On $\mathbb{Z}_4$-linear Reed-Muller like codes

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Abstract

For each $r$, $0 \leq r \leq m$, it is presented the class of quaternary linear codes $LRM(r, m)$ whose images under the Gray map are binary codes with parameters of Reed-Muller $RM(r, m)$ code of order $r$.

1 Preliminaries

First consider some necessary definitions and notions. Consider the ring $\mathbb{Z}_4$ of integers modulo 4. The set $\mathbb{Z}_4^n$ is a module with addition operation over the ring $\mathbb{Z}_4$. The Lee weight, $w_L(\cdot)$, of a quaternary vector is the sum of weights of its coordinate positions:

$$w_L(0) = 0, \quad w_L(1) = w_L(3) = 1, \quad w_L(2) = 2.$$  

The Lee distance, $d_L(\cdot, \cdot)$, between any quaternary vectors $x, y \in \mathbb{Z}_4^n$ is defined as $d_L(x, y) = w_L(x - y)$. The set $\mathbb{Z}_4^n$ is a metric space with respect to the Lee metric.

A quaternary code of length $n$ is a subset of the metric space $\mathbb{Z}_4^n$. A quaternary code of length $n$ is linear if it is a subgroup of the additive group of the ring $\mathbb{Z}_4^n$. We use further capital letters for binary codes and calligraphic for quaternary.

Let us remind the standard maps $\alpha, \beta$ and $\gamma$ from $\mathbb{Z}_4$ to $\mathbb{Z}_2$:

\begin{align*}
| & Z_4 & \alpha & \beta & \gamma \\
0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 1 \\
2 & 0 & 1 & 1 & 1 \\
3 & 1 & 1 & 0 & 0
\end{align*}

These maps can be extended in the usual way to maps from $\mathbb{Z}_4^n$ to $\mathbb{Z}_2^n$. The Gray map $\phi: \mathbb{Z}_4^n \rightarrow \mathbb{Z}_2^{2n}$ is defined by

$$\phi(x) = (\beta(x), \gamma(x)), \quad \text{for any } x \in \mathbb{Z}_4^n.$$  

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It is well known that $\phi$ is an isometry of the metric spaces $\mathbb{Z}_4^n$ and $\mathbb{Z}_2^{2n}$.

Two quaternary codes, $C$ and $D$, of length $n$ are equivalent if there exist a vector $x \in \mathbb{Z}_4^n$, a permutation $\pi$ of $n$ coordinate positions and an inversion $\sigma$ on $n$ coordinates such that $C = \pi(\sigma(D)) + x$. A binary code is called $\mathbb{Z}_4$-linear with respect to the Gray map if there exists an equivalent code $C$ such that its preimage $\phi^{-1}(C)$ is linear.

There are several known classes of nonlinear binary codes with good properties which can be represented as linear quaternary codes. Among them there are such prominent codes as Preparata, Kerdock, Delsarte-Goethals, Goethals-Delsarte, some perfect codes, some Hadamard codes, see the list of references [10, 8, 4, 11, 6, 7, 14].

In [6, 7] the classifications of $\mathbb{Z}_4$-linear perfect and $\mathbb{Z}_4$-linear Hadamard codes are presented. It is established that for any $n = 2^k$, $n \geq 16$ the number of nonequivalent $\mathbb{Z}_4$-linear perfect (Hadamard) codes is $[(k + 1)/2]$. All quaternary linear codes whose images under the Gray map are perfect codes can be described using Mollard construction, see [6].

The representation of $\mathbb{Z}_4$-linear Preparata codes is done in [13]. Using switching approach it is established that the set of all quaternary linear Preparata codes of length $n = 2^m$, $m$ odd, $m \geq 3$, is nothing more than the set of codes of the form $\mathcal{H}_{\lambda,\psi} + \mathcal{M}$ with

$$\mathcal{H}_{\lambda,\psi} = \{ y + T_\lambda(y) + S_\psi(y) \mid y \in H^n \}, \quad \mathcal{M} = 2H^n,$$

where $T_\lambda(\cdot)$ and $S_\psi(\cdot)$ are vector fields of a special form defined over the binary extended linear Hamming code $H^n$ of length $n$. An upper bound on the number of nonequivalent quaternary linear Preparata codes of length $n$ is obtained, namely, $2^{n \log_2 n}$.

There are several papers devoted to quaternary Reed-Muller $RM(r, m)$ codes of order $r$, $0 \leq r \leq m$. In [4] it is established that binary Reed-Muller $RM(r, m)$ codes of order $r$, $r \in \{0, 1, 2, m - 1, m\}$ are $\mathbb{Z}_4$-linear and conjectured that all other Reed-Muller codes are not $\mathbb{Z}_4$-linear. The conjecture is proved in [5].

