SHARPNESS OF THE BRASCAMP–LIEB INEQUALITY IN LORENTZ SPACES

NEAL BEZ, SANGHYUK LEE, SHOHEI NAKAMURA, AND YOSHIHIRO SAWANO

Abstract. We provide necessary conditions for the refined version of the Brascamp–Lieb inequality where the input functions are allowed to belong to Lorentz spaces, thereby establishing the sharpness of the range of Lorentz exponents in the subcritical case. Using similar considerations, some sharp refinements of the Strichartz estimates for the kinetic transport equation are established.

1. Introduction

The Brascamp–Lieb inequality originating in [9] takes the form

$$\int_{\mathbb{R}^d} \prod_{j=1}^m f_j \circ L_j \leq BL(L, p) \prod_{j=1}^m \|f_j\|_{L^{p_j}}$$

(1)

for nonnegative input functions $f_j$ in $L^{p_j}(\mathbb{R}^{d_j})$. Each such inequality has an associated Brascamp–Lieb datum $(L, p)$, where $L = (L_j)_{j=1}^m$ is an $m$-tuple of surjective linear mappings and $p = (p_j)_{j=1}^m$ is an $m$-tuple of exponents belonging to $[1, \infty]$, and an associated Brascamp–Lieb constant $BL(L, p) \in [0, \infty]$ (taken to be the optimal constant). It is clear that additional assumptions on the datum are needed in order to avoid the uninteresting case where $BL(L, p)$ is infinite; for example, a necessary scaling condition is

$$\sum_{j=1}^m d_j \frac{1}{p_j} = d.$$

(2)

It was shown in [6] (see also [7]) that the finiteness of $BL(L, p)$ is equivalent to the scaling condition combined with the dimension condition, which states that

$$\dim(V) \leq \sum_{j=1}^m \frac{1}{p_j} \dim(L_j V)$$

(3)

for all subspaces $V$ of $\mathbb{R}^d$. 

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Given the generality of the setup, it is not surprising that the Brascamp–Lieb inequality has featured in a wide spectrum of fields. As an example, important stimulus to the theory of the inequality was provided by its applicability to problems in convex geometry (see, for example, [2] and [3]). The geometric nature of the inequality may also be seen in the case of the Loomis–Whitney inequality, where each $L_j$ is the projection onto the $j^{th}$ coordinate hyperplane, and far reaching generalizations in the form of multilinear Kakeya-type inequalities (see, for example, [8], [16], and [4]).

By now, the theory of the Brascamp–Lieb inequality for input functions in classical Lebesgue spaces is well advanced, with other important issues such as an understanding of the optimal value of the Brascamp–Lieb constant and the extremizing input functions having been addressed (see, for example, [6], [11], [18], [26]). This short note is concerned with refinements of the Brascamp–Lieb inequality with input functions $f_j$ belonging to Lorentz spaces

$$\int_{\mathbb{R}^d} \prod_{j=1}^m f_j \circ L_j \leq C \prod_{j=1}^m \|f_j\|_{L^{p_j,r_j}}, \tag{4}$$

where $(r_j)_{j=1}^m$ is an $m$-tuple of exponents belonging to $[1, \infty]$. It is for this reason that we formulated the classical Brascamp–Lieb inequality in (1) for input functions $f_j \in L^{p_j}$; we caution that whilst we follow some of the notation in [6] to a degree, the inequality there is studied in its equivalent formulation for input functions in $L^1$ (and hence $\frac{1}{p_j}$ here plays the role of $p_j$ in [6]).

The Lorentz space $L^{p,r}$ is the space of measurable functions $f$ such that the quantity (in general, a quasi-norm)

$$\|f\|_{L^{p,r}} = \left( \int_0^\infty \left( t^{1/p} f^*(t) \right)^r \frac{dt}{t} \right)^{1/r}$$

is finite. Here, $f^*$ is the decreasing rearrangement of $f$ and, when $r = \infty$, the above quantity is to be interpreted as $\|f\|_{L^{p,r}} = \sup_{t > 0} t^{1/p} f^*(t)$. This is not a normable space whenever $p = 1$ and $r \in (1, \infty]$; for simplicity, from herein we shall focus on the case $p \in (1, \infty)$ and $r \in [1, \infty]$, in which case $L^{p,r}$ is normable and becomes a Banach space. We refer the reader to [25] for details about Lorentz spaces.

