MAGNUS EMBEDDING AND ALGORITHMIC PROPERTIES OF GROUPS $F/N^{(d)}$

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ABSTRACT. In this paper we further study properties of Magnus embedding, give a precise reducibility diagram for Dehn problems in groups of the form $F/N^{(d)}$, and provide a detailed answer to Problem 12.98 in Kourovka notebook. We also show that most of the reductions are polynomial time reductions and can be used in practical computation.

Keywords. Magnus embedding, word problem, power problem, conjugacy problem, free solvable groups.

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1. Introduction

Let $F = F(X)$ be the free group on generators $X$, $N$ a normal subgroup of $F$, $N'$ the derived subgroup of $N$, and $N^{(d)}$ the $d$th derived subgroup of $N$. The Magnus embedding is the main tool to study groups of the form $F/N'$. It was introduced in [7] by W. Magnus who showed that the elements of $F/N'$ can be encoded by $2 \times 2$ matrices:

$$M(X; N) = \left\{ \begin{pmatrix} g & \pi \\ 0 & 1 \end{pmatrix} \right| g \in F/N, \pi \in F_{\Gamma} \right\},$$

where $F_{\Gamma}$ is a free module over the group ring $\mathbb{Z}F/N$. In this paper we study algorithmic properties of Magnus embedding and decidability of the following problems for groups of the form $F/N^{(d)}$.

WP($G$), word problem in $G$. Given a word $w$ in the generators of $G$ decide if $w = 1$ in $G$, or not.

CP($G$), conjugacy problem in $G$. Given words $u, v$ in the generators of $G$ decide if $u \sim v$ in $G$, or not.

PP($G$), power problem in $G$. Given words $u, v$ in the generators of $G$ decide if $v = u^k$ in $G$ for some $k \in \mathbb{Z}$, or not.

The Magnus embedding proved to be especially robust in the study of free solvable groups. Indeed, free solvable group naturally appear in the context because $F/F^{(d)} = F/(F^{(d-1)}')$ is the free solvable group of rank $n$ and degree $d$. It immediately follows from the work of Magnus that decidability of the word problem in $F/N$ implies decidability of the word problem in $F/N'$. The conjugacy problem in groups of the type $F/N'$ was first approached by J. Matthews in [8] who proved that:

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(a) \( u, v \in F/N' \) are conjugate (for free abelian \( F/N \)) if and only if their images under Magnus embedding are conjugate in \( M(X; N) \);
(b) conjugacy problem in \( M(X; N) \) is decidable if and only if conjugacy problem in \( F/N \) is decidable and power problem in \( F/N \) is decidable.

These two facts imply that free metabelian groups have decidable conjugacy problem. Later Remeslennikov and Sokolov in [15] extended (a) to any torsion free group \( F/N \), showed that the power problem is decidable in free solvable groups, and deduced that free solvable groups have decidable conjugacy problem. Finally, C. Gupta in [4] proved that (a) holds for groups with torsion as well and hence decidability of the conjugacy and power problems in \( F/N \) implies decidability of the conjugacy problem in \( F/N' \). We use the following notation for reducibility of decision problems in the sequel:

\[
\begin{align*}
\text{CP}(F/N) & \Rightarrow \text{CP}(F/N') \\
\text{PP}(F/N) & \Rightarrow \text{CP}(F/N').
\end{align*}
\]

In the light of these results, V. Shpilrain raised the following questions in [9, Problem 12.98]. Is it correct that:

(a) \( \text{WP}(F/N) \) is decidable if and only if \( \text{WP}(F/N') \) is decidable.
(b) \( \text{CP}(F/N) \) is decidable if and only if \( \text{CP}(F/N') \) is decidable.
(c) \( \text{WP}(F/N') \) is decidable if and only if \( \text{CP}(F/N') \) is decidable.

It was shown by Anokhin in [1] that only 12.98(a) has an affirmative answer. He constructed a group \( F/N \) such that \( \text{CP}(F/N) \) is decidable and \( \text{CP}(F/N') \) is undecidable. Such a group is clearly a counterexample to both 12.98(b) and 12.98(c).

We also would like to mention several results related to practical computations in free solvable groups in which the Magnus embedding plays a crucial role. S. Vassileva showed in [19] that the power problem in free solvable groups can be solved in \( O(rd(|u| + |v|)^6) \) time and used that result to show that the Matthews-Remeslennikov-Sokolov approach can be transformed into a polynomial time \( O(rd(|u| + |v|)^8) \) algorithm for the conjugacy problem. In [18] those complexity bounds were further improved and randomized algorithms were developed. Another generalization was done by Lysenok and Ushakov in [6]. It was shown that the Diophantine problem for spherical quadratic equations, i.e., equations of the form:

\[
z_1^{-1}c_1z_1 \ldots z_k^{-1}c_kz_k = 1,
\]

in free metabelian groups is decidable. Recall that for every \( n \geq 2 \) the Diophantine problem in free metabelian groups is undecidable (see [10]). Recently Vassileva proved in [20] that the Magnus embedding is a quasi-isometry.

1.1. **Our contribution.** Here we shortly outline the main results of the paper (somewhat simplifying the statements). Let \( F \) be a free group of rank at least 2 and \( N \) a recursively enumerable normal subgroup of \( F \).

**Theorem 6.10.** \( \text{WP}(F/N) \Rightarrow \text{PP}(F/N') \).

**Theorem 6.13.** \( \text{PP}(F/N) \Rightarrow \text{CP}(F/N') \).

**Theorem 6.15.** \( \text{PP}(F/N) \leftarrow \text{CP}(F/N') \).

**Theorem 6.17.** \( \text{CP}(F/N) \not\Rightarrow \text{CP}(F/N') \).
Our results give the following reducibility diagram for the decision problems in groups of the form $F/N(d)$:

$$
\begin{array}{c}
\text{WP}(F/N) & \leftrightarrow & \text{WP}(F/N') & \leftrightarrow & \text{WP}(F/N'') & \leftrightarrow & \ldots \\
\text{PP}(F/N) & \leftrightarrow & \text{PP}(F/N') & \leftrightarrow & \text{PP}(F/N'') & \leftrightarrow & \ldots \\
\text{CP}(F/N) & \leftrightarrow & \text{CP}(F/N') & \leftrightarrow & \text{CP}(F/N'') & \leftrightarrow & \ldots \\
\end{array}
$$

In particular, the following theorem holds.

**Corollary 6.16.** For every recursively enumerable $N \leq F$ the following holds.

(a) $\text{PP}(F/N') \Leftrightarrow \text{PP}(F/N'')$.
(b) $\text{CP}(F/N'') \Leftrightarrow \text{CP}(F/N'')$.
(c) $\text{WP}(F/N'') \Leftrightarrow \text{PP}(F/N'') \Leftrightarrow \text{CP}(F/N'')$. \hfill $\square$

Furthermore, most of the reductions are polynomial time computable. Denote by $P$ the class of decision problems decidable in polynomial time.

