RANDOM DYNAMICS OF FRACTIONAL NONCLASSICAL DIFFUSION EQUATIONS DRIVEN BY COLORED NOISE

RENhai Wang and YANGrong Li*

School of Mathematics and statistics, Southwest University
Chongqing 400715, China

BIXIANG Wang

Department of Mathematics, New Mexico Institute of Mining and Technology
Socorro, NM 87801, USA

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ABSTRACT. The random dynamics in $H^s(\mathbb{R}^n)$ with $s \in (0, 1)$ is investigated for the fractional nonclassical diffusion equations driven by colored noise. Both existence and uniqueness of pullback random attractors are established for the equations with a wide class of nonlinear diffusion terms. In the case of additive noise, the upper semi-continuity of these attractors is proved as the correlation time of the colored noise approaches zero. The methods of uniform tail-estimate and spectral decomposition are employed to obtain the pullback asymptotic compactness of the solutions in order to overcome the non-compactness of the Sobolev embedding on an unbounded domain.

1. Introduction. This paper is concerned with the random dynamics of the following fractional nonclassical diffusion equation driven by colored noise on $\mathbb{R}^n$:

\[
\begin{aligned}
&\begin{cases}
    u_t + (-\Delta)^s u_t + (-\Delta)^s u + \lambda u = f(t, x, u) + g(t, x) + h(t, x, u) \zeta_\delta(\theta_t \omega), \\
    u(\tau, x) = u_\tau(x), \quad x \in \mathbb{R}^n, \quad t > \tau, \quad \tau \in \mathbb{R},
\end{cases}
\end{aligned}
\]

where $\lambda > 0$, $s \in (0, 1)$, $g \in \mathcal{L}^2_{\text{loc}}(\mathbb{R}, \mathcal{L}^2(\mathbb{R}^n))$ and $f, h : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ are smooth functions satisfying certain growth conditions. The term $\zeta_\delta(\theta_t \omega)$ is the so-called colored noise with correlation time $\delta > 0$.

To describe the colored noise, we introduce a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\}$ equipped with the compact-open topology, $\mathcal{F} = \mathcal{B}(\Omega)$ is the Borel sigma-algebra of $\Omega$, $\mathbb{P}$ is the Wiener measure. We define the measure-preserving transformation group by $\theta_t \omega(\cdot) = \omega(\cdot + t) - \omega(t)$ for all $(\omega, t) \in \Omega \times \mathbb{R}$. Let $W$ be a two-sided real-valued Wiener process on $(\Omega, \mathcal{F}, \mathbb{P})$, we define

\[
\zeta_\delta : \Omega \to \mathbb{R} \quad \text{by} \quad \zeta_\delta(\omega) = \frac{1}{\delta} \int_{-\infty}^0 e^{s/\delta} dW(s) \quad \text{for each } \delta > 0.
\]

Then the process $\zeta_\delta(\theta_t \omega)$ is called an Ornstein-Uhlenbeck process (or colored noise) which is the unique stationary solution of the one-dimensional stochastic differential equation $d\zeta_\delta + \frac{1}{\delta} \zeta_\delta dt = \frac{1}{\delta} dW$. The colored noise was first introduced in [46] and...
then widely used in many applications to study the dynamics of physical systems, see e.g., \cite{6,22} and the references therein.

The nonclassical diffusion equation arises in physics and fluid mechanics, see, e.g., \cite{2,3,32}. If \( s = 1 \), then the fractional operator \((-\Delta)^s\) becomes the standard Laplace operator. In this special case, the attractor of the nonclassical diffusion equation has been studied in \cite{4,5,52,54,56,61} for the deterministic systems, and in \cite{7,59,60} for the stochastic systems driven by linear white noise. However, even for \( s = 1 \), the attractors of the nonclassical diffusion equations driven by colored noise have not been discussed in the literature. The purpose of the present paper is to investigate this problem and prove the existence of attractors of (1) driven by colored noise with \( s \in (0,1) \).

Fractional partial differential equations can be derived from a variety of applications in physics, biology, chemistry, finance and other fields of science, see e.g., \cite{11,29,43,44,45} and the references therein. If the term \((-\Delta)^s u_t\) is absent, then equation (1) reduces to a fractional parabolic equation. In this case, the existence of random attractors of the fractional equation driven by linear white noise has been studied in \cite{27,39,40,41,50}. But, for the fractional equation like (1) involving \((-\Delta)^s u_t\), there is no result available on the existence of such attractors. We will deal with this problem in the present article.

We remark that for stochastic PDEs with standard Laplacian, random attractors have been investigated in \cite{8,12,13,14,15,18,20,21,23,31,35,36,47,48,53,57,58} for the autonomous case, and in \cite{1,9,10,24,25,19,30,34,37,49,62,63} for the non-autonomous case. The reader is referred to \cite{26,28,38,55} for attractors of random systems with standard Laplacian driven by colored noise or approximations of white noise.

In this paper, we will first prove the existence of attractors of the random equation (1) in \( H^s(\mathbb{R}^n) \) with \( s \in (0,1) \) when the nonlinear diffusion term \( h \) satisfies some growth conditions. We then examine the limiting behavior of these attractors as the correlation time \( \delta \) of the colored noise tends to zero. More precisely, we will also consider the following fractional nonclassical diffusion equation driven by additive white noise:

\[
\begin{aligned}
    du + d\left(\left(-\Delta\right)^s u + \left(-\Delta\right)^s u_t\right) dt + \lambda u dt &= f(t, x, u) dt + g(t, x) dt + h(x) dW, \\
    u(t, x) &= u_0(x), \quad x \in \mathbb{R}^n, \quad t > \tau, \quad \tau \in \mathbb{R},
\end{aligned}
\]  

(2)

where \( h \in L^2(\mathbb{R}^n) \). We prove the stochastic equation (2) has a random attractor \( \mathcal{A}_0 = \{ \mathcal{A}_0(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \) in \( H^s(\mathbb{R}^n) \) which, in a sense, can be considered as the limit of the attractors of the following random equation driven by colored noise as \( \delta \to 0 \):

\[
\begin{aligned}
    u_t + \left(-\Delta\right)^s u_t + \left(-\Delta\right)^s u + \lambda u &= f(t, x, u) + g(t, x) + h(x) \zeta_\delta(\theta t \omega), \\
    u(\tau, x) &= u_\tau(x), \quad x \in \mathbb{R}^n, \quad t > \tau, \quad \tau \in \mathbb{R},
\end{aligned}
\]  

(3)

Indeed, we will show the random attractors \( \mathcal{A}_\delta = \{ \mathcal{A}_\delta(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \) of (3) are upper semi-continuous at \( \delta = 0 \) and the limit is given by \( \mathcal{A}_0 = \{ \mathcal{A}_0(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \) in \( H^s(\mathbb{R}^n) \). Furthermore, we prove that such upper semi-continuity is uniform in a probability subspace \( \Omega_\varepsilon \subseteq \Omega \) with \( \mathbb{P}(\Omega_\varepsilon) > 1 - \varepsilon \) for any small \( \varepsilon > 0 \), see Theorem 4.8.

To show the existence of random attractors in \( H^s(\mathbb{R}^n) \), we need to establish the pullback asymptotic compactness of the solutions in \( H^s(\mathbb{R}^n) \). The main difficulties for proving such compactness for (1) come from the following aspects:
(i) The uniform estimates of the solutions for the fractional equations in \( H^s(\mathbb{R}^n) \) are more complicated than that for the equations with the standard Laplace operator.

(ii) The Sobolev embedding \( H^r(\mathbb{R}^n) \hookrightarrow H^s(\mathbb{R}^n) \) with \( r > s \) is not compact.

(iii) Due to the term \((-\Delta)^s u_t\), the fractional equation (1) has no smoothing effect on its solutions.

(iv) Because the Wiener process is nowhere differentiable with respect to time, we cannot use the method by differentiating the equation with respect to \( t \), as given in the deterministic case [5].

To overcome these difficulties, in the present paper, we will combine the methods of uniform tail-estimate and spectral decomposition to establish the desired pullback asymptotic compactness in \( H^s(\mathbb{R}^n) \), see Lemma 2.6 for more details.

This paper is organized as follows. In the next section, we show that problem (1) has a unique random attractor in \( H^s(\mathbb{R}^n) \) for a wide class of nonlinear functions \( f \) and \( h \). In Section 3, we prove that problem (2) driven by additive noise has also a unique random attractor in \( H^s(\mathbb{R}^n) \). In the last section, we establish the upper semi-continuity of these attractors in \( H^s(\mathbb{R}^n) \) as \( \delta \) approaches zero.

2. Attractors of fractional equations driven by colored noise. In this section, we study the existence and uniqueness of random attractors for the fractional nonclassical diffusion equations driven by colored noise on \( \mathbb{R}^n \):

\[
\begin{aligned}
\begin{cases}
  u_t + (-\Delta)^s u_t + (-\Delta)^s u + \lambda u = f(t, x, u) + g(t, x) + h(t, x) \zeta_t(\theta_t \omega), \\
  u(\tau, x) = u_\tau(x), \quad x \in \mathbb{R}^n, \quad t > \tau, \quad \tau \in \mathbb{R},
\end{cases}
\end{aligned}
\] (4)

where \( \lambda > 0, \ s \in (0, 1), \ g \in L^2_{loc}(\mathbb{R}, L^2(\mathbb{R}^n)) \). The nonlinear functions \( f, h : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \) satisfy the following conditions: for all \( (t, x, u) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \),

\[
\begin{aligned}
  f(t, x, u) &\leq -\alpha_1 |u|^p + \psi_1(t, x), \quad \psi_1 \in L^1_{loc}(\mathbb{R}, L^1(\mathbb{R}^n)), \\
  |f(t, x, u)| &\leq \alpha_2 |u|^{p-1} + \psi_2(t, x), \quad \psi_2 \in L^{\frac{p}{p-1}}_{loc}(\mathbb{R}, L^{\frac{p}{p-1}}(\mathbb{R}^n)), \\
  \frac{\partial}{\partial u} f(t, x, u) &\leq \psi_3(t, x), \quad \psi_3 \in L^\infty_{loc}(\mathbb{R}, L^\infty(\mathbb{R}^n)),
\end{aligned}
\] (5)

\[
\begin{aligned}
  |f(t, x, u)| &\leq \alpha_4 |t| |u|^q, \quad \psi_4 \in L^\infty_{loc}(\mathbb{R}, L^\infty(\mathbb{R}^n)), \\
  |h(t, x, u)| &\leq \beta_1(t, x) |u|^{q-1} + \beta_2(t, x), \quad \beta_3, \beta_4 \in L^\infty(\mathbb{R}, L^2(\mathbb{R}^n)),
\end{aligned}
\] (6)

where \( \alpha_1, \alpha_2, \alpha_3, \alpha_4 > 0 \) are constants, \( \beta_1 \in L^\infty(\mathbb{R}, L^{\frac{2p-2}{p-2}}(\mathbb{R}^n) \cap L^{\frac{p}{p-q}}(\mathbb{R}^n) \cap L^{\frac{2p}{n-2s}}(\mathbb{R}^n)) \) and

\[
2 \leq q < p < \infty \text{ if } n = 1 \text{ and } s \in [\frac{1}{2}, 1); \text{ otherwise, } 2 \leq q < p \leq \frac{2n-2s}{n-2s}.
\] (11)

Note that, by [28], there exists a \( \{\theta_t\}_{t \in \mathbb{R}} \)-invariant subset (still denoted by \( \Omega \)) of full measure such that

\[
\lim_{t \to \pm \infty} \left| \frac{\zeta_t(\theta_t \omega)}{t} \right| = 0 \text{ for each } \delta \in (0, 1] \text{ and } \omega \in \Omega.
\] (12)

Next we briefly review the concepts of fractional derivatives and fractional Sobolev spaces. Let \((-\Delta)^s\) with \( s \in (0, 1) \) be the non-local, fractional Laplace operator defined by
\((-\Delta)^s u(x) = C(n, s) \text{ P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy, \ x \in \mathbb{R}^n,\)

where P.V. means the principal value of the integral and \(C(n, s)\) is a constant given by
\[
C(n, s) = \frac{s^4 \Gamma \left( \frac{n+2s}{2} \right)}{\pi^{\frac{n}{2}} \Gamma \left( 1 - s \right)} > 0.
\]

Let \(H^s(\mathbb{R}^n)\) with \(s \in (0, 1)\) be the fractional Sobolev space given by
\[
H^s(\mathbb{R}^n) = \left\{ u \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dx dy < \infty \right\},
\]
which is a Hilbert space equipped with the inner product and the norm:
\[
(u, v)_{H^s(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}} dx dy,
\]
\[
\|u\|_{H^s(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dx dy \right)^{\frac{1}{2}}, \ \ u, v \in H^s(\mathbb{R}^n).
\]

For convenience, we use \((\cdot, \cdot)_{H^s(\mathbb{R}^n)}\) and \(\| \cdot \|_{H^s(\mathbb{R}^n)}\) to denote
\[
(u, v)_{H^s(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}} dx dy,
\]
\[
\|u\|_{H^s(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dx dy \right)^{\frac{1}{2}}, \ \ u, v \in H^s(\mathbb{R}^n).
\]

Then \(\|u\|_{H^s(\mathbb{R}^n)}^2 = \|u\|_{L^2(\mathbb{R}^n)}^2 + \|u\|_{H^s(\mathbb{R}^n)}^2\) for all \(u \in H^s(\mathbb{R}^n)\). By [17], one can verify
\[
\|u\|_{H^s(\mathbb{R}^n)}^2 = \|u\|_{L^2(\mathbb{R}^n)}^2 + \frac{2}{C(n, s)} \|(-\Delta)^{\frac{s}{2}} u\|_{L^2(\mathbb{R}^n)}^2, \ \ u \in H^s(\mathbb{R}^n),
\]
which means that \(\left( \| \cdot \|_{L^2(\mathbb{R}^n)}^2 + \|(-\Delta)^{\frac{s}{2}} \cdot \|_{L^2(\mathbb{R}^n)}^2 \right)^{\frac{1}{2}}\) is equivalent to \(\| \cdot \|_{H^s(\mathbb{R}^n)}\) and that
\[
H^s(\mathbb{R}^n) = \left\{ u \in L^2(\mathbb{R}^n) : (-\Delta)^{\frac{s}{2}} u \in L^2(\mathbb{R}^n) \right\}.
\]

It follows from [17, Prop. 6.5] and (11) that
\[
\|u\|_{L^r(\mathbb{R}^n)} \leq c_1 \|u\|_{H^s(\mathbb{R}^n)}, \ \forall 2 \leq r \leq 2p - 2 \text{ and } 0 < s < \min(1, \frac{n}{2}), \quad (13)
\]
where \(p\) is the number in (5). In addition, by [42, Prop. 5.9 and 5.14], we have
\[
\|u\|_{L^r(\mathbb{R}^n)} \leq c_2 \|u\|_{H^s(\mathbb{R}^n)}, \ \forall 2 \leq r < \infty, \ \frac{1}{2} \leq s \leq 1 \text{ and } n = 1,
\]
\[
\|u\|_{L^{\frac{2n}{n+2s}}(\mathbb{R}^n)} \leq c_3 \|u\|_{L^2(\mathbb{R}^n)} \|(-\Delta)^{\frac{s}{2}} u\|_{L^2(\mathbb{R}^n)}^2, \ \forall 0 < s < 1 \text{ and } n \geq 1. \quad (14)
\]

Note that (4) can be viewed as a deterministic random equation parameterized by \(\omega \in \Omega\), then one can show that for each \(\tau \in \mathbb{R}, \ \omega \in \Omega\) and \(u_{\tau} \in H^s(\mathbb{R}^n)\), problem (4) has a unique solution \(u(\cdot, \tau, \omega, u_{\tau}) \in C([\tau, \infty), H^s(\mathbb{R}^n))\) such that the solution continuously depends on \(u_{\tau} \in H^s(\mathbb{R}^n)\). In addition, one can show the \((\mathcal{F}, \mathcal{B}(H^s(\mathbb{R}^n)))-\text{measurability of the solution}. Hence the mapping \(\Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times H^s(\mathbb{R}^n) \to H^s(\mathbb{R}^n)\) given by
\[
\Phi(t, \tau, \omega, u_{\tau}) = u(t + \tau, \tau, \omega, u_{\tau}), \ (t, \tau, \omega, u_{\tau}) \in \mathbb{R}^+ \times \mathbb{R} \times \Omega \times H^s(\mathbb{R}^n) \quad (15)
\]
is a continuous cocycle in the sense of [49, Def. 1.1].

Next we study the existence and uniqueness of $\mathcal{D}$-pullback random attractors of $\Phi$ in $H^s(\mathbb{R}^n)$, where $\mathcal{D}$ is a family of bi-parametric sets in $H^s(\mathbb{R}^n)$ such that $\mathcal{D} = \{\mathcal{D}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \subset \mathcal{D}$ if and only if
\[
\lim_{t \to +\infty} e^{-\gamma t}||D(\tau - t, \theta, \omega)||^2_{H^s(\mathbb{R}^n)} = 0, \quad \forall \gamma > 0, \tau \in \mathbb{R}, \omega \in \Omega.
\] (16)

In this article, we let $\kappa = \min(\lambda, 1)$. We use $\| \cdot \|_r$ (resp. $\| \cdot \|$) to denote the norm of $L^r(\mathbb{R}^n)$ (resp. $L^2(\mathbb{R}^n)$) and make the following assumptions:
\[
\int_{-\infty}^{\tau} e^{\kappa r} (\|g(r)\|^2 + \|\psi_1(r)\|_1 + \|\psi_2(r)\|^2) dr < \infty, \quad \forall \tau \in \mathbb{R},
\] (17)
\[
\lim_{t \to +\infty} e^{-\gamma t} \int_{-\infty}^{\tau} e^{\kappa r} (\|g(r - t)\|^2 + \|\psi_1(r - t)\|_1 + \|\psi_2(r - t)\|^2) dr = 0, \quad \forall \gamma > 0.
\] (18)

2.1. Uniform estimates. This subsection is devoted to the uniform estimates of the solutions of problem (4), which are crucial for constructing $\mathcal{D}$-pullback random absorbing sets for the cocycle $\Phi$.

