An Optimal Penalty Constant For Discrete Optimal Control Regulator Problems

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Abstract: Indiscriminate and imprecise use of penalty constants can lead to substantial computational problems and consequently erroneous conclusions and deductions. This paper view this, with utmost seriousness and thus establish an optimal penalty constant $\varphi$ that optimizes the minimization of the cost functional $\langle z, Hz \rangle$ while solving discrete optimal control problems using extended conjugate gradient method (ECGM). I employed some measures that examine optimal control of dynamic processes that can be described by Differential Algebraic Equations (DAEs) that entails integer restriction on some or all of the control functions. I established the construction of an optimal penalty constant which can be employed in the extended conjugate gradient method algorithm for discrete optimal control regulator problems. Comparative results emanating from the use of different penalty constants/algorithms on some problems are given. The obtained numerical results reveal that the proposed optimal penalty constant expression is efficient.

Key word: Optimal; Penalty constant; DOCP; Functional; Operator; DOCRP.

1. Introduction

Recall vividly that [12] developed the extended conjugate gradient method (ECGM) algorithm, as a penalty optimization technique for solving a class of continuous optimal control problem (COCP) governed by a system of differential equation. [13] generalized the concept of the extended conjugate gradient method algorithm and [1] provided a numerical behaviour of the algorithm while solving a significant one-dimensional problem. [17] improved the efficiency of the ECGM algorithm for solving a class of regulator problem in the continuous optimal control problem with emphasis on the mathematical theory underlying the algorithm. Furthermore [16] proposed an ECGM algorithm and the control operator $\hat{H}$ to enhance solving a class of discrete optimal control regulator problems (DOCRP). [2], [3] and [4] addressed the ECGM algorithm for DOCRP, with kin interest on some of its features, the sequence of
gradients and search directions generated from the algorithm and the computing techniques for the gradients and search directions. I observed that by taking the inner product of any two alternate gradients, \( \langle g_i, g_j \rangle, i \neq j \), I saw that the gradients are orthogonal. Also the search directions emanating from the algorithm were conjugate with respect to the operator \( \tilde{H} \). These characteristics enhance / justify the applicability of the algorithm to problems in optimal control.

Thus our desire here is to improve the efficiency of this algorithm for DOCRP by finding a penalty constant \( \varphi (\varphi > 0) \) that optimizes the functional \( \langle z, \tilde{H}z \rangle \) and use it to solve the minimization of the functional. In the continuous case, I noticed the [14] remarked that a decreasing rate of convergence to the solution under study is obtained through the use of very large penalty constants. Alternately, it is observed that once penalty constant approaches zero, the rate of convergence deteriorate sharply. Hence, the need to balance the situation is necessary to enable us implement this algorithm successfully. To substantiate this desire, [12] observed that indiscriminate and imprecise use of penalty constants can lead to substantial computational problems and consequently to erroneous conclusions and deductions. This is the basis of our contribution, which focused on modeling the problem of optimal control for a discrete optimal control by establishing the construction of an optimal penalty constant, which can be employed in the ECGM algorithm. This will no doubt further enhance the application of this algorithm to problems in optimal control.

Optimal Control Problems can be categorized into: (1) Minimum-Time Problems, (2) Terminal-Control Problems (3) Minimum-Control effort Problems (4) Tracking Problems (5) Regulator Problems. For the purpose of subsequent discussions, problem P(1), which will be described within the context of the ECGM algorithm as the minimization of penalized performance index \( J(x, u, \varphi) \) of linear processes with quadratic cost is given as follows

\[
P(1) \quad \text{Minimize} \sum_{i=1}^{k} [x_i^T P x_i + u_i^T Q u_i]
\]

subject to the dynamic constraints

\[
x_i = A x_{i-1} + D u_{i-1}, x(0) = x_0, \text{ specified}
\]

where \( x_i \in \mathbb{R}^n \) is an \( n \) – dimensional state vector, \( u_i \in \mathbb{R}^m \), is an \( m \) – dimensional column vector; \( P \) and \( Q \) are \( n \times n \), and \( n \times m \) symmetric positive definite constant matrices respectively with \( A \) and \( D \) constant matrices, \( x^T \) denotes the transpose of \( x \), an \( n \times 1 \) state vector, \( u \) is \( m \times 1 \) control vector. The discrete optimal control regulator problem performance index is classified into three forms viz [19]:

