Higher order coefficients of the inverse mass expansion of one–loop effective actions are obtained from a one–dimensional path integral representation. For the evaluation of the path integral with Wick contractions a suitable Green function has to be chosen. We consider the case of a massive scalar loop in the background of both a scalar potential and a (non–abelian) gauge field. For the pure scalar case the method yields the coefficients of the expansion in a minimal set of basis terms whereas complicated ordering problems arise in gauge theory. An appropriate reduction scheme is discussed.

In recent years, the inverse mass expansion of one–loop effective actions has been applied in fields as different as the calculation of quark determinants, bubble nucleation during the electroweak phase transition or baryon number violation by instanton or sphaleron processes. A variety of approaches and results can be found in the literature [1]. The actual computation of higher order coefficients, however, suffers from practical limitations and was worked out only to order $O(T^6)$ for the ungauged theory [2] and $O(T^5)$ for the gauged case [3]. In our approach, instead of using conventional heat–kernel techniques we start from a representation of the one–loop effective action as a one–dimensional path integral. For the evaluation of the path integral we make use of recent progress in calculating one–loop amplitudes inspired by string theory [4,5,6], which enables us to compute the inverse mass expansion to high orders.

Formally, the one-loop effective action for a scalar particle with mass $m$ in the background of a gauge field $A_{\mu}$ (abelian or non–abelian) and a matrix–valued scalar potential $V$ is obtained as the determinant of a fluctuation operator $M$, which can be written in the Schwinger proper time formalism as

$$\Gamma_{\text{eff}}[A, V] = -\ln(\det M) = -\text{Tr}(\ln M) = \int_0^{\infty} \frac{dT}{T} \text{Tr} e^{-TM}. \quad (1)$$

For the case we are considering the operator $M$ reads (in $d$ Euclidean dimensions)

$$M = -D^2 + m^2 + V(x) \quad \text{with} \quad D_{\mu} = \partial_{\mu} - igA_{\mu}(x). \quad (2)$$

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The expansion of $\Gamma_{\text{eff}}$ in inverse powers of the heavy mass $m$ is obtained from expanding the exponential in Eq. (1) in powers of the proper time $T$. (Since we will not evaluate the proper time integration, we do not comment on its regularization.)

The operator trace is nothing but the diagonal element of the heat kernel for the operator $M$, integrated over space–time. It can be represented as a one–dimensional path integral on the space of closed loops in space-time with fixed proper time circumference $T$. The action then takes the form

$$\Gamma[A, V] = \int_0^\infty \frac{dT}{T} e^{-m^2 T} \text{tr} \mathcal{P} \int_{x(T)=x(0)} \mathcal{D}x \exp \left[-\int_0^T d\tau \left(\frac{\dot{x}^2}{4} + ig\dot{x}^\mu A_\mu + V(x)\right)\right],$$

(3)

where $\mathcal{P}$ denotes path ordering and the operator trace $\text{Tr}$ has reduced to an ordinary matrix trace $\text{tr}$.

For the Wick contractions one needs the Green function $G_B$ for the Laplacian on the circle with periodic boundary conditions [5]. However, without the introduction of a “background charge” $\rho$ on the wordline, the defining equation for $G_B$ has no solution. A convenient choice is a uniformly distributed $\rho = 1/T$, which yields the modified defining equation

$$\frac{1}{2} \frac{\partial^2}{\partial \tau_1^2} G_B(\tau_1, \tau_2) = \delta(\tau_1 - \tau_2) - \rho(\tau_1) \delta(\tau_1 - \tau_2) - \frac{1}{T},$$

(4)

with the solution

$$G_B(\tau_1, \tau_2) = |\tau_1 - \tau_2| - \frac{(\tau_1 - \tau_2)^2}{T}. \quad (5)$$

It is precisely this choice of the background charge which distinguishes our approach from previous uses of path integrals in this context [7].

As a consequence of the introduction of a background charge, the Green function $G_B$ fulfills the identity (integration by parts)

$$\int_0^T d\tau_2 \frac{1}{2} G_B(\tau_1, \tau_2) \ddot{x}(\tau_2) = x(\tau_1) - \frac{1}{T} \int_0^T d\tau_2 x(\tau_2),$$

where the second term on the r.h.s. should vanish. This suggests that one should introduce a loop center of mass $x_0$ and a relative loop coordinate $y$,

$$x^\mu(\tau) = x_0^\mu + y^\mu(\tau) \quad \text{with} \quad \int_0^T d\tau y^\mu(\tau) = 0,$$

