AN APPROACH FOR COMPUTING FAMILIES OF MULTI-BRANCH-POINT COVERS AND APPLICATIONS FOR SYMPLECTIC GALOIS GROUPS

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Abstract. We propose an approach for the computation of multi-parameter families of Galois extensions with prescribed ramification type. More precisely, we combine existing deformation and interpolation techniques with recently developed strong tools for the computation of 3-point covers. To demonstrate the applicability of our method in relatively large degrees, we compute several families of polynomials with symplectic Galois groups, in particular obtaining the first totally real polynomials with Galois group $\operatorname{PSp}_6(2)$.

1. Introduction and overview of results

This article deals with the explicit computation of families of Galois extensions of $\mathbb{Q}(t)$ with prescribed Galois group and ramification type. Techniques for such computations as well as examples of interest have been exhibited in several papers, notably [15], [7], [11], [14], [13]. Methods used include Gröbner basis calculations (often referred to as “direct methods”), complex and $p$-adic deformation methods, interpolation techniques, Riemann–Roch space computations etc. Since direct methods become expensive quite quickly as the degree and number of branch points increase, an idea used extensively was the deformation of covers with small branch point number into covers with larger branch point number, ideally reducing the computation of a cover with many branch points to computation of covers with only three branch points. While this procedure has been applied successfully in several special cases, it also has obvious downsides, notably possible problems with numerical instability as well as its length in case of iterative application. Here, we present a modified approach, reducing the computation of Galois covers with group $G$ and $r$ branch points directly to the computation of 3-branch point covers. This comes at the cost of increasing the degree of the cover; however, in light of recent rapid improvements regarding the calculation of 3-point covers, this is a price worth paying.

This paper is structured as follows: Section 2 introduces the key idea of reducing multi-branch point covers to those with three branch points, and on how this helps
to compute the entire family of polynomials with prescribed Galois group. The required theoretical background will be established in Section 3, including the theory of Hurwitz spaces, as well as computational techniques. In Section 4 we demonstrate our methods by computing a two-parameter family of polynomials with symplectic Galois group $\text{PSp}_6(2)$ of degree 28 with infinitely many totally real specializations. Further results are collected in the final Section 5, which contains a two-parameter family of polynomials of degree 27 with $\text{PSp}_4(3)$ as Galois group, a family of degree 36 polynomials with group $\text{PSp}_6(2)$, and an example of a complex approximation of a five-branch-point covering with $\text{PSp}_6(2)$ as monodromy group.

2. Main ideas

2.1. Reducing to Belyi maps. Recall that a covering $f: X \to \mathbb{P}_C^1$ is called a Belyi map if it is unramified outside of $\{0, 1, \infty\}$. Belyi’s famous theorem asserts that every compact Riemann surface which can be defined over $\mathbb{Q}$ admits a Belyi map to $\mathbb{P}^1$. Belyi’s proof uses a clever composition of covers, successively reducing the number of branch points. It was first suggested to us by Peter Müller to use such an idea to reduce calculation of multi-branch point covers to Belyi maps. This can be achieved efficiently due to the following result.

**Proposition 2.1.** Let $r \geq 3$, and let $C = (C_1, ..., C_r)$ be a ramification type for the finite group $G$ with non-empty Nielsen class, i.e. there are elements $c_i \in C_i$ for $i = 1, ..., r$ such that $\prod_i c_i = 1$ and $G = \langle c_1, ..., c_r \rangle$. Then for every Belyi map $g: \mathbb{P}^1 \to \mathbb{P}^1$ of degree at least $r - 2$, there exists a cover $f: X \to \mathbb{P}^1$ of type $C$ such that $g \circ f: X \to \mathbb{P}^1$ is a Belyi map. Furthermore, $r - 2$ is the minimal degree with this property.

**Proof.** Assume that $g$ is as above with $\deg(g) \geq r - 2$. From the Riemann–Hurwitz genus formula, it follows that the set $g^{-1}(\{0, 1, \infty\}) \subset \mathbb{P}^1$ is of cardinality exactly $\deg(g) + 2 \geq r$. Using Riemann’s existence theorem, we may pick a cover $f: X \to \mathbb{P}^1$ of type $C$ ramified only inside $g^{-1}(\{0, 1, \infty\})$. By construction, $g \circ f$ is then unramified outside $\{0, 1, \infty\}$. Conversely, if $\deg(g) < r - 2$, then the above shows that any cover of type $C$ has to be ramified at some point outside $g^{-1}(\{0, 1, \infty\})$, implying that $g \circ f$ is not a Belyi map. 

The point of Proposition 2.1 is that, assuming that we have an efficient algorithm to compute explicit equations for Belyi maps with a prescribed ramification type, we automatically get, as a component of the resulting map $g \circ f$, an explicit equation for some cover in a prescribed family with more than three branch points. The technique for computation of Belyi maps which we use (see Section 4) has been developed by the first and third author, and has previously been applied to

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1We also allow less than three branch points here, to simplify notation below.
calculate Belyi maps with interesting monodromy groups of degree up to 280 (see [2], [3], [4], [5]). Of course, any other method that allows the computation of high degree Belyi maps works as well. Alternative techniques are discussed in [24], [22], [25], [19], [12], [20], [21].

