Energy-Resolved Information Scrambling in a Weakly-Interacting Fermi Gas

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Weakly interacting Fermi gases provide a rich platform for investigating information spreading and spin coherence in a large many-body quantum system with effective long-range interactions and conserved single-atom energies. We demonstrate a new protocol for the study of information scrambling, employing an inverse Abel-transform to extract energy-resolved out-of-time-order correlation functions and many-body coherence from the spatial profiles of the spin density.

Recently, it has been shown that measurements of certain out-of-time-order correlation (OTOC) functions can serve as entanglement witnesses and to quantify coherence and information scrambling in quantum many-body systems. Here, “scrambling” denotes the spread of quantum information over the many-body degrees of freedom, which becomes inaccessible to local probes. OTOC measurements have been performed by reversing the time evolution of the many-body state, which is achieved by reversing the sign of the Hamiltonian. Such protocols were first implemented in nuclear magnetic resonance experiments at high temperatures, where the initial state is described by a density operator and high order quantum coherence has been observed. New OTOC studies have been performed in trapped ion systems containing relatively small numbers of atoms, where the individual sites are nearly equivalent, and the initial state is pure. Related methods have been developed for systems containing up to 100 atoms, but the application to very large numbers remains a challenge.

OTOC measurements in trapped, weakly interacting Fermi gases offer new prospects for understanding information spreading in large quantum systems containing \(N \approx 10^5\) atoms with a tunable, reversible Hamiltonian. In weakly interacting Fermi gases, the s-wave scattering length \(a\) is adjusted to nearly vanish. The corresponding collision rate \(\propto |a|^2\) is negligible, so that single atom energies are conserved over the time scale set by the mean field frequency \(\propto |a|\), which can be 1 sec or more. The single particle energies label the “sites” of an effective energy-space lattice, which is suitable for exploring a wide variety of spin-lattice models. Interactions are effectively long range, which is well-suited for studies of fast information scrambling. However, measurements of the total spin vector obscure the contributions of non-equivalent spins, which exhibit energy-dependent evolution.

In this Letter, we report the demonstration of an energy-resolved OTOC protocol that exploits single atom energy conservation in a weakly interacting Fermi gas, which is confined in a spin-dependent harmonic trap. The method employs an inverse Abel-transform to extract the energy-dependent spin vector \(\mathbf{S}(E)\) from the spatial profile of the spin density \(\mathbf{S}(x)\), providing a new platform for the study of information scrambling in large many-body systems.

For a pure initial state \(|\psi_0\rangle\), the utility of the OTOC can be simply understood. Let \(\hat{W}\) and \(\hat{V}\) be two, generally time-dependent, unitary operators, \(U^\dagger U = 1\). Consider the two normalized states \(|\psi_1\rangle = \hat{W}|\psi_0\rangle\) and \(|\psi_2\rangle = \hat{V}|\psi_0\rangle\), where the operators are applied in reverse order. We define the overlap \(F \equiv \langle \psi_2|\psi_1\rangle = \langle \psi_0|\hat{W}^\dagger \hat{V}^\dagger \hat{V} \hat{W}|\psi_0\rangle\), which is of the OTOC form. Suppose that \(\hat{W}\) and \(\hat{V}\) commute at time \(t = 0\). Then \(F(0) = 1\). Using \(2 \text{Re}\{F\} = F + F^*\), it is straightforward to obtain \(F(t) = 1 - \text{Re}\{F\}\). From this, we see that if the operators \(\hat{W}\) and \(\hat{V}\) do not commute for \(t > 0\), then \(\text{Re}\{F(t)\} < 1\). Thus, a measurement of \(\text{Re}\{F\}\) determines how two initially commuting operators fail to commute at a later time in a many-body system, providing a measure of information scrambling.

The Hamiltonian for a weakly interacting Fermi gas, confined in a cigar-shaped optical trap along the x-axis, takes the form of a spin “lattice” in energy space, with

\[
H(a) = -\sum_i \Omega_i \mathbf{s}_i + a \sum_{i,j \neq i} g_{ij} \mathbf{s}_i \cdot \mathbf{s}_j
\]

