Determination of Bayesian optimal warranty length under Type-II unified hybrid censoring scheme

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ABSTRACT
We consider determination of optimal warranty length for the combined free replacement and pro-rata warranty (FRW-PRW) policy based on the Type-II unified hybrid censored data by Bayesian approach. A non-linear pro-rata rebate cost function is proposed based on which warranty cost is computed. It is assumed that the lifetime follows log normal distribution. The optimal warranty length is obtained by maximizing an expected utility function consisting of three cost functions such as economic benefit function, warranty cost function and dissatisfaction cost function. The expectation is taken with respect to the posterior predictive model for the time-to-failure data. It is observed that the non-linear pro-rata rebate cost function gives a larger warranty length with maximum profit as compared to linear pro-rata rebate cost function. A real-data set is analyzed in order to illustrate the proposed methodology of finding optimal warranty length.

1 Introduction

List of notations:

| Symbol | Description |
|--------|-------------|
| \( X_i \) | Lifetime of the \( i \)th testing unit |
| \( T_1, T_2 \) | Time truncating parameters of the censoring scheme |
| \( r, l \) | Failure truncating parameters of the censoring scheme |
| \( X_{in} \) | \( i \)th order statistic from size \( n \) |
| \( D \) | Number of failures |
| \( \xi \) | Duration of testing |
| \( \mu, \tau \) | Parameters of the log-normal distribution |
| \( a_1, b_1 \) | Parameters of the gamma prior |
| \( c_1, d_1 \) | Parameters of the normal prior |
| \( a \) | Parameter which controls non-linearity of the rebate function |
| \( S \) | Sales price of a product |
| \( w_1, w_2 \) | Parameters of the warranty length |
| \( A_1 \) | The parameter to control the speed of increment in cost benefit function |
| \( A_2 \) | Manufacturer’s profit of a product |
| \( L \) | Consumer’s expected lifetime on a product |

Warranty analysis of a manufactured product is an integral part of statistical quality improvement. Improper warranty analysis may affect the business goal and the perceived quality can be in turmoil. Warranty is defined (see Blischke et al., 2011) as a contractual agreement between

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manufacturer (or seller) and consumer (or buyer) that is entered into upon sale of a product. This contract defines the compensation available to the buyer if the performance of the product is found to be unsatisfactory. Therefore, by providing warranties, manufacturer tries to gain the consumer’s belief about the quality of the product. However, if the quality of the product is not satisfactory, but manufacturer offers an unrealistically large warranty period, then it may incur high penalty cost to the manufacturer. Also, if the warranty period is smaller in comparison with the other competitors in the market, then sales volume of the product may decrease. Therefore, finding an appropriate warranty period is an important task for the manufacturer. The usual way to find the warranty length is based on the assessment of product reliability. The reliability assessment is typically done through a life-testing experiment. In practice, life-tests are conducted under various censoring schemes in order to save time and cost of the experimentation.

In this article, we consider Type-II unified hybrid censoring scheme, abbreviated as Type-II UHCS, (see Balakrishnan et al., 2008), which is the generalization of the generalized Type-I and Type-II hybrid censoring schemes (see Chandrasekar et al., 2004). The Type-II UHCS can be described as follows. The testing starts with n units and alongside two integers l, r ∈ {1, 2, ⋯ , n} and two time points T_1, T_2 ∈ (0, ∞) are chosen such that l < r and T_1 < T_2. If the rth failure occurs before time T_1, terminate the test at T_1. If the lth failure occurs before T_1 and rth failure occurs between T_1 and T_2, terminate the test at rth failure time. If the lth failure occurs before T_1 and rth failure occurs after T_2, terminate the test at T_2. If the lth failure occurs after T_1 and rth failure occurs before T_2, terminate the experiment at rth failure time. If the lth failure occurs after T_1 and rth failure occurs after T_2, terminate the test at T_2. Finally, if the lth failure occurs after time T_2, terminate the experiment at lth failure time. The advantage of Type-II UHCS is that it ensures at least l failures and the test duration does not exceed max\{X_l, T_2\}. A schematic representation of Type-II UHCS is presented in Figures 1 and Figure 2. Note that the values of the design parameters

![Figure 1](image-url)
are chosen by the experimenter based on convenience or by some optimal methods. The determination of optimal design parameter values is beyond the scope of this article. Therefore, we assume some convenient values of these parameters for the illustrative purpose in the subsequent sections.

