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Cosmological equations for a thick brane

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Generalized Friedmann equations governing the cosmological evolution inside a thick brane embedded in a five-dimensional anti–de Sitter spacetime are derived. These equations are written in terms of four-dimensional effective brane quantities obtained by integrating, along the fifth dimension, over the brane thickness. In the case of a Randall-Sundrum type cosmology, different limits of these effective quantities are considered yielding cosmological equations which interpolate between the thin brane limit (governed by unconventional brane cosmology), and the opposite limit of an “infinite” brane thickness corresponding to the familiar Kaluza-Klein approach. In the more restrictive case of a Minkowski bulk, it is shown that no effective four-dimensional reduction is possible in the regimes where the brane thickness is not small enough.

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I. INTRODUCTION

The idea of extra dimensions where ordinary matter is confined within a lower dimensional submanifold has received an enormous amount of attention during the past few years (see, e.g., [1] for a recent review). This idea has been explored, in particular, in the cosmological context.

In the case of a single extra dimension, a simplifying assumption has been to consider our brane universe as infinitely thin along this extra dimension. The purpose of the present paper is to explore the modification of the equations yielding the cosmological evolution within the brane when the finite thickness of the brane is taken into account. We consider this question in a purely phenomenological approach by assuming that the matter energy-momentum tensor describing the brane has some distribution over a finite length interval along the fifth dimension. Our approach (similar to that of [2]) is thus more crude, but also more general, than the domain wall description of a brane, for which the brane is embodied by a scalar field, such as in [3,4] (see also [5] for a recent study in 4D gravity and [6] for a review on domain walls, in particular in supergravity).

Two limiting cases are already known. The case of an infinitely thin brane gives the unconventional Friedmann equation of [7], which can be made compatible with ordinary cosmology at late times if one introduces both a negative cosmological constant in the bulk spacetime and a related tension (unmeasurable by cosmological observers) in the brane [8–11]. The other and opposite limit is that of an infinitely thick brane, in a sense defined below. This limit effectively corresponds to a Kaluza-Klein picture where matter is homogeneously distributed over the extra-dimension(s) and, as has been known for a long time, ordinary Friedmann equations are recovered. Our analysis gives access to a description of the intermediate cases.

The paper is organized as follows. Our effective Friedmann equation is established in Sec. II and written in terms of four-dimensional effective quantities. Section III is devoted to the cosmology for a brane with a tension, the cosmological effect of which is canceled by a negative bulk cosmological constant. We consider the limits of a low brane energy (for all brane thickness) and of a small brane thickness (for all brane energy), and we propose uniform expressions for the cosmological equations valid over the whole parameter domain covered by these two limits. In Sec. IV, we determine the higher order corrections (in the normalized brane thickness) to the effective cosmological equations in the simpler case of a Minkowski bulk. In particular, we show that no effective four-dimensional reduction is possible in the regimes of order one normalized brane thickness.

II. EFFECTIVE FRIEDMANN EQUATION FOR A THICK BRANE

We consider a five-dimensional spacetime, which is homogeneous and isotropic along three spatial dimensions, and which contains a thick brane. For simplicity, we will restrict ourselves to the case of a mirror symmetric brane (and thus spacetime) along the extra dimension. It is then always possible to find a Gaussian coordinate system, starting from the hypersurface representing the “center” of the brane, in which the line element takes the simple diagonal form

\[
ds^2 = g_{AB} dx^A dx^B = -n^2(t,y) dt^2 + a^2(t,y) \delta_{ij} dx^i dx^j + r_b^2 dy^2,
\]

where \(y\) is the coordinate of the fifth dimension and \(r_b\) is the brane thickness which will be assumed to be time independent. The brane is localized between \(y = -1/2\) and \(y = 1/2\). To be compatible with the spacetime symmetries, the energy-momentum tensor of the matter content in the brane is necessarily of the form

\[
T^A_{\;B} = \text{diag} \left[ \frac{1}{r_b} (-p_b, p_b, p_b, p_b) \right].
\]
where \( \rho_b, p_b, \) and \( P_T \) are functions of \( t \) and \( y \). The presence of a negative cosmological constant in the bulk, \( \Lambda \), is accounted for by an energy-momentum tensor of the form

\[
T^{A B}_{\text{Bulk}} = - \Lambda g^{A B}.
\]

(3)

Note that, strictly speaking, \( \Lambda \) as defined here does not have the dimension of a cosmological constant but rather that of a five-dimensional energy density.

The purpose of this paper is to establish effective cosmological equations for an observer living in the brane. Because of the finite thickness of the brane, there is some arbitrariness in the definition of what the effective four-dimensional quantities should be. The simplest prescription one can think of consists in defining the four-dimensional effective four spatial quantities associated to a five-dimensional quantity \( Q(t,y) \) as its spatial average over the brane thickness

\[
\langle Q \rangle(t) = \int_{-1/2}^{1/2} Q(t,y) dy.
\]

(4)

In this paper, we will adopt this simple prescription. The four-dimensional “observable” counterparts of \( a, \rho_b \), and \( p_b \) are thus \( \langle a \rangle = \int_{-1/2}^{1/2} a \, dy, \langle \rho_b \rangle = \int_{-1/2}^{1/2} \rho_b \, dy, \) and \( \langle p_b \rangle = \int_{-1/2}^{1/2} p_b \, dy \). Note that, in contrast with the usual dimensional reduction in field theory, we do not integrate over the whole (compact) extra dimension but only over the extension of the brane in the extra dimension.

