Heat Semigroups on Weyl Algebra

Ivan G. Avramidi

Department of Mathematics
New Mexico Institute of Mining and Technology
Socorro, NM 87801, USA
E-mail: ivan.avramidi@nmt.edu

We study the algebra of semigroups of Laplacians on the Weyl algebra. We consider two sets of operators $\nabla_\pm^i$ forming the Lie algebra $[\nabla_\pm^j, \nabla_\pm^k] = i R^\pm_{jk}$ and $[\nabla_+^j, \nabla_-^k] = i \frac{1}{2} (R^+_{jk} + R^-_{jk})$ with some anti-symmetric matrices $R^\pm_{ij}$ and define the corresponding Laplacians $\Delta_\pm = g^{ij}_\pm \nabla_\pm^i \nabla_\pm^j$ with some positive matrices $g^{ij}_\pm$. We show that the heat semigroups $\exp(t \Delta_\pm)$ can be represented as a Gaussian average of the operators $\exp(\langle \xi, \nabla_\pm \rangle)$ and use these representations to compute the product of the semigroups, $\exp(t \Delta_+) \exp(s \Delta_-)$ and the corresponding heat kernel.
1 Introduction

Elliptic partial differential operators on manifolds play a crucial role in global analysis, spectral geometry and mathematical physics \[13, 11, 1, 3, 4, 5, 6\]. The spectrum of elliptic operators does, of course, depend on the geometry of the manifold. Therefore, one can ask the question: “Does the spectrum of an elliptic operator describe the geometry?” It is well known now that the answer to this question is negative, that is, there are non-isometric manifolds that have the same spectrum (see, e.g. \[14, 16, 11\]). That is why, it makes sense to study more general invariants of partial differential operators, or, even, a collection of operators, that might contain more information about the geometry of the manifold. Such invariants are not necessarily spectral invariants that only depend on the eigenvalues of the operators; they depend, rather, on both the eigenvalues and the eigensections.

In our paper \[10\] we (with B. J. Buckman) initiated the study of a new invariant of second-order elliptic partial differential operators that we called heat determinant. In our paper \[7\] we studied so called relativistic heat trace of a Laplace type operator \(L\),

\[
\Theta_r(\beta) = \text{Tr} \exp(-\beta \omega),
\]

where \(\omega = \sqrt{L}\) and \(\beta\) is a positive parameter (that plays the role of the temperature), as well as the quantum heat traces,

\[
\Theta_{b,f}(\beta, \mu) = \text{Tr} E_{b,f}[\beta(\omega - \mu)].
\]

where \(\mu\) is another real parameter (not necessarily positive, that plays the role of the chemical potential) and \(E_{b,f}\) are functions defined by

\[
E_{b,f}(x) = \frac{1}{e^x \mp 1}.
\]

These functions come from the quantum statistical physics with the indices \(b\) and \(f\) standing for the bosonic and fermionic cases.

In the paper \[8\] we studied so called relative spectral invariants of two operators \(L_{\pm}\), of the form

\[
\Psi(t, s) = \text{Tr} \left\{ \exp(-tL_+) - \exp(-tL_-) \right\} \left\{ \exp(-sL_+) - \exp(-sL_-) \right\}.
\]

In our recent paper \[9\] we introduced another invariant, called the Bogolyubov invariant,

\[
B_b(\beta) = \text{Tr} \left\{ E_f(\beta \omega_+) - E_f(\beta \omega_-) \right\} \left\{ E_b(\beta \omega_+) - E_b(\beta \omega_-) \right\},
\]

where \(\omega = \sqrt{L}\) and \(\beta\) is a positive parameter (that plays the role of the temperature), as well as the quantum heat traces,
and applied it to the study of particle creation in quantum field theory and quantum gravity. The long term goal of this project is to develop a comprehensive methodology for such invariants in the same way as the theory of the standard heat trace invariants.

We describe our ideas rather formally. This should serve just as a motivation for the further study. Let \( M \) be a closed manifold (compact without boundary) and \( \mathcal{V} \) a vector bundle over \( M \). Let \( \mathcal{V}^* \) be the dual vector bundle and \( \mathcal{V} \otimes \mathcal{V}^* \) be the external tensor product of the bundles \( \mathcal{V} \) and \( \mathcal{V}^* \) over the product manifold \( M \times M \). Let \( \mathcal{K}(\mathcal{V}) = C^\infty(\mathcal{V} \otimes \mathcal{V}^*) \) be the set of all smooth sections of the vector bundle \( \mathcal{V} \otimes \mathcal{V}^* \); we will call them kernels. The convolution of kernels

\[
(K \star K')(x,x') = \int_M dy \ K(x,y)K'(y,x')
\]  

defines an associative binary operation on \( \mathcal{K}(\mathcal{V}) \)

\[
\circ : \mathcal{K}(\mathcal{V}) \times \mathcal{K}(\mathcal{V}) \to \mathcal{K}(\mathcal{V}),
\]

making it a semigroup.

Let \( \mathcal{L}(\mathcal{V}) \) be the set of all operators \( L : C^\infty(\mathcal{V}) \to C^\infty(\mathcal{V}) \) acting on \( \mathcal{V} \) and \( \text{Op}(\mathcal{V}) \subseteq \mathcal{L}(\mathcal{V}) \) be the subset of operators with the well defined exponential \( \exp(-L) \in \mathcal{L}(\mathcal{V}) \). Assume that for any two operators, \( L, L' \in \text{Op}(\mathcal{V}) \) there is an operator \( L \star L' \in \text{Op}(\mathcal{V}) \) such that

\[
\exp(-L \star L') = \exp(-L) \exp(-L'),
\]

which defines an associative binary operation

\[
\star : \text{Op}(\mathcal{V}) \times \text{Op}(\mathcal{V}) \to \text{Op}(\mathcal{V}),
\]

that we call the Campbell-Haussdorf (CH) product; this makes \( \text{Op}(\mathcal{V}) \) a semigroup. Obviously, this product is non-commutative. In the trivial case when the operators commute the CH product is nothing but just the sum, \( L \star L' = L + L' \).

So, in general, the CH product has the form

\[
L \star L' = L + L' + \text{commutators};
\]

that is why, it can be viewed as a non-commutative deformation of addition.

Of course, every kernel \( K \in \mathcal{K}(\mathcal{V}) \) defines an operator \( L_K \in \mathcal{L}(\mathcal{V}) \) on the bundle \( \mathcal{V} \). In particular, a one parameter family of kernels, \( K_t \in \mathcal{K}(\mathcal{V}) \), with a positive real parameter \( t > 0 \), defines an operator \( L \in \mathcal{L}(\mathcal{V}) \) by

\[
(L \phi)(x) = -\frac{\partial}{\partial t} \int_M dx' \ K_t(x,x') \phi(x') \bigg|_{t=0}.
\]
Conversely, every operator $L \in \text{Op}(\mathcal{V})$ defines the kernel $U_L \in \mathcal{H}(\mathcal{V})$, called the heat semigroup kernel (or just the heat kernel) of $L$, by

$$U_L(x, x') = \exp(-L)\delta(x, x'),$$

which defines a map

$$F : \text{Op}(\mathcal{V}) \to \text{Hk}(\mathcal{V}),$$

where $\text{Hk}(\mathcal{V}) \subseteq \mathcal{H}(\mathcal{V})$ is the set of heat kernels of all operators from $\text{Op}(\mathcal{V})$.

It should be clear now that the convolution of the heat kernels of the operators $L$ and $L'$ is the heat kernel of the operator $L \ast L'$,

$$U_{L \ast L'} = U_L \circ U_{L'},$$

that is, the map $F$ is a homomorphism,

$$F(L \ast L') = F(L) \circ F(L').$$

In this paper we study various properties of the homomorphism $F$, In the present paper we study the product of the semigroups of two operators $L_+$ and $L_-$,

$$U(t, s) = \exp(-tL_+)\exp(-sL_-),$$

and the corresponding kernels by using purely algebraic tools. For that to work we consider a rather simple non-geometric setup of operators on $\mathbb{R}^n$, such that they form a nilpotent algebra described below.

In Sec. 2 we describe the standard theory of Gaussian integrals in $\mathbb{R}^n$ in the form that will be convenient for us later. In particular, we introduced a Gaussian average of functions on $\mathbb{R}^n$ and study its properties. In Sec. 3 we study so called Gaussian kernels. We show that the set of Gaussian kernel is a semigroup with respect to the convolution and study some of its sub-semigroups. In Sec. 4 we explore some of the well known formulas related to the Campbell-Hausdorff series and prove a couple of useful lemmas. In Sec. 5 we introduce a real antisymmetric matrix $R_{ij}$ that we call curvature study some canonical functions of this matrix.

In Sec. 6 we consider the Heisenberg algebra and its universal enveloping algebra. We introduce a particular representation of the Heisenberg algebra related to the Weyl algebra of differential operators with polynomial coefficients, $(\nabla_1, \ldots, \nabla_n, x^1, \ldots, x^n, i)$, where $\nabla_j = \partial_j - \frac{1}{2}iR_{jk}x^k$, Given a positive matrix $g$ we introduce an operator called the Laplacian by $\Delta_g = g^{ij}\nabla_i \nabla_j$, where $g^{ij}$ is the inverse matrix.

In Sec. 7 we compute integrals of functions of the operators $\nabla$. We prove the following theorem.
**Theorem 1** Let $\mathcal{R} = (\mathcal{R}_{ij})$ be an anti-symmetric matrix and $g = (g_{ij})$ be a positive symmetric matrix. Let $D(t) = (D_{ij})$ be a symmetric matrix defined by

\[ D(t) = i\mathcal{R} \coth tg^{-1}i\mathcal{R}, \tag{1.17} \]

$T(t)$ be a matrix defined by

\[ T(t) = D(t) + i\mathcal{R}, \tag{1.18} \]

and $\Omega(t)$ be a function defined by

\[ \Omega(t) = \det T(t)^{1/2} = \det \left( g^{1/2} \sinh\left( \frac{tg^{-1}i\mathcal{R}}{2} \right) \right)^{1/2}. \tag{1.19} \]

Let $\nabla_i$ be operators forming the Lie algebra

\[ [\nabla_j, \nabla_k] = i\mathcal{R}_{jk}, \tag{1.20} \]

and $\Delta_g$ be the operator defined by

\[ \Delta_g = g^{ij} \nabla_i \nabla_j. \tag{1.21} \]

Then

\[ \exp(t\Delta_g) = (4\pi)^{-n/2} \Omega(t) \int_{\mathbb{R}^n} d\xi \exp \left\{ -\frac{1}{4} \langle \xi, D(t)\xi \rangle \right\} \exp \langle \xi, \nabla \rangle. \tag{1.22} \]

In Sec. 8 we consider two sets of operators $\nabla_i^+$ and $\nabla_i^-$ and prove the following theorem.

