DECOMPOSITIONS OF GELFAND-SHILOV KERNELS INTO KERNELS OF SIMILAR CLASS

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Abstract. We prove that any linear operator with kernel in a Gelfand-Shilov space is a composition of two operators with kernels in the same Gelfand-Shilov space. We also give links on numerical approximations for such compositions. We apply these composition rules to establish Schatten-von Neumann properties for such operators.

0. Introduction

In this paper we investigate possibilities to decompose linear operators into operators in the same class. It is obvious that for any topological vector space \( B \), the set \( \mathcal{M} = \mathcal{L}(B) \) of linear and continuous operators on \( B \) is a decomposition algebra. That is, any operator \( T \) in \( \mathcal{M} \) is a composition of two operators in \( \mathcal{T}_1, \mathcal{T}_2 \in \mathcal{M} \), since we may choose \( \mathcal{T}_1 \) as the identity operator, and \( \mathcal{T}_2 = \mathcal{T} \). If in addition \( B \) is a Hilbert space, then it follows from spectral decomposition that the set of compact operators on \( B \) is a decomposition algebra, where the decomposition property are obtained by straight-forward applications of the spectral theorem.

An interesting subclass of linear and continuous operators on \( L^2 \) concerns the set of all linear operators whose kernels belong to the Schwartz space. There are several proofs of the fact that this operator class is a decomposition algebra (cf. e.g. [17, 12, 15] and the references therein).

We remark that there are operator algebras which are not decomposition algebras. For example, if \( B \) is an infinite-dimensional Hilbert space and \( 0 < p < \infty \), then the set of all Schatten-von Neumann operators of order \( p \) is not a decomposition algebra.

In this paper we consider the case when \( \mathcal{M} \) is the set of all linear operators with distribution kernels in the Schwartz class, or in a Gelfand-Shilov space. We note that these operator classes are small, because the restrictions on corresponding kernels are strong. For example, it is obvious that the identity operator does not belong to any of these operator classes.

We prove that any such \( \mathcal{M} \) is a decomposition algebra. Furthermore, in the end of the paper we apply the result and prove that any operator in \( \mathcal{M} \) as a map between appropriate Banach spaces, or quasi-Banach spaces, belongs to every Schatten-von Neumann class between these spaces.

Key words and phrases. matrices, Hermite functions, kernel theorems, Schatten-von Neumann operators, singular values.
1. Preliminaries

In this section we recall some facts on Gelfand-Shilov spaces and pseudo-differential operators.

We start by defining Gelfand-Shilov spaces and recalling some basic facts.

Let $0 < h, s \in \mathbb{R}$ be fixed. Then we let $S_{s,h}(\mathbb{R}^d)$ be the set of all $f \in C^\infty(\mathbb{R}^d)$ such that

$$
\|f\|_{S_{s,h}} \equiv \sup_{h \|\alpha\| + |\beta| (\alpha! \beta!)^s} \left| \frac{x^\beta \partial^\alpha f(x)}{h^{\|\alpha\|} \alpha! \beta!} \right|
$$

is finite. Here the supremum should be taken over all $\alpha, \beta \in \mathbb{N}^d$ and $x \in \mathbb{R}^d$.

Obviously $S_{s,h} \subseteq \mathcal{S}$ is a Banach space which increases with $h$ and $s$. Furthermore, if $s \geq 1/2$ and $h$ is sufficiently large, then $S_{s,h}$ contains all finite linear combinations of Hermite functions. Since such linear combinations are dense in $\mathcal{S}$, it follows that the dual $S'_{s,h}(\mathbb{R}^d)$ of $S_{s,h}(\mathbb{R}^d)$ is a Banach space which contains $\mathcal{S}'(\mathbb{R}^d)$.

The Gelfand-Shilov spaces $S_s(\mathbb{R}^d)$ and $\Sigma_s(\mathbb{R}^d)$ are the inductive and projective limits respectively of $S_{s,h}(\mathbb{R}^d)$. This means that

$$
S_s(\mathbb{R}^d) = \bigcup_{h > 0} S_{s,h}(\mathbb{R}^d) \quad \text{and} \quad \Sigma_s(\mathbb{R}^d) = \bigcap_{h > 0} S_{s,h}(\mathbb{R}^d),
$$

$S_s(\mathbb{R}^d)$ is equipped with the strongest topology such that each inclusion map from $S_{s,h}(\mathbb{R}^d)$ to $S_s(\mathbb{R}^d)$ is continuous, and that $\Sigma_s(\mathbb{R}^d)$ is a Fréchet space with semi norms $\| \cdot \|_{S_{s,h}}$, $h > 0$.

We remark that $S_s(\mathbb{R}^d)$ equals $\{0\}$ if and only if $s < 1/2$, and that $\Sigma_s(\mathbb{R}^d)$ equals $\{0\}$ if and only if $s \leq 1/2$. For each $\varepsilon > 0$ and $s \geq 1/2$, we have

$$
\Sigma_s(\mathbb{R}^d) \hookrightarrow S_s(\mathbb{R}^d) \hookrightarrow \Sigma_{s+\varepsilon}(\mathbb{R}^d).
$$

Here, if $A$ and $B$ are topological spaces, then $A \hookrightarrow B$ means that $A$ is continuously embedded in $B$. On the other hand, in [11] there is an alternative elegant definition of $S_{s_1}(\mathbb{R}^d)$ and $S_{s_2}(\mathbb{R}^d)$ such that these spaces agrees with the definitions above when $s_1 > 1/2$ and $s_2 \geq 1/2$, but $\Sigma_{1/2}(\mathbb{R}^d)$ is non-trivial and contained in $S_{1/2}(\mathbb{R}^d)$.

From now on we assume that $s \geq 1/2$ when considering $\Sigma_s(\mathbb{R}^d)$.

The Gelfand-Shilov distribution spaces $S'_s(\mathbb{R}^d)$ and $\Sigma'_s(\mathbb{R}^d)$ are the projective and inductive limit respectively of $S'_{s,h}(\mathbb{R}^d)$. This means that

$$
S'_s(\mathbb{R}^d) = \bigcap_{h > 0} S'_{s,h}(\mathbb{R}^d) \quad \text{and} \quad \Sigma'_s(\mathbb{R}^d) = \bigcup_{h > 0} S'_{s,h}(\mathbb{R}^d).
$$

We remark that already in [11] it is proved that $S'_s(\mathbb{R}^d)$ is the dual of $S_s(\mathbb{R}^d)$, and if $s > 1/2$, then $\Sigma'_s(\mathbb{R}^d)$ is the dual of $\Sigma_s(\mathbb{R}^d)$ (also in topological sense).