The most commonly used warranty policies are free replacement warranty (FRW) policy, pro-rata warranty (PRW) policy and combined FRW-PRW policy (see Blischke et al., 2011; Murthy & Blischke, 2006; Zhu et al., 2018). An important feature of a warranty policy is that if the product fails during the warranty period, consumer will get full or pro-rated compensation from the manufacturer. Under the FRW policy, if the product fails during the warranty period, a non-repairable product is replaced by an identical one free of charge. In case of repairable product, the manufacturer will repair the product free of cost. However, if the product fails under PRW policy, the manufacturer will provide a pro-rated compensation to the consumer. Sometimes, a combination of both the policies is also considered, which is termed as combined FRW-PRW policy. Now onwards, we will use the abbreviation ‘FRW-PRW policy’ to denote combined FRW-PRW policy. We consider determination of warranty length for a FRW-PRW policy based on data observed under Type-II UHCS. Although there are many works on determination of warranty length for different policies based on complete data (see C.-C. Wu et al., 2006; DeCroix, 1999; Menezes & Currim, 1992), there are few works under censored data. Gutiérrez-Pulido et al. (2006) determined Bayesian optimal warranty length under pro-rata warranty policy where they considered two-parameter Weibull distribution as product lifetime. Wu and Huang (2010) investigated a decision problem under FRW-PRW policy. They used a Bayesian approach to determine the optimal warranty lengths under Type-II progressive censoring scheme for a Rayleigh distribution. Chakrabarty et al. (2020a) investigated optimal reliability acceptance sampling plans under Type-I hybrid censoring schemes by taking warranty cost as constraint. Budhiraja and Pradhan (2019) developed optimal reliability acceptance sampling plans under progressive Type-I interval censoring with random removal using a cost model, which consists of warranty cost as a component. C. C. Wu et al. (2007) obtained optimal burn-in time and warranty length under FRW-PRW policy in classical.
setup. Few more relevant articles on Bayesian procedures and warranty issues can be found in the following references Kwon (1996), Tsai (2008), Patel and Patel (2017), Chakrabarty et al. (2020b), Salem et al. (2020), and Aslam et al. (2020).

The aim of this article is two-fold. First, a generalized censoring scheme is used to develop the proposed methodologies; therefore, it can be easily extended to the other censoring schemes which are the special cases of Type-II UHCS. For the purpose of illustration, log-normal distribution is considered as lifetime model. The log-normal distribution is quite popular in reliability studies because of the flexibility of its shape (see Johnson et al., 1994). Moreover, this distribution was not studied in the optimal warranty length determination problems. Second, this article proposes a non-linear pro-rated rebate cost and compared it with linear pro-rated rebate cost proposed by Wu and Huang (2010). It has been shown that the proposed non-linear rebate function gives a larger warranty period with maximum profit in comparison with the linear rebate function.

The organization of the paper is as follows. In Section 2, lifetime model and posterior predictive distribution based on the data obtained through Type-II UHCS are discussed. In Section 3, we have derived various non-linear cost functions such as rebate function, economic benefit function, warranty cost function and dissatisfaction cost function. Using the cost functions, an utility function is constructed in this section, which is maximized to compute optimal warranty length. The computational methodology to obtain optimal warranty lengths is discussed in Section 4. A real-life data are analyzed to illustrate the proposed method in Section 5, and finally, some concluding remarks are made in Section 6.

2 Lifetime model and posterior distribution

Suppose that \( X_1, X_2, \ldots, X_n \) are the lifetimes of \( n \) testing units, which follow a log-normal distribution \( \text{LN}(\mu, \tau) \). The probability density function (PDF) and the cumulative distribution function (CDF) of \( \text{LN}(\mu, \tau) \) are given by

\[
f_X(x; \mu, \tau) = \sqrt{\frac{\tau}{2\pi x}} e^{-\frac{(\ln x - \mu)^2}{2\tau}}, \quad x > 0, \quad -\infty < \mu < \infty, \quad \tau > 0, \tag{1}
\]

and

\[
F_X(x; \mu, \tau) = \Phi\left(\sqrt{\tau} (\ln x - \mu)\right), \quad x > 0,
\]