In connection to various related problems it seems entirely natural to consider extension of the classical Brascamp–Lieb inequality to Lorentz spaces. An example which may help solidify this view is the fundamental inequality of Hardy–Littlewood–Sobolev which, in its classical form, belongs to the family of inequalities (4). In this case, we take $m = 3$ with two arbitrary input functions and the remaining input function of the form $|\cdot|^{-\lambda}$. In this case, the Lorentz refinement of the classical version of the Hardy–Littlewood–Sobolev inequality states that

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f_1(x_1)f_2(x_2)}{|x_1 - x_2|^\lambda} \, dx_1 dx_2 \leq C \|f_1\|_{L^{p_1,r_1}} \|f_2\|_{L^{p_2,r_2}}, \tag{5}$$

where $n \geq 1$, $\lambda \in (0, n)$, $p_j \in (1, \infty)$ satisfy $\frac{1}{p_1} + \frac{1}{p_2} + \frac{\lambda}{n} = 2$, and $\frac{1}{r_1} + \frac{1}{r_2} \geq 1$ (as far as we are aware, this is due to O’Neil [20]). Such Lorentz refinements have important consequences which are not achievable from the classical form; we mention recent work of Quilodrán [24] on compactness properties of extremizing sequences for the adjoint restriction inequality for the cone in $\mathbb{R}^3$ (or Strichartz estimates for the wave equation with $\dot{H}^{1/2}(\mathbb{R}^2)$ initial data). In this application, (5) played a decisive role
in establishing a refinement of the classical adjoint restriction inequality (a so-called “cap estimate”) which in turn was fundamental to the compactness argument.

Recently, significantly more involved instances of (4) arose in the work of Brown [10] (see also [1], [19], [23]) in inverse scattering theory, and in particular the continuity of the scattering maps connected with the Davey–Stewartson II system. Such Lorentz refinements were obtained by Brown from the classical Brascamp–Lieb inequalities in combination with a multilinear interpolation argument. Subsequently Christ [23] developed the theory along broadly similar lines, establishing the following rather general result (in particular, containing the result of Brown [10]).

**Theorem 1.** Suppose that \( L_j : \mathbb{R}^d \to \mathbb{R}^{d_j} \) is a surjective linear transformation, \( p_j \in (1, \infty) \) and \( r_j \in [1, \infty] \) for each \( 1 \leq j \leq m \). If the scaling condition (2) holds and the dimension condition (3) holds with strict inequality for all nonzero proper subspaces \( V \) of \( \mathbb{R}^d \), then (4) holds whenever

\[
\sum_{j=1}^{m} \frac{1}{r_j} \geq 1.
\]  

(6)

For fixed \( L \), subspaces are said to be subcritical with respect to \( p \) if (3) holds with strict inequality for nontrivial proper subspaces, and one can verify that subcriticality is stable under small enough perturbations of \( p \). Recalling that (2) and (3) are equivalent to the finiteness of \( BL(L, p) \), such stability along with Christ’s multilinear interpolation argument from [13] is enough to establish Theorem 1. Our main purpose here is to show that (6) is always a necessary condition for (4) to hold, and in particular, we may conclude that the above result of Christ is sharp in this regard. Moreover, we shall see that (4) does not, in general, hold in the range (6) without the subcriticality assumption; this will be demonstrated by considering the case of Loomis–Whitney data and serves as another viewpoint from which Theorem 1 can be considered sharp.

**Theorem 2.** Suppose that \( L_j : \mathbb{R}^d \to \mathbb{R}^{d_j} \) is a linear transformation, \( p_j \in (1, \infty) \) and \( r_j \in [1, \infty] \) for each \( 1 \leq j \leq m \). If the Lorentz refinement (4) of the Brascamp–Lieb inequality holds, then necessarily (2), (3), and (6) hold. On the other hand, (2), (3), and (6) are not, in general, sufficient for (4).

The proof of Theorem 2 is presented in Section 2 along with further additional remarks. In Section 3 we provide a Lorentz space refinement of the Strichartz estimates for the solution of the kinetic transport equation based on (5), and the sharpness of the resulting estimate is shown using similar considerations to those in Section 2.