The Gelfand-Shilov spaces are invariant under several basic transformations. For example they are invariant under translations, dilations, tensor products and to some extent under any Fourier transformation.

From now on we let $\mathcal{F}$ be the Fourier transform which takes the form

$$(\mathcal{F} f)(\xi) = \hat{f}(\xi) \equiv (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x) e^{-i(x,\xi)} \, dx.$$
when \( f \in L^1(\mathbb{R}^d) \). Here \( \langle \cdot, \cdot \rangle \) denotes the usual scalar product on \( \mathbb{R}^d \). The map \( F \) extends uniquely to homeomorphisms on \( \mathcal{S}'(\mathbb{R}^d), S'_s(\mathbb{R}^d) \) and \( \Sigma'_s(\mathbb{R}^d) \), and restricts to homeomorphisms on \( \mathcal{S}(\mathbb{R}^d), S_s(\mathbb{R}^d) \) and \( \Sigma_s(\mathbb{R}^d) \), and to a unitary operator on \( L^2(\mathbb{R}^d) \).

The following lemma shows that elements in Gelfand-Shilov spaces can be characterized by estimates of the form

\[
|f(x)| \leq C e^{-\varepsilon |x|^{1/s}} \quad \text{and} \quad |\hat{f}(\xi)| \leq C e^{-\varepsilon |\xi|^{1/s}}. \tag{1.2}
\]

The proof is omitted, since the result can be found in e. g. [2, 4].

**Lemma 1.1.** Let \( f \in S'_{1/2}(\mathbb{R}^d) \). Then the following is true:

1. if \( s \geq 1/2 \), then \( f \in S_s(\mathbb{R}^d) \), if and only if there are constants \( \varepsilon > 0 \) and \( C > 0 \) such that (1.2) holds;
2. if \( s > 1/2 \), then \( f \in \Sigma_s(\mathbb{R}^d) \), if and only if for each \( \varepsilon > 0 \), there is a constant \( C \) such that (1.2) holds.

Gelfand-Shilov spaces can also easily be characterized by Hermite functions. We recall that the Hermite function \( h_\alpha \) with respect to the multi-index \( \alpha \in \mathbb{N}^d \) is defined by

\[
h_\alpha(x) = \pi^{-d/4}(-1)^{|\alpha|/2}|\alpha|!^{-1/2} e^{x^2/2}(\partial^\alpha e^{-|x|^2}).
\]

The set \( \{h_\alpha \}_{\alpha \in \mathbb{N}^d} \) is an orthonormal basis for \( L^2(\mathbb{R}^d) \). In particular,

\[
f = \sum_\alpha c_\alpha h_\alpha, \quad c_\alpha = \langle f, h_\alpha \rangle_{L^2(\mathbb{R}^d)}, \tag{1.3}
\]

and

\[
\|f\|_{L^2} = \|\{c_\alpha\}_\alpha\|_{l^2} < \infty,
\]
when \( f \in L^2(\mathbb{R}^d) \). Here and in what follows, \( \langle \cdot, \cdot \rangle_{L^2(\mathbb{R}^d)} \) denotes any continuous extension of the \( L^2 \) form on \( S'_{1/2}(\mathbb{R}^d) \).

Let \( p \in [1, \infty] \) be fixed. Then it is well-known that \( f \) here belongs to \( \mathcal{S}(\mathbb{R}^d) \), if and only if

\[
\|\{c_\alpha(\alpha)\}_\alpha\|_{l^p} < \infty \tag{1.4}
\]

for every \( t \geq 0 \). Here we let \( \langle x \rangle = (1 + |x|^2)^{1/2} \) when \( x \in \mathbb{R}^d \). Furthermore, for every \( f \in \mathcal{S}'(\mathbb{R}^d) \), the expansion (1.3) still holds, where the sum converges in \( \mathcal{S}' \), and (1.4) holds for some choice of \( t \in \mathbb{R} \), which depends on \( f \).

The following proposition, which can be found in e. g. [5], shows that similar conclusion for Gelfand-Shilov spaces holds, after the estimate (1.4) is replaced by

\[
\|\{c_\alpha e^{t|\alpha|^{1/2s}}\}_\alpha\|_{l^p} < \infty. \tag{1.5}
\]

We refer to the proof of formula (2.12) in [5] for its proof.

**Proposition 1.2.** Let \( p \in [1, \infty], f \in S'_{1/2}(\mathbb{R}^d), s_1 \geq 1/2, s_2 \geq 1/2 \) and let \( c_\alpha \) be as in (1.3). Then the following is true:

1. \( f \in \mathcal{S}(\mathbb{R}^d) \), if and only if (1.4) holds for every \( t > 0 \). Furthermore, (1.3) holds where the sum converges in \( \mathcal{S} \);
(2) $f \in \mathcal{S}_{s_1}(\mathbb{R}^d)$, if and only if (1.5) holds for some $t > 0$. Furthermore, (1.3) holds where the sum converges in $\mathcal{S}_{s_1}$;

(3) $f \in \mathcal{S}_{s_2}(\mathbb{R}^d)$, if and only if (1.5) holds for every $t > 0$. Furthermore, (1.3) holds where the sum converges in $\mathcal{S}_{s_2}$.

(4) $f \in \mathcal{S}'(\mathbb{R}^d)$, if and only if (1.4) holds for some $t < 0$. Furthermore, (1.3) holds where the sum converges in $\mathcal{S}'$;

(5) $f \in \mathcal{S}'_{s_1}(\mathbb{R}^d)$, if and only if (1.5) holds for every $t < 0$. Furthermore, (1.3) holds where the sum converges in $\mathcal{S}'_{s_1}$;

(6) $f \in \mathcal{S}'_{s_2}(\mathbb{R}^d)$, if and only if (1.5) holds for some $t < 0$. Furthermore, (1.3) holds where the sum converges in $\mathcal{S}'_{s_2}$;

Proposition 1.2 is fundamental in the proof of Theorem 2.2 which in turn is a cornerstone in the proof of Theorem 1.1. It also gives links on how properties valid for tempered distributions or Schwartz functions can be carried over to Gelfand-Shilov spaces of functions or distributions by passing from estimates of the form (1.4) to estimates of the form (1.5), and vice versa.