Mayer form: \( J = \frac{1}{2} X_i^T H X_i \)

Lagrange form: \( J = \sum_{i=1}^{k} [X_i^T P x_i + U_i^T Q u_i] \)
Boltzmann form: \[ J = \frac{1}{2} X^T H X + \sum_{i=1}^{k} \{ X^T P X + U^T Q U \} \] (1.2c)

By introducing the penalty constant \( \varphi(\varphi > 0) \) our constrained problem becomes an unconstrained problem given as

\[ J(x, u, \varphi) = \sum_{i=1}^{k} [x_i^T P x_i + u_i^T Q u_i + \varphi(x_i - A x_{i-1} - D u_{i-1})^2] \] (1.3)

where \( \varphi(\varphi > 0) \) is the penalty constant parameter. The term \((x_i - A x_{i-1} - D u_{i-1})^2\) is called the penalty functional, since it violates the constraints. The ECGL algorithm designed by [17], using similar ideas of [12] and the explicit knowledge of the control operator \( H \), satisfies

\[ \langle z, \tilde{H} z \rangle_w = \sum_{i=1}^{k} [x_i^T P x_i + u_i^T Q u_i + \varphi(x_i - A x_{i-1} - D u_{i-1})(x_i - A x_{i-1} - D u_{i-1})] \] (1.4)

where \( W \) is the Hilbert space, with norm denoted by \( \| \cdot \| \) and defined by

\[ \|x\|^2 = \left[ \sum_{i=1}^{k} x_i \right]^{1/2} \] (1.5)

and \( \langle \cdot, \cdot \rangle \) is the scalar product in Euclidean n-dimensional space.

The rest of this paper will comprise of the following sections: section two contains a proposition, the main thrust of the paper and necessary tools for the proof of theorem (2.1). Section three dwell on the expansion of \( \langle z, \tilde{H} z \rangle \) as we consider a one-dimensional case of problem (3). Section four presents a formal proof of the theorem (2.1), that will enable us find the optimal value and the application of the expression in equation (2.1). Finally, we discuss the comments and conclusion emanating from our study so far.

2. The main thrust of the paper

The main thrust in this paper, will be formalized in the ensuing theorem, but first we state without proof the following proposition which will create an enabling environment for us to prove our theorem.

2.1. Proposition 2.0
Given a discrete optimal control regulator problem, such as in (1.1), then the optimal performance index, \( J(x, u, \varphi) \) emanating from the use of \( u = K^T x \), is quadratic in \( x_0 \), i.e. \( J(x, u, \varphi) = x^T H x \).

2.2. Theorem 2.1
The optimal penalty constant \( \varphi^* \) that optimizes the functional \( \langle z, \tilde{H} z \rangle \) as in the problem \( P(1) \), equation (3) is
\[ \varphi^* = \frac{-R \pm \sqrt{N}}{T} \]  

\[ R = [(v^2 - v + vs)x_0 + sv^2 + sv^2 + s^2v + 2rs - 2rs + 4s^2 - s + 2r] \]  

\[ N = [(v^4 - 2v^3(1-s) - v^2(2s - s^2 - 1))x_0 + 41s^4 + 20vs^4 + s^4v^2 + 15s^2v^4 + 2s^3v^3 + 3s^3v^2 + 10s^3v + 2s^3v^2 + 8s^3v + 2s^2 + 4s^2v^2 - 4r^2s^2v - 2rs^2v + 22rs^2 + 18rs^2v + 8r^2sv - 4r^2s + 4rs^2v + 4rs^3v^2 + 4rsv + 4rs^3v^3 + 4rs^2v^3 + 2rs^2v^2 + r^2s^2 - 4rs + 4r^2 + 2[v^4s + 2sv^2 + 2s^2v^2 - sv^2 + vs^2 - vs^2]x_0 + [4rs^3v - 6rs^2v + 2rs^2v]x_0 + (4v^3r - 4vr + 4rsv)x_0 + (8v^3s^2 - 10vs^2 + 8vs^3 - 2s^2v + 2sv)x_0, \]  

\[ T = 4v^3s + 2v^2 - 6vs^2 - 2v + 6s. \]

provided T is not zero and we are guaranteed of having a nonzero T since the sum of the individual terms with the arbitrary s and v is nonzero.

