A classical counting process and non-Wiener stochastic differential equations in quantum optics

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Abstract. The von Neumann projection postulate under the registration of a spontaneously emitted photon is described through the stochastic counting process as a non-Wiener linear stochastic differential equation for a non-normalized atomic state vector.

1. Introduction
A lot of physical objects are described by the so-called stochastic differential equations (SDEs). For instance, quantum SDEs are used to describe open quantum systems in the Markov approximation. Ordinary differential equations become mathematically incorrect if, in terms of laws of physics, they are added by stochastic values, and a simple case of delta-correlation is considered. Nevertheless, physicists themselves are not very familiar with even the elementary theory of these equations. In addition, the same symbols used to denote ordinary and Ito differentials contribute to more confusion about the significance of being able to handle the mechanism of SDEs. Research practice has shown that using SDEs leads both to a more natural and simple derivation of master equations, as well as the analysis of dynamics of interconnected open systems. SDE textbooks for physicists [1–4] usually follow an inverse chronology of presenting the material – the introduction of a master equation into the Markov approximation and the subsequent introduction of a Wiener stochastic process, a Wiener SDE, etc. Work [5] proposes a simple and straightforward way of introducing primarily the counting process (number process), the non-Wiener SDE based on it, and the subsequent derivation of the kinetic equation (master equation). The present work will demonstrate the simplest consequences of such an approach – a simple derivation of a Wiener SDE from a non-Winner one without using the central limit theorem, and the derivation of a linear Schrödinger SDE for a radiating atom when the photodetector records the emitted photons. In this case, it is followed by a Lindblad-type standard kinetic equation for the atomic density matrix, which preserves its norm.

2. The Ito differential algebra for counting processes
A counting process \( N(t) \) is the number of stochastic events that occurred in a time interval \([0,t)\), and its increment \( \Delta N(t) \) gives the number of such events in a time interval \([t,t+\Delta t)\). A natural division of this time interval into smaller sub-intervals \( t = t_0 < t_1 < t_2 < \ldots < t_M = t + \Delta t \) has the property of unlimited divisibility, where

\[
\Delta N(t) = \sum_{i=1}^{M} \Delta N_i = N(t+\Delta t) - N(t) \ , \ \Delta N_i = N(t_i) - N(t_{i-1}) \ ,
\]

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and leads to a condition when \( \Delta N \) takes only two values: either zero or 1 in the absence of accumulation of stochastic events. The absence of accumulations is characteristic of time-uniform distributions. In this case, we do not consider events that occurred simultaneously - they are described by other counting processes. It all leads to the exact ratio \( (\Delta N)^2 = \Delta N \) which holds for sufficiently small time intervals. We designate both the time intervals, where these relations are valid, and the corresponding differentials of a stochastic variable as the Ito differentials, and we assume stochastic variables \( dN(t) \) and \( dN(t') \) to be statistically independent at \( t \neq t' \). So \( dN(t) \) and \( N(t) \) are independent. We write \cite{5} \[
\begin{align*}
dN(t) &= N(t + dt) - N(t) , \quad (dN(t))^2 = dN(t) , \quad <dN(t) >= \gamma dt , \quad dtdN(t) = dt^2 = 0 . \quad (1)
\end{align*}
\]
Here the angle brackets denote the averaging of a stochastic variable \( dN(t) = dN(t, \omega) \) over probability space \( \Omega = \{ \omega \} \), where a stochastic variable \( N(t) \) is defined, and other equalities have to be interpreted as an equality of stochastic integrals for any non-anticipatory (statistically independent from \( dN(t) \) ) function \( f(t) \), for example, \[
\int_{0}^{t} f(t) dN(t) dN(t) = \int_{0}^{t} f(t) dN(t) .
\]
Meanwhile, the stochastic integrals are defined as the limit in the root mean square of the Ito integral sums \[
\int_{0}^{t} f(t) dN(t) = ms \lim_{M \to \infty} \sum_{i=1}^{M} f(t_i) \Delta N_i .
\]
The choice of the value of the integrand over division subintervals corresponds to the Ito choice and is self-consistent with a simple algebra of differentials \( (dN(t))^2 = dN(t) \).