Next we recall some properties of pseudo-differential operators. Let $t \in \mathbb{R}$ be fixed and let $a \in \mathcal{S}'_{1/2}(\mathbb{R}^{2d})$. Then the pseudo-differential operator $\text{Op}_a(t)$ with symbol $a$ is the linear and continuous operator on $\mathcal{S}_{1/2}(\mathbb{R}^d)$, defined by the formula

$$\text{Op}_a(t)f(x) = (2\pi)^{-d} \int \int a((1 - t)x + ty, \xi) f(y) e^{i(x-y, \xi)} dyd\xi. \quad (1.6)$$

The definition of $\text{Op}_a(t)$ extends to each $a \in \mathcal{S}'_{1/2}(\mathbb{R}^{2d})$, and then $\text{Op}_a(t)$ is continuous from $\mathcal{S}_{1/2}(\mathbb{R}^d)$ to $\mathcal{S}'_{1/2}(\mathbb{R}^d)$. (Cf. e.g. [3], and to some extent [6].) More precisely, for any $a \in \mathcal{S}'_{1/2}(\mathbb{R}^{2d})$, the operator $\text{Op}_a(t)$ is defined as the linear and continuous operator from $\mathcal{S}_{1/2}(\mathbb{R}^d)$ to $\mathcal{S}'_{1/2}(\mathbb{R}^d)$ with distribution kernel given by

$$K_{a,t}(x, y) = (\mathcal{F}_2^{-1}a)((1 - t)x + ty, x - y). \quad (1.7)$$

Here $\mathcal{F}_2F$ is the partial Fourier transform of $F(x, y) \in \mathcal{S}'_{1/2}(\mathbb{R}^{2d})$ with respect to the $y$ variable. This definition makes sense, since the mappings $\mathcal{F}_2$ and $F(x, y) \mapsto F((1 - t)x + ty, y - x)$ are homeomorphisms on $\mathcal{S}'_{1/2}(\mathbb{R}^{2d})$.

On the other hand, let $T$ be an arbitrary linear and continuous operator from $\mathcal{S}_{1/2}(\mathbb{R}^d)$ to $\mathcal{S}'_{1/2}(\mathbb{R}^d)$. Then it follows from Theorem 2.2 in [9] that for some $K = K_T \in \mathcal{S}'_{1/2}(\mathbb{R}^{2d})$ we have

$$(Tf, g)_{L^2(\mathbb{R}^d)} = (K, g \otimes \mathcal{F}T)_{L^2(\mathbb{R}^{2d})},$$

for every $f, g \in \mathcal{S}_{1/2}(\mathbb{R}^d)$. Now by letting $a$ be defined by (1.7) after replacing $K_{a,t}$ with $K$ it follows that $T = \text{Op}_a(t)$. Consequently, the map $a \mapsto \text{Op}_a(t)$ is bijective from $\mathcal{S}'_{1/2}(\mathbb{R}^{2d})$ to $\mathcal{S}'(\mathcal{S}_{1/2}(\mathbb{R}^d), \mathcal{S}'_{1/2}(\mathbb{R}^d))$.

If $t = 1/2$, then $\text{Op}_a(t)$ is equal to the Weyl quantization $\text{Op}_a^w(t)$ of $a$. If instead $t = 0$, then the standard (Kohn-Nirenberg) representation $a(x, D)$ is obtained.
In particular, if \( a \in S_{1/2}'(\mathbb{R}^{2d}) \) and \( s,t \in \mathbb{R} \), then there is a unique \( b \in S_{1/2}'(\mathbb{R}^{2d}) \) such that \( \text{Op}_s(a) = \text{Op}_t(b) \). By straight-forward applications of Fourier’s inversion formula, it follows that

\[
\text{Op}_s(a) = \text{Op}_t(b) \iff b(x, \xi) = e^{i(t-s)(D_x, D_\xi)}a(x, \xi). \tag{1.8}
\]

(Cf. Section 18.5 in [6].) Note here that the right-hand side makes sense, because \( e^{i(t-s)(D_x, D_\xi)} \) on the Fourier transform side is a multiplication by the function \( e^{i(t-s)(x, \xi)} \), which is a continuous operation on \( S_{1/2}'(\mathbb{R}^{2d}) \), in view of the definitions.

Next let \( t \in \mathbb{R} \) be fixed and let \( a, b \in S_{1/2}'(\mathbb{R}^{2d}) \). Then the product \( a \# b \) is defined by the formula

\[
\text{Op}_t(a \# b) = \text{Op}_t(a) \circ \text{Op}_t(b),
\]

provided the right-hand side makes sense as a continuous operator from \( S_{1/2}(\mathbb{R}^d) \) to \( S_{1/2}'(\mathbb{R}^d) \). We note that the element \( a \# b \) is uniquely defined and belongs to \( c \in S_{1/2}'(\mathbb{R}^{2d}) \).

2. GELFAND-SHILOV KERNELS AND PSEUDO-DIFFERENTIAL OPERATORS

In what follows we use the convention that if \( T_0 \) is a linear and continuous operator from \( S_{1/2}(\mathbb{R}^{d_1}) \) to \( S_{1/2}'(\mathbb{R}^{d_2}) \), and \( g \in S_{1/2}'(\mathbb{R}^{d_2}) \), then \( T_0 \otimes g \) is the linear and continuous operator from \( S_{1/2}(\mathbb{R}^{d_1}) \) to \( S_{1/2}'(\mathbb{R}^{d_2+d_0}) \), given by

\[
(T_0 \otimes g) : f \mapsto (T_0 f) \otimes g.
\]

In the following definition we recall that an operator \( T \) from \( S_{1/2}(\mathbb{R}^d) \) to \( S_{1/2}'(\mathbb{R}^d) \) is called positive semi-definite, if \( (Tf, f)_{L^2} \geq 0 \), for every \( f \in S_{1/2}(\mathbb{R}^d) \). Then we write \( T \geq 0 \).

**Definition 2.1.** Let \( d_2 \geq d_1 \) and let \( T \) be a linear operator from \( S_{1/2}(\mathbb{R}^{d_1}) \) to \( S_{1/2}'(\mathbb{R}^{d_2}) \). Then \( T \) is said to be a Hermite diagonal operator if \( T = T_0 \otimes g \), where the Hermite functions are eigenfunctions to \( T_0 \), and either \( d_2 = d_1 \) and \( g = 1 \), or \( d_2 > d_1 \) and \( g \) is a Hermite function.

