Monogamy of quantum discord

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Abstract

The original quantum discord (QD) is shown to be not monogamous except for the three-qubit states. Recently, a complete monogamy relation for multiparty quantum system was established for entanglement in (Guo and Zhang 2020 Phys. Rev. A \textbf{101} 032301), and in addition, a new multipartite generalization of QD was proposed in (Radhakrishnan \textit{et al} 2020 Phys. Rev. Lett. \textbf{124} 110401). In this work, we firstly define the complete monogamy for this multipartite quantum discord (MQD) and the global quantum discord (GQD). MQD, with the same spirit as the complete monogamy of entanglement, is said to be completely monogamous (i) if it does not increase under coarsening of subsystems and (ii) if some given combination of subsystems reach the total amount of the correlation, then all other combinations of subsystems that do not include all the given subsystems do not contain such a correlation any more. Here, coarsening of subsystems means discarding or combining some subsystems up to the given partition. Simultaneously, the complete monogamy of GQD is also defined with slight modification on the coarsening relation. Consequently, we explore all the coarsening relations of MQD and show that it is completely monogamous with a modicum assumption. In addition, with the same spirit, we investigate all the coarsening relations for GQD and show that GQD is not completely monogamous. That is, in the sense of the complete monogamy relation, MQD as a generalization of the original QD captures the nature of such a quantum correlation, and thus it is nicer than GQD as a generalization.

1. Introduction

Quantum discord, as the foremost one of the quantum correlations beyond entanglement, has been extensively explored in the last two decades due to its remarkable applications in quantum information protocols [1–13]. It is originally defined for bipartite system as the minimized difference between the quantum mutual information with and without a von Neumann projective measurement applied on one of the subsystems [1, 2]. Consequently, several multipartite generalizations have been proposed based on different scenarios [7, 8, 11]. In reference [7], by means of the multipartite mutual information, the global quantum discord (GQD) is defined as the minimal difference between the mutual information the pre- and post-state under local von Neumann measurements. In reference [8], the total quantum discord and the genuine quantum discord were investigated based on the total information and the total classical correlation. Very recently, Radhakrishnan \textit{et al} [11] gave another way of extending quantum discord to multipartite case up to a fixed conditional local von Neumann measurement. The GQD, total quantum discord and the genuine quantum discord were investigated based on the total information and the total classical correlation. Very recently, Radhakrishnan \textit{et al} [11] gave another way of extending quantum discord to multipartite case up to a fixed conditional local von Neumann measurement. The GQD, total quantum discord and the genuine quantum discord are symmetric under exchange the subsystems, while the multipartite quantum discord (MQD) in reference [11] is asymmetric.

An important issue closely related to a nonlocal correlation measure of composite quantum system (such as entanglement measure, quantum discord, and Einstein–Podolsky–Rosen steering, etc) is to explore the distribution of the correlation over many parties [14–22]. In this context, there are two ways to describe the distribution: one is the monogamy or polygamy relation by the bipartite measure [14–18] and the other
one is the relation based on the multipartite measure [19, 20, 23]. Among these attempts, numerous efforts have been made for entanglement. However, there are very few studies on quantum discord. For the bipartite measure of quantum discord, it is proved that the square of quantum discord obeys the monogamy relation for three qubit states [24] while other systems always display polygamous behavior [24–28]. In reference [23], it has been shown that GQD acts as a monogamy bound for pairwise quantum discord in which one of the subsystem is fixed while the other part runs over all subsystems provided that bipartite discord does not increase under discard of subsystems. According to the framework of complete monogamy relation (for entanglement), when we deal with the multipartite systems, the traditional approach of exploring monogamy relation based the bipartite measure is not substantial. There are something missing [19]. Therefore, for MQD and GQD, we need to examine renewedly with the complete monogamy protocol.

The aim of this article is to explore the distribution of discord contained in a multipartite state whenever it is measured by MQD and GQD respectively in the framework of the complete monogamy relation. However, quantum discord is far different from entanglement: quantum discord is not monotonic under local operation and classical communication (LOCC) while it is a basic feature for entanglement. So we need at first to consider the coarsening relations when we consider monogamy of MQD and GQD and establish the framework of complete monogamy relation for MQD and GQD.

The rest of this paper is organized as follows. We recall the definitions of the original quantum discord, MQD, GQD, and investigate coarser relation of multiparty partitions by which we then introduce the monogamy laws in literatures in section 2. In section 3, we put forward the definition of complete multiparty quantum discord and by which we discuss and establish the framework of complete monogamy relation of the MQD. It is divided into three subsections. The first subsection discusses the tripartite case, the second subsection deals with the four-partite case and the main conclusion is given in the last subsection for the general multipartite case. In section 5, we deal with GQD in the framework of complete monogamy relation. Finally, we conclude. For convenience, throughout this paper, we call the n-partite discord in reference [11] MQD, and the one in references [7, 23] GQD, and we call both MQD and GQD multiparty quantum discord in order to differentiate these different concepts.

2. Preliminaries

In this section, we review at first the definitions of MQD and GQD, and then recall the monogamy relation and complete monogamy relation in literatures. Throughout this paper, we let $H^{A_1A_2...A_n}$ be the Hilbert space corresponding to the n-parite quantum system with finite dimension. And let $S^n$ be the set of density operators acting on $H^n$. The tripartite and the four-partite systems are always denoted by $H^{ABC}$ and $H^{ABCD}$, respectively.

2.1. Quantum discord and its generalization

For any bipartite state $\rho \in S^{AB}$, the original quantum discord is defined as [1, 2]

$$D_{A:B}(\rho) = \min_{\Pi^A} \left[ S_{B|\Pi^A}(\rho) - S_{B|A}(\rho) \right],$$

(1)

where $S_{B|A}(\rho) = S_{AB}(\rho) - S_A(\rho)$, $S(\rho) = -\text{Tr}(\rho \log \rho)$ is the von Neumann entropy,

$$S_{B|\Pi^A}(\rho) = \sum_j p_j S_{AB} \left( \Pi^A_j \rho \Pi^A_j / p_j \right) = S_{AB}(\rho_{\Pi^A}) - S_A(\rho_{\Pi^A}).$$

$\Pi^A_j$ is a one-dimensional von Neumann projection operator on subsystem $A$ and $p_j = \text{Tr}(\Pi^A_j \rho_{\Pi^A})$, $\rho_{\Pi^A} = \sum_j \Pi^A_j \rho_{\Pi^A}$. Hereafter, we denote $S(\rho^A)$ by $S_X(\rho)$ sometimes for simplicity.

For multipartite systems, the $(n-1)$-partite measurement is written [11]

$$\Pi^{A_1...A_{n-1}}_{j_1...j_{n-1}} = \Pi^{A_1}_{j_1} \otimes \Pi^{A_2}_{j_2} \otimes \ldots \otimes \Pi^{A_{n-1}}_{j_{n-1}},$$

(2)

where the $n$ subsystems are labeled as $A_i$, the measurements take place in the order $A_1 \rightarrow A_2 \rightarrow \ldots \rightarrow A_{n-1}$. For $\rho \in S^{A_1A_2...A_n}$, the $n$-partite MQD is defined by [11]

$$D_{A_1:A_2...A_n}(\rho) = \min_{\Pi^{A_1...A_{n-1}}} \left[ -S_{A_{n-1}:A_n}(\rho) + S_{A_{n-1}|\Pi^{A_1}}(\rho) \ldots + S_{A_1|\Pi^{A_1...A_{n-1}}}(\rho) \right]$$