**Lemma 2.1.** Let (5)-(10) and (17) be satisfied. Then for every $\tau \in \mathbb{R}, \omega \in \Omega$ and $\mathcal{D} = \{\mathcal{D}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \subset \mathcal{D}$, there is a $T := T(\tau, \omega, D) > 0$ such that for all $t \geq T$, the solution of (4) with $u_{\tau - t} \in \mathcal{D}(\tau - t, \theta, \omega)$ satisfies
\[
\|u(\tau, \tau - t, \theta, \omega, u_{\tau - t})\|_{H^s(\mathbb{R}^n)}^2 \leq MR(\tau, \omega),
\] (19)
\[
\int_{\tau - t}^{\tau} e^{\kappa(r - \tau)} \|u(r, \tau - t, \theta, \omega, u_{\tau - t})\|_{H^s(\mathbb{R}^n)}^{2p - 2} dr \leq M \hat{R}^{p - 1}(\tau, \omega),
\] (20)
where $M > 0$ is a constant independent of $\tau$, $\omega$ and $\mathcal{D}$, $R(\tau, \omega)$ and $\hat{R}(\tau, \omega)$ are given by
\[
R(\tau, \omega) = \int_{-\infty}^{0} e^{\kappa r} (\|g(r + \tau)\|^2 + \|\psi_1(r + \tau)\|_1 + \|\psi_2(r + \tau)\|^2 + |\zeta(\theta, \omega)|^2 + |\zeta(\theta, \omega)|^2) dr,
\] (21)
\[
\hat{R}(\tau, \omega) = \int_{-\infty}^{0} e^{\kappa r} (\|g(r + \tau)\|^2 + \|\psi_1(r + \tau)\|_1 + \|\psi_2(r + \tau)\|^2 + |\zeta(\theta, \omega)|^2 + |\zeta(\theta, \omega)|^2) dr.
\] (22)

**Proof.** Taking the inner product of (4) with $u$ in $L^2(\mathbb{R}^n)$, we obtain
\[
\frac{1}{2} \frac{d}{dt} (\|u\|^2 + \|(-\Delta)^{\frac{s}{2}} u\|^2) + \lambda \|u\|^2 + \|(-\Delta)^{\frac{s}{2}} u\|^2
= (f(t, u), u) + \zeta(\theta, \omega)(h(t, u), u) + (g(t, u)).
\] (23)

Note that $\beta_1 \in L^\infty(\mathbb{R}, L^{\frac{p}{p - q}}(\mathbb{R}^n))$ and $\beta_2 \in L^\infty(\mathbb{R}, L^2(\mathbb{R}^n))$, we have
\[
(f(t, u), u) + \zeta(\theta, \omega)(h(t, u), u) \quad (by \ (5) \ and \ (9))
\leq -\alpha_1 \|u\|^p + \|\psi_1(t)\|_1 + \|\zeta(\theta, \omega)| \int_{\mathbb{R}^n} |\beta_1(t, x)||u|^q dx + \|\zeta(\theta, \omega)| \int_{\mathbb{R}^n} |\beta_2(t, x)||u|dx
\leq \|\psi_1(t)\|_1 + \frac{\lambda}{4} \|u\|^2 + c |\zeta(\theta, \omega)|^p \|\beta_1(t)\|^p \|\beta_2(t)\|^{\frac{p}{p - q}} + c |\zeta(\theta, \omega)|^2 \|\beta_2(t)\|^2
\leq \frac{\lambda}{4} \|u\|^2 + \|\psi_1(t)\|_1 + c |\zeta(\theta, \omega)|^p \|\beta_1(t)\|^p + c |\zeta(\theta, \omega)|^2.
\] (24)
Moreover, we have
\[
\frac{d}{dt}(\|u\|^2 + \|(-\Delta)\frac{3}{2} u\|^2) + \kappa (\|v\|^2 + \|(-\Delta)\frac{3}{2} u\|^2) \leq c\|g(t)\|^2 + c\psi_1(t)\|1 + c|\zeta_\delta(\theta_\omega)|^{\frac{2}{\gamma - 1}} + c|\zeta_\delta(\theta_\omega)|^2,
\]
which along with (28) implies (19).\]

which further implies
\[
\|u(\sigma, \tau - t, \theta_{-\tau}\omega, u_{\tau-t})\|_{H^s(\mathbb{R}^n)}^2 \leq ce^{-\kappa \tau(\sigma - t)}\|u_{\tau-t}\|_{H^s(\mathbb{R}^n)}^2 + cR(\tau, \omega),
\]
where \(R(\tau, \omega)\) is finite due to (12) and (17). By \(u_{\tau-t} \in \mathcal{D}(\tau - t, \theta_{-t}\omega)\) with \(\mathcal{D} \in \mathcal{D}\),
\[
ce^{-\kappa \tau t}\|u_{\tau-t}\|_{H^s(\mathbb{R}^n)}^2 \leq ce^{-\kappa \tau t}\|\mathcal{D}(\tau - t, \theta_{-t}\omega)\|_{H^s(\mathbb{R}^n)}^2 \rightarrow 0 \text{ as } t \rightarrow +\infty,
\]
which along with (28) implies (19).

Next we show (20). We infer from (26) that for all \(\sigma \in [\tau - t, \tau]\) with \(t \geq 0,\)
\[
\|u(\sigma, \tau - t, \theta_{-\tau}\omega, u_{\tau-t})\|_{H^s(\mathbb{R}^n)}^2 \leq ce^{-\kappa \tau(\sigma - t)}\|u_{\tau-t}\|_{H^s(\mathbb{R}^n)}^2 + c\int_{\tau-t}^{\tau} e^{-\kappa \tau \sigma} (\|g(r + \tau)\|^2 + \|\psi_1(r + \tau)\|_1 + |\zeta_\delta(\theta_\omega)|^{\frac{2}{\gamma - 1}} + |\zeta_\delta(\theta_\omega)|^2) dr
\]

where \(\hat{R}(\tau, \omega)\) is finite due to (12) and (17). Taking \((p - 1)\)th power of (29) and multiplying by \(e^{\kappa(\sigma - t)}\), we integrate the result over \((\tau - t, \tau)\) with \(t \geq 0\) to get
\[
\int_{\tau-t}^{\tau} e^{\kappa(\sigma - t)}\|u(\sigma, \tau - t, \theta_{-\tau}\omega, u_{\tau-t})\|_{H^s(\mathbb{R}^n)}^{2p-2} d\sigma
\]

By \(u_{\tau-t} \in \mathcal{D}(\tau - t, \theta_{-t}\omega)\) with \(\mathcal{D} \in \mathcal{D}\), we have
\[
ce^{-\kappa \tau t}\|u_{\tau-t}\|_{H^s(\mathbb{R}^n)}^{2p-2} \leq ce^{-\kappa \tau t}\|\mathcal{D}(\tau - t, \theta_{-t}\omega)\|_{H^s(\mathbb{R}^n)}^{2p-2} \rightarrow 0 \text{ as } t \rightarrow +\infty,
\]
which together with (30) implies the desired result (20).\]
2.2. Uniform tail-estimates. To derive uniform tail-estimates of solutions of (4), we need the following auxiliary estimate.

**Lemma 2.2.** Let (5)-(10) hold. Then for every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $u_\tau \in H^s(\mathbb{R}^n)$, the derivative of the solution $u(t, \tau, \omega, u_\tau)$ of (4) satisfies, for all $t \geq \tau$,

$$
u_t(t, \tau, \omega, u_\tau)^2 \leq c(1 + \|u\|_{H^{2p-2}\mathbb{R}^n}^2 + \|g(t)\|^2 + \|\psi_2(t)\|^2 + \|\zeta_\delta(\delta_t \omega)\|^2_{\mathcal{F}^{2p-2}}).$$

(31)

**Proof.** Multiplying (4) by $u_t$ to find

$$
\|u_t\|^2 + \|(-\Delta)^{\frac{s}{2}} u_t\|^2 + \lambda(u, u_t) + ((-\Delta)^{\frac{s}{2}} u, (-\Delta)^{\frac{s}{2}} u_t) = (f(t, u), u_t) + \zeta_\delta(\delta_t \omega)(h(t, u), u_t) + (g(t), u_t).
$$

(32)

By (6) and the Sobolev embedding inequality given in (13), we see

$$
|f(t, u)| \leq \int_{\mathbb{R}^n} (\alpha_2 |u|^{p-1} + |\psi_2(t)|)|u_t| dx \leq \frac{1}{4} \|u_t\|^2 + c \|u\|_{H^{2p-2}\mathbb{R}^n}^2 + c \|\psi_2(t)\|^2.
$$

(33)

By (13), $\beta_1 \in L^\infty(\mathbb{R}, L^{2p-2})$ and Young’s inequality, we have

$$
\zeta_\delta(\delta_t \omega)(h(t, u), u_t) \quad \text{(by (9))}
$$

$$
\leq |\zeta_\delta(\delta_t \omega)| \int_{\mathbb{R}^n} (|\beta_1(t, x)| |u|^{q-1} + |\beta_2(t, x)|)|u_t|dx
$$

$$
\leq \frac{1}{4} \|u_t\|^2 + c \int_{\mathbb{R}^n} \left|\zeta_\delta(\delta_t \omega)\right|^2 |\beta_1(t, x)|^2 |u|^{2q-2} dx + c \|u\|_{H^{2p-2}\mathbb{R}^n}^2 \|\beta_2(t)\|^2
$$

$$
\leq \frac{1}{4} \|u_t\|^2 + c \|u\|_{H^{2p-2}\mathbb{R}^n}^2 + c \|\zeta_\delta(\delta_t \omega)\|^2_{\mathcal{F}^{2p-2}} + c \|\beta_2(t)\|^2
$$

$$
\leq \frac{1}{4} \|u_t\|^2 + c \|u\|_{H^{2p-2}\mathbb{R}^n}^2 + c \|\zeta_\delta(\delta_t \omega)\|^2_{\mathcal{F}^{2p-2}} + c.
$$

(34)

It follows from Young’s inequality that

$$
\lambda(u, u_t) + |((-\Delta)^{\frac{s}{2}} u, (-\Delta)^{\frac{s}{2}} u_t)| + |(g(t), u_t)|
$$

$$
\leq \frac{1}{4} \|u_t\|^2 + \frac{3}{4} \|(-\Delta)^{\frac{s}{2}} u_t\|^2 + c \|u\|_{H^{2p-2}\mathbb{R}^n}^2 + c \|g(t)\|^2.
$$

(35)

The desired result (31) follows from (32)-(35) immediately. \qed

Next we derive uniform tail-estimates of the solutions. To this end, we let $\rho_k(x) = \rho(\frac{1}{k}x)$ for $x \in \mathbb{R}^n$ and $k \in \mathbb{N}$, where $\rho : \mathbb{R}^+ \to [0, 1]$ is a smooth function with $\rho(t) \equiv 0$ for all $0 \leq t \leq \frac{1}{2}$ and $\rho(t) \equiv 1$ for all $t \geq 1$. By [27, Lemma 3.4], for every $s \in (0, 1)$ and $k \in \mathbb{N}$,

$$
\sup_{y \in \mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|\rho_k(x) - \rho_k(y)|^2}{|x - y|^{n+2s}} dx \leq \frac{c}{k^{2s}} \quad \text{and} \quad \sup_{y \in \mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} \rho_k(y)| \leq \frac{c}{k^{2s}}.
$$

(36)

**Lemma 2.3.** Let (5)-(10) and (17) be satisfied. Then for every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, the solution of (4) with $u_{\tau-\cdot} \in D(\tau - t, \theta_{-t}\omega) \equiv D(\tau - t, \theta_{-t}\omega)$ satisfies

$$
limit_{k,t \to \infty} \int_{\mathcal{C}_k} |u(\tau, \tau - t, \theta_{-t}\omega, u_{\tau-\cdot})(x)|^2 dx = 0.
$$

(37)
\[
\lim_{k,t \to +\infty} \int_{(O_k)^c} \frac{|u(\tau, \tau - t, \theta, \omega, u_{\tau-\omega})(x) - u(\tau, \tau - t, \theta, \omega, u_{\tau-\omega})(y)|^2}{|x - y|^{n+2s}} \, dx \, dy = 0,
\]

where \( O_k = \{ x \in \mathbb{R}^n : |x| < k \} \), \( (O_k)^c = \mathbb{R}^n \setminus O_k \) and \((O_k^2)^c = \mathbb{R}^n \times \mathbb{R}^n \setminus O_k \times O_k \).

**Proof.** Multiplying (4) by \( \rho_k(x)u \), we get

\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} \rho_k |u|^2 \, dx + \int_{\mathbb{R}^n} \rho_k u(-\Delta)^s u_t \, dx + \lambda \int_{\mathbb{R}^n} \rho_k |u|^2 \, dx + \int_{\mathbb{R}^n} \rho_k (-\Delta)^s u dx = \int_{\mathbb{R}^n} \rho_k f(t, x, u) u dx + \zeta(\theta) \int_{\mathbb{R}^n} \rho_k h(t, x, u) u dx + \int_{\mathbb{R}^n} \rho_k g(t, x) u dx.
\]

We calculate the second term as follows:

\[
\int_{\mathbb{R}^n} \rho_k(x)u(-\Delta)^s u_t \, dx = \frac{1}{2} C(n, s) \left( \rho_k u, u_t \right)_{H^s(\mathbb{R}^n)}
\]

\[
= \frac{1}{2} C(n, s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(\rho_k(x)u(x) - \rho_k(y)u(y))(u_t(x) - u_t(y))}{|x - y|^{n+2s}} \, dx \, dy
\]

\[
= \frac{1}{2} C(n, s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\rho_k(x)(u(x) - u(y))(u_t(x) - u_t(y))}{|x - y|^{n+2s}} \, dx \, dy
\]

\[
+ \frac{1}{2} C(n, s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\rho_k(y)(u_t(x) - u_t(y))}{|x - y|^{n+2s}} \, dx \, dy
\]

\[
= \frac{1}{2} C(n, s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(\rho_k(x) - \rho_k(y))(u_t(x) - u_t(y))}{|x - y|^{n+2s}} \, dx \, dy =: I_1 + I_2.
\]

By (36), we obtain the lower bound of \( I_2 \):

\[
I_2 \geq \frac{1}{2} C(n, s) \|u\| \left( \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} \frac{(\rho_k(x) - \rho_k(y))(u_t(x) - u_t(y))}{|x - y|^{n+2s}} \, dx \right)^2 \, dy \right)^{\frac{1}{2}}
\]

\[
\geq \frac{1}{2} C(n, s) \|u\| \left( \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} \frac{|\rho_k(x) - \rho_k(y)|}{|x - y|^{n+2s}} \, dx \int_{\mathbb{R}^n} \frac{|u_t(x) - u_t(y)|}{|x - y|^{n+2s}} \, dx \right) \, dy \right)^{\frac{1}{2}}
\]

\[
\geq -ck^{-s}\|u\|\|u_t\|_{H^s(\mathbb{R}^n)} \geq -ck^{-s}(\|u\|^2_{H^s(\mathbb{R}^n)} + \|u_t\|^2_{H^s(\mathbb{R}^n)}).
\]

Therefore, the second term on the left-hand side of (39) satisfies

\[
\int_{\mathbb{R}^n} \rho_k(x)u(-\Delta)^s u_t \, dx \geq I_1 - ck^{-s}(\|u\|^2_{H^s(\mathbb{R}^n)} + \|u_t\|^2_{H^s(\mathbb{R}^n)}).
\]

We then calculate the last term on the left-hand side of (39):

\[
\int_{\mathbb{R}^n} \rho_k(x)u(-\Delta)^s u dx = \frac{1}{2} C(n, s) \left( \rho_k u, u \right)_{H^s(\mathbb{R}^n)}
\]

\[
= \frac{1}{2} C(n, s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(\rho_k(x)u(x) - \rho_k(y)u(y))(u(x) - u(y))}{|x - y|^{n+2s}} \, dx \, dy
\]

\[
= \frac{1}{2} C(n, s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\rho_k(x)|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dx \, dy
\]

\[
+ \frac{1}{2} C(n, s) \int_{\mathbb{R}^n} u(y) \left( \int_{\mathbb{R}^n} \frac{(\rho_k(x) - \rho_k(y))(u(x) - u(y))}{|x - y|^{n+2s}} \, dx \right) \, dy =: I_3 + I_4.
\]
By (36), $I_4$ has the following lower bound:

\[
I_4 \geq -\frac{1}{2} C(n, s)\|u\| \left( \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} \frac{\rho_k(x) - \rho_k(y)}{|x - y|^{n+2s}} |u(x) - u(y)| \, dx \right)^2 \, dy \right)^{\frac{1}{2}} 
\]

\[
\geq -\frac{1}{2} C(n, s)\|u\| \left( \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} \frac{|\rho_k(x) - \rho_k(y)|^2}{|x - y|^{n+2s}} \, dx \right) \int_{\mathbb{R}^n} |u(x) - u(y)|^2 \, dy \right)^{\frac{1}{2}} 
\]

\[
\geq -ck^{-s}\|u\|_{H^s(\mathbb{R}^n)} \geq -ck^{-s}\|u\|_{H^s(\mathbb{R}^n)}. 
\]

So, the last term on the left-hand side of (39) enjoys

\[
\int_{\mathbb{R}^n} \rho_k(x)u(-\Delta)^s u \, dx \geq I_3 - ck^{-s}\|u\|_{H^s(\mathbb{R}^n)}. 
\] (41)

By (5) and (9), the first and second terms on the right-hand side of (39) are controlled by

\[
\int_{\mathbb{R}^n} \rho_k f(t, u)u \, dx + \zeta_\delta(\theta, \omega) \int_{\mathbb{R}^n} \rho_k h(t, u)u \, dx 
\]

\[
\leq -\alpha \int_{\mathbb{R}^n} \rho_k |u|^p + \int_{\mathbb{R}^n} \rho_k |\psi_1(t)| + |\zeta_\delta(\theta, \omega)| \int_{\mathbb{R}^n} \rho_k (|\beta_1(t)| |u|^q + |\beta_2(t)||u|) 
\]

\[
\leq \frac{\lambda}{4} \int_{\mathbb{R}^n} \rho_k |u|^2 + \int_{\mathbb{R}^n} \rho_k |\psi_1(t)| 
\]

\[
+ c (|\zeta_\delta(\theta, \omega)| \frac{p}{p-1} + |\zeta_\delta(\theta, \omega)|^2) \int_{\mathbb{R}^n} \rho_k (|\beta_1(t)| \frac{p}{p-1} + |\beta_2(t)|^2). 
\] (42)

Moreover, the last term on the right-hand side of (39) is estimated by

\[
\int_{\mathbb{R}^n} \rho_k(x)g(t, x)u \, dx \leq \frac{\lambda}{4} \int_{\mathbb{R}^n} \rho_k(x)|u|^2 \, dx + c \int_{\mathbb{R}^n} \rho_k(x)|g(t, x)|^2 \, dx. 
\] (43)

Substituting (40)-(43) into (39), we see from Lemma 2.2 that

\[
\frac{d}{dt} \left( \int_{\mathbb{R}^n} \rho_k(x)|u|^2 \, dx + I_3 \right) + \kappa \left( \int_{\mathbb{R}^n} \rho_k(x)|u|^2 \, dx + I_3 \right) 
\]

\[
\leq ck^{-s} (1 + \|u\|_{H^s(\mathbb{R}^n)}^{2p-2} + \|g(t)||^2 + \|\psi_2(t)||^2 + |\zeta_\delta(\theta, \omega)|^{\frac{2p-2}{p-1}}) 
\]

\[
+ c \int_{|x| \geq \frac{1}{2} k} (|g(t, x)|^2 + |\psi_1(t, x)|) \, dx 
\]

\[
+ c (|\zeta_\delta(\theta, \omega)| \frac{p}{p-1} + |\zeta_\delta(\theta, \omega)|^2) \int_{|x| \geq \frac{1}{2} k} (|\beta_1(t, x)| \frac{p}{p-1} + |\beta_2(t, x)|^2) \, dx. 
\] (44)

where $I_3$ is just defined as above, that is,

\[
I_3 = \frac{1}{2} C(n, s) \int_{\mathbb{R}^n} \rho_k(x) (u(x) - u(y))^2 \, dx \, dy. 
\]

