Odd degree number fields with odd class number

Wei Ho, Arul Shankar, and Ila Varma

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Abstract

For every odd integer \( n \geq 3 \), we prove that there exist infinitely many fields of degree \( n \) whose class number is odd. To do so, we study the ideal class groups of families of degree \( n \) number fields whose rings of integers arise as the coordinate rings of the subschemes of \( \mathbb{P}^1 \) cut out by integral binary \( n \)-ic forms. By obtaining upper bounds on the mean number of 2-torsion elements in the class groups of fields in these families, we prove that a positive proportion (tending to 1 as \( n \) tends to \( \infty \)) of such fields have trivial 2-torsion subgroup in their class groups and narrow class groups. Conditional on a tail estimate, we also prove the corresponding lower bounds and obtain the exact values of these averages. These mean numbers coincide with the predictions from the Cohen-Lenstra-Martinet-Malle heuristics for the family of all degree \( n \) \( S_n \)-fields.

Additionally, for any degree \( n \) order \( O_f \) arising from an integral binary \( n \)-ic form \( f \), we compare the sizes of \( \text{Cl}_2(O_f) \), the 2-torsion subgroup of ideal classes in \( O_f \), and \( \mathcal{I}_2(O_f) \), the 2-torsion subgroup of ideals in \( O_f \). For the family of orders arising from integral binary \( n \)-ic forms and contained in fields with fixed signature \((r_1,r_2)\), we prove that the mean value of the difference \( |\text{Cl}_2(O_f)| - 2^{1-r_1-r_2}|\mathcal{I}_2(O_f)| \) is equal to 1, generalizing a result of Bhargava and the third-named author for cubic fields. Conditional on certain tail estimates, we also prove that the mean value of \( |\text{Cl}_2(O_f)| - 2^{1-r_1-r_2}|\mathcal{I}_2(O_f)| \) remains 1 for families obtained by imposing local splitting and maximality conditions.

1 Introduction

The Cohen-Lenstra heuristics [14] give precise predictions for the distribution of ideal class groups in families of quadratic fields. Very few cases of these conjectures have been proven; among them are the celebrated results of Davenport–Heilbronn [17] on the average number of 3-torsion elements in the class groups of quadratic fields, and of Fouvry–Kluners [19] on the 4-ranks of the class groups of quadratic fields. These heuristics were later generalized by Cohen–Martinet [15] to describe the distribution of ideal class groups in fixed degree families of number fields. In 2010, Malle [21] proposed a modification of Cohen–Martinet’s heuristics [15] to account for observed variations in the asymptotic behavior of the \( p \)-part of the class groups of families of number fields over a base field containing the \( p \)th roots of unity; for example, for \( p = 2 \), the modified heuristics yield predictions for the 2-part of the class groups of number fields over \( \mathbb{Q} \). In particular, Conjecture 2.1 and Proposition 2.2 of [21] predict the mean number of 2-torsion ideal classes of certain odd degree families of number fields:

**Conjecture 1 (Cohen-Lenstra-Martinet-Malle).** Fix an odd integer \( n \geq 3 \) and a pair of nonnegative integers \((r_1,r_2)\) such that \( r_1 + 2r_2 = n \). Consider the set of isomorphism classes of degree \( n \) number fields with signature \((r_1,r_2)\), i.e., with \( r_1 \) real embeddings and \( r_2 \) pairs of complex embeddings. The average number of 2-torsion elements in the ideal class groups of such fields is

\[
1 + 2^{1-r_1-r_2}
\]

when these fields are ordered by discriminant.
The only verified cases of Conjecture 1 are for \( n = 3 \), due to Bhargava [2]. The original proof involves associating quartic fields to index 2 subgroups in the class groups of cubic fields, and then proving results on the density of quartic fields to conclude that the mean number of 2-torsion elements in the class groups of totally real (respectively, complex) cubic fields ordered by discriminant is \( 3/2 \) (resp. \( 5/4 \)). A more direct proof is given by Bhargava and the third-named author [10], using the parametrization of 2-torsion ideal classes of cubic fields (and cubic orders) by pairs of ternary quadratic forms due to Bhargava [1, Thm. 4]. In this paper, we give theoretical evidence for Conjecture 1 for odd \( n \geq 3 \) by generalizing the strategy in [10], using Wood’s parametrization of 2-torsion ideal classes in certain number fields by pairs of \( n \)-ary quadratic forms from [26]. To state our main results, we first describe the families of number fields we study.

For an integer \( n \geq 3 \), to a nonzero integral binary \( n \)-ic form \( f \in \text{Sym}_n(\mathbb{Z}^2) \), we may naturally associate the coordinate ring \( R_f \) of the subscheme of \( \mathbb{P}^1_{\mathbb{Z}} \) cut out by \( f \) (see Nakagawa [22] and Wood [25]). Define the family \( \mathcal{R}_H \) to be the multiset of rings

\[
\mathcal{R}_H = \{ R_f \mid f \in \text{Sym}_n(\mathbb{Z}^2) \},
\]

equipped with the \textit{height} ordering \( H \), where \( H(R_f) = H(f) \) is defined as the maximum absolute value of the coefficients of the binary \( n \)-ic form \( f \). It is important to note that although two rings in \( \mathcal{R}_H \) may be isomorphic, their heights need not be equal. For example, if \( \gamma \in \text{SL}_2(\mathbb{Z}) \), and we define the action \( \gamma f(x, y) := f((x, y)\gamma) \) on the space of integral binary \( n \)-ic forms, then \( R_f \cong R_{\gamma f} \), but it is not in general true that \( H(f) = H(\gamma f) \). Nevertheless, there is a well-defined isomorphism class of rings \( R_{\gamma f} \) associated to an \( \text{SL}_2(\mathbb{Z}) \)-orbit \([f] \in \text{SL}_2(\mathbb{Z})\backslash \text{Sym}_n(\mathbb{Z}^2)\) since \( R_{\gamma f} \) is isomorphic to \( R_f \) for any \( \gamma \in \text{SL}_2(\mathbb{Z}) \). Such orbits \([f] \) may be ordered by their \textit{Julia invariant} (see [20]). Thus, we also define the family \( \mathcal{R}_J \) to be the multiset of rings

\[
\mathcal{R}_J = \{ R_{[f]} \mid [f] \in \text{SL}_2(\mathbb{Z})\backslash \text{Sym}_n(\mathbb{Z}^2) \},
\]

ordered by Julia invariant \( J \), where \( J(R_{[f]}) = J([f]) = J(\gamma f) \) for any \( \gamma \in \text{SL}_2(\mathbb{Z}) \) is defined in §3.3. Asymptotics on the size of \( \mathcal{R}_J \) were obtained by Bhargava–Yang [12].

In this paper, we compute averages taken over certain families contained in \( \mathcal{R}_H \) or \( \mathcal{R}_J \). For any integer \( n \geq 3 \), a \textit{signature} is a pair of nonnegative integers \((r_1, r_2)\) satisfying \( r_1 + 2r_2 = n \), and a degree \( n \) number field has signature \((r_1, r_2)\) if and only if it has \( r_1 \) real embeddings and \( r_2 \) pairs of conjugate complex embeddings. Let \( \mathcal{R}_{H,r_1,r_2} \subset \mathcal{R}_H \) and \( \mathcal{R}_{J,r_1,r_2} \subset \mathcal{R}_J \) be the respective subfamilies consisting of all Gorenstein\(^1\) integral domains whose fraction field has signature \((r_1, r_2)\). Also, let \( \mathcal{R}_{H,r_1,r_2}^{\text{max}} \subset \mathcal{R}_{H,r_1,r_2} \) (resp. \( \mathcal{R}_{J,r_1,r_2}^{\text{max}} \subset \mathcal{R}_{J,r_1,r_2} \)) be the subfamily containing all maximal orders. It is worthwhile to note that a given order \( \mathcal{O} \) in a number field with signature \((r_1, r_2)\) may occur in \( \mathcal{R}_{H,r_1,r_2} \) or \( \mathcal{R}_{J,r_1,r_2} \), but not in \( \mathcal{R}_{H,r_1,r_2}^{\text{max}} \) or \( \mathcal{R}_{J,r_1,r_2}^{\text{max}} \) by a result of Birch–Merriman [13].

For any \( \Sigma_H \subset \mathcal{R}_{H,r_1,r_2}^{\text{max}} \) and \( \Sigma_J \subset \mathcal{R}_{J,r_1,r_2}^{\text{max}} \), we denote the average number of 2-torsion elements of class groups over \( \Sigma_H \) ordered by height and over \( \Sigma_J \) ordered by Julia invariant as follows:

\[
\text{Avg}_H(\Sigma_H, \text{Cl}_2) = \lim_{X \to \infty} \frac{\sum_{R_f \in \Sigma_H} |\text{Cl}_2(R_f)|}{\sum_{R_f \in \Sigma_H} 1} \quad \text{and} \quad \text{Avg}_J(\Sigma_J, \text{Cl}_2) = \lim_{X \to \infty} \frac{\sum_{R_{[f]} \in \Sigma_J} |\text{Cl}_2(R_{[f]})|}{\sum_{R_{[f]} \in \Sigma_J} 1}, \tag{1}
\]

\(^1\)Note that the ring \( R_f \) is Gorenstein if and only if \( f \) is \textit{primitive}, i.e., the coefficients of \( f \) do not share any common prime factors.
where $\text{Cl}_2(R_f)$ denotes the 2-torsion subgroup of the class group of $R_f$. By replacing $\text{Cl}_2(R_f)$ with the 2-torsion subgroup $\text{Cl}_2^+(R_f)$ of the narrow class group in the right hand side of the equalities in (1), we may analogously study the average number of 2-torsion elements of narrow class groups over $\Sigma_H$ ordered by height and over $\Sigma_J$ ordered by Julia invariant; we denote these means by $\text{Avg}_H(\Sigma_H, \text{Cl}_2^+)$ and $\text{Avg}_J(\Sigma_J, \text{Cl}_2^+)$, respectively. The notation $\text{Avg}_H(\ast, \ast) \leq c$ or $\text{Avg}_J(\ast, \ast) \leq c$ denotes that the limsup of the fractions analogous to those in (1) are bounded by the constant $c$.

**Theorem 2.** Fix an odd integer $n > 3$ and a corresponding signature $(r_1, r_2)$. Then

(a) $\text{Avg}_H(\mathcal{R}_{H,\max}^{r_1,r_2}, \text{Cl}_2) \leq 1 + 2^{1-r_1-r_2}$ and $\text{Avg}_J(\mathcal{R}_{J,\max}^{r_1,r_2}, \text{Cl}_2) \leq 1 + 2^{1-r_1-r_2}$, and

(b) $\text{Avg}_H(\mathcal{R}_{H,\max}^{r_1,r_2}, \text{Cl}_2^+) \leq 1 + 2^{-r_2}$ and $\text{Avg}_J(\mathcal{R}_{J,\max}^{r_1,r_2}, \text{Cl}_2^+) \leq 1 + 2^{-r_2}$.

If the tail estimates in (31) hold, then both (a) and (b) are equalities. Additionally, the same upper bounds (and conditional equalities) hold when imposing any finite set of local conditions on the fields in $\mathcal{R}_{H,\max}^{r_1,r_2}$ and $\mathcal{R}_{J,\max}^{r_1,r_2}$.

When $n = 3$, the Julia invariant of a ring $R_f$ associated to a binary cubic form $f$ coincides with its discriminant, and the family $\mathcal{R}_f$ is essentially the same as the family of all cubic rings ordered by discriminant. The family $\mathcal{R}_H$ contains all cubic rings, and each such ring occurs infinitely often. We prove that the Cohen-Lenstra-Martinet-Malle heuristics are true for the multiset of cubic fields ordered by height and further determine the mean size of 2-torsion subgroups for narrow class groups over $\mathcal{R}_{H,\max}^{r_1,r_2}$.

**Theorem 3.** We have

(a) $\text{Avg}_H(\mathcal{R}_{H,\max}^{3,0}, \text{Cl}_2) = 5/4$,

(b) $\text{Avg}_H(\mathcal{R}_{H,\max}^{1,1}, \text{Cl}_2) = 3/2$, and

(c) $\text{Avg}_H(\mathcal{R}_{H,\max}^{3,0}, \text{Cl}_2^+) = 2$.

The values in Theorem 3 agree with those in [2] and [10], where the cubic fields are ordered by discriminant; this gives concrete evidence that the Cohen-Lenstra and Cohen-Martinet-Malle heuristics hold for any natural ordering of fields. Additionally, Theorems 2 and 3 suggest that for the set of all isomorphism classes of odd degree number fields with signature $(r_1, r_2)$, we might expect the average number of 2-torsion elements in the narrow class groups of such fields to be $1 + 2^{-r_2}$, as conjectured in forthcoming work by Dummit–Voight [18]. The results of [10] combined with Theorem 3(c) prove this fact in the cubic case, whether ordering by discriminant or height.

Theorems 2 and 3 in combination with results of [13] immediately imply that most number fields within these families have no nontrivial 2-torsion elements in their ideal class groups. More precisely, we have:

**Theorem 4.** Fix an odd integer $n \geq 3$ and signature $(r_1, r_2)$.

(a) There is an infinite number of fields with signature $(r_1, r_2)$ that have odd class number.

(b) If $r_2 \geq 1$, then there is an infinite number of fields with signature $(r_1, r_2)$ that have odd narrow class number. Thus, there is an infinite number of fields with signature $(r_1, r_2)$ for which the narrow class number equals the class number.
Additionally, Theorem 4(b) implies that there exists an infinite number of odd degree fields with at least one complex embedding that have units of all signatures.\(^2\)

Our methods are not limited to studying maximal orders in number fields; we also study the ideal class groups of general orders in \(\mathcal{R}_{r_1}^{r_1,r_2}\) and \(\mathcal{R}_{r_1}^{r_1,r_2}\). Specifically, for each odd \(n \geq 3\), we compute on average how many 2-torsion ideal classes in the class groups of such orders arise from nontrivial elements of order 2 in the ideal groups of such orders. More precisely, if \(\mathcal{O}\) is an order in a number field, let the ideal group \(\mathcal{I}(\mathcal{O})\) be the group of invertible fractional ideals of \(\mathcal{O}\); recall that the class group \(\text{Cl}(\mathcal{O})\) is defined as the quotient of \(\mathcal{I}(\mathcal{O})\) by the subgroup of principal ideals in \(\mathcal{I}(\mathcal{O})\). Denote the \(p\)-torsion subgroups of \(\text{Cl}(\mathcal{O})\) and \(\mathcal{I}(\mathcal{O})\) by \(\text{Cl}_p(\mathcal{O})\) and \(\mathcal{I}_p(\mathcal{O})\) for any prime \(p\). Although \(\mathcal{I}_p(\mathcal{O})\) is trivial for maximal orders \(\mathcal{O}\), this is not always true for non-maximal orders \(\mathcal{O}\).

In [10], the authors show that the mean value of the difference \(|\text{Cl}_2(\mathcal{O})| - \frac{1}{2^{r_1+r_2-1}}|\mathcal{I}_2(\mathcal{O})|\) is equal to 1, when averaging over maximal orders \(\mathcal{O}\) in cubic fields of a fixed signature \((r_1, r_2)\), over all orders in such cubic fields, or even over certain acceptable families of orders defined by local conditions (in all cases ordered by discriminant). An analogous result is also known for 3-torsion ideal classes of acceptable families of quadratic orders and fields (see [11]). In this paper, we obtain a similar statement for \(\mathcal{R}_{r_1}^{r_1,r_2}\) and \(\mathcal{R}_{r_1}^{r_1,r_2}\):

**Theorem 5.** Fix an odd integer \(n \geq 3\) and signature \((r_1, r_2)\).

(a) The average size of

\[
|\text{Cl}_2(\mathcal{O})| - \frac{1}{2^{r_1+r_2-1}}|\mathcal{I}_2(\mathcal{O})|
\]

over \(\mathcal{O} \in \mathcal{R}_{r_1}^{r_1,r_2}\) ordered by height or over \(\mathcal{O} \in \mathcal{R}_{r_1}^{r_1,r_2}\) ordered by Julia invariant is 1.

(b) The average size of

\[
|\text{Cl}_2^{+}(\mathcal{O})| - \frac{1}{2^{r_2}}|\mathcal{I}_2(\mathcal{O})|
\]

over \(\mathcal{O} \in \mathcal{R}_{r_1}^{r_1,r_2}\) ordered by height or over \(\mathcal{O} \in \mathcal{R}_{r_1}^{r_1,r_2}\) ordered by Julia invariant is 1.

In fact, we prove a much stronger statement indicating that the above averages remain equal to 1 when taken over any very large family in \(\mathcal{R}_{r_1}^{r_1,r_2}\) or \(\mathcal{R}_{r_1}^{r_1,r_2}\) (see Definition 6.1). For any acceptable family (as defined in [3.1] in \(\mathcal{R}_{r_1}^{r_1,r_2}\) or \(\mathcal{R}_{r_1}^{r_1,r_2}\), the analogous averages are shown to have an upper bound equal to 1; furthermore, conditional on the tail estimates in (31), averages over acceptable families in \(\mathcal{R}_{r_1}^{r_1,r_2}\) and \(\mathcal{R}_{r_1}^{r_1,r_2}\) also have lower bound equal to 1 (see Theorem 6.2). Some important acceptable families include \(\mathcal{R}_{H,\text{max}}^{r_1,r_2}\) and \(\mathcal{R}_{J,\text{max}}^{r_1,r_2}\), as well as subfamilies of \(\mathcal{R}_{H,\text{max}}^{r_1,r_2}\) and \(\mathcal{R}_{J,\text{max}}^{r_1,r_2}\) that are defined by local conditions at any finite set of primes.

The main strategy to prove Theorems 2, 3, and 5 relies on Wood’s parametrization [26] of 2-torsion ideal classes of rings in \(\mathcal{R}_{r_1}^{r_1,r_2}\) and \(\mathcal{R}_{r_1}^{r_1,r_2}\) by certain integral orbits of the representation \(\mathbb{Z}^2 \otimes \text{Sym}^2(\mathbb{Z}^n)\), and then computing asymptotic counts of the relevant orbits using geometry-of-numbers techniques developed by [2, 3, 7]. However, the necessary geometry-of-numbers arguments are complicated by the fact that we simultaneously consider an infinite set of representations, namely one for each odd \(n \geq 3\). Similar infinite sets have been handled previously in [4, 6].

