On the equity-efficiency trade-off in food-bank network operations

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ABSTRACT
In this paper, we present a novel modeling perspective to the food-bank donation allocation problem under equity and efficiency performance measures. Using a penalty factor in the objective function, our model explicitly accounts for both efficiency and equity, simultaneously. We give the tightest lower and upper bounds of the penalty factor, which can conveniently characterize closed-form optimal solutions for the perfect efficiency and perfect equity cases. Testing our model on the full spectrum of our penalty factor, using real data from Feeding America, we demonstrate that the solutions from our model dominate those of a benchmark from the literature in terms of both equity and efficiency, being on the Pareto frontier. Our sensitivity analysis demonstrates that assisting the food-banks should go hand-in-hand with helping eliminate poverty in the demand population. This will ensure that adding more capacity to the network will always lead to a decrease in the price of equity for the food-banks. On the other hand, our results reveal that encouraging charitability is always beneficial for the food-banks, albeit with diminishing returns. Finally, we extend our model to the case with stochastic receiving capacities and derive additional insights with regards to the inherent trade-offs between equity, efficiency, and reliability in the network.

1. Introduction
In some countries, non-profit organizations fight hunger by collecting and distributing the otherwise wasted food to the impoverished population. For instance, Feeding America is the United States’ largest non-profit hunger-relief organization, operating through a nationwide network of 200 food-banks to provide food assistance to the poverty population. The organization collects food donations from national food and grocery stores, retailers, farmers and governmental agencies and distributes them in an equitable manner to its member food-banks. Subsequently, Feeding America’s food-banks distribute the food to charitable agencies in their service regions. The decision to establish a new agency in a county depends mainly on the census data on the poverty level in that county. Therefore, each county faces a demand proportional to the poverty population it serves.

The next tier of the supply chain, the charitable agencies, distribute the food they receive from the food-banks to the demand population in their network. Agencies are typically run by fully volunteer staff and have a certain capacity that is expressed in terms of pounds of food they can process. This capacity is generally a function of several factors such as their volunteer manpower, storage, and loading/unloading capacity. Therefore an agency’s capacity may be treated as a stochastic parameter, being driven by volunteers’ choices to participate in the distribution of food. A county, containing several agencies, has a capacity equivalent to the aggregate capacity of the corresponding agencies within its area. In our paper, we first consider capacity as a known parameter for the purposes of computational simplicity and comparison with the benchmark, and later extend this deterministic model to address the stochastic nature of the capacities.

Essentially, in this paper, we address the problem of food allocation from a food-bank to counties facing county-specific demands and capacities by developing policies that are (i) equitable: each county should get its fair-share of the supply, (ii) efficient: waste of food across the network should be minimized, and (iii) reliable: when county capacities are stochastic, the allocation should meet the capacities of the counties with a county-specific confidence level.

Figure 1 shows the dynamics of the two performance measures of equity and efficiency in non-profit settings. The center of each circle represents perfect accomplishment of the corresponding performance measure and the line connecting the centers represents the trade-off between equity and efficiency. The choice of the policy depends on the flexibility of the organization’s desired trade-off between the two performance measures.
measures, as well as practical considerations in the network. In other words, the organization may choose to focus solely on efficiency (center of the top circle in Figure 1) or it may choose to sacrifice some efficiency for the sake of achieving equity and move down in the trade-off line towards more equitable policies. The policy that focuses solely on and achieves absolute equity is represented by the center of the bottom circle in Figure 1. Any policy outside of the straight blue line (and including the two extreme center points) is not Pareto optimal, i.e., it sacrifices from one measure without fully exploiting the gains from the other. A goal of this research is to provide a flexible model to enable the food-bank decision-makers choose where in the Pareto spectrum of equity-efficiency trade-off they would like to perform.

Our work contributes to the literature on decision-making within the context of food-bank operations at the tactical level. Specifically, we, (i) develop a flexible model that explicitly accounts for both efficiency and equity in the objective function, (ii) derive closed-form solutions for perfect equity and perfect efficiency and, through a thorough numerical study, show that our results, being on the Pareto frontier, dominate those of a benchmark from the literature in terms of efficiency and equity across the full spectrum of model parameters (iii) derive managerial insights with regards to society’s charitability, wealth disparity, and food-bank volunteer levels which enables managers at the food-banks to make informed decisions on their operations, and (iv) extend our model to the case with stochastic capacities via a chance-constrained programming approach and derive additional insights with regards to the trade-offs between equity, efficiency, and reliability.

The remainder of this paper is organized as follows: Section 2 reviews the research most related to our paper. Sections 3 and 4 introduce and analyze our model for the problem, respectively. Section 5 demonstrates the performance of our model compared to a benchmark from the literature and highlights our insights on the decision-making of the food-bank. Section 6 extends our model to the case with stochastic capacities using a chance-constrained approach. Finally, Section 7 concludes the paper with critical insights from the study and future extensions of our work.

2. Relevant literature

Food-banks attract a major portion of research in non-profit literature. Some of the works in food-bank research focus on operational level decisions (Biswal, Jenamani, & Kumar, 2018; Mohan, Gopalakrishnan, & Mizzi, 2013). Our paper differs from this stream in that our work focuses on tactical level decisions of the food-bank. On the tactical level, two particular sub-problems have received major attention in food-bank research, i.e., (i) routing and (ii) allocation. The stream of research that focuses on routing includes scheduling of either pick-ups (from the donors) or drop-offs (at the counties) or both, while optimizing criteria like travel distance and freshness. Some of the works in this stream are Davis, Sengul, Ivy, Brock, and Miles (2014), Solak, Scherrer, and Ghoniem (2014), Reihaneh and Ghoniem (2018), Buisman, Hajjema, Akkerman, and Bloemhof (2019), Balcik, Iravani, and Smilowitz (2014), Eisenhandler and Tzur (2019a, 2019b), Nair, Rey, and Dixit (2017), Eisenhandler and Tzur (2019b), Stauffer, Vanajakumari, Kumar, and Mangapora (2022). However, all of these works study the food-bank problem with a focus on the upstream issues related to scheduling and routing. Additionally, the nature of the objective function in our setting is a fill-rate based one which accounts for both goals of efficiency and equity while these works focus on scheduling-based objective functions.

In contrast to for-profit organizations, the performance measures that non-profit firms seek are not solely efficiency-based. Particularly, the need for understanding the trade-off between efficiency and equity has been emphasized in the literature (Savas, 1978). A stream of research to address this need in the literature has been addressed in the context of humanitarian and disaster relief operations (Balcik, Beamon, & Smilowitz, 2008; Huang, Smilowitz, & Balcik, 2012; Orgut et al., 2016a; Park & Berenguer, 2020; Taskin & Lodree, 2010; Yang et al., 2023). Most of the works in this context only consider a disaster-related decision and not day-to-day operations as in our setting.

Equity is a major consideration for many supply chains and its position has been accentuated in the literature (Dos-Santos, 2020; Rea et al., 2021). Within the context of food-bank research, Orgut, Ivy, Uzsoy, and Wilson (2016b) scrutinize the equity-efficiency trade-off, with a thorough analysis of the importance of equity in food-bank decision-making. While our work follows the same path, they account for efficiency through minimization of leftover food and for equity through maximum deviation from a perfectly equitable allocation. Our results in this model take Orgut et al. (2016b) as a benchmark and demonstrate the
dominance of our solutions to that of Orgut et al. (2016b) in terms of efficiency and equity. Islam and Ivy (2021) minimize the costs of food processing and waste at the food-bank and, similar to Orgut et al. (2016b), account for equity in the model through a maximum deviation tolerance from perfect equity. Orgut, Ivy, and Uzsoy (2017) focus on the effects of changes in the capacities in their previous model in Orgut et al. (2016b) and conclude that the structure of the solution to the problem has a newsvendor behaviour. Similar to Orgut et al. (2016b) and Orgut et al. (2017), Orgut, Ivy, Uzsoy, and Hale (2018) consider equity and efficiency as major elements of their model, and divide their analysis into two types of perfect equity and another that accounts for a maximum allowed deviations from perfect equity. Hasnain, Sengul Orgut, and Ivy (2021) consider the inclusion of competing objective functions such as equity and efficiency where efficiency also includes a transportation cost. They develop an algorithm to reflect the preference of the food-bank towards each one of the criteria considered in the objective function. However, they also include the consideration of equity in the constraints through deviations from a perfect equity instead of directly accounting for it in the objective function. Similar to our work in terms of consideration of a fill-rate based objective function, Lien, Iravani, and Smilowitz (2014) solve a sequential fair allocation problem, but apply a maxi-min approach to raise the minimum fill-rate across the network. Their results focus on developing near-optimal solutions for the sequence of agency visits. Fianu and Davis (2018) consider fair allocation of unknown supply among the agencies of a food-bank and identify optimal allocation rules under varying supply and demand scenarios. Similar to Fianu and Davis (2018), Alkaabneh, Diabat, and Gao (2020) develop a dynamic programming approach to the food allocation problem by a food-bank to the agencies in its network and explicitly account for nutritional value of the food delivered to each agency as well as the utility and equity among the served agencies. Fairness in allocation of supply to demands across the network is studied by Spiliotopoulou and Conte (2021), whose numerical results suggest that when a distribution network is supply-constrained using a fill-rate based measure of efficiency is ideal. Hynninen, Vilkkumaa, and Salo (2021) suggest that in a resource allocation problem, cost of equity is an important measure to analyze within the contexts of utilitarian (efficiency-based) and egalitarian (equity-based) objectives. To reduce the burden imposed by the COVID-19 pandemic on the food-bank supply chain, Blackmon et al. (2021) develop a decision support system in order to coordinate supply and demand between suppliers and agencies, with the food banks serving as “virtual intermediaries”.

Consideration of stochastic parameters for the food banks leads to further complications in the decision making. Orgut et al. (2018) consider such a setting and develop a robust programming approach to handle the stochastic receiving capacity through the definition of the budget of uncertainty and the deviation of each county’s capacity from its nominal value. In another effort to handle the uncertainty in food supply, Alkaabneh, Diabat, and Gao (2021) model it as a Markov Decision Process and approximate the Markov Chain through a policy iteration scheme and derive insights. Our contribution to this stream of work is that we use a novel chance-constrained programming approach to handle the uncertainty associated with capacity, enabling us to derive critical insights with regards to reliability in addition to the regular efficiency-equity trade-offs.

From a methodological perspective, our solution approach belongs to the weighting technique methods. However, although many previous works belong to this category of deriving the Pareto frontier, the weights used by such literature do not have the closed-form bounds based on the realization of the feasible region and objective function (Hasnain et al., 2021; Kabadurmus, Kazançoğlu, Yüksel, & Pala, 2022; Mejia-Argueta, Gaytán, Caballero, Molina, & Vitoriano, 2018; Ozdemir et al., 2021; Raimundo, Ferreira, & Von Zuben, 2020; Smith, Harper, & Potts, 2013; Yılmaz & Kabak, 2020). We theoretically analyze the speciality of our problem and derive the Pareto spectrum of the weights. In Section 4, we will show that with minor adjustments, the results in our paper could be easily extended to the general cases.