2.2. Outline of our algorithm. Here, we give a brief description of our algorithmic application of Proposition 2.1.

First, note that maps $g$ as required always exist; in fact, we may and will choose the cyclic cover $g : x \mapsto x^{r-2}$ ramified only at 0 and $\infty$. The cover $g \circ f$ then can be chosen with rather simple group-theoretical invariants:

i) Its Galois group embeds naturally into the intransitive wreath product $G \wr C_{r-2} = G^{r-2} \rtimes C_{r-2}$.

ii) Its monodromy $(\sigma_0, \sigma_1, \sigma_\infty)$ has the following properties: $\sigma_0^{r-2}$ (resp., $\sigma_\infty^{r-2}$) is an element of $G^{r-2}$ projecting into class $C_1$ (resp., $C_r$) in every component; and $\sigma_1$ is an element of $C_2 \times \ldots \times C_{r-1} \subseteq G^{r-2}$.

We therefore begin by finding such a permutation triple $(\sigma_0, \sigma_1, \sigma_\infty) \in G \wr C_{r-2}$ and calculating a complex approximation of the corresponding Belyi map. This automatically yields an approximate equation for the $(r$-branch point) subcover $f : X \to \mathbb{P}^1$ in Proposition 2.1.

Next, we use Newton’s method to deform the cover $f$, moving the branch points in small steps in $\mathbb{P}^1$ and thus obtaining approximate equations for covers in our family with any desired branch point locus. It is vital for numerical stability of this method that the branch points should not be too close to each other; this makes the cyclic cover $g$ a good choice, since $f$ then has branch points 0, $\infty$ and $(r-2)$-th roots of unity. Next, it is important to note that the coefficients of the equations thus obtained are specializations of certain algebraic functions, parameterized by an algebraic variety called the Hurwitz space of our prescribed ramification type. This means that any sufficiently large number of these coefficients (viewed as functions) will satisfy an algebraic dependency. Finding these dependencies is possible e.g. via interpolation between sufficiently many different covers, and will eventually yield defining equations for the Hurwitz space itself.

After such dependencies are obtained (and, ideally, improved in order to get nice equations) for all occurring coefficients we may search for rational values, corresponding to rational points on the Hurwitz space, i.e. (via Theorem 3.1) to covers of the prescribed ramification type defined over the rationals. We explain the above steps in some more detail with a concrete example in Section 4.2. We also refer to [13] for more on these and related techniques.

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2For this to be true, we implicitly assume the existence of a universal family; e.g., the assumption $Z(G) = \{1\}$ is sufficient for this, cf. the following section.
2.3. Comparison with previous approaches. There have been many previous papers on the computation of families of covers, notably by Malle ([15]), Couveignes ([7]) and the second author ([14], [13]). In these, either the permutation degree of the group in question is sufficiently low to allow finding an equation for at least one cover of the prescribed ramification type “directly” (e.g., via a Gröbner basis approach, or via brute-force search modulo a small prime, followed by $p$-adic lifting), or the starting point of the calculation is an $(r - 1)$-branch point cover, with monodromy $(\sigma_1\sigma_2, \sigma_3, ..., \sigma_r)$, which is then deformed into an $r$-point cover with monodromy $(\sigma_1, ..., \sigma_r)$ — possibly iteratively, to eventually get from a 3-point cover to an $r$-point one. In somewhat large permutation degrees, these methods have obvious downsides. Firstly, direct methods like the Gröbner basis approach become very expensive as the number of variables (which is roughly proportional to the permutation degree times the number of branch points) grows. Also, for $r \geq 5$, the iterated deformation process to obtain larger number of branch points is quite time-consuming. Finally, the complex deformation techniques turn out to be numerically rather delicate in many cases. Especially where there is no transitive tuple $(\sigma_1\sigma_2, \sigma_3, ..., \sigma_r)$ available, experiments showed numerically unstable behaviour in many examples. The main improvement in the use of Proposition 2.1 — which can be considered a “vertical” approach, compared to the “horizontal” one of deformation and moving branch points — is to circumvent the lengthy process of deformation and get directly into the prescribed family of $r$-point covers, after which the remaining calculations are rather smooth. The obvious price is that the degree of the initial Belyi map is increased by a factor of $r - 2$. However, due to the far-developed methods in computation of Belyi maps, this is (often) worth the effort.

3. Theoretical background on Hurwitz spaces and universal families

Here we recall some basic facts about Hurwitz spaces of Galois covers, which should be helpful for the understanding of the computations in the next sections. For a deeper introduction, cf. [10], [23] or [26].

Let $G$ be a finite group and $r \in \mathbb{N}$ be such that $G$ can be generated by $r - 1$ elements. Then by Riemann’s existence theorem the set of all Galois covers of $\mathbb{P}^1\mathbb{C}$ with Galois group $G$ and with exactly $r$ branch points is non-empty. The set of equivalence classes of these covers is denoted by $\mathcal{H}^r_{\text{in}}(G)$ (we refer to [10, Section 1.2] for the precise definition of the equivalence relation).

The set $\mathcal{H}^r_{\text{in}}(G)$ carries a natural topological structure, and also the structure of an algebraic variety. More precisely, if $\mathcal{U}_r$ denotes the space of $r$-sets in $\mathbb{P}^1$, then the branch point reference map $\Psi : \mathcal{H}^r_{\text{in}}(G) \to \mathcal{U}_r$ is a dominant morphism. This directly links the inverse Galois problem with the existence of rational points on
certain algebraic varieties. The main result is the following (cf. [26] Cor. 10.25 and [9, Th. 4.3]):

**Theorem 3.1.** Let $G$ be a finite group with $Z(G) = 1$. There is a universal family of ramified coverings $\mathcal{F} : \mathcal{T}_r(G) \to \mathcal{H}^\text{in}_r(G) \times \mathbb{P}^1$, such that for each $h \in \mathcal{H}^\text{in}_r(G)$, the fiber cover $\mathcal{F}^{-1}(h) \to \mathbb{P}^1$ is a ramified Galois cover with group $G$. This cover is defined regularly over a field $K \subseteq \mathbb{C}$ if and only if $h$ is a $K$-rational point. In particular, the group $G$ occurs regularly as a Galois group over $\mathbb{Q}$ if and only if $\mathcal{H}^\text{in}_r(G)$ has a rational point for some $r$.