in units of s\(^{-1}\). Here, \(\mathbf{s}(E_i) \equiv \mathbf{s}_i\) is the dimensionless collective spin vector for atoms of energy \(E_i\), the energy of the \(i\)th axial harmonic oscillator state, which plays the role of the “site” \(i\). The site-to-site interaction is governed by the tunable s-wave scattering length \(a\) and by \(g_{ij} \propto \int dx |\phi_{E_i}(x)|^2 |\phi_{E_j}(x)|^2\), the overlap of the probability densities for colliding atoms of energies \(E_i\) and \(E_j\), which produces an effective long range interaction \(\propto 1/\sqrt{|E_i - E_j|}\). The effective Zeeman interaction is site-dependent, \(\Omega(E_i) \equiv \Omega_i = \Omega'(E_i - E) + \Delta\). Here, \(\Omega'(E_i - E)\) scales linearly with energy, and arises from the curvature of the bias magnetic field, which produces a difference between the harmonic oscillation frequencies of the \(\uparrow\) and \(\downarrow\) states in the net weakly confining axial potential. \(\Delta\) is a global radio-frequency detuning, where \(\Delta = 0\) corresponds to resonance for atoms of mean energy \(E\). As the bias field curvature has a negligible effect on the tight transverse (y, z) confining potential, the system is effectively one-dimensional.

Our experimental protocol for measuring \(F\) is shown in Fig. \(\text{H}\) and defines the operators \(\hat{W}\) and \(\hat{V}\). Physically, \(\hat{W}\) applies a rotation to the total interacting spin system in between the forward and time-reversed evolu-
tions. The operator $\hat{V}$ performs a measurement to diag-
nose the effects of the rotation on the spins of energy $E_i$, i.e., at "site" $i$ in energy space. We employ a degener-
ate, weakly interacting cloud of $^4$Li, in a bias magnetic field near 527.18 G, where the $s$-wave scattering length $a$ vanishes [11]. The two lowest hyperfine-Zeeman states are denoted by $|1\rangle \equiv |↑_z\rangle$ and $|2\rangle \equiv |↓_z\rangle$. We start with a fully $z$-polarized state $|z_{1z_2\ldots z_N}\rangle \equiv |ψ_0\rangle$ in a bias magnetic field $B_1 = 528.53$ G, which selects an initial scattering length $a_1 \equiv a = 4.24 a_0$. In the ideal case, for negligible detuning $Δ = 0$, Fig. 1(a), we apply a 0.5 ms radio-frequency ($π/2$)$_y$ pulse (defined to be about the $y$-
axis), which is resonant with the $|↓_z\rangle \rightarrow |↑_z\rangle$ transition at the bias field $B_1$, to produce an initial $x$-polarized N-
atom state $|ψ_0\rangle = e^{-iπS_y} |ψ_0\rangle = |↑_{x_1}↑_{x_2}\ldots ↑_{x_N}\rangle$. The system evolves for a time $τ = 200$ ms at the initial bias magnetic field $B_1 = 528.53$ G. Then, a resonant radio-
frequency pulse $|φ\rangle_x$, shifted in phase from the first pulse by $π/2$, rotates the N-atom state about the $x$-axis [13] by a chosen angle $φ$. Immediately following this rotation, we reverse the sign of the Hamiltonian by applying a $(π)_{y}$ pulse and tuning the bias magnetic field to a value $B_2 = 525.83$ G, where the scattering length $a_2 = -a$, i.e., $e^{iπS_y}H(-a)e^{-iπS_y} = -H(a)$, from Eq. 1. After the system evolves for an additional time $τ$, the bias field is ramped back to $B_1$, and a final $(π/2)_{y}$ pulse is applied [13]. The final state of the N-atom system after the pulse sequence of Fig. 1(a) can be written as

$$|ψ_f\rangle = e^{-iπS_y}\hat{W}_φ(τ)|ψ_0\rangle,$$

where the $\hat{W}$-operator is defined by

$$\hat{W}_φ(τ) = e^{iH(a)τ}e^{-iφS_y}e^{-iH(a)τ},$$

with $S_x = \sum_{j=1}^{N} s_{xj}$ the $x$-component of the total spin vector and $|ψ_0\rangle$ the fully $x$-polarized state. After the pulse sequence, the spin densities $n_{↑}(x)$ and $n_{↓}(x)$ are measured for a single cloud using two resonant absorption images, separated in time by 10 μs. We define one repeti-
tion of this experimental sequence as a "single-shot," in Fig. 1(b) and (c).

The measured $z$-component of the $i$th spin is

$$\langle ψ_f|\hat{s}_z|ψ_f\rangle = \langle ψ_0|\hat{W}_φ(τ)^{*}\hat{s}_z\hat{W}_φ(τ)|ψ_0\rangle$$

from Eq. 2 with $e^{iπS_y}\hat{s}_z e^{-iπS_y} = \hat{s}_z$. $\langle ψ_f|\hat{s}_z|ψ_f\rangle$ is of the same form as $F$ above, if we choose $V = 2 \hat{s}_x = \hat{σ}_{x_i}$, i.e., the $x$-
Pauli matrix for the $i$th spin, which commutes with $S_x$ and hence with $\hat{W}_φ(0)$. With this choice, $\hat{V}^\dagger = \hat{V}$ and

$$\hat{V}^\dagger \hat{V} = \hat{σ}_x^2 = 1$$

are required. Further, $\hat{V}|ψ_0\rangle = |ψ_0\rangle$, so that $F_i \equiv 2 \langle ψ_0|\hat{W}_φ(τ)^{*}\hat{s}_x\hat{W}_φ(τ)|ψ_0\rangle = 2\langle ψ_f|\hat{s}_x|ψ_f\rangle$ for the $i$th spin of axial energy $E_i$, and $F_i$ is real.