In this research, we focus on allocation of the available supply from a single food-bank to the counties in its network while explicitly accounting for efficiency and equity in the objective function, simultaneously. We provide closed-form solutions to the cases of perfect equity and perfect efficiency and provide numerical results to test the performance of our model while deriving managerial insights. Through a chance-constrained programming technique, we further extend our model to the case with stochastic capacities and derive equivalent theoretical bounds and managerial insights for that setting as well.

3. Model formulation

In this section, we consider the deterministic problem faced by a food-bank that sources n counties in its network. The food-bank receives an amount of donation (S) which needs to be allocated to its network of counties, where county i has a capacity level (Ci) to handle and distribute its share of the allocated donation among its demand population (Di). The aim of the food-bank is to allocate the available supply among the counties in the most equitable
and efficient manner. We discuss the case with stochastic capacities in Section 6.

Before proceeding to the model, we present the formal structure of the supply chain under consideration in Figure 2:

In Figure 2, $S$, $C_i$, and $D_i$ are supply at the food-bank, capacity at county $i$, and demand at county $i$, respectively. The decision variable $x_i$ represents the amount of donation allocated to county $i$. These three parameters along with the decision variable $x_i$ are calculated in pounds of food. Further mathematical notations are defined in Table 1.

We present the mathematical programming model to the problem (Model 1) as follows:

\[
\begin{align*}
\text{Max} & \sum_{i \in I} (\beta_i - \theta z_i) \\
\text{s.t.} & \sum_{i \in I} x_i \leq S \\
& x_i \leq C_i \quad \forall i \in I \\
& \beta - \beta_i = z_i \quad \forall i \in I \\
& \beta_i = \frac{x_i}{D_i} \quad \forall i \in I \\
& 0 \leq \beta \leq 1 \\
& x_i, z_i \geq 0 \quad \forall i \in I,
\end{align*}
\]

where $\beta_i$ is the fill-rate at county $i$ and $\beta$ represents the maximum fill-rate among all counties. We define fill-rate as the ratio of the allocated supply to the demand at each county (Zipkin, 2000).

In Model 1, the objective function in Equation (1) maximizes fill-rates across the network (efficiency) while penalizing the deviations from the maximum fill-rate by parameter $\theta$ (equity). Constraint (1a) states that the total allocation across the counties cannot exceed the total supply. Constraint (1b) sets the upper-bound on the allocation to each county as the capacity of that county to process food. The deviation from the maximum fill-rate for each county is set by constraint (1c). Constraint (1d) defines the fill-rate for each county. Finally, constraints (1e) and (1f) set the upper-bound and lower-bound for each decision variable in Model 1.

Using $\beta_i$ facilitates accounting for both efficiency and equity in our model. Specifically, since $\beta_i = \frac{x_i}{D_i}$, maximizing $\beta_i$ leads to maximizing $x_i$ and thereby minimizing the waste across the network in terms of food-waste both at the food-bank and at the county levels. Additionally, when all the counties have exactly the same fill-rate such that $\beta_1 = \beta_2 = \cdots = \beta_n$, the allocation is perfectly equitable. As the disparity among $\beta_i$'s increases, the allocation becomes more and more inequitable. Therefore, $\theta$ is introduced in Model 1, indicating the degree of penalty for the deviation $\beta - \beta_i$.

Table 1. Summary of the mathematical notation.

| Notation | Definition | Type |
|----------|------------|------|
| $n$      | number of counties. | Parameter |
| $I$      | set of counties, i.e., $I = \{1, \ldots, n\}$. | Set |
| $S$      | supply of donations arrived at the food-bank. | Parameter |
| $C_i$    | capacity at county $i$. | Parameter |
| $D_i$    | demand at county $i$. | Parameter |
| $\tilde{D}_i$ | effective demand at county $i$, $\tilde{D}_i = \min(C_i, D_i), \forall i \in I$. | Expression |
| $D$      | the total demand across all counties, $D = \sum_{i \in I} D_i$. | Expression |
| $x_i$    | amount of donation allocated to county $i$. | Decision Variable |
| $\beta_i$ | fill-rate at county $i$, $\beta_i = \frac{x_i}{D_i}$, $\forall i \in I$. | Expression |
| $\beta$  | maximum fill-rate among all counties, $\beta = \max_{i \in I} \beta_i$. | Decision Variable |
| $z_i$    | deviation from the maximum fill-rate at county $i$, $z_i = \beta - \beta_i$. | Decision Variable |
| $\theta$ | penalty for deviations from the maximum fill-rate. | Parameter |
| $\theta^U$ | the tightest upper-bound of $\theta$. | Expression |
| $\theta^L$ | the tightest lower-bound of $\theta$. | Expression |

![Figure 2. The overall problem setting.](image-url)
Since only $\beta_i$'s are involved in the objective function, we notice that it is more convenient to use $\beta_i$'s as decision variables and remove $x_i$'s from the model. To do so, we first define the effective demand and a binding variable as follows:

**Definition 1.** Let $\tilde{D}_i = \min \{C_i, D_i\}$ and $\tilde{\beta}_i = \tilde{D}_i/D_i$. $\tilde{D}_i$ and $\tilde{\beta}_i$ are called the effective demand and the maximum possible fill-rate of county $i$, respectively.

**Definition 2.** Given a feasible solution $\beta = (\beta_1, \ldots, \beta_n)$, $\beta_i$ is called a binding variable if $\beta_i = \tilde{\beta}_i$.

By Definition 1, it is convenient to combine constraints (1b) and (1d) and exclude $x_i$ from our model. Model 1 can be rewritten as the following concise form in which $\beta_i$ and $\beta$ are the only decision variables. Our analysis will use this concise form hereafter. Furthermore, when there is no confusion, $\beta = (\beta_1, \ldots, \beta_n)$ and $(\beta, \beta)$ are interchangeably called the solution to Model 2 because $\beta = \max_{i \in I} \beta_i$ which can be calculated using $\beta$.

$$
\max \sum_{i \in I} [\beta_i - \theta(\beta - \beta_i)] \\
\text{s.t.} \quad \sum_{i \in I} \tilde{\beta}_i \tilde{D}_i \leq S \\
\beta_i \leq \tilde{\beta}_i \quad \forall i \in I \\
\beta_i \leq \beta \quad \forall i \in I \\
\beta \leq 1
$$

(2)

Since $\theta$ is positive, the bi-objective programming of Model 2 actually derives the Pareto frontier between the efficiency and the equity. In other words, the optimal solutions from different values of $\theta$ are Pareto optimal and lie on the Pareto frontier. In fact, since our model uses the weighting method as the generating technique, only extreme (corner) points on the efficient frontier are obtained and not the interior points as happens with other generating techniques as the constraint method. The details are discussed in Appendix A.

Hereafter, our discussion will be based on Model 2 and focuses on quantifying the equity-efficiency trade-off within the context of food-bank operations (Bertsimas, Farias, & Trichakis, 2011, 2012). In our setting, $\theta = 0$ indicates that the sole priority of the food-bank is efficiency, i.e., the center of the efficiency circle in Figure 1. On the other hand, when $\theta$ is the arbitrarily large value $M$, it means that the sole priority of the food-bank is equity, i.e., the center of the equity circle in Figure 1. Finally, when $0 < \theta < M$, it means that the food-bank chooses to operate with a combined approach towards efficiency and equity, i.e., the line connecting the centers of the two circles in Figure 1. The values 0 and $M$ are chosen hypothetically at this point. The tightest values for the range of $\theta$ as $\theta^U$ and $\theta^L$, which will replace 0 and $M$, respectively, are discussed in Section 4.

### 4. Model analysis

In this section, we discuss some properties of our model. The goal of our analysis is to develop the tightest upper and lower bounds for $\theta$ in Model 2 which are denoted by $\theta^U$ and $\theta^L$, respectively. We use bold face letters to indicate vectors. Additionally, all missing proofs for our theorems can be found in the appendices.

We know that $\theta \in [0, M]$, where 0 and $M$ represent perfectly efficient and perfectly equitable policies, respectively. However, in real applications, the food-bank administrators may have concerns on using $M$. Particularly, the concern arises with respect to the arbitrary nature of $M$. In what follows, we give a closed-form solution for finding $\theta^U$ which conveniently enables the food-bank administrators to set the upper-bound to this value instead of $M$. In fact, we rigorously show that the value of $\theta^U$ is at most $n - 1$.

Now, we discuss the related definitions which culminate in the closed-form solution for $\theta^U$ in Theorem 1.

**Definition 3.** Let $\beta^EQ = (\beta^EQ_1, \ldots, \beta^EQ_n)$ be the optimal solution of Model 2 under the criterion of perfect equity, such that $\beta^EQ_i = \beta^EQ \ (\forall i \in I)$ and $\sum_{i \in I} \beta^EQ_i = n \beta^EQ \geq \sum_{i \in I} \beta_i - M \sum_{i \in I} (\beta - \beta_i)$ for any feasible solution $(\beta_1, \ldots, \beta_n)$ and an arbitrarily large value $M$. $\beta^EQ$ is called the perfectly equitable fill-rate.

Based on the definition of $\beta^EQ$, now we can rigorously define the upper-bound of $\theta$ as follows:

**Definition 4.** $\theta^U$ is called the tightest upper-bound of $\theta$ if both of the following conditions hold:

a. For any $\theta \geq \theta^U$, $\beta^EQ$ is the optimal solution.

b. For any $\theta < \theta^U$, there exists a feasible solution $\beta$ dominating $\beta^EQ$ such that $\sum_{i \in I} \beta^EQ_i = n \beta^EQ < \sum_{i \in I} \beta_i - \theta \sum_{i \in I} (\beta - \beta_i)$

Next, we discuss a special case where the solution $\beta^EQ$ can be found superior in both equity and efficiency. We call the problem setting under such case as utopian, rigorously defined in Definition 5.

**Definition 5.** Given the optimal solution $\beta^EQ$ under the criterion of perfect equity, $\beta^EQ$ is called utopian if the total supply is depleted, i.e., $\sum_{i \in I} x_i = \beta^EQ D = S$.

If a problem instance is utopian, it implies both perfect equity and perfect efficiency is on-hand.
This is not interesting, because: (i) it is hardly true in real applications, and (ii) the closed-form solution of $\beta^{EQ}$ and the condition under which it is utopian are given by Theorem 2.

Theorem 1 gives the closed-form of $\theta^U$ when the problem is not utopian.

