Wormholes as Basis for the Hilbert Space in Lorentzian Gravity

Guillermo A. Mena Marugán

Instituto de Matemáticas y Física Fundamental, C.S.I.C., Serrano 121, 28006 Madrid, Spain,

and

Center for Gravitational Physics and Geometry, Dpt. of Physics, Pennsylvania State University, University Park, PA 16802, USA.

Abstract

We carry out to completion the quantization of a Friedmann-Robertson-Walker model provided with a conformal scalar field, and of a Kantowski-Sachs spacetime minimally coupled to a massless scalar field. We prove that the Hilbert space determined by the reality conditions that correspond to Lorentzian gravity admits a basis of wormhole wave functions. This result implies that the vector space spanned by the quantum wormholes can be equipped with an unique inner product by demanding an adequate set of Lorentzian reality conditions, and that the Hilbert space of wormholes obtained in this way can be identified with the whole Hilbert space of physical states for Lorentzian gravity. In particular, all the normalizable quantum states can then be interpreted as superpositions of wormholes. For each of the models considered here, we finally show that the physical Hilbert space is separable by constructing a discrete orthonormal basis of wormhole solutions.

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I Introduction

Classical wormholes are solutions to the Euclidean equations of General Relativity that connect two asymptotic regions of large three-volume by a throat [1-4]. This type of instanton may exist only for matter fields that allow the Ricci tensor to have negative eigenvalues [5,6]. Therefore, classical wormholes cannot play a fundamental role in physics, because they are ruled out for generic kinds of matter content. However, from the quantum mechanical point of view, wormholes can be realized as physical states with an appropriate asymptotic behavior for large three-geometries, such that they can be interpreted as tubes connected to a region of spacetime with large three-volume [7]. As opposed to the situation found in classical gravity, quantum wormholes may exist regardless of the properties of the matter fields.

In 1990, Hawking and Page proposed to characterize the quantum wormholes as the wave functions that decrease exponentially in the limit of large three-geometries and present no singularities for finite matter fields when the three-geometry is regular or collapses to zero because of an ill-defined slicing of the spacetime [7]. These conditions on the quantum wormhole states were reformulated by Garay in an adequate language for the complex path integral approach to General Relativity. Garay proved that (at least in some minisuperspace models) the wormhole wave functions can be defined as path integrals over Euclidean manifolds that match asymptotic large three-geometries with fixed gravitational momenta and no gravitational excitations [8].

Quantum wormholes can significantly affect the low energy physics in our Universe, because they allow field interactions between distant points of the spacetime [1]. Assuming that the space of wormholes admits a Hilbert structure, it has been shown in the literature that the existence of quantum wormholes may alter the values of the effective physical constants of nature and, in particular, explain why we
observe a vanishing cosmological constant [9-10]. The conjecture that wormholes form a Hilbert space has, nonetheless, never been proved in full gravity, although it has been demonstrated in a number of minisuperspace models for which one can explicitly find a complete set of wormhole solutions [11]. In these minisuperspaces, the Hilbert structure of the wormhole wave functions was obtained by restricting to the space of physical states an inner product that can be introduced in a natural way in the corresponding unconstrained models [11]. In more general gravitational systems, however, one should not expect the physical states to be normalizable with respect to any inner product defined before imposing the first-class constraints of the model. A more rigorous prescription is therefore needed in order to fix the Hilbert structure in the space spanned by the wormhole solutions. In a recent paper [12], we argued that such a structure can in fact be uniquely determined by imposing a set of Lorentzian reality conditions in the quantum theory, i.e., adopting Ashtekar’s proposal for the non-perturbative quantization of systems with constraints [13,14].

We recall that reality conditions are adjointness relations between observables\(^1\) of the quantum theory that capture the complex conjugation relations between the classical variables of the system [13,14]. According to Ashtekar, these complex conjugation relations must be promoted under quantization into an abstract \(\star\)-involution of operators [13,15]. Once one has chosen a representation for the quantum theory, what includes the selection of a (possibly over-)complete set of elementary operators that is closed under commutation relations and under the introduced \(\star\)-involution [13,14], the physical inner product in the space of solutions to all quantum constraints (the physical states) is fixed by imposing the \(\star\)-relations between a sufficiently large number of observables as adjointness requirements (reality conditions) [16]. In other words, given a representation for the quantum theory, the Hilbert space of physical states is entirely determined by the set of reality conditions im-

\(^1\) We understand observable to refer to operators that commute with all the first-class constraints.
posed on the system.

As a consequence, if the space of wormhole solutions admits in a certain representation a Hilbert structure that can be found by demanding Lorentzian reality conditions, as we propose [12], such Hilbert space of wormholes has then to be a subspace of the Hilbert space of the Lorentzian gravitational model in the adopted representation. Moreover, if we assume that the vector space spanned by the wormhole wave functions is stable under the action of the observables and restrict our attention to irreducible representations, we conclude that the Hilbert space of wormholes can be identified with that of the considered Lorentzian model [12]. In particular, this implies that the quantum wormhole solutions provide a basis for the Hilbert space of the Lorentzian theory.

The above statements were rigorously proved in Ref. [12] for the case of a Friedmann-Robertson-Walker (FRW) spacetime in the presence of a minimal massless scalar field. Our aim in this work is to present a similar proof for the other two minisuperspace models for which we know a complete set of wormhole wave functions, namely, a FRW model provided with a conformal scalar field [8,11] and a Kantowski-Sachs (KS) spacetime minimally coupled to a massless scalar field [8,17].

The outline of the rest of this paper is as follows. In Sec. II we quantize to completion the conformally coupled FRW spacetime. We find the Hilbert space of physical states determined by the Lorentzian reality conditions and show that it is spanned by a set of wormhole wave functions. The quantization of the KS model is carried out in Sec. III. For this minisuperspace, we prove in Sec. IV that the physical Hilbert space admits a basis of generalized wormhole solutions, which are asymptotically damped for large three-geometries but fail to satisfy the requirement of regularity when the three-geometry degenerates. In Sec. V we consider other choices of basis of wormholes for the Hilbert space of the KS model. In particular, we show that there exists a discrete basis formed by normalizable states. Finally,
we summarize the results in Sec. VI.

