Identification of Sparse Continuous-Time Linear Systems with Low Sampling Rate: Optimization Approaches

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Abstract—This paper addresses identification of sparse linear and noise-driven continuous-time state-space systems, i.e., the right-hand sides in the dynamical equations depend only on a subset of the states. The key assumption in this study, is that the sample rate is not high enough to directly infer the continuous time system from the data. This assumption is relevant in applications where sampling is expensive or requires human intervention (e.g., biomedicine applications). We propose an iterative optimization scheme with \( l_1 \)-regularisation, where the search directions are restricted those that decrease prediction error in each iteration. We provide numerical examples illustrating the proposed method; the method outperforms the least squares estimation for large noise.

I. INTRODUCTION

This paper addresses identification of sparse linear and noise-driven continuous-time state-space systems, i.e., the right-hand sides in the dynamical equations depend only on a subset of the states. We assume that the entire state of the system can be measured. The key assumption in this study is that the sample period is not small enough to directly infer the continuous-time system from the data. This study is pertinent, as many dynamical systems have a sparse structure. Applications range from internet protocols \cite{1} to biological systems such as the interactions of genes and proteins in human cells \cite{2}–\cite{4}.

For a sparse dynamical system, properties such as connectivity and stability (and rate of convergence) can be studied. In particular, the property of reaching consensus has been thoroughly studied, see for example \cite{5}, \cite{6}. Such properties can only be investigated if the system is known (or at least the sparse structure thereof). If the system is unknown, we need to estimate it from data samples, the task conducted in this paper. This area, i.e., estimating continuous time system from discrete data is an important part of of the system identification field, see e.g. \cite{7}, \cite{8}, or \cite{9}. Also the MATLAB System Identification Toolbox, \cite{10} by default returns continuous time models from most of its estimation routines. The typical approaches is a direct method to directly infer a continuous-time system from data and an indirect method, where a discrete time model is first estimated and then converted to continuous time (e.g. “c2d” in MATLAB).

The assumption on the sampling period of being “small enough” is important for the direct method to work. The assumption here, in this work, is that the sampling period is not small enough. To also address this case is important, especially in applications where sampling is expensive or requires human intervention (medical applications). Many sparse systems are nonlinear. However, as we show, even for the restricted class we consider here – linear noise-driven dynamical systems evolving in continuous time with full state measurements – there are issues to address.

We propose an optimization scheme for the problem at hand. First we formulate a nonlinear non-convex optimization problem where sparsity is penalized in the \( l_1 \)-norm sense. This problem is, to a large extent, intractable in the sense of obtaining a global optimal solution. Thus we propose an iterative optimization procedure with \( l_1 \)-regularisation that achieves optimality. In the procedure, we restrict the search directions to descent directions for a least squares problem without the \( l_1 \) penalty, hence guaranteeing to decrease prediction error in each iteration. The restriction of the search directions corresponds to a linear constraint in the optimization problem.

The paper proceeds as follows. In Section II, the type of systems or models we consider are introduced. In Section III the problems we are considering are (mathematically) formulated. In section IV the main algorithm is provided. In Section V fundamental properties about the optimization problem(s) is provided (the solvability). The reader primarily interested in the proposed method and the numerical simulations can skip this section. In Section VI numerical simulations are provided and in Section VII the paper is concluded.

II. PRELIMINARIES

This paper addresses estimation of sparse \( A \)-matrices in linear noise-driven dynamical systems on state-space form. In the estimation we use samples of the entire states. The main assumption is that the sampling frequency is assumed to be low. Formally, the following system is considered,

\[
\dot{x}(t) = Ax(t)dt + dw(t).
\]

At time \( t \) the state \( x(t) \) is an element of \( \mathbb{R}^n \) and at time \( t = 0 \) the state is equal to \( x_0 \). The matrix \( A \in \mathbb{R}^{n \times n} \) is stable and \( w(t) \) is an \( n \)-dimensional Wiener process with incremental covariance matrix \( \Sigma \).\( dt \).

