\
I_n & 0
\end{pmatrix} \in M_{2n}(\mathbb{C})
\]

Let \( \omega_\lambda \) be the symplectic form defined on \( \mathcal{C}_\lambda \) by identifying it with a coadjoint orbit of \( Sp(n) \). Then,

\[ \text{Gwidth}(\mathcal{C}_\lambda, \omega_\lambda) \leq \lambda_1 \]

Proof. We have that \( \mathcal{C}_\lambda \cong IG(k, 2n) \cong Sp(2n, \mathbb{C}) / P_{\alpha_k} \) where \( P_{\alpha_k} \subset Sp(2n, \mathbb{C}) \) is the maximal parabolic subgroup associated with the simple root \( \alpha_k \). Let \( A \in H_2(A, \mathbb{Z}) \) be the effective cyclic generator. The symplectic area of \( A \) is equal to

\[ \omega_\lambda(A) = (\lambda_1, \check{\alpha}_k) = \lambda_1 \]

The result now follows from Theorem 5.2 and Remark 2.6.

6. Upper bounds for the Gromov width of Orthogonal Grassmannians

In this section we are going to write the statement of Theorem 4.9 for Grassmannian manifolds of type B and D, better known as Orthogonal Grassmannians.

For a positive integer \( m \), let \( SO(m) \) be the group of special orthogonal transformations on \( \mathbb{R}^m \) which preserves the standard symmetric bilinear form defined on \( \mathbb{R}^m \). We will write \( m \) as \( 2n \) if \( m \) is even, and as \( 2n + 1 \) if \( m \) is odd (here \( n \) is a non-negative integer number). For \( \theta = (\theta_1, \cdots, \theta_n), \lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n \), we define the matrices \( I_n(\lambda), R_n(\theta) \in M_{2n}(\mathbb{R}) \) as

\[
I_n(\lambda) := \begin{pmatrix}
0 & \lambda_1 \\
-\lambda_1 & 0 \\
& \ddots & \ddots \\
0 & \lambda_n \\
-\lambda_n & 0
\end{pmatrix}, \quad R_n(\theta) := \begin{pmatrix}
\cos \theta_1 & \sin \theta_1 & & \\
-\sin \theta_1 & \cos \theta_1 & & \\
& & \ddots & \\
& & & \cos \theta_n & \sin \theta_n \\
& & & -\sin \theta_n & \cos \theta_n
\end{pmatrix}
\]

We make the following choice of maximal compact torus for \( SO(m) \) depending on \( m \) being either of the form \( 2n + 1 \) or \( 2n \):
Let \( R_n(\theta) \in M_{2n}(\mathbb{R}) : \theta \in \mathbb{R}^n \), \( T_{SO(2n+1)} = \left\{ \begin{pmatrix} R_n(\theta) & 0 \\ 0 & 0 \end{pmatrix} : \theta \in \mathbb{R}^n \right\} \) with corresponding Lie algebras

\[
\mathfrak{t}_{SO(2n)} := \left\{ I_n(\lambda) \in M_{2n}(\mathbb{R}) : \lambda \in \mathbb{R}^n \right\}, \quad \mathfrak{t}_{SO(2n)} := \left\{ \begin{pmatrix} I_n(\lambda) & 0 \\ 0 & 0 \end{pmatrix} : \lambda \in \mathbb{R}^n \right\}
\]

These Lie algebras are identified with its corresponding duals \( \mathfrak{t}^*_{SO(2n+1)}, \mathfrak{t}^*_{SO(2n)} \) via an Ad-invariant inner product. Let \( \{e_i\}_{i=1}^n \) be the dual basis in \( \mathfrak{t}^*_{SO(m)} \) associated to the standard basis of \( t \cong \mathbb{R}^n \).