(3)
up to the measurement ordering $A_1 \rightarrow A_2 \rightarrow \cdots \rightarrow A_n$, where

$$S_{A_k|\Pi^{i_1\cdots i_{t+1}}}(\rho) = \sum_{j_{i_{t+1}}} p_j^{(k-1)} S_{A_{i_{t+1}}|A_k}(\Pi_j^{(k-1)} \rho \Pi_j^{(k-1)})$$

with $\Pi_j^{(k)} \equiv \Pi_{j_{i_{t+1}}|A_k} \rho_j^{(k)} = \text{Tr}(\Pi_j^{(k)} \rho \Pi_j^{(k)})$. For example, the tripartite case $\rho \in S_{ABC}$,

$$D_{A:B:C}(\rho) = \min_{\Pi^{AB}} \left[-S_{BCA}(\rho) + S_{B|\Pi^{AC}}(\rho) + S_{C|\Pi^{AB}}(\rho)\right]$$

$$= \min_{\Pi^{AB}} \left[S(\rho^{AB}) - S(\rho^A) + S(\rho^B) - S(\rho^A) + S(\rho^B) - S(\rho) + S(\rho^A) \right]$$

up to the measurement ordering $A \rightarrow B$, where $\Pi^I(\rho) = \rho'$, $\Pi^{AB}(\rho) = \rho''$, and $\rho^X = \text{Tr}_X(\rho)$ with $\bar{X}$ denotes the complementary of $X$ in $\{A, B, C\}$ (e.g. $\rho^{AB} = \text{Tr}_C(\rho)$).

The QMD, denoted by $D_{A_1\cdots A_n}$, for an arbitrary state $\rho \in S^{A_1\cdots A_n}$ under a set of local measurements $\{\Pi_{j_1}, \ldots, \Pi_{j_m}\}$ is defined as [7, 23]

$$D_{A_1\cdots A_n}(\rho) = \min_{\Phi} \left[ I(\rho) - I(\Phi(\rho)) \right],$$

where

$$\Phi(\rho) = \sum_k \Pi_k \rho \Pi_k,$$

with $\Pi_k = \Pi_{j_1} \cdots \Pi_{j_m}$ and $k$ denoting the index string $(j_1, \ldots, j_m)$, the mutual information $I(\rho)$ is defined by [29]

$$I(\rho) = \sum_{k=1}^{n} S_{A_k}(\rho) - S_{A_1\cdots A_n}(\rho),$$

where

$$\Phi(\rho_j^{(k)}) = \sum_{j'} \Pi^{j'}_{j} \rho_j^{(k)} \Pi^{j}_{j'}.$$

Note here that, the difference between $D_{A_1;A_2;\cdots;A_n}$ and $D_{A_1\cdots A_n}$ is that the former is semicolon in the subscript $A_1; A_2; \ldots; A_n$ while the latter one is with colon in the subscript $A_1 : \cdots : A_n$.

2.2. Coarser relation of multipartite partition

For any partition $X_1|X_2|\ldots|X_s$ of $A_1A_2\ldots A_n$ with $X_t = A_{i(1)|A_{i(2)}\cdots A_{i(t)}}$, $s(i) < s(j)$ whenever $i < j$, and $s(\rho) < t(\rho)$ whenever $s < t$ for any possible $p$ and $q$, $1 \leq s, t \leq k$. For instance, partition $AB|C|DE$ is a three-partition of $ABCD$. Let $X_1|X_2|\ldots|X_k$ and $Y_1|Y_2|\ldots|Y_l$ be two partitions of $A_1A_2\ldots A_n$ or subsystem of $A_1A_2\ldots A_n$, $Y_1|Y_2|\ldots|Y_l$ be said to be coarser than $X_1|X_2|\ldots|X_k$, denoted by

$$X_1|X_2|\ldots|X_k \succ Y_1|Y_2|\ldots|Y_l,$$

if $Y_1|Y_2|\ldots|Y_l$ can be obtained from $X_1|X_2|\ldots|X_k$ by some or all of the following ways:

- (C1) Discarding some subsystem(s) of $X_1|X_2|\ldots|X_k$;
- (C2) Combining some subsystems of $X_1|X_2|\ldots|X_k$;
- (C3) Discarding some subsystem(s) of the last subsystem $X_k$ provided that the last subsystem $X_k$ in

$$X_1|X_2|\ldots|X_k$$

is $X_k = A_{i(1)}A_{i(2)}\cdots A_{i(k)}$ with $f(k) \geq 2$.

For example, $A|B|C|D|E \succ A|B|C|DE \succ A|B|C|D \succ AB|CD \succ AB|C|DE \succ A|B|C|D|E$. Clearly,

$$X_1|X_2|\ldots|X_k \succ Y_1|Y_2|\ldots|Y_l$$

and $Y_1|Y_2|\ldots|Y_l$ imply $X_1|X_2|\ldots|X_k \succ Z_1|Z_2|\ldots|Z_l$. For any partition $X_1|X_2|\ldots|X_k$ of $A_1A_2\ldots A_n$ or subsystem of $A_1A_2\ldots A_n$, the associated QMD is $D_{X_1|X_2|\ldots|X_k}$ up to the measurement $\Pi^{1\cdots k-1}$. For example, for partition $A|B|C|D$ of $ABCD$, the associated QMD is $D_{A:B:C}(\rho^{ABC})$ up to measurement $\Pi^{AB}$, while for partition $AB|C$ of the subsystem $ABC$ in $ABCD$, the associated QMD is $D_{AB:C}(\rho^{ABC})$ up to measurement $\Pi^{AB}$, where $\Pi^{AB}$ is a von Neumann measurement acting on $AB$ with $AB$ is regarded as a single particle, $\rho^{ABC} = \text{Tr}_D(\rho^{ABC})$. One can easily see that

$$X_1|X_2|\ldots|X_k \succ Y_1|Y_2|\ldots|Y_l$$

if and only if $Y_1|Y_2|\ldots|Y_l$ can be induced by $X_1|X_2|\ldots|X_k$ but not vice versa. In such a case, we say $Y_1|Y_2|\ldots|Y_l$ and $X_1|X_2|\ldots|X_k$ are compatible. Otherwise, it is incompatible. For example, $\Pi^{ABC}$ and $\Pi^{AB}$ are incompatible on the system $H^{ABCD}$ since the projection on part $BC$ (regarded as a single part) is not necessarily a projection on $B/C$.

Furthermore, if $X_1|X_2|\ldots|X_k \succ Y_1|Y_2|\ldots|Y_l$, we denote by $\Xi(X_1|X_2|\ldots|X_k - Y_1|Y_2|\ldots|Y_l)$ the set of all the partitions that are coarser than $X_1|X_2|\ldots|X_k$ and either exclude any subsystem of $Y_1|Y_2|\ldots|Y_l$ or
include some but not all subsystems of $Y_1|Y_2|\ldots|Y_l$. We take the five-partite system $ABCDE$ for example,

$$\Xi(A|B|CD|E-A|B) = \{CD|E,A|CD|E,B|CD|E,A|CD, A|E, B|E, A|C, A|D, B|C, B|D\},$$

$$\Xi(A|B|C|D|E-A|C) = \{B|D|E, B|D|B|E, B|E, A|B, A|D, A|E, B|C|D, C|D,E, A|B|D, A|B|E, A|BE, AB|E, A|DE, AD|E, B|C|D, B|DE, BD|E, C|D|E, C|DE, CD|E, A|BD|E, B|CD, BCD|E, A|BD|E, A|BD, A|BD, A|BD, A|BD, A|BD\},$$

For more clarity, we fix the following notations. Let $X_1|X_2|\ldots|X_k$ and $Y_1|Y_2|\ldots|Y_l$ be partitions of $A_1A_2\ldotsA_n$ or subsystem of $A_1A_2\ldotsA_n$. We denote by

$$X_1|X_2|\ldots|X_k =^a Y_1|Y_2|\ldots|Y_l \quad (9)$$

for the case of (C1) and by,

$$X_1|X_2|\ldots|X_k =^b Y_1|Y_2|\ldots|Y_l \quad (10)$$

for the case of (C2), and in addition by

$$X_1|X_2|\ldots|X_k =^c Y_1|Y_2|\ldots|Y_l \quad (11)$$

for the case of (C3). For example, $A|B|C =^a A|B, A|B|C =^b A|BC, A|BC =^c A|B, A|BC =^c A|C$.