respectively, where \( \mu \) and \( \tau \) denote unknown parameters of the distribution. Here, \( \Phi(\cdot) \) is the CDF of standard normal distribution. Suppose \( X_{1:n} < X_{2:n} < \cdots < X_{n:n} \) represent corresponding ordered lifetimes. Let \( D \) and \( \xi \) represent the number of failures and the duration of the life-testing, respectively, under a Type-II UHCS. Therefore, \((X_{1:n}, X_{2:n}, \ldots, X_{D:n}, \xi)\) represents a Type-II UHCS data defined as

\[
(D, \xi) = \begin{cases} 
(D_1, T_1) & \text{if } X_{1:n} < X_{r:n} < T_1 < T_2, \quad \text{where } D_1 = r, r + 1, \ldots, n, \\
(r, X_{r:n}) & \text{if } X_{1:n} < T_1 < X_{r:n} < T_2, \\
(D_2, T_2) & \text{if } X_{1:n} < T_1 < T_2 < X_{r:n}, \quad \text{where } D_2 = l, l + 1, \ldots, r - 1, \\
(r, X_{r:n}) & \text{if } T_1 < X_{1:n} < X_{r:n} < T_2, \\
(D_2, T_2) & \text{if } T_1 < X_{1:n} < T_2 < X_{r:n}, \quad \text{where } D_2 = l, l + 1, \ldots, r - 1, \\
(l, X_{l:n}) & \text{if } T_1 < T_2 < X_{1:n} < X_{r:n}.
\end{cases}
\]

Based on the data obtained by a Type-II UHCS, the likelihood function is given by

\[
L(\mu, \tau|\text{data}) \propto \prod_{i=1}^{d} f_X(x_{i:n}; \mu, \tau) \{1 - F_X(\xi_0; \mu, \tau)\}^{n-d}, \tag{2}
\]
where, \( d, \xi_0 \) and \( x_{i,n} \) are the observed values of \( D, \xi \) and \( X_{i,n} \), respectively. It is assumed that the joint prior distribution of \((\mu, \tau)\) follows a normal-gamma distribution with probability density function

\[
\pi(\mu, \tau) = \frac{b_0^2}{\Gamma(d)} \omega^{a_1-\frac{1}{2}} e^{-\frac{d_1}{2} \omega^{2} (\mu - c_1)^2 - b_1 \tau}
\]

\[
= \frac{b_0^2}{\Gamma(d)} \omega^{a_1-\frac{1}{2}} e^{-b_1 \tau} \times \frac{1}{\sqrt{2\pi}} \frac{1}{\tau^{d_1}} e^{-\frac{\omega^2}{2\tau}}
\]

\[
= \pi(\tau) \times \pi(\mu|\tau), \tag{3}
\]

where, \( \pi(\tau) \sim \text{Gamma}(a_1, b_1) \) and \( \pi(\mu|\tau) \sim N_{\mu|\tau}(c_1, 1/\tau d_1) \). The hyper parameters \( a_1, b_1, c_1 \) and \( d_1 \) reflect prior knowledge about unknown parameters of interest, where \( a_1, b_1 > 0, d_1 > 0 \) and \(-\infty < c_1 < \infty\). The posterior distribution is given as

\[
\pi(\mu, \tau|\text{data}) = L(\mu, \tau|\text{data}) \times \pi(\mu, \tau).
\]

The posterior predictive distribution, which represents the current beliefs of the decision maker about the failure time, is given as

\[
f(t|\text{data}) = \int_{\infty}^{\infty} \int_{\infty}^{\infty} f(t; \mu, \tau) \pi(\mu, \tau|\text{data}) d\mu d\tau.
\]

Now we are presenting the expression of the Fisher information \( I(\theta) \) about \( \theta = (\mu, \tau) \), which will be used in the next section for applying Metropolis–Hastings (MH) algorithm to compute optimal warranty length. The Fisher information \( I(\theta) \) under Type-II UHCS is given by (see Sen et al., 2020)

\[
I(\theta) = I_{T_1}(\theta) + I_{1,...,l,n}(\theta) + I_{X_{i,n}\wedge T_2}(\theta) - I_{X_{i,n}\wedge T_1}(\theta) - I_{X_{i,n}\wedge T_2}(\theta),
\]

where \( I_{T_1}(\theta), I_{1,...,l,n}(\theta), \) and \( I_{X_{i,n}\wedge T_2}(\theta) \) represent the Fisher information about \( \theta \) under Type-I censoring, Type-II censoring and Type-I hybrid censoring schemes, respectively. The expressions of each of them are given as follows

