Fractional Poisson Analysis in Dimension one

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Abstract

In this paper, we use a biorthogonal approach (Appell system) to construct and characterize the spaces of test and generalized functions associated to the fractional Poisson measure $\pi_{\lambda,\beta}$, that is, a probability measure in the set of natural (or real) numbers. The Hilbert space $L^2(\pi_{\lambda,\beta})$ of complex-valued functions plays a central role in the construction, namely, the test function spaces $(\mathbb{N})^\kappa_{\pi_{\lambda,\beta}}$, $\kappa \in [0, 1]$ is densely embedded in $L^2(\pi_{\lambda,\beta})$. Moreover, $L^2(\pi_{\lambda,\beta})$ is also dense in the dual $((\mathbb{N})^\kappa_{\pi_{\lambda,\beta}})' = (\mathbb{N})^{-\kappa}_{\pi_{\lambda,\beta}}$. Hence, we obtain a chain of densely embeddings $(\mathbb{N})^\kappa_{\pi_{\lambda,\beta}} \subset L^2(\pi_{\lambda,\beta}) \subset (\mathbb{N})^{-\kappa}_{\pi_{\lambda,\beta}}$. The characterization of these spaces is realized via integral transforms and chain of spaces of entire functions of different types and order of growth. Wick calculus extends in a straightforward manner from Gaussian analysis to the present non-Gaussian framework. Finally, in Appendix B we give an explicit relation between (generalized) Appell polynomials and Bell polynomials.

Keywords: Fractional Poisson measure, Appell polynomials, test functions, generalized functions, Wick product, Bell polynomials.

Contents

1 Introduction 2

2 Fractional Poisson Measure 3

3 Generalized Appell Polynomials 5
1 Introduction

In 1830, the Poisson process was named after the French mathematician Simeon Denis Poisson which describes the number of occurrences of a certain event occurring in a given time period, given the average number of times the event occurs over that time period. At present, it is one of the most useful statistical distribution applied in a wide range of fields, including astronomy, business, finance, medicine and sports, just to name a few.

In 2000, O. N. Repin and A. I. Saichev [24] initiated the study of fractional Poisson process as a process with long-memory effect which results from the non-exponential waiting time probability distribution function. The fractional Poisson process is a natural generalization of the Poisson process. In this case, a parameter $\beta$, $0 < \beta \leq 1$, is introduced in the probability distribution function of the fractional Poisson process wherein at $\beta = 1$, the fractional Poisson process becomes the Poisson process.

In this work, the starting point is the marginal distribution of the fractional Poisson process as a measure on $N_0$ or on $\mathbb{R}$. Moreover, we investigate an abstract fractional Poisson measure $\pi_{\lambda,\beta}$ which gives rise to a fractional Poisson process $N_{\lambda,\beta}$ with time dependent rate. The two key properties of the fractional Poisson measure are its analytic Laplace transform $l_{\pi_{\lambda,\beta}}$ and to assign positive measure to a nonempty set of its support, see Remark 3.2 below. The polynomials associated with the Poisson measure are the well-known orthogonal system of Charlier polynomials. However, the polynomials associated with the fractional Poisson measure, a generalization to the Charlier polynomials, are not orthogonal. Thus a biorthogonal approach (see [15, 12, 14, 9]) to the fractional Poisson measure will be constructed, which involves the system of Appell polynomials and the dual Appell system of the measure $\pi_{\lambda,\beta}$. More precisely, the Appell polynomials are generated by the normalized (or Wick) exponential

$$e_{\pi_{\lambda,\beta}}(z; x) := \frac{e^{xz}}{l_{\pi_{\lambda,\beta}}(z)} = \frac{e^{xz}}{E_\beta(\lambda(e^z - 1))}, \quad z \in \mathbb{C}, \; x \in \mathbb{R},$$

(1.1)

where $E_\beta$ is the Mittag-Leffler function, see (2.2) for the definition. Interestingly, these Appell polynomials may be expressed in terms of the Bell polynomials, see Appendix B.2. These polynomials play a key role in the construction of test function spaces associated to $\pi_{\lambda,\beta}$.
Below we use a modification of \( e_{\nu}(z) \) (see (3.2)) adapted to the present framework, which produce the generalized Appell polynomials. On the other hand, to construct and describe generalized functions we need the so-called generalized dual Appell system. The two systems thus introduced, that is, generalized Appell polynomials and the generalized dual Appell system are biorthogonal with respect to \( L^2(\pi_{l,\beta}) \).

The paper is organized as follows. In Section 2, we introduce the fractional Poisson measure. We show that the fractional Poisson measure is a mixture of Poisson measures with a certain probability measure \( \nu_{l}\beta \) on \( \mathbb{R}_+ \). In Section 3, we define and emphasize certain properties of the generalized Appell polynomials. The generalized dual Appell system in then introduced in Section 4 which forms a biorthogonal system together with the generalized Appell polynomials, cf. Proposition 4.2. The construction of test and generalized functions associated to \( \pi_{l,\beta} \) are presented in Section 5 and in Section 6 their characterization via integral transforms. Finally, in Section 7, we introduce the Wick product (and related calculus) in the larger space of generalized functions associated to \( \pi_{l,\beta} \), that is, \( (N_{l})^{-1}_{\pi_{l,\beta}} \). In Appendix A and B, we show the equivalence of certain norms in the space of entire functions (Appendix A) and express the generalized Appell polynomials in terms of Bell polynomials (Appendix B).

## 2 Fractional Poisson Measure

The Poisson measure (probability distribution) \( \pi_{l} \) on \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \) (or \( \mathbb{R} \)) with rate \( l > 0 \) is defined on the \( \sigma \)-algebra of all subsets of \( \mathbb{N} \), denoted by \( \mathcal{P}(\mathbb{N}_0) \), as

\[
\pi_{l}(B) := \sum_{k \in B} \frac{\lambda^k}{k!} e^{-\lambda} = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda} \delta_k(B), \quad B \in \mathcal{P}(\mathbb{N}_0),
\]

where \( \delta_k \) is the Dirac measure at \( k \in \mathbb{N}_0 \). In particular, for every \( k \in \mathbb{N} \) and \( B = \{k\} \in \mathcal{P}(\mathbb{N}_0) \), we have

\[
\pi_{l}(\{k\}) = \frac{\lambda^k}{k!} e^{-\lambda}.
\]

It is easy to obtain the Laplace transform \( l_{\pi_{l}} \) of the measure \( \pi_{l} \), namely, for any \( s \in \mathbb{R} \), we have

\[
l_{\pi_{l}}(s) := \int_{\mathbb{R}} e^{sx} d\pi_{l}(x) = e^{-\lambda} \sum_{k=0}^{\infty} e^{sk} \frac{\lambda^k}{k!} = \exp(\lambda(e^s - 1)).
\]

**Remark 2.1.** The measure \( \pi_{\lambda t} \), \( t > 0 \), corresponds to the marginal distribution of a standard Poisson process \( N_{\lambda} = (N_{\lambda}(t))_{t \geq 0} \) with parameter \( \lambda t > 0 \) defined on a probability space \((\Omega, \mathcal{F}, P)\). More precisely, we have

\[
\pi_{\lambda t}(\{k\}) = P(N_{\lambda}(t) = k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}, \quad k \in \mathbb{N}_0.
\]

The number \( \pi_{\lambda t}(\{k\}) \) is the probability that \( k \) events occur in the time interval of length \( t \).

The Laplace transform \( l_{\pi_{l}} \) of \( \pi_{l} \) may be extended to complex arguments \( z \in \mathbb{C} \) and we obtain

\[
l_{\pi_{l}}(z) = \int_{\mathbb{R}} e^{zx} d\pi_{l}(x) = e^{-\lambda} \sum_{k=0}^{\infty} e^{zk} \frac{\lambda^k}{k!} = \exp(\lambda(e^z - 1)). \tag{2.1}
\]

Now we would like to introduce the fractional Poisson measure (fPm). At first we introduce the Mittag-Leffler function \( E_{\beta} \) of parameter \( \beta \in (0, 1] \). The Mittag-Leffler function is an entire function defined on the complex plane by the power series

\[
E_{\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\beta n + 1)}, \quad z \in \mathbb{C}. \tag{2.2}
\]
The Mittag-Leffler function plays the same role for the fPm as the exponential function plays for Poisson measure.

For $0 < \beta \leq 1$, the fractional Poisson measure $\pi_{\lambda,\beta}$ on $\mathbb{N}_0$ (or $\mathbb{R}$) with rate $\lambda > 0$ is defined for any $B \in \mathcal{P}(\mathbb{N}_0)$ by

$$\pi_{\lambda,\beta}(B) := \sum_{k \in B} \frac{\lambda^k}{k!} E^{(k)}_{\beta}(-\lambda) = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} E^{(k)}_{\beta}(-\lambda) \delta_k(B),$$

where $E^{(k)}_{\beta}(z) := \frac{d^k}{dz^k} E_{\beta}(z)$ is the $k$th derivative of the Mittag-Leffler $E_{\beta}$ function. In particular, if $B = \{k\} \in \mathcal{P}(\mathbb{N}_0)$, we obtain

$$\pi_{\lambda,\beta}(\{k\}) := \frac{\lambda^k}{k!} E^{(k)}_{\beta}(-\lambda).$$

The Laplace transform of the measure $\pi_{\lambda,\beta}$ is given for any $z \in \mathbb{C}$ by

$$l_{\pi_{\lambda,\beta}}(z) = \int_{\mathbb{R}} e^{zx} d\pi_{\lambda,\beta}(x) = \sum_{k=0}^{\infty} \frac{(e^z\lambda)^k}{k!} E^{(k)}_{\beta}(-\lambda) = E_{\beta}(\lambda(e^z - 1)). \quad (2.3)$$

**Remark 2.2.** The measure $\pi_{\lambda,\beta}$ corresponds to the marginal distribution of the fractional Poisson process $N_{\lambda,\beta} = (N_{\lambda,\beta}(t))_{t \geq 0}$ with rate $\lambda t^\beta > 0$ defined on a probability space $(\Omega, \mathcal{F}, P)$. Thus, we obtain

$$\pi_{\lambda,\beta}(\{k\}) = P(N_{\lambda,\beta}(t) = k) = \frac{(\lambda t^\beta)^k}{k!} E^{(k)}_{\beta}(-\lambda t^\beta), \quad k \in \mathbb{N}_0.$$  

**Remark 2.3.** The fractional Poisson process $N_{\lambda,\beta}$ was proposed by O. N. Repin and A. I. Saichev [24]. Since then, it was studied and applied by many authors, see N. Laskin [16], F. Mainardi et al. [18, 19, 7], V. V. Uchaikin et al. [26], L. Beghin and E. Orsingher [2] M. Politi et al. [23], M. M. Meerschaert et al. [21], R. Biard and B. J. Saussereau [4] and references therein.

An interesting property of the fPm $\pi_{\lambda,\beta}$ is given by a relation with the Poisson measure $\pi_{\lambda}$, namely, $\pi_{\lambda,\beta}$ is a mixture of Poisson measures with respect to a probability measure $\nu_{\beta}$ on $\mathbb{R}_+ := [0, \infty)$. That probability measure $\nu_{\beta}$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}_+$ with a probability density $M_{\beta}$. The Laplace transform of the density $M_{\beta}$ is given by, (see [20, Eq. 4.10] or [9, Cor. A.5])

$$\int_0^\infty e^{-z\tau} d\nu_{\beta}(\tau) = \int_0^\infty e^{-z\tau} M_{\beta}(\tau) d\tau = E_{\beta}(-z), \quad \forall z \in \mathbb{C}.$$ 

We have the following lemma which gives the fPm as a mixture of Poisson measures with different rates.

**Lemma 2.4.** For $0 < \beta \leq 1$, the fPm $\pi_{\lambda,\beta}$ is an integral (or mixture) of Poisson measure $\pi_{\lambda}$ with respect to the probability measure $\nu_{\beta}$, i.e.,

$$\pi_{\lambda,\beta} = \int_0^\infty \pi_{\lambda\tau} d\nu_{\beta}(\tau), \quad \forall \lambda > 0. \quad (2.4)$$

**Proof.** Denote the right hand side of (2.4) by $\mu := \int_0^\infty \pi_{\lambda\tau} M_{\beta}(\tau) d\tau$. We compute the Laplace transform of $\mu$ and use Fubini’s theorem to obtain

$$\int_0^\infty e^{zx} d\mu(x) = \int_0^\infty e^{zx} \int_0^\infty d\pi_{\lambda\tau}(x) M_{\beta}(\tau) d\tau$$

$$= \int_0^\infty \left( \int_0^\infty e^{zx} d\pi_{\lambda\tau}(x) \right) M_{\beta}(\tau) d\tau$$

$$= \int_0^\infty e^{\tau(e^z - 1)} M_{\beta}(\tau) d\tau$$

$$= E_{\beta}(\lambda(e^z - 1)).$$
Thus, we conclude that both the Laplace transforms of $\mu$ and $\pi_{\lambda,\beta}$ (cf. (2.3)) coincides. The result follows by the uniqueness of the Laplace transform.

**Theorem 2.5** (Moments of $\pi_{\lambda,\beta}$, cf. [17]). The fPm $\pi_{\lambda,\beta}$ has moments of all order. More precisely, the $n$th moment of the measure $\pi_{\lambda,\beta}$ is given by

$$M_{\lambda,\beta}(n) := \int_{\mathbb{R}} x^n \, d\pi_{\lambda,\beta}(x) = \sum_{m=0}^{n} \frac{m!}{\Gamma(m\beta+1)} S(n,m) \lambda^m,$$

where $S(n,m)$ is the Stirling number of the second kind, see Definition B.2, on page 26.

Here are the first few moments of the measure $\pi_{\lambda,\beta}$,

$$M_{\lambda,\beta}(0) = 1, \quad M_{\lambda,\beta}(1) = \frac{\lambda}{\Gamma(\beta+1)},$$
$$M_{\lambda,\beta}(2) = \frac{\lambda}{\Gamma(\beta+1)} + \frac{2\lambda^2}{\Gamma(2\beta+1)},$$
$$M_{\lambda,\beta}(3) = \frac{\lambda}{\Gamma(\beta+1)} + \frac{2\lambda^2}{\Gamma(2\beta+1)} + \frac{6\lambda^3}{\Gamma(3\beta+1)},$$
$$M_{\lambda,\beta}(4) = \frac{\lambda}{\Gamma(\beta+1)} + \frac{2\lambda^2}{\Gamma(2\beta+1)} + \frac{14\lambda^2}{\Gamma(3\beta+1)} + \frac{24\lambda^4}{\Gamma(4\beta+1)}.$$

When $\beta = 1$, these moments become the moments of the Poisson measure $\pi_{\lambda}$:

$$M_{\lambda,1}(0) = 1, \quad M_{\lambda,1}(1) = \lambda, \quad M_{\lambda,1}(2) = \lambda + \lambda^2, \quad M_{\lambda,1}(3) = \lambda + 3\lambda^2 + \lambda^3, \quad M_{\lambda,1}(4) = \lambda + 7\lambda^2 + 6\lambda^3 + \lambda^4.$$

### 3 Generalized Appell Polynomials

In this section we introduce the system of generalized Appell polynomials associated with the fPm $\pi_{\lambda,\beta}$ in $\mathbb{R}$. Our first concern is the analytic property of the Laplace transform given in (2.3), that is,

$$l_{\pi_{\lambda,\beta}}(z) = E_\beta(\lambda(e^z - 1)), \quad z \in \mathbb{C}.$$

In fact, $l_{\pi_{\lambda,\beta}}(\cdot)$ is the composition of two entire functions, thus it is entire.