## 2. Proof of Theorem 2

**Preliminaries.** In this section, we write \( A \lesssim B \) and \( B \gtrsim A \) to mean \( A \leq CB \) for some constant \( C \) which is allowed to depend on the Brascamp–Lieb datum \((L, p)\), and \( A \sim B \) means both \( A \lesssim B \) and \( A \gtrsim B \). Also, we write \( \mathbf{1}_E \) for the characteristic function of \( E \).

As noted above, \( \|f\|_{L^{p,r}} \) is in general a quasi-norm, however it will be important in the argument below that the space \( L^{p,r} \) is normable for \( p \in (1, \infty) \) and \( r \in [1, \infty] \).
by replacing \( \| \cdot \|_{L^{p,r}} \) with \( \| \cdot \|^*_{L^{p,r}} \) given by
\[
\| f \|^*_{L^{p,r}} = \left( \int_0^\infty \left( \frac{t^{1/p} f^*(t)}{t} \right)^r \frac{dt}{t} \right)^{1/r},
\]
where
\[
f^*(t) = \frac{1}{t} \int_0^t f^*(s) \, ds.
\]
The quantities \( \| \cdot \|_{L^{p,r}} \) and \( \| \cdot \|^*_{L^{p,r}} \) are equivalent in the sense that
\[
\| f \|_{L^{p,r}} \leq \| f \|^*_{L^{p,r}} \leq \frac{p}{p-1} \| f \|_{L^{p,r}}
\]
holds for all \( f \in L^{p,r} \); this can be found, for example, in [25, Theorem 3.21] along with further details concerning Lorentz spaces.

Moreover, we shall use the scaling property
\[
\| f(R\cdot) \|^*_{L^{p,r}} = R^{-n/p} \| f \|^*_{L^{p,r}}
\]
for functions on \( \mathbb{R}^n, n \geq 1 \), and any \( R > 0 \). This follows from the fact that \( f(R\cdot)^*(t) = f^*(R^nt) \) and hence \( f(R\cdot)^{**}(t) = f^{**}(R^nt) \).

**Necessity of (2).** One consequence of (8) is that (2) is also necessary for (4). Indeed, by replacing each \( f_j \) with \( f_j(R\cdot) \) and using (8) it can quickly be seen that (2) must hold.

**Necessity of (3).** The necessity of the dimension condition follows from the argument in [6] to prove the necessity for (1) (even in the weakest case where each \( r_j = 1 \)), since the argument is based on testing the inequality on Gaussian inputs for which there is no essential difference between the Lebesgue space norm and the Lorentz norm; thus, we simply refer the reader to [6] for the details.

**Necessity of (6).** For each \( 1 \leq j \leq m \), fix a nonnegative and nonincreasing sequence \( (a^j_k)_{k \geq 1} \) and set
\[
f_j = \sum_{k=1}^\infty a^j_k 1_{L_j(Q_k)},
\]
where \( Q_k = (-2^k, 2^k)^d \). Estimating the left-hand side of (4) from below and taking only the diagonal terms, we obtain
\[
\int_{\mathbb{R}^d} \prod_{j=1}^m f_j \circ L_j \geq \sum_{k=1}^\infty \left( \prod_{\ell=1}^m a^\ell_k \right) \int_{\mathbb{R}^d} \prod_{j=1}^m 1_{L_j(Q_k)}(L_jx) \, dx
\]
\[
= \sum_{k=1}^\infty \left( \prod_{\ell=1}^m a^\ell_k \right) \left| \prod_{j=1}^m L_j^{-1}(L_j(Q_k)) \right|
\]
and hence
\[
\int_{\mathbb{R}^d} \prod_{j=1}^m f_j \circ L_j \gtrsim \sum_{k=1}^\infty 2^{kd} \left( \prod_{\ell=1}^m a^\ell_k \right).
\]
For the right-hand side of (4), we claim that
\[
\| f_j \|_{L^{p_j,r_j}} \lesssim \left( \sum_{k=1}^\infty \left( a^j_k 2^{kdj} \right)^{r_j} \right)^{1/r_j}
\]
(10)
for each $1 \leq j \leq m$. To see this, first observe that $L_j(\mathcal{Q}_k) \subseteq \mathcal{B}^j_k$, where