2.3. Necessary Tools For The Proof Of Theorem (2.1)

Before presenting the proof of theorem (2.1), consider the functional

\[ J(x, u, \varphi) = \left\langle z, \tilde{H}z \right\rangle_w \]  

and state the following proposition and theorem, which will no doubt be useful tools in the study.

2.4. Proposition 2.1: [19]

A linear operator \( D \) admits the inverse \( D^{-1} \) if and only if \( Dz = 0 \) implies \( z = 0 \). This proposition is essential in conjunction with the theorem below because they specifies that all algorithms which accommodates the ECGM generates a sequence of solutions which converges to a unique solution, that involves the inverse of the operator D, a linear symmetric positive definite matrix operator, in the functional

\[ f(z) = f_0 + \left\langle a, z \right\rangle_w + \frac{1}{2} \left\langle z, Dz \right\rangle_w \]  

2.5. Theorem 2.2 [11]

Suppose \( z \in W \) be specified. Then the sequence \( \{z_n\} \) generated by the Conjugate Gradient Method, converges to a unique solution of the functional in equation (2.6), where D is a linear symmetric positive definite matrix operator, defined on the Hilbert space \( w \), \( \left\langle z, z \right\rangle_w \) is the inner product on the space. Hence the optimal solution for (2.6) is

\[ z^* = D^{-1}a \]
However, in this paper, consider the functional in equation (2.5), which possesses the same qualities as $D$. Similarly, in optimal control of dynamic processes described by differential algebraic equations (DAEs), integer restrictions can be placed on some or all the control functions $[6]$. Consequently, I set the following which will be useful in the proof of theorem (2.1) and thus are in order.

\[ x_i = \phi v_i s, i = 1(2)k, \]

Now set \( x_i = r + \phi(1 + v)i, i = 2(2)k \), \[ u_i = 1, i = 0(1)k \]

3. Expansion Of The Functional \( \langle z, H \rangle \)

First consider the one–dimensional case of problem (P1) such as

\[ P(2) \text{ Minimize } \sum_{i=1}^{k} r x_i^2 + q u_i^2 \]  

Subject to the dynamic constraint

\[ x_i = v x_{i-1} + s u_{i-1} \]
\[ x(0) = x_0 \]

where $r$ and $q$ are positive constants while $v$ and $s$ are constants not necessarily positive. Define the unconstrained problem associated with equation (3.1) through the penalized cost functional $J(x, u, \phi)$ in the following manner

\[ P(3) \text{ Min } J(x, u, \phi) = \sum_{i=1}^{k} \left[ r x_i^2 + q u_i^2 + \phi(x_i - v x_{i-1} - s u_{i-1})^2 \right] \]

The ECGM algorithm for DOCP due to $[17]$, utilizes the explicit knowledge of the control operator $H$ which satisfies

\[ \langle z, H z \rangle = \sum_{i=1}^{k} \left[ r x_i^2 + q u_i^2 + \phi(x_i - v x_{i-1} - s u_{i-1})^2 \right] \]

where \( z = (x_0, x_1, x_2, \ldots, x_k, u_0, u_1, u_2, \ldots, u_k)^T \) and $H$ is the control operator by $[17]$ to enhance the application of the ECGM on DOCP. The operator $H$ is a block matrix of order $2(k+1)$ and is defined as

\[ \tilde{H} = \begin{pmatrix} F & N \\ N^T & B \end{pmatrix} \]  

Where,

F is a tridiagonal matrix of order $k+1$ with entries
\[ f_{ij} = \varphi v^2, \quad f_{ij} = -\varphi v, \quad \text{provided } |i - j| = 1 \]
\[ f_{ij} = r + \varphi(1 + v), \quad f_{jk+1} = r + \varphi, \quad f_{ij} = 0, \quad \text{otherwise} \]

\( N \) is a square matrix of order \((k+1)\) with entries

\[ n_{ij} = \varphi s, \quad n_{k+1} = 0, \quad n_{j} = -\varphi s, \quad \forall i, j \text{ such that } i - j = 1 \]
\[ n_{ij} = 0, \quad \text{otherwise} \]

\( N^T \) is the transpose of the matrix \( N \).