The analytic property of the function $l_{\pi_{\lambda,\beta}}(\cdot)$ is equivalent and characterized by the following proposition.

**Proposition 3.1.** Let $\pi_{\lambda,\beta}$ be the fPm in $\mathbb{R}$. The following statements are equivalent:

1. $\exists C, K > 0$ such that $\forall n \in \mathbb{N}_0$, $| \int_{\mathbb{R}} x^n \, d\pi_{\lambda,\beta}(x) | < n! C^n K$,

2. $\exists \varepsilon > 0$, such that $\int_{\mathbb{R}} e^{\varepsilon |x|} \, d\pi_{\lambda,\beta}(x) < \infty$.

**Proof.** 1. $\Rightarrow$ 2. Let $\pi_{\lambda,\beta}$ be the fPm in $\mathbb{R}$. The Taylor expansion of $l_{\pi_{\lambda,\beta}}(\cdot)$ is given by

$$l_{\pi_{\lambda,\beta}}(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \left[ \frac{d^n}{dz^n} l_{\pi_{\lambda,\beta}}(z) \right]_{z=0}, \quad z \in \mathbb{C}.$$
Then by definition of $l_{\pi,\lambda,\beta}(\cdot)$, we have

$$l_{\pi,\lambda,\beta}(z) = \int_{\mathbb{R}} e^{zx} \, d\pi_{\lambda,\beta}(x) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \int_{\mathbb{R}} x^n \, d\pi_{\lambda,\beta}(x).$$

Now, note that

$$\int_{\mathbb{R}} e^{|x|} \, d\pi_{\lambda,\beta}(x) = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} \int_{\mathbb{R}} |x|^n \, d\pi_{\lambda,\beta}(x).$$

By the Cauchy-Schwarz inequality, we have

$$\int_{\mathbb{R}} |x|^n \, d\pi_{\lambda,\beta}(x) \leq \left( \int_{\mathbb{R}} x^{2n} \, d\pi_{\lambda,\beta}(x) \right)^{1/2}$$

and so

$$\int_{\mathbb{R}} e^{|x|} \, d\pi_{\lambda,\beta}(x) \leq \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} \left( \int_{\mathbb{R}} x^{2n} \, d\pi_{\lambda,\beta}(x) \right)^{1/2}.$$

By the hypothesis, we have

$$\int_{\mathbb{R}} x^n \, d\pi_{\lambda,\beta}(x) \leq \left| \int_{\mathbb{R}} x^n \, d\pi_{\lambda,\beta}(x) \right| < n! C^n K$$

for all $n \in \mathbb{N}_0$, therefore

$$\int_{\mathbb{R}} x^{2n} \, d\pi_{\lambda,\beta}(x) \leq (2n)! C^{2n} K.$$

Since $(2n)! \leq 2^{2n}(n!)^2$ for all $n \in \mathbb{N}_0$, we have

$$\int_{\mathbb{R}} x^{2n} \, d\pi_{\lambda,\beta}(x) \leq 2^{2n}(n!)^2 C^{2n} K.$$

Thus,

$$\int_{\mathbb{R}} e^{|x|} \, d\pi_{\lambda,\beta}(x) \leq \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} \left( \int_{\mathbb{R}} x^{2n} \, d\pi_{\lambda,\beta}(x) \right)^{1/2}$$

$$\leq \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} 2^{2n}(n!)^2 C^n \sqrt{K}$$

$$= \sqrt{K} \sum_{n=0}^{\infty} (2C\varepsilon)^n$$

which is finite provided $\varepsilon$ is chosen such that $2C\varepsilon < 1$.

2. $\Rightarrow$ 1. Suppose that there exists $\varepsilon > 0$ and $K_\varepsilon > 0$ such that

$$\int_{\mathbb{R}} e^{\varepsilon|x|} \, d\pi_{\lambda,\beta}(x) = K_\varepsilon.$$

Then

$$\sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} \int_{\mathbb{R}} |x|^n \, d\pi_{\lambda,\beta}(x) = K_\varepsilon.$$

This implies that each term in this series is less than or equal to $K_\varepsilon$, that is

$$\frac{\varepsilon^n}{n!} \int_{\mathbb{R}} |x|^n \, d\pi_{\lambda,\beta}(x) \leq K_\varepsilon.$$
Later on, we consider the Taylor series function spaces (to be introduced in Section 5) in $L^2(\pi_{\lambda,\beta})$, that is, the space of complex-valued measurable functions whose modulus square is integrable with respect to $\pi_{\lambda,\beta}$, see [13].

**Remark 3.2.** For every nonempty open set $O \subset \mathbb{R}$ such that $\mathbb{N} \cap O \neq \emptyset$ we have $\pi_{\lambda,\beta}(O) > 0$.

We define the following entire function $f$ on a neighborhood of $0 \in \mathbb{C}$ by

$$f(z) := \log(1 + z).$$

Recall from (1.1), the Wick exponential with respect to the measure $\pi_{\lambda,\beta}$ defined by

$$e_{\pi_{\lambda,\beta}}(z; \cdot) : \mathbb{R} \to \mathbb{C}, x \mapsto e_{\pi_{\lambda,\beta}}(z; x) = \frac{e^{xz}}{l_{\pi_{\lambda,\beta}}(z)} = \frac{e^{xz}}{E_{\beta}(\lambda(e^z - 1))}. \quad (3.1)$$

Since $l_{\pi_{\lambda,\beta}}(0) = 1$, there is a neighborhood $V$ of $0 \in \mathbb{C}$ where $e_{\pi_{\lambda,\beta}}(f(\cdot); x)$ is given by

$$e_{\pi_{\lambda,\beta}}(f(z); x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} C_n^{\lambda,\beta}(x), \quad \forall z \in V, \quad (3.2)$$

with

$$C_n^{\lambda,\beta}(x) = \left. \frac{d^n}{dz^n} e_{\pi_{\lambda,\beta}}(f(z); x) \right|_{z=0}. \quad (3.3)$$

These functions $C_n^{\lambda,\beta}(\cdot), n \in \mathbb{N}$, are polynomials of degree $n$ which are also known as the generalized Appell polynomials. The set $\{C_n^{\lambda,\beta}(\cdot), n \in \mathbb{N}_0\}$ is called the generalized Appell polynomial system associated with the fPm $\pi_{\lambda,\beta}$ and we denote it by $\mathbb{P}_{\pi_{\lambda,\beta}}$. The following are the first four generalized Appell polynomials $C_n^{\lambda,\beta}(x)$ associated with the fPm:

- $C_0^{\lambda,\beta}(x) = 1$,
- $C_1^{\lambda,\beta}(x) = x - \frac{\lambda}{\Gamma(\beta + 1)}$,
- $C_2^{\lambda,\beta}(x) = x^2 - \left(\frac{2\lambda}{\Gamma(\beta + 1)} + 1\right)x + \frac{2\lambda^2}{\Gamma(2\beta + 1)} + 2 \left(\frac{\lambda}{\Gamma(\beta + 1)}\right)^2$,
- $C_3^{\lambda,\beta}(x) = x^3 - 3 \left(\frac{\lambda}{\Gamma(\beta + 1)} + 1\right)x^2 + \left[6 \left(\frac{\lambda}{\Gamma(\beta + 1)}\right)^2 - \frac{6\lambda^2}{\Gamma(2\beta + 1)} + \frac{3\lambda}{\Gamma(\beta + 1)} + \frac{2\lambda}{\Gamma(\beta + 1)}\right]x$
  \[= -\frac{6\lambda^3}{\Gamma(3\beta + 1)} + \frac{12\lambda^3}{\Gamma(\beta + 1)\Gamma(2\beta + 1)} - 6 \left(\frac{\lambda}{\Gamma(\beta + 1)}\right)^3. \]

At $\beta = 1$ these polynomials become the well known Charlier polynomials, i.e.,

- $C_0^{\lambda,1}(x) = 1$,
- $C_1^{\lambda,1}(x) = x - \lambda$,
- $C_2^{\lambda,1}(x) = x^2 - (1 + 2\lambda)x + \lambda^2$,
- $C_3^{\lambda,1}(x) = x^3 - (3 + 3\lambda)x^2 + (2 + 3\lambda + 3\lambda^2)x - \lambda^3. \quad (3.4)$

Later on, we consider the Taylor series

$$e_{\pi_{\lambda,\beta}}(z; x) = \frac{e^{xz}}{l_{\pi_{\lambda,\beta}}(z)} = \sum_{n=0}^{\infty} \frac{x^n}{n!} A_n^{\lambda,\beta}(x), \quad (3.5)$$
for all $z \in W$, a neighborhood of zero in $\mathbb{C}$. The functions $A_n^{\lambda,\beta}(\cdot)$, $n \in \mathbb{N}$, are polynomials of degree $n$ which are also known as the Appell polynomials. In Appendix B.2, we obtain the explicit form of the Appell polynomials generated by the fP\(m\) in terms of the Bell polynomials, see Theorem B.12.

The next proposition summarizes the most important properties of the polynomials $C_n^{\lambda,\beta}(\cdot)$, $n \in \mathbb{N}_0$.

**Proposition 3.3.** For any $x, y \in \mathbb{R}$, the polynomials $C_n^{\lambda,\beta}(\cdot)$, $n \in \mathbb{N}$, satisfy the following properties

(P1) \[ C_n^{\lambda,\beta}(x) = \sum_{m=0}^{n} \frac{A^m_m}{m!} A_m^\lambda(x), \text{ where } A_0^\lambda := 1 \text{ and } A_n^\lambda := \sum_{l_1+\ldots+l_m=n} \frac{(-1)^{n+m}}{l_1! \ldots l_m!}, l_i \in \{1, \ldots, n\} \text{ for all } i = 1, \ldots, m. \]

(P2) \[ x^n = \sum_{k=0}^{n} \sum_{m=0}^{k} \binom{n}{k} \frac{B^m_m}{m!} C_m^{\lambda,\beta}(x) M_{\lambda,\beta}(n-k), \text{ where } B_0^m := 1 \text{ and } B_k^m := \sum_{l_1+\ldots+l_m=k} \frac{k!}{l_1! \ldots l_m!}. \]

(P3) \[ C_n^{\lambda,\beta}(x+y) = \sum_{k+l+m=n} \frac{n!}{k!l!m!} C_k^{\lambda,\beta}(x) C_l^{\lambda,\beta}(y) \tilde{M}_{\lambda,\beta}(m), \text{ where } \tilde{M}_{\lambda,\beta}(m) := \frac{m! \lambda^m}{\Gamma(m\beta + 1)}. \]

(P4) \[ C_n^{\lambda,\beta}(x+y) = \sum_{k=0}^{n} \binom{n}{k} C_k^{\lambda,\beta}(x) (y)_{n-k}, \text{ where } (y)_m \text{ are the falling factorials, for all } m \in \mathbb{N}_0. \]

(P5) $\mathbb{E}(C_n^{\lambda,\beta}(\cdot)) = \delta_{n,0}$, where $\delta_{n,m}$ is the Kronecker delta and $\mathbb{E}(\cdot)$ is the expectation with respect to the measure $\pi_{\lambda,\beta}$.

(P6) For every $\varepsilon > 0$, there exists $C_\varepsilon, \sigma_\varepsilon > 0$ such that

\[ |C_n^{\lambda,\beta}(x)| \leq C_\varepsilon n! \sigma_\varepsilon^{-n} e^{\varepsilon |x|}. \]

**Proof.** Let $C_n^{\lambda,\beta}(x)$, $n \in \mathbb{N}_0$, be the generalized Appell polynomials generated by the measure $\pi_{\lambda,\beta}$ and let $x, y \in \mathbb{R}$ be given.

(P1) In view of equation (3.5), we have

\[ e_{\pi_{\lambda,\beta}}(f(z); x) := \mathbb{E}_{\pi_{\lambda,\beta}}(f(z)) = \frac{e^{\mathbb{E}(f(z))}}{\mathbb{E}_{\pi_{\lambda,\beta}}(f(z))} = \sum_{m=0}^{\infty} \frac{f(z)^m}{m!} A_m^{\lambda,\beta}(x), \]

for any $z \in V$ (neighborhood of zero in $\mathbb{C}$). Denote $A_l$ the polynomials of degree $l$ generated by $f(z)$, i.e.,

\[ f(z) = \log(1 + z) = \sum_{l=0}^{\infty} \frac{z^l}{l!} A_l \]

where

\[ A_l = \left. \frac{d^l}{dz^l} f(z) \right|_{z=0} = (-1)^{l+1}(l-1)! \]
for all \( l > 0 \) and \( A_0 = 0 \). Hence,
\[
\sum_{n=0}^{\infty} \frac{z^n}{n!} \lambda_n^{\lambda, \beta}(x) = A_0^{\lambda, \beta}(x) + \sum_{m=1}^{\infty} \frac{1}{m!} \left( \sum_{l=0}^{\infty} \frac{z^l}{l!} A_l \right)^m A_m^{\lambda, \beta}(x)
\]
\[
= A_0^{\lambda, \beta}(x) + \sum_{m=1}^{\infty} \frac{1}{m!} \sum_{n=m}^{\infty} \frac{z^n}{n!} \sum_{l_1+\ldots+l_m=n} \frac{n!}{l_1! \ldots l_m!} \prod_{i=1}^{m} A_i A_m^{\lambda, \beta}(x)
\]
\[
= A_0^{\lambda, \beta}(x) + \sum_{m=1}^{\infty} \frac{1}{m!} \sum_{n=m}^{\infty} \frac{z^n}{n!} \sum_{l_1+\ldots+l_m=n} \frac{(-1)^{n+m} n!}{l_1! \ldots l_m!} A_m^{\lambda, \beta}(x)
\]
\[
= \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{m=0}^{n} \frac{A_m}{m!} A_m^{\lambda, \beta}(x),
\]
where \( A_0 = 1 \) and \( l_i \in \{1, \ldots, n\} \) for all \( i = 1, \ldots, m \). The result follows by comparing both sides of the equation.