(6)

and separate the integration over the center of mass from the path integral:

$$\int \mathcal{D}x = \int d^d x_0 \int \mathcal{D}y,$$

(7)

resulting in a path integral over the space of all loops with a common center of mass $x_0$. With $\dot{y}^\mu = \ddot{x}^\mu$, the inverse mass expansion of $\Gamma_{\text{eff}}$ follows from expanding the path ordered exponential in Eq. (3).
Choosing the background gauge field to be in Fock-Schwinger gauge with reference point \(x_0\) [6], it can be written as

\[
A_\mu(x_0 + y) = y^\rho \int_0^1 d\eta \, \eta \, F_{\rho\mu}(x_0 + \eta y) .
\]

(8)

The main advantage of the Fock-Schwinger gauge is that the Taylor expansion of the background fields around \(x_0\) can be done in a covariant way,

\[
F_{\rho\mu}(x_0 + \eta y) = e^{\eta y D_{\rho\mu}} F_{\rho\mu}(x_0) , \quad V(x_0 + y) = e^{y D} V(x_0) .
\]

(9)

With these ingredients, Eq. (3) takes the manifestly covariant form

\[
\Gamma[F, V] = \int_0^\infty \frac{dT}{T} e^{-m^2 T} \, \text{tr} \int d^d x_0 \sum_{n=0}^\infty (-1)^n \int D y \, \exp \left[ - \int_0^T d\tau \, \frac{y^2}{4} \right] \\
\times \int_0^T d\tau_1 \int_0^{\tau_1} d\tau_2 \cdots \int_0^{\tau_{n-1}} d\tau_n \prod_{j=1}^n \left[ e^{y(\tau_j) D,j} V^{(j)}(x_0) + \\
+ i g y^{\mu_1}(\tau_j) y^{\mu_2}(\tau_j) \int_0^1 d\eta_j \eta_j \, e^{\eta_j y(\tau_j) D,j} F^{(j)}_{\rho_1 \rho_2}(x_0) \right] .
\]

(10)

For the calculation of the coefficients of the inverse mass expansion to a given order, the following steps have to be performed:

(i) Wick-contractions: One truncates the sum in Eq. (10) and expands the exponentials. All possible Wick contractions have to be evaluated using the contraction rules

\[
\langle x^\mu(\tau_1) x^\nu(\tau_2) \rangle = -g^{\mu\nu} G_B(\tau_1, \tau_2) = -g^{\mu\nu} \left[ (\tau_1 - \tau_2) \frac{2}{T} \right] ,
\]

\[
\langle \dot{x}^\mu(\tau_1) x^\nu(\tau_2) \rangle = -g^{\mu\nu} \dot{G}_B(\tau_1, \tau_2) = -g^{\mu\nu} \left[ \text{sign}(\tau_1 - \tau_2) - \frac{2(\tau_1 - \tau_2)}{T} \right] ,
\]

\[
\langle \dot{x}^\mu(\tau_1) \dot{x}^\nu(\tau_2) \rangle = +g^{\mu\nu} \ddot{G}_B(\tau_1, \tau_2) = +g^{\mu\nu} \left[ 2\delta(\tau_1 - \tau_2) - \frac{2}{T} \right] .
\]

(11)

(From these elementary prescriptions it is possible to derive contraction rules for exponentials, which can be used equivalently.)

(ii) Integrations: Next, the polynomial \(\tau\)- and \(\eta\)-integrations have to be performed. After the expansion in step (i) the integration over the \(\eta\)'s is trivial. The integrands of the \(\tau\)-integrations are constructed from the worldline Green function \(G_B\) and its derivatives and are thus polynomials in the variables \(\tau_j\).

(iii) Cyclic reduction: The number of terms will be drastically reduced if one identifies structures which differ only by cyclic permutation under the trace. Since all cyclic permutations of a given term actually occur during the calculation and since the polynomial integrations in step (ii) yield the same factor for terms equivalent under cyclic permutation (due to the translational invariance of our Green function), the identification of equivalent terms can easily be done.
For the ungauged case, this already completes the program. Let us therefore cite the results for a pure scalar background [7]. The “master equation” (10) simplifies

\[ \Gamma[V] = \int_0^\infty \frac{dT}{T} [4\pi T]^{-d/2} e^{-m^2 T} \text{tr} \int d^d x_0 \sum_{n=0}^\infty (-T)^n \int_0^1 du_1 \int_0^1 du_2 \ldots \int_0^1 du_n \]

\[ \times \exp \left[ -T \sum_{i<k} G(u_i, u_k) \partial_{(i)} \partial_{(k)} \right] V^{(1)}(x_0) \ldots V^{(n)}(x_0) , \] 

where a rescaling to the unit circle, \( \tau_i = T u_i \) (i = 1, ..., n), has been performed using the scaling property of the Green function, \( G(\tau_1, \tau_2) = T G(u_1, u_2) \). The factor \( [4\pi T]^{-d/2} \) arises from the normalization of the free path integral. After expanding the exponential, performing the integrations and identifying terms equivalent under cyclic permutation one obtains the final result in the form

\[ \Gamma[V] = \int_0^\infty \frac{dT}{T} [4\pi T]^{-d/2} e^{-m^2 T} \sum_{n=1}^\infty \frac{(-T)^n}{n!} \int d x_0 \text{tr} O_n . \] 

