1. Introduction

The purpose of this article is to study the Chow groups and Chow motives of the so-called wonderful compactifications of an arrangement of subvarieties, in particular the Fulton-MacPherson configuration spaces.

All the varieties in the paper are over an algebraically closed field. Let $Y$ be a nonsingular quasi-projective variety. Let $S$ be an arrangement of subvarieties of $Y$ (cf. Definition 2.2). Let $G$ be a building set of $S$, i.e., a finite set of nonsingular subvarieties in $S$ satisfying Definition 2.3. The wonderful compactification $Y_G$ is constructed by blowing up $Y$ along subvarieties in $G$ successively (cf. Definition 2.5). There are different orders in which the blow-ups can be carried out, for example we can blow up along the centers in any order that is compatible with the inclusion relation. There are many important examples of such compactifications: De Concini and Procesi’s wonderful model of a subspace arrangement, the Fulton-MacPherson configuration spaces, the moduli space $\overline{M}_{0,n}$ of stable rational curves with $n$ marked points, etc. These spaces have many properties in common. Studying them by a uniform method gives us better understanding of these spaces. In this article, we study their Chow groups and Chow motives.

If we assume that $Y$ is projective, then the Chow motive of $Y_G$, denoted by $h(Y_G)$, can be decomposed canonically into a direct sum of the motive of $Y$ and the twisted motives of the subvarieties in the arrangement (cf. §2.1 for a review of Chow motives). We will prove the following theorem, where the precise definition of the set $M_T$ and the subvarieties $Y_0T$ of $Y$ are in §3.

Main Theorem (Theorems 3.1, 3.2). Let $Y$ be a nonsingular quasi-projective variety, $G$ be a building set and $Y_G$ be the wonderful compactification $Y_G$. Then we have the Chow group decomposition

$$A^*Y_G = A^*Y \oplus \bigoplus_{T \in M_T} A^{*-\|\mu\|}(Y_0T)$$

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where \( T \) runs through all \( G \)-nests. Moreover, when \( Y \) is projective we have a Chow motive decomposition

\[
h(Y_G) \cong h(Y) \oplus \bigoplus_T \bigoplus_{\mu \in M_T} h(Y_T(||\mu||))
\]

where \( T \) runs through all \( G \)-nests. In this case the correspondences giving the above isomorphism are canonical in the following sense: although there is no canonical order of blow-ups (in general) to construct \( Y_G \), the correspondences turn out to be independent of the order we choose.

The Fulton-MacPherson configuration space \( X[n] \) is one of the most interesting examples of the wonderful compactification \( Y_G \) where \( Y = X^n \) and \( G \) is the the set of all the diagonals in \( X^n \) (cf. §4.1). Applying the main theorem to \( X[n] \), we obtain the following theorem, where the precise definit ion of the nests \( S \), the polydiagonals \( \Delta_S \), the integers \( c(S) \), the sets of lattice points \( M_S \), and the correspondences \( \alpha_{S,\mu} \) and \( \beta_{S,\mu} \) are in §4.1.

**Theorem 4.2.** Let \( X \) be a nonsingular projective variety. Then there is a canonical isomorphism of Chow motives

\[
\bigoplus_S \bigoplus_{\mu \in M_S} \alpha_{S,\mu} : h(X[n]) \cong \bigoplus_S \bigoplus_{\mu \in M_S} h(\Delta_S(||\mu||))
\]

with the inverse \( \sum_S \sum_{\mu \in S} \beta_{S,\mu} \). Equivalently, we have the following decomposition of the Chow motive of \( X[n] \):

\[
h(X[n]) \cong \bigoplus_S \bigoplus_{\mu \in M_S} h(X^{c(S)}(||\mu||))
\]

Here are two consequences of this theorem. One is that we can easily express the decomposition of \( h(X[n]) \) using a generating function \( N(x,t) \), as follows.

**Theorem 4.3.** Define \( f_i(x) \) to be the polynomials whose exponential generating function \( N(x,t) = \sum_{i \geq 1} f_i(x)\frac{t^i}{i!} \) satisfies the identity

\[(1 - x)x^d + (1 - x^{d+1}) = \exp(x^d N) - x^{d+1} \exp(N)\]

where \( d = \dim X \). Then

\[
h(X[n]) = \bigoplus_{1 \leq k \leq n} \bigoplus_{i \geq 0} (h(X^k)(i)) \frac{x_i^k}{i!}
\]

The other consequence is a decomposition of the Chow motive of the quotient variety \( X[n]/S_n \) obtained from the natural symmetric group \( S_n \) action on \( X[n] \). To make sense of the motive of a quotient variety, we assume the base field is of characteristic 0. The correspondences appeared in Theorem 4.2 are canonical, and therefore symmetric with respect to the
symmetric group \( \mathfrak{S}_n \). It is then possible to compute the \( \mathfrak{S}_n \)-invariant part of \( h(X[n]) \), which is the Chow motive of \( X[n]/\mathfrak{S}_n \). As pointed out by [FM94], unlike the isotropy groups of a point in \( X^n \), the isotropy group of any point in \( X[n] \) is always solvable, therefore the singularity of \( X[n]/\mathfrak{S}_n \) is “better” than the singularity of the symmetric product \( X^{(n)} := X^n/\mathfrak{S}_n \). It would be interesting to see how different is the Chow motive \( h(X[n]/\mathfrak{S}_n) \) from \( h(X^{(n)}) \). In the following theorem, an unlabeled weighted forest is a forest whose nodes are not labeled and that each non-leaf node is attached by a positive integer called weight; we call an unlabeled weighted forest of type \( \nu := \{n_1, \ldots, n_r\} \) if the forest is of the form \( n_1T_1 + \cdots + n_rT_r \), where \( T_i \) are mutually distinct unlabeled weighted tree.

**Theorem 5.3** For any unordered set of positive integers \( \nu = \{n_1, \ldots, n_r\} \) and any non-negative integer \( m \), let \( \lambda(\nu, m) \) to be the number of unlabeled weighted forest with \( n \) leaves, of type \( \nu \) and of total weight \( m \), such that at each non-leaf \( v \) with \( c_v \) children, the weight \( m_v \) satisfies \( 1 \leq m_v \leq (c_v - 1) \dim X - 1 \). Then

\[
h(X[n]/\mathfrak{S}_n) = \bigoplus_{\nu, m} \left( h(X^{(n_1)} \times \cdots \times X^{(n_r)})(m) \right)^{\otimes \lambda(\nu, m)}.
\]

The importance of all the above results of Chow motives can be seen through a working principle:

**Principle:** A result proved for Chow motives is valid if we replace them by homological/numerical motives, Chow groups \( A^*_\mathbb{Q} \), cohomology groups \( H^*_\mathbb{Q} \), Grothendieck groups (the aforementioned groups are taken with \( \mathbb{Q} \)-coefficients), Hodge structures, etc.

Thus for example, we have a decomposition for the \( \mathbb{Q} \)-coefficient singular cohomology of \( Y_\mathfrak{G}, X[n] \) and \( X[n]/\mathfrak{S}_n \).

The paper is organized as follows. §2 contains a review of motives and the wonderful compactifications of arrangement of subvarieties. In §3 a motivic decomposition for the wonderful compactifications is proved. In §4 we give a motivic decomposition for the Fulton-MacPherson configuration spaces. §5 gives a motivic decomposition for the quotient variety \( X[n]/\mathfrak{S}_n \).

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2. Preliminaries

2.1. Motives. Given an algebraic variety \( X \) of dimension \( d \), let \( A^iX = A_{d-i}X \) be the Chow group of codimension \( i \), i.e., the group of algebraic
cycles of codimension $i$ in $X$ modulo rational equivalence. Define $A_i^r X = A^r X \otimes \mathbb{Q}$.

Let $X, Y$ be two non-singular projective varieties. The group of correspondences of degree $r$ from $X$ to $Y$ is defined as

$$Corr^r(X, Y) := A^{\dim X + r}(X \times Y).$$

The group $Corr^r_\mathbb{Q}(X, Y)$ denotes the tensor of $Corr^r(X, Y)$ with $\mathbb{Q}$.

The composition of two correspondences $f \in Corr^r(X_1, X_2)$ and $g \in Corr^s(X_2, X_3)$ is a correspondence in $Corr^{r+s}(X_1, X_3)$ defined as

$$g \circ f := \pi_{13*}(\pi_{12*} f \cdot \pi_{23*} g)$$

where $\pi_{ij}$ is the projection from $X_1 \times X_2 \times X_3$ to $X_i \times X_j$.

A correspondence $p \in Corr^0(X, X)$ is called a projector of $X$ if $p^2 := p \circ p = p$.

Let $V$ denote the category of (not necessarily connected) non-singular projective varieties over a field $k$.

**Definition 2.1** (CH00). The category of Chow motives over $k$, denoted by $CHM$, is defined as follows: an object of $CHM$, called a Chow motive, is a triple $(X, p, r)$, where $X$ is a nonsingular projective variety, $p$ is a projector of $X$ and $r$ is an integer. The morphisms in $CHM$ are defined as

$$\text{Hom}_{CHM}((X, p, r), (Y, q, s)) := q \circ Corr^{s-r}(X, Y) \circ p.$$

The composition of morphisms is defined as the composition of correspondences.

For a Chow motive $M = (X, p, r)$ and an integer $\ell$, we define

$$M(\ell) := (X, p, r + \ell).$$

There is a natural contravariant functor $h$ from $V$ to $CHM$, which sends $X$ to $(X, id_X, 0)$ and sends a morphism $f : X \to Y$ to $\Gamma_f : h(Y) \to h(X)$, the transpose of the graph of $f$. Naturally, $h(X)(\ell)$ stands for the Chow motive $(X, id_X, \ell)$.

According to [Hib98], we can generalize the theory of Chow motives on nonsingular projective varieties to the one on varieties which are quotients of smooth projective varieties by finite group actions. To be more precisely, let $V'$ be the category of varieties of type $X/G$ with $X \in \text{Ob} V$ and $G$ a finite group. We can define the group of correspondences $Corr^r_\mathbb{Q}(X', Y')$ for $X', Y' \in V'$ and the category of Chow motives $CHM'$ similar to the nonsingular case. (The difference is that we have to use $\mathbb{Q}$-coefficients).

There is a natural contravariant functor $h : V' \to CHM'$.

Define the $G$-average correspondence $\text{ave}_G$ as

$$\text{ave}_G := \frac{1}{|G|} \sum [g] \in Corr^0_\mathbb{Q}(X, X)$$
where $|g|$ is given by the graph of $g$ in $X \times X$. By [IBV98] Proposition 1.2, there is an isomorphism

$$h(X/G) \cong (X, \text{ave}\Delta) \cong h(X)^G.$$ 

Such a definition is consistent with the realization functors and $\mathbb{Q}$-coefficient Chow groups.

2.2. Wonderful compactification of an arrangement of subvarieties.

The wonderful compactification of an arrangement of subvarieties is introduced in [Li06] as a generalization of De Concini and Procesi’s wonderful model of subspace arrangements. We briefly review the definition and some properties of such compactifications. For details we refer to [Li06].

**Definition 2.2.** A (simple) arrangement of subvarieties of $Y$ is a finite set $\mathcal{S} = \{S_i\}$ of nonsingular closed subvarieties of $Y$ satisfying the following conditions:

1. $S_i$ and $S_j$ intersect cleanly (i.e. their intersection is nonsingular and $T(S_i \cap S_j) = T(S_i) \cap T(S_j)$).
2. $S_i \cap S_j$ is either empty or equal to some $S_k \in \mathcal{S}$.

**Definition 2.3.** Let $\mathcal{S}$ be an arrangement of subvarieties of $Y$. A subset $G \subseteq \mathcal{S}$ is called a building set of $\mathcal{S}$ if $\forall S \in \mathcal{S}$, the minimal elements in $G$ which contains $S$ intersect transversally and their intersection is $S$ (this condition is always satisfied if $S \in G$). These minimal elements are called the $G$-factors of $S$. We call a finite set $G$ of subvarieties a building set if the set

$$\mathcal{S} := \{ \bigcap_{V \in \mathcal{F}} V \} _{\mathcal{F}},$$

where $\mathcal{F}$ runs through all subsets of $G$, is an arrangement and $G$ is a building set of $\mathcal{S}$ (for $\mathcal{F} = \emptyset$ we set $\bigcap_{V \in \mathcal{F}} V = Y$). In this case we call $\mathcal{S}$ the induced arrangement of $G$.

**Definition 2.4.** Let $G$ be a building set. A subset $\mathcal{T} \subseteq G$ is called $G$-nested (or a $G$-nest) if it satisfies one of the following equivalent relations:

1. There is a flag of elements in $\mathcal{S}$: $S_1 \subseteq S_2 \subseteq \cdots \subseteq S_k$, such that

$$\mathcal{T} = \bigcup_{i=1}^{k} \{ A : A \text{ is a } G\text{-factor of } S_i \}.$$ (We say that $\mathcal{T}$ is induced by the flag $S_1 \subseteq S_2 \subseteq \cdots \subseteq S_k$.)

2. Let $A_1, \ldots, A_k$ be the minimal elements of $\mathcal{T}$, then they are all the $G$-factors of a certain element in $\mathcal{S}$, and for each $1 \leq i \leq k$, the set $\{ A \in \mathcal{T} : A \supseteq A_i \}$ is also $G$-nested defined by induction.

The wonderful compactification is defined as follows:
Definition 2.5. Denote $Y^o = Y \setminus \cup_{G \in \mathcal{G}} G$. There is a natural locally closed embedding

$$Y^o \hookrightarrow Y \times \prod_{G \in \mathcal{G}} \text{Bl}_{G} Y.$$ 

The closure of this embedding, denoted by $Y_{\mathcal{G}}$, is called the wonderful compactification of $\mathcal{G}$.