When ordering by height, we study the orbits of \(\text{SL}_n(\mathbb{Z})\) acting on the space \(\mathbb{Z}^2 \otimes \text{Sym}_2(\mathbb{Z}^n)\) of pairs \((A, B)\) of integral \(n\)-ary quadratic forms. Each such pair gives rise to an invariant binary \(n\)-ic form

\[
f_{(A,B)}(x, y) := \det(Ax - By)
\]

\(^2\)Recall that for any number field \(K\) with \(r_1\) distinct real embeddings, there is a signature homomorphism \(\mathcal{O}_K^\times \to \{\pm 1\}^{r_1}\) that takes a unit to its signature, i.e., the product of its image under each real embedding in \(\{\pm 1\}^{r_1}\).
when $A$ and $B$ are viewed as symmetric $n \times n$ matrices. If $R_f \in \mathfrak{R}_H^{r_1,r_2}$ for some signature $(r_1,r_2)$, then certain \textit{projective} $\text{SL}_n(\mathbb{Z})$-orbits of pairs $(A,B)$ with invariant binary $n$-ic $f_{(A,B)} = f$ are equipped with a composition law coming from the group structure on the 2-torsion subgroup of the class group of $R_f$. This implies that the number of such orbits is determined by the number of 2-torsion ideal class elements of $R_f$. Thus, to compute the averages when ordering by height in Theorem 5, we compare the number of rings (with multiplicity) in $\mathfrak{R}_H^{r_1,r_2}$ of bounded height to the number of relevant $\text{SL}_n(\mathbb{Z})$-orbits whose binary $n$-ic invariant is bounded by the same height. To obtain Theorems 2 and 3, we restrict to maximal orders, namely those rings $R_f \in \mathfrak{R}_{H,\max}^{r_1,r_2}$; however, a conjectural tail estimate is required to obtain a lower bound.

When ordering by Julia invariant, we count the number of $\text{SL}_2(\mathbb{Z}) \times \text{SL}_n(\mathbb{Z})$-orbits of $\mathbb{Z}^2 \otimes \text{Sym}_2(\mathbb{Z}^n)$ relative to the number of $\text{SL}_2(\mathbb{Z})$-orbits of $\text{Sym}_n(\mathbb{Z}^2)$. As described above, the rings $R_f$ associated to a binary $n$-ic form $f$ are invariant under the action of $\text{SL}_2(\mathbb{Z})$ on $f$, i.e., for any $\gamma f \in [f] = \text{SL}_2(\mathbb{Z}) \cdot f$, we have $R_{\gamma f} \cong R_f$. It follows from [26] that if $O_f \in \mathfrak{R}_j^{r_1,r_2}$ for some signature $(r_1,r_2)$, then \textit{projective} $\text{SL}_2(\mathbb{Z}) \times \text{SL}_n(\mathbb{Z})$-orbits of pairs of $n$-ary quadratic forms $(A,B)$ with $[f_{(A,B)}] = [f]$ are in bijection with 2-torsion elements of the class group of $O_f$. We then use the same geometry-of-numbers methods utilized when ordering by height to conclude Theorems 2 and 5 when ordering by Julia invariant. Note that when $n = 3$, the Julia invariant coincides with the discriminant of a binary cubic form, and so our argument can be viewed as a generalization of that given in [10].

We now give a short description of the organization of the paper. In Section 2, we recall and expand on the details of the construction of rank $n$ rings $R_f$ from binary $n$-ic forms $f$ given in [22, 25] as well as a correspondence between $\text{SL}_n$-orbits of pairs of $n$-ary quadratic forms and order 2 ideal classes of such rings $R_f$ from [26]. Section 4 focuses on using geometry-of-numbers methods to count the appropriate projective integral orbits of pairs of $n$-ary quadratic forms, i.e., those with reducible and nondegenerate binary $n$-ic invariant $f$, which is equivalent to restricting to rings in $\mathfrak{R}_H^{r_1,r_2}$ or $\mathfrak{R}_J^{r_1,r_2}$. In Section 5, we describe several sieves that allow us to restrict our count from Section 4 to orbits that correspond to invertible ideal classes in orders (or maximal orders). Finally, in Section 6, the analytic methods in Sections 4 and 5 are combined with the algebraic interpretation of the orbits given in Section 2 to conclude the main results.

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2 Parametrizations of 2-torsion ideal classes and composition laws

Let $n \geq 3$ be a fixed odd integer. In this section, we begin by recalling from [22, 25] how rings of rank $n$ naturally arise from integral binary $n$-ic forms. We then recall the parametrization given in [26] of 2-torsion ideal classes in such rings by orbits of pairs of $n$-ary quadratic forms. In §2.3, we describe a composition law for certain orbits of pairs of $n$-ary quadratic forms, arising from the group law on ideal classes in rings. In §2.4, we discuss “reducible” elements in the space of such integral pairs and the properties of the corresponding 2-torsion ideal classes via the parametrization; these are elements that will be excluded in the volume computations in later sections. Finally, in §2.5, we use a rigidified version of the parametrization theorem in [26] over principal ideal domains to explicitly describe the stabilizers and orbits of these representations for a few specific base rings.
2.1 Rings associated to binary $n$-ic forms

We first describe the construction of a rank $n$ ring over $\mathbb{Z}$ and ideals from an integral binary $n$-ic form. Let $f(x, y) = f_0 x^n + f_1 x^{n-1} y + \cdots + f_n y^n$, where $f_i \in \mathbb{Z}$. We begin with the case where $f_0 \neq 0$, and let $B_{f_0} = \mathbb{Z}[\frac{1}{f_0}]$. Define the ring $R_f$ as a subring of $B_{f_0}[\theta]/f(\theta, 1)$, generated as a $\mathbb{Z}$-module as

$$R_f = \langle 1, f_0 \theta, f_0 \theta^2 + f_1 \theta, \ldots, f_0 \theta^{n-1} + f_1 \theta^{n-2} + \cdots + f_{n-2} \theta \rangle. \quad (2)$$

For $k > 0$, define $\zeta_k = f_0 \theta^k + \cdots + f_{k-1} \theta$, and let $\zeta_0 = 1$. (Both [22] and [25] show that $R_f = \langle \zeta_0, \ldots, \zeta_{n-1} \rangle$ is closed under multiplication.) We define the following $\mathbb{Z}$-submodule of $B_{f_0}[\theta]/f(\theta, 1)$:

$$I_f = \langle 1, \theta, \zeta_2, \ldots, \zeta_{n-1} \rangle. \quad (3)$$

As shown in [22, 25], the module $I_f$ is closed under multiplication by elements of $R_f$ and thus is an ideal of $R_f$. It is easy to check that for $0 \leq k \leq n - 1$, we have

$$I_f^k = \langle 1, \theta, \theta^2, \ldots, \theta^k, \zeta_{k+1}, \ldots, \zeta_{n-1} \rangle \quad (4)$$

as a $\mathbb{Z}$-submodule of $B_{f_0}[\theta]/f(\theta, 1)$. For odd $n$, the ideal $I_f^{n-3}$ is a square of the ideal $I_f^{n-3}$, which has the following explicit basis as a $\mathbb{Z}$-module:

$$I_f^{n-3} = \langle 1, \theta, \theta^2, \cdots, \theta^{n-2}, \zeta_{n-3+1}, \ldots, \zeta_{n-1} \rangle.$$  

Additionally, there is a natural action of $\gamma \in \text{GL}_2(\mathbb{Z})$ on the set of binary $n$-ic forms $f$ sending $\gamma \cdot f(x, y) = f((x, y)\gamma)$; under this action, the ring $R_f$ and the ideal $I_f$ (and its powers) are invariant (up to isomorphism). If $f$ is irreducible, then $R_f$ is an order of $\mathbb{Q}[\theta]/f(\theta, 1)$, and the discriminants of $R_f$ and $f$ coincide [22, Proposition 1.1]. In addition, the form $f$ is primitive (i.e., the gcd of its coefficients is 1) if and only if $R_f$ is Gorenstein, which is equivalent to the property that $I_f$ is an invertible fractional ideal [25, Prop. 2.1 and Cor. 2.3].

In fact, by recording the basis (4), the ideals $I_f^k$ may be considered as based ideals of $R_f$, i.e., ideals of $R_f$ along with an ordered basis as a rank $n$ $\mathbb{Z}$-module. The norm $N(I)$ of a based ideal $I$ of $R_f$ is the determinant of the $\mathbb{Z}$-linear transformation taking the chosen basis of $I$ to the basis of $R_f$ given by (2).

We also introduce dual elements to $\theta^k$ for all $0 \leq k \leq n - 1$. Let $\{\hat{\theta}_0, \hat{\theta}_1, \ldots, \hat{\theta}_{n-1}\}$ be the $B_{f_0}$-module basis of $\text{Hom}_{B_{f_0}}(B_{f_0}[\theta]/f(\theta, 1), B_{f_0})$ dual to $\{1, \theta, \theta^2, \ldots, \theta^{n-1}\}$. Additionally, define $\hat{\zeta}_{n-1} := \frac{\hat{\theta}_0}{f_0}$, and note that $\hat{\zeta}_{n-1}(\zeta_k) = \delta_{k, n-1}$ for all $0 \leq k \leq n - 1$. In [26, Proposition 2.1], Wood computes that for any $r \in B_{f_0}[\theta]/f(\theta, 1)$ and $0 \leq k \leq n - 2$,

$$\hat{\theta}_k(r) = \hat{\zeta}_{n-1}(\zeta_{n-1-k} r) + f_{n-1-k} \hat{\zeta}_{n-1}(r), \quad (5)$$

which will be useful for computations in the following section.

Remark 2.1. If $f_0 = 0$ but $f \neq 0$, there exists a $\text{GL}_2(\mathbb{Z})$-transformation that takes $f$ to another binary $n$-ic form $f'$ with a nonzero leading coefficient. To obtain the ring $R_f$ and the ideal class $I_f$ (which are, up to isomorphism, $\text{GL}_2(\mathbb{Z})$-invariant), one may use the above constructions for $f'$ (see [25, §2]).

The above construction holds if one replaces $\mathbb{Z}$ with any integral domain $T$ (see [25]); this gives an explicit way of associating a ring $R_f$, which is rank $n$ as a $T$-module, and a distinguished (based) ideal $I_f$ of $R_f$ to a binary $n$-ic form over $T$. We refer to $R_f$ as the ring associated to $f$ and $I_f$ as
the distinguished ideal of \( R_f \) or \( f \). Geometrically, for nonzero forms \( f \), the ring \( R_f \) is the ring of functions on the subscheme \( X_f \) of \( \mathbb{P}^1_T \) cut out by the binary \( n \)-ic form \( f \), and the ideal \( I^k_f \) is the pullback of \( \mathcal{O}(k) \) from \( \mathbb{P}^1_T \) to \( X_f \) (see [25, Theorem 2.4]).

We are interested in counting the 2-torsion ideal classes of the rings \( R_f \) associated to irreducible forms \( f \) when \( n \) is odd. A key ingredient is a parametrization of such ideal classes in terms of pairs of \( n \times n \) symmetric matrices, which we recall next.

### 2.2 Parametrization of order 2 ideal classes in \( R_f \)

For any base ring \( T \), let \( U(T) = \text{Sym}_n(T^2) \) denote the space of binary \( n \)-ic forms with coefficients in \( T \). Let \( V(T) = T^2 \otimes \text{Sym}_2(T^n) \) denote the space of pairs \( (A, B) \) of symmetric \( n \times n \) matrices with coefficients \( a_{ij} \) of \( A \) and \( b_{ij} \) of \( B \) in \( T \) (for \( 1 \leq i, j \leq n \)) where \( a_{ij} = a_{ji} \) and \( b_{ij} = b_{ji} \). The group \( \text{SL}_n(T) \) acts naturally on \( V(T) \), where \( \gamma \in \text{SL}_n(T) \) acts on \( (A, B) \) by

\[
\gamma(A, B) = (\gamma A \gamma^t, \gamma B \gamma^t).
\]

The map \( \pi : V(T) \to U(T) \) sending \( (A, B) \mapsto \det(Ax - By) \) is clearly \( \text{SL}_n(T) \)-equivariant. We call \( f_{(A,B)} := \pi(A, B) \) the binary \( n \)-ic invariant or resolvent form of the pair \( (A, B) \) (or of the \( \text{SL}_n(T) \)-equivalence class of \( (A, B) \)). Recall that a binary \( n \)-ic form \( f \) is nondegenerate if and only if its discriminant \( \Delta(f) \) is nonzero, and we will call the pair \( (A, B) \) nondegenerate if and only if \( f_{(A,B)} \) is. In [26, Thm. 1.3], Wood describes the \( \text{SL}_n(\mathbb{Z}) \)-orbits of \( V(\mathbb{Z}) \) in terms of fractional ideals of the rings \( R_f \) from \S 2.1:

**Theorem 2.2.** Let \( f \in U(\mathbb{Z}) \) be a nondegenerate primitive binary \( n \)-ic form with integral coefficients. Then there is a bijection between \( \text{SL}_n(\mathbb{Z}) \)-orbits of \( (A, B) \in V(\mathbb{Z}) \) with \( f_{(A,B)} = f \) and equivalence classes of pairs \((I, \delta)\) where \( I \) is a fractional ideal of \( R_f \) and \( \delta \in (R_f \otimes_{\mathbb{Z}} \mathbb{Q})^\times \) with \( I^2 \subseteq \delta I_f^{n-3} \) as ideals and \( N(I)^2 = N(\delta)N(I_f^{n-3}) \). Two pairs \((I, \delta)\) and \((I', \delta')\) are equivalent if there exists \( \kappa \in (R_f \otimes_{\mathbb{Z}} \mathbb{Q})^\times \) such that \( I' = \kappa I \) and \( \delta' = \kappa^2 \delta \).

We now explicitly describe the bijective map of Theorem 2.2, as some of these computations will be needed in \S 2.4. We start with how to associate an element of \( V(\mathbb{Z}) \) to a pair \((I, \delta)\). Fix a primitive nondegenerate \( f = f_0 x^n + f_1 x^{n-1} y + \cdots + f_n y^n \in U(\mathbb{Z}) \) with \( f_0 \neq 0 \) and thus \( R_f \), and let \( f \) be a fractional ideal of \( R_f \) and \( \delta \) be an invertible element of \( R_f \otimes_{\mathbb{Z}} \mathbb{Q} \) such that \( I^2 \subseteq \delta I_f^{n-3} \) and \( N(I)^2 = N(\delta)N(I_f^{n-3}) \). This implies that there is a map

\[
\varphi : I \otimes I \to I_f
\]

\[
\alpha \otimes \alpha' \mapsto \frac{\alpha \alpha'}{\delta}.
\]

Composing \( \varphi \) with the quotient map \( I_f \to I_f/\langle 1, \theta, \cdots, \theta^{n-3} \rangle \) gives a symmetric bilinear map and corresponds to an element of \( (A, B) \in V(\mathbb{Z}) \). Equivalently, we can explicitly write a \( \mathbb{Z} \)-basis for \( I \) as \( \langle \alpha_1, \cdots, \alpha_n \rangle \) with the same orientation as \( \langle 1, \zeta_1, \zeta_2, \cdots, \zeta_{n-1} \rangle \), i.e., the change-of-basis matrix from \( \langle \zeta_1 \rangle \) to \( \langle \alpha_i \rangle \) has positive determinant. Because \( I^2 \subseteq \delta I_f^{n-3} \), we have that for all \( i, j \in \{1, \ldots, n\} \),

\[
\varphi(\alpha_i \otimes \alpha_j) = c_{ij}^{(0)} + c_{ij}^{(1)} \theta + \cdots + c_{ij}^{(n-3)} \theta^{n-3} + b_{ij} \zeta_{n-2} + a_{ij} \zeta_{n-1}
\]

where \( a_{ij}, b_{ij}, c_{ij}^{(k)} \in \mathbb{Z} \) for \( 0 \leq k \leq n - 3 \). Then \((A, B) = ((a_{ij}), (b_{ij})) \) yields the desired pair of integral symmetric \( n \times n \) matrices.
Conversely, let \((A, B) \in V(\mathbb{Z})\) satisfy \(\pi(A, B) = f\) with coefficients \(a_{ij}\) for \(A\) and \(b_{ij}\) for \(B\). Note that \(\det A = f_0\), so the desired condition that \(f_0 \neq 0\) is equivalent to requiring \(A\) be invertible.

We want to construct a fractional ideal \(I\) of \(R_f\) along with an element \(\delta \in (R_f \otimes_{\mathbb{Z}} \mathbb{Q})^\times\) such that the map \(\varphi\) described in equation (7) is well defined. We describe \(I\) explicitly in terms of a \(\mathbb{Z}\)-basis \(\langle \alpha_1, \ldots, \alpha_n \rangle\).

First, we consider the action of \(\theta\) on \(\alpha_i\). By [26, Proposition 3.3], if we write elements of \(I\) as row vectors relative to the basis \(\langle \alpha_1, \ldots, \alpha_n \rangle\), then \(\theta\) acts on \(I\) by right multiplication by \(BA^{-1}\), i.e.,

\[
\theta \cdot (\alpha_1, \alpha_2, \ldots, \alpha_n)^t = (\alpha_1, \alpha_2, \ldots, \alpha_n)^t \cdot BA^{-1}.
\]

This completely determines the action of \(R_f\) (and even the action of \(Bf_0[\theta]/f(\theta, 1)\)) on \(I\). We now define the map \(\varphi\) by explicitly writing its image on basis elements \(\langle \alpha_i \otimes \alpha_j \rangle\) of \(I \otimes I\). For \(0 \leq k \leq n - 3\), let \(C^{(k)}\) be the matrix

\[
C^{(k)} := (f_0 \cdot (BA^{-1})^{n-k-2} + f_1 \cdot (BA^{-1})^{n-k-3} + \ldots + f_{n-k-3} \cdot BA^{-1})B,
\]

and let \(c_{ij}^{(k)}\) denote the \((i, j)^{th}\) entry of \(C^{(k)}\). Then

\[
\varphi : I \otimes I \rightarrow I_f
\]

\[
\alpha_i \otimes \alpha_j \mapsto c_{ij}^{(0)} + c_{ij}^{(1)} \theta + \ldots + c_{ij}^{(n-3)} \theta^{n-3} + b_{ij} \zeta_{n-2} + a_{ij} \zeta_{n-1}.
\]

Finally, we set \(\delta = \frac{\alpha_i \alpha_j}{\varphi(\alpha_i \otimes \alpha_j)}\). Equation (9) shows that for each \(1 \leq j \leq n\), we have that the \(\alpha_i\) satisfy the following ratios:

\[
\alpha_1 : \alpha_2 : \ldots : \alpha_{n-1} : \alpha_n = c_{1,j}^{(0)} + \ldots + c_{1,j}^{(n-3)} \theta^{n-3} + b_{1,j} \zeta_{n-2} + a_{1,j} \zeta_{n-3} : \ldots : c_{n-1,j}^{(n-3)} \theta^{n-3} + b_{n-1,j} \zeta_{n-2} + a_{n-1,j} \zeta_{n-3} : c_{n,j}^{(n-3)} \theta^{n-3} + b_{n,j} \zeta_{n-2} + a_{n,j} \zeta_{n-3}.
\]

The ratios are independent of the choice of \(j\), so this determines \(\langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle\) up to a scalar factor in \(R_f \otimes_{\mathbb{Z}} \mathbb{Q}\). Once we fix a choice of \(\langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle\), then \(\delta\) is uniquely determined.

