ON WINDOW THEOREM FOR CATEGORICAL DONALDSON-THOMAS
THEORIES ON LOCAL SURFACES AND ITS APPLICATIONS

YUKINOBU TODA

Abstract. In this paper, we prove a window theorem for categorical Donaldson-Thomas theories on local surfaces as an analogue of window theorem for GIT quotient stacks. We give two applications of our main result. The first one is a proof of wall-crossing equivalences of DT categories for one dimensional stable sheaves on local surfaces, under some technical condition on strictly semistable sheaves. The second one is to show the existence of fully-faithful functors from categorical PT theories to categorical MNOP theories, when the curve class is reduced. These results indicate categorifications of wall-crossing formulas of numerical DT invariants, and also regarded as d-critical analogue of D/K conjecture in birational geometry.

Contents

1. Introduction 1
2. Variation of GIT quotients and derived factorization categories 6
3. Categorical DT theory via singular supports 11
4. Window theorem for DT categories 18
5. Wall-crossing equivalence of DT categories for one dimensional stable sheaves 34
6. Categorical MNOP/PT correspondence 47
Appendices
Appendix A. Some auxiliary results in derived algebraic geometry 52
Appendix B. Formal neighborhood theorem for moduli stacks of semistable sheaves 55
Appendix C. Comparisons of DT categories 59
References 61

1. Introduction

1.1. Motivation and background. The Donaldson-Thomas (DT) invariants [Tho00] virtually count stable coherent sheaves on Calabi-Yau 3-folds, and play important roles in curve counting theories, representation theory, mathematical physics, etc. They are defined to be the integrations of the virtual fundamental classes on moduli spaces of stable sheaves on CY 3-folds, or alternatively integrations of Behrend’s constructible functions [Beh09] on these moduli spaces. Recently several refinements of DT invariants have been defined and studied in details, e.g. motivic DT invariants, cohomological DT invariants (see [KS, KSII, BJM, BBD+15]). The basic fact behind these refinements is the existence of (∓1)-shifted symplectic structures on derived moduli spaces of sheaves on Calabi-Yau 3-folds [PTVV13], which induce Joyce’s d-critical structures on classical moduli spaces [Joy15].

Let $X$ be a smooth projective Calabi-Yau 3-fold over $\mathbb{C}$. For a stability condition $\sigma$ (e.g. Gieseker stability, Bridgeland stability) on $D^b_{\text{coh}}(X)$, we denote by $M_{X,\sigma}(v)$ the moduli space of $\sigma$-stable objects with Chern character $v$. In general there is a wall-chamber structure on the space of stability conditions such that $M_{X,\sigma}(v)$ is constant if $\sigma$ lies on a chamber, but may change when $\sigma$ crosses a wall. The change of the associated DT invariants is described by the wall-crossing formula [JST12]...
which is one of the important developments in DT theory. For example, wall-crossing formula yields several properties of generating series of curve counting DT invariants \cite{Toda10a, Toda12}.

Suppose that $\sigma$ lies on a wall and $\sigma^\pm$ lie on its adjacent chambers. In \cite{Toda}, the author formulated the notion of d-critical flips (flops) which describes the change of moduli spaces $M_{X,\sigma^+(v)} \to M_{X,\sigma^-(v)}$ under wall-crossing. These are d-critical analogue of usual flips (flops) in birational geometry, which are not honest birational maps and should be interpreted as virtual birational maps. On the other hand, Bondal-Orlov [BO] and Kawamata [Kaw02] conjectured that for a flip (flop) of smooth varieties there exists a fully-faithful functor (equivalence) of bounded derived categories of coherent sheaves. Their conjecture is called $D/K$ equivalence conjecture. In \cite{Toda}, we proposed the following picture which relates wall-crossing of DT invariants and $D/K$ equivalence conjecture.

**Problem 1.1.** (i) There exists a triangulated category $\mathcal{DT}(M_{X,\sigma}(v))$, obtained as a gluing of locally defined categories of matrix factorizations, which recover the numerical/cohomological DT invariants from its categorical invariants. We call the category $\mathcal{DT}(M_{X,\sigma}(v))$ as DT category.

(ii) If $M_{X,\sigma^+(v)} \to M_{X,\sigma^-(v)}$ is a d-critical flip (flop), there exists a fully-faithful functor (equivalence) of DT categories

$\mathcal{DT}(M_{X,\sigma^-(v)}) \to (\wedge)\mathcal{DT}(M_{X,\sigma^+(v)})$. (1.1)

(iii) The semi-orthogonal complements of the fully-faithful functor \cite{Toda} recover the wall-crossing formula of numerical DT invariants.

A construction of the DT category for an arbitrary CY 3-fold $X$ is not yet available. In \cite{Todb}, we proposed a definition of $C^*$-equivariant DT categories for local surfaces, i.e. $X = \text{Tot}S(\omega_S)$ for a smooth projective surface $S$, where $C^*$ acts on fibers of $X \to S$. The purpose of this paper to prove a window theorem for variants of $C^*$-equivariant DT categories, and apply it for Problem 1.1 (ii), that is a d-critical analogue of $D/K$ equivalence conjecture.

### 1.2. Window theorem

Let $Y$ be a smooth affine variety with an action of a reductive algebraic group $G$. For a given $G$-equivariant line bundle $l$ on $Y$, we have the $G$-invariant open subset $Y^{ss}(l) \subset Y$ of $l$-semistable points. The window theorem proved in \cite{HL15, BF19} is the existence of a triangulated subcategory $\mathcal{W}^l(\{Y/G\}) \subset D_{\text{coh}}(\{Y/G\})$ such that the composition

$\mathcal{W}^l(\{Y/G\}) \to D_{\text{coh}}^{b}(\{Y/G\}) \to D_{\text{coh}}\left(\{Y^{ss}(l)/G\}\right)$

is an equivalence. The window theorem has been used to show several derived equivalences of birational varieties given as variations of GIT quotients, showing many evidence for original $D/K$ equivalence conjecture.

Let $\mathcal{M}$ be a quasi-smooth derived stack and $\Omega_{\mathcal{M}}[-1] \to \mathcal{M}$ its $(-1)$-shifted cotangent stack. Let us take a conical closed substack $Z \subset t_0(\Omega_{\mathcal{M}}[-1])$. In this paper, we consider the following model for $C^*$-equivariant DT categories

$$\lim_{\mathcal{U} \supset \mathcal{M}} \left( D_{\text{coh}}^{b}(\mathcal{U}/C_{\alpha^*}Z) \right).$$ (1.2)

Here $\alpha$ is a smooth morphism from a suitable affine derived scheme $\mathcal{U}$, $C_{\alpha^*}Z \subset D_{\text{coh}}^{b}(\mathcal{U})$ consists of objects whose singular supports \cite{AG15} are contained in $\alpha^*Z \subset t_0(\Omega_{\mathcal{U}}[-1])$, and the limit is taken for dg-enhancements of the quotient categories $D_{\text{coh}}^{b}(\mathcal{U}/C_{\alpha^*}Z$ in the $\infty$-category of dg-categories. In this introduction we will not discuss its precise definition, and refer to Subsection 5.3 for details. A key point of the above construction is that, by Koszul duality, the limit (1.2) is regarded as a gluing of some $C^*$-equivariant derived factorization categories. We also remark that the model for the DT categories (1.2) is slightly different from the one studied in \cite{Todb} (see Remark 5.10), and we discuss their relations in Appendix C.

The following is our main result, proving a window theorem for the category (1.2) when the classical truncation $\mathcal{M} = t_0(\mathcal{M})$ admits a good moduli space.
Theorem 1.2. (Theorem 3.11) For a quasi-smooth derived stack \( \mathcal{M} \), suppose that \( \mathcal{M} = t_0(\mathcal{M}) \) admits a good moduli space, a symmetric structure \( \mathcal{S} \) of \( \mathcal{M} \) is given, and \( \mathcal{M} \) satisfies formal neighborhood theorem. Let us take \( l, \delta \in \text{Pic}(\mathcal{M})_{\mathbb{R}} \) which are \( \mathcal{S} \)-generic, and \( \delta \) is \( l \)-generic. Then there exists a triangulated subcategory \( \mathcal{W}_{\delta}^{\text{int}/\mathcal{S}}(\mathcal{M}) \subset D^b_{\text{coh}}(\mathcal{M}) \) such that, for the \( l \)-unstable locus \( Z_{l,\text{us}} \subset t_0(\Omega_{\mathcal{M}}[-1]) \), the composition

\[
\mathcal{W}_{\delta}^{\text{int}/\mathcal{S}}(\mathcal{M}) \hookrightarrow D^b_{\text{coh}}(\mathcal{M}) \to \lim_{\Delta_{\mathcal{M}} \to \mathcal{M}} (D^b_{\text{coh}}(\mathcal{M})/C_{\alpha^*Z_{l,\text{us}}})
\]

is fully-faithful, which is an equivalence if \( l \) is compatible with \( \mathcal{S} \).

Here several notions ‘symmetric structure’, ‘formal neighborhood theorem’, ‘\( \mathcal{S} \)-generic’, ‘\( l \)-generic’, ‘compatible with \( \mathcal{S} \)’ appear in the statement of the above theorem. We do not explain these technical notions here and refer to Subsection 3.3 for details. As a corollary of the above theorem, we have the following:

Corollary 1.3. (Corollary 3.12) Under the assumption of Theorem 3.11, suppose that \( l_1, l_2 \in \text{Pic}(\mathcal{M})_{\mathbb{R}} \) are \( \mathcal{S} \)-generic and \( l_1 \) is compatible with \( \mathcal{S} \). Then there exists a fully-faithful functor

\[
\lim_{\Delta_{\mathcal{M}} \to \mathcal{M}} (D^b_{\text{coh}}(\mathcal{M})/C_{\alpha^*Z_{l_1,\text{us}}}) \hookrightarrow \lim_{\Delta_{\mathcal{M}} \to \mathcal{M}} (D^b_{\text{coh}}(\mathcal{M})/C_{\alpha^*Z_{l_2,\text{us}}})
\]

which is an equivalence if \( l_2 \) is also compatible with \( \mathcal{S} \).

The proofs of Theorem 1.2 and Corollary 1.3 are much inspired by ideas and techniques of Halpern-Leistner [HL], where he proves that Bridgeland moduli spaces of stable objects on K3 surfaces are derived equivalent under wall-crossing. Indeed Corollary 1.3 is applied for (some rigidified version of) derived moduli stacks of semistable objects on K3 surfaces should recover the above Halpern-Leistner’s result (see Remark 5.18), though we will not discuss this case in this paper.

On the other hand, the proof of Theorem 1.4 works other than moduli stacks of semistable objects on K3 surfaces. The latter moduli stacks are 0-shifted symplectic, and this fact is essential in the proof of derived equivalence in [HL2]. In our situation the derived stacks are defined by applying the construction (1.2) to derived moduli stacks of sheaves on surfaces. A symmetric structure \( \mathcal{S} \) is a choice of direct sum decompositions of \( \text{Aut}(x) \)-representations

\[
\mathcal{H}^0(T_{\mathcal{M}}|_x) \oplus \mathcal{H}^1(T_{\mathcal{M}}|_x)^\vee = S_x \oplus U_x
\]

at each closed point \( x \in \mathcal{M} \) such that \( S_x \) is a symmetric \( \text{Aut}(x) \)-representation. An element \( l \in \text{Pic}(\mathcal{M})_{\mathbb{R}} \) is compatible with \( \mathcal{S} \) if the \( l \)-semistable locus in the LHS of (1.4) is the pull-back of the \( l \)-semistable locus in \( S_x \). We will use the symmetric structure \( \mathcal{S} \) together with an auxiliary data \( \delta \in \text{Pic}(\mathcal{M})_{\mathbb{R}} \) to define the intrinsic window subcategory (see Definition 4.23)

\[
\mathcal{W}_{\delta}^{\text{int}/\mathcal{S}}(\mathcal{M}) \subset D^b_{\text{coh}}(\mathcal{M})
\]

which gives a desired subcategory in Theorem 1.2.

Locally on the good moduli space, the intrinsic window subcategory is constructed so that it coincides with the magic window subcategory on the derived factorization via Koszul duality. Here the magic window subcategory is defined by Halpern-Leistner and Sam [HLS] which give stability independent descriptions of window subcategories in [HL15, BFK19], and it is based on combinatorial arguments by Spenko and Van den Bergh [SvdB17]. So the subcategory (1.5) is interpreted as a gluing of magic window subcategories in the Koszul dual side, rather than those on \( \mathcal{M} \) itself.

Let \( \mathcal{S} \) be a smooth projective surface over \( \mathbb{C} \). We consider the total space of its canonical line bundle (called local surface)

\[
\pi : X := \text{Tot}_{\mathcal{S}}(\omega_{\mathcal{S}}) \to \mathcal{S}
\]

which is a non-compact Calabi-Yau 3-fold. The \( \mathbb{C} \)-equivariant DT categories for local surfaces are defined by applying the construction (1.2) to derived moduli stacks of sheaves on surfaces. Applying Corollary 1.3, we will study Problem 1.1 (ii) in the following two cases: moduli spaces of one dimensional stable sheaves and MNOP/PT moduli spaces.
1.3. Wall-crossing equivalence of DT categories for one dimensional stable sheaves. We first discuss wall-crossing of moduli spaces of one dimensional stable sheaves on local surfaces. For a smooth projective surface $S$, let $A(S)_C$ be the complexified ample cone
\[
A(S)_C := \{ B + \sqrt{-1}H \in \text{NS}(S)_C : H \text{ is ample} \}.
\]

Then each element $\sigma \in A(S)_C$ determines a Bridgeland stability condition on the abelian category of compactly supported one or zero dimensional coherent sheaves on $X$ (see Subsection 5.2). For a choice of primitive $v = (\beta, n)$ with $\beta \in \text{NS}(S)$, $n \in \mathbb{Z}$, and also a choice of $\sigma \in A(S)_C$, we have the moduli stack $M_{X,\sigma}(v)$ of compactly supported $\sigma$-semistable sheaves $E$ on $X$ with $\text{ch}(\pi_*E) = v$. It fits into the commutative diagram
\[
\begin{array}{c}
\mathcal{O}_{S,\sigma}(v) & \xrightarrow{\pi_*} & \mathcal{O}_{S}(v) \\
\mathcal{O}_{S}(v) & \xrightarrow{\mathcal{O}_{S,\sigma}(v)} & \mathcal{O}_{S}(v) \\
\end{array}
\]

Here $\mathcal{M}_S(v)$ is the derived moduli stack of coherent sheaves $F$ on $S$ with $\text{ch}(F) = v$ and $\mathcal{M}_S(v) = t_0(\mathcal{M}_S(v))$ is the classical truncation. The top left horizontal arrow is an open immersion and the right horizontal arrows are closed immersions. We also have the closed substack $Z_{\sigma,\text{us}} = t_0(\mathcal{M}_S(v)) \setminus \mathcal{M}_{X,\sigma}(v)$ consisting of $\sigma$-unstable sheaves.

The $\mathbb{C}^*$-equivariant DT category of one dimensional stable sheaves on $X$ is defined by applying the construction [12] for $\mathcal{M} = \mathcal{M}^{\mathbb{C}^*-\text{rig}}(v)^{\text{fin}}$
\[
\mathcal{D}^C_{\mathbb{C}^*}(M_{X,\sigma}(v)) := \lim_{\mathcal{M}_{\mathbb{C}^*-\text{rig}}(v)^{\text{fin}}} \left( D^b_{\text{coh}}(\mathcal{U})/\mathcal{C}_{\mathbb{C}^*-\text{rig}} \right).
\]

Here $\mathbb{C}^*$-rig means the $\mathbb{C}^*$-rigidification and $\mathcal{M}_{\mathbb{C}^*-\text{rig}}(v)^{\text{fin}} \subset \mathcal{M}_{\mathbb{C}^*-\text{rig}}(v)$ is a finite type derived open substack which contains $\pi_*M_{X,\sigma}(v)$ (see Definition [5.9] for details). The following result gives a variant of [Toda] Conjecture 5.9) under some technical assumption [18], and is proved by applying Corollary [1.9].

**Theorem 1.4.** (Theorem 5.12) Let $\sigma \in A(S)_C$ lies on a wall and $\sigma^\pm \in A(S)_C$ lie on its adjacent chambers. Moreover assume that
\[
M_{X,\sigma,\text{sss}}(v) \subset \pi_{\sigma}^{-1}(M_{S,\sigma}(v)),
\]
where $\sigma$-sss indicates strictly $\sigma$-semistable locus. Then we have an equivalence
\[
\mathcal{D}^C_M(M_{X,\sigma}(v)) \sim \mathcal{D}^C_M(M_{X,\sigma^\pm}(v)).
\]

In [Toda], it is observed that the moduli spaces $M_{X,\sigma^+}(v) \rightarrow M_{X,\sigma^-}(v)$ are related by a d-critical flop. Therefore the result of Theorem [1.4] realizes d-critical analogue of D/K equivalence conjecture in Problem [1.1] (ii). On the other hand, the integrations of the Behrend functions on $M_{X,\sigma^\pm}(v)$ are genus zero Gopakumar-Vafa invariants, which are known to be invariant under wall-crossing (see [JS12] Theorem 6.16], [MTT] Section 3.3). Therefore the result of Theorem [1.4] also indicates a categorification of wall-crossing invariance of genus zero Gopakumar-Vafa invariants.

In addition to [18], suppose furthermore that the following conditions hold:
\[
M_{X,\sigma}(v) \subset \pi_{\sigma}^{-1}(M_{S,\sigma}(v)).
\]

In this case, the equivalence [13] implies a derived equivalence of derived moduli spaces of one dimensional stable sheaves on surfaces (see Corollary 5.13). The conditions [1.8], (1.10) are satisfied if $\beta$ is a reduced class, i.e. any effective divisor $C$ on $S$ with $|C| = \beta$ is reduced. So we have the following corollary of Theorem [1.4]:
Corollary 1.5. (Corollary [5.15]) For \( v = (\beta, n) \), suppose that \( \beta \) is a reduced class. Then for any generic \( \sigma^\pm \in A(S)_C \), there exists an equivalence
\[
D^b_{\text{coh}}(\mathfrak{M}^\sigma_{S,\sigma^\pm}(v)) \sim D^b_{\text{coh}}(\mathfrak{M}^\beta_{S,\sigma^\pm}(v)).
\]

In some cases, e.g. \( S \) is an Enriques surface or del-Pezzo surface, the conditions (1.3), (1.10) are automatically satisfied and derived stacks in (1.11) are equivalent to their classical truncations \( M_{S,\sigma^\pm}(v) \). Below for \( H \in A(X)_E \), we write \( M_{S,H}(v) := M_{S,\sqrt{-1}H}(v) \), which is nothing but the Gieseker moduli space of \( H \)-semistable sheaves. In the case of Enriques surface, we have the following corollary showing a particular case of the usual D/K equivalence conjecture.

Corollary 1.6. (Corollary [5.20]) Let \( S \) be a general Enriques surface and \(|C|\) be a linear system on it which contains an irreducible divisor \( C \subset S \) with arithmetic genus \( g \geq 2 \). We take \( v = ([C], n) \in \text{NS}(S) \oplus \mathbb{Z} \) such that \( n \neq 0 \) and \( ([C], 2n) \) is coprime. Then for generic \( H^\pm \in A(S)_E \), Sacca [Sac10] proved that \( M_{S,H}(v) \) are smooth \((\beta^2+1)\)-dimensional birational Calabi-Yau manifolds. There exists a derived equivalence of Sacca’s Calabi-Yau manifolds
\[
D^b_{\text{coh}}(M_{S,H}(v)) \sim D^b_{\text{coh}}(M_{S,H^+}(v)).
\]

In the case of del-Pezzo surface, we have the following corollary:

Corollary 1.7. (Corollary [5.21]) Let \( S \) be a del-Pezzo surface and take \( \sigma = -\sqrt{-1}K_S \). Then for any primitive \( v \in N_{\leq 1}(S) \) and generic perturbations \( \sigma^\pm \) of \( \sigma \), the moduli spaces \( M_{S,\sigma^\pm}(v) \) are smooth projective varieties and there exists a derived equivalence
\[
D^b_{\text{coh}}(M_{S,\sigma^\pm}(v)) \sim D^b_{\text{coh}}(M_{S,\sigma^-}(v)).
\]

When \( S \) is a K3 surface, then \( M_{S,\sigma^\pm}(v) \) are holomorphic symplectic manifolds [Muk87, BM14]. In this case, we also have a derived equivalence of \( M_{S,\sigma^\pm}(v) \) from a slight modification of Theorem 1.4 (see Remark 5.18). Since this is a special case of [HLM5], we will not discuss in detail.

If the condition (1.8) is satisfied but the conditions (1.10) are not satisfied, then Theorem 1.4 does not necessary imply a derived equivalence of \( \mathfrak{M}^\sigma_{S,\sigma^\pm}(v) \), and the relevant categorical equivalence may be only formulated using our DT categories (1.2). We will discuss one of such examples in Subsection 5.6.

1.4. Categorical MNOP/PT correspondences. We next discuss categorical MNOP/PT wall-crossing. The MNOP conjecture [MNOP06] (which is proved in many cases [PP17]) is a conjectural relationship between generating series of Gromov-Witten invariants on a 3-fold and those of DT invariants counting one or zero dimensional subschemes on it. In our situation of the local surface, the relevant moduli space in the DT side is
\[
I_n(X, \beta), \quad (\beta, n) \in \text{NS}(S) \oplus \mathbb{Z}
\]
which parametrizes ideal sheaves \( I_C \subset O_X \) for a compactly supported one or zero dimensional subschemes \( C \subset X \) satisfying \( \pi_*[C] = \beta \) and \( \chi(O_C) = n \).

The notion of stable pairs was introduced by Pandharipande-Thomas [PT09] in order to give a better formulation of MNOP conjecture. By definition a PT stable pair is a pair \((E, \xi)\), where \( E \) is a compactly supported pure one dimensional sheaf on \( X \), and \( \xi: O_X \to E \) is surjective in dimension one. The moduli space of stable pairs is denoted by
\[
P_n(X, \beta), \quad (\beta, n) \in \text{NS}(S) \oplus \mathbb{Z}
\]
and it parametrizes stable pairs \((E, \xi)\) such that \( \pi_*[E] = \beta \) and \( \chi(E) = n \). The relationship between the corresponding DT type invariants is conjectured in [PT09] and proved in [Bri11, Tod10a, ST11] for CY 3-folds using wall-crossing arguments.

In this paper, we construct variants of \( \mathbb{C}^* \)-equivariant MNOP/PT categories defined in [Todb, Section 6], applying the construction (1.2) (see Definition 6.2)
As an application of Corollary 1.3, we show the following:

**Theorem 1.8.** (Theorem 6.9) Suppose that \( \beta \) is a reduced class. Then there exists a fully-faithful functor

\[
\widehat{\mathcal{D}}^\mathbb{C}^* (P_n(X, \beta)) \hookrightarrow \widehat{\mathcal{D}}^\mathbb{C}^* (I_n(X, \beta)).
\]

In [Toda], it is observed that the two moduli spaces \( I_n(X, \beta) \to P_n(X, \beta) \) are related by a d-critical flip. Therefore the result of Theorem 1.8 implies a d-critical analogue of D/K equivalence conjecture in Problem 1.1 (ii).

1.5. **Notation and convention.** In this paper, all the schemes or derived stacks are defined over \( \mathbb{C} \). For a scheme or derived stack \( Y \) and a quasi-coherent sheaf \( \mathcal{F} \) on it, we denote by \( S(\mathcal{F}) \) its symmetric product \( \oplus_{i \geq 0} \text{Sym}^i_{\mathcal{O}_Y}(\mathcal{F}) \). For an object \( A \in \mathbb{D}^{\text{coh}}(BC^*) \), we denote by \( \text{wt}(A) \subset \mathbb{Z} \) the set of \( \mathbb{C}^* \)-weights of \( \mathcal{H}^\bullet(A) \). If \( A \in \text{Pic}(BC^*) \), then \( \text{wt}(A) \) consists of one element so we identify it as an element \( \text{wt}(A) \in \mathbb{Z} \). We also denote by \( A_{\text{wt} > 0} \) (resp. \( A_{\text{wt} < 0} \)) the direct summand of \( A \) corresponding to positive (resp. negative) \( \mathbb{C}^* \)-weights. For a triangulated category \( \mathcal{D} \) and a set of objects \( \mathcal{S} \subset \mathcal{D} \), we denote by \( \langle \mathcal{S} \rangle_{\text{ex}} \) the extension closure of \( \mathcal{S} \), i.e. the smallest extension closed subcategory which contains \( \mathcal{S} \).

1.6. **Acknowledgements.** The author would like to thank Hsueh-Yung Lin for informing the reference [Sac19]. The author is supported by World Premier International Research Center Initiative (WPI initiative), MEXT, Japan, and Grant-in Aid for Scientific Research grant (No. 19H01779) from MEXT, Japan.

2. **Variation of GIT quotients and derived factorization categories**

In this section, we recall Kempf-Ness (KN) stratifications associated with GIT quotients, the triangulated categories of derived factorizations, and the theory of window subcategories developed in [HL15, BFK19]. We then recall magic window theorem proved in [HLS], and give a variant of it which will be required later.

2.1. **Kempf-Ness stratification.** Let \( G \) be a reductive algebraic group, with maximal torus \( T \subset G \). We always denote by \( M \) the character lattice of \( T \) and \( N \) the cocharacter lattice of \( T \), i.e.

\[
M = \text{Hom}_\mathbb{Z}(T, \mathbb{C}^*), \quad N = \text{Hom}_\mathbb{Z}(\mathbb{C}^*, T).
\]

The subspace \( M^W_\mathbb{R} \subset M_\mathbb{R} \) is defined to be the Weyl-invariant subspace, which is identified with \( \text{Pic}(BG)_\mathbb{R} \). Note that \( M, N \) are finitely generated free abelian groups with a perfect pairing

\[
\langle -, - \rangle : M \times N \to \mathbb{Z}.
\]

Below we follow the convention of [HL15 Section 2.1] for Kempf-Ness stratification associated with GIT quotients. Let \( Y \) be a smooth affine variety with a \( G \)-action. For an element \( l \in \text{Pic}([Y/G])_\mathbb{R} \), we have the open subset of \( l \)-semistable points

\[
Y^{ss}(l) \subset Y.
\]

By the Hilbert-Mumford criterion, \( Y^{ss}(l) \) is characterized by the set of points \( y \in Y \) such that for any one parameter subgroup \( \lambda : \mathbb{C}^* \to G \) such that the limit \( z = \lim_{t \to 0} \lambda(t)(y) \) exists in \( Y \), we have \( \text{wt}(l|_z) \geq 0 \).

We will often take \( l = M^W_\mathbb{R} = \text{Pic}(BG)_\mathbb{R} \) and regard it as an element of \( \text{Pic}([Y/G])_\mathbb{R} \) by the pull-back of the natural morphism

\[
a : [Y/G] \to BG.
\]

In this case, the condition \( \text{wt}(l|_z) \geq 0 \) is equivalent to \( \langle l, \lambda \rangle \geq 0 \). In some cases we may assume that \( l \) is pulled back from \( \text{Pic}(BG)_\mathbb{R} \) by the following lemma:
Lemma 2.1. Let \( x \in Y \) be fixed by \( G \) and set \( y = p(x) \) for the quotient morphism \( p: Y \to Y/G \). Let \( \mu: BG \to [Y/G] \) be the map sending a point to \( x \) and identity on stabilizer groups. Then for any \( l \in \text{Pic}(Y/G) \) there exists a Zariski open subset \( x \in U \subset Y/G \) such that \( l|_{p^{-1}(U)/G} \) is isomorphic to \( a^*\mu^*(l)|_{p^{-1}(U)/G} \).

Proof. Let us set \( \mathcal{L} := l \otimes a^*\mu^*(l)^{-1} \in \text{Pic}(Y/G) \). Then \( \mu^*\mathcal{L} \) is a trivial line bundle on \( BG \), or in other word \( H^0(\mathcal{L}|_x)^G = \mathbb{C} \). Since \( Y \) is affine and \( G \) is reductive, the functor \( H^0(-)^G \) on \( \text{Coh}(Y/G) \) is an exact functor. Therefore by applying the above functor to the surjection \( p_0 \in U \subset Y/G \), then \( (2.4) \) is equivalent to \( \text{Orlov's triangulated category} \) \[\text{Orl09}\] of \( (\text{the homotopy category of the factorizations} \ (2.5) \text{by its subcategory of acyclic factorizations). If} \ G \text{is reductive, the functor} \ H^0(\cdot)^G \text{on} \ Coh(\text{Y/G}) \text{is an exact functor. Therefore by applying the above functor to the surjection} \ \mathcal{L} \to \mathcal{L}|_x, \text{we obtain a surjection} \ H^0(\mathcal{L})^G \to H^0(\mathcal{L}|_x)^G. \text{In particular there is} \ s \in H^0(\mathcal{L})^G \text{which is non-zero on} \ x. \text{It gives a} \ G \text{-equivariant map} \ \mathcal{O}_Y \to \mathcal{L}, \text{and let} \ Z \subset Y \text{be the zero locus of} \ s. \text{Then} \ Z \text{is a} \ G \text{-invariant closed subset of} \ Y \text{which does not contain} \ x. \text{However the map} \ [Y/G] \to Y/G \text{is a good moduli space for} \ [Y/G], \text{so in particular it is universally closed (see} \ [\text{Alp13, Theorem 4.16}]. \text{Therefore there exists a Zariski open subset} \ y \in U \subset Y/G \text{such that} \ Z \cap p^{-1}(U) = \emptyset, \text{which implies that} \ s \text{is an isomorphism on} \ p^{-1}(U). \square

By fixing a Weyl-invariant norm \( \|\cdot\| \) on \( N_R \), we have the associated Kempf-Ness (KN) stratification

\[
Y = Y^\text{ss}(l) \sqcup S_1 \sqcup S_2 \sqcup \ldots.
\]

Here for each \( \alpha \) there exists an one parameter subgroup \( \lambda_\alpha: \mathbb{C}^* \to T \subset G \), a connected component \( Z_\alpha \) of the \( \lambda_\alpha \)-fixed part of \( Y \setminus \cup_{\alpha' \prec \alpha} S_{\alpha'} \) such that

\[
S_\alpha = G \cdot Y_\alpha, \quad Y_\alpha := \{ y \in Y : \lim_{t \to 0} \lambda_\alpha(t)(y) \in Z_\alpha \}.
\]

Moreover by setting

\[
\mu_\alpha = -\frac{\text{wt}(l|_{Z_\alpha})}{|\lambda_\alpha|} \in \mathbb{R}
\]

we have the inequalities \( \mu_1 > \mu_2 > \ldots > 0 \). By taking the quotient stacks of the stratification (2.2), we have the stratification of the quotient stack \( \mathcal{Y} = [Y/G] \)

\[
\mathcal{Y} = \mathcal{Y}^\text{ss}(l) \sqcup S_1 \sqcup S_2 \sqcup \ldots.
\]

Using Hilbert-Mumford criterion, the notion of semistability can be generalized to an arbitrary Artin stack \( \mathcal{Y} \) and \( l \in \text{Pic}(\mathcal{Y})_R \) (see [HLc, Definition 1.13]). Namely a point \( p \in \mathcal{Y} \) is \( l \)-semistable if for any map \( f: [\mathbb{A}^1/C^*] \to \mathcal{Y} \) with \( f(1) \sim p \), we have \( \text{wt}(f(0)^*(l)) \geq 0 \). Here \( C^* \) acts on \( \mathbb{A}^1 \) by weight one. The set of \( l \)-semistable points is denoted by

\[
\mathcal{Y}^\text{ss}(l) \subset \mathcal{Y}.
\]

When \( \mathcal{Y} = [Y/G] \), then \( \mathcal{Y}^\text{ss}(l) = [Y^\text{ss}(l)/G] \) where \( Y^\text{ss}(l) \) is the GIT semistable locus (2.1).

2.2. The triangulated categories of derived factorizations. Let \( Y \) be a scheme which admits an action of a reductive algebraic group \( G \), and also a \( C^* \)-action which commutes with the above \( G \)-action. Let \( \tau: G \times C^* \to C^* \) be a character given by the second projection, and let \( w \in \Gamma(Y, \mathcal{O}_Y) \) be \( \tau \)-semi invariant of weight two, i.e. \( g^*w = \tau(g)^2w \) for any \( g \in G \times C^* \). Given data as above, the category

\[
\text{MF}_{\text{coh}}^G([Y/G], w)
\]

is defined to be the triangulated category whose objects consist of \((G \times C^*)\)-equivariant factorizations of \( w \), i.e. a pair

\[
(\mathcal{P}, dp), \quad dp: \mathcal{P} \to \mathcal{P}(1), \quad dp \circ dp = w
\]

where \( \mathcal{P} \) is a \((G \times C^*)\)-equivariant coherent sheaf on \( Y \), \( dp \) is a \((G \times C^*)\)-equivariant morphism. Here \( \langle n \rangle \) means the twist by the \((G \times C^*)\)-character \( \tau^n \). The category (2.4) is defined to be the localization of the homotopy category of the factorizations (2.5) by its subcategory of acyclic factorizations. If \( Y \) is affine, then (2.4) is equivalent to Orlov’s triangulated category \[\text{Orl09}\] of \((G \times C^*)\)-equivariant matrix factorizations of \( w \). For details on the definition of the category (2.4), we refer to [BFK14, Section 3], [EP15, Section 2.2].
For a \((G \times \mathbb{C}^*)\)-invariant closed subscheme \(Z \subset Y\), the subcategory
\[
\text{MF}^{\mathbb{C}^*}_{\text{coh}}([Y/G], w)|_Z \subset \text{MF}^{\mathbb{C}^*}_{\text{coh}}([Y/G], w)
\]
is defined to be the kernel of the restriction functor
\[
\text{MF}^{\mathbb{C}^*}_{\text{coh}}([Y/G], w) \to \text{MF}^{\mathbb{C}^*}_{\text{coh}}([Y \setminus Z]/G, w).
\]
By taking the Verdier quotients, we have an equivalence (cf. [Che10, Theorem 1.3])
\[
(\text{MF}^{\mathbb{C}^*}_{\text{coh}}([Y/G], w)|_Z)/\lambda \cong \text{MF}^{\mathbb{C}^*}_{\text{coh}}([Y \setminus Z]/G, w).
\]
Suppose furthermore that \(Y\) is a smooth affine variety. Given \(l \in \text{Pic}([Y/G])_{\mathbb{R}}\), we have a KN-stratification (2.2). For each \(\alpha\), let \(i_{\alpha}\) be the morphism of stacks
\[
i_{\alpha} : [Z_{\alpha}/\lambda_{\alpha}] \to [Y/G].
\]
Here \([Z_{\alpha}/\lambda_{\alpha}] := [Z_{\alpha}/\mathbb{C}^*]\) for the trivial \(\mathbb{C}^*\)-action on \(Z_{\alpha}\), and \(i_{\alpha}\) is induced by the embedding \(Z_{\alpha} \hookrightarrow Y\) together with the one parameter subgroup \(\lambda_{\alpha} : \mathbb{C}^* \to G\). Note that we have the decomposition of the derived factorization category on \([Z_{\alpha}/\lambda_{\alpha}]\) into the \(\lambda_{\alpha}\)-weight parts
\[
\text{MF}^{\mathbb{C}^*}_{\text{coh}}([Z_{\alpha}/\lambda_{\alpha}], w)|_{z_{\alpha}} = \bigoplus_{k \in \mathbb{Z}} \text{MF}^{\mathbb{C}^*}_{\text{coh}}([Z_{\alpha}/\lambda_{\alpha}], w|_{z_{\alpha}})_k.
\]
Let \(\eta_{\alpha} \in \mathbb{Z}\) be defined by
\[
(\eta_{\alpha} = \text{wt}_{\lambda_{\alpha}}(\det(N^\vee_{S_{\alpha}/Y}|_{z_{\alpha}}))).
\]
We will use the following version of window subcategory.

**Definition 2.2.** For \(l, \delta \in \text{Pic}([Y/G])_{\mathbb{R}}\), the triangulated subcategory
\[
(\mathbb{W}_l^\delta([Y/G], w) \subset \text{MF}^{\mathbb{C}^*}_{\text{coh}}([Y/G], w)
\]
is defined to be consisting of factorizations \((\mathcal{P}, d_P)\) as in (2.7) such that each derived restriction \((\mathcal{P}, d_P)|_{Z_{\alpha}}\) satisfies that
\[
(\mathcal{P}, d_P)|_{Z_{\alpha}} \in \bigoplus_{k \in \text{wt}_{\lambda_{\alpha}}(\delta|_{z_{\alpha}}) + \{-\eta_{\alpha}/2, \eta_{\alpha}/2\}} \text{MF}^{\mathbb{C}^*}_{\text{coh}}([Z_{\alpha}/\lambda_{\alpha}], w|_{z_{\alpha}})_k.
\]

The following is a version of window theorem for derived factorization categories of GIT quotients.

**Theorem 2.3.** ([HL15] [BFK19]) The composition
\[
\mathbb{W}_l^\delta([Y/G], w) \subset \text{MF}^{\mathbb{C}^*}_{\text{coh}}([Y/G], w) \overset{\text{res}}{\longrightarrow} \text{MF}^{\mathbb{C}^*}_{\text{coh}}([Y/\mathbb{S}]^{\text{ss}}(l)/G, w)
\]
is an equivalence of triangulated categories. Here the right arrow is the restriction functor to the open subset \(Y^{\text{ss}}(l) \subset Y\).

