Remarks on random dynamical systems with inputs and outputs,
and a small-gain theorem for monotone RDS

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Abstract

This note introduces a new notion of random dynamical systems with inputs and outputs, and sketches a small-gain theorem for monotone systems which generalizes a similar theorem known for deterministic systems.

1 Introduction

Monotone systems, whose mathematical development was pioneered by Hirsch [9, 10] and Smith [13, 11], play a key role in many application areas, and particularly in the modeling and analysis of biological systems. They constitute a class of dynamical systems for which a rich theory exists, endowing them with very robust dynamical characteristics. In order to analyze interconnections of monotone systems, however, it is necessary to extend the notion and introduce the concept of monotone systems with inputs and outputs, as standard in control theory [14], so as to incorporate input and output channels. This was done in [1], which also presented a result that guarantees stability of monotone systems under negative feedback (so that the closed-loop system is no longer monotone); this result may be viewed as a “small gain theorem” in terms of an appropriate notion of system gain.

An interesting question is: to what extent do results for monotone I/O systems extend to the analysis of systems subject to stochastic uncertainty or random inputs? This paper proposes an approach based upon the notion of “random dynamical system” (RDS) due to Arnold [2], and more specifically the subclass of monotone RDS studied in [5]. Even more than when passing from deterministic monotone systems to monotone systems with inputs and outputs, the generalization is not entirely straightforward, and many subtle mathematical and conceptual details have to be worked out. We introduce the necessary formalism in this note (in a more general context of not necessarily monotone systems) and sketch an analogue of the small gain theorem proved in [1].

2 Random Dynamical Systems

We first review the random dynamical systems framework of Arnold [2]. Along the way we introduce a couple of pieces of terminology not found in [2] to facilitate the discussion. Suppose given a measure preserving dynamical system (MPDS)

\[ \theta = (\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t \in \mathbb{T}}) \]

Arnold [2, page 635] and Chueshov [5, Definition 1.1.1 on page 10] refer to such an object primarily as a metric dynamical system. We find measure preserving, which Arnold also uses as a synonym, less confusing and more informative.
that is, a probability space \((\Omega, \mathcal{F}, \mathbb{P})\), a topological group \((\mathcal{T}, +)\), and a measurable flow \(\{\theta_t\}_{t \in \mathcal{T}}\) of measure preserving maps \(\Omega \rightarrow \Omega\) satisfying (T1)–(T3):

(T1) \((t, \omega) \mapsto \theta_t \omega, (t, \omega) \in \mathcal{T} \times \Omega\), is \((\mathcal{B}(\mathcal{T}) \otimes \mathcal{F})\)-measurable,

(T2) \(\theta_{t+s} = \theta_t \circ \theta_s\) for every \(t, s \in \mathcal{T}\) (semigroup property),

(T3) \(\mathbb{P} \circ \theta_t = \mathbb{P}\) for each \(t \in \mathcal{T}\) (measure preserving).

In this work \(\mathcal{T}\) will always refer to either \(\mathbb{R}\) or \(\mathbb{Z}\), depending on whether one is talking about continuous or discrete time, respectively. In either case \(\mathcal{T}_{\geq 0}\) refers to the nonnegative elements of \(\mathcal{T}\). We will occasionally need to make measure-theoretic considerations about \(\mathcal{T}\) or Borel subsets of it. If \(\mathcal{T} = \mathbb{R}\), that is, in continuous time, then we tacitly equip any Borel subset of \(\mathcal{T}\) with the measure induced by the Lebesgue measure on \(\mathbb{R}\). If \(\mathcal{T} = \mathbb{Z}\), or in discrete time, then we think of the counting measure in \(\mathbb{Z}\). When \(\mathcal{T} = \mathbb{Z}\), it follows from (T2) that \(\theta\) is completely determined by \(\theta_1 = \theta(1, \cdot)\). In that case we will abuse the notation and use the same \(\theta\) to denote both the underlying MPDS and \(\theta_1\).

In the context of a given MPDS \(\theta\), a set \(B \in \mathcal{F}\) is said to be \(\theta\)-invariant if \(\theta_t(B) = B\) for all \(t \in \mathcal{T}\). We say that an MPDS \(\theta\) is ergodic (under \(\mathbb{P}\)) if, whenever \(B \in \mathcal{F}\) is \(\theta\)-invariant, then we have either \(\mathbb{P}(B) = 0\) or \(\mathbb{P}(B) = 1\).

Let \(X\) be a metric space constituting the measurable space \((X, \mathcal{B})\) when equipped with the \(\sigma\)-algebra \(\mathcal{B}\) of Borel subsets of \(X\). A \((\text{continuous})\) random dynamical system (RDS) on \(X\) is a pair \((\theta, \varphi)\) in which \(\theta\) is an MPDS and \(\varphi : \mathcal{T}_{\geq 0} \times \Omega \times X \rightarrow X\) is a \((\text{continuous})\) cocycle over \(\theta\); that is, a \((\mathcal{B}(\mathcal{T}_{\geq 0}) \otimes \mathcal{F} \otimes \mathcal{B})\)-measurable map such that

(S1) \(\varphi(t, \omega) := \varphi(t, \omega, \cdot) : X \rightarrow X\) is continuous for each \(t \in \mathcal{T}_{\geq 0}, \omega \in \Omega\),

(S2) \(\varphi(0, w) = id_X\) for each \(\omega \in \Omega\), and (cocycle property)

\[\varphi(t + s, \omega) = \varphi(t, \theta_s \omega) \circ \varphi(s, \omega), \quad \forall s, t \in \mathcal{T}_{\geq 0}, \forall \omega \in \Omega.\]

The cocycle property generalizes the semigroup property of deterministic dynamical systems. Thus RDS’s include deterministic dynamical systems as the special case in which \(\Omega\) is a singleton.

### 2.1 Trajectories, Equilibria, and \(\theta\)-Stationary Processes

In the context of RDS’s, the analogue to points in the state space \(X\) for a deterministic system are random variables \(\Omega \rightarrow X\), that is, \(\mathcal{B}\)-measurable maps \(\Omega \rightarrow X\). We denote the set of all random variables on a metric space \(X\) by \(X^\Omega\). We refer to a \((\mathcal{B}(\mathcal{T}_{\geq 0}) \otimes \mathcal{F})\)-measurable map \(q : \mathcal{T}_{\geq 0} \times \Omega \rightarrow X\) as a \(\theta\)-stochastic process\(^2\) on \(X\), and denote \(q_t := q(t, \cdot)\) for each \(t \in \mathcal{T}_{\geq 0}\). The set of all \(\theta\)-stochastic processes on a metric space \(X\) is denoted by \(\mathcal{S}_\theta^X\).

Let \((\theta, \varphi)\) be an RDS. Given \(x \in X^\Omega\), we define the \((\text{forward})\) trajectory starting at \(x\) to be the \(\theta\)-stochastic process \(\xi_x \in \mathcal{S}_\theta^X\) defined by

\[\xi_x^t(\omega) := \varphi(t, \omega, x(\omega)), \quad (t, \omega) \in \mathcal{T}_{\geq 0} \times \Omega.\]  

\(^2\) Property (T3) is normally\(^{15}\) Definition 1.1] stated as

\[\mathbb{P}(\theta_t^{-1}(B)) = \mathbb{P}(B), \quad \forall B \in \mathcal{F}, \forall t \in \mathcal{T}.\]

But since it follows from (T2) that \(\theta_t\) is invertible with \(\theta_t^{-1} = \theta_{-t}\) for each \(t \in \mathcal{T}\), the two formulations are equivalent in this context.

\(^3\)A “\(\theta\)-stochastic process” is indeed a stochastic process in the traditional sense. We use the prefix “\(\theta\)” to emphasize the underlying probability space, as well as the time semigroup.
The pullback trajectory starting at $x$ is in turn defined to be the $\theta$-stochastic process $\tilde{\xi}^x : T_{\geq 0} \times \Omega \rightarrow X$ defined by

$$
\tilde{\xi}^x_t(\omega) := \varphi(t, \theta^{-t}\omega, x(\theta^{-t}\omega)), \quad (t, \omega) \in T_{\geq 0} \times \Omega.
$$

More generally, the pullback of a $\theta$-stochastic process $q \in S_\theta^X$ is the $\theta$-stochastic process $\tilde{q} \in S_\theta^X$ defined by

$$
\tilde{q}_t(\omega) := q_t(\theta^{-t}\omega), \quad (t, \omega) \in T_{\geq 0} \times \Omega.
$$

So the pullback trajectory starting at $x$ is simply the pullback of the forward trajectory starting at $x$. We will always use the accent $\tilde{}$ to indicate the pullback of the $\theta$-stochastic process being accented.

We slightly modify the standard notion of equilibrium for RDS’s (see, for instance, [5, Definition 1.7.1 on page 38]) to allow for the defining property to hold only almost everywhere, as opposed to everywhere. So an equilibrium of an RDS $(\theta, \varphi)$ is a random variable $x \in X^\Omega_B$ such that

$$
\xi^x_t(\omega) = \varphi(t, \omega, x(\omega)) = x(\theta_t\omega), \quad \forall t \in T_{\geq 0}, \forall \omega \in \tilde{\Omega},
$$

for some $\theta$-invariant $\tilde{\Omega} \subseteq \Omega$ of full measure. It is often not necessary to specify the said $\tilde{\Omega}$. So we say “for $\theta$-almost all $\omega \in \Omega$” and write

$\forall \omega \in \tilde{\Omega}$

to mean ‘for all $\omega \in \tilde{\Omega}$, for some $\theta$-invariant set $\tilde{\Omega} \subseteq \Omega$ of full measure’.

In view of the notion of pullback convergence with which we will be working (see Subsection 2.3), it is more natural to think of the concept of equilibrium in terms of pullback trajectories. Observe that a $\theta$-stationary $\bar{\varphi}$ of a shift operator in the set $S_\theta^X$ is $\theta$-stationary if and only if

$$
\tilde{\xi}^\bar{\varphi}_t(\omega) = \varphi(t, \theta^{-t}\omega, x(\theta^{-t}\omega)) = x(\omega), \quad \forall t \in T_{\geq 0}, \forall \omega \in \tilde{\Omega}.
$$

The remaining of this section is devoted to interpreting the concept of equilibrium for an RDS in terms of all $\theta$-stochastic processes on $X$. For each $s \in T_{\geq 0}$, let

$$
\rho_s : S_\theta^X \longrightarrow S_\theta^X
$$

be defined by

$$
(\rho_s(q)_t(\omega) := q_{t+s}(\theta^{-s}\omega)), \quad (t, \omega) \in T_{\geq 0} \times \Omega.
$$

**Definition 2.1.** A $\theta$-stochastic process $\tilde{q} \in S_\theta^X$ is said to be $\theta$-stationary if

$$
(\rho_s(\tilde{q})_t(\omega) = \tilde{q}_t(\omega),
$$

for all $s, t \in T_{\geq 0}$, for $\theta$-almost all $\omega \in \Omega$. △

We use the prefix “$\theta$-” in “$\theta$-stationary” to emphasize the dependence on the underlying MPDS $\theta$.

**Lemma 2.2.** The $\theta$-stochastic process $\tilde{q} \in S_\theta^X$ is $\theta$-stationary if and only if there exists a random variable $q \in X^\Omega_B$ such that

$$
\tilde{q}_t(\omega) = q(\theta_t\omega), \quad \forall t \in T_{\geq 0}, \forall \omega \in \Omega.
$$

**Proof.** (Sufficiency) Suppose that (5) holds for some $q \in X^\Omega_B$. Pick any $s \in T_{\geq 0}$. For any $t \in T_{\geq 0}$ and $\theta$-almost all $\omega \in \Omega$,

$$
(\rho_s(\tilde{q})_t(\omega) = \tilde{q}_{t+s}(\theta^{-s}\omega) = q(\theta_{t+s}\theta^{-s}\omega) = q(\theta_t\omega) = \tilde{q}_t(\omega).
$$

So $\tilde{q}$ is $\theta$-stationary.

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4That is, $\theta_t\tilde{\Omega} = \tilde{\Omega}$ for all $t \in T$, and $P(\tilde{\Omega}) = 1$. 

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(Necessity) Suppose that $\bar{q} \in S^\theta$ is $\theta$-stationary and define $q \in X^\Omega_B$ by
\[
q(\omega) := \bar{q}_0(\omega), \quad \omega \in \Omega.
\]
We have
\[
\bar{q}_{t+s}(\theta^{-s}\omega) = (\rho_s(q))_t(\omega) = \bar{q}_t(\omega), \quad \forall s, t \in \mathcal{T}_{\geq 0}, \bar{\omega} \in \Omega.
\]
Setting $t = 0$ and renaming $s$ as $t$ we then have
\[
\bar{q}_t(\theta^{-t}\bar{\omega}) = \bar{q}_0(\bar{\omega}) = q(\bar{\omega}), \quad \forall t \in \mathcal{T}_{\geq 0}, \bar{\omega} \in \Omega.
\]
Given any $\omega \in \hat{\Omega}$ and any $t \in \mathcal{T}_{\geq 0}$, we may apply this property with $\bar{\omega} = \theta_t \omega$ due to the $\theta$-invariance of $\hat{\Omega}$, thus obtaining
\[
\bar{q}_t(\omega) = q(\theta_t \omega).
\]
Therefore (6) holds. □

Note that the random variable $q$ associated to $\bar{q}$ is unique up to a $\theta$-invariant set of measure zero. Indeed, it is determined $\theta$-almost everywhere by Equation (6). Thus, we have:

**Corollary 2.3.** Given an RDS $(\theta, \varphi)$ over a metric space $X$ and a random state $x \in X^\Omega_B$, the following two properties are equivalent:

1. $x$ is an equilibrium;
2. the trajectory $\xi^x$, as defined in Equation (1), is $\theta$-stationary;
3. the map $t \mapsto \hat{\xi}_t^x \in X^\Omega_B$, $t \in \mathcal{T}_{\geq 0}$, is constant.

