ON STABILITY OF THE ERDŐS-RADEMACHER PROBLEM

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Abstract. Mantel’s theorem states that every $n$-vertex graph with \( \left\lfloor \frac{n^2}{4} \right\rfloor + t \) edges, where $t > 0$, contains a triangle. The problem of determining the minimum number of triangles in such a graph is usually referred to as the Erdős-Rademacher problem. Lovász and Simonovits proved that there are at least $t\lfloor n/2 \rfloor$ triangles in each of those graphs. Katona and Xiao considered the same problem under the additional condition that there are no $s - 1$ vertices covering all triangles. They settled the case $t = 1$ and $s = 2$. Solving their conjecture, we determine the minimum number of triangles for every fixed pair of $s$ and $t$, when $n$ is sufficiently large. Additionally, solving another conjecture of Katona and Xiao, we extend the theory for considering cliques instead of triangles.

1. Introduction

A classical result in extremal combinatorics is Mantel’s theorem [11]. It says that every $n$-vertex graph with at least $\left\lfloor \frac{n^2}{4} \right\rfloor + 1$ edges contains a triangle. There have been various extensions of Mantel’s theorem. Turán [13] generalized it for cliques: Every $n$-vertex graph with at least $t_r(n) + 1$ edges contains a clique of size $r + 1$, where $t_r(n)$ is the number of edges of the complete balanced $r$-partite graph on $n$ vertices. A result of Rademacher, see [3], says that every graph on $n$ vertices with $\left\lfloor \frac{n^2}{4} \right\rfloor + 1$ edges contains at least $\left\lfloor \frac{n}{2} \right\rfloor$ triangles and this is best possible. Erdős [3, 4] conjectured that an $n$-vertex graph with $\left\lfloor \frac{n^2}{4} \right\rfloor + t$ edges for $t < \frac{n}{2}$ contains at least $t\lfloor \frac{n}{2} \rfloor$ triangles. This was proven by Lovász and Simonovits [10]. Xiao and Katona [14] proved a stability variant of the Lovász and Simonovits result for $t = 1$: If there is no vertex contained in all triangles, then there are at least $n - 2$ triangles in $G$. We will prove two extensions of this statement, one verifying a conjecture by Xiao and Katona [14] and the other proving a slight modification of another conjecture by Xiao and Katona [14].

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For a graph $G$ denote $e(G)$ the number of edges of $G$ and $T_r(G)$ the number of copies of $K_r$ in $G$. A $K_r$-covering set in $V(G)$ is a vertex set that contains at least one vertex of every copy of $K_r$ in $G$. The $r$-clique covering number $\tau_r(G)$ is the size of the smallest $K_r$-covering set. A triangle covering set is a $K_3$-covering set and the triangle covering number is the 3-clique covering number. Given a vertex partition $V(G) = V_1 \cup V_2 \cup \ldots \cup V_r$, we call an edge $e \in E(G)$ a class-edge if $e \in E(G[V_i])$ for some $i$ and a cross-edge otherwise. A set of two vertices $\{x, y\}$ is a missing cross-edge if $xy \notin E(G)$ and $x \in V_i, y \in V_j$ for some $i \neq j$. Denote $e_G(V_1, V_2, \ldots, V_r)$ the number of cross-edges and $e^c_G(V_1, V_2, \ldots, V_r)$ the number of missing cross-edges. We drop the index $G$ and just write $e(V_1, V_2, \ldots, V_r)$ and $e^c(V_1, V_2, \ldots, V_r)$, respectively, if $G$ is clear from context.

**Conjecture 1** (Xiao, Katona [14]). Let $t, s \in \mathbb{N}$ such that $0 < t < s$. Suppose the graph $G$ has $n$ vertices and $\left\lceil \frac{n^2}{4} \right\rceil + t$ edges satisfying $\tau_3(G) \geq s$ and $n \geq n(s, t)$ is large. Then $G$ contains at least

$$(s - 1) \left\lceil \frac{n}{2} \right\rceil + \left\lceil \frac{n}{2} \right\rceil - 2(s - t)$$

many triangles.

Xiao and Katona [14] proved their conjecture for $t = 1$ and $s = 2$ and that it is best possible. However, their conjecture in full generality holds only up to an additive constant. We will determine precisely the minimum number of triangles a graph on $n$ vertices with $\lfloor n^2/4 \rfloor + t$ edges and $\tau_3(G) \geq s$ can have. Depending on $s, t$ and $n$ one of the following two constructions will be an extremal example.