Elements in a given fiber under the branch point reference map $\Psi$ can be described in a purely group-theoretical way, as follows.

Let $G$ be a finite group, $r \geq 2$ and

$$\mathcal{E}_r(G) := \{ (\sigma_1, \ldots, \sigma_r) \in (G \setminus \{1\})^r \mid \sigma_1 \cdot \ldots \cdot \sigma_r = 1, \langle \sigma_1, \ldots, \sigma_r \rangle = G \}$$

the set of all generating $r$-tuples in $G \setminus \{1\}$ with product 1. Furthermore let $\mathcal{E}_r^\text{in}(G)$ be the quotient of $\mathcal{E}_r(G)$ modulo conjugating the tuples simultaneously with elements of $G$. For any $r$-tuple $C := (C_1, \ldots, C_r)$ of non-trivial conjugacy classes of $G$, the Nielsen class $Ni(C)$ is defined as the set of all $(\sigma_1, \ldots, \sigma_r) \in \mathcal{E}_r(G)$ such that for some permutation $\pi \in S_r$, it holds that $\sigma_i \in C_{\pi(i)}$ for all $i \in \{1, \ldots, r\}$. The straight Nielsen class $SNi(C)$ is defined as $\{(\sigma_1, \ldots, \sigma_r) \in \mathcal{E}_r(G) \mid \sigma_i \in C_i \text{ for } i = 1, \ldots, r\}$. As above, one may also define $Ni^\text{in}(C)$ and $SNi^\text{in}(C)$.

Elements of a given fiber under $\Psi$ are then in a natural 1-1 correspondence with elements of $\mathcal{E}_r^\text{in}(G)$. Every Nielsen class $Ni^\text{in}(C)$ is a union of orbits under the action of the braid group (cf. [16], Chapter III.1.1 and III.1.2 for a definition of the braid group and its action), and each orbit corresponds to a connected component of $\mathcal{H}^\text{in}_r(G)$. The union of all connected components corresponding to a non-empty Nielsen class $Ni^\text{in}(C)$ is called the (inner) Hurwitz space of $C$.

By pullback with the space $\mathcal{U}^r$ of ordered $r$-subsets of $\mathbb{P}^1$, one gets an “ordered version” of the branch point reference map $\psi' : (\mathcal{H}^\text{in})^\prime(C) \to \mathcal{U}^r$, of degree $|SNi^\text{in}(C)|$, with the elements of a given fiber corresponding to the elements of a straight Nielsen class. Next, via restriction of $\psi'$ to the subvariety $\mathcal{H}^{\text{red}}(C)$ of $(\mathcal{H}^\text{in})^\prime(C)$ consisting of covers with the first three branch points equal to 0, 1, and $\infty$, we get a finite morphism of $(r - 3)$-dimensional varieties $\mathcal{H}^{\text{red}}(C) \to \mathcal{U}^{r-3}$. Particularly in the case $r = 4$, $C := \mathcal{H}^{\text{red}}(C)$ is a curve — it corresponds to the set of all covers with monodromy in $C$ and ordered branch point set $(0, 1, \infty, \lambda)$, for some $\lambda \in \mathbb{C} \setminus \{0, 1\}$. Of course, this choice of branch points is not always without loss over a minimal field of definition; therefore one may consider covers with partially symmetrized branch point sets as well — cf. Chapter III.7 in [16]. Theorem 3.1 thus links the existence of Galois covers defined over $K$ to the existence of $K$-points on certain curves. We also refer to these as Hurwitz curves. There are well known theoretical criteria to determine the genus of these Hurwitz curves, cf.
e.g. Thm. III.7.8 in [16], as well as our application in Proposition 4.1, and thereby in some cases to guarantee the existence of rational points.

Finally, restriction of the universal family in Theorem 3.1 (in the case \(Z(G) = 1\)) yields a universal family \(F : T \to \mathcal{H}^m(C) \times \mathbb{P}^1\) of covers in the Nielsen class \(N_i^m(C)\) (cf. Theorem 4.3 in [9]), and same for straight Nielsen classes. Defining polynomials for this family, whose coefficients lie in the function field of the respective Hurwitz space, are usually the main object of computations; compare the following sections.

4. Example: Totally real PSp_6(2)-realizations

4.1. Theoretical results. A classical variant of the inverse Galois problem is the question whether, for a given finite group \(G\), there exists a Galois extension \(F \mid \mathbb{Q}\) with \(F \subset \mathbb{R}\) such that \(\text{Gal}(F \mid \mathbb{Q}) \cong G\). It is known that if every finite group is a Galois group, then also every finite group is a Galois group of such a totally real extension.

Computation of “multi-branch-point” covers is particularly important for the computation of totally real Galois extensions. This is due to the fact that, with very few exceptions, a \(\mathbb{Q}\)-regular Galois extension of \(\mathbb{Q}(t)\) with totally real specializations must have at least 4 branch points, see Example 10.2 in [16, Chapter I].