The method was completely computerized using the algebraic language FORM and for identification of cyclic redundancies – PERL. The calculation was performed up to \( O_{11} \). In the following the results to \( O(T^6) \) are quoted [8], using the shorthand notation \( V_{\kappa\lambda} \equiv \partial_\kappa \partial_\lambda V(x_0) \):

\( O_1 = V, \quad O_2 = V^2, \quad O_3 = \left( V^3 + \frac{1}{2} V_{\kappa} V_{\kappa} \right), \quad O_4 = \left( V^4 + 2 V V_{\kappa} V_{\kappa} + \frac{1}{5} V_{\kappa\lambda} V_{\kappa\lambda} \right), \)

\( O_5 = \left( V^5 + 3 V^2 V_{\kappa} V_{\kappa} + 2 V V_{\kappa} V V_{\kappa} + V V_{\kappa\lambda} V_{\kappa\lambda} + \frac{5}{3} V_{\kappa} V_{\lambda} V_{\kappa\lambda} + \frac{1}{14} V_{\kappa\lambda\mu} V_{\kappa\lambda\mu} \right), \)

\( O_6 = \left( V^6 + 4 V^3 V_{\kappa} V_{\kappa} + 6 V^2 V_{\kappa} V V_{\kappa} + \frac{12}{7} V^2 V_{\kappa\lambda} V_{\kappa\lambda} + \frac{9}{7} V V_{\kappa\lambda} V V_{\kappa\lambda} + \frac{26}{7} V_{\kappa\lambda\mu} V_{\kappa\lambda\mu} + \frac{26}{7} V V_{\kappa\lambda\mu} V_{\kappa\lambda\mu} + \frac{17}{14} V_{\kappa\lambda} V_{\kappa\lambda} V_{\lambda\mu} + \frac{18}{7} V V_{\kappa\lambda} V_{\kappa\lambda} V_{\lambda\mu} + \frac{9}{7} V_{\kappa\lambda} V_{\kappa\lambda} V_{\lambda\mu} \right) \)

\[ + \frac{3}{7} V_{\kappa\lambda\mu} V_{\kappa\lambda\mu} + V_{\mu} V_{\kappa\lambda\mu} V_{\kappa\lambda\mu} + V_{\mu} V_{\kappa\lambda\mu} V_{\kappa\lambda\mu} + \frac{11}{21} V_{\kappa\lambda} V_{\kappa\lambda} V_{\lambda\mu} + \frac{1}{42} V_{\kappa\lambda\mu} V_{\kappa\lambda\mu} \] 

For the higher order coefficients we simply quote the number of terms in Table 1.

| n  | 7   | 8   | 9   | 10  | 11  |
|----|-----|-----|-----|-----|-----|
| # of terms | 37 | 114 | 380 | 1373 | 5301 |

It should be emphasized that no partial integrations have to be used to obtain the result in a unique minimal basis, which does not contain any box operators \( \Box = \partial^2 \) (this is due to the fact that the Green function obeys the property \( G(u_i, u_i) = 0 \), i.e. there are no self–contractions and box operators never occur).
In a gauged theory, however, the worldline path integral method does not yield a minimal basis after identification of cyclic permutations only. This is due to Bianchi identities and the antisymmetry of the field strength tensor. This gives additional relations among the coefficients which have to be used to reduce the result into a minimal basis. To show the principle of the reduction scheme let us restrict to the pure gauge case in the following. For a discussion of the general case including a proof of minimality see [9]. Like in the pure scalar case there are no self–contractions, i.e. the coefficients have the generic form

$$(D \ldots DF) \ldots (D_{\mu_1} \ldots D_{\mu_k} F_{\mu_{k+1}\mu_{k+2}}) \ldots (D \ldots DF)$$  \hspace{1cm} (14)$$

where there is no contraction among the indices $\mu_i$ within a factor $(D \ldots DF)$. Starting from this particular form of the coefficients the reduction is performed with the following steps:

**Step 1:** Consider terms with the following index structure:

$$(D \ldots DF) \ldots (D_{\mu_1} \ldots D_{\mu_i} \ldots D_{\mu_k} F_{\alpha\beta}) \ldots X_{\alpha} \ldots X_{\mu_i} \ldots X_{\beta} \ldots ,$$

where $X$ denotes $D$’s or $F$’s in different factors. In a first step the derivatives are exchanged (producing additional terms with a higher number of $F$’s) until $D_{\mu_i}$ acts on $F_{\alpha\beta}$ directly. Then the Bianchi identity and the antisymmetry of $F$ is used to convert the term into two terms of the structure

$$(D \ldots DF) \ldots (D_{\mu_1} \ldots D_{\mu_{i-1}} D_{\mu_{i+1}} \ldots D_{\mu_k} D_\alpha F_{\mu_i\beta}) \ldots X_{\alpha} \ldots X_{\mu_i} \ldots X_{\beta} \ldots (D \ldots DF) \ldots (D_{\mu_1} \ldots D_{\mu_{i-1}} D_{\mu_{i+1}} \ldots D_{\mu_k} D_\beta F_{\alpha\mu_i}) \ldots X_{\alpha} \ldots X_{\mu_i} \ldots X_{\beta} \ldots .$$

In the first term the derivative $D$ acting on $F$ directly is contracted with the first $X$, whereas it is contracted with the last $X$ in the second term. This has to be done iteratively with all indices $\mu_i$ whose contraction is “nested” between the contractions of $F$.

**Step 2:** Multiple contractions between factors can be converted into a standard form by exchanging $D$’s (again producing terms with a higher number of $F$’s) and using Bianchi identities and the antisymmetry of $F$,

$$(D \ldots DF) \ldots (D \ldots F_{\alpha\beta}) \ldots (D \ldots D_\alpha D_\beta \ldots D_{\epsilon\lambda}) \ldots (D \ldots DF) \rightarrow$$

$$(D \ldots DF) \ldots (D \ldots F_{\alpha\beta}) \ldots (D \ldots D_\alpha F_{\beta\lambda}) \ldots (D \ldots DF) \rightarrow$$

$$(D \ldots DF) \ldots (D \ldots F_{\alpha\beta}) \ldots (D \ldots D_{\alpha\beta}) \ldots (D \ldots DF) .$$

yielding a “double contraction” of $F$’s. However, this kind of reduction is not always possible. There is a special case resulting in two terms:

$$(D \ldots DF) \ldots (D \ldots D_\alpha F_{\beta\kappa}) \ldots (D \ldots D_\beta F_{\alpha\lambda}) \ldots (D \ldots DF) \rightarrow$$

$$(D \ldots DF) \ldots (D \ldots D_\alpha F_{\alpha\beta}) \ldots (D \ldots D_\lambda F_{\alpha\beta}) \ldots (D \ldots DF) +$$

$$(D \ldots DF) \ldots (D \ldots D_\alpha F_{\beta\kappa}) \ldots (D \ldots D_\alpha F_{\beta\lambda}) \ldots (D \ldots DF) .$$
Again there is a double contraction in the first term. The contractions are fixed such that the $D$’s and the first indices of the $F$’s are contracted in the second term.

**Step 3:** One can use the antisymmetry of $F$ and a symmetrization/antisymmetrization of the derivatives for a further reduction. Consider a (remaining) index structure

$$(D \ldots DF) \ldots (D \ldots D_\mu D_\nu F) \ldots (D \ldots DF)(D \ldots DF_\mu) \ldots (D \ldots DF).$$

The antisymmetric part of the term results in a term of higher order in $F$, whereas the symmetric part vanishes due to the antisymmetry of $F_{\mu\nu}$.

**Step 4:** The last step is to fix the ordering of indices. For the derivatives this can be done by exchanging them (producing terms with a higher number of $F$’s again) whereas one can use the antisymmetry in case of the $F$’s.

Because terms with a smaller number of derivatives and higher number of field strength tensors are produced throughout this procedure, one has to start the reduction with the terms containing a maximum number of derivatives. To give an example we quote the seven basis terms left in $O(T^4)$ for the pure gauge case:

$$F_{\mu\nu}F_{\mu\nu}F_{\kappa\lambda}F_{\kappa\lambda}, F_{\mu\nu}F_{\kappa\lambda}F_{\mu\nu}F_{\kappa\lambda}, F_{\mu\nu}F_{\kappa\lambda}F_{\kappa\lambda}F_{\lambda\mu}, F_{\mu\nu}F_{\kappa\lambda}F_{\lambda\mu},$$

$$D_\kappa F_{\mu\nu}D_\lambda F_{\kappa\lambda}, D_\kappa F_{\mu\nu}D_\lambda F_{\kappa\lambda}, D_\kappa D_\lambda F_{\mu\nu}D_\lambda D_\lambda F_{\mu\nu}.$$

In conclusion, we have obtained the inverse mass expansion of the one-loop effective action within a worldline path integral formalism. With a complete computerization of the algorithm we computed the expansion coefficients up to order $O(T^{11})$ for the ungauged case. The application of the reduction scheme to the coefficients in gauge theories is in progress, as well as inclusion of background gravitational fields. The method can be generalized to multiloop calculations in scalar theories and quantum electrodynamics [10].

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