**Theorem 1.** Assume $\beta^{EQ}$ is the optimal non-utopian solution of Model 1 under the criterion of perfect equity. Letting $m^U$ denote the number of unbinding variables in $\beta^{EQ} = \{\beta_1, \ldots, \beta_n\}$, $\theta^U$ is calculated as follows:

$$\theta^U = \frac{m^U}{n - m^U}$$

**Theorem 2.** Let $\beta^a = \xi$, $\beta^b = \min_{i \in I} \beta_i$, and $J = \{j \mid d\beta_j = \beta^b, \forall j \in I\}$. The following results hold.

i. If $\beta^a \leq \beta^b$, the problem is utopian and $\beta^{EQ} = \beta^b$.

ii. If $\beta^a > \beta^b$, then $\beta^{EQ} = \beta^b$ and $m^U = n - |J|$ where $| \cdot |$ represents the cardinality of a set.

Theorem 2 actually gives the closed-form of $\beta^{EQ}$.

So far, we have discussed how to find the tightest upper-bound $\theta^U$. Instead of an unknown arbitrarily large “$M$”, $\theta^U$ conveniently enables the food-bank decision-makers to know where exactly in the spectrum of values of $\theta$ perfect equity begins.

Similar to the case of the upper-bound on $\theta$, we next discuss how the decision-maker can adopt $\theta^L$ (the lower-bound of $\theta$) to replace $0$. Essentially, $\theta^L$ marks where exactly in the spectrum of values of $\theta$ perfect efficiency begins. Before proceeding to the closed-form of $\theta^L$, we first characterize the nature of $\theta^L$ in Definition 6.

**Definition 6.** Assuming $\xi$ is the optimal objective function value when $\theta = 0$, $\theta^L$ is called the tightest lower-bound of $\theta$ if both of the following conditions hold:

a. For any $\theta < \theta^L$, if $\beta = (\beta_1, \ldots, \beta_n)$ is the optimal solution, then $\sum_{i \in I} \beta_i = \xi$.

b. For any $\theta > \theta^L$, if $\beta = (\beta_1, \ldots, \beta_n)$ is the optimal solution, then $\sum_{i \in I} \beta_i < \xi$.

The reason that we do not specify the optimal solution for $\theta = 0$ is because there may exist multiple or infinitely many optimal solutions when efficiency is the sole criterion in the objective function. The unique optimal solution $\beta^{EQ}$ provided the clue to find the tightest upper-bound. Similarly, in what follows, we show that although the problem may not have a unique optimal solution for $\theta = 0$, there exists a dominant solution which facilitates our analysis. Specifically, we develop Algorithm 1 to find this unique optimal dominant solution. The algorithm and the discussion on its rationale can be found in Appendix D.

The optimal solution constructed by Algorithm 1 dominates all other optimal solutions that solely focus on efficiency. In other words, if we collect all the optimal solutions for $\theta = 0$ in the set $B$, then this unique optimal solution dominates the rest of the solutions in $B$ for a positive $\theta$. This is formally stated in Theorem 3.

**Theorem 3.** Let $B$ be the set of all optimal solutions when $\theta = 0$ and $\beta^{EF} = (\beta_1^{EF}, \ldots, \beta_n^{EF})$ the dominant solution constructed by Algorithm 1. For any $\beta \in B$ and $\theta \geq 0$, $\beta^{EF}$ dominates $\beta$ such that $\sum_{i \in I} [\beta_i^{EF} - \theta(\beta_i^{EF} - \beta_i)] \geq \sum_{i \in I} [\beta_i - \theta(\beta_i - \beta_i)]$.

Since the optimal solution $\beta^{EF}$ constructed by Algorithm 1 dominates all the other optima for $\theta = 0$, to find the tightest lower-bound, we only have to show that while $\theta$ becomes positive whether $\beta^{EF}$ is still optimal. Based on this point and Definition 6, we present Theorem 4, which gives the tightest lower-bound. The proof is similar to that of the tightest upper-bound.

**Theorem 4.** Let $\beta^{EF}$ be the dominant optimal solution constructed by Algorithm 1, $\beta^{EF} = \max_{i \in I} \beta_i^{EF}$ and $K = \{i \mid d\beta_i^{EF} = \beta_i^{EF}, i \in I\}$. Let $D_m = \min_{i \in I} D_i$ where $L = \{i \mid d\beta_i^{EF} < \beta_i, i \in I/K\}$ - the set of counties not fully filled except those in $K$. If the problem is non-utopian and supply-constrained such that $S \leq \sum_{i \in I} D_i$, the tightest lower-bound $\theta^L$ is calculated as follows:

$$\theta^L = \frac{m^L}{n - m^L},$$

where,

$$m^L = \begin{cases} |K| - \sum_{i \in K} D_i/D_m \quad & \text{if } L \text{ is not empty} \\ |K| \quad & \text{if } L \text{ is empty} \end{cases}$$

Theorem 4 applies to the case where the total supply cannot satisfy all the counties’ effective demands. The case with sufficient supply such that $S \geq \sum_{i \in I} D_i$, is similar to the utopian problem and rarely realistic. In early 2020, the novel coronavirus (COVID-19) began to spread across the United States, and one of the consequences was an economic recession that resulted in more food insecurity the lack of access to sufficient food because of limited financial resources. Feeding America estimates that 45 million people (1 in 7), including 15 million...
children (1 in 5), may have experienced food insecurity in 2020 (Feeding America, 2021). If supply is abundant, food-bank administrators can simply satisfy all the counties at their effective demands.

In summary, this section presents the Pareto spectrum \([\theta^L, \theta^U]\) of policies for food-bank decision-makers to operate in. We also give the closed-form solutions corresponding to \(\theta^U\) and \(\theta^L\). As a result, the food-bank administrators can pick a policy (defined by a \(\theta \in [\theta^L, \theta^U]\)) according to their organization’s preference towards efficiency and equity. Next, in Section 5, using real data from Feeding America, we demonstrate the performance of our model against a benchmark from the literature and provide critical insights for the food-bank decision-makers.

5. Computational study and insights

In this section, we first, summarize the performance of our model compared to a benchmark from the literature (Section 5.1). Then we proceed with describing the implications of our results to the decision-making procedure of Feeding America describing the implications of our results to the solutions corresponding to the capacity of the food-bank whose data is utilized for our study. Subsequently, in Section 5.2, we demonstrate how improvement of the bottleneck county’s capacity affects equity and efficiency. Finally, in Section 5.3, we further demonstrate how changes in charitability and volunteer level in the counties affect the equity-efficiency trade-off for the food-bank. Specifically, we provide guidelines on the policies the decision-makers can adopt in order to facilitate the food-bank’s path towards achieving higher levels of equity (efficiency) while sacrificing the least from their efficiency (equity).

The data we use throughout our computational study is provided by Feeding America Research Team. Specifically, we use the Map-the-Meal-Gap and Food Insecurity Projections data sets from 2011 to 2021. Although, to keep consistency with our benchmark, the branch whose data is used in here is the Food Bank of Central and Eastern North Carolina (FBCENC), we also communicated with the Community Food Bank of Central Alabama to verify the accuracy of the data set. For instance, in October 2021, the Community Food Bank of Central Alabama distributed 1,356,799 lbs. of food, which provided 1,130,666 meals. This figure implies exactly 1.2 lbs./meal, and matches the data from The Feeding America Research Team. In our computational study, we use the data from 2020 for the service area served by FBCENC composed of 34 counties: Brunswick, Carteret, Chatham, Columbus, Craven, Duplin, Durham, Edgecombe, Franklin, Granville, Greene, Halifax, Harnett, Johnston, Jones, Lee, Lenoir, Moore, Nash, New Hanover, Onslow, Orange, Pamlico, Pender, Person, Pitt, Richmond, Sampson, Scotland, Vance, Wake, Warren, Wayne, Wilson.

Finally, we program the models in Python under Google Colaboratory platform.

5.1. Efficient frontier insights

We use two performance measures of equity and efficiency defined as follows:

- Efficiency: the ratio of the total food distributed towards satisfying the demand to the total available supply, calculated according to the following:

\[
\text{Efficiency} = \frac{S - (\text{County Waste} + \text{Food Bank Waste})}{S}
\]

(5)

We define county waste as the amount of food which has been sent to a county, but not distributed towards satisfying the demand, while food-bank waste is defined as the amount of food that is left at the food-bank and not allocated to the counties. We would like to point out that county waste can happen only if the food-bank allocates food to a particular county beyond its demand. Our model avoids county waste via constraint (2b).

- Equity: the degree to which each county serving the impoverished population receives its fair-share of the supply \(S\), calculated according to an equity measure.

Our model takes equity into account in a general way by penalizing the difference among the county fill-rates. To evaluate how equitable the solutions to our model are, we need to utilize a measure of equity. However, to avoid divisions by zero, in the literature, various measures of inequity have been utilized instead of equity. In this paper, we utilize some of the most widely used measures of inequity for a given vector of fill-rates \((\beta_1, \beta_2, \ldots, \beta_n)\) having the mean of \(\bar{\beta}\), maximum of \(\beta_{\text{max}}\), and minimum of \(\beta_{\text{min}}\) as follows:

\[
\text{Gini Coefficient} = \frac{\sum_{i \in I} \sum_{j \in I} |\beta_i - \beta_j|}{2n^2 \bar{\beta}}
\]

(6)

\[
\text{Variance} = \frac{\sum_{i \in I} (\beta_i - \bar{\beta})^2}{n}
\]

(7)

\[
\text{Mean Absolute Deviation} = \frac{\sum_{i \in I} |\beta_i - \bar{\beta}|}{n}
\]

(8)

In this section, to calculate inequity we put our focus on Gini coefficient, the most widely used...
measure of inequity in social welfare (Marsh & Schilling, 1994). The rest of the measures in Equations (7) and (8) used for comparing our model to that of Orgut et al. (2016b) as our benchmark (henceforward referred to as “ORG”) are reported in the appendices which show similar significant improvements. These measures are the Variance as well as the Mean Absolute Deviation, given in Equations (7) and (8), respectively.

Gini coefficient measures inequity as the ratio of the area between the line of perfect equity and the Lorenz curve and the total area under the line of perfect equity, thereby it is a measure between zero and one (Gini, 1912). Therefore, to calculate equity in this section, we subtract the Gini coefficient from one.

To calculate the inequity measures in Equations (6)–(8) for our setting, \( x \) refers to the set of counties and parameter \( n \) corresponds to the number of counties in the problem. Furthermore, \( \beta_i \) is the fill-rate at county \( i \). Then, each measure of inequity calculates the level of unfairness among \( \beta_i \)’s.

The fill-rate \( \beta_i = \frac{x_i}{D_i} \) is a ratio as opposed to the absolute shipment value \( x_i \). Note that using the absolute shipment value \( x_i \) to calculate equity is not reasonable because different counties face different scales of demand according to their poverty population. For instance, the county Wake has a demand of 4,272,480 lbs./month while the county Jones’ demand is 71,640 lbs./month.