2.3. Complete multipartite entanglement measure

The counterpart to MQD for multipartite entanglement is the complete multipartite entanglement measure. A function $E^{(n)}: S^A_{A_1A_2\ldotsA_n} \rightarrow \mathbb{R}_+$ is called an $n$-partite entanglement measure in literature [30] if it satisfies:

- (E1) $E^{(n)}(\rho) = 0$ if $\rho$ is fully separable;
- (E2) $E^{(n)}$ cannot increase under $n$-partite LOCC.

When we take into consideration an $n$-partite measure of entanglement or other quantum correlation, we need discuss whether it is defined uniformly for any $k$-partite system at first, $k < n$. Let $Q^{(n)}$ be a multipartite measure (for entanglement or quantum discord, etc). If $Q^{(k)}$ is uniquely determined by $Q^{(n)}$ for any $2 \leq k < n$, then we call $Q^{(n)}$ a uniform measure. For example, MQD and GQD are uniquely defined for any $k$, thus they are uniform measures. The $n$-partite entanglement of formation [19] defined as $E_{ij}^{(n)}(\rho^{ij}) = \frac{1}{2} \sum_{i=1}^{m} S(p^A_i)$ for pure state and via the convex-roof extension for mixed states (i.e. $E^{(n)}(\rho) := \min_{\sum_{ij} p_{ij} \rho^{ij}} \sum_{i=1}^{m} p_{ij} E_{ij}^{(n)}(\rho^{ij})$ for any mixed state $\rho$, where the minimum is taken over all pure-state decompositions $\{p_{ij}, \rho_i^{ij}\}$ of $\rho$), is a uniform multipartite entanglement measure. That is, a uniform measure is series of measures that have uniform expressions definitely. A uniform multipartite entanglement measure $E^{(n)}$ is called a unified multipartite entanglement measure if it also satisfies the following condition [19]:

- (E3) the unification condition, i.e. $E^{(n)}$ is consistent with $E^{(k)}$ for any $2 \leq k < n$.

The unification condition should be comprehended in the following sense [19]. Let $|\psi^{A_1A_2\ldotsA_n}| = |\psi^{A_1A_2\ldotsA_k}| |\psi^{A_{k+1}\ldotsA_n}|$, then

$$E^{(n)}(|\psi^{A_1A_2\ldotsA_n}|) = E^{(k)}(|\psi^{A_1A_2\ldotsA_k}|) + E^{(n-k)}|\psi^{A_{k+1}\ldotsA_n}|,$$

and

$$E^{(n)}(\rho^{A_1A_2\ldotsA_n}) = E^{(n)}(\rho^{\pi(A_1A_2\ldotsA_n)})$$

for any $\rho^{A_1A_2\ldotsA_n} \in S^{A_1A_2\ldotsA_n}$, where $\pi$ is a permutation of the subsystems. In addition,

$$E^{(k)}(X_1|X_2|\ldots|X_k) \geq E^{(l)}(Y_1|Y_2|\ldots|Y_l)$$
for any $\rho^{A_1A_2\ldots A_n} \in S^{A_1A_2\ldots A_n}$ whenever $X_1|X_2|\ldots|X_k > a Y_1|Y_2|\ldots|Y_l$, where the vertical bar indicates the split across which the entanglement is measured. A uniform MEM $E^{(0)}$ is called a complete multipartite entanglement measure if it satisfies both (E3) above and the following [19]:

- (E4) $E^{(0)}(X_1|X_2|\ldots|X_k) \geq E^{(0)}(Y_1|Y_2|\ldots|Y_l)$ holds for all $\rho \in S^{A_1A_2\ldots A_n}$ whenever $X_1|X_2|\ldots|X_k > b Y_1|Y_2|\ldots|Y_l$.

It is easy to see that $E^{(0)}$ is a complete multipartite entanglement measure [19].

2.4. Monogamy relation

It is known that classical correlations can be freely shared among many parties, i.e. A party $A$ can have maximal classical correlations with two parties $B$ and $C$ simultaneously. But this is no longer the case if quantum entanglement or other nonlocal correlations are concerned [16]. The impossibility of sharing those types of nonclassical correlations unconditionally across many parties are known as monogamy constraints. For a given bipartite measure of nonlocal correlation $Q$, $Q$ is said to be monogamous (we take the tripartite case for example) if [14, 16]

$$Q(A|BC) \geq Q(AB) + Q(AC).$$  

(12)

However, equation (12) is not valid for many entanglement measures [14, 16] and quantum discord [24] but some power function of $Q$ admits the monogamy relation, i.e. $Q^\alpha(A|BC) \geq Q^\alpha(AB) + Q^\alpha(AC)$ for some $\alpha > 0$. In reference [17], we address this issue by proposing an improved definition of monogamy (without inequalities) for entanglement measure: a measure of entanglement $E$ is monogamous if for any $\rho \in S^{ABC}$ that satisfies the disentangling condition, i.e.

$$E(\rho^{ABC}) = E(\rho^{AB}),$$  

(13)

we have that $E(\rho^{AC}) = 0$. With respect to this definition, a continuous measure $E$ is monogamous according to this definition if and only if there exists $0 < \alpha < \infty$ such that

$$E^\alpha(\rho^{ABC}) \geq E^\alpha(\rho^{AB}) + E^\alpha(\rho^{AC}),$$  

(14)

for all $\rho$ acting on the state space $\mathcal{H}^{ABC}$ with fixed dim $\mathcal{H}^{ABC} = d < \infty$ (see theorem 1 in reference [17]). With this improved definition of monogamy, we proved that almost all the bipartite entanglement measures so far are monogamous [17, 18]. Notice that, for these bipartite measures, only the relation between $A|BC$ and $AB$ and $AC$ are revealed, the global correlation in $ABC$ and the correlation contained in part $BC$ is missed [19]. That is, the monogamy relation in such a sense is not ‘complete’. Recently, we established a complete monogamy relation for entanglement in reference [19]. For a unified tripartite entanglement measure $E^{(2)}$, it is said to be completely monogamous if it satisfies the complete disentangling condition [19], i.e. for any $\rho \in S^{ABC}$ that satisfies

$$E^{(2)}(\rho^{ABC}) = E^{(2)}(\rho^{AB}),$$  

(15)

we have that $E^{(2)}(\rho^{AC}) = E^{(2)}(\rho^{BC}) = 0$. If $E^{(2)}$ is a continuous unified tripartite entanglement measure. Then, $E^{(3)}$ is completely monogamous if and only if there exists $0 < \alpha < \infty$ such that [19]

$$E^\alpha(\rho^{ABC}) \geq E^\alpha(\rho^{AB}) + E^\alpha(\rho^{AC}) + E^\alpha(\rho^{BC}),$$  

(16)

for all $\rho^{ABC} \in S^{ABC}$ with fixed dim $\mathcal{H}^{ABC} = d < \infty$, here we omitted the superscript $(2,3)$ of $E^{(2,3)}$ for brevity. Let $E^{(3)}$ be a complete MEM. $E^{(3)}$ is defined to be tightly complete monogamous if it satisfies the tight disentangling condition, i.e. for any state $\rho^{ABC} \in S^{ABC}$ that satisfying [19]

$$E^{(3)}(\rho^{ABC}) = E^{(3)}(\rho^{A|BC}),$$  

(17)

we have $E^{(2)}(\rho^{BC}) = 0$, which is equivalent to

$$E^\alpha(\rho^{ABC}) \geq E^\alpha(\rho^{A|BC}) + E^\alpha(\rho^{BC})$$

for some $\alpha > 0$, here we omitted the superscript $(2,3)$ of $E^{(2,3)}$ for brevity. For the general case of $E^{(m)}$, one can similarly process with the same spirit.

