New Phenomenon of Nonlinear Regge Trajectory and Quantum Dual String Theory

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Abstract

The relation between the spin and the mass of an infinite number of particles in a $q$-deformed dual string theory is studied. For the deformation parameter $q$ a root of unity, in addition to the relation of such values of $q$ with the rational conformal field theory, the Fock space of each oscillator mode in the Fubini-Veneziano operator formulation becomes truncated. Thus, based on general physical grounds, the resulting spin-(mass)$^2$ relation is expected to be below the usual linear trajectory. For such specific values of $q$, we find that the linear Regge trajectory turns into a square-root trajectory as the mass increases.

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As is well known string theory provides a promising approach to describe the different forces and particles observed in nature. The development of this theory is, more or less, directly related to the Veneziano’s discovery \[1\] of a four-point crossing-symmetric scattering amplitude with linear Regge trajectories. Afterwards Fubini, Gordon and Veneziano \[2\] provided an elegant operator formalism for the dual amplitudes, which further led to the interpretation of the Veneziano model as a theory of interacting strings in physical spacetime.

In recent years the mathematical structure of the quantum groups (see, e.g. \[3\]) has been extensively explored in connection with several important aspects of physical phenomena. Among them, the classification of two dimensional conformally invariant field theories plays an important role in understanding the structure of string theory (see, e.g. \[4\]). In particular, for rational conformal field theories, described by a finite number of primary fields, the fundamental properties of fusing and braiding of conformal blocks can be casted within the product of representations of \(q\)-deformed algebras for the specific values of the deformation parameter \(q\) being a root of unity \[1\]. For such values of \(q\) quantum algebras exhibit rich representation behaviours and are important in two-dimensional conformal field theories and in statistical mechanics models \[3\].

In Ref. \[7\] the operator formalism proposed in \[2\] was followed to present a \(q\)-deformed dual string model which possesses crossing symmetry in exchanging the \(s\)- and \(t\)-channels and factorization of the dual amplitudes in such a way that they can be constructed as a field theory with Feynman-like diagrams out of products of vertices and propagators as the building blocks. However, as we shall see below, the \(q\)-deformed dual amplitude proposed there leads to a relation between the spin and the mass spectrum of the particles which may not be of physical interest and in principle could be rule out by experimental results.

In this letter we propose an alternative \(q\)-deformed dual string amplitude, which has not only the required properties of crossing symmetry and factorization, the correct pole structure and a suitable asymptotic behaviour, but also leads to a spin-(mass)\(^2\) relation, which in our opinion possesses features more interesting from the physical point of view. As in \[7\] we shall introduce an infinite number of \(q\)-oscillators which build up a Fock space with well-known properties. However, for the specific values of the deformation parameter \(q\) being a root of unity, only a finite number of oscillators for each (harmonic) mode possesses non-vanishing norm. This is a fundamental property which is responsible for a drastic change in the high energy behaviour of the amplitudes and, therefore, in the mass spectrum of the physical particles. Earlier attempts by introducing a logarithmic behaviour for the Regge trajectories with only a finite number of oscillators \[8\] or by replacing the ordinary gamma functions by their \(q\)-analogs \[9\] have been proposed. However, they cannot bring to a field theoretical formulation of the dual string model.

Let us consider first the usual Veneziano \[1\] 4-point dual amplitude given by

\[
A_4 = \int_0^1 z^{-\alpha(t)-1} (1 - z)^{-\alpha(s)-1} dz,
\]

where \(\alpha(s) = \alpha' s + \alpha_0\) is the linear Regge trajectory, with \(\alpha'\) and \(\alpha_0\) the Regge slope and intercept, respectively. We will take our units so that \(\alpha' = 1\). The amplitude \(\langle \rangle\) can be
factorized by introducing an infinite set of oscillators \( \mathbb{I} \) satisfying \( [a_m^+, a_n^-] = \delta_{mn} g_{\mu \nu} \); \( \mu \) and \( \nu \) are Lorentz indices; \( n, m = 1, 2, \ldots, \infty \) correspond to the different oscillator modes. We have then the identity

\[
(1 - z)^{-AA} = \prod_{n=1}^{\infty} \langle 0 | e^{\frac{Aa_n^+}{\sqrt{\nu}} z^{nN_n} e^{\frac{Aa_n^-}{\sqrt{\nu}}}} | 0 \rangle_q ,
\]

where the contraction of the Lorentz indices is understood in all the scalar products. Here the four-vectors \( A_\mu \) and \( \bar{A}_\mu \) correspond to incoming and outgoing momenta respectively. In what follows we shall denote \( a \equiv \alpha(s) + 1 \) and \( b \equiv \alpha(t) + 1 \) and omit the Lorentz indices.