**Theorem.** Suppose that $\text{WP}(F/N) \in P$. Then the problems $\text{WP}(F/N^{(d)})$, $\text{PP}(F/N^{(d)})$, and $\text{CP}(F/N^{(d)})$ are in $P$ for every $d \geq 2$. Moreover, each of those problems has a unique polynomial bound that does not depend on $d$. \hfill $\square$

Finally, in Section 6.3 we consider two combinatorial problems for groups $F/N'$: subset sum problem $\text{SSP}(F/N')$ and acyclic graph problem $\text{AGP}(F/N')$. The main results of that sections are:

**Theorem 6.20** $\text{SSP}(F/N') \in P$ if and only if $\text{WP}(F/N) \in P$ and either $N = \{1\}$ or $[F : N] < \infty$. \hfill $\square$

**Theorem 6.18** If $\text{WP}(F/N) \in P$, $N \neq \{1\}$, and $[F : N] = \infty$, then $\text{SSP}(F/N')$ and $\text{AGP}(F/N')$ are $\text{NP}$-complete. \hfill $\square$

**Corollary 6.21** $\text{AGP}(F/N') \in P$ if and only if $\text{WP}(F/N) \in P$ and either $N = \{1\}$ or $[F : N] < \infty$. \hfill $\square$

2. Preliminaries: $X$-digraphs

Let $X$ be a set (called an *alphabet*) and $F = F(X)$ the free group on $X$. By $X^*$ we denote the set of formal inverses of elements in $X$ and put $X^\pm = X \cup X^*$. An $X$-labeled directed graph $\Gamma$ (or an $X$-digraph) is a pair of sets $(V, E)$ where the set $V$ is called the *vertex set* and the set $E \subseteq V \times V \times X^\pm$ is called the *edge set*. An element $e = (v_1, v_2, x) \in E$ designates an edge with the *origin* $v_1$ (also denoted by $\alpha(e)$), the *terminus* $v_2$ (also denoted by $\omega(e)$), labeled with $x$ (also denoted by $\mu(e)$). We often use notation $v_1 \xrightarrow{x} v_2$ to denote the edge $(v_1, v_2, x)$. A *path* in $\Gamma$ is a sequence of edges $p = e_1, \ldots, e_k$ satisfying $\omega(e_i) = \alpha(e_{i+1})$ for every $i = 1, \ldots, k - 1$. The *origin* $\alpha(p)$ of $p$ is the vertex $\alpha(e_1)$, the *terminus* $\omega(p)$ is the vertex $\omega(e_k)$, and the *label* $\mu(p)$ of $p$ is the word $\mu(e_1) \ldots \mu(e_k)$. We say that an $X$-digraph $\Gamma$ is:

- **rooted** if it has a special vertex, called the root;
- **folded** (or deterministic) if for every $v \in V$ and $x \in X$ there exists at most one edge with the origin $v$ labeled with $x$;
• **X-complete** (or simply complete) if for every \( v_1 \in V \) and \( x \in X^\pm \) there exists an edge \( v_1 \xrightarrow{x} v_2 \);

• **inverse** if with every edge \( e = g_1 \xrightarrow{x} g_2 \) the graph \( \Gamma \) also contains the inverse edge \( g_2 \xrightarrow{x^{-1}} g_1 \), denoted by \( e^{-1} \).

All X-digraphs in this paper are connected. A *morphism* of two rooted X-digraphs is a graph morphism which maps the root to the root and preserves labels. For more information on X-digraphs we refer to \cite{17}.

**Example 2.1.** Let \( F = F(X) \) and \( H \leq F \). The Schreier graph of the subgroup \( H \), denoted by \( \text{Sch}(X; H) \), is an X-digraph \((V, E)\), where \( V \) is the set of right cosets

\[
V = \{ Hg \mid g \in F \}
\]

and

\[
E = \{ Hg \xrightarrow{x} Hgx \mid g \in F, \ x \in X^\pm \}.
\]

By definition, \( \text{Sch}(X; H) \) is a folded complete inverse X-digraph. We always assume that \( H \) is the root of \( \text{Sch}(X; H) \). A special case of the Schreier graph is when \( H \leq F \), called a *Cayley graph* of the group \( F/H \) denoted by \( \text{Cay}(X; H) \). \( \square \)

Let \( \Gamma = (V, E) \) be an inverse X-digraph. The set of edges \( E \) can be split into a disjoint union \( E = E^+ \sqcup E^- \), where

\[
E^+ = \{ e \in E \mid \mu(e) \in X \}
\]

is called the set of *positive edges*, and

\[
E^- = \{ e \in E \mid \mu(e) \in X^- \}
\]

is called the set of *negative edges*. Clearly, \((E^+)^{-1} = E^-\) and \((E^-)^{-1} = E^+\).

The *rank* \( r(\Gamma) \) of an inverse X-digraph \( \Gamma \) is defined as \(|E^+| - |T|\), where \( T \) is any spanning subtree of \( \Gamma \). The fundamental group \( \pi_1(\Gamma) \) is the group of labels of all cycles at the root; it is naturally a subgroup of \( F(X) \) of the rank \( r(\Gamma) \), see \cite{5}.

### 3. Preliminaries: Computational model and data representation

All computations are assumed to be performed on a random access machine. We use base 2 positional number system in which presentations of integers are converted into integers via the rule:

\[
(a_{k-1} \ldots a_3a_2a_1a_0)_2 = a_{k-1}2^{k-1} + \ldots + a_22^2 + a_12 + a_0,
\]

where we assume that \( a_{k-1} = 1 \). The number \( k \) is called the *bit-length* of the presentation.

Let \( G \) be a group generated by a finite set \( X = \{ x_1, \ldots, x_n \} \). We formally encode the word problem for \( G \) as a subset of \( \{0, 1\}^* \) as follows. We first encode elements of the set \( X^\pm = \{ x_1^\pm, \ldots, x_n^\pm \} \) by unique bit-strings of length \( \lceil \log_2 n \rceil + 1 \). The code for a word \( w = w(X^\pm) \) is a concatenation of codes for letters and, formally:

\[
\text{WP}(F/N) = \{ \text{code}(w) \mid w \in N \}.
\]

Thus, the bit-length of the representation for a word \( w \in F \) is:

\[
|\text{code}(w)| = |w|(|\log_2 n| + 1).
\]

We encode the power and conjugacy problems in a similar fashion. For both of these problems instances are pairs of words and the encoding can be done by introducing a new letter "\(^**\)" into the alphabet \( X^\pm \).
3.1. **Quasi-linear time complexity.** An algorithm is said to run in *quasi-linear time* if its time complexity function is $O(n \log^k n)$ for some constant $k \in \mathbb{N}$. We use notation $\Theta(n)$ to denote quasi-linear time complexity. Quasi-linear time algorithms are also $o(n^{1+\varepsilon})$ for every $\varepsilon > 0$, and thus run faster than any polynomial in $n$ with exponent strictly greater than 1. See [13] for more information on quasi-linear time complexity theory. Similarly, one can define quasi-quadratic $\tilde{O}(n^2)$, quasi-cubic $\tilde{O}(n^3)$ time complexity as $O(n^2 \log^k n)$, $O(n^3 \log^k n)$, etc.

4. **Flows on inverse $X$-digraphs**

Let $\Gamma = (V, E)$ be an inverse $X$-digraph. We say that a function $f : E \to \mathbb{Z}$ is *balanced* if:

(F1) $f(e) = -f(e^{-1})$ for any $e \in E$.

All functions in this paper are balanced. A function $f : \mathbb{N} \to \mathbb{Z}$ defines the function $N_f : V \to \mathbb{Z}$:

$$N_f(v) = \sum_{\alpha(e) = v} f(e),$$

called the *net-flow* function of $f$. We say that $f$ is a *flow* if it satisfies the conditions (F1), (F2), and (F3).

(F2) $f$ has a finite support $\text{supp}(f) = \{e \in E \mid f(e) \neq 0\}$.

(F3) Either $N_f(v) = 0$ for every $v \in V$ in which case we say that $f$ is a *circulation*, or there exist $s, t \in V$ such that $N_f(v) = 0$ for all $v \in V \setminus \{s, t\}$, and $N_f(s) = 1$ and $N_f(t) = -1$ and we say that $f$ is a flow from the *source* $s$ to the *sink* $t$.