The next step is now to obtain evolution equations for the four-dimensional effective quantities from the five-dimensional Einstein’s equations

\[
G_{AB} - \frac{1}{2} R g_{AB} = \kappa^2 T_{AB},
\]

(5)

where \( R_{AB} \) is the Ricci tensor and \( R \) its trace. The components of the Einstein equations, with the metric (1) and the energy-momentum tensors (2) and (3) can be obtained from the components of the Einstein tensor given in the Appendix. In particular, the 05 component of the Einstein equations yields

\[
n(t,y) = \xi(t) \dot{a}(t,y),
\]

(6)

where the dot stands for a partial derivative with respect to time \( t \) and \( \xi \) depends on the normalization prescription for \( n \). In the following, we will take the normalization \( \langle n \rangle = 1 \) which gives \( \xi = \langle \dot{a} \rangle^{-1} \). The 00 component of the Einstein equations then reads

\[
\langle a \rangle^2 = - \frac{2 \kappa^2}{3} (r_b \rho_b + r_b^2 \Lambda) a^2 + 2 r_b^2 \langle \dot{a} \rangle^2,
\]

(7)

where a prime denotes a partial derivative with respect to the coordinate \( y \). Integrating this equation over the brane, one obtains

\[
a(1/2)' = \frac{1}{a(1/2)} \left[ - \frac{\kappa^2}{6} r_b \rho_b a^2 - \frac{\kappa^2}{6} r_b^2 \Lambda a^2 \right.
\]

\[
+ \frac{1}{2} \frac{r_b^2}{a} \langle \dot{a} \rangle^2 \left. \right] \frac{1}{\alpha} \left[ - \varepsilon \eta \frac{r_b}{l_\Lambda^2} \eta + \frac{1}{2} \frac{r_b^2}{a} \eta \right],
\]

(8)

where we have introduced the effective four-dimensional Hubble parameter

\[
H(\alpha) = \langle \dot{a} \rangle / \langle a \rangle,
\]

(9)

the anti–De Sitter length scale associated to the (negative) cosmological constant in the bulk

\[
l_\Lambda = \sqrt{\frac{6}{\kappa^2 \Lambda}},
\]

(10)

and the dimensionless quantities \( \varepsilon, \alpha, \eta, \) and \( \tilde{\eta} \), respectively defined by

\[
\varepsilon = \frac{\kappa^2}{6} r_b \rho_b,
\]

(11)

and

\[
\alpha = \frac{a(1/2)}{\langle a \rangle}, \quad \eta = \frac{\langle \rho_b a^2 \rangle}{\langle \rho_b \rangle \langle a \rangle^2}, \quad \tilde{\eta} = \frac{\langle a^2 \rangle}{\langle a \rangle^2}.
\]

(12)

Whereas \( \varepsilon \) characterizes the thickness of the brane, the quantities \( \alpha, \eta, \) and \( \tilde{\eta} \) characterize the inhomogeneity of the brane along the fifth dimension (in the case of a homogeneous brane, one has \( \alpha = \eta = \tilde{\eta} = 1 \)). The 55 component of the Einstein equations can be written as

\[
F = \frac{2}{3} \kappa^2 \dot{a} a^3 P_T,
\]

(13)

with \( F = a^2[(a''/r_b)^2 - \langle \dot{a} \rangle^2 - \kappa^2 a^2 \Lambda / 6] \). Imposing the boundary condition \( P_T(\pm 1/2) = 0 \), one has \( F(\pm 1/2) = 0 \) which gives, after time integration,

\[
r_b^2 H^2(\alpha) = \left[ \frac{a(1/2)^2}{\langle a \rangle} \right]^2 \left[ \frac{C r_b}{a^2 (\alpha)} + \frac{\kappa^2}{6} a^2 r_b \Lambda \right],
\]

(14)

where \( C \) is a constant of integration. Injecting Eq. (8) into Eq. (14), one obtains

\[
a^2 r_b^2 H^2(\alpha) = - \varepsilon (\alpha) \left[ \frac{r_b}{l_\Lambda^2} \eta + \frac{1}{2} \frac{r_b^2}{a} \eta \right]^2 + \frac{C r_b}{a (\alpha)^4} - \frac{r_b^2}{a^2} \alpha^4.
\]

(15)

This equation is one of the main results of this paper. It provides the generalization of the (first) Friedmann equation relating the effective four-dimensional Hubble parameter to
the effective brane energy density and the brane thickness. It of course also includes some dependence on the exact profile of the energy-momentum tensor but this dependence has been conveniently reduced to the three dimensionless quantities (12), which in typical cases are of order unity. From Eqs. (7) and (15), one finds that a necessary condition for the brane to be homogeneous along the fifth dimension reads

$$\frac{C}{\langle a \rangle^4} = \frac{2\epsilon}{r_b^2} - \frac{1}{l_A^2},$$

(16)

which, in general, cannot be satisfied uniformly in time unless the energy-momentum content of the brane reduces to a mere brane tension (as, e.g., in the original static Randall-Sundrum model [12]). It follows that, in the cosmological context, a brane with a constant finite thickness cannot remain homogeneous along the fifth dimension.

Equation (15) can be rewritten in the form

$$H_{(a)}^2 = \frac{2}{r_b^2} \left( \alpha^2 + \epsilon \eta \frac{r_b^2}{l_A^2} \right) \times \left[ 1 \pm \sqrt{1 - \frac{(\epsilon \eta - \frac{2}{\alpha^2} + \frac{C}{\langle a \rangle^4})^2 + \frac{2}{\alpha^2} + \frac{C}{\langle a \rangle^4}}{(\epsilon \eta - \frac{2}{\alpha^2} + \frac{C}{\langle a \rangle^4})^2} \right].$$

(17)

which admits real solutions if the arbitrary constant $C$ satisfies the condition

$$\frac{Cr_b^2}{\langle a \rangle^4} \leq 2\alpha^2 \left( \epsilon \eta - \frac{r_b^2}{l_A^2} \right) + \alpha^4 \left( 1 + \frac{r_b^2}{l_A^2} \right).$$

(18)

In the following, we will consider expression (17) with the minus sign only, which corresponds to a well defined cosmology in the $r_b \to 0$ limit. It is important to notice that this equation should be regarded as an implicit equation for $H_{(a)}$, because $\alpha$, $\eta$, and $\eta$ can depend on $H_{(a)}$.