The wonderful compactification $Y_{\mathcal{G}}$ of $\mathcal{G}$ has the following properties, where (1) and (2) are in Theorem 1.2 in [Li06] and (3) is clear from the proof there.

Theorem 2.6. The variety $Y_{\mathcal{G}}$ is nonsingular. For each $G \in \mathcal{G}$ there is a nonsingular divisor $D_{G}$ on $Y_{\mathcal{G}}$ such that:

1. The union of the divisors $D_{G}$ is $Y_{\mathcal{G}} \setminus Y^o$.
2. Any collection of the divisors $D_{G}$ intersects transversally. An intersection of divisors $D_{T_1} \cap \cdots \cap D_{T_r}$ is nonempty exactly when $\{T_1, \ldots, T_r\}$ forms a $\mathcal{G}$-nest.
3. Each $D_{G}$ is the unique connected component of $\pi^{-1}(G)$ that maps surjectively to the subvariety $G$, where $\pi$ is the natural morphism $Y_{\mathcal{G}} \to Y$. (This $D_{G}$ is called the dominant transform of $G$ and denoted by $\tilde{G}$ in [Li06].)

The dominant transform can also be defined as follows. Let $\pi : \tilde{Y} \to Y$ be the blow-up along a nonsingular subvariety $G \subseteq Y$. For any irreducible subvariety $V$ in $Y$, we define the dominant transform of $V$, denoted by $\tilde{V}$ or $V^\sim$, to be the strict transform of $V$ when $V \not\subseteq G$, and $\pi^{-1}(V)$ when $V \subseteq G$. For a sequence of $N$ blow-ups $Y_0 \to Y_{N-1} \to \cdots \to Y_1 \to Y_0$ and a subvariety $V \subseteq Y_0$ we define the dominant transform $\tilde{V} \subseteq Y_{N}$ (or denoted by $V^\sim$) to be the $N$-th iterated dominant transform $(\cdots ((V^\sim)^\sim) \cdots)^\sim$.

It is known (cf. [Li06]) that $Y_{\mathcal{G}}$ can be constructed by a sequence of blow-ups as follows. Let $Y$ be a nonsingular variety, $\mathcal{S}$ be an arrangement of subvarieties and

$$\mathcal{G} = \{G_1, \ldots, G_N\}$$

be a building set with respect to $\mathcal{S}$. Suppose the subvarieties in $\mathcal{G} = \{G_1, \ldots, G_N\}$ are indexed in an order compatible with inclusion relations, i.e. $i \leq j$ if $G_i \subseteq G_j$. We define the triple $(Y_k, \mathcal{S}^{(k)}, \mathcal{G}^{(k)})$ inductively with respect to $k$, where $Y_k$ is a nonsingular variety, $\mathcal{S}^{(k)}$ is an arrangement of subvarieties of $Y_k$ and $\mathcal{G}^{(k)}$ is a building set with respect to $\mathcal{S}^{(k)}$:

1. For $k = 0$, define $Y_0 = Y$, $\mathcal{S}^{(0)} = \mathcal{S}$, $\mathcal{G}^{(0)} = \mathcal{G} = \{G_1, \ldots, G_N\}$, $G_i^{(0)} = G_i$ for $1 \leq i \leq N$.
2. Assume the triple $(Y_k, \mathcal{S}^{(k)}, \mathcal{G}^{(k)})$ has been constructed. Define $Y_k$ to be the blow-up of $Y_{k-1}$ along the nonsingular subvariety $G_k^{(k-1)}$. Define $G^{(k)}$ to be the dominant transform $(G^{(k-1)})^\sim$ for all $G \in \mathcal{G}$. Then $\mathcal{G}^{(k)} := \{G_1^{(k)}, \ldots, G_N^{(k)}\}$.
\(\{G^{(k)}\}_{G \in \mathcal{G}}\) is a building set (by \[Li06\] Proposition 2.8). We denote the induced arrangement by \(S^{(k)}\).

(3) Continue the inductive construction until \(k = N\). We get a nonsingular variety \(Y_N\) and all elements in the building set \(\mathcal{G}^{(N)}\) are divisors. The resulting variety is isomorphic to \(Y_G\).

For any \(G\)-nest \(T\), define
\[
Y_kT = \bigcap_{G \in T} G^{(k)}.
\]

The following property of \(Y_kT\) is used often throughout the paper.

**Proposition 2.7.** Let \(0 \leq k \leq N - 2\) and let \(T \subseteq \{G_{k+2}, G_{k+3}, \ldots, G_N\}\) be a \(G\)-nest. Then \(Y_{k+1}T\) is an irreducible nonsingular subvariety of \(Y_{k+1}\) with the following property:

If \(\{G_{k+1}\} \cup T\) is not a \(G\)-nest, then \(G_{k+1}^{(k)} \cap Y_kT = \emptyset\) and \(Y_{k+1}T \cong Y_kT\); otherwise, the intersection \(G_{k+1}^{(k)} \cap Y_kT\) is clean, \(Y_{k+1}T\) is isomorphic to the blow-up of \(Y_kT\) along \(G_{k+1}^{(k)} \cap Y_kT\) with exceptional divisor \(G_{k+1}^{(k+1)} \cap Y_{k+1}T\) (where the intersection is transverse), and the codimension of \(G_{k+1}^{(k)} \cap Y_kT\) in \(Y_kT\) equals to
\[
\begin{cases}
\dim \cap_{G_{k+1} \subseteq G \in \mathcal{T} \cap \mathcal{G}} G - \dim G_{k+1}, & \text{if } \{G : G_{k+1} \subseteq G \in T\} \neq \emptyset; \\
\dim Y - \dim G_{k+1}, & \text{otherwise}.
\end{cases}
\]

**Proof.** We prove the statement by induction on \(k\). The case \(k = 0\) is obvious. Now assume that the statement is true for \(k\).

(i) Suppose that \(\{G_{k+1}\} \cup T\) is not a \(G\)-nest. We will show that \(G_{k+1}^{(k)} \cap Y_kT = \emptyset\). As a consequence we have \(Y_{k+1}T \cong Y_kT\), since the center of the blow-up is away from \(Y_kT\).

We prove by contradiction. Assume \(G_{k+1}^{(k)} \cap Y_kT \neq \emptyset\). Since \(T\) is a \(G\)-nest, \(\{G^{(k)}\}_{G \in \mathcal{T}}\) is a \(G^{(k)}\)-nest by \[Li06\] Proposition 2.8 (3). By Definition 2.2 (1), the nest \(\{G^{(k)}\}_{G \in \mathcal{T}}\) is induced by a flag
\[
S'_1 \subseteq S'_2 \subseteq \cdots \subseteq S'_l
\]
where \(S'_i \in \mathcal{S}^{(k)}\). We claim that \(\{G_{k+1}^{(k)}\} \cup \{G^{(k)}\}_{G \in \mathcal{T}} \subseteq \mathcal{S}^{(k)}\) is a \(G^{(k)}\)-nest induced by the flag
\[
(G_{k+1}^{(k)} \cap S'_1') \subseteq S'_1' \subseteq S'_2' \subseteq \cdots \subseteq S'_l'.
\]
Indeed, since \(Y_kT = S'_1\), we know \(G_{k+1}^{(k)} \cap S'_1' \neq \emptyset\). By \[Li06\] Lemma 2.4 (ii), the \(G^{(k)}\)-factors of \(G_{k+1}^{(k)} \cap S'_1'\) are \(G^{(k)}\) and some \(G^{(k)}\)-factors of \(S'_1'\), hence our claim follows. Then \[Li06\] Proposition 2.8 (3) asserts that, since \(\{G_{k+1}^{(k)}\} \cup \{G^{(k)}\}_{G \in \mathcal{T}}\) is a \(G^{(k)}\)-nest, \(\{G_{k+1}\} \cup T\) must be a \(G\)-nest. But by assumption \(\{G_{k+1}\} \cup T\) is not a \(G\)-nest, contradiction.

(ii) Suppose that \(T \cup \{G_{k+1}\}\) is a \(G\)-nest. Let the \(G^{(k)}\)-factors of \(Y_kT\) be \(G'_1, \ldots, G'_p\). Then they are minimal elements in the \(G^{(k)}\)-nest \(\{G^{(k)}\}_{G \in \mathcal{T}}\), by
the definition of nest. Assume without loss of generality that the first $m$
subvarieties $G'_1, \ldots, G'_m$ contain $G'^{(k)}_{k+1}$. Define $A = \cap_{i=1}^m G'_i$, $B = \cap_{i=m+1}^r G'_i$,
then $Y_kT = A \cap B$ is the $G'^{(k)}_{k+1}$-factorization of $Y_kT$ by \cite{Li06} Definition-
Lemma 2.6.

Notice that for $p, q \geq k + 2$ and $G^{(k)}_p \subseteq G^{(k)}_q$, we have $G^{(k+1)}_p \subseteq G^{(k+1)}_q$ be-
cause strict transforms keep the inclusion relation. Moreover, since $G'_1, \ldots, G'_r$
are the minimal elements in $G^{(k)}$ which contain $Y_kT$, the subvariety $Y_{k+1}T$
is the intersection $\cap_{i=1}^r G'_i$. Then

$$A = \bigcap_{i=1}^m \widetilde{G}'_i, \quad B = \bigcap_{i=m+1}^r \widetilde{G}'_i, \quad (A \cap B)^{\sim} = \widetilde{A} \cap \widetilde{B} = \bigcap_{i=1}^m \widetilde{G}'_i$$

by \cite{Li06} Lemma 2.9. Thus $Y_{k+1}T = (Y_kT)^{\sim}$. By the definition of arrange-
ment we know that $Y_kT$ and $G^{(k)}_{k+1}$ intersect cleanly, so $Y_{k+1}T$ is the blow-up
of $Y_kT$ along the center $G^{(k)}_{k+1} \cap Y_kT$. The exceptional divisor is the preimage
of the center, hence is $G^{(k+1)}_{k+1} \cap Y_{k+1}T$. Since $G^{(k+1)}_{k+1}$ and $Y_{k+1}T$
intersect cleanly and since the divisor $G^{(k+1)}_{k+1}$ does not contain $Y_{k+1}T$, we can see that
the intersection $G^{(k)}_{k+1} \cap Y_kT$ is actually transversal.

The codimension of the center $Y_kT \cap G^{(k)}_{k+1}$ in $Y_kT$ equals

$$\text{codim}(A \cap B \cap G^{(k)}_{k+1}, A \cap B) = \text{codim}(G^{(k)}_{k+1} \cap B, A \cap B) = \text{codim}(G^{(k)}_{k+1}, A),$$

where the second equality is because of the transversality of the intersection
$G^{(k)}_{k+1} \cap B$. If no elements in $T$ contain $G_{k+1}$, then $A = Y$ and

$$\text{codim}(G^{(k)}_{k+1}, A) = \dim Y - \dim G_{k+1};$$

otherwise

$$\text{codim}(G^{(k)}_{k+1}, A) = \text{codim}(G^{(k)}_{k+1}, \bigcap_{i=1}^m G'_i) = \text{codim}(G_{k+1}, \bigcap_{G_{k+1} \subseteq G \in T} G)$$

$$= \dim \bigcap_{G_{k+1} \subseteq G \in T} G - \dim G_{k+1}.$$ 

Thus the proof is complete. \hfill \square

3. The motive of wonderful compactifications

Notations:

- Let $Y$ be a nonsingular quasi-projective variety with an arrangement of
subvarieties $\mathcal{S}$. Let $\mathcal{G}$ be a building set with respect to $\mathcal{S}$. Let $Y_\mathcal{G}$ be the
wonderful compactification. Let $\mathcal{T}$ be a $\mathcal{G}$-nest.
- For $T \in \mathcal{G}$, define $D_T$ to be the divisor $T^{(N)}$ in $Y_\mathcal{G}$. When no confusion
arise, we use the same notation $D_T$ for its restriction to a subvariety of $Y_\mathcal{G}$.
- Denote $j_T : Y_\mathcal{G} \to Y_G$ to be the natural imbedding. Denote $g_T : Y_\mathcal{G} \to
Y_0T$ to be the restriction of the natural morphism $Y_\mathcal{G} \to Y$. 
Suppose $j : B \to C$ and $g : B \to D$ are two morphisms of varieties. Denote by $(j, g) : B \to C \times D$ the composition of the diagonal map $\Delta$ with $f \times g$:

$$(j, g) : B \to B \times B \xrightarrow{f \times g} C \times D.$$ 

Given $a \in A(P)$, denote by $\{ a \}_{i}$ the image of the projection $A(P) \to A^{i}(P)$ of the Chow ring to its degree $i$ direct summand, i.e., taking the codimension $i$ part of $a$.

We set $\bigcap_{G \subseteq T \in \mathcal{T}} T = Y$ if no $T$ satisfies $G \subseteq T \in \mathcal{T}$. Define

$$r_{G} := \dim (\bigcap_{G \subseteq T \in \mathcal{T}} T) - \dim G.$$ 

Define

$$N_{G} := N_{G}(\bigcap_{G \subseteq T \in \mathcal{T}} T)|_{Y_{0}}$$

the restriction to $Y_{0}$ of the normal bundle of $G$ in the ambient space $(\bigcap_{G \subseteq T \in \mathcal{T}} T)$. Define

$$M_{\mathcal{T}} := \{ \mu = \{ \mu_{G} \}_{G \in \mathcal{G}} : 1 \leq \mu_{G} \leq r_{G} - 1, \mu_{G} \in \mathbb{Z} \}$$

and define $||\mu|| := \sum_{G \in \mathcal{G}} \mu_{G}$ for $\mu \in M_{\mathcal{T}}$.