In the experiments, we measure $F_i$, with $i$ summed over a subset $N_x$ of spins with nearly the same energy. This determines the $φ$-dependent, energy-resolved, mean-
square commutator,

$$1 \sum_{i=1}^{N_x} \langle ψ_0|[\hat{W}_φ(τ), \hat{s}_x]|ψ_0\rangle = \frac{1}{N_x} \sum_{i=1}^{N_x} \langle ψ_f|\hat{s}_x|ψ_f\rangle = 2.$$  (4)

Restricting the OTOC measurement to atoms with energies within $∆E$ of a chosen energy $E$, the second term on the righthand side of Eq. 4 is $S_z(∆E)/|n(E)∆E|$, and

$$F(E, φ) \equiv \frac{1}{2}\frac{n_{↑}(E, φ) - n_{↓}(E, φ)}{n_{↑}(E, φ) + n_{↓}(E, φ)} = 1/2.$$  (5)

where $n(E) = n_{↑}(E, φ) + n_{↓}(E, φ)$ is independent of $φ$ and $F(E, 0) = 1/2$.

Absorption images determine the spatial profiles $n_{↑}(x, φ)$ and $n_{↓}(x, φ)$ of single clouds, after each repeti-
tion of the pulse sequence of Fig. 1(a) with a chosen rotation angle $φ$. To reveal the corresponding profiles in energy space, as needed to measure $F(E, φ)$, we as-
sume that there is no energy-space coherence, i.e., for each spin state $s$, $n_{s}(x, φ) = \sum_{E} |φ_{E}(x)|^2 n_{E}(E, φ)$. This assumption has been used previously in our mean field model of a weakly interacting Fermi gas [8, 11], which very well fits the small scale spatial structure in the measured spin-density profiles [11] and is consistent with the observed time-independence of the spatial profile for the total atom density [11, 19], which remains thermal. For a
harmonic trap, the $\phi_E(x)$ are harmonic oscillator states, and the equation for $n_\pi(x,\phi)$, in a continuous limit, corresponds to an integral Abel-transform that can be inverted [19, 20].

Fig. 1(b) shows the difference of the measured single-shot spin-up and spin-down densities, i.e., $S_\pi(x,\phi)$ in units of the central total spin density, for $a = 4.24 a_0$ and $\phi = \pi$. Fig. 1(c) shows the extracted single-shot $S_\pi(E,\phi)$ from the inverse Abel transform. We see that $S_\pi(E,\phi)$ appears smooth compared to the single shot spin density $S_\pi(x,\phi)$, which requires averaging over several shots to obtain a smooth profile. To check that the inverse Abel transform has adequate energy resolution, we generate a smooth profile. To check that the inverse Abel transform has adequate energy resolution, we generate a profile consistent with the input spatial profile [19].

We can extract information about the many-body coherence from Eq. 4 by writing the sum on the right-hand side as

$$\frac{1}{N_s} \sum_{\tau=1}^{N_s} \langle \psi_0 | \hat{W}_\phi(\tau) \hat{s}_\pi(\tau) | \psi_0 \rangle = \sum_m e^{im\phi} B_m.$$  

(6)

Non-vanishing coefficients $B_m$ correspond to coherence between states for which the $x$-component $S_x$ of the total angular momentum differs by $m$ [11]. Since the left side of Eq. 6 is real, $B_{-m} = B_m^\ast$. We can expand Eq. 5 for the measured, energy-selected average in the form

$$F(E,\phi) = B_0 + \sum_{m \geq 1} 2|B_m| \cos(m\phi + \varphi_m).$$  

(7)

In fitting the data with Eq. 7, we restrict the range of $m$ to 4. We find that the fits are not improved by further increase of $m$, consistent with the limited number of $\phi$ values measured in the experiments.