Define $F_{\Gamma}$ to be the set of all balanced integral functions on $E$ with finite support:

$$F_{\Gamma} = \{f : E \to \mathbb{Z}\}.$$ 

For $f, g \in F_{\Gamma}$ define $f + g \in F_{\Gamma}$ as follows:

$$(f + g)(e) = f(e) + g(e).$$

Clearly, $(F_{\Gamma}, +)$ is an abelian group. The function $\| \cdot \| : F_{\Gamma} \to \mathbb{Z}$ defined by:

$$\|\pi\| = \sum_{e \in E^+} |\pi(e)|$$

called a *norm* on $F_{\Gamma}$. It is easy to see that every $X$-digraph morphism $\varphi : \Gamma \to \Delta$ induces a homomorphism of abelian groups $\rho_{\varphi} : F_{\Gamma} \to F_{\Delta}$ defined as follows:

$$\rho_{\varphi}(f)(\varphi(e)) = \sum_{\varphi(e) = e'} f(e),$$

for $f \in F_{\Gamma}$ and $e' \in E(\Delta)$. Clearly, $\|\pi\| \geq \|\rho_{\varphi}(\pi)\|$ for every $\pi \in F_{\Gamma}$.

4.1. **Flows defined by words.** Let $\Gamma = (V, E)$ be an rooted folded complete inverse $X$-digraph and $w = x_1^{i_1} \ldots x_k^{i_k} \in F(X)$. The word $w$ defines a unique path $p_w$ in $\Gamma$:

$$v_0 \xrightarrow{x_1^{i_1}} v_1 \xrightarrow{x_2^{i_2}} v_2 \xrightarrow{x_3^{i_3}} \ldots \xrightarrow{x_k^{i_k}} v_k$$

where $v_0$ is the root of $\Gamma$, and a function $\pi_w : E \to \mathbb{Z}$ which associates to an edge $e$ the number of times $e$ is traversed minus the number of times $e^{-1}$ is traversed by $p_w$. It is easy to check that $\pi_w^\Gamma$ is a flow in $\Gamma$. We call $\pi_w^\Gamma$ the *flow* of $w$ in $\Gamma$. 

Figure 1. The Cayley graph of $S_3 = \langle a, b \rangle$ with $|a| = 3$ and $|b| = 2$ and the Schreier graph of $H = \langle b \rangle \leq S_3$. The straight edges correspond to $b$ and dashed ones to $a$. The values of $\pi_w$ for $w = [a^2, b]$ are shown on the edges.

Lemma 4.1 (See [11, Lemma 2.5]). For any flow $\pi : E(\Gamma) \to \mathbb{Z}$ there exists $w \in F(X)$ satisfying $\pi = \pi_{\Gamma}^w$. \hfill $\square$

In general, if $\Gamma$ is not complete, then some words can not be traced in $\Gamma$. Suppose that a reduced nontrivial word $w$ can be traced in $\Gamma$. The set of edges traversed by $w$ in $\Gamma$ forms a connected $X$-digraph called the support graph of $w$ in $\Gamma$.

Lemma 4.2. Let $\Gamma$ be a rooted folded inverse $X$-digraph and $m$ the length of a shortest cycle in $\Gamma$ (not necessarily at the root). Suppose that a reduced nontrivial word $w$ can be traced in $\Gamma$ and $\pi_{\Gamma}^w = 0$. Then $|w| \geq 3m$.

Proof. It follows from our assumption $\pi_w = 0$ that the path $p_w$ is a cycle in $\Gamma$. Let $\Delta$ be the support graph of $w$ in $\Gamma$. The rank of $\Delta$ can not be 0 ($w$ is not reduced in this case) and can not be 1 (either $w$ is not reduced or $\pi_w \neq 0$). Therefore, the rank of $\Delta$ is at least 2. Each edge of $\Delta$ is traversed by $w$ at least twice. Hence, it is sufficient to prove that $2|E(\Delta)| \geq 3m$. Let $\Delta'$ be a minimal subgraph of $\Delta$ of rank exactly 2. There are exactly two distinct configurations possible for $\Delta'$, shown in Figure 2.

Figure 2. Two configurations for support graphs in Lemma 4.2.
Let $a, b, c$ be the lengths of arcs as shown in the figure. Since, the length of a shortest cycle in $\Gamma$ is $m$, we get the following bounds for our cases:

$$\begin{cases} a + b \geq m, \\ a + c \geq m, \\ b + c \geq m. \end{cases}$$

In both cases we have $2(a + b + c) \geq 3m$ which proves that $2|E(\Delta)| \geq 3m$. Thus, $|w| \geq 3m$. \qed

**4.2. Flows on Schreier graphs.** In this section we study properties of flows on Schreier graphs. The following lemma is the most important tool in the study of groups of the type $F/N$ and is the foundation of the Magnus embedding discussed in Section 5. It is a well-known result and can be proven using algebraic topology techniques. Here we provide a proof using elementary properties of Stallings’ graphs.

**Lemma 4.3.** Let $H \leq F$, $\Delta = \text{Sch}(X; H)$, and $w \in F$. Then $\pi^\Delta_w = 0$ if and only if $w \in [H, H]$.

**Proof.** “⇒” If $w \in [H, H]$, then $\pi^\Delta_w = 0$.

“⇒” Assume that $\pi^\Delta_w = 0$. Then $w \in H$. Taking a spanning tree in $\text{Sch}(X; H)$ we can choose a good (perhaps infinite) set of generators $Y$ for $H$ corresponding to the cycles defined by positive edges outside of the spanning tree. The word $w$ can be (uniquely) expressed as a word $w = u(Y)$ in the generators $Y$. Since $\pi^\Delta_w = 0$ we should have $\sigma_g(u) = 0$ (algebraic sum of powers of $y$’s in the expression for $u$ is 0) for every $y \in Y$. This means that $u$ can be expressed as a product of commutators of elements from $\langle Y \rangle = H$. Hence $w \in [H, H]$. \qed

**Corollary 4.4.** $w = 1$ in $F/N'$ if and only if $\pi^\Gamma_w = 0$ where $\Gamma = \text{Cay}(X; N)$.

**Corollary 4.5.** Let $H \leq F$ and $m$ is the length of a shortest nonempty word in $H$. Then the length of a shortest nonempty word in $[H, H]$ is at least $3m$.

**Proof.** By Lemma 4.3 if $w \in [H, H] \setminus \{\varepsilon\}$, then $\pi^\Delta_w = 0$ in $\Delta = \text{Sch}(X; H)$. By Lemma 4.2 $|w| \geq 3m$. \qed

**Lemma 4.6.** Let $N \leq F$, $w \in F$, and $\Delta = \text{Sch}(X; \langle N, w \rangle)$. If $\pi^\Delta_w = 0$, then $w \in N$.

**Proof.** If $\pi^\Delta_w = 0$ then, by Lemma 4.3, $w \in \langle \langle N, w \rangle \rangle$. Hence, $w$ can be expressed as a product of commutators over $\langle N, w \rangle$. That expresses $w$ as a product of elements from $N$ and $w$’s with the trivial algebraic sum of powers for $w$. That product belongs to $N$ because $N$ is normal in $F$. \qed

**Corollary 4.7.** Let $N \leq F$, $w \in F \setminus N$, and $\Delta = \text{Sch}(X; \langle N, w \rangle)$. Then $\pi^\Delta_w \neq 0$.