These considerations are sufficient for elementary calculations, which will be illustrated with two examples. Work \cite{5} states that it follows from the listed requirements that \( N(t) \) is a simple Poisson intensity process \( \gamma t \), but, for the sake of convenience, the value \( \gamma \) has to be called the intensity of a counting process.

3. The non-Wiener and Wiener SDE for electron current

The flow of electrons across the valve can serve as an example how to use the counting process. An Ito differential for the electric current \( I(t) \) is described by SDEs \[
\begin{align*}
dI(t) &= -\Gamma I(t) dt + \beta dN(t) \quad (2)
\end{align*}
\]
whose parameters are coupled with the value \( q_0 \) of elementary charge by the ratio \( -\Gamma q_0 + \beta = 0 \), and ineritance \( \Gamma \) is due to the system inductance, determining time scale \( \tau \) of change in the given quantity \( I(t) \), regardless of the behavior of the stochastic process \( N(t) \): \( \tau = 1/\Gamma \).

Equation (2) is referred to as a non-Wiener one, since it is determined by the Ito differential of the counting process. Let be \( \gamma \tau >> 1 \), time intervals \( \Delta t_i = t_i - t_{i-1} \) and their sum \( \Delta t << \tau \). Then \[
\Delta I(t) = I(t + \Delta t) - I(t) \approx -\Gamma I(t) \Delta t + \beta \sum_{i=1}^{M} \Delta N_i . \quad (3)
\]
Let us emphasize that the sign of approximate equality in this equation is associated only with the assumption of an approximate constant value \( I(t) \) over the time interval \( \Delta t \).

We introduce a stochastic variable \( W(t) \), defining its increment as \( \Delta W(t) = \gamma^{-1/2} \sum_{i=1}^{M} (\Delta N_i - \gamma \Delta t_i) \). Then \( W(t) \) and \( \Delta W(t) \), \( \Delta W(t) \) and \( \Delta W(t') \) are statistically independent (the latter ones at \( t \neq t' \)), having characteristic mean values \( <\Delta W(t)> = 0 \) and \[
<\Delta W(t)^2> = \gamma^{-1} <\sum_{i,j=1}^{M} (\Delta N_i - \gamma \Delta t_i)(\Delta N_j - \gamma \Delta t_j)> = \gamma^{-1} <\sum_{i=1}^{M} (\gamma (\Delta t_i) - \gamma^2 (\Delta t_i)^2)> \to \Delta t .
\]
In defining the stochastic integral $\int_{0}^{t} f(t')dW(t')$ as an rms limit of Ito integral sums, we obtain the algebra for Ito differentials

$$dW(t)dW(t) = dt, \quad dW(t)dt = 0, \quad <dW(t)> = 0.$$  \hspace{1cm} (4)

Using this algebra, similar to [1,5], we obtain kinetic equations (master equation) which testify that $W(t)$ is a standard Wiener process. Under the conditions of an intensive counting process equation (3) takes the form of a Wiener SDE:

$$dI(t) = -(\Gamma I(t) - \beta \gamma)dt + \beta \gamma^{1/2}dW(t).$$  \hspace{1cm} (5)

4. The non-Wiener and Wiener SDE for photon registration

Quantum mechanics postulates state the wave vector evolution of a system, according to the Schrödinger equation, in the absence of quantum system measurement and the reduction of the density matrix of a system to a state determined by the measurement [6]. If the measurement of a quantum system is the registration of the system’s emitted photons, then the corresponding description of a posteriori states of a quantum system was repeatedly investigated [7, 8]. Here, it is natural to use a classical counting process to describe the triggering of a photodetector and the corresponding SDEs. In the process, nonlinear SDEs would emerge since we considered reduction of the system’s density matrix to the normalized state. Further below we are going to derive a SDE for non-normalized state vectors.