First, we deduce the existence of totally real PSp_6(2)-realizations theoretically from known criteria.

Proposition 4.1. Let \(G := \text{PSp}_6(2) \leq S_{28}\) be given in its 2-transitive permutation action on 28 points. Let \(C_1\) be the (unique) conjugacy class of involutions of cycle structure \((2^6.1^16)\) in \(G\), let \(C_2\) be the class of involutions of cycle structure \((2^{12}.1^4)\) and length 3780, and let \(C_3\) be the class of elements of order 7 (and cycle structure \((7^4)\)). Then there exist \(\mathbb{Q}\)-regular Galois extensions \(E \mid \mathbb{Q}(t)\) of ramification type \((C_1, C_2, C_2, C_3)\) which possess totally real specializations. Furthermore, the degree-28 subfield of \(E \mid \mathbb{Q}(t)\) is a rational function field.

Proof. Computation e.g. with Magma [6] yields the following: The Nielsen class \(SN_i^m(C)\) for the type \(C := (C_1, C_2, C_2, C_3)\) is of length 70 and forms a single orbit under braid group action. Moreover, the \((2, 3)\)-symmetrized Hurwitz curve \(\mathcal{C}\) for the type \((C_1, C_2, C_2, C_3)\) is of genus 0; more precisely, Theorem 7.8a) in Chapter III of [16] yields that the degree-70 cover \(\mathcal{C} \to \mathbb{P}^1\) induced by the branch point reference map is branched at three points, with inertia group generators of cycle structures \((15.12^2.9.8.7^2), (3^{13}.2^{14}.1^3)\) and \((2^{35})\). Since there is, e.g., a unique cycle
Furthermore, there are tuples \((\sigma_1, \sigma_2, \sigma_3)\) in the above Nielsen class fulfilling \(\sigma_3 = (\sigma_3^{-1})^{\sigma_{2,2}}\) and \(\sigma_1 = (\sigma_1^{-1})^{\sigma_{2,1}}\) \(^3\). Theorem 10.3 in Chapter I of \([16]\) implies that there are \(G\)-covers \(X \to \mathbb{P}^1\) of ramification type \((C_1, C_2, C_3)\), defined over \(\mathbb{R}\), such that all four branch points are real and the complex conjugation in the segment between the two branch points of class \(C_2\) is induced by the identity element of \(G\). Since the rational points of (the rational genus-0 curve) \(\mathcal{C}\) are dense in the set of real points, there are also infinitely many \(G\)-covers with the above property which correspond to rational points on \(\mathcal{C}\), i.e., which are defined over \(\mathbb{Q}\). Each of these yields a \(\mathbb{Q}\)-regular Galois extension \(E \mid \mathbb{Q}(t)\) of ramification type \((C_1, C_2, C_2, C_3)\) such that any specialization \(t_0 \in \mathbb{Q}\) in the segment between the two \(C_2\)-branch points yields a totally real extension. Of course Hilbert’s irreducibility theorem ensures that many of these specializations preserve the Galois group \(G\).

The last assertion follows from the fact that the tuples in our Nielsen class are of genus 0 and that the normalizer in \(G\) of a cyclic subgroup generated by an element of \(C_3\) fixes one of the 7-cycles. This last claim implies that in a \(\mathbb{Q}\)-regular \(G\)-extension \(E \mid \mathbb{Q}(t)\) as above, the degree-28 subfield corresponding to a point stabilizer has a place of degree 1 extending the \(C_3\)-branch point. This ensures that this field is a rational genus-0 function field. \(\square\)

4.2. Steps of the computation. We now turn the theoretical result of Proposition 4.1 into an explicit polynomial, using the approach outlined in Section 2.

The Belyi map. As explained before, we start by computing an approximate equation for a decomposable genus zero Belyi map \(X \to \mathbb{P}^1\) such that the following holds: for a degree-2 subcover \(Y\) of \(X \to \mathbb{P}^1\), the (genus zero) cover \(X \to Y\) has ramification type \((C_1, C_2, C_2, C_3)\). Then \(X \to \mathbb{P}^1\) is of degree 56 with Galois group (of the Galois closure) embedding into \(\operatorname{PSp}_6(2) \wr C_2\), and if \((x, y, (xy)^{-1}) \in S_{56}^3\) is a triple describing the ramification of this Belyi map, the elements \(x, y,\) and \((xy)^{-1}\) are of cycle structure \((14^4), (2^{24}, 1^8),\) and \((4^6, 2^{16})\), respectively. The permutations can be chosen to be

\[
x = (1, 55, 27, 36, 8, 54, 26, 32, 4, 50, 22, 30, 2, 29, 3, 34, 6, 35, 7, 56, 28, 42, 14, 52, 24, 40, 12, 31, 5, 47, 19, 45, 17, 37, 9, 43, 15, 49, 21, 44, 16, 33, 10, 51, 23, 39, 11, 46, 18, 53, 25, 48, 20, 41, 13, 38),
\]

\(^3\)Note that all conjugacy classes \(C_1, C_2, C_3\) are rational conjugacy classes of \(G\), which ensures that \(\mathcal{C}\) is defined over \(\mathbb{Q}\).