\[
I_{T_1}(\theta) = \int_0^{T_1} \left\langle \frac{\partial}{\partial \theta} \ln h(x; \theta) \right\rangle f_X(x; \theta) dx,
\]

\[
I_{1,...,l,n}(\theta) = \int_0^{\infty} \left\langle \frac{\partial}{\partial \theta} \ln h(x; \theta) \right\rangle \sum_{i=1}^{l} f_{i,n}(x; \theta) dx,
\]

\[
I_{X_{i,n}\wedge T_2}(\theta) = \int_0^{T_2} \left\langle \frac{\partial}{\partial \theta} \ln h(x; \theta) \right\rangle \sum_{i=1}^{r} f_{i,n}(x; \theta) dx,
\]

where \( h(x; \theta) \) is the hazard function of \( X, f_{i,n}(x; \theta) \) is the density of \( X_{i,n}, (\partial / \partial \theta) \ln h(x; \theta) \) is the vector \(( (\partial \mu) \ln h(x; \theta), (\partial \tau) \ln h(x; \theta) \) and \( \langle A \rangle \) is defined as the matrix \( AA' \), for \( A \in \mathbb{R}^2 \). It may be noted that the expressions of the Fisher information about \( \theta \) under Type-I hybrid censored data with schemes \( (n, r, T_1) \) and \( (n, l, T_2) \) are similar as presented in equation (6).

### 3 Cost functions

FRW-PRW policy can be viewed as choosing two positive time points \( w_1 \) and \( w_2 \) such that \( w_1 < w_2 \), in which FRW policy is applicable in period \([0, w_1]\) and PRW policy is applicable in period \([w_1, w_2]\). Different choices of \( w_1 \) and \( w_2 \) raise to FRW or PRW policies as sub-case of FRW-PRW policy. For
instance, when \( w_1 = w_2 \), it reduces to FRW policy and, when \( w_1 = 0 \), it reduces to PRW policy. Wu and Huang (2010) considered a linear pro-rata rebate function which is the function of the remaining time of the warranty period. Assuming \( S \) be the sales price of a certain product, Wu and Huang (2010) defined the cost of reimbursing an item, which is linear in nature under FRW-PRW policy, as

\[
R_{\text{cost}}(t) = \begin{cases} 
S, & \text{if } 0 \leq t < w_1, \\
S\left(\frac{w_1 - t}{w_2 - w_1}\right), & \text{if } w_1 \leq t < w_2, \\
0, & \text{if } t \geq w_2.
\end{cases}
\]

A pictorial diagram of reimbursing an item under FRW-PRW policy can be visualized in Figure 3. In this article, we propose a non-linear rebate function under FRW-PRW policy, which is defined as

\[
R_{\text{cost}}(t) = \begin{cases} 
S, & \text{if } 0 \leq t \leq w_1, \\
S\left(1 - e^{-a\left(\frac{t}{w_2 - w_1}\right)}\right), & \text{if } w_1 < t \leq w_2, \\
0, & \text{if } t \geq w_2,
\end{cases}
\]

where the parameter \( a (0 < a < 1) \) controls the non-linearity of the pro-rated rebate function. To compute the optimal warranty length of the products, it is required to construct an utility function, which will be optimized. In this article, three cost functions such as economic benefit function, warranty cost function and dissatisfaction cost function are considered, which were proposed by Gutiérrez-Pulido et al. (2006). In the subsequent sections, we have discussed those cost functions.

### 3.1 Economic benefit function

By providing a suitable warranty period, the manufacturer may expect an increase in sales volume of the products, which results in economic benefit. Therefore, the economic benefit function is considered as the monotone increasing function of the average of two-stage warranty lengths. An economic benefit function, denoted as \( b(w_1, w_2) \), is considered here as (see Wu & Huang, 2010)

\[
b(w_1, w_2) = A_2 M\left(1 - e^{-A_1\left(\frac{w_1 + w_2}{2}\right)}\right),
\]

where \( A_2 \) is the manufacturer's profit for one product, \( M \) is the potential number of products to be sold with this warranty policy and \( A_1 \) is the parameter to control the speed of increment in benefit. \( A_1 \) can be uniquely determined from the ratio of two special quantities in the combined FRW/PRW policy. Assuming \( t_w \) is the standard market warranty under FRW policy, let us consider the following ratio

\[
R(t) = \frac{t_w}{w_2 - w_1}
\]
\[ b(0, t_w) = \frac{1 - e^{-A_1 t_w}}{1 - e^{-A_1 t_w}}. \]