$$\mathcal{B}^j_k = \{x \in \mathbb{R}^d : |x| < \lambda_j 2^k\}$$

with a sufficiently large $\lambda_j \sim 1$. If we write

$$\mathcal{A}^j_k = \begin{cases} \mathcal{B}^j_k \setminus \mathcal{B}^j_{k-1} & k \geq 2, \\ \mathcal{B}_1^j & k = 1, \end{cases}$$

then since

$$\left\| \sum_{k=1}^{\infty} a_k^j 1_{\mathcal{A}^j_k} \right\|_{L^{p,j,r_j}} \sim \left( \sum_{k=1}^{\infty} \left( a_k^j 2^{j k} \right)^{-r_j} \right)^{1/r_j}$$

we may obtain (10) from the following lemma.

**Lemma 3.** Let $n \geq 1$, $\lambda > 0$, $p \in (1, \infty)$ and $r \in [1, \infty]$. Then, there is a constant $C$ such that

$$\left\| \sum_{k=1}^{\infty} a_k 1_{\mathcal{B}^j_k} \right\|_{L^{p,r}} \leq C \left\| \sum_{k=1}^{\infty} a_k 1_{\mathcal{A}^j_k} \right\|_{L^{p,r}}$$

holds for all sequences $(a_k)_{k \geq 1}$ of nonnegative numbers, where $\mathcal{B}_k = \{x \in \mathbb{R}^n : |x| < \lambda 2^k\}$ and

$$\mathcal{A}^j_k = \begin{cases} \mathcal{B}^j_k \setminus \mathcal{B}^j_{k-1} & k \geq 2, \\ \mathcal{B}_1^j & k = 1. \end{cases}$$

**Proof.** Clearly, we have $1_{\mathcal{B}^j_k} = 1_{\mathcal{A}^j_k} + 1_{\mathcal{B}^j_{k-1}}$ for $k \geq 2$, and therefore

$$\sum_{k=1}^{\infty} a_k 1_{\mathcal{B}^j_k} = \sum_{k=1}^{\infty} a_k 1_{\mathcal{A}^j_k} + \sum_{k=2}^{\infty} a_k 1_{\mathcal{B}^j_{k-1}} \leq \sum_{k=1}^{\infty} a_k 1_{\mathcal{A}^j_k} + \sum_{k=1}^{\infty} a_k 1_{\mathcal{B}^j_{k-1}}.$$ 

Since $1_{\mathcal{B}^j_{k-1}}(x) = 1_{\mathcal{B}_k}(2x)$, the above inequality implies that

$$f(x) \leq g(x) + f(2x) \quad (11)$$

for each $x \in \mathbb{R}^n$, where $f = \sum_{k=1}^{\infty} a_k 1_{\mathcal{B}^j_k}$ and $g = \sum_{k=1}^{\infty} a_k 1_{\mathcal{A}^j_k}$. Using that $\| \cdot \|_{L^{p,r}}$ is a norm, along with the scaling property (8), we obtain

$$\|f\|_{L^{p,r}}^* \leq (1 - 2^{-n/p})^{-1} \|g\|_{L^{p,r}}^*$$

from (11). (Strictly speaking, to ensure finiteness of the norm of $f$ we should first truncate the sums and use a limiting process to deduce the above norm inequality.) The lemma now follows from (7). \qed

From (10), we may obtain the desired conclusion in Theorem 2. Indeed, if we assume that (4) holds, from (9) and (10), then

$$\sum_{k=1}^{\infty} \left( \prod_{j=1}^{m} a_k^j \right)^{2^{kd}} \lesssim \prod_{j=1}^{m} \left( \sum_{k=1}^{\infty} \left( a_k^j 2^{j k} \right)^{r_j} \right)^{\frac{1}{r_j}}$$

for all nonnegative and nonincreasing sequences $(a_k^j)_{k \geq 1}$, $1 \leq j \leq m$. We now take any nonnegative and nonincreasing sequence $b^j = (b_k^j)_{k \geq 1}$ for each $1 \leq j \leq m$, and let $a_k^j = b_k^j 2^{-kd_j/p_j}$. From the above, using the scaling condition (2), we obtain