Multiplying (44) by $e^{\kappa t}$ and integrating over $(\tau - t, \tau)$ with $t \geq 0$, then we replace $\omega$ by $\theta_{-\tau} \omega$ in the resulting inequality to obtain

\[
\int_{\mathbb{R}^n} \rho_k(x)|u(\tau, \tau - t, \theta_{-\tau} \omega, u_{\tau - t})(x)|^2 \, dx 
\]

\[
+ \frac{1}{2} C(n, s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \rho_k(x)|u(\tau, \tau - t, \theta_{-\tau} \omega, u_{\tau - t})(x) - u(\tau, \tau - t)(y)|^2 \, dx \, dy 
\]

\[
\leq J_1 + J_2 + J_3 + J_4 + J_5, 
\] (45)
where $J_t$ are defined and estimated as follows. By $u_{\tau-t} \in \mathcal{D}(\tau-t, \theta_{-t}\omega)$ with $\mathcal{D} \in \mathcal{D}$,

$$J_1 := e^{-\kappa t} \left( \int_{\mathbb{R}^n} \rho_k(x) |u_{\tau-t}|^2 dx + \frac{1}{2} C(n, s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \rho_k(x) |u_{\tau-t}(x) - u_{\tau-t}(y)|^2 dxdy \right) \leq ce^{-\kappa t} \|u_{\tau-t}\|_{H^s(\mathbb{R}^n)}^2 \leq ce^{-\kappa t} \|\mathcal{D}(\tau-t, \theta_{-t}\omega)\|_{H^s(\mathbb{R}^n)}^2 \to 0 \text{ as } t \to +\infty. \quad (46)$$

By Lemma 2.1, there exists $T := T(\tau, \omega; \mathcal{D}) > 0$ such that for all $t \geq T$, as $k \to \infty$,

$$J_2 := ck^{-s} \int_{-t}^\tau e^{\kappa(r-t)} \|u(r, \tau-t, \theta_{-t}\omega, u_{\tau-t})\|_{H^s(\mathbb{R}^n)}^{2p-2} dr \leq ck^{-s} R^{p-1}(\tau, \omega) \to 0. \quad (47)$$

By (17) and (12), as $k \to \infty$,

$$J_3 := ck^{-s} \int_{-\infty}^0 e^{\kappa r} \left( 1 + \|g(r+\tau)\|^2 + \|\psi_2(r+\tau)\|^2 + \|\zeta_\delta(\theta_t, \omega)\|_{H^s(\mathbb{R}^n)}^{2p-2} \right) dr \leq ck^{-s} \int_{-\infty}^0 e^{\frac{1}{2} \kappa r} \left( 1 + \|g(r+\tau)\|^2 + \|\psi_2(r+\tau)\|^2 + \|\zeta_\delta(\theta_t, \omega)\|_{H^s(\mathbb{R}^n)}^{2p-2} \right) dr \to 0. \quad (48)$$

For the fourth term on the right-hand side of (45). We deduce from (17) that as $k \to \infty$

$$J_4 := c \int_{-\infty}^0 e^{\kappa r} \int_{|x| \geq \frac{1}{2}} \left( \|g(r+\tau, x)\|^2 + \|\psi_1(r+\tau, x)\| \right) dx dr \to 0. \quad (49)$$

By (12), the Lebesgue controlled convergence theorem gives that

$$J_5 := \int_{-\infty}^0 e^{\kappa r} \left( \|\zeta_\delta(\theta_t, \omega)\|_{H^s(\mathbb{R}^n)}^{2p-2} + \|\zeta_\delta(\theta_t, \omega)\|_{H^s(\mathbb{R}^n)}^{2p-2} \right) \int_{|x| \geq \frac{1}{2}} \left( \|\beta_1(r+\tau)\|_{H^s(\mathbb{R}^n)}^{2p-2} + \|\beta_2(r+\tau)\|_{H^s(\mathbb{R}^n)}^{2p-2} \right) dx dr \quad (50)$$

tends to zero as $k \to \infty$.

Therefore, by all estimates given in (46)-(50), we see from (45) that

$$\lim_{k,t \to +\infty} \int_{\mathbb{R}^n} \rho_k(x) |u(\tau, \tau-t, \theta_{-t}\omega, u_{\tau-t})(x)|^2 dx = 0, \quad (51)$$

$$\lim_{k,t \to +\infty} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \rho_k(x) \left( |u(\tau, \tau-t, \theta_{-t}\omega, u_{\tau-t})(x) - u(\tau, \tau-t)(y)| \right)^2 \frac{|x-y|^{n+2s}}{dx dy} = 0. \quad (52)$$

And thereby by (51), we obtain the first result (37). Moreover it follows from (52) that

$$\lim_{k,t \to +\infty} \int_{|x| \geq k} \frac{|u(\tau, \tau-t, \theta_{-t}\omega, u_{\tau-t})(x) - u(\tau, \tau-t)(y)|^2}{|x-y|^{n+2s}} dx dy = 0. \quad (53)$$

Interchanging $x$ and $y$ in (53) yields

$$\lim_{k,t \to +\infty} \int_{|y| \geq k} \frac{|u(\tau, \tau-t, \theta_{-t}\omega, u_{\tau-t})(x) - u(\tau, \tau-t)(y)|^2}{|x-y|^{n+2s}} dy dx = 0. \quad (54)$$

Combining (53) and (54), we obtain the second result (38). The proof is concluded.
2.3. Uniform estimates on bounded domains. To verify the pullback asymptotic compactness of solutions in $H^s(\mathbb{R}^n)$, we derive uniform estimates of solutions on bounded domains. For every $x \in \mathbb{R}^n$ and $k \in \mathbb{N}$, we put

$$\hat{u}(t, \tau, \omega, \hat{u}_{\tau})(x) = \xi_k(x)u(t, \tau, \omega, u_{\tau})(x), \quad \text{where } \xi_k(x) = 1 - \rho\left(\frac{|x|}{R}\right).$$

Then $\hat{u}(t, \tau, \omega, \hat{u}_{\tau})(x) = 0$ for $k \in \mathbb{N}$ and $x \in O_k$, and $\|\hat{u}\|_{H^s(\mathbb{R}^n)} \leq c\|u\|_{H^s(\mathbb{R}^n)}$ for some constant $c > 0$ independent of $k \in \mathbb{N}$. By the definition of $(-\Delta)^s$, we can verify

$$(-\Delta)^s \hat{u}(x) = C(n, s) \text{ P.V.} \int_{\mathbb{R}^n} \frac{\xi_k(x)u(t, \tau, \omega, u_{\tau})(x) - \xi_k(y)u(t, \tau, \omega, u_{\tau})(y)}{|x - y|^{n+2s}} dy$$

$$= C(n, s) \text{ P.V.} \int_{\mathbb{R}^n} \frac{\xi_k(x)u(t, \tau, \omega, u_{\tau})(x) - \xi_k(y)u(t, \tau, \omega, u_{\tau})(y)}{|x - y|^{n+2s}} dy$$

and similarly

$$(-\Delta)^s \hat{u}(x) = \xi_k(x)(-\Delta)^s u(x) + u(x)(-\Delta)^s \xi_k(x)$$

$$+ C(n, s) \text{ P.V.} \int_{\mathbb{R}^n} \frac{\left(\xi_k(x) - \xi_k(y)\right)\left(u(t, \tau, \omega, u_{\tau})(x) - u(t, \tau, \omega, u_{\tau})(y)\right)}{|x - y|^{n+2s}} dy.$$  

Multiplying (4) by $\xi_k(x)$ and substituting (56)-(57) into the obtained result, we find

$$\hat{u}_t + (-\Delta)^s \hat{u}_t + (-\Delta)^s \hat{u} + \lambda \hat{u}$$

$$= \xi_k(x)f(t, x, u) + \xi_k(\theta, \omega)\xi_k(x)h(t, x, u) + \xi_k(x)g(t, x)$$

$$+ u(t, \tau, \omega, u_{\tau})(x)(-\Delta)^s \xi_k(x) + C(n, s) \text{ P.V.} \int_{\mathbb{R}^n} \frac{\left(\xi_k(x) - \xi_k(y)\right)\left(u(t, \tau, \omega, u_{\tau})(x) - u(t, \tau, \omega, u_{\tau})(y)\right)}{|x - y|^{n+2s}} dy$$

$$+ u(t, \tau, \omega, u_{\tau})(x)(-\Delta)^s \xi_k(x) + C(n, s) \text{ P.V.} \int_{\mathbb{R}^n} \frac{\left(\xi_k(x) - \xi_k(y)\right)\left(u(t, \tau, \omega, u_{\tau})(x) - u(t, \tau, \omega, u_{\tau})(y)\right)}{|x - y|^{n+2s}} dy,$$

with initial-boundary conditions:

$$\hat{u}(\tau, x) = \xi_k(x)u_{\tau}(x), \quad \forall x \in \mathbb{R}^n \quad \text{and thus} \quad \hat{u}(\tau, x) = 0, \quad \forall x \in O_k.$$  

Now let $H = \{u \in L^2(\mathbb{R}^n) : u = 0 \text{ on } O_k^c\}$ and $V = \{u \in H^s(\mathbb{R}^n) : u = 0 \text{ on } O_k^c\}$. Note that the following eigenvalue problem:

$$(-\Delta)^s u = \lambda u \quad \forall x \in \mathcal{O}_k \quad \text{and} \quad u = 0 \quad \forall x \in \mathcal{O}_k^c$$

has a family of eigenvalues $\{\lambda_i\}_{i=1}^{\infty}$ such that $0 < \lambda_1 \leq \lambda_2 \leq \cdots \lambda_i \to \infty$ as $i \to \infty$ and the corresponding eigenfunctions $\{e_i\}_{i=1}^{\infty}$ in $V$ form an orthonormal basis of $H$. Let $\mathcal{P}_m : H \to \text{span}\{e_1, e_2, \cdots, e_m\}$ be the canonical projection.

**Lemma 2.4.** Let (5)-(10) and (17) be satisfied. Then for every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $\mathcal{D} = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \subset \mathcal{D}$, the solution of (4) with $u_{\tau-\ell} \in D(\tau - \theta, -\ell, \omega)$ satisfies

$$\lim_{m, t \to +\infty} \|(I - \mathcal{P}_m)\xi_k u(\tau, \tau - t, \theta, -\ell, u_{\tau-\ell})\|_{H^s(\mathbb{R}^n)} = 0 \quad \text{for each } k \in \mathbb{N}.$$
Proof. Let \( \hat{u}_{m,2} = (I - \mathcal{P}_m)\hat{u} \). Applying \( I - \mathcal{P}_m \) to (58) and multiplying by \( \hat{u}_{m,2} \), we get

\[
\frac{1}{2} \frac{d}{dt} \| \hat{u}_{m,2} \|_2^2 + \| (-\Delta) \hat{u}_{m,2} \|_2^2 + \lambda \| \hat{u}_{m,2} \|_2^2 + \| (-\Delta) \hat{u}_{m,2} \|_2^2
\]

\[
= \int_{\mathbb{R}^n} \xi_k f(t, x, u) \hat{u}_{m,2} \, dx
\]

\[
+ \zeta \zeta (\theta \omega) \int_{\mathbb{R}^n} \xi_k h(t, x, u) \hat{u}_{m,2} \, dx + \left( \xi_k g(t) + u(\Delta)^{\frac{1}{2}} \xi_k + u(-\Delta)^{\frac{1}{2}} \xi_k, \hat{u}_{m,2} \right)
\]

\[
+ C(n, s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(\xi_k(x) - \xi_k(y))(u(x) - u(y))}{|x - y|^{n+2s}} \, dy \, dx
\]

\[
+ C(n, s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(\xi_k(x) - \xi_k(y))(u(x) - u(y))}{|x - y|^{n+2s}} \, dy \, dx.
\]  

(61)

By (6) and (13), the first term on the right-hand side of (61) is bounded by

\[
\int_{\mathbb{R}^n} \xi_k f(t, x, u) \hat{u}_{m,2} \, dx \leq c \| \hat{u}_{m,2} \|_2 \| u \|_{L^p(\mathbb{R}^n)}^{p-1} + c \| \hat{u}_{m,2} \|_2 \| \psi(t) \|_2
\]

\[
\leq c \lambda_{m+1}^{-\frac{1}{2}} \| (-\Delta)^{\frac{1}{2}} \hat{u}_{m,2} \|_2 \| u \|_{L^p(\mathbb{R}^n)}^{p-1} + c \lambda_{m+1}^{-\frac{1}{2}} \| (-\Delta)^{\frac{1}{2}} \hat{u}_{m,2} \|_2 \| \psi(t) \|_2
\]

\[
\leq \frac{1}{16} \| (-\Delta)^{\frac{1}{2}} \hat{u}_{m,2} \|_2^2 + c \lambda_{m+1}^{-1} \| u \|_{L^p(\mathbb{R}^n)}^{2p-2} + \| \psi(t) \|_2^2.
\]  

(62)

By \( \beta_1 \in L^\infty(\mathbb{R}, L^{2p}(\mathbb{R}^n)) \) and \( \beta_2 \in L^\infty(\mathbb{R}, L^2(\mathbb{R}^n)) \), the first term in the second line of (61) satisfies

\[
\zeta (\theta \omega) \int_{\mathbb{R}^n} \xi_k h(t, x, u) \hat{u}_{m,2} \, dx \quad \text{(by (9))}
\]

\[
\leq |\zeta (\theta \omega)| \int_{\mathbb{R}^n} (|\beta_1(t, x)| |u|^{q-1} + |\beta_2(t, x)|) \| \hat{u}_{m,2} \|_2 \, dx \quad \text{(by Hölder’s inequality)}
\]

\[
\leq |\zeta (\theta \omega)| \| \hat{u}_{m,2} \|_2 \left[ \frac{1}{2p} \| \beta_1(t) \|_{L^p(\mathbb{R}^n)}^{2p-1} + |\zeta (\theta \omega)| \| \hat{u}_{m,2} \|_2 \| \beta_2(t) \|_2 \right] \quad \text{(by (13)-(14))}
\]

\[
\leq c |\zeta (\theta \omega)| \| \hat{u}_{m,2} \|_2 \| (-\Delta)^{\frac{1}{2}} \hat{u}_{m,2} \|_2 + c |\zeta (\theta \omega)| \| \hat{u}_{m,2} \|_2
\]

\[
\leq c \lambda_{m+1}^{-\frac{1}{2}} |\zeta (\theta \omega)| \| (-\Delta)^{\frac{1}{2}} \hat{u}_{m,2} \|_2 \| u \|_{L^p(\mathbb{R}^n)}^{p-1} + c \lambda_{m+1}^{-\frac{1}{2}} |\zeta (\theta \omega)| \| (-\Delta)^{\frac{1}{2}} \hat{u}_{m,2} \|_2
\]

\[
\leq \frac{1}{16} \| (-\Delta)^{\frac{1}{2}} \hat{u}_{m,2} \|_2^2 + c \lambda_{m+1}^{-1} + \lambda_{m+1}^{-1} \| (-\Delta)^{\frac{1}{2}} \hat{u}_{m,2} \|_2 \| u \|_{L^p(\mathbb{R}^n)}^{2p-2} + \| \zeta (\theta \omega) \|_2^2.
\]  

(63)

By \( (-\Delta)^{\frac{1}{2}} \xi_k = (-\Delta)^{\frac{1}{2}} \rho_k \) and (36), the remaining terms in the second line of (61) satisfy

\[
(\xi_k g(t) + u(\Delta)^{\frac{1}{2}} \xi_k + u(-\Delta)^{\frac{1}{2}} \xi_k, \hat{u}_{m,2})
\]

\[
\leq c \lambda_{m+1}^{-\frac{1}{2}} \| (-\Delta)^{\frac{1}{2}} \hat{u}_{m,2} \|_2 \| u \|_{L^p(\mathbb{R}^n)} + \| g(t) \|_2 + \| u \|_{L^p(\mathbb{R}^n)}
\]

\[
\leq \frac{1}{8} \| (-\Delta)^{\frac{1}{2}} \hat{u}_{m,2} \|_2^2 + c \lambda_{m+1}^{-1} \| u \|_{L^p(\mathbb{R}^n)}^2 + \| u \|_{L^p(\mathbb{R}^n)}^2 + \| g(t) \|_2^2.
\]  

(64)

For each \( k \in \mathbb{N} \), by (36), the term in the third line of (61) is bounded by

\[
I_5 := C(n, s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(\xi_k(x) - \xi_k(y))(u(x) - u(y))}{|x - y|^{n+2s}} \, dy \, dx
\]
Substituting (62)-(66) into (61), we see from Lemma 2.2 that it follows that

\[ I_5 \leq \frac{1}{8} \|(-\Delta)^{\frac{1}{2}} \tilde{u}_{m,2}\|^2 + c\lambda_{m+1}^{-1} \|u_t\|_{H^1(\mathbb{R}^n)}^2. \]  

By the same method, the last term in (61) is bounded by

\[ I_6 := C(n, s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (\xi_k(x) - \xi_k(y))(u(x) - u(y)) \, dy \, dx \leq \frac{1}{8} \|(-\Delta)^{\frac{1}{2}} \tilde{u}_{m,2}\|^2 + c\lambda_{m+1}^{-1} \|u_t\|_{H^1(\mathbb{R}^n)}^2. \]

Substituting (62)-(66) into (61), we see from Lemma 2.2 that

\[ \frac{d}{dt}(\|\tilde{u}_{m,2}\|^2 + \|(-\Delta)^{\frac{1}{2}} \tilde{u}_{m,2}\|^2) + \kappa (\|\tilde{u}_{m,2}\|^2 + \|(-\Delta)^{\frac{1}{2}} \tilde{u}_{m,2}\|^2) \leq c(\lambda_{m+1}^{\frac{1}{2}} + \lambda_{m+1}^{-1}) \left(1 + \|u\|_{H^{2p_2}(\mathbb{R}^n)}^2 + \|g(t)\|^2 + \|\psi_2(t)\|^2 + |\zeta_\delta(\theta_{\tau}\omega)| \frac{2p_2-2}{p-\frac{4}{3}} \right). \]  

Applying Gronwall’s inequality to (67) over \((\tau - t, \tau)\) with \(t \geq 0\) and replacing \(\omega\) by \(\theta_{\tau}\omega\) in the resulting inequality, we obtain

\[ \|\tilde{u}_{m,2}(\tau, \tau - t, \theta_{\tau}\omega, u_{\tau-t})\|^2 + \|(-\Delta)^{\frac{1}{2}} \tilde{u}_{m,2}(\tau, \tau - t, \theta_{\tau}\omega, u_{\tau-t})\|^2 \leq e^{-\kappa t} \left(\|(I - P_m)\xi_k u_{\tau-t}\|^2 + \|(-\Delta)^{\frac{1}{2}} (I - P_m)\xi_k u_{\tau-t}\|^2\right) + c(\lambda_{m+1}^{\frac{1}{2}} + \lambda_{m+1}^{-1}) \int_{\tau-t}^\tau e^{\kappa (r - \tau)} \|u(r, \tau - t, \theta_{\tau}\omega, u_{\tau-t})\|_{H^{2p_2}(\mathbb{R}^n)}^2 dr \]

\[ + c(\lambda_{m+1}^{\frac{1}{2}} + \lambda_{m+1}^{-1}) \int_{\tau-t}^\tau e^{\kappa (r - \tau)} \left(1 + \|g(r)\|^2 + \|\psi_2(r)\|^2 + |\zeta_\delta(\theta_{\tau}\omega)| \frac{2p_2-2}{p-\frac{4}{3}} \right) dr. \]

By \(|I - P_m| \leq 1, \|\xi_k\|_{L^\infty} \leq 1\) and \(u_{\tau-t} \in \mathcal{D}(\tau - t, \theta_{\tau}\omega)\) with \(\mathcal{D} \in \mathcal{D}\), we have

\[ e^{-\kappa t} \left(\|(I - P_m)\xi_k u_{\tau-t}\|^2 + \|(-\Delta)^{\frac{1}{2}} (I - P_m)\xi_k u_{\tau-t}\|^2\right) \leq ce^{-\kappa t} \|u_{\tau-t}\|_{H^1(\mathbb{R}^n)}^2 \leq ce^{-\kappa t} \|\mathcal{D}(\tau - t, \theta_{\tau}\omega)\|_{H^1(\mathbb{R}^n)}^2 \to 0 \text{ as } t \to +\infty. \]
Assume Lemma 2.5.establish the existence of $D$ by colored noise has a unique
on the right-hand side of (68) converge to zero as $m, t \to +\infty$. Thus we have
$$\lim_{m, t \to +\infty} \| \hat{u}_{m, 2}(\tau, \tau - t, \theta - \tau \omega, (I - \mathcal{P}_m) \xi_{k, m}) \|_{H^s(\mathbb{R}^n)} = 0,$$
which implies (60) as desired.