3. Framework of complete monogamy relation for quantum discord

All the previous discussions on the monogamy for quantum discord in literatures are based on the bipartite quantum discord as in equation (12). The complete monogamy relation in equation (15) displays the
distribution of the correlation throughly. We thus adopt this scenario to describe the monogamy of MQD and GQD, namely, we consider the complete monogamy relation as equations (15) and (17) with MQD/GQD replacing entanglement measure E.

According to the complete multipartite entanglement measure, for a uniform measure of quantum correlation $Q^{(m)}$, we expect it satisfies the following conditions when we discuss the complete monogamy relation [we take the tripartite case for example, the $m$-partite ($m \geq 4$) case can be argued similarly]:

$$\begin{align*}
(U1) & : Q^{(2)}(\rho\otimes\rho^C) = Q^{(2)}(\rho^{AB}), & & \forall \rho^{AB} \in \mathcal{S}^{ABC}. \\
(U2) & : Q^{(3)}(\rho^{ABC}) = Q^{(3)}(\rho^{ABC}), & & \forall \rho^{ABC} \in \mathcal{S}^{ABC}. \\
(U3) & : Q^{(3)}(ABC) \geq Q^{(2)}(XY), & & \forall \rho^{ABC} \in \mathcal{S}^{ABC}. \\
(U4) & : Q^{(3)}(ABC) \geq Q^{(2)}(X|YZ), & & \forall \rho^{ABC} \in \mathcal{S}^{ABC}.
\end{align*}$$

We now begin to check whether MQD and GQD are unified. It is proved in reference [11] that $D_{AB;C} = D_{X;Y}$ for any $\rho = \rho^{XY} \otimes \rho^Z \in \mathcal{S}^{ABC}$, $\{X, Y, Z\} = \{A, B, C\}$. Namely, MQD satisfies (U1). Going further, for any bipartite state $\rho \in \mathcal{S}^{AB}$, for any given decomposition $\rho = \sum_i p_i \rho_i^{AB}$, it can be extended by adding an auxiliary system $C$ that does not correlate with $AB$ as

$$\rho = \sum_i p_i|ii\rangle \langle ii| \otimes \rho_i^{AB}.$$  

That is, $D_{C;A:B} = 0$. One can easily show that

$$D_{C;A:B} = D_{A:B}$$

for such a state. We can also show that (see in propositions 1 and 2 in the next section)

$$D_{A;C:B} \geq D_{A:B}$$

with the equality holds if and only if $D_{A;C} = 0$ for such a state. Equations (23) and (24) imply that: (i) when an auxiliary particle classically correlated with the state is added and it does not disturb the measurement ordering $A \rightarrow B$, then the MQD equals to the pre-state; (ii) when the auxiliary particle disturb the previous measurement ordering $A \rightarrow B$, then the multipartite discord does not decrease. That is MQD obeys more than (U1). But MQD violates (U2) obviously. The GQD satisfies (U1) since $D_{A;B;C} = D_{A:B} = D_{A:B}(\rho^{AB})$ and (U2) is clear since it is symmetric. However, for these measurement-induced correlations, whether they obey conditions (U3) and (U4), is not straightforward.

We are now ready to discuss the complete monogamy of these two generalizations of quantum discord. With the same principle as the unified/complete multiparty entanglement measure and the complete monogamy of entanglement introduced in the previous section, we can now give the definition of complete multiparty quantum discord and complete monogamy for MQD and GQD.

**Definition 1** Let $D_{A_1|A_2:...:An}$ be $D_{A_1:A_2:...:An}$ or $D_{A_1:A_2:...:An}$ defined on $\mathcal{S}^{A_1:A_2:...:An}$. With the notations aforementioned, $D_{A_1|A_2:...:An}$ is said to be complete if it is monotonic under coarsening of subsystem(s), i.e. for any $\rho \in \mathcal{S}^{A_1:A_2:...:An}$, we have

$$D_{X_i|X_{i-1}:...:X_1} \geq D_{Y_j|Y_{j-1}:...:Y_1}$$

holds for any partitions $X_1:X_2:...:|X_k$ and $Y_1:Y_2:...:|Y_l$ of $A_1:A_2:...:An$ or subsystem of $A_1:A_2:...:An$ with $X_1:X_2:...:X_k \succ Y_1:Y_2:...:Y_l$ whenever $D_{A_1|A_2:...:An} = D_{A_1:A_2:...:An}$ and with $X_1:X_2:...:X_k \succ a_{ij} Y_1:Y_2:...:Y_l$ whenever $D_{A_1|A_2:...:An} = D_{A_1:A_2:...:An}$.

**Definition 2** Let $D_{A_1|A_2:...:An}$ be $D_{A_1:A_2:...:An}$ or $D_{A_1:A_2:...:An}$ defined on $\mathcal{S}^{A_1:A_2:...:An}$. With the notations aforementioned, $D_{A_1|A_2:...:An}$ is said to be completely monogamous if it is complete and satisfies the dis-correlated condition, i.e. for any state $\rho \in \mathcal{S}^{A_1:A_2:...:An}$ that satisfies

$$D_{X_i|X_{i-1}:...:X_1} = D_{Y_j|Y_{j-1}:...:Y_1}$$

we have that

$$D_{Y_i} = 0$$

holds for all $\Gamma \in \Xi(X_1:X_2:...:X_k - Y_1:Y_2:...:Y_l)$, where $X_1:X_2:...:X_k$ and $Y_1:Y_2:...:Y_l$ are arbitrarily given partitions of $A_1:A_2:...:An$ or subsystem of $A_1:A_2:...:An$, and where $X_1:X_2:...:X_k \succ Y_1:Y_2:...:Y_l$ whenever $D_{A_1|A_2:...:An} = D_{A_1:A_2:...:An}$, $X_1:X_2:...:|X_k \succ a_{ij} Y_1:Y_2:...:Y_l$ whenever $D_{A_1|A_2:...:An} = D_{A_1:A_2:...:An}$.

We immediately get the definition of complete monogamous for a uniform measure of quantum correlation $Q^{(m)}$ of any $\rho \in \mathcal{S}^{A_1:A_2:...:An}$.

**Definition 3** Let $D_{A_1|A_2:...:An}$ be $D_{A_1:A_2:...:An}$ or $D_{A_1:A_2:...:An}$ defined on $\mathcal{S}^{A_1:A_2:...:An}$. With the notations aforementioned, $D_{A_1:A_2:...:An}$ is said to be completely monogamous if it is complete and satisfies

$$D_{X_i|X_{i-1}:...:X_1} = D_{Y_j|Y_{j-1}:...:Y_1}$$

we have that

$$D_{Y_i} = 0$$

holds for any $\Gamma \in \Xi(X_1:X_2:...:X_k - Y_1:Y_2:...:Y_l)$, where $X_1:X_2:...:X_k$ and $Y_1:Y_2:...:Y_l$ are arbitrarily given partitions of $A_1:A_2:...:An$ or subsystem of $A_1:A_2:...:An$, and where $X_1:X_2:...:X_k \succ Y_1:Y_2:...:Y_l$ whenever $D_{A_1:A_2:...:An} = D_{A_1:A_2:...:An}$, $X_1:X_2:...:|X_k \succ a_{ij} Y_1:Y_2:...:Y_l$ whenever $D_{A_1:A_2:...:An} = D_{A_1:A_2:...:An}$.
That is, MQD/GQD is complete if and only if it satisfies the conditions (U3) and (U4). Namely, for MQD and GQD, we do not distinguish conditions (U3) and (U4) which are corresponding to the unified measure and complete measure for entanglement respectively, and call it complete for both of them uniformly. In other words, MQD/GQD is a complete multiparty quantum discord if equation (25) hold true. Condition (26) is the counterpart to the complete disentangling condition together with the tight disentangling condition for multipartite entanglement. It is not necessary to distinguish these two disentangling conditions for MQD and GQD. In what follows, both the complete disentangling condition and the tight disentangling condition for multipartite entanglement are called complete disentangling condition for simplicity.

