Bounding the Solutions to Some SDEs via Ergodic Theory

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Abstract

In this note we consider autonomous SDEs admitting smooth invariant measures. We present a method in finding (almost everywhere) good bounds for \( \sup\{\|X_t\| : t \in [0, T]\} \) for strong solutions \( X \) to such SDEs, which in many cases are optimal bounds. In some situation (especially in one-dimensional SDEs’ cases), the discarded measure-zero set can be chosen to be a measure-zero set of the underlying Brownian motion uniform for all initial points \( X_0 = x \).

1 Introduction

It’s well known that, for a given one-dimensional stationary Ornstein-Uhlenbeck (OU for short) process \( X = \{X_t : t \geq 0\} \) there exist \( \lambda, \sigma > 0, \mu \in \mathbb{R} \) and a standard Brownian Motion (BM for short) \( B(\cdot) \) such that \( X \) has the same distribution as \( \{\sigma \cdot e^{-\lambda t} \cdot B(e^{\lambda t}) + \mu : t \geq 0\} \). Therefore the law of iterated logarithm for BM (see, e.g., [7]) leads us to the conclusion \( X_t = O(\sqrt{\log t}) \) almost surely. In a previous note [8] we have proved the validity of such a bound for general OU processes \( X = \{X_t : t \geq 0\} \) via elementary arguments in Ito’s stochastical analysis theory. In this note, we will consider the following SDE with smooth coefficients

\[
\frac{dX_t}{dt} = b(X_t)dt + \sigma(X_t)dW_t
\]  \hspace{1cm} (1.1)

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which admits stationary strong solutions with steady distribution denoted by \( \mu \); we will denote by \( \mathbb{P} \) the distribution of Wiener process \( W \), and \( \mathbb{P}_\mu \) the distribution of the stationary strong solution \( X \). And we are interested in the growth of \( \sup\{\|X_t\| : t \in [0, T]\} \) in terms of \( T \), where \( \| \cdot \| \) is the Euclidean norm. We will present an ergodic theoretic method in solving such problems. We recall that, the system \((1.1)\) is called strong complete \([3]\), if its solutions \( X \), with arbitrarily initial value \( X_0 = x \) is continuous in \( t \in [0, +\infty) \) for all Wiener process orbits \( W \), in a common full standard Wiener-measure set; see, for instance, \([1] [2] [3] [4]\) for results relating the property of strong completeness.

Our main result may be stated as the following.

**Theorem 1** Suppose that the smooth coefficients of \((1.1)\) satisfy

\[
\lim_{\|x\| \to +\infty} \frac{\|b(x)\| + \|\sigma(x)\|}{\|x\|^m} < \infty
\]

for some \( m \in \mathbb{N} \). Assume the steady distribution \( \mu \) is such that there exists a smooth positive function \( V \) with \( \int e^{\delta V(x)} d\mu < \infty \) for all \( \delta \in (0, 1) \) and

\[
\lim_{\|x\| \to +\infty} \frac{V(x)}{\log \|x\|} = \infty, \quad \lim_{\|x\| \to +\infty} \frac{\|\nabla V(x)\| + \|\text{Hess}_V(x)\|}{\|x\|^m} < \infty.
\]

Here \( \text{Hess}_V(x) \) denotes the Hessian of \( V \); we always assume the monotonicity of \( V(x) \) in \( \|x\| \) for large \( \|x\| \). Then the solution to \((1.1)\) always satisfies

\[
\lim_{t \to \infty} \frac{V(X_t)}{\log t} \leq 1
\]

\( \mathbb{P}_\mu \) almost surely.

If furthermore both \( \mu \) and the transition probability semigroup of \((1.1)\) have smooth densities, then the statement \((1.2)\) holds true for all initial values \( X_0 = x \) and \( \mathbb{P} \)-a.e. Wiener orbits \( W \); in one-dimensional case with the assumption of the strong completeness of the system, the validity of this statement can even be strengthened to be valid for all initial values \( X_0 = x \) and all Wiener process orbits \( W \) in a \( \mathbb{P} \)-full measure set.
By choosing a suitable smooth function $V(\cdot)$, it is possible to get good bounds for the growth of $\sup\{\|X_t\| : t \in [0, T]\}$ in terms of $T$ as the examples reveal. Such result seems to be new in literature as to our knowledge and deserves a publication somewhere.

2 Proof of the Main Theorem

First assume $X_t$ to be a stationary strong solution to (1.1). Consider

$$Y_t := e^{\delta V(X_t)}$$

with $\delta \in (0, 1/2)$.