2.4. Existence of pullback random attractors. In this section, we show that the cocycle $\Phi$ generated by the fractional nonclassical diffusion equations driven by colored noise has a unique $\mathcal{D}$-pullback random attractors in $H^s(\mathbb{R}^n)$. We first establish the existence of $\mathcal{D}$-pullback absorbing sets in $H^s(\mathbb{R}^n)$.

**Lemma 2.5.** Assume (5)-(10) and (17)-(18) hold. Then the cocycle $\Phi$ for problem (4) has a closed $\mathcal{F}$-measurable $\mathcal{D}$-pullback absorbing set $\mathcal{K} = \{ \mathcal{K}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \in \mathcal{D}$ given by
$$\mathcal{K}(\tau, \omega) := \{ u \in H^s(\mathbb{R}^n) : \| u \|_{H^s(\mathbb{R}^n)} \leq MR(\tau, \omega), \quad (\tau, \omega) \in \mathbb{R} \times \Omega, \}$$
where $M$ and $R(\tau, \omega)$ are the same as given in Lemma 2.1.

**Proof.** The $\mathcal{F}$- measurability of $\mathcal{K}$ follows from the $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurability of the mapping $\omega \to R(\tau, \omega)$. By Lemma 2.1, we know that for every $(\tau, \omega, \mathcal{D}) \in \mathbb{R} \times \Omega \times \mathcal{D}$, there exists $T := T(\tau, \omega, \mathcal{D}) > 0$ such that for all $t \geq T$ and $u_{\tau - t} \in D(\tau - t, \theta - \tau \omega)$,
$$\| u(\tau, \tau - t, \theta - \tau \omega, u_{\tau - t}) \|_{H^s(\mathbb{R}^n)} \leq M R(\tau, \omega),$$
which along with the definition of $\Phi$ implies that for all $t \geq T$,
$$\Phi(t, \tau - t, \theta - \tau \omega, D(\tau - t, \theta - \tau \omega)) = u(\tau, \tau - t, \theta - \tau \omega, D(\tau - t, \theta - \tau \omega)) \subset \mathcal{K}(\tau, \omega).$$
Then $\mathcal{K}$ is a pullback absorbing set. We then show $\mathcal{K} \in \mathcal{D}$. Given $\gamma > 0$, by (18), we have
$$\lim_{t \to +\infty} e^{-\gamma t} \int_{-\infty}^{0} e^{\kappa \tau} (\| g(r + \tau - t) \| + \| \psi_{1}(r + \tau - t) \|_1) \, dr$$
$$= e^{-\gamma t} \lim_{t \to +\infty} e^{-\gamma t} \int_{-\infty}^{0} e^{\kappa \tau} (\| g(r - t) \| + \| \psi_{1}(r - t) \|_1) \, dr = 0. \quad (69)$$
Let $\hat{\kappa} = \min(\gamma, \kappa)$. By (12), we know $\int_{-\infty}^{0} e^{\hat{\kappa} \tau} (|\xi_{\delta}(\theta - \tau \omega)|^{\frac{2}{\beta - 2}} + |\xi_{\delta}(\theta - \omega)|^{2}) < \infty$, which implies that
$$e^{-\gamma t} \int_{-\infty}^{0} e^{\kappa \tau} (|\xi_{\delta}(\theta - \tau \omega)|^{\frac{2}{\beta - 2}} + |\xi_{\delta}(\theta - \omega)|^{2}) \, dr \leq \int_{-\infty}^{0} e^{\hat{\kappa}(\tau - t)} (|\xi_{\delta}(\theta - \tau \omega)|^{\frac{2}{\beta - 2}} + |\xi_{\delta}(\theta - \omega)|^{2}) \, dr \leq \int_{-\infty}^{t} e^{\hat{\kappa} \tau} (|\xi_{\delta}(\theta - \omega)|^{\frac{2}{\beta - 2}} + |\xi_{\delta}(\theta - \omega)|^{2}) \, dr \to 0 \text{ as } t \to +\infty. \quad (70)$$
Combining (69) and (70), we find
$$e^{-\gamma t} \| \mathcal{K}(\tau - t, \theta - \tau \omega) \|_{H^s(\mathbb{R}^n)} \leq M e^{-\gamma t} R(\tau - t, \theta - \tau \omega) = M e^{-\gamma t} \int_{-\infty}^{0} e^{\kappa \tau} (\| g(r + \tau - t) \| + \| \psi_{1}(r + \tau - t) \|_1 + |\xi_{\delta}(\theta - \tau \omega)|^{\frac{2}{\beta - 2}} + |\xi_{\delta}(\theta - \omega)|^{2}) \, dr,$$
which tends to zero as $t \to +\infty$. Then we have $\mathcal{K} \in \mathcal{D}$ as required. \qed
We then show the $\mathcal{D}$-pullback asymptotic compactness of $\Phi$.

**Lemma 2.6.** Let (5)-(10) and (17) be satisfied. Then the continuous cocycle $\Phi$ associated with problem (4) is $\mathcal{D}$-pullback asymptotically compact in $H^s(\mathbb{R}^n)$, that is, for every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $\mathcal{D} = \{\mathcal{D}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, the sequence \( \{\Phi(t_n, \tau - t_n, \theta_{-t_n}, u(0))\}_{n=1}^{\infty} \) has a convergent subsequence in $H^s(\mathbb{R}^n)$ whenever $t_n \to +\infty$ and $u(0) \in \mathcal{D}(\tau - t_n, \theta_{-t_n}, \omega)$.

**Proof.** Let $\varepsilon > 0$ be an arbitrary number, we show that the sequence
\[
\Phi(t_n, \tau - t_n, \theta_{-t_n}, u(0)) \to 0
\]
has a finite open cover with radius less then $\varepsilon$ in $H^s(\mathbb{R}^n)$ provided $t_n \to +\infty$ and $u(0) \in \mathcal{D}(\tau - t_n, \theta_{-t_n}, \omega)$. By Lemma 2.3, there are $N_1 = N_1(\tau, \omega, \mathcal{D}, \varepsilon) \geq 1$ and $k_0 = k_0(\tau, \omega, \mathcal{D}, \varepsilon) \geq 1$ such that for all $n \geq N_1$,\[
\int_{\mathcal{O}_{k_0}^c} |u(\tau, \tau - t_n, \theta_{-t_n}, u(0))(x)|^2 \, dx < \frac{\varepsilon^2}{8},
\]
(71)
\[
\int_{(\mathcal{O}_{k_0}^c)^c} \frac{|u(\tau, \tau - t_n, \theta_{-t_n}, u(0))(x) - u(\tau, \tau - t_n, \theta_{-t_n}, u(0))(y)|^2}{|x - y|^{n+2s}} \, dx \, dy < \frac{\varepsilon^2}{8}.
\]
(72)
Then we see from (71) and (72) that for all $n \geq N_1$,
\[
\|u(\tau, \tau - t_n, \theta_{-t_n}, u(0))\|_{H^s(\mathcal{O}_{k_0}^c)} < \frac{\varepsilon}{2}.
\]
(73)
By Lemma 2.4 with $k = 2k_0$, there are $N_2 \geq N_1$ and $m_0 \geq 1$ such that for all $n \geq N_2$,
\[
\|(I - \mathcal{P}_{m_0})\xi_{2k_0} u(\tau, \tau - t_n, \theta_{-t_n}, u(0))\|_{H^s(\mathbb{R}^n)} < \frac{\varepsilon}{4}.
\]
(74)
By Lemma 2.1, there are $N_3 \geq N_2$ and $c_1 = c_1(\tau, \omega) > 0$ such that for all $n \geq N_3$,
\[
\|u(\tau, \tau - t_n, \theta_{-t_n}, u(0))\|_{H^s(\mathbb{R}^n)} \leq c_1,
\]
which further implies that there exists $c_2 = c_2(\tau, \omega) > 0$ such that for all $n \geq N_3$,
\[
\|\xi_{2k_0} u(\tau, \tau - t_n, \theta_{-t_n}, u(0))\|_{H^s(\mathbb{R}^n)} \leq c_2.
\]
(75)
Then \( \{\mathcal{P}_{m_0} \xi_{2k_0} u(\tau, \tau - t_n, \theta_{-t_n}, u(0))\}_{n=1}^{\infty} \) is pre-compact in $\mathcal{P}_{m_0} H^s(\mathbb{R}^n)$ due to (75) and the finite-dimensional range of $\mathcal{P}_{m_0}$. By (74), \( \{\xi_{2k_0} u(\tau, \tau - t_n, \theta_{-t_n}, \omega, u(0))\}_{n=1}^{\infty} \) has a finite $\frac{\varepsilon}{2}$-net in $H^s(\mathbb{R}^n)$. Note that \( u(\tau, \tau - t_n, \theta_{-t_n}, \omega, u(0))(x) = \xi_{2k_0}(x) u(\tau, \tau - t_n, \theta_{-t_n}, \omega, u(0))(x) \) for all \( x \in \mathcal{O}_{k_0} \).

The sequence \( \{u(\tau, \tau - t_n, \theta_{-t_n}, \omega, u(0))\}_{n=1}^{\infty} \) has a finite $\varepsilon$-net in $H^s(\mathcal{O}_{k_0})$. By (73), \( \{u(\tau, \tau - t_n, \theta_{-t_n}, \omega, u(0))\}_{n=1}^{\infty} \) has a finite $\varepsilon$-net in $H^s(\mathbb{R}^n)$. The proof is completed.

Finally, we present the main result of this section as follows.

**Theorem 2.7.** Let (5)-(10) and (17)-(18) be satisfied. Then the cocycle $\Phi$ for problem (4) has a unique $\mathcal{D}$-pullback random attractor $\mathcal{A} = \{\mathcal{A}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ in $H^s(\mathbb{R}^n)$.

**Proof.** Based on Lemmas 2.5 and 2.6, the result follows from [49] and [51] immediately. \qed
3. Attractors of fractional equations driven by additive white noise. In this section, we show the existence and uniqueness of random attractors for the following fractional nonclassical diffusion equations driven by additive noise:

\[
\begin{aligned}
&du + d((-\Delta)^s u + (-\Delta)^s u dt + \lambda u dt = f(t, x, u) dt + g(t, x) dt + h(x) dW, \\
&u(\tau, x) = u_\tau(x), x \in \mathbb{R}^n, t > \tau, \tau \in \mathbb{R},
\end{aligned}
\]  

(76)

where \( \lambda > 0, s \in (0, 1), f \) and \( g \) are the same as in Section 2 and \( W \) is the Wiener process on the probability space \((\Omega, \mathcal{F}, \mathbb{P})\). Denote by

\[
v(t, \tau, \omega, v_\tau) = u(t, \tau, \omega, u_\tau) - z(\theta_t \omega) \quad \text{with} \quad v_\tau = u_\tau - z(\theta_\tau \omega),
\]

(77)

where \( z(\theta_t \omega) = (I + (-\Delta)^s)^{-1} h g(\theta_t \omega) \) with \( g(\theta_t \omega) = -\int_{-\infty}^{\tau} e^s(\theta_t \omega)(s) ds \) being the stationary solution of the one-dimensional Ornstein-Uhlenbeck equation \( du + y dt = dW(t) \). By [16], we know

\[
\lim_{t \to \pm \infty} \frac{y(\theta_t \omega)}{t} = 0 \quad \text{for every} \quad \omega \in \Omega.
\]

(78)

Note that

\[
h(x) dW = dz(\theta_t \omega) + d((-\Delta)^s z(\theta_t \omega)) + z(\theta_t \omega) dt + (-\Delta)^s z(\theta_t \omega) dt.
\]

(79)

Substituting (77) and (79) into (76), we get

\[
\begin{aligned}
&v_t + (-\Delta)^s v_t + (-\Delta)^s v + \lambda v = f(t, x, v + z(\theta_t \omega)) + g(t, x) + (1 - \lambda) z(\theta_t \omega), \\
&v(\tau, x) = u_\tau(x) - z(\theta_\tau \omega), \quad x \in \mathbb{R}^n, \quad t > \tau, \quad \tau \in \mathbb{R}.
\end{aligned}
\]

(80)

For each \((\tau, \omega, v_\tau) \in \mathbb{R} \times \Omega \times H^s(\mathbb{R}^n)\), we can show that problem (80) under (5)-(8) has an unique solution \(v(\cdot, \tau, \omega, v_\tau) \in C([\tau, \infty), H^s(\mathbb{R}^n))\) such that \(v(t, \tau, \omega, v_\tau)\) is continuous in \(v_\tau \in H^s(\mathbb{R}^n)\) and is \((\mathcal{F}, \mathcal{B}(H^s(\mathbb{R}^n)))\)-measurable for all \(t \geq \tau\). Define \(\Phi_0: \mathbb{R}^+ \times \mathbb{R} \times \Omega \times H^s(\mathbb{R}^n) \to H^s(\mathbb{R}^n)\) given by

\[
\Phi_0(t, \tau, \omega, u_\tau) = u(t + \tau, \tau, \theta_{-\tau} \omega, u_\tau) = v(t + \tau, \tau, \theta_{-\tau} \omega, v_\tau) + z(\theta_t \omega),
\]

(81)

where \(v_\tau = u_\tau - z(\omega)\). Then \(\Phi_0\) is a continuous cocycle over \(\mathbb{R}\) and \((\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t \in \mathbb{R}})\).

3.1. Uniform estimates.

**Lemma 3.1.** Let (5)-(8) and (17) be satisfied. Then for every \(\tau \in \mathbb{R}, \omega \in \Omega\) and \(D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}\), there is a \(T := T(\tau, \omega, D) > 0\) such that for all \(t \geq T\), the solution of (76) with \(u_{\tau - \tau} \in D(\tau - t, \theta_{-\tau} \omega)\) satisfies

\[
\begin{aligned}
&\|u(\tau, \tau - t, \theta_{-\tau} \omega, u_{\tau - \tau})\|^2_{H^s(\mathbb{R}^n)} \leq M_1 R_0(\tau, \omega), \\
&\int_{\tau - t}^{\tau} e^{\kappa(r-t)} \|u(r, \tau - t, \theta_{-\tau} \omega, u_{\tau - \tau})\|^2_{H^s(\mathbb{R}^n)} dr \leq M_1 \left( \int_{-\infty}^{\tau} e^{\kappa \omega} |y(\theta_t \omega)|^{2p-2} d\sigma + \tilde{R}_0^{-1}(\tau, \omega) \right),
\end{aligned}
\]

(82)

(83)

where \(M_1\) is a positive constant independent of \(\tau, \omega\) and \(D\),

\[
R_0(\tau, \omega) = |y(\omega)|^2 + \int_{-\infty}^{0} e^{\kappa \tau} (1 + G(r + \tau) + |y(\theta_t \omega)|^p) dr,
\]

(84)

\[
\tilde{R}_0(\tau, \omega) = \int_{-\infty}^{0} e^{\kappa \tau} (1 + G(r + \tau) + |y(\theta_t \omega)|^p) dr.
\]

(85)

and \(G(s) = \|g(s)\|^2 + \|\psi_1(s)\|_1 + \|\psi_2(s)\|^2\) for all \(s \in \mathbb{R}\).
Proof. Taking the inner product of (80) with \( v \) in \( L^2(\mathbb{R}^n) \) to find

\[
\frac{1}{2} \frac{d}{dt} (\|v\|^2 + \|(-\Delta)\hat{v}\|^2) + \lambda \|v\|^2 + \|(-\Delta)\hat{v}\|^2
= (f(t, u), v) + (g(t), v) + (1 - \lambda) (z(\theta \omega), v).
\] (86)

It yields from (5)-(6) and (13) that

\[
(f(t, u), v) = \int_{\mathbb{R}^n} f(t, x, u) ud\mathcal{L} - \int_{\mathbb{R}^n} f(t, x, u) z(\theta \omega) d\mathcal{L}
\leq -\alpha_1 \|u\|^p_p + \|\psi_1(t)\|_1 + \int_{\mathbb{R}^n} (\alpha_2 |u|^{p-1} + |\psi_2(t, x)|) z(\theta \omega) d\mathcal{L}
\leq \|\psi_1(t)\|_1 + c \|z(\theta \omega)\|^p_p + c \|\psi_2(t)\|^2 + c (|g(t)|^2 + c(1 + |y(\theta \omega)|^p)).
\] (87)

Moreover, by Young’s inequality,

\[
\int_{\mathbb{R}^n} g(t, x) v d\mathcal{L} + (1 - \lambda) (z(\theta \omega), v) \leq \frac{\lambda}{2} \|v\|^2 + c \|g(t)\|^2 + c(1 + |y(\theta \omega)|^p).
\] (88)

Recall that \( \kappa = \min(\lambda, 1) \), then we see from (86)-(88) that

\[
\frac{d}{dt} (\|v\|^2 + \|(-\Delta)\hat{v}\|^2) + \kappa (\|v\|^2 + \|(-\Delta)\hat{v}\|^2) \leq c(1 + G(t) + |y(\theta \omega)|^p). \] (89)

Multiplying (89) by \( e^{\kappa t} \) and integrating over \((\tau - t, \sigma)\) with \( \sigma \geq \tau - t \) and \( t \geq 0 \), then we replace \( \omega \) by \( \theta_{-\tau} \omega \) to find

\[
\|\psi(\sigma, \tau - t, \theta_{-\tau} \omega, v_{\tau-t})\|_{H^s(\mathbb{R}^n)} \leq c e^{-\kappa(\sigma - \tau + t)} \|v_{\tau-t}\|_{H^s(\mathbb{R}^n)}^2
+ c \int_{-\tau}^{\sigma-\tau} e^{\kappa(\tau + \sigma - \tau)} (1 + G(r + \tau) + |y(\theta_{-\tau} \omega)|^p) dr.
\] (90)

By the change (77) of variables, we have

\[
u(\sigma, \tau - t, \theta_{-\tau} \omega, u_{\tau-t}) = v(\sigma, \tau - t, \theta_{-\tau} \omega, v_{\tau-t}) + z(\theta_{\sigma-\tau} \omega),
\] (91)

and thus \( u_{\tau-t} = v_{\tau-t} + z(\theta_{-\tau} \omega) \). Hence, we can rewritten (90) as follows:

\[
\|u(\sigma, \tau - t, \theta_{-\tau} \omega, u_{\tau-t})\|_{H^s(\mathbb{R}^n)}^2
\leq c e^{-\kappa(\sigma - \tau)} (\|u_{\tau-t}\|_{H^s(\mathbb{R}^n)}^2 + |y(\theta_{-\tau} \omega)|^2)
+ c \int_{-\tau}^{\sigma-\tau} e^{\kappa(\tau + \sigma - \tau)} (1 + G(r + \tau) + |y(\theta_{-\tau} \omega)|^p) dr.
\] (92)

In particular, we let \( \sigma = \tau \) in (92) to see

\[
\|u(\tau, \tau - t, \theta_{-\tau} \omega, u_{\tau-t})\|_{H^s(\mathbb{R}^n)}^2 \leq c e^{-\kappa t} (\|u_{\tau-t}\|_{H^s(\mathbb{R}^n)}^2 + |y(\theta_{-\tau} \omega)|^2) + c R_0(\tau, w),
\] (93)

which tends to \( c R_0(\tau, w) \) as \( t \to +\infty \). Hence, (82) holds true.