II FRW Model

Let us study first the case of a homogeneous scalar field conformally coupled to a FRW spacetime. The FRW metric can be written in the form [8]

$$ds^2 = \frac{2G}{3\pi} q^2(t) \left(-N^2(t) dt^2 + d\Omega_3^2\right),\quad (2.1)$$

where $N(t)$ and $q(t)$ are, respectively, the rescaled lapse function and the scale factor of the model, $G$ is the gravitational constant, and $d\Omega_3^2$ is the metric of the unit three-sphere. Since the line element in (2.1) depends only on the square of $q$, we will restrict the scale factor to be positive, $q > 0$, so that each Lorentzian metric be considered once. Redefining then the scalar field $\Phi(t)$ as [8]

$$\Phi(t) = \sqrt{\frac{3}{4\pi G}} \frac{\chi(t)}{q(t)},\quad (2.2)$$

we arrive at the following expression for the Hamiltonian constraint of the minisuperspace model:

$$H \equiv \frac{1}{2} \left(q^2 + \Pi^2_q - \chi^2 - \Pi^2_\chi\right) = 0 \quad (q > 0),\quad (2.3)$$

with $\Pi_q$ and $\Pi_\chi$ the momenta canonically conjugate to $q$ and $\chi$.

Apart from the restriction $q > 0$, the above Hamiltonian is exactly the difference of the Hamiltonians of two harmonic oscillators. It therefore seems natural to adopt as representation space for the quantum theory the vector space of complex functions $\Psi$ on $\mathbb{R}^+ \times \mathbb{R}$ spanned by the basis

$$\tilde{\Psi}_{nm}(q, \chi) = \varphi_n(q)\varphi_m(\chi) \quad (q \in \mathbb{R}^+, \chi \in \mathbb{R}),\quad (2.4)$$

with $\varphi_n(x)$ the harmonic-oscillator wave functions,

$$\varphi_n(x) = \frac{(-1)^n}{\sqrt{2^n n!\sqrt{\pi}}} e^{\frac{x^2}{2}} \frac{d^n}{dx^n}(e^{-x^2}), \quad n = 0, 1, \ldots, \quad (2.5)$$
As our complete set of elementary operators we will select the analogue of the annihilation and creation operators of the two harmonic oscillators:

\[ \hat{a}_\alpha \Psi = \frac{1}{\sqrt{2}}(\alpha + \partial_\alpha)\Psi, \quad \hat{a}_\alpha^\dagger \Psi = \frac{1}{\sqrt{2}}(\alpha - \partial_\alpha)\Psi, \]

(2.6)

where the variable \( \alpha \) denotes both \( q \) and \( \chi \), and we have let \( \hbar = 1 \). These operators represent the following complex functions on phase space:

\[ a_\alpha = \frac{1}{\sqrt{2}}(\alpha + i\Pi_\alpha), \quad a_\alpha^\dagger = \frac{1}{\sqrt{2}}(\alpha - i\Pi_\alpha), \]

(2.7)

They form \[18\] a closed algebra under commutation relations, the only non-vanishing commutators being \([\hat{a}_\alpha, \hat{a}_\alpha^\dagger] = 1 \quad (\alpha = q \text{ or } \chi)\). On the other hand, recalling that \( q, \chi, \Pi_q, \text{ and } \Pi_\chi \) are real in Lorentzian gravity, it is easy to check that the algebra of basic operators given by (2.6) is also closed under the \(\star\)-involution that is induced by the complex conjugation relations between classical variables, namely, \( \hat{a}_\alpha^* = \hat{a}_\alpha^\dagger \).

In the constructed representation, the Hamiltonian constraint (2.3) can be rewritten as the Wheeler-DeWitt equation:

\[ \hat{H}\Psi(q, \chi) \equiv (\hat{a}_q^\dagger \hat{a}_q - \hat{a}_\chi^\dagger \hat{a}_\chi)\Psi(q, \chi) = \frac{1}{2} \left( q^2 - \partial_q^2 - \chi^2 + \partial_\chi^2 \right) \Psi(q, \chi) = 0. \]

(2.8)

Hence, the physical states of the system turn out to be linear combinations of the wave functions

\[ \Psi_n(q, \chi) = \tilde{\Psi}_{nn}(q, \chi) = \varphi_n(q)\varphi_n(\chi), \quad n = 0, 1, ... \]

(2.9)

From the expression of the Hamiltonian \( \hat{H} \) in eq. (2.8) one can prove that the operators

\[ \hat{N}_q = \hat{a}_q^\dagger \hat{a}_q, \quad \hat{N}_\chi = \hat{a}_\chi^\dagger \hat{a}_\chi, \quad \hat{J}_+ = \hat{a}_q^\dagger \hat{a}_\chi^\dagger, \quad \hat{J}_- = \hat{a}_q \hat{a}_\chi, \]

(2.10)

provide an over-complete set of observables for the model \[13\]. These observables \[18\] form a closed algebra under commutation relations, with non-vanishing commutators given by

\[ [\hat{N}_\alpha, \hat{J}_+] = \hat{J}_+, \quad [\hat{N}_\alpha, \hat{J}_-] = -\hat{J}_- \quad (\alpha = q \text{ or } \chi), \]

(2.11)
The relations $\hat{a}_\alpha^* = \hat{a}_\alpha^\dagger$ imply on the other hand that
\[ N_q^* = \hat{N}_q, \quad N_\chi^* = \hat{N}_\chi, \quad \hat{J}_+^* = \hat{J}_-, \tag{2.13} \]
while the Hamiltonian constraint (2.8) demands that the operators $\hat{N}_q$ and $\hat{N}_\chi$ coincide on the physical states. Therefore, the Lorentzian reality conditions amount to require that, on the space of physical states, $\hat{N}_\chi$ (or $\hat{N}_q$) be self-adjoint and $\hat{J}_-$ be the adjoint of $\hat{J}_+$. We expect the inner product in such a space to adopt the generic expression
\[ \langle \Upsilon, \Psi \rangle = \int_{\mathbb{R}^+} dq \int_{\mathbb{R}} d\chi \ g(q, \chi) \Upsilon^*(q, \chi) \Psi(q, \chi), \tag{2.14} \]
where the symbol $\ast$ denotes complex conjugation, $g(q, \chi)$ is a measure to be determined by imposing the reality conditions stated above, and
\[ \Psi(q, \chi) = \sum_{n=0}^{\infty} f_n \varphi_n(q) \varphi_n(\chi), \quad \Upsilon(q, \chi) = \sum_{n=0}^{\infty} h_n \varphi_n(q) \varphi_n(\chi), \quad f_n, h_n \in \mathcal{C}. \tag{2.15} \]