**P2** Note that \( g(z) = e^z - 1 \) is the inverse of \( f(z) = \log(1 + z) \). Similar as in (P1), we use equation (3.1) such that for any \( z \in V \) (neighborhood of zero in \( \mathbb{C} \)), we have
\[
\frac{e^{xz}}{l_{\pi, \beta}(z)} = \sum_{m=0}^{\infty} \frac{g(z)^m}{m!} C_m^{\lambda, \beta}(x).
\]
Note that
\[
g(z) = e^z - 1 = \sum_{l=1}^{\infty} \frac{z^l}{l!}.
\]
Thus,
\[
\sum_{n=0}^{\infty} \frac{z^n}{n!} A_n^{\lambda, \beta}(x) = C_0^{\lambda, \beta}(x) + \sum_{m=1}^{\infty} \frac{1}{m!} \left( \sum_{l=1}^{\infty} \frac{z^l}{l!} \right)^m C_m^{\lambda, \beta}(x)
\]
\[
= C_0^{\lambda, \beta}(x) + \sum_{m=1}^{\infty} \frac{1}{m!} \sum_{n=m}^{\infty} \frac{z^n}{n!} \sum_{l_1+\ldots+l_m=n} \frac{n!}{l_1! \ldots l_m!} C_m^{\lambda, \beta}(x)
\]
\[
= \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{m=0}^{n} \frac{A_m}{m!} C_m^{\lambda, \beta}(x),
\]
where \( B_0 = 1 \) and \( l_i \in \{1, \ldots, n\} \) for all \( i = 1, \ldots, m \). Comparing both sides of the equation, we obtain
\[
A_n^{\lambda, \beta}(x) = \sum_{m=0}^{n} \frac{B_m}{m!} C_m^{\lambda, \beta}(x). \tag{3.6}
\]
Now, we use the equality for any \( z \in W \) (neighborhood of zero in \( \mathbb{C} \)),
\[
e^{xz} = e_{\pi, \beta}(z; x) l_{\pi, \beta}(z)
\]
so that
\[
\sum_{n=0}^{\infty} \frac{z^n}{n!} A_n^{\lambda, \beta}(x) \cdot \sum_{k=0}^{n} \frac{z^k}{k!} M_{\lambda, \beta}(n)
\]
\[
= \sum_{n=0}^{\infty} \frac{z^n}{n!} \left( \sum_{k=0}^{n} \binom{n}{k} A_k^{\lambda, \beta}(x) M_{\lambda, \beta}(n - k) \right).
\]
This implies that

\[ z^n = \sum_{k=0}^{n} \binom{n}{k} A_k^{\lambda,\beta}(x) M_{\lambda,\beta}(n-k). \]  

(3.7)

The result follows by applying equation (3.6) to (3.7).

(P3) By definition of the Wick exponential,

\[ e_{\pi_{\lambda,\beta}}(f(z); x + y) = e_{\pi_{\lambda,\beta}}(f(z); x) e_{\pi_{\lambda,\beta}}(f(z); y) l_{\pi_{\lambda,\beta}}(f(z)). \]

The Taylor expansion of \( l_{\pi_{\lambda,\beta}}(f(z)) \) around \( z = 0 \) is given by

\[ l_{\pi_{\lambda,\beta}}(f(z)) = E_{\beta}(\lambda z) = \sum_{m=0}^{\infty} \frac{(\lambda z)^m}{\Gamma(m\beta + 1)} \]

\[ = \sum_{m=0}^{\infty} z^m \frac{\lambda^m m!}{\Gamma(m\beta + 1)} = \sum_{m=0}^{\infty} \frac{z^m}{m!} M_{\lambda,\beta}(m). \]

Then, we have

\[ \sum_{n=0}^{\infty} z^n \frac{n!}{n!} C_n^{\lambda,\beta}(x + y) = \sum_{k=0}^{\infty} \frac{z^k}{k!} C_k^{\lambda,\beta}(x) \cdot \sum_{l=0}^{\infty} \frac{z^l}{l!} C_l^{\lambda,\beta}(y) \cdot \sum_{m=0}^{\infty} \frac{z^m}{m!} M_{\lambda,\beta}(m) \]

\[ = \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{k+l+m=n} \frac{n!}{k!l!m!} C_k^{\lambda,\beta}(x) C_l^{\lambda,\beta}(y) M_{\lambda,\beta}(m). \]

The result follows by comparing the coefficients in both sides of the equation.

(P4) Again, by definition of the Wick exponential,

\[ e_{\pi_{\lambda,\beta}}(f(z); x + y) = e_{\pi_{\lambda,\beta}}(f(z); x) \exp(yf(z)). \]

The Taylor expansion of \( \exp(yf(z)) \) around \( z = 0 \) is given by

\[ \exp(yf(z)) = \sum_{m=0}^{\infty} \frac{z^m}{m!} (y)_m, \]

where \((y)_m\) are the falling factorials (see for example [8]) given by

\[ (y)_m := \begin{cases} 1 & \text{if } m = 0 \\ y(y-1)\ldots(y-m+1) & \text{if } m \in \mathbb{N}. \end{cases} \]

Hence,

\[ \sum_{n=0}^{\infty} \frac{z^n}{n!} C_n^{\lambda,\beta}(x + y) = \sum_{k=0}^{\infty} \frac{z^k}{k!} C_k^{\lambda,\beta}(x) \cdot \sum_{m=0}^{\infty} \frac{z^m}{m!} (y)_m \]

\[ = \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{k+l+m=n} \frac{n!}{k!l!m!} C_k^{\lambda,\beta}(x)(y)_m \]

\[ = \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{k=0}^{n} \binom{n}{k} C_k^{\lambda,\beta}(x)(y)_{n-k}. \]

The claim follows again by comparing the coefficients in both sides of the equation.
(P5) Note that
\[
\mathbb{E}(e^{\pi_{\lambda,\beta}(f(z); \cdot)}) = \frac{1}{l_{\pi_{\lambda,\beta}}(f(z))} \mathbb{E}(e^{f(z)}) = \frac{l_{\pi_{\lambda,\beta}}(f(z))}{l_{\pi_{\lambda,\beta}}(f(z))} = 1
\]
and using (3.2) we obtain
\[
\mathbb{E}(e^{\pi_{\lambda,\beta}(f(z); \cdot)}) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \int_{\mathbb{R}} C_n^{\lambda,\beta}(x) d\pi_{\lambda,\beta}(x)
\]
which implies the result comparing the coefficients.

(P6) Given \( \varepsilon > 0 \) let \( \sigma_{\varepsilon} > 0 \) be such that \( |f(z)| < \varepsilon \) for any \( z \in \{z' \mid |z'| = \sigma_{\varepsilon}\} \). The following bound follows from the definition of the polynomials \( C_n^{\lambda,\beta}(\cdot) \), \( n \in \mathbb{N} \) and the Cauchy inequality
\[
|C_n^{\lambda,\beta}(x)| \leq \frac{n!}{2\pi} \int_{|z|=\sigma_{\varepsilon}} \frac{|e^{\pi_{\lambda,\beta}(f(z); x)}|}{|z|^{n+1}} |dz|
\leq \frac{n!}{2\pi} \int_{|z|=\sigma_{\varepsilon}} \frac{e^{f(z)|x|}}{|z|^{n+1} |l_{\pi_{\lambda,\beta}}(f(z))|} |dz|
\leq n! \sup_{|z|=\sigma_{\varepsilon}} \frac{1}{|l_{\pi_{\lambda,\beta}}(f(z))|} e^{\varepsilon|x|}
= C_\varepsilon n! \sigma_{\varepsilon}^{-n} e^{\varepsilon|x|}.
\]
This concludes the proof. \( \square \)

An alternative representation of the properties (P1) and (P2) of \( C_n^{\lambda,\beta}(\cdot) \) from Proposition 3.3 is given in the following corollary.

**Corollary 3.4.** For any \( x, y \in \mathbb{R} \), the polynomials \( C_n^{\lambda,\beta}(\cdot) \), \( n \in \mathbb{N} \), satisfy the following properties

(P1') \( C_n^{\lambda,\beta}(x) = \sum_{m=0}^{n} s(n, m) A_m^{\lambda,\beta}(x) \), where \( s(m, n) \) is the Stirling numbers of the first kind.

(P2') \( x^n = \sum_{k=0}^{n} \sum_{m=0}^{k} \binom{n}{k} C_m^{\lambda,\beta}(x) S(k, m) M_{\lambda,\beta}(n - k) \), where \( S(m, n) \) is the Stirling numbers of the second kind.

**Proof.** The claim in (P1') (resp. (P2')) follows directly from Proposition 3.3–(P1) (resp. (P2)) and Proposition B.4–1 (resp. 2) in Appendix B.1. \( \square \)

**Remark 3.5.** In Appendix B.2, we also obtain the explicit form of the generalized Appell polynomials generated by the fractional Poisson measure in terms of the Bell polynomials, see Theorem B.15.

4 Generalized Dual Appell System

Let us consider the Hilbert space \( L^2(\pi_{\lambda,\beta}) := L^2(\mathbb{R}, \mathcal{B}(\mathbb{R}), \pi_{\lambda,\beta}; \mathbb{C}) \) of all complex-valued measurable functions whose modulus square is integrable with respect to \( \pi_{\lambda,\beta} \). In addition, we denote
by \( \mathcal{P}(\mathbb{R}) \) the set of all polynomials in \( \mathbb{R} \) with complex coefficients. Using the Proposition 3.3–(P2), we can write \( \mathcal{P}(\mathbb{R}) \) in the form

\[
\mathcal{P}(\mathbb{R}) = \left\{ \varphi : \mathbb{R} \to \mathbb{C} \mid \varphi(x) = \sum_{k=0}^{n} \varphi_k C^k_{\lambda,\beta}(x), \ \varphi_k \in \mathbb{C}, \ n \in \mathbb{N}_0 \right\}.
\]

Since \( \pi_{\lambda,\beta} \) admits the moments of all orders, then we have \( \mathcal{P}(\mathbb{R}) \subset L^2(\pi_{\lambda,\beta}) \). On the other hand, Remark 3.2 ensures that the inclusion \( \mathcal{P}(\mathbb{R}) \hookrightarrow L^2(\pi_{\lambda,\beta}) \) is dense.

In \( \mathcal{P}(\mathbb{R}) \) the following notion of convergence is defined: a sequence of polynomials \( (\varphi_i)_{i \in \mathbb{N}} \subset \mathcal{P}(\mathbb{R}) \) converges to \( \varphi \in \mathcal{P}(\mathbb{R}) \) if and only if the sequence \( (n_{\varphi_i})_{i \in \mathbb{N}} \) of the degree of the polynomials \( \varphi_i \) is bounded and the coefficients converge term by term.

Let us consider the differential operator of the order \( k \), \( k \in \mathbb{N}_0 \), denoted by \( \nabla^k \), defined in \( \mathcal{P}(\mathbb{R}) \) by

\[
(\nabla^k \varphi)(x) = \frac{d^k \varphi(x)}{dx^k}, \ \varphi \in \mathcal{P}(\mathbb{R}).
\]

Note that \( \nabla^k \) is a continuous linear operator with respect to the above convergence.

Let \( \mathcal{P}_{\pi_{\lambda,\beta}}(\mathbb{R}) \) be the dual of \( \mathcal{P}(\mathbb{R}) \) with respect to \( L^2(\pi_{\lambda,\beta}) \), that is, \( \mathcal{P}_{\pi_{\lambda,\beta}}(\mathbb{R}) \) is the set of all continuous linear functionals defined in \( \mathcal{P}(\mathbb{R}) \) which are given in terms of the inner product in \( L^2(\pi_{\lambda,\beta}) \). More precisely, for every \( \Phi \in L^2(\pi_{\lambda,\beta}) \subset \mathcal{P}_{\pi_{\lambda,\beta}}(\mathbb{R}) \) and any \( \varphi \in \mathcal{P}(\mathbb{R}) \) we use the notation,

\[
\Phi(\varphi) := \langle \varphi, \Phi \rangle_{\pi_{\lambda,\beta}} := \langle \varphi, \Phi \rangle_{L^2(\pi_{\lambda,\beta})}.
\]

The bilinear map \( \langle \cdot, \cdot \rangle_{\pi_{\lambda,\beta}} \) is also called dual pair between \( \mathcal{P}_{\pi_{\lambda,\beta}}(\mathbb{R}) \) and \( \mathcal{P}(\mathbb{R}) \). The elements of \( \mathcal{P}_{\pi_{\lambda,\beta}}(\mathbb{R}) \) are called generalized functions. So, we obtain a chain of continuous and dense embeddings

\[
\mathcal{P}(\mathbb{R}) \subset L^2(\pi_{\lambda,\beta}) \subset \mathcal{P}_{\pi_{\lambda,\beta}}(\mathbb{R}).
\]

Consider the function \( g(z) = e^z - 1 \) which is the inverse of \( f(z) = \log(1 + z) \). For \( k \in \mathbb{N} \), recall that the Taylor expansion of \( (g(z))^k \) and \( (f(z))^k \) at \( z = 0 \) are given by the following series

\[
g(z)^k = k! \sum_{n=k}^{\infty} S(n,k) \frac{z^n}{n!} \quad \text{and} \quad f(z)^k = k! \sum_{n=k}^{\infty} s(n,k) \frac{z^n}{n!},
\]

where \( s(n,k) \) and \( S(n,k) \) are the Stirling numbers of the first kind and Stirling numbers of the second kind, respectively. Then, for any \( \varphi \in \mathcal{P}(\mathbb{R}) \), we have

\[
(g(\nabla)^k \varphi)(x) = k! \sum_{n=k}^{\infty} S(n,k) \frac{d^n}{dx^n} \varphi(x).
\]

It follows from [22, Eq. (26.8.37)] that

\[
g(\nabla)^k = \Delta^k,
\]

where \( \Delta \) is the difference operator defined by

\[
(\Delta f)(x) := f(x + 1) - f(x).
\]

It is easy to see that \( g(\nabla)^k \) is a continuous operator on \( \mathcal{P}(\mathbb{R}) \), hence its adjoint

\[
(g(\nabla)^k)^* : \mathcal{P}_{\pi_{\lambda,\beta}}(\mathbb{R}) \to \mathcal{P}_{\pi_{\lambda,\beta}}(\mathbb{R})
\]

is well-defined and is given by

\[
\langle \varphi, (g(\nabla)^k)^* \Phi \rangle_{\pi_{\lambda,\beta}} = \langle g(\nabla)^k \varphi, \Phi \rangle_{\pi_{\lambda,\beta}}, \quad \forall \Phi \in \mathcal{P}_{\pi_{\lambda,\beta}}(\mathbb{R}), \ \varphi \in \mathcal{P}(\mathbb{R}).
\]
By Proposition 3.3–(P5), we obtain

\[ Q_k^\pi_{\lambda,\beta} := (g(\nabla)^{k})^* 1, \quad k \in \mathbb{N}_0. \]

The system \( Q_k^\pi_{\lambda,\beta} := \{ Q_k^\pi_{\lambda,\beta}, k \in \mathbb{N}_0 \} \) is called the generalized dual Appell system associated with \( \pi_{\lambda,\beta} \). The pair \( Q^\pi_{\lambda,\beta} := (\mathbb{P}^\pi_{\lambda,\beta}, Q^\pi_{\lambda,\beta}) \) is called generalized Appell system generated by the measure \( \pi_{\lambda,\beta} \).