**Theorem 3.1.** We have the Chow group decomposition

$$A^{*}Y_{G} = A^{*}Y \bigoplus \bigoplus_{\mathcal{T}, \mu \in M_{\mathcal{T}}} A^{*-||\mu||}(Y_{0})$$

where $\mathcal{T}$ runs through all $\mathcal{G}$-nests.

Moreover, when $Y$ is complete, we have the Chow motive decomposition

$$h(Y_{G}) = h(Y) \bigoplus \bigoplus_{\mathcal{T}, \mu \in M_{\mathcal{T}}} h(Y_{0})(||\mu||)$$

where $\mathcal{T}$ runs through all $\mathcal{G}$-nests.

**Theorem 3.2.** The correspondence that gives each of the above direct summand can be explicitly expressed as follows,

$$\alpha : h(Y_{G}) \to h(Y_{0})(||\mu||)$$

$$\alpha = (j_{\mathcal{T}}, g_{\mathcal{T}})_{*} \prod_{G \in \mathcal{T}} \left\{ c \left( g_{\mathcal{T}}^{*}(N_{G}) \otimes \mathcal{O}(\sum_{(\star)} - D_{G'}) \right) \frac{1}{1 + D_{G}} \right\}_{r_{G} - 1 - \mu_{G}},$$

here $c$ is total Chern class, the subscript $r_{G} - 1 - \mu_{G}$ means the codimension $r_{G} - 1 - \mu_{G}$ part, and condition $(\star)$ is: $G' \subseteq G$ and $\mathcal{T} \cup \{ G' \}$ is a $\mathcal{G}$-nest.

The inverse correspondence is

$$\beta : h(Y_{0})(||\mu||) \to h(Y_{G})$$

$$\beta = (g_{\mathcal{T}}, j_{\mathcal{T}})_{*} \prod_{G \in \mathcal{T}} \left( - D_{G} \right)^{\mu_{G} - 1}.$$
3.1. Proof of the Theorem [3.1]

**Lemma 3.3.** Given a $G$-nest $T \subseteq \{G_{k+2}, \ldots, G_N\}$. Suppose $T' := T \cup \{G_{k+1}\}$ is also a $G$-nest. Define $r = r_{k,T}$ to be

\[
\begin{cases} \dim \cap_{G_{k+1} \subseteq G \in T} G - \dim G_{k+1}, & \text{if } \{G : G_{k+1} \subsetneq G \in T\} \neq \emptyset; \\ \dim Y - \dim G_{k+1}, & \text{otherwise}. \end{cases}
\]

Then the following Chow group decomposition holds:

\[
A^*(Y_{k+1}T) = A^*(Y_kT) \oplus \bigoplus_{t=1}^{r-1} A^{*-t}(Y_kT').
\]

When $Y$ is complete, we also have the motivic decomposition

\[
h(Y_{k+1}T) = h(Y_kT) \oplus \bigoplus_{t=1}^{r-1} h(Y_kT')(t).
\]

**Proof.** Apply the well known blow-up formula for the Chow group and for the Chow motive (Theorem A.2) to Proposition 2.7 immediately gives the conclusion. \qed

Iteratively applying the above lemma gives the proof of Theorem 3.1:

**Proof of Theorem 3.1.** Define

\[
M^{(k)}_T = \{ \mu = \{\mu_G\}_{G \in G} : 1 \leq \mu_G \leq \dim(\bigcap_T T^{(k)}) - \dim G^{(k)} - 1, \mu_G \in \mathbb{Z} \}
\]

where $T$ runs through the subvarieties in $T$ such that $G^{(k)} \subsetneq T^{(k)}$. Define $\|\mu\| := \sum_{G \in G} \mu_G$ for $\mu \in M^{(k)}_T$.

We prove the following statement using a downward induction on $k$:

\[
A^*Y_G = A^*Y_k \oplus \bigoplus_{T \in M^{(k)}_T} A^{*-\|\mu\|}(Y_kT).
\]

where $T$ runs through all $G$-nest such that $T \subseteq \{G_{k+1}, G_{k+2}, \ldots, G_N\}$.

The assertion for $k = N$ is trivial because all $G^{(N)}$ are divisors in $Y_G$ hence of codimension 1 and $M^{(k)}_T = \emptyset$.

Assume (2) has been proved for $k + 1$, i.e.,

\[
A^*Y_G = A^*Y_{k+1} \oplus \bigoplus_{T \in M^{(k+1)}_T} A^{*-\|\mu\|}(Y_{k+1}T)
\]
where $\mathcal{T}$ runs through all $\mathcal{G}$-nest such that $\mathcal{T} \subseteq \{G_{k+2}, G_{k+3}, \ldots, G_N\}$. Apply Lemma 3.3, we have

\begin{equation}
A^*Y_G = A^*Y_k \bigoplus \left( \bigoplus_{t=1}^{\text{codim}(G_{k+1},Y)-1} A^{*+t}(G_{k+1}^{(k)}) \right) \bigoplus \left( \bigoplus_{T \in \mathcal{M}_{k+1}} \bigoplus_{\mu \in \mathcal{M}_{T,k+1}} A^{*+\|\mu\|}(Y_k^T) \right) \bigoplus \left( \bigoplus_{T \in \mathcal{M}_{k+1}} \bigoplus_{t=1}^{r_{k+1,T}-1} A^{*+\|\mu\|-t}(Y_k(\{G_{k+1}\} \cup T)) \right). 
\end{equation}

This immediately gives the Chow group decomposition \((2)\) for $k$. Indeed, any $\mathcal{G}$-nest contained in $\{G_{k+1}, G_{k+2}, \ldots, G_N\}$ must be one of the three: $\{G_{k+1}\}$, a $\mathcal{G}$-nest $\mathcal{T}$ contained in $\{G_{k+2}, G_{k+3}, \ldots, G_N\}$, or $\{G_{k+1}\} \cup \mathcal{T}$. They correspond to the second, third and last summands in \((3)\) respectively. (Notice that $Y_k(\{G_{k+1}\} \cup \mathcal{T}) = \emptyset$ if $\{G_{k+1}\} \cup \mathcal{T}$ is not a $\mathcal{G}$-nest by Proposition 2.7.)

Therefore, the Chow group decomposition \((2)\) holds for all $k$, in particular the case $k = 0$ gives the desired Chow group decomposition. For the proof of the Chow motive decomposition, we can either repeat the above proof almost word by word or, as the referee pointed out, notice that the Chow motive decomposition follows from the result on the Chow groups and Manin’s identity principle. \qed

3.2. Proof of Theorem 3.2. First, we introduce some notations. For a given $\mathcal{G}$-nest $\mathcal{T}$,
- Define $\mathcal{T}_k := \mathcal{T} \cap \{G_{k+1}, G_{k+2}, \ldots, G_N\}$ for $0 \leq k \leq N$. Then we have a chain of $\mathcal{G}$-nests $\mathcal{T}_0 \supseteq \mathcal{T}_1 \supseteq \cdots \supseteq \mathcal{T}_N$, where $\mathcal{T}_0 = \mathcal{T}$ and $\mathcal{T}_N = \emptyset$.
- For $\mu \in \mathcal{M}_{\mathcal{T}}$ and $1 \leq i \leq N$, define

$$
\mu_i := \begin{cases} 
\mu_{G_i}, & \text{if } G_i \in \mathcal{T}; \\
0, & \text{otherwise}.
\end{cases}
$$
Lemma 3.4. Denote by \( g : Y_k \to Y_{k-1} \) the natural morphism. Then for \( l \leq k - 1 \), we have

\[
g^{-1}(G_{l}^{(k-1)}) = G_{l}^{(k)}.
\]

Proof. First, we claim that \( G_{l}^{(k-1)} \nsubseteq G_{k}^{(k-1)} \). Otherwise \( G_{l} \supseteq G_{k} \) since they are the respective images of \( G_{l}^{(k-1)} \) and \( G_{k}^{(k-1)} \) under \( Y_k \to Y_0 \). But then by the assumption that the order of \( \{G_i\} \) is compatible with inclusion relations, we obtain a contradiction \( l \geq k \).

Next, it is easy to see that \( G_{l}^{(k-1)} \nsubseteq G_{k}^{(k-1)} \) since \( G_{l}^{(k-1)} \) is a divisor. Now we know that the two nonsingular subvarieties \( G_{l}^{(k-1)} \) and \( G_{k}^{(k-1)} \) intersect cleanly and neither one contains the other, therefore they must intersect transversally. Then it is standard to show by local coordinates calculation that the following isomorphism between ideal sheaves holds:

\[
g^{-1}\mathcal{I}(G_{l}^{(k-1)}) \cdot \mathcal{O}_{Y_k} \cong \mathcal{I}(G_{l}^{(k)}).
\]

The desired conclusion follows from this. \( \square \)

Lemma 3.5. In Diagram (4), all squares are fiber squares. Moreover, for any \( N \geq k > l \geq 0 \), we have

(i) \( j_{kl} \) is injective;

(ii) If \( G_k \in \mathcal{T} \), then \( g_{kl} \) is the projection of a projective bundle with fiber isomorphic to a projective space of dimension \( r_k \cdot r - 1 \);

(iii) If \( G_k \notin \mathcal{T} \) but \( \{G_k\} \cup \mathcal{T} \) is a \( \mathcal{G} \)-nest, then \( g_{kl} \) is the blow-up of \( Y_{k-1} \mathcal{T}_l \) along the center \( G_{k}^{(k-1)} \cap Y_{k-1} \mathcal{T}_l \);

(iv) If \( \{G_k\} \cup \mathcal{T} \) is not a \( \mathcal{G} \)-nest, then \( g_{kl} \) is an isomorphism.

Proof. It is obvious that \( j_{kl} \) is injective.
Then we have \( (5) \)
\[
\alpha_1 \alpha_2 = (j_2 j_3, g_1 g_3)_* (j_3^* \gamma_2 \cdot g_3^* \gamma_1),
\]
\( (6) \)
\[
\beta_2 \beta_1 = (g_1 g_3, j_2 j_3)_* (g_3^* \gamma_1' \cdot j_3^* \gamma_2').
\]
Next we show that for any \( l \leq k - 1 \), \( g_{kl} \) is the restriction of \( g_{k,k-1} \) to a smaller base \( Y_{k-1} \), which will then show that \( g_{kl} \) is also a projective bundle with fiber of the same dimension \( r_{k,T} - 1 \). Fix \( k \) and use downward induction on \( l \). By inductive assumption, \( g_{k,l+1} \) is a restriction of \( g_{k,k-1} \). Since
\[
g_{k,l+1}^{-1}(G_{l+1}^{k-1} \cap Y_{k-1} T_l) = G_{l+1}^{k} \cap Y_k T_l
\]
by Lemma 3.6, the restriction of the projective bundle \( g_{k,l+1} \) to a smaller base space \( Y_{k-1} T_l = Y_{k-1} T_{l+1} \cap G_{l+1}^{k-1} \) is exactly \( g_{kl} \).

Next, we show \( g_{kl} \) is birational if \( G_k \notin T \). This is again implied by Proposition 2.7. Notice that \( G_{k-1} \) is minimal in \( T' := \{ G_{k}^{(k-1)} \} \cup \{ G_{k-1}^{(k-1)} \}_{G \in \mathcal{T}}. \)

If \( T' \) is a \( G_{k-1} \)-nest, then \( g_{kl} : Y_k T_l \rightarrow Y_{k-1} T_l \) is a blow-up along the center \( G_{k-1} \cap Y_{k-1} T_l \); otherwise, \( g_{kl} \) is an isomorphism. In both cases, \( g_{kl} \) is birational.

Finally, all squares in Diagram 4.1 are fiber squares since \( \forall l \leq k - 2 \), \( g_{kl} \) is a restriction of \( g_{k,l+1} \). The proof is complete. □

The following lemma computes the composition of correspondences in certain diagrams. The author thanks the referee to suggest a proof much simpler than the original proof given by the author.

**Lemma 3.6.** Let \( W,U,V,X,Y,Z \) be nonsingular quasi-projective varieties. Suppose the square in the following diagram is a fiber square.

\[
\begin{array}{ccc}
W & \xrightarrow{j_3} & U \xrightarrow{j_2} X \\
g_3 & \xrightarrow{g_2} & \square \xrightarrow{g_1} Y \\
V & \xrightarrow{j_1} & Y \\
g_1 & \xrightarrow{g_1} & Z
\end{array}
\]

and suppose that \( \dim W - \dim V = \dim U - \dim Y \) and that \( j_k, g_k (1 \leq k \leq 3) \) are proper. Take \( \gamma_1, \gamma_1' \in A(V), \gamma_2, \gamma_2' \in A(U) \) and define correspondences
\[
\alpha_k = (j_k, g_k)_* \gamma_k, \quad \beta_k = (g_k, j_k)_* \gamma_k', \quad \text{for } k = 1, 2.
\]
Then we have
\[
(5) \quad \alpha_1 \alpha_2 = (j_2 j_3, g_1 g_3)_* (j_3^* \gamma_2 \cdot g_3^* \gamma_1),
\]
\[
(6) \quad \beta_2 \beta_1 = (g_1 g_3, j_2 j_3)_* (g_3^* \gamma_1' \cdot j_3^* \gamma_2').
\]
Proof. By abuse of notation, for \( \gamma \in A(V) \) we use the same \( \gamma \) to denote the correspondence \( \Delta V, (\gamma) \in A(V \times V) \) where \( \Delta V : V \to V \times V \) is the diagonal embedding. For a map \( j : U \to X \), we denote by \( j_k \) the correspondence \( \Gamma_j \) (i.e. the graph of \( j \)) and by \( j^* \) the correspondence \( \Gamma'_j \) (i.e. the transpose of \( \Gamma_j \)).