We will always use an overbar to denote the $\theta$-stationary $\theta$-stochastic process $\bar{q}$ associated with a given random variable $q$.

### 2.2 Perfection of Crude Cocycles

We briefly review the theory of perfection of crude cocycles discussed in Arnold’s [2, Section 1.2]. It is customary for the definition of an RDS to require that the cocycle property of $\varphi$ in (S2) holds for every $s, t \in \mathcal{T}_{\geq 0}$ and every $\omega \in \Omega$. If we want to emphasize this fact we shall say that $\varphi$ is a perfect cocycle (over the underlying MPDS $\theta$).

**Definition 2.4 (Crude Cocycle).** We say that $\varphi: \mathcal{T}_{\geq 0} \times \Omega \times X \to X$ is a crude cocycle (over $\theta$) if it is a $(\mathcal{B}(\mathcal{T}) \otimes \mathcal{F} \otimes \mathcal{B})$-measurable map satisfying (S1) and

- $(S2') \varphi(0, w) = id_X$ for each $\omega \in \Omega$, and for every $s \in \mathcal{T}_{\geq 0}$, there exists a subset $\Omega_s \subseteq \Omega$ of full measure such that
  \[
  \varphi(t+s, \omega) = \varphi(t, \theta_s \omega) \circ \varphi(s, \omega), \quad \forall t \in \mathcal{T}_{\geq 0}, \forall \omega \in \Omega_s.
  \]

The $\Omega_s$’s need not be $\theta$-invariant. △

As Arnold points out, there are circumstances where this flexibility in the requirements for a cocycle is desirable. For instance, the flow of a stochastic differential equation is only guaranteed to be a crude cocycle [2, Section 2.3]. Another example will come up below after we introduce random dynamical systems with inputs. Consider (deterministic) controlled dynamical systems. Such systems yield a (deterministic) dynamical system when restricted to a constant input. One would expect a sensible extension of the concept to random dynamical systems to have an analogous property. However we shall see in the proof of Lemma 3.10 in the next section that the restriction of the flow of an RDS with inputs to a $\theta$-stationary input is not necessarily a perfect cocycle.
**Definition 2.5** (Indistinguishable Crude Cocycles). Let \( \theta \) be an MPDS and \( \varphi, \psi : \mathcal{T}_{\geq 0} \times \Omega \times X \rightarrow X \) crude cocycles over \( \theta \). If there exists a subset \( N \in \mathcal{F} \) such that \( P(N) = 0 \) and
\[
\{ \omega \in \Omega ; \, \varphi(t, \omega) \neq \psi(t, \omega) , \text{ for some } t \in \mathcal{T}_{\geq 0} \} \subseteq N ,
\]
then \( \varphi \) and \( \psi \) are said to be **indistinguishable**. \( \triangle \)

**Definition 2.6** (Perfection of Crude Cocycles). A **perfection** of a crude cocycle \( \varphi \) is a perfect cocycle \( \psi \) (with respect to the same MPDS and evolving in the same state space) such that \( \varphi \) and \( \psi \) are indistinguishable. In this case we say that \( \varphi \) is **perfected by \( \psi \)**, or that \( \psi \) is a **perfection of \( \varphi \)**. \( \triangle \)

In this work we will not need the full power of Arnold’s theory of perfection of crude cocycles. The \( \Omega \)'s of the crude cocycles we shall have to deal with will be actually \( \theta \)-invariant, and so it will be enough to simply redefine the flow on a \( \theta \)-invariant set of measure zero. We nevertheless state the proposition below for the sake of completeness.

**Proposition 2.7.** Let \( \theta = (\mathcal{F}, \Omega, P, (\theta_t)_{t \in \mathcal{T}}) \) be an MPDS with \( \mathcal{T} = \mathbb{Z} \) or \( \mathcal{T} = \mathbb{R} \). Suppose \( \varphi : \mathcal{T}_{\geq 0} \times \Omega \times X \rightarrow X \) is a crude cocycle over \( \theta \) evolving in a locally compact, locally connected, Hausdorff topological space \( X \). Then \( \varphi \) can be perfected; in other words, there exists a perfect cocycle \( \psi : \mathcal{T}_{\geq 0} \times \Omega \times X \rightarrow X \) such that \( \varphi \) and \( \psi \) are indistinguishable.

**Proof.** See Arnold [2, Theorem 1.2.1] for the discrete case, which actually holds with weaker hypotheses and yields stronger conclusions. For the continuous case, see Arnold [2, Theorem 1.2.2 and Corollary 1.2.4]. \( \square \)

### 2.3 Pullback Convergence

We work with the notion of pullback convergence developed in the literature and canonized in the works of Arnold and Chueshov [2, 5].

**Definition 2.8.** (Pullback Convergence) A \( \theta \)-stochastic process \( \xi \in \mathcal{S}^X_\theta \) is said to **converge to a random variable** \( \xi_\infty \in X^Z_\theta \) in the pullback sense if
\[
\tilde{\xi}_t(\omega) := \xi_t(\theta_{-t}\omega) \rightarrow \xi_\infty(\omega) \quad \text{as} \quad t \rightarrow \infty ,
\]
for \( \theta \)-almost all \( \omega \in \Omega \). \( \triangle \)

**Proposition 2.9.** Let \((\theta, \varphi)\) be an RDS evolving on a metric space \( X \). Suppose there exists a random initial state \( x \in X^Z_\theta \) and a map \( x_\infty : \Omega \rightarrow X \) such that
\[
\tilde{\xi}_t^x(\omega) = \varphi(t, \theta_{-t}\omega, x(\theta_{-t}\omega)) \rightarrow x_\infty(\omega) \quad \text{as} \quad t \rightarrow \infty , \quad \forall \omega \in \Omega . \tag{7}
\]
Then \( x_\infty \) is an equilibrium.

**Proof.** For each \( t \in \mathcal{T}_{\geq 0} \), the map \( \omega \mapsto \varphi(t, \theta_{-t}\omega, x(\theta_{-t}\omega)) , \omega \in \Omega \), is measurable, since it is the composition of measurable maps:
\[
\omega \mapsto \varphi(t, \theta_{-t}\omega),
\]
\[
(\theta_{-t}\omega, x(\theta_{-t}\omega)) \mapsto \varphi(t, \theta_{-t}\omega, x(\theta_{-t}\omega)).
\]
So it follows from [12, Chapter 11, §1, Property M7 on page 248] that \( x_\infty \) is measurable. (If \( \mathcal{T} \) is continuous time, just pick a subsequence \((t_n)_{n \in \mathbb{N}}\) going to infinity.)

In addition, for each \( \omega \in \Omega \) such that the limit in **(7)** exists, and each \( \tau \in \mathcal{T}_{\geq 0} \), we have
\[
\lim_{t \rightarrow \infty} \varphi(t - \tau, \theta_{t - \tau}\omega, x(\theta_{t - \tau}\omega)) = x_\infty(\omega)
\]
also. By $\theta$-invariance, the limit in (7) exists for $\theta \tau \omega$ as well. Hence
\begin{align*}
x_\infty(\theta \tau \omega) &= \lim_{t \to \infty} \varphi(t, \theta t \theta \tau \omega, x(\theta t \theta \tau \omega)) \\
&= \lim_{t \to \infty} \varphi(\tau, \theta t \tau - t \omega, \varphi(t - \tau, \theta t - t \omega, x(\theta t - t \omega))) \\
&= \varphi(\tau, \omega, x_\infty(\omega))
\end{align*}
by continuity (property (S1) in the definition of an RDS).

3 RDS’s with Inputs and Outputs

We now define a new concept. It extends the notion of RDS’s to systems in which there is an external input or forcing function. A contribution of this work is the precise formulation of this concept, particularly the way in which the argument of the input is shifted in the semigroup (cocycle) property.

As in the previous section, given a metric space $U$, we equip it with its Borel $\sigma$-algebra $\mathcal{B}(U)$ and denote by $U^\Omega$ the set of Borel measurable maps $\Omega \to U$. Let $S^U_0$ be the set of all $\theta$-stochastic processes $T_{\geq 0} \times \Omega \to U$. Given $u, v \in S^U_0$ and $s \in T_{\geq 0}$, we define $u \circ_v v: T_{\geq 0} \times \Omega \to U$ by
\[
(u \circ_v v)_s(\omega) = \begin{cases} 
   u_s(\omega), & 0 \leq \tau < s \\
   v_{\tau - s}(\theta s \omega), & s \leq \tau 
\end{cases}, \quad \tau \in T_{\geq 0}, \omega \in \Omega.
\]
Given $\hat{u} \in U$, we denote by $c(\hat{u})$ the trivial $\theta$-stochastic process defined by $(c(\hat{u}))_t(\omega) := \hat{u}$ for every $t \in T_{\geq 0}$ and every $\omega \in \Omega$.

**Definition 3.1 (θ-Inputs).** We say that a subset $U \subseteq S^U_0$ is a set of $\theta$-inputs if $c(\hat{u}) \in U$, for every $\hat{u} \in U$, and $u \circ_v v \in U$, for any $u, v \in U$ and any $s \in T_{\geq 0}$. In other words, a set of $\theta$-inputs is a subset of $S^U_0$ which contains all constant inputs and is closed under concatenation.

**Definition 3.2.** A random dynamical system with inputs (RDSI) is a triple $(\theta, \varphi, U)$ consisting of an MPDS $\theta = (\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t \in T})$, a set of $\theta$-inputs $U \subseteq S^U_0$, and a map
\[
\varphi: T_{\geq 0} \times \Omega \times X \times U \to X
\]
satisfying

(i1) $\varphi(\cdot, \cdot, \cdot, u): T_{\geq 0} \times \Omega \times X \to X$ is $(\mathcal{B}(T_{\geq 0}) \otimes \mathcal{F} \otimes \mathcal{B})$-measurable for each fixed $u \in U$;

(i1') the map $\tilde{\varphi}: T_{\geq 0} \times \Omega \times X \times U \to X$ defined by
\[
\tilde{\varphi}(t, \omega, x, \hat{u}) := \varphi(t, \omega, x, c(\hat{u})), \quad (t, \omega, x, \hat{u}) \in T_{\geq 0} \times \Omega \times X \times U,
\]
is $(\mathcal{B}(T_{\geq 0}) \otimes \mathcal{F} \otimes \mathcal{B} \otimes \mathcal{B}(U))$-measurable;

(i2) $\varphi(t, \omega, \cdot, u): X \to X$ is continuous, for each fixed $(t, \omega, u) \in T_{\geq 0} \times \Omega \times U$;

(i3) $\varphi(0, \omega, x, u) = x$ for each $(\omega, x, u) \in \Omega \times X \times U$;

(i4) given $s, t \in T_{\geq 0}$, $\omega \in \Omega$, $x \in X$, $u, v \in U$, if
\[
\varphi(s, \omega, x, u) = y
\]
and
\[
\varphi(t, \theta_x \omega, y, v) = z,
\]
then
\[
\varphi(s + t, \omega, x, u \circ_v v) = z;
\]
We refer to the elements \( u \in \mathcal{U} \) as \( \theta \)-inputs, or simply inputs. Whenever we talk about an RDSI \((\theta, \varphi, \mathcal{U})\), we tacitly assume the notation laid above, unless otherwise specified. △

\((I1), (I1')\) and \((I2)\) are regularity conditions. \((I3)\) means that nothing has “happened” if one is still at time \( t = 0 \). \((I4)\) generalizes the cocycle property and \((I5)\) states that the evolution of an RDS subject to an input \( u \) is, so to speak, independent of “irrelevant” random input values.

**Remark 3.3.** Notice that for each \( s, t \in \mathcal{T}_{\geq 0}, x \in X, \omega \in \Omega, \)

\[ \varphi(t+s, \omega, x, u) = \varphi(t, \theta_s \omega, \varphi(s, \omega, x, u), \rho_s(u)), \quad \forall u \in \mathcal{U}, \]

where \( \rho_s : \mathcal{S}^U_\theta \to \mathcal{S}^U_\theta \) is defined by Equation \((8)\):

\[ (\rho_s(u))_t(\theta_s \omega) \equiv u_{t+s}(\omega). \]

This follows from \((I4)\) with \( v = \rho_s(u) \), which then yields \( u \bowtie_s v = u \). □

The shift operator \( \rho_s \) has a physical interpretation. The right-hand side is the input as interpreted by an observer of the RDSI \( \varphi \) who started at time \( t_1 = 0 \), while the left-hand side is how someone who started observing the system at time \( t_2 = s \) would describe it at time \( t \) \((+ t_2) \). Following this interpretation, a \( \theta \)-stationary input would then be an input which is observed to be just the same, regardless of when one started observing it.

We also introduce a notion of outputs.