As mentioned, we assume that the continuous time signal \( x(t) \) is not available, but only samples thereof. The samples are obtained with a “small” sampling frequency or equivalently “large” sampling period \( h \). We assume that \( X = \)

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\[ x(0), x(h), x(2h), \ldots, x((N+1)h) \] is available, where 0 is the initial time and \( N \) is a positive integer.

The discrete time system that relates the values of the state variable \( x \) in (1) at the sampling times [11, p. 82-85] \[8, \text{chap. 2}, \] in the weak sense [12], is given by
\[
x(t + 1) = A_d x(t) + v(t),
\]
where \( A_d(A) = e^{hA} \) and the white noise \( v(t) \) has mean zero and covariance matrix
\[
\Sigma_d(A, \Sigma) = \int_0^h e^{A\tau} \Sigma e^{A^T\tau} \, d\tau.
\]

Let
\[
X_1 \triangleq [x(h), x(2h), \ldots, x(Nh)]
\]
\[
X_2 \triangleq [x(0), x(h), \ldots, x((N-1)h)]
\]
and \( D := (X_1, X_2) \), where \( N \) is a positive integer, \( X_1, X_2 \in \mathbb{R}^{n \times N} \).

### III. Problem Formulations

An \( l_1 \)-regularised nonlinear least squares problem is formulated as
\[
\min_{A} \|X_1 - \exp(hA)X_2\|_F^2 + \lambda \|A\|_1, \quad (4)
\]
where \( \lambda \in \mathbb{R}^+ \) and \( h \in \mathbb{R}^+ \) is the fixed and known sampling period. Our definition of the 1-norm is \( \|A\|_1 := \sum_{i=1}^n \sum_{j=1}^n |a_{ij}| \). Thus, (4) has a Kronecker form as follows:
\[
\min_{A} \|\text{vec}(X_1) - (X_2^T \otimes I_n) \text{vec}(\exp(hA))\|_2^2 + \lambda \|\text{vec}(A)\|_1,
\]
where \( \text{vec}(X_1) \in \mathbb{R}^{nN} \), \( (X_2^T \otimes I_n) \in \mathbb{R}^{nN \times n^2} \), and \( \text{vec}(\exp(hA)) \in \mathbb{R}^{n^2} \).

Note that a sparse \( A \) in (1) could correspond to a dense \( A_d \) in (2). Conversely, a sparse \( A_d \) might correspond to a dense \( A \). Thus, we gain little when \( h \) is large, by imposing sparsity constraints directly on \( A_d \) to identify a sparse \( A \). This motivates the considered problem.

**Remark 1.** The parameter \( \lambda \) is used to scale penalization on sparsity in the \( l_1 \)-norm sense, which could be selected via cross-validation in practice. To avoid the issue of system aliasing, we assume from now on, that the principal matrix logarithm is always chosen as initialization for the method we present (such logarithm exists and is unique if the matrix does not have any eigenvalues on the negative real axis). Equivalently we could have assumed that some other branch of the matrix logarithm is used, but for simplicity we choose the principal one.

**Remark 2.** Problem (4) is formulated with the Prediction Error Minimisation perspective as a foundation, with a least squares cost function. If the Maximum Likelihood perspective used, the objective function without penalisation is given by
\[
\min_{A \in \mathbb{R}^{m \times n}, \Sigma \in \mathcal{S}^n} \|P(A, \Sigma)(X_1 - A_d(A)X_2)\|_F^2 - N \log \det (P(A, \Sigma)^T P(A, \Sigma)),
\]
where \( \Sigma_d^{-1}(A, \Sigma) = P(A, \Sigma)^T P(A, \Sigma) \). This problem is difficult to solve due to its non-convex structure. However, by noting that \( \Sigma_d \) is also parametrized by \( \Sigma \), which is fully unknown, the Prediction Error Minimisation method (the problem (4) without \( l_1 \)-regularisation) gives the same consistent estimation as the problem (6). For more details, we refer to [7].

Before we move on to the next section and a method for (4), we introduce the concept of Fréchet derivatives and their Kronecker representations.