Fig. 2(a) shows $F(\phi)$ for the total number of atoms, measured without energy resolution. In this case, Eq. 5 for $F(E,\phi) \rightarrow F(\phi) \equiv \frac{1}{2} [N_{\uparrow\pi}(\phi) - N_{\downarrow\pi}(\phi)]/[N_{\uparrow\pi}(\phi) + N_{\downarrow\pi}(\phi)]$, with $N_{\uparrow\pi}$ and $N_{\downarrow\pi}$ the total numbers in each spin state. The data (blue dots) are an average of 6 repetitions of the entire $\phi$ sequence, with a fixed scattering length $a = 4.24 a_0$. Fig. 2(a) (red curve) shows the fit of Eq. 7 to the measured $F(\phi)$, which determines the magnitude (b) and phase (c) of the coherence coefficients $B_m$.

We can compare the $\phi$-dependent data of Fig. 2 to the mean field model of Ref. [11], which predicts the red curve shown in Fig. 2(d). For the mean field model, the global detuning is $\Delta = 0$. However, the model requires a scattering length that is 2.63 times larger than the measured value to fit the observed $\phi$ dependence.

Fig. 3 shows the energy-resolved measurements obtained from the same data, using the inverse Abel-transform to determine $F(E,\phi)$. The top row shows the $\phi$ dependence for several different energies $E$. The shapes of the profiles vary significantly with energy $E$, changing in symmetry and structure as the energy is varied from $E = 0$ to $E = 0.7 E_F$. The red curves in the first row show the fit of Eq. 7 which yields the magnitudes of the coherence coefficients $|B_m|$ shown in the second row and the corresponding phases $\varphi_m$ shown in the third row. In the last row, we again compare the data to the predictions of the mean field model of Ref. [11]. Using the detuning as a free parameter, the mean field model is able to capture the complex $\phi$-dependent shapes of the data. However, we again find that the scattering length must be increased over the measured value by a factor of 2.63 and a different detuning is needed for each energy to produce the structure observed in the data obtained using the OTOC protocol.

In summary, energy-resolved measurements of OTOC’s in a spin-dependent harmonic trap reveal a rich coherence structure that is hidden in measurements of the total collective spin vector. Remarkably, the measured $\phi$-dependent structure of the energy-dependent collective spin vector is consistent with the predictions of the mean field model of Ref. [11], using the scattering length and the detuning as free parameters. However, we find that the $\phi$-dependence predicted using the measured scattering length is much too smooth. In contrast, using the measured scattering length without adjustment, the same model is in close agreement with the complex spatial structure of the spin density...
FIG. 3. Energy-resolved collective spin projection $S_z(E)$ versus rotation angle $\phi$ for spins of selected energies (left to right) $E/E_F = 0, 0.15, 0.25, 0.5, 0.7$. Here, $\mathcal{F}(\phi) = \frac{1}{2} |n_\uparrow(E) - n_\downarrow(E)| / [n_\uparrow(E) + n_\downarrow(E)]$. The top row shows the data (blue dots) for a measured scattering length $a = 4.24 a_0$. The red curve is the fit of Eq. 7, which determines the magnitudes of the coherence coefficients $|B_m|$ (second row) and corresponding phases $\varphi_m$ (third row); The bottom row shows the predictions (red curves) of the mean field model of Ref. [11] to the data (blue dots), using a scattering length 2.63 times the measured value and global detunings, ordered in energy, of $\Delta(\text{Hz}) = 0, 0.8, 0.65, -0.8, 0.15$.

Profiles $S_z(x)$ observed in single pulse experiments [11], which are independent of the detuning. These results suggest that a beyond mean field treatment is needed to correctly predict the observed OTOC measurements. For the measurements presented here, the OTOC was measured as a function of rotation angle $\phi$ at a fixed time $\tau = 200$ ms. By performing the same measurements for fixed $\phi$ as a function of $\tau$, it will be possible in future work to study the dynamics of information scrambling by mapping out the OTOC as a function of $\tau$.

The new methods pave the way for microscopic measurements of time-dependent information scrambling in large quantum systems, for example by observing the build-up of correlations between spins in different energy-space partitions. The ability to access energy-resolved OTOCs under a wide variety of conditions in a large system with long-range interactions will motivate new cold atom approaches to simulate “out-of-equilibrium” dynamics in spin-lattice systems [21], information propagation by site-resolved measurements [22], “fast scrambling” [17], and quantum gravity [23].

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Appendix A: Supplemental Material

In this supplemental material, we describe the experiments, the coherence measurement method, and the inverse-Abel transform method for extracting the energy-dependent collective spin vector from the spatial profile of the spin density.