**Theorem 2** Let $\mathcal{R}^\pm = (\mathcal{R}_{ij}^\pm)$ be two anti-symmetric matrices, $g^\pm = (g_{ij}^\pm)$ be two positive symmetric matrices and let $D^\pm(t)$ be two symmetric matrices defined by

\[ D^\pm(t) = i\mathcal{R}_\pm \coth(tg^{-1}i\mathcal{R}_\pm). \tag{1.23} \]

Let $\tilde{\mathcal{F}} = (\tilde{\mathcal{F}}_{AB})$ be a $2n \times 2n$ anti-symmetric matrix defined by

\[ \tilde{\mathcal{F}} = \begin{pmatrix} \mathcal{R}^+ & \mathcal{R}^- \\ \mathcal{R}^- & \mathcal{R}^+ \end{pmatrix}, \tag{1.24} \]

where

\[ \mathcal{R} = \frac{1}{2} (\mathcal{R}^+ + \mathcal{R}^-). \tag{1.25} \]
and \( \tilde{Q}(t,s) = (\tilde{Q}_{AB}) \) and \( \tilde{G}^{-1}(t,s) = (\tilde{G}^{AB}) \) be the \( 2n \times 2n \) symmetric matrices defined by

\[
\tilde{Q}(t,s) = \begin{pmatrix} D_+(t) & -iR \\ iR & D_-(s) \end{pmatrix}, \tag{1.26}
\]

\[
\tilde{G}^{-1}(t,s) = \tanh^{-1}(\tilde{Q}^{-1}(t,s)i\tilde{F})(i\tilde{F})^{-1}. \tag{1.27}
\]

Let \((\tilde{\mathcal{D}}_1, \ldots, \tilde{\mathcal{D}}_n, \tilde{\mathcal{D}}_{n+1}, \ldots, \tilde{\mathcal{D}}_{2n}) = (\nabla^+_1, \ldots, \nabla^+_n, \nabla^-_1, \ldots, \nabla^-_n)\) be operators forming the Lie algebra

\[
[\nabla^+_i, \nabla^+_j] = iR^+_{ij}, \tag{1.28}
\]

\[
[\nabla^-_i, \nabla^-_j] = iR^-_{ij}, \tag{1.29}
\]

\[
[\nabla^+_i, \nabla^-_j] = iR_{ij}, \tag{1.30}
\]

and \( \Delta_{\pm} \) be two operators defined by

\[
\Delta_{\pm} = g_{\pm}^{ij}\nabla^\pm_i \nabla^\pm_j. \tag{1.31}
\]

Let \( \mathcal{H}(t,s) \) be the operator defined by

\[
\mathcal{H}(t,s) = \langle \tilde{\mathcal{D}}; \tilde{G}^{-1}(t,s)\tilde{\mathcal{D}} \rangle. \tag{1.32}
\]

Then

\[
\exp(t\Delta_+) \exp(s\Delta_-) = \exp \mathcal{H}(t,s). \tag{1.33}
\]

We should remark here that eq. (1.27) should be understood in terms of a power series; it is well defined even if the matrix \( \tilde{F} \) is not invertible.

We also prove the following theorem. Let the matrices \( R_{\pm}, g_{\pm}, D_{\pm}(t) \), and the operators \( \nabla^\pm \) be defined as in Theorem 2.

**Theorem 3** Let \( T_{\pm}(t) \), \( D(t,s) \) and \( Z(t,s) \) be the matrices defined by

\[
T_{\pm}(t) = D_{\pm}(t) + iR, \tag{1.34}
\]

\[
D(t,s) = D_+(t) + D_-(s), \tag{1.35}
\]

\[
Z(t,s) = D_+(t) - D_-(s) - 2iR_- \tag{1.36}
\]

and \( \Omega(t,s) \) be a function defined by

\[
\Omega(t,s) = \det T_+^{1/2} \det T_-^{1/2} \det D^{-1/2}(t,s). \tag{1.37}
\]
Let $H(t,s)$ be a symmetric matrix defined by

$$H(t,s) = \frac{1}{4} \left( D(t,s) - Z^T(t,s)D^{-1}(t,s)Z(t,s) \right). \quad (1.38)$$

Let $\nabla_i$ and $X_j$ be the operators defined by

$$\nabla_i = \frac{1}{2}(\nabla_i^+ + \nabla_i^-), \quad (1.39)$$
$$X_i = \nabla_i^+ - \nabla_i^- . \quad (1.40)$$

Then

$$\exp(t\Delta_+) \exp(s\Delta_-) = (4\pi)^{-n/2} \Omega(t,s) \exp \left\{ -\frac{1}{4} \langle \alpha, H(t,s) \alpha \rangle - \frac{1}{2} \langle \alpha, Z^T(t,s)D^{-1}(t,s)X \rangle \right\} \exp \langle \alpha, \nabla \rangle . \quad (1.41)$$

In Sec. 9 we compute the convolution of the heat kernels and prove the following theorem. Let the matrices $R_\pm$, $g_\pm$, $D_\pm(t)$, $T_\pm(t)$ and the operators $\nabla^\pm$ be defined as in Theorem 3.

**Theorem 4** Let $A_\pm(t,s)$ and $B(t,s)$ be matrices defined by

$$A_+(t,s) = D_+(t) - T_+^T(t)D^{-1}(t,s)T_+(t), \quad (1.42)$$
$$A_-(t,s) = D_-(s) - T_-^T(s)D^{-1}(t,s)T_-(s), \quad (1.43)$$
$$B(t,s) = T_+(t)D^{-1}(t,s)T_-(s), \quad (1.44)$$

and $S$ be a function defined by

$$S(t,s;x,x') = \frac{1}{4} \langle x, A_+(t,s) x \rangle + \frac{1}{4} \langle x', A_-(t,s)x' \rangle - \frac{1}{2} \langle x, B(t,s)x' \rangle . \quad (1.45)$$

Then the kernel of the product of the semigroups is

$$U(t,s;x,x') = \exp(t\Delta_+) \exp(s\Delta_-) \delta(x-x'),$$
$$= \det \left( -\frac{S_{xx'}(t,s)}{2\pi} \right)^{1/2} \exp \left\{ -S(t,s;x,x') \right\} . \quad (1.46)$$
2 Gaussian Integrals

We will make extensive use of Gaussian integrals. We denote by $\langle \cdot, \cdot \rangle$ the standard pairing in $\mathbb{R}^n$. Let $\gamma$ be a symmetric $n \times n$ matrix with positive definite real part. Then for any vector $A$ there holds (see, e.g. [15])

$$\int_{\mathbb{R}^n} d\xi \exp \left\{ -\frac{1}{4} \langle \xi, \gamma \xi \rangle + \langle A, \xi \rangle \right\} = (4\pi)^{n/2} \det \gamma^{-1/2} \exp \langle A, \gamma^{-1} A \rangle. \quad (2.1)$$

Let $S : \mathbb{R}^n \to \mathbb{C}$ be a quadratic polynomial with positive definite real quadratic part. Such a polynomial can always be written in the form

$$S(\xi) = S_0 + \frac{1}{2} \langle S_\xi, S_\xi^{-1} S_\xi \rangle, \quad (2.2)$$

where $S_0$ is a constant, $S_\xi$ and $S_{\xi \xi}$ are the vector of first partial derivatives and the matrix of the second partial derivatives,

$$S_\xi = \left( \frac{\partial S}{\partial \xi^j} \right), \quad (2.3)$$

$$S_{\xi \xi} = \left( \frac{\partial^2 S}{\partial \xi^i \partial \xi^j} \right). \quad (2.4)$$

Therefore, the Gaussian integral takes the form

$$\int_{\mathbb{R}^n} d\xi \exp \left\{ -S(\xi) \right\} = \det \left( \frac{S_{\xi \xi}}{2\pi} \right)^{-1/2} \exp(-S_0)$$

$$= \det \left( \frac{S_{\xi \xi}}{2\pi} \right)^{-1/2} \exp \left\{ -S(\xi) + \frac{1}{2} \langle S_\xi, S_\xi^{-1} S_\xi \rangle \right\}$$

$$= \exp \left\{ -\hat{S}(\xi) \right\}, \quad (2.5)$$

where

$$\hat{S} = S - \frac{1}{2} \langle S_\xi, S_\xi^{-1} S_\xi \rangle + \frac{1}{2} \text{tr} \log \left( \frac{S_{\xi \xi}}{2\pi} \right); \quad (2.6)$$

which can be evaluated at an arbitrary point $\xi$.

Let $t > 0$ be a positive parameter. We introduce a one-parameter family of Gaussian averages of functions $f : \mathbb{R}^n \to \mathbb{C}$ by

$$\langle f(\xi) \rangle_t = (4\pi t)^{-n/2} (\det \gamma)^{1/2} \int_{\mathbb{R}^n} d\xi \exp \left\{ -\frac{1}{4t} \langle \xi, \gamma \xi \rangle \right\} f(\xi). \quad (2.7)$$
Notice that this average depends on the parameter $t$ and
\[
\langle f(\xi) \rangle_t = \langle f(\sqrt{t}\xi) \rangle_1.
\] (2.8)

Therefore, for any smooth function, as $t \to 0^+$,
\[
\langle f(\xi) \rangle_0 = f(0).
\] (2.9)

By integration by parts we get useful equations
\[
\langle \xi^i f(\xi) \rangle_t = 2t \gamma^{ij} \left\langle \frac{\partial}{\partial \xi^j} f(\xi) \right\rangle_t,
\] (2.10)
\[
\langle \xi^j \xi^i f(\xi) \rangle_t = 2t \gamma^{ij} \langle f(\xi) \rangle_t + 4t^2 \gamma^{jk} \gamma^{im} \left\langle \frac{\partial}{\partial \xi^m} \frac{\partial}{\partial \xi^k} f(\xi) \right\rangle_t,
\] (2.11)

where $\gamma^{ij}$ is the inverse of the matrix $\gamma_{ij}$. Here and everywhere below, we denote the elements of the inverse matrix by the same letter with upper indices. Also, it is easy to see that
\[
\partial_t \langle f(\xi) \rangle_t = \frac{1}{2t} \left\langle \xi^i \frac{\partial}{\partial \xi^i} f(\xi) \right\rangle_t
= \langle \Delta_\xi f(\xi) \rangle_t,
\] (2.12)

where
\[
\Delta_\xi = \gamma^{ij} \frac{\partial}{\partial \xi^i} \frac{\partial}{\partial \xi^j}.
\] (2.13)

We compute the Gaussian average of the function $\exp \langle \xi, ip \rangle$. It satisfies the differential equation
\[
\partial_t \langle \exp \langle \xi, ip \rangle \rangle_t = -\langle p, \gamma^{-1} p \rangle \langle \exp \langle \xi, ip \rangle \rangle_t,
\] (2.14)

and, therefore,
\[
\langle \exp \langle \xi, ip \rangle \rangle_t = \exp \left( -t \langle p, \gamma^{-1} p \rangle \right).
\] (2.15)

By expanding both sides in the Taylor series we see that Gaussian averages of odd order homogeneous polynomials vanish and for even order homogeneous polynomials we obtain
\[
\langle \xi^{i_1} \ldots \xi^{2k} \rangle_t = \frac{(2k)!}{k!} i_{k}^{i_1 i_2 \ldots i_{2k-1} i_{2k}}.
\] (2.16)
Therefore, for any analytic function the Gaussian average is

\[ \langle f(\xi) \rangle_t = \sum_{k=0}^{\infty} \frac{t^k}{k!} \Delta_\xi^k f(\xi) \bigg|_{\xi=0} = \exp(t\Delta_\xi) f(\xi) \bigg|_{\xi=0}. \tag{2.17} \]

Notice that this average only contains the inverse matrix \( \gamma^{-1} \). So, strictly speaking, this average is defined also in the limiting case when the matrix \( \gamma^{-1} \) is degenerate, that is, has zero eigenvalues.