### 2.3. Magic window subcategories.
Suppose that \(Y\) is an affine space, i.e. \(Y = \mathbb{A}^n\) for some \(n\), and it is a \(G\)-representation. Let us take a decomposition into a direct sum of \(G\)-representations
\[
Y = \mathbb{S} \oplus \mathbb{U}
\]
such that \(\mathbb{S}\) is a symmetric, i.e. \(\mathbb{S} \cong \mathbb{S}^\vee\) as \(G\)-representations. We call such a decomposition \(Y = \mathbb{S} \oplus \mathbb{U}\) as a symmetric structure of \(Y\), and refer to it as \(\mathbb{S}\).

For each one parameter subgroup \(\lambda : \mathbb{C}^* \to T\) and an element \(L \in K_0(\text{Rep}(T)) = \mathbb{Z}[M]\), we define \(L^\lambda_{>0}\) to be the projection of this class onto the subspace spanned by weights which pair positively with \(\lambda\). We define \(\nabla_{\mathbb{S}} \subset M_{\mathbb{R}}\) to be
\[
(\nabla_{\mathbb{S}} := \{\chi \in M_{\mathbb{R}} : \langle \chi, \lambda \rangle \in \left[ -\frac{1}{2}, \frac{1}{2} \right] \langle L^\lambda_{>0}, [\mathbb{S}/G]|_0, \lambda \rangle \}. \text{for all } \lambda : \mathbb{C}^* \to T\}
\]
Here \(L^\lambda_{>0}|_0\) is the cotangent complex of the quotient stack \([\mathbb{S}/G]\) restricted to the origin, whose \(K\)-theory class is
\[
L^\lambda_{>0}|_0 = [\mathbb{S}^\vee] - [\mathbb{g}^\vee] \in K_0(\text{Rep}(T)).
\]
Here $g$ is the Lie algebra of $G$, and regarded as a $T$-representation by the adjoint representation. If $\lambda = \lambda_\alpha$ for the KN stratification \cite[(2.2)]{HLS}, then we have the identity (see \cite[Equation (4)]{HLS})

\begin{equation}
\eta_\alpha = \left\langle L^{\lambda_\alpha\circ 0}_{[\Sigma /|G|]}, \lambda \right\rangle.
\end{equation}

Here $\eta_\alpha$ is defined in \cite[(2.7)]{HLS}. We introduce some conditions for elements in $M^W_R$. Let $\Sigma_S \subset M_R$ be the convex hull of the $T$-characters of $\Lambda^\ast(S)$.

**Definition 2.4.** For $l, \delta \in M^W_R$, we say

(i) $l$ is $S$-generic if it is contained in the linear span of $\Sigma_S$ but is not parallel to any face of $\Sigma_S$. We denote by $(M^W_R)^{\text{gen}/S} \subset M^W_R$ the subset of $S$-generic elements.

(ii) $\delta$ is $l$-generic if $\langle \delta, \lambda \rangle \notin \mathbb{Q}$ for any one parameter subgroup $\lambda: \mathbb{C}^\ast \to T$ such that $\langle l, \lambda \rangle \neq 0$.

(iii) $l$ is compatible with the symmetric structure $S$ in \cite[(2.9)]{HLS} if we have

\[ Y^\text{sw}(l) = \mathcal{S}^\circ(l) \oplus U. \]

The genericity condition for (i) is introduced in \cite[Section 2]{HLS}. As for (ii), for example if $l \in M^W_R$ then $\delta = \varepsilon \cdot l$ is $l$-generic for any $\varepsilon \in \mathbb{R} \setminus \mathbb{Q}$. Note that if $\delta$ is $l$-generic, then $\langle \delta, \lambda_\alpha \rangle \notin \mathbb{Q}$ for any one parameter subgroup $\lambda_\alpha$ which appears in a KN stratification \cite[(2.2)]{HLS}. Later we will use the following lemma:

**Lemma 2.5.** For $l_1, l_2 \in (M^W_R)^{\text{gen}/S}$, there is an uncountable dense subset $U \subset \mathbb{R}^2$ such that for $(\alpha_1, \alpha_2) \in U$, $\alpha_1 l_1 + \alpha_2 l_2$ is $l_1$-generic and $l_2$-generic.

**Proof.** If both of $l_1, l_2$ are generic, then obviously the set of $(\alpha_1, \alpha_2) \in \mathbb{R}^2$ such that $\alpha_1 l_1 + \alpha_2 l_2$ is generic is a dense open subset in $\mathbb{R}^2$. For a one parameter subgroup $\lambda: \mathbb{C}^\ast \to T$ such that either $\langle l_1, \lambda \rangle \neq 0$ or $\langle l_2, \lambda \rangle \neq 0$, the set of $(\alpha_1, \alpha_2) \in \mathbb{R}^2$ such that $\alpha_1 l_1 + \alpha_2 l_2 \neq 0 \notin \mathbb{Q}$ is a complement of countable number of lines in $\mathbb{R}^2$, so it is uncountable and dense. As such one parameter subgroups $\lambda$ are also countable many, we obtain the lemma. \hfill \Box

For $\delta \in M^W_R$, the subcategory (called magic window over $\mathcal{S}$)

\[ \mathcal{W}^\text{mag}/S([Y/G], w) \subset \text{MF}^\ast_{\text{coh}}([Y/G], w) \]

is defined to be split generated by factorizations $(\mathcal{P}, d_\mathcal{P})$ as in \cite[(2.3)]{HLS}, such that $\mathcal{P}$ is isomorphic to $W \otimes_{\mathcal{O}Y} W$ for a $(G \times \mathbb{C}^\ast)$-representation $W$ whose $T$-weights are contained in $\delta + \nabla_S$. The following result is proved in the proof of \cite[Proposition 2.6]{KT}, which is itself based on magic window theorem in \cite[GVdB17]{HLS}.

**Proposition 2.6.** (\cite[Proposition 2.6]{KT}) Suppose that $l, \delta \in (M^W_R)^{\text{gen}/S}$ and $\delta$ is $l$-generic. Then we have the inclusion

\[ \mathcal{W}^\text{mag}/S([Y/G], w) \subset \mathcal{W}^l_d([Y/G], w) \]

which is identity if $l$ is compatible with the symmetric structure $\mathcal{S}$ in \cite[(2.4)]{HLS}. In particular the composition

\[ \mathcal{W}^\text{mag}/S([Y/G], w) \subset \text{MF}^\ast_{\text{coh}}([Y/G], w) \overset{\text{res}}{\longrightarrow} \text{MF}^\ast_{\text{coh}}([Y^\text{sw}(l)/G], w) \]

is fully-faithful, which is an equivalence if $l$ is compatible with $\mathcal{S}$.

**2.4. Window subcategories for formal completions.** Let $Y$ be a smooth affine scheme with an action of a reductive algebraic group $G$, and $V \to Y$ a $G$-equivariant vector bundle. In this paper, we often work with the formal fiber of the map

\[ [V/G] \to [Y/G] \overset{\pi}{\rightarrow} Y//G. \]

We will use the following notation. For $y \in Y//G$, we denote by $\hat{Y}_{y//G}$ be the formal completion of $Y//G$ at $y$,

\[ \hat{Y}_{y//G} := \text{Spec} \hat{\mathcal{O}}_{Y//G,y}. \]
Then by setting 
\[\hat{Y}_y := Y \times_{Y/G} \hat{Y}/G, \hat{V}_y := V \times_{Y/G} \hat{Y}/G,\]
we have the Cartesian squares
\[
\begin{array}{ccc}
[\hat{V}_y/G] & \to & [\hat{Y}/G] \to \hat{Y}/G \\
\downarrow & & \downarrow \\
[V/G] & \to & [Y/G] \to Y/G.
\end{array}
\tag{2.12}
\]

The KN stratification of \(V\) with respect to the above \(G\)-action on \(V\) and \(l \in \mathcal{M}_W^W\) is restricted to a stratification on \(\hat{V}_y\)
\[
\hat{V}_y = \hat{V}^{\text{ss}}(l) \cup \hat{S}_1 \cup \hat{S}_2 \cup \ldots.
\tag{2.13}
\]

Suppose furthermore that \(Y\) is a finite dimensional \(G\)-representation, and \(V \to Y\) a \(G\)-equivariant vector bundle. Then the total space of \(V\) is a direct sum of \(G\)-representations \(V|_0 \oplus Y\) and the projection \(V \to Y\) is identified with the second projection. Below we take a symmetric structure \(\mathcal{S}\) of \(V\),
\[
V|_0 \oplus Y = \mathcal{S} \oplus U
\tag{2.14}
\]
and prove a version of Theorem 2.6 for the formal fiber \([\hat{V}_y/G]\) at \(y = 0 \in Y/G\). Let \(C^*\) acts on the fibers of \(V \to Y\) and \(\hat{w}_0 : [\hat{V}_0/G] \to A^1\) a morphism which is \(\tau\)-semi invariant of weight two. Then for each \(l, \delta \in \mathcal{M}_W^W\), the window subcategory
\[
\mathcal{W}_\delta^\mathcal{S}([\hat{V}_0/G], \hat{w}_0) \subset MF^C_{\text{coh}}([\hat{V}_0/G], \hat{w}_0)
\tag{2.15}
\]
is defined in the same way as Definition 2.8 using the induced stratification (2.13). The magic window subcategory
\[
\mathcal{W}_\delta^\mathcal{mag}/\mathcal{S}([\hat{V}_0/G], \hat{w}_0) \subset MF^C_{\text{coh}}([\hat{V}_0/G], \hat{w}_0)
\]
is also defined to be split generated by \((P, dP)\) where \(P\) is isomorphic to \(W \otimes_C \mathcal{O}_{\hat{V}_0}\) for a \((G \times C^*)\)-representation \(W\) whose \(T\)-weights are contained in \(\delta + \nabla_\mathcal{S}\). We will use the following variant of Theorem 2.6

**Proposition 2.7.** Suppose that \(l, \delta \in (\mathcal{M}_W^W)^{\text{gen}/\mathcal{S}}\) and \(\delta\) is \(l\)-generic. Then we have the inclusion
\[
\mathcal{W}_\delta^\mathcal{mag}/\mathcal{S}([\hat{V}_0/G], \hat{w}_0) \subset \mathcal{W}_\delta([\hat{V}_0/G], \hat{w}_0)
\tag{2.16}
\]
which is identity if \(l\) is compatible with the symmetric structure \(\mathcal{S}\) in (2.14). In particular the composition
\[
\mathcal{W}_\delta^\mathcal{mag}/\mathcal{S}([\hat{V}_0/G], \hat{w}_0) \subset MF^C_{\text{coh}}([\hat{V}_0/G], \hat{w}_0) \overset{\text{res}}{\to} MF^C_{\text{coh}}([\hat{V}_0^{\text{ss}}(l)/G], \hat{w}_0)
\]
is fully-faithful, which is an equivalence if \(l\) is compatible with \(\mathcal{S}\).

**Proof.** The proof is almost same as in [KT], Proposition 2.6. Let \(\lambda_\alpha : C^* \to T\) be a one parameter subgroup which appears in a KN stratification of \(V\) with respect to the \(G\)-action on it. By the assumption that \(\delta\) is \(l\)-generic, we have \(\langle \delta, \lambda_\alpha \rangle \pm \eta_\alpha/2 \notin \mathbb{Z}\). Together with the inequality
\[
\langle \frac{\eta_\alpha > 0}{\lambda_\alpha > 0} \rangle |_0, \lambda_\alpha \rangle \leq \langle \frac{\eta_\alpha > 0}{\lambda_\alpha > 0} \rangle |_0, \lambda_\alpha \rangle = \eta_\alpha
\tag{2.17}
\]
we have the inclusion
\[
\mathcal{W}_\delta^\mathcal{mag}([\hat{V}_0/G], \hat{w}_0) \subset \mathcal{W}_\delta([\hat{V}_0/G], \hat{w}_0)
\tag{2.18}
\]
by the definition of these window subcategories. The composition
\[
\mathcal{W}_\delta([\hat{V}_0/G], \hat{w}_0) \subset MF^C_{\text{coh}}([\hat{V}_0/G], \hat{w}_0) \overset{\text{res}}{\to} MF^C_{\text{coh}}([\hat{V}_0^{\text{ss}}(l)/G], \hat{w}_0)
\]
is an equivalence by Theorem 2.3. Therefore the composition (2.18) is fully-faithful.
Now suppose that $l$ is compatible with the symmetric structure $S$ in (2.14). Let $\Delta_S$ be the set of isomorphism classes of finite dimensional $(G \times \mathbb{C}^*)$-representations $W$ whose $(T \times \{1\})$-weights are contained in $\delta + \nabla_S$. Using the assumption of the proposition, it is proved in [HLS, Proposition 3.11] that any vector bundle $W' \otimes O_S$ on $S$ for a finite dimensional $(G \times \mathbb{C}^*)$-representation $W'$ admits a $(G \times \mathbb{C}^*)$-equivariant resolution by vector bundles $W \otimes O_{\Delta_S}$ for $W \in \Delta_S$, modulo sheaves on $S$ supported on $l$-unstable locus. By pulling it back to $V$ via the projection $V \to S$ and using the assumption that $l$ is compatible with $S$, any vector bundle $W' \otimes O_V$ for a finite dimensional $(G \times \mathbb{C}^*)$-representation $W'$ admits a $(G \times \mathbb{C}^*)$-equivariant resolution by vector bundles $W \otimes O_V$ for $W \in \Delta_S$, modulo sheaves on $V$ supported on $l$-unstable locus. By restricting it to $\hat{V}_0$, the same also applies to $W' \otimes O_{\hat{V}_0}$. Therefore $D_{coh}^b(\hat{V}_0^{ss}(l)/(G \times \mathbb{C}^*))$ is split generated by vector bundles $W \otimes O_{\hat{V}_0^{ss}(l)}$ for $W \in \Delta_S$. It follows that for any object $(P, d_P)$ in $MF_{coh}^C(\hat{V}_0^{ss}(l)/G, \hat{w}_0)$, the underlying $(G \times \mathbb{C}^*)$-equivariant sheaf $P$ is a direct summand of a bounded complex $W = W^* \otimes O_{\hat{V}_0^{ss}(l)}$ for $W^* \in \Delta_S$ in the derived category $D_{coh}^b(\hat{V}_0^{ss}(l)/(G \times \mathbb{C}^*))$).

Note that we have

$$\Ext^>_{\hat{V}_0^{ss}(l)/(G \times \mathbb{C}^*)}(W_1 \otimes O_{\hat{V}_0^{ss}(l)}, W_2 \otimes O_{\hat{V}_0^{ss}(l)}) \cong \Ext^>_{\hat{V}_0^{ss}(l)/(G \times \mathbb{C}^*)}(W_1 \otimes O_{\hat{V}_0}, W_2 \otimes O_{\hat{V}_0}) = 0$$

for $W_1, W_2 \in \Delta_S$. Here the first isomorphism follows since the composition (2.18) without superpotential case is also fully-faithful, and the second identity follows since $\hat{V}_0$ is affine and $G$ is reductive. Therefore using the resolution property of factorizations (see [BDF+16, Theorem 3.9], [NAS, Lemma 4.10]), the differential $d_P$ can be lifted to a differential $d_W$ on the totalization of $W$ so that $(W, d_W) \in MF_{coh}^C(\hat{V}_0^{ss}(l)/G, \hat{w}_0)$ and $(P, d_P)$ is a direct summand of $(W, d_W)$. Therefore the composition (2.16) is essentially surjective, so the inclusion (2.17) is identity.

3. Categorial DT theory via singular supports

In this section, we recall the notion of singular supports of coherent sheaves on quasi-smooth derived stacks introduced in [AGI15]. We then recall the model for $\mathbb{C}^*$-equivariant DT categories associated with quasi-smooth derived stacks introduced in [Ted19], and their variant. We also give several properties on derived stacks with good moduli spaces, and introduce notions which appear in the statement of Theorem 1.2.

3.1. Singular supports of coherent sheaves. Let $Y$ be a smooth affine scheme, $V \to Y$ a vector bundle on it and $s$ a section of it:

$$V \overset{s}{\longrightarrow} Y.$$

We then consider the affine derived scheme $\mathfrak{U}$ given by the derived zero locus of $s$

$$\mathfrak{U} = \Spec \mathcal{R}(V \to Y, s)$$

where $\mathcal{R}(V \to Y, s)$ is the Koszul complex

$$\mathcal{R}(V \to Y, s) := \left(\cdots \to \bigwedge^2 V^\vee \xrightarrow{\nabla} V^\vee \xrightarrow{\delta} \mathcal{O}_Y\right).$$

The classical truncation of $\mathfrak{U}$ is the closed subscheme of $Y$ given by

$$\mathfrak{U} := t_0(\mathfrak{U}) = (s = 0) \subset Y.$$

On the other hand, let $w : V^\vee \to \mathbb{A}^1$ be the function defined by

$$w(x, v) = \langle s(x), v \rangle, \quad x \in Y, \quad v \in V^\vee|_x.$$

Then its critical locus is the classical truncation of $(-1)$-shifted cotangent scheme over $\mathfrak{U}$ (or called dual obstruction cone, see [JT17])

$$\text{Crit}(w) = t_0(\Omega_{\mathfrak{U}}[-1]) = \Spec S(\mathcal{H}^1(T_{\mathfrak{U}|_\mathfrak{U}})).$$
Here $T_{\mathcal{U}|\mathcal{U}}$ is the tangent complex of $\mathcal{U}$ restricted to $\mathcal{U}$,

$$T_{\mathcal{U}|\mathcal{U}} = (T_Y|_{\mathcal{U}} \xrightarrow{d} V|_{\mathcal{U}}).$$

Note that there is a $\mathbb{C}^*$-action on $V^\vee$ which acts on fibers of $V^\vee \to Y$, and $\text{Crit}(w) \subset V^\vee$ is preserved by the above $\mathbb{C}^*$-action.

Let $\text{HH}^*(\mathcal{U})$ be the Hochschild cohomology of $\mathcal{U}$

$$\text{HH}^*(\mathcal{U}) := \text{Hom}_{\mathcal{U} \times \mathcal{U}}(\Delta_* \mathcal{O}_{\mathcal{U}}, \Delta_* \mathcal{O}_{\mathcal{U}}).$$

Here $\Delta: \mathcal{U} \to \mathcal{U} \times \mathcal{U}$ is the diagonal. Then it is shown in [AG15, Section 4] that there exists a canonical map $\mathcal{H}^1(T_{\mathcal{U}|\mathcal{U}}) \to \text{HH}^2(\mathcal{U})$. So for $F \in D^b_{\text{coh}}(\mathcal{U})$, we have the map of graded rings

$$\mathcal{O}_{\text{Crit}(w)} = S(\mathcal{H}^1(T_{\mathcal{U}|\mathcal{U}})) \to \text{HH}^2(\mathcal{U}) \to \text{Hom}^2(\mathcal{F}, \mathcal{F}).$$

Here the last arrow is defined by taking Fourier-Mukai transforms associated with morphisms $\Delta_* \mathcal{O}_{\mathcal{U}} \to \Delta_* \mathcal{O}_{\mathcal{U}}[2s]$. The above map defines a $\mathbb{C}^*$-equivariant $\mathcal{O}_{\text{Crit}(w)}$-module structure on $\text{Hom}^2(\mathcal{F}, \mathcal{F})$, which is finitely generated by [AG15, Theorem 4.1.8]. The singular support of $F$

$$(3.4) \quad \text{Supp}^s(F) \subset \text{Crit}(w)$$

is defined to be the support of $\text{Hom}^2(\mathcal{F}, \mathcal{F})$ as a graded $\mathcal{O}_{\text{Crit}(w)}$-module. Note that the singular support is a conical (i.e. $\mathbb{C}^*$-invariant) closed subset of $\text{Crit}(w)$. For a conical closed subset $Z \subset \text{Crit}(w)$, we denote by

$$\mathcal{C}_Z \subset D^b_{\text{coh}}(\mathcal{U})$$

the triangulated subcategory of objects $F \in D^b_{\text{coh}}(\mathcal{U})$ whose singular supports are contained in $Z$.

### 3.2. Quasi-smooth derived stacks.

Below, we denote by $\mathfrak{M}$ a derived Artin stack over $\mathbb{C}$. This means that $\mathfrak{M}$ is a contravariant $\infty$-functor from the $\infty$-category of affine derived schemes over $\mathbb{C}$ to the $\infty$-category of simplicial sets

$$\mathfrak{M}: \text{Aff}^{\text{op}} \to \text{SSets}$$

satisfying some conditions (see [Toe14, Section 3.2] for details). Here $\text{Aff}^{\text{op}}$ is defined to be the $\infty$-category of commutative simplicial $\mathbb{C}$-algebras, which is equivalent to the $\infty$-category of commutative differential graded $\mathbb{C}$-algebras with non-positive degrees. The classical truncation of $\mathfrak{M}$ is denoted by

$$\mathcal{M} := t_0(\mathfrak{M}) : \text{Aff}^{\text{op}} \hookrightarrow \text{Aff}^{\text{op}} \to \text{SSets}$$

where the first arrow is a natural functor from the category of affine schemes to affine derived schemes. The triangulated category of quasi-coherent sheaves on $\mathfrak{M}$ is defined as (see [Toe14, Section 4.1])

$$D_{\text{qcoh}}(\mathfrak{M}) := \lim_{\mathcal{U} \in \mathfrak{M}} D_{\text{qcoh}}(\mathcal{U}).$$

Here $\mathcal{U} = \text{Spec } A$ is an affine derived scheme for a cdga $A$, $\alpha: \mathcal{U} \to \mathfrak{M}$ is a smooth morphism and $D_{\text{qcoh}}(\mathcal{U})$ is the derived category of dg-modules over $A$. The limit is taken for the $\infty$-category of smooth morphisms $\alpha: \mathcal{U} \to \mathfrak{M}$ with 1-morphisms given by commutative diagrams

$$\begin{tikzcd}
\mathcal{U} \arrow{rr}{f} \arrow{dr}{\alpha} & & \mathcal{U}' \arrow{dl}{\alpha'} \\
& \mathfrak{M} &
\end{tikzcd}$$

Here $\mathcal{U}$, $\mathcal{U}'$ are affine derived schemes, $f$ is a smooth morphism and $\alpha' \circ f$ is equivalent to $\alpha$.

Here we note that a limit of triangulated categories cannot be defined in general, and in order to take it we have to replace triangulated categories with their dg-enhancements and define the limit in the $\infty$-category of dg-categories (see [GR17, Toe11]). By abuse of notation, we use the following
notation for the limit. For an \( \infty \)-category \( I \), let \( \{ D_{dg}^i \}_{i \in I} \) be the \( I \)-diagram of dg-categories \( D_{dg}^i \) for \( i \in I \), and denote by \( D_i = \text{Ho}(D_{dg}^i) \) the homotopy category of \( D_{dg}^i \). Then we write

\[
\lim_{i \in I} D_i := \text{Ho} \left( \lim_{i \in I} D_{dg}^i \right).
\]

In the above convention, the category (3.7) is more precisely defined as follows. Let \( L_{qcoh}(U) \) be the dg-category of dg-modules over \( A \) localized by quasi-isomorphisms (see [To¨ e11, Section 2.4]), so that its homotopy category is equivalent to the derived category \( D_{qcoh}(U) \). Then the category (3.7) is defined by taking the limit for \( L_{qcoh}(U) \), and then take its homotopy category. We have the triangulated subcategory \( D^b_{coh}(M) \subset D_{qcoh}(M) \) consisting of objects which have bounded coherent cohomologies. We note that there is a bounded t-structure on \( D^b_{coh}(M) \) whose heart coincides with \( \text{Coh}(M) \).

A derived stack \( \mathcal{M} \) is called quasi-smooth if the cotangent complex \( L_{\mathcal{M}} \) is perfect such that for any point \( x \to \mathcal{M} \) the restriction \( L_{\mathcal{M}}|_x \) is of cohomological amplitude \([-1, 1]\). By [BBBBJ15, Theorem 2.8], the quasi-smoothness of \( \mathcal{M} \) is equivalent to that \( \mathcal{M} \) is a 1-stack, and any point of \( \mathcal{M} \) lies in the image of a 0-representable smooth morphism (3.9)

\[
\alpha: \mathcal{U} \to \mathcal{M}
\]

where \( \mathcal{U} \) is an affine derived scheme of the form (3.2). In this case, we have

\[
D^b_{coh}(\mathcal{M}) = \lim_{\mathcal{U} \to \mathcal{M}} D^b_{coh}(\mathcal{U})
\]

where the limit is taken for the \( \infty \)-category \( I \) of smooth morphisms (3.9) from \( \mathcal{U} \) of the form (3.2) with 1-morphisms given by (3.8). In this paper when we write \( \lim_{\mathcal{U} \to \mathcal{M}} (-) \) for a quasi-smooth \( \mathcal{M} \), the limit is always taken for the \( \infty \)-category \( I \) as above.

Following [DG13, Definition 1.1.8], a derived stack \( \mathcal{M} \) is called QCA (quasi-compact and with affine automorphism groups) if the following conditions hold:

(i) \( \mathcal{M} \) is quasi-compact;
(ii) The automorphism groups of its geometric points are affine;
(iii) The classical inertia stack \( I_M := \Delta \times_{\mathcal{M} \times \mathcal{M}} \Delta \) is of finite presentation over \( \mathcal{M} \).

In this paper we always assume that a derived stack \( \mathcal{M} \) is QCA. We will often use the following derived stack:

**Example 3.1.** Let \( G \) be a reductive algebraic group. A \( G \)-equivariant tuple is a tuple

\[
(Y, V, s)
\]

where \( Y \) a smooth affine scheme with \( G \)-action, \( V \to Y \) is a \( G \)-equivariant vector bundle and \( s \) is a \( G \)-invariant section of \( V \). Given \( (Y, V, s) \) as above, we have the associated quasi-smooth and QCA derived stack

\[
[\mathcal{U}/G] = [\text{Spec} \mathcal{R}(V \to Y, s)/G].
\]

**3.3 DT categories via singular supports.** Let \( \mathcal{M} \) be a quasi-smooth derived stack. We denote by \( \Omega_{\mathcal{M}}[-1] \) the \((-1\)-shifted cotangent stack over \( \mathcal{M} \), i.e.

\[
\Omega_{\mathcal{M}}[-1] = \text{Spec} \text{Sym}(T_{\mathcal{M}}[1]).
\]

Here \( T_{\mathcal{M}} \) is the tangent complex of \( \mathcal{M} \). We have the projection of the classical truncations

\[
p: t_0(\Omega_{\mathcal{M}}[-1]) \to \mathcal{M}.
\]

We have the \( \mathbb{C}^* \)-action on \( t_0(\Omega_{\mathcal{M}}[-1]) \) which acts on fibers of \( p \). A closed substack of \( t_0(\Omega_{\mathcal{M}}[-1]) \) is called conical if it is closed under the above fiberwise \( \mathbb{C}^* \)-action.
Note that a smooth morphism (3.9) induces the diagram

\[ t_0(\Omega_U[-1]) \xrightarrow{\alpha^\heartsuit} t_0(\Omega_M[1] \times \Omega_L) \xrightarrow{\alpha^\diamondsuit} t_0(\Omega_M[-1]). \]

Here \( \alpha^\heartsuit \) is the projection and \( \alpha^\diamondsuit \) is induced by the natural morphism \( \alpha^*\Omega_M \to \Omega_L \). For a conical closed substack \( Z \subset t_0(\Omega_M[-1]) \), we have the conical closed subscheme
\[
\alpha^*Z := \alpha^\diamondsuit(\alpha^\heartsuit)^{-1}(Z) \subset t_0(\Omega_U[-1]) = \text{Crit}(w).
\]

We define the triangulated subcategory
\[ (3.13) \quad C_Z \subset D^b_{\text{coh}}(\mathcal{M}) \]
to be consisting of objects \( E \in D^b_{\text{coh}}(\mathcal{M}) \) such that for any map \( \alpha \) as in (3.9), we have
\[
\text{Supp}^E(\alpha^*E) \subset \alpha^*Z.
\]
For a conical closed substack \( Z \subset t_0(\Omega_M[-1]) \), the quotient category
\[ (3.14) \quad D^b_{\text{coh}}(\mathcal{M})/C_Z \]
was our model for the definition of \( C^* \)-equivariant DT category in (Todb). On the other hand, suppose that we have a diagram (3.8). Then the pull-back \( f^*: D^b_{\text{coh}}(\mathcal{U}) \to D^b_{\text{coh}}(\mathcal{U}') \) takes \( C_{\alpha^*Z} \) to \( C_{\alpha^*Z'} \) by [AG15 Corollary 7.5.5]. Therefore we have the induced functor
\[
\alpha^*: D^b_{\text{coh}}(\mathcal{U})/C_{\alpha^*Z} \to D^b_{\text{coh}}(\mathcal{U}')/C_{\alpha^*Z'}.
\]
Then we can take their limit
\[ (3.15) \quad \lim_{\alpha: \mathcal{M} \to \mathcal{M}} \left( D^b_{\text{coh}}(\mathcal{U})/C_{\alpha^*Z} \right) \]
where \( \alpha: \mathcal{U} \to \mathcal{M} \) is a smooth morphism as in (3.9). Again the category (3.15) is more precisely defined to be the homotopy category of the limit of the Drinfeld dg-quotients \( L_{\text{coh}}(\mathcal{U})/L_{C_{\alpha^*Z}} \) defined in [Todb Section 3.3], which are dg-enhancements of \( D^b_{\text{coh}}(\mathcal{U})/C_{\alpha^*Z} \). By the equivalence (3.9) explained later, the limit (3.15) may be regarded as a gluing of derived factorization categories
\[
\lim_{\mathcal{U} \to \mathcal{M}} \text{MF}^C_{\text{coh}}(V^\vee \setminus \alpha^*Z, w).
\]
A relationship between the categories (3.14) and (3.15) will be addressed in Section C. In the case that \( Z = p^{-1}(\mathcal{W}) \) for a closed substack \( \mathcal{W} \subset \mathcal{M} \), we have equivalences (see Corollary C.4)
\[ (3.16) \quad D^b_{\text{coh}}(\mathcal{M})/C_Z \cong \lim_{\alpha: \mathcal{M} \to \mathcal{M}} \left( D^b_{\text{coh}}(\mathcal{U})/C_{\alpha^*Z} \right) \cong D^b_{\text{coh}}(\mathcal{M}^o). \]

Here \( \mathcal{M}^o \subset \mathcal{M} \) be the derived open substack whose classical truncation is \( \mathcal{M} \setminus \mathcal{W} \).

For an element \( l \in \text{Pic}(\mathcal{M})_\mathbb{Q} \), we use the same notation \( l \) for its pull-back to \( t_0(\Omega_M[-1]) \) by the projection (3.11). We will apply the above construction for the conical closed substack (called \( l \)-unstable locus)
\[ (3.17) \quad Z_{l\text{-us}} := t_0(\Omega_M[-1]) \setminus t_0(\Omega_M[-1])^m(l) \subset t_0(\Omega_M[-1]). \]

We remark that, for the projection \( p \) in (3.11), we always have the inclusion
\[ (3.18) \quad Z_{l\text{-us}} \subset p^{-1}(\mathcal{M} \setminus \mathcal{M}^m(l)). \]

However in general they do not coincide. If (3.18) is the identity, then by (3.16) the categories (3.14), (3.15) are equivalent to \( D^b_{\text{coh}}(\mathcal{M}^m(l)) \), where \( \mathcal{M}^m(l) \subset \mathcal{M} \) is the derived open substack of \( l \)-semistable points.
3.4. **Good moduli spaces of Artin stacks.** In general for an Artin stack $\mathcal{M}$, its *good moduli space* is an algebraic space $M$ together with a quasi-compact morphism, 

$$\pi_M : \mathcal{M} \to M$$

satisfying the following conditions (cf. [Alp13, Section 1.2]):

(i) The push-forward $\pi_{M*} : \text{QCoh}(\mathcal{M}) \to \text{QCoh}(M)$ is exact.

(ii) The induced morphism $O_M \to \pi_{M*}O_M$ is an isomorphism.

The good moduli space morphism $\pi_M$ is universally closed. Moreover for each closed point $y \in M$, there exists a unique closed point $x \in \pi_M^{-1}(y)$, and its automorphism group $\text{Aut}(x)$ is reductive (see [Alp13, Theorem 4.16, Proposition 12.14]).

Let $\mathcal{M}$ be a quasi-smooth derived stack and take its classical truncation $\mathcal{M} = t_0(\mathcal{M})$. Suppose that it admits a good moduli space $\pi_M : \mathcal{M} \to M$. We will use the following étale local structure result for good moduli spaces proved in [AHRc].

**Theorem 3.2. ([AHRc, Theorem 2.9])** For any closed point $y \in M$, there exists an étale neighborhood $\iota : (U, 0) \to (M, y)$ and a Cartesian diagram of the form

$$
\begin{array}{ccc}
[\mathcal{U}/G] & \longrightarrow & \mathcal{M}_U \\
\downarrow \hspace{1cm} \Pi_M \downarrow & \searrow & \downarrow \pi_M \\
\mathcal{U}/\hspace{1cm} G & \longrightarrow & U \\
\end{array}
\begin{array}{ccc}
\mathcal{U}/G & \longrightarrow & \mathcal{M}
\end{array}
\begin{array}{ccc}
\iota & \longrightarrow & M
\end{array}
$$

Here $G = \text{Aut}(x)$ for a unique closed point $x \in \pi_M^{-1}(y)$, $U = \text{Spec } \mathcal{R}$ is an affine $\mathbb{C}$-scheme with $G$-action, and $\mathcal{U}/G = \text{Spec } \mathcal{R}^G$.

The above result can be extended to derived stacks as follows:

**Proposition 3.3. (cf. [HLb, Lemma 2.4, Lemma 2.5])** In the situation of Theorem 3.2 there is a unique (up to equivalence) derived stack $\mathcal{M}_U$ with $\mathcal{M}_U = t_0(\mathcal{M}_U)$ satisfying the following: it fits into a Cartesian diagram

$$(3.19)$$

$$
\begin{array}{ccc}
\mathcal{M}_U & \longrightarrow & \mathcal{M}
\end{array}
\begin{array}{ccc}
\iota_U & \longrightarrow & M
\end{array}
\begin{array}{ccc}
\mathcal{U}/G & \longrightarrow & \mathcal{M}
\end{array}
\begin{array}{ccc}
\iota & \longrightarrow & M
\end{array}
$$

such that $\mathcal{M}_U$ is equivalent to $[\mathcal{U}/G]$, where $\mathcal{U} = \text{Spec } \mathcal{R}(V \to Y, s)$ is a quasi-smooth affine derived scheme associated with a $G$-equivariant tuple $(Y, V, s)$ as in Example 3.1.

Moreover for $l \in \text{Pic}(\mathcal{M})_R$, by replacing $U$ with a Zariski open neighborhood of $0 \in U$, the pull-back $\iota_U^*(l) \in \text{Pic}(\mathcal{U}/G)_R$ is extended to a $\mathbb{R}$-line bundle on $[Y/G]$.

**Proof.** Since the category of étale morphisms to $\mathcal{M}$ is equivalent to that of étale morphisms to $\mathcal{M}$, there is a unique derived stack $\mathcal{M}_U$ which fits into the Cartesian diagram (3.19). Then by Proposition A.1, the derived stack $\mathcal{M}_U$ is equivalent to $[\mathcal{U}/G]$ for $\mathcal{U} = \text{Spec } \mathcal{R}(V \to Y, s)$ associated with a $G$-equivariant tuple $(Y, V, s)$.

The final statement follows from Lemma 2.1. Indeed by shrinking $U$ if necessary, the $\mathbb{R}$-line bundle $\iota_U^*(l)$ on $[\mathcal{U}/G]$ is pulled back via $[\mathcal{U}/G] \to BG$, so can be extended to $[Y/G]$ by taking the pull-back via $[Y/G] \to BG$. □

Let $\mathcal{J}$ be the category consisting of étale maps $\iota : U \to M$ from an affine scheme $U$ satisfying the conditions in Theorem 3.2 and Proposition 3.3. The set of morphisms in $\mathcal{J}$ consists of étale maps $U' \to U$ commuting with maps to $M$. For each morphism $U' \to U$ in $\mathcal{J}$, we have the Cartesian
In the case that \( M \) admits a good moduli space as above, we have another description of the \( C^\ast \)-equivariant DT category (3.15):

**Lemma 3.4.** For a conical closed substack \( Z \subset t_0(\Omega_M[-1]) \), we have an equivalence

\[
\lim_{U \to M} \left( D^b_{\text{coh}}(U)/C_{\alpha\ast}Z \right) \sim \lim_{(U \to M) \in J} \left( D^b_{\text{coh}}(\mathcal{M}_U)/C_{\iota_U\ast}Z \right).
\]

**Proof.** By \cite[Corollary 4.2.3.10]{Lur}, we have an equivalence

\[
\lim_{U \to M} \left( D^b_{\text{coh}}(U)/C_{\alpha\ast}Z \right) \sim \lim_{(U \to M) \in J} \left( D^b_{\text{coh}}(U\ '/)/C_{\alpha'\ast\iota_U\ast}Z \right).
\]

Here \( \alpha, \alpha' \) are smooth morphisms from affine derived schemes \( U, U' \) of the form \( \mathbb{C}_x \). By Corollary \cite[Corollary 4.3]{C}, we also have an equivalence

\[
D^b_{\text{coh}}(\mathcal{M}_U)/C_{\iota_U\ast}Z \sim \lim_{U \to M} \left( D^b_{\text{coh}}(U\ '/)/C_{\alpha'\ast\iota_U\ast}Z \right).
\]

Therefore we obtain the lemma. \( \square \)

### 3.5. Symmetric structures of derived stacks

Let \( \mathfrak{M} \) be a quasi-smooth derived stack such that \( M = t_0(\mathfrak{M}) \) admits a good moduli space \( \pi_M : M \to M \). Recall that for any closed point \( x \in M \), its automorphism group \( \text{Aut}(x) \) is a reductive algebraic group by \cite[Proposition 12.14]{Alp}. We introduce the notion of symmetric structures for derived stacks.