To perform the \( q \)-deformation within the operator formalism, it seems most natural to use the \( q \)-deformed oscillators \( \mathbb{I} \) instead of the usual ones in the factorization procedure, as this was done in \( \mathbb{I} \), since \( q \)-oscillators have many features in common with the usual harmonic oscillators and from the quantum group point of view are the straightforward generalization of the latters. Therefore, we will replace the identity (2) by the \( q \)-deformed expression

\[
F(a, z) = \prod_{n=1}^{\infty} q \langle 0 | e_q^{\sqrt{\nu} N_n} e_q^{\sqrt{\nu}} | 0 \rangle_q = \prod_{n=1}^{\infty} \sum_{\ell=0}^{\infty} \left( \frac{a}{n} \right) \ell \frac{z^{n \ell}}{[\ell]!} = \prod_{n=1}^{\infty} e_q^{\frac{a}{n} z^n} ,
\]

where \( [\ell] = (q^\ell - q^{-\ell})/(q - q^{-1}) \), \( [\ell]! = [1][2] \cdots [\ell] \) and \( [0]! = 1 \); \( e_q^n = \sum_{\ell=0}^{\infty} \frac{x^n}{[\ell]!} \) is the \( q \)-exponential function. The \( q \)-oscillators entering in (3) satisfy \( \mathbb{I} \)

\[
a_m a_m^+ - qa_m^+ a_m = q^{-N_m} , \quad [N_m, a_m^+] = a_m^+ , \quad [N_m, a_m] = -a_m ,
\]

with all the other commutation relations corresponding to different indices of oscillators, vanishing. The operator \( N_m \) is the number operator corresponding to the mode \( m \).

The main difference between the expression (3) and the corresponding one proposed in \( \mathbb{I} \) (cf. Eq. (8) in \( \mathbb{I} \)) is that we have assumed the total energy operator of the system to be \( H = \sum_{n=1}^{\infty} N_n \) instead of \( H = \sum_{n=1}^{\infty} n a_n^+ a_n \) and therefore, we have replaced \( a_n^+ a_n \) by the number operator \( N_n \) in the exponent of \( z \). As a consequence of this assumption, the amplitudes defined in the present model will exhibit poles in \( s \) when \( \alpha(s) \) is a positive integer, or poles in \( t \) when \( \alpha(t) \) is a positive integer as in the undeformed case. For the amplitudes defined in \( \mathbb{I} \), as we shall see below, the poles are in general located at non-integer values.

In order to preserve duality, expressed by the symmetry \( z \rightarrow 1 - z \) in \( \mathbb{I} \), we define then the \( q \)-deformed 4-point amplitude as

\[
A_4^q = \int_0^1 F(a, z) F(b, 1 - z) dz = \prod_{n=1}^{\infty} q \langle 0 | e_q^{\sqrt{\nu} N_n} \int_0^1 z^{nN_n} F(b, 1 - z) dz e_q^{\sqrt{\nu} N_n} | 0 \rangle_q .
\]

Inserting a complete set of orthonormal states \( 1 = \sum_{\lambda} |\lambda\rangle_q q \langle \lambda| \), with \( |\lambda\rangle_q = \prod_i \frac{(a_i^+)^{\lambda_i}}{\sqrt{[\lambda_i]!}} |0\rangle_q \) we find \( A_4^q \) in its factorized form

\[
A_4^q = \sum_{\lambda} V(A, \lambda) D(\lambda, b) V^+(\bar{A}, \lambda) ,
\]
where
\[ V(A, \lambda) = q\langle 0 | \prod_{n=1}^{\infty} e_{\sqrt{n}}^{A_n^+} | \lambda \rangle_q \tag{7} \]
is the vertex operator (with only one leg off shell) and
\[ D(\lambda, b) = q\langle \lambda | \int_0^1 dz \ z^{\infty} \sum_{n=1}^{\infty} nN_n F(b, 1 - z) | \lambda \rangle_q \tag{8} \]
is the propagator.