### 3 Gaussian Kernels

Let \( S \) be a quadratic polynomial on \( \mathbb{R}^n \times \mathbb{R}^n \) of the form

\[ S(x,y) = \frac{1}{4} \langle x, Ax \rangle - \frac{1}{2} \langle x, Cy \rangle + \frac{1}{4} \langle y, By \rangle - \frac{1}{2} \langle v, x \rangle - \frac{1}{2} \langle w, y \rangle + r, \tag{3.1} \]

where \( A \) and \( B \) are real symmetric positive matrices, \( C \) is a complex matrix, \( v, w \) are some vectors and \( r \) is a complex number. We introduce the following notation for the derivatives of the function \( S \)

\[ S_1(x,y) = S_x(x,y) = \frac{1}{2} (Ax - Cy - v) \tag{3.2} \]
\[ S_2(x,y) = S_y(x,y) = \frac{1}{2} (-C^T x + By - w) \tag{3.3} \]
\[ S_{11}(x,y) = S_{xx}(x,y) = \frac{1}{2} A, \tag{3.4} \]
\[ S_{12}(x,y) = S_{xy}(x,y) = -\frac{1}{2} C, \tag{3.5} \]
\[ S_{22}(x,y) = S_{yy}(x,y) = \frac{1}{2} B. \tag{3.6} \]

Let \( \mathcal{A} \) be the \( 2n \times 2n \) matrix

\[ \mathcal{A} = \begin{pmatrix} A & -C \\ -C^T & B \end{pmatrix}. \tag{3.7} \]

We assume that the matrix \( \text{Re} \mathcal{A} \) is non-negative, \( \text{Re} \mathcal{A} \geq 0 \). This can be achieved by parametrizing the matrices \( A \) and \( B \) by

\[ A = D^2, \tag{3.8} \]
\[ B = E^2, \tag{3.9} \]
\[ \text{Re} C = DAE. \tag{3.10} \]
where $D$ and $E$ are real symmetric positive matrices and $\Lambda$ is an orthogonal matrix, that is,

$$\text{Re} \mathcal{A} = \begin{pmatrix} D^2 & -DAE \\ -E\Lambda^T D & E^2 \end{pmatrix},$$

(3.11)

Then the quadratic part of the function $S$ is nonnegative; indeed, in this case

$$S(x, y) = \frac{1}{4}||Dx - \Lambda Ey||^2 + \cdots.$$  

(3.12)

Each such quadratic polynomial $S$ defines a function $U_S$ (that we call a Gaussian kernel) on $\mathbb{R}^n \times \mathbb{R}^n$ of the form

$$U_S(x, y) = \det \left( -\frac{S_{12}}{2\pi} \right)^{1/2} \exp \{ -S(x, y) \}. $$

(3.13)

Let $\mathcal{G}$ be the set of all Gaussian kernels. We define the convolution of kernels by

$$(U_S \circ U_{S'}) (x, z) = \int_{\mathbb{R}^n} dy U_S(x, y) U_{S'}(y, z).$$

(3.14)

It is easy to show that the convolution of Gaussian kernels is again a Gaussian kernel, that is, the set $\mathcal{G}$ is closed under convolution. Since the convolution is associative, the set $\mathcal{G}$ is a semigroup. The group multiplication $\ast$ is defined by

$$U_S \circ U_{S'} = U_{S \ast S'}.$$ 

(3.15)

It is not a group since it does not have the identity and the inverses.

Notice that a Gaussian kernel $U_\Sigma$ with the function $\Sigma$ of the form

$$\Sigma(x, y) = \frac{1}{4} \langle (x - y), g(x - y) \rangle,$$ 

(3.16)

that is, $A = B = C = g$, with $g$ a symmetric positive matrix and $v = w = r = 0$, plays the role of the asymptotic identity as

$$U_{n\Sigma}(x, y) \xrightarrow{n \to \infty} \delta(x - y).$$

(3.17)

So, even though $\mathcal{G}$ is not a group there is a sequence of kernels that converge to the identity.

It is easy to see that the subset $\mathcal{G}_0$ of Gaussian kernels of the form $U_\Sigma$ is closed under convolution and forms an Abelian sub-semigroup with the group multiplication defined as follows: the function $\tilde{\Sigma} = \Sigma \ast \Sigma'$ is defined by the matrix

$$\tilde{g} = (g^{-1} + g'^{-1})^{-1} = g' (g + g')^{-1} g.$$ 

(3.18)
In general, the group multiplication has the following form. Let $\tilde{S} = S \ast S'$; then

$$
\tilde{S}(x,z) = S(x,y) + S'(y,z) - \frac{1}{2} \left\langle \left[ S_2(x,y) + S'_1(y,z) \right], \left( S_{22} + S'_{11} \right) \right\rangle^{-1} \left[ S_2(x,y) + S'_1(y,z) \right],
$$

with an arbitrary $y$, or, more explicitly,

$$
\tilde{S}(x,z) = \frac{1}{4} \left\langle x, \left[ A - C(B + A')^{-1}C^T \right] x \right\rangle + \frac{1}{4} \left\langle z, \left[ B' - C'^T(B + A')^{-1}C' \right] z \right\rangle - \frac{1}{2} \left\langle \left[ v + C(B + A')^{-1}(w + v') \right], x \right\rangle - \frac{1}{2} \left\langle \left[ w' + C'^T(B + A')^{-1}(w + v') \right], z \right\rangle + r + r' - \frac{1}{4} \left\langle \left( w + v' \right), \left( B + A' \right)^{-1}(w + v') \right\rangle.
$$

Thus, finally, we the group transformation reads

\begin{align*}
\tilde{A} &= A - C(B + A')^{-1}C^T, \\
\tilde{B} &= B' - C'^T(B + A')^{-1}C', \\
\tilde{C} &= C(B + A')^{-1}C', \\
\tilde{v} &= v + C(B + A')^{-1}(w + v'), \\
\tilde{w} &= w' + C'^T(B + A')^{-1}(w + v'), \\
\tilde{r} &= r + r' - \frac{1}{4} \left\langle \left( w + v' \right), \left( B + A' \right)^{-1}(w + v') \right\rangle.
\end{align*}

**Remarks.** Notice that this product is non-Abelian, in general. The semigroup of Gaussian kernels $\mathcal{G}$ has various subsemigroups; these are the subsets closed under the group multiplication.

1. The subset $\mathcal{G}_c$ with $C = 0$. In this case $\tilde{A} = A$, $\tilde{B} = B'$, $\tilde{v} = v$, $\tilde{w} = w'$ and

$$
\tilde{r} = r + r' - \frac{1}{4} \left\langle \left( w + v' \right), \left( B + A' \right)^{-1}(w + v') \right\rangle.
$$

2. The subset $\mathcal{G}_2$ with $v = w = r = 0$, that is, with quadratic homogeneous polynomial $S$. In this case

\begin{align*}
\tilde{A} &= A - C(B + A')^{-1}C^T, \\
\tilde{B} &= B' - C'^T(B + A')^{-1}C', \\
\tilde{C} &= C(B + A')^{-1}C'.
\end{align*}
3. The subset $\mathcal{G}_0$ with $C = B = A$ and $v = w = r = 0$. In this case the group transformation law is Abelian; it takes a particularly simple form,\[ \tilde{A}^{-1} = A^{-1} + A'^{-1}. \] (3.31)
This is exactly the transformation (3.18) discussed above.

4 Campbell-Hausdorff Formula

We describe some of the well-known facts about the Campbell-Hausdorff formula (for a detailed exposition see, e.g. [12]). Let $\mathfrak{g}$ be a Lie algebra. For any operator $X \in \mathfrak{g}$ we define the operator $\text{ad}_X : \mathfrak{g} \to \mathfrak{g}$ by\[ \text{ad}_X Y = [X, Y]. \] (4.1)

Let $P$ and $X$ be some operators in some Lie algebra $\mathfrak{g}$. Our goal is to compute the product $(\exp P)(\exp X)$. We proceed rather formally.

We consider a one-parameter family of operators in a Lie algebra, $Q_t$, $t \in [0, 1]$. We prove a useful lemma.

**Lemma 1** The derivative of the exponential $\exp Q_t$ is given by\[ \partial_t \exp Q_t = \exp(-Q_t) \left\{ 1 - \frac{\exp(-\text{ad}_{Q_t})}{\text{ad}_{Q_t}} \partial_t Q_t \right\}. \] (4.2)
\[ = \left\{ \frac{\exp(\text{ad}_{Q_t}) - 1}{\text{ad}_{Q_t}} \partial_t Q_t \right\} \exp Q_t. \] (4.3)

**Proof.** Let $F(t, s)$ be a function defined by\[ F(t, s) = \exp(-sQ_t) \partial_t \exp(sQ_t), \] (4.4)
so that $F(t, 0) = 0$. Then it is not difficult to see that\[ \partial_s F = \exp(-sQ_t)(\partial_t Q) \exp(sQ_t) = \exp(-s\text{ad}_{Q_t}) \partial_t Q. \] (4.5)

Therefore, by integrating over $s$ from 0 to 1 we get\[ F(t, 1) = \int_0^1 ds \exp(-s\text{ad}_{Q_t}) \partial_t Q = \frac{1 - \exp(-\text{ad}_{Q_t})}{\text{ad}_{Q_t}} \partial_t Q, \] (4.6)
which proves (4.2). Eq. (4.3) is proved similarly. \(\Box\)

The expression in (4.2) and (4.3) are understood as power series, that is,

\[
\exp(-Q_t) \partial_t \exp Q_t = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!} \text{ad}_Q^k \partial_t = \partial_t Q_t + \text{commutators}, \tag{4.7}
\]

so, in some sense, this is a “non-commutative derivative”.

**Corollary 1** The operators \(Q_1\) and \(Q_0\) are related by

\[
Q_1 = Q_0 + \int_0^1 dt \frac{\text{ad}_{Q_t}}{1 - \exp(-\text{ad}_{Q_t})} A(t), \tag{4.8}
\]

where

\[
A(t) = \exp(-Q_t) \partial_t \exp Q_t. \tag{4.9}
\]

**Proof.** This follows directly from the differential equation (4.2). \(\Box\)

This enables one to prove the general Campbell-Haussdorff theorem. Let \(P\) and \(X\) be two operators in some Lie algebra.

**Theorem 5** Let \(\psi(z)\) be a function defined by

\[
\psi(z) = \frac{z}{z-1} \log z. \tag{4.10}
\]

Then

\[
\exp P \exp X = \exp V, \tag{4.11}
\]

where

\[
V = P + \int_0^1 dt \psi \left( \exp(\text{ad}_P) \exp(t \text{ad}_X) \right) X. \tag{4.12}
\]

**Proof.** Let \(Q_t\) be a one-parameter family of operators defined by

\[
\exp Q_t = (\exp P) \exp(tX), \tag{4.13}
\]

so that \(Q_0 = P\) and \(Q_1 = V\). It is easy to see that in this case

\[
A(t) = \exp(-Q_t) \partial_t (\exp Q_t) = X \tag{4.14}
\]

Therefore, by using (4.8) we get

\[
V = P + \int_0^1 dt \frac{\text{ad}_{Q_t}}{1 - \exp(-\text{ad}_{Q_t})} X. \tag{4.15}
\]
Next, we can rewrite this in the form
\[ \frac{\text{ad}_{Q_t}}{1 - \exp(-\text{ad}_{Q_t})} = \exp(\text{ad}_{Q_t}) \exp(\text{ad}_{Q_t}) = \psi(\exp(\text{ad}_{Q_t})). \] (4.16)

Further, by using the fact that
\[ \exp(\text{ad}_{Q_t}) = \exp(\text{ad}_P) \exp(t \text{ad}_X). \] (4.17)
we obtain the result. □

Of course, this theorem is understood in terms of a power series in the operators \( P \) and \( X \) by using the Taylor series
\[ \psi(z) = \frac{z}{z - 1} \log z = 1 - \sum_{k=0}^{\infty} \frac{1}{k(k+1)} (1 - z)^k. \] (4.18)
and by rescaling the operators \( P \mapsto sP, X \mapsto sX \) and expanding everything in powers of \( s \).