The root system for the group \( SO(2n+1) \), with respect to the chosen maximal torus, is the set \( \{\pm e_i, \pm(e_j \pm e_k) : j \neq k\}_{1 \leq i,j \leq n} \subset \mathfrak{t}^*_{SO(2n+1)} \cong \mathbb{R}^n \) with a choice of simple roots given by \( S = \{\alpha_1 = e_1 - e_2, \ldots, \alpha_n = e_n, \alpha_n = e_n\} \), and Dynkin diagram

\[
\begin{array}{cccccccc}
e_1 - e_2 & e_2 - e_3 & e_3 - e_{n-1} & e_n \end{array}
\]

The root system for the group \( SO(2n) \) is the set \( \{\pm(e_j \pm e_k) : j \neq k\}_{1 \leq i,j \leq n} \) with simple roots given by \( S = \{\alpha_1 = e_1 - e_2, \ldots, \alpha_n = e_n, \alpha_n = e_n\} \), in the Lie algebra \( \mathfrak{t}^*_{SO(2n)} \cong \mathbb{R}^n \), and with Dynkin diagram

\[
\begin{array}{cccccccc}
e_1 - e_2 & e_2 - e_3 & e_3 - e_{n-1} & e_{n-1} + e_n \end{array}
\]

Every real skew-symmetric matrix in \( \mathfrak{so}(m) \) can be diagonalized by orthogonal transformations to a matrix in the Lie algebra \( \mathfrak{t}_{SO(m)} \). For \( \lambda \in \mathbb{R}^n \), we denote by \( S_\lambda \) the set of real skew-symmetric matrices of size \( m \times m \) that can be diagonalized by orthogonal transformations to the matrix \( I_n(\lambda) \in M_{2n}(\mathbb{R}) \), if \( m = 2n \); or to the matrix \( \begin{pmatrix} I_n(\lambda) & 0 \\ 0 & 0 \end{pmatrix} \in M_{2n+1}(\mathbb{R}) \), if \( m = 2n + 1 \).

If \( m = 2n + 1 \) is odd, the matrices in \( S_\lambda \) can be diagonalized with special orthogonal transformations to the corresponding matrix in \( \mathfrak{t}_{SO(m)} \). The same is true if \( m = 2n \) is even and at least one component of \( \lambda \in \mathbb{R}^n \) is zero. If \( m = 2n \) is even, and all the components of \( \lambda \) are different from zero, the set of matrices \( S_\lambda \) has two \( SO(m) \)-orbits which consist of the skew symmetric matrices in \( S_\lambda \) with positive and negative Pfaffian. We denote these two orbits by \( S_\lambda^\pm \).

Let \( O(m, \mathbb{C}) \) be the group of complex orthogonal matrices which preserves the standard nondegenerate symmetric bilinear form defined on \( \mathbb{C}^m \), and let \( SO(m, \mathbb{C}) \) be the set of complex orthogonal matrices in \( O(m, \mathbb{C}) \) with determinant one.
Let $k \leq m/2$ be a positive integer. We denote by $OG(k, m)$ the **Orthogonal Grassmannian manifold** of $k$-dimensional isotropic subspaces in $\mathbb{C}^m$ with respect to the standard nondegenerate symmetric bilinear form defined on $\mathbb{C}^m$.

Witt’s theorem states that any isometry between two subspaces of $\mathbb{C}^m$ can be extended to an isometry of the whole space (see e.g. [44]). As a consequence, given any two isotropic subspaces in $\mathbb{C}^m$ of the same dimension, they can be mapped to the other by a complex orthogonal transformation of $\mathbb{C}^m$. Thus the complex orthogonal group $O(m, \mathbb{C})$ acts transitively on the Orthogonal Grassmannian manifold $OG(k, m)$.

When $k \neq m/2$, the group $SO(m, \mathbb{C})$ acts transitively on $OG(k, m)$ and the orthogonal Grassmannian $OG(k, m)$ is isomorphic to the quotient $SO(m, \mathbb{C})/P_{\alpha_k}$.

When $k = m/2 = n$, the orthogonal Grassmannian $OG(n, 2n)$ is the union of two $SO(2n, \mathbb{C})$-orbits. These two $SO(2n, \mathbb{C})$-orbits are isomorphic to $SO(2n, \mathbb{C})/P_{\alpha_{n-1}}$ and $SO(2n, \mathbb{C})/P_{\alpha_n}$.