Taking into account the asymptotic behavior of the harmonic-oscillator wave functions $\varphi_n$ at infinity and that $\varphi_{2p+1}(q = 0) = 0$ for all non-negative integers $p$, a straightforward calculation shows that the Lorentzian reality conditions are satisfied if and only if $g(q, \chi)$ is constant. Letting thus $g(q, \chi) = 1$, we arrive at the inner product
\[ \langle \Upsilon, \Psi \rangle = \int_{\mathbb{R}^+} dq \int_{\mathbb{R}} d\chi \ \Upsilon^*(q, \chi) \Psi(q, \chi) = \frac{1}{2} \sum_{n=0}^{\infty} h_n^* f_n. \tag{2.16} \]
On the right hand side of this equation we have substituted eq. (2.15) and used that
\[ \langle \Psi_n, \Psi_m \rangle = \frac{1}{2} \delta_{nm}, \tag{2.17} \]
with $\delta_{nm}$ the Kronecker delta (see eqs. (2.5) and (2.9)). The physical Hilbert space of the model is then isomorphic to $l^2$, the space of square summable sequences.

Notice that, in the representation that we have chosen, the Hilbert space of the Lorentzian theory is spanned by the set of wave functions (2.9). These functions
correspond in fact to quantum wormhole states, because they are exponentially damped for large three-geometries \( q \gg 1 \), present no singularities in the region \( \{q > 0, \phi \in \mathbb{R}\} \), and have a well-defined limit when \( q \) tends to zero, i.e., when the three-geometry degenerates. Hence, in agreement with our discussion in the Introduction, the space spanned by the wormhole wave functions (2.9) does not only admit a Hilbert structure, which can be selected by imposing an adequate set of Lorentzian reality conditions, but, in addition, the Hilbert space of wormholes determined in this way turns out to coincide with that of the Lorentzian model under study.

The different elements of the basis (2.9) are described by the wormhole parameter \( n \), which is the eigenvalue reached by the observable \( \hat{N}_x \) (or \( \hat{N}_q \)) in these wormhole solutions. Let us also note that, given eq. (2.17), the wormhole wave functions \( \sqrt{2} \Psi_n(q, \chi) \ (n = 0, 1, \ldots) \) provide an orthonormal basis for the physical Hilbert space, which results then in being separable.

The wormhole states (2.9) were first considered by Garay in Refs. [8,11]. Following different arguments, he also found the inner product (2.16) in the space of quantum wormholes, except for that he allowed the scale factor \( q \) to run over the whole real axis [11]. Since extending the domain of definition of \( q \) in (2.16) to \( \mathbb{R} \) only changes the right hand side of that equation by a factor of 2, the Hilbert space of wormholes determined in Ref. [11] is nonetheless isomorphic to that analyzed in this section, and the two constructed quantizations equivalent. In particular, the prediction that (for normalizable physical states) the effective gravitational constant observed in the asymptotic regions of large volume is positive [11] remains still valid in our quantization.
III KS Model: Quantization

Let us study now the other minisuperspace model for which we know a complete set of quantum wormhole solutions, i.e., a KS spacetime minimally coupled to an homogeneous massless scalar field $\Phi [8,17]$. The KS metric can be expressed in the form

$$ds^2 = \frac{G}{2\pi} \left( -q^4(t)e^{-2\alpha(t)} N^2(t) dt^2 + e^{2\alpha(t)} dr^2 + q^2(t)e^{-2\alpha(t)} d\Omega^2_2 \right),$$  \hspace{1cm} (3.1)

where $d\Omega^2_2$ is the metric of the unit two-sphere, $r$ is a periodic coordinate, with period equal to $2\pi$, $N(t)$ is the rescaled lapse function, and $e^{\alpha(t)}$ and $q(t)$ are the two scale factors of the model. As in the case of the FRW spacetime, we will restrict the scale factors to take only positive values, so that each Lorentzian metric is considered only once. Thus, we will impose $q > 0$ and $\alpha \in \mathbb{R}$.

The model is subject only to a first-class constraint, provided by the Hamiltonian of the system [17],

$$H \equiv \frac{1}{2} \left( q^2 \Pi_q^2 + q^2 - \Pi^2_{\alpha} - \Pi^2_{\chi} \right) = 0 \quad (q > 0).$$ \hspace{1cm} (3.2)

Here, $\Pi_q$, $\Pi_{\alpha}$, and $\Pi_{\chi}$ are the momenta canonically conjugate to $q$, $\alpha$, and

$$\chi(t) = \sqrt{4\pi G} \Phi(t),$$ \hspace{1cm} (3.3)

respectively. For Lorentzian gravity and real fields we will have $\chi, \Pi_q, \Pi_{\alpha}, \Pi_{\chi} \in \mathbb{R}$.

The canonical transformation defined by

$$q = \frac{\Pi}{\cosh X}, \quad q \Pi_q = -\Pi \tanh X, \hspace{1cm} (3.4)$$

$$\chi = \phi \sin \theta + \frac{z}{\Pi_{\phi}} \cos \theta, \quad \Pi_{\chi} = \Pi_{\phi} \sin \theta, \hspace{1cm} (3.5)$$

$$\alpha = \phi \cos \theta - \frac{z}{\Pi_{\phi}} \sin \theta, \quad \Pi_{\alpha} = \Pi_{\phi} \cos \theta, \hspace{1cm} (3.6)$$

allows us then to rewrite the constraint (3.2) as

$$H \equiv \frac{1}{2} (\Pi^2 - \Pi_{\phi}^2) = 0.$$ \hspace{1cm} (3.7)
In the above equations, $X, \phi, \text{ and } z$ are the new configuration variables and $\Pi, \Pi_\phi$, and $\theta$ their respective canonically conjugate momenta. A generating function for the introduced transformation is given by

$$F(q, \alpha, \chi, X, \Pi_\phi, \theta) = -q \sinh X + \Pi_\phi (\alpha \cos \theta + \chi \sin \theta).$$

(3.8)

Recalling that $q > 0$, it is not difficult to check from eqs. (3.4-6) that, in the new set of variables, the domains

$$X, \phi, z \in \mathbb{R}, \quad \Pi, \Pi_\phi > 0, \quad \theta \in S^1,$$

(3.9)

cover the whole phase space of the model, except for those points with $\Pi_\alpha = \Pi_\chi = 0$ ($\Pi_\phi = 0$), that can never be reached in the physical solutions owing to the Hamiltonian constraint (3.7). The canonical transformation (3.4-6) turns out then to be analytic and invertible in the region (3.9).

