A REMARK ON POST-CRITICALLY FINITE COMPOSITIONS
OF POLYNOMIALS

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Abstract. The second author proved that the set of post-critically finite polynomials of given degree is a set of bounded height, up to change of variables. Motivated by an observation about unicritical polynomials, we complement this by proving that the set of monic polynomials $g(z) \in \mathbb{Q}[z]$ of given degree with the property that there exists a $d \geq 2$ such that $g(z^d)$ is post-critically finite, is also a set of bounded height. Moreover, we establish a lower bound on the critical height of $g(z^d)$.

A polynomial $f(z) \in \mathbb{C}[z]$ is post-critically finite (PCF) if and only if the orbit of every critical point is finite, and in this case one can show that $f$ is defined over $\mathbb{Q}$, after a change of variables. The first author showed that the set of PCF polynomials of given degree is a set of bounded height [1], depending strongly on the degree. So for example, for each $d \geq 2$ and each number field $K$ there exist only finitely many $c \in K$ such that $z^d + c$ is PCF. In this very special case, though, one can do much better, and with elementary tools. If $z^d + c$ is PCF, then it is easy to show that $h(c) \leq \frac{\log 2}{d}$, and so in particular there are only finitely many $c \in K$ such that there exists a $d \geq 2$ with $z^d + c$ PCF. This note generalizes this phenomenon from unicritical polynomials, viewed as post-compositions of a power map by a linear map, to arbitrary polynomial post-compositions of a power map, including a lower bound on the critical height.

Theorem 1. For monic polynomials $g(z) \in \mathbb{Q}[z]$ and $d \geq 2$, we have

$$\hat{h}_{\text{crit}}(g(z^d)) \geq \frac{1}{d \deg(g)} \left( h(g) + O(1) \right),$$

where the implied constant depends just on $\deg(g)$. In particular, the set of monic $g(z) \in \mathbb{Q}[z]$ of given degree such that there exists a $d \geq 2$ with $g(z^d)$ PCF is a set of bounded height.

The main result of [1] is a similar lower bound on the critical height for polynomials of fixed degree (i.e., the $d = 1$ case of Theorem 1), but the height bound there is only up to change of coordinates, since $f(z + a) - a$ will be PCF whenever $f$ is. In contrast, a monic polynomial of the form $g(z^d)$ is only conjugate to finitely many other polynomials of the same form, when $d \geq 2$ (since the change of variables would need to move $z = 0$ to another critical point).

It is well known that there are no algebraic families of PCF polynomials in characteristic zero, and the tools in [1] allow one to extend this result to characteristic $p$, but only for polynomials of degree less than $p$ (see also work of Levy [3]). Indeed, the family $z^p + t$ is PCF in characteristic $p$, for any $m$ as long as $t \neq 0$, so one cannot hope to completely remove the condition on the degree. One can, however, obtain results in the flavour of Theorem 1.
**Theorem 2.** Let $k$ be an algebraically closed field of characteristic $p > 0$. Then there are no non-constant PCF algebraic families of the form $g(z^d)$ with $d \geq 2$, and $\deg(g) < p$.

One must specify what one means by a PCF algebraic family, since it is of course the case that any rational function defined over $\mathbb{F}_p$ is PCF. By a PCF algebraic family we mean a family parametrized by an irreducible quasi-projective variety, whose generic fibre is PCF. It is easy to check that this is equivalent to saying that the postcritical sets of the specializations are of uniformly bounded size.

1. **Local considerations**

In this section we fix $m \geq 2$, and work over a field $K$ of characteristic 0 or $p > m$, equipped with a set $M$ of absolute values $|\cdot|_v$, extended in some way to the algebraic closure of $K$. For any tuple $x_1, \ldots, x_m$ we set $x = x_1, \ldots, x_m$ and

$$\|x\| = \|x_1, \ldots, x_n\|_v = \max\{\|x_1\|_v, \ldots, \|x_n\|_v\},$$

and we concatenate tuples with a comma.

We also consider monic polynomials $g(z)$ of degree $m$, and write

$$g(z) = (z - a_1) \cdots (z - a_m)$$

and

$$g'(z) = mz^{m-1} - (m-1)\sigma_1(a)z^{m-2} + \cdots + \pm \sigma_{m-1}(a).$$

over $\overline{K}$.

**Lemma 3.** We have

$$\left| \log \|a\|_v - \log \|c, g(0)^{1/m}\|_v \right| \leq C_1, v,$$

where $C_1, v = 0$ when $v$ is non-archimedean, or some fixed non-negative value depending just on $m$ otherwise.