$$\left\| \prod_{j=1}^{m} b^j \right\|_{L^t} \lesssim \prod_{j=1}^{m} \|b^j\|_{L^{t_j}},$$
from which it follows that $\sum_{j=1}^m \frac{1}{r_j} \geq 1$. To see this, observe that if we assume $\sum_{j=1}^m \frac{1}{r_j} \in (0,1)$ and we set $\eta = (\sum_{j=1}^m \frac{1}{r_j})^{-1}$, then taking $b_k^j = k^{-\eta/r_j}$ we see that the right-hand side coincides with the $\ell^\eta$ norm of the harmonic series, whilst the left-hand side coincides with the $\ell^1$ norm of the harmonic series; this contradiction implies $\sum_{j=1}^m \frac{1}{r_j} \geq 1$.

**Insufficiency of (2), (3), and (6).** We provide an example where (2), (3), and (6) are not sufficient to obtain a Lorentz refinement of the form (4). Of course, given Theorem 1, we are led to consider datum $(L, p)$ for which (3) holds with equality for some nonzero proper subspaces; such subspaces are said to be critical (with respect to $p$).

Specifically, we consider the classical Loomis–Whitney inequality; this is the special case of (1) where $m = d \geq 3$, $d_j = d - 1$ for each $j$, and $L_j : \mathbb{R}^d \to \mathbb{R}^{d-1}$ deletes the $j$th component of its input (e.g., $L_1x = (x_2, x_3, \ldots, x_d)$ and so on). In this case, $BL(L, p) < \infty$ if and only if $p_j = d - 1$ for each $j$.

For such $L$, we claim that
\[
\sum_{j=1}^d \frac{1}{r_j} \geq \frac{d}{d - 1}
\]  
(12)
is a necessary condition for (4) to hold. To see this, fix the parameter $N \gg 1$ and take each $f_j$ to be the $(d-1)$-fold tensor product function $f_j = g = g_N \otimes \cdots \otimes g_N$, where
\[
g_N = \sum_{k=1}^N \frac{1}{2^k} 1_{[2(d-1)k, 2(d-1)(k+1))}.
\]
Then a calculation shows that
\[
g^* = \sum_{\ell=d-1}^{(d-1)N} \frac{1}{2\ell} 1_{[\alpha_{\ell-1}, \alpha_\ell)},
\]
where $(\alpha_\ell)_{\ell \geq d-2}$ is a strictly increasing sequence with $\alpha_\ell \lesssim \ell^{d-2} 2^{(d-1)\ell}$, and thus $\|g\|_{L^{d-1-\tau}} \lesssim N^{\frac{d-2}{d-1}} + \frac{1}{\tau}$. Therefore the right-hand side of (4) satisfies
\[
\prod_{j=1}^d \|f_j\|_{L^{d-1-\frac{1}{r_j}}} \lesssim N^{\frac{d(d-2)}{d-1} + \sum_{j=1}^d \frac{1}{r_j}}
\]
whilst the left-hand side satisfies
\[
\int_{\mathbb{R}^d} \prod_{j=1}^d f_j \circ L_j \sim \|g_N\|_{L^{d-1}}^{d(d-1)} \sim N^d.
\]
This forces (12) to hold, as claimed. \hfill \Box

**Remark.** The multilinear Hölder inequality is also an instance of (1), by taking each $L_j$ to be the identity operator on $\mathbb{R}^d$, in which case $BL(L, p) < \infty$ if and only if $\sum_{j=1}^m \frac{1}{p_j} = 1$. Clearly this is a more elementary example where the subcritical hypothesis in Theorem 1 fails to hold (indeed, all subspaces are critical under the given restriction on the $p_j$). However, under the assumption (6), it is known that (4) holds in this case (this goes back at least to [20]).
It would, of course, be desirable to understand precisely when (4) holds for
Loomis–Whitney data, and more widely, in the case of arbitrary critical data; we
hope to address this problem in future work.