It remains to show (83). We deduce from (92) that for all \( \sigma \in [\tau - t, \tau] \) with \( t \geq 0 \),

\[
\|u(\sigma, \tau - t, \theta_{-\tau} \omega, u_{\tau-t})\|_{H^s(\mathbb{R}^n)}^2
\leq c e^{-\kappa(\sigma - \tau + t)} (\|u_{\tau-t}\|_{H^s(\mathbb{R}^n)}^2 + |y(\theta_{-\tau} \omega)|^2) + c |y(\theta_{\sigma-\tau} \omega)|^2
\]
\[ D \in \mathfrak{D}, \text{the first term in the last line of (95) tends to zero as } t \to +\infty \text{ and thus (83) holds true.} \]

3.2. Uniform tail-estimates. The proof of the following auxiliary estimate is similar to Lemma 2.2 and so omitted.

\textbf{Lemma 3.2.} Let (5)-(8) be satisfied. Then for every \( \tau \in \mathbb{R} \), \( \omega \in \Omega \) and \( v_\tau \in H^s(\mathbb{R}^n) \), the derivative of the solution \( v \) of (80) satisfies

\[ ||v_\tau(t, \tau, \omega, v_\tau)||_{H^s}^2 \leq c(1 + ||u(t, \tau, \omega, u_\tau)||_{H^s}^{2p-2} + ||g(t)||^2 + ||v_2(t)||^2 + ||y(\theta_\omega)||^2). \]

\textbf{Lemma 3.3.} Let (5)-(8) and (17) be satisfied. Then for every \( \tau \in \mathbb{R} \), \( \omega \in \Omega \) and \( \mathcal{D} = \{ \mathcal{D}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \in \mathfrak{D} \), the solution of (76) with \( u_{\tau-t} \in \mathcal{D}(\tau-t, \theta_{-\tau} \omega) \) satisfies

\[ \lim_{k,t \to +\infty} \int_{\mathcal{O}_k^t} |u(\tau, \tau-t, \theta_{-\tau} \omega, u_{\tau-t})|^2 dx = 0, \]

\[ \lim_{k,t \to +\infty} \int_{(\mathcal{O}_k^t)^c} \frac{|u(\tau, \tau-t, \theta_{-\tau} \omega, u_{\tau-t})(x) - u(\tau, \tau-t, \theta_{-\tau} \omega, u_{\tau-t})(y)|^2}{|x-y|^{n+2s}} dxdy = 0. \]

\textbf{Proof.} Multiplying (80) by \( \rho_k(x)v \) to find

\[ \frac{d}{dt} \int_{\mathbb{R}^n} \rho_k(x)|v|^2 dx + \int_{\mathbb{R}^n} \rho_k(x)v(-\Delta)^s v dx + \lambda \int_{\mathbb{R}^n} \rho_k(x)|v|^2 dx \]

\[ + \int_{\mathbb{R}^n} \rho_k(x)v(-\Delta)^s v dx \]

\[ = \int_{\mathbb{R}^n} \rho_k(x)f(t, x, u)v dx + \int_{\mathbb{R}^n} \rho_k(x)g(t, x)v dx + (1-\lambda) \int_{\mathbb{R}^n} \rho_k(x)z(\theta_\omega)v dx. \]
By the argument of (40), we have
\begin{align*}
\int_{\mathbb{R}^n} \rho_k(x)v(-\Delta)^sv dx & \geq \frac{1}{4} C(n, s) \frac{d}{dt} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \rho_k(x)|v(x) - v(y)|^2 dx dy \\
& \quad - ck^{-s} \left( \|v\|_{H^s(\mathbb{R}^n)}^2 + \|v_t\|_{H^s(\mathbb{R}^n)}^2 \right).
\end{align*}
(100)

By the argument of (41), we obtain
\begin{equation}
\int_{\mathbb{R}^n} \rho_k(x)v(-\Delta)^sv dx \geq \frac{1}{2} C(n, s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \rho_k(x)|v(x) - v(y)|^2 dx dy - ck^{-s} \|v\|_{H^s(\mathbb{R}^n)}^2.
\end{equation}
(101)

By (5) and (6) and Young’s inequality, the nonlinear term in (99) is bounded by
\begin{align*}
\int_{\mathbb{R}^n} \rho_k(x)f(t, x, u)vdx &= \int_{\mathbb{R}^n} \rho_k(x)f(t, x, u)vdx - \int_{\mathbb{R}^n} \rho_k(x)f(t, x, u)z(\theta_t \omega)dx \\
& \leq \int_{\mathbb{R}^n} \rho_k(x)(-\alpha_1 |u|^p + \psi_1(t, x))dx + \int_{\mathbb{R}^n} \rho_k(x)(\alpha_2 |u|^{p-1} + |\psi_2(t, x)|)z(\theta_t \omega)dx \\
& \leq c \int_{\mathbb{R}^n} \rho_k(x)\left(1 + |\psi_1(t, x)| + |\psi_2(t, x)|^2 + |z(\theta_t \omega)|^p\right)dx.
\end{align*}
(102)

Moreover, we use Young’s inequality to estimate
\begin{align*}
\int_{\mathbb{R}^n} \rho_k(x)g(t, x)vdx + (1 - \lambda) \int_{\mathbb{R}^n} \rho_k(x)z(\theta_t \omega)vdx \\
& \leq \frac{\lambda}{2} \int_{\mathbb{R}^n} \rho_k(x)|v|^2 dx + c \int_{\mathbb{R}^n} \rho_k(x)\left(1 + |g(t, x)|^2 + |z(\theta_t \omega)|^p\right)dx.
\end{align*}
(103)

Substituting (100)-(103) into (99), we see from Lemma 3.2 that
\begin{align*}
\frac{d}{dt} \left( \int_{\mathbb{R}^n} \rho_k(x)|v|^2 dx + \frac{1}{2} C(n, s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \rho_k(x)|v(x) - v(y)|^2 dx dy \right) \\
& \quad + \kappa \left( \int_{\mathbb{R}^n} \rho_k(x)|v|^2 dx + \frac{1}{2} C(n, s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \rho_k(x)|v(x) - v(y)|^2 dx dy \right) \\
& \leq ck^{-s} \left(1 + \|u\|_{H^s(\mathbb{R}^n)}^{2p-2} + \|g(t)\|^2 + \|\psi_2(t)\|^2 + |y(\theta_t \omega)|^2\right) \\
& \quad + c \int_{|x| \geq \frac{1}{2} k} \left(1 + |g(t, x)|^2 + |\psi_1(t, x)| + |\psi_2(t, x)|^2 + |z(\theta_t \omega)|^p\right)dx.
\end{align*}
(104)

Applying Gronwall’s inequality to (104) over ($\tau - t, \tau$) with $t \geq 0$, replacing $\omega$ by $\theta_{-\tau} \omega$ and using (91) with $\sigma = \tau$, we obtain
\begin{align*}
& \int_{\mathbb{R}^n} \rho_k(x)|u(\tau, \tau - t, \theta_{-\tau} \omega, u_{\tau - t})|^2 dx \\
& \quad + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \rho_k(x)|(u(\tau, \tau - t, \theta_{-\tau} \omega, u_{\tau - t}) - u(\tau, \tau - t, \theta_{-\tau} \omega, u_{\tau - t})(y)|^2 dx dy \\
& \leq ce^{-\kappa t} \left( \int_{\mathbb{R}^n} \rho_k(x)|u_{\tau - t}|^2 dx + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \rho_k(x)|u_{\tau - t}(x) - u_{\tau - t}(y)|^2 dx dy \right) \\
& \quad + ce^{-\kappa t} \left( \int_{\mathbb{R}^n} \rho_k(x)|z(\theta_{-\tau} \omega)(x) - z(\theta_{-\tau} \omega)(y)|^2 dx dy \right).
\end{align*}
\[ + c k^{-s} \int_{\tau-t}^{\tau} e^{\kappa(r-\tau)} \| u(r, \tau-t, \theta_{\tau}, \omega, u_{\tau-t}) \|_{H^s(\mathbb{R}_x^n)}^{2p-2} dr \]
\[ + c \left( \int_{\mathbb{R}_x^n} \rho_k(x) |z(\omega)(x)|^2 dx + \int_{\mathbb{R}_x^n} \int_{\mathbb{R}_x^n} \frac{\rho_k(x) |z(\omega)(x) - z(\omega)(y)|^2}{|x - y|^{n+2s}} dxdy \right) \]
\[ + c k^{-s} \int_{-\infty}^{0} e^{\kappa r} \left( 1 + \| g(r+\tau) \|^2 + \| \psi_2(r+\tau) \|^2 + \| y(\theta, \omega) \|^2 \right) dr \]
\[ + c \int_{-\infty}^{0} e^{\kappa r} \int_{|x| \geq \frac{1}{2} k} \left( 1 + |g(r+\tau, x)|^2 + |\psi_1(r+\tau, x)| + |\psi_2(r+\tau, x)|^2 \right) dx dr \]
\[ + c \int_{-\infty}^{0} e^{\kappa r} \int_{|x| \geq \frac{1}{2} k} |z(\theta, \omega)|^p dxdx =: \sum_{i=1}^{3} I_i(k, t) + \sum_{i=4}^{7} I_i(k). \] (105)

By the same methods as in (46) and (46), we can show that \( I_1(k, t) + I_2(k, t) \to 0 \) as \( t \to +\infty \) for all \( k \geq 1 \). By Lemma 3.1, there is a \( T := T(\tau, \omega, D) > 0 \) such that for all \( t \geq T \),
\[ I_3(k, t) \leq c k^{-s} \left( \int_{-\infty}^{0} e^{\kappa r} |y(\theta, \omega)|^{2p-2} dr + \hat{R}^{p-1}_\omega(\tau, \omega) \right) \to 0 \) as \( k \to \infty \). (106)

By \( h \in L^2(\mathbb{R}_x^n) \), we have, as \( k \to \infty \),
\[ I_4(k) \leq c |y(\omega)|^2 \int_{|x| \geq \frac{1}{2} k} |(I + (-\Delta)^s)^{-1} h(x)|^2 dx \]
\[ + c |y(\omega)|^2 \int_{\mathbb{R}_x^n} \int_{|x| \geq \frac{1}{2} k} \frac{|(I + (-\Delta)^s)^{-1} h(x) - (I + (-\Delta)^s)^{-1} h(y)|^2}{|x - y|^{n+2s}} dxdy \to 0. \] (107)

By (17) and (78), we have \( I_5(k) + I_6(k) \to 0 \) as \( k \to \infty \). Finally, we estimate the last term on the right-hand side of (105). By \( h \in L^2(\mathbb{R}_x^n) \),
\[ \int_{\mathbb{R}_x^n} |(I + (-\Delta)^s)^{-1} h(x)|^p dx \leq c \| (I + (-\Delta)^s)^{-1} h \|_{H^s(\mathbb{R}_x^n)}^p \leq c, \] (108)

which together with \( \int_{-\infty}^{0} e^{\kappa r} |y(\theta, \omega)|^p dr < \infty \) implies that as \( k \to \infty \),
\[ I_7(k) \leq c \int_{|x| \geq \frac{1}{2} k} |(I + (-\Delta)^s)^{-1} h(x)|^p dx \int_{-\infty}^{0} e^{\kappa r} |y(\theta, \omega)|^p dr \to 0. \] (109)

We substitute all above estimates into (105) to obtain (97) and (98) as desired. \( \square \)

3.3. Flattening property. Let \( \tilde{u}(t, \tau, \omega, \hat{u}, \tilde{v}) = \xi_k(x) u(t, \tau, \omega, u_{\tau}) \) and \( \hat{v}(t, \tau, \omega, \hat{v}, \tilde{v}) = \xi_k(x) v(t, \tau, \omega, v_{\tau}) \) where \( \xi_k(x) = 1 - \rho(|x|/k) \) with the same \( \rho \) as in Section 2. By (56) and (57), we similarly obtain
\[ (-\Delta)^s \hat{v}(t) = \xi_k(-\Delta)^s v(t)(x)(-\Delta)^s \xi_k(x) \]
\[ + C(n, s) \text{ P.V.} \int_{\mathbb{R}_x^n} \frac{\{\xi_k(x) - \xi_k(y)\} (v(t)(x) - v(t)(y))}{|x - y|^{n+2s}} dy, \] (110)

\[ (-\Delta)^s \hat{v}(x) = \xi_k(-\Delta)^s v(x) \]
\[ + C(n, s) \text{ P.V.} \int_{\mathbb{R}_x^n} \frac{\{\xi_k(x) - \xi_k(y)\} (v(t)(x) - v(t)(y))}{|x - y|^{n+2s}} dy. \] (111)
We multiply (80) by $\xi_k(x)$ and use (110)-(111) to find that $\hat{v}$ satisfies

$$
\hat{v}_t + (-\Delta)^s \hat{v} + (1-\tau_s)^s \hat{v} + \lambda \hat{v} = \xi_k(x)f(t, x, u) + \xi_k(x)g(t, x) + (1 - \lambda)\xi_k(x)z(\theta t, \omega) + v_t(x)(-\Delta)^s \xi_k(x) + C(n, s) P.V. \int_{\mathbb{R}^n} \frac{(\xi_k(x) - \xi_k(y))(v_t(x) - v_t(y))}{|x-y|^{n+2s}}dy 
$$

(112)

\textbf{Lemma 3.4.} Let (5)-(8) and (17) be satisfied. Then for every $k \in \mathbb{N}$, $t \in \mathbb{R}$, $\omega \in \Omega$ and $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, the solution of (76) with $u_{\tau-t} \in D(\tau-t, \theta \omega)$ satisfies

$$
\lim_{m,z \to +\infty} \| (I - P_m)\xi_k u(\tau - t, \theta_{-\tau\omega}, u_{\tau-t}) \|_{H^s(\mathbb{R}^n)} = 0,
$$

where $P_m$ is the canonical projection operator as in Section 2.

\textbf{Proof.} Let $\hat{u}_{m,2} = (I - P_m)\hat{u}$ and $\hat{v}_{m,2} = (I - P_m)\hat{v}$ for each $m \in \mathbb{N}$. We apply $I - P_m$ to (112) and multiply the resulting equation by $\hat{v}_{m,2}$ to yield

$$
\frac{1}{2} \frac{d}{dt} \left( \| \hat{v}_{m,2} \|^2 + \| (\Delta)^{\frac{s}{2}} \hat{v}_{m,2} \|^2 \right) + \lambda \| \hat{v}_{m,2} \|^2 + \| (\Delta)^{\frac{s}{2}} \hat{v}_{m,2} \|^2 
$$

$$
= \int_{\mathbb{R}^n} \xi_k f(t, x, u)\hat{v}_{m,2} dx + (\xi_k g(t, \hat{v}_{m,2}) + (1-\lambda)(\xi_k(x)z(\theta t, \omega), \hat{v}_{m,2}) + v_t(x)(-\Delta)^s \xi_k(x) + C(n, s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(\xi_k(x) - \xi_k(y))(v_t(x) - v_t(y))}{|x-y|^{n+2s}}dy \hat{v}_{m,2} dx 
$$

(113)

As in (62), the nonlinear term in (113) is bounded by

$$
\int_{\mathbb{R}^n} \xi_k f(t, x, u)\hat{v}_{m,2} dx \leq \frac{1}{8}\|(-\Delta)^{\frac{s}{2}} \hat{v}_{m,2} \|^2 + c\lambda_{m+1}^{-1} (\|u\|_{L^{6p-2}(\mathbb{R}^n)}^2 + \|\psi(t)\|^2) 
$$

(114)

By $(-\Delta)^s \xi_k = (-\Delta)^s \rho_k$ and (36), the terms in the second line of (113) are bounded by

$$
(\xi_k g(t, \hat{v}_{m,2}) + (1-\lambda)(\xi_k(x)z(\theta t, \omega), \hat{v}_{m,2}) + v_t(x)(-\Delta)^s \xi_k, \hat{v}_{m,2}) + (v_t(x)(-\Delta)^s \xi_k, \hat{v}_{m,2}) 
$$

$$
\leq c\|\hat{v}_{m,2}\| (\|v_t\| + \|v\| + \|g(t)\| + \|z(\theta t, \omega)\|) 
$$

$$
\leq c\lambda_{m+1}^{-\frac{1}{2}} (\|(-\Delta)^{\frac{s}{2}} \hat{v}_{m,2} \|^2 + \|v_t\|^2 + \|v\|^2 + \|g(t)\|^2 + \|z(\theta t, \omega)\|) 
$$

$$
\leq \frac{1}{8}\|(-\Delta)^{\frac{s}{2}} \hat{v}_{m,2} \|^2 + c\lambda_{m+1}^{-1} (\|v_t\|^2_{L^{6p-2}(\mathbb{R}^n)} + \|v\|^2_{L^{6p-2}(\mathbb{R}^n)} + \|g(t)\|^2 + \|z(\theta t, \omega)\|^2) 
$$

(115)

By the argument of (65), the term in the third line of (113) is estimated by

$$
C(n, s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(\xi_k(x) - \xi_k(y))(v_t(x) - v_t(y))}{|x-y|^{n+2s}}dy \hat{v}_{m,2} dx 
$$

$$
\leq \frac{1}{8}\|(-\Delta)^{\frac{s}{2}} \hat{v}_{m,2} \|^2 + c\lambda_{m+1}^{-1} \|v_t\|^2_{L^{6p-2}(\mathbb{R}^n)}. 
$$

(116)
Similar to (66), the last term in (113) is bounded by
\[
C(n,s) \int \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(\xi_k(x) - \xi_k(y))(v(x) - v(y))}{|x-y|^{n+2s}} \, dy \, dx
\]
\[
\leq \frac{1}{8} \|(-\Delta)^{\frac{1}{2}} \hat{v}_{m,2}\|^2 + c\lambda_{m+1}^{-1} \|u\|^2_{\mathcal{H}^s(\mathbb{R}^n)}. \tag{117}
\]
Substituting (114)-(117) into (113), we see from Lemma 3.2 that
\[
\frac{d}{dt} (\|v_{m,2}\|^2 + \|(-\Delta)^{\frac{1}{2}} v_{m,2}\|^2) + \kappa \left( \|v_{m,2}\|^2 + \|(-\Delta)^{\frac{1}{2}} v_{m,2}\|^2 \right)
\leq c\lambda_{m+1}^{-1} \left( 1 + \|u\|^{2m-2}_{\mathcal{H}^s(\mathbb{R}^n)} + \|g(t)\|^2 + \|\psi_2(t)\|^2 + |y(\theta_t \omega)|^2 \right). \tag{118}
\]
Applying Gronwall’s inequality to (118) over \((\tau - t, \tau)\) with \(t \geq 0\) and replacing \(\omega\) by \(\theta_{-\tau} \omega\), we obtain from (91) that
\[
\|\hat{u}_{m,2}(\tau, \tau - t, \theta_{-\tau} \omega, u_{\tau-t})\|^2 + \|(-\Delta)^{\frac{1}{2}} \hat{u}_{m,2}(\tau, \tau - t, \theta_{-\tau} \omega, u_{\tau-t})\|^2
\leq c \exp(\kappa) \left( \|v_{m,2}(\tau, \tau - t, \theta_{-\tau} \omega, u_{\tau-t})\|^2 + \|(-\Delta)^{\frac{1}{2}} v_{m,2}(\tau, \tau - t, \theta_{-\tau} \omega, u_{\tau-t})\|^2 \right)
\leq c \exp(\kappa) \left( \|v_{m,2}(\tau, \tau - t, \theta_{-\tau} \omega, u_{\tau-t})\|^2 + \|(-\Delta)^{\frac{1}{2}} v_{m,2}(\tau, \tau - t, \theta_{-\tau} \omega, u_{\tau-t})\|^2 \right)
\]
\[
+ \int_{\tau-t}^{\tau} e^{\kappa(r-\tau)} \left( \|u(r, \tau - t, \theta_{-\tau} \omega, u_{\tau-t})\|^{2m-2}_{\mathcal{H}^s(\mathbb{R}^n)} \right) dr.
\tag{119}
\]
By (78), (83) and (119), we complete the proof.