The ratio indicates whether the percentage of benefit increases if the manufacturer changes warranty policy from FRW to PRW. Note that \( b(w_1, w_2) \) cease to \( A_2M \) when either \( w_1 \) or \( w_2 \) approaches infinity. It interprets the fact that the economic benefit to the manufacturer is bounded above by some positive number. Let \( g(A_1) = (1 - e^{-\frac{A_1 w}{2}})/(1 - e^{-A_1 t_w}) \). Then, \( g \) is a strictly monotone increasing function, which can take any value between 0.5 and 1. Note that \( g(0^+) = 0.5 \) and \( g(\infty) = 1 \). Therefore, \( A_1 \) can be determined uniquely by solving the equation \( g(A_1) = p^* \) for given \( p^* \), where \( 0.5 < p^* < 1 \).

### 3.2 Warranty cost function

Warranty cost is the direct cost to the manufacturer for reimbursing the products which fail during warranty period. Let us denote warranty cost by \( W(t, w_1, w_2) \). It is defined as

\[
W(t, w_1, w_2) = \{ \text{cost of reimbursing an item } R_{\text{cost}}(t) \} \times \{ \text{the expected number of items that fail under the warranty period} \},
\]

where \( R_{\text{cost}}(t) \) is defined in Section 3. To find the expected number of failures during the warranty period, we consider the probabilities of failure before time period \( w_1 \), between time period \( w_1 \) and \( w_2 \) and after time period \( w_2 \). Thus, the warranty cost function can be formulated as

\[
W(t, w_1, w_2) = MF(w_1|\text{data})SI_{[0, w_1]}(t) + MF(w_2|\text{data})SI_{[w_1, w_2]}(t) - MF(w_1|\text{data})SI_{[0, w_1]}(t),
\]

where \( I_{[\cdot]}(\cdot) \) denotes the indicator function and \( F(t|\text{data}) \) represents the posterior predictive cumulative distribution function.

### 3.3 Dissatisfaction cost function

We consider another cost function which is the manufacturer’s indirect cost to the product. This is called as dissatisfaction cost or penalty cost. Typically, the consumers have certain expectation about the product lifetime. Suppose that the consumer’s expected lifetime of the product is \( L \), which can be considered as the mean, median or percentile of the posterior predictive distribution. Now, if the product fails during the two-stage warranty period or if it fails immediately after the expiration of combined warranty period, then the consumers have certain dissatisfaction about the product. This can indirectly affect on company’s reputation to the buyer and also it can reduce the future sale volumes of the product. Therefore, we split the total dissatisfaction cost into three time intervals as follows.

**Case I:** Item fails in the time period \( [0, w_1] \).

**Case II:** Item fails in the time period \( (w_1, w_2] \).

**Case III:** Item fails in the time period \( (w_2, L] \).

Note that we assume the consumer’s expected lifetime \( L \) greater than second stage warranty period \( w_2 \). Customers often seek for higher reliable products, and thus, this assumption is quite practical in nature.

The dissatisfaction cost in Case I (that is, the product fails in the FRW policy) is defined as

\[
D_1(t, w_1) = \{ \text{Proportion } q_1(0 < q_1 < 1) \text{of the sales price} \} \times
\]
Therefore, consumer’s cost, incurred during the time of the PRW period. At \( w_1 \), per unit cost of dissatisfaction is \( Sq_1 \) and, at \( w_2 \), per unit cost of dissatisfaction is \( Sq_2 \) where \( 0 < q_2 < 1 \) with \( q_2 < q_1 \). Therefore, the dissatisfaction cost in Case II is defined as

\[
D_2(t, w_1, w_2) = M\{F(w_2|\text{data}) - F(w_1|\text{data})\} \times \\
\left\{ Sq_2 - (Sq_2 - Sq_1)\left(1 - e^{-a\frac{t}{w_1}}\right) \right\}I_{(w_1, w_2)}(t).
\]

Finally, suppose that the product fails during immediate expiration of warranty and before the consumer’s expected lifetime \( L(>w_2) \) of the product. Therefore, being an unsatisfied customer, it incurred some dissatisfaction cost. We propose that the dissatisfaction cost decreases non-linearly with remaining time of consumer’s expected lifetime of the product and the dissatisfaction reaches to zero at \( L \). Hence, the dissatisfaction cost in Case III is defined as

\[
D_3(t, w_2) = M\{F(L|\text{data}) - F(w_2|\text{data})\}\left\{ Sq_2\left(1 - e^{-a\frac{L}{w_2}}\right) \right\}I_{(w_2, L)}(t).
\]

Therefore, the total dissatisfaction cost is given by summing the above three costs and defined as

\[
D(t, w_1, w_2) = D_1(t, w_1) + D_2(t, w_1, w_2) + D_3(t, w_2).
\]

A pictorial diagram of the dissatisfaction cost structure can be visualized in Figure 4.

