Homology and manifolds with corners

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Abstract

We define a model for the homology of manifolds and use it to describe the intersection product on the homology of compact oriented manifolds and to define homological quantum field theories which generalizes the notions of string topology introduced by Chas and Sullivan and homotopy quantum field theories introduced by Turaev.

1 Introduction

In this paper we propose a chain model for the homology groups of smooth manifolds and show that this model is suited for the study of the intersection product on the homology of compact oriented manifolds. Homology groups of a manifold $M$ are usually defined through the simplicial chain model $C^s(M)$ where

$$C^s(M) = \bigoplus_{i=0}^{\infty} C^s_i(M),$$

and $C^s_i(M)$ denotes the free vector space generated by smooth maps from the $i$-dimensional simplex into $M$, i.e.,

$$C^s_i(M) = \langle c \mid c: \Delta_i \to M \text{ is a smooth map} \rangle.$$

Above $\Delta_i = \{(x_0, \ldots, x_i) \in (\mathbb{R}_{\geq 0})^{i+1} \mid x_0 + \ldots + x_i = 1\}$. For $0 \leq k \leq i$ we have maps $e_k: \Delta_{i-1} \to \Delta_i$ given by $e_k(x_0, \ldots, x_{i-1}) = (x_0, \ldots, x_{k-1}, 0, x_k, \ldots, x_{i-1})$. The differential $\partial: C^s_i(M) \to C^s_{i-1}(M)$ is given by $\partial(c) = \sum_{k=0}^{i} (-1)^k c \circ e_k$ for all $c: \Delta_i \to M$. The singular homology $H(M)$ of the manifold $M$ are by definition the homology groups of the complex $C^s(M)$.

We are interested in the intersection product on homology. Despite its simplicity and good properties the simplicial chains model have major shortcomings when it comes to define the intersection product at the chain level. The main difficulties can be summarized in the following facts

- Given chains $c: \Delta_i \to M$ and $d: \Delta_j \to M$ one obtains a map $c \times d: \Delta_i \times \Delta_j \to M \times M$ which unfortunately does not define a chain on $M \times M$ since $\Delta_i \times \Delta_j$ is a not a simplex.

The way around is to triangulate $\Delta_i \times \Delta_j$ which can be done in a canonical but not

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1All vector spaces in this paper are defined over the complex numbers.
unique way. The choice of triangulations introduces a combinatorics somewhat foreign to the geometric setting on which homology should lie.

- The natural domain for the intersection product $c \cap d$ of transversal chains $c$ and $d$ is $(c \times d)^{-1}(\Delta_M)$, where $\Delta_M = \{(m,m) \in M \times M \mid m \in M\}$. Even if we assume as given canonical triangulations for $\Delta_i \times \Delta_j$ this will be of little help in order to define triangulations on $(c \times d)^{-1}(\Delta_M)$. Such triangulations exist but are neither unique nor canonical.

These facts explain why it is so hard to define the intersection product at the chain level using the simplicial chain model. The main problem lies in the severe restrictions on the domain of simplicial chains, indeed only standard simplices are allowed. In order to overcome this difficulties we propose an alternative model for the homology of manifolds which allows a wide variety of domains for its chains. For each manifold we define a complex $C(M) = \bigoplus_{i=0}^{\infty} C_i(M)$.

The space of degree $i$ chains $C_i(M)$ is constructed as follows

- Let $\overline{C_i(M)}$ be the vector space freely generated by pairs $(K,c)$ such that $K$ is a compact oriented manifold with corners and $c: K \to M$ is a smooth map.

- $C_i(M)$ is the quotient of $\overline{C_i(M)}$ by the relations

1) $(K^{op}, c) = -(K, c)$ where $K^{op}$ is the manifold $K$ provided with the opposite orientation.

2) $(K_1 \sqcup K_2, c_1 \sqcup c_2) = (K_1, c_1) + (K_2, c_2)$.

The differential $\partial: C_i(M) \to C_{i-1}(M)$ is given by

$$\partial(K,c) = \sum_{L \in \pi_0(\partial_1 K)} (\overline{L}, c|_L).$$

The sum above ranges over the connected components $L$ of the first boundary strata $\partial_1 K$ of $K$ provided with the induced boundary orientation. $c|_L$ denotes the restriction of $c$ to the closure of $L$. Complexes $C(M)$ enjoy the following properties that make them worthy of study

- The homology groups $H(M)$ of the complex $C(M)$ are isomorphic to the singular homology groups of $M$. The isomorphism map is induced by the inclusion $i: C(M) \to C(M)$, which is well defined since every simplex is a manifold with corners.

- Given chains $c: K \to M$ and $d: L \to M$ we have a chain $c \times d: K \times L \to M \times M$, since the Cartesian product of oriented manifolds with corners is an oriented manifold with corner. Moreover one can show that

$$\partial_1(K \times L) = (\partial_1 K) \times L + (-1)^{\dim(K)} K \times (\partial_1 L).$$

- If $c: K \to M$ and $d: L \to M$ are transversal chains then $(c \times d)^{-1}(\Delta_M)$ is a submanifold with corners of $K \times L$. Letting $\pi: K \times L$ be the projection onto $K$, the intersection product $c \cap d$ of transversal chains $c$ and $d$ is defined to be $((c \times d)^{-1}(\Delta_M), c \circ \pi)$.  


The purpose of this paper is to systematically use the manifold with corners chain model to study the intersection product on the homology of compact connected oriented manifolds and its various generalizations. The paper is organized as follows:

- In Section 2, we formalize our construction of the complex \( C(M) \) for each manifold \( M \). We show that homology of \( C(M) \) agrees with the singular homology of \( M \).
- In Section 3, we extend the notion of transversality for maps whose domain is a manifold with corners and whose target is a smooth manifold. We use this construction to give a characterization of the intersection product on the homology groups of compact oriented manifolds.
- In Section 4, we use our model for homology to show that the operad \( H(D_d) \) of little discs in dimension \( d \) acts on the homology groups \( H(M^{S^d}) \) of the space of free \( d \)-spheres on a compact connected oriented manifold \( M \), thus obtaining a new proof of the following result due to Voronov [22], see also the book by Cohen, Hess, and Voronov [7].