3.4. Existence of pullback random attractors. Now, we are in the position to present the main results of this section.

**Theorem 3.5.** Let (5)-(8) and (17)-(18). Then the cocycle \(\Phi_0\) associated with problem (76) has a unique \(\mathcal{D}\)-pullback random attractor \(\mathcal{A}_0 = \{A_0(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}\) in \(H^s(\mathbb{R}^n)\).

**Proof.** As in Lemma 2.5, we infer from Lemma 3.1 that the cocycle \(\Phi_0\) has a closed \(\mathcal{F}\)-measurable \(\mathcal{D}\)-pullback absorbing set \(K_0 = \{K_0(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}\) given by
\[
K_0(\tau, \omega) = \{u \in H^s(\mathbb{R}^n) : \|u\|^2_{H^s(\mathbb{R}^n)} \leq M_1 R_0(\tau, \omega), \quad (\tau, \omega) \in \mathbb{R} \times \Omega\}, \tag{120}
\]
where \(M_1\) and \(R_0(\tau, \omega)\) are the same as in Lemma 3.1.

By Lemmas 3.3 and 3.4, we can use the same method as in Lemma 2.6 to prove that \(\Phi_0\) is \(\mathcal{D}\)-pullback asymptotically compact in \(H^s(\mathbb{R}^n)\). Therefore, the existence and uniqueness of pullback random attractors of \(\Phi_0\) follows from the abstract results given in [49] and [51] immediately.

4. Convergence of random attractors from colored noise to white noise. In this section, we approximate the random attractor \(\mathcal{A}_0\) of the stochastic equation (76) by the random attractor \(\mathcal{A}_\delta\) of the following pathwise random equation:
\[
\begin{cases}
u_t + (-\Delta)^{\delta} u_t + (-\Delta)^{\delta} u + \lambda u = f(t, x, u) + g(t, x) + h(x) \zeta_\delta(\theta_t \omega), & u(\tau, x) = u_\tau(x), \quad x \in \mathbb{R}^n, \quad t > \tau, \quad \tau \in \mathbb{R}.
\end{cases}
\tag{121}
\]
Now we write the solution of (121) as $u_\delta$ to indicate the dependence of the solution on the parameter $\delta$. Note that (121) is a special case of (4). Therefore, the continuous cocycle $\Phi_\delta$ associated with (121) in $H^s(\mathbb{R}^n)$ has a unique $\mathcal{D}$-pullback random attractor $A_\delta$. Next we introduce a transformation:

$$v_\delta(t, \tau, \omega, v_{\delta, \tau}) = u_\delta(t, \tau, \omega, u_{\delta, \tau}) - z_\delta(\theta_t \omega)$$ with $v_{\delta, \tau} = u_{\delta, \tau} - z_\delta(\theta_\tau \omega),$ \hspace{1cm} (122)

where $z_\delta(\theta_t \omega) = (I + (-\Delta)^s)^{-1} h_y(\theta_t \omega)$ with $y_\delta(\theta_t \omega) = e^{-t} \int_0^t e^{s} \zeta_\delta(\theta_s \omega) ds$ being the stationary solution of the random equation: $dy + ydt = \zeta(\theta_t \omega)$. By [26], we have

$$\lim_{t \to \pm \infty} \sup_{\delta \in (0, \frac{1}{2}] \in (0, \frac{1}{2}]} \frac{|y_\delta(\theta_t \omega)|}{T} = 0$$ for every $\omega \in \Omega,$ \hspace{1cm} (123)

$$\lim_{\delta \to 0} \sup_{\delta \in (0, \frac{1}{2}]} |y_\delta(\theta_t \omega) - y(\theta_t \omega)| = 0$$ for every $T > 0$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$, \hspace{1cm} (124)

where $y(\theta_t \omega) = - \int_{-\infty}^0 e^{s}(\theta_{s+}\omega)(s)ds$ is the stationary solution of the one-dimensional Ornstein-Uhlenbeck equation $dy + ydt = dW(t)$. We infer from (121)-(122) that

$$\begin{cases}
\frac{d}{dt}(v_\delta + (-\Delta)^s v_\delta) + (-\Delta)^s v_\delta + \lambda v_\delta = f(t, u_\delta) + g(t) + (1 - \lambda) z_\delta(\theta_t \omega), \\
v_\delta(\tau, x) = v_{\delta, \tau}(x) = u_{\delta, \tau}(x) - z_\delta(\theta_\tau \omega), \ x \in \mathbb{R}^n, \ t > \tau, \ \tau \in \mathbb{R}.
\end{cases}$$ \hspace{1cm} (125)

4.1. Convergence of the absorbing radii.

Lemma 4.1. Assume (5)-(8) and (17) hold. Then for every $\delta \in (0, \frac{1}{2}], \tau \in \mathbb{R}$, $\omega \in \Omega$ and $\mathcal{D} = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, there exists $T := T(\delta, \tau, \omega, \mathcal{D}) > 0$ such that for all $t \geq T$, the solution of (121) with $u_{\delta, \tau-t} \in D(\tau - t, \theta_{-\tau} \omega)$ satisfies

$$\|u_\delta(\tau, \tau - t, \theta_{-\tau} \omega, u_{\delta, \tau-t})\|^2_{H^s(\mathbb{R}^n)} \leq M_2 R_\delta(\tau, \omega),$$

where $M_2$ is a constant independent of $\delta$, $\tau$, $\omega$ and $\mathcal{D}$, and $R_\delta(\tau, \omega)$ is given by

$$R_\delta(\tau, \omega) = |y_\delta(\omega)|^2 + \int_{-\infty}^{0} e^{\kappa(r+\tau)}(1 + G(r + \tau) + |y_\delta(\theta_r \omega)|^p)dr.$$ \hspace{1cm} (126)

Proof. By formally repeating the procedure of deriving (89), we obtain

$$\frac{d}{dt} \left(\|v_\delta\|^2 + \|(-\Delta)^s v_\delta\|^2\right) \leq \kappa \left(\|v_\delta\|^2 + \|(-\Delta)^s v_\delta\|^2\right) \leq c(1 + G(t) + |y_\delta(\theta_t \omega)|^p).$$ \hspace{1cm} (127)

Multiplying (127) by $e^\kappa t$ and integrating over $(\tau - t, \sigma)$ with $\sigma \geq \tau - t$ and $t \geq 0$, then we replace $\omega$ by $\theta_{-\tau} \omega$ to find

$$\|u_\delta(\sigma, \tau - t, \theta_{-\tau} \omega, v_{\delta, \tau-t})\|^2_{H^s} \leq c e^{-\kappa(\sigma - t + \tau)} \|v_{\delta, \tau-t}\|^2_{H^s} + c \int_{\tau-t}^{\sigma - \tau} e^{\kappa(r + t - \sigma)} (1 + G(r + \tau) + |y_\delta(\theta_r \omega)|^p)dr.$$ \hspace{1cm} (128)

It yields from (122) that for all $\sigma \geq \tau - t$ with $t \geq 0$,

$$u_\delta(\sigma, \tau - t, \theta_{-\tau} \omega, u_{\delta, \tau-t}) = v_\delta(\sigma, \tau - t, \theta_{-\tau} \omega, v_{\delta, \tau-t}) + z_\delta(\theta_{\tau-\sigma} \omega), \ u_{\delta, \tau-t} = v_{\delta, \tau-t} + z_\delta(\theta_{\tau-\omega}).$$ \hspace{1cm} (129)
Therefore we see from (128) and (129) that for all \( \sigma \geq \tau - t, \)
\[
\| u_\delta(\sigma, \tau - t, \theta_\tau \omega, u_{\delta, \tau - t}) \|_H^2 \\
\leq c e^{-\kappa (\sigma - t)} (\| u_{\delta, \tau - t} \|_H^2 + | y_\delta(\theta_\tau \omega) |^2 ) + c | y_\delta(\theta_\tau \omega) |^2 \\
+ c \int_{-t}^{\tau - \sigma} e^{\kappa (r+\tau-\sigma)} (1 + G(r+\tau) + | y_\delta(\theta_r \omega) |^p) dr.
\] (130)

In particular, we let \( \sigma = \tau \) in (130) to see
\[
\| u_\delta(\tau, \tau - t, \theta_\tau \omega, u_{\delta, \tau - t}) \|_H^2 \\
\leq c e^{-\kappa t} (\| u_{\delta, \tau - t} \|_H^2 + | y_\delta(\theta_\tau \omega) |^2 ) + c R_\delta(\tau, \omega),
\] (131)
where \( R_\delta(\tau, \omega) \) is well-defined due to (123) and (17). By (123) and \( u_{\delta, \tau - t} \in D(\tau - t, \theta_\tau \omega) \) with \( D \in \mathcal{D} \), we have, for every \( \delta \in (0, \frac{1}{2}] \), as \( t \to +\infty, \)
\[
e^{-\kappa t} (\| u_{\delta, \tau - t} \|_H^2 + | y_\delta(\theta_\tau \omega) |^2 ) \leq e^{-\kappa t} (\| D(\tau - t, \theta_\tau \omega) \|_H^2 + | y_\delta(\theta_\tau \omega) |^2 ) \to 0,
\]
which together with (131) implies the desired result.

By Lemma 4.1, we establish the existence of \( \mathcal{D} \)-pullback absorbing set \( \mathcal{K}_\delta \) for \( \Phi_\delta \). Furthermore, we show the convergence of the absorbing radii of \( \mathcal{K}_\delta \) as \( \delta \to 0 \).

**Lemma 4.2.** Let (5)-(8) and (17)-(18) be satisfied. Then the cocycle \( \Phi_\delta \) for problem (121) has a closed \( \mathcal{F} \)-measurable \( \mathcal{D} \)-pullback absorbing set \( \mathcal{K}_\delta = \{ \mathcal{K}_\delta(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \in \mathcal{D} \) given by

\[
\mathcal{K}_\delta(\tau, \omega) = \{ u \in H^s(\mathbb{R}^n) : \| u \|_{H^s(\mathbb{R}^n)}^2 \leq M_2 R_\delta(\tau, \omega) \} \quad \text{for every } \tau \in \mathbb{R}, \omega \in \Omega.
\] (132)

where \( M_2 \) and \( R_\delta(\tau, \omega) \) are the same as in Lemma 4.1. Moreover, we have

\[
\lim_{\delta \to 0} R_\delta(\tau, \omega) = R_0(\tau, \omega), \quad \forall \tau \in \mathbb{R}, \omega \in \Omega, \quad \text{where } R_0(\tau, \omega) \text{ is given by (84)}.
\] (133)

**Proof.** By the argument of Lemma 2.5, we can show that \( \mathcal{K}_\delta \) is a closed \( \mathcal{F} \)-measurable \( \mathcal{D} \)-pullback absorbing set for \( \Phi_\delta \). We only need to show (133). By (124), we have

\[
\lim_{\delta \to 0} | y_\delta(\theta_r \omega) |^m = | y(\theta_r \omega) |^m \quad \text{for each fixed } m \geq 1, r \in \mathbb{R} \text{ and } \omega \in \Omega.
\] (134)

It follows from (123) that there exists \( r_0 = r_0(\tau, \omega) > 0 \) such that

\[
e^{s r} | y_\delta(\theta_r \omega) |^p \leq e^{s r} | r |^p \quad \text{for all } \delta \in (0, \frac{1}{2}] \text{ and } r \leq -r_0,
\]
which along with \( \int_{-\infty}^{r_0} e^{s r} | r |^p dr < \infty \) and (134) with \( m = p \) implies that

\[
\lim_{\delta \to 0} \int_{-\infty}^{r_0} e^{s r} | y_\delta(\theta_r \omega) |^p dr = \int_{-\infty}^{r_0} e^{s r} | y(\theta_r \omega) |^p dr.
\] (135)

By (124) and the continuity of the mapping \( r \to y_\delta(\theta_r \omega) \), we have

\[
\lim_{\delta \to 0} \sup_{r \in [-r_0, 0]} | y_\delta(\theta_r \omega) |^p = | y(\theta_r \omega) |^p \quad \text{for every } \omega \in \Omega.
\]

Then one can verify

\[
\lim_{\delta \to 0} \int_{-r_0}^{0} e^{s r} | y_\delta(\theta_r \omega) |^p dr = \int_{-r_0}^{0} e^{s r} | y(\theta_r \omega) |^p dr.
\] (136)
Thus by (135)-(136), we obtain
\[
\lim_{\delta \to 0} \int_{-\infty}^{0} e^{r\tau} |y_\delta(\theta, \omega)|^p \, dr = \lim_{\delta \to 0} \int_{-\infty}^{-\infty} e^{r\tau} |y_\delta(\theta, \omega)|^p \, dr + \lim_{\delta \to 0} \int_{-\infty}^{0} e^{r\tau} |y_\delta(\theta, \omega)|^p \, dr \\
= \int_{-\infty}^{-\infty} e^{r\tau} |y(\theta, \omega)|^p \, dr + \int_{-\infty}^{0} e^{r\tau} |y(\theta, \omega)|^p \, dr = \int_{-\infty}^{0} e^{r\tau} |y(\theta, \omega)|^p \, dr.
\]
(137)

Then (133) follows from (137) and (134) with \( m = 2 \) and \( r = 0 \).

4.2. Convergence of solutions.

Lemma 4.3. Suppose (5)-(8) hold and \( \delta_n \to 0 \). Let \( u_{\delta_n} \) and \( u \) be the solutions of (76) and (121) with initial dates \( u_{\delta_n, \tau} \) and \( u, \) respectively. If \( u_{\delta_n, \tau} \to u_{\tau} \) in \( H^s(\mathbb{R}^n) \), then
\[
\lim_{n \to \infty} \sup_{t \in [\tau, \tau + T]} \|u_{\delta_n}(t, \tau, \omega, u_{\delta_n, \tau}) - u(t, \tau, \omega, u_{\tau})\|_{H^s} = 0, \quad \text{for each } T > 0, \omega \in \Omega.
\]

Proof. Let \( \tau \in \mathbb{R}, T > 0 \) and \( \omega \in \Omega \) be fixed. For \( t \in [\tau, \tau + T] \) and \( \delta \in (0, \frac{1}{2}] \), denote by
\[
U_\delta(t) := u_\delta(t, \tau, \omega, u_{\delta, \tau}) - u(t, \tau, \omega, u_{\tau}) \quad \text{and} \quad V_\delta(t) := v_\delta(t, \tau, \omega, u_{\delta, \tau}) - v(t, \tau, \omega, v_{\tau}).
\]

We deduce from (80) and (125) that
\[
\frac{1}{2} \frac{d}{dt} (|V_\delta|^2 + \|(-\Delta)^{\frac{1}{2}} V_\delta\|^2 + \lambda |V_\delta|^2 + \|(-\Delta)^{\frac{1}{2}} V_\delta\|^2)
\]
\[
= \int_{\mathbb{R}^n} (f(t, x, v_\delta + z_\delta(\theta t \omega)) - f(t, x, v + z(\theta t \omega))) V_\delta \, dx
\]
\[
+ (1 - \lambda) \int_{\mathbb{R}^n} (z_\delta(\theta t \omega) - z(\theta t \omega)) V_\delta \, dx.
\] (138)

Let \( \tilde{f}(t, x, s) = \frac{\partial}{\partial \tau} f(t, x, s) \), there exists \( \xi \) between \( v_\delta + z_\delta(\theta t \omega) \) and \( v + z(\theta t \omega) \) such that
\[
\int_{\mathbb{R}^n} \left( f(t, x, v_\delta + z_\delta(\theta t \omega)) - f(t, x, v + z(\theta t \omega)) \right) V_\delta \, dx
\]
\[
\leq \int_{\mathbb{R}^n} \tilde{f}(t, x, \xi) |V_\delta|^2 + \int_{\mathbb{R}^n} |z_\delta(\theta t \omega) - z(\theta t \omega)| |\tilde{f}(t, x, \xi)| |V_\delta| \, dx \quad \text{by (7)-(8)}
\]
\[
\leq \|\psi_\delta(t)\|_{L^\infty} \|V_\delta\|^2 + \|\psi_\delta(t)\|_{L^\infty} \int_{\mathbb{R}^n} |z_\delta(\theta t \omega) - z(\theta t \omega)| (1 + |\xi|^{p-2}) (|v_\delta| + |v|) \, dx
\]
\[
\leq \|\psi_\delta(t)\|_{L^\infty} \|V_\delta\|^2 + c \|\psi_\delta(t)\|_{L^\infty} \int_{\mathbb{R}^n} |z_\delta(\theta t \omega) - z(\theta t \omega)| (|v_\delta| + |v|)
\]
\[
+ |v_\delta|^{p-1} + |v|^{p-1} + |z(\theta t \omega)|^{p-1} + |z(\theta t \omega)|^{p-1} \, dx
\]
\[
\leq \|\psi_\delta(t)\|_{L^\infty} \|V_\delta\|^2 + c \|\psi_\delta(t)\|_{L^\infty} \|y_\delta(\theta t \omega) - y(\theta t \omega)\| \|((I + (-\Delta)^{s})^{-1} h\|^2 + |v_\delta|^2
\]
\[
+ |v|^2 + |v_\delta|^{2p-2} + |v|^{2p-2} + |z(\theta t \omega)|^{2p-2} + |z(\theta t \omega)|^{2p-2} \quad \text{(by (13))}
\]
\[
\leq \|\psi_\delta(t)\|_{L^\infty} \|V_\delta\|^2 + c \|\psi_\delta(t)\|_{L^\infty} \|y_\delta(\theta t \omega) - y(\theta t \omega)\| (1 + \|v_\delta\|_{H^s}^{2p-2} + \|v\|_{H^s}^{2p-2})
\]
\[
+ |y(\theta t \omega)|^{2p-2} + |y_\delta(\theta t \omega)|^{2p-2} \quad \text{(139)}
\]

Moreover, we use Young’s inequality to estimate
\[
(1 - \lambda) \int_{\mathbb{R}^n} (z_\delta - z) V_\delta \, dx \leq c |y_\delta(\theta t \omega) - y(\theta t \omega)| (1 + \|v_\delta\|_{H^s}^{2p-2} + \|v\|_{H^s}^{2p-2}).
\] (140)
By (124) and the continuity of \(y(\theta_t \omega)\) on \([\tau, \tau + T]\), we find that for every \(\varepsilon \in (0, 1)\), there exist \(\delta_0 = \delta_0(\varepsilon, \tau, \omega, T) \in (0, \frac{1}{2}]\) and \(c = c(\tau, \omega, T) > 0\) such that for all \(t \in [\tau, \tau + T]\) and \(\delta \in (0, \delta_0)\),

\[
|y_\delta(\theta_t \omega) - y(\theta_t \omega)| \leq \varepsilon \quad \text{and} \quad |y_\delta(\theta_t \omega)| \leq |y_\delta(\theta_t \omega) - y(\theta_t \omega)| + |y(\theta_t \omega)| \leq c.
\] (141)