**Definition 3.4.** A random dynamical system with inputs and outputs (RDSIO) is a quadruple \((\theta, \varphi, \mathcal{U}, h)\) such that \((\theta, \varphi, \mathcal{U})\) is an RDSI, and

\[ h : \Omega \times X \to Y \]

is an \((\mathcal{F} \otimes \mathcal{B})\)-measurable map into a metric space \( Y \) such that \( h(\omega, \cdot) \) is continuous for each \( \omega \in \Omega \). In this context we call \( h \) an output function and \( Y \) an output space.

It may sometimes be useful to refer to a random dynamical system with outputs (RDSO) only, by which we mean a triple \((\theta, \varphi, h)\) where \((\theta, \varphi)\) is an RDS and \( h \) is an output function. △

The \( \Omega \)-component in the domain of output functions is important. It allows for the concept to model uncertainties in the readout as well. We will return to systems with outputs further down, in the context of RDSIO’s which can be realized as cascades of RDSO’s and RDSIO’s.

**Example 3.5.** (Linear Example) Suppose that \( \mathcal{T} = \mathbb{R} \), and also that \( X = \mathcal{U} = \mathbb{R} \). Let \( \mathcal{U} : = \mathcal{S}^U_c \subseteq \mathcal{S}^U_0 \) be the set of \( \theta \)-inputs consisting of all \( \theta \)-stochastic processes \( u \in \mathcal{S}^U_0 \) such that

\[ t \longmapsto u_t(\omega), \quad t \geq 0, \]

is locally essentially bounded for each \( \omega \in \Omega \). For each \( u \in \mathcal{U} = \mathcal{S}^U_c \), consider the linear Random Differential Equation with Inputs (RDEI)

\[ \dot{\xi} = a(\theta_t \omega) \xi + b(\theta_t \omega) u_t(\omega), \quad t \geq 0, \]

(9)

where \( a, b \in X^\Omega_B \) are such that

\[ t \longmapsto a(\theta_t \omega) \quad \text{and} \quad t \longmapsto b(\theta_t \omega), \quad t \geq 0, \]

\footnote{We will use the same notation \( \rho_s \) for the shift operator \( \mathcal{S}^V_\theta \to \mathcal{S}^V_0 \) defined by Equation \((11)\), irrespective of the underlying metric space \( V \). Since the domain of any \( \theta \)-stochastic process is always \( \mathcal{T}_{\geq 0} \times \Omega \), this will not be a source of confusion.}
are locally essentially bounded for all $\omega \in \Omega$. Then for each $\omega \in \Omega$ and any initial state $x \in X$, the ordinary differential equation (9) has a unique solution $\varphi(\cdot; \omega, x, u)$ defined by

$$\varphi(t; \omega, x, u) := x e^{\int_0^t a(\theta, \omega) \, d\tau} + \int_0^t b(\theta, \omega) u(\omega) e^{\int_\sigma^t a(\theta, \omega) \, d\tau} \, d\sigma, \quad (10)$$

for each $t \geq 0$. This defines an RDSI $\varphi: \mathcal{T}_{\geq 0} \times \Omega \times X \times \mathcal{U} \to X$ over $\theta$. Indeed, (I1) and (I1') follow from the fact that the limit of a sequence of measurable functions is measurable. properties (I2) and (I3) follow directly from (10). And (I4) and (I5) follow from uniqueness of solutions applied for each fixed $\omega \in \Omega$—one basically verifies that both sides of each equation we want to prove to be true, when looked at as functions of $t$, define solutions of the same differential equation with the same initial condition. $\Diamond$

We use $X^0_{\infty}$ instead of the more traditional $L^\infty(\Omega; X)$ to denote the space of essentially bounded, measurable maps $\Omega \to X$. This way we can be more consistent with the notations $X^U_B$ for random variables and $X^U_{\infty}$ for tempered random variables $\Omega \to X$ (see definition in Section 3.3). Similarly, we denote by $S^U_{\infty}$ the subset of $\theta$-inputs consisting of the globally essentially bounded $\theta$-stochastic processes in $S^U_B$.

**Definition 3.6.** (Bounded RDSI) An RDSI $(\theta, \varphi, \mathcal{U})$ is said to be bounded if the random state

$$\omega \longmapsto \varphi(t, \omega, x(\omega), u), \quad \omega \in \Omega,$$

is essentially bounded for every fixed $(t, x, u) \in \mathcal{T}_{\geq 0} \times X^0_{\infty} \times (\mathcal{U} \cap S^U_{\infty})$. $\triangle$

We emphasize the fact that we only test the condition in the definition above for bounded random initial states $x \in X^0_{\infty}$.

### 3.1 Pullback trajectories

Let $(\theta, \varphi, \mathcal{U}, h)$ be an RDSIO with output space $Y$. Given $x \in X^0_{\infty}$ and $u \in \mathcal{U}$, we define the *(forward)* trajectory starting at $x$ and subject to $u$ to be the $\theta$-stochastic process $\xi^{x,u} \in S^X_B$ defined by

$$\xi^{x,u}_t(\omega) := \varphi(t, \omega, x(\omega), u), \quad (t, \omega) \in \mathcal{T}_{\geq 0} \times \Omega.$$

We then define the *(pullback)* trajectory starting at $x$ and subject to $u$ to be the $\theta$-stochastic process $\hat{\xi}^{x,u} \in S^X_B$ defined by

$$\hat{\xi}^{x,u}_t(\omega) := \xi^{x,u}_t(\theta_{-t} \omega) = \varphi(t, \theta_{-t} \omega, x(\theta_{-t} \omega), u), \quad (t, \omega) \in \mathcal{T}_{\geq 0} \times \Omega.$$

The *(forward)* output trajectory corresponding to initial state $x$ and input $u$ is defined to be the $\theta$-stochastic process $\eta^{x,u} \in S^Y_B$, where

$$\eta^{x,u}_t(\omega) := h(\theta_{-t} \omega, \varphi(t, \omega, x(\omega), u)) = h(\theta_{-t} \omega, \xi^{x,u}_t(\omega)), \quad (t, \omega) \in \mathcal{T}_{\geq 0} \times \Omega,$$

while the *(pullback)* output trajectory corresponding to initial state $x$ and input $u$ is analogously defined to be the $\theta$-stochastic process $\hat{\eta}^{x,u} \in S^Y_B$, where

$$\hat{\eta}^{x,u}_t(\omega) := \eta^{x,u}_t(\theta_{-t} \omega) = h(\omega, \varphi(t, \theta_{-t} \omega, x(\theta_{-t} \omega), u)) = h(\omega, \hat{\xi}^{x,u}_t(\omega)), \quad (t, \omega) \in \mathcal{T}_{\geq 0} \times \Omega.$$

For RDSI’s the definitions of forward and pullback trajectories are the same and we also use the notations $\xi^{x,u}$ and $\hat{\xi}^{x,u}$. For RDSO’s the definitions are analogous, except that they of course do not depend on any inputs. So forward and pullback trajectories are defined as for RDS’s and we also use the notations

8
\( \xi^x \) and \( \xi^{x,u} \), respectively. We denote the forward and pullback output trajectories corresponding to initial state \( x \) by \( \eta^x \) and \( \hat{\eta}^x \), respectively:

\[
\eta^x_t(\omega) := h(\theta_t \omega, \varphi(t, \omega, x(\omega))) = h(\theta_t \omega, \xi^t(\omega))
\]

and

\[
\hat{\eta}^x_t(\omega) := h(\omega, \varphi(t, \theta_{-t} \omega, x(\theta_{-t} \omega))) = h(\omega, \xi^t(\omega))
\]

for every \((t, \omega) \in T_{\geq 0} \times \Omega\).

Note that the input \( u \) is not shifted in the argument of \( \varphi \) in the pullback, while at first one might intuitively think it should have been. There are several reasons why this is so. First notice that \( \hat{\xi}^x,u \) is just the pullback of the \( \theta \)-stochastic process \( \xi^x,u \), as it should be the case. However we are more concerned with what happens in the context of cascades and feedback interconnections of RDSIO’s. But before we get to that we first discuss discrete RDSIO’s. This will further motivate axioms (I1)–(I5) in the definition of an RDSI, provide —and completely characterize— a whole class of examples, and provide the framework for said discussion of pullback trajectories and cascades.

We say that an RDSI (or RDSIO) is discrete when \( T = \mathbb{Z} \). We first note that, just like RDS’s \([2, \text{Section 2.1}]\), RDSI’s also have their flows completely determined by their state at time \( t = 1 \).

**Theorem 3.7.** For every discrete RDSI \((\theta, \varphi, U)\), there exists a unique map \( f : \Omega \times X \times U \to X \) such that

\[
\begin{aligned}
(\text{G1}) & \quad f : \Omega \times X \times U \to X \text{ is } (\mathcal{F} \otimes \mathcal{B} \otimes \mathcal{B}(U))\text{-measurable,} \\
(\text{G2}) & \quad f(\omega, \cdot, \tilde{u}) : X \to X \text{ is continuous for each } (\omega, \tilde{u}) \in \Omega \times U,
\end{aligned}
\]

and

\[
\varphi(n + 1, \omega, x, u) = f(\theta_n \omega, \varphi(n, \omega, x, u), u_n(\omega)),
\]

for every \((n, \omega, x, u) \in T_{\geq 0} \times \Omega \times X \times U\).

Conversely, given an MPDS \( \theta \), a set of \( \theta \)-inputs \( U \) and a map 

\[
f : \Omega \times X \times U \to X
\]

satisfying (G1) and (G2), define \( \varphi : T_{\geq 0} \times \Omega \times X \times U \to X \) recursively by

\[
\varphi(0, \omega, x, u) := x, \quad (\omega, x, u) \in \Omega \times X \times U,
\]

and Equation (11). Then \((\theta, \varphi, U)\) is an RDSI.

We refer to the map \( f \) as the generator of the RDSI \((\theta, \varphi, U)\).

**Proof.** Define \( f \) by setting

\[
f(\omega, x, \tilde{u}) := \varphi(1, \omega, x, c(\tilde{u})), \quad (\omega, x, \tilde{u}) \in \Omega \times X \times U.
\]

Then (G1) and (G2) follow directly from (I1’) and (I2), respectively. Equation (11) follows from (I4) (see Remark 3.3 and (I5):

\[
\varphi(n + 1, \omega, x, u) = \varphi(1, \theta_n \omega, \varphi(n, \omega, x, u), \rho_n(u))
\]

\[
= \varphi(1, \theta_n \omega, \varphi(n, \omega, x, u), c(\rho_n(u) \rho_0(\theta_n \omega)))
\]

\[
= f(\theta_n \omega, \varphi(n, \omega, x, u), (\rho_n(u))_0(\theta_n \omega))
\]

\[
= f(\theta_n \omega, \varphi(n, \omega, x, u), u_n(\omega))
\]
for any \((n, ω, x, u) \in T_{≥0} \times Ω \times X \times U\). Uniqueness follows from (I3) and (I5), together with the computations above performed backwards for \(t = 0\).

Now suppose \(f\) satisfies (G1) and (G2), and that \(ϕ\) is defined recursively by (12) and (11). For (I1), pick any \(u \in U\). One first shows using induction on \(n\) that

\[
ϕ(n, ω, x, u) = f(θ_{n-1}, ϕ(n-1, ω, x, u), u_{n-1}(\cdot))
\]

is \((F ⊗ B)\)-measurable for each \(n \in \mathbb{Z}_{≥0}\). Indeed, at \(n = 1\) we have

\[
ϕ(1, ω, x, u) = f(θ_0, ϕ(0, ω, x, u), u_0(\cdot)) = f(\cdot, u_0(\cdot)),
\]

which is \((F ⊗ B)\)-measurable, since \(f\) satisfies (G1) and \(u_0\) is \(F\)-measurable. Now Equation (13) gives us the inductive step, since the righthand side is a composition of measurable functions and, hence, itself measurable. Now pick any \(A \in B\). We then have

\[
ϕ(⋅, ω, x, u)^{-1}(A) = \bigcup_{n=0}^{∞} \{n\} \times ϕ(n, ω, x, u)^{-1}(A) \in 2^{2^{≥0}} \otimes F \otimes B,
\]

since it is a countable union of \((2^{2^{≥0}} \otimes F \otimes B)\)-measurable sets. Thus (I1) holds. One can prove (I1’), in the same way, by noting that

\[
ϕ(⋅, ω)^{-1}(A) = \bigcup_{n=0}^{∞} \{n\} \times ϕ(n, ω)^{-1}(A)
\]

for each \(A \in B\), and that

\[
ϕ(⋅, ω)^{-1}(A) = f(θ_{n-1}, ϕ(n-1, ω, x, u), u_n(ω))
\]

is continuous for any \(ω \in Ω\) and any \(u \in U\) as well.

Property (I3) follows from (G2), (12) and (11), again by induction on \(n \in \mathbb{Z}_{≥0}\). Indeed, at \(n = 0\), \(ϕ(0, ω, x, u)\) is continuous for every \(ω \in Ω\) and every \(u \in U\). So once (I2) has been proved for a certain value of \(n \in \mathbb{Z}_{≥0}\), we conclude that

\[
ϕ(n + 1, ω, x, u) = f(θ_{nω}, ϕ(n, ω, x, u), u_n(ω))
\]

is continuous for any \(ω \in Ω\) and any \(u \in U\) as well.