**Definition 3.5.** A symmetric structure \( S \) of \( \mathfrak{M} \) is a choice of symmetric structures of \( \mathcal{H}^0(\mathcal{T}_{\mathfrak{M}|x}) \oplus \mathcal{H}^1(\mathcal{T}_{\mathfrak{M}|x})^\vee \) at each closed point \( x \in M \), i.e., a direct sum of \( \text{Aut}(x)\)-representations

\[
(3.20) \quad \mathcal{H}^0(\mathcal{T}_{\mathfrak{M}|x}) \oplus \mathcal{H}^1(\mathcal{T}_{\mathfrak{M}|x})^\vee = S_x \oplus U_x
\]

such that \( S_x \) is a symmetric \( \text{Aut}(x)\)-representation.

A derived stack \( \mathfrak{M} \) is called symmetric if \( \mathcal{H}^0(\mathcal{T}_{\mathfrak{M}|x}) \oplus \mathcal{H}^1(\mathcal{T}_{\mathfrak{M}|x})^\vee \) is a symmetric \( \text{Aut}(x)\)-representation for any closed point \( x \in M \). In this case, we have a symmetric structure (3.20) such that \( U_x = 0 \), which we call a maximal symmetric structure.

We will use the following lemma on symmetric structures.

**Lemma 3.6.** Suppose that for a \( G \)-equivariant tuple \((Y, V, s)\) in Example (3.1), a symmetric structure of the derived stack \( \mathfrak{M} = [U/G] \) is given as in (3.20). Then for any \( G \)-fixed point \( x \in U \), there is a symmetric structure of the \( G \)-representation \( T_x Y \oplus V|_x \) of the form

\[
T_x Y \oplus V|_x = (S_x \oplus P \oplus P^\vee) \oplus U_x
\]

for some \( G \)-representation \( P \). Here \( S_x \oplus P \oplus P^\vee \) is the symmetric part.

**Proof.** The tangent complex of \( \mathfrak{M} \) at \( x \) is given by

\[
\mathcal{T}_{\mathfrak{M}|x} = (\mathfrak{g} \overset{0}{\to} T_x Y \overset{d_1}{\to} V|_x).
\]

Here the first map is zero because \( x \) is fixed by \( G \). Since \( G \) is reductive, we have an isomorphism of complexes of \( G \)-representations

\[
(T_x Y \overset{d_1}{\to} V|_x) \cong (\text{Ker } ds|_x \overset{0}{\to} \text{Cok } ds|_x) \oplus (P \overset{id}{\to} P)
\]
for some $G$-representation $P$. Then we have

$$T_xY \oplus V^\vee_x \cong H^0(T_{\mathfrak{M}|x}) \oplus H^1(T_{\mathfrak{M}|x})^\vee \oplus P \oplus P^\vee.$$ 

Therefore the lemma holds.

We next introduce the notion of ‘formal neighborhood theorem’ for quasi-smooth derived stacks with good moduli spaces.

**Definition 3.7.** We say that $\mathfrak{M}$ satisfies formal neighborhood theorem if for any closed point $x \in \mathcal{M}$ with $y = \pi_{\mathcal{M}}(x) \in M$, there exists an $\text{Aut}(x)$-equivariant morphism

$$\kappa: \hat{H}^0(T_{\mathfrak{M}|x})_0 \to H^1(T_{\mathfrak{M}|x})$$

with $\kappa(0) = 0$ such that, by setting $\hat{N}_0 \hookrightarrow H^0(T_{\mathfrak{M}|x})_0$ to be the classical zero locus of $\kappa$, we have commutative isomorphisms

$$\hat{N}_0/\text{Aut}(x) \sim \to \mathcal{M} \times_M \text{Spec} \hat{O}_{M,y}$$

Here the top isomorphism sends $0$ to $x$, and identity on stabilizer groups at these points.

**Remark 3.8.** Similarly to (2.12), the scheme $\hat{H}^0(T_{\mathfrak{M}|x})_0$ is defined to be the formal fiber of the morphism

$$H^0(T_{\mathfrak{M}|x}) \to H^0(T_{\mathfrak{M}|x})/\text{Aut}(x)$$

at the origin. This should not be confused with the formal completion of $H^0(T_{\mathfrak{M}|x})$ at the origin.

The following lemma is proved along with the similar argument of Proposition A.2, so we omit its proof.

**Lemma 3.9.** Suppose that $\mathfrak{M}$ satisfies formal neighborhood theorem. Then the diagram (3.22) can be extended to a Cartesian diagram

$$\begin{array}{ccc}
[\hat{N}_0/\text{Aut}(x)] & \sim \rightarrow & \mathcal{M} \\
\downarrow & & \downarrow \\
\hat{N}_0/\text{Aut}(x) & \sim \rightarrow & \text{Spec} \hat{O}_{M,y}
\end{array}$$

Here the vertical arrows are closed immersions given by taking the classical truncations and $\hat{N}_0$ is the derived zero locus of (3.21).

We also introduce some conditions for $\mathbb{R}$-line bundles on the classical stack $\mathcal{M} = t_0(\mathfrak{M})$. For a closed point $x \in \mathcal{M}$, we denote by $\mu_x: B\text{Aut}(x) \to \mathcal{M}$ the map sending a point to $x$ and identity on the automorphism groups. For $l \in \text{Pic}(\mathcal{M})_\mathbb{R}$, we set $l_x := \mu_x^*l \in \text{Pic}(B\text{Aut}(x))_\mathbb{R}$. Recall the conditions for the characters of reductive algebraic groups in Definition 2.4.

**Definition 3.10.** For $l, \delta \in \text{Pic}(\mathcal{M})_\mathbb{R}$, we say

(i) $l$ is $\mathcal{S}$-generic for a symmetric structure $\mathcal{S}$ given in (3.20) if for any closed point $x \in \mathcal{M}$, the $\mathbb{R}$-line bundle $l_x$ on $B\text{Aut}(x)$ is $\mathcal{S}_x$-generic. We denote by $\text{Pic}(\mathcal{M})^{\text{gen}/S}_\mathbb{R} \subset \text{Pic}(\mathcal{M})_\mathbb{R}$ the subset of $\mathcal{S}$-generic elements.

(ii) $\delta$ is $l$-generic if $\delta_x$ is $l_x$-generic for any closed point $x \in \mathcal{M}$.

(iii) $l$ is compatible with the symmetric structure $\mathcal{S}$ in (3.20) if $l_x$ is compatible with the symmetric structure (3.20) for any closed point $x \in \mathcal{M}$.

We now state the main result in this paper:
Theorem 3.11. Let $\mathcal{M}$ be a quasi-smooth derived stack with a good moduli space $\mathcal{M} = t_0(\mathcal{M}) \rightarrow M$. Suppose that a symmetric structure $\mathcal{S}$ of $\mathcal{M}$ is given as \cite{320}, and $\mathcal{M}$ satisfies formal neighborhood theorem. Let us take $l, \delta \in \text{Pic}(\mathcal{M})^\text{gen/S}$ such that $\delta$ is $l$-generic. Then there exists a triangulated subcategory $\mathcal{W}_\delta^\text{int/S}(\mathcal{M}) \subset D^b_{\text{coh}}(\mathcal{M})$ such that, for the $l$-unstable locus $Z_{l\text{-un}} \subset t_0(\Omega_{\mathcal{M}}[-1])$ defined in \cite{3.7}, the composition
\[
\Theta_l: \mathcal{W}_\delta^\text{int/S}(\mathcal{M}) \hookrightarrow D^b_{\text{coh}}(\mathcal{M}) \rightarrow \lim_{\mathfrak{U} \in \mathcal{S}} \left( D^b_{\text{coh}}(\mathfrak{U}) / \mathcal{C}_{\mathfrak{U}^*} Z_{l\text{-un}} \right)
\]
is fully-faithful, which is an equivalence if $l$ is compatible with $\mathcal{S}$.

Proof. The proof will be given in Subsection 4.7 (see Theorem 4.24).

We have the following corollary of the above theorem.

Corollary 3.12. Under the assumption of Theorem 3.11 let us take $l_1, l_2 \in \text{Pic}(\mathcal{M})^\text{gen/S}$ such that $l_1$ is compatible with the symmetric structure $\mathcal{S}$. Then there exists a fully-faithful functor
\[
(3.23) \quad \Theta_{l_1 l_2}: \lim_{\mathfrak{U} \in \mathcal{S}} \left( D^b_{\text{coh}}(\mathfrak{U}) / \mathcal{C}_{\mathfrak{U}^*} Z_{l_1\text{-un}} \right) \rightarrow \lim_{\mathfrak{U} \in \mathcal{S}} \left( D^b_{\text{coh}}(\mathfrak{U}) / \mathcal{C}_{\mathfrak{U}^*} Z_{l_2\text{-un}} \right)
\]
which is an equivalence if $l_2$ is also compatible with $\mathcal{S}$. In particular if $\mathcal{M}$ is symmetric, we have an equivalence \cite{123} for any $l_1, l_2 \in \text{Pic}(\mathcal{M})^\text{gen/S}$, where $\mathcal{S}$ is a maximal symmetric structure.

Proof. By Lemma \ref{2.26} for each closed point $x \in \mathcal{M}$ there is an uncountable many ($\varepsilon_1, \varepsilon_2) \in \mathbb{R}^2$ such that for $\delta = \varepsilon_1 l_1 + \varepsilon_2 l_2$, $\delta$ is $\mathcal{S}_x$-generic, $l_1, x$-generic and $l_2, x$-generic. Since the set of isomorphism classes of reductive algebraic groups $\text{Aut}(x)$ together with their representations $\text{H}^1(\mathcal{T}_{\mathcal{M}}|_x) \oplus \text{H}^1(\mathcal{T}_{\mathcal{M}}|_x)'$ and their symmetric structures is at most countable many, we can take $(\varepsilon_1, \varepsilon_2)$ to be independent of $x$. Then the desired fully-faithful functor $\Theta_{l_1 l_2}$ is given by $\Theta_{l_2} \circ \Theta_{l_1}^{-1}$ for the above choice of $\delta$. In the case that $\mathcal{M}$ is symmetric, then each $l_i$ is compatible with the maximal symmetric structure, so $\Theta_{l_i}$ are equivalences. Therefore $\Theta_{l_1 l_2}$ is an equivalence.

Remark 3.13. In the situation of Corollary 3.12 let $\mathcal{M}' \subset \mathcal{M}$ be an open immersion with $\mathcal{M}' = t_0(\mathcal{M}')$, and set $Z'_1 := Z_{l_1\text{-un}} \times_{\mathcal{M}} \mathcal{M}'$ which are conical closed substacks in $t_0(\Omega_{\mathcal{M}}' [-1])$. If $Z'_1 = Z'_2$, then we have the commutative diagram
\[
(3.24) \quad \begin{array}{c}
\lim_{\mathfrak{U} \in \mathcal{S}} \left( D^b_{\text{coh}}(\mathfrak{U}) / \mathcal{C}_{\mathfrak{U}^*} Z_{l_1\text{-un}} \right) \\
\downarrow \\
\lim_{\mathfrak{U} \in \mathcal{S}'} \left( D^b_{\text{coh}}(\mathfrak{U}) / \mathcal{C}_{\mathfrak{U}^*} Z'_2 \right)
\end{array} \rightarrow \begin{array}{c}
\lim_{\mathfrak{U} \in \mathcal{S}} \left( D^b_{\text{coh}}(\mathfrak{U}) / \mathcal{C}_{\mathfrak{U}^*} Z_{l_2\text{-un}} \right) \\
\downarrow \\
\lim_{\mathfrak{U} \in \mathcal{S}'} \left( D^b_{\text{coh}}(\mathfrak{U}) / \mathcal{C}_{\mathfrak{U}^*} Z'_1 \right)
\end{array}
\]
Here the vertical arrows are restriction functors, and the bottom arrow is a natural equivalence given by $Z'_1 = Z'_2$. The commutative diagram (3.24) follows since the compositions
\[
\mathcal{W}_\delta^\text{int/S}(\mathcal{M}) \hookrightarrow D^b_{\text{coh}}(\mathcal{M}) \rightarrow \lim_{\mathfrak{U} \in \mathcal{S}} \left( D^b_{\text{coh}}(\mathfrak{U}) / \mathcal{C}_{\mathfrak{U}^*} Z_{l_1\text{-un}} \right) \rightarrow \lim_{\mathfrak{U} \in \mathcal{S}'} \left( D^b_{\text{coh}}(\mathfrak{U}) / \mathcal{C}_{\mathfrak{U}^*} Z'_1 \right)
\]
are identified if $Z'_1 = Z'_2$.

4. Window theorem for DT categories

The purpose of this section is to introduce ‘intrinsic window subcategories’ in the derived categories of quasi-smooth derived stacks, and use them to prove Theorem 3.11. We first define intrinsic window subcategories for derived stacks in Example 3.1. We then compare them with window subcategories in the derived factorization categories under Koszul dualities, and show that Theorem 3.11 holds étale locally on the good moduli space. The global intrinsic window subcategories are defined by gluing the above local intrinsic window subcategories, and Theorem 3.11 is proved by gluing the above étale local results.
4.1. Koszul duality equivalence. Let \((Y, V, s)\) be a \(G\)-equivariant tuple as in Example 3.1 so that we have the derived stack

\[
\mathcal{U}/G = \text{[Spec } \mathcal{R}(V \to Y, s)/G].
\]

We take the weight two \(C^*\)-action on the fibers of \(V^\vee \to Y\). Then we have the \((G \times C^*)\)-action on \(V^\vee\) and the function \(w\) defined in (4.3) is \(\tau\)-semi invariant of weight two, where \(\tau\) is the projection \(G \times C^* \to C^*\). For a \((G \times C^*)\)-invariant closed subset \(Z \subset \text{Crit}(w)\), we have the subcategory defined in (4.1)

\[
\mathcal{C}_Z \subset D^b_{\text{coh}}([U/G])
\]

consisting of objects whose singular supports are contained in the closed substack

\[
Z = [Z/G] \subset \text{[Spec } \mathcal{O}_Y[1]/G].
\]

Note that we have the following commutative diagram, which we will often use below.

\[
\begin{array}{ccc}
\mathcal{U}/G & \xrightarrow{j} & [Y/G] \xrightarrow{\pi_V} Y/G \\
\downarrow{\pi_U} & & \downarrow{\pi_V} \\
Y/G & \xrightarrow{s} & Y/G
\end{array}
\]

Here 0 is the zero section of \(V^\vee \to Y\), and \(\pi_U, \pi_V\) are good moduli space morphisms. In the above situation, the following result is proved in several references (see [Hir17, Proposition 4.8], [Todb, Theorem 2.6] and also [Isi13, Shi12, OR, Todb]).

**Theorem 4.1.** ([Isi13, Shi12, Hir17, OR, Todb]) We have an equivalence

\[
\Phi: D^b_{\text{coh}}([U/G]) \xrightarrow{\sim} \text{MF}^C_{\text{coh}}([V^\vee/G], w).
\]

The equivalence \(\Phi\) is constructed in the following way. Let \(K_s\) be the following \((G \times C^*)\)-equivariant factorization of \(w\) (called Koszul factorization)

\[
K_s := (\mathcal{O}_{V^\vee} \otimes_{\mathcal{O}_Y} \mathcal{O}_U, d_{K_s}).
\]

Here \(G\) acts on \(\mathcal{O}_{V^\vee} \otimes_{\mathcal{O}_Y} \mathcal{O}_U\) diagonally, the \(C^*\)-action is given by the grading

\[
\mathcal{O}_{V^\vee} \otimes_{\mathcal{O}_Y} \mathcal{O}_U = S(V[-2]) \otimes_{\mathcal{O}_V} S(V^\vee[1]),
\]

and the weight one map \(d_{K_s}\) is given by

\[
d_{K_s} = 1 \otimes d_{\mathcal{O}_U} + \eta: \mathcal{O}_{V^\vee} \otimes_{\mathcal{O}_Y} \mathcal{O}_U \to \mathcal{O}_{V^\vee} \otimes_{\mathcal{O}_Y} \mathcal{O}_U(1),
\]

where \(\eta \in V \otimes_{\mathcal{O}_V} V^\vee \subset \mathcal{O}_{V^\vee} \otimes_{\mathcal{O}_V} \mathcal{O}_U\) corresponds to \(id \in \text{Hom}(V, V)\). The object \(K_s\) also admits a \(G\)-equivariant dg \(\mathcal{O}_U\)-module structure by the right factor of the tensor product. The functor \(\Phi\) is defined by

\[
\Phi: D^b_{\text{coh}}([U/G]) \to \text{MF}^C_{\text{coh}}([V^\vee/G], w), \quad (-) \mapsto K_s \otimes_{\mathcal{O}_U} (-).
\]

**Remark 4.2.** The functor \(\Phi\) is more precisely formulated in terms of curved dg-modules [PP12]. See [Todb, Remark 2.5].

Under the Koszul duality equivalence in Theorem 4.1, we can compare the singular supports and the usual supports in the derived factorization category. The following proposition was claimed in [AG15, Section H], and proved in [Todb, Proposition 2.10].
Lemma 4.4. The following lemma will be useful later.

For the zero section 0: $Y \to V^\vee$, we have $\Phi(C_Z) = MF_{coh}^C([V^\vee/G], w)_Z$.

In particular by taking Verdier quotients and using $(2.7)$, we also have an equivalence
\begin{equation}
(4.5) \quad \Phi: D_{coh}^b([\mathfrak{U}/G])/C_Z \sim \to MF_{coh}^C(([V^\vee \setminus Z]/G), w).
\end{equation}

4.2. Some functorial properties of Koszul duality equivalence. Here we show some functorial properties of Koszul duality equivalence in Theorem 4.1. First let us consider the diagram (4.2). The diagram (4.7) also induces the following diagram
\begin{equation}
(4.6) \quad D_{coh}^b([\mathfrak{U}/G]) \xrightarrow{\Phi} \to MF_{coh}^C([V^\vee/G], w).
\end{equation}

Here the bottom arrow is a tautological equivalence (see [Hir17] Proposition 2.14) for example.

Proof. The lemma follows from the construction of $\Phi$ and noting that $O_Y \otimes_{O_{V^\vee}} K_s = O_{\mathfrak{U}}$,
\begin{equation}
0^* \Phi(F) = O_Y \otimes_{O_{V^\vee}} K_s \otimes_{O_{\mathfrak{U}}} F \cong F.
\end{equation}

We next show the functorial property of the Koszul duality equivalence (4.3) under push-forward of affine derived schemes. Let $(Y', V', s')$ be another $G$-equivariant tuple, and suppose that we have the following commutative diagrams
\begin{equation}
(4.7) \quad \begin{array}{ccc}
Y & \xrightarrow{k} & Y' \\
\downarrow{s} & & \downarrow{s'} \\
Y' & \xrightarrow{i} & Y',
\end{array}
\quad \begin{array}{ccc}
V & \xrightarrow{j} & V' \\
\downarrow{\gamma} & & \downarrow{\gamma'} \\
V' & \xrightarrow{i} & V'.
\end{array}
\end{equation}

Here the left diagram is a $G$-equivariant commutative diagram such that the induced morphism $V \to i^*V'$ is a vector bundle morphism. The right diagram is obtained from the left one by taking the quotients by $G$, where $\mathcal{Y} = [Y/G], \mathcal{Y}' = [Y'/G], V = [V/G]$ and $\mathcal{V}' = [V'/G]$. We assume that the right diagram in (4.7) induces an equivalence of derived stacks
\begin{equation}
(4.8) \quad f: [\mathfrak{U}/G] \sim \to [\mathfrak{U}'/G]
\end{equation}
where $\mathfrak{U}, \mathfrak{U}'$ are derived zero loci of $s, s'$, i.e. $\mathfrak{U} = \text{Spec} \mathcal{R}(V \to Y, s), \mathfrak{U}' = \text{Spec} \mathcal{R}(V' \to Y', s')$. The diagram (4.7) also induces the following diagram
\begin{equation}
(4.9) \quad \begin{array}{ccc}
C & \xrightarrow{w} & Y' \\
\downarrow{\pi} & & \downarrow{i} \\
V^\vee & \xrightarrow{g} & V^\vee
\end{array}
\quad \begin{array}{ccc}
V^\vee & \xrightarrow{i} & V' \\
\downarrow{\pi'} & & \downarrow{i} \\
Y & \xrightarrow{i} & Y'.
\end{array}
\end{equation}

Here $w'$ is defined as in (3.3) from $(Y', s')$ and $\pi$ is determined by
\begin{equation}
\pi = i^*s' \in \Gamma(Y, i^*V') \subset \Gamma(Y, S(i^*V')).
\end{equation}
Remark 4.7. The commutativity of (4.7) implies that the diagram (4.9) is also commutative.

**Lemma 4.5.** Suppose that $i$ is a closed immersion. Then the composition functor

$$i_*g^*: \MF^c_{\coh}(V^\vee, w) \xrightarrow{g^*} \MF^c_{\coh}(i^*V^\vee, \overline{\pi}) \xrightarrow{i_*} \MF^c_{\coh}(V^\vee, w')$$

is an equivalence and the following diagram is commutative:

$$\begin{array}{ccc}
D^b_{\coh}([\mathcal{U}/G]) & \xrightarrow{\Phi} & \MF^c_{\coh}([V^\vee/G], w) \\
| & f | & | i_*g^* |
\end{array}$$

$$\begin{array}{ccc}
D^b_{\coh}([\mathcal{U}'/G]) & \xrightarrow{\Phi'} & \MF^c_{\coh}([V'^{\vee}/G], w').
\end{array}$$

Here $\Phi'$ is defined by (4.4) for $(Y', s')$.

**Proof.** An object $F^\star \in D^b_{\coh}([\mathcal{U}/G])$ is represented by a complex of $G$-equivariant coherent sheaves $F^\star$ on $Y$ together with $G$-equivariant morphisms $F^\star \otimes V^\vee \to F'^{\vee -1}$ which are compatible with the differentials on $F^\star$ and $\mathcal{O}_G$. Then from the description of $\Phi$ in (4.4), the underlying coherent sheaf on $V^\vee$ of the factorization $\Phi(F^\star)$ is given by $p^\star F^\star$. On the other hand for a coherent sheaf $F$ on $Y$, by the commutative diagram (4.9) we have natural isomorphisms

$$i_*g^*p^\star F \cong i_*p'^\star F \cong p'^\star i_*F.$$

Here we note that the above sheaves are coherent sheaves on $V'^{\vee}$ as $i$ is a closed immersion. Therefore both of factorizations $(i_*g^*)\Phi(F^\star)$ and $\Phi'(f_*F^\star)$ have isomorphic underlying coherent sheaves on $V'^{\vee}$. Using the commutativity of (4.7), it is straightforward to check that the above isomorphism is compatible with the differentials of factorizations $(i_*g^*)\Phi(F^\star)$ and $\Phi'(f_*F^\star)$. Therefore we obtain the commutative diagram (4.11). Then the functor (4.10) is an equivalence by the commutative diagram (4.11) together with the fact that $f_*$ is an equivalence since $f$ is an equivalence of derived stacks.

\[\square\]

### 4.3. Intrinsic window subcategories

In this subsection, we define the intrinsic window subcategory of $D^b_{\coh}([\mathcal{U}/G])$ for a derived stack $[\mathcal{U}/G]$ as in Example 3.11 in terms of weight conditions for objects in $D^b_{\coh}([\mathcal{U}/G])$ under the push-forward to $[Y/G]$.

We first prepare some notation. Let $\mathfrak{M}$ be a quasi-smooth derived stack such that $\mathcal{M} = t_0(\mathfrak{M})$ admits a good moduli space $\mathcal{M} \to M$. Assume that $\mathfrak{M}$ satisfies the formal neighborhood theorem (see Definition 3.7) and its symmetric structure (3.20). For each map $\lambda: B\mathcal{C}^\ast \to \mathcal{M}$, it induces the map of good moduli spaces $\Spec \mathbb{C} \to M$ whose image is denoted by $y(\lambda) \in M$. We define $x(\lambda) \in \mathcal{M}$ to be the unique closed point of the fiber of $\pi_{\mathcal{M}}: \mathcal{M} \to M$ at $y(\lambda)$. Then by the diagram (3.22), $\lambda$ factors through the map

$$\lambda: B\mathcal{C}^\ast \to [\mathcal{N}_0/\Aut(\lambda(x))].$$

Note that any $\Aut(\lambda(x))$-representation induces a vector bundle on $[\mathcal{N}_0/\Aut(\lambda(x))]$, so in particular $U_{x(\lambda)}$ in the decomposition (3.20) is regarded as a vector bundle on $[\mathcal{N}_0/\Aut(\lambda(x))]$. Under the above preparation, we introduce the following definition:

**Definition 4.6.** We define $\mu_{\lambda}^\pm \in \mathbb{Z}$ to be

$$\mu_{\lambda}^+ := \wt \det((\lambda^\ast U_{\lambda(x)})^{wt>0}), \quad \mu_{\lambda}^- := \wt \det((\lambda^\ast U_{\lambda(x)})^{wt<0}).$$

**Remark 4.7.** The integers $\mu_{\lambda}^\pm$ are independent of choices of isomorphisms in (4.6) by Lemma 4.20.
Let \((Y, V, s)\) be a \(G\)-equivariant tuple in Example \ref{example} and consider the associated derived stack \([U/G]\). Note that we have the following diagram

\[
(4.12) \quad \begin{array}{ccc}
\mathcal{V} & \xrightarrow{\lambda} & [V/G] \\
\mathcal{U}/G & \xleftarrow{j} & \mathcal{Y} & \xrightarrow{s} & [Y/G]
\end{array}
\]

such that \([U/G]\) is the derived zero locus of \(s\). Below we also fix a symmetric structure \(S\) of \([U/G]\), i.e. decompositions

\[
(4.13) \quad \mathcal{H}^0(T_{[U/G]}|_{x}) \oplus \mathcal{H}^1(T_{[U/G]}|_{x}) \cong S_x \oplus U_x
\]

of \(\text{Aut}(x)\)-representations for each closed point \(x \in [U/G]\) such that \(S_x\) is symmetric. The intrinsic window subcategory with respect to the above symmetric structure is given as follows.

**Definition 4.8.** For an element \(\delta \in \text{Pic}([U/G])_R\), we define

\[
(4.14) \quad \mathcal{W}_{\delta}^{\text{int}/S}([U/G]) \subset D^b_{\text{coh}}([U/G])
\]

to be the triangulated subcategory consisting of \(E \in D^b_{\text{coh}}([U/G])\) such that for any morphism \(\lambda: BC^* \to [U/G]\) we have

\[
(4.15) \quad \text{wt}(\lambda^* j_* E) \subset \text{wt}(\lambda^* \delta) + \left[\frac{1}{2}\text{wt det}((\lambda^* L_{V|Y})^{\text{wt}<0}) - \frac{1}{2}H_\lambda^- \cdot \frac{1}{2}\text{wt det}((\lambda^* L_{V|Y})^{\text{wt}>0}) - \frac{1}{2}H_\lambda^+ \right].
\]

Here \(L_{V|Y}\) is the cotangent complex on \(V\) restricted to the zero section of \(V \to Y\), and we have used the same notation \(\lambda\) for the composition \(BC^* \xrightarrow{j} [U/G] \xrightarrow{\lambda} Y\).

In what follows, for an equivalence of derived stacks \(f: [U/G] \xrightarrow{\sim} [U'/G']\), we always take the symmetric structure \(S'\) of \([U'/G']\) induced by the symmetric structure \(\delta\) and isomorphisms \(f^* \mathcal{H}^i(T_{[U'/G']}|_{x}) \xrightarrow{\sim} \mathcal{H}^i(T_{[U/G]}|_{x})\). We have the following lemma.

**Lemma 4.9.** Suppose that we have a commutative diagram

\[
(4.16) \quad \begin{array}{ccc}
[U/G] & \xrightarrow{j} & [Y/G] \\
\mathcal{U}/G & \xleftarrow{s} & \mathcal{V}/G' \\
\mathcal{U}'/G' & \xleftarrow{s'} & [V'/G']
\end{array}
\]

\(\lambda\) is a \(G'\)-equivariant tuple and \([U'/G']\) is the associated derived stack in Example \ref{example}. The right vertical arrow is an isomorphism of vector bundles, the equivalence \(f\) is induced by the right commutative isomorphisms, and the left vertical arrow is the induced isomorphism on classical truncations.

For \(\delta' \in \text{Pic}([U'/G'])_R\), we take \(\delta = f^* \delta' \in \text{Pic}([U/G])_R\). Then we have the equivalences

\[
f_*: \mathcal{W}_{\delta}^{\text{int}/S}([U/G]) \xrightarrow{\sim} \mathcal{W}_{\delta'}^{\text{int}/S}([U'/G']), \quad f^*: \mathcal{W}_{\delta'}^{\text{int}/S}([U'/G']) \xrightarrow{\sim} \mathcal{W}_{\delta}^{\text{int}/S}([U/G]).
\]

**Proof.** The lemma is obvious since the category \(\mathcal{W}_{\delta}^{\text{int}/S}([U/G])\) is defined in terms of intrinsic properties of isomorphism classes of the diagram of stacks \ref{presentation}. \(\square\)

Below we show that the category \(\mathcal{W}_{\delta}^{\text{int}/S}([U/G])\) is also independent of possibly non-isomorphic presentation \ref{presentation} for a fixed \(G\). We prepare two lemmas.
Lemma 4.10. Suppose that we have commutative diagram (4.7) which induces an equivalence of derived stacks (4.8). Moreover suppose that the morphism \( i: \mathcal{Y} \to \mathcal{Y'} \) satisfies the condition
\[
\omega_Y = i^* \omega_{\mathcal{Y'}}.
\]
for any \( \lambda: BC^* \to \mathcal{Y} \) such that \( i \circ \lambda = \lambda' \).

For \( E \in \text{Coh}(\mathcal{Y}) \), we have the identity in 
\[
\text{wt}(i^* E) = \text{wt}(E).
\]
for any \( \lambda: BC^* \to \mathcal{Y} \) such that \( i \circ \lambda = \lambda' \).

Proof. For \( E \in \text{Coh}(\mathcal{Y}) \), we have
\[
\text{wt}^{-k}(i^* E) = T_{\text{o}^k \mathcal{Y}}(i^* E, i^* \mathcal{O}_{\mathcal{Y}}) \cong E \otimes k \bigwedge N^Y_{\mathcal{Y}}.
\]
By the above isomorphism, for any \( E \in \mathcal{D}^b(\mathcal{Y}) \) the object \( i^* E \in \mathcal{D}^b(\mathcal{Y}) \) fits into a finite sequence of distinguished triangles
\[
(4.17) \quad \cdots \xrightarrow{} \mathcal{E}^{-2} \xrightarrow{[1]} \mathcal{E}^{-1} \xrightarrow{[1]} \mathcal{E}^0 = i^* i_* E
\]
such that \( P^{-k} \cong \mathcal{E} \otimes \bigwedge N^Y_{\mathcal{Y}} \). For an object \( F \in \mathcal{D}^b(\mathcal{Y}) \), we denote by \( \text{wt}^{\text{max}}(F) \in \mathbb{Z} \) (resp. \( \text{wt}^{\text{min}}(F) \in \mathbb{Z} \)) the maximal (resp. minimal) \( C^* \)-weight of \( \mathcal{H}^*(\mathcal{F}) \). By the distinguished triangles (4.17), for any map \( \lambda: BC^* \to \mathcal{Y} \) we have
\[
\text{wt}^{\text{max}}(\lambda^* i_* E) = \text{wt}^{\text{max}}(\lambda^* E) + \text{wt}(\lambda^* N^Y_{\mathcal{Y}}). \tag{4.18}
\]
Moreover since the diagram (4.7) induces an equivalence of derived zero loci (4.8), by comparing the cotangent complexes of \( [U/G] \) and \( [U'/G] \) we have the identity in 
\[
\lambda^* L_{\mathcal{Y'}} = \lambda^* L_{\mathcal{Y}} \wedge \lambda^* \mathcal{V} \wedge \mathcal{V}.
\]
Therefore we have the identities in 
\[
\lambda^* L_{\mathcal{Y'}} = \lambda^* L_{\mathcal{Y}} + \lambda^* \mathcal{V} \wedge \mathcal{V}.
\]
From (4.18), it follows that
\[
\text{wt}^{\text{max}}(\lambda^* i_* E) = \text{wt}^{\text{max}}(\lambda^* E) + \frac{1}{2} \text{wt}(\lambda^* L_{\mathcal{Y}}) - \frac{1}{2} \text{wt}(\lambda^* L_{\mathcal{Y}}) \wedge \mathcal{V}.
\]
Similarly we have
\[
\text{wt}^{\text{min}}(\lambda^* i_* E) = \text{wt}^{\text{min}}(\lambda^* E) + \frac{1}{2} \text{wt}(\lambda^* L_{\mathcal{Y}}) - \frac{1}{2} \text{wt}(\lambda^* L_{\mathcal{Y}}) \wedge \mathcal{V}.
\]
The lemma follows from the above two identities. ∎

Later we will reduce some statements to the case that \( i \) is a closed immersion using the following lemma:
Lemma 4.11. Suppose that we have commutative diagrams (4.19) which induce an equivalence of derived stacks (4.8). Then there exist $G$-equivariant commutative diagrams

\[
\begin{array}{c}
\begin{array}{ccc}
V & \xrightarrow{k''} & V'' \\
s & \downarrow & s'' \\
Y & \xrightarrow{i''} & Y''
\end{array} \\
\begin{array}{ccc}
V' & \xrightarrow{k'} & V'' \\
s' & \downarrow & s'' \\
Y' & \xrightarrow{i''} & Y''
\end{array}
\end{array}
\]

where $(Y'', V'', s'')$ is a $G$-equivariant tuple and the top arrows are vector bundle morphisms. They satisfy the followings:

(i) The diagrams (4.19) induce equivalences of derived stacks

\[
f'': [Y/G] \xrightarrow{\sim} [Y''/G], \quad f': [Y'/G] \xrightarrow{\sim} [Y''/G]
\]

which commute with the equivalence (4.8) in the $\infty$-category of derived stacks, i.e. $f' \circ f \sim f''$. Here $Y''$ is a derived zero locus of $s''$, i.e. $Y'' = \text{Spec} \mathcal{R}(V'' \to Y'', s'')$.

(ii) The morphisms $i', i''$ are closed immersions.

Proof. The following proof will be given in Subsection A.2 \[ \square \]

The following proposition shows that the intrinsic window subcategories are independent of presentations as derived critical loci for a fixed $G$.

Proposition 4.12. Suppose that we have a commutative diagram

\[
\begin{array}{ccc}
BG & \xrightarrow{j} & [Y/G] \\
\downarrow \cong \quad h & \sim \quad f & \downarrow \\
BG & \xrightarrow{j'} & [Y'/G]
\end{array}
\]

\[
\begin{array}{ccc}
BG & \xrightarrow{s} & [V/G] \\
\quad \downarrow & \quad & \downarrow s' \\
BG & \xrightarrow{s''} & [V'/G]
\end{array}
\]

where $f$ is an equivalence of derived stacks and $[Y/G], [Y'/G]$ are derived zero loci of $s, s'$, and the left horizontal arrows are given by canonical $G$-torsors $\mathcal{U} \to [Y/G], \mathcal{U}' \to [Y'/G]$. Then by setting $\delta = f^* \delta'$ for $\delta' \in \text{Pic}(\mathcal{U}'/G)|_{\mathbb{R}}$, we have the equivalences

\[
f_* : W^{\text{int}/S}_{\delta}(\mathcal{U}/G) \xrightarrow{\sim} W^{\text{int}/S}_{\delta'}(\mathcal{U}'/G), \quad f^* : W^{\text{int}/S}_{\delta'}(\mathcal{U}'/G) \xrightarrow{\sim} W^{\text{int}/S}_{\delta}(\mathcal{U}/G).
\]

Proof. We first remark that since $f$ is an equivalence, the functors

\[
f_* : D^b_{\text{coh}}(\mathcal{U}/G) \to D^b_{\text{coh}}(\mathcal{U}'/G), \quad f^* : D^b_{\text{coh}}(\mathcal{U}'/G) \to D^b_{\text{coh}}(\mathcal{U}/G)
\]

are equivalences which are quasi-inverse each other. So we have the left equivalence in (4.22) if and only if we have the right equivalence in (4.22). Moreover let

\[
f : [\mathcal{U}/G] \xrightarrow{f''} [\mathcal{U}'/G] \xrightarrow{f'} [\mathcal{U}'/G]
\]

be a factorization of $f$, i.e. $f \sim f'' \circ f'$, such that $f', f''$ are equivalences of derived stacks. Then if two of three pairs $(f_*, f^*)$, $(f'_*, f'^*)$, $(f''_*, f''^*)$ satisfy the equivalences (4.22), then the rest of them also satisfies (4.22).
By taking the pull-back of the bottom horizontal diagram in (4.21) via \( h: BG \to BG \), we have the commutative diagram

\[
\begin{array}{ccc}
BG & \xleftarrow{[U/G]} & [Y/G] & \xrightarrow{s} & [V/G] \\
\text{id} & \sim & f_{oh^{-1}} & \sim & \\
BG & \xleftarrow{[U'/G]} & [Y'/G] & \xrightarrow{s'} & [V'/G] \\
h & \sim & h & \sim & h & \sim \\
BG & \xleftarrow{[U'/G]} & [Y'/G] & \xrightarrow{s'} & [V'/G].
\end{array}
\]

Here the \( G \)-actions on \( U', Y' \) and \( V' \) in the middle horizontal diagram are twisted by \( h \in \text{Aut}(G) \).