**Definition 1** (Fréchet Derivatives [13]). The Fréchet derivative of the matrix function \( f : \mathbb{C}^{n \times n} \rightarrow \mathbb{C}^{n \times n} \) at a point \( X \in \mathbb{C}^{n \times n} \) is a linear mapping
\[
\mathbb{C}^{n \times n} \xrightarrow{L} \mathbb{C}^{n \times n}
\]
\[
E \mapsto L(A, E)
\]
such that for all \( E \in \mathbb{C}^{n \times n} \)
\[
f(A + E) - f(A) - L(A, E) = o(\|E\|).
\]

The Fréchet derivative is unique if it exists, and for matrix functions \( \exp \) (matrix exponential) and \( \log \) (principal matrix logarithm) it exists. The Fréchet derivative of the function \( \exp \) [13] is
\[
L_{\exp}(X, E) = \int_0^1 e^{X(1-s)E} X s \, ds,
\]
which can be efficiently calculated by the Scaling-Pade-Squaring method in [14]. It gives a linear approximation of \( \exp \) at a given point \( A_c \) in the direction \( E \)
\[
e^{hA} = e^{hA_c E} = e^{hA_c} + L(hA_c, hE) + O(\|hE\|^2).
\]

**Theorem 1** (Kronecker representation [13, Thm. 10.13]). For \( A \in \mathbb{C}^{n \times n} \), \( \text{vec}(L(A, E)) = K(A)\text{vec}(E) \), where \( K(A) \in \mathbb{C}^{n^2 \times n^2} \) has the representations
\[
K(A) = \begin{pmatrix}
(I \otimes e^{A})\psi(A^T \oplus (-A)) \\
(e^{A^T/2} \otimes e^{A/2}) \text{sinc} \left( \frac{1}{2}[A^T \oplus (-A)] \right) \\
\frac{1}{2} (e^{A^T} \oplus e^A) \tau \left( \frac{1}{2}[e^T \oplus (-A)] \right)
\end{pmatrix}
\]
where \( \psi(x) = (e^x - 1)/x \) and \( \tau(x) = \tanh(x)/x \). The third expression is valid if \( \frac{1}{2} \|A^T \oplus (-A)\| < \pi/2 \) for some consistent matrix norm.

Here \( \text{vec}(E) \) denotes the vectorization of the matrix \( E \) formed by stacking the columns of \( E \) into a single column vector. The operators \( \otimes \) and \( \oplus \) are the Kronecker product and the Kronecker sum, respectively. The latter one is given by the former via
\[
(A \oplus B) = A \otimes I_n + I_m \otimes B,
\]
for \( A \in \mathbb{R}^{m \times n} \) and \( B \in \mathbb{R}^{n \times n} \).
IV. METHODS

A numerical method designed for problem (4) is presented here. As already mentioned, (4) is not a convex optimization problem due to the term \(\exp(hA)\). The general idea is to use Fréchet derivatives to approximate (4) at each point by a constrained \(l_1\)-regularised least square problem, which is efficiently solved by convex optimisation methods.

To avoid tensor operations, we will adopt the vectorized form (5), which is equivalent to (4) but easier for programming purposes. Let \(r(A)\) denote

\[
r(A) \triangleq \text{vec}(X_1) - (X_2^T \otimes I_n) \text{vec} (\exp(hA)),
\]

and \(\phi(A) := r(A)^T r(A)\). Then \(\min_A \phi(A)\) denotes the problem (5) without \(l_1\)-penalisation. The gradient of \(\phi(A)\) is

\[
\nabla \phi(A) = 2J(A)^T r(A),
\]

where

\[
J(A) = -h(X_2^T \otimes I_n)K(hA),
\]

and \(K(A)\) is defined in Theorem 1. A linear approximation of \(r(A)\) in a neighbourhood of a given point \(A_0\) is given as

\[
r_c(A) = r(A_0) + J(A_0) \text{vec}(A - A_0).
\]

One may then use this approximation and formulate a \(l_1\)-regularised linear least squares problem

\[
\min_{A} \| r(A_0) + J(A_0) \text{vec}(A - A_0) \|_2^2 + \lambda \| \text{vec}(A) \|_1,
\]

which can be solved to obtain an approximate solution to (5). Resolving it in an iterative manner, amounts to a Gauss-Newton method. However, \(\text{vec}(A - A_0)\) is not necessarily a \textit{decent direction} of either (5) or \(\phi\). Therefore, by letting \(A_k\) denote the current approximation, a new search direction \(p_k\) is instead computed from the solution of a constrained \(l_1\)-regularised linear least squares problem

\[
\min_{p_k \in \mathbb{R}^{n^2}} \| r(A_k) + J(A_k) \text{vec}(A - A_k) \|_2^2 + \mu_k \| p_k \|_2^2 + \lambda \| \text{vec}(A_k) + p_k \|_1,
\]

subject to \(\nabla \phi(A_k)^T p_k \leq 0\).