1. Experiment

To implement the many-body echo protocol, shown in Fig. 1 of the main text, we prepare a z-polarized initial state in a cloud of $^6$Li, with all atoms in the spin-down hyperfine state [2]. To prepare this state, we begin with a 50-50 mixture of the two lowest hyperfine states, denoted as $|1\rangle$ and $|2\rangle$, which is evaporatively cooled to degeneracy near
the $|1\rangle - |2\rangle$ Feshbach resonance at 832.2 G. The magnetic field is then ramped to the weakly interacting regime near 1200 G, and the $|1\rangle$ spin component is eliminated by means of a resonant optical pulse. Then the bias magnetic field is ramped near 527.18 G, where the s-wave scattering length vanishes \[11\].

We employ a cigar-shaped optical trap with parameters close to those employed in our previous work \[11\]. The typical total atom number is $N = 6.5 \times 10^4$. By fitting a finite temperature Thomas-Fermi profile to the measured total spatial density, we find that $T/T_F = 0.38$, where the Fermi temperature for our trap parameters is $T_F = 0.58 \mu K$ and the corresponding Thomas-Fermi radius is $\sigma_{TF} = 278 \mu m$. By fitting a zero-temperature spatial profile to the measured total spatial density, we find the effective Thomas-Fermi radius, $\sigma = 360 \mu m$, which is used in the mean field model \[11\]. The scattering length is determined by measured tuning rate of $3.14a_0/G$ \[11\] and the bias magnetic field, which is precisely measured by rf spectroscopy.

At a bias magnetic field $B_1 = 528.53$ G, where the s-wave scattering length $a = 4.24a_0$, we apply a 0.5 ms radiofrequency pulse to rotate the spin state by $\pi/2$ about the y-axis, preparing an x-polarized state. After an evolution time $\tau = 200$ ms, we rotate the state by an angle $\phi$ about the x-axis, using a radio-frequency (rf) pulse, shifted in phase from the first pulse by $90^\circ$. Immediately following this pulse, the Hamiltonian is inverted by applying a $\pi$ rotation about the y-axis and sweeping of the bias magnetic field over 5 ms to a value $B_2 = 525.83$ G where the scattering length is $-a$. This sweep, over a few gauss, is accomplished using a set of low inductance coils, wound concentric with the primary bias field coils. After an additional $\tau = 200$ ms, the bias magnetic field is swept back to its original value $B_1$ over 5 ms and a final $\pi/2$ rotation about the negative y-axis is applied. The density profiles of both spin components are then immediately measured for a single cloud, using two camera shots separated by $10 \mu s$. This defines a single-shot measurement. Subtraction yields the single-shot z-component of the collective spin vector density $S_z(x)$, which, in the ideal case, corresponds to the x-component just prior to the final $\pi/2$ pulse.

a. Detuning

The ideal implementation of the protocol of Fig. 1 of the main text, as described above, assumes a global detuning $\Delta = 0$. In the actual experiments, the global detuning $\Delta$ is near resonance at the initial bias magnetic field $B_1$, but changes by several kHz when the bias magnetic field is tuned to $B_2$ for 200 ms. This results in a large, but reproducible phase shift. To compensate, we choose the time for the final $\pi/2$ pulse to be delayed by a time $\tau_f$ of several ms after we begin the sweep of the magnetic field from $B_2$ back toward its original value $B_1$. This provides adequate time for the frequency detuning to return to a nonzero value well-within the pulse bandwidth. In this final low frequency detuning region, we find that a delay of $\tau_f = 10$ ms produces a stable net phase shift of $180^\circ$ (modulo $2\pi$) and a maximum transfer of atoms from the initially populated state 2 to the initially unpopulated state 1 for $\phi = 0$, i.e., a $-\pi/2$ pulse about the y-axis as noted above. The negative sign is taken into account in the data analysis.

To set the radiofrequency detuning close to resonance at the field $B_1$, we initially find the resonance frequency for the radiofrequency pulses, by observing the transfer of atoms from state 2 to state 1 using a single 50 ms pulse. The observed linewidth is 8 Hz half width at half maximum, enabling an approximate determination of the $\Delta = 0$ frequency within 1 Hz. To keep the rf frequency nominally on resonance as data is collected, for each choice of $\phi$, we consistently check that the $\phi = 0$ configuration produces maximum transfer of atoms from state 2 to state 1 at the end of the 400 ms total sequence. If not, the rf frequency is slightly changed to compensate for magnetic field drift, which changes the resonance frequency by $\sim 3.6$ Hz/mG.

For the data used in the measurements, we estimate the effective detuning from resonance for single shots at several values of $\phi$ by fitting our mean field model \[11\] to the spatial profiles, as shown in Fig. S1. Fig. S2 shows the sensitivity of the fits to $\Delta$ for $\phi = \pi$. From the fits to single shots, we estimate that the maximum detuning $\Delta$ is $< \pm 1.0$ Hz for the data shown in the main text.