**Construction 1:** Let $G_1$ be the graph on $n$ vertices, where $V(G_1) = A \cup B$ with

$|A| = \left\lceil \frac{n}{2} \right\rceil + a, \quad |B| = \left\lfloor \frac{n}{2} \right\rfloor - a,$

where $a$ is chosen later to minimize the number of triangles in this construction. Pick $2(s - 1)$ vertices $x_1, y_1, \ldots, x_{s-1}, y_{s-1}$ in $A$ and two vertices $u_1, u_2$ in $B$. Add the edges $\{x_i, y_i\}$ and $\{u_1, u_2\}$ to $K_{|A|, |B|}$ and delete the edges $\{u_1, x_1\}, \ldots, \{u_1, x_n\}$, where $\alpha := s - t - a^2 - 1_{\{2|n|\}}a$. Then, $G_1$ has $\left\lceil \frac{n^2}{4} \right\rceil + t$ edges, triangle covering number $\tau_3(G_1) = s$ and

$$(1) \quad T_3(G_1) = (s - 1) \left( \left\lfloor \frac{n}{2} \right\rfloor - a \right) + \left( \left\lceil \frac{n}{2} \right\rceil + a \right) - 2 \left( s - t - a^2 - 1_{\{2|n|\}}a \right).$$

Now, choose $a \geq 0$ satisfying $\alpha \geq 0$ and minimizing (1). Note that $G_1$ is similar to the construction of Xiao and Katona [14] resulting in Conjecture 1, except that the class sizes are different in some cases.

**Construction 2:** Let $G_2$ be the graph on $n$ vertices, where $V(G_2) = A \cup B$
with

$$|A| = \left\lfloor \frac{n}{2} \right\rfloor + a, \quad |B| = \left\lceil \frac{n}{2} \right\rceil - a,$$

where $a$ is chosen later to minimize the number of triangles in this construction. Pick $2s$ vertices $x_1, y_1, \ldots, x_s, y_s \in A$ and one vertex $u \in B$. Add the edges $\{x_i, y_i\}$ to $K_{|A|, |B|}$ and delete the edges $\{u, x_i\}, \ldots, \{u, x_{a}\}$, where $\alpha := s - t - a^2 - \mathbf{1}_{\{2|n|\}a}$. The graph $G_2$ has $\left\lfloor \frac{n^2}{4} \right\rfloor + t$ edges, triangle covering number $\tau_3(G_2) = s$ and

$$T_3(G_2) = s \left( \left\lfloor \frac{n}{2} \right\rfloor - a \right) - (s - t - a^2 - \mathbf{1}_{\{2|n|\}a}).$$

Now, choose $a \geq 0$ satisfying $\alpha \geq 0$ and minimizing (2). Our first main result is the following.

**Theorem 1.** Let $t, s \in \mathbb{N}$ such that $0 < t < s$. Suppose the graph $G$ has $n$ vertices and $\left\lfloor \frac{n^2}{4} \right\rfloor + t$ edges, $\tau_3(G) \geq s$ and $n \geq n(s, t)$ is large. Then $G$ contains at least $\min\{T_3(G_1), T_3(G_2)\}$ triangles.

Note that both $G_1$ and $G_2$ achieve this minimum for some values of $s$ and $t$. Xiao and Katona [14] also conjectured that the minimum number of copies of $K_{r+1}$ in a graph $G$ on $n$ vertices with $t_{r-1}(n) + 1$ edges and $\tau_{r+1}(G) \geq 2$ could be. They conjectured the following graph to be an extremal example.

**Construction 3:** The graph $G_3$ is constructed as follows from the complete $r$-partite graph with vertex partition $V(G_3) = V_1 \cup V_2 \cup \ldots \cup V_r$ where $|V_1| \geq |V_2| \geq \ldots \geq |V_r|$ and $|V_1| - |V_r| \leq 1$. Let $v_1, v_2 \in V_1$ and $u_1, u_2 \in V_2$. Remove the edge $\{v_1, u_1\}$ and add the edges $\{v_1, v_2\}$ and $\{u_1, u_2\}$. The graph $G_3$ satisfies $\tau_{r+1}(G_3) = 2$, $e(G_3) = t_r(n) + 1$ and

$$T_{r+1}(G_3) = (|V_1| + |V_2| - 2) \prod_{i=3}^r |V_i|.$$

We will verify Xiao and Katona’s [14] conjecture.