By the above remark together with Lemma 4.9, we may assume that \( h = \text{id} \).

By Lemma 4.13 below and the first remark in the proof of this proposition, we can assume that \( f \) fits into a commutative diagram

(4.23) \[
\begin{array}{ccc}
[U/G] & \xleftarrow{j} & [Y/G] & \xrightarrow{s} & [V/G] \\
\sim & & \sim & & \\
[U'/G] & \xleftarrow{j'} & [Y'/G] & \xrightarrow{s'} & [V'/G].
\end{array}
\]

Then using Lemma 4.11 and the above mentioned remark, we may also assume that \( i \) is a closed immersion. Then the left equivalence in (4.22) follows from Lemma 4.10 therefore the proposition holds.

We have used the following lemma:

**Lemma 4.13.** Suppose that \( h = \text{id} \) in the diagram (4.21). Then there exists a \( G \)-equivariant tuple \((Y, V, s)\) and a \( G \)-equivariant commutative diagram

(4.24) \[
\begin{array}{ccc}
V & \xleftarrow{\tilde{s}} & \tilde{V} & \xrightarrow{\tilde{g}} & V' \\
\sim & & \sim & & \\
Y & \xleftarrow{s} & \tilde{Y} & \xrightarrow{\tilde{g}'} & Y'.
\end{array}
\]

such that, by setting \( \tilde{U} \) to be the derived zero locus of \( s \), the above diagram induces equivalences \( g: [\tilde{U}/G] \xrightarrow{\sim} [U/G] \), \( g': [\tilde{U}/G] \xrightarrow{\sim} [U'/G] \) which commute with \( f \), i.e. \( f \circ g \sim g' \).

**Proof.** The proof will be given in Subsection 4.3.

### 4.4 Window subcategories under Koszul duality (linear case)

In this subsection and the next two subsections, we compare intrinsic window subcategories with the original window subcategories on derived factorization categories under Koszul duality in Theorem 4.1. This subsection is devoted to the linear case.

Let \((Y, V, s)\) be a \( G \)-equivariant tuple in Example 3.1 such that \([U/G]\) is the associated derived stack. In this subsection, we assume that \( Y = \mathbb{A}^n \) is a \( G \)-representation and \( s(0) = 0 \). Let \( S \) be a symmetric structure of \([U/G]\) as in (4.13). By writing the total space of \( V \to Y \) as a direct sum of \( G \)-representations \( V|_0 \oplus Y \), by Lemma 3.6 we have the induced symmetric structure on the total space of \( V^\vee \to Y \)

(4.25) \[
V|_0^\vee \oplus Y = (S_0 \oplus P \oplus P^\vee) \oplus U_0
\]
for some G-representation P. We denote by \( \widetilde{S}_0 = S_0 \oplus P \oplus P^\vee \) its symmetric part.

As in the diagram \ref{fig:12}, we set \( \mathcal{Y} = [Y/G] \) and \( V = [V/G] \). For a one parameter subgroup \( \lambda : S^* \to T \), we use the same notation \( \lambda : BC^* \to \mathcal{Y} \) for the corresponding map sending a point to \( 0 \in Y \), and the map on stabilizer groups is given by \( \lambda : S^* \to T \). We have the following lemma:

**Lemma 4.14.** For a one parameter subgroup \( \lambda : \mathbb{C}^* \to T \) and the corresponding map \( \lambda : BC^* \to \mathcal{Y} \), we have the identities

\[
\begin{align*}
(4.26) & \quad \text{wt det}((\lambda^*L^Y_0|_Y)^{\text{wt} > 0}) - \mu^+_\lambda = \left( \lambda \right)_{[\lambda]} + \text{wt} \lambda^* Y, \\
(4.27) & \quad \text{wt det}((\lambda^*L^Y_0|_Y)^{\text{wt} < 0}) - \mu^-_\lambda = \left( \lambda \right)_{[\lambda]} + \text{wt} \lambda^* Y.
\end{align*}
\]

Here \( K_Y := \text{det}(L_Y) \in \text{Pic}(Y) \).

**Proof.** Since \( \tilde{S}_0 \) is symmetric, we have the identity in \( K(BG) \)
\[
V_0^Y + Y - U_0 = V|_0 + Y^Y - U_0^Y.
\]

Therefore for any \( \lambda : \mathbb{C}^* \to T \) we have \( \langle Y, \lambda \rangle = \langle V|_0 + U_0, \lambda \rangle \). Using the above identity, we have the identities
\[
\begin{align*}
\text{wt det}((\lambda^*L^Y_0|_Y)^{\text{wt} > 0}) &= \langle (V_0^Y)^\lambda > 0 + (Y^Y)^\lambda > 0 - (g^Y)^\lambda > 0, \lambda \rangle \\
&= \langle (V_0^Y)^\lambda > 0 + (Y^Y)^\lambda > 0 - (g^Y)^\lambda > 0, \lambda \rangle - \langle V|_0, \lambda \rangle \\
&= \langle (\tilde{S}_0)^\lambda > 0 - (g^Y)^\lambda > 0, \lambda \rangle + \langle (U_0^Y)^\lambda > 0, \lambda \rangle - \langle Y, \lambda \rangle + \langle U_0, \lambda \rangle \\
&= \left( \lambda \right)_{[\lambda]} + \text{wt} \lambda^* Y + \mu^+_\lambda.
\end{align*}
\]

Therefore the identity \ref{4.26} holds. The identity \ref{4.27} also holds by the same computation. \( \square \)

The following proposition gives a comparison of window subcategories under Koszul duality in Theorem 4.1.

**Proposition 4.15.** We take \( l, \delta \in M^W_\mathbb{R} \) such that \( \delta \) is \( l \)-generic. Then under the equivalence \( \Phi \) in Theorem 4.1, we have
\[
(4.28) \quad \mathcal{W}_{\delta + K_Y/2}^{\text{mag}/\mathbb{R}}([V^Y/G], w) \subset \Phi(\mathcal{W}_{\delta}^{\text{int}/\mathbb{R}}([\mathbb{L}/G])) \subset \mathcal{W}_{\delta + K_Y/2}^{l}([V^Y/G], w).
\]

In particular if \( l, \delta \in (M^W_\mathbb{R})^{\text{gen}/\mathbb{R}} \) and \( l \) is compatible with \( \tilde{S}_0 \), then
\[
\mathcal{W}_{\delta + K_Y/2}^{\text{mag}/\mathbb{R}}([V^Y/G], w) = \Phi(\mathcal{W}_{\delta}^{\text{int}/\mathbb{R}}([\mathbb{L}/G])) = \mathcal{W}_{\delta + K_Y/2}^{l}([V^Y/G], w).
\]

**Proof.** We first show the inclusion
\[
(4.29) \quad \Phi(\mathcal{W}_{\delta}^{\text{int}/\mathbb{R}}([\mathbb{L}/G])) \subset \mathcal{W}_{\delta + K_Y/2}^{l}([V^Y/G], w).
\]

Let \( \lambda = \lambda_0 : \mathbb{C}^* \to T \) be a one parameter subgroup which appears in a KN stratification \ref{2.2} for the G-action on \( V^Y \) with respect to \( l \in M^W_\mathbb{R} \), and \( (V^Y)^\lambda \to Y^\lambda \) the restriction of the projection \( V \to Y \) to the \( \lambda \)-fixed loci. Let \( \eta_\lambda \) be defined as in \ref{2.7} for the G-action on \( V^Y \). We have the inequality
\[
\left( \lambda \right)_{[\lambda]} \leq \left( \lambda \right)_{[V^Y/G]} \leq \eta_\lambda,
\]

where the second identity is \ref{2.11}. By the \( l \)-genericity of \( \delta \), we have \( \langle \delta, \lambda \rangle + \eta_\lambda/2 \notin \mathbb{Z} \). Therefore for an object \( E \in \mathcal{W}_{\delta}^{\text{int}/\mathbb{R}}([\mathbb{L}/G]) \), it is enough to show that
\[
(4.30) \quad \Phi(E)|_{(V^Y)^\lambda} \in \bigoplus_{k \in I} \text{MF}^C_{\text{coh}}(\mathbb{R}, [V^Y/G], w|_{(V^Y)^\lambda})_k.
\]
Here \( I \subset \mathbb{R} \) is the interval
\[
I = \left\{ \delta + \frac{K_y}{2}, \lambda \right\} + \left\{ -\frac{1}{2} \langle L^{\lambda>0}_{[\delta_0/G]} |_0, \lambda \rangle, \frac{1}{2} \langle L^{\lambda>0}_{[\delta_0/G]} |_0, \lambda \rangle \right\}
= \langle \delta, \lambda \rangle + \left\{ \frac{1}{2} \det((\lambda^*L_Y|_Y)^{wt<0}) - \frac{1}{2} \mu_{\lambda - \lambda}, \frac{1}{2} \det((\lambda^*L_Y|_Y)^{wt>0}) - \frac{1}{2} \mu_{\lambda - \lambda} \right\}.
\]

Here the second identity follows from Lemma \( \ref{lem:4.14} \). By restricting the left hand side of \( \ref{eq:4.30} \) to the zero section \( Y^\lambda \hookrightarrow (V^\vee)^\lambda \) and noting that \( \omega|_{Y^\lambda} = 0 \), we obtain the object
\[
\Phi|_{(V^\vee)^\lambda} \in \text{MF}^c_{\text{coh}}([Y^\lambda/\lambda], 0).
\]

By Lemma \( \ref{lem:4.14} \), under the tautological equivalence \( D^b_{\text{coh}}([Y^\lambda/\lambda]) \to \text{MF}^c_{\text{coh}}([Y^\lambda/\lambda], 0) \) we have \( \Phi|_{(V^\vee)^\lambda} \cong (j_*\mathcal{E})|_{Y^\lambda} \). Therefore the condition \( \ref{eq:4.15} \) implies that
\[
\Phi(\mathcal{E})|_{Y^\lambda} \in \bigoplus_{k \in I} \text{MF}^c_{\text{coh}}([Y^\lambda/\lambda], 0).
\]

By comparing \( \ref{eq:4.30} \) with \( \ref{eq:4.31} \), it is enough to show that the pull-back by the zero section
\[
0^* : \text{MF}^c_{\text{coh}}([[(V^\vee)^\lambda/\lambda], w]|_{(V^\vee)^\lambda}) \to \text{MF}^c_{\text{coh}}([Y^\lambda/\lambda], 0)
\]
has trivial kernel. This also follows from Lemma \( \ref{lem:4.14} \) since the push-forward \( j_* \) in the diagram \( \ref{diagram:4.6} \) has trivial kernel as \( j \) is a closed immersion.

We next show the inclusion
\[
\mathcal{W}^\text{mag}_{\delta+K_y/2}([V^\vee/G], w) \subset \Phi(\mathcal{W}_{\delta+K_y/2}^{\text{int}}([U/G])).
\]

Let us take an object \( \mathcal{P} \in \mathcal{W}^\text{mag}_{\delta+K_y/2}([V^\vee/G], w) \). By Lemma \( \ref{lem:4.14} \) and the definition of the magic window subcategory, the object \( j_*\Phi^{-1}(\mathcal{P}) \in D^b_{\text{coh}}([Y/G]) \) is split generated by \( W \otimes O_Y \) for \( G \)-representations \( W \) whose \( T \)-weights are contained in \( \delta + K_y/2 + \nabla_{\delta_0} \). It follows that, by Lemma \( \ref{lem:4.14} \) for any map \( \lambda : BC^* \to [U/G] \) the object \( \lambda^*j_*\Phi^{-1}(\mathcal{P}) \) satisfies the weight condition \( \ref{eq:4.15} \). Therefore we have \( \Phi^{-1}(\mathcal{P}) \in \mathcal{W}^{\text{int}}_{\delta+K_y/2}([U/G]) \) from the definition of intrinsic window subcategory.

Finally the last statement holds from the inclusions \( \ref{eq:4.25} \) together with Proposition \( \ref{prop:2.6} \). \( \square \)

In the next lemma, we show that if the Koszul duality equivalence restricts to the equivalence of window subcategories for some presentation as a derived zero locus, then the same property also holds for other presentations.

**Lemma 4.16.** In the situation of Proposition \( \ref{prop:4.12} \), suppose that \( Y, Y' \) are \( G \)-representations. Let \( \Phi, \Phi' \) be equivalences in Theorem \( \ref{thm:4.1} \) applied for \([U/G], [U'/G] \). We take \( i, \delta \in M^W_\mathbb{R} \) such that \( \delta \) is \( l \)-generic. Then \( \Phi \) restricts to the equivalence
\[
\Phi : \mathcal{W}_{\delta}^{\text{int}}([U/G]) \cong \mathcal{W}_{\delta+K_y/2}([V^\vee/G], w)
\]
if and only if \( \Phi' \) restricts to the equivalence
\[
\Phi' : \mathcal{W}_{\delta}^{\text{int}}([U'/G]) \cong \mathcal{W}_{\delta+K_{y'}/2}([V'^\vee/G], w').
\]

**Proof.** As in the proof of Proposition \( \ref{prop:4.12} \), we may assume that \( f \) is induced by a \( G \)-equivariant diagram \( \ref{eq:4.7} \) such that \( i \) is a closed immersion. Then we have the commutative diagram \( \ref{diagram:4.11} \).
Together with Proposition 4.15 we have the commutative diagram

\[
\begin{array}{ccc}
\mathcal{W}_{\delta}^{int/G}([\mathcal{U}/G]) & \xrightarrow{\phi} & \mathcal{W}_{\delta+K_Y/2}([V^\vee/G], w) \\
\downarrow_{f_*} & & \downarrow_{\sim} \\
\mathcal{W}_{\delta}^{int/G'}([\mathcal{U}'/G]) & \xrightarrow{\phi'} & \mathcal{W}_{\delta+K_{Y'}/2}([V'^{\vee}/G], w')
\end{array}
\]

\[
\begin{array}{ccc}
\text{res} & & \\
\downarrow & & \\
\text{res}'
\end{array}
\]

\[
\begin{array}{c}
\text{MF}_{coh}^+(([V^\vee/G], w) \rightsquigarrow \text{MF}_{coh}^+(([V'^{\vee}/G], w')).
\end{array}
\]

We show that there is an equivalence $\Theta$ in the dotted arrow which makes the above diagram commutative. We set

\[
Z_{l\text{-us}} = \text{Crit}(w) \setminus \text{Crit}(w)^{ss}(l), \quad Z'_{l\text{-us}} = \text{Crit}(w') \setminus \text{Crit}(w')^{ss}(l).
\]

Since we have $\text{Crit}(w)^{ss}(l) = (V^\vee)^{ss}(l) \cap \text{Crit}(w)$, we have the open immersion

\[
(V^\vee)^{ss}(l) \hookrightarrow V^\vee \setminus Z_{l\text{-us}}
\]

such that we have

\[
\text{Crit}(w) \cap (V^\vee)^{ss}(l) = \text{Crit}(w) \cap (V^\vee \setminus Z_{l\text{-us}}) = \text{Crit}(w)^{ss}(l).
\]

Since a derived factorization category depends only on an open neighborhood of the critical locus (for example see [HLS, Lemma 5.5]), the restriction along the open immersion (4.34) gives an equivalence

\[
\text{MF}_{coh}^+(([V^\vee \setminus Z_{l\text{-us}})/G], w) \cong \text{MF}_{coh}^+(([V'^{\vee}/G], w').
\]

On the other hand, the equivalence of derived stacks $f$ in the diagram (4.21) induces the isomorphism

\[
|\text{Crit}(w)/G| \cong |\text{Crit}(w')/G|
\]

which sends a conical closed substack $[Z_{l\text{-us}}/G]$ to $[Z'_{l\text{-us}}/G]$. Therefore the equivalence $f_* : D^b_{coh}([\mathcal{U}/G]) \cong D^b_{coh}([\mathcal{U}'/G])$ restricts to the equivalence

\[
f_* : \mathcal{C}_{[Z_{l\text{-us}}/G]} \cong \mathcal{C}_{[Z'_{l\text{-us}}/G]}.
\]

By Proposition 4.3 and Lemma 4.5 the equivalence $i_*g^*$ in the diagram (4.33) restricts to the equivalence

\[
i_*g^* : \text{MF}_{coh}^+(([V^\vee/G], w)_{Z_{l\text{-us}}} \cong \text{MF}_{coh}^+(([V'^{\vee}/G], w')_{Z'_{l\text{-us}}}.
\]

By taking the Verdier quotients as in 4.6 and using 4.35, we obtain the desired equivalence $\Theta$.

Note that the functors $\text{res}, \text{res}'$ in the diagram (4.33) are equivalences by Theorem 2.3. Using the equivalence $\Theta$, we have the commutative diagram

\[
\begin{array}{ccc}
\mathcal{W}_{\delta}^{int/G}([\mathcal{U}/G]) & \xrightarrow{\phi} & \mathcal{W}_{\delta+K_Y/2}([V^\vee/G], w) \\
\downarrow & & \downarrow_{\sim} \\
\mathcal{W}_{\delta}^{int/G'}([\mathcal{U}'/G]) & \xrightarrow{\phi'} & \mathcal{W}_{\delta+K_{Y'}/2}([V'^{\vee}/G], w')
\end{array}
\]

The lemma follows from the above commutative diagram. \qed
4.5. **Window subcategories under Koszul duality (formal fiber case).** We also have the formal fiber version of the results in the previous subsections. Let $Y$ be a smooth affine scheme with $G$-action, and take the formal completion of $Y//G$ at $y$, that is $\hat{Y}_y//G := \text{Spec} \hat{\mathcal{O}}_{Y//G,y}$ as in Subsection 2.4. Let $V \to Y$ be a $G$-equivariant vector bundle, and

\[
[\bar{V}_y//G] \to [\hat{Y}_y//G]
\]

the formal fibers at $y$ as in the diagram (2.12). Let $\hat{s}_y$ be a section of the above vector bundle and $[\hat{U}_y//G]$ the derived zero locus of $\hat{s}_y$. Similarly to (1.12), we have the commutative diagram

(4.36)

\[
\begin{array}{c}
\hat{U}_y//G \ar[r]_{\pi_U} \ar[d]_{\pi_U} & \hat{Y}_y//G \ar[d]_{\pi_Y} \\
\bar{U}_y//G \ar[r]_{\pi_U} & \bar{Y}_y//G
\end{array}
\]

By Theorem 4.1 we have the equivalence

\[
\Phi_y : D^b_{\text{coh}}([\hat{U}_y//G]) \sim \text{MF}^C_{\text{coh}}([\hat{V}_y//G], \hat{\omega}_y).
\]

Let $x$ be the unique closed point in $[\hat{U}_y//G]$. Then for a symmetric structure $S$ of $[U//G]$ as in (4.13), its restriction to $x$ determines the symmetric structure $S_x$ of $[\hat{U}_y//G]$. For $\delta \in \text{Pic}([U//G]_{R})$, the intrinsic window subcategory

\[
W^\text{int/\delta}_x([\hat{U}_y//G]) \subset D^b_{\text{coh}}([\hat{U}_y//G])
\]

is defined similarly to (1.14), using the closed immersion $[\hat{U}_y//G] \hookrightarrow [\hat{Y}_y//G]$ and the symmetric structure (1.13) at $x$. We have the following lemma which relates window subcategories and those on formal fibers.

**Lemma 4.17.** Let $s$ be a section of $[V//G] \to [Y//G]$, and $[U//G] \hookrightarrow [Y//G]$ its derived zero locus. For each $y \in Y//G$, let $\hat{s}_y$ be the section of $[\bar{V}_y//G] \to [\hat{Y}_y//G]$ induced from $s$. Then for an object $E \in D^b_{\text{coh}}([U//G])$, we have

\[
E \in W^\text{int/\delta}_x([U//G]), \quad (\text{resp. } \Phi(E) \in W^l_{\delta+K_Y/2}([V^\vee//G], w))
\]

if and only if for any $y \in U//G$ we have

\[
\widehat{E}_y \in W^\text{int/\delta}_x([\hat{U}_y//G]), \quad (\text{resp. } \Phi_y(\widehat{E}_y) \in W^l_{\delta+K_Y/2}([\hat{V}_y^\vee//G], \hat{\omega}_y)).
\]

Here $\widehat{E}_y$ is the pull-back of $E$ to $[\hat{U}_y//G]$, and the pull-backs of $l, \delta, K_Y$ to the formal fibers are also denoted as $l, \delta, K_y$.

**Proof.** By the construction of the equivalence in Theorem 4.1 we have the commutative diagram

(4.39)

\[
\begin{array}{ccc}
D^b_{\text{coh}}([U//G]) \ar[r]_{\Phi} \ar[d] & \text{MF}^C_{\text{coh}}([V^\vee//G], w) \ar[d] \\
D^b_{\text{coh}}([\hat{U}_y//G]) \ar[r]_{\Phi_y} & \text{MF}^C_{\text{coh}}([\hat{V}_y^\vee//G], \hat{\omega}_y).
\end{array}
\]

Here the vertical arrows are pull-back functors. The lemma follows from the above commutative diagram, since the defining conditions of the relevant window subcategories are local on $U//G$.  □
Suppose that $Y = \mathbb{A}^n$ is a $G$-representation and take the formal fibers at $0 \in Y/G$. Let $\tilde{s}_0$ be the symmetric structure as in (4.20). In this case, we have the following formal fiber version of Proposition 4.15.

**Proposition 4.18.** We take $l, \delta \in M^W$ such that $\delta$ is $l$-generic. Then under the equivalence $\tilde{\Phi}_0$ in (4.37), we have

$$W_{\delta + K Y/2}(\tilde{V}_0^\vee/G, \tilde{w}_0) \supseteq \tilde{\Phi}_0(W_{\delta}^{\text{int}}/\tilde{s}_0(\tilde{U}_0^0/G)) \supset W_{\delta + K Y/2}(\tilde{V}_0^\vee/G, \tilde{w}_0).$$

In particular if $l, \delta \in (M^W)^{\text{gen/30}}$ and $l$ is compatible with $\tilde{s}_0$, then

$$W_{\delta + K Y/2}(\tilde{V}_0^\vee/G, \tilde{w}_0) = \tilde{\Phi}_0(W_{\delta}^{\text{int}}/\tilde{s}_0(\tilde{U}_0^0/G)) = W_{\delta + K Y/2}(\tilde{V}_0^\vee/G, \tilde{w}_0).$$

**Proof.** The argument of Proposition 4.15 applies verbatim. For the second statement, we use Proposition 2.7 instead of Theorem 2.6. \qed

We also have the following formal fiber version of Lemma 4.16.

**Lemma 4.19.** Let $Y, Y'$ be $G$-representations and $V \rightarrow Y$, $V' \rightarrow Y'$ be $G$-equivariant vector bundles. Suppose that we have the following diagram

$$\begin{array}{ccc}
\tilde{U}_0^0/G & \overset{j}{\longrightarrow} & \tilde{V}_0^0/G \\
| & | & | \\
\tilde{U}_0'/G & \overset{j'}{\longrightarrow} & \tilde{V}_0'/G
\end{array}$$

where $j$ is an equivalence of derived stacks and $\tilde{U}_0^0/G, \tilde{U}_0'/G$ are derived zero loci of the sections $\tilde{s}_0, \tilde{s}'_0$, respectively. We assume that $\tilde{s}_0(0) = 0, \tilde{s}'_0(0) = 0$ and $\tilde{f}(0) = 0$. We take $l, \delta \in M^W$ such that $\delta$ is $l$-generic. Then $\tilde{\Phi}_0$ restricts to the equivalence

$$\tilde{\Phi}_0 : W_{\delta}^{\text{int}}/\tilde{s}_0(\tilde{U}_0^0/G) \sim W_{\delta + K Y/2}^{l}(\tilde{V}_0^\vee/G, \tilde{w}_0)$$

if and only if $\tilde{\Phi}_0'$ restricts to the equivalence

$$\tilde{\Phi}_0' : W_{\delta}^{\text{int}}/\tilde{s}_0'(\tilde{U}_0'/G) \sim W_{\delta + K Y/2}^{l'}(\tilde{V}_0^\vee/G, \tilde{w}_0').$$

**Proof.** The argument of Lemma 4.16 almost applies verbatim, using Proposition 4.18 instead of Proposition 4.15. One subtle difference is that in the formal fiber case the condition $\tilde{f}(0) = 0$ implies that $\tilde{f}$ fits into a left commutative diagram in (4.40). Namely, let $\tilde{f}(0) : BG \rightarrow BG$ be the induced morphism at the origins. Then the diagram

$$\begin{array}{ccc}
\tilde{U}_0^0/G & \longrightarrow & BG \\
\| & \| & \| \\
\tilde{U}_0'/G & \longrightarrow & BG
\end{array}$$

commutes. Here the horizontal arrows are given by canonical $G$-torsors $\tilde{U}_0 \rightarrow [\tilde{U}_0^0/G], \tilde{U}_0' \rightarrow [\tilde{U}_0'/G]$. The commutative diagram (4.40) follows from Lemma 4.20 below. \qed

We have used the following lemma:

**Lemma 4.20.** For morphisms $f, f' : [\tilde{U}_0^0/G] \rightarrow BG$, suppose that $f \circ \mu \cong f' \circ \mu$ as morphisms $BG \rightarrow BG$, where $\mu : BG \rightarrow [\tilde{U}_0^0/G]$ sends the point to 0 and identity on the stabilizer groups. Then we have $f \sim f'$. **Proof.** The proof will be given in Subsection A.4. \qed
4.6. Window subcategories under Koszul duality (affine case). For a $G$-equivariant tuple $(Y, V, s)$ as in Example 3.1, we consider the derived stack $[\mathcal{U}/G]$ as in (4.1) with a symmetric structure $S$ as in (4.13). In Proposition 4.15, we compared window subcategories under Koszul duality when $Y$ is a $G$-representation. By applying the results for the formal fibers and using étale slice theorem, we prove a similar comparison result for an affine $Y$. We have the following proposition:

**Proposition 4.21.** For a $G$-equivariant tuple $(Y, V, s)$ in Example 3.1, suppose that the derived stack $[\mathcal{U}/G]$ satisfies formal neighborhood theorem. Let us take $l, \delta \in \text{Pic}(\mathcal{U}/G)_{\text{gen}}$ such that $\delta$ is $l$-generic, and they are extended to $\mathbb{R}$-line bundles on $[Y/G]$ which use the same notation $l, \delta$. Then the equivalence $\Phi$ in Theorem 4.1 restricts to the fully-faithful functor

\[
(4.41) \quad \Phi: W_{\delta}^{\text{int/S}}([\mathcal{U}/G]) \hookrightarrow W_{\delta + K_{\gamma}/Z}^l([V^V/G], w)
\]

which is an equivalence if $l$ is compatible $S$. We have already regarded $\mathbb{R}$-line bundles on $Y$ as $\mathbb{R}$-line bundles on $V^V$ by the pull-back of the projection $V^V \to Y$.

**Proof.** Let us take a closed point $y \in \mathcal{U}/G$. By [Alp13] Theorem 4.16 (iii), there is a unique closed point in the fiber of $[\mathcal{U}/G] \to \mathcal{U}/G$ at $y$, i.e. there is a unique closed $G$-orbit in $\mathcal{U}$ which is mapped to $y$. Let $x \in \mathcal{U}$ be a closed point contained in the above unique closed $G$-orbit. We denote by $G_x \subset G$ the stabilizer subgroup of $x$, which is a reductive algebraic group by [Alp13] Proposition 12.14]. By Luna’s étale slice theorem for the $G$-action on $Y$ (see [Lun73, AHRc]), there is a $G_x$-invariant locally closed subscheme $x \in Y \subset Z$ and Cartesian diagrams

\[
(4.42) \quad [(V, y)/G] \xrightarrow{\delta} [(V, y)/G] \xrightarrow{\pi_Y} [(Y, y)/G] \xrightarrow{\pi_Y} (Y/G, y) \quad [(T_x Z, y)/G_x] \xrightarrow{\pi_T} (T_x Z/G_x, y).
\]

Here each horizontal arrows are étale morphisms, and $T_x Z, T_x(V|z)$ are the Zariski tangent spaces of $Z, V|z$ at $x$, where we regard $x$ as a point of $V|z$ by the zero section of $V|z \to Z$. Note that $Z$ is smooth since $Y$ is smooth and $[Z/G_x] \to [Y/G]$ is étale. Also note that $T_x(V|z) = V|z + T_x Z$ as $G_x$-representations.

By taking the formal fibers of arrows at $y \in Y/G$ and the right arrows at $0 \in T_x Z/G_x$, we obtain the commutative diagram (see the diagram (4.36) for the notation)

\[
(4.43) \quad [\hat{Y}_y/G] \xrightarrow{\hat{\gamma}_y} [(V_x \times (T_x Z)_0)/G_x] \xrightarrow{\hat{\iota}_0} [(T_x Z)_0/G_x].
\]

Here $\hat{\gamma}_y$ is induced by the section $s: Y \to V$, and $\hat{\iota}_0$ is defined by the commutative diagram (4.43). In particular, $[\hat{U}_y/G]$ is equivalent to the derived zero locus of $\hat{\iota}_0$. By the commutative diagram (4.43) together with Lemma 4.9 and Lemma 4.17 we can apply Proposition 4.18 to conclude that the functor $\Phi$ in Theorem 4.1 restricts to the functor (4.41), which is fully-faithful.

Below we show that the functor (4.41) is an equivalence if $l$ is compatible with $S$. We set

$$Y' := \mathcal{H}^0(\mathcal{U}_l|x), \quad V' := \mathcal{H}^1(\mathcal{U}_l|x) \oplus \mathcal{H}^0(\mathcal{U}_l|x).$$

Note that $Y'$ is a $G_x$-representation, and $V'$ is regarded as a $G_x$-equivariant vector bundle on $Y'$ by the second projection $V' \to Y'$. As we assume that $[\mathcal{U}/G]$ satisfies the formal neighborhood
theorem, Lemma 3.9 implies the following: there exists a diagram
\begin{equation}
(4.44) \quad \begin{array}{c}
\widehat{V}_0/G_z \\
\Upsilon_0 \\
\widehat{V}_0'/G_x
\end{array}
\end{equation}
where \([\widehat{U}_0'/G_x]\) is the derived zero locus of the section \(\Upsilon_0\), such that there is an equivalence
\begin{equation}
(4.45) \quad [\widehat{U}/G] \sim [\widehat{U}/G_x]
\end{equation}
where \(x \in U\) corresponds to \(0 \in Y'\). By Lemma 2.1, the \(\mathbb{R}\)-line bundles \(l_{[\widehat{U}/G]}\), \(\delta_{[\widehat{U}/G]} \in \text{Pic}(\widehat{U}/G)_{\mathbb{R}}\) correspond to \(l_x, \delta_x \in \text{Pic}(BG)_{\mathbb{R}}\) under the equivalence \((4.45)\). Note that by the genericity assumption on \(l, \delta\), the elements \(l_x, \delta_x\) are \(S_x\)-generic and \(\delta_x\) is \(l_x\)-generic.

Let \(\widehat{\Phi}_0\) be the Koszul duality equivalence in Theorem 4.11 applied for the diagram \((4.44)\). Here \(\widehat{\Phi}_0'\) is a morphism associated with the diagram \((4.44)\) as in Subsection 3.1.

By Proposition 4.18 and the genericity condition on \(l, \delta\), the equivalence \(\widehat{\Phi}_y\) in \((4.44)\) restricts to \(\widehat{\Phi}_y\) and \(\widehat{\Phi}_y'\) restricts to \(\widehat{\Phi}_y'\) as in \((4.45)\) and \((4.46)\). Then the functor \((4.41)\) is essentially surjective by Lemma 4.17 hence it is an equivalence.

We have the following corollary of the above proposition:

**Corollary 4.22.** In the situation of Proposition 4.21 let \(Z_{t, \text{us}} \subset \text{Crit}(w)/G\) be the conical closed substack of \(t\)-unstable points. Then the composition
\begin{equation}
(4.46) \quad \begin{array}{c}
\mathcal{W}^{{\text{int/}S}}_{\delta}(\mathbb{G}/G) \\
\mathcal{D}^b_{\text{coh}}(\mathbb{G}/G) \\
\mathcal{D}^b_{\text{coh}}(\mathbb{G}/G)/C_{Z_{t, \text{us}}}
\end{array}
\end{equation}
is fully-faithful, which is an equivalence if \(l\) is compatible with \(S\).

**Proof.** By Proposition 4.21 we have the commutative diagram
\begin{equation}
(4.45) \quad \begin{array}{c}
\mathcal{W}^{{\text{int/}S}}_{\delta}(\mathbb{G}/G) \\
\mathcal{D}^b_{\text{coh}}(\mathbb{G}/G) \\
\mathcal{D}^b_{\text{coh}}(\mathbb{G}/G)/C_{Z_{t, \text{us}}}
\end{array}
\end{equation}
Since have an equivalence \((4.45)\), the bottom composition is an equivalence by Theorem 2.3. Therefore the corollary follows from Proposition 4.21.

### 4.7. Proof of window theorem for DT categories
Finally, in this section, we give a proof of Theorem 3.11 by taking the limits of the results in the previous subsections. Let \(\mathcal{M}\) be a quasi-smooth derived stack such that \(\mathcal{M} = t_0(\mathcal{M})\) admits a good moduli space \(\mathcal{M} \to M\) and satisfies the formal neighborhood theorem. Let \(S\) be a symmetric structure of \(\mathcal{M}\) as in 3.20. Note that for an \(\text{étale}\) morphism \(\iota_U: \mathcal{M}_U \to \mathcal{M}\) in the diagram \((3.19)\), we have the induced symmetric structure \(S_\iota\) since \(\iota_U\) induces the equivalences of tangent complexes at each closed points. Using Proposition 3.12 the definition of intrinsic window subcategory in \(D^b_{\text{coh}}(\mathcal{M})\) is defined as a globalization of Definition 2.2.
Definition 4.23. For $\delta \in \text{Pic}(\mathcal{M})_R$, we define the triangulated subcategory
\[ \mathcal{W}^\text{int/S}_\delta(\mathcal{M}) \subset D^b_{\text{coh}}(\mathcal{M}) \]
to be consisting of objects $\mathcal{E} \in D^b_{\text{coh}}(\mathcal{M})$ such that for any étale morphism $\iota: U \to M$ from an affine scheme $U$ which fits into a diagram \([3.19]\), we have $\iota^*_U \mathcal{E} \in \mathcal{W}^\text{int/\mathcal{S}U}_\iota^*\delta(\mathcal{M}_U)$.

In the following, we show that the above intrinsic window subcategory gives a desired subcategory in Theorem 4.24.

Theorem 4.24. Let us take $l, \delta \in \text{Pic}(\mathcal{M})^\text{gen/S}_R$ such that $\delta$ is $l$-generic. Then the composition
\[ \Theta_l: \mathcal{W}^\text{int/S}_\delta(\mathcal{M}) \hookrightarrow D^b_{\text{coh}}(\mathcal{M}) \to \lim_{U \to \mathcal{M}} \left( D^b_{\text{coh}}(\mathcal{U})/\mathcal{C}_{\alpha^*Z_{\mathcal{U}}} \right) \]
is fully-faithful, which is an equivalence if $l$ is compatible with $S$.}

Proof. Let $\mathcal{J}$ be the category of étale morphisms $\iota: U \to M$ satisfying the conditions in Theorem 3.2 and Proposition 3.3. For each $(\iota: U \to M) \in \mathcal{J}$, we have the induced étale morphism $\iota_U^*: \mathcal{M}_U \to \mathcal{M}$ in the diagram \([3.19]\). Since we have the étale cover $\coprod_{(U \to M) \in \mathcal{J}} \mathcal{M}_U \xrightarrow{\iota} \mathcal{M}$ of $\mathcal{M}$, we have an equivalence
\[ (4.47) \quad D^b_{\text{coh}}(\mathcal{M}) \cong \lim_{(U \to M) \in \mathcal{J}} D^b_{\text{coh}}(\mathcal{M}_U). \]

By Lemma 4.25 below, the pull-back $\iota^*_{U \to M}$ takes $\mathcal{W}^\text{int/\mathcal{S}U}_{\iota^*\delta}(\mathcal{M}_U)$ to $\mathcal{W}^\text{int/\mathcal{S}U}_{\iota'_*\delta}(\mathcal{M}_U)$. Therefore from (4.47) and the definition of $\mathcal{W}^\text{int/S}_\delta(\mathcal{M})$, the equivalence (4.47) restricts to the equivalence
\[ (4.48) \quad \mathcal{W}^\text{int/S}_\delta(\mathcal{M}) \cong \lim_{(U \to M) \in \mathcal{J}} \mathcal{W}^\text{int/\mathcal{S}U}_{\iota^*\delta}(\mathcal{M}_U). \]

On the other hand the assumption on $\mathcal{M}$ together with the genericity of $l, \delta$ imply that, for each $(U \to M) \in \mathcal{J}$, the derived stack $\mathcal{M}_U$ together with $\iota^*_U, \iota^*_U \delta$ satisfy the assumption of Proposition 4.21. Therefore by Corollary 4.22, the composition
\[ \mathcal{W}^\text{int/\mathcal{S}U}_{\iota^*\delta}(\mathcal{M}_U) \hookrightarrow D^b_{\text{coh}}(\mathcal{M}_U) \to D^b_{\text{coh}}(\mathcal{M}_U)/\mathcal{C}_{\iota^*Z_{\mathcal{M}_U}} \]
is fully-faithful, and an equivalence if $l$ is compatible with $S$. By taking the limit for $(U \to M) \in \mathcal{J}$ and using the equivalences (4.47), (4.48), the composition
\[ \mathcal{W}^\text{int/S}_\delta(\mathcal{M}) \to D^b_{\text{coh}}(\mathcal{M}) \to \lim_{(U \to M) \in \mathcal{J}} \left( D^b_{\text{coh}}(\mathcal{M}_U)/\mathcal{C}_{\iota^*Z_{\mathcal{M}_U}} \right) \]
is full-faithful, and an equivalence if $l$ is compatible with $S$. Then the theorem follows by using Lemma 3.3.