Moreover, if \( T = T_0 \otimes g \) is on Hermite diagonal form and \( T_0 \) is positive semi-definite, then \( T \) is said to be a positive Hermite diagonal operator.

The first part of the following result can be found in [1, 15] (see also [7, 12] and the references therein for an elementary proof).

**Theorem 2.2.** Let \( T \) be a linear and continuous operator from \( S_{1/2}(\mathbb{R}^{d_1}) \) to \( S_{1/2}'(\mathbb{R}^{d_2}) \) with the kernel \( K \), and let \( d_0 \geq \min(d_1, d_2) \). Then the following is true:

1. If \( K \in \mathcal{S}(\mathbb{R}^{d_2+d_1}) \), then there are operators \( T_1 \) and \( T_2 \) with kernels \( K_1 \in \mathcal{S}(\mathbb{R}^{d_0+d_1}) \) and \( K_2 \in \mathcal{S}(\mathbb{R}^{d_2+d_0}) \) respectively such that \( T = T_2 \circ T_1 \). Furthermore, at least one of \( T_1 \) and \( T_2 \) can be chosen as positive Hermite diagonal operator;

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(2) If \( s \geq 1/2 \) and \( K \in \mathcal{S}_s(\mathbb{R}^{d_2 + d_1}) \), then there are operators \( T_1 \) and \( T_2 \) with kernels \( K_1 \in \mathcal{S}_s(\mathbb{R}^{d_0 + d_1}) \) and \( K_2 \in \mathcal{S}_s(\mathbb{R}^{d_2 + d_0}) \) respectively such that \( T = T_2 \circ T_1 \). Furthermore, at least one of \( T_1 \) and \( T_2 \) can be chosen as positive Hermite diagonal operator;

(3) If \( s > 1/2 \) and \( K \in \Sigma_s(\mathbb{R}^{d_2 + d_1}) \), then there are operators \( T_1 \) and \( T_2 \) with kernels \( K_1 \in \Sigma_s(\mathbb{R}^{d_0 + d_1}) \) and \( K_2 \in \Sigma_s(\mathbb{R}^{d_2 + d_0}) \) respectively such that \( T = T_2 \circ T_1 \). Furthermore, at least one of \( T_1 \) and \( T_2 \) can be chosen as positive Hermite diagonal operator.

Remark 2.3. An operator with kernel in \( \mathcal{S}_s(\mathbb{R}^{2d}) \) is sometimes called a regularizing operator with respect to \( \mathcal{S}_s \), because it extends uniquely to a continuous map from (the large space) \( \mathcal{S}'(\mathbb{R}^d) \) into (the small space) \( \mathcal{S}_s(\mathbb{R}^d) \). Analogously, an operator with kernel in \( \Sigma_s(\mathbb{R}^{2d}) \) \( (\mathcal{S}(\mathbb{R}^{2d})) \) is sometimes called a regularizing operator with respect to \( \Sigma_s \).

Proof of Theorem 2.2. We only prove (2) and (3). The assertion (1) follows by similar arguments as in the proof of (3). Furthermore, a proof of the first part of (1) can be found in [1, 12].

First we assume that \( d_0 = d_1 \), and start to prove (2). Let \( h_{d,\alpha}(x) \) be the Hermite function on \( \mathbb{R}^d \) of order \( \alpha \in \mathbb{N}^d \). Then \( K \) posses the expansion

\[
K(x, y) = \sum_{\alpha \in \mathbb{N}^{d_2}} \sum_{\beta \in \mathbb{N}^{d_1}} a_{\alpha, \beta} h_{d_2, \alpha}(x) h_{d_1, \beta}(y),
\]

where the coefficients \( a_{\alpha, \beta} \) satisfies

\[
\sup_{\alpha, \beta} |a_{\alpha, \beta} e^{r(|\alpha|^{1/2s} + |\beta|^{1/2s})}| < \infty,
\]

for some \( r > 0 \).

Now we let \( z \in \mathbb{R}^{d_1} \), and

\[
K_{0,2}(x, z) = \sum_{\alpha \in \mathbb{N}^{d_2}} \sum_{\beta \in \mathbb{N}^{d_1}} b_{\alpha, \beta} h_{d_2, \alpha}(x) h_{d_1, \beta}(z),
\]

\[
K_{0,1}(z, y) = \sum_{\alpha, \beta \in \mathbb{N}^{d_1}} c_{\alpha, \beta} h_{d_1, \alpha}(z) h_{d_1, \beta}(y),
\]

where

\[
b_{\alpha, \beta} = a_{\alpha, \beta} e^{r|\beta|^{1/2s}/2} \quad \text{and} \quad c_{\alpha, \beta} = \delta_{\alpha, \beta} e^{-r|\alpha|^{1/2s}/2}.
\]

Here \( \delta_{\alpha, \beta} \) is the Kronecker delta. Then it follows that

\[
\int K_{0,2}(x, z) K_{0,1}(z, y) \, dz = \sum_{\alpha \in \mathbb{N}^{d_2}} \sum_{\beta \in \mathbb{N}^{d_1}} a_{\alpha, \beta} h_{d_2, \alpha}(x) h_{d_1, \beta}(y) = K(x, y).
\]

Hence, if \( T_j \) is the operator with kernel \( K_{0,j} \), \( j = 1, 2 \), then \( T = T_2 \circ T_1 \). Furthermore,

\[
\sup_{\alpha, \beta} |b_{\alpha, \beta} e^{r(|\alpha|^{1/2s} + |\beta|^{1/2s})/2}| \leq \sup_{\alpha, \beta} |a_{\alpha, \beta} e^{r(|\alpha|^{1/2s} + |\beta|^{1/2s})}| < \infty
\]
and
\[ \sup_{\alpha, \beta} |c_{\alpha, \beta} e^{r(|\alpha|^{1/2s} + |\beta|^{1/2s})}| = \sup_{\alpha} |e^{-r|\alpha|^{1/2s}} e^{r|\alpha|^{1/2s}}| < \infty. \]

This implies that \( K_{0,1} \in \mathcal{S}(\mathbb{R}^{d_1 + d_1}) \) and \( K_{0,2} \in \mathcal{S}(\mathbb{R}^{d_2 + d_1}) \) in view of Proposition 1.2 and (2) follows with \( K_1 = K_{0,1} \) and \( K_2 = K_{0,2} \) in the case \( d_0 = d_1 \).