**Theorem 2.** Let $r \in \mathbb{N}$ and $n \geq n(r)$ be a sufficiently large integer. If a graph $G$ on $n$ vertices has $t_r(n) + 1$ edges and the copies of $K_{r+1}$ have an empty intersection, then the number of copies of $K_{r+1}$ is at least as many as in $G_3$.

The key ingredient for proving Theorems 1 and 2 is the following general structural result.

**Theorem 3.** Let $r, t, s \in \mathbb{N}$ with $0 < t < s$ and let $n \geq n(s, t, r)$ be sufficiently large. Let $G$ be an $n$-vertex graph with $t_r(n) + t$ edges satisfying $\tau_{r+1}(G) \geq s$. Denote $\mathcal{H}$ the family of graphs $H$ on $n$ vertices with $t_r(n) + t$ edges such that there exists a vertex partition $V(H) = V_1 \cup V_2 \cup \ldots \cup V_r$ satisfying

- the class-edges form a matching of size $s$,
• \(e^c_H(V_1, \ldots, V_r) \leq s - t\),
• all missing cross-edges are incident to one of the class-edges,
• if the class-edges are not all in the same class, then each missing cross-edge is incident to class-edges on both endpoints.

Then \(T_{r+1}(G) \geq T_{r+1}(H)\) for some \(H \in \mathcal{H}\).

We will observe (Lemma 4) that \(|V_i| = n/r + O(1)\) for all \(1 \leq i \leq r\). Hence, the family \(\mathcal{H}\) is small. An optimization argument will reduce the family \(\mathcal{H}\) to \(G_1\) and \(G_2\) in the case \(r = 3\), and to \(G_3\) in the case \(r > 3, s = 2, t = 1\). A similar cleaning argument could potentially be done in the general case, however the computational effort needed seems not worth the outcome. Also, it is likely that for some parameters \(r, t, s\), the extremal family realizing the minimum number of cliques of size \(r + 1\) contains several graphs.

For \(t \geq s\) we know from a result of Erdős [5] that the number of cliques of size \(r + 1\) is at least \((1 + o(1))t(\frac{n}{r})^{r-1}\) and the graph \(G_4\) constructed by adding a matching of size \(t\) to one of the classes of the complete balanced \(r\)-partite graph on \(n\) vertices satisfies \(\tau_{r+1}(G_4) = t\) and \(T_{r+1}(G_4) = (1 + o(1))t(\frac{n}{r})^{r-1}\). Hence this problem is interesting only when \(0 < t < s\).

Our result can be extended to consider \(s\) and \(t\) as functions of \(n\). In particular, in the triangle case \(r = 3\), our proof allows \(s\) to be linear in \(n\). However, the methods we use, especially our main tool Theorem 6, will not give us the entire range of \(s\) where our theorem should hold. Thus, we do not attempt to optimize our proof with respect to the dependency of \(n\) and \(s\).

Our main tool is a stability-supersaturation result by Balogh, Bushaw, Colleares, Liu, Morris and Sharifzadeh [1], which extends on the Erdős-Simonovits stability theorem [6] and Füredi’s [7] proof’s of it.

We would like to point out that Liu and Mubayi [9] independently obtained similar results to ours.

Our paper is organized as follows. In Section 2 we prove Theorem 3, in Section 3 we conclude Theorem 1 and in Section 4 we conclude Theorem 2.

2. Proof of Theorem 3

The well-known Turán’s theorem [13] determines the maximum number of edges in a \(K_{r+1}\)-free graph to be the number of edges \(t_r(n)\) of the complete balanced \(r\)-partite graph. We will make use of the following bound on \(t_r(n)\).

\[
\frac{n^2}{2} \left(1 - \frac{1}{r}\right) - \frac{r}{2} \leq t_r(n) \leq \frac{n^2}{2} \left(1 - \frac{1}{r}\right).
\]

The classical Erdős-Simonovits stability theorem [6] asserts that any \(n\)-vertex \(K_{r+1}\)-free graph with almost \(t_r(n)\) edges is close to the complete balanced \(r\)-partite graph. There are many different quantitative versions [2, 8, 12] of this. A standard calculation shows that \(n\)-vertex graphs with almost \(t_r(n)\) edges
and a vertex partition with few class-edges need to have almost balanced class sizes.