Experts in nonlinear and quantum optics are familiar with an optical model of the connection between the reduction postulate of quantum theory and the counting process. That is an atom in two states $|2\rangle$ of energy $E_2$ and $|1\rangle$ energy $E_1$. Other atomic levels are considered to be unpopulated. Then the model Hamiltonian of the atom is expressed as $H_n = \sum_k E_k \langle k | E_n < E_k | k \rangle$. While being in the upper energy state, an atom can spontaneously emit a photon. Generally, the probability of emission in a small time interval $[t,t+dt]$ is proportional to the magnitude of this interval, so that the relative decrease in the population of the upper level

$$\frac{d\rho_{22}(t)}{\rho_{22}(t)} = -\gamma dt.$$  

with the vacuum electromagnetic field surrounding the atom. Emission causes the atom to move from an excited $|2\rangle$ to a lower (unexcited) state $|1\rangle$.

We believe that all spontaneously emitted photons at the moments of their emission $t_i$ may be registered. According to the postulate of quantum theory, at the time of the photon counter triggering - registration of spontaneous emission - the atomic wave function is reduced (collapses) to the lower energy state $|1\rangle$. The quantum reduction postulate indicates the atomic density matrix, but we will model it by reducing the wave function as follows. Photon counter triggering is to be comparable to the stochastic counting process $N(t)$ with intensity $\gamma$. We cannot describe a state vector by an ordinary Schrödinger equation due to the registration of the emitted photon and reduction postulate. We will consider the change in the atomic state vector in the interaction representation associated with the registration or non-registration of a photon.

Let a wave vector $|\Psi(t)\rangle$ of the system, which is made up of a two-level atom and its surrounding electromagnetic field, takes a form at the time $t$ [7]

$$|\Psi(t)\rangle = (c_1(t)|1\rangle + c_2(t)|2\rangle) \otimes |0\text{photon}\rangle = \psi(t) \otimes |0\text{photon}\rangle.$$  

Then at $t+dt$ the wave vector of the system will be different depending on whether the photon emission occurred in the time interval $[t,t+dt]$ or not, and, accordingly, whether the detector tripped. These capabilities are quantum alternatives to the evolution of the system. If the final a posteriori wave function is associated with value $dN(t)$, then, when $dN(t)$ used in its record, it should be...
expressed as the sum of these alternatives. \(|\tilde{\Psi}(t + dt)\rangle = |\tilde{\Psi}^{(0)}(t + dt)\rangle + |\tilde{\Psi}^{(i)}(t + dt)\rangle\). Generally speaking, the tilde sign of the general wave vector indicates non-normalization of the wave vector. The probabilities of these alternatives are proportional to the values (with a common coefficient of proportionality) \(<\tilde{\Psi}^{(i)}(t + dt)|\tilde{\Psi}^{(i)}(t + dt)\rangle\) and \(<\tilde{\Psi}^{(0)}(t + dt)|\tilde{\Psi}^{(0)}(t + dt)\rangle\).

For the emitted photon and its registration \(dN(t) = 1\), a non-normalized vector \(|\tilde{\Psi}^{(i)}(t + dt)\rangle\) has to be proportional to \(|1\rangle \otimes |1\text{ photon}\rangle\). Assume that

\[
|\tilde{\Psi}^{(i)}(t + dt)\rangle = \sqrt{p} |1\rangle \otimes |1\text{ photon}\rangle.
\]

(6)

Here \(p = |c_2|^2\) is the probability of photon emission of an excited level where \(|c_2|^2 = \rho_{22}\).