**4.3. Flows on Cayley graphs.** Let $X = \{x_1, \ldots, x_n\}$, $F = F(X)$, $N \leq F$, $G = F/N$, and $\Gamma = \text{Cay}(X; N)$. The group $G$ acts on its Cayley graph $\Gamma$ by shifts:

$$e^g = g^{-1}h \rightarrow g^{-1}hx.$$ 

for $g \in G$ and $e = h \xrightarrow{x} hx \in E$. The following action of $ZG$ on $F_\Gamma$ turns the later into a $ZG$-module. For $c_1g_1 + \ldots + c_kg_k \in ZG$ and $f \in F_\Gamma$ define $f' = (c_1g_1 + \ldots + c_kg_k)f$ on $e = g \xrightarrow{x} gx$ to be:

$$f'(e) = c_1f(e^{g_1}) + \ldots + c_kf(e^{g_k}).$$

Denote $\pi_{x_i}$ by $\pi_i$ for $i = 1, \ldots, n$. The next lemma is straightforward.
Lemma 4.8. \( F_\Gamma \) is a free \( \mathbb{Z}G \)-module of rank \( n \) with a free basis \( \{\pi_1, \ldots, \pi_n\} \). In particular, every \( \pi \in F_\Gamma \) can be uniquely expressed as a \( \mathbb{Z}G \) linear combination of \( \pi_1, \ldots, \pi_n \). \( \square \)

The set of circulation \( C_\Gamma \) in \( F_\Gamma \) is closed under addition and the scalar \( \mathbb{Z}G \)-multiplication and hence \( C_\Gamma \) is a \( \mathbb{Z}G \)-submodule of \( F_\Gamma \).

Lemma 4.9. If \( G \) is finitely presented, then \( C_\Gamma \) is a finitely generated \( \mathbb{Z}G \)-module.

Proof. If \( N = \text{ncl}_F(r_1, \ldots, r_k) \), then \( C_\Gamma = \langle \pi_{r_1}, \ldots, \pi_{r_k} \rangle \). \( \square \)

There exists an algebraic way to define the net-flow function of \( \pi \in F_\Gamma \). Define a map \( N : F_\Gamma \to \mathbb{Z}G \) by:

\[
\pi = \alpha_1 \pi_1 + \ldots + \alpha_n \pi_n \xrightarrow{\mathcal{N}} \sum_{i=1}^{n} \alpha_i (1 - \pi_i).
\]

If \( \pi = \sum_{i=1}^{n} \sum_{g \in G} a_{g,i} g \pi_i \), then the coefficient for \( g \) in \( \mathcal{N}(\pi) \) is

\[
a_{g,1} + \ldots + a_{g,n} - a_{g \cdot (x_1) \cdot 1} - \ldots - a_{g \cdot (x_n) \cdot 1,n},
\]

which is exactly the value of the net flow at \( g \) defined by \( \pi \). Hence, \( \mathcal{N}(\pi) \) is a description of \( \mathcal{N}_\pi \) as an element of \( \mathbb{Z}G \). The next propositions are obvious.

Proposition 4.10. \( \mathcal{N} : F_\Gamma \to \mathbb{Z}G \) is a \( \mathbb{Z}G \)-module homomorphism. \( \square \)

Proposition 4.11. For any \( \pi \in F_\Gamma \) the following holds.

(a) \( \pi \in C_\Gamma \) if and only if \( \mathcal{N}(\pi) = 0 \).

(b) \( \pi \) is a flow on \( \Gamma \) if and only if \( \mathcal{N}(\pi) = 1 - g \) for some \( g \in G \). \( \square \)

Consider the augmentation map \( \epsilon : \mathbb{Z}G \to \mathbb{Z} \), defined by:

\[
\sum_{g \in G} \alpha_g g \xmapsto{\epsilon} \sum_{g \in G} \alpha_g.
\]

Recall that \( \epsilon \) is a ring homomorphism and therefore a homomorphism of \( \mathbb{Z}G \)-modules once \( \mathbb{Z} \) is assumed to be a \( \mathbb{Z}G \)-module with trivial \( G \) action.

Lemma 4.12. The sequence \( F_\Gamma \xrightarrow{\mathcal{N}} \mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z} \) is an exact sequence of \( \mathbb{Z}G \)-modules.

Proof. The inclusion \( \text{im}(\mathcal{N}) \subseteq \ker(\epsilon) \) can be easily shown by induction on \( ||\pi|| \). To show the opposite inclusion pick an arbitrary \( \sum \alpha_g g \in \ker(\epsilon) \). Then

\[
\sum \alpha_g g = \sum \alpha_g g - \sum \alpha_g = \sum \alpha_g (g - 1).
\]

If \( G \ni g = x_{i_1}^{e_1} \ldots x_{i_k}^{e_k} \), then \( g - 1 \) can be written as follows:

\[
(x_{i_1}^{e_1} \ldots x_{i_k}^{e_k} - x_{i_1}^{e_1} \ldots x_{i_k}^{e_k-1}) + (x_{i_1}^{e_1} \ldots x_{i_{k-1}}^{e_{k-1}} - x_{i_1}^{e_1} \ldots x_{i_{k-2}}^{e_{k-2}}) + \ldots + (x_{i_1}^{e_1} - 1) = \sum_{j=0}^{k-1} x_{i_j}^{e_j} (x_{i_{j+1}}^{e_{j+1}} - 1).
\]

Therefore, \( -\sum \alpha_g g = \sum \alpha_g (1 - g) \) is a linear \( \mathbb{Z}G \)-combination of the elements of the form \( (1 - \pi_i) \) and hence belongs to \( \text{im}(\mathcal{N}) \). Thus, \( \sum \alpha_g g \in \text{im}(\mathcal{N}) \). \( \square \)
5. Magnus embedding

Let $X = \{x_1, \ldots, x_n\}$, $F = F(X)$, $N \triangleleft F$, and $\Gamma = \text{Cay}(X; N)$. In this section we study relations between groups $G = F/N$ and $F/N'$. It is easy to check that the set of matrices:

$$M(X; N) = \left\{ \begin{pmatrix} g & \pi \\ 0 & 1 \end{pmatrix} \mid g \in G, \pi \in \mathcal{F}_\Gamma \right\}$$

forms a group with respect to the matrix multiplication which can be easily recognized as the wreath product $\mathbb{Z}^n \wr G$.

Let $\varphi : F \to F/N$ be the canonical epimorphism. Define a homomorphism $\mu : F \to M(X; N)$ by:

$$x_i \xrightarrow{\mu} \begin{pmatrix} 1 & \pi_i \\ 0 & 1 \end{pmatrix}, \quad x_i^{-1} \xrightarrow{\mu} \begin{pmatrix} -\pi_i & -1 \\ 0 & 1 \end{pmatrix}.$$ 

It is easy to check by induction on $|w|$ that:

$$\mu(w) = \begin{pmatrix} \overline{w} & \pi_w \\ 0 & 1 \end{pmatrix}.$$ 

**Proposition 5.1.** For any $w \in F$ if $\pi_w = 0$ then $\overline{w} = 1$.

**Proof.** Assume that $\overline{w} \neq 1$ in $F/N$. Tracing $w$ in $\Gamma$ we obtain a path $p_w$ from 1 to $wN \neq 1$. The path $p_w$ is not a circuit and the corresponding flow is not a circulation, i.e.

$$N_{\pi_w}(1) = \sum_{\alpha(e)=1} \pi_w(e) = 1.$$ 

Therefore, $\pi_w(e) \neq 0$ for some edge $e$ adjacent to 1. Thus, $\pi_w \neq 0$. □

Now note that for every $w \in F$:

$$w \in \ker(\varphi) \Leftrightarrow \pi_w = 0 \quad \text{and} \quad \overline{w} = 1$$

$$\Leftrightarrow \pi_w = 0 \quad \text{by Proposition 5.1}$$

$$\Leftrightarrow w \in N' \quad \text{by Lemma 1.3},$$

which proves the following theorem.

