LP Decoding of Regular LDPC Codes in Memoryless Channels

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Low-Density Parity-Check Codes

Factor graph representation of LDPC codes:

- Code $\mathcal{C}(G)$ and codewords $x$:
  \[ x \in \mathcal{C}(G) \iff \forall c_j. \sum_{x_i \in N(c_j)} x_i = 0 \pmod{2} \]

- Local-codes $C_j = C_j(G)$:
  \[ x \in C_j \iff \sum_{x_i \in N(c_j)} x_i = 0 \pmod{2} \]

- $(d_L,d_R)$-regular LDPC code:
  \[ \forall v \in \text{Variables. } \deg_G(v) = d_L \]
  \[ \forall c \in \text{Checks. } \deg_G(c) = d_R \]
Maximum-Likelihood (ML) Decoding

- Log-likelihood ratio (LLR) $\lambda_i$ for a received observation $y_i$:

\[
\lambda_i(y_i) = \ln \left( \frac{\mathbb{P}_{Y_i/X_i}(y_i / x_i = 0)}{\mathbb{P}_{Y_i/X_i}(y_i / x_i = 1)} \right)
\]

- Any memoryless binary-input output-symmetric (MBIOS) channel can be described by an LLR function.

- Maximum-likelihood (ML) decoding for any binary-input memory-less channel:

\[
\hat{x}^{ML}(y) = \arg\min_{x \in \mathcal{C}} \langle \lambda(y), x \rangle
\]
Linear Programming (LP) Decoding

- **Maximum-likelihood (ML) decoding** formulated as a linear program:

\[ \hat{x}^{ML}(y) = \arg\min_{x \in \mathcal{C}} \left\langle \lambda(y), x \right\rangle = \arg\min_{x \in \text{conv}(\mathcal{C})} \left\langle \lambda(y), x \right\rangle \]
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- **Linear Programming (LP) decoding** [Fel03, FWK05] – relaxation of the polytope \( \text{conv}(C) \)

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\hat{x}^{LP}(y) = \arg \min_{x \in \bigcap \text{check nodes } j \text{ conv}(C_j)} \langle \lambda(y), x \rangle
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Linear Programming (LP) Decoding

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- Linear Programming (LP) decoding [Fel03, FWK05] – relaxation of the polytope \(\text{conv}(C)\)

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\(\hat{x}^{LP}\) integral \(\Rightarrow\) success!

We also know \(\hat{x}^{LP} = \hat{x}^{ML} \in C\) (“ML certificate”)

\(\hat{x}^{LP}\) fractional \(\Rightarrow\) fail

\text{Solve LP}
Previous Bounds for LP Decoding (1)

- No tree assumption! \( \Rightarrow \) Bounds relevant for finite lengths

- Bounds for specific families of codes:
  - Cycle codes / RA(2) codes over memoryless channels [FK02,HE03].
  - Expander LDPC codes over bit flipping channels (e.g., BSC, adversarial) [FMSSW04, DDKW07].
  - Capacity achieving binary expander codes over memoryless channels [FS05].
  - Non-binary expander codes [Ska09].
Previous Bounds for LP Decoding (2)

- $(d_L, d_R)$-regular LDPC codes [KV06, ADS09]
  - Form of finite length bounds: $\exists c > 1. \exists t. \forall$ noise $< t$.
    \[
    \Pr(\text{LP decoder success}) \geq 1 - \exp(-c^{\text{girth}})
    \]
  - If girth $= \Theta(\log n)$, then
    \[
    \Pr(\text{LP decoder success}) \geq 1 - \exp(-n^{\gamma}), \text{ for } 0 < \gamma < 1
    \]

$n \to \infty : t$ is a lower bound on the threshold of LP decoding
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| Technique | Koetter and Vontobel ’06 |
|-----------|--------------------------|
| Channels  | Dual witness technique   |
| Example for (3,6)-regular LDPC code | Memoryless channels |
|          | BSC(\(p\)) threshold: \(p^{LP} > 0.01\) |
|          | BI-AWGNC(\(\sigma\)) threshold: \(\sigma^{LP} > 0.5574, E_b/N_0^{LP} < 5.07\text{dB}\) |

\(\sigma^{\text{Max-Product}} = 0.8223\)
\(E_b/N_0^{\text{Max-Product}} \sim 1.7\text{dB}\)
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| Technique          | Koetter and Vontobel ’06                                      | Arora, Daskalakis and Steurer ’09                           |
|--------------------|----------------------------------------------------------------|-------------------------------------------------------------|
| Channels           | Dual witness technique                                        | Primal LP analysis                                          |
| Example for        | Memoryless channels                                           | BSC                                                         |
| (3,6)-regular LDPC code | BSC(\( p \)) threshold: \( p^{LP} > 0.01 \)               | BSC(\( p \)) threshold: \( p^{LP} > 0.05 \)               |
|                    | BI-AWGNC(\( \sigma \)) threshold: \( \sigma^{LP} > 0.5574 \) \( E_b/N_0^{LP} < 5.07\text{dB} \) | \( p^{BP} = 0.084 \) \( \sigma^{\text{Max-Product}} = 0.8223 \) \( E_b/N_0 \text{ Max-Product} \sim 1.7\text{dB} \) |
Our Results