**Lemma 4.1.** Let \( n, k \in \mathbb{N}_0 \) be given. Then

\[
\int_{\mathbb{R}} \nabla^k C_n^{\lambda,\beta}(x) d\pi_{\lambda,\beta}(x) = \begin{cases} 
k! s(n, k), & k \leq n, \\
0, & k > n, \end{cases}
\]

where \( s(n, k) \) is the Stirling numbers of the first kind.

**Proof.** The case \( k > n \) is clear since \( \nabla^k C_n^{\lambda,\beta}(x) = 0 \). For the case \( k \leq n \), we first note that

\[
\nabla^k C_n^{\lambda,\beta}(x) = \frac{d^k}{dx^k} \left( \frac{e^{x f(z)}}{E_{\beta}(\lambda z)} \right) = \frac{d^n}{dz^n} \left[ f(z)^k \frac{e^{x f(z)}}{E_{\beta}(\lambda z)} \right]_{z=0} = \sum_{i=0}^{n} \binom{n}{i} z^i \frac{d^n}{dz^n} \sum_{j=0}^{\infty} \frac{z^j}{j!} s(j, k) C_{n-i}^{\lambda,\beta}(x).
\]

By Proposition 3.3–(P5), we obtain

\[
\int_{\mathbb{R}} \nabla^k C_n^{\lambda,\beta}(x) d\pi_{\lambda,\beta}(x) = k! \sum_{i=k}^{n} \binom{n}{i} s(i, k) \int_{\mathbb{R}} C_{n-i}^{\lambda,\beta}(x) d\pi_{\lambda,\beta}(x)
= k! \sum_{i=k}^{n} \binom{n}{i} s(i, k) \delta_{n,i}
= k! s(n, k).
\]

**Theorem 4.2.** The generalized Appell polynomial system \( \mathbb{P}^\pi_{\lambda,\beta} \) and the generalized dual Appell system \( Q^\pi_{\lambda,\beta} \) are biorthogonal with respect to \( L^2(\pi_{\lambda,\beta}) \) and satisfies

\[
\langle C_n^{\lambda,\beta}, Q_m^{\pi,\lambda,\beta} \rangle_{\pi_{\lambda,\beta}} = n! \delta_{n,m}.
\]

**Proof.** It follows from Lemma 4.1 that

\[
\langle C_n^{\lambda,\beta}, Q_m^{\pi,\lambda,\beta} \rangle_{\pi_{\lambda,\beta}} = \int_{\mathbb{R}} m! \sum_{k=m}^{\infty} \frac{S(k, m)}{k!} \left( \nabla^k C_n^{\lambda,\beta}(x) \right) d\pi_{\lambda,\beta}(x)
= m! \sum_{k=m}^{\infty} \frac{S(k, m)}{k!} k! s(n, k)
= m! \sum_{k=m}^{n} s(n, k) S(k, m)
= n! \delta_{n,m}.
\]

The last equality holds due to, see [22, Eq. (28.8.39)]

\[
\sum_{j=k}^{n} s(j, k) S(n, j) = \sum_{j=k}^{n} s(n, j) S(j, k) = \delta_{n,k}, \quad \forall n, k \in \mathbb{N}_0.
\]

13
Theorem 4.3. For every element $\Phi \in \mathcal{P}_{\pi, \lambda, \mu}(\mathbb{R})$, there exist a unique sequence $(\Phi_k)_{k=0}^{\infty} \subset \mathbb{C}$ such that
\[
\Phi = \sum_{k=0}^{\infty} \Phi_k Q_{\pi, \lambda, \mu}. \tag{4.1}
\]
Conversely, the entire series of the form (4.1) generates a generalized function in $\mathcal{P}_{\pi, \lambda, \mu}(\mathbb{R})$.

Proof. Let $\Phi \in \mathcal{P}_{\pi, \lambda, \mu}(\mathbb{R})$ be arbitrary. For each $k \in \mathbb{N}_0$, let us consider the complex numbers given by $\Phi_k := \frac{1}{n!} \langle C_{\lambda, \mu}^k, \Phi \rangle_{\pi, \lambda, \mu}$, and the functional in $\mathcal{P}(\mathbb{R})$ defined by
\[
\mathcal{P}(\mathbb{R}) \ni \varphi \mapsto \sum_{k=0}^{\infty} k! \varphi_k \Phi_k \in \mathbb{C}.
\]
Since this is a continuous linear functional which is given by the inner product in $L^2(\pi, \lambda, \mu)$, it defines an element in $\mathcal{P}_{\pi, \lambda, \mu}(\mathbb{R})$. We denote it by $\Psi = \sum_{k=0}^{\infty} \Phi_k Q_{\pi, \lambda, \mu}$. So, $\Psi$ is such that
\[
\forall \varphi \in \mathcal{P}(\mathbb{R}), \ \langle \varphi, \Psi \rangle_{\pi, \lambda, \mu} = \sum_{k=0}^{\infty} k! \varphi_k \Phi_k = \langle \varphi, \Phi \rangle_{L^2(\pi, \lambda, \mu)}.
\]
Hence, it follows that $\Psi = \Phi$, since the representation for $\Phi$ is unique.

Conversely, suppose that $\Phi = \sum_{k=0}^{\infty} \Phi_k Q_{\pi, \lambda, \mu}$, $\Phi_k \in \mathbb{C}$, $k \in \mathbb{N}_0$. We are going to show that $\Phi \in \mathcal{P}_{\pi, \lambda, \mu}(\mathbb{R})$. Let $\varphi \in \mathcal{P}(\mathbb{R})$ be of the form $\varphi = \sum_{k=0}^{\infty} \varphi_k C_{\lambda, \mu}^k$. By Theorem 4.2,
\[
\langle \varphi, \Phi \rangle_{\pi, \lambda, \mu} = \sum_{k=0}^{\infty} k! \varphi_k \Phi_k = \langle \varphi, \hat{\Phi} \rangle_{L^2(\pi, \lambda, \mu)}.
\]
This is clearly a linear map and is also continuous in the topology of $\mathcal{P}(\mathbb{R})$, being given by the inner product in $L^2(\pi, \lambda, \mu)$. These then defines $\Phi$ an element in $\mathcal{P}_{\pi, \lambda, \mu}(\mathbb{R})$. □

5 Test and Generalized Function Spaces

Given $\kappa \in [0, 1]$ and $q \in \mathbb{N}_0$, let $\varphi \in \mathcal{P}(\mathbb{R})$ be such that $\varphi = \sum_{n=0}^{k} \varphi_n C_{\lambda, \mu}^n$. We introduce in $\mathcal{P}(\mathbb{R})$ a Hilbert norm by
\[
\| \varphi \|_{q, \kappa, \pi, \lambda, \mu}^2 := \sum_{n=0}^{k} (n!)^{1+\kappa} 2^{nq} |\varphi_n|^2.
\]
The completion of $\mathcal{P}(\mathbb{R})$ in the norm $\| \cdot \|_{q, \kappa, \pi, \lambda, \mu}$ is denoted by $(H)_{q, \pi, \lambda, \mu}^\kappa$, so $\mathcal{P}(\mathbb{R}) \hookrightarrow (H)_{q, \pi, \lambda, \mu}^\kappa$ densely. The space $(H)_{q, \pi, \lambda, \mu}^\kappa$ is a Hilbert space with inner product given by
\[
\langle \varphi, \psi \rangle_{q, \pi, \lambda, \mu}^\kappa := \sum_{n=0}^{\infty} (n!)^{1+\kappa} 2^{nq} \varphi_n \overline{\psi}_n,
\]
admitting the representation
\[
(H)_{q, \pi, \lambda, \mu}^\kappa := \left\{ \varphi : \mathbb{R} \to \mathbb{C} \bigg| \varphi = \sum_{n=0}^{\infty} \varphi_n C_{\lambda, \mu}^n, \ |\varphi|_{q, \kappa, \pi, \lambda, \mu}^2 = \sum_{n=0}^{\infty} (n!)^{1+\kappa} 2^{nq} |\varphi_n|^2 < \infty \right\}.
\]
We also have that the inclusion $(H)_{q, \pi, \lambda, \mu}^\kappa \subset L^2(\pi, \lambda, \mu)$ is dense which results from Remark 3.2 on $\pi, \lambda, \mu$. In this way we obtain the following dense chain of Hilbert spaces:
\[
\cdots \subset (H)_{q, \pi, \lambda, \mu}^{\kappa+1} \subset (H)_{q, \pi, \lambda, \mu}^{\kappa} \subset \cdots \subset (H)_{0, \pi, \lambda, \mu}^{\kappa} \subset L^2(\pi, \lambda, \mu). \tag{5.1}
\]
For $p > q$, the injection operator $I_{p,q} : (H)_{p,\pi,\lambda,\beta}^\kappa \rightarrow (H)_{q,\pi,\lambda,\beta}^\kappa$ is Hilbert-Schmidt. In fact, the set \( \{ C_n^{\lambda,\beta} : ((n!)^{1+\kappa}2^n)^{-\frac{1}{2}} C_n^{\lambda,\beta} \mid n \in \mathbb{N}_0 \} \) is a total orthonormal set in \((H)_{p,\pi,\lambda,\beta}^\kappa\). Then the Hilbert-Schmidt norm of $I_{p,q}$ is given by

\[
\|I_{p,q}\|_{HS}^2 = \sum_{n=0}^{\infty} \|I_{p,q}C_n^{\lambda,\beta}\|_{q,\pi,\lambda,\beta}^2 = \sum_{n=0}^{\infty} (n!)^{1+\kappa}2^n ((n!)^{1+\kappa}2^n)^{-1} = \sum_{n=0}^{\infty} \left( \frac{1}{2^{p-q}} \right)^n < \infty.
\]

Given $\kappa \in [0,1]$ the test function space associated with $\pi_{\lambda,\beta}$ is defined by

\[
(N)_{\pi_{\lambda,\beta}}^\kappa := \bigcap_{q=0}^{\infty} (H)_{q,\pi_{\lambda,\beta}}^\kappa,
\]

which is a nuclear space.

**Example 5.1.** Let us consider the Wick exponential $e_{\pi,\lambda,\beta}(f(z), \cdot), z \in \mathbb{C}$. For $q \in \mathbb{N}_0$,

\[
\|e_{\pi,\lambda,\beta}(f(z), \cdot)\|_{q,\pi,\lambda,\beta}^2 = \sum_{n=0}^{\infty} (n!)^{1+\kappa}2^n \frac{|z|^{2n}}{(n!)^2} = \sum_{n=0}^{\infty} \frac{1}{2^{2n\kappa}} \frac{(2^{\kappa}2^q|z|^2)^n}{(n!)^{1-\kappa}}.
\]

If $\kappa = 0$, we have

\[
\|e_{\pi,\lambda,\beta}(f(z), \cdot)\|_{q,\pi,\lambda,\beta}^2 = \exp(2^q|z|^2) < \infty, \forall z \in \mathbb{C}.
\]

For $\kappa \in (0,1)$, we use Hölder’s inequality with the pair \((\frac{1}{\kappa}, \frac{1}{1-\kappa})\) and obtain

\[
\|e_{\pi,\lambda,\beta}(f(z), \cdot)\|_{q,\pi,\lambda,\beta}^2 \leq \left( \sum_{n=0}^{\infty} \left( \frac{1}{2^{2n\kappa}} \right)^{\frac{1}{\kappa}} \right)^{\frac{\kappa}{2}} \left( \sum_{n=0}^{\infty} \frac{(2^{\kappa}2^q|z|^2)^n}{(n!)^{1-\kappa}} \right)^{\frac{1}{1-\kappa}}
\]

\[
= 2^q \exp \left( (1-\kappa)2^{\frac{\kappa-q}{1-\kappa}}|z|^2 \right) < \infty,
\]

for all $z \in \mathbb{C}$. Hence $e_{\pi,\lambda,\beta}(f(z), \cdot) \in (N)_{\pi_{\lambda,\beta}}^\kappa$, $\kappa \in [0,1)$. For $\kappa = 1$ and $q \in \mathbb{N}_0$, we have

\[
\|e_{\pi,\lambda,\beta}(f(z), \cdot)\|_{q,\pi,\lambda,\beta}^2 = \sum_{n=0}^{\infty} (2^q|z|^2)^n
\]

which is convergent if and only if $|z| < 2^{-q/2}$. Thus, $e_{\pi,\lambda,\beta}(f(z), \cdot) \notin (N)_{\pi_{\lambda,\beta}}^1$, $z \in \mathbb{C}\{0\}$. However, for each $q \in \mathbb{N}_0$, $e_{\pi,\lambda,\beta}(f(z), \cdot) \in (H)_{q,\pi,\lambda,\beta}^1$ provided that $|z| < 2^{-q/2}$.

**Proposition 5.2.**

1. Every test functions $\varphi \in (N)_{\pi_{\lambda,\beta}}^1$ has a unique extension to the set $\mathbb{C}$ such that $\varphi$ is an entire function of minimal type and order of growth one, that is, $\varphi : \mathbb{C} \rightarrow \mathbb{C}$ is entire and for all $\varepsilon > 0$ there is $K > 0$ such that $|\varphi(z)| \leq Ke^{\varepsilon|z|}$.