The other vertex operator \( \Gamma \) with two legs off shell, which is needed for higher \( n \)-point functions \((n \geq 5)\), is given now by
\[ \Gamma(\lambda, \lambda', p, \bar{p}) = q\langle \lambda | \prod_{n=1}^{\infty} e_{\sqrt{n}}^{p_n^+} e_{\sqrt{n}}^{\bar{p}_n^+} | \lambda' \rangle_q , \tag{9} \]
where \( p = \bar{p} \) is the momentum of the unexcited leg in the vertex. The 5-point \( q \)-deformed amplitude, e.g., now can be written as a Feynman-like diagram in terms of products of vertices and propagators in the tree approximation:
\[ A_q^5 = \sum_{\lambda, \lambda'} V(A, \lambda) D(\lambda, b_1) \Gamma(\lambda, \lambda', p, \bar{p}) D(\lambda', b_2) V^+(A, \lambda'), \tag{10} \]
where \( b_1, b_2 \) are the Mandelstam variables of the corresponding channels.

Notice that the vertices (7) and (8) and the propagator (8) differ from those proposed in [7] since there, two infinite sets of independent unphysical oscillators \((b_i, b_i^+, i = 1, 2)\), which were just auxiliary and not necessary, were introduced. The corresponding \( q \)-deformed \( n \)-point amplitudes, however, are identical whether one uses these auxiliary oscillators or not. Notice also that (7), (8) and (9) lead to the usual undeformed expressions for the vertices and propagators in the limit \( q \to 1 \). The latter feature was not present in the vertices and propagators defined in [7].

Let us first examine the singularities of \( A_q^4 \) as a function of \( s \) and \( t \). Since the integral in (5) is symmetric in \( s \) and \( t \), we need only to analyse the singularities in one of the two variables, e.g. in the variable \( t \). We first observe that near the singular point \( z = 0 \) the integrand \( F(b, 1 - z) \) in (5) can be approximated as
\[ F(b, 1 - z) \sim z^{-b} , \tag{11} \]
and therefore the integral (5) is not defined when \( \alpha(t) \geq 0 \) (or \( \alpha(s) \geq 0 \) ). Following the standard procedure [11] we can define it by analytic continuation to show that the amplitude \( A_q^4 \) exhibits poles in \( t \) at any integer. The residue at the \( n \)-th pole in the \( t \)-channel is a polynomial in \( s \) of degree \( J \leq n \), which when decomposed into spherical harmonics describes the exchange of a set of particles of spins \( \leq J \).
Let us consider now the effects of the deformation in the high energy behaviour of $A_4^q$. In the undeformed case the behaviour of the 4-point amplitude (1) for $a$ large can be easily obtained (see, e.g., [12]) from the contour integral,

$$A_4 \sim \oint_{z=0} \frac{dz}{2\pi i} (1 + az + \cdots + \frac{a(a+1)\cdots(a+n-1)}{n!}z^n + \cdots)z^{-b},$$

leading to

$$A_4 \sim \frac{a(a+1)\cdots(a+b-2)}{\Gamma(b)} \sim \frac{a^{b-1}}{\Gamma(b)},$$

where $\Gamma$ is the gamma function.

In order to discuss the high energy behaviour of the $q$-deformed 4-point amplitude (5), we will follow a procedure similar to the one used for the undeformed case, namely, we write $A_4^q$ as the contour integral

$$A_4^q \sim \oint_{z=0} \frac{dz}{2\pi i} F(a, z)z^{-b}.$$  

Consider now a generic term in the product (3) which can be written as

$$C_{n_1,\ldots,n_m} a^{n_1+n_2+\cdots+n_m} z^{n_1+2n_2+\cdots+mn_m},$$

where $m = 1, 2, \ldots, \infty$ and $C_{n_1,\ldots,n_m}$ are coefficients which depend on $q$. Since we are interested in the high energy behaviour of (14), i.e. in the case when $a$ is large, we should pick in $F(a, z)$ the highest power of $a$ which gives nonvanishing contribution to the contour integral (14). We are therefore led to solve the algebraic equation

$$n_1 + 2n_2 + \cdots + mn_m = b - 1,$$

with the condition that the spin

$$J = n_1 + n_2 + \cdots + n_m,$$

which is the power of $a$ in (15), takes its maximum value.