**Theorem 14.** For \( d \geq 1 \), the graded vector space \( H(M^{S^d}) \) is

a. A differential graded associative algebra if \( d = 1 \).

b. A differential graded twisted Poisson algebra with the commutative associative product of the degree 0 and with the Lie bracket of degree \( (1 - d) \) in the case of odd \( d \geq 3 \).

c. A differential graded twisted Gerstenhaber algebra with the commutative associative product of the degree 0 and with the Lie bracket of degree \( (1 - d) \) in the case of even \( d \geq 2 \).

- In Section 5, we construct the category \( H(M^{S(Y)}) \) of dynamical \( Y \) branes living on a manifold \( M \) and show that this category admits analogues of the notion of transposition and trace for matrices.
- In Section 6, we apply our model for homology to study homological quantum field theories HQLF which lies in the crossroad of two lines of thought. On one hand we have the notion of topological quantum field theory as axiomatized by Atiyah in [1] and further generalized to the notion of homotopical quantum field theory by Turaev in [19] and [20]. On the other hand we have the already mentioned loop product on \( H(M^{S^1}) \) defined by Chas and Sullivan in [6] and further studied by Cohen and Jones [8], and Cohen, Jones and Jun [9]. Combining both constructions we arrived to the notion of HQLF. We give an example of homological quantum field theory in arbitrary dimension.

## 2 Homology using manifolds with corners

Let us start this Section with a brief introduction to manifolds with corners. For \( 0 \leq k \leq n \), we denote by \( H_k^k \) the subspace of \( \mathbb{R}^n \) given by

\[
H_k^k = [0, \infty)^k \times \mathbb{R}^{n-k} = \{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid x_1 \geq 0, \cdots, x_k \geq 0\}.
\]
A subset \( V \subset H^n_k \) is open if there exists an open subset \( W \subset \mathbb{R}^n \) such that \( V = W \cap H^n_k \). Let \( \partial_0 (H^n_k) = (0, \infty)^k \times \mathbb{R}^{n-k} \), and for a set \( V \subset H^n_k \), let \( \partial_0 (V) = V \cap \partial_0 (H^n_k) \). A map \( f: V \to \mathbb{R} \) is said to be smooth if there exits open set \( W \subset \mathbb{R}^n \) and smooth map \( F: W \to \mathbb{R}^n \) such that \( V = W \cap H^n_k \) and \( F|_V = f \). Given open sets \( V_i \subset H^n_{k_i}, \; 0 \leq k_i \leq n, \; i = 1, 2 \), we say that a map \( f: V_1 \to V_2 \) is a diffeomorphism if it is a homeomorphism with inverse \( g: V_2 \to V_1 \) and each coordinate component of \( f \) or \( g \) is a smooth map.

Let \( M \) be an Hausdorff topological space. \( M \) is a \( n \)-manifold with corners if it is locally homeomorphic to \( H^n_k \) for some \( 0 \leq k \leq n \), that is, there exists an open cover \( \mathcal{U} = \{ U_i \}_{i \in \Lambda} \) of \( M \) such that for each \( i \in \Lambda \), there is a map \( \varphi_i: U_i \to H^n_{k_i}, \; 0 \leq k_i \leq n \), which maps \( U_i \) homeomorphically onto an open subset of \( H^n_{k_i} \). We call \((\varphi_i, U_i)\) a chart with domain \( U_i \). The set of charts \( \Phi = \{ (\varphi_i, U_i) \}_{i \in \Lambda} \) is an atlas. Two charts \((\varphi_i, U_i), (\varphi_j, U_j)\) are said to have a smooth overlap if the coordinate changes
\[
\varphi_j \circ \varphi_i^{-1}: \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)
\]
are smooth diffeomorphisms. An atlas \( \Phi \) on \( M \) is called smooth if every pair of charts in it have smooth overlaps. There is a unique maximal smooth atlas which contains \( \Phi \). A maximal atlas \( \Phi \) on \( M \) defines a structure of a smooth manifold with corners on \( M \). The pair \((M, \Phi)\) is called a \( n \)-manifold with corners.

All manifold with corner are naturally stratified spaces with smooth strata. The smooth strata of \( H^n_k \) are given for \( 0 \leq l \leq k \) by
\[
\partial_l H^n_k = \{ x \in H^n_k \mid x_i = 0 \text{ for exactly } l \text{ of the first } k \text{ indices} \}.
\]
Notice that
\[
\partial_l H^n_k = \bigcup_{I \subset \{1, \ldots, k\}} H^n_l,
\]
where \( H^n_l = \{ (x_1, \ldots, x_n) \mid x_i = 0 \text{ if and only if } i \in I \} \). For a manifold with corners \( M \) we set
\[
\partial_l M = \{ m \in M \mid \text{there exists local coordinates mapping } m \text{ to } \partial_l H^n_k \}.
\]
One can checks that \( M = \bigcup_{0 \leq l \leq n} \partial_l M \) and that each \( \partial_l M \) is a submanifold of \( M \). A manifold with corners \( M \) is a manifold if \( M = \partial_0 M \). A manifold with corners is a manifold with boundaries if \( \partial_2 M = \emptyset \). Given a manifold \( M \) we define the graded vector space
\[
\mathcal{C}(M) = \bigoplus_{i=0}^{\infty} \mathcal{C}_i(M),
\]
where \( \mathcal{C}_i(M) \) denotes the vector space
\[
\left\langle (K, c) : \begin{align*}
&K \text{ is an oriented } i\text{-manifold with corners and } c: K \to M \text{ is a smooth map} \\
&\langle (K^{op}, c) - (K, c), (K \sqcup L, c \sqcup d) - (K, c) - (L, d) \rangle
\end{align*} \right\rangle.
\]
We recall that \( M^{op} \) denotes the manifold \( M \) provided with the opposite orientation. Figures 1 and 2 below show a couple of examples of chains whose domain are manifold with corners.
Figure 1: Chain with domain a manifold with corners.