We illustrate definitions 1 and 2 with the four partite case. $D_{A:B,C:D}$ is complete if: for any state $\rho \in S^{A:B:C:D}$, $D_{A:B,C:D} \geq D_{X,Y,Z} \geq D_{M,N}$ for any $\{M, N\} \subseteq \{X, Y, Z\} \subseteq \{A, B, C, D\}$ with ordering $A \to B \to C \to D$ for both $M, N$ and $X, Y, Z$. In addition, $D_{A:B,C:D} \geq D_{A:B:C:D} \geq D_{A:B:C:D} \geq D_{A:B,C:D} \geq D_{A:B,C:D} \geq D_{A:B,C:D} \geq D_{A:B,C:D}$. $D_{A:B,C:D} \geq D_{A:B,C:D} \geq D_{A:B,C:D} \geq D_{A:B,C:D} \geq D_{A:B,C:D} \geq D_{A:B,C,D} \geq D_{A:B,C,D} \geq D_{A:B,C,D} \geq D_{A:B,C,D} \geq D_{A:B,C,D} \geq D_{A:B,C,D}$. $D_{A:B,C,D} \geq D_{A:B,C,D} \geq D_{A:B,C,D} \geq D_{A:B,C,D} \geq D_{A:B,C,D}$ is completely monogamous if: for any state $\rho \in S^{A:B:C:D}$ that satisfies $D_{A:B,C:D} = D_{A:B,C,D} \geq D_{A:B,C,D} = D_{A:B,C,D} = D_{A:B,C,D} = 0$. For any state $\rho \in S^{A:B:C:D}$ that satisfies $D_{A:B,C:D} = D_{A:B,C,D}$, we have that $D_{A:B,C,D} = D_{A:B,C,D} \geq D_{A:B,C,D} \geq D_{A:B,C,D} \geq D_{A:B,C,D} \geq D_{A:B,C,D} \geq D_{A:B,C,D} \geq D_{A:B,C,D} \geq D_{A:B,C,D} \geq D_{A:B,C,D} \geq D_{A:B,C,D} \geq D_{A:B,C,D}$.

Remark 1. We call it complete monogamy here for MQD and GQD since both MQD and GQD are multipartite measures, and this is consistent with the complete and tightly complete monogamy for multipartite entanglement measure. The previous research in this field is based on the bipartite measure of quantum discord [24–28] and it is called monogamy but not complete monogamy. This is analogous to that of bipartite entanglement measure in which it is called monogamy but not complete monogamy. That is, for both quantum discord and entanglement, ‘complete’ refers to the uniform multipartite measure and the one without ‘complete’ refers to the bipartite measure.

Remark 2. Comparing with the definition of the complete monogamy for entanglement, the difference is that, for entanglement we only require the condition (26), i.e. the complete disentangling condition for entanglement (see equations (13) and (15)). Condition (25) is true naturally for any unified multipartite entanglement measure $E^{(m)}$ since it is decreasing under tracing out any subsystem. But this fact is not obvious for other quantum correlation. So the monogamy of quantum discord need this assumption necessarily.

Remark 3. For MQD, equation (25) in definition 1 and equation (26) in definition 2 are presented under the fixed ordering $A_1 \to A_2 \to \cdots \to A_{n-1} \to A_n$ since MQD strictly depends on the ordering of the measurements on subsystems. But it is superfluous for the GQD since it admits (U2).

Remark 4. In definitions 1 and 2, MQD and GQD are different: the coarser relation of class (C3) is meaningless for GQD since in such a case the corresponding measurements are not compatible as we argued in subsection 2.2. It has been shown that there exists three-qubit state $\rho \in S^{A:B:C}$ such that $D_{A:B,C} < D_{A:B}$. However, although $D_{A:B,C} < D_{A:B}$ for some state, it seems true that $D_{A:B,C} \geq D_{M,N}$ for any tripartite state (also see the assumption in proposition 9), $\{M, N\} \subseteq \{A, B, C\}$. That is, the monotonicity should holds whenever the measurement on both sides are the same one. But for the case of MQD such a case cannot occur since the last subsystem is not measured, so the remaining measurements are always compatible. This is why we require three classes of coarsening relations when we comparing $D_{X_1,Z_2,\cdots,Z_n}$ and $D_{Y_1,Y_2,\cdots,Y_n}$ and only require coarsening classes of (C1) and (C2) for that of GQD in these definitions.
4. Monogamy of the multipartite quantum discord

For convenience, we fix some notations. For any local measurement $\Pi_{A_i-A_k}$ acting on the reduced state $\rho^{A_1-\cdots-A_k}$ of $\rho^{A_1-A_k}$, the conditional mutual information changes by an amount

$$d_{i_1,\ldots,i_{k+1}} = S_{A_{i_{k+1}}|A_{i_1}\cdots A_k} - S_{A_{i_{k+1}}|A_{i_1},\ldots,A_k},$$

where $1 \leq i_1 < i_2 \ldots < i_{k+1} \leq n$.

4.1. The tripartite case

Different from multipartite entanglement, it is unknown whether the correlation decreases under coarsening the subsystems.

**Proposition 1** For any state $\rho$ in $S_{ABC}$, the following coarsening relations hold:

- $D_{A:B:C} \geq D_{A:B} + D_{A:B:C} - D_{A:C}$
- $D_{A:B:C} \geq D_{B:C}$ provided that $d_{A:B:C} \geq d_{B:C}$.
- $D_{A:B:C} \geq D_{A:B:C} \geq D_{A:B}$.
- $D_{A:B:C} \geq D_{A:C}$ provided that $d_{A:B:C} \geq d_{A:C}$.

That is, if $d_{A:B:C} \geq d_{B:C}$ and $d_{A:B:C} \geq d_{A:C}$, then $D_{A:B:C}$ is a complete multipartite quantum discord.

**Proof.** We assume that $\rho^{A_1-A_k}$ is a complete multiparty quantum discord.

Note that, even though the optimal local measurement does not exist, we have that, for any given $\epsilon > 0$, there exist $\Pi^{AB}$ and $\Pi^{ABC}$ such that $D_{A:B:C} \geq S(\rho^{AB}) - S(\rho^{A}) + S(\rho^{ABC}) - S(\rho^{A}) - S(\rho^{ABC}) + S(\rho^{A}) - \epsilon$ and $S(\rho^{AB}) - S(\rho^{A}) - S(\rho^{ABC}) + S(\rho^{A}) - \epsilon \geq 0$ is equivalent to $S(\rho^{AB}) - S(\rho^{A}) + S(\rho^{ABC}) - S(\rho^{A}) - S(\rho^{ABC}) + S(\rho^{A}) \geq 0$ since $\epsilon$ can be arbitrarily small. So we can assume with no loss of generality that the optimal local measurement always exists hereafter. Similarly,

$$d_{A:B:C} \geq (S_{AB} - S_{A}^\epsilon + S_{AB} - S_{AB} + S_A + S_{A}) - (S_{AB}^\epsilon - S_{A}^\epsilon + S_{AB} + S_A)$$