To conclude this section, it can be instructive to consider the two opposite limits of small and large $r_b$. In the limit $r_b \to 0$, Eq. (17) reduces to

$$H_{(a)}^2 = \frac{1}{\alpha^2} \left( \epsilon \eta - \frac{r_b^2}{l_A^2} + \frac{C}{\langle a \rangle^4} - \frac{\alpha^4}{l_A^2} \right)$$

$$= \frac{1}{\alpha^2} \left( \frac{\eta(\eta)}{\langle a \rangle^4} - \frac{\alpha^4}{l_A^2} \right).$$

(19)

It will be shown in the following that, in this limit, both $\alpha$ and $\eta$ tend to 1. Thus, one exactly recovers the unconventional Friedmann equation of thin brane cosmology, where the brane energy density enters quadratically [7,10]. In the opposite limit $r_b \to +\infty$, Eq. (17) reduces to

$$H_{(a)}^2 = \frac{2\epsilon}{r_b^2} - \frac{\eta}{l_A^2} + \frac{2}{r_b} \sqrt{\frac{\alpha^2(\alpha^2 - 2\eta)}{l_A^2} \frac{C}{\langle a \rangle^4}}.$$

(20)

where terms up to $O(r_b^{-1})$ have been kept, assuming that the inhomogeneity parameters (12) remain bounded as $r_b \to +\infty$. In the case of a Minkowski bulk, $l_A^2 = 0$, Eq. (18) yields $\langle a \rangle^4 \to 0$ as $r_b \to +\infty$ and one obtains

$$H_{(a)}^2 = \frac{2\epsilon \eta}{r_b^2} \frac{\kappa(\eta)}{3r_b},$$

which corresponds to the standard Friedmann equation, in which the energy density enters linearly with the effective four-dimensional gravitational constant $\kappa(\eta) = \kappa(\eta)/r_b$. Note that, in this limit, both sides of the necessary condition (16) vanish, which allows the possibility of a homogeneous brane along the fifth dimension ($\eta = 1$), in agreement with the usual Kaluza-Klein picture.

### III. RANDALL-SUNDRUM TYPE COSMOLOGY OF A THICK BRANE

In thin brane cosmology, the unconventional Friedmann equation can be made compatible with the current cosmological observations if one assumes that the brane energy density consists of ordinary cosmological matter living in the brane and of a tension $\lambda$ adjusted so as to compensate the negative cosmological constant in the bulk [8,10]. This adjustment corresponds to the fine-tuning condition of Randall-Sundrum [12], which can be reexpressed as the simple relation

$$l_k = l_A^\lambda,$$

(19)

where the length scale $l_k$ is defined from the brane tension $\lambda$ as

$$l_k = \frac{6}{\kappa(\eta)}.$$  

(20)

In this section, we will also assume that the energy-momentum content of the brane can be decomposed into two parts,

$$\rho_b(t, y) = \lambda + \rho_m(t, y),$$

(21a)

$$p_b(t, y) = -\lambda + p_m(t, y),$$

(21b)

where $\lambda$, which is strictly constant, corresponds to a tension of the brane. For the sake of simplicity, we assume here that $\lambda$ is independent of $y$. The case of a non-trivial profile of the brane tension along the fifth dimension could also be easily considered by writing $\lambda = \langle \lambda \rangle + \delta \lambda (y)$ and by defining an effective “matter” energy-momentum content $\bar{\rho}_m(t, y) = \rho_m(t, y) + \delta \lambda (y)$ and $\bar{p}_m(t, y) = p_m(t, y) - \delta \lambda (y)$. Since $\langle \delta \lambda (y) \rangle = 0$, it can be seen that the only modification induced by such a brane tension profile reduces to adding the geometrical term $\langle \delta \lambda (y) \rangle / \langle \lambda \rangle$ to the parameter $\epsilon\rho$ defined in Eq. (24). We will not analyze further on the effects of this extra term and restrict ourselves to the case of a $y$-independent brane tension. Injecting Eq. (21a) into the effective Friedmann equation (17), it can be easily seen that
the generalization of the Randall-Sundrum fine-tuning allowing for a finite brane thickness is given by

\[ l_\lambda = \frac{\tilde{\eta}(1-u)}{\alpha^2}l_\lambda, \]  

where we have introduced the dimensionless ratio

\[ u = \frac{r_p l_\lambda}{l_\lambda} = - \frac{r_b l_\lambda}{\lambda}. \]  

In the limit where \( r_b \) goes to zero, it can be checked that Eq. (22) reduces to the Randall-Sundrum condition (19), as it should be. From Eqs. (17), (21a), and (22), and introducing the dimensionless parameters

\[ \varepsilon_\lambda = \frac{r_b}{l_\lambda}, \quad \varepsilon_p = \frac{\eta(r_m)}{\lambda}, \]

where \( \eta = (\rho_m a^2)/(a^2) \), one finds that Eq. (17) (with the minus sign) can be rewritten as

\[ H_{a} = \frac{1}{2} \left[ a^2(1 + e_\lambda l_\lambda/l_\lambda) + e_\lambda e_p \right] \]

\[ \times \left[ 1 - \sqrt{1 - \frac{\varepsilon_\lambda^2 (r_b l_\lambda + \varepsilon_\lambda e_p)^2}{\varepsilon_\lambda^2 (1 + e_\lambda l_\lambda/l_\lambda) + e_\lambda e_p}^2} \right]. \]

(25)

