INVESTIGATION OF SELF-PRESERVATION THEORY IN TWO DIMENSIONAL TURBULENT MIXING LAYERS

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Abstract
The behavior of a two dimensional, steady turbulent mixing layer was investigated. Besides the usual velocity components, all the contributing components of the Reynolds Stresses are also calculated and presented. The results indicated that in the two dimensional steady turbulent mixing layers the theory of self-preservation is not valid for all the flow domain, and that the flow is turbulent regime is divided into two areas. Through calculation of Reynolds Stress components at the point of initiation of the flow, it was shown that the turbulent flow in the area containing the point of singularity of the flow is not, as previously believed, self-preserved.

Key Words: Turbulent, Plane Mixing Layer, Reynolds Stress Equations, Self-Preservation

1 Introduction

Turbulence is the most frequently occurring mode of fluid motion. It is characterized by high fluid particle mixing, and high energy dissipation [1-3]. The exact time-dependent nature of events leading to the onset and development of turbulence is not fully known; however, by the aid of time averaging techniques we can obtain meaningful and practically useful statistical definitions [3]. Nearly all the problems that are studied in the field of Turbulent Fluid Dynamics are numerically extensive and expensive [3-5], and there is

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no single concept to model different classes of turbulent flow regimes [3-5]. In general a turbulent flow is characterized by random variation of flow variables, high levels of fluid particle mixing, and high energy dissipation [2, 3].

Turbulent flow regimes may be divided into two main categories: Free Turbulence, where no physical boundary limits the development of flow; Bounded Turbulence, where flow is restricted by physical boundaries [2]. Experimentally, Free turbulent flows are easy to generate, and that is primarily why they have been so extensively studied. From a theoretical standpoint, their formulation and simulation is also much simpler, since for normal incompressible flows, no pressure gradient term has relevance [3]. In studying free turbulent flow fields it is desirable for practical reasons to confine oneself, at first, to rather simple but important elementary flow patterns [1, 2]. These elementary flow patterns may be classified into two main groups: (1) free jets, and (2) wake flows behind obstacles. The flow pattern under investigation here is a plane, two dimensional mixing layer [3]. In this flow pattern it is possible to distinguish one main flow direction with velocity substantially greater than in any other direction [1, 2].

Throughout the past few decades, turbulent plane mixing layer has been the subject of numerous experimental and theoretical research, and it is widely used to investigate various aspects of phenomena related to free turbulence, including process and nature of mixing [3], structure of turbulent components and their self-preservation [3-6], stability of the flow [4], transfer of momentum and energy [5], intensity, and rates of temporal and spatial expansions of free turbulent flows [6].

The spatial and temporal development of the turbulence is closely correlated to the development of Reynolds Stress Components. The existing literature indicates that for free turbulent flows, the entire flow domain was considered to be self-preserved [7]. However, in this investigation it is shown that the self-preservation theory is not applicable to all the flow field, and that at areas very close to the point of initiation of turbulent flow regime the said theory is not applicable, i.e. the flow is not self-preserved.

Following the introduction the fluid dynamics equations are indicated and the formulation is completed. In section three, the simulation results and discussions are presented. And in the final section discussion and concluding remarks are presented.

2 Turbulent plane mixing layer’s governing equations

For a 2-dimensional incompressible turbulent plane mixing layer, the governing equations are Reynolds equations and the continuity equation. They may be written in the following form:

\[
\frac{\partial \bar{u}_1}{\partial t} + \bar{u}_1 \frac{\partial \bar{u}_1}{\partial x} + \bar{u}_2 \frac{\partial \bar{u}_1}{\partial y} = \nu \frac{\partial^2 \bar{u}_1}{\partial y^2} - \frac{\partial \bar{u}_1 \bar{u}_2'}{\partial x} - \frac{\partial \bar{u}_1' \bar{u}_2}{\partial y},
\]

(1)

\[
\frac{\partial \bar{u}_2}{\partial t} + \bar{u}_1 \frac{\partial \bar{u}_2}{\partial x} + \bar{u}_2 \frac{\partial \bar{u}_2}{\partial y} = \nu \frac{\partial^2 \bar{u}_2}{\partial y^2} - \frac{\partial \bar{u}_1' \bar{u}_2'}{\partial x},
\]

(2)

\[
\frac{\partial \bar{u}_1}{\partial x} + \frac{\partial \bar{u}_2}{\partial y} = 0,
\]

(3)
where bar represents time-averaged quantities, and \( u_1, u_2 \) are the velocity components in the \( x \) and \( y \) directions respectively; also \( u'_1 \) and \( u'_2 \) are the fluctuating components of the \( u_1 \) and \( u_2 \) velocity components respectively.