\(^4\)In fact, the second equality is automatic from the first due to the product-1 condition.
Let \( a := \text{ord}(x) \), \( b := \text{ord}(y) \), and \( c := \text{ord}((xy)^{-1}) \), and
\[
\Delta := \langle \delta_a, \delta_b, \delta_c \mid \delta_a^a = \delta_b^b = \delta_c^c = \delta_a \delta_b \delta_c = 1 \rangle.
\]

Note that \((x, y, (xy)^{-1})\) is hyperbolic since \(1/a + 1/b + 1/c < 1\). We now consider the embedding \(\Delta \hookrightarrow \text{PSL}_2(\mathbb{R})\) described in [12, Proposition 2.5], where \(\delta_a\) (resp. \(\delta_b\)) is mapped to a hyperbolic rotation around \(i\) (resp. \(\mu i\) for some \(\mu > 1\)) of angle \(\pi/a\) (resp. \(\pi/b\)). Thus \(\Delta\) acts on the upper half-plane \(\mathbb{H} := \{z \in \mathbb{C} : \text{Im}(z) > 0\}\) via the natural action of \(\text{PSL}_2(\mathbb{R})\) on \(\mathbb{H}\), that is
\[
\text{PSL}_2(\mathbb{R}) \rightarrow \text{Aut}(\mathbb{H}) : \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mapsto \begin{pmatrix} \alpha z + \beta \\ \gamma z + \delta \end{pmatrix},
\]
and its fundamental domain can be chosen to be the hyperbolic kite with vertices \(i, h, \mu i, -\bar{h}\) for some \(h \in \mathbb{H}\). Furthermore, let \(\varphi\) denote the homomorphism from \(\Delta\) onto \(G := \langle x, y \rangle\) such that \(\delta_a \mapsto x\) and \(\delta_b \mapsto y\). With the notation \(\Gamma := \varphi^{-1}(\text{Stab}_G(1))\) we define
\[
\Phi : \mathbb{H}/\Gamma \rightarrow \mathbb{H}/\Delta, \quad z \; \text{mod} \; \Gamma \mapsto z \; \text{mod} \; \Delta.
\]
This is a three-point branched cover of degree 56 with monodromy group isomorphic to \(G\), see for example [12] for more details. Because \(\mathbb{H}/\Delta\) is homeomorphic to \(\mathbb{P}^1\) we may assume that the ramification locus of \(\Phi\) is given by \(\{0, 1, \infty\}\), i.e., \(\Phi\) is a Belyi map. We now study the dessin of \(\Phi\), i.e. the set \(\Phi^{-1}([0, 1])\) which is visualized in Figure 4.

Pick a connected fundamental domain \(D\) of \(\mathbb{H}/\Gamma\), see Figure 1. This is possible, because \(G = \langle x, y \rangle\) is transitive. Since \(D\) is irregularly shaped, we conformally map it to \(\mathbb{H}\). For example, this can be achieved by using the Schwarz–Christoffel Toolbox [8] for MATLAB [18], see Figure 2.

Now \(\partial \mathbb{H} = \mathbb{R} \cup \{\infty\}\) inherits the structure of \(D\) induced by the quotient \(\mathbb{H}/\Gamma\). In order to glue corresponding (adjacent) real line segments we apply slit maps of type (Figure 3)
\[
\text{slit}_A : \mathbb{H} \rightarrow \mathbb{H} : z \mapsto (z - A)^A(z + 1 - A)^{A-1}
\]
where \(0 < A < 1\) is a real number, see [17] and [4] for a thorough analysis. This step requires the permutation triple \((x, y, (xy)^{-1})\) to be of genus zero, otherwise

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5For a topological space \(X\) and a group \(G\) acting on \(X\) let \(X/G\) denote the corresponding orbit space.
corresponding line segments will not appear to be adjacent to each other at some point.

We end up with an approximate dessin in $\mathbb{P}^1$ with the same ramification structure as $\Phi$, see Figure 4. This approximation can be combined with Newton’s method to obtain coefficients of the desired Belyi map with sufficiently high precision.

Towards the universal family. Up to Möbius transformation, the computed Belyi map of degree 56 is of the form $f_0(X)^2$ with a rational function $f_0$ of degree 28, see section A in the extra file; this function has four branch points, group $\text{PSp}_6(2)$ and ramification type $(C_1, C_2, C_2, C_3)$. Up to rescaling, the branch points of $f_0$
are 0, 1, −1 and ∞. We use this function $f_0$ as a starting point to compute
the Hurwitz curve of all covers with ramification type $(C_1, C_2, C_2, C_3)$ and branch
points $0, 1 + \sqrt{\lambda}, 1 - \sqrt{\lambda}, \infty$, with $\lambda \in \mathbb{Q} \setminus \{0, 1\}$.
This can be done by complex deformation techniques. The point is that this is
numerically stable, since the starting point is already a cover with 4 branch points.
We thus assemble equations (or equivalently, rational functions $f_\lambda(X)$) for many
covers $C_\lambda$ with ramification type and branch point set as above, for values $\lambda \in \mathbb{C}$.
The function field of the Hurwitz curve is finite over $\mathbb{Q}(\lambda)$ (where $\lambda$ is viewed as
a transcendental). Thus, the universal family corresponding to our Hurwitz space
can be parameterized by $f_1(X)^2f_2(X) - tf_3(X)^7$, where $\deg(f_1) = 6$, $\deg(f_2) = 16$,
$\deg(f_3) = 3$, and where each coefficient of the $f_i$ is a function fulfilling some
algebraic dependency with $\lambda$. To make the equations unique, we may apply linear
transformations in $X$ and, e.g., fix the coefficient at $X^2$ in $f_3$ to be 0 and the one
at $X^5$ in $f_1$ to be 1. In fact, two “random” coefficients can usually be expected
to generate the entire function field of the Hurwitz curve. We use this and let the
coefficient $\beta$ at $X^4$ in $f_1$ converge to a rational value, using Newton approximation.
Then any further coefficient \( \gamma \) will converge to an algebraic number of degree at most \([\mathbb{Q}(\beta, \gamma) : \mathbb{Q}(\beta)]\) (which is bounded by the index of \( \mathbb{Q}(\beta) \) in the function field of the Hurwitz curve), and given a sufficient complex precision, we can recognize this algebraic number using the LLL algorithm. Doing this for several rational values of \( \beta \), we obtain (by interpolation) the algebraic dependency between \( \beta \) and \( \gamma \) parameterizing the function field of our Hurwitz curve.