**Theorem 5.2 (See [7]).** Let $F = F(x_1, \ldots, x_n)$, $N \triangleleft F$, and $\varphi : F \to F/N$ be the canonical epimorphism. The homomorphism $\mu : F \to M(X; N)$ defined by

$$x \xrightarrow{\mu} \begin{pmatrix} \pi & \pi_i \\ 0 & 1 \end{pmatrix}$$

satisfies $\ker(\mu) = N'$. Therefore, $F/N' \simeq \mu(F) \leq M(X; N)$. The induced embedding $F/N' \to M(X; N)$ is called the Magnus embedding. □

5.1. Properties of Magnus embedding. The following proposition was proved in [15] using Fox derivatives. Let $g \in G$, $\pi = \sum_{i=1}^{n} \alpha_i \pi_i \in \mathcal{F}_\Gamma$, and

$$A = \begin{pmatrix} g & \sum_{i=1}^{n} \alpha_i \pi_i \\ 0 & 1 \end{pmatrix}.$$ 

**Proposition 5.3** (See [15] Theorem 2]). The following holds.

(a) $A \in \mu(F)$ if and only if $\sum_{i=1}^{n} \alpha_i(1 - \pi_i) = 1 - g$.

(b) $A \in \mu(N)$ if and only if $\sum_{i=1}^{n} \alpha_i(1 - \pi_i) = 0$. 
Proof. Follows from (2) and Proposition 4.11.

For a nontrivial $g \in G$ define a map $\tau_g : \mathcal{F}_\Gamma \rightarrow \mathcal{F}_\Gamma$ by:
\[
\pi \mapsto (1-g)\pi.
\]

Denote $\text{Sch}(N; (N, g))$ by $\Delta$. The natural projection $\varphi : \Gamma \rightarrow \Delta$ induces an abelian group homomorphism $\rho_g : \mathcal{F}_\Gamma \rightarrow \mathcal{F}_\Delta$. Properties of the functions $\tau_g$ and $\rho_g$ are very important in the study of the conjugacy problem in $F/N'$.

Lemma 5.4. The sequence $\mathcal{F}_\Gamma \xrightarrow{\tau_g} \mathcal{F}_\Gamma \xrightarrow{\rho_g} \mathcal{F}_\Delta$ is an exact sequence of abelian groups.

Proof. The image of $\tau_g$ is a subgroup of $\mathcal{F}_\Gamma$ generated by the elements $(1-g)h\pi_i$ for $h \in G$ and $i = 1, \ldots, n$. Clearly, $\rho_g((1-g)h\pi_i) = 0$.

Conversely, assume that $\pi' = \rho_g(\pi) = 0$. Let $H = \langle N, g \rangle$. Consider any edge $e' = Hh \xrightarrow{\tau_g} Hhx_i$ in $\Delta$. By definition of $\rho_g$ we get
\[
0 = \pi'(e') = \sum_{\varphi(e) = e'} \pi(e) = \sum_{g^i} \pi(Ng^ih \xrightarrow{\tau_g} Ng^ihx_i),
\]
where $g^i$ ranges over all distinct powers of $g$. It is easy to see that such $\pi$ is an integral linear combination of the elements $(1-g)h\pi_i$ and, hence, belongs to $\text{im}(\tau_g)$.

Lemma 5.5 (Cf. [4, Lemma 4]). $\ker(\tau_g)$ is not trivial if and only if $|g| = k < \infty$, in which case it is an abelian subgroup of $\mathcal{F}_\Gamma$:
\[
\ker(\tau_g) = \langle (1+g+\ldots+g^{k-1})h\pi_i \mid h \in G \text{ and } i = 1, \ldots, n \rangle.
\]

Proof. Pick any $\pi \in \mathcal{F}_\Gamma$ such that $(1-g)\pi = 0$. Then $g^j\pi = \pi$ for every $j \in \mathbb{Z}$, which can not happen if $|g| = \infty$ since $\pi$ has finite support. Assume that $|g| = k < \infty$.

It is straightforward to check that $(1+g+\ldots+g^{k-1})h\pi_i \in \ker(\tau_g)$. On the other hand if $g^j\pi = \pi$ for every $j \in \mathbb{Z}$, then the coefficients $\alpha_{h,i}$ are constant on right $\langle g \rangle$-cosets and hence are linear combinations of the generators.

Lemma 5.6. Let $\pi \in \mathcal{F}_\Gamma$ and $1 \neq g \in G$. If $(1-g)\pi \in \mathcal{C}_\Gamma$, then there exists $\pi^* \in \mathcal{C}_\Gamma$ satisfying $(1-g)\pi = (1-g)\pi^*$.

Proof. If $(1-g)\pi \in \mathcal{C}_\Gamma$, then
\[
\mathcal{N}((1-g)\pi) = \mathcal{N}(\pi) - g\mathcal{N}(\pi) = 0.
\]
Therefore, $\mathcal{N}(\pi) = g^j\mathcal{N}(\pi)$ for every $j \in \mathbb{Z}$.

CASE-I. Assume that $|g| = \infty$. Since $\mathcal{N}(\pi)$ has finite support, it must be the case that $\mathcal{N}(\pi) = 0$. Thus, $\pi \in \mathcal{C}_\Gamma$.

CASE-II. Assume that $|g| = k < \infty$. Then:
\[
\mathcal{N}(\pi) = g\mathcal{N}(\pi) = \ldots = g^{k-1}\mathcal{N}(\pi),
\]
i.e., $\mathcal{N}(\pi)$ is constant on right $\langle g \rangle$-cosets. Hence,
\[
\mathcal{N}(\pi) = (1+g+\ldots+g^{k-1})N', \quad \text{for some } N' \in \mathbb{Z}G.
\]

By Lemma 4.12, we have:
\[
0 = \epsilon(\mathcal{N}(\pi)) = \epsilon(1+g+\ldots+g^{k-1})\epsilon(N') = k\epsilon(N') \quad \text{in } \mathbb{Z},
\]
which implies that $\epsilon(N') = 0$. Hence $N' \in \ker(\epsilon)$ and by Lemma 4.12
\[
N' = \sum \alpha_i(1-\pi_i) \quad \text{for some } \alpha_i \in \mathbb{Z}G.
Put $\pi' = \alpha_1 \pi_1 + \ldots + \alpha_n \pi_n \in \mathcal{F}_\Gamma$. Notice that $N' = \mathcal{N}(\pi')$ and:

$$\mathcal{N}(\pi) = (1 + g + \ldots + g^{k-1})\mathcal{N}(\pi').$$

Define $\pi^* = \pi - (1 + g + \ldots + g^{k-1})\pi'$ and observe that:

$$\mathcal{N}(\pi^*) = 0 \text{ and } (1 - g)\pi = (1 - g)\pi^*,$$

i.e., $\pi^*$ is a required element. \qed

6. Algorithmic properties of groups $F/N(d)$

In this section we study relations between algorithmic problems in groups $\{F/N(d)\}_{d=0}^\infty$ where $N(d)$ denotes the $d$th derived subgroup of $N$.