**Lemma 4.** Let \( s, n \in \mathbb{N}, t \in \mathbb{Z} \) and \( G \) a graph on \( n \) vertices and \( t_r(n) + t \) edges with vertex partition \( V(G) = V_1 \cup V_2 \cup \ldots \cup V_r \) such that the number of class-edges is \( s \). Then \( |V_i| = \frac{n}{r} + O(1) \) for all \( 1 \leq i \leq r \).

**Proof.** Without loss of generality we can assume that \( |V_1| \geq |V_2| \geq \ldots \geq |V_r| \). Now, let \( x = |V_1| - \frac{n}{r} \). Then,

\[
\sum_{i=1}^{r} |V_i|^2 = \left( \frac{n}{r} + x \right)^2 + \sum_{i=2}^{r} |V_i|^2 \geq \left( \frac{n}{r} + x \right)^2 + \frac{\left( \sum_{i=2}^{r} |V_i| \right)^2}{r-1} \\
\geq \left( \frac{n}{r} + x \right)^2 + \frac{n}{r} \left( 1 - \frac{1}{r} \right) - x^2 \geq \frac{n^2}{r} + x^2.
\]

(4)

Thus, we can conclude

\[
t_r(n) + t = e(G) = \sum_{i=1}^{r} e(G[V_i]) + e(V_1, V_2, \ldots, V_r) \leq s + \sum_{1 \leq i < j \leq r} |V_i||V_j| \\
= s + \frac{1}{2} \left( n^2 - \sum_{i=1}^{r} |V_i|^2 \right) \leq s + \frac{n^2}{2} \left( 1 - \frac{1}{r} \right) - \frac{x^2}{2} \leq t_r(n) + \frac{r}{2} + s - x^2,
\]

implying \( x = O(1) \). Note that in the second to last inequality we used (4) and in the last inequality we used (3). We conclude \( |V_1| = \frac{n}{r} + O(1) \). Similarly, we can conclude \( |V_r| = \frac{n}{r} + O(1) \).

**Lemma 5.** Let \( s, n \in \mathbb{N}, t \in \mathbb{Z} \) and \( G \) a graph on \( n \) vertices and \( t_r(n) + t \) edges with vertex partition \( V(G) = V_1 \cup V_2 \cup \ldots \cup V_r \) such that the number of class-edges is \( s \). Then

\[
T_{r+1}(G) = (1 + o(1))s \left( \frac{n}{r} \right)^{r-1}.
\]

**Proof.** Any copy of \( K_{r+1} \) needs to contain at least one class-edge. There are exactly \( (\frac{n}{r} + O(1))^{r-1} \) copies of \( K_{r+1} \) containing one given class-edge but no others, since the other \( r - 1 \) vertices need to be chosen from different classes. Since the number of missing cross-edges is \( O(1) \), they do not have an impact in this counting. There are \( s \) class-edges and thus we conclude that the number of copies of \( K_{r+1} \) containing exactly one class-edge is

\[
(1 + o(1))s \left( \frac{n}{r} \right)^{r-1}.
\]

The number of copies containing at least two class-edges is \( O(n^{r-2}) \), because there are \( O(1) \) ways to chose three vertices among the vertices being incident
to the class-edges and at most $O(n^{r-2})$ ways to chose the remaining $r-2$ vertices. We conclude

$$T_{r+1}(G) = (1 + o(1))s \left( \frac{n}{r} \right)^{r-1}.$$  

We say that a graph $G$ is $x$-far from being $r$-partite if $\chi(G') > r$ for every subgraph $G' \subset G$ with $e(G') > e(G) - x$. Balogh, Bushaw, Collares, Liu, Morris and Sharifzadeh [1] proved that graphs which are $x$-far from being $r$-partite, contain many cliques of size $r + 1$.

**Theorem 6** ([1]). *For every $n, x, r \in \mathbb{N}$, the following holds. Every graph $G$ on $n$ vertices which is $x$-far from being $r$-partite contains at least

$$\frac{n^{r-1}}{r!e^{2r}} \left( e(G) + x - \left( 1 - \frac{1}{r} \right) \left( \frac{n^2}{2} \right) \right)$$

copies of $K_{r+1}$.***

With this tool in hand, we now prove Theorem 3.