**Lemma 2** Suppose that for any positive integer \( k \) the operator \( X \) commutes with all commutators \([P, \ldots, [P, X] \ldots] \), that is,
\[ \text{ad}_X \text{ad}_P^k X = 0. \] (4.19)

Then
\[ \exp(P + X) = \exp \left\{ \frac{1 - \exp(-\text{ad}_P)}{\text{ad}_P} X \right\} \exp P \]
\[ = (\exp P) \exp \left\{ \frac{\exp(\text{ad}_P) - 1}{\text{ad}_P} X \right\}. \] (4.20)

**Proof.** Let \( Q_t = P + tX \). Then \( \partial_t Q_t = X \) and by using (4.2) we get
\[ A(t) = \exp(-Q_t) \partial_t \exp Q_t = \frac{1 - \exp(-\text{ad}_{Q_t})}{\text{ad}_{Q_t}} X. \] (4.21)
We obviously have
\[ \text{ad}_{Q_t} = \text{ad}_P + t \text{ad}_X. \] (4.22)
Therefore, under the assumptions of the lemma
\[ \text{ad}_{Q_t}^k X = (\text{ad}_P + t \text{ad}_X)^k X = \text{ad}_P^k X, \] (4.23)
and, therefore, the operator \( A(t) \) does not depend on \( t \),

\[
A(t) = \exp(-Q_t) \partial_t \exp Q_t = \frac{1 - \exp(-\text{ad}_P) X}{\text{ad}_P}.
\] (4.24)

Therefore, we can integrate this differential equation with the initial condition \( \exp(Q_t) \big|_{t=0} = \exp P \) to obtain

\[
\exp Q_t = \exp \left\{ t \frac{1 - \exp(-\text{ad}_P) X}{\text{ad}_P} \right\} \exp P; \tag{4.25}
\]

and by setting \( t = 1 \) we prove the lemma. The second equation is obtained by taking the inverse of the first and changing the signs of \( P \) and \( X \). □

**Corollary 2** Suppose that for any positive integer \( k \) the operator \( X \) commutes with all commutators \( [P,\ldots,[P,X]\ldots] \), that is,

\[
\text{ad}_X \text{ad}_P^k X = 0. \tag{4.26}
\]

Then

\[
\exp(X) \exp(P) = \exp \left\{ P + \frac{1 - \exp(\text{ad}_P) X}{\text{ad}_P} \right\} \tag{4.27}
\]

\[
\exp(P) \exp(X) = \exp \left\{ P + \frac{\exp(-\text{ad}_P) - 1}{\text{ad}_P} X \right\}. \tag{4.28}
\]

**Proof.** Let \( Y = P + X \). This operator commutes with all commutators \( \text{ad}_P^k Y \) and for any \( k > 1 \)

\[
\text{ad}_P^k X = \text{ad}_P^k Y, \tag{4.29}
\]

and, therefore,

\[
\frac{1 - \exp(-\text{ad}_P) X}{\text{ad}_P} X = -P + \frac{1 - \exp(-\text{ad}_P) Y}{\text{ad}_P}. \tag{4.30}
\]

Therefore,

\[
\exp Y = \exp \left\{ -P + \frac{1 - \exp(-\text{ad}_P) Y}{\text{ad}_P} \right\} \exp P. \tag{4.31}
\]

Now, by multiplying by \( \exp(-P) \) on the right and changing the sign of \( P \) we prove eq. (4.27). The second equation is obtained by taking the inverse. □
In the case when the commutator \([A, B]\) commutes with both operators \(A\) and \(B\), that is, \([B, [A, B]] = [A, [A, B]] = 0\), this reduces to the well known special case of the Campbell-Hausdorff formula
\[
(expA)(expB) = \exp \left( \frac{1}{2} [A, B] \right) \exp(A + B),
\]
which also means
\[
(expA)(expB) = \exp[A, B](expB)(expA).
\]

5 Curvature

Let \(M_n(\mathbb{R})\) be the algebra \(n \times n\) matrices with the standard matrix product and \(\mathcal{R} = (\mathcal{R}_{ij}) \in M_n(\mathbb{R})\) be a fixed real antisymmetric matrix that we call the curvature. We define a bilinear binary operation (that we will call a \(\mathcal{R}\)-bracket)
\[
\{ , \} : M_n(\mathbb{R}) \times M_n(\mathbb{R}) \rightarrow M_n(\mathbb{R})
\]
as follows: for any two matrices \(A = (A^{ij})\) and \(B = (B^{ij})\)
\[
\{A, B\} = A\mathcal{R}B - B\mathcal{R}A.
\]

Obviously, this bracket is anti-symmetric, \(\{A, B\} = -\{B, A\}\) and satisfies the Jacobi identity
\[
\{A, \{B, C\}\} + \{B, \{C, A\}\} + \{C, \{A, B\}\} = 0
\]

Let \(S_n \subseteq M_n(\mathbb{R})\) and \(L_n \subseteq M_n(\mathbb{R})\) be the subspaces of symmetric and anti-symmetric matrices. It is easy to see that they are closed under the \(\mathcal{R}\)-bracket which defines the Lie brackets \(\{ , \} : S_n \times S_n \rightarrow S_n\) and \(\{ , \} : L_n \times L_n \rightarrow L_n\) turning them into Lie algebras of symmetric and anti-symmetric matrices.

Let \(\gamma = (\gamma_{ij})\) be a positive symmetric matrix. We will be considering analytic functions \(f(\gamma^{-1}i\mathcal{R})\) of the matrix \(\gamma^{-1}i\mathcal{R}\). It is easy to see that for any non-negative integer \(k\)
\[
\left[(\gamma^{-1}i\mathcal{R})^k\right]^T = \gamma(-\gamma^{-1}i\mathcal{R})^k \gamma^{-1};
\]
therefore, for any analytic function \(f\)
\[
\left[f(t\gamma^{-1}i\mathcal{R})\right]^T = \gamma f(-t\gamma^{-1}i\mathcal{R})\gamma^{-1};
\]
hence, for any even function \( f \) the matrices \( \gamma f(\gamma^{-1}iR) \) and \( f(\gamma^{-1}iR)\gamma^{-1} \) are symmetric. More generally, let \( \Lambda \) be a nondegenerate matrix; then for any analytic function we have

\[
f(t\gamma^{-1}iR)\gamma^{-1} = \Lambda f(t\gamma'^{-1}iR')\gamma'^{-1}\Lambda^T, \tag{5.6}
\]

where

\[
\gamma' = \Lambda^T \gamma \Lambda, \tag{5.7}
\]
\[
R' = \Lambda^T R \Lambda. \tag{5.8}
\]

The matrix \( \Lambda \) is arbitrary and can be chosen from convenience.

We can always write the matrix \( \gamma \) in the form

\[
\gamma^{-1} = \omega \omega^T, \tag{5.9}
\]

where \( \omega \) is some non-degenerate matrix. Then the matrix \( \gamma^{-1}iR \) has the following canonical form

\[
\gamma^{-1}iR = \omega \sum_{\alpha=1}^{m} B_{\alpha}E_{\alpha} \omega^{-1}, \tag{5.10}
\]

where \( m \leq n/2, B_{\alpha} \) are some real invariants, and \( E_{\alpha} \) are irreducible anti-symmetric matrices satisfying

\[
E_{\alpha}^2 = P_{\alpha}, \tag{5.11}
\]
\[
E_{\alpha}E_{\beta} = 0, \quad \text{for} \quad \alpha \neq \beta, \tag{5.12}
\]

with the corresponding symmetric projections \( P_{\alpha} \) satisfying

\[
P_{\alpha}^2 = P_{\alpha}, \tag{5.13}
\]
\[
P_{\alpha}P_{\beta} = 0, \quad \text{for} \quad \alpha \neq \beta, \tag{5.14}
\]
\[
\text{tr} P_{\alpha} = 2. \tag{5.15}
\]

Therefore, for any even analytic function \( f \) we have

\[
f(t\gamma^{-1}iR) = \omega \left\{ f(0)I + \sum_{\alpha=1}^{m} [f(tB_{\alpha}) - f(0)]P_{\alpha} \right\} \omega^{-1}. \tag{5.16}
\]
In this paper we will be using extensively two even functions

\[
\Phi(z) = \frac{\tanh^{-1}(z)}{z} = \frac{1}{2z} \log \left( \frac{1+z}{1-z} \right)
= \sum_{k=0}^{\infty} \frac{1}{2k+1} z^{2k} = 1 + \frac{1}{3} z^2 + \cdots,
\]

(5.17)

\[
\Psi(z) = z \coth z
= \sum_{k=0}^{\infty} \frac{2^{2k} B_{2k}}{(2k)!} z^{2k} = 1 + \frac{1}{3} z^2 + \cdots,
\]

(5.18)

where \(B_n\) are Bernoulli numbers. Note that the function \(\Phi(z)\) is analytic with cuts along the real axis from \(-\infty\) to \(-1\) and from \(1\) to \(\infty\), whereas the function \(\Psi(z)\) is meromorphic with simple poles on the imaginary axis at \(z = 2\pi i n\), where \(n\) is a non-zero integer, \(n \neq 0\). That is, the function \(\Phi(x)\) is singular at \(x = \pm 1\) and \(\Psi(x)\) is well defined on the whole real line. These functions are related by

\[
\Psi(z) = \Phi(\tanh z).
\]

(5.19)

In particular, we have

\[
\Phi(t^\gamma^{-1} i \mathcal{R}) = \omega \left\{ I + \sum_{\alpha=1}^{m} \left[ \frac{\tanh^{-1}(t B_\alpha)}{t B_\alpha} - 1 \right] P_\alpha \right\} \omega^{-1},
\]

(5.20)

\[
\Psi(t^\gamma^{-1} i \mathcal{R}) = \omega \left\{ I + \sum_{\alpha=1}^{m} [t B_\alpha \coth(t B_\alpha) - 1] P_\alpha \right\} \omega^{-1}.
\]

(5.21)

6 Weyl Algebra

The Lie algebra \(h_n\) of the Heisenberg group \(H_{2n+1}\) is generated by the \((2n + 1)\) operators \((p_1, \ldots, p_n, x^1, \ldots, x^n, i)\) satisfying the commutation relations

\[
[p_k, x^l] = i \delta^l_k,
\]

(6.1)

\[
[p_k, p_j] = [x^k, x^l] = [p_k, i] = [x^k, i] = 0.
\]

(6.2)

We will just call it the Heisenberg algebra. We will use another basis of the Heisenberg algebra \((P_1, \ldots, P_n, x^1, \ldots, x^n, i)\) defined by

\[
P_k = p_k + \frac{1}{2} \mathcal{R}_{kj} x^j.
\]

(6.3)
satisfying the commutation relations

\[
\begin{align*}
[P_k, x^j] &= i \delta^j_k, \\
[P_k, P_j] &= -i R_{kj}, \\
[x^k, x^j] &= [P_k, i] = [x^k, i] = 0.
\end{align*}
\] (6.4)

\[
\begin{align*}
[P_k, x^j] &= i \delta^j_k, \\
[P_k, P_j] &= -i R_{kj}, \\
[x^k, x^j] &= [P_k, i] = [x^k, i] = 0.
\end{align*}
\] (6.5)

\[
\begin{align*}
[x^k, x^j] &= [P_k, i] = [x^k, i] = 0.
\end{align*}
\] (6.6)

Obviously, the operators \((P_j, i)\) form a subalgebra of the Heisenberg algebra.

Its universal enveloping algebra \(U(\mathfrak{h}_n)\) is the set of all polynomials in these
operators subject to these commutation relations. Let \((x^1, \ldots, x^n)\) be the coor-
dinates of the Euclidean space \(\mathbb{R}^n\) and \((\partial_1, \ldots, \partial_n)\) be the corresponding partial
derivatives. Obviously, the Heisenberg algebra can be represented by the first or-
der order differential operators \(p_k = i \partial_k\). Then the universal enveloping algebra \(U(\mathfrak{h}_n)\)
is simply the ring of all differential operators with polynomial coefficients, also
called the Weyl algebra \(A_n\). The operators \(P_k\) then have the form

\[
P_k = \partial_k - \frac{1}{2} i R_{kj} x^j.
\] (6.7)

These operators satisfy the commutation relations

\[
\begin{align*}
[\nabla_k, x^j] &= \delta^j_k, \\
[\nabla_k, \nabla_j] &= i R_{kj}, \\
[\nabla_k, i] &= [x^k, i] = 0.
\end{align*}
\] (6.8)

\[
\begin{align*}
[\nabla_k, x^j] &= \delta^j_k, \\
[\nabla_k, \nabla_j] &= i R_{kj}, \\
[\nabla_k, i] &= [x^k, i] = 0.
\end{align*}
\] (6.9)

\[
[\nabla_k, \nabla_j] = i R_{kj}.
\] (6.10)

We consider analytic functions \(f(i, x, \nabla)\) defined by power series with the coef-
ficients in the Weyl algebra, \(A_n\). We will be mostly interested in analytic functions
functions \(f(i, \nabla)\) that only depend on the operators \(\nabla_j\) and \(i\) but not on \(x^k\). We will
need the following lemma.