**A. Cosmological equations in the low energy limit**

In this subsection, we will consider the low energy regime in which the ordinary cosmological energy (and pressure) is small with respect to the brane tension. For the sake of simplicity, we will take \( C = 0 \). Consider the low energy limit in the regime defined by \( \varepsilon_p \ll \min(1,1/\varepsilon_\lambda) \). Then, at lowest order in \( \varepsilon_p \), Eq. (25) reduces to

\[ r_b^2 H_{a}^2 = \frac{2 \varepsilon_\lambda^2 e_p}{e_\lambda + l_\lambda l_\lambda}, \]

and the quantities \( \alpha, \eta, \) and \( \tilde{\eta} \) must be determined at zeroth order in \( \varepsilon_p \). In this limit, the 00 component of the Einstein equations [see Eq. (7)] reduces to

\[ \left( \frac{a^2}{(a')^2} \right)'' = -4\varepsilon_\lambda (1-u) \frac{a^2}{(a')^2}, \]

where we have neglected the term \( 2r_b^2H_{a}^2 \) which is first order in \( \varepsilon_p \). Restricting ourselves to the case of physical interest (\( 1-u \))>0 [it can be checked a posteriori, cf. Eq. (33b), that this inequality is always satisfied], and integrating Eq. (27) with the boundary condition \( a'(y=0) = 0 \), one finds

\[ a = \langle a \rangle [\mathcal{A} \cos(ky)]^{1/2}, \]

with

\[ k^2 = 4\varepsilon_\lambda (1-u) \]

and

\[ \mathcal{A} = (\cos^{1/2}(ky))^{-2}. \]

Note that here we necessarily have \( k < \pi \), and the scale factor \( a \) is always positive throughout the brane, reaching the extremity of the brane at \( y = 1/2 \). It is possible to determine the metric outside the brane by matching, at \( y = 1/2 \), the scale factor and its derivative with respect to \( y \) to the general solution for the bulk with cosmological constant, obtained in [10], which is in fact the Schwarzschild-AdS solution expressed in Gaussian normal coordinates.

These expressions can be used to determine the evolution of \( \rho_m \) at lowest order in \( \varepsilon_p \). Since neither \( \mathcal{A} \) nor \( k \) depend on \( t \), the conservation equation (A6) (see the Appendix) simply yields

\[ \dot{\rho}_m + 3H_{a} \rho_m + p_m = 0, \]

from which it follows that, at this order, \( \rho_m \) factorizes as \( \rho_m(y,t) = f(y)\rho_m(t) \).

The effective four-dimensional brane cosmology in the limit \( \varepsilon_p \ll \min(1,1/\varepsilon_\lambda) \) is thus given by

\[ H_{a}^2 = \frac{\kappa_5^2}{3} \frac{\eta}{r_b + l_\lambda} \langle \rho_m \rangle, \]

\[ \langle \dot{\rho}_m \rangle + 3H_{a} \langle \rho_m \rangle + \langle p_m \rangle = 0, \]

with

\[ \eta = \frac{\langle f(y) \cos(ky) \rangle}{\langle f(y) \rangle \langle \cos^{1/2}(ky) \rangle^2}, \]

where \( k \) and \( \mathcal{A} \) are respectively given by Eqs. (29) and (30). Using the fine-tuning condition (22) with \( \alpha = [\mathcal{A} \cos(k/2)]^{1/2} \) and \( \tilde{\eta} = (2\mathcal{A}k)\sin(k/2), \) one can reexpress the constant \( k \) as a function of \( \varepsilon_\lambda \) in the parametric form

\[ k = 2 \tan^{-1} \sqrt{\frac{u}{1-u}}, \]

\[ \varepsilon_\lambda = \frac{\tan^{-1} \sqrt{\frac{u}{1-u}}^2}{1-u}, \]

with \( 0 \leq u < 1 \). Note that since, in this limit, \( \eta \) does not depend on \( t \), the only effect of the finite brane thickness is a modification of the relation between \( \kappa_5 \) and \( \kappa_4 \). Namely, one has

\[ \kappa_4^2 = \frac{\eta \kappa_5^2}{r_b + l_\lambda}. \]

Taking into account the constraint (22), one has the limits \( \kappa_4^2 = \kappa_5^2 \sqrt{\frac{\lambda}{6}} = \kappa_5^2 \lambda/6 \) for \( r_b/l_\lambda \ll 1 \), and \( \kappa_4^2 = \eta \kappa_5^2/r_b \) for \( r_b/l_\lambda \gg 1 \). In the simplest case \( f(y) = \text{const} \), \( \rho_m = \rho_m(t) \), and \( \lim_{r_b \rightarrow +\infty} \eta = 1.094 \). In Fig. 1 we have plotted \( k \) as a function of \( u \) (solid line), and \( \varepsilon_\lambda \) as a function of \( u \) (dashed-
brane profile, as a function of $u$. The infinitely thin limit corresponds to $\varepsilon_\Lambda \to 0$, in which case $u \to 0$ and $k \to 0$. The opposite limit $\varepsilon_\Lambda \to +\infty$, yields $u \to 1$ and $k \to k_{\text{max}} = \pi$. Figure 2 shows the dimensionless coefficients, $A$, $\alpha$ and $\bar{\eta}$, characterizing the brane profile, as a function of $k/k_{\text{max}}$. In the infinitely thin brane limit ($k \to 0$), these three coefficients tend to 1. It can be seen that $\bar{\eta}$ stays almost constant over the full range of $k$, with $\bar{\eta}(k_{\text{max}}) \approx 1.094$. As the brane thickness increases, $\alpha$ decreases down to zero for $k = k_{\text{max}}$ and $A$ increases up to the value $A(k_{\text{max}}) \approx 1.719$.