**The case** \( p_j = 1 \). We end this section by offering an alternative proof of (10)
which is somewhat less self-contained but has the advantage that it avoids use of
the norm \( \| \cdot \|_{L^p,\cdot}^r \) and thus we may include the case \( p_j = 1 \). The argument also uses
the surjectivity of the \( L_j \); we note that is a necessary condition for (4). Indeed, we
have already observed that (2) and (3) are necessary for (4), and by testing (3) on
\( V = \mathbb{R}^d \) and comparing the resulting inequality with (2), the surjectivity of each
\( L_j \) can be deduced.

Assume (4) holds. Since each \( L_j \) is linear and surjective, we can find small
\( \eta_j \in (0,1) \) such that \( L_j((-2,2)^d) \supseteq [\eta_j,2\eta_j)^d \). This implies the pointwise estimate
\[
1_{L_j((-2,2)^d)} \lesssim (M_{1\{\eta_j,2\eta_j)^d\_j})^2,
\]
where \( M \) denotes the Hardy–Littlewood maximal operator. Furthermore, using the
linearity of the \( L_j \) and rescaling, we obtain
\[
1_{L_j(B_1)}(x_j) = 1_{L_j((-2,2)^d)}(2^{-k}x_j) \lesssim (M_{1\{2^{-k-1}\eta_j,2\eta_j)^d\_j})^2.
\]
Then, if we write \( g_k^j = (a_k^j)^{1/2} 1_{d\{2^{-k-1}\eta_j,2\eta_j)^d\_j} \) and use the fact that \( 2p_j \in (1,\infty) \),
we may employ the vector-valued type boundedness of \( M \) on Lorentz spaces (which
follows, for example, from ([14])) to obtain
\[
\| f_j \|_{L^{p_j,r_j}} \lesssim \left\| \sum_{k=1}^{\infty} (Mg_k^j)^2 \right\|_{L^{2p_j,2r_j}}^{1/2} \lesssim \left\| \sum_{k=1}^{\infty} (g_k^j)^2 \right\|_{L^{2p_j,2r_j}}^{1/2} \sim \left( \sum_{k=1}^{\infty} \left( a_k^j 2^{d/2k} \right)^{r_j} \right)^{1/r_j}
\]
as desired.

**Remark.** The condition that \( p_j \geq 1 \) for each \( j \) may also be shown to be necessary
for (4), using an argument which works equally well for the classical Brascamp–Lieb
inequality (1). Indeed, fixing \( 1 \leq j \leq m \), we take \( f_j = 1_{B(0,\delta)} \) and \( f_k = 1_{B(0,1)} \) for
every \( k \neq j \), where \( 0 < \delta \ll 1 \) and where we write \( B(x,R) \) for the Euclidean ball
of radius \( R > 0 \) centered at \( x \) (in the appropriate Euclidean space). Then one may
check that the left-hand side of (4) is \( \sim \delta^{d_j} \) whilst the right-hand side is \( \sim \delta^{d_j/p_j} \);
clearly we may obtain the necessary condition \( p_j \geq 1 \).

3. A Lorentz refinement of Strichartz estimates for the
kinetic transport equation

An example which played a role in the motivation of this work is the family of
Strichartz estimates
\[
\| \rho f \|_{L^q_t L^r_x} \lesssim \| f \|_{L_{p,v}^{2p,\infty}}^{2p \rho_v} \tag{13}
\]
for the velocity average
\[
\rho f(t,x) = \int_{\mathbb{R}^d} f(x-tv,v) \, dv
\]
of the solution of the kinetic transport equation
\[
\partial_t F(t,x,v) + v \cdot \nabla_x F(t,x,v) = 0, \quad F(0,x,v) = f(x,v)
\]
for \((t, x, v) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d\). In this context, we write \(A \lesssim B \) and \(B \gtrsim A \) to mean \(A \leq CB \) for some constant \(C \) which is allowed to depend on \((p, q, d)\), and \(A \sim B \) means both \(A \lesssim B \) and \(B \gtrsim A \).