To close the above set of equations the Reynolds Stress Equation Model ("RSM") is used \([4]\). The exact equation for the transport of Reynolds Stresses, \( R_{ij} \), takes the following form:

\[
\frac{\partial R_{ij}}{\partial t} + \text{div}(R_{ij}V) = P_{ij} + D_{ij} - \varepsilon_{ij} + \Pi_{ij} + \Omega_{ij} ,
\]

where \( P_{ij}, D_{ij}, \varepsilon_{ij}, \Pi_{ij}, \) and \( \Omega_{ij} \) represent rate of production of \( R_{ij} \), transport of \( R_{ij} \) by diffusion, rate of dissipation of \( R_{ij} \), transport of \( R_{ij} \) due to turbulent pressure-strain interactions, and transport of \( R_{ij} \) due to rotation respectively. These terms are represented by the following equations:

\[
P_{ij} = -(R_{im} \frac{\partial \bar{V}_j}{\partial x_m} + R_{jm} \frac{\partial \bar{V}_i}{\partial x_m}) ,
\]

\[
D_{ij} = \frac{\partial}{\partial x_m} \left( \frac{\nu_t}{\sigma_k} \frac{\partial R_{ij}}{\partial x_m} \right) ,
\]

\[
\varepsilon_{ij} = (2/3)\varepsilon \delta_{ij} ,
\]

\[
\Pi_{ij} = -C_1 \left( \frac{\varepsilon}{k} \right) (R_{ij} - (2/3)k \delta_{ij}) - C_2 (P_{ij} - (2/3)P \delta_{ij}) ,
\]

\[
\Omega_{ij} = -2\omega_k (R_{jm}e_{ikm} + R_{im}e_{jkm}) .
\]

Turbulent kinetic energy \( k \) is needed in the above formula and can be found by adding the three normal stresses together:

\[
k = 0.5(R_{11} + R_{22} + R_{33})
\]
and the equation for the transport of scalar dissipation rate $\varepsilon$ is:

$$
\frac{\partial (\rho \varepsilon)}{\partial t} + \text{div}(\rho \varepsilon \mathbf{V}) = \text{div}(\frac{\mu_t}{\sigma_{\varepsilon}} \text{grad} \varepsilon) + 2(\varepsilon/k)C_{1\varepsilon}\mu_t E_{ij} \cdot E_{ij} - \rho(\varepsilon^2/k)C_{2\varepsilon},
$$

(12)

where $C_{1\varepsilon} = 1.44$ and $C_{2\varepsilon} = 1.92$, and $E_{ij}$ is the mean rate of deformation of a fluid element and $E_{ij} \cdot E_{ij}$ is their scalar product [4]. Note that for the free stream velocities $R_{ij} = 0$ and $\varepsilon = 0$.

Here the values of the above-defined relationships are presented. For $R_{12}$ we have:

$$
P_{12} = -(R_{11} \frac{\partial \bar{u}_2}{\partial x_1} + R_{12} \frac{\partial \bar{u}_1}{\partial x_1} + R_{12} \frac{\partial \bar{u}_2}{\partial x_2} + R_{22} \frac{\partial \bar{u}_1}{\partial x_2}),
$$

(13)

and

$$
D_{12} = \frac{\partial}{\partial x_1}(\frac{\nu_t}{\sigma_k} \frac{\partial R_{12}}{\partial x_1}) + \frac{\partial}{\partial x_2}(\frac{\nu_t}{\sigma_k} \frac{\partial R_{12}}{\partial x_2}).
$$

(14)