All the results of this section have been obtained in the limit $\varepsilon_\rho \approx \min(1,1/\varepsilon_\Lambda)$ by keeping terms explicitly of lowest order in $\varepsilon_\rho$ and of any order in $\varepsilon_\Lambda$. The actual small parameter appearing in the expansion of the square root on the right-hand side of Eq. (25) is $\varepsilon_\Lambda^2 \varepsilon_\rho (\alpha^2 l_\Lambda/l_\Lambda) \sim \varepsilon_\Lambda^2 \varepsilon_\rho \min(1,1/\varepsilon_\Lambda)$. These equations are thus correct, in the limit $\varepsilon_\rho \approx \min(1,1/\varepsilon_\Lambda)$, at lowest order in $\varepsilon_\Lambda^2 \varepsilon_\rho \min(1,1/\varepsilon_\Lambda)$ and at any order in $\varepsilon_\Lambda$ and $\varepsilon_\Lambda = r_\rho/l_\Lambda$.

**B. Cosmological equations for a small brane thickness**

In the limit of a small brane thickness, $\varepsilon_\Lambda \ll 1$, Eq. (25) with $\zeta = 0$ reduces to

$$ r_\rho^2 H_{(a)}^2 = \frac{\varepsilon_\Lambda^2 (2 \varepsilon_\rho \alpha^2 l_\Lambda/l_\Lambda + \varepsilon_\rho^2)}{\alpha^2 (1 + \varepsilon_\Lambda l_\Lambda/l_\Lambda) + \varepsilon_\Lambda \varepsilon_\rho}, $$

which is correct at third order in $\varepsilon_\Lambda$. At this order, $\alpha$ must be expressed at first order, and $\eta$ and $\bar{\eta}$ at zeroth order [note that $\eta$ and $\bar{\eta}$ behave as $1 + O(\varepsilon_\lambda^2)$]. To determine $\alpha$ perturbatively in $\varepsilon_\Lambda$, we will use the following $y$ expansion:

$$ a(t,y) = \langle a(t) \rangle \sum_{n=0}^{+\infty} \bar{a}_n(t) y^{2n} = \langle a(t) \rangle \Sigma_a(t,y), $$

$$ \rho_m(t,y) = \langle \rho_m(t) \rangle \sum_{n=0}^{+\infty} \bar{\rho}_n(t) y^{2n} = \langle \rho_m(t) \rangle \Sigma_{\rho}(t,y). $$

Inserting Eqs. (36) in Eq. (7), using the constraint (22), and neglecting the $O(\varepsilon_\Lambda^2)$ terms, one obtains

$$ (\Sigma_a^2)^n = -4 \varepsilon_\Lambda (\varepsilon_\rho \Sigma_{\rho} + l_\Lambda/l_\Lambda) \Sigma_{\alpha}^2. $$

Since one needs $a$ at first order in $\varepsilon_\Lambda$, one can consider the constant (in $y$) component of Eq. (37) only. One finds $\bar{a}_1 = -\varepsilon_\Lambda \bar{a}_0 (\varepsilon_\rho \rho_0 + l_\Lambda/l_\Lambda)$ and $a = \langle a \rangle \bar{a}_0 [1 - \varepsilon_\Lambda (\varepsilon_\rho \rho_0 + l_\Lambda/l_\Lambda) y^2]$. In this expression, $\bar{\rho}_0$ must be determined at zeroth order in $\varepsilon_\Lambda$, which yields $\bar{\rho}_0 = 1$ and $\langle \rho_m \rangle = \rho_m$ solution to the usual 4D energy-momentum conservation equation. Thus, at first order in $\varepsilon_\Lambda$ one has $a = \langle a \rangle [1 - \varepsilon_\Lambda (\varepsilon_\rho \rho_0 + l_\Lambda/l_\Lambda) (y^2 - 1/12)]$, $\alpha = 1 - \varepsilon_\Lambda (\varepsilon_\rho + l_\Lambda/l_\Lambda)$ and $\eta = \bar{\eta} = 1$. Inserting these expressions in Eq. (22), one obtains (at first order in $\varepsilon_\Lambda$)

$$ \frac{l_\Lambda}{l_\Lambda} = 1 - \varepsilon_\Lambda (\varepsilon_\rho - 2) \frac{3}{3}, $$

and $\alpha$ reduces to

$$ \alpha = 1 - \varepsilon_\Lambda (\varepsilon_\rho + 1) \frac{6}{6}. $$

From Eqs. (35), (38), and (39) it follows that the effective four-dimensional brane cosmology in the limit $\varepsilon_\Lambda \ll \min(1,1/\varepsilon_\rho)$ is given, at third order in $\varepsilon_\Lambda$, by

$$ H_{(a)}^2 = \frac{\kappa_{(5)}^2}{3} \left[ \frac{(3 + r_\rho l_\Lambda) + (3 - 4 r_\rho l_\Lambda) \langle \rho_m \rangle (2\lambda)}{2 r_\sigma (1 + \langle \rho_m \rangle/\lambda) + 3 l_\Lambda} \right] $$

$$ \times \langle \rho_m \rangle, $$

$$ \langle \rho_m \rangle + 3 H_{(a)} \langle \rho_m \rangle + \langle \rho_m \rangle = 0. $$

These equations have been obtained in the limit $\varepsilon_\Lambda \ll \min(1,1/\varepsilon_\rho)$, by keeping terms explicitly of third order in $\varepsilon_\Lambda$. The actual small parameter appearing in the expansion of the square root on the right-hand side of Eq. (25) is $\varepsilon_\Lambda^2 \varepsilon_\rho \max(1,1/\varepsilon_\rho)$. It follows that these equations are correct, in the limit $\varepsilon_\Lambda \ll \min(1,1/\varepsilon_\rho)$, at third order in $\varepsilon_\Lambda \varepsilon_\rho$ for $\varepsilon_\rho > 1$, and up to terms of order $\varepsilon_\Lambda^2 \varepsilon_\rho$ for $\varepsilon_\rho \ll 1$. Note that in
the domain \(\min(1,1/e_\rho)/e_\rho \leq \varepsilon_\lambda \leq \min(1,1/e_\rho)\), only the third order (in \(\varepsilon_\lambda\)) correction corresponding to the term \(\varepsilon_\lambda e_\rho\) must be kept; the other corrections are smaller than (or of the same order as) terms already neglected. It can be checked that in the limit \(\varepsilon_\lambda \equiv 1\) and \(e_\rho \equiv 1\), Eqs. (26) and (35) coincide at first order in \(e_\rho\) and third order in \(\varepsilon_\lambda\), as it should be.