In order to prove (3), we assume that \( K \in \mathcal{S}(\mathbb{R}^{d_2 + d_1}) \), and we let \( a_{\alpha, \beta} \) be the same as the above. Then (2.2) holds for any \( r > 0 \), which implies that if \( N \geq 0 \) is an integer, then

\[ \Theta_N \equiv \sup \{ |\beta| ; |a_{\alpha, \beta}| \geq e^{-2(N+1)(|\alpha|^{1/2s} + |\beta|^{1/2s}} \text{ for some } \alpha \in \mathbb{N}^{d_2} \} \quad (2.4) \]

is finite.

We let
\[ I_1 = \{ \beta \in \mathbb{N}^{d_1} ; |\beta| \leq \Theta_1 + 1 \} \]

and define inductively
\[ I_j = \{ \beta \in \mathbb{N}^{d_1} \setminus I_{j-1} ; |\beta| \leq \Theta_j + j \}, \quad j \geq 2. \]

Then
\[ I_j \cap I_k = \emptyset \quad \text{when} \quad j \neq k, \quad \text{and} \quad \bigcup_{j \geq 0} I_j = \mathbb{N}^{d_1}. \]

We also let \( K_{0,1} \) and \( K_{0,2} \) be given by (2.3), where
\[ b_{\alpha_2, \beta} = a_{\alpha_2, \beta} e^{j|\beta|^{1/2s}} \quad \text{and} \quad c_{\alpha_1, \beta} = \delta_{\alpha_1, \beta} e^{-j|\beta|^{1/2s}}, \]
when \( \alpha_1 \in \mathbb{N}^{d_1}, \alpha_2 \in \mathbb{N}^{d_2} \) and \( \beta \in I_j \). If \( T_j \) is the operator with kernel \( K_{0,j} \) for \( j = 1, 2 \), then it follows that \( T_2 \circ T_1 = T \). Furthermore, if \( r > 0 \), then we have
\[ \sup_{\alpha, \beta} |b_{\alpha, \beta} e^{r(|\alpha|^{1/2s} + |\beta|^{1/2s})}| \leq J_1 + J_2, \]
where
\[ J_1 = \sup_{j \leq r+1} \sup_{\alpha} \sup_{\beta \in I_j} |b_{\alpha, \beta} e^{r(|\alpha|^{1/2s} + |\beta|^{1/2s})}| \quad (2.5) \]
and
\[ J_2 = \sup_{j > r+1} \sup_{\alpha} \sup_{\beta \in I_j} |b_{\alpha, \beta} e^{r(|\alpha|^{1/2s} + |\beta|^{1/2s})}| \quad (2.6) \]

Since only finite numbers of \( \beta \) is involved in the suprema in (2.5), it follows from (2.2) and the definition of \( b_{\alpha, \beta} \) that \( J_1 \) is finite.

For \( J_2 \) we have
\[ J_2 = \sup_{j > r+1} \sup_{\alpha} \sup_{\beta \in I_j} |a_{\alpha, \beta} e^{r(|\alpha|^{1/2s} + (r+j)|\beta|^{1/2s})}| \]
\[ \leq \sup_{j > r+1} \sup_{\alpha} \sup_{\beta \in I_j} |e^{-2(j(|\alpha|^{1/2s} + |\beta|^{1/2s}) e^{r(|\alpha|^{1/2s} + (r+j)|\beta|^{1/2s})}| < \infty, \]
where the first inequality follows from (2.4). Hence
\[ \sup_{\alpha,\beta} |b_{\alpha,\beta} e^{r(|\alpha|^{1/2}+|\beta|^{1/2})}| < \infty, \]
which implies that \( K_{0,2} \in \Sigma_s(\mathbb{R}^{d_2+d_1}) \).

If we now replace \( b_{\alpha,\beta} \) with \( c_{\alpha,\beta} \) in the definition of \( J_1 \) and \( J_2 \), it follows by similar arguments that both \( J_1 \) and \( J_2 \) are finite, also in this case. This gives
\[ \sup_{\alpha,\beta} |c_{\alpha,\beta} e^{r(|\alpha|^{1/2}+|\beta|^{1/2})}| < \infty. \]
Hence \( K_1 \in \Sigma_s(\mathbb{R}^{d_1+d_1}) \), and (3) follows in the case \( d_0 = d_1 \).

Next assume that \( d_0 > d_1 \), and let \( d = d_0 - d_1 \geq 1 \). Then we set
\[ K_1(z, y) = K_{0,1}(z_1, y) h_{d,0}(z_2) \quad \text{and} \quad K_2(x, z) = K_{0,2}(x, z_1) h_{d,0}(z_2), \]
where \( K_{0,j} \) are the same as in the first part of the proofs, \( z_1 \in \mathbb{R}^{d_1} \) and \( z_2 \in \mathbb{R}^d \), giving that \( z = (z_1, z_2) \in \mathbb{R}^{d_0} \). We get
\[ \int_{\mathbb{R}^{d_0}} K_2(x, z) K_1(z, y) \, dz = \int_{\mathbb{R}^{d_1}} K_{0,2}(x, z_1) K_{0,1}(z_1, y) \, dz_1 = K(x, y). \]
The assertions (2) now follows in the case \( d_0 > d_1 \) from the equivalences
\[ K_1 \in \mathcal{S}_s(\mathbb{R}^{d_0+d_1}) \iff K_{0,1} \in \mathcal{S}_s(\mathbb{R}^{d_1+d_1}) \]
and
\[ K_2 \in \mathcal{S}_s(\mathbb{R}^{d_2+d_0}) \iff K_{0,2} \in \mathcal{S}_s(\mathbb{R}^{d_2+d_1}), \]
Since the same equivalences hold after the \( \mathcal{S}_s \) spaces have been replaced by \( \Sigma_s \) spaces, the assertion (3) also follows in the case \( d_0 > d_1 \), and the theorem follows in the case \( d_0 \geq d_1 \).