To estimate the demand in pounds of food at each county (represented by \( D_i \) in our model), we use 1.2 lbs. of food consumption per person per serving (Feeding America, 2020) multiplied by the population in poverty served. Without loss of generality, we assume that the counties serve a single meal per day, every day throughout the month. Note that the magnitude of the demand does not change the equity-efficiency trade-off as both equity and efficiency are ratios.

To calculate the capacity at each county (represented by \( C_i \) in our model), we use the \( C_i/D_i \) ratios from ORG. The total amount of food donations to be allocated to the 34 counties in the area under study is 2,838,584 lbs., which is represented by parameter \( S \) in our model. These three parameters form our three nominal parameters required to generate the test cases for our analysis.

Demand may change across the counties, resulting in some counties with very high and some with comparatively lower demands. To test the performance of the models against the locational demand variability in the data, using the nominal demand vector, we randomly generate the cases through the following procedure:

1. Letting \( D_{i,\text{nom}} \) be the nominal demand of county \( i \) and \( D_{i,\text{nom}}^\ast \) the mean value of \( D_{i,\text{nom}} \)’s, we calculate the variance \( \sigma_{D,\text{nom}}^2 = \frac{\sum_{i=1}^{n} (D_{i,\text{nom}} - D_{i,\text{nom}}^\ast)^2}{n} \).
2. Based on the mean value \( D_{\text{nom}}^\ast \) and the variance \( \sigma_{D,\text{nom}}^2 \), we construct a normal (Gaussian) distribution, such that \( D \sim \text{Gaussian}(D_{\text{nom}}^\ast, \sigma_{D,\text{nom}}^2) \). The parameter \( \sigma \) is used to adjust the variability of this random distribution. For the low variability category, \( \sigma \) is set to 0.8, while for the high variability category, \( \sigma \) is set to 1.8.
3. Using \( D \), we randomly generate 1000 cases (realizations) for each of the low and high demand variability categories (2000 cases in total). More specifically, in each case, we use \( D \) to generate \( n \) different demands \( D = (D_1, D_2, \ldots, D_n) \) for \( n \) counties. The generated \( D_i \)’s must be positive and the variance of \( D \) must be less than or equal to \( 0.8 \sigma_{D,\text{nom}}^2 \) for the low variability category, and more than or equal to \( 1.8 \sigma_{D,\text{nom}}^2 \) for the high variability category. Otherwise, the case should be regenerated.

Additionally, we made sure that all our instances satisfy the two criteria for non-triviality discussed in Section 4: (i) they are not utopian (from Definition 5); and (ii) they are supply-constrained (from Theorem 4). Every point in Figures 3, 7(a), and 7(b), along with the figures in the appendices are averaged over the 1000 demand realizations. In this
section, the capacity vector remains constant and equal to the nominal capacity across all of these 2000 demand realizations.

Next, we determine the range of the auxiliary parameters $\theta$ (for our model) and $K$ (for ORG model). In detail, the lower-bound of $\theta$ in our model considers only efficiency as the goal of the objective function and it is calculated using Theorem 4. On the other hand, the upper-bound of $\theta$ only takes equity into account in our objective function and is calculated based on Theorem 1 in our model.

To determine the corresponding range for $K$, we note that the mathematical formulation of the problem according to ORG is as follows:

\[
\begin{align*}
\text{Min } & P \\
\text{s.t. } & \left| \frac{x_i}{\sum_{i \in I} x_i} - \frac{D_i}{\sum_{i \in I} D_i} \right| \leq K \quad \forall i \in I \\
& S - \sum_{i \in I} x_i = P = 0, \\
& x_i \leq C_i \quad \forall i \in I, \\
& x_i, P \geq 0 \quad \forall i \in I,
\end{align*}
\]

where $P$ represents the leftover food after the allocation of the donated supply $S$. We note that $P$ represents Food-Bank Waste in Equation (5). In this model, parameter $K$ is between 0 and 1 and controls the level of equity in the allocation of the total supply. Lower values of $K$ impose stricter (therefore higher) priority on equity and vice versa, wherein $K = 0$ is the lower-bound of $K$ and corresponds to absolute equity. On the other hand, higher values of $K$ give more freedom to the objective function to choose solutions with higher efficiency. In order to find the upper-bound of $K$ for ORG model (denoted by $K^U$), we note that in perfect efficiency $p = 0$. Therefore, from constraint (11b), we have $\sum_{i \in I} x_i = S$. Thus, $K^U$ is the value of the objective function to the following mathematical model:

\[
\begin{align*}
\text{Min } & K^U \\
\text{s.t. } & \left| \frac{x_i}{\sum_{i \in I} x_i} - \frac{D_i}{\sum_{i \in I} D_i} \right| \leq K^U \quad \forall i \in I \\
& \sum_{i \in I} x_i = S, \\
& x_i \leq C_i \quad \forall i \in I, \\
& x_i, K^U \geq 0 \quad \forall i \in I.
\end{align*}
\]

Finally, to show the difference between the models more clearly, we divided the length between $\theta^L$ and $\theta^U$ as well as 0 and $K^U$ to 50 pieces. Therefore, every point in Figures 3 corresponds to a value of $\theta$ ($K$) in our model (ORG model). We point out that some of the values of $\theta$ in our model yield the same optimal solutions, thus gaining identical equity and efficiency values, leading to reduced number of points for our model in the figures. Essentially, Figure 3 shows the efficient frontiers of our model and ORG using Gini coefficient to calculate the equity level under three demand scenarios: low variability of demand across the counties, nominal variability (actual existing data), and high variability of demand across the counties.

Inclusion and exclusion of the demand constraint in the food-bank donation allocation problem has been argued against and in-favor-of in the literature. Of course, when demand is deterministic (such as in our setting in this section), exclusion of the demand constraint does not have either logical or practical sensibility. However, in order to perform a thorough comparison between the performance of the models, we run both models with and without consideration of the demand constraints. Specifically, the efficient frontier without the inclusion of the demand constraint is represented by red circles [o] and light-blue circles [●] for ORG and our model, respectively. On the other hand, the efficient frontier with inclusion of the demand constraint is represented by gray-filled disks [●] and blue-filled disks [●] for ORG and our model, respectively.

In all three subfigures of Figure 3, the points resulting in perfect efficiency (Efficiency = 1) correspond to $K = K^U$ in ORG model and $\theta = \theta^L$ in our model. On the other hand, the points resulting in perfect equity (Equity = 1) correspond to $K = 0$ in ORG model and $\theta = \theta^U$ in our model.

Our first observation from Figure 3 is that while efficiency drops in an exponential manner as the value of $K$ decreases from $K^U$ in ORG model, the efficiency in our model drops in an almost linear manner while $\theta$ increases from its lower-bound $\theta^L$. This behavior holds under all settings of inclusion or exclusion of the demand constraint in either of the models. This results in the fact that for the same levels of equity (efficiency), our model yields much higher levels of efficiency (equity). This is resulted from the fact that solutions from our model strictly lie on the Pareto frontier. Therefore, all three players in the food-bank supply chain simultaneously benefit from using our model: the food-bank, counties, and the demand population.

Furthermore, we observed that exclusion (or inclusion) of the demand constraint does not affect the performance of either of the models significantly. The only minor exception is regarding ORG model’s performance under high demand variability, where inclusion of the demand constraint has slightly improved the efficient frontier. The purpose for including the demand constraint in either of the models is to prevent the county wastes. Therefore, the implication of this observation for the food-bank supply chains is that majority of the waste
arises in potential leftovers at the food-banks rather than the counties as also pointed out by ORG.

As demand variability increases from left to right in Figure 3, the gap between the efficient frontiers of the two models increases. Specifically, while the variability increases, our model maintains almost the same equity-efficiency trade-off, making it robust against the variability in the data.

Regardless of the level of variations in the demand realizations, the two extremes in both models correspond to the cases with perfect equity (Equity $= 1$) and perfect efficiency (Efficiency $= 1$). At perfect equity, both models result in the same efficiency. This is expected, because although the models have different objective functions, at perfect equity, the solutions to the problems are unique, resulting in the same amount of waste at both models. However, at perfect efficiency the models may have infinitely many optimal solutions (please see our example in Section 4 corresponding to Definition 6). In other words, since equity is not considered in the objective function under perfect efficiency, there may be infinitely many perfectly efficient solutions (Efficiency $= 1$), but with varying levels of equity. Among these solutions, our model chooses a dominant solution with considerable improvements in equity compared to that of ORG, especially under high demand variability.

Finally, Figure 3 suggests that for achieving a unit of equity, the food-bank needs to sacrifice a high amount of efficiency. This is while sacrificing a small amount from equity can achieve much more improvement in efficiency in the allocation of the supply. In other words, our equity-efficiency trade-off suggests that achieving equity is more difficult of a goal for the food-banks than achieving efficiency.

In order to demonstrate the implications of the results from our paper in practice, next, we explain the nominal demand scenario from Figure 3 in more details. Essentially, the nominal scenario corresponds to the outcome of our model applied to the current nominal data from Feeding America’s North Carolina food-bank. The results for this analysis are presented in Figure 4. Note that Figure 4 suggests that any policy taken by Feeding America outside of the range $\theta \in [0^L = 0.34, 0^U = 33]$ represents an inferior policy, unnecessarily sacrificing one performance measure without gaining from the other. The superior policies are represented by blue-filled disks [●] in Figure 4 obtained from solving our model with the nominal data. Essentially, the range $\theta \in [0^L = 0.34, 0^U = 33]$ gives perfect guidelines to Feeding America on where to set their preference on the equity-efficiency trade-off.

### 5.2. Bottleneck analysis insights

In this section, we solve the problem with the nominal data for 51 different $\theta$’s, i.e., $\theta = 0^L + \frac{k}{50} (0^U - 0^L)$ for $k = 0, 1, \ldots, 50$. The average fill-rates across different counties are then presented in Figure 5. Figure 5 shows that although the counties have different levels of effective fill-rate, our model tries to equitably allocate the food supply. The fill-rates of all 34 counties are around 9%, with a minimum of...
7% for Wilson and a maximum of 10.58% for Halifax. Note that Wilson is the bottleneck county with the minimum effective fill-rate 7%. The average fill-rate for Wilson is also 7%. This means that no matter how the penalty factor \( \theta \) changes, the model always allocates the maximum possible food supply to Wilson. If more available capacity could be added to Wilson, the deviation between the maximum fill-rate and the minimum fill-rate will be immediately reduced and result in an equity improvement. This is also what Theorem 2 implies.

Specifically, if the problem is utopian, solving our model will result in an optimal solution with both absolute efficiency and absolute equity. In other words, the solution has zero waste such that \( S - \sum_{i \in I} x_i = 0 \) and all the counties have exactly the same fill-rates such that \( \beta_1 = \beta_2 = \cdots = \beta_n \).