This convex problem can be recast and solved as a second-order-cone program [15]. For example it can be effectively solved by SDPT3 [16]. The updates are computed via

\[
\text{vec}(A_{k+1}) = \text{vec}(A_k) + \alpha_k p_k,
\]

where the step length \(\alpha_k\) is determined by line search methods, e.g. the Armijo-Goldstein step length principle [17], which takes \(\alpha_k\) to be the largest number in the sequence \(1, \frac{1}{2}, \frac{1}{4}, \ldots\) for which the inequality

\[
\| r(A_k) \|_2^2 - \| r(A_k + \alpha_k p_k) \|_2^2 \geq \frac{1}{2} \alpha_k \| J(A_k) p_k \|_2^2
\]

holds. The above iterative method is summarised in Algorithm 1.

Remark 3. The proposed method can be considered as a version of the \textit{damped Gauss-Newton} method\footnote{It exactly follows the damped Gauss-Newton method, if removing the penalty term.}. If the

\[
J(A_k) + J(A_k) p_k \|_2^2 + \mu_k \| p_k \|_2^2 + \lambda \| \text{vec}(A_k) + p_k \|_1,
\]

Algorithm 1 Iterative method for sparse continuous-time system identification with low sampling rate

\begin{enumerate}
   \item Initialise \(A_0 = \gamma \log(\tilde{A}_0^2)\), where \(\tilde{A}_0^2 = X_1 X_2^T (X_2 X_2^T)^{-1}\) and \(\gamma \in (0, 1)\);
   \item for \(k = 0, 1, 2, \ldots\) do
      \begin{enumerate}
         \item Calculate \(r(A_k), J(A_k), \nabla \phi(A_k)\) using (10), (11), (12);
         \item Solve the following \(l_1\)-regularised linear least squares problem subject to linear constraints:
            \[
            p_k = \arg \min_{p_k} \| r(A_k) + J(A_k) p_k \|_2^2 + \lambda \| \text{vec}(A_k) + p_k \|_1,
            \]
            subject to \(\nabla \phi(A_k)^T p_k \leq 0\);
         \item Find \(\alpha_k\) using (17);
         \item Update \(A_{k+1}\) using (16);
      \end{enumerate}
   \item \textbf{if} a stopping criterion is satisfied \textbf{then}
      \begin{enumerate}
         \item \textbf{return} \(A \leftarrow A_{k+1}\);
      \end{enumerate}
   \item \textbf{end if}
\end{enumerate}

Jacobian matrix \(J(A_k)\) does not have full column rank, one could adopt the \textit{Levenberg-Marquardt} method and solve

\[
\min_{p_k \in \mathbb{R}^{n^2}} \| r(A_k) + J(A_k) p_k \|_2^2 + \mu_k \| p_k \|_2^2 + \lambda \| \text{vec}(A_k) + p_k \|_1,
\]

subject to \(\nabla \phi(A_k)^T p_k \leq 0\).

Moreover, if the matrices \(J(A_k)^T J(A_k)\) are not uniformly bounded or well conditioned, the damped Gauss-Newton method may not work effectively.

V. STRICTLY LOCAL OPTIMALITY

The concept of “strictly local optimality” is introduced to show whether a globally optimal point is possible to be achieved, if initial points in numerical methods are well chosen. This is particularly important in network inference since locally optimal points might give different estimation of sparse structures of \(A\) in (1).