If the photon is not emitted \(dN(t) = 0\) and the state of an atom which did not emit a photon has to take the following form

\[
|\tilde{\Psi}^{(0)}(t + dt)\rangle = (c_1(t + dt)|1\rangle + c_2(t + dt)|2\rangle) \otimes |0\text{ photon}\rangle.
\]

The condition for the normalization of the wave vector should adequately lead to the relation \(<\tilde{\Psi}^{(0)}(t + dt)|\tilde{\Psi}^{(0)}(t + dt)\rangle = 1 - p\), since we have accepted (6). In this context, it is to be emphasized that it is necessary to work with a normalized wave vector in order to calculate mathematical expectations and use usual formulas of quantum theory. The amplitude dependence \(c_i(t)\) on time has to be \(c_i(t) \sim \exp(-\gamma t/2)\) in order for the evolution of the wave vector to allow for the decay probability and the dependence of the density matrix \(\rho_{22}(t) \sim \exp(-\gamma t)\). In other words,

\[
dc_i(t) = -\frac{\gamma}{2} c_i(t) dt.
\]

In observing the quantum system we deal with the a posteriori density matrix, and notice that we get \(p^{-1} |\Psi^{(i)}(t + dt)\rangle < |\Psi^{(0)}(t + dt)\rangle\) at a photon emission, and in the other case we have \((1 - p)^{-1} |\Psi^{(0)}(t + dt)\rangle < |\Psi^{(0)}(t + dt)\rangle\). The derived equations will be nonlinear due to the dependence of the value \(p\) on the atomic system state [7,8]. So will a SDE for the normalized a posteriori state vector [8] be nonlinear, which is obtained by purifying the density matrix. However, it is more convenient to deal with linear equations for the wave vector.

A linear SDE can be obtained, but only for non-normalized a posteriori state vector. It should be noted that the requirements of the reduction postulate can be satisfied as a first approximation if the Ito differential of the atomic wave vector is described using the operator \(V(t)dt = -i\hbar \gamma S^+ S / 2 dt + i\hbar (S - 1) dN(t)\). In this case a posteriori lies in a specific implementation of the counting process \(N(t)\).

Then

\[
d |\psi(t)\rangle \propto [-\gamma S^+ S / 2 dt + (S - 1) dN(t)] |\psi(t)\rangle,
\]

(7)

where \(S = |1\rangle \otimes |2\rangle\). We indicate the first approximation and use another symbol, \(\propto\), instead of the equality sign, keeping in mind the following considerations. On the one hand, we can consider only the atomic wave vector, because at the time of the detector triggering, the photon is absorbed by it in the electromagnetic field surrounding the atom. The above representation for wave vectors \(|\tilde{\Psi}^{(0)}\rangle\) and \(|\tilde{\Psi}^{(i)}\rangle\) emphasizes their orthogonality. On the other hand, the state of a one-photon electromagnetic field is inextricably linked with the value of a stochastic variable \(dN(t)\). Averaging over the states of the electromagnetic field also implies averaging over a stochastic process \(N(t)\), which means the departure from the notion of SDEs. That is why we remain the counting process, considering only the atomic subsystem. Although a non-normalized vector of the a posteriori state is used, we still need to obtain a standard equation for the atomic density matrix after averaging over the counting process. Therefore, we will require that, on average, the norm of the atomic wave vector be kept. But in order to keep a SDE form for an atomic wave vector and to avoid premature averaging over a stochastic...
It is possible that the equation for the wave vector be corrected to satisfy the condition \( <d < \psi(t) \mid \psi(t) >= 0 \). In this case a differential is meant as an Ito differential. The nature of the correction can be established quite definitely, if one obtains the equation for the atomic density matrix from equation (7) in a standard way [2], with the account of the Ito algebra for the counting process differential.