**Proof of Theorem 3.** Let $G$ be a graph on $n$ vertices and $t_r(n) + t$ edges such that $\tau_{r+1}(G) \geq s$ and $n$ is large enough. If $G$ is $2sr!e^{2r}$-far from being $r$-partite, then by applying Theorem 6, the number of copies of $K_{r+1}$ in $G$ is at least

$$T_{r+1}(G) \geq \frac{n^{r-1}}{r!e^{2r}} \left( t_r(n) + t + 2sr!e^{2r} - \left( 1 - \frac{1}{r} \right) \left( \frac{n^2}{2} \right) \right) \geq sn^{r-1},$$

where we used (3) to lower bound $t_r(n)$. By Lemma 5, $T_{r+1}(H) = (1 + o(1))s\left( \frac{n}{r} \right)^{r-1}$ for every $H \in \mathcal{H}$, and thus $T_{r+1}(H) \leq T_{r+1}(G)$. Hence, we can assume that $G$ is not $2sr!e^{2r}$-far from being $r$-partite. Then there exists a subgraph $G' \subset G$ with $\chi(G') \leq r$ and

$$e(G') > e(G) - 2sr!e^{2r}s > t_r(n) - 2sr!e^{2r}.$$  

Let $V(G') = V_1 \cup V_2 \cup \ldots \cup V_r$ be a partition such that $V_1, V_2, \ldots, V_r$ are independent sets in $G'$. By Lemma 4, the class sizes are roughly balanced, that is $|V_i| = \frac{n}{r} + O(1)$ for all $1 \leq i \leq r$.

Since $G' \subset G$, and $e(G') > e(G) - 2sr!e^{2r}$, the vertex partition $V(G') = V_1 \cup V_2 \cup \ldots \cup V_r$ contains at most $2sr!e^{2r}$ class-edges in $G$. If the number of class-edges in $G$ is less than $s$, then there is a $K_{r+1}$-covering set of size at most $s - 1$, contradicting $\tau_{r+1}(G) \geq s$. If the number of class-edges is more than $s$, then by Lemma 5

$$T_{r+1}(G) = (1 + o(1))(s + 1)\left( \frac{n}{r} \right)^{r-1} \geq (1 + o(1))s\left( \frac{n}{r} \right)^{r-1} = T_{r+1}(H)$$

for every $H \in \mathcal{H}$. Therefore, we can assume that the number of class-edges in $G$ is exactly $s$. These $s$ edges need to form a matching, otherwise there
is a $K_{r+1}$-covering set of size $s-1$, contradicting $\tau_{r+1}(G) \geq s$. Further, $e^c(V_1, \ldots, V_r) \leq s-t$, because otherwise
\[
e(G) = e(V_1, V_2, \ldots, V_r) + \sum_{i=1}^{r} e(G[V_i]) \leq t_{r+1}(n) - e^c(V_1, V_2, \ldots, V_r) + s
\]
\[< t_{r+1}(n) + t = e(G).
\]
Next, we assume that there is a missing cross-edge $uv \notin E(G)$ not incident to any of the class-edges. Take a cross-edge $xy \in E(G)$ which is incident to exactly one class-edge. We will show that $T_{r+1}(G + uv - xy) < T_{r+1}(G)$. Adding $uv$ to $G$ increases the number of cliques of size $r+1$ by at most $snr^{-3}$, because for each of the $s$ class-edges $e$ the number of cliques of size $r+1$ containing $e$ and $uv$ is increased by at most $n^r$. Removing $xy$ from $G + uv$ decreases the number of cliques of size $r+1$ by at least
\[
\min(|V_i| - s)r^{-2} \geq \left(\frac{n}{r} + O(1) - s\right)r^{-2} = \Omega(n^{-2}),
\]
where we used that in each class $V_i$ the number of vertices incident to no missing cross-edge is at least $|V_i| - s$. Thus, performing this edge flip decreases the number of cliques of size $r+1$, i.e. $T_{r+1}(G + uv - xy) < T_{r+1}(G)$. We repeat this process until we end up with a graph $\bar{G}$ such that $T_{r+1}(\bar{G}) \leq T_{r+1}(G)$ and every missing cross-edge is incident to a class-edge.