**Proof.** If $\sigma_i$ is the basic symmetric polynomial of degree $i$ in the relevant number of variables, then

$$g(z) = z^m - \sigma_1(a)z^{m-1} + \cdots \pm \sigma_m(a),$$

and so

$$g'(z) = mz^{m-1} - (m-1)\sigma_1(a)z^{m-2} + \cdots + \sigma_{m-1}(a).$$

In the non-archimedean case, $g'(c_i) = 0$ implies

$$|mc_i^{m-1}|_v \leq |(m-j)\sigma_j(a)c_i^{m-j-1}|_v \leq |c_i|_v^{m-j-1}\|a\|_v^{j},$$

for some $j$, whence $|c_i|_v \leq \|a\|_v|a|_v^{-1}$. Since

$$|g(0)|_v = |a_1 \cdots a_m|_v \leq \|a\|_v^m,$$

we have one direction of our bound, and the archimedean case is similar, using

$$\|\sigma_j(a)\|_v \leq \binom{m}{j} \|a\|_v^j.$$

In the other direction, we may use a similar argument applying $g(a_i) = 0$ to the expression

$$g(z) = z^m - \frac{1}{m-1}\sigma_1(c)z^{m-1} + \cdots \pm \sigma_{m-1}(c)z + g(0).$$
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Now, for any polynomial \( f(z) \in K[z] \) of degree at least 2, set

\[
G_{f,v}(z) = \lim_{k \to \infty} \log^+ |f^k(z)|_v / \deg(f^k),
\]

noting that \( G_{f,v}(f(z)) = \deg(f) G_{f,v}(z) \), and put

\[
\lambda_{\text{crit},v}(f) = \max_{f(c)=0} \{ G_{f,v}(c) \}.
\]

Lemma 4. Set \( f(z) = g(z^d) \), and suppose that

\[
d \log |z|_v > \log^+ \|a\|_v + C_{2,v},
\]

where \( C_{2,v} = \frac{m}{m-1} \log 2 \) for \( v \) archimedean, and \( C_{2,v} = 0 \) otherwise. Then

\[
G_{f,v}(z) = \log |z|_v + \varepsilon_v(z),
\]

where \( \varepsilon_v(z) = 0 \) for non-archimedean \( v \), and

\[
-\frac{m}{dm - 1} \log 2 \leq \varepsilon_v(z) \leq \frac{m}{dm - 1} \log(3/2)
\]

for \( v \) archimedean.

Proof. First consider \( v \) non-archimedean. Then \( \log |z^d|_v > \log^+ \|a\|_v \) implies \( |z^d - a_i|_v = |z|_v^d \) for all \( i \), whence

\[
\log |f(z)|_v = \deg(f) \log |z|_v \geq \log |z^d|_v > \log^+ \|a\|_v.
\]

Iterating gives the desired result.

For an archimedean \( v \),

\[
\log |z^d|_v > \log^+ \|a\|_v + \frac{m}{m-1} \log 2 > \log^+ \|a\|_v + \log 2
\]

gives \( |z^d - a_i|_v > |z|_v^d/2 \), whence

\[
\log |f(z)|_v \geq \deg(f) \log |z|_v - m \log 2 \geq \log |z|_v + (m-1)d \log |z|_v - m \log 2 \geq \log |z|_v + (m-1) \log^+ \|a\|_v \geq \log |z|_v \geq \log \|a\|_v + C_{2,v},
\]

and so iterating gives

\[
G_{f,v}(z) \geq \log |z|_v - \frac{m}{dm - 1} \log 2.
\]

The other direction is obtained using the conclusion \( |z^d - a_i|_v \leq \frac{3}{2} |z|_v^d \) from (2). \( \square \)

We now prove the key lemma.

Lemma 5. We have

\[
\lambda_{\text{crit},v}(g(z^d)) \geq \frac{1}{d} \left( \log^+ \|a\|_v - C_{3,v} \right),
\]

where \( C_{3,v} \) depends only on \( m \) and the absolute value, and can be taken to be 0 if \( v \) is non-archimedean, and not \( p \)-adic for any \( p \leq m \).
Proof. Note first, as a general comment, that \( \lambda_{\text{crit}}(g(z^d)) \geq 0 \), and so the desired result holds trivially if ever \( \log^+ \|a\|_v \leq C_{3,v} \). We will assume that

\[
\log^+ \|a\|_v = \log \|a\|_v > \frac{m}{m-1}C_{1,v} + \frac{1}{m}C_{2,v} \geq 0,
\]

since requiring \( C_{3,v} \geq \frac{m}{m-1}C_{1,v} + \frac{1}{m}C_{2,v} \) will mean that our claim is trivial otherwise.

Write \( f(z) = g(z^d) \), and note by the chain rule that the branch points of \( f \) (the images of the critical points) are exactly \( g(0) \) and the branch points of \( g \). Let \( \beta \) be any branch point of \( f \) and suppose for now that that \( \log |\beta|_v > \log \|a\|_v + C_{2,v} \), so that \( z = \beta \) certainly satisfies the hypothesis (2). Then we have

\[
\lambda_{\text{crit}}(f) \geq 1 \frac{\partial G_{f,v}(\beta)}{\partial \beta} \geq 1 \frac{\partial}{\partial \beta} \left( \log |\beta|_v + \varepsilon_v(z) \right) \geq \frac{1}{d} \left( \log \|a\|_v + C_{2,v} + \varepsilon_v(z) \right),
\]

which implies the claimed bound, as long as we eventually take \( C_{3,v} \geq \frac{m}{2m-1} \log 2 \).