To construct the matrices \(C^{(k)}\), recall that \(\zeta_{n-2} = f_0 \theta^{n-2} + f_1 \theta^{n-3} + \ldots + f_{n-3} \theta\) and \(\zeta_{n-1} = f_0 \theta^{n-1} + f_1 \theta^{n-2} + \ldots + f_{n-2} \theta\), so from the definition of \(\delta\) and \(\varphi\), we have

\[
\frac{\alpha_i \cdot \alpha_j}{\delta} = c_{ij}^{(0)} + (c_{ij}^{(1)} + f_{n-3} b_{ij} + f_{n-2} a_{ij}) \theta + (c_{ij}^{(2)} + f_{n-4} b_{ij} + f_{n-3} a_{ij}) \theta^2 + \ldots + (c_{ij}^{(n-3)} + f_{1} b_{ij} + f_{2} a_{ij}) \theta^{n-3} + (b_{ij} + f_{1} a_{ij}) \theta^{n-2} + a_{ij} \theta^{n-1}.
\]

Thus, the coefficients \(c_{ij}^{(k)}\) must satisfy

\[
c_{ij}^{(k)} = \hat{\theta}_k \left(\frac{\alpha_i \cdot \alpha_j}{\delta}\right) - f_{n-k-2} \cdot b_{ij} - f_{n-k-1} \cdot a_{ij}.
\]

Using Equation (5), we then have that \(c_{ij}^{(k)}\) must satisfy

\[
c_{ij}^{(k)} = f_{n-k-1} \hat{\zeta}_{n-1} \left(\frac{\alpha_i \cdot \alpha_j}{\delta}\right) + \hat{\zeta}_{n-1} \left(\frac{\alpha_i \cdot \alpha_j}{\delta}\right) - f_{n-k-2} \cdot b_{ij} - f_{n-k-1} \cdot a_{ij}
\]

\[
= \hat{\zeta}_{n-1} \left(\frac{\alpha_i \cdot \alpha_j}{\delta}\right) - f_{n-k-2} \cdot b_{ij}.
\]
\[
\zeta_{n-1} = \left( f_0 \theta^{n-k-1} + f_1 \theta^{n-k-2} + \ldots + f_{n-k-2} \theta \right) \cdot \alpha_i \cdot \alpha_j - f_{n-k-2} \cdot b_{ij}.
\]

The middle equality follows from the fact that \( \zeta_{n-1} \left( \frac{\alpha_i \alpha_j}{\delta} \right) = a_{ij} \) by Equation (9). In matrix notation, we thus have

\[
C^{(k)} = (f_0 \cdot (BA^{-1})^{n-k-1} + f_1 \cdot (BA^{-1})^{n-k-2} + \ldots + f_{n-k-2} \cdot BA^{-1}) A - f_{n-k-2} B \quad (11)
\]

with entries \( c_{ij}^{(k)} \). It is clear that the \( c_{ij}^{(k)} \) are determined by the pair \((A, B)\).

The action of \( SL_n(\mathbb{Z}) \) on \( V(\mathbb{Z}) \) corresponds to the action \( g_n \in SL_n(\mathbb{Z}) \) on the chosen basis for \( I \) which sends

\[
\langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle \mapsto \langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle \cdot g_n^t.
\]

Thus, the ideal \( I \) is invariant under the action of \( SL_n(\mathbb{Z}) \).

### 2.3 Composition of elements of \( V(\mathbb{Z}) \) with the same binary \( n \)-ic invariant

Let \( \mathcal{O} \) be an \( S_n \)-order in a degree \( n \) number field \( K \) over \( \mathbb{Q} \). Consider the set of pairs \((I, \delta)\), where \( I \) is a fractional ideal of \( \mathcal{O} \), \( \delta \in K^\times \), \( I^2 \subset \delta \), and \( N(I) I = N(\delta) \). Recall that we called two such pairs \((I, \delta)\) and \((I', \delta')\) equivalent if there exists \( \kappa \in K^\times \) such that \( I' = \kappa I \) and \( \delta' = \kappa^2 \delta \). We have a natural law of composition on equivalence classes of such pairs given by

\[
(I, \delta) \circ (I', \delta') = (II', \delta \delta'). \quad (13)
\]

We say that a pair \((I, \delta)\) is projective if \( I \) is projective as an \( \mathcal{O} \)-module, i.e., if \( I \) is invertible as a fractional ideal of \( \mathcal{O} \); the pair \((I, \delta)\) is projective if and only if \( I^2 = \delta \). The set of equivalence classes of projective pairs \((I, \delta)\) for \( \mathcal{O} \) forms a group under the composition law (13), which we denote by \( H(\mathcal{O}) \).

There exists a natural group homomorphism from \( H(\mathcal{O}) \) to \( Cl_2(\mathcal{O}) \), given by sending the pair \((I, \delta)\) to the ideal class of \( I \). This map is clearly well defined and surjective. The kernel consists of equivalence classes of pairs \((I, \delta)\) where \( I \) is a principal ideal; each such equivalence class has a representative of the form \((\mathcal{O}, \delta)\) where \( \delta \) is a norm 1 unit. Therefore, we obtain the exact sequence

\[
1 \to \mathcal{O}_N^\times \to H(\mathcal{O}) \to Cl_2(\mathcal{O}) \to 1, \quad (14)
\]

which implies that \( H(\mathcal{O}) \) is an extension of the 2-torsion subgroup of the class group of \( \mathcal{O} \). Using Dirichlet’s unit theorem and the fact that \(-1 \in \mathcal{O}^\times\) has norm \(-1\), we immediately obtain the following lemma:

**Lemma 2.3.** Let \( \mathcal{O} \) be an order in a number field of degree \( n \) with Galois group \( S_n \) and signature \((r_1, r_2)\). Then \( |H(\mathcal{O})| = 2^{r_1+r_2-1}|Cl_2(\mathcal{O})| \).

We next compare certain elements of \( H(\mathcal{O}) \) to the 2-torsion subgroup \( Cl_2^+(\mathcal{O}) \) of the narrow class group \( Cl^+(\mathcal{O}) \) of \( \mathcal{O} \). Recall that \( Cl^+(\mathcal{O}) \) is the quotient of the ideal group \( I(\mathcal{O}) \) of \( \mathcal{O} \) by the group \( P^+(\mathcal{O}) \) of totally positive principal fractional ideals of \( \mathcal{O} \), i.e., ideals of the form \( a\mathcal{O} \) where \( a \) is an element of \( \text{Frac}(\mathcal{O})^\times \) such that \( \sigma(a) > 0 \) for every embedding \( \sigma : \text{Frac}(\mathcal{O}) \to \mathbb{R} \). We say that such an element \( a \) is totally positive and denote this condition by \( a \gg 0 \).

**Lemma 2.4.** Let \( \mathcal{O} \) be an order in a number field of degree \( n \) with Galois group \( S_n \) and signature \((r_1, r_2)\). If \( H^+(\mathcal{O}) \) denotes the subgroup of \( H(\mathcal{O}) \) consisting of projective pairs \((I, \delta)\) such that \( \delta \gg 0 \), then

\[
|H^+(\mathcal{O})| = 2^{r_2} |Cl_2^+(\mathcal{O})|. \quad (15)
\]
Proof. Let $\mathcal{O}_{\geq 0}^\times$ denote the totally positive units of $\mathcal{O}$, and define $\text{sgn} : \mathcal{O}^\times \to \{\pm 1\}^{r_1}$ as the signature homomorphism, which takes a unit to the sign of its image under each real embedding $\sigma : \text{Frac} \mathcal{O} \to \mathbb{R}$. Let $r$ be the nonnegative integer satisfying $|\text{Image}(\text{sgn})| = 2^r$ and let \[
abla_{\geq 0}^{\times 2}(\mathcal{O}) = \{[I] : \text{there exists } \delta \gg 0 \text{ such that } I^2 = (\delta)\}\]
be the set of equivalence class of ideals whose square is totally positive, where two ideals are equivalent if they differ by a principal ideal (in the usual sense). We then have the following commutative diagram of exact sequences:

\[
\begin{array}{ccccccccc}
1 & \rightarrow & \mathcal{O}_{\geq 0}^\times/(\mathcal{O}^\times)^2 & \rightarrow & \mathcal{O}^\times/(\mathcal{O}^\times)^2 & \rightarrow & \{\pm 1\}^{r_1} & \rightarrow & \{\pm 1\}^{r_1}/\text{sgn}(\mathcal{O}^\times) & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \rightarrow & \mathcal{O}_2^{2\times} & \rightarrow & \mathcal{O}_2^{\times} & \rightarrow & \mathcal{O}_2^{\times} & \rightarrow & \mathcal{O}_2^{\times}/\text{sgn}(\mathcal{O}^\times) & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \rightarrow & \mathcal{H}^+(\mathcal{O}) & \rightarrow & \mathcal{Cl}_2^{\times}(\mathcal{O}) & \rightarrow & \mathcal{C}_{\geq 0}^{\times 2}(\mathcal{O}) & \rightarrow & 1 \\
\end{array}
\]

where the map $\alpha$ sends a pair $(I, \delta)$ with $\delta \gg 0$ to the equivalence class $[I]$, and the map $\beta$ sends a coset $I + P^+(\mathcal{O})$ to the equivalence class $[I]$. We have that $|\mathcal{O}^\times/(\mathcal{O}^\times)^2| = 2^{r_1+r_2}$ and $|\{\pm 1\}^{r_1}/\text{sgn}(\mathcal{O}^\times)| = 2^{r_1-r}$, so $|\mathcal{O}_{\geq 0}^\times/(\mathcal{O}^\times)^2| = 2^{r_1-r+r_2}$. The equality (15) follows immediately. 

We now relate projective orbits of $V(\mathbb{Z})$ to the size of the 2-torsion subgroup of the ideal class group of the corresponding rings. Assume $f \in U(\mathbb{Z})$ is nondegenerate and irreducible so that $R_f$ is an order in a degree $n$ number field. Further assume that $f$ is primitive, which implies that $I_f$ is invertible. We say that a pair $(A,B) \in V(\mathbb{Z}) \cap \pi^{-1}(f)$ is projective if the corresponding pair $(I, \delta)$ under the bijection of Theorem 2.2 is projective. We then have the following result:

**Proposition 2.5.** Let $\mathcal{O}$ be an $S_n$-order corresponding to an integral, nondegenerate, irreducible, and primitive binary $n$-ic form $f$. Then $\mathcal{H}(\mathcal{O})$ is in natural bijection with the set of projective $\text{SL}_n(\mathbb{Z})$-orbits on $V(\mathbb{Z}) \cap \pi^{-1}(f)$. The number of such projective orbits is equal to \[
2^{r_1+r_2-1} |\mathcal{C}_{\geq 0}^{\times 2}(\mathcal{O})|,
\]
where $(r_1, r_2)$ is the signature of the fraction field of $\mathcal{O}$.

**Proof.** From Theorem 2.2, projective orbits in $V(\mathbb{Z}) \cap \pi^{-1}(f)$ are in bijection with pairs $(I, \delta)$, where $I$ is a fractional ideal of $\mathcal{O}$, $\delta \in K^\times$ and $I_f^2 = \delta I_f^{n-3}$. The set of such pairs is clearly in bijection with $\mathcal{H}(\mathcal{O})$ by simply sending $(I, \delta)$ to $\left(I \cdot I_f^{n-3}, \delta\right)$. The second assertion of the proposition now follows immediately from Lemma 2.3. 

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2.4 Reducible elements in $V(\mathbb{Z})$

We say that an element $(A, B) \in V(\mathbb{Q})$ is reducible if the quadrics in $\mathbb{P}^{n-1}(\mathbb{Q})$ corresponding to $A$ and $B$ have a common rational isotropic subspace of dimension $(n-3)/2$ in $\mathbb{P}^{n-1}(\mathbb{Q})$. The condition of reducibility has the following arithmetic significance:

**Theorem 2.6.** Let $(A, B)$ be a projective element of $V(\mathbb{Z})$ whose binary $n$-ic invariant is primitive, irreducible, and nondegenerate, and let $(I, \delta)$ denote the corresponding pair as given by Theorem 2.2. Then $(A, B)$ is reducible if and only if $\delta$ is a square in $(R_f \otimes \mathbb{Q})^\times$.

**Proof.** Suppose first that $\delta = r^2$ is the square of an invertible element in $(R_f \otimes \mathbb{Q})^\times$. By replacing $I$ with $r^{-1}I$ and $\delta$ with $r^{-2}\delta$, we may assume that $\delta = 1$. Let $\alpha_1, \ldots, \alpha_{n-1}$ be a $\mathbb{Z}$-basis for $I \cap (\mathbb{Z} \oplus \mathbb{Z}\theta \oplus \cdots \oplus \mathbb{Z}\theta^{n-2})$, and extend it to a basis $\alpha_1, \ldots, \alpha_n$ of $I$. It follows from (8) that, with these coordinates, we have $a_{ij} = b_{ij} = 0$ for $1 \leq i, j \leq (n-1)/2$, which is sufficient for $(A, B)$ to be reducible.

Now assume that $(A, B)$ is reducible; we would like to prove that $\delta$ is a square. Let $x_1, x_2, \ldots, x_n$ denote a set of coordinates for $\mathbb{P}^{n-1}$. By replacing $(A, B)$ with an $\text{SL}_n(\mathbb{Q})$-translate if necessary, we may assume that the common isotropic subspace is the one generated by $x_1, \ldots, x_{(n-1)/2}$. This implies that $a_{ij} = b_{ij} = 0$ for $1 \leq i, j \leq (n-1)/2$. From (10) and (11), we see that the quantity $\alpha_i \alpha_j / \delta$ is given by the $ij$th coordinate of the matrix

$$D := \sum_{k=0}^{n-1} \left( C^{(k)} + f_{n-k-2}B + f_{n-k-1}A \right) \cdot \theta^k$$

$$= \sum_{k=0}^{n-1} \sum_{j=0}^{n-k-1} \left( \sum_{j+k \leq n-1} f_{n-j-1}(BA^{-1})^j A \right) \cdot \theta^k$$

$$= \sum_{j,k \geq 0} f_{n-j-1}(BA^{-1})^j A \cdot \theta^k$$

(16)

where $f = f_0 x^n + f_1 x^{n-1} y + \cdots + f_n y^n$ is the binary $n$-ic invariant of $(A, B)$. (Note that $A$ is invertible because $f$ is assumed to be irreducible, so $f_0 = \det A \neq 0$.)

We now prove that the 11-coefficient $d_{11}$ of $D$ is a square using the fact that $a_{ij} = b_{ij} = 0$ for $1 \leq i, j \leq (n-1)/2$. This implies that $\delta = \alpha_1^2/d_{11}$ is a square as well. First, from (16), note that the coefficients of $\theta^{n-1}$ and $\theta^{n-2}$ of $d_{11}$ are 0, since $a_{11} = b_{11} = 0$. We start with the following lemma:

**Lemma 2.7.** The coefficient of $\theta^{n-3}$ in $d_{11}$ is a square.

**Proof.** From (16) and the fact that $a_{11} = b_{11} = 0$, the coefficient of $\theta^{n-3}$ in $d_{11}$ is equal to the 11-coefficient of the matrix $f_0(BA^{-1})^2 A = f_0BA^{-1} B$. Let $M$ denote the cofactor matrix of $A$, i.e., the $ij$-coefficient $m_{ij}$ of $M$ is equal to $(-1)^{i+j}$ times the determinant of the matrix obtained by removing the $i$th row and the $j$th column of $A$. Then the coefficient of $\theta^{n-3}$ in $d_{11}$ is equal to the 11-coefficient of $BMB$, which is easily computed to be a square. □

Next, we show that the constant coefficient of $d_{11}$ (considered as a polynomial in $\theta$) is a square.

**Lemma 2.8.** The constant coefficient $d_{11}(0)$ of $d_{11}(\theta)$ is a square.
Proof. Because the binary $n$-ic invariant of $(A, B)$ is $f$, we have $\det(Ax - By) = \det(Ix - BA^{-1}y)\det(A) = f(x, y)$. Since $BA^{-1}$ satisfies its characteristic polynomial, we obtain

$$\sum_{j=0}^{n} f_{n-j}(BA^{-1})^j = 0.$$ 

By (16), we compute $d_{11}(0)$ to be the 11-coefficient of the matrix

$$\left(\sum_{j=0}^{n-1} f_{n-j-1}(BA^{-1})^j\right)A = \left(\sum_{j=0}^{n-1} f_{n-(j+1)}(BA^{-1})^{j+1}\right)AB^{-1}A$$

$$= \left(\sum_{j=0}^{n} f_{n-j}(BA^{-1})^j\right)AB^{-1}A - f_nAB^{-1}A.$$

Note that $B$ is invertible because $\det B = f_n \neq 0$ since $f$ is irreducible. The lemma now follows from the proof of Lemma 2.7 and symmetry (and the fact that $n$ is odd). \qed

We next show that $d_{11}(m)$ is a square for every integer $m$, by applying Lemma 2.8 on the pair $(A, B - mA)$. Let $g$ denote the binary $n$-ic invariant of the pair $(A, B - mA)$, and let $g_k$ denote the coefficient of $x^{n-k}y^{k}$ in $g(x, y)$. We have

$$g(x, y) = \det(Ax - (B - mA)y) = \det(A(x + my) - By) = f(x + my, y).$$

As a consequence, we compute the $g_k$ to be

$$g_k = \sum_{j=0}^{k} \binom{n-j}{k-j} f_{j}m^{k-j}.$$

By applying Lemma 2.8 to $(A, B - mA)$, we see that the 11-coefficient of the following matrix is a square:

$$\left(\sum_{j=0}^{n-1} g_{n-j-1}(BA^{-1} - mI)^j\right)A$$

$$= \left(\sum_{k=0}^{n-1} g_k(BA^{-1} - mI)^{n-k-1}\right)A$$

$$= \left(\sum_{k=0}^{n-1} \sum_{j=0}^{k} \binom{n-j}{k-j} f_{j}m^{k-j}\right)(BA^{-1} - mI)^{n-k-1}A$$

$$= \left(\sum_{k=0}^{n-1} \sum_{j=0}^{k} \binom{n-j}{k-j} f_{j}m^{k-j}\right)\left(\sum_{i=0}^{n-k-1} (-1)^{k+i} \binom{n-k-1}{i}(BA^{-1})^{i}m^{n-k-1-i}\right)A$$

$$= \sum_{k=0}^{n-1} \sum_{j=0}^{k} \sum_{i=0}^{n-k-1} (-1)^{i+k} f_{j}(BA^{-1})^{i}m^{n-k-1-i-1}\binom{n-j}{k-j}\binom{n-k-1}{i}A$$
\[
\begin{align*}
&= \sum_{i,j \geq 0} f_j(BA^{-1})^i m^{n-i-j-1} \left( \sum_{k=j}^{n-i-1} (-1)^{k+i} \binom{n-j}{k-j} \binom{n-k-1}{i} \right) A \\
&= \sum_{i,j \geq 0} f_j(BA^{-1})^i m^{n-i-j-1} A,
\end{align*}
\]

where the last equality is a consequence of the following lemma:

**Lemma 2.9.** For nonnegative integers \( n, i, \) and \( j \) satisfying \( i + j \leq n - 1 \), we have

\[
\sum_{k=j}^{n-i-1} (-1)^{k+i} \binom{n-j}{k-j} \binom{n-k-1}{i} = (-1)^{n+1}.
\]

**Proof.** By taking the \( i \)th derivative of both sides of the identity

\[
(1 + x)^{n-j} - 1 = \sum_{k=j}^{n-1} x^n \binom{n-j}{n-k} \]

and setting \( x = -1 \), we obtain the lemma.