We have used the following lemma:

Lemma 4.25. Let $[\mathcal{U}/G], [\mathcal{U}'/G']$ be derived stacks of the form \([4.1]\). Suppose that we have a commutative diagram
\[ (4.49) \quad \mathcal{U}'//G' \longrightarrow [\mathcal{U}'/G'] \longleftarrow [\mathcal{U}/G'] \]
\[ \mathcal{U}//G \longrightarrow [\mathcal{U}/G] \]
where each square is a Cartesian and the vertical arrows are étale. Then for $\delta \in \text{Pic}([\mathcal{U}/G])_R$ and $\delta' = f^*\delta \in \text{Pic}([\mathcal{U}'/G'])_R$, the functor $f^*: D^b_{\text{coh}}([\mathcal{U}/G]) \to D^b_{\text{coh}}([\mathcal{U}'/G'])$ restricts to the functor
\[ f^*: \mathcal{W}^\text{int/S}_\delta([\mathcal{U}/G]) \to \mathcal{W}^\text{int/S'}_{\delta'}([\mathcal{U}'/G']). \]
Here $S'$ is induced from $S$ by the étale morphism $f$.\qed
Proof. For a closed point \( y' \in \mathcal{U}' \parallel G' \) and \( y = f(y') \in \mathcal{U} \parallel G \), the diagram (4.49) induces an equivalence
\[
\widehat{f}_y : [\widehat{\mathcal{U}}'_{y'}/G'] \to [\widehat{\mathcal{U}}_y/G].
\]
Here we have used the notation in (4.39). Let \( x \in \mathcal{U}, \ x' \in \mathcal{U}' \) be closed points in the closed orbits of \( \mathcal{U} \to \mathcal{U} \parallel G, \mathcal{U}' \to \mathcal{U}' \parallel G' \) at \( y, y' \), respectively. Let \( x \in Z \subset Y, \ x' \in Z' \subset Y' \) be étale slices as in the proof of Proposition 4.24. Then by the diagram (4.43), we have the diagram
\[
\begin{array}{ccc}
\widehat{\mathcal{U}}_{y'}/G' & \xrightarrow{j'} & \left[ (T_xZ')_0/G'_{x'} \right] \\
\downarrow \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \frac{y}{x} \quad \frac{y'}{x'}
\end{array}
\]

such that \([\widehat{\mathcal{U}}'_{y'}/G'], \ [\widehat{\mathcal{U}}_y/G] \) are equivalent to derived zero loci of \( \widehat{p}_0, \widehat{t}_0 \), respectively.

Let us take an object \( E \in W_s^{int/\int}(\lfloor \mathcal{U}/G \rfloor) \). Then we have \( \widehat{E}_y \in W_{\delta}^{int/\int}(\lfloor \widehat{\mathcal{U}}_y/G \rfloor) \) by Lemma 4.17. Since \( f \) induces the isomorphism \( G'_{x'} \sim G_x \), by Lemma 4.9 and Proposition 4.12 we conclude that
\[
(f^*E)_y' = \widehat{f}_y(\widehat{E}_y) \in W_{\delta}^{int/\int}(\lfloor \widehat{\mathcal{U}}_{y'}/G' \rfloor).
\]

Since this holds for any \( y' \in \mathcal{U}' \parallel G' \), we have \( f^*E \in W_{\delta}^{int/\int}(\lfloor \mathcal{U}' \parallel G' \rfloor) \) by Lemma 4.17.

5. Wall-crossing equivalence of DT categories for one dimensional stable sheaves

In this section, we apply the result of Corollary 3.12 to prove Theorem 1.4. The result of this section may be regarded as a categorification of wall-crossing invariance of genus zero Gopakumar-Vafa invariants (see [IS12, Theorem 6.16], [MT18, Section 3.3], [Toda, Conjecture 5.9]).

5.1. Derived moduli stacks of one dimensional sheaves on surfaces. Let \( S \) be a smooth projective surface over \( \mathbb{C} \). We consider the derived Artin stack constructed in [TV07]
\[
\PsiRef_S : dAff^{op} \to SSets
\]
whose \( T \)-valued points for \( T \in dAff \) form the \( \infty \)-groupoid of perfect complexes on \( T \times S \). We have the open substack
\[
\mathcal{M}_S \subset \PsiRef_S
\]
corresponding to perfect complexes on \( S \) quasi-isomorphic to coherent sheaves on \( S \) whose supports have dimensions less than or equal to one. Since any object in \( \text{Coh}(S) \) is perfect as \( S \) is smooth, the derived Artin stack \( \mathcal{M}_S \) is the derived moduli stack of objects in \( \text{Coh}_{1}(S) \).

Let \( \mathcal{M} := t_0(\mathcal{M}_S) \) and take the universal families
\[
\mathcal{F} \in D^b_{coh}(S \times \mathcal{M}_S), \ \mathcal{F} := \mathcal{F}|_{S \times \mathcal{M}_S} \in D^b_{coh}(S \times \mathcal{M}).
\]
Then the cotangent complex of \( \mathcal{M}_S \) restricted to \( \mathcal{M}_S \) is given by
\[
L_{\mathcal{M}_S|\mathcal{M}_S} = (R\text{Hom}_{S \times \mathcal{M}_S}(\mathcal{F}, \mathcal{F}[1])^{\vee}.
\]
Here \( p_{\mathcal{M}} : S \times \mathcal{M}_S \to \mathcal{M}_S \) is the projection. By the above description of the cotangent complex, the derived moduli stack \( \mathcal{M}_S \) is quasi-smooth.

Let \( N_{\leq 1}(S) \) be defined by
\[
N_{\leq 1}(S) := \text{NS}(S) \oplus \mathbb{Z}.
\]
We have the Chern character map
\[ \text{ch}: K(\text{Coh}_{\leq 1}(S)) \to N_{\leq 1}(S), \quad F \mapsto ([F], \chi(F)). \]
Here \([F]\) is the fundamental one cycle associated with \(F\). We have the decompositions into open and closed substacks
\[ \mathcal{M}_S = \coprod_{v \in N_{\leq 1}(S)} \mathcal{M}_S(v), \quad \mathcal{M}_S = \coprod_{v \in N_{\leq 1}(S)} \mathcal{M}_S(v) \]
where each component corresponds to sheaves \(F\) with \(\text{ch}(F) = v\).

Note that the automorphism group of a sheaf \(F\) on \(S\) contains a one dimensional torus \(\mathbb{C}^* \subset \text{Aut}(F)\) given by the scalar multiplication, which acts on \(\text{RHom}(F, F)\) by weight zero. Therefore the inertia stack \(I_{\mathcal{M}_S}\) of \(\mathcal{M}_S\) admits an embedding \((\mathbb{C}^*)_{\mathcal{M}_S} \subset I_{\mathcal{M}_S}\) which acts on \(L_{\mathcal{M}_S}|_{\mathcal{M}_S}\) by weight zero. Then we have the \(\mathbb{C}^*\)-gerbe (called \(\mathbb{C}^*\)-rigidification)
\[ \mathcal{M}_S(v) \to \mathcal{M}^\mathbb{C}^*\text{-rig}(v) \]
such that the automorphism group at a point \([E] \in \mathcal{M}^\mathbb{C}^*\text{-rig}(v)\) is \(\text{Aut}(E)/\mathbb{C}^*\) (see [HLb, Proposition 4.2], [Todb Section 3.5]).

5.2. Derived moduli stacks of one dimensional semistable sheaves on surfaces. Let \(A(S)_\mathbb{C}\) be the complexified ample cone
\[ A(S)_\mathbb{C} := \{ B + \sqrt{-1}H \in \text{NS}(S)_\mathbb{C} : H \text{ is ample } \}. \]
Let us take \(\sigma = B + \sqrt{-1}H \in A(S)_\mathbb{C}\). For \(F \in \text{Coh}_{\leq 1}(S)\), its \(\mu_\sigma\)-slope is defined by
\[ \mu_\sigma(F) = \frac{\chi(F) - B \cdot [F]}{H \cdot [F]} \in \mathbb{R} \cup \{ \infty \}. \]
Here \(\mu_\sigma(F) = \infty\) if the denominator is zero. By definition, an object \(F \in \text{Coh}_{\leq 1}(S)\) is called \(\sigma\)-semistable if and only if it is Bridgeland (semi)stable with respect to the central charge
\[ Z_{B,H}: K(\text{Coh}_{\leq 1}(S)) \to \mathbb{C}, \quad F \mapsto -\chi(F) + (B + \sqrt{-1}H)[F]. \]
If \(B = 0\), then \(\sigma\)-(semi)stability is equivalent to classical Gieseker \(H\)-(semi)stability (see [HL97]).

We denote by
\[ \mathcal{M}_{S,\sigma\text{-st}}(v) \subset \mathcal{M}_{S,\sigma}(v) \subset \mathcal{M}_S(v), \quad \mathcal{M}^\mathbb{C}^*\text{-rig}(v) \subset \mathcal{M}_{S,\sigma\text{-st}}^\mathbb{C}^*\text{-rig}(v) \subset \mathcal{M}_S^\mathbb{C}^*\text{-rig}(v) \]
the derived open substacks corresponding to \(\sigma\)-stable sheaves, \(\sigma\)-semistable sheaves, respectively. Their classical truncations are denoted by \(\mathcal{M}_{S,\sigma\text{-st}}(v), \mathcal{M}_{S,\sigma}(v), \mathcal{M}^\mathbb{C}^*\text{-rig}(v), \mathcal{M}_S^\mathbb{C}^*\text{-rig}(v)\) respectively. Also the strictly semistable locus is denoted by
\[ \mathcal{M}_{S,\sigma\text{-ss}}(v) := \mathcal{M}_{S,\sigma}(v) \setminus \mathcal{M}_{S,\sigma\text{-st}}(v). \]
There is a projective scheme \(M_{S,\sigma}(v)\) parameterizing \(\sigma\)-polystable sheaves with Chern character \(v\) together with morphisms
\[ \mathcal{M}_{S,\sigma}(v) \to M_{S,\sigma}^\mathbb{C}^*\text{-rig}(v) \to M_{S,\sigma}(v) \]
which realize good moduli spaces for both of \(M_{S,\sigma}(v)\) and \(M_{S,\sigma}^\mathbb{C}^*\text{-rig}(v)\) (see [Tod18 Lemma 7.4]).

Lemma 5.2. Both of derived stacks \(\mathcal{M}_{S,\sigma}(v)\) and \(\mathcal{M}_{S,\sigma}^\mathbb{C}^*\text{-rig}(v)\) are symmetric and satisfy formal neighborhood theorem.
Proof. A closed point \( x \in \mathcal{M}_{S,\sigma}(v) \) corresponds to a \( \sigma \)-polystable sheaf \( F \) on \( S \), which is of the form

\[
(5.6) \quad F = \bigoplus_{i=1}^{m} V_i \otimes F_i
\]

where each \( F_i \) is a \( \sigma \)-stable sheaf on \( S \), \( V_i \) is a finite dimensional vector space such that \( F_i \) is not isomorphic to \( F_j \) for \( i \neq j \), and \( \mu_\sigma(F_i) = \mu_\sigma(F_j) \) for all \( i, j \). By the description of the cotangent complex \([5.3]\), we have

\[
\mathcal{H}^0(T_{\mathcal{M}_{S,\sigma}(v)}|_x) + \mathcal{H}^1(T_{\mathcal{M}_{S,\sigma}(v)}|_x)^{\vee} = \text{Ext}_S^1(F, F) \oplus \text{Ext}_S^2(F, F)^{\vee}
\]

(5.7)

\[
= \bigoplus_{a, b} \text{Hom}(V_a, V_b) \otimes (\text{Ext}_S^1(F_a, F_b) \oplus \text{Ext}_S^2(F_a, F_b)).
\]

The automorphism group of \( \mathcal{M}_{S,\sigma}(v) \) at \( x \) is given by

\[
\text{Aut}(x) = \text{Aut}(F) = \prod_{i=1}^{m} \text{GL}(V_i)
\]

and its acts on \((5.7)\) by the conjugation. The dual representation of \((5.7)\) is given by

\[
\bigoplus_{a, b} \text{Hom}(V_a, V_b) \otimes (\text{Ext}_S^1(F_a, F_b)^{\vee} \oplus \text{Ext}_S^2(F_a, F_b)).
\]

Therefore in order to show that \((5.7)\) is a dual representation of \( \text{Aut}(x) \), we need to show that

\[
\text{ext}_S^1(F_a, F_b) + \text{ext}_S^2(F_b, F_a) = \text{ext}_S^1(F_b, F_a) + \text{ext}_S^2(F_a, F_b).
\]

By the Riemann-Roch theorem and the stability for \( F_a \), we have

\[
\text{ext}_S^1(F_a, F_b) - \text{ext}_S^2(F_a, F_b) = \delta_{ab} + [F_a] \cdot [F_b]
\]

which is symmetric in \( a \) and \( b \). Therefore \((5.8)\) holds, and \( \mathcal{M}_{S,\sigma}(v) \) is symmetric. Similarly \((5.7)\) is a symmetric \( \text{Aut}(x)/C^* \)-representation, so \( \mathcal{M}_{S,\sigma}^{C^*\text{-rig}}(v) \) is symmetric.

The fact that the derived stack \( \mathcal{M}_{S,\sigma}(v) \) satisfies the formal neighborhood theorem follows from Theorem \([3.2]\). Here note that Theorem \([3.2]\) is formulated for Gieseker stability (i.e. \( \sigma = \sqrt{-1}H \) for an ample divisor \( H \)), but the same argument applies for one dimensional \( \mu_\sigma \)-semistable sheaves by the existence of the good moduli space for \( \mathcal{M}_{S,\sigma}(v) \) (see \([Tod18, \text{Lemma 7.4}]\)). Then the derived stack \( \mathcal{M}_{S,\sigma}^{C^*\text{-rig}}(v) \) also satisfies the formal neighborhood theorem by taking the \( C^* \)-rigidifications of top isomorphism in the diagram \((3.12)\).

\[\square\]

Remark 5.3. Let \( X = \text{Tot}_S(\omega_S) \) and \( i: S \hookrightarrow X \) the zero section. The \( \text{Aut}(x) \)-representation \((5.7)\) is isomorphic to the conjugate \( \text{Aut}(x) \)-action on

\[
\text{Ext}_X^1(i_*F, i_*F) = \bigoplus_{e \in E_\bullet} \text{Hom}(V_{e(c)}, V_{e(c)})
\]

where \( E_\bullet \) is the \( \text{Ext} \)-quiver associated with the collection \( E_\bullet = (i_*F_1, \ldots, i_*F_k) \) (see Subsection \([2.2]\)). This is because of the isomorphisms

\[
\text{Ext}_X^1(i_*F_a, i_*F_b) \cong \text{Ext}_S^1(i^*i_*F_a, F_b)
\]

\[
\cong \text{Ext}_S^1(F_a, F_b) \oplus \text{Hom}(F_a, F_b \otimes \omega_S)
\]

\[
\cong \text{Ext}_S^1(F_a, F_b) \oplus \text{Ext}_S^2(F_b, F_a)^{\vee}.
\]
5.3. **Moduli stacks of compactly supported sheaves on local surfaces.** For a smooth projective surface $S$, we consider its total space of the canonical line bundle:
\[ X = \text{Tot}_S(\omega_S) \xrightarrow{\pi} S. \]
Here $\pi$ is the projection. We denote by $\text{Coh}_{\leq 1}(X) \subset \text{Coh}(X)$ the subcategory of compactly supported coherent sheaves on $X$ whose supports have dimensions less than or equal to one. We consider the classical Artin stack
\[ M_X : \text{Aff}^{\text{op}} \rightarrow \text{Groupoid} \]
whose $T$-valued points for $T \in \text{Aff}$ form the groupoid of $T$-flat families of objects in $\text{Coh}_{\leq 1}(X)$. We have the decomposition into open and closed substacks
\[ M_X = \bigsqcup_{v \in N_{\leq 1}(S)} M_X(v) \]
where $M_X(v)$ corresponds to compactly supported sheaves $E$ on $X$ with $\text{ch}(\pi_* E) = v$. By pushing forward to $S$, we have the natural morphism
\[ \pi_* : M_X(v) \rightarrow M_S(v), \ E \mapsto \pi_* E. \]
By [Todh] Lemma 5.1, the stack $M_X(v)$ is the classical truncation of $(-1)$-shifted cotangent stack over $M_S(v)$
\[ M_X(v) \xrightarrow{\cong} t_0(\Omega M_S(v)[-1]). \]
We also have the $\mathbb{C}^*$-rigidification of $M_X(v)$ by [AOV08] Theorem A.1
\[ M_X(v) \rightarrow M_X^{\mathbb{C}^*\text{-rig}}(v). \]
By taking the $\mathbb{C}^*$-rigidifications of the isomorphism (5.10), we have the isomorphism
\[ M_X^{\mathbb{C}^*\text{-rig}}(v) \xrightarrow{\cong} t_0(\Omega M_S(v)^{\mathbb{C}^*\text{-rig}})[-1]). \]
For $\sigma = B + \sqrt{-1}H \in A(S)_{\mathbb{C}}$ and $E \in \text{Coh}_{\leq 1}(X)$, we define $\mu_{\sigma}(E) := \mu_{\sigma}(\pi_* E)$. By definition, $E \in \text{Coh}_{\leq 1}(X)$ is $\sigma$-(semi)stable if for any non-zero subsheaf $E' \subsetneq E$, we have
\[ \mu_{\sigma}(E') < (\leq) \mu_{\sigma}(E). \]
We have open substacks
\[ M_{X, \sigma, \text{st}}(v) \subset M_{X, \sigma}(v) \subset M_X(v), \ M_{X, \sigma, \text{st}}^{\mathbb{C}^*\text{-rig}}(v) \subset M_{X, \sigma}^{\mathbb{C}^*\text{-rig}}(v) \]
corresponding to $\sigma$-stable sheaves, $\sigma$-semistable sheaves, respectively. The strictly $\sigma$-semistable locus is denoted by
\[ M_{X, \sigma, \text{sss}}(v) := M_{X, \sigma}(v) \setminus M_{X, \sigma, \text{st}}(v). \]
Note that we have the inclusion
\[ (5.12) \quad \pi_*^{-1}(M_{S, \sigma}(v)) \subset M_{X, \sigma}(v) \]
which is not an identity in general. Also similarly to (5.5), there is a quasi-projective scheme $\tilde{M}_{X, \sigma}(v)$ which parametrizes $\sigma$-polystable sheaves on $X$ and morphisms
\[ M_{X, \sigma}(v) \rightarrow M_{X, \sigma}^{\mathbb{C}^*\text{-rig}}(v) \rightarrow M_{X, \sigma}(v) \]
giving good moduli spaces for both of $M_{X, \sigma}(v)$ and $M_{X, \sigma}^{\mathbb{C}^*\text{-rig}}(v)$. 
5.4. **Wall-chamber structure.** Below we fix a primitive element \( v = (\beta, n) \in N_{\leq 1}(S) \) such that \( \beta > 0 \). Here we write \( \beta > 0 \) if \( \beta = [C] \) for a non-zero effective divisor \( C \) on \( S \). For each decomposition
\[
v = v_1 + v_2, \quad v_i = (\beta_i, n_i), \quad \beta_i > 0
\]
we define
\[
W_{v_1, v_2} := \{ \sigma \in A(S)_C : \mu_{\sigma}(v_1) = \mu_{\sigma}(v_2) \}
\]
\[
= \{ B + \sqrt{-1}H \in A(S)_C : (n_1\beta_2 - n_2\beta_1) \cdot H = B\beta_1 \cdot H\beta_2 - B\beta_2 \cdot H\beta_1 \}. 
\]
Since \( v \) is primitive, \( W_{v_1, v_2} \subseteq A(S)_C \) and \( W_{v_1, v_2} \) is a real codimension one hypersurface in \( A(S)_C \). For a fixed \( v \), the set of hypersurfaces \( W_{v_1, v_2} \) are called **walls**. It is easy to see that the walls are locally finite. Also each connected component
\[
\mathcal{C} \subset A(S)_C \setminus \bigcup_{v_1 + v_2 = v} W_{v_1, v_2}
\]
is called a **chamber**. From the construction of walls, the moduli stacks \( \mathcal{M}_{S,\sigma}(v) \), \( \mathcal{M}_{X,\sigma}(v) \) are constant if \( \sigma \) is contained in a chamber, but may change when \( \sigma \) crosses a wall. Moreover if \( \sigma \) lies in a chamber, they consist of \( \sigma \)-stable sheaves, i.e. we have
\[
\mathcal{M}_{S,\sigma,ss}(v) = \mathcal{M}_{X,\sigma,ss}(v) = \emptyset.
\]

We now define some line bundle on \( \mathcal{M}_S(v) \) associated with an integral class \( \sigma \in A(S)_C \).

**Definition 5.4.** For an integral class \( \sigma = B + \sqrt{-1}H \in A(S)_C \) such that \( H \) is an effective class, we define \( l(\sigma) \in \text{Pic}(\mathcal{M}_S(v)) \) by
\[
l(\sigma) = (\det R_{p_M_*}(F \boxtimes \mathcal{O}_S(-B)))^{-\beta \cdot H} \otimes (\det R_{p_M_*}(F \boxtimes \mathcal{O}_H))^{n-B \cdot \beta}.
\]

Here \( F \) is a universal sheaf \((5.2)\). Its pull-back to \( \mathcal{M}_X(v) \), and also its restriction to an open substack of \( \mathcal{M}_X(v) \) are also denoted by \( l(\sigma) \).

The line bundle in Definition 5.4 descends to the line bundle in the \( \mathbb{C}^* \)-rigidification:

**Lemma 5.5.** The line bundles \( l(\sigma) \) on \( \mathcal{M}_S(v) \), \( \mathcal{M}_X(v) \) descend to line bundles on \( \mathcal{M}_{S,\text{rig}}(v) \), \( \mathcal{M}_{X,\text{rig}}(v) \).

**Proof.** At each point \([F] \in \mathcal{M}_S(v)\), the inertial \( \mathbb{C}^* \)-weight of \( l(\sigma)|_{[F]} \) is
\[
-(\beta \cdot H)\chi(F \boxtimes \mathcal{O}_S(-B)) + (n - B \cdot \beta)\chi(F \boxtimes \mathcal{O}_H) = -(\beta \cdot H)(n - B \cdot \beta) + (n - B \cdot \beta)(\beta \cdot H) = 0.
\]

Therefore \( l(\sigma) \) descends to \( \mathcal{M}_{S,\text{rig}}(v) \). The case for \( \mathcal{M}_X(v) \) follows from the same argument. \( \square \)

Suppose that \( \sigma \in A(S)_C \) lies on a wall and take \( \sigma^\pm = B^\pm + \sqrt{-1}H^\pm \in A(S)_C \) which lie on its adjacent chambers. Note that we have open immersions
\[
\mathcal{M}_{X,\sigma^\pm}(v) \subset \mathcal{M}_{X,\sigma}(v).
\]
Since each chamber contains dense rational points, by taking small deformations of \( \sigma^\pm \) and rescaling we may assume that \( B^\pm, H^\pm \) are integral and \( H^\pm \) are effective without changing \( \mathcal{M}_{X,\sigma^\pm}(v) \). Then by Definition 5.4 we have the line bundles \( l(\sigma^\pm) \) on \( \mathcal{M}_{S,\sigma}(v) \), \( \mathcal{M}_{X,\sigma}(v) \). By Lemma 5.5 they descend to line bundles on \( \mathcal{M}_{S,\text{rig}}(v) \), \( \mathcal{M}_{X,\sigma}(v), \) which we also denote by \( l(\sigma^\pm) \).

The following lemma shows that the open substacks \( \mathcal{M}_{X,\sigma^\pm}(v) \subset \mathcal{M}_{X,\sigma}(v) \) coincide with \( l(\sigma^\pm) \)-semistable loci.

**Lemma 5.6.** We have the identity of open substacks in \( \mathcal{M}_{X,\sigma}(v) \),
\[
(5.13) \quad \mathcal{M}_{X,\sigma}(v)^{\text{ss}}(l(\sigma^\pm)) = \mathcal{M}_{X,\sigma^\pm}(v).
\]
Proof. It is enough to prove the identity \((5.13)\) on each fiber of the good moduli space morphism \(\pi_{M_X}: M_{X,\sigma} \to M_{X,\sigma}(v)\). Let \(y \in M_{X,\sigma}(v)\) corresponds to a \(\sigma\)-polystable sheaf on \(X\) of the form

\[
E = \bigoplus_{i=1}^{m} V_i \otimes E_i. \tag{5.14}
\]

Here each \(V_i\) is a finite dimensional vector space and \(\{E_1,\ldots,E_m\}\) are mutually non-isomorphic \(\sigma\)-stable sheaves. Let \(Q_{E_i}\) be the Ext-quiver associated with the collection \((E_1,\ldots,E_m)\) (see Subsection B.2). Then the fiber of \(\pi_{M_X}\) at \(y\) is the closed substack of the nilpotent \(Q_{E_i}\)-representations with dimension vector \((\dim V_i)_{1 \leq i \leq m}\) (see Subsection B.2)

\[
\left\{ \bigoplus_{i \to j \in Q_{E_i}} \operatorname{Hom}(V_i, V_j) \right\}^{\text{nil}} / G. \tag{5.15}
\]

Here \(G = \prod_{i=1}^{m} \operatorname{GL}(V_i)\) and the subscript ‘nil’ means nilpotent \(Q_{E_i}\)-representations. Let us write \(\chi^{(\pi, E_i)} = (\beta_i, n_i)\). We define the following group homomorphisms \(W^\pm\)

\[
W^\pm : K(Q_{E_i}) \xrightarrow{\dim} \bigoplus_{i=1}^{m} \mathbb{Z} \cdot e_i \to \mathbb{C}.
\]

Here the first arrow is taking the dimension vector, and the second arrow is given by

\[
e_i \mapsto -n_i + B^\pm \cdot \beta_i + (H^\pm \cdot \beta_i)\sqrt{-1} \in \mathbb{C}.
\]

Then \(W^\pm\) determine Bridgeland stability conditions [Bri07] on the abelian category of finite dimensional \(Q_{E_i}\)-representations: a finite dimensional \(Q_{E_i}\)-representation \(R\) is \(W^\pm\)-(semi)stable if for any non-zero subrepresentation \(R' \subsetneq R\), we have

\[
\arg W^\pm(R') < (\leq) \arg W^\pm(R)
\]

in \((0, \pi]\). By [Tod18, Lemma 7.8], the intersection \(\pi_{M_X}^{-1}(p) \cap M_{X,\sigma \pm}(v)\) corresponds to \(W^\pm\)-semistable \(Q_{E_i}\)-representations inside the stack \((5.15)\). An easy calculation shows that a \(Q_{E_i}\)-representation \(R\) of dimension vector \((\dim V_i)_{1 \leq i \leq m}\) is \(W^\pm\)-(semi)stable if and only if for any non-zero subrepresentation \(R' \subsetneq R\), we have

\[
\theta^\pm(R') := \sum_{i=1}^{m} \theta^\pm_i \cdot r_i > (\geq) 0 = \theta^\pm(R).
\]

Here \((r_i)_{1 \leq i \leq m}\) is the dimension vector of \(R'\) and \(\theta^\pm_i \in \mathbb{Z}\) is given by

\[
\theta^\pm_i = (B^\pm \cdot \beta_i - n_i) \cdot (H^\pm \cdot \beta) + (n - B^\pm \cdot \beta) \cdot (H^\pm \cdot \beta_i).
\]

By the relation of \(\theta\)-stability and GIT stability proved by King [Kin94, Theorem 4.1], the \(\theta^\pm\)-(semi)stable loci in \((5.15)\) correspond to GIT (semi)stable loci with respect to the characters

\[
G \to \mathbb{C}^*, \quad (g_i)_{1 \leq i \leq m} \mapsto \prod_{i=1}^{m} \det(g_i)^{\theta^\pm_i}. \tag{5.17}
\]

On the other hand, let \(x \in M_{X,\sigma}(v)\) be the closed point corresponding to the polystable sheaf \((5.14)\). Then we have \(G = \operatorname{Aut}(x)\). In the notation of Definition 3.10 the pull-backs \(l(\sigma^\pm)x \in \)
Pic(BG) are described as
\[
\begin{aligned}
l(\sigma^\pm)_x &= \det R^! \left( \bigoplus_{i=1}^m V_i \otimes E_i \otimes \mathcal{O}_S(-B^\pm) \right)^{-H^+ \cdot \beta} \otimes \det R^! \left( \bigoplus_{i=1}^m V_i \otimes E_i \otimes \mathcal{O}_H^{\pm} \right)^{n-B^\pm \cdot \beta} \\
&= \bigotimes_{i=1}^m (\det V_i)^{-H^+ \cdot \beta_i} \cdot (\det(\chi_{E_i} \otimes \mathcal{O}_S(-B^\pm))) \otimes \bigotimes_{i=1}^m (\det V_i)^{(H^+ \cdot \beta_i) \cdot (n-B^\pm \cdot \beta)} \\
&= m \cdot \det(V_i)^{\theta^\pm}.
\end{aligned}
\]

Therefore \(l(\sigma^\pm)_x\) are induced by the \(G\)-characters \((5.17)\). Together with using Lemma 2.1, the line bundles \(l(\sigma^\pm)\) on \(\mathcal{M}_{X,\sigma}(v)\) are induced by the \(G\)-characters \((5.17)\) on the fiber \(\pi^{-1}_{\mathcal{M}_X}(p)\), so the identity \((5.13)\) holds on \(\pi^{-1}_{\mathcal{M}_X}(p)\). \(\square\)

Since the derived stacks \(\mathcal{M}_{S,\sigma}(v), \mathcal{M}^{G\text{-rig}}_{S,\sigma}(v)\) are symmetric by Lemma 5.2, we take their maximal symmetric structures \(S\) as in Definition 3.10. The following lemma shows that the line bundles \(l(\sigma^\pm)\) satisfy the genericity condition in Definition 3.10.

**Lemma 5.7.** The line bundles \(l(\sigma^\pm)\) in \(\mathcal{M}^{G\text{-rig}}_{S,\sigma}(v)\) are \(S\)-generic.

**Proof.** Let us take a closed point \(x \in \mathcal{M}_{S,\sigma}^{G\text{-rig}}(v)\) corresponding to a polystable sheaf \((5.10)\). Then \(G' := \text{Aut}(x) = G/\mathbb{C}^*\) where \(\mathbb{C}^* \subset G = \prod_{i=1}^m \text{GL}(V_i)\) is the diagonal torus. From the proof of Lemma 5.6, the element \(l(\sigma^\pm)_x \in \text{Pic}(B \text{Aut}(x))\) corresponds to a \(G'\)-character of the form \((5.17)\), where \(\theta^\pm\) are given as in \((5.10)\) for \(\beta_i = [F_i]\) and \(v_i = \chi(F_i)\). Here note that the \(G\)-character \((5.17)\) descends to the \(G'\)-character since it restricts to the trivial character on the diagonal torus \(\mathbb{C}^* \subset G\) (see Lemma 5.5). By the assumption that \(\sigma^\pm\) do not lie on walls, we have \(\theta^\pm(\vec{v}') \neq 0\) for any \(0 < \vec{v}' < \vec{v}\) where \(\vec{v} = (\dim V_i)_{1 \leq i \leq m}\). Then the lemma follows from Remark 5.3 and Lemma 5.8 below. \(\square\)

In the above lemma, we have used the following lemma on symmetric quiver representations. Here a quiver \(Q\) is called symmetric if for any vertices \(i, j\), the number of arrows from \(i\) to \(j\) is the same as that from \(j\) to \(i\).

**Lemma 5.8.** Let \(Q\) be a symmetric quiver whose vertex set, edge set, are denoted by \(V(Q), E(Q)\), respectively. Let \(\vec{v} = (v_i)_{i \in V(Q)}\) be a dimension vector of \(Q\), and \(\{\theta_i\}_{i \in V(Q)}\) with \(\theta_i \in \mathbb{Z}\) satisfy that
\[
\sum_{i \in V(Q)} \theta_i \cdot v_i = 0, \quad \sum_{i \in V(Q)} \theta_i \cdot v'_i \neq 0
\]
for any \(\vec{v}' = (v'_i)_{i \in V(Q)}\) such that \(0 < \vec{v}' < \vec{v}\). Here \(0 < \vec{v}'\) means \(v'_i \geq 0\) for any \(i \in V(Q)\) and \(\vec{v}' \neq 0\), and \(\vec{v}' < \vec{v}\) means \(\vec{v} - \vec{v}' > 0\). Let \(V_i\) for \(i \in V(Q)\) be vector spaces with dimension \(v_i\). Then for \(G = \prod_{i \in V(Q)} \text{GL}(V_i)\) and \(G' = G/\mathbb{C}^*\), the \(G'\)-character
\[
\chi_0: G' \to \mathbb{C}^*, \quad (g_i)_{i \in V(Q)} \mapsto \prod_{i \in V(Q)} \det(g_i)^{\theta_i}
\]
is \(S\)-generic with respect to the symmetric \(G'\)-representation
\[
\bigoplus_{e \in E(Q)} \text{Hom}(V_{s(e)}, V_{t(e)}).
\]

Here \(S\) is the maximal symmetric structure, \(G'\) acts on \((5.19)\) by conjugation, \(s(e)\) is the source of \(e\) and \(t(e)\) is the target of \(e\).

**Proof.** The proof is a slight modification of [KT Lemma 3.3]. The maximal torus of \(G'\) is given by \(T' = T/\mathbb{C}^*\) where \(T = \prod_{i \in V(Q)} T_i\) for the maximal torus \(T_i \subset \text{GL}(V_i)\), and the character lattice \(M'\)
of $M$ is given by the kernel of $M \to \mathbb{Z}$ dual to the diagonal embedding $\mathbb{C}^* \to T$. Let $\{e_{i1}, \ldots, e_{iv_i}\}$ be a basis of $V_j$. By fixing $i_0 \in V(Q)$ and $1 \leq k_0 \leq v_{i_0}$, we can write $M'_{R}$ as

$$M'_{R} = \bigoplus_{i \in V(Q)} \bigoplus_{1 \leq k \leq v_i} \mathbb{R}(e_{ik} - e_{i_0 k_0}).$$

By [HLS Proposition 2.1], the genericity of $\chi_\theta$ is equivalent to that for any proper subspace $H \subset M'_{R}$, there is a one parameter subgroup $\lambda : \mathbb{C}^* \to T'$ such that $\langle \chi_\theta, \lambda \rangle = 0$ for any $\gamma_j \in H$ and $\langle \chi_\theta, \lambda \rangle \neq 0$. Then $H \subset M'_{R}$ be a proper linear subspace. Let $\gamma_1, \ldots, \gamma_d \in M'$ be the $T'$-weights of the $G'$-representation $\rho(5.19).$ Then any non-zero $T'$-character $\gamma_j$ is of the form $e_{ik} - e_{i'k'}$. We set

$$\lambda(t) = (e^{\lambda_i})_{i \in V(Q), 1 \leq k \leq v_i}, \lambda_{ik} = \begin{cases} 0, & \text{if } e_{ik} - e_{i_0 k_0} \in H, \\ 1, & \text{if } e_{ik} - e_{i_0 k_0} \notin H. \end{cases}$$

Then $\langle \gamma_j, \lambda \rangle = 0$ for any $\gamma_j \in H$. As $\chi_\theta = \sum_{i,k} \theta_i \cdot e_{ik}$, we have

$$\langle \chi_\theta, \lambda \rangle = \sum_{i \in V(Q)} \theta_i \cdot 2\{1 \leq k \leq v_i : e_{ik} - e_{i_0 k_0} \notin H\} \neq 0.$$

Here the latter inequality follows from (5.18). Therefore $\chi_\theta$ is $S$-generic.