Theorem 2 actually demonstrates that the impediment of achieving the utopian solution comes from those bottleneck counties, i.e., the minimum effective fill-rates \( \beta^b \). We note that \( \beta^b = \min_{i \in I} \beta_i \) and \( \tilde{\beta}_i = \frac{\min(C_i, D_i)}{D_i} \). From the nominal data, we know that the counties are capacity constrained, such that \( C_i \leq D_i \). Hence, \( \tilde{\beta}_i = \frac{\min(C_i, D_i)}{D_i} = \frac{C_i}{D_i} \). Therefore, improving the capacity of the bottleneck counties will increase the value of \( \beta^b \) and make the problem approach the utopian solution.

To visualize this, we create a case such that the demand and the capacity of the counties are same as the nominal but the supply \( S \) is 50% of the total demand (hence \( \beta^b = 50\% \)). We do not use the nominal supply because it only occupies 12.91% of the total demand which cannot show a wide range of change. The parameter \( \theta \) is set to be almost the same as the lower-bound, i.e., the lower-bound plus 0.3% of the difference between the upper and the lower bounds. In this case, we gradually add capacity to those bottleneck counties, i.e., the counties with lowest capacity-to-demand ratio \( (C_i/D_i) \) and solve our model to investigate the corresponding change of both efficiency and equity. The results are shown in Figure 6.

As we can see from Figure 6, while more capacity becomes available for those bottleneck counties with the minimum effective fill-rate (\( \beta^b \)), both efficiency and equity are improved. Increasing capacity for the bottleneck counties results in higher value of \( \beta^b \). Once the effective fill-rate of the bottleneck counties (\( \beta^b \)) reaches 50% (which is the value of \( \beta^b \)), the solution becomes utopian, that is, both efficiency and equity become 1 (simultaneously perfect equity and efficiency). This is exactly what Theorem 2 implies.

### 5.3. Price of equity insights

Particular sources of data which are of interest to the managers of food-bank operations are the supply at the food-bank level and capacity at the county level and the dynamics they create in the non-profit network operations. In this section, we quantify the effects of change in supply and capacity levels on the price of equity under two levels of variability in demand realizations and discuss some of our managerial insights. Price of equity is defined as the units of efficiency that need to be sacrificed in order to observe a unit change in equity. Bertsimas et al. (2011).

Figure 7(a) shows the changes in price of equity as a function of available supply (denoted by \( S \) in our model). As it is evident from Figure 7(a), higher availability of supply reduces the amount of efficiency that should be sacrificed to achieve a single unit of increase in equity. In other words, the more a society becomes charitable, the easier it is for food-banks to become equitable in their distribution of food to the public. Another point of attention in Figure 7(a) is that the price of equity reduces with a diminishing rate as the supply increases, which indicates that initial level of increase in supply has the greatest impact in achieving a more equitable allocation.

Figure 7(a) suggests that, at least at low supply, achieving equity is more difficult than achieving efficiency (price of equity is greater than one). In detail, price of efficiency can be defined as the reciprocal of the price of equity. Therefore a general observation from Figure 7(a) is that achieving higher equity requires more sacrifice from efficiency for the food-banks, especially when supplies are more scarce. Our final observation from Figure 7(a) is that as the variability in demand increases, the food-bank has to sacrifice more from its efficiency to achieve a single unit of equity for the same amount of supply.
In traditional econometric models for capacity planning, the cost-capacity function has a unique minimizer. For instance, considering a traditional model for a warehousing system, cost of low capacity is the opportunity cost for the customer demands that could have been served, but are lost because there is not enough capacity for holding inventory. On the other hand, cost of high capacity is the idle space that is not used. Similarly, Figure 7(b) shows the changes in price of equity when average capacity increases in the network. Figure 7(b) suggests that there is a unique capacity at which price of equity is at its minimum. Specifically, a low capacity network of counties in our setting behaves similar to having low supply, which, as already discussed for Figure 7(a), causes the price of equity to rise. This is intuitively correct as having low capacity to handle the donated food at the counties limits the amount of supply they can receive and process.

The cost of high capacity follows a similar behaviour. In detail, in our data from Feeding America, capacity is always a percentage of the demand in the network. This fact is a common phenomenon in practice as the capacity of the counties typically falls short of their demand. Therefore, the cost of high capacity is due to the fact that an increase in capacities results in higher effective demands in the network (from Definition 1). When effective demand increases and supply stays constant, it is as if the supply has become more scarce. Therefore, according to Figure 7(a), the price of equity starts to rise after it meets its minimum in Figure 7(b).

We note that, as discussed in our explanation of Figure 6, addition of capacity to the bottleneck county improves both equity and efficiency until they both reach perfection, resulting in a utopian solution. However, we emphasize that Figure 7(b) discusses how the ratio of change in efficiency to change in equity (price of equity) behaves with respect to changes in average capacity across the counties. In other words, although both efficiency and equity improve with improvement of average capacity, the change of equity cannot catch up with the change of efficiency. Moreover, we note that a change in the average capacity changes the capacity of all the counties simultaneously. This means that not all the increased capacity is added to the bottleneck counties. For instance, Duplin has the highest capacity-to-demand ratio. Once the capacity-to-demand ratio of Duplin reaches one, adding more capacity will have no effect and will be wasted. In practice, Figure 7(b) demonstrates that it is not always the case that expanding the capacity of food-banks will lead to ease in meeting higher equity levels. In other words, encouragement for volunteer involvement in food-bank operations and therefore enhancing their capacities, comes at a secondary level of priority when compared to creating a more charitable society or reduction of poverty in the society. This is because a more charitable society supplies more food donation which lowers the price of equity as shown in Figure 7(a). Additionally, reducing poverty leads to lower effective demands in the network and extends the decreasing behaviour of price of equity as the result of increase in capacities (before meeting the minimum) in Figure 7(b). Our final observation from Figure 7(b) is that high variability in demand causes a shift in the price of equity as a function of capacity. In other words, expansion of capacities in food-banks leads to higher gains if the wealth is distributed more evenly in the society.

6. Extension to stochastic capacities

In this section, we extend our model to the case with stochastic receiving capacities. Capacity of a county is generally a function of several factors such as volunteer manpower, storage, and loading/unloading capacity. However, due to extensive reliance on volunteer manpower and the inherent uncertainty associated with human choice, food-bank managers generally have to deal with
uncertainty in the capacity of the counties to receive and handle food. It is interesting to consider our problem subject to such uncertainty.

Hereafter, we assume $C_i$, i.e., the capacity for county $i$, is a random variable. An arbitrary allocation $x_i$ is feasible for the stochastic capacity $C_i$ with probability $Pr\{x_i \leq C_i\}$. This probability represents the reliability level of decision variable $x_i$, which we assume is imposed to be at least $\eta_i$. Hence, by applying the idea of the chance-constrained method, we revise our Model 1 in the form of the following stochastic programming model.

$$\begin{align*}
& \text{Max} \sum_{i \in \mathcal{I}} (\beta_i - \theta z_i) \\
& \text{s.t.} \quad \sum_{i \in \mathcal{I}} x_i \leq S \quad (13a) \\
& \quad Pr\{x_i \leq C_i\} \geq \eta_i \quad \forall i \in \mathcal{I} \quad (13b) \\
& \quad \beta - \beta_i \geq z_i \quad \forall i \in \mathcal{I} \quad (13c) \\
& \quad \beta_i = \frac{x_i}{D_i} \quad \forall i \in \mathcal{I} \quad (13d) \\
& \quad 0 \leq \beta \leq 1 \quad (13e) \\
& \quad x_i, z_i \geq 0 \quad \forall i \in \mathcal{I}. \quad (13f)
\end{align*}$$

In model (13), chance constraint (13b) guarantees a minimum reliability level that has to be met for each county. The other constraints are similar to that of Model 1.

Constraint (13b) is a stochastic inequality. Specifically, if the decision-maker knows the exact theoretical distribution of $C_i$, then Constraint (13b) has an equivalent linear form and the model degenerates to a linear program. However, if such a distribution is not in hand, using historical records (empirical distribution) is the next best recourse to estimate $Pr\{x_i \leq C_i\}$. Denote $\mathcal{R}$ and $R = |\mathcal{R}|$ the set and the number of historical records on capacities, respectively. Then, reliability can be estimated by the proportion of the historical records in which $x_i \leq C_i$, where $r \in \mathcal{R}$. Obviously, as $R$ increases, the decision-maker is more confident in its estimation of the corresponding reliability level, albeit with an increasing computational burden.

In order to derive the deterministic equivalent of chance constraint (13b) and account for the desired reliability and tolerance levels, we define the binary decision variable $y_{ir}$ over the set of historical records of the stochastic capacity $C_i$ as follows:

$$y_{ir} = \begin{cases} 
1 & \text{if } x_i - C_i \leq 0 \\
0 & \text{otherwise,} 
\end{cases} \quad (14)$$

where $C_i$ is historical record $r$ from the empirical data available on $C_i$. The definition in Equation (14) states that the binary decision variable $y_{ir}$ can take a value of 1 if and only if the capacity constraint $x_i \leq C_i$ is not violated.

Additionally, under the circumstances, if the constraint is to be violated at county $i$, the decision-maker may wish such violation not to exceed the county-specific tolerance level $\delta_i C_i$. The value of $\delta_i$, essentially, reflects the ability of county $i$ in recourse actions in case the capacity constraint is violated. Obviously, higher values of $\delta_i$ indicate that county $i$ is capable to deal with more extreme situations in its capacity violation. However, such capability always incurs additional costs (e.g., overtime payment) and therefore consideration of the finite positive tolerance parameter $\delta_i$ is a natural way to address such conditions.

With these definitions, we are ready to give the deterministic equivalent form of Model (13) while simultaneously accounting for both reliability and tolerance as follows:

$$\begin{align*}
& \text{Max} \sum_{i \in \mathcal{I}} (\beta_i - \theta z_i) \\
& \text{s.t.} \quad \sum_{i \in \mathcal{I}} x_i \leq S \quad (15a) \\
& \quad x_i - C_i \leq \delta_i C_i (1 - y_{ir}) \quad \forall i \in \mathcal{I}, \forall r \in \mathcal{R} \quad (15b) \\
& \quad \frac{\sum_{r \in \mathcal{R}} y_{ir}}{R} \geq \eta_i \quad \forall i \in \mathcal{I} \quad (15c) \\
& \quad \beta - \beta_i = z_i \quad \forall i \in \mathcal{I} \quad (15d) \\
& \quad \beta_i = \frac{x_i}{D_i} \quad \forall i \in \mathcal{I} \quad (15e) \\
& \quad 0 \leq \beta \leq 1 \quad (15f) \\
& \quad x_i, z_i \geq 0, y_{ir} \in \{0, 1\} \quad \forall i \in \mathcal{I}, \forall r \in \mathcal{R}. \quad (15g)
\end{align*}$$

The key characteristic of Model (15) lies in the definition of constraints (15b) and (15c). Specifically, when the capacity constraint $x_i - C_i \leq 0$ is not violated, $y_{ir}$ in Constraint (15c) may accept either a value of 0 or 1. However, due to Constraint (6), it guarantees a minimum reliability level that has to be met by the model. Essentially, Constraint (15c) mandates the minimum number of constraints of the form $x_i - C_i \leq 0$ that have to be met over the historical records for each county (minimum number of $y_{ir}$ variables that have to be set to a value of 1 over the historical records for each county), that is, $[\eta_i R]$.