Thus we derive from (138)-(141) that for all \(t \in [\tau, \tau + T]\) and \(\delta \in (0, \delta_0)\),

\[
\frac{d}{dt} \|V_\delta\|_{H^s}^2 \leq c\|\nabla y_\delta\|_{H^s}^2 + \varepsilon c \left( 1 + \|\nabla y_\delta\|_{H^s}^{2p-2} + \|\nabla y(\nabla)\|_{H^s}^{2p-2} \right).
\] (142)

Then Gronwall’s inequality shows that for all \(t \in [\tau, \tau + T]\) and \(\delta \in (0, \delta_0)\),

\[
\|V_\delta(t)\|_{H^s}^2 \leq e^{cT} \left( \|V_\delta(\tau)\|_{H^s}^2 + \varepsilon c \int_\tau^{\tau + T} \left( 1 + \|\nabla y_\delta(\sigma)\|_{H^s}^{2p-2} + \|\nabla y(\sigma)\|_{H^s}^{2p-2} \right) d\sigma \right).
\] (143)

Now we multiply (127) by \(e^{\varepsilon t}\) and integrate over \((\tau, \sigma)\) with \(\sigma \in [\tau, \tau + T]\) to see

\[
\|v_\delta(\sigma, \tau, \omega, v_{\delta, \tau})\|_{H^s}^2 \leq ce^{-\kappa(\sigma-\tau)}\|v_\delta(\tau, \omega)\|_{H^s}^2 + c \int_\tau^{\sigma} e^{\kappa(\tau-\sigma)} \left( 1 + G(r) + |y_\delta(\theta_r \omega)|^p \right) dr,
\] (144)

which along with (141) implies that for all \(\delta \in (0, \delta_0)\),

\[
\int_\tau^{\tau + T} \|v_\delta(\sigma, \tau, \omega, v_{\delta, \tau})\|_{H^s}^{2p-2} d\sigma \\
\leq c\|v_\delta(\tau, \omega)\|_{H^s}^{2p-2} + c \left( \int_\tau^{\tau + T} \left( 1 + G(r) + |y_\delta(\theta_r \omega)|^p \right) dr \right)^{p-1} \leq c\|v_{\delta, \tau}\|_{H^s}^{2p-2} + c.
\] (145)

Based on the energy inequality (89), it is similar to show

\[
\int_\tau^{\tau + T} \|v(\sigma, \tau, \omega, v_{\tau})\|_{H^s}^{2p-2} d\sigma \leq c\|v_{\tau}\|_{H^s}^{2p-2} + c \left( \int_\tau^{\tau + T} \left( 1 + G(r) + |y(\theta_r \omega)|^p \right) dr \right)^{p-1} \leq c\|v_{\tau}\|_{H^s}^{2p-2} + c.
\] (146)

Thus it follows from (143) and (145)-(146) that for all \(t \in [\tau, \tau + T]\) and \(\delta \in (0, \delta_0)\),

\[
\|V_\delta(t)\|_{H^s}^2 \leq c\|v_{\delta, \tau} - v_{\tau}\|_{H^s}^2 + \varepsilon c \left( 1 + \|v_{\delta, \tau}\|_{H^s}^{2p-2} + \|v_{\tau}\|_{H^s}^{2p-2} \right),
\] (147)

which along with (77) and (122) implies, in view of (141), that for all \(t \in [\tau, \tau + T]\) and \(\delta \in (0, \delta_0)\),

\[
\|U_\delta(t)\|_{H^s} \leq c\|u_{\delta, \tau} - u_{\tau}\|_{H^s}^2 + \varepsilon c \left( 1 + \|u_{\delta, \tau}\|_{H^s}^{2p-2} + \|u_{\tau}\|_{H^s}^{2p-2} \right).
\]

This completes the proof. \(\square\)

4.3. **Uniform compactness of random attractors.** We first derive the following uniform estimates with respect to small \(\delta\).

**Lemma 4.4.** Let (5)-(8) and (17) be satisfied. Then for every \(\tau \in \mathbb{R}\) and \(\omega \in \Omega\), there exist \(\delta_0 = \delta_0(\omega) > 0\) and \(T := T(\tau, \omega) > 0\) (independent of \(\delta\)) such that for
all \( \delta \in (0, \delta_0) \), \( t \geq T \) and \( u_{\delta, \tau-t} \in K_\delta(\tau-t, \theta-t) \),

\[
\int_{\tau-t}^{\tau} e^{\kappa(r-\tau)} \| u_{\delta}(r, \tau-t, \theta-t, u_{\delta, \tau-t}) \|^2_{H^s} dr \\
\leq c \int_{-\infty}^{\delta} e^{\kappa \sigma} |y(\theta_{\omega})|^{2p-2} d\sigma + c\tilde{R}_\delta^{p-1}(\tau, \omega) + c, \tag{148}
\]

where \( K_\delta \) is given by (132) and \( \tilde{R}_\delta(\tau, \omega) \) is defined by

\[
\tilde{R}_\delta(\tau, \omega) = \int_{-\infty}^{\delta} e^{\frac{1}{2} \kappa r} (G(r+\tau) + |y_\delta(\theta, \omega)|^p) dr. \tag{149}
\]

**Proof.** We deduce from (130) that for all \( \sigma \in [\tau-t, \tau] \) with \( t \geq 0 \),
\[
\| u_{\delta}(\sigma, \tau-t, \theta-t, u_{\delta, \tau-t}) \|^2_{H^s} \\
\leq ce^{-\kappa (\sigma-r-t)} \left( \| u_{\delta, \tau-t} \|^2_{H^s} + |y_\delta(\theta-t)|^2 \right) + c|y_\delta(\theta_{\omega})|^2 \\
+ c \int_{\tau-t}^{\sigma-t} e^\kappa (r+\tau-\sigma) (1 + G(r+\tau) + |y_\delta(\theta, \omega)|^p) dr \\
\leq ce^{-\kappa (\sigma-r-t)} \left( \| u_{\delta, \tau-t} \|^2_{H^s} + |y_\delta(\theta-t)|^2 \right) + c|y_\delta(\theta_{\omega})|^2 \\
+ ce^{-\frac{1}{p} \kappa (\sigma-r-t)} \tilde{R}_\delta(\tau, \omega) + c. \tag{150}
\]

By the integrals of the \( p-1 \)-power of (150) in \( \sigma \in [\tau-t, \tau] \),
\[
\int_{\tau-t}^{\sigma-t} e^{\kappa (\sigma-r-t)} \| u_{\delta}(\sigma, \tau-t, \theta-t, u_{\delta, \tau-t}) \|^2_{H^s} d\sigma \\
\leq ce^{-\kappa t} \left( \| u_{\delta, \tau-t} \|^2_{H^s} + |y_\delta(\theta-t)|^2 \right) \\
+ c \int_{-\infty}^{0} e^{\kappa \sigma} |y(\theta_{\omega})|^{2p-2} d\sigma + c\tilde{R}_\delta^{p-1}(\tau, \omega) + c. \tag{151}
\]

By (123) and (124), there exist \( \delta_1 > 0 \) and \( C_0(\omega) > 0 \) such that
\[
|y_\delta(\theta, \omega)| \leq -t + C_0(\omega) \quad \text{for all} \quad t \leq 0 \quad \text{and} \quad \delta \in (0, \delta_0). \tag{152}
\]

By (152), we have, for every \( m_1, m_2, m_3 \geq 1 \),
\[
\lim_{t \to +\infty} e^{-\kappa t} \sup_{\delta \in (0, \delta_0)} |y_\delta(\theta-t, \omega)|^{m_1} = 0 \quad \text{and} \quad \int_{-\infty}^{0} e^{\frac{1}{2} \kappa r} \sup_{\delta \in (0, \delta_0)} |y_\delta(\theta, \omega)|^{m_3} dr < \infty. \tag{153}
\]

W see from (17) and (153) with \( m_1 = 2p-2 \) and \( m_2 = m_3 = p \) that for all \( \delta \in (0, \delta_0) \),
\[
ce^{-\kappa t} \left( \| u_{\delta, \tau-t} \|^2_{H^s} + |y_\delta(\theta-t, \omega)|^{2p-2} \right) \\
\leq ce^{-\kappa t} \left( \| K_\delta(\tau-t, \theta-t) \|^2_{H^s} + |y_\delta(\theta-t)|^{2p-2} \right) \\
\leq ce^{-\kappa t} \sup_{\delta \in (0, \delta_0)} |y_\delta(\theta-t, \omega)|^{2p-2} + ce^{-\frac{1}{p} \kappa t} (\int_{-\infty}^{t} e^{\frac{1}{p} \kappa r} \sup_{\delta \in (0, \delta_0)} |y_\delta(\theta, \omega)|^p dr)^{p-1} \\
+ ce^{-\frac{1}{p} \kappa t} (\int_{-\infty}^{t} e^{\frac{1}{p} \kappa r} (1 + G(r+\tau)) dr)^{p-1} \to 0. \tag{154}
\]

as \( t \to +\infty \). Then (148) follows from (151) and (154). \( \square \)

We then derive the uniform tail-estimates of solutions of (125) with respect to small \( \delta \).
Lemma 4.5. Suppose (5)-(8) and (17) hold. Then for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$, there exists $\delta_0 > 0$ such that for all $u_{\delta, \tau - t} \in \mathcal{K}_\delta(\tau - t, \theta - t\omega)$,

$$\lim_{k,t \to +\infty} \sup_{\delta \in (0, \delta_0)} \int_{C^\delta_k} |u_\delta(\tau, \tau - t, \theta - t\omega, u_{\delta, \tau - t})|^2\ dx = 0,$$

$$\lim_{k,t \to +\infty} \sup_{\delta \in (0, \delta_0)} \int_{(O^\delta_k)^c} \frac{|u_\delta(\tau, \tau - t, \theta - t\omega, u_{\delta, \tau - t}) - u_\delta(\tau, \tau - t)(y)|^2}{|x - y|^{n+2s}}
\cdot \ dx\ dy = 0.$$

Proof. Repeating the procedure of deriving (104), we have

$$\frac{d}{dt} \left( \int_{\mathbb{R}^n} \rho_k(x)|v_\delta|^2\ dx + \frac{1}{2} C(n, s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\rho_k(x)(v_\delta(x) - v_\delta(y))^2}{|x - y|^{n+2s}}\ dxdy \right)$$

$$+ \kappa \left( \int_{\mathbb{R}^n} \rho_k(x)|v_\delta|^2\ dx + \frac{1}{2} C(n, s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\rho_k(x)(v_\delta(x) - v_\delta(y))^2}{|x - y|^{n+2s}}\ dxdy \right)$$

$$\leq ck^{-s} (||u_\delta||_{H^{2p-2}}^2 + 1 + ||g(t)||^2 + ||\psi_2(t)||^2 + |y_\delta(\theta t\omega)|^2)$$

$$+ c \int_{|x| \geq \frac{\delta}{2}} \left( 1 + |g(t, x)|^2 + |\psi_1(t, x)| + |\psi_2(t, x)|^2 + |z_\delta(\theta t\omega)|^p \right) dx.$$

(157)

Applying Gronwall’s inequality to (157) over $(\tau - t, \tau)$ and replacing $\omega$ by $\theta - t\omega$ to obtain

$$\int_{\mathbb{R}^n} \rho_k(x)|v_\delta(\tau, \tau - t, \theta - t\omega, u_{\delta, \tau - t})|^2\ dx$$

$$+ \frac{1}{2} C(n, s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\rho_k(x)(v_\delta(\tau, \tau - t, \theta - t\omega, u_{\delta, \tau - t}) - v_\delta(\tau, \tau - t)(y))^2}{|x - y|^{n+2s}}\ dxdy$$

$$\leq e^{-\kappa t} \left( \int_{\mathbb{R}^n} \rho_k(x)|v_\delta, \tau - t\omega(\tau - t, \theta - t\omega, u_{\delta, \tau - t})|^2\ dx + 1 + \frac{1}{2} C(n, s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\rho_k(x)(v_\delta, \tau - t\omega(\tau - t, \theta - t\omega, u_{\delta, \tau - t}) - v_\delta(\tau, \tau - t)(y))^2}{|x - y|^{n+2s}}\ dxdy \right)$$

$$+ ck^{-s} \int_{\tau - t}^\tau e^{\kappa(r - \tau)} ||u_\delta(r, \tau - t, \theta - t\omega, u_{\delta, \tau - t})||_{H^{2p-2}}^2 \ dr$$

$$+ ck^{-s} \int_0^\infty e^{\kappa r} \left( 1 + |g(r + \tau)|^2 + ||\psi_2(r + \tau)||^2 + |y_\delta(\theta r\omega)|^2 \right) \ dr$$

$$+ c \int_{-\infty}^0 e^{\kappa r} \left( 1 + |g(r + \tau)|^2 + |\psi_1(r + \tau)| + |\psi_2(r + \tau)|^2 + |z_\delta(\theta r\omega)|^p \right),$$

(158)

which along with (129) with $\sigma = \tau$ further implies

$$\int_{\mathbb{R}^n} \rho_k(x)|u_\delta(\tau, \tau - t, \theta - t\omega, u_{\delta, \tau - t})|^2\ dx$$

$$+ \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \rho_k(x)(u_\delta(\tau, \tau - t, \theta - t\omega, u_{\delta, \tau - t}) - u_\delta(\tau, \tau - t)(y))^2\ dxdy$$

$$\leq ce^{-\kappa t} \left( \int_{\mathbb{R}^n} \rho_k(x)|u_\delta, \tau - t\omega(\tau - t, \theta - t\omega, u_{\delta, \tau - t})|^2\ dx + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \rho_k(x)(u_\delta, \tau - t\omega(\tau - t, \theta - t\omega, u_{\delta, \tau - t}) - u_\delta(\tau, \tau - t)(y))^2\ dxdy \right)$$

$$+ ce^{-\kappa t} \left( \int_{\mathbb{R}^n} \rho_k|z_\delta(\theta - t\omega)|^2\ dx + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \rho_k(x)|z_\delta(\theta - t\omega)(x) - z_\delta(\theta - t\omega)(y)|^2\ dx\ dy \right).$$
Next we show that every term on the right-hand side of (159) tends to zero as \( t, k \to +\infty \) uniformly for small \( \delta \). Let \( \delta_0 \) be the number as in Lemma 4.4. Then by (132) and \( u_{\delta, t-\tau} \in K_{\delta}(\tau-t,\theta_{t-\omega}), \) we see from (17) and (153) with \( m_1 = 2 \) and \( m_2 = m_3 = p \) that for all \( \delta \in (0, \delta_0], as \( t \to +\infty, \)

\[
e^{-\kappa t} \left( \int_{\mathbb{R}^n} \rho_k(x) |u_{\delta, t-\tau}(x) - u_{\delta, t-\tau}(y)|^2 \, dx \right) \\
\leq c e^{-\kappa t} \| u_{\delta, t-\tau} \|^2_{H^p(\mathbb{R}^n)} \\
\leq c e^{-\kappa t} \sup_{\delta \in (0, \delta_0]} \| y_{\delta}(\theta_{t-\omega}) \|^2 + c \int_{-\infty}^{-t} e^{\frac{\kappa \tau}{2}} \sup_{\delta \in (0, \delta_0]} \| y_{\delta}(\theta_{t-\omega}) \|^2 \, dt \\
+ c \int_{-\infty}^{-t} e^{\frac{\kappa \tau}{2}} \left( 1 + \| g(r + \tau) \|^2 + \| \psi_1(r + \tau) \|^2 + \| \psi_2(r + \tau) \|^2 \right) \, dt 
\]

(160)

It yields from (153) with \( m_1 = 2 \) that for all \( \delta \in (0, \delta_0], \)

\[
e^{-\kappa t} \int_{\mathbb{R}^n} \rho_k(x) |u_{\delta, t-\tau}(x) - u_{\delta, t-\tau}(y)|^2 \, dx \]

\[
\leq c e^{-\kappa t} \| y_{\delta}(\theta_{t-\omega}) \|^2 \to 0 as \( t \to +\infty. \) \]

(161)

It follows from (152) with \( t = 0 \) that for all \( \delta \in (0, \delta_0], as \( k \to \infty, \)

\[
\int_{\mathbb{R}^n} \rho_k(x) |z_{\delta}(\omega)|^2 \, dx + \frac{1}{2} C(n, s) \int_{\mathbb{R}^n} \rho_k(x) |z_{\delta}(\omega)(x) - z_{\delta}(\omega)(y)|^2 \, dx \]

\[
\leq c \sup_{\delta \in (0, \delta_0]} |y_{\delta}(\omega)|^2 \int_{|x| \geq \frac{1}{k}} |(I + (-\Delta)^s)^{-1} h(x)|^2 \, dx \\
+ c \sup_{\delta \in (0, \delta_0]} |y_{\delta}(\omega)|^2 \int_{\mathbb{R}^n} \int_{|x| \geq \frac{1}{k}} |(I + (-\Delta)^s)^{-1} h(x) - (I + (-\Delta)^s)^{-1} h(y)|^2 \, dx \, dy,
\]

(162)

which tends to zero as \( k \to \infty. By (153) with \( m_2 = 1 \) and \( m_3 = 2p - 2, (153) \) with \( m_2 = m_3 = p \) and (17), we see from Lemma 4.4 that there exists \( T = T(\tau, \omega) > 0 \) (independent of \( \delta \)) such that for all \( t \geq T \) and \( \delta \in (0, \delta_0], as \( k \to \infty, \)

\[
k^{-s} \int_{\tau-t}^{\tau} e^{\kappa(r-\tau)} \| u_{\delta}(r, \tau-t, \theta_{t-\omega}, u_{\delta, t-\tau}) \|^2 H^p(\mathbb{R}^n) \, dr
\]
Let \( \hat{\delta} \) uniformly in \( u \in \Omega \). By (17) and (153) with \( m_2 = 1 \) and \( m_3 = 2 \), we know that for all \( \delta \in (0, \delta_0) \), as \( k \to \infty \),

\[
  k^{-s} \int_{-\infty}^{0} e^{\kappa r} \left( 1 + |g(r+\tau)|^2 + |\psi_2(r+\tau)|^2 + \sup_{\delta \in (0, \delta_0)} |y_\delta(\tau,\omega)|^2 \right) dr \to 0
\]

By (17), we have, as \( k \to \infty \),

\[
  \int_{-\infty}^{0} e^{\kappa r} \int_{|x| \geq \frac{1}{k}} \left( 1 + |g(r+\tau, x)|^2 + |\psi_1(r+\tau)| + |\psi_2(r+\tau)|^2 \right) dx dr \to 0.
\]

By (108) and (153) with \( m_2 = 1 \) and \( m_3 = p \), we see that for all \( \delta \in (0, \delta_0) \), as \( k \to +\infty \),

\[
  \int_{-\infty}^{0} e^{\kappa r} \int_{|x| \geq \frac{1}{k}} |z_\delta(\tau,\omega)|^p dx dr \leq \int_{|x| \geq \frac{1}{k}} |(I + (-\Delta)^s)^{-1} h|^p dx \int_{-\infty}^{0} e^{\kappa r} \sup_{\delta \in (0, \delta_0)} |y_\delta(\tau,\omega)|^p dr \to 0.
\]

By (160)-(166), we see from (159) that as \( k, t \to +\infty \),

\[
  \int_{\mathbb{R}^n} \rho_k(x) u_\delta(t, \tau - t, \theta, \omega, \theta_{s, \tau-t})^2 dx \to 0,
\]

\[
  \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \rho_k(x) |(u_\delta(t, \tau - t, \theta, \omega, \theta_{s, \tau-t}) - u_\delta(t, \tau - t, \tau - t)(y)|^2 dy dx \to 0,
\]

uniformly in \( \delta \in (0, \delta_0) \). Therefore, (155) and (156) holds true.