Drifts in the radiofrequency detuning $\Delta$ are partially mitigated by the $\pi$ pulse at the center of the protocol of Fig. 1 of the main paper, which reverses the net accumulated phase at time $\tau$ for a fixed detuning. If the detuning is stable over the 400 ms duration of the sequence, this accumulated phase is cancelled. Further, we compensate for the phase shift arising from the magnetic field sweep between $B_1$ and $B_2$, as discussed above. In the following, we discuss the effect of the detuning on the determination of the coherence coefficients from the $\phi$-dependent spin density.

b. Many-Body Coherence Measurement

To understand the effect of finite global detuning $\Delta$ on the coherence coefficients, consider the measurement of the z-projection of the $i^{th}$ spin after the pulse sequence, where $i$ denotes an atom of axial energy $E_i$. As in the main text,
FIG. S1. Estimating the radiofrequency detuning from the measured single-shot spin density profiles. Here, the spin density \( \Delta n(x) = n(x) - n(0) \) is given in units of the central density \( n(0) \) and the measured scattering length is \( a_{\text{meas}} = 4.24 \, a_0 \). The single shot data (blue dots) are fit with the mean field model (red curves), using the detuning \( \Delta \) as a fit parameter. (a) \( \phi = 0.64 \, \pi, \, a_{\text{fit}} = 2.35 \, a_{\text{meas}}, \, \Delta_{\text{fit}} = 0 \times 2\pi \, \text{rad/s}; \) (b) \( \phi = 1.18 \, \pi, \, a_{\text{fit}} = 2.5 \, a_{\text{meas}}, \, \Delta = 0.1 \times 2\pi \, \text{rad/s}; \) (c) \( \phi = 1.63 \, \pi, \, a = 2.4 \, a_{\text{meas}}, \, \Delta = 0.1 \times 2\pi \, \text{rad/s}. \) Note that the model requires a scattering length that is nominally 2.4 times the measured value to fit the data.

for the final many-body state \( |\psi_f\rangle \), we define

\[
\mathcal{F}_i(\phi) \equiv 2 \langle \psi_f | \hat{s}_{zj} | \psi_f \rangle = 2 \langle \psi_0 | \hat{W}_\phi^{(i)}(0) \hat{s}_{zj} W_\phi(0) | \psi_0 \rangle.
\]

(S1)

On the righthand side, \( |\psi_0\rangle \) is the fully \( x \)-polarized state, obtained after the first \( \pi/2 \) pulse and \( \hat{W} \) is defined by Eq. 3 of the main text,

\[
\hat{W}_\phi(\tau) = e^{iH(a)\tau} e^{-i\phi S_z} e^{-iH(a)\tau}.
\]

(S2)

To explicitly display the detuning dependence of the measurement, we write the Hamiltonian in Eq. 1 of the main text as \( H(a, \Delta) \equiv H(a, 0) - \Delta S_z \), where \( S_z = \sum_{j=1}^{N} \hat{s}_{zj} \) is the \( z \)-component of the total spin vector. Then, since \( \{H(a, 0), S_z\} = 0 \), we have

\[
\hat{W}_\phi(\tau) = e^{iH(a, 0)\tau} e^{-i\Delta \tau S_z} e^{-i\phi S_z} e^{i\Delta \tau S_z} e^{-iH(a, 0)\tau} = e^{iH(a, 0)\tau} e^{-i\phi S_z} e^{iH(a, 0)\tau}.
\]

(S3)

Here, the phase shift \( \Delta \tau \) is accumulated during the time \( \tau \) between the first \( \pi/2 \) pulse and the \( \phi \) rotation. We see that a nonzero detuning changes the axis for the \( \phi \) rotation from \( x \) to \( x' \), with \( S_{x'} = S_x \cos(\Delta \tau) + S_y \sin(\Delta \tau) \). Note that the angle between the \( x' \) and \( x \) axes for a typical single-shot with \( \Delta = 0.4 \) Hz and \( \tau = 0.2 \) s is \( \Delta \tau \approx 0.5 \) rad.