### 3.4 Utility function

Utilizing the three proposed cost functions in Sections 3.2 and 3.3, we define an utility function as (see Gutiérrez-Pulido et al., 2006)

\[
U(t, w_1, w_2) = b(w_1, w_2) - W(t, w_1, w_2) - D(t, w_1, w_2).
\]

It is of note that the time to failure \( t \) is the random quantity in utility function. Therefore, the expected value of the utility function \( U(t, w_1, w_2) \) is given by

\[
u^*(w_1, w_2) = E_{\text{data}}[U(t, w_1, w_2)|\text{data}]
\]
where

\[ I_1(\mu, \tau) = \int_0^\infty U(t, w_1, w_2)f(t; \mu, \tau)dt \]

\[ = \int_0^\infty \{b(w_1, w_2) - W(t, w_1, w_2) - D(t, w_1, w_2)\}f(t; \mu, \tau)dt \]

\[ = b(w_1, w_2) - \int_0^\infty W(t, w_1, w_2)f(t; \mu, \tau)dt - \int_0^\infty D(t, w_1, w_2)f(t; \mu, \tau)dt, \]

\[ \int_0^\infty W(t, w_1, w_2)f(t; \mu, \tau)dt = S[F(w_1|data)]^2 + S[F(w_2|data) - F(w_1|data)] \]

\[ \times \int_{w_1}^{w_2} (1 - e^{-a(\frac{t}{w_1})})f(t; \mu, \tau)dt, \]

\[ \int_0^\infty D(t, w_1, w_2)f(t; \mu, \tau)dt = \int_0^\infty \{D_1(t, w_1) + D_2(t, w_1, w_2) + D_3(t, w_2)\}f(t; \mu, \tau)dt \]

\[ = S[F(w_1|data)]^2 + S[F(w_2|data) - F(w_1|data)] \]

\[ \times \int_{w_1}^{w_2} (1 - e^{-a(\frac{t}{w_1})})f(t; \mu, \tau)dt \]

\[ + S[F(L|data) - F(w_2|data)] \]

\[ \times \int_{w_2}^{L} \{q_2 - q_2(1 - e^{-a(\frac{t}{w_1})})\}f(t; \mu, \tau)dt. \]

4 Optimal warranty length

In order to compute the optimal warranty length, that is, the optimal values of \(w_1\) and \(w_2\), the expected utility function \(u^*(w_1, w_2)\) given in equation (7) is maximized with respect to \(w_1\) and \(w_2\). Therefore, the optimal warranty \((w_1^*, w_2^*)\) is the solution of the following optimization problem:

\[(w_1^*, w_2^*) = \arg\max_{w_1 < w_2; w_i \in R^+, i=1,2} u^*(w_1, w_2), \quad (8)\]
where \( R^+ \) is the set of all positive real numbers. In general, the optimization problem in (8) does not have a closed form analytical solution. Nevertheless, in this article, we propose to use Metropolis-Hasting (MH) algorithm (see Ntzoufras, 2009) to compute the Bayes estimates of (7) and, then, using that estimate in (8), we compute the optimal solution. The procedure of MH algorithm suggests that the samples from a posterior distribution can be generated using some proposal density. Commonly, a symmetric type proposal density such as \( I(\mu^*, \tau^*)|I(\mu, \tau)) = I(\mu^*, \tau^*) \) can be taken into consideration. Here, we consider a bivariate normal \( N_2(\mu, \ln \tau, I^{-1}(\mu, \tau)) \) proposal density where \( I^{-1}(\mu, \tau) \) denotes the inverse of the information matrix. Since we are generating samples from a bivariate normal distribution, few negative observations for \( \tau \) may appear, which is not acceptable. In this regard, we propose the following steps of the MH algorithm to draw samples from the corresponding posterior density.