If the components of $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ are of the form

$$\lambda_1 = \cdots = \lambda_k > \lambda_{k+1} = \cdots = \lambda_n = 0,$$

the set of skew-symmetric matrices $S_\lambda$ is isomorphic to the orthogonal Grassmannians $OG(k, 2n)$ and $OG(k, 2n+1)$. When the components of $\lambda$ are of the form

$$\lambda_1 = \cdots = \lambda_n > 0,$$

the two $SO(2n, \mathbb{C})$-orbits of the orthogonal Grassmannian $OG(n, 2n)$ are isomorphic with the two $SO(2n)$-orbits of $S_\lambda$, $S_\lambda^+$, and $S_\lambda^-$. Both of them are isomorphic to the orthogonal Grassmannian $OG(n, 2n + 1)$.

**6.1. Theorem.** Let $m$ be a positive integer number that we will denote by $2n$ if it is even, and by $2n + 1$ if it is odd. Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ such that its components are of the form

$$\lambda_1 = \cdots = \lambda_k > \lambda_{k+1} = \cdots = \lambda_n = 0,$$

where $k$ is an integer such that $0 < k \leq m/2$. Let $S_\lambda$ be the set of $m \times m$ skew-symmetric matrices that can be diagonalized by orthogonal transformations to the matrix

$$\begin{pmatrix}
0 & \lambda_k I_{k \times k} \\
-\lambda_k I_{k \times k} & 0 \\
0 & 0_{(m-2k) \times (m-2k)}
\end{pmatrix} \in M_m(\mathbb{R})$$

Let $\omega_\lambda$ be the symplectic form defined on $S_\lambda$ by identifying it with a coadjoint orbit of the orthogonal group of matrices via an Ad-invariant inner product. Then,

$$\text{Gwidth}(S_\lambda, \omega_\lambda) \leq \begin{cases}
\lambda_k & \text{if } k < (m-1)/2 \\
2\lambda_k & \text{if } k = (m-1)/2
\end{cases}$$

If $k = m/2$, the set of skew symmetric matrices $S_\lambda$ has two connected $S_\lambda^+$ and $S_\lambda^-$, in this case

$$\text{Gwidth}(S_\lambda^+, \omega_\lambda) = \text{Gwidth}(S_\lambda^-, \omega_\lambda) \leq 2\lambda_k$$

**Proof.** When $k \neq m/2$, the set of skew symmetric matrices $S_\lambda$ is isomorphic with $SO(m, \mathbb{C})/P_{\alpha_k}$, where $\alpha_k$ is the simple root corresponding to the $k$-node in the Dynkin diagram of $SO(m)$.
If $k < (m-1)/2$, the simple root $\alpha_k$ is long and by Theorem 4.9
\[ \text{Gwidth}(S_\lambda, \omega_\lambda) \leq \omega_\lambda(A) = \langle \lambda, \alpha_k \rangle = \lambda_k \]
When $k = (m-1)/2$, the set of skew symmetric matrices $S_\lambda$ is isomorphic with $SO(2n, \mathbb{C})/P_{\alpha_n}$.
Since the root $\alpha_n$ is long, we have that
\[ \text{Gwidth}(S_\lambda, \omega_\lambda) \leq \omega_\lambda(A) = \langle \lambda, \alpha_n \rangle = 2\lambda_n = 2\lambda_k \]
Finally, when $k = m/2$, the two connected components $S^\pm_\lambda$ of $S_\lambda$ are isomorphic with $SO(2n, \mathbb{C})/P_{\alpha_n} \sim SO(2n, \mathbb{C})/P_{\alpha_n-1}$, and as consequence
\[ \text{Gwidth}(S^+_\lambda, \omega_\lambda) = \text{Gwidth}(S^-_\lambda, \omega_\lambda) \leq \langle \lambda, \alpha_n \rangle = 2\lambda_n = 2\lambda_k \]
\[ \square \]

7. Upper bound for the Gromov width of coadjoint orbits of the exceptional group $G_2$

Let $G = G_2$ and let $T \subset G$ be the maximal torus whose Lie algebra $t$ is identified with $\mathbb{R}^2$ and such that the set
\[ S = \{ \alpha_1 = (-\frac{3}{2}, \frac{\sqrt{3}}{2}), \alpha_2 = (1, 0) \} \subset t^* \cong \mathbb{R}^2 \]
defines a set of simple root systems for $G$ with Dynkin diagram

\[ \begin{array}{c}
\alpha_1 \\
\longrightarrow \\
\alpha_2
\end{array} \]

