Generation of Kerr non-Gaussian motional states of trapped ions

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Abstract – Non-Gaussian states represent a powerful resource for quantum information protocols in the continuous variables regime. Cat states, in particular, have been produced in the motional degree of freedom of trapped ions by controlled displacements dependent on the ionic internal state. An alternative method harnesses the Kerr nonlinearity naturally present in this kind of system. We perform detailed calculations confirming its feasibility for typical experimental conditions. Additionally, this method permits generation of all other complex non-Gaussian states with negative Wigner functions resulting from Kerr nonlinear interaction. Especially, superpositions of several coherent states are achieved at a fraction of the time necessary to produce the cat state.

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Introduction. – Quantum information processing relying on physical observables with continuous spectra (continuous variables, or CV) offer new possibilities of implementation and employment when compared to their discrete alternatives. In particular, states with non-Gaussian Wigner function are essential for the speedup and universality of CV quantum computation [1–3]. According to the Gottesman-Knill theorem, quantum computing based only on components described by quadratic Hamiltonians, Gaussian inputs, and measurements on canonical variables can be simulated by a classical computation [4]. Thus, non-Gaussian states allow to exploit the advantage of the quantum algorithms over the classical ones [2]. Construction of a CV universal quantum computer for transformations that are polynomial in those variables requires at least a cubic (Kerr) nonlinear operations [1]. Non-Gaussian states also find important applications in high-precision metrology [5], novel tests of Bell-like inequalities [6,7], and fundamental investigations [8–10]. Therefore, finding the effective and deterministic way of generation and manipulation of non-Gaussian states and interactions is crucial for further development of quantum based applications as well as for better understanding of physics.

Position and momentum of trapped ions represent a good CV quantum system candidate, especially taking into account the recent achievements regarding low heating rates [11]. So far, this system is much less exploited than the ionic discrete variables [12,13]. In fact, motional degrees of freedom have been mostly used as qubits, fulfilling the role of a quantum information “bus” among the ionic internal states [14]. However, recently proof-of-principle quantum walks using the quadratures of the harmonic-oscillator state were performed [15,16] and CV quantum simulations were demonstrated [17].

The current ion trap technology allows an arbitrary quantum manipulation of the motional states [18–20]. In practice, however, the fidelity of operations changing the phonon number stepwise (e.g. blue sideband pulses) are comparatively low, limiting the maximum achievable phonon number to just a few quanta in a coherent superposition [21]. Alternatively, “single-shot” methods relying on a single laser pulse to directly produce the desired final state seem to have an advantage from the experimental point of view. For instance, bichromatic quantum gates allow for high fidelities in the generation of two-qubit entangled states based on precise manipulation of the vibrational degree of freedom [22–24], as well as coherent superpositions involving tens of phonons [15,16].

In this paper we investigate a deterministic method to directly harness the Kerr nonlinearity present in the motion of trapped ions and thus to produce non-Gaussian vibrational states [25]. The method by employing a single laser pulse allows for creation of several types of...
“quasi-macroscopic” coherent state superpositions, such as the cat states, and all other non-Gaussian states emerging from Kerr nonlinear interaction. While operations capable of producing such states exist in principle, we believe the simplicity of our scheme will further push the border of what can be experimentally realized with high fidelities in CV systems.

This paper is organized as follows. First we introduce the Kerr non-Gaussian state. In the next section we present the method of direct generation of this state on a vibrational mode of a trapped ion. Then the ideal Kerr state is compared to the result of the calculation using the complete Hamiltonian of the trapped ions. We finish the paper with our conclusions.

The Kerr state. – The lowest-order Hamiltonian in powers of the harmonic-oscillator annihilation \( a \) and creation \( a^\dagger \) operators capable of producing non-Gaussian states describes a Kerr nonlinear medium. The evolution of an initial coherent state \( |\alpha\rangle \) subject to this Hamiltonian gives rise to the so-called “Kerr states”

\[
|\Psi(\alpha, \tau)\rangle = e^{-i\alpha^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} e^{i\pi(n-1)}|n\rangle.
\]  

They are parametrized in terms of only one quantity, the effective time \( \tau \) [26]. In general, the Kerr states are highly nonclassical and for certain values of \( \tau \) their Wigner function assumes negative values [27]. Their most prominent examples are the cat states, formed by superpositions of two coherent states shifted in phase by \( \pi \). For instance, the cat state \( e^{-i\pi/4}|i\alpha\rangle + e^{i\pi/4}|-i\alpha\rangle \) results from eq. (1) taken for \( \tau = \pi \). After this time the evolution effectively reverses to reach the original coherent state at \( \tau = 2\pi \).