It is clear that for suitable choice of $\tilde{b}(\cdot)$ and $\tilde{\sigma}$

$$dY_t = Y_t [\tilde{b}(X_t) dt + \tilde{\sigma}(X_t) dW_t]$$

with

$$\lim_{\|x\| \to +\infty} \frac{|\tilde{b}(x)| + |\tilde{\sigma}(x)|}{\|x\|^{4m}} < \infty.$$ 

This guarantees the integrability of $\tilde{b} \cdot e^{\delta V}$ and $|\tilde{\sigma}|^2 \cdot e^{2\delta V}$ with respect to $\mu$. Define $M_t := \int_0^t \tilde{\sigma}(X_s) dW_s$. It is easy to see that $M_t$ is an $L_2$-martingale with

$$< M >_t := \int_0^t |\tilde{\sigma}(X_s)|^2 \cdot e^{2\delta V(X_s)} ds.$$  \hspace{1cm} \text{(2.1)}

By Birkhoff’s ergodic theorem

$$\lim_{t \to \infty} \frac{< M >_t}{t} = \mathbb{E}_\mu[|\tilde{\sigma}(X_0)|^2 \cdot e^{2\delta V(X_0)}] < \infty.$$ 

The law of iterated logarithm (abbr. LIL) for continuous martingale [7] then tells us

$$\lim_{t \to \infty} \frac{M_t}{t} = 0$$

almost surely.

It is also clear that

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t Y_s \cdot \tilde{b}(X_s) ds = \mathbb{E}_\mu[Y_0 \cdot \tilde{b}(X_0)]$$

almost surely.
Therefore \( e^{\delta V(X_t)} \leq a \cdot t + b \) for all \( t \geq 0 \) almost surely, where \( a \) is a positive constant and \( b \) is a measurable function of \( X \), independent of \( t \). Hence

\[
\lim_{t \to \infty} \frac{V(X_t)}{\log t} \leq 2 \quad (2.2)
\]

\( \mathbb{P}_\mu \) almost surely.

Now we are going to lower the bound 2 in the right hand side of (2.2) down into 1 as (1.2) says. Assume that we have already proved

\[
\lim_{t \to \infty} \frac{V(X_t)}{\log t} \leq \beta
\]

for some constant \( \beta \); therefore for any \( \varepsilon \in (0, 1) \) there is \( C \) such that

\[
e^{V(X_t)} \leq C(t^{\beta+\varepsilon} + 1), \quad \forall t \geq 0.
\]

Now fix a number \( \delta \in (1/2, 1) \) arbitrarily and define \( Y_t, M_t \) as above. Then using the above arguments once again, we find

\[
\frac{< M >_t}{t} = \frac{1}{t} \int_0^t \tilde{\sigma}(X_s)^2 e^{(2\delta+\varepsilon-1)V(X_s)} \cdot e^{(1-\varepsilon)V(X_s)} ds \\
\leq \frac{1}{t} \int_0^t e^{(1-\varepsilon)V(X_s)} ds \cdot [C(t^{\beta+2\varepsilon} + 1)]^{2\delta+2\varepsilon-1}
\]

for sufficiently large \( t \). In view of LIL for continuous martingale [7] and Birkhoff’s ergodic theorem, this implies

\[
\lim_{t \to +\infty} \frac{M_t}{t^{\frac{1}{2} + (\delta + 2\varepsilon - \frac{1}{2}) \cdot (\beta + 2\varepsilon)}} = 0.
\]

On the other hand, we still have \( \tilde{b} \cdot e^{\delta V} \in L_1(\mu) \). Therefore we have

\[
\lim_{t \to +\infty} \frac{e^{\delta V(X_t)}}{t^\gamma} = 0
\]

for \( \gamma := \max\{1 + \varepsilon, \frac{1}{2} + (\delta + 2\varepsilon - \frac{1}{2}) \cdot (\beta + 2\varepsilon)\} \). This proves

\[
\lim_{t \to +\infty} \frac{V(X_t)}{\log t} \leq \frac{\gamma}{\delta},
\]
Letting $\delta \to 1$ and then $\varepsilon \to 0$, we obtain

$$\lim_{t \to +\infty} \frac{V(X_t)}{\log t} \leq \beta' := \max\{1, \frac{1 + \beta}{2}\} = \frac{1 + \beta}{2}$$

with initial $\beta = 2$. This machinery leads us finally to (1.2).