Figure 2: Chain with domain a manifold with corners.

**Definition 1.** Let the map \( \partial: C^i(M) \to C^{i-1}(M) \) be given by \( \partial(K, c) = \sum L (\overline{L}, c|_\overline{L}) \), where the sum runs over the connected components of \( \partial_1 K \) provided with the induced boundary orientation and \( c|_\overline{L} \) denotes the restriction of \( c \) to the closure of \( L \).

With this notation we have the following

**Theorem 2.** \((C(M), \partial)\) is a differential complex. Moreover \( H(C(M), \partial) = H(M) \).

**Proof.** \( \partial^2 = 0 \) since

\[
\partial^2(K, c) = \sum_{L \in \pi_0(\partial_2 K)} [(\overline{L}, c|_{\overline{L}}) + (\overline{L}^p, c|_{\overline{L}})].
\]

There is an obvious inclusion \( i: C^s(M) \to C(M) \). The map \( i \) is quasi-isomorphism since any manifold with corners can be triangulated \( \Box \) and thus any chain in \( C(M) \) is homologous to a chain in \( C^s(M) \).

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3 Transversal intersection of manifolds with corners

In this Section we study transversal intersection for manifold with corners. Let us first state a useful fact.

**Lemma 3.** Let \( M \) and \( N \) be manifolds with corners then \( M \times N \) is also a manifold with corners. Moreover if \( x \in \partial_k M \) and \( y \in \partial_k N \) then \( (x, y) \in \partial_{k+k}(M \times N) \).

**Definition 4.** Let \( N_0 \) \((N_1)\) be a \( n_0 \) \((n_1)\)-manifold with corners and \( M \) be a smooth oriented manifold. Let \( f_0: N_0 \to M \) and \( f_1: N_1 \to M \) be smooth maps. We say that \( f_0 \) and \( f_1 \) are

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\(^2\)We thank J. P. Brasselet, M. Goresky, and R. Melrose for helpful comments on the triangulation of manifold with corners. See references \[10\], \[12\] and \[21\] for more information on triangulability.
transversal, $f_0 \pitchfork f_1$, if for $0 \leq k \leq n_0$, $0 \leq s \leq n_1$, $f_0 |_{\partial_k(N_0)}$ and $f_1 |_{\partial_k(N_1)}$ are transversal, i.e., for $x_0 \in \partial_k(N_0)$ and $x_1 \in \partial_k(N_1)$ such that $f_0(x_0) = f_1(x_1) = m$ we have that $d(f_0)(T_{x_0}\partial_k(N_0)) + d(f_1)(T_{x_1}\partial_s(N_1)) = T_mM$.

The next couple of results are generalization of the corresponding results for manifold with boundaries proved by Guillemin and Pollack in [11].

**Lemma 5.** Let $M$ be a smooth manifold and $f = (f_1, \cdots, f_s): M \to \mathbb{R}^s$ a smooth map with regular value $(0, \cdots, 0)$. Then $\{x \in M \mid f_i(x) \geq 0, \ i = 1, \cdots, s\}$ is a manifold with corners.

**Proof.** $\{x \in M \mid f(x) > 0\}$ is a open subspace of $M$ thus it is a smooth manifold. Let $x \in M$ be such that $f(x) = 0$. Since 0 is a regular value then $f$ is locally equivalent to the submersion $\pi^s: \mathbb{R}^n \to \mathbb{R}^s$ given by $\pi^s(x_1, \cdots, x_n) = (x_1, \cdots, x_s)$. The desired result holds since locally $\{x \in M \mid f_i(x) \geq 0, \ i = 1, \cdots, s\} = \{x \in \mathbb{R}^n \mid x_i \geq 0, \ i = 1, \cdots, s\}$. $\square$

**Theorem 6.** Let $M$ be a $m$-manifold and $N$ be a $n$-manifold with corners. Let $f: N \to M$ be a smooth map and $i: P \hookrightarrow M$ be an embedded $p$-submanifold of $M$. If $f \pitchfork i$ then $f^{-1}(P)$ is a submanifold with corners of $N$.

**Proof.** Since $f |_{\partial_0(N)}$ is transversal to $i$ we have that $f^{-1}(P) \cap \partial_0(N)$ is a manifold of codimension $m - p$. Consider $x \in f^{-1}(P) \cap \partial_k(N)$ and let $l$ be a submersion from a neighborhood of $f(x)$ in $M$ to $\mathbb{R}^{n-p}$ such that in this neighborhood $P = l^{-1}(0)$. Choose a local parametrization $h: U \to N$ around $x$ where $U$ is an open subset of $H^t_0$, $0 \leq k \leq t \leq n$, and set $g = l \circ f \circ h$. Since $h$ is a diffeomorphism $f^{-1}(P)$ is a manifold with corners in a neighborhood of $x$ if and only if $(f \circ h)^{-1}(0) = g^{-1}(0)$ is a manifold with corners near $y = h^{-1}(x) \in \partial_k(U)$. The transversality condition $d(f)(T_x\partial_k(N)) + T_f(x)(P) = T_f(x)(M)$ implies that $x$ is a regular point of $l \circ f$, i.e., that $g$ is regular at $y$. Since $g$ is smooth then it can be extended to a smooth map $G$ on a neighborhood of $y$ in $\mathbb{R}^n$. As $dG = dg$, $G$ is also regular at $y$ and thus $G^{-1}(0)$ is a smooth submanifold of $\mathbb{R}^n$ in a neighborhood of $y$.