Cat states were used in studies of decoherence mechanisms and the quantum-classical boundary [8,9]. The “large cats” for which the two components \( |i\alpha\rangle \) and \( |-i\alpha\rangle \) are nearly orthogonal (\( \alpha > 1.5 \)) also find applications in quantum information processing [28,29]. Efforts were made to produce such states using the Kerr nonlinearity in optical fibers [30]. However, optical nonlinearities are far too small to enter the highly nonlinear regime and thus to obtain nonclassical Wigner functions before dissipation effects destroy the coherences. Superpositions of multiple coherent states are obtained for \( \tau = \pi/2n \), where \( n \) is the number of coherent states in the superposition [31,32]. For instance, at \( \tau_{\text{eff}} = \pi/2 \) one achieves the superposition of four of them (the compass state) and for \( \tau_{\text{eff}} = \pi/3 \) six coherent states are superposed [33]. The maximum number of participating coherent states relies solely on the physical possibility of distinguishing them, which depends on the amplitude of the initial coherent state. Larger values of \( \alpha \) will allow more complex superpositions to be constructed, but will have support on a larger number of Fock states as well. Such examples of Kerr states were not experimentally investigated so far.

Time evolution of mean values and variances of the position \( X = a + a^\dagger \) and momentum \( Y = -i(a - a^\dagger) \) operators in Kerr medium is depicted in fig. 1(a) [26,34]. The evolution spreads the initial coherent state \( \langle \Delta^2X = \Delta^2Y = 1 \rangle \) around the origin of phase space, soon producing negative values in the Wigner function. Negative values with amplitude around 10% of the maximum positive value are already observed for \( \tau \approx \pi/20 \). In fig. 1(b) they reach nearly the same magnitude in sub-Planck areas of phase space for \( \tau \approx \pi/5 \). Since the Kerr interaction does not couple different Fock states, but instead only dephases them independently, the Wigner function cannot occupy higher phonon numbers than originally present. For intermediate times, fig. 1(c) shows one possible superposition of multiple coherent states, forming in this case a triangular pattern. The cat state is obtained at the turning point of the evolution (fig. 1(d)), after which the states produced appear once more in reversed order.

Generation of motional non-Gaussian Kerr states. – Let us consider a single ion trapped in the effective harmonic potential of a Paul trap interacting with a resonant laser field. By proper choice of geometry, only one of the three existing vibrational modes can be made relevant to the dynamics. In this case the interaction Hamiltonian reads [35]

\[
H = \hbar \frac{\Omega}{2} \left\{ \sigma^+ e^{i\omega(t-a^\dagger a)} + h.c. \right\},
\]

where \( \Omega \) is the Rabi frequency, \( \sigma^+ \) is the electronic state rising operator, \( \omega \) is the quantum oscillator harmonic
frequency, and $\eta$ is the Lamb-Dicke parameter. We will further assume the Lamb-Dicke regime and thus $\langle a^\dagger a \rangle \eta \lesssim 1$. It is justified in this case to expand the exponentials of eq. (2) in $\eta$ and disregard the higher-order terms. The Kerr effect is described by the term $(a^\dagger a)^2$. It dephases each Fock state proportionally to its eigenvalue (self-phase modulation) [25]. Keeping only the terms resonant to the atomic transition (carrier) up to the fourth order in $\eta$, we obtain in the interaction picture

$$H^{\text{int}} = \hbar \frac{\Omega}{2} \left( 1 - \frac{\eta^2}{2} + \frac{\eta^4}{8} \right) \sigma_z.$$  

(3)

The first three terms describe the corrected internal transition Rabi frequency, and do not influence the motional state. The terms proportional to $a^\dagger a$ represent a rotation in phase space and can be omitted by considering another rotating frame. Therefore, the lowest-order term to nontrivially influence the motional dynamics is indeed the Kerr self-phase modulation. Higher-order terms, although contributing as well, are much smaller by the factor of order of $\eta^2$. Similarly, off-resonant terms will also hinder the ideal dynamics. All these effects are disregarded in the present qualitative discussion, but are fully taken into account in the numerical calculation presented in the next section.

The inclusion of additional vibrational modes brings other nonlinear effects such as cross-phase modulation [36]. Here we will disregard those interesting but more complicated effects by supposing that the Lamb-Dicke parameters of such modes are much smaller than for the main mode considered. If more ions participate in the dynamics, all the vibrational modes in a given direction must be included for consistency, in which case cross-phase modulation is unavoidable.