6.1. Word problem. Here we review the relations between the word problems in groups $\{F/N(d)\}$.

**Proposition 6.1.** $\text{WP}(F/N) \Rightarrow \text{WP}(F/N')$.

**Proof.** Let $\Gamma = \text{Cay}(X; N)$. By Lemma 4.2, $w = x_{z_1}^{z_1} \ldots x_{z_k}^{z_k}$ represents the identity in $F/N'$ if and only if $\pi^\Gamma_w = 0$. To describe the function $\pi^\Gamma_w$ one needs to distinguish edges in $\Gamma$ traversed by $w$:

$$x_{i_1} \rightarrow x_{i_2} \rightarrow \ldots \rightarrow x_{i_k},$$

which can be done because by assumption $\text{WP}(F/N)$ is decidable. \qed

**Proposition 6.2.** Let $w \in F \setminus \{\varepsilon\}$. If $w = 1$ in $F/N(d)$, then $|w| \geq 3^d$.

**Proof.** Easy induction on $d$ using Lemma 1.2. \qed

**Corollary 6.3.** For any $N \leq F$ the sequence of groups $F/N(d)$ converges to $F$ in Gromov-Hausdorff topology. \qed

**Theorem 6.4.** If $\text{WP}(F/N) \in \tilde{O}(T(n))$, then $\text{WP}(F/N') \in \tilde{O}(T(n)n^2)$. Furthermore, $\text{WP}(F/N(d)) \in \tilde{O}(T(n)n^2)$ for every $d \in \mathbb{N}$.

**Proof.** It requires $|w|^2$ calls of the $\text{WP}(F/N)$ to construct the support graph for a given word $w$. Given the support graph for $w$ it is straightforward to construct the flow for $w$ in $\text{Cay}(F/N)$ as described in [18 Section 2]. By Proposition 6.2 to solve $\text{WP}(F/N(d))$ one must iterate the procedure at most $\log_3 |w|$ times. \qed

The converse to Proposition 6.4 also holds. It was first proven in [1] using Bronstein monotonocity theorem. Here we use Auslander-Lyndon theorem which gives the most straightforward algorithm.

**Theorem 6.5.** Assume that $N$ is a recursively enumerable normal subgroup of $F$ and $N'$ is recursive, then $N$ is recursive.

**Proof.** The statement is obvious for abelian $F$ or $N = \{1\}$. Assume that $F$ is not abelian and $N$ is not trivial. Then $N$ has rank at least 2. By [2] Theorem 1, for any $v \in F \setminus N$ there exists $w \in N$ such that $[v, w] \notin N'$. That gives a procedure for testing if $v \notin N$. Thus, $N$ is recursive. \qed

**Corollary 6.6.** Assume that $N$ is a recursively enumerable normal subgroup of $F$ and $\text{WP}(F/N(d))$ is decidable for some $d \in \mathbb{N}$. Then $\text{WP}(F/N(d))$ is decidable for every $d \in \mathbb{N}$. \qed
6.2. Power problem. In this section we use properties of Magnus embedding to prove that the group \( F/N \) is torsion free and to solve the power problem in \( F/N' \). Let \( \Gamma = \text{Cay}(X; N) \).

**Lemma 6.7.** For every \( w \notin N' \) and \( k \in \mathbb{N} \) we have \( \| \pi_{u,k}^\Gamma \| \geq k \).

**Proof.** Assume that \( w \notin N' \) and consider two cases.

**Case-I:** If \( w \in N \), then \( \| \pi_{u,k}^\Gamma \| \geq 1 \) and \( \| k \pi_{u,k} \| \geq k \) for every \( k \in \mathbb{N} \).

**Case-II:** Assume that \( w \notin N \) and denote \( \text{Sch}(X; (N, w)) \) by \( \Delta \). By Lemma 4.6, \( \| \pi_{u,k}^\Gamma \| \geq 1 \) and \( \| k \pi_{u,k}^\Delta \| \geq k \) for every \( k \in \mathbb{N} \). The later implies that \( \| \pi_{u,k}^\Gamma \| \geq k \). \( \square \)

**Theorem 6.8.** For every \( N \trianglelefteq F \) the group \( F/N' \) is torsion free.

**Proof.** By Lemma 6.7 if \( w \notin N' \) and \( k \in \mathbb{N} \), then \( \| \pi_{u,k}^\Gamma \| \geq k \), i.e., \( w^k \notin N' \). \( \square \)

**Lemma 6.9.** Let \( u, v \in F \) and \( u \notin N' \). If \( u^k = v \) in \( F/N' \), then \( k \leq |v| \).

**Proof.** If \( |v| < k \), then \( \| \pi_{v} \| < k \leq \| \pi_{u,k} \| \), which means that \( u^k \neq v \) in \( F/N' \). \( \square \)

**Theorem 6.10.** If \( \text{WP}(F/N) \) is decidable, then \( \text{PP}(F/N') \) is decidable. Furthermore, if \( \text{WP}(F/N) \in \tilde{O}(T(n)) \), then \( \text{PP}(F/N'(d)) \in \tilde{O}(T(L^2)L) \) for every \( d \in \mathbb{N} \), where \( L = |u| + |v| \) is the size of the input.

**Proof.** By Lemma 6.9 given \( u, v \in F \) it is sufficient to check if \( v = u^k \) in \( F/N' \) for \( k = -|v|, \ldots, |v| \) which reduces to \( 2|v| + 1 = O(L) \) calls of the word problem in \( F/N' \) for words \( v^{-1}u^{-|v|}, \ldots, v^{-1}u^{|v|} \) whose lengths are bounded by \( |v| + |u| \cdot |v| = O(L^2) \). \( \square \)

6.3. Conjugacy problem. Matthews proved in [8] Theorem B] that:

\[
\text{CP}(M(X; N)) \leftrightarrow \left\{ \begin{array}{ll}
\text{CP}(F/N), \\
\text{PP}(F/N).
\end{array} \right.
\]

The main result of this section states that restricting the conjugacy problem from \( M(X; N) \) to \( F/N' \) gives a problem equivalent to \( \text{PP}(F/N) \). In general, decidability of \( \text{CP}(F/N) \) is irrelevant to decidability of \( \text{CP}(F/N') \).

The theorem below was first proved by Remeslennikov and Sokolov in [15] for a torsion free group \( F/N \) and by C. Gupta in [8] for any finitely generated group \( F/N \).

**Theorem 6.11.** For any \( u, v \in F \) the matrices

\[
\mu(u) = \begin{pmatrix} \pi & \pi_u^\Gamma \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \mu(v) = \begin{pmatrix} \pi & \pi_v^\Gamma \\ 0 & 1 \end{pmatrix}
\]

are conjugate in \( M(X; N) \) if and only if they are conjugate in \( \mu(F) \).

**Proof.** Assume that for some \( c \in F \) and \( \pi \in F \) the matrix

\[
w = \begin{pmatrix} \pi & \pi \\ 0 & 1 \end{pmatrix} \in M
\]

conjugates \( \mu(u) \) into \( \mu(v) \). Then \( \pi^{-1}w \pi c = \pi \).

**Case-I.** If \( \pi = \pi = 1 \in F/N' \), then it is easy to check that the matrix

\[
\begin{pmatrix} \pi & \pi_v^\Gamma \\ 0 & 1 \end{pmatrix} \in \mu(F)
\]
is a conjugator for \( \mu(u) \) and \( \mu(v) \) as well.