Consider the constrained version of (4)

\[
\min_{A} \| X_1 - \exp(hA) X_2 \|_F,
\]

subject to \(\| A \|_1 \leq \kappa\),

and the unconstrained optimisation problem

\[
\min_{A_d} \| X_1 - A_d X_2 \|_F.
\]

Definition 2 (strictly local optimality). Let \(A^*\) be one of the optimal points of (19) and \(B_\delta(A^*) = \{ A : \| A - A^* \|_1 \leq \delta \}\). We say that \(A^*\) is \textit{strictly local optimal} if the following statement is false:

For any \(\delta > 0\), there always exists a feasible point \(A \in B_\delta(A^*)\) such that \(A \neq A^*\) and \(A\) is locally optimal.

Roughly speaking, the \textit{strictly local optimality} indicates that the matrix exponential \(\exp(\cdot)\) is not an “extremely terrible” function to deal with in optimisation.

Assumption 2. Let \(A^*\) be the \textit{global optimal point} of (19). We have \(-\pi/h < \text{Im}(\lambda_i(A^*)) < \pi/h\), \(\forall i = 1, \ldots, n\), where \(\text{Im}(\lambda_i(A^*))\) denotes the imaginary part of \(i\)-th eigenvalue of \(A^*\).
Let $\theta^*$ be the optimal point of (20) (using $\theta^*$ to avoid ambiguities by using $A_*^{\theta}$ that better refers to $A_d(A^*)$), $f(\theta) := \|X_1 - \theta X_2\|_F$, and $\Omega$ is the feasible set in (19), i.e.

$$\Omega := \{ A \in \mathbb{R}^{n \times n} : \|A\|_1 \leq \kappa \}.$$  \hspace{1cm} (21)

The boundary and interior set of the feasible set $\Omega$ of (19) are denoted by

$$\partial \Omega := \{ A : \|A\|_1 = \kappa \},$$

$$\Omega^\circ := \{ A : \|A\|_1 < \kappa \}.$$  

Using the principal logarithm as the inverse of the matrix exponential $\log : \exp(\cdot)$, the matrix exponential is a $C^\infty$-diffeomorphism, as illustrated in Figure 1:

$$\exp : \begin{cases} U \to V \\ A \to \exp(A) \end{cases},$$

where

$$U := \{ A \in \mathbb{R}^{n \times n} : -\pi < \text{Im}(\lambda_i(A)) < \pi, i = 1, \ldots, n \},$$

$$V := \{ A \in \mathbb{R}^{n \times n} : \lambda_i(A) \notin \mathbb{R}^-, i = 1, \ldots, n \}\{0\}.$$  \hspace{1cm} (22)

![Fig. 1: Illustration of the diffeomorphism map. $h$ is a fixed known positive real number.](image)

**Lemma 3.** Suppose the global optimal point $A^*$ is on the boundary, i.e. $A^* \in \partial \Omega$. Then there exists no locally optimal point $A^*_L$ of (19) in $\Omega^\circ$.

**Proof.** Assume that there exists a locally optimal point $A^*_L$ in $\Omega^\circ$. By the definition of locally optimal points, there exists a neighbourhood of $A^*_L$, denoted as $B_\delta(A^*_L)$, such that for any $A \in B_\delta(A^*_L)$ we have $f(e^{A^* h}) \geq f(e^{A_L h})$. Let $\theta^*_L := \exp(A^*_L h)$. Since the matrix exponential is $C^\infty$-diffeomorphism, there exists $\delta_0 > 0$ such that $f(\theta) \geq f(\theta^*_L)$ for any $\theta \in B_\delta(\theta^*_L)$. Here $\delta_0$ can be chosen in such a way that the open ball $B_\delta(\theta^*_L) \subseteq N(\theta^*_L)$, where $N(\theta^*_L)$ is the image of $B_\delta(A^*_L)$ under $\exp(A^* h)$ and is open. Consider $\theta^* := \exp(A^*)$, which satisfies $f(\theta^*) \geq f(\theta^*_L)$ with $\theta^* \neq \theta^*_L$ by noticing that $\exp$ is a diffeomorphism and $A^*$ is global optimal. However, this is impossible since (20) has a unique optimal point (any locally optimal points are global optimal).