First observe that \( \alpha_k = g_k \circ \gamma \circ j_k^* \) for \( k = 1, 2 \). Indeed, by properties of correspondences (cf. [Fu98] Prop 16.1.1(c)), we have \( \Gamma_j \circ \gamma = (1_U \times j)_* \gamma \), \( \gamma \circ \Gamma'_j = (g \times 1_U)_* \gamma \), so \( g_k \circ \gamma_k \circ j_k^* = \Gamma_{g_k} \circ \gamma_k \circ \Gamma'_{j_k} = (g_k \times j_k)_* \gamma_k = (g_k, j_k)_* \gamma_k = \alpha_k \) for \( k = 1, 2 \).

With the above observation, (5) is equivalent to

\[
\gamma_1 j_1^* g_1 \gamma_2^* g_2 = \gamma_3 (j_3^* \gamma_2 \cdot g_3^* \gamma_1) j_1^*.
\]

So it suffices to prove

\[
\gamma_1 j_1^* g_2 \gamma_2 = g_3 (j_3^* \gamma_2 \cdot g_3^* \gamma_1) j_1^*.
\]

For any \( u \in A(U) \), we have

\[
\gamma_1 j_1^* g_2 \gamma_2 (u) = \gamma_1 g_3 j_3^* \gamma_2 (u) = g_3 (j_3^* \gamma_1 \cdot j_3^* (\gamma_2 u)) = g_3 (j_3^* \gamma_1 \cdot j_3^* \gamma_2) j_1^*(u)
\]

where the first “=” is because of \( \dim W - \dim V = \dim U - \dim Y \), the second “=” is because of the projection formula. Then we apply Manin’s Identity Principle to obtain (7), hence (5). The identity (6) can be obtained by transposing (5).

Now we state a simple lemma and omit the proof.

Lemma 3.7. If \( A, B_i, C_{ij} \) are motives such that

(i) \( \bigoplus_i \alpha_i : A \cong \bigoplus_i B_i \) is an isomorphism with inverse \( \sum_i \beta_i \), and

(ii) \( \bigoplus_j \alpha_{ij} : B_i \cong \bigoplus_j C_{ij} \) is an isomorphism with inverse \( \sum_j \beta_{ij} \),

then the correspondence \( \bigoplus_{i,j} \alpha_{ij} \circ \alpha_i \) gives an isomorphism \( A \cong \bigoplus_{i,j} C_{ij} \) with inverse \( \sum_{i,j} \beta_{i,j} \circ \beta_{ij} \).

For \( G_k \in \mathcal{T} \), define \( h_k \in A^1(Y_k \mathcal{T}_{k-1}) \) to be first Chern class of the invertible sheaf \( \mathcal{O}(1) \) of the projective bundle \( g_{k,k-1} \). Define

\[
\alpha_k = \begin{cases} (j_{k,k-1}, g_{k,k-1})_* 1, & \text{if } G_k \notin \mathcal{T}; \\ (j_{k,k-1}, g_{k,k-1})_* \left( \left\{ g_{k,k-1}^* c(N_k) \frac{1}{1 - h_k} \right\} r_{k-1} - \mu_k \right), & \text{if } G_k \in \mathcal{T}, \end{cases}
\]

where \( N_k := N_{Y_{k-1} \mathcal{T}_{k-1} Y_{k-1} \mathcal{T}_k} \). Define

\[
\beta_k = \begin{cases} (g_{k,k-1}, j_{k,k-1})_* 1, & \text{if } G_k \notin \mathcal{T}; \\ (g_{k,k-1}, j_{k,k-1})_* h_{k-1}^{\mu_k - 1}, & \text{if } G_k \in \mathcal{T}. \end{cases}
\]

Thanks to the blow-up formula of motives (Theorem A.2), the correspondence

\[
a_k : h(Y_k \mathcal{T}_k) \left( \sum_{i=k+1}^N \mu_i \right) \to h(Y_{k-1} \mathcal{T}_{k-1}) \left( \sum_{i=k}^N \mu_i \right)
\]
expresses $h(Y_{k-1}T_{k-1})(\sum_{k}^{N} \mu_{i})$ as a direct summand of $h(Y_{k}T_{k})(\sum_{k+1}^{N} \mu_{i})$ with right inverse $\beta_{k}$.

By Lemma 3.7, the correspondence

$$\alpha_{T,\mu} : h(Y_{G}) \rightarrow h(Y_{0}T)(||\mu||)$$

that gives the direct summand $h(Y_{0}T)(||\mu||)$ in Theorem 3.1 can be expressed as the composition $\alpha_{1} \circ \alpha_{2} \circ \cdots \circ \alpha_{N}$, with right inverse $\beta_{N} \circ \cdots \circ \beta_{1}$. Therefore we have

**Proposition 3.8.** Denote by $f_{k} : Y_{N}T_{0} \rightarrow Y_{k}T_{k-1}$ the natural map in Diagram (4). (i.e. $g_{k+1,k-1} \cdots \circ g_{N,k-1} \circ j_{N,k-2} \circ \cdots \circ j_{N0}.)$ Then

$$\alpha_{1} \circ \cdots \circ \alpha_{N} = (j_{T},g_{T}) \prod_{G_{k} \in T} \{f^{*}_{k}g_{k,k-1}c(N_{k}) \frac{1}{1-f^{*}_{k}h_{k}}\}_{r_{k}-1-\mu_{k}}^{1-1}$$

$$\beta_{N} \circ \cdots \circ \beta_{1} = (g_{T},j_{T}) \prod_{G_{k} \in T} f^{*}_{k}h_{k}^{r_{k}-1}.$$

**Proof.** Combine Lemma 3.5 and Lemma 3.6 with the above discussion. □

The following two standard facts about normal bundles of subvarieties are used in the proof of Theorem 3.2.

**Fact 3.9.** Let $Z$ be a nonsingular variety. Let $Y, W$ be nonsingular proper subvarieties of $Z$ and assume $Y$ intersects transversally with $W$. Let $\pi : \tilde{Z} \rightarrow Z$ be the blow-up of $Z$ along $W$ and let $\tilde{Y}$ be the strict transform of $Y$. Then

$$N_{Y}_{\tilde{Z}} \simeq \pi^{*}N_{Y}Z.$$

**Fact 3.10.** Let $W \subset Y \subset Z$ be nonsingular varieties and $\pi : \tilde{Z} \rightarrow Z$ be the blow-up of $Z$ along $W$. Denote by $\tilde{Y}$ the strict transform of $Y$, and denote by $E$ the exceptional divisor on $\tilde{Y}$. Then

$$N_{Y}_{\tilde{Z}} \simeq \pi^{*}N_{Y}Z \otimes \mathcal{O}(-E).$$

**Proof of the above two facts.** Prove by local coordinates. Or see [Fu98]. □

**Proof of Theorem 3.2.** To conclude Theorem 3.2 from Proposition 3.8 we prove in three steps.

Step 1: Show $f^{*}_{k}h_{k} = -D_{G_{k}}|Y_{N}T_{0}$.

Recall that for $G_{k} \in T$, $h_{k}$ is first Chern class of the invertible sheaf $\mathcal{O}(1)$ of the projective bundle $g_{k,k-1}$.

Consider the following diagram (not necessary a fiber square) where $\pi$ and $j$ are the natural morphisms:

$$\begin{array}{ccc}
Y_{N}T_{0} & \xrightarrow{j_{T}} & Y_{N} \\
\downarrow f_{k} & & \downarrow \pi \\
Y_{k}T_{k-1} & \xrightarrow{j} & Y_{k}.
\end{array}$$
By Proposition 2.7, $Y_k T_{k-1}$ is the exceptional divisor of the blow-up $g_{k,k-1} : Y_k T_{k-1} \to Y_{k-1} T_{k-1}$, so $h_k = -j^*_{k,k-1}[Y_k T_{k-1}]$. Since $Y_k T_{k-1}$ is the transversal intersection $Y_k T_k \cap G(k)^{t}$, $h_k = -j^* G(k)^{t}$, so the intersection of the third equality can be expressed by successively applying Lemma 3.4.

Step 2: Let $0 \leq s < k \leq N$. Denote $g_{k} : Y_s T_k \to Y_{s-1} T_k$ to be the natural map induced from $Y_s \to Y_{s-1}$. We claim the following:

If $G_k \in \mathcal{T}$ (hence $T_{k-1} = T_k \cup \{G_k\}$), then the normal bundle $N_{Y_s T_{k-1}, Y_s T_k}$ is isomorphic to

$$g^*_{s,k-1}(N_{Y_{s-1} T_{k-1}, Y_{s-1} T_k}) \otimes (-[G(s)]|_{Y_{s-1} T_{k-1}}),$$

if (** holds; otherwise.

where condition (** is: $G_s \subseteq G_k$ and $T_k \cup \{G_s\}$ is a $\mathcal{G}$-nest.

For the proof, we discuss three cases.

Case (i): condition (** holds. It is a direct conclusion of Fact 3.10. Indeed, to apply Fact 3.10 we need to show that

$$Y_{s-1} T_k \cap G(s-1)|^{s-1} \subseteq Y_{s-1} T_k \cap G(s-1) \subseteq Y_{s-1} T_k.$$

The second inequality is obvious. The first inclusion is strict because of the following reason. $G(s-1)$ is a $\mathcal{G}(s-1)$-factor of $Y_{s-1} T_k \cap G(s-1)$, therefore $G(s-1)$ is not a $\mathcal{G}(s-1)$-factor because it strictly contains $G(s-1)$. On the other hand, $G(s-1)$ is a $\mathcal{G}(s-1)$-factor of $Y_{s-1} T_k \cap G(s-1)$. So the first inclusion is strict.

Case (ii): $T_k \cup \{G_s\}$ is not $\mathcal{G}$-nested. In this case, $G(s-1) \cap Y_{s-1} T_k = \emptyset$ by Proposition 2.7. Hence no twisting is needed for the normal bundle.

Case (iii): $T_k \cup \{G_s\}$ is $\mathcal{G}$-nested but $G_s$ is not strictly contained in $G_k$. If $T_{k-1} \cup \{G_s\}$ is not a $\mathcal{G}$-nest, then $G(s-1) \cap Y_{s-1} T_{k-1} = \emptyset$ by Proposition 2.7. Hence blowing up along $G(s-1)$ will not affect the normal bundle of $Y_{s-1} T_{k-1}$, so no twisting is needed. Otherwise, assume $T_{k-1} \cup \{G_s\}$ is a $\mathcal{G}$-nest. Both $G_s$ and $G_k$ are minimal in the $\mathcal{G}$-nest $T_{k-1} \cup \{G_s\}$. Then $G(s-1)$ and $G(s-1)$ are minimal in a nest and neither one contains the other, therefore they intersect transversally by the definition of nest. Thus, $Y_{s-1} T_k \cap G(s-1)$ and $Y_{s-1} T_k \cap G(s-1)$, regarded as subvarieties of the ambient space $Y_{s-1} T_k$, intersect transversally. Therefore Fact 3.3 applies, and no twisting is needed for the normal bundle.

Step 3: Apply the result of Step 2 successively for $s = 1, 2, \ldots, k-1$. The normal bundle $N_{Y_{k-1} T_{k-1}, Y_{k-1} T_k}$ is isomorphic to

$$(g^*_{k-1,k-1} \cdots g^*_{1,k-1}(N_{Y_{0} T_{k-1}, Y_{0} T_k}) \otimes (- \sum_{(*)} [G(s)]|_{Y_{s-1} T_{k-1}})$$
where the sum is over all \( s \) that satisfying condition (**). (Here we have used Lemma 3.4.) Therefore

\[
f^*_k g^*_k c(N_{Y_{k-1}T_{k-1}} Y_{k-1} T_k) = c\left(g^*_T (N_{Y_{0}T_{k-1}} Y_{0} T_k | Y_{0} T) \otimes O\left(-\sum_{s} [D_s]|_{Y_{N}T_{k-1}}\right)\right).
\]

Notice that

\[
(N_{Y_{0}T_{k-1}} Y_{0} T_k)|_{Y_{0} T} = N_{G_k}(\bigcap_{G_k \subseteq G \in T} G)|_{Y_{0} T}
\]

which is denoted by \( N_{G_k} \) by our notation. (The proof is as follows: Suppose \( T_1, \ldots, T_m, T_{m+1}, \ldots, T_r \) are the minimal elements of the nest \( T_{k} \), where the first \( m \) elements contain \( G_k \). Then the minimal element of the nest \( T_{k-1} \) are \( G_k, T_{m+1}, \ldots, T_r \). By the definition of nest, \( Y_0 T_k \) is the transversal intersection \( T_1 \cap \cdots \cap T_m \cap T_{m+1} \cap \cdots \cap T_r \), and \( Y_0 T_{k-1} \) is the transversal intersection \( G_k \cap T_{m+1} \cap \cdots \cap T_r \). Therefore

\[
N_{Y_{0}T_{k-1}} Y_{0} T_k = N_{G_k}(T_1 \cap \cdots \cap T_m)|_{Y_{0} T_{k-1}}.
\]

Since \( T_1 \cap \cdots \cap T_m = \bigcap_{G_k \subseteq G \in T} G \), the conclusion follows immediately.)

