Strength Distribution in Derivative Networks

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Abstract

This article describes a complex network model whose weights are proportional to the difference between uniformly distributed “fitness” values assigned to the nodes. It is shown both analytically and experimentally that the strength density (i.e. the weighted node degree) for this model, called derivative complex networks, follows a power law with exponent $\gamma < 1$ if the fitness has an upper limit and $\gamma > 1$ if the fitness has no upper limit but a positive lower limit. Possible implications for neuronal networks topology and dynamics are also discussed.

1 Introduction

Great part of the interest focused on complex networks [1, 2, 3] recently stems from scale free or power law distributions of respective topological measurements, such as the node degree. At the same time, weighted complex networks have attracted growing interest because of their relevance as models of several natural phenomena, with special attention given to systems used for distribution/collection of materials or information.

There are two main ways to approach the degrees of weighted networks: (i) by thresholding the weights and using the traditional node degree [1]; and (ii) by adding the weights of the edges attached to each node, yielding the respective node strength [4]. Related recent works include the identification of strength power law in word association networks [5, 6], studies of scientific collaborations and air-transportation networks [7], the analysis of amino acid sequences in terms of weighted networks [8], the investigation of weighted networks defined by dynamical coupling between topology and weights [9], analytical characterization of thresholded networks [8], as well as the characterization of motifs [10] and shortest paths in weighted networks [11].

The current article describes a network whose nodes have a respective “fitness” value and every node is connected to all other nodes with higher fitness through an edge whose weight is determined by the difference between the fitness of the nodes. Because of such a construction, these networks are henceforth called derivative complex networks. Previous works considering the difference of fitness values associated to the network nodes include the image analysis approach reported in [11], which involves the thresholding of the fitness difference values in order to perform image segmentation and the gradient networks discussed in [12, 13], which take into account the edges corresponding to the highest differences, i.e. to the gradient of a scalar field associated along the network nodes.

The developments in the current article represent a continuation and extension of preliminary investigations reported in [14]. We show both analytically and numerically that the strength densities for the derivative network model follow a power law with exponent $\gamma < 1$ if the fitness has an upper limit and $\gamma > 1$ if the fitness has no upper limit but a positive lower limit.

We start by describing the derivative network model and follow by calculating its respective strength distribution and discussing possible implications for neuronal networks.
2 Derivative Networks

Consider a network with $N$ nodes. To each node $i$, a “fitness” value $\varphi_i > 0$ is randomly assigned following a distribution $\rho(\varphi)$. The fitness derivative determines the connectivity of the nodes in the network. A directed arc linking node $j$ to node $i$ is drawn iff $\varphi_i > \varphi_j$; in that case, the arc has weight given by a function of the fitness difference:

$$w_{ji} = \sigma(\varphi_i - \varphi_j). \quad (1)$$

We are here interested in the study of the distribution of input strengths in the network under some assumptions on the functions $\rho$ and $\sigma$. The input strength of node $i$, $s_i$, is the sum of the weights of the arcs linked to $i$:

$$s_i = \sum_j w_{ji}, \quad (2)$$

where the sum is taken over all nodes $j$ that have an arc linked to node $i$.

From Eqs. (2) and (1), the strength of a node is determined by its fitness $\varphi$:

$$s(\varphi) = \sum_{\varphi' < \varphi} \sigma(\varphi - \varphi'); \quad (3)$$

where the sum is taken for all nodes with $\varphi'$ smaller than $\varphi$. Considering the distribution of fitness $\rho(\varphi)$, for the limit of large number of nodes $N$ we can write:

$$s = N \int_0^\varphi \rho(\varphi)\sigma(\varphi - x)\, dx. \quad (4)$$

For the weight function, we assume a power law as:

$$\sigma(x) = bx^{\beta-1}. \quad (5)$$

We are interested in the case where the weight grows with the difference in fitness, implying $\beta > 1$.

For the fitness distribution, a power law distribution will be also considered. In the following sections, distributions with an upper limit and with a positive lower limit will be considered.

2.1 Upper bound fitness

First we study the case where the values of fitness have a maximum value that we assume, without loss of generality, to be unitary: $0 < \varphi < 1$. For a power law distribution

$$\rho(\varphi) = \alpha \varphi^{\alpha-1} \quad (6)$$

to be a valid probability distribution in the interval $0 < \varphi < 1$, we must have $\alpha > 0$. From Eqs. (4), (5) and (6) we have:

$$s = N \int_0^\varphi \alpha x^{\alpha-1}(\varphi - x)^{\beta-1} \, dx, \quad (7)$$

resulting in:

$$s = N b a B(\alpha, \beta) \varphi^{\alpha + \beta - 1}, \quad (8)$$

where $B(a, b)$ is the Euler beta function:

$$B(a, b) = \int_0^1 t^{a-1}(1 - t)^{b-1} \, dt.$$ 

Note that this expression implies an upper limit for $s$ given by

$$s_{\text{max}} = N b a B(\alpha, \beta). \quad (9)$$

The probability density of $s$, $p(s)$, is given by:

$$p(s) = \rho(\varphi) \frac{ds}{ds}. \quad (10)$$

The derivative can be obtained from Eq. (4), giving:

$$p(s) = \frac{\alpha}{\alpha + \beta - 1} \left( \frac{1}{s_{\text{max}}} \right) s^{\beta - 1} \left( \frac{1}{s_{\text{max}}} \right) \varphi = \frac{\alpha}{\alpha + \beta - 1} s^{\beta - 1} s^{\beta - 1}.$$

This expression is only valid for $0 \leq s < s_{\text{max}}$, and is a power law $p(s) \sim s^{-\gamma_U}$ with

$$\gamma_U = 1 - \frac{\alpha}{\alpha + \beta - 1}. \quad (12)$$

Considering $\alpha > 0$ and $\beta > 1$ we have $0 < \gamma < 1$.