5.5. DT categories for one dimensional stable sheaves on local surfaces. We fix a primitive $v \in N_{\leq 1}(S)$ and take $\sigma \in A(S)_C$. We also take a derived open substack of finite type

$$\mathcal{M}_S(v)^{\text{fin}} \subset \mathcal{M}_S(v)$$

such that its classical truncation $\mathcal{M}_S(v)^{\text{fin}}$ contains $\pi_* \mathcal{M}_{X,\sigma}(v)$, where $\pi_*$ is defined in (5.4). Then we have the inclusion

$$\mathcal{M}_{X,\sigma}(v) \subset \pi_*^{-1}(\mathcal{M}_S(v)^{\text{fin}}) = t_0(\Omega^{\mathcal{M}_S(v)^{\text{fin}}}_{\mathcal{M}_{X,\sigma}(v)}[-1]).$$

We have the following conical closed substack of $\sigma$-unstable locus

$$\mathcal{Z}_{\sigma,\text{un}} := t_0(\Omega^{\mathcal{M}_S(v)^{\text{fin}}}_{\mathcal{M}_{X,\sigma}(v)}[-1]) \setminus \mathcal{M}_{X,\sigma}(v).$$

As we mentioned in Section 1, the following is a version of $\mathbb{C}^*$-equivariant DT category we consider:

**Definition 5.9.** We define the $\mathbb{C}^*$-equivariant DT category of $\mathcal{M}_{X,\sigma}(v)$ to be

$$\widehat{\mathcal{D}\mathcal{T}}^{\mathbb{C}^*}(\mathcal{M}_{X,\sigma}(v)) := \lim_{\Delta \to \mathcal{M}_{\mathbb{C}^*\text{-rig}}(v)^{\text{fin}}} \left( D^b_{\mathcal{C}(\Delta)}(\mathbb{C}^*/\mathcal{Z}_{\sigma,\text{un}}^{\mathbb{C}^*-\text{rig}}) \right) \cdot \mathcal{C}(\mathcal{M}_{X,\sigma}(v)).$$

Here $\Delta$ is a smooth morphism from an affine derived scheme $\Delta$ of the form $\mathcal{M}_{X,\sigma}(v)$.

**Remark 5.10.** In [Todb] Definition 5.4, we defined $\mathbb{C}^*$-equivariant DT category to be the Verdier quotient

$$\mathcal{D}\mathcal{T}^{\mathbb{C}^*}(\mathcal{M}_{X,\sigma}(v)) := D^b_{\mathcal{C}(\mathcal{M}_{\mathbb{C}^*\text{-rig}}(v)^{\text{fin}})}(\mathcal{Z}_{\sigma,\text{un}}^{\mathbb{C}^*-\text{rig}}).$$

There is a natural functor

$$\mathcal{D}\mathcal{T}^{\mathbb{C}^*}(\mathcal{M}_{X,\sigma}(v)) \to \widehat{\mathcal{D}\mathcal{T}}^{\mathbb{C}^*}(\mathcal{M}_{X,\sigma}(v)),$$

however we don’t know whether this is an equivalence or not in general. In Section C we will compare these categories, and show that the above functor is fully-faithful with dense image, assuming that $\text{Ind} \mathcal{C}_{\mathcal{Z}_{\sigma,\text{un}}}$ is compactly generated.

**Remark 5.11.** In [Todb] Lemma 3.10, Remark 5.5, it is proved that $\mathcal{D}\mathcal{T}^{\mathbb{C}^*}(\mathcal{M}_{X,\sigma}(v))$ is independent of a choice of $\mathcal{M}_S(v)^{\text{fin}}$ satisfying $\pi_* \mathcal{M}_{X,\sigma}(v) \subset \mathcal{M}_S(v)^{\text{fin}}$, up to equivalence. A similar argument easily shows that our version $\widehat{\mathcal{D}\mathcal{T}}^{\mathbb{C}^*}(\mathcal{M}_{X,\sigma}(v))$ is also independent of such a choice of $\mathcal{M}_S(v)^{\text{fin}}$ up to equivalence.

The following is our main result in this section:
Theorem 5.12. Let $\sigma \in A(S)_C$ lies on a wall with respect to $v \in N_{\leq 1}(S)$ and $\sigma^\pm \in A(S)_C$ lie on its adjacent chambers. Moreover assume that

$$M_{X,\sigma,\text{ss}(v)} \subset \pi_*^{-1}(M_S(v)).$$

Then there exists an equivalence

$$\tilde{D^+}(M_{X,\sigma^+}(v)) \sim \tilde{D^+}(M_{X,\sigma^-}(v)).$$

Proof. We first remark that the condition (5.20) implies that any $\sigma$-strictly semistable sheaf $E$ on $X$ with $\text{ch}(\pi_* E) = v$ is push-forward to a $\sigma$-strictly semistable sheaf $F = \pi_* E$ on $S$. Indeed we have an exact sequence $0 \rightarrow E_1 \rightarrow E \rightarrow E_2 \rightarrow 0$ in $\text{Coh}_{\leq 1}(X)$ such that $E_1 \neq 0$ and $\mu_{\sigma}(E_1) = \mu_{\sigma}(E_2)$. By applying $\pi_*$, we obtain an exact sequence $0 \rightarrow F_1 \rightarrow F \rightarrow F_2 \rightarrow 0$ in $\text{Coh}_{\leq 1}(S)$, where $F_1 = \pi_* E_1 \neq 0$ satisfy $\mu_{\sigma}(F_1) = \mu_{\sigma}(F_2)$. Since $F$ is $\sigma$-semistable by the condition (5.20), $F_1$ is also $\sigma$-semistable, so $F$ is strictly $\sigma$-semistable.

Let $M_S(v)_{\text{fin}} \subset M_S(v)_{\text{fin}}$ be a derived open substack whose classical truncation is $M_S(v)_{\text{fin}} \setminus M_{X,\sigma,\text{ss}(v)}$. Then we have an open cover of $M_S(v)_{\text{fin}}$,

$$M_S(v)_{\text{fin}} = M_{S,\sigma}(v) \cup M_S(v)_{\text{fin}}^\prime \cap M_S(v)_{\text{fin}} = M_{S,\sigma,\text{at}}(v).$$

Below we set $M = M_S^{\text{C-rig}}(v)_{\text{fin}}$ for simplicity. Let $I$ be the $\infty$-category consisting of smooth morphisms $\alpha : \mathcal{U} \rightarrow M$ from affine derived schemes $\mathcal{U}$ of the form (5.2), and the 1-morphisms in $I$ are given by the commutative diagrams (5.8). We have the full subcategories $I_1, I_2 \subset I$ consisting of smooth morphisms $\alpha : \mathcal{U} \rightarrow M$ which factor through $M_S^{\text{C-rig}}(v)$, $M_S^{\text{C-rig}}(v)_{\text{fin}}$, respectively. Then the family of smooth morphisms $(\mathcal{U} \rightarrow M) \in I_1 \cup I_2$ form a smooth covering of $M$. Therefore we have equivalences

$$\tilde{D^+}(M_{X,\sigma^+}(v)) \sim \lim_{(\mathcal{U} \rightarrow M) \in I} \left( \frac{D^b_{\text{coh}}(\mathcal{U})/C^{\text{C-rig}}_{\alpha^* \mathcal{Z}^{\text{C-rig}}}}{ \times \lim_{(\mathcal{U} \rightarrow M) \in I_1 \cap I_2} \left( \frac{D^b_{\text{coh}}(\mathcal{U})/C^{\text{C-rig}}_{\alpha^* \mathcal{Z}^{\text{C-rig}}}}{\lim_{(\mathcal{U} \rightarrow M) \in I_2} \left( D^b_{\text{coh}}(\mathcal{U})/C^{\text{C-rig}}_{\alpha^* \mathcal{Z}^{\text{C-rig}}}) \right) \right) \right).$$

Here the second equivalence is due to [Lur], Corollary 4.2.3.10, and the fiber product is taken in their dg-enhancements (see the convention in Subsection 5.2). Now we have

$$\mathcal{Z}_{\sigma^\pm, \text{us}} \times_{M_S(v)_{\text{fin}}} M_S(v)_{\text{fin}} = (t_0(\Omega_{M_S(v)_{\text{fin}}}/[\sigma^\pm]) \setminus M_{X,\sigma^\pm}(v)) \times M_S(v)_{\text{fin}} \setminus M_S(v)_{\text{fin}}$$

$$= t_0(\Omega_{M_S(v)_{\text{fin}}}/[\sigma^\pm]) \setminus (M_{X,\sigma^\pm}(v) \times M_S(v)_{\text{fin}} \setminus M_S(v)_{\text{fin}})$$

Here for the third identity, we have used the remark mentioned in the first part of the proof, and the inclusions

$$M_{X,\sigma,\text{at}}(v) \subset M_{X,\sigma,\text{at}}(v) = M_{X,\sigma^\pm}(v) \subset M_{X,\sigma}(v).$$

Therefore for any $(\mathcal{U} \rightarrow M) \in I_2$ we have $\alpha^* \mathcal{Z}_{\sigma^\pm, \text{us}} = \alpha^* \mathcal{Z}_{\sigma^\pm, \text{us}}$, so for $I' = I_2$ or $I' = I_1 \cap I_2$ we have a natural equivalence

$$\Theta_{I'} : \lim_{(\mathcal{U} \rightarrow M) \in I'} \left( \frac{D^b_{\text{coh}}(\mathcal{U})/C^{\text{C-rig}}_{\alpha^* \mathcal{Z}^{\text{C-rig}}}}{\times \lim_{(\mathcal{U} \rightarrow M) \in I_1 \cap I_2} \left( \frac{D^b_{\text{coh}}(\mathcal{U})/C^{\text{C-rig}}_{\alpha^* \mathcal{Z}^{\text{C-rig}}}}{\lim_{(\mathcal{U} \rightarrow M) \in I_2} \left( D^b_{\text{coh}}(\mathcal{U})/C^{\text{C-rig}}_{\alpha^* \mathcal{Z}^{\text{C-rig}}}) \right) \right) \right).$$

Below we show that an equivalence (5.22) also holds for $I' = I_1$. We claim that the following identity holds

$$M_{X,\sigma}(v)_{\text{ss}}(l(\sigma^\pm)) \cap \pi_*^{-1}(M_S(v)) = \pi_*^{-1}(M_S(v))_{\text{ss}}(l(\sigma^\pm)).$$
Since we have the inclusion (5.12), it is obvious that the left hand side is contained in the right hand side. As for the converse direction, let us take \( x \in \pi_*^{-1}(M_{S,\sigma}(v))^{ss}(l(\sigma^\pm)) \) and a map
\[
f: [\mathbb{A}^1/\mathbb{C}^*] \to M_{X,\sigma}(v)
\]
with \( f(1) \sim x \). Suppose that \( \text{wt}(f(0)^*l(\sigma^\pm)) < 0 \). Then \( x \in M_{X,\sigma}(v) \) is not semistable with respect to \( l(\sigma^\pm) \), hence by Lemma 5.6 it corresponds to a strictly \( \sigma \)-semistable sheaf on \( X \), i.e. \( x \in M_{X,\sigma}(v) \). As \( M_{X,\sigma}(v) \) is a closed substack of \( M_{X,\sigma}(v) \), we have \( f(0) \in M_{X,\sigma}(v) \).

By the assumption (5.20), we conclude that \( x \in M_{X,\sigma}(v) \). If we impose some further assumption, the result of Theorem 5.12 is described in terms of derived moduli spaces of stable sheaves on \( S \).

Corollary 5.13. Under the assumption of Theorem 5.12, suppose furthermore that the following condition also holds:
\[
M_{X,\sigma^\pm}(v) \subset \pi_*^{-1}(M_{S,\sigma^\pm}(v)).
\]

Then we have an equivalence
\[
\Theta_{\sigma^+, \sigma^-} : D^b_{\text{coh}}(M_{S,\sigma^+}^{\text{C-rig}}(v)) \sim D^b_{\text{coh}}(M_{S,\sigma^-}^{\text{C-rig}}(v))
\]
such that we have the commutative diagram
\[
\begin{array}{ccc}
D^b_{\text{coh}}(M_{S,\sigma^+}^{\text{C-rig}}(v)) & \xrightarrow{\Theta_{\sigma^+, \sigma^-}} & D^b_{\text{coh}}(M_{S,\sigma^-}^{\text{C-rig}}(v)) \\
\downarrow & & \downarrow \\
D^b_{\text{coh}}(M_{S,\sigma^+}^{\text{C-rig}}(v)) & \sim & D^b_{\text{coh}}(M_{S,\sigma^-}^{\text{C-rig}}(v)).
\end{array}
\]

Here the vertical arrows are restriction functors.

Proof. If the inclusion (5.25) holds, then it is identity by [Toda, Lemma 5.6], and we have equivalences
\[
D^b_{\text{coh}}(M_{S,\sigma^\pm}^{\text{C-rig}}(v)) \sim \mathcal{D}^{\mathbb{C}^*} (M_{X,\sigma^\pm}(v))
\]
by Corollary [C], (also see [Toda, Lemma 5.7]). Therefore by Theorem 5.12 we have the equivalence (5.26). The commutative diagram (5.27) follows from (5.24). □
5.6. **Examples and applications.** In this subsection, we give several examples where the condition \(5.20\) holds so that we can apply Theorem \(5.12\). In several cases we also have the condition \(5.20\), so we can apply Corollary \(5.13\). But there also exists an example where \(5.20\) holds but \(5.25\) does not hold.

We first consider the reduced class. A divisor class \(\beta \in \text{NS}(S)\) on a smooth projective surface \(S\) is called reduced if any effective divisor \(D\) on \(S\) with \([D] = \beta\) is a reduced divisor. In this case, for any \(n \in \mathbb{Z}\) the element \(v = (\beta, n) \in N_{\leq 1}(S)\) is primitive.

**Lemma 5.14.** For \(v = (\beta, n) \in N_{\leq 1}(S)\), suppose that \(\beta\) is reduced. Then for any \(\sigma \in A(S)_C\), we have
\[
(5.29) \quad \mathcal{M}_{X,\sigma}(v) \subset \pi_{\ast}^{-1}(\mathcal{M}_{S,\sigma}(v)).
\]

**Proof.** Note that giving a compactly supported coherent sheaf on \(X\) is equivalent to giving a pair \((F, \theta)\), where \(F \in \text{Coh}(S)\) and \(\theta \in \text{Hom}(F, F \otimes \omega_S)\). The push-forward \(\pi_{\ast}\) sends such a pair \((F, \theta)\) to \(F\).

Let \(E \in \text{Coh}_{\leq 1}(X)\) be a \(\sigma\)-semistable sheaf which corresponds to a pair \((F, \theta)\) such that \(\text{ch}(F) = (\beta, n)\). Suppose that \(F\) is not \(\sigma\)-semistable. Then there exists an exact sequence \(0 \to F' \to F \to F'' \to 0\) in \(\text{Coh}(S)\) such that \(F'\), \(F''\) are pure one dimensional sheaves and \(\mu_{\sigma}(F') > \mu_{\sigma}(F'')\). We consider the following diagram
\[
\begin{array}{ccc}
0 & \to & F' \\
\downarrow & & \downarrow \\
F'' & \to & F & \to & F'' \\
\downarrow & & \downarrow & & \downarrow \\
0 & \to & F' \otimes \omega_S & \to & F \otimes \omega_S & \to & F'' \otimes \omega_S & \to & 0.
\end{array}
\]

By the assumption, the sheaf \(F\) is scheme theoretically supported on a reduced divisor on \(S\). Therefore the supports of \(F'\) and \(F''\) do not have common irreducible components, hence we have \(\text{Hom}(F', F'' \otimes \omega_S) = 0\). Then there exist dotted arrows in the diagram \(5.30\) which makes the diagram \(5.30\) commutative. However then the diagram \(5.30\) destabilizes \(E\), which is a contradiction. \(\square\)

By the above lemma, in the reduced case the conditions \(5.20\), \(5.25\) are satisfied for any \(\sigma \in A(S)_C\) and \(\sigma^\pm \in A(S)_C\). Therefore by Corollary \(5.13\) we have the following:

**Corollary 5.15.** Let \(S\) be a smooth projective surface, and take \(v = (\beta, n) \in N_{\leq 1}(S)\) such that \(\beta\) is a reduced class. Then for any \(\sigma \in A(S)_C\) which do not lie walls, we have an equivalence
\[
D^{b}_{\text{coh}}(\mathfrak{M}^{C, \text{rig}}_{S, \sigma}(v)) \sim D^{b}_{\text{coh}}(\mathfrak{M}^{C, \text{rig}}_{S, \sigma^\pm}(v)).
\]

We also have the following lemma for the conditions \(5.20\), \(5.25\) to hold:

**Lemma 5.16.** Suppose that \(c_1(\omega_S) \in \mathbb{R}_{\leq 0} \cdot H\) for some \(H \in A(S)_R\). Then for \(\sigma = \sqrt{-1} H\), both of the conditions \(5.20\), \(5.25\) hold for any primitive \(v\).

**Proof.** We follow the proof of Lemma \(5.14\). As for the condition \(5.20\), let \(E \in \text{Coh}_{\leq 1}(X)\) be a \(\sigma\)-semistable sheaf which corresponds to a pair \((F, \theta)\) such that \(\text{ch}(F) = (\beta, n)\). Suppose that \(F\) is not \(\sigma\)-semistable. Then by taking Harder-Narasimhan filtration in \(\sigma\)-stability, there exists an exact sequence
\[
(5.31) \quad 0 \to F' \to F \to F'' \to 0
\]
in \(\text{Coh}(S)\) such that \(\mu_{\sigma}^-(F') > \mu_{\sigma}^+(F'')\). Here \(\mu_{\sigma}^+(\cdot)\) (resp. \(\mu_{\sigma}^-(\cdot)\)) is the maximal (resp. minimal) slope among the Harder-Narasimhan factors of \(\cdot\). By the assumption \(c_1(\omega_S) \in \mathbb{R}_{\leq 0} \cdot H\), taking the tensor product with \(\omega_S\) preserves \(\sigma\)-stability. Therefore we have
\[
\mu_{\sigma}^-(F') > \mu_{\sigma}^+(F'') \geq \mu_{\sigma}^+(F'' \otimes \omega_S),
\]
hence the vanishing \(\text{Hom}(F', F'' \otimes \omega_S) = 0\) holds. Then we obtain the diagram \(5.30\), and a contradiction.
As for the condition (5.25), the same argument as above applies to the case of \(c_1(\omega_S) = 0\). So we assume that \(c_1(\omega_S) \in \mathbb{R}_{<0} \cdot H\). Suppose that \(E = (F, \theta)\) is \(\sigma^\pm\)-semistable but \(F\) is not \(\sigma^\pm\)-semistable. Then similarly to above, we have an exact sequence (5.31) such that \(\mu_{\sigma^\pm}(F') > \mu_{\sigma^\pm}(F'')\). Since \(E\) is \(\sigma\)-semistable, \(F\) is also \(\sigma\)-semistable by the above argument, therefore \(F'\) and \(F''\) are also \(\sigma\)-semistable with the same \(\mu_{\sigma}\)-slope. Then \(F'' \otimes \omega_S\) is also \(\sigma\)-semistable with
\[
\mu_{\sigma}(F') \geq \mu_{\sigma}(F'') > \mu_{\sigma}(F'' \otimes \omega_S),
\]
hence \(\text{Hom}(F', F'' \otimes \omega_S) = 0\) holds. Similarly to above, we obtain a contradiction. 

A surface \(S\) with \(c_1(\omega_S) = 0\) is classified into four types: K3 surface, abelian surface, Enriques surface and bielliptic surface. In the above case, by Corollary 5.13 and Lemma 5.16 we have the following:

**Corollary 5.17.** Suppose that \(S\) is a smooth projective surface satisfying \(c_1(\omega_S) = 0\) in \(H^2(S, \mathbb{R})\). Then for any primitive \(v \in N_{\leq 1}(S)\) and \(\sigma^{\pm} \in A(S)_C\) which do not lie on walls, there exists an equivalence
\[
D^{b, \text{coh}}(\mathcal{M}^{\sigma^{\pm}, \text{rig}}_S(v)) \sim D^{b, \text{coh}}(\mathcal{M}^{\sigma^{\pm}, \text{rig}}_S(v)).
\]

**Remark 5.18.** In the case that \(S\) is a K3 surface, the classical truncations
\[
M_{S, \sigma^{\pm}} := t_0(\mathcal{M}_{S, \sigma^{\pm}}(v))
\]
are holomorphic symplectic manifolds (Ma17, BM14). A derived equivalence
\[
D^{b, \text{coh}}(M_{S, \sigma^+(v)}) \sim D^{b, \text{coh}}(M_{S, \sigma^-(v)})
\]
is a special case of Halpern-Leistner’s result [HL13]. Since the closed immersion \(M_{S, \sigma^{\pm}}(v) \hookrightarrow \mathcal{M}_{S, \sigma^{\pm}}(v)\) is not an equivalence of derived stacks, the equivalence (5.33) does not directly imply (5.32). This is caused by the existence of surjections \(\text{Ext}^2_S(E, E) \to \mathbb{C}\) for any coherent sheaf \(E\) on \(S\). In order to obtain an equivalence (5.33), we replace \(\mathcal{M}^{\sigma^{\pm}, \text{rig}}_S(v)\) with another derived stack \(\mathcal{M}^{\sigma^{\pm}, \text{rig}}_S(v)\) obtained from \(\mathcal{M}^{\sigma^{\pm}, \text{rig}}_S(v)\) by getting rid of the above one dimensional obstruction space from \(\mathcal{M}^{\sigma^{\pm}, \text{rig}}_S(v)\) as in [HL13, Proposition 4.2]. Then we can recover the Halpern-Leistner’s equivalence (5.33) from the argument of Theorem 7.12.

In the case of Enriques surface, Sacca [Sac19] proved the following:

**Theorem 5.19.** [Sac19] Let \(S\) be a general Enriques surface and \([C]\) be a linear system on it which contains an irreducible divisor \(C \subset S\) with arithmetic genus \(g \geq 2\). We take \(v = ([C], n) \in \text{NS}(S) \oplus \mathbb{Z}\) such that \(n \neq 0\) and \(([C], 2n)\) is coprime. Then for generic \(H^{\pm} \in A(S)_C\), the moduli spaces \(M_{S, H^{\pm}}(v)\) are smooth \((\beta^2 + 1)\)-dimensional birational Calabi-Yau manifolds.

By Corollary 5.17, we have an equivalence of Sacca’s Calabi-Yau manifolds for different polarizations:

**Corollary 5.20.** For a general Enriques surface \(S\), let \(v \in N_{\leq 1}(S)\) be as in Theorem 5.19. Then for generic \(H^{\pm} \in A(S)_C\), there exists an equivalence
\[
D^{b, \text{coh}}(M_{S, H^+(v)}) \sim D^{b, \text{coh}}(M_{S, H^-(v)}).
\]

**Proof.** Let \(H \in A(S)_C\) lies on a wall, and \(H^{\pm}\) lie on its adjacent chambers. It is enough to show an equivalence (5.34) for such \(H^{\pm}\). It is proved in [Sac19, Lemma 2.5] that there is no obstruction space for \(H^{\pm}\)-stable sheaf \(E\) such that \([E] \in M_{S, H^{\pm}}(v)\), i.e. \(\text{Ext}^2(E, E) = 0\). Therefore for \(\sigma^{\pm} = \sqrt{-1}H^{\pm}\), the closed immersions
\[
M_{S, H^{\pm}}(v) = t_0(\mathcal{M}^{\sigma^{\pm}, \text{rig}}_S(v)) \hookrightarrow \mathcal{M}^{\sigma^{\pm}, \text{rig}}_S(v)
\]
are equivalences. Therefore the Corollary follows from Corollary 5.17. 

Let \(S\) be a del-Pezzo surface, i.e. \(-K_S\) is ample. In this case, we have the following:
Corollary 5.21. Let $S$ be a del-Pezzo surface. Then for any primitive $v \in N_{\leq 1}(S)$ and generic perturbations $\sigma^{\pm}$ of $-\sqrt{-1}K_S$, we have a derived equivalence of smooth projective varieties

$$D^{b}_{\text{coh}}(M_{S,\sigma^{+}}(v)) \cong D^{b}_{\text{coh}}(M_{S,\sigma^{-}}(v)).$$

Proof. The moduli stack $M_{S,\sigma}(v)$ is a smooth stack without obstruction space, i.e. for any Gieseker $-K_S$-semistable sheaf $E$ on $S$ with $\text{ch}(E) = v$, we have

$$\text{Ext}^2(E, E) = \text{Hom}(E, E \otimes \omega_S)^\vee = 0.$$  

Therefore for generic perturbations $\sigma^{\pm}$ of $-K_S$, the moduli spaces $M_{S,\sigma^{\pm}}(v)$ are smooth projective varieties such that the closed immersions

$$M_{S,\sigma^{\pm}}(v) = t_0(\mathfrak{M}_{S,\sigma^{\pm}}^*(v)) \hookrightarrow \mathfrak{M}_{S,\sigma^{\pm}}^*(v)$$

are equivalences. Therefore the corollary follows from Corollary 5.13 and Lemma 5.16.

As we observed in Corollary 5.13, if we assume both of (5.20) and (5.25) then the equivalence in Theorem 5.12 can be reduced to an equivalence of derived moduli spaces of stable sheaves on $S$. On the other hand, there exist examples where (5.20) holds but (5.25) does not hold, so that the possibility for such $(\beta, \sigma_i, \eta)$ is ($\beta_1, \eta_1$). Then we have

$$\beta_1 + \beta_2 = 2C + E_1 + E_2, \quad n_1 + n_2 = 2.$$  

Since $\mu_\sigma(F_1) > 0$, we have $n_1 > 0$, therefore $n_1 = n_2 = 1$. Then we have $H \cdot \beta_1 = H \cdot \beta_2$. The only possibility for such $(\beta_1, \beta_2)$ is $(C + E_1, C + E_2)$ or $(C + E_2, C + E_1)$. Because $H^0(\mathcal{O}_S(C + E_1))$ is one dimensional, $C + E_i$ is the unique element of the linear system $|C + E_i|$, so $C + E_i$ is a reduced class. Therefore $\pi_*F_1$ is $\sigma$-semistable by Lemma 5.13 so $\pi_*E$ is strictly $\sigma$-semistable.

We next show that (5.25) is not satisfied. Let us consider exact sequences

$$0 \to \mathcal{O}_{E_i}(-1) \to \mathcal{O}_{C + E_i} \to \mathcal{O}_C \to 0, \quad 0 \to \mathcal{O}_C(-1) \to \mathcal{O}_{C + E_i} \to \mathcal{O}_{E_i} \to 0.$$  

Since we have

$$\text{Hom}(\mathcal{O}_C, \mathcal{O}_C(-1) \otimes \omega_S) = \text{Hom}(\mathcal{O}_C, \mathcal{O}_C(k - 4)) \neq 0$$

we have the non-vanishing

$$\text{Hom}(\mathcal{O}_{C + E_1}, \mathcal{O}_{C + E_2} \otimes \omega_S) \neq 0, \quad \text{Hom}(\mathcal{O}_{C + E_2}, \mathcal{O}_{C + E_1} \otimes \omega_S) \neq 0.$$  

Let $\theta$ be a morphism

$$\theta : \mathcal{O}_{C + E_1} \oplus \mathcal{O}_{C + E_2} \to (\mathcal{O}_{C + E_1} \oplus \mathcal{O}_{C + E_2}) \otimes \omega_S$$
whose \( \text{Hom}(\mathcal{O}_{C+E_1}, \mathcal{O}_{C+E_2} \otimes \omega_S) \)-factor is non-zero and the other factors are zero. Then the above pair \( (\mathcal{O}_{C+E_1} \oplus \mathcal{O}_{C+E_2}, \theta) \) determines an object \( E \in \text{Coh}_{\leq 1}(X) \). We have the non-split exact sequence
\[
0 \to i_* \mathcal{O}_{C+E_2} \to E \to i_* \mathcal{O}_{C+E_1} \to 0.
\]
Here \( i: S \hookrightarrow X \) is the zero section of \( \pi: X \to S \). Note that \( \mathcal{O}_{C+E_1} \) are \( \sigma \)-stable with \( \mu_{\sigma}(\mathcal{O}_{C+E_1}) = \mu_{\sigma}(\mathcal{O}_{C+E_2}) < \mu_{\sigma+}(\mathcal{O}_{C+E_1}) \). Together with the above exact sequence, we can easily check that \( E \) is \( \sigma^+ \)-stable. However \( \pi_* \mathcal{E} = \mathcal{O}_{C+E_1} \oplus \mathcal{O}_{C+E_2} \) is not \( \sigma^+ \)-semistable, so (5.25) is not satisfied for \( \sigma^- \)-stability. The case for \( \sigma^- \)-stability is similar by replacing \( E_1 \) with \( E_2 \). \( \square \)

6. Categorical MNOP/PT correspondence

In this section, we prove Theorem 1.8 as an application of Corollary 3.12.

6.1. Derived moduli stacks of pairs. For a smooth projective surface \( S \), let \( \mathcal{M}_S \) be the derived moduli stack of coherent sheaves on \( S \) considered in (6.1), and \( \mathfrak{g} \) the universal object (5.2). The derived stack \( \mathcal{M}_S^\dagger \) is defined to be
\[
\rho^\dagger: \mathcal{M}^\dagger_S := \text{Spec}_{\mathcal{M}_S}(\text{Sym}(p_{2M} \mathfrak{g})^\vee) \to \mathcal{M}_S.
\]
Here \( p_{2M}: S \times \mathcal{M}_S \to \mathcal{M}_S \) is the projection. For \( T \in \text{dAff} \), the \( T \)-valued points of \( \mathcal{M}_S^\dagger \) form the \( \infty \)-groupoid of pairs
\[
(F_T, \xi): \mathcal{O}_{S \times T} \to F_T
\]
where \( F_T \) is a \( T \)-valued point of \( \mathcal{M}_S \).

The classical truncation of \( \mathcal{M}_S^\dagger \) is a 1-stack
\[
\mathcal{M}_S := \mathfrak{t}_0(\mathcal{M}_S^\dagger) = \text{Spec}_{\mathcal{M}_S}(\text{Sym}(\mathcal{H}^0((p_{2M} \mathcal{F})^\vee))).
\]
We have the universal pair on \( S \times \mathcal{M}_S^\dagger \)
\[
\mathcal{I}^\bullet = (\mathcal{O}_{S \times \mathcal{M}_S^\dagger} \to \mathcal{F}).
\]
Then we have the following description of the cotangent complex of \( \mathcal{M}_S^\dagger \)
\[
\mathcal{L}_{\mathcal{M}_S^\dagger}|_{\mathcal{M}_S} = (\text{Hom}_{\mathcal{M}_S^\dagger}(\mathcal{I}^\bullet, \mathcal{F}))^\vee.
\]
Here \( p_{\mathcal{M}^\dagger}: S \times \mathcal{M}_S^\dagger \to \mathcal{M}_S^\dagger \) is the projection. Also we have the decompositions into open and closed substacks
\[
\mathcal{M}_S^\dagger = \coprod_{v \in \mathcal{N}_{\leq 1}(S)} \mathcal{M}_S^\dagger(v), \quad \mathcal{M}_S = \coprod_{v \in \mathcal{N}_{\leq 1}(S)} \mathcal{M}_S(v),
\]
where each component corresponds to pairs \( (F, \xi) \) such that \( \text{ch}(F) = v \).

For \( v \in \mathcal{N}_{\leq 1}(S) \), the derived stack \( \mathcal{M}_S^\dagger(v) \) is quasi-smooth (see [Loddb Lemma 6.1]). We have the \((-1)\)-shifted cotangent stack, and its classical truncation
\[
\Omega_{\mathcal{M}_S^\dagger(v)}[-1] \to \mathcal{M}_S^\dagger(v), \quad t_0(\Omega_{\mathcal{M}_S^\dagger(v)}[-1]) \to \mathcal{M}_S(v).
\]

6.2. MNOP/PT categories. Let \( \overline{X} \) be the projective compactification of \( X \)
\[
X \subset \overline{X} := \mathbb{P}(\omega_S \oplus \mathcal{O}_S) = X \cup S_{\infty}.
\]
Here \( S_{\infty} \) is the divisor at the infinity. The category of D0-D2-D6 bound states on the non-compact CY 3-fold \( X = \text{Tot}_S(\omega_S) \) is defined to be the extension closure in \( D^{\text{coh}}(\overline{X}) \)
\[
\mathcal{A}_{X} := (\mathcal{O}_{\overline{X}}, \text{Coh}_{\leq 1}(X)[-1])_{\text{ex}} \subset D^{\text{coh}}(\overline{X}).
\]
Here \( \text{Coh}_{\leq 1}(X) \) is the abelian category of compactly supported coherent sheaves on \( X \) whose supports have dimensions less than or equal to one. We regard \( \text{Coh}_{\leq 1}(X) \) as a subcategory of \( \text{Coh}(\overline{X}) \) by the push-forward of the open immersion \( X \subset \overline{X} \). There is a group homomorphism
\[
\text{ch}: K(\mathcal{A}_{X}) \to \mathbb{Z} \oplus \mathcal{N}_{\leq 1}(S)
\]
characterized by the condition that \( \text{ch}(\mathcal{O}_X) = (1, 0) \) and \( \text{ch}(F) = (0, \text{ch}(\pi_* F)) \) for \( F \in \text{Coh}_{\leq 1}(X) \).

The following result is proved in [Todb]:

**Theorem 6.1.** ([Todb] Theorem 6.3) Let \( \mathcal{M}_X^\dagger(v) \) be the classical moduli stack of objects \( \mathcal{E} \in \mathcal{A}_X \) with \( \text{ch}(\mathcal{E}) = (1, -v) \) together with a trivialization \( \mathcal{E} \otimes \mathcal{O}_{S_{\infty}} \cong \mathcal{O}_{S_{\infty}} \). Then there is a natural isomorphism of stacks over \( \mathcal{M}_S^\dagger(v) \)

\[
\mathcal{M}_X^\dagger(v) \xrightarrow{\cong} [0](\mathcal{O}_{2\mathcal{M}_S^\dagger(v)}[-1]).
\]

Let \( \pi_* \) be the morphism

\[
\pi_* : \mathcal{M}_X^\dagger(v) \to \mathcal{M}_S^\dagger(v)
\]

given by the composition of the isomorphism (6.4) and the projection \( [0](\mathcal{O}_{2\mathcal{M}_S^\dagger(v)}[-1]) \to \mathcal{M}_S^\dagger(v) \).

Note that a two term complex \((\mathcal{O}_{X} \to E)\) for \( E \in \text{Coh}_{\leq 1}(X) \) is an object in \( \mathcal{A}_X \), and \( \pi_* \) sends the above complex to \((\mathcal{O}_S \to \pi_* E)\) given by the adjunction (see [Todb] Remark 6.4).

For \( v = (\beta, n) \in N_{\leq 1}(S) \), we have the MNOP moduli space \( \text{MNOP}^{06} \)

\[
I_n(X, \beta)
\]

which is a quasi-projective scheme parameterizing ideal sheaves \( I_C \) for a compactly supported subschemes \( C \subset X \) with \( \dim C \leq 1 \), \( \pi_*[C] = \beta \) and \( \chi(\mathcal{O}_C) = n \). The moduli space \( I_n(X, \beta) \) is an open substack of \( \mathcal{M}_X^\dagger(\beta, n) \) by regarding it as the moduli space of surjections \((\mathcal{O}_{X} \to \mathcal{O}_C)\) with \( \text{ch}(\pi_* \mathcal{O}_C) = (\beta, n) \).

We also have the Pandharipande-Thomas moduli space \( \text{PT}^{09} \)

\[
P_n(X, \beta)
\]

which is a quasi-projective scheme parameterizing pairs \((E, \xi)\) where \( E \in \text{Coh}_{\leq 1}(X) \) is pure one dimensional and \( \xi : \mathcal{O}_X \to E \) is surjective in dimension one. The moduli space \( P_n(X, \beta) \) is also an open substack of \( \mathcal{M}_X^\dagger(\beta, n) \) by regarding it as the moduli space of two term complexes \((\mathcal{O}_{X} \to E)\) with \( \text{ch}(\pi_* E) = (\beta, n) \).

As we mentioned above, both of moduli spaces \( I_n(X, \beta), P_n(X, \beta) \) are open substacks of \( \mathcal{M}_X^\dagger(\beta, n) \), so we have the complements

\[
Z_{I-\text{us}} := \mathcal{M}_X^\dagger(\beta, n) \setminus I_n(X, \beta), \quad Z_{P-\text{us}} := \mathcal{M}_X^\dagger(\beta, n) \setminus P_n(X, \beta)
\]

which are conical closed substacks in \( \mathcal{M}_X^\dagger(\beta, n) \). Also there exists a derived open substack of finite type

\[
\mathcal{M}_S^\dagger(\beta, n)_{\text{fin}} \subset \mathcal{M}_S^\dagger(\beta, n)
\]

which contains both of \( \pi_* I_n(X, \beta) \) and \( \pi_* P_n(X, \beta) \). The \( \mathbb{C}^* \)-equivariant MNOP/PT categories in our version are defined as follows:

**Definition 6.2.** We define the triangulated categories \( \mathcal{D}^\mathbb{C}^* \text{-equivariant MNOP/PT categories}\)

\[
\mathcal{D}^\mathbb{C}^* \left( I_n(X, \beta) \right) := \lim_{\mathcal{U} \vdash \mathcal{M}_S^\dagger(\beta, n)_{\text{fin}}} \left( D^b_{\mathcal{Coh}}(\mathcal{U})/\mathcal{C}_{\alpha^* Z_{I-\text{us}}} \right),
\]

\[
\mathcal{D}^\mathbb{C}^* \left( P_n(X, \beta) \right) := \lim_{\mathcal{U} \vdash \mathcal{M}_S^\dagger(\beta, n)_{\text{fin}}} \left( D^b_{\mathcal{Coh}}(\mathcal{U})/\mathcal{C}_{\alpha^* Z_{P-\text{us}}} \right).
\]

Here \( \alpha \) is a smooth morphism from an affine derived scheme \( \mathcal{U} \) of the form [3.2].