Since $g^{-1}(0) = G^{-1}(0) \cap H^t_0$ in a neighborhood of $y$, we must show that $G^{-1}(0) \cap H^t_0$ is a manifold with corners in a neighborhood of $y$. The map $\pi = (\pi_1, \cdots, \pi_t): G^{-1}(0) \subseteq \mathbb{R}^n \to \mathbb{R}^t$, is such that $G^{-1}(0) \cap H^t_0 = \{s \in G^{-1}(0) \mid \pi_i(s) \geq 0, \ i = 1, \cdots, t\}$.

We claim that $(0, \cdots, 0)$ is a regular value for $\pi$. Otherwise, there is a point $z \in G^{-1}(0)$ such that $\pi(z) = (0, \cdots, 0)$ and $\dim(d_z\pi) < t$. Thus, there exist at least a linear relation $a_1d_z\pi_1 + \cdots + a_td_z\pi_t = 0$. Making a further change of variables we may assume that the linear relation is $d_z\pi_1 = 0$.

The fact that $d_z\pi_1$ is zero on $T_z(G^{-1}(0))$ means that the first coordinate of every vector in $T_z(G^{-1}(0))$ is zero, i.e., that $T_z(G^{-1}(0)) \subset \{0\} \times \mathbb{R}^{n-1}$. The kernel of $dg = dG: \mathbb{R}^n \to \mathbb{R}$ is $T_z(G^{-1}(0))$ and $d_z(\partial_{11}g)$ is the restriction of $d_zg: \mathbb{R}^n \to \mathbb{R}$ to $\mathbb{R}^{n-1}$. Since the kernel of $dg$ is contained in $\mathbb{R}^{n-1}$ then the linear maps $dg: \mathbb{R}^n \to \mathbb{R}$ and $d(\partial_{11}g): \mathbb{R}^{n-1} \to \mathbb{R}$ must have the same kernel. By transversality both maps are surjective, so the kernel of $dg$ has dimension $n - 1$ whereas the kernel of $d(\partial_{11}g)$ has dimension $n - 2$, which is a contradiction. Lemma 5 finishes the proof of Theorem 6. $\square$
Lemma 7. Let $K$ and $L$ be manifolds with corners and $M$ be an oriented manifold. Assume that $c: K \to M$ and $d: L \to M$ are transversal maps, then
\[ K \times_M L = \{(a, b) \in K \times L \mid x(a) = y(b)\} \]
is an oriented manifold with corners. Moreover $T(K \times_M L) = TK \times_T M TL$.

Proof. $K \times L$ is a manifold with corners. By Lemma 3. Since $c$ and $d$ are transversal the map $c \times d: K \times L \to M \times M$ is transversal to $\Delta: M \to M \times M$. Theorem 6 then guarantees that $(c \times d)^{-1}(\Delta(M)) = K \times_M L$ is a manifold with corners.

Lemma 8. Let $K$ and $L$ be manifolds with corners and $M$ be a oriented manifold. Assume also that $c: K \to M$ and $d: L \to M$ are transversal maps. Then
\begin{itemize}
  \item $\partial_x (K \times L) = (\partial_1 K) \times L + (-1)^{\dim(K)} K \times (\partial_1 L)$.
  \item $\partial_x (K \times_M L) = (\partial_1 K) \times_M L + (-1)^{\dim(K)} K \times_M (\partial_1 L)$.
\end{itemize}

Lemma 8 guarantees that the intersection product on homology $H(M)$ to be defined below through the intersection of transversal chains is indeed well defined. The following result gives us information on the algebraic structure on the homology of a compact connected oriented manifold $M$.

Theorem 9. $H(M)[\dim(M)]$ is a Frobenius algebra.

The Frobenius algebra structure on $H(M)[\dim(M)]$ is given
\begin{itemize}
  \item An intersection product $\cap: H(M)[\dim(M)] \otimes H(M)[\dim(M)] \to H(M)[\dim(M)]$, given by $[K, c] \cap [L, d] = [K \times_M L, c \circ \pi]$ for transversal chains $c: K \to M$ and $d: L \to M$.
  \item There is a trace map $\text{tr}: H(M)[\dim(M)] \to \mathbb{C}$ defined to be zero in non-zero degrees and the identity in degree $-d$.
  \item The bilinear form $\langle \cdot, \cdot \rangle: H(M)[\dim(M)] \otimes H(M)[\dim(M)] \to \mathbb{C}$ defined by $\langle a, b \rangle = \text{tr}(a \cap b)$, for $a, b \in H(M)[\dim(M)]$, is non-degenerated by Poincaré duality.
  \item The unit in $H(M)[\dim(M)]$ is defined by the identity map $I: M \to M$.
\end{itemize}