Since this function field is a rational field, it is now easy to find a parameter \( \alpha \) such that \( \mathbb{Q}(\beta, \gamma) = \mathbb{Q}(\alpha) \) using a Riemann–Roch space computation.

Finally, we use Newton approximation again to let \( \alpha \) converge to several rational values. Then all the coefficients of \( f_1, f_2, f_3 \) must also be rational values. These can easily be recognized from their complex approximation, and interpolating between these several values once again yields dependencies between \( \alpha \) and each coefficient, i.e. expressions of each coefficient as a rational function in \( \alpha \).

We then obtain a polynomial \( f = f(\alpha, t, X) \in \mathbb{Q}[\alpha, t, X] \) whose Galois group over \( \mathbb{Q}(\alpha, t) \) is isomorphic to \( \text{PSp}_6(2) \). Since the coefficients of \( f \) are too large to present it here, we refer to section B in the extra file.

**Theorem 4.2.** The polynomial \( f(\alpha, t, X) = p(\alpha, X) - tq(\alpha, X) \in \mathbb{Q}(\alpha, t)[X] \), where \( p \) and \( q \) are given in the extra file (section B), has Galois group \( \text{PSp}_6(2) \leq S_{28} \) over \( \mathbb{Q}(\alpha, t) \) and possesses infinitely many totally real specializations. The ramification with respect to \( t \) is of type \( (2^6.1^1, 2^12.1^4, 2^{12}.1^4, 7^4) \).

*Proof.* Let \( f_2, p_2, q_2 \in \mathbb{Q}(t)[X] \) denote the specializations of \( f, p, q \) at the place \( a \mapsto 2 \), and \( \tilde{f}_2, \tilde{p}_2, \tilde{q}_2 \) their images in \( \mathbb{F}_{31}(t)[X] \) under the canonical projection.

By computing the discriminant \( \Delta \) of \( f \) we see that \( f \) and \( f_2 \) have exactly four branch points with respect to \( t \). Furthermore the branch cycle structure of \( f \) can be derived by inspecting the inseparability behaviour of \( f \) evaluated at the places \( t \leftrightarrow 0 \), \( t \leftrightarrow \infty \), and \( t \leftrightarrow r_i \) for \( i = 1, 2 \) where \( r_1 \) and \( r_2 \) denote the non-zero roots of \( \Delta \in \mathbb{Q}(\alpha)[t] \).

By a result of Malle (see [13] Lemma 3.1), the two groups \( \text{Gal}(f \mid \mathbb{Q}(\alpha, t)) \) and \( \text{Gal}(f_2 \mid \mathbb{Q}(t)) \) coincide. It remains to show that \( \text{Gal}(f_2 \mid \mathbb{Q}(t)) \) is isomorphic to \( \text{PSp}_6(2) \).

Since \( \frac{1}{X-t} \cdot \tilde{f}_2 \left( \frac{\tilde{p}_2(t)}{\tilde{q}_2(t)}, X \right) \in \mathbb{F}_{31}(t)[X] \) is irreducible the Galois group of \( \tilde{f}_2 \) over \( \mathbb{F}_{31}(t) \) must be 2-transitive of permutation degree 28, implying \( \text{Gal}(\tilde{f}_2 \mid \mathbb{F}_{31}(t)) \in \{\text{PSp}_6(2), A_{28}, S_{28}\} \) due to the classification of finite 2-transitive groups. Dedekind reduction yields \( \text{Gal}(f_2 \mid \mathbb{Q}(t)) \in \{\text{PSp}_6(2), A_{28}, S_{28}\} \). Because both discriminants of \( f_2 \) and \( \tilde{f}_2 \) are squares, \( \text{Gal}(f_2 \mid \mathbb{Q}(t)) \) and \( \text{Gal}(\tilde{f}_2 \mid \mathbb{F}_{31}(t)) \) are not \( S_{28} \). In particular, \( \text{Gal}(f_2 \mid \mathbb{Q}(t)) \) is simple, and the corresponding function field extension must be regular, allowing us to apply a theorem of Beckmann, see [16] Chapter I, Proposition 10.9], to obtain \( \text{Gal}(f_2 \mid \mathbb{Q}(t)) \cong \text{Gal}(\tilde{f}_2 \mid \mathbb{F}_{31}(t)) \).
Let $r(t, X) \in \mathbb{F}_{31}(t)[X]$ be the irreducible polynomial of degree 63 in the ancillary file (see section C), then $r \left( \frac{p_2(t)}{q_2(t)}, X \right)$ becomes reducible over $\mathbb{F}_{31}(t)$\footnote{The polynomial $r$ was obtained by using the \texttt{Magma} command \texttt{GaloisSubgroup} for the index 63 subgroup of $\text{PSp}(6, 2)$}. This guarantees the existence of an index $d \neq 1$ subgroup of $\text{Gal}(\bar{f}_1 \mid \mathbb{F}_{31}(t))$ where $d$ is a divisor of 63. Since $A_{28}$ does not contain such a subgroup we end up with $\text{Gal}(\bar{f}_2 \mid \mathbb{F}_{31}(t)) = \text{PSp}_6(2)$, thus $\text{Gal}(\bar{f}_2 \mid \mathbb{Q}(t)) = \text{PSp}_6(2)$.