On the other hand, the motivation for violating the capacity constraint in Model (15) is in increasing the objective (efficiency and equity) by higher values of $x_i$. However, when the capacity constraint is violated, i.e., $x_i - C_i > 0$, $y_{ir}$ in Constraint (15b) is forced to accept a value of 0, resulting in the form $x_i \leq (1 + \delta_i) C_i$. In other words, $x_i$ is allowed to exceed the capacity limit $C_i$ with tolerance level of $\delta_i \neq 0$. It is trivial to see that $y_{ir} = 0$ directly implies that $x_i - C_i > 0$ is allowed.

In Model (15), higher values of $\eta_i$ impose a more conservative approach in meeting the capacity constraint at county $i$ and vice versa. On the other hand, for those counties whose capacity constraint is allowed to be violated (potentially in some of the records), smaller values of $\delta_i$ set a more conservative
view towards the violation tolerance and vice versa. In summary, the values of the two parameters \( \eta_i \) and \( \delta_i \) set in the model, reflect how conservative the decision-maker wants to be in allocation of the food supply to the counties.

The complexity of Model (15) highly depends on the scale of \( R \) and the computational time increases exponentially. We notice that Constraints (15b) and (15c) hold is equivalent to the stochastic capacity variability, although it is used as a constraint to be violated. We consider \( \delta_i \) in [0.02, 0.04, 0.06, 0.08, 0.1]. For simplicity, in each case generated, we let \( \eta_i, \delta_i \) and \( \delta_i \) be identical among all 34 counties. In total 5 \( \times \) 5 \( \times \) 5 = 125 cases (problem settings) are generated. Finally, for \( \theta \) we divide the range \([\theta^L, \theta^U]\) into 10 chunks where the first value is \( \theta = \theta^L \) and the last value is \( \theta = \theta^U \).

Therefore, every point in the Pareto frontier curves corresponds to a value of \( \theta \). For each value of \( \theta \in (\theta^L, \theta^U) \), we use the Pulp library in Python with CPLEX as the solver engine under Google Colaboratory platform to solve the corresponding model (125 \( \times \) 10 = 1250 models in total). We note that since capacity is generally the limiting factor for the problem, achieving 100\% efficiency is not always possible. Therefore, in the experimental results, we may see that efficiency of 1 is not always achieved.

The results with regards to capacity variability are presented in Figure 8.

The interesting insight is regarding the effect of capacity variability at the counties (captured by parameter \( a_i \)) on the efficiency-equity trade-off. This effect is depicted in Figure 8(a) for different values of \( a_i \). It is clear that when capacity variability is less in the counties (indicated by lower values of \( a_i \)), the food-bank can achieve a more superior equity-efficiency trade-off. The more subtle insight from Figure 8(a) is that the effect of this variability is not similar for equity and efficiency individually.

In other words, we observed that at high equity
levels, the amount of increase in efficiency, by a decrease in the level of capacity variability, is less when compared to lower equity levels.

Our last set of insights in Figure 8(b) concern the trade-off between the objective value and reliability at different levels of capacity variability at the counties (captured by parameter \( x_i \)). To estimate the reliability of a solution, we used the solution from each problem setting to calculate the percentage of capacity constraints for each county that are not violated. Averaging these values for all the counties results in the overall reliability for the corresponding experiment. Our main takeaway from Figure 8(b) is that as the level of this variability decreases (smaller values of \( x_i \)), the food-bank achieves a better trade-off among the two performance measures with higher objective value and higher reliability level, simultaneously. These insights indicate the value of volunteer retention and engagement policies at the food-banks with the goal of achieving a more stable capacity level.

### 7. Conclusions

Food-banks strive to distribute donations they receive from the public, government agencies, and grocery stores to the counties within their network in an equitable and efficient manner. In this paper, we present a new model to the food-bank donation allocation problem under equity and efficiency performance measures. Our model explicitly accounts for both efficiency and equity in the objective function and is capable of offering closed-form optimal solutions in perfect efficiency and perfect equity, while providing Pareto optimal solutions for the spectrum in between.

Using the real data from Feeding America (one of the largest food donation distributors in North America) for their particular food-bank in North Carolina, we compare the results of our model against a benchmark from the literature in terms of equity and efficiency. Our numerical study demonstrates considerable improvements in terms of both performance measures, simultaneously. This means concurrently benefitting all three players in the food-bank supply chain: the food-bank, the counties, and the demand population.

We present a bottleneck analysis and demonstrate how changes in the capacity of the bottleneck counties affect equity and efficiency. Our sensitivity analysis demonstrates several interesting observations regarding the changes in price of equity when donated supply and counties’ capacities change. In particular, our sensitivity analysis demonstrates that the priority must be put on reducing demand before investing on capacity expansions in the food-banks. This will ensure that adding more capacity to the counties associated with a food-bank will always lead to decreasing the price of equity. Additionally, we observed that encouraging charitability is always beneficial for the food-banks, albeit with a diminishing rate. Finally, our experiments demonstrate that decreasing disparity of wealth in the demand population will lead to lower costs for the food-banks to distribute public donations equitably to their network.

Considering the stochastic nature of the capacities at the counties based on their dependence on the volunteer choices, we extend our model to a case where such a condition is allowed. Modeling the problem as a chance-constrained program, we derive additional insights with respect to the trade-offs between equity, efficiency, and reliability. Specifically, we observe that when capacity variability is less among the counties (which implies that the capacities are closer to their mean), the food-bank can achieve more superior equity-efficiency as well as equity-efficiency-reliability trade-offs. These insights indicate the value of volunteer retention and engagement policies at the food-banks with the goal of achieving a more stable capacity level.

Our research can be extended in multiple directions. In this work, we consider a deterministic setting for our donated supply, because, from a chronological point of view, the food-bank in our setting makes the allocation decisions after observing the donated quantity. In settings wherein the collection of donated supply from pick-up points and their allocation to drop-off locations in the counties happens concurrently, supply should be treated as a stochastic parameter. Such models, however, involve routing and scheduling type formulations which deserve a separate investigation. In another extension, we would like to refer the readers to the role of perishability in the decisions a food-bank makes. Particularly, food-banks typically receive close-to-expiry products from their donors and therefore integration of time (and perhaps a dynamic programming approach) into our model will further advance the insights in the science of decision-making in non-profit operations.

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Appendix A: Our optimal solutions are pareto optimal

If we let \( f_i(\beta) = \sum_{i \in \mathcal{I}} \beta_i \) be the portion of the objective function corresponding to efficiency and \( f_j(\beta) = \sum_{i \in \mathcal{I}} (\beta_i - \bar{\beta}) \) corresponding to equity, the dominance of this bi-objective problem can be defined as follows (Collette & Siarry, 2005), where \( \mathcal{S} \) is the feasible region by the constraints of Model 2 in the manuscript.

Definition 7. Assuming \( \beta_i, \beta_j \in \mathcal{S} \), \( \beta_i \) dominates \( \beta_j \) if the following two conditions hold:

1. \( f_i(\beta_i) \leq f_i(\beta_j), \quad \forall i \in \{1, 2\} \)
2. \( \exists i \in \{1, 2\}, \text{ s.t. } f_i(\beta_i) < f_i(\beta_j) \)

Based on the above definition, a solution \( \beta^* \) is called Pareto optimal if there does not exist any feasible solution \( \beta \neq \beta^* \) which dominates \( \beta^* \). The weighting techniques combine the two objective functions with positive weights \( w_1 \) and \( w_2 \) and solve the problem as follows:

\[
\max f(\beta) = w_1 f_1(\beta) + w_2 f_2(\beta)
\]

s.t. \( \beta \in \mathcal{S} \)

The optimal solution \( \beta^* \) to the above optimization is \( \beta^* \in \arg\max_{\beta \in \mathcal{S}} f(\beta) \). Because all weights are positive, it is trivial that \( \beta^* \) must be Pareto optimal, otherwise we get contradiction if there exists Pareto improvement. Letting \( w_1 = 1 \) and \( w_2 = 0 \), the above optimization is exactly our Model 2.

Appendix B: Proof of Theorem 1

Proof We prove Theorem 1 by parts. If \( m^U = 0 \), it means that all the decision variables are binding such that \( \bar{\beta}_i = \beta_i^{EQ} = \beta_i^{EQ} \). On the other hand, by constraints (2b) and (2c), the objective function value of Model 2 is \( \sum_{i \in \mathcal{I}} (\beta_i - \bar{\beta}_i) \leq \sum_{i \in \mathcal{I}} \beta_i = \sum_{i \in \mathcal{I}} \beta_i^{EQ} \). Hence, \( \beta_i^{EQ} \) is the optimal solution for any \( \theta \geq \theta^U = \frac{m^U}{m^U - m^L} = 0 \).

Now, let’s consider \( m^U > 0 \). We prove this by the definition of \( \theta^U \) (Definition 4). Notice that \( \beta_i^{EQ} \) is not utopian, so \( \sum_{i \in \mathcal{I}} \beta_i^{EQ} D_i < S \).

i. Given a \( \theta < \frac{m^L}{m^U - m^L} \), there exists a \( \beta \neq \beta^{EQ} \) that dominates \( \beta^{EQ} \).
Denote by \( A = \{ i \mid d \theta_i = \beta^{EO} \} \), the set of binding indices. We can construct a solution \( \beta \) by a sufficiently small variable \( z > 0 \) such that \( \beta_i = \beta^{EO} + z \) for any \( i \in A \), and \( \beta = \beta^{EO} + z \) is sufficiently small, it is trivial that \((\beta, \beta)\) is a feasible solution for Model 2. On the other hand, letting \( |I| \) be the cardinality of a set, we have

\[
\sum_{i \in I} [\beta_i - 0(\beta - \beta_i)] = \sum_{i \in I} \beta^{EO}_i - \sum_{i \in I} [\beta_i - 0(\beta - \beta_i)] = \sum_{i \in I} \left[ \beta_i - \beta^{EO}_i \right] - \sum_{i \in I} 0(\beta - \beta_i) = |I| \beta^{EO} - |I| \beta = |I| \beta^{EO} + z.
\]

Hence, we found a feasible solution \( \beta \) dominating \( \beta^{EO} \), for any \( 0 < \frac{m^u}{n-m^v} \).

Given a \( \theta \geq \frac{m^u}{n-m^v} \), \( \beta^{EO} \) is the optimal solution.