Now put everything into Corollary 3.8, we have

\[
\alpha_1 \circ \cdots \circ \alpha_N = (j_T, g_T)_* \prod_{G_k \in T} \left\{ c(g^*_T (N_{G_k}) \otimes O\left(-\sum_{s} [D_s]|_{Y_{N}T}\right) \frac{1}{1 + D_{G_k}|_{Y_{N}T}}\right\}^{r_k-1-\mu_k}.
\]

\[
\beta_N \circ \cdots \circ \beta_1 = (g_T, j_T)_* \prod_{G_k \in T} (-D_{G_k})^{\mu_k-1}|_{Y_{N}T}.
\]

Finally, we show that the condition (***) can be replaced by the following condition:

\( \star \) : \( G_s \subseteq G_k \) and \( T \cup \{G_s\} \) is a \( G \)-nest.

Indeed, \( \star \) is stronger than (**). However, for those \( G_s \) satisfying (***) but not \( \star \), the divisor \([D_{G_s}]|_{Y_{N}T}\) would be trivial because \( D_{G_s} \cap Y_{N}T = \emptyset \). Therefore, replacing (***) by \( \star \) will not affect the resulting correspondence.

Hence the proof is complete.

We write a direct conclusion from Step 3 for later usage:

**Corollary 3.11.** Denote \( \pi : G^{(k)}_{k+1} \rightarrow G_{k+1} \). Then

\[
c(N_{G_{k+1}^{(k)}} Y_{k}) = c(\pi^* N_{G_{k+1}^{(k)}} Y \otimes \sum_{G_{k+1} \supsetneq G \in T} (-[D_G])|_{G_{k+1}^{(k)}}).
\]

**Proof.** Apply Step 3 to the nest \( T = \{G_{k+1}\} \).
4. Fulton-MacPherson configuration spaces

Fix a nonsingular variety \( X \) of dimension \( d \). The configuration space of \( n \) distinct ordered points on \( X \), denoted by \( F(X, n) \), can be naturally identified with an open subvariety of the Cartesian product \( X^n \):

\[ F(X, n) := \{ (x_1, x_2, \ldots, x_n) \in X^n : x_i \neq x_j \text{ for } i \neq j \}. \]

In their celebrated paper [FM94], Fulton and MacPherson have discovered an interesting compactification \( X[n] \) of the configuration space \( F(X, n) \). The compactification is obtained by replacing the diagonals of \( X^n \) by a simple normal crossing divisor. It has many attractive properties, for example the geometry when \( n \) points collide, i.e. the degenerate configuration, can be explicitly described using \( X[n] \). \( X[n] \) is closely related to the well known compactification \( \overline{M}_{0,n} \) of the moduli space of stable rational curves with \( n \) marked points. The reader is referred to the beautiful paper [FM94] for the original construction and various applications of the Fulton-MacPherson configuration space.

The Fulton-MacPherson configuration space \( X[n] \) can be realized as a wonderful compactification of an arrangement of subvarieties by taking \( Y = X^n, \mathcal{G} \) the collection of all diagonals of \( X^n \) and therefore the induced arrangement is the set of intersections of diagonals which is called polydiagonals (cf. [Li06]).

4.1. Main theorems. First we fix some notations:

(i) Denote \([n] := \{1, 2, \ldots, n\}\). We call two subsets \( I, J \subseteq [n] \) overlapped if \( I \cap J \) is a nonempty proper subset of \( I \) and \( J \). For a set \( S \) of subsets of \([n]\), we call \( I \) is compatible with \( S \) (denote by \( I \sim S \)) if \( I \) does not overlap any element in \( S \).

A nest \( S \) is a set of subsets of \([n]\) such that any two elements \( I \neq J \in S \) are not overlapped, and all singletons \( \{1\}, \ldots, \{n\} \) are in \( S \). Notice that the nest defined here, unlike the one defined in [FM94], is allowed to contain singletons.

Given a nest \( S \), define \( S^\circ = S \setminus \{\{1\}, \ldots, \{n\}\} \). In the description of nests by forests below, \( S^\circ \) correspond to the forest \( S \) cutting of all leaves.

A nest \( S \) naturally corresponds to forest (i.e. a not necessarily connected tree), each node of which is labeled by an element in \( S \). For example, the following forest corresponds to a nest \( S = \{1, 2, 3, 23\} \).

\[
\begin{array}{c}
123 \\
\downarrow \\
23 \\
\downarrow \\
2 \\
\downarrow \\
3
\end{array}
\]

Denote by \( c(S) \) the number of connected components of the forest, i.e., the number of maximal elements of \( S \). Denote by \( c_I(S) \) (or \( c_J \) if no ambiguity arise) the number of maximal elements of the set \( \{J \in S | J \subset I\} \), i.e. the
number of sons of the node $I$. In the above example, $c(S) = 1$, $c_{123} = c_{23} = 2$.

(ii) For a subset $I \subseteq [n]$ consisting of at least two elements, define the diagonal

$$\Delta_I := \{(x_1, \ldots, x_n) \in X^n : x_i = x_j \text{ if } i, j \in I\}.$$  

It is shown in [FM94] that complement of $F(x, n)$ in the Fulton-MacPherson compactification $X[n]$ is a union of normal crossing nonsingular divisors $D_I$, indexed by subsets $I \subseteq [n]$ with at least two elements. More precisely, $D_I$ is the dominant transform $\tilde{\Delta}_I$ under the natural morphism $X[n] \to X^n$.

For every nest $S$, $X(S) := \cap_{I \in S} D_I$ is a nonsingular subvariety of $X[n]$. Define $j_S : X(S) \hookrightarrow X[n]$ to be the natural inclusion.

Define $\Delta_S := \cap_{I \in S} \Delta_I$. Define $g_S : X(S) \to \Delta_S$ to be the restriction of the morphism $\pi : X[n] \to X^n$ to the subvariety $X(S)$.

(iii) Let $p_I : X[n] \to X$ be the composition of $\pi : X[n] \to X^n$ with the projection $X^n \to X$ to the $i$-th factor for an arbitrary $i \in I$. (The choice of $i \in I$ is not essential: indeed, the only place we need $p_I$ is in the formulation of $\alpha_{S, \mu}$ below, where need the composition $j_S^* p_I^*$. By the following diagram

$$\begin{array}{ccc}
X(S) & \xrightarrow{j_S} & X[n] \\
\downarrow g_S & & \downarrow p_i \\
\Delta_S & \xrightarrow{q_i} & X
\end{array}$$

where $i \in I$, we have $j_S^* p_I^* = g_S q_i^*$, but $q_i$ is independent of the choice of $i \in I$ since $\Delta_S \subseteq \Delta_I$, so $j_S^* p_I^*$ is independent of the choice of $i \in I$ for $p_I$.)

(iv) For a nest $S \neq \{\{1\}, \ldots, \{n\}\}$ (i.e. $S^c \neq \emptyset$), define

$$M_S := \{\mu = \{\mu_I\}_{I \in S^c} : 1 \leq \mu_I \leq d(c_I - 1) - 1, \mu_I \in \mathbb{Z}\}.$$  

(recall that $d = \dim X$ and $c_I = c_I(S)$ is defined in (i)) and define

$$\|\mu\| := \sum_{I \in S^c} \mu_I, \quad \forall \mu \in M_S.$$  

For $S = \{\{1\}, \ldots, \{n\}\}$, assume $M_S = \{\mu\}$ with $\|\mu\| = 0$.

We will show in the proof of Theorem 4.1 that $M_S$ is the special case of $M_T$ defined in [R] where $Y$ is $X^n$, $G$ is the set of diagonals of $X^n$ and $T$ is the set of $G$-nests.

Define function $\zeta(x) := \sum_{i=0}^{d} (1 + x)^{d-i} c_i(T_X)$.  

Define $\alpha_{S,\mu} \in \text{Corr}^{-||\mu||}(X[n], \Delta_S)$, $\beta_{S,\mu} \in \text{Corr}^{||\mu||}(\Delta_S, X[n])$, $p_{S,\mu} \in \text{Corr}^{0}(X[n], \bar{X}[n])$ as follows,

$$
\alpha_{S,\mu} = (j_S, g_S) \ast j_S^+ \left( \prod_{I \in S^0} \left\{ - p_I^* \zeta \left( - \sum_{j \sim S} D_j \right) c_{c_I^{-1} - \mu_I} \right\} \right),
$$

$$
\beta_{S,\mu} = (g_S, j_S) \ast j_S^+ \left( \prod_{I \in S^0} D_I^{-\mu_I - 1} \right),
$$

$$
p_{S,\mu} = \beta_{S,\mu} \circ \alpha_{S,\mu}.
$$

(In the above definition of $\alpha_{S,\mu}$ and $\beta_{S,\mu}$, the products are set to be $1_{X(S)} \in A^0(X(S))$ if $S^0 = \emptyset$.)

The following are the main theorems on the Chow groups and Chow motives of Fulton-MacPherson configuration spaces.

**Theorem 4.1.** Let $X$ be a nonsingular quasi-projective variety. There is an isomorphism of Chow groups:

$$
A^*(X[n]) = \bigoplus_{S \in MS} \bigoplus_{\|\mu\|} A^{*-\|\mu\|}(X^c(S)),
$$

where $S$ runs through all nests of $[n]$.

**Theorem 4.2.** Let $X$ be a nonsingular projective variety. Then there is a canonical isomorphism of Chow motives

$$
\bigoplus_{S \in MS} \bigoplus_{\|\mu\|} \alpha_{S,\mu} : h(X[n]) \cong \bigoplus_{S \in MS} \bigoplus_{\|\mu\|} h(\Delta_S)(\|\mu\|)
$$

with the inverse $\sum_S \sum_{\|\mu\|} \beta_{S,\mu}$. Equivalently, we have

$$
h(X[n]) \cong \bigoplus_{S \in MS} \bigoplus_{\|\mu\|} h(X^c(S))(\|\mu\|).
$$

**Remark:** Observe that the two sets of correspondences $\{\alpha_{S,\mu}\}, \{\beta_{S,\mu}\}$ are $\mathfrak{S}_n$-symmetric in the sense that the following diagram commutes for any $\sigma \in \mathfrak{S}_n$,

$$
\begin{array}{ccc}
X[n] & \xrightarrow{\alpha_{S,\mu}} & X^c(S)(\|\mu\|) \\
\downarrow{\sigma} & & \downarrow{\sigma} \\
X[n] & \xrightarrow{\alpha_{\sigma(S,\mu)}} & X^c(S)(\|\mu\|) \\
\end{array}
$$

$$
\begin{array}{ccc}
X[n] & \xrightarrow{\beta_{S,\mu}} & X[n] \\
\downarrow{\sigma} & & \downarrow{\sigma} \\
X[n] & \xrightarrow{\beta_{\sigma(S,\mu)}} & X[n] \\
\end{array}
$$

**Proof of Theorem 4.1.** Apply Theorem 3.1 with the ambient space $Y = X^n$ and the building set

$$
\mathcal{G} = \{\Delta_I\}_{I \subseteq [n], |I| \geq 2}
$$
First notice that a nest $S$ of $[n]$ gives a $G$-nest $T = \{\Delta_I\}_{I \in S}$. Moreover, the inverse is also true: a $G$-nest will give a nest of $[n]$. Indeed, given a partition $\Pi = (I_1, \ldots, I_t)$ of $[n]$, a $G$-factor of $\Delta_\Pi$ by definition is a minimal element in $\{G \in G : G \supseteq \Delta_\Pi\}$. So $\{\Delta_{I_1}, \ldots, \Delta_{I_t}\}$ are all the $G$-factors of $\Delta_\Pi$. By the definition of $G$-nest, $T$ is induced from a flag of strata $\Delta_{\Pi_1} \supseteq \Delta_{\Pi_2} \supseteq \cdots \supseteq \Delta_{\Pi_t}$.

Then $\Pi_1 \supseteq \Pi_2 \supseteq \cdots \supseteq \Pi_k$.

(Here $\Pi \supseteq \Pi'$ means $\Pi$ is a finer partition than $\Pi'$, e.g. $(12, 3, 4) \supseteq (123, 4)$.) The nest $T$ is induced by “taking the union of all factors of each $\Delta_\Pi$”, which corresponds to “taking all $I$’s that appear in any of the partition $\Pi_i$”. Since the partitions are totally ordered, the set of $I$’s forms a nest of $[n]$.

Next we prove the range of $\mu$ is as stated. Theorem 3.1 asserts that $1 \leq \mu_G \leq r_G - 1$. Now $G = \Delta_I$ is a diagonal, so by definition

$$r_G := \dim(\bigcap_{G \subseteq T \in T} T) - \dim T$$

$$= \dim(\bigcap_{I \supseteq \Pi \in \mathcal{S}} \Delta_I) - \dim \Delta_I$$

$$= d(c_I - 1).$$

Finally, observe that

$$Y_0 T = \bigcap_{G \in T} G = \bigcap_{I \in S} \Delta_I = \Delta_S \cong X^{c(S)}.$$

Therefore the expected conclusion is implied by Theorem 3.1. □

Proof of Theorem 4.2. The statement of the motive decomposition is proved exactly as the above proof.

The correspondences are induced from Theorem 3.2. The improvement of this theorem than Theorem 3.2 is: we can say more about the Chern classes appeared in the correspondence $\alpha_{S, \mu}$ in Theorem 4.2.

First, given $G = \Delta_I$, let $\Pi = (I_1, \ldots, I_{c_I})$ be the partition containing all sons of $I$ in $S$. We compute the normal bundle $N_G := N_{\Delta_I} \Delta_\Pi$. Without loss of generality, assume $I = (12 \ldots m)$, where $m \leq n$.