**Lemma 4.** Suppose $A^* \in \partial \Omega$, then either $\exp(A^* h) = \theta^*$ or the following statement is false:

there exists a sequence of locally optimal points of (19) $\{ A^{(k)} \} \ (k = 0, 1, 2, \ldots)$ in $\Omega$ that converges to $A^*$.

**Proof.** (For easier understanding, one could consider the vectorized form (19) to avoid tensor operations.) In a proof by contradiction, one assume that the statement in Lemma 4 is true provided that $\exp(A^* h) \neq \theta^*$. Let $g(A)$ denote the objective function of (19) and $\varphi(A) := \exp(A h)$; hence $g(A) = (f \circ \varphi)(A)$. Considering $A^* \in \partial \Omega$, each element in $\{ A^{(k)} \}$ is on $\partial \Omega$, as illustrated in Figure 2. By the necessary conditions on local optimality [18], all $\{ A^{(k)} \}$ and $A^*$ are stationary for (19), which means

$$\nabla g(\bar{A})(A - \bar{A}) \geq 0$$

for all $A \in \Omega$, where $\bar{A} \in \{ A^{(k)} \} \cup \{ A^* \}$, and $\nabla g(\bar{A})$ is the gradient of $g$ at $\bar{A}$. Therefore we have

$$\nabla g(A^*)(A^{(k)} - A^*) \geq 0,$$

$$\nabla g(A^{(k)})(A^* - A^{(k)}) \geq 0,$$

for all $k \in \mathbb{N}$, and hence

$$\left( \nabla g(A^*)(A^{(k)} - A^*) \right)^T \left( \nabla g(A^{(k)})(A^* - A^{(k)}) \right) \geq 0,$$

or equivalently

$$- (A^{(k)} - A^*)^T \left( \nabla g(A^*)^T \nabla g(A^{(k)}) \right) (A^{(k)} - A^*) \geq 0.$$  \hspace{1cm} (22)

Noticing that $g(A)$ is continuously differentiable, we could evaluate (22) when $k \to \infty$, resulting in

$$\nabla g(A^*)^T \nabla g(A^*) \leq 0,$$

and hence $\nabla g(A^*) = 0$. By general forms of the chain rule, we have

$$\nabla g(A^*) = \nabla f(\phi(A^*)) J_\phi(A^*).$$

Moreover, $\phi(A)$ is a diffeomorphism, so $J_\phi(A)$ is nonsingular [19]. Hence $\nabla f(\phi(A^*)) = 0$ that implies $\theta^* = \exp(A^* h)$, which contradicts with our assumption. \hfill $\square$

**Theorem 5** (strictly local optimality). Suppose that (19) is solvable up to the principal logarithm, and $X_2$ is full row rank. Let $A^*$ and $\theta^*$ be the globally optimal points of (19) and (20) respectively, $\partial \Omega$ the boundary of the feasible set of (19), and $B_\delta(A^*) := \{ A : \|A - A^*\|_1 \leq \delta \}$. Then either $\exp(A^* h) = \theta^*$ with $A^* \in \partial \Omega$, or there exists $\delta > 0$ such that for any $A \neq A^* \in B_\delta(A^*)$, the feasible point $A$ is not locally optimal.

**Proof.** In a proof by contradiction, one assume that for any $\delta > 0$, there always exists $A \in B_\delta(A^*)$ and $A \neq A^*$ such that $A$ is locally optimal.
CASE 1 When $A^* \in \Omega^\circ$, there exists small enough $\delta_A > 0$ such that $B_{\delta_A}(A^*) \subseteq \Omega^\circ$. By assumption there exists $A^*_L \in B_{\delta_A}(A^*)$ such that $A^*_L$ is locally optimal. Hence there exists $\delta_{A_L} > 0$ such that $f(e^{Ah}) \geq f(e^{A_Lh})$ for any $A \in B_{\delta_{A_L}}(A^*_L)$. Letting $\theta^* := \exp(A^*)$, $\theta^*_L := \exp(A^*_L)$ and $N(\theta^*_L) := \exp(B_{\delta_{A_L}}(A^*_L))$, one obtains that $f(\theta^*) \geq f(\theta^*_L)$ for any $\theta \in N(\theta^*_L)$. This implies $\theta^*_L$ is a locally optimal point of (19), and $f(\theta^*_L) \geq f(\theta^*)$ with $\theta^*_L \neq \theta^*$. However, this is impossible since (20) is a convex optimisation problem, whose locally optimal point is globally optimal, and (20) has a unique optimal point since $X_2$ is full row rank.