**Lemma 3** For any \(\eta, \xi \in \mathbb{R}^n\) there holds

\[
\exp \langle (\xi + \eta), \nabla \rangle = \exp \left\langle \eta, \left(\nabla - \frac{1}{2} i R \xi \right) \right\rangle \exp \langle \xi, \nabla \rangle.
\] (6.11)

**Proof.** By using the Campbell-Hausdorff formula (4.32) we have

\[
\exp \langle (\xi + \eta), \nabla \rangle = \exp \left\langle \eta, \left(\nabla - \frac{1}{2} \left[\langle \eta, \nabla \rangle, \langle \xi, \nabla \rangle \right] \right) \right\rangle \exp \langle \xi, \nabla \rangle.
\] (6.12)

Now, by using the commutator (6.5) we obtain the result. \(\square\)
Corollary 3 The partial derivatives of the operator $\exp \langle \xi, \nabla \rangle$ are

$$\frac{\partial}{\partial \xi^i} \cdots \frac{\partial}{\partial \xi^k} \exp \langle \xi, \nabla \rangle = \left( \nabla_{(i} \frac{1}{2} i R_{(i_1j_1)} \xi^{j_1} \right) \cdots \left( \nabla_{k)} \frac{1}{2} i R_{k)j_2} \xi^{j_2} \right) \exp \langle \xi, \nabla \rangle,$$

(6.13)
in particular,

$$\frac{\partial}{\partial \xi^i} \exp \langle \xi, \nabla \rangle = \left( \nabla_{i} \frac{1}{2} i R_{ij} \xi^j \right) \exp \langle \xi, \nabla \rangle$$

(6.14)

$$\frac{\partial}{\partial \xi^i} \frac{\partial}{\partial \xi^j} \exp \langle \xi, \nabla \rangle = \left( \nabla_{(i} \frac{1}{2} i R_{(j|k)} \xi^{k} \right) \left( \nabla_{i)} \frac{1}{2} i R_{i)m} \xi^{m} \right) \exp \langle \xi, \nabla \rangle.$$

(6.15)

Proof. This is proved by expanding (6.11) in Taylor series in $\eta$.
Alternatively, eq. (6.14) this can also be proved following [2, 6]. Let

$$F_i(t) = - \exp \langle t \langle \xi, \nabla \rangle \rangle \frac{\partial}{\partial \xi^i} \exp \langle -t \langle \xi, \nabla \rangle \rangle.$$

(6.16)

This operator satisfies the equation

$$\frac{\partial}{\partial t} F_i(t) = \mathrm{ad}_{\langle \xi, \nabla \rangle} F_i(t) + \nabla_i,$$

(6.17)

with the initial condition $F_i(0) = 0$. The solution of this equation is

$$F_i(t) = \frac{\exp (t \mathrm{ad}_{\langle \xi, \nabla \rangle}) - 1}{\mathrm{ad}_{\langle \xi, \nabla \rangle}} \nabla_i + \sum_{k=1}^{\infty} \frac{t^k \mathrm{ad}_{\langle \xi, \nabla \rangle}^{k-1} \nabla_i}{k!}.$$

(6.18)

The function $F_i(t)$ can be evaluated by expanding it in the Taylor series. First, we compute

$$\mathrm{ad}_{\langle \xi, \nabla \rangle} \nabla_i = i R_{ij} \xi^j,$$

(6.19)

$$(\mathrm{ad}_{\langle \xi, \nabla \rangle})^k \nabla_i = 0, \quad \text{for any } k \geq 2;$$

(6.20)
therefore,

$$F_i(t) = t \nabla_i - \frac{1}{2} t^2 i R_{ij} \xi^j.$$

(6.21)

Next, we have

$$\frac{\partial}{\partial t} \exp \langle t \langle \xi, \nabla \rangle \rangle = F_i(t) \exp \langle t \langle \xi, \nabla \rangle \rangle$$

$$= - \exp \langle t \langle \xi, \nabla \rangle \rangle F_i(-t).$$

(6.22)
By setting $t = 1$ in (6.22) we get eq. (6.14). Eq. (6.15) follows by differentiation and symmetrization. □

Given a real symmetric positive matrix $g = (g_{ij})$ we define the operator (that we call the \textit{Laplacian})

$$\Delta_g = \langle \nabla, g^{-1} \nabla \rangle = g^{ij} \nabla_i \nabla_j,$$

where $g^{-1} = (g^{ij})$ is the inverse of the matrix $g$. We can always write the matrix $g^{-1}$ in the form $g^{-1} = \omega \omega^T$ so that so that

$$\Delta_g = \langle \nabla', \nabla' \rangle,$$

where

$$\nabla' = \omega^T \nabla.$$

These operators satisfy the same commutation relations (6.5) with

$$\mathcal{R}' = \omega^T \mathcal{R} \omega.$$  \hspace{1cm} (6.26)

Furthermore, we can still transform the operators $\nabla'$ by an orthogonal transformation

$$\tilde{\nabla} = O^T \nabla' = O^T \omega^T \nabla$$

with the orthogonal matrix $O$ so that $\Delta_g = \langle \tilde{\nabla}, \tilde{\nabla} \rangle$ and the corresponding curvature

$$\tilde{\mathcal{R}} = O^T \omega^T \mathcal{R} \omega O,$$  \hspace{1cm} (6.28)

to bring the matrix $\tilde{\mathcal{R}}$ to a canonical form. Therefore, without loss of generality we can restrict in the following to the case $g = I$ and the matrix $\mathcal{R}$ in the canonical form (5.10); so, we will drop the prime and the tilde below and just assume that $g = I$ and $\mathcal{R}$ has the form (5.10).

Our goal is computing the product of the semigroups $\exp(t\Delta_{g^+}) \exp(s\Delta_{g^-})$ for two symmetric positive matrices $g_{\pm}$.

\textbf{Lemma 4} \textit{The set of Laplacian operators is closed under commutation. That is, the commutator of Laplacians $\Delta_{g^+}$ and $\Delta_{g^-}$ is again a Laplacian}

$$[\Delta_{g^+}, \Delta_{g^-}] = 2i\Delta_G,$$  \hspace{1cm} (6.29)

where $G^{-1} = \{g_{\pm}^{-1}, g_{\pm}^{-1}\} = g_{\pm}^{-1} \mathcal{R} g_{\pm}^{-1} - g_{\pm}^{-1} \mathcal{R} g_{\pm}^{-1}$ is given by the bracket $\{ , \}$ defined by (5.2), that is, $G^{ij} = g_{+}^{ik} \mathcal{R} g_{-}^{mj} - g_{-}^{ik} \mathcal{R} g_{+}^{mj}$. 

Proof. We compute the commutator of these operators with the operators $\nabla_i$,

$$[\nabla_i, \Delta_{g_+}] = 2i\mathcal{R}_{ij}g_{ik}\nabla_k. \quad (6.30)$$

The statement follows. $\Box$

By using this lemma we have an immediate corollary.

**Corollary 4** The operators $\Delta_{g_+}$ and $\Delta_{g_-}$ commute if and only if the matrices $g_+$ and $g_-$ satisfy the condition

$$\{g_{+1}, g_{-1}\} = 0. \quad (6.31)$$

## 7 Noncommutative Gaussian Integrals

Let $f(\xi)$ be a function depending on the operators $\nabla_i$ and $\langle f(\xi) \rangle_t$ be the Gaussian average defined by (2.7). Suppose that we can find operators $L(t)$ such that this average satisfies the differential equation

$$\partial_t \langle f(\xi) \rangle_t = L(t) \langle f(\xi) \rangle_t. \quad (7.1)$$

Recall that $\langle f(\xi) \rangle_0 = f(0)$, which serves as the initial condition. Then, if the operators $L(t)$ and $L(s)$ commute for any $t$ and $s$, one can solve this equation to obtain

$$\langle f(\xi) \rangle_t = \exp \left\{ \int_0^t d\tau L(\tau) \right\} f(0). \quad (7.2)$$

We use this idea to study the Gaussian average of the operator $\exp \langle \xi, \nabla \rangle$,

$$\langle \exp \langle \xi, \nabla \rangle \rangle_t = (4\pi t)^{-n/2}(\det \gamma)^{1/2} \int_{\mathbb{R}^n} d\xi \exp \left\{ -\frac{1}{4t} \langle \xi, \gamma \xi \rangle \right\} \exp \langle \xi, \nabla \rangle. \quad (7.3)$$

By expanding this operator in the Taylor series and using (2.16) it is easy to obtain

$$\langle \exp \langle \xi, \nabla \rangle \rangle_t = \sum_{k=0}^{\infty} \frac{t^k}{k!} \gamma^{(i_1i_2 \ldots \gamma^{(i_{2k-1}i_{2k})} \nabla_{(i_1} \ldots \nabla_{i_{2k})}. \quad (7.4)$$

Notice that in the trivial case when $\mathcal{R} = 0$, that is, when the operators $\nabla_i$ commute, we have $\langle \exp \langle \xi, \nabla \rangle \rangle_t = \exp(t\Delta_\gamma)$, where $\Delta_\gamma = \gamma^{ij}\nabla_i \nabla_j$. 
In the following lemma we compute this Gaussian average for non-commuting operators \( \nabla_i \) forming the Lie algebra (6.10) following [2, 6]. Let \( G^{-1}(t) = (G^{ij}) \) be a symmetric matrix defined by

\[
G^{-1}(t) = \Phi(t\gamma^{-1}iR)\gamma^{-1} = \frac{\tanh^{-1}(t\gamma^{-1}iR)}{t\gamma^{-1}iR} \gamma^{-1}
\]

and \( \Delta_{G(t)} \) be an operator defined by

\[
\Delta_{G(t)} = \langle \nabla, G^{-1}(t) \nabla \rangle = G^{ij}(t)\nabla_i \nabla_j.
\]

For the future reference we explore the behavior of the metric \( G(t) \) for small \( t \),

\[
G(t) = \gamma - \frac{1}{3} t^2 iR \gamma^{-1} iR + O(t^4).
\]

**Lemma 5** For sufficiently small \( t > 0 \)

\[
\langle \exp(\xi, \nabla) \rangle_t = \det \left( I + t\gamma^{-1}iR \right)^{-1/2} \exp \left( t\Delta_{G(t)} \right). \tag{7.8}
\]

**Proof.** Let \( I(t) = \langle \exp(\xi, \nabla) \rangle_t \). We compute the derivative of the function \( I(t) \) by using (2.12)

\[
\partial_t I(t) = \frac{1}{2t} \nabla_i \langle \xi^i \exp(\xi, \nabla) \rangle_t. \tag{7.9}
\]

Next, by using (2.10) we get

\[
\langle \xi^i \exp(\xi, \nabla) \rangle_t = 2t\gamma^{ik} \left\langle \frac{\partial}{\partial \xi^k} \exp(\xi, \nabla) \right\rangle_t \tag{7.10}
\]

Further, by using eqs. (2.10) and (6.14) we have

\[
\Gamma_{ij} \gamma^{ik} \left\langle \frac{\partial}{\partial \xi^k} \exp(\xi, \nabla) \right\rangle_t = \nabla_i \langle \exp(\xi, \nabla) \rangle_t, \tag{7.11}
\]

where \( \Gamma = (\Gamma_{ij}) \) is the matrix

\[
\Gamma = \gamma + tiR; \tag{7.12}
\]

and, therefore,

\[
\gamma^{ik} \left\langle \frac{\partial}{\partial \xi^k} \exp(\xi, \nabla) \right\rangle_t = \Gamma^{ji} \nabla_i \langle \exp(\xi, \nabla) \rangle_t, \tag{7.13}
\]
where $\Gamma^{-1} = (\Gamma^{ij})$ is the inverse of the matrix $\Gamma = (\Gamma_{ij})$. By substituting (7.13) in (7.10) we get

$$\langle \xi^j \exp \langle \xi, \nabla \rangle \rangle_t = 2t\Gamma^{ji} \nabla_j \langle \exp \langle \xi, \nabla \rangle \rangle_t,$$

and, finally, by using this equation in (7.9) we obtain a differential equation for the function $I(t)$,

$$\partial_t I(t) = L(t)I(t),$$

(7.15)

where

$$L = \Gamma^{ij} \nabla_i \nabla_j,$$

(7.16)

with the initial condition $I(0) = I$.