Comparing the formulas (17) and (16) with \( \theta = m \) shows that \( d_{11}(m) \) is a square for any integer \( m \). Since a polynomial that takes only square values on integers must itself be a square, it follows that the 11-coefficient of \( D \) is a square. This concludes the proof of Theorem 2.6.

**Remark 2.10.** Theorem 2.6 also follows from a different interpretation of orbits of \( V(\mathbb{Q}) \) in terms of Jacobians of hyperelliptic curves, found in Wang’s dissertation [24].

For an order \( \mathcal{O} \), let \( \mathcal{I}_2(\mathcal{O}) \) denote the 2-torsion subgroup of the ideal group of \( \mathcal{O} \), i.e., the group of invertible fractional ideals \( I \) of \( \mathcal{O} \) such that \( I^2 = \mathcal{O} \). Note that the group \( \mathcal{I}_2(\mathcal{O}) \) is trivial when \( \mathcal{O} \) is maximal. We have the following result parametrizing elements of \( \mathcal{I}_2(\mathcal{O}) \) for all primitive orders \( \mathcal{O} \) arising from integral binary \( n \)-ic forms.

**Proposition 2.11.** Let \( \mathcal{O}_f \) be an order corresponding to the integral primitive irreducible and nondegenerate binary \( n \)-ic form \( f \). Then \( \mathcal{I}_2(\mathcal{O}_f) \) is in natural bijection with the set of projective reducible \( \text{SL}_n(\mathbb{Z}) \)-orbits on \( V(\mathbb{Z}) \cap \pi^{-1}(f) \).

**Proof.** Theorem 2.6 shows that a projective \( \text{SL}_n(\mathbb{Z}) \)-orbit on \( V(\mathbb{Z}) \) corresponding to the pair \((I, \delta)\) is reducible exactly when \( \delta \) is a square, say \( \delta = \kappa^2 \). The map from projective reducible \( \text{SL}_n(\mathbb{Z}) \)-orbits on \( V(\mathbb{Z}) \cap \pi^{-1}(f) \) to \( \mathcal{I}_2(\mathcal{R}) \) that sends such an orbit to \( \kappa^{-1} I \cdot I_f^{-\frac{n-2}{2}} \) is clearly a bijection.

**2.5 Parametrizations over other rings**

Let \( T \) be a principal ideal domain. We now describe an analogue of Theorem 2.2 over \( T \), and we study a rigidified version of the parametrization to better understand the orbits and stabilizers of the group action.

The following theorem describes how \( \text{SL}_n(T) \)-orbits of \( V(T) \) are related to rank \( n \) rings and ideal classes; it is a restatement of [26, Thm. 6.3], using the fact that our base ring \( T \) is a principal ideal domain:
Theorem 2.12 ([26]). Let $f \in U(T)$ be a nondegenerate primitive binary $n$-ic form. Then there is a bijection between $\text{SL}_n(T)$-orbits of $(A,B) \in V(T)$ with $f_{(A,B)} = f$ and equivalence classes of pairs $(I,\delta)$ where $I \subset K_f := T[x]/(f(x,1))$ is an ideal of $R_f$ and $\delta \in K_f^\times$ satisfying $I^2 \subset \delta I_f^{n-3}$ as ideals and $N(I)^2 = N(\delta)N(I_f^{n-3})$. Two pairs $(I,\delta)$ and $(I',\delta')$ are equivalent if there exists $\kappa \in K_f^\times$ such that $I' = \kappa I$ and $\delta' = \kappa^2 \delta$.

Note that in [26, §6] the theorems are stated for $\text{SL}_n(T)$-orbits instead of $\text{SL}_n(T)$-orbits, where $\text{SL}_n(T)$ denotes the elements of determinant $\pm 1$ in $\text{GL}_n(T)$. However, since $n$ is odd here, we have $\text{SL}_n(T) \cong \{\pm 1\} \times \text{SL}_n(T)$, and since $-1$ acts trivially on pairs $(A,B)$ by (6), the $\text{SL}_n(T)$-orbits are precisely the same as the $\text{SL}_n(T)$-orbits.

In order to understand the stabilizer of the action of $\text{SL}_n(T)$ on an element $(A,B) \in V(T)$, we now discuss precisely with what the elements (instead of $\text{SL}_n(T)$-orbits) of $V(T)$ are in correspondence, in terms of the pair $(I,\delta)$ along with a basis for $I$.

Proposition 2.13 ([26]). Let $f \in U(T)$ be a nondegenerate primitive binary $n$-ic form. Let $K_f := T[x]/(f(x,1))$. Then the nonzero elements $(A,B) \in V(T)$ with $f_{(A,B)} = f$ are in bijection with equivalence classes of triples $(I,B,\delta)$ where $I \subset K_f$ is a basis ideal of $R_f$, with an ordered basis given by an isomorphism $B : I \rightarrow T^n$ of $T$-modules, and $\delta \in K_f^\times$, satisfying $I^2 \subset \delta I_f^{n-3}$ as ideals and $N(I)^2 = N(\delta)N(I_f^{n-3})$. Two such triples $(I,B,\delta)$ and $(I',B',\delta')$ are equivalent if and only if there exists $\kappa \in K_f^\times$ such that $I' = \kappa I$, $B \circ (\times \kappa) = B'$, and $\delta' = \kappa^2 \delta$.

As stated, Proposition 2.13 is a “symmetric” version of the first part of [26, Thm. 6.1]. For any $(A,B) \in V(T)$ corresponding to $(I,B,\delta)$ in Proposition 2.13, the action of $\text{SL}_n(T)$ on $(A,B)$ as in Equation (6) induces an action of $\text{SL}_n(T)$ on the basis $B$ through the correspondence, namely as given in (12). This action of $\text{SL}_n(T)$ takes $I$ to itself and does not affect $\delta$, so $\text{SL}_n(T)$ acts on the triples $(I,B,\delta)$. Quotienting both sides of the correspondence in Proposition 2.13 by $\text{SL}_n(T)$ yields precisely Theorem 2.12.

For the computations in later sections, we are interested in the stabilizer of $(A,B) \in V(T)$ in $\text{SL}_n(T)$. Any $g \in \text{SL}_n(T)$ that fixes $(A,B)$ must correspond to an automorphism of the corresponding triple $(I,B,\delta)$ that also preserves $I$ and $\delta$, in other words, an element $\kappa \in K_f^\times$ such that $\kappa^2 = 1$ (in fact, such $\kappa$ lie in $R_f^\times$). Furthermore, since multiplication on $B$ by $\kappa$ exactly corresponds to multiplication by the matrix $g$, we must have and $N(\kappa) = \det(g) = 1$. It is also easy to check that any such $\kappa$ yields an element $g \in \text{SL}_n(T)$ that stabilizes $(A,B)$. We thus have the following description of the stabilizers:

Corollary 2.14. Fix a principal ideal domain $T$. Let $(A,B) \in V(T)$ be a nondegenerate element with primitive binary $n$-ic invariant $f$, corresponding to the ring $R_f$ and the pair $(I,\delta)$ under Theorem 2.12. Then the stabilizer group in $\text{SL}_n(T)$ of $(A,B)$ corresponds to the norm 1 elements $R_f^\times[2] \cong 1$ of the 2-torsion in $R_f^\times$.

In the cases where $T$ is a field or $\mathbb{Z}_p$, we may also describe the $\text{SL}_n(T)$-orbits of $V(T)$ corresponding to a given binary $n$-ic invariant in a simple way. We restrict to projective orbits, i.e., those corresponding to $(I,\delta)$ where $I$ is projective as an $R_f$-module. (In the case where $T$ is a field, this will be no restriction.)

Corollary 2.15. Let $T$ be a field or $\mathbb{Z}_p$. Let $f$ be a separable nondegenerate binary $n$-ic form with coefficients in $T$. Then the projective $\text{SL}_n(T)$-orbits of $V(T)$ with invariant binary $n$-ic form $f$ are in bijection with elements of $(R_f^\times/R_f^\times)^2 \cong 1$. 

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Proof. Let $T = k$ be a field and let $f$ be a separable nondegenerate binary $n$-ic form over $k$. Then $R_f$ is a commutative $k$-algebra of dimension $n$, and in particular, a direct product of field extensions of $k$ and thus a principal ideal ring. It is easy to check that $I_f = R_f$. In this case, Theorem 2.12 implies that $\text{SL}_n(k)$-orbits on $V(k)$ with binary $n$-ic invariant $f$ correspond to equivalence classes of pairs $(I, \delta)$, where $I$ is a fractional ideal of $R_f$ and $\delta \in R_f^\times$ such that $I^2 = \delta I_f^{n-3} = \delta R_f$. The only ideals in $R_f$ are products of either the unit ideal or the zero ideal in each of the factors; since $\delta$ must be invertible, we have $I = R_f$ and so $N(\delta) = 1$. Thus, the equivalence classes of the pairs $(I, \delta)$ are parametrized by norm 1 elements $\delta$ of $R_f^\times/(R_f^\times)^2$.

Now let $T = \mathbb{Z}_p$. The ring $R_f$ is a direct product of finite extensions of $\mathbb{Z}_p$ and is thus a principal ideal ring. For projective pairs $(I, \delta)$ as in Theorem 2.12, the norm condition implies that $I^2 = \delta I_f^{n-3}$. As a result, the ideal $I$ is again determined by the element $\delta$ of $R_f^\times$. Furthermore, since $n - 3$ is even, we obtain that

$$N(\delta) = \left( \frac{N(I)}{N(I_f^{(n-3)/2})} \right)^2$$

is a square, so the set of equivalence classes of pairs $(I, \delta)$ are parametrized by $(R_f^\times/(R_f^\times)^2)_{n=1}$. □

Example 2.16. For $k = \mathbb{R}$, for a given $f$ as above, we have that $R_f$ is isomorphic to $\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$ for some nonnegative integers $r_1$ and $r_2$ with $r_1 + 2r_2 = n$. Then the number of $\text{SL}_n(\mathbb{R})$-orbits with invariant binary $n$-ic form $f$ is $2^{n_1-1}$, and the order of the stabilizer in $\text{SL}_n(\mathbb{R})$ is $2^{n_1+r_2-1}$.

3 Counting binary $n$-ic forms in acceptable families

Our goal in this section is to determine asymptotics for the number of irreducible elements in acceptable families of binary $n$-ic forms having bounded height, as well as to determine asymptotics for the number of irreducible $\text{SL}_2(\mathbb{Z})$-orbits on $\text{SL}_2(\mathbb{Z})$-invariant acceptable families having bounded Julia invariant. We first define an acceptable family of binary $n$-ic forms, as well as how to compute the size of such families when ordered by height. We then define the Julia invariant, and recall a result of [12] on the asymptotics of orbits of binary $n$-ic forms ordered by Julia invariant.

3.1 Acceptable families of binary $n$-ic forms

Recall that $U(T) = \text{Sym}_n(T^2)$ denotes the space of binary $n$-ic forms over a ring $T$, and an element $\gamma \in \text{SL}_2(T)$ acts on $f \in U(T)$ via $\gamma f(x, y) = f((x, y)\gamma)$. Let $\Delta(f)$ denote the discriminant of a form $f \in U(T)$. Let $U(\mathbb{R})^{(r_2)}$ denote the set of binary $n$-ic forms with coefficients in $\mathbb{R}$ that have nonzero discriminant and $r_2$ pairs of complex conjugate roots for some fixed $r_2 \in \{0, \ldots, (n - 1)/2\}$.

Definition 3.1. For each finite prime $p$, let $\Sigma_p \subset U(\mathbb{Z}_p) \setminus \{\Delta = 0\}$ be a nonempty open set whose boundary has measure 0, and let $\Sigma_\infty = U(\mathbb{R})^{(r_2)}$ for some such $r_2$. We say that a collection $\Sigma = (\Sigma_p)_p \cup \Sigma_\infty$ is acceptable if, for all large enough primes $p$, the set $\Sigma_p$ contains all elements $f \in V(\mathbb{Z}_p)$ with $p^2 \not\mid \Delta(f)$. We refer to each $\Sigma_\nu$ where $\nu$ is any finite or infinite place of $\mathbb{Q}$ as a local specification of $\Sigma$ at $\nu$. To a collection $\Sigma$, we associate a family $\mathcal{U}(\Sigma)$ of integral binary $n$-ic forms given by

$$\mathcal{U}(\Sigma) = \{ f \in U(\mathbb{Z}) : f \in \Sigma_\nu \text{ for all places } \nu \},$$

and say that $\mathcal{U}(\Sigma)$ is acceptable if $\Sigma$ is.
Note that if $\Sigma_p$ is $\text{SL}_2(\mathbb{Z}_p)$-invariant for every prime $p$ (the set $\Sigma_\infty$ is automatically $\text{SL}_2(\mathbb{R})$-invariant), then $U(\Sigma)$ is $\text{SL}_2(\mathbb{Z})$-invariant. In this case, we say that such a collection $\Sigma$ is $\text{SL}_2$-invariant. Additionally, for any $U(\Sigma)$, note that there is a multi-subset $\Sigma_H = \{ R_f \mid f \in U(\Sigma) \}$ inside $\mathfrak{N}_H$. Similarly, for any $\text{SL}_2$-invariant $U(\Sigma)$, there is also a multi-subset $\Sigma_J = \{ R[f] \mid [f] \in \text{SL}_2(\mathbb{Z}) \setminus U(\Sigma) \}$. We say that a family $\Sigma_H$ or $\Sigma_J$ is acceptable if it is defined by an acceptable family $U(\Sigma)$ of integral binary $n$-ic forms.

### 3.2 Binary $n$-ic forms ordered by height

In this subsection, we order real and integral binary $n$-ic forms by the following height function:

$$H(f_0x^n + \cdots + f_ny^n) := \max |f_i|.$$  \hfill (18)

For any subset $S$ of $U(\mathbb{R})$ or $U(\mathbb{Z})$, we denote the set of elements in $S$ having height less than $X$ by $S_{H<X}$. For a subset $S$ of $U(\mathbb{Z})$, we denote the subset of irreducible elements in $S$ by $S^{\text{irr}}$. Asymptotics for the number of integral irreducible binary $n$-ic forms having squarefree discriminant and bounded height is determined in [9]. The key ingredient in that result is a tail estimate on the number of integral binary $n$-ic forms having bounded height whose discriminants are divisible by $p^2$ for large primes $p$. Namely, let $W_p \subset U(\mathbb{Z})$ denote the set of integral binary $n$-ic forms with $p^2 \mid \Delta(f)$. Then the following tail estimate is proved in [9]:

**Proposition 3.2.** We have

$$\# \left( \bigcup_{p>M} W_p \right)_{H<X} = O \left( \frac{X^{n+1}}{\sqrt{M}} \right) + o(X^{n+1}).$$

The next theorem follows from Proposition 3.2 just as [8, Theorem 2.21] follows from [8, Theorem 2.13].

**Theorem 3.3.** Let $\Sigma$ be an acceptable collection of local specifications. Then we have

$$\# U(\Sigma)_{H<X}^{\text{irr}} = \text{Vol}(\Sigma_\infty, H<X) \prod_p \text{Vol}(\Sigma_p) + o(X^{n+1}).$$ \hfill (19)

Note that since $\text{Vol}(\Sigma_\infty, H<X)$ grows like a nonzero constant times $X^{n+1}$, the error term in the right hand side of (19) is indeed smaller than the main term.

### 3.3 $\text{SL}_2(\mathbb{Z})$-orbits on binary $n$-ic forms ordered by Julia invariant

Every binary $n$-ic form with real coefficients whose leading coefficient $a_0$ is nonzero can be written as

$$f(x, y) = a_0(x - \alpha_1y) \cdots (x - \alpha_ny),$$

with $\alpha_i \in \mathbb{C}$. For $t = (t_1, \ldots, t_n) \in \mathbb{R}^n$, consider the positive definite binary quadratic form

$$Q_t(x, y) = \sum_{i=1}^n t_i^2(x - \alpha_iy)(x - \alpha_iy).$$

In [20] Julia proved that if $t$ is chosen to minimize the quantity

$$\vartheta(f) = \frac{a_0^2 |\text{Disc} \, Q_t|^{n/2}}{t_1^2 \cdots t_n^2},$$ \hfill (20)
then $\vartheta$ is an $SL_2(\mathbb{R})$-invariant of $f$, i.e., $\vartheta(f) = \vartheta(\gamma \cdot f)$ for any $\gamma \in SL_2(\mathbb{R})$. We call $\vartheta$ the Julia invariant of the binary $n$-ic form $f(x,y)$. The Julia invariant is not a polynomial invariant, but it is homogeneous of degree 2, in the sense that $\vartheta(\lambda f) = \lambda^2 \vartheta(f)$ for $\lambda \in \mathbb{R}$. Indeed, the roots of $f$ and $\lambda f$ are the same; when we replace $f$ with $\lambda f$, the $a_0$ in the right hand side of (20) is replaced with $\lambda^2 a_0$ while the remaining quantities stay the same. In this section, we will order $SL_2(\mathbb{Z})$-orbits $[f]$ of $U(\mathbb{Z})$ by the degree 1 invariant
\[
J(f) = \sqrt{\vartheta(f)}.
\] (21)
Note that we may define the Julia invariant for forms $f$ with leading coefficient 0 by using an $SL_2(\mathbb{R})$-equivalent form with nonzero leading coefficient.

Asymptotics for the number of irreducible $SL_2(\mathbb{Z})$-orbits on integral binary $n$-ic forms were recently computed by Bhargava and Yang [12]. The following theorem is a rewording of [12, Theorem 9]:

**Theorem 3.4.** Let $n$ be a positive integer and $r_2 \in \{0,1,\ldots,\lfloor n/2 \rfloor\}$. Let $\Sigma$ be a collection of local specifications such that the family $\mathcal{U}(\Sigma)$ is defined by finitely many congruence conditions, and $\Sigma_\infty = U(\mathbb{R})^{(r_2)}$. Then there exists a constant $c_{n,r_2}$, depending only on $n$ and $r_2$, such that
\[
\#(SL_2(\mathbb{Z}) \setminus \mathcal{U}(\Sigma)_{J<X}) = c_{n,r_2} \prod_p \text{Vol}(\Sigma_p) X^{n+1} + o(X^{n+1-\frac{2}{n}}).
\] (22)

To prove Theorem 3.4, the authors construct a fundamental domain $F$ for the action of $SL_2(\mathbb{Z})$ on $U(\mathbb{R})^{(r_2)}$. This fundamental domain has the property that $F_{J<X} = XF_{J<1}$. Estimating the number of irreducible integral binary $n$-ic forms in $F_{J<X}$ is difficult because $F_{J<X}$ is not compact and has a cusp going to infinity. Using an averaging technique, they prove that the cuspidal region of $F_{J<X}$ contains negligibly many irreducible integral binary $n$-ic forms, while the non-cuspidal region has negligibly many reducible binary $n$-ic forms. This allows them to prove that the left hand side of (22) is well approximated by the volume of $F_{J<X}$, yielding the result. In fact, the constant $c_{n,k}$ in Theorem 3.4 is simply $\text{Vol}(F_{J<1})$. We now prove the following theorem.