7.1. Theorem. For $\lambda = (\lambda_1, \lambda_2) \in t^*$, let $O_\lambda$ be the $G_2$-coadjoint orbit that passes through $\lambda$ and let $\omega_\lambda$ be the Kostant-Kirillov-Souriau form defined on it. Then
\[ \text{Gwidth}(O_\lambda, \omega_\lambda) \leq \begin{cases} 
\frac{\sqrt{3}\lambda_2}{3} & \text{if } \lambda_1 = 0 \\
2\lambda_1 & \text{if } \lambda_2 = \sqrt{3}\lambda_1 
\end{cases} \]

Proof. If $\lambda_1 = 0$, then $O_\lambda \cong G_2/P_{\alpha_1}$, where $P_{\alpha_1} \subset G_2$ is the maximal parabolic subgroup associated with the simple root $\alpha_1 \in S$. Since the root $\alpha_1$ is long, we have by Theorem 4.9 that
\[ \text{Gwidth}(O_\lambda, \omega_\lambda) \leq \langle \lambda, \alpha_1 \rangle = \frac{\sqrt{3}\lambda_2}{3} \]

On the other hand, if $\sqrt{3}\lambda_2 = 3\lambda_1$, then $O_\lambda \cong G_2/P_{\alpha_2}$, where $P_{\alpha_2} \subset G_2$ is the maximal parabolic subgroup associated with the simple root $\alpha_2 \in S$. The homogeneous space $G_2/P_{\alpha_2}$ can be considered as a homogenous space of type $SO(7, \mathbb{C})$: Let $w_1 \in t^*$ be the fundamental weight associated with $\alpha_1$. Let $L(w_1) = G_2 \times \mathbb{C}(w_1)$ be the line bundle defined over $G_2/P_{\alpha_2}$ associated with the fundamental weight $w_1$. The irreducible representation $H^0(G_2/P_{\alpha_2}, L(w_1))$ has dimension 7 (this computation can be made by using for instance the Weyl dimensional formula). Thus, $G_2/P_{\alpha_2}$ is embedded as a non-degenerate quadric in the 6 dimensional projective space $\mathbb{P}(H^0(G_2/P_{\alpha_2}, L(w_1))) \cong \mathbb{C}P^6$. 

A quadric in $\mathbb{CP}^6$ is a complete homogeneous space for the special orthogonal group $SO(7, \mathbb{C})$. Now, if we consider our quadric as a homogeneous space of $SO(7, \mathbb{C})$, then by Theorem 4.9,

$$\text{Gwidth}(\mathcal{O}_\lambda, \omega_\lambda) \leq \langle \lambda, \tilde{\alpha}_2 \rangle = 2\lambda_1$$

□

8. Upper bound for the Gromov width of coadjoint orbits of compact Lie groups

The problem of finding upper bounds for the Gromov width of coadjoint orbits of compact Lie groups has already been addressed by Masrour Zoghi in his Ph.D thesis [55] where he has considered the problem of determining the Gromov width of regular coadjoint orbits of compact Lie groups.

We now recall Zoghi’s upper bound for the Gromov width of regular coadjoint orbits of compact Lie groups.

8.1. Theorem (Zoghi [55]). Let $G$ be a compact simple Lie group. Let $\lambda \in \mathfrak{g}^*$ and let us assume that $\mathcal{O}_\lambda \subset \mathfrak{g}^*$ is a regular coadjoint orbit of $G$. Let $B \subset G_C$ be a Borel subgroup such that $\mathcal{O}_\lambda \cong G_C/B$ and $S$ be a system of simple roots compatible with $B$. If there exists $\alpha \in S$ such that for any $\beta \in S$, $\langle \lambda, \tilde{\beta} \rangle$ is an integer multiple of $\langle \lambda, \tilde{\alpha} \rangle$; then

$$\text{Gwidth}(\mathcal{O}_\lambda, \omega_\lambda) \leq \langle \lambda, \tilde{\alpha} \rangle,$$

where $\omega_\lambda$ denotes the Kostant-Kirillov-Souriau form defined on $\mathcal{O}_\lambda$.