The Hamiltonian of eq. (3) couples the internal and the external degrees of freedom and results in an internal state-dependent evolution of the motion. To remove this undesired effect, the atomic internal state must be initially prepared in an eigenstate of $\sigma_z$. Usual $\pi/2$ Rabi pulse with $\pi/2$ phase with respect to the main Kerr pulse accomplishes the atomic state superposition on the time scale of $\Omega^{-1}$, i.e. much faster than the Kerr evolution time scale $\tau_{\text{eff}}$ of eq. (5). Thus, it has a negligible effect on the ion motional state. With this procedure, the internal atomic state is separable from the motional state. The evolution operator acting on the vibrational mode then equals

$$U(t) = \exp \left( i \frac{1}{8} \eta \Omega t \right).$$  

(4)

From this expression we identify the effective value of the Kerr evolution parameter

$$\tau_{\text{eff}} = \frac{1}{4} \eta \Omega t.$$  

(5)

The highly non-Gaussian Kerr states require a coherent state as the starting point of the nonlinear evolution. It can be prepared in a number of ways [12,13]. Once prepared, any state can be reconstructed by a standard technique using displacements and parity measurements [37].

A special feature of the method under discussion is that the cat state “size” (with respect to $\langle a^\dagger a \rangle = |\alpha|^2$) is independent of the time necessary to produce it. Since the Kerr evolution recurrence time does not depend on the initial state, a “small cat” and a “large cat” are generated by the same light pulse. This is in sharp contrast to approximative methods where the desired final superposition is composed of several laser pulses each raising the phonon number by one unity [18]. In the scheme we investigate here, the maximum cat size is only limited by the trap anharmonicities and decoherence (as all other methods ultimately are). Furthermore, no additional truncation at a maximum phonon occupation number is necessary.

A simple estimate using eq. (5) shows the feasibility of the scheme within the typical experimental conditions. We take $\omega = 2\pi \times 3$ MHz as the quantum oscillator frequency. The Rabi frequency $\Omega = 2\pi \times 200$ kHz would result in off-resonant excitation on the order of 0.1%. For these parameters, $\eta = 0.1$ would allow achieving the cat state in $t_{\text{cat}} \approx 100$ ms. This value lies on the boundary of the expected motional coherence time, typically less than 1 ms for small traps and above 100 ms for larger traps. However, because of the strong dependence of $t_{\text{cat}}$ on the Lamb-Dicke parameter, $\eta = 0.2$ would already result in a pulse duration of only $t_{\text{cat}} \approx 6$ ms. In any case, other complex Kerr states would be accessible for even shorter pulses, as previously discussed, lying well within the expected motional coherence time. The Lamb-Dicke parameter can be manipulated to a certain extent by changing the oscillator frequency as well as by small adjustments in the angle between the oscillator axis and the laser propagation vector (or two lasers, in the case of Raman excitation) [38].

**Fidelity of the Kerr state produced by the complete Hamiltonian.** – The complete internal and external dynamics of the trapped ion using the full Hamiltonian of eq. (2) was computed. The effects of the higher-order and off-resonant terms were included in the calculation. As the initial condition for the ion motion a coherent state of amplitude $\alpha = 2$ was taken, corresponding to mean phonon number $\langle a^\dagger a \rangle = 4$. The oscillator frequency $\omega = 2\pi \times 3$ MHz was assumed.

For the numerical calculation the Fock basis was truncated at $|n = 21\rangle$ in order to obtain the probability smaller than $10^{-19}$ for the population of the higher Fock states. The initial electronic state $(|g\rangle + |e\rangle)/\sqrt{2}$ was chosen as the eigenstate of $\sigma_z$ (where $|g\rangle$ and $|e\rangle$ stand for the ground and the excited internal state, respectively). The evolution of the vibrational state was calculated by applying the evolution operator based on the Hamiltonian of eq. (2) in steps of $dt = 10^{-5}$ ms or smaller. The states obtained
in this manner were compared to the ideal Kerr state of eq. (1) using the fidelity defined as the square of their scalar product.

An example of pulse operation as a function of time for \( \eta = 0.2 \) is shown in fig. 2. For this particular choice of parameters, the cat state was obtained in 13 ms interaction time with 98.5% fidelity. More complex states were generated faster with higher fidelity. The fast ripples on the internal state population (fig. 2(b)) reach maximum amplitudes of 1% (although typically 0.1%), whilst the phonon state populations (fig. 2(c)) fluctuates within 0.1%. Those off-resonant excitations, however, do not accumulate dramatically over time. Those results confirm that the dynamics using higher-order terms of the Hamiltonian does not hinder itself the attainability of high fidelities.