**Theorem 3.5.** Let $\Sigma$ be an acceptable $SL_2$-invariant collection of local specifications. Then we have
\[
\#(SL_2(\mathbb{Z}) \setminus \mathcal{U}(\Sigma)_{J<X}) = \text{Vol}(SL_2(\mathbb{Z}) \setminus \Sigma_{\infty,J<X}) \prod_p \text{Vol}(\Sigma_p) + o(X^{n+1}).
\]

**Proof.** For every $\epsilon > 0$ there exists an acceptable collection $(\Sigma'_\nu)_\nu$ such that $\Sigma_\infty = \Sigma'_\infty$, $\Sigma_p \subset \Sigma'_p$ for each prime $p$, $\prod_p \text{Vol}(\Sigma_p) \geq \prod_p \text{Vol}(\Sigma'_p) - \epsilon$, and the set $\mathcal{U}(\Sigma')$ is defined by finitely many congruence conditions. From Theorem 3.4, we obtain
\[
\#(SL_2(\mathbb{Z}) \setminus \mathcal{U}(\Sigma)_{J<X}) \leq \#(SL_2(\mathbb{Z}) \setminus \mathcal{U}(\Sigma')_{J<X})
\]
\[
= \text{Vol}(SL_2(\mathbb{Z}) \setminus \Sigma'_{\infty,J<X}) \prod_p \text{Vol}(\Sigma'_p) + o(X^{n+1})
\]
\[
\leq \text{Vol}(SL_2(\mathbb{Z}) \setminus \Sigma_{\infty,J<X}) \prod_p \text{Vol}(\Sigma_p) + \epsilon + o(X^{n+1}).
\]

Letting $\epsilon$ tend to 0, we obtain the required upper bound on $\#(SL_2(\mathbb{Z}) \setminus \mathcal{U}(\Sigma)_{J<X})$.

To obtain the lower bound, we proceed as follows. For $\epsilon > 0$, we take sets $F_{J<1}^{(\epsilon)}$ to be a semi-algebraic bounded subset of $F_{J<1}$ such that $\text{Vol}(F_{J<1}^{(\epsilon)}) \geq (1-\epsilon)\text{Vol}(F_{J<1})$. We denote $XF_{J<1}^{(\epsilon)}$ by
$F_{J < X}^{(e)}$. Just as [8, Theorem 2.21] follows from [8, Theorem 2.13], we obtain from Proposition 3.2 the estimate

$$
\#(F_{J < X}^{(e)} \cap U(\Sigma)^{irr}) = \text{Vol}(F_{J < X}^{(e)}) \prod_p \text{Vol}(\Sigma_p) + o(X^{n+1}).
$$

(23)

From the proof of [12, Theorem 9], we have the following estimate on the number of integral elements in the “cuspidal region”:

$$
\#((F_{J < X} \setminus F_{J < X}^{(e)}) \cap U(\Sigma)^{irr}) \leq \epsilon X^{n+1} + O(X^{n+1-\frac{2}{m}}).
$$

(24)

Combining (23) and (24) yields the required lower bound on \#(SL_2(\mathbb{Z}) \setminus U(\Sigma)^{irr}_{J < X}) and completes the proof of Theorem 3.5.

\[
\square
\]

4 Counting orbits of pairs of $n \times n$ symmetric matrices

The main goal of this section is to determine asymptotics for the number of (absolutely) irreducible $\text{SL}_n(\mathbb{Z})$-orbits of pairs of $n \times n$ symmetric matrices having bounded height and the number of irreducible $\text{SL}_2(\mathbb{Z}) \times \text{SL}_n(\mathbb{Z})$-orbits of pairs of $n \times n$ symmetric matrices having bounded Julia invariant. We first construct fundamental domains for the action of $\text{SL}_n(\mathbb{Z})$ and $\text{SL}_2(\mathbb{Z}) \times \text{SL}_n(\mathbb{Z})$ on pairs of real $n \times n$ symmetric matrices. We then show that the cusps of these fundamental domains have a negligible number of irreducible integral points. Additionally, we show that the number of reducible integral points in the main body of these fundamental domains are also negligible. A theorem of Davenport [16] allows us to conclude that the number of irreducible integral points of bounded height in the fundamental domain for the action of $\text{SL}_n(\mathbb{Z})$ or the number of irreducible integer points of bounded Julia invariant in the fundamental domain for the action of $\text{SL}_2(\mathbb{Z}) \times \text{SL}_n(\mathbb{Z})$ is asymptotically equal to the volumes of their respective main bodies.

Fix an odd integer $n \geq 3$ and let $m = (n-1)/2$. Recall that $V(T) = T^2 \otimes \text{Sym}_2(T^n)$ is the space of pairs of $n \times n$ symmetric matrices $(A, B)$ over a ring $T$. The group $G(T) := \text{SL}_2(T) \times \text{SL}_n(T)$ acts on $V(T)$ via the action

$$(\gamma_2, \gamma_n) \cdot (A, B) = (\gamma_n A \gamma_n^{-1}, \gamma_n A \gamma_n^{-1}) \gamma_2^\dagger$$

for all $(\gamma_2, \gamma_n) \in G(T).$ \hspace{1cm} (25)

It is easy to verify that we have

$$\pi((\gamma_2, \gamma_n) \cdot (A, B)) = \gamma_2^\dagger(\pi(A, B))$$

for all $(\gamma_2, \gamma_n) \in G(T),$ \hspace{1cm} (26)

where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^\dagger := \begin{pmatrix} a & -c \\ b & -d \end{pmatrix}.$$

The space $V(\mathbb{R})$ inherits a height function $H$ and Julia invariant $J$ via $\pi$:

$$H(A, B) := H(\pi(A, B))$$

$$J(A, B) := J(\pi(A, B))$$

where $H$ and $J$ are defined on $U(\mathbb{R})$ as in §3. From (26), it follows that $H$ is $\text{SL}_n(\mathbb{R})$-invariant and $J$ is $G(\mathbb{R})$-invariant on $V(\mathbb{R})$.

We say that an element $(A, B) \in V(\mathbb{Z})$ with $\pi(A, B) = f$ is absolutely irreducible if

1. $f$ corresponds an order in an $S_n$-field, and
2. $(A, B)$ is not reducible in the sense of Theorem 2.6.

We denote the set of absolutely irreducible elements in $V(\mathbb{Z})$ by $V(\mathbb{Z})^{irr}$. 

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4.1 Construction of fundamental domains

For $0 \leq r_2 \leq m = (n - 1)/2$, recall that $U(\mathbb{R})^{(r_2)}$ denotes the set of binary n-ic forms in $U(\mathbb{R})$ that have nonzero discriminant and $r_2$ distinct pairs of complex conjugate roots in $\mathbb{P}^1(\mathbb{C})$. Let $V(\mathbb{R})^{(r_2)}$ denote the set of elements in $V(\mathbb{R})$ whose image under $\pi$ lies in $U(\mathbb{R})^{(r_2)}$. In this subsection, we construct fundamental domains for the actions of $SL_n(\mathbb{Z})$ and $G(\mathbb{Z})$ on $V(\mathbb{R})^{(r_2)}$ for $0 \leq r_2 \leq m$.

Fundamental sets for the action of $SL_n(\mathbb{R})$ and $G(\mathbb{R})$ on $V(\mathbb{R})^{(r_2)}$

Fix an integer $r_2$ with $0 \leq r_2 \leq m$, and let $r_1 = n - 2r_2$. For $f \in U(\mathbb{R})^{(r_2)}$, the $\mathbb{R}$-algebra $R_f$ corresponding to $f$ is isomorphic to $\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$. Corollary 2.15 states that the $SL_n(\mathbb{R})$-orbits of $\pi^{-1}(f)$ are in bijection with elements $\delta \in (R_f^e / R_f^e)^{N=1}$, which in turn is in natural bijection with the subset $\mathcal{T}(r_2) \subset \{ \pm 1 \}^{r_1} \times \{ 1 \}^{r_2}$ of elements having an even number of $-1$ factors (independent of the choice of $f \in U(\mathbb{R})^{(r_2)}$). For an element $\delta \in \mathcal{T}(r_2)$, let $V(\mathbb{R})^{(r_2),\delta}$ denote the set of $v \in V(\mathbb{R})^{(r_2)}$ such that $v$ corresponds to the pair $(R_{\pi(v)}, \delta)$ under the bijection of Theorem 2.12. It follows that for $f \in U(\mathbb{R})^{(r_2)}$ and $\delta \in \mathcal{T}(r_2)$, the set $\pi^{-1}(f) \cap V(\mathbb{R})^{(r_2),\delta}$ consists of a single $SL_n(\mathbb{R})$-orbit.

Therefore, to construct a fundamental domain for the action of $SL_n(\mathbb{R})$ on $V(\mathbb{R})^{(r_2),\delta}$, it is enough to pick one element $v_f \in V(\mathbb{R})^{(r_2),\delta}$ for each $f \in U(\mathbb{R})^{(r_2)}$. However, we require our fundamental set to be semialgebraic in order to apply our geometry-of-numbers techniques. We give an explicit description of such a fundamental set when $\delta = (1, 1, \ldots, 1)$ by describing a section $e : U(T) \to V(T)$ of $\pi$ for any ring $T$. When $T = \mathbb{R}$, it is easy to check that $e(f) \in V(\mathbb{R})^{(r_2),\delta}$ for $f \in U(\mathbb{R})^{(r_2)}$. For $n = 3$, the section $e$ takes a binary cubic form $f(x, y) = f_0x^3 + f_1x^2y + f_2xy^2 + f_3y^3$ to the pair

$$\left( \begin{array}{ccc} 0 & 0 & 1 \\ 0 & -f_0 & 0 \\ 1 & 0 & -f_2 \end{array} \right), \left( \begin{array}{ccc} 0 & 1 & 0 \\ 1 & f_1 & 0 \\ 0 & 0 & f_3 \end{array} \right).$$

For $n = 5$, the map $e$ sends a binary quintic form $f(x, y) = f_0x^5 + f_1x^4y + f_2x^3y^2 + f_3x^2y^3 + f_4xy^4 + f_5y^5$ to

$$\left( \begin{array}{ccc} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & f_0 & 0 & 0 \\ 0 & 1 & 0 & f_2 & 0 \\ 1 & 0 & 0 & 0 & f_4 \end{array} \right), \left( \begin{array}{ccc} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -f_1 & 0 & 0 \\ 1 & 0 & 0 & -f_3 & 0 \\ 0 & 0 & 0 & 0 & -f_5 \end{array} \right).$$

For general $n$, a binary $n$-ic form $f(x, y) = f_0x^n + f_1x^{n-1}y + f_2x^{n-2}y^2 + \cdots + f_ny^n$ is mapped under $e$ to $(a_{i,j}, (b_{i,j}))$ where:

- $a_{k,k-n} = 1$ for $1 \leq k < \frac{n-1}{2}$ or $\frac{n-1}{2} < k < n$
- $a_{\frac{n-1}{2}+k, \frac{n-1}{2}+k} = (-1)^{\frac{n-1}{2}}f_{2k}$ for $0 \leq k \leq \frac{n-1}{2}$
- $b_{\frac{n-1}{2}+k, \frac{n-1}{2}+k} = (-1)^{\frac{n+1}{2}}f_{2k+1}$ for $0 \leq k \leq \frac{n-1}{2}$
- $a_{i,j} = 0$ otherwise
- $b_{i,j} = 0$ otherwise.

We now handle the case of general $\delta$. For a fixed $\delta \in \mathcal{T}(r_2)$ and an element $f = f_0x^n + \cdots + f_ny^n \in U(\mathbb{R})^{(r_2)}$ with $f_0 \neq 0$, consider the pair $(R_f, \delta)$. Given the basis $(1, \theta, \ldots, \theta^{n-1})$ for $R_f$, the corresponding pair $(A, B)$ may be written explicitly using (7) and (8). From the definitions of $\theta$ and $\delta$, it follows that $\phi(\theta \otimes \theta')$ may be written as polynomials of degree less than $n$ in $\theta$, whose coefficients are polynomials in the $f_i$ and $1/f_0$. Since $\zeta_{n-2}$ and $\zeta_{n-1}$ are polynomials in $\theta$ both with leading coefficient $f_0$, the coefficients of $A$ and $B$ are polynomials in the $f_i$ and $1/f_0$. We define the function $s_\delta : U(\mathbb{R})^{(r_2)} \to V(\mathbb{R})$ by sending such a binary $n$-ic form $f$ to this pair $(A, B)$.
Lemma 4.1. Let $S \subset U(\mathbb{R})$ be a compact semialgebraic set that does not contain 0. Then there exists a finite subset $T \subset SO_2(\mathbb{R})$ and semialgebraic subsets $S_\tau \subset S$ for each $\tau \in T$, such that the leading coefficients of $\tau \cdot f$ are bounded away from 0 independent of $f \in S_\tau$, and that the union of the $S_\tau$ is $S$.

Proof. The set $\tilde{S} = S \times \{(x, y) : x^2 + y^2 = 1\} \subset U(\mathbb{R}) \times \mathbb{R}^2$ is semialgebraic. The function $S \to \mathbb{R}_{\geq 0}$ given by
\[ f \mapsto \max_{x^2+y^2=1} |f(x, y)| \]
is continuous and nonzero. Hence its image is bounded away from 0 by some $\epsilon > 0$. Therefore, the set
\[ S_1 := \{(f, (x, y)) : f \in S, (x, y) \in \mathbb{R}^2, x^2 + y^2 = 1, |f(x, y)| \geq \epsilon/2\} \]
is semialgebraic and its projection to $S$ is all of $S$. Given an element $\lambda = (x, y) \in \mathbb{R}^2$ with $x^2 + y^2 = 1$, let $S_\lambda$ denote the set of elements $f$ in $S$ such that $(f, \lambda) \in S_1$. Since the projections of semialgebraic sets are semialgebraic, it follows that $S_\lambda$ is semialgebraic. Since $S$ is compact, there exists a finite subset $T'$ of $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ such that the union of $S_\lambda$ over all $\lambda$ in this finite set is $S$. Given $\lambda = (x, y)$, choose $\tau \in SO_2(\mathbb{R})$ to be the matrix $\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$, where $\cos t = x$ and $\sin t = y$. The leading coefficient of $\tau \cdot f$ is $(\tau \cdot f)(1,0) = f(x, y) > \epsilon/2$. The lemma follows by taking $T$ to be the finite set of matrices $\tau$ in $SO_2(\mathbb{R})$ corresponding to the finite set $T'$ of pairs $\lambda = (x, y)$ in $\mathbb{R}^2$, and setting $S_\tau$ to be $S_\lambda$, for $\tau$ corresponding to $\lambda$.

We can clearly choose the sets $S_\tau$ to be disjoint in the above lemma. The set $S = U(\mathbb{R})_{H=1}$ satisfies the conditions of the above lemma. For a fixed $r_2$, we may write $U(\mathbb{R})_{H=1}^{(r_2)}$ as a finite disjoint union of the sets $S_\tau^{(r_2)} = S_\tau \cap U(\mathbb{R})^{(r_2)}$. We now take our fundamental set for the action of $SL_n(\mathbb{R})$ on $V(\mathbb{R})^{(r_2),\delta}$ to be the finite union
\[ R_{H}^{(r_2),\delta} := \bigcup_{\tau > 0} \tau^{-1} s_\delta(\tau \cdot S_\tau^{(r_2)}). \]

We define a fundamental set $R_{J}^{(r_2),\delta}$ for the action of $G(\mathbb{R})$ on $V(\mathbb{R})^{(r_2),\delta}$ in exactly the same way by considering the set $S = L_n$, where $L_n$ is constructed in [12, §3] to be a semialgebraic bounded fundamental set for the action of $SL_2(\mathbb{R})$ on the set of elements in $U(\mathbb{R})$ having Julia invariant 1.

Let $R_{H}^{(r_2),\delta}(X)$ (respectively, $R_{J}^{(r_2),\delta}(X)$) denote the set of elements in $R_{H}^{(r_2),\delta}$ (resp. $R_{J}^{(r_2),\delta}$) having height (resp. Julia invariant) bounded by $X$. The sets $\tau^{-1} s_\delta(\tau \cdot S_\tau^{(r_2)})$ are bounded for $S = U(\mathbb{R})_{H=1}$ and $S = L_n$ because every $f \in \tau \cdot S_\tau$ has bounded coefficients and has leading coefficient bounded away from 0. Since both height and Julia invariant on $V(\mathbb{R})$ have degree $n$, the coefficients of elements $(A, B)$ in $R_{H}^{(r_2),\delta}(X)$ and $R_{J}^{(r_2),\delta}(X)$ are bounded by $O(X^{1/n})$, where the implied constant is independent of $(A, B)$.

**Fundamental domains for $SL_n(\mathbb{Z}) \backslash SL_n(\mathbb{R})$ and $G(\mathbb{Z}) \backslash G(\mathbb{R})$**

Let $SL_n(\mathbb{R}) = N_n T_n K_n$ be the Iwasawa decomposition of $SL_n(\mathbb{R})$, where $N_n \subset SL_n(\mathbb{R})$ denotes the set of unipotent lower triangular matrices, $T_n \subset SL_n(\mathbb{R})$ denotes the set of diagonal matrices, and $K_n = SO_n(\mathbb{R}) \subset SL_n(\mathbb{R})$ is the maximal compact subgroup. Let $G_H$ be a Siegel domain in $SL_n(\mathbb{R})$ defined as
\[ G_H := N'_n T'_n K_n, \]
where \( N'_n \subset N_n \) is the set of elements in \( N_n \) whose coefficients are bounded by 1 in absolute value and \( T'_n \subset T_n \) is given by

\[
T'_n := \{ \text{diag}(t_1^{-1}, t_2^{-1}, \ldots, t_n^{-1}) : t_1/t_2 > c, \ldots, t_{n-1}/t_n > c \},
\]

for some constant \( c > 0 \) that is sufficiently small to ensure the existence of a fundamental domain \( \mathcal{F}_H \) for the action of \( \text{SL}_n(\mathbb{Z}) \) on \( \text{SL}_n(\mathbb{R}) \) that is contained in \( \mathcal{S}_H \).

Next, we pick \( N'_2 \subset N \) to be the set of elements whose coefficients are bounded by 1 in absolute value and \( T'_2 \subset T_2 \) to be the set

\[
T'_2 := \{ \text{diag}(t^{-1}, t) : t > 1/4 \}.
\]

Let

\[
\mathcal{S}_J := (N'_2, N'_n)(T'_2, T'_n)(K_2, K_n)
\]

be a Siegel domain. Then \( \mathcal{S}_J \) contains a fundamental domain \( \mathcal{F}_J \) for the action of \( G(\mathbb{Z}) \) on \( G(\mathbb{R}) \).