Before proving (I4) we first prove (I5) by induction on \(n \in \mathbb{Z}_{≥0}\). Fix \(ω \in Ω\), \(x \in X\). Equation (12) gives us the base of the induction. Now assume (I5) holds for a certain value of \(n \in \mathbb{Z}_{≥0}\). If \(u, v \in U\) are such that \(u_j(ω) = v_j(ω)\) for \(j = 0, 1, \ldots, n\), then \(ϕ(n, ω, x, u) = ϕ(n, ω, x, v)\) by the induction hypothesis. So it follows from (11) that

\[
ϕ(n + 1, ω, x, u) = f(θ_{nω}, ϕ(n, ω, x, u), u_n(ω)) = f(θ_{nω}, ϕ(n, ω, x, v), v_n(ω)) = ϕ(n + 1, ω, x, v).
\]

This proves (I5).

It remains to prove (I4). For each arbitrarily fixed \(p \in \mathbb{Z}_{≥0}\), we use induction on \(n \in \mathbb{Z}_{≥0}\). For \(n = 0\), (I4) holds in virtue of (I3) and (I5). For any \(ω \in Ω\), we have \(u_j(ω) = (u_p)_j(ω)\) for \(j = 0, 1, \ldots, p - 1\). Therefore

\[
ϕ(0, θ_{pω}, ϕ(p, ω, x, u), v) = ϕ(p, ω, x, u) = ϕ(0 + p, ω, x, u_{pω}),
\]

for any \(x \in X\). Now suppose (I4) holds for some \(n \in \mathbb{Z}_{≥0}\). Given \(ω \in Ω\) and \(x \in X\), set \(y := ϕ(n, θ_{pω}, x, u)\). Then

\[
ϕ(n + 1, θ_{pω}, y, v) = f(θ_{nθ_{pω}}, ϕ(n, θ_{pω}, y, v), v_n(θ_{pω})) = f(θ_{nθ_{pω}}, ϕ(n + p, ω, x, u_{pω}), u_{pω}(ω)) = ϕ(n + p + 1, ω, x, u_{pω}).
\]

This completes the proof that \((θ, ϕ, U)\) is an RDSI.
Observe that we did not need (I1) in order to prove the first half of the theorem. So we could have in principle dropped this axiom from the definition of an RDSI and an analogous result would still hold. We remind the reader that (I1) was nevertheless used in showing that RDSI’s restricted to \( \theta \)-stationary inputs are RDS’s (see Lemma 3.10 below).

From the construction of the generator \( f \) of an RDSI \((\theta, \varphi, \mathcal{U})\), it is clear how the dependence of the flow \( \varphi \) at time \( n \in \mathbb{Z}_{\geq 0} \) and subject to \( \omega \in \Omega \) on the input \( u \) is really through the value \( u_n(\omega) \) of the input \( u \). So when one shifts the \( \Omega \)-argument \( \omega \) of \( \varphi \) in the pullback trajectory to \( \theta^{-n}\omega \), there is no need to change the input, since \( \varphi(n, \theta^{-n}\omega, x(\theta^{-n}\omega), u) \) depends on \( u_n(\theta^{-n}\omega) \) already. This is our second reason for defining the pullback trajectories of systems with inputs like so.

We now discuss the third and most important reason this is the mathematically sensible way of defining pullback trajectories for RDSI’s. Let \((\theta, \psi)\) be a discrete RDS evolving on the state space \( Z = X_1 \times X_2 \):

\[
\psi: \mathbb{Z}_{\geq 0} \times \Omega \times (X_1 \times X_2) \rightarrow (X_1 \times X_2).
\]

Let \( g: \Omega \times Z \rightarrow Z \) be the generator of \((\theta, \psi)\). Suppose \( g \) can be written as

\[
g(\omega, (x_1, x_2)) \equiv \begin{pmatrix} f_1(\omega, x_1) \\ f_2(\omega, x_2, h_1(\omega, x_1)) \end{pmatrix},
\]

where \( f_1: \Omega \times X_1 \rightarrow X_1 \) is the generator of some RDSO \((\theta, \varphi_1, h_1)\) with output space \( Y_1 \), and \( f_2: \Omega \times X_2 \times U_2 \rightarrow X_2 \) is the generator of some RDSI \((\theta, \varphi_2, \mathcal{U}_2)\) with input space \( U_2 = Y_1 \). Let \( \pi_2: X_1 \times X_2 \rightarrow X_2 \) be the projection onto the second coordinate. We use \( \eta_1 \) to denote the output trajectories of \((\theta, \varphi_1, h_1)\), \( \xi \) for the state trajectories of \( \psi \), and \( \xi_2 \) for the state trajectories of \((\theta, \varphi_2, \mathcal{U}_2)\).

**Theorem 3.8.** For any random initial state

\[ z = (x_1, x_2) \in Z^\Omega_{B(Z)} = (X_1)^\Omega_{B(X_1)} \times (X_2)^\Omega_{B(X_2)}, \]

the following two identities hold:

1. \( \psi(n, \omega, z(\omega)) \equiv \begin{pmatrix} \varphi_1(n, \omega, x_1(\omega)) \\ \varphi_2(n, \omega, x_2(\omega), (\eta_1)^{x_1}(\omega)) \end{pmatrix}, \) and

2. \( \pi_2(\xi_n^z(\omega)) \equiv (\xi_2)^{x_2(\omega), (\eta_1)^{x_1}(\omega)}(\omega). \)

**Proof.** (1) For each fixed \( \omega \in \Omega \) and \( z \in Z^\Omega_{B(Z)} \), we use induction on \( n \in \mathbb{Z}_{\geq 0} \). At \( n = 0 \) we have

\[
\psi(0, \omega, z(\omega)) = z(\omega) = \begin{pmatrix} x_1(\omega) \\ x_2(\omega) \end{pmatrix} = \begin{pmatrix} \varphi_1(0, \omega, x_1(\omega)) \\ \varphi_2(0, \omega, x_2(\omega), (\eta_1)^{x_1}(\omega)) \end{pmatrix}.
\]

Now suppose that (1) holds for some \( n \in \mathbb{Z}_{\geq 0} \). Since

\[
h_1(\theta_n \omega, \varphi_1(n, \omega, x_1(\omega))) = (\eta_1)^{x_1}(\omega)
\]

by definition, it follows that

\[
\psi(n + 1, \omega, z(\omega)) = g(\theta_n \omega, \psi(n, \omega, z(\omega)))
\]

\[
= \begin{pmatrix} f_1(\theta_n \omega, \varphi_1(n, \omega, x_1(\omega))) \\ f_2(\theta_n \omega, \varphi_2(n, \omega, x_2(\omega), (\eta_1)^{x_1}(\omega))) \end{pmatrix}
\]

\[
= \begin{pmatrix} \varphi_1(n + 1, \omega, x_1(\omega)) \\ \varphi_2(n + 1, \omega, x_2(\omega), (\eta_1)^{x_1}) \end{pmatrix}.
\]

This completes the induction.
(2) We prove by induction that (2) holds, for each $n \in \mathbb{N}$, for all random initial states $z = (x_1, x_2) \in Z^0_{B(Z)}$, and all $\omega \in \Omega$. At $n = 0$ we have

$$\pi_2(\xi^0_n(\omega)) = \pi_2(\varphi(0, \omega, (x_1(\omega), x_2(\omega))))$$

$$= x_2(\omega)$$

$$= \varphi_2(0, \omega, x_2(\omega), (\eta_1)^{x_1})$$

$$= (\xi^0_n)^{x_2, (\eta_1)^{x_1}}.$$ 

Now assume (2) has been proved to hold for all integer values of $n$ up to some $n_0 \geq 0$, for all random initial states $z = (x_1, x_2) \in Z^0_{B(Z)}$ and all $\omega \in \Omega$. Given $z = (x_1, x_2) \in Z^0_{B(Z)}$, define $\hat{z} = (\hat{x}_1, \hat{x}_2) \in Z^0_{B(Z)}$ by

$$\hat{z}(\omega) = g(\theta_{-1}\omega, z(\theta_{-1}\omega))$$

$$:= \left(f_1(\theta_{-1}\omega, x_1(\theta_{-1}\omega)), f_2(\theta_{-1}\omega, x_2(\theta_{-1}\omega), h_1(\theta_{-1}\omega, x_1(\theta_{-1}\omega)))\right), \quad \omega \in \Omega. \quad (15)$$

We have $(\eta_1)^{\hat{z}_1} = \rho_1((\eta_1)^{x_1})$ by Lemma 3.9 below, and also

$$h_1(\theta_{-(n_0+1)}\omega, x_1(\theta_{-(n_0+1)}\omega)) = (\eta_1)^{\hat{z}_1}(\theta_{-(n_0+1)}\omega), \quad \omega \in \Omega.$$ 

Fix $\omega \in \Omega$ arbitrarily and denote $\hat{\omega} := \theta_{-(n_0+1)}\omega$. Then

$$\pi_2(\xi^{n_0+1}_{\hat{z}_1}(\omega)) = \pi_2(\varphi(n_0 + 1, \hat{\omega}, z(\hat{\omega})))$$

$$= \pi_2(\varphi(n_0, \theta_{-n_0}\omega, \psi(1, \hat{\omega}, z(\hat{\omega})))))$$

$$= \pi_2(\varphi(n_0, \theta_{-n_0}\omega, g(\hat{\omega}, z(\hat{\omega}))))$$

$$= \pi_2(\varphi(n_0, \theta_{-n_0}\omega, \hat{z}(\theta_{-n_0}\omega)))$$

$$= \pi_2(\xi^{\hat{z}_{n_0}}(\omega))$$

$$= (\xi^{\hat{z}_{n_0}}(\eta_1)^{x_1}(\omega).$$ 

by the induction hypothesis. Now

$$(\xi^{\hat{z}_{n+1}}(\eta_1)^{x_1}(\omega) = \varphi_2(n_0, \theta_{-n_0}\omega, \hat{x}_2(\theta_{-n_0}\omega), (\eta_1)^{\hat{z}_1})$$

$$= \varphi_2(n_0, \theta_{-n_0}\omega, f_2(\hat{\omega}, x_2(\hat{\omega}), (\eta_1)^{\hat{z}_1}(\hat{\omega})), (\eta_1)^{\hat{x}_2})$$

$$= \varphi_2(n_0, \theta_{-n_0}\omega, \varphi_2(1, \hat{\omega}, x_2(\hat{\omega}), (\eta_1)^{\hat{z}_1}), (\eta_1)^{\hat{z}_1})$$

$$= \varphi_2(n_0 + 1, \theta_{-(n_0+1)}\omega, x_2(\theta_{-(n_0+1)}\omega), (\eta_1)^{\hat{z}_1})$$

$$= (\xi^{\hat{z}_{n+1}}_{n+1}(\eta_1)^{x_1}(\omega).$$ 

So

$$\pi_2(\xi^{\hat{z}_{n+1}}(\omega)) = (\xi^{\hat{z}_{n+1}}_{n+1}(\eta_1)^{x_1}(\omega).$$ 

Since $z = (x_1, x_2) \in Z^0_{B(Z)}$ and $\omega \in \Omega$ were arbitrary, this completes the inductive step. \qed

The left hand side of (2) in the proposition above is the projection over the second coordinate of the pullback trajectory starting at $z = (x_1, x_2)$ of the RDS $(\theta, \varphi)$. The right hand side is the pullback trajectory of the RDSI $(\theta, \varphi_2, U_2)$ starting at $x_2$ and subject to the input $(\eta_1)^{\hat{z}_1}$, the output trajectory of $(\theta, \varphi_1, h_1)$ starting at $x_1$. Theorem 3.8 then says that they coincide. An analogous result holds in continuous time for systems generated by random differential equations. These provide the motivation for the definition of cascades of systems with inputs and outputs, an introductory discussion of which is carried out in Subsection 4.2.

We now state and prove the technical lemma referred to in the proof of item (2) in Theorem 3.8.
for every \( n \)

The concept of RDSI subsumes that of an RDS, as we shall see below. Denote the subset of \( \theta \) convention that an overbar is used to indicate the \( \theta, \phi \),

Let \( \) and \( \phi \)

\[ \text{Proof.} \] Indeed, we have \( \]

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Proof. Let \( \bar{\Omega} \) be the \( \theta \)-invariant subset of full measure of \( \Omega \) from the proof of Lemma 3.10 and denote \( N := \Omega \setminus \bar{\Omega} \). Note that \( N \) is \( \theta \)-invariant and \( \mathbb{P}(N) = 0 \). Define \( \psi_u : \mathcal{T}_{\geq 0} \times \Omega \times X \to X \) by

\[
\psi_u(t, \omega, x) := \begin{cases} 
\varphi_u(t, \omega, x), & \text{if } \omega \in \bar{\Omega}, \\
x, & \text{if } \omega \in N.
\end{cases}
\]  

(17)

Pick any open subset \( A \subseteq X \). Then

\[ \psi_u^{-1}(A) = (\varphi_u^{-1}(A)) \cup \mathcal{T}_{\geq 0} \times N \times X \cup \mathcal{T}_{\geq 0} \times N \times A. \]

Thus \( \psi_u^{-1}(A) \in \mathcal{B}(\mathcal{T}_{\geq 0}) \otimes \mathcal{F} \otimes \mathcal{B} \), proving that \( \psi_u \) is \( (\mathcal{B}(\mathcal{T}_{\geq 0}) \otimes \mathcal{F} \otimes \mathcal{B}) \)-measurable.