**Case II.** Assume that \( \overline{u}, \overline{v} \neq 1 \) in \( F/N \). Conjugating \( \mu(u) \) by \( \mu(c) \) we obtain an equivalent instance of the conjugacy problem with the conjugator \( \overline{k} \) having trivial upper-left entry. Hence, from the beginning we may assume that the \( \overline{u} = 1 \) and \( \overline{v} = \overline{v} \). That gives the equality:

\[
\begin{pmatrix}
1 & -\pi \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
\overline{u} & \pi \overline{u} \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & \pi \\
0 & 1
\end{pmatrix}
= \begin{pmatrix}
\overline{v} & \pi \overline{v} \\
0 & 1
\end{pmatrix}
\]

which implies that \( \pi \overline{u} - \pi \overline{v} = (1 - \overline{u})\pi \). Since \( \pi \overline{u} - \pi \overline{v} \in C_\Gamma \), by Lemma 5.6 there exists a circulation \( \pi^* \) satisfying \( \pi \overline{u} - \pi \overline{v} = (1 - \overline{v})\pi^* \). The matrix

\[
\begin{pmatrix}
1 & \pi^* \\
0 & 1
\end{pmatrix}
\in \mu(F)
\]

is a required conjugator for \( \mu(u) \) and \( \mu(v) \).

**Theorem 6.12** (Geometry of conjugacy problem). Let \( N \subseteq F, u, v \in F, \) and \( \Delta = \text{Sch}(X, \langle N, u \rangle) \). Then \( u \sim v \) in \( F/N' \) if and only if there exists \( c \in F \) satisfying the conditions:

(a) \( \pi_u^\Delta = \pi v^\Delta \), i.e., \( \pi_u \) can be obtained by a \( \pi \)-shift of \( \pi v \) in \( \Delta \);

(b) \( \pi^{-1} \overline{u} \overline{c} = \overline{v} \) in \( F/N \).

**Proof.** By Theorem 6.11 \( u \sim v \) in \( F/N' \) if and only if \( \mu(u) \sim \mu(v) \) in \( M(X; N) \).

The conjugacy equation for \( \mu(u) \) and \( \mu(v) \) is:

\[
\begin{pmatrix}
\overline{u}^{-1} & -\overline{u}^{-1} \pi \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
\pi u^\Delta \\
0
\end{pmatrix}
= \begin{pmatrix}
\overline{v}^{-1} & -\overline{v}^{-1} \pi \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
\pi v^\Delta \\
0
\end{pmatrix}
\]

for some \( c \in F \) and \( \pi \in \mathcal{F}_\Gamma \), which is equivalent to the system:

\[
\begin{cases}
\overline{u}^{-1} \overline{c} = \overline{v}, \\
\pi u^\Delta - \pi v^\Delta = (1 - \overline{u})\pi.
\end{cases}
\]

By Lemma 5.6 the equality \( \pi u^\Delta - \pi v^\Delta = (1 - \overline{u})\pi \) holds if and only if \( \pi u^\Delta - \pi v^\Delta = 0 \).

**Theorem 6.13.** If \( \text{PP}(F/N) \) is decidable, then \( \text{CP}(F/N') \) is decidable. Furthermore, if \( \text{PP}(F/N) \in \tilde{O}(T(n)) \), then \( \text{CP}(F/N'(d)) \in \tilde{O}(T(L^d)L^4) \) for every \( d \in \mathbb{N} \), where \( L = |u| + |v| \) is the size of the input.

**Proof.** We may assume that \( u \neq 1 \) and \( v \neq 1 \) in \( F/N' \). If \( u = 1 \) in \( F/N \), then \( \pi u^\Delta = \pi v^\Delta = 0 \). If \( u \neq 1 \) in \( F/N \), then by Corollary 4.7 we get the same:

\[
\pi u^\Delta \neq 0.
\]

It shows that Case (2) in the proof of [8] Theorem B is impossible in \( F/N' \) and allows to drop decidability of \( \text{CP}(F/N) \). The rest of the proof is essentially the same as that of Case (3) in the proof of [8] Theorem B.

Let \( u = x_1 \ldots x_s \in F \) and \( v = y_1 \ldots y_t \in F \), where \( x_i, y_j \in X^\pm \). For \( i = 1, \ldots, s \) and \( j = 1, \ldots, t \) define words:

\[
u_i = x_1 \ldots x_i, \quad v_j = y_1 \ldots y_j, \quad \text{and} \quad c_{i,j} = \nu_i v_j^{-1}.
\]

Fix \( i \) such that \( \pi u^\Delta(c_{i-1} \rightarrow u_i) \neq 0 \). The set of solutions of the equation \( \pi u = \pi v \) is a finite (perhaps trivial) union of some cosets \( \langle u \rangle c_{i,j} \) in \( F/N \). Since \( \text{PP}(F/N) \) is decidable we can directly check the equalities \( \pi u^\Delta = \pi c_{i,j} \pi v^\Delta \) and \( \pi^{-1} \overline{u} \overline{c}_{i,j} \overline{v} = \overline{v} \).

Hence, \( \text{CP}(F/N') \) is decidable.
For every \( c_{i,j} \) the described algorithm for \( \text{CP}(F'/N') \) makes \( O(L^2) \) calls of the subroutine for \( \text{PP}(F/N) \) with instances \((u, c_{i,j}v_ku_l)\) of size \( O(L) \). Hence overall complexity of testing \( \pi_u = \pi v \) is \( O(T(L^2)L^2L^2L) \). Thus the claimed complexity bound. \( \square \)

Our next goal is to prove the converse of Theorem 6.13.

**Proposition 6.14.** Let \( N \subseteq F \) and \( u, v \in F \setminus N \) satisfy \([u, v] = 1 \) in \( F/N \). Then \( v \in \langle u \rangle \) in \( F/N \) if and only if \( u \sim u[w, v] \) in \( F/N' \) for every \( w \in \langle N, u \rangle \).

**Proof.** Let \( \Delta = \text{Sch}(X; \langle N, u \rangle) \). Since \([u, v] = 1 \) in \( F/N \), we have \( v^{-1}\langle N, u \rangle v = \langle N, u \rangle \), i.e., \( v \) normalizes \( \langle N, u \rangle \). Therefore, \( \langle N, u \rangle \subseteq \langle N, u, v \rangle \) and the following holds.

\[
v \in \langle u \rangle \text{ in } F/N \iff v \in \langle N, u \rangle \text{ in } F
\]

\[
\iff \pi_{[w, v]} = 0 \text{ and } [w, v] = 1 \text{ in } F/N, \quad \forall w \in \langle N, u \rangle
\]

\[
\iff \pi_{u}^{\Delta} = \pi_{u[w, v]}^{\Delta} \text{ and } u = u[w, v] \text{ in } F/N, \quad \forall w \in \langle N, u \rangle.
\]

By Theorem 6.12 the later holds if and only if \( u \sim u[w, v] \) in \( F/N' \) for every \( w \in \langle N, u \rangle \).

**Theorem 6.15** (Cf. [1] Lemma 1). Assume that \( N \) is recursively enumerable. Then \( \text{CP}(F'/N') \Rightarrow \text{PP}(F/N) \).

**Proof.** By assumption \( \text{CP}(F'/N') \) is decidable. Hence \( \text{WP}(F'/N') \) is decidable and \( \text{WP}(F/N) \) is decidable. Consider an arbitrary instance \( u, v \in F \) of \( \text{PP}(F/N) \). Our goal is to decide if \( v \in \langle u \rangle \) in \( F/N \), or not.

- If \( v = 1 \) in \( F/N \), then the answer is YES.
- If \( u = 1 \) in \( F/N \) and \( v \neq 1 \), then the answer is NO.
- If \([u, v] \neq 1 \) in \( F/N \), then the answer is NO.

Hence, we may assume that \( u \neq 1 \), \( v \neq 1 \), and \([u, v] = 1 \) in \( F/N \).