4 Algebraic structures on $H(M^{S^d})[\dim(M)]$

Let $D^d = \{x \in \mathbb{R}^d \mid \|x\| \leq 1\}$. A little disc in $D^d$ is an affine transformation $T_{a,r}: D^d \to D^d$ given by $T_{a,r}(x) = rx + a$, where $0 < r < 1$ and $a \in D^d$. For $n \geq 0$, let us introduce the topological spaces
\[ D_d(n) = \left\{ (T_{x_1, r_1}, \ldots, T_{x_n, r_n}) \mid \frac{x_i, x_j \in D^d, 0 \leq r_i < 1 \text{ such that } i \neq j}{\im(T_{x_i, r_i}) \cap \im(T_{x_j, r_j}) = \emptyset \text{ for all } 1 \leq i, j \leq n} \right\}.
\]
Notice that the disc with center $a$ and radius $r$ is obtained as the image of the transformation $T_{a,r}$ applied the standard disc $D^d$. The little discs operad was introduced by J. M. Boardman and R. M. Vogt [3], and Peter May [15].
Definition 10. The little discs operad in dimension \(d\), \(D_d = \{D_d(n)\}_{n \geq 0}\) is the topological operad with the following structure

a. The composition \(\gamma_k : D_d(k) \times D_d(n_1) \times \cdots \times D_d(n_k) \to D_d(j)\), with \(j = \sum n_k\), are defined as follows: given \((T_{x_1,r_1}, \ldots, T_{x_k,r_k}) \in D_d(k)\) and \((T_{x_{i_1},r_{i_1}}, \ldots, T_{x_{i_{n_i}},r_{i_{n_i}}}) \in D_d(n_i)\) for \(1 \leq i \leq k\), by

\[
\gamma_k((T_{x_1,r_1}, \ldots, T_{x_k,r_k}); (T_{x_{i_1},r_{i_1}}, \ldots, T_{x_{i_{n_i}},r_{i_{n_i}}}), \ldots, (T_{x_{k_1},r_{k_1}}, \ldots, T_{x_{k_n},r_{k_n}})) = (T_{x_1,r_1} \circ T_{x_{i_1},r_{i_1}}, \ldots, T_{x_1,r_1} \circ T_{x_{i_{n_i}},r_{i_{n_i}}}, \ldots, T_{x_k,r_k} \circ T_{x_{k_1},r_{k_1}}, \ldots, T_{x_k,r_k} \circ T_{x_{k_n},r_{k_n}}).
\]

b. The action of the symmetric group \(S_n\) on \(D_d(n)\) is given by

\[
\sigma(T_{x_1,r_1}, \ldots, T_{x_n,r_n}) = (T_{x_{\sigma^{-1}(1)},r_{\sigma^{-1}(1)}}, \ldots, T_{x_{\sigma^{-1}(n)},r_{\sigma^{-1}(n)}}),
\]

for all \(\sigma \in S_n\) and \((T_{x_1,r_1}, \ldots, T_{x_n,r_n}) \in D_d(n)\).

Since \(D_d\) is a topological operad its homology \(H(D_d)\) is an operad in the category of graded vector spaces. So the following definition makes sense

Definition 11. A \(d\)-algebra is an algebra over \(H(D_d)\) in the category of graded vector spaces.

Let us fix a compact oriented manifold \(M\).

Definition 12. We denote by \(M^{\ast d}\) the set of all smooth maps \(\alpha : D^d \to M\) such that \(\alpha\) is constant in an open neighborhood of \(\partial(D^d)\). \(M^{\ast d}\) is given the compact-open topology.

Let \(C(M^{\ast d}) = \bigoplus_{i=0}^{\infty} C_i(M^{\ast d})\) be the vector space generated by chains \(c : K \to M^{\ast d}\) such that the associated map \(\hat{c} : K \times D^d \to M\) given by \(\hat{c}(k,p) = (c(k))(p)\) for \((k,p) \in K \times D^d\) is smooth. Let \(c : M^{\ast d} \to M\) be the map given by \(c(\alpha) = \alpha(\partial(D^d))\). We denote by \(e_*\) the induced maps \(e_* : C(M^{\ast d}) \to C(M)\) and \(e_* : H(M^{\ast d}) \to H(M)\).

Theorem 13. \(H(M^{\ast d})[\dim(M)]\) is a \(d\)-algebra.

Proof. We construct maps \(\theta_n : H(D_d(n)) \otimes H(M^{\ast d})^n \to H(M^{\ast d})\). Assume that \([K,c] \in H(D_d(n))\) and \([K_i,c_i] \in H(M^{\ast d})\) for \(1 \leq i \leq n\). Let \([L,d] = \theta_n([K,c]; [K_1,c_1], \ldots, [K_n,c_n])\) be defined as follows

- \(L = K \times K_1 \times_M \cdots \times_M K_n\).
- In order to define \(d : L \to M^{\ast d}\) let \((k_1, \ldots, k_n) \in L\) and assume that \(c(k)\) is such that \(c(k) = (T_{p_1(k),r_1(k)}, \ldots, T_{p_n(k),r_n(k)})\), then the map \(d(k_1, \ldots, k_n) : S^d \to M\) is given for \(y \in S^d\) by

\[
d(k_1, \ldots, k_n)(y) = \begin{cases} 
  e(c_1(k_1)) & \text{if } y \notin \overline{\text{im}(T_{p_1(k),r_1(k)})} \\
  c_i(k_i)(y - p_i(k) / r_i(k)) & \text{if } y \in \overline{\text{im}(T_{p_i(k),r_i(k)})}
\end{cases}
\]

\[\Box\]
Next result follows from Theorem 13 above and Theorem 3 of [13].

**Theorem 14.** For \( d \geq 1 \), \( H(M^{S^d})[\dim(M)] \) is

- a. A differential graded associative algebra if \( d = 1 \).
- b. A differential graded twisted Poisson algebra with the commutative associative product of the degree 0 and with the Lie bracket of degree \((1 - d)\) in the case of odd \( d \geq 3 \).
- c. A differential graded twisted Gerstenhaber algebra with the commutative associative product of degree 0 and with the Lie bracket of degree \((1 - d)\) in the case of even \( d \geq 2 \).

## 5 Branes category

Let \( M \) be a compact connected oriented manifold. Let \( N_0 \) and \( N_1 \) be connected oriented embedded submanifolds of \( M \). Let \( Y \) be a compact manifold.