If \( \omega \in \bar{\Omega} \), then \( \psi_u(t, \omega, \cdot) = \varphi_u(t, \omega, \cdot) = \varphi(t, \omega, \cdot, u) \), which is continuous for any \( t \geq 0 \) by (I2). And if \( \omega \in N \), then \( \psi_u(t, \omega, \cdot) = \text{id}_X \), and thus also continuous for any \( t \geq 0 \). This shows \( \psi_u \) satisfies (S1).

It is clear from (17) that \( \psi_u(0, \omega, \cdot) = \text{id}_X \) for any \( \omega \in \Omega \). We already know from the proof of Lemma 3.10 that \( \psi_u \) satisfies the cocycle property for every \( \omega \in \Omega \). Similar computations using (17) together with the fact that \( N \) is also \( \theta \)-invariant show that \( \psi_u \) also satisfies the cocycle property for values of \( \omega \) in \( N \). This proves (S2), completing the proof that \( (\theta, \psi_u) \) is an RDS.

Since \( \mathbb{P}(N) = 0 \), we conclude that \( \varphi_u \) and \( \psi_u \) are indistinguishable. Hence \( \psi_u \) is a perfection of \( \varphi_u \). □

Whenever the state space \( X \) is such that \( \varphi_u \) can be perfected, we shall assume that \( \varphi_u \) has already been replaced by an indistinguishable perfection and then refer to the resulting RDS \( (\theta, \varphi_u) \).

3.3 Input to State Characteristics

**Definition 3.12.** Let \( (\theta, \varphi, \mathcal{U}) \) be an RDSI, and suppose that \( \bar{u} \in \mathcal{U} \cap \bar{\mathcal{S}}^U_0 \), with corresponding random variable \( u \) (see Lemma 2.2). An **equilibrium** associated to \( u \) (or to \( \bar{u} \)) is any equilibrium \( \xi \) of the RDS \( (\theta, \varphi_u) \). The set of all equilibria associated to \( \bar{u} \) is denoted as \( \mathcal{E}(\bar{u}) \) (or \( \mathcal{E}(u) \)). △

In other words, an element \( x \in \mathcal{E}(\bar{u}) \) is a random variable \( \Omega \to X \) such that

\[ \varphi_u(t, \theta_{-t}\omega, x(\theta_{-t}\omega)) = x(\omega), \quad \forall t \in \mathcal{T}_{\geq 0}, \bar{\nu}_\omega \in \Omega. \]  

(18)

When we have a “proper” RDS (no inputs), we write simply \( \mathcal{E} \) for the set of equilibria.

Though not really used in this work, we take advantage of the concepts and notation being introduced to present the following quick observation:

**Proposition 3.13.** Let \( (\theta, \varphi, h) \) be an RDSIO and suppose that \( \xi \in \mathcal{E}(\mu) \). Then the output trajectory \( \eta^\xi : \mathcal{T}_{\geq 0} \times \Omega \to Y \) starting at \( \xi \) is \( \theta \)-stationary.

**Proof.** Indeed, it follows from (18) that, for any \( t, s \geq 0 \) and any \( \omega \in \Omega \),

\[
(\rho_s(\eta^\xi))_t(\omega) = \eta^\xi_{t+s}(\theta_{-s}\omega) = h(\theta_{t+s}\theta_{-s}\omega, \varphi(t+s, \theta_{-s}\omega, \xi(\theta_{-s}\omega), \bar{\mu})) = h(\theta_t\omega, \varphi(t+s, \theta_{-(s+t)}\theta_t\omega, \xi(\theta_{-(s+t)}\theta_t\omega), \bar{\mu})) = h(\theta_t\omega, \xi(\theta_t\omega)) = h(\theta_t\omega, \varphi(t, \omega, \xi(\omega), \bar{\mu})) = \eta^\xi_t(\omega),
\]

proving that \( \rho_s(\eta^\xi) = \eta^\xi \) for all \( s \in \mathcal{T}_{\geq 0} \), that is, \( \eta^\xi \) is \( \theta \)-stationary. □

**Definition 3.14.** Let \( (\theta, \varphi, h) \) be an RDSIO, and suppose that \( \xi \in \mathcal{E}(\mu) \). Then \( \eta^\xi_0 \) is an **output equilibrium** associated to \( \mu \) (or \( \bar{\mu} \)). The set of all output equilibria associated to \( \mu \) is denoted as \( \mathcal{K}(\mu) \). △
For deterministic systems ($\Omega$ is a singleton), when the set $\mathcal{E}(\bar{u})$ consists of a single globally attracting equilibrium, the mapping $u \mapsto \mathcal{E}(\bar{u})$ is the object called the “input to state characteristic” in the literature on monotone i/o systems. For systems with outputs, composition with the output map $h$ provides the “input to output” characteristic. We extend this notion to RDSI’s. For reasons which will be illustrated in Example 3.20 and become clearer in the proofs of Theorems 4.2 and 4.3 (converging input to output characteristic [1]. We extend this notion to RDSI’s. For reasons which will be illustrated in Example 3.20 and become clearer in the proofs of Theorems 4.2 and 4.3 (converging input to output characteristic [1]. We extend this notion to RDSI’s.

In what follows, given an MPDS $\theta$ and a normed space $(X, \| \cdot \|)$, we denote by $X_\theta^\Omega$ the space of tempered random variables $\Omega \to X$; that is, the space of $\mathcal{F}$-measurable maps $r: \Omega \to X$ such that

$$\sup_{s \in T} \| r(\theta_s \omega) \| e^{-\gamma|s|} < \infty, \quad \forall \gamma > 0, \quad \forall \omega \in \bar{\Omega}.$$ 

For ease of reference we note a few easy properties of tempered random variables:

**Lemma 3.15.** Suppose $\theta$ is an MPDS, $(X, \| \cdot \|)$ is a normed space over $\mathbb{R}$, and let $R_1, R_2 \in X_\theta^\Omega$, $r \in \mathbb{R}_\theta^\Omega$, and $c \in \mathbb{R}$. Then

1. $R_1 + R_2$ is tempered.
2. $cr_1$ is tempered.
3. $rR_1$ is tempered; in particular, the product of two real-valued tempered random variables is tempered.

**Proof.** (1) Indeed, for any $\gamma > 0$ and any $\omega \in \bar{\Omega}$, we have

$$\sup_{s \in T} \| (R_1 + R_2)(\theta_s \omega) \| e^{-\gamma|s|} = \sup_{s \in T} \| R_1(\theta_s \omega) \| e^{-\gamma|s|} + \sup_{s \in T} \| R_2(\theta_s \omega) \| e^{-\gamma|s|} < \infty,$$

where we write $(R_1 + R_2)(\theta_s \omega)$ for $R_1(\theta_s \omega) + R_2(\theta_s \omega)$. So both $R_1 + R_2$ is tempered.

(2) follows from (3), which we now prove. Given $\gamma > 0$ and $\omega \in \bar{\Omega}$, apply the definition of tempered random variable for $\gamma/2$:

$$\sup_{s \in T} \| r(\theta_s \omega) R_1(\theta_s \omega) \| e^{-\gamma|s|} = \sup_{s \in T} \| r(\theta_s \omega) e^{-\frac{\gamma}{2}|s|} \| R_1(\theta_s \omega) \| e^{-\frac{\gamma}{2}|s|} \leq \left( \sup_{s \in T} \| r(\theta_s \omega) e^{-\frac{\gamma}{2}|s|} \| \right) \left( \sup_{s \in T} \| R_1(\theta_s \omega) \| e^{-\frac{\gamma}{2}|s|} \right) < \infty.$$ 

Thus $rR_1$ is tempered.

In other words, $X_\theta^\Omega$ is a real vector space, and a module over the ring of real-valued tempered random variables.

**Definition 3.16.** (Tempered Convergence) Let $\theta$ be an MPDS, $X$ be a normed space, $(\xi_\alpha)_{\alpha \in A}$ be a net in $X_\theta^\Omega$ and $\xi_\infty$ any random variable in $X_\theta^\Omega$. We say that $(\xi_\alpha)_{\alpha \in A}$ converges to $\xi_\infty$ in the tempered sense if there exists a nonnegative, tempered random variable $r: \Omega \to \mathbb{R}_{\geq 0}$ and an $\alpha_0 \in A$ such that

1. $\xi_\alpha(\omega) \to \xi_\infty(\omega)$ as $\alpha \to \infty$ for $\theta$-almost all $\omega \in \Omega$, and
2. $\|\xi_\alpha(\omega) - \xi_\infty(\omega)\| \leq r(\omega)$ for all $\alpha \geq \alpha_0$, for $\theta$-almost all $\omega \in \Omega$.

In this case we denote $\xi_\alpha \xrightarrow{\theta} \xi_\infty$ (as $\alpha \to \infty$).
**Definition 3.17.** (Tempered Continuity) Let $\theta$ be an MPDS and $X, U$ normed spaces. A map $K: U \subseteq \Omega \to X^\Omega_{\theta}$ is said to be tempered continuous if, whenever $(u_\alpha)_{\alpha \in A}$ is a net in $U$ convergent to $u_\infty \in U$ in the tempered sense, then $K(u_\alpha) \to_\theta K(u_\infty)$ as $\alpha \to \infty$ as well.

In what follows, when we speak of a given RDS ({$\theta, \varphi$}), or of an RDSI ({$\theta, \varphi, U$}), etc, the underlying state space $X$, input space $U$ and output space $Y$ are all assumed to be normed spaces, unless otherwise specified.

**Definition 3.18.** (I/S Characteristic) An RDSI ({$\theta, \varphi, U$}) is said to have an input to state (i/s) characteristic $K: U^\Omega_{\theta} \to X^\Omega_{\theta}$ if

$$U^\Omega_{\theta} \subseteq U$$

and

$$\varphi_u(t, \theta-t\omega, x(\theta-t\omega)) \to (K(u))(\omega), \quad \text{as } t \to \infty, \quad \forall \omega \in \Omega,$$

for each $u \in U^\Omega_{\theta}$, and each $x \in X^\Omega_{\theta}$.

**Definition 3.19.** Let ({$\theta, \varphi, U$}) be an RDSI with an i/s characteristic $K: U^\Omega_{\theta} \to X^\Omega_{\theta}$. We say that $K$ is a tempered i/s characteristic if

$$\xi^x_{t, u} \to_\theta K(u), \quad \forall x \in X^\Omega_{\theta}, \quad \forall u \in U^\Omega_{\theta}. $$

If

$$\xi^x_{t, u} \to_\theta K(u), \quad \forall x \in X^\Omega_{\theta}, \quad \forall u \in U^\Omega_{\infty}, $$

then $K$ is said to be a bounded i/s characteristic.

Notice that $U^\Omega_{\infty} \subseteq U^\Omega_{\theta}$, so that the definition of bounded i/s characteristic is well-posed.

**Example 3.20.** (Linear Example) Consider the RDSI ({$\theta, \varphi, U$}) from Example 3.5 generated by the random differential equation with inputs

$$\dot{\xi} = a(\theta, \omega)\xi + b(\theta, \omega)u(t, \omega), \quad t \geq 0, \quad u \in U := S^U_{c}.$$

Suppose that, in addition to the hypotheses in Example 3.5, $a, b$ also satisfy

(L1) $b$ is tempered; and

(L2) there exist a $\lambda > 0$ and nonnegative, tempered random variables $\gamma, \tilde{\gamma} \in (R_+)^\Omega_{\theta}$ such that

$$e^{\int_{s}^{t} a(\theta, \omega) d\tau} \leq \gamma(\theta, \omega) e^{-\lambda \tau}, \quad \forall s \in \mathbb{R}, \quad \forall r \geq 0, \quad \forall \omega \in \Omega,$$

and

$$e^{\int_{s}^{t} a(\theta, \omega) d\tau} \leq \tilde{\gamma}(\theta, \omega) e^{-\lambda \tau}, \quad \forall s \in \mathbb{R}, \quad \forall r \geq 0, \quad \forall \omega \in \Omega.$$

**Remark 3.21.** If we assume $\theta$ to be ergodic and

$$\mathbb{E}a := \int_{\Omega} a(\omega) d\mathbb{P}(\omega) < 0,$$

then one can show (see [5]) that (L2) holds with $\lambda := -\mathbb{E}a$. □

We summarize the point of this example in the lemma and proposition below.

**Lemma 3.22.** Suppose that, in addition to the hypotheses in Example 3.5, the coefficients $a$ and $b$ also satisfy (L1) and (L2). Then the RDSI ({$\theta, \varphi, U$}) has a tempered continuous input to state characteristic.

**Proposition 3.23.** Suppose that, in addition to satisfying the hypotheses in Example 3.5, plus (L1) and (L2), $b$ is also essentially bounded, and $a$ is essentially bounded from above (as functions of $\omega \in \Omega$). Then the RDSI ({$\theta, \varphi, U$}) is bounded (in the sense of Definition 3.6), and has a continuous and bounded input to state characteristic (in the sense of Definitions 3.17, 3.18 and 3.19).
We prove both these results together.