For each detuning \( \Delta \), we can expand Eq. (S3) for \( \hat{W}_\phi(\tau) \) in a total angular momentum eigenstate basis \( |J, M\rangle_{x'}, \) with \( S_{x'} |J, M\rangle_{x'} = M |J, M\rangle_{x'}, \) where we suppress all other quantum numbers that define the states, such as intermediate angular momenta. Then, Eq. (S1) for the \( i \)th spin can be written as

\[
\mathcal{F}_i(\phi) = 2 \sum_m B_{m}^{(i)} e^{im\phi}.
\]

(S4)
where the integer \( m = M' - M \) is the difference of the total angular momentum projections along the \( x' \) axis, and

\[
B_m^{(i)} = \sum_{J, J', M} x' \langle J M | \hat{\rho}(\tau) | J' M + m \rangle_{x'} x' \langle J' M + m | \hat{s}_{x1}(\tau) | J M \rangle_{x'}. \tag{S5}
\]

Here, \( \hat{\rho}(\tau) = e^{-iH(a, 0)\tau} |\psi_0\rangle \langle \psi_0| e^{iH(a, 0)\tau} \) is the density operator at time \( \tau \) and \( \hat{s}_{x1}(\tau) = e^{-iH(a, 0)\tau} \hat{s}_{x1} e^{iH(a, 0)\tau} \). For \( \phi = 0 \), using the completeness of the total angular momentum states, we have \( 2 \sum_m B_m^{(0)} = 2 \langle \psi_0 | \hat{s}_{x1} | \psi_0 \rangle = 1 \). Further, \( B_m^{(0)} = B_m^{(0)*} \), as required for real \( F_1(\phi) \).

Without interactions, \( a = 0 \), the Hamiltonian reduces to an energy-dependent rotation about the \( z \)-axis. As \( \hat{s}_{x1} \) is a rank one operator, for \( a = 0 \), \( m = 0, \pm 1 \) only, corresponding to the \( \phi \)-dependent projection of each spin along the \( x \)-axis. However, for the interacting system, \( a \neq 0 \), collisions create coherence between spins with different energies, and hence between states with \( |m| = |M' - M| > 1 \).

In the experiments, we measure the sum of Eq. S4 over atoms with an energy near \( E \), given by Eqs. 6 and 7 of the main text, which contain the coefficient

\[
B_m = \frac{1}{N_s} \sum_{i=1}^{N_s} B_m^{(i)}. \tag{S6}
\]

For an average of several shots with varying detunings, as utilized in the experiments to measure the \( \phi \) dependence of the spin density, the expansion coefficients \( B_m^{(i)} \) of Eq. S5 are simply averaged over a range of rotation axes \( x' \). This axis averaging, and the sum over a small range of spin energies near \( E \) in Eq. S6, will not change the general \( \phi \)-dependent structure of Eq. S4 which enables measurements of the average coherence coefficients, as shown in the main text.

### 2. Inverse Abel-Transform Method

The energy-dependent collective spin vector \( \mathbf{S}(E, t) \) is determined from the measured spatial profile of the spin density \( S(x, t) \) by employing an inverse Abel-transform method. In using this method, we are assuming that the measured axial spin density profiles are given in the continuum limit by \[11\],

\[
S(x, t) = \int dE |\phi_E(x)|^2 \mathbf{S}(E, t), \tag{S7}
\]

which is defined so that \( \int dx S(x, t) = \int dE \mathbf{S}(E, t) \) is the total collective spin vector. An important feature of Eq. S7 is the assumption that there is no coherence between states of different energy, which is justified in the energy-conserving regime of a very weakly interacting Fermi gas. Physically, each atom remains on its respective energy “site,” \( E_i \). As shown in Ref. \[11\], this assumption yields predictions in very good agreement with the small scale spatial structure observed in the spin density \( S_z(x) \) for single pulse experiments.

To illustrate these ideas, Fig. S3 shows the single-shot spin density profiles taken after the full OTOC pulse sequence of Fig. 1 of the main paper with \( a = 4.24 a_0 \) and \( \phi = \pi \) (blue dots). Despite the complex structure observed in the spatial profiles for the individual spin densities, which arises from spin coherence, the total density, shown on the right hand side, remains in a thermal distribution, consistent with the assumption of no energy-space coherence.

In the continuum limit, where the harmonic oscillator energy level spacing is small compared to the energy scale, the harmonic oscillator wave functions can be evaluated using a WKB approximation. In this case, the probability densities take the simple form \[11\],

\[
|\phi_E(x)|^2 = \frac{\tilde{\omega}_x}{\pi} \int_0^{\infty} dp_x \delta \left( E - \frac{p_x^2}{2m} - \frac{m \tilde{\omega}_x^2}{2} x^2 \right). \tag{S8}
\]