Let \((\beta, \beta)\) be any feasible solution. Because \( n - m^v \) variables are binding and \( \beta_i = \beta^{EO}_i \) for the binding variables (in set \( A \)), we have \( \sum_{i \in A} \beta_i = \sum_{i \in A} \beta^{EO}_i \geq \sum_{i \in I} \theta(\beta - \beta_i) \geq (n-m^v)\theta(\beta - \beta^{EO}) \)

In addition, \( \sum_{i \in I} (\beta_i - \beta^{EO}_i) \leq \sum_{i \in I} (\beta_i - \beta^{EO}_i) \leq m^u(\beta - \beta^{EO}) \)

Hence, \( \beta^{EO} \) dominates any feasible solution \( \beta \) and is of course the optimal.

Combining the above two parts of the proof, we have shown that \( \theta^{EO} = \frac{m^u}{n-m^v} \) is the tightest upper-bound by the definition.

Q.E.D.

**Appendix C: Proof of Theorem 2**

**Proof** Here, we prove Theorem 2, the closed-form of \( \beta^{EO} \).

First, if \( \beta^b \leq \beta^o \), let \( \beta^{EO} = (\beta^b, \ldots, \beta^b) \) and \( \beta = \beta^o \). It is trivial to verify that \((\beta^{EO}, \beta)\) is a feasible solution because \( \beta^o \leq \beta^b \leq \beta_i \) for any \( i \in I \). On the other hand, there does not exist another \( \beta^{EO} > \beta^b \) that satisfies the perfect equity criterion, because \( \beta^{EO} D > \beta^b D = S \) is impossible. Hence, \( \beta^{EO} = \beta^b \).

Second, if \( \beta^b > \beta^o \), then \( \beta^{EO} \) cannot be greater than \( \beta^b \). If \( \beta^{EO} > \beta^b \) is true, then for any \( j \in J \), we have \( \beta_j = \beta^{EO}_j > \beta^b_j \) which violates constraint (2b). Letting \( \beta^{EO} = (\beta^o, \ldots, \beta^b) \) and \( \beta = \beta^o \), it is trivial to see that this solution is feasible. Since \( \beta^b \) is the maximum possible \( \beta^{EO} \), \( \beta^{EO} = \beta^b \).

The above two claims complete the proof.

Q.E.D.

**Appendix D: The rationale of Algorithm 1**

**Algorithm 1** The Dominant Efficient Solution

**Input:** the subsets \( I = \bigcup_{i=1}^{j} I_i \) and \( I = \bigcup_{i=1}^{j} I_i \); the parameters \( S, D_0, D_j \), and \( \beta_i \) \((i = 1, \ldots, n; j = 1, \ldots, n_i)\).

**Output:** \( \beta^{EF} = (\beta^{EF}_1, \ldots, \beta^{EF}_n) \)

**Initialization:** let \( \beta_i^{EF} = 0 \) for any \( i \in I \).

1. procedure \textsc{update}(\( S, I, D, D_j, \beta_j \))
2. for \( i = 1 \) to \( n \) do
3. if \( S = 0 \) then
4. break
5. else
6. for \( j = 1 \) to \( n_i \) do
7. \( i = \bigcup_{i=1}^{j} I_i \)
8. if \( |I| \leq S \) then
9. for \( \forall k \in I \) do \( \beta_k^{EF} = \beta_k^{EF} + \beta_j \)
10. for \( \forall k \in I \) do \( \beta_k^{EF} = \beta_k^{EF} + \frac{|I|}{|I|} \)
11. \( S = S - |I_j| D_j \)
12. else
13. for \( \forall k \in I \) do \( \beta_k^{EF} = \beta_k^{EF} + \frac{|I|}{|I|} D_j \)
14. for \( \forall k \in I \) do \( \beta_k^{EF} = \beta_k^{EF} + S/|I| D_j \)
15. \( S = 0 \)
16. if \( S = 0 \) then
17. break

return \( \beta^{EF} \)

Intuitively, the rationale of the algorithm is motivated by that of the greedy method. Specifically, when \( \theta = 0 \), we only focus on the efficiency, that is, maximizing the fill-rates \( \beta_i \)'s. Considering \( \beta_i = \frac{S_i}{D_i} \), the county with the smallest demand \( D_i \) should be filled first since the same amount of food supply generates a higher fill-rate. So we divide country set \( I \) into subsets denoted by \( I_j \). In each subset \( I_j \), all the counties have the same demand such that \( D_{k_i} = D_{k_j} = D_j \) for any \( k_1, k_2 \in I \). Assuming there are \( n \) different subsets, we have \( I = \bigcup_{i=1}^{j} I_j \), which reduces \( n \) counties to \( n \) groups. Without loss of generality, assume \( D_1 \leq D_2 \leq \cdots \leq D_n \). Our algorithm attempts to fill the first group (subset) of counties, then the second, third, until there is no food supply is left-on-hand. Next task is to characterize the order by which the counties in the same group \( I_i \) are filled. Although the counties in the same group have identical demands, they differ in effective demand which is the minimum value of demand and capacity for each county. Therefore, we can further divide \( I_i \) into subsets \( I_j \). In each subset \( I_j \), all the counties have not only the same demand but also the same effective demand such that \( D_{k_i} = D_{k_j} = D_j \) for any \( k_1, k_2 \in I \). Without loss of generality, assume there are \( n_i \) different subsets in group \( I_j \) and \( D_{k_j} = D_{k_j} = D_j \). When filling the counties in the group \( I_i \), however, we do not fill \( I_j \) one by one in ascending order. This is because even if two counties belong to different subsets (i.e., different effective demands) in the group \( I_j \), they still have the same demand. Hence, allocating all the supply exclusively to one subset \( I_j \) is identical to spreading the supply over different \( I_j \)'s simultaneously. However, the second approach is preferred as it favors equity in addition to efficiency. This will benefit the solution when the penalty factor \( \theta \) becomes positive to penalize the inequity.
This procedure is formally delineated in Algorithm 1. However, the summary of how Algorithm 1 fills the subset $I_0$ is given in the following three steps:

i. Initialize $j = 1$

ii. Denote by $I_{r} = \cup_{i=1}^{k_{r}} I_k$ the subsets not yet fully filled. Check whether the remaining supply can increase the fill-rate of all the counties in $I_{r}$ with the same amount $\bar{y}_g$. If yes, allocate food to all the counties in $I_{r}$ to raise the fill-rates by $\bar{y}_g$. If no, then distribute the remaining supply and equally raise the fill-rates for all the counties in $I_{r}$.

iii. Adjust S and $\bar{y}_g$. If the remaining supply S reaches 0, stop; otherwise, $j = j + 1$ and go to step (ii).

Appendix E: Proof of Theorem 3

**Proof** Denote $I_{r} \subset \bar{I}$ the first subset not fully filled, such that $\beta^{EF}_{jk} D_k < D_k$ for any $k \in I_{r}$; otherwise, if $S > \sum_{k \in I_{r}} D_k$, then we have a unique optimal solution $\beta^{EF} = D_k / D_k$ ($\forall i \in \bar{I}$) which means all the counties are filled at their maximum level—the effective demand.

First, letting $\beta$ be the set of all optimal solutions when $\theta = 0$, we show that for any optimal solution $\beta = (\beta_1, \ldots, \beta_n) \in \mathcal{B}$, $\beta_k = \frac{\beta_k^{EF}}{D_k}$ if $k \in I_0$ and $i > r$; $\beta_k = \frac{\beta_k^{EF}}{D_k}$ if $k \in I_r$ and $i < r$. In other words, the only different part between the dominant solution $\beta^{EF}$ and other $\beta$s in $\mathcal{B}$, if exists, is the fill-rates of counties in the group $I_r$. Denoting $I_0 \cup I_r = I_{r'}$ and $J_r = \cup_{j=1}^{k_r} J_r$ because $\beta_k^{EF} = \frac{\beta_k}{D_k}$ for any $k \in J_0$ and $\beta_k^{EF} = 0 \leq \beta_k$ for any $k \in J_r$, we have:

$$
\sum_{k \in J_0} (\beta_k^{EF} - \beta_k)D_k + \sum_{k \in J_r} (\beta_k^{EF} - \beta_k)D_k + \sum_{k \in J_r} (\beta_k^{EF} - \beta_k)D_k = 0
$$

Denote $J_0 = I_0 \cup J_r \cup J_0$. We have:

$$
\sum_{k \in J_0} (\beta_k^{EF} - \beta_k)D_k = \sum_{k \in J_r} (\beta_k^{EF} - \beta_k)D_k = \sum_{k \in J_r} (\beta_k^{EF} - \beta_k)D_k.
$$

The first equation comes from the observation $S < \sum_{k \in I_{r}} D_k$. When there is no penalty for inequity, all the food supply will be shipped since the total effective demand is larger than the supply. On the other hand, notice that $\beta^{EF} \not\in \mathcal{B}$, so $\sum_{k \in \bar{I}} (\beta_k^{EF} - \beta_k) = 0$. However, $\mathcal{I} = J_0 \cup I_r \cup J_0$,

we have $\sum_{k \in J_0} (\beta_k^{EF} - \beta_k) = \sum_{k \in J_r} (\beta_k^{EF} - \beta_k)$. Therefore, if $\sum_{k \in J_0} (\beta_k^{EF} - \beta_k) > 0$, then

$$
\sum_{k \in J_0} (\beta_k^{EF} - \beta_k)D_k \geq D_{r+1} \sum_{k \in J_0} (\beta_k^{EF} - \beta_k) > D_r \sum_{k \in J_0} (\beta_k^{EF} - \beta_k) = D_r \sum_{k \in J_0} (\beta_k^{EF} - \beta_k)
$$

gets contradiction. Hence, $\beta_k = \beta_k^{EF} = 0$ if $k \in J_1$. Furthermore, if $\sum_{k \in J_0} (\beta_k^{EF} - \beta_k) > 0$, we have

$$
\sum_{k \in J_0} (\beta_k^{EF} - \beta_k) = \sum_{k \in J_0} (\beta_k^{EF} - \beta_k) + \sum_{k \in J_0} (\beta_k^{EF} - \beta_k) + D_r \sum_{k \in J_0} (\beta_k^{EF} - \beta_k)
$$

gets contradiction. Hence, $\beta_k = \beta_k^{EF} = D_k / D_k$ if $k \in J_0$ and $\sum_{k \in J_0} (\beta_k^{EF} - \beta_k) = 0$.

Now, based on the results above, we have $\beta_k = \beta_k^{EF}$ when $k \in I / I_r$. On the other hand, by how Algorithm 1 (Greedy method) constructs $\beta^{EF}$, it is obvious max$_{k \in I_r} \beta_k \geq$ max$_{k \in \bar{I}} \beta_k^{EF}$. Therefore, $\beta \geq \beta^{EF}$. When $\theta > 0$, consider the difference between objective function values as follows:

$$
\sum_{k \in I} (\beta_k - \beta_k^{EF}) + (1 + \theta) \sum_{k \in I} (\beta_k^{EF} - \beta_k)
$$

completes the proof.

Q.E.D.