CASE 2 When $A^* \in \partial \Omega$, assume that $\exp(A^*\eta) \neq \theta^*$ (under which (19) becomes a trivial case that can be solved by (20) equipped with the principal logarithm). Consider a sequence of positive real numbers $\{\delta_A^{(k)}\}$ ($k = 0, 1, 2, \ldots$) that converges to 0. By assumption, for each $\delta_A^{(k)}$, there exists a locally optimal point $A^{(k)}_L$ in $B_{\delta_A^{(k)}}(A^*)$, and hence we obtain a sequence of locally optimal points $A_L^{(k)}$ ($k = 0, 1, 2, \ldots$) that converges to $A^*$. However, Lemma 4 tells that this is not possible.

In a sum, we show that the network inference problem of (1) is locally solvable by a proof of contradiction. \qed

Remark 4. It is not necessary to assume that (19) has a unique global optimal point up to the principal logarithm to guarantee Theorem 5 being true. The essential requirement is the unconstrained optimisation problem (20) has a unique optimal solution, i.e. $X_2$ is required to be full row rank.

VI. NUMERICAL EXAMPLES AND DISCUSSION

In this section, we use an numerical example to show the effectiveness of the proposed algorithm. The data set for identification was sampled with $h = 2$ from simulation of stochastic differential equations (state-space), in which $A$ is a stable sparse matrix generated randomly (not truly random; using “sprandn(4, 4, 0.8)”, checking stability, and then put into diagonal blocks). The initial values of states were randomly generated, and the process noise is normal i.i.d. Here we chose a low sampling frequency, large noise and limited samples to generate a challenging time series for identification, which gives a typical profile of time series data in biomedicine (e.g. microarray data [20]), as illustrated in Figure 5.

The results are shown in Figure 6, together with their corresponding $A_d$’s computed via matrix exponential. Note that in order to estimate $A$, we have to use the matrix exponential and matrix logarithm due to the large sampling period. The simplest way to estimate $A$ is taking the principal matrix logarithm of least square solution $\hat{A}_d$, which is, however, contaminated by process noise and unable to give reasonable sparse structure of $A$, clearly shown in Figure 6a. This is mainly due to the effect of process noise on the estimation through matrix logarithms. However, one need to know that, referring to the simulations, the direct logarithm of $\hat{A}_d$ works pretty well when the dimension is small (e.g. $\dim(A) \leq 6$).

Convergence of Algorithm 1 is illustrated in Figure 7, with respect to $\lambda = 0.04$. Similar to LASSO, $\lambda$ cannot be chosen too large, otherwise $A$ inclined to be a zero matrix. A better $\gamma$ could be obtained by running the cross-validation technique. In order to show convergence being almost irrelevant to initial points in simulation, we set $\gamma$ be an random between $[0, 1]$. When $\gamma$ approaches to 1, the algorithm converges faster. However, $\lambda$ cannot be too close to 1, e.g. $\gamma \geq 0.95$. As a powerfully local optimum, $\log(A_d)$ disables the method to obtain a sparse solution, i.e. the algorithm will converge to certain points that are really close to $\log(A_d)$ and tends to be non-sparse. One might compare it with Figure 8 to understand different behaviours of $\gamma$’s.

The numerical computation was performed in MATLAB, and the codes will be released soon in public on the github. Considering computational efficiency, we directly used vector/matrix norms instead of quadratic forms in implementation (cf. [21, chap. 11.1]).
samples on which the proposed algorithm still work, another being the maximal Signal-to-Noise ratio that causes failure of identification. Also parallelism in the implementation is interesting to investigate.

**APPENDIX**

**REFERENCES**

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Fig. 8: Matrix heatmap of the estimated $A$’s and $A_d$’s with different $\gamma$’s.