2. For $\varepsilon > 0$, there exists $\sigma_\varepsilon > 0$ in which $2^q > \sigma_\varepsilon^{-2}$, for $q \in \mathbb{N}_0$ and $K' > 0$ such that we have the following bound

\[
|\varphi(z)| \leq K' \|\varphi\|_{q,\pi,\lambda,\beta} e^{\varepsilon|z|}, \quad z \in \mathbb{C}.
\]

**Proof.** Let $\varphi \in (N)_{\pi_{\lambda,\beta}}^1$ be given. Then $\varphi$ can be expressed as

\[
\varphi = \sum_{n=0}^{\infty} \varphi_n C_n^{\lambda,\beta}, \quad \varphi_n \in \mathbb{C}, \ n \in \mathbb{N}_0,
\]

where $C_n^{\lambda,\beta}$ are given.
with
\[ \| \varphi \|_{q, \pi, \lambda, \beta}^2 = \sum_{n=0}^{\infty} (n!)^2 2^{nq} |\varphi_n|^2 \]
for all \( q \in \mathbb{N}_0 \). Let \( \varepsilon > 0 \) and \( z \in \mathbb{C} \). By Proposition 3.3-(P6), there exist \( C_\varepsilon, \sigma_\varepsilon > 0 \) such that
\[ |\varphi(z)| \leq \sum_{n=0}^{\infty} |\varphi_n C_{n, \lambda, \beta}(z)| \leq \sum_{n=0}^{\infty} |\varphi_n| \| C_{n, \lambda, \beta}(z) \| \leq \varepsilon |z| C_\varepsilon \sum_{n=0}^{\infty} n! |\varphi_n| \sigma_\varepsilon^{-n}. \]
Using Hölder’s inequality, we have
\[ |\varphi(z)| \leq C_\varepsilon |z|^1 \left( \sum_{n=0}^{\infty} (n!)^2 2^{nq} |\varphi_n|^2 \right)^{1/2} \left( \sum_{n=0}^{\infty} 2^{-nq} \sigma_\varepsilon^{-2n} \right)^{1/2}, \]
for \( 2^q > \sigma_\varepsilon^{-2} \). By the Weierstrass M-test, the series \( \varphi(z) = \sum_{n=0}^{\infty} \varphi_n C_{n, \lambda, \beta}(z) \) converges uniformly and absolutely in any neighborhood of zero in \( \mathbb{C} \). Since each term \( \varphi_n C_{n, \lambda, \beta}(z) \) is entire in \( z \), the uniform convergence implies that \( z \mapsto \sum_{n=0}^{\infty} \varphi_n C_{n, \lambda, \beta}(z) \) is entire on \( \mathbb{C} \). On the other hand, take \( K' = C_\varepsilon (1 - 2^{-q} \sigma_\varepsilon^{-2})^{-1/2} \) in equation (5.2) and so we obtain the required bound
\[ |\varphi(z)| \leq K' \| \varphi \|_{q, \pi, \lambda, \beta} |z|^1, \quad z \in \mathbb{C}. \]
This completes the proof. \( \square \)

As the inclusion \((H)^\kappa_{q, \pi, \lambda, \beta} \hookrightarrow L^2(\pi, \lambda, \beta)\) is dense, we may compute the dual of \((H)^\kappa_{q, \pi, \lambda, \beta}\) with respect to \( L^2(\pi, \lambda, \beta) \), that is, the functionals are represented in terms of the inner product \(( \cdot, \cdot \))_{\pi, \lambda, \beta}. In the literature this process is known as rigged Hilbert spaces, see for example \([3]\). We are not going to reproduce this process here. The resulting triplet of Hilbert spaces is
\[ (H)^\kappa_{q, \pi, \lambda, \beta} \subset L^2(\pi, \lambda, \beta) \subset (H)^{-\kappa}_{q, \pi, \lambda, \beta}. \]

The Hilbert space \((H)^{-\kappa}_{q, \pi, \lambda, \beta}\) which corresponds to the completion of \( L^2(\pi, \lambda, \beta) \) with respect to the norm \( \| \cdot \|_{-q, -\kappa, \pi, \lambda, \beta} \), admit the following representation
\[ (H)^{-\kappa}_{q, \pi, \lambda, \beta} = \left\{ \Phi = \sum_{n=0}^{\infty} \Phi_n Q_{n, \pi, \lambda, \beta}, \Phi_n \in \mathbb{C}, \| \Phi \|_{-q, -\kappa, \pi, \lambda, \beta}^2 = \sum_{n=0}^{\infty} (n!)^{1-\kappa} 2^{-nq} |\Phi_n|^2 < \infty \right\}. \]
From the general theory of duality (see for example \([25]\)) the dual of \((N)^\kappa_{\pi, \lambda, \beta}\) with respect to \( L^2(\pi, \lambda, \beta) \) is given by
\[ (N)^{\kappa}_{\pi, \lambda, \beta} = \bigcup_{q=0}^{\infty} (H)^{-\kappa}_{-q, \pi, \lambda, \beta}. \]
As a result we obtain the chain of continuous embeddings
\[ (N)^{\kappa}_{\pi, \lambda, \beta} \subset \cdots \subset (H)^{\kappa}_{0, \pi, \lambda, \beta} \subset \cdots \subset L^2(\pi, \lambda, \beta) \subset \cdots \subset (H)^{-\kappa}_{0, \pi, \lambda, \beta} \subset \cdots \subset (N)^{-\kappa}_{\pi, \lambda, \beta}. \]

**Example 5.3** (Generalized Radon-Nikodým derivative). We want to define the generalized function \( \rho_{\pi, \lambda, \beta}(z, \cdot) \in (N)^{-1}_{\pi, \lambda, \beta}, \ z \in \mathbb{C}, \) such that
\[ \langle \varphi, \rho_{\pi, \lambda, \beta}(z, \cdot) \rangle_{\pi, \lambda, \beta} = \int_{\mathbb{R}} \varphi(x - z) \, d\pi_{\lambda, \beta}(x), \quad \varphi \in (N)^1_{\pi, \lambda, \beta}. \]
Taking into account the result of Proposition 5.2–(2), it turns out that
\[
\| \langle \varphi, \rho_{\pi,\lambda,\beta}(z, \cdot) \rangle \|_{\pi,\lambda,\beta} \leq K' \| \varphi \|_{q,1,\pi,\lambda,\beta} e^{\varepsilon|z|} \int_{\mathbb{R}} e^{\varepsilon|x|} d\pi_{\lambda,\beta}(x).
\]
The integral on the right hand side is finite by Proposition 3.1. Thus, we have in fact \( \rho_{\pi,\lambda,\beta}(z, \cdot) \) \( \in \) \( (N)^{-1}_{\pi,\lambda,\beta} \). By Proposition 3.3–(P4) and (P5), we have
\[
\langle C_n^{\lambda,\beta}, \rho_{\pi,\lambda,\beta}(z, \cdot) \rangle_{\pi,\lambda,\beta} = \int_{\mathbb{R}} C_n^{\lambda,\beta}(x-z) d\pi_{\lambda,\beta}(x)
= \sum_{k=0}^{\infty} \left( \begin{array}{c} n \\ k \end{array} \right) (-z)_{n-k} \int_{\mathbb{R}} C_k^{\lambda,\beta}(x) d\pi_{\lambda,\beta}(x)
= \sum_{k=0}^{\infty} \left( \begin{array}{c} n \\ k \end{array} \right) (-z)_{n-k} \delta_{k,0}
= (-z)_n
= \left\langle C_n^{\lambda,\beta}, \sum_{k=0}^{\infty} \frac{(-z)_k}{k!} Q_k^{\pi,\lambda,\beta} \right\rangle_{\pi,\lambda,\beta}
\]
where we used the biorthogonal property of \( P^{\pi,\lambda,\beta} \) and \( Q^{\pi,\lambda,\beta} \). This implies that \( \rho_{\pi,\lambda,\beta}(z, \cdot) \) has the following representation
\[
\rho_{\pi,\lambda,\beta}(z, \cdot) = \sum_{k=0}^{\infty} \frac{(z)_k}{k!} Q_k^{\pi,\lambda,\beta}(\cdot).
\] (5.4)

In other words, \( \rho_{\pi,\lambda,\beta}(z, \cdot) \) is the generating generalized function of the system \( Q^{\pi,\lambda,\beta} \).

**Example 5.4** (Delta distribution). For \( z \in \mathbb{C} \), we define the distribution \( \delta_z \) by the following \( Q^{\pi,\lambda,\beta} \)-decomposition:
\[
\delta_z = \sum_{n=0}^{\infty} \frac{C_n^{\lambda,\beta}(z)}{n!} Q_n^{\pi,\lambda,\beta}.
\]
By Proposition 3.3-(P6), given \( \varepsilon > 0 \), there exist \( C_{\varepsilon}, \sigma_{\varepsilon} > 0 \) such that
\[
\| \delta_z \|_{q,-\kappa,\pi,\lambda,\beta}^2 = \sum_{n=0}^{\infty} (n!)^{-1-\kappa} 2^{-nq} |C_n^{\lambda,\beta}(z)|^2
\leq C_{\varepsilon} e^{2z|z|} \sum_{n=0}^{\infty} 2^{-nq} \sigma_{\varepsilon}^{-2n} (n!)^{1-\kappa}
\leq C_{\varepsilon} e^{2z|z|} \left( \sum_{n=0}^{\infty} \left( \frac{2^{-nq}}{n^{1-\kappa}} \right) \right)^{2-\kappa} \left( \sum_{n=0}^{\infty} \left( \frac{\sigma_{\varepsilon}^{-2n}}{(n!)^{1-\kappa}} \right) \right)^{\kappa-1}
\leq C_{\varepsilon} e^{2z|z|} \left( \sum_{n=0}^{\infty} \left( \frac{1}{2^q/2-\kappa} \right)^n \right)^{2-\kappa} \exp \left( (\kappa - 1) \sigma_{\varepsilon}^{2/(1-\kappa)} \right)
\]
which is finite for \( q \geq 2 - \kappa, \kappa \in [0,1] \). Hence, \( \delta_z \in (H)^{-\kappa}_{-q,\pi,\lambda,\beta} \subset (N)^{-\kappa}_{\pi,\lambda,\beta}, \kappa \in [0,1] \). Also, for \( \kappa = 1 \), we have
\[
\| \delta_z \|_{q,-1,\pi,\lambda,\beta}^2 = \sum_{n=0}^{\infty} (n!)^{-2} 2^{-nq} |C_n^{\lambda,\beta}(z)|^2
\leq C_{\varepsilon} e^{2z|z|} \sum_{n=0}^{\infty} 2^{-nq} \sigma_{\varepsilon}^{-2n},
\]
which is finite for sufficiently large $q \in \mathbb{N}$. Thus, $\delta_z \in (N)_{\pi_{\lambda, \beta}}^{-1}$. For $\varphi = \sum_{n=0}^{\infty} \varphi_n C_n^{\lambda, \beta}$, the action of $\delta_z$ is given by

$$
\langle \langle \delta_z, \varphi \rangle \rangle_{\pi_{\lambda, \beta}} = \left\langle \left\langle \sum_{m=0}^{\infty} \varphi_m C_m^{\lambda, \beta}, \sum_{n=0}^{\infty} \frac{C_n^{\lambda, \beta}(z)}{n!} Q_{n \lambda, \beta} \right\rangle \right\rangle_{\pi_{\lambda, \beta}} = \sum_{n=0}^{\infty} \varphi_n C_n^{\lambda, \beta}(z) = \varphi(z)
$$

by the biorthogonal property of $\mathcal{P}_{\pi_{\lambda, \beta}}$ and $Q_{\pi_{\lambda, \beta}}$. This implies that $\delta_z$ (in particular for $z$ real) plays the role of a “$\delta$-function” (evaluation map) in calculus.

## 6 Characterization Theorems

In this section, we define two integrals transforms, called $S_{\pi_{\lambda, \beta}}$-transform and convolution $C_{\pi_{\lambda, \beta}}$, which are used to characterize the test function spaces $(N)^{\kappa}_{\pi_{\lambda, \beta}}$ and generalized function spaces $(N)^{-\kappa}_{\pi_{\lambda, \beta}}$. The $S_{\pi_{\lambda, \beta}}$-transform of $\varphi \in (N)^{\kappa}_{\pi_{\lambda, \beta}}$ is defined by

$$(S_{\pi_{\lambda, \beta}} \varphi)(z) := \int_{\mathbb{R}} \varphi(x) e_{\pi_{\lambda, \beta}}(z, x) \, d\pi_{\lambda, \beta}(x).$$

The $S_{\pi_{\lambda, \beta}}$-transform may be extended to $\Phi \in (N)^{-\kappa}_{\pi_{\lambda, \beta}}$ (in a consistent way) as follows

$$(S_{\pi_{\lambda, \beta}} \Phi)(z) := \langle \langle e_{\pi_{\lambda, \beta}}(z, \cdot), \Phi \rangle \rangle_{\pi_{\lambda, \beta}}.$$

Note that for $\kappa = 1$, $S_{\pi_{\lambda, \beta}} \Phi$ is defined only in a neighborhood of the zero in $\mathbb{C}$, because if $\Phi \in (H)^{-1}_{\pi_{\lambda, \beta}}$, then $e_{\pi_{\lambda, \beta}}(z, \cdot) \in (H)^{1}_{q, \pi_{\lambda, \beta}}$ for $z \in \mathbb{C}$ such that $|z| < 2^{-\eta/2}$ as shown in Example 5.1. Now, we introduce the second integral transform which is more appropriate to characterize the test function spaces $(N)^{\kappa}_{\pi_{\lambda, \beta}}$. The convolution $C_{\pi_{\lambda, \beta}}$ of $\varphi \in (N)^{\kappa}_{\pi_{\lambda, \beta}}$ is defined by

$$(C_{\pi_{\lambda, \beta}} \varphi)(z) := \int_{\mathbb{R}} \varphi(x + z) \, d\pi_{\lambda, \beta}(x).$$

and using Example 5.3 may be written as

$$(C_{\pi_{\lambda, \beta}} \varphi)(z) = \langle \langle \varphi, \rho_{\pi_{\lambda, \beta}}(-z, \cdot) \rangle \rangle_{\pi_{\lambda, \beta}}.$$ 

In Gaussian analysis, the convolution $C_{\pi_{\lambda, \beta}}$ and the $S_{\pi_{\lambda, \beta}}$-transform coincide, however these two transformations are different in the fractional Poisson analysis, or more generally, in non-Gaussian analysis.

### 6.1 Characterization of Test Functions

For every $l \in \mathbb{N}_0$, we denote by $\mathcal{E}^{k}_{2^{-l}}(\mathbb{C})$ the set of entire functions of order of growth $k \in [1, 2]$ and type $2^{-l}$, i.e.,

$$\mathcal{E}^{k}_{2^{-l}}(\mathbb{C}) = \{ u : \mathbb{C} \to \mathbb{C} \text{ is entire } | u(z) | \leq C \exp(2^{-l}|z|^k), C > 0 \}.$$ 

For any $l \in \mathbb{N}_0$, the map

$$| \cdot |_{l, k} : \mathcal{E}^{k}_{2^{-l}}(\mathbb{C}) \to \mathbb{R}_+, \ u \mapsto |u|_{l, k} := \sup_{z \in \mathbb{C}} \{ |u(z)| \exp(-2^{-l}|z|^k) \},$$

is a norm in $\mathcal{E}^{k}_{2^{-l}}(\mathbb{C})$ and $(\mathcal{E}^{k}_{2^{-l}}(\mathbb{C}), | \cdot |_{l, k})$ is a Banach space. The space of entire functions of minimal type and order of growth $k$ is defined by

$$\mathcal{E}^{k}_{\text{min}}(\mathbb{C}) := \bigcap_{l \in \mathbb{N}_0} \mathcal{E}^{k}_{2^{-l}}(\mathbb{C}).$$
Any entire function \( u \) can be represented in Taylor series in the form
\[
    u(z) = \sum_{n=0}^{\infty} u_n z^n, \quad u_n = \frac{1}{n!} \frac{d^n}{dz^n} u(z) \bigg|_{z=0}, \quad z \in \mathbb{C}.
\]

Let us consider the family \((\|\cdot\|_{q,\kappa})_{q \in \mathbb{N}_0}, \kappa \in [0, 1]\) of Hilbert norms in \( \mathcal{E}_\text{min}^k(\mathbb{C}) \), defined by
\[
    \|u\|_{q,\kappa}^2 := \sum_{n=0}^{\infty} (n!)^{1+\kappa} 2^{nq} |u_n|^2 < \infty, \quad u \in \mathcal{E}_\text{min}^k(\mathbb{C}).
\]

This family of Hilbert norms is equivalent to the family of norms \((|\cdot|_{l,k})_{l \in \mathbb{N}_0}\), see Appendix A.