**Remark 6.3.** Similarly to Remark 5.11, the \( \mathbb{C}^* \)-equivariant MNOP/PT categories in Definition 6.2 are independent of a choice of \([\mathcal{E}, \beta]\) up to equivalence.
6.3. The moduli stack of semistable objects on MNOP/PT wall. We define the open substacks
\[(6.6) \quad \mathcal{M}^I_S(\beta, n) \subset \mathcal{M}^I_X(\beta, n), \quad \mathcal{M}^I_P(\beta, n) \subset \mathcal{M}^I_X(\beta, n)\]
corresponding to pairs \((\mathcal{O}_S \xrightarrow{\xi} \mathcal{E})\) such that \(\text{Cok}(\xi)\) is at most zero dimensional, objects \(\mathcal{E} \in \mathcal{A}_X\) such that \(\mathcal{H}^1(\mathcal{E})\) is at most zero dimensional, respectively. We have the following lemma:

**Lemma 6.4.** (i) The stack \(\mathcal{M}^I_X(\beta, n)\) is the moduli stack of pairs \((\mathcal{O}_X \xrightarrow{\xi} \mathcal{E})\) such that \(\text{Cok}(\xi)\) is at most zero dimensional.

(ii) We have the inclusion
\[(6.7) \quad \pi^*_i(\mathcal{M}^I_S(\beta, n)) \subset \mathcal{M}^I_X(\beta, n).\]

**Proof.** (i) follows from [Tod10a, Lemma 3.11 (ii)]. As for (ii), let \(\mathcal{E} \in \mathcal{A}_X\) be an object corresponding to a point in the left hand side of \((6.1)\), i.e. \(\pi_* \mathcal{E}\) is a pair \((\mathcal{O}_S \xrightarrow{\xi} \mathcal{E})\) such that \(\text{Cok}(\xi)\) is at most zero dimensional. By pushing forward the surjection \(\mathcal{E} \to \mathcal{H}^1(\mathcal{E})[-1]\) in \(\mathcal{A}_X\) to \(\mathcal{S}\), we obtain the surjection \(\text{Cok}(\xi) \to \pi_* \mathcal{H}^1(\mathcal{E})\). Therefore \(\pi_* \mathcal{H}^1(\mathcal{E})\) is at most zero dimensional, so \(\mathcal{H}^1(\mathcal{E})\) is also at most zero dimensional.

Note that we have the open immersions
\[I_n(X, \beta) \subset \mathcal{M}^I_X(\beta, n) \supset P_n(X, \beta),\]
and the stack \(\mathcal{M}^I_S(\beta, n)\) is the moduli stack of semistable objects on the MNOP/PT wall [Tod10a, Lemma 3.11]. By the GIT construction of the above wall-crossing [ST11], the open substacks \((6.6)\) are of finite type with good moduli spaces
\[(6.8) \quad \pi_{\mathcal{M}^I_X} : \mathcal{M}^I_X(\beta, n) \to \mathcal{M}^I_S(\beta, n), \quad \pi_{\mathcal{M}^I_P} : \mathcal{M}^I_P(\beta, n) \to \mathcal{M}^I_S(\beta, n).\]

We then define some line bundles on the above moduli stacks.

**Definition 6.5.** We define \(l_I, l_P \in \text{Pic}(\mathcal{M}^I_S)\) to be
\[l_I := \text{det}(R_{p_{\mathcal{M}^I_P}}^* \mathcal{F}), \quad l_P := \text{det}(R_{p_{\mathcal{M}^I_P}}^* \mathcal{F})^{-1}.\]
Their restrictions to \(\mathcal{M}^I_S(\beta, n)\) and pull-backs via \(\pi_* : \mathcal{M}^I_X(\beta, n) \to \mathcal{M}^I_S\) are also denoted by \(l_I, l_P\).

We have the following proposition:

**Proposition 6.6.** For \(l_I, l_P \in \text{Pic}(\mathcal{M}^I_X(\beta, n))\), we have the identities
\[(6.9) \quad I_n(X, \beta) = \mathcal{M}^I_X(\beta, n)^{\text{ss}}(l_I), \quad P_n(X, \beta) = \mathcal{M}^I_X(\beta, n)^{\text{ss}}(l_P).\]

**Proof.** It is enough to show the identities \((6.9)\) on the fibers of good moduli space morphisms \(\pi_{\mathcal{M}^I_S} : \mathcal{M}^I_X(\beta, n) \to \mathcal{M}^I_S(\beta, n)\). A closed point \(y \in \mathcal{M}^I_X(\beta, n)\) corresponds to a polystable object on MNOP/PT wall (see [Toda, Appendix B])
\[(6.10) \quad I_C \oplus \bigoplus_{i=1}^m V_i \otimes \mathcal{O}_{p_i}[-1].\]

Here \(I_C \subset \mathcal{O}_X\) is the ideal sheaf of a Cohen-Macaulay curve \(C \subset X, p_1, \cdots, p_m \in X\) are distinct points and \(V_i\) is a finite dimensional vector space. Let \(Q\) be the Ext-quiver associated with the collection \(\{I_C, \mathcal{O}_{p_1}[-1], \ldots, \mathcal{O}_{p_m}[-1]\}\). Then the vertex set \(V(Q)\) is \(\{0, 1, \ldots, m\}\), and the edge set \(E(Q)\) is given by
\[\sharp(0 \to i) = \text{hom}(I_C, \mathcal{O}_{p_i}), \quad \sharp(i \to 0) = \text{ext}^2(\mathcal{O}_{p_i}, I_C), \quad \sharp(0 \to 0) = \text{ext}^1(I_C, I_C)\]
and also for \(1 \leq i, j \leq m\)
\[\sharp(i \to j) = \begin{cases} 3, & (i = j) \\ 0, & (i \neq j). \end{cases}\]
Similarly to the argument in Subsection 6.2 we have the closed immersion

\[(6.11) \quad \pi_{\mathcal{M}^1}(y) \hookrightarrow \left\{ (V_i)^{(0 \rightarrow i)} \oplus (V_i^{'})^{(i \rightarrow 0)} \oplus \bigoplus_{i=1}^{m} \text{End}(V_i)^{\otimes 3} \oplus \mathbb{C}^{(0 \rightarrow 0)} \right\} / G. \]

Here the RHS is the moduli stack of nilpotent $Q$-representations with dimension vector $(1, \{\dim V_i\}_{1 \leq i \leq k})$. The algebraic group $G = \prod_{i=1}^{m} \text{GL}(V_i)$ acts on $V_i$, $V_i'$ in a standard way, on $\text{End}(V_i)$ by the conjugation and on $\mathbb{C}^{(0 \rightarrow 0)}$ trivially.

Let $x \in \mathcal{M}_X^0(\beta, n)$ be a closed point corresponding to the object (6.10). Then we have $G = \text{Aut}(x)$ and

\[(6.12) \quad (l_I)_x = \det R\Gamma \left( \mathcal{O}_C \oplus \bigoplus_{i=1}^{m} V_i \otimes \mathcal{O}_{p_i} \right) \cong \bigotimes_{i=1}^{m} \det V_i\]

as an element of $\text{Pic}(BG)$, i.e. they are the determinant character

\[\chi_0: G \to \mathbb{C}^*, (g_i)_{1 \leq i \leq m} \mapsto \prod_{i=1}^{m} \det(g_i).\]

Let $e_i$ for $1 \leq i \leq m$ be the simple $Q$-representation corresponding to the vertex $i$, and $Q^0 \subset Q$ be the full subquiver whose vertex set is $\{1, \ldots, m\}$. By [Toda, Lemma 7.10] and the proof of [KT, Lemma 3.4], a $Q$-representation $R$ with dimension vector $(1, \{\dim V_i\}_{1 \leq i \leq m})$ is $\chi_0$-(semi)stable if and only if the images of $\mathbb{C} \to V_i$ for $(0 \to i)$ generate $\oplus_{i=1}^{m} V_i$ as a $\mathbb{C}[Q^0]$-module. This is equivalent to that $\text{Hom}(R, e_i) = 0$ for all $1 \leq i \leq m$. Under the embedding (6.11), such a representation $R$ corresponds to an object in the extension closure

\[(6.13) \quad \mathcal{E} \in \langle I_C, \mathcal{O}_{p_1}[-1], \ldots, \mathcal{O}_{p_m}[-1]\rangle_{\text{ex}} \]

such that $\text{Hom}(\mathcal{E}, \mathcal{O}_{p_i}[-1]) = 0$. By Lemma 6.1 (i), $\mathcal{E}$ is represented by a pair $(\mathcal{O}_\mathbb{C} \xrightarrow{\xi} \mathcal{E})$ for a generically surjective $\xi$, and the above condition is equivalent to that $\xi$ is surjective. Therefore the identity for $I_n(X, \beta)$ holds on $\pi^{-1}_{\mathcal{M}_X^0}(y)$.

The identity for $P_n(X, \beta)$ is similar. By [Toda, Lemma 7.10] and the proof of [KT, Lemma 3.4], a $Q$-representation $R$ with dimension vector $(1, \{\dim V_i\}_{1 \leq i \leq m})$ is $\chi_0$-(semi)stable if and only if the images of duals of $V_i \to \mathbb{C}$ for $(i \to 0)$ generate $\oplus_{i=1}^{m} V_i^\prime$ as a $\mathbb{C}[Q^0]$-module, which is equivalent to that $\text{Hom}(e_i, R) = 0$ for all $1 \leq i \leq m$. Such a representation corresponds to an object (6.13) such that $\text{Hom}(\mathcal{O}_{p_i}[-1], \mathcal{E}) = 0$. As $\mathcal{E}$ is represented by a pair $(\mathcal{O}_\mathbb{C} \xrightarrow{\xi} \mathcal{E})$ for a generically surjective $\xi$, the above condition is equivalent to that $\mathcal{E}$ is pure. Therefore the identity for $P_n(X, \beta)$ holds on $\pi^{-1}_{\mathcal{M}_X^0}(y)$. \hfill \Box

We have the derived open substack

\[(6.14) \quad \mathcal{M}^0_S(\beta, n) \subset \mathcal{M}^1_S(\beta, n)\]

whose classical truncation is $\mathcal{M}^{0}_{S}(\beta, n)$. In the following proposition, we show the existence of a symmetric structure on $\mathcal{M}^{0}_{S}(\beta, n)$ such that $l_P$ is compatible with it.

**Proposition 6.7.** There exists a symmetric structure $\mathcal{S}$ on $\mathcal{M}^0_{S}(\beta, n)$ such that $l_I, l_P$ are $\mathcal{S}$-generic and $l_P$ is compatible with $\mathcal{S}$.

**Proof.** A closed point $x$ of the stack $\mathcal{M}^0_S(\beta, n)$ corresponds to a direct sum of pairs

\[(\mathcal{O}_S \to F) = (\mathcal{O}_S \to \mathcal{O}_C) \oplus \bigoplus_{i=1}^{m} V_i \otimes (0 \to \mathcal{O}_{p_i}).\]
Here $C \subset S$ is a Cohen-Macaulay curve, $p_1, \ldots, p_m \in S$ are distinct points and $V_i$ is a finite dimensional vector space. Then $\text{Aut}(x) = \prod_{i=1}^{m} \text{GL}(V_i)$. From the description of the cotangent complex $\mathcal{O}_S$, we have

$$T^{m}_S(\beta,n)|_x = R\text{Hom}\left(\mathcal{O}_S(-C) \oplus \bigoplus_{i=1}^{m} (V_i \otimes \mathcal{O}_{p_i})[-1], \bigoplus_{i=1}^{m} V_i \otimes \mathcal{O}_{p_i}\right).$$

By the above, an easy computation shows that

$$(6.15) \quad H^0(T^{m}_S(\beta,n)|_x) \oplus H^1(T^{m}_S(\beta,n)|_x)^\vee$$

$$= \left( \bigoplus_{i=1}^{m} \text{Ext}^2_S(\mathcal{O}_{p_i}, \mathcal{O}_C)^\vee \oplus H^0(\mathcal{O}_S|_{p_i}) \right) \otimes V_i \oplus \left( \bigoplus_{i=1}^{m} \text{Ext}^1_S(\mathcal{O}_{p_i}, \mathcal{O}_C) \right) \otimes V_i^\vee \oplus \bigoplus_{i=1}^{m} \text{End}(V_i)^{\oplus 3} \oplus H^0(\mathcal{O}_C|_{p_i}) \oplus H^1(\mathcal{O}_C|_{p_i})^\vee.$$

We take the decomposition $S_x \oplus U_x$ of $(6.15)$ to be

$$S_x = \left( \bigoplus_{i=1}^{m} \text{Ext}^2_S(\mathcal{O}_{p_i}, \mathcal{O}_C)^\vee \right) \otimes V_i \oplus \left( \bigoplus_{i=1}^{m} \text{Ext}^1_S(\mathcal{O}_{p_i}, \mathcal{O}_C) \right) \otimes V_i^\vee \oplus \bigoplus_{i=1}^{m} H^0(\mathcal{O}_S|_{p_i}) \otimes V_i,$$

$$U_x = \bigoplus_{i=1}^{m} H^0(\mathcal{O}_S|_{p_i}) \otimes V_i.$$

As $C$ is Cohen-Macaulay we have $\text{Hom}(\mathcal{O}_{p_i}, \mathcal{O}_C) = 0$, so the Riemann-Roch theorem shows that $\text{ext}^2_S(\mathcal{O}_{p_i}, \mathcal{O}_C) = \text{ext}^2_S(\mathcal{O}_{p_i}, \mathcal{O}_C)$. Therefore $S_x$ is a symmetric $\text{Aut}(x)$-representation, and the decomposition $S_x \oplus U_x$ of $(6.15)$ gives its symmetric structure.

Similarly to Remark 5.3, the $\text{Aut}(x)$-representation $(6.15)$ is obtained as the space of $Q$-representations with dimension vector $\{1, \{\text{dim } V_i\}_{1 \leq i \leq m}\}$, where $Q$ is the $\text{Ext}$-quiver associated with the collection

$$\{ I_{i(C)}, i_* \mathcal{O}_{p_i}[-1], \ldots, i_* \mathcal{O}_{p_m}[-1] \}.$$

Here $i : S \hookrightarrow X$ is the zero section. Therefore from the proof of Proposition 5.6 the $(l_I)_x$-stability on $(6.15)$ is given by the determinant character $\chi_0$ in $(6.12)$, and $(l_P)_x$ is given by $\chi_0^{-1}$. These characters are $S_x$-generic by [KT] Lemma 3.3. Therefore $l_I, l_P$ are $S$-generic.

The proof that $l_P$ is compatible with $S$ is the same as [KT] Lemma 3.4. By the proof of Proposition 6.6 the $(l_P)_x$-stability on $(6.15)$ imposes constraints only on $V_i^\vee$ and $\text{End}(V_i)$-factors, and does not impose any constraint on $V_i$-factors. The same also applies to the $(l_P)_x$-stability on $S_x$. Since $S_x$ is obtained from $(6.15)$ by extracting some of $V_i$-factors, the condition

$$\left( H^0(T^{m}_S(\beta,n)|_x) \oplus H^1(T^{m}_S(\beta,n)|_x)^\vee \right)^{ss}((l_P)_x) = (S_x)^{ss}((l_P)_x) \oplus U_x$$

is satisfied. Since this holds for any $x$, we conclude that $l_P$ is compatible with $S$. \hfill \Box

6.4. **Proof of categorical MNOP/PT correspondence.** Suppose that $\beta$ is a reduced class. Then we have the following:

**Lemma 6.8.** If $\beta$ is a reduced class, then the inclusion $(6.7)$ is the identity.

**Proof.** Suppose that $\beta$ is reduced and let $(\mathcal{O}_X, \xi : E)$ corresponds to a point in the right hand side of $(6.7)$, i.e. $\text{Cok}(\xi)$ is at most zero dimensional. Then $\xi$ is non-zero at each generic point of the support of $E$. Therefore $\pi_* \xi : \mathcal{O}_S \to \pi_* E$ is non-zero at each generic point of the support of $\pi_* E$. Since the fundamental one cycle of $\pi_* E$ is reduced, this implies that $\text{Cok}(\pi_* \xi)$ is at most zero dimensional. \hfill \Box
The following is the main result in this section.

**Theorem 6.9.** Suppose that $\beta$ is a reduced class. Then there exists a fully-faithful functor

$$\overline{\mathcal{D}T}^G(P_n(X, \beta)) \hookrightarrow \overline{\mathcal{D}T}^G(I_n(X, \beta)).$$

**Proof.** By Lemma 6.8 and Remark 6.3, we can take a finite type derived open substack $(\mathcal{U})$ to be

$$\mathcal{M}^1_S(\beta, n)_{\text{fin}} = \mathcal{M}^2_S(\beta, n).$$

As we mentioned before $\mathcal{M}^1_S(\beta, n)$ admits a good moduli space, and $\mathcal{M}^2_S(\beta, n)$ satisfies formal neighborhood theorem by a similar argument in Appendix B applied for the abelian category of pairs $(O_S^k \to F)$ for $k \in \mathbb{Z}_{\geq 0}$ and $F \in \text{Coh}_{\leq 1}(S)$. Then the theorem is a consequence of Corollary 6.12 Proposition 6.6 and Proposition 6.7.

**Remark 6.10.** For a stable pair $(O_X \to E)$ on $X$, if $E$ has a reduced support then it push-forwards to a stable pair $(O_S \to \pi_*E)$ on $S$. Therefore for a reduced class $\beta$, we have an equivalence by Corollary 6.4

$$D^b_{\text{coh}}(\mathfrak{M}_n(S, \beta)) \sim \overline{\mathcal{D}T}^G(P_n(X, \beta)),$$

where $\mathfrak{M}_n(S, \beta)$ is the derived moduli space of stable pairs on $S$. On the other hand, a surjection $O_X \to E$ does not push-forward to a surjection $O_S \to \pi_*E$ in general, even if the support of $E$ is reduced. So for a reduced class $\beta$, the $G^*$-equivariant MNOP category $\overline{\mathcal{D}T}^G(I_n(X, \beta))$ is not necessary equivalent to $D^b_{\text{coh}}(\mathfrak{M}_n(S, \beta))$, where $\mathfrak{M}_n(S, \beta)$ is the derived moduli space of closed subschemes in $S$.

**Appendix A. Some auxiliary results in derived algebraic geometry**

In this section, we prove some auxiliary results in derived algebraic geometry which was used in Section 4.

**A.1. Equivariant affine derived schemes.** Let $G$ be a reductive algebraic group. We denote by $\text{cdga}^G$ the $\infty$-category of $G$-equivariant cdga’s consisting of non-positive degrees. Here a $G$-equivariant cdga is a cdga which admits a $G$-action and satisfies obvious compatibility with differentials and products (in a usual strict sense, not in a homotopy sense). The $\infty$-category of $G$-equivariant affine derived schemes is defined by

$$d\text{Aff}^G := W^{-1}(\text{cdga}^G)^{\text{op}}.$$

Here $W^{-1}(-)$ means $\infty$-categorical localization by weak homotopy equivalences (cf. [Toë14, Section 2.2]). Let $\text{dSt}/BG$ be the $\infty$-category of derived stacks over $BG$. We have the natural $\infty$-functor

$$\Pi: d\text{Aff}^G \to \text{dSt}/BG, \, \mathfrak{U} \mapsto [\mathfrak{U}/G].$$

We use the following two propositions:

**Proposition A.1.** Let $\mathfrak{M}$ be a quasi-smooth derived stack over $BG$ such that $t_0(\mathfrak{M}) = [\mathfrak{U}/G]$ for an affine scheme $\mathfrak{U}$ with $G$-action. Then there exists a $G$-equivariant tuple $(Y, V, s)$ as in Example 3.1 and an equivalence $\mathfrak{M} \sim \Pi(\mathfrak{U})$ where $\mathfrak{U} = \text{Spec} R(V \to Y, s)$ is an object in $d\text{Aff}^G$.

**Proof.** Note that we have a sequence of square-zero extensions

$$\mathcal{M} = t_0(\mathfrak{M}) \hookrightarrow t_{\leq 1}(\mathfrak{M}) \hookrightarrow t_{\leq 2}(\mathfrak{M}) \hookrightarrow \cdots$$

such that $t_{\leq n}(\mathfrak{M}) \to \mathfrak{M}$ is an equivalence for $n \gg 0$. Let us take distinguished triangles

$$\mathcal{I}_n \to O_{t_{\leq n+1}(\mathfrak{M})} \to O_{t_{\leq n}(\mathfrak{M})}.$$

Then the square zero extension $t_{\leq n}(\mathfrak{M}) \to t_{\leq n+1}(\mathfrak{M})$ in $\text{dSt}/BG$ corresponds to an element in (see [GRT1] Section 1)

$$\text{Hom}_{t_{\leq n}(\mathfrak{M})}(\mathbb{I}_{t_{\leq n}(\mathfrak{M})/BG}, \mathcal{I}_n[1]).$$
Suppose that \( t_{\leq n}(\mathcal{M}) \) is equivalent to \( \Pi(\mathcal{U}_{\leq n}) \) for some \( \mathcal{U}_{\leq n} \in d\text{Aff}^G \). Then as \( G \) is reductive, the Hom space in \( \mathcal{A}_1 \) is isomorphic to \( \text{Hom}_{U_{\leq n}}(L_{U_{\leq n}}(\mathcal{I}_n[1]))^G \). An element of the above corresponds to a square zero extension \( \mathcal{U}_{\leq n} \to \mathcal{U}_{\leq n+1} \) in \( d\text{Aff}^G \). Therefore \( t_{\leq n+1}(\mathcal{M}) \) is equivalent to \( \Pi(\mathcal{U}_{\leq n+1}) \) for some \( \mathcal{U}_{\leq n+1} \in d\text{Aff}^G \). By the induction, we see that \( \mathcal{M} \) is equivalent to \( \mathcal{M}/G \) for some \( \mathcal{M} \in d\text{Aff}^G \).

We are left to prove that \( \mathcal{U} \) is equivalent in \( d\text{Aff}^G \) to an affine derived scheme \( \text{Spec} \mathcal{R}(V \to Y, s) \) associated with a \( G \)-equivariant tuple \( (Y, V, s) \) as in Example \( 5.7 \). We prove this by following an argument of [BBJ19, Theorem 4.1]. Similarly to loc. cit., below we will not fix a particular model for the \( G \)-equivariant cdga \( \mathcal{O}_{\mathcal{U}} \), and regard \( \mathcal{U} \) as an object in the \( \infty \)-category \( d\text{Aff}^G \). Also any map \( \mathcal{U} \to \mathcal{U}' \) is regarded as a morphism in the \( \infty \)-category \( d\text{Aff}^G \).

As \( G \) is reductive, we can find a \( G \)-equivariant closed embedding \( \mathcal{U} \to Y \) for some smooth affine scheme \( Y \) with a \( G \)-action, and let \( I \subset \mathcal{O}_Y \) be the ideal sheaf which defines \( \mathcal{U} \). As \( \mathcal{U} \) is quasi-smooth, the natural morphism of cotangent complexes \( \phi: L_{\mathcal{U}/Y} \to \tau_{\geq -1} L_{\mathcal{U}} \) is a \( G \)-equivariant perfect obstruction theory on \( \mathcal{U} \) (see [BF97]), i.e. \( \phi \) is a morphism in the derived category of \( G \)-equivariant coherent sheaves on \( \mathcal{U} \), \( \mathcal{H}^0(\phi) \) is an isomorphism and \( \mathcal{H}^{-1}(\phi) \) is a surjection. Since \( \mathcal{U} \) and \( Y \) are affine, the morphism \( \phi \) is represented by a morphism of complexes of \( G \)-equivariant sheaves on \( \mathcal{U} \).

\[
\begin{array}{ccc}
V^\vee|_U & \longrightarrow & \Omega_Y|_U \\
\downarrow & & \downarrow \\
I/I^2 & \longrightarrow & \Omega_Y|_U.
\end{array}
\]

Here \( V \to Y \) is a \( G \)-equivariant vector bundle, and the left arrow \( V^\vee|_U \to I/I^2 \) is a surjection. One can lift the surjection \( V^\vee|_U \to I/I^2 \) to a \( G \)-equivariant map \( s: V^\vee \to I \), which we can assume to be surjective by shrinking \( Y \) if necessary. Since \( Y \) is smooth, we can lift the closed immersion \( \mathcal{U} \to Y \) to a map \( j: \mathcal{U} \to Y \) in \( d\text{Aff}^G \). Then the diagram

\[
\begin{array}{ccc}
\mathcal{U} & \xrightarrow{j} & Y \\
\downarrow & & \downarrow \\
Y & \xrightarrow{s} & V
\end{array}
\]

is commutative in \( d\text{Aff}^G \) up to equivalence. Therefore the above diagram induces a map \( \mathcal{U} \to \text{Spec} \mathcal{R}(V \to Y, s) \) in \( d\text{Aff}^G \). The above map induces the isomorphism on the classical truncation, and the induced map on cotangent complexes is a quasi-isomorphism by the construction. Therefore \( \mathcal{U} \) is equivalent to \( \text{Spec} \mathcal{R}(V \to Y, s) \) in \( d\text{Aff}^G \).

**Proposition A.2.** For \( \mathcal{U}, \mathcal{U}' \in d\text{Aff}^G \), let \( f: \Pi(\mathcal{U}) \to \Pi(\mathcal{U}') \) be a morphism in \( d\text{St}/BG \). Then there exists a morphism \( \tilde{f}: \mathcal{U} \to \mathcal{U}' \) in \( d\text{Aff}^G \) such that \( f \sim \Pi(g) \).

**Proof.** The proof is similar to Proposition A.1 using deformation argument. We set

\[ f_{\leq n} := f|_{t_{\leq n}(\mathcal{U}(\mathcal{U})): t_{\leq n}(\Pi(\mathcal{U}))/\Pi(\mathcal{U})} \to \Pi(\mathcal{U}) \].

For \( n = 0 \), \( f_{\leq 0} \) factors through a morphism of classical Artin stacks \( [t_0(\mathcal{U})/G] \to [t_0(\mathcal{U}')/G] \) over \( BG \). By pulling back via \( \text{Spec} \mathbb{C} \to BG \), it lifts to a \( G \)-equivariant morphisms of affine schemes \( t_0(\mathcal{U}) \to t_0(\mathcal{U}') \). By setting \( f_{\leq 0} \) to be the composition of the above morphism with \( t_0(\mathcal{U}') \to \mathcal{U}' \), we have \( f_{\leq 0} \sim \Pi(f_{\leq 0}) \).

Suppose that there exists \( \tilde{f}_{\leq n}: t_{\leq n}(\mathcal{U}) \to \mathcal{U}' \) in \( d\text{Aff}^G \) such that \( f_{\leq n} \sim \Pi(\tilde{f}_{\leq n}) \). We take the distinguished triangle

\[ J_n \to O_{t_{\leq n+1}(\Pi(\mathcal{U}))} \to O_{t_{\leq n}(\Pi(\mathcal{U}))}. \]

Note that \( J_n \) is an object in \( \mathcal{D}^b_{\text{coh}}(\Pi(\mathcal{U})) \) for \( \mathcal{U} = t_0(\mathcal{U}) \) push-forward along with the closed immersion \( \Pi(\mathcal{U}) \hookrightarrow \Pi(\mathcal{U}_{\leq n+1}) \). We set \( S_n, T_n \) to be

\[
S_n^i := \text{Hom}_{\mathcal{U}}(\tilde{f}_{\leq n}(L_{\mathcal{U}}|_U), p^* J_n[i]),
T_n^i := \text{Hom}_{\mathcal{U}/G}(f_{\leq n}(L_{\Pi(\mathcal{U}')/BG}|_{\Pi(\mathcal{U})}), J_n[i]).
\]
Here $\mathcal{U}' = t_0(\mathcal{U}')$ and $p: \mathcal{U} \to [\mathcal{U}/G]$. Since $G$ is reductive, we have natural isomorphisms

(A.2) \[ S_n^1 \cong T_n^1. \]

The obstruction of extending $\bar{f}_{\leq n}$ to $\bar{f}_{\leq n+1}$ in $\text{dAff}^G$ lies in $S_n^1$, which corresponds to the obstruction in $T_n^1$ of extending $\bar{f}_{\leq n}$ to $\bar{f}_{\leq n+1}$ in $dSt/BG$ under the isomorphism (A.2). As $f_{\leq n}$ is extended to $\bar{f}_{\leq n+1}$ by the assumption, the above obstruction vanishes so that there exists an extension $\bar{f}_{\leq n}$ of $\bar{f}_{\leq n}$ in $\text{dAff}^G$. An extension of $f_{\leq n}$ to $\bar{f}_{\leq n+1}$ in $dSt/BG$ is classified by $S_0^n$, while an extension of $f_{\leq n}$ to $\bar{f}_{\leq n+1}$ in $dSt/BG$ is classified by $T_0^n$. Therefore by the isomorphism (A.2) there exists an extension $\bar{f}_{\leq n+1}$ such that $\Pi(\bar{f}_{\leq n+1}) \sim f_{\leq n+1}$. By the induction, we obtain $f: \mathcal{U} \to \mathcal{U}'$ in $\text{dAff}^G$ such that $\Pi(g) \sim f$. \qed

A.2. Proof of Lemma 4.11

Proof. We take a $G$-equivariant closed immersion $i: Y \hookrightarrow W$ for a $G$-representation $W$. We set

$$ Y'' = W \times Y', \quad V'' = W \times W \times V' $$

and regard $V''$ as a $G$-equivariant vector bundle on $Y''$ by the projection onto the last two factors composed with the projection $W \times V' \to W \times Y'$. We have the following $G$-equivariant commutative diagram

\[
\begin{array}{ccc}
V'' & \xrightarrow{k''} & V'' \\
\downarrow{s''} & & \downarrow{s''} \\
V' & \xrightarrow{k'} & V' \\
\downarrow{s} & & \downarrow{s} \\
Y'' & \xrightarrow{p} & Y''
\end{array}
\]

(A.3)

Here the middle horizontal arrows are projections, and

$i''(y) = (i(y), i(y)), \quad s''(w, y') = (w, w, s'(y')), \quad i'(y') = (0, y'), \quad k'(v') = (0, 0, v').$

Note that $i'$, $i''$ are closed immersions.

Let $\mathcal{U}''$ be the derived zero locus of $s''$. By the constructions of $(Y'', V'', s'')$, the projection $p$ and the closed immersion $i'$ in the diagram (A.3) induce equivalences

(A.4) \[ f'': [\mathcal{U}''/G] \simto [\mathcal{U}'/G], \quad f': [\mathcal{U}'/G] \simto [\mathcal{U}'/G] \]

which are inverse each other in the infinity-category $dSt/BG$. Since we assumed that the diagram (A.3) induces an equivalence $[\mathcal{U}'/G] \simto [\mathcal{U}'/G]$, by composing with (A.4) we obtain an equivalence $[\mathcal{U}'/G] \simto [\mathcal{U}'/G]$ in $dSt/BG$. By Proposition A.2, the equivalence $[\mathcal{U}'/G] \simto [\mathcal{U}'/G]$ lifts to an equivalence $\mathcal{U} \simto \mathcal{U}''$ in $\text{dAff}^G$. Now we have the morphisms $\mathcal{U} \to Y \xleftarrow{s''} Y''$ in $\text{dAff}^G$, and as $\mathcal{O}_{\mathcal{U}''}$ is cofibrant over $\mathcal{O}_{Y''}$ in $\text{cdga}^G$, the equivalence $\mathcal{U} \simto \mathcal{U}''$ is given by an actual morphism of $G$-equivariant cdgas, i.e., we have a $G$-equivariant vector bundle morphism $k''$ in the dotted arrow in (1.10) which induces an equivalence $\mathcal{U} \simto \mathcal{U}''$. Then it induces an equivalence $f'': [\mathcal{U}/G] \simto [\mathcal{U}''/G]$ as desired. \qed

A.3. Proof of Lemma 4.13

Proof. By Proposition A.2, the equivalence $f$ lifts to an equivalence $\tilde{f}: \mathcal{U} \simto \mathcal{U}'$ in $\text{dAff}^G$. We then construct the diagram (1.21) following the argument of [BBJ19] Theorem 4.2. By the equivalence $\tilde{f}: \mathcal{U} \simto \mathcal{U}'$ together with closed immersions $\mathcal{U} \hookrightarrow Y$, $\mathcal{U}' \hookrightarrow Y'$, we have the $G$-equivariant closed immersion $\mathcal{U} \hookrightarrow Y \times Y'$. Then similarly to the proof Proposition A.1 there exist a $G$-invariant affine open subset $\tilde{Y} \subset Y \times Y'$ which contains $\mathcal{U}$, a $G$-equivariant vector bundle $\tilde{V} \to \tilde{Y}$ with a $G$-invariant section $\tilde{s}$, such that $\mathcal{U} \hookrightarrow \tilde{Y}$ factors through an equivalence $\mathcal{U} \simto \tilde{U}$ in $\text{dAff}^G$, where $\tilde{U}$ is the derived zero locus of $\tilde{s}$. Let us consider the composition $\tilde{U} \hookrightarrow \tilde{Y} \to Y$, where the latter map is the projection. Since $\mathcal{O}_{\tilde{U}}$ is cofibrant over $\mathcal{O}_Y$ in $\text{cdga}^G$, the above map factors through an
equivalence \( \tilde{\mathcal{U}} \sim \mathcal{U} \) in \( \text{dA}f / G \), which is induced by an actual morphism of \( G \)-equivariant cdga's. So we obtain a left diagram in (4.24) and an equivalence \( \tilde{g} \). The existence of a right diagram in (4.24) and \( \tilde{g}' \) also follow similarly.

A.4. Proof of Lemma 4.20

Proof. Let \( \mathfrak{m} \subset O_{\tilde{\mathcal{U}}_0} \) be the maximal ideal of the origin and \( \mathcal{U}^{[n]} \subset \tilde{\mathcal{U}}_0 \) the closed subscheme defined by \( \mathfrak{m}^n \). We claim that the compatible isomorphisms

\[
(A.5) \quad f|_{\mathcal{U}^{[n]}/G} \cong f'|_{\mathcal{U}^{[n]}/G}
\]

exist by the induction on \( n \). The \( n = 1 \) case follows from \( f \circ \mu \cong f' \circ \mu \). Suppose that the claim holds for \( n \). By a standard deformation theory of morphisms of stacks, the set of possible extensions of \( f|_{\mathcal{U}^{[n]}/G} \) to \( [\mathcal{U}^{[n+1]}/G] \) is a torsor over

\[
\text{Hom}_{[\mathcal{U}^{[n]}/G]}(f|_{[\mathcal{U}^{[n]}/G]} \mathbb{L}_{BG}, \mathfrak{m}^n/\mathfrak{m}^{n+1}) = 0
\]

since \( \mathbb{L}_{BG} = \mathfrak{g}^\vee [-1] \). \( \mathcal{U}^{[n]} \) is affine and \( G \) is reductive. Therefore the isomorphism \( \text{(A.5)} \) lifts to an isomorphism for \( n + 1 \), so the claim holds.

Since \( \tilde{\mathcal{U}}_0/G \) is complete local, by [AHRc, Corollary 3.6] a morphism \( [\tilde{\mathcal{U}}_0/G] \rightarrow BG \) is determined by its restriction to \( \lim_n [\mathcal{U}^{[n]}/G] \). Therefore we conclude that \( f|_{[\tilde{\mathcal{U}}_0/G]} \cong f'|_{[\tilde{\mathcal{U}}_0/G]} \). We then apply the same deformation argument above for the square zero extensions

\[
[\tilde{\mathcal{U}}_0/G] = \iota_0([\tilde{\mathcal{U}}^0/G]) \hookrightarrow \iota_{\leq 1}([\tilde{\mathcal{U}}^1/G]) \hookrightarrow \iota_{\leq 2}([\tilde{\mathcal{U}}^2/G]) \hookrightarrow \cdots
\]

and conclude that \( f \sim f' \).

Appendix B. Formal neighborhood theorem for moduli stacks of semistable sheaves

In this section, we prove the formal neighborhood theorem for moduli stacks of semistable sheaves on smooth projective varieties. A similar result is proved in [Tod18] on the good moduli spaces, using rough theory argument. Instead of rough theory, we use formal GAGA theorem to give a purely algebraic proof for the formal neighborhood theorem.

B.1. Formal GAGA for good moduli spaces. Let \( \mathcal{X} \) be a classical Artin stack with affine diagonal, and

\[
\pi : \mathcal{X} \rightarrow \text{Spec} \, R
\]

its good moduli space. We assume that \( R \) is a complete local Noetherian ring with maximal ideal \( \mathfrak{m} \subset R \), and \( \pi \) is of finite type. Let \( \mathcal{I} \subset O_{\mathcal{X}} \) be the ideal sheaf generated by the pull-back of \( \mathfrak{m} \). We define the closed substack \( \mathcal{X}_n \subset \mathcal{X} \) to be defined by the ideal \( \mathcal{I}^n \). We have the formal stack \( \lim_n \mathcal{X}_n \). The formal GAGA theorem for \( \mathcal{X} \) is stated as follows:

**Theorem B.1.** ([GZB15 Theorem 1.1], [AHRd Corollary 1.7]) The restriction functor

\[
\text{Coh}(\mathcal{X}) \rightarrow \text{Coh}(\lim_n \mathcal{X}_n)
\]

is an equivalence of categories.