$$= (S_{ABC} - S_{ABC} + S_{ABC}) + (S_{ABC} - S_{ABC} + S_{ABC})$$

$$\geq 0.$$
Observe that
\[
D_{A:B:C} - D_{B:C} \geq (S''_{AB} - S''_{A} + S''_{ABC} - S''_{AB} - S_{ABC} + S_{A}) - (S''_{B} - S''_{A} + S''_{BC} + S_{B})
\]
\[
= (S''_{AB} - S''_{A} + S''_{ABC} - S''_{AB} - S_{ABC} + S_{A}) - (S''_{B} - S''_{A} + S''_{BC} + S_{B})
\]
\[
= (S''_{ABC} - S''_{AB} - S_{ABC} + S_{A}) + (S''_{A} - S''_{A} - S_{AB} + S_{A})
\]
\[
- (S''_{B} - S''_{B} - S_{BC} + S_{B}),
\]
thus \(D_{A:B:C} \geq D_{B:C}\) since \(d_{A:B:C} \geq d_{B:C}\) by assumption. The following calculation are clear.
\[
D_{A:B:C} - D_{A:B} \geq (S''_{AB} - S''_{A} + S''_{ABC} - S''_{AB} - S_{ABC} + S_{A}) - (S''_{A} - S''_{A} - S_{ABC} + S_{A}) = (S''_{AB} - S''_{A} + S''_{ABC} - S''_{AB} - S_{ABC} + S_{A}) - (S''_{A} - S''_{A} - S_{ABC} + S_{A})
\]
implies \(D_{A:B:C} \geq D_{A:B}\) as well.

Thus, we complete the proof.

**Proposition 2**

\(D_{A:B:C}\) is completely monogamous if it is complete.

**Proof.** We use the notations in the proof of proposition 1.

Case 1: \(D_{A:B:C} = D_{A:B}\). If \(D_{A:B:C}(\rho^{ABC}) = D_{A:B}(\rho^{AB})\), we get \(S(\rho''^{ABC}) - S(\rho^{ABC}) = S(\rho^{ABC})\) for some von Neumann measurement \(\Pi^{ABC}(\rho^{ABC}) = \rho^{ABC}\), which implies that \(\rho^{ABC} = \sum_{i}p_{i}|i\rangle\langle i| \otimes \rho_{i}^{B}\). It is clear that \(D_{A:B:C} = D_{A:B} = 0\) for such a state.

Case 2: \(D_{A:B:C} = D_{B:C}\). According to equation (29), for any state \(\rho \in S^{ABC}\), if \(D_{A:B:C} = D_{B:C}\), then \(S''_{ABC} - S''_{A} - S_{ABC} + S_{A} = I_{BCA} - I_{BCA}^{B} = 0\). Therefore \(\rho = \sum_{i}p_{i}|i\rangle\langle i| \otimes \rho_{i}^{B}\) for some basis \(\{|i\rangle\rangle\}\) of \(\mathcal{H}^{A}\), and thus \(D_{A:B} = D_{A:C} = 0\).

Case 3: if \(D_{A:B:C} = D_{A:C}\), one can easily check that \(\rho = \sum_{i}p_{i}|i\rangle\langle i| \otimes \rho_{i}^{A}\) for some basis \(\{|i\rangle\rangle\}\) in \(\mathcal{H}^{B}\). It follows that \(D_{A:B} = D_{B:C} = 0\). The other two cases \(D_{A:B:C} = D_{A:B}\) and \(D_{A:B:C} = D_{A:B}\) can be checked analogously.

Using the same notations as in the proof of proposition 1, it is clear that, for any tripartite state \(\rho \in S^{ABC}\),
\[
D_{A:B:C} - D_{A:B} - D_{A:C} \geq (S''_{ABC} - S''_{AB} - S_{ABC} + S_{A}) - (S''^{AC} - S''^{A} - S_{AC} + S_{A}) \geq 0
\]
whenever \(d_{A:B:C} \geq d_{A:C}\). That is

**Proposition 3**

For any state \(\rho \in S^{ABC}\) that satisfies \(d_{A:B:C} \geq d_{A:C}\), we have
\[
D_{A:B:C} \geq D_{A:B} + D_{A:C}.
\]

### 4.2. The four-partite case

With the increasing of particles involved, the hierarchy of coarsening relations become more complicated.

**Proposition 4**

For any state \(\rho \in S^{ABCD}\), the following coarsening relations hold:

- \(D_{A:B:C:D} \geq D_{A:B:C} + D_{ABC:D}\).
- \(D_{A:B:C:D} \geq D_{A:B:D}\).
- \(D_{A:B:C:D} \geq D_{A:B}, D_{A:B:C,D} \geq D_{A:C}, D_{A:B:C,D} \geq D_{A:D}\).
- \(D_{A:B:C:D} \geq D_{A:C,D} \) provided that \(d_{A:B:C:D} \geq d_{A:C,D}\).
- \(D_{A:B:C:D} \geq D_{A:C,D} + D_{A:B} \) provided that \(d_{A:B:C:D} \geq d_{A:C,D} \) and \(d_{A:B,C} \geq d_{A:C}\).
- \(D_{A:B:C:D} \geq D_{B:C,D} + D_{A:B} \) provided that \(d_{A:B:C:D} \geq d_{B:C,D} \) and \(d_{A:B,C} \geq d_{B:C}\).
- \(D_{A:B:C:D} \geq D_{A:B,D} + D_{A:B,C} \) provided that \(d_{A:B:C:D} \geq d_{A:B,D}\).
- \(D_{A:B:C,D} \geq D_{B:C,D} \) provided that \(d_{A:B:C,D} \geq d_{B:C}\) or \(d_{A:B,C:D} \geq d_{B:C}\).
- \(D_{A:B:C,D} \geq D_{B:D} \) provided that \(d_{A:B:C,D} \geq d_{B:D}\).
- \(D_{A:B,C,D} \geq D_{C,D} \) provided that \(d_{A:B:C,D} \geq d_{C,D}\).
- \(D_{A:B,C,D} \geq D_{A:B} + D_{A:B,C} \).
- \(D_{A:B,C,D} \geq D_{A:B,D} \).
- \(D_{A:B,C,D} \geq D_{A:B,C,D} \).
- \(D_{A:B,C,D} \geq D_{A:B,C,D} + D_{A:B} \).
- \(D_{A:B,C,D} \geq D_{A:B,C,D} \).
- \(D_{A:B,C,D} \geq D_{A:B,C,D} \).
That is, \( D_{A;B;C;D} \) is a complete multiparty quantum discord if \( d_{ABC} \geq d_{AB;D}, d_{ABC;D} \geq d_{AC;D}, d_{ABC;D} \geq d_{BC;D}, d_{AB;C} \geq d_{A;C}, \) and \( d_{ABC} \geq d_{B;C} \).