We decompose the matrix $\Gamma^{-1} = (\Gamma^{ij})$ in the symmetric and the anti-symmetric parts and use the commutator of the operators $\nabla_i$ to get

$$L(t) = \Delta_{S(t)} + P(t),$$

(7.17)

where

$$P(t) = -\frac{1}{2} \text{tr} \left\{ (I + t\gamma^{-1} i\mathcal{R})^{-1} \gamma^{-1} i\mathcal{R} \right\},$$

$$\Delta_{S(t)} = S^{ij}(t) \nabla_i \nabla_j,$$

(7.18)

(7.19)

with $S^{-1} = (S^{ij})$ being a symmetric matrix defined by

$$S^{-1} = \frac{1}{2} \left\{ (I + t\gamma^{-1} i\mathcal{R})^{-1} + (I - t\gamma^{-1} i\mathcal{R})^{-1} \right\} \gamma^{-1}.$$  

(7.20)

Next, by using the fact that the matrix $S^{-1}(t)$ at different times satisfies the equation

$$S^{-1}(t)\mathcal{R}S^{-1}(s) = S^{-1}(s)\mathcal{R}S^{-1}(t),$$

(7.21)

we can use Lemma 4 to show that the operators $\Delta_{S(t)}$ and, therefore, the operators $L(t)$, at different times commute [2],

$$[L(t), L(s)] = 0.$$  

(7.22)

This enables one to solve the differential equation (7.15) with the initial condition $I(0) = I$ to obtain

$$I(t) = \exp \left\{ \int_0^t d\tau L(\tau) \right\} = \exp \left\{ t\Delta_{G(t)} + M(t) \right\},$$

(7.23)
where
\[
M(t) = -\frac{1}{2} \text{tr} \log \left( I + t\gamma^{-1}i\mathcal{R} \right),
\] (7.24)
and \(\Delta_{G(t)}\) is given by (7.6) with
\[
G^{-1}(t) = \frac{\text{tanh}^{-1} \left( t\gamma^{-1}i\mathcal{R} \right)}{t\gamma^{-1}i\mathcal{R}} \gamma^{-1}.
\] (7.25)
Thus, we obtain from (7.23)
\[
I(t) = \det \left( I + t\gamma^{-1}i\mathcal{R} \right)^{-1/2} \exp \left( t\Delta_{G(t)} \right).
\] (7.26)

\[
\text{Corollary 5} \, \text{Let } A_i \text{ be a vector commuting with the operators } \nabla_j. \text{ Then for sufficiently small } t > 0
\]
\[
\int d\xi \exp \left\{ -\frac{1}{4t} \left( \xi, \gamma\xi \right) + \left( A, \xi \right) \right\} \exp \left( \xi, \nabla \right) = (4\pi t)^{n/2} \text{det}(\gamma + ti\mathcal{R})^{-1/2}
\]
\[
\times \exp \left\{ t\Delta_{G(t)} + 2t \left( G^{-1}(t)A, \nabla \right) + t \left( A, G^{-1}(t)A \right) \right\}.
\] (7.27)
\[
\text{Proof.} \, \text{We notice that the operators } \nabla_i + A_i \text{ form the same Lie algebra as the operators } \nabla_i. \text{ Therefore, by replacing } \nabla_i \mapsto \nabla_i + A_i \text{ we obtain (7.27).} \]

This lemma enables one to prove the following theorem for the heat semigroup \[2, 6\]. Notice that, although Lemma \[5\] is valid, strictly speaking, only for small \(t\), this theorem holds for any \(t\). Let \(g^{-1} = (g^{-1}) \) be a symmetric positive matrix, and \(\Delta_g = g^{-1}i\nabla \nabla \). Let \(D(t) = (D_{ij})\) be a symmetric matrix defined by
\[
D(t) = \frac{1}{t} g\Psi(tg^{-1}i\mathcal{R}) = i\mathcal{R} \coth \left( tg^{-1}i\mathcal{R} \right)
\] (7.28)
and \(\Omega(t)\) be a function defined by
\[
\Omega(t) = \text{det} \left( D(t) + ti\mathcal{R} \right)^{1/2} = \text{det} \left( g^{-1} \sinh(tg^{-1}i\mathcal{R}) \right)^{-1/2}.
\] (7.29)
For the future reference we explore the Taylor expansion of these objects for small \(t\)
\[
D(t) = \frac{1}{t} \left\{ g + \frac{1}{3} t^2 i\mathcal{R} g^{-1}i\mathcal{R} + O(t^4) \right\},
\] (7.30)
\[
\Omega(t) = t^{-n/2} (\text{det} g)^{1/2} \left\{ 1 - \frac{1}{12} t^2 \text{tr} \left( g^{-1}i\mathcal{R} g^{-1}i\mathcal{R} \right) + O(t^4) \right\}.
\] (7.31)
7.1 Proof of Theorem 1

Let \( J(t) \) be the right hand side of eq. (1.22). By using Lemma 5 to compute the integral over \( \xi \) (with \( \gamma = tD(t) \)) we obtain from (7.8) or (7.27)

\[
J(t) = \Omega(t) \det (D(t) + i\mathcal{R})^{-1/2} \exp(t\Delta_{G(t)}), \tag{7.32}
\]

where the matrix \( G^{-1}(t) \) is given by

\[
G^{-1}(t) = \frac{1}{t} \Phi(D^{-1}(t)i\mathcal{R}) D^{-1}(t). \tag{7.33}
\]

It is easy to see that

\[
D^{-1}(t)i\mathcal{R} = \tanh(tg^{-1}i\mathcal{R}), \tag{7.34}
\]

therefore, by using (5.19) we get

\[
G^{-1}(t) = \frac{1}{t} \Phi(tg^{-1}i\mathcal{R}) D^{-1}(t) = g^{-1}, \tag{7.35}
\]

and \( \Delta_{G(t)} = \Delta_{g} \). Also, we notice that

\[
\det(D(t) + i\mathcal{R}) = \det \left( (D + i\mathcal{R})(D - i\mathcal{R}) \right)^{1/2}
\]

\[
= \det \left( \frac{i\mathcal{R}}{\sinh(tg^{-1}i\mathcal{R})} \right) = \Omega^{2}(t). \tag{7.36}
\]

Therefore, by using eq. (7.36) in (7.32) we obtain \( J(t) = \exp(t\Delta_{g}) \). \( \square \)

By using Lemma 3 and Theorem 1 one can prove the following corollary.

**Corollary 6** Let \( T(t) \) be the matrix defined by

\[
T(t) = D(t) + i\mathcal{R}. \tag{7.37}
\]

Then

\[
\nabla_{k} \exp(t\Delta_{g}) = \frac{1}{2}(4\pi)^{-n/2} \Omega(t) \int_{\mathbb{R}^{n}} d\xi \exp \left\{ -\frac{1}{4} \langle \xi, D(t)\xi \rangle \right\} T_{kj}(t)\xi^{j} \exp(\xi, \nabla), \tag{7.38}
\]

\[
\exp(t\Delta_{g})\nabla_{k} = \frac{1}{2}(4\pi)^{-n/2} \Omega(t) \int_{\mathbb{R}^{n}} d\xi \exp \left\{ -\frac{1}{4} \langle \xi, D(t)\xi \rangle \right\} T_{jk}(t)\xi^{j} \exp(\xi, \nabla). \tag{7.39}
\]

**Proof.** This is proved by using the Lemma 3 and integrating by parts. \( \square \)
8 Product of Semigroups

We consider the set of operators \((\nabla^+_1, \ldots, \nabla^+_n, \nabla^-_1, \ldots, \nabla^-_n, i)\) forming the Lie algebra

\[
\left[ \nabla^+_i, \nabla^+_j \right] = i\mathcal{R}^+_{ij},
\]

(8.1)

\[
\left[ \nabla^-_i, \nabla^-_j \right] = i\mathcal{R}^-_{ij},
\]

(8.2)

\[
\left[ \nabla^+_i, \nabla^-_j \right] = i\mathcal{R}_{ij},
\]

(8.3)

where

\[
\mathcal{R}_{ij} = \frac{1}{2} \left( \mathcal{R}^+_{ij} + \mathcal{R}^-_{ij} \right).
\]

(8.4)

This algebra can be written in a more compact form by introducing the operators \((\hat{\mathcal{D}}_A) = (\hat{\mathcal{D}}_1, \ldots, \hat{\mathcal{D}}_n, \hat{\mathcal{D}}_{n+1}, \ldots, \hat{\mathcal{D}}_{2n})\) by

\[
\hat{\mathcal{D}}_1 = \nabla^+_1, \ldots, \hat{\mathcal{D}}_n = \nabla^+_n,
\]

(8.5)

\[
\hat{\mathcal{D}}_{n+1} = \nabla^-_1, \ldots, \hat{\mathcal{D}}_{2n} = \nabla^-_n.
\]

(8.6)

We will use the convention that the capital Latin indices run over \(1, \ldots, 2n\). The operators \(\hat{\mathcal{D}}_A\) form the algebra

\[
[\hat{\mathcal{D}}_A, \hat{\mathcal{D}}_B] = i\mathcal{F}_{AB},
\]

(8.7)

where \(\mathcal{F} = (\mathcal{F}_{AB})\) is a \(2n \times 2n\) anti-symmetric matrix defined by

\[
\mathcal{F} = \begin{pmatrix} \mathcal{R}^+ & \mathcal{R}^- \\ \mathcal{R}^- & \mathcal{R}^+ \end{pmatrix}.
\]

(8.8)

The same algebra can be also rewritten in more convenient form by explicitly exhibiting an Abelian subalgebra. Let

\[
\nabla_i = \frac{1}{2}(\nabla^+_i + \nabla^-_i),
\]

(8.9)

\[
X_i = \nabla^+_i - \nabla^-_i,
\]

(8.10)

so that

\[
\nabla^\pm_i = \nabla_i \pm \frac{1}{2} X_i.
\]

(8.11)

These operators form the algebra

\[
[\nabla_i, \nabla_j] = i\mathcal{R}_{ij},
\]

(8.12)

\[
[\nabla_i, X_j] = iF_{ij},
\]

(8.13)

\[
[X_i, X_j] = 0,
\]

(8.14)
where
\[ F_{ij} = \frac{1}{2} (\mathcal{R}_{ij}^+ - \mathcal{R}_{ij}^-). \] (8.15)

Let us also define \((\mathcal{D}_A) = (\mathcal{D}_1, \ldots, \mathcal{D}_n, \mathcal{D}_{n+1}, \ldots, \mathcal{D}_{2n})\),
\[ \mathcal{D}_1 = \nabla_1, \ldots, \mathcal{D}_n = \nabla_n, \] (8.16)
\[ \mathcal{D}_{n+1} = X_1, \ldots, \mathcal{D}_{2n} = X_n, \] (8.17)

These operators satisfy the commutation relations
\[ [\mathcal{D}_A, \mathcal{D}_B] = i \mathcal{F}_{AB}, \] (8.18)
where \(\mathcal{F} = (\mathcal{F}_{AB})\) is a \(2n \times 2n\) anti-symmetric matrix defined by
\[ \mathcal{F} = \begin{pmatrix} \mathcal{R} & F \\ F & 0 \end{pmatrix}. \] (8.19)

These bases are related by
\[ \tilde{\mathcal{D}} = \Lambda \mathcal{D}, \] (8.20)
where \(\Lambda\) is a \(2n \times 2n\) matrix defined by
\[ \Lambda = \begin{pmatrix} I & \frac{1}{2}I \\ I & -\frac{1}{2}I \end{pmatrix}. \] (8.21)
and, therefore,
\[ \tilde{\mathcal{F}} = \Lambda \mathcal{F} \Lambda^T. \] (8.22)