**Definition 15.** Let \( M^{S(Y)}(N_0, N_1) \) be the set of smooth maps \( f: Y \times [-1,1] \to M \) such that

- \( f(y, -1) \in N_0, \ f(y, 1) \in N_1 \) for \( y \in Y \).
- \( f \) is constant on open neighborhoods of \( Y \times \{-1\} \) and \( Y \times \{1\} \), respectively.

We give \( M^{S(Y)}(N_0, N_1) \) the compact-open topology. Notice that \( M^{S(Y)}(N_0, N_1) \) is a subspace of \( \text{Map}(S(Y), M) \) where \( S(Y) = Y \times [-1,1] / \sim \) and \( \sim \) is the equivalence relation given by \( y_1 \times \{-1\} \sim y_2 \times \{-1\} \), and \( y_1 \times \{1\} \sim y_2 \times \{1\} \) for \( y_1, y_2 \in Y \).

Let \( C(M^{S(Y)}(N_0, N_1)) = \bigoplus_{i=0}^{\infty} C_i(M^{S(Y)}(N_0, N_1)) \) be the vector space generated by chains \( c: K \to M^{S(Y)}(N_0, N_1) \) such that the maps

\[
e_{-1}(x): K \times Y \to N_0 \quad \text{and} \quad e_1(x): K \times Y \to N_1
\]
given by \( e_i(c)(x, y) = c(x)(y, i) \) for \( i = -1, 1 \) are smooth. Let

\[
e_{-1}: M^{S(Y)}(N_0, N_1) \to N_0 \quad \text{and} \quad e_1: M^{S(Y)}(N_0, N_1) \to N_1
\]
be the maps given by \( e_{-1}(f) = f(y, -1) \in N_0 \) and \( e_1(f) = f(y, 1) \in N_1 \), respectively. We denote by \( e_i^* \) the induced maps

\[
e_i^*: C(M^{S(Y)}(N_0, N_1)) \to C(N_0) \quad \text{and} \quad e_i^*: H(M^{S(Y)}(N_0, N_1)) \to H(N_0)
\]
for \( i = -1, 1 \).

Let us introduce the category \( H(M^{S(Y)}) \) given by

- Objects in \( H(M^{S(Y)}) \) are oriented connected compact embedded submanifolds of \( M \).
- Morphisms in \( H(M^{S(Y)}) \) are given by \( H(M^{S(Y)}(N_0, N_1)) = H(M^{S(Y)}(N_0, N_1)) \).
• The identity morphism $[N, I_N] \in \text{H}(M^{S(Y)})(N, N)$ is given by the map

$$\bar{T}_N: N \times S(Y) \to M,$$

given by $\bar{T}_N(n, s) = n$ for $n \in N$ and $s \in S(Y)$.

• Given $[K, c] \in \text{H}(M^{S(Y)})(N_0, N_1)$ and $[L, d] \in \text{H}(M^{S(Y)})(N_1, N_2)$ the composition morphism $[L, d] \circ [K, c] \in \text{H}(M^{S(Y)})(N_0, N_1)$ is given for transversal cycles $c$ and $d$ by the map

$$[L, d] \circ [K, c]: L \times_M K \times Y \times [-1, 1] \to M$$

defined by

$$[L, d] \circ [K, c](l, k, y, t) = \begin{cases} \tilde{c}(k, y, t) & \text{if } t \in [0, \frac{1}{2}] \\ \tilde{d}(l, y, t) & \text{if } t \in [\frac{1}{2}, 1] \end{cases}$$

Summing up we obtain

**Theorem 16.** $\text{H}(M^{S(Y)})$ is a category.

An interesting feature of the category $\text{H}(M^{S(Y)})$ is that it comes equipped with analogues of the operation of transposition and trace on matrices.

**Theorem 17.** There is a contravariant functor $r: \text{H}(M^{S(Y)}) \to \text{H}(M^{S(Y)})$ identical on objects, given for $N_0$ and $N_1$ by

$$r: \text{H}(M^{S(Y)})(N_0, N_1) \to \text{H}(M^{S(Y)})(N_1, N_0)$$

taking $[K, c]$ into $[K, r(c)]$ defined as follows: for $k \in K$, $y \in Y$ and $-1 \leq t \leq 1$, we have that

$$r(c)(k)(y, t) = c(k)(y, -t).$$

Let $M^{Y \times S^1}$ be the space of smooth maps $f: Y \times S^1 \to M$ provided with the compact-open topology. We denote by $\text{H}(M^{Y \times S^1}) = \bigoplus_{i=0}^{\infty} \text{H}_i(M^{Y \times S^1})$ the vector space generated by chains $c: K \to M^{Y \times S^1}$ such that the induced map $\tilde{c}: K \times Y \times S^1 \to M$ is smooth.

**Theorem 18.** For each object $N$ in $\text{H}(M^{Y \times S^1})$ there is a map

$$tr: \text{H}(M^{S(Y)})(N, N) \to \text{H}(M^{Y \times S^1}),$$

where for $[K, c]$ such that $e_{-1*}(c)$ is transversal to $e_{1*}(c)$ we define

$$tr([K, c]) = [(e_{-1*}(c) \times e_{1*}(c)) \circ \Delta]^{-1}(\Delta_M, c \circ \pi),$$

where $\Delta: M \to M \times M$ is the diagonal map.
6 Homological quantum field theory

The theory of cobordism was introduced by Rene Thom [17]. Based on the notion of cobordisms Michael F. Atiyah [1] and G. B. Segal [16] (see also [2], [13], and [18]) introduced the axioms for topological quantum field theories TQFT and conformal field theories CFT, respectively. V. Turaev [19] introduced the axioms for homotopical quantum field theories. Essentially the axioms of Atiyah may be summarized as follows: 1) Considerer the monoidal category Cob\(_n\) of \(n\)-dimensional cobordism. 2) Define the category of TQFT as the category MFunc(Cob\(_n\), vect) of monoidal functors \(F: \text{Cob}_n \to \text{vect}\). Segal’s definition of CFT may also be recast as a category of monoidal functors.