In [4] the class of quaternary codes $QRM(r, m)$ for each $r$, $0 \leq r \leq m$ is introduced. The image of the code $QRM(r, m)$ under the map $\alpha$ (see the definition of the map above) is linear Reed-Muller $RM(r, m)$ code for all $r$, $0 \leq r \leq m$. The class of the codes includes the quaternary linear Kerdock codes and its dual the quaternary linear Preparata code from [4]. The generalization of the result is given in [11]. The class of the codes obtained in [11] includes all the quaternary linear Kerdock codes and the quaternary linear Preparata codes. Thus, another representation of the quaternary linear Preparata codes is given. The images of all these codes under the map $\alpha$ are also linear Reed-Muller codes.

In [12] the additive Reed-Muller code $ARM(r, m)$ of order $r$, $0 \leq r \leq m$ is defined. The code is an additive subgroup of $\mathbb{Z}_2^{k_1} \times \mathbb{Z}_4^{k_2}$, $k_1 = 2^{m-1}$, $k_2 = 2^{m-2}$, $m \geq 2$ and it is announced that its binary image is linear for $r \in \{0, 1, m - 1, m\}$ and nonlinear for $r = m - 2$, $m > 3$.

### 2 Reed-Muller-like codes

In this section for every integer $r$, $r \in \{0, 1, \ldots, m\}$, we construct the class of quaternary linear codes of length $2^{m-1}$, code distance $d = 2^{m-r}$ and size $2^k$, where

$$k = 1 + \binom{m}{1} + \cdots + \binom{m}{r}.$$
The image of any such code under the Gray map is a binary (not necessary linear) code with parameters of Reed-Muller $RM(r, m)$ code of order $r$.

Let $v = (v_1, \ldots, v_m)$ range over $\mathbb{Z}_2^m$. The binary Reed-Muller code $RM(r, m)$ of order $r$ is generated by all binary vectors of length $2^m$ corresponding to the Boolean functions $f(v)$ equaled to monomials of degree not more than $r$. The code has the following parameters:

- length of the code is $n = 2^m$;
- the size of the code is $2^k$, where $k = 1 + \left( \begin{array}{c} m \\ 1 \end{array} \right) + \cdots + \left( \begin{array}{c} m \\ r \end{array} \right)$;
- the code distance $d = 2^{m-r}$.

It is known, see [9], that the binary Reed-Muller code $RM(r, m)$ of order $r$ can be described by Plotkin (doubling) construction:

$$RM(r, m) = \{(x, x + y) \mid x \in RM(r, m - 1), y \in RM(r - 1, m - 1)\}.$$

Binary not necessary linear code whose parameters coincide with parameters of the binary linear Reed-Muller code $RM(r, m)$ of order $r$ we will call Reed-Muller-like code of order $r$. We are going to prove that among them there are $\mathbb{Z}_4$-linear codes. Such codes have some regular properties. For example, all such binary codes are transitive. Preimages of them under the Gray map are quaternary codes with parameters:

- length of the code is $n = 2^{m-1}$;
- the size of the code is $2^k$, where $k = 1 + \left( \begin{array}{c} m \\ 1 \end{array} \right) + \cdots + \left( \begin{array}{c} m \\ r \end{array} \right)$;
- the code distance $d = 2^{m-r}$.

We will denote any such quaternary linear code by $\mathcal{LRM}(r, m)$, its binary image – by $LRM(r, m)$.

We construct a sequence of quaternary linear $\mathcal{LRM}(r, m)$ codes that includes the following classes of quaternary linear codes: the quaternary repetition code $\mathcal{LRM}(0, m)$, some quaternary linear Hadamard codes from [7], some quaternary linear extended perfect codes from [6, 7], full-even weight code $\mathcal{LRM}(m, m)$ and $\mathcal{LRM}(m, m) = \mathbb{Z}_4^2$. We construct the $\mathcal{LRM}(r, m)$ codes by induction on $m$, where $m = \log n, m > 1$. For $m = 1$ there exist the following trivial quaternary linear codes:

- a) the quaternary linear code $\mathcal{LRM}(0, 1) = \mathcal{RM}(0, 1) = \{(0), (2)\}$, its binary image is $LRM(0, 1) = \phi(\mathcal{LRM}(0, 1)) = RM(0, 1) = \{(0, 0), (1, 1)\}$,
- b) the quaternary linear code $\mathcal{LRM}(1, 1) = \mathcal{RM}(1, 1) = \{(0), (1), (2), (3)\}$, its binary image is $LRM(1, 1) = \phi(\mathcal{LRM}(1, 1)) = RM(1, 1) = \{(0, 0), (0, 1), (1, 1), ((1, 0))\}$.