Pick any \( u \in U^\Omega_0 \). Then (L1) and (L2) imply (32) that the limit

\[
\lim_{t \to -\infty} \varphi(t, \theta^{-t} \omega, x(\theta^{-t} \omega), \bar{u}) = \int_{-\infty}^{0} b(\theta_{s} \omega) u(\theta_{s} \omega) e^{\int_{s}^{0} a(\theta_{r} \omega) \, dr} \, d\sigma \tag{19}
\]

exists for \( \theta \)-almost every \( \omega \in \Omega \), for any tempered initial state \( x \in X^\Omega_0 \). Let \( \mathcal{K}(u) : \Omega \to X \) be the map defined \( \theta \)-almost everywhere in \( \Omega \) by

\[
(\mathcal{K}(u))(\omega) := \int_{-\infty}^{0} b(\theta_{s} \omega) u(\theta_{s} \omega) e^{\int_{s}^{0} a(\theta_{r} \omega) \, dr} \, d\sigma .
\tag{20}
\]

By Equation (19) and Proposition 2.9, \( \mathcal{K}(u) \) is an equilibrium of \( (\theta, \varphi_u) \).

\( \mathcal{K}(u) \) is tempered. Fix \( \delta > 0 \) arbitrarily. By (L2), for each \( s \in \mathbb{R} \), we have

\[
|\mathcal{K}(u)(\theta_s \omega)| e^{-\delta |s|} \leq \int_{-\infty}^{0} |b(\theta_{s+r} \omega) u(\theta_{s+r} \omega) \gamma(\theta_{s+r} \omega)| e^{-(\lambda|\sigma|+\delta|s|)} \, d\sigma \\
\leq \int_{-\infty}^{0} |b(\theta_{s+r} \omega) u(\theta_{s+r} \omega) \gamma(\theta_{s+r} \omega)| e^{-m|\sigma|} \, d\sigma \\
\leq K_{\omega, \delta} \int_{-\infty}^{0} e^{-m|\sigma|/2} \, d\sigma \\
\leq K_{\omega, \delta} \int_{-\infty}^{0} e^{-m\sigma/2} \, d\sigma,
\]

which is finite and depends only on \( \omega \) and \( \delta \)—in the computations above we used \( m := \min\{\lambda, \delta\} > 0 \) and

\[
K_{\omega, \delta} := \sup_{s \in \mathbb{R}} |b(\theta_{s} \omega) u(\theta_{s} \omega) \gamma(\theta_{s} \omega)| e^{-m|s|/2} < \infty.
\]

\( K_{\omega, \delta} \) being finite follows from the hypotheses that \( b, u, \gamma \) are tempered and the fact that the product of tempered random variables is also tempered (Lemma 3.15 (3)).

Observe that \( u \in U^\Omega_0 \) was chosen arbitrarily. Therefore this shows that (20) defines an input to state characteristic \( \mathcal{K} : U^\Omega_0 \to X^\Omega_0 \) for \( (\theta, \varphi, \mathcal{U}) \). We now show that \( \mathcal{K} \) is tempered continuous in the sense of Definition 3.17 and bounded in the sense of Definition 3.19.

Continuity. Suppose that \( u \in \mathcal{U} \) converges to \( u_{\infty} \in U^\Omega_0 \) in the tempered sense, with corresponding \( r \in (\mathbb{R}_{\geq 0})^\theta \), and \( \alpha = t_0 \geq 0 \) (see Definition 3.18). Then \( \mathcal{K}(\bar{u}_t) \to \mathcal{K}(u_{\infty}) \) as well. Indeed, for \( t \geq t_0 \), we have

\[
|\mathcal{K}(\bar{u}_t)(\omega) - \mathcal{K}(u_{\infty})(\omega)| = \int_{-\infty}^{0} |b(\theta_{s} \omega)||u_t(\theta_{s} \omega) - u_{\infty}(\theta_{s} \omega)| e^{\int_{s}^{0} a(\theta_{r} \omega) \, dr} \, d\sigma \\
\leq \int_{-\infty}^{0} |b(\theta_{s} \omega)||\gamma(\theta_{s} \omega) e^{-\lambda|\sigma|} \, d\sigma \\
=: R(\omega), \quad \forall \omega \in \Omega.
\]

Now computations along the lines of the ones above showing that \( \mathcal{K}(u) \) is tempered will show that the random variable \( R \) so defined is also tempered.

This completes the proof of Lemma 3.22. We now discuss Proposition 3.23.

\( \mathcal{K} \) is bounded. Fix \( u \in U^\Omega_\infty \) arbitrarily. If \( x \in X^\Omega_\infty \), then

\[
|x(\theta^{-t} \omega) e^{\int_{0}^{t} a(\theta^{-r} \omega) \, dr} - 0| = |x(\theta^{-t} \omega) e^{\int_{0}^{t} a(\theta^{-r} \omega) \, dr} | \\
\lesssim \|x\|_{\infty} \gamma(\omega) e^{-\lambda t} \\
\lesssim \|x\|_{\infty} \gamma(\omega) e^{-\lambda t}, \quad \forall t \geq 0, \quad \forall \omega \in \Omega.
\]

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Whenever

Most often the underlying partial order will be clear from the context and we shall use simply “partially ordered (\(\leq\))”.

Furthermore, \[ |\int_0^t b(\theta_{\tau-t}\omega) u(\theta_{\tau-t}\omega) e^{\int_0^\tau a(\theta_{\tau-t}\omega) d\tau} \, d\sigma| \]

\[ - |\int_{-\infty}^0 b(\theta_{\tau}\omega) u(\theta_{\tau}\omega) e^{\int_0^\tau a(\theta_{\tau}\omega) d\tau} \, d\sigma| \]

\[ = \left| \int_{-\infty}^{-t} b(\theta_{\tau}\omega) u(\theta_{\tau}\omega) e^{\int_0^\tau a(\theta_{\tau}\omega) d\tau} \, d\sigma \right| \]

\[ \leq \int_{-\infty}^0 |b(\theta_{\tau}\omega) u(\theta_{\tau}\omega)| \gamma(\theta_{\tau}\omega) e^{-\lambda |\tau|} \, d\sigma \]

\[ =: R_3(\omega), \quad \forall \omega \in \Omega, \]

the random variable \(R_1\) so defined being tempered as per computations such as above. This shows that \(K(\xi_t^{\omega,a}) \to_\theta K(u)\) as \(t \to \infty\), for every \(x \in X_{\Omega}^\infty\), and for every \(u \in U_{\Omega}^\infty\). Therefore \(K\) is bounded.

Finally, we show that \(\varphi\) is bounded. Now suppose that \(b\) is essentially bounded and, in addition to satisfying (L2), \(a\) is bounded from above almost everywhere by a constant \(M\). Then for any \(x \in X_{\Omega}^\infty\),

\[ |x(\theta_{-t}\omega) e^{\int_0^t a(\theta_{-t}\omega) d\tau}| = |x(\theta_{-t}\omega) e^{\int_t^0 a(\theta_{-t}\omega) d\tau}| \]

\[ \leq \|x\|_\infty e^M t, \quad \forall t \geq 0, \quad \forall \omega \in \Omega. \]

And if \(u \in U \cap S_{\Omega}^U\), then

\[ |\int_0^t b(\theta_{\tau-t}\omega) u(\theta_{\tau-t}\omega) e^{\int_0^\tau a(\theta_{\tau-t}\omega) d\tau} \, d\sigma| \leq \|b\|_\infty \|u\|_\infty \int_0^t e^{M(t-\sigma)} \, d\sigma < \infty, \]

for all \(t \geq 0\) and \(\theta\)-almost all \(\omega \in \Omega\). This shows that \(\varphi\) is bounded in the sense of Definition 4.6. This completes the proof of Proposition 3.23. \(\diamondsuit\)

4 Monotone RDSI’s

If \((X, \leq)\) is a partially ordered space and \(p, q \in S_{\theta}^X\), we write \(p \leq q\) to mean that \(p(t, \omega) \leq q(t, \omega)\) for all \(t \in T_{\geq 0}\) and all \(\omega \in \Omega\).

Definition 4.1. An RDSI \((\theta, \varphi, U)\) is said to be monotone if the underlying state and input spaces are partially ordered \((X, \leq_X), (U, \leq_U)\), and

\[ \varphi(t, \omega, x, u) \leq_X \varphi(t, \omega, z, v) \]

whenever \(x, z \in X\) and \(u, v \in S_{\theta}^U\) are such that \(x \leq_X z\) and \(u \leq_U v\). \(\triangle\)

Most often the underlying partial order will be clear from the context and we shall use simply “\(\leq\)” to denote either of “\(\leq_X\)”, “\(\leq_U\)” or “\(\leq_Y\)”.

4.1 Converging Input to Converging State

Recall the convention of using a check mark \(\check{}\) above the symbol for a given \(\theta\)-stochastic process to denote its corresponding pullback flow.
We first show that \( \varphi \) trajectories of \( K \) and \( \omega \) for each \( \tau \), then \( u \in U \) and \( u_\infty \in U_\theta^\Omega \) are such that

\[
\tilde{u}_t \longrightarrow_\theta u_\infty, \quad (21)
\]

then

\[
\xi_t^{x,u} \longrightarrow_\theta K(u_\infty), \quad \forall x \in X_\theta^\Omega.
\]

In other words, if the pullback trajectory of \( u \) converges to \( u_\infty \) in the tempered sense, then the pullback trajectories of \( \varphi \) subject to the given input \( u \) and starting at any tempered random state \( x \) will converge to \( K(u_\infty) \) as well.

Proof. For each \( \tau \geq 0 \), let \( a_\tau, b_\tau \in U_\theta^\Omega \) be defined by

\[
a_\tau(\omega) := \inf_{t \geq \tau} u_t(\theta^{-t}\omega) \quad (22)
\]

and

\[
b_\tau(\omega) := \sup_{t \geq \tau} u_t(\theta^{-t}\omega) \quad (23)
\]

for each \( \omega \in \Omega \), where the inf and sup are taken coordinatewise. It follows from (21) that

\[
a_\tau(\omega), b_\tau(\omega) \longrightarrow u_\infty(\omega), \quad \text{as} \quad \tau \rightarrow \infty, \quad \forall \omega \in \Omega.
\]

Let \( t_0 \geq 0 \) and \( r \in (\mathbb{R}_{\geq 0})_\theta \) be such that

\[
|u_t(\theta^{-t}\omega) - u_\infty(\omega)| \leq r(\omega), \quad \forall \tau \geq t_0, \quad \forall \omega \in \Omega.
\]

So it follows from the continuity of the Euclidean norm in \( \mathbb{R}^n \) that

\[
|a_\tau(\omega) - u_\infty(\omega)|, |b_\tau(\omega) - u_\infty(\omega)| \leq r(\omega), \quad \forall \tau \geq t_0, \quad \forall \omega \in \Omega.
\]

also. In other words, \( a_\tau, b_\tau \longrightarrow u_\infty \) as \( \tau \rightarrow \infty \). Moreover,

\[
a_\tau(\theta^t\omega) \leq u_t(\theta^t\omega) \leq b_\tau(\theta^t\omega), \quad t \geq \tau \geq 0, \quad \omega \in \Omega.
\]

For each \( \tau \geq 0 \), let \( \bar{a}_\tau, \bar{b}_\tau \) be the \( \theta \)-stationary stochastic processes generated by \( a_\tau, b_\tau \), respectively. Then

\[
(\bar{a}_\tau)_s(\omega) = a_\tau(\theta^s\omega) = a_\tau(\theta^{s+\tau}\theta^{-\tau}\omega) \leq u_{\tau+s}(\theta^{-\tau}\omega) = (\rho_\tau(u))_s(\omega),
\]

and, similarly,

\[
(\rho_\tau(u))_s(\omega) \leq (\bar{b}_\tau)_s(\omega), \quad s, \tau \geq 0, \quad \omega \in \Omega.
\]

Thus

\[
\bar{a}_\tau \leq \rho_\tau(u) \leq \bar{b}_\tau, \quad \tau \geq 0. \quad (24)
\]

Fix arbitrarily \( x_0 \in X_\theta^\Omega \). For any \( \omega \in \Omega \) and any \( t \geq \tau \geq 0 \), we have

\[
|\xi_t^{x_0,u}(\omega) - (K(u_\infty))(\omega)| \leq |\tilde{\xi}_t^{x_0,u}(\omega) - \xi_t^{x_0,\bar{a}_\tau}(\omega)| + |\xi_t^{x_0,\bar{a}_\tau}(\omega) - (K(a_\tau))(\omega)| + |(K(a_\tau))(\omega) - (K(u_\infty))(\omega)|.
\]