Denote by $p_i : \Delta_I \to X$ and $q_i : \Delta_\Pi \to X$ the projections induced from the projection of $X^n$ to the $i$-th factor. For each $1 \leq i \leq c_I$, pick an $a_i \in I_i$.

$$T_{\Delta_I} = p_1^* T_X \oplus p_{m+1}^* T_X \oplus \cdots \oplus p_{n}^* T_X,$$

$$T_{\Delta_\Pi} = q_{a_1}^* T_X \oplus \cdots \oplus q_{a_{c_I}}^* T_X \oplus q_{m+1}^* T_X \oplus \cdots \oplus q_{n}^* T_X,$$

$$T_{\Delta_\Pi \Delta_I} = p_1^* T_X \oplus \cdots \oplus p_{a_1}^* T_X \oplus q_{m+1}^* T_X \oplus \cdots \oplus q_{n}^* T_X,$$

therefore, $c(N_G) = p_1^* c(T_X)^{c_I - 1}$. 

Therefore the expected conclusion is implied by Theorem 3.1. □
To compute the Chern classes of $N_G$ twisted by a line bundle $L$, we use the Chern root technique. For any vector bundle $N$ on $X$, define the Chern polynomial as

$$c_y(N) := c_0(N) + c_1(N)y + c_2(N)y^2 + \ldots.$$ 

Define $x = c_1(L)$. Recall that the rank of $N_G$ is $r_G = d(c_1 - 1)$. Then

$$c(N_G \otimes L) = c_{c_1}(N_G) + c_{c_1-1}(N_G)(1 + x) + \ldots + c_0(N_G)(1 + x)^{r_G}$$

$$= (x + 1)^{r_G} c_{1 + \frac{1}{x+1}}(N_G)$$

$$= (x + 1)^{d(c_1-1)} p_1^* c_{1 + \frac{1}{x+1}}(T_X)^{c_1-1}$$

$$= p_1^* [(x + 1)^d c_{1 + \frac{1}{x+1}}(T_X)]^{c_1-1} = p_1^* \zeta(x)^{c_1-1}.$$ 

Finally, by restricting to $\Delta_S$ and pulling back to $X(S)$ we get the expected formula for correspondences $\alpha_{S, \mu}$. \hfill $\square$

4.2. A formula for the generating function of Chow groups and Chow motive of $X[n]$. In this section, we express the decompositions of the Chow groups (Theorem 4.1) and the Chow motive (Theorem 4.2) in terms of exponential generating functions.

Define $[x^i]$ to be the function that picks up the coefficient of $x^i$ from a power series. Define $[\frac{x^i t^n}{n!}]$ to be the function that picks up the coefficient of $\frac{x^i t^n}{n!}$ from a power series with two variables $x$ and $t$, i.e.,

$$[\frac{x^i t^n}{n!}] \sum_{j,m} a_{jm} \frac{x^j t^m}{m!} := a_{in}.$$ 

The main theorem of this section is the following:

**Theorem 4.3.** Define $f_i(x)$ to be the polynomials whose exponential generating function $N(x, t) = \sum_{i \geq 1} f_i(x) \frac{t^i}{i!}$ satisfies the identity

$$(1 - x) x^d t + (1 - x^{d+1}) = \exp(x^d N) - x^{d+1} \exp(N).$$

Then for a nonsingular $d$-dimensional quasi-projective variety $X$,

$$A^*(X[n]) = \bigoplus_{1 \leq k \leq n} A^* - i(X^k) \oplus [\frac{x^i t^n}{n!}] \frac{N^k}{t^i}.$$ 

Moreover, if $X$ is projective, we have the motive decomposition

$$h(X[n]) = \bigoplus_{\Pi = (I_1, \ldots, I_k)} \text{partition of } [n] (h(\Delta_\Pi)(i)) \oplus [x^i] (f_{I_1}(x) \ldots f_{I_k}(x))$$

$$= \bigoplus_{1 \leq k \leq n} h(X^k)(i) \oplus [\frac{x^i t^n}{n!}] \frac{N^k}{t^i}.$$
Remark: One can write down by hand the first several terms of $N$. Define
\[ \sigma_j = \sum_{i=1}^{d_j} x^i \text{ (when } d = 1, \text{ define } \sigma_1 = 0). \]
Then
\[ N = t + \sigma_1 \frac{t^2}{2!} + (\sigma_2 + 3\sigma_1^2) \frac{t^3}{3!} + (\sigma_3 + 10\sigma_1\sigma_2 + 15\sigma_1^3) \frac{t^4}{4!} \]
\[ + (\sigma_4 + 15\sigma_1\sigma_3 + 10\sigma_2^2 + 105\sigma_1^2\sigma_2 + 105\sigma_1^4) \frac{t^5}{5!} + \ldots. \]

Proof of Theorem 4.3. We prove only the statement for motives, since the statement for Chow groups can be proved by exactly the same method.

By Theorem 4.2, we want to count for any given $i$ and $k$, how many possible $S$ and $\mu \in S$ satisfy $c(S) = k$ and $\|\mu\| = i$. First, consider the case when $c(S) = 1$, i.e. $S$ is a connected forest.

Define
\[ f_n(x) := \sum_{S: c(S) = 1} \sum_{\mu \in M_S} x^{\|\mu\|} \]
and define $f_1(x) = 1$.

For a nest $S$ of $[n]$ with $c(S) = 1$, we have
\[ \sum_{\mu \in M_S} x^{\|\mu\|} = \prod_{I \in S^0} \sigma_{c(I)-1}, \]
i.e., $I$ goes through all non-leaves of $S$ (if $n = 1$, then the sum is set to be 1). Since the sons of the root of $S$ correspond to a partition $\{I_1, \ldots, I_k\}$ of $[n]$, we have following formula for $n \geq 2$,
\[ f_n(x) = \sum_{\{I_1, \ldots, I_k\} \text{ partition of } [n]} f_{|I_1|} f_{|I_2|} \ldots f_{|I_k|} \sigma_{k-1}. \]
where $\sigma_k = \sum_{i=1}^{d_k} x^i$ for $k > 0$, and $\sigma_0 = 0$. Since the equality does not hold for $n = 1$ where $f_1(x) = 1$ but the right side is 0, so one define
\[ \tilde{f}_n(x) = \begin{cases} f_n(x), & \text{if } n > 1; \\ 0, & \text{if } n = 1. \end{cases} \]

Then the following holds for any $n \geq 1$:
\[ \tilde{f}_n(x) = \sum_{\{I_1, \ldots, I_k\} \text{ partition of } [n]} f_{|I_1|} f_{|I_2|} \ldots f_{|I_k|} \sigma_{k-1}. \]

Recall the Compositional Formula of exponential generating functions (cf. [St99], Theorem 5.1.4), which asserts that if an equation as above holds, then
\[ E_{\tilde{f}}(t) = E_\sigma(E_f(t)), \]
where
\[ E_f(t) = 1 + \tilde{f}_1 t + \tilde{f}_2 t^2/2! + \tilde{f}_3 t^3/3! + \ldots \]
\[ E_\sigma(t) = 1 + \sigma_0 t + \sigma_1 t^2/2! + \sigma_2 t^3/3! + \ldots \]
\[ E_f(t) = \tilde{f}_1 t + \tilde{f}_2 t^2/2! + \tilde{f}_3 t^3/3! + \ldots \]
By the definition of $\tilde{f}$, $E_{\tilde{f}} = E_f - t + 1$. Denote $N = E_f$, one has

$$N - t + 1 = E_g(N),$$

Standard Computation shows

$$E_g(N) = 1 + N + \frac{1}{x - 1} \left[ \frac{1}{x^d} (e^{x^d N} - 1) - xe^N + x \right].$$

Therefore

$$(1 - x)x^dt + (1 - x^{d+1}) = \exp(x^d N) - x^{d+1} \exp(N).$$

Now consider the case when $c(S)$ is not necessarily 1, i.e., the forest $S$ is not necessarily connected. For a partition $\Pi = \{I_1, ..., I_k\}$ of $[n]$, the number of times that $h(\Delta_\Pi(i))$ appears in the decomposition of $h(X[n])$ is equal to $[x^k](f_{|I_1(x)}...f_{|I_k(x)}(x))$, the coefficient of $x^k$ in the product. Denote by $a_{k,i}$ the sum of these numbers for all partitions with $k$ blocks. Then $a_{k,i}$ is the number of times that $h(X^k(i)$ appears in the decomposition of $H(X[n])$.

Define

$$F_n(y) = \sum_{\{I_1, ..., I_k\} \text{ partition of } [n]} f_{|I_1}f_{|I_2}...f_{|I_k}y^k.$$

Then the coefficient $[y^k]F_n(y) = \sum a_{k,i}x^i$. Use the Compositional Formula again,

$$F_n = \left[\frac{t^n}{n!}\right] \exp(yN).$$

Therefore

$$[y^k]F_n(y) = \left[\frac{t^n}{n!}\right] \exp(yN)$$

$$= \left[\frac{t^n}{n!}\right] [y^k] \exp(yN)$$

$$= \left[\frac{t^n}{n!}\right] \frac{N^k}{k!}.$$ 

This yields the formula for the decomposition of the Chow motive $h(X[n])$. 

4.3. Description of $X[n]$ for small $n$. In this section we explain the previous Theorems (4.1, 4.2, and 4.3) about Fulton-MacPherson configuration space $X[n]$ for small $n = 2, 3, 4$.

For unification of expression, assume $d > 1$ in the following examples. (The case $d = 1$ is simpler but the expression needs to be modified.)

**Example** $n = 2$. The morphism $\pi : X[2] \to X^2$ is a blow-up along the diagonal $\Delta_{12}$. Theorem 4.3 asserts

$$h(X[2]) \cong h(X^2) \oplus \bigoplus_{i=1}^{d-1} h(\Delta_{12})(i) \cong h(X^2) \oplus \bigoplus_{i=1}^{d-1} h(X)(i).$$

There are 2 possible nests: $S = \{1, 2\}$ and $S = \{1, 2, 12\}$. Theorem 4.2 asserts the following.
For the first nest, $M_{S}$ contains only one element $\mu$ with $\|\mu\| = 0$. Therefore $\alpha = \Gamma_{\pi}, \beta = \Gamma_{\pi}'$, $p = \Gamma_{\pi}' \circ \Gamma_{\pi}$. They give the first direct summand in the decomposition (8).

For the second nest, $S^{\circ} = \{12\}$, $1 \leq \mu_{12} \leq d - 1$, so there are $d - 1$ direct summands for this nest. Denote $j : D_{12} \hookrightarrow X[2], g : D_{12} \rightarrow \Delta_{12}$ as the natural map, we have

$$\alpha_{S, \mu} = -(j, g)_{\ast} j^{\ast} \left( \sum_{i=0}^{d-1-\mu_{12}} p_{i}^{\ast} c_{i}(T_{X})(-D_{12})^{d-1-\mu_{12} - i} \right),$$

$$\beta_{S, \mu} = (g, j)_{\ast} j^{\ast} (D^{\mu_{12} - 1}),$$

$$\pi_{S, \mu} = \beta_{S, \mu} \circ \alpha_{S, \mu}.$$  

They give the direct summand $h(\Delta_{12})(\mu_{12})$ in the decomposition (8).

**Example** $n = 3$. Apply Theorem 4.3

$$h(X[3]) \cong h(X^{3}) \oplus \bigoplus_{i=1}^{d-1} h(\Delta_{12})(i) \oplus \bigoplus_{i=1}^{d-1} h(\Delta_{13})(i) \oplus \bigoplus_{i=1}^{d-1} h(\Delta_{23})(i) \oplus \bigoplus_{i=1}^{2d-1} \left( h(\Delta_{123})(i) \right)^{\oplus \min\{3i-2, 6d-3i-2\}}$$

$$\cong h(X^{3}) \oplus \bigoplus_{i=1}^{d-1} (h(X^{2})(i))^{\oplus 3} \oplus \bigoplus_{i=1}^{2d-1} (h(X)(i))^{\oplus \min\{3i-2, 6d-3i-2\}}$$

Now we write out all the correspondences that give the decomposition of motives. There are 8 possible nests, correspond to 8 trees (see the right side of Figure 4.3).

The tree on the left side of Figure 4.3 helps us to understand the relation between subvarieties of different $Y_{i}$’s (i.e. at different levels): each node with label $I$ at level $k$ correspond to the subvariety $Y_{i}I := (\Delta_{I})^{(k)}$ in $Y_{k}$. The node at level $k$ without label correspond to $Y_{k}$. For example, the root at level 4 corresponds to $Y_{4}$, its two successors correspond to $Y_{3}$ and $Y_{3}(23)$, and the relation is that $Y_{4}$ is the blow-up of $Y_{3}$ along $Y_{3}(23)$.

We list below those correspondences $\alpha, \beta, p$ for the 8 trees:

1. gives $\alpha = \Gamma_{\pi}, \beta = \Gamma_{\pi}'$, $p = \Gamma_{\pi}' \circ \Gamma_{\pi}$.