Then the spin density is given by

\[
S(x, t) = \frac{\tilde{\omega}_x}{\pi} \int_0^{\infty} dp_x S \left( \frac{p_x^2}{2m} + \frac{m \tilde{\omega}_x^2}{2} x^2, t \right), \tag{S9}
\]

which is an Abel-transform, i.e., the \( y \)-integral of a function of \( x^2 + y^2 \). Hence, an inverse Abel-transform enables a determination of the energy-dependent collective spin component \( S_z(E, t) \) from the measured spatial profile \( S_z(x, t) \). To extract \( S_z(E) \) from the data, the measured spatial profile is first symmetrized by folding about \( x = 0 \) and then
FIG. S3. Spin density profiles measured for a single shot with $a = 4.24\ a_0$ and $\phi = \pi$ (blue dots) in units of the central density $n(0)$. (a) $n_\uparrow(x, \phi = \pi)$; (b) $n_\downarrow(x, \phi = \pi)$; (c) Difference of the density profiles $S_z(x, \phi = \pi) = [n_\uparrow(x, \phi = \pi) - n_\downarrow(x, \phi = \pi)]/n(0)$; (d) Total density $n(x) = n_\uparrow(x, \pi) + n_\downarrow(x, \pi)$ in units of the central density $n(0)$. Despite the complex spatial structure in the individual spin density profiles, the total density remains thermal. The red curves show the predictions of the mean field model of Ref. [11] using a scattering length 2.35 times the measured value of $4.24\ a_0$ and a global detuning of $0.27\ Hz$, i.e., $\Delta = 2\pi \times 0.27\ rad/s$.

FIG. S4. Testing the inverse Abel-transform method. Using a mean field model, spin density “data” (a) for $S_z(x, \phi = \pi)$ are generated for the protocol of Fig. 1 of the main paper, with the same x-spacing as the actual data. Inverse Abel-transformation (right) yields $S_z(E, \phi = \pi)$ (b), which closely matches the input $S_z(E, \phi = \pi)$ (red curve) from the mean field model, which was used to generate the model data for the spin density spatial profile. An inverse Abel-transform is implemented without employing derivatives by using the method described in Ref. [20]. For this method, the unknown radial distribution is expanded in a series of cosine-functions, the amplitudes of which are calculated by least-squares-fitting of the Abel-transformed series to the measured data. In our analysis, we use up to 20 cosine terms.

To test the inversion method, we generate model “data” for $S_z(x, \phi = \pi)$, Fig. S4, with the same x-spacing as the real data. Here, $S_z(x, \phi = \pi)$ is given by Eq. S9 using the mean field model of Ref. [11] to determine $S_z(E, \phi = \pi)$ for $\phi = \pi$, scattering length $a = 2.35 \times 4.24\ a_0$, and global detuning $\Delta = 2\pi \times 0.27\ rad/s$, as used in the fits of Fig. S3. Inverting the model data for $S_z(x, \phi = \pi)$, we find the result shown as the blue curve of Fig. S4. For the inversion, we start with a small number of cosine terms and increase the number until the agreement with the exact input $S_z(E, \phi = \pi)$ (red curve) shows no further improvement. Using 20 cosine terms, we find that the $S_z(E, \phi = \pi)$ obtained from the spatial profile by inversion (blue) is in close agreement with the exact input from the mean field model (red curve) that was used to generate the spatial profile.

Next, we apply the Abel-transform method to find the energy-dependent spin component $S_z(E, \phi)$ from the measured spin density $S_z(x, \phi)$, Fig. S5. We check the consistency of the extracted $S_z(E, \phi)$ by Abel-transformation, which generates $S_z(x, \phi = \pi)$ for comparison to the input data Fig. S5(c).

For single-shot measurements, the shape of the $S_z(E, \phi = \pi)$ curve predicted by the mean field model is in reasonable agreement with the data at low energy, but fails to fit the slower oscillation observed at higher energy as shown in Fig. S6. Further, the scattering length used in the model is 2.35 times the measured value.
FIG. S5. Extracting the energy-dependent collective spin component $S_z(E, \phi = \pi)$ for a single shot. (a) Measured single-shot spin density $S_z(x, \phi = \pi)$ for the protocol of Fig. 1 with $a = 4.24 \ a_0$ and $\phi = \pi$. (b) Inverse Abel-transformation of (a) yields $S_z(E, \phi = \pi)$ (red dots). (c) $S_z(x, \phi = \pi)$ (red curve) generated from the extracted $S_z(E, \phi = \pi)$ is consistent with the input spin density data (blue dots).

FIG. S6. Comparison of the extracted energy-dependent collective spin component $S_z(E, \phi = \pi)$ for a single shot with the mean field model. (a) Measured single-shot spin density $S_z(x, \phi = \pi)$ for the protocol of Fig. 1 with $a = 4.24 \ a_0$ and $\phi = \pi$. (b) Inverse Abel-transformation of (a) yields $S_z(E, \phi = \pi)$ (blue dots). The red curve is the prediction of the mean field model of Ref. [11] $S_z(E, \phi = \pi)$ (red curve) with an increased scattering length $a = 2.35 \times 4.24 \ a_0$ and a global detuning $\Delta = 2\pi \times 0.27$ rad/s.