Let \(g_{ij}^{\pm}\) be two positive symmetric matrices defining the Laplacians
\[ \Delta_\pm = \langle \nabla_\pm, g_\pm^{-1} \nabla_\pm \rangle = g_{ij}^{\pm} \nabla_i^\pm \nabla_j^\pm. \] (8.23)

Our goal is the computation of the convolution
\[ U(t, s) = \exp(t \Delta_+) \exp(s \Delta_-). \] (8.24)

We write the matrices \(g_{\pm}^{-1}\) in the form \(g_{\pm}^{-1} = \omega_{\pm} \omega_{\pm}^T\). Then we have \(\Delta_\pm = \langle \nabla'_\pm, \nabla'_\pm \rangle\), with \(\nabla'_\pm = \omega_{\pm}^T \nabla_\pm\) satisfying the commutation relations with the curvatures \(\mathcal{R}_{\pm} = \omega_{\pm}^T \mathcal{R}_\pm \omega_{\pm}\). Further, we can subject the operators \(\nabla'_\pm\) to orthogonal transformations \(\tilde{\nabla}_\pm = O_\pm \nabla'_\pm\) with orthogonal matrices \(O_\pm\) without changing the operators \(\Delta_\pm\). Then \(\Delta_\pm = \langle \tilde{\nabla}_\pm, \tilde{\nabla}_\pm \rangle\) and the transformed curvatures are \(\tilde{\mathcal{R}}_\pm = O_\pm \omega_{\pm}^T \mathcal{R}_\pm \omega_{\pm} O_{\pm}^T\); the matrices \(O_\pm\) can be chosen so that the curvatures \(\tilde{\mathcal{R}}_\pm\) have the canonical form (5.10). Thus, without loss of generality we can assume that the matrices \(g_\pm\) are equal to the unit matrix, \(g_\pm = I\), and the matrices \(\mathcal{R}_\pm\) have the canonical form.
8.1 Method I

In the following we use the functions $\Phi(z)$ and $\Psi(z)$ defined by (5.17) and (5.18) and the matrix $\Lambda$ defined by (8.21). Let $D_\pm$ be two positive symmetric matrices defined by

$$D_\pm(t) = \frac{1}{t} g_\pm \Psi(t g_\pm^{-1} i \mathbb{R}_\pm).$$  \hfill (8.25)

Let $\tilde{Q}(t, s) = (\tilde{Q}_{AB})$ and $Q(t, s) = (Q_{AB})$ be the $2n \times 2n$ symmetric matrices defined by

$$\tilde{Q}(t, s) = \begin{pmatrix} D_+(t) & -i \mathbb{R} \\ i \mathbb{R} & D_-(s) \end{pmatrix},$$  \hfill (8.26)

$$Q(t, s) = \Lambda^{-1} \tilde{Q}(t, s) \Lambda^{-1T} = \begin{pmatrix} \frac{1}{2} (D_+ + D_-) & \frac{1}{2} (D_+ - D_- + 2i \mathbb{R}) \\ \frac{1}{2} (D_+ - D_- - 2i \mathbb{R}) & D_+ + D_- \end{pmatrix},$$  \hfill (8.27)

and $\mathcal{G}^{-1}(t, s) = (\mathcal{G}^{AB})$ and $\mathcal{G}^{-1}(t, s) = (\mathcal{G}^{AB})$ be the $2n \times 2n$ symmetric matrices defined by

$$\mathcal{G}^{-1}(t, s) = \Phi(\tilde{Q}^{-1}(t, s) i \mathcal{F}) \tilde{Q}^{-1}(t, s),$$  \hfill (8.28)

$$\mathcal{G}^{-1}(t, s) = \Lambda^T \mathcal{G}^{-1}(t, s) \Lambda = \Phi(Q^{-1}(t, s) i \mathcal{F}) Q^{-1}(t, s).$$  \hfill (8.29)

Let $\mathcal{H}(t, s)$ be the operator defined by

$$\mathcal{H}(t, s) = \langle \mathcal{D}, \mathcal{G}^{-1}(t, s) \mathcal{D} \rangle;$$  \hfill (8.30)

it is easy to see that it is also equal to

$$\mathcal{H}(t, s) = \langle \tilde{\mathcal{D}}, \mathcal{G}^{-1}(t, s) \tilde{\mathcal{D}} \rangle.$$  \hfill (8.31)

We can write the operator $\mathcal{H}$ in a more explicit form. We represent the metric $\mathcal{G}^{-1}$ in the block diagonal form

$$\mathcal{G}^{-1} = \begin{pmatrix} G^{-1} & Y \\ Y^T & M \end{pmatrix},$$  \hfill (8.32)

where $G$ and $M$ are symmetric matrices. Then the operator $\mathcal{H}$ takes the form

$$\mathcal{H} = \langle \nabla, G^{-1} \nabla \rangle + \langle \nabla, Y X \rangle + \langle X, Y^T \nabla \rangle + \langle X, M X \rangle,$$

$$= \langle \tilde{\nabla}, G^{-1} \tilde{\nabla} \rangle + \langle X, V X \rangle,$$  \hfill (8.33)
where
\begin{align}
\hat{\nabla} &= \nabla + GYX = \frac{1}{2}(I + 2GY)\nabla_+ + \frac{1}{2}(I - 2GY)\nabla_- \quad \text{(8.34)} \\
V &= M - Y^T GY. \quad \text{(8.35)}
\end{align}

### 8.1.1 Proof of Theorem\textsuperscript{2}

By using Theorem\textsuperscript{1} we have
\begin{align}
U(t, s) &= (4\pi)^{-n} \Omega_+(t) \Omega_-(s) \int_{\mathbb{R}^{2n}} d\xi_+ d\xi_- \exp \left\{ -\frac{1}{4} \langle \xi_+, D_+(t)\xi_+ \rangle - \frac{1}{4} \langle \xi_-, D_-(s)\xi_- \rangle \right\} \\
&\quad \times \exp \left\{ \langle \xi_+, \nabla^+ \rangle \exp \langle \xi_-, \nabla^- \rangle \right\}, \quad \text{(8.36)}
\end{align}

where the functions \( \Omega_\pm \) are defined by (7.29). By using the special case of the Campbell-Hausdorff formula (4.32) and the commutators (8.3) we get
\begin{align}
U &= (4\pi)^{-n} \Omega_+ \Omega_- \int_{\mathbb{R}^{2n}} d\eta \exp \left\{ -\frac{1}{4} \langle \eta_+, D_+ \eta_+ \rangle - \frac{1}{4} \langle \eta_-, D_- \eta_- \rangle + \frac{1}{2} \langle \eta_+, iR \eta_- \rangle \right\} \\
&\quad \times \exp \left\{ \langle \eta_+, \nabla^+ \rangle + \langle \eta_-, \nabla^- \rangle \right\}. \quad \text{(8.37)}
\end{align}

We introduce new integration variables \((\eta^A) = (\xi_1^+, \ldots, \xi_n^+, \xi_1^-, \ldots, \xi_n^-)\) and use the operators \(\tilde{\Theta}_A\) to rewrite this in the form
\begin{align}
U &= (4\pi)^{-n} \Omega_+ \Omega_- \int_{\mathbb{R}^{2n}} d\eta \exp \left\{ -\frac{1}{4} \langle \eta, \tilde{\Theta} \eta \rangle \right\} \exp \langle \eta, \tilde{\Theta} \rangle. \quad \text{(8.38)}
\end{align}

This integral can be computed by using Corollary\textsuperscript{5} to obtain
\begin{align}
U(t, s) = \frac{\Omega_+(t)\Omega_-(s)}{\Omega(t, s)} \exp \langle \tilde{\Theta}, \tilde{\Theta}^{-1}(t, s) \tilde{\Theta} \rangle, \quad \text{(8.39)}
\end{align}

where
\begin{align}
\tilde{\Omega}(t, s) &= \det (\tilde{D}(t, s) + i\tilde{\mathcal{F}})^{1/2}. \quad \text{(8.40)}
\end{align}

Finally, we compute the determinant
\begin{align}
det (\tilde{D}(t, s) + i\tilde{\mathcal{F}}) = det(D_+ + i\mathcal{F}_+) \det(D_- + i\mathcal{F}_-) = \Omega_+^2(t)\Omega_-^2(s); \quad \text{(8.41)}
\end{align}

therefore, \(\tilde{\Omega}(t, s) = \Omega_+(t)\Omega_-(s)\), which proves the theorem by taking into account (8.31). \(\square\)
8.1.2 Examples

Notice that for small $t,s$ the metric factorizes

$$\tilde{g}^{-1} = \begin{pmatrix} tI & 0 \\ 0 & sI \end{pmatrix} + \cdots,$$

so that

$$\mathcal{H} = t\Delta_+ + s\Delta_- + \cdots. \quad (8.43)$$

Let us study another particular case, when the operators are equal, that is, $\nabla_+ = \nabla_- = \nabla$, $\mathcal{R}_+ = \mathcal{R}_- = \mathcal{R}$. Then we should have obviously

$$\tilde{\mathcal{H}} = (t+s)\Delta. \quad (8.44)$$

Then there is only one matrix in the problem, $\mathcal{R}$. Therefore, all matrices commute. To simplify notation let $x = ti\mathcal{R}$ and $y = si\mathcal{R}$ and

$$a = \coth x, \quad b = \coth y. \quad (8.45)$$

Then

$$i\tilde{\mathcal{F}} = \frac{x+y}{t+s} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \tilde{Q} = \frac{x+y}{t+s} \begin{pmatrix} a & -1 \\ 1 & b \end{pmatrix}. \quad (8.46)$$

The inverse of the matrix $\tilde{D}$ has the form

$$\tilde{Q}^{-1} = \frac{(t+s)}{(x+y)(ab+1)} \begin{pmatrix} b & 1 \\ -1 & a \end{pmatrix}; \quad (8.47)$$

therefore,

$$\tilde{Q}^{-1}i\tilde{\mathcal{F}} = c\Pi, \quad (8.48)$$

where and $c = \frac{a+b}{ab+1} = \tanh(x+y)$ and

$$\Pi = \frac{1}{a+b} \begin{pmatrix} b+1 & b+1 \\ a-1 & a-1 \end{pmatrix}. \quad (8.49)$$

It is easy to see that the matrix $\Pi$ is idempotent

$$\Pi^2 = \Pi, \quad (8.50)$$

and, therefore, for any analytic function of $\Pi$

$$f(t\Pi) = f(0)(I-\Pi) + f(t)\Pi. \quad (8.51)$$
Therefore, by using (5.19) we have
\[
\Phi(\tilde{Q}^{-1}i\tilde{R}) = I + \left[\frac{(x+y)}{c} - 1\right] \Pi. \tag{8.52}
\]

Further, we compute the metric
\[
\tilde{G}^{-1} = \Phi(\tilde{Q}^{-1}i\tilde{R}) \tilde{Q}^{-1} = (t+s) \left\{ \frac{1}{(a+b)^2} \left( \begin{array}{cc} b^2 - 1 & (a+1)(b+1) \\ (a-1)(b-1) & a^2 - 1 \end{array} \right) \right. \\
+ \frac{1}{(x+y)(a+b)} \left( \begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array} \right) \right\}. \tag{8.53}
\]

By using this metric (and noticing that the sum of all its elements is equal to \((t+s)\)) it is easy to show that indeed the operator \(\mathcal{H}\) takes the form
\[
\mathcal{H} = \langle \tilde{\mathcal{D}}, \tilde{G}^{-1} \tilde{\mathcal{D}} \rangle = (t+s)\Delta. \tag{8.54}
\]

### 8.2 Method II

Finally, we present a method that will be useful to compute the kernel of the product \(\exp(t\Delta_+)\exp(s\Delta_-)\). We recall the definition of the matrices \(T(t) = D(t) + i\mathcal{R}\) by (7.37). Also, we define two matrices
\[
D(t,s) = D_+(t) + D_-(s), \tag{8.55}
\]
\[
Z(t,s) = D_+(t) - D_-(s) - 2i\mathcal{R}, \tag{8.56}
\]
and a function \(\Omega(t,s)\) by
\[
\Omega(t,s) = \Omega_+(t)\Omega_-(s) \det D^{-1/2}(t,s). \tag{8.57}
\]