**Algorithm 1**

**Step 1:** Set initial value of \((\mu, \tau) = (\mu_0, \tau_0)\)

**Step 2:** For \( i = 1, 2, \ldots, N \) repeat the following steps

(a) Set \((\mu, \tau) = (\mu_{i-1}, \tau_{i-1})\)

(b) Generate a new candidate parameter value \( \delta \) from \( N_2((\mu, \ln \tau, I^{-1}(\mu, \tau)) \)

(c) Set \((\mu^*, \tau^*) = (\delta_1, \exp(\delta_2))\)

(d) Calculate \( a^* = \min\left(1, \frac{N(\mu^*, \tau^*)}{N(\mu, \tau)} \right) \)

(e) Update \((\mu_i, \tau_i) = (\mu^*, \tau^*) \) with probability \( a^* \); otherwise set \((\mu_i, \tau_i) = (\mu, \tau)\)

Algorithm 1 will generate \( N \) observations of \((\mu, \tau)\). Some initial observations of size \( N_0 \), say, are discarded as burn-in observations and the remaining observations \( N - N_0 (= k, \text{say}) \) can be used to compute the Bayes estimate of \( u^*(w_1, w_2) \) in (7). Subsequently, the corresponding Bayes estimate can be computed as

\[
\hat{u}^*_{\text{Bayes}}(w_1, w_2) = \frac{1}{k} \sum_{i=1}^{k} I_1(\mu_i, \tau_i).
\]

Finally, the optimal warranty period \((w_1^*, w_2^*)\) in (8) is computed by solving the optimization problem

\[
\text{Maximize } \frac{1}{k} \sum_{i=1}^{k} I_1(\mu_i, \tau_i).
\]

This is a non-linear optimization problem with two real continuous decision variables. Newton-Raphson method can be used to solve this problem.

## 5 Numerical illustration with real-life data analysis

In this section, a real-life data set is considered for illustrative purpose. The data set is taken from Proschan (1963). The data represent the intervals of successive failure times in hours of the air conditioning system of Boeing 7912 jet airplane. The corresponding ordered times are listed as

|   |   |   |   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 3 | 5 | 7 | 11 | 11 | 11 | 12 | 14 | 14 |
| 14 | 16 | 16 | 20 | 21 | 23 | 42 | 47 | 52 | 62 |
| 71 | 71 | 87 | 90 | 95 | 120 | 120 | 225 | 246 | 261 |

Gutiérrez-Pulido et al. (2005) fitted the data with log-normal distribution with parameters \((\mu, \tau)\). They also demonstrated a method to calculate the hyper parameter values of normal-gamma prior of \((\mu, \tau)\) as \(a_1 = 36.9, b_1 = 29.1, c_1 = 3.3\) and \(d_1 = 287.9\). By using the censoring scheme \(n = 30, r = 20, l = 7, T_1 = 100, T_2 = 120\), we have generated Type-II UHCS data as
Suppose that the sales price of this product is $S = 700$ and the production cost of the product is $C = 500$. Therefore, the profit per unit product is $A_2 = 200$. The manufacturer gives a standard warranty, which is the 0.1th quantile of the posterior predictive distribution under the FRW policy i.e. standard warranty is $t_w = 7.245$ hours. Since we are considering combined FRW/PRW policy, manufacturer is interested to change the warranty policy from FRW to PRW and assumed that the percentage of benefit remains to be $p^* = 0.75$ i.e., $g(A_1) = 0.75$. In this case, the unique solution to the equation $g(A_1) = 0.75$ is $A_1 = 0.303$. Since the customer dissatisfaction indirectly affect on the image of the company, so, the product sales may be reduced. Therefore, we assume that proportions of customer dissatisfaction are $(q_1, q_2) = (0.09, 0.04)$. The consumer will satisfy with the product if its lifetime reaches customer’s expectation over the product lifetime. Here, we assume customer’s
Expected lifetime of the product is $L = 27.263$, which is the median of the posterior predictive distribution. Assuming the number of selling products as $M = 15$, we observed that for linear rebate function, the optimal warranty length under combined FRW-PRW policy is $(w_1^*, w_2^*) = (7.317, 11.641)$ and the maximum value is $\$2629.215$. Using the proposed non-linear rebate function defined in Section 4.1, the optimal warranty lengths under the combined FRW/PRW policy are computed and presented in Table 1. Trace plots of the parameters $\mu$ and $\tau$ in one simulation in MH algorithm are presented in Figure 5, which confirms that the Markov chain Monte Carlo convergences in distribution. It is observed that the optimal warranty lengths under non-linear rebate function are wider than that of linear rebate function. Also, optimal warranty length and optimal value both decrease with increasing $a$. For $a = 0.01, 0.05, 0.09, 0.2, 0.5$ and $0.9$, the corresponding non-linear rebate functions are plotted in Figure 6. From Figure 6, it is observed that when $a$ increases, the non-linear pro-rated rebate function looks like a linear pro-rated rebate function.