In what follows we shall consider two cases:

(i) Assume that $q$ is real. Since in this case the $n_i$ in (16) range from 0 to $\infty$, we obtain that the maximum value of $J$ in (17) will be given by $J = b - 1$ when $n_1 = b - 1, n_i = 0$ for $i \geq 2$. Thus we obtain from (13) that for high energies

$$A_4^q \sim a^{b-1}.$$  

We notice that the high $s$-behaviour of the 4-point $q$-deformed amplitude is proportional to $s^j$, which leads to a mass spectrum with a linearly-rising Regge trajectory as in the undeformed case.
Let us consider now the case when the deformation parameter $q$ is a $K$-th root of unity, i.e. $q = \exp(2i\pi/K)$. The $q$-analogs $[\ell]$ thus become $[\ell] = \sin(2\pi\ell/K) / \sin(2\pi/K)$. We should point out here the main difference with respect to the undeformed case: due to the truncation of the Fock space for each oscillator mode in the Fubini-Veneziano operator formulation, each term in the product (3) consists of a finite series ending at $\ell = \tilde{K}$, where

$$\tilde{K} = \begin{cases} K - 1, & \text{for } K \text{ odd} \\ K/2 - 1, & \text{for } K \text{ even} \end{cases}.$$  \hfill (19)

In this case $0 \leq n_i \leq \tilde{K}$ and we have two possibilities:

If $b \leq \tilde{K} + 1$, then it is always possible to find a solution of (16) such that $n_1 = b - 1 \leq \tilde{K}$ and $n_i = 0, i \geq 2$, which gives the maximum value of $J$ in (17). Then as before the high energy behaviour will be given by eq. (18) and, thus, the trajectories will be linear.

If $b > \tilde{K} + 1$, it is easy to see that for a generic $m$ the highest power of $a$ in (15) is $J = m\tilde{K}$ and is obtained when all the $n_i$ are equal to their maximum value $\tilde{K}$. Then according to (16) $J$ will satisfy the second-order algebraic equation

$$\frac{J(J + \tilde{K})}{2\tilde{K}} = b - 1,$$  \hfill (20)

which has the positive solution

$$J = \frac{\tilde{K}}{2} \left\{ -1 + \sqrt{1 + \frac{8(b - 1)}{\tilde{K}}} \right\}. \hfill (21)$$

Thus finally we obtain that for large values of $a$ and for $b > \tilde{K} + 1$

$$A_q^b \sim a^J,$$ \hfill (22)

where $J$ is given by (21).

We notice now from (21) and (22) that the high $s$-behaviour of the $q$-deformed amplitudes is proportional to $s^{\sqrt{t}}$ which leads to the mass spectrum with a square-root Regge trajectory $\alpha(t) = \sqrt{t} + \text{const}$. It appears, as was expected by physical arguments, that the crucial change in the high energy behaviour of $A_q^b$ occurs due to the truncation in the series in (3) and as mentioned before, is a direct consequence of $q$ being a root of unity. This change can be understood by the finite character of the Fock space as a consequence of the assumed values of $q$.

It is worth mentioning that Eq. (21) gives the position of the poles in the leading Regge trajectory for the values of the spin $J = m\tilde{K}$. In general, for an arbitrary value of the spin $J = m\tilde{K} + r$ with $r = 0, 1, \ldots, K - 1$, it is possible to show that the poles of the amplitude are located at the points given by the solution of the equation

$$\frac{(J + r)(J + \tilde{K} - r)}{2\tilde{K}} = \frac{(2J - m\tilde{K})(m + 1)}{2} = b - 1.$$  \hfill (23)
In Fig. 1 the leading Regge trajectories for both cases, when \( q \) is real and \( q = \exp(2i\pi/K) \), are shown. The dots denote the position of the poles and the linearly-rising Regge trajectory corresponds to the case when the deformation parameter \( q \) is real. We observe that when \( q \) is a \( K \)-th root of unity, the trajectory is linear for \( t \leq \tilde{K} \) and turns into a square-root trajectory for \( t > \tilde{K} \), with \( \tilde{K} \) given by Eq.(19). We also notice that for smaller values of the parameter \( K \) (i.e. when the truncation of the series in (3) occurs earlier), the effect of the deformation is enhanced. As \( K \to \infty \), then \( q \to 1 \) and we recover the usual linear trajectory. Since the amplitude is symmetric in \( s \) and \( t \), the resonances in the \( s \)-channel also lie on the same Regge trajectory.