C. Uniform expressions in the regime: \(\varepsilon_\lambda e_\rho \ll 1\)

The two previous regimes cover the whole domain \(\varepsilon_\lambda e_\rho = \kappa^2_5 r_b(\rho_m)/(6 \ll 1)\) (with, of course, compatibility in the overlapping domain \(\varepsilon_\lambda \ll 1\) and \(e_\rho \ll 1\) as mentioned at the end of the previous section). This suggests the following uniform expressions valid in the whole domain \(\varepsilon_\lambda e_\rho \ll 1\):

\[
H^2_{(a)} = \frac{\kappa^2_5}{3} \left[ \alpha^2 (l_\lambda/l_\Lambda) + \eta(\rho_m)/(2 \lambda) \right] + \frac{\alpha^2 l_\lambda}{r_b(\rho_m)/(2 \lambda)} \eta(\rho_m),
\]

(41a)

\[
\dot{\rho}_m + 3H_{(a)}(\dot{\rho}_m + \langle \dot{p}_m \rangle) = 0,
\]

(41b)

where

\[
\alpha^2 = \cos(k/2) / \cos(1/2(ky))^2,
\]

(42a)

\[
\eta = \left( f(y) \cos(ky) \right) / \left( f(y) \cos(1/2(ky))^2 \right),
\]

(42b)

and

\[
k = 2 \left( 1 + \frac{(\rho_m)/(\lambda)}{1 - u} \right)^{1/2} \sqrt{\frac{u}{1 - u}},
\]

(43a)

\[
\frac{l_\lambda}{l_\Lambda} = \sqrt{\frac{u(1 - u)}{u/(1 - u)}},
\]

(43b)

\[
\varepsilon_\lambda = \frac{\left( \tan^{-1} \sqrt{u/(1 - u)} \right)}{1 - u},
\]

(43c)

with \(0 \leq u < 1\).

IV. EFFECTIVE COSMOLOGICAL EQUATIONS AT HIGHER ORDERS IN THE CASE OF A MINKOWSKI BULK

In this section, we wish to extend the analysis of the previous section to higher orders. In order to simplify the calculations, we will restrict ourselves to the case where the bulk cosmological constant vanishes (\(\Lambda = 0\)), i.e. the spacetime outside the brane effectively is a four-dimensional Minkowski spacetime. In this case, one has \(l_\Lambda^2 = 0\) and Eq. (17) reads

\[
H^2_{(a)} = \frac{2(\alpha^2 + e \eta)}{r_b^2} \left[ 1 - \sqrt{1 - \frac{\varepsilon^2 \eta^2 + C^2 r^2/(a)^4}{(\alpha^2 + e \eta)^2}} \right].
\]

(44)

As previously mentioned, in the limit of a vanishing brane thickness, Eq. (44) reduces to the unconventional Friedmann equation of thin brane cosmology

\[
H^2_{(a)} = \frac{\kappa^4_5 (\rho_b)^2 + C \langle a \rangle^4}{36(\langle a \rangle^4)}
\]

(45)

(N.B. \(\alpha \to 1\) and \(\eta \to 1\) as \(r_b \to 0\).) In the following, we will consider the limit \(\varepsilon \ll 1\), \(r_b C^2/(a)^4 = O(\varepsilon)\), and determine the corrections to Eq. (45) due to a finite brane thickness (i.e. a finite \(\varepsilon\)).

A. \(r_b^2 H^2_{(a)}\) at third order in \(\varepsilon\)

At third order in \(\varepsilon\), Eq. (44) can be written as

\[
r_b^2 H^2_{(a)} = \frac{\varepsilon^2 \eta^2 + C r^2/(a)^4}{(\alpha^2 + e \eta)^2}.
\]

(46)

At this order, \(\alpha\) and \(\eta\) must be expressed at first and zeroth order respectively [note that \(\eta\) behaves as \(1 + O(\varepsilon^2)\)]. To determine \(\alpha\) perturbatively in \(\varepsilon\), we will use the \(y\) expansion (36a), (36b) with \(\rho_b\) replacing \(\rho_m\). Inserting this expansion in Eq. (7) with \(\Lambda = 0\), one obtains

\[
(\Sigma_a^2) = -4 \varepsilon \Sigma_a \Sigma_a^2 + 2 r_b^2 H^2_{(a)}.
\]

(47)

Since one needs \(a\) at first order in \(\varepsilon\), one can neglect the second term on the right-hand side of Eq. (47) which is \(O(\varepsilon^3)\), and consider the constant (in \(y\)) component of Eq. (47) only. One finds \(a_1 = -e \rho_0 a_0\) and \(a = \langle a \rangle a_0(1 - e \rho_0 y^2)\). In this expression, \(\rho_0\) must be determined at zeroth order in \(\varepsilon\), which yields \(\rho_0 = 1\) and \(\langle \rho_b \rangle\) solution to the usual 4D energy-momentum conservation equation. Thus, at first order in \(\varepsilon\) one has \(a = \langle a \rangle [1 - \varepsilon - \varepsilon^2/12]\), \(\alpha = 1 - \varepsilon/6\), and \(\eta = 1\). It follows that, at third order in \(\varepsilon\), the quantity \(r_b^2 H^2_{(a)}\) reads

\[
r_b^2 H^2_{(a)} = \frac{\varepsilon^2 + C r^2/(a)^4}{1 + 2 \varepsilon/3},
\]

or, equivalently,

\[
H^2_{(a)} = \frac{\kappa^4_5 (\rho_b)^2/36 + C \langle a \rangle^4}{1 + \kappa^2_5 r_b(\rho_b)^2/9},
\]