Let us consider also more interesting case $m = 2$. Here we have the following quaternary linear codes and their binary images under the Gray map:

- a) the quaternary linear code $\mathcal{LRM}(0, 2) = \mathcal{RM}(0, 2) = \{(0, 0), (2, 2)\}$ and its binary image $LRM(0, 2) = \phi(\mathcal{LRM}(0, 2)) = RM(0, 2) = \{(0, 0, 0, 0), (1, 1, 1, 1)\}$;
- b) the quaternary linear code $\mathcal{LRM}(1, 2) = \mathcal{RM}(0, 2) = \{(0, 0), (1, 1), (2, 2), (3, 3), (1, 3), (3, 1), (0, 2), (2, 0)\}$ and its binary image full-even weight code of length 4: $LRM(1, 2) = \phi(\mathcal{LRM}(1, 2)) = RM(1, 2)$;
c) the quaternary linear code $\mathcal{LRM}(2, 2) = \mathbb{Z}_4^2$ and its binary image

$$\mathcal{LRM}(2, 2) = \phi(\mathcal{LRM}(2, 2)) = \mathcal{RM}(2, 2).$$

Let $\mathcal{LRM}(r, m - 1)$ and $\mathcal{LRM}(r - 1, m - 1)$ be any two quaternary linear codes with parameters

$$(n = 2^{m-2}, 2^k, d = 2^{m-r-1}) \text{ and } (n = 2^{m-2}, 2^{k'}, d = 2^{m-r}),$$

where

$$k = \sum_{i=0}^{r} \left( \begin{array}{c} m-1 \\ i \end{array} \right), \quad k' = \sum_{i=0}^{r-1} \left( \begin{array}{c} m-1 \\ i \end{array} \right).$$

It is not difficult to show that Plotkin construction applied to these codes gives us a quaternary linear $\mathcal{LRM}(r, m)$ code of order $r$, i.e.

$$\mathcal{LRM}(r, m) = \{(x, x + y) \mid x \in \mathcal{LRM}(r, m - 1), y \in \mathcal{LRM}(r - 1, m - 1)\}.$$

So we get the following result.

**Theorem 1.** For any $r$, $0 \leq r \leq m$, $m \geq 1$, the set $\mathcal{LRM}(r, m)$ is a quaternary linear code with parameters

$$(n = 2^{m-1}, 2^k, d = 2^{m-r}), \text{ where } k = \sum_{i=0}^{r} \left( \begin{array}{c} m \\ i \end{array} \right),$$

(1)

whose image under the Gray map is a binary code with parameters of Reed-Muller $\mathcal{RM}(r, m)$ code of order $r$.

Taking into account that there exist quaternary linear Hadamard codes, all quaternary linear extended perfect codes from [6, 7] whose binary images under the Gray map are nonlinear codes and all $\mathcal{RM}(r, m)$ codes of order $r$, $r \in \{3, \ldots, m-2\}$ are not $\mathbb{Z}_4$-linear we get a sequence of quaternary linear codes $\mathcal{LRM}(r, m)$ of order $r$ such that all their binary images are nonlinear codes with parameters of Reed-Muller codes $\mathcal{RM}(r, m)$ for any order $r \in \{3, \ldots, m-2\}$.

Let us show that the codes from Theorem 1 do not equivalent to codes from [1] for $r \in \{3, \ldots, m-2\}$. According to [1] a quaternary linear code $\mathcal{QRM}(r, m - 1)$ of length $2^{m-1}$ has size $2^{2k}$, where $k = \sum_{i=0}^{r} \left( \begin{array}{c} m-1 \\ i \end{array} \right)$. From this fact and (1) we conclude that any quaternary linear codes $\mathcal{LRM}(r, m)$ and $\mathcal{QRM}(r, m)$ of the same length have different sizes, so they do not equivalent to each other. All binary images of the codes $\mathcal{LRM}(r, m)$ under the Gray map are nonlinear codes with parameters of Reed-Muller codes $\mathcal{RM}(r, m)$ of order $r \in \{3, \ldots, m-2\}$, but all binary images of codes $\mathcal{QRM}(r, m)$ are linear $\mathcal{RM}(r, m)$ codes of the same order. Therefore their binary images having the same parameters are also nonequivalent to each other.

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