**Proof.** For any \( \rho \in S^{ABCD} \), we assume that \( D_{A;B;C;D}(\rho^{ABCD}) = S(\rho^{AB}) - S(\rho^{ABC}) - S(\rho^{AB}) + S(\rho^{ABCD}) - S(\rho^{ABC}) - S(\rho^{ABCD}) \) for some \( \Pi^{ABCD} \) with \( \Pi^{AB}(\rho^{ABCD}) = \rho^{ABCD} \) and \( \Pi^{ABC}(\rho^{ABCD}) = \rho^{ABC} \). (Hereafter, in this subsection, we denote by \( \rho^{AB} \) the state after the optimal local measurement on one particle of the four-partite state, by \( \rho^{ABC} \) the state after the optimal local measurement on two particles of the four-partite state, and by \( \rho^{ABCD} \) the state after the optimal local measurement on three particles of the four-partite state.) Obviously, \( D_{A;B;C}(\rho^{ABCD}) \leq S_{AB}^{m} - S_{A}^{m} + S_{ABC}^{m} - S_{AB}^{m} + S_{A}^{m} + S_{ABC}^{m} - S_{AB}^{m} \) and

\[
D_{ABC} = S_{ABC}^{m} - S_{A}^{m} - S_{AB}^{m} + S_{ABC}^{m} \\
\geq 0.
\]

Similarly,

\[
D_{A;B;C;D} - D_{A;B;C} \geq (S_{ABC}^{m} - S_{A}^{m} - S_{AB}^{m} + S_{ABC})+(S_{ABC}^{m} - S_{A}^{m} - S_{AB}^{m} + S_{ABC}) \\
\geq 0.
\]

provided that \( d_{ABC} \geq d_{AC} \). Evidently, \( D_{A;B;C;D} \geq D_{A;B;C} \) whenever \( d_{ABC;D} \geq d_{AC;D} \) and \( d_{ABC} \geq d_{A;C} \). With the same argument, one can check that \( D_{A;B;C;D} \geq D_{AB;C} \) and \( D_{A;B;C} \geq D_{B;C} \). Since \( D_{A;B;C;D} \geq D_{A;B;C} \) and \( D_{A;B;C;D} \geq D_{A;C} \), we get \( D_{A;B;C;D} \geq D_{A;B} \) and \( D_{A;B;C;D} \geq D_{A;C} \) directly. We also have \( D_{A;B;C;D} \geq D_{A;D} \) since

\[
D_{A;B;C;D} - D_{A;D} \geq (S_{ABC}^{m} - S_{A}^{m} - S_{AB}^{m} + S_{ABC})+(S_{ABC}^{m} - S_{A}^{m} - S_{AB}^{m} + S_{ABC}) \\
\geq 0.
\]
\[ (S''_{ABC} - S''_{AB} - S''_{ABCD} + S'_{ABC}) + (S''_{ABCD} - S''_{AD} - S_{ABCD} + S_{AD}) \]
\[ + (S''_{ABC} - S''_{AB} - S''_{ABCD} + S'_{AB}) \geq 0. \]

The other items can also be easily checked, and thus the proof is completed. \( \square \)

**Proposition 5** \( D_{A,B,C,D} \) is completely monogamous if it is complete, i.e.

- \( d_{ABCD} \geq d_{XYD} \) for any state \( \rho \in S^{ABCD} \).
- \( d_{ABCD} \geq d_{ED} \) for any state \( \rho \in \{A, B, C\} \).
- \( d_{AB} \geq d_{BC} \) hold for any state \( \rho \in S^{ABCD} \).

**Proof.** For convenience, we use the same notations as that of the proof of proposition 4. If \( D_{ABCD} = d_{ABCD}^{\rho ABCD} = D_{AB}^{\rho AB} \), then \( S''_{ABC} - S''_{AB} - S_{ABCD} + S_{ABC} = 0 \) and \( S''_{AB} - S_{ABCD} + S_{AB} = 0 \), which implies that \( \rho^{ABCD} = \sum_{i,j,k} p_{ijkl} |k\rangle \langle j| \otimes |i\rangle \langle l| \otimes \rho_{ijkl}^D \). Thus \( D_{A,C} = D_{A,D} = D_{B,C} = D_{B,D} = 0 \). If \( D_{ABCD} = d_{ABCD}^{\rho ABCD} = D_{ABC}^{\rho ABC} \), then \( S''_{ABC} - S''_{AB} - S_{ABCD} + S_{ABC} = 0 \), which implies that \( \rho^{ABCD} = \sum_{i,j,k} p_{ijkl} |k\rangle \langle j| \otimes |i\rangle \otimes \rho_{ijkl}^D \). Thus \( D_{A,D} = D_{B,D} = D_{C,D} = 0 \). Similarly, one can easily verify that all the other dis-correlate conditions are valid. \( \square \)

### 4.3. The \( n \)-partite case

Moving to general \( n \)-partite system, we can conclude the following theorems which are main results of this paper by similar arguments as that of propositions 4 and 5.

**Theorem 6** Let \( D_{A_1,A_2,\ldots,A_n} \) be the \( n \)-partite quantum discord defined on \( S^{A_1A_2\ldots A_n} \). Then the following holds true for any state \( \rho \in S^{A_1A_2\ldots A_n} \):

- \( D_{A_1,A_2,\ldots,A_n} \geq D_{A_1,A_2,\ldots,A_n}^{\rho A_1A_2,\ldots,A_n} + D_{A_1A_2,\ldots,A_n}^{A_1,A_2,\ldots,A_n} \).
- \( D_{A_1,A_2,\ldots,A_n}^{\rho A_1A_2,\ldots,A_n} \geq D_{A_1,\ldots,A_n}^{A_1,\ldots,A_n} \geq 0 \).
- \( D_{A_1,A_2,\ldots,A_n}^{\rho A_1A_2,\ldots,A_n} \geq D_{A_1,A_2,\ldots,A_n}^{A_1,A_2,\ldots,A_n} \geq 0 \).
- \( D_{A_1,A_2,\ldots,X_l} \geq D_{Y_1,Y_2,\ldots,Y_l} \) for any \( X_l \) \( \ll Y_1 \) \( Y_2 \) \( Y_l \).
- \( D_{X_1,X_2,\ldots,X_l} \leq D_{Y_1,Y_2,\ldots,Y_l} \) whenever \( X_1 \) \( Y_1 \) \( X_2 \) \( Y_2 \) \( X_l \) \( Y_l \).
- \( D_{A_1,A_2,\ldots,A_n} \) is complete multiparty quantum discord if it is complete.

\[ D_{X_1,X_2,\ldots,X_l} \leq D_{Y_1,Y_2,\ldots,Y_l} \]

\[ D_{X_1,X_2,\ldots,X_l} \geq D_{Y_1,Y_2,\ldots,Y_l} \]

\[ D_{X_1,X_2,\ldots,X_l} \geq D_{Y_1,Y_2,\ldots,Y_l} \]

\[ D_{X_1,X_2,\ldots,X_l} \geq D_{Y_1,Y_2,\ldots,Y_l} \]

That is, if equation (30) holds true, then \( D_{A_1,A_2,\ldots,A_n} \) is a complete multiparty quantum discord.

**Proof.** For any \( n \), analogous to that of tripartite and four-partite cases, all these coarsening relations can be derived by taking the optimal measurement for the former one [e.g. in relation \( D_{A,B,C,D} \geq D_{A,B,C,D} \)], we assume that \( D_{A,B,C,D} \) is obtained by the optimal measurement, i.e. \( D_{A,B,C,D} = -S_{ABCD} + S_A + S_{AB} - S_{ABCD} + S''_{ABCD} - S''_{ABCD} = (S''_{ABCD} - S''_{ABCD} + S_{ABCD} + S_{ABCD}) + (S''_{ABCD} - S''_{ABCD} + S_{ABCD} + S_{ABCD}) \), and then by using either the fact that the mutual information is decreased under local operation (which is equivalent to \( S_{XY} - S_X - S_{XY} + S_X \geq 0 \), or such a fact together with the assumption (30). The former four cases can be derived without the assumption (30) since all these cases can be reduced to \( S_{XY} - S_X - S_{XY} + S_X \geq 0 \). The last case can be easily processed under the assumption (30).

Using similar arguments as that of propositions 2 and 5, we can conclude the following theorem easily.

**Theorem 7** The multiparticle discord \( D_{A_1,A_2,\ldots,A_n} \) is completely monogamous if it is complete, or equivalently, if it is monotonic under coarsening of subsystem(s).

The former four monotonicity relations in theorem 6 are true automatically. So, in theorem 7, we only need the other monotonicity conditions as equation (31). (Here, we should note that equation (31)
contains the second and the third monotonicity relations in theorem 6 as special cases.) We thus obtain the following proposition.

**Proposition 8** The multipartite discord $D_{A_1;A_2;...;A_n}$ is completely monogamous provided that the condition (31) is valid.