We then derive uniform estimates of solutions of (121) for small \( \delta \) on bounded domains.

**Lemma 4.6.** Suppose (5)-(8) and (17) hold. Then for every \( k \in \mathbb{N} \), \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \), there exists \( \delta_0 > 0 \) such that the solution of (121) satisfies

\[
  \lim_{m, t \to +\infty} \sup_{\delta \in (0, \delta_0)} \| (I - \mathcal{P}_m) \xi_k u_\delta(\tau, \tau - t, \theta, \omega, u_{\delta, \tau-t}) \|_{H^s(\mathbb{R}^n)} = 0,
\]

uniformly in \( u_{\delta, \tau-t} \in \mathcal{K}_\delta(\tau - t, \theta, \omega) \) with \( \mathcal{K}_\delta \) given by (132).

**Proof.** Let \( \hat{u}_{k,m,2} = (I - \mathcal{P}_m) \hat{u}_k \) with \( \hat{u}_k = \xi_k(x) u_k \) and \( \hat{v}_{k,m,2} = (I - \mathcal{P}_m) \hat{v}_k \) for each \( m \in \mathbb{N} \). By formally repeating the procedure of deriving (118), we have

\[
  \frac{d}{dt} (\| \hat{v}_{\delta,m,2} \|^2 + \|(-\Delta)^{\frac{s}{2}} \hat{v}_{\delta,m,2} \|^2) + \kappa (\|\hat{v}_{\delta,m,2} \|^2 + \|(-\Delta)^{\frac{s}{2}} \hat{v}_{\delta,m,2} \|^2)
  \leq c \lambda_{m+1}^{-1} (1 + \|u_\delta\|^2_{H^s(\mathbb{R}^n)} + \|g(t)\|^2 + \|\psi_2(t)\|^2 + \|y_\delta(\tau,\omega)\|^2).
\]
Applying Gronwall’s inequality to (169) over \((\tau - t, \tau)\) and replacing \(\omega\) by \(\theta - \tau\omega\) to produce
\[
\|u_{\delta,m,2}(\tau, \tau - t, \theta - \tau\omega, (I - P_m)\xi_k v_{\delta, \tau - t})\|^2
+ \|(-\Delta)^{\frac{\sigma}{2}} u_{\delta,m,2}(\tau, \tau - t, \theta - \tau\omega, (I - P_m)\xi_k v_{\delta, \tau - t})\|^2
\leq e^{-\eta t} \left( \|u_{\delta}(\tau, \tau - t, \theta - \tau\omega, u_{\delta, \tau - t})\|^2 + \|(-\Delta)^{\frac{\sigma}{2}} u_{\delta}(\tau, \tau - t, \theta - \tau\omega, u_{\delta, \tau - t})\|^2 \right)
+ c\lambda_{m+1}^{-1} \int_{\tau-t}^{\tau} e^{\kappa(r - \tau)} \|u_{\delta}(r, \tau - t, \theta - \tau\omega, u_{\delta, \tau - t})\|_{H^s(\mathbb{R}^n)}^{2p - 2} dr
+ c\lambda_{m+1}^{-1} \int_{-\infty}^{0} e^{\kappa t} \left( 1 + \|g(r + \tau)\|^2 + \|\psi_2(r + \tau)\|^2 + |y_{\delta}(\theta - \tau\omega)|^2 \right) dr,
\]
which along with (129) with \(\sigma = \tau\) further implies
\[
\|u_{\delta,m,2}(\tau, \tau - t, \theta - \tau\omega, (I - P_m)\xi_k u_{\delta, \tau - t})\|^2
+ \|(-\Delta)^{\frac{\sigma}{2}} u_{\delta,m,2}(\tau, \tau - t, \theta - \tau\omega, (I - P_m)\xi_k u_{\delta, \tau - t})\|^2
\leq ce^{-\eta t} \left( \|(I - P_m)\xi_k u_{\delta, \tau - t}\|^2 + \|(-\Delta)^{\frac{\sigma}{2}} (I - P_m)\xi_k u_{\delta, \tau - t}\|^2 \right)
+ ce^{-\eta t} \left( \|(I - P_m)\xi_k z_{\delta}(\theta - \tau\omega)\|^2 + \|(-\Delta)^{\frac{\sigma}{2}} (I - P_m)\xi_k z_{\delta}(\theta - \tau\omega)\|^2 \right)
+ c\lambda_{m+1}^{-1} \int_{\tau-t}^{\tau} e^{\kappa(r - \tau)} \|u_{\delta}(r, \tau - t, \theta - \tau\omega, u_{\delta, \tau - t})\|_{H^s(\mathbb{R}^n)}^{2p - 2} dr
+ c\lambda_{m+1}^{-1} \int_{-\infty}^{0} e^{\kappa t} \left( 1 + \|g(r + \tau)\|^2 + \|\psi_2(r + \tau)\|^2 + |y_{\delta}(\theta - \tau\omega)|^2 \right) dr.
\]
Let \(\delta_0\) be the number as in Lemma 4.4. Since \(\|I - P_m\| \leq 1, \|\xi_k\|_{\infty} \leq 1\) and \(u_{\delta, \tau - t} \in K_{\delta}(\tau - t, \theta - \tau\omega)\), then by the argument of (160), we have, for all \(\delta \in (0, \delta_0]\), as \(t \to +\infty\),
\[
e^{-\eta t} \left( \|(I - P_m)\xi_k u_{\delta, \tau - t}\|^2 + \|(-\Delta)^{\frac{\sigma}{2}} (I - P_m)\xi_k u_{\delta, \tau - t}\|^2 \right)
\leq ce^{-\eta t} \|u_{\delta, \tau - t}\|_{H^s(\mathbb{R}^n)}^2 \leq ce^{-\eta t} \|K_{\delta}(\tau - t, \theta - \tau\omega)\|_{H^s(\mathbb{R}^n)}^2
\leq c \sup_{\delta \in (0, \delta_0]} \|z_{\delta}(\theta - \tau\omega)\|_{H^s(\mathbb{R}^n)}^2 + c \int_{-\infty}^{-t} e^{\frac{\kappa}{2} r} \sup_{\delta \in (0, \delta_0]} \|y_{\delta}(\theta - \tau\omega)\|^2 dr
+ c \int_{-\infty}^{-t} e^{\frac{\kappa}{2} r} \left( 1 + \|g(r + \tau)\|^2 + \|\psi_1(r + \tau)\|_1 + \|\psi_2(r + \tau)\|^2 \right) dr \to 0.
\]
It follows from (153) with \(m = 1\) that for all \(\delta \in (0, \delta_0]\),
\[
e^{-\eta t} \left( \|(I - P_m)\xi_k z_{\delta}(\theta - \tau\omega)\|^2 + \|(-\Delta)^{\frac{\sigma}{2}} (I - P_m)\xi_k z_{\delta}(\theta - \tau\omega)\|^2 \right)
\leq ce^{-\eta t} \|z_{\delta}(\theta - \tau\omega)\|_{H^s(\mathbb{R}^n)}^2 \leq ce^{-\eta t} \|y_{\delta}(\theta - \tau\omega)\|^2 \to 0 \text{ as } t \to +\infty.
\]
By the property of \(P_m\) and (152) with \(t = 0\), we have, for all \(\delta \in (0, \delta_0]\),
\[
\|(I - P_m)\xi_k z_{\delta}(\omega)\|^2 + \|(-\Delta)^{\frac{\sigma}{2}} (I - P_m)\xi_k z_{\delta}(\omega)\|^2 \leq c \|(I - P_m)\xi_k z_{\delta}(\omega)\|_{H^s(\mathbb{R}^n)}^2 \leq c \sup_{\delta \in (0, \delta_0]} \|y_{\delta}(\omega)\|^2 \|(I - P_m)(I + (-\Delta)^{\frac{s}{2}})^{-1}\xi_k h\|_{H^s(\mathbb{R}^n)}^2 \to 0 \text{ as } m \to \infty.
\]
Since \(\lambda_{m+1}^{-1} \to 0\) as \(m \to \infty\), then similar to (163)-(164), the remaining terms on the right-hand side of (170) tend to zero as \(m, t \to +\infty\) uniformly for \(\delta \in (0, \delta_1]\).
Thus
\[ \lim_{m,t \to +\infty} \sup_{\delta \in (0, \delta_0)} \| \tilde{u}_{\delta,m,2}(\tau, \tau - t, \theta_{-\tau} \omega, (I - P_m)\xi_k u_{\delta,\tau-t}) \|_{H^s(\mathbb{R}^n)} = 0. \]

This completes the proof. \( \square \)

Finally, we show the uniform compactness of random attractors.

**Lemma 4.7.** Suppose (5)-(8) and (17)-(18) hold. Then for each \((\tau, \omega) \in \mathbb{R} \times \Omega\), every sequence \(\{u_i\}_{i=1}^{\infty}\) is pre-compact in \(H^s(\mathbb{R}^n)\) whenever \(u_i \in \mathcal{A}_{\delta_i}(\tau, \omega)\) with \(\delta_i \to 0\).

**Proof.** Let \(\delta_0 > 0\) be the number as given in Lemmas 4.4-4.6. It suffices to show that 
\[ \bigcup_{\delta \in (0, \delta_0]} \mathcal{A}_{\delta}(\tau, \omega) \text{ has a finite } \varepsilon \text{-net in } H^s(\mathbb{R}^n) \text{ for any } \varepsilon > 0. \]

Indeed, by invariance of \(\mathcal{A}_{\delta}\),
\[ \mathcal{A}_{\delta}(\tau, \omega) = \Phi_{\delta}(t, \tau - t, \theta_{-\tau} \omega) \mathcal{A}_{\delta}(\tau - t, \theta_{-\tau} \omega) \subseteq \Phi_{\delta}(t, \tau - t, \theta_{-\tau} \omega) K_{\delta}(\tau - t, \theta_{-\tau} \omega). \]

By Lemma 4.5, there are \(T_1 \geq 1\) and \(k_0 \geq 1\) such that for all \(t \geq T_1\),
\[ \sup_{\delta \in (0, \delta_0]} \| u_{\delta}(\tau, \tau - t, \theta_{-\tau} \omega, u_{\delta,\tau-t}) \|_{H^s(\mathcal{O}_{\delta_0})} < \frac{\varepsilon}{2} \]
for all \(u_{\delta,\tau-t} \in K_{\delta}(\tau - t, \theta_{-\tau} \omega)\).

This along with (171) implies that
\[ \| u \|_{H^s(\mathcal{O}_{\delta_0})} < \frac{\varepsilon}{2} \]
for all \(u \in \bigcup_{\delta \in (0, \delta_0]} \mathcal{A}_{\delta}(\tau, \omega)\).

By Lemma 4.6 with \(k = 2k_0\), there are \(T_2 \geq T_1\) and \(m_0 \geq 1\) such that for all \(t \geq T_2\) and \(u_{\delta,\tau-t} \in K_{\delta}(\tau - t, \theta_{-\tau} \omega)\),
\[ \sup_{\delta \in (0, \delta_0]} \| (I - P_{m_0})\xi_{2k_0} u_{\delta}(\tau, \tau - t, \theta_{-\tau} \omega, u_{\delta,\tau-t}) \|_{H^s(\mathbb{R}^n)} < \frac{\varepsilon}{4}. \]

which along with (171) implies that
\[ \| (I - P_{m_0})\xi_{2k_0} u \|_{H^s(\mathbb{R}^n)} < \frac{\varepsilon}{2} \]
for all \(u \in \bigcup_{\delta \in (0, \delta_0]} \mathcal{A}_{\delta}(\tau, \omega)\).

By (152), we see
\[ \sup_{\delta \in (0, \delta_0]} |y_{\delta}(\omega)|^2 + \sup_{\delta \in (0, \delta_0]} \int_{-\infty}^{0} e^{\kappa r} |y_{\delta}(\theta, \omega)|^p dr < \infty. \]

which together with (132) implies
\[ \| u \|_{H^s(\mathbb{R}^n)} \leq \sup_{\delta \in (0, \delta_0]} \sqrt{M_2 R_{\delta}(\tau, \omega)} < \infty \]
for all \(u \in \bigcup_{\delta \in (0, \delta_0]} \mathcal{A}_{\delta}(\tau, \omega)\).

This further implies
\[ \| \xi_{2k_0} u \|_{H^s(\mathbb{R}^n)} < c \sup_{\delta \in (0, \delta_0]} \sqrt{M_2 R_{\delta}(\tau, \omega)} < \infty \]
for all \(u \in \bigcup_{\delta \in (0, \delta_0]} \mathcal{A}_{\delta}(\tau, \omega)\). (176)

Note that \(\xi_{2k_0} u = u\) on \(O_{k_0}\) and the range of \(P_{m_0}\) is finite-dimensional, then it follows from (175) and (176) that the restricted set of \(\bigcup_{\delta \in (0, \delta_0]} \mathcal{A}_{\delta}(\tau, \omega)\) on \(O_{k_0}\) has a finite \(\frac{\varepsilon}{4}\)-net in \(H^s(O_{k_0})\), which together with (173) proves that \(\bigcup_{\delta \in (0, \delta_0]} \mathcal{A}_{\delta}(\tau, \omega)\) has a finite \(\varepsilon\)-net in \(H^s(\mathbb{R}^n)\). This completes the proof. \( \square \)
4.4. Convergence of random attractors. In this subsection, we prove the upper semi-continuity of random attractors $A_\delta$ as $\delta$ tends to zero.

**Theorem 4.8.** Suppose (5)-(8) and (17)-(18) hold. Then the random attractor $A_\delta$ of problem (121) is upper semi-continuous to the random attractor $A$ of problem (76) in $H^s(\mathbb{R}^n)$, more precisely,

$$\lim_{\delta \to 0} \text{dist}_{H^s}(A_\delta(\tau, \omega), A(\tau, \omega)) = 0 \quad \text{for each } \tau \in \mathbb{R} \text{ and a.e. } \omega \in \Omega.$$  

Furthermore, for any $\delta_n \to 0$ and $\varepsilon > 0$, there exists $\Omega_\varepsilon \in \mathcal{F}$ with $P(\Omega_\varepsilon) > 1 - \varepsilon$ such that

$$\lim_{n \to \infty} \sup_{\omega \in \Omega_\varepsilon} \text{dist}_{H^s}(A_{\delta_n}(\tau, \omega), A(\tau, \omega)) = 0 \quad \text{for each } \tau \in \mathbb{R}.$$  

**Proof.** Let $\|u_{\delta_n, \tau} - u_\tau\|_{H^s(\mathbb{R}^n)} \to 0$ as $\delta_n \to 0$, we see from Lemma 4.3 that

$$\lim_{n \to \infty} \|\Phi_{\delta_n}(t, \tau, \omega)u_{\delta_n, \tau} - \Phi_0(t, \tau, \omega)u_\tau\|_{H^s(\mathbb{R}^n)} = 0, \quad \forall t \geq 0, \ \tau \in \mathbb{R}, \ \omega \in \Omega.$$  

Define a new set

$$B_0(\tau, \omega) = \{u \in H^s(\mathbb{R}^n) : \|u\|^2_{H^s(\mathbb{R}^n)} \leq M_0R_0(\tau, \omega)\} \quad \text{for every } \tau \in \mathbb{R}, \ \omega \in \Omega,$$

where $R_0(\tau, \omega)$ is given in Lemma 3.1 and $M_0 = \max(M_1, M_2)$ with $M_1$ and $M_2$ being the same numbers as in Lemma 3.1 and Lemma 4.1, respectively. By Lemma 3.1, we see that $B_0 = \{B_0(\tau, \omega) : \tau \in \mathbb{R}, \ \omega \in \Omega\}$ is also a $\mathcal{D}$-pullback random absorbing set of $\Phi_0$. It yields from (133) that

$$\lim_{\delta \to 0} \|K_\delta(\tau, \omega)\|_{H^s(\mathbb{R}^n)} \leq \|B_0(\tau, \omega)\|_{H^s(\mathbb{R}^n)} \quad \text{for all } (\tau, \omega) \in \mathbb{R} \times \Omega.$$  

Then by (179)-(180) and Lemma 4.7, we know that all conditions in [51, Theorem 3.1] are fulfilled, and thereby (177) follows immediately.

To show (178), we let $\tau \in \mathbb{R}$ be fixed and consider the following set:

$$\Omega_1 = \bigcup_{k=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \{\omega \in \Omega : \chi_n(\omega) \geq \frac{1}{k}\},$$

where $\chi_n(\omega) = \text{dist}_{H^s}(A_{\delta_n}(\tau, \omega), A(\tau, \omega))$. Then, by (177), we have $P(\Omega_1) = 0$. For each $k \in \mathbb{N}$, $\cup_{n=m}^{\infty} \{\omega \in \Omega : \chi_n(\omega) \geq \frac{1}{k}\}$ is decreasing in $m$, we have

$$\lim_{m \to \infty} P\left(\bigcup_{n=m}^{\infty} \{\omega \in \Omega : \chi_n(\omega) \geq \frac{1}{k}\}\right) = P\left(\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \{\omega \in \Omega : \chi_n(\omega) \geq \frac{1}{k}\}\right) \leq P(\Omega_1) = 0,$$

which implies that for each $\varepsilon > 0$ and $k \in \mathbb{N}$, there is an $m(k) \in \mathbb{N}$ such that

$$P\left(\bigcup_{n=m(k)}^{\infty} \{\omega \in \Omega : \chi_n(\omega) \geq \frac{1}{k}\}\right) < \frac{\varepsilon}{2^k}.$$
Denote by $\Omega_\varepsilon = \cap_{k=1}^\infty \cap_{n=m(k)}^\infty \{ \omega \in \Omega : \chi_n(\omega) < \frac{1}{k} \}$ for each $k \in \mathbb{N}$. Then it is easy to verify
\[
P(\Omega_\varepsilon) = P\left( \bigcup_{k=1}^\infty \bigcup_{n=m(k)}^\infty \{ \omega \in \Omega : \chi_n(\omega) \geq \frac{1}{k} \} \right)
\leq \sum_{k=1}^\infty P\left( \bigcup_{n=m(k)}^\infty \{ \omega \in \Omega : \chi_n(\omega) \geq \frac{1}{k} \} \right) < \sum_{k=1}^\infty \frac{\varepsilon}{2^k} = \varepsilon,
\]
which shows $P(\Omega_\varepsilon) > 1 - \varepsilon$. Moreover, for every $\eta > 0$, there is a $k_0 = k_0(\eta) \in \mathbb{N}$ such that $\frac{1}{k_0} < \eta$, then we find an $m(k_0)$ such that $\Omega_\varepsilon \subset \cap_{n=m(k_0)}^\infty \{ \omega \in \Omega : \chi_n(\omega) < \frac{1}{k_0} \}$. Then
\[
\sup_{\omega \in \Omega_\varepsilon} \text{dist}_{H^s}(A_{h_n}(\tau, \omega), A(\tau, \omega)) = \sup_{\omega \in \Omega_\varepsilon} \chi_n(\omega) < \frac{1}{k_0} < \eta,
\]
for all $n \geq m(k_0)$. This completes the proof.

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E-mail address: rwang-math@outlook.com
E-mail address: liyr@swu.edu.cn
E-mail address: bwang@nmt.edu