Therefore,

$$
D_{12} = \frac{\partial \nu_t}{\partial x_1} \frac{\partial R_{12}}{\partial x_1} + \frac{\partial \nu_t}{\partial x_2} \frac{\partial R_{12}}{\partial x_2} + \nu_t \frac{\partial^2 R_{12}}{\partial x_1^2} + \nu_t \frac{\partial^2 R_{12}}{\partial x_2^2},
$$

(15)

where,

$$
\frac{\partial \nu_t}{\partial x_1} = (0.09/\varepsilon)[\frac{\partial (R_{11} + R_{22})^2}{\partial x_1} - \frac{(R_{11} + R_{22})^2}{\varepsilon} \frac{\partial \varepsilon}{\partial x_1}]
$$

(16)

and

$$
\frac{\partial \nu_t}{\partial x_2} = (0.09/\varepsilon)[\frac{\partial (R_{11} + R_{22})^2}{\partial x_2} - \frac{(R_{11} + R_{22})^2}{\varepsilon} \frac{\partial \varepsilon}{\partial x_2}]
$$

(17)

also,

$$
\Pi_{12} = (-1.8\varepsilon/k)R_{12} - 0.6P_{12}
$$

(18)

$$
\Omega_{12} = 0
$$

(19)

$$
\varepsilon_{12} = 0.
$$

(20)

For $R_{11}$ components we have:

$$
P_{11} = -2(R_{11} \frac{\partial \bar{u}_1}{\partial x_1} + R_{12} \frac{\partial \bar{u}_1}{\partial x_2}),
$$

(21)

$$
D_{11} = \frac{\partial}{\partial x_1}(\frac{\nu_t}{\sigma_k} \frac{\partial R_{11}}{\partial x_1}) + \frac{\partial}{\partial x_2}(\frac{\nu_t}{\sigma_k} \frac{\partial R_{11}}{\partial x_2}),
$$

(22)

$$
\varepsilon_{11} = (2/3)\varepsilon,
$$

(23)

$$
\Pi_{11} = (-1.8\varepsilon/k)(R_{11} - (2/3)k) - 0.6P_{11}
$$

(24)

$$
\Omega_{11} = 0.
$$

(25)

And for $R_{22}$ components we have:

$$
P_{22} = -2(R_{22} \frac{\partial \bar{u}_2}{\partial x_1} + R_{22} \frac{\partial \bar{u}_2}{\partial x_2}),
$$

(26)
\[
D_{22} = \frac{\partial}{\partial x_1} \left( \nu_t \frac{\partial R_{22}}{\partial x_1} \right) + \frac{\partial}{\partial x_2} \left( \nu_t \frac{\partial R_{22}}{\partial x_2} \right), \\
\varepsilon_{22} = \frac{2}{3} \varepsilon, \\
\Pi_{22} = \left( -\frac{1.8}{k} R_{22} - \frac{2}{3} k \right) - 0.6 P_{22}, \\
\Omega_{22} = 0.
\] (27)

For \( \varepsilon \) we have the following expressions:

\[
\frac{\partial \varepsilon}{\partial t} + \bar{u}_1 \frac{\partial \eta}{\partial x_1} + \bar{u}_2 \frac{\partial \varepsilon}{\partial x_2} = \frac{\partial}{\partial x_1} \left( \frac{\nu_t}{1.3} \frac{\partial \varepsilon}{\partial x_1} \right) + \frac{\partial}{\partial x_2} \left( \frac{\nu_t}{1.3} \frac{\partial \varepsilon}{\partial x_2} \right) + \left( \frac{2.88 \nu_t \varepsilon}{k} \right) \left[ E_{11}^2 + 2E_{12}^2 + E_{22}^2 \right] - \frac{1.92 \varepsilon^2}{k},
\] (31)

where

\[
E_{11} = \frac{\partial \bar{u}_1}{\partial x_1}, \\
E_{22} = \frac{\partial \bar{u}_2}{\partial x_2}, \\
E_{12} = 0.5 \left( \frac{\partial \bar{u}_1}{\partial x_2} + \frac{\partial \bar{u}_2}{\partial x_1} \right),
\] (32)

and the kinetic energy of turbulence is

\[
k = 0.5(R_{11} + R_{22}).
\] (35)

Therefore we have obtained six equations which must be solved simultaneously to provide the two unknown velocity components, three Reynolds stress components and the scalar turbulent dissipation rate.