\( B \) is a diagonal matrix of order \((k+1)\) with entries

\[ b_{1} = \varphi s^2, \quad b_{k+1} = q, \quad b_{j} = q + \varphi s^2, \quad b_{j} = 0, \quad \text{otherwise} \]

Thus the matrix operator \( \tilde{H} \) will consists

\[
\tilde{H} = \begin{bmatrix}
\varphi s^2 & -\varphi v & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
-\varphi v & r + \phi(1 + v) & -\varphi v & 0 & \cdots & 0 & 0 & -\varphi s & \varphi s \\
0 & -\varphi v & r + \phi(1 + v) & -\varphi v & \cdots & 0 & 0 & -\varphi s & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & -\varphi v & r + \phi & 0 & 0 & 0 & 0 & 0 & \cdots & -\varphi s & 0 \\
\varphi v s & -\varphi s & 0 & 0 & \cdots & 0 & 0 & \varphi s^2 & 0 & 0 & \cdots & 0 & 0 \\
0 & \varphi v s & -\varphi s & 0 & \cdots & 0 & 0 & 0 & q + \varphi s^2 & 0 & \cdots & 0 & 0 \\
0 & 0 & \varphi v s & -\varphi s & \cdots & 0 & 0 & 0 & 0 & q + \varphi s^2 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & q
\end{bmatrix}
\]

Hence by taking the matrix – vector product of \( H \) and \( z \), yield,
\[ \tilde{H}_z = \begin{bmatrix} \varphi v^2 x_0 - \varphi v x_1 - \cdots + \varphi su_{k-1} \\ - \varphi v x_0 + [r + \varphi(1 + v)]x_1 - \varphi v x_2 - \varphi s u_0 + \varphi sv u_0 \\ - \varphi v x_1 + [r + \varphi(1 + v)]x_2 - \varphi v x_3 - \varphi s u_1 + \varphi sv u_1 \\ \vdots \\ - \varphi v x_{k-1} + (r + \varphi)x_k - \varphi s u_{k-1} \\ + \varphi sv x_0 - \varphi s x_1 - \cdots + \varphi s^2 u_0 \\ + \varphi sv x_1 - \varphi s x_2 - \cdots + (q + \varphi s^2) u_1 \\ + \varphi sv x_2 - \varphi s x_3 - \cdots (q + \varphi s^2) u_2 \\ \vdots \\ - \varphi s x_k \\ + q u_k \end{bmatrix} \] (3.4)

Next take the inner product of \( z \) and \( \tilde{H}_z \),

\[ \langle z, \tilde{H}_z \rangle = \left[ x_0, x_1, \ldots, x_k, u_0, u_1, \ldots, u_k \right] \begin{bmatrix} \varphi v^2 x_0 - \varphi v x_1 - \cdots + \varphi sv u_0 \\ - \varphi v x_0 + [r + \varphi(1 + v)]x_1 - \varphi v x_2 - \varphi sv u_0 + \varphi sv u_1 \\ - \varphi v x_1 + [r + \varphi(1 + v)]x_2 - \varphi v x_3 - \varphi sv u_1 + \varphi sv u_2 \\ \vdots \\ - \varphi v x_{k-1} + (r + \varphi)x_k - \varphi sv u_{k-1} \\ + \varphi sv x_0 - \varphi s x_1 - \cdots + \varphi s^2 u_0 \\ + \varphi sv x_1 - \varphi s x_2 - \cdots + (q + \varphi s^2) u_1 \\ + \varphi sv x_2 - \varphi s x_3 - \cdots (q + \varphi s^2) u_2 \\ \vdots \\ - \varphi s x_k \\ + q u_k \end{bmatrix} \] (3.5)

\[ \begin{align*}
    x_0 \left( \varphi v^2 x_0 - \varphi v x_1 - \cdots + \varphi sv u_0 \right) + x_1 \left( -\varphi v x_0 + [r + \varphi(1 + v)]x_1 - \varphi v x_2 - \varphi s u_0 + \varphi sv u_1 \right) \\
    x_2 \left( -\varphi v x_1 + [r + \varphi(1 + v)]x_2 - \varphi v x_3 - \varphi s u_1 + \varphi sv u_2 \right) + \cdots + \\
    + x_k \left( -\varphi v x_{k-1} + (r + \varphi)x_k - \varphi sv u_{k-1} \right) + u_0 \left( \varphi sv x_0 - \varphi s x_1 - \cdots + \varphi s^2 u_0 \right) + \\
    + u_1 \left( \varphi sv x_1 - \varphi s x_2 - \cdots + (q + \varphi s^2) u_1 \right) + \\
    + u_2 \left( \varphi sv x_2 - \varphi s x_3 - \cdots (q + \varphi s^2) u_2 \right) - u_{k-1} (\varphi s x_k) \\
    + u_k (q u_k)
\end{align*} \]
Recall equation (2.5),