**Theorem 6.1.** The convolution \( C_{\pi,\beta} \) is a homeomorphism between the test function space \( (N)_{\pi,\beta}^1 \) and the space \( \mathcal{E}_\text{min}^k(\mathbb{C}) \) of entire functions of minimal type and order of growth one.

**Proof.** Let \( \varphi \in (N)_{\pi,\beta}^1 \) be of the form \( \varphi = \sum_{n=0}^{\infty} \varphi_n C_{\lambda,\beta}^{n} \) with
\[
    \|\varphi\|_{q,1,\pi,\beta}^2 = \sum_{n=0}^{\infty} (n!)^2 2^{nq} |\varphi_n|^2 < \infty
\]
for each \( q \in \mathbb{N}_0 \). So we have
\[
    |\varphi_n| \leq (n!)^{-1/2} 2^{-nq/2} \|\varphi\|_{q,1,\pi,\beta}.
\]

It follows from Example 5.3 that
\[
    (C_{\pi,\beta} \varphi)(z) = \sum_{n=0}^{\infty} \varphi_n(z) = \sum_{n=0}^{\infty} \varphi_n \sum_{k=0}^{n} s(n, k) z^k = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \varphi_n s(n, k) z^k,
\]
where in the second equality we use Proposition B.5–1. From Proposition B.4–1, we have
\[
    s(n, k) = \frac{A_n}{k!}.
\]

For \( \varepsilon > 0 \), we define
\[
    F_{\varepsilon} := \sup_{|z|=\varepsilon} |f(z)|,
\]
where \( f(z) = \log(1+z) \), so that by Cauchy inequality
\[
    |s(n, k)| \leq \frac{1}{k!} \sum_{l_1+\cdots+l_k=n} \frac{n!}{l_1! \cdots l_k!} |f^{(l_1)}(0)| \cdots |f^{(l_k)}(0)| \leq \frac{1}{k!} \sum_{l_1+\cdots+l_k=n} \frac{n! l_1! \cdots l_k!}{l_1! \cdots l_k!} F_{\varepsilon}^{k \varepsilon^{-n}} \leq \frac{n!}{k!} F_{\varepsilon}^{k \varepsilon^{-n}} \varepsilon^{-n}.
\]

Now, we estimate \( \sum_{n=0}^{\infty} \varphi_n s(n, k) \) as follows
\[
    \left| \sum_{n=0}^{\infty} \varphi_n s(n, k) \right| \leq \sum_{n=0}^{\infty} |\varphi_n| |s(n, k)| \leq \sum_{n=0}^{\infty} (n!)^{-1/2} 2^{-nq/2} \|\varphi\|_{q,1,\pi,\beta} \frac{n!}{k!} F_{\varepsilon}^{k \varepsilon^{-n}} \leq \frac{\|\varphi\|_{q,1,\pi,\beta} F_{\varepsilon}^{k}}{k!} \sum_{n=0}^{\infty} \left( \frac{2^{2^{2} \varepsilon^{-1}}}{\varepsilon^{-1}} \right)^n = \frac{\|\varphi\|_{q,1,\pi,\beta} F_{\varepsilon}^{k}}{k!} \frac{1}{(1 - \frac{2^{2^{2} \varepsilon^{-1}}}{\varepsilon^{-1}})^{-1}}
\]

19
for $2^{\frac{2}{1-\varepsilon}} < 1$. It follows that

$$
\| (C_{\lambda,\beta} \varphi)(z) \|_{q,1}^2 \leq \sum_{k=0}^{\infty} (k!)^2 2^{kq} \| \varphi \|_{q,1,\pi,\lambda,\beta}^2 F_{\varepsilon}^{2k} \left( \frac{1}{k!} \right)^2 (1 - 2^{\frac{2}{1-\varepsilon}} \varepsilon^{-1})^{-2}
$$

$$
= \| \varphi \|_{q,1,\pi,\lambda,\beta}^2 (1 - 2^{\frac{2}{1-\varepsilon}} \varepsilon^{-1})^{-2} \sum_{k=0}^{\infty} (2^q F_{\varepsilon}^2)^k
$$

$$
= \| \varphi \|_{q,1,\pi,\lambda,\beta}^2 (1 - 2^{\frac{2}{1-\varepsilon}} \varepsilon^{-1})^{-2} (1 - 2^q F_{\varepsilon}^2)^{-1},
$$

where $\varepsilon > 0$ is such that $2^q F_{\varepsilon}^2 < 1$. So $C_{\lambda,\beta}$ is continuous.

Conversely, let $u \in \mathcal{E}_0^k(\mathbb{C})$ be given such that

$$
u(z) = \sum_{n=0}^{\infty} u_n z^n
$$

with

$$
\| u \|_{q,1}^2 = \sum_{n=0}^{\infty} (n!)^2 2^{nq} |u_n|^2 < \infty
$$

for each $q \in \mathbb{N}_0$. Hence,

$$
|u_n| \leq (n!)^{-1/2} 2^{-nq/2} \| u \|_{q,1}.
$$

Note that

$$
u(z) = \sum_{n=0}^{\infty} u_n z^n = \sum_{n=0}^{\infty} u_n \sum_{k=0}^{\infty} S(n, k)(z)_k = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} u_n S(n, k)(z)_k
$$

by Proposition B.5–2. From Proposition B.4–2, we have

$$
S(n, k) = \frac{B_n^k}{k!}
$$

For $\varepsilon > 0$, we define

$$
G_{\varepsilon} := \sup_{|z| = \varepsilon} |g(z)|
$$

where $g(z) = e^z - 1$ such that by Cauchy inequality

$$
|S(n, k)| \leq \frac{1}{k!} \sum_{l_1 + \ldots + l_k = n} \frac{n!}{l_1! \ldots l_k!} |g^{(l_1)}(0)| \ldots |g^{(l_k)}(0)|
$$

$$
\leq \frac{1}{k!} \sum_{l_1 + \ldots + l_k = n} \frac{n! l_1! \ldots l_k!}{l_1! \ldots l_k!} G_{\varepsilon}^{l_1} \ldots G_{\varepsilon}^{l_k} \varepsilon^{-n}
$$

$$
\leq \frac{n!}{k!} G_{\varepsilon}^{n} \varepsilon^{-n}.
$$

Now, we estimate $\sum_{n=0}^{\infty} u_n S(n, k)$ as follows

$$
\left| \sum_{n=0}^{\infty} u_n S(n, k) \right| \leq \sum_{n=0}^{\infty} |u_n| S(n, k)
$$

$$
\leq \sum_{n=0}^{\infty} (n!)^{-1} 2^{-nq/2} \| u \|_{q,1} \frac{n!}{k!} G_{\varepsilon}^{n} \varepsilon^{-n}
$$

$$
\leq \| u \|_{q,1} \frac{G_{\varepsilon}^{k}}{k!} \sum_{n=0}^{\infty} \left( \frac{2^{\frac{2}{1-\varepsilon}} \varepsilon^{-1}}{2^q F_{\varepsilon}^2} \right)^n
$$

$$
= \| u \|_{q,1} \frac{G_{\varepsilon}^{k}}{k!} \left( 1 - \frac{2^{\frac{2}{1-\varepsilon}} \varepsilon^{-1}}{2^q F_{\varepsilon}^2} \right)^{-1}
$$

20
for $2^{2\frac{q}{1-k}}\varepsilon^{-1} < 1$. Define $\psi$ of the form

$$\psi = \sum_{k=0}^{\infty} \psi_k C_k^{\lambda,\beta}$$

where $\psi_k = \sum_{n=0}^{\infty} u_n S(n, k)$. Then we have

$$\|\psi\|_{q,1,\pi,\lambda,\beta}^2 \leq \sum_{k=0}^{\infty} (k!)^{2q} \left\| u \right\|_{q,1}^2 \frac{G^q_k \varepsilon^{-1}}{(k!)^2} (1 - 2^{2\frac{q}{1-k}}\varepsilon^{-1})^{-2}$$

$$= \frac{\| u \|_{q,1}^2 (1 - 2^{2\frac{q}{1-k}}\varepsilon^{-1})^{-2} \sum_{k=0}^{\infty} (2^q G^q_k)^k}{(1 - 2^{2\frac{q}{1-k}}\varepsilon^{-1})^{-2}(1 - 2^q G^q_k)^{-1}},$$

(6.1)

where $\varepsilon > 0$ is such that $2^q G^q_k < 1$. Hence, $\psi \in (N)^{1,\pi,\lambda,\beta}$ and $C_{\pi,\lambda,\beta} \psi = u$. Moreover, if $C_{\pi,\lambda,\beta} \psi = 0$, equation (6.1) implies that $\psi = 0$ and so $C_{\pi,\lambda,\beta}$ is one-to-one. On the other hand, we have $C_{\pi,\lambda,\beta}^{-1} u = \psi$ which implies $C_{\pi,\lambda,\beta}$ is onto. Now, it follows that

$$\|C_{\pi,\lambda,\beta}^{-1} u\|_{q,1,\pi,\lambda,\beta} \leq \frac{\| u \|_{q,1}^2 (1 - 2^{2\frac{q}{1-k}}\varepsilon^{-1})^{-2}(1 - 2^q G^q_k)^{-1}}{\sum_{\lambda,\beta} \| u \|_{q,1}^2} < \infty$$

which proves the continuity of $C_{\pi,\lambda,\beta}^{-1}$. 

\[\square\]

**6.2 Characterization of $(N)^{-\kappa}_{\pi,\lambda,\beta}$, $\kappa \in [0, 1)$**

For every $l \in \mathbb{N}_0$, we denote by $E_{2^l}^{k'}(\mathbb{C})$ the set of entire functions of growth $k' \in [2, \infty)$ and type $2^l$, i.e.,

$$E_{2^l}^{k'}(\mathbb{C}) = \{ U : \mathbb{C} \to \mathbb{C} \text{ is entire } | |U(z)| \leq C \exp(2^l |z|^{k'}), \ C > 0 \}.$$  

For all natural $l$, the map

$$| \cdot |_{l,k'} : E_{2^l}^{k'}(\mathbb{C}) \to \mathbb{R}, U \mapsto |U|_{l,k} := \sup_{z \in \mathbb{C}} \{ |U(z)| \exp(-2^l |z|^{k'}) \}$$

is a norm in $E_{2^l}^{k'}(\mathbb{C})$ and $(E_{2^l}^{k'}(\mathbb{C}), | \cdot |_{l,k'})$ is a Banach space. The space of entire functions of maximal type and order of growth $k'$, is defined by

$$E_{\text{max}}^{k'}(\mathbb{C}) := \bigcup_{l \in \mathbb{N}_0} E_{2^l}^{k'}(\mathbb{C}).$$

Taking into account Taylor’s representation around the origin of $U \in E_{\text{max}}^{k'}(\mathbb{C})$, we can consider in $E_{\text{max}}^{k'}(\mathbb{C})$ the family $(| \cdot |_{q,\pi,\lambda,\beta})_{q \in \mathbb{N}_0, \ \kappa \in [0, 1)}$, of Hilbert norms, given by

$$\|U\|_{q,\pi,\lambda,\beta}^2 = \sum_{n=0}^{\infty} (n!)^{1-\beta} 2^{-nq} |U_n|^2 < \infty,$$

which is equivalent to the family of norms $(| \cdot |_{l,k'})_{l \in \mathbb{N}_0}$, see Appendix A.

**Theorem 6.2.** The $S_{\pi,\lambda,\beta}$-transform is a homeomorphism between the generalized function space $(N)^{-\kappa}_{\pi,\lambda,\beta}$, $\kappa \in [0, 1)$ and the space $E_{\text{max}}^{k'}(\mathbb{C})$ of entire functions of maximal type and order of growth $k' = \frac{2}{1-\beta}$.

**Proof.** The proof of this theorem is analogous to that of the previous theorem, using Proposition A.2 in Appendix A. \[\square\]
6.3 Characterization of \((N)^{-1}_{\pi_\lambda,\beta}\)

Let \(\text{Hol}_0(\mathbb{C})\) be the set of holomorphic functions in a neighborhood of the origin. Let us consider the family \((|\cdot|_l)_{l \in \mathbb{N}_0}\) of norms in \(\text{Hol}_0(\mathbb{C})\), defined as follows. If \(U \in \text{Hol}_0(\mathbb{C})\), then there is \(l \in \mathbb{N}_0\), such that
\[
|U|_{l,\infty} := \sup_{|z| \leq 2^{-l}} |U(z)| < \infty.
\]
Under these conditions, \(U\) admits a Taylor series representation in \(D = \{z \in \mathbb{C} \mid |z| \leq 2^{-l}\}\),
\[
U(z) = \sum_{n=0}^{\infty} U_n z^n, \quad \forall z \in D,
\]
there exist \(q \in \mathbb{N}_0\), such that
\[
\|U\|_{2^{-q},-1} = \sum_{n=0}^{\infty} 2^{-nq} |U_n|^2 < \infty.
\]
Thus, we have introduced in \(\text{Hol}_0(\mathbb{C})\) another family of norms, \((|\cdot|_{2^{-q},-1})_{q \in \mathbb{N}_0}\), equivalent to the first \((|\cdot|_l)_{l \in \mathbb{N}_0}\), see Proposition A.3 in Appendix A.

**Theorem 6.3.** The \(S_{\pi_\lambda,\beta}\)-transform is a homeomorphism between the generalized function space \((N)^{-1}_{\pi_\lambda,\beta}\) and the \(\text{Hol}_0(\mathbb{C})\) space.