- Extension of ADS’09 from BSC to MBIOS channels:
  - Combinatorial characterization: Local Opt. ⇒ LP Opt.
  - Alternative proofs using graph covers [VK05]
  - Finite length bound: decoding errors decrease doubly exponential in the girth of the factor graph
    - Example: for (3,6)-regular LDPC code, ∀ σ ≤ 0.605
      \[ P_{err} < \frac{1}{125} e^{\frac{3}{2}} \frac{1}{\sigma^2} n \cdot c^2 \frac{1}{4^g} \]
      for some constant \( c < 1 \).

- Lower bound on thresholds of LP decoding for regular LDPC codes
  - Analytic bounds for MBIOS
  - “Density evolution” bounds on thresholds for BI-AWGNC
    - Example: for (3,6)-regular LDPC code
      \[ \sigma^{\text{LP}} > 0.735 \]
      \[ E_b/N_0^{\text{LP}} < 2.67 \text{dB} \]
      \( (\sigma^{\text{Max-Product}} = 0.8223) \)
      \( (E_b/N_0^{\text{Max-Product}} \sim 1.7 \text{dB}) \)
Skinny Trees Embedded in Factor Graphs

Consider a subgraph $\tau$ of $G$:

- root $= v_0 \in V_L$
- $\tau \subseteq \text{Ball}(v_0, 2h)$
- $\forall v \in \tau \cap V_L : \deg_{\tau}(v) = \deg_G(v)$.
- $\forall c \in \tau \cap V_R : \deg_{\tau}(c) = 2$.

- $\text{girth}(G) > 4h \Rightarrow \tau$ is a tree – Skinny Tree

Moreover, in a $d_L$ left regular graph all skinny trees are isomorphic to:
Cost of a Weighted Skinny Tree [ADS09]

- Given layer weights $\omega : \mathbb{N} \rightarrow \mathbb{R}$, define $\omega$-weighted skinny tree $\tau$ of height $2h$.

- Given assignment of LLR values $\lambda$ to variable nodes, define the cost of an $\omega$-weighted skinny tree $\tau$.

$$\text{val}_\omega(\tau, \lambda) \triangleq \sum_{l=0}^{h-1} \sum_{v \in \tau \cap V_{2l}} \omega_l \cdot \lambda_v$$
Proving Error Bounds using Local Optimality [following ADS09]

- **Local optimality** – sufficient condition for the (global) optimality of a decoded codeword based on skinny trees

- **Theorem:** Fix $h < \frac{1}{4} \text{girth}(G)$ and $\omega \in \mathbb{R}^h$. Then
  \[
  \mathbb{P}\{\text{LP decoding fails}\} \leq \mathbb{P}\{\exists \text{skinny tree } \tau. \text{val}_\omega(\tau, \lambda) \leq 0 \mid x = 0^n\}. 
  \]

- **Task:** bound the probability that there exists a weighted skinny tree with non-positive cost.
Computing $\mathbb{P} [ \min_{\tau} val_\omega (\tau; \lambda) \leq 0 ]$

- $\mathcal{T}$ – induced graph of factor graph $G$ on $\text{Ball}(v_0, 2h)$
- $\{ \gamma \}$ – values associated with variable nodes.
- $Y_l$ – variable nodes of $\mathcal{T}$ at height $2l$.
- $X_l$ – check nodes of $\mathcal{T}$ at height $2l+1$.
- Dyn. Prog. recurrence for computing min cost skinny tree in $\mathcal{T}$:

**Basis:** leaves: $Y_0 = \omega_0 \gamma$

**Step:** checks: $X_l = \min \{ Y_l^{(1)}, ..., Y_l^{(d_R-1)} \}$

**vars:** $Y_l = \omega_l \gamma + X_{l-1}^{(1)} + ... + X_{l-1}^{(d_L-1)}$

$\mathcal{T}$ for $(3,6)$-regular graph, $h=2$
Computing $\mathbb{P}[\min \text{val}_\omega(\tau;\lambda) \leq 0]$

- $\mathcal{T}$ – induced graph of factor graph $G$ on Ball($v_0$, $2h$)
- $\{\gamma\}$ – values associated with variable nodes.
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- Dyn. Prog. recurrence for computing min cost skinny tree in $\mathcal{T}$:

  **Basis:** leaves: $Y_0 = \omega_0 \gamma$
  **Step:** checks: $X_l = \min \left\{ Y^{(1)}_l, \ldots, Y^{(d_R-1)}_l \right\}$
  vars: $Y_l = \omega_l \gamma + X^{(1)}_{l-1} + \ldots + X^{(d_L-1)}_{l-1}$

- **Process:** let $\{\gamma\}$ = components of LLR random vector $\lambda$.
- BI-AWGN($\sigma$) + all zeros assumption:
  $$\lambda_i = 1 + \phi_i \text{ where } \phi_i \sim \mathcal{N}(0, \sigma^2).$$

$\mathcal{T}$ for (3,6)-regular graph, $h=2$
Density Evolution Based Bound for BI-AWGNC(\(\sigma\))

**Theorem**: Let \(G\) denote a \((d_L,d_R)\)-regular bipartite graph with girth \(\Omega(\log n)\), and let \(\mathcal{C}(G)\) denote the LDPC code defined by \(G\). Consider the BI-AWGNC(\(\sigma\)). Then, LP decoding succeeds with probability at least \(1 - \exp(-n^\gamma)\) for some constant \(0 < \gamma < 1\), provided that:

1. \(s < \frac{1}{4} \text{girth}(G)\), and
2. \[\min_{t \geq 0} \mathbb{E} e^{-tX_s} < \left((d_R - 1)e^{-\frac{1}{2\sigma^2}}\right)^{-\frac{1}{d_L-2}}\]

**Condition (2)** holds for \(\sigma < \sigma_0\), where

\[\sigma_0 \triangleq \sup \left\{ \sigma > 0 \mid \min_{t \geq 0} \mathbb{E} e^{-tX_s} \cdot \left((d_R - 1)e^{-\frac{1}{2\sigma^2}}\right)^{-\frac{1}{d_L-2}} < 1 \right\}\]
Gaussian PDFs’ Evolution

- Probability density functions of $X_l$ for $l = 0, \ldots, 4$
  $(d_L, d_R) = (3, 6)$, and $\sigma = 0.7$.

$$Y_0 = \omega_0 \gamma$$
$$X_l = \min \{ Y_l^{(1)}, \ldots, Y_l^{(d_R-1)} \}$$
$$Y_l = \omega_l \gamma + X_{l-1}^{(1)} + \ldots + X_{l-1}^{(d_L-1)}$$

Numeric computation based on quantization following methods used in implementations of density evolution
Threshold bound values for finite $s$, $(d_L,d_R)=(3,6)$

$$\sigma_0 \triangleq \sup \left\{ \sigma > 0 \mid \min_{t \geq 0} \mathbb{E}e^{-tX_s} \cdot \left( (d_R - 1)e^{-\frac{1}{2\sigma^2}} \right)^{\frac{1}{d_L-2}} < 1 \right\}$$

| $s$ | $\sigma_0$ | $E_b/N_0$ [dB] |
|-----|-------------|----------------|
| 0   | 0.605       | 4.36           |
| 1   | 0.635       | 3.94           |
| 2   | 0.66        | 3.61           |
| 3   | 0.675       | 3.41           |
| 4   | 0.685       | 3.29           |
| 6   | 0.7         | 3.1            |
| 10  | 0.715       | 2.91           |
| 22  | 0.735       | 2.67           |

Region for which $5e^{-\frac{1}{2\sigma^2}} \mathbb{E}e^{-tX_4} < 1$ as a function of $t$ and $\sigma$ for $(d_L, d_R) = (3, 6)$.

Max-Product threshold: $\sigma = 0.82$, $E_b/N_0 \sim 1.7$ dB
Summary

- Extended analysis of ADS’09 to MBIOS channels:
  - We saw a sketch of one of the main results:
    - Bound on the threshold of LP decoding for regular LDPC codes with log girth over BI-AWGNC.
      - “Density evolution” bounds: a step towards closing the gap to BP-based threshold
  - More in the paper:
    - Reformulations of some results of ADS’09 in terms of graph covers [VK’05]
    - Combinatorial characterization:
      - Local Opt. $\Rightarrow$ LP Opt.
    - Derivation of finite length bound

“LP Decoding of Regular LDPC Codes in Memoryless Channels” @ arXiv
Future Directions

■ Further understanding the gap to BP-based algorithms thresholds

■ For BI-AWGNC, applying Gaussian approximation techniques to “density evolution” bound
  ⇒ Better thresholds (?)

■ Vontobel [Von10] generalized the geometrical aspects of ADS’09 via normal graphs. Can a modified “DE style” analysis improve performance guarantees?
Thank You!