With the aid of standard Computational Fluid Dynamic (CFD) schemes and formulations, namely the Central Finite Difference Discretization, and using a forward time-marching scheme, the equations are numerically simulated and solved. The time marching is carried out until the steady state conditions were established. The \( x, y \) and \( t \) steps are determined by trial and error and are fixed and equal to 0.0001/40 m, 0.00001/40 m, and \( 10^{-7} \) seconds respectively.

### 3 Simulation results and discussions

The respective computer codes for the CFD formulation of the turbulent plane mixing layer are run until the steady state solution is obtained. Using the approximation, for free stream velocities of \( U_{max} = 10.0(m/s) \), \( U_{min} = 5.0(m/s) \), the velocity component distributions and the non-dimensional Reynolds stress components at various flow sections are obtained, and for sections located at about \( x = 0.00001(m) \), near the initiation point of the flow, where inherently a critical point of discontinuity exists, and about \( x = 0.000036(m) \), far down-stream from the point of initiation of the flow, are plotted and presented in the following figures. In the following figures \( U_{ref} = U_{max} - U_{min} \).
Figure 1 indicates the experimental results for the spatial expansion of the turbulent mixing layer obtained by D. Oster and I. Wygnanski [7]. In the same figure, the simulation results obtained from the utilized CFD code are indicated.

In Figure 2, the Reynolds Stress distribution obtained experimentally [7], at \( x = 300mm \), \( x = 500 \) mm through 1700 mm, and by the utilized CFD formulation, at the area close to the point of initiation of the turbulent flow, is shown.

Figures 3 and 4 indicate the variation of u velocity component against the non-dimensional vertical distance at two intersections close to the point of initiation of the mixing layer, as calculated by the utilized CFD scheme.

Figures 5 and 6 show the distribution of non-dimensional \( u'^2 \) Reynolds Stress Components at cross sections close to the point of initiation of turbulent flow, calculated by the used CFD scheme.

From these figures, the reduction in the peak value of turbulent intensity in the flow direction indicates that the in the region under consideration the flow is not self-preserving.

Figures 7 and 8 present the variation of the non-dimensional \( v'^2 \) Reynolds Stress Components at cross sections close to the point of initiation of turbulent flow, determined by the used CFD code.

Figures 7 and 8 also indicate that the turbulent flow at the desired area is not self-preserving. In addition, comparing the order of magnitudes of \( u'^2 \) and \( v'^2 \) Reynolds Stress components in figures 8 through 10, although the relative contribution of \( u'^2 \) component to the generation of turbulence is more, its effect on further development of turbulence regime downstream of the point of initiation of the plane mixing layer is reduced.

Figures 9 and 10 show the distribution of the time averaged, non-dimensional auto-correlated components of the turbulent velocity fluctuations, very close to the point of emergence of the mixing layer, obtained by the CFD solution.

From figures 9 and 10, comparing the maximum value of the turbulence intensity with the experimental values measured by Oster and Wygnanski [7], Yule, and Spencer, both presented at the same reference and indicated in Table 1, the acceptable performance of the CFD solution and our results is established.

**Table 1.** The experimental and theoretical maximum value of intensity

| Source                          | \( \overline{u'v'/U_{ref}} \) |
|---------------------------------|---------------------------------|
| Spencer (1970)                  | 0.011                           |
| Yule (1971)                     | 0.013                           |
| Oster and Wygnanski (1982)      | 0.013                           |
| Present result                  | 0.013                           |

In addition, Figures 5 through 10 indicate that turbulence Reynolds Stresses, which are themselves responsible for turbulence generation, have their peak value at the center
of symmetry of the flow. This fact together with the order of magnitude of Reynolds Stress variation complies with the known behavior of such turbulent flow regimes [3-7].