It remains to prove the result in the case \( d_0 \geq d_2 \). The rules of \( d_1 \) and \( d_2 \) are interchanged when taking the adjoints. Hence, the result follows from the first part of the proof in combination with the facts that \( \mathcal{S}_s \) and \( \Sigma_s \) are invariant under pullbacks of bijective linear transformations. The proof is complete. \( \square \)

**Remark 2.4.** From the construction of \( K_1 \) and \( K_2 \) in the proof of Theorem 2.2 it follows that it is not so complicated for using numerical methods when obtaining approximations of candidates to \( K_1 \) and \( K_2 \). In fact, \( K_1 \) and \( K_2 \) are formed explicitly by the elements of the matrix for \( T \), when the Hermite functions are used as basis for \( \mathcal{S}_s \), \( \mathcal{S}_s \) and \( \Sigma_s \).

The following result is an immediate consequence of Theorem 2.2 and the fact that the map \( a \mapsto K_{a,t} \) is continuous and bijective on \( \mathcal{S}_s(\mathbb{R}^{2d}) \), and on \( \Sigma_s(\mathbb{R}^{2d}) \), for every \( s_1 \geq 1/2, s_2 > 1/2 \) and \( t \in \mathbb{R} \).

**Theorem 2.5.** Let \( t \in \mathbb{R}, s_1 \geq 1/2 \) and \( s_2 > 1/2 \). Then the following is true:
1. if \( a \in \mathcal{S}_s(\mathbb{R}^{2d}) \), then there are \( a_1, a_2 \in \mathcal{S}_s(\mathbb{R}^{2d}) \) such that \( a = a_1 \#_t a_2 \);
2. if \( a \in \Sigma_s(\mathbb{R}^{2d}) \), then there are \( a_1, a_2 \in \mathcal{S}_s(\mathbb{R}^{2d}) \) such that \( a = a_1 \#_t a_2 \).
3. Schatten-von Neumann properties for operators with Gelfand-Shilov kernels

In this section we use Theorem 2.2 to prove that if \( T \) is a linear operator with kernel in \( S_s \), and \( B_1, B_2 \subseteq S_s^* \) are such that \( S_s \subseteq B_1, B_2 \), then \( T \) belongs to any Schatten-von Neumann class of operators between \( B_1 \) and \( B_2 \). In particular it follows that the singular values of \( T \) fulfill strong decay properties.

We start by recalling the definition of Schatten-von Neumann operators in the (quasi-)Banach space case. Let \( B \) be a vector space. A quasi-norm \( \| \cdot \|_B \) on \( B \) is a non-negative and real-valued function on \( B \) which is non-degenerate in the sense

\[
\|f\|_B = 0 \iff f = 0, \quad f \in B,
\]

and fulfills

\[
\|\alpha f\|_B = |\alpha| \cdot \|f\|_B, \quad f \in B, \quad \alpha \in \mathbb{C}
\]

and

\[
\|f + g\|_B \leq D(\|f\|_B + \|g\|_B), \quad f, g \in B,
\]

for some constant \( D \geq 1 \) which is independent of \( f, g \in B \). Then \( B \) is a topological vector space when the topology for \( B \) is defined by \( \| \cdot \|_B \), and \( B \) is called a quasi-Banach space if \( B \) is complete under this topology.

Let \( B_1 \) and \( B_2 \) be (quasi-)Banach spaces, and let \( T \) be a linear map between \( B_1 \) and \( B_2 \). For every integer \( j \geq 1 \), the singular values of order \( j \) of \( T \) is given by

\[
\sigma_j(T) = \sigma_j(B_1, B_2, T) \equiv \inf \|T - T_0\|_{B_1 \to B_2},
\]

where the infimum is taken over all linear operators \( T_0 \) from \( B_1 \) to \( B_2 \) with rank at most \( j - 1 \). Therefore, \( \sigma_1(T) \) equals \( \|T\|_{B_1 \to B_2} \), and \( \sigma_j(T) \) are non-negative which decreases with \( j \).

For any \( p \in (0, \infty] \) we set

\[
\|T\|_{\mathcal{S}_p} = \|T\|_{\mathcal{S}_p(B_1, B_2)} \equiv \left(\sum_{j=1}^{\infty} (\sigma_j(B_1, B_2, T))^p\right)^{1/p}
\]

(which might attain \( +\infty \)). The operator \( T \) is called a Schatten-von Neumann operator of order \( p \) from \( B_1 \) to \( B_2 \), if \( \|T\|_{\mathcal{S}_p} \) is finite, i.e. \( (\sigma_j(B_1, B_2, T))^p \) should belong to \( l^p \). The set of all Schatten-von Neumann operators of order \( p \) from \( B_1 \) to \( B_2 \) is denoted by \( \mathcal{S}_p = \mathcal{S}_p(B_1, B_2) \). We note that \( \mathcal{S}_\infty(B_1, B_2) \) agrees with \( B(B_1, B_2) \), the set of linear and bounded operators from \( B_1 \) to \( B_2 \), and if \( p < \infty \), then \( \mathcal{S}_p(B_1, B_2) \) is contained in \( K(B_1, B_2) \), the set of linear and compact operators from \( B_1 \) to \( B_2 \). If \( B_1 = B_2 \), then the shorter notation \( \mathcal{S}_p(B_1) \) is used instead of \( \mathcal{S}_p(B_1, B_2) \), and similarly for \( B(B_1, B_2) \) and \( K(B_1, B_2) \).

Schatten-von Neumann classes posses several convenient properties. For example, if \( B_1, B_2 \) and \( B_3 \) are Banach spaces, \( p_1, p_2, r \in (0, \infty] \) satisfy the Hölder condition \( 1/p_1 + 1/p_2 = 1/r \), and \( T_k \in \mathcal{S}_{p_k}(B_k, B_{k+1}) \), then \( T_2 \circ T_1 \in \mathcal{S}_r(B_1, B_3) \), and

\[
\|T_2 \circ T_1\|_{\mathcal{S}_r(B_1, B_3)} \leq C_r \|T_1\|_{\mathcal{S}_{p_1}(B_1, B_2)} \|T_2\|_{\mathcal{S}_{p_2}(B_2, B_3)},
\]

where \( C_r = 1 \) when \( B_j, j = 1, 2, 3 \), are Hilbert spaces, and \( C_r = 2^{1/r} \) otherwise (cf. e.g. [10][4]). We refer to [10][3] for a brief analysis of Schatten-von Neumann operators.
The following theorem, which is the main result in this section, concerns Schatten-von Neumann properties for an operator $T_K$ when the operator kernel $K$ belongs to Gelfand-Shilov spaces.