Let us fix a few conventions. \(\pi_0(M)\) denotes the set of connected components of \(M\). We define the completion of a manifold \(N\) to be \(\overline{N} = \prod_{c \in \pi_0(N)} c\). Figure 3 gives an example of a manifold and its completion. We denote by \(D(M)\) the set of embedded connected oriented submanifolds of \(M\). By convention the empty set is assumed to be a \(n\)-dimensional manifold for all \(n \in \mathbb{N}\).

Following the pattern discussed above we define the category HLQFT\(_d(M)\) of homological quantum field theories as follows

a. First we construct a category Cob\(_d^M\), object of which are called extended cobordisms.

b. Second we define HLQFT\(_d(M)\) to be the category of monoidal functors MFunc(Cob\(_d^M\), vect).

Objects in Cob\(_d^M\) are triples \((N, f, \prec)\) such that

- \(N\) is a compact oriented manifold of dimension \(d - 1\).
- \(f: \pi_0(N) \to D(M)\) is a map.
- \(\prec\) is a linear ordering on \(\pi_0(N)\).

For objects \((N_0, f_0, \prec_0)\) and \((N_1, f_1, \prec_1)\) in Cob\(_d^M\) we set

\[
\text{Cob}_d^M((N_0, f_0, \prec_0), (N_1, f_1, \prec_1)) = \text{Cob}_d^M((N_0, f_0, \prec_0), (N_1, f_1, \prec_1)) / \sim
\]

where \(\text{Cob}_d^M((N_0, f_0, \prec_0), (N_1, f_1, \prec_1))\) is the set of triples \((P, \alpha, |c|)\) such that
a. $P$ is a compact oriented $d$-manifold with boundaries.

b. $\alpha: N_0 \sqcup N_1 \times [0, 1) \to \text{im}(\alpha) \subseteq P$ is a diffeomorphism. $\alpha|_{N_0 \sqcup N_1} \to \partial P$ is such that $\alpha|_{N_0}$ reverses the orientation and $\alpha|_{N_1}$ preserves the orientation.

c. If $\alpha \in H(\text{Map}(P, M)_{f_0, f_1})[\text{dim}(N_1)]$, where $\text{Map}(P, M)_{f_0, f_1}$ denotes the set of smooth maps $g: P \to M$ such that $g$ is constant on a neighborhood of each connected component of its boundary $\partial P$. If $\alpha \in \text{Map}(P, M)_{f_0, f_1}$ we define $e_i(\alpha) \in \overline{N_i}$ by $e_i(\alpha)(t) = \alpha(t)$ for $t \in \pi_0(N_i)$ and $i = 0, 1$.

Triples $(P, \alpha, \xi)$ and $(P', \alpha', \xi')$ in $\text{Cob}_d^M((N_0, f_0, <0), (N_1, f_1, <1))$ are equivalent according to $\sim$ if and only if

- There exists an orientation preserving diffeomorphism $\varphi: P_1 \to P_2$ such that:
- $\varphi \circ \alpha = \alpha'$.
- $\varphi_\ast(\xi) = \xi'$.

Assume we are given $(P_1, \alpha_1, [c_1]) \in \text{Cob}_d^M((N_0, f_0, <0), (N_1, f_1, <1))$ and $(P_2, \alpha_2, [c_2]) \in \text{Cob}_d^M((N_1, f_1, <1), (N_2, f_2, <2))$.

Assume $[c] \in H(\text{Map}(P, M)_{f_0, f_1})$ is given by a smooth map $c: K \times P \to M$ for a compact oriented manifold with corners $K$.

The composition morphism

$$(P_2, \alpha_2, [c_2]) \circ (P_1, \alpha_1, [c_1]) \in \text{Cob}_d^M((N_0, f_0, <0), (N_2, f_2, <2)))$$

is the triple $(P_2 \circ P_1, \alpha_2 \circ \alpha_1, [c_2 \circ c_1])$ where

- $P_2 \circ P_1 = P_1 \sqcup_{N_1} P_2$.
- $\alpha_2 \circ \alpha_1 = \alpha_2|_{N_2} \sqcup \alpha_1|_{N_0}$.

- The domain of $c_2 \circ c_1$ is $L \times \overline{N_1} K$. The map $c_2 \circ c_1: L \times \overline{N_1} K \to \text{Map}(P_1 \circ P_2, M)_{f_0, f_2}$ has associated map

$$\overline{c_1 \circ c_2}: L \times \overline{N_1} K \times (P_1 \circ P_2) \to M$$

given by $\overline{c_2 \circ c_1}(l, k, p) = \overline{c_1}(k, p)$ if $p \in P_1$ and $\overline{c_2 \circ c_1}(k, l, p) = \overline{c_2}(l, p)$ if $p \in P_2$. Figure 4 represents a $n$-cobordism enriched over $M$ and Figure 5 shows a composition of $n$-cobordism enriched over $M$. 

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The identity morphism $(N \times [0,1], \alpha, [c]) \in \text{Cob}_d^M((N,f,<(N,f,<))$ is determined by the map $\tilde{c}: N \times N \times [0,1] \rightarrow M$ given by $\tilde{c}(n,m,t) = n$ for $n, m \in N$ and $t \in [0,1]$.

Thus we obtain

**Theorem 19.** $(\text{Cob}_n^M, \sqcup, \emptyset)$ is a monoidal category with disjoint union $\sqcup$ as product and empty set as unit.

**Definition 20.** $(\text{Cob}_{n,r}^M, \sqcup, \emptyset)$ is the full monoidal subcategory of $\text{Cob}_n^M$ without the unit.