**Fundamental domains for the action of \( \text{SL}_n(\mathbb{Z}) \) on \( V(\mathbb{R})^{(r_2)} \)**

The size of the stabilizer in \( \text{SL}_n(\mathbb{R}) \) of \( v \in V(\mathbb{R})^{(r_2,\delta)} \) can be computed from Corollary 2.14. This size depends only on \( r_2 \) and we denote it by \( \sigma'(r_2) \). It is well known that the size of the stabilizer of a generic element \( f \in U(\mathbb{R})^{(r_2)} \) is 3 if \( n = 3 \) and \( r_2 = 0 \), and 1 otherwise. It follows that the size of the stabilizer in \( G(\mathbb{R}) \) of a generic element in \( V(\mathbb{R})^{(r_2,\delta)} \) is \( \sigma'(r_2) \), where \( \sigma'(r_2) = 3\sigma(r_2) \) if \( n = 3 \) and \( r_2 = 0 \) and \( \sigma'(r_2) = \sigma(r_2) \) otherwise. By arguments identical to those in [8, §2.1], we see that \( \mathcal{F}_H \cdot \mathcal{R}_H^{(r_2,\delta)} \) is a \( \sigma(r_2) \)-fold cover of a fundamental domain for the action of \( \text{SL}_n(\mathbb{Z}) \) on \( V(\mathbb{R})^{(r_2,\delta)} \) and that \( \mathcal{F}_J \cdot \mathcal{R}_J^{(r_2,\delta)} \) is a \( \sigma'(r_2) \)-fold cover of a fundamental domain for the action of \( G(\mathbb{Z}) \) on \( V(\mathbb{R})^{(r_2,\delta)} \), where \( \mathcal{F}_H \cdot \mathcal{R}_H^{(r_2,\delta)} \) and \( \mathcal{F}_J \cdot \mathcal{R}_J^{(r_2,\delta)} \) are regarded as multisets. More precisely, the \( \text{SL}_n(\mathbb{Z}) \)-orbit of any \( v \in V(\mathbb{R})^{(r_2,\delta)} \) is represented \( \#\text{Stab}_{\text{SL}_n(\mathbb{R})}(v)/\#\text{Stab}_{\text{SL}_n(\mathbb{Z})}(v) \) times in \( \mathcal{F}_H \cdot \mathcal{R}_H^{(r_2,\delta)} \), with the analogous statement also holding for the multiset \( \mathcal{F}_J \cdot \mathcal{R}_J^{(r_2,\delta)} \).

For an \( \text{SL}_n(\mathbb{Z}) \)-invariant set \( S \subset V(\mathbb{Z})^{(r_2,\delta)} := V(\mathbb{R})^{(r_2,\delta)} \cap V(\mathbb{Z}) \), let \( N_H(S; X) \) denote the number of absolutely irreducible \( \text{SL}_n(\mathbb{Z}) \)-orbits on \( S \) that have height bounded by \( X \). For a \( (G(\mathbb{Z})) \)-invariant set \( S' \subset V(\mathbb{Z})^{(r_2,\delta)} \), let \( N_J(S'; X) \) denote the number of absolutely irreducible \( G(\mathbb{Z}) \)-orbits on \( S' \) whose Julia invariant is bounded by \( X \). Let \( v \in V(\mathbb{Z}) \) be absolutely irreducible with resolvent form \( f \). Then \( f \) corresponds to an order \( \mathcal{O} \) in an \( S_n \)-number field and \( \mathcal{O}^\times[2]_{N=1} \) is trivial. Furthermore, \( f \) has trivial stabilizer in \( \text{SL}_2(\mathbb{Z}) \) since \( \text{Aut}(\mathcal{O}) \) is trivial. Therefore, \( v \) has trivial stabilizer in \( \text{SL}_n(\mathbb{Z}) \) and \( G(\mathbb{Z}) \). For any set \( L \subset V(\mathbb{Z}) \), let \( L^{\text{irr}} \) denote the set of absolutely irreducible elements in \( L \). Let \( \mathcal{R}_H^{(r_2,\delta)}(X) \) (respectively, \( \mathcal{R}_J^{(r_2,\delta)}(X) \)) denote the set of elements in \( \mathcal{R}_H^{(r_2)} \) (resp. \( \mathcal{R}_J^{(r_2)} \)) having height (resp. Julia invariant) bounded by \( X \). Then we have the following:

**Proposition 4.2.** Let notation be as above. We have

\[
N_H(S; X) = \frac{1}{\sigma(r_2)} \# \{ \mathcal{F}_H^{(r_2,\delta)}(X) \cap S^{\text{irr}} \};
\]

\[
N_J(S'; X) = \frac{1}{\sigma'(r_2)} \# \{ \mathcal{F}_J^{(r_2,\delta)}(X) \cap S^{\text{irr}} \}.
\]

(27)
4.2 Averaging and cutting off the cusp

Let $G_0$ (resp. $G'_0$) be a bounded open nonempty $K_n$-invariant (resp. $K_2 \times K_n$-invariant) set in $\text{SL}_n(\mathbb{R})$ (resp. $G(\mathbb{R})$). We abuse notation and refer to Haar measures in both groups $\text{SL}_n(\mathbb{R})$ and $G(\mathbb{R})$ by $dh$. From Proposition 4.2 and by an argument identical to the proof of [8, Theorem 2.5], we obtain

$$N_H(S; X) = \frac{1}{\sigma(r_2) \text{Vol}(G_0)} \int_{h \in \mathcal{F}_H} \#\{hG_0 \cdot \mathcal{R}_H^{(r_2)}(X) \cap S^{\text{irr}}\} \, dh,$$

$$N_J(S'; X) = \frac{1}{\sigma'(r_2) \text{Vol}(G'_0)} \int_{h \in \mathcal{F}_J} \#\{hG'_0 \cdot \mathcal{R}_J^{(r_2)}(X) \cap S^{\text{irr}}\} \, dh,$$

(28)

where the volumes of $G_0$ and $G'_0$ are computed with respect to $dh$. We use (28) to define $N_H(S; X)$ (resp. $N_J(S'; X)$) even when $S$ (resp. $S'$) is not $\text{SL}_n(\mathbb{Z})$-invariant (resp. $G(\mathbb{Z})$-invariant).

Let $\mathcal{F}'_H \subset \mathcal{F}_H$ and $\mathcal{F}'_J \subset \mathcal{F}_J$ denote the sets of elements $\gamma \in \mathcal{F}_H$ and $\gamma \in \mathcal{F}_J$ such that $|a_{11}(v)| < 1$ for every element $v \in \gamma \cdot G_0 \mathcal{R}_H^{(r_2)}(X)$ and $v \in \gamma \cdot G'_0 \mathcal{R}_J^{(r_2)}(X)$, respectively. We will refer to the integrals of the integrands in (28) over $\mathcal{F}'_H$ and $\mathcal{F}'_J$ as the “cuspidal” part of the integral, and to the integrals over $\mathcal{F}_H \setminus \mathcal{F}'_H$ and $\mathcal{F}_J \setminus \mathcal{F}'_J$ as the “main body” of the integral.

Absolutely irreducible points in the cusp

We shall prove that the number of absolutely irreducible integral points in the cusp is negligible:

**Proposition 4.3.** We have

$$\int_{h \in \mathcal{F}'_H} \#\{hG_0 \cdot \mathcal{R}_H^{(r_2)}(X) \cap \mathcal{V}(\mathbb{Z})^{\text{irr}}\} \, dh = O(X^{n+1-\frac{1}{2}}), \text{ and}$$

$$\int_{h \in \mathcal{F}'_J} \#\{hG'_0 \cdot \mathcal{R}_J^{(r_2)}(X) \cap \mathcal{V}(\mathbb{Z})^{\text{irr}}\} \, dh = O(X^{n+1-\frac{1}{2}}).$$

First, we list sufficient conditions to guarantee that an element $(A, B) \in \mathcal{V}(\mathbb{Z})$ is not absolutely irreducible:

**Lemma 4.4.** Let $(A, B) \in \mathcal{V}(\mathbb{Z})$ be such that all the variables in one of the following sets vanish:

(a) $\{a_{ij}, b_{ij} : 1 \leq i \leq k, 1 \leq j \leq n - k\}$ for some $1 \leq k \leq n - 1$.

(b) $\{a_{ij}, b_{ij} : 1 \leq i, j \leq (n - 1)/2\}$.

Then $(A, B)$ is not absolutely irreducible.

**Proof.** If $(A, B)$ satisfies condition 4.4, then it is easy to see that the binary $n$-ic invariant of $(A, B)$ has a repeated factor over $\mathbb{Q}$. Thus, the discriminant of the form vanishes. If $(A, B)$ satisfies condition 4.4, then clearly the quadratic forms $A$ and $B$ have a common isotropic subspace of dimension $(n - 1)/2$. In either case, the pair $(A, B)$ is not absolutely irreducible. \qed

Recall that the condition for $t = (t_1^{-1}, \ldots, t_n^{-1})$ to be an element of $T'_n$ is that $t_i/t_{i+1} > c$ for $1 \leq i \leq n - 1$. To simplify this condition, we use a change of variables: let $s_i = t_i/t_{i+1}$ for $1 \leq i \leq n - 1$. Then $s = (s_1, \ldots, s_{n-1})$ is contained in $T'$ if and only if $s_i > c$ for each $i$. The action of the torus $T_2 \times T_0$ of $G(\mathbb{R})$ on $\mathcal{V}(\mathbb{R})$ multiplies each coefficient by a monomial in $t, s_1, \ldots, s_{n-1}$. We denote the set of coefficients of $\mathcal{V}(\mathbb{R})$ by $\text{Var}$; we have

$$\text{Var} := \{a_{ij}, b_{ij} : 1 \leq i \leq j \leq n\}.$$
To each variable $c_{ij}$ in $\text{Var}$, we associate two weights: first, the monomial $w_H(c_{ij})$ in the $s_i$ by which the action of $T_n$ scales $c_{ij}$ and second, the monomial $w_J(c_{ij})$ in $t$ and the $s_i$ by which the action of $T_2 \times T_n$ scales $c_{ij}$. We multiplicatively extend the function $w_H$ and $w_J$ to products of integral powers of elements in $\text{Var}$. We define a partial ordering on $\text{Var}$ by setting $\alpha_1 \preceq_H \alpha_2$ (resp. $\alpha_1 \preceq_J \alpha_2$) whenever $w_H(\alpha_2)/w_H(\alpha_1)$ (resp. $w_J(\alpha_2)/w_J(\alpha_1)$) is a product of nonnegative powers of $s_i$ for each $i$ (resp. of $t$ and $s_i$ for each $i$). The variable $a_{11}$ has minimal weight under both these partial orderings. For a subset $\text{Var}' \subset \text{Var}$, let $V(\mathbb{Z})(\text{Var}')$ denote the set of $v \in V(\mathbb{Z})$ such that $\alpha(v) = 0$ for $\alpha \in \text{Var}'$. Then we have the following immediate consequence of Lemma 4.4:

**Lemma 4.5.** Let $\text{Var}' \subset \text{Var}$ be a set that is closed under one of the partial orderings $\preceq_H$ and $\preceq_J$. If $V(\mathbb{Z})(\text{Var}')$ is nonempty, then $\text{Var}'$ must be contained in the following set:

$$\text{Var}_0 := \{ a_{ij} \in \text{Var} : i + j \leq n \} \cup \{ b_{ij} \in \text{Var} : i + j \leq n - 1 \} \setminus \{ b_{mm} \}.$$

**Proof of Proposition 4.3.** By the arguments of [7, §3], it suffices to display the following data in order to prove the part of Proposition 4.3 regarding the height (resp. the Julia invariant): a function $\psi : \text{Var}_0 \setminus a_{11} \rightarrow \text{Var} \setminus \text{Var}_0$ such that

1. $\alpha \preceq_H \psi(\alpha) \forall \alpha \in \text{Var}_0 \setminus a_{11}$ (resp. $\alpha \preceq_J \psi(\alpha) \forall \alpha \in \text{Var}_0 \setminus a_{11}$), and
2. $w_H\left( \prod_{\alpha \in \text{Var}_0} \alpha^{-1} \psi(\alpha) \right) \cdot h_H$ (resp. $w_J\left( \prod_{\alpha \in \text{Var}_0} \alpha^{-1} \psi(\alpha) \right) \cdot h_J$) is a product of negative powers of $s_i$ (resp. negative powers of $t$ and the $s_i$),

where $\psi(a_{11})$ is defined to be 1 and $h_H$ and $h_J$ are factors arising the Haar-measures of $\text{SL}_n(\mathbb{R})$ and $G(\mathbb{R})$ and are given by

$$h_H := \prod_{k=1}^{n-1} s_k^{-nk(n-k)} \quad \text{and} \quad h_J := t^{-2} \prod_{k=1}^{n-1} s_k^{-nk(n-k)}.$$

First note that such a function $\psi$ satisfying the required conditions regarding the Julia invariant automatically satisfies the required conditions regarding the height (since $\alpha \preceq_J \beta$ implies $\alpha \preceq_H \beta$.) We define $\psi$ as follows:

$$\psi(a_{ij}) := \begin{cases} a_{1n} & \text{for } i = 1; \\ a_{i(n-i+1)} & \text{for } i > 1 \text{ and } j \neq m; \\ a_{(m+1)(m+1)} & \text{for } i > 1 \text{ and } j = m; \\ \end{cases}$$

$$\psi(b_{ij}) := \begin{cases} b_{j(n-j)} & \text{for } j < m; \\ b_{mm} & \text{for } j = m; \\ b_{(n-j-1)(j+1)} & \text{for } j > m. \end{cases} \quad (29)$$

The function $\psi$ clearly satisfies the first of the two required conditions. From an elementary computation, we see that

$$w_J\left( \prod_{\alpha \in \text{Var}_0} \alpha^{-1} \psi(\alpha) \right) \cdot h_J = t^{-1} \prod_{k=1}^{m} s_k^{-2k} \prod_{k=m+1}^{n-1} s_k^{-2(k-m)+1}.$$

This concludes the proof of Proposition 4.3.
Reducible points in the main body

We say that an element \( v \in V(\mathbb{Z}) \) is bad if \( v \) is not absolutely irreducible. Denote the set of bad elements in \( V(\mathbb{Z}) \) by \( V(\mathbb{Z})^{\text{bad}} \). We have the following theorem proving that the number of bad elements in the main body is negligible.

**Proposition 4.6.** We have

\[
\int_{h \in F_H \setminus F_H'} \#\{hG_0 \cdot R_H^{(r_2),\delta}(X) \cap V(\mathbb{Z})^{\text{bad}}\} \, dh = o(X^{n+1}), \quad \text{and}
\]

\[
\int_{h \in F_J \setminus F_J'} \#\{hG_0' \cdot R_J^{(r_2),\delta}(X) \cap V(\mathbb{Z})^{\text{bad}}\} \, dh = o(X^{n+1}).
\]

**Proof.** For an integer \( k \) with \( 2 \leq k \leq n \), let \( V(\mathbb{Z})^{\neq k} \) denote the set of elements \( v \in V(\mathbb{Z}) \) such that, for each prime \( p \), the reduction modulo \( p \) of the resolvent of \( v \) does not factor into a product of an irreducible degree \( k \) factor and \( n - k \) linear factors. We claim that if the resolvent \( f \) of an element \( v \in V(\mathbb{Z}) \) does not correspond to an order in an \( S_n \)-field, then \( v \) belongs to \( V(\mathbb{Z})^{\neq k} \) for some \( k \).

Indeed, if \( v \) lies in the complement of \( V(\mathbb{Z})^{\neq n} \), then the reduction modulo \( p \) of \( f \) is irreducible for some prime \( p \), implying that \( f \) is irreducible and hence \( R_f \) is an order. Furthermore, the Galois group of the Galois closure of the fraction field of \( R_f \) contains a \( k \)-cycle for each \( k \), implying that this Galois group is \( S_n \).

Hence we may write

\[ V(\mathbb{Z})^{\text{bad}} = (\cup V(\mathbb{Z})^{\neq k}) \bigcup V(\mathbb{Z})^{\text{red}} \]

where \( V(\mathbb{Z})^{\text{red}} \) denotes the set of elements that are reducible in the sense of Theorem 2.6.

For each prime \( p \), let \( V(\mathbb{F}_p)^{=k} \) denote the set of elements whose cubic resolvents factor into a product of a degree \( k \) irreducible factor and \( n - k \) distinct linear factors. Let \( V(\mathbb{F}_p)^{\text{irr}} \) denote the set of elements in \( v \in V(\mathbb{F}_p) \) such that every lift \( \tilde{v} \in V(\mathbb{Z}) \) is not reducible in the sense of Theorem 2.6. Let \( V(\mathbb{F}_p)^{\text{nostab}} \) denote the set of elements which have trivial stabilizer in \( G(\mathbb{F}_p) \). Then, from [7, §3], it suffices to prove the following estimates:

\[
\#V(\mathbb{F}_p)^{=k} \gg \#V(\mathbb{F}_p), \quad \text{and} \quad \#V(\mathbb{F}_p)^{\text{irr}} \gg \#V(\mathbb{F}_p).
\]

(30)

\[
\#V(\mathbb{F}_p)^{=k} \gg \#U(\mathbb{F}_p)^{=k} \cdot \#\text{SL}_n(\mathbb{F}_p) \gg \#V(\mathbb{F}_p),
\]

as desired.

The proof of the inequality (30) is similar. It follows from the observation that every element in \( V(\mathbb{F}_p)^{=n} \) that corresponds to a nonidentity element in \( \mathbb{F}_p^* / (\mathbb{F}_p^*)^2 \) under the bijection of Corollary 2.15, belongs to \( V(\mathbb{F}_p)^{\text{irr}} \).
Absolutely irreducible points in the main body

Let \( L \subset V(\mathbb{Z}) \) be a lattice or a translate of a lattice in \( V(\mathbb{R}) \), and let \( L^{(r_2),\delta} \) denote \( L \cap V(\mathbb{Z})^{(r_2),\delta} \). We have already proven that the number of irreducible integral points in the cusp is negligible and that the number of reducible integral points in the main body is negligible. Therefore, from (28), Proposition 4.3, and Proposition 4.6, we have

\[
N_H(L^{(r_2),\delta}, X) = \frac{1}{\sigma(r_2)\text{Vol}(G_0)} \int_{h \in \mathcal{F}_H \backslash \mathcal{F}'_H} \# \{ hG_0 \cdot \mathcal{R}_H^{(r_2),\delta}(X) \cap L \} \, dh + o(X),
\]

\[
N_J(L^{(r_2),\delta}, X) = \frac{1}{\sigma'(r_2)\text{Vol}(G_0')} \int_{h \in \mathcal{F}_J \backslash \mathcal{F}'_J} \# \{ hG_0' \cdot \mathcal{R}_J^{(r_2),\delta}(X) \cap L \} \, dh + o(X).
\]

To estimate the number of lattice points in \( hG_0 \cdot \mathcal{R}_H^{(r_2),\delta}(X) \) and \( hG_0' \cdot \mathcal{R}_J^{(r_2),\delta}(X) \), we have the following result of Davenport [16].