**Proof.** The proof of this theorem is similar to that of the two previous theorems, using Proposition A.3 in Appendix A. \(\square\)

7 Wick Calculus

It is easy to see that the set \(\text{Hol}_0(\mathbb{C})\) forms an algebra of functions, for the usual operations of addition and multiplication by a scalar. So, if \(\Phi, \Psi \in (N)_{\pi_\lambda,\beta}^{-1}\), then \(S_{\pi_\lambda,\beta} \Phi, S_{\pi_\lambda,\beta} \Psi \in \text{Hol}_0(\mathbb{C})\) and also \(S_{\pi_\lambda,\beta} \Phi \cdot S_{\pi_\lambda,\beta} \Psi \in \text{Hol}_0(\mathbb{C})\). By Theorem 6.3, there exists \(\Theta \in (N)_{\pi_\lambda,\beta}^{-1}\) such that \(S\Theta = S_{\pi_\lambda,\beta} \Phi \cdot S_{\pi_\lambda,\beta} \Psi\). We denote the generalized function \(\Theta\) by \(\Phi \diamond \Psi\) which we call **Wick product** of \(\Phi\) and \(\Psi\). So if
\[
\Phi = \sum_{n=0}^{\infty} \Phi_n Q_n^{-\pi_\lambda,\beta}, \quad \Psi = \sum_{n=0}^{\infty} \Psi_n Q_n^{-\pi_\lambda,\beta},
\]
then \(\Phi \diamond \Psi\) is represented by
\[
\Phi \diamond \Psi = \sum_{n=0}^{\infty} \Theta_n Q_n^{-\pi_\lambda,\beta}, \quad \Theta_n = \sum_{k=0}^{n} \Phi_k \Psi_{n-k}.
\]

The following proposition tells us that the Wick product is a continuous map.

**Proposition 7.1.** The Wick product is a continuous mapping in \((N)_{\pi_\lambda,\beta}^{-1}\), and given \(\Phi \in (H)_{-q,\pi_\lambda,\beta}^{-1}\), \(\Psi \in (H)_{-p,\pi_\lambda,\beta}^{-1}\) we have the following:
\[
\|\Phi \diamond \Psi\|_{-r,\pi_\lambda,\beta} \leq \|\Phi\|_{-q,\pi_\lambda,\beta} \|\Psi\|_{-p,\pi_\lambda,\beta}, \quad r = p + q + 1.
\]
Proof. We can estimate the norm as follows:

\[
\|\Phi \diamond \Psi\|_{\tau, -1, \pi_{\lambda, \beta}}^2 = \sum_{n=0}^{\infty} 2^{-nr} |\Theta_n|^2 \\
\leq \sum_{n=0}^{\infty} 2^{-nr} (n + 1) \sum_{k=0}^{n} |\Phi_k|^2 |\Psi_{n-k}|^2 \\
\leq \sum_{n=0}^{\infty} \sum_{k=0}^{n} 2^{-nq} 2^{-np} |\Phi_k|^2 |\Psi_{n-k}|^2 \\
\leq \left( \sum_{n=0}^{\infty} 2^{-nq} |\Phi_n|^2 \right) \left( \sum_{n=0}^{\infty} 2^{-np} |\Psi_n|^2 \right) \\
= \|\Phi\|_{q, -1, \pi_{\lambda, \beta}}^2 \|\Psi\|_{p, -1, \pi_{\lambda, \beta}}^2,
\]

where we use the fact \( \frac{n+1}{2^n} \leq 1 \).

The Wick powers \( \Phi^{\diamond n} := \Phi \diamond \Phi \diamond \ldots \diamond \Phi \), \( n \in \mathbb{N}_0 \) of \( \Phi \in (N)^{-1}_{\pi_{\lambda, \beta}} \) are defined as

\[
\Phi^{\diamond n} = S_{\pi_{\lambda, \beta}}^{-1} ((S_{\pi_{\lambda, \beta}} \Phi)^n).
\]

In general, if \( L(z) = \sum_{n=0}^{\infty} L_n z^n \) is an analytic function, so we can study the Wick version of this function, i.e., \( L^{\diamond}(\Phi) \), \( \Phi \in (N)^{-1}_{\pi_{\lambda, \beta}} \). We have the following theorem.

**Theorem 7.2.** Let \( L \) be an analytic function in a neighborhood of \( z_0 = E_{\pi_{\lambda, \beta}}(\Phi) \), \( \Phi \in (N)^{-1}_{\pi_{\lambda, \beta}} \). Then \( L^{\diamond}(\Phi) \) defined by \( L^{\diamond}(\Phi) := S_{\pi_{\lambda, \beta}}^{-1} (L(S_{\pi_{\lambda, \beta}} \Phi)) \) is an element in \( (N)^{-1}_{\pi_{\lambda, \beta}} \).

Proof. In order to apply the characterization theorem (cf. Theorem 6.3) it is enough to check that \( L(S_{\pi_{\lambda, \beta}} \Phi) \) is holomorphic around the origin. But this follows easily by choosing a sufficiently small neighborhood around the origin so that the composition \( L \circ S_{\pi_{\lambda, \beta}} \Phi \) is holomorphic.

**Example 7.3.** Let \( \Phi, \Psi \in (N)^{-1}_{\pi_{\lambda, \beta}} \) be two generalized functions.

1. Then \( \exp^{\diamond}(\Phi) \) can be written as

\[
\exp^{\diamond}(\Phi) = \sum_{n=0}^{\infty} \frac{1}{n!} \Phi^{\diamond n}.
\]

2. It is easy to check the following property:

\[
\exp^{\diamond}(\Phi) \exp^{\diamond}(\Psi) = \exp^{\diamond}(\Phi + \Psi).
\]

3. If \( \Phi \) is such that \( E_{\pi_{\lambda, \beta}}(\Phi) > 0 \), then \( \log^{\diamond}(\Phi) \in (N)^{-1}_{\pi_{\lambda, \beta}} \) which is the solution of the equation

\[
\exp^{\diamond}(X) = \Phi.
\]

4. If \( \Phi \) is such that \( E_{\pi_{\lambda, \beta}}(\Phi) \neq 0 \), then \( \Phi^{\diamond -1} := S_{\pi_{\lambda, \beta}}^{-1} ((S_{\pi_{\lambda, \beta}} \Phi)^{-1}) \in (N)^{-1}_{\pi_{\lambda, \beta}} \) and the solution of the equation

\[
X \diamond \Phi = \Psi
\]

is \( X = \Phi^{\diamond -1} \diamond \Psi \).

The Wick product easily generalizes to generalized function spaces \( (N)^{-\kappa}_{\pi_{\lambda, \beta}} \), \( \kappa \in [0, 1) \), as well as test function spaces \( (N)^{\kappa}_{\pi_{\lambda, \beta}} \), \( \kappa \in [0, 1] \). The concept of Wick product is often used in models of stochastic differential equations, the solutions being obtained through the \( S_{\pi_{\lambda, \beta}} \)-transform, see for example [10].
8 Conclusion and Outlook

In this paper, we have developed the biorthogonal system to investigate the spaces of test and generalized functions associated to the fractional Poisson measure $\pi_{\lambda,\beta}$ on $\mathbb{N}_0$. The system of generalized Appell polynomials $P_{\pi,\lambda,\beta}$ describes the spaces of test functions while the generalized dual Appell system $Q_{\pi,\lambda,\beta}$ is suited to description of generalized functions rising from $\pi_{\lambda,\beta}$. In addition, we have characterized both test and generalized function spaces through two suitable integral transforms. The Wick calculus extends in a straightforward manner from Gaussian analysis to the present fractional Poisson analysis. In a future work, we plan to extend these results to an infinite dimensional framework. More precisely, the fractional Poisson measure on the linear space $\mathcal{D}'(\mathbb{R}^d) = C^\infty_0(\mathbb{R}^d)$, the dual of the nuclear space of Schwartz test functions, or on the configuration space $\Gamma_{\mathbb{R}^d} \subset \mathcal{D}'(\mathbb{R}^d)$ over $\mathbb{R}^d$.

A Proofs. Equivalence of Norms

For the completeness of this work we provide the proofs of the following propositions.

**Proposition A.1.** The two system of norms $\{\cdot\}_{l,k}$ and $\{\cdot\}_{q,\kappa}$, $k = \frac{2}{1+\kappa}$ and $\kappa \in [0, 1]$ defined in the space $\mathcal{E}_{\min}^k(\mathbb{C})$ are equivalent.

**Proof.** Let $l \in \mathbb{N}_0$ and $u \in \mathcal{E}_{\min}^k(\mathbb{C})$ be arbitrary. We are going to show that there exist $C > 0$ and $q \in \mathbb{N}_0$, such that $|u|_{l,k} \leq C \|u\|_{q,\kappa}$. As $u \in \mathcal{E}_{\min}^k(\mathbb{C})$, we have that $\|u\|_{q,\kappa} < \infty$, $\forall q \in \mathbb{N}_0$. In addition, it turns out also that $|u|_{l,k} < \infty$, $\forall l \in \mathbb{N}_0$ because $\mathcal{E}_{\min}^k(\mathbb{C}) \subset \mathcal{E}_{\min}^k(\mathbb{C})$, $\forall l \in \mathbb{N}_0$. Since $u$ is entire, it admits the Taylor series representation around the origin, $u(z) = \sum_{n=0}^{\infty} u_n z^n$. Thus, $\forall q \in \mathbb{N}_0$, $\forall z \in \mathbb{C}$, by the Cauchy-Schwarz inequality we obtain

$$|u(z)| \leq \sum_{n=0}^{\infty} |u_n| |z|^n \leq \sum_{n=0}^{\infty} \left( (n!)^{1+\kappa} 2^{nq} \right)^{1/2} |u_n| \left( (n!)^{1+\kappa} 2^{nq} \right)^{-1/2} |z|^n \leq \|u\|_{q,\kappa} \sum_{n=0}^{\infty} \left( \frac{2^{-q} |z|^2 n}{(n!)^{1+\kappa}} \right) \leq \|u\|_{q,\kappa} \sum_{n=0}^{\infty} \left( \frac{2^{-q} |z|^k n}{n!} \right)^{1+\kappa}.$$

Note that by the Hölder's inequality, we have

$$\sum_{n=0}^{\infty} \left( \frac{x^n}{n!} \right) \leq \left( \sum_{n=0}^{\infty} \frac{x^n}{n!} \right)^{2-\gamma} \left( \sum_{n=0}^{\infty} \left( \frac{x^n}{n!} \right)^2 \right)^{\gamma-1} \leq (e^x)^{2-\gamma} (e^{2x})^{\gamma-1} = e^{\gamma x}$$

for $x \geq 0$ and $1 < \gamma < 2$. This yields

$$\sum_{n=0}^{\infty} \left( \frac{2^{q/2} |z|^k n}{n!} \right)^{1+\kappa} \leq \exp \left( k^{-1} 2^{-q/2} |z|^k \right).$$

Thus, we have
Given $K > 0$, let $u \in \mathcal{E}_{\text{max}}^k(\mathbb{C})$ be arbitrary. Now we are going to show that there exist $K > 0$ and $l \in \mathbb{N}_0$, such that $\|u\|_{q,\kappa} \leq K \|u\|_{l,k}$. Note that, for all $l \in \mathbb{N}_0$, we have

$$|u(z)| \leq |u|_{l,k} \exp(2^{-l}|z|^k), \quad \forall z \in \mathbb{C}.$$ 

Thus, it turns out that

$$|u_n| \leq \frac{1}{2\pi} \int_{|z|=\rho>0} |u(z)| \frac{e^{2i\rho n}}{|z|^{n+1}} |dz| \leq |u|_{l,k} \frac{e^{2i\rho^k}}{\rho^n}, \quad \rho > 0.$$ 

This bound may be optimized by taking $\rho = (n2^k)^{-1}$. Substituting, we get

$$|u_n| \leq |u|_{l,k}(e^k - n)^{-k-1}(k^2 n^{-l})^{-k^{-1}}.$$ 

Finally, taking into the account the Stirling formula,

$$e^n n^{-n} \leq 1, \quad n \in \mathbb{N},$$

we obtain

$$|u_n| \leq |u|_{l,k} \left( \frac{e\sqrt{2\pi}(n+1)}{n!} \right)^{-k^{-1}} (k^2 n^{-l})^{-k^{-1}}.$$ 

Thus, the norm $\|u\|_{q,\kappa}$ is given by

$$\|u\|_{q,\kappa}^2 = \sum_{n=0}^{\infty} (n!)^{1+\kappa} 2^n |u_n|^2 \leq |u|_{l,k}^2 \left( \frac{e\sqrt{2\pi}}{n!} \right)^{1+\kappa} \sum_{n=0}^{\infty} (n+1)^2 (k^2 n-1)^{1+\kappa}. \quad (A.1)$$ 

Given $q \in \mathbb{N}_0$, we take $l \in \mathbb{N}_0$ such that $2^q (k2^{-l})^{1+\kappa} < 1$ and use the equality

$$\sum_{n=0}^{\infty} (n+1)^2 x^n \leq \sum_{n=0}^{\infty} (n+1)(n+2)x^n = (1-x)^{-3}, \quad |x| < 1,$$

the series in (A.1) is finite. The inequality (A.1) also proves that the norms of the family $(\|\cdot\|_{q,\kappa})_{q \in \mathbb{N}_0}$ are also well defined.

**Proposition A.2.** The two system of norms $(\|\cdot\|_{l,k'})_{l \in \mathbb{N}_0}, \ k' = \frac{2}{1+\kappa}$ and $(\|\cdot\|_{-q,-\kappa})_{q \in \mathbb{N}_0}, \ k \in [0, 1)$ defined in the space $\mathcal{E}_{\text{max}}^{k'}(\mathbb{C})$ are equivalent.

**Proof.** Following a process analogous to the previous proof, taking $q \in \mathbb{N}_0$ and $U \in \mathcal{E}_{\text{max}}^{k'}(\mathbb{C})$ as arbitrary, we show that there is a natural number $l$, such that $2^l \geq (k')^{-1} 2^{q-1} (k+\kappa)/(1-\kappa)$ and a positive number $C = 2^{\kappa/2}$ such that $|U|_{l,k'} \leq C \|U\|_{-q,-\kappa}$.

Conversely, and also analogously to the previous proof, given $l \in \mathbb{N}_0$ and $U \in \mathcal{E}_{\text{max}}^{k'}(\mathbb{C})$ such that $|U|_{l,k'} < \infty$, we prove that there is a natural number $Q$ where $2^{-q} (\frac{1}{1-\kappa})^{-1} < 1$ and a positive number $K = (e\sqrt{2\pi})^{(1-\kappa)/2} (1 - 2^{-q} (\frac{2^{q+1}}{1-\kappa})^{-1})$, such that $\|U\|_{-q,-\kappa} \leq K \|U|_{l,k'}$.

**Proposition A.3.** The two system of norms $(\|\cdot\|_{l,\infty})_{l \in \mathbb{N}_0}$ and $(\|\cdot\|_{-q,-1})_{q \in \mathbb{N}_0}$, introduced in $\text{Hol}_0(\mathbb{C})$ are equivalent.