It is known ([12], [17]) that (13) holds if \(d, p, q \geq 1 \) satisfy the scaling condition \(\frac{d}{p} + \frac{2}{q} = d \) and \(p \in [1, \frac{d+1}{2})\); the scaling condition is easily verified to be necessary, and examples show that the range of \(p \) cannot be extended (the failure of the endpoint case \(p = \frac{d+1}{2} \) was established in [15] and [21] when \(d = 1 \) and in [5] for all \(d \geq 2 \)). In terms of the dual operator

\[
\rho^*g(x, v) = \int_\mathbb{R} g(t, x + tv) \, dt,
\]

these results are succinctly expressible as

\[
\|\rho^*g\|_{L^2_{x,v}} \lesssim \|g\|_{L^1_{t}L^2_{x,v}}
\]

whenever \(\frac{d}{p} + \frac{2}{q} = 2 \) and \(p \in (\frac{d+1}{2}, \infty] \), with the endpoint case \(p = \frac{d+1}{2} \) being false. Here we prove that these results may be refined in the following way.

**Theorem 4.** Let \(d \geq 1 \) and \(p, q, r \in (1, \infty) \). Then

\[
\|\rho^*g\|_{L^r_{x,v}} \lesssim \|g\|_{L^p_{t}L^q_{x,v}}
\]

holds if and only if \(\frac{d}{p} + \frac{2}{q} = 2 \), \(p \in (\frac{d+1}{2}, \infty] \) and \(r \leq 2p \).

**Proof.** The sufficiency of the given conditions on \((p, q, r)\) can be shown by following the \(TT^*\) proof of (13) by Ovcharov [22] and, at the appropriate stage in the proof, using (5) rather than the classical form of the Hardy–Littlewood–Sobolev inequality; for completeness, we include a very brief outline. First, observe that it suffices to prove (15) when \(r = 2p \) and, if we use duality and the fact that the set of solutions of the kinetic transport equation is invariant under \(f \mapsto f^\lambda \) for \(\lambda > 0 \) (to shift the problem to initial data in \(L^2_{x,v}(\mathbb{R}) \)), we may express the desired estimate as

\[
\|f(x - tv, v)\|_{L^{p_0}_{t}L^{q_0}_{x,v}} \lesssim \|f\|_{L^2_{x,v}},
\]

where \(p_0 = p' + 1 \) and \(\frac{d}{p_0} + \frac{1}{q_0} = \frac{d}{2} \). Equivalently

\[
\|TT*g\|_{L^{q_0}_{t}L^{p_0}_{x,v}} \lesssim \|g\|_{L^{q_0}_{t}L^{p_0}_{x,v}},
\]

where \(T \) is the solution operator \(Tf(t, x, v) = U(t)f(x, v) = f(x - tv, v) \). To show this, note that the dispersive estimate

\[
\|U(t)f\|_{L^{p_0}_{t}L^{q_0}_{x,v}} \lesssim \frac{1}{|t|^{d(1 - \frac{1}{p_0})}} \|f\|_{L^{p_0}_{t}L^{q_0}_{x,v}}
\]

quickly yields

\[
\|TT*g\|_{L^{q_0}_{t}L^{p_0}_{x,v}} \lesssim \|g\|_{L^{q_0}_{t}L^{p_0}_{x,v}},
\]

where \(G(s) = \|g(s, x, v)\|_{L^{q_0}_{t}L^{p_0}_{x,v}} \). It is now clear that (5) implies (16) as desired (one may readily verify that \(p \in (\frac{d+1}{2}, \infty) \) implies that \(d(1 - \frac{2}{p_0}) \in (0, 1) \)).

Next, we establish that the conditions \(\frac{d}{p} + \frac{2}{q} = 2 \), \(p \in (\frac{d+1}{2}, \infty] \) and \(r \leq 2p \) are necessary for (15) to hold. The first condition \(\frac{d}{p} + \frac{2}{q} = 2 \) follows from (8) by a scaling argument in the \(t\)-variable. To see that \(p \in (\frac{d+1}{2}, \infty) \) is necessary, we test...
on an isotropic Gaussian input $g(t, x) = e^{-\pi(t^2 + x^2)}$. In this case, a calculation leads to
\[ \rho^* g(x, v) = (1 + |v|^2)^{-\frac{1}{2}} e^{-\pi|v|^2} e^{\frac{x(v) \cdot (x, v)}{1 + |v|^2}} \]
and hence
\[ \|\rho^* g(\cdot, v)\|_{L^p_{\mathbb{R}^d}}^2 = (2p)^{-\frac{1}{2}} (1 + |v|^2)^{\frac{1}{2} - p}. \]
It follows that if (15) holds then $p > \frac{d+1}{2}$.