The Gromov width of arbitrary coadjoint orbits of compact Lie group would be estimated by computing Gromov-Witten invariants on holomorphic fibrations whose fibers are isomorphic to Grassmannian manifolds. The following result found in Li-Ruan [38, Proposition 2.10] is the key point of this argument:

8.2. Theorem. Let $(M, \omega)$ be a symplectic manifold and $J$ be a regular $\omega$-compatible almost complex structure on $M$, and suppose that $M$ admits a $J$-holomorphic fibration $\pi : M \to Y$. Let $i : \pi^{-1}(y) \to M$ be the inclusion map for a generic fiber over $y \in Y$. Then, for $A \in H_2(\pi^{-1}(y), \mathbb{Z})$ and $\alpha_2, \ldots, \alpha_k \in H^*(M, \mathbb{R})$

$$\text{GW}_{A,k}^{\pi^{-1}(y)}(\text{PD}[pt], i^*\alpha_2, \ldots, i^*\alpha_k) = \text{GW}_{A,k}^M(\text{PD}[pt], \alpha_2, \ldots, \alpha_k).$$

Now, we are ready to prove the Main Theorem:

8.3. Theorem. Let $G$ be a compact connected simple Lie group with Lie algebra $\mathfrak{g}$. Let $T \subset G$ be a maximal torus and let $\tilde{T} \subset \mathfrak{t}$ be the corresponding system of co-roots. We identify the dual Lie algebra $\mathfrak{t}^*$ with the fixed points of the coadjoint action of $T$ on $\mathfrak{g}^*$. Let $\lambda \in \mathfrak{t}^* \subset \mathfrak{g}^*$, $\mathcal{O}_{\lambda}$ be the coadjoint orbit passing through $\lambda$ and $\omega_\lambda$ be the Kostant-Kirillov-Souriau form defined on $\mathcal{O}_{\lambda}$, then

$$\text{Gwidth}(\mathcal{O}_\lambda, \omega_\lambda) \leq \min_{\tilde{\alpha} \in \tilde{T}} |\langle \lambda, \tilde{\alpha} \rangle|$$
Proof. For the coadjoint orbit \( \mathcal{O}_\lambda \) there exists a parabolic subgroup \( P \subset G_C \) such that \( \mathcal{O}_\lambda \cong G_C/P \). For each \( \alpha \in S\setminus S_P \), we have a parabolic subgroup \( Q \subset P \) with \( S_Q = S_P \sqcup \{ \alpha \} \), and a holomorphic fibration

\[
\pi_\alpha : G_C/P \to G_C/Q.
\]

The fiber \( Q/P \) can be identified with the quotient of a simple Lie group and a maximal parabolic subgroup. Let \( A \in H_2(Q/P, \mathbb{Z}) \) be the effective cyclic generator of the second homology group of the Grassmannian \( Q/P \). The Schubert variety \( X_P(s_\alpha) = Bs_\alpha P/P \) is totally contained in \( Q/P \) and its fundamental class is the same as the second homology class \( A \in H_2(Q/P, \mathbb{Z}) \). The symplectic area of \( A \) with respect to \( \omega_\lambda \) is equal to \( \langle \lambda, \tilde{\alpha} \rangle \).

By the analysis done in the previous sections, there exists a Schubert variety \( X \subset Q/P \) such that

\[
GW^{Q/P}_{A,2}(\text{PD}[\text{pt}], \text{PD}[X]) \neq 0.
\]

The inclusion map \( \iota : Q/P \hookrightarrow G_C/P \) is cohomologically surjective, and as a consequence there exists \( \beta \in H^*(G_C/P, \mathbb{Z}) \) such that \( \iota^* \beta = \text{PD}[X] \). Thus, by Theorem 8.2

\[
GW^{Q/P}_{A,2}(\text{PD}[\text{pt}], \text{PD}[X]) = GW^{G_C/P}_{A,2}(\text{PD}[\text{pt}], \beta) \neq 0,
\]

so that we would have by Remark 2.4 that

\[
\text{Gwidth}(\mathcal{O}_\lambda, \omega_\lambda) \leq \omega_\lambda(A) = \langle \lambda, \tilde{\alpha} \rangle.
\]

The above inequality holds for any \( \alpha \in S\setminus S_P \), and as consequence for any \( \alpha \in R^+\setminus R_P^+ \), and we are done.