If $\bar{G}$ has all class-edges in the same class, then $\bar{G} \in \mathcal{H}$. Thus, we can assume that $\bar{G}$ has the property that not all class-edges are in the same class and there is a missing cross-edge not adjacent to class-edges on both ends. Denote $G'$ the graph constructed from $\bar{G}$ by adding all missing cross-edges. This graph satisfies
\[
T_{r+1}(G') \leq T_{r+1}(\bar{G}) + (2e_G^c(V_1, V_2, \ldots, V_r) - 1) \left(\frac{n}{r} + O(1)\right)r^{-2}
\]
\[+ se_G^c(V_1, V_2, \ldots, V_r)n^{-3}
\]
\[\leq T_{r+1}(G) + (1 + o(1)) \left(2e_G^c(V_1, V_2, \ldots, V_r) - 1\right) \left(\frac{n}{r}\right)r^{-2},
\]
because for each class-edge $e$ and each added edge $e'$, the number of cliques of size $r+1$ containing $e$ and $e'$ increases by at most $\left(\frac{n}{r} + O(1)\right)r^{-2}$ if $e$ and $e'$ share a vertex and by at most $n^{-3}$ if $e$ and $e'$ are disjoint.

There is a set $M$ of $e_G^c(V_1, V_2, \ldots, V_r) \leq s-1$ cross-edges such that each $e \in M$ is incident to class-edges on both endpoints and every class-edge is incident to edges from $M$ on at most one endpoint. Denote $H$ the graph constructed from $G'$ by removing those edges. This graph satisfies $H \in \mathcal{H}$ and
\[
T_{r+1}(H) \leq T_{r+1}(G') - (2 + o(1))e_G^c(V_1, V_2, \ldots, V_r) \left(\frac{n}{r} + O(1) - s\right)r^{-2}
\]
\[\leq T_{r+1}(G),
\]
completing the proof. □

3. Proof of Theorem 1

Proof of Theorem 1. By Theorem 3, we can assume that $G$ has a vertex partition $V(G) = A \cup B$ where $|A| \geq |B|$ with exactly $s$ class-edges forming a matching. Let $|A| = \lceil n/2 \rceil + a, |B| = \lfloor n/2 \rfloor - a$ for some $a \geq 0$. We have

$$\left\lfloor \frac{n^2}{4} \right\rfloor + t = e(G) = |A||B| - e^c(A, B) + s$$

$$= \left(\left\lfloor \frac{n}{2} \right\rfloor + a\right) \left(\left\lceil \frac{n}{2} \right\rceil - a\right) - e^c(A, B) + s,$$

and thus the number of missing cross-edges is

$$e^c(A, B) = s - t - a^2 - 1_{\{2 \nmid n\}}.$$

Let $M_A$ be the matching of class-edges inside $A$ and $M_B$ the matching of class-edges inside $B$. Denote $G'$ the graph constructed from $G$ by adding all missing cross-edges $e^c(A, B)$. The number of triangles in this graph is $|M_A||B| + |M_B||A|$.  

Case 1: $|M_B| > 0$.

The number of triangles in $G$ is at least $|M_A||B| + |M_B||A| - 2e^c(A, B)$, because each missing cross-edge is in at most two triangles in $G'$. Since $|A| \geq |B|$, we get

$$T_3(G) \geq (s - 1) \left(\left\lfloor \frac{n}{2} \right\rfloor - a\right) + \left(\left\lceil \frac{n}{2} \right\rceil + a\right) - 2e^c(A, B)$$

$$= (s - 1) \left(\left\lfloor \frac{n}{2} \right\rfloor - a\right) + \left(\left\lceil \frac{n}{2} \right\rceil + a\right) - 2(s - t - a^2 - 1_{\{2 \nmid n\}}a)$$

$$\geq T_3(G_1).$$

Case 2: $|M_B| = 0$.