**Proposition 4.7.** Let \( \mathcal{R} \) be a bounded, semi-algebraic multiset in \( \mathbb{R}^n \) having maximum multiplicity \( m \), and that is defined by at most \( k \) polynomial inequalities each having degree at most \( \ell \). Then the number of integral lattice points (counted with multiplicity) contained in the region \( \mathcal{R} \) is

\[
\text{Vol}(\mathcal{R}) + O(\max\{\text{Vol}(\mathcal{R}), 1\}),
\]

where \( \text{Vol}(\mathcal{R}) \) denotes the greatest \( d \)-dimensional volume of any projection of \( \mathcal{R} \) onto a coordinate subspace obtained by equating \( n-d \) coordinates to zero, where \( d \) takes all values from 1 to \( n-1 \). The implied constant in the second summand depends only on \( n, m, k, \) and \( \ell \).

The coefficient \( a_{11} \) has minimal weight among all the coefficients. Furthermore, for \( h \in \mathcal{F}_H \backslash \mathcal{F}'_H \), the volume of the projection of \( hG_0 \cdot \mathcal{R}^{(r_2)}(X) \) onto the \( a_{11} \)-coordinate is bounded away from 0 by the definition of \( \mathcal{F}'_H \). Therefore, for \( h \in \mathcal{F}_H \backslash \mathcal{F}'_H \), all proper projections of \( hG_0 \cdot \mathcal{R}^{(r_2)}(X) \) are bounded by a constant times its projection onto the \( a_{11} = 0 \) hyperplane. Proposition 4.7 thus implies that

\[
N_H(L^{(r_2),\delta}, X) = \frac{1}{\sigma(r_2)\text{Vol}(G_0)} \int_{h \in \mathcal{F}_H \backslash \mathcal{F}'_H} \# \{ hG_0 \cdot \mathcal{R}_H^{(r_2),\delta}(X) \cap L \} \, dh + o(X^{n+1})
\]

\[
= \frac{1}{\sigma(r_2)\text{Vol}(G_0)} \int_{h \in \mathcal{F}_H \backslash \mathcal{F}'_H} \text{Vol}_L(hG_0 \cdot \mathcal{R}_H^{(r_2),\delta}(X)) \, dh + o(X^{n+1})
\]

\[
= \frac{1}{\sigma(r_2)\text{Vol}(G_0)} \text{Vol}(\mathcal{F}_H)\text{Vol}_L(G_0 \cdot \mathcal{R}_H^{(r_2),\delta}(X)) + o(X^{n+1})
\]

\[
= \frac{1}{\sigma(r_2)\text{Vol}(\mathcal{F}_H \cdot \mathcal{R}_H^{(r_2),\delta}(X)) + o(X^{n+1}),
\]

where the volume \( \text{Vol}_L \) of sets in \( V(\mathbb{R}) \) is computed with respect to Euclidean measure on \( V(\mathbb{R}) \) normalized so that \( L \) has covolume 1, and where the third equality follows since \( \text{Vol}(\mathcal{F}') \) tends to zero as \( X \) tends to infinity, and \( \text{Vol}_L(hG_0 \cdot \mathcal{R}^{(r_2)}(X)) \) is independent of \( h \), and the final equality follows from the Jacobian change of variables in Theorem 6.3.

An identical argument yields the analogous estimate for \( N_J(L^{(r_2),\delta}, X) \). Let \( L_p \) denote the closure of \( L \) in \( V(\mathbb{Z}_p) \). Then for measurable sets \( B \) in \( V(\mathbb{R}) \), we have

\[
\text{Vol}_L(B) = \text{Vol}(B) \cdot \text{Vol}(L_p),
\]

25
where \( \text{Vol}(B) \) is computed with respect to Euclidean measure in \( V(\mathbb{R}) \) normalized so that \( V(\mathbb{Z}) \) has covolume 1, and the volumes of \( L \subset V(\mathbb{Z}_p) \) are computed with respect to Haar measure on \( V(\mathbb{Z}_p) \) normalized so that \( V(\mathbb{Z}_p) \) has volume 1. We thus have the following theorem:

**Theorem 4.8.** Let notation be as above. Then we have

\[
N_H(L(r_2), X) = \frac{1}{\sigma(r_2)} \text{Vol}(\mathcal{F}_H : \mathcal{R}_H^{(r_2), \delta}(X)) \prod_p \text{Vol}(L_p) + o(X^{n+1}),
\]

\[
N_J(L(r_2), X) = \frac{1}{\sigma'(r_2)} \text{Vol}(\mathcal{F}_J : \mathcal{R}_J^{(r_2), \delta}(X)) \prod_p \text{Vol}(L_p) + o(X^{n+1}).
\]

**Remark 4.9.** Using the Selberg sieve identically as in [23, §3], we may improve the error term in Proposition 4.6, and thus in Theorem 4.8, to \( O(X^{n+1 - \frac{1}{m}}) \). However, this additional saving will not be necessary for the results in this paper.

## 5 Sieving to projective elements and acceptable sets

In this section, we first determine asymptotics for \( \text{SL}_n(\mathbb{Z}) \)-orbits and \( G(\mathbb{Z}) \)-orbits on certain families having bounded height. Second, we determine asymptotics for \( \text{SL}_n(\mathbb{Z}) \)-orbits and \( G(\mathbb{Z}) \)-orbits on acceptable sets conditional on a tail estimate. This tail estimate is unknown for \( n \geq 5 \), but is known when \( n = 3 \) (see [2, Proposition 23]). We begin by describing the very large and acceptable families we study.

For each prime \( p \), let \( \Lambda_p \subset V(\mathbb{Z}_p) \setminus \{ \Delta = 0 \} \) be a nonempty open set whose boundary has measure 0. Let \( \Lambda_\infty \) denote \( V(\mathbb{R})^{(r_2), \delta} \) for some integer \( r_2 \) with \( 0 \leq r_2 \leq (n - 1)/2 \) and some \( \delta \in \{ \pm 1 \}^{n - 2r_2} \times \{ 1 \}^{r_2} \). To a collection \( \Lambda = (\Lambda_p)_p \), of these local specifications, we associate the set

\[
\mathcal{V}(\Lambda) := \{ v \in V(\mathbb{Z}) : v \in \Lambda_p \text{ for all } p \}.
\]

We say that the collection \( \Lambda = (\Lambda_p)_p \) is very large (resp. acceptable) if, for all large enough primes \( p \), the set \( \Lambda_p \) contains all elements \( v \in V(\mathbb{Z}_p) \) such that \( v \) is projective and has primitive invariant form (resp. \( p^2 \nmid \Delta(v) \)). We say that \( \mathcal{V}(\Lambda) \) is very large or acceptable if \( \Lambda \) is.

### 5.1 Sieving to projective elements

We define \( V(\mathbb{Z}_p)^{\text{proj}} \) to be the set of elements \( (A, B) \in V(\mathbb{Z}_p) \) whose binary \( n \)-ic invariants are not divisible by \( p \) and correspond to a pair \( (I, \delta) \) such that \( I^2 = (\delta) \). Then

\[
V(\mathbb{Z})^{(r_2), \text{proj}} = V(\mathbb{Z})^{(r_2)} \cap (\bigcap_p V(\mathbb{Z}_p)^{\text{proj}}).
\]

For a prime \( p \), let \( W_p \) now denote the set of elements in \( V(\mathbb{Z}) \) that do not belong to \( V(\mathbb{Z}_p)^{\text{proj}} \). We would like to estimate the number of elements in \( W_p \) for large \( p \). We have the following theorem:

**Theorem 5.1.** We have

\[
N_H(\cup_{p \geq M} W_p, X) = O(X^{n+1}/M^{1-\epsilon}) + o(X^{n+1}), \text{ and}
\]

\[
N_J(\cup_{p \geq M} W_p, X) = O(X^{n+1}/M^{1-\epsilon}) + o(X^{n+1}),
\]

where the implied constant is independent of \( X \) and \( M \).
Proof. If \((A, B) \in W_p\) gives rise to the binary \(n\)-ic form \(f\), then the ring \(R_f\) is nonmaximal at \(p\), which implies that \(p^2 \mid \Delta(A, B) = \Delta(f)\). Let \((A, B) \in W_p\), regarded as an element of \(V(\mathbb{Z}_p)\), correspond to a pair \((I, \delta)\) with \(I^2 \neq (\delta)I_{f_I}^{-3}\). Then the reduction of \((A, B)\) modulo \(p\) corresponds to the pair \((f \otimes \mathbb{F}_p, \overline{\delta})\), where \(\overline{\delta}\) is the reduction of \(\delta\) modulo \(p\). From Nakayama’s lemma, it follows that \(I^2 \otimes \mathbb{F}_p \neq (\overline{\delta})I_{f_I}^{-3} \otimes \mathbb{F}_p\).

Let \((A_1, B_1) \in V(\mathbb{Z})\) be any element congruent to \((A, B)\) modulo \(p\). Denote the binary \(n\)-ic form associated to \((A_1, B_1)\) by \(f_1\). If \((A_1, B_1)\) corresponds to the pair \((I_1, \delta_1)\), then it follows (again from Nakayama’s lemma) that \(I_{f_1}^2 \neq (\delta_1)I_{f_1}^{-3}\). Thus \((A_1, B_1) \in W_p\).

Also, the set of elements in \(W_p\) whose binary \(n\)-ic invariants are divisible by \(p\) is the preimage under \(V(\mathbb{Z}_p) \to V(\mathbb{F}_p)\) of the set of elements in \(V(\mathbb{F}_p)\) having binary \(n\)-ic invariant 0. It follows that \(W_p\) is defined via congruence conditions modulo \(p\), i.e., the set \(W_p\) is the preimage of some subset of \(V(\mathbb{F}_p)\) under the reduction modulo \(p\) map.

To prove the theorem, we start with the fundamental domain \(\mathcal{F}_H\) chosen in §4.1. For every \(0 < \epsilon < 1\), we pick a set \(\mathcal{F}(\epsilon) \subset \mathcal{F}_H\) which is open and bounded and whose measure is \((1 - \epsilon)\) times the measure of \(\mathcal{F}_H\). Let \(\mathcal{R}\) be the union of the \(\mathcal{R}(r_2, \delta)\) over all possible \(r_2\) and \(\delta\), and let \(\mathcal{R}_X\) denote the set of elements in \(\mathcal{R}\) having height bounded by \(X\). Then we have

\[
\#\{\mathcal{F}(\epsilon) \cdot \mathcal{R}_X \cap (\cup_{p \geq M} W_p)\} = O(X^{n+1}/M \log M)
\]

from an immediate application of [5, Theorem 3.3]. We further obtain

\[
\#\{(\mathcal{F} \setminus \mathcal{F}(\epsilon)) \cdot \mathcal{R}_X \cap V(\mathbb{Z})^{\text{irr}}\} = O(\epsilon X^{n+1})
\]

from the methods of the previous section. The first assertion of the theorem follows. The second assertion follows in an identical fashion by starting with \(\mathcal{F}_J\) instead of \(\mathcal{F}_H\).

We now have the following theorem.

**Theorem 5.2.** Let \(r_2\) be an integer such that \(0 \leq r_2 \leq (n - 1)/2\) and let \(\delta \in \{\pm 1\}^{n-2r_2} \times \{1\}^{r_2}\) be fixed. Let \(\Lambda\) be a very large collection of local specifications such that \(\Lambda_\infty = V(\mathbb{R})^{(r_2, \delta)}\). Then we have

\[
N_H(V(\Lambda), X) = \frac{1}{\sigma(r_2)} \Vol(\mathcal{F}_H \cdot \mathcal{R}_H^{(r_2, \delta)}(X)) \prod_p \Vol(\Lambda_p) + o(X^{n+1}), \text{ and}
\]

\[
N_J(V(\Lambda), X) = \frac{1}{\sigma'(r_2)} \Vol(\mathcal{F}_J \cdot \mathcal{R}_J^{(r_2, \delta)}(X)) \prod_p \Vol(\Lambda_p) + o(X^{n+1}),
\]

where the volumes of sets in \(V(\mathbb{Z}_p)\) are computed with respect to the Euclidean measure normalized so that \(V(\mathbb{Z}_p)\) has measure 1.

The first estimate asserted by Theorem 5.2 follows from Theorem 5.1 just as [8, Theorem 2.21] follows from [8, Theorem 2.13]. The second estimate follows from a proof identical to that of Theorem 3.5 (which itself uses the methods of the proof of [8, Theorem 2.21]).

### 5.2 Sieving to acceptable sets (conditional on a tail estimate)

Let \(\Lambda\) be an acceptable collection of local specifications with \(\Lambda_\infty = V(\mathbb{R})^{(r_2, \delta)}\). Then we have the following theorem whose proof is identical to the proof of the upper bound in [8, Theorem 2.21]:
Theorem 5.3. We have
\[ N_H(V(\Lambda), X) \leq \frac{1}{\sigma(r_2)} Vol(F_H \cdot R_H^{(r_2), \delta}(X)) \prod_p \text{Vol}(\Lambda_p) + o(X^{n+1}), \quad \text{and} \]
\[ N_J(V(\Lambda), X) \leq \frac{1}{\sigma(r_2)} Vol(F_J \cdot R_J^{(r_2), \delta}(X)) \prod_p \text{Vol}(\Lambda_p) + o(X^{n+1}), \]
where the volumes of sets in \( V(\mathbb{R}) \) are computed with respect to Euclidean measure normalized so that \( V(\mathbb{Z}) \) has covolume 1 and the volumes of sets in \( V(\mathbb{Z}_p) \) are computed with respect to the Euclidean measure normalized so that \( V(\mathbb{Z}_p) \) has volume 1.

For a prime \( p \), let \( W_p \) denote the set of elements in \( V(\mathbb{Z}) \) such that \( p^2 \mid \Delta \). The following estimates are unknown but likely to be true:
\[ N_H(\bigcup_{p \geq M} W_p, X) = O(X^{n+1}/M^{1-\epsilon}) + o(X^{n+1}) \]
\[ N_J(\bigcup_{p \geq M} W_p, X) = O(X^{n+1}/M^{1-\epsilon}) + o(X^{n+1}) \]  

We now have the following theorem.

Theorem 5.4. Assume that one of the equations in (31) holds. Let \( \Lambda \) be an acceptable collection of local specifications with \( \Lambda_\infty = V(\mathbb{R})^{(r_2), \delta} \). Then we have
\[ N_H(V(\Lambda), X) = \frac{1}{\sigma(r_2)} Vol(F_H \cdot R_H^{(r_2), \delta}(X)) \prod_p \text{Vol}(\Lambda_p) + o(X^{n+1}), \quad \text{and} \]
\[ N_J(V(\Lambda), X) = \frac{1}{\sigma(r_2)} Vol(F_J \cdot R_J^{(r_2), \delta}(X)) \prod_p \text{Vol}(\Lambda_p) + o(X^{n+1}), \]
where the volumes of sets in \( V(\mathbb{R}) \) are computed with respect to Euclidean measure normalized so that \( V(\mathbb{Z}) \) has covolume 1 and the volumes of sets in \( V(\mathbb{Z}_p) \) are computed with respect to the Euclidean measure normalized so that \( V(\mathbb{Z}_p) \) has volume 1.

Proof. We first assume that the first equation in (31) holds. Then the first assertion of the Theorem follows just as [8, Theorem 2.21] follows from [8, Theorem 2.13]. The second estimate follows from a proof identical to that of Theorem 3.5.

We now assume that the second equation in (31) holds. Then the second assertion of the Theorem follows just as [8, Theorem 2.21] follows from [8, Theorem 2.13]. To prove the first assertion, we use methods from the proof of [5, Lemma 3.7]. The set \( F_H \cdot R_H^{(r_2), \delta}(X) \setminus \{\Delta = 0\} \) can be covered with countably many fundamental domains for the action of \( G(\mathbb{Z}) \) on \( V(\mathbb{R})^{(r_2), \delta} \). Therefore, for any \( \epsilon > 0 \), there exist \( s \) fundamental domains for the action of \( G(\mathbb{Z}) \) on \( V(\mathbb{R})^{(r_2), \delta} \) whose union covers all but measure \( \epsilon X^{n+1} \) of the finite measure multiset \( F_H \cdot R_H^{(r_2), \delta}(X) \), where \( s \) is independent of \( X \). (To ensure that \( s \) is independent of \( X \), we merely choose \( s \) fundamental domains when \( X = 1 \), and then scale these fundamental domains for large \( X \).) Once again arguments in the proof of [8, Theorem 2.21] imply the bound
\[ \frac{N_H(V(\Lambda), X)}{X^{n+1}} \geq \frac{1}{\sigma(r_2)} (\text{Vol}(F_H \cdot R_H^{(r_2), \delta}(1)) - \epsilon) \prod_{p < M} \text{Vol}(\Lambda_p) + O(s/M^{1-\delta}) + o(s). \]

Letting \( M \) tend to \( \infty \), and then \( \epsilon \) to 0, and then \( s \) to \( \infty \) yields the required lower bound. The upper bound follows from Theorem 5.3. This concludes the proof of Theorem 5.4. \( \square \)
6 Proof of the main theorems

We are now ready to prove Theorems 2-5. To do so, we establish Theorem 6.2, which determines an upper bound for the average sizes of the 2-torsion subgroup in the class groups of acceptable families of orders of fixed signature ordered by height or by Julia invariant. For certain very large families, we obtain that the average sizes are in fact equal to 1; for all other acceptable families, the lower bound being equal to 1 is dependent on the tail estimates described in (31). The proof of Theorem 6.2 involves the computation of local volumes in order to determine the number of absolutely irreducible lattice points in \( \mathcal{F}_H \) of bounded height and \( \mathcal{F}_J \) of bounded Julia invariant. The results of §2 then allow us to conclude the theorem, and it immediately implies Theorems 2, 3, and 5. We obtain Theorem 4 from combining Theorems 2 and 3 with the results of [13].

We adopt the notation of the introduction. Recall that for an infinite collection \( \Sigma \) of local specifications, \( U(\Sigma) \) is the associated set of integral binary \( n \)-ic forms, and acceptable sets \( U(\Sigma) \) give rise to acceptable families \( \Sigma_H \subseteq \mathcal{R}_H \) (and acceptable families \( \Sigma_J \subseteq \mathcal{R}_J \) if \( U(\Sigma) \) is also \( \text{SL}_2(\mathbb{Z}) \)-invariant). We now describe the collections for which we obtain equalities on the average sizes in Theorem 5.

**Definition 6.1.** We say that \( \Sigma = (\Sigma_p)_p \) and \( U(\Sigma) \) are **very large** if, for all sufficiently large primes \( p \), the set \( \Sigma_p \) is precisely \( U(\mathbb{Z}_p) \setminus pU(\mathbb{Z}_p) \). We say that a family \( \Sigma_H \subseteq \mathcal{R}_H \) is **very large** if it is defined by a very large family \( U(\Sigma) \), i.e., \( \mathcal{R}_H = \{ R_f \mid f \in U(\Sigma) \} \). A family \( \Sigma_J \subseteq \mathcal{R}_J \) is very large if it is defined by a very large \( \text{SL}_2(\mathbb{Z}) \)-invariant family \( U(\Sigma) \).