---

**Figure 1.** $X[3]$ by the symmetric construction.
Remark: If we use Fulton and MacPherson’s nonsymmetric construction of \(X[3]\), we would get another set of correspondences which also gives a decomposition of the motive \(h(X[n])\). This set of correspondences turns out to be different than the ones given above: a straightforward calculation shows that, by the nonsymmetric construction of \(X[3]\), the correspondence that gives the direct summand \(h(\Delta_{12})(\mu_{12})\) is

\[
\alpha : \ h(X[3]) \rightarrow h(\Delta_{12})(\mu_{12}),
\]

\[
\alpha = (j_{12}, g_{12}) \ast j_{12}^* \left( \{ p_1^* \zeta (-D_{123}) \frac{1}{1 + D_{12}} \}^{d-1-\mu_{12}} \right),
\]

where \(j_{12} : D_{12} \hookrightarrow X[3]\) and \(g_{12} : D_{12} \rightarrow \Delta_{12}\) are the natural morphisms. However, the correspondence giving the direct summand \(h(\Delta_{13})(\mu_{13})\) is

\[
\alpha' : \ h(X[3]) \rightarrow h(\Delta_{13}) \otimes_\mathbb{P} \mu_{13},
\]

\[
\alpha' = (j_{13}, g_{13}) \ast j_{13}^* \left( \{ p_1^* \zeta (-D_{123}) \frac{1}{1 + D_{13}} \}^{d-1-\mu_{13}} \right).
\]

where \(j_{13} : D_{13} \hookrightarrow X[3], g_{13} : D_{13} \rightarrow \Delta_{13}\) are the natural morphisms. Notice that \(\alpha\) and \(\alpha'\) are not of similar forms (Compare \(\zeta(\mathcal{O})\) with \(\zeta(-D_{123})\)). Therefore the non-symmetry of the construction of \(X[3]\) induces the non-symmetry of correspondences. Actually, this is one reason why we choose the symmetric construction of \(X[n]\) (cf. Remark 4.1).

Example \(n = 4\). we only look at one nest \(S\):

\[
\alpha_{S_n} = (j_{12}, g_{12}) \ast j_{12}^* \left( \{ -p_1^* \zeta (-D_{123}) \frac{1}{1 + D_{12}} \}^{d-1-\mu_{12}} \right),
\]

\[
\beta_{S_n} = (g_{12}, j_{12}) \ast j_{12}^* \left( D_{123}^{\mu_{12}} \right).
\]

where \(X(S) = D_{12}, 1 \leq \mu_{12} \leq d - 1\).

\[
\alpha_{S_n} = (j_{12}, g_{12}) \ast j_{12}^* \left( \{ -p_1^* \zeta (\mathcal{O}) \frac{1}{1 + D_{12}} \}^{2d-1-\mu_{12}} \right),
\]

\[
\beta_{S_n} = (g_{12}, j_{12}) \ast j_{12}^* \left( D_{123}^{\mu_{12}} \right).
\]

where \(X(S) = D_{123}, 1 \leq \mu_{123} \leq 2d - 1\).

\[
\alpha_{S_n} = (j_{12}, g_{12}) \ast j_{12}^* \left( \{ -p_1^* \zeta (-D_{123}) \frac{1}{1 + D_{12}} \}^{d-1-\mu_{12}} \right),
\]

\[
\beta_{S_n} = (g_{12}, j_{12}) \ast j_{12}^* \left( D_{123}^{\mu_{12}} \right).
\]

where \(X(S) = D_{12} \cap D_{123}, 1 \leq \mu_{12}, \mu_{123} \leq d - 1\).
We have $X(S) = D_{12} \cap D_{34}$, $1 \leq \mu_{12}, \mu_{34} \leq d - 1$ and
\[
\alpha_{S, \mu} = (j_S, g_S), \quad j_S^* = (\{p_1^* \zeta(-D_{1234}) 1 \over 1 + D_{12}\}_{d - 1 - \mu_{12}} \{p_3^* \zeta(-D_{1234}) 1 \over 1 + D_{34}\}_{d - 1 - \mu_{34}}),
\]
\[
\beta_{S, \mu} = (g_S, j_S) = j_S^*(D_{12}^{\mu_{12} - 1}D_{34}^{\mu_{34} - 1}).
\]

Since $\Delta_{12}$ and $\Delta_{34}$ would not be disjoint in the procedure of blow-ups, a priori we have to make a choice of order that whether blow up along (the strict transform of) $\Delta_{12}$ first, or along (the strict transform of) $\Delta_{34}$ first.

Although we have to choose (non-canonically) an order to compute the correspondences, it turns out that the correspondences (hence projectors) which give the motive decomposition in Theorem 4.2 are actually independent of the choice, therefore “canonical”. This independence is a special case of Remark 4.1: for $\sigma = (13)(24) \in S_4$, the above correspondences is invariant under the action induced by $\sigma$.

An application of Theorem 4.3 is: we can compute the rank of $A(X[n])$ (as an abelian group) once given the ranks of $A(X^k)$ for all $1 \leq k \leq n$ (assuming that the ranks of $A(X^k)$’s are finite).

Let us take $\mathbb{P}^d[5]$ for example. Since the rank of $A((\mathbb{P}^d)^k)$ is $(d + 1)^k$, Theorem 4.3 implies that the rank of $A(\mathbb{P}^d[5])$ is
\[
\sum_{1 \leq k \leq 5} (d + 1)^k \left( \frac{t^5}{t!} \left( \frac{N^k}{k!} \bigg|_{x=1} \right) \right).
\]

By Remark 4.2, we can compute the following
\[
\begin{align*}
\frac{N^2}{2!} &= \frac{t^2}{2!} + 3\sigma_1 \frac{t^3}{3!} + (15\sigma_1^2 + 4\sigma_2) \frac{t^4}{4!} + (105\sigma_1^3 + 60\sigma_1\sigma_2 + 5\sigma_3) \frac{t^5}{5!} + \ldots, \\
\frac{N^3}{3!} &= \frac{t^3}{3!} + 6\sigma_1 \frac{t^4}{4!} + (45\sigma_1^2 + 10\sigma_2) \frac{t^5}{5!} + \ldots, \\
\frac{N^4}{4!} &= \frac{t^4}{4!} + 10\sigma_1 \frac{t^5}{5!} + \ldots, \\
\frac{N^5}{5!} &= \frac{t^5}{5!} + \ldots.
\end{align*}
\]

Now plug in $x = 1$, we have $\sigma_j = dj - 1$. The above sum is a polynomial of $d$ as follows
\[
\begin{align*}
& (d + 1)^5 + (d + 1)^4 10\sigma_1 + (d + 1)^3 (45\sigma_1^2 + 10\sigma_2) \\
& + (d + 1)^2 (105\sigma_1^3 + 60\sigma_1\sigma_2 + 5\sigma_3) \\
& + (d + 1) (\sigma_4 + 15\sigma_1\sigma_3 + 10\sigma_2^2 + 105\sigma_1^2\sigma_2 + 105\sigma_1^4).
\end{align*}
\]

In particular, the rank of $A(\mathbb{P}^1[5])$ is 178, the rank of $A(\mathbb{P}^2[5])$ is 7644.
Remark: For the example $X = \mathbb{P}^d$, since $X[n]$ has an affine cell decomposition, the rank of the Chow group $A_k(X[n])$ coincides with the $2k$-th Betti number of $X[n]$. Therefore we could also get the above rank by the Poincaré polynomial of $X[n]$ computed in [FM94]. However, the rank of $A(X[n])$ for a general variety $X$ is not implied by the Poincaré polynomial of $X[n]$.

5. Chow motives of $X[n]/\mathcal{S}_n$

It is proved in [FM94] that the isotropy group of any point in $X[n]$ is a solvable group. It is natural to consider the quotient space $X[n]/\mathcal{S}_n$. In this section, we compute its Chow motive in terms of the Chow motives of the Cartesian products of symmetric products of $X$.

The base field is of characteristic 0 throughout this section.

Lemma 5.1. Suppose a finite group $G$ acts on a nonsingular projective variety $Y$. If $p_1, \ldots, p_k$ are orthogonal projectors of $Y$ that

i) $\sigma p_i = p_i \sigma$, $\forall 1 \leq i \leq k, \forall \sigma \in G$.

ii) $p_1 + p_2 + \cdots p_k = \Delta_Y$.

Then $\text{ave} \Delta_Y = \sum \text{ave} \circ p_i$ where $\text{ave} \circ p_1, \ldots, \text{ave} \circ p_k$ are orthogonal projectors. Consequently, $h(Y) = \oplus (Y, \text{ave} \circ p_i)$.

Proof. Since

$$(\text{ave} p_i)(\text{ave} p_j) = \left( \frac{1}{|G|} \sum_\sigma \sigma p_i \right) \left( \frac{1}{|G|} \sum_\tau \tau p_j \right)$$

$$= \frac{1}{|G|^2} \sum_{\sigma, \tau} \sigma \tau p_i p_j = \frac{1}{|G|} \sum_{\sigma} \sigma \delta_{ij} p_i = \delta_{ij} (\text{ave} p_j).$$

Then the lemma follows. \qed

Lemma 5.2. Suppose $Y, Z$ are nonsingular (not necessary connected) projective varieties with finite group $G$ actions. Suppose that $\alpha \in \text{Corr}^{-m}(Y, Z)$ has an inverse $\beta \in \text{Corr}^m(Z, Y)$, and $\alpha$ gives an isomorphism of Chow motives

$$(Y, p) \cong h(Z)(m)$$

where $p = \beta \alpha$, and $\alpha \sigma = \sigma \alpha, \beta \sigma = \sigma \beta, \forall \sigma \in G$. Then

$$(Y, \text{ave} \circ p) \cong h(Z/G)(m).$$

Proof. Similar to the proof of Lemma 5.1 we have $(\text{ave} p)^2 = \text{ave} p$ and the following commutative diagram

$$\begin{array}{ccc}
Y & \xrightarrow{\text{ave}} & Z \xrightarrow{\text{ave} \beta} Y \\
\downarrow \text{ave} p & & \downarrow \text{ave} \Delta_Z \\
Y & \xrightarrow{\text{ave} \alpha} & Z \xrightarrow{\text{ave} \beta} Y.
\end{array}$$

Therefore, $(Y, \text{ave} p) \cong (Z, \text{ave} \Delta_Z)(m) \cong h(Z/G)(m)$. \qed
Now we consider the quotient variety $X[n]/\mathcal{G}_n$. For convenience, define $G := \mathcal{G}_n$. There is a natural action of $G$ on the set $\{ (S, \mu) \}$ where $S$ are nests and $\mu \in M_S$. Define the subgroup $G_{S, \mu}$ of $G$ as

$$G_{S, \mu} = \{ \sigma \in \mathcal{G}_n : \sigma(S, \mu) = (S, \mu) \}.$$ 

Define $\overline{(S, \mu)}$ to be the class of $G$-orbit $G \cdot (S, \mu)$. Then

$$\Delta_Y = \sum_{S, \mu} p_{S, \mu} = \sum_{\overline{(S, \mu)}} \sum_{\sigma \in G/G_{S, \mu}} p_{\sigma(S, \mu)}.$$ 

Since $\{ \alpha_{S, \mu} \}$, $\{ \beta_{S, \mu} \}$ are $\mathcal{G}_n$-symmetric (cf. the Remark after Theorem 4.2), it is easy to check that $\sum_{\sigma \in G/G_{S, \mu}} p_{\sigma(S, \mu)}$ commutes with every $\tau \in G$. By Lemma 5.1, $h(X[n]/G) \cong (Y, \text{ave} \circ \Delta_Y) \cong \bigoplus_{\overline{(S, \mu)}} \left( Y, \text{ave} \circ \sum_{\sigma \in G/G_{S, \mu}} p_{\sigma(S, \mu)} \right)\).$

Since

$$\left( Y, \sum_{\sigma \in G/G_{S, \mu}} p_{\sigma(S, \mu)} \right) \cong \left( \bigsqcup_{\sigma \in G/G_{S, \mu}} \Delta_{\sigma(S)}(|\mu|) \right),$$

by Lemma 5.2 we have

$$\left( Y, \sum_{\sigma \in G/G_{S, \mu}} p_{\sigma(S, \mu)} \right) \cong h\left( \bigsqcup_{\sigma \in G/G_{S, \mu}} \Delta_{\sigma(S)}|/G (|\mu|) \right) \cong h(\Delta_S/G_{S, \mu}(|\mu|)).$$

The space $\Delta_S/G_{S, \mu}$ can be described as follows. Each $(S, \mu)$ corresponds to a labeled “weighted” forest, the correspondence is given by attaching an integer $\mu_I$ to each non-leaf node $I$ of the labeled forest $S$. Forgetting all the labels on the nodes of $S$, we get an unlabeled weighted forest of the form $n_1T_1 + \cdots + n_rT_r$, where $T_i$ are mutually distinct unlabeled weighted tree (we call such a tree is of type $\{ n_1, \ldots, n_r \}$). Then

$$\Delta_S/G_{S, \mu} \cong X^{(n_1)} \times \cdots \times X^{(n_r)}.$$ 

Figure 2 gives an example of a labeled weighted forest and the corresponding unlabeled weighted forest. The weight $a, b$ are integers.

![Figure 2. Labeled and unlabeled weighted forests.](image-url)
Therefore we have proved the following decomposition of the Chow motive of $X[n]/\mathfrak{S}_n$:

**Theorem 5.3.** For any unordered set of integers $\nu = \{n_1, \ldots, n_r\}$ and any integer $m$, let $\lambda(\nu, m)$ to be the number of unlabeled weighted forest with $n$ leaves, of type $\nu$ and total weight $m$, such that at each non-leaf $v$ with $c_v$ children, the weight $m_v$ satisfies $1 \leq m_v \leq (c_v - 1) \dim X - 1$. Then

$$h(X[n]/\mathfrak{S}_n) = \bigoplus_{\nu,m} \left[ h(X^{(n_1)} \times \cdots \times X^{(n_r)})(m) \right] \otimes \lambda(\nu,m).$$

**Remark:** An application of this theorem. MacDonald proved a formula that relates the Betti number of $X$ and its symmetric powers:

$$\sum_{n=0}^{\infty} P_1 X^{(n)} \cdot T^n = \frac{(1 + tT)^{b_1} (1 + t^2T)^{b_2} \cdots}{(1 - T)^{b_0} (1 - t^2T)^{b_2} \cdots}$$

where $b_i$ is the $i$-th Betti number of $X$. By the decomposition of the de Rham cohomology of $X[n]/\mathfrak{S}_n$ induced by the motivic decomposition formula in the above theorem, we can compute the Betti number of $X[n]/\mathfrak{S}_n$ (modulo the combinatorial difficulty of calculating $\lambda(\nu,m)$).