Given monoidal categories $\mathcal{C}$ and $\mathcal{D}$ we denote by $\text{MFunc}(\mathcal{C}, \mathcal{D})$ the category of monoidal functors from $\mathcal{C}$ to $\mathcal{D}$. We are finally ready to introduce our main definition.

**Definition 21.** $\text{HLQFT}_d^d(M) = \text{MFunc}(\text{Cob}_n^M, \text{vect})$ and $\text{HLQFT}_{d,r}(M) = \text{MFunc}(\text{Cob}_{d,r}^M, \text{vect})$. $\text{HLQFT}_d^d(M)$ is the category of $d$ dimensional homological quantum field theories. $\text{HLQFT}_{d,r}(M)$ is the category of the $d$ dimensional restricted homological quantum field theories.

Explicitly the category $\text{HLQFT}_d^d(M)$ is given by

- Objects in $\text{HLQFT}_d^d(M)$ are monoidal functors $F: \text{Cob}_n^M \rightarrow \text{vect}$.
- Morphisms in $\text{HLQFT}_d^d(M)(F,G)$ are natural transformations $T: F \rightarrow G$ for objects $F, G$ in $\text{HLQFT}_d^d(M)$.

We now give a construction which yields our main example of homological field theory. Consider the map $H: \text{Cob}_{d,r}^M \rightarrow \text{vect}$ given on objects by

$$H: \text{Ob}(\text{Cob}_{d,r}^M) \rightarrow \text{Ob}(\text{vect})$$

$$(N,f,<) \mapsto H(N,f,<) = H(N)$$

The image under $H$ of a morphism $(P,\alpha, [c]) \in \text{Cob}_{n,r}^M((N_0,f_0,<_0),(N_1,f_1,<_1))$ is the linear map

$$H(P,\alpha, [c]): H(N_0,f_0,<_0) \rightarrow H(N_1,f_1,<_1)$$

$$[h] \mapsto [H(P,\alpha, [c])(h)]$$
Let \([d] \in H(\text{Map}(P,M),_{f_0,f_1})\) be given by the chain \(d: L \to \text{Map}(P,M)_{f_0,f_1}\). Below we use the map \(e_0(d): L \to \overline{N}_0\). Assume that \(h\) is given by a map \(h: O \to \overline{N}_0\). We define the domain of \(H(P,\alpha,d)(h)\) by \(L \times \overline{N}_0 \times O\) and let the map

\[
H(P,\alpha,d)(h): L \times \overline{N}_0 \times O \to \overline{N}_1
\]

be given by

\[
H(P,\alpha,d)(h)(l,o) = e_1(d(l)),
\]

for \((l,o) \in L \times \overline{N}_0 \times O\).

**Theorem 22.** \(H\) defines a restricted homological quantum field theory.

**Proof.** Let

\[
(P_1,\alpha_1,[c_1]) \in \text{Cob}^M_{n,r}((N_0,f_0,<), (N_1, f_1,<))
\]

and

\[
(P_2,\alpha_2,[c_2]) \in \text{Cob}^M_{n,r}((N_1,f_1,<), (N_2, f_2,<)),
\]

where \(c_1: K_1 \to \text{Map}(P,M)_{f_0,f_1}\) and \(c_2: K_2 \to \text{Map}(P,M)_{f_1,f_2}\). We want to check that

\[
H((P_2,\alpha_2,[c_2]) \circ (P_1,\alpha_1,[c_1])) = H(P_2,\alpha_2,[c_2]) \circ H(P_1,\alpha_1,[c_1]).
\]

The domain of \((P_2,\alpha_2,c_2) \circ (P_1,\alpha_1,c_1)\) is \(K_2 \times \overline{N}_1 K_1\), thus for \([h] \in H(\overline{N}_0)\) given by a chain \(h: L \to \overline{N}_0\) the domain of

\[
H((P_2,\alpha_2,c_2) \circ (P_1,\alpha_1,c_1))(h)
\]

is \((K_2 \times \overline{N}_1 K_1) \times \overline{N}_0 L\). On the other hand the domain of

\[
H(P_2,\alpha_2,c_2)(H(P_1,\alpha_1,c_1)(h))
\]

is \(K_2 \times \overline{N}_1 (K_1 \times \overline{N}_0 L)\). Thus we see that the domains agree. It is easy to check that the corresponding maps also agree. \(\square\)

We close this paper studying the functorial properties of the correspondence

\[M \to \text{HLQFT}_d(M).\]

Let \(\text{Oman}\) be the groupoid whose objects are compact oriented smooth manifolds and whose morphisms are orientation preserving diffeomorphisms. By \(\text{Cat}\) we denote the category of small categories.

**Theorem 23.** The map \(\text{HLQFT}_d(-): \text{Oman}^{\text{op}} \to \text{Cat}\) given on objects by \(M \to \text{HLQFT}_d(M)\) is functorial.

**Proof.** Let \(M,N\) be objects in \(\text{Oman}\) and \(\varphi \in \text{Oman}^{\text{op}}(M,N)\) an orientation preserving diffeomorphism \(\varphi: N \to M\). \(\varphi\) induces a functor \(\varphi_*: \text{Cob}^N_d \to \text{Cob}^M_d\) which is given on objects by \(\varphi_*(N,f,<) = (N,\varphi \circ f,\varphi_*(<))\) and similarly on morphisms. Thus we get a functor \(\varphi^*: \text{MFunc}(\text{Cob}^M_d,\text{vect}) \to \text{MFunc}(\text{Cob}^N_d,\text{vect})\) given by \(\varphi^*(\mathcal{F}) = \mathcal{F} \circ \varphi_*\) for any monoidal functor \(\mathcal{F}: \text{Cob}^N_d \to \text{vect}\). \(\square\)
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