So we are done unless

\[
\log |\beta|_v \leq \log \|a\|_v + C_{2,v}
\]

for every branch point \( \beta \) of \( f \). In particular, since \( g(0) \) is a branch point, and

\[
m \log \|a\|_v - mC_{1,v} > \log \|a\|_v + C_{2,v},
\]

from (3), we have \( \log |g(0)| < m \log \|a\| - mC_{1,v} \), whence from (1)

\[
\left| \log \|a\| - \log \|c\| \right| \leq C_{1,v}.
\]

Now write \( \psi(z) = \frac{1}{m} (g(z) - g(0)) \). By the main result of [1], there is a constant \( C_{4,v} \), depending just on the absolute value (and to be taken as 0 if \( v \) is non-archimedean and not \( p \)-adic for \( p \leq m \); see e.g., [2], and the comment therein that arguments pursued over \( \mathbb{Q} \) in [1] really work over \( \mathbb{Z}[1/2,1/3,\ldots,1/m] \)), such that

\[
\log |\psi(c_j)| \geq m \log \|c\| - C_{4,v}
\]

for some \( j \), noting that the critical points of \( \psi \) are also the \( c_i \). Thus we have

\[
m \log \|a\| - mC_{1,v} - C_{4,v} - \log |m| \leq m \log \|c\| - C_{4,v} - \log |m| \leq \log |g(c_j) - g(0)| \leq \log \max\{|g(c_j)|, |g(0)|\} \leq \log \|a\| + C_{2,v},
\]

by (5), which gives

\[
\log \|a\| \leq \frac{1}{m-1} \left( mC_{1,v} + C_{2,v} + C_{4,v} + \log |m| \right).
\]

Again our bound is trivial as long as we take \( C_{3,v} \) to be at least as large as the right-hand-side of (6). \( \square \)
2. Proofs of Theorems 1 and Theorem 2

It is now relatively easy to prove Theorem 1 from Lemma 5.

**Proof of Theorem 1.** Recall that for a tuple \( a \in K^m \), we have

\[
h(a) = \sum_{v \in M_K} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} \log^+ ||a||_v.
\]

We define, as is now standard for polynomials,

\[
\hat{h}_{\text{crit}}(f) = \sum_{v \in M_K} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} \lambda_{\text{crit},v}(f).
\]

Summing Lemma 5 over all places, we have

\[
\hat{h}_{\text{crit}}(g(z^d)) \geq \frac{1}{d} \left( h(a) - C \right),
\]

where \( C = \sum_{v \in M_K} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} C_{3,v} \). Using the estimate

\[
h(g) \leq \deg(g)(h(a) + \log 2)
\]

(obtained by bounding the coefficients of \( g \) as symmetric polynomials in the roots) completes the proof of Theorem 1. \( \square \)

**Proof of Theorem 2.** Recall (e.g., [2, Lemma 10]) that if \( V \) is any quasi-projective variety over an algebraically closed field \( k \), then the function field \( k(V) \) admits a set of absolute values satisfying the product formula, and such that the points of height zero are precisely the points with constant coordinates. Now, the error terms in Lemma 5 vanish for each of these absolute values, since they are neither archimedean nor \( p \)-adic, and so we have

\[
\lambda_{\text{crit},v}(g(z^d)) \geq \frac{1}{d \deg(g)} \log^+ ||a||_v
\]

for all places. So if \( g(z^d) \) is PCF, we must have \( ||a||_v \leq 1 \) for all \( v \), and so the \( a_i \) are constant. \( \square \)

We end with a remark, namely: Let \( V \) be a quasi-projective variety over \( k = \mathbb{F}_p \), and suppose that \( f(z) \in k[V,z] \) has degree \( d \geq 2 \) in \( z \). Then \( f \) is PCF if and only if there is uniform bound on the size of the postcritical set of the specialization \( f_t \), for \( t \in V(k) \).

In one direction this is clear, since if the postcritical set on the generic fibre contains at most \( N \) elements, the same is true of every specialization. On the other hand, for every \( N \), there is a Zariski closed subset of \( V \) corresponding to specializations in which the postcritical set contains at most \( N \) elements. If the specializations have uniformly bounded postcritical set sizes, then since \( \mathbb{F}_p \) points are Zariski dense in \( V \), it must be that some irreducible component of this closed subset is all of \( V \). But, marking the critical points and describing their orbits partitions these components, so all orbits must have the same combinatorics on \( V \).
References

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