(48)

with

\[
\langle \rho_b \rangle + 3H_{(a)}(\langle \rho_b \rangle + \langle \rho_m \rangle) = 0.
\]

(49)

B. \(r_b^2 H^2_{(a)}\) at fourth order in \(\varepsilon\)

At fourth order in \(\varepsilon\), Eq. (44) reads

\[
r_b^2 H^2_{(a)} = \frac{\varepsilon^2 \eta^2 + C r^2/(a)^4}{\alpha^2 + e \eta} \left[ 1 + \frac{\varepsilon^2 \eta^2 + C r^2/(a)^4}{4(\alpha^2 + e \eta)^2} \right].
\]

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\[ e^2 \eta^2 + C r_b^2/(a)^4 \frac{1 + \frac{1}{4} r_b^2 H^2(a) + O(\varepsilon^3)}{a^2 + e \eta} \]

which gives

\[ r_b^2 H^2(a) = \frac{e^2 \eta^2 + C r_b^2/(a)^4}{a^2 + e \eta - (e^2 \eta^2 + C r_b^2/(a)^4)/4}. \]

In the numerator \( \eta \) must be determined at second order in \( \varepsilon \).
In the denominator \( \alpha \) must be determined at second order in \( \varepsilon \) and \( \eta = 1 \) [N.B. \( \eta = 1 + O(\varepsilon^2) \)]. Since one needs \( a \) at second order in \( \varepsilon \), one must consider the components of Eq. (47) up to terms proportional to \( y^2 \). One finds \( \tilde{a}_1 = -\bar{\rho}_0 a_0 + r_b^2 H^2(a)/(2 \tilde{a}_0) \) and \( \tilde{a}_2 = -(\varepsilon/6)(2 \bar{\rho}_0 \tilde{a}_1 + \bar{\rho}_1 \tilde{a}_0) - \tilde{a}_1/(2 \tilde{a}_0) \), which gives after some straightforward algebra

\[
\begin{align*}
\frac{a}{(a)} &= 1 - e \left( y^2 - \frac{1}{12} \right) + C r_b \left( y^2 - \frac{1}{12} \right) + e^2 \left[ 2 \left( y^2 - \frac{1}{12} \right) \right] \\
&\quad - 2 \left( y^4 - \frac{1}{80} \right) + e \bar{\rho}_1 \left[ \left( y^2 - \frac{1}{12} \right) - 2 \left( y^4 - \frac{1}{80} \right) \right].
\end{align*}
\]

Thus, at second order in \( \varepsilon \) one has

\[
\alpha = 1 - \frac{e}{6} + C \frac{r_b^2}{12(a)^4} + \frac{e(11e + \bar{\rho}_1)}{180},
\]

\[
\eta = 1 + \frac{e(\varepsilon - 2 \bar{\rho}_1)}{180}.
\]

To obtain \( \eta \) one has used the expression \( \rho_b/\langle \rho_b \rangle = 1 + \bar{\rho}_1(y^2 - 1/12) + O(e^2) \) and the fact that the \( O(e^2) \) term, the average of which must vanish, does not contribute to \( \langle \rho_a^2 \rangle \) at second order in \( \varepsilon \). Inserting these expressions of \( \alpha \) and \( \eta \) in Eq. (51), one obtains at fourth order in \( \varepsilon \)

\[ r_b^2 H^2(a) = \frac{e^2 \left( 1 + e(\varepsilon - 2 \bar{\rho}_1)/90 + C r_b^2/(a)^4 \right)}{1 + 2e/3 - e(9e - \bar{\rho}_1)/90 - C r_b^2/(12(a)^4)}. \]

The equation for \( \langle \rho_b \rangle \)

\[ \langle \dot{\rho}_b \rangle + 3 \frac{H(a)}{\langle \rho_b \rangle} + \langle \rho_b \rangle = -3 \left( \langle \rho_b \rangle \Sigma \rho + \langle \rho_b \rangle \frac{\partial \log \Sigma_a}{\partial t} \right). \]

Writing the right-hand side of this equation at second order in \( \varepsilon \), one obtains

\[ \langle \dot{\rho}_b \rangle = - \frac{3 H(a) \langle \rho_b \rangle + \langle \rho_b \rangle}{1 - [e(\varepsilon + \bar{\rho}_1) + \langle \rho_b \rangle](e + \bar{\rho}_1)/\langle \rho_b \rangle]}/60. \]

To obtain this equation we have used \( \Sigma \rho = 1 + \bar{\rho}_1(y^2 - 1/12) + O(e^2) \), \( \Sigma \rho = 1 + \bar{\rho}_1(y^2 - 1/12) + O(e^2) \), and the fact that, at second order in \( \varepsilon \), there is no contribution of the \( O(e^2) \) terms. It remains to determine the evolution equation for \( \bar{\rho}_1 \) at lowest order in \( \varepsilon \), i.e., in which one keeps terms at most of first order. To this end let us assume an equation of state of the form \( p = w(t) \rho_b \) (which implies \( \Sigma \rho = \Sigma \rho \)). In this case, the energy-momentum conservation reads

\[ \langle \dot{\rho}_b \rangle \langle \rho_b \rangle + [\langle \dot{\rho}_b \rangle + 3 \langle H(a) \rangle (1 + w) \langle \rho_b \rangle] \langle \Sigma \rho \rangle = -3 \langle 1 + w \rangle \langle \rho_b \rangle \frac{\partial \log \Sigma_a}{\partial t}, \]

which yields, at lowest order in \( \varepsilon \),

\[ \bar{\rho}_1 = -3 \langle 1 + w \rangle \bar{\rho}_1 = 3 \langle 1 + w \rangle \dot{\varepsilon}. \]