8.4. Remark. In the previous proof, if we assume that for any \( \tilde{\beta} \in \tilde{R} \) the pair \( \langle \lambda, \tilde{\beta} \rangle \) is an integer multiple of \( \langle \lambda, \tilde{\alpha} \rangle \), the effective cyclic generator \( A \) of the second homology group of the fiber \( H_2(Q/P, \mathbb{Z}) \) would be \( \omega_\lambda \)-indecomposable in \( H_2(G_C/P, \mathbb{Z}) \). This simple assumption will imply that the evaluation map

\[
ev_J : \mathcal{M}_{A,2}(\mathcal{O}_\lambda, J) \to \mathcal{O}_\lambda^2
\]

is a pseudocycle (see [13, Lemma 7.1.8]) and as a consequence the Gromov-Witten invariant

\[
GW^{G_C/P}_{A,2}(\text{PD}[\text{pt}], \beta)
\]

can be computed using \( \ev_J : \mathcal{M}_{A,2}(\mathcal{O}_\lambda, J) \to \mathcal{O}_\lambda^2 \) without having to define virtual fundamental classes on it.

On the other hand, Cieliebak-Mohnke have provided an alternative definition of the Gromov-Witten invariant in [12] with the assumption that \( [\omega_\lambda] \in H^2(\mathcal{O}_\lambda, \mathbb{Z}) \). This definition makes use of Donaldson’s divisors as auxiliary data to define the Gromov-Witten invariant. This definition of the Gromov-Witten invariant is one of the most widely accepted by the symplectic community.

8.5. Corollary. Let \( \lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n \), and

\[
\mathcal{H}_\lambda = \{ A \in M_n(\mathbb{C}) : A^* = A, \text{ spectrum } A = \lambda \}.
\]

Let \( \omega_\lambda \) be the symplectic form defined on \( \mathcal{H}_\lambda \) obtained by identifying \( \mathcal{H}_\lambda \) with a coadjoint orbit of \( U(n) \) via an \( \text{Ad} \)-invariant inner product. Then

\[
\text{Gwidth}(\mathcal{H}_\lambda, \omega_\lambda) \leq \min_{\lambda_i \neq \lambda_j} |\lambda_i - \lambda_j|.
\]
8.6. Corollary. Let $m$ be a positive integer number that we will denote by $2n$ if it is even, and by $2n + 1$ if it is odd. Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}_{\geq 0}^n$ and let $\{\lambda_i, \ldots, \lambda_i\}$ be the non-zero components of $\lambda$ counted with multiplicity. Let $S_\lambda$ be the set of real skew-symmetric matrix of size $m \times m$ that can be diagonalized by orthogonal transformations to a matrix of the form
\[
\begin{pmatrix}
0 & \lambda_i \\
-\lambda_i & 0 \\
& & \ddots \\
& & & 0 & \lambda_i \\
& & & -\lambda_i & 0 \\
& & & & & 0
\end{pmatrix}
\]
Let $\omega_\lambda$ be the Kirillov-Kostant-Souriau form defined on $S_\lambda$ by identifying it with a coadjoint orbit of the special orthogonal group $SO(n, \mathbb{R})$. Then,
\[
\text{Gwidth}(S_\lambda, \omega_\lambda) \leq \begin{cases}
\min_{\lambda_k \neq 0} \{ |2\lambda_k|, |\lambda_i \pm \lambda_j| \} & \text{if } m = 2n + 1 \\
\min_{\lambda_i \neq \pm \lambda_j} |\lambda_i \pm \lambda_j| & \text{if } m = 2n
\end{cases}
\]

8.7. Corollary. Let $\lambda \in \mathbb{R}^n$ and
\[
C_\lambda = \{ A \in M_{2n}(\mathbb{C}) : A = QA^TQ, A^* = -A, \text{ spectrum } A = i(\lambda, -\lambda) \}
\]
where $Q$ is the matrix
\[
\begin{pmatrix}
0 & -I_n \\
I_n & 0
\end{pmatrix}
\]
Let $\omega_\lambda$ be the symplectic form defined on $C_\lambda$ by identifying it with a coadjoint orbit of $Sp(n)$ via an Ad-invariant inner product. Then
\[
\text{Gwidth}(C_\lambda, \omega_\lambda) \leq \min_{\lambda_k \neq 0} \{ |\lambda_k|, |\lambda_i \pm \lambda_j| \}
\]

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