Notice that, from eq. (3.9), the phase space of the system can be regarded as the cotangent bundle over $\mathbb{R}^+ \times \mathbb{R}^+ \times S^1$. The inclusion of the points with $\Pi_\phi = 0$ would have not altered this result, because such points are just part of the boundary of the considered phase space. On the other hand, taking into account the Hamiltonian constraint (3.7), it is straightforward to see that the reduced phase space of the model is given by the cotangent bundle over $\mathbb{R}^+ \times S^1$ ($\Pi_\phi \in \mathbb{R}^+, \theta \in S^1$).

To quantize the theory, we will select as representation space the vector space of complex functions $\Psi(X, \Pi_\phi, \theta)$ on $\mathbb{R} \times \mathbb{R}^+ \times S^1$, and define the following set of elementary operators (with $\hbar = 1$):

$$\hat{\Pi} \Psi = -i \partial_X \Psi(X, \Pi_\phi, \theta), \quad \hat{X} \Psi = X \Psi(X, \Pi_\phi, \theta),$$

(3.10)

$$\hat{\Pi}_\phi \Psi = \Pi_\phi \Psi(X, \Pi_\phi, \theta), \quad \widehat{\Pi_\phi} \Psi = i \left( \Pi_\phi \partial_{\Pi_\phi} - \frac{1}{2} \right) \Psi(X, \Pi_\phi, \theta),$$

(3.11)

$$\widehat{\cos \theta} \Psi = \cos \theta \Psi(X, \Pi_\phi, \theta), \quad \widehat{\sin \theta} \Psi = \sin \theta \Psi(X, \Pi_\phi, \theta),$$

(3.12)

$$\hat{z} \Psi = i \partial_\theta \Psi(X, \Pi_\phi, \theta).$$

(3.13)
The action of $\hat{\Pi}_\phi\phi$ in (3.11) has been chosen in a convenient way to simplify the rest of our calculations. The above set of operators forms a closed algebra [18] under commutation relations:

$$[\hat{X}, \hat{\Pi}] = i, \quad [\hat{\Pi}_\phi, \hat{\Pi}_\phi\phi] = -i\hat{\Pi}_\phi,$$

(3.14)

$$[\hat{z}, \cos \theta] = -i \sin \theta, \quad [\hat{z}, \sin \theta] = i \cos \theta,$$

(3.15)

all other commutators in the algebra being equal to zero. In the quantum theory, the operators (3.10-13) represent the classical variables $(\Pi, X, \Pi_\phi, \Pi_\phi\phi, \cos \theta, \sin \theta, z)$, which can be seen to provide a complete set of functions on the phase space of the system. Recalling eq. (3.9), one can easily show then that, for Lorentzian gravity, the $\star$-involution induced in the algebra of elementary operators (3.10-13) by the complex conjugation relations between classical variables coincides with the identity operation.

Given the constraint (3.7) and that $\Pi, \Pi_\phi > 0$, the physical states of the model must satisfy the equation

$$\hat{\Pi}\Psi(X, \Pi_\phi, \theta) = \hat{\Pi}_\phi\Psi(X, \Pi_\phi, \theta),$$

(3.16)

whose solutions adopt the generic expression

$$\Psi(X, \Pi_\phi, \theta) = e^{iX\Pi_\phi} f(\Pi_\phi, \theta),$$

(3.17)

with $f(\Pi_\phi, \theta)$ a complex function on $\mathbb{R}^+ \times S^1$.

Since the reduced phase space of the system is the cotangent bundle over $\mathbb{R}^+ \times S^1$, a complete set of observables is provided by the operators $\cos \theta, \sin \theta, \hat{z}, \hat{\Pi}_\phi$ and [12]

$$\hat{w} = \hat{\Pi}_\phi\phi + \hat{\Pi}_\phi^2 \left( \hat{X}\hat{\Pi}^{-1} + \frac{i}{2}\hat{\Pi}^{-2} \right).$$

(3.18)

This last observable corresponds to the classical variable $\Pi_\phi\phi + \Pi_\phi^2 X\Pi^{-1}$. The term in brackets in (3.18) is simply the symmetrized product of $\hat{X}$ and $\hat{\Pi}^{-1}$. Using eqs.
(3.10,11) and (3.16) it is possible to check that the operator \( \hat{w} \) acts in the following way on the physical states:

\[
\hat{w} \Psi(X, \Pi_\phi, \theta) = e^{iX\Pi_\phi} i\Pi_\phi \partial_{\Pi_\phi} f(\Pi_\phi, \theta).
\]  

(3.19)

The inner product in the space of physical states can then be determined by requiring the self-adjointness of the above set of observables, because they all represent classical variables that are real in the Lorentzian theory. In this way, one arrives at the inner product

\[
< \Upsilon, \Psi > = \int_{\mathbb{R}^+} \int_{S^1} d\theta h^*(\Pi_\phi, \theta) f(\Pi_\phi, \theta),
\]

(3.20)

where \( \Upsilon = e^{iX\Pi_\phi} h(\Pi_\phi, \theta) \). Thus, the Hilbert space of the studied Lorentzian model consists of all functions (3.17) with \( f(\Pi_\phi, \theta) \in L^2(\mathbb{R}^+ \times S^1, d\Pi_\phi d\theta/\Pi_\phi) \).