We first show that

\[
|\xi_t^{x_0,u}(\omega) - (K(u_\infty))(\omega)| \longrightarrow 0, \quad \text{as} \quad t \rightarrow \infty, \quad \forall \omega \in \Omega. \quad (25)
\]
Given $\epsilon > 0$, it follows from the continuity of $K$ that there exists $\tau_0 \geq 0$ such that
\[
|\langle K(a_\tau) \rangle(\omega) - (K(u_\infty))(\omega)\rangle, |\langle K(b_\tau) \rangle(\omega) - (K(u_\infty))(\omega)\rangle| < \epsilon/6, \quad \tau \geq \tau_0.
\]
Now we can use the definition of input to state characteristic to choose $t_0 \geq \tau_0$ such that
\[
|\xi_t^{x_0, a_{t_0}}(\omega) - (K(a_{t_0}))(\omega)\rangle| < \epsilon/6, \quad t \geq t_0.
\]
Using the cocycle property we may write
\[
\xi_t^{x_0, u}(\omega) = \varphi(t - \tau_0, \theta_{-t} - \tau_0)\omega, \varphi(\tau_0, \theta_{-t} - \tau_0)\omega, x_0(\theta_{-t} - \tau_0)\omega, u, \rho_{\tau_0}(u)) = \varphi(s, \theta_{-s} - \tau_0)\omega, x_1(\theta_{-s} - \tau_0)\omega, \rho_{\tau_0}(u)) = \xi_s^{x_1, \rho_{\tau_0}(u)}(\omega)
\]
where $x_1 \in X_0^\Omega$ is defined by $x_1(\omega) := \varphi(\tau_0, \theta_{\tau_0} - \tau_0, x_0(\theta_{\tau_0} - \tau_0)\omega, u, \omega \in \Omega$, and we write $s := t - \tau_0$. Now by monotonicity
\[
\xi_s^{x_1, \rho_{\tau_0}(u)}(\omega) \leq \xi_s^{x_1, \rho_{\tau_0}(u)}(\omega) \leq \xi_s^{x_1, \rho_{\tau_0}(u)}(\omega),
\]
hence, using that $x \leq y \leq z$ in the positive orthant order implies $|y - x| \leq |z - x|$, we have
\[
|\xi_s^{x_1, \rho_{\tau_0}(u)}(\omega) - \xi_s^{x_1, \rho_{\tau_0}(u)}(\omega)\rangle| \leq |\xi_s^{x_1, \rho_{\tau_0}(u)}(\omega) - \xi_s^{x_1, \rho_{\tau_0}(u)}(\omega)\rangle| + |\langle K(b_{\tau_0}) \rangle(\omega) - (K(u_\infty))(\omega)\rangle| + |\langle K(u_0) \rangle(\omega) - (K(a_{\tau_0}))\rangle(\omega)\rangle| + |\langle K(a_{\tau_0}) \rangle(\omega) - \xi_s^{x_1, \rho_{\tau_0}(u)}(\omega)\rangle| + |\langle K(b_{\tau_0}) \rangle(\omega) - \xi_s^{x_1, \rho_{\tau_0}(u)}(\omega)\rangle| + |\langle K(a_{\tau_0}) \rangle(\omega) - \xi_s^{x_1, \rho_{\tau_0}(u)}(\omega)\rangle| + \epsilon/3,
\]
for every $s \geq 0$. Again from the definition of input to state characteristic, one can choose $s_0 \geq 0$ large enough so that
\[
|\xi_s^{x_1, \rho_{\tau_0}(u)}(\omega) - (K(b_{\tau_0}))(\omega))|, |\langle K(a_{\tau_0}) \rangle(\omega) - \xi_s^{x_1, \rho_{\tau_0}(u)}(\omega)\rangle| < \epsilon/6, \quad s \geq s_0.
\]
It then follows that
\[
|\xi_t^{x_0, u}(\omega) - (K(u_\infty))(\omega)\rangle| < \epsilon/2 + \epsilon/3 + \epsilon/3 = \epsilon, \quad t \geq \max\{t_0, \tau_0 + s_0\}.
\]
And since $\epsilon > 0$ was arbitrary, (26) holds.

It remains to show that the convergence occurs in the tempered sense. Indeed, pick $\tau_1, \tau_2, l_3, s_4, s_5 \geq 0$ and $r_1, r_2, r_3, r_4, r_5 \in (\mathbb{R}_{\geq 0})^\Omega$ such that
\[
|\langle K(a_{\tau}) \rangle(\omega) - (K(u_\infty))(\omega)\rangle| \leq r_1(\omega) \quad \tau \geq \tau_1, \quad \nu_\omega \in \Omega,
\]
\[
|\langle K(b_{\tau}) \rangle(\omega) - (K(u_\infty))(\omega)\rangle| \leq r_2(\omega) \quad \tau \geq \tau_2, \quad \nu_\omega \in \Omega,
\]
\[
|\xi_t^{x_0, a_{\tau_0}}(\omega) - (K(a_{\tau_0}))(\omega))\rangle| \leq r_3(\omega) \quad t \geq t_3, \quad \nu_\omega \in \Omega,
\]
where $\tau_0 := \max\{\tau_1, \tau_2\}$, and where we may assume that $t_3 \geq \tau_0$,
\[
|\xi_s^{x_1, \rho_{\tau_0}(u)}(\omega) - (K(b_{\tau_0}))(\omega))\rangle| \leq r_4(\omega) \quad s \geq s_4, \quad \nu_\omega \in \Omega,
\]
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variable Lemma 3.15, the theorem follows by setting $R$, where $x_2 \in X^\Omega$ is defined by $x_2(\omega) := \varphi(\bar{\theta}_0, \theta_0\omega, x_0(\theta_0\omega)), \omega \in \Omega$. Then similar estimates as the ones we just carried out for the almost everywhere pointwise convergence give us

$$|\xi^{x_2,a_\tau}(\omega) - (\mathcal{K}(u_\infty))(\omega)| \leq r_1(\omega) + r_3(\omega) + (r_4(\omega) + r_2(\omega) + r_1(\omega) + r_5(\omega)),$$

for all $t \geq T_0 := \max\{t_3, \bar{\tau}_0 + s_4, \bar{\tau}_0 + s_5\}$, and for almost every $\omega \in \Omega$. Since the sum of tempered random variables is still a tempered random variable—Lemma 3.15 (1)—, the theorem follows by setting $R := 2r_1 + r_2 + r_3 + r_4 + r_5$.

**Theorem 4.3 (Bounded CICS).** Let $(\theta, \varphi, U)$ be a monotone RDSI with state space $X = \mathbb{R}^n$ and input space $U = \mathbb{R}^k$, both equipped with the usual positive orthant-induced partial order. Suppose that $\varphi$ is bounded—see Definition 3.19—and has a continuous and bounded i/s characteristic $\mathcal{K}: U^\Omega \rightarrow X^\Omega$. If $u \in U \cap S^U_\infty$ and $u_\infty \in U^\Omega_\infty$ are such that

$$\hat{u}_t \rightarrow u_\infty,$$

then

$$\xi^{x,u}_t \rightarrow_{\theta} \mathcal{K}(u_\infty), \quad \forall x \in X^\Omega.$$

In other words, if the pullback trajectory of $u$ converges to $u_\infty$ in the tempered sense, then the pullback trajectories of $\varphi$ subject to the given input $u$ and starting at any bounded random state $x$ will converge to $\mathcal{K}(u_\infty)$.

**Proof.** The proof is essentially the same as the proof of Theorem 4.2 with just a couple of extra observations. The assumption that $\varphi$ is bounded will cause the random variable $x_2$ defined along said proof to be also bounded. So pullback trajectories starting at $x_2$ and subject to a $\theta$-stationary input will converge to the appropriate state characteristic. Moreover, the hypothesis that $u \in U \cap S^U_\infty$ implies that the random variables $a_\tau, b_\tau$ are also bounded. So

$$\xi^{x,a_\tau}_t \rightarrow_{\theta} \mathcal{K}(a_\tau)$$

and

$$\xi^{x,b_\tau}_t \rightarrow_{\theta} \mathcal{K}(b_\tau)$$

as $t \rightarrow \infty$ for any $x \in X^\Omega_\infty$, for any $\tau \geq 0$. And so the estimates at the end of the proof of temperedness of the convergence

$$\xi^{x,u}_t \rightarrow_{\theta} \mathcal{K}(u_\infty)$$

in Theorem 4.2 still hold. The result follows.

Other than the tempered convergence and continuity notions in the hypotheses, the key element in the proofs of Theorems 4.2 and 4.3 is the existence and measurability of the inf’s and sup’s in Equations (22) and (23). Therefore these results can be generalized to a wider class of spaces and orders satisfying enough geometric conditions for that to happen.

### 4.2 Cascades

We now discuss a few corollaries of the ‘converging input to converging state’ theorems just proved. Separate work in preparation deals with a small-gain theorem for random dynamical systems.

Let $(\theta, \psi)$ be an autonomous RDS evolving on a space $Z = X_1 \times X_2$. We say that $(\theta, \psi)$ is *cascaded* if the flow $\psi$ can be decomposed as

$$\psi(t, \omega, (x_1(\omega), x_2(\omega))) \equiv \left( \begin{array}{c} \varphi_1(t, \omega, x_1(\omega)) \\ \varphi_2(t, \omega, x_2(\omega), (\eta_1)^x_1) \end{array} \right),$$

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for some RDSO \((\theta, \varphi_1, h_1)\) with state space \(X_1\) and output space \(Y_1\), and some RDSI \((\theta, \varphi_2, U_2)\) with state space \(X_2\), input space \(U_2 = Y_1\), and set of \(\theta\)-inputs \(U_2\) containing all (forward) output trajectories of \((\theta, \varphi_1, h_1)\). In this case we write \(\psi = \varphi_1 \otimes \varphi_2\). Recall from item (1) in Theorem 4.3 that if the generator of a discrete RDS can be decomposed as in Equation (14), then the said RDS is a cascade. A similar decomposition can be done for systems generated by random differential equations with inputs whose generator satisfies the natural analogues of Equation (14).

**Example 4.4 (Bounded Outputs).** Let \((\theta, \psi) = (\theta, \varphi_1 \otimes \varphi_2)\) be a cascaded RDS as above. Suppose that \((\theta, \varphi_1, h_1)\) is an RDSO evolving on a normed space \(X_1\) such that \((\theta, \varphi_1)\) has a unique, globally attracting equilibrium \((\xi_1)_\infty \in X_1^\Omega\):

\[
(\xi_1)^{t_1}_\xi(\omega) \rightarrow (\xi_1)_\infty(\omega), \quad \text{as } t \to \infty, \quad \forall \omega, x_1 \in (X_1)^\Omega.
\]

Suppose further that the output function \(h_1\) is bounded; that is, there exists \(M \geq 0\) such that

\[
h_1(\omega, x_1) \leq M, \quad \forall x_1 \in X_1, \quad \forall \omega \in \Omega.
\]

We have

\[
(\bar{h}_1)^{t_1}_\xi(\omega) = h_1(\omega, (\xi_1)^{t_1}_\xi(\omega)) \rightarrow h_1(\omega, (\xi_1)_\infty(\omega)), \quad \text{as } t \to \infty, \quad \forall \omega, x_1 \in (X_1)^\Omega.
\]

And since \(h_1\) is bounded, the convergence is automatically tempered. We denote \((u_2)_\infty := h_1(\cdot, (\xi_1)_\infty(\cdot))\). The hypothesis that \(h_1\) is bounded also guarantees that \((u_2)_\infty\) is tempered.

Now assume that that \((\theta, \varphi_2, U_2)\) satisfies the hypotheses of Theorem 4.2. It then follows that \((\theta, \psi)\) has a globally attracting equilibrium:

\[
\bar{\xi}^t_\xi(\omega) \rightarrow \left(\frac{(\xi_1)_\infty(\omega)}{(\xi_2)_\infty(\omega)}\right) \quad \text{as } t \to \infty, \quad \forall \omega, z \in \mathcal{Z}_\theta^\Omega,
\]

where \((\xi_2)_\infty := \mathcal{K}((u_2)_\infty)\). In particular, the convergence in the second coordinate is tempered.

For conditions guaranteeing that an RDS \((\theta, \varphi)\) would have a unique, globally attracting equilibrium in the sense above, see [4, Theorem 3.2]. The assumption that the output is bounded is very reasonable in biological applications, since there is often a cut off or saturation on the reading of the strength of a signal.

Before we consider the next example, we develop a stronger notion of regularity for output functions. We seek a property which preserves tempered convergence, and which we could check it holds in specific examples.

**Definition 4.5 (Tempered Lipschitz).** An output function \(h: \Omega \times X \to Y\) is said to be tempered Lipschitz (with respect to a given MPDS \(\theta\)) if there exists a tempered random variable \(L \in (\mathbb{R}_{\geq 0})_\theta^\Omega\) such that

\[
\|h(\omega, x_1) - h(\omega, x_2)\| \leq L(\omega)\|x_1 - x_2\|, \quad \forall x_1, x_2 \in X, \quad \forall \omega \in \Omega.
\]

We refer to \(L\) as a Lipschitz random variable for \(h\).

For example, suppose that \(X \subseteq \mathbb{R}^n\), and that \(h: \Omega \times X \to \mathbb{R}^k\) is an output function such that \(h(\omega, \cdot)\) is differentiable for all \(\omega\) in a \(\theta\)-invariant set of full measure \(\Omega \subseteq \Omega\). If the norm of the Jacobian with respect to \(x\),

\[
\omega \mapsto \|D_x h(\omega, \cdot)\| := \sup_{x \in X} |D_x h(\omega, x)|, \quad \omega \in \Omega,
\]

is tempered, then \(h\) is tempered Lipschitz.

**Lemma 4.6.** Let \(h: \Omega \times X \to Y\) be a tempered Lipschitz continuous output function, \(p \in \mathcal{S}_\theta^X\) be a \(\theta\)-stochastic process in \(X\), and let \(p_\infty \in X_1^\Omega\). Let \(q: \mathcal{T}_{\geq 0} \times \Omega \to Y\) be the \(\theta\)-stochastic process in \(Y\) defined by

\[
q_t(\omega) := h(\omega, p_t(\omega)), \quad (t, \omega) \in \mathcal{T}_{\geq 0} \times \Omega,
\]

where \(\mathcal{T}_{\geq 0}\) is the set of starting times.
and \( q_\infty \in Y_{\mathbb{R}^\Omega} \) be the random variable in \( Y \) defined by
\[
g_\infty(\omega) := h(\omega, p_\infty(\omega)), \quad \omega \in \Omega.
\]
If \( p_t \to \theta p_\infty \), then \( q_t \to \theta q_\infty \).