**Theorem 6.2.** Fix an integer \( n \) and a signature \((r_1, r_2)\) with \( r_1 + 2r_2 = n \). Let \( \mathcal{R}_1 \subseteq \mathcal{R}_H^{r_1, r_2} \) be a family of rings that arises from an acceptable set of integral binary \( n \)-ic forms and let \( \mathcal{R}_2 \subseteq \mathcal{R}_J^{r_1, r_2} \) be a family of rings that arises from an acceptable \( \text{SL}_2(\mathbb{Z}) \)-invariant set of binary \( n \)-ic forms. Then:

(a) The average sizes of
\[
|\text{Cl}_2(O)| - \frac{1}{2^{r_1+r_2-1}}|\mathcal{I}_2(O)|
\]
over \( O \in \mathcal{R}_1 \) ordered by height and over \( O \in \mathcal{R}_2 \) ordered by Julia invariant are bounded above by 1.

(b) The average sizes of
\[
|\text{Cl}_2^+(O)| - \frac{1}{2^{r_2}}|\mathcal{I}_2(O)|
\]
over \( O \in \mathcal{R}_1 \) ordered by height and over \( O \in \mathcal{R}_2 \) ordered by Julia invariant are bounded above by 1.

If we assume that \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \) arise from very large sets of binary \( n \)-ic forms, then the average sizes in (a) and (b) are equal to 1, independent of the choice of very large set. Furthermore, conditional on the tail estimates in (31), the average sizes in (a) and (b) are indeed equal to 1 for all \( \mathcal{R}_1 \) or \( \mathcal{R}_2 \) arising from any acceptable set of binary \( n \)-ic forms.

We will prove Theorem 6.2 in the following sections.

6.1 Computing the product of local volumes

We first prove a statement about the “compatibility of measures”. Let \( dv \) and \( df \) denote Euclidean measures on \( V \) and \( U \), respectively, normalized so that \( V(\mathbb{Z}) \) and \( U(\mathbb{Z}) \) have covolume 1. Let \( \omega \) be an algebraic differential form that generates the rank 1 module of top degree left-invariant differential forms on \( \text{SL}_n \) over \( \mathbb{Z} \). We have the following theorem, whose proof is identical to that of [8, Props. 3.11 & 3.12].
Theorem 6.3. Let $T$ be $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{Z}_p$ for some prime $p$. Let $s : U(T) \rightarrow V(T)$ be a continuous section for $\pi$, i.e., a continuous function such that the invariant binary $n$-ic of $w_f := s(f)$ is $f$. Then there exists a rational nonzero constant $J$ such that for any measurable function $\phi$ on $V(T)$, we have

$$
\int_{v \in SL_n(T) : s(U(T))} \phi(v) \, dv = |J| \int_{U(T)} \int_{SL_n(T)} \phi(g \cdot w_f) \omega(g) \, df \int_{V(T)} \phi(v) \, dv = |J| \int_{U(T)} \sum_{v \in \frac{\Delta(f) \not \equiv 0}{\Delta(f) \equiv 0}} \frac{1}{\text{Stab}_{SL_n(T)}(v)} \int_{g \in SL_n(T)} \phi(g \cdot v \omega(g)) \, df
$$

where we regard $SL_n(T) \cdot s(R)$ as a multiset and $\frac{V(T)(f)}{SL_n(T)}$ denotes a set of representatives for the action of $SL_n(T)$ on elements in $V(T)$ having invariant $f$.

For $r_2 \in \{1, \ldots, (n - 1)/2\}$ and for $f \in V(\mathbb{Z}_p)$ we define local masses

$$
m_p(f) := \frac{|(R_f^r \cdot R_f^r)^N_{N=1}|}{|R_f^r|_{N=1}|} \quad \text{and} \quad m_\infty(r_2) := \frac{\left|\left(\left(\mathbb{R}^{r_2} / \mathbb{C}^{r_2}\right)^{2 r_2} / \left(\mathbb{R}^{n - 2 r_2} \times \mathbb{C}^{r_2}\right)^2\right)_{N=1}\right|}{|\left(\mathbb{R}^{n - 2 r_2} \times \mathbb{C}^{r_2}\right)_{N=1}|}.
$$

We denote the numerator and the denominator of the right hand side in the equation defining $m_\infty(r_2)$ by $\tau(r_2)$ and $\sigma(r_2)$, respectively. For a prime $p$, let $\Sigma_p \subset U(\mathbb{Z}_p) \setminus pU(\mathbb{Z}_p)$ be a non-empty open set whose boundary has measure 0. Let $\Lambda_p$ denote the set of projective elements in $V(\mathbb{Z}_p)$ whose invariant binary form belongs to $\Sigma_p$. We have the following corollary to Theorem 6.3:

Corollary 6.4. Let notation be as above. We have

$$
\text{Vol}(\mathcal{F}_H \cdot \mathcal{R}_H^{(r_2),\delta}(X)) = |J| \text{Vol}(\mathcal{F}_H) \text{Vol}(U(\mathbb{R})^{(r_2)}_{H < X}),
$$

$$
\text{Vol}(\mathcal{F}_f \cdot \mathcal{R}_f^{(r_2),\delta}(X)) = \frac{\sigma'(r_2)}{\sigma(r_2)} |J| \text{Vol}(\mathcal{F}_H) \text{Vol}(SL_2(\mathbb{Z}) \setminus U(\mathbb{R})^{(r_2)}_{f < X}), \quad \text{and} \quad \text{Vol}(\Lambda_p) = |J| \text{Vol}(SL_n(\mathbb{Z}_p)) \int_{f \in \Sigma_p} m_p(f) \, df,
$$

where the volumes of $\mathcal{F}_H$ and $SL_n(\mathbb{Z}_p)$ are computed with respect to $\omega$, and $\sigma'(r_2)$ denotes the size of the stabilizer in $G(\mathbb{R})$ of a generic element of $V(\mathbb{R})^{(r_2)}$.

Proof. The first equality follows immediately from Theorem 6.3. Next, note that we have $\mathcal{F}_f = \mathcal{F}_2 \times \mathcal{F}_H$, where $\mathcal{F}_2$ is a fundamental domain for the action of $SL_2(\mathbb{Z})$ on $SL_2(\mathbb{R})$. Let the multiset $I \subset U(\mathbb{R})$ denote the invariants of the multiset $\mathcal{F}_2 \cdot \mathcal{R}_f^{(r_2),\delta}(X)$. Then $I$ generically represents each element of $SL_2(\mathbb{Z}) \setminus U(\mathbb{R})^{(r_2)}_{f < X}$ exactly $\sigma'(r_2)/\sigma(r_2)$ times, since $\sigma(r_2)$ is the size of the stabilizer in $SL_2(\mathbb{R})$ of an element in $U(\mathbb{R})^{(r_2)}$. (We have already seen that $s(r_2) = 3$ when $n = 3$ and $r_2 = 0$ and $s(r_2) = 1$ otherwise.) The second equality now follows immediately from Theorem 6.3.

To obtain the final equality, note that Theorem 6.3 implies

$$
\int_{\Lambda_p} dv = |J| \text{Vol}(SL_n(\mathbb{Z}_p)) \int_{f \in \Sigma_p} \frac{1}{\text{Stab}_{SL_n(\mathbb{Z}_p)}(v)} \sum_{v \in \text{det}^{-1}(f)} \text{Stab}_{SL_n(\mathbb{Z}_p)}(v) \, dv,
$$

where the sum runs over representatives in projective $SL_n(\mathbb{Z}_p)$-orbits of $\text{det}^{-1}(f)$. The result now follows from Corollary 2.14.
Denote \( n - 2r_2 \) by \( r_1 \) so that \( r_1 + 2r_2 = n \). By Corollaries 2.14 and 2.15 and Example 2.16, we have
\[
\tau(r_2) = 2^{r_1-1}, \quad \sigma(r_2) = 2^{r_1+r_2-1}, \quad \text{and} \quad m_\infty(r_2) = 2^{-r_2}. \tag{32}
\]
In [10, Lemma 22], the values of \( m_p(f) \) are computed for cubic rings. We now compute these values for degree \( n \) rings using a similar argument.

**Lemma 6.5.** Let \( R \) be a nondegenerate ring of degree \( n \) over \( \mathbb{Z}_p \). Then
\[
\frac{|(R^\times/(R^\times)^2)_{N=1}|}{|R^\times[2]_{N=1}|} \tag{33}
\]
is 1 if \( p \neq 2 \) and \( 2^{n-1} \) if \( p = 2 \).

**Proof.** The unit group of \( R^\times \) is the direct product of a finite abelian subgroup and \( \mathbb{Z}_p^n \), and the norm 1 part \( R^\times_{N=1} \) is also a direct product of a finite abelian group and \( \mathbb{Z}_p^{n-1} \). For \( G \) a finite abelian group or \( G = \mathbb{Z}_p^n \) when \( p \neq 2 \), we have
\[
\frac{|G/G^2|}{|G[2]|} = 1,
\]
so the value of (33) is 1 for \( p \neq 2 \). When \( p = 2 \), because 2 is not a unit in \( \mathbb{Z}_2 \), the \( \mathbb{Z}_2 \)-module \( 2\mathbb{Z}_2^{n-1} \) has index \( 2^{n-1} \) in \( \mathbb{Z}_2^{n-1} \) instead, implying that (33) evaluates to \( 2^{n-1} \).

It follows that for a fixed prime \( p \), the value of \( m_p(f) \) is independent of \( f \in U(\mathbb{Z}_p)^{\text{prim}} \). We denote this value by \( m_p \). We conclude with the following theorem:

**Theorem 6.6.** We have
\[
\frac{1}{\sigma(r_2)} \text{Vol}(\mathcal{F}_H \cdot \mathcal{R}_H^{(r_2),\delta}(X)) \prod_p \text{Vol}(\Lambda_p) = 2^{r_2} \text{Vol}(U(\mathbb{R})^{(r_2)}_{H < X}) \prod_p \text{Vol}(\Sigma_p) \quad \text{and}
\]
\[
\frac{1}{\sigma'(r_2)} \text{Vol}(\mathcal{F}_J \cdot \mathcal{R}_J^{(r_2),\delta}(X)) \prod_p \text{Vol}(\Lambda_p) = 2^{r_2} \text{Vol}(\text{SL}_2(\mathbb{Z})\setminus U(\mathbb{R})^{(r_2)}_{J < X}) \prod_p \text{Vol}(\Sigma_p) \tag{34}
\]

**Proof.** From Corollary 6.4 and Lemma 6.5, we obtain
\[
\frac{1}{\sigma(r_2)} \text{Vol}(\mathcal{F}_H \cdot \mathcal{R}_H^{(r_2),\delta}(X)) \prod_p \text{Vol}(\Lambda_p) = \frac{1}{\sigma(r_2)} |\mathcal{J}| \text{Vol}(\mathcal{F}_H) \text{Vol}(U(\mathbb{R})^{(r_2)}_{H < X}) \prod_p |\mathcal{J}| \text{Vol}(\text{SL}_n(\mathbb{Z}_p)) m_p \text{Vol}(\Sigma_p), \tag{34}
\]
and
\[
\frac{1}{\sigma'(r_2)} \text{Vol}(\mathcal{F}_J \cdot \mathcal{R}_J^{(r_2),\delta}(X)) \prod_p \text{Vol}(\Lambda_p) = \frac{1}{\sigma'(r_2)} |\mathcal{J}| \text{Vol}(\mathcal{F}_H) \text{Vol}(\text{SL}_2(\mathbb{Z})\setminus U(\mathbb{R})^{(r_2)}_{J < X}) \prod_p |\mathcal{J}| \text{Vol}(\text{SL}_n(\mathbb{Z}_p)) m_p \text{Vol}(\Sigma_p). \tag{35}
\]

We simplify the right hand side of these expressions by noting that
\[
|\mathcal{J}| \prod_p |\mathcal{J}| = 1, \tag{36}
\]

31
Vol(\mathcal{F}_H) \prod_p \text{Vol}(\text{SL}_n(\mathbb{Z}_p)) = 1, \quad (37)
$$
$$
$$
\frac{1}{\sigma(r_2)} \prod_p m_p = 2^{r_2}, \quad (38)
$$

where (36) follows from the product formula, (37) comes from the Tamagawa number of \text{SL}_n(\mathbb{Q}) being 1, and (38) follows from (32) and Lemma 6.5. Combining these with (34) and (35) yields the theorem.

\[ 6.2 \ \text{Proof of Theorem 6.2} \]

Let \( \mathfrak{R} \subseteq \mathfrak{R}_H \) be an acceptable family of rings having fixed signature \((r_1, r_2)\). Then the rings in \( \mathfrak{R} \) are in bijection with an acceptable set \( \mathcal{U}(\Sigma) \subset U(\mathbb{Z}) \) of binary \( n \)-ic forms with \( \Sigma_\infty = U(\mathbb{R})^{(r_2)} \). Let \( \Lambda^{(\delta)} \) be a collection of local specifications for \( V \), where \( \Lambda_p \) consists of projective elements in \( V(\mathbb{Z}_p) \) whose invariants belong to \( \Sigma_p \) and \( \Lambda_\infty = V(\mathbb{R})^{(r_2)} \). Then \( \Lambda = (\Lambda_\nu)_{\nu} \) is acceptable. Furthermore, if \( \mathfrak{R} \) is very large, then so is \( \Lambda \).

From Propositions 2.5 and 2.11 and Lemma 2.4, we know that

\[
\sum_{O \in \mathcal{R}} 2^{r_1+r_2-1}|\text{Cl}_2(O)| - |I_2(O)| = \sum_\delta N_H(V(\Lambda^{(\delta)}), X), \quad \text{and}
\]

\[
\sum_{O \in \mathcal{R}} 2^{r_2}|\text{Cl}_2^+(O)| - |I_2(O)| = N_H(V(\Lambda^{(\delta_{>0})}), X),
\]

where the first sum is over all possible \( \delta \) and \( \delta_{>0} \) denotes the element \((1, 1, \ldots, 1) \in \mathbb{R}^{r_1} \times \mathbb{C}^{r_2} \). As a result, we have

\[
\lim_{X \to \infty} \frac{\sum_{O \in \mathcal{R}} 2^{r_1+r_2-1}|\text{Cl}_2(O)| - |I_2(O)|}{\sum_{O \in \mathcal{R}} 1} \leq \lim_{X \to \infty} \frac{\sum_\delta N_H(V(\Lambda^{(\delta)}), X)}{\# \mathcal{U}(\Sigma)_{H < X}} = 2^{r_1+r_2-1},
\]

and

\[
\lim_{X \to \infty} \frac{\sum_{O \in \mathcal{R}} 2^{r_2}|\text{Cl}_2^+(O)| - |I_2(O)|}{\sum_{O \in \mathcal{R}} 1} \leq \lim_{X \to \infty} \frac{N_H(V(\Lambda^{(\delta_{>0})}), X)}{\# \mathcal{U}(\Sigma)_{H < X}} = 2^{r_2},
\]

where we use Theorems 5.3 and 3.3 to evaluate the numerators and the denominators of the middle terms in the above equation, and Theorem 6.6 to evaluate the product of local volumes that arise.

Similarly, let \( \mathfrak{R} \subset \mathfrak{R}_J \) be an acceptable family of rings having fixed signature \((r_1, r_2)\). Then the rings in \( \mathfrak{R} \) are in bijection with \( \text{SL}_2(\mathbb{Z}) \)-orbits on an acceptable set \( \mathcal{U}(\Sigma) \subset U(\mathbb{Z}) \) of binary \( n \)-ic forms with \( \Sigma_\infty = U(\mathbb{R})^{(r_2)} \). We define \( \Lambda^{(\delta)} \) as above, and obtain...
now follows Corollary 4 and Theorems 2 and 3 in \([\ldots]\) to evaluate the product of local volumes that arise.

If the families \(\mathfrak{R}\) are very large, then from Theorem 5.4, the inequalities in (39) and (40) can be replaced with equalities. Likewise, if we assume that one of the estimates in (31) hold, then from Theorem 5.4, the inequalities in (39) and (40) can be replaced with equalities. This concludes the proof of Theorem 6.2.

\section{Proof of Theorem 4}

Since Theorem 6.2 implies Theorems 2, 3, and 5, it remains to prove Theorem 4. We first prove a corollary of Theorem 2 and Theorem 3 on the proportion of maximal orders in \(\mathfrak{R}_{J, \text{max}}^{r_1, r_2}\) which have odd (narrow) class number.

\begin{corollary}
Fix an odd integer \(n \geq 3\) and signature \((r_1, r_2)\). If \(\mathfrak{R} \subset \mathfrak{R}_{J, \text{max}}^{r_1, r_2}\) corresponds to an acceptable set of degree \(n\) polynomials, then:
\begin{enumerate}[(a)]
  \item A positive proportion (at least \(1 - 2^{1-r_1-r_2}\)) of maximal orders in \(\mathfrak{R}\) have odd class number.
  \item If \(r_2\) is also assumed to be nonzero, then a positive proportion (at least \(1 - 2^{-r_2}\)) of \(\mathfrak{R}\) have odd narrow class number. Thus, at least a proportion of \(1 - 2^{-r_2}\) of \(\mathfrak{R}\) have narrow class number equal to the class number.
\end{enumerate}
\end{corollary}

\begin{proof}
Fix a signature \((r_1, r_2)\), and suppose a lower proportion than \(1 - 2^{1-r_1-r_2}\) of rings of integers of number fields with signature \((r_1, r_2)\) that correspond to integral binary \(n\)-ic forms have odd class number. This implies that a larger proportion than \(2^{1-r_1-r_2}\) of such maximal orders would have nontrivial 2-torsion subgroup in their class group and thus have \(|\text{Cl}_2| \geq 2\). Then the limsup of the mean number of 2-torsion elements in class groups of such maximal orders would be strictly larger than \(1 + \frac{1}{2^{n-1-r_2}}\), contradicting Theorem 2(a), Theorem 3(a), Theorem 3(b), or Corollary 3 in [10].

Now suppose a lower proportion than \(1 - 2^{-r_2}\) of maximal orders in number fields of signature \((r_1, r_2)\) in \(\mathfrak{R}\) have odd narrow class number. We would then be able to conclude that a larger proportion than \(2^{-r_2}\) of such maximal orders would have at least two distinct 2-torsion elements in its narrow class group. Then the limsup of the mean number of 2-torsion elements in the narrow class groups of such maximal orders would be strictly larger than \(1 + 2^{-r_2}\), contradicting Theorem 2(b). When \(n = 3\), note that the narrow class group of a complex cubic field is always equal to its class group.
\end{proof}

Theorem 4 now follows Corollary 6.7 in conjunction with Theorem 2 in [13], which implies that a ring is represented only finitely often in \(\mathfrak{R}_{J, \text{max}}^{r_1, r_2}\).
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