The number of triangles in $G$ is at least $|M_A||B| - e^c(A, B)$, because each missing cross-edge is in at most one triangle in $G'$. Therefore

$$T_3(G) \geq |M_A||B| - e^c(A, B) = s \left(\left\lfloor \frac{n}{2} \right\rfloor - a\right) - (s - t - a^2 - 1_{\{2 \nmid n\}}a)$$

$$\geq T_3(G_2).$$

□

4. Proof of Theorem 2

Let $\ell = n \mod r$ and $m$ be an integer such that $n = rm + \ell$. The number of cliques of size $r + 1$ in $G_3$ is

$$T_{r+1}(G_3) = (|V_1| + |V_2| - 2) \prod_{i=3}^{r} |V_i| = \begin{cases} (2m - 2) m^{r-2}, & \text{iff } \ell = 0, \\ (2m - 1) m^{r-2}, & \text{iff } \ell = 1, \\ 2(m+1)\ell^{-2}m^r, & \text{otherwise}. \end{cases}$$
Proof of Theorem 2. Let $G$ be an $n$-vertex graph with $t_r(n) + 1$ edges satisfying $\tau_{r+1}(G) \geq 2$. By applying Theorem 3 for $s = 2$ and $t = 1$, we can assume without loss of generality that $G \in \mathcal{H}$ and thus has a vertex partition $V(G) = V_1 \cup V_2 \cup \ldots \cup V_r$ where $|V_1| \geq |V_2| \geq \ldots \geq |V_r|$ with exactly two class-edges $e_1, e_2$ and at most one missing cross-edge. We have

$$\sum_{1 \leq i < j \leq r} |V_i||V_j| - e^c(V_1, V_2, \ldots, V_r) + 2 = e(G) = t_r(n) + 1. \tag{5}$$

This means we have two cases:

Case 1: $e^c(V_1, V_2, \ldots, V_r) = 0$ or $e^c(V_1, V_2, \ldots, V_r) = 1$.

In this case

$$\sum_{1 \leq i < j \leq r} |V_i||V_j| = t_r(n) - 1$$

by (5). Note that $|V_1| - |V_r| = 2$, because if $|V_1| - |V_r| \geq 3$, then moving one vertex from $V_1$ to $V_r$ increases the sum $\sum_{1 \leq i < j \leq r} |V_i||V_j|$ by at least two and thus

$$\sum_{1 \leq i < j \leq r} |V_i||V_j| \leq t_r(n) - 2, \tag{6}$$

a contradiction. If $|V_1| - |V_r| \leq 1$, then $\sum_{1 \leq i < j \leq r} |V_i||V_j| = t_r(n)$, giving a contradiction as well. Thus, we have $|V_1| - |V_r| = 2$. Similarly, $|V_2| - |V_r-1| \leq 1$, as otherwise (6) holds. Conclusively, there are only the following types of combination of class sizes. We list them depending on $\ell = n \mod r$. If $\ell \in \{0, 1\}$, then

- $|V_1| = \ldots = |V_{\ell+1}| = m + 1$, $|V_{\ell+2}| = \ldots = |V_{r-1}| = m$, $|V_r| = m - 1$.

If $\ell = 2$, then

- Type 1: $|V_1| = m + 2$, $|V_2| = \ldots = |V_{\ell-1}| = m$, $|V_{\ell+1}| = |V_3| = m + 1$, $|V_4| = \ldots = |V_{r-1}| = m$, $|V_r| = m - 1$.

If $3 \leq \ell \leq r - 3$, then

- Type 1: $|V_1| = m + 2$, $|V_2| = \ldots = |V_{\ell-1}| = m + 1$, $|V_{\ell+1}| = \ldots = |V_{r-1}| = m$, $|V_r| = m - 1$.

If $\ell = r - 2$, then

- Type 1: $|V_1| = m + 2$, $|V_2| = \ldots = |V_{r-3}| = m + 1$, $|V_{r-2}| = \ldots = |V_r| = m$.

If $\ell = r - 1$, then

- Type 2: $|V_1| = \ldots = |V_{r-1}| = m + 1$, $|V_r| = m - 1$. 
Thus, in Case 1, $|V_1| = m + 2$, $|V_2| = \ldots = |V_{r-2}| = m + 1$, $|V_{r-1}| = |V_r| = m$.