B.2. Ext-quivers associated with simple collections. Let \( Z \) be a smooth projective variety over \( \mathbb{C} \). A collection of coherent sheaves \( (E_1, \ldots, E_m) \) on \( Z \) is called a simple collection if \( \text{Hom}(E_i, E_j) = \mathbb{C} \cdot \delta_{ij} \). The Ext-quiver \( Q_{E^\circ} \) associated with the collection \( (E_1, \ldots, E_m) \) is defined as follows: the vertex set \( V(Q_{E^\circ}) \) and the edge set \( E(Q_{E^\circ}) \) are given by

\[
V(Q_{E^\circ}) = \{1, \ldots, m\}, \quad E(Q_{E^\circ}) = \bigcup_{1 \leq i, j \leq m} E_{i,j}.
\]

Here \( E_{i,j} \) the set of edges from \( i \) to \( j \) with \( \sharp E_{i,j} = \text{ext}^1(E_i, E_j) \). Let \( E_{i,j} \) be the \( \mathbb{C} \)-vector space spanned by \( E_{i,j} \). We set

\[
(E.1) \quad E_{i,j}^\vee := \{e^\vee : e \in E_{i,j}\} \subset E_{i,j}^\vee.
\]
Here for $e \in E_{i,j}$, the element $e^\vee \in E^\vee_{i,j}$ is defined by the condition $e^\vee(e) = 1$ and $e^\vee(e') = 0$ for any $e \neq e' \in E_{i,j}$, i.e. $E^\vee_{i,j}$ is the dual basis of $E_{i,j}$.

By setting $E^\vee = \oplus_{i=1}^m E_i$, we have linear maps

$$m_n: \text{Ext}^1(E, E)^{\otimes n} \to \text{Ext}^2(E, E)$$

given by a minimal $A_\infty$-structure of the dg-algebra $R\text{Hom}(E, E)$. The map (B.2) only consists of the direct sum factors of the form (see [Tod18, Section 5.1])

$$m_n: \text{Ext}^1(E_{\psi(1)}, E_{\psi(2)}) \otimes \text{Ext}^1(E_{\psi(2)}, E_{\psi(3)}) \otimes \cdots$$

(B.3)

\[ \cdots \otimes \text{Ext}^1(E_{\psi(n)}, E_{\psi(n+1)}) \to \text{Ext}^2(E_{\psi(1)}, E_{\psi(n+1)}). \]

Here $\psi$ is a map $\psi: \{1, \ldots, n+1\} \to \{1, \ldots, m\}$, and the above $\{m_n\}_{n \geq 2}$ give a minimal $A_\infty$-category structure on the dg-category generated by $(E_1, \ldots, E_m)$. By taking the dual and the products of (B.3) for all $n \geq 2$, we obtain the linear map

$$m^\vee := \prod_{n \geq 2} m_n^\vee: \text{Ext}^2(E, E)^\vee \to \bigoplus_{n \geq 2 \{1, \ldots, n+1\}} \text{Ext}^1(E_{\psi(1)}, E_{\psi(2)})^\vee \otimes \cdots$$

\[ \cdots \otimes \text{Ext}^1(E_{\psi(n)}, E_{\psi(n+1)})^\vee. \]

Note that an element of the RHS is an element of the completed path algebra $\mathbb{C}[Q_{E^\bullet}]$. Let $I \subset \mathbb{C}[Q_{E^\bullet}]$ be the topological closure of the ideal generated by the image of $m^\vee$. By [Tod18, Section 6.4], the quotient algebra

$$A := \mathbb{C}[Q_{E^\bullet}]/I$$

is a pro-representable hull for the NC deformation functor associated with the collection $(E_1, \ldots, E_m)$.

Let $E \in \text{Coh}(X)$ be given by

$$E = \bigoplus_{i=1}^m V_i \otimes E_i$$

for finite dimensional vector spaces $V_i$. Then

$$[\text{Ext}^1(E, E)/ \text{Aut}(E)] = \left[ \bigoplus_{(i,j) \in Q_{E^\bullet}} \text{Hom}(V_i, V_j)/\prod_{i=1}^m \text{GL}(V_i) \right]$$

is the moduli stack of $Q_{E^\bullet}$-representations with dimension vector $\vec{v} = (\dim V_i)_{1 \leq i \leq m}$. Here $\text{Aut}(E)$ acts on $\text{Ext}^1(E, E)$ by the conjugation. The fiber of the morphism to the good moduli space

(B.4)

$$[\text{Ext}^1(E, E)/ \text{Aut}(E)] \to \text{Ext}^1(E, E)/\!\!/ \text{Aut}(E)$$

at the origin consists of nilpotent $Q_{E^\bullet}$-representations. More precisely, let $I \subset \mathcal{O}_{\text{Ext}^1(E, E)}$ be the ideal sheaf which defines the fiber (B.4) at the origin, and $T_n \to \text{Ext}^1(E, E)$ the closed subscheme defined by $I^n$. Then the formal stack $\lim T_n/\text{Aut}(E)$ represents the 2-functor

(B.5)

$$\mathcal{M}^{nil}_{Q_{E^\bullet}}(\vec{v}) : \text{Aff}^{op} \to \text{Groupoid}$$

which sends an affine $\mathbb{C}$-scheme $T$ to the groupoid of data

(B.6)

$$\{(V_i, \phi_e)_{e \in V(Q_{E^\bullet}), e \in E(Q_{E^\bullet})}, \phi_e : V_{s(e)} \to V_{t(e)}\}.$$
we consider the following Ext\(^2(E, E)\)-valued Aut\((E)\)-equivariant formal function
\[
\kappa(u) = \sum_{n \geq 2} \sum_{c_i \in E(\psi(1), \psi(1 + i))} m_n(e^\psi_1, \ldots, e^\psi_n) \cdot u_{c_n} \circ \cdots \circ u_{c_2} \circ u_{c_1}. \tag{B.7}
\]
The above sum is a finite sum if \(u\) corresponds to a nilpotent \(Q_{E_n}\)-representation. Therefore the restriction \(\kappa|_{T_n}\) determines the Aut\((E)\)-equivariant algebraic map
\[
\kappa_n := \kappa|_{T_n} : T_n \to \text{Ext}^2(E, E).
\]
By the above arguments, we obtain sections of vector bundles over \(T_n\)
\[
\xymatrix{ \left(\text{Ext}^2(E, E) \times T_n\right)/\text{Aut}(E) \ar[r]^-{\kappa_n} & [T_n/\text{Aut}(E)]. }
\]
Let \(N_n \hookrightarrow T_n\) be the closed subscheme defined by \(\kappa_n = 0\). Then the formal stack \(\lim_{\to} [N_n/\text{Aut}(E)]\) represents the sub 2-functor of \([\mathcal{B}_\ast]\),
\[
\mathcal{M}^\text{nil}(Q_{E_n}, \psi)(\tilde{v}) \subset \mathcal{M}^\text{nil}_n(Q_{E_n}, \psi)
\]
consisting of groupoids \([B.6]\) such that \(\{\phi_e\}_{e \in E(Q_{E_n})}\) satisfy the relation \(I \subset \mathbb{C}[Q_{E_n}]\).

By Theorem \([B.1]\) the system of sections \(\{\kappa_n\}_{n \geq 1}\) above uniquely lifts to a section of a vector bundle on \([\text{Ext}^1(E, E)_0/\text{Aut}(E)]\)
\[
\xymatrix{ \left(\text{Ext}^2(E, E) \times \text{Ext}^1(E, E)_0\right)/\text{Aut}(E) \ar[r]^-{\kappa} & \text{Ext}^1(E, E)_0/\text{Aut}(E). }
\]
Here as in Subsection \([2.4]\) \(\text{Ext}^1(E, E)_0\) is the formal fiber at the origin \(0 \in \text{Ext}^1(E, E)\). Let \(\tilde{N}_0 \subset \text{Ext}^1(E, E)_0\) be the closed subscheme defined by \(\kappa = 0\). We have the quotient stack with good moduli space
\[
[\tilde{N}_0/\text{Aut}(E)] \arr \tilde{N}_0/\text{Aut}(E).
\]
The good moduli space \(\tilde{N}_0/\text{Aut}(E)\) is a closed subscheme of \(\text{Ext}^1(E, E)_0/\text{Aut}(E)\), hence written as Spec \(R\) for a complete local Noetherian ring \(R\). The formal stack \(\lim_{\to} [N_n/\text{Aut}(E)]\) is obtained by taking the colimit of thickened fibers of the morphism \([B.9]\) at the origin as in Subsection \([B.1]\).

B.3. Moduli stacks of semistable sheaves. For an ample divisor \(H\) on \(Z\) and \(v \in H^{2*}(Z, \mathbb{Q})\), we denote by \(\mathcal{M}_{Z,H}(v)\) the moduli stack of Gieseker \(H\)-semistable sheaves \(E\) on \(Z\) with \(\text{ch}(E) = v\). By the GIT construction of the moduli stack \(\mathcal{M}_{Z,H}(v)\) (see \([HL97]\)), it admits a good moduli space
\[
\pi : \mathcal{M}_{Z,H}(v) \arr M_{Z,H}(v).
\]
The good moduli space \(M_{Z,H}(v)\) is a projective scheme which parametrizes \(H\)-polystable sheaves, i.e. a closed point \(y \in M_{Z,H}(v)\) corresponds to a direct sum
\[
E = \bigoplus_{i=1}^m V_i \otimes E_i
\]
where each \(E_i\) is a Gieseker \(H\)-stable sheaf, \(E_i\) is not isomorphic to \(E_j\) for \(i \neq j\), and \(E_i\) has the same reduced Hilbert polynomial with \(E_j\). Note that \((E_1, \ldots, E_m)\) is a simple collection. Let \(R = \hat{\mathcal{O}}_{M_{Z,H}(v), y}\), which is a complete local Noetherian ring, and set
\[
\hat{\mathcal{M}}_{Z,H}(v)_y := \mathcal{M}_{Z,H}(v) \times_{M_{Z,H}(v)} \text{Spec } R \arr \text{Spec } R.
\]
Note that the above map is a good moduli space morphism of \(\hat{\mathcal{M}}_{Z,H}(v)_y\), since the good moduli morphism is preserved by the base change (see \([Alp13]\) Proposition 4.7)).
Let $\left[\hat{N}_0/\operatorname{Aut}(E)\right]$ be the quotient stack constructed in (B.9) associated with the above collection $(E_1, \ldots, E_m)$ for the polystable sheaf (B.10). The following result gives the formal neighborhood theorem for Gieseker moduli spaces.

**Theorem B.2.** We have the commutative isomorphisms

\[
\left[\hat{N}_0/\operatorname{Aut}(E)\right] \xrightarrow{\cong} \hat{M}_{Z,H}(v)_y
\]

\[
\hat{N}_0/\operatorname{Aut}(E) \xrightarrow{\cong} \operatorname{Spec} R.
\]

**Proof.** Note that the closed fiber of the morphism (B.11) corresponds to $H$-semistable sheaves which are $S$-equivalent to $E$, so in particular they are objects in $\langle E_1, \ldots, E_m\rangle_{ex}$. Here $\langle - \rangle_{ex}$ is the extension closure of $(-)$. Let $I \subset O_{\hat{M}_{Z,H}(v)_y}$ be the ideal sheaf which is the pull-back of the maximal ideal $m \subset R$ by the morphism (B.11). Let $M_n \hookrightarrow \hat{M}_{Z,H}(v)_y$ be the closed substack defined by $I^n$. Then the formal stack $\varinjlim M_n$ represents the functor $M_{E_{Z,H}}(v) : \mathbf{Aff}^{\text{op}} \rightarrow \mathbf{Groupoid}$ sending an affine $C$-scheme $T$ to the groupoid of flat families of coherent sheaves in $\langle E_1, \ldots, E_m\rangle_{ex}$.

By [Tod18, Corollary 6.7], we have an equivalence of categories

\[
\Phi : \text{mod}_{nil}(A) \xrightarrow{\sim} \langle E_1, \ldots, E_m\rangle_{ex}.
\]

Here the left hand side is the abelian category of finitely generated nilpotent right $A$-modules. The above functor is given by

\[
\Phi(M) = M \otimes_A \mathcal{E}, \quad \mathcal{E} \in \operatorname{Coh}(O_X \otimes A)
\]

Therefore the functor $\Phi$ gives an isomorphism of 2-functors

\[
\Phi : M_{nil}(\mathcal{O}_{\mathcal{X},1})^{\overline{v}}(\overline{v}) \xrightarrow{\cong} M_{Z,H}^{E}(v).
\]

Therefore the functor $\Phi$ also induces the isomorphism of formal stacks

\[
\Phi : \varinjlim \left[\mathcal{N}_n/\operatorname{Aut}(E)\right] \xrightarrow{\cong} \varinjlim M_n.
\]

Then applying Theorem B.1 we have equivalences

\[
\operatorname{Coh} \left(\hat{M}_{Z,H}(v)_y\right) \xrightarrow{\sim} \operatorname{Coh} \left(\left[\hat{N}_0/\operatorname{Aut}(E)\right]\right)
\]

\[
\xrightarrow{\sim} \operatorname{Coh} \left(\varinjlim M_n\right) \xrightarrow{\cong} \operatorname{Coh} \left(\varinjlim \left[\mathcal{N}_n/\operatorname{Aut}(E)\right]\right).
\]

Here the vertical arrows are restriction functors and the top arrow is defined by the above commutative diagram. The top arrow sends $O_{\hat{M}_{Z,H}(v)_y}$ to $O_{\left[\hat{N}_0/\operatorname{Aut}(E)\right]}$ and preserves the tensor products, as these properties are satisfied in the bottom arrow. Therefore by the Tannaka duality for Artin stacks (see [AHRa, Theorem 1.1]), there exists an unique isomorphism of stacks

\[
\Psi : \left[\hat{N}_0/\operatorname{Aut}(E)\right] \xrightarrow{\cong} \hat{M}_{Z,H}(v)_y
\]

such that the top arrow of (B.14) is isomorphic to $\Psi^\ast$. Therefore we obtain the top isomorphism in the diagram (B.12). The bottom isomorphism in the diagram (B.12) follows from the uniqueness of good moduli spaces. □
APPENDIX C. Comparisons of DT categories

In this section, we give a comparison of DT categories considered in this paper with those studied in [Ioddb].

C.1. **Ind-completions.** Let $\mathcal{M}$ be a quasi-smooth derived stack which is QCA, and set $\mathcal{M} = t_0(\mathcal{M})$. Our aim here is to compare the categories $\text{Ind} D^b_{\text{coh}}(\mathcal{M})$ and $\text{Ind} D^b_{\text{coh}}(\mathcal{U})$, $\text{Ind} C_Z := \lim_{\mathcal{U} \leftarrow \mathcal{M}} \text{Ind} D^b_{\text{coh}}(\mathcal{U})$. Here $\alpha$ is a smooth morphism from an affine derived scheme $\mathcal{U}$ of the form $\text{[3.2]}$, and $Z \subset t_0(\mathcal{M})$ is a conical closed substack. It is proved in [DG13, Theorem 3.3.5] that $\text{Ind} D^b_{\text{coh}}(\mathcal{M})$ is compactly generated whose compact objects coincide with $D^b_{\text{coh}}(\mathcal{M})$. In particular, we have

$$\text{Ind} D^b_{\text{coh}}(\mathcal{M}) = \text{Ind} D^b_{\text{coh}}(\mathcal{U}).$$

We show that, assuming a similar property for $\text{Ind} C_Z$, the category $\text{[3.14]}$ is a dense subcategory of $\text{Ind} D^b_{\text{coh}}(\mathcal{M})$. For a triangulated category $\mathcal{D}$, its triangulated subcategory $\mathcal{S} \subset \mathcal{D}$ is called dense if any object in $\mathcal{D}$ is a direct summand of an object in $\mathcal{S}$. We will use the following general result from [Kra10]:

**Theorem C.1.** ([Kra10, Theorem 7.2.1]) Let $\mathcal{D}$ be a triangulated category and $\mathcal{D}' \subset \mathcal{D}$ a thick triangulated subcategory closed under taking small coproducts. Suppose that $\mathcal{D}$ and $\mathcal{D}'$ are compactly generated. Then $\mathcal{D}/\mathcal{D}'$ is also compactly generated, and we have the fully-faithful functor $\mathcal{D}^e/\mathcal{D}'^e \hookrightarrow (\mathcal{D}/\mathcal{D}')^e$ with dense image. Here $\mathcal{D}^e \subset \mathcal{D}$ is the subcategory of compact objects.

C.2. **Equivalences of DT categories.** We have the following proposition.

**Proposition C.2.** Suppose that $\text{Ind} C_Z$ is compactly generated whose compact objects coincide with $C_Z$. Then the natural functor

$$(C.1) D^b_{\text{coh}}(\mathcal{M})/C_Z \to \lim_{\mathcal{U} \leftarrow \mathcal{M}} (D^b_{\text{coh}}(\mathcal{U})/C_{\alpha \cdot Z})$$

is fully-faithful with dense image. In particular if furthermore $D^b_{\text{coh}}(\mathcal{M})/C_Z$ is idempotent complete, then the functor $\text{[C.1]}$ is an equivalence.

**Proof.** Applying Theorem C.1 for the subcategory $\text{Ind} C_Z \subset \text{Ind} D^b_{\text{coh}}(\mathcal{M})$ and using the assumption, we have the fully-faithful functor

$$D^b_{\text{coh}}(\mathcal{M})/C_Z \hookrightarrow (\text{Ind} D^b_{\text{coh}}(\mathcal{M})/\text{Ind} C_Z)^e$$

with dense image. Moreover $\text{Ind} D^b_{\text{coh}}(\mathcal{M})/\text{Ind} C_Z$ is compactly generated, so we have an equivalence

$$(C.2) \text{Ind} D^b_{\text{coh}}(\mathcal{M})/\text{Ind} C_Z \stackrel{\sim}{\to} \text{Ind} (D^b_{\text{coh}}(\mathcal{M})/C_Z).$$

Below we say that a sequence of triangulated categories $\mathcal{D}_1 \hookrightarrow \mathcal{D}_2 \to \mathcal{D}_3$ is exact if $j_1$ is fully-faithful, $j_2 \circ j_1 \cong 0$ and $j_2$ induces an equivalence $\mathcal{D}_2/\mathcal{D}_1 \sim \mathcal{D}_3$. Applying the equivalence $\text{[C.2]}$ for each smooth morphism $\alpha: \mathcal{U} \to \mathcal{M}$, we have the exact sequence of triangulated categories

$$(C.3) \text{Ind}(C_{\alpha \cdot Z}) \stackrel{i_{\mathcal{U}}}{\to} \text{Ind}(D^b_{\text{coh}}(\mathcal{U})) \stackrel{j_{\mathcal{U}}}{\to} \text{Ind}(D^b_{\text{coh}}(\mathcal{U})/C_{\alpha \cdot Z}).$$

By [AG15, Section 4.3], the functor $i_{\mathcal{U}}$ admits a right adjoint $i_{\mathcal{U}}^R$. Therefore there also exists a right adjoint $j_{\mathcal{U}}^R$ such that for each $E \in \text{Ind}(D^b_{\text{coh}}(\mathcal{U}))$ we have the exact triangle

$$i_{\mathcal{U}} \circ i_{\mathcal{U}}^R(E) \to E \to j_{\mathcal{U}}^R \circ j_{\mathcal{U}}(E).$$

By the above triangle, we have $j_{\mathcal{U}} \circ j_{\mathcal{U}}^R \cong \text{id}$. On the other hand by taking the limit of the sequence $\text{[C.3]}$, we obtain the sequence

$$(C.4) \text{Ind} C_Z \to \text{Ind} D^b_{\text{coh}}(\mathcal{M}) \to \lim_{\mathcal{U} \leftarrow \mathcal{M}} \text{Ind} (D^b_{\text{coh}}(\mathcal{U})/C_{\alpha \cdot Z}).$$
Since \( j^R_\U \) is functorial for \( \U \), the functor \( j \) also admits a right adjoint \( j^R \) by taking the limit of \( j^R_\U \) such that \( j \circ j^R \cong \id \). Also by the construction of \( j \), we have that \( \ker(j) = \ind C_Z \). By [HR Lemma 3.4], the above properties of the sequence (C.1) implies that the sequence (C.1) is exact, i.e.

we have an equivalence

\[
\ind D^b_{\coh}(\mathcal{M})/\ind C_Z \sim \lim_{\U \to \mathcal{M}} \ind \left( D^b_{\coh}(\U)/C_{\alpha^* Z} \right).
\]

We have the commutative diagram

\[
\begin{array}{ccc}
D^b_{\coh}(\mathcal{M})/C_Z & \longrightarrow & \lim_{\U \to \mathcal{M}} \left( D^b_{\coh}(\U)/C_{\alpha^* Z} \right) \\
\downarrow & & \downarrow \\
(\ind D^b_{\coh}(\mathcal{M})/\ind C_Z)^e & \sim & (\lim_{\U \to \mathcal{M}} \ind \left( D^b_{\coh}(\U)/C_{\alpha^* Z} \right))^e.
\end{array}
\]

Here the left vertical arrow is fully-faithful with dense image, the right vertical arrow exists by Lemma (C.3) and it is fully-faithful. Therefore the top horizontal arrow is also full-faithful with dense image. \( \square \)

In the proof of the above proposition, we have used the following lemma:

**Lemma C.3.** Any object in the subcategory

\[
\lim_{\U \to \mathcal{M}} \left( D^b_{\coh}(\U)/C_{\alpha^* Z} \right) \subset \lim_{\U \to \mathcal{M}} \ind \left( D^b_{\coh}(\U)/C_{\alpha^* Z} \right)
\]

is a compact object.

**Proof.** The lemma is proved for \( Z = \emptyset \) in [DG13 Proposition 3.4.2 (b)]. Since the action of \( D^b_{\qcoh}(\U) \) on \( \ind D^b_{\coh}(\U) \) by taking tensor products preserves \( \ind(C_{\alpha^* Z}) \) (see [AG15 Lemma 4.2.2]), the quotient category \( \ind D^b_{\coh}(\U)/\ind (C_{\alpha^* Z}) = \ind(D^b_{\coh}(\U)/C_{\alpha^* Z}) \) is a module over \( D^b_{\qcoh}(\U) \). Therefore the proof of [DG13 Proposition 3.4.2 (b)] applies verbatim. \( \square \)

In general, it is not known whether \( \ind C_Z \) is compactly generated or not (see [AG15 Remark 8.12]). We give two cases where this holds so that we have the comparison of DT categories.

**Corollary C.4.** Suppose that \( Z = p^{-1}(W) \) for a closed substack \( W \subset \mathcal{M} \), where \( p \) is the projection (3.11). Let \( \mathcal{M}_o \subset \mathcal{M} \) be the derived open substack whose classical truncation is \( \mathcal{M} \setminus W \). Then the functor (C.1) is an equivalence, and both sides are equivalent to \( D^b_{\coh}(\mathcal{M}_o) \).

**Proof.** Let

\[
D^b_{\coh}(\mathcal{M})_W \subset D^b_{\coh}(\mathcal{M}), \ind D^b_{\coh}(\mathcal{M})_W \subset \ind D^b_{\coh}(\mathcal{M})
\]

be the subcategories consisting of objects supported on \( W \). By [Toda Lemma 3.9] and [AG15 Corollary 4.5.2], for \( Z = p^{-1}(W) \) we have \( C_Z = D^b_{\coh}(\mathcal{M})_W \) and \( \ind C_Z = \ind D^b_{\coh}(\mathcal{M})_W \). As we already mentioned, it is proved in [DG13 Theorem 3.3.5] that \( \ind D^b_{\coh}(\mathcal{M}) \) is compactly generated with compact objects \( D^b_{\coh}(\mathcal{M})_W \)). Using the fact that \( \coh(\mathcal{M})_W := \ind D^b_{\coh}(\mathcal{M})_W \cap \coh(\mathcal{M})_W \) is the heart of a t-structure on \( D^b_{\coh}(\mathcal{M})_W \), the same argument of [DG13 Proposition 3.5.1] shows that \( \coh(\mathcal{M})_W \) generates \( \ind D^b_{\coh}(\mathcal{M})_W \). Then the arguments of [DG13 Theorem 3.3.5] apply verbatim to show that \( \ind D^b_{\coh}(\mathcal{M})_W \) is compactly generated with compact objects \( D^b_{\coh}(\mathcal{M})_W \). Therefore the first claim follows from Proposition (C.2). The latter claim follows from [Toda Lemma 3.7]. \( \square \)

**Remark C.5.** For a conical closed substack \( Z \subset t_0(\Omega_{\mathcal{M}}[-1]) \), the subcategory \( \ind C_Z \cap \coh(\mathcal{M}) \subset C_Z \) is not the heart of a t-structure on \( C_Z \) in general, so the argument in [DG13 Theorem 3.3.5] does not apply to the compact generation of \( \ind C_Z \).

We also have the following corollary:

\[
\]
Corollary C.6. Let \( \mathcal{M} = [\mathcal{U}/G] \) be a derived stack as in the diagram (4.2), and \( \mathcal{Z} \subset t_0(\Omega_{\mathcal{M}}[-1]) \) a conical closed substack. Then we have an equivalence

\[
D^b_{\text{coh}}([\mathcal{U}/G])/\mathcal{C}_\mathcal{Z} \cong \lim_{\mathcal{U}' \in [\mathcal{U}/G]} (D^b_{\text{coh}}(\mathcal{U}')/\mathcal{C}_{\alpha'_*\mathcal{Z}}).
\]

Here \( \mathcal{U}' \) is an affine derived scheme of the form (4.2), and \( \alpha' \) is a smooth morphism.

**Proof.** The stack \([\mathcal{U}/G]\) is a global complete intersection stack in the sense of [AG15, Section 9.1]. Therefore by [AG15, Corollary 9.2.7, Corollary 9.2.8], \( \text{Ind} \mathcal{C}_\mathcal{Z} \) is compactly generated whose compact objects coincide with \( \mathcal{C}_\mathcal{Z} \). Moreover \( W^\text{int}_G([\mathcal{U}/G]) \) is obviously idempotent closed by its definition, therefore \( D^b_{\text{coh}}([\mathcal{U}/G])/\mathcal{C}_\mathcal{Z} \) is also idempotent closed by Corollary 4.22. Therefore the corollary follows from Proposition C.2.

**References**

[AOV08] D. Abramovich, M. Olsson and Angelo Vistoli, *Tame stacks in positive characteristic*, Ann. Inst. Fourier (Grenoble) **58** (2008), no. 4, 1057–1091. MR 2427954

[Alp13] J. Alper, *Good moduli spaces for Artin stacks*, Ann. Inst. Fourier (Grenoble) **63** (2013), no. 6, 2349–2402. MR 3234751

[AHRa] J. Alper, J. Hall, and D. Rydh, *Coherent Tannaka duality and algebraicity of Hom stacks*, arXiv:1405.7680.

[AHRb] ———, *The étale local structure of algebraic stacks*, arXiv:1912.06162.

[AG15] D. Arinkin and D. Gaitsgory, *Singular support of coherent sheaves and the geometric Langlands conjecture*, Selecta Math. (N.S.) **21** (2015), no. 1, 1–199. MR 3300415

[BDF+16] M. Ballard, D. Deliu, D. Favero, M. U. Isik and L. Katzarkov, *Resolutions in factorization categories*, Adv. Math. **295** (2016), 195–249. MR 3488035

[BFK14] M. Ballard, D. Favero and L. Katzarkov, *A category of kernels for equivariant factorizations and its implications for Hodge theory*, Publ. Math. Inst. Hautes Études Sci. **120** (2014), 1–111. MR 3270588

[BFK19] ———, *Variation of geometric invariant theory quotients and derived categories*, J. Reine Angew. Math. **746** (2019), 235–303. MR 3895631

[BBBBJ15] O. B. Bassat, C. Brav, V. Bussi and D. Joyce, *A Darboux theorem for shifted symplectic structures on derived Artin stacks, with applications*, Geom. Topol. **19** (2015), no. 3, 1287–1359. MR 3352237

[BM14] A. Bayer and E. Macrì, *Projectivity and birational geometry of Bridgeland moduli spaces*, J. Amer. Math. Soc. **27** (2014), no. 2, 707–752. MR 3194493

[Beh09] K. Behrend, *Donaldson-Thomas type invariants via microlocal geometry*, Ann. of Math **170** (2009), 1307–1338.

[BF97] K. Behrend and B. Fantechi, *The intrinsic normal cone*, Invent. Math. **128** (1997), 45–88.

[BO] A. Bondal and D. Orlov, *Semicontinuity for algebraic varieties*, preprint, arXiv:9506012.

[BBJ19] C. Brav, V. Bussi and D. Joyce, *A Darboux theorem for derived schemes with shifted symplectic structure*, J. Amer. Math. Soc. **32** (2019), no. 2, 399–443. MR 3904157

[BBD+15] C. Brav, V. Bussi, D. Dupont, D. Joyce and B. Szendr˝oi, *Symmetries and stabilization for sheaves of vanishing cycles*, J. Singul. **11** (2015), 85–151. With an appendix by Jörg Schürmann. MR 3353002

[Bri07] T. Bridgeland, *Stability conditions on triangulated categories*, Ann. of Math **166** (2007), 317–345.

[Bri11] ———, *Hall algebras and curve-counting invariants*, J. Amer. Math. Soc. **24** (2011), no. 4, 969–998. MR 2813335

[BJM] V. Bussi, D. Joyce and S. Meinhardt, *On motivic vanishing cycles of critical loci*, preprint, arXiv:1305.6428.

[Che10] X. W. Chen, *Unifying two results of Orlov on singularity categories*, Abh. Math. Semin. Univ. Hambg. **80** (2010), no. 2, 207–212. MR 2734686

[NAS] W. Donovan, N. Addington and E. Segal, *The Pfaffian-Grassmannian Equivalence Revisited*, arXiv:1401.3661.

[DG13] V. Drinfeld and D. Gaitsgory, *On some finiteness questions for algebraic stacks*, Geom. Funct. Anal. **23** (2013), no. 1, 149–294. MR 3037900

[EP15] A. I. Efimov and L. Positselski, *Coherent analogues of matrix factorizations and relative singularity categories*, Algebra Number Theory **9** (2015), no. 5, 1159–1292. MR 3366002

[Gai13] Dennis Gaitsgory, *ind-coherent sheaves*, Mosc. Math. J. **13** (2013), no. 3, 399–528, 553. MR 3136100

[GR17] D. Gaitsgory and N. Rozenblyum, *A study in derived algebraic geometry. Vol. II. Deformations, Lie theory and formal geometry*, Mathematical Surveys and Monographs, vol. 221, American Mathematical Society, Providence, RI, 2017. MR 3701353
[GZB15] A. Geraschenko and D. Zureick-Brown, Formal GAGA for good moduli spaces, Algebraic Geometry 2 (2015), 214–230.

[GV] R. Gopakumar and C. Vafa, M-theory and topological strings II, hep-th/9812127.

[HR] J. Hall and D. Rydh, Perfect complexes on algebraic stacks, arXiv:1405.1887.

[HL15] D. Halpern-Leistner, The derived category of a GIT quotient, J. Amer. Math. Soc. 28 (2015), no. 3, 871–912. MR 3327537

[HLe] D. Halpern-Leistner, The D-equivalence conjecture for moduli spaces of sheaves on a K3 surface, available in http://www.math.columbia.edu/~dahjl/

[HLS] D. Halpern-Leistner and S. V. Sam, Combinatorial constructions of derived equivalences, arXiv:1601.02030.

[Hir17] Y. Hirano, Derived Knörrer periodicity and Orlov’s theorem for gauged Landau-Ginzburg models, Compos. Math. 153 (2017), no. 5, 973–1007. MR 3631231

[HL97] D. Huybrechts and M. Lehn, The geometry of moduli spaces of sheaves, Aspects of Mathematics, E31, Friedr. Vieweg & Sohn, Braunschweig, 1997. MR 1450870

[Isi13] M. U. Isik, Equivalence of the derived category of a variety with a singularity category, Int. Math. Res. Not. IMRN (2013), no. 12, 2787–2808. MR 3071664

[JK17] Y. Jiang and R. P. Thomas, Virtual signed Euler characteristics, J. Algebraic Geom. 26 (2017), no. 2, 379–397. MR 3607000

[Joy15] D. Joyce, A classical model for derived critical loci, J. Differential Geom. 101 (2015), no. 2, 289–367. MR 3399099

[JS12] D. Joyce and Y. Song, A theory of generalized Donaldson-Thomas invariants, Mem. Amer. Math. Soc. 217 (2012).

[Kat08] S. Katz, Genus zero Gopakumar-Vafa invariants of contractible curves, J. Differential. Geom. 79 (2008), 185–195.

[Kaw02] Y. Kawamata, D-equivalence and K-equivalence, J. Differential Geom. 61 (2002), no. 1, 147–171. MR 1949787

[Kin94] A. King, Moduli of representations of finite-dimensional algebras, Quart. J. Math. Oxford Ser. (2) 45 (1994), 515–530.

[KS] M. Kontsevich and Y. Soibelman, Stability structures, motivic Donaldson-Thomas invariants and cluster transformations, preprint, arXiv:0811.2435.

[KS11] M. Kontsevich and Y. Soibelman, Cohomological Hall algebra, exponential Hodge structures and motivic Donaldson-Thomas invariants, Commun. Number Theory Phys. 5 (2011), no. 2, 231–352. MR 2851153

[KT] N. Koseki and Y. Toda, Derived categories of Thaddeus pair moduli spaces via d-critical flips, arXiv:1904.04949.

[Kra10] H. Krause, Localization theory for triangulated categories, Triangulated categories, London Math. Soc. Lecture Note Ser., vol. 375, Cambridge Univ. Press, Cambridge, 2010, pp. 161–235. MR 2681709

[Lan73] D. Luna, Slices étalés, 81–105, Bull. Soc. Math. France, Paris, Mém. 33. MR 0342523

[Lur] J. Lurie, Higher topos theory.

[MMOP06] D. Maulik, N. Nekrasov, A. Okounkov, and R. Pandharipande, Gromov-Witten/Donaldson-Thomas theory and wall crossings, Mem. Amer. Math. Soc. 190 (2008), no. 889, viii+154 pp. MR 2387477

[MNP06] D. Maulik, N. Nekrasov, A. Okounkov, and R. Pandharipande, Gromov-Witten/Pairs correspondence for the quintic 3-fold, J. Amer. Math. Soc. 20 (2007), no. 2, 429–514. MR 2312006

[Muk87] S. Mukai, On the moduli space of bundles on K3 surfaces I, Vector Bundles on Algebraic Varieties, M. F. Atiyah et al., Oxford University Press (1987), 341–413.

[OR] A. Oblomkov and L. Rozansky, Categorical Chern character and braid groups, arXiv:1811.03257

[Orl09] D. Orlov, Derived categories of coherent sheaves and triangulated categories of singularities, Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. II, Progr. Math., vol. 269, Birkhäuser Boston, Inc., Boston, MA, 2009, pp. 503–531. MR 2641200

[PP17] R. Pandharipande and A. Pixton, Gromov-Witten/Pairs correspondence for the quintic 3-fold, J. Amer. Math. Soc. 30 (2017), no. 2, 389–449. MR 3600040

[PT09] R. Pandharipande and R. P. Thomas, Curve counting via stable pairs in the derived category, Invent. Math. 178 (2009), 407–447.

[PTVV13] T. Pantev, B. Toën, M. Vaquié and G. Vezzosi, Shifted symplectic structures, Publ. Math. Inst. Hautes Études Sci. 117 (2013), 271–328. MR 3090262

[PP12] A. Polishchuk and L. Positselski, Hochschild (co)homology of the second kind I, Trans. Amer. Math. Soc. 364 (2012), no. 10, 5311–5368. MR 2931331

[Sac19] G. Sacca, Relative compactified Jacobians of linear systems on Enriques surfaces, Trans. AMS. 371 (2019), 7791–7843.

[Shi12] I. Shipman, A geometric approach to Orlov’s theorem, Compos. Math. 148 (2012), no. 5, 1365–1389. MR 2982435
ON WINDOW THEOREM FOR CATEGORICAL DT THEORIES

[vvdB17] Š. Špenko and M. Van den Bergh, Non-commutative resolutions of quotient singularities for reductive groups, Invent. Math. 210 (2017), no. 1, 3–67. MR 3698338

[ST11] J. Stoppa and R. P. Thomas, Hilbert schemes and stable pairs: GIT and derived category wall crossings, Bull. Soc. Math. France 139 (2011), 297–339.

[Tho00] R. P. Thomas, A holomorphic Casson invariant for Calabi-Yau 3-folds, and bundles on K3 fibrations, J. Differential Geom. 54 (2000), no. 2, 367–438. MR 1818182

[Toda] Y. Toda, Birational geometry for d-critical loci and wall-crossing in Calabi-Yau 3-folds, arXiv:1806.00182

[Tod12] ——, On categorical Donaldson-Thomas theory for local surfaces, arXiv:1907.09076

[Tod18] ——, Stability conditions and curve counting invariants on Calabi-Yau 3-folds, Kyoto J. Math. 52 (2012), no. 1, 1–50. MR 2892766

[Tod10a] ——, Curve counting theories via stable objects I: DT/PT correspondence, J. Amer. Math. Soc. 23 (2010), 1119–1157.

[Toe11] B. Toën, Lectures on dg-categories, Topics in algebraic and topological K-theory, Lecture Notes in Math., vol. 2008, Springer, Berlin, 2011, pp. 243–302. MR 2762557

[Toe14] ——, Derived algebraic geometry, EMS Surv. Math. Sci. 1 (2014), no. 2, 153–240. MR 3285853

[TV07] B. Toën and Michel Vaquié, Moduli of objects in dg-categories, Ann. Sci. École Norm. Sup. (4) 40 (2007), no. 3, 387–444. MR 2493386

Kavli Institute for the Physics and Mathematics of the Universe (WPI), University of Tokyo, 5-1-5 Kashiwanoha, Kashiwa, 277-8583, Japan.

E-mail address: yukinobu.toda@ipmu.jp