**Theorem 3.1.** Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be quasi-Banach spaces such that
$$S_{1/2}(\mathbb{R}^{d_1}) \hookrightarrow \mathcal{B}_j \hookrightarrow S_{1/2}'(\mathbb{R}^{d_1}), \quad j = 1, 2,$$
and let $p \in (0, \infty]$. Then the following is true:

1. if $s \geq 1/2$, $\mathcal{B}_1 \hookrightarrow S'_s(\mathbb{R}^{d_1})$, $\mathcal{S}_s(\mathbb{R}^{d_2}) \hookrightarrow \mathcal{B}_2$, and $K \in \mathcal{S}_s(\mathbb{R}^{d_2+d_1})$, then $T_K \in \mathcal{I}_p(\mathcal{B}_1, \mathcal{B}_2)$;

2. if $s > 1/2$, $\mathcal{B}_1 \hookrightarrow \Sigma'_s(\mathbb{R}^{d_1})$, $\Sigma_s(\mathbb{R}^{d_2}) \hookrightarrow \mathcal{B}_2$, and $K \in \Sigma_s(\mathbb{R}^{d_2+d_1})$, then $T_K \in \mathcal{I}_p(\mathcal{B}_1, \mathcal{B}_2)$;

3. if $\mathcal{B}_1 \hookrightarrow \mathcal{S}'(\mathbb{R}^{d_1})$, $\mathcal{S}(\mathbb{R}^{d_2}) \hookrightarrow \mathcal{B}_2$, and $K \in \mathcal{S}(\mathbb{R}^{d_2+d_1})$, then $T_K \in \mathcal{I}_p(\mathcal{B}_1, \mathcal{B}_2)$.

We need some preparations for the proof. First we note that if $\mathcal{B}_j, \mathcal{C}_j, j = 1, 2$, are quasi-Banach spaces and $T : \mathcal{B}_1 \to \mathcal{B}_2$, then
$$\|T\|_{\mathcal{B}_1 \to \mathcal{B}_2} \leq C\|T\|_{\mathcal{B}_1 \to \mathcal{B}_2}, \quad \text{when } \mathcal{B}_1 \hookrightarrow \mathcal{B}_1 \text{ and } \mathcal{B}_2 \hookrightarrow \mathcal{C}_2,$$
for some constant $C$. Here and in what follows we use the convention that if $T$ is a linear operator from $\mathcal{B}_1$ to $\mathcal{B}_2$, $\mathcal{B}_1 \subseteq \mathcal{B}_1$ and $\mathcal{B}_2 \subseteq \mathcal{C}_2$, then the restriction of $T$ to an operator from $\mathcal{B}_1$ to $\mathcal{C}_2$ is still denoted by $T$.

**Lemma 3.2.** Let $\mathcal{B}_k, \mathcal{C}_k, k = 1, 2$, be quasi-Banach spaces such that $\mathcal{C}_1 \hookrightarrow \mathcal{B}_1$ and $\mathcal{B}_2 \hookrightarrow \mathcal{C}_2$. Also let $p \in (0, \infty]$ and $T : \mathcal{B}_1 \to \mathcal{B}_2$ be linear and continuous. Then
$$\sigma_j(\mathcal{C}_1, \mathcal{C}_2; T) \leq C\sigma_j(\mathcal{B}_1, \mathcal{B}_2; T), \quad j \geq 1,$$
and
$$\|T\|_{\mathcal{B}_1 \hookrightarrow \mathcal{B}_2} \leq C\|T\|_{\mathcal{B}_1 \hookrightarrow \mathcal{B}_2},$$
where $C$ is the same constant as in (3.1).

**Proof.** It suffices to prove the first inequality in (3.2). Let
$$\Omega_j = \{ T_0 \in \mathcal{B}(\mathcal{B}_1, \mathcal{B}_2) : \text{rank } T_0 < j \},$$
$$\Omega_{1,j} = \{ T_0 \in \mathcal{B}(\mathcal{C}_1, \mathcal{C}_2) : \text{rank } T_0 < j \},$$
and let $\Omega_{2,j}$ be the set of all $T_0$ in $\Omega_{1,j}$ such that $T_0$ is a restriction of an element in $\Omega_j$. Then $\Omega_{2,j} \subseteq \Omega_{1,j}$, and the restrictions of the elements in $\Omega_j$ to $\mathcal{C}_1$ belong to $\Omega_{2,j}$. This gives
$$\sigma_j(\mathcal{C}_1, \mathcal{C}_2; T) = \inf_{T_0 \in \Omega_j} \|T - T_0\|_{\mathcal{B}_1 \to \mathcal{B}_2} \leq \inf_{T_0 \in \Omega_{1,j}} \|T - T_0\|_{\mathcal{C}_1 \to \mathcal{C}_2} \leq C \inf_{T_0 \in \Omega_{2,j}} \|T - T_0\|_{\mathcal{C}_1 \to \mathcal{C}_2} = C\sigma_j(\mathcal{B}_1, \mathcal{B}_2; T),$$
where the last inequality follows from (3.1). Hence (3.2) follows, and the proof is complete. □
Before stating the next results we need some notions from [14] concerning Hilbert spaces $\mathcal{H}$, which satisfy

$$S_{1/2}(\mathbb{R}^d) \hookrightarrow \mathcal{H} \hookrightarrow S'_1(\mathbb{R}^d).$$

We let

$$(S_\pi f)(x) \equiv f(x_{\pi(1)}, \ldots, x_{\pi(d)}) \in \mathcal{H} \quad \text{when} \quad f \in S'_{1/2}(\mathbb{R}^d),$$

when $\pi$ is a permutation of $\{1, \ldots, d\}$. The Hilbert space $\mathcal{H}$ is said to be of Hermite type, if $(h_\alpha/\|h_\alpha\|_\mathcal{H})_\alpha$ is an orthonormal basis for $\mathcal{H}$, and $\|S_\pi f\|_\mathcal{H} = \|f\|_\mathcal{H}$ for every $f \in \mathcal{H}$ and every permutation $\pi$ on $\{1, \ldots, d\}$.

The $L^2$-dual $\mathcal{H}'$ of $\mathcal{H}$ consists of all $f \in S'_{1/2}(\mathbb{R}^d)$ such that

$$\|f\|_{\mathcal{H}'} \equiv \sup \|f, \varphi\|_{L^2}$$

is finite. Here the supremum is taken over all $\varphi \in S_{1/2}$ such that $\|\varphi\|_{\mathcal{H}'} \leq 1$.