Finally, we specialize $\alpha \mapsto 0$ (which does not decrease the number of branch points) and verify that for some (and hence, for all!) specialization of $t$ in the interval $[-2.8 \cdot 10^{12}, 0]$ (the left bound being approximately the only negative branch point of $f(0, t, X)$), the number of real roots of $f(0, t, X)$ is equal to 28. \hfill \Box

In particular, specializing $\alpha \mapsto 0$ and applying some linear transformations to decrease the coefficients, we obtain the following:

\textbf{Corollary 4.3.} Let $\tilde{f}(t, X) := (X^6 - 33/2X^5 - 42924X^4 - 1525664X^3 + 477587712X^2 + 4047878536X + 86354742768)^2 \cdot (X^{16} + 271X^{15} - 430719/4X^{14} - 35366300X^{13} + 3314214496X^{12} + 1797598385556X^{11} + 28249865746816X^{10} - 425773969378944X^9 - 35468841715160408X^8 + 388165289642365195520X^7 + 67637298931930365811712X^6 + 1157375979002203859189760X^5 - 370365044650038661036441600X^4 - 30197279842907494819422011392X^3 - 81483048856896074491713272576X^2 + 1626666895113353417111909978112X + 256038325580946715804749139017728) - t(X^3 - 21952X - 1229312)^2 \in \mathbb{Q}(t, X)$.

Then every specialization of $t$ in the interval $[-4.9 \cdot 10^{15}, 0]$ which preserves the Galois group yields a totally real PSp$_6(2)$-polynomial.

\section{Further examples}

\subsection{A two-parameter family of polynomials with Galois group PSp$_4(3)$}

As a further application of our algorithm, we calculate families of polynomials $f(\alpha, t, X)$ with Galois groups PSp$_4(3)$, resp. PSp$_4(3)$.2. The group $G := \text{PSp}_4(3).2$ happens to possess a \textit{rigid} genus-0 four-tuple of rational conjugacy classes. More precisely, if $G$ is viewed in its transitive permutation action on 27 points, the tuple $(C_1, C_1, C_2, C_3)$ is rigid, where $C_1$ denotes the class of involutions of cycle structure $(2^6.1^{15})$, $C_2$ denotes the class of cycle structure $(4^6.1^3)$, and $C_3$ denotes the class of length 720 whose elements have cycle structure $(6^4.3^3)$. The rigidity criterion (see, e.g., \cite{16} Theorem I.4.8) then yields that for every 4-set of rational points $p_1, \ldots, p_4$, there exists a $\mathbb{Q}$-regular Galois extension of $\mathbb{Q}(t)$ with ramification type $(C_1, C_1, C_2, C_3)$ and branch points $p_1, \ldots, p_4$. Below, we turn this theoretical result into an explicit (and particularly nice!) parametric family of polynomials with group PSp$_4(3).2$. This family and the corresponding Belyi map of degree 54 is given in section D in the extra file.
Theorem 5.1. The polynomial \( f(\alpha, t, X) := (2X^6 - 10\alpha X^4 + 10\alpha X^3 - 10\alpha^2 X^2 + 2\alpha^2 X + 2\alpha^3 - \alpha^2)^4(4X^3 - 4\alpha X + \alpha) - t(3X^4 - 6\alpha X^2 + 3\alpha X - \alpha^2)^6 \in \mathbb{Q}(\alpha, t)[X] \) has regular Galois group \( \text{PSp}_4(3).2 \leq S_{27} \) over \( \mathbb{Q}(\alpha, t) \), and branch cycle structure \((2^6.1^{15}, 2^6.1^{15}, 4^6.1^{13}, 6^4.3^1)\) with respect to \( t \). Furthermore, the polynomial \( f(3\alpha^2, t(s), X) \in \mathbb{Q}(\alpha, t)[X] \) where \( t(s) := \frac{((-8\alpha/3)^3 + (-8\alpha/3)^2(3s^2-(\alpha+3/8)/(\alpha-3/8))}{d^2 + 1} \) has Galois group \( \text{PSp}_4(3) \) over \( \mathbb{Q}(\alpha, s) \).

Proof. Define \( f_1, f_2 \in \mathbb{Q}(\alpha)[X] \) such that \( f = f_1 - tf_2 \). The branch cycle structure can be easily computed by observing the inseparability behaviour of \( f_1, f_2 \) and of the specialized polynomials \( f(\alpha, t_0, X) \), where \( t_0 \in \mathbb{Q} \setminus \{0\} \) is a root of the discriminant \( \Delta(\alpha, t) \in \mathbb{Q}(\alpha)[t] \) of \( f \). Next, computer calculation shows that \( f_1(X)f_2(Y) - f_2(X)f_1(Y) \) is reducible in \( \mathbb{Q}(\alpha)[X, Y] \), with three factors of \( X \)-degree 1, 10 and 16. This means that \( f \) factors into degree 1, 10 and 16 over \( \mathbb{Q}(\alpha)(y) \), where \( y \) is a root of \( f \). In other words, the point stabilizer in \( \text{Gal}(f \mid \mathbb{Q}(\alpha, t)) \) has three orbits on the roots of sizes 1, 10 and 16. This implies \( \text{Gal}(f \mid \mathbb{Q}(\alpha, t)) \) is a primitive group, since otherwise a suitable union of orbits of the point stabilizer would have to be of cardinality a non-trivial divisor of 27. One now verifies that \( \text{PSp}_4(3) \) and \( \text{PSp}_4(3).2 \) are the only primitive groups of degree 27 with stabilizers having orbits of size 1, 10 and 16. However, the inseparability behaviour at one of the finite, non-zero branch points shows that \( \text{Gal}(f \mid \mathbb{Q}(\alpha, t)) \) contains an element of cycle structure \((2^6.1^{15})\), which \( \text{PSp}_4(3) \) does not. This shows \( \text{Gal}(f \mid \mathbb{Q}(\alpha, t)) \cong \text{PSp}_4(3).2 \).