\[ \begin{align*}
&\{ \varphi x_0^2 - \varphi x_0 x_1 - \cdots + \varphi sx_0 u_0 - \varphi x_0 x_1 + [r + \varphi(1 + v)]x_1^2 - \varphi x_1 x_2 - \varphi sx_1 u_0 + \varphi sx_1 u_1 \\
&- \varphi x_1 x_2 + [r + \varphi(1 + v)]x_2^2 - \varphi x_2 x_3 - \varphi sx_2 u_1 + \varphi sx_2 u_2 + \cdots + \\
&- \varphi x_k x_{k-1} + (r + \varphi)x_k^2 - \varphi sx_k u_{k-1} + \varphi sx_k u_k + \varphi sx_k u_{k+1} + \cdots + \varphi s^2 u_k^2 + \\
&+ \varphi s u_k x_1 - \varphi s u_1 x_2 - \cdots + (q + \varphi s^2)u_1^2 + \\
&+ \varphi s u_k x_2 - \varphi s u_2 x_3 - \cdots + (q + \varphi s^2)u_2^2 - \varphi sx_k u_{k-1} \\
&+ q u_k^2 \} \\
\end{align*} \]

By rearranging the terms in powers of \( x_i \) and \( u_i \), produces,

\[ \begin{align*}
\langle z, \tilde{H}z \rangle &= \left\{ \begin{array}{l}
\varphi x_0^2 - 2\varphi x_0 x_1 - 2\varphi x_1 x_2 + [r + \varphi(1 + v)]x_1^2 - 2\varphi x_1 x_2 + [r + \varphi(1 + v)]x_2^2 \\
- 2\varphi sx_0 u_0 - 2\varphi sx_1 u_1 + 2\varphi sx_2 u_2 - \varphi sx_1 u_{k-1} + (r + \varphi)x_k^2 - \varphi sx_k u_{k-1} + 2\varphi sx_k u_0 + \\
+ \varphi s^2 u_k^2 + 2\varphi s u_k x_1 + (q + \varphi s^2)u_1^2 + 2\varphi s u_2 x_2 - 2\varphi su_2 x_3 + (q + \varphi s^2)u_2^2 \\
- \varphi sx_k u_{k-1} + q u_k^2 \end{array} \right\} \\
\end{align*} \]

4. A Formal Proof Of Theorem (2.1)

The proof of theorem (2.1) is formally presented as follows:

4.1. Proof of theorem (2.1)

Let \( z = (x_0, x_1, x_2, \ldots, x_k, u_0, u_1, u_2, \ldots, u_k)^T \) where \( x_i \) and \( u_i \) are the state and control vector specified and \( z^* = (x_0^*, x_1^*, x_2^*, \ldots, x_k^*, u_0^*, u_1^*, u_2^*, \ldots, u_k^*)^T \) where \( x_i^* \) and \( u_i^* \) are the optimal values of the state and control vectors.

Recall equation (2.5), \( J(x, u, \varphi) = \langle z, \tilde{H}z \rangle \) and from it obtain

\[ \nabla J(x, u, \varphi) = 2\tilde{H}z \]

At optimum, \( \nabla J(x, u, \varphi) = 0 \), \[1\] which implies that

\[ \tilde{H}z^* = 0, \]

where \( z^* = (x_0^*, x_1^*, x_2^*, \ldots, x_k^*, u_0^*, u_1^*, u_2^*, \ldots, u_k^*)^T \)

Substituting equation (4.2) and (2.8) into (3.4) and setting the expression to zero, yield,
Rearranging equation (4.3) yields

$$\phi^2 x_0 - \phi v(\phi v s) - \cdots + \phi v s - \phi v x_0 + [r + \phi(1 + \nu)](\phi v s - \phi v[r + (1 + \nu)]) - \phi v + \phi v s$$

$$- \phi v[r + \phi(1 + \nu)](r + \phi)(\phi v s) - \phi v + \phi v x_0 - \phi v s(\phi v s) + \phi s^2 + (\phi v s)^2$$

$$- \phi v[r + \phi(1 + \nu)] + 2(q + q s^2) + \phi v s[r + \phi(1 + \nu)] - \phi v(\phi v s) - q s(\phi v s)$$

$$+ \phi v s[r + \phi(1 + \nu)] + 10q = 0$$

Equation (4.4) is quadratic in $\phi$ and solving for $\phi$, yields the result in equation (2.4) and this completes the proof of our theorem.