**Examples:** Here are some examples of $h(X[n]/\mathfrak{S}_n)$ for small $n$. Let $d = \dim X$.

i) $n=2$. There are $d$ different forests as follows, where each weight $a \in \mathbb{Z}$ ($1 \leq a \leq d - 1$) gives a forest:

$$\nu = \{2\} \quad \nu = \{1\}$$

Therefore

$$h(X[2]/\mathfrak{S}_2) \cong h(X^{(2)}) \oplus \bigoplus_{a=1}^{d-1} h(X)(a).$$

ii) $n=3$. The forests are:

$$\nu = \{3\} \quad \nu = \{1, 1\} \quad \nu = \{1\} \quad \nu = \{1\}$$

where the weights $a, b, c, e \in \mathbb{Z}$ satisfy $1 \leq a, c, e \leq d - 1$, and $1 \leq b \leq 2d - 1$. We have

$$h(X[3]/\mathfrak{S}_3) \cong h(X^{(3)}) \oplus \bigoplus_{i=1}^{d-1} (h(X^2)(i))^3 \oplus \bigoplus_{i=1}^{2d-1} (h(X)(i))^{\min\{i, 2d-i\}}.$$
iii) \( n=4 \). The varieties appear in the decomposition of \( h(X[4]/\mathbb{G}_4) \) are:

\[
X^{(4)}, X \times X^{(2)}, X^2, X^{(2)}, X.
\]

The decomposition is a bit nasty to be written here. Therefore we only point out a fact. Consider the forest in Figure 3 where \( a, b \in \mathbb{Z} \) and \( 1 \leq a, b \leq d-1 \). For any \( a < b \), the weighted forest is of type \( \nu = \{1, 1\} \) and therefore

\[
\cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \·
where the subscript $r - 1 - k$ in the definition of $\alpha_k$ means taking the codimension $(r - 1 - k)$ component. We will give the proof of the following proposition at the end of this section.

**Proposition A.1.** Define $\alpha_k, \beta_k, p_k, \alpha_0, \beta_0, p_0$ as above. The following holds.

(i) $\alpha_0 \beta_0 = \Delta_Y$, $\alpha_k \beta_k = \Delta_V$ for $1 \leq k \leq r - 1$; $\alpha_i \beta_j = 0$ for $i \neq j$.

(ii) $p_0, p_1, p_2, ..., p_{r-1}$ are mutually orthogonal projectors of $\bar{Y}$, and

\[
\sum_{i=0}^{r-1} p_i = \Delta_{\bar{Y}} \text{ in } A(\bar{Y} \times \bar{Y}),
\]

i.e. equality holds up to rational equivalence.

(iii) We have the following isomorphisms of motives,

\[
\alpha_0 : (\bar{Y}, p_0, 0) \simeq h(Y), \text{ with inverse morphism } \beta_0,
\]

\[
\alpha_k : (\bar{Y}, p_k, 0) \simeq h(V)(k), \text{ with inverse morphism } \beta_k, \text{ for } 1 \leq k \leq r - 1.
\]

Define $\Gamma := \bigoplus_{i=0}^{r-1} \alpha_i$, $\Gamma' := \sum_{i=0}^{r-1} \beta_i$, then Proposition A.1 can be conveniently reformulated as follows:

**Theorem A.2.** The correspondence $\Gamma$ gives a canonical isomorphism in $\mathcal{CHM}$,

\[
\Gamma : h(\bar{Y}) \cong h(Y) \oplus \bigoplus_{k=1}^{r-1} h(V)(k).
\]

with an inverse isomorphism given by $\Gamma'$.

**Remark:** When the normal bundle $N$ of $V$ in $Y$ is trivial (for example, when $V$ is a point), $P$ is isomorphic to a product space $V \times \mathbb{P}^{r-1}$ and $h = c_1(O_P(1))$ can be represented (not canonically) by a product space $H = V \times \mathbb{P}^{r-2}$ in $P$. In this case, we have simple forms for the projectors:

\[
p_k = -(j \times j)_*(H^{r-1-k} \times VH^{k-1}), \text{ for } 1 \leq k \leq r - 1;
\]

\[
p_0 = \Delta + \sum_{k=1}^{r-1} (j \times j)_*(H^{r-1-k} \times VH^{k-1}).
\]

In general, for a nontrivial normal bundle $N$, more terms involving the Chern classes of $N$ are needed, and the correspondences cannot be represented by explicit and natural algebraic cycles.

**Remark:** The isomorphism of motives in Theorem A.2 is also a consequence of “Theorem on the additive structure of the motif” of $\bar{Y}$ in [Man68, §9, which states, in our notation, that there is a split exact sequence

\[
0 \to h(V)(r) \xrightarrow{a} h(Y) \oplus h(P)(1) \xrightarrow{b} h(\bar{Y}) \to 0.
\]
The correspondences appeared in our theorem are not given, at least not explicitly, in Manin’s paper. In order to clarify this point, define

\[ \Phi = c_{r-1}(g^*N/O_N(-1)) \in A^{r-1}(P), c_{\Phi} = \delta_{P^*}(\Phi) \in \text{Corr}(P, P) , \]

\[ a = (i_* c_{\Phi} \circ g^*), a' = g_* , \]

\[ b = f^* + j_*, b' \text{ its right inverse}, \]

\[ d = \Delta_{Y \times P} - aa', d' = \Delta_Y \otimes (\Delta_P - p_0^P) \text{ (where } p_0^P = c_{h^{r-1} \circ g^* \circ g_*}), \]

denote by \( e : \bigoplus_{k=1}^{r-1} V(k) \to (P, \Delta_P - p_0^P) \) the isomorphism implicitly defined in [Man68] §7, and denote by \( e' \) the inverse of \( e \).

We have the following isomorphisms

\[ h(Y) \oplus \bigoplus_{k=1}^{r-1} h(V)(k) \xrightarrow{\Delta_Y \otimes e} \bigoplus_{k=1}^{r-1} (Y \sqcup P, (\Delta_Y, \Delta_P - p_0^P)) \xrightarrow{d} \]

\[ (Y \sqcup P, \Delta_{Y \sqcup P} - aa') \xrightarrow{b} (\widetilde{Y}, \Delta_{\widetilde{Y}}) . \]

Hence the following is an isomorphism of Chow motives

\[ (\Delta_Y \otimes e') \circ d' \circ b' : h(\widetilde{Y}) \cong h(Y) \oplus \bigoplus_{k=1}^{r-1} h(V) \otimes L^k \]

with inverse \( b \circ d \circ (\Delta_Y \otimes e) \).

Therefore, to write down the correspondence \((\Delta_Y \otimes e') \circ d' \circ b'\), we need to find explicitly the right inverse \( b' \) of \( b \). However in [Man68] the construction of \( b' \) is based on the surjectivity of \( \gamma : A(\widetilde{Y} \times (Y \sqcup P)) \to A(\widetilde{Y} \times \widetilde{Y}) \) as follows: by the surjectivity of \( \gamma \), there is a cycle class \( c \in A(\widetilde{Y} \times (Y \sqcup P)) \) (which is not given, at least explicitly, in [Man68]) such that \( \gamma(c) = \Delta_{\widetilde{Y}} \in A(\widetilde{Y} \times \widetilde{Y}). \) Then \( b' \) is defined to be \((1 - aa')c\).

On the other hand, the correspondences \( \Gamma \) and \( \Gamma' \) we have constructed in Theorem A.2 give an explicit construction of \( b' \). Indeed, \( b' = d \circ (\Delta_Y \otimes e) \circ \Gamma \).

**Proof of Proposition A.7.** In the proof, we assume \( 1 \leq k \leq r - 1, \quad 0 \leq i, j \leq r - 1 \).

The idea is as follows: we study the morphisms \( \alpha_{i*}, \beta_{i*} \) and \( p_{i*} \) of Chow groups induced by the correspondences \( \alpha_i, \beta_i \) and \( p_i \). As a consequence, the identities of morphisms of Chow groups which are induced by the identities in Proposition A.1 (i) (ii) hold. On the other hand, Manin’s Identity Principle asserts that the identities of morphisms of Chow groups imply the identities of correspondences, providing that the correspondences are universal in certain sense.
By Theorem 9.27, an element $\tilde{y} \in A(\tilde{Y})$ can be expressed uniquely as

$$\tilde{y} = \sum_{i=1}^{r-1} j_* (g^* a_i \cdot h^{i-1}) + f^* y.$$ 

It is standard to verify

(\(\alpha_k\)) The morphism \(\alpha_k : A(\tilde{Y}) \to A(V)\) maps \(\tilde{y} \mapsto a_k\).

(\(\beta_k\)) The morphism \(\beta_k : A(V) \to A(\tilde{Y})\) maps \(x \mapsto j_*(g^* x \cdot h^{k-1})\).

(\(\alpha_0\)) The morphism \(\alpha_{0*} : A(\tilde{Y}) \to A(Y)\) maps \(\tilde{y} \mapsto y\).

(\(\beta_0\)) The morphism \(\beta_{0*} : A(Y) \to A(\tilde{Y})\) maps \(y \mapsto f^* y\).

To give a flavor, we prove only the statement \((\alpha_k)\), that is, \(\alpha_{k*}(\tilde{y}) = a_k\). Define \(a_0 = -i^* y\). Since \(j^* j_* z = -h \cdot z\) for \(\forall z \in A(P)\), we have

$$j^* \tilde{y} = \sum_{i=1}^{r-1} j^* j_*(g^* a_i \cdot h^{i-1}) + j^* f^* y = \sum_{i=0}^{r-1} g^* a_i \cdot h^i + g^* i^* y = \sum_{i=0}^{r-1} g^* a_i \cdot h^i.$$ 

By definition (see \([Fu98]\) §3), the \(i\)-th Segre class of \(N\) is

$$s_i(N) := g_*(h^{i+r-1}),$$

hence

$$\alpha_{k*}(\tilde{y}) = -g_* (j^* \tilde{y} \cdot \sum_{l=0}^{r-1-k} g^* c_{r-1-k-l}(N) \cdot h^l)$$

$$= -g_* \left( \left( -\sum_{i=0}^{r-1} g^* a_i \cdot h^i \right) \cdot \left( \sum_{l=0}^{r-1-k} g^* c_{r-1-k-l} \cdot h^l \right) \right)$$

$$= g_* \left( \sum_{i=0}^{r-1} \sum_{l=0}^{r-1-k} g^*(a_i c_{r-1-k-l}) h^{i+l} \right)$$

$$= \sum_{i=0}^{r-1} a_i \left( \sum_{l=0}^{r-1-k} c_{r-1-k-l}s_{i+l+1-r} \right).$$

Since \(c(N)s(N) = 1\) where \(c(N) := \sum c_i(N)\) is the total Chern class and \(s(N) := \sum s_i(N)\) is the total Segre class, we have

$$\sum_{l=0}^{r-1-k} c_{r-1-k-l}s_{i+l+1-r} = \sum_{l=-\infty}^{+\infty} c_{r-1-k-l}s_{i+l+1-r} = \{c(N)s(N)\}_{i-k} = \delta_{ik},$$

the first equality is because \(s_{i+l+1-r} = 0\) for \(l < 0\) and \(c_{r-1-k-l} = 0\) for \(l > r - 1 - k\). It follows that \(\alpha_{k*}(\tilde{y}) = a_k\), as we claimed.
The statements \((\alpha_k), (\beta_k), (\alpha_0), (\beta_0)\) immediately imply the following identities:

\[
\begin{align*}
\alpha_k \beta_k &= id_{A(V)}, \\
\alpha_0 \beta_0 &= id_{A(Y)}, \\
\alpha_i \beta_j &= 0 \text{ for } i \neq j,
\end{align*}
\]

\[
(p_i p_j)_* = \delta_{ij} p_i^*,
\]

\[
\sum_{i=0}^{r-1} p_i^* = id_{A(\tilde{Y})}.
\]

For any smooth scheme \(T\), \(T \times \tilde{Y}\) is the blow-up of \(T \times Y\) along the smooth subvariety \(T \times V\). Denote \(j' = id_T \times j\), \(g' = id_T \times g\), \(f' = id_T \times f\), \(i' = id_T \times i\), we have the following fiber square:

\[
\begin{array}{ccc}
T \times P & \xrightarrow{j'} & T \times \tilde{Y} \\
\downarrow g' & & \downarrow f' \\
T \times V & \xrightarrow{i'} & T \times Y
\end{array}
\]

We can construct the correspondences \(\alpha'_i, \beta'_i, p'_i\) for this fiber square as we did in (9). we have

\[
\begin{align*}
\alpha'_i &= id_T \otimes \alpha_i, \\
\beta'_i &= id_T \otimes \beta_i, \\
p'_i &= id_T \otimes p_i.
\end{align*}
\]

Then (i) and (ii) follows from Manin’s Identity Principle.

For (iii), to show that \(\alpha_k\) gives an isomorphism \((\tilde{Y}, p_k, 0) \simeq h(V) \otimes \mathbb{L}^k\) with inverse \(\beta_k\), we need to show that \(p_k = p_k \circ \beta_k \circ \alpha_k\) and \(id = id \circ \alpha_k \circ \beta_k\). but they are direct consequences of the fact that \(\alpha_k \circ \beta_k = \Delta_V\) from (i). The proof for \((\tilde{Y}, p_0, 0) \simeq h(Y)\) is similar.

\[\square\]

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Li Li
Department of Mathematics
University of Illinois at Urbana-Champaign
Email: llpku@math.uiuc.edu