Denote the two class-edges $e_1, e_2$. Let $1 \leq \alpha \leq \beta \leq r$ such that $e_1 \subset V_\alpha$ and $e_2 \subset V_\beta$. The number of copies of $K_{r+1}$ containing $e_1$ but not $e_2$ is $\prod_{i \neq \alpha} |V_i|$. Similarly, the number of copies of $K_{r+1}$ containing $e_2$ but not $e_1$ is $\prod_{i \neq \beta} |V_i|$. Thus, the total number of copies of $K_{r+1}$ in $G$ is at least

$$T_{r+1}(G) \geq \prod_{i \neq \alpha} |V_i| + \prod_{i \neq \beta} |V_i| \geq 2 \prod_{i=2}^{r} |V_i|.$$  

We will now check that for any possible combination of class sizes the right hand side in (7) is at least $T_{r+1}(G_3)$. We distinguish cases depending on $\ell = n \mod r$. If $\ell = 0$, then

$$2 \prod_{i=2}^{r} |V_i| = 2m^{r-2}(m-1) \geq (2m - 2)m^{r-2} = T_{r+1}(G_3).$$

If $\ell = 1$, then

$$2 \prod_{i=2}^{r} |V_i| = 2(m+1)m^{r-3}(m-1) \geq (2m-1)m^{r-2} = T_{r+1}(G_3).$$

If $\ell = 2$, then

$$2 \prod_{i=2}^{r} |V_i| = \begin{cases} 2m^{r-1}, & \text{for Type 1} \\ 2(m+1)^2m^{r-4}(m-1), & \text{for Type 2} \end{cases} \geq 2m^{r-1} = T_{r+1}(G_3).$$

If $3 \leq \ell \leq r-3$, then

$$2 \prod_{i=2}^{r} |V_i| = \begin{cases} 2(m+1)^{\ell-2}m^{r-\ell+1}, & \text{for Type 1} \\ 2(m+1)^{\ell}m^{r-\ell-2}(m-1), & \text{for Type 2} \end{cases} \geq 2m^{r-\ell+1}(m+1)^{\ell-2} = T_{r+1}(G_3).$$

If $\ell = r-2$, then

$$2 \prod_{i=2}^{r} |V_i| = \begin{cases} 2(m+1)^{r-4}m^3, & \text{for Type 1} \\ 2(m+1)^{r-2}(m-1), & \text{for Type 2} \end{cases} \geq 2m^3(m+1)^{r-4} = T_{r+1}(G_3).$$

Finally, if $\ell = r-1$, then

$$2 \prod_{i=2}^{r} |V_i| = 2(m+1)^{r-3}m^2 = T_{r+1}(G_3).$$

Thus, in Case 1, $T_{r+1}(G) \geq T_{r+1}(G_3)$.

**Case 2:** $e^c(V_1, V_2, \ldots, V_r) = 1$.

Observe that in this case the class sizes of $G$ and $G_3$ are the same. Further, the number of missing cross-edges is exactly one. Now, we distinguish two cases depending on where $e_1$ and $e_2$ are. Case 2a is when both class-edges are
inside the same class. In Case 2b, \( e_1 \) and \( e_2 \) are in different classes.

**Case 2a:** \( e_1, e_2 \subseteq V_\alpha \) for some \( 1 \leq \alpha \leq r \).

The missing edge is incident to \( e_1 \) or \( e_2 \). The second endpoint of the missing edge is in \( V_\beta \) for some \( \beta \neq \alpha \). Then

\[
T_{r+1}(G) = 2 \prod_{i \neq \alpha} |V_i| - \prod_{i \neq \alpha, \beta} |V_i| = (2|V_\beta| - 1) \prod_{i \neq \alpha} |V_i| 
\geq (|V_\alpha| + |V_\beta| - 2) \prod_{i \neq \alpha, \beta} |V_i| \geq (|V_1| + |V_2| - 2) \prod_{i=3}^r |V_i| = T_{r+1}(G_3),
\]

where the sum of the factors are the same in the two products, hence the product is larger when the factors are ‘closer’ to each other.

**Case 2b:** \( e_1 \subseteq V_\alpha, e_2 \subseteq V_\beta \) for some \( \alpha \neq \beta \).

Since the missing edge is incident to both class-edges,

\[
T_{r+1}(G) = (|V_\beta| - 1) \prod_{i \neq \alpha, \beta} |V_i| + (|V_\alpha| - 1) \prod_{i \neq \alpha, \beta} |V_i| 
= (|V_\alpha| + |V_\beta| - 2) \prod_{i \neq \alpha, \beta} |V_i| \geq (|V_1| + |V_2| - 2) \prod_{i=3}^r |V_i| = T_{r+1}(G_3),
\]

where the last inequality holds by the same reasoning as in Case 2a.

\[\square\]

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