We also define a symmetric matrix \(H\) by
\[
H(t,s) = \frac{1}{4} \left( D(t,s) - Z^T(t,s)D^{-1}(t,s)Z(t,s) \right). \tag{8.58}
\]

By rewriting the matrix \(Z\) in the form
\[
Z = D - 2T_- \tag{8.59}
\]
the matrix $H$ takes the form

$$H(t,s) = D_-(s) - T_T^- (s) D^{-1} (t,s) T_-(s)$$  \hspace{1cm} (8.60)$$

and by writing the matrix $Z$ in the form

$$Z = -D + 2T_T^+ + 4iF$$  \hspace{1cm} (8.61)$$

we can represent the matrix $H$ in terms of the matrices $D_+$ and $T_+$ by

$$H(t,s) = D_+(t) - T_+(t) D^{-1} (t,s) T_T^+ (t)$$
$$- 2T_+(t) D^{-1} (t,s) iF + 2iFD^{-1} (t,s) T_T^+ (t) + 4iFD^{-1} (t,s) iF.$$  \hspace{1cm} (8.62)$$

We exhibit the behavior of these objects for small $t, s$,

$$D(t,s) = \left(1 + \frac{1}{t} + \frac{1}{s}\right) I + \cdots$$  \hspace{1cm} (8.63)$$

$$Z(t,s) = \left(1 - \frac{1}{t} - \frac{1}{s}\right) I + \cdots$$  \hspace{1cm} (8.64)$$

$$\Omega(t,s) = (t+s)^{-n/2} + \cdots$$  \hspace{1cm} (8.65)$$

$$H(t,s) = (s+t)^{-1} I + \cdots$$  \hspace{1cm} (8.66)$$

8.2.1 Proof of Theorem \[3\]

We change the integration variables in (8.37) by

$$\xi_+ = \frac{1}{2} \alpha + \beta, \hspace{1cm} \xi_- = \frac{1}{2} \alpha - \beta,$$  \hspace{1cm} (8.67)$$

so that

$$\alpha = \xi_+ + \xi_-, \hspace{1cm} \beta = \frac{1}{2} (\xi_+ - \xi_-),$$  \hspace{1cm} (8.68)$$

to obtain

$$U(t,s) = (4\pi)^{-n} \Omega_+ \Omega_- \int_{\mathbb{R}^{2n}} d\alpha d\beta \exp \left\{ -\frac{1}{16} \langle \alpha, D\alpha \rangle - \frac{1}{4} \langle \beta, D\beta \rangle \right\}$$
$$\times \exp \left\{ -\frac{1}{4} \langle \beta, (D_+ - D_- - 2i\mathcal{R}) \alpha \rangle \right\} \exp \{ \langle \alpha, \nabla \rangle + \langle \beta, X \rangle \}.  \hspace{1cm} (8.69)$$
By using the Campbell-Hausdorff formula (4.32) we obtain
\[
\exp\{\langle \alpha, \nabla \rangle + \langle \beta, X \rangle\} = \exp\left\{-\frac{1}{2} \langle \beta, iF\alpha \rangle \right\} \exp \langle \beta, X \rangle \exp \langle \alpha, \nabla \rangle.
\] (8.70)

Therefore, eq. (8.69) takes the form
\[
U = (4\pi)^{-n} \Omega_{+} \Omega_{-} \int_{\mathbb{R}^{2n}} d\alpha d\beta \exp\left\{-\frac{1}{4} \langle \beta, D\beta \rangle - \frac{1}{4} \langle \beta, Z\alpha \rangle + \langle \beta, X \rangle\right\} \times \exp\left\{-\frac{1}{16} \langle \alpha, D\alpha \rangle\right\} \exp \langle \alpha, \nabla \rangle,
\] (8.71)

with the matrices \(D\) and \(Z\) defined by (8.55) and (8.56). The integral over \(\beta\) can be computed by using Corollary 5 to obtain (1.41) with the matrix \(H\) in the form (8.58).

\[\blacksquare\]

9 Convolution of Heat Kernels

We work with operators
\[
\nabla_{i}^{\pm} = \partial_{i} - \frac{1}{2} i R_{ij}^{\pm} x^{j}, \quad (9.1)
\]
acting on smooth functions in \(\mathbb{R}^{n}\). These operators form the Lie algebra (8.3). The operators \(\nabla_{i}\) and \(X_{i}\) defined in (8.9) and (8.10) then take the form
\[
\nabla_{i} = \partial_{i} - \frac{1}{2} i R_{ij} x^{j}, \quad (9.2)
\]
\[
X_{i} = -i F_{ij} x^{j}, \quad (9.3)
\]
with the matrices \(R_{ij}\) and \(F_{ij}\) defined in (8.4) and (8.15). The corresponding Laplacians are \(\Delta_{\pm} = g_{ij}^{\pm} \nabla_{i}^{\pm} \nabla_{j}^{\pm}\).

Recall the definition of the matrices \(D_{\pm}(t)\) and \(T_{\pm}(t)\) by (8.25) and (7.37). Also, notice that the functions \(\Omega_{\pm}(t)\) defined by (7.29) are determined by the determinant of the matrices \(T_{\pm}\)
\[
\Omega_{\pm}(t) = \det T_{\pm}^{1/2}. \quad (9.4)
\]

Let \(S_{\pm}\) be functions defined by
\[
S_{\pm}(t; x, x') = \frac{1}{4} \langle (x - x'), D_{\pm}(t)(x - x') \rangle - \frac{1}{2} \langle x, i R_{\pm} x' \rangle
\]
\[
= \frac{1}{4} \langle x, D_{\pm}(t)x \rangle + \frac{1}{4} \langle x', D_{\pm}(t)x' \rangle - \frac{1}{2} \langle x, T_{\pm}(t)x' \rangle, \quad (9.5)
\]
Notice that the matrix of second mixed partial derivatives of the function \(S_\pm\) is proportional to the matrix \(T_\pm\)

\[
S_{xx'}^\pm = -\frac{1}{2} T_\pm(t), \quad (9.6)
\]

and, therefore,

\[
\Omega_\pm(t) = \det \left( -2S_{xx'}^\pm \right)^{1/2}. \quad (9.7)
\]

**Lemma 6** The heat kernel of the operator \(\Delta_\pm\) is

\[
U_\pm(t; x, x') = \det \left( -\frac{S_{xx'}^\pm}{2\pi} \right)^{1/2} \exp \left\{ -\frac{1}{4} \langle \xi_\pm, D_\pm(t) \xi_\pm \rangle \exp \langle \xi_\pm, \nabla_\pm \rangle \delta(x - x') \right\}. \quad (9.8)
\]

**Proof.** The heat kernel is given by \(U_\pm(t; x, x') = \exp(t\Delta_\pm)\delta(x - x').\) We use eq. (1.22) to compute it,

\[
U_\pm(t; x, x') = (4\pi)^{-n/2} \Omega_\pm(t) \int d\xi_\pm \exp \left\{ -\frac{1}{4} \langle \xi_\pm, D_\pm(t) \xi_\pm \rangle \right\} \exp \langle \xi_\pm, \nabla_\pm \rangle \delta(x - x'). \quad (9.9)
\]

We notice that the operators \(\langle \xi_\pm, \partial \rangle\) and \(\langle \xi_\pm, \mathcal{R}_\pm x \rangle\) commute; therefore, we have

\[
\exp \langle \xi_\pm, \nabla_\pm \rangle = \exp \left\{ -\frac{1}{2} \langle \xi_\pm, i\mathcal{R}_\pm x \rangle \right\} \exp \langle \xi_\pm, \partial \rangle \quad (9.10)
\]

and, hence,

\[
\exp \langle \xi_\pm, \nabla_\pm \rangle \delta(x - x') = \exp \left\{ -\frac{1}{2} \langle \xi_\pm, i\mathcal{R}_\pm x \rangle \right\} \delta(x - x' + \xi_\pm). \quad (9.11)
\]

This immediately gives the heat kernel

\[
U_\pm(t; x, x') = (4\pi)^{-n/2} \Omega_\pm(t) \exp \left\{ -\frac{1}{4} \langle x, A_\pm(t, s)x \rangle + \frac{1}{4} \langle x', A_\pm(t, s)x' \rangle - \frac{1}{2} \langle x, B(t, s)x \rangle \right\}, \quad (9.12)
\]

which proves the lemma by taking into account (9.7). \(\Box\)

We proceed as follows. Let \(S\) be a function defined by

\[
S(t, s; x, x') = \frac{1}{4} \langle x, A_+(t, s)x \rangle + \frac{1}{4} \langle x', A_-(t, s)x' \rangle - \frac{1}{2} \langle x, B(t, s)x \rangle, \quad (9.13)
\]
where
\[ A_+ = H+Z^TD^{-1}iF-iFD^{-1}Z+4iFD^{-1}iF, \]  
\[ A_- = H, \]  
\[ B = H-iFD^{-1}Z+i\mathcal{H}. \]

By using the definition of the matrix $H$, (8.60) and (8.62), and of the matrix $Z$, (8.56) and (8.61), one can show that
\[ A_+(t,s) = D_+(t)-T_+^T(t)D_{-1}^{-1}(t,s)T_+(t), \]  
\[ A_-(t,s) = D_-(s)-T_-^T(s)D_{-1}^{-1}(t,s)T_-(s), \]  
\[ B(t,s) = T_+(t)D_{-1}(t,s)T_-(s). \]

9.1 Proof of Theorem 4

The kernel of the product of the semigroups $U(t,s;x,x')$ can be computed by using the equation (1.41). Since the operators $\langle \alpha, \partial \rangle$ and $\langle \alpha, \mathcal{H}x \rangle$ commute we have
\[ \exp \langle \alpha, \nabla \rangle \delta(x-x') = \exp \left\{ -\frac{1}{2} \langle \alpha, i\mathcal{H}x \rangle \right\} \delta(x-x'+\alpha). \]  

Therefore, we immediately obtain from (1.41)
\[ U(t,s;x,x') = (4\pi)^{-n/2}\Omega(t,s)\exp \left\{ -S(t,s;x,x') \right\}, \]  
where $\Omega(t,s)$ is defined by (8.57) and $S$ is a function given by
\[ S(t,s;x,x') = \frac{1}{4} \langle (x-x'), H(x-x') \rangle - \frac{1}{2} \langle (x-x'), Z^TD^{-1}X \rangle \]  
\[ -\langle X, D^{-1}X \rangle - \frac{1}{2} \langle x, i\mathcal{H}x' \rangle. \]  

with $H$ and $Z$ being the matrices defined by (8.60) and (8.56). It is not difficult to simplify this to the form (9.13). Further, we see that
\[ \Omega(t,s) = \frac{\det \Omega_+(t)\Omega_-(s)}{\det D_{1/2}(t,s)} = \det B^{1/2} = \det (-2S_{xx'}(t,s))^{1/2}, \]
which gives finally (1.46). \(\blacksquare\)
As an independent check we compute the convolution of the two heat kernels directly by computing the integral
\[
U(t,s;x,x') = \int_{\mathbb{R}^n} dz \, U_+(t;x,z)U_-(s;z,x'). \tag{9.24}
\]
This integral is Gaussian and can be easily computed to obtain
\[
U(t,s;x,x') = (4\pi)^{-n/2}\Omega(t,s)\exp\{-S(t,s;x,x')\}, \tag{9.25}
\]
where \(\Omega(t,s)\) is defined by (8.57) and \(S\) is a function given by
\[
S(t,s;x,x') = \frac{1}{4} \langle x, D_+(t)x \rangle + \frac{1}{4} \langle x', D_-(s)x' \rangle - \frac{1}{4} \langle y, D^{-1}(t,s)y \rangle, \tag{9.26}
\]
with
\[
y = T_+^T(t)x + T_-(s)x'. \tag{9.27}
\]
It is easy to see that it is indeed equal to (9.13).

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