Note that the second term on the left-hand side of Eq. (55) is second order and must be neglected, as it can be seen from Eq. (54). If \( w \) is constant over some time interval \( t_1 \leq t \leq t_2 \), one can integrate Eq. (56), which gives

\[ \bar{\rho}_1(t_1 \leq t \leq t_2) = \bar{\rho}_1(t_1) + 3 \langle 1 + w \rangle [\varepsilon(t) - \varepsilon(t_1)]. \]

To summarize, at fourth order in \( \varepsilon \), the quantity \( r_b^2 H^2(a) \) reads

\[ r_b^2 H^2(a) = \frac{e^2 \left[ 1 + e(\varepsilon - 2 \bar{\rho}_1)/90 + C r_b^2/(a)^4 \right]}{1 + 2e/3 - e(9e - \bar{\rho}_1)/90 - C r_b^2/(12(a)^4)}, \]

with [Eq. (54)]

\[ \dot{\varepsilon} = - \frac{3 H(a) (1 + w) \varepsilon}{1 - e \varepsilon (1 + w)}/60, \]

and [Eqs. (56) and (54)]

\[ \bar{\rho}_1 = 9 H(a)(1 + w)^2 \varepsilon. \]

This shows that at this order, and for the next orders as well, one must introduce auxiliary quantities, here \( \bar{\rho}_1 \), in order to describe adequately the effective cosmology from the brane point of view. This is a sign that in the limit of a small but finite brane thickness (\( \varepsilon < 1 \)), one should not expect any effective four-dimensional reduction to be possible if terms of high order in \( \varepsilon \) must be kept.

V. CONCLUSION

In the present work, we have attempted to address in a phenomenological way the question of finite brane thickness and its influence on the effective four-dimensional cosmological equations. In order to simplify our analysis, we have made a number of assumptions, which might be interesting to explore further or to relax in future investigations.

First we have defined in a heuristic way the notion of four-dimensional quantities by simply integrating over the
fifth coordinate \( y \), which is defined in a Gaussian normal coordinate system centered on the middle layer of the brane, supposed to be mirror symmetric. Although this averaging procedure corresponds to the standard one, one could maybe envisage, when having a warping factor, other types of averaging.

Another important assumption was to take the brane thickness constant in time. It would be worthwhile to generalize our results when the brane thickness is allowed to evolve in time.

After emphasizing a few limits of the present work, let us now summarize the main results. A very instructive result of our work is a generalized brane Friedmann equation, which interpolates between the familiar Kaluza-Klein picture, where the effective Friedmann equations after the averaging procedure yield exactly the usual four-dimensional Friedmann equations, and the initially surprising thin brane Friedmann equations, where the energy density enters quadratically. We hope that our generalized Friedmann equation will help in the intuitive understanding of the unconventional thin brane result, and clarify the delimitations of the various regimes where different Friedmann equations have to be applied.

Assuming the existence of a strict (in both partial and temporal senses) cosmological constant, we have also obtained a generalized version of the Randall-Sundrum condition of cancellation between the bulk cosmological constant and the brane tension term in the Friedmann equation. In our case, this cancellation condition depends explicitly on the brane thickness and yields back the familiar condition in the thin brane limit. Moreover, when this cancellation condition is assumed to hold and when the cosmological matter content of the brane is assumed to be small with respect to its tension, we were able to solve for the brane profile and obtain the effective cosmological equations.

Finally, we show that this is only in this case, as well as in the limit where the brane thickness is sufficiently small, that one can make sense of a four-dimensional effective cosmology. In the other cases, one can try to extend the regime of validity of the effective four-dimensional description but at the price of introducing auxiliary quantities, which is the sign that the full five-dimensional description is more adequate.

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APPENDIX: FIVE-DIMENSIONAL EINSTEIN EQUATIONS

The non-vanishing components of the Einstein tensor \( G_{AB} \) corresponding to the ansatz (1) for the metric are

\[
G_{00} = 3 \left( \frac{\dot{a}}{a} \right)^2 - \frac{n^2}{r_b^2} \left[ \frac{a''}{a} + \frac{(a')^2}{a} \right],
\]

\[
G_{ij} = \frac{a^2}{r_b^2} \delta_{ij} \left[ \frac{a'}{a} \left( \frac{a'}{a} + \frac{2n'}{n} \right) + 2 \frac{a''}{a} + \frac{n''}{n} \right] \quad (A1)
\]

\[
+ \frac{a^2}{n^2} \delta_{ij} \left[ \frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a} - \frac{2n'}{n} \right], \quad (A2)
\]

\[
G_{05} = 3 \left( \frac{\dot{a}}{a} - \frac{\dot{n}}{n} \right), \quad (A3)
\]

\[
G_{55} = 3 \left( \frac{a'}{a} \frac{a'}{a} + \frac{n'}{n} \right) - \frac{r_b^2}{n^2} \left[ \frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a} + \frac{n'}{n} \right] \quad (A4)
\]

In the above expressions, a prime stands for a derivative with respect to \( y \), and a dot for a derivative with respect to \( t \). The conservation of the energy-momentum tensor follows from Einstein equations as a consequence of Bianchi identity. Since, according to Eq. (3), one has \( \nabla_A T^A_{\phantom{A}B} \big|_{\text{Bulk}} = 0 \) trivially, it reduces to

\[
\nabla_A T^A_{\phantom{A}B} = 0, \quad (A5)
\]

where \( T^A_{\phantom{A}B} \) is the energy-momentum tensor of the matter content in the brane as given by Eq. (2). Using the metric (1) where \( \dot{r}_b = 0 \), the time component of Eq. (A5) yields

\[
\dot{\rho}_b + 3H_a (\rho_b + p_b) = 0, \quad (A6)
\]

with \( H_a = \dot{a}/a \).

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