For comparison we shall consider the amplitude proposed in Ref. [7]. As we already mentioned, the main difference consists on the choice of the Hamiltonian of the model, which in [7] was taken as \( H = \sum_{n=1}^{\infty} n a_n^\dagger a_n \). In this case Eq. (3) reads as

\[
F(a, z) = \prod_{n=1}^{\infty} \sum_{\ell=0}^{\infty} \left( \frac{a_n^\ell}{n!} \right) \frac{z^n[\ell]}{[\ell]!},
\]

so that the deformation parameter \( q \) enters explicitly in the exponent of \( z \). Therefore, in order to obtain the pole structure of the amplitude, Eq.(16) should be replaced by

\[
[n_1] + 2[n_2] + \cdots + m[n_m] = b - 1,
\]

where as before the \( n_i \) (\( i = 1, 2, \ldots, m \)) can take any integer value for \( q \) real, and \( n_i \leq \tilde{K} \) when \( q \) is a \( K \)-th root of unity. It is possible to show that in this case the poles of the \( q \)-deformed amplitude are defined by the solutions of Eq. (25), which in general are given by real (and not only integer) values of \( b \). Nevertheless, the residue at each pole is a polynomial in \( a \) of degree \( J \) given by Eq. (17). Therefore, even when \( q \) is real the Regge trajectories appear to be deformed in this case (Fig. 2). Another peculiarity of the amplitude defined in [7] appears in the case when \( q = \exp(2i\pi/K) \). Due to the relation \([K - \ell] = [\ell] \), there exists a degeneracy in the solutions of Eq. (25), which leads to a splitting of the Regge trajectories, so that only spins of the form \( J = m\tilde{K} \), with \( m \) being an integer number, lie on the leading trajectory. In Fig. 3 we show the splitting of the trajectories for the particular case of \( K = 3 \), which is the same as for \( K = 6 \). We see that only the poles corresponding to even spins lie in the leading Regge trajectory. These features make the model proposed in [7] less appealing than the one presented here.

Thus quantum groups and the \( q \)-deformation provide with a new phenomenon of a linear Regge trajectory to turn into a square-root trajectory for higher masses in the case when the deformation parameter is a root of unity. This case is of upmost physical interest and in particular, appears in rational conformal field theories. An ultimate aim to combine the quantum group ideas with the (super)string theory in order to obtain a \( q \)-deformed (super)string amplitude, can be pursued along similar lines as presented in this letter.
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Figure Captions

Fig. 1 The behaviour of the Regge trajectories for the $q$-deformed dual string model proposed in this letter. The dots denote the position of the poles and the straight (dashed) line corresponds to the case when $q$ is real, with linearly-rising Regge trajectory. When $q$ is a $K$-th root of unity, $q = \exp(2i\pi/K)$, the trajectory is linear for $t \leq \tilde{K}$ and turns into a square-root trajectory for $t > \tilde{K}$, with $\tilde{K} = K - 1$ for $K$ odd and $\tilde{K} = K/2 - 1$ for $K$ even (see Eq. (19)). The curves are shown for the particular values of $K = 3, 4, 6, 8, 12$.

Fig. 2 The Regge trajectories in the case of the model proposed in [7] for different values of $q$ real. The straight (dashed) line corresponds to the undeformed case. As $q$ increases the deviation of the trajectories from the linear one is enhanced.

Fig. 3 Splitting of the Regge trajectories in the model of Ref. [7] for the case when $q$ is a $K$-th root of unity. The trajectories are shown for the particular case of $K = 3$, which is the same as for $K = 6$. Only the poles corresponding to even spins lie in the leading trajectory. The trajectories behave as a square-root.
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