Qian and Zhang’s argument [6, page 1637] tells us that, when $\mu$ and the transition probability semigroup of $X$ have densities, (1.2) holds true for all $X_0 = x$ and all Wiener orbits $W \in \Lambda_x$ with $P(\Lambda_x) = 1$. In one dimensional case, we can say more. Let

$$\Lambda := \bigcap_{r \in \mathbb{Q}} \Lambda_r,$$

so $P(\Lambda) = 1$. For any two solutions $X^x, X^y$ to (1.1) with initial values $X_0^x = x, X_0^y = y, x \neq y \in \mathbb{R}$, write $Z_t := X^x_t - X^y_t$. It is easy to see that

$$dZ_t = Z_t \cdot \left[\bar{b}(X^x_t, X^y_t)dt + \bar{\sigma}(X^x_t, X^y_t)dW_t\right]$$

for some smooth functions $\bar{b}(x, y), \bar{\sigma}(x, y)$. Then one clearly has

$$Z_t = Z_0 \cdot \exp\left(\int_0^t \left[\bar{b}(X^x_s, X^y_s) - \frac{1}{2}\bar{\sigma}(X^x_s, X^y_s)^2\right]ds + \int_0^t \bar{\sigma}(X^x_s, X^y_s)dW_s\right),$$

which implies that

$$X^x_t \leq X^y_t \text{ for all } t \geq 0 \text{ if } x < y. \quad (2.3)$$

If $\varphi$ is a continuous monotonic function in $L_1(\mu)$, then it is easy to see that Birkhoff’s ergodic theorem holds

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t \varphi(X_s)ds = \int \varphi(x)d\mu$$

for all $x \in \mathbb{R}$ and all $W \in \Lambda$. The same argument applies to the limit

$$\lim_{t \to \infty} \frac{V(X_t)}{\log t} \leq 1$$

yielding its validity for all $x \in \mathbb{R}$ and all $W \in \Lambda$, if $V(x)$ is increasing in $|x|$ for large $|x|$.
3 Examples and Discussions

The first example is the standard OU process \(\{X_t := e^{-t}B(e^{2t}) : t \geq 0\}\), where \(B(\cdot)\) is a standard BM. By LIL for BM, we have

\[
\lim_{t \to \infty} \frac{X_t}{\sqrt{2 \log t}} = -1, \quad \lim_{t \to \infty} \frac{X_t}{\sqrt{2 \log t}} = 1.
\]

While our argument in this note yields

\[
\lim_{t \to \infty} \frac{|X_t|}{\sqrt{2 \log t}} \leq 1.
\]

Hence this example indicates that our method can give optimal bounds in some cases (hopefully always so).

The second example is the following one dimensional SDE:

\[
dX_t = -U'(X_t)dt + \sqrt{2\varepsilon}dW_t
\]

with

\[
\int e^{-U(x)/\varepsilon}dx < \infty.
\]

Here \(U(x)\) is a polynomial with leading term being \(cx^{2p}\) for some \(c > 0, p \geq 1\). Then our argument gives

\[
\lim_{t \to \infty} \frac{|X_t|}{(\log t)^{1/2p}} \leq \left(\frac{\varepsilon}{c}\right)^{1/2p}
\]

for all \(X_0 = x\) and all BM orbits \(W \in \Lambda\) with \(P(\Lambda) = 1\), since the strong completeness of the model is guaranteed by \([4]\).

We would like to give some discussions. As is well known, Birkhoff’s ergodic theorem is an extension of Kolmogrov’s strong law of large numbers (abbr. SLLN); It also holds for stationary processes with continuous-time parameter under suitable \(L_1\)-integrability condition. In probability theory, when the \(L_1\)-integrability condition is replaced by \(L_p\)-integrability condition (with \(p \in (0, 2)\)), Marcinkiewicz-Zygmund’s SLLN would take place.
of Kolmogrov’s SLLN for i.i.d. random variables sequence. It is easy to see that the following result holds, which is a generalization of one part of Marcinkiewicz-Zygmund’s SLLN.

**Theorem 2** Let \( \{X_n : n \geq 0\} \) be an stationary process with \( \mathbb{E}|X_0|^p < \infty \) for some \( p \in (0, 1) \). Then

\[
\lim_{n \to +\infty} \frac{1}{n^{1/p}} \sum_{k=0}^{n-1} X_k = 0 \quad \text{almost surely.}
\]

The counterpart of \( p \in (1, 2) \) as the above theorem to Marcinkiewicz-Zygmund’s SLLN seems still unknown. Also, it is interesting to ask the validity of the continuous-time counterpart of the above theorem. It seems to us that, a proper statement might be as the following: Let \( \{X_t : t \geq 0\} \) be an stationary process with \( \mathbb{E}|X_0|^p < \infty \) for some \( p \in (0, 1) \). Assume the continuity of \( X_t \) in \( t \). Then for all \( \varepsilon > 0 \)

\[
\lim_{T \to +\infty} \frac{1}{T^{1/p+\varepsilon}} \int_0^T X_s ds = 0 \quad \text{almost surely.}
\]

It is the deficiency of such a result that forces us to find the machinery mentioned in the second paragraph of Sect. 2.

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