**IV  KS Model: Quantum Wormholes**

In order to discuss the role played in the quantum theory by the wormhole states, we will translate now the results of the previous section into the \((q, \alpha, \chi)\) representation that has been employed in the literature to find the wormhole wave functions of the considered KS model [8,17]. We expect that (at least for the physical states) the change to such a representation be given by an integral transform \( \sigma \) of the type [12,19]:

\[
\Psi(q, \alpha, \chi) = \sigma (\Psi(X, \Pi_\phi, \theta)) = \int_{S^1} d\theta \int_{\mathbb{R}^+} d\Pi_\phi \int_\mathbb{R} dX g(X, \Pi_\phi, \theta) e^{iF(q, \alpha, \chi, X, \Pi_\phi, \theta)} e^{iX\Pi_\phi} f(\Pi_\phi, \theta),
\]

(4.1)

with \( F \) the generating function (3.8), and \( g \) a certain function on \( \mathbb{R} \times \mathbb{R}^+ \times S^1 \). Notice that on the right hand side of (4.1) we have substituted the explicit form of the physical states (3.17).

\[\text{[In fact, the definition of } \Pi_\phi \text{ in (3.11) was chosen to guarantee that eq. (3.19) holds.} \]
To fix the function $g$ that appears in the integral transform $\sigma$ we will demand the following conditions:

\[
\sigma \left( \frac{1}{\cosh X} \Psi(X, \Pi_\phi, \theta) \right) \equiv \hat{q} \Psi = q \Psi(q, \alpha, \chi), \quad (4.2)
\]

\[
\sigma \left( -\Pi \tanh X \Psi(X, \Pi_\phi, \theta) \right) \equiv \hat{\Pi} q \Psi = -i q \partial_q \Psi(q, \alpha, \chi), \quad (4.3)
\]

\[
\sigma (\Pi_\phi \cos \theta \Psi(X, \Pi_\phi, \theta)) \equiv \hat{\Pi}_\alpha \Psi = -i \partial_\alpha \Psi(q, \alpha, \chi), \quad (4.4)
\]

\[
\sigma (\Pi_\phi \sin \theta \Psi(X, \Pi_\phi, \theta)) \equiv \hat{\Pi}_\chi \Psi = -i \partial_\chi \Psi(q, \alpha, \chi), \quad (4.5)
\]

\[
\sigma \left( (\Pi_\phi \phi \cos^2 \theta - G(\theta, \hat{z})) \Psi(X, \Pi_\phi, \theta) \right) \equiv \alpha \hat{\Pi}_\alpha \Psi = -i \left( \alpha \partial_\alpha + \frac{1}{2} \right) \Psi(q, \alpha, \chi), \quad (4.6)
\]

\[
\sigma \left( (\Pi_\phi \phi \sin^2 \theta + G(\theta, \hat{z})) \Psi(X, \Pi_\phi, \theta) \right) \equiv \chi \hat{\Pi}_\chi \Psi = -i \left( \chi \partial_\chi + \frac{1}{2} \right) \Psi(q, \alpha, \chi), \quad (4.7)
\]

where we have introduced the notation

\[
G(\theta, \hat{z}) \Psi(X, \Pi_\phi, \theta) = \frac{1}{2} (\cos \theta \sin \theta \hat{z} + \hat{z} \cos \theta \sin \theta) \Psi(X, \Pi_\phi, \theta) \quad (4.8)
\]

for the symmetrized product of the functions $\cos \theta$ and $\sin \theta$ with the operator $\hat{z}$.

On the left hand side of eqs. (4.2-7) we have used the classical relations (3.4-6) to define $(\hat{q}, q \hat{\Pi}_q, \hat{\Pi}_\alpha, \alpha \hat{\Pi}_\alpha, \hat{\Pi}_\chi, \chi \hat{\Pi}_\chi)$ as functions of the elementary operators (3.10-13) in the $(X, \Pi_\phi, \theta)$ representation. In addition, the right hand sides of eqs. (4.2) and (4.4,5) provide the standard action of the operators $(\hat{q}, \hat{\Pi}_\alpha, \hat{\Pi}_\chi)$ in the $(q, \alpha, \chi)$ representation. The action of $\alpha \hat{\Pi}_\alpha$ in eq. (4.6), on the other hand, can be regarded as that of the symmetrized product of $\hat{\Pi}_\alpha$ and the multiplicative operator $\hat{\alpha}$ (with $\hat{\alpha} \Psi = \alpha \Psi(q, \alpha, \chi)$). Similar considerations apply to the definition of $\chi \hat{\Pi}_\chi$ in eq. (4.7). Finally, the action of $q \hat{\Pi}_q$ and the factor ordering on the left hand side of eqs. (4.2,3) have been chosen in such a way that, under the change of representation, the quantum version of the Hamiltonian constraint (3.7) translates into [12]

\[
\hat{H} \Psi(q, \alpha, \chi) = \frac{1}{2} \left( q^2 - (q \partial_q)^2 + \partial_\alpha^2 + \partial_\chi^2 \right) \Psi(q, \alpha, \chi) = 0, \quad (4.9)
\]

The integral transform $\sigma$ maps then physical states in the $(X, \Pi_\phi, \theta)$ representation into solutions to eq. (4.9), which is precisely the Wheeler-DeWitt equation (obtained
from constraint (3.2)) for which we know explicitly a complete set of wormhole wave functions.

Requirements (4.2-7) determine the function \( g(X, \Pi \phi, \theta) \) in (4.1) up to a constant:

\[
g = \frac{1}{\sqrt{\Pi \phi}}. \tag{4.10}
\]