Based upon their experimental set up, erected to study free and forced turbulent plane mixing layers, D. Oster and I. Wygnanski had deducted that for free stream velocity ratios of less than 0.6 the flow remains self-preserving [7]. However, their area of interest started 200 mm down stream of the point of initiation of the turbulent flow.

The results obtained by the utilized formulation indicated that for the area of interest in such flow patterns the auto-correlated Reynolds Stress components have closer spatial behavior, but the cross-correlation of turbulent velocity components has a different contribution to the generation of turbulence, as it shows a larger peak at 0.00001 m, and decays faster at 0.000036 m from the point of initiation of turbulent flow regime.

Referring to the our results, D. Oster and I. Wygnanski’s deduction is invalid for the area of the flow close to the point of initiation of turbulence regime. In addition, figures 5 through 10 indicate that at points further away from, but close to, the point of initiation of turbulence flow, the auto-correlated components of Reynolds Stresses play the dominant role in generating and maintaining the turbulence behavior. As a further expansion to such deduction it can be noted that at initial stages of such flow regimes, turbulence starts and expands more due to contribution of cross-correlated component of velocity fluctuations, and as the influenced region of turbulent flow increases, most of the generated momentum and energy of turbulent components needed to generate and maintain the turbulent flow pattern is supported through the influence of the auto-correlated fluctuating velocity components. In other words, there are two regions of Reynolds Stress contribution to the generation and expansion of turbulence. In the first region, the turbulence is generated through a higher correlation of vertical fluctuating velocity components; such close correlation decays yielding a second mechanism of turbulence generation, forming a region at which turbulence is mainly maintained by higher temporal auto-correlation of fluctuating velocity components.

4 Conclusion

The particular area under consideration includes a point of discontinuity, critical point, where all the flow variables are constantly zero [3-7]. As the starting point of the turbulent flow regime, the consequent development of the turbulent mixing layer is dependent to events that take place in this very small region. Comparing the results with similar experimental research on free turbulent plane mixing layers, we deduct that the obtained results are acceptable [5, 7]. However, the complex nature of events in this region and their ultimate effect on the development of turbulent flow deserve much more theoretical and experimental investigation.
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Figure captions

Fig. 1. Comparison of the spreading of unforced mixing layer obtained by the utilized CFD and experimental set up of D. Oster and I. Wygnanski.

Fig. 2. Distribution of time averaged $|u'u'|/U_{ref}^2$ for $U_{max}/U_{min}$ of less than 0.6, provided by D. Oster and I. Wygnanski, and calculated by the utilized CFD scheme.

Fig. 3. Graph of Velocity Component $u$, as calculated by the CFD scheme for flow cross-section at $x = 0.00001$ m, for the free stream velocity ratio of 1/2.

Fig. 4. Graph of velocity component $u$, as calculated by the CFD scheme for flow cross section at $x = 0.000036$ m, for the free stream velocity ratio of 1/2.

Fig. 5. Graph of variation of average $|u'u'|/U_{ref}^2$, against vertical distance, as calculated by CFD scheme, for flow cross section at $x = 0.00001$ m, for free stream velocity ratio of 1/2.

Fig. 6. Graph of variation of average $|u'u'|/U_{ref}^2$, against vertical distance, as calculated by CFD scheme, for flow cross section at $x = 0.000036$ m, for free stream velocity ratio of 1/2.

Fig. 7. Graph of variation of average $|v'v'|/U_{ref}^2$, against vertical distance, as calculated by CFD scheme, for flow cross section at $x = 0.00001$ m, for free stream velocity ratio of 1/2.

Fig. 8. Graph of variation of average $|v'v'|/U_{ref}^2$, against vertical distance, as calculated by CFD scheme, for flow cross section at $x = 0.000036$ m, for free stream velocity ratio of 1/2.

Fig. 9. Graph of variation of average $|u'v'|/U_{ref}^2$, against vertical distance, as calculated by CFD scheme, for flow cross section at $x = 0.00001$ m, for free stream velocity ratio of 1/2.

Fig. 10. Graph of variation of average $|u'v'|/U_{ref}^2$, against vertical distance, as calculated by CFD scheme, for flow cross section at $x = 0.000036$ m, for free stream velocity ratio of 1/2.
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