Finally we show that $r \leq 2p$ is necessary. Take any nonnegative and nonincreasing sequence $(a_k)_{k \geq 0}$ and let
\[ g(t, x) = 1_{B(0, 10)}(x) \left( a_0 1_{[0, 1)}(t) + \sum_{k=1}^{\infty} a_k 1_{[2^{k-1}, 2^k)}(t) \right) \tag{17} \]
for $(t, x) \in \mathbb{R} \times \mathbb{R}^d$. As above, we write $B(x, R)$ for the Euclidean ball of radius $R > 0$ centered at $x$. Furthermore, we write
\[ \mathcal{A}_k^x = 2^{-k} B(-x, 1) \setminus 2^{-(k-1)} B(-x, 1). \]

For any $k \geq 1$, $x \in B(0, 1)$, and $v \in \mathcal{A}_k^x$, we have that $x + tv \in B(0, 10)$ whenever $t \in [2^{k-1}, 2^k)$, and hence $\rho^* g(x, v) \gtrsim a_k 2^k$. Since the $\mathcal{A}_k^x$ are disjoint in $k$, and using the scaling condition $\frac{d}{p} + \frac{2}{q} = 2$, it follows that
\[ \|\rho^* g\|_{L^p_{\mathbb{R}^d}} \gtrsim \left( \sum_{k=0}^{\infty} (a_k 2^k)^{2p - k} d \right)^{\frac{1}{2p}} = \left( \sum_{k=0}^{\infty} (a_k 2^k)^{2p} \right)^{\frac{1}{2p}}. \]

On the other hand,
\[ \|g\|_{L^q_{\mathbb{R}^d}} \sim \left\| a_0 1_{[0, 1)}(t) + \sum_{k=1}^{\infty} a_k 1_{[2^{k-1}, 2^k)}(t) \right\|_{L^q_{\mathbb{R}^d}} \sim \left( \sum_{k=0}^{\infty} (a_k 2^k)^{2p} \right)^{\frac{1}{2}} \]
and hence
\[ \left( \sum_{k=0}^{\infty} (a_k 2^k)^{2p} \right)^{\frac{1}{p}} \lesssim \left( \sum_{k=0}^{\infty} (a_k 2^k)^{q} \right)^{\frac{1}{q}} \]
for any nonnegative and nonincreasing sequence $(a_k)_{k \geq 0}$. This implies $r \leq 2p$, as required. \hfill \Box

Remarks. (1) An alternative proof of (13) was given in [5]. The argument ultimately rested on an application of the $(d+1)$-linear generalization of the Hardy–Littlewood–Sobolev inequality (this was proved in [13] and is another special case of (4)) in the time variable. In light of the necessary condition (6), as it stands, this multilinear approach leads to the weaker Lorentz space estimate (15) with $r \leq d+1$.

(2) Regarding the existing literature for the classical estimates (14), the necessary condition $p > \frac{d+1}{2}$ was established through two separate proofs; it was shown in [17] that $p \geq \frac{d+1}{2}$ is necessary, whereas the endpoint $p = \frac{d+1}{2}$ was shown to fail in [5] by completely different means. We remark that the above Gaussian example on its own generates the necessary condition $p > \frac{d+1}{2}$. Our motivation for testing on Gaussians came from [5], where it was shown that the endpoint fails rather generically.
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Department of Mathematics, Graduate School of Science and Engineering, Saitama University, Saitama 338-8570, Japan
E-mail address: nealbez@mail.saitama-u.ac.jp

Department of Mathematical Sciences, Seoul National University, Seoul 151-747, Korea
E-mail address: shklee@snu.ac.kr

Department of Mathematics and Information Sciences, Tokyo Metropolitan University, 1-1 Minami-Ohsawa, Hachioji, Tokyo, 192-0397, Japan
E-mail address: pokopoko9131@icloud.com

Department of Mathematics and Information Sciences, Tokyo Metropolitan University, 1-1 Minami-Ohsawa, Hachioji, Tokyo, 192-0397, Japan
E-mail address: ysawano@tmu.ac.jp