With this choice for \( g \), it is possible to prove that the change of representation (4.1) is well-defined, at least, for all physical states (3.17) in the Hilbert space of the model (i.e., with \( f(\Pi \phi, \theta) \in L^2(\mathbb{R}^+ \times S^1, d\Pi \phi d\theta/\Pi \phi) \)). Substituting then eqs. (3.8) and (4.10) in (4.1) and performing the integration over the variable \( X \) [12], we arrive at the following formula for the integral transform \( \sigma \):

\[
\Psi(q, \alpha, \chi) = \int_{S^1} d\theta \int_{\mathbb{R}^+} \frac{d\Pi \phi}{\sqrt{\Pi \phi}} e^{i\Pi \phi (\alpha \cos \theta + \chi \sin \theta)} 2K_{i\Pi \phi}(q) e^{\frac{\pi}{2} \Pi \phi} f(\Pi \phi, \theta), \tag{4.11}
\]

where \( K_{i\Pi \phi} \) is a modified Bessel function of imaginary order [20]. This transformation can be inverted to regain the wave function \( f(\Pi \phi, \theta) \) that characterizes the physical states in the \((X, \Pi \phi, \theta)\) representation [12]:

\[
f(\Pi \phi, \theta) = \int_{\mathbb{R}^+} dq R(\Pi \phi) K_{i\Pi \phi}(q) \int_{\mathbb{R}} d\alpha \int_{\mathbb{R}} d\chi e^{-i\Pi \phi (\alpha \cos \theta + \chi \sin \theta)} \Psi(q, \alpha, \chi), \tag{4.12}
\]

\[
R(\Pi \phi) = \frac{2}{\pi^2} \Pi \phi \sqrt{\Pi \phi} \cosh(\pi \Pi \phi) e^{-\frac{\pi}{2} \Pi \phi}. \tag{4.13}
\]

Employing eqs. (3.20) and (4.12,13) one can also obtain the expression of the inner product in the constructed \((q, \alpha, \chi)\) representation.

Given the form of the integral transform (4.11), it is clear that, in the \((X, \Pi \phi, \theta)\) representation, all the physical states can be interpreted as superpositions of the wave functions

\[
\Psi_{\tilde{\Pi} \phi, \tilde{\theta}} = 2K_{i\tilde{\Pi} \phi}(q) e^{\frac{\pi}{2} \tilde{\Pi} \phi} e^{i\tilde{\Pi} \phi (\alpha \cos \tilde{\theta} + \chi \sin \tilde{\theta})}, \tag{4.14}
\]

with \( \tilde{\Pi} \phi \in \mathbb{R}^+ \) and \( \tilde{\theta} \in S^1 \). These wave functions were discovered (apart from an overall constant factor) by Campbell and Garay [17], and provide a complete set of wormhole states for the model. They are asymptotically damped for large
scale factors \((q \to \infty)\) and regular for non-degenerate three-geometries \((q > 0)\). However, they are singular when the three-geometry collapses to zero \((q \to 0)\), so that, in a rigorous way, they can only be considered as generalized quantum wormhole solutions.

In conclusion, what we have shown is that the Hilbert space of physical states determined in the \((q, \alpha, \chi)\) representation by the Lorentzian reality conditions is spanned by the basis of generalized wormholes \((4.14)\). In this sense, the Hilbert space of the Lorentzian theory coincides with the Hilbert space of wormhole states in the adopted representation.

Using eqs. \((4.12,13)\) it is not difficult to check that the generalized wormholes \((4.14)\) are described in the \((X, \Pi_\phi, \theta)\) representation by the wave functions

\[
f(\Pi_\phi, \theta) = \sqrt{\Pi_\phi} \delta(\Pi_\phi - \bar{\Pi}_\phi) \delta(\theta - \bar{\theta}) \equiv f_{\bar{\Pi}_\phi, \bar{\theta}}(\Pi_\phi, \theta),
\]

\((4.15)\)

\(\delta(\theta)\) being the delta function on \(S^1\). The wormhole solutions \((4.14)\) are then mutually orthogonal with respect to the physical inner product \((3.20)\), although none of them is normalizable:

\[
\langle \Psi_{\bar{\Pi}_\phi, \bar{\theta}'}, \Psi_{\bar{\Pi}_\phi, \bar{\theta}} \rangle = \delta(\bar{\Pi}_\phi' - \bar{\Pi}_\phi) \delta(\bar{\theta}' - \bar{\theta}).
\]

\((4.16)\)

Employing eqs. \((4.11)\) and \((4.14)\), on the other hand, we can rewrite the wave functions in the \((q, \alpha, \chi)\) representation as

\[
\Psi(q, \alpha, \chi) = \int_{S^1} d\bar{\theta} \int_{\mathbb{R}^+} d\bar{\Pi}_\phi \bar{f}(\bar{\Pi}_\phi, \bar{\theta}) \Psi_{\bar{\Pi}_\phi, \bar{\theta}}(q, \alpha, \chi),
\]

\((4.17)\)

where \(\bar{f}(\bar{\Pi}_\phi, \bar{\theta}) = f(\bar{\Pi}_\phi, \bar{\theta}) / \sqrt{\bar{\Pi}_\phi}\) is the contribution of the quantum wormhole \(\Psi_{\bar{\Pi}_\phi, \bar{\theta}}\) to the physical state \(\Psi\). The inner product \((3.20)\) then takes the following form:

\[
\langle \Upsilon, \Psi \rangle = \int_{S^1} d\bar{\theta} \int_{\mathbb{R}^+} d\Pi_\phi \bar{h}^*(\Pi_\phi, \theta) \bar{f}(\Pi_\phi, \theta)
\]

\((4.18)\)

(with \(\bar{h} = h / \sqrt{\bar{\Pi}_\phi}\)). As a consequence, the function \(|\bar{f}(\bar{\Pi}_\phi, \bar{\theta})|^2\) can be interpreted, for states of unit norm, as the probability to find a wormhole with parameters \(\bar{\Pi}_\phi\) and \(\bar{\theta}\), the sum of all probabilities being equal to one.
Finally, we notice that the parameters \((\Pi_\phi, \tilde{\theta})\) that describe the different wormhole wave functions (4.15) (or, equivalently, (4.14)) are determined by the eigenvalues reached in such solutions by the complete set of compatible observables \(\{\Pi_\phi, \cos \theta, \sin \theta\}\).

V KS Model: Other Wormhole Wave Functions

We now wish to consider other known sets of wormhole wave functions for the KS model under analysis.