Although $D_{A_1;A_2;...;A_n}$ is not continuous (since $D_{A_1;A_2}$ is not continuous [6, 32]), $n > 2$, we still can, taking the tripartite case for example, get the following monogamy relation: if $D_{A,B,C}$ is monotonous under discord of subsystems, then for any given $\rho \in \mathcal{S}^{ABC}$ there exists $0 < \alpha < \infty$ such that

$$D_{R}^{\alpha} \geq D_{A,B}^{\alpha} + D_{A,C}^{\alpha} + D_{B,C}^{\alpha}.$$  

(32)

That is, $\alpha$ is dependent not only on $\dim \mathcal{H}_{ABC}$ but also on the given state.

5. The coarser relation of global quantum discord

In reference [23], the following monogamy bound is proved provided that the bipartite discord does not increase under loss of subsystems, i.e. $D_{A_1;A_2;A_{k+1}} \geq D_{A_1;A_{k+1}}$, $2 \leq k < n$:

$$D_{A_1;...;A_{n}} \geq \sum_{k=1}^{n-1} D_{A_1;A_{k+1}}.$$  

(33)

However, the assumption $D_{A_1;...;A_{k+1}} \geq D_{A_1;A_{k+1}}$ is not valid in general. For example, consider a three qubit state $\rho_{ABC} = (1/2)(|000\rangle \langle 000| + |111\rangle \langle 111|)$, where $|+\rangle = ((|0\rangle + |1\rangle)/\sqrt{2}$, it is evident that $D_{ABC} = 0 < D_{A,B,C}$. That is, $D_{A_1;A_2;A_{k+1}}$ and $D_{A_1;A_{k+1}}$ are uncomparable since the associated measurements are not compatible.

For any state $\rho \in \mathcal{S}^{ABC}$, we let $D_{A,B,C}(\rho) = S_{ABC} - S_A - S_B - S_C - S_{ABC} + S_A + S_B + S_C$ for some local von Neumann measurement $\Phi(\rho) = \rho'$. It follows that

$$D_{A,B,C} - D_{ABC} \geq (S_{ABC} - S_A - S_B - S_C - S_{ABC} + S_A + S_B + S_C)$$

$$- (S_{ABC} - S_A' - S_B' - S_{ABC} + S_A + S_B + S_C')$$

$$= (S_B + S_C - S_{BC}) - (S_B' + S_C' - S_{BC}') \geq 0$$

since mutual information is decreased under local von Neumann measurement. With the notations as in section 4, we let $X_1, X_2, ..., X_k$ and $Y_1, Y_2, ..., Y_l$ be two arbitrarily given partitions of $A_1, A_2, ..., A_n$ or subsystem of $A_1, A_2, ..., A_n$, and assume that $X_1, X_2, ..., X_k \succ Y_1, Y_2, ..., Y_l$. Then we can conclude $D_{X_1, X_2, ..., X_k} \geq D_{Y_1, Y_2, ..., Y_l}$.

We now begin to discuss the coarsening relation of GQD as a complement to the results in reference [23]. For any state $\rho^{XY} \in \mathcal{S}^{XY}$, we denote $S(\rho^{XY} || (\rho^{X})^{\otimes Y})$ by $S_{XY}(\rho^{XY})$ for brevity, where $S(\rho || \sigma) = \text{tr}(\rho \log \rho - \rho \log \sigma)$ is the relative entropy. For any state $\rho \in S(A_1;A_2;...;A_n)$, we write state after measurement $\Phi$ as in equation (5) as $\rho'$, write $S(\rho^{XY} || (\rho^{X})^{\otimes Y})$ as $S_{XY}(\rho^{XY})$, and write $I(\rho) - I(\rho')$ as $d_{A_1;A_2;...;A_n}^{\rho}$. Then, it is easy to argue that, for any $k < n$,

$$d_{A_1;A_2;...;A_n}^{\rho} - d_{A_1;A_2;...;A_k}^{\rho} = S_{A_1A_2...A_n|A_1A_2...A_k} = S_{A_1A_2...A_n|A_1A_2...A_k} - S_{A_1A_2...A_n|A_1A_2...A_k}$$

(34)

where $j = 1, 2, ..., n$. With these notations in mind, we can easily conclude the following coarsening relations.

**Proposition 9 (i)** *Using the notations as equation (34), if $d_{A_1;A_2;...;A_n}^{\rho} \geq d_{A_1;A_2;...;A_k}^{\rho}$ for any $k < n$, then $D_{A_1;A_2;...;A_n}$ is monotonic under discord of subsystems.* (ii) $D_{X_1, X_2, ..., X_k} \geq D_{Y_1, Y_2, ..., Y_l}$ whenever $X_1, X_2, ..., X_k \succ Y_1, Y_2, ..., Y_l$. Namely, $D_{A_1;A_2;...;A_n}$ is complete whenever $d_{A_1;A_2;...;A_n}^{\rho} \geq d_{A_1;A_2;...;A_k}^{\rho}$ for any $k < n$.

Next, we consider the complete monogamy of $D_{A_1;A_2;...;A_n}$. Take a three qubit state $\rho_{ABC} = (1/2)(|000\rangle \langle 000| + |111\rangle \langle 111|)$ as in reference [23], where $|\pm\rangle = ((|0\rangle + |1\rangle)/\sqrt{2}$. For this state, it is shown in reference [23] that $D_{ABC} = D_{AB} \approx 0.204$ and $D_{AC} = 0$. It is easy to see that $D_{BC} = D_{AB}$. That is, $D_{ABC} = D_{AB}$ but $D_{BC} \neq 0$. Namely, GQD violates the dis-correlated condition. We thus obtain the following theorem.
Table 1. Comparison between MQD, GQD and complete multipartite entanglement measure for tripartite case. CM is the abbreviation of ‘complete monogamous’.

| Measure | U1 | U2 | U3 | U4 | CM |
|---------|----|----|----|----|----|
| $D_{A_1;A_2;A_3}$ | ✓ | × | Partially, conditionally$^a$ | ✓ | Conditionally$^b$ |
| $D_{A_1;A_2;A_3}^{A_4}$ | ✓ | ✓ | Conditionally$^c$ | ✓ | × |

$^a$ See theorem 6.
$^b$ See theorem 7.
$^c$ See proposition 9.

**Theorem 10** $D_{A_1;...;A_n}$ is not completely monogamous.

That is, even though GQD is complete, it is not completely monogamous. Comparing with MQD, MQD $D_{A_1;A_2;...;A_n}$ is better than GQD $D_{A_1;A_2;...;A_n}$ in such a sense. Of course that, $D_{A_1;A_2;...;A_n}$ has some merit, e.g. it is symmetric but $D_{A_1;A_2;...;A_n}$ is not symmetric (see table 1 for more detail).

6. Conclusions and discussions

In this paper we have discussed the monogamy relation of MQD and GQD in detail. Different from the monogamy scenario of quantum discord discussed in the previous literatures, we defined complete multiparty quantum discord and by which we put forward the complete monogamy framework for both MQD and GQD. In such a framework, to characterize the distribution of quantum discord, the first issue is to check whether the quantity is monotonic under coarsening of subsystems and the other issue it to investigate the dis-correlate condition. The dis-correlate condition of quantum discord is the counterpart to the disentanglement condition for entanglement. With the assumption of monotonicity under coarsening of subsystems, we proved that the MQD is completely monogamous. We presented counterexamples which implies that GQD is not completely monogamous. This fact also supports that the MQD in reference [11] is an excellent generalization of quantum discord. Going further, we conjecture that the assumptions in theorems 6, 7, propositions 8 and 9 are true but it seems difficult to prove and remains further investigation in the future. In addition, our framework can be used for any multiparty nonlocal quantum correlation whenever the associated monogamy relation is concerned.

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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