An interesting special case is that of uniform distribution, which corresponds to $\alpha = 1$ in Eq. (4), resulting in

$$\gamma_U = 1 - \frac{1}{\beta}.$$ 

In case we also have that the weight is proportional to the fitness difference, i.e. $\beta = 2$ in Eq. (12), then $\gamma_U = \frac{1}{2}$. 

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2.2 Lower bound fitness

We consider now the case where the fitness value has no upper limit, but a positive lower limit, that we assume, without loss of generality, to be unitary. In this case, we write the probability density as

$$\rho(\varphi) = \alpha \varphi^{-\alpha-1},$$

which is correctly normalized in the interval $1 < \varphi < \infty$ for $\alpha > 0$.

Using Eqs. (4), (5), and (13) we have:

$$s = N \int_{1}^{\varphi} \alpha x^{-\alpha-1} b(\varphi - x)^{\beta-1} \, dx,$$

giving:

$$s = N b \alpha \int_{1}^{\varphi} (\frac{1}{\varphi})^{1-\alpha, \beta} \varphi^{\beta-1-\alpha}.$$  \hspace{1cm} (15)

where $B(x, y, a, b)$ is the generalized incomplete beta function:

$$B(x, y, a, b) = \int_{x}^{y} t^{a-1} (1 - t)^{b-1} \, dt.$$  \hspace{1cm} (16)

For $\varphi \approx 1$, $B(\frac{1}{\varphi}, 1, -\alpha, \beta) \approx 0$ giving $s \approx 0$. For large $\varphi$, $B(\frac{1}{\varphi}, 1, -\alpha, \beta) \sim \varphi^\alpha$. Then

$$N b \alpha \int_{1}^{\varphi} (\frac{1}{\varphi})^{1-\alpha, \beta} \varphi^{\beta-1-\alpha} \approx c \varphi^\alpha.$$  \hspace{1cm} (17)

and therefore we get:

$$s \approx c \varphi^{\beta-1}$$

for large $\varphi$.

Proceeding in a similar fashion to the previous case we get:

$$p(s) \approx \frac{\alpha}{\beta - 1} \varphi^{\alpha - 1} s^{\frac{\beta - 1 - \alpha}{\beta - 1}}.$$  \hspace{1cm} (18)

valid for large $s$.

Considering that $\beta > 1$ and $\alpha > 0$, this expression corresponds to a power law $p(s) \sim s^{-\gamma_L}$ with $\gamma_L > 1$.

3 Simulation results

We simulated derivative networks with upper and lower limit for three sets of $\alpha$ and $\beta$ parameters each. Figure 2 shows the strength distribution for networks with upper fitness limit; figure 2 shows the results for networks with lower limited fitness. The results assumed $N = 2000$ nodes and $b = 0.0005$; a total of 100 networks were considered for each set of parameters, and the results show the mean values and respective standard deviations.

The curves obtained by the simulations resulted remarkably close to the analytical results for upper limited networks, except for the smallest strengths. This is because the strength distribution $p(s) \sim s^{-\gamma}$ approaches a discontinuity at zero.

Less adherence between theoretical and experimental values was observed for the lower limited networks as a consequence of the fact that the approximation in Eq. (18) is only valid for large values of $s$, which corresponds to small values of the probability $p(s)$, resulting in poor statistics.

4 Discussion

We have described two simple models of derivative networks and shown that their node instrength distributions follow a power law under some assumptions on distribution of fitness and weight assignment. This result implies that although several nodes of a derivative network have low strength values, there are hubs characterized by a variety of large instrength values. Because the instrength of a node can be understood as the total weighted influence it receives from the adjacent nodes, it follows that in case the node state is directly proportional to its instrength, the linear dynamics of such networks will be characterized by a near power law with similar parameters as those of the instrength density. Such an interpretation is particularly interesting from the perspective of integrating neuronal network and complex network research (e.g. [13]), where each neuron is represented by a node and the synaptic connections by directed edges whose weights reflect the respective synaptic strengths. The “fitness” values in this case could be related to gradient of neurotrophic growth factors or depolarization bias facilitating the action potential [16][17]. Another approach worth further attention would be to consider derivative networks as models of perceptual and cognitive processes, which are known to explore derivatives as a means of diminishing stimuli redundancy (e.g. [15]).
Figure 1: Strength distribution in upper limited networks. Experimental results are represented by points with error bars (one standard deviation) and the continuous curves are the theoretical expressions. Results for: (a) $\alpha = 2, \beta = 2$ ($\gamma = 1/3$); (b) $\alpha = 1, \beta = 2$ ($\gamma = 1/2$); (c) $\alpha = 1, \beta = 3$ ($\gamma = 2/3$).

Figure 2: Strength distribution in lower limited networks. Experimental results (points with error bars) are represented together with a continuous line displaying the theoretical asymptotic inclination. Results for: (a) $\alpha = 0.5, \beta = 1.5$ ($\gamma = 2$); (b) $\alpha = 1.5, \beta = 2$ ($\gamma = 2.5$); (c) $\alpha = 1, \beta = 1.5$ ($\gamma = 3$).
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