The cat state fidelity as a function of Rabi frequency for three values of \( \eta \) is depicted in fig. 3. The smaller values of \( \eta \), the higher fidelities for the cat state. The physical time necessary for the formation of the cat state is shown on the right axis. In principle, fidelity of the state can be arbitrarily close to unity although the time required for its creation increases. In practice, decoherence will limit the maximum operation time. Thus, the optimum pulse duration will depend on the particular experimental conditions. The tradeoff between increasing Rabi frequency and decreasing the ideal fidelity results in shorter pulse duration. In fact, the operation could be faster even for low values of \( \eta \) if the Rabi frequency would be increased to values comparable to the oscillator frequency. It is important to note however, that eq. (5) is no longer valid in the limit of small \( \eta \) and large \( \Omega \) due to saturation effects and off-resonant excitations.

The more realistic experimental situation where the cat state generation has to be completed given a fixed interaction time was also examined. The cat state fidelity as a function of the Lamb-Dicke parameter \( \eta \) for a 10 ms pulse duration is shown in fig. 4. Rabi frequencies are
Generation of Kerr non-Gaussian motional states of trapped ions

Fig. 4: (Colour on-line) Cat state fidelity as a function of the Lamb-Dicke parameter \( \eta \) for a fixed cat state creation time of 10 ms (black circles). The required Rabi frequencies relative to the oscillator frequency \( (\Omega/\omega) \) are indicated by the red squares. Each point results from a numerical calculation of the quantum evolution.

indicated on the right axis of the figure. The optimum cat state fidelity of 99\% was found for \( \eta \approx 0.15 \) for our choice of parameters. The smaller \( \eta \) is, the lower the fidelity becomes due to the large values of \( \Omega \) which drive undesired transitions. As expected, the higher-order terms in eq. (2) contribute significantly for larger \( \eta \) and hinder the ideal Kerr dynamics.

Although the Kerr Hamiltonian allows one to generate either “small” or “big” cats using the same interaction time, bigger cats will decohere faster, since coupling to a thermal reservoir results in loss of visibility of the interference pattern in phase space (with exponential dependence on the cat size) [35]. However, the cat state is the final product of the Kerr interaction, and as such takes the longest time to be created. Other interesting macroscopic quantum superpositions could therefore coherently occupy larger areas in phase space albeit for shorter durations.

Comparison between the quantum evolution of some of the states presented in fig. 4 and the ideal Kerr states is presented in fig. 5. For the optimum \( \eta \), a superposition of four coherent states is produced with 99.6\% fidelity, whilst three superposed coherent states can be obtained with 99.3\% fidelity. The first negativities of the Wigner function are observed already after 500 \( \mu \)s interaction time, and dramatic effects in sub-Planck regions of phase space appear after 2 ms with more than 99.9\% fidelity. We stress that these nearly ideal fidelities indicate that the actual experimental limitations will not be a consequence of the form of the Hamiltonian, but rather decoherence and imperfections which will inherently depend on specific details of the experimental setup. For instance, larger cats can be expected to experience stronger decoherence [39], effectively requiring faster nonlinear dynamics.

From the perspective of the interaction Hamiltonian, the upper limit on the fidelity for a fixed cat state creation time results either from off-resonant excitation involving the phonon sidebands or from higher-order terms in the Hamiltonian. Off-resonant excitations cause noisier evolution for smaller \( \eta \), while for larger values of \( \eta \) the evolution becomes smoother (fig. 5). These excitations can be thus avoided by increasing the oscillator frequency. Taking all necessary compensations into account, our calculations indicate that a higher oscillator frequency has a positive effect on the state fidelity.

Conclusion. – We have presented a detailed investigation of a practical method of involving Kerr non-Gaussian state generation in the vibrational mode of trapped ions [25]. The method employs only one laser pulse, directly connecting an initial coherent state to the desired highly nonclassical Kerr state. Since only a resonant carrier pulse is needed to harness the natural Kerr nonlinearity of trapped ions, this scheme can be performed even in very simple setups, such as the stylus trap [40]. An optimized setup would be desirable, however, where the trap size should be large enough to avoid motional heating and decoherence at the same time that a high oscillator frequency would hinder off-resonant excitations even for fast dynamics.

Special attention has been paid to the cat states, for their importance in quantum information applications. However, other complex quantum states such as superpositions of several coherent states can also be generated in the process. Highly non-Gaussian states with negative values on the Wigner function are obtained for very short interaction times.

The scheme produces large superpositions in phase space using the same time resources necessary for small superpositions, without the need for artificially truncating the population in the Fock state basis.
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