4.2. Application

In this section, I considered two discrete optimal control problems, in each generate the optimal penalty constant using the expression in equation (2.1) and employed the value generated in the ECGM algorithm to solve the problems.

4.2.1. Problem I

Minimize $\sum_{i=1}^{k} [x_i^2 + u_i^2]$

Subject to $x_i = 0.5x_{i-1} + u_{i-1}, x_0 = 1$

This problem has been solved by [18], employing even and uneven parametrization partitions, before introducing a simple constraint transcription to approximate, each of the all-step constraints, into a single constraint and cast in a canonical form, similar to that of the cost functional to simplify computation of gradients. Similarly, [17] solved the same problem using the ECGM algorithm at even value of $k$, using penalty constant ($\phi = 0.5, 0.05$. The concern here, in this paper, is solving the problem, with the ECGM algorithm, while employing the optimal penalty constant generated from equation (2.1). I have the following results in Table 1.

4.2.2. Problem II

Minimize $\sum_{i=1}^{k} (x_i^2 + u_i^2)$

Subject to $x_i = 2.095x_{i-1} + 1.904u_{i-1}, x_0 = 1.$

The exact analytical solution is 1.0647 given by [10]

The problem has been solved by some other numerical methods such as Function Space Algorithm (FSA), Extended Conjugate Gradient Method, Discretized Constrained Algorithm and Multiplier Imbedding Extended Conjugate Gradient Method (MECGM). The solution of the problem (II) using the ECGM algorithm while employing $x_0 = 1, u_0 = 0.5, \phi = 0.7$ (the optimal penalty constant computed from equation (2.1)) and basic programming language, produced the results in Table 2.
Table 1. Showing ECGM algorithm, [18] and [17]

| K(φ) | Algorithm   | Iters | Teo et al(1990) | K(φ) | Otunta(1991) |
|------|-------------|-------|----------------|------|--------------|
| 1(φ) | 0.75422     | 10    | 1.2934773      | 1(0) | 0.47222      |
| 2(φ) | 0.841383    |       | 1.29998        | 2(0.5)| 0.81720134  |
| 3(φ) | 0.891234    | 16    | 1.2484346      | 2(0.05)| 0.97584404  |
| 4(φ) | 1.1612422   | 31    | 1.1327822      | 4(0.5)| 1.15931778  |
|      |             |       |                | 4(0.0)| 1.22022399  |

Table 2. Depicts the algorithm (ECGM), DCA[15] and FSA[17].

| Penalty Const | Algorithm | Iteration | OBJ | CSAT | PenFunctnal |
|---------------|-----------|-----------|-----|------|-------------|
| μ = 0.5       | DCA       | 22        | 1.205379 | 5.231375 | 3.821067    |
| φ* = 0.45     | ECGM      | 1         | 7.202491076 | 0.187526557 | 6.295       |
| μ = 0.5       | FSA       | 50        | 1.6517  | 11.6227|             |
| μ = 1.5       | DCA       | 25        | 0.760914 | 3.265003 | 5.658419   |
| φ* = 0.45     | ECGM      | 2         | 11.91387514 | 4.26885551 | 2.355493991|
| μ = 1.5       | FSA       | 50        | 1.60017 | 10.9884|             |

5. Comments And Conclusion

I have successfully derived an expression for computing the optimal penalty constant φ*. This will enable me to generate a value for the penalty constant and avoid indiscriminate and arbitrary use of penalty values. When the optimal penalty constant computed as φ* = 0.7 from equation (2.1), is employed in the ECGM algorithm while solving problem (I), I observed good comparative performance with other values, such as 0.5, 0.05 used by [17] and [18] in solving problem (I). The trend is not different as in problem (II). I also noticed that the constraint satisfaction is minimal, when the optimal penalty value is in use. This implies lesser penalty payment. Conclusively, I wish to state here that, the use of the optimal penalty constant only introduces another measure of getting these values, whenever one want to solve problems I and II. It does not exclude the use of larger penalty constants and so it is a matter of choice as to which approach one may wish to employ. The optimal penalty constant will go a long way to strike a balance between the two extremes – very large and very low penalty constants and enhance the need for appropriate deductions and conclusions as at when due.

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