We also let $\mathcal{H}^r$ be the set of all $f \in S'_{1/2}$ such that $f \in \mathcal{H}$. Then $\mathcal{H}$ and $\mathcal{H}^r$ with norms $f \mapsto \|f\|_{\mathcal{H}}$ and $f \mapsto \|f\|_{\mathcal{H}^r}$ respectively, are Hilbert spaces.

The following two results are immediate consequences of Propositions 3.8 and 4.9 in [14]. The proofs are therefore omitted.

**Proposition 3.3.** Let $\mathcal{H}_j$ be Hilbert space of Hermite types on $\mathbb{R}^d_j$ for $j = 1, 2$, and let $T$ be a linear and continuous map from $\mathcal{H}_1$ to $\mathcal{H}_2$. Also let $\mathcal{H} = \mathcal{H}_2 \otimes (\mathcal{H}_1^*)^r$ (Hilbert tensor product). If $K_T$ is the kernel of $T$, then $T \in \mathcal{J}_2(\mathcal{H}_1, \mathcal{H}_2)$, if and only if $K_T \in \mathcal{H}$, and

$$\|T\|_{\mathcal{J}_2(\mathcal{H}_1, \mathcal{H}_2)} = \|K_T\|_{\mathcal{H}}. \quad (3.3)$$

**Proposition 3.4.** Let $s \geq 1/2$ and let $\mathcal{B}$ be a quasi-Banach space such that

$$S_{1/2}(\mathbb{R}^d) \hookrightarrow \mathcal{B} \hookrightarrow S'_{1/2}(\mathbb{R}^d)$$

holds. Then the following is true:

1. if $S_s(\mathbb{R}^d) \hookrightarrow \mathcal{B}$, then there is a Hilbert space $\mathcal{H}$ of Hermite type such that $S_s(\mathbb{R}^d) \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{B}$;

2. if $\mathcal{B} \hookrightarrow S'_s(\mathbb{R}^d)$, then there is a Hilbert space $\mathcal{H}$ of Hermite type such that $\mathcal{B} \hookrightarrow \mathcal{H} \hookrightarrow S'_s(\mathbb{R}^d)$.

The same conclusions hold after $s \geq 1/2$, $S_s$ and $S'_s$ are replaced by $s > 1/2$, $\Sigma_s$ and $\Sigma'_s$ respectively, or after $S_s$ and $S'_s$ are replaced by $\mathcal{I}$ and $\mathcal{I}'$ respectively.

**Proof of Theorem 3.1.** We only prove (1). The other cases follow by similar arguments and are left for the reader. By Lemma 2.2 and Proposition 3.4 it follows that we may assume that $\mathcal{B}_1$ and $\mathcal{B}_2$ are Hilbert spaces of Hermite type.

For every integer $N \geq 1$, it follows by repeated application of Theorem 2.2 that

$$T_K = T_{K_N} \circ \cdots \circ T_{K_1},$$

for some $K_j \in \mathcal{S}_s$, $j = 1, \ldots, N$. Then Proposition 3.3 shows that $T_{K_j} \in \mathcal{J}_2$, for every $j = 1, \ldots, N$. Hence, if $N \geq 2/p$, then Hölder’s inequality for Schatten-von Neumann operators give

$$T_K = T_{K_N} \circ \cdots \circ T_{K_1} \in \mathcal{J}_2 \circ \cdots \circ \mathcal{J}_2 \subseteq \mathcal{J}_{2/N} \subseteq \mathcal{J}_p.$$
which gives the result. The proof is complete. □

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REFERENCES

[1] R. Beals Characterization of pseudodifferential operators and applications, Duke Math. J. 44 (1977), 45–57.
[2] J. Chung, S.-Y. Chung, D. Kim Characterizations of the Gelfand-Shilov spaces via Fourier transforms, Proc. Amer. Math. Soc. 124 (1996), 2101–2108.
[3] E. Cordero, S. Pilipović, L. Rodino, N. Teofanov Quasianalytic Gelfand-Shilov spaces with applications to localization operators, Rocky Mt. J. Math. 40 (2010), 1123–1147.
[4] I. M. Gelfand, G. E. Shilov Generalized functions, I–III, Academic Press, NewYork London, 1968.
[5] T. Gramchev, S. Pilipović, L. Rodino Classes of degenerate elliptic operators in Gelfand-Shilov spaces in: L. Rodino, M. W. Wong (Eds) New developments in pseudo-differential operators, Operator Theory: Advances and Applications 189, Birkhäuser Verlag, Basel 2009, pp. 15–31.
[6] L. Hörmander The Analysis of Linear Partial Differential Operators, vol I–III, Springer-Verlag, Berlin Heidelberg NewYork Tokyo, 1983, 1985.
[7] Czesław Klis A simple proof of the identity \( S \circ S = S \), in: Proc. of the Conference on Convergence and Generalized Functions, Katowice 1983, Preprint of the Institute of Math., Polish Academy of Sci., pp. 71-74.
[8] Z. Lozanov-Crvenković, D. Perišić Kernel theorems for the spaces of tempered ultradistributions, Integral Transform Spec. Funct. 18 (2007), 699–713.
[9] Z. Lozanov-Crvenković, D. Perišić, M. Tasković Gelfand-Shilov spaces structural and kernel theorems, (preprint). arXiv:0706.2298v2.
[10] A. Pietsch Operator ideals, Mathematische Monographien, vol 16, VEB Deutscher Verlag der Wissenschaften, Berlin, 1978.
[11] S. Pilipović Generalization of Zemanian spaces of generalized functions which have orthonormal series expansions, SIAM J. Math. Anal. 17 (1986), 477484.
[12] Z. Sadlok On uniform convergence of Hermite series, Ann. Polon. Math. 43 (1983), 207–210.
[13] B. Simon Trace ideals and their applications, I, London Math. Soc. Lecture Note Series, Cambridge University Press, Cambridge London New York Melbourne, 1979.
[14] J. Toft Multiplication properties in Gelfand-Shilov pseudo-differential calculus in: S. Mohanty, S. Pilipović, J. Toft, M. W. Wong, H. Zhu (eds) Pseudo-Differential Operators, Generalized Functions and Asymptotics, Operator Theory: Advances and Applications, Birkhäuser Verlag, Basel, 56 pages (to appear).
[15] J. Voigt Factorization in some Fréchet algebras of differentiable functions, Studia Math. 77 (1984), 333–348.
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