Appendix F: Proof of Theorem 4

**Proof** Because the problem is not utopian, it is easy to see $\theta \neq 0$. Hence, $\theta^l$ is well defined. We prove this theorem by parts based on Definition 6.

i. Given a $\theta > \theta^l = \frac{m^l}{n - m^l}$, there exists a feasible solution $(\beta, \beta^l)$ such that $\sum_{k \in I} \beta_k < \xi$ and it dominates $(\beta^{EF}, \beta^{EF})$ created by Algorithm 1.

Denote $\mathcal{K} = \{I \cap \beta^{EF} = \beta^{EF}, i \in I\}$, the set of counties with the highest level of fill-rate; and $\mathcal{A} = \{I \cap \beta = \beta^{EF} / m \cap \beta / m\}$, the set of counties with minimum demand among those that can be further filled.

Assume $\mathcal{L}$ is not empty. Denoting $\rho = \sum_{k \in \mathcal{L}} D_k$, we construct a solution $\beta$ by a sufficiently small $x > 0$ such that $\beta_i = \beta_i^{EF} - x$ for any $i \in \mathcal{K}$, $\beta_i = \beta^{EF} + x \rho / |A|$ for any $i \in \mathcal{A}$ and $\beta_i = \beta_i^{EF}$ for any $i \in I / (K \cup A)$. Since $x$ is sufficiently small and $S < \sum_{i \in I_{r}} D_i$, it is trivial that $(\beta, \beta^l)$ is a feasible solution for Model 2. Therefore, $\beta = \beta^{EF} - x$, and we have:

$$
\sum_{i \in I} (\beta_i - \beta^{EF}_i) - \sum_{i \in I_0} (\beta_i^{EF} - \beta_i^{EF}_i) = \sum_{i \in I_0} (\beta_i^{EF} - \beta_i^{EF}_i) + \sum_{i \in I_0} (\beta_i^{EF} - \beta_i^{EF}_i)
$$

gets contradiction. Hence, $\beta_k = \beta_k^{EF} = 0$ if $k \in J_1$. Furthermore, if $\sum_{k \in J_0} (\beta_k^{EF} - \beta_k) > 0$, we have

$$
\sum_{k \in J_0} (\beta_k^{EF} - \beta_k) = \sum_{k \in J_0} (\beta_k^{EF} - \beta_k) + D_r \sum_{k \in J_0} (\beta_k^{EF} - \beta_k)
$$

gets contradiction. Hence, $\beta_k = \beta_k^{EF} = D_k / D_k$ if $k \in J_0$ and $\sum_{k \in J_0} (\beta_k^{EF} - \beta_k) = 0$. Now, based on the results above, we have $\beta_k = \beta_k^{EF}$ when $k \in I / I_r$. On the other hand, by how Algorithm 1 (Greedy method) constructs $\beta^{EF}$, it is obvious max$_{k \in I_r} \beta_k \geq$ max$_{k \in \bar{I}} \beta_k^{EF}$. Therefore, $\beta \geq \beta^{EF}$. When $\theta > 0$, consider the difference between objective function values of $\beta^{EF}$ and $\beta$ is

$$
\Delta(\beta) = (1 + \theta) \sum_{i \in I} (\beta_i^{EF} - \beta_i) + \theta \sum_{i \in I} (\beta_i - \beta_i^{EF})
$$

if $\beta \geq \beta^{EF}$ then $\beta^{EF}$ dominates $\beta$ since $\Delta \geq 0$. Now, consider the case $\beta < \beta^{EF}$.
For a $\mathbf{b}$, if $\exists i \in K$ such that $\beta_i < \beta$, we can always find a solution $\mathbf{b}'$ with $\beta_i' = \beta$ but dominating $\mathbf{b}$, i.e., $\Delta(\mathbf{b}') \leq \Delta(\mathbf{b})$. To obtain $\beta_i$, we have to first reduce $\beta_i^{EF}$ (which is equal to $\beta_i^{EF}$) to $\beta_i$ and then further reduce it to $\beta_i$. However, the further reduction does not change $\beta$ but only saves the supply by $D_i(\beta - \beta_i)$. Based on the way Algorithm 1 constructs $\mathbf{b}^{EF}$, the saved supply could at most increase the total fill-rates of $\mathbf{b}$ by $D_i/D_m(\beta - \beta_i)$ which is less than or equal to $\beta - \beta_i$, because $D_i \leq D_k$ for any $k \in \mathcal{L} \cup K$. So the further reduction cannot reduce the value of first term of $\Delta$ and does not change the value of the second term. This implies that a $\mathbf{b}'$ without further reducing the fill-rate of county $i$ from $\beta$ to $\beta_i$ dominates $\mathbf{b}$. Without loss of generality, we assume $\beta_i = \beta, \forall i \in K$.

Let $K_1 = \{(i \in \mathcal{I} | \bar{d}_i < \beta_i^{EF}, i \in \mathcal{I} \}$ and $K_2 = \{(i \in \mathcal{I} | \bar{d}_i > \beta_i^{EF}, i \in \mathcal{I} \}$. Trivially, $K_1 \supseteq K$ notice that $\beta < \beta_i^{EF}$. Furthermore, because $\sum_{i \in K_1} D_i(\beta_i - \beta_i^{EF}) = \sum_{i \in K_2} D_i(\beta_i - \beta_i^{EF})$ by the assumption that the problem is supply-constrained, we observe

$$\sum_{i \in K_1} (\beta_i - \beta_i^{EF}) \leq \sum_{i \in K_1} D_i/D_m(\beta_i^{EF} - \beta_i) = \rho(\beta_i^{EF} - \beta) + \sum_{i \in K_1} D_i/D_m(\beta_i^{EF} - \beta_i),$$

where the second equality comes from the fact that $\beta_i^{EF} - \beta_i = \beta_i^{EF} - \beta$ for any $i \in K$. Hence, we have:

$$\Delta(\mathbf{b}) = (1 + \theta) \sum_{i \in K} (\beta_i^{EF} - \beta_i) + \theta \sum_{i \in K} (\beta_i - \beta_i^{EF})$$

$$= (1 + \theta) \left[ |K| (\beta_i^{EF} - \beta_i) + \sum_{i \in K_1} (\beta_i^{EF} - \beta_i) - \sum_{i \in K_2} (\beta_i^{EF} - \beta_i) \right]$$

$$+ n \theta (\beta_i^{EF} - \beta_i)$$

$$\geq (1 + \theta) \left[ (|K| - \rho) (\beta_i^{EF} - \beta_i) + \sum_{i \in K_1} (1 - D_i/D_m)(\beta_i^{EF} - \beta_i) \right]$$

$$+ n \theta (\beta_i^{EF} - \beta_i)$$

$$\geq \frac{n m^2 t}{n - m t} (\beta_i^{EF} - \beta_i) - \frac{n m}{n - m t} (\beta_i^{EF} - \beta_i) = 0$$

Therefore, $\beta_i^{EF}$ is the optimal solution. For the case when $\mathcal{L}$, we have $K_2 = \emptyset$. The proof of this case is similar and more straightforward.

In summary, the above two parts together complete the proof.

Q.E.D.

**Appendix G: Proof of Theorem 5**

**Proof** We prove Theorem 5 by parts.

1. **Constraints (15b) and (15c) ⇒ $x_i \leq C_i$.**

   First, assuming $x_i > C_i^{EF}$, Constraint (15c) is no longer feasible. This is because $C_i \leq C_i^{EF} \leq \cdots \leq C_R$. If $x_i > C_i^{EF}$, then $y_r = 0$ for any $r \leq r'$, and as a result,

$$\sum_{r} y_r \leq \frac{R - \sum_{r} y_r}{R} < \frac{R - (1 - 1/R)}{R} < \eta_i,$$

gets contradiction.

Hence, $x_i \leq C_i^{EF}$. On the other hand, if $x_i > (1 + \delta)C_i$, it is obvious that Constraint (15b) is infeasible. Therefore, Constraints (15b) and (15c) imply $x_i \leq C_i$.

2. **$x_i \leq C_i \Rightarrow$ Constraints (15b) and (15c)**

   First, $x_i \leq C_i^{EF}$ and $C_i \leq C_i^{EF} \leq \cdots \leq C_R$ imply that

$$\sum_{r \in \mathcal{R}} y_r = \frac{n}{R} \sum_{r \in \mathcal{R}} y_r \leq \frac{R}{n} \sum_{r \in \mathcal{R}} y_r = \eta_i$$

by letting $y_r = 1$ for any $r \geq r'$ and $y_r = 0$ otherwise. So, Constraint (15c) holds. On the other hand, $x_i \leq (1 + \delta)C_i$ and $C_i \leq C_i^{EF} \leq \cdots \leq C_R$ imply that Constraint (15b) holds by letting $y_r = 1$ for any $r \geq r'$ and $y_r = 0$ otherwise.

The above two parts show that the feasible regions of $x_i$ in Models (15) and (1) are exactly the same, which completes the proof.

Q.E.D.

**Appendix H: Performance of the models under other measures of inequity**

In this section, we report the comparison between the two models based on the remaining two measures of inequity, variance and mean absolute deviation:

\[ \text{Variance} = \frac{\sum_{i \in \mathcal{L}} (\bar{y} - \bar{y})^2}{n} \quad (H.1) \]

\[ \text{Mean Absolute Deviation} = \frac{\sum_{i \in \mathcal{L}} |y_i - \bar{y}|}{n} \quad (H.2) \]

To obtain the efficient frontier we need to convert the measures to return equity. However, while Gini Coefficient returns a value between 0 and 1, Variance and Mean Absolute Deviation (MAD) return a non-negative value which is not necessarily between 0 and 1. Therefore, similar to Orgut et al. (2016b), we first standardize our data for each point in each measure to be between 0 and 1 in its standard form by subtracting each value from the minimum and dividing it by the distance of minimum and maximum. Our equity then is calculated as one minus the standardized values.

Figure H1 shows the efficient frontiers of both our model and ORG model using $\bar{P}$’s to calculate the two equity measures Variance and MAD. However, Orgut et al. (2016b) defines $E_i$ capturing the distance from absolute equity for each county as

\[ E_i = \frac{\sum_{i \in \mathcal{L}} x_i}{\sum_{i \in \mathcal{L}} D_i} \leq K. \]

Figure H2 shows the comparison between our model and ORG using this term instead of $\bar{P}$’s to calculate equity.

As both Figures H1 and H2 suggest, regardless of using $\bar{P}$’s or $E_i$’s and regardless of the measure used to calculate the equity in the allocation of the food supply, our model reports a superior efficient frontier in majority of the cases. This demonstrates the robustness of our model compared to our benchmark since our solutions strictly satisfy the Pareto optimality.
Figure H1. Efficient frontiers using Variance and MAD to calculate equity based on $\beta_i$'s.
Figure H2. Efficient frontiers using Variance and MAD to calculate equity using $E_i$ defined in ORG.