Finally, a study on sensitivity of the optimal solution with respect to sales price as input parameter is carried out. We define the relative efficiency of a input parameter value $(S)$ compared to the true input parameter value $(S_0)$ based on utility function as

$$REff(S) = \frac{\text{Expected utility under } S \text{ corresponding to optimal warranty length under } S_0}{\text{Expected utility under } S_0 \text{ corresponding to optimal warranty length under } S}.$$
In 2010, Wu and Huang proposed the optimal rebate policy, which combines pro-rated rebate with perturbations and reduces the sales cost. It is observed that the optimal warranty length is less sensitive with respect to sales price as input parameter.

In order to establish the advantage of using combined FRW-PRW policy, warranty lengths and the corresponding optimal costs under individual FRW and PRW policies are computed in Table 3 and then compared with the combined FRW-PRW policies in Table 1. Note that, by setting \( a = 0 \) and \( w_1 = w_2 \), FRW-PRW policy reduces to FRW policy. However, by setting \( w_1 = 0 \), FRW-PRW policy reduces to PRW policy. Table 3 shows that the maximum profit is lesser in the case of both individual policies as compared to combined FRW-PRW policy in Table 1. This indicates the advantage of using combined FRW-PRW policy over the individual policies.

### 6 Conclusion

In our study, we have considered log-normal as the lifetime distribution of the product; however, the proposed methodologies can be extended easily to other lifetime distributions. This article also proposes a non-linear pro-rated rebate cost and compared it with linear pro-rated rebate cost proposed by Wu and Huang (2010). For non-linear pro-rated rebate cost, a larger warranty period with maximum profit is obtained in comparison with the linear rebate cost. This is the prime advantage of choosing a non-linear pro-rated rebate cost function. Another advantage of choosing combined FRW-PRW policy over individual FRW or PRW policies is verified through a numerical study. In the construction of non-linear pro-rated rebate cost function, the parameter \( a \) controls the non-linearity of the rebate function (see Figure 6). A further study could be carried out to find the optimal value of \( a \) for which the warranty length is optimized. Presently we are working on this and hope to report our findings in the future articles.

| Table 2. Sensitivity analysis of optimal warranty length with respect to sales price as input parameter. |
| --- |
| \( a \) | \( S \) | Optimal warranty length | REff(\( S \)) |
| 0.01 | 680 | (5.449, 17.307) | 0.999 |
| 700 | (5.491, 17.281) | 0.998 |
| 710 | (5.432, 17.152) | 1 |
| 720 | (5.421, 17.149) | 1 |
| 0.05 | 680 | (6.124, 14.786) | 1 |
| 690 | (6.149, 14.772) | 0.998 |
| 700 | (6.136, 14.722) | 1 |
| 710 | (6.151, 14.686) | 1 |
| 720 | (6.127, 14.588) | 0.999 |
| 0.2 | 680 | (6.814, 12.688) | 1 |
| 690 | (6.931, 12.783) | 0.998 |
| 700 | (6.926, 12.750) | 1 |
| 710 | (6.838, 12.649) | 0.999 |
| 720 | (6.803, 12.590) | 0.999 |

| Table 3. Optimal warranty lengths under individual FRW and PRW policies. |
| --- |
| Policy | Warranty length | Maximum profit |
| FRW: \( a = 0 \) and \( w_1 = w_2 \) | \( (0, 8.344) \) | \( \$2441.610 \) |
| PRW: \( a = 0.01 \) and \( w_1 = 0 \) | \( (0, 21.525) \) | \( \$2555.790 \) |
| \( a = 0.09 \) and \( w_1 = 0 \) | \( (0, 17.469) \) | \( \$2424.525 \) |
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Disclosure statement

No potential conflict of interest was reported by the author(s).

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**Appendix**

The R codes associated with this article are uploaded in the homepage of the corresponding author: https://sites.google.com/view/ritwikbhatta/home