One of these sets was first found by Campbell and Garay [17], and its elements can be obtained as Euclidean path integrals over manifolds with asymptotic large three-volume and no gravitational excitations [8]:

\[
\Psi_{\tilde{\theta}, \alpha_0, \chi_0}(q, \alpha, \chi) = e^{-q \cosh \left[ (\alpha - \alpha_0) \cos \tilde{\theta} + (\chi - \chi_0) \sin \tilde{\theta} \right]}.
\]  

(5.1)

Here, \(\alpha_0\) and \(\chi_0\) run over the real axis, and coincide with the asymptotic values of the fields \(\alpha\) and \(\chi\), respectively, in the region of large three-volume [8]. On the other hand, given that the parameters \(\tilde{\theta}\) and \(\tilde{\theta} + \pi\) lead to the same wave function in (5.1), we will restrict \(\tilde{\theta} \in [0, \pi]\) from now on.

The physical states (5.1) are obviously regular for all finite fields \(\alpha\) and \(\chi\) and non-negative scale factors \(q\) (including \(q = 0\)). They are also exponentially damped when the scale factor tends to infinity. Thus, they can be considered truly wormhole solutions of the model.

A careful calculation using formulas (4.12,13) shows that, in the \((X, \Pi_\phi, \theta)\) representation constructed in Sec. III, these wormhole states are described by the wave functions:

\[
f(\Pi_\phi, \theta) = \sqrt{\Pi_\phi} e^{-\frac{\pi}{2} \Pi_\phi} e^{-i \Pi_\phi (\alpha_0 \cos \theta + \chi_0 \sin \theta)} \left[ \delta(\theta - \tilde{\theta}) + \delta(\theta - \tilde{\theta} - \pi) \right]
\]

\[
\equiv f_{\tilde{\theta}, \alpha_0, \chi_0}(\Pi_\phi, \theta).
\]  

(5.2)
This result can be easily checked by substituting eq. (5.2) in (4.11) and employing that \[\int_R dp e^{-ip\omega} K_{ip}(q) = \pi e^{-q \cosh \omega} \quad \text{for } q > 0. \tag{5.3}\]

From eq. (5.2) and the form of the physical inner product (3.20) it is then straightforward to conclude that the wormhole wave functions (5.1) do not correspond to normalizable physical states.

Another set of solutions to the constraint (4.9) is provided by [17]

\[\Psi_{n,\tilde{\theta}}(q, \alpha, \chi) = \varphi_n(u) \varphi_n(v) = \frac{1}{2^n n! \sqrt{\pi}} H_n(u) H_n(v) e^{-q \cosh (\alpha \cos \tilde{\theta} + \chi \sin \tilde{\theta})}, \tag{5.4}\]

\[u = \sqrt{2q} \cosh \left(\frac{\alpha \cos \tilde{\theta} + \chi \sin \tilde{\theta}}{2}\right), \quad v = \sqrt{2q} \sinh \left(\frac{\alpha \cos \tilde{\theta} + \chi \sin \tilde{\theta}}{2}\right), \tag{5.5}\]

where \(\varphi_n\) are the harmonic-oscillator wave functions (2.5), \(H_n\) the Hermite polynomials [21], and \(n\) is a non-negative integer (\(n = 0, 1, ...\)). Given the parity properties of the Hermite polynomials, the quantum states \(\Psi_{n,\tilde{\theta}}\) and \(\Psi_{n,\tilde{\theta}+\pi}\) differ just by a factor of \((-1)^n\). In order to consider only linearly independent solutions, it will thus suffice to analyze the sector \(\tilde{\theta} \in [0, \pi)\).

The above wave functions represent proper quantum wormholes, because they are regular everywhere in the configuration space \(\{q > 0, \alpha, \chi \in R\}\), decrease exponentially for large values of the scale factor \(q\), and have a well-defined limit when the three-geometry degenerates (i.e., when \(q \to 0\)). Moreover, they actually form a basis for the Hilbert space of the Lorentzian model, for all the elements of the basis (4.14) can be expressed as superpositions of the wormhole solutions (5.4,5) [11,12]:

\[\Psi_{\tilde{\pi},\tilde{\theta}}(q, \alpha, \chi) = \sum_{n=0}^{\infty} F_n(\tilde{\pi},\tilde{\theta}) \Psi_{n,\tilde{\theta}}(q, \alpha, \chi), \tag{5.6}\]

\[F_n(\tilde{\pi},\tilde{\theta}) = 2\sqrt{\pi} e^{\pi \tilde{\pi}} \int_R d\eta e^{2\tilde{\pi}\eta} \frac{\sinh^n \eta}{\cosh^{n+1} \eta}. \tag{5.7}\]

We conclude in this way that the Hilbert space of the constructed quantum theory admits indeed a basis of wormhole states, in agreement with our comments in the
Introduction.

The wave functions (3.17) that correspond to the wormholes \( \Psi_{n,\tilde{\theta}} \) in the \((X, \Pi_\phi, \theta)\) representation can be obtained using eqs. (4.12,13):

\[
f(\Pi_\phi, \theta) = \sqrt{\frac{\Pi_\phi}{2\pi^2}} e^{-\pi \Pi_\phi F_n^*(\Pi_\phi)} [\delta(\theta - \tilde{\theta}) + (-1)^n \delta(\theta - \tilde{\theta} - \pi)] \equiv f_{n,\tilde{\theta}}(\Pi_\phi, \theta),
\]

with \( \tilde{\theta} \in [0, \pi) \). It is then easy to check that these wormhole solutions are not normalizable with respect to the physical inner product (3.20).

From the wormhole wave functions (5.4,5), nevertheless, one can arrive at a new basis of states whose elements have all finite norm, as we will show in the rest of this section. Let us first substitute relation (5.6) (and (4.14)) in eq. (4.11). Assuming that we can interchange the order of summation and integration, we get the following expression for all physical states in the Hilbert space of the model:

\[
\Psi(q, \alpha, \chi) = \sum_{n=0}^{\infty} \int_0^\pi d\theta \, 2h_n(\theta) \Psi_{n,\theta}(q, \alpha, \chi),
\]

(5.9)

where

\[
h_n(\theta) = \int_{\mathbb{R}^+} \frac{d\Pi_\phi}{2\sqrt{\Pi_\phi}} F_n(\Pi_\phi) [f(\Pi_\phi, \theta) + (-1)^n f(\Pi_\phi, \theta + \pi)].
\]

(5.10)

From eqs. (5.4,5) and (5.10), it is not difficult to prove that the functions

\[
\tilde{\Psi}_{n,\theta}(q, \alpha, \chi) = e^{in\theta} \Psi_{n,\theta}(q, \alpha, \chi), \quad \tilde{h}_n(\theta) = e^{-in\theta} h_n(\theta)
\]