**Proof.** The proof is analogous to the two previous ones. Taking $q \in \mathbb{N}_0$ and $U \in \text{Hol}_0(\mathbb{C})$, so that $\|U\|_{-q,-1} < \infty$, we show that there exist $C = (1 - 2^q |z|^2)^{-1-2} > 0$ and $l \in \mathbb{N}_0$ ($l > \frac{q}{2}$), such that $|U|_{l,\infty} \leq C \|U\|_{-q,-1}$. In turn, given $l \in \mathbb{N}_0$ and $U \in \text{Hol}_0(\mathbb{C})$, with $|U|_{l,\infty} < \infty$, we obtain $q \in \mathbb{N}_0$ ($q > 2l$), and $K = (1 - 2^{-q+2})^{-1-2} > 0$ with $\|U\|_{-q,-1} < K |U|_{l,\infty}$. \qed
B  Appell Polynomials and Bell Polynomials

In this section, we recall some fundamental concepts and results well known in the literature and needed in what follows. More precisely, the notion of Stirling numbers and Bell polynomials as well as Appell sequences are introduced (see Subsection B.1) and their connection with Appell polynomials are developed in Subsection B.2.

B.1 Stirling Numbers and Bell Polynomials

Definition B.1 (Stirling numbers of the first kind, see [22]). Given $n, m \in \mathbb{N}_0$, with $0 \leq m \leq n$, the Stirling numbers of the first kind, denoted by $s(n, m)$, is defined as $(-1)^n$ times the number of ways to arrange $n$ distinct objects around $m$ (indistinguishable) circles such that each circle has at least one object.

Definition B.2 (Stirling numbers of the second kind, see [6]). Given two nonnegative integers $n$ and $m$, the Stirling numbers of the second kind, denoted by $S(n, m)$, is defined as the number of ways of distributing $n$ distinct objects into $m$ identical boxes such that no box is empty.

Proposition B.3 (see [22]). The Stirling numbers have the following generating functions

$$(e^z - 1)^k = k! \sum_{n=k}^{\infty} S(n, k) \frac{z^n}{n!}$$

and

$$(\log(1 + z))^k = k! \sum_{n=k}^{\infty} s(n, k) \frac{z^n}{n!}.$$

Proposition B.4. The Stirling numbers can be expressed as

1. $s(n, m) = \frac{A_m}{m!}$, where $A_0 := 1$ and $A_n := \sum_{l_1 + \cdots + l_m = n} \frac{(-1)^{n+m} n!}{l_1! \cdots l_m!}$, $l_i \in \{1, \ldots, n\}$ for all $i = 1, \ldots, m$.

2. $S(n, m) = \frac{B_m}{m!}$, where $B_0 := 1$ and $B_n := \sum_{l_1 + \cdots + l_m = n} \frac{n!}{l_1! \cdots l_m!}$, $l_i \in \{1, \ldots, n\}$ for all $i = 1, \ldots, m$.

Proof. Let $k, n, m \in \mathbb{N}_0$ be given.

1. First, note that $s(0, 0) = 1$. By Proposition B.3,

$$\sum_{n=m}^{\infty} s(n, m) \frac{z^n}{n!} = \frac{(\log(1 + z))^m}{m!}$$

$$= \frac{1}{m!} \left( \sum_{l=1}^{\infty} \frac{z^l}{l!} (-1)^{l+1} (l-1)! \right)^m$$

$$= \frac{1}{m!} \sum_{n=m}^{\infty} \frac{z^n}{n!} \sum_{l_1 + \cdots + l_m = n} \frac{n!}{l_1! \cdots l_m!} \prod_{i=1}^{m} (-1)^{l_i+1} (l_i - 1)!$$

$$= \frac{1}{m!} \sum_{n=m}^{\infty} \frac{z^n}{n!} \sum_{l_1 + \cdots + l_m = n} \frac{(-1)^{n+m} n!}{l_1 \cdots l_m}$$

$$= \sum_{n=m}^{\infty} \frac{A_m}{m!} \frac{z^n}{n!}.$$

The result follows by comparing the coefficients in both sides of the equation.
Note that \( S(0,0) = 1 \). By Proposition B.3,

\[
\sum_{n=m}^{\infty} S(n,m) \frac{z^n}{n!} = \frac{(e^z - 1)^m}{m!}
\]

\[
= \frac{1}{m!} \left( \sum_{l=1}^{\infty} \frac{z^l}{l!} \right)^m
\]

\[
= \frac{1}{m!} \sum_{n=m}^{\infty} \frac{z^n}{n!} \sum_{l_1 + \cdots + l_m = n} \frac{n!}{l_1! \cdots l_m!}
\]

\[
= \sum_{n=m}^{\infty} \frac{B_m^n z^n}{m! n!}
\]

The claim follows again by comparing the coefficients in both sides of the equation.

\[\square\]

**Proposition B.5** (see [22]). For \( n, k \in \mathbb{N}_0 \), we have

1. \( (x)_n = \sum_{k=0}^{n} s(n,k)x^k \),
2. \( x^n = \sum_{k=0}^{n} S(n,k)(x)_k \).

**Definition B.6** (Appell sequences, see [1, 27]). A polynomial sequence \( A_n(x) \) is said to be *Appell sequence for \( g(t) \)* if and only if

\[
\frac{1}{g(t)} \exp(xt) = \sum_{n=0}^{\infty} A_n(x) \frac{t^n}{n!},
\]

where

\[
g(t) = \sum_{n=0}^{\infty} g_n \frac{t^n}{n!}
\]

and \( g_0 \neq 0 \).

**Theorem B.7** (General Leibniz Rule, see [6]). If \( f \) and \( g \) are \( n \)-times differentiable functions, then the product is also \( n \)-times differentiable and its \( n \)-th derivative is given by

\[
(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)}g^{(k)},
\]

where \( \binom{n}{k} = \frac{n!}{k!(n-k)!} \) is the binomial coefficient and \( f^{(0)} = f \).

**Theorem B.8** (Faà di Bruno’s Formula, see [5, 11]). If \( f \) and \( g \) are functions for which all necessary derivatives are defined, then

\[
\frac{d^n}{dt^n} f(g(t)) = \sum m_1! m_2! \cdots m_k! f^{(m_1+\cdots+m_k)}(g(t)) \prod_{j=1}^{n} \left( \frac{g^{(j)}(t)}{j!} \right)^{m_j},
\]

where the sum is over all \( n \)-tuples of non-negative integers \((m_1, \ldots, m_n)\) satisfying the constraint

\[
1 \cdot m_1 + 2 \cdot m_2 + \cdots + n \cdot m_n = n.
\]
**Definition B.9** (Bell polynomials, see [6]). The Bell polynomials $B_{n,k}(\cdot)$, $n, k \in \mathbb{N}_0$ are defined for any $x_1, x_2, \ldots, x_{n-k+1} \in \mathbb{R}$ by

$$B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) := \sum_{j_1! \cdots j_{n-k+1}!} \frac{n!}{j_1! \cdots j_{n-k+1}!} \left(\frac{x_1}{1!}\right)^{j_1} \left(\frac{x_2}{2!}\right)^{j_2} \cdots \left(\frac{x_{n-k+1}}{(n-k+1)!}\right)^{j_{n-k+1}},$$

where the sum is taken over all sequences $j_1, j_2, \ldots, j_{n-k+1}$ of non-negative integers such that the following conditions are satisfied

1. $j_1 + j_2 + \cdots + j_{n-k+1} = k$,
2. $j_1 + 2j_2 + \cdots + (n-k+1)j_{n-k+1} = n$.

For convenience, we use the short notation

$$B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) := B_{n,k}(x_1^n, x_2^n, \ldots, x_{n-k+1}^n).$$

**Theorem B.10** (Faà di Bruno’s formula using Bell polynomial, see [5, 11]). If $f(t)$ and $g(t)$ are functions for which all necessary derivatives are defined, then

$$\frac{d^n}{dt^n} f(g(t)) = \sum_{k=0}^{n} f^{(k)}(g(t)) B_{n,k}(g'(t), g''(t), \ldots, g^{(n-k+1)}(t)).$$

**B.2 Connections Between Appell and Bell Polynomials**

In general, the Appell polynomials $A_{\lambda,\beta}^n(\cdot)$, $n \in \mathbb{N}$, may be explicitly written in terms of the Bell polynomials and the moments of the fractional Poisson measure $\pi_{\lambda,\beta}$, see Theorem B.15 below.

At first recall the definition of the moments of $\pi_{\lambda,\beta}$, for any $n \in \mathbb{N}_0$,

$$M_{\lambda,\beta}(n) := \frac{d^n}{dz^n} \pi_{\lambda,\beta}(z) \bigg|_{z=0} = \frac{d^n}{dz^n} E_{\beta}(\lambda(e^z - 1)) \bigg|_{z=0}.$$

**Lemma B.11.** For every $n \in \mathbb{N}_0$, we have

$$A_{\lambda,\beta}^n(0) = \sum_{k=0}^{n} (-1)^k k! B_{n,k}(M_{\lambda,\beta}(j))_{j=1}^{n-k+1}).$$

**Proof.** By definition of $A_{\lambda,\beta}^n(0)$ we have

$$A_{\lambda,\beta}^n(0) := \frac{d^n}{dz^n} e^{\pi_{\lambda,\beta}(z; x)} \bigg|_{z=x=0}.$$

We denote by $f$ and $g$ the functions

$$f(z) := e^{\pi_{\lambda,\beta}(z; 0)} = \frac{1}{l_{\pi_{\lambda,\beta}}(z)} = \frac{1}{E_{\beta}(\lambda(e^z - 1))},$$

$$g(z) := e^{zx}.$$ 

Hence, using the general Leibniz rule (B.3), $A_{\lambda,\beta}^n(0)$ may be expression as

$$A_{\lambda,\beta}^n(0) = \frac{d^n}{dz^n} (f(z)g(z)) \bigg|_{z=x=0} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)}(z)g^{(k)}(z) \bigg|_{z=x=0}.$$
On one hand, it is clear that
\[ g^{(k)}(z) = x^k g(z). \]

On the other hand, to compute \( f^{(n-k)}(z) \) we use the Faà di Bruno formula given in terms of Bell polynomials, see Theorem B.10. Namely, we represent
\[ f(z) = h\left(l_{\lambda,\beta}(z)\right), \]
where \( h(z) = \frac{1}{z} \) and obtain
\[
\begin{align*}
f^{(n-k)}(z) &= \sum_{j=0}^{n-k} \frac{(-1)^j j!}{(l_{\lambda,\beta}(z))^{j+1}} B_{n-k,j}\left(\left(l_{\lambda,\beta}(z)\right)^{n-k-j+1}\right). \\
&= \sum_{j=0}^{n-k} (-1)^j j! B_{n-k,j}\left(\left(l_{\lambda,\beta}(z)\right)^{n-k-j+1}\right).
\end{align*}
\]

Putting together yields
\[
\begin{align*}
\frac{d^n}{dz^n}(f(z)g(z)) &= \sum_{k=0}^{n} \binom{n}{k} \left[ \sum_{j=0}^{n-k} (-1)^j j! B_{n-k,j}\left(\left(l_{\lambda,\beta}(z)\right)^{n-k-j+1}\right) \right] x^k g(z)
\end{align*}
\]
and evaluating this expression at \( z = x = 0 \) gives the claimed result. \( \square \)

**Theorem B.12.** The Appell polynomials \( A_{\lambda,\beta}^n(\cdot) \), \( n \in \mathbb{N} \), generated by the fractional Poisson measure \( \pi_{\lambda,\beta} \) are given for any \( x \in \mathbb{R} \) in terms of the Bells polynomials as
\[
A_{\lambda,\beta}^n(x) = \sum_{k=0}^{n} \binom{n}{k} \left[ \sum_{j=0}^{n-k} (-1)^j j! B_{n-k,j}\left(\left(M_{\lambda,\beta}(j)\right)^{n-k-j+1}\right) \right] x^k. \quad (B.4)
\]

**Proof.** It follows from (3.7), Lemma B.11 and the uniqueness of the representation in terms of the usual powers \( x^k \). \( \square \)

**Remark B.13.** The property (B.5) of the Appell polynomials \( A_{\lambda,\beta}^n \), \( n \in \mathbb{N}_0 \), is not specific of the fractional Poisson measure \( \pi_{\lambda,\beta} \). In fact, given a probability space \( (\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu) \) such that the Laplace transform \( l_{\mu} \) is an analytic function, then the Appell polynomial sequence generated by the corresponding Wick exponential \( e_\mu(\cdot; x) \) also satisfies the mentioned property. This follows from the definition of the Wick exponential and the same arguments as in the proof of Lemma B.14 and again Proposition 3.3–(P2) from Proposition 3.3.

### B.3 Connections Between Generalized Appell Polynomials and Bell Polynomials

We also give an explicit representation of the generalized Appell polynomials \( C_{\lambda,\beta}^n(\cdot) \), \( n \in \mathbb{N} \), in terms of the Bell polynomials, see Theorem B.15 below.

**Lemma B.14.** For every \( n \in \mathbb{N}_0 \), we have
\[
C_{\lambda,\beta}^n(0) = \sum_{j=0}^{n} (-1)^j j! B_{n,j} \left(\left(M_{\lambda,\beta}(m)\right)^{n-j+1}\right). 
\]
Proof. Following a process analogous to the proof of Lemma B.11, we use the general Leibniz rule on the polynomials $C_n^{\lambda,\beta}(\cdot)$, $n \in \mathbb{N}$ i.e.,

$$C_n^{\lambda,\beta}(0) = \frac{d^n}{dz^n} \left( f(z)g(z) \right) \big|_{z=x} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)}(z)g^{(k)}(z) \big|_{z=x}$$

where

$$f(z) := e^{\pi \lambda,\beta(z;0)} = \frac{1}{\pi \lambda,\beta(\log(1+z))} = \frac{1}{E_{\beta}(\lambda z)},$$

$$g(z) := e^{x \log(1+z)}.$$

Here, we use the Faà di Bruno formula to obtain

$$f^{(n-k)}(z) = \sum_{j=0}^{n-k} \frac{(-1)^j j!}{(l \pi \lambda,\beta(z))^{j+1}} B_{n-k,j} \left( \left( \sum_{p=i}^{\infty} \frac{p!}{(p-i)!} \frac{\lambda^{p-i}}{\Gamma(p\beta + 1)} \right)_{i=1}^{n-k-j+1} \right)$$

and

$$g^{(k)}(z) = \sum_{l=0}^{k} x^l e^{x \log(1+z)} B_{k,l} \left( \left( \frac{(-1)^{n+1}(n-1)!}{(1+z)^n} \right)_{m=1}^{k-l+1} \right).$$

Theorem B.15. The generalized Appell polynomials $C_n^{\lambda,\beta}(\cdot)$, $n \in \mathbb{N}$, generated by the fractional Poisson measure $\pi_{\lambda,\beta}$ are given for any $x \in \mathbb{R}$ in terms of the Bells polynomials as

$$C_n^{\lambda,\beta}(x) = \sum_{k=0}^{n} \binom{n}{k} \left[ \sum_{i=0}^{n-k} (-1)^{i} i! B_{n-k,i} \left( \left( \tilde{M}_{\lambda,\beta}(j) \right)_{j=1}^{n-k-i+1} \right) \right] (x)_k,$$

where $(x)_k$ are the falling factorials.

Proof. The proof follows from Proposition 3.3–(P4) and Lemma B.14.

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