The first approximation of the SDE for a non-normalized wave vector gives rise to the following Ito differential

\[
d < \psi(t) \mid \psi(t) > = < \psi(t) \mid S^* S \mid \psi(t) > (dN(t) - \gamma dt) - < \psi(t) \mid \psi(t) > dN(t).
\]

To satisfy the condition \( < d < \psi(t) \mid \psi(t) >= 0 \) when averaging over \( N(t) \) (according to the photodetector’s response time), for an atomic wave vector it is sufficient to take a SDE in the following form

\[
d \mid \psi(t) >= [-\gamma \frac{S^* S - 1}{2} dt + (S - 1) dN(t)] \mid \psi(t) >
\]

Then the SDE (8) correctly determines the equation for the atomic density matrix [6] and can be used for the numerical simulation of the a posteriori wave vector. In this case, equation (8) is linear over the state vector.

The equation for the stochastic density matrix follows from (8) in a standard way

\[
\begin{align*}
\frac{d \rho(t)}{dt} &= \{[\psi(t) > + d \mid \psi(t)] \} (\mid \psi(t) > + d \mid \psi(t)] - \mid \psi(t) > < \psi(t) = \\
&= -\frac{\gamma}{2} r(t) S^* S dt - \frac{\gamma}{2} S^* S r(t) dt + S r(t) S^* dN(t) + (\gamma dt - dN(t)) r(t).
\end{align*}
\]

It can be easily seen that normalization of the density matrix at averaging is kept

\[\frac{dT}{dt} < r(t) > = 0.\]

By averaging over a stochastic counting process, we obtain a standard kinetic equation for the atomic density matrix \( \rho(t) = < r(t) > \) in the Lindblad-type form

\[
\frac{d \rho(t)}{dt} = -\frac{\gamma}{2} \rho(t) S^* S - \frac{\gamma}{2} S^* S \rho(t) + \gamma S \rho(t) S^*.
\]

It is no wonder that the equation (8) differs from the known ones. Different SDEs can correspond to the same kinetic equation. Within the equation (8), it is also possible to pass on from a non-Wiener equation to a Wiener equation, as was demonstrated in the previous section, i.e. on substituting \( \gamma^{-1/2} (dN(t) - \gamma dt) = dW(t) \), which is quite fair under consideration of multi-atomic system. But then there will be lost a connection with the measurement procedure, and there will occur a specific averaging over the measurement procedures.

5. Conclusion

The operator \( H_{int}(t) \) of the interaction of the emitting atom with the classical resonant pump field can be introduced into equations (8) and (9), and then the process of emission and registration of the emitted photons should be stationary in time if the pumping effect is stationary

\[
\begin{align*}
d \mid \psi(t) >= -\frac{i}{\hbar} H_{int}(t) \mid \psi(t) > + & \{[\gamma \frac{S^* S - 1}{2} dt + (S - 1) dN(t)] \} \mid \psi(t) >, \\
\frac{d \rho(t)}{dt} &= \frac{i}{\hbar} (\rho(t) H_{int}(t) - H_{int}(t) \rho(t)) - \frac{\gamma}{2} \rho(t) S^* S - \frac{\gamma}{2} S^* S \rho(t) + \gamma S \rho(t) S^*.
\end{align*}
\]

However, the following thing is to be borne in mind. A joint action of the classical pump field and the quantized vacuum field leads to a hierarchy of various interference effects that determine new channels for energy relaxation and dephasing [9, 10]. Therefore, the author strongly believes that the traditional approach (where the general-form kinetic equation [3] is compared to an open quantum system, which is an emitting atom and an electromagnetic emission field [3]), the theory of open
quantum systems requires a differently new approach which is aimed at comparing its effective Hamiltonian with an open quantum system. The effective Hamiltonian determines the SDE of an open quantum system, on the basis of which its own kinetic equation is derived for a specific physical problem [9, 10]. Such an approach is free from paradoxes [11] when considering thermodynamic issues, as exemplified by the “paradoxical” work [12] and the calculation in the effective Hamiltonian method [13]. It is to be noted that the effective Hamiltonian method as an algebraic perturbation theory [9, 10, 13] is only an algebraic form of the Krylov – Bogolyubov – Mitropolsky averaging method [14, 15], which is necessary in optical applications owing to a lot of variables that slowly and fast change over time.

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