(5.11)

are periodic in \( \theta \), with period equal to \( \pi \). Therefore, they both admit a Fourier expansion in the unit circle described by the angle \( \beta = 2\theta \). As a consequence, eq. (5.9) can be rewritten in the form

\[
\Psi(q, \alpha, \chi) = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} \tilde{h}_{nm} \tilde{\Psi}_{nm}(q, \alpha, \chi),
\]

(5.12)

with

\[
\tilde{h}_{nm} = \int_{S^1} d\beta \frac{e^{-im\beta}}{\sqrt{2\pi}} \tilde{h}_n \left( \frac{\beta}{2} \right),
\]

(5.13)

\(^3\text{We recall that } f(\Pi_\phi, \theta) \text{ is defined on } \mathbb{R}^+ \times S^1.\)
\[
\tilde{\Psi}_{nm}(q, \alpha, \chi) = \int_{S^1} \frac{d\beta}{\sqrt{2\pi}} e^{im\beta} \tilde{\Psi}_{n,\beta/2}(q, \alpha, \chi). \tag{5.14}
\]

Eq. (5.12) implies that all the normalizable physical states are superpositions of the wave functions \{\tilde{\Psi}_{nm}, n = 0, 1, \ldots, m = 0, \pm 1, \ldots\}, which thus provide a discrete basis for the Hilbert space of the Lorentzian model.

To find the expression of the states \tilde{\Psi}_{nm} in the \((X, \Pi_\phi, \theta)\) representation one can use eqs. (4.12,13), (5.4,5), (5.11) and (5.14). In fact, that amounts to replace the function \tilde{\Psi}_{n,\beta/2}(q, \alpha, \chi) on the right hand side of (5.14) by \(e^{in\beta/2} f_{n,\beta/2}(\Pi_\phi, \theta)\) (with \(f_{n,\beta/2}\) given by eq. (5.8)) and perform then the integration over \(\beta\). In doing so, one arrives at the result:

\[
f(\Pi_\phi, \theta) = e^{i(2m+n)\theta} \sqrt{\Pi_\phi} \cosh (\pi \Pi_\phi) e^{-\pi \Pi_\phi} F_n^*(\Pi_\phi) \equiv f_{nm}(\Pi_\phi, \theta), \tag{5.15}
\]

and, from eq. (3.17),

\[
\tilde{\Psi}_{nm}(X, \Pi_\phi, \theta) = e^{iX\Pi_\phi} f_{nm}(\Pi_\phi, \theta). \tag{5.16}
\]

Substituting now eq. (5.15) in the formula of the inner product (3.20), we get that

\[
< \tilde{\Psi}_{pq}, \tilde{\Psi}_{nm} > = \delta_{pq+2m+n+2m} \int_{\mathbb{R}^+} \frac{d\Pi_\phi}{\pi^4} \cosh^2(\pi \Pi_\phi) e^{-2\pi \Pi_\phi} F_p(\Pi_\phi) F_n^*(\Pi_\phi), \tag{5.17}
\]

with \(\delta_{nm}\) the Kronecker delta. The integral over \(\Pi_\phi\) in the above equation converges for all possible values of \(p\) and \(n\), because the integrand is continuous \(\forall \Pi_\phi \in (0, \infty)\), has a finite limit when \(\Pi_\phi\) tends to zero, and decrease faster than \(\Pi_\phi^{-1}\) at infinity.

This last statement follows from the form of the function \(F_n(\Pi_\phi)\) (5.7) and the fact that [12]

\[
\left| \lim_{\Pi_\phi \to \infty} e^{\pi \Pi_\phi} \Pi_\phi^2 \int_{\mathbb{R}} d\eta e^{\pm i2\Pi_\phi \eta} \frac{\sinh^n \eta}{\cosh^{n+1} \eta} \right| < \infty. \tag{5.18}
\]

Therefore, the wave functions \(\tilde{\Psi}_{nm}\) turn out to be normalizable with respect to the physical inner product (3.20). Since the basis \{\tilde{\Psi}_{nm}\} is discrete, we conclude that it can be orthonormalized following the standard Gram-Schmidt technique. Finally, we notice that the Hilbert space of the considered Lorentzian model is then separable, for it admits a discrete basis of orthonormal states.
VI Conclusions

We have analyzed the Hilbert structure of the space of wave functions spanned by the quantum wormhole solutions in a FRW model provided with a conformally coupled scalar field and in a KS spacetime minimally coupled to a massless scalar field. We have proved that the inner product in such a space can be uniquely fixed by imposing a set of Lorentzian reality conditions. The Hilbert spaces determined in this way have been shown to coincide, for the adopted representations, with those of the respective Lorentzian models under study. This implies in particular that all the normalizable physical states can be interpreted in the Lorentzian theory as superpositions of wormholes.

The results of this work, together with those presented in Ref. [12] (where the case of a FRW spacetime with a minimally coupled scalar field was considered), prove that, in all the minisuperspace models in which a complete set of wormhole wave functions is known, one can carry out to completion the quantization of the Lorentzian theory following Ashtekar’s program, and that, with an adequate choice of representation, the obtained Hilbert space turns out to admit a basis of quantum wormhole states. In all these models, we have seen in addition that it is possible to find a discrete orthonormal basis of wormholes, so that the Hilbert spaces of the corresponding Lorentzian systems are separable.

For Lorentzian General Relativity, we expect that the Hilbert space of wormholes can be identified as the physical Hilbert space if: 1) one can find a representation for full gravity in which there exist wormhole wave functions, and 2) the vector space spanned by those wave functions is stable under the action of the gravitational observables. If this is the case, one can choose the vector space of wormholes as the (restricted) representation space for the quantum theory of gravity and complete the quantization by demanding an appropriate set of reality conditions. By construction,
the Hilbert space of physical states will then be spanned by a basis of wormholes. We notice that such a quantization will be meaningful only if the space of wormholes is neither empty nor trivial, therefore the first condition. The second condition, on the other hand, guarantees that the reality conditions on the observables can be imposed as adjointness relations in the Hilbert space of wormhole solutions.

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