Next, note that only the conjugacy class \( C_1 \) lies outside the index-2 normal subgroup \( \text{PSp}_4(3) \). Furthermore, upon replacing \( \alpha \) by \( 3\alpha^2 \), computation of the discriminant of \( f \) shows that the two branch points with inertia group generator in \( C_1 \) become \( \mathbb{Q}(\alpha) \)-rational, say \( t \mapsto k \) and \( t \mapsto \ell \) (with \( k, \ell \in \mathbb{Q}(\alpha) \)). The quadratic extension corresponding to the fixed field of \( \text{PSp}_4(3) \) is thus ramified at exactly two points, both rational. It is therefore a rational function field, given by an equation \( cY^2 = (t - k)(t - \ell) \) (with some constant \( c \)), and fractional linear transformation easily yield a parameter \( s \) for such a function field, providing the equation \( t = t(s) \) as above.

\[
\text{5.2. Another two-parameter family with group PSp}_6(2) \text{ of degree 36.}
\]
Let \( (C_1, C_2, C_2, C_3) \) be the class vector of the group \( \text{PSp}_6(2) \) acting 2-transitively on 36 elements where the conjugacy classes \( C_1, C_2 \) and \( C_3 \) are unique of type \((3^{12})\), \((1^{12}.2^{12})\), and \((1^6.2.4^7)\). In the same fashion as before one can show theoretically that \( \text{PSp}_6(2) \) occurs as a Galois group over \( \mathbb{Q}(\alpha, t) \) where the ramification type is given by the above class vector. Surprisingly, the explicit two-parameter family turns out have rather small coefficients:
Theorem 5.2. Let \( f(\alpha, t, X) = p(\alpha, X) - tq(\alpha, X) \in \mathbb{Q}(\alpha, t)[X] \) where
\[
p(\alpha, X) = (X^{12} + X^{11} + (144\alpha + \frac{1}{8})X^{10} + 40\alpha X^9 + \left(-1728\alpha^2 + \frac{21}{4}\alpha\right)X^8
+ \left(-576\alpha^2 + \frac{3}{8}\alpha\right)X^7 - 84\alpha^2 X^6 - 6\alpha^2 X^5 + \left(144\alpha^3 - \frac{3}{64}\alpha^2\right)X^4
+ 40\alpha^3 X^3 + \frac{13}{4}\alpha^3 X^2 + \frac{1}{8}\alpha^3 X + \alpha^4)^3,
\]
and
\[
q(\alpha, X) = \left(X^6 - 12\alpha X^4 + \frac{1}{2}\alpha^2\right) \cdot \left(X^3 - 24\alpha X - 2\alpha\right)^4
\cdot \left(X^4 + \frac{1}{6}X^3 + \frac{1}{24}\alpha\right)^4.
\]
Then the Galois group of \( f \) over \( \mathbb{Q}(\alpha, t) \) is isomorphic to \( \text{PSp}_6(2) \leq S_{36} \), and the branch cycle structure of \( f \) with respect to \( t \) is given by \( (3^{12}, 1^{12}, 2^{12}, 1^{12}, 2^{12}, 3^{12}) \).

Proof. One can mimic the proof of Theorem 4.2 with some minor changes. □

The degree-36 polynomial appearing in the theorem as well as the corresponding degree-72 Belyi map are also listed in the ancillary file, see section E.

5.3. A 5-branch point cover for \( \text{PSp}_6(2) \). The advantage of our approach compared to previous ones increases as the number of branch points grows. We give just one example of a complex approximation for a 5-branch point cover with Galois group \( \text{PSp}_6(2) \). Using the techniques of the previous sections, one could again use this to obtain an equation for a family of covers. This time, we take a genus-0 tuple of type \( (2^{10}, 1^{16}, 2^{12}, 1^{12}, 2^{12}, 1^{12}, 2^{12}, 1^{12}, 3^{12}) \) in the 2-transitive degree-36 permutation action of \( \text{PSp}_6(2) \). Using Proposition 2.1, we turn this into a Belyi function of degree 108, with imprimitive Galois group contained in \( \text{PSp}_6(2) \wr C_3 \), by composing with the rational function \( x \mapsto x^3 \), see Figure 5. The third root of this Belyi map then gives the desired 5-branch point \( \text{PSp}_6(2) \)-cover. We have included it in the file, see section F.

The monodromy of the computed complex cover can be checked numerically with the path lifting algorithm in [14, Chapter 11.1].

Acknowledgements

We are indebted to Peter Müller for several valuable suggestions.
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Figure 5. approximate dessin of degree 108 in \( \mathbb{P}^1 \)

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