**Proof.** It follows from continuity with respect to \( x \in X \) that
\[
q_t(\omega) = h(\omega, p_t(\omega)) \to h(\omega, p_\infty(\omega)) = p_\infty(\omega), \quad \omega \in \Omega.
\]
Because \( p_t \to \theta p_\infty \), there exist \( r \in (\mathbb{R}_{\geq 0})^\Omega \) and \( t_0 \geq 0 \) such that
\[
\|p_t(\omega) - p_\infty(\omega)\| \leq r(\omega), \quad \forall t \geq t_0, \quad \omega \in \Omega.
\]
Let \( L \) be a Lipschitz random variable for \( h \). Then
\[
\|q_t(\omega) - q_\infty(\omega)\| = \|h(\omega, p_t(\omega)) - h(\omega, p_\infty(\omega))\|
\leq L(\omega)\|p_t(\omega) - p_\infty(\omega)\|
\leq L(\omega)r(\omega), \quad \forall t \geq t_0, \quad \omega \in \Omega.
\]
By item (3) in Lemma 3.15 \( Lr \) is tempered, which completes the proof. \( \square \)

Now suppose that \((\theta, \psi, U)\) is an RDSI evolving on a state space \( Z = X_1 \times X_2 \). In this case we say that \((\theta, \psi, U)\) is *cascaded* if the flow \( \psi \) can be decomposed as
\[
\psi(t, \omega, (x_1(\omega), x_2(\omega)), u) \equiv \begin{pmatrix} \varphi_1(t, \omega, x_1(\omega), \eta_1) \\ \varphi_2(t, \omega, x_2(\omega), \eta_2) \end{pmatrix},
\]
for some RDSIO \((\theta, \varphi_1, U_1, h_1)\) with state space \( X_1 \), set of \( \theta \)-inputs \( U_1 = U \) and output space \( Y_1 \), and some RDSI \((\theta, \varphi_2, U_2)\) with state space \( X_2 \), input space \( U_2 = Y_1 \), and set of \( \theta \)-inputs \( U_2 \) containing all (forward) output trajectories of \((\theta, \varphi_1, U_1, h_1)\). In this case we also write \( \psi = \varphi_1 \otimes \varphi_2 \). Item (1) in Theorem 3.8 can be generalized to contemplate this kind of cascades for discrete systems, as well as systems generated by random differential equations.

**Example 4.7 (Tempered Lipschitz Outputs).** Suppose that \((\theta, \varphi_1, U_1)\) and \((\theta, \varphi_2, U_2)\) from the decomposition above both satisfy the hypotheses of Theorem 4.2. If the output function \( h_1 \) is Lipschitz continuous, then \((\theta, \psi, U)\) also has the ‘converging input to converging state’ property; that is, if \( u \in U \) is such that \( \bar{u}_t \to \theta u_\infty \) for some \( u_\infty \in U_\theta^0 \), then there exists a \( \xi_\infty \in Z_\theta^\Omega \) such that
\[
\tilde{\xi}_t^Z \to \theta \xi_\infty, \quad \forall z \in Z_\theta^\Omega,
\]as well.

To see this, let \( K_1: (U_1)^\Omega \to (X_1)^\Omega \) and \( K_2: (U_2)^\Omega \to (X_2)^\Omega \) be the \( \i/s \) characteristics of \((\theta, \varphi_1, U_1)\) and \((\theta, \varphi_2, U_2)\), respectively. Fix
\[
z = (x_1, x_2) \in Z_\theta^\Omega = (X_1)^\Omega \times (X_2)^\Omega
\]arbitrarily. From Theorem 4.2, we have
\[
(\xi_1)^{x_1, u} \to \theta K_1(u_\infty).
\]
Since \( h_1 \) is tempered Lipschitz, it follows that
\[
(\eta_1)^{x_1, u} \to \theta (u_2)_\infty,
\]
where
\[
(u_2)_\infty := h_1(\cdot, K_1(u_\infty)(\cdot)).
\]

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It follows, again from Theorem 4.2, that
\[
(\xi_2_t^x, y_1, y_2^x) \xrightarrow{\mu} \mathcal{K}_2((u_x)_\infty).
\]
Hence
\[
(\xi_t^{x, u}) = \left( (\xi_1_t^{x, u}) \mathcal{K}_2((u_x)_\infty) \right).
\]
Since \( z \in \mathbb{Z}^2 \) was picked arbitrarily, this establishes (26).

The procedure above can be generalized to cascades of three or more systems to show that the ‘converging input to converging state’ property will hold provided that it holds for its components—and the intermediate outputs are tempered Lipschitz. Even if is possible to check directly that \((\theta, \varphi, U)\) already satisfies the hypotheses of Theorem 4.2, it might be easier to check them for each component. \(\triangle\)

Examples 4.4 and 4.7 illustrate how one can obtain global convergence results for systems decomposable into cascades. In the next section we discuss “closed loop” systems.

5 Small-Gain Theorem

5.1 Feedback Interconnections

We start by establishing what it means for the feedback interconnection of two RDSIO’s to be well-posed. Given two RDSIO’s, we want to consider an RDS obtained by plugging the output of one of the systems in as an input for the other, and viceversa. Let \((\theta, \varphi, U, h)\) be a discrete RDSIO with state space \(X\), input space \(U\), and output space \(Y\). Suppose further that the set \(U\) of \(\theta\)-inputs of \((\theta, \varphi, U, h)\) contains all (forward) output trajectories of \((\theta, \varphi, U, h)\), \(i = 1, 2, j = 2, 1\). Even if is possible to check directly that \((\theta, \varphi, U)\) already satisfies the hypotheses of Theorem 4.2, it might be easier to check them for each component.

\textbf{Definition 5.1.} We say that the feedback (interconnection) of the RDSIO’s \((\theta, \varphi, U, h_1)\) and \((\theta, \varphi, U, h_2)\) is well-posed provided that, for each \(x_1 \in X_1\) and each \(x_2 \in X_2\),

1. there exist \(\mu = \mu_{x_1, x_2} \in U_1\) and \(\nu = \nu_{x_1, x_2} \in U_2\) such that
   \[
   \mu_t(\omega) = h_1(\theta_t, \varphi_1(t, \omega), x_1, \mu),
   \]
   \[
   \nu_t(\omega) = h_2(\theta_t, \varphi_2(t, \omega, x_2, \nu)),
   \]
   for all \(t \in \mathbb{Z}^2\) and all \(\omega \in \Omega\);

2. if \(\mu' \in U_1, \nu' \in U_2\) are such that
   \[
   \mu'_t(\omega) = h_1(\theta_t, \varphi_1(t, \omega, x_1, \mu')),
   \]
   \[
   \nu'_t(\omega) = h_2(\theta_t, \varphi_2(t, \omega, x_2, \nu')),
   \]
   then \(\mu'(\omega) = \mu(\omega)\) and \(\nu'(\omega) = \nu(\omega)\) for \(\theta\)-almost all \(\omega \in \Omega\).

That is, \(\mu_{x_1, x_2}, \nu_{x_1, x_2}\) must exist, and be unique in a certain sense for the feedback interconnection to be well-posed. \(\triangle\)

One can show using arguments along the lines of the constructions in Section 5.1 that for discrete systems the feedback interconnections are always well-posed. The same is true of RDSIO’s generated by RDSE’s.

\textbf{Proposition 5.2.} Suppose that the feedback interconnection of \((\theta, \varphi, U, h_1)\) and \((\theta, \varphi, U, h_2)\) is well-posed. Then \(\psi: \mathbb{T}_{\geq 0} \times \Omega \times (X_1 \times X_2) \rightarrow X_1 \times X_2\), defined by
\[
\psi(t, \omega, (x_1, x_2)) := (\varphi_1(t, \omega, x_1, \mu_{x_1, x_2}), \varphi_2(t, \omega, x_2, \nu_{x_1, x_2})),
\]
\((t, \omega, (x_1, x_2)) \in \mathbb{T}_{\geq 0} \times \Omega \times (X_1 \times X_2),\)

satisfies the cocycle property and \((\theta, \psi)\) is an RDS with state space the Cartesian product \(X_1 \times X_2\).
Lemma 5.3. Suppose that the feedback interconnection of \((\theta, \varphi_1, U_1, h_1)\) and \((\theta, \varphi_2, U_2, h_2)\) is well-posed. Then, the following two properties are equivalent:

1. for the feedback system, \((\bar{\xi}, \bar{\zeta}) \in \mathcal{E}\);
2. there exists a pair of \(\theta\)-constant inputs \(\mu\) and \(\nu\) such that 
   \[ \bar{\mu} \in \mathcal{K}(\bar{\nu}), \ \bar{\nu} \in \mathcal{K}(\bar{\mu}), \ \bar{\xi} \in \mathcal{E}(\bar{\mu}), \ \bar{\zeta} \in \mathcal{E}(\bar{\nu}). \]

5.2 Small-Gain Theorem: Discussion

A small-gain theorem for anti-monotone systems follows by the same steps as in the deterministic case \([1, 7]\). The key idea is to assume that characteristics are continuous in our tempered sense and that a small gain condition is satisfied. The main theorem will be for an anti-monotone RDSIO so that iterations of characteristics converge to a unique equilibrium, which amounts (subject to mild technical conditions on cones and the space \(X\)) to the requirement that \(\mathcal{K}\) has no period two points except for a unique equilibrium, in the sense that there exists some \(\mu_0\) so that

\[ \bar{\mu} \in \mathcal{K}(\mathcal{K}(\bar{\mu})) \Rightarrow \bar{\mu} = \bar{\mu}_0. \]

We assume, further, that the feedback connection of this system with itself is well-posed and monotone. Then (almost) every solution of the closed-loop system converges to \(\mathcal{E}(\bar{\mu})\). A proof as in \([1, 7]\) begins by appealing to the CICS (Converging Input/Converging State) property for monotone RDSIO’s, and establishes a contraction property on the “limsup” and “liminf” (defined as in these references) of external signals. An alternative approach is based on the idea in \([8]\), in which one first proves that the feedback of two monotone or two anti-monotone systems is monotone. More precisely, if \((\theta_1, \varphi_1, U_1, h_1)\) and \((\theta_2, \varphi_2, U_2, h_2)\) are both anti-monotone RDSIO’s, we consider the following order in \(X_1 \times X_2\):

\[ (x, z) \leq (\bar{x}, \bar{z}) \iff x \leq \bar{x} \text{ and } \bar{z} \leq z \]

(that is, the order on the second component is reversed). If, instead, \((\theta_1, \varphi_1, U_1, h_1)\) and \((\theta_2, \varphi_2, U_2, h_2)\) are both monotone RDSIO’s, we consider the product order in \(X_1 \times X_2\):

\[ (x, z) \leq (\bar{x}, \bar{z}) \iff x \leq \bar{x} \text{ and } z \leq \bar{z}. \]

One then shows that the feedback system is monotone, and the small-gain condition assures that there is a unique equilibrium (a.e.) for the composite system, allowing one to appeal to the theorem in \([3]\).

References

[1] D. Angeli and E.D. Sontag. Monotone control systems. *IEEE Trans. Automat. Control*, 48(10):1684–1698, 2003.
[2] L. Arnold. *Random Dynamical Systems*. Springer, 2010.
[3] F. Cao and J. Jiang. On the global attractivity of monotone random dynamical systems. *Proceedings of The American Mathematical Society*, 138(3):891–898, March 2010.
[4] F. Cao and Jifa Jiang. On the global attractivity of monotone random dynamical systems. *Proceedings of the American Mathematical Society*, 138(3):891–898, 2010.
[5] I. Chueshov. *Monotone Random Systems—Theory and Applications*. Springer, 2002.
[6] G.A. Enciso, H.L. Smith, and E.D. Sontag. Non-monotone systems decomposable into monotone systems with negative feedback. *J. of Differential Equations*, 224:205–227, 2006.
[7] G.A. Enciso and E.D. Sontag. Global attractivity, I/O monotone small-gain theorems, and biological delay systems. *Discrete Contin. Dyn. Syst.*, 14(3):549–578, 2006.

[8] Gerald B. Folland. *Real Analysis: Modern Techniques and Their Applications*. Wiley, second edition, 1999.

[9] M. Hirsch. Differential equations and convergence almost everywhere in strongly monotone flows. *Contemporary Mathematics*, 17:267–285, 1983.

[10] M. Hirsch. Systems of differential equations that are competitive or cooperative ii: Convergence almost everywhere. *SIAM J. Mathematical Analysis*, 16:423–439, 1985.

[11] M. Hirsch and H.L. Smith. Monotone dynamical systems. In *Handbook of Differential Equations, Ordinary Differential Equations (second volume)*. Elsevier, Amsterdam, 2005.

[12] S. Lang. *Real Analysis*. Addison-Weasley, second edition, 1983.

[13] H. Smith. *Monotone Dynamical Systems: An Introduction to the Theory of Competitive and Cooperative Systems*, Mathematical Surveys and Monographs, vol. 41. AMS, Providence, RI, 1995.

[14] E.D. Sontag. *Mathematical Control Theory. Deterministic Finite-Dimensional Systems*, volume 6 of *Texts in Applied Mathematics*. Springer-Verlag, New York, second edition, 1998.

[15] P. Walters. *An Introduction to Ergodic Theory*. Springer, 2000.