To test if \( v \in \langle u \rangle \) in \( F/N \) we run a process that checks if \( v = u^k \) in \( F/N \) for some \( k \in \mathbb{Z} \). To test if \( v \notin \langle u \rangle \) in \( F/N \) we enumerate all words \( w \in \langle N, u \rangle \) and solve the conjugacy problem for words \( u \) and \( u[w, v] \) in \( F/N' \). By Proposition 6.14 if \( v \notin \langle u \rangle \) then a negative instance will be found eventually. \( \square \)

6.4. **Relations among the algorithmic problems.** Theorems 6.10, 6.13 and 6.15 give the following diagram of problem reducibility for a finitely generated recursively presented group \( F/N \):

\[
\begin{array}{cccccc}
\text{WP}(F/N) & \leftarrow & \text{WP}(F'/N') & \leftarrow & \text{WP}(F''/N'') & \leftarrow & \ldots \\
\uparrow & & \uparrow & & \uparrow & & \\
\text{PP}(F/N) & \leftarrow & \text{PP}(F'/N') & \leftarrow & \text{PP}(F''/N'') & \leftarrow & \ldots \\
\downarrow & & \downarrow & & \downarrow & & \\
\text{CP}(F/N) & \leftarrow & \text{CP}(F'/N') & \leftarrow & \text{CP}(F''/N'') & \leftarrow & \ldots
\end{array}
\]

Below we deduce a few corollaries in the spirit of [9] Problem 12.98.

**Corollary 6.16.** For every recursively enumerable \( N \subseteq F \) the following holds.
(a) $\text{PP}(F/N')$ is decidable if and only if $\text{PP}(F/N'')$ is decidable.
(b) $\text{CP}(F/N'')$ is decidable if and only if $\text{CP}(F/N''')$ is decidable.
(c) $\text{WP}(F/N'')$ is decidable if and only if $\text{PP}(F/N'')$ is decidable if and only if $\text{CP}(F/N'')$ is decidable. □

Existence of a group $F/N$ with decidable $\text{CP}(F/N)$ and undecidable $\text{CP}(F/N')$ was shown by Anokhin in [1] (his result is somewhat similar to Theorem 6.15, but is weaker).

**Theorem 6.17.** There exists a recursive $N \leq F$ with undecidable $\text{CP}(F/N)$ and decidable $\text{CP}(F/N')$.

**Proof.** C. Miller introduced the following construction in [12]. Let $U$ be a group given by a finite presentation:

$$U = \langle s_1, \ldots , s_n \mid R_1, \ldots , R_m \rangle.$$

Define a new group $G(U)$ with generators $q, s_1, \ldots , s_n, t_1, \ldots , t_m, d_1, \ldots , d_m$ and relations of several types:

- $t_i^{-1}qt_i = qR_i$;
- $t_i^{-1}s_ks_i = s_k$;
- $d_j^{-1}qd_j = s_j^{-1}qs_j$;
- $d_j^{-1}s_ed_j = s_e$;

for all $1 \leq i \leq m, 1 \leq j \leq n$, and $1 \leq k \leq n$. The group $G(U)$ can be easily recognized as a multiple HNN-extension of a free group on generators $\{q, s_1, \ldots , s_n\}$ with stable letters $t_1, \ldots , t_m, d_1, \ldots , d_m$. Hence, $\text{WP}(G(U))$ is decidable (by Britton lemma) by computing reduced forms. Cyclically reduced forms are computable as well and hence $\text{PP}(G(U))$ is decidable. Miller proved in [12] that $\text{CP}(G(U))$ is decidable if and only if $\text{WP}(U)$ is decidable. Thus, choosing a finitely presented group $U = \langle X \mid R \rangle$ with undecidable word problem we obtain a group $G(U)$ with the required property. □

Examples of finitely presented groups $F/N$ with solvable word problem and undecidable power problem exist (can be deduced from [14, Corollary 1]). For such groups $\text{CP}(F/N')$ is undecidable and $\text{CP}(F/N'')$ is decidable. This shows that, in general, both implications of [9, Problem 12.98(b)] fail.

6.5. Some combinatorial problems for groups $F/N(d)$. In [10], the authors introduce a number of certain decision, search and optimization algorithmic problems in groups, such as subset sum problem, knapsack problem, and bounded submonoid membership problem. These problems are collectively referred to as knapsack-type problems and deal with different generalizations of the classical knapsack and subset sum problems over $\mathbb{Z}$ to the case of arbitrary groups. Here we consider the subset sum problem and its generalization called acyclic graph membership problem as defined in [3].

**SSP**$(G)$, subset sum problem in $G$: Given $g_1, \ldots , g_k, g \in G$ decide if

$$g = g_1^{\varepsilon_1} \cdots g_k^{\varepsilon_k}$$

for some $\varepsilon_1, \ldots , \varepsilon_k \in \{0, 1\}$.
AGP($G$), acyclic graph membership problem in $G$: Given $g \in G$ and a finite acyclic oriented graph $\Gamma$ labeled by words in $X^\pm$, decide whether there is a directed path in $\Gamma$ labeled by a word $w$ such that $w = g$ in $G$.

These problems were shown to be hard in a vast class of groups. Below we prove that they are hard in most groups of the type $F/N(d)$.

**Theorem 6.18.** If $\text{WP}(F/N) \in \text{P}$, $N \neq \{1\}$, and $[F : N] = \infty$, then $\text{SSP}(F/N')$ and hence $\text{AGP}(F/N')$ are $\text{NP}$-complete.

**Proof.** We claim that under our assumptions the following holds.

(a) $N/N'$ is free abelian of infinite rank.

(b) It requires polynomial time to find $n$ linearly independent elements in $N/N'$.

Clearly, (a) and (b) allow us to reduce zero-one equation problem (ZOE) known to be $\text{NP}$-complete to $\text{SSP}(F/N')$.

Choose a shortest nontrivial relation $u$ in $F/N$. By $B_c$ we denote the ball of radius $|u|$ in $F/N$ centered at $c$. Put $c_1 = 1$. Choose $c_2 \in F/N \setminus B_{c_1}$ and, in general, choose:

$$c_{n+1} \in F/N \setminus (B_{c_1} \cup \ldots \cup B_{c_n}).$$

It is clear from the choice of $c_i$'s that the set of elements:

$$\{c_iuc_i^{-1} \mid i = 1, \ldots, n\},$$

freely generates a free abelian group of rank $n$ in $F/N'$. Furthermore, it requires polynomial time in $n$ to construct such a set. □

**Proposition 6.19.** If $\text{WP}(F/N) \in \text{P}$ and $[F : N] < \infty$, then $\text{SSP}(F/N')$ is in $\text{P}$.

**Proof.** The condition $[F : N] < \infty$ implies that the group $F/N'$ is virtually abelian (of finite rank). Then by [10], Theorem 3.3, $\text{SSP}(F/N') \in \text{P}$. □

**Theorem 6.20.** Let $N \triangleleft F$. Then $\text{SSP}(F/N') \in \text{P}$ if and only if $\text{WP}(F/N) \in \text{P}$ and either $N = \{1\}$ or $[F : N] < \infty$.

**Corollary 6.21.** Let $N \triangleleft F$. Then $\text{AGP}(F/N') \in \text{P}$ if and only if $\text{WP}(F/N) \in \text{P}$ and either $N = \{1\}$ or $[F : N] < \infty$.

**Corollary 6.22.** Let $\{1\} \lhd N \lhd F$ and $\text{WP}(F/N) \in \text{P}$. Then $\text{SSP}(F/N(d))$ is $\text{NP}$-complete for every $d \geq 2$.

This gives an example of a class of groups with $\text{NP}$-hard subset sum problem which